COMPLETE SYSTEMS OF PARTIAL DERIVATIVES OF ENTIRE FUNCTIONS AND FREQUENTLY HYPERCYCLIC OPERATORS

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ABSTRACT. We find some sufficient conditions for a system of partial derivatives of an entire function to be complete in the space \( H(\mathbb{C}^d) \) of all entire functions of \( d \) variables. As an application of this result we describe new classes of frequently hypercyclic operators on \( H(\mathbb{C}^d) \).

1. INTRODUCTION

In this paper we consider the space \( H(\mathbb{C}^d) \) of all entire functions on \( \mathbb{C}^d \) with the topology of uniform convergence on compact subsets. As usual, \( \mathbb{N}_0 \) is the set of non-negative integers. For \( z = (z_1, \cdots, z_d), w = (w_1, \cdots, w_d) \in \mathbb{C}^d \) and \( n = (n_1, \cdots, n_d) \in \mathbb{N}_0^d \) we shall use the following notations: \( z^n = z_1^{n_1}z_2^{n_2}\cdots z_d^{n_d}, n! = n_1!n_2!\cdots n_d!, \|n\| = n_1 + \cdots + n_d \), \( D^n = \frac{\partial^{\|n\|}}{\partial z_1^{n_1}\cdots \partial z_d^{n_d}}, \langle w, z \rangle = w_1z_1 + \cdots + w_dz_d, |z| = \max \{|z_j|, 1 \leq j \leq d\} \).

Recall that a continuous linear operator \( T \) on a topological vector space \( X \) is called hypercyclic if there exists an element \( x \in X \) such that its orbit \( \text{Orb}(T, x) = \{T^nx, n = 0, 1, \cdots\} \) is dense in \( X \). Recently, F. Bayart and S. Grivaux \( \text{I} \) have introduced a stronger notion of
hypercyclicity, namely the concept of frequently hypercyclic operators. Recall that the lower density of the set \( A \subseteq \mathbb{N}_0 \) is defined as

\[
\text{dens}(A) = \liminf_{N \to \infty} \frac{\#\{n \in A : n \leq N\}}{N},
\]

where \( \# \) denotes the cardinality of a set.

**Definition.** A continuous linear operator \( T \) on a topological vector space \( X \) is called frequently hypercyclic if there exists an element \( x \in X \) such that, for any non-empty open subset \( U \) of \( X \),

\[
\text{dens}\{n \in \mathbb{N}_0 : T^n x \in U\} > 0.
\]

We refer to [2], [9] for additional information on hypercyclic and frequently hypercyclic operators.

In the present paper we prove the following result.

**Theorem 1.1.** Let \( T_j : H(\mathbb{C}^d) \to H(\mathbb{C}^d), j = 1, \cdots, d \) be continuous linear operators such that

1. \( \bigcap_{j=1}^d \ker T_j \neq \{0\} \);
2. \( T_j \) satisfy the commutation relations:

\[
[T_j, \frac{\partial}{\partial z_k}] = \delta_{jk} a_j I; j, k = 1, \cdots, d,
\]

where \( \delta_{jk} \) is Kronecker's delta, \( a_j \in \mathbb{C} \setminus \{0\} \) are some constants and \( I \) is identity operator.

Then each of the operators \( T_j, j = 1, \cdots, d \) is frequently hypercyclic.
Remark 1.2. In Theorem 1.1 we consider constants $a_j \neq 0$. Note that if $a_j = 0$ in (1.1) for some $j$ then operator $T_j$ commutes with each partial differentiation operator. Recall that a continuous linear operator on $H(\mathbb{C}^d)$ that commutes with each partial differentiation operator is called a convolution operator. A theorem of Godefroy and Shapiro [7] states that every convolution operator on $H(\mathbb{C}^d)$ that is not a scalar multiple of the identity is hypercyclic. In [3] it was shown that these operators are even frequently hypercyclic.

Remark 1.3. Theorem 1.1 is an extension of author’s result in [10] that establishes hypercyclicity of operators satisfying the conditions of Theorem 1.1 for $d = 1$.

The paper is organised as follows. In Section 2 we obtain results on completeness of systems translates and systems of partial derivatives of entire functions. In Section 3 we prove Theorem 1.1 by application of the results of Section 2. In Section 4 we provide some examples of operators that are frequently hypercyclic by Theorem 1.1.

2. Complete systems of entire functions

In this section we study completeness of some systems of entire functions in the space $H(\mathbb{C}^d)$. More precisely, we shall consider the systems of translates and the systems of all partial derivatives of an entire function. Recall that the system of elements of a topological vector space...
X is said to be complete in X if the linear span of this system is dense in X. For $\lambda \in \mathbb{C}^d$ and $f \in H(\mathbb{C}^d)$ let us denote $S_\lambda f(z) \equiv f(z + \lambda)$.

In this section we shall need some facts from the theory of convolution operators on $H(\mathbb{C}^d)$. These facts can be found, for example, in [11]. Recall that every convolution operators on $H(\mathbb{C}^d)$ is defined by some linear continuous functional on $H(\mathbb{C}^d)$. Denote by $H'(\mathbb{C}^d)$ the strong dual space of $H(\mathbb{C}^d)$. Let $F \in H'(\mathbb{C}^d)$. Then the corresponding convolution operator $M_F$ has the form:

$$M_F[f](\lambda) = (F, S_\lambda f), \ \lambda \in \mathbb{C}^d, \ f \in H(\mathbb{C}^d).$$

A convolution operator $M_F$ is a linear continuous operator from $H(\mathbb{C}^d)$ to $H(\mathbb{C}^d)$. A Laplace transform

$$(2.1) \quad L : F \rightarrow \hat{F}(\lambda) = (F, \exp(z\lambda)),$$

where $n = (n_1, \ldots, n_d)$, establishes one-to-one correspondence (and moreover, topological isomorphism) between $H'(\mathbb{C}^d)$ and the space $P_{\mathbb{C}^d}$ of entire functions of exponential type on $\mathbb{C}^d$. A function $\hat{F}$ is said to be a characteristic function for operator $M_F$. Consider the Taylor expansion:

$$\hat{F}(\lambda) = \sum_{\|n\|=0}^{\infty} \frac{b_n}{n!} \lambda^n.$$

Then $(F, f) = \sum_{\|n\|=0}^{\infty} a_n b_n$, where $f(z) = \sum_{\|n\|=0}^{\infty} a_n z^n$. Since

$$f(\xi + \lambda) = \sum_{\|n\|=0}^{\infty} \frac{D^n f(\lambda)}{n!} \xi^n,$$
then $M_F$ can be represented in the form:

\[ M_F[f](\lambda) = \sum_{|n|=0}^{\infty} b_n D^n f(\lambda) n! \quad (2.2) \]

On the other hand, since

\[ f(\xi + \lambda) = \sum_{|n|=0}^{\infty} D^n f(\xi) \frac{\lambda^n}{n!} \]

then

\[ M_F[f](\lambda) = \sum_{|n|=0}^{\infty} (F, D^n f(\xi)) \frac{\lambda^n}{n!} \quad (2.3) \]

In the following theorem we establish necessary and sufficient conditions for systems of partial derivatives and systems of translates to be complete in $H(\mathbb{C}^d)$.

**Theorem 2.1.** Let $f \in H(\mathbb{C}^d)$, $f \neq 0$. Then the following conditions are equivalent:

1. $f \notin \text{Ker } M_F$ for any $F \in H'(\mathbb{C}^d) \setminus \{0\}$;
2. A system of translates $\{S_\lambda f, \lambda \in \Lambda\}$ is complete in $H(\mathbb{C}^d)$ for each open set $\Lambda \subset \mathbb{C}^d$;
3. A system of partial derivatives $\{D^n f, n \in \mathbb{N}_0\}$ is complete in $H(\mathbb{C}^d)$.

**Proof.** $1 \Rightarrow 2$. Assume that the system $\{S_\lambda f, \lambda \in \Lambda\}$ is not complete in $H(\mathbb{C}^d)$ for some open set $\Lambda \subset \mathbb{C}^d$. Then according to the Approximation Principle (see e.g. [4, Ch.2]) there is a non-zero continuous linear
functional $F$ on $H(\mathbb{C}^d)$ such that $(F, S_\lambda f) = 0$, $\forall \lambda \in \Lambda$. Since $\Lambda$ is open set, then $M_F[f](\lambda) = (F, S_\lambda f) = 0$, $\forall \lambda \in \mathbb{C}^d$. Hence, $f \in \text{Ker} M_F$.

$2 \Rightarrow 3$. Assume that the system $\{D^n f, n \in \mathbb{N}_0^d\}$ is not complete in $H(\mathbb{C}^d)$. Then there is $0 \neq F \in H'(\mathbb{C}^d)$ such that $(F, D^n f) = 0$, $\forall n \in \mathbb{N}_0^d$. Let $\Lambda \subset \mathbb{C}^d$ be some open set. Then

$$(F, S_\lambda f) = M_F[f](\lambda) = \sum_{|n| = 0}^{\infty} \frac{(F, D^n f)}{n!} \lambda^n = 0, \forall \lambda \in \Lambda.$$ 

Hence, the system $\{S_\lambda f, \lambda \in \Lambda\}$ is not complete in $H(\mathbb{C}^d)$.

$3 \Rightarrow 1$. Assume that $f \in \text{Ker} M_F$ for some $F \in H'(\mathbb{C}^d) \setminus \{0\}$. Then by (2.3)

$$M_F[f](\lambda) = \sum_{|n| = 0}^{\infty} \frac{(F, D^n f)}{n!} \lambda^n = 0, \forall \lambda \in \mathbb{C}^d.$$ 

Hence, $(F, D^n f) = 0$, $\forall n \in \mathbb{N}_0^d$. Thus, the system $\{D^n f, n \in \mathbb{N}_0^d\}$ is not complete in $H(\mathbb{C}^d)$. □

We need the following:

**Lemma 2.2.** Let operators $T_j$, $j = 1, \ldots, d$ satisfy the conditions of the Theorem 1.1. Let $f \in \bigcap_{j=1}^{d} \ker T_j$, $f \neq 0$. Then, for all $k = (k_1, \ldots, k_d), n = (n_1, \ldots, n_d) \in \mathbb{N}_0^d$,

$$(2.4) \quad T^k D^n f = \begin{cases} a^k \frac{n!}{(n-k)!} D^{n-k} f, & \text{if } k_j \leq n_j, j = 1, \ldots, d, \\ 0, & \text{otherwise,} \end{cases}$$

where $T^k = T_1^{k_1} T_2^{k_2} \ldots T_d^{k_d}$, $a^k = a_1^{k_1} a_2^{k_2} \ldots a_d^{k_d}$.
Proof. Let us fix some arbitrary \( j \in \mathbb{N} : 1 \leq j \leq d \) and \( n \in \mathbb{N}_0^d \). From (1.1) it follows (see e.g. [5, §16]) that

\[
[T_j, \frac{\partial^{n_j}}{\partial z_j^{n_j}}] = a_j n_j \frac{\partial^{n_j-1}}{\partial z_j^{n_j-1}}.
\]

Let us put

\[\hat{\tilde{f}}(z) = \frac{\partial^{||n||-n_j}}{\partial z_1^{n_1} \cdots \partial z_j^{n_j-1} \partial z_{j+1}^{n_{j+1}} \cdots \partial z_d^{n_d}} f(z).\]

Then \( D^n f(z) = \frac{\partial^{n_j}}{\partial z_j^{n_j}} \hat{\tilde{f}}(z) \). Since \( f \in \text{Ker} T_j \), then it follows from (1.1) that \( \hat{\tilde{f}} \in \text{Ker} T_j \). Hence, (2.5) implies that

\[
T_k^{j} \frac{\partial^{n_j}}{\partial z_j^{n_j}} \hat{\tilde{f}} = a_k^{j} \frac{n_j!}{(n_j - k)!} \frac{\partial^{n_j-k}}{\partial z_j^{n_j-k}} \hat{\tilde{f}}, \quad \text{if } k_j \leq n_j,
\]

and

\[
T_k^{j} \frac{\partial^{n_j}}{\partial z_j^{n_j}} \hat{\tilde{f}} = 0, \quad \text{if } k_j > n_j.
\]

Since (2.6), (2.7) hold for an arbitrary \( j \in \mathbb{N} : 1 \leq j \leq d \), the Lemma then follows.

Now we are ready to state and prove the main result of this section.

**Theorem 2.3.** Let operators \( T_j, j = 1, \cdots, d \) satisfy the conditions of the Theorem 1.1. Let \( f \in \bigcap_{j=1}^d \text{ker} T_j, f \not\equiv 0 \). Then the system \( \{D^n f, n \in \mathbb{N}_0^d \} \) is complete in \( H(\mathbb{C}^d) \).

Proof. Assume that there exists \( F \in H'(\mathbb{C}^d) \setminus \{0\} \) such that \( f \in \text{Ker} M_F \). Then \( T^k M_F[f] = 0, \forall k \in \mathbb{N}_0^d \). Then by (2.1), (2.2) we have

\[
T^k M_F[f](z) = \sum_{||n||=k} \frac{D^n \tilde{F}(0)}{n!} T^k D^n f(z) \equiv 0, \forall k \in \mathbb{N}_0^d.
\]
From (2.4), (2.8) we obtain
\[ \sum_{\|n\|=k} \frac{D^n \hat{F}(0)}{(n - k)!} D^{n-k} f(z) \equiv 0, \ \forall k \in \mathbb{N}_0, \]
or
\[ (2.9) \sum_{\|n\|=0} \frac{D^{n+k} \hat{F}(0)}{n!} D^n f(z) \equiv 0, \ \forall k \in \mathbb{N}_0. \]

Next, using the Taylor series expansion of $D^n f(z)$ we obtain
\[ (2.10) \sum_{\|n\|=0} \frac{D^{n+k} \hat{F}(0)}{n!} \sum_{\|m\|=0} \frac{D^{n+m} f(0)}{m!} z^m \equiv 0, \ \forall k \in \mathbb{N}_0. \]

Since the series in (2.10) converges absolutely, then we can interchange the order of summation:
\[ (2.11) \sum_{\|m\|=0} \sum_{\|n\|=0} \frac{D^{n+m} f(0) D^{n+k} \hat{F}(0)}{n!} \equiv 0, \ \forall k \in \mathbb{N}_0. \]

From (2.11) it follows that
\[ \sum_{\|n\|=0} \frac{D^{n+m} f(0) D^{n+k} \hat{F}(0)}{n!} = 0, \ \forall m, k \in \mathbb{N}_0. \]

Hence,
\[ \frac{\lambda^k}{k!} \sum_{\|n\|=0} \frac{D^{n+m} f(0) D^{n+k} \hat{F}(0)}{n!} = 0, \ \forall m, k \in \mathbb{N}_0, \ \forall \lambda \in \mathbb{C}^d, \]
and
\[ (2.12) \sum_{\|k\|=0} \frac{\lambda^k}{k!} \sum_{\|n\|=0} \frac{D^{n+m} f(0) D^{n+k} \hat{F}(0)}{n!} = 0, \ \forall m \in \mathbb{N}_0, \ \forall \lambda \in \mathbb{C}^d. \]
Since the series in (2.12) converges absolutely, then we can interchange the order of summation:

\[
\sum_{||n||=0}^{\infty} \frac{D^{n+m}f(0)}{n!} \sum_{||k||=0}^{\infty} \frac{D^{n+k}\hat{F}(0)}{k!} \lambda^k = 0, \ \forall m \in \mathbb{N}_0^d, \ \forall \lambda \in \mathbb{C}^d.
\]

Finally, we can rewrite (2.13) in the following form:

\[
\sum_{||n||=0}^{\infty} \frac{D^{n+m}f(0)}{n!} D^n\hat{F}(\lambda) \equiv 0, \ \forall m \in \mathbb{N}_0^d.
\]

Consider the infinite system of infinite-order homogeneous linear partial differential equations with constant coefficients

\[
\sum_{||n||=0}^{\infty} \frac{D^{n+m}f(0)}{n!} D^n g(\lambda) = 0, \ g \in P_{\mathbb{C}^d}, \ \forall m \in \mathbb{N}_0^d.
\]

with characteristic functions

\[
D^m f(z) = \sum_{||n||=0}^{\infty} \frac{D^{n+m}f(0)}{n!} z^n, \ m \in \mathbb{N}_0^d.
\]

Note that the series in (2.15) converges absolutely and uniformly for any function \(g \in P_{\mathbb{C}^d}\). For each of the equations in the system (2.15) denote by \(W_m\) the space of solutions, \(m \in \mathbb{N}_0^d\). Denote by \(W\) the space of solutions of the system (2.15). Then, obviously, \(W = \bigcap W_m\). It is easy to see that \(W\) is a translation-invariant subspace of \(P_{\mathbb{C}^d}\) (i.e. if \(g \in W\), then \(S_ug \in W, \ \forall u \in \mathbb{C}^d\)).

In [6] it was shown that every translation-invariant subspace of \(P_{\mathbb{C}^d}\) admits spectral synthesis. In particular, this means that a translation-invariant subspace \(V \subset P_{\mathbb{C}^d}\) is not trivial (i.e. \(V \neq \{0\}\)) if and only if
it contains some exponential element of the form

\[(2.16) \quad \varphi_\mu(z) = P(z) \exp(\mu, z), \; \mu \in \mathbb{C}^d,\]

where \(P(z) \in H(\mathbb{C}^d)\) is some polynomial. From (2.14) it follows that the function \(\hat{F} \in P_{\mathbb{C}^d}\) satisfy the system (2.15). Since \(\hat{F} \in W\), then our initial assumption that \(\hat{F} \not\equiv 0\) implies that \(W \neq \{0\}\), and hence \(W\) contains a function \(\varphi_\mu\) of the form (2.16). Therefore, \(\varphi_\mu \in W_m, \; \forall m \in \mathbb{N}_0^d\). This implies that \(D^m f(\mu) = 0, \; \forall m \in \mathbb{N}_0^d\). But this is impossible since \(f \not\equiv 0\). Hence, \(W = \{0\}\). But this contradicts our initial assumption that \(\hat{F} \not\equiv 0\). Hence, \(f \not\in \text{Ker} \, M_\Phi\) for any \(\Phi \in H'(\mathbb{C}^d) \setminus \{0\}\). The Theorem then follows from Theorem 2.1. \(\Box\)

**Corollary 2.4.** Let operators \(T_j, \; j = 1, \cdots, d\) satisfy the conditions of the Theorem 1.1. Let \(f \in \bigcap_{j=1}^d \ker T_j, \; f \not\equiv 0\). Then the system of translates \(\{S_\lambda f, \; \lambda \in \Lambda\}\) is complete in \(H(\mathbb{C}^d)\) for each open set \(\Lambda \subset \mathbb{C}^d\).

**Remark 2.5.** The analogue of Theorem 2.3 for \(d = 1\) was previously proved in author’s paper [10] with the help of the result of L. Schwartz [12] which says that every translation-invariant subspace of \(H(\mathbb{C})\) admits spectral synthesis. The situation is more difficult if \(d > 1\) because there is a counter example on spectral synthesis in \(H(\mathbb{C}^d), \; d > 1\), by D. Gurevich [8]. In particular Gurevich proved that there exists translation-invariant subspace \(U \neq \{0\}\) of \(H(\mathbb{C}^d)\) that does not contain
any exponential element. This is why we had to transform a system (2.9) of equations in $H(\mathbb{C}^d)$ to a system (2.14) in $P_{\mathbb{C}^d}$.

3. The proof of the main result

In this section we give a proof of Theorem 1.1.

Proof. Let us fix some arbitrary $j : 1 \leq j \leq d$. We are going to show that operator $T_j$ is frequently hypercyclic. The proof will follow by an application of the frequent hypercyclicity criterion [1] (see also e.g. [9, Ch. 9]): it is enough to show that there is a dense subset $H_0$ of $H(\mathbb{C}^d)$ and a map $S_j : H_0 \to H_0$ such that, for any $x \in H_0$,

(i): $\sum_{k=0}^{\infty} T_j^k x$ converges unconditionally;

(ii): $\sum_{k=0}^{\infty} S_j^k x$ converges unconditionally;

(iii): $T_j S_j x = x$.

Consider some function $f \in \bigcap_{j=1}^{d} \text{ker } T_j$, $f \neq 0$. Let us put

$$H_0 = \text{span}\{D^n f, n \in \mathbb{N}_0^d\}.$$ 

Then by Theorem 2.3, $H_0$ is a dense subset of $H(\mathbb{C}^d)$. From (2.7) it follows that for any $x \in H_0$ there is $K \in \mathbb{N}$ such that $T_j^k x = 0$, if $k \geq K$. Thus, condition (i) is fulfilled.

Let us define a map $S_j$ in the following way:

$$S_j[D^n f] = \frac{1}{a_j(n_j + 1)} \frac{\partial}{\partial z_j} D^n f, \ n \in \mathbb{N}_0^d.$$
Therefore,
\[
S^k_j[D^n f] = \frac{n_j!}{a_j^k(n_j + k)!} \frac{\partial^k}{\partial z^k_j} D^n f, \quad n \in \mathbb{N}_0^d, \quad k \in \mathbb{N}_0.
\]

Let us fix some \( \varepsilon \in \mathbb{R} \) such that

\[
(3.1) \quad \varepsilon > \max_{1 \leq s \leq d} \frac{1}{|a_s|}.
\]

The topology on \( H(\mathbb{C}^d) \) can be defined by a family of semi-norms

\[
p_m(g) = \sup_{z \in \Omega_m} |g(z)|, \quad m \in \mathbb{N},
\]

where \( \Omega_m = \{z \in \mathbb{C}^d : |z| \leq m\varepsilon\} \). In order to prove (ii), it is enough to show that the series

\[
(3.2) \quad \sum_{k=0}^{\infty} p_m(S^k_jD^n f)
\]

converges for any \( m \in \mathbb{N}, n \in \mathbb{N}_0^d \).

Using the Cauchy estimate for derivatives, we obtain for all \( n \in \mathbb{N}_0^d, k \in \mathbb{N}_0, m \in \mathbb{N} \):

\[
p_m(S^k_jD^n f) = \frac{n_j!}{a_j^k(n_j + k)!} p_m \left( \frac{\partial^k}{\partial z^k_j} D^n f \right) \leq \frac{n_j!k!p_m(D^n f)}{|a_j|k(n_j + k)! (m\varepsilon)^k}.
\]

Then by (3.1) we have

\[
\lim_{k \to \infty} \sqrt[k]{p_m(S^k_jD^n f)} \leq \frac{1}{|a_j|m\varepsilon} < 1, \quad m \in \mathbb{N}, n \in \mathbb{N}_0^d.
\]

Therefore, the series (3.2) converges for any \( m \in \mathbb{N}, n \in \mathbb{N}_0^d \). This means that condition (ii) is fulfilled. Furthermore, \( T_j S_j x = x \) for any \( x \in H_0 \) and thus, condition (iii) is fulfilled. \( \square \)
4. Examples

In this section we provide some examples of operators that are frequently hypercyclic by Theorem 1.1.

1) Let

\[ T_1 g(z) = \frac{\partial g(z)}{\partial z_1} - z_1 g(z), \quad T_2 g(z) = \frac{\partial g(z)}{\partial z_2} - z_2 g(z), \quad g \in H(\mathbb{C}^2). \]

Then \( \bigcap_{j=1}^2 \ker T_j \) contain, for instance, a function \( \exp(z_1^2/2) \cdot \exp(z_2^2/2) \).

Besides,

\[ (4.1) \quad [T_j, \frac{\partial}{\partial z_k}] = \delta_{jk} I; \quad j, k = 1, 2. \]

Therefore, \( T_1 \) and \( T_1 \) are frequently hypercyclic by Theorem 1.1.

2) Let

\[ T_1 g(z) = \frac{\partial^2 g(z)}{\partial z_1^2} - z_1 g(z), \quad T_2 g(z) = \frac{\partial^2 g(z)}{\partial z_2^2} - z_2 g(z), \quad g \in H(\mathbb{C}^2). \]

Then \( \bigcap_{j=1}^2 \ker T_j \) contain, for example, a function \( \text{Ai}(z_1) \cdot \text{Ai}(z_2) \), where \( \text{Ai} \) is Airy function. Moreover, (4.1) holds. Hence, \( T_1 \) and \( T_1 \) are frequently hypercyclic by Theorem 1.1.

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