Design of Polar Codes in 5G New Radio
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Abstract—Polar codes have attracted the attention of academia and industry alike in the past decade, such that the 5th generation wireless systems (5G) standardization process of the 3rd generation partnership project (3GPP) chose polar codes as a channel coding scheme. In this tutorial, we provide a description of the encoding process of polar codes foreseen by the 5G standard. We illustrate the struggles of designing a family of polar codes able to satisfy the demands of 5G systems, with particular attention to rate flexibility and low decoding latency. The result of these efforts is an elaborate framework that applies novel coding techniques to provide a solid channel code for NR requirements.

I. INTRODUCTION

Polar codes are a class of capacity-achieving codes introduced in [1]. In the past decade, the interest and research effort on polar codes has been constantly rising in academia and industry alike. Within the ongoing 5th generation wireless systems (5G) standardization process of the 3rd generation partnership project (3GPP), polar codes have been adopted as channel coding for uplink and downlink control information for the enhanced mobile broadband (eMBB) communication service. 5G foresees two other frameworks, namely ultra-reliable low-latency communications (URLLC) and massive machine-type communications (mMTC), for which polar codes are among the possible coding schemes.

The construction of a polar code involves the identification of channel reliability values associated to each bit to be encoded. This identification can be effectively performed given a code length and a specific signal-to-noise ratio. However, within the 5G framework, various code lengths, rates and channel conditions are foreseen, and having a different reliability vector for each parameter combination is unfeasible. Thus, substantial effort has been put in the design of polar codes that are easy to implement, having low description complexity, while maintaining good error-correction performance over multiple code and channel parameters.

The majority of available literature does not take in account the specific codes designed for 5G and their encoding process; given their upcoming widespread utilization, the research community would benefit from considering them within error-correction performance evaluations and encoder/decoder designs. Both the encoding and the decoding process can in fact incur substantial speed and complexity overhead, while the performance of decoders is tightly bound to the characteristics of the polar code. Works focusing on hardware and software implementations can effectively broaden their audience by including compliance to the 5G standard.

In this paper, we provide a tutorial for the polar code encoding process foreseen by 5G in [2], from the code concatenation, through interleaving functions, to the polar-code specific subchannel allocation and rate-matching schemes. The purpose of this work is to provide the reader with a straightforward, self-contained guide to the understanding and implementation of 5G-compliant encoding of polar codes. Not willing to substitute the reading of the standard, we aim at assisting the reader in its comprehension, restructuring its presentation and reformulating some of the contents to improve readability. Moreover, we analyze the foreseen communication chain and give insights about its operating steps, along with a deep introduction of all the techniques employed in the standard definition.

The remainder of the paper is organized as follows. Section II introduces polar codes, along with concepts used in the 5G encoding process, such as interleaving and rate-matching. Section III details a step-by-step guide to 5G polar code encoding. Conclusions are drawn in Section IV.

II. PRELIMINARIES

In this Section, we introduce the basic concepts on polar codes. In particular, we review various approaches to frozen set design, decoding and rate matching. Moreover, considerations on the cyclic redundancy check code (CRC), distributed CRC and assistant bits used in the 5G are given as well, along with the description of 5G polar code use cases.

A. Polar codes definition

The mathematical foundations of polar codes lay on the polarization effect [1] of the matrix $G_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$. In an $(N, K)$ polar code of code length $N = 2^n$, the polarization effect establishes $N$ virtual channels, and through each channel a single bit $u_i$ is transmitted. Each of these bit-channels, or subchannels, has a different reliability; message bits are allocated to the $K$ most reliable channels. The polar code is hence defined by the transformation matrix $G_N = G_2^\otimes n$, i.e. as the $n$-th Kronecker power of the polarizing matrix, and either the frozen set $\mathcal{F}$ of size $N - K$, or its complementary information set $\mathcal{I} = \mathcal{F}^C$ of size $K$, where $\mathcal{I}$ and $\mathcal{F}$ are subsets of the index set $\{0, 1, \ldots, 2^n-1\}$. A codeword $d = \{d_0, d_1, \ldots, d_{N-1}\}$ is calculated as

$$d = uG_N,$$

where the input vector $u = \{u_0, u_1, \ldots, u_{N-1}\}$ is generated by assigning $u_i = 0$ if $i \in \mathcal{F}$, and storing information in the remaining elements. Each index $i$ identifies a different bit-channel.

B. Frozen set design

As $N$ goes toward infinity, the polarization phenomenon influences the reliability of bit-channels, that are either completely noisy or completely noiseless; even more, the fraction of noiseless bit-channels equals the channel capacity [1]. More
formally, let $W$ be a binary memoryless symmetric channel with input alphabet $\mathcal{X} = \{0, 1\}$ and output alphabet $\mathcal{Y}$, and let $\{W(y \mid x) : x \in \mathcal{X}, y \in \mathcal{Y}\}$ be the transition probabilities. In order to quantify the reliability, i.e. the goodness, of the channel $W$, we use the Bhattacharyya parameter $Z(W) \in [0,1]$, that is defined as

$$Z(W) = \sum_{y \in \mathcal{Y}} \sqrt{W(y \mid 0)W(y \mid 1)}. \quad (2)$$

Hence, the good bit-channels are the ones that have the lowest Bhattacharyya parameter.

For finite practical code lengths, the polarization of bit-channels is incomplete, therefore, there are bit-channels that are partially noisy. The polar encoding process consists in the classification of the bit-channels in $u$ into two groups: the $K$ good bit-channels that will carry the information bits and are indexed by the information set $\mathcal{I}$, and the $N-K$ bad bit-channels that are fixed to a predefined value (usually 0) and are indexed by the frozen set $\mathcal{F}$. Bit-channels are sorted in order of reliability. In case of finite code lengths, the $K$ best bit-channels, i.e. the ones with the highest reliability, are selected to form the information set, while the remaining bit-channels are frozen.

In general, the order of reliabilities of the bit-channels depends on the channel condition and on the code length, and therefore is not universal. This non-universality of polar codes poses huge practical problems in the construction of polar codes, when a large range of code lengths and code rates need to be supported. Many methods to design the frozen sets on-the-fly with limited complexity have been proposed [3]. Along with the Bhattacharyya parameter, Arikan initially proposed to use Monte-Carlo simulations to estimate bit-channel reliabilities [1]. The density evolution (DE) method, initially proposed in [4] and improved in [5], can provide theoretical guarantees on the estimation accuracy, however at a high computational cost. A bit-channel reliability estimation method for AWGN channels based on Gaussian approximation (GA) of DE has been proposed in [6], giving accurate results with limited complexity. On-the-fly design, however, increases the latency on encoder and decoder side too much to meet the 5G requirements.

Recent attempts to study the partial order imposed by the polarization effect in order to design universal polar codes have been useful for practical application of polar codes [7]–[9]. Moreover, taking into account distance properties in short polar codes design may give superior error-correction performance [10], [11]. Finally, bit-channel analysis usually does not take into account the use of list decoders or assistant bits in the decoding [12]. As we will see, these studies and intensive simulations lead the 5G standardization to propose a universal bit channel reliability sequence. As a result, this universal reliability sequence is used as a basis to extract the individual reliability sequence for all the polar codes considered in 5G.

C. Decoding

Polar codes have been specifically designed for Successive Cancellation (SC) decoding [1], with which they achieve channel capacity at infinite code length. This algorithm decodes bits sequentially using both the soft information received from the channel and the hard decisions taken on the previously decoded bits. It can be represented as a binary tree search, where the leaf nodes are the $N$ bits to be estimated, and soft information about the received vector is input at the root node. Fig. 1 shows the SC decoding tree for an $(8, 4)$ polar code. We consider the soft values received from the channel and the internally exchanged information to be logarithmic likelihood ratios (LLRs). At each stage $t$, the LLR values $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_{2^t-1})$ are sent from parent to child nodes, while the hard decision values $\beta = (\beta_0, \beta_1, \ldots, \beta_{2^t-1})$ go from child nodes to the parent node.

The $2^t-1$-element LLR vector going to the left child node $\alpha^l = (\alpha^l_0, \alpha^l_1, \ldots, \alpha^l_{2^t-1})$, and the one going to the right child node $\alpha^r = (\alpha^r_0, \alpha^r_1, \ldots, \alpha^r_{2^t-1})$, are computed as

$$\alpha^l_i = 2 \arctanh \left( \tanh \left( \frac{\alpha_i}{2} \right) \tanh \left( \frac{\alpha_{i+2^t-1}}{2} \right) \right), \quad (3)$$

$$\approx \text{sgn}(\alpha_i) \text{sgn}(\alpha_{i+2^t-1}) \min(|\alpha_i|, |\alpha_{i+2^t-1}|), \quad (4)$$

The $2^t$ elements of $\beta$ are instead computed through $\beta^l = (\beta^l_0, \beta^l_1, \ldots, \beta^l_{2^t-1})$ and $\beta^r = (\beta^r_0, \beta^r_1, \ldots, \beta^r_{2^t-1})$ as

$$\beta^l_i = \beta^r_i \oplus \beta^l_i, \quad \text{if } i < 2^{t-1},$$

$$\beta^l_{2^t-1} = \beta^r_{2^t-1}, \quad \text{otherwise}, \quad (6)$$

where $\oplus$ represents the bitwise exclusive OR (XOR) operation. The estimation of the $i$-th bit $\hat{u}_i$ at leaf nodes is performed as

$$\hat{u}_i = \begin{cases} 0, & \text{if } i \in \mathcal{F} \text{ or } \alpha_i \geq 0, \\ 1, & \text{otherwise}. \end{cases} \quad (7)$$

The SC algorithm can be implemented in both software and hardware with low complexity [13], [14], but its error-correction performance is mediocre when decoding practical code lengths. Thus, many attempts have been made to overcome this shortcoming [15], [16].

Eventually, a list-based decoding approach to polar codes (SCL) was introduced in [17]. A set of SC decoders works in parallel maintaining different codeword candidates at the same time. Every time a leaf node is reached, the bit is estimated as both 1 and 0, doubling the number of codeword candidates. A path metric for each candidate is updated, so that the less likely candidates are discarded to limit the growth in complexity of the algorithm. SCL substantially improves the error-correction
performance of SC at moderate code lengths, especially when the code is concatenated to an outer code acting as a genie, e.g. a cyclic redundancy check (CRC), at the cost of an augmented complexity. In fact, the minimum distance of polar codes can be dramatically improved by adding such an outer code to the original polar code [13]. This improved distance spectrum is fully exploited by SCL decoders, and the resulting performance improvement has contributed to the selection of polar codes for 5G. This algorithm has been taken as a baseline in 5G error-correction performance evaluations, in particular with a list size equal to 8.

While the effectiveness of SCL improves as the list size increases, its implementation complexity increases as well. To limit the rise in complexity, various approaches have been proposed for software and hardware decoders alike. Partitioned SCL [19] and its evolutions [20], [21] consider different list sizes at different stages of the SC tree, reducing the memory requirements at higher stages. SC-Stack decoding [22] expands only the most probable candidate thanks to a priority queue. Adaptive SCL decoding [23] foresees increasing list sizes in case of failed decoding, while a hardware decoder with flexible list size has been proposed in [24].

SC-based decoding algorithms suffer from long latency, due to their serial nature. Fast decoding algorithms for SC, SC-Flip and SCL have been proposed in [25]–[31]. They rely on the identification of special nodes, i.e., patterns of frozen and information bits, and the efficient decoding of these nodes. These techniques prune the SC decoding tree, and substantially decrease latency, at the expense of more complex hardware implementations. Multi-bit decoding [32], [33] traverses the whole tree, but allows to estimate a higher number of bits in parallel, effectively reducing the number of stages of the tree.

Belief propagation (BP) decoding of polar codes has been proposed as well [1]. The algorithm iterates on the polar code Tanner graph, exchanging soft information in both directions. BP decoding of polar codes can potentially achieve faster decoding than SC-based algorithms, since it is inherently possible to parallelize operations, at the cost of a higher implementation complexity [34]. However, polar codes constructed for SC suffer severe error-correction performance degradation when decoded with BP, and construction of polar codes for BP is difficult, due to the complexity of the factor graph [35]. In limited cases, degradation can be recovered augmenting the number of iterations, however leading to poor throughput [36].

D. Rate matching

According to their definition, the length $N$ of a polar code is limited to powers of two, while the code dimension $K$ can assume any value smaller than $N$, since only the $K$ most reliable bits will be used to carry the information. This is a limitation for typical 5G applications, where the amount of information $A$ is fixed and a codeword of length $E$ is needed to achieve the desired rate $R = A/E$. Rate matching for polar codes becomes thus a length matching problem, and can be faced through classical coding theory techniques as puncturing and shortening [37].

Both puncturing and shortening reduce the length of a mother code by not transmitting code bits in a predetermined pattern, called matching pattern; the difference lies in the meaning of the code bits belonging to the matching pattern. In puncturing, one or more code bits are not transmitted, which are treated as erased at the decoder. In shortening, a sub-code is introduced such that one or more code bits assume a fixed value, typically zero, and not transmitted since they are known at the decoder. Rate matching alters the reliability of the subchannels, with an impact that depends on the strategy adopted. As a rule of thumb, it has been observed that for polar codes, shortening works better for high rates, puncturing for low rates [38].

Puncturing deteriorates subchannel reliabilities; moreover, the erasures introduced by puncturing cause some bit channels, called incapable bits, to be completely unreliable. It can be shown that $U$ punctured code bits make exactly $U$ subchannels incapable; the position of these bits can be calculated on the basis of the matching pattern [39]. In order to avoid catastrophic error-correction performance degradation under SC decoding, incapable bits must be frozen to prevent random decisions at the decoder. Shortening, on the contrary, improves the bit channel reliabilities by introducing overcapable bits, i.e., bits with conceptually infinite reliability: those bits are surely correctly decoded under SC, if the previous bits have been correctly decoded [38]. However, given the particular structure of polar codes, code bits in the matching pattern must depend on frozen bits only, which forces to freeze the usually most reliable subchannels.

Three main strategies have been proposed to design the matching pattern. The first option is to design the frozen set of the mother polar code based on the matching pattern. According to this strategy, the matching pattern is initially generated according to a some heuristic, then the subchannel reliabilities are calculated in order to find the optimal frozen set. This method has been proposed in [40] for shortening and in [39], [41] for puncturing of polar codes, where the DE/GA algorithm is run to find the optimal frozen set on the basis of different matching patterns. The result is a code with good block error rate (BLER) performance, at the cost of a higher complexity due to the reliability calculation. An alternative approach, proposed in [42], [43], is to design the matching pattern on the basis of the frozen set. This significantly reduces the code design complexity, albeit at the cost of an increased BLER. As we will see, this is the approach selected by 3GPP for the standardization of polar codes in 5G. Finally, joint optimization, presented in [44], proposes to design frozen set and matching pattern at the same time. Symmetries in the polar code structure reduce the number of matching patterns to be tested, however not enough to make this technique practical for application in 5G.

E. Assistant bits design

It has been noticed that the introduction of an outer CRC code improves the error-correction performance [45], for example when used to help the selection of the correct candidate in SCL decoding. In general, it has been proven that the minimum distance of polar codes can be dramatically improved by adding an outer code to polar codes [18]. This improved
code spectrum is fully used by SCL decoders [17], and it has contributed to the selection of polar codes for 5G.

Assistant bits can be broadly identified as additional bits that help the decoding of the polar code, either increasing the error-correction performance or improving a metric, like speed or complexity. The introduction of an outer code can be seen as the addition of assistant bits to information bits. After the initial proposal of CRC as assistant bits made in [17], many other outer codes have been proposed [46]. Parity-checks (PC) have emerged as an alternative to design assistant bits due to their simplicity and flexibility [18], [47]. Introducing parity check bits in the middle of the decoding instead of a unique CRC check at the end makes it easier to tune the polar code spectrum. It has been shown that subchannels corresponding to minimum weight rows are the best candidates to host parity checks [48]. In [49], a low complexity parity check design based on shift registers is proposed, showing that good performance can be obtained adding really simple parity checks to the information set. A further step is to mix CRC and parity checks, as proposed in [50], to improve the flexibility of the polar code design.

The insertion of an interleaver between the CRC encoder and the polar encoder may have a positive impact on the performance of the code due to the induced change in the number of minimum weight codewords [51]. The interleaver is used to turn the CRC into a distributed CRC, in that a CRC remainder bit is assigned to a bit-channel as soon as all the information bits involved in its parity check have been assigned as well. This feature can be used to reduce the decoding complexity by early terminating the decoding if an incorrect check is met, providing an additional false alarm rate (FAR) mitigation [52]. Furthermore, the distributed CRC bits scheme can be used to prune the SCL decoding tree [18].

F. Polar coded modulation

If low-order modulation schemes like 4QAM are used, the BLER is not affected by the modulation scheme since all bits in modulated symbols have uniform reliability. Polar-coded modulation (PCM) was introduced in [53] for larger constellations, exploiting the polarization effect in the construction of polar codes for higher-order modulation [54]. However, the canonical PCM requires the introduction of an additional polarization matrix whose size depends on the modulation scheme used [55]. This results in increased latency due to the additional decoding step [56].

The introduction of a channel interleaver emerged as a low-complexity alternative to PCM; this technique, termed as bit-interleaved polar-coded modulation (BIPCM) [57], proved to improve the diversity gain under high-order modulation without increasing the code complexity [58]. In BIPCM, the channel is considered as a set of parallel bit channels which can be combined with polar coded bits. Carefully mapping coded bits to modulation symbols offers a certain gain over the conventional random interleaving for high-order QAM over AWGN channels [59]. Moreover, interleavers designed to be adaptive to channel selectivity can achieve a remarkable diversity gain compared to random interleaving for polar-coded OFDM transmission [60]. The correlation between coded bits mapped into the same symbol allows to combine the demapping and deinterleaving units with the SC decoder to perform the decoding directly on the LLRs of the received symbols instead of the ones of the coded bits [61].

III. POLAR CODE ENCODING IN 5G

In this section we describe in detail the framework agreed for the encoding of polar codes in 5G standard. In the following, the notation introduced in the 3GPP technical specification [62] will be used. Polar codes in the uplink are used to encode the uplink control information (UCI) over the physical uplink control channel (PUCCH) and the physical uplink shared channel (PUSCH). In the downlink, polar codes are used to encode the downlink control information (DCI) over the physical downlink control channel (PDCCH), and the payload in the physical broadcast channel (PBCH). Table I summarizes the encoding chain parameters depending on channel and code parameters.
According to the definition of polar codes, the code length is limited to powers of two, \( N = 2^n \), while the code dimension \( K \), i.e. the number of information bits, can assume any value \( K < N \). In 5G applications, on the other hand, typically the number of information bits, \( A \), is fixed and a codeword of length \( E \) is needed to achieve the desired rate \( R = A/E \), as required by upper communication layers. To accommodate polar codes to this requirement, a mother polar code of length \( N = 2^n \) is initially constructed, and the desired code length \( E \) is matched via puncturing, shortening or repetition of the mother polar code. The mother code length \( N \) is lower bounded by \( N_{\text{min}} = 32 \), while the value of the upper bound \( N_{\text{max}} \) depends on the channel used, being \( N_{\text{max}} = 512 \) for downlink and \( N_{\text{max}} = 1024 \) for uplink. An ulterior lower bound is imposed by the minimal accepted code rate \( \frac{1}{8} \).

Parameters \( A \) and \( E \) are bounded according to the channel used, obviously having \( A \leq E \). In uplink, \( 12 \leq A \leq 1706 \), while for \( A \leq 11 \) ad-hoc block codes are used. Codeword length is upper bounded by \( E \leq 8192 \), however payload length \( G \) can be larger: in this case, information bits may be divided into two polar codewords through segmentation. In downlink, \( A \) is upper bounded by 140 for PDCCH, however if \( A \leq 11 \) the message is padded with zeros to reach \( A = 12 \). Due to the presence of CRC, \( E \) is lower bounded by 25, while \( E \leq 8192 \) as for uplink. For PBCH, only one code is accepted with parameters \( A = 32 \) and \( E = 864 \).

Figure 2 portrays the set of operations that information encoded with polar codes goes through within the 5G framework. Vector \( \mathbf{a} \) contains the \( A \) information bits to be transmitted using a payload of \( G \) code bits. Depending on code parameters, message may be split and segmented into two parts, that are encoded separately and transmitted together. Each segmented vector \( \mathbf{a}' \) of length \( A' \) will be encoded in an \( E \)-bit polar codeword. To every \( A' \)-bit vector, an \( L \)-bit CRC is attached. The resulting vector \( \mathbf{e} \), constituted of \( K = A' + L \) bits, is passed through an interleaver. On the basis of the desired code rate \( R \) and codeword length \( E \), a mother polar code of length \( N \) is designed, along with the relative bit channel reliability sequence and frozen set. The interleaved vector \( \mathbf{e}' \) is assigned to the information set along with ad-hoc parity-check bits, while the remaining bits in the \( N \)-bit vector \( \mathbf{y} \) are frozen. Vector \( \mathbf{u} \) is encoded with \( \mathbf{d}=\mathbf{uG}\mathbf{N} \), where \( \mathbf{G}\mathbf{N} = \mathbf{G}_2^{N/32} \) is the generator matrix for the selected mother code. After encoding, a sub-block interleaver divides \( \mathbf{d} \) in 32 equal-length blocks, scrambling them and creating \( \mathbf{y} \), that is fed into the circular buffer. For rate matching, puncturing, shortening or repetition are applied to change the \( N \)-bit vector \( \mathbf{y} \) into the \( E \)-bit vector \( \mathbf{e} \). A channel interleaver is finally applied to compute the vector \( \mathbf{f} \), that is now ready to be modulated and transmitted as \( \mathbf{g} \) after concatenation, if required.

In the following, we detail the operations necessary in each step of the 5G encoding chain.

### A. Message segmentation

Given message length \( A \) and payload length \( G \), the information may be decomposed in two blocks and encoded separately. Segmentation is activated by flag \( I_{\text{seg}} \), and in particular it may be activated PUCCH and PUSCH UCIs \( (I_{\text{seg}} = 1) \) while it is always bypassed for PBCH payloads and PDCCH DCIs \( (I_{\text{seg}} = 0) \). When code parameters satisfy the condition \( (A \geq 1013) \lor (A \geq 360 \land G \geq 1088) \), segmentation is activated. In this case, message is divided into two parts of length \( A' = \lceil A/2 \rceil \); if \( A \) is odd, the first message is composed by the first \( \lfloor A/2 \rfloor \) bits with the addition of a zero padding at the beginning. Code length, usually set to \( E = G \), has to be changed accordingly, namely setting \( E = \lfloor G/2 \rfloor \).

### B. Code parameters and rate matching selection

The 5G polar code encoding process relies on several parameters that depend on the amount and type of information to be transmitted and on the used channel. The first parameter that needs to be identified is the code length of the mother polar code, \( N = 2^n \). The number \( n \) is calculated as

\[
n = \max(\min(n_1, n_2, n_{\text{max}}), n_{\text{min}})
\]
where \( n_{\text{min}} \) and \( n_{\text{max}} \) give a lower and an upper bound on the mother code length, respectively. In particular, \( n_{\text{min}} = 5 \), while \( n_{\text{max}} = 9 \) for the downlink control channel, and \( n_{\text{max}} = 10 \) for the uplink. Parameter \( n_2 \) gives an upper bound on the code based on the minimum code rate admitted by the encoder, i.e. \( \frac{1}{2} \); as a consequence, \( n_2 = \lceil \log_2 (8K) \rceil \). Finally, the value of \( n_1 \) is bound to the selection of the rate-matching scheme. It is in fact usually calculated as \( n_1 = \lfloor \log_2 (E) \rfloor \), so that \( 2^{n_1} \) is the smallest power of two larger than \( E \). However, a correction factor is introduced to avoid a too severe rate matching: if \( \lfloor \log_2 (E) \rfloor < 0.17 \), i.e. if the smallest power of two larger than \( E \) is too far from \( E \), the parameter is set to \( n_1 = \lfloor \log_2 (E) \rfloor \). In this case an additional constraint on the code dimension is added, imposing \( K < \frac{9}{16} E \), to assure that \( K < N \).

If a mother polar code of length \( N > E \) is selected, the mother polar code will be punctured or shortened, depending on the code rate, before the transmission. In particular, if \( \frac{K}{N} \leq \frac{7}{16} \), the code will be punctured, otherwise it will be shortened. On the contrary, if \( N < E \), repetition is applied and some encoded bits will be transmitted twice; in this case, the code construction assures that \( K < N \).

As shown in Table 1, a set of flags and parameters assume different values depending on the type of transmission. The flags \( I_{IL} \) and \( I_{ILL} \) refer to the activation of the input bits interleaver and the channel interleaver respectively. The number of the two types of assistant PC bits are given by \( n_{PC} \) and \( n_{PC}^\text{IL} \) (see Section II-E for details). Figure 3 shows, for both uplink and downlink channels, the rate matching scheme and segmentation option used for different combinations of payload length \( G \) and number of information bits \( A \), along with the length \( N \) of the mother polar code used in the encoding.

### C. CRC encoding

A CRC of \( L \) bits is appended to the \( A' \) message bits stored in \( a' \), resulting in a vector \( c \) of \( A' + L \) bits. The possible CRC generator polynomials are the following:

\[
g_0(x) = x^6 + x^5 + 1
\]
\[
g_{11}(x) = x^{11} + x^{10} + x^9 + x^5 + 1
\]
\[
g_{24}(x) = x^{24} + x^{23} + x^{21} + x^{20} + x^{17} + x^{15} + x^{13} + x^{12} + x^8 + x^4 + x^2 + x + 1
\]

The polynomial \( g_{24}(x) \) is used for the payload in PBCH and DCIs in the PDCCH, where a larger number of assistant bits are necessary to enable early termination in the case of failures. Polynomials \( g_6(x) \) and \( g_{11}(x) \) are used for UCIs, in the case \( 12 \leq A \leq 19 \) and \( A \geq 20 \), respectively. The CRC shift register is initialized by all zeros for UCIs and for the PBCH payloads, and to all ones for the DCIs. Moreover, for DCIs, the CRC parity bits are “scrambled” according to a radio network temporary identifier (RNTI) \( x_0^{\text{rnti}}, x_1^{\text{rnti}}, \ldots, x_{15}^{\text{rnti}} \), i.e. the RNTI is masked in the last 16 CRC bits calculated by \( g_{24}(x) \) as \( c_{A + 8 + k} = c_{A + 8 + k} \oplus x_k^{\text{rnti}} \) for \( k = 0, \ldots, 15 \).

### D. Input bits Interleaver

The \( K \) bits obtained from the CRC encoder are interleaved before being inserted into the information set of the mother polar code. This feature can be used to reduce the decoding complexity by early terminating the decoding if an incorrect check is met, providing an additional false alarm rate (FAR) mitigation. The interleaver is activated through a flag \( I_{IL} \). In particular, the input bit interleaver is activated for PBCH payloads and PDCCH DCIs \( (I_{IL} = 1) \), while it is bypassed in the case of PUCCH and PUSCH UCIs \( (I_{IL} = 0) \) since FAR mitigation is not required.

By construction, the number of interleaved bits is upper bounded by \( K_{IL}^{\text{max}} = 164 \). The \( K \) bits from the previous

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**Table 1: Channel parameters.**

|            | Uplink (PUCCH/PUSCH) | Downlink (PDCCH) | Downlink (PBCH) |
|------------|----------------------|------------------|------------------|
| \( A \geq 20 \) | \( A \geq 1013 \uparrow \) \( \land G \geq 1088 \) | \( A < 1013 \) \( \land G < 1088 \) | \( E - A \leq 175 \) | \( E - A > 175 \) |
| \( n_{\text{max}} \) | 10 | 9 |
| \( \bar{I}_{\text{IL}} \) | 0 | 1 |
| \( \bar{I}_{\text{ILL}} \) | 1 | 0 |
| \( I_{\text{seg}} \) | 1 | 0 |
| \( G_{\text{max}} \) | 16384 | 8192 |
| \( G_{\text{min}} \) | 31 | 18 |
| \( A_{\text{max}} \) | 1706 | 25 |
| \( A_{\text{min}} \) | 12 | 140 |
| \( A_{\text{min}} \) | 12 | 32 |

\( K \)

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\( \lfloor x \rfloor \) represents the fractional part of a real number \( x \).
The interleaving function is applied to make possible by the fact that after interleaving, every CRC decoding and on broadcast and DCI blind detection. This is proposed to facilitate early termination, both during normal step are padded to reach the length of 164 bits and permuted according to the sequence $\Pi_{IL}^{max}$ presented in Table II. In more detail, the parameter $h = K_{IL}^{max} - K$ is calculated. Starting from the entry at index 0, all elements of $\Pi_{IL}^{max}$ are compared to $h$, and in the case that they are larger, they are stored in $\Pi$. Finally, $h$ is subtracted from all the entries of $\Pi$, such that $\Pi$ contains all the integers smaller than $K$ in permuted order. The construction of the interleaving pattern is illustrated in Figure 4. This scrambling sequence has been compared to $\Pi$ according to the sequence $\Pi(0)$, ..., $\Pi(K-1)$, all elements of $\Pi$ are finally, all elements of $\Pi$ are subtracted from all the entries of $\Pi$. Finally, $h$ is subtracted from all the entries of $\Pi$, such that $\Pi$ contains all the integers smaller than $K$ in permuted order. The construction of the interleaving pattern $\Pi$ is illustrated in Figure 4. This scrambling sequence has been proposed to facilitate early termination, both during normal decoding and on broadcast and DCI blind detection. This is made possible by the fact that after interleaving, every CRC remainder bit is placed after its relevant information bits. The interleaving function is applied to $c$, and the $K$-bit vector $c' = \{c_{\Pi(0)}, \ldots, c_{\Pi(K-1)}\}$ is obtained.

### E. Subchannel allocation and PC bits calculation

In this procedure, vector $c'$ is expanded in the $N$-bit input vector $u$ with the addition of assistant bits and frozen bits. As a first step, $n_{PC}$ parity-check bits are inserted within the $K$ information and CRC bits. The mother polar code is hence a $(N, K')$ code, with $K' = K + n_{PC}$.

To create the input vector $u$ to be encoded, the frozen set of subchannels needs to be identified. The number and position of frozen bits depend on $N$, $E$, and the selected rate-matching scheme. Initially, the frozen set $Q_{E}^{\emptyset}$ and the complementary information set $Q_{E}^{\emptyset}$ are computed based on the universal reliability sequence $Q_{E}^{N_{max}-1}$ [63] and the rate matching strategy. Later, information bits are assigned to $u$ according to the information set. Finally, assistant parity check bits are calculated and stored in $u$, if necessary. In the following, we examine every step of the creation of the input vector $u$ in more detail.

1) Frozen set $Q_{E}^{\emptyset}$: The first bits identified in the frozen set correspond to the indices of the $U = N - E$ untransmitted bits, i.e. the bits eliminated from the codeword by the rate-matching scheme. These indices correspond to the first $U$ or the last $U$ codeword bits in the case of puncturing and shortening, respectively, as explained in Section III-H. Due to the presence of an interleaver $J$ between the encoding and the rate matching, the actual indices to be added to the frozen set correspond to the first or the last after interleaving; details on this sub-block interleaver can be found in Section III-G. If $K_{E} \leq \frac{T}{N}$ and hence the mother polar code has to be punctured, additional indices are included in the frozen set such that $\{0, \ldots, T\} \subset Q_{E}^{\emptyset}$, with

$$T = \left\lfloor \frac{3}{4} N - \frac{E}{4} \right\rfloor - 1 \quad \text{if } E \geq \frac{3}{4} N \quad \left\lfloor \frac{9}{16} N - \frac{E}{4} \right\rfloor - 1 \quad \text{otherwise} \quad (9)$$

This extra freezing is necessary to prevent bits in the information set to become incapable due to puncturing. Finally, new indices are added to the frozen set from the universal reliability sequence, starting from the least reliable, until $|Q_{E}^{\emptyset}| = N - K'$. The universal reliability sequence $Q_{E}^{N_{max}-1}$ is a list of integers smaller than 1024 sorted in reliability order, from the least reliable to the most reliable; indices smaller than $N$ are extracted neatly in the creation of the frozen set of a mother polar code of length $N$. A qualitative depiction of the reliability sequence and the subchannel selection process is illustrated in Figure 5. The 1024 squares represent all the subchannels of the mother code, from the least reliable in the top-left corner, to the most reliable in the bottom-right corner. There are 32 magenta subchannels, that are relative to the bit indices 0 to 31, and 32 red subchannels, referring to bit indices 32 to 63. The 64 yellow ones are relative to bit indices 64 to 127, the 128 green ones to bits 128 to 255, the 256 azure ones to bits 256 to 511, and the 512 blue ones to bits 512 to 1023. Darker shades of each color represent larger indices, while lighter shades are smaller ones. In the case that $N = 512$ is selected, the 512 blue subchannels are excluded, while the red, yellow, green and azure ones are extracted maintaining their relative order, for a total of 512 ordered indices. In the case

| Table II: Input bits interleaver pattern mother sequence (bold integers represent CRC bit indices). |
|-----------------------------------------------|
| $\Pi_{IL}^{max}$                                      |
| 0 2 4 7 9 14 19 20 24 25 26 28                   |
| 31 34 42 45 49 50 51 53 54 56 58 59           |
| 61 62 65 66 67 69 70 71 72 76 77 81           |
| 82 83 87 88 89 91 93 95 98 101 104 106      |
| 108 110 111 113 115 118 119 120 122 123 126 127 |
| 129 132 134 138 139 140 1 3 5 8 10 15       |
| 21 27 29 32 35 43 46 52 55 57 60 63        |
| 68 73 78 84 90 92 94 96 99 102 105 107     |
| 109 112 114 116 121 124 128 130 133 135 141 16 |
| 11 16 22 30 33 36 44 47 64 74 79 85       |
| 97 100 103 117 125 131 136 142 12 17 23 37   |
| 48 75 80 86 137 143 13 18 38 144 39 145   |
| 40 146 41 147 148 149 150 151 152 153 154 155 |
| 156 157 158 159 160 161 162 163             |
that \( N = 256 \) is selected, only the red, yellow and green ones are extracted, and so on.

To summarize, the frozen set \( \bar{Q}_F^N \) is designed in three steps:
1) Pre-freezing: \( Q_1 = \{ J(\gamma), \ldots , J(\gamma + U - 1) \} \) where \( \gamma = 0 \) if \( K_F \leq \frac{7}{16} \) and \( \gamma = E \) otherwise.
2) Extra freezing: \( Q_2 = \{ 0, \ldots , T \} \) where \( T \) is calculated according to (9) if \( K_F \leq \frac{7}{16} \), otherwise \( Q_2 = \emptyset \).
3) Reliability freezing: \( Q_3 \) contains the first \( N - K' - \max \{ Q_1, Q_2 \} \) elements of \( Q_N^{\max-1} \) smaller than \( N \) not already included in \( Q_1 \cup Q_2 \).

Finally, the frozen set is given by \( \bar{Q}_F^N = Q_1 \cup Q_2 \cup Q_3 \). The bits of \( u \) corresponding to these indices are set to zero, i.e. \( u_i = 0 \) for all \( i \in \bar{Q}_F^N \).

2) Subchannel allocation: The information set \( \bar{Q}_I^N \) is calculated as the complement of \( \bar{Q}_F^N \), and contains \( K' = K + n_{PC} \) elements, corresponding to the bit indices that will contain the message bits and the parity check (PC) bits. The subchannels to be assigned to PC bits are calculated according to two different strategies: \( n_{PC}^\text{rem} \) are bound to the weight of rows of the generator matrix, while \( n_{PC}^\text{rem} = n_{PC} - n_{PC}^\text{rem} \) are bound to the subchannel reliability. The set of the parity check indices is called \( \bar{Q}_{PC}^N \), with \( \bar{Q}_{PC}^N \subset \bar{Q}_I^N \). To begin with, \( n_{PC}^\text{rem} \) bit indices are initially assigned as the \( n_{PC}^\text{rem} \) least reliable subchannels in \( \bar{Q}_I^N \). The index of the remaining \( n_{PC}^\text{rem} \) PC bit is selected as the subchannels corresponding to the row of minimum weight in the transformation matrix among the \( K \) most reliable bit indices in \( \bar{Q}_I^N \). In the case of uncertainty due to the presence of too many rows with the same weight, the index with the highest reliability is selected. The row weight \( w(g_i) \) of subchannel \( i \) corresponds to the number of ones of the \( i \)-th row \( g_i \) of the transformation matrix \( G_N \), and it can be easily calculated as \( w(g_i) = 2^{\alpha_i} \), where \( \alpha_i \) denotes the number of ones in the binary expansion of \( i \) [64]. After the subchannels have been allocated, the \( K \) message bits are stored in vector \( u \), i.e. the message is stored in the \( K \) indices of \( \bar{Q}_I^N \setminus \bar{Q}_{PC}^N \), and the values of the remaining \( n_{PC}^\text{rem} \) indices are assigned as follows.

3) PC bit calculation: The calculation of the PC bits is performed through a cyclic shift register of length 5, initialized to 0. Each PC bit is calculated as the XOR of the message bits assigned to preceding subchannels, modulo 5, excluding the previously calculated parity check bits. To summarize, a PC bit \( u_i \), with \( i \in \bar{Q}_{PC}^N \), is calculated as

\[
u_i = \bigoplus_{j=\lfloor i_{PC}/5 \rfloor}^{q-1} u_{5j+p}, \tag{10}\]

where \( q = \lfloor i/5 \rfloor \), \( p = i \mod 5 \) and \( i_{PC} \in \bar{Q}_{PC}^N \), is the highest index smaller than \( i \) for which \( i_{PC} \mod 5 = p \). If no such index exists, \( i_{PC} = 0 \).

F. Encoding

The encoding is performed by the multiplication in \( \mathbb{F}_2 \)

\[
d = u \cdot G_N, \tag{11}\]

where \( G_N = G_2 \otimes n \), with \( G_2 = [1\ 0] \). Encoding complexity can be proved to be \( O(N \log N) \) [1]. However, the recursive structure of the transformation matrix suggests the possibility to have parallel implementation. If \( N/2 \) processing units are available, encoding latency can be reduced to \( O(\log N) \). A tradeoff between hardware complexity and encoding latency can be found in between.

G. Sub-block interleaver

The \( N \) encoded bits are then interleaved before performing the rate matching. This interleaver \( J \) divides the \( N \) encoded bits stored in \( d \) into 32 blocks of length \( B = \frac{N}{16} \), interleaveing the blocks according to a list of 32 integers \( P \) and obtaining the vector \( y \) as illustrated in Figure 6. In practice, the index of an interleaved bit \( y_j \) is given by

\[
i = J(j) = \lfloor B \cdot P(\lfloor j/B \rfloor) \rfloor + q, \tag{12}\]

where \( q = j \mod B \), for all \( j = 0, \ldots , N - 1 \).

H. Rate matching

Rate matching is performed by a circular buffer, and the codeword \( e \) of length \( E \) bits is calculated. As already mentioned in Section III-B, three possible rate-matching schemes are foreseen:

- Puncturing: if \( E \leq N \) and \( R \leq \frac{7}{16} \), the mother code is punctured. In this case, the first \( U = N - E \) bits are not transmitted, hence \( e_i = y_{i+U} \) for \( i = 0, \ldots , E - 1 \).
- Shortening: if \( E \leq N \) and \( R > \frac{7}{16} \) the mother code is shortened. In this case, the last \( U = N - E \) bits are not transmitted, hence \( e_i = y_{i} \) for \( i = 0, \ldots , E - 1 \).
- Repetition: if \( E > N \), the first \( U = N - E \) bits are transmitted twice, hence \( e_i = y_{i \mod N} \) for \( i = 0, \ldots , E - 1 \).

The operating principle of the rate-matcher based on the circular buffer is illustrated in Figure 7.
I. Channel interleaver

Before passing the rate-matched codeword to the modulator, the bits in $e$ are interleaved one more time using a triangular bit interleaver. This interleaver has been considered necessary to improve the coding performance of the coding scheme for high-order modulation; it is not applied for every use case, hence it is triggered by a parameter $I_{BIL}$. In particular, the channel interleaver is activated for PUCCH and PUSCH UCIs ($I_{BIL} = 1$), while it is bypassed in the case of PBCH payloads and PDCCH DCIs ($I_{BIL} = 0$).

The channel interleaver is formed by an isosceles triangular structure of length $T$ bits, where $T$ is the smallest integer such that $\frac{T(T+1)}{2} \geq E$; its value can be be calculated as $T = \left\lceil \sqrt{8E+1} - 1 \right\rceil$. The encoded bits in $e$ are written into the rows of the triangular structure, while the interleaved vector $f$ is obtained by reading bits out of the structure in columns. The construction of the interleaving pattern is illustrated in Figure 8. In more detail, an auxiliary $T \times T$ matrix $V$ is created on the basis of $e$ with

$$V_{i,j} = \begin{cases} \text{NULL} & \text{if } i + j \geq T \text{ or } r(i) + j \geq E \\ e_{r(i)+j} & \text{otherwise} \end{cases}$$

where $r(i) = \frac{i(2T-i+1)}{2}$. The interleaved vector $f$ is created by appending the columns of $V$ while skipping the NULL entries. This triangular interleaver has been proposed in 5G standardization because of the practical advantages provided by its high parallelism factor, due to its maximum contention-free property, and its high flexibility.

J. Block concatenation

If segmentation has been activated at the beginning of the process, the two codewords of length $E$ are appended in order to obtain a unique block of length $G$. If $G = 2E + 1$, a zero bit is appended at the end of the second codeword.

IV. Conclusion

In this work, we have detailed the polar code encoding process within the 5th generation wireless systems standard, providing the reader with a user-friendly description to understand, implement and simulate 5G-compliant polar code encoding. This encoding chain showcases the successful efforts of the 3GPP standardization body to meet the various requirements on the code for the eMBB control channel: low description complexity and low encoding complexity, while covering a wide range of code lengths and code rates. Throughout this work, we hinted that the standardization process also took the receiver side into account. Typical for modern channel coding, the encoder was designed such that the decoder can be implemented with feasible complexity and operate at the required latency, assuming state-of-the-art decoders and hardware. New decoding principles or decoding architectures, however, may now be developed in order to optimize decoding complexity or also increase error-rate performance.

With the 5G eMBB control channel, polar codes have found their first adoption into a standard only 10 years after their invention. This standardization has triggered further academic and industrial research into polar coding, and adoption in future standards and systems can be foreseen, given the flexibility of code and decoder design that polar codes offer. Our detailed description of the 5G polar codes, including the individual components of the encoding chain, may serve as a reference to further development of polar codes for future applications.

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