Cylindrically symmetric perfect-fluid universes

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Abstract. The aim of this paper is to examine some obtained exact solutions of the Einstein-Maxwell equations, especially their properties from a chronological point of view. Each our spacetime is stationary cylindrically symmetric and it is filled up with an perfect fluid that is electrically charged. There are two classes of solutions and examples of each of them are investigated. We give examples of the first class both for the vanishing as well as for the non-vanishing Lorentz force.

1. Introduction

Thanks to its remarkable properties the Gödel solution ([1], [2], [10]) became subject to several interesting generalizations. Banerji and Banerjee in [3] have found solutions of the Einstein-Maxwell equations for a charged perfect fluid. Also generalization to the case with or without the electromagnetic field and when non-constant component $g_{zz}$ or $g_{tt}$ of the metric tensor is admitted is possible ([7] and references cited therein).

In this paper we give a number of solutions of the Einstein-Maxwell equations with the vanishing cosmological constant which are generalizations of the metrics discussed in [3] and which involve electromagnetic field as well as the non-constancy of $g_{zz}$ or $g_{tt}$. These solutions result from two classes of exact solutions of the Einstein-Maxwell equations for a spacetime filled up with the charged perfect fluid. The subject of this work directly generalizes the paper given by Mitskievič and Tsalakou [4], who used the H-M conjecture ([5]), and therefore their designations are partly preserved.

The plan of the paper is as follows. In section 2 we obtain the Einstein-Maxwell equations for the stationary and cylindrically symmetric spacetime with vector potential (2.3). Through the whole paper we systematically use an orthonormal basis in which the stress-energy tensor of the perfect fluid is diagonal. In section 3 we get a general solution of the first class that corresponds to the non-constant component $g_{zz}$ of the metric tensor. In the case of vanishing Lorentz force we write down the inequalities implied by the energy conditions as well as the values of the Riemann tensor for this solution. Section 4 derives a second class corresponding to the non-constancy of $g_{tt}$ component of the metric tensor.
2. The Einstein-Maxwell perfect-fluid universes

In this paper, we consider a spacetime filled up with the charged perfect fluid of the pressure $p$ and the energy density $\mu$. Let us choose local coordinate system $x^\mu = (t, \varphi, z, r)$ of the comoving coordinates, where $\varphi$ is angular, $r$ radial and $z$ ordinary Cartesian coordinates. We search for the metric which is stationary and cylindrically symmetric and depends on the only coordinate $r$, so we will restrict our attention to the case when the spacetime admits Killing vector fields in remaining directions, i.e. in $t$, $\varphi$ and $z$ directions. Since we demand the metric to be invariant over simultaneous time reversion $t \rightarrow -t$ and reflection $\varphi \rightarrow -\varphi$, and over the inversion $z \rightarrow -z$, the spacetime metric tensor has the form

$$ds^2 = e^{2\alpha}(dt + f\, d\varphi)^2 - l^2 \, d\varphi^2 - e^{2\gamma} \, dz^2 - e^{2\delta} \, dr^2 ,$$

where all metric functions as well as the pressure and the charge density depend only on the coordinate $r$. The function $\delta$ can achieve any value. The velocity vector is given by $u = e^{-\alpha} \partial_t$ and the acceleration of the particles of the fluid is

$$\dot{u} = \nabla_u u = \alpha' e^{-2\delta} \partial_3 ,$$

where comma means derivative with respect to $r$. The expansion tensor as well the shear tensor of the spacetime with the metric (2.1) is vanishing

$$\Theta_{\mu\nu} = \frac{1}{2} \partial_{\mu} h_{\nu\sigma} - \frac{1}{2} \partial_{\nu} h_{\mu\sigma} - \dot{u}(\partial_{\mu} u_{\nu}) = 0 .$$

Finally, let the one-form of the vector potential be expressed like (see [11])

$$\Lambda = m(r) \, d\varphi + n(r) \, dt .$$

Let us introduce an orthonormal basis with the basic one-forms as follows

$$\Theta^0 = e^\alpha (dt + f\, d\varphi) , \quad \Theta^1 = l\, d\varphi , \quad \Theta^2 = e^\gamma dz , \quad \Theta^3 = e^\delta dr .$$

Using (2.3) for mixed components of the two-form of the electromagnetic field one gets

$$F_{\mu\nu} = l^{-2} \delta^3 \left\{ \left[ n' l'^2 e^{-2\alpha} + (m' - f n') f \right] \delta_{\nu}^0 - (m' - f n') \delta_{\nu}^1 + e^{-2\delta} \delta_{\nu}^3 \left[ (n' \delta_{\mu}^0 + m' \delta_{\mu}^3) \right] \right\} \right) ,$$

The Maxwell equations with sources $\delta F = 4\pi j$ give

$$l^{-1} e^{-\alpha - \gamma - \delta} \left\{ e^{\alpha + \gamma - \delta} \left[ n' l'^2 e^{-2\alpha} + (m' - f n') f \right] \delta_{\nu}^0 - (m' - f n') \delta_{\nu}^1 \right\} \right) ,$$

where $j(r)$ is the current density, $\dot{\delta} j = 0$. We postulate that the fluid particles carry the charge $j = \rho u$, $\rho$ is the charge density. It will be accomplished if

$$(m' - f n') e^{\alpha + \gamma - \delta} = Bl ,$$

$B$ is constant. The equations (2.6) and (2.7) result in the following expression for the charge density

$$4\pi \rho = -\frac{B^2}{m' - f n'} \frac{d}{dr} \left( \frac{l^2 n' e^{-2\alpha}}{m' - f n'} + f \right) e^{-\alpha - 2\gamma} .$$

The choice (2.3) corresponds to the formulae for the electric and the magnetic fields (1)

$$E_z = n' e^{-\alpha - \delta} , \quad B_z = Be^{-\alpha - \gamma} .$$

§ The coordinates will be numbered $(t, \varphi, z, r) = (0, 1, 2, 3)$.

¶ We use the units with the speed of light $c$ and the Newtonian gravity constant $G$ equaled to one.
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With the help of the constraint (2.7) the vorticity one-form is given by

$$\omega = \frac{1}{2} (u \wedge du) = \frac{Bf'}{2(m' - fn')} \, dz.$$  

The total stress-energy tensor of the spacetime $T_{\text{total}}$ arises as a sum of the stress-energy tensor of the electromagnetic field

$$T_{\text{elmag}} = (8\pi)^{-1} l^{-2} e^{-2\delta} \left\{ 2n'(m' - fn') \left( e^{-\alpha}(\Theta^0 \otimes \Theta^1) + \Theta^1 \otimes \Theta^0 \right) + [n'^2 l^2 e^{-2\alpha} + (m' - fn')^2] \Theta^0 \otimes \Theta^0 + [n'^2 l^2 e^{-2\alpha} + (m' - fn')^2] \Theta^1 \otimes \Theta^1 + [n'^2 l^2 e^{-2\alpha} - (m' - fn')^2] \Theta^2 \otimes \Theta^2 - [n'^2 l^2 e^{-2\alpha} - (m' - fn')^2] \Theta^3 \otimes \Theta^3 \right\}$$  

and the stress-energy tensor of the perfect fluid

$$T_{\text{fluid}} = \mu \Theta^0 \otimes \Theta^0 + p \left( \Theta^1 \otimes \Theta^1 + \Theta^2 \otimes \Theta^2 + \Theta^3 \otimes \Theta^3 \right).$$  

The Bianchi identities reduce to the single equation of the motion of the fluid

$$p' + \alpha' (p + \mu) + \rho m' = 0.$$  

The non-zero components of the Einstein tensor in the orthonormal basis (2.4) can be written in the form

$$G_{\hat{t}\hat{r}} = \frac{1}{2} e^{-2\alpha - \gamma - \delta} \frac{d}{dr} \left( l^{-1} f'e^{2\sigma} e^{\alpha + \gamma - \delta} \right),$$  

$$G_{\hat{t}\hat{r}} = l e^{-\alpha - \gamma - \delta} \frac{d}{dr} \left( l^{-1} (\alpha + \gamma) e^{\alpha + \gamma - \delta} - 2\alpha' e^{2\delta} \right),$$  

$$G_{\hat{t}\hat{t}} + G_{\hat{z}\hat{z}} = \frac{1}{2} l^{-1} e^{-\alpha - \gamma - \delta} \frac{d}{dr} \left[ l^{-1} e^{-\alpha + \gamma - \delta} \frac{d}{dr} (l^2 e^{2\alpha}) \right],$$  

$$G_{\hat{t}\hat{r}} - G_{\hat{t}\hat{r}} = \frac{1}{2} l^{-1} e^{-\alpha - \gamma - \delta} \frac{d}{dr} \left[ l^{-1} e^{-\alpha - \gamma - \delta} \frac{d}{dr} (l^2 e^{2\alpha}) \right] + \frac{1}{2} l^{-2} f'^2 e^{2(\alpha - \delta)},$$  

$$G_{\hat{t}\hat{t}} = (\alpha' l' + l' \gamma' + l' \alpha') l^{-1} e^{-2\delta} + \frac{1}{4} l^{-2} f'^2 e^{2(\alpha - \delta)},$$

and the Einstein-Maxwell equations $G_{\hat{\mu}\hat{\nu}} = 8\pi T_{\hat{\mu}\hat{\nu}}$ are

$$\frac{d}{dr} \left( \frac{e^{2\alpha} f'}{m' - fn'} \right) = 4n',$$  

$$\frac{d}{dr} \left( \frac{e^{2\sigma} (\alpha + \gamma)'}{m' - fn'} - \frac{2\alpha' \gamma'}{m' - fn'} \right) = \frac{2n'^2 l^2 e^{-2\alpha}}{m' - fn'},$$  

$$\frac{1}{2} \frac{d}{dr} \left( \frac{e^{-2\alpha}}{m' - fn'} \frac{d}{dr} (l^2 e^{2\alpha}) \right) = \frac{16\pi l^2 p e^{2\delta}}{m' - fn'},$$  

$$\frac{1}{2} \frac{d}{dr} \left( \frac{e^{-2\gamma}}{m' - fn'} \frac{d}{dr} (l^2 e^{2\gamma}) \right) = \frac{8\pi (p - \mu)}{m' - fn'} l^2 e^{2\delta} + \frac{f'^2 e^{2\alpha}}{2 m' - fn'} - \frac{2n'^2 l^2 e^{-2\alpha}}{m' - fn'},$$  

$$(\alpha' + \gamma') \frac{dl'^2}{dr} + 2f' l'^2 = 16\pi l^2 p e^{2\delta} - \frac{1}{2} f'^2 e^{2\alpha} - 2n'^2 l^2 e^{-2\alpha} + 2(m' - fn')^2.$$  

In summary, we obtained the system of six equations (2.13a)-(2.13c) and (2.6), minus (2.13d), which is an integral of the (2.12). After eliminating $p$ from (2.13a)-(2.13c) with the help of the (2.13d) one gets the system of six equations for totally nine unknown functions $\alpha, \gamma, \delta, f, l, \mu, p, m$ and $n$. It means that three out of nine functions can be chosen arbitrarily. These three degrees of freedom are equivalent to introducing $\delta, \mu$ and $\mu(p)$. In the next we will consider the $m$ and $n$ fixed, and moreover we impose, as in (2.6), additional condition: either $\alpha$ or $\gamma$ are constant, which is in accordance with two classes of solutions.
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3. First class – α is constant

When α = const the Einstein–Maxwell equations (2.13) can be easily integrated in terms of the electric and the magnetic potential m and n. Equation (2.12) now says that the Lorentz force is balanced by the pressure gradient, \( \text{grad} p + \rho E = 0 \), resulting in the geodesic motion of the fluid particles. We can, without loss of generality, suppose that \( \alpha \) is zero. The metric components are

\[
f e^2 n^2 = 4 \int m' n e^2 n^2 \, dr + F, \tag{3.1a}
\]

\[
\gamma = \int \left[ \int \frac{2n'^2}{m' - fn'} \, dr + C \right] (m' - fn') \, dr, \tag{3.1b}
\]

\[
l^2 e^{-2\gamma} = E - 4 \int (m - fn + k) (m' - fn') e^{-2\gamma} \, dr. \tag{3.1c}
\]

In (3.1a) \( C, E, F, k \) are constants of integration. Inserting (3.1a) into (2.13e) and (2.13d) yields for the pressure and the energy density

\[
8\pi p = B^2 \left[ 4n^2 + \frac{\gamma'^2 + n'^2}{(m' - fn')^2} - 2\gamma \frac{m - fn + k}{m' - fn'} - 1 \right] e^{-2\gamma},
\]

\[
8\pi \mu = B^2 \left[ 4n^2 - \frac{3\gamma'^2 + 5n'^2}{(m' - fn')^2} + 6\gamma \frac{m - fn + k}{m' - fn'} + 1 \right] e^{-2\gamma}. \tag{3.2}
\]

3.1. Example of the first class with non-vanishing Lorentz force

We give here explicitly the solution when \( m = \tau n + \frac{2}{3} \beta n^3 \), \( \tau \) and \( \beta \) being constants, and \( k, F \) and \( C \) in (3.1a)-(3.1c) are zero. In this case (according to the (2.5) the Lorentz force does not vanish) the metric can be written like this

\[
ds^2 = \left[ dt + (2\beta n^2 + \tau - \beta) \, d\varphi \right]^2 - \frac{\beta^2}{3} \left( 3E\beta^{-2} e^{2n^2} - 4n^2 + 1 \right) \, dr^2
\]

\[- e^{2n^2} \, dz^2 - \frac{1}{B^2} \frac{3n'^2 e^{2n^2}}{3E\beta^{-2} e^{2n^2} - 4n^2 + 1} \, dr^2, \tag{3.3}
\]

and for the charge density (2.8), for the the pressure and the energy density (3.2) one has

\[
8\pi p = B^2 \left[ \frac{E}{\beta^2} (4n^2 + 1) - \frac{2}{3} e^{-2n^2} \right], \quad \pi \rho = -B^2 n \left( \frac{E}{\beta^2} + \frac{1}{3} e^{-2n^2} \right),
\]

\[
8\pi \mu = B^2 \left\{ \left[ \frac{56}{3} n^2 - \frac{2}{3} \right] e^{-2n^2} - \frac{E}{\beta^2} (12n^2 + 5) \right\}. \tag{3.4}
\]

3.2. Solution with the vanishing Lorentz force

Metric given by (3.1a)-(3.1d) is not too transparent. In the rest of this section we restrict ourselves to purely magnetic field when the electric potential n is constant, which will be marked as \( 1 \), i.e. when according to (2.5) or (2.7) the Lorentz force \( F'\mu u' \) vanishes. The result written in terms of the magnetic potential m is
Einstein equations in our convention are

\[ G_{\mu\nu} = 8\pi T_{\mu\nu} \]

solution describing charged dust in the spacetime with the cosmological constant \(\Lambda = \pm 1\). Thanks to the homogeneity of the pressure \(3.6\), the solution \(3.5\) can be reinterpreted also as an electromagnetic field variable by rescaling the coordinate \(r\).

In fact, there is another way of deriving locally the same metric as \(3.5\) that introduces new variable by rescaling the coordinate \(r\) by the definition \(m'(r) = m\). Then for the two form of the electromagnetic field \(F = dm \wedge d\varphi\). In effect, this substitution results in inserting unity except \(m'\).

Our fluid moves geodesically, without shear and expansion \(7\). For \(m\) being increasing or decreasing function defined on \((0, \infty)\) and covering the whole range \((0, \infty)\), it can be treated as a new radial variable and one obtains the first family of the solutions in \(\m 4\), if we interpret the coordinate \(\varphi\) as an ordinary Cartesian coordinate (i.e. one abandons the periodicity of the \(\varphi\) coordinate). The same will be true in the section \(\m 4\). The solution without the electromagnetic field \((B = 0)\) was recovered by Wright \((6)\).

3.2.1. Energy conditions. Dominant energy condition implies inequalities

\[
\begin{align*}
(b^2 - 2)e^{-2Cm} - 2C^2E & \geq 0, \\
(b^2 - 2)e^{-2Cm} - 4C^2E & \geq 0.
\end{align*}
\]

In our case the strong energy condition is satisfied if the dominant energy condition is. Altogether the energy conditions will be fulfilled if \((3.7a)\) or \((3.7b)\) is satisfied depending on whether \(E\) is negative or positive respectively, or \(b^2 \geq 2\), if \(E\) is zero.

3.2.2. Curvature. For the future convenience we mention the concrete values of the Riemann curvature tensor for the metric \((3.5)\). All non-vanishing components of the Riemann tensor are given by

\[
\begin{align*}
R_{\tilde{t}\tilde{z}\tilde{z}} &= R_{\tilde{\varphi}\tilde{\varphi}\tilde{z}} = \frac{1}{2} B^2 e^{-2\alpha} \left( \lambda C e^{-2Cm} + 2C^2E \right), \\
R_{\tilde{r}\tilde{t}\tilde{r}} &= R_{\tilde{t}\tilde{r}\tilde{t}} = -\frac{1}{4} B^2 b^2 e^{-4\alpha} e^{-2Cm}, \\
R_{\tilde{r}\tilde{\varphi}\tilde{r}} &= R_{\tilde{\varphi}\tilde{r}\tilde{r}} = \frac{1}{2} B^2 b C e^{-3\alpha} (E e^{2Cm} + \lambda m + \nu) \frac{1}{2} e^{-2Cm}, \\
R_{\tilde{\varphi}\tilde{\varphi}\tilde{r}} &= R_{\tilde{r}\tilde{\varphi}\tilde{\varphi}} = -\frac{1}{4} B^2 e^{-2\alpha} (3b^2 e^{-2\alpha} + 2\lambda C) e^{-2Cm} + B^2 C^2 E e^{-2\alpha}.
\end{align*}
\]

In fact, there is another way of deriving locally the same metric as \(3.5\) that introduces new variable by rescaling the coordinate \(r\) by the definition \(m'(r) = m\). Then for the two form of the electromagnetic field \(F = dm \wedge d\varphi\). In effect, this substitution results in inserting unity except \(m'\).

\(\dagger\) Thanks to the homogeneity of the pressure \(3.4\), the solution \(3.5\) can be reinterpreted also as a solution describing charged dust in the spacetime with the cosmological constant \(\Lambda = -B^2 C^2 E\) (the Einstein equations in our convention are \(G - \Lambda g = 8\pi T\)).
3.2.3. Closed timelike curves. For the analysis of potential chronology violation we use the Carter theorem (3.8, 3.9). In our case (3.3) the isometry group, generated by the Killing vector fields $\partial_t$, $\partial_\varphi$, $\partial_z$, is Abelian and timewise orthogonally transitive, since the hypersurfaces of transitivity given by constant $r$ are everywhere timelike, $g^{rr} < 0$. The only exception could be the case $C = 0$, but then the chronological structure follows from the continuity and we will meet such an example in subsection 3.2.4. Chronology will be preserved when one is able to find such constants $p, q, s$ that the differential form $\psi = p\,dt + q\,d\varphi + s\,dz$ is everywhere timelike or null. Since the contribution of $s$ is always negative, it can be treated as zero.

3.2.4. Example of the first class with vanishing Lorentz force. Let the vector potential be $A = 2a^2B\sh^2(r/2a)\,d\varphi$. Here $a$ is a length characteristic, $B$ is according to (2.9) a value of the magnetic field on the rotation axis. The choice of the constant of integration $E = -\nu = (2a^2B^2C^2)^{-1}$, $b^2 = 4 - 4CB^{-1} + 2(aB)^{-2}$ yields the metric

$$
\begin{align*}
\frac{ds^2}{e^{2Cm}} & = \left[ \frac{dt + 2a(4a^2B^2 + 2 - 2C)\,d\varphi}{e^{2Cm} - 2C(1 - C)m - 1} \right]^2 - \frac{2a^2C}{C^2} \left[ e^{2Cm} - 2C(1 - C)m - 1 \right] d\varphi^2 \\
& \quad - e^{2Cm} \, dz^2 - \frac{2a^2C^2m^2e^{2Cm}}{e^{2Cm} - 2C(1 - C)m - 1} \, d\varphi^2,
\end{align*}
$$

(3.9)

where, for convenience, we have denoted $2a^2BC \to C$ and $m = \sh^2(r/2a)$. For the physical quantities $\mu$, $p$ and $\rho$ one obtains

$$
\begin{align*}
8\pi\mu & = 2 \left( B^2 + \frac{1 - C}{a^2} \right) e^{-2Cm} - \frac{3}{2a^2}, \\
2\pi\rho & = -B \left( B^2 + \frac{1 - C}{2a^2} \right) \frac{1}{2} e^{-2Cm}, \quad 8\pi p = \frac{1}{2a^2}.
\end{align*}
$$

(3.10)

Signature in (3.9) is always correct and the energy condition (3.7) requires $C \leq 0$. From equations (3.8) we can see that for negative $C$ the physical singularity occurs when $r \to \infty$. From (3.9) one can also see that no event horizon is present for any finite value of $r$. In the limit $C = 0$ we get Banerji-Banerjee solution (formulae (11) and (12) in the alternative interpretation) in which is spacetime homogeneous and singularity free. So we can interpret $C$ as an indicator of the difference of the spacetime from the spacetime homogeneity.

In fact, the choice $m = \sh^2(r/2a)$ represents the family of models in which $m$ is increasing or decreasing function with domain $(0, \infty)$ and image $(0, \infty)$. Each such choice after the introducing of the new variable $m = m(r)$ leads to the (3.9). From this point of view it is obvious that for example the choice $m \propto r^{-1}$ has the same physical content as $m = \sh^2(r/2a)$. The same remark will be true in subsection 3.2.3.

The condition $\psi_\alpha\psi^\alpha \geq 0$ of the subsubsection 3.2.3 reads

$$
\frac{1}{2C^2} [e^{2Cm} - 2C(1 - C)m - 1] - \left[ (4a^2B^2 + 2 - 2C)\frac{1}{2} m - \frac{q}{2\alpha p} \right]^2 \geq 0.
$$

(3.11)

Because $q$ must be zero in order not to predominate in (3.11) for small $r$, the condition for the non-existence of the closed timelike curves (hereafter CTC) has the familiar form $g_{\varphi\varphi} \leq 0$ (or $g_{\varphi\varphi} < 0$ for the non-existence of the closed causal curves), and we see that CTC will appear always for sufficiently large $r$.

* With respect to the previous footnote, $a^2 = -(2\Lambda)^{-1}$ holds in the alternative reinterpretation.
Table 1. Numerical dates that relate the quantity of the magnetic field $B$ on the rotation axis to the radius $R$ of the first null circle $(t, z, r = \text{const})$, and to the cosmological constant $\Lambda$ in the alternative interpretation. For simplicity, dates are given only for $C = 0$. The matter density used is $10^{-26} \text{kg} \cdot \text{m}^{-3}$, which corresponds to the period of the rotation $7 \cdot 10^{10}$ years.

| $B$ $[10^4 \text{Gauss}]$ | 0,0 | 3,7 | 4,8 | 6,7 | 8,2 | 10,5 |
|--------------------------|-----|-----|-----|-----|-----|------|
| $\Lambda$ $[10^{-53} \text{m}^{-2}]$ | -9,31 | -8,20 | -7,45 | -5,59 | -3,73 | -0,28 |
| $R$ $[10^9 \text{ly}]$ | 137 | 132 | 129 | 123 | 118 | 110 |

Thanks to absence of any event horizons, for any two spacetime points $p$ and $q$, one has $p \gg q$ and simultaneously $q \gg p$ (10). Especially $I^+(p) = I^-(p) = J^+(p) = J^-(p) = M$. Although through every point passes some CTC, each such curve must inevitable cross the region $r > R$, where $R$ is determined as a root of (3.11). The CTC we are interested in are nontrivial (8). Moreover, because the Carter theorem can be applied independently on whether or not $\varphi$ is periodic, one can see that CTC occur even if we interpret $\varphi$ as the ordinary Cartesian coordinate. (The chronology could be preserved in the case that we took only part of the (3.9) confined to the region $r < R$ and tried to match it to some other chronologically well behaved solution.)

The magnetic field has always positive influence on the chronology violation, precisely the larger value of the magnetic field on the rotation axis the closer to it will be the "first" null circle followed by CTC (see table 1).

Similarly, another Banerjee-Banerji solution (expression (15) in 3) or the Som and Raychaudhuri solution [11] can be got by suitable choice of the constant of integration.

4. Second class of the solutions – $\gamma$ is constant

Solution with constant $\gamma$ will be refered as second class and here we restricted ourselves only to the case when the Lorentz force vanishes, with the electric potential $n = b/4$. Equation (3.1c) remains valid in this case too. $\gamma$ can be put to zero and the solution of the Einstein-Maxwell equations (2.13a)-(2.13d) and (2.6) can be written in the form

\[
\begin{align*}
f &= -\frac{b}{2C} e^{-2Cm} + H , \quad e^{2\alpha} = e^{2Cm} , \\
l^2 &= \frac{b^2}{4C^2} e^{-2Cm} - 2m^2 + Dm + E ,
\end{align*}
\]  

(4.1)

with constants of integration $C$, $H$ and $D = Hb - 4k$. For the energy density, the pressure and the charge density one has

\[
\begin{align*}
16\pi \mu &= B^2 e^{-2Cm} [CD + 2(1 - 2Cm)] , \\
16\pi p &= B^2 e^{-2Cm} [CD - 2(1 + 2Cm)] , \\
4\pi \rho &= -B^2 be^{-3Cm} .
\end{align*}
\]  

(4.2)

Thanks to inhomogeneity of the pressure (4.2), the fluid already does not move along geodesic lines. Its acceleration (2.2) is

\[
\nabla_\alpha u = C m' e^{-2\delta} \partial_3 .
\]

In contrary with the corresponding solution of the first class (3.9), because of the inhomogeneity of the pressure, the metrics of the second class cannot be reinterpreted as describing dust with the non-zero cosmological constant.
4.1. Energy conditions

In the case of the second class (4.1) and (4.2) the energy conditions altogether will be fulfilled if following inequality will hold

\[ CD \geq 4Cm. \] (4.3)

4.2. Curvature

The non-zero components of the Riemann tensor are given by

\[ R_{\hat{t}\hat{r}\hat{t}\hat{r}} = B^2 C e^{-2\alpha} \left( 2m - \frac{D}{2} \right) e^{-2Cm}, \]
\[ R_{\hat{t}\hat{r}\hat{\phi}\hat{\phi}} = B^2 e^{-2\alpha} \left[ C \left( 2m - \frac{D}{2} \right) - 2 \right] e^{-2Cm}. \] (4.4)

4.3. Example of the second class of the solutions

This special solution corresponds to that discussed in 3.2.4 in the sense that when \( C \) goes to zero, we obtain the same solution as in 3.2.4 when \( C \) goes to zero, i.e. (11) and (12) in [3]. For let us choose \( A = 2a^2 B \text{sh}^2(\pi/2a) \right_\hat{\varphi} \) and constants of integration \( b^2 = 4 + 2(aB)^{-2}, D = 2B^{-1} + 2C^{-1} + (a^2B^2C)^{-1}, E = -b^2(2C)^{-2}, H = b(2C)^{-1} \). Components of the metric tensor read (denoting \( 2a^2BC \to C \) and \( m = \text{sh}^2(\pi/2a) \))

\[ e^{2\alpha} = e^{2Cm}, \quad f = a \left( \frac{4a^2B^2 + 1}{C} \right)^{\frac{1}{2}} (1 - e^{-2Cm}), \]
\[ l^2 = 4a^2 \left[ \frac{2a^2B^2 + 1}{2C^2} e^{-2Cm} - 2a^2B^2m^2 + \frac{(2a^2B^2 + 1 + C)m}{C} - \frac{2a^2B^2 + 1}{2C^2} \right]. \] (4.5)

The energy density, the pressure and the charge density are given respectively by

\[ 8\pi\mu = \left( 2B^2 + \frac{1+C}{2a^2} - 2B^2Cm \right) e^{-2Cm}, \]
\[ 8\pi p = \left( \frac{1+C}{2a^2} - 2B^2Cm \right) e^{-2Cm}, \]
\[ 2\pi\rho = -B \left( B^2 + \frac{1}{2a^2} \right)^{\frac{1}{2}} e^{-3Cm}. \] (4.6)

It can be seen from the form of metrics (4.3) that the signature is correct for \( C \leq 0 \) and, in case when \( B \) is zero, then for every \( C \). The absence of the event horizons for any finite value of \( r \) is apparent from the form of the metric (4.5). From analysis of the energy conditions (4.3) it follows that these will be fulfilled for intervals \(- (2a^2B^2 + 1) \leq C \leq 0 \), and if the electromagnetic field vanishes for \( C \geq -1 \). One can convince oneself that CTC will occur if \(- (2a^2B^2 + 1) \leq C \leq 0 \) when \( B \neq 0 \), and \( C \geq -1 \) when \( B = 0 \). Every two points of the spacetime can be connected each other by both future and past directed timelike curve which is non-trivial, so that the spacetime is totally vicious ([8]). Formulae (4.4) show that the physical singularity occurs for \( C < 0 \) when \( r \) tends to the infinity. The only exception is the case with the vanishing electromagnetic field (\( B = 0 \)) and \( C = -1 \), which is after transformation \( e^{-2m} = 1 - \omega^2u^2 \), with \( \omega^{-2} = 2a^2 \), the Minkowski spacetime in the rotating cylindrical coordinates \( \varphi \rightarrow \varphi - \omega t \).
Cylindrically symmetric perfect-fluid universes

5. Conclusion

From the requirement of the stationarity and cylindrical symmetry and from the assumption that the spacetime is filled up with a perfect fluid that is charged, we have obtained the system of ordinary differential equations (2.13a)-(2.13d) plus (2.6), that was solved in terms of the functions $m$ and $n$ in two cases: the first class (3.1a)-(3.2), with constant $\alpha$ and the second class (4.1), (4.2) with constant $\gamma$ (but we gave the solution of the second class only for the vanishing Lorentz force). These two general classes are both shear-free. Fluid of the first class moves geodesically and its velocity is the Killing vector field. It is not true in the second class thanks to the inhomogeneity of the pressure (4.3). We shown explicit special solution of the first class for the non-vanishing Lorentz force (formulae (3.3) and (3.4)), but without detailed analysis, and in some details we discussed the example of the first class (subsubsection 3.2.3) and the corresponding example of the second class (subsection 4.3) with the vanishing Lorentz force, that are generalizations to the solution given in [3]. These metrics with three parameters (the length characteristic $a$, the value of the magnetic field on the rotation axis $B$, and $C$ determining the difference of the solutions from the homogeneity) contain CTC at least for some interval of the values of $C$. It turns out that the magnetic field has positive influence on the appearance of the non-trivial chronology violation (in fact by sufficiently large magnetic field we can always ensure the chronology violating).

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References

[1] Gödel K 1949 Rev. Mod. Phys. 21 447
[2] Kramer D, Stephani H, MacCallum M A H and Hertl E 1980 Exact solutions of Einstein’s field equations VEB DAW, Berlin
[3] Banerjee A and Banerji S 1968 J. Phys. A (Proc. Roy. Soc.) 1 188
[4] Mitskievič N V and Tsalakou G A 1991 Class. Quantum Grav. 8 209
[5] Horský J and Mitskievič N V 1989 Czech. J. Phys. B 39 957
[6] Wright J P 1965 J. Math. Phys. 6 103
[7] Krasiński A 1999 in On Einstein’s path ed. Harvey A (Berlin: Springer Verlag)
[8] Carter B 1968 Phys. Rev. 174 1559
[9] Tipler J F 1974 Phys. Rev. D 9 2203
[10] Hawking S W and Ellis G F R 1973 The Large Scale Structure of the Spacetime, (Cambridge: Cambridge University Press)
[11] Bonnor W B 1980 J. Phys. A: Math. Gen. 13 3465