On the Degenerate Multiplicity of the $sl_2$ Loop Algebra for the 6V Transfer Matrix at Roots of Unity

Tetsuo DEGUUCHI

Department of Physics, Faculty of Science, Ochanomizu University, 2-1-1 Ohtsuka, Bunkyo-Ku, Tokyo 112-8610, Japan
E-mail: deguchi@phys.ocha.ac.jp

Received October 31, 2005, in final form February 06, 2006; Published online February 17, 2006
Original article is available at http://www.emis.de/journals/SIGMA/2006/Paper021/

Abstract. We review the main result of cond-mat/0503564. The Hamiltonian of the XXZ spin chain and the transfer matrix of the six-vertex model has the $sl_2$ loop algebra symmetry if the $q$ parameter is given by a root of unity, $q_{0}^{2N} = 1$, for an integer $N$. We discuss the dimensions of the degenerate eigenspace generated by a regular Bethe state in some sectors, rigorously as follows: We show that every regular Bethe ansatz eigenvector in the sectors is a highest weight vector and derive the highest weight $\bar{d}^{\pm}$, which leads to evaluation parameters $a_j$. If the evaluation parameters are distinct, we obtain the dimensions of the highest weight representation generated by the regular Bethe state.

Key words: loop algebra; the six-vertex model; roots of unity representations of quantum groups; Drinfeld polynomial

2000 Mathematics Subject Classification: 81R10; 81R12; 81R50; 81V70

1 Introduction

The XXZ spin chain is one of the most important exactly solvable quantum systems. The Hamiltonian under the periodic boundary conditions is given by

$$H_{XXZ} = \frac{1}{2} \sum_{j=1}^{L} \left( \sigma_{j}^{X} \sigma_{j+1}^{X} + \sigma_{j}^{Y} \sigma_{j+1}^{Y} + \Delta \sigma_{j}^{Z} \sigma_{j+1}^{Z} \right).$$

Here the XXZ anisotropy $\Delta$ is related to the $q$ parameter by $\Delta = (q + q^{-1})/2$. The XXZ Hamiltonian is derived from the logarithmic derivative of the transfer matrix of the six-vertex model, and hence they have the same set of eigenvectors.

Recently, it was explicitly shown that when $q$ is a root of unity the XXZ Hamiltonian commutes with the generators of the $sl_2$ loop algebra. Let $q_0$ be a primitive root of unity satisfying $q_0^{2N} = 1$ for an integer $N$. We introduce operators $S^{\pm(N)}$ as follows

$$S^{\pm(N)} = \sum_{1 \leq j_1 < \cdots < j_N \leq L} q_0^{\frac{N}{2} \sigma_z} \otimes \cdots \otimes q_0^{\frac{N}{2} \sigma_z} \otimes \sigma_{j_1}^{\pm} \otimes \sigma_{j_1}^{\pm} \otimes \sigma_{j_1}^{\pm} \otimes q_0^{\frac{N-1}{2} \sigma_z} \otimes \cdots \otimes q_0^{\frac{N-1}{2} \sigma_z} \otimes \sigma_{j_1}^{\pm} \otimes q_0^{\frac{N-1}{2} \sigma_z} \otimes \cdots \otimes q_0^{\frac{N-1}{2} \sigma_z}. $$

They are derived from the $N$th power of the generators $S^{\pm}$ of the quantum group $U_q(sl_2)$ or $U_q(\hat{sl}_2)$. We also define $T^{\pm}$ by the complex conjugates of $S^{\pm(N)}$, i.e. $T^{\pm(N)} = (S^{\pm(N)})^*$. The operators, $S^{\pm(N)}$ and $T^{\pm(N)}$, generate the $sl_2$ loop algebra, $U(L(sl_2))$, in the sector

$$S^z \equiv 0 \pmod{N}.$$
Here the value of the total spin $S^Z$ is given by an integral multiple of $N$. In the sector $\mathfrak{sl}_2$, the operators $S^{\pm(N)}$ and $T^{\pm(N)}$ (anti-)commute with the transfer matrix of the six-vertex model $\tau_{0V}(v)$, and they commute with the Hamiltonian of the XXZ spin chain $\mathfrak{sl}_2$:

$$[S^{\pm(N)}, H_{XXZ}] = [T^{\pm(N)}, H_{XXZ}] = 0.$$ 

One of the most important physical questions is to obtain the degenerate multiplicity of the $\mathfrak{sl}_2$ loop algebra. Let us denote by $|R\rangle$ a regular Bethe state with $R$ down spins. Here we define regular Bethe states by such Bethe ansatz eigenvectors that are constructed from finite and distinct solutions of the Bethe ansatz equations, whose precise definition will be given in Section 2. For any given regular Bethe state in the sector $\mathfrak{sl}_2$, we may have the following degenerate eigenvectors of the XXZ Hamiltonian

$$S^{-(N)}|R\rangle, \quad T^{-(N)}|R\rangle, \quad (S^{-(N)})^2|R\rangle, \quad T^{-(N)}S^{+(N)}T^{-(N)}|R\rangle, \quad \ldots.$$ 

However, it is nontrivial how many of them are linearly independent. The number should explain the degree of the spectral degeneracy. We thus want to know the dimensions of the degenerate eigenspace generated by the Bethe state $|R\rangle$. Here we note that some of the spectral degeneracies of the XYZ spin chain (the eight-vertex model) at roots of unity were first discussed by Baxter \cite{2} (see also \cite{3, 4}). In fact, there are such spectral degeneracies of the XYZ spin chain at roots of unity that are closely related to the $\mathfrak{sl}_2$ loop algebra symmetry of the XXZ spin chain \cite{3, 11, 17, 18}. Quite interestingly, various aspects of the spectral degeneracies of the XXZ spin chain have been discussed by several authors from different viewpoints \cite{11, 21, 22, 27, 29}.

Recently, there has been some progress on the degenerate multiplicity as far as regular Bethe states in some sectors such as $\mathfrak{sl}_2$ are concerned \cite{11}. For the XXZ spin chain at roots of unity, Fabricius and McCoy had made important observations on degenerate multiplicities of the $\mathfrak{sl}_2$ loop algebra, and conjectured the ‘Drinfeld polynomials of Bethe ansatz eigenvectors’ \cite{14, 15, 16}. However, it was not clear whether the representation generated by a Bethe state is irreducible or not. Here we remark that a Drinfeld polynomial is defined for an irreducible finite-dimensional representation. Motivated by the previous results \cite{11, 15, 16}, an algorithm for determining the dimensions of the representation generated by a given regular Bethe state has been rigorously formulated in some sectors such as $\mathfrak{sl}_2$ and some restricted cases of evaluation parameters $a_j$, which will be defined in Section 5. We show rigorously \cite{11} that every regular Bethe state in the sectors is highest weight, assuming a conjecture that the Bethe roots are continuous with respect to the $q$ parameter at roots of unity. We evaluate the highest weight $d_k^{x, y}$ for the Bethe state, which leads to evaluation parameters $a_j$. Here $d_k^{x, y}$ will be defined in Section 4. Independently, it has been shown \cite{5} that if the evaluation parameters of a highest weight representation of the $\mathfrak{sl}_2$ loop algebra are distinct, then it is irreducible. It thus follows that the ‘Drinfeld polynomial of a regular Bethe state’ corresponds to the standard Drinfeld polynomial defined for the irreducible representation generated by the regular Bethe state, if the evaluation parameters are distinct.

The conjecture of Fabricius and McCoy has been proved at least in the sectors such as $\mathfrak{sl}_2$ and in the case of distinct evaluation parameters. Thus, the purpose of the paper is to review the main points of the complete formulation of \cite{11} briefly with some illustrative examples. Some other aspects of the $\mathfrak{sl}_2$ loop algebra symmetry have also been discussed \cite{5, 24, 30}.

The contents of the paper is given as follows. In Section 2, we introduce Bethe ansatz equations and regular solutions of the Bethe ansatz equations. We define regular Bethe states. In Section 3, we discuss briefly the $\mathfrak{sl}_2$ loop algebra symmetry of the XXZ spin chain at roots of unity. Here, the conditions of roots of unity are specified precisely. In Section 4, we explain the Drinfeld realization of the $\mathfrak{sl}_2$ loop algebra, i.e. the classical analogues of the Drinfeld realization of the quantum affine group $U_q(\mathfrak{sl}_2)$. In Section 5, we review the algorithm for determining the degenerate multiplicity of a regular Bethe state in the sectors. Here we note that Theorem 2
generalizes the $su(2)$ symmetry of the XXX spin chain shown by Takhtajan and Faddeev. In Section 6 we discuss some examples of the Drinfeld polynomials explicitly.

## 2 Bethe ansatz equations and the transfer matrix

### 2.1 Regular solutions of the Bethe ansatz equations

Let us assume that a set of complex numbers, $\tilde{t}_1, \tilde{t}_2, \ldots, \tilde{t}_R$ satisfy the Bethe ansatz equations at a root of unity:

$\left( \frac{\sinh(\tilde{t}_j + \eta_0)}{\sinh(\tilde{t}_j - \eta_0)} \right)^L = \prod_{k=1; k\neq j}^M \frac{\sinh(\tilde{t}_j - \tilde{t}_k + 2\eta_0)}{\sinh(\tilde{t}_j - \tilde{t}_k - 2\eta_0)}$, for $j = 1, 2, \ldots, R$. \hspace{1cm} (3)

Here the parameter $\eta$ is defined by the relation $q = \exp(2\eta)$, and $\eta_0$ is given by $q_0 = \exp(2\eta_0)$. If a given set of solutions of the Bethe ansatz equations are finite and distinct, we call them regular. We call a set of solutions of the Bethe ansatz equations Bethe roots.

A set of regular solutions of the Bethe ansatz equations leads to an eigenvector of the XXZ Hamiltonian. We call it a regular Bethe state of the XXZ spin chain or a regular XXZ Bethe state, briefly. Explicit expressions of regular Bethe states are derived through the algebraic Bethe ansatz method.

### 2.2 Transfer matrix of the six-vertex model

We now introduce $L$ operators for the XXZ spin chain. Let $V_n$ be two-dimensional vector spaces for $n = 0, 1, \ldots, L$. We define an operator-valued matrix $L_n(z)$ by

$L_n(z) = \begin{pmatrix} L_n(z)^{11} & L_n(z)^{12} \\ L_n(z)^{21} & L_n(z)^{22} \end{pmatrix} = \begin{pmatrix} \sinh (z I_n + \eta \sigma^x_n) & \sinh 2\eta \sigma_n^- \\ \sinh 2\eta \sigma_n^+ & \sinh (z I_n - \eta \sigma^x_n) \end{pmatrix}$.

Here $L_n(z)$ is a matrix acting on the auxiliary vector space $V_0$, where $I_n$ and $\sigma^a_n$ ($a = z, \pm$) are operators acting on the $n$th vector space $V_n$. The symbol $I$ denotes the two-by-two identity matrix, $\sigma^\pm$ denote $\sigma^+ = E_{12}$ and $\sigma^- = E_{21}$, and $\sigma^x$, $\sigma^y$, $\sigma^z$ are the Pauli matrices.

We define the monodromy matrix $T$ by the product:

$T(z) = L_L(z) \cdots L_2(z)L_1(z)$. 

Here the matrix elements of $T(z)$ are given by

$T(z) = \begin{pmatrix} A(z) & B(z) \\ C(z) & D(z) \end{pmatrix}$.

We define the transfer matrix of the six vertex model $\tau_{6V}(z)$ by the following trace:

$\tau_{6V}(z) = \text{tr} \ T(z) = A(z) + D(z)$. 

We call the transfer matrix homogeneous. It is invariant under lattice translation.

## 3 The $sl_2$ loop algebra symmetry at roots of unity

We shall show the $sl_2$ loop algebra symmetry of the XXZ spin chain at roots of unity in some sectors. We shall discuss two cases with even $L$ and odd $L$. 

3.1 Roots of unity conditions

Let us explicitly formulate roots of unity conditions as follows.

**Definition 1 (Roots of unity conditions).** We say that \( q_0 \) is a root of unity with \( q_0^{2N} = 1 \), if one of the conditions hold: (i) \( q_0 \) is a primitive \( N \)th root of unity with \( N \) odd \( (q_0^N = 1) \); (ii) \( q_0 \) is a primitive \( 2N \)th root of unity with \( N \) odd \( (q_0^N = -1) \); (iii) \( q_0 \) is a primitive \( 2N \)th root of unity with \( N \) even \( (q_0^N = -1) \). We call the cases (i) and (iii) type I, the case (ii) type II.

In the case of \( S^Z \equiv 0 \pmod{N} \) we consider all the three conditions of roots of unity. However, in the case of \( S^Z \equiv N/2 \pmod{N} \) with \( N \) odd, we consider only the condition (i) of roots of unity, i.e. \( q_0 \) is a primitive \( N \)th root of unity with \( N \) odd \( (q_0^N = 1) \).

3.2 (Anti-)commutation relations at roots of unity

We show the \( sl_2 \) loop algebra symmetry of the XXZ spin chain in the following two sectors: (a) in the sector \( S^Z \equiv 0 \pmod{N} \) where \( q_0 \) is a root of unity with \( q_0^{2N} = 1 \), as specified in Definition 1; (b) in the sector \( S^Z \equiv N/2 \pmod{N} \) with \( N \) odd where \( q_0 \) is a primitive \( N \)th root of unity.

Let us assume that there exists a set of regular solutions of Bethe ansatz equations with \( R \) down-spins, i.e. \( R \) regular Bethe roots. We also assume that the lattice size \( L \), the number of regular Bethe roots \( R \) and the integer \( N \) satisfy the following relation:

\[
L - 2R = nN, \quad n \in \mathbb{Z}. \tag{4}
\]

If \( n \) is even, the regular Bethe state \( |R \rangle \) is in the sector \( S^Z \equiv 0 \pmod{N} \), while if \( n \) is odd and \( N \) is also odd, then it is in the sector \( S^Z \equiv N/2 \pmod{N} \). Here we recall that the symbol \( |R \rangle \) denotes the regular Bethe state constructed from the given \( R \) regular Bethe roots.

It has been shown [13] that operators \( S^{\pm(N)} \) and \( T^{\pm(N)} \) (anti-)commute with the transfer matrix of the six-vertex model \( \tau_{6V}(v) \) in the sector \( S^Z \equiv 0 \pmod{N} \) at \( q_0 \) with \( q_0^{2N} = 1 \)

\[
S^{\pm(N)} \tau_{6V}(z) = q_0^N \tau_{6V}(z) S^{\pm(N)}, \quad T^{\pm(N)} \tau_{6V}(z) = q_0^N \tau_{6V}(z) T^{\pm(N)}. \tag{5}
\]

Furthermore, it is also shown [13] that in the sector \( S^Z \equiv N/2 \pmod{N} \) when \( N \) is odd and \( q_0 \) satisfies \( q_0^N = 1 \), operators \( S^{\pm(N)} \) and \( T^{\pm(N)} \) (anti-)commute with the transfer matrix of the six-vertex model \( \tau_{6V}(v) \).

From the (anti-)commutation relations [3] it follows that the operators \( S^{\pm(N)} \) and \( T^{\pm(N)} \) commute with the XXZ Hamiltonian in the sector \( S^Z \equiv 0 \pmod{N} \) when \( q_0 \) satisfies \( q_0^{2N} = 1 \), and in the sector \( S^Z \equiv N/2 \) when \( N \) is odd and \( q_0^N = 1 \). Here we recall that the XXZ Hamiltonian \( H_{XXZ} \) is given by the logarithmic derivative of the (homogeneous) transfer matrix \( \tau_{6V}(v) \).

3.3 The algebra generated by \( S^{\pm(N)} \) and \( T^{\pm(N)} \)

Let us discuss the algebra generated by the operators [13], \( S^{\pm(N)} \) and \( T^{\pm(N)} \). When \( q_0 \) is of type I, we have the following identification [13]:

\[
E_0^+ = T^{-(N)}, \quad E_0^- = T^{+(N)}, \quad E_1^+ = S^{+(N)} - S^{-N}, \quad E_1^- = S^{-N},
\]

\[
-H_0 = H_1 = \frac{2}{N} S^Z. \tag{6}
\]

When \( q \) is of type II, we have the following [9]:

\[
E_0^+ = \sqrt{-1} T^{-(N)}, \quad E_0^- = \sqrt{-1} T^{+(N)}, \quad E_1^+ = \sqrt{-1} S^{+(N)},
\]
\[ E_1^- = \sqrt{-1} S^{-(N)}, \quad -H_0 = H_1 = \frac{2}{N} S^Z. \] (7)

Here \( \sqrt{-1} \) denotes the square root of \(-1\) (cf. (A.13) of [9]; see also [21]). The operators \( E_j^\pm \) and \( H_j \) for \( j = 0, 1 \), are the Chevalley generators of the affine Lie algebra \( \hat{sl}_2 \). In fact, the operators \( E_j^\pm, H_j \) for \( j = 0, 1 \), satisfy the defining relations [20] of the \( sl_2 \) loop algebra [13]:

\[
\begin{align*}
H_0 + H_1 &= 0, \quad [H_j, E_j^\pm] = \pm a_{ij} E_j^\pm, \quad i, j = 0, 1, \\
[E_i^+, E_j^-] &= \delta_{ij} H_j, \quad i, j = 0, 1, \\
[E_i^\pm, [E_i^\pm, E_j^\pm]] &= 0, \quad i, j = 0, 1, \quad i \neq j.
\end{align*}
\]

Here, the Cartan matrix \((a_{ij})\) of \( A_1^{(1)} \) is defined by

\[
\begin{pmatrix}
  a_{00} & a_{01} \\
  a_{10} & a_{11}
\end{pmatrix} = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}.
\]

The Serre relations [14] hold if \( q_0 \) is a primitive \( 2N \)th root of unity, or a primitive \( N \)th root of unity with \( N \) odd [13]. We derive it through the higher order quantum Serre relations due to Lusztig [20]. The Cartan relations [3] hold for generic \( q \). The relation [9] holds for the identification [6] when \( q_0 \) is a root of unity of type I, and for the identification [7] when \( q_0 \) is a root of unity of type II.

In the sector \( S^Z \equiv 0 \pmod{N} \) we have the commutation relation [13]:

\[
[S^{+(N)}, S^{-(N)}] = (-1)^{N-1} q^N \frac{2}{N} S^Z.
\]

Here the sign factor \((-1)^{N-1} q^N\) is given by 1 or \(-1\) when \( q \) is a root of unity of type I or II, respectively. In the case of the sector \( S^Z \equiv N/2 \pmod{N} \) with \( N \) odd and \( q_0 \) a primitive \( N \)th root of unity, we have the following commutation relation:

\[
[S^{+(N)}, S^{-(N)}] = \frac{2}{N} S^Z.
\]

### 3.4 Some remarks on quantum groups at roots of unity

Let the symbol \( U_q^{\text{res}}(g) \) denote the algebra generated by the \( q \)-divided powers of the Chevalley generators of a Lie algebra \( g \) [4]. The correspondence of the algebra \( U_q^{\text{res}}(g) \) at a root of unity, \( q_0 \), to the Lie algebra \( U(g) \) was obtained essentially through the machinery introduced by Lusztig [25, 20] both for finite-dimensional simple Lie algebras and infinite-dimensional affine Lie algebras. In fact, by using the higher order quantum Serre relations [26], it has been shown that the affine Lie algebra \( U(\hat{sl}_2) \) is generated by the operators such as \( S^{\pm(N)} \) at roots of unity. However, in the case of the affine Lie algebras \( \hat{g} \), the highest weight conditions for the Drinfeld generators are different from those for the Chevalley generators. Through the highest weight vectors of the Drinfeld generators, finite-dimensional representations were discussed by Chari and Pressley for \( U_q^{\text{res}}(\hat{g}) \) [7].

### 4 The Drinfeld realization of the \( sl_2 \) loop algebra

Finite-dimensional representations of the \( sl_2 \) loop algebra, \( U(L(sl_2)) \), are derived by taking the classical analogues of the Drinfeld realization of the quantum \( sl_2 \) loop algebra, \( U_q(L(sl_2)) \) [6, 4]. The classical analogues of the Drinfeld generators, \( \bar{x}_k^\pm \) and \( \bar{h}_k \) (\( k \in \mathbb{Z} \)), satisfy the defining relations in the following:

\[
[h_j, \bar{x}_k^\pm] = \pm 2 \bar{x}^\pm_{j+k}, \quad [\bar{x}_j^+, \bar{x}_k^-] = \bar{h}_{j+k}, \quad \text{for } j, k \in \mathbb{Z}.
\]

Here \( [\bar{h}_j, \bar{h}_k] = 0 \) and \( [\bar{x}_j^\pm, \bar{x}_k^\pm] = 0 \) for \( j, k \in \mathbb{Z} \).
Let us now define highest weight vectors. In a representation of $U(L(sl_2))$, a vector $\Omega$ is called a highest weight vector if $\Omega$ is annihilated by generators $\bar{x}_k^+$ for all integers $k$ and such that $\Omega$ is a simultaneous eigenvector of every generator of the Cartan subalgebra, $\bar{h}_k$ ($k \in \mathbb{Z}$) \cite{8}:
\[
\bar{x}_k^+ \Omega = 0, \quad \text{for } k \in \mathbb{Z},
\]
\[
\bar{h}_k \Omega = \bar{d}_k^+ \Omega, \quad \bar{h}_{-k} \Omega = \bar{d}_{-k}^- \Omega, \quad \text{for } k \in \mathbb{Z}_{\geq 0}.
\]

We call a representation of $U(L(sl_2))$ highest weight if it is generated by a highest weight vector. The set of the complex numbers $\bar{d}_k^\pm$ given in \cite{12} is called the highest weight. It is shown \cite{6} that every finite-dimensional irreducible representation is highest weight. To a finite-dimensional irreducible representation $V$ we associate a unique polynomial through the highest weight $\bar{d}_0^\pm$. We call it the Drinfeld polynomial. Here the degree $r$ is given by the weight $\bar{d}_0^\pm$.

It is easy to see that the highest weight vector of a finite-dimensional irreducible representation $V$ is a simultaneous eigenvector of operators $(\bar{x}_0^+)^k(\bar{x}_0^-)^k/(k!)^2$ for $k > 0$, and the Drinfeld polynomial of the representation $V$ has another expression as follows \cite{13}:
\[
P(u) = \sum_{k=0}^{r} \lambda_k (-u)^k,
\]
where $\lambda_k$ denote the eigenvalues of operators $(\bar{x}_0^+)^k(\bar{x}_0^-)^k/(k!)^2$. It is noted that the author learned the expression of the Drinfeld polynomial \cite{13} from Jimbo \cite{19} (see also \cite{14, 16}).

## 5 Algorithm for evaluating the degenerate multiplicity

### 5.1 A useful theorem on the $sl_2$ loop algebra

Let $\Omega$ be a highest weight vector and $V$ the representation generated by $\Omega$. Here $V$ is not necessarily irreducible. Suppose that $V$ is finite-dimensional and $\bar{h}_0 \Omega = r \Omega$. We define a polynomial $P_\Omega(u)$ by the relation \cite{13} with $\lambda_k$. We show that the roots of the polynomial $P_\Omega(u)$ are nonzero and finite, and the degree of $P_\Omega(u)$ is given by $r$ \cite{11}. Let us factorize $P_\Omega(u)$ as
\[
P_\Omega(u) = \prod_{k=1}^{s} (1 - a_k u)^{m_k},
\]
where $a_1, a_2, \ldots, a_s$ are distinct, and their multiplicities are given by $m_1, m_2, \ldots, m_s$, respectively. Then, we call $a_j$ the evaluation parameters of $\Omega$. Here we note that $r$ is given by the sum: $r = m_1 + \cdots + m_s$. If all the multiplicities are given by 1, i.e. $m_j = 1$ for $j = 1, 2, \ldots, s$, we say that evaluation parameters $a_1, a_2, \ldots, a_r$ are distinct. In the case of distinct evaluation parameters (i.e. $m_j = 1$ for all $j$), we have the following theorem \cite{8}:

**Theorem 1.** Every finite-dimensional highest weight representation of the $sl_2$ loop algebra with distinct evaluation parameters $a_1, a_2, \ldots, a_r$ is irreducible. Furthermore, it has dimensions $2^r$.

Theorem \cite{11} plays an important role in connecting the polynomial $P_\Omega(u)$ with an irreducible finite-dimensional representation. In fact, if $V$ has distinct evaluation parameters, then it is irreducible, and the polynomial $P_\Omega(u)$ is equivalent to the Drinfeld polynomial of $V$. Moreover, we rigorously obtain the dimensions of $V$.

Theorem \cite{11} has been shown in \cite{8} by using the fact that the specialized irreducible modules for the quantum algebra are quotients of the Weyl module. Here we note that theorem \cite{11} is also derived by constructing explicitly a basis of a highest weight representation with distinct evaluation parameters \cite{12}.
5.2 Regular Bethe states as highest weight vectors

For the XXZ spin chain at roots of unity, Fabriiucis and McCoy made important observations on the highest weight conjecture \[\textit{[14, 15, 16]}\]. Motivated by them, we discuss the following:

**Theorem 2.** (i) Every regular Bethe state \(|R\rangle\) in the sector \(S^Z \equiv 0 \mod N\) at \(q_0\) is a highest weight vector of the \(sl_2\) loop algebra. Here \(q_0\) is a root of unity with \(q_0^{2N} = 1\), as specified in Definition \[\textit{1}]. (ii) Every regular Bethe state \(|R\rangle\) in the sector \(S^Z \equiv N/2 \mod N\) at \(q_0\) is a highest weight vector of the \(sl_2\) loop algebra. Here \(N\) is odd and \(q_0\) is a primitive \(N\)th root of unity.

The Theorem \[\textit{2}\] is proved in \[\textit{[11]}\] by assuming the conjecture that for a given regular Bethe state in the sector \(S^Z \equiv 0 \mod N\) (or \(S^Z \equiv N/2 \mod N\)) the set of solutions of the Bethe ansatz equations are continuous with respect to the parameter \(q\) at the root of unity \(q_0\). In some cases such as \(R = 0\) or 1, the conjecture is trivial.

By the method of the algebraic Bethe ansatz, we derive the following relations \[\textit{[11]}\]:

\[
S^{+(N)} |R\rangle = T^{+(N)} |R\rangle = 0, \\
(S^{+(N)})^k (T^{-(N)})^k / (k!)^2 |R\rangle = Z_k^+ |R\rangle \quad \text{for } k \in \mathbb{Z}_{\geq 0}, \\
(T^{+(N)})^k (S^{-(N)})^k / (k!)^2 |R\rangle = Z_k^- |R\rangle \quad \text{for } k \in \mathbb{Z}_{\geq 0}.
\]

(14)

Here, the operators \(S^{\pm(N)}, T^{+(N)}, T^{-(N)}\) and \(2S^Z/N\) satisfy the same defining relations of the \(sl_2\) loop algebra as generators \(\tilde{x}_0^\pm, \tilde{x}_1^+, \tilde{x}_1^-\) and \(\tilde{h}_0\), respectively, and hence the relations \[\textit{[11]}\] correspond to \[\textit{[11]}\] and \[\textit{[12]}\].

In equations \[\textit{[13]}\] eigenvalues \(Z_k^\pm\) are explicitly evaluated as follows:

\[
Z_k^+ = (-1)^{kN} \tilde{\chi}_{kN}^+, \quad Z_k^- = (-1)^{kN} \tilde{\chi}_{kN}^-.
\]

Here the \(\tilde{\chi}_{kN}^\pm\) have been defined by the coefficients of the following expansion with respect to small \(x\):

\[
\phi(x) = \sum_{j=0}^{\infty} \tilde{\chi}_j^+ x^j,
\]

where \(\phi(x) = (1 - x)^L\) and \(\tilde{F}^\pm(x) = \prod_{j=1}^{R} (1 - x \exp(\pm 2i\ell_j))\).

5.3 Drinfeld polynomials of regular Bethe states and the degenerate multiplicity

Let \(|R\rangle\) be a regular Bethe state at \(q_0\) in one of the sectors specified in Theorem \[\textit{2}\]. The Drinfeld polynomial of the regular Bethe state \(|R\rangle\) is explicitly derived by putting \(\lambda_k = (-1)^{kN} \tilde{\chi}_{kN}^+\) into equation \[\textit{[13]}\]. Here, the coefficients \(\tilde{\chi}_{kN}^\pm\) are explicitly evaluated as \[\textit{[11]}\]

\[
\tilde{\chi}_{kN}^\pm = \sum_{n=0}^{\min(L,kN)} (-1)^n \binom{L}{n} \sum_{n_1 + \cdots + n_R = kN - n} \bigg[ \sum_{j=1}^{R} 2n_j \ell_j \bigg]^{R} [n_j + 1]_{[q_0]}.
\]

(15)

Here \([n]_q = (q^n - q^{-n})/(q - q^{-1})\) and the sum is taken over all nonnegative integers \(n_1, n_2, \ldots, n_R\) satisfying \(n_1 + \cdots + n_R = kN - n\): when \(R = 0, n\) is given by \(n = kN\).

For the case of distinct evaluation parameters, we obtain the algorithm for the degeneracy of a regular Bethe state as follows.

**Corollary 1.** Let \(|R\rangle\) be a regular Bethe state such as specified in Theorem \[\textit{2}\]. If the Drinfeld polynomial of the representation \(V\) generated by \(|R\rangle\) gives evaluation parameters \(a_j\) with multiplicities \(m_j = 1\) for \(j = 1, 2, \ldots, r\), then we have \(\dim V = 2^r\), where \(r = (L - 2R)/N\).
6 Examples of Drinfeld polynomials of Bethe states

6.1 The vacuum state with even \( L \)

We now calculate the Drinfeld polynomial \( P(u) \) for the the vacuum state \(|0\rangle\) where \( L = 6 \) and \( N = 3 \) with \( q_0^3 = 1 \). When \( N \) is odd and \( q_0^N = 1 \), we have \( \lambda_k^+ = Z_k^+ = (-1)^k \tilde{\chi}_{kN}^+ \). From the formula (15) we have

\[
\tilde{\chi}_3^+ = (-1)^3 \binom{6}{3} = -20, \quad \tilde{\chi}_6^+ = (-1)^6 \binom{6}{6} = 1.
\]

Thus we have \( \lambda_1^+ = 20 \) and \( \lambda_2^+ = 1 \), and the Drinfeld polynomial is given by

\[
P(u) = 1 - 20u + u^2.
\]

Here, the evaluation parameters are given by

\[
a_1, a_2 = 10 \pm 3\sqrt{11}.
\]

We note that the two evaluation parameters are distinct, and the degree of \( P \) is two, i.e. \( r = 2 \), \( m_1 = m_2 = 1 \). Therefore, the degenerate multiplicity is given by \( 2^2 = 4 \).

6.2 The vacuum state with odd \( L \)

Let us calculate the Drinfeld polynomial \( P(u) \) for the odd \( L \) case. We consider the the vacuum state \(|0\rangle\) where \( L = 9 \) and \( N = 3 \) with \( q_0^3 = 1 \). The vacuum state \(|0\rangle\) is in the sector \( S^Z \equiv 3/2 \) (mod 3), since \( S^Z = 9/2 = 3/2 + 3 \).

From the formula (15) we have

\[
\tilde{\chi}_3^+ = (-1)^3 \binom{9}{3} = -84, \quad \tilde{\chi}_6^+ = (-1)^6 \binom{9}{6} = 84, \quad \tilde{\chi}_9^+ = (-1)^9 \binom{9}{9} = -1.
\]

Thus we have \( \lambda_1^+ = 84 \), \( \lambda_2^+ = 84 \), and \( \lambda_3^+ = 1 \). The Drinfeld polynomial is given by

\[
P(u) = 1 - 84u + 84u^2 - u^3.
\]

Here, the evaluation parameters are given by

\[
a_1, a_2 = \frac{1}{2} \left( 83 \pm 9\sqrt{85} \right).
\]

We note that the three evaluation parameters are distinct, and the degree of \( P \) is three, i.e. \( r = 3 \), \( m_1 = m_2 = m_3 = 1 \). Therefore, the degenerate multiplicity is given by \( 2^3 = 8 \).

6.3 The regular Bethe state with one down-spin \((R = 1)\)

For the case of \( R = 1 \), the Bethe ansatz equations at generic \( q \) are given by

\[
\left( \frac{\sinh(t_j + \eta)}{\sinh(t_j - \eta)} \right)^L = 1 \quad \text{for} \quad j = 0, 1, \ldots, L - 1.
\]

We solve the Bethe ansatz equations in terms of variable \( \exp(2t_j) \) as follows

\[
\exp(2t_j) = \frac{1 - \omega_j q}{q - \omega_j} \quad \text{for} \quad j = 0, 1, \ldots, L - 1,
\]

where \( \omega_j \) denotes an \( L \)th root of unity: \( \omega_j = \exp \left( \frac{2\pi \sqrt{-1} j}{L} \right) \), for \( j = 0, 1, \ldots, L - 1 \).
Let us assume that the regular Bethe state with one down-spin is in the sector \( S^Z \equiv 0 \) (mod \( N \)) and \( q_0 \) be a root of unity with \( q_0^N = 1 \), or in the sector \( S^Z \equiv N/2 \) (mod \( N \)) where \( q_0 \) is a primitive \( N \)th root of unity with \( N \) odd. Then we have from (15)

\[
\tilde{\chi}_{kN}^+ = \sum_{\ell=0}^{\min(nN+2,kN)} (-1)^j[kN + 1 - \ell]q_0^n \frac{nN+2}{q_0 - \omega_j}^{kN-\ell}.
\]

Here we have from (4) that \( L = nN + 2 \) when \( R = 1 \).

Let us consider the case of \( N = 3 \), and \( q_0 = \exp(\pm 2\pi \sqrt{-1}/3) \). Here \( L = 8 \). The regular Bethe state with rapidity \( \tilde{t}_2 \) (the case of \( j = 2 \) in equation (16)) has the Drinfeld polynomial in the following:

\[
P(u) = 1 - 13 \left( 2 - \sqrt{3} \right) u + \left( 7 - 4\sqrt{3} \right) u^2,
\]

where the evaluation parameters \( a_1 \) and \( a_2 \) are given by

\[
a_1, a_2 = \frac{1}{2} \left( 13 \pm \sqrt{165} \right) \left( 2 - \sqrt{3} \right).
\]

The dimensions of the highest weight representation generated by the Bethe state are therefore given by \( 2^2 = 4 \).

Acknowledgements

The author would like to thank K. Fabricius and B.M. McCoy for useful discussions. He would also like to thank the organizers for their kind invitation to the Sixth International Conference “Symmetry in Nonlinear Mathematical Physics”, June 20–26, 2005, Institute of Mathematics, Kyiv, Ukraine. He is grateful to many participants of the conference for useful comments and discussions. This work is partially supported by the Grant-in-Aid (No. 17540351).

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