Closed-form evaluation of 2D static lattice sums

Semyon Yakubovich\textsuperscript{1}, Piotr Drygas\textsuperscript{2}, Vladimir Mityushev\textsuperscript{3}

May 8, 2018

\textsuperscript{1}Department of Mathematics, Faculty of Sciences, University of Porto, Campo Alegre st., 687, 4169-007 Porto, Portugal, syakubov@fc.up.pt
\textsuperscript{2}Department of Differential Equations and Statistics, Faculty of Mathematics and Natural Sciences, University of Rzeszow, Pigiona 1, 35-959 Rzeszow, Poland, drygaspi@ur.edu.pl
\textsuperscript{3}Institute of Computer Sciences and Computer Methods, Pedagogical University, ul.Podchorazych 2, Krakow, 30-084, Poland, mityu@up.krakow.pl

\textsuperscript{3}The corresponding author

Key words: Lattice sum; Eisenstein summation method; Effective properties of 2D composites; Complete elliptic integrals of the first and second kind; Riemann zeta-function; Mellin transform

MSC (2000): 30E25, 65N21, 44A15

1 Introduction

The mathematical questions of convergence, numerically effective algorithm and closed-form evaluation of the lattice sums were discussed in the fundamental book Borwein et al \cite{1} and works cited therein. The present paper is devoted to closed-form evaluation of the conditionally convergent 2D lattice sums. One of them, \( S_2 \), defined by \cite{20}, was considered by Lord Rayleigh \cite{2}. Numerically effective series for Rayleigh’s sum based on the elliptic functions are outlined in Borwein et al \cite{1} Sec.3.2]. McPhedran et al \cite{3}, Movchan et al \cite{5} and Greengard et al \cite{4} developed the Rayleigh method to elastostatic (see lattice sum \cite{20}) and elastodynamic problems having paid the
main attention to computationally convenient and accurate expressions for
the lattice sums constructed for the square array.

The effective properties of unidirectional fibrous composites can be ex-
pressed in terms of the series in concentration \( f \). The famous Clausius-
Mossotti approximation also known as the Maxwell formula \([6, \text{ Ch. 10}]\) is
valid in the first order approximation. The second order approximation in-
cludes the value of \( S_2 \). In order to shortly describe these approximations
following Rayleigh we consider a doubly periodic rectangular array of dis-
sks of conductivity \( \lambda_1 \) embedded in matrix of conductivity \( \lambda \). Let \( \lambda_{xx} \) and \( \lambda_{yy} \)
be the principal components of the effective conductivity tensor. Then \([2, 7, 8]\)

\[
\frac{\lambda_{xx}}{\lambda} = 1 + 2\rho f + 2\rho^2 f^2 \frac{S_2}{\pi} + O((|\rho f|^3), \\
\frac{\lambda_{yy}}{\lambda} = 1 + 2\rho f + 2\rho^2 f^2 \left( 2 - \frac{S_2}{\pi} \right) + O((|\rho f|^3), 
\tag{1.1}
\tag{1.2}
\]

where \( \rho = \frac{\lambda_1 - \lambda}{\lambda_1 + \lambda} \) denotes the contrast parameter. For the square array, the
medium becomes macroscopically isotropic, i.e., \( \lambda_e = \lambda_{xx} = \lambda_{yy} \) and the
above formulae becomes the Clausius-Mossotti approximation

\[
\frac{\lambda_e}{\lambda} = 1 + 2\rho f + 2\rho^2 f^2 + O((|\rho f|^3) = \frac{1 + \rho f}{1 - \rho f} + O((|\rho f|^3). \quad \tag{1.3}
\]

Actually, the approximation in the right part of (1.3) holds up to \( O((|\rho f|^5)) \)
(see formula (28) from [11] where the correction in the fifth order term should
be \( 6S_4^2\pi^{-2}f^5 \)).

The same rule holds for elastostatic problems [12]. An analytic formula for
the macroscopic elastic constants must include lattice sums (2.1) and (2.2) in
the second order term \( O(f^2) \). Such formulae for elastic problems are similar
to (1.1)-(1.3). They are described in Sec.4 Therefore, analytical formulae
for the lattice sums (2.1) and (2.2) have the fundamental applications in 2D
composites.

In the present paper, employing properties of the complete elliptic in-
tegrals of the first and second kind, we deduce closed-form formulae for the
lattice sums (2.1), (2.2) and other new formulae. Applications to the effective
properties of regular and random composites are discussed.
2 Eisenstein summation method and Rayleigh integral

Let $\mathbb{Z}$ and $\mathbb{C}$ denote the sets of integer and complex numbers, respectively, $i$ the imaginary unit. Consider a lattice $\{m\omega_1 + n\omega_2 \in \mathbb{C} : m, n \text{ run over } \mathbb{Z}\}$ determined by two fundamental translation vectors expressed by complex numbers $\omega_1, \omega_2$. Without loss of generality we assume that $\omega_1 > 0$ and $\text{Im} \tau > 0$ where $\tau = \frac{\omega_2}{\omega_1}$. Let the area of the fundamental parallelogram be normalized to unity, hence, $\omega_1 \omega_2 \text{Im} \tau = 1$. The main object of the present paper is the conditionally convergent lattice sums

$$S_2 = \sum_{(m,n) \in \mathbb{Z}^2 \setminus \{(0,0)\}} \frac{1}{(m\omega_1 + n\omega_2)^2} = \text{Im} \tau \sum_{(m,n) \in \mathbb{Z}^2 \setminus \{(0,0)\}} \frac{1}{(m + n\tau)^2} \quad (2.1)$$

and

$$T_2 = \sum_{(m,n) \in \mathbb{Z}^2 \setminus \{(0,0)\}} \frac{m\omega_1 + n\omega_2}{(m\omega_1 + n\omega_2)^2} = \text{Im} \tau \sum_{(m,n) \in \mathbb{Z}^2 \setminus \{(0,0)\}} \frac{m + n\tau}{(m + n\tau)^3}, \quad (2.2)$$

where the Eisenstein summation method [15] is used

$$\sum^e = \lim_{M_2 \to \infty} \sum_{n=-M_2}^{M_2} \left( \lim_{M_1 \to \infty} \sum_{m=-M_1}^{M_1} \right). \quad (2.3)$$

Having used the summation method (2.3) Rayleigh deduced the formula

$$S_2(\tau) = \pi^2 \text{Im} \tau \left[ \frac{1}{3} + 2 \sum_{m=1}^{\infty} \frac{1}{\sin^2(\pi m\tau)} \right] \quad (2.4)$$

for an rectangular array. It can be easily extended to other shapes of the fundamental cell. A physical justification of the Eisenstein summation method was presented in [9] and [10]. A rigorous mathematical proof can be found in [13]. Rayleigh (1892) did not cite Eisenstein’s result (1847) and addressed to Weierstrass investigations (1856). Perhaps, it is related to that Eisenstein treated formally his series without uniform convergence introduced by Weierstrass.

Moreover, Rayleigh [2] found the beautiful formula $S_2(i) = \pi$ where $\tau = i$ corresponds to the square array. The method of calculation was based on the reduction of the sum $S_2(i)$ to the integral

$$S_2(i) = 2 \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} \frac{dy}{(x + iy)^2} = \pi. \quad (2.5)$$
The integral over the central square \((-v, v) \times (-v, v)\) is eliminated in (2.5) since it vanishes. Further, the equality \(S_2 = \pi\) was proved in [10] for the hexagonal array (under the normalization of the area of the fundamental cell to unity).

The following formula was independently deduced in [14]

\[
S_2(\tau) = \frac{2}{\omega_1} \zeta\left(\frac{\omega_1}{2}\right) = \pi^2 \text{Im} \tau \left[\frac{1}{3} - 8 \sum_{m=1}^{\infty} \frac{m \exp(2i\pi m\tau)}{1 - \exp(2i\pi m\tau)}\right],
\]

where the \(\zeta\)-Weierstrass function is used. The equality \(S_2(i) = \pi\) was proved by Legendre’s identity. Though formulae (2.4) and (2.6) are similar a reduction of one to other can be justified only through the Eisenstein summation method applied to the \(\zeta\)-Weierstrass function.

We now proceed to discuss the lattice sum (2.2) beginning from the relation [15]

\[
\sum_{m \in \mathbb{Z} \setminus \{0\}} \frac{1}{(m + \tau)^{3}} = \pi^{3} \frac{\cos(\pi \tau)}{\sin^{3}(\pi \tau)},
\]

Consider the general term of the series (2.2)

\[
\frac{m + n\tau}{(m + n\tau)^{3}} = \frac{1}{(m + n\tau)^{2}} - 2i\text{Im}\tau \frac{n}{(m + n\tau)^{3}}.
\]

Substitution of (2.4) and (2.8) into (2.2) and use of (2.1) yields the computationally effective formula

\[
T_2(\tau) = S_2(\tau) - 4\pi^3 (\text{Im} \tau)^2 \sum_{n=1}^{\infty} n \frac{\cos(n\pi \tau)}{\sin^{3}(n\pi \tau)}.
\]

Surprisingly, that Rayleigh’s integral gives a wrong result

\[
T_2(i) = 2 \int_{-v}^{v} dx \int_{-v}^{v} \frac{x - iy}{(x + iy)^{3}} dy = 2
\]

though it formally corresponds to the Eisenstein summation method. This is because the Rayleigh reduction to the integral is formal. Moreover, by simple substitution we observe that iterated integrals (2.5), (2.10) do not depend on \(v\) and the corresponding double integrals diverge. Therefore, only formula (2.9) was in our disposal to get numerical values of \(T_2(\tau)\). However, in the sequel we will give a rigorous proof of the closed-form formulae for the lattice sums (2.1), (2.2) on the imaginary axis and on the vertical lines \(\text{Re} \tau = \pm 1/2\), basing on the theory of the complete elliptic integrals of the first and second kind.
3 Closed-form formulae

Let \( \mathbb{R}, \mathbb{R}_+ \) be the sets of real and real positive numbers, respectively. Let \( x \in \mathbb{R}_+ \) be given by the formula

\[
x \equiv x(k) = \frac{K(k')}{K(k)}, \quad k \in (0, 1), \quad k' = \sqrt{1 - k^2}, \quad (3.1)
\]

where \( K(k) \) is the complete elliptic integral of the first kind \([19], [22], \text{Vol. II, [26]}\)

\[
K(k) = \int_0^1 \frac{dt}{\sqrt{(1 - t^2)(1 - k^2t^2)}}. \quad (3.2)
\]

The parameter \( k \) is called the elliptic modulus and \( k' \) is the complimentary modulus. As we see the function \( x \) as a function of \( k \in (0, 1) \) is monotone decreasing and continuously differentiable bijective map \( x : (0, 1) \to \mathbb{R}_+ \). Therefore any \( x > 0 \) is uniquely defined by the corresponding modulus \( k \).

The complete elliptic integral \( K(k) \) satisfies the Legendre relation

\[
E(k)K(k') + E(k')K(k) - K(k')K(k) = \frac{\pi}{2}, \quad (3.3)
\]

where \( E(k) \) is the complete elliptic integral of the second kind

\[
E(k) = \int_0^1 \sqrt{\frac{1 - k^2t^2}{1 - t^2}} dt. \quad (3.4)
\]

Its derivative can be calculated by formula

\[
\frac{dE}{dk} = \frac{E(k) - K(k)}{k}. \quad (3.5)
\]

It is known \([26]\), that \( K(k), K(k') \) satisfy the differential equation

\[
\frac{d}{dk} \left( k(k')^2 \frac{du}{dk} \right) = ku \quad (3.6)
\]

and \( E(k), E(k') - K(k') \) are, in turn, solutions of the differential equation

\[
(k')^2 \frac{d}{dk} \left( k \frac{du}{dk} \right) + ku = 0. \quad (3.7)
\]

The derivative of \( K(k) \) can be calculated by the formula

\[
\frac{dK}{dk} = \frac{E(k) - (k')^2 K(k)}{k(k')^2}. \quad (3.8)
\]
Let \( k_r \) be an elliptic modulus such that \( x(k_r) = \sqrt{r} \) (see (3.1)). In the sequel we will use such values for small \( r \) and the corresponding elliptic integral singular values \( K(k_r) \) (see [20], [21]), namely

\[
k_1 = \frac{1}{\sqrt{2}}, \quad k_2 = \sqrt{2} - 1, \quad k_3 = \frac{1}{4}\sqrt{2}(\sqrt{3} - 1), \quad k_4 = 3 - 2\sqrt{2},
\]

(3.9)

\[
K(k_1) = \frac{\Gamma^2(1/4)}{4\sqrt{\pi}}, \quad K(k_2) = \frac{(\sqrt{2} + 1)^{1/2}\Gamma(1/8)\Gamma(3/8)}{2^{13/4}\sqrt{\pi}},
\]

(3.10)

\[
K(k_3) = \frac{3^{1/4}\Gamma^3(1/3)}{2^{7/3}\pi}, \quad K(k_4) = \frac{(\sqrt{2} + 1)\Gamma^2(1/4)}{2^{7/2}\sqrt{\pi}},
\]

(3.11)

where \( \Gamma(z) \) is Euler’s gamma-function [22], Vol. I. According to [21] the so-called elliptic alpha function for the integral singular values

\[
\alpha(r) = \frac{E(k'_r)}{K(k_r)} - \frac{\pi}{4[K(k_r)]^2} = \frac{\pi}{4[K(k_r)]^2} + \sqrt{r} \left[ 1 - \frac{E(k_r)}{K(k_r)} \right]
\]

is calculated, in particular, for small values and we have

\[
\alpha(1) = \frac{1}{2}, \quad \alpha(2) = \sqrt{2} - 1, \quad \alpha(3) = \frac{1}{2}(\sqrt{3} - 1), \quad \alpha(4) = 2(\sqrt{2} - 1)^2.
\]

(3.13)

Meanwhile, appealing to relations (2.4.3.1), (2.4.3.3) in [24] and the inverse Mellin transform [25], we derive the following integral representations, related to the hyperbolic functions which will be useful in the sequel

\[
\frac{1}{\sinh^2(cx)} = \frac{2}{\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \Gamma(s)\zeta(s - 1)(2cx)^{-s} ds, \quad c > 0, \gamma > 2,
\]

(3.14)

\[
\frac{1}{\cosh^2(cx)} = \frac{2}{\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} (1 - 2^{-s})\Gamma(s)\zeta(s - 1)(2cx)^{-s} ds, \quad c > 0, \gamma > 0,
\]

(3.15)

where \( \zeta(s) \) is the Riemann zeta-function [22], Vol. I, which satisfies the familiar functional equation

\[
\zeta(s) = 2^s\pi^{s-1}\sin\left(\frac{\pi s}{2}\right)\Gamma(1 - s)\zeta(1 - s).
\]

(3.16)
We begin, recalling the Rayleigh formula (2.4) and recently obtained formula (2.9) in order to give a rigorous proof of the following functional equations for \( S_2(\tau), T_2(\tau) \) on the imaginary positive half-axis and positive half-lines \( \text{Re} \tau = \pm 1/2. \)

**Theorem 1.** Let \( x \in \mathbb{R}_+ \). Then

\[
S_2(ix) + S_2(ix^{-1}) = 2\pi, \quad (3.17)
\]

\[
S_2\left(\frac{\pm 1 + ix}{2}\right) + S_2\left(\frac{\pm 1 + ix^{-1}}{2}\right)
= S_2\left(\frac{\pm 1 + ix}{2}\right) + S_2\left(\frac{\mp 1 + ix^{-1}}{2}\right) = 2\pi, \quad (3.18)
\]

\[
T_2(ix) = T_2(ix^{-1}), \quad (3.19)
\]

\[
T_2\left(\frac{\pm 1 + ix}{2}\right) - T_2\left(\frac{\pm 1 + ix^{-1}}{2}\right)
= 4 \left( S_2\left(\frac{\pm 1 + ix}{2}\right) - S_2(ix) \right) + \frac{2\pi^2}{3} \left( x - \frac{1}{x} \right). \quad (3.20)
\]

**Proof.** Indeed, employing integral representation (3.14), we substitute it into (2.4) to write for \( \tau = ix \)

\[
S_2(ix) = \pi^2 x \left[ \frac{1}{3} - 4 \sum_{m=1}^{\infty} \int_{\gamma - i\infty}^{\gamma + i\infty} \Gamma(s) \zeta(s-1)(2\pi mx)^{-s} ds \right], \quad (3.21)
\]

Since \( \gamma > 2 \) and the zeta-function is bounded on the vertical line \((\gamma - i\infty, \gamma + i\infty)\), i.e. \( |\zeta(s - 1)| \leq \zeta(\gamma - 1) \), the interchange of the order of summation and integration is allowed for each \( x > 0 \) via the absolute and uniform convergence by virtue of the estimate

\[
\sum_{m=1}^{\infty} \int_{\gamma - i\infty}^{\gamma + i\infty} |\Gamma(s)\zeta(s-1)(2\pi mx)^{-s} ds| \\
\leq (2\pi x)^{-\gamma} \zeta(\gamma - 1) \sum_{m=1}^{\infty} \frac{1}{m^\gamma} \int_{\gamma - i\infty}^{\gamma + i\infty} |\Gamma(s)| ds \\
= (2\pi x)^{-\gamma} \zeta(\gamma - 1) \zeta(\gamma) \int_{\gamma - i\infty}^{\gamma + i\infty} |\Gamma(s)| ds < \infty,
\]
where the convergence of the latter integral can be easily verified, appealing to the Stirling asymptotic formula for gamma-function when $|\text{Im} s| \to \infty$ (see \cite{22}, Vol. I). Hence with the definition of the Riemann zeta-function in terms of the series, equality (3.21) becomes

$$S_2(ix) = \pi^2 x \left[ \frac{1}{3} - \frac{4}{\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \Gamma(s)\zeta(s)\zeta(s-1)(2\pi x)^{-s} ds \right]. \quad (3.22)$$

On the other hand, the product of zeta-functions $\zeta(s)\zeta(s-1)$ can be represented by the Ramanujan identity \cite{27}

$$\zeta(s)\zeta(s-1) = \sum_{m=1}^{\infty} \frac{\sigma(m)}{m^s}, \quad \gamma > 2, \quad (3.23)$$

where $\sigma(m)$ is the arithmetic function \cite{19}, denoting the sum of divisors of $m$. Hence, substituting in (3.22) and inverting the order of integration and summation owing to the same motivation, we obtain

$$S_2(ix) = \pi^2 x \left[ \frac{1}{3} - 8 \sum_{m=1}^{\infty} \sigma(m)e^{-2\pi mx} \right], \quad (3.24)$$

where the inverse Mellin transform of the gamma-function \cite{25} is used

$$e^{-x} = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \Gamma(s)x^{-s} ds, \quad x > 0.$$

In the meantime, the Nasim identity \cite{23} says that

$$\sum_{m=1}^{\infty} \sigma(m)e^{-2\pi mx} + x^{-2} \sum_{m=1}^{\infty} \sigma(m)e^{-2\pi mx} = \frac{1}{24} \left( 1 + \frac{1}{x^2} \right) - \frac{1}{4\pi x}, \quad x > 0. \quad (3.25)$$

Consequently, from (3.22) we find

$$S_2(ix) + S_2(ix^{-1}) = \frac{\pi^2}{3} \left( x + \frac{1}{x} \right) - 8\pi^2 \left[ x \sum_{m=1}^{\infty} \sigma(m)e^{-2\pi mx} + \frac{1}{x} \sum_{m=1}^{\infty} \sigma(m)e^{-2\pi mx} \right] = 2\pi,$$
proving equation (3.17). In order to prove equations (3.18), we invoke representation (3.15), motivating all passages analogously to the previous case. Moreover, as we will see it is sufficient to prove (3.18) for positive real parts. So, we have (see (2.4))

\[
S_2 \left( \frac{1+ix}{2} \right) = \frac{\pi^2 x}{2} \left[ \frac{1}{3} - 2 \sum_{m=1}^{\infty} \frac{1}{\sinh^2(\pi mx)} + 2 \sum_{m=1}^{\infty} \frac{1}{\cosh^2(\pi (m-1/2)x)} \right]
\]

\[
= \frac{1}{2} S_2(ix) - 2\pi ix \int_{\gamma - \infty}^{\gamma + \infty} (1 - 2^{2-s}) \Gamma(s) \zeta(s-1)(\pi x)^{-s} \sum_{m=1}^{\infty} \frac{1}{(2m-1)^s} ds.
\]

But the latter series is easily calculated for $\gamma > 1$ via the definition of the Riemann zeta-function and we obtain

\[
\sum_{m=1}^{\infty} \frac{1}{(2m-1)^s} = (1 - 2^{-s}) \zeta(s).
\]

Hence, recalling the Ramanujan identity (3.23), changing the order of integration and summation and calculating the inverse Mellin transform of the gamma-function of different arguments, we deduce

\[
S_2 \left( \frac{1+ix}{2} \right) = \frac{1}{2} S_2(ix) - 2\pi ix \int_{\gamma - \infty}^{\gamma + \infty} (1 - 2^{2-s}) (1 - 2^{-s})
\]

\[
\times \Gamma(s) \zeta(s) \zeta(s-1)(\pi x)^{-s} ds = \frac{1}{2} S_2(ix)
\]

\[
+ 4\pi^2 x \sum_{m=1}^{\infty} \sigma(m) \left[ e^{-\pi mx} - 5e^{-2\pi mx} + 4e^{-4\pi mx} \right].
\]

Meanwhile, from (3.24) we find

\[
x \sum_{m=1}^{\infty} \sigma(m) e^{-2\pi mx} = \frac{x}{24} - \frac{1}{8\pi^2} S_2(ix).
\]

Therefore, it yields

\[
S_2 \left( \frac{1+ix}{2} \right) = S_2(ix) - \frac{\pi^2 x}{6} + 4\pi^2 x \sum_{m=1}^{\infty} \sigma(m) \left[ e^{-\pi mx} - 2e^{-2\pi mx} \right]
\]

\[9\]
In the meantime, appealing to another Nasim’s formula (see [23], formula (5.1) with $a = x/2, b = x$)

$$\sum_{m=1}^{\infty} \frac{\sigma(m)}{m} \left[ e^{-\pi mx} - e^{-2\pi mx} \right] = \sum_{m=1}^{\infty} \frac{\sigma(m)}{m} \left[ e^{-4\pi m/x} - e^{-2\pi m/x} \right]$$

$$+ \frac{\pi}{12} \left( \frac{1}{x} + \frac{x}{2} \right) - \frac{1}{2} \log 2,$$

we differentiate it with respect to $x$, which is permitted via the absolute and uniform convergence and multiply by $x$ the obtained equality. Thus we get

$$x \sum_{m=1}^{\infty} \sigma(m) \left[ 2e^{-2\pi mx} - e^{-\pi mx} \right] = 2x \sum_{m=1}^{\infty} \sigma(m) \left[ 2e^{-4\pi m/x} - e^{-2\pi m/x} \right]$$

$$+ \frac{x}{24} - \frac{1}{12x} \quad (3.28)$$

Substituting in (3.27), we derive

$$S_2 \left( \frac{1 + ix}{2} \right) = S_2(ix) + \frac{8\pi^2}{x} \sum_{m=1}^{\infty} \sigma(m) \left[ e^{-2\pi m/x} - 2e^{-4\pi m/x} \right]$$

$$- 8\pi^2 x \sum_{m=1}^{\infty} \sigma(m) \left[ e^{-2\pi mx} - 2e^{-4\pi mx} \right] . \quad (3.29)$$

Now, changing in (3.29) $x$ by $1/x$ and adding these two equalities with the use of (3.17), we obtain (3.18).

Let us prove (3.19). To do this, we let $\tau = ix, x > 0$ in (2.9) and write it in the form

$$T_2(ix) = S_2(ix) + 4\pi^3 x^2 \sum_{n=1}^{\infty} \frac{\cosh(n\pi x)}{\sinh^3(n\pi x)} . \quad (3.30)$$

However, the series in (3.30) can be obtained by termwise differentiation with respect to $x$ of the series $\sum \cosech^2(\pi nx)$ for $x \geq x_0 > 0$ due to the absolute and uniform convergence. Hence
\( T_2(ix) = S_2(ix) - 2\pi^2 x^2 \frac{d}{dx} \sum_{n=1}^{\infty} \frac{1}{\sinh^2(n\pi x)}. \quad (3.31) \)

But from (2.4), (3.23), (3.24) and termwise differentiation of the series with arithmetic function \( \sigma(m) \) in (3.24) by the same reasons, we obtain

\[
T_2(ix) = S_2(ix) + 16\pi^3 x^2 \sum_{n=1}^{\infty} m \, \sigma(m) e^{-2\pi mx}. 
\quad (3.32)
\]

Meanwhile, differentiating the Nasim identity (3.25) with respect to \( x \) and then multiplying both sides of the obtained equality by \( -x^2/(2\pi) \), we find

\[
x^2 \sum_{m=1}^{\infty} m \, \sigma(m) e^{-2\pi mx} = x^{-2} \sum_{m=1}^{\infty} m \, \sigma(m) e^{-2\pi mx} + \frac{1}{24\pi x} \\
- \frac{1}{8\pi^2} - \frac{1}{\pi x} \sum_{m=1}^{\infty} \sigma(m) e^{-2\pi mx}. 
\quad (3.33)
\]

Substituting the left-hand side of the latter equality in (3.32) and appealing to (3.26), we write

\[
T_2(ix) - S_2(ix) = T_2(ix^{-1}) + S_2(ix^{-1}) - 2\pi. 
\]

Therefore, equality (3.17) leads us to (3.19). In order to establish (3.20), we employ again (2.9) to get as in (3.31), (3.32)

\[
T_2 \left( \frac{\pm 1 + ix}{2} \right) = S_2 \left( \frac{\pm 1 + ix}{2} \right) + (\pi x)^2 \frac{d}{dx} \left[ \sum_{m=1}^{\infty} \frac{1}{\cosh^2(\pi(m-1/2)x)} \right] \\
- \sum_{m=1}^{\infty} \frac{1}{\sinh^2(\pi mx)} \right] = S_2 \left( \frac{\pm 1 + ix}{2} \right) + \pi^3 x^2 \left[ \frac{1}{\pi x} \sum_{m=1}^{\infty} \frac{1}{\cosh^2(\pi(m-1/2)x)} \right] \\
+ 8 \sum_{n=1}^{\infty} m \, \sigma(m) e^{-2\pi nx} 
\]

In the meantime, recalling (3.15), (3.23) and termwise differentiation, we deduce

\[
\frac{d}{dx} \sum_{m=1}^{\infty} \frac{1}{\cosh^2(\pi(m-1/2)x)} = 4\pi \sum_{n=1}^{\infty} m \, \sigma(m) [10e^{-2\pi nx} 
\]
Thus

\[ T_2 \left( \frac{\pm 1 + ix}{2} \right) = S_2 \left( \frac{\pm 1 + ix}{2} \right) + 4\pi^3 x^2 \sum_{n=1}^{\infty} m \sigma(m) \left[ 12e^{-2\pi mx} - 16e^{-4\pi mx} - e^{-\pi mx} \right]. \]

Hence,

\[
\frac{1}{4\pi^3} \left( T_2 \left( \frac{\pm 1 + ix}{2} \right) - S_2 \left( \frac{\pm 1 + ix}{2} \right) \right) = 8x^2 \sum_{n=1}^{\infty} m \sigma(m)e^{-2\pi mx} - 16x^2 \sum_{n=1}^{\infty} m \sigma(m)e^{-4\pi mx} - x^2 \sum_{n=1}^{\infty} m \sigma(m) \left[ e^{-\pi mx} - 4e^{-2\pi mx} \right].
\]

(3.34)

Returning to (3.28) and making the termwise differentiation and simple manipulations, we derive

\[ x^2 \sum_{m=1}^{\infty} m \sigma(m) \left[ e^{-\pi mx} - 4e^{-2\pi mx} \right] = -4 \frac{x}{\pi} \sum_{m=1}^{\infty} \sigma(m) \left[ 2e^{-4\pi m/x} - e^{-2\pi m/x} \right] \]

\[ + \frac{1}{6\pi x} + \frac{4}{x^2} \sum_{m=1}^{\infty} m \sigma(m) \left[ 4e^{-4\pi m/x} - e^{-2\pi m/x} \right]. \]

Therefore with the use of (3.32), equality (3.34) becomes

\[
\frac{1}{4\pi^3} \left( T_2 \left( \frac{\pm 1 + ix}{2} \right) - S_2 \left( \frac{\pm 1 + ix}{2} \right) \right) = \frac{1}{2\pi^3} \left( T_2 (ix) - S_2 (ix) \right)
\]

\[ -16x^2 \sum_{n=1}^{\infty} m \sigma(m)e^{-4\pi mx} + \frac{4}{\pi x} \sum_{m=1}^{\infty} \sigma(m) \left[ e^{-2\pi m/x} - 2e^{-4\pi m/x} \right] \]

\[ - \frac{1}{6\pi x} - \frac{16}{x^2} \sum_{n=1}^{\infty} m \sigma(m)e^{-4\pi m/x} + \frac{1}{4\pi^3} \left( T_2 (ix^{-1}) - S_2 (ix^{-1}) \right). \]
Hence,

\[
\frac{1}{4\pi^3} \left(T_2 \left(\frac{\pm 1 + ix}{2}\right) - S_2 \left(\frac{\pm 1 + ix}{2}\right)\right) - \frac{1}{2\pi^3} (T_2 (ix) - S_2 (ix))
\]

\[-\frac{1}{4\pi^3} (T_2 (ix^{-1}) - S_2 (ix^{-1})) + \frac{1}{6\pi x} - \frac{4}{\pi x} \sum_{m=1}^{\infty} \sigma(m) \left[ e^{-2\pi m/x} - 2e^{-4\pi m/x} \right] \]

\[= \frac{1}{4\pi^3} \left(T_2 \left(\frac{\pm 1 + ix^{-1}}{2}\right) - S_2 \left(\frac{\pm 1 + ix^{-1}}{2}\right)\right) - \frac{1}{2\pi^3} (T_2 (ix^{-1}) - S_2 (ix^{-1})) \]

\[-\frac{1}{4\pi^3} (T_2 (ix) - S_2 (ix)) + \frac{x}{6\pi} - \frac{4x}{\pi} \sum_{m=1}^{\infty} \sigma(m) \left[ e^{-2\pi mx} - 2e^{-4\pi mx} \right].\]

Meanwhile, appealing to (3.29), we find

\[
\frac{4}{\pi x} \sum_{m=1}^{\infty} \sigma(m) \left[ e^{-2\pi m/x} - 2e^{-4\pi m/x} \right] - \frac{4x}{\pi} \sum_{m=1}^{\infty} \sigma(m) \left[ e^{-2\pi mx} - 2e^{-4\pi mx} \right]
\]

\[= \frac{1}{2\pi^3} \left(S_2 \left(\frac{1 + ix}{2}\right) - S_2 (ix)\right).\]

Thus, accounting (3.19),

\[T_2 \left(\frac{\pm 1 + ix}{2}\right) - T_2 \left(\frac{\pm 1 + ix^{-1}}{2}\right) + 3S_2 (ix) - S_2 (ix^{-1})\]

\[= 3 S_2 \left(\frac{\pm 1 + ix}{2}\right) - 3 S_2 \left(\frac{\pm 1 + ix^{-1}}{2}\right) + \frac{2\pi^2}{3} \left(x - \frac{1}{x}\right).\]

Finally, equalities (3.17), (3.18) drive us at (3.20), completing the proof of Theorem 1.

The explicit expressions of \(S_2(\tau)\) on the imaginary axis and the lines \(\text{Re} \tau = \pm 1/2\) are given by

**Theorem 2.** Let \(x \in \mathbb{R}\setminus\{0\}\). Then the following formulae hold

\[S_2(ix) = \frac{4}{3} \text{sign}(x) K(k') \left[3E(k) + (k^2 - 2)K(k)\right], \quad (3.35)\]

13
\[
S_2 \left( \frac{\pm 1 + ix}{2} \right) = 2 \, \text{sign}(x) \, K(k') \left[ 2 \, E(k) + \frac{4k^2 - 5}{3} \, K(k) \right], \quad (3.36)
\]

where

\[
|x| = \frac{K(k')}{K(k)}, \quad k \in (0, 1)
\]

and \( k' \) is defined by (3.1).

Proof. Let us first consider a positive \( x \) being defined by (3.1). Fortunately, the series in (2.4) for \( \tau = ix \) is calculated in [24], relation (5.3.4.5), and we have

\[
\sum_{m=1}^{\infty} \frac{1}{\sinh^2(\pi mx)} = \frac{1}{6} + \frac{2(2 - k^2)}{3\pi^2} \, K^2(k) - \frac{2}{\pi^2} K(k) E(k). \quad (3.37)
\]

Therefore,

\[
S_2(ix) = \frac{4}{3} \left( k^2 - 2 \right) xK^2(k) + 4xK(k)E(k)
\]

\[
= \frac{4}{3} \, K(k') \left[ 3E(k) + (k^2 - 2)K(k) \right].
\]

Hence it proves (3.35) for positive \( x \), and for negative \( x \) it can be easily extended via (2.4). In order to prove (3.36), we employ relation (5.3.6.6) in [24]

\[
\sum_{m=1}^{\infty} \frac{1}{\cosh^2(\pi x(m - 1/2))} = \frac{2}{\pi^2} K(k) E(k) - \frac{2(1 - k^2)}{\pi^2} K^2(k). \quad (3.38)
\]

Then for positive \( x \) we find from [24], (3.37), (3.38)

\[
S_2 \left( \frac{\pm 1 + ix}{2} \right) = \pi^2 x \left[ \frac{1}{6} + \sum_{m=1}^{\infty} \frac{1}{\sin^2(\pi m(\pm 1 + ix))} \right]
\]

\[
+ \sum_{m=1}^{\infty} \frac{1}{\sin^2(\pi (2m - 1)(\pm 1 + ix)/2)} \right] = \pi^2 x \left[ \frac{1}{6} - \sum_{m=1}^{\infty} \frac{1}{\sinh^2(\pi mx)} \right]
\]

14
\[ + \sum_{m=1}^{\infty} \frac{1}{\cosh^2(\pi x(m - 1/2))} \] = 4K(k')E(k) - \frac{2(2 - k^2)}{3} K(k')K(k)

\[ -2(1 - k^2) K(k')K(k) = 2K(k') \left[ 2E(k) + \frac{4k^2 - 5}{3} K(k) \right]. \]

Hence spreading the latter equalities for negative \( x \), we get (3.36).

**Corollary 1.** Formula (3.20) can be written in the form

\[ T_2 \left( \frac{\pm 1 + ix}{2} \right) - T_2 \left( \frac{\pm 1 + ix^{-1}}{2} \right) = \frac{4}{3} (4k^2 - 2) K(k) + \frac{2\pi^2}{3} \left( x - \frac{1}{x} \right). \]

As we could see above, the only value \( S_2(i) = \pi \) was known explicitly. Now we are able to calculate more interesting particular values of (3.35), (3.36). Indeed, we have

**Corollary 2.** The following values take place

\[ S_2(\pm i) = S_2 \left( \frac{1 \pm i}{2} \right) = \pm \pi, \quad (3.39) \]

\[ S_2(\pm i\sqrt{2}) \pm \left[ \pi + \frac{\Gamma^2(1/8)\Gamma^2(3/8)}{48\pi \sqrt{2}} \right], \quad (3.40) \]

\[ S_2 \left( \frac{1 \pm i\sqrt{2}}{2} \right) = \pm \left[ \pi + \frac{(2\sqrt{2} - 3) \Gamma^2(1/8)\Gamma^2(3/8)}{96 \pi} \right], \quad (3.41) \]

\[ S_2(\pm i\sqrt{3}) \pm \left[ \pi + \frac{\sqrt{3} \Gamma^6(1/3)}{16\pi^2 2^{2/3}} \right], \quad (3.42) \]

\[ S_2 \left( \frac{1 \pm i\sqrt{3}}{2} \right) = \pm \pi, \quad (3.43) \]

\[ S_2 (\pm 2i) = \pm \left[ \pi + \frac{\Gamma^4(1/4)}{16 \pi} \right], \quad (3.44) \]

\[ S_2 \left( \frac{1}{2} \pm i \right) = \pm \left[ \pi + \frac{(3 - 2\sqrt{2}) \Gamma^4(1/4)}{32 \pi} \right]. \quad (3.45) \]
Proof. As we observe from (3.35), (3.36), it is sufficient to establish the above constants for a positive imaginary part of the corresponding \( \tau \). To do this we employ particular cases (3.9) of the modulus \( k_r \) and the corresponding singular values (3.10), (3.11) \( K(k_r), \ r = 1, 2, 3, 4 \). In fact, letting \( x = 1, \ \sqrt{2}, \ \sqrt{3}, \ 2 \) and taking in mind (3.12), (3.13), we derive, respectively,

\[
S_2(i) = \frac{4}{3}K(k_1) \left[ \frac{3}{2}K(k_1) + \frac{3\pi}{4K(k_1)} - \frac{3}{2}K(k_1) \right] = \pi;
\]

\[
S_2 \left( \frac{1+i}{2} \right) = 2K(k_1) \left[ K(k_1) + \frac{\pi}{2K(k_1)} - K(k_1) \right] = \pi;
\]

\[
S_2(i\sqrt{2}) = \frac{4\sqrt{2}}{3} \left[ \frac{3}{\sqrt{2}}K^2(k_2) + \frac{3\pi}{4\sqrt{2}} + (1-2\sqrt{2})K^2(k_2) \right]
= \pi + \frac{4}{3} \left( \sqrt{2} - 1 \right) K^2(k_2) = \pi + \frac{\Gamma^2(1/8)\Gamma^2(3/8)}{48\pi\sqrt{2}};
\]

\[
S_2 \left( \frac{1+i\sqrt{2}}{2} \right) = \pi + \frac{2(7\sqrt{2} - 10)}{3} K^2(k_2) = \pi + \frac{(2\sqrt{2} - 3) \Gamma^2(1/8)\Gamma^2(3/8)}{96\pi};
\]

\[
S_2(i\sqrt{3}) = \pi + K^2(k_3) = \pi + \frac{\sqrt{3} \Gamma^6(1/3)}{16\pi^2 \ 2^{2/3}};
\]

\[
S_2 \left( \frac{1+i\sqrt{3}}{2} \right) = \pi + 2K^2(k_3)(\sqrt{3} + 1) - \frac{2}{\sqrt{3}}(3 + \sqrt{3})K^2(k_3) = \pi;
\]

\[
S_2(2i) = \pi + 8K^2(k_4)(3 - 2\sqrt{2}) = \pi + \frac{\Gamma^4(1/4)}{16\pi};
\]

\[
S_2 \left( \frac{1}{2} + i \right) = \pi + 4(17 - 12\sqrt{2})K^2(k_4) = \pi + \frac{(3 - 2\sqrt{2}) \Gamma^4(1/4)}{32\pi}.
\]

A more technically difficult task is to find explicit expressions for \( T_2(\tau) \) on the same lines in the complex plane. To achieve our goal we will involve the method of termwise differentiation of the series in (2.4) with respect to the elliptic modulus (for \( \tau = ix(k) \) or \( \tau = (\pm 1 + ix(k))/2 \)). Indeed, as we mentioned above, \( x(k) \) by formula (3.1) is continuously differentiable.
and when \( k \in [a_0, b_0], \ 0 < a_0 < b_0 < 1 \), the corresponding series (2.4) is absolutely and uniformly convergent. Moreover, it is not difficult to show that the series of the derivatives with respect to \( k \) converges absolutely and uniformly on the segment \([a_0, b_0]\). Thus the known theorem from calculus says that the termwise differentiation of the series is allowed. This leads us to

**Theorem 3.** Under conditions of Theorem 2 the following formulae hold valid

\[
T_2(ix) = \frac{4}{3} \text{sign}(x) K(k') \left[ 1 - \frac{2}{\pi} K(k') E(k) \right] \left[ 3E(k) + (k^2 - 2)K(k) \right] \\
- \frac{2}{\pi} K(k') K(k) \left[ (1 - k^2) [K(k) - E(k)] - E(k) \right]
\]

\[
T_2 \left( \frac{\pm 1 + ix}{2} \right) = \frac{2}{3} \text{sign}(x) K(k') \left[ (6E(k) + K(k)(4k^2 - 5)) \right] \\
\times \left[ 1 - \frac{2}{\pi} K(k') \left( E(k) + K(k)(k^2 - 1) \right) \right]
\]

\[
- \frac{2}{\pi} K(k') K(k) \left[ (1 - 2k^2)E(k) + (4k^2 - 1)(1 - k^2)K(k) \right].
\]

**Proof.** Indeed, concerning the proof of formula (3.37), we let \( \tau = ix, \ x > 0 \) in (2.9) and write it in the form

\[
T_2(ix) = S_2(ix) + 4\pi^3 x^2 \sum_{n=1}^{\infty} n \frac{\cosh(n \pi x)}{\sinh^4(n \pi x)},
\]

where \( x \) is a function of \( k \) by (3.1) and since the termwise differentiation is permitted, we obtain

\[
\sum_{n=1}^{\infty} n \frac{\cosh(n \pi x)}{\sinh^4(n \pi x)} = -\frac{1}{2\pi x'(k)} \frac{d}{dk} \sum_{n=1}^{\infty} \frac{1}{\sinh^2(n \pi x)}. \tag{3.49}
\]

Meanwhile, the derivative \( x'(k) \) can be calculated explicitly, employing twice (3.8). Hence we find

\[
x'(k) = -\frac{K(k')}{K^2(k)} \frac{dK(k)}{dk} - \frac{k}{(1 - k^2)K(k)} \left[ \frac{E(k')}{k^2} - K(k') \right]
\]
\[
\frac{1}{k(1-k^2)K(k)} \left[ K(k') \left[ 1 - \frac{E(k)}{K(k)} \right] - E(k') \right]
\]

and the Legendre identity (3.3) leads us to the final result

\[
x'(k) = -\frac{\pi}{2k(1-k^2)K^2(k)}.
\] (3.50)

Therefore, recalling (3.5), (3.8), (3.37), we deduce from (3.49)

\[
4\pi^3 x^2 \sum_{n=1}^{\infty} \frac{n \cosh(n \pi x)}{\sinh^3(n \pi x)} = \frac{8}{\pi} k(1-k^2) K^2(k') \frac{d}{dk} \left[ K(k) \left( \frac{2-k^2}{3} K(k) - E(k) \right) \right]
\]

\[
= \frac{8}{3\pi} K^2(k') \left( E(k) - (1-k^2) K(k) \right) \left( (2-k^2) K(k) - 3E(k) \right)
\]

\[
+ \frac{8}{3\pi} K^2(k') K(k) \left[ E(k)(2k^2-1) + K(k)(k^2-1)^2 \right]
\]

\[
= \frac{8}{3\pi} K^2(k') \left[ 2E(k)K(k)(2-k^2) + K^2(k)(k^2-1) - 3E^2(k) \right].
\]

Hence, appealing to (3.35) and combining with (3.48), we arrive at (3.46) being valued for positive \(x\). Then we extend it on negative numbers as in Theorem 2.

In order to establish identity (3.47), we write (2.9) for \(\tau = (\pm 1 + ix)/2, \; x > 0\) in the same manner as in the proof of identity (3.20). Nevertheless, we will employ explicit expressions (3.37) and (3.38) and make the termwise differentiation with respect to the elliptic modulus. Hence, taking in mind (3.36), (3.50), we obtain

\[
T_2 \left( \frac{\pm 1 + ix}{2} \right) = S_2 \left( \frac{\pm 1 + ix}{2} \right) + \frac{(\pi x)^2}{x'(k)} \frac{d}{dk} \left[ \sum_{m=1}^{\infty} \frac{1}{\cosh^2(\pi m - 1/2) x} \right]
\]

\[
- \sum_{m=1}^{\infty} \frac{1}{\sinh^2(\pi mx)} = \frac{2}{3} K(k') \left( 6E(k) + K(k)(4k^2 - 5) \right)
\]

\[
- \frac{4}{3\pi} k(k')^2 K^2(k') \frac{d}{dk} \left[ K(k) \left( 6E(k) + K(k)(4k^2 - 5) \right) \right].
\]

18
Fulfilling the differentiation with the aid of (3.5), (3.6), (3.8), we find

\[
T_2 \left( \frac{\pm 1 + ix}{2} \right) = \frac{2}{3} K(k') \left[ (6E(k) + K(k)(4k^2 - 5)) \right]
\times \left[ 1 - \frac{2}{\pi} K(k') (E(k) + K(k)(k^2 - 1)) \right]
\]

\[-\frac{2}{\pi} K(k') K(k) \left[ (1 - 2k^2)E(k) + (4k^2 - 1)(1 - k^2)K(k) \right], \]

arriving at (3.47) after the same extension on negative numbers \( x \) as in Theorem 3.

As a corollary we calculate particular values of \( T_2 \) on the mentioned vertical lines, recalling \( k_r \) in (3.9) and \( K(k_r) \) in (3.10), (3.11), letting \( r = 1, 2, 3, 4 \). In particular, the value \( x = 1 \), corresponding \( k_1 = k_1' = \frac{1}{\sqrt{2}} \), gives the interesting and important constant numerical value of which coincides with the numerical value of \( T_2(i) = 4.078451 \) computed with (2.9)

\[
T_2(i) = \frac{\pi}{2} + \frac{\Gamma^8(1/4)}{384 \pi^3}. \quad (3.51)
\]

We note that this numerical result \( T_2(i) = 4.078451 \) coincides with the numerical value obtained by other approaches \cite{3}, \cite{4}.

**Corollary 3.** Certain explicit constants related to \( T_2(\tau) \) are the following values

\[
T_2(\pm i) = \pm \left[ \frac{\pi}{2} + \frac{\Gamma^8(1/4)}{384 \pi^3} \right];
\]

\[
T_2 \left( \frac{1 \pm i}{2} \right) = \pm \left[ \frac{\pi}{2} - \frac{\Gamma^8(1/4)}{384 \pi^3} \right];
\]

\[
T_2(\pm i\sqrt{2}) = \pm \left[ \frac{\pi}{2} + \frac{\Gamma^4(1/8)\Gamma^4(3/8)}{1024 \pi^3} \right];
\]

\[
T_2 \left( \frac{1 \pm i\sqrt{2}}{2} \right) = \pm \left[ \frac{\pi}{2} - \frac{\Gamma^4(1/8)\Gamma^4(3/8)(\sqrt{2} - 1)}{1024 \pi^3} \right];
\]

\[
T_2(\pm i\sqrt{3}) = \pm \left[ \frac{\pi}{2} - \frac{2^{2/3}\Gamma^{12}(1/3)(9 + 4\sqrt{3})}{512 \pi^5} \right];
\]
\[ T_2 \left( \frac{1 \pm i \sqrt{3}}{2} \right) = \pm \frac{\pi}{2}; \]
\[ T_2 (\pm 2i) = \pm \left[ \frac{\pi}{2} + \frac{\Gamma^8(1/4)}{192 \pi^3} \right]; \]
\[ T_2 \left( \frac{1}{2} \pm i \right) = \pm \left[ \frac{\pi}{2} + \frac{\Gamma^8(1/4)(5 - 3\sqrt{2})}{768 \pi^3} \right]; \]

4 Random lattice sums

Consider a lattice with the fixed periods \( \omega_1, \omega_2 \) and the corresponding fundamental parallelogram

\[ G_{(0,0)} := \{ t_1 \omega_1 + t_2 \omega_2 \in \mathbb{C} : 0 < t_1, t_2 < 1 \} \]

The Eisenstein function of second order \cite{15} is defined by the series

\[ E_2(z) = \sum_{(m,n) \in \mathbb{Z}^2} \frac{1}{(z - m\omega_1 - n\omega_2)^2}. \quad (4.1) \]

It is related to the \( \wp \)-Weierstrass function by formula \cite{15}

\[ E_2(z) = \wp(z) + S_2. \quad (4.2) \]

Following (4.1) we introduce the function

\[ G_2(z) = \sum_{(m,n) \in \mathbb{Z}^2} \frac{z - m\omega_1 - n\omega_2}{(z - m\omega_1 - n\omega_2)^3}. \quad (4.3) \]

This function is related to the Natanzon function \cite{16}

\[ \wp'_1(z) = -2 \sum_{(m,n) \in \mathbb{Z}^2 \setminus \{(0,0)\}} \left[ \frac{z - m\omega_1 - n\omega_2}{(z - m\omega_1 - n\omega_2)^3} + \frac{m\omega_1 + n\omega_2}{(m\omega_1 + n\omega_2)^3} \right] \quad (4.4) \]

by formula

\[ G_2(z) = -\frac{1}{2} \wp'_1(z) + \frac{1}{2} \wp'_1(z) + T_2. \quad (4.5) \]

Filshtinsky \cite{17} Appendix 2] found a relation between the Natanzon and Weierstrass functions which can be written in our case as

\[ \pi \wp'_1(z) = \frac{1}{3} \wp''(z) + [\zeta(z) - (S_2 - \pi)z] \wp'(z) - 2(S_2 - \pi) \wp(z) - 10S_4, \quad (4.6) \]
where \( \zeta(z) \) is the \( \zeta \)-Weierstrass function and \( S_4 \) is defined by the absolutely convergent series

\[
S_4 = \sum_{(m,n) \in \mathbb{Z}^2 \backslash \{(0,0)\}} \frac{1}{(m\omega_1 + n\omega_2)^2}.
\]  

(4.7)

Substitution of (4.6) into (4.5) yields

\[
G_2(z) = -\frac{1}{2} \wp'(z) + \frac{1}{6\pi} \wp''(z) + \frac{1}{2} \left[ \frac{\zeta(z)}{\pi} - \left( \frac{S_2}{\pi} - 1 \right) \right] \wp'(z)
- \left( \frac{S_2}{\pi} - 1 \right) \wp(z) - \frac{5}{\pi} S_4 + T_2.
\]  

(4.8)

Consider \( N \) non-overlapping circular disks \( D_k \) of radius \( r \) with the centers \( a_k \in \mathcal{G}_{(0,0)} \). These centers can be considered as random variables. Introduce the sums

\[
e_2 = \frac{1}{N^2} \sum_{k=1}^{N} \sum_{m=1}^{N} E_2(a_k - a_m),
\]

(4.9)

\[
g_2 = \frac{1}{N^2} \sum_{k=1}^{N} \sum_{m=1}^{N} F_2(a_k - a_m),
\]

(4.10)

where it is assumed that \( E_2(0) := S_2 \) and \( F_2(0) := T_2 \). Such a consideration implies that for \( N = 1 \) \( e_2 \) becomes \( S_2 \) and \( g_2 \) becomes \( T_2 \).

The sums (4.9)-(4.10) play the fundamental role in the theory of random 2D composites, since the effective conductivity tensor of the composite represented by \( N \) discs per periodicity cell can be calculated by the asymptotic formula [7] (cf. (1.1))

\[
\frac{\lambda_{xx} - i\lambda_{xy}}{\lambda} = 1 + 2\rho f + 2\rho^2 f^2 \frac{e_2}{\pi} + O((|\rho f|)^3),
\]

(4.11)

\[
\frac{\lambda_{yy} + i\lambda_{xy}}{\lambda} = 1 + 2\rho f + 2\rho^2 f^2 \left( 2 - \frac{e_2}{\pi} \right) + O((|\rho f|)^3),
\]

(4.12)

In the case of macroscopically isotropic composites, \( \lambda_{xx} = \lambda_{yy} \) and \( \lambda_{xy} = 0 \). This implies that \( e_2 \) must be equal to \( \pi \). One can consider this assertion as a physical prove of the identity \( e_2 = \pi \) for a macroscopically isotropic distribution of \( a_k \).

Analogous formulae take place for the elastic constants. Let elastic fibers \( D_k \) with the shear modulus \( \mu_1 \) and the Poisson ration \( \nu_1 \) are distributed in the matrix with the constants \( \mu \) and \( \nu \). Let \( \kappa = 3 - 4\nu \) and \( \kappa_1 = 3 - 4\nu_1 \) be the corresponding Muskhelishvili constants for the plane strain. Consider the
averaged constant \( \mu_e = \frac{\langle \sigma_{xx} - \sigma_{yy} \rangle}{2(\epsilon_{xx} - \epsilon_{yy})} \) where \( \sigma_{\alpha\beta} \) and \( \epsilon_{\alpha\beta} \) denote the components of the stress and deformation tensors, respectively (\( \alpha \) and \( \beta \) can be \( x \) and \( y \)). Here, \( \langle \cdot \rangle \) denotes the average value (double integral over the periodicity cell). In particular, for macroscopically isotropic composites \( \mu_e \) yields the effective shear modulus. The value \( \mu_e \) can be estimated by asymptotic formula deduced in [12]

\[
\frac{\mu_e}{\mu(1 + \kappa)} = \frac{1}{1 + \kappa} + \frac{\mu_1 - \mu}{\kappa \mu_1 + \mu} f + \left( \frac{\mu_1 - \mu}{\kappa \mu_1 + \mu} \right)^2 \left( \kappa - \frac{2 \text{Re} g_2}{\pi} \right) f^2 + O(f^3). \tag{4.13}
\]

One can see that the value \( g_2 \) from (4.10) occurs in the coefficient on \( f^2 \).

Numerical simulations of \( e_2 \) were performed in [18] for macroscopically isotropic composites generated by the RSA algorithm and by random walks. Using the RSA protocol we compute 100 times \( g_2 \) for \( r = 0.003 \) when \( f \) is about 0.09, \( N \) is about 3250. More precisely, \( f \) and \( N \) slightly change in each simulation of location in accordance with the RSA protocol [18]. The mean value of \( g_2 - \pi \) holds 0.00457824 + 0.0121335i, the variance 0.0286453. For \( e_2 - \pi \) we get the mean value 0.000723263 + 0.00575626i and the variance 0.0296968.

5 Conclusion

Explicit formulae of Section 3 deduced in this paper yield asymptotic analytical formulae for the effective tensors of 2D composites with circular inclusions. The obtained fundamental values \( S_2 \) and \( T_2 \) give a possibility to pass through the approximation \( O(f^2) \) terms to get high order analytical formulae for the effective elastic constants of fibrous composites [12].

References

[1] Borwein JM, Glasser L, McPhedran R, Wan JG, Zucker IJ. Lattice Sums: Then and Now. Cambridge: Cambridge University Press; 2013. 390p.

[2] Rayleigh L. On the influence of obstacles arranged in rectangular order upon the properties of medium. Phil.Mag. 1892;34:481-502.

[3] Movchan AB, Nicorovici NA, McPhedran. Green’s tensors and lattice sums for elastostatics and elastodynamics. Proc. R. Soc. Lond. A 1997Mar;453(1958):643-662.
[4] Greengard L, Helsing J. On the numerical evaluation of elastostatic fields in locally isotropic two-dimensional composites. Journal of the Mechanics and Physics of Solids. 1998 Jan;46(8):1441-1462.

[5] Movchan AB, Movhan NV, Poulton CG. Asymptotic Models of Fields in Dilute and Densely Packed Composites. London: Imperial College Press; 2002. 204p.

[6] Milton GW. The Theory of Composites. Cambridge: Cambridge University Press; 2002. 719p.

[7] Mityushev V. Transport properties of doubly periodic arrays of circular cylinders and optimal design problems, Appl. Math. Optimization. 2001;44:17-31.

[8] Mityushev V, Rylko N. Maxwell’s approach to effective conductivity and its limitations. The Quarterly Journal of Mechanics and Applied Mathematics. 2013;66(2):241-251. DOI:10.1093/qjmam/hbt003

[9] McPhedran RC, McKenzie DR. The conductivity of lattices of spheres. The simple cubic lattice. Proceedings of the Royal Society of London. A. 1978 Jan;359(1696):45-63.

[10] Perrins WT, McKenzie DR, McPhedran RC. Transport Properties of Regular Arrays of Cylinders. Proc. R. Soc. Lond. A. 1979 Dec;369(1737):207-225.

[11] Rylko N. Transport properties of the regular array of highly conducting cylinders. J. Eng. Math. 2000;38:1-12.

[12] Drygański P, Mityushev V. Effective elastic properties of random two-dimensional composites. 2016. Arxiv

[13] Mityushev VV. Transport properties of finite and infinite composite materials and Rayleigh’s sum. Arch. Mech. 1997;49(2):345-358.

[14] Mityushev VV. Transport Properties of Double-Periodic Arrays of Circular Cylinders. ZAMM. 1997;77(2):115-120.

[15] Weil A. Elliptic Functions According to Eisenstein and Kronecker. Berlin etc.: Springer-Verlag; 1976. 92p.

[16] Natanzon VY. On the stresses in a stretched plate weakened by identical holes located in chessboard arrangement. Mat. Sb. 1935;42(5):616-636.
[17] Grigolyuk EI, Filshtinsky LA. Periodic piece-wise constant structures. Moscow: Nauka; 1992. 288p. [in Russian]

[18] Czapla R, Nawalaniec W, Mityushev V. Effective conductivity of random two-dimensional composites with circular non-overlapping inclusions. Computational Materials Science. 2012;63:118-126.

[19] Apostol TM. Modular Functions and Dirichlet Series in Number Theory, 2nd ed. Springer, New York (1990).

[20] Borwein JM, Borwein PB. Pi and the AGM: A Study in Analytic Number Theory and Computational Complexity, Wiley, New York (1987).

[21] Borwein JM, Zucker IJ. Elliptic integral evaluation of the Gamma-function at rational values of small denominators, IMA J. Numerical Analysis, 12 (1992), 519-526.

[22] Erdélyi A, Magnus W, Oberhettinger F, Tricomi FG. Higher Transcendental Functions, Vols I, II and III, McGraw-Hill, New York, London and Toronto (1953).

[23] Nasim C. A summation formula involving $\sigma(n)$, Transactions of the American Mathematical Society, 192 (1974), 307-317.

[24] Prudnikov AP, Brychkov YA, Marichev OI. Integrals and Series: Vol. I: Elementary Functions, Gordon and Breach, New York (1986).

[25] Titchmarsh EC. An Introduction to the Theory of Fourier Integrals, Chelsea, New York (1986).

[26] Whittaker ET, Watson GN. A Course in Modern Analysis, 4th ed. Cambridge University Press, Cambridge (1990).

[27] Yakubovich S. Integral and series transformations via Ramanujan’s identities and Salem’s type equivalences to the Riemann hypothesis, Integral Transforms and Special Functions, 25, N 4 (2014), 255-271.