RESEARCH ARTICLE

Row-column factorial designs with multiple levels

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Abstract
An $m \times n$ row-column factorial design is an arrangement of the elements of a factorial design into a rectangular array. Such an array is used in experimental design, where the rows and columns can act as blocking factors. Formally, for any integer $q$, let $[q] = \{0, 1, ..., q - 1\}$. The $q^k$ (full) factorial design with replication $\alpha$ is the multiset consisting of $\alpha$ occurrences of each element of $[q]^k$; we denote this by $\alpha \times [q]^k$. A regular $m \times n$ row-column factorial design is an arrangement of the elements of $\alpha \times [q]^k$ into an $m \times n$ array (which we say is of type $I_k(m, n; q)$) such that for each row (column) and fixed vector position $i \in [k]$, each element of $[q]$ occurs $n/q$ times (respectively, $m/q$ times). Let $m \leq n$. We show that an array of type $I_k(m, n; q)$ exists if and only if (a) $q \mid m$ and $q \mid n$; (b) $q^k \mid mn$; (c) $(k, q, m, n) \neq (2, 6, 6, 6)$, and (d) if $(k, q, m) = (2, 2, 2)$ then $4$ divides $n$. Godolphin showed the above is true for the case $q = 2$ when $m$ and $n$ are powers of 2. In the case $k = 2$, the above implies necessary and sufficient conditions for the existence of a pair of mutually orthogonal frequency rectangles (or $F$-rectangles) whenever each symbol occurs the same number of times in a given row or column.

KEYWORDS
blocking factor, double confounding, frequency rectangle or $F$-rectangle, MOFS, row-column factorial design

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1 | INTRODUCTION

For any integer $q$, let $[q] = \{0, 1, \ldots, q - 1\}$. Consider the following example of an experimental design from [8]. Suppose we wish to study the effects of four drugs at two dosage levels while controlling for the effects of four breeds and four age groups. We could conduct 16 experiments based on the row-column factorial design given in Table 1.

Here the rows and columns correspond to age groups and breeds, respectively, of calves; with the binary vector in a cell indicating the dosage of each of the four drugs as one of two levels.

In the above the 16 vectors from $[2]^4$ are arranged in a $4 \times 4$ array, in such a way that for each row (column) and $i \in [4]$, the entries 0 and 1 each appear twice in position $i$ of a vector in that row (respectively, column). In the context of experimental design, these properties of regularity mean that if we consider the set of six effects (the four drugs together with breed and age group), there is no confounding between any pair of these effects. On the other hand, if we ignore age and breed effects, the underlying factorial design allows us to also estimate the effects of any subset of the four types of drug (these are called interactions) without confounding. We refer the reader to Chapter 9 of [15] for more detail of the application of row-column factorial designs to statistical experimental design.

Formally, the $q^k$ (full) factorial design with replication $\alpha$ is the multiset consisting of $\alpha$ occurrences of each element of $[q]^k$; we denote this by $\alpha \times [q]^k$. An $m \times n$ row-column factorial design $q^k$ is any arrangement of the elements of $\alpha \times q^k$ into an $m \times n$ array. Necessarily, $q^k$ must divide $mn$. Without loss of generality, we always assume $m \leq n$. We call such a design regular if for each row (column) and $i \in [q]$, each element of $[q]$ occurs $n/q$ times (respectively, $m/q$ times). Furthermore we denote the type of such an array to be $I_k(m, n; q)$, where regularity is always assumed to hold. Observe that regularity implies that $q$ divides both $m$ and $n$. The above example is thus a regular $4 \times 4$ row-column factorial design $2^4$, or equivalently an array of type $I_4(4, 4; 2)$. Note that an array of type $I_4(n, n; n)$ is equivalent to a pair of orthogonal Latin squares of order $n$.

We first review the impact of row-column factorial designs within experimental design literature. A blocking factor can be thought of as a partition of the blocks of a design, typically an equipartition (all subsets in the partition have equal size) with further properties of regularity to minimize confounding within the design structure. Blocking factors for factorial designs have been well-studied ([1], [2], [10], [9], [17]). However, as mentioned in [13], having two forms of blocking for a factorial design is less well-studied.

Within experimental design, a row-column design can refer to a variety of combinatorial designs, all with the property of being arranged in a rectangular array, where the rows and columns are typically (but not always) blocking factors. This is sometimes referred to as double confounding [13]. To ensure that certain effects can be estimated without confounding, regularity conditions are imposed. For example, in a Latin square each symbol occurs once per row and once per column.

| TABLE 1 | A regular row-column factorial design of type $I_4(4, 4; 2)$ |
| --- | --- | --- | --- |
| 1111 | 0100 | 0010 | 1001 |
| 0001 | 1010 | 1100 | 0111 |
| 1000 | 0011 | 0101 | 1110 |
| 0110 | 1101 | 1011 | 0000 |
In practice nonregular row-column factorial designs are also sometimes of use. In [25], a
nonregular row-column factorial design is given which was used by the CSIRO Division of
Forestry for a glasshouse experiment. Here the physical distance to the edge of the glasshouse
is an important effect to consider.

A quasi-Latin square is an \( n \times n \) array such that for some \( k > n \) which divides \( n^2 \), each entry
from \([k]\) occurs \( n^2/k \) times in the array, with no entry occurring more than once per row or
column. Some of the literature on quasi-Latin squares features row-column factorial designs
[3]. Here if we consider the vectors as the entries, a row-column factorial design can be thought
of as a quasi-Latin square if no vector occurs more than once in a row or column (necessarily,
\( m, n < 2^k \)). John and Lewis [16] describe a technique to cyclically generate some regular row-
column factorial designs. Examples of regular row-column factorial designs are also given in
[6], [7] and [5]. Wang [24] constructs \( I_k(2^M, 2^N; 2) \) whenever \( k = M + N \). A variation of row-
column factorial designs is considered by [8]: a generalized confounded row-column design can
be thought of as a factorial design arranged into a rectangular array where each cell contains a
constant number of vectors.

Row-column factorial designs with two levels (i.e., \( q = 2 \)) are studied in [13]. As well as the
result in Theorem 1 below, designs are also constructed to estimate paired interactions without
confounding by row and column blocking factors.

**Theorem 1** (Godolphin [13]). Let \( 1 \leq M \leq N \). An array of type \( I_k(2^M, 2^N; 2) \) (i.e., a
regular \( 2^M \times 2^N \) row-column factorial design \( 2^k \)) exists if and only if \( k \leq M + N \)
and \( (k, M, N) \neq (2, 1, 1) \).

We next describe the connection between regular row-column factorial designs and fre-
quency rectangles. Given two vectors \( v = (v_0, v_1, ..., v_{\ell−1}) \) and \( w = (w_0, w_1, ..., w_{\ell−1}) \), we define
\( v \oplus w \) to be the concatenation of \( v \) and \( w \), that is:

\[
v \oplus w := (v_0, v_1, ..., v_{\ell−1}, w_0, w_1, ..., w_{\ell−1}).
\]

Next, let \( A = [a_{ij}] \) and \( B = [b_{ij}] \) be matrices of the same dimensions, with each entry of \( A \) is a
vector of dimension \( k \) and each entry of \( B \) is a vector of dimension \( \ell \). Then we define
\( C = A \oplus B \) to be the matrix given by \( C = [c_{ij} = a_{ij} \oplus b_{ij}] \).

Now, an array of type \( I_k(m, n; q) \) can be written in the form \( F_0 \oplus F_1 \oplus \cdots \oplus F_{k−1} \), where
each entry of each \( F_i, i \in [k] \), has dimension 1. Since regularity is assumed, each element of \([q]\)
occurs precisely \( n/q \) times per row and \( m/q \) times per column, for each of the arrays \( F_i, i \in [k] \).
These arrays are thus frequency rectangles.

Formally, a frequency rectangle (sometimes known as an \( F \)-rectangle) of type \( FR(m, n; q) \) is an
\( m \times n \) array such that each element of \([q]\) occurs \( n/q \) times per row and \( m/q \) times per column.
Thus, we may write any array of type \( I_k(m, n; q) \) as \( F_0 \oplus F_1 \oplus \cdots \oplus F_{k−1} \), where for each \( i \in [k], F_i \)
is a frequency rectangle of type \( FR(m, n; q) \). We note here that frequency rectangles in the literature
(most often frequency squares or \( F \)-squares when \( m = n \)) may have different row/column frequencies
for distinct symbols. In this paper we restrict ourselves to the regular case.

Two frequency rectangles of type \( FR(m, n; q) \) are orthogonal if, when superimposed, each
ordered pair from \([q] \times [q]\) occurs exactly \( mn/q^2 \) times in the array. A set of pairwise ortho-
gonal frequency rectangles are called mutually orthogonal frequency rectangles. These have
mostly been studied in the case \( m = n \), where such structures are called mutually orthogonal
frequency squares or \([\text{MOFS}]\), and in particular the case \( m = n = q \), where such structures are
mutually orthogonal latin squares (MOLS).
The existence problem for pairs of MOFS has been completely classified; the following theorem is a special case of [18, p. 67]. The exceptions are precisely the two orders for which pairs of MOLS which do not exist, as known by Euler.

**Theorem 2.** There exists a pair of MOFS of type $F(n, n; q)$ (equivalently, an array of type $I_2(n, n; q)$) if and only if $(n, q) \notin \{(2, 1), (6, 1)\}$.

Hedayat et al. [14] showed that if a set of $k$ MOFS of type $FR(n, n; q)$ exists then $k \leq (n - 1)^2/(q - 1)$. When $k$ meets this upper bound such a set is called complete. Complete sets of MOFS exist when $q = 2$ and if there exists a Hadamard matrix of order $n$ [11]; otherwise they are only known to exist when $n$ is a prime power [19–21,23]. A complete set of MOFS does not exist when $q = 2$ and $n \equiv 2 \pmod{4}$ [4].

Note that while an array of type $I_k(m, n; q)$ yields a set of $k$ mutually orthogonal frequency rectangles, the converse is not always true for $k \geq 3$, as seen below. Here we see a set of three mutually orthogonal frequency rectangles (overlapped) which is not a regular row-column factorial design.

However, if $k = 2$ an $I_k(m, n; q)$ is equivalent to a pair of mutually orthogonal frequency rectangles of type $F(m, n; q)$.

**Theorem 3** (Federer et al. [12]). Let $q$ divide $m$ and $n$. If $q \notin \{2, 6\}$ or at least one of $n/q, m/q$ is even, there exists a pair of mutually orthogonal frequency rectangles of type $F(m, n; q)$ (equivalently, an array of type $I_2(m, n; q)$).

A set of mutually orthogonal frequency rectangles can also be thought of as a type of mixed orthogonal array. In general, a mixed orthogonal array OA $(N, s_1^{k_1}s_2^{k_2}...s_v^{k_v}, t)$ is an array of size $N \times k$, where $k = \sum_{i=1}^{v} k_i$ in which $k_i$ columns have symbols from the set $[s_i]$, such that in any $N \times t$ subarray every possible $t$-tuple occurs the same number of times. The parameter $t$ is called the strength of the orthogonal array.

Given a set $F_1, F_2, ..., F_v$ of $k$ mutually orthogonal frequency rectangles of type $F(m, n; q)$, for each cell $(i, j) \in [m] \times [n]$, create a row of an $mn \times (k + 2)$ array by placing the entry in cell $(i, j)$ of $F_\ell$ in column $\ell$, with $i$ and $j$ the entries, respectively, in the final two columns. The result is a mixed orthogonal array OA $(mn, q^k, m^1, n^1, 2)$; in fact there is equivalence between the two combinatorial structures.

Thus, from above, an $I_k(m, n; q)$ is equivalent to a mixed orthogonal array only in the case $k = 2$. Mixed orthogonal arrays for $N \leq 100$ are classified in [22]. Consequently, it is known whether $I_2(m, n; q)$ exists for any $m$ and $n$ such that $mn \leq 100$.

In this paper our main result is to classify the parameters for which there exists a regular row-column factorial design, generalizing the results of Theorems 1, 2, and 3 above.
Theorem 4. Let $m \leq n$. There exists an array of type $I_k(m, n; q)$ (i.e., a regular $m \times n$ row-column factorial design $q^k$) if and only if $q$ divides $m$, $q$ divides $n$, $q^k$ divides $mn$ unless:

(i) $k = q = m = 2$ and $n \equiv 2 \pmod{4}$.
(ii) $k = 2$ and $q = m = n = 6$.

In Section 4, we generalize Theorem 3 to find necessary and sufficient conditions for the existence of an array of type $I_2(m, n; q)$. We prove the remaining cases of Theorem 4 in Section 5, using the recursive constructions from Section 2 and the finite field constructions from Section 3.

2 | RECURSIVE CONSTRUCTIONS

In this section, we discuss ways in which row-column factorial designs can be built recursively. We begin with a straightforward lemma.

Lemma 1. If there exist arrays of type $I_k(m, n; q)$ and $I_k(m', n; q)$ there exists an array of type $I_k(m + m', n; q)$. If there exist arrays of type $I_k(m, n; q)$ and $I_k(m, n'; q)$ there exists an array of type $I_k(m, n + n'; q)$.

Next we consider a type of Kronecker Product. Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be two arrays of sizes $m \times n$ and $u \times v$, respectively. The Kronecker product, $A \otimes B$, of $A$ and $B$ is an $mu \times nv$ array defined by:

$$A \otimes B = \left( \begin{array}{c} a_{11}B \ldots a_{1n}B \\ \vdots \\ a_{m1}B \ldots a_{mn}B \end{array} \right)$$

where $a_{ik}B$ is a $u \times v$ array with the entry $(i, j)$ given by $(a_{ik}, b_{ij})$.

It is easy to see that if $F$ and $F'$ are two frequency rectangles of type $FR(m, n; q)$ and $FR(m', n'; q')$, respectively, then their Kronecker product $F \otimes F'$ is a frequency rectangle of type $FR(mm', nn'; qq')$, where the entries of $[q] \times [q']$ are mapped to $[qq']$ by some bijection $f$. In this fashion, let $D = F_0 \oplus F_1 \oplus \cdots \oplus F_{k-1}$ and $D' = F'_0 \oplus F'_1 \oplus \cdots \oplus F'_{k-1}$ be two row-column factorial designs of types $I_k(m, n; q)$ and $I_k(m', n'; q')$, respectively, where $F_i$ and $F'_i$ are frequency rectangles for each $i \in [k]$. Then we define $D \boxtimes D'$ to be the array given by $(F_0 \otimes F'_0) \oplus (F_1 \otimes F'_1) \oplus \cdots \oplus (F_{k-1} \otimes F'_{k-1})$.

Lemma 2. If $D$ and $D'$ are arrays of type $I_k(m, n; q)$ and $I_k(m', n'; q')$, respectively, then $D \boxtimes D'$, as defined above, is an array of type $I_k(mm', nn'; qq')$.

Proof. It suffices to show that the entries of the cells of $D \boxtimes D'$ form a regular factorial design. Let $D = F_0 \oplus F_1 \oplus \cdots \oplus F_{k-1}$ and $D' = F'_0 \oplus F'_1 \oplus \cdots \oplus F'_{k-1}$ as above.

Consider any $(\alpha_0, \alpha_1, \ldots, \alpha_{k-1}) \in [qq']$ in a cell of $D \boxtimes D'$. Then for each $i \in [k]$, $\alpha_i = f(a_i, b_i)$, for some $a_i$ and $b_i$ belong to the symbol sets of $F_i$ and $F'_i$ respectively. Since $D$ is of type $I_k(m, n; q)$, there are precisely $mn/q^k$ cells containing $a_i$ in $F_i$ for each $i \in [k]$. Similarly, there are exactly $m'n'/q'(q')^k$ cells containing $b_i$ in $F'_i$ for each $i \in [k]$. From the definition of the Kronecker product, the sequence $(\alpha_0, \alpha_1, \ldots, \alpha_{k-1})$ appears exactly $mm'n'n'/qq'(q')^k$ times in $D \boxtimes D'$. \qed
Example 1. Consider the two arrays $D$ and $D'$ of type $I_3(4, 2; 2)$ and $I_3(3, 9; 3)$, respectively.

Then by taking the product $D \boxtimes D'$ and using the bijection $(a, b) \mapsto 3a + b$ to transform the symbol set, we get an array of type $I_3(12, 18; 6)$:

|   |   |   |   |   |   |   |   |   |
|---|---|---|---|---|---|---|---|---|
| 000 | 011 | 011 | 110 | 011 | 022 | 012 | 112 | 120 |
| 111 | 122 | 121 | 122 | 202 | 210 | 221 | 211 | 210 |
| 144 | 155 | 133 | 245 | 253 | 234 | 043 | 054 | 035 |
| 255 | 233 | 244 | 053 | 034 | 045 | 154 | 135 | 143 |
| 303 | 314 | 325 | 404 | 415 | 423 | 505 | 513 | 524 |
| 414 | 425 | 403 | 515 | 523 | 504 | 313 | 324 | 305 |
| 525 | 503 | 514 | 323 | 304 | 315 | 424 | 405 | 413 |
| 330 | 341 | 352 | 431 | 422 | 410 | 532 | 540 | 551 |
| 441 | 452 | 430 | 542 | 550 | 531 | 340 | 351 | 332 |
| 552 | 530 | 541 | 350 | 331 | 342 | 451 | 432 | 440 |

An array of type $I_3(12, 18; 6)$

Corollary 1. If there exist $r$ arrays of types $I_k(m_i, n_i; q_i)$, where $i \in [r]$, then there exist an array of type $I_k\left(\prod_{i=0}^{r-1} m_i, \prod_{i=0}^{r-1} n_i, \prod_{i=0}^{r-1} q_i\right)$.

Trivially there exists an array of type $I_k(m, n; 1)$ for any integers $k, m, n$. The following corollary is then immediate.

Corollary 2. If there exists an array of type $I_k(m, n; q)$, then there exists an array of type $I_k(m'm', n'n'; q)$ for any integers $m', n' \geq 1$.

3 | PRIME POWER CONSTRUCTIONS

In this section, we construct row-column factorial designs via finite fields of prime power order $q$. It is implicitly understood that field elements are relabelled to elements of $[q]$ as a final step in construction.

Lemma 3. Let $M, N \geq 1$ and $q \geq 2$, with $(M, N, q) \neq (1, 1, 2)$. Then there exists a linearly independent set of $M + N$ polynomials:

$$f_r(x_0, \ldots, x_{M+N-1}) = a_{r,0}x_0 + a_{r,1}x_1 + \cdots + a_{r,M+N-1}x_{M+N-1}; \quad r \in [M+N]$$

over the field $\mathbb{F}_q$ which satisfy the following two conditions for each $r \in [M+N]$:

(i) $(a_{r,0}, \ldots, a_{r,M-1}) \neq (0, \ldots, 0)$;
(ii) $(a_{r,M}, \ldots, a_{r,M+N-1}) \neq (0, \ldots, 0)$. 

Proof. We split the proof into cases.

Case I: When $M = N = 1$ and $q > 2$.

In this case we can take the following two polynomials:

$$f_0(x_0, x_1) = x_0 + x_1$$
$$f_1(x_0, x_1) = x_0 + \alpha x_1,$$

where $\alpha$ is a nonzero element in $\mathbb{F}_q$ other than identity.

Case II (a): When $N \geq 2$ and $q$ is a power of 2.

We remind the reader that since we are working over a field of order $q$, $1 + 1 = 0$ in this case. Consider the identity matrix $I_{M+N}$ of order $M + N$. By performing the following two row operations sequentially:

$$R_0 + R_s \rightarrow R_t \quad \text{for each } s \in \{1, 2, \ldots, M + N - 1\};$$
$$R_s + (R_{M+N-1} + R_{M+N-2}) \rightarrow R_s \quad \text{for each } s \in [M],$$

we get the following matrix:

Now corresponding to each row $R_s = (r_{s0}, \ldots, r_{s(M+N-1)})$ of the above matrix we define a polynomial $f_s = r_{s0}x_0 + \cdots + r_{s(M+N-1)}x_{M+N-1}$ in $\mathbb{F}_q$, where, $s \in [M + N]$. Then these polynomials satisfy the conditions (i) and (ii) and are linearly independent.

Case II (b): When $N \geq 2$ and $q$ is not a power of 2.

In this case again take the identity matrix $I_{M+N}$ and by replacing the second row operation in (1) by $R_s + R_{M+N-1} \rightarrow R_s$ for each $s \in [M]$, we get the following matrix:
It is easy to see that the corresponding polynomials are linearly independent in \( \mathbb{F}_q \) and satisfy the conditions (i) and (ii).

\[ \square \]

**Theorem 5.** Let \( q \geq 2 \) be a prime power. Let \( M, N \geq 1 \) and \( (M, N, q) \neq (1, 1, 2) \). There exists an array of type \( I_{M+N}(q^M, q^N; q) \).

**Proof.** First we describe a method to construct a frequency rectangle of type \( FR(q^M, q^N; q) \) corresponding to polynomial \( f_r \) for each \( r \in [M+N] \), as given by Lemma 3.

Label the rows and columns of a \( q \times q \) array, respectively, by using the set of all \( M \)-tuples and \( N \)-tuples over the field \( \mathbb{F}_q \). Now consider a polynomial

\[ f_r(x_0, ..., x_{M+N-1}) = a_{r,0}x_0 + \cdots + a_{r,M+N-1}x_{r,M+N-1} \]

over the field \( \mathbb{F}_q \) that satisfies the conditions given in Lemma 3. We place the element \( f_r(b_0, ..., b_{M-1}, c_0, ..., c_{N-1}) \) in the intersection of row \( (b_0, ..., b_{M-1}) \) and column \( (c_0, ..., c_{N-1}) \) of the \( q^M \times q^N \) array.

Now we show that the array obtained in this way is a frequency rectangle of type \( FR(q^M, q^N; q) \), that is every element of \( \mathbb{F}_q \) appears exactly \( q^{N-1} \) times in each row and \( q^{M-1} \) times in each column. Consider a row which is labelled by \( (b_0, ..., b_{M-1}) \) and take an element \( \alpha \in \mathbb{F}_q \). For this row, the equation

\[ f_r(b_0, ..., b_{M-1}, x_0, ..., x_{M+N-1}) = \alpha \]

reduces to the equation

\[ K + a_{r,M}x_M + \cdots + a_{r,M+N-1}x_{M+N-1} = \alpha \]

where \( K \) is a constant. Now by axiom (ii) of Lemma 3 there exists \( i \in [M, M+1, ..., M+N-1] \) such that \( a_{r,i} \neq 0 \). We solve the Equation (2) for \( x_i \):

\[ x_i = \frac{1}{a_{r,i}}(\alpha - K - a_{r,M}x_M - \cdots - a_{r,i-1}x_{i-1} - a_{r,i+1}x_{i+1} - \cdots - a_{r,M+N-1}x_{M+N-1}). \]

Since there are \( q \) elements in \( \mathbb{F}_q \) and the \( N-1 \) variables on the right side of (3) can take any value from \( \mathbb{F}_q \), the Equation (3) has exactly \( q^{N-1} \) solutions in \( \mathbb{F}_q \). This implies that the symbol \( \alpha \) appears exactly at \( q^{N-1} \) places in the row \( (b_0, ..., b_{M-1}) \). By a similar
argument we can prove that each symbol appears exactly $q^{M-1}$ times in each column. Thus the resulting array is a frequency rectangle of type $FR(q^M, q^N; q)$.

Now to construct an array of type $I_{M+N}(q^M, q^N; q)$. Consider a set of $M + N$ linearly independent polynomials

$$f_r(x_0, ..., x_{M+N-1}) = a_{r,0}x_0 + \cdots + a_{r,M+N-1}x_{M+N-1}; \quad r \in [M + N]$$

over the field $\mathbb{F}_q$, such that the coefficients satisfy the conditions (i) and (ii) of Lemma 3. As above, for each $r \in [M + N]$ we obtain a frequency rectangle $F_r$ of type $FR(q^M, q^N; q)$.

It remains to show that $F_0 \oplus F_1 \oplus \cdots \oplus F_{M+N-1}$ is an array of type $I_{M+N}(q^M, q^N; q)$. To this end, consider any $(\alpha_0, \alpha_1, ..., \alpha_{M+N-1}) \in (\mathbb{F}_q)^{M+N}$. Since the polynomials above are linearly independent, the system of equations:

$$a_{0,0}x_0 + \cdots + a_{0,M+N-1}x_{M+N-1} = \alpha_0$$
$$a_{1,0}x_0 + \cdots + a_{1,M+N-1}x_{M+N-1} = \alpha_1$$
$$\vdots$$
$$a_{M+N-1,0}x_0 + \cdots + a_{M+N-1,M+N-1}x_{M+N-1} = \alpha_{M+N-1}$$

has rank $M + N$ and therefore has a unique solution in $\mathbb{F}_q$, which shows that $(\alpha_0, \alpha_1, ..., \alpha_{M+N-1})$ appears in exactly one cell of the array constructed. □

Corollary 2 implies the following:

**Corollary 3.** If $(q, M, N) \neq (2, 1, 1)$, there exist an array of type $I_{M+N}(q^M b_1, q^N b_2; q)$, for any prime power $q$.

### 3.1 “Sudoku” frequency rectangles

In this section, we take the construction above and take it one step further. Specifically, we show in Theorem 7 that if $q$ divides $b_1 b_2$ and $q$ is a prime power, there exists an array of type $I_{M+N+1}(q^M b_1, q^N b_2; q)$.

First we describe a Latin square which has a Sudoku-type property with $q = q_1 q_2$ symbols, where $q_1$ and $q_2$ are positive integers and the symbol set is taken to be $[q]$. That is, such a Latin square can be partitioned into $q_1 \times q_2$ subarrays containing each element of $[q]$.

**Theorem 6.** Let $q_1, q_2 \geq 1$. Then there exists a Latin square $L(q_1, q_2)$ of order $q_1 q_2$ such that for each $i \in [q_1]$ and $j \in [q_2]$, the set of cells

$$\{(i', j') | i \equiv i' \pmod{q_1}, j \equiv j' \pmod{q_2}\}$$

contain each entry from $[q_1 q_2]$ exactly once.

**Proof.** In what follows, $q = q_1 q_2$. Let $S_0$ be the $q_1 \times q_2$ array where cell $(i, j)$ of $S$ contains the integer $i + j q_1$, for each $i \in [q_1]$ and $j \in [q_2]$. Thus the entries of $S_0$ are the elements of $[q]$, listed in ascending order from the first column:
Now we define $S_i$ to be the array obtained by adding the symbol $i$ (mod $q$) to each cell of $S_0$. Finally, define $L(q_1, q_2)$ to be the following array of order $q$:

$$
\begin{array}{cccc}
\uparrow & q_2 & \rightarrow & q_2 & \rightarrow & \ldots & \rightarrow & q_2 & \rightarrow \\
q_1 & S_0 & S_1 & \ldots & S_{q_1 - 1} \\
\downarrow & S_{q_1} & S_{q_1 + 1} & \ldots & S_{2q_1 - 1} \\
\downarrow & \vdots & \vdots & \ddots & \vdots \\
q_1 & S_{(q_1 - q_2)} & S_{(q_1 - q_2) + 1} & \ldots & S_{q - 1} \\
\end{array}
$$

Observe that each entry of $[q]$ occurs once per row and once per column; hence $L(q_1, q_2)$ is a Latin square. □

**Example 2.** We exhibit the construction in the previous theorem in the case $q = 6$, $q_1 = 2$ and $q_2 = 3$:

$$
\begin{array}{ccc}
0 & 2 & 4 \\
1 & 3 & 5 \\
2 & 4 & 0 \\
3 & 5 & 1 \\
4 & 0 & 2 \\
5 & 1 & 3 \\
\end{array}
$$

The Latin square $L(2, 3)$ as defined in Theorem 6

Now, a Latin square of order $q$ is also an array of type $I_1(q, q; q)$. Thus, from Corollary 2, we have the following corollary.

**Corollary 4.** For any integers $\mu, \lambda, q_1, q_2 \geq 1$, there exists a frequency rectangle of type $FR(q_1 q_2; q)$ (where $q = q_1 q_2$) such that for each $i \in [\mu q_1]$ and $j \in [\lambda q_2]$, the set of cells

$$
\{(i', j') | i \equiv i' \pmod{\mu q_1}, j \equiv j' \pmod{\lambda q_2}\}
$$

contain each entry from $[q_1 q_2]$ exactly once.
Example 3. If \( L \) is the Latin square \( L(2, 3) \), then \( L \boxtimes I_1(2, 3; 1) \) yields a frequency rectangle of type \( FR(12, 18; 6) \). The entries in bold show the elements of \([6]\) occurring in cells of the form \((i, j)\), where \( i \equiv 1 \pmod{4} \) and \( j \equiv 2 \pmod{9} \).

\[
\begin{array}{cccccccccccccccc}
0 & 0 & 0 & 2 & 2 & 2 & 4 & 4 & 4 & 1 & 1 & 1 & 3 & 3 & 3 & 5 & 5 & 5 \\
0 & 0 & 0 & 2 & 2 & 2 & 4 & 4 & 4 & 1 & 1 & 1 & 3 & 3 & 3 & 5 & 5 & 5 \\
1 & 1 & 1 & 3 & 3 & 3 & 5 & 5 & 5 & 2 & 2 & 2 & 4 & 4 & 4 & 0 & 0 & 0 \\
1 & 1 & 1 & 3 & 3 & 3 & 5 & 5 & 5 & 2 & 2 & 2 & 4 & 4 & 4 & 0 & 0 & 0 \\
2 & 2 & 2 & 4 & 4 & 4 & 0 & 0 & 0 & 3 & 3 & 3 & 5 & 5 & 5 & 1 & 1 & 1 \\
2 & 2 & 2 & 4 & 4 & 4 & 0 & 0 & 0 & 3 & 3 & 3 & 5 & 5 & 5 & 1 & 1 & 1 \\
3 & 3 & 3 & 5 & 5 & 5 & 1 & 1 & 1 & 4 & 4 & 4 & 0 & 0 & 0 & 2 & 2 & 2 \\
3 & 3 & 3 & 5 & 5 & 5 & 1 & 1 & 1 & 4 & 4 & 4 & 0 & 0 & 0 & 2 & 2 & 2 \\
4 & 4 & 4 & 0 & 0 & 0 & 2 & 2 & 2 & 5 & 5 & 5 & 1 & 1 & 1 & 3 & 3 & 3 \\
4 & 4 & 4 & 0 & 0 & 0 & 2 & 2 & 2 & 5 & 5 & 5 & 1 & 1 & 1 & 3 & 3 & 3 \\
5 & 5 & 5 & 1 & 1 & 1 & 3 & 3 & 3 & 0 & 0 & 0 & 2 & 2 & 2 & 4 & 4 & 4 \\
5 & 5 & 5 & 1 & 1 & 1 & 3 & 3 & 3 & 0 & 0 & 0 & 2 & 2 & 2 & 4 & 4 & 4 \\
\end{array}
\]

A frequency rectangle of type \( FR(12, 18; 6) \) by Corollary 4

Before we prove Theorem 7, we require the following number-theoretic observation.

Lemma 4. Let \( b_1, b_2, \) and \( q \) be positive integers such that \( q \) divides the product \( b_1 b_2 \). Then there exist positive integers \( q_1 \) and \( q_2 \) such that \( q_1 q_2 = q \) and \( q_1 \) divides \( b_2 \) and \( q_2 \) divides \( b_1 \).

Proof. Let \( q = p_0^{s_0} p_1^{s_1} \ldots p_{m-1}^{s_{m-1}} \) be the prime factorization of \( q \). Since \( q \) divides \( b_1 b_2 \), \( b_1 \) and \( b_2 \) must be of the form:

\[
b_1 = B_1 p_0^{\alpha_0} p_1^{\alpha_1} \ldots p_{m-1}^{\alpha_{m-1}},
\]

\[
b_2 = B_2 p_0^{\beta_0} p_1^{\beta_1} \ldots p_{m-1}^{\beta_{m-1}},
\]

where \( p_i \) does not divide \( B_j \) and \( \alpha_i + \beta_i \geq s_i \) for all \( i \in [m] \) and \( j \in \{1, 2\} \). Let

\[
q_1 = p_0^{u_0} p_1^{u_1} \ldots p_{m-1}^{u_{m-1}}
\]

and

\[
q_2 = p_0^{t_0} p_1^{t_1} \ldots p_{m-1}^{t_{m-1}},
\]

where \( t_i := \max\{0, s_i - \beta_i\} \) and \( u_i = s_i - t_i \) for all \( i \in [m] \).

Since \( \alpha_i + \beta_i \geq s_i \) and \( t_i \leq \alpha_i \) for each \( i \in [m] \), which implies that \( q_2 \) divides \( b_1 \). Also \( s_i - \beta_i \leq t_i \) implies that \( u_i = s_i - t_i \leq \beta_i \) and thus \( q_1 \) divides \( b_2 \). Finally observe that \( q_1 q_2 = q \). \( \Box \)

Theorem 7. Let \( q \) be a divisor of \( b_1 b_2 \). If there exist an array of type \( I_{M+N}(q^M, q^N; q) \), then there exists an array of type \( I_{M+N+1}(q^M b_1, q^N b_2; q) \).

Proof. By Lemma 4 we can choose \( q_1, q_2 \) such that \( q_1 q_2 = q \) and \( q_1 \) divides \( b_2 \) and \( q_2 \) divides \( b_1 \).

Let \( I' \) be the array \( I_1(q_2, q_1; 1) \boxtimes I_{M+N}(q^M, q^N; q) \) as shown in Table 2.
Applying Corollary 4 with $\mu = q^{M-1}q_2$ and $\lambda = q^{N-1}q_1$, there exists a rectangular array $J'$ of type $I_1(q^Mq_2, q^Nq_1; q)$ such that for each $i \in [q^M]$ and $j \in [q^N]$, the set of cells

$$\{(i', j') | i \equiv i' \pmod{q^M}, j \equiv j' \pmod{q^N}\}$$

contain each entry from $[q_1, q_2]$ exactly once. It follows that the array $I' \oplus J'$ contains each sequence of length $M + N + 1$ exactly once. Thus $I' \oplus J'$ is of type $I_{M+N+1}(q^Mq_2, q^Nq_1; q)$. Finally, by Corollary 2, $I_1(b_1/q_2, b_2/q_1; 1) \boxtimes (I' \oplus J')$ is an array of type $I_{M+N+1}(q^Mb_1, q^Nb_2; q)$. □

| 4 | THE CASE $k = 2$ |

In this section, we prove Theorem 4 in the case $k = 2$.

Given Theorem 3, it suffices to consider the existence of an array of type $I_2(m, n; q)$ only in the case $q \in \{2, 6\}$ and $m/q$ and $n/q$ odd.

From Theorem 2, arrays of type $I_2(2, 2; 2)$ and $I_2(6, 6; 6)$ do not exist. We next give another nonexistence result.

**Lemma 5.** There does not exist an array of type $I_2(2, n; 2)$ whenever $n/2$ is odd.

**Proof.** Consider a frequency rectangle $F$ of type $FR(2, n; 2)$. By permuting columns we can assume $F$ is in the following form:

$$
\begin{array}{cccc}
0 & 0 & \ldots & 0 \\
1 & 1 & \ldots & 1 \\
1 & 1 & \ldots & 0 \\
0 & 0 & \ldots & 0
\end{array}
$$

Now consider any other frequency square $F'$ of type $FR(2, n; 2)$. Now since $n \equiv 2 \pmod{4}$, $F'$ contains at least $\left\lfloor \frac{n}{4} \right\rfloor + 1$ symbols of the same type (say 0) in the first $n/2$ cells of its first row. This implies there are at least $\left\lfloor \frac{n}{4} \right\rfloor + 1$ occurrences of 1 in the second half of the first row and consequently we have $\left\lfloor \frac{n}{4} \right\rfloor + 1$ occurrences of 0 in the second half of its second row. Thus if we superimpose $F$ and $F'$, we get at...
least $2 \times \left\lceil \frac{n}{4} \right\rceil + 2 > n/2 = 2n/4$ ordered pairs of type $(0, 0)$. Which shows $F$ and $F'$ are not orthogonal. \hfill \Box

Lemma 6. \textit{Let }m/2 \text{ and } n/2 \text{ be odd, where } n \geq m > 2. \text{ Then there exists an array of type } I_2(m, n; 2).

\textbf{Proof.} \text{Let } m = 2l_1 \text{ and } n = 2l_2, \text{ where } l_1 \text{ and } l_2 \text{ are odd and } l_2 \geq l_1 > 1. \text{ Let } l_2 = l_1 + 2t. \text{ Now by Theorem 2 there exists an array of type } I_2(2l_1, 2l_1; 2). \text{ By Theorem 5 and Corollary 2 there exists an array of type } I_2(2l_1, 4t; 2). \text{ Thus by Lemma 1, there exists an array of type } I_2(2l_1, 2l_2; 2). \hfill \Box

Next we consider when $q = 6$. An array of type $I_2(6, 12; 6)$ exists by Theorem 3. We also exhibit an array of type $I_2(6, 18; 6)$:

\begin{center}
\begin{tabular}{cccccccccccccccc}
13 & 24 & 35 & 40 & 51 & 02 & 15 & 24 & 30 & 43 & 51 & 02 & 10 & 24 & 33 & 45 & 51 & 02 \\
34 & 43 & 01 & 52 & 20 & 15 & 34 & 45 & 01 & 52 & 23 & 10 & 34 & 40 & 01 & 52 & 25 & 13 \\
43 & 32 & 10 & 25 & 04 & 53 & 41 & 32 & 13 & 20 & 04 & 55 & 41 & 32 & 15 & 23 & 04 & 50 \\
22 & 11 & 54 & 03 & 45 & 30 & 22 & 11 & 54 & 05 & 40 & 33 & 22 & 11 & 54 & 00 & 43 & 35 \\
50 & 05 & 23 & 31 & 12 & 44 & 53 & 00 & 25 & 31 & 12 & 44 & 55 & 03 & 20 & 31 & 12 & 44 \\
04 & 50 & 42 & 14 & 33 & 21 & 00 & 53 & 42 & 14 & 35 & 21 & 03 & 55 & 42 & 14 & 30 & 21 \\
\end{tabular}
\end{center}

An array of type $I_2(6, 18; 6)$.

By Lemma 1, we thus obtain the following.

Lemma 7. \textit{There exists an array of type } I_2(6l_1, 6l_2; 6) \text{ if and only if } (l_1, l_2) \neq (1, 1).

5 \hspace{1cm} \textbf{THE CASE } k \geq 3

It now suffices to prove the case $k \geq 3$ to prove Theorem 4.

\textbf{Theorem 8.} \textit{Let } k \geq 3, q|m, q|n \text{ and } q^k|m n. \text{ Then there exist an array of type } I_k(m, n; q).

\textbf{Proof.} \text{Trivially, if an array of type } I_k(m, n; q) \text{ exists, then an array of type } I_\ell(m, n; q) \text{ exists for each } 1 \leq \ell < k. \text{ Thus we may assume that } k = \max\{t : q^t|mn\}. \text{ Let } mn = q^k b.

Consider the prime factorization of $q$:

$$q = p_0^{s_0} p_1^{s_1} \ldots p_{l-1}^{s_{l-1}}.$$

For each $r \in [l]$, let $i_r = \max\{t : q_r^t|m\}$ and $j_r = \max\{t : q_r^t|n\}$, where $q_r = p_r^{s_r}$. Thus $m$ and $n$ can be expressed as follows:

$$m = q_0^{i_0} \ldots q_{l-1}^{i_{l-1}} \quad p_0^{\alpha_0} \ldots p_{l-1}^{\alpha_{l-1}} \quad a_1,$$

$$n = q_0^{j_0} \ldots q_{l-1}^{j_{l-1}} \quad p_0^{\beta_0} \ldots p_{l-1}^{\beta_{l-1}} \quad a_2$$

with $\alpha_r, \beta_r < s_r$ and $p_r t a_1$ and $p_r t a_2$ for each $r \in [l]$. Now for any $c \in [l]$ we have the following two cases:
Case I: When $\alpha_c + \beta_c < s_c$.

In this case, $i_c + j_c$ is the largest power of $q_c$ which divides $mn$, and thus $i_c + j_c$ is the largest power of $q_c$ which divides $k$. By Theorem 5 and Corollary 3, if $(i_c, j_c, q_c) \neq (1, 1, 2)$, there exists an array of type $I_k(q_c^{i_c} p_c^{a_c}, q_c^{j_c} p_c^{b_c}; q_c)$, where $k = i_c + j_c$. However if $(i_c, j_c, q_c) = (1, 1, 2)$, then $s_c = 1, \alpha_c = \beta_c = 0$ and $2^3$ does not divide $mn$, contradicting $k \geq 3$.

Case II: When $\alpha_c + \beta_c \geq s_c$.

Since $\alpha_c, \beta_c < s_c$, this implies $\alpha_c + \beta_c < 2s_c$ and therefore $i_c + j_c + 1$ is the largest power of $q_c$ which divides $k$. By combining Theorems 5 and 7 we obtain an array of type $I_k(q_c^{i_c} p_c^{a_c}, q_c^{j_c} p_c^{b_c}; q_c)$ where $k = i_c + j_c + 1$.

Thus in both cases for each $c \in [l]$ we obtain an array of type $I_k(q_c^{i_c} p_c^{a_c}, q_c^{j_c} p_c^{b_c}; q_c)$ and by taking their Kronecker product (see Corollary 1), we can construct an array of type $I_k(m, n; q)$ where $a_1$ and $a_2$ are defined in Equation (4). Finally, by applying Corollary 2 we obtain an array of type $I_k(m, n; q)$, which completes the proof. □

The previous section and Theorem 8 together imply Theorem 4.

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