THE JULIA-CARATHÉODORY THEOREM ON THE
BIDISK REVISITED

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Abstract. The Julia quotient measures the ratio of the distance
of a function value from the boundary to the distance from the
boundary. The Julia-Carathéodory theorem on the bidisk states
that if the Julia quotient is bounded along some sequence of non-
tangential approach to some point in the torus, the function must
have directional derivatives in all directions pointing into the bidisk.
The directional derivative, however, need not be a linear function
of the direction in that case. In this note, we show that if the Julia
quotient is uniformly bounded along every sequence of nontangen-
tial approach, the function must have a linear directional deriv-
ative. Additionally, we analyze a weaker condition, corresponding
to being Lipschitz near the boundary, which implies the existence
of a linear directional derivative for rational functions.

1. Introduction

Consider the following four holomorphic functions. They all map the
bidisk $D^2$ to $D$, and they all have a singularity at $(1, 1)$. However, the
nature of the singularity gets progressively worse.

\[
\begin{align*}
\phi_1(z) &= \frac{-4z_1z_2^2 + z_2^2 + 3z_1z_2 - z_1 + z_2}{z_2^2 - z_1z_2 - z_1 - 3z_2 + 4} \\
\phi_2(z) &= \frac{(z_2 - z_1) - 2(1 - z_1)(1 - z_2)\log\left(\frac{1+z_2}{1-z_2}\right)}{(z_2 - z_1) + 2(1 - z_1)(1 - z_2)\log\left(\frac{1+z_2}{1-z_2}\right)} \\
\phi_3(z) &= \frac{3z_1z_2 - 2z_1 - z_2}{3 - z_1 - 2z_2} \\
\phi_4(z) &= \prod_{n=1}^{\infty} \frac{2(1 - 2^{-n}) - z_1 - z_2}{2 - (1 - 2^{-n})(z_1 + z_2)}.
\end{align*}
\]
In one variable, the Julia-Carathéodory theorem asserts that for a holomorphic function that maps $\mathbb{D}$ to $\mathbb{D}$, a relatively mild regularity assumption at a boundary point results in seemingly stronger forms of regularity [6, 8]. Variants of their results on the bidisk have been studied in [1–5, 7, 9]. In [3] two notions of regularity, a weaker one called a B point and a stronger one called a C point, were studied (we shall give definitions in Section 2). Letting $\chi$ denote the point $(1,1)$ on the distinguished boundary of $\mathbb{D}^2$, the function $\phi_1$ above is the only one with a C point at $\chi$; the first three functions have a B point at $\chi$, and $\phi_4$ does not.

In this note, we give an alternative characterization of C points (Theorem 2.8). We also introduce a new notion of regularity, called a B+ point, which is intermediate between B point and C point. We analyze these points in Theorem 2.6), and show that the function $\phi_2$ above has a B+ point at $\chi$, and $\phi_3$ does not. We prove that if a function is rational, every B+ point is actually a C point (Theorem 2.9).

2. B, B+ and C points

In the bidisk, we shall measure distances with the supremum norm. If $\tau \in \partial \mathbb{D}^2$, we shall say that a set $S \subset \mathbb{D}^2$ is non-tangential at $\tau$ if there exists a constant $M > 0$ such that

\[(2.1) \quad \text{dist}(z, \tau) \leq M \text{dist}(z, \partial \mathbb{D}^2) \quad \forall z \in S.\]

The least $M$ that satisfies (2.1) we shall call the aperture of $S$. We shall say that a statement holds non-tangentially at $\tau$ if, for every set $S$ that is non-tangential at $\tau$, there exists $\varepsilon > 0$ such that the statement holds on $S \cap B(\tau, \varepsilon)$.

We shall let $\Pi$ denote the upper half plane. A Pick function in $d$ variables is a holomorphic function from $\Pi^d$ to $\overline{\Pi}$.

For the rest of this section, we shall assume that $\varphi : \mathbb{D}^2 \to \overline{\mathbb{D}}$ is analytic and $\tau \in \mathbb{T}^2$ is in the distinguished boundary of the bidisk.

The point $\tau$ is called a B point for $\varphi$ if either (A) or (B) of the following equivalent conditions hold:

**Theorem 2.2.** [3] The following are equivalent:

(A) \[ \liminf_{z \to \tau} \frac{1 - |\varphi(z)|}{1 - \|z\|} < \infty. \]

(B) There exists an $\omega \in \mathbb{T}$ such that for every nontangential set $S$ there exists $\alpha > 0, \varepsilon > 0$, such that for every $z \in S \cap B(\tau, \varepsilon)$,

\[(2.3) \quad \|\varphi(z) - \omega\| \leq \alpha \|\tau - z\|. \]
Moreover, if these conditions hold, then for every \( h \in \mathbb{C}^2 \) such that \( \tau + th \in \mathbb{D}^2 \) for small positive \( t \), then the directional derivative

\[
D\varphi(\tau)[h] := \lim_{t \to 0} \frac{\varphi(\tau + th) - \omega}{t}
\]

exists, and is given by

\[
(2.4) \quad D\varphi(\tau)[h] = -\omega \tau_2 h_2 \eta \left( \frac{\tau_2 h_2}{\tau_1 h_1} \right),
\]

where \( \eta(w) \) and \(-w \eta(w)\) are Pick functions in one variable.

We shall say that a point is a B+ point if the constant \( \alpha \) in (2.3) can be chosen independently of the non-tangential set \( S \).

**Definition 2.5.** We say that \( \varphi \) has a B\(^+\)-point at \( \tau \) if there exists an \( \omega \in \mathbb{T} \), \( \alpha > 0 \), such that for every nontangential set \( S \) there exists an \( \varepsilon > 0 \), such that for every \( z \in S \cap B(\tau, \varepsilon) \),

\[
\|\varphi(z) - \omega\| \leq \alpha \|\tau - z\|.
\]

We show the following theorem characterizing \( B^+ \) points.

**Theorem 2.6.** The following are equivalent:

1. \( \varphi \) has a \( B^+ \) point at \( \tau \).
2. \( \varphi \) has a \( B^- \) point at \( \tau \) and for some constant \( \alpha \)

\[
|D\varphi(\tau)[h]| \leq \alpha\|h\|.
\]

3. There exist an \( \omega \in \mathbb{T} \), non-negative real numbers \( A \) and \( B \), and a bounded analytic function \( g \) on the domain \( \mathbb{C} \setminus [-1,1] \) such that

\[
D\varphi(\tau)[h] = Ah_1 + Bh_2 + (h_2 - h_1)g \left( \frac{h_1 + h_2}{h_1 - h_2} \right).
\]

The point \( \tau \) is called a C point for \( \varphi \) if either condition (C) or (D) below holds. They say that \( D\varphi(\tau)[h] \) is a linear function of \( h \).

**Theorem 2.7.** [3] The following are equivalent:

(C) There exist \( \omega \in \mathbb{T} \) and \( \lambda \in \mathbb{C}^2 \) such that for every nontangential set \( S \) and every \( \beta > 0 \), there exists \( \varepsilon > 0 \) such that for every \( z \in S \cap B(\tau, \varepsilon) \),

\[
\|\varphi(z) - \omega - \lambda \cdot z\| \leq \beta \|\tau - z\|.
\]

(D) There exist \( \omega \in \mathbb{T} \) and \( \lambda \in \mathbb{C}^2 \) such that \( \lim_{z \to \tau} \varphi(z) = \omega \) and \( \lim_{z \to \tau} \nabla \phi(z) = \lambda \).

Thus:
• At a $B$-point,
  \[ \varphi(z) = \omega + O(\tau - z) \text{ non-tangentially.} \]
• At a $B^+$-point,
  \[ \varphi(z) = \omega + O(\tau - z) \text{ uniformly non-tangentially.} \]
• At a $C$-point
  \[ \varphi(z) = \omega + \eta \cdot z + o(\|\tau - z\|) \text{ non-tangentially.} \]

We prove that if the lim inf in condition (A) is replaced by a non-tangential \( \limsup \), we get another characterization of $C$ points:

**Theorem 2.8.** Let \( \varphi : \mathbb{D}^2 \to \overline{\mathbb{D}} \) be analytic and \( \tau \in \mathbb{T}^2 \). If
\[
\limsup_{z \to \tau} \frac{1 - |\varphi(z)|}{1 - \|z\|} < \infty,
\]
then \( \varphi \) has a $C$-point at \( \tau \). (Here the \( \limsup \) is over all sequences which approach \( \tau \) non-tangentially.)

We also show that for rational functions, $B^+$ points are $C$ points.

**Theorem 2.9.** Let \( \varphi : \mathbb{D}^2 \to \overline{\mathbb{D}} \) be rational. Then if \( \tau \in \partial \mathbb{D}^2 \) is a $B^+$ point for \( \varphi \), it is a $C$ point.

3. Proofs

In [4, Thm. 5.3], Agler, Tully-Doyle and Young gave the following characterization of the functions \( \eta \) arising in Theorem 2.2. In [4, Thm 6.2], they showed that all such \( \eta \) can arise. (We introduce a factor of 4 in the following theorem just for notational convenience later).

**Theorem 3.1.** [4] The functions \( \eta(w) \) and \( -w \eta(w) \) are both in the Pick class of one variable if and only if there is a positive measure \( \mu \) supported on \([-1, 1]\) such that
\[
(3.2) \quad \eta(w) = -4 \int_{-1}^{1} \frac{1}{(1-t)+(1+t)w}d\mu(t).
\]
Moreover, for any \( \eta \) satisfying (3.2), there exists a holomorphic \( \varphi : \mathbb{D}^2 \to \mathbb{D} \) with a $B$ point at \( \chi \) satisfying (2.4).

Note that any function \( \eta \) given by (3.2) is analytic on \( \mathbb{C} \setminus (-\infty, 0] \) and is real-valued on the positive real axis.

A Pick function of two variables is called homogeneous of degree one if
\[
f(wz_1, wz_2) =wf(z_1, z_2)
\]
whenever \((z_1, z_2), (wz_1, wz_2) \in \Pi^2\). There is a one-to-one correspondence between homogeneous Pick functions of two variables and functions \(\eta\) of the form (3.2).

**Proposition 3.3.** Let \(f\) be a homogeneous Pick function of degree one in 2 variables. Define \(\eta\) by

\[
\eta(w) = e^{is} f(-\frac{e^{-is}}{w}, -e^{-is})
\]

whenever \((-\frac{e^{-is}}{w}, -e^{-is})\) is in \(\Pi^2\). Then \(\eta\) is well-defined, and both \(\eta(w)\) and \(-w\eta(w)\) are in the Pick class of one variable. Conversely, if \(\eta(w)\) and \(-w\eta(w)\) are in the Pick class, then the function \(f\) defined by

\[
f(z_1, z_2) = \begin{cases} 
-z_2 \eta(\frac{z_2}{z_1}) & \text{if } \frac{z_2}{z_1} \in \Pi \\
-z_2 \eta(\frac{\bar{z}}{z_1}) & \text{if } \frac{\bar{z}}{z_1} \in \Pi
\end{cases}
\]

is a homogeneous Pick function of degree one.

**Proof:** Let \(f\) be given. Homogeneity means (3.4) is independent of \(s\). For any \(w \in \Pi\), if \(s\) is positive and small enough, the point \((-\frac{e^{-is}}{w}, -e^{-is})\) is in \(\Pi^2\), and letting \(s \downarrow 0\), we get that \(\eta(w) \in \Pi\). Similarly, for \(s\) positive and sufficiently small,

\[-w\eta(w) = e^{is/2} f(e^{-is/2}, e^{-is/2}w)\]

is in \(\Pi\).

Conversely, let \(\eta\) be given. Since \(\eta\) is real-valued on the positive real-axis, both definitions of \(f\) agree when \(z_2\) is a positive multiple of \(z_1\), so by continuity (3.5) defines \(f\) for any \(z \in \Pi^2\). By inspection, the formula is homogeneous of degree one. It remains to show that \(f\) maps \(\Pi^2\) to \(\Pi\).

Let \(z \in \Pi^2\), and first assume that \(\arg(z_2/z_1) \in [0, \pi]\). Let \(\theta_r = \arg(z_r), r = 1, 2\). Then as both \(\eta(z_2/z_1)\) and \(-z_2/z_1)\eta(z_2/z_1)\) are in \(\Pi\), we have

\[0 < \arg \eta(z_2/z_1) < \pi\]

\[0 < \arg \eta(z_2/z_1) + \theta_2 - \theta_1 - \pi < \pi.\]

By (3.5), we have

\[\arg f(z_1, z_2) = \theta_2 + \arg \eta(z_2/z_1) - \pi,\]

and this is sandwiched between \(\theta_1\) and \(\theta_2\). Similarly, if \(\arg(z_2/z_1) \in (-\pi, 0]\), we get that \(\arg f(z_1, z_2)\) is between \(\theta_2\) and \(\theta_1\). \(\Box\)
Combining Proposition 3.3 with Theorem 3.1, we get that a function \( f \) is in the Pick class of two variables and homogeneous of degree one if and only if it can be represented as

\[
(3.6) \quad f(z_1, z_2) = 4 \int_{-1}^{1} \frac{z_1 z_2}{(1 + t)z_1 + (1 - t)z_2} d\mu(t).
\]

Moreover, if \( \phi \) has a B point at \( \tau \), then (2.4) becomes

\[
(3.7) \quad D\phi(\tau)(i\tau_1 z_1, i\tau_2 z_2) = i\phi(\tau)f(z_1, z_2).
\]

We shall rewrite (3.6) in a way that makes it easier to read off regularity.

**Lemma 3.8.** If \( f \) is of the form (3.6), then \( f \) can be written

\[
(3.9) \quad f(z_1, z_2) = (\mu_0 - \mu_1)z_1 + (\mu_0 + \mu_1)z_2 + (z_2 - z_1)g \left( \frac{z_1 + z_2}{z_2 - z_1} \right),
\]

where \( \mu_i \) is the \( i \)-th moment of \( \mu \) and \( g \) is a Pick function of one variable such that \( \text{Im } g \) is the Poisson integral of the measure \( (1 - t^2)\mu(t) \).

**Proof.** Let us calculate \( f(\zeta - 1, \zeta + 1) \) for \( \zeta \in \Pi \).

\[
f(\zeta - 1, \zeta + 1) = 4 \int_{-1}^{1} \frac{(\zeta - 1)(\zeta + 1)}{(1 + t)(\zeta - 1) + (1 - t)(\zeta + 1)} d\mu(t)
\]

\[
= 2 \int_{-1}^{1} \frac{1 - \zeta^2}{t - \zeta} d\mu(t)
\]

\[
= 2 \int_{-1}^{1} \frac{1 - t^2}{t - \zeta} + \frac{t^2 - \zeta^2}{t - \zeta} d\mu(t)
\]

\[
= 2 \int_{-1}^{1} \frac{1 - t^2}{t - \zeta} + \zeta + td\mu(t)
\]

\[
= 2\mu_0 \zeta + 2\mu_1 + 2 \int_{-1}^{1} \frac{1 - t^2}{t - \zeta} d\mu(t)
\]

\[
= 2\mu_0 \zeta + 2\mu_1 + 2 \int_{-1}^{1} \frac{1}{t - \zeta} (1 - t^2) d\mu(t)
\]

Define the Pick function \( g \) by

\[
(3.10) \quad g(\zeta) = \int_{-1}^{1} \frac{1}{t - \zeta} (1 - t^2) d\mu(t).
\]
Notice that $g$ is holomorphic on $\mathbb{C} \setminus [-1, 1]$, and satisfies $g(\zeta) = g(\overline{\zeta})$.

Let $z \in \Pi^2$. If $z_1 \neq z_2$, define

\[ w = \frac{z_2 - z_1}{2}, \]
\[ \zeta = \frac{z_2 + z_1}{z_2 - z_1}, \]

so $z_1 = w(\zeta - 1)$ and $z_2 = w(\zeta + 1)$. Since $z_2$ cannot be a negative multiple of $z_1$, we get that $\zeta$ is never in $[-1, 1]$, so (3.9) holds provided $z_1 \neq z_2$. By continuity, it holds for all $z \in \Pi^2$.

Note that from (3.10), we see that $\limsup_{s \to \infty} |isg(is)| < \infty$. Using Lemma 3.8, we obtain that a rational homogeneous Pick function which is linearly bounded must be linear, which in turn will imply that for rational functions, a $B^+$ point is a $C$-point.

**Lemma 3.11.** Let $f$ be a homogeneous Pick function of degree 1 in two variables represented as in (3.9). There exists a $C > 0$ such that

\[ \|f(z_1, z_2)\| \leq C\|z\| \]

if and only if $g$ is a bounded function on $\mathbb{C} \setminus [-1, 1]$. If in addition $f$ is rational, then $g$ is identically 0.

**Proof.** The ($\Rightarrow$) implication is trivial. For the converse, assume $f$ is a homogeneous Pick function of degree 1 in two variables, and represent it as in (3.9). Then we have

\[
C\|z\| \geq \|f(z_1, z_2)\| \\
\geq -\| (\mu_0 - \mu_1)z_1 \| - \| (\mu_0 + \mu_1)z_2 \| + \| (z_2 - z_1) g \left( \frac{z_1 + z_2}{z_2 - z_1} \right) \| \\
\geq -|\mu_0 - \mu_1| \|z\| - |\mu_0 + \mu_1| \|z\| + \| (z_2 - z_1) g \left( \frac{z_1 + z_2}{z_2 - z_1} \right) \|.
\]

So

\[
(C + 2\mu_0)\|z\| \geq \| (z_2 - z_1) g \left( \frac{z_1 + z_2}{z_2 - z_1} \right) \|.
\]

Making the substitution $z = (\zeta - 1, \zeta + 1)$, we get that

\[
(C + 2\mu_0)(\|\zeta\| + 1) \geq \|2g(\zeta)\|
\]

for all $\zeta \in \mathbb{C} \setminus [-1, 1]$. Since $g$ is the Cauchy transform of a compactly supported measure, it vanishes at $\infty$, and so (3.12) yields that $g$ is bounded on $\mathbb{C} \setminus [-1, 1]$.

If $f$ is rational, then

\[
g(\zeta) = \frac{1}{2}[f(\zeta - 1, \zeta + 1)] - \mu_0 \zeta - \mu_1.
\]
is also rational, and since it vanishes at $\infty$ and has no poles, it is is identically zero.

**Proof of Theorem 2.6:** (1) $\iff$ (2): By Theorems 2.2 and 3.1, the derivative of $\varphi$ at $\tau$ is a Pick function of two variables, homogeneous of degree 1, which can therefore be represented as in (3.9). From (3.7), we have that for every $h$ that points into $D^2$, for $t > 0$:

$$\varphi(\tau + th) - \varphi(\tau) = i \varphi(\tau) t f(-i\bar{\tau}h_1, -i\bar{\tau}h_2) + o(t).$$

At a B+ point, the left-hand side is bounded by $\alpha t \|h\|$, which means

$$|D\varphi(\tau)[h]| = |f(-i\bar{\tau}h_1, -i\bar{\tau}h_2)|$$

is bounded by $\alpha \|h\|$.

Conversely, assume

$$|D\varphi(\tau)[h]| \leq \alpha \|h\|,$$

and fix a set $S$ that is non-tangential at $\tau$. Since $S$ is non-tangential, it is enclosed in a cone with apex at $\tau$ and some aperture $M$, so there exists some $\varepsilon > 0$ and some compact connected set $K$ of vectors in $\mathbb{C}^2$ with unit norm such that

$$S \cap B(\tau, \varepsilon) \subseteq \{\tau + th : 0 < t < \varepsilon, \ h \in K\} \subseteq D^2.$$

For each $0 < t < \varepsilon$, let

$$\Psi_t(h) = \frac{\varphi(\tau + th) - \varphi(\tau)}{t}.$$  

The functions $\Psi_t(h)$ are continuous on $K$, and

$$\lim_{t \downarrow 0} \Psi_t(h) = D\varphi(\tau)[h]$$

exists. Since $K$ is compact, the convergence is uniform. So for all $\beta > 0$, there exists $\delta > 0$ such that

$$|\Psi_t(h) - D\varphi(\tau)[h]| < \beta \ \forall \ 0 < t < \delta.$$

Therefore

$$|\varphi(\tau + th) - \varphi(\tau)| \leq (\alpha + \beta)t \ \forall \ 0 < t < \delta,$$

which means

$$|\varphi(z) - \varphi(\tau)| \leq (\alpha + \beta)\|\tau - z\| \ \forall \ z \in S \cap B(\tau, \delta).$$

(2) $\iff$ (3) This follows from (3.13) and Lemma 3.11. 

**Proof of Theorem 2.9:** If $\varphi$ is rational, then $D\varphi(\tau)[h]$ is a rational function of $h$, so $f$ is rational. By Theorem 2.6 and Lemma 3.11, if $f$ is rational then $g$ is zero, so $f$ is linear, and hence $\tau$ is actually a C point. 

$\square$
We now prove two more lemmata for Theorem 2.8.

**Lemma 3.14.** Let $f$ be a homogeneous Pick function of degree 1 in two variables which satisfies

$$(3.15) \quad \text{Im} f(z_1, z_2) \leq \gamma \max(\text{Im} z_1, \text{Im} z_2)$$

for some $\gamma \geq 0$. Then $f$ is linear.

*Proof.* Let

$$f(z_1, z_2) = (\mu_0 - \mu_1)z_1 + (\mu_0 + \mu_1)z_2 + (z_2 - z_1)g\left(\frac{z_1 + z_2}{z_2 - z_1}\right)$$

as in (3.9), where $g$ is the Cauchy transform of the measure $(1-t^2)d\mu(t)$ on $[-1, 1]$, and $\text{Im} g$ is the Poisson transform of this measure.

We will show $g$ must equal to 0.

Since $(\mu_0 - \mu_1)z_1 + (\mu_0 + \mu_1)z_2$ satisfies (3.15) for some $\gamma$, we get that

$$h(z_1, z_2) = (z_2 - z_1)g\left(\frac{z_1 + z_2}{z_2 - z_1}\right)$$

must satisfy, for some perhaps different $\gamma$,

$$\text{Im} h(z_1, z_2) \leq \gamma \max(\text{Im} z_1, \text{Im} z_2).$$

Under the substitution $z_1 = \zeta - 1, z_2 = \zeta + 1$ we get that

$$(3.16) \quad 2\text{Im} g(\zeta) \leq \gamma \text{Im} \zeta.$$

As $\text{Im} g$ is bounded in $\Pi$, and $\text{Im} g(t + iy)dt$ converges weak-star to $(1-t^2)d\mu(t)$ as $y \to 0^+$, (3.16) shows that $\text{Im} g$, and hence $g$, is identically 0. □

**Lemma 3.17.** Suppose

$$\gamma = \limsup_{z \to \tau} \frac{1 - |\varphi(z)|}{1 - \|z\|^2}$$

is finite. The Pick function in two variables

$$f(z_1, z_2) = -i\varphi(\tau)D\varphi(\tau)(\tau_1 z_1, \tau_2 z_2)$$

satisfies

$$(3.18) \quad \text{Im} f(z_1, z_2) \leq \gamma \max(\text{Im} z_1, \text{Im} z_2).$$

*Proof.* Without loss of generality, assume $\tau = \chi = (1, 1)$ and $f(\tau) = 1$. Note $\limsup_{z \to \tau} \frac{1 - |\varphi(z)|^2}{1 - |\varphi(z)|^2}$ is finite for $r = 1, 2$. Consider the limit along
the ray $z = 1 - th$ as $t \downarrow 0$, for any $h$ pointing into $\mathbb{D}^2$. Since we chose $\tau = \chi$, this means $\text{Re } (h_r) < 0$ for $r = 1, 2$.

$$\limsup_{z \rightarrow \tau} \frac{1 - |\varphi(1 - th)|^2}{1 - |1 - th_r|^2} \leq \gamma.$$  

So

$$\limsup_{t \rightarrow 0^+} \frac{1 - (\text{Re } \varphi(1 - th))^2 - (\text{Im } \varphi(1 - th))^2}{1 - (\text{Re } 1 - th_r)^2 - (\text{Im } 1 - th_r)^2} \leq \gamma.$$  

Applying l’Hospital’s rule, we get

$$\frac{\text{Re } D\varphi(\tau)[h]}{\text{Re } h_r} \leq \gamma.$$  

That is,

$$\text{Re } D\varphi(\tau)[h] \leq \gamma \text{Re } h_r.$$  

As $z = ih$, we get (3.18).  

\begin{proof}
\end{proof}

\textbf{Proof of Theorem 2.8:} This now follows from Lemmata 3.17 and 3.14.

\begin{proof}
\end{proof}

4. Examples

Consider the functions $\phi_1$ to $\phi_4$ from the Introduction. Note that

$$\phi_4(r, r) = B(r)$$

where $B$ is the Blaschke product with zeroes at $(1 - 2^{-n})$. This is an interpolating sequence, so even the limit $\lim_{r \uparrow 1} |\phi_4(r, r)|$ does not exist. In particular, $\chi$ fails to be a B point for $\phi_4$.

For $\phi_3$, we get

$$\lim_{r \uparrow 1} \frac{1 - |\varphi(r\chi)|}{1 - r} = 1,$$

so $\chi$ is a B point. A calculation yields

$$D\phi_3(\chi)[h] = -\frac{3h_1h_2}{h_1 + 2h_2}.$$  

Choosing $h = (-\varepsilon + 2i, -\varepsilon - i)$, we see that this is not $O(||h||)$, so $\chi$ is not a B+ point.

The function $\phi_1$ is given in [3] as an example of a function with a C point. It is easy to check that

$$\phi_1(r, r) = r^2,$$

so $\chi$ is a B point. By a straightforward but lengthier calculation,

$$D\phi_1(\chi)[h] = 2h_2,$$

which is linear, so $\chi$ is a C point.
The formula for $\phi_2$ reduces to $0/0$ when $z_1 = z_2$; on the diagonal $\phi_2(z)$ should be defined to equal $-\frac{3+5z_1}{5-3z_1}$. We claim that this will give a function in the Schur class that has a B+ point at $\chi$.

To show this, we shall start with $g$ from (3.10), where we choose $\mu$ to be linear Lebesgue measure on $[-1,1]$; so

$$g(\zeta) = \int_{-1}^{1} \frac{1-t^2}{t-\zeta} dt = (1-\zeta^2) \log \left( \frac{\zeta-1}{\zeta+1} \right) - 2\zeta.$$  

The function $g$ is bounded on $\mathbb{C} \setminus [-1,1]$. From (3.9),

$$f(z_1, z_2) = 2z_1 + 2z_2 + (z_2 - z_1) g \left( \frac{z_2 + z_1}{z_2 - z_1} \right)$$

$$= \left\{ \begin{array}{ll}
-\frac{1+z_1}{z_2 - z_1} \log \left( \frac{z_1}{z_2} \right) & z_1 \neq z_2 \\
-4z_1 & z_1 = z_2.
\end{array} \right.$$  

From (3.2), we have

$$\eta(w) = -4 \int_{-1}^{1} \frac{dt}{(1-t) + (1+t)w}.$$  

Now, we follow the recipe from [4] to produce a function with a B point at $\chi$ and slope function $\eta$. For each $t \in [-1,1]$, let

$$\psi_t(z) = \left( (1-t) \frac{1+z_1}{1-z_1} + (1+t) \frac{1+z_2}{1-z_2} \right)^{-1},$$

and let

$$\psi(z) = 4 \int_{-1}^{1} \psi_t(z) dt.$$  

Each $\psi_t$, and hence also $\psi$, maps $\mathbb{D}^2$ to the right half-plane. Moreover $\chi$ is a B point for $\psi$, and

$$D\psi(\chi)[h] = -2 \int_{-1}^{1} \frac{h_1 h_2}{(1-t)h_1 + (1+t)h_2} dt.$$  

Finally, we let

$$\phi_2(z) = \frac{1 - \psi(z)}{1 + \psi(z)}$$

$$= \left\{ \begin{array}{ll}
\frac{z_2 - z_1 - 2(1-z_1)(1-z_2) \log \left( \frac{1+z_1}{1-z_1} \frac{1+z_2}{1-z_2} \right)}{1+z_2 + z_1} & z_1 \neq z_2 \\
\frac{(z_2 - z_1) + 2(1-z_1)(1-z_2) \log \left( \frac{1+z_2}{1-z_2} \frac{1+z_1}{1-z_1} \right)}{1+z_2 + z_1} & z_1 = z_2.
\end{array} \right.$$
Then \( \lim_{r \to 1} \phi_2(r, r) = 1 \) and
\[
D\phi_2(\chi)[h] = -2D\phi(\chi)[h] = -h_2\eta(h_2/h_1),
\]
so \( \phi_2 \) has the required slope function, and hence \( \chi \) is a B+ point.

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