The Sokoban Random Walk

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Random motion in disordered media is sensitive to the presence of obstacles which prevent atoms, molecules, and other particles from moving freely in space. When obstacles are static, a sharp transition between confined motion and free diffusion occurs at a critical obstacle density: the percolation threshold. Surprisingly, this conventional wisdom breaks in the presence of simple interactions. To show this, we introduce the Sokoban random walk which — akin to the protagonist of an eponymous video game — has some limited ability to push away obstacles that block its path. While one expects that this added capacity would allow the walk to venture further away, we find that this is only true above the percolation threshold. Indeed, we show that the Sokoban always confines itself to a finite region whose mean size is uniquely determined by the initial obstacle density. The observed relation obeys an equation that we derive based on a conservative rule for the onset of confinement and by accounting for the emergent fractal nature of Sokoban trajectories. Our findings depart markedly from the ruling “ant in a labyrinth” paradigm, and vividly illustrate that even weak and localized particle-media interactions cannot be neglected when coming to understand transport phenomena.

More than a century after their introduction to the readers of Nature by Karl Pearson, random walks continue to fascinate and draw attention. While initially motivated by the theory of gambling and financial speculation, random walks became important in the natural sciences following the pioneering works of Einstein, Smoluchowski, and others on diffusion of atoms and molecules. In the time following the publication of these seminal works, random walks were further established as a versatile modelling tool with applications in physics, chemistry, biology, movement ecology, finance, and economics.

One paradigmatic random walk is the “ant in a labyrinth” which was introduced by Pierre-Gilles de Gennes, as a simple model for diffusion in disordered media. Consider an ant walking on a two-dimensional square lattice, where a fraction of the lattice sites are occupied with obstacles, and all other sites are empty. Each time unit, the ant takes a step onto an empty neighbouring site that is chosen randomly. Given a specific lattice size and density, one can ask how does the mean squared displacement (MSD) of the ant depend on time? When is small, i.e., most sites are empty, the ant’s motion is almost unobstructed. In this case, the MSD scales linearly with time.

On the other extreme, when is large, most sites are occupied by obstacles and the ant’s motion is highly restricted. In this limit the ant will find itself caged in the labyrinth, resulting in an MSD that saturates asymptotically. As it turns out, for large enough systems, the transition between restricted motion and free diffusion is sharp, occurring at a critical density .

The basic assumption made in de Gennes’ model is that obstacles comprising the media are immobile. In particular, it is assumed that the ant’s motion has no effect on the distribution of obstacles around it. Yet, this assumption is often violated, e.g. when considering active particles that burn energy and exert strong forces on their surroundings. Ironically, the ant — which can lift many times its body weight — is a quintessential example for such an active particle. Assuming immobile obstacles is thus far when considering a hydrogen atom diffusing in a solid, but the validity of this assumption is questionable for animals and microorganisms plowing their way through crowded environments. Validity is also questionable on the microscopic scale where one may encounter immobile obstacles that cannot be nudged by thermal fluctuations (on relevant time scales), e.g. since they are reversibly anchored or simply too large. Yet, such obstacles may nevertheless be pushed around by active particles in the media, and little is known on how transport properties of the latter are affected as a result.

Here, we introduce a minimalist model to show that particle-media interactions from the type mentioned above could result in a drastic, qualitative, change of transport properties. To this end, we consider a random walker that has ability to push away obstacles that block its path. We imagine an square arena where a fraction of the available sites are occupied with obstacles. Taking to be odd, we place a random walker at the center of this arena. The random walk then takes place according to the following rules which are illustrated in Fig. 1a. The walker can move into an unoccupied neighbouring site, placed horizontally or vertically relative to its position.
Figure 2: Sokoban vs. simple random walk with obstacles. Panel (a): Mean squared displacement (MSD) of a simple random walk (yellow) and the Sokoban random walk (purple) as a function of time, for three different obstacle densities: $\rho = 0.45, 0.5, 0.55$. In the long time limit, the MSD of the Sokoban is orders of magnitude higher compared to the MSD of the simple random walk. Panel (b): Trajectories of the simple (yellow) and Sokoban (purple) random walks, starting in an identical arena with $\rho = 0.5$. The trajectory of the simple random walk is plotted over the trajectory of the Sokoban random walk. The difference in MSD is evident.

In addition, even when a site is occupied by an obstacle, the walker can move into this site while pushing the obstacle one site forward, in its direction of motion. Yet, this can only be done provided that the next site (in the direction of motion) is not occupied by another obstacle. Thus, the walker cannot push more than one obstacle at a time. Finally, at each time step, the walker chooses between all feasible moves with equal probability.

The model presented herein is inspired by the video game Sokoban (Japanese for warehouse keeper), which was created in 1981 by Hiroyuki Imabayashi. The premise of the game is simple: The player, playing as the keeper, pushes boxes around in a warehouse, in attempt to transport them to marked storage locations. The rules of the game are similar to the rules of the Sokoban presented in Fig. 1. While being fairly simple to play, solving Sokoban puzzles turns out to be a difficult computational task. It was first proved to be NP-hard and was later shown to be PSPACE-complete.

An illustration of a trajectory of the Sokoban random walk is given in panels (b) and (c) of Fig. 1. The initial configuration of the arena is given in panel (b), and the trajectory of the walk is illustrated in panel (c). Note that a simple random walk, i.e., one that cannot push obstacles that stand in its path, would have actually been caged by the initial configuration of the arena. In contrast, the Sokoban was able to escape this cage by pushing some of the obstacles surrounding it (highlighted in gray). More generally, we expect that the ability to push obstacles will enable the Sokoban random walk to venture further away from its initial position when compared to a simple random walk without this pushing ability (“ant in a labyrinth”).

To test this hypothesis, we simulate the Sokoban and simple random walks, for a large number of randomly generated and sufficiently large arenas, so as to completely avoid boundary effects. In Fig. 2 we plot the mean squared displacement, given by $MSD(t) = \langle r^2(t) \rangle$, where $r(t)$ is the Euclidean distance to the initial position at time $t$, and $\langle \cdot \rangle$ indicates an ensemble average over all generated walks. Plots are made for the Sokoban (purple) and simple (yellow) random walks at three different obstacle densities. As expected, in the long time limit, the MSD of the Sokoban is significantly higher compared to the MSD of the simple random walk. As a result, the Sokoban explores larger portions of the arena as illustrated by the trajectories given in Fig. 2. Further illustration of the Sokoban walk is provided in a supplementary video (SV1 avi).

All densities in Fig. 2 were taken to be above the 2D site-percolation threshold, namely, the critical density $\rho_C \approx 0.407$ above which the simple random walk eventually becomes restricted (caged). The fact that for these densities the Sokoban was able to explore larger portions of the arena, hints that the critical density for this walk should be higher than, or equal to, the percolation threshold; allowing the Sokoban to roam unbounded when the density of obstacles drops below $\rho_C$. However, when simulating the Sokoban for $\rho < \rho_C$, we surprisingly find that its MSD saturates in the long time limit. An example is given in panels (a-c) of Fig. 2 where we take $\rho = 0.4$, and present a time evolution of a typical Sokoban trajectory. Snapshots are taken for $t = 10^6, 10^7$ and $10^8$. For $t \geq 10^7$ the walk does not visit new sites, indicating it is indeed confined (see Fig. S1).

Further evidence that the Sokoban random walk dynamically confines itself at densities lower than the percolation threshold comes from extensive numerical simulations that we perform for this system. Defining the exploration radius $r_\infty = \lim_{t \to \infty} \sqrt{MSD(t)}$, i.e., the level at which the root MSD sat-
The results presented in panels (a-d) of Fig. 3 assert that the Sokoban random walk is effectively caged and does not visit new sites (see comparison at $t = 10^9$ in Fig. S1). Panel (d): The exploration radius $r_\infty = \lim_{t \to \infty} \sqrt{\text{MSD}(t)}$ for the Sokoban (purple markers) and simple (yellow markers) random walks vs. the obstacle density $\rho$ (log-log scale). Markers coming from simulations show that while the exploration radius of the simple random walk diverges as $\rho \to \rho_c$, this radius remains finite for the Sokoban random walk. Simulations are in excellent agreement with Eq. (4) in the main text whose prediction is given by a dashed line surrounded by a grayed out sleeve which indicates an error of ±2.5% in the estimates made for the parameters $\gamma$ and $C$. Panel (e): The mean number of sites $\langle A \rangle$ visited by the Sokoban and mean number of perimeter sites $\langle P \rangle$ follow the power-law relations of Eqs. (2) and (3) [note the log-log scale, also see panel (f)]. Markers come from simulations and best fits yield the following estimates of the relevant parameters $A = 0.2277, B = 0.5736, \alpha = 1.936$, and $\beta = 1.784$. Panel (f): Area (purple) and perimeter (green) of the trajectory from panel (c). The area of the trajectory is defined as the number of visited sites. The perimeter of the trajectory is taken as the double layer of unvisited sites that surround area sites (share a mutual edge).

To further proceed, we formulate a simple condition that leads to caging. We define $\mathcal{A}$ to be the area covered by the Sokoban trajectory, i.e., the total number of sites visited in the long time limit. In addition, we let $\mathcal{P}$ stand for the number of sites in the double-layered perimeter that surrounds visited sites. For example, the double-layered perimeter of the sites visited by the trajectory in Fig. 3d is indicated in Fig. 3f. Recall that area and perimeter sites were initially occupied by

\begin{align*}
\text{(a)} & \quad t = 10^6 \\
\text{(b)} & \quad t = 10^7 \\
\text{(c)} & \quad t = 10^8 \\
\text{(d)} & \quad \rho = 0.4 \\
\text{(e)} & \quad \rho = 0.55 \\
\text{(f)} & \quad \rho = 0.55
\end{align*}
obstacles with the same probability $\rho$. Thus, on average, $\mathcal{A}\rho$ area sites were initially occupied while $\mathcal{P}(1 - \rho)$ perimeter sites were vacant. Now, to guarantee caging, we demand

$$\mathcal{A}\rho = \mathcal{P}(1 - \rho).$$  

(1)

In other words, Eq. (1) states that the Sokoban will surely get caged when $\mathcal{A}\rho$ obstacles are pushed clear from its path to occupy the $\mathcal{P}(1 - \rho)$ perimeter sites that were initially empty.

Due to the inherent randomness of the Sokoban walk the $\mathcal{A}$ and $\mathcal{P}$ defined above are random, taking different values with every realization. However, averaging over many realizations we observe that these quantities obey a power-law scaling

$$\langle \mathcal{A} \rangle \simeq A r_{\omega},$$

(2)

and

$$\langle \mathcal{P} \rangle \simeq B r_{\omega}^\gamma,$$

(3)

as shown in Fig. 3c. Underlying these relations is what seems to be a fractal shape of the Sokoban’s trajectories (Fig. 3b). Substituting $\mathcal{A}$ and $\mathcal{P}$ in Eq. (1) by their averages and rearranging we obtain

$$r_{\omega} = \left( \frac{1 - \rho}{C \rho} \right)^{1/\gamma},$$

(4)

where $\gamma = \alpha - \beta$ and $C = A/B$.

Equation (4) conveys a relation between the exploration radius, $r_{\omega}$, of the Sokoban random walk and the obstacle density $\rho$. To test it, we fit the simulations data in Fig. 3 and estimate the parameters $\{A, B, \alpha, \beta\}$ which govern the power law scaling of $\langle \mathcal{A} \rangle$ in Eq. (2) and $\langle \mathcal{P} \rangle$ in Eq. (3). Using these estimates we obtain $\gamma \approx 0.182$ and $C \approx 0.397$. Substituting these numbers back into Eq. (4), we plot the predicted relation (dashed line, Fig. 3b). We compare this prediction to direct estimates of the mean exploration radii that were obtained from the asymptotic MSDs at different obstacle densities (markers, Fig. 3b). Very good agreement is found between the prediction of Eq. (4) and data coming from simulations.

While the agreement between Eq. (4) and data coming from simulations is very good, it is not perfect. One source of error comes from the conservative assumption that was made while writing Eq. (1). Namely, that the Sokoban gets caged only when all visited (area) sites are empty and all perimeter sites are occupied. However, we find that caging usually occurs earlier, with some visited sites still occupied by obstacles and some perimeter sites still empty. This can happen as the Sokoban may get trapped in a small micro-environment that becomes isolated from the rest of the arena after caging occurs. Yet, modifying Eq. (1) to state that the Sokoban gets caged when a fraction $f_A$ of the obstacles that resided in visited sites were pushed to occupy a fraction $f_P$ of perimeter sites that were initially vacant, $\mathcal{A} f_A \rho = \mathcal{P} f_P (1 - \rho)$, yields $C = A f_A / B f_P$ in Eq. (4) and does not change $\gamma$.

Interestingly, for the obstacle densities examined here, we find that the average fraction $f_A$ is only slightly larger than $f_P$, thus explaining the slight overestimate in the theoretical prediction of $r_{\omega}$ compared with simulations data (Fig. 3b). Whether larger deviations from Eq. (4) arise for smaller obstacle densities is currently unknown; but cannot be entirely ruled out since $f_A$ and $f_P$ also show some dependence on $\rho$.

Summarizing, in this paper we introduced a new model for random walks in disordered media. Contrary to the canonical “ant in a labyrinth” model, the Sokoban random walk considered herein actively interacts and modifies its surroundings by pushing obstacles in its course of motion. We studied the dynamics of the Sokoban using extensive Monte-Carlo simulations, measured its MSD, and compared it to that obtained for a simple random walk in the presence of obstacles. At obstacle densities above the percolation threshold, we find that the Sokoban typically roams much further than a simple random walk that cannot push away obstacles that block its path. However, at obstacle densities that are close to the percolation threshold and lower, there is a striking change of trend: while the simple random walk becomes unbounded, the Sokoban random walk remains confined (caged).

A conservative rule regarding the onset of caging was used, in tandem with fractal scaling laws, to derive Eq. (4) which relates the density of obstacles to the asymptotic root MSD of the Sokoban. This equation explained the observed density dependence of the mean exploration radius. A prime corollary coming from Eq. (4) is that the exploration radius remains finite for any positive obstacle density $\rho > 0$, suggesting that the Sokoban undergoes dynamical caging at all obstacle densities. However, Eq. (4) is not exact. In addition, numerical determination of the mean exploration radius at extremely low obstacle densities is very challenging computationally and beyond our reach. Thus, the existence of a critical density in the Sokoban model, or lack of it thereof, remains to be proven.

Despite their superficial similarity, the Sokoban and simple random walks exhibit qualitatively different behaviours. While the simple random walk becomes unbounded below a critical obstacle density, the Sokoban random walk undergoes dynamical caging well beyond this density. From this we learn that the ability to push away obstacles is not always beneficial for a random walker seeking to explore its surroundings. Indeed, depending on the obstacle density, the asymptotic MSDs of the Sokoban and simple random walks may differ by orders of magnitude. However, for a very narrow range of densities near the percolation threshold the asymptotic MSDs are similar, thus making it difficult to discriminate the two walks based solely on this static measure. To this end, we recall that a simple random walk on a percolation cluster displays sub-diffusive behaviour. In contrast, we find that the short time behaviour of the Sokoban is diffusive, showing MSDs that grow linearly with time (Fig. 5).

Short-ranged particle-media interactions are often neglected as they are not believed to significantly impact transport properties at the macro scale. However, the findings presented herein vividly demonstrate that even a limited ability of a random walker to dynamically modify its local environment, could drastically alter its long-ranged transport behaviour. In such cases, where strong deviations from the inert “ant in a labyrinth” paradigm occur, the Sokoban provides an alternative null model that could be further adapted to capture essential physics and details that were absent from the treatment and discussion presented above.

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Figure S1: Dynamical caging of the Sokoban random walk. Sites visited by a Sokoban random walk on a lattice with obstacle density of $\rho = 0.4$ (below percolation threshold). The observed configurations after $10^7$ and $10^8$ steps are identical indicating that the Sokoban is caged.

Figure S2: Fractals formed by the Sokoban random walk. Snapshots of all sites visited by the Sokoban random walk on a lattice with obstacle density $\rho = 0.3484$ (below the percolation threshold).
Figure S3: The *Sokoban* random walk displays diffusive mean squared displacement (MSD). The MSD of the *Sokoban* random walk as a function of time, for five different obstacle densities: $\rho = 0.37, 0.40, 0.45, 0.50, 0.55$. At short times the MSD scales linearly with time before saturating to its asymptotic value.