NAMBU BRACKETS WITH CONSTRAINT FUNCTIONALS
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Abstract
If a Hamiltonian dynamical system with \( n \) degrees of freedom admits \( m \) constants of motion more than \( 2n - 1 \), then there exist some functional relations between the constants of motion. Among these relations the number of functionally independent ones are \( s = m - (2n - 1) \). It is shown that for such a system in which the constants of motion constitute a polynomial algebra closing in Poisson bracket, the Nambu brackets can be written in terms of these \( s \) constraint functionals. The exemplification is very rich and several of them are analyzed in the text.

1 Introduction

The concept of generalized Hamiltonian dynamics arose in 1973 with an article by Y. Nambu. [1] In his proposal, Nambu employed an \( N \)-ary bracket, generically called Nambu bracket (NB), to describe the time evolution of the dynamical system in \( N \)-dimensional (\( ND \)) phase space. His bracket includes \( N - 1 \) functionally independent constants of motion, the so-called generalized Hamiltonians. As an illustrative example, Nambu considered the Euler equations of free rigid body for a 3D phase space and this was the only example given. Finding examples in higher odd-dimensions is still very tedious matter.

In Nambu formalism, dynamical systems produce inevitably a nontrivial normalization factor \( C \) at least when \( N \) is an even integer grater than three. [2] In words, in order to get the correct Hamiltonian dynamics the NBs must be normalized properly. The explicit general form of \( C \) has been derived in detail for superintegrable systems. [3] The aim of this paper is to obtain \( C \) for a Hamiltonian system with \( n \) degrees of freedom and \( m \) constants of motion \( C_1, \cdots, C_m \), with \( m \geq 2n \).

First, we begin by reviewing the basic features of the Nambu formalism. Let \( M \) be an \( ND \) smooth manifold and let \( C^\infty(M) \) be the linear space of smooth real-valued functions on \( M \). The real multilinear map

\[
\{, \cdots, \} : \underbrace{C^\infty(M) \times \cdots \times C^\infty(M)}_{N \text{ times}} \rightarrow C^\infty(M)
\]  

(1)

defines NB of \( N \)-th order satisfying the properties skew-symmetry, Leibniz rule and generalized Jacobi identity (fundamental identity). [4] When one considers this NB structure \( C^\infty(M) \) admits another algebra structure.
The time evolution of $f \in C^\infty(M)$ is determined by $N-1$ Hamiltonian functions $C_1, \ldots, C_{N-1} \in C^\infty(M)$ and is described by the Nambu-Hamilton (NH) equations of motion

$$\frac{df}{dt} = X_{NH}(f) = \{f, C_1, \ldots, C_{N-1}\}, \quad (2)$$

where $X_{NH}$ is called the NH vector field corresponding to $C_1, \ldots, C_{N-1}$.

Finally, we recall that when $M$ is a symplectic manifold of dimension $N = 2n$, $C^\infty(M)$ has also an infinite-dimensional Lie algebra structure defined with respect to the Poisson bracket (PB)

$$\{f, g\}_P = \sum_{j=1}^{n} (\partial_{q_j} f \partial_{p_j} g - \partial_{p_j} f \partial_{q_j} g), \quad (3)$$

where $(q, p) = (q_1, \ldots, q_n, p_1, \ldots, p_n)$ are the local Darboux canonical coordinates.

### 2 Canonical Nambu Bracket and its Decomposition

A concrete realization of NB was embodied with the following form

$$\{f_1, \ldots, f_N\} = \frac{\partial(f_1, \ldots, f_N)}{\partial(x_1, \ldots, x_N)} \quad (4)$$

by Y. Nambu in the case of $M = \mathbb{R}^N$ and it is called the canonical NB. In (4) $x = (x_1, \ldots, x_N)$ denotes the local coordinates of $\mathbb{R}^N$ and the right hand side stands for the Jacobian of the mapping $f = (f_1, \ldots, f_N) : \mathbb{R}^N \to \mathbb{R}^N$.

For the systems with $n$ degrees of freedom, i.e., $2n$-dimensional phase space, the canonical Nambu bracket is determined explicitly by

$$\{f_1, \ldots, f_{2n}\} = \frac{\partial(f_1, \ldots, f_{2n})}{\partial(q_1, p_1, \ldots, q_n, p_n)} = \epsilon_{i_1 \ldots i_{2n}} \partial_{i_1} f_1 \cdots \partial_{i_{2n}} f_{2n} \quad (5)$$

and it can be decomposed into a skew-symmetric product of PBs which is a useful identity

$$\{f_1, \ldots, f_{2n}\} = \{f_1, f_{[2]} \}_P \{f_3, \ldots, f_{2n}\}, \quad (6)$$

where the bracket $[\ ]$ indicates the cyclic sum with respect to its elements.\[5\] (Here and hereafter we will use the Einstein summation convention for all repeated indices.)

For coordinate-free expression of the canonical NB we associate the $(N-1)$-form

$$\Gamma = dC_1 \wedge \cdots \wedge dC_{N-1} \quad (7)$$
to given $N-1$ Hamiltonian functions $C_i$. In (7), $d$ and $\wedge$ denote the usual exterior derivative and exterior product of Cartan calculus. If we employ the Hodge map $\star : \Lambda^p(\mathbb{R}^N) \rightarrow \Lambda^{N-p}(\mathbb{R}^N)$ between the space of $p$-forms $\Lambda^p(\mathbb{R}^N)$, $p \leq N$ and the space of $(N-p)$-forms $\Lambda^{N-p}(\mathbb{R}^N)$, an easy calculation shows that

$$\star(d f \wedge \Gamma) = \{f, C_1, \ldots, C_{N-1}\} = \frac{\partial(f, C_1, \ldots, C_{N-1})}{\partial(x_1, \ldots, x_{2^n})}. \quad (8)$$

Finally, for the purpose of this study, we write the NH equation for any $f$ as

$$\star(df \wedge \Gamma) = C \dot{f}, \quad (9)$$

where $C$ is also a constant of motion. $C$ will be referred to as normalization constant corresponding to the set $C_1, \ldots, C_{N-1}$ and will be specified from the requirement that (9) produces the correct Hamiltonian equations of motion in the case of $N = 2n$. The requirement of nontrivial $C$ is inevitable at least when $n \geq 2$. The NB given in (9) will be used in the rest of the text.

### 3 Nambu Brackets and Constraint Functionals

While some authors have investigated the connection of Nambu dynamics to Dirac’s constraint formalism, [6, 7, 8] our approach will depend on functions of these constraints. We consider a system with $n$ degrees of freedom, i.e., a $2n$-dimensional phase space with coordinates $(q_1, p_1, \ldots, q_n, p_n)$, $n \geq 2$. We suppose that the system acquires $m \geq 2n$ constants of motion $C_1, \ldots, C_m$ constituting a polynomial algebra closing in PB. Though these restrictions seem a bit rigorous, it may be remarkable to emphasize that most of the examples in the literature are of this kind. We impose no restriction on involutive properties of the constants of motion. In this sense, the system need not be integrable or superintegrable. For such a system, when possible, the number of functionally independent constants of motion is $2n-1$ and therefore there are independent $m-(2n-1) = s$ functional relations

$$F_j = F_j(C_1, \ldots, C_m) = 0, \quad (j = 1, \ldots, s) \quad (10)$$

between $C_i$s [9]. We will call the $F_j$s constraint functionals to avoid any confusion. In some cases, too many constraint functionals may appear, but any independent $s$ of them are enough to operate the formalism, hence the choice of the constraints for these systems is not unique. We divide the derivation into two main cases to obtain a full treatment. First, for the case $s \leq 2n-1$, let us choose the independent set of $F_1, \ldots, F_s$ and construct the $s$-form

$$\alpha = dF_1 \wedge \cdots \wedge dF_s = \frac{\partial F_1}{\partial C_{i_1}} \cdots \frac{\partial F_s}{\partial C_{i_s}} dC_{i_1} \wedge \cdots \wedge dC_{i_s}. \quad (11)$$
Now we take the arbitrary set $C_{i_{s+1}}, \ldots, C_{i_{2n-1}}$ and construct the $((2n-1)-s)$-form
\[
\beta = dC_{i_{s+1}} \wedge \cdots \wedge dC_{i_{2n-1}},
\]
and then the $(2n-1)$-form
\[
\alpha \wedge \beta = dF_1 \wedge \cdots \wedge dF_{s} \wedge dC_{i_{s+1}} \wedge \cdots \wedge dC_{i_{2n-1}}
= \frac{\partial F_1}{\partial C_{i_1}} \cdots \frac{\partial F_s}{\partial C_{i_s}} dC_{i_1} \wedge \cdots \wedge dC_{i_s} \wedge dC_{i_{s+1}} \wedge \cdots \wedge dC_{i_{2n-1}}.
\]
(13)
When we multiply (13) by $df$ and apply the Hodge map, we get a $2n$-th order NB
\[
*(df \wedge \alpha \wedge \beta) = \{f, F_1, \ldots, F_s, C_{i_{s+1}}, \ldots, C_{i_{2n-1}}\}.
\]
(14)
Obviously, if the $F_j$s are written in terms of the phase space coordinates then they get zero, therefore
\[
*(df \wedge \alpha \wedge \beta) = \frac{\partial F_1}{\partial C_{i_1}} \cdots \frac{\partial F_s}{\partial C_{i_s}} *(df \wedge dC_{i_1} \wedge \cdots \wedge dC_{i_{2n-1}}) = 0.
\]
(15)
Each NB in 15, by the definition (9), implies a normalization constant associated with $C_{i_1}, \ldots, C_{i_{2n-1}}$, i.e.,
\[
*(df \wedge dC_{i_1} \wedge \cdots \wedge dC_{i_{2n-1}}) = \dot{f} C_{i_1 \cdots i_{2n-1}}.
\]
(16)
Thus this leads to a linear homogeneous system
\[
\frac{\partial F_1}{\partial C_{i_1}} \cdots \frac{\partial F_s}{\partial C_{i_s}} C_{i_1 \cdots i_{2n-1}} = 0
\]
(17)
including $\binom{m}{2n-1-s}$ equations. It is easy to see by the basic solution techniques that (17) has infinitely many solutions for the unknowns $C_{i_1 \cdots i_{2n-1}}$. But we require a solution which is nontrivial and compatible with the decomposition (6). The suitable solution can be chosen as
\[
C_{i_1 \cdots i_{2n-1}} = \epsilon_{i_1 \cdots i_{2n-1} i_{2n} \cdots i_m} \frac{\partial F_1}{\partial C_{i_{2n}}} \cdots \frac{\partial F_s}{\partial C_{i_m}} = \pm \frac{\partial (F_1, \ldots, F_s)}{\partial (C_{i_{2n}}, \ldots, C_{i_m})},
\]
(18)
where the sign $\pm$ is determined by the Levi-Civita tensor in the second term. Indeed, this choice argues with (17),
\[
\epsilon_{i_1 \cdots i_s i_{s+1} \cdots i_{2n-1} i_{2n} \cdots i_m} \frac{\partial F_1}{\partial C_{i_1}} \cdots \frac{\partial F_s}{\partial C_{i_s}} \frac{\partial F_1}{\partial C_{i_{2n}}} \cdots \frac{\partial F_s}{\partial C_{i_m}} = \pm \frac{\partial (F_1, \ldots, F_s)}{\partial (C_{i_1}, \ldots, C_{i_s}, C_{i_{2n}}, \ldots, C_{i_m})} = 0.
\]
(19)
An illustrative example may be convenient to be more explicit. Consider the system with two degrees of freedom including the constants of motion $C_1, \ldots, C_5$, which is also considered in Subsec.4.1. Thus there exist two independent constraint functionals, $F_1$, $F_2$. Therefore

$$\alpha = dF_1 \wedge dF_2 = \frac{\partial F_1}{\partial C_{i_1}} \frac{\partial F_2}{\partial C_{i_2}} \partial C_{i_1} \wedge \partial C_{i_2}$$

$$= \frac{\partial (F_1, F_2)}{\partial (C_{i_1}, C_{i_2})} \partial C_{i_1} \wedge \partial C_{i_2}, \quad (i_1 < i_2); \quad \text{(20)}$$

and $\beta = dC_{i_3}$. For an arbitrary $C_{i_3}$, say $C_1$, the condition $\ast (dF_1 \wedge dF_2 \wedge dC_1) = 0$ implies

$$\frac{\partial (F_1, F_2)}{\partial (C_2, C_3)} C_{231} + \frac{\partial (F_1, F_2)}{\partial (C_2, C_4)} C_{241} + \frac{\partial (F_1, F_2)}{\partial (C_2, C_5)} C_{251}$$

$$+ \frac{\partial (F_1, F_2)}{\partial (C_3, C_4)} C_{341} + \frac{\partial (F_1, F_2)}{\partial (C_3, C_5)} C_{351} + \frac{\partial (F_1, F_2)}{\partial (C_4, C_5)} C_{451} = 0. \quad \text{(21)}$$

Referring back to (18), one computes easily

$$2 \frac{\partial (F_1, F_2)}{\partial (C_2, C_3)} \frac{\partial (F_1, F_2)}{\partial (C_4, C_5)} - 2 \frac{\partial (F_1, F_2)}{\partial (C_2, C_4)} \frac{\partial (F_1, F_2)}{\partial (C_3, C_5)} + 2 \frac{\partial (F_1, F_2)}{\partial (C_2, C_5)} \frac{\partial (F_1, F_2)}{\partial (C_3, C_4)}$$

$$= \frac{\partial (F_1, F_2, F_1, F_2)}{\partial (C_2, C_3, C_4, C_5)} = 0. \quad \text{(22)}$$

At the second stage, i.e., for the case $s > 2n - 1$, we choose any $2n - 1$ constraints from the set $F_1, \ldots, F_s$. With the same argument followed in the previous case, it is easy to construct the 2n-form

$$\ast (dF_1 \wedge dF_{i_1} \wedge \cdots \wedge dF_{i_{2n-1}})$$

$$= \frac{\partial F_{j_1}}{\partial C_{i_1}} \cdots \frac{\partial F_{j_{2n-1}}}{\partial C_{i_{2n-1}}} \ast (dF_{i_1} \wedge \cdots \wedge dF_{i_{2n-1}}) = 0 \quad \text{(23)}$$

generating the system with $\binom{s}{2n-1}$ equations

$$\frac{\partial F_{j_1}}{\partial C_{i_1}} \cdots \frac{\partial F_{j_{2n-1}}}{\partial C_{i_{2n-1}}} C_{i_1 \cdots i_{2n-1}} = 0. \quad \text{(24)}$$

Since the choice of the set $F_{j_1}, \ldots, F_{j_{2n-1}}$ is arbitrary, (18) is always a solution to (24):

$$\epsilon_{i_1 \cdots i_m} \frac{\partial F_{j_1}}{\partial C_{i_1}} \cdots \frac{\partial F_{j_{2n-1}}}{\partial C_{i_{2n-1}}} \frac{\partial F_1}{\partial C_{i_2}} \cdots \frac{\partial F_s}{\partial C_{i_m}}$$

$$= \pm \frac{\partial (F_{j_1}, \ldots, F_{j_{2n-1}}, F_1, \ldots, F_s)}{\partial (C_{i_1}, \ldots, C_{i_{2n-1}}, C_{i_2}, \ldots, C_{i_m})} = 0. \quad \text{(25)}$$
Consequently, if (16) is recombined with (18), one concludes that
\[* (df \wedge dC_{i_1} \wedge \cdots \wedge dC_{i_{2n-1}}) = \{ f, C_{i_1}, \ldots, C_{i_{2n-1}} \} = \varepsilon_{i_1 \cdots i_{2n-1}} \partial F_1 \cdots \partial F_s \frac{\partial}{\partial F_s} \dot{f} = \pm \frac{\partial (F_1, \ldots, F_s)}{\partial (C_{i_{2n-1}}, \ldots, C_{i_m})} \dot{f}, \]

The formalism (26) possesses several NBs all are in accordance with correct equations of motion. It may be remarkable to emphasize that we have no any restriction about the independence of the set $C_{i_{2n-1}}, \ldots, C_{i_m}$. Thus, even if they are not independent, this does not destroy the validity of the formalism. On the other hand, the general result (26) also justifies the statement: If a NB includes any dependent subset of the constants of motion, then it vanishes. We shall make these remarks more precise in the discussion of the examples.

Now, as a corollary, we conclude the statement: If one of the constants of motion, say $C_k$, is taken as the Hamiltonian, then the decomposition (6) can, by discarding the Hamiltonian, be written in terms of the constraint functionals. The proof is straightforward: By the decomposition (6),
\[ \{ f, C_k, C_{i_1}, \ldots, C_{i_{2n-2}} \} = \{ f, C_k \} P \{ C_{i_1}, \ldots, C_{i_{2n-2}} \} = \dot{f} \{ C_{i_1}, \ldots, C_{i_{2n-2}} \}, \]
on the other hand, by (26),
\[ \{ f, C_k, C_{i_1}, \ldots, C_{i_{2n-2}} \} = \varepsilon_{k i_1 \cdots i_{m-1}} \frac{\partial F_1}{\partial C_{i_{2n-1}}} \cdots \frac{\partial F_s}{\partial C_{i_{m-1}}} \dot{f}, \]

thus
\[ \{ C_{i_1}, \ldots, C_{i_{2n-2}} \} = \pm \frac{\partial (F_1, \ldots, F_s)}{\partial (C_{i_{2n-1}}, \ldots, C_{i_{n-1}})}. \]

In particular, for the case $n = 2$, (29) holds for the PBs.

Finally, we talk about determination of the $F_j$ s. Note that, as a general result of (6), for the NBs including the Hamiltonian, the normalization constant $C$ is obtained easily in terms of the PBs. Therefore this observation supplies a helpful guide in determining the constraint functionals. Under this circumstance, $s - 1$ functional relations can be taken as the constraint functionals without any rearrangement. Thus, the construction of the last one is reduced to the problem of finding a function whose derivatives are known. Since the PBs of the constants of motion close to constitute a polynomial algebra, this gives us an easy integration process.
Having shown how to construct such a formalism we will give examples to confirm its correctness. To be more clear, the first example (harmonic oscillator) has been analyzed in detail. The examples have been chosen in a variety so that they include various alternatives when considering the dimension of the phase space and the number of the constraint functionals. In all examples, although the construction of constants of motion is not unique, we kept the forms and numbers of them just as appeared in the literature cited in the text.

4 Systems with Two and Three Degrees of Freedom

4.1 Harmonic oscillator

Although we take the system as the one with two degrees of freedom, its $2n$, $(n \geq 2)$ dimensional extension is obtainable as a special case from the Winternitz system given in Subsec.5.1. The system is described by the Hamiltonian

$$C_1 = H = \frac{p^2}{2} + kq^2/2,$$

(30)

where $q^2 = q_1^2 + q_2^2$ and $p^2 = p_1^2 + p_2^2$ [2]. Suppose that in addition to the Hamiltonian we are given the following set of equations as the constants of motion,

$$C_2 = \frac{p_1^2}{2} + kq_1^2/2, \quad C_3 = \frac{p_2^2}{2} + kq_2^2/2,$n

$$C_4 = q_1p_2 - q_2p_1, \quad C_5 = p_1p_2 + kq_1q_2.$$n

(31)

Their nonvanishing PBs are given by

$$\{C_2, C_4\}_P = -\{C_3, C_4\}_P = -C_5,$n

$$\{C_2, C_5\}_P = -\{C_3, C_5\}_P = kC_4,$n

$$\{C_4, C_5\}_P = -2(C_2 - C_3).$$

(32)

If the functional relation $C_1 = C_2 + C_3$ is taken as the first constraint such that $F_1 = C_1 - C_2 - C_3$, then the other which is compatible with the all PBs via (29) can be constructed as the following

$$F_2 = 2C_2C_3 - \frac{1}{2}kC_4^2 - \frac{1}{2}C_5^2$$

(33)

so that

$$\{C_2, C_4\}_P = -\frac{\partial(F_1, F_2)}{\partial(C_3, C_5)} = \frac{\partial F_2}{\partial C_5} = -C_5,$n

$$\{C_2, C_5\}_P = \frac{\partial(F_1, F_2)}{\partial(C_3, C_4)} = -\frac{\partial F_2}{\partial C_4} = kC_4,$n

$$\{C_4, C_5\}_P = \frac{\partial(F_1, F_2)}{\partial(C_2, C_3)} = -\frac{\partial F_2}{\partial C_3} + \frac{\partial F_2}{\partial C_2} = -2(C_2 - C_3).$$

(34)
Now we list two of the NBs as the sample,
\[
\{ f, C_1, C_2, C_4 \} = -\frac{\partial(F_1, F_2)}{\partial(C_3, C_5)} \dot{f} = -C_5 \dot{f},
\]
\[
\{ f, C_1, C_2, C_3 \} = \frac{\partial(F_1, F_2)}{\partial(C_4, C_5)} \dot{f} = 0
\]  \tag{35}
which are consistent with
\[
\{ f, C_1, C_2, C_4 \} = \{ f, C_1 \}_P \{ C_2, C_4 \}_P = -C_5 \dot{f},
\]
\[
\{ f, C_1, C_2, C_3 \} = \{ f, C_1 \}_P \{ C_2, C_3 \}_P = 0.
\]  \tag{36}

On the other hand, for the NBs not including the Hamiltonian, \( C \) is not so evident via (6). For example
\[
\{ f, C_3, C_4, C_5 \} = \{ f, C_3 \}_P \{ C_4, C_5 \}_P + \{ f, C_5 \}_P \{ C_3, C_4 \}_P + \{ f, C_4 \}_P \{ C_5, C_3 \}_P
\]
\[
= -2(C_2 - C_3) \{ f, C_3 \}_P + C_5 \{ f, C_5 \}_P + kC_4 \{ f, C_4 \}_P,
\]  \tag{37}
but it is straightforward to obtain it by the virtue of
\[
\{ f, C_3, C_4, C_5 \} = \frac{\partial(F_1, F_2)}{\partial(C_1, C_2)} \dot{f} = 2C_3 \dot{f}.
\]  \tag{38}

If we turn off the Hamiltonian, the set \( C_2, C_3, C_4, C_5 \) is still closed in PB and we need only one constraint functional, \( F = -F_2 \), thus
\[
\{ f, C_3, C_4, C_5 \} = -\frac{\partial F}{\partial C_2} \dot{f} = 2C_3 \dot{f}.
\]  \tag{39}

Now consider the bracket
\[
\{ f, C_1, C_4, C_5 \} = \frac{\partial(F_1, F_2)}{\partial(C_2, C_3)} \dot{f} = -2(C_2 - C_3) \dot{f},
\]  \tag{40}
and take artificially \( C_6 = C_2 - C_3 \) so that \( C_2, C_3 \) and \( C_6 \) are not independent. Again
\[
\{ f, C_1, C_4, C_5 \} = \frac{\partial(F_1, F_2, F_3)}{\partial(C_6, C_2, C_3)} \dot{f} = -2(C_2 - C_3) \dot{f},
\]  \tag{41}
where \( F_3 = C_6 - C_2 + C_3 \).

Finally, for the case \( s > 2n - 1 \), let \( C_7 = C_4 C_5 \). Of much NBs, we have chosen
\[
\{ f, C_3, C_4, C_5 \} = \frac{\partial(F_1, F_2, F_3, F_4)}{\partial(C_6, C_7, C_1, C_2)} = 2C_3 \dot{f},
\]  \tag{42}
where \( F_4 = C_7 - C_4 C_5 \).
4.2 Smorodinsky-Winternitz system

Smorodinsky-Winternitz system consists of a set of four Hamiltonians which have potential form, i.e., \( H = \frac{p^2}{2} + V \) [10]. All potentials are separable into at least two coordinate systems and they also admit superintegrable structure. We will consider symbolically only the system

\[
C_1 = H = \frac{p^2}{2} + \omega^2 (4q_1^2 + q_2^2) + \alpha_1 q_1 + \alpha_2 q_2^2,
\]

where all Greek letters are some real constants, \( q = (q_1^2 + q_2^2)^{1/2} \) and \( p^2 = p_1^2 + p_2^2 \). All constants of motions are at most quadratic in momenta,

\[
\begin{align*}
C_2 &= p_1^2/2 + \alpha_1 q_1 + 4\omega^2 q_1^2, \\
C_3 &= 2p_2 L_3 - 4\omega^2 q_1 q_2^2 + 4\alpha_2 q_1/q_2^2 - \alpha_1 q_2^2, \\
C_4 &= -2(\alpha_1 + 8\omega^2 q_1)q_2 p_2 - p_1(2p_2^2 - 4\omega^2 q_2^2 + 4\alpha_2/q_2^2),
\end{align*}
\]

where \( L_3 = q_1 p_2 - q_2 p_1 \) is the third component of angular momentum. Their nonvanishing PBs close in a Poisson algebra

\[
\begin{align*}
\{C_2, C_3\}_P &= C_4, \\
\{C_2, C_4\}_P &= 4\alpha_1 C_2 - 8\omega^2 C_3 - 4\alpha_1 C_1, \\
\{C_3, C_4\}_P &= -48C_2^2 + 64C_1 C_2 - 4\alpha_1 C_3 + 64\omega^2 \alpha_2 - 16C_1^2,
\end{align*}
\]

admitting a Casimir

\[
F = \frac{C_1^2}{2} - 4\alpha_1 C_2 C_3 + 4\omega^2 C_3^2 + 4\alpha_1 C_1 C_3 - 16C_1^2 C_2^2 + 32C_1 C_2^2 + 64\omega^2 \alpha_2 C_2 - 16C_1 C_2 + 4\alpha_1^2 \alpha_2
\]

as the constraint functional [11]. Note that the constraint (46) can be obtained as the integration of the PBs in (45). This observation justifies the expressions

\[
\begin{align*}
\{C_2, C_3\}_P &= \frac{\partial F}{\partial C_4}, \\
\{C_2, C_4\}_P &= -\frac{\partial F}{\partial C_3}, \\
\{C_3, C_4\}_P &= \frac{\partial F}{\partial C_2}.
\end{align*}
\]

And hence two of possible NBs are

\[
\begin{align*}
\{f, C_1, C_2, C_4\} &= -\frac{\partial F}{\partial C_3} \dot{f} = -(4\alpha_1 C_2 - 8\omega^2 C_3 - 4\alpha_1 C_1) \dot{f}, \\
\{f, C_2, C_3, C_4\} &= -\frac{\partial F}{\partial C_1} \dot{f} = -(4\alpha_1 C_3 + 32C_2^2 - 32C_1 C_2) \dot{f}.
\end{align*}
\]

4.3 Kepler-Coulomb system

Let us concentrate now on the 6D Kepler-Coulomb Hamiltonian

\[
H = \frac{p^2}{2} - \frac{\alpha}{q},
\]

(49)
where $\alpha$ is a real constant, $p^2 = p_1^2 + p_2^2 + p_3^2$ and $q = (q_1^2 + q_2^2 + q_3^2)^{1/2}$ [2]. Because of the rotational symmetry of the system, the angular momentum $L$ is integral invariant and hence its components can be taken as constants of motion, i.e.,

$$L_1 = q_2 p_3 - q_3 p_2, \quad L_2 = q_3 p_1 - q_1 p_3, \quad L_3 = q_1 p_2 - q_2 p_1. \quad (50)$$

(From now on, we will write the constants of motion just as appeared in the system without corresponding any $C_i$ to them to avoid any confusion in mind and to keep their symmetries in writing). Moreover, there also exists an extra invariant arising from that the particle has a closed orbit. This invariant is called the Runge-Lenz vector $A$ given by

$$A = p \times L - \alpha q / q. \quad (51)$$

In addition to the previous constants of motion,

$$A_1 = p_2 L_3 - p_3 L_2 - \alpha q_1 / q, \quad A_2 = p_3 L_1 - p_1 L_3 - \alpha q_2 / q, \quad A_3 = p_1 L_2 - p_2 L_1 - \alpha q_3 / q. \quad (52)$$

Consequently we have seven constants of motion satisfying the commutations

$$\{L_a, L_b\}_P = \epsilon_{abc} L_c, \quad \{A_a, A_b\}_P = -2H \epsilon_{abc} L_c, \quad \{L_a, A_b\}_P = \epsilon_{abc} A_c, \quad (53)$$

and the following functional relations

$$A \cdot L = 0, \quad A^2 = 1 + 2HL^2 \quad (54)$$

which are candidates for the constraint functionals. Thus if we choose these functions as follows

$$F_1 = A_1 L_1 + A_2 L_2 + A_3 L_3, \quad F_2 = \frac{1}{2} + H(L_1^2 + L_2^2 + L_3^2) - \frac{1}{2}(A_1^2 + A_2^2 + A_3^2). \quad (55)$$

(29) implies several brackets such as

$$\{L_1, L_2, L_3, A_1\} = \frac{\partial (F_1, F_2)}{\partial (A_2, A_3)} = A_2 L_3 - A_3 L_2. \quad (56)$$

This is compatible with the decomposition

$$\{L_1, L_2, L_3, A_1\} = \{L_1, L_2\}_P \{L_3, A_1\}_P + \{L_1, A_1\}_P \{L_2, L_3\}_P + \{L_1, L_3\}_P \{A_1, L_2\}_P. \quad (57)$$

So, one of the NBs is

$$\{f, H, L_1, L_2, L_3, A_1\} = \{f, H\}_P \{L_1, L_2, L_3, A_1\}$$

$$= \frac{\partial (F_1, F_2)}{\partial (A_2, A_3)} \dot{f} = (A_2 L_3 - A_3 L_2) \dot{f}. \quad (58)$$
Other two of them may be chosen as the following

\[ \{ f, L_1, L_2, L_3, A_1, A_2 \} = \frac{\partial (F_1, F_2)}{\partial (A_3, L)} \frac{\dot{f}}{\partial (A_3, H)} = L_3 (L_1^2 + L_2^2 + L_3^2) \dot{f}, \]

\[ \{ f, H, L_1, L_2, A_2, A_3 \} = \frac{\partial (F_1, F_2)}{\partial (L_3, A_1)} \frac{\dot{f}}{\partial (L_3, A_1)} = -(A_1 A_3 + 2HL_1L_3) \dot{f}. \] (59)

5 Systems with \( n \) Degrees of Freedom

5.1 Winternitz system

Winternitz system is the arbitrary dimensional generalization of one of the Smorodinsky - Winternitz Hamiltonians mentioned above. Their constants of motion are constructed by using the Lax matrix representation [12]. The particle’s Hamiltonian in \( n \) degrees of freedom is given by

\[ H = \frac{1}{2} \sum_{i=1}^{n} \left( p_i^2 + k_i^2 x_i^2 + \frac{k_i^2}{x_i^2} \right), \] (60)

where \( k \) and the \( k_i \) are real constants. (Throughout this section all subscripts range from 1 to \( n \)). The elements of degeneracy group \( SU(n) \) of the Winternitz system are taken as the constants of motion all commuting with the Hamiltonian. First group of these \( n^2 \) functions has the form

\[ T_{ii} = \frac{1}{2k} (H_i - kk_i), \] (61)

where the conserved quantity \( H_i \) is taken as the energy in the \( i \)-th direction. Second group is given by the ansatz

\[ T_{ij} = f(H_i) f(H_j) A_i A_j^*, \quad i \neq j, \] (62)

where

\[ f(H_i) = \left( \frac{2k}{H_i + kk_i} \right)^{1/2}, \] (63)

and

\[ A_i = \frac{1}{4k} \left( p_i^2 + \frac{k_i^2}{x_i^2} - k_i^2 x_i^2 + 2ik x_i p_i \right). \] (64)

The PBs of the functions \( T_{ij} \) argue

\[ \{ T_{ij}, T_{rs} \}_P = i\delta_{jr} T_{is} - i\delta_{is} T_{rj}. \] (65)
With this argument, there exist totally \( n^2 + 1 \) constants of motion and there must be functional relations between the invariants. One of the functional relations, so by (61), is the simplest one

\[
H = \sum_{i=1}^{n} (2kT_{ii} + kk_i). \tag{66}
\]

After a series of calculations, the others can be expressed as

\[
T_{ij}T_{jk} = T_{jj}T_{ik}. \tag{67}
\]

Note, by referring to the discussion in Sec.3, that the number of independent constraint functionals is \( n^2 - 2n + 2 \).

After having been defined the Winternitz system and its invariants, we now perform the case of three degrees of freedom as an example. Despite the fact that it is possible to study with ten constants of motion and five constraint functionals (or ten functional relations), for the sake of simplicity, we prefer the set of constants of motion \( H, T_{11}, T_{22}, T_{33}, T_{12}, T_{13} \) which is also closed in the PB. The suitable choice for the only one constraint functional is

\[
F = T_{12}T_{13} \left[ -\frac{H}{2k} + T_{11} + T_{22} + T_{33} + \frac{1}{2}(k_1 + k_2 + k_3) \right]. \tag{68}
\]

Consequently among the all possible four nonvanishing NBs two of them are listed in the following

\[
\{f, H, T_{11}, T_{22}, T_{12}, T_{13}\} = \frac{\partial F}{\partial T_{33}} \dot{f} = T_{12}T_{13}\dot{f},
\]

\[
\{f, T_{11}, T_{22}, T_{33}, T_{12}, T_{13}\} = -\frac{\partial F}{\partial H} \dot{f} = \frac{1}{2k}T_{12}T_{13}\dot{f}. \tag{69}
\]

5.2 Free particle on \( n \)-sphere

Our last example is a free particle moving on the surface of an \( n \)-sphere with the radius \( q = (q_1^2 + \cdots + q_n^2)^{1/2} \) [5]. The PB Lie algebra of the integral invariants (charges of \( so(n+1) \)) is generated by the \( n(n - 1)/2 \) rotation elements \( L_{\alpha\beta} = q_\alpha p_\beta - q_\beta p_\alpha, \ \alpha, \beta = 1, \ldots n \), and the \( n \) momenta \( P_\alpha = (1 - q^2)^{1/2}p_\alpha \). The PBs of the invariants are

\[
\{L_{\alpha\beta}, L_{\gamma\xi}\}_P = \delta_{\beta\xi}L_{\alpha\gamma} + \delta_{\alpha\gamma}L_{\beta\xi} - \delta_{\beta\gamma}L_{\alpha\xi} - \delta_{\alpha\xi}L_{\beta\gamma}, \tag{70}
\]

\[
\{P_\alpha, P_\beta\}_P = L_{\alpha\beta}, \quad \{L_{\alpha\beta}, P_\gamma\}_P = \delta_{\alpha\gamma}P_\beta - \delta_{\beta\gamma}P_\alpha. \tag{71}
\]
The Hamiltonian of the particle is given by

\[ H = \frac{1}{2}(P_\alpha P_\alpha + L_{\beta \gamma} L_{\beta \gamma}), \quad \beta < \gamma. \]  

(72)

Unlike the previous example, the algebra is closed by all of the \( n(n + 1)/2 + 1 \) constants of motion, and then we need \( n(n - 3)/2 + 2 \) independent functionals to proceed the formalism. For this aim we will use the following concluded functional relations as the source of the constraint functionals;

\[ L_{[\alpha_1 \alpha_2 P_{\alpha_3}] = 0, \]
\[ L_{[\alpha_1 \alpha_2 P_{\alpha_3} P_{\alpha_4}] = 0, \]
\[ \vdots \]
\[ L_{[\alpha_1 \alpha_2 P_{\alpha_3} P_{\alpha_4} \cdots P_{\alpha_n}] = 0.} \]

(73)

Additionally a second group appears as

\[ L_{\alpha_1 [\alpha_2 L_{\alpha_3 \alpha_4}] = 0.} \]

(74)

For each of the relations in (73) and (74), the \( \alpha_i \)s can be chosen freely provided they are all different from one another. It is clear to see that the number of functional relations listed above exceeds the needed one too much, but any set of independent, suitable rearranged \( n(n - 3)/2 + 2 \) functionals does work. As was the case for the previous example, we restricted ourselves to a particular, say \( n = 4 \), degrees of freedom. In that case, we have 11 constants of motion and therefore we need four independent constraints. First one is the simplest one, i.e., (72),

\[ F_1 = H - \frac{1}{2}(P_1^2 + P_2^2 + P_3^2 + P_4^2) \]
\[ - \frac{1}{2}(L_{12}^2 + L_{13}^2 + L_{14}^2 + L_{23}^2 + L_{24}^2 + L_{34}^2). \]

(75)

For the other three, a suitable choice can be written explicitly as the following

\[ F_2 = L_{12} L_{34} + L_{14} L_{23} - L_{13} L_{24}, \]
\[ F_3 = L_{12} P_3 - L_{13} P_2 + L_{23} P_1, \]
\[ F_4 = P_4 - \frac{L_{14}}{L_{13}} P_3 + \frac{L_{34}}{L_{13}} P_1. \]

(76)

Now, as before, we choose a sample from the NBs,

\[ \{ f, H, P_1, P_2, P_3, P_4, L_{12}, L_{13} \} = \frac{\partial (F_1, F_2, F_3, F_4)}{\partial (L_{14}, L_{23}, L_{24}, L_{34})} \dot{f} \]
\[ = \left[ L_{12} P_2 P_4 + L_{13} P_3 P_4 - L_{14}(P_1^2 + P_2^2 + P_3^2) \right] \dot{f}, \]

(77)
here we used the facts that $L_{12}P_3 = 0$, $L_{12}P_4 = 0$ and $L_{13}P_4 = 0$. As a further consequence, we remark that another set

$$
F'_1 = F_1,
F'_2 = L_{14}P_2P_3 - L_{13}P_2P_4 - L_{24}P_1P_3 + L_{23}P_1P_4,
F'_3 = \frac{L_{23}}{P_3} - \frac{L_{24}}{P_4} + \frac{L_{34}}{P_3P_4}P_2,
F'_4 = \frac{P_3P_4}{P_2}L_{12} - P_3L_{14} + P_1L_{34} + \frac{P_1P_4}{P_2}L_{23}
$$

(78)
gives the same equations of motion. For the readers who may wonder of other brackets, here we list two of them,

$$
\{f, P_1, P_2, P_3, P_4, L_{12}, L_{13}, L_{24}\} = -P_1P_2\dot{f},
\{f, H, L_{12}, L_{13}, L_{14}, L_{23}, L_{24}, L_{34}\} = 0.
$$

(79)

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