THE $\gamma$-FILTRATION AND THE ROST INVARIANT

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Abstract. Let $X$ be the variety of Borel subgroups of a simple and strongly inner linear algebraic group $G$ over a field $k$. We prove that the torsion part of the second quotient of Grothendieck’s $\gamma$-filtration on $X$ is a cyclic group of order the Dynkin index of $G$.

As a byproduct of the proof we obtain an explicit cycle $\theta$ that generates this cyclic group; we provide an upper bound for the torsion of the Chow group of codimension-3 cycles on $X$; we relate the cycle $\theta$ with the Rost invariant and the torsion of the respective generalized Rost motives; we use $\theta$ to obtain a uniform lower bound for the essential dimension of (almost) all simple linear algebraic groups.

Grothendieck’s celebrated $\gamma$-filtration of the ring $K_0(X)$ gives a way to estimate the Chow group $\text{CH}^i(X)$ of algebraic cycles on $X$ modulo rational equivalence when $X$ is a smooth projective variety over a field $k$. Namely, by the Riemann-Roch theorem without denominators [8, §15.3] the $i$-th Chern class provides a well-defined group homomorphism $c_i : \gamma^{i+1}(X) \rightarrow \text{CH}^i(X)$, $i \geq 0$ from the $i$-th quotient of the $\gamma$-filtration to the Chow group of codimension-$i$ cycles on $X$. Observe that for $i = 0$ it is the identity map and for $i = 1$ it is an isomorphism identifying $\text{CH}^1(X)$ with the Picard group of $X$.

In the present paper we study these homomorphisms in the cases $i = 2, 3$ and $X$ is a generically split projective homogeneous variety under a semisimple linear algebraic group $G$. Our core determines and bounds respectively the torsion subgroup of $\gamma^{2/3}(\mathcal{B})$ and $\gamma^{3/4}(\mathcal{B})$ for the variety of Borel subgroups $\mathcal{B}$ of strongly inner $G$ (Theorem 3.1). For instance, we show that the torsion subgroup $\text{Tors} \gamma^{2/3}(\mathcal{B})$ is cyclic of order the Dynkin index of $G$ and exhibit a generator $\theta$ for it (Definition 3.3).

This fact together with the Riemann-Roch theorem imply (see 4.4) that the surjection

$$\langle \theta \rangle = \text{Tors} \gamma^{2/3}(\mathcal{B}) \rightarrow \text{Tors} \gamma^{3/4}(\mathcal{B})$$

can be viewed as a substitute of the key map $Q(V) \rightarrow H^3(k, \mathbb{Q}/\mathbb{Z}(2))$ in the definition of the Rost invariant [10] pp. 126-127. Indeed, a theorem of Peyre-Merkurjev [27] shows that $\text{Tors} \gamma^{2/3}(\mathcal{B})$ can be identified with the kernel of the restriction $H^3(k, \mathbb{Q}/\mathbb{Z}(2)) \rightarrow H^3(k(\mathcal{B}), \mathbb{Q}/\mathbb{Z}(2))$. Furthermore, the order of $c_2(\theta)$ in $\text{Tors} \gamma^{2/3}(\mathcal{B})$ equals to the order of the Rost invariant of $G$ (see Prop. 3.2).

Our result gives bounds for the torsion in $\text{CH}^3$ for generically split $X$ (see 4.5) and provides explicit generators of torsion subgroups of $\text{CH}^2$ of certain generalized Rost-Voevodsky motives. Note that typically, one does not even know a priori if the torsion subgroup of $\text{CH}^i(X)$, $i \geq 3$, is finitely generated. However, determining the torsion subgroup determines $\text{CH}^i(X)$ as an abelian group, since the dimension of its free part $\text{CH}^i(X) \otimes \mathbb{Q}$ can be easily computed.
where

In section [6] we study the behaviour of the image of $c_2$ under field extensions. In particular, we show that this image is non-trivial if $G$ has Tits algebras of index 2 (Prop. [65]). Using this fact we prove (Prop. [74]) that the essential dimension $ed(G)$ of any absolutely almost simple linear algebraic group not of type $A$ nor isomorphic to $Sp_{2n}$ is greater or equal than 3.

1. Preliminaries

We now provide several facts and observations concerning Chow groups, characteristic maps, invariants, Dynkin indices and filtrations on $K_0$ for varieties of Borel subgroups of split simple linear algebraic groups.

§1A. Two filtrations on $K_0$. All facts provided here can be found in [17 §2], [8 §15] and [9 Ch. 3.5]. Let $X$ be a smooth projective variety over a field $k$. Consider the $\gamma$-filtration on $K_0(X)$. It is given by the subgroups

$$
\gamma^i(X) = \left\langle c_n^K(b_1) \cdots c_m^K(b_m) \mid n_1 + \cdots + n_m \geq i \text{ and } b_1, \ldots, b_m \in K_0(X) \right\rangle,
$$

where $c_n^K$ denote the $n$-th Chern class with values in $K_0$. For example, for the class of a line bundle we have $c_1^K([L]) = 1 - [L^*]$. Let $\gamma^{i/i+1}(X) = \gamma^i(X)/\gamma^{i+1}(X)$ denote the respective quotient. Consider the topological filtration on $K_0(X)$ given by the subgroups

$$
\tau^i(X) = \left\langle [O_V] \mid V \hookrightarrow X \text{ and codim } V \geq i \right\rangle,
$$

where $[O_V]$ is the class of the structure sheaf of a closed subvariety $V$. Let $\tau^{i/i+1}(X) = \tau^i(X)/\tau^{i+1}(X)$ denote the corresponding quotient.

There is an obvious surjection $pr : CH^i(X) \twoheadrightarrow \tau^{i/i+1}(X)$ from the Chow group of codimension $i$ cycles given by $V \mapsto [O_V]$. By the Riemann-Roch Theorem without denominators the $i$-th Chern class induces the map in the opposite direction

$$
c_i : \tau^{i/i+1}(X) \to CH^i(X)
$$

and the composite $c_i \circ pr$ is the multiplication by $(-1)^{i-1}(i-1)!$ which is an isomorphism for $i \leq 2$ [8 Ex.15.3.6]. For example, by the very definition we have

$$
c_i(\prod_{j=1}^i c_1^K([L_j])) = (-1)^{i-1}(i-1)! \prod_{j=1}^i c_1^{CH}(L_j),
$$

where $L_j$ is a line bundle. Observe also that $c_i$ becomes an isomorphism after tensoring with $\mathbb{Q}$.

There is an embedding $\gamma^i(X) \subset \tau^i(X)$ for all $i$. Moreover, $\gamma^i(X) = \tau^i(X)$ for $i \leq 2$. Observe that $\gamma^{1/2}(X) = \tau^{1/2}(X) = CH^1(X)$ is the Picard group and by [17 Cor. 2.15] there is an exact sequence

$$
0 \to \tau^{3/2}(X)/\gamma^{3/2}(X) \to \text{Tors } \gamma^{2/3}(X) \xrightarrow{c_2} \text{Tors } CH^2(X) \to 0,
$$

where we have written $c_2$ for the composition $\gamma^{2/3}(X) \to \tau^{2/3}(X) \xrightarrow{c_2} CH^2(X)$.
§1B. Characteristic maps and invariants. Let $G_s$ be a split simply connected simple linear algebraic group of rank $n$ over a field $k$. We fix a split maximal torus $T$ and a Borel subgroup $B$ such that $T \subset B \subset G_s$. Let $\mathfrak{B}_s$ denote the variety of Borel subgroups of $G_s$ and let $T^*$ denote the group of characters of $T$. We fix a basis of $T^*$ given by the fundamental weights $\omega_1, \ldots, \omega_n$.

Let $S(T^*)$ be the symmetric algebra of $T^*$. Its elements are polynomials in the fundamental weights $\omega_i$ with coefficients in $\mathbb{Z}$. Let $\mathbb{Z}[T^*]$ be the integral group ring of $T^*$. Its elements are integral linear combinations $\sum a_i e^{\lambda_i}$, $\lambda_i \in T^*$. Consider the characteristic maps for $\text{CH}$ and $K_0$ (see [3] §8, 9 and [6] §1.5, 1.6)

$$c: S(T^*) \to \text{CH}(\mathfrak{B}_s) \text{ and } c': \mathbb{Z}[T^*] \to K_0(\mathfrak{B}_s)$$

given by

$$c: \omega_i \mapsto c_1^{\text{CH}}(L(\omega_i)) \text{ and } c': e^\lambda \mapsto [L(\lambda)]$$

where $L(\lambda)$ is the line bundle over $\mathfrak{B}_s$ associated to the character $\lambda$.

There are obvious augmentation maps $S(T^*) \to \mathbb{Z}$ and $\text{aug}: \mathbb{Z}[T^*] \to \mathbb{Z}$ given by $\omega_i \mapsto 0$ and $e^\lambda \mapsto 1$ respectively. The Weyl group acts naturally on $T^*$, hence also on $S(T^*)$ and $\mathbb{Z}[T^*]$. Consider the subgroups of invariants $S(T^*)^W$ and $\mathbb{Z}[T^*]^W$. Let $I$ (resp. $I'$) be the ideal generated by the elements of $S(T^*)^W$ (resp. $\mathbb{Z}[T^*]^W$) from the kernel of the augmentation map. Then we have

$$\ker c = I \text{ and } \ker c' = I'.$$

Therefore we have embeddings

$$c: S(T^*)/I \hookrightarrow \text{CH}(\mathfrak{B}_s) \text{ and } c': \mathbb{Z}[T^*]/I' \xrightarrow{\sim} K_0(\mathfrak{B}_s),$$

where the second map is surjective since $G_s$ is simply connected (see [28]).

By [5] §2 and Cor.2 the kernel $I$ of $c$ consists of elements $g$ such that

$$m \cdot g = \sum_i g_i \cdot f_i,$$

for $m \in \mathbb{Z}$, $f_i$ the basic polynomial invariants, and $g_i \in S(T^*)$.

There is a $W$-invariant quadratic form $q$ on $T^* \otimes \mathbb{Q}$ that is uniquely determined up to a scalar multiple [2] §VI.1.1–2]. We normalize $q$ so that it takes the value 1 on every short coroot; as $q$ is indivisible, it can be taken as the generator of $I$ of degree 2. To say it differently, each element of $I$ of degree 2 is a multiple of $q$ by an integer.

The form $q$ should be familiar. Its polar bilinear form $b_q$ amounts to the restriction of the “reduced Killing form” to the Cartan subalgebra of the Lie algebra of $G_s$ as described in [15] §5. In the case where the roots are all one length, an explicit formula for $b_q$ is well known: its Gram matrix is the Cartan matrix of the root system.

§1C. Degree 3 elements of $I$. If $G_s$ is not of type $A_n$ ($n \geq 2$), then there is no basic invariant of degree 3 [16, p. 59]. Then by (1.2) and the indivisibility of $q$, every $g \in I$ of degree 3 can be written as $g = (\sum a_i \omega_i)q$ for some $a_i \in \mathbb{Z}$.

If $G_s$ is of type $A_n$ for some $n \geq 2$, then $S(T^*)^W$ has a basic generator $f_3$ of degree 3. We view the weight lattice as in [2], meaning that it is contained in a lattice with basis $\varepsilon_1, \ldots, \varepsilon_{n+1}$ so that the embedding is defined by $\varepsilon_1 = \omega_1, \varepsilon_{n+1} = -\omega_n$. 


and \( \varepsilon_i = \omega_i - \omega_{i-1} \) for \( 2 \leq i \leq n \), and \( W \) is the permutation group of the \( \varepsilon \)'s. Then \( q \) and

\[
f_3 := (\varepsilon_1^3 + \cdots + \varepsilon_{n+1})/3 = \sum_{i=2}^{n} \omega_{i-1}^2 \omega_i - \omega_{i-1} \omega_i^2
\]

are members of a set of basic invariants \([16] \S 3.12\).

For \( g \in I \) of degree 3, we have \( mg = g_2q + g_3f_3 \) for some \( g_3, m \in \mathbb{Z} \) and \( g_2 = \sum a_i \omega_i \) with \( a_i \in \mathbb{Z} \). On the right side, the monomial \( \omega_i \) occurs only in \( g_2q \) and has coefficient \( ai \), hence \( m \) divides \( a_i \) for all \( i \), hence \( m \) divides \( g_2 \) and also \( g_3 \).

In summary, \( g = (g_2/m)q + (g_3/m)f_3 \) for some \( g_2/m, g_3/m \in \mathbb{S}(T^*) \).

§1D. The \( \gamma \)-filtration on the variety of Borel subgroups. Consider the \( \gamma \)-filtration on the variety \( \mathfrak{B}_s \) of Borel subgroups of \( G_s \). Let \( \gamma^m \) denote the subgroup of \( \mathbb{Z}[T^*] \) generated by products of at least \( m \) elements of the form \( (1 - e^{-\omega_i}) \), where \( \omega_i \) is a fundamental weight. Then the isomorphism \( \epsilon' \) induces an isomorphism

\[
\gamma^{m/m+1}(\mathfrak{B}_s) \simeq \gamma^m/(\gamma^{m+1} + I') \text{ for each } i.
\]

For example \( \gamma^{1/2}(X) \simeq \gamma^1/(\gamma^2 + I') \) is a free abelian group with a basis given by the classes of the elements

\[
(1 - e^{-\omega_i}) \in \gamma^1, \quad i = 1, \ldots, n.
\]

Indeed, \( c_1^K([L(\omega_i)]) = 1 - [L(-\omega_i)] \), the map \( c_1 : \gamma^{1/2}(\mathfrak{B}_s) \to \text{CH}^1(\mathfrak{B}_s) \) is an isomorphism and the elements \( c_1([L(\omega_i)]) \) for \( i = 1, \ldots, n \) form a basis of the Picard group \( \text{CH}^1(\mathfrak{B}_s) \).

Since \( K_0(\mathfrak{B}_s) \) is generated by classes of line bundles (see \([28]\)), so is \( \gamma^1(\mathfrak{B}_s) \). Therefore, we have

\[
\gamma^i(\mathfrak{B}_s) = \langle c_1^K([L_1]) \cdots c_1^K([L_m]) \mid m \geq i \text{ and } L_j \text{ is a line bundle over } \mathfrak{B}_s \rangle.
\]

Let \( \lambda = \sum a_i \omega_i \) be a presentation of a character \( \lambda \) in terms of the fundamental weights. Then \( L(\lambda) = \otimes_i L(\omega_i)^{a_i} \). Since for any two line bundles \( L_1 \) and \( L_2 \) we have

\[
c_1^K([L_1 \otimes L_2]) = c_1^K([L_1]) + c_1^K([L_2]) - c_1^K([L_1])c_1^K([L_2])
\]

applying this formula recursively we can express any element of \( \gamma^{i+i+1}(\mathfrak{B}_s) \) as a linear combination of the products of the first Chern classes of the bundles \( L(\omega_i) \), \( i = 1, \ldots, n \). For instance, any element of \( \gamma^{2/3}(\mathfrak{B}_s) \) can be written as a class of

\[
\sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{i=1}^{m} \sum_{j=1}^{m} a_{ij}(1 - e^{-\omega_i})(1 - e^{-\omega_j}) \in \gamma^2 \text{ mod } \gamma^3 + I', \text{ where } a_{ij} \in \mathbb{Z}.
\]

§1E. The Dynkin index. Let \( N \) denote the map \( \mathbb{Z}[T^*]^W \to \mathbb{Z} \) defined by fixing a long root \( \alpha \) and setting

\[
N \left( \sum_{i} a_i e^{\lambda_i} \right) := \frac{1}{2} \sum_{i} a_i (\lambda_i, \alpha)^2.
\]

This does not depend on the choice of \( \alpha \) and takes values in \( \mathbb{Z} \) (and not merely in \( \frac{1}{2} \mathbb{Z} \)), cf. Lemma \([2, 25]\) below. The number \( N(\chi) \) is called the Dynkin index of \( \chi \). Note that for \( m \in \mathbb{Z} \), we have \( N(m) = N(me^0) = 0 \), so \( N(\chi) \) only depends on the image of \( \chi \) in the kernel of the augmentation map.
In case $G_s$ has two root lengths, it is natural to wonder what one would find if one used a short root, say, $\delta$ in the definition of $N$ instead of the long root $\alpha$. We claim that

$$\frac{1}{2} \sum a_i \langle \lambda_i, \delta' \rangle^2 = q(\delta') \left[ \frac{1}{2} \sum a_i \langle \lambda_i, \alpha' \rangle^2 \right],$$

where $q$ is the form introduced in §1B. In other words, one obtains something that differs by a factor of $q(\delta')$. (We will use this observation later.) To prove it, define quadratic forms $n_\alpha$ and $n_\delta$ on $T^*$ via $n_\alpha(\lambda) = \sum_{w \in W} \langle w \lambda, \alpha' \rangle^2$ and similarly for $\delta'$. For example, $n_\delta(\alpha) = q(\delta') n_\alpha(\delta)$. But $n_\alpha$ is a $W$-invariant quadratic form on $T^*$, hence it is a scalar multiple of $q$. As $q(\alpha) = q(\delta') q(\delta)$, we have $n_\alpha(\alpha) = q(\delta') n_\alpha(\delta)$. But $n_\delta$ is also a scalar multiple of $q$, so we conclude that $n_\delta = q(\delta') n_\alpha$, proving the claim.

The Dynkin index $N(G_s)$ is defined to be the gcd of $N(\chi)$ as $\chi$ varies over the characters of finite-dimensional representations of $G_s$. The number $N(G_s)$ is calculated in [10], [18], or [19]:

| type of $G_s$ | $A$ or $C$ | $B_n$ ($n \geq 3$), $D_n$ ($n \geq 4$), $G_2$ | $F_4$ or $E_6$ | $E_7$ | $E_8$ |
|---------------|------------|---------------------------------|----------------|-------|-------|
| $N(G_s)$      | 1          | 2                               | 6              | 12    | 60    |

If $G$ is a simple and strongly inner group, then, for the purposes of this paper, we define the Dynkin index $N(G)$ of $G$ to be the Dynkin index $N(G_s)$ of the split simply connected group of the same Killing-Cartan type.

2. Dynkin indices and the map $\phi$

Let $G_s$ denote a split simply connected simple linear algebraic group of rank $n$ over a field $k$. We fix a pinning for $G_s$ and in particular a split maximal torus $T$ and fundamental weights $\omega_1, \ldots, \omega_n$. As $G_s$ is simply connected, $T_s (= \text{Hom}(G_m, T))$ and $T^*$ are canonically identified with the coroot and weight lattices respectively.

2.1. Definition. Put $\mathbb{Z}[T^*] := \mathbb{Z}[e^{\omega_1}, \ldots, e^{\omega_n}]$, the integral group ring, and $S(T^*) := \mathbb{Z}[\omega_1, \ldots, \omega_n]$, the symmetric algebra of $T^*$. We define a ring homomorphism

$$\phi_m : \mathbb{Z}[T^*]/\gamma^{m+1} \rightarrow S(T^*)/(S^{m+1}(T^*)), \ m \geq 2,$$

via $\phi_m (e^{\sum_{i=1}^n a_i \omega_i}) = \prod_{i=1}^n (1 - \omega_i)^{-a_i}.$

In particular, $\phi_m (e^{\omega_i}) = 1 + \omega_i + \cdots + \omega_i^m$ and $\phi_m (e^{-\omega_i}) = 1 - \omega_i$. (Note that $\mathbb{Z}[T^*]$ can be viewed as Laurent polynomials in the variables $\omega_1, \ldots, \omega_n$, and from this perspective it is clear that the formula for $\phi$ gives a well-defined ring homomorphism on $\mathbb{Z}[T^*]$ and $\phi_m (\gamma^{m+1})$ is zero in $S(T^*)/(S^{m+1}(T^*))$.)

The homomorphism $\phi_m$ is an isomorphism. To see this, define a homomorphism $S(T^*) \rightarrow \mathbb{Z}[T^*]/\gamma^{m+1}$ via $\psi_m (\omega_i) = 1 - e^{-\omega_i}$ for all $i$; it induces a homomorphism $S(T^*)/(S^{m+1}(T^*)) \rightarrow \mathbb{Z}[T^*]/\gamma^{m+1}$ that we also denote by $\psi_m$. As the compositions $\phi_m \psi_m$ and $\psi_m \phi_m$ are both the identity on generators, the claim is proved.

The goal of this section is to give a formula for $\phi_2$ on $I'$.

2.2. Proposition. If $G_s$ is simple, then for $\chi \in \mathbb{Z}[T^*]^W$, we have:

$$\phi_2(\chi) = \text{aug}(\chi) + N(\chi) \cdot q \in (S(T^*)/(S^3(T^*)))^W,$$

where $q$ is the invariant form introduced in §1B.
The proof would be much easier if we already knew that $\phi_2$ takes $W$-invariant elements to $W$-invariant elements, but this only comes as a consequence of the proof of the proposition. We give some preliminary material before the proof.

2.3. Example ($SL_2$). In case $G_s = SL_2$, write $\omega$ for the unique fundamental weight. For $n > 0$, we have:

$$\phi_2(e^{n\omega} + e^{-n\omega}) = (1 + \omega + \omega^2)^n + (1 - \omega)^n = 2 + n^2\omega^2,$$

which verifies Prop. 2.2 for this group.

2.4. Example ($SL_2 \times SL_2$). In case $G_s = SL_2 \times SL_2$ there are two fundamental weights $\omega_1, \omega_2$ and the Weyl group $W$ is the Klein four-group; it acts by flipping the signs of $\omega_1$ and $\omega_2$. The definition of $\phi_2$ above makes sense here even though $G_s$ is not simple. We find:

$$\phi_2(W;e^{\omega_1 \omega_2 + a_2 \omega_2}) = 4 + 2 \left[a_1^2 \omega_1^2 + a_2^2 \omega_2^2\right].$$

One final observation about Weyl group actions. We write $W\lambda$ for the $W$-orbit of $\lambda \in T^*$.

2.5. Lemma. For every root $\alpha$ and weight $\lambda \in T^*$, the map $W\lambda \to \mathbb{Z}$ defined by $\pi \mapsto \langle \pi, \alpha' \rangle$ hits $x$ and $-x$ the same number of times, for every $x \in \mathbb{Z}$. If $\alpha, \beta$ are orthogonal roots, then for every weight $\lambda \in T^*$, the map $W\lambda \to \mathbb{Z} \times \mathbb{Z}$ defined by $\pi \mapsto \langle \langle \pi, \alpha' \rangle, \langle \pi, \beta' \rangle \rangle$ hits $(x, y), (-x, y), (x, -y)$ and $(-x, -y)$ the same number of times, for every $x, y \in \mathbb{Z}$.

Sketch of proof. It is an exercise to show the analogous statements for the map $W \to \mathbb{Z}$ defined by $w \mapsto \langle w\lambda, \alpha' \rangle$ and similarly for the second claim. The lemma follows. □

Proof of Prop. 2.2 We may assume that $\chi = \sum e^{\lambda_j}$ where $\lambda_1, \ldots, \lambda_r$ is the Weyl orbit of some $\lambda \in T^*$. Put $\lambda_j = \sum_{i=1}^n a_{ij} \omega_i$, so $\phi(\chi) = \sum_{j=1}^r \prod_{i=1}^n (1 - \omega_i)^{-a_{ij}}$. Obviously, the degree 0 component of $\phi(\chi)$ is $r = \text{aug}(\chi)$.

The degree 1 component of $\phi(\chi)$ is $t_j \sum_{i} a_{ij} \omega_i = \sum_i \left(\sum_j a_{ij}\right) \omega_i$. Here the claim is that $\sum_j a_{ij} = 0$ for each $i$. The $a_{ij}$’s are the images of $W\lambda$ in $\mathbb{Z}$ under the map $\lambda_j \mapsto \langle \lambda_j, \alpha_i' \rangle$ where $\alpha_i$ denotes the simple root corresponding to the fundamental weight $\omega_i$, hence the claim follows from Lemma 2.5.

The crux is to check the claim on the degree 2 component $q_2$ of $\phi(\chi)$: it is an integer-valued quadratic form on the coroot lattice $T_s$ and we check that it equals $q_2 := N(\chi)q$. We write out for $\ell = 1, 2$:

$$q_\ell(\sum_i d_i \alpha_i') = \sum_i d_i^2 q_\ell(\alpha_i') + \sum_{i<j} d_i d_j b_{ij}(\alpha_i', \alpha_j'),$$

where $b_{ij}$ is the polar bilinear form of $q_\ell$. We will check that the value of this expression is the same for $\ell = 1, 2$.

First suppose that $\delta' := \sum_i d_i \alpha_i'$ is a coroot and every $d_i$ is 0 or 1. Then it defines a homomorphism $\eta: SL_2 \to G_s$ so that, roughly speaking, the simple coroot $\alpha'$ of $SL_2$ (viewed as a map $\mathbb{G}_m \to T_1 := \eta^{-1}(T)$) satisfies $\eta(\alpha') = \delta'$. We check that the diagram

$$\begin{array}{ccc}
\mathbb{Z}[T^*] & \xrightarrow{\phi_2} & S(T^*)/(S^3(T^*)) \\
\eta^* & \downarrow & \eta^* \\
\mathbb{Z}[T_1^*] & \xrightarrow{\phi_2} & S(T_1^*)/(S^3(T_1^*))
\end{array}$$

(2.7)
commutes. Since \( \omega_j(\delta^\vee) = d_j \), we have \( \eta^*(\omega_j) = d_j \omega \) for \( \omega \) the fundamental weight of \( SL_2 \) dual to \( \alpha^\vee \). We find:
\[
\eta^* \varphi_2(e^{\sum c_j \omega_j}) = \prod_j (1 - d_j \omega)^{-c_j} = (1 - \omega)^{-\sum c_j d_j},
\]
because the \( d_j \) are all 0 or 1. As this is \( \varphi_2(e^{\sum d_j c_j \omega}) = \varphi_2 \eta^*(e^{\sum c_j \omega}) \), we have confirmed the commutativity of (2.7).

Put \( \delta^2 \) for the composition of \( \varphi_2 \) with the projection onto the degree 2 component \( \mathbb{S}^2 \), so \( q_1 = \delta^2(\chi) \). Then \( q_1(\delta^\vee) = (\eta^* \varphi_2(\chi))(\alpha^\vee) \) obviously, which is \( (\delta^2 \eta^*(\chi))(\delta^\vee) \) by commutativity of (2.7).

We have \( \eta^*(\chi) = \sum_j e^{\sum \alpha_j \omega} \) and by Lemma 2.5, the multiset of the \( j \) integers \( \sum_i a_{ij} d_i \) is symmetric under multiplication by \(-1 \), hence by Example 2.6 we find:
\[
q_1(\delta^\vee) = \frac{1}{2} \left( \sum_j \left( \sum_i a_{ij} b_{ij} \right)^2 \right) = \frac{1}{2} \sum_j (\lambda_j, \delta^\vee)^2.
\]
By (2.6) this equals \( q(\delta^\vee) N(\chi) = q_2(\delta^\vee) \).

Returning to equation (2.6), this shows that the term \( q(\alpha_i^\vee) \) does not depend on \( \ell \). Similarly, if \( \alpha_i^\vee \) and \( \alpha_j^\vee \) are not orthogonal coroots, then \( \alpha_i^\vee \) and \( \alpha_j^\vee \) are adjacent in the Dynkin diagram and \( \alpha_i^\vee + \alpha_j^\vee \) is a coroot \([2 \text{ VI.1.6, Cor. 3b}]\). The preceding two paragraphs show that the value of
\[
b_{q_1}(\alpha_i^\vee, \alpha_j^\vee) = q_1(\alpha_i^\vee + \alpha_j^\vee) - q_1(\alpha_i^\vee) - q_1(\alpha_j^\vee)
\]
does not depend on \( \ell \).

It remains to consider \( b_{q_1}(\alpha_i^\vee, \alpha_j^\vee) \) where \( \alpha_i^\vee \) and \( \alpha_j^\vee \) are orthogonal (relative to the polar form of \( q \)) – it follows that they are orthogonal relative to \( b_{q_2} \). We use \( \alpha_i^\vee \) and \( \alpha_j^\vee \) to define a homomorphism \( \tau : SL_2 \times SL_2 \to G_s \) and as we did for \( SL_2 \) above – we fix a torus \( T_2 = T_1 \times T_1 \subset SL_2 \times SL_2 \) such that \( \tau(T_2) = \text{im}(\alpha_i^\vee \times \alpha_j^\vee) \subset T \).

Arguing using a commutative diagram analogous to (2.7), it suffices to check that the simple roots of \( SL_2 \times SL_2 \) are orthogonal relative to \( \tau q_1 = \delta^2 \tau(\chi) \), which follows from Example 2.4 and Lemma 2.5.

In view of (2.1E) Prop. 2.2 gives:

2.8. Corollary. \( \varphi_2(I') = \mathbb{Z} \cdot N(G_s) \cdot q \).

2.9. Example. Suppose \( G \) has type \( A_n \) for some \( n \geq 2 \). We continue the notation of 2.1C and we compute:
\[
\varphi_3(W e^{\omega_1}) = \varphi_3(e^{\omega_1} + e^{-\omega_n} + \sum_{i=2}^n e^{\omega_1 - \omega_{i-1}}) = (n + 1) + q + \sum_{j=1}^n \omega_j^2 - \sum_{i=2}^n \omega_{i-1} \omega_i^2.
\]
The element \( \Delta := W e^{\omega_1} - W e^{\omega_n} \) is in \( I' \), and by symmetry we see that \( \varphi_3(\Delta) = f_3 \).

3. Torsion in the \( \gamma \)-filtration

Let \( \mathfrak{B} \) denote the variety of Borel subgroups of a strongly inner simple linear algebraic group \( G \) over \( k \). Recall that \( G \) is strongly inner if the simply connected cover of \( G \) is isomorphic to \( G_s \) twisted by a cocycle \( \xi \in H^1(k, G_s) \), where \( G_s \) denotes the simply connected split group of the same Killing-Cartan type as \( G \). Observe that the variety \( \mathfrak{B} \) is always defined over \( k \) by [21 Cor. XXVI.3.6]; it is a
twisted form of the variety of Borel subgroups $\mathfrak{B}_s$ of $G_s$, i.e., $\mathfrak{B}$ and $\mathfrak{B}_s$ become isomorphic over the algebraic closure of $k$.

In the present section we determine and bound respectively the torsion parts of the second and the third quotients of the $\gamma$-filtration on the variety $\mathfrak{B}$. The main result is the following.

3.1. Theorem. Let $\mathfrak{B}$ be the variety of Borel subgroups of a strongly inner simple linear algebraic group $G$ over a field $k$. Then

(i) $\text{Tors} \gamma^{2/3}(\mathfrak{B})$ is a cyclic group of order the Dynkin index $N(G)$ and generated by $\hat{c}^r(\theta)$ for $\theta$ as in Def. 3.3.

(ii) The subgroup $\tau^3(\mathfrak{B})/\gamma^3(\mathfrak{B})$ of $\text{Tors} \gamma^{2/3}(\mathfrak{B})$ is generated by $o(r(G)) \hat{c}^r(\theta)$; it is cyclic of order $N(G)/o(r(G))$.

(iii) $2 \text{Tors} \gamma^{3/4}(\mathfrak{B})$ is a quotient of $(\mathbb{Z}/N(G))^{\oplus \text{rank } G}$.

Proof. It is a straightforward calculation, using the definition of Panin [25, Thm. 2.2.(1)].

3.2. Proposition. Let $X$ be a projective homogeneous variety under $G$. If $G$ is split by $k(X)$, then $\text{Tors} \text{CH}^2(X)$ is a cyclic group whose order is the same as the order of $r(G)$ in $H^3(k, \mathbb{Q}/\mathbb{Z}(2))$; in particular its order divides $N(G)$.

Proof. We view $\xi$ as a principal homogeneous $G$-variety. The kernel of the scalar extension map $H^3(k, \mathbb{Q}/\mathbb{Z}(2)) \to H^3(kG, \mathbb{Q}/\mathbb{Z}(2))$ is the cyclic group generated by $r(G)$ by [10] p. 129. For every extension $L/k$, $\xi$ has an $L$-point if and only if $G$ is split, if and only if $X$ has an $L$-point. Therefore, this kernel is the same as the kernel of $H^3(k, \mathbb{Q}/\mathbb{Z}(2)) \to H^3(k(X), \mathbb{Q}/\mathbb{Z}(2))$. A theorem of Peyre-Merkurjev [27] shows that this kernel is isomorphic to $\text{Tors} \text{CH}^2(X)$. \qed

Obviously, one can take $X = \mathfrak{B}$ in the proposition. Furthermore, the same proof shows that the proposition still holds if one replaces "$G$ is strongly inner" and "$G$ is split by $k(X)$" with "$G$ has trivial Tits algebras" and "$G$ becomes quasi-split over $k(X)$".

Also, statement 3.1. makes use of the following definition.

3.3. Definition. Write the form $q$ from subsection 3.1B (relative to the split group $G_s$) as $q = \sum_{i\leq j} c_{ij} \omega_i \omega_j \in \mathbb{S}(T^*)$. We call the element

$$\theta := \sum_{i\leq j} c_{ij} (1 - e^{-\omega_i})(1 - e^{-\omega_j}) \in \mathbb{Z}[T^*]$$

the special cycle. Its image in $\mathbb{Z}[T^*]/\gamma^{m+1}$ is $\psi_m(q)$ for all $m \geq 2$.

Proof of Theorem 3.1. By the result of Panin [25, Thm. 2.2.(2)] since $G$ is strongly inner, the restriction map

$$\text{res}: K_0(\mathfrak{B}) \to K_0(\mathfrak{B} \times k_{\text{alg}}) \simeq K_0(\mathfrak{B}_s \times k_{\text{alg}}) \simeq K_0(\mathfrak{B}_s)$$

is an isomorphism, where $k_{\text{alg}}$ denotes an algebraic closure of $k$. Since the $\gamma$-filtration is defined in terms of Chern classes and the latter commute with restrictions, it induces an isomorphism between the $\gamma$-quotients, i.e.,

$$\text{res}: \gamma^{i+1}(\mathfrak{B}) \simeq \gamma^{i+1}(\mathfrak{B}_s).$$
Therefore, we may reduce to the split case $G = G_s$. Let $T$, $T^*$, etc., be as in subsection §1B.

There is a commutative diagram

\[
\begin{array}{cccc}
\gamma^m/m+1(\mathcal{B}_s) & \xrightarrow{c} & \text{CH}^m(\mathcal{B}_s) & \xrightarrow{\epsilon} \\
\gamma^m/m+1 & \xrightarrow{(-1)^{m-1}(m-1)!\phi_m} & S^m(T^*)/(S^{m+1}(T^*)) & \xrightarrow{c}
\end{array}
\]

First take $m = 2$ and suppose that $x \in \gamma^{2/3}$ maps to a torsion element in $\gamma^{2/3}(\mathcal{B}_s)$. As $\text{CH}^2(\mathcal{B}_s)$ has zero torsion, the commutativity of (3.5) shows that $\phi_2(x)$ is in the kernel $I$ of $c$. Writing $x = \sum_{i,j} a_{ij}(1 - e^{ai})(1 - e^{bj}) \mod \gamma^3$, we have $\phi_2(x) = \sum a_{ij}\omega_i\omega_j$ of degree 2 in $I$, hence $\phi_2(x) = aq$ for some $a \in \mathbb{Z}$. Then modulo $\gamma^3$, we have $x \equiv \psi_2\phi_2(x) \equiv a\theta$, so $\text{Tors} \gamma^{2/3}(\mathcal{B}_s)$ is a cyclic group generated by the class of the special cycle $\theta$ modulo $\gamma^3 + I'$.

By Corollary 2.8, there exists $\chi \in I'$ such that $\phi_2(\chi) = N(G_s) \cdot q$. Applying $\psi_2$ we obtain that

\[0 \equiv \chi \equiv \frac{N(G_s)}{\theta} \mod \gamma^3 + I',\]

hence, the order of $\theta$ modulo $\gamma^3 + I'$ divides the Dynkin index $N(G_s)$. This shows that $\text{Tors} \gamma^{2/3}(\mathcal{B})$ is a cyclic group of order dividing $N(G)$ with generator $c'(\theta)$.

Let $\xi' \in H^1(k',G_s)$ be a versal $G_s$-torsor for some extension $k'$ of $k$, and write $\mathcal{B}'$ for the Borel variety (over $k'$) of the group $G_s$ twisted by $\xi'$. The element $r(\xi')$ has order $N(G_s)$ in $H^3(k',\mathbb{Q}/\mathbb{Z}(2))$ by [10] pp. 31, 133. But $\text{Tors} \gamma^{2/3}(\mathcal{B}')$ is cyclic of order dividing $N(G_s)$, hence Prop. 3.2 and the exactness of (1.1) give that $\text{Tors} \gamma^{2/3}(\mathcal{B}')$ also has order $N(G_s)$. Now take $K$ to be an algebraically closed field containing $k'$. The restriction maps for $k \to K$ and $k' \to K$ give isomorphisms $\text{Tors} \gamma^{2/3}(\mathcal{B}_{jk}) \simeq \text{Tors} \gamma^{2/3}(\mathcal{B}_s)_K \simeq \text{Tors} \gamma^{2/3}(\mathcal{B}'_{jk})$, which is itself $\mathbb{Z}/N(G)$, completing the proof of [11]. Claim [11] follows from the exactness of (1.1).

Now take $m = 3$ and suppose that $x \in \gamma^{3/4}$ maps to a torsion element in $\gamma^{3/4}(\mathcal{B}_s)$. As $\text{CH}^3(\mathcal{B}_s)$ has zero torsion, diagram (3.5) shows that $2\phi_3(x)$ is in the kernel $I$ of $c$. As in the $m = 2$ case, $2\phi_3(x)$ has degree 3.

Suppose $G_s$ is not of type $A_n$ for $n \geq 2$. Then by [11C] $2\phi_3(x) = q \cdot f$, where $f = \sum_{i=1}^n a_i\omega_i$. Applying $\psi_3$ we obtain that $2x = \theta \cdot f'$, where $f' = \sum_{i=1}^n a_i(1 - e^{-\omega_i})$. In other words, the torsion part of $2\gamma^{3/4}(\mathcal{B}_s)$ is generated by the elements $c'(\theta \cdot (1 - e^{-\omega_i}))$ for $i = 1, \ldots, n$.

By Corollary 2.8 there exists $\chi \in I'$ such that $\phi_3(\chi \cdot (1 - e^{-\omega_i})) \equiv N(G_s) \cdot q \cdot \omega_i \mod (\mathbb{S}(T^*))$. Applying $\psi_3$ we obtain that

\[0 \equiv \chi \cdot (1 - e^{-\omega_i}) \equiv \frac{N(G_s)}{\theta} \cdot (1 - e^{-\omega_i}) \mod \gamma^4 + I',\]

hence, the torsion part of $2\gamma^{3/4}(\mathcal{B}_s)$ is a product of $n$ cyclic groups of orders dividing $N(G_s)$.

It remains to consider the case $m = 3$ and $G_s$ of type $A_n$ for $n \geq 2$; the claim is that $2x$ is in $I'$. As in the preceding paragraph, [11C] and Example 2.9 show that $2\phi_3(x) = q \cdot f + b\phi_3(\Delta)$ for some $b \in \mathbb{Z}$. Applying $\psi_3$, we find $2x = \theta \cdot f' + \Delta \cdot b$. As $\theta$ and $\Delta$ belong to $I'$, the proof of (III) is complete. \qed
Remark. Observe that in the proof we used the structure of the ideal of invariants \( I \) in degrees 2 and 3. In principle it is possible to extend the proof to higher degrees, but then one must extend the arguments of §1C and Example 2.9.

The next lemma can be used to extend the bounds obtained in Th. 3.1 to the case of a semisimple group.

**Lemma.** Let \( G_1, \ldots, G_m \) be simple and strongly inner groups and write \( B_j \) for the Borel variety of \( G_j \). The Borel variety for \( \prod G_j \) is \( \prod B_j \), and we have:

\[
\text{Tors} \gamma^{2/3}(\prod B_j) \simeq \bigoplus \text{Tors} \gamma^{2/3}(B_j),
\]

and

\[
\text{Tors} \gamma^{3/4}(\prod B_j) \simeq m \bigoplus \left( \text{Tors} \gamma^{3/4}(B_j) \oplus \text{Tors} \gamma^{2/3}(B_j) \right).
\]

**Proof.** Apply the Künneth decomposition and the fact that \( \gamma^i \) has no torsion for \( i = 0 \) and 1.

\[ \square \]

4. Examples of torsion in \( \text{CH}^2 \)

We now make a few remarks regarding torsion in \( \text{CH}^2 \). We maintain the notation of the previous section.

Using Prop. 3.2 one can view the assignment \( G \mapsto \text{Tors} \text{CH}^2(B) \) as a replacement for the Rost invariant of \( G \). Furthermore, the group \( \text{Tors} \text{CH}^2(B) \) is generated by the image of the special cycle \( \theta \in \mathbb{Z}[T^*] \) which is defined in purely combinatorial terms.

The next two examples show that the image of \( \theta \) generates the torsion in \( \text{CH}^2 \) for certain (generalized) Rost motives.

**Example.** Assume that the motive of \( B \) splits as a direct sum of twisted copies of the Rost motive \( R_2 \) corresponding to a 3-fold Pfister form. According to [26, §7] this can happen for \( G \) of type \( B_n, D_n, G_2, F_4, \) or \( E_6 \). Then we have

\[
(\text{Tors} \text{CH}^2(B)) \otimes \mathbb{Z}/2\mathbb{Z} \simeq \text{Ch}^2(R_2),
\]

where \( \text{Ch} \) denotes the Chow group with \( \mathbb{Z}/2\mathbb{Z} \)-coefficients and the image of the special cycle \( \theta \) generates \( (\text{Tors} \text{CH}^2(B)) \otimes \mathbb{Z}/2\mathbb{Z} \) and, hence, the group \( \text{Ch}(R_2) \).

**Example.** Suppose \( G \) is an anisotropic group of type \( F_4 \) over \( k \) that is split by a cubic extension of \( k \). (Such a group exists if and only if \( H^3(k, \mathbb{Z}/3\mathbb{Z}(2)) \) is not zero.) By [24, Rem. 4.5] and the main theorem of [26] the motive of the Borel variety \( B \) splits as a direct sum of generalized Rost motives \( R_3 \) corresponding to the Rost-Serre invariant \( g_3 \) (the mod-3 portion of the Rost invariant) of \( G \). Therefore, we have

\[
(\text{Tors} \text{CH}^2(B)) \otimes \mathbb{Z}/3\mathbb{Z} \simeq \text{Ch}^2(R_3),
\]

where \( \text{Ch} \) denotes the Chow group with \( \mathbb{Z}/3\mathbb{Z} \)-coefficients.

By the recent results of Merkurjev-Suslin [23] and Yagita [34] Thm. 10.5, Cor. 10.8 we have \( \text{Ch}^2(R_3) \simeq \mathbb{Z}/3\mathbb{Z} \). On the other hand, the image of the special cycle \( \theta \) generates \( (\text{Tors} \text{CH}^2(B)) \otimes \mathbb{Z}/3\mathbb{Z} \) and, hence, the group \( \text{Ch}^2(R_3) \).
5. Torsion in $\text{CH}^3$

Let $X$ be a projective homogeneous $G$-variety such that $G$ is split over $k(X)$. ("$X$ is generically split.") Thanks to Proposition 5.2 we may view $\text{Tors} \text{CH}^2(X)$ as known, so we now investigate $\text{Tors} \text{CH}^3(X)$. We retain the meaning of $G$ and $\mathfrak{B}$ from the previous section. Let $n$ denote the rank of $G$ and let $r$ denote the rank of the Picard group of $X$ over an algebraic closure of $k$.

We remark that the results in this section only use the fact that $\text{Tors} \text{CH}^2(X)$ is cyclic of order dividing $N(G)$, which follows from our Theorem 3.1.1 and Eq. (1.1). They do not need the finer result of Prop. 3.2, hence also do not need material from [10] and [27].

For an abelian group $A$ and a prime $p$, write $\text{Tors}_p A$ for the subgroup of $A$ consisting of elements of order a power of $p$.

5.1. Lemma. The restriction of the $m$-th Chern class gives a surjection

$$\text{Tors} \tau^{m/m+1}(X) \rightarrow (m-1)! \text{Tors} \text{CH}^m(X)$$

and for each prime $p$ not dividing $(m-1)!$, $c_m$ is an isomorphism

$$\text{Tors}_p \tau^{m/m+1}(X) \xrightarrow{\cong} \text{Tors}_p \text{CH}^m(X).$$

Proof. By Riemann-Roch (see subsection 3.1A), the composition

$$\text{CH}^m(X) \xrightarrow{\text{pr}} \tau^{m/m+1}(X) \xrightarrow{\tau^m} \text{CH}^m(X)$$

is multiplication by $(-1)^{m-1}(m-1)!$, hence $c_m(\tau^{m/m+1}(X)) = (m-1)! \text{CH}^m(X)$. For $x \in \text{Tors} \text{CH}^m(X)$, we have $(m-1)! \cdot x = c_m(\text{pr}(x))$, where $\text{pr}(x)$ is in $\text{Tors} \tau^{m/m+1}(X)$. This proves the first claim, from which the second claim follows immediately. \qed

5.2. Proposition. If $\tau^3(\mathfrak{B}) = \gamma^3(\mathfrak{B})$, then $\text{Tors} 4 \cdot \text{CH}^3(\mathfrak{B})$ is a quotient of the direct sum $\mathbb{Z}/N(G)\mathbb{Z}^{\otimes n}$. In particular, the torsion part of $\text{CH}^3(\mathfrak{B})$ can consist only of subgroups $\mathbb{Z}/2^s\mathbb{Z}$ for $s \leq 4$, $\mathbb{Z}/3\mathbb{Z}$, or $\mathbb{Z}/5\mathbb{Z}$.

Theorem 3.1.11 gives a way to check the hypothesis on the filtration.

Proof. By the hypothesis, the map $\gamma^{3/4}(\mathfrak{B}) \rightarrow \tau^{3/4}(\mathfrak{B})$ is surjective. Now combine Lemma 5.1 and Theorem 3.1.11. \qed

As an alternative to making a hypothesis on the filtrations, we may control the torsion on $\text{CH}^3(X)$ based on information about the torsion in $\text{CH}^2(X)$ and the motivic decomposition of $X$, as we now illustrate.

Fix a prime $p$. In the category of Chow motives with $\mathbb{Z}/p\mathbb{Z}$-coefficients, the motive of $X$ is a direct sum of shifts of an indecomposable motive $\mathcal{R}$, see [26 Th. 5.17], where $\mathcal{R}$ depends on $G$ but not the choice of $X$ (ibid., Th. 3.7). We write $\text{CH}^m(\mathcal{R})$ for the $m$-th Chow group of $\mathcal{R}$ with $\mathbb{Z}/p\mathbb{Z}$ coefficients.

5.3. Lemma. We have:

(i) $(\text{Tors}_p \text{CH}^2(X)) \otimes \mathbb{Z}/p\mathbb{Z} \simeq \text{Ch}^2(\mathcal{R})$;
(ii) $(\text{Tors}_p \text{CH}^3(X)) \otimes \mathbb{Z}/p\mathbb{Z} \simeq (\text{Ch}^2(\mathcal{R}))^\oplus r \oplus \text{Ch}^3(\mathcal{R})$.
(iii) $\text{Tors} \text{CH}^3(\mathfrak{B}) \simeq (\text{Tors} \text{CH}^2(X))^{\oplus (n-r)} \oplus \text{Tors} \text{CH}^3(X)$. 

As an alternative to making a hypothesis on the filtrations, we may control the torsion on $\text{CH}^3(X)$ based on information about the torsion in $\text{CH}^2(X)$ and the motivic decomposition of $X$, as we now illustrate.
Proof. The expression of the motive of $X$ from [26] gives:

$$(\text{Tors}_p \text{CH}^m(X)) \otimes \mathbb{Z}/p\mathbb{Z} \simeq \overline{\text{CH}}^m(\mathcal{R}) \oplus (\text{CH}^{m-1}(\mathcal{R}))^{\oplus p} \oplus (\text{CH}^{m-2}(\mathcal{R}))^{\oplus r} \oplus \ldots$$

where $\overline{\text{CH}}^m(\mathcal{R})$ denotes the kernel of the restriction $\text{CH}^m(\mathcal{R}) \to \text{CH}^m(\mathcal{R} \times k\bar{k})$ to the algebraic closure $\bar{k}$. By the formula for the generating function [26, Thm. 5.13(3)] and table 4.13 in ibid., we have $\text{CH}^i(\mathcal{R}) = \overline{\text{CH}}^i(\mathcal{R}) = 0$ and $\text{CH}(\mathcal{R}) = \text{CH}(\mathcal{R})$ for $i = 2, 3$. This implies claims (i) and (ii).

Claim (iii) is proved similarly, but using the integral motivic decomposition from [26, Th. 3.7] with $Y = \mathfrak{X}$.

5.4. Proposition. Fix an odd prime $p$. If $\text{Tors}_p \text{CH}^2(X) \neq 0$, then

1. $p = 3$ or 5;
2. $\text{CH}^2(\mathcal{R}) \simeq \mathbb{Z}/p\mathbb{Z}$ and $\text{CH}^3(\mathcal{R}) = 0$.
3. $\text{Tors}_p \text{CH}^2(X) \simeq \mathbb{Z}/p\mathbb{Z}$ and $\text{Tors}_p \text{CH}^3(X) \simeq (\mathbb{Z}/p\mathbb{Z})^{\oplus r}$.

Proof. By Prop. 5.3 (or [26, Th. 3.7]), $\text{CH}^2(X)$ and $\text{CH}^2(\mathfrak{X})$ have the same $p$-torsion. As $\text{Tor}_p \text{CH}^2(\mathfrak{X})$ has order dividing $N(G)$ by Theorem 5.11, the list of Dynkin indexes in [31] gives that $p = 3$ or 5 and $\text{Tors}_p \text{CH}^2(\mathfrak{X}) \simeq \mathbb{Z}/p\mathbb{Z}$. Combining this with Lemma 5.3(iii), it only remains to prove the claims about $\text{CH}^3(\mathcal{R})$ and $\text{CH}^3(X)$.

Tensoring sequence (1.1) with $\mathbb{Z}/p\mathbb{Z}$, we find that

$$\gamma^3(\mathfrak{X}) \otimes \mathbb{Z}/p\mathbb{Z} = \tau^3(\mathfrak{X}) \otimes \mathbb{Z}/p\mathbb{Z}.$$  

Combining Lemma 5.1 and Theorem 5.11 gives that $\text{Tors}_p \text{CH}^3(\mathfrak{X})$ is a product of at most $n$ copies of $\mathbb{Z}/p\mathbb{Z}$. By Lemma 5.3(iii) applied to $X = \mathfrak{X}$ we obtain

$$(\text{Tors}_p \text{CH}^3(\mathfrak{X})) \otimes \mathbb{Z}/p\mathbb{Z} \simeq (\mathbb{Z}/p\mathbb{Z})^{\oplus n} \oplus \text{CH}^3(\mathcal{R}).$$

Since the right hand side already contains $n$ copies of $\mathbb{Z}/p\mathbb{Z}$, $\text{Tors}_p \text{CH}^3(\mathfrak{X}) = (\mathbb{Z}/p\mathbb{Z})^{\oplus n}$ and $\text{CH}^3(\mathcal{R})$ is zero. The second part of (i) now follows by Lemma 5.3(iii).

6. Cohomological invariants and the Tits algebras

So far, we have studied the case where $G$ is strongly inner and we constructed the special cocycle $c^i(\theta)$ in $K_0(\mathfrak{X})$, cf. Example 6.3 below. We now relax our hypothesis on $G$ and ask if $c^i(\theta)$ is still defined over $k$.

In the present section $G_\ast$ denotes an adjoint split linear algebraic group over a field $k$. As it is adjoint, the character group $T^*$ of a split maximal torus of $G_\ast$ is naturally identified with the root lattice $\Lambda_r$.

We fix a pinning for $G_\ast$, which includes a set of simple roots $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ in $\Lambda_r$. Write $\omega_i$ for the fundamental weight corresponding to $\alpha_i$ and $s_i$ for the reflection of the weight lattice $\Lambda$ in the hyperplane orthogonal to $\alpha$.

The Steinberg basis. For each element $w$ of the Weyl group $W$ of $G$ we define

$$\rho_w := \sum_{\{ i \in \{1, \ldots, n\} \mid w^{-1}(\alpha_i) < 0 \}} w^{-1}(\omega_i) \in \Lambda.$$  

Let $\mathbb{Z}[\Lambda]^W$ denote the subring of $W$-invariant elements. Then the integral group ring $\mathbb{Z}[\Lambda]$ is a free $\mathbb{Z}[\Lambda]^W$-module with the basis $\{e^w \mid w \in W\}$ by [32, Th. 2.2].
Let \( \mathfrak{B}_s \) denote the variety of Borel subgroups of \( G_s \). Consider the characteristic map for the simply connected cover of \( G_s \)

\[ \epsilon' : Z[\Lambda] \to K_0(\mathfrak{B}_s). \]

Since the kernel of the surjection \( \epsilon' \) is generated by elements \( x \in Z[\Lambda]^W \) in the kernel of the augmentation map, there is an isomorphism

\[ Z[\Lambda] \otimes_{Z[\Lambda]^W} Z \cong Z[\Lambda]/\ker(\epsilon') \cong K_0(\mathfrak{B}_s). \]

The elements

\[ \{ g_w := \epsilon'(e^{\rho w}) = [\mathcal{L}(\rho_w)] | w \in W \} \]

form a free \( Z \)-basis of \( K_0(\mathfrak{B}_s) \) called the Steinberg basis.

Observe that the quotient group \( \Lambda/\Lambda_r \) coincides with the group of characters of the center of the simply connected cover of \( G_s \). Consider the surjective ring homomorphism induced by the restriction \( Z[\Lambda] \to Z[\Lambda/\Lambda_r] \). Since \( W \) acts trivially on \( \Lambda/\Lambda_r \), we obtain that

\[ \tilde{\rho}_w = \sum_{\{i \in \ldots | w^{-1}(\alpha_i) < 0\}} \tilde{\omega}_i, \]

where \( \tilde{\rho} \) means the restriction to \( \Lambda/\Lambda_r \).

6.1. Example. (a) For a simple reflection \( s_j \) we have

\[ \rho_{s_j} = \sum_{\{i \in \ldots | s_j(\alpha_i) < 0\}} s_j(\omega_i) = s_j(\omega_j) = \omega_j - \omega_j, \]

(b) More generally, let \( w := s_{i_1} s_{i_2} \ldots s_{i_m} \) be a product of \( m \) distinct simple reflections such that the simple roots \( \alpha_{i_j}, \alpha_{i_k} \) are orthogonal for all \( j \neq \ell \). Then

\[ \rho_{s_{i_1} s_{i_2} \ldots s_{i_m}} = \rho_{s_{i_1}} + \rho_{s_{i_2}} + \ldots + \rho_{s_{i_m}}, \]

because \( w^{-1}(\alpha_{i_j}) \) is negative if and only if \( i = i_j \) for some \( j \).

(c) For a product of two simple reflections \( s_i s_j \) such that \( c_{ij} = \alpha_i^\vee(\alpha_j) < 0 \) we obtain

\[ \rho_{s_i s_j} = \rho_{s_i} + c_{ij} \alpha_j. \]

The Tits algebras and the base change. Let \( G \) be a twisted form of \( G_s \), i.e. \( G \) is obtained by twisting \( G_s \) by a cocycle \( \xi \in Z^1(k, \text{Aut}(G_s)) \). More specifically, our choice of pinning for \( G_s \) defines a section \( s \) of the quotient map \( \pi : \text{Aut}(G_s) \to \text{Aut}(\Delta) \). Twisting \( G_s \) by \( \xi' := s \pi(\xi) \) gives a quasi-split group \( G_{\bar{q}} \) and we pick \( \xi'' \in Z^1(k, G_{\bar{q}}) \) that maps via twisting to \( \xi' \)—i.e., we pick \( \xi'' \) so that \( G \) is isomorphic to \( \xi'' G_{\bar{q}} \).

Let \( \mathfrak{B} = \xi \mathfrak{B}_s \) be the variety of Borel subgroups of \( G \). Let \( \Gamma \) denote the absolute Galois group of \( k \); it acts on the weight lattice \( \Lambda \) via the cocycle \( \xi' \).

Following [33] (see also [25, §3.1, §11.7] and [22, §2]) we associate with each \( \chi \in \Lambda/\Lambda_r \) the field of definition \( k_\chi \) of \( \chi \) and the central simple algebra \( A_{\chi, \xi} \) over \( k_\chi \) called the Tits algebra. Here \( k_\chi \) is a fixed subfield for the stabilizer

\[ \Gamma_\chi = \{ \tau \in \Gamma | \tau(\chi) = \chi \}. \]

There is a group homomorphism

\[ \beta : (\Lambda/\Lambda_r)^{\Gamma_\chi} \to \text{Br}(k_\chi) \text{ with } \beta(\chi') = [A_{\chi', \xi}]. \]
By [27, Thm. 2.1] there is an isomorphism
\[
\text{Tors} \, \text{CH}^2(\mathfrak{B}) \cong \ker \left( H^3(k, \mathbb{Q}/\mathbb{Z}(2)) \to H^3(k(\mathfrak{B}), \mathbb{Q}/\mathbb{Z}(2)) \right) / \bigoplus_{\chi \in \Lambda/\Lambda'} N_{k_{\chi}/k}(k_{\chi}^* \cup \beta(\chi)),
\]
where the numerator is the kernel of the restriction map to the field of fractions \(k(\mathfrak{B})\) of \(\mathfrak{B}\) and \(N_{k_{\chi}/k}\) is the norm. Let \(H^3_2(k, \mathbb{Q}/\mathbb{Z}(2))\) denote the cohomology quotient
\[
H^3_3(k, \mathbb{Q}/\mathbb{Z}(2)) = H^3(k, \mathbb{Q}/\mathbb{Z}(2)) / \bigoplus_{\chi \in \Lambda/\Lambda'} N_{k_{\chi}/k}(k_{\chi}^* \cup \beta(\chi))
\]
so that \(\text{Tors} \, \text{CH}^2(\mathfrak{B}) \subseteq H^3_3(k, \mathbb{Q}/\mathbb{Z}(2))\).

Let \(l/k\) be a field extension. Since the Chern classes commute with restrictions, there is the induced map
\[
\text{res}_{l/k} : \gamma^{i+1}(\mathfrak{B}_l) \to \gamma^{i+1}(\mathfrak{B}_l),
\]
where \(\mathfrak{B}_l = \mathfrak{B} \times_k l\), with the image generated by the products
\[
\langle c_{n_1}(x_1) \cdots c_{n_m}(x_m) | n_1 + \cdots + n_m = i, \ x_1, \ldots, x_m \in \text{res}_{l/k}(K_0(\mathfrak{B})) \rangle
\]
and there is a commutative diagram
\[
\begin{align*}
\text{Tors} \gamma^{2/3}(\mathfrak{B}) & \xrightarrow{c_2} \text{Tors} \, \text{CH}^2(\mathfrak{B}) \subseteq H^3_3(k, \mathbb{Q}/\mathbb{Z}(2)) \\
\text{res}_{l/k} & \\
\text{Tors} \gamma^{2/3}(\mathfrak{B}_l) & \xrightarrow{c_2} \text{Tors} \, \text{CH}^2(\mathfrak{B}_l) \subseteq H^3_3(l, \mathbb{Q}/\mathbb{Z}(2))
\end{align*}
\]

Observe that the image \(\text{res}_{l/k}(K_0(\mathfrak{B}))\) can be computed using [25, Thm. 4.2]. For instance, if \(G\) is an inner group, i.e., \(\xi = 0\), then \(\Gamma\) acts trivially on \(\Lambda/\Lambda_r\), i.e. \(k_{\chi} = k\) for all \(\chi\) and by [25, Thm. 4.2] the image of the restriction map \(K_0(\mathfrak{B}) \to K_0(\mathfrak{B}_l)\) from [3.4] coincides with the sublattice
\[
\langle \text{ind}(A_{\rho, \xi}) : g_w | w \in W \rangle.
\]

Using (6.2) one can provide a non-trivial element in \(H^3_3(k, \mathbb{Q}/\mathbb{Z}(2))\) as follows:

- Assume that we are given non-trivial elements over \(l\), i.e. that there is a non-trivial element \(\theta \in \text{Tors} \gamma^{2/3}(\mathfrak{B}_l)\) such that \(c_2(\theta) \in H^3_3(l, \mathbb{Q}/\mathbb{Z}(2))\) is non-trivial.
- Assume that we know that \(\theta\) is defined over \(k\), i.e. that \(\theta = \text{res}_{l/k}(\theta')\) for some \(\theta' \in \text{Tors} \gamma^{2/3}(\mathfrak{B})\).

Then the image \(c_2(\theta')\) provides a non-trivial element in \(H^3_3(k, \mathbb{Q}/\mathbb{Z}(2))\).

6.3. Example (strongly inner case). If \(G\) is strongly inner—i.e., if \(G\) is inner and \(\beta\) is the trivial homomorphism—then for any field extension \(l/k\) the left vertical arrow in (6.2) is an isomorphism, hence, identifying \(\text{Tors} \gamma^{2/3}(\mathfrak{B})\) with the cyclic group generated by the special cycle \(\theta\). As in Prop. [3.2] and its proof \(\text{Tors} \, \text{CH}^2(\mathfrak{B})\) coincides with the usual unramified cohomology generated by the Rost invariant \(r(G)\) of \(G\) and \(\langle c_2(\theta) \rangle = \langle r(G) \rangle\) in \(H^3(k, \mathbb{Q}/\mathbb{Z}(2))\).

6.4. Lemma. Assume that \(G\) is inner.
Suppose that \( i \), i.e., is in the image of \( \gamma \) in the image of \( \gamma_2 \) in the image of \( \gamma_3 \). The Whitney Sum Formula gives that 
\[
c_1^\gamma(\{\mathcal{L}(\omega_i)\}) = c_1^\gamma(\{\mathcal{L}(\alpha_i)\}) \equiv c_1^\gamma(\sum_j g_{s_i}) \equiv c_1^\gamma(g_w) \mod \gamma^3(\mathfrak{B}_s); 
\]

In particular, it holds for the classes \( [\mathcal{L}(\alpha_i)] \) of simple roots \( \alpha_i \).

Under the notation of Example 6.4(b) we have
\[
(\sum_j c_1^\mathcal{K}_i(\{\mathcal{L}(\omega_i)\})) \equiv c_1^\mathcal{K}_i(\{\mathcal{L}(\alpha_i)\}) \equiv c_1^\mathcal{K}_i(\sum_j g_{s_i}) \equiv c_1^\mathcal{K}_i(g_w) \mod \gamma^3(\mathfrak{B}_s); 
\]

(c) If \( \beta(\sum_j \omega_{i_j}) = 0 \), then \( \sum_j c_1^\gamma(\{\mathcal{L}(\omega_{i_j})\}) \) is in the image of

\[
\text{res: } \gamma^{1/2}(\mathfrak{B}) \to \gamma^{1/2}(\mathfrak{B}_s). 
\]

Proof. (a) follows by [14, Cor. 3.1]. (b) follows by the formula for the first Chern class \( c_1^\gamma(\{\mathcal{L}(\alpha_i)\}) \) is in the image of the restriction map which implies (c). \( \square \)

6.5. Proposition (quaternionic inner case). Assume that \( G \) is inner. If every Tits algebra of \( \mathcal{G}_s \) has index 1 or 2, then the special cycle \( \theta \) is in the image of the restriction map 

\[
\text{res: } \gamma^{2/3}(\mathfrak{B}) \to \gamma^{2/3}(\mathfrak{B}_s). 
\]

In other words, if \( l/k \) is an extension that kills \( \text{im } \beta \), then the image of \( c_2 \) over \( l \) coincides with the subgroup generated by the respective Rost invariant, i.e. we have

\[
\text{im}(c_2) = \langle r(G_1) \rangle \subseteq H^3(l, \mathbb{Q}/\mathbb{Z}(2)). 
\]

Proof of Prop. 6.5. We may assume that \( N(G) \) is not 1 (otherwise \( \theta \) maps to zero in \( \gamma^{2/3}(\mathfrak{B}_s) \) by Th. 6.1) and \( \Lambda/\Lambda_r \) has even order (otherwise Example 6.3 applies), i.e., we may assume that \( G \) has type \( B, C, D, \) or \( E_7 \).

We first make a general observation. Mod \( \gamma^3(\mathfrak{B}_s) \), we have:

\[
c_1^\mathcal{K}_i(\{\mathcal{L}(\omega_i)\})^2 \equiv (c_1^\mathcal{K}_i(g_{s_i}) + c_1^\mathcal{K}_i(\{\mathcal{L}(\alpha_i)\}))^2 
\equiv c_1^\mathcal{K}_i(g_{s_i})^2 + 2c_1^\mathcal{K}_i(g_{s_i})c_1^\mathcal{K}_i(\{\mathcal{L}(\alpha_i)\}) + c_1^\mathcal{K}_i(\{\mathcal{L}(\alpha_i)\})^2.
\]

The Whitney Sum Formula gives that 
\[
c_2^\mathcal{K}_i(2g_{s_i}) = c_1^\mathcal{K}_i(g_{s_i})^2 \quad \text{and} \quad c_3^\mathcal{K}_i(2g_{s_i}) = 2c_1^\mathcal{K}_i(g_{s_i}) \mod \gamma^3(\mathfrak{B}_s). 
\]

Our hypothesis on the Tits algebras gives that \( 2g_{s_i} \) is in the image of \( K_0(\mathfrak{B}) \to K_0(\mathfrak{B}_s) \), and it follows that \( c_3^\mathcal{K}_i(\{\mathcal{L}(\omega_i)\})^2 \) is rational – i.e., is in the image of \( \gamma^{2/3}(\mathfrak{B}) \to \gamma^{2/3}(\mathfrak{B}_s) \) – for all \( i \).

Type \( E_7 \): Suppose that \( G \) has type \( E_7 \). Then

\[
q = \left( \sum_{i=1}^{7} \omega_i^2 \right) - \omega_1\omega_3 - \omega_2\omega_4 - \omega_4\omega_5 - \omega_5\omega_6 - \omega_6\omega_7 
\]

where we have numbered the roots following [2]. Each \( \omega_i^2 \) contributes a term of the form \( c_1^\mathcal{K}_i(\{\mathcal{L}(\omega_i)\})^2 \) to the image of \( c'(\theta) \) in \( \gamma^{2/3}(\mathfrak{B}_s) \), and such a term is rational by the preceding paragraph. The weights \( \omega_1, \omega_3, \omega_4, \omega_5 \) belong to the root lattice and so the term \( \omega_1\omega_3 \) contributes a rational term \( c_1^\mathcal{K}_i(\{\mathcal{L}(\omega_i)\}) c_1^\mathcal{K}_i(\{\mathcal{L}(\omega_3)\}) \) to \( c'(\theta) \), and similarly for the term \( \omega_3\omega_4 \). Next we observe that \( \omega_4\omega_2 + \omega_4\omega_5 \) contributes

\[
c_1^\mathcal{K}_i(\{\mathcal{L}(\omega_4)\}) \left( c_1^\mathcal{K}_i(\{\mathcal{L}(\omega_2)\}) + c_1^\mathcal{K}_i(\{\mathcal{L}(\omega_5)\}) \right)
\]

to \( c'(\theta) \). But \( \omega_4 \) and \( \omega_2 + \omega_5 \) both lie in the root lattice, so both terms in the product are rational by Lemma 6.3. The same argument handles \( \omega_3\omega_6 + \omega_6\omega_7 \), and we are done with the \( E_7 \) case.
**Type D:** Suppose that $G$ has type $D_n$. Then

$$q = \sum_{i=1}^{n} \omega_i^2 - \sum_{i=1}^{n-2} \omega_i \omega_{i+1} - \omega_{n-2} \omega_n.$$  

The terms $\omega_i^2$ are treated as in the $E_7$ case. For the terms in the second sum, we collect around terms with even subscripts: for even $i < n - 2$, consider $\omega_i(\omega_{i-1} + \omega_{i+1})$. As $\omega_i$ and $\omega_{i-1} + \omega_{i+1}$ belong to the root lattice, we see as in the $E_7$ case that they contribute rational terms to $c'(\theta)$.

Suppose now that $n$ is even. Then we have not accounted for $\omega_{n-2}(\omega_{n-3} + \omega_{n-1} + \omega_n)$ from $q$. As both terms in the product belong to the root lattice, we are finished as in the $E_7$ case.

If $n$ is odd, then we have not accounted for $\omega_{n-2}(\omega_{n-1} + \omega_n)$ in $q$. Here $\Lambda/\Lambda_r$ is isomorphic to $\mathbb{Z}/4$ and $\omega_{n-2}, \omega_{n-1}, \omega_n$ map to $2, \pm 1, \pm 3$ respectively. In particular, $\beta(\omega_{n-2}) = 2\beta(\omega_n)$, which is zero by our hypothesis on the Tits algebras, so $[\mathcal{L}(\omega_{n-2})]$ is in the image of $\text{res}: K_0(\mathfrak{B}) \to K_0(\mathfrak{B}_s)$. Similarly, $\beta(\omega_{n-1} + \omega_n) = \beta(\omega_{n-1}) + \beta(\omega_n) = 0$, and as in the $E_7$ case, we see that $c'(\theta)$ is rational.

**Type B or C:** If $G$ has type $B_n$ or $C_n$, $\Lambda/\Lambda_r$ equals $\mathbb{Z}/2$. In either case,

$$q = \sum_{i=1}^{n} c_{ii} \omega_i^2 - \sum_{i=1}^{n-1} 2\omega_i \omega_{i+1},$$

where the $c_{ii}$ are 1 or 2.

For type $C_n$, the map $\Lambda \to \Lambda/\Lambda_r$ sends $\omega_i$ to the class of $i$. Previous arguments easily handle the $n$ odd case. If $n$ is even, previous arguments leave us to consider the term $2\omega_{n-1} \omega_n$. But

$$2c_1^{K_0}([\mathcal{L}(\omega_{n-1})]) \equiv 2(c_1^{K_0}(g_{s_{n-1}}) + c_1^{K_0}([\mathcal{L}(\alpha_{n-1})])) \mod \gamma^2(\mathfrak{B}_s),$$

$$\equiv c_1^{K_0}(2g_{s_{n-1}}) + 2c_1^{K_0}([\mathcal{L}(\alpha_{n-1})]),$$

and again we find that $c'(\theta)$ is rational.

For type $B_n$, the map $\Lambda \to \Lambda/\Lambda_r$ sends $\omega_n$ to 1 and all other fundamental weights to zero. Consequently, it suffices to consider the term $2\omega_{n-1} \omega_n$ in $q$. For this we can apply the argument in the preceding paragraph. □

### 7. Application to essential dimension

We now apply results from the previous section to give a lower bound on the essential dimension $\text{ed}(G)$ for some algebraic groups $G$. We refer to Reichstein’s 2010 ICM lecture \[3\] for a definition and survey of this notion. Roughly speaking, it gives the number of parameters required to specify a $G$-torsor.

**Proposition.** Let $G$ be an absolutely almost simple algebraic group. Then $\text{ed}(G) \geq 3$ unless $G$ is isomorphic to $\text{Sp}_{2n}$ for some $n \geq 2$ (in which case $\text{ed}(G) = 0$) or $G$ has type $A$.

The lower bound of 3 is in many cases very weak, but it has the advantage of being uniform and having a proof that is almost as uniform. The existence of the Rost invariant gives the same lower bound on $\text{ed}(G)$ when $G$ is simply connected, so our proposition can be viewed as removing the hypothesis “simply connected” from that result.
7.2. **Remark.** For groups of type $A$, the lower bound is more complicated and we do not know what the answer is in every case. Of course $\deg(SL_n) = 0$. Some other known cases are: For $n$ divisible by the square of a prime, we know that $\deg(PGL_n) \geq 4$ by [29] Th. 16.1(b)]; and for “intermediate” groups of type $A$ in good characteristic, the essential dimension is at least 4 by [31] Th. 8.13]. On the other hand, A.A. Albert conjectured that central simple algebras of prime degree are cyclic, which would imply that $\deg(PGL_n) = 2$ for square-free $n$. However, this is only currently known for $n = 2, 3, \text{and} 6$.

**Proof of Prop. 7.1**  **Main case:** Suppose first that $G$ is not of type $A$, $C$, nor $E_6$. As essential dimension only goes down with field extensions, we may assume that $k$ is algebraically closed and bound $\deg(G_s)$ where $G_s$ is a split simple group not of type $A$ or $C$. Put $G_s$ for the simply connected cover of $G_s$. Fix a versal $G$-torsor $\xi \in H^1(L, G_s)$ for some extension $L/k$. Let $K$ be a field between $k$ and $L$ of minimal transcendence degree such that there is a $\xi \in H^1(K, G_s)$ whose image in $H^1(L, G_s)$ is the same as the image of $\xi$.

For sake of contradiction, suppose that $K$ has transcendence degree at most 2 over $k$. By the hypothesis on the type of $G$, the Tits algebras of $\xi G$ have exponent a power of 2 and so are actually of index 1 or 2 over $K$ by De Jong, see [4] or [20] Th. 4.2.2.3]. By Proposition 6.3 there is a class $\psi \in \gamma^2(\mathbb{B})$ whose image under restriction to $L$ is $\varepsilon'(\theta)$. Now $\text{Tor}(\mathbb{B}, \mathbb{Z})$ is zero by Prop. 5.3 (2) because $H^3(K, \mathbb{Q}/\mathbb{Z}(2))$ is zero, and it follows that $\varepsilon'(\theta)$ is zero in $\text{Tor}(\mathbb{B}, \mathbb{Z})$. But $\varepsilon'(\theta)$ has order $N(G_s) \neq 1$, a contradiction.

**Type C:** If $G$ of type $C_n$ ($n \geq 2$) is simply connected, by hypothesis it is not $Sp_{2n}$, so the Rost invariant is not zero on a versal $G$-torsor and the claim follows. So suppose $G$ is adjoint. (We give an argument that is characteristic-free; if char $k \neq 2$, then $\deg(G) \geq n + 1$ by [3] (1.1)\).

If $n$ is odd then we can construct a nonzero normalized cohomological invariant of $PSp_{2n}$ of degree 4 as in [21] Th. 4.1] and that case is settled.

It remains to show that $\deg(PSp_{2n}) \geq 3$ when $k$ is algebraically closed and $n$ is even. Let $k'$ be an extension of $k$ that has a quaternion division algebra $D$ and fix a class $\xi \in H^1(k', PSp_{2n})$ with image $[D] \in H^2(k', \mu_2)$ under the natural connecting homomorphism $\partial_k$. Fix a versal torsor $\xi \in H^1(L, \xi) Sp_{2n}$ for some extension $L/k'$. Let $K$ be a field between $k$ and $L$ so that there is a class $\xi \in H^1(K, PSp_{2n})$ with the same image as $\xi$ in $H^1(L, PSp_{2n})$. For sake of contradiction, suppose that $K$ has transcendence degree at most 2 over $k$.

By De Jong, $\partial_K(\xi)$ is the class of a quaternion algebra in $H^2(K, \mu_2)$. So we could arrange from the beginning that $k' = K$ and $[D] = \partial_K(\xi)$. We have a well-defined invariant

$$\text{im} \left[ H^1(*, \xi) Sp_{2n} \to H^1(*, \xi PSp_{2n}) \right] \to H^3(*, \mathbb{Z}/2\mathbb{Z})$$

defined on extensions of $K$ because the Rost invariant vanishes on $H^1$ of the center of every simply connected group of type $C_n$ with $n$ even [12]. On the other hand, the class of $\xi$ belongs to the domain of this map and the invariant does not vanish on it; this contradicts the hypothesis that $K$ has transcendence degree at most 2 over $k$.

The remaining case of type $E_6$ is known by arguments as in [13] 9.5, 9.7]. Alternatively, one can repeat the “main case” argument focusing on essential 2-dimension instead of essential dimension. □
Although we stated the proposition for absolutely almost simple groups, it quickly leads to the lower bound $\text{ed}(G) \geq 3$ for more groups $G$. We mention:

(A) to obtain a lower bound on $\text{ed}(G)$ for any linear group $G$, it suffices to give a lower bound on the essential dimension of the identity component of $G$ \cite[6.19]{1}, so one needn’t assume that $G$ is connected. (B) If $k/k_0$ is a finite separable extension, one has $\text{ed}(R_{k/k_0}(G)) \geq \text{ed}(G)$, so the proposition immediately gives a similar (but slightly more complicated to state) result for groups that are simple but not absolutely simple. (C) If $G \cong G_1 \times G_2$, then $\text{ed}(G) \geq \max\{\text{ed}(G_1), \text{ed}(G_2)\}$, and in this way we can weaken the hypothesis “simple” to “semisimple” at the cost of demanding that $G$ be adjoint or simply connected.

Acknowledgments. We thank Zinovy Reichstein and Mark MacDonald for their comments on an earlier version of this paper. The first author’s research was partially supported by the NSF under grant DMS-0653502. The second author was supported by NSERC Discovery 385795-2010 and Accelerator Supplement 396100-2010 grants

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