GLOBAL CLASSICAL SOLUTIONS TO THE FREE BOUNDARY PROBLEM OF PLANAR FULL MAGNETOHYDRODYNAMIC EQUATIONS WITH LARGE INITIAL DATA

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ABSTRACT. The free boundary problem of planar full compressible magnetohydrodynamic equations with large initial data is studied in this paper, when the initial density connects to vacuum smoothly. The global existence and uniqueness of classical solutions are established, and the expanding rate of the free interface is shown. Using the method of Lagrangian particle path, we derive some $L^\infty$ estimates and weighted energy estimates, which lead to the global existence of classical solutions. The main difficulty of this problem is the degeneracy of the system near the free boundary, while previous results (cf. [4, 30]) require that the density is bounded from below by a positive constant.

1. Introduction. Magnetohydrodynamics (MHD) concerns the motion of conducting fluids in an electromagnetic field. The applications of MHD cover a broad range of physical areas, for example, liquid metals and cosmic plasmas. The dynamics of MHD are affected obviously by the magnetic field, thus the hydrodynamic and electrodynamic effects are strongly coupled. The planar magnetohydrodynamic flows with magnetic diffusion are governed by the following MHD equations (cf. [4])

\[
\begin{align*}
\rho_t + (\rho u)_\eta &= 0, \quad \eta \in \mathbb{R}, \quad t > 0, \\
(\rho u)_t + (\rho u^2 + p + \frac{1}{2}|b|^2)_\eta &= (\lambda uu)_\eta, \\
(\rho w)_t + (\rho uw - b)_\eta &= (\mu w)_\eta, \\
b_t + (ub - w)_\eta &= (\nu b)_\eta, \\
E_t + (u(E + p + \frac{1}{2}|b|^2) - w \cdot b)_\eta &= (\lambda uu + \mu w \cdot w + \nu b \cdot b + \kappa T)_\eta,
\end{align*}
\]

where $\eta, t, \rho, u \in \mathbb{R}, \ w \in \mathbb{R}^2, \ b \in \mathbb{R}^2, \ T$ denote, respectively, the spatial variable, time variable, density of the flow, longitudinal velocity, transverse velocity, the transverse magnetic field, and the temperature. The longitudinal magnetic field is a constant which is taken to be unity in [11]. The total energy of the planar...
magnetohydrodynamic flows is

\[ E = \rho \left( e + \frac{1}{2}(u^2 + |w|^2) \right) + \frac{1}{2}|b|^2, \]

where both the pressure \( p \) and the internal energy \( e \) are related to the density and temperature of the flow via the equations of states for perfect gases:

\[ p = R\rho T, \quad e = c_v T. \] (1.2)

Here \( R \) is the generic gas constant and \( c_v \) is the specific heat per volume. In what follows, \( c_v \) and \( R \) are set to be unity for convenience. The constants \( \lambda \) and \( \mu \) are the viscosity coefficients of the flow, \( \nu \) is the magnetic diffusivity constant, \( \kappa = \kappa(T) \) is the heat conductivity coefficient, and all these coefficients are independent of the magnitude and direction of the magnetic field.

We are interested in a free-boundary problem of (1.1) for \( \eta \in \mathcal{I}(t), \ t > 0 \) with the following initial-boundary conditions:

\[
\begin{align*}
(\rho, u, w, b, T)|_{t=0} &= (\rho_0, u_0, w_0, b_0, \theta_0)(\eta), \quad \eta \in \mathcal{I}(0) = (0, 1), \\
(\mathbf{w}, \mathbf{b}, \mathcal{T}_n)|_{\partial\mathcal{I}(t)} &= 0, \quad u|_{\eta=0} = 0, \quad (p - \lambda u_\eta)|_{\eta=\Gamma(t)} = 0,
\end{align*}
\] (1.3)

where \( \mathcal{I}(t) = (0, \Gamma(t)) \) is the free interval occupied by the fluids with the boundary \( \partial\mathcal{I}(t) = \{0, \Gamma(t)\} \), and \( \Gamma(t) \) represents the moving interface between the fluid and vacuum with speed \( \dot{\Gamma}(t) \).

The free boundary problems of compressible fluids, which arise in several important physical situations such as astrophysics, shallow water waves, magnetohydrodynamics and so on, have been intensively studied in the literature. When the density of fluid is always bounded from below by a positive constant, the problem is extensively studied and many important progresses are achieved during the past years. The readers may refer to [2, 11, 12, 19, 27, 28, 29, 33, 35, 38, 43] for the results of Navier-Stokes equations, for instance. For planar MHD equations, the global solution was established in [4] when the initial datum was in \( H^1 \), and the global well-posedness of classical solutions was studied in [30]. Related initial-boundary value problems for planar MHD equations in Lagrangian coordinates are studied in [15, 31].

When the fluid connects to vacuum states continuously at the free interface, strong singularity occurs near the free boundary. This case is the so-called vacuum free boundary problem. There have been some local-in-time results for weak solutions of compressible Navier-Stokes equations (cf. [21, 39]) and for smooth solutions of various compressible hydrodynamic equations (cf. [5, 6, 16, 17, 18, 14, 22]). Concerning the global-in-time results, most of the known results are for weak solutions or strong solutions, in particular for Navier-Stokes equations. Please refer to [3, 7, 8, 9, 13, 20, 23, 26, 34, 39] for instance. However, to have a better understanding on the behaviors of solutions to free boundary problems, it is very important to achieve the global-in-time regularity. For example, the velocity is required to be Lipschitz continuous so that the interface separating fluids and vacuum states is well-defined in the classical sense. The well-posedness of global smooth solutions to vacuum free boundary problems has been established very recently in [24, 42] for isentropic flows. However, to our best knowledge, the global existence of classical solution for vacuum free boundary problems of MHD equations, especially in the non-isentropic regime, is still open, due to the difficulties from the combined effects of the temperature, magnetic field and the degeneracy of the system near the free boundary.
In this paper, we show the global existence and uniqueness of classical solutions to the vacuum free boundary problems of full compressible planar MHD equations with large initial data. Since the density always connects to vacuum smoothly, the system is degenerate near the free boundary, while previous results [4, 30] require that the density of fluid is strictly positive at the free surface. We remark that the method in this paper also applied to the situations in [4, 30].

In the aforementioned studies of vacuum free boundary problems for viscous flows, the method of Lagrangian mass coordinates was often used, by which the global existence of weak solutions was given. However, in order to trace the free boundary, the regularity of velocity field is usually required to be Lipschitz with respect to spatial variables, for instance. But it is hard to derive this estimate in the case that the system is degenerate near the free boundary. As in [5, 6], we use a method of Lagrangian trajectory, so that the problem can be transformed into an equivalent initial-boundary problem in the usual sense. Based on the conservation of total energy, we first show the positive upper and lower bounds of \( \eta_x(x,t) \), where the Lagrangian variable \( \eta(x,t) \) is defined in (2.1). These crucial estimates ensure the equivalency of the original free boundary problem and resulting initial-boundary value problem. For the isentropic Navier-Stokes equations, \( \eta_x(x,t) \) can be estimated straightforwardly in terms of initial energy (cf. [42]), however, for the non-isentropic magnetohydrodynamics, there are some essential difficulty in the crucial estimates for \( \eta_x \) and \( \eta_x^{-1} \), due to the combined effects of the temperature and magnetic field. We overcome the trouble by showing first the estimates for \( \int_I \eta_x(x,t)dx \) (indeed, it give the expanding rate for the free interval), and then some new estimates for the temperature and magnetic field, that is, the bounds for \( \int_0^t \| \theta(\cdot, s) \|_{L^\infty} \, ds \) and \( \int_0^t |B|^2(x,s) \, ds \). Next, we show the weighted energy estimates for lower order derivatives of the solutions, which can give the global existence of strong solutions to (1.1)-(1.3). In particular, we obtain the estimate for \( \| v_x(\cdot, t) \|_{L^\infty} \), which guarantees the regularity of particle path \( \eta(x,t) \) up to the boundary. Finally, we estimate the higher order derivatives of solutions by delicate analysis, so that the estimates for each term in (1.1) can be improved to be non-weighted and the global existence of smooth solutions for (1.1)-(1.3) is shown. The non-weighted estimates are based on an inequality obtained in [36].

The rest of this paper is organized as follows. In Section 2, we will reformulate the free boundary problem (1.1)-(1.3) by the method of Lagrangian particle path and state the main results of this article. Section 3 will be devoted to some preliminaries. In Section 4, we derive the uniform estimates for the solutions, which gives the global existence of the classical solution.

### 2. Lagrangian reformulation and main results.

The main idea to solve the problem (1.1)-(1.3) is to transform it into an equivalent initial-boundary value problem with fixed boundary. To this end, we use \( x \) as the reference variable and define the \textit{Lagrangian variable} \( \eta(x,t) \) by

\[
\eta_t(x,t) = u(\eta(x,t),t) \quad \text{for} \quad t > 0 \quad \text{and} \quad \eta(x,0) = x, \quad x \in I. \tag{2.1}
\]

For simplicity of presentation, the initial domain is denoted as \( I := (0,1) \).

We set the \textit{Lagrangian density}, \textit{longitudinal velocity}, \textit{transverse velocity}, \textit{temperature}, and \textit{magnetic field}, respectively, by

\[
f(x,t) = \rho(\eta(x,t),t), \quad v(x,t) = u(\eta(x,t),t), \quad W(x,t) = w(\eta(x,t),t),
\]

\[
\theta(x,t) = T(\eta(x,t),t) \quad \text{and} \quad B(x,t) = b(\eta(x,t),t), \tag{2.2}
\]
for any \((x, t) \in I \times [0, T]\). We consider the situation that the fluid always occupy the region \(I(t)\), and the density vanishes at the free boundary \(\Gamma(t)\). Then the Lagrangian version of (1.1) can be written equivalently as

\[
\begin{cases}
  f_t + f_x v_x = 0 & \text{in } I \times [0, T], \\
  f \xi_t + \left(\frac{f \xi}{\eta_x} + \frac{1}{2\eta_x} (|B|^2)_x - \frac{1}{\eta_x} \left(\frac{\lambda v_x}{\eta_x}\right)_x\right) = 0 & \text{in } I \times [0, T], \\
  f W_t - \frac{B_x}{\eta_x} - \frac{1}{\eta_x} \left(\frac{\mu W_x}{\eta_x}\right)_x = 0 & \text{in } I \times [0, T], \\
  B_t + B_x \frac{v_x}{\eta_x} - \frac{W_x}{\eta_x} - \frac{1}{\eta_x} \left(\frac{\nu B_x}{\eta_x}\right)_x = 0 & \text{in } I \times [0, T], \\
  f \xi_t + f \xi \frac{v_x}{\eta_x} - \frac{1}{\eta_x} \left(\frac{\nu f \xi}{\eta_x}\right)_x = \frac{\lambda v_x^2}{\eta_x^2} + \mu \frac{|W_x|^2}{\eta_x^2} + \nu \frac{|B_x|^2}{\eta_x^2} & \text{in } I \times [0, T], \\
  f > 0 & \text{in } I \times [0, T], \\
  B = 0, \ W = 0, \ \theta_x = 0 & \text{on } \partial I \times [0, T], \\
  v(0, t) = v_x(1, t) = f(1, t) = 0 & \text{for } t \in [0, T], \\
  (f, v, W, B, \theta) = (\rho_0, u_0, w_0, b_0, \theta_0) & \text{on } I \times \{t = 0\},
\end{cases}
\]

(2.3)

Thus the system (2.3) can be rewritten as

\[
\begin{cases}
  \rho_0 v_t + \left(\frac{\rho_0 \xi}{\eta_x} + \frac{1}{2\eta_x} |B|^2\right)_x = \frac{\lambda v_x}{\eta_x} & \text{in } I \times [0, T], \\
  \rho_0 W_t - B_x = \frac{\mu W_x}{\eta_x} & \text{in } I \times [0, T], \\
  B_t + B_x \frac{v_x}{\eta_x} - W_x = \frac{\nu B_x}{\eta_x} & \text{in } I \times [0, T], \\
  \rho_0 \xi_t + \rho_0 \xi \frac{v_x}{\eta_x} = \frac{\kappa \xi}{\eta_x} + \frac{v_x^2}{\eta_x} + \mu \frac{|W_x|^2}{\eta_x} + \nu \frac{|B_x|^2}{\eta_x} & \text{in } I \times [0, T], \\
  B = 0, \ W = 0, \ \theta_x = 0 & \text{on } \partial I \times [0, T], \\
  v(0, t) = 0, \ v_x(1, t) = 0 & \text{for } t \in [0, T], \\
  (v, W, B, \theta) = (u_0, w_0, b_0, \theta_0) & \text{on } I \times \{t = 0\}.
\end{cases}
\]

(2.5)

It is well known that, for the heat-conductive fluids, \(\kappa\) is a non-decreasing function of the temperature \(\theta\). In particular, \(\kappa(\theta) \approx 1 + \theta^q, \ q > 0\) for some important physical regimes (see [1, 11]), for instance, \(q \in (4.5, 5.5)\) for the fluids with high temperature (cf. [10]). In this paper, we assume that the heat conductivity coefficient \(\kappa(\theta)\) satisfies

\[
\kappa(\theta) \in C^3 \quad \text{for any } \theta > 0,
\]
\[ c_1(1 + \theta^q) \leq \kappa(\theta) \leq c_2(1 + \theta^q), \quad q > 0, \] (2.6)

where \( q, c_1 \) (i = 1, 2) are positive constants.

Now, we are ready to state the main result of this paper.

**Theorem 2.1.** (i) (existence) Assume that \( \rho_0 \in H^1(I) \), \((u_0, w_0, b_0, \theta_0) \in H^2(I) \) satisfy \( \rho_0 > 0 \) in \( I \cup \{0\} \) and \( \theta_0 \geq \theta_0 > 0 \) in \( I \cup \partial I \).

\( \rho_0(0) = 0, \quad u_0(0) = u_{0x}(1) = 0, \quad \text{and} \quad w_0 = \theta_{0x} = b_0 = 0 \quad \text{on} \quad \partial I. \)

Set \( s_0 := \lambda u_{0x} - \rho_0 \theta_0 - \frac{1}{2} |b_0|^2, \quad \omega_0 := \mu w_{0x} + b_0, \quad \phi_0 := w_{0x} + \nu b_{0xx} - b_0 u_{0x} \) and \( \tau_0 := (\kappa \theta_{0x})_x + \lambda u_{0x}^2 + \mu |w_{0x}|^2 + \nu |b_{0x}|^2 - \rho_0 \theta_0 u_{0x}. \) Moreover, we suppose

\[ \int \rho_0^{-1} (|s_{0x}|^2 + |\omega_{0x}|^2 + |\phi_{0x}|^2 + |(\rho_0^{-1} \tau_0)_x|^2) \, dx \leq C, \] (2.7)

where \( C \) is a given constant. Then for any \( T > 0 \), there exists a unique global strong solution \((v, W, B, \theta)\) to (2.5) satisfying \( \theta > 0 \) in \( I \times [0, T] \) and

\[ (v, W, B, \theta) \in L^\infty(0, T; H^2(I)), \quad (v_t, W_t, B_t, \theta_t) \in L^2(0, T; H^1(I)), \]

\( (\sqrt{\rho_0} v_t, \sqrt{\rho_0} W_t, B_t, \sqrt{\rho_0} \theta_t) \in L^\infty(0, T; L^2(I)). \)

(ii) (regularity) Suppose additionally that \( \rho_0 \in H^3(I) \), \((u_0, w_0, b_0, \theta_0) \in H^4 \) with the following requirements: \( ||(\sqrt{\rho_0})_x||_{L^\infty(I)} \leq C, \)

\[ \int \left( |(\rho_0^{-1} s_{0x})_x|^2 + |(\rho_0^{-1} \omega_{0x})_x|^2 + |\phi_{0x}|^2 + |(\rho_0^{-1} \tau_0)_x|^2 \right) \, dx \leq C, \]

(2.9)

\[ \int |(\rho_0^{-1} s_{0x})_x - \lambda u_{0x}^2 + \rho_0 \theta_0 u_{0x} - \tau_0 - b_0 \phi_0|^2 \, dx \]

\[ + \int |(\rho_0^{-1} \omega_{0x})_x + \phi_0 - \mu u_{0x} w_{0x}|^2 \, dx \leq C, \]

(2.10)

where \( C \) is a given constant. Then the solution \((v, W, B, \theta)\) obtained in (i) satisfies

\[ (v, W, B) \in L^\infty(0, T; H^4(I)), \quad \theta \in L^\infty(0, T; H^3(I)), \]

\[ (v_t, W_t, B_t) \in L^\infty(0, T; H^2(I)), \quad \theta_t \in L^\infty(0, T; H^1(I)), \]

\( (\sqrt{\rho_0} v_{tt}, \sqrt{\rho_0} W_{tt}, B_{tt}) \in L^\infty(0, T; L^2(I)), \)

\( (v_{txt}, W_{txt}, B_{txt}, \theta_{txt}, \sqrt{\rho_0} \theta_{tt}, \theta_{txxx}) \in L^2(0, T; L^2(I)). \)

(iii) (Asymptotic behaviors) The free interface \( \Gamma(t) = \eta(1, t) \) satisfies

\[ \mathcal{C} \leq \Gamma(t) \leq d_1(1 + t), \quad t \in [0, +\infty), \] (2.12)

for some positive constants \( \mathcal{C} \) and \( d_1 \).

The proof of this theorem is given in Section 4.

**Remark 2.2.** The requirements (2.7) come from the first-order compatibility condition for \((v, W, \theta)\), which give the "initial data" for \( ||(\sqrt{\rho_0} v_t, \sqrt{\rho_0} W_t, \sqrt{\rho_0} \theta_t)||_{L^2(I)} \); while (2.9) and (2.10) are equivalent to the high-order compatibility conditions.

**Remark 2.3.** The regularity in (2.11) and Lemma 3.3 below give that

\[ (v, W, B, \theta) \in C([0, T], C^{2+\alpha}), \quad (\rho_0 v_t, \rho_0 W_t, B_t, \rho_0 \theta_t) \in C([0, T], C^\alpha), \]

with \( \eta \in C^1([0, T], C^{2+\alpha}), \) for any \( \alpha \in (0, 1/2) \), thus the global solution \((v, W, B, \theta)\) obtained in Theorem 2.1 is a classical solution to (2.5). Therefore \((\rho, u, w, b, T)\) is a classical solution to (2.3) by using (2.4) and Lemma 4.3 i.e., the transform \((\eta, t) \mapsto (x, t)\) is a bijection.
3. Preliminaries.

**Lemma 3.1.** (See [37]) Assume that $\rho_0(x) \geq 0$ for $x \in I = [a, b]$, $\int_I \rho_0 dx = M > 0$ and $| \int_I \rho_0 v dx | \leq L$. Then for $q > 0$, there exists a constant $C = C(M, L, q) > 0$, such that

$$\|v^q\|_{L^\infty(I)} \leq C\|v^q\|_{L^1(I)} + C,$$

for any $v^q \in H^1(I)$. Here $M$, $L$ and $q$ are positive constants independent of $v$.

**Remark 3.2.** The multi-dimensional version of Lemma 3.1 can be found in [10]. Moreover, when $q = 1$ and $\rho_0(x) = constant$, this lemma reduces to the Poincaré inequality.

**Lemma 3.3.** (See [32, 37]) Suppose that $X \subset Z \subset Y$ are Banach spaces and the embedding $X \subset Z$ is compact. Let $E := \{u(x,t)|u \in L^\infty(0,T;X), u_t \in L^r(0,T;Y)\}$ for $r > 1$. Then the embedding $E \subset C([0,T],Z)$ is compact.

**Notation.**

1) Throughout the rest of the paper, $C$ will denote a positive constant which does not depend on the data, but possibly on the given constant $T$. They are referred as universal and can change from one inequality to another one. Also we use $C(\delta)$ to denote a certain positive constant depending on the quantity $\delta$.

2) We will employ the notation $a \lesssim b$ to denote $a \leq Cb$, where $C$ is the universal constant as defined above.

3) In the rest of the paper, we will use the notations

$$\|\cdot\|_{L^p} := \|\cdot\|_{L^p(I)} \quad \text{and} \quad \|\cdot\|_{H^k} := \|\cdot\|_{H^k(I)},$$

for any $1 \leq p \leq +\infty$ and $1 \leq k < +\infty$.

4. Proof of Theorem 2.1. The local existence and uniqueness to (2.5) can be established similarly as in [8] by the method of finite difference scheme, thus we omit the details here. To show the global existence of strong or classical solutions, we only need to obtain uniform-in-time estimates with regularity, which is the main task of this section.

Suppose that $(v, W, B, \theta)$ is the solution to (2.5) in $I \times [0,T]$, which belong to the classes of functions in (2.11). Moreover, we assume

$$\eta_x(x,t) > 0 \quad \text{for any } (x,t) \in I \times [0,T],$$

which will be recovered later in Lemma 4.3.

4.1. Lower order estimates. In this subsection, we establish the uniform estimates in the part (i) of Theorem 2.1.

**Lemma 4.1.** Assume that (4.1) holds. Then for any $t \in [0,T]$, we have

$$\int_I \left( \frac{\rho_0}{2} v^2 + \frac{\rho_0}{2} W^2 + \rho_0 \theta + \frac{\eta_x}{2} |B|^2 \right) dx$$

$$= \int_I \left( \frac{\rho_0}{2} u_0^2 + \frac{\rho_0}{2} |w_0|^2 + \rho_0 \theta_0 + \frac{1}{2} |b_0|^2 \right) dx.$$  \hspace{1cm} (4.2)

**Proof.** Integrating the sum of $v \cdot (2.5)_1$ and $W \cdot (2.5)_2$ and $B \cdot (2.5)_3$ and $(2.5)_4$ over $I$, and using the boundary conditions $(2.5)_{5,6}$, we have

$$\frac{d}{dt} \int_I \left( \rho_0 \theta + \frac{1}{2} \rho_0 v^2 + \frac{1}{2} \rho_0 |W|^2 + \frac{1}{2} \eta_x |B|^2 \right) dx = 0$$  \hspace{1cm} (4.3)

which gives (4.2) immediately. \hspace{1cm} $\Box$
Remark 4.2. Note that, by an argument similar as in Chapter 3, \cite{10}, we can obtain that $\eta(x,t) \geq 0$ for any $(x,t) \in I \times [0,T]$. To be precise, we introduce a non-positive function $H(\theta) := \min\{-\theta, 0\}$. Observing that $\Theta H(\theta) = H(\theta)$, $H'(\theta) \leq 0$ and $H''(\theta) = 0$ a.e. $(x,t) \in I \times [0,T]$, we multiply (4.1) by $H'(\theta)$ and integrate over $I$ to get

$$
\frac{d}{dt} \int_I \rho_0 H(\theta)dx = - \int_I \rho_0 \Theta H'(\theta) \frac{v_x}{\eta_x} dx - \int_I H''(\theta) \frac{\rho_0 \theta^2}{\eta_x} dx + \int_I H'(\theta) \left( \frac{v^2}{\eta_x} + \mu \frac{|W_x|^2}{\eta_x} + \nu \frac{|B_x|^2}{\eta_x} \right) dx 
$$

$$
\leq - C \int_I \rho_0 H(\theta)dx,
$$

where $C$ is a positive constant depending on $\eta_x$ and $v_x$. Thus one gets

$$
\int_I \rho_0 H(\theta)dx(t) = 0 \quad \text{for any } t \in [0,T],
$$

using the non-positiveness of $H(\theta)$ and the assumption $\theta_0 \geq 0$. It follows that $H(\theta)(x,t) = 0$ for a.e. $(x,t) \in I \times [0,T]$, by the assumptions that $\rho_0 > 0$ in $I$. Due to the regularity of $\theta$, we obtain $\theta(x,t) \geq 0$, $(x,t) \in I \times [0,T]$.

Lemma 4.3. Suppose that (4.1) holds. Then for any $(x,t) \in I \times [0,T]$, one has

$$
C \leq \eta_x(x,t) \leq \bar{C}(1 + t)^2, \quad \text{(4.4)}
$$

$$
0 \leq \eta(x,t) \leq \bar{C}(1 + t), \quad \text{(4.5)}
$$

where $C, \bar{C}$ are positive constants independent of $T$.

Proof. Step 1. Lower bounds of $\eta_x(x,t)$ and $\eta(x,t)$. By (4.1) and the boundary condition (2.5), we find

$$
\eta(x,t) = \eta(0,t) + \int_0^x \eta_y(y,t)dy \geq \eta(0,t) = \int_0^t v(0,s)ds = 0. \quad \text{(4.6)}
$$

Integrating (2.5) from $x$ to 1 yields

$$
\lambda (\log \eta_x)_t = - \int_x^1 (\rho_0 v)_t dy + p + \frac{1}{2} |B|^2,
$$

where $p = \rho_0 \theta/\eta_x$.

Next, we also integrate the above equation in $t$ and use the initial condition that $\eta(x,0) = x$ to get

$$
\lambda \log \eta_x = - \int_x^1 (\rho_0 v - \rho_0 u_0)dy + \int_0^t p(x,s)ds + \int_0^t \frac{1}{2} |B|^2 ds,
$$

which reads

$$
\eta_x = \exp \left\{ - \frac{1}{\lambda} \int_x^1 (\rho_0 v - \rho_0 u_0)dy \right\} \exp \left\{ \frac{1}{\lambda} \int_0^t p(x,s)ds \right\} \exp \left\{ \frac{1}{\lambda} \int_0^t \frac{1}{2} |B|^2 ds \right\}.
$$

In virtue of the fact that $p = \rho_0 \theta_0^{-1} \geq 0$, $|B|^2 \geq 0$, and Lemma 4.1 one gets

$$
\eta_x \geq \exp \left\{ - \frac{1}{\lambda} \int_x^1 (\rho_0 v - \rho_0 u_0)dy \right\}.
$$
Then with the help of (4.7), we have

$$\geq \exp \left\{ -\frac{1}{\lambda} \left( \int_I \rho_0 v dx \right)^{\frac{1}{2}} \left[ \left( \int_I \rho_0 v^2 dx \right)^{\frac{1}{2}} + \left( \int_I \rho_0 \theta^2 dx \right)^{\frac{1}{2}} \right] \right\}$$

(4.8)

$$\geq C > 0.$$  

**Step 2. Upper bounds of \( \eta_x(x,t) \) and \( \eta(x,t) \).** Let

$$\Phi(x,t) := \exp \left\{ \frac{1}{\lambda} \int_0^t \left( p(x,s) + \frac{1}{2} |B|^2 \right) ds \right\}.$$  

Then with the help of (4.7), we have

$$\frac{\partial}{\partial t} \Phi(x,t) = \frac{1}{\lambda} \left( \rho_0 \theta + \frac{1}{2} \eta_x |B|^2 \right) \exp \left\{ -\frac{1}{\lambda} \int_x^t (\rho_0 v - \rho_0 u_0) dy \right\},$$

which gives

$$\Phi(x,t) = 1 + \int_0^t (\rho_0 \theta + \frac{1}{2} \eta_x |B|^2) \exp \left\{ -\frac{1}{\lambda} \int_x^t (\rho_0 v - \rho_0 u_0) dy \right\} ds.$$  

(4.9)

It follows from (4.2), (4.7) and (4.9) that

$$\eta_x = \exp \left\{ -\frac{1}{\lambda} \int_x^t (\rho_0 v - \rho_0 u_0) dy \right\} \Phi(x,t)$$

$$\leq C \left[ 1 + \int_0^t (\rho_0 \theta + \frac{1}{2} \eta_x |B|^2) ds \right].$$

(4.10)

Thus by Gronwall’s inequality

$$\eta_x \leq \exp \left\{ C \int_0^t |B|^2 (\cdot, s) ds \right\} \left[ 1 + \int_0^t \|\theta(\cdot, s)\|_{L^\infty} ds \right].$$

Moreover,

$$\int_I \eta_x dx \leq C \int_I \left[ 1 + \int_0^t (\rho_0 \theta + \frac{1}{2} \eta_x |B|^2) ds \right] dx$$

$$= C \left[ 1 + \int_0^t \left( \int_I (\rho_0 \theta + \frac{1}{2} \eta_x |B|^2) dx \right) ds \right]$$

$$\leq C (1+t).$$

(4.11)

It follows that for any \((x,t) \in I \times [0,T],\)

$$\eta(x,t) = \eta(1,t) - \int_1^x \eta_y dy \leq \eta(1,t) \leq \eta(0,t) + \int_0^t \eta_x dx \leq C (1+t).$$

(4.12)

Now it suffices to evaluate \( \int_0^t \|\theta(\cdot,s)\|_{L^\infty} ds \) and \( \int_0^t |B|^2 ds \), which is discussed in two cases as follows.

**Case (i).** \( q \geq 1. \) To avoid the possible zero denominator in the calculations, we multiply (2.5) by \( \theta_0^{-1} \) to get

$$\rho_0 (\log \theta_0) + \frac{\rho_0 \theta_x}{\theta_0}\frac{\theta_0}{\eta_x} = \left( \frac{\kappa(\theta)\theta_x}{\eta_x} \right)_x + \frac{\kappa(\theta)\theta_x^2}{\eta_x \theta_0} + \frac{\lambda \nu_x^2 + \mu |W_x|^2 + \nu |B_x|^2}{\eta_x \theta_0},$$

where \( \theta_0 := \theta + \delta \) for any constant \( \delta \in (0,1). \) Integrating the above equation in \( I \times (0,t) \) and using (4.8) and the assumption (2.6), we reach

$$\int_I \rho_0 |\log \theta_0| dx(t) + \int_0^t \int_I \left( \frac{1 + \theta^q \theta_x^2}{\eta_x} + \frac{\lambda \nu_x^2 + \mu |W_x|^2 + \nu |B_x|^2}{\eta_x \theta_0} \right) dx ds$$
\[
\begin{align*}
2 \int_{\{\theta_{\delta} \geq 1\}} \rho_0 |\log \theta_{\delta}| dx(t) &- \int_0^t \rho_0 \log(\theta_0 + \delta) dx + \int_0^t \int_I \frac{\rho_0 v_x}{\theta_\delta} dx ds \\
\leq & 2 \int_{\{\theta_{\delta} \geq 1\}} \rho_0 \theta_{\delta} dx(t) + C + \frac{\lambda}{2} \int_0^t \int_I \frac{v_x^2}{\theta_\delta} dx ds \\
& + C ||\eta_x^{-1}||_{L^\infty} \int_0^t \int_I \frac{\rho_0 \theta_{\delta}^2}{\eta_x} dx ds \\
\leq & C(1 + t) \int \rho_0 \theta_{\delta} dx(s) + C + \frac{1}{2} \int_0^t \int_I \frac{\lambda v_x^2 + |W_x|^2 + \nu |B_x|^2}{\eta_x \theta_{\delta}} dx ds.
\end{align*}
\]

Therefore, by (4.2), we have, for any \( t \in [0, T] \),
\[
\int \rho_0 \log \theta_{\delta} dx(t) + \int_0^t \int_I \left( \frac{(1 + \theta_\delta) \theta_{\delta}^2}{\eta_x \theta_{\delta}^2} + \frac{\lambda v_x^2 + |W_x|^2 + \nu |B_x|^2}{\eta_x \theta_{\delta}} \right) dx ds \\
\leq C(1 + t).
\]

From (4.2), Lemma 3.1 and the Young inequality, one has, for \( q \geq 1 \),
\[
\begin{align*}
\int_0^t \theta_{\delta}(\cdot, s) \eta_x dx \leq & C(q)(1 + t) + \int_0^t \theta_{\delta}^2(\cdot, s) \eta_x dx, \\
\leq & C(q)(1 + t) + \int_0^t \theta_{\delta}^2(\cdot, s) \eta_x dx, \\
\leq & C(q)(1 + t) + \sup_{s \in [0, t]} \int_I \theta_{\delta}(x, s) dx \int_0^t \int_I \eta_x \eta_x \theta_{\delta}^{-2} \theta_{\delta}^2 dx ds.
\end{align*}
\]

Plugging (4.11) and (4.13) into (4.14), then taking the limit \( \delta \to 0 \), we find, for any \( t \in [0, T] \),
\[
\int_0^t \theta_{\delta}(\cdot, s) \eta_x dx \leq C(q)(1 + t)^2, \quad q \geq 1,
\]
where \( C(q) \) is a positive constant independent of \( \delta \). Note that \( |B|^2(x, t) = \int_0^\pi |\theta_{\delta}| dx dy \), due to the boundary condition (2.5). We discover
\[
\begin{align*}
\int_0^t \|\theta_{\delta}(\cdot, s)\|_{L^\infty} ds &\leq C(q)(1 + t)^2, \quad q \geq 1.
\end{align*}
\]

Case (ii). \( q \in (0, 1) \). Multiplying (2.3) by \( \theta_{\delta}^{\alpha-\beta} \), where \( \alpha \in (0, q) \) and \( \beta \in (0, 1) \), we get
\[
\begin{align*}
\left( \frac{\rho_0 \theta_{\delta}^{\alpha-\beta}}{1 - \alpha} \right)_t + \rho_0 v_x + \frac{\kappa(\theta) \theta_{\delta}^2}{\eta_x \theta_{\delta}^{\alpha+\beta}} = \left( \frac{\kappa(\theta) \theta_{\delta}^2}{\eta_x \theta_{\delta}^{\alpha+\beta}} \right)_x + \frac{\alpha \kappa(\theta) \theta_{\delta}^2}{\eta_x \theta_{\delta}^{\alpha+\beta}} + \frac{\lambda v_x^2 + |W_x|^2 + \nu |B_x|^2}{\eta_x \theta_{\delta}^{\alpha+\beta}}.
\end{align*}
\]
Then we integrate this equation over \( I \times (0, t) \), using (4.2), (4.8) and the fact \( \delta \in (0, 1) \), to get

\[
\int_0^t \int_I \left( \frac{\theta_\delta^{q-1} - \alpha \theta_\delta^2}{\eta_x} + \frac{\lambda \nu_x + \mu |W_x|^2 + \nu |B_x|^2}{\eta_x \theta_\delta^q} \right) \, dx \, ds
\]

\[
\leq \sup_{s \in [0, t]} \int_I \rho_0 \theta_\delta^{q-1} \, dx(s) + \int_0^t \int_I \rho_0 \theta_\delta^{q-1} \eta_x^{-1} |v_x| \, dx \, ds
\]

\[
\leq C(\alpha) + \frac{1}{2} \int_0^t \int_I \frac{\nu_x^2}{\eta_x \theta_\delta^q} \, dx \, ds + C \sup_{s \in [0, t]} \int_I \rho_0 \theta_\delta \, dx(s) \int_0^t \| \theta_\delta(\cdot, s) \|_{L^\infty} \, ds,
\]

which gives

\[
\int_0^t \int_I \left( \frac{\theta_\delta^{q-1} - \alpha \theta_\delta^2}{\eta_x} + \frac{\lambda \nu_x + \mu |W_x|^2 + \nu |B_x|^2}{\eta_x \theta_\delta^q} \right) \, dx \, ds
\]

\[
\leq C(\alpha) \left( 1 + \int_0^t \| \theta + \delta \|_{L^\infty} \right). \tag{4.17}
\]

Here the constant \( C(\alpha) \) is independent of \( \delta \). Thus by Lemmas 3.1 and 4.1 and (4.17), we have

\[
\int_0^t \| \theta_\delta(\cdot, s) \|_{L^\infty} \, ds
\]

\[
\leq \int_0^t \int_I \rho_0 \theta_\delta \, dx dt + \int_0^t \int_I \theta_\delta^{q-1} \frac{\eta_x}{\eta_x} \cdot \eta_x^{-\frac{q}{2}} \theta_\delta^{1+\alpha-\alpha} \, dx dt
\]

\[
\leq C(1 + t) + \epsilon \max_{s \in [0, t]} \int_I \eta_x(x, s) \, dx \int_0^t \| \theta_\delta \|_{L^\infty} \, ds + C(\epsilon) \int_0^t \int_I \frac{\theta_\delta^{q-1} - \alpha \theta_\delta^2}{\eta_x} \, dx \, ds
\]

\[
\leq C(\alpha)(1 + t) + C(\alpha)(1 + t) \int_0^t \| \theta_\delta \|_{L^\infty} \, ds + C(\alpha, \epsilon) \int_0^t \| \theta_\delta \|_{L^\infty} \, ds.
\]

Note that \( 0 < 1 - \alpha < 1 \) and \( 0 < 1 + \alpha - q < 1 \), and take \( \alpha = q/2 \). Then we choose \( \epsilon = \epsilon(t) \) to be small enough and use the Young inequality, then let \( \delta \to 0 \) to derive

\[
\int_0^t \| \theta(\cdot, s) \|_{L^\infty} \, ds \leq C(q)(1 + t)^2, \quad q \in (0, 1). \tag{4.18}
\]

Similarly to (4.16), we derive

\[
\int_0^t |B|^2(x, s) \, ds \leq \int_0^t \left( \int_I |B|^2 \, dx \right)^{\frac{1}{2}} \left( \int_I \frac{|B_x|^2}{\eta_x \theta_\delta^q} \, dx \right)^{\frac{1}{2}} \| \theta_\delta \|_{L^\infty} \, ds
\]

\[
\leq C \left( \sup_{s \in [0, t]} \int_I \eta_x |B|^2 \, dx(s) \right)^{\frac{1}{2}} \int_0^t \left( \int_I \frac{|B_x|^2}{\eta_x \theta_\delta^q} \, dx + \| \theta_\delta \|_{L^\infty}^q \right) \, ds
\]

\[
\leq C(q)(1 + t)^2, \tag{4.19}
\]

since \( \delta, \alpha \in (0, 1) \). Submitting (4.15), (4.16), (4.18) and (4.19) into (4.10), we obtain that for any \( (x, t) \in I \times [0, T] \),

\[
\eta_x(x, t) \leq C(q)(1 + t)^2, \quad q > 0. \tag{4.20}
\]

Therefore, we show the estimates in (4.4) by combining (4.8) and (4.20). \( \square \)
Lemma 4.6. For any \((x,t) \in I \times [0,T]\). Moreover, from now on, we always use the conclusion that \(C \leq \eta_x \leq C(T)\) without additional claim.

In what follows, we always denote by \(C\) a generic constant depending possibly on \(T\) but not on the unknowns, for the sake of simplicity.

**Corollary 4.5.** The following estimates are satisfied:

\[
\int_0^t \|\theta(\cdot,s)\|_{L^\infty} ds + \int_0^t \|B(\cdot,s)\|_{L^\infty} ds \leq C, \quad t \in [0,T], \tag{4.21}
\]

\[
\theta(x,t) > 0, \quad (x,t) \in I \times [0,T]. \tag{4.22}
\]

**Proof.** The estimates \((4.21)\) follows from \((4.15)\), \((4.16)\), \((4.18)\) and \((4.19)\). Moreover, we can get \((4.22)\) from the proof of Lemma 4.3. Indeed, we derive from \((4.13)\) and \((4.4)\) that

\[
\int \rho_0 |\log \theta| dx(t) + \int_0^t \int I (|\log \theta|) \theta^2 dx ds \leq C, \quad t \in [0,T],
\]

where \(C\) is a positive constant independent of \(\delta\). Note that this inequality holds for any \(q \in (0,\infty)\). Thus by using Lemma 3.1, we obtain

\[
\int_0^t \|\log \theta(\cdot,s)\|_{L^\infty} ds \leq C.
\]

It follows that

\[
\|\log \theta(\cdot,t)\|_{L^\infty} \leq C(t) < \infty, \quad a.e. \ t \in [0,T],
\]

where \(C(t)\) is a constant depending on \(t\), but not on \(\delta\). Taking the limit \(\delta \to 0\), we discover

\[
\|\log \theta(\cdot,t)\|_{L^\infty} \leq C(t), \quad a.e. \ t \in [0,T],
\]

which gives

\[
\theta(x,t) > 0, \quad a.e. \ (x,t) \in I \times [0,T].
\]

Due to the regularity of \(\theta\), we obtain \(\theta(x,t) > 0\), for any \((x,t) \in I \times [0,T]\). The idea of this argument is inspired by [10].

**Lemma 4.6.** For any \(t \in [0,T]\), we have

\[
\int_I (v^2_x + |W|_x^2 + |B|_x^2 + \rho_0 \theta^{q+2}) dx(t) \tag{4.23}
\]

\[
+ \int_0^t \int_I (\rho_0 v^2_t + |W_t|^2 + |B_t|^2 + (1 + \theta^q \theta_2^2) dx dt \leq C,
\]

\[
\|v(\cdot,t)\|_{L^\infty} + \|W(\cdot,t)\|_{L^\infty} + \|B(\cdot,t)\|_{L^\infty} \leq C. \tag{4.24}
\]

**Proof.** Multiplying \(2.5)_{1}\) by \(v\) and integrating over \(I \times (0,t)\), we obtain

\[
\lambda \int_0^t \int_I \eta^{-1} v^2_x dx dt
\]

\[
\leq \int_I \rho_0 v^2 dx(t) + \int_I \rho_0 u^2 dx + \int_0^t \int_I \rho_0 \eta^{-1} |v| dx dt + \int_0^t \int_I |B|^2 v_x dx ds
\]

\[
\leq C + 2\delta \int_0^t \int_I \eta^{-1} v^2_x dx dt + \frac{C}{\delta} \sup_{s \in [0,t]} \int_I \rho_0 \theta dx(s) \int_0^t \|\theta(\cdot,s)\|_{L^\infty} ds
\]

\[
+ \frac{C}{\delta} \sup_t \int_I |B|^2 dx(t) \int_0^t \|B(\cdot,s)\|_{L^\infty}^2 ds.
\]
Choosing $\delta < \lambda/4$ and using (4.2), (4.4) and (4.21), we obtain
\[ \int_0^t \int v_x^2 dx dt \lesssim \| \eta_x \|_{L^\infty} \int_0^t \int \eta_x^{-1} v_x^2 dx dt \lesssim \int_0^t \| \theta(\cdot, s) \|_{L^\infty} ds \leq C. \] (4.25)

Integrating (2.5) from $x$ to 1, we get
\[ \lambda \frac{v_x}{\eta_x} = \frac{\rho_0 \theta}{\eta_x} + \frac{1}{2} |B|^2 - \int_x^1 \frac{\rho_0 \theta v_x}{\eta_x} + \frac{1}{2} |B|^2 - \frac{d}{dt} \int_x^1 \rho_0 v d y. \] (4.26)

Thus we obtain by using (4.2) and (4.21)
\[ \int_0^t \| v_x \|_{L^\infty} \leq \int_0^t (\| \theta \|_{L^\infty} + \| B \|_{L^\infty}^2) ds + \left( \int \rho_0 v^2 d x \right)^{\frac{1}{2}} + \left( \int \rho_0 v_0^2 d x \right)^{\frac{1}{2}} \leq C. \] (4.27)

Next, we integrate $v_t$ (2.5) over $I$ to get
\[ \frac{\lambda}{2} \frac{d}{dt} \int I \eta_x^{-1} v_x^2 d x + \int I \rho_0 v_t^2 d x = - \frac{\lambda}{2} \int I \eta_x^{-2} v_x^3 d x + \int I \eta_x^{-1} \rho_0 \theta v_t x d x + \frac{1}{2} \int I |B|^2 v_t x d x := J_1 + J_2 + J_3. \] (4.28)

We estimate $J_1$ and $J_2$ respectively as follows:

\[ |J_1| \lesssim \| \eta_x^{-1} v_x \|_{L^\infty} \int I \eta_x^{-1} v_x^2 d x; \]
\[ J_2 = \frac{d}{dt} \int I \eta_x^{-1} \rho_0 \theta v_x d x - \int I v_x (\eta_x^{-1} \rho_0 \theta), d x = \frac{d}{dt} \int I \eta_x^{-1} \rho_0 \theta v_x d x - \int I (v_x - \lambda^{-1} \rho_0 \theta) (\eta_x^{-1} \rho_0 \theta), d x - \frac{1}{2 \lambda} \int I \eta_x^{-1} (\rho_0 \theta)^2 d x - \frac{1}{2 \lambda} \int I \eta_x^{-2} v_x \rho_0 \theta^2 d x \]
\[ := J_{21} + J_{22} + J_{23} + J_{24}, \]

where
\[ |J_{24}| \lesssim \| \eta_x^{-1} v_x \|_{L^\infty} \int I \rho_0 \theta^2 d x, \]

and by (2.5) and (2.5),
\[ |J_{22}| = \frac{1}{\lambda} \int I (\lambda v_x \eta_x^{-1} - \rho_0 \theta \eta_x^{-1})(\eta_x^{-1} v_x \rho_0 \theta - \rho_0 \theta t), d x \]
\[ \lesssim \left| \int I [ (\lambda v_x \eta_x^{-1} - \rho_0 \theta \eta_x^{-1}) (2 \eta_x^{-1} v_x \rho_0 \theta - \lambda \eta_x^{-2} v_x^2) + \rho_0 v \eta_x^{-1} \kappa(\theta) \theta_x ] d x \right| \]
\[ \lesssim (\| \eta_x^{-1} v_x \|_{L^\infty} + \| \theta \|_{L^\infty}) \int I (\eta_x^{-1} v_x^2 + \rho_0 \theta^2) d x + \delta \int I \rho_0 v_t^2 d x + C(\delta) \int I \rho_0 \kappa^2(\theta) \theta_x^2 d x. \]

Moreover,
\[ J_3 = \frac{d}{dt} \int I |B|^2 v_x d x - 2 \int I B \cdot B_t v_x d x \]
\[ \leq \frac{d}{dt} \int I |B|^2 v_x d x + \delta \int I |B|^2 d x + C(\delta) \| B \|_{L^\infty} \int I \eta_x^{-1} v_x^2 d x. \]
Then it follows from \((4.27), (4.28)\) and the above calculations that
\[
\frac{d}{dt} \left( \eta_x^{-1} (\sqrt{v_x} - \rho_0 \theta)^2 + |B|^2 \right) dx + \int \rho_0 v_t^2 dx
\]
\[
\lesssim \left( \| \eta_x^{-1} v_x \|_{L^\infty} + \| \theta \|_{L^\infty} \right) \int \left( \eta_x^{-1} v_x^2 + \rho_0 \theta^2 \right) dx + \int \rho_0 \kappa^2(\theta) \partial_x^2 \theta dx
\]
\[
+ \delta \int |B_t|^2 dx + C(\delta) \| B \|^2_{L^\infty} \int v_x^2 dx.
\] (4.29)

On the other hand, we multiply \((2.5)\) by \(\int_0^\theta \kappa(y) dy\) and integrate to get
\[
\frac{d}{dt} \int_0^\theta \int_0^\xi \kappa(y) dy d\xi dx + \int_0^\theta \eta_x^{-1} \kappa^2(\theta) \partial_x^2 \theta dx
\]
\[
= - \int_0^\theta \int_0^\theta \rho_0 \theta \eta_x^{-1} v_x \int \kappa(y) dy dx + \int \left( \lambda \eta_x^2 \eta_x + \frac{\mu |W_x|^2}{\eta_x} + \nu \frac{|B_x|^2}{\eta_x} \right) \int \kappa(y) dy dx
\]
\[
\lesssim \| \theta(1 + \theta^2) \|_{L^\infty} \left( \int_0^\theta \rho_0 \theta |v_x| dx + \int (v_x^2 + |W_x|^2 + |B_x|^2) dx \right)
\]
\[
\lesssim \left( \int \kappa^2(\theta) \partial_x^2 \theta dx \right)^{\frac{1}{2}} \left( \int_0^\theta \rho_0 \theta |v_x| dx + \int (v_x^2 + |W_x|^2 + |B_x|^2) dx \right)
\]
\[
\leq \delta \int \kappa^2(\theta) \partial_x^2 \theta dx + C(\delta) \left[ \int_0^\theta \rho_0 \theta^2 dx + \left( 1 + \int v_x^2 dx \right) \int v_x^2 dx \right]
\]
\[
+ C(\delta) \left( \left( \int |B_x|^2 dx \right)^2 + \left( \int |W_x|^2 dx \right)^2 \right).
\] (4.30)

From \((4.2), (4.29), (4.30), (4.21), (4.25)\) and the Gronwall inequality, we have
\[
\int \left[ v_x^2 + \rho_0 (\theta^2 + \theta^2 + \theta^2) \right] dx + \int_0^t \left[ \rho_0 v_t^2 + (1 + \theta^2) \theta_x^2 \right] dx dt
\]
\[
\lesssim \int \left[ v_x^2 + \rho_0 (\theta^2 + \theta^2 + \theta^2) \right] dx + \int_0^t \left( \| \theta \|_{L^\infty} + \| B \|^2_{L^\infty} + 1 + \int v_x^2 dx \right) ds
\]
\[
+ \int B^4(t) + \delta \int_0^t \int |B|^2 dx dt + C(\delta)
\] (4.31)
\[
\leq C + C \int |B|^2 dx \int (|B|^2 + |B_x|^2) dx + \delta \int_0^t \int |B|^2 dx dt + C(\delta)
\]
\[
\leq C + C \int |B|^2 dx + \delta \int_0^t \int |B|^2 dx dt + C(\delta).
\]

Similar to the procedure of \((4.28)\), we derive
\[
\frac{\mu}{2} \frac{d}{dt} \int \eta_x^{-1} |W_x|^2 dx + \int \rho_0 |W_t|^2 dx
\]
\[
= - \frac{\mu}{2} \int \eta_x^{-2} |W_x|^2 v_x dx - \int B \cdot W_{tx} dx := L_1 + L_2.
\] (4.32)

We evaluate the two terms on the right hand side as follows:
\[
|L_1| \lesssim \| \eta_x^{-1} v_x \|_{L^\infty} \int \eta_x^{-1} |W_x|^2 dx,
\]
\[ L_2 = -\frac{d}{dt} \int_I \mathbf{B} \cdot \mathbf{W}_x dx + \int_I \mathbf{B}_t \cdot \mathbf{W}_x dx \]
\[ \leq -\frac{d}{dt} \int_I \mathbf{B} \cdot \mathbf{W}_x dx + \delta \int_I \eta_x |\mathbf{B}_t|^2 dx + C(\delta) \int_I \eta_x^{-1} |\mathbf{W}_x|^2 dx. \]

Thus we apply the Gronwall inequality to (4.32) and use (4.2) and (4.4) to get
\[ \mu \int_I \eta_x^{-1} |\mathbf{W}_x|^2 dx + \int_0^t \int_I \rho_0 |\mathbf{W}_0|^2 dx dt \]
\[ \lesssim \int_I |\mathbf{w}|^2 dx + \delta \int_0^t \int_I \eta_x |\mathbf{B}_t|^2 dx dt + C(\delta). \] (4.33)

To absorb the first term on the right side, we multiply (4.31) by \( \mathbf{B}_t \) and integrate to obtain
\[ \frac{\nu}{2} \frac{d}{dt} \int_I \eta_x^{-1} |\mathbf{B}_x|^2 dx + \int_I \eta_x |\mathbf{B}_t|^2 dx \]
\[ = -\frac{\nu}{2} \int_I \eta_x^{-2} |\mathbf{B}_x|^2 v_x dx - \int_I \mathbf{B}_t \cdot \mathbf{W}_x dx - \int_I \mathbf{B} \cdot \mathbf{B}_t v_x \] (4.34)
\[ := M_1 + M_2 + M_3. \]

Clearly,
\[ |M_1| \lesssim \|\eta_x^{-1} v_x\|_{L^\infty} \int_I \eta_x^{-1} |\mathbf{B}_x|^2 dx, \]
\[ |M_2| \leq \frac{1}{4} \int_I \eta_x |\mathbf{B}_t|^2 dx + \int_I \eta_x^{-1} |\mathbf{W}_x|^2 dx, \]
\[ |M_3| \leq \frac{1}{4} \int_I \eta_x |\mathbf{B}_t|^2 dx + \|\mathbf{B}\|_{L^\infty}^2 \int_I \eta_x^2 v_x^2 dx. \]

Applying the Gronwall inequality to (4.32) and using the above estimates, we obtain
\[ \nu \int_I \eta_x^{-1} |\mathbf{B}_x|^2 dx + \int_0^t \int_I \eta_x |\mathbf{B}_t|^2 dx \]
\[ \lesssim \int_I |\mathbf{B}_0|^2 dx + \int_0^t \int_I \eta_x^{-1} |\mathbf{W}_x|^2 dx dt + \int_0^t \|\mathbf{B}\|_{L^\infty}^2 \int_I \eta_x^2 v_x^2 dx dt. \] (4.35)

Finally, we combine (4.35) and (4.33) and employ the Gronwall inequality of integral type to get
\[ \int_I (\nu |\mathbf{B}_x|^2 + \mu |\mathbf{W}_x|^2) dx + \int_0^t \int_I (|\mathbf{B}_t|^2 + \rho_0 |\mathbf{W}_t|^2) dx \]
\[ \leq C + C \int_0^t \|\mathbf{B}\|_{L^\infty}^2 \int_I \eta_x^2 v_x^2 dx dt. \] (4.36)

Therefore we obtain (4.23) by (4.31), (4.36), (4.21) and the Gronwall inequality, and (4.24) follows from (4.2), (4.23) and Lemma 4.3 \( \square \)

**Lemma 4.7.** For any \( t \in [0, T] \), we have
\[ \int_I \left( \rho_0 (v_x^2 + |\mathbf{W}_t|^2)^2 + |\mathbf{B}_t|^2 + \kappa^2 (\theta)(\theta_t^2) \right) dx(t) \]
\[ + \int_0^t \int_I (v_{tx}^2 + |\mathbf{W}_{tx}|^2 + |\mathbf{B}_{tx}|^2 + \rho_0 \kappa(\theta)(\theta^2_t)) dx dt \leq C, \] (4.37)
\[ \|\theta(\cdot, t)\|_{L^\infty} \leq C, \] (4.38)
\[ \|v_x(\cdot, t)\|_{L^\infty} + \|\mathbf{W}_x(\cdot, t)\|_{L^\infty} + \|\mathbf{B}_x(\cdot, t)\|_{L^\infty} \leq C. \] (4.39)
Proof. Differentiating (2.5) with respect to $t$, we obtain
\[
\rho_0 v_t - \lambda (n_x^{-1} v_{tx})_x = -\lambda (n_x^{-2} v_x^2)_x + (\rho_0 \theta n_x^{-2} v_x^2 - \rho_0 \theta \eta_x^{-2} - \mathbf{B} \cdot \mathbf{B}_t)_x. 
\] (4.40)
Then we multiply the above equation by $v$ and integrate over $I$ to get
\[
\frac{1}{2} \frac{d}{dt} \int_I \rho_0 v_t^2 dx + \lambda \int_I n_x^{-1} v_{tx}^2 dx 
\leq \delta \int_I v_t^2 dx + C(\delta) \int_I (n_x^{-4} v_x^4 + \rho_0^2 \eta_x^{-2} \theta_t^2 + \rho_0^2 \theta^2 n_x^{-4} v_x^2 + |\mathbf{B}|^2 |\mathbf{B}_t|^2) dx 
\leq \delta \int_I v_t^2 dx + C(\delta) \left( \|v_x\|_{L^\infty}^2 \int_I (v_x^2 + \rho_0 \theta^2) dx + \int_I \rho_0 \theta_t^2 dx + \|\mathbf{B}\|_{L^\infty}^2 \int_I |\mathbf{B}_t|^2 dx \right). 
\] (4.41)
Note that from (4.2), (4.26) and Lemma 4.3, one has
\[
\|v_x\|_{L^\infty}^2 \lesssim \|\theta\|_{L^\infty}^2 + \int_I \rho_0 v_t^2 dx + \|\mathbf{B}\|_{L^\infty}^2 \int_I |\mathbf{B}|^2 dx, 
\] (4.42)
and
\[
\|\theta\|_{L^\infty}^2 \lesssim \int_I \theta_x^2 dx + \int_I \rho_0 \theta dx \leq \int_I \theta_x^2 dx + C. 
\] (4.43)
Therefore, submitting (4.42) and (4.43) into (4.41), we have
\[
\frac{1}{2} \frac{d}{dt} \int_I \rho_0 v_t^2 dx + \lambda \int_I n_x^{-1} v_{tx}^2 dx 
\lesssim \left( \int_I \rho_0 v_t^2 dx + \int_I \theta_x^2 dx + 1 + \|\mathbf{B}\|_{L^\infty}^2 \right) \int_I (v_x^2 + \rho_0 \theta^2) dx 
+ \int_I \rho_0 \theta_t^2 dx + \|\mathbf{B}\|_{L^\infty}^2 \int_I |\mathbf{B}_t|^2 dx, 
\]
which gives
\[
\int_I \rho_0 v_t^2 dx + \lambda \int_0^t \int_I v_{tx}^2 dx dt \lesssim \int_I \rho_0^{-1} s_x^2 dx + \int_0^t \int_I \rho_0 \theta_t^2 dx dt + C, 
\] (4.44)
by using (4.23), (4.24) and the Gronwall inequality. On the other hand, we integrate $\kappa(\theta) \theta_t \cdot (2.5)_4$ over $I$ to get
\[
\frac{1}{2} \frac{d}{dt} \int_I n_x^{-1} \kappa^2(\theta) \theta_x^2 dx + \int_I \rho_0 \kappa(\theta) \theta_t^2 dx 
= - \frac{1}{2} \int_I n_x^{-2} v_x \kappa^2(\theta) \theta_x^2 dx - \int_I n_x^{-1} v_x \rho_0 n \theta \kappa(\theta) \theta_t dx 
+ \int_I \eta_x^{-1} (\lambda v_x^2 + v_i |\mathbf{B}_x|^2 + \mu |\mathbf{W}_x|^2) \kappa(\theta) \theta_i dx 
:= A_1 + A_2 + A_3, 
\] (4.45)
where
\[
|A_1| \lesssim \|v_x\|_{L^\infty} \int_I \kappa^2(\theta) \theta_x^2 dx, 
\]
\[
|A_2| \leq \delta \int_I \rho_0 \kappa(\theta) \theta_t^2 dx + C(\delta) \|\kappa(\theta) \theta_t\|_{L^\infty} \int_I v_x^2 dx 
\leq \delta \int_I \rho_0 \kappa(\theta) \theta_t^2 dx + C(\delta) \left( \int_I \kappa^2(\theta) \theta_x^2 dx + 1 \right), 
\]
and
by (4.23) and Lemma 4.3, and
\[ A_3 = \frac{d}{dt} \int_I \eta_x^{-1}(\lambda v_x^2 + \nu |B_x|^2 + \mu |W_x|^2) \int_0^\theta \kappa(y)dydx \]
\[ - \int_I \left( \int_0^\theta \kappa(y)dy \right) [2\eta_x^{-1}(\lambda v_x v_{tx} + \nu B_x \cdot B_{tx} + \mu W_x \cdot W_{tx}) \]
\[ - \eta_x^{-2}v_x(\lambda v_x^2 + \nu |B_x|^2 + \mu |W_x|^2) \] \[ dx. \]

Note that using (4.23), (4.42), (4.43) and Lemma 4.3, we have
\[ \int_0^t |A_3|dt \lesssim \|\kappa(\theta)\|_{L^\infty} \int_I (\lambda v_x^2 + \nu |B_x|^2 + \mu |W_x|^2)dx + C \]
\[ + \delta \int_0^t \int_I (v_{tx}^2 + |B_{tx}|^2 + |W_{tx}|^2)dxdt \]
\[ + C(\delta) \int_0^t \|\kappa(\theta)\|_{L^\infty} dt \cdot \sup_t \int_I (v_x^2 + |B_x|^2 + |W_x|^2)dx(t) \]
\[ + \int_0^t \|v_x\|_{L^\infty} dt \cdot \sup_t \|\kappa(\theta)\|_{L^\infty}(t) \int_I (v_x^2 + |B_x|^2 + |W_x|^2)dx(t) \]
\[ \lesssim \left( \int_I \kappa^2(\theta)\theta_x^2 dx \right)^{\frac{1}{2}} + C + \delta \int_0^t \int_I (v_{tx}^2 + |B_{tx}|^2 + |W_{tx}|^2)dxdt \]
\[ + C(\delta) \int_0^t \int_I \kappa^2(\theta)\theta_x^2 dxdt \]
\[ \leq \delta \left( \int_I \kappa^2(\theta)\theta_x^2 dx + \int_0^t \int_I (v_{tx}^2 + |B_{tx}|^2 + |W_{tx}|^2)dxdt \right) + C_{\delta}. \]

Integrating (4.45) in (0, t) and using the above estimates of A_1 through A_3, we have, for suitably small \( \delta \),
\[ \int_I \kappa^2(\theta)\theta_x^2 dx(t) + \int_0^t \int_I \rho_0 \kappa(\theta)\theta_x^2 dxdt \]
\[ \leq \int_I \kappa^2(\theta_0)\theta_0^2 dx + \delta \int_0^t \int_I (v_x^2 + |B_x|^2 + |W_x|^2)dxdt + C(\delta). \]

Now it suffices to estimate \( \int_0^t \int_I (|B_{tx}|^2 + |W_{tx}|^2)dxdt \) to close the energy estimates. Integrating (2.5) from 0 to x gives
\[ \mu \eta_x^{-1}W_x(x,t) = -B_t(x,t) + \int_0^x \rho_0 W_t(y,t)dy, \]
\[ \nu \eta_x^{-1}B_x(x,t) = -W_t(x,t) + \int_0^x B_y\eta_ydy + \int_0^x B_vydv. \]

Thus
\[ \mu \|W_x\|_{L^\infty}^2 = \|B\|_{L^\infty}^2 + \int_I \rho_0 \|W_t\|^2dx, \]
\[ \mu \int_I \|W_x(\cdot,s)\|_{L^\infty}^2 ds \leq \int_I \|B(\cdot,s)\|_{L^\infty}^2 ds + \int_0^t \int_I \rho_0 \|W_s\|^2dxds \leq C. \]

Next, we apply \( \partial_t \) to (2.5) to get
\[ \rho_0 W_{tt} - (\mu \eta_x^{-1}W_{tx})_x = B_{tx} - \mu (\eta_x^{-2}v_x W_x)_x. \]
Similarly, we differentiate (2.5) with respect to $t$ and multiply the resulting equation by $B_t$ to get

$$
\frac{1}{2} \frac{d}{dt} \int_{I} \rho_0 |W_t|^2 dx + \mu \int_{I} \eta_x^{-1} |W_{tx}|^2 dx \\
= -\int_{I} B_t \cdot W_{tx} dx + \mu \int_{I} \eta_x^{-2} v_x W_x \cdot W_{tx} dx \\
\leq \delta \int_{I} |W_{tx}|^2 dx + C(\delta) \int_{I} (|B_t|^2 + |v_x W_x|^2) dx,
$$

(4.51)

which gives

$$
\int_{I} \rho_0 |W_t|^2 dx + \mu \int_{0}^{t} \int_{I} \eta_x^{-1} |W_{tx}|^2 dx dt \\
\leq \int_{I} \rho_0^{-1} |\omega_0|^2 dx + \int_{0}^{t} \int_{I} |B_t|^2 dx dt + \int_{0}^{t} \|W_x\|_{L^\infty}^2 dt \leq C.
$$

(4.52)

Similarly, we differentiate (4.3) with respect to $t$ and multiply the resulting equation by $B_t$ to get

$$
\frac{1}{2} \frac{d}{dt} \int_{I} \eta_x |B_t|^2 dx + \nu \int_{I} \eta_x^{-1} |B_{tx}|^2 dx \\
= -\frac{3}{2} \int_{I} v_x |B_t|^2 dx - \int_{I} B_t \cdot (B v_{tx} - W_{tx}) dx + \nu \int_{I} \eta_x^{-2} v_x B_x \cdot B_{tx} dx \\
\leq C\|v_x\|_{L^\infty} \int_{I} |B_t|^2 dx + \delta \int_{I} (v_{tx}^2 + |W_{tx}|^2 + |B_{tx}|^2) dx \\
+ C(\delta) \left(1 + \|B\|_{L^\infty}^2 \right) \int_{I} |B_t|^2 dx + \|v_x\|_{L^\infty}^2 \int_{I} |B_x|^2 dx,
$$

(4.53)

Choosing $\delta$ to be small and using the Gronwall inequality, we get

$$
\int_{I} \eta_x |B_t|^2 dx + \nu \int_{0}^{t} \int_{I} \eta_x^{-1} |B_{tx}|^2 dx dt \leq \delta \int_{0}^{t} \int_{I} (v_{tx}^2 + |W_{tx}|^2) dx dt + C(\delta),
$$

(4.54)

since

$$
\int_{I} |B_t(x,0)|^2 dx \lesssim \|b_0\|_{H^1}^2 \|u_0\|_{H^1}^2 + \|w_0\|_{H^1}^2 + \|b_0\|_{H^2}^2.
$$

Therefore (4.37) follows from (4.44), (4.46), and (4.38) is a direct consequence of (4.37) and Lemma 4.3. From (4.26), (4.37), (4.38) and (4.47), we obtain (4.39).

**Lemma 4.8.** For any $t \in [0, T]$, we have

$$
\int_{I} \rho_0 \kappa(\theta) \theta_t^2 dx(t) + \int_{0}^{t} \int_{I} \kappa^2(\theta) \theta_{tx}^2 dx dt \leq C,
$$

(4.55)

$$
\|\kappa(\theta) \theta_x(\cdot, t)\|_{L^\infty} \leq C.
$$

(4.56)

**Proof.** Applying $\partial_t$ to (2.5), we have

$$
\rho_0 \theta_t - (\eta_x^{-1} \kappa(\theta) \theta_x)_{tx} = (\rho_0 \theta_x)_t + \lambda (\eta_x^{-1} v_x^2)_t + \mu (\eta_x^{-1} |W_x|^2)_t + \nu (\eta_x^{-1} |B_x|^2)_t.
$$

(4.57)

Multiplying the above equality by $\kappa(\theta) \theta_t$ and integrating the resulting equation over $I$ yield

$$
\frac{1}{2} \frac{d}{dt} \int_{I} \rho_0 \kappa(\theta) \theta_t^2 dx + \int_{I} \eta_x^{-1} |(\kappa(\theta) \theta_t)|^2 dx,
$$
Lemma 4.3:

\[-\frac{1}{2} \int_\Omega \rho_0 \kappa'(\theta) \theta_t^3 dx + \int_\Omega \eta_x^{-2} v_x \kappa(\theta) \theta_x (\kappa(\theta) \theta_t)_x dx \]
\[-\int_\Omega \rho_0 \kappa'(\theta) \theta_t \eta_x^{-1} v_x + \theta \eta_x^{-1} v_{tx} - \theta \eta_x^{-2} v_x^2 dx \]
\[+ \lambda \int_\Omega \kappa(\theta) \theta_t (2 \eta_x^{-1} v_{tx} - \eta_x^{-2} v_x^2) dx \]
\[+ \mu \int_\Omega \kappa(\theta) \theta_t (2 \eta_x^{-1} \mathbf{W}_x \cdot \mathbf{W}_{tx} - \eta_x^{-2} |\mathbf{W}_x|^2 v_x) dx \]
\[+ \nu \int_\Omega \kappa(\theta) \theta_t (2 \eta_x^{-1} \mathbf{B}_x \cdot \mathbf{B}_{tx} - \eta_x^{-2} |\mathbf{B}_x|^2 v_x) dx \]
\[= B_1 + B_2 + B_3 + B_4 + B_5 + B_6. \]

We estimate $B_1$ through $B_6$ as follows, by use of (2.6), (4.37), (4.38), (4.39) and Lemma 4.3.

\[-\frac{1}{2} \int_\Omega \rho_0 \kappa'(\theta) \theta_t^3 dx + \int_\Omega \eta_x^{-2} v_x \kappa(\theta) \theta_x (\kappa(\theta) \theta_t)_x dx \]
\[-\int_\Omega \rho_0 \kappa'(\theta) \theta_t \eta_x^{-1} v_x + \theta \eta_x^{-1} v_{tx} - \theta \eta_x^{-2} v_x^2 dx \]
\[+ \lambda \int_\Omega \kappa(\theta) \theta_t (2 \eta_x^{-1} v_{tx} - \eta_x^{-2} v_x^2) dx \]
\[+ \mu \int_\Omega \kappa(\theta) \theta_t (2 \eta_x^{-1} \mathbf{W}_x \cdot \mathbf{W}_{tx} - \eta_x^{-2} |\mathbf{W}_x|^2 v_x) dx \]
\[+ \nu \int_\Omega \kappa(\theta) \theta_t (2 \eta_x^{-1} \mathbf{B}_x \cdot \mathbf{B}_{tx} - \eta_x^{-2} |\mathbf{B}_x|^2 v_x) dx \]
\[= B_1 + B_2 + B_3 + B_4 + B_5 + B_6. \]
which gives, by using Lemma 4.3 again,
\[ \int_0^t \| \kappa(t) \theta \|_\infty^2 dt \leq C, \quad t \in [0, T]. \tag{4.60} \]

Note that from (4.37), (4.38), (4.59) and (4.60)
\[ \int_0^t \int_I \kappa^2(\theta) \theta_x^2 dx dt \lesssim \int_0^t \int_I (\kappa(\theta) \theta_x)_x^2 dx dt + \int_0^t \| (\kappa(\theta))_x \theta_x \|_\infty^2 dt \]
\[ \lesssim C + \| \kappa'(\theta) \|_{L^\infty}^2 \| \theta_x(\cdot, t) \|_{L^2}^2 \int_0^t \| \theta_x \|_{L^2}^2 dt \]
\[ \leq C. \tag{4.61} \]

Therefore, (4.55) follows from (4.59) and (4.61). Next, we integrate (2.5) from 0 to \( t \) to get
\[ \kappa(\theta) \theta_x \eta_x^{-1} = \int_0^x \rho_0 \theta_1 dy + \int_0^x \rho_0 \theta_v v_y \eta_y^{-1} dy 
\quad - \lambda \int_0^x v_y^2 \eta_y^{-1} dy - \mu \int_0^x |W_y|^2 \eta_y^{-1} dy - \nu \int_0^x |B_y|^2 \eta_y^{-1} dy \tag{4.62} \]

Thus (4.38), (4.39) and (4.59) imply that
\[ \| \kappa(\theta) \theta_x \|_{L^\infty} \lesssim \left( \int_I \rho_0 \theta_1^2 dx \right)^{\frac{1}{2}} + \| \theta \|_{L^\infty} \| v_x \|_{L^\infty} + \| v_x \|_{L^2}^2 + \| W_x \|_{L^\infty} + \| B_x \|_{L^\infty}^2 \]
\[ \leq C. \]

**Lemma 4.9.** For any \( t \in [0, T] \), we have
\[ \| v_{xx} \|_{L^2(t)} + \| W_{xx} \|_{L^2(t)} + \| B_{xx} \|_{L^2(t)} + \| \eta_{xx} \|_{L^2(t)} + \| \theta_{xx} \|_{L^2(t)} \leq C. \tag{4.63} \]

**Proof.** First we rewrite (2.5) as
\[ \lambda v_{xx} = \eta_x^{-1} (\lambda v_x - \rho_0 \theta) \eta_{xx} + \rho_0 \theta v_x \eta_x + (\rho_0 \theta)_x + B \cdot B_x \eta_x^{-1}. \tag{4.64} \]

Then by (4.37), (4.38) and (4.39) we have
\[ \| v_{xx} \|_{L^2} \lesssim \left( \| v_x \|_{L^\infty} + \| \theta \|_{L^\infty} \right) \| \eta_{xx} \|_{L^2} + \| \sqrt{\rho_0} \theta_t \|_{L^2} \]
\[ + \| \theta_x \|_{L^2} + \| \theta \|_{L^2} + \| B \|_{L^\infty} \| B_x \|_{L^2} \]
\[ \leq C \int_0^t \| v_{xx} \|_{L^2} dt + C, \]

which gives
\[ \| v_{xx}(\cdot, t) \|_{L^2} \leq C, \quad t \in [0, T], \]

by the Gronwall inequality and the fact that
\[ \eta_{xx}(x, t) = \int_0^t v_{xx}(x, s) ds. \]

It follows that \( \| \eta_{xx}(\cdot, t) \|_{L^2} \leq C, \quad t \in [0, T] \). Similarly, we rewrite (2.5) as
\[ \mu W_{xx} = \mu W_x \eta_x \eta_x^{-1} + \rho_0 W_t \eta_x - B_x \eta_x, \tag{4.65} \]
\[ \nu B_{xx} = \nu B_x \eta_x \eta_x^{-1} + B_t \eta_x^2 + B_v v_x \eta_x - W_x \eta_x, \tag{4.66} \]
\[ \kappa(\theta) \theta_{xx} = \rho_0 \theta v_x \eta_x + \rho_0 \theta v_x + \kappa(\theta) \theta_x \eta_x^{-1} \eta_{xx} 
\quad - \kappa'(\theta) \eta_x^2 - \lambda v_x^2 - \mu W_x^2 - \nu B_x^2. \tag{4.67} \]
Then it is easy to show that
\[
\|W_{xx}\|_{L^2} \lesssim \|W_x\|_{L^\infty} \|\eta_{xx}\|_{L^2} + \|W_t\|_{L^2} + \|B_x\|_{L^\infty} \leq C
\]
\[
\|B_{xx}\|_{L^2} \lesssim \|B_x\|_{L^\infty} \|\eta_{xx}\|_{L^2} + \|B_t\|_{L^2} + \|B\|_{L^\infty} \|v_x\|_{L^\infty} + \|W_x\|_{L^\infty} \leq C
\]
\[
\|\theta_{xx}\|_{L^2} \lesssim \|\kappa(\theta)\theta_{xx}\|_{L^2}
\lesssim \|\sqrt{\rho_0}\theta_t\|_{L^2} + \|\theta\|_{L^\infty} \|v_x\|_{L^\infty} + \|\kappa(\theta)\|_{L^\infty} \|\theta_x\|_{L^\infty} \|\eta_{xx}\|_{L^2}
+ \|\kappa'(\theta)\|_{L^\infty} \|\theta_x\|_{L^\infty}^2 + \|v_x\|_{L^\infty}^2 + \|W_x\|_{L^\infty}^2 + \|B_x\|_{L^\infty}^2
\leq C.
\]

From Lemmas 4.6, 4.7, 4.8 and 4.9, we obtain the following result.

**Proposition 4.10.** For any \(t \in [0, T]\), we have
\[
\|\eta, v, W, B, \theta\|_{H^2(I)(t)} + \|\sqrt{\rho_0}v_t, \sqrt{\rho_0}W_t, \sqrt{\rho_0}\theta_t, B_t\|_{L^2(I)(t)} \leq C,
\]
\[
\|v_{tx}, W_{tx}, B_{tx}, \theta_{tx}\|_{L^2(0,T;L^2(I))} \leq C.
\]
This proposition gives the energy estimates for the part (i) of Theorem 2.1.

4.2. Higher order estimates. In what follows, we will deal with the higher order estimates for the solutions, so that the system (2.5), is satisfied in the classical sense.

**Lemma 4.11.** For any \(t \in [0, T]\), we have
\[
\int_{I} (v_{tx}^2 + |W_{tx}|^2 + |B_{tx}|^2) \, dx \, dt + \int_{0}^{t} \int_{I} (\rho_0(v_{tt}^2 + |W_{tt}|^2) + |B_{tt}|^2) \, dx \, dt \leq C,
\]
\[
\|(v_t, W_t, B_t)\|_{L^\infty} \leq C.
\]

**Proof.** We first multiply (4.50) by \(W_t\) and integrate over \(I \times (0, t)\) to get
\[
\int_{0}^{t} \int_{I} \mu |W_{tx}|^2 \, dx \, dt + \frac{\mu}{2} \int_{I} \eta_x^{-1} |W_{tx}|^2 \, dx \bigg|_{0}^{t}
= -\frac{\mu}{2} \int_{I} |W_{tx}|^2 \eta_x^{-2} v_x \, dx \, dt + \mu \int_{0}^{t} \int_{I} W_x \cdot W_{tx} v_x \eta_x^{-2} \, dx \, dt
+ \int_{0}^{t} \int_{I} B_{tx} \cdot W_{tt} \, dx \, dt
\]
\[
:= D_1 + D_2 + D_3,
\]
where
\[
|D_1| \leq \frac{\mu}{2} \|\eta_x^{-1} v_x\|_{L^\infty_t} \int_{0}^{t} \int_{I} |W_{tx}|^2 \eta_x^{-1} \, dx \, dt,
\]
\[
D_2 = \mu \int_{I} W_x \cdot W_{tx} v_x \eta_x^{-2} \, dx \bigg|_{0}^{t} - \mu \int_{I} |W_{tx}|^2 \eta_x^{-2} \, dx \, dt
- \mu \int_{I} \int_{0}^{t} W_x \cdot W_{tx} v_x \eta_x^{-2} \, dx \, dt + 2\mu \int_{I} W_x \cdot W_{tx} v_x^2 \eta_x^{-3} \, dx \, dt
:= D_{21} + D_{22} + D_{23} + D_{24},
\]
\[
D_{21} \lesssim \left( \delta \int_{I} |W_{tx}|^2 \eta_x^{-1} \, dx + C(\delta) \int_{I} |W_x|^2 \, dx \right) \|v_x\|_{L^\infty_t}^2
\]
Next, we differentiate (2.5)

\[ \nu \sum_{j} \int_{I} \eta_{x}^{2} \| \mathbf{B}_{tx} \|^{2} |dx| + \int_{I} \eta_{x} |B_{tt}|^{2} |dx| \\
= -\frac{\nu}{2} \sum_{j} \int_{I} \mathbf{B}_{tx}^{2} \eta_{x}^{2} v_{x} dx + \int_{I} \mathbf{W}_{tx} \cdot \mathbf{B}_{tt} dx \\
- 2 \int_{I} \mathbf{B}_{tt} \cdot \mathbf{B}_{tv} dx - \int_{I} \mathbf{B} \cdot \mathbf{B}_{tt} v_{x} dx \\
+ \nu \int_{I} \mathbf{B}_{x} \cdot \mathbf{B}_{tx} v_{x} \eta_{x}^{2} dx - \nu \int_{I} \mathbf{B}_{tx}^{2} v_{x} \eta_{x}^{2} dx \\
- \nu \int_{I} \mathbf{B}_{x} \cdot \mathbf{B}_{tx} v_{x} \eta_{x}^{2} dx + 2 \nu \int_{I} \mathbf{B}_{x} \cdot \mathbf{B}_{tx} v_{x}^{2} \eta_{x}^{2} dx \\
\leq \delta \left( \int_{I} \eta_{x}^{-1} \mathbf{B}_{tx}^{2} |dx| + \int_{I} \eta_{x} |B_{tt}|^{2} |dx| \right) \\
+ C(\delta) \left( \| \mathbf{B}_{x} \|^{2}_{L^{2}} + \| \mathbf{B}_{tx} \|^{2}_{L^{2}} \| v_{x} \|^{2}_{L^{2}} + \int_{I} \mathbf{W}_{tx}^{2} dx + \| \mathbf{B} \|_{L^{\infty}} \int_{I} v_{tx}^{2} dx \right) \\
+ C \int_{I} \mathbf{B}_{tx}^{2} dx \left( \| v_{x} \|_{L^{\infty}} + \| \mathbf{B}_{x} \|_{L^{\infty}} \int_{I} v_{tx}^{2} dx + \| \mathbf{B}_{x} \|^{2}_{L^{\infty}} \| v_{x} \|^{4}_{L^{\infty}} \right). \\
\]

Observing that

\[ \int_{I} |\mathbf{B}_{tx}^{2} | dx \leq \| \mathbf{b}_{0} \|^{2}_{H^{1}} \| u_{0} \|^{2}_{H^{2}} + \| \mathbf{w}_{0} \|^{2}_{H^{2}} + \| \mathbf{b}_{0} \|^{2}_{H^{3}} \leq C, \]

and using Lemmas 4.6 and 4.7, we get

\[ \int_{I} |\mathbf{B}_{tx}^{2} | dx + \int_{I} |\mathbf{B}_{tt}^{2} | dx \leq C, \]

(4.72)

where the estimates in Proposition 4.1 are used.
Then, we multiply (4.40) by $v_{tx}$ and integrate over $I$ to get
\[
\int I \rho_0 v_{tx}^2 dx + \frac{\lambda}{2} \frac{d}{dt} \int I \eta_x^{-1} v_{tx}^2 dx = -\lambda \frac{d}{dt} \int I \eta_x^{-2} v_x v_{tx}^2 dx + \lambda \int I \eta_x^{-2} v_x v_{tx}^2 dx \nonumber \\
- \int I \rho_0 (\eta_x^{-2} \theta v_x - \eta_x^{-1} \theta_t) v_{tx} dx + \int I \mathbf{B} \cdot \mathbf{B}_t v_{tx} 
= M_1 + M_2 + M_3 + M_4. \tag{4.73}
\]

We estimate $M_1$ through $M_4$ as follows.

$$|M_1| \lesssim \|v_x\|_{L^\infty} \|v_{tx}\|_{L^2}^2,$$

$$M_2 = \lambda \frac{d}{dt} \int I \eta_x^{-2} v_x v_{tx} dx + 2 \lambda \int I v_{tx} (\eta_x^{-3} v_x^3 - \eta_x^{-2} v_x v_{tx}) dx \nonumber \\
\leq \lambda \frac{d}{dt} \int I \eta_x^{-2} v_x v_{tx} dx + C \|v_x\|_{L^\infty} \int I \eta_x^{-1} v_{tx}^2 dx + C \|v_x\|_{L^\infty}^5,$$

and

$$M_3 = -\frac{d}{dt} \int I \rho_0 (\eta_x^{-2} \theta v_x - \eta_x^{-1} \theta_t) v_{tx} dx \nonumber \\
- \int I v_{tx} [\eta_x^{-1} (\rho_0 \theta_t) v_x - \rho_0 (2 \eta_x^{-2} \theta_t v_x + \eta_x^{-2} \theta v_{tx})] dx \nonumber \\
:= M_{311} + M_{321}.$$

Note that by using (2.5) and (2.5), we have

$$M_{321} \leq C \int I |v_{tx}| ((\sqrt{\rho_0} \theta_t ||v_x|| + \theta |v_x| + \theta v_x^2) dx 
- \int I \eta_x^{-1} v_{tx} [\eta_x^{-1} (\kappa(\theta_x) v_x + \theta_0 \eta_x^{-1} v_x + \eta_x^{-1} v_x^2 + \mu \eta_x^{-1} W_x^2 + \nu \eta_x^{-1} B_x^2)] dx 
\leq C [(1 + \|\theta\|_{L^\infty}) ||v_{tx}||_{L^2}^2 + (||\sqrt{\rho_0} \theta_t||_{L^2} ||v_x||_{L^2}^2 + ||\theta||_{L^\infty} ||v_x||_{L^\infty}] 
- \int I \eta_x^{-1} v_{tx} [\eta_x^{-1} v_x^2 + \mu \eta_x^{-1} W_x^2 + \nu \eta_x^{-1} B_x^2 - \rho_0 \theta_0 \eta_x^{-1} v_x] dx 
- \int I [\eta_x^{-1} (\rho_0 \theta_t) v_x + \eta_x^{-2} v_x^2] (\eta_x^{-1} \kappa(\theta_x) v_x) v_{tx} dx 
- \int I (\eta_x^{-1} v_x + \rho_0 \eta_x^{-1} \theta_t) (\eta_x^{-1} \kappa(\theta_x) v_x) v_{tx} dx \nonumber \\
:= M_{321} + M_{322} + M_{323} + M_{324},$$

where

$$|M_{321}| \leq C (1 + ||v_{tx}||_{L^2}^2),$$

$$|M_{322}| \leq C \int I |v_{tx}| [||v_x||^3 + ||v_x|| |v_{tx}| + ||W_x||^2 |v_x| + ||W_x|| B_{tx}] dx 
+ ||B_{tx}||^2 |v_x| + ||B_x|| B_{tx} + \rho_0 (||\theta_t|| |v_x| + \theta v_x^2 + \theta |v_{tx}|) dx \nonumber \\
\leq C ||v_{tx}||_{L^2} \left[ ||v_x||_{L^\infty}^3 + ||v_x||_{L^\infty} (||v_{tx}||_{L^2} + ||W_{tx}||_{L^2} + ||B_{tx}||_{L^2} \right. 
+ ||W_x||_{L^\infty} + ||B_x||_{L^\infty}) + \sqrt{\rho_0} \theta_t ||v_x||_{L^\infty} + ||v_x||_{L^\infty}^3 + 1 
\leq C (1 + ||v_{tx}||_{L^2}^2).$$
\[ |M_{323}| = \left| \int_I \left[ (\eta_x^{-1} \rho_0 \theta_t + \eta_x^{-2} v_x^2 \eta_x^{-1} \kappa(\theta) \theta_x) \right] dt \right| \]
\[ \leq C \int_I \left[ |\eta_x| \left( |\theta_t| + v_x^2 + |\theta_x| + |v_{xx}| \right) \right. \]
\[ \cdot \left. \left( |v_x| |\theta_x| + |\theta_t| |\theta_x| + |\theta_t| \right) dx \right] \]
\[ \leq C \left[ |\eta_x| \|L^2 \| (\|\theta_t\| \|L^\infty\| + \|v_x\|_{L^\infty}^2) + \|\theta_t, \theta_x, v_x, v_{xx}\|_{L^2} \right] \]
\[ \leq C (1 + \|\theta_t\|^2_{H^1}). \]

Finally,
\[ |M_{324}| = \left| \int_I \rho_0 v_{tt} \left[ -\eta_x^{-2} v_x \kappa(\theta) \theta_x + \eta_x^{-1} (\kappa'(\theta) \theta_t \theta_x + \kappa(\theta) \theta_{tx}) \right] dx \right| \]
\[ \leq \delta \int_I \rho_0 v_{tt}^2 dx + C(\delta) (1 + \|\theta_t\|^2_{H^1}). \]

Thus we integrate (4.73) in \((0, t)\) to get
\[ \int_0^t \int_I \rho_0 v_{tt}^2 dx dt + \int_I v_{tt}^2 dx \]
\[ \leq \int_I (\rho_0^{-1} s_{0x}) x |^2 dx + C \left( \int_I \eta_x^{-2} v_x^{-2} v_{tx} dx \right) + \int_I B_t \cdot B_{tt} v_{tx} dx \]
\[ \leq C + \delta \int_I v_{tt}^2 dx + C(\delta) \left( \|v_x\|_{L^\infty}^2 + \|B_t\|_{L^2}^2 \right). \]

From (4.71), (4.72), (4.74) and Lemma 3.1, we get (4.68), (4.69) immediately. \( \square \)

**Lemma 4.12.** For any \( t \in [0, T] \), we have
\[ \int_I (\rho_0 v_{tt}^2 + |W_{tt}|^2 + |B_{tt}|^2 + |(\kappa \theta_t)_{x}^2|) \ dx(t) \]
\[ + \int_0^t \int_I (v_{tx}^2 + |W_{txx}|^2 + |B_{txx}|^2 + \rho_0 \kappa^2 \theta_{tt}^2) \ dx dt \leq C. \]  

**Proof.** Applying \( \partial_t \) to (4.50), we get
\[ \rho_0 W_{ttt} - \mu (\eta_x^{-1} W_{txx})_x = B_{ttx} + (2 W_x \eta_x^{-3} v_x^2 - 2 \eta_x^{-2} W_{tx} v_x - \eta_x^{-2} v_{tx} W_x)_x. \]

Multiply the above equation by \( W_{tt} \) and integrate over \( I \times [0, t] \) to get
\[ \int_I \rho_0 |W_{tt}|^2 dx + \mu \int_0^t \int_I \eta_x^{-1} |W_{txx}|^2 dx dt \]
\[ = - \int_0^t \int_I B_{tt} \cdot W_{txx} dx dt \]
\[ + \int_0^t \int_I (2 \eta_x^{-2} W_{tx} v_x + \eta_x^{-2} v_{tx} W_x - 2 W_x \eta_x^{-3} v_x^2) \cdot W_{txx} dx dt \]
\[ \leq C. \]

Thus we have
\[ \int_I (\rho_0 v_{tt}^2 + |W_{tt}|^2 + |B_{tt}|^2 + |(\kappa \theta_t)_{x}^2|) \ dx(t) \]
\[ + \int_0^t \int_I (v_{tx}^2 + |W_{txx}|^2 + |B_{txx}|^2 + \rho_0 \kappa^2 \theta_{tt}^2) \ dx dt \leq C. \]
\begin{align*}
&\leq \int_0^t \left( \int |W_{tt}|^2 dx dt + C(\delta) \int_0^t \int |B_{tt}|^2 dx dt \right) \\
&\quad + C \int_0^t \|v_x\|_{L^\infty}^2 \int \left( \|W_{tx}\|^2 + |W_x|^2 \right) dx dt + C \int_0^t \|W_x\|_{L^\infty}^2 \int \|v_{tx}\|^2 dx dt,
\end{align*}

which gives

\[ \int_0^t \rho_0 |W_{tt}|^2 dx + \int_0^t \int |W_{tx}|^2 dx dt \leq C, \quad (4.77) \]

since

\[ \int_0^t \rho_0 |W_{tt}|^2(x,0) dx \leq \int \rho_0^{-1} \left( \mu \left( \rho_0^{-1} \omega_0 \right)_{tx} + \phi_0 - \mu u_0 w_0 \right)_{tx}^2 dx \leq C. \]

Applying \( \partial_t \) to \eqref{eq:5.3}, we get

\[ \eta_x B_{ttx} - \nu(\eta_x^{-1} B_{txx} x) = B_{tt} \eta_x v_x + W_{tx} - 2(B_{tt} v_x + B_t v_{tx}) - B_t v_{xx} \]
\[ \quad - B_{txx} + \nu(2B_x \eta_x^{-3} v_x^2 - 2B_{txx} \eta_x^{-2} v_x - B_x \eta_x^{-2} v_{tx})_{tx}. \quad (4.78) \]

Integrate \( B_{txx} \) \eqref{eq:5.78} over \( I \), we have

\[
\frac{d}{dt} \int \eta_x |B_{txx}|^2 dx + \nu \int \eta_x^{-1} |B_{txx}|^2 dx \\
= - \int \eta_x \frac{d}{dt} |B_{txx}|^2 + 2 \int |B_{txx}|^2 v_x dx - 2 \int B_t \cdot B_{txx} dx \\
- \int B_t \cdot B_{txx} dx + \int v_{tx} B_x \cdot B_{txx} dx + \int v_{tx} B \cdot B_{txx} dx \\
- \int (2B_x \eta_x^{-3} v_x^2 - 2B_{txx} \eta_x^{-2} v_x - B_x \eta_x^{-2} v_{tx}) \cdot B_{txx} dx \\
\leq \delta \int B_{txx}^2 dx + C(\delta) \left( \|W_{ttx}\|_{L^2}^2 + \|v_x\|_{L^\infty}^2 \|B_x\|_{L^2}^2 + \|v_{tx}\|_{L^\infty}^2 \|B_{txx}\|_{L^2}^2 \right) \\
+ \left( \|B\|_{L^\infty} \|B_{txx}\|_{L^2} \right) \left( \|B_{txx}\|_{L^2}^2 \right) + \|v_x\|_{L^\infty} \|B_{txx}\|_{L^2}^2 \\
+ \|B_t\|_{L^\infty} \|B_{txx}\|_{L^2}^2 \|v_{tx}\|_{L^2}^2 + \|B_t\|_{L^\infty} \|B_{txx}\|_{L^2}^2 \|v_{tx}\|_{L^2}^2.
\]
Noting that \( (\frac{\partial \rho \theta}{\partial t})_t \) yields We estimate each term as follows. Integrating \((\kappa \theta)_{tt} \cdot \) over \( I \times [0,T] \). One gets

\[
0 = \int_0^t \int_I (\rho \theta)_t (\kappa \theta)_t dx dt - \int_0^t \int_I (\eta_x^{-1} \kappa \theta)_x (\kappa \theta)_t dx dt \\
+ \int_0^t \int_I \rho (\theta \eta_x^{-1} v_x)_t (\kappa \theta)_t dx dt - \lambda \int_0^t \int_I (\eta_x^{-1} v_x)_t (\kappa \theta)_t dx dt \\
- \mu \int_0^t \int_I (\eta_x^{-1} (|W_x|^2)_x (\kappa \theta)_t dx dt - \nu \int_0^t \int_I (\eta_x^{-1} (|B_x|^2)_x (\kappa \theta)_t dx dt \\
:= L_1 + L_2 + L_3 + L_4 + L_5 + L_6.
\]

We estimate each term as follows.

\[
L_1 = \int_0^t \int_I \rho \theta (\kappa \theta)_t^2 dx dt + \int_0^t \int_I \rho \theta (\theta)_t^2 dx \\
\geq \int_0^t \int_I \rho \theta (\kappa \theta)_t^2 dx dt - \frac{1}{2} \int_0^t \int_I \rho \theta (\theta)_t^2 dx dt - C \int_0^t \| \theta \|^2_{H^1} dt \cdot \sup_t \int_I \rho \theta^2 dx(t) \\
\geq \frac{1}{2} \int_0^t \int_I \rho \theta (\kappa \theta)_t^2 dx dt - C.
\]

Noting that \((\kappa \theta)_t = (\kappa \theta)_x\), we integrate by parts to get

\[
L_2 = \int_I \left[ \frac{1}{2} \eta_x^{-1} |(\kappa \theta)_x|^2 - \eta_x^{-2} v_x \kappa \theta_x (\kappa \theta)_x \right] dx \\
+ \frac{1}{2} \int_0^t \int_I \eta_x^{-2} v_x |(\kappa \theta)_x|^2 dx dt + \int_0^t \int_I (\eta_x^{-2} v_x \kappa \theta_x)_t (\kappa \theta)_t dx dt \\
\geq \frac{1}{2} \int I \eta_x^{-1} |(\kappa \theta)_x|^2 dx - C \int_0^t \left( (\rho_0^{-1} \tau_0) x + |(\rho_0^{-1} \tau_0) x|^2 \right) dx - \frac{1}{4} \int I \eta_x^{-1} |(\kappa \theta)_x|^2 dx \\
- C \| v_x \|^2_{L^\infty_{x,t}} \| \theta \|^2_{H^1_{x,t}} - C - C \int_0^t (1 + \| v_x \|_{L^\infty}) \int I \eta_x^{-1} |(\kappa \theta)_x|^2 dx dt \\
- C \| v_x \|^2_{L^\infty_{x,t}} \left( \| v_x \|_{L^\infty_{x,t}} + \int_0^t v_x^2 dx dt \right) \\
\geq \frac{1}{4} \int I \eta_x^{-1} |(\kappa \theta)_x|^2 dx dt - C \int_0^t \int I \eta_x^{-1} |(\kappa \theta)_x|^2 dx dt - C,
\]

which yields

\[
\int_I \rho \theta^2 dx(t) + \int_0^t \int_I \rho \theta^2 (\kappa \theta)_t dx dt \leq C \left( 1 + \int_0^t \int_I \rho \theta^2 (\theta)_t dx dt \right), \quad t \in [0,T].
\]
\[ L_3 = -\int_I \rho_0 \kappa_1 (\theta \eta^{-1} v_x)_t dx \bigg|^t_0 + \int_0^t \int_I \rho_0 \kappa_1 (\theta \eta^{-1} v_x)_{xt} dx dt \]
\[ \leq C \left( \sup_t \int_I \rho_0 \theta^2 dx(t) + \sup_t \| \sqrt{\rho_0} \theta \|_{L^2(t)} (1 + \sup_t \| v_x \|_{L^2(t)}) \right) + C \]
\[ + \delta \int_0^t (\rho_0 \theta^2_1 + v_x^2_{xt}) dx dt + C(\delta) \int_0^t (1 + \| \theta \|_{L^\infty}) \int_I (\rho_0 \theta^2_1 + v_x^2_{xt}) dx dt, \]
\[ L_4 = \lambda \int_I \kappa_1 (2 \eta^{-1} v_x v_{xt} - \eta^{-2} v_x^3) dx \bigg|^t_0 - \lambda \int_0^t \int_I \kappa_1 (2 \eta^{-1} v_x v_{xt} - \eta^{-2} v_x^3) dx dt \]
\[ \leq C \| \kappa \|_{L^2} (\| v_x \|_{L^\infty(t)} \| v_{xt} \|_{L^2(t)} + \| v_x \|_{L^2(t)}) + C \]
\[ + C \int_0^t \| \kappa \|_{L^\infty} (\| v_{xt} \|_{L^2}^2 + \| v_x \|_{L^\infty}^2 \| v_{xt} \|_{L^2} + \| v_x \|_{L^\infty} \| v_{xt} \|_{L^2}) dt \]
\[ \leq \delta \| \kappa \|_{L^2}^2(t) + C \delta + \delta \int_0^t \| v_{xt} \|_{L^2}^2 dt + C(\delta) \int_0^t \| \kappa \|_{L^2}^2 dt + C \]
\[ \leq \delta \left( \sup_t \| (\kappa \eta) x \|_{L^2}^2 + \int_0^t \| v_{xt} \|_{L^2}^2 dt \right) + C(\delta), \]
\[ L_5 = \mu \int_I \kappa_1 (2 \eta^{-1} W_x \cdot W_{xt} - \eta^{-2} |W_x|^2 v_x) dx \bigg|^t_0 \]
\[ - \mu \int_0^t \int_I \kappa_1 (2 \eta^{-1} W_x \cdot W_{xt} - \eta^{-2} |W_x|^2 v_x) dx dt \]
\[ \leq C \| \kappa \|_{L^2} (\| W_x \|_{L^\infty} \| W_{xt} \|_{L^2(t)} + \| v_x \|_{L^\infty} \| W \|_{L^\infty}^2) + C \]
\[ + C \int_0^t \| \kappa \|_{L^\infty} (\| v_{xt} \|_{L^2}^2 + \| W_{xt} \|_{L^2} \| v_x \|_{L^\infty} \| W \|_{L^\infty} + \| W_x \|_{L^\infty} \| v_x \|_{L^\infty} + \| W_{xt} \|_{L^2} \| W \|_{L^\infty}) dt \]
\[ \leq \delta \| \kappa \|_{L^2}^2(t) + C \delta + \delta \int_0^t \| W_{xt} \|_{L^2}^2 dt + C(\delta) \int_0^t \| \kappa \|_{L^2}^2 dt + C \]
\[ \leq \delta \left( \sup_t \| (\kappa \eta) x \|_{L^2}^2 + \int_0^t \| W_{xt} \|_{L^2}^2 dt \right) + C(\delta), \]

and
\[ L_6 = \nu \int_I \kappa_1 (2 \eta^{-1} B_x \cdot B_{xt} - \eta^{-2} |B_x|^2 v_x) dx \bigg|^t_0 \]
\[ - \nu \int_0^t \int_I \kappa_1 (2 \eta^{-1} B_x \cdot B_{xt} - \eta^{-2} |B_x|^2 v_x) dx dt \]
\[ \leq C \| \kappa \|_{L^2} (\| B_x \|_{L^\infty} \| B_{xt} \|_{L^2(t)} + \| v_x \|_{L^\infty} \| B \|_{L^\infty}^2) + C \]
\[ + C \int_0^t \| \kappa \|_{L^\infty} (\| v_{xt} \|_{L^2}^2 + \| B_{xt} \|_{L^2} \| v_x \|_{L^\infty} \| B \|_{L^\infty} + \| B_x \|_{L^\infty} \| v_x \|_{L^\infty} + \| B_{xt} \|_{L^2} \| B \|_{L^\infty}) dt \]
\[ \leq \delta \| \kappa \|_{L^2}^2(t) + C \delta + \delta \int_0^t \| B_{xt} \|_{L^2}^2 dt + C(\delta) \int_0^t \| \kappa \|_{L^2}^2 dt + C \]
\[ \leq \delta \left( \sup_{t \in [0,T]} \| (\kappa \eta) x \|_{L^2}^2(t) + \int_0^t \| B_{xt} \|_{L^2}^2 dt \right) + C(\delta), \]
Lemma 4.13. For any $t \in [0, T]$, one has

\[-\frac{1}{2} + \frac{1}{2} \| (v_{xx}, W_{xxx}, B_{xxx}, \eta_{xxx}, \theta_{xxx}) \|_{L^2}(t) \leq C.\]  

\[\| (v_{txt}, W_{txt}, B_{txt}) \|_{L^2}(t) + \| (v_{xxx}, W_{xxx}, B_{xxx}) \|_{L^2}(t) \leq C, \quad t \in [0, T],\]  

\[\| \theta_{xxx} \|_{L^2([0, T]; L^2(\Omega))} + \| \theta_{xxxx} \|_{L^2([0, T]; L^2(\Omega))} \leq C.\]  

Proof. From (4.26), we have

\[\lambda v_{xxx} = (\rho_0 \theta)_{xx} + \left( \frac{1}{2} \eta_x \| B \|^2 \right)_{xx} + (\eta_x \int_x^1 \rho_0 v_t dy)_{xx} \]

\[= (\rho_0 \theta)_{xx} + \frac{1}{2} \eta_{xxx} \| B \|^2 + 2 \eta_{xx} B \cdot B_x + \eta_x (B \cdot B_{xx} + |B_x|^2) \]

\[+ \eta_{xxx} \int_x^1 \rho_0 v_t dy + 2 \eta_{xx} \rho_0 v_t + \eta_x (\rho_0 v_t).\]

It follows that

\[\lambda \| v_{xxx} \|_{L^2} \leq \| \rho_0 \|_{H^1} \| \theta \|_{H^2} + \| \eta_{xxx} \|_{L^2} \| \| B \|^2 \|_{L^2} + \| \eta_{xx} \|_{L^2} \| B_x \|_{L^2} \| B \|_{L^\infty} \]

\[+ \| \eta_x \|_{L^\infty} \| B \|_{L^\infty} \| B_{xxx} \|_{L^2} + (\| \eta_{xx} \|_{L^2} + \| \eta_{xx} \|_{L^2}) \left( \int_0^t \rho_0 v_t^2 dt \right)^{\frac{1}{2}} \]

\[+ \| \rho_0 \|_{L^\infty} \| v_{txt} \|_{L^2} + \| (\rho_0 x) \|_{L^\infty} \| v_t \|_{L^2} \]

\[\leq \int_0^t \| v_{xxx}(\cdot, s) \|_{L^2} ds + 1.\]

Utilizing the Gronwall inequality, we have

\[\| v_{xxx} \|_{L^2}(t) \leq C, \quad t \in [0, T],\]

which gives immediately

\[\| \eta_{xxx} \|_{L^2}(t) \leq C, \quad t \in [0, T].\]

We differentiate (2.5) twice with respect to $x$ to get

\[\mu \eta_x^{-1} W_{xxx} = (\rho_0 W_t)_x - B_{xxx} + 2 \mu \eta_x^{-1} \eta_{xx} W_{xx} \]

\[+ 2 \mu \eta_x^{-3} \eta_{xx}^2 W_x - \mu \eta_x^{-2} \eta_{xxx} W_x.\]  

(4.86)

Then, we have

\[\| W_{xxx} \|_{L^2} \leq \| \rho_0 \|_{H^1} \| W_t \|_{H^1} + \| B_{xxx} \|_{L^2} + \| W_{xxx} \|_{L^2} \| \eta_{xxx} \|_{L^\infty} \]

\[+ \| W_x \|_{L^\infty} (\| \eta_{xx} \|_{L^2}^2 + \| \eta_{xxx} \|_{L^2}).\]

\[\leq C.\]
Similar to (4.86), we find
\[
\nu B_{txx} = B_{tx} \eta_x + B_t \eta_{xx} + B_x v_x + B v_{xx} - W_{xx} + 2 \nu \eta_x^{-2} \eta_{xx} B_{xx} \\
- 2 \nu \eta_x^{-3} \eta_{xx}^2 B_x + \nu \eta_x^{-2} \eta_{xx} B_x,
\]
thus
\[
\|B_{xxx}\|_{L^2} \lesssim \|B_{tx}\|_{L^2} + \|\eta_{xx}\|_{L^\infty} (\|B_t\|_{L^\infty} + \|B_{tx}\|_{L^2} + \|\eta_{xx2}\|_{L^\infty} \|B_x\|_{L^\infty}) \\
\quad + \|B_x\|_{L^\infty} \|v_x\|_{L^\infty} + \|B\|_{L^\infty}\|v_{xx}\|_{L^2} + \|W_{xx}\|_{L^2} + \|\eta_{xx2}\|_{L^2} \|B_x\|_{L^\infty} \\
\quad \leq C.
\]
Next, from (4.62), we derive
\[
\theta_{xxx} = \left[ \eta_x \kappa^{-1}(\theta) \int_0^x (\rho_0 \theta_t + \rho_0 \theta v_y \eta_y^{-1} - \lambda v_y^2 - \mu |W_y|^2 - \nu |B_y|^2) \, dy \right]_{xx} \\
= \left[ \eta_x \kappa^{-1}(\theta) \right]_{xx} \int_0^x (\rho_0 \theta_t + \rho_0 \theta v_y \eta_y^{-1} - \lambda v_y^2 - \mu |W_y|^2 - \nu |B_y|^2) \, dy \\
\quad + \left[ \eta_x \kappa^{-1}(\theta) \right]_x (\rho_0 \theta_t + \rho_0 \theta v_x \eta_x^{-1} - \lambda v_x^2 - \mu |W_x|^2 - \nu |B_x|^2) \\
\quad + \eta_x \kappa^{-1}(\theta) \left( (\rho_0)_x \theta_t + \rho_0 \theta v_x (\rho_0 \eta_x^{-1})_x \right) + 2 \lambda v_x v_{xx} - 2 \mu W_x \cdot W_{xx} - 2 \nu B_x \cdot B_{xx}.
\]
It yields that
\[
\|\theta_{xxx}\|_{L^2} \lesssim \|\eta_x\|_{H^2} \|\theta\|_{H^2} \left( \|\rho_0 \theta_t\|_{L^2} + \|\theta\|_{L^\infty} \|v_x\|_{L^\infty} + \|v_{xx}\|_{L^2} + \|W_x\|_{L^2} + \|B_x\|_{L^\infty} \right) \\
\quad + \|((\rho_0)_x)\|_{L^\infty} \|\theta_t\|_{L^2} + \|\rho_0 \theta_{tx}\|_{L^2} + \|\theta\|_{H^1} \|v_x\|_{H^1} \|\eta_x\|_{H^1} + \|v_{xx}\|_{L^\infty} \|v_{xx}\|_{L^2} \\
\quad + \|W_x\|_{L^\infty} \|W_{xx}\|_{L^2} + \|W_x\|_{L^\infty} \|W_{xx}\|_{L^2} \leq C,
\]
by using Lemma 4.12.
From (4.40), we have
\[
\|v_{txx}\|_{L^2} \lesssim \|\rho_0 \theta_t\|_{L^2} + \|v_x\|_{L^2} \|\eta_{xx}\|_{L^\infty} + \|\eta_x\|_{H^1} \|v_x\|_{H^1} \\
\quad + \|\rho_0\|_{H^2} \|\theta\|_{H^2} \|\eta_x\|_{H^2} \|v_{xx}\|_{H^2} + \|\rho_0\|_{H^2} \|\theta_t\|_{H^2} \|\eta_{xx}\|_{H^2} \\
\quad + \|B_t\|_{L^2} \|B_x\|_{L^\infty} + \|B\|_{L^\infty} \|B_{tx}\|_{L^2} \lesssim C.
\]
From (4.50) we have
\[
\mu \eta_x^{-2} W_{tx} = \rho_0 W_{tt} + \mu \eta_x^{-2} \eta_{xx} W_{tx} - B_{tx} + \mu (\eta^{-2} v_x W_x).
\]
Therefore,
\[
\|W_{txx}\|_{L^2} \lesssim \|\rho_0 W_{tt}\|_{L^2} + \|\eta_{xx}\|_{L^\infty} \|W_{tx}\|_{L^2} + \|B_{tx}\|_{L^2} + \|v_x\|_{L^\infty} \|W_{xx}\|_{L^2} \\
\quad + \|\eta_{xx}\|_{H^1} \|v_x\|_{L^\infty} \|W_x\|_{L^\infty} + \|v_{xx}\|_{L^2} \|W_x\|_{L^\infty} \leq C.
\]
Differentiating (2.5) with respect to $t$, we get
\[
\|B_{txx}\|_{L^2} \lesssim \|B_t\|_{L^2} + \|\eta_{xx}\|_{L^\infty} \|B_{tx}\|_{L^2} + \|W_{tx}\|_{L^2} + \|B_t\|_{L^2} \|v_x\|_{L^2} \\
\quad + \|B\|_{L^2} \|v_{xx}\|_{L^2} + \|\eta_{xx}\|_{H^1} \|v_x\|_{L^\infty} \|B_x\|_{L^\infty} \\
\quad + \|v_{xx}\|_{L^2} \|B_x\|_{L^\infty} + \|v_x\|_{L^\infty} \|B_{xx}\|_{L^2} \leq C.
\]
Then, we derive from (4.57) that
\[ \| \theta_{xx} \|_{L^2(0,T; L^2(1))} \lesssim \sqrt{\rho_0 \| \theta_t \|_{L^2(0,T; L^2(1))}} \]
\[ + \int_0^T \left\{ \| v_{xx} \|_{L^2} \| \theta_x \|_{L^\infty} + \| v_x \|_{L^\infty} \left( \| \theta_x \|_{L^2}^2 + \| \theta_{xx} \|_{L^2} \right) \\
+ \| \eta_{xx} \|_{L^\infty} \left( \| \theta_t \|_{L^2} \| \theta_x \|_{L^\infty} + \| \theta_{xx} \|_{L^2} \right) \\
+ \| \rho_0 \|_{L^\infty} \left[ \| \theta_t \|_{L^2} \| v_x \|_{L^\infty} + \| \theta \|_{L^\infty} \left( \| v_x \|_{L^2}^2 + \| v_{xx} \|_{L^2} \right) \right] \\
+ \| v_x \|_{L^2}^2 + \| \rho \|_{L^\infty} \| v_{xx} \|_{L^2} + \| v_x \|_{L^\infty} \| W_x \|_{L^\infty}^2 + \| W_x \|_{L^\infty} \| W_{xx} \|_{L^2} \\
+ \| v_x \|_{L^\infty} \| B_x \|_{L^2}^2 + \| B_{xx} \|_{L^2} \right\} dt \]
\[ \lesssim C. \]

To get the \( H^4 \) estimate of \( v \), we differentiate (2.5) twice with respect to \( x \) to achieve
\[ \lambda v_x^{-1} v_{xxxx} + \lambda v_x \eta_x^{-1} v_{xxx} - \rho_0 \theta (\eta_x^{-1} v_{xx} - \eta_x^{-2} v_{xxx} + 3 \beta_x \cdot B_{xx} + B \cdot B_{xx} \phi_{xx} \right) \\
= \left( \rho_0 v \right)_{xx} + 3 \left( \rho_0 \theta - \lambda v_x \right) (\eta_x^{-1} v) + 3 \left( \rho_0 \theta - \mu v_x \right) (\eta_x^{-1} v_{xx} \right) + \left( \rho_0 \theta \right) \phi_{xxx} \eta_x^{-1}. \]

It follows that
\[ \lambda \| v_{xxxx} \|_{L^2} \leq C \left( \| \rho_0 \|_{L^\infty} \| \theta \|_{L^\infty} + \| v_x \|_{L^\infty} \right) \int_0^t \| v_{xxxx} \|_{L^2} dt \]
\[ + \| \rho_0 \|_{H^1} \| \theta \|_{H^1} \| \eta_x \|_{H^2} + \| \rho_0 \|_{H^2} \| v_x \|_{H^2} \]
\[ + \| B_x \|_{L^\infty} \| B_{xx} \|_{L^2} + \| B \|_{L^\infty} \| B_{xxx} \|_{L^2}. \]

Then, utilizing the Gronwall inequality, we obtain
\[ \| v_{xxxx} \|_{L^2(t)} + \| \eta_{xxxx} \|_{L^2(t)} \leq C. \]

Differentiating (2.5) twice with respect to \( x \), we have
\[ \mu \left[ W_{xxxx} \eta_x^{-1} - 3 W_{xxx} \eta_x^{-2} \eta_x + 3 W_{xx} (2 \eta_x^{-3} \eta_x - \eta_x^{-2} \eta_{xxx}) \right] \\
+ \left[ W (-6 \eta_x^{-4} \eta_x + 2 \eta_x^{-3} \eta_{xxx} + 2 \eta_x^{-3} \eta_{xxx} \eta_{xxx} - \eta_x^{-2} \eta_{xxxx}) \right] \]
\[ = \left( \rho_0 W_t \right)_{xx} - B_{xxx}, \]
which gives
\[ \mu \| W_{xxxx} \|_{L^2} \lesssim \| W \|_{H^1} \| \eta_x \|_{H^2} + \| \eta_{xxxx} \|_{L^2} + \| \rho_0 \|_{H^2} \| W_t \|_{H^2} + \| B_{xxx} \|_{L^2} \leq C. \]

Similarly, we get \( H^4 \) estimate of \( B \) through
\[ \nu \| B_{xxxx} \|_{L^2} \lesssim \| B \|_{H^1} \| \eta_x \|_{H^2} + \| \eta_{xxxx} \|_{L^2} + \| \rho_0 \|_{H^2} \| v \|_{H^3} \]
\[ + \| B_t \|_{H^2} \| \eta_x \|_{H^2} + \| \eta_{xxx} \|_{L^2} \]
\[ \leq C. \]

Next, we apply \( \partial_{xx} \) to (2.5), and then estimate the resulting equation to get
\[ \| \theta_{xxxx} \|_{L^2(0,T; L^2(1))} \lesssim \| \theta_{xx} \|_{L^2(0,T; L^2(1))} \]
\[ + \| \eta_x \|_{H^2} \| v_x \|_{H^2}^2 + \| \eta_x \|_{H^3} \left( \| \theta_x \|_{H^2}^2 + 1 \right) \right\} dt \]
\[ \leq C. \]
Due to Lemmas 4.11, 4.12, 4.13 and Proposition 4.10, the following proposition is acquired.

**Proposition 4.14.** For any $t \in [0, T]$, we have

\[
\begin{align*}
\| & (\eta, v, W, B) \|_{H^4(I)(t)} + \| (\theta, W, B) \|_{H^2(I)(t)} \\
& + \| \theta \|_{H^3(I)(t)} + \| \theta_t \|_{H^2(I)(t)} + \| \sqrt{\rho v_t}, \sqrt{\rho} W_t, B_t \|_{L^2(I)(t)} \leq C,
\end{align*}
\]

\[
\| (v_{ttt}, W_{ttt}, B_{ttt}, \theta_{xxxx}, \sqrt{\rho_0} \theta_{ttt}, \theta_{xxxx}) \|_{L^2(0, T; L^2(I))} \leq C.
\]

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