ISOMETRIC ACTION OF $SL_2(\mathbb{R})$ ON HOMOGENEOUS SPACES

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Abstract. We investigate the $SL_2(\mathbb{R})$ invariant geodesic curves with the associated invariant distance function in parabolic geometry. Parabolic geometry naturally occurs in the study of $SL_2(\mathbb{R})$ and is placed in between the elliptic and the hyperbolic (also known as the Lobachevsky half-plane and 2-dimensional Minkowski half-plane space-time) geometries. Initially we attempt to use standard methods of finding geodesics but they lead to degeneracy in this setup. Instead, by studying closely the two related elliptic and hyperbolic geometries we discover a unified approach to a more exotic and less obvious parabolic case. With aid of common invariants we describe the possible distance functions that turn out to have some unexpected, interesting properties.

1. Introduction

In this paper we will explore the isometric action of the semi-simple Lie group $G = SL_2(\mathbb{R})$ ($2 \times 2$ real matrices with determinant one) on the homogeneous spaces $G/H$ where $H$ is one dimensional subgroup of $G$. There are only three such subgroups up to conjugacy, proved in [13, §III.1]:

$$K' = \left\{ \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}, \text{ where } 0 \leq \theta < 2\pi \right\}$$

$$N' = \left\{ \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}, \text{ where } t \in \mathbb{R} \right\}$$

$$A' = \left\{ \begin{pmatrix} \cosh \alpha & \sinh \alpha \\ \sinh \alpha & \cosh \alpha \end{pmatrix}, \text{ where } \alpha \in \mathbb{R} \right\}. \quad (1)$$

We can represent the action of $SL_2(\mathbb{R})$ on the homogeneous spaces $G/H$ by Möbius transformations:

$$g \cdot w = \frac{aw + b}{cw + d} \text{ where } w \in \mathbb{U}, \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R}). \quad (2)$$

This is a group homomorphism from $SL_2(\mathbb{R})$ to Möbius transformations of the upper half-plane due to the fact that composition of two such Möbius maps is represented by the product of two respective matrices. Here $\mathbb{U}$ are numbers of the form $w = x + iy$ with $i^2 = \sigma = -1, 0, 1$ and are called complex, dual and double numbers respectively see [4,7].

Those three subgroups give rise to elliptic, parabolic and hyperbolic geometries (abbreviated $EPH$). The name comes about from the shape of the equidistant orbits which are ellipses, parabolas, hyperbolas respectively. Thus the Lobachevsky geometry is elliptic (not hyperbolic) in our terminology.

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EPH are 2-dimensional Riemannian, non-Riemannian and pseudo-Riemannian geometries on the upper half-plane (or on the unit disc). Non-Riemannian geometries, see [3, 4, 12, 14], are a growing field with geometries like Finsler in [6] gaining more influence. The Minkowski geometry is formalised in a sector of a flat plane by means of double numbers [4], see also [11].

Subgroups $H$ from (1) fix the imaginary unit $i$ under the action of (2) and thus are known as EPH rotations (around $i$). Consider a distance function invariant under the $SL_2(\mathbb{R})$ action. Then the orbits of $H$ will be equidistant points from $i$, giving some indication on what the distance function should be. But this does not determine the distance entirely since there is freedom in assigning values to the orbits. Review a well-known standard definition of distance $d : X \times X \to \mathbb{R}^+$ with:

\begin{align*}
(1) \quad & d(x, y) \geq 0, \\
(2) \quad & d(x, y) = 0 \text{ iff } x = y, \\
(3) \quad & d(x, y) = d(y, x), \\
(4) \quad & d(x, y) \leq d(x, z) + d(z, y),
\end{align*}

for all $x, y, z \in X$. Although adequate in many cases, the defined concept does not cover all interesting distance functions. Examples include the recent study of distances with omission of symmetry or the triangle inequality as in [12]. In the hugely important Minkowski space-time the reverse of the triangle inequality holds. We will be referring to this later.

Our consideration will be based on equidistant orbits, which physically correspond to wavefronts with a constant velocity. For example, if you drop a stone in the pond the ripples you see will be waves, which travelled the same distance from a dropping point. A dual description to wavefronts uses rays—the paths along which waves travels, i.e. the geodesics in the case of a constant velocity. The duality between wavefronts and rays is provided by Huygens’ principle in [1, §46].

Geodesics also play a central role in differential geometry generalising the notion of a straight lines. They are closely related to a distance function: geodesics are often defined as curves which extremize distance. As a consequence, along geodesics the distance function is additive.

In the next section we will describe the invariant metrics in EPH and the Riemannian approach to geodesics. In Section [3] we describe all invariant distance functions satisfying a mild assumption. In Section [4] we deduce geodesics from invariant distances using the additivity property. An alternative construction of invariant distances from invariant geodesics is described in Section [5]. The triangle inequality for those distance functions is studied in Section [6].

2. Metric, Length and Extrema

Recall the established procedure of constructing geodesics in Riemannian geometry (two-dimensional case) as in [15 §7]:

(1) Define the metric of the space: $Edu^2 + Fdudv + Gdv^2$.

(2) Define length for a curve $\Gamma$ as:

$$\text{length}(\Gamma) = \int_{\Gamma} (Edu^2 + Fdudv + Gdv^2)^{\frac{1}{2}}.$$

(3) Then geodesics will be defined as the curves which give a stationary point for length.

(4) Lastly the distance between two points is the length of a geodesic joining those two points.

In this respect, to obtain the $SL_2(\mathbb{R})$ invariant (referred simply as invariant in the rest of the paper) distance we require the invariant metric.
Theorem 2.1. The invariant metric in EPH cases is
\[ ds^2 = \frac{du^2 - \sigma dv^2}{v^2}, \] (4)
where \( \sigma = -1, 0, 1 \) respectively.

In the proof below we will follow the same procedure as in \([5, \S\,10]\).

**Proof.** In order to calculate the metric consider the subgroups \([\Gamma]\) of Möbius transformations that fix \(i\). Denote an element of those rotations by \(E_\sigma\). We require an isometry so:
\[ d(i, i + \delta v) = d(i, E_\sigma(i + \delta v)). \] (5)
Using the Taylor series we get:
\[ E_\sigma(i + \delta v) = i + J_\sigma(i) \delta v + o(\delta v), \] (6)
where the Jacobian denoted \(J_\sigma\) respectively is:
\[ \begin{pmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 2t & 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \cosh 2\alpha & \sinh 2\alpha \\ \sinh 2\alpha & \cosh 2\alpha \end{pmatrix}. \] (7)
A metric is invariant under the above rotations if it is preserved under the linear transformation:
\[ \begin{pmatrix} du \\ dv \end{pmatrix} = J_\sigma \begin{pmatrix} du \\ dv \end{pmatrix}, \] (8)
which turns out to be, \(du^2 - \sigma dv^2\) in the three cases.

To calculate the metric at an arbitrary point \(w = u + iv\) we map \(w\) to \(i\) by an affine Möbius transformation, which acts transitively on the upper half-plane
\[ r^{-1}: \quad w \mapsto \frac{w - u}{v}, \] (9)
hence there is a factor of \((\frac{1}{v})^2\) multiplying the metric giving \(ds^2 = \frac{du^2 - \sigma dv^2}{v^2}\). \(\square\)

**Corollary 2.2.** With the notation from above, for an arbitrary curve \(\Gamma\):
\[ \text{length}(\Gamma) = \int_\Gamma \frac{(du^2 - \sigma dv^2)^{\frac{1}{2}}}{v}. \] (10)

In the two non-degenerate cases (elliptic and hyperbolic) to find the geodesics is straightforward, it is now the case of solving the Euler-Lagrange equations and hence finding the minimum or the maximum respectively. The Euler-Lagrange equations for the metric \([\Gamma]\) take the form:
\[ \frac{d}{dt} \left( \frac{\gamma_1}{y^2} \right) = 0, \quad \frac{d}{dt} \left( \frac{\sigma \gamma_2}{y^2} \right) = \frac{\gamma_1^2 - \sigma \gamma_2}{y^3}. \] (11)
where \(\gamma\) is a smooth curve \(\gamma(T) = (\gamma_1(T), \gamma_2(T))\) and \(T \in (a, b)\).

For \(\sigma = -1\) the solution is well-known: semicircles orthogonal to the real axes or vertical lines, as in \([21, \S\,15]\). Equations of the ones passing though \(i\) are, see \([15, \S\,7]\):
\[ (x^2 + y^2) \sin 2t - 2x \cos 2t - \sin 2t = 0 \] (12)
where \(t \in \mathbb{R}\). And the distance function is then:
\[ d(z, w) = \sinh^{-1} \frac{|z - w|}{2\sqrt{\Im[z]\Im[w]}} \] (13)
where \(\Im[z]\) is the imaginary part of \(z\).

In the hyperbolic case when \(\sigma = 1\) there are two families of solutions, one space-like, one time-like:
\[ x^2 - y^2 - 2tx + 1 = 0 \quad \text{and} \quad (x^2 - y^2) \sinh 2t - 2x \cosh 2t + \sinh 2t = 0 \] (14) (15)
with \( t \in \mathbb{R} \). Those are again orthogonal see \([11, 10]\) to the real axes. And the distance functions are:

\[
d(z, w) = \begin{cases} 
2 \sin^{-1} \sqrt{\frac{|\Re(z-w)|^2 - \Im(z-w)^2}{2\sqrt{\Im(z-w)^2}}} & \text{when space-like;} \\
2 \sinh^{-1} \sqrt{\frac{|\Re(z-w)|^2 - \Im(z-w)^2}{2\sqrt{\Im(z-w)^2}}} & \text{when time-like,}
\end{cases}
\]

(16)

where \( \Re[z] \) and \( \Im[z-w] \) are the real and imaginary part of \( z \).

But in the parabolic framework the only solution of (11) are vertical lines, as in \([16, \S \text{3}]\). Note that they are again orthogonal to the real axes. They indeed minimise the distance between two points \( w_1, w_2 \) since the geodesic is up the line \( x = \Re[w_1] \) through infinity and down \( x = \Re[w_2] \). Any points on the same vertical lines have distance zero, so \( d(w_1, w_2) = 0 \) for all \( w_1, w_2 \) which is a very degenerate function. Hence we go on to study further the algebraic and geometric invariants to find a more adequate answer.

Remark 2.3. The same geodesic equations can be obtained by Beltrami’s method \([4]\)

3. Algebraic Invariants

We seek all real valued functions invariant under the group action of \( SL_2(\mathbb{R}) \) \([2]\):

\[
f(g(z), g(w)) = f(z, w) \quad \text{for all} \quad z, w \in \mathbb{U} \quad \text{and} \quad g \in SL_2(\mathbb{R})
\]

where \( \mathbb{U} \) is the complex, dual or double numbers. One such function is, similarly to \([13, 16]\):

\[
F(z, w) = \frac{|z-w|_\sigma}{\sqrt{\Im[z-w]^2}}
\]

(17)

which can be shown by a simple direct calculation. Note that

\[
|z-w|_\sigma = \Re[z-w]^2 - \sigma \Im[z-w]^2
\]

(18)

in analogy with the metric in EPH geometries, similarly to what is done in \([16, \text{App. C}]\). We will need the following definition:

Definition 3.1. A function \( f : X \times X \to \mathbb{R}^+ \) is called a monotonous distance if \( f(\Gamma(0), \Gamma(t)) \) is a continuous monotonically increasing function of \( t \) where \( \Gamma : [0, 1] \to X \) is a smooth curve with \( \Gamma(0) = z_0 \) that intersects all equidistant orbits of \( z_0 \) exactly once.

For example function \( F(z, w) \) is monotonous.

Theorem 3.2. A monotonous function \( f(z, w) \) is invariant under \( g \in SL_2(\mathbb{R}) \) if and only if there exists a monotonically increasing continuous real function \( h \) such that \( f(z, w) = h \circ F(z, w) \).

Proof. Given \( f(z, w) = h \circ F(z, w) \) then:

\[
f(g(z), g(w)) = h \circ F(g(z), g(w)) = h \circ F(z, w) = f(z, w)
\]

with \( g \in SL_2(\mathbb{R}) \). Also \( F(z, w) \) is monotonous and so \( h \circ F(z, w) \) is.

Conversely, suppose there exists another function with such a property say, \( H(z, w) \). Due to invariance under \( SL_2(\mathbb{R}) \) this can be viewed as a function in one variable if we apply \( r^{-1} \) (cf. \([20]\) which sends \( z \) to \( i \) and \( w \) to \( r^{-1}(w) \). Now by considering a fixed smooth curve \( \Gamma \) from \([4, 11]\) we can completely define \( H(z, w) \) as a function of a single real variable \( h(t) = H(i, \Gamma(t)) \) and similarly for \( F(z, w) \):

\[
H(z, w) = H(i, r^{-1}(w)) = h(t) \quad \text{and} \quad F(z, w) = F(i, r^{-1}(w)) = f(t)
\]
where \( h \) and \( f \) are both continuous and monotonically increasing since they represent distance. Hence the inverse \( f^{-1} \) exists everywhere by the inverse function theorem. So:

\[
H(i, r^{-1}(w)) = h \circ f^{-1} \circ F(i, r^{-1}(w)).
\]

(19)

Note that \( hf^{-1} \) is monotonic as its the composition of two monotonically increasing functions and this ends the proof.

**Remark 3.3.** The above proof carries over to a more general theorem stating: If there exist two monotonous functions \( F(x, y) \) and \( H(x, y) \) invariant under a transitive action of the group \( G \) then there exists a monotonically increasing real function \( h \) such that \( H(z, w) = h \circ F(z, w) \).

As discussed in the previous section, in elliptic and hyperbolic geometries the function \( h \) from above is either \( \sinh^{-1} t \) or \( \sin^{-1} t \). Hence it is reasonable to try inverse trigonometric and hyperbolic functions in the intermediate parabolic case too.

**Remark 3.4.** The above result sheds light on the freedom we have; we can either (i) “label” the equidistant orbits with numbers i.e choose a function \( h \) which will then determine the geodesic; (ii) or choose a geodesic which will then determine \( h \). Those two approaches are reflected in the next two chapters.

4. Additivity

As pointed out, earlier there might not be a distance function which satisfies all the traditional properties. But we still need the key ones, in light of this we make the following definition:

**Definition 4.1.** Geodesics are smooth curves along which the distance function is additive.

**Remark 4.2.** It is important that this definition is relevant in all EPH cases, i.e. in the elliptic and hyperbolic cases it would produce the well-known geodesics defined by the extremality condition.

Schematically the proposed approach is:

\[
\begin{align*}
\text{invariant distance} & \xrightarrow{\text{additivity}} \text{invariant geodesic} \\
\text{local metric} & \xrightarrow{\text{extrema}} \text{geodesic} \xrightarrow{\text{integration}} \text{distance}.
\end{align*}
\]

(20)

(21)

Let us now proceed with finding geodesics from a distance function.

Suppose given a function \( d(w, w') \) with \( w, w' \in U \) there exists a smooth curve joining every two points along which \( d \) is additive. We can view the function \( d(w, w') \) as a real function in 4 variables say \( f(u, v, u', v') \). Take two points \( w_1 = u_1 + iv_1 \) and \( w_2 = u_2 + iv_2 \) on such a curve. Then consider a nearby point \( w_3 = u_2 + \delta u + i(v_2 + \delta v) \) and write the additivity condition:

\[
d(w_1, w_2) + d(w_2, w_3) = d(w_1, w_3).
\]

(22)

Use the Taylor series, viewing \( f \) as a function of four real variables, we obtain:

\[
d(w_2, w_3) = f(u_2, v_2, u_2 + \delta u, v_2 + \delta v) = f(u_2, v_2, u_2, v_2) + J_f(u_2, v_2, u_2, v_2) \delta w + o(\delta w)
\]

(23)

where \( J_f \) is the Jacobian. Since \( f \) is a distance function we demand that distance from a point to itself is zero hence \( f(u_2, v_2, u_2, v_2) = 0 \). Also:

\[
d(w_1, w_3) = f(u_1, v_1, u_2 + \delta u, v_2 + \delta v) = f(u_1, v_1, u_2, v_2) + J_f(u_1, v_1, u_2, v_2) \delta w + o(\delta w),
\]

(24)
substituting this into (22) gives:

\[ J_f(u_2, v_2, u_2, v_2) \delta w = J_f(u_1, v_1, u_2, v_2) \delta w, \]

\[ f_3(u_2, v_2, u_2, v_2) \delta u + f_4(u_2, v_2, u_2, v_2) \delta v = f_3(u_1, v_1, u_2, v_2) \delta u + f_4(u_1, v_1, u_2, v_2) \delta v. \]

And finally obtain the differential equation:

\[ \frac{\delta v}{\delta u} = -\frac{f_3(u_1, v_1, u_2, v_2) + f_3(u_2, v_2, u_2, v_2)}{f_4(u_1, v_1, u_2, v_2) - f_4(u_2, v_2, u_2, v_2)}, \]

(25)

where \( f_n \) stands for partial derivative with respect to the \( n \)-th variable.

A natural choice for \( \sin^{-1}_\sigma \) is \( \sin^{-1}_\sigma t \) where:

\[ \sin^{-1}_\sigma t = \begin{cases} \sin^{-1} t, & \text{if } \sigma = -1; \\ 2t, & \text{if } \sigma = 0; \\ \sin^{-1} t, & \text{if } \sigma = 1. \end{cases} \]

(26)

known as elliptic, parabolic and hyperbolic inverse sine, for motivation see [7][9]. Note that \( \sigma \) is entirely different from \( \sigma \) although it takes the same values. It is used to denote the possible sub-cases within the parabolic geometry alone. Substituting those functions in (25) gives:

\[ \delta v = \frac{2v_2}{u_2 - u_1} \sqrt{[\sigma(u_1 - u_2)^2 + 4v_1v_2]/(u_2 - u_1)}. \]

(27)

where \( \sigma = -1, 0, 1 \) correspond to elliptic, parabolic and hyperbolic inverse sine of the distance function. Now taking \( u_1 = 0 \) and \( v_1 = 1 \) and varying \( u_2 + iv_2 \) gives the differential equation:

\[ \frac{\delta v}{\delta u} = \frac{2v}{u} = \sqrt{[\sigma u^2 + 4v]/u}. \]

(28)

Solutions are families of parabolas \((\sigma + 4t^2)u^2 - 8tu - 4v + 4 = 0\). This is a subset of a very important invariant class of curves called cycles defined in [16 § 7], and further studied in [10]. Cycles could be thought of the natural curves in those geometries, in the EPH cases they are circles, parabolas and hyperbolas respectively.

Summarising:

**Theorem 4.3.** The geodesics through \( i \) with the distance function \( \sin^{-1}_\sigma \) are parabolas of the form \((\bar{\sigma} + 4t^2)u^2 - 8tu - 4v + 4 = 0\).

5. **Geometric Invariants**

As we discussed earlier the invariant metric alone within Riemannian framework produces only the trivial distance function. In this section we work out the distance from the metric and geodesics. Schematically:

\[
\text{invariant metric + invariant geodesics} \xrightarrow{\text{integration}} \text{distance function}. \]

(29)

Again looking back at (21) we can see that those two approaches are similar.

Geodesics should form an invariant subset of an invariant class of curves with no more than one curve joining every two points. Here this class is cycles as they are the most and almost the only key objects in EPH, shown in [8]. Such a subset may be characterised by an invariant algebraic condition, which in analogy with elliptic and hyperbolic geometries is taken to be focal orthogonality (f-orthogonality) to the real axes defined in [10 § 4.3]. In brief a cycle is f-orthogonality to the real axes if the real axes inverted in a cycle is orthogonal (in the usual sense) to the real
axes. Explicitly parabola \( ku^2 - 2lu - 2nv + m = 0 \) is \( f \)-orthogonal to the real axes if:

\[
l^2 + \sigma n^2 - mk = 0,
\]

where \( \sigma = -1, 0, 1 \) similarly to (but independently of!) \( \sigma \).

As a starting point consider the cycles that pass through \( i \). It is enough to specify only one such \( f \)-orthogonal cycle; the rest will be obtained by Möbius transformations fixing \( i \) i.e parabolic rotations \( (1) \). Within those constraints there are three different families of parabolas. They are obtained by parabolic rotation of principal parabolas:

\[
y = \frac{\sigma}{4} x^2 + 1,
\]

where \( \sigma = -1, 0, 1 \). Explicitly they are given by equations: \( \sigma + 4t^2x^2 - 8tx + 4 = 4y \). Note that those are exactly the same geodesics as in the previous section. Hence we know what the distance function has to be. But we still give a sketch of how to work out the distance below. This calculation does not involve anything from the previous section and is in a way more elementary and intuitive.

Remark 5.1. The length taken along an arbitrary parabola \( ax^2 + bx + c = y \) (parametrised as \( t + i(at^2 + bt + c) \)) using the standard definition of length is:

\[
\text{length}(\Gamma) = \int_{\Gamma} \frac{dt}{at^2 + bt + c}.
\]

Depending on whether the discriminant of the denominator is positive, zero or negative the results are trigonometric, rationals or hyperbolic functions respectively. This gives an inside to why there are only three distinct types of geodesics.

Consider specifically the \( f \)-orthogonal parabolas described above and use a trick of moving one point to \( i \) and the second one to the principle parabola as in \( (31, 15, 7) \). Then the distance along the principle parabola is:

\[
\int_{0}^{\varphi} \frac{dt}{4\sigma t^2 + 1} = \begin{cases} 
4\log \frac{\varphi}{2\sigma}, & \text{if } \sigma = -1; \\
x, & \text{if } \sigma = 0; \\
\tan^{-1} \frac{x}{2\sigma}, & \text{if } \sigma = 1.
\end{cases}
\]

Given an arbitrary point \((u, v)\) the distance to \( i \) can be calculated by finding where the orbit of \((u, v)\) intersects the principle parabola, which is a matter of solving simultaneous equations:

\[
v = \frac{\sigma}{4} u^2 + 1, \quad v = ku,
\]

where \( k = \frac{u}{\sigma} \) giving:

\[
u = \frac{u_0}{\sqrt{v_0 - \frac{\sigma u_0^2}{4}}}.
\]

Finally the distance from \((u_1, v_1)\) to \((u_2, v_2)\) is calculated by applying the Möbius transformation \( r^{-1}(w) \) (cf. \([9]\)) which sends \( w_1 \mapsto i \) and \( w_2 \mapsto \left( \frac{u_2 - u_1}{v_2}, \frac{v_2}{v_1} \right) \). So:

\[
d(w_1, w_2) = \begin{cases} 
8 \sin^{-1} \frac{|u_2 - u_1|}{2\sqrt{v_1 v_2}}, & \text{if } \sigma = -1; \\
\frac{|u_2 - u_1|}{\sqrt{v_1 v_2}}, & \text{if } \sigma = 0; \\
\sin^{-1} \frac{|u_2 - u_1|}{2\sqrt{v_1 v_2}}, & \text{if } \sigma = 1;
\end{cases}
\]

Although the answer above is of no surprise, in light of previous section, it is nevertheless gives more inside how \( \sin^{-1} \) and \( \sin^{-1} \) appears. Diffusion of the parabolic geometry into three different sub-cases is known and got the names \( P_e, P_p \) and \( P_h \) to stand for elliptic, parabolic and hyperbolic flavours, for further examples see \([9]\). The geodesics have been drawn in Figure \([11]\) and it is striking how there
seems to be a “continuous” transformation between the geometries. We can see the transitions from the elliptic case to $P_e$ then to $P_p$ to $P_h$ to the hyperbolic space-like and finally to light-like.

There is one more pleasant parallel between all the geometries. In the Lobachevsky and Minkowski geometries the centre of geodesics lies on the real axes. In the parabolic geometry the respective foci of geodesic parabolas lie on the real axes. Here $P_h$ focus is just the usual focus of the parabola, the $P_p$ is the vertex of the parabola and the $P_e$ is the point of directrix nearest to the parabola. Those foci are closely linked [8 § 4.3] to the f-orthogonality we used earlier.

$$E = (\sin(t) \times \cos(t), [0.5] \times \cos(t)^2 - (0.5) \times \sin(t)^2, -\sin(t) \times \cos(t))$$

$$P_e = (1 - 4 * t^2, [-4 * t, -2], -4)$$

$$P_p = ([t, 0.5], 1)$$

$$P_h = [(t, 0.5), 1]$$

$$P_e = (0, 0.5 + \cosh(t)^2, \sinh(t) \times \cosh(t))$$

$$P_p = (l, [t, 0.5], 1)$$

$$P_h = (\sinh(t) \times \cosh(t), [-0.5 + \cosh(t)^2, \sinh(t) \times \cosh(t)]$$

**Figure 1.** Showing geodesics (blue) and equidistant orbits (green) in EPH geometries. Above are written $(k, [l, n], m)$ in $kx^2 - 2lx - 2nw + m = 0$ giving the equation of geodesics.
Remark 5.2. We can translate all the above results from the upper half plane model we have been using to the unit disc model. This is done via the Cayley transform which in parabolic geometries is given by, derived in [8]:

\[ w \mapsto \frac{2w - i}{i\bar{\sigma}w + 2} \]  

(37)

Again note that the values of \( \bar{\sigma} = -1, 0, 1 \) correspond to \( P_\ell, P_\rho \) and \( P_h \) flavours of the parabolic unit discs. Then applying the inverse Cayley transform we can see that the invariant distance between two points \( u_1 + iv_1 \) and \( u_2 + iv_2 \) in the disc is:

\[ \sin^{-1}_\sigma \frac{1}{2} \frac{|u_2 - u_1|}{\sqrt{(1 + 2v_1 + \bar{\sigma}u_1^2)(1 + 2v_2 + \bar{\sigma}u_2^2)}}. \]  

(38)

An interesting feature of the Cayley transform (37) is as follows: in all three flavours of the parabolic unit disk the real line is a geodesic passing the origin for the respective distance (38) with the same value of \( \bar{\sigma} \) as in (37).

6. Properties of the distance function

In the introduction, we listed properties of the standard distance functions which we are going to re-visit in the context of obtained invariant functions \( d(z, w)_\bar{\sigma} = \sin^{-1}_\sigma \frac{|z - w|}{2\sqrt{|z||\bar{\sigma}w|}} \). Two of the four properties hold: it is clearly symmetric and positive for every two points. But the distance of any point to a point on the same vertical line is zero so \( d(z, w) = 0 \) does not imply \( z = w \). This can be overcome by introducing a different distance function just for the points on the vertical lines as is done in [16, §3]. Note that we still have \( d(z, z) = 0 \) for all \( z \).

Also the triangle inequality does not hold but an interesting variation on it is true:

**Theorem 6.1.** Take any \( SL_2(\mathbb{R}) \) invariant distance function and take two points \( w_1, w_2 \) and the geodesic (in the sense of section 4) through points. Consider the strip \( \mathbb{R}[w_1] < u < \mathbb{R}[w_2] \) and take a point \( z \) in it. Then the geodesic divides the strip into two regions where \( d(w_1, w_2) \leq d(w_1, z) + d(z, w_2) \) and where \( d(w_1, w_2) \geq d(w_1, z) + d(z, w_2) \).

**Remark 6.2.** This is a kind of intermediate theorem between the elliptic case where \( d(w_1, w_2) \leq d(w_1, z) + d(z, w_2) \) for all \( w_1, w_2, z \in \mathbb{C} \) and the hyperbolic geometry where the converse is true \( d(w_1, w_2) \geq d(w_1, z) + d(z, w_2) \).

**Proof.** The only possible invariant distance function in parabolic geometry is of the form \( d(z, w) = h \circ \frac{|z - w|}{2\sqrt{|z||\bar{\sigma}w|}} \) where \( h \) is a monotonically increasing continuous real function by Thm. 5.2. Fix two points \( w_1, w_2 \) and the geodesic though them. Now consider some point \( z = a + ib \) in the strip. The distance function is additive along a geodesic so \( d(w_1, w_2) = d(w_1, w(a)) + d(w(a), w_2) \) where \( w(a) \) is a point on the geodesic with real part equal to \( a \). But if \( \Im[w(a)] < b \) then \( d(w_1, w(a)) > d(w_1, z) \) and \( d(w(a), w_2) > d(z, w_2) \) which implies \( d(w_1, w_2) > d(w_1, z) + d(z, w_2) \). Similarly if \( \Im[w(a)] > b \) then \( d(w_1, w_2) < d(w_1, z) + d(z, w_2) \). \( \square \)

**Remark 6.3.** The reason for the ease with which the result falls out is the fact that the distance function is additive along the geodesics. This justifies the definition of geodesic in terms of additivity.

To illustrate those ideas look at the region where the converse of the triangular inequality holds for \( d(z, w)_\bar{\sigma} = \sin^{-1}_\sigma \frac{|z - w|}{2\sqrt{|z||\bar{\sigma}w|}} \) marked red on Figure 2. It is enclosed by two parabolas both of the form \((\bar{\sigma} + 4t^2)u^2 - 8tu - 4u + 4 = 0 \) (which is the general equation of geodesics) and both go through the two fixed point. They
arise from taking $\pm$ when solving the quadratic equation to find $t$. Both of them separate the region where the triangle inequality fails but one of them in between two points and the second outside.

7. Conclusion

In summary, in this paper, we managed to find non-degenerate “straight lines” in parabolic geometries, showing that parabolic geometry possesses many non-trivial features. Also some of its properties have been discovered and this reveals its importance as “the missing link” between well known geometries. We opened new options for further study since now it is possible to create objects like triangles. This gives opportunities to discover the corresponding parabolic theorems to famous ones like Pythagoras. In other words finding the “lines” gives a solid footing for further investigation into this exciting subject. Parabolic geometry is a promising area because of the interest, refreshingly different from all other geometries distance functions.

The approach in this paper applies not only to $SL_2(\mathbb{R})$ but can be used on other semi-simple Lie groups $G$. By considering their action on homogeneous spaces $G/H$ where $H$ its subgroup, it is possible to create higher dimensional geometries.

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