Orlicz–Lorentz centroid bodies

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Abstract

We extend the definition of the centroid body operator to an Orlicz–Lorentz centroid body operator on the star bodies in $\mathbb{R}^n$, and establish the sharp affine isoperimetric inequality that bounds (from below) the volume of the Orlicz–Lorentz centroid body of any convex body containing the origin in its interior by the volume of this convex body.

1 Introduction

The concepts of centroid body and projection body are the central notions in convex geometry (or, in Brunn–Minkowski theory). The classical affine isoperimetric inequalities that relate the volume of a convex body with that of its centroid body or its projection body were established in a landmark works of Petty [40] and nowaday are known as the Busemann-Petty centroid inequality and Busemann–Petty projection inequality. (See, e.g., the books of Gardner [8], Schneider [42], and Thompson [44] for references.)

The Brunn–Minkowski theory has a natural extension to the $L_p$ Brunn–Minkowski theory and its dual. This new theory was initiated in the early 1960s when Firey introduced his concept of $L_p$ composition of convex bodies (see, e.g., the book of Schneider [42]). These Firey–Minkowski $L_p$ combinations were shown to lead to an embryonic $L_p$ Brunn–Minkowski theory in the works of Lutwak [26, 27]. This new theory (and its dual) has witnessed a rapid growth. Its central concepts (and its dual) are the $L_p$ centroid body and $L_p$ projection body (an $L_p$ analogue of the centroid body and projection body) which was

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introduced by Lutwak, Yang and Zhang [28]. The $L_p$ analogues of the Busemann–Petty centroid inequality and Busemann–Petty projection inequality were also established in [28] by using Steiner’s symmetrization method, and nowadays are named as the $L_p$ Busemann–Petty centroid inequality and $L_p$ Busemann–Petty projection inequality or shortly $L_p$ affine isoperimetric inequalities (see [2, 5, 39] for the other proofs of these inequalities based on the shadow system which was introduced by Rogers and Shephard [41, 43], or the random method). It was shown in [4, 29] that $L_p$ affine isoperimetric inequalities are crucial tools to establish the sharp affine $L_p$ Sobolev inequalities and the affine Pólya–Szegő principle which are stronger than the usual sharp Sobolev inequalities and the usual Pólya–Szegő principle in Euclidean space (see [15, 16, 18] for the strengthened asymmetric counterparts of these results). The $L_p$-centroid bodies recently found some important applications in the field of asymptotic geometric analysis (see, e.g., [6, 7, 11, 14, 20, 35–38] and references therein, especially, in establishing the concentration of mass on convex bodies of Paouris [36, 37] and in thin–shell estimates of Guédon and Milman) and even in the theory of stable distributions [33].

Recently, Lutwak, Yang and Zhang extended the $L_p$ Brunn–Minkowski theory to an Orlicz–Brunn–Minkowski theory by introducing the concepts of Orlicz centroid body (see [30]) and the Orlicz projection body (see [31]) for any convex body. They also established the affine isoperimetric inequalities revealing the volume of a convex body with the volume of its Orlicz centroid body and the volume of its Orlicz projection body which are called the Orlicz Busemann–Petty centroid inequality and Orlicz Busemann–Petty projection inequality, respectively (see [5, 22, 39] for the other proofs of these affine isoperimetric inequalities, and see also [48] for the Orlicz Busemann–Petty centroid inequality on the star bodies). The reverse Orlicz Busemann–Petty centroid inequality was proved in [3]. Since the works of Lutwak, Yang and Zhang, the Orlicz–Brunn–Minkowski theory were developed very fast by many authors (see, e.g., [3, 9, 10, 18, 19, 21, 23, 45, 46, 48–50] and references therein). For example, in [9], Gardner, Hug and Weil developed a general framework for this new theory by introducing the definition of Orlicz addition. They show that Orlicz addition is intimately related to a natural and fundamental generalization of Minkowski addition called $M$–addition. They also proved some inequalities of Brunn–Minkowski type (such as the Orlicz–Brunn–Minkowski inequality and Orlicz–Minkowski inequality) for both Orlicz addition and $M$–addition. These new inequalities are generalizations of the ones in the $L_p$–Brunn–Minkowski theory, and have a connection with the conjectured log–Brunn–Minkowski inequality of Böröczky, Lutwak, Yang and Zhang [1]. Another proof of the Orlicz–Brunn–Minkowski inequality using Steiner symmetrization method can be found in [45]. In [18], Haberl, Lutwak, Yang and Zhang posed the Orlicz–Minkowski problem asking the necessary and sufficient conditions of a given Borel measure on sphere for which this measure is the Orlicz surface area of a convex body. This problem was solved in [18] when the given measure is even. For the discrete measure, this problem was solved in [19]. The dual Orlicz–Brunn–Minkowski theory was recently developed in [10, 46, 48].

In this paper, we extend the definition of Orlicz centroid body of Lutwak, Yang and Zhang to a more general situation of the Orlicz–Lorentz spaces which are generalization of both Orlicz spaces introduced by Orlicz [34] (see also [32]) and Lorentz spaces introduced
by Lorentz (see [24,25]). To do this, let us recall some basic elements of these spaces. Let
\((\Omega, \Sigma, \mu)\) be a measure space with an \(\sigma\)-finite, non atom measure \(\mu\). For any measurable
function \(f : \Omega \to \mathbb{R}\), we define the distribution function of \(f\) by
\[
\mu_f(t) = \mu(\{x : |f(x)| > t\}), \quad \forall t > 0,
\]
and the decreasing rearrangement of \(f\) by
\[
f^*(t) = \inf\{\lambda > 0 : \mu_f(\lambda) \leq t\},
\]
for any \(t > 0\) (for convention \(\inf\emptyset = \infty\)).

We denote \(I = (0, \mu(\Omega))\). A function \(\phi : [0, \infty) \to [0, \infty)\) is called an Orlicz function
if \(\phi\) is a convex function such that \(\phi(t) > 0\) if \(t > 0\), \(\phi(0) = 0\) and \(\lim_{t \to \infty} \phi(t) = \infty\). A
function \(\omega : I \to (0, \infty)\) is called a weight function if \(\omega\) is nonincreasing function which is
locally integrable with respect to the Lebesgue measure on \(I\) such that \(\int_I \omega(t)dt = \infty\) if
\(I = (0, \infty)\). For an Orlicz function \(\phi\) and a weight function \(\omega\), we define the Orlicz-Lorentz
space \(\Lambda_{\phi, \omega}\) on \((\Omega, \Sigma, \mu)\) to be the set of all measurable functions \(f\) on \(\Omega\) such that
\[
\int_I \phi\left(\frac{f^*(t)}{\lambda}\right) \omega(t)dt < \infty,
\]
for some \(\lambda > 0\). If the function \(f \in \Lambda_{\phi, \omega}\), its Orlicz norm is defined by
\[
\|f\|_{\Lambda_{\phi, \omega}} = \inf\left\{\lambda > 0 : \int_I \phi\left(\frac{f^*(t)}{\lambda}\right) \omega(t)dt \leq 1\right\}.
\]  
(1.1)

It is obvious from this definition that if \(f\) and \(g\) have the same distribution function then \(\|f\|_{\Lambda_{\phi, \omega}} = \|g\|_{\Lambda_{\phi, \omega}}\). When \(\omega \equiv 1\), the Orlicz–Lorentz space \(\Lambda_{\phi, \omega}\) is the Orlicz space.
Especially, when \(\phi(t) = t^p\) and \(\omega \equiv 1\), we obtain the Lebesgue space \(L_p(\Omega, \mu)\). When
\(\phi(t) = t\), we obtain the Lorentz space \(\Lambda_\omega\).

Let \(K\) be a star body (see section \(\S 2\) for precise definition) with respect to the origin
in \(\mathbb{R}^n\) with volume \(|K|\). We consider the measure space \((\Omega, \Sigma, \mu) = (K, \mathcal{B}_K, \mu^K)\) here and
thereafter \(\mathcal{B}_A\) denotes \(\sigma\)-algebra of all Lebesgue measurable subset of \(A\), and \(\mu^A\) denotes
the normalized measure on \(A\) whose density is \(1_{A}(x)dx/|A|\) for any Lebesgue measurable
\(A \subset \mathbb{R}^n\) of positive measure. For any vector \(x \in \mathbb{R}^n\), we define the function \(f_{x,K}\) on \(K\) by
\(f_{x,K}(y) = x \cdot y\), with \(y \in K\) where \(x \cdot y\) denotes the standard inner product of vectors \(x\) and
\(y\) in \(\mathbb{R}^n\). Given an Orlicz function \(\phi\) and a weight function \(\omega\) on \(I = (0,1)\), we define the
Orlicz–Lorentz centroid body of \(K\) denoted by \(\Gamma_{\phi, \omega}K\) to be the convex body in \(\mathbb{R}^n\) whose
support function is given by
\[
h(\Gamma_{\phi, \omega}K, x) = \|f_{x,K}\|_{\Lambda_{\phi, \omega}} = \inf\left\{\lambda > 0 : \int_0^1 \phi\left(\frac{f_{x,K}(t)}{\lambda}\right) \omega(t)dt \leq 1\right\}.
\]
When \(\omega \equiv 1\), our definition of Orlicz–Lorentz centroid body coincides with the definition of
Orlicz centroid body given by Lutwak, Yang and Zhang [30] for even convex function
\( \phi \) in \( \mathbb{R} \). Note that Lutwak, Yang and Zhang defined the Orlicz centroid body for any function convex function \( \phi : \mathbb{R} \rightarrow (0, \infty) \) such that \( \phi \) is nonincreasing on \((-\infty, 0]\), \( \phi \) is nondecreasing on \([0, \infty) \) and one of these monotonicity is strict. Their definition is more general than ours in this case. However, when \( \phi(t) = t^p \) and \( \omega = 1 \), we again obtain the definition of the \( L_p \) centroid body given in [28].

We will establish the following affine isoperimetric inequality for Orlicz–Lorentz centroid bodies.

**Theorem 1.1.** If \( \phi \) is an Orlicz function, \( \omega \) is a weight function on \((0, 1)\) and \( K \) is a convex body in \( \mathbb{R}^n \) containing the origin in its interior, then the volume ratio

\[
\frac{|\Gamma_{\phi, \omega} K|}{|K|}
\]

is minimized if and only if \( K \) is an origin–centered ellipsoid.

This theorem contains as a special case the classical Busemann–Petty centroid inequality for convex bodies [40], as well as the \( L_p \) Busemann–Petty centroid inequality for convex bodies (even for star bodies with respect to the origin) that established in [28], and the Orlicz Busemann–Petty centroid inequality for convex bodies that established in [30] for even convex function \( \phi \). Our proof of Theorem 1.1 use the traditional approach to establish the \( L_p \) Busemann–Petty centroid inequality [28] and the Orlicz–Busemann–Petty centroid inequality [30] by using Steiner’s symmetrization (see Section §2 for its definition). However, the appearance of the weight function \( \omega \) and working with the decreasing rearrangement function make our proof more complicate. We strongly believe that the shadow system approach (see [2, 22]) or radom approach [5, 39] would give the another proof of Theorem 1.1.

The rest of this paper is organized as follows. In Section §1, we list some basic and well-known facts of convex bodies. Some basic properties of the Orlicz–Lorentz centroid body will be given in Section §3. Section §4 is devoted to prove Theorem 1.1.

## 2 Background material

Schneider’s book [42] is an excellent reference on theory of convex bodies. Our setting will be Euclidean \( n \)-space \( \mathbb{R}^n \). We write \( e_1, e_2, \ldots, e_n \) for the standard orthonormal basis of \( \mathbb{R}^n \) and when we write \( \mathbb{R}^n = \mathbb{R}^{n-1} \times \mathbb{R} \), we always assume that \( e_n \) is associated with the last factor. We will attempt to use \( x, y \) for vectors in \( \mathbb{R}^n \) and \( x', y' \) for vectors in \( \mathbb{R}^{n-1} \). We will also attempt to use \( a, b, s, t \) for numbers in \( \mathbb{R} \) and \( \lambda \) for strictly positive reals. If \( Q \) is a Borel subset of \( \mathbb{R}^n \) and \( Q \) is contained in an \( i \)-dimensional affine subspace of \( \mathbb{R}^n \) but in no affine subspace of lower dimension, then \( |Q| \) will denote the \( i \)-dimensional Lebesgue measure of \( Q \). If \( x \in \mathbb{R}^n \) then by abuse of notation we will write \( |x| \) for the norm of \( x \). For any \( r > 0 \), we denote by \( B_r \) the ball centered at the origin of radius \( r \). The unit ball \( B_1 \) will be written by \( B \) for simplicity. Its volume is \( \omega_n = |B| = \pi^{n/2}/\Gamma(1 + n/2) \). The unit sphere in \( \mathbb{R}^n \) will be denoted by \( S^{n-1} \).
For $A \in GL(n)$ (the set of all invertible $n \times n$ matrices), we write $A^t$ for the transpose of $A$ and $A^{-t}$ for the inverse of the transpose (contragradient) of $A$. Write $|A|$ for the absolute value of the determinant of $A$.

Let $\mathcal{C}$ denote the set of all Orlicz functions $\phi$ on $[0, \infty)$. It is remarkable from its definition that any Orlicz function $\phi \in \mathcal{C}$ is strict increasing in $[0, \infty)$, and hence its inversion function $\phi^{-1}$ exists and is continuous. We say that the sequence $\{\phi_i\}_i$ of Orlicz functions is such that $\phi_i \rightarrow \phi_0 \in \mathcal{C}$ if

$$|\phi_i - \phi_0|_I = \max_{t \in I} |\phi_i(t) - \phi_0(t)| \rightarrow 0,$$

for any compact interval $I \subset [0, \infty)$.

A subset $K \subset \mathbb{R}^n$ is a star-shaped about the origin if for any $x \in K$ then the segment $\{tx : t \in [0, 1]\}$ is contained in $K$. For a star-shaped about the origin $K$, its radial function $\rho_K : \mathbb{R}^n \setminus \{0\} \rightarrow [0, \infty]$ is defined by

$$\rho_K(x) = \max\{\lambda > 0 : \lambda x \in K\}.$$

If $\rho_K$ is strict positive and continuous, then we call $K$ a star body. Let $\mathcal{K}_n^0$ denote the set of all convex bodies containing the origin in its interior of $\mathbb{R}^n$. Note that on $\mathcal{K}_n^0$ the radial distance and Hausdorff distance are equivalent.

For $K \in \mathcal{K}_n^0$, denote

$$R_K = \max_{u \in S^{n-1}} \rho_K(u), \quad r_K = \min_{u \in S^{n-1}} \rho_K(u). \quad (2.1)$$

Since $K \in \mathcal{K}_n^0$ then $0 < r_K \leq R_K < \infty$.

For a convex body $K$ and a direction $u \in S^{n-1}$, let $K_u$ denote the image of the orthogonal projection of $K$ on $u^\perp$, the subspace of $\mathbb{R}^n$ orthogonal to $u$. Let $f_u$ and $g_u$
denote the undergraph and overgraph functions of $K$ in the direction $u$, i.e., $K$ is described by

$$K = \{ y' + tu : -f_u(y') \leq t \leq g_u(y') ; \ y' \in K_u \}.$$ 

Note that $f_u, g_u : K_u \to \mathbb{R}$ are concave functions. For $y' \in K_u$, we define

$$\sigma(y') = \frac{f_u(y') + g_u(y')}{2} \quad \text{and} \quad m(y') = \frac{g_u(y') - f_u(y')}{2}, \quad (2.2)$$

that is, $\sigma(y')$ is a half of the length of the chord $K \cap \{ y' + \mathbb{R}u \}$, and $y' + m(y')u$ is the midpoint of this chord. With these notations, we have another description of $K$ as follows

$$K = \{ y' + (m(y') + t)u : y' \in K_u, |t| \leq \sigma(y') \}.$$ 

The Steiner symmetrization of $K$ in the direction $u$ denoted by $S_uK$ is the convex body defined by

$$S_uK = \{ y' + tu : |t| \leq \sigma(y'), y' \in K_u \}.$$ 

It follows from Fubini’s theorem that $|S_uK| = |K|$ for any $u \in S^{n-1}$. Moreover, for any convex body $K$, there exists a sequence $\{u_i\}_{i \geq 1} \subset S^{n-1}$ such that the sequence of convex bodies $\{S_{u_i}\}_{i \geq 1}$ converges to an origin-centered ball of volume $|K|$. This is the content of Blaschke’s selection theorem.

When considering the convex body $K \subset \mathbb{R}^{n-1} \times \mathbb{R}$, for $(x', t) \in \mathbb{R}^{n-1} \times \mathbb{R}$ we will usually write $h(K, x', t)$ rather than $h(K; (x', t))$. The following Lemma is, in fact, an immediate consequence of Fubini’s theorem.

**Lemma 2.1.** Let $K$ be a convex body in $\mathbb{R}^n$ and $u$ is a direction in $S^{n-1}$. Then the following maps $S : K \to S_uK$ defined by

$$S(y' + (m(y') + t)u) = y' + tu,$$

and $T : K \to K$ defined by

$$T(y' + (m(y') + t)u) = y' + (m(y') - t)u$$

with $y' \in K_u$ and $|t| \leq \sigma(y')$ are volume preserving maps.

The following is well known (see [2]).

**Lemma 2.2.** Suppose $K \in \mathcal{K}_0^n$ and $u \in S^{n-1}$. For any $y' \in \text{relint}(K_u)$, the overgraph and undergraph functions of $K$ in direction $u$ are given by

$$g_u(y') = \min_{x' \in u^\perp} h(K, x' + u) - x' \cdot y', \quad (2.3)$$

and

$$f_u(y') = \min_{x' \in u^\perp} h(K, x' - u) - x' \cdot y'. \quad (2.4)$$
The following estimate was proved in [30].

**Lemma 2.3.** Suppose $K \in \mathcal{K}_0^n$ and $u \in S^{n-1}$. If $y' \in (r_K/2)B \cap u^\perp$ and $x_1', x_2' \in u^\perp$ are such that

$$g_u(y') = h(K, x_1' + tu) - x_1' \cdot y' \quad \text{and} \quad f_u(y') = h(K, x_2' - tu) - x_2' \cdot y',$$

then both

$$|x_1'|, |x_2'| \leq \frac{2R_K}{r_K}.$$

Finally, in order to establish the equality case in our Orlicz–Lorentz Busemann–Petty centroid inequality in Theorem 1.1, we need to know which characterizations of $K \in \mathcal{K}_0^n$ are to be an origin–centered ellipsoid. A classical result says that a convex body $K \in \mathcal{K}_0^n$ is an origin–centered ellipsoid if and only if for any direction $u \in S^{n-1}$ all of the midpoints of the chords of $K$ parallel to $u$ lie in a subspace of $\mathbb{R}^n$. In our proof, we need the following characterization of the ellipsoid due to Gruber and Ludwig [12, 13].

**Lemma 2.4.** A convex body $K \in \mathcal{K}_0^n$ is an origin–centered ellipsoid if and only if there exists an $\epsilon_K > 0$ such that for any direction $u \in S^{n-1}$ all of the chords of $K$ that come within a distance of $\epsilon_K$ of the origin and are parallel to $u$, have midpoints that lie in a subspace of $\mathbb{R}^n$.

### 3 Basic properties of Orlicz–Lorentz centroid bodies

Recall that if $\phi \in \mathcal{C}$ is an Orlicz function, $\omega : (0, 1) \to (0, \infty)$ is a weight function and $K \in \mathcal{S}_0^n$, then the Orlicz–Lorentz centroid body of $K$ denoted by $\Gamma_{\phi, \omega}K$ is defined to be a convex body whose support function is

$$h(\Gamma_{\phi, \omega}K, x) = \|f_{x, K}\|_{\lambda_{\phi, \omega}} = \inf \left\{ \lambda > 0 : \int_0^1 \phi \left( \frac{f_{x, K}(t)}{\lambda} \right) \omega(t) dt \leq 1 \right\}. \quad (3.1)$$

Since $\phi \in \mathcal{C}$, and $\omega$ is strictly positive then $h(\Gamma_{\phi, \omega}K, x) > 0$ for any $x \neq 0$.

Note that the function

$$\Phi : \lambda \in (0, 1) \mapsto \int_0^1 \phi \left( \frac{f_{x, K}(t)}{\lambda} \right) \omega(t) dt$$

is strictly decreasing, continuous on $(0, \infty)$ since $\int_0^1 \omega(t) dt < \infty$. Moreover, it satisfies

$$\lim_{\lambda \to 0^+} \Phi(\lambda) = \infty, \quad \lim_{\lambda \to \infty} \Phi(\lambda) = 0.$$

Thus we easily get the following.
Lemma 3.1. Suppose that $K \in S^n_0$ and $u_0 \in S^{n-1}$. Then

$$\int_0^1 \phi \left( \frac{f_{u_0,K}(t)}{\lambda_0} \right) \omega(t)dt = 1$$

if and only if

$$h(\Gamma_{\phi,\omega}K, u_0) = \lambda_0.$$

Since $\| \cdot \|_{\Lambda_{\phi,\omega}}$ is a norm on $\Lambda_{\phi,\omega}$, thus we have:

Lemma 3.2. If $K \in S^n_0$ then $h(\Gamma_{\phi,\omega}K, \cdot)$ is the support function of an origin–centered convex body in $K^n_0$.

Next lemma gives us the upper and lower bounds for the support function $h(\Gamma_{\phi,\omega}K, \cdot)$.

Lemma 3.3. If $K \in S^n_0$ then

$$\frac{1}{r_K f_{u,B}^*(1/2) \phi^{-1} \left( \frac{1}{\int_0^1 \omega(t)dt} \right)} \leq h(\Gamma_{\phi,\omega}K, u) \leq \frac{R_K}{\phi^{-1} \left( \frac{1}{\int_0^1 \omega(t)dt} \right)}$$

for any $u \in S^{n-1}$, where $r_K, R_K$ is defined by (2.1), $\phi^{-1}$ denotes the inverse function of $\phi$, $f_{u,B}^*$ is the decreasing rearrangement function of the function $x \mapsto u \cdot x$ on $B$ and is defined on the measure space $(B, \mathcal{B}_B, \mu^B)$ and

$$c(n, K) = \frac{r_K^2 \omega_n}{2|K|}.$$

Proof. We follow the argument in the proof of Lemma 2.3 in [30]. Given $u \in S^{n-1}$ and suppose that $h(\Gamma_{\phi,\omega}K, u) = \lambda_0$, by Lemma 3.1 we have

$$\int_0^1 \phi \left( \frac{f_{u,K}(t)}{\lambda_0} \right) \omega(t)dt = 1. \quad (3.2)$$

We first prove the upper bound. By the definition of $R_K$, we have $K \subseteq R_K B$ which implies $|f_{u,K}(x)| \leq R_K$ for any $x \in K$, hence $f_{u,K}^*(t) \leq R_K$ for any $t \in (0, 1)$. Thus, by (3.2) and the strict increasing monotonicity of $\phi$, we obtain

$$\phi \left( \frac{R_K}{\lambda_0} \right) \int_0^1 \omega(t)dt \geq 1,$$

or equivalently,

$$\lambda_0 \leq \frac{R_K}{\phi^{-1} \left( \frac{1}{\int_0^1 \omega(t)dt} \right)}.$$

We next prove the lower bound. By the definition of $r_K$, we have $r_K B \subseteq K$ then we have

$$\{ x \in K : |x \cdot u| > t \} \supseteq r_K \left\{ x \in B : |x \cdot u| > \frac{t}{r_K} \right\}, \quad \forall t > 0,$$
which then implies
\[ \mu^K_{f_{u,K}}(t) \geq \frac{r^n_K \omega_n}{|K|} \mu^B_{f_{u,B}} \left( \frac{t}{r_K} \right), \quad \forall t > 0. \]
Thus, by the definition of the decreasing rearrangement function, we readily obtain
\[ f^*_{u,K}(t) \begin{cases} r_K f^*_u(\frac{|K|}{r_K} t) & \text{if } t < \frac{r^n_K \omega_n}{|K|}, \\ 0 & \text{if } \frac{r^n_K \omega_n}{|K|} \leq t < 1. \end{cases} \]
Denote \( c(n,K) = \frac{r^n_K \omega_n}{(2|K|)} \) we then have
\[ f^*_{u,K}(t) \geq r_K f^*_u \left( \frac{1}{2} \right), \quad \forall t \in (0, c(n,K)]. \] (3.3)
It follows from (3.2), (3.3) and the strictly increasing monotonicity of \( \phi \) that
\[ \phi \left( \frac{r_K f^*_u(1/2)}{\lambda_0} \right) \int_0^{c(n,K)} \omega(t) dt \leq 1, \]
or equivalently,
\[ \lambda_0 \geq \frac{1}{r_K f^*_u(1/2) \phi^{-1} \left( \frac{1}{\int_0^{c(n,K)} \omega(t) dt} \right)}. \]

Since \( f^*_{u,B} \) does not depend on \( u \in S^{n-1} \), hence Lemma 3.3 gives a lower bound of \( h(\Gamma_{\phi,\omega} K, u) \) which is independent of \( u \in S^{n-1} \). In the next lemma, we show that the Orlicz-Lorentz centroid operator \( \Gamma_{\phi,\omega} \) commutes with any \( A \in GL(n) \).

**Lemma 3.4.** If \( K \in S^n_0 \) and \( A \in GL(n) \) then
\[ \Gamma_{\phi,\omega}(AK) = A \Gamma_{\phi,\omega} K. \]

**Proof.** Let \( u \in \mathbb{R}^n \), it is evident that \( f_{u,AK}(x) = f_{A^t u, K}(x) \) for any \( x \in \mathbb{R}^n \). Hence
\[ \{ x \in AK : |f_{u,AK}(x)| > s \} = A \{ x \in K : |f_{u,K}(x)| > s \}, \]
which then implies \( \mu^K_{f_{u,A^t u, K}}(s) = \mu^K_{f_{u,K}}(s) \) for any \( s > 0 \). Consequently, we get
\[ f^*_{u,AK}(t) = f^*_{A^t u, K}(t), \quad \forall t \in (0, 1). \]
The definition of the Orlicz–Lorentz centroid body (3.1) then yields
\[ h(\Gamma_{\phi,\omega}(AK), u) = h(\Gamma_{\phi,\omega} K, A^t u) = h(A \Gamma_{\phi,\omega} K, u), \]
for any \( u \in \mathbb{R}^n \). This finishes our proof. \( \square \)
We next prove that the Orlicz-Lorentz centroid operator \( \Gamma_{\phi,\omega} : S_0^n \to K_0^n \) is continuous.

**Lemma 3.5.** Let \( \phi \in C \) be an Orlicz function and \( \omega \) is a weight function on \((0,1)\). If \( K_i, K \in S_0^n, i \geq 1 \) and \( K_i \to K \) in \( S_0^n \) then \( \Gamma_{\phi,\omega} K_i \to \Gamma_{\phi,\omega} K \) in \( K_0^n \).

**Proof.** Since \( \Gamma_{\phi,\omega} K_i, \Gamma_{\phi,\omega} K \in K_0^n \), \( i \geq 1 \), it is enough to show that
\[
\lim_{i \to \infty} h(\Gamma_{\phi,\omega} K_i, u) = h(\Gamma_{\phi,\omega} K, u),
\]
for any \( u \in S_{n-1} \). Fix a vector \( u \in S_{n-1} \), denote \( \lambda_i = h(\Gamma_{\phi,\omega} K_i, u) \).

Lemma 3.3 says that
\[
1 \frac{1}{r_{K_i} f^*_{u,B}(1/2)(\phi^{-1}\left(\int_0^t \omega(s) dt\right))} \leq \lambda_i \leq \frac{R_{K_i}}{\phi^{-1}\left(\int_0^t \omega(s) dt\right)} , \quad \forall i \geq 1.
\]

Since \( K_i \to K \) in \( S_0^n \) then we have
\[
|K_i| \to |K| , \quad r_{K_i} \to r_K > 0 , \quad \text{and} \quad R_{K_i} \to R_K < \infty .
\]
Thus, we get \( c(n, K_i) \to c(n, K) > 0 \). Hence there exists positive constants \( a, b \) such that
\[
a \leq \lambda_i \leq b , \quad \forall i \geq 1 . \tag{3.4}
\]

We next show that
\[
f^*_{u,K_i}(t) \to f^*_{u,K}(t) \quad \text{for a.e } t \in (0,1) . \tag{3.5}
\]

For \( s > 0 \), denote
\[
A_i(s) = \{ x \in K_i : |u \cdot x| > s \} \quad \text{and} \quad A(s) = \{ x \in K : |u \cdot x| > s \} .
\]

Let \( A \Delta B \) denote the symmetric difference of two measurable subsets \( A, B \subset \mathbb{R}^n \), i.e.,
\[
A \Delta B = (A \setminus B) \cup (B \setminus A) .
\]
We claim that
\[
A_i(s) \Delta A(s) \subset K_i \Delta K , \quad \forall i \geq 1 . \tag{3.6}
\]
Indeed, if \( x \in A_i(s) \setminus A(s) \), we must have \( x \in K_i \) and \( |u \cdot x| > s \). This implies that \( x \not\in K \) since if \( x \in K \) then \( x \in A(s) \) which is a contradiction. Hence \( x \in K_i \setminus K \). Similarly, if \( x \in A(s) \setminus A_i(s) \) then \( x \in K \setminus K_i \). Our claim (3.6) is proved.

By our assumption \( K_i \to K \) in \( S_0^n \), we have \( |K_i \Delta K| \to 0 \). This fact and our claim (3.6) yield
\[
\lim_{i \to \infty} |A_i(s) \Delta A(s)| = 0 . \tag{3.7}
\]
Let \( t \in (0, 1) \) be an arbitrary point. For any \( s < f^*_{u,K}(t) \), we must have \( \mu^K_{f_{u,K}}(s) > t \). It is obvious that \( A(s) \subset A_i(s) \cup (A(s) \setminus A_i(s)) \), then
\[
|A(s)| \leq |A_i(s)| + |A(s) \setminus A_i(s)|.
\]
Combining this inequality and (3.7) proves that
\[
\liminf_{i \to \infty} \mu^K_{f_{u,K_i}}(s) = \liminf_{i \to \infty} \frac{|A_i(s)|}{|K_i|} \geq \frac{|A(s)|}{|K|} = \mu^K_{f_{u,K}}(s) > t.
\]
Therefore, there exists \( i_0 \) such that
\[
\mu^K_{f_{u,K_i}}(s) > t, \quad \forall i \geq i_0.
\]
This implies
\[
f^*_{u,K_i}(t) \geq s, \quad \forall i \geq i_0,
\]
and hence
\[
\liminf_{i \to \infty} f^*_{u,K_i}(t) \geq s.
\]
Since \( s < f^*_{u,K}(t) \) is arbitrary, then
\[
\liminf_{i \to \infty} f^*_{u,K_i}(t) \geq f^*_{u,K}(t). \tag{3.8}
\]
Let \( t \in (0, 1) \) be a left continuous point of \( f^*_{u,K} \). For any \( s > f^*_{u,K}(t) \), there exists \( t' \in (0, t) \) such that \( f^*_{u,K}(t') < s \) by the left continuity of \( f^*_{u,K} \) at \( t \). Consequently, by the definition of \( f^*_{u,K} \), we have
\[
\mu_{f_{u,K}}(s) \leq t' < t.
\]
Note that \( A_i(s) \subset A(s) \cup (A_i(s) \setminus A(s)) \), hence
\[
|A_i(s)| \leq |A(s)| + |A_i(s) \setminus A(s)|.
\]
Combining this inequality and (3.7) proves that
\[
\limsup_{i \to \infty} \mu^K_{f_{u,K_i}}(s) = \limsup_{i \to \infty} \frac{|A_i(s)|}{|K_i|} \leq \frac{|A(s)|}{|K|} = \mu_{f_{u,K}}(s) < t,
\]
here we use (3.7). Thus, there exists \( i_1 \) such that
\[
\mu^K_{f_{u,K_i}}(s) < t, \quad \forall i \geq i_1.
\]
By definition of \( f^*_{u,K_i} \), we obtain
\[
f^*_{u,K_i}(t) \leq s, \quad \forall i \geq i_1,
\]
which implies
\[
\limsup_{i \to \infty} f^*_{u,K_i}(t) \leq s.
\]
Since $s > f^*_u,K(t)$ is arbitrary, then
\[
\limsup_{i \to \infty} f^*_{u,K_i}(t) \leq f^*_u,K(t). \tag{3.9}
\]

(3.8) and (3.9) show that $f^*_{u,K_i}(t) \to f^*_u,K(t)$ for any left continuous point $t$ of $f^*_u,K$. This proves (3.5) since $f^*_u,K$ is left continuous a.e in $(0,1)$ because of the nonincreasing monotonicity.

Let $\{\lambda_i\}_k$ be an arbitrary subsequence of $\{\lambda_i\}_i$. Since $\{\lambda_i\}_k$ is bounded (by $a$ and $b$, see (3.4)), it possesses a subsequence (still denoted by $\{\lambda_i\}_k$) converging to $\lambda_0$. Obviously, we have $a \leq \lambda_0 \leq b$. We have from definition of $h(\Gamma_{\phi,\omega}K_K,u)$ that
\[
\int_0^1 \phi \left( \frac{f^*_{u,K_i}(t)}{\lambda_{i_k}} \right) \omega(t) dt = 1,
\]
for any $k \geq 1$. Since $R_0 = \sup\{R_i : i \geq 1\} < \infty$, hence
\[
\phi \left( \frac{f^*_{u,K_i}(t)}{\lambda_{i_k}} \right) \omega(t) \leq \phi \left( \frac{R_0}{a} \right) \omega(t) \in L_1((0,1)).
\]

Letting $k \to \infty$ and using the dominated convergent theorem and (3.5), we get
\[
\int_0^1 \phi \left( \frac{f^*_{u,K}(t)}{\lambda_0} \right) \omega(t) dt = 1,
\]
or $\lambda_0 = h(\Gamma_{\phi,\omega}K,u)$ by Lemma 3.1. Since $\{\lambda_i\}_k$ is an arbitrary subsequence of $\{\lambda_i\}_i$, then we have
\[
\lim_{i \to \infty} h(\Gamma_{\phi,\omega}K_i,u) = h(\Gamma_{\phi,\omega}K,u).
\]
This Lemma is completely proved. \hfill \Box

We next show that the Orlicz-Lorentz centroid operator is continuous in $\phi$. Recall that $\phi_i \to \phi \in C$ if $\phi_i$ uniformly converges to $\phi$ on any compact interval of $[0, \infty)$.

**Lemma 3.6.** If $\phi_i \to \phi \in C$, then $\Gamma_{\phi_i,\omega}K \to \Gamma_{\phi,\omega}K$ for any $K \in S^n_0$ and the weight function $\omega$ on $(0,1)$.

**Proof.** Suppose that $K \in S^n_0$, it is enough to prove that
\[
h(\Gamma_{\phi_i,\omega}K,u) \to h(\Gamma_{\phi,\omega}K,u) \tag{3.10}
\]
for any $u \in S^{n-1}$. Denote
\[
\lambda_i = h(\Gamma_{\phi_i,\omega}K,u), \quad \forall i \geq 1.
\]
It implies from Lemma 3.3 that
\[
\frac{1}{r_K f^*_u(i/2) \phi_i^{-1} \left( \frac{1}{\int_0^{(n,K)} \omega(t)dt} \right)} \leq \lambda_i \leq \frac{R_K \phi_i^{-1} \left( \frac{1}{\int_0^1 \omega(t)dt} \right)}{\phi_i^{-1} \left( \frac{1}{\int_0^{(n,K)} \omega(t)dt} \right)}, \quad \forall i \geq 1. \tag{3.11}
\]

We first prove that
\[
\phi_i^{-1}(a) \rightarrow \phi^{-1}(a) \tag{3.12}
\]
for all \(a > 0\). Indeed, we can choose \(M, m > 0\) such that
\[
0 < 2\phi(m) < a < \frac{1}{2} \phi(M) < \infty.
\]
Since \(\phi_i \rightarrow \phi\) in \(C\) then there exists \(i_0\) such that
\[
\phi_i(m) < a < \phi_i(M), \quad \forall \ i \geq i_0,
\]
or equivalently
\[
m < \phi_i^{-1}(a) < M, \quad \forall \ i \geq i_0.
\]
Hence \(\{\phi_i^{-1}(a)\}_i\) is bounded. Suppose that \(\{\phi_i^{-1}(a)\}_k\) is a subsequence of \(\{\phi_i^{-1}(a)\}\) and converges to \(b\). Obviously, we have \(m \leq b \leq M\). Since \(\phi_i \rightarrow \phi\) in \(C\), then
\[
a = \lim_{k \rightarrow \infty} \phi_i_k(\phi_i^{-1}(a)) = \phi(b),
\]
or \(b = \phi^{-1}(a)\). We have shown that \(\{\phi_i^{-1}(a)\}_i\) has at most one accumulation point which is \(\phi^{-1}(a)\) if it exists. Since \(\{\phi_i^{-1}(a)\}_i\) is bounded, then (3.12) holds.

Combining (3.11) and (3.12) implies the existence of \(a, b > 0\) such that \(a < \lambda_i < b\) for any \(i \geq 1\). Suppose that \(\{\lambda_{i_k}\}_k\) is a subsequence of \(\{\lambda_i\}_i\) and converges to \(\lambda_0\). From the definition of \(h(\Gamma_{\phi,\omega}K, u)\) we have
\[
\int_0^1 \phi_{i_k} \left( \frac{f_{u,K}^*(t)}{\lambda_{i_k}} \right) \omega(t)dt = 1,
\]
for any \(k \geq 1\). Moreover, it holds
\[
\phi_{i_k} \left( \frac{f_{u,K}^*(t)}{\lambda_{i_k}} \right) \omega(t) \leq \phi_{i_k} \left( \frac{R_K}{a} \right) \omega(t) \leq \left( \sup_{i \geq 1} \phi_i \left( \frac{R_K}{a} \right) \right) \omega(t) \in L_1((0, 1)).
\]
Letting \(k \rightarrow \infty\) and using the dominated convergence theorem and the assumption \(\phi_i \rightarrow \phi\) in \(C\), we get
\[
\int_0^1 \phi \left( \frac{f_{u,K}^*(t)}{\lambda_0} \right) \omega(t)dt,
\]
or \(\lambda_0 = h(\Gamma_{\phi,\omega}K, u)\) by Lemma 3.1. We thus have shown that the sequence \(\{\lambda_i\}_i\) has at most one accumulation point which is \(h(\Gamma_{\phi,\omega}K, u)\) if it exists. Since \(\{\lambda_i\}_i\) is bounded, then (3.10) holds as desired. This finishes our proof.
4 Proof of Theorem 1.1

The next lemma plays crucial role in our proof of Theorem 1.1.

Lemma 4.1. Let \( \phi \in \mathcal{C} \) be an Orlicz function, \( \omega \) is a weight function on \((0, 1)\), and \( K \in K_0^n \).

If \( u \in S^{n-1} \) and \( x_1', x_2' \in u_\perp \), then

\[
h \left( \Gamma_\phi, \omega(S_u K), \frac{1}{2} x_1' + \frac{1}{2} x_2' + u \right) \leq \frac{1}{2} h(\Gamma_\phi, \omega K, x_1' + u) + \frac{1}{2} h(\Gamma_\phi, \omega K, x_2' - u). \quad (4.1)
\]

Equality in (4.1) implies that all of the chords of \( K \) parallel to \( u \), whose distance from the origin is less than \( r_K/(2 \max\{1, |x_1'|, |x_2'|\}) \) have the midpoints that lie in the subspace

\[
\left\{ y' + \frac{1}{2}(x_2' - x_1'), y' : y' \in u_\perp \right\}
\]
of \( \mathbb{R}^n \).

Proof. By the Lemma 3.4 we can assume, without of loss generality, that \( |K| = |S_u K| = 1 \). Let \( K_u \) denote the image of the orthogonal projection of \( K \) onto the subspace \( u_\perp \). If \( y' \in K_u \), define \( \sigma(y') \) and \( m(y') \) as in (2.2). Denote

\[
x_1 = x_1' + u, \quad x_2 = x_2' - u, \quad x = \frac{1}{2} x_1' + \frac{1}{2} x_2' + u, \quad \text{and} \quad \lambda_i = h(\Gamma_\phi, \omega K, x_i), \quad i = 1, 2.
\]

From the Lemma 3.1 we have

\[
\int_0^1 \phi \left( \frac{f_{x_1,K}(t)}{\lambda_i} \right) \omega(t) dt = 1, \quad i = 1, 2.
\]

If \( y = y' + (m(y') + t)u \in K \) then

\[
f_{x_1,K}(y) = x_1' \cdot y' + m(y') + t, \quad f_{x_2,K}(Ty) = x_2' \cdot y' - m(y') + t, \quad \text{and} \quad Sy = y' + tu,
\]

where \( S, T \) are maps given in the Lemma 2.1. Hence we have

\[
f_{x,S_u K}(Sy) = \frac{1}{2} (x_1' \cdot y' + x_2' \cdot y') + t = \frac{1}{2} f_{x_1,K}(y) + \frac{1}{2} f_{x_2,K}(Ty) =: f(y).
\]

The volume preserving property of \( S \) (by Lemma 2.1) yields \( f_{x,S_u K}^* = f^* \). Similarly, the volume preserving property of \( T \) implies that \( f_{x_2,K} \circ T \) and \( f_{x_2,K} \) have the same decreasing rearrangement function.

Denote \( \lambda = (\lambda_1 + \lambda_2)/2 \). Since \( \phi \in \mathcal{C} \), then \( \phi(g^*) = (\phi(|g|))^* \) holds for any measurable function \( g \). This identity implies

\[
\phi \left( \frac{f_{x,S_u K}^*}{\lambda} \right) = \phi \left( \frac{(f_{x_1,K} + f_{x_2,K} \circ T)^*}{\lambda_1 + \lambda_2} \right) = \left( \phi \left( \frac{|f_{x_1,K} + f_{x_2,K} \circ T|}{\lambda_1 + \lambda_2} \right) \right)^*. \quad (4.2)
\]

14
It is obvious that
\[
\frac{|f_{x_1,K} + f_{x_2,K} \circ T|}{\lambda_1 + \lambda_2} \leq \frac{\lambda_1}{\lambda_1 + \lambda_2} \frac{|f_{x_1,K}|}{\lambda_1} + \frac{\lambda_2}{\lambda_1 + \lambda_2} \frac{|f_{x_2,K} \circ T|}{\lambda_2}. \tag{4.3}
\]

The increasing monotonicity and convexity of \( \phi \) together (4.3) imply
\[
\phi\left(\frac{|f_{x_1,K} + f_{x_2,K} \circ T|}{\lambda_1 + \lambda_2}\right) \leq \frac{\lambda_1}{\lambda_1 + \lambda_2} \phi\left(\frac{|f_{x_1,K}|}{\lambda_1}\right) + \frac{\lambda_2}{\lambda_1 + \lambda_2} \phi\left(\frac{|f_{x_2,K} \circ T|}{\lambda_2}\right). \tag{4.4}
\]

The decreasing rearrangement preserves the order on the positive functions. This fact together (4.2) and (4.4) prove that
\[
\phi\left(\frac{f_{x,Su,K}^*}{\lambda}\right) \leq \left(\frac{\lambda_1}{\lambda_1 + \lambda_2} \phi\left(\frac{|f_{x_1,K}|}{\lambda_1}\right) + \frac{\lambda_2}{\lambda_1 + \lambda_2} \phi\left(\frac{|f_{x_2,K} \circ T|}{\lambda_2}\right)\right)^*. \tag{4.5}
\]

Multiplying both sides of (4.5) by \( \omega \), then integrating the obtained inequality on \((0, 1)\) and using the known fact
\[
\int_0^1 (g_1 + g_2)\* (t) \omega(t) dt \leq \int_0^1 g_1^\* (t) \omega(t) dt + \int_0^1 g_2^\* (t) \omega(t) dt, \tag{4.6}
\]
and again the equality \( \phi(g^\*) = (\phi(|g|))^\* \), we obtain
\[
\int_0^1 \phi\left(\frac{f_{x,Su,K}^*}{\lambda}(t)\right) \omega(t) dt
\leq \int_0^1 \left(\frac{\lambda_1}{\lambda_1 + \lambda_2} \phi\left(\frac{|f_{x_1,K}|}{\lambda_1}\right) + \frac{\lambda_2}{\lambda_1 + \lambda_2} \phi\left(\frac{|f_{x_2,K} \circ T|}{\lambda_2}\right)\right)^*(t) \omega(t) dt
\leq \frac{\lambda_1}{\lambda_1 + \lambda_2} \int_0^1 \phi\left(\frac{f_{x_1,K}^*}{\lambda_1}(t)\right) \omega(t) dt + \frac{\lambda_2}{\lambda_1 + \lambda_2} \int_0^1 \phi\left(\frac{f_{x_2,K}^*}{\lambda_2}(t)\right) \omega(t) dt
= 1, \tag{4.7}
\]
here we used the property that \( f_{x_2,K} \) and \( f_{x_2,K} \circ T \) have the same decreasing rearrangement function. The latter inequality (4.7) and the definition of \( h(\Gamma_{\phi,\omega}SuK, \cdot) \) prove
\[
h(\Gamma_{\phi,\omega}(SuK), x) \leq \lambda = \frac{1}{2} h(\Gamma_{\phi,\omega}K, x_1) + \frac{1}{2} h(\Gamma_{\phi,\omega}K, x_2),
\]
as our desired inequality (4.1).

Suppose that equality holds in (4.1). Thus we must have equalities in (4.3), (4.4) for a.e \( y \in K \) and equality in (4.6) for \( f_{x_1,K} \) and \( f_{x_2,K} \circ T \). The continuity of \( f_{x_1,K} \) and \( f_{x_2,K} \circ T \) on \( K \) implies that (4.3) holds on whole \( K \). Hence, for any fixed \( y' \in K_u \), the signs of \( x_1' \cdot y' + m(y') + t \) and \( x_2' \cdot y' - m(y') + t \) coincide for all \( |t| \leq \sigma(y') \).

For any \( y' \in K_u \) and \( |y'| \leq \frac{\sqrt{3}}{2}r_Ku \), we have
\[
y' \pm \frac{\sqrt{3}}{2}r_Ku \in r_KB \subset K,
\]
15
hence \( g_u(y') \geq \sqrt{3}r_K/2 > r_K/2 \) and \(-f_u(y') \leq -\sqrt{3}r_K/2 < -r_K/2\). Thus we have proved the following inclusion

\[
\left(-\frac{r_K}{2}, \frac{r_K}{2}\right) \subset (m(y') - \sigma(y'), m(y') + \sigma(y'))
\]

(4.8)

and hence

\[
\left(-\frac{r_K}{2}, \frac{r_K}{2}\right) \subset (-m(y') - \sigma(y'), -m(y') + \sigma(y'))
\]

(4.9)

Suppose that \( y' \in K_u \) and

\[
|y'| \leq \frac{r_K}{2\max\{1, |x'_1|, |x'_2|\}}.
\]

The inclusions (4.8) and (4.9) imply

\[
x'_1.y' + m(y') \in (-\sigma(y'), \sigma(y')) \quad \text{and} \quad x'_2.y' - m(y') \in (-\sigma(y'), \sigma(y')).
\]

Hence the affine functions

\[
t \mapsto x'_1.y' + m(y') + t \quad \text{and} \quad t \mapsto x'_2.y' - m(y') + t
\]

both have their root in \((-\sigma(y'), \sigma(y'))\). However, we know that they have the same sign on this interval, then they must have a root at the same \( t(y') \in (-\sigma(y'), \sigma(y')) \). This assertion yields

\[
(x'_2 - x'_1).y' = 2m(y').
\]

Therefore, we have proved that for any \( y' \in K_u \) with \( |y'| \leq r_K/(2\max\{1, |x'_1|, |x'_2|\}) \), the midpoints

\[
\left\{ y' + m(y')u : y' \in K_u \right\}
\]

of the chords of \( K \) parallel to \( u \) lie in the subspace

\[
\left\{ y' + \frac{1}{2}(x'_2 - x'_1).y'u : y' \in u^\perp \right\}
\]

of \( \mathbb{R}^n \) as our desired.

From the inequality (4.1), we deduce the following inequality

\[
h\left(\Gamma_{\phi,\omega}(S_uK), \frac{1}{2}x'_1 + \frac{1}{2}x'_2 - u\right) \leq \frac{1}{2}h(\Gamma_{\phi,\omega}K, x'_1 + u) + \frac{1}{2}h(\Gamma_{\phi,\omega}K, x'_2 - u).
\]

Lemma 4.2. Let \( \phi \in \mathcal{C} \) be an Orlicz function, \( \omega \) is a weight function on \((0, 1)\) and \( K \in \mathcal{K}_0^n \).

If \( u \in S^{n-1} \) then

\[
\Gamma_{\phi,\omega}(S_uK) \subset S_u(\Gamma_{\phi,\omega}K).
\]

(4.10)

If the inclusion is an identity then all chords of \( K \) parallel to \( u \), whose distance from the origin is less than \( r_K/4\Gamma_{\phi,\omega}K/(4R_{\Gamma_{\phi,\omega}K}) \) have the midpoints that lie in a subspace of \( \mathbb{R}^n \).
Proof. For any compact subset \( L \subset \mathbb{R}^n \) and any unit vector \( u \in S^{n-1} \), we denote by \( L_u \) the orthogonal image of \( L \) on \( u^\perp \). For any \( y' \in L_u \), we define
\[
g_u(L, y') = \sup \{ t : y' + tu \in L \}
\]
and
\[
f_u(L, y') = -\inf \{ t : y' + tu \in L \} = \sup \{ -t : y' + tu \in L \}.
\]
Given \( y' \in \text{relint}(\Gamma_{\phi,\omega}K)_u \), by Lemma 2.2, there exist \( x'_1, x'_2 \in u^\perp \) such that
\[
g_u(\Gamma_{\phi,\omega}K, y') = h(\Gamma_{\phi,\omega}K, x'_1 + u) - x'_1.y',
\]
and
\[
f_u(\Gamma_{\phi,\omega}K, y') = h(\Gamma_{\phi,\omega}K, x'_2 - u) - x'_2.y'.
\]
Combining (4.11) and (4.12) together Lemma 4.1 imply
\[
g_u(S_u(\Gamma_{\phi,\omega}K), y') = \frac{1}{2}(g_u(\Gamma_{\phi,\omega}K, y') + f_u(\Gamma_{\phi,\omega}K, y'))
\]
\[
= \frac{1}{2}(h(\Gamma_{\phi,\omega}K, x'_1 + u) + h(\Gamma_{\phi,\omega}K, x'_2 - u)) - \frac{1}{2}(x'_1 + x'_2).y'
\]
\[
\geq h \left( \Gamma_{\phi,\omega}(S_uK), \frac{1}{2}(x'_1 + x'_2) + u \right) - \frac{1}{2}(x'_1 + x'_2).y'
\]
\[
\geq \min_{x' \in u^\perp} \{ h(\Gamma_{\phi,\omega}(S_uK), x' + u) - x'.y' \}
\]
\[
= g_u(\Gamma_{\phi,\omega}(S_uK), y'),
\]
and
\[
f_u(S_u(\Gamma_{\phi,\omega}K), y') = \frac{1}{2}(g_u(\Gamma_{\phi,\omega}K, y') + f_u(\Gamma_{\phi,\omega}K, y'))
\]
\[
= \frac{1}{2}(h(\Gamma_{\phi,\omega}K, x'_1 + u) + h(\Gamma_{\phi,\omega}K, x'_2 - u)) - \frac{1}{2}(x'_1 + x'_2).y'
\]
\[
\geq h \left( \Gamma_{\phi,\omega}(S_uK), \frac{1}{2}(x'_1 + x'_2) - u \right) - \frac{1}{2}(x'_1 + x'_2).y'
\]
\[
\geq \min_{x' \in u^\perp} \{ h(\Gamma_{\phi,\omega}(S_uK), x' - u) - x'.y' \}
\]
\[
= f_u(\Gamma_{\phi,\omega}(S_uK), y').
\]
These two inequalities prove the inclusion (4.10).

Now suppose that the inclusion (4.10) is an identity, then for any \( y' \in (\Gamma_{\phi,\omega}K)_u \) we have
\[
g_u(S_u(\Gamma_{\phi,\omega}K), y') = g_u(\Gamma_{\phi,\omega}(S_uK), y') \quad \text{and} \quad g_u(S_u(\Gamma_{\phi,\omega}K), y') = f_u(\Gamma_{\phi,\omega}(S_uK), y').
\]
(4.13)

For each \( y' \in (\Gamma_{\phi,\omega}K)_u \) and \(|y'| \leq r_{\Gamma_{\phi,\omega}K}/2\), by Lemma 2.2, there exist \( x'_1, x'_2 \in u^\perp \) such that
\[
g_u(\Gamma_{\phi,\omega}K, y') = h(\Gamma_{\phi,\omega}K, x'_1 + u) - x'_1.y',
\]

17
and

\[ f_u(\Gamma_{\phi,\omega}K, y') = h(\Gamma_{\phi,\omega}K, x'_2 - u) - x'_2 y'. \]

Lemma 2.3 implies

\[ |x'_1| \leq \frac{2R_{\phi,\omega}}{r_{\phi,\omega}} \quad \text{and} \quad |x'_2| \leq \frac{2R_{\phi,\omega}}{r_{\phi,\omega}}. \]

Equalities in (4.13) deduce that

\[ h\left( \Gamma_{\phi,\omega}(S_uK), \frac{1}{2}(x'_1 + x'_2) + u \right) = \frac{1}{2} h(\Gamma_{\phi,\omega}K, x'_1 + u) + \frac{1}{2} h(\Gamma_{\phi,\omega}K, x'_2 - u). \]

By Lemma 4.1, all the chords of \( K \) parallel to \( u \) whose distance from the origin is less than \( r_K/(2 \max\{1, |x'_1|, |x'_2|\}) \) have midpoints that lie in a subspace of \( \mathbb{R}^n \). However, the following estimate

\[ \frac{r_K}{2 \max\{1, |x'_1|, |x'_2|\}} \geq \frac{r_K r_{\phi,\omega}}{4R_{\phi,\omega}}, \]

holds, which then proves the conclusion of this Lemma.

An immediate consequence of Lemma 4.11 and Lemma 2.4 reads as follows.

**Corollary 4.3.** Let \( \phi \in C \) be an Orlicz function, \( \omega \) is a weight function on \((0, 1)\) and \( K \in K_0^n \). If \( u \in S^{n-1} \) then

\[ \Gamma_{\phi,\omega}(S_uK) \subset S_u(\Gamma_{\phi,\omega}K). \]

If the inclusion is an identity for any \( u \in S^{n-1} \), then \( K \) is an ellipsoid centered at the origin.

With Corollary 4.3 in hand, we now ready prove our main theorem (i.e., Theorem 1.1) by using Steiner’s symmetrization method.

**Proof of Theorem 1.1.** Let \( K \in K_0^n \). By using Blaschke’s selection principle, we can take a sequence of unit vectors \( \{u_i\}_i \subset S^{n-1} \) such that the sequence of convex bodies defined by

\[ K_0 := K, \quad K_i = S_{u_i}K_{i-1}, \quad i \geq 1 \]

converges to \((|K|/\omega_n)^{1/n} B \) in \( K_0^n \). Thank to Lemma 3.4 and Lemma 3.5, we have

\[ \lim_{i \to \infty} \Gamma_{\phi,\omega}K_i = \left( \frac{|K|}{\omega_n} \right)^{\frac{1}{n}} \Gamma_{\phi,\omega}B \quad \text{in} \quad K_0^n. \quad (4.14) \]

The volume preserving property of Steiner’s symmetrization and Corollary 4.3 imply that

\[ |\Gamma_{\phi,\omega}K_i| \leq |\Gamma_{\phi,\omega}K_{i-1}| \leq \cdots \leq |\Gamma_{\phi,\omega}K|, \quad \forall i \geq 1. \]

Letting \( i \to \infty \) and using (4.14), we get

\[ \frac{|K|}{\omega_n}|\Gamma_{\phi,\omega}B| = \lim_{i \to \infty} |\Gamma_{\phi,\omega}K_i| \leq |\Gamma_{\phi,\omega}K|, \]

18
which then proves that the volume ratio $|\Gamma_{\phi,\omega} K|/|K|$ is minimized at $B$ in $\mathcal{K}_0^n$. Using again Lemma 3.4, this volume ratio is minimized at all origin–centered ellipsoids.

Suppose that the volume ratio $|\Gamma_{\phi,\omega} K|/|K|$ is minimized at $K$. By Corollary 4.3, we have

$$\Gamma_{\phi,\omega}(S_u K) \subset S_u(\Gamma_{\phi,\omega} K), \quad \forall u \in S^{n-1},$$

which proves $|\Gamma_{\phi,\omega}(S_u K)| \leq |\Gamma_{\phi,\omega} K|$ for any $u \in S^{n-1}$. Since $|S_u K| = |K|$ and $|\Gamma_{\phi,\omega} K|/|K|$ is minimized at $K$, it holds

$$\frac{|\Gamma_{\phi,\omega} K|}{|K|} \leq \frac{|\Gamma_{\phi,\omega}(S_u K)|}{|S_u K|} = \frac{|\Gamma_{\phi,\omega}(S_u K)|}{|K|} \leq \frac{|\Gamma_{\phi,\omega} K|}{|K|},$$

which forces $\Gamma_{\phi,\omega}(S_u K) = S_u(\Gamma_{\phi,\omega} K)$ for any $u \in S^{n-1}$. Thank to Corollary 4.3, we conclude that $K$ is an origin–centered ellipsoid.

Despite Theorem 1.1 states only for convex bodies in $\mathcal{K}_0^n$, we hope that it could hold for any star bodies in $\mathcal{S}_0^n$. The $L_p$ Busemann–Petty centroid inequality on star bodies in $\mathcal{S}_0^n$ is reduced from the one on convex bodies in $\mathcal{K}_0^n$ by using a special class–reduction argument (see [28]). We do not know the existence of this argument in proving of the Orlicz Busemann–Petty centroid inequality until now as mentioned in [30]. Recently, Zhu [47] established the Orlicz Busemann–Petty centroid inequality for all star bodies in $\mathcal{S}_0^n$ by extending the method of Lutwak, Yang and Zhang in [30]. We believe that Zhu’s proof could be used to extend Theorem 1.1 to all star bodies in $\mathcal{S}_0^n$.

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