Isotopic Symmetry Breaking in the $\eta(1405) \to f_0(980)\pi^0 \to \pi^+\pi^-\pi^0$. Decay through a $K \bar{K}$ Loop Diagram and the Role of Anomalous Landau Thresholds

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Anomalous isotopic symmetry breaking in the $\eta(1405) \to f_0(980)\pi^0 \to \pi^+\pi^-\pi^0$ decay through a mechanism featuring anomalous Landau thresholds in the form of logarithmic triangle singularities, i.e., through the $\eta(1405) \to (K^*\bar{K} + K^*\bar{K}) \to (K^+K^- + K^0\bar{K}^0)\pi^0 \to f_0(980)\pi^0 \to \pi^+\pi^-\pi^0$ transition, has been analyzed. It has been shown that this effect can be correctly quantified only by taking into account the nonzero $K^*$ width. Different scales of isotopic symmetry breaking associated with the $K^+ - K^0$ mass difference are compared.

In the measurements of the $J/\psi \to \gamma\pi^+\pi^-\pi^0$ and $J/\psi \to \gamma\pi^0\pi^0\pi^0$ decays carried out by the BESIII collaboration in 2012, a resonant peak at $\sim 1.4$ GeV with a width near 50 MeV was revealed in the three-pion mass spectra [1]. Additionally, a narrow structure with a width near 10 MeV was observed in the corresponding $\pi^+\pi^-$ and $\pi^0\pi^0$ mass spectra at $\sim 990$ MeV near the $K^+K^-$ and $K^0\bar{K}^0$ thresholds [1]. Thereby, the isospin-violating decay $J/\psi \to \gamma\eta(1405) \to \gamma f_0(980)\pi^0$ was with the subsequent $f_0(980) \to \pi^+\pi^-, \pi^0\pi^0$ decay was observed for the first time (with a statistical significance more than 10σ). Its branching ratio was measured in [1] as

$$BR(J/\psi \to \gamma\eta(1405) \to \gamma f_0(980)\pi^0 = \pi^+\pi^-\pi^0) = (1.50 \pm 0.11 \pm 0.11) \times 10^{-5}.$$  \hspace{1cm} (1)

Taking into account the PDG data, the BESIII collaboration obtained the ratio [1]

$$\frac{BR(\eta(1405) \to f_0(980)\pi^0 \to \pi^+\pi^-\pi^0)}{BR(\eta(1405) \to a_0^0(980)\pi^0 \to \eta\pi^0\pi^0)} = (17.9 \pm 4.2)\%,$$  \hspace{1cm} (2)

which practically rules out attributing the observed isotopic symmetry breaking to the $a_0^0(980) - f_0(980)$ mixing through the $a_0^0(980) \to (K^+K^- + K^0\bar{K}^0)$ transition to $f_0(980)$ transition. At the same time, the observation of a narrow resonance-like structure near the $K^+K^-$ and $K^0\bar{K}^0$ thresholds in the $\pi^+\pi^-$ and $\pi^0\pi^0$ mass spectra in the $\eta(1405) \to \pi^+\pi^-\pi^0$, $\pi^0\pi^0\pi^0$ mass spectra suggests that the mechanism of $f_0(980)$ formation in the $\eta(1405) \to f_0(980)\pi^0 \to 3\pi$ decay is similar to that of the $a_0^0(980) - f_0(980)$ mixing [2, 3].

In other words, this mechanism is described by the $\eta(1405) \to (K^+K^- + K^0\bar{K}^0)\pi^0 \to a_0(980)\pi^0 \to f_0(980)\pi^0 \to 3\pi$ transition, whose amplitude is nonzero owing to the $K^+K^-$ mass difference and is significantly large in a narrow region between the $K^+K^-$ and $K^0\bar{K}^0$ thresholds.

Comparing the BESIII result [1] with the PDG data [3] for the dominant decay channel $J/\psi \to \gamma\eta(1405/1475) \to \gamma K\bar{K}\pi$, we obtain

$$\frac{BR(J/\psi \to \gamma\eta(1405/1475) \to \gamma K\bar{K}\pi)}{BR(J/\psi \to \gamma\eta(1405/1475) \to \gamma K\bar{K}\pi)} = (0.53 \pm 0.13)\%.$$  \hspace{1cm} (4)

This value also implies that isospin symmetry is very strongly broken in the $\eta(1405) \to f_0(980)\pi^0$ decay.

In what follows, we will try to theoretically describe the strong isotopic symmetry breaking in the $\eta(1405) \to f_0(980)\pi^0 \to \pi^+\pi^-\pi^0$ decay in terms of anomalous Landau thresholds (or logarithmic triangle singularities) present in the amplitude of the $\eta(1405) \to (K^*\bar{K} + K^*\bar{K}) \to K\bar{K}\pi$ decay near the $KK$ thresholds (see Fig. 1). The authors of [4] attempted to describe the $\eta(1405) \to f_0(980)\pi^0 \to \pi^+\pi^-\pi^0$ decay by this mechanism with the $K^*(892)$ vector meson in the intermediate state treated as a stable particle. Our subsequent analysis [10] demonstrated that, if its finite width $\Gamma_{K^*} \approx \Gamma_{K^*K^*K\pi} \approx 50$ MeV is taken into account, logarithmic singularities in the amplitude are smoothed and the computed probability of the $\eta(1405) \to f_0(980)\pi^0 \to \pi^+\pi^-\pi^0$ decay is reduced.

![Figure 1: Diagram of the $\eta(1405) \to (K^*\bar{K} + K^*\bar{K}) \to K\bar{K}\pi$ transition.](https://example.com/figure1.png)
by a factor of 6–8 as compared to that for \( \Gamma_{K^0} = 0 \). Also assuming the dominance of the \( \eta(1405) \to (K^*K + K\bar{K}) \to K\bar{K}\pi^0 \) decay, we obtained in \(^{10}\) the estimate

\[
BR(J/\psi \to \gamma\eta(1405) \to \gamma f_0(980)\pi^0 \to \gamma\pi^+\pi^-\pi^0) \approx 1.12 \cdot 10^{-5},
\]

(5)

which is in reasonably good agreement with the BESII measurement (1).

In contrast to \(^{10}\), we demonstrate here in detail how the inclusion of the nonzero \( K^* \) width affects the imaginary and real parts of the isospin-violating amplitude, effectively removing the logarithmic singularity, and how a narrow resonance structure arises in the \( \pi^+\pi^- \) mass spectrum of the \( \eta(1405) \to f_0(980)\pi^0 \to \pi^+\pi^-\pi^0 \) decay. Apart from that, we demonstrate for the first time that the phase of the \( f_0(980) \) transition amplitude changes abruptly by 90° between the \( K^+K^- \) and \( K^0\bar{K}^0 \) thresholds.

To elucidate the impact of the nonzero \( K^* \) width on the isospin-violating transition diagrammatically shown in Fig. \( \text{I} \) we neglect the spin effects that significantly complicate the intermediate calculations \(^{10}\) but weakly affect the final results. In other words, in what follows, the \( K^* \) meson is assumed to be a spinless particle.\(^3\)

We denote the triangle-loop amplitude (see Fig. \( \text{I} \)) as

\[
T = \frac{g_1 g_2 g_3}{16\pi} |F_+(s_1, s_2) - \mathcal{F}_0(s_1, s_2)| \, ,
\]

(6)

where \( g_1, g_2, \) and \( g_3 \) are the coupling constants in the three vertices assumed to be equal for the charged and neutral channels; the amplitudes \( F_+(s_1, s_2) \) and \( \mathcal{F}_0(s_1, s_2) \) describe the contributions of the charged and neutral intermediate states, respectively; and a factor of 2 appears because there are two such contributions. For the discussed diagram, we assume that isotopic symmetry breaking arises only from the mass difference between the charged and neutral stable \( K \) mesons, and set \( m_{K^{*+}} = m_{K^{0\alpha}} = 0.8955 \) GeV. The amplitude \( F_+ \equiv F_+(s_1, s_2) \) has the form

\[
F_+ = \frac{i}{\pi^2} \int \frac{d^4k}{D_1 D_2 D_3} \, ,
\]

(7)

where \( D_1 = (k^2 - m_{K^{*+}} - i\varepsilon), \) \( D_2 = ((p_1 - k)^2 - m_{K^-} - i\varepsilon), \) and \( D_3 = ((k - p_2)^2 - m_{K^+} - i\varepsilon) \) are the inverse propagators of the particles forming the loop. In the region of \( s_1 \geq (m_{K^{*+}} + m_{K^+})^2 \) and \( s_2 \geq 4m_{K^0}^2 \), the imaginary part of \( F_+ \) includes the term determined by the jump across the \( K^+K^- \) cut in the \( s_1 \) variable and the term determined by the jump across the \( K^0\bar{K}^0 \) cut in the \( s_2 \) variable:

\[
\text{Im}F_+ = \text{Im}F_+^{(K^+K^-)} + \text{Im}F_+^{(K^0\bar{K}^0)} \, .
\]

(8)

Here,

\[
\text{Im}F_+^{(K^+K^-)} = \frac{1}{\sqrt{\Delta}} \ln \frac{\alpha_+ + \sqrt{\Delta \delta_+}}{\alpha_+ - \sqrt{\Delta \delta_+}},
\]

(9)

\[
\text{Im}F_+^{(K^0\bar{K}^0)} = \frac{1}{\sqrt{\Delta}} \ln \frac{\alpha'_+ + \sqrt{\Delta \delta'_+}}{\alpha'_+ - \sqrt{\Delta \delta'_+}},
\]

(10)

\[
\Delta = s_1^2 - 2s_1(s_2 + m_{K^0}^2) + (s_2 - m_{K^0}^2)^2, \]

\[
\alpha_+ = s_1^2 - s_1(s_2 + m_{K^0}^2 + m_{K^{*+}}^2 - m_{K^+}^2) + (s_2 - m_{K^0}^2)(m_{K^{*+}}^2 - m_{K^+}^2), \]

\[
\delta_+ = s_2^2 - 2s_2(m_{K^{*+}}^2 + m_{K^+}^2) + (m_{K^{*+}}^2 - m_{K^+}^2)^2, \]

\[
\alpha'_+ = s_2(s_2 - s_1 - m_{K^0}^2 - 2m_{K^{*+}}^2 + 2m_{K^+}^2), \]

\[
\delta'_+ = s_2(s_2 - 4m_{K^0}^2).
\]

The amplitude \( F_0 \equiv F_0(s_1, s_2) \) is obtained by replacing the subscript + by the subscript 0 for the functions and substituting the masses of neutral partners for the masses of charged intermediate particles in Eqs. \( \text{I} \) - \( \text{III} \).

The specificity of the considered case is that all intermediate states in the triangular diagram in Fig. \( \text{I} \) do not contribute to the \( \eta(1405) \) resonance region can be on the mass shell. This occurs for such values of the kinematic variables \( s_1 \) and \( s_2 \) for which

\[
\alpha_{+0} = \pm \sqrt{\Delta \delta_{+0}},
\]

(16)

or, equivalently,

\[
\alpha'_{+0} = \pm \sqrt{\Delta \delta'_{+0}}.
\]

(17)

This implies that, as soon as the \( K^* \) meson is hypothetically assumed to be stable, the amplitude of this triangular diagram has a logarithmic singularity in its imaginary part \( \text{II} \), \( \text{III} \). For the contributions of the intermediate \( K^{*+}K^- \) and \( K^{*0}\bar{K}^0 \) states, the loci of logarithmic singularities in the \( (\sqrt{s_2}, \sqrt{s_1}) \) plane are shown in Fig. \( \text{II} \). In the \( \eta(1405) \) mass region, these are seen to be very close to the \( K\bar{K} \) thresholds (depicted by dotted vertical lines in this and subsequent figures). Thus, the singularities of the \( K^{*+}K^- \) and \( K^{*0}\bar{K}^0 \) intermediate-state contributions at \( \sqrt{s_1} = 1.420 \) GeV sit at the \( \pi^+\pi^- \) invariant masses of \( \sqrt{s_2} \approx 0.989 \) and 0.998 GeV, respectively (see Fig. \( \text{II} \). For \( \sqrt{s_2} \) in the \( \eta(1405) \) mass region (\( \sqrt{s_1} = 1.420 \) GeV for concreteness), typical \( \sqrt{s_2} \) dependences of the real and imaginary parts of the amplitudes \( F_+(s_1, s_2) \) and

\(^3\) We also note that the discussed isospin-violating effect is independent of whether or not the triangular diagram is convergent (this equally applies to \( K\bar{K} \) loops in the case of the \( \phi_0(980) \to (K^*K^- + K^0\bar{K}^0) \to f_0(980) \) transition). This is because in the dispersive representation of the isospin-violating amplitude, the sum of subtraction constants for the contributions of charged and neutral intermediate states has a natural smallness of \( \sim (m_{K^0} - m_{K^+}) \) and, therefore, cannot enhance the isotopic symmetry breaking in the narrow mass region near the \( K^+K^- \) and \( K^0\bar{K}^0 \) thresholds.
$\sqrt{s_1}$ states is positioned at respectively. The curves touch each other, the discussed mechanism seems to induce an thresholds in the $F_3$. The dependences of $\text{Im} F_0$ have singularities and jumps, respectively.

Since the singularities of the $K^+ K^-$ and $K^0 \bar{K}^0$ contributions have different locations and do not cancel each other, the discussed mechanism seems to induce an abrupt isotopic symmetry breaking in the $\eta(1405) \to \pi^+ \pi^- \pi^0$ decay as is illustrated in Fig. 4. However, this singularity-dominated pattern is not realistic. This is because one has to take into account the nonzero $K^*$ width by averaging the amplitude over the resonant Breit-Wigner mass distribution according to the Källén-Lehmann spectral representation for the unstable-$K^*$ propagator. This effectively smooths the logarithmic singularities, thereby increasing the mutual compensation of the contributions from the $(K^+ K^- + K^0 \bar{K}^0)$ and $(K^+ K^- + K^0 \bar{K}^0)$ intermediate states. As a result, the computed $\eta(1405) \to \pi^+ \pi^- \pi^0$ width proves to be several times less than that for $\Gamma_{K^* \to K\pi} = 0$, and the isospin-violating effect is largely restricted to $\pi^+ \pi^- \pi^0$ invariant masses between the $K\bar{K}$ thresholds.

Following this strategy, we substitute the unstable-$K^*$ propagator in the Källén-Lehmann spectral form. 

$$ \frac{1}{m_{K^*}^2 - k^2 - i\epsilon} \to \int_0^\infty dm^2 \frac{\rho(m^2)}{(m_{K^*}^2 - m^2 - i\epsilon)} , $$

(18)
The ρ meson for the charged and neutral intermediate states in the triangular loop, respectively, computed taking into account the nonzero width of the intermediate K* meson. (b) Phase of the amplitude $\tilde{F}(s_1, s_2)$.

\[ \rho(m^2) = \frac{1}{\pi} \frac{m_{K^*} \Gamma_{K^*}}{(m^2 - m_{K^*}^2)^2 + \left(m_{K^*} \Gamma_{K^*}\right)^2}. \] (19)

Then, in the expressions for the amplitudes $F_{+,0}(s_1, s_2)$, the $K^*$ mass squared $m_{K^*}^2$, is replaced by the variable-mass squared $m^2$ and the amplitudes are weighted with the spectral density $\rho(m^2)$ [11-13] according to

\[ \tilde{F}_{+,0}(s_1, s_2) = \int_{(m_{K^*} + m_\pi)^2} \rho(m^2) F_{+,0}(s_1, s_2; m^2) \, dm^2. \] (20)

The $\sqrt{s_2}$ dependences of the real and imaginary parts of the weighted amplitudes $\tilde{F}_+(s_1, s_2)$ and $\tilde{F}_0(s_1, s_2)$ in the $K\bar{K}$ threshold region are illustrated in Fig. 5 for $\sqrt{s_1} = 1.420$ GeV. The singularities of the unweighted amplitudes $F_+(s_1, s_2)$ and $F_0(s_1, s_2)$ and shown in Fig. 3 are seen to be practically eliminated by taking into account the instability of the intermediate $K^*$ meson. The absolute value, imaginary and real parts, and phase of the isospin-violating triangle-loop amplitude $\tilde{F}(s_1, s_2) = \tilde{F}_+(s_1, s_2) - \tilde{F}_0(s_1, s_2)$, computed taking into account the intermediate-$K^*$ instability, are shown in Fig. 4. All characteristic irregularities of the amplitude $\tilde{F}(s_1, s_2)$ are seen to occur at the $K\bar{K}$ thresholds, and its absolute value and phase behave in much the same way as those of the $c_0^0(980) - f_0(980)$ mixing amplitude [2, 4].

It is interesting to compare the squared absolute value of the amplitude $\tilde{F}(s_1, s_2) = \tilde{F}_+(s_1, s_2) - \tilde{F}_0(s_1, s_2)$ weighted with the $K^*$ spectral function to that obtained under the assumption $\Gamma_{K^*} = 0$ (cf. Figs. 7 and 4, respectively). Note that the areas under the corresponding curves differ by nearly an order of magnitude, and that this difference arises from a nonzero $K^*$ width of 50 MeV. We also note that logarithmic triangle singularities are fully determined by conditions (16) and (17) irrespective of particle spins, and their modifications arising from the nonzero width are practically unaffected by the spin effects in the $\eta(1405) \rightarrow (K^+\bar{K} + K^*\bar{K}) \rightarrow (K^+\bar{K} - K^0\bar{K}^0)\pi^0 \rightarrow f_0(980)\pi^0 \rightarrow \pi^+\pi^-\pi^0$ decay. This has been explicitly demonstrated in [10].

The general pattern remains the same for all $\sqrt{s_1}$ values in the $\eta(1405)$ mass region. Figure 8 shows the general form of the $\pi^+\pi^-$ mass spectrum in the $\eta(1405) \rightarrow \pi^+\pi^-\pi^0$ decay obtained for the $\eta(1405)$ nominal mass, or for $\sqrt{s_1} = 1.405$ GeV, by the formula

\[ \frac{dN}{d\sqrt{s_2}} = C \sqrt{\frac{\Delta}{s_1}} \left| \tilde{F}_+(s_1, s_2) - \tilde{F}_0(s_1, s_2) \right|^2 \times \]
In the processes with isotopic symmetry breaking in the region between the $K^+K^-$ and $K^0\bar{K}^0$ thresholds induced by any mechanism for the production of an S-wave $KK$ pair with a definite isospin that is free of anomalous Landau thresholds [4, 14], such as the $a_0^0(980) - f_0(980)$ mixing [2, 4], the discussed degree reaches

$$\simeq \sqrt{2(m_{K^0} - m_{K^+})/m_{K^0}} \approx 0.127.$$ (23)

For isotopic symmetry breaking in the $\eta(1405) \to f_0(980)\pi^0 \to \pi^+\pi^-\pi^0$ decay amplitude arising from logarithmic triangle singularities in the contributions from the $(K^*\bar{K} + K^*\bar{K}^*)$ intermediate states in the $\sqrt{s_2}$ region between the $K^0\bar{K}^0$ and $K^+\bar{K}^-$ thresholds, the discussed degree is estimated [10] as

$$\simeq \left| \ln \left( \frac{\Gamma_{K^*}/2}{\sqrt{m_{K^0}^2 - m_{K^+}^2 + m_{K^*}^2}/4} \right) \right| \approx 1.$$ (24)

For the nonvanishing sum of the contributions of triangular diagrams with charged and neutral intermediate states, this estimate consistent with Fig. 8 can be obtained from, e.g., Eq. (10) upon substituting $m_{K^*}^2 - im_{K^*}\Gamma_{K^*}$ for $m_{K^*}^2$ at the singularity point. In all cases of anomalous isotopic symmetry breaking corresponding to Eqs. (23) and (24), the phase of the isotopic symmetry-violating amplitude varies by nearly $90^\circ$ across the region between the $K^+K^-$ and $K^0\bar{K}^0$ thresholds [4, 10, 14].

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