Strong gravitational lensing by Sgr A*

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Abstract
In recent years, there has been increasing recognition of the potential to use the galactic center as a probe of general relativity in the strong field. There is almost certainly a black hole at Sgr A* in the galactic center, and this would allow us to have the opportunity to probe dynamics near the exterior of the black hole. In the last decade, there has been theoretical research into extreme gravitational lensing in the galactic center. Unlike in most applications of gravitational lensing, where the bending angle is of the order of, at most, an arc minute, very large bending angles are possible for light that closely approaches a black hole. Photons may even loop multiple times around a black hole before reaching the observer. There have been many proposals to use light’s close approach to the black hole as a probe of the black hole metric. Of particular interest are the properties of images formed from the light of S stars orbiting in the galactic center. This paper will review some of the attempts made to study extreme lensing as well as extend the analysis of S star lensing. In particular, we are interested in the effect of a Reissner–Nordstrom like \(1/r^2\) term in the metric and how this would affect the properties of relativistic images.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction
Over the last generation, increasingly precise observations of the compact radio source Sgr A* at the Galactic center and its surrounding stars have convinced physicists that only a very gross deviation from general relativity (GR) could allow for the absence of a black hole there [1]. The black hole is estimated to have a mass of about \(4.31 \times 10^6 M_\odot\) and be at a distance of about 8.33 kpc from Earth (these values are used for computations in this paper). Some proposed methods for testing GR in the ‘laboratory’ of Sgr A* include imaging the accretion flow around the black hole as well as frame dragging effects on stars orbiting close to the black hole [2]. In
Figure 1. The weak field approximation (the bending angle is $2/x_0$, where $x_0$ is the point of closest approach in Schwarzschild radii) and exact bending angle as a function of the closest approach. The $x$ axis is in terms of Schwarzschild radii. This figure shows that, except for lensing in the immediate vicinity of very compact objects, the weak field approximation is close to exact. However, close to the black hole, the approximation breaks down.

Figure 1 shows the comparison between the weak field approximation and the exact bending angle as a function of the closest approach. The weak field approximation is close to exact for points away from the black hole, but it breaks down near the black hole due to the strong gravitational effects. This figure is crucial for understanding how lensing works near black holes and how it can be used to test general relativity (GR) in the galactic center.

In a Schwarzschild chart (in which the coordinate $r$ corresponds to one of the nested round spheres filling the spacetime) on a spherically symmetric, static spacetime, the line element takes the form

$$ds^2 = -A(r)\,dt^2 + B(r)\,dr^2 + r^2\,d\Omega^2.$$  

In this general case, the bending angle is an elliptic integral based on the functions of the metric [5] and is

$$\alpha(r_0) = 2\int_{r_0}^{\infty} \sqrt{B(r)} \left[ \left( \frac{r}{r_0} \right)^2 \frac{A(r_0)}{A(r)} - 1 \right]^{-1/2} \frac{dr}{r} - \pi. \tag{2}$$

As $r_0$ gets smaller, the bending angle becomes larger, diverging at the photon sphere at $r_m$, where photons are in a circular orbit. The orbit is unstable if the function $r^2/A(r)$ has a positive second derivative at its minimum (where this minimum represents the radius of the photon sphere and it is assumed that there is one local minimum [18]).

Although the possibility of very large bending angles leading to multiple images was first mentioned in the 1950s [6], the potential for studying lensing effects due to a large bending angle is only recently being more fully explored. In this paper, we will discuss the potential for using lensing in the galactic center as a probe of the metric around a black hole, particularly looking for deviations from the Schwarzschild metric that take the form of a $1/r^2$ term.
black hole with charge as well as some modifications of GR can be represented by the ‘tidal Reissner–Nordstrom’ (TRN) metric \[7\], which is

\[
dS^2 = - \left( 1 - \frac{2M}{r} + \frac{Q}{r^2} \right) dt^2 + \left( 1 - \frac{2M}{r} + \frac{Q}{r^2} \right)^{-1} dr^2 + r^2 d\Omega^2, \tag{3}
\]

with \( Q = q^4 M^2 \) and \( q \) being a free parameter. Note that, in this case, \( Q \) should be seen as a free parameter as opposed to a physical quantity such as electrical charge, as in the case of the Reissner–Nordstrom metric. The TRN metric differs from the Reissner–Nordstrom metric in that \( Q \) does not represent charge, and is therefore allowed to be both positive and negative. While this metric should be seen as a generic modification of gravity, it is intriguing because the \( 1/r^2 \) correction term is suggestive of higher dimensional theories, and the TRN metric is a possible solution for the black hole on the brane in the Randall–Sundrum II model. Background on the RS braneworlds, black holes in RS braneworlds, and the suitability of the TRN metric for use in a large black hole can be found in \[8–11\].

In section 2, we will discuss the recent studies of relativistic images and their theoretical use in probes to distinguish a Schwarzschild black hole from a black hole in modified gravity, particularly a braneworld black hole. As can be seen from the form of the bending angle integral in equation (2), the exact properties of gravitational lensing are dependent on the metric functions. An excellent review of the mathematical formalism for lensing by black holes is in \[12\]. This paper will concentrate more on the astrophysical predictions made by recent studies of black hole lensing as well as derive some new results. In section 3, we will discuss the more recent application of lensing to the secondary images of S stars orbiting in the galactic center and present new results regarding the relativistic images of these stars. S stars are members of a cluster of stars, mainly bright O, B, and Wolf–Rayet stars that have been found in the central arcsecond of our galaxy. These stars are remarkably fast moving sources and are numbered in the order of their discovery \[13\]. We conclude in section 4.

2. Relativistic images

In 2000, Virbhadra and Ellis \[14\] revived interest in the study of very strong gravitational lensing. Because black holes are extremely compact, they are able to cause photons to loop around the black hole one or more times before they reach the observer. This is schematically illustrated in figure 2. This differentiates relativistic images from primary and secondary images; the primary image is the outermost image that appears on the same side of the lens as the source. The secondary image is the outermost image on the opposite side of the source. The images that cluster close to the lens are the relativistic images. They \[14\] calculated the location and magnification of the outermost two relativistic images on both sides of the source for a variety of image positions. They showed that for all realistic source positions, relativistic images are extremely demagnified. In later work, the authors of \[5, 15\] showed that relativistic image properties near Sgr A* will be different if the Schwarzschild metric is replaced by a Janis–Newman–Winicour metric associated with a static, spherically symmetric real scalar field. They catalogue the effect of the scalar charge on relativistic image properties. This demonstrated the potential for extreme lensing to differentiate between different types of matter in the galactic center. The results in these studies were derived by numerically solving the Virbhadra–Ellis lens equation for the image position and for the image magnification. Details can be found in \[9, 11, 14\].

Early attempts at developing an analytic approach to relativistic images were made by \[6\]. More recently, \[16\] expressed equation (2) as an expansion of an elliptic integral diverging at the photon sphere. They found the first-order expansion of the elliptic equation from its
Figure 2. A sketch of the null geodesics forming the first two relativistic images on the primary image side of the optic axis. The figure on the left represents the first relativistic image, which forms when a photon loops around the black hole once. The figure on the right represents the second relativistic image which forms even closer to the black hole, after looping around twice. In reality, both images are seen by the observer to be very close to the black hole’s photon sphere.

divergence and used it in place of equation (2). This approach was updated in [17, 18] to include arbitrary source and observer positions. This approach is later used to calculate lensing observables in the braneworld scenario [19]. In [9, 17], analysis showed that for large bending angles ($\alpha > \pi$), the analytic approach and numerical approach are nearly identical except in very special spacetimes.

Later studies showed the results of lensing of light around Sgr A* if the mass at Sgr A* was assumed to be a Reissner–Nordstrom black hole [20], a braneworld black hole [9, 19, 21–24], or other exotic objects [25–27]. Work on braneworld black holes found that a negative value for $Q$ in the TRN metric creates a larger photon sphere, so relativistic images are further from the optic axis than in a Schwarzschild spacetime. In addition, they find that using the TRN metrics causes the relativistic images to cluster closer to the black hole; there is not as much of a separation between the first two relativistic images as there is in a Schwarzschild spacetime. Finally, having a negative value of $Q$ makes the ratio between the brightness of first relativistic image and the sum of the rest of the images larger. This means that the first relativistic image is brighter compared to the second image in the TRN spacetime. In [9], we studied the lensing effects of primordial black holes which had fixed quantities of the tidal charge.

The upcoming Multi-AO Imaging Camera for Deep Observations (MICADO) telescope at the European Extremely Large Telescope (E-ELT) [28] is projected to come online in about 2018 and have a maximum sensitivity of about the 30th magnitude and resolution of about 10 mas in the near infrared (and an accuracy of about 10 $\mu$ arcsec in astrometric mode). While the astrometric measurements are close to the scale on which relativistic images operate, MICADO will not be nearly sensitive enough to observe relativistic images. In addition, the galactic center is not a clean environment for observations. Clearly, relativistic images will not be a part of observational astronomy for the foreseeable future. If the observation of relativistic images is nearly impossible, using the properties of relativistic images to differentiate between theories will be even more difficult and is even further in the future.
3. Using S stars as sources

While it may not be feasible to observe relativistic images, there are other lines of inquiry into lensing where the bending angle is large and observational prospects are not as dismal. Holz and Wheeler [29] consider the idea that a small black hole of roughly a solar mass can lens the light from the sun and redirect it back to Earth. In the case when the Earth is directly in between the Sun and the black hole, the bending angle is $\pi$. They termed such a black hole a ‘retro-MACHO’ and they found that a $10M_\odot$ black hole at $10^{-2}$ pc from the sun (and on the opposite side of the Earth) and nearly perfectly aligned will produce a ring with a magnitude of 26.1 in the visual part of the spectrum. However, this value drops off sharply as the black hole falls out of the maximum alignment. In [30], the effect of ‘retro-lensing’, or lensing with a bending angle of about $\pi$, is studied for the case of the star S2, which orbits around the galactic center. They find a maximum brightness in the $K$ band of $m_K = 30$ before extinction of light from the dust in the galactic center is considered. Finally, by using orbital parameters of the S stars in the galactic center, [31] study the properties of secondary images of those S stars. We will examine this possibility more closely.

Modeling a source orbiting around a lens requires a slightly different formalism for the lens equation and source angles. The formalism used in this paper is illustrated in figure 3. At each point in time, the star is treated as a source being lensed by the black hole. Using orbital parameters and the intrinsic brightness of the stars in the K-band, [31] calculated the position and magnitude of the secondary and relativistic images of many stars, assuming a Schwarzschild metric. Over time, the brightness of the images would change, corresponding to the changing values of $D_{LS}$ and $\gamma$. In [10], we extended the analysis of secondary images of S stars to the TRN metric. We showed that if the value of $Q$ in equation (3) is negative, the brightness of the secondary image is increased at all times relative to the $Q = 0$ (Schwarzschild) case. For a large value of $-Q$, this increase in brightness can be
Table 1. Orbital parameters of S stars studied in this paper: \( a \) is the semimajor axis given by angular size on the sky, \( e \) is the eccentricity, \( i \) is the inclination of the normal of the orbit with respect to the line of sight, \( \Omega \) is the position angle of the ascending node, \( \omega \) is the periapse anomaly with respect to the ascending node, \( t_p \) is the epoch of either the last or next periapse, \( T \) is the orbital period, and \( K \) is the apparent magnitude in the \( K \) band (data taken from [1]).

| Star | \( a \) (\( \prime\prime \)) | \( e \) | \( i \) (\( ^\circ \)) | \( \Omega \) (\( ^\circ \)) | \( \omega \) (\( ^\circ \)) | \( t_p \) (yr) | \( T \) (yr) | \( K \) |
|------|-----------------|-----|------|--------|------|-------|-------|----|
| S2   | 0.123 ± 0.001   | 0.88 ± 0.003 | 135.25 ± 0.47 | 225.39 ± 0.84 | 63.56 ± 0.84 | 2002.32 ± 0.01 | 15.8 ± 0.11 | 14  |
| S14  | 0.256 ± 0.01    | 0.963 ± 0.006 | 99.4 ± 1.    | 227.74 ± 0.7  | 339 ± 1.6   | 2000.07 ± 0.06 | 47.3 ± 2.9  | 15.7|
| S6   | 0.436 ± 0.153   | 0.886 ± 0.0026 | 86.44 ± 0.59 | 83.46 ± 0.69  | 129.5 ± 3.1 | 2063 ± 21   | 105± 34     | 15.4|

significant. In addition, we considered positive values of \( Q \), including an extremal Reissner–Nordstrom (ERN) black hole in the case \( Q = M^2 \). We showed that, for positive \( Q \), the secondary images will sometimes be brighter and sometimes fainter than the \( Q = 0 \) case, but there will not be a very significant difference between the brightness of the two images, even for an extremal black hole \( (q = 0.25) \). Detection of secondary images and their properties can therefore be considered a probe of the black hole metric. While it is more likely that other methods, including direct imaging, will fix the size of the black hole and its photon sphere, gravitational lensing can be a complementary probe of the galactic center and its black hole. For further elaboration, see [10].

In this paper, we extend this analysis to relativistic images and examine the first relativistic image of the stars S2, S6, and S14. Image positions are obtained by solving a modification of the Ohanian lens equation [10, 32]:

\[
\gamma = \alpha(\theta) - \frac{D_L}{D_{LS}} \theta. \tag{4}
\]

All of the terms in this equation are explained in the caption to figure 3. In [10, 31], the Keplerian orbits are solved, yielding the functions \( \gamma(t) \) and \( D_{LS}(t) \). We are particularly interested in S stars that have a close approach to Sgr A*, or, alternatively, with orbits well aligned with the galactic plane. This means that the minimum value for \( \gamma \) over the course of its orbit is small and its secondary and relativistic images will be relatively bright. The orbital parameters of several of these stars are in table 1. The magnification of an image at any given point is given by [31]

\[
\mu = \frac{D_L^2}{D_{LS}^2} \frac{\sin \theta}{\sin \gamma} \frac{d\theta}{d\gamma}. \tag{5}
\]

In the strong deflection limit, for very large bending angles \((\alpha > \pi)\), the bending angle is of the form:

\[
\alpha(\theta) = \bar{\alpha} \log \left( \frac{\theta}{\theta_m} - 1 \right) + \bar{b}, \tag{6}
\]

where \( \theta_m \) is the angular size of the photon sphere on the observer’s sky and can be calculated theoretically if we know the parameters of the black hole metric \((\theta_m \text{ can also be possibly observed by imaging of accretion disks—see [4]})\). The quantities \( \bar{\alpha} \) and \( \bar{b} \) are functions of the metric determined for an arbitrary static spherically symmetric, metric in [18]. The values of the functions for the three cases considered in this paper are in table 2. The time dependence of \( \theta \) for large \( \alpha \) is solved and expressed as

\[
\theta(t) = \theta_m[1 + e^{(\bar{b} - \gamma(t))/\bar{\alpha} \pi}]. \tag{7}
\]

6
Table 2. $a$ and $b$ are functions of the metric that change for different values of $q$. In this table we list the values of these functions for the TRN ($q = -1.6$), ERN ($q = 0.25$), and Schwarzschild cases ($q = 0$).

| Case | $a$  | $b$  |
|------|------|------|
| TRN  | 0.833| -0.508|
| Sch  | 1    | -0.400|
| ERN  | 1.414| -0.733|

The magnification is derived from equation (5). The form of $\theta(t)$ in equation (7) makes it straightforward to calculate $d\theta/d\gamma$. The magnification is then

$$\mu(t) = -\frac{D_S^2}{D_{LS}^2(t)} \frac{\theta_m^2 e^{(\delta-\gamma(t))/\pi}[1 + e^{(\delta-\gamma(t))/\pi}]}{a \sin \gamma(t)}. \quad (8)$$

The formulations in equations (7) and (8) are particularly powerful because the time dependence of $D_{LS}$ and $\gamma$ are easily determined by known orbital parameters.

To consider the case of a relativistic image, we consider the source to be at $\gamma(t) + 2\pi$. This reflects the scenario of light looping around the black hole before reaching the observer. We consider this to be the first relativistic image, and we examine its properties in the next section.

3.1. Relativistic image positions and brightness

Relativistic images form very close to the angular position of the photon sphere. This is because light from a star would need to pass close enough to the photon sphere to generate a large enough deflection for it to loop around the black hole before reaching the observer. The greater the bending angle required to reach the observer (equivalently, the larger $\gamma$ is), the closer light must approach to the black hole. We would like to analyze where relativistic images appear relative to the photon sphere in the Schwarzschild, TRN ($q = -1.6$), and the ERN ($q = 0.25$) spacetimes. Using our adopted values for the mass and distance of the black hole at the galactic center, the size of the photon sphere depends on the value of $q$. For an observer on Earth, the size of the photon sphere is $\theta_m = 27.5 \mu$ arcsec in a Schwarzschild spacetime, $\theta_m = 44.8 \mu$ arcsec in a TRN spacetime, and $\theta_m = 21.2 \mu$ arcsec in an ERN spacetime. In figure 4, we consider lensing of the S2 star considering three different metrics. Our plot shows the ratio of the image position to $\theta_m$ over time for three different values of $q$.

There are several interesting results in figure 4. Firstly, although the image position in the ERN spacetime is smaller than the image position in the other two spacetimes, the image shifts considerably more relative to the photon sphere (meaning, how far the image appears from the photon sphere as a percentage of the size of the photon sphere. As the photon sphere size is itself a function of $q$, this offers us a way to compare image properties with different values of $q$ in the metric). In terms of the different spacetimes, relativistic images appear closest to the photon sphere in the TRN spacetime, a little more distant from the photon sphere in the Schwarzschild spacetime, and the most shifted images appear in an ERN spacetime. Secondly, the shift relative to the photon sphere, for all spacetimes, in smallest for S2, greater for S14, and greatest for S6 (the plots are not shown for S6 and S14). Understanding these effects can give us insight into the strong lensing that takes place near a black hole.

The second effect, images appearing close to the photon sphere for S2 (when compared to S6 or S14), is due to S2’s large angular distance from the optic axis even at its peak alignment.
Figure 4. The ratio of $\theta/\theta_m$ over time for S2 for slightly more than one orbital period. Each plot represents the ratio of $\theta(t)/\theta_m$ for that value of $q$. So the plot corresponding to the TRN spacetime actually plots $\theta(t)/(44.8 \mu \text{arcsec})$ when $q = -1.6$ and the ERN plot represents $\theta(t)/(21.2 \mu \text{arcsec})$ when $q = 0.25$.

Figure 5. This figure displays $\alpha(r)$ for the three metrics considered in this paper. The point of closest approach is given in coordinates normalized by $r_m$. Here, we show the bending angle very close to the photon sphere. In this way, we see the dependence of the bending angle on distance from the photon sphere. As can be seen, the bending angle for the ERN is noticeably larger than the other two near the photon sphere.

($\gamma_{\text{min}} = 45.3^\circ$). Since S2 is at a large angular position, light must pass close to the photon sphere to generate the extra deflection required to make up for the large value of $\gamma$.

The first effect noted above, the relationship between the spacetime and position of the relativistic image relative to the photon sphere, can be understood by considering the plot of the bending angle in each spacetime. In figure 5, we plot the bending angle of the three spacetimes as a function of the closest approach. The bending angle goes to 0 as $r_0 \to \infty$ for all the metrics. However, for the ERN metric, this happens the slowest. The bending angle for the Schwarzschild spacetime asymptotes to 0 more quickly, and the asymptote is quickest for the TRN spacetime. By inspection of figure 5, the value of $r_0/r_m$ required to
generate a particular bending angle is smallest for the TRN spacetime and largest for the ERN spacetime. Hence, the image in the ERN spacetime appears furthest from the photon sphere in figure 4.

Next, we used the formula for image magnification in equation (8) to calculate the apparent brightness of the first relativistic image of S2 over the course of its orbit around Sgr A*. This is shown in figure 6.

At peak brightness, and at all other times, the image is brightest if the metric is ERN, which contrasts with the case of the secondary image, in which the image is brightest for the TRN metric. For relativistic images, the radial magnification of each image is greatest for the ERN spacetime because of the slower asymptote of the $\alpha(r_0)$ curve in the ERN spacetime. The flatter nature of this curve makes the term $dy/\phi \theta$ smaller in equation (5) and, therefore, makes the magnification larger. At peak, if the spacetime is ERN, the image is about 1.2 magnitudes brighter than if the spacetime would be Schwarzschild, and about 1.7 magnitudes brighter than if the spacetime would be TRN. When the images are fainter, the differences are even greater; if the spacetime is ERN, the image can be up to 2.5
magnitudes brighter than if the spacetime is TRN and as many as 1.7 magnitudes brighter than if the spacetime is Schwarzschild. The peak brightness of all images, both secondary and relativistic, occurs during the period between periapse and maximum alignment about 13 days later.

Interestingly, the peak brightness of relativistic images occurs a bit earlier than the peak brightness of the secondary image. And even within the set of relativistic images, peak brightness occurs at slightly different times for different spacetimes. The image in the ERN spacetime peaks about 7.9 days after periapse, the Schwarzschild image 8.4 days, the TRN image 8.7 days, and the secondary image 10.4 days. As S2 moves away from periapse, the $1/D_{LS}^2$ term in the magnification contributes toward dimming the image (as $D_{LS}$ is increasing). However, as $\gamma$ continues to grow smaller as S2 approaches its maximum alignment, the quantities $\theta$ and $1/(d\theta/d\gamma)$ grow and contribute to brightening the image. For part of the period between periapse and maximum alignment, the increase in $\theta$ and $1/(d\theta/d\gamma)$ offset the decrease in $1/D_{LS}^2$ and the image continues to brighten. However, as $\gamma$ grows smaller, the photons do not approach as close to the black hole as they require a smaller bending angle.
Since the bending angle function in all spacetimes flattens as $r_0$ gets larger, $1/(d\theta/d\gamma)$ and $\theta$ grow at a smaller rate; eventually, the decreasing $1/D_{LS}^2$ term dominates and the image starts to grow fainter. For an image in an ERN spacetime $r_0/r_m$ is greater than it is for a TRN or Schwarzschild spacetime—for a particular source and required value of $\alpha$, light in the TRN requires a path that brings it closer to the photon sphere. In an ERN spacetime, however, a larger value of $r_0/r_m$ generates the same bending angle. Therefore, the image in the ERN spacetime is in a flatter part of the bending angle function in figure 5, and the bending angle does not decrease as quickly as $\gamma$ decreases. Hence, the radial magnification does not go up as quickly as $\gamma$ approaches periapse, and the image in the ERN reaches peak brightness quicker. Since the image in the TRN spacetime forms closer to the black hole, near the photon sphere, as $\gamma$ decreases, the bending angle decreases quicker, leading to a greater increase in radial magnification, and the image continues to grow brighter. A similar dynamic plays out for other stars we have analyzed, and the light curves for the first relativistic image of S6 and S14 are displayed in figure 7.

4. Conclusion

We have analyzed the difference between relativistic images in extremal Reissner–Nordstrom, tidal Reissner–Nordstrom, and Schwarzschild spacetimes. Although observational capabilities are nowhere close to being able to see these images, and the uncertainties in these stars’ orbital parameters are larger than some of the effects discussed, the study of these details challenges and expands our understanding of the magnification mechanism, dynamics in the vicinity of a black hole, and the subtle effects that come into play for relativistic images. These lessons can be used for further studies of images in the galactic center. The fact that the properties of images are sensitive to the choice of metric indicates that gravitational lensing in the galactic center may one day be an important probe of GR.

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