NETWORK OF TANDEM AND BI-TANDEM QUEUEING PROCESS WITH RENEGING AND JOCKEYING

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Abstract - The steady state behaviour of a queueing model where two bi-tandem channels are linked in tandem with a common channel has been studied using the concept of reneging and jockeying.

Keywords - Reneging, Jockeying, Bi-tandem channels.

1. INTRODUCTION

O'Brien [1], Jackson [2], Suzuki [3], Maggu [4] etc. studied the model of tandem (series) queues. In 1957 the concept of reneging in queueing system was introduced by Barrer [5]. Further this concept has been discussed in different ways by many researchers as Haight [6] Blackburn [7] etc. in different models.

The concept of jockeying was first discussed by Glazer. Then using this concept in different models many researchers discussed as Keonigsberg [8], Disney and Mitchell [9] etc. The network of queues was studied by Finch [10], Kelly [11], Melamed [12] and recently Chandramouli [13] discussed a model in which two bi-tandem channels are linked with a common channel taking the concept of non-linear service growth rate. In the present paper the concept of reneging and jockeying has been introduced in this model when the service rates do not depend upon the queue length. The steady behaviour of this model has been discussed. The practical situation corresponding to this model can be realized in banks or in a publishing company etc. For example, consider a publishing company which has three types of machines say $S_1$, $S_2$ and $S_3$. Let $S_1$ print the matter in red ink and $S_2$ in blue ink. and $S_3$ denote the binding machine. We suppose that the arrivals (matters for printing) are printed in two colours (red or blue) and finally go to the binding process at $S_3$. It has also been assumed that the binding machine $S_3$ undertakes outside printed matter for binding. Now in this situation, the reneging and jockeying at the arrivals may also be observed.

2. FORMULATION AND SOLUTION

Let $S_1$, $S_2$ and $S_3$ denote the three service channels in which it is supposed that $S_1 \rightarrow S_2$, that is $S_1$ and $S_2$ are in bi-tandem and $S_1 \rightarrow S_3$ or $S_2 \rightarrow S_3$, that is, each is further linked in tandem with $S_3$. An arriving unit for service at either $S_1$ or $S_2$ may follow one of the following routes for terminal services:

$S_1 \rightarrow S_2 \rightarrow S_3$ or $S_2 \rightarrow S_1 \rightarrow S_3$.

This unit which arrives directly at $S_3$ departs from the system after servicing at $S_3$. Let $Q_1$, $Q_2$ and $Q_3$ be waiting line formed before $S_1$, $S_2$ and $S_3$. If they are busy. It
has been supposed that an arriving unit after intolerable waiting time in the queue $Q_i$ or $Q_2$ may renege (leave) at $S_i$ or $S_2$ without service. Also it has been assumed that units may jockey (move) from $Q_1 \rightarrow Q_2$ or from $Q_2 \rightarrow Q_1$ for personal economic gains.

Let $\lambda_i$ denote the Possion mean rate of arrivals at $Q_i$ before $S_i$ ($i = 1, 2, 3$), we assume that the input source is infinite. Let $\mu_i$ denote the Possion mean departure rates at $S_i$. Also let $b_r$ denote the constant rate of reneging from queues $Q_r$ ($r = 1, 2$).

Further, let $J_{ir}$ ($i \neq r, i, r = 1, 2$) denote the constant rates of jockeying from $Q_i \rightarrow Q_r$. Let $p_{12}$ and $p_{13}$ denote the probabilities that a unit after service at $S_1$ departs to join the respective queues $Q_2$ and $Q_3$. Again let $p_{21}$ and $p_{23}$ denote the probabilities that a unit after service at $S_2$ join the respective queues $Q_1$ and $Q_3$, where $p_{ij} \geq 0$ ($i \neq j, i = 1, 2, j = 1, 2, 3$) and $p_{12} + p_{13} = 1$, $p_{21} + p_{23} = 1$. Let $P(k, m, n)$ denote the steady-state probability that there are waiting $k$ units in $Q_1$, $m$ units in $Q_2$ and $n$ units in $Q_3$. Each queue includes service also and $k, m, n \leq 0$.

![Figure:- Queue model with Reneging and Jockeying](image)

For steady state situation the following difference equation exists for $k, m, n > 0$

$$\left( \lambda_1 + \lambda_2 + \lambda_3 + \mu_i + \mu_2 + \mu_3 + b_1 + b_2 + J_{12} + J_{21} \right) P(k, m, n) = \lambda_1 P(k-1, m, n) + \lambda_2 P(k, m-1, n) + \lambda_3 P(k, m, n-1) + b_1 (k+1, m, n) + b_2 (k, m+1, n) + J_{12} P(k+1, m-1, n) + J_{21} P(k-1, m+1, n) + \mu_i P(k, m, n-1) + \mu_2 P(k-1, m+1, n) + \mu_3 P(k, m+1, n-1) + \mu_i P(k, m, n-1)$$

for $k, m, n > 0$ ..................................(2.1)

If one of $k, m, n$ is zero and other two are non-zero e.g. $k = 0$, $m, n > 0$ then in this case for $P(0, m, n)$, $b_1 = 0 = \mu_i = J_{12}$ and negative of $P(k, m, n)$ is zero. Substituting these value is (1) we get the equation. Similarly for $m = 0$, $n > 0$ and also for $n = 0$, $k m > 0$. We get three equations in this manner. Again, if two of $k, m, n$ are zero and other one is non-zero e.g. $k = 0 = m$, $n > 0$ then in this case for $P(0, 0, n)$, $b_1 = 0 = b_2 = J_{12} = J_{21} = \mu_i = \mu_2$ and negative of $P(k, m, n)$ is zero. Substituting these values in (1) we get the equation. Similarly, for $k = 0 = n$, $m > 0$ and also $m = 0 = n$, $k > 0$. We get three equations in this manner also.
Again, if \( k, m, n \) are also zero. Then in this case for \( P(0, 0, 0) \),
\( b_1 = b_2 = b_3 = \mu_1 = \mu_2 = J_{12} = J_{21} \) and negative of \( P(k, m, n) \) in zero and substitute
these values in (1) we get one equation.

Hence, the above set of eight difference equations govern the model in steady
state situation.

To solve the above set of difference equations we use the generating function
technique and similar steps as Chandramouli [14] has taken in his paper. Now, define
the generating function as

\[
F(x, y, z) = \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} P(k, m, n) x^k y^m z^n \quad \text{Where} \quad |x| \leq 1 \text{ and } |z| \leq 1 \quad \text{(2.2)}
\]

Using the following partial generating functions and simplification

\[
F_{m,n}(x) = \sum_{k=0}^{\infty} P(k, m, n) x^k
\]

\[
G_{k,n}(y) = \sum_{m=0}^{\infty} P(k, m, n) y^m \quad \text{(2.3)}
\]

\[
I_k(x, y) = \sum_{m=0}^{\infty} F_{m,n}(x) y^m
\]

\[
A_k(y, z) = \sum_{n=0}^{\infty} G_{k,n}(y) z^n \quad \text{(2.4)}
\]

\[
B_m(x, z) = \sum_{n=0}^{\infty} F_{m,n}(x) z^n
\]

We get the following equation : \( F(x, y, z) = \)

\[
\left[ \left( \mu_1 \left( 1 - \frac{z}{x} p_{13} - \frac{z}{x} p_{12} \right) + J_{12} \left( 1 - \frac{y}{x} \right) \right) A_0(y, z) + \mu_2 \left( 1 - \frac{z}{y} p_{23} - \frac{x}{y} p_{21} \right) + b_1 \left( 1 - \frac{1}{x} \right) + J_{21} \left( 1 - \frac{y}{x} \right) \right] B_0(x, z) + \mu_0 \left( 1 - \frac{1}{z} \right) I_0(x, y)
\]

\[
+ \mu_1 \left( 1 - \frac{z}{x} p_{13} - \frac{y}{z} p_{12} \right) + \mu_2 \left( 1 - \frac{z}{y} p_{23} - \frac{x}{y} p_{21} \right) + \mu_3 \left( 1 - \frac{1}{z} \right) \quad \text{(2.5)}
\]

Using L’ Hospital’s rule for indeterminate form \% , and using \( F(x, 1, 1) = 1 \) as
\( x \to 1 \) and similarly other also we have the following set of equations :

\[
1 = \frac{\mu_1 p_{13} A_0(1, 1) + \mu_2 p_{23} B_0(1, 1) + \mu_1 I_0(1, 1)}{\lambda_3 + \mu_1 p_{13} + \mu_2 p_{23} - \mu_3} \quad \text{(2.6)}
\]
1 \(= \frac{\left(\mu_1 + b_1 + J_{12}\right) A_0 (1,1) - (\mu_3 p_{21} + J_{21}) B_0 (1,1)}{-\lambda_1 + b_1 + J_{12} - J_{21} + \mu_1 - \mu_2 p_{21}} \) ........................(2.7)

1 \(= \frac{-\left(\mu_1 p_{12} + J_{12}\right) A_0 (1,1) - \left(\mu_3 + b_2 + J_{21}\right) B_0 (1,1)}{-\lambda_2 + b_2 - J_{12} + J_{21} - \mu_1 p_{12} + \mu_2} \) ........................(2.8)

In matrix notations, the equations (2.6), (2.7) and (2.8) can be written as

\[ AX = B; \] ........................(2.9)

where

\[
A = \begin{bmatrix}
\mu_1 & -\mu_1 p_{13} & -\mu_2 p_{23} \\
0 & \mu_1 + b_1 + J_{12} & -\mu_2 p_{21} - J_{12} \\
0 & -\mu_1 p_{12} - J_{12} & \mu_2 + b_2 + J_{21}
\end{bmatrix}
\]

\[
X = \begin{bmatrix}
I_0 (1,1) \\
A_0 (1,1) \\
B_0 (1,1)
\end{bmatrix}
\] and

\[
B = \begin{bmatrix}
-\lambda_3 - \mu_1 p_{13} - \mu_2 p_{23} + \mu_3 \\
-\lambda_1 + b_1 + J_{12} - J_{21} - \mu_2 p_{21} + \mu_1 \\
-\lambda_2 + b_2 + J_{21} - J_{12} - \mu_1 p_{12} + \mu_2
\end{bmatrix}
\]

The augmented matrix \([A : B]\) after the elementary row transformation,

\[
R_3 \rightarrow R_3 + \left[ \frac{\mu_1 p_{13} + J_{12}}{\mu_1 + b_1 + J_{12}} \right] R_2 \]

becomes

\[
\begin{bmatrix}
\mu_3 & -\mu_1 p_{13} & -\mu_2 p_{23} & -\lambda_3 - \mu_1 p_{13} - \mu_2 p_{23} + \mu_3 \\
0 & \mu_1 + b_1 + J_{12} & -\mu_2 p_{21} - J_{12} & -\lambda_1 + b_1 + J_{12} - J_{21} - \mu_2 p_{21} + \mu_1 \\
0 & 0 & M & M - N
\end{bmatrix}
\]

Where

\[
M = \mu_1 \mu_2 (1 - p_{12} p_{21}) + \mu_1 J_{12} p_{23} + \mu_1 J_{21} p_{13} + \mu_1 b_1 + b_2 J_{12} + b_2 J_{21} + b_1 b_2
\]

And

\[
N = \lambda_1 (\mu_1 p_{12} + J_{12}) + \lambda_2 (\mu_2 + J_{21} + b_1)
\]

By matrix algebra, the system of equation (2.9) are consistent. Thus, the value of three unknowns \(B_0 (1,1), A_0 (1,1)\) and \(I_0 (1,1)\), after simplification are as follows:

\[
B_0 (1,1) = 1 - \frac{\lambda_1 \left(\mu_1 p_{12} + J_{21}\right) + \lambda_2 \left(\mu_1 b_1 + J_{21}\right)}{M},
\]

\[
A_0 (1,1) = 1 - \frac{\lambda_1 \left(\mu_2 + b_2 + J_{21}\right) + \lambda_2 \left(\mu_2 p_{21} + J_{21}\right)}{M},
\]

and

\[
I_0 (1,1) = 1 \left[ 1 - \left(\frac{\lambda_1 \left(\mu_2 p_{12} + p_{23} p_{12}\right) + \mu_1 \left(J_{12} + b_1 p_{13} + \mu_2 J_{12} p_{23}\right) + \lambda_2 \left(\mu_2 p_{23} + p_{23} p_{12}\right) + \mu_1 J_{21} p_{13} + \mu_2 \left(J_{12} + b_1\right) p_{23}\right)}{\mu_1 M} \right]
\]

........................(2.11)

Now, the steady state solution of M/M/1, when there are \(h\) persons (including service) in the queue is given by:
\[ p_h = p_0 (1 - p_0)^h \]
\[ p_0 = (1 - \rho) \text{ and } \rho = \frac{\lambda}{\mu} < 1 \text{ with } h \geq 0 \]

…..(2.12)

Now, if \( p_k, q_m, r_n \) denote the probabilities that there are \( k \) units in \( Q_1 \), \( m \) units in \( Q_2 \) and \( n \) units in \( Q_3 \) and since in our model all the probability distribution are mutually independent, therefore, the joint probability that there are \( k \) units in \( Q_1 \), \( m \) units in \( Q_2 \) and \( n \) units in \( Q_3 \), including service, if any, in the system is given by :

\[ P(k, m, n) = p_k q_m r_n \]

….. (2.13)

Hence, by virtue of (12) we have :

\[ P_k = p_0 (1 - p_0)^k, p_k \text{ converge if } 1 - p_0 < 1 \]

and similarly \( q_m \) and \( r_n \) also.

Now, using (2.13) and (2.14), we obtain

\[ P(k, m, n) = p_0 q_0 r_0 (1 - p_0)^k (1 - q_0)^m (1 - r_0)^n \]

…..(2.15)

Where \( p_0, q_0, r_0 > 0 \)

Now, \( A_0(1,1) \) denotes the marginal probability generating function (m.p.g.f.) of 0 units in \( Q_1 \) when \( Q_2 \) and \( Q_3 \) have been eliminated from the consideration in the system. Similarly \( B_0(1,1) \) and \( I_0(1,1) \) for consideration in the system. Therefore, we can easily see that :

\[ A_0(1,1) = p_0, B_0(1,1) = q_0 \text{ and } I_0(1,1) = r_0 \]

…..(2.16)

Therefore, using (2.11) and (2.16) the steady state solution in (15) can be written as

\[ P(k, m, n) = P(0, 0, 0) \rho_1^k \rho_2^m \rho_3^n \]

…..(2.17)

Where

\[ \rho_1 = \frac{\lambda_1 (\mu_2 + b_2 + J_{21}) + \lambda_2 (\mu_3 p_{21} + J_{21})}{M} \]

\[ \rho_2 = \frac{\mu_1 p_{12} (\mu_2 + J_{12}) + \lambda_2 (\mu_1 + b_1 + J_{12})}{M} \]

\[ \rho_3 = \frac{\lambda_1 \{\mu_2 (p_{13} + p_{12}p_{23}) + \mu_1 (J_{21} + b_2) p_{13} + \mu_2 J_{21} p_{23}\} + \lambda_2 \{\mu_2 (p_{23} + p_{21}p_{13}) + \mu_2 J_{21} p_{13} + \mu_2 p_{23} (J_{12} + b_1)\} + \lambda_3 M}{\mu_1 M} \]

where

\[ M = \mu_1 p_1 (1 - p_{12} p_{21}) + \mu_2 J_{12} p_{23} + \mu_3 J_{21} p_{13} + b_2 (\mu_4 + J_{12} + b_1) + b_1 (\mu_2 + J_{21}) \]

with \( P(0, 0, 0) = (1 - \rho_1)(1 - \rho_2)(1 - \rho_3) \)

…..(2.18)

Under the assumption that \( p_0, q_0, r_0 > 0 \) otherwise (2.17) may diverge to \( \infty \).

The marginal probability \( P(k...) \) of \( k \) units are in waiting and in service in \( Q_1 \) can be obtained by using the value of \( P(k, m, n) \) from (17) in the formula :

\[ P(k...) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} P(k, m, n) \]
\[ = P(0, 0, 0) \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \rho_1^k \rho_2^m \rho_3^n \]
\[ = \rho_1^k (1 - \rho_1), \text{ for } k = 0, 1, 2 \quad \ldots \ldots (2.19) \]

Similarly, the marginal probabilities \( P(., m, .) \) of \( m \) units in \( Q_2 \) and \( P(., ., n) \) of \( n \) units in \( Q_3 \) are

\[ P(., m, .) = \rho_2^m (1 - \rho_2), \text{ for } m = 0, 1, 2 \ldots \]
\[ P(., ., n) = \rho_3^n (1 - \rho_3), \text{ for } m = 0, 1, 2 \ldots \]

3. SOME CHARACTERISTICS OF THE SYSTEM

3.1. Mean queue length: It is denoted by \( L \) and is equal to the sum of marginal queue lengths of the queues \( Q_1, Q_2 \) and \( Q_3 \) which are denoted by \( L_1, L_2 \) and \( L_3 \) respectively.

Hence, \( L = L_1 + L_2 + L_3 \) \( \ldots \ldots (3.1) \)

Now, the marginal queue length \( L_1 \) in the queue \( Q_1 \) is obtained by the formula

\[ L_1 = \sum_{k=0}^{\infty} kP(k, .) \]

Using (2.19) for \( P(k, .) \) in the above relation and simplifying, we have

\[ L_1 = \frac{\rho_1}{1 - \rho_1} \quad \ldots \ldots (3.2) \]

Similarly, the marginal mean queue lengths \( L_2 \) and \( L_3 \) for the queues, \( Q_2 \) and \( Q_3 \) respectively are:

\[ L_2 = \frac{\rho_2}{1 - \rho_2} \quad \ldots \ldots (3.3) \]

And \( L_3 = \frac{\rho_3}{1 - \rho_3} \quad \ldots \ldots (3.4) \)

Now, using (21), (22) and (23) in (20), we have

\[ L = \frac{\rho_1}{1 - \rho_1} + \frac{\rho_2}{1 - \rho_2} + \frac{\rho_3}{1 - \rho_3} \quad \ldots \ldots (3.5) \]

Where \( P_1, P_2 \) and \( P_3 \) are defined in (17),

3.2. Fluctuation in the queue length: Fluctuation is denoted by \( \text{Var } \theta \) for \( \theta = k + m + n \) and is evaluated as

\[ \text{Var } \theta = \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (\theta - L)^2 \rho(k, m, n) \]

\[ = \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (k + m + n)^2 \rho(k, m, n) - L^2 \quad \ldots \ldots (3.6) \]
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using $\rho (k, m, n)$, from (2.17) in (3.6) and the value of $L$ from (3.5), we have

$$Var \theta = P(0,0,0) \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (k+m+n)^2 \rho_k^2 \rho_m^2 \rho_n^2 - L^2$$

$$= \frac{3}{(1-\rho_i)^2} \left[ \sum_{j=1}^{3} \rho_j (1-\rho_j) \right]^2 - \left[ \sum_{j=1}^{3} \rho_j \right]^2 \ldots \ldots (3.7)$$

4. PARTICULAR CASES

Case-I. If we take $\lambda_3 = 0 = b_1 = b_2 = J_{12} = J_{21} = p_{12} = p_{21}$

Then, equation (3.5) becomes

$$L = \frac{\lambda_1}{\mu_i - \lambda_1} + \frac{\lambda_2}{\mu_2 - \lambda_2} + \frac{\lambda_1 + \lambda_2}{\mu_3 - (\lambda_1 + \lambda_2)} \ldots \ldots (4.1)$$

Which coincides with the result given by Maggu [4].

Case-II. If we consider, $\lambda_2 = 0 = \lambda_3 = b_1 = b_2 = J_{12} = J_{21} = p_{21}$ and $p_{12} = 1$ with

$\mu_3 \to \infty$ in the equation (3.5),

we have:

$$L = \frac{\lambda_1}{\mu_i - \lambda_1} + \frac{\lambda_2}{\mu_2 - \lambda_1} \ldots \ldots (4.2)$$

This result coincides with the result given by Jackson [2].

Case-III. If we assume $\lambda_2 = 0 = \lambda_3 = b_1 = b_2 = J_{12} = J_{21}$ and $p_{12} = 1$ with

$\mu_3 \to \infty$ in equation (2.17), we have

$$P(k,m) = \left[ 1 - \frac{\lambda_1}{\mu_i (1-p_{21})} \right] \left[ 1 - \frac{\lambda_1}{\mu_2 (1-p_{21})} \right] \left[ \frac{\lambda_1}{\mu_i (1-p_{21})} \right]^k \left[ \frac{\lambda_1}{\mu_2 (1-p_{21})} \right]^m \ldots \ldots (4.3)$$

This results gives the solution of the cyclic queues with terminal feedback which was given by Finch [10].
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