COMPACTNESS OF SEMIGROUPS OF EXPLOSIVE
SYMMETRIC MARKOV PROCESSES

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Abstract. In this paper, we investigate spectral properties of explosive symmetric Markov processes. Under a condition on its 1-resolvent, we prove the \( L^1 \)-semigroups of Markov processes become compact operators.

1. Introduction

Let \( E \) be a locally compact separable metric space and \( \mu \) a positive Radon measure on \( E \) with topological full support. Let \( X = (\{X_t\}_{t \geq 0}, \{P_x\}_{x \in E}, \zeta) \) be a \( \mu \)-symmetric Hunt process on \( E \). Here \( \zeta \) is the life time of \( X \). We assume \( X \) satisfies the irreducible property, resolvent strong Feller property, in addition, tightness property, namely, for any \( \varepsilon > 0 \), there exists a compact subset \( K \subset E \) such that \( \sup_{x \in E} R_1 \mathbf{1}_{E\setminus K}(x) < \varepsilon \). Here \( R_1 \) is the 1-resolvent of \( X \). The family of symmetric Markov processes with these three properties is called Class (T).

In [13], the spectral properties of a Markov process in Class (T) are studied. For example, if \( \mu \)-symmetric Hunt process \( X \) belongs to Class (T), the semigroup becomes a compact operator on \( L^2(E, \mu) \). This implies the corresponding non-positive self-adjoint operator has only discrete spectrum. Furthermore, it is shown that the eigenfunctions have bounded continuous versions. The self-adjoint operator is extended to linear operators \( (L^p, D(L^p)) \) on \( L^p(E, \mu) \) for any \( 1 \leq p \leq \infty \). In [11], it is shown that the spectral bounds of the operators \( (L^p, D(L^p)) \) are independent of \( p \in [1, \infty] \). Then, a question arises: if a \( \mu \)-symmetric Hunt process \( X \) belongs to Class (T), the spectra of \( (L^p, D(L^p)) \) are independent of \( p \in [1, \infty] \)?

In this paper, we answer this question by showing that the semigroup of \( X \) becomes a compact operator on \( L^1(E, \mu) \) under some additional conditions. These include the condition that \( \lim_{x \to \partial} R_1 \mathbf{1}_E(x) = 0 \) which are more restrictive than Class (T). However, it will be proved that for the symmetric \( \alpha \)-stable process \( X^D \) on an open subset \( D \subset \mathbb{R}^d \) the following assertions are equivalent (Theorem 4.2):

(i) for any \( 1 \leq p \leq \infty \), the semigroup of \( X^D \) is a compact operator on \( L^p(D, m) \);
(ii) the semigroup of \( X^D \) is a compact operator on \( L^2(D, m) \);
(iii) \( \lim_{|x| \to \infty} E_x[\tau_D] = 0 \);
(iv) \( \lim_{|x| \to \infty} \int_0^\infty e^{-t} P_x[\tau_D > t] dt = 0 \).

Here, \( m \) is the Lebesgue measure on \( D \) and \( \tau_D = \inf\{t > 0 \mid X_t^D \notin D\} \). The above conditions are equivalent to

(iii)' \( \lim\inf_{x \in D, |x| \to \infty} E_x[\tau_D] = 0 \).

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provided $D$ is unbounded. In fact, the assertion (iv) is equivalent to the tightness property of $X$. Thus, for the symmetric $\alpha$-stable process $X^D$ on an open subset $D$, the tightness property is equivalent to all assertions in the Theorem 4.2 mentioned above and implies that the spectra are independent of $p \in [1, \infty]$. The key idea is to give an approximate estimate by the semigroup of part processes by employing Dynkin’s formula (Proposition 3.4).

In [14, Theorem 4.2], the authors consider the rotationally symmetric $\alpha$-stable process on $\mathbb{R}^d$ with a killing potential $V$. Under a suitable condition on $V$, they proved the tightness property of the killed stable process. In Example 4.4 below, we will prove the semigroup of the process becomes a compact operator on $L^1(\mathbb{R}^d, m)$ under the assumption on $V$ essentially equivalent to [14, Theorem 4.2].

In Example 4.7 below, we will consider the time-changed process of the rotationally symmetric $\alpha$-stable process on $\mathbb{R}^d$ by the additive functional $A_t = \int_0^t W(X_s)^{-1} \, ds$. Here $\alpha \in (0, 2)$ and $W$ is a nonnegative Borel measurable function on $\mathbb{R}^d$. The Revuz measure of $A$ is $W^{-1}m$ and the time-changed process $X^W$ becomes a $W^{-1}m$-symmetric Hunt process on $\mathbb{R}^d$. The life time of $X^W$ equals to $A_\infty$. To investigate the spectral property of $X^W$ is just to investigate the spectral properties of the operator of the form $L^W = -W(x)(-\Delta)^{\alpha/2}$ on $L^2(\mathbb{R}^d, W^{-1}m)$. When $W(x) = 1 + |x|^\beta$ and $\alpha = 2$, it is shown in [10, Proposition 2.2] that the spectrum of $L^W$ is discrete in $L^2(\mathbb{R}^d, W^{-1}m)$ if and only if $\beta > 2$. When $\alpha \in (0, 2)$, $d > \alpha$, and $W(x) = 1 + |x|^\beta$ with $\beta \geq 0$, it is shown in [14, Proposition 3.3] that the spectrum of $L^W$ in $L^2(\mathbb{R}^d, W^{-1}m)$ is discrete if and only if $\beta > \alpha$. This is equivalent to that the semigroup of $X^W$ is a compact operator on $L^2(\mathbb{R}^d, W^{-1}m)$ if and only if $\beta > \alpha$. In Theorem 4.8 below, we shall prove that if $\beta > \alpha$, the semigroup becomes a compact operator on $L^1(\mathbb{R}^d, W^{-1}m)$.

2. Main results

Let $E$ be a locally compact separable metric space and $\mu$ a positive Radon measure on $E$. Let $E_0$ be the its one-point compactification $E_0 = E \cup \{\partial\}$. A $[-\infty, \infty]$-valued function $u$ on $E$ is extended to a function on $E_0$ by setting $u(\partial) = 0$.

Let $X = \{(X_t)_{t \geq 0}, (P_x)_{x \in E}, \zeta\}$ be a $\mu$-symmetric Hunt process on $E$. The semigroup $\{p_t\}_{t \geq 0}$ and the resolvent $\{R_\alpha\}_{\alpha \geq 0}$ are defined as follows:

$$p_t f(x) = E_x[f(X_t)] = E_x[f(X_t) : t < \zeta],$$

$$R_\alpha f(x) = E_x \left[ \int_0^\zeta \exp(-\alpha t) f(X_t) \, dt \right], \quad f \in \mathcal{B}_b(E), \; x \in E.$$

Here, $\mathcal{B}_b(E)$ is the space of bounded Borel measurable functions on $E$. $E_x$ denotes the expectation with respect to $P_x$. By the symmetry and the Markov property of $\{p_t\}_{t \geq 0}$ and $\{R_\alpha\}_{\alpha \geq 0}$ are canonically extended to operators on $L^p(E, \mu)$ for any $1 \leq p \leq \infty$. The extensions are also denoted by $\{p_t\}_{t \geq 0}$ and $\{R_\alpha\}_{\alpha \geq 0}$, respectively.

For an open subset $U \subset E$, we define $\tau_U$ by $\tau_U = \inf\{t > 0 \mid X_t \notin U\}$ with the convention that $\inf \emptyset = \infty$. We denote by $X^U$ the part of $X$ on $U$. Namely, $X^U$ is defined as follows.

$$X^U_t = \begin{cases} X_t, & t < \tau_U \\ \partial, & t \geq \tau_U. \end{cases}$$
X^U = \{X^U_t\}_{t \geq 0}, \{P_x\}_{x \in U}\) also becomes a Hunt process on \(U\) with life time \(\tau_U\).

The semigroup \(\{p^U_t\}_{t \geq 0}\) is identified with
\[p^U_t f(x) = \mathbb{E}_x [f(X^U_t)] = \mathbb{E}_x [f(X_t) : t < \tau_U]\]
\(\{p^U_t\}_{t \geq 0}\) is also symmetric with respect to the measure \(\mu\) restricted to \(U\). \(\{p^U_t\}_{t \geq 0}\) and \(\{R^U_\alpha\}_{\alpha > 0}\) are also extended to operators on \(L^p(U, \mu)\) for any \(1 \leq p \leq \infty\) and the extensions are also denoted by \(\{p^U_t\}_{t \geq 0}\) and \(\{R^U_\alpha\}_{\alpha > 0}\), respectively.

We now make the following conditions on the symmetric Markov process \(X\).

I. (Semigroup strong Feller) For any \(t > 0\), \(p_t(B_b(E)) \subset C_b(E)\), where \(C_b(E)\) is the space of bounded continuous functions on \(E\).

II. (Tightness property) \(\lim_{x \to 0} R^U_1E(x) = 0\).

III. (Local \(L^\infty\)-compactness) For any \(t > 0\) and open subset \(U \subset E\) with \(\mu(U) < \infty\), \(p^U_t\) is a compact operator on \(L^\infty(U, \mu)\).

Remark 2.1. (i) By the condition I, the semigroup kernel of \(X\) is absolutely continuous with respect to \(\mu\):
\[p_t(x, dy) = p_t(x, y) \, d\mu(y)\]
Furthermore, the resolvent of \(X\) is strong Feller: for any \(\alpha > 0\), \(R_\alpha(B_b(E)) \subset C_b(E)\).

(ii) The conditions I and II lead us to the tightness property in the sense of [12][13]: for any \(\varepsilon > 0\), there exists a compact subset \(K \subset E\) such that \(\sup_{x \in E} R^U_1E_{\setminus K}(x) < \varepsilon\). See [12] Remark 2.1 (ii)] for details. We denote by \(C_\infty(E)\) the space of continuous functions on \(E\) vanishing at infinity. Under the condition I and the invariance \(R^U_1(C_\infty(E)) \subset C_\infty(E)\) of \(X\), the condition II is equivalent to the tightness property in the sense of [12][13]. See [12] Remark 2.1 (iii)] for details. In addition to the conditions I and II, we assume \(X\) is irreducible in the sense of [12]. Then, by using [12] Lemma 2.2 (ii), Lemma 2.6, Corollary 3.8], we can show \(\sup_{x \in E} \mathbb{E}_x [\exp(\lambda \zeta)] < \infty\) for some \(\lambda > 0\) and thus \(R_01_E\) is bounded on \(E\).

(iii) The conditions I and II imply \(p_t(C_\infty(E)) \subset C_\infty(E)\) for any \(t > 0\), and thus \(X\) is doubly Feller in the sense of [3]. This implies that for any \(t > 0\) and open \(U \subset E\), \(p^U_t\) is strong Feller: \(p^U_t(B_b(U)) \subset C_b(U)\). See [3] Theorem 1.4] for the proof.

(iv) Let \(U \subset E\) be an open subset with \(\mu(U) < \infty\). The condition III is satisfied if the semigroup of \(X^U\) is ultracontractive: for any \(t > 0\) and \(f \in L^1(U, \mu)\), \(p^U_t f\) belongs to \(L^\infty(U, \mu)\). Indeed, we see from [4] Theorem 1.6.4] that \(p^U_t\) is a compact operator on \(L^1(U, \mu)\) and so is on \(L^\infty(U, \mu)\). In particular, if the semigroup of \(X\) is ultracontractive, the condition III is satisfied.

We are ready to state the main result of this paper.

**Theorem 2.2.** Assume \(X\) satisfies the conditions from I to III. Then, for any \(t > 0\), \(p_t\) becomes a compact operator on \(L^\infty(E, \mu)\).

By the symmetry of \(X\), each \(p_t : L^\infty(E, \mu) \to L^\infty(E, \mu)\) is regarded as the dual-operator of \(p_t : L^1(E, \mu) \to L^1(E, \mu)\). By using Schauder’s theorem, we obtain the next corollary.
Corollary 2.3. Assume $X$ satisfies the conditions from I to III. Then, for any $t > 0$, $p_t$ becomes a compact operator on $L^1(E, \mu)$.

Let $(\mathcal{L}^p, D(\mathcal{L}^p))$ be the generator of $\{p_t\}_{t > 0}$ on $L^p(E, \mu)$, $1 \leq p \leq \infty$. By using \cite[Theorem 1.6.4]{[4]}, we can show the next theorem.

Theorem 2.4. Assume $X$ satisfies the conditions from I to III. Then,

(i) for any $1 \leq p \leq \infty$ and $t > 0$, $p_t$ is a compact operator on $L^p(E, \mu)$;

(ii) spectra of $(\mathcal{L}^p, D(\mathcal{L}^p))$ are independent of $p \in [1, \infty]$ and the eigenfunctions of $(\mathcal{L}^2, D(\mathcal{L}^2))$ belong to $L^p(E, \mu)$ for any $1 \leq p \leq \infty$.

3. Proof of Theorem 2.4

Since $E$ is a locally compact separable metric space, there exist increasing bounded open subsets $\{U_n\}_{n=1}^{\infty}$ and compact subsets $\{K_n\}_{n=1}^{\infty}$ such that for any $n \in \mathbb{N}$, $K_n \subset U_n \subset K_{n+1}$ and $E = \bigcup_{n=1}^{\infty} U_n = \bigcup_{n=1}^{\infty} K_n$. We write $\tau_n$ for $\tau_{U_n}$. The semigroup of the part process of $X$ on $U_n$ is simply denoted by $\{p_t^n\}_{t > 0}$.

The quasi-left continuity of $X$ yields the next lemma.

Lemma 3.1. For any $x \in E$, $P_x(\lim_{n \to \infty} \tau_n = \zeta) = 1$.

The following formula is called Dynkin’s formula.

Lemma 3.2. It holds that

$$p_t f(x) = p_t^U f(x) + E_x[p_{t - \tau_U} f(X_{\tau_U}) : \tau_U \leq t]$$

for any $x \in E$, $f \in B_b(E)$, $t > 0$, and any open subset $U$ of $E$.

Proof. It is easy to see that (3.1)

$$p_t f(x) = p_t^U f(x) + E_x[f(X_t) : \tau_U \leq t].$$

Let $n \in \mathbb{N}$. On $\{\tau_U \leq t\}$, we define $s_n$ by

$$s_n |_{\{(k-1)/2^n \leq t - \tau_U < k/2^n\}} = k/2^n, \quad k \in \mathbb{N}.$$  

We note that $\lim_{n \to \infty} s_n = t - \tau_U$. By the strong Markov property of $X$,

$$E_x[f(X_{\tau_U + s_n}) : \tau_U \leq t] = \sum_{k=1}^{\infty} E_x[f(X_{\tau_U + k/2^n}) : (k-1)/2^n \leq t - \tau_U < k/2^n]$$

$$= \sum_{k=1}^{\infty} E_x[E_{X_{\tau_U}}[f(X_{k/2^n})] : (k-1)/2^n \leq t - \tau_U < k/2^n]$$

$$= E_x[p_{s_n} f(X_{\tau_U}) : \tau_U \leq t].$$

Letting $n \to \infty$ in (3.2), we obtain (3.3)

$$E_x[f(X_t) : \tau_U \leq t] = E_x[p_{t - \tau_U} f(X_{\tau_U}) : \tau_U \leq t]$$

Combining (3.1) with (3.3), we complete the proof.

By using Dynkin’s formula and the semigroup strong Feller property, we obtain the next lemma.

Lemma 3.3. Let $K$ be a compact subset of $E$. Then, for any $t > 0$ and a nonegative $f \in B_b(E)$,

$$\lim_{n \to \infty} \sup_{x \in K} E_x[p_{t - \tau_n} f(X_{\tau_n}) : \tau_n \leq t] = 0.$$
Proof: We may assume \( K \subset U_1 \). By the condition I and Remark 2.1 (iii), both \( p_t f \) and \( p_t^n f \) are continuous on \( K \). Hence, we see from Dynkin’s formula (Lemma 3.2) that
\[
E_x[|p_{t-\tau_n}f(X_{\tau_n}) : \tau_n \leq t|] = p_t f(x) - p_t^n f(x)
\]
is continuous on \( K \). For any \( t > 0 \) and \( x \in E \), \( p_t^n f(x) \leq p_t^{n+1} f(x) \). Hence, (LHS) of (3.4) is non-increasing in \( n \). By Lemma 3.1 (LHS) of (3.4) converges to
\[
\lim_{n \to \infty} E_x[p_{t-\tau_n}f(X_{\tau_n}) : \tau_n \leq t] = \lim_{n \to \infty} (p_t f(x) - p_t^n f(x))
\]
and the proof is complete by Dini’s theorem. \( \square \)

For each \( n, m \in \mathbb{N} \) and \( t > 0 \), we define the operator \( T_{n,t} \) on \( L^\infty(E, \mu) \) by
\[
L^\infty(E, \mu) \ni f \mapsto E_{(\cdot)}[p_{t-\tau_n}f(X_{\tau_n}) : \tau_n \leq t].
\]
The operator norm of \( T_{n,t} \) is estimated as follows.

**Proposition 3.4.** Let \( n, m \in \mathbb{N} \) with \( m < n \). Then, for any \( t > 0 \),
\[
\|T_{n,t}\|_{L^\infty(E, \mu) \to L^\infty(E, \mu)} \leq \sup_{x \in K_m} E_x[p_{t-\tau_n}1_E(X_{\tau_n}) : \tau_n \leq t] + (4/t) \times \sup_{x \in E \setminus K_m} E_x[\zeta].
\]
Here, \( \| \cdot \|_{L^\infty(E, \mu) \to L^\infty(E, \mu)} \) denotes the operator norm from \( L^\infty(E, \mu) \) to itself.

**Proof.** Let \( f \in L^\infty(E, \mu) \) with \( \| f \|_{L^\infty(E, \mu)} = 1 \). Then, we have
\[
\|E_{(\cdot)}[p_{t-\tau_n}f(X_{\tau_n}) : \tau_n \leq t]\|_{L^\infty(E, \mu)} \leq \|f\|_{L^\infty(E, \mu)} \times \text{ess sup}_{x \in E} E_x[p_{t-\tau_n}1_E(X_{\tau_n}) : \tau_n \leq t]
\]
\[
\leq \text{ess sup}_{x \in E \setminus K_m} E_x[p_{t-\tau_n}1_E(X_{\tau_n}) : \tau_n \leq t/2] + \text{ess sup}_{x \in E \setminus K_m} E_x[p_{t-\tau_n}1_E(X_{\tau_n}) : t/2 < \tau_n \leq t]
\]
\[
+ \text{ess sup}_{x \in E \setminus K_m} E_x[p_{t-\tau_n}1_E(X_{\tau_n}) : \tau_n \leq t/2]
\]
\[
\leq \sup_{x \in E \setminus K_m} E_x[p_{t-\tau_n}1_E(X_{\tau_n}) : \tau_n \leq t] + \sup_{x \in E \setminus K_m} P_x(t/2 < \tau_n)
\]
Here, ess sup denotes the essential supremum with respect to \( \mu \). Moreover, we see
\[
P_x(t/2 < \tau_n) \leq P_x(t/2 < \zeta) \leq (2/t) \times E_x[\zeta]
\]
and
\[
p_x1_E(x) = P_x(X_s \in E) = P_x(s < \zeta) \leq (1/s) \times E_x[\zeta].
\]
Combining these estimates, we obtain the following estimate
\[
\|E_{(\cdot)}[p_{t-\tau_n}f(X_{\tau_n}) : \tau_n \leq t]\|_{L^\infty(E, \mu)} \leq \sup_{x \in K_m} E_x[p_{t-\tau_n}1_E(X_{\tau_n}) : \tau_n \leq t] + (4/t) \times \sup_{x \in E \setminus K_m} E_x[\zeta].
\]
\( \square \)
Lemma 3.2 leads us to that for any \( p_t^{(1)} \) is given by
\[
p_t^{(1)}(x) := E_x^{(1)}[f(X^{(1)}_t)] = E_x[e^{-t}f(X_t)], \quad t > 0, \ x \in E, \ f \in B_0(E),
\]
where \( E_x^{(1)} \) is the expectation with respect to \( p_t^{(1)} \). For each \( n \in \mathbb{N} \), we denote by \( X^{(1),n} \) the part process of \( X^{(1)} \) on \( U_n \). The semigroup is denoted by \( \{ p_t^{(1),n} \}_{t \geq 0} \). It is easy to see
\[
p_t^{(1)}(x) - p_t^{(1),n}(x) = e^{-t}(p_t f(x) - p_t^{n} f(x))
\]
for any \( t > 0, \ x \in E, \ f \in B_0(E) \), and \( n \in \mathbb{N} \).

For each \( n \in \mathbb{N} \) and \( t > 0 \), we define the operator \( T_{n,t}^{(1)} \) on \( L^\infty(E, \mu) \) by
\[
L^\infty(E, \mu) \ni f \mapsto E^{(1)}[p_t^{(1)} f(X^{(1)}_{\tau_n^t}) : \tau_n^t \leq t],
\]
where we define \( \tau_n = \inf\{ t > 0 \mid X^{(1)}_t \notin U_n \} \). By using (3.6) and applying Lemma 3.2 to \( X \) and \( X^{(1)} \), we have
\[
T_{n,t}^{(1)}(f) = p_t^{(1)}(f) - p_t^{(1),n}(f) = e^{-t}(p_t f(x) - p_t^{n} f(x))
\]
for any \( t > 0, n \in \mathbb{N}, \ x \in E \) and \( f \in B_0(E) \). By using (3.6) and Lemma 3.3 we obtain the next lemma.

**Lemma 3.5.**

(i) It holds that
\[
\lim_{n \to \infty} \sup_{x \in K} T_{n,t}^{(1)}(f)(x) = 0
\]
for any compact subset \( K \subset E \), \( t > 0 \) and nonnegative \( f \in B_0(E) \).

(ii) It holds that
\[
\| T_{n,t}^{(1)} \|_{L^\infty(E, \mu) \to L^\infty(E, \mu)} = e^t \times \| T_{n,t}^{(1)} \|_{L^\infty(E, \mu) \to L^\infty(E, \mu)}
\]
for any \( t > 0 \) and \( n \in \mathbb{N} \).

**Proof of Theorem 2.2.** By the condition III, each \( p_t^{n} \) is regarded as a compact operator on \( L^\infty(E, \mu) \). Therefore it is sufficient to prove
\[
\lim_{n \to \infty} \| p_t^{n} - p_t \|_{L^\infty(E, \mu) \to L^\infty(E, \mu)} = 0.
\]

Lemma 3.2 lead us to that for any \( n \in \mathbb{N} \) and \( t > 0 \)
\[
\| p_t^{n} - p_t^{(1)} \|_{L^\infty(E, \mu) \to L^\infty(E, \mu)} = \sup_{f \in L^\infty(E, \mu), \ \| f \|_{L^\infty(E, \mu)} = 1} \| E^{(1)}[p_t^{(1)} f(X^{(1)}_{\tau_n^t}) : \tau_n \leq t] \|_{L^\infty(E, \mu)}
\]
\[
= \| T_{n,t}^{(1)} \|_{L^\infty(E, \mu) \to L^\infty(E, \mu)}.
\]

It holds that \( E_x^{(1)}[\zeta^{(1)}] = R_t 1_E(x) \) for any \( x \in E \). Applying Proposition 3.3 to \( X^{(1)} \), we have
\[
\| T_{n,t}^{(1)} \|_{L^\infty(E, \mu) \to L^\infty(E, \mu)} \leq \sup_{x \in K_m} E_x^{(1)}[p_t^{(1)} 1_E(X^{(1)}_{\tau_n^t}) : \tau_n \leq t] + (4/t) \sup_{x \in E \setminus K_m} E_x^{(1)}[\zeta^{(1)}]
\]
\[
= \sup_{x \in K_m} T_{n,t}^{(1)} 1_E(x) + (4/t) \sup_{x \in E \setminus K_m} R_t 1_E(x).
\]
Combining (3.7), (3.8) and Lemma 3.5 (ii), we have
\[
\|p_t - p^k_t\|_{L^\infty(E, \mu) \to L^\infty(E, \mu)} \leq e^{t} \times \left\{ \sup_{x \in K_m} T^{(1)}_{n,t} \mathbf{1}_E(x) + \frac{4}{t} \times \sup_{x \in E \setminus K_m} R_1 \mathbf{1}_E(x) \right\}.
\]
Letting \(n \to \infty\) and then \(m \to \infty\), the proof is complete by Lemma 3.5 (i). \(\square\)

4. Examples

Example 4.1. Let \(\alpha \in (0, 2]\) and \(X\) be the rotationally symmetric \(\alpha\)-stable process on \(\mathbb{R}^d\). If \(\alpha = 2\), \(X\) is identified with the \(d\)-dimensional Brownian motion. Let \(D \subset \mathbb{R}^d\) be an open subset of \(\mathbb{R}^d\) and \(X^D\) be the \(\alpha\)-stable process on \(D\) with Dirichlet boundary condition. Since \(X\) is semigroup doubly Feller in the sense of [3], the condition I is satisfied for \(X^D\). Since the semigroup of \(X\) is ultracontractive, so is the semigroup of \(X^D\). Thus, the condition III is also satisfied. It is shown in [9, Lemma 1] that the semigroup of \(X^D\) is a compact operator on \(L^2(D, m)\) if and only if \(\lim_{|x| \to \infty} E_x [\tau_D] = 0\).

Hence, by using Theorem 2.3 and Theorem 2.4, we obtain the next theorem.

Theorem 4.2. The following are equivalent:

(i) for any \(1 \leq p \leq \infty\), the semigroup of \(X^D\) is a compact operator on \(L^p(D, m)\);

(ii) the semigroup of \(X^D\) is a compact operator on \(L^2(D, m)\);

(iii) \(\lim_{|x| \to \infty} E_x [\tau_D] = 0\);

(iv) \(\lim_{|x| \to \infty} \int_0^\infty e^{-t} P_x [\tau_D > t] \, dt = 0\).

Remark 4.3. The semigroup of \(X^D\) is not necessarily a Hilbert-Schmidt operator but can be a compact operator on \(L^1(D, m)\). Namely, there exists an open subset \(D \subset \mathbb{R}^d\) which satisfies the following conditions:

(D.1) \(\lim_{|x| \to \infty} E_x [\tau_D] = 0\);

(D.2) the trace of the semigroup of \(X^D\) is infinite.

For example, let \(\alpha = 2\), \(d \in \mathbb{N}\), and
\[
D = \bigcup_{n=1}^\infty D_n = \bigcup_{n=1}^\infty B(e_n, r_n)
\]
Here, \(B(e_n, r_n) \subset \mathbb{R}^d\) denotes the open ball centered at \(e_n = (n, 0, \cdots, 0) \in \mathbb{R}^d\) with radius \(r_n = \{(\log \log (n + 3))^{-1/2}\}. It is easy to see \(r_n > 1\) for \(n > 24\). We shall check \(D\) satisfies the conditions (D.1) and (D.2). We denote by \(p^{D_n}_t(x, y)\) the heat kernel density of \(X^{D_n}\) with respect to \(m\). By [4] Theorem 1.9.3,
\[
\int_D p^{D_n}_t(x, x) \, dm(x) \geq \sum_{n=25}^\infty \int_{D_n} p^{D_n}_t(x, x) \, dm(x)
\]
\[
\geq \sum_{n=25}^\infty (8\pi t)^{-d/2} \times r_n \times \exp(-8\pi^2 dt/r_n^2)
\]
\[
\geq (8\pi t)^{-d/2} \sum_{n=25}^\infty \left\{ \log(n + 3) \right\}^{-1/2 - 8\pi^2 dt} = \infty.
\]
Therefore, the trace of the semigroup of $X^D$ is infinite. On the other hand, for any $x \in D_n$,

$$E_x[\tau_D] = E_x[\tau_{D_n}] \leq E_o[\tau_{B(|e_n - x| + r_n)}].$$

Here, $o$ denotes the origin of $\mathbb{R}^d$ and $B(|e_n - x| + r_n)$ denotes the open ball centered at the origin with radius $|e_n - x| + r_n$. $|e_n - x|$ is the length of $e_n - x$. Since $|e_n - x| \leq r_n$, it holds that

$$E_o[\tau_{B(|e_n - x| + r_n)}] = (|e_n - x| + r_n)^2/d \leq 4r_n^2/d.$$

Since $r_n \to 0$ as $n \to \infty$, $\lim_{|x| \to \infty} E_x[\tau_D] = 0$.

**Example 4.4.** Let $\alpha \in (0, 2]$ and $X = \{X_t\}_{t \geq 0}, \{P_x\}_{x \in \mathbb{R}^d}, \zeta$ be the rotationally symmetric $\alpha$-stable process on $\mathbb{R}^d$. The semigroup of $X$ is denoted by $\{p_t\}_{t \geq 0}$. Let $V$ be a positive Borel measurable function on $\mathbb{R}^d$ with the following properties:

1. $V$ is locally bounded. Namely, for any relatively compact open subset $G \subset \mathbb{R}^d$, $\sup_{x \in G} V < \infty$;
2. $\lim_{x \to \mathbb{R}^d, |x| \to \infty} V(x) = \infty$.

We set $A_t = \int_0^t V(X_s) \, ds$. Let $X^V = \{X_t\}_{t \geq 0}, \{P_x^V\}_{x \in \mathbb{R}^d}, \zeta$ be the subprocess of $X$ defined by $dP^V_x = \exp(-A_t) dP_x$. The semigroup $\{p^V_t\}_{t \geq 0}$ is identified with $p_t^V f(x) = E_x[\exp(-A_t) f(X_t)], \quad f \in B_b(\mathbb{R}^d), \quad x \in \mathbb{R}^d$.

**Theorem 4.5.** $X^V$ satisfies the conditions from I to III.

Before proving Theorem 4.5, we give a lemma. We denote by $B(n)$ the open ball of $\mathbb{R}^d$ centered at the origin $o$ and radius $n \in \mathbb{N}$. The semigroup of $X$ is doubly Feller in the sense of [3]. Thus, for any $n \in \mathbb{N}$, the semigroup of $X^{B(n)}$ is strong Feller.

**Lemma 4.6.** It holds that

$$\lim_{n \to \infty} \sup_{t \leq t} P_x(\tau_{B(n)} \leq t) = 0$$

for any $t > 0$ and compact subset $K \subset \mathbb{R}^d$. Here, $\tau_{B(n)} = \inf\{t > 0 \mid X_t \in \mathbb{R}^d \setminus B(n)\}$.

**Proof.** Without loss of generality, we may assume $K \subset B(1)$. For any $t > 0, n \in \mathbb{N}$, and $x \in \mathbb{R}^d$,

$$P_x(\tau_{B(n)} \leq t) = 1_{\mathbb{R}^d}(x) - P_x(\tau_{B(n)} > t)$$

$$= 1_{\mathbb{R}^d}(x) - p_t^{B(n)} 1_{\mathbb{R}^d}(x).$$

Thus, we see from the strong Feller property of $X^{B(n)}$ that for any $n \in \mathbb{N}$, $P_x(\tau_{B(n)} \leq t)$ is a continuous function on $K$. It follows from the conservativeness of $X$ and Lemma 3.1 that for any $x \in \mathbb{R}^d$,

$$\lim_{n \to \infty} P_x(\tau_{B(n)} \leq t) \leq P_x(\zeta \leq t) = 0$$

and the convergence is non-increasing. The proof is complete by Dini’s theorem. \hfill \Box

**Proof of Theorem 4.5.** Since the semigroup of $X$ is ultracontractive, so is the semigroup of $X^V$. Hence, the condition III is satisfied. We will check $X^V$ satisfies the
condition I. Let $K$ be a compact subset of $\mathbb{R}^d$ and take $n_0 \in \mathbb{N}$ such that $K \subset B(n_0)$. Then, for any $s \in (0, 1)$ and $n > n_0$,

$$
\sup_{x \in K} E_x[1 - \exp(-A_s)] \\
\leq \sup_{x \in K} E_x[A_s \land \tau_B(n)] + \sup_{x \in K} P_x(\tau_B(n) \leq s) \\
= \sup_{x \in K} E_x \left[ \int_0^{\tau_B(n)} V(X_t) dt \right] + \sup_{x \in K} P_x(\tau_B(n) \leq 1) =: I_1 + I_2.
$$

By the condition (V.1), $\lim_{s \to 0} I_1 = 0$. By Lemma 4.6, $\lim_{n \to \infty} I_2 = 0$. Thus, (4.1)

$$
\lim_{s \to 0} \sup_{x \in K} E_x[1 - \exp(-A_s)] = 0.
$$

Let $t > 0$ and $f \in B_0(\mathbb{R}^d)$. Since the semigroup of $X$ is strong Feller, for any $s \in (0, t)$, $p_s p_{t-s} f$ is continuous on $\mathbb{R}^d$. By using (4.1), we have

$$
\lim_{s \to 0} \sup_{x \in K} [p_s^V f(x) - p_s p_{t-s} f(x)] \\
= \lim_{s \to 0} \sup_{x \in K} \left| E_x[\exp(-A_s) f(X_t)] - E_x[p_s^V f(X_s)] \right| \\
= \lim_{s \to 0} \sup_{x \in K} \left| E_x[\exp(-A_s) E_{X_s}[\exp(-A_{t-s}) f(X_{t-s})]] - E_x[p_s^V f(X_s)] \right| \\
\leq \|f\|_{L^\infty(\mathbb{R}^d, m)} \times \lim_{s \to 0} \sup_{x \in K} E_x[1 - \exp(-A_s)] = 0.
$$

This means that the semigroup of $X^V$ is strong Feller and the condition I is satisfied.

Finally, we shall show the condition II. Let $x \in \mathbb{R}^d$ and $t > 0$. Since $X$ is spatially homogeneous,

$$
P_x^V(\zeta > t) = E_x \left[ \exp \left( - \int_0^t V(X_s) ds \right) \right] = E_x \left[ \exp \left( - \int_0^t V(x + X_s) ds \right) \right].
$$

It follows from the condition (V.2) that for any $t > 0$, $\lim_{x \to \infty} \|X_t\| = 0$. By the positivity of $V$, we can show that $\sup_{x \in \mathbb{R}^d} P_x^V(\zeta > t) < 1$ for any $t > 0$. By the additivity of $\{A_t\}_{t \geq 0}$,

$$
P_x^V(\zeta > t + s) = E_x[\exp(-A_{t+s}) : t + s < \zeta] \\
= E_x[\exp(-A_s) E_{X_s}[\exp(-A_t) : t < \zeta] : s < \zeta] \\
\leq \sup_{x \in \mathbb{R}^d} P_x^V(\zeta > t) \times \sup_{x \in \mathbb{R}^d} P_x^V(\zeta > s)
$$

for any $x \in \mathbb{R}^d$ and $s, t > 0$. Hence, letting $p = \sup_{x \in \mathbb{R}^d} P_x^V(\zeta > 1) < 1$, we have

$$
\sup_{x \in \mathbb{R}^d} E_x^V[\zeta] = \sup_{x \in \mathbb{R}^d} \int_0^\infty P_x^V(\zeta > t) dt \leq \sum_{n=0}^{\infty} \int_0^{n+1} \sup_{x \in \mathbb{R}^d} P_x^V(\zeta > n) dt \\
\leq 1 + \sum_{n=1}^{\infty} p^n = 1/(1 - p).
$$

We denote by $p_t^V(x, y)$ the heat kernel density of $X^V$. For any $\varepsilon > 0$,

$$
E_x^V[\zeta] \leq \varepsilon + E_x^V[E_{X_x}^V[\zeta]] \leq \varepsilon + \int_{\mathbb{R}^d} p_t^V(x, y) E_y^V[\zeta] dm(y) \\
\leq \varepsilon + \frac{1}{1 - p} \times P_x^V(\zeta > \varepsilon).
$$
By letting \( x \to \infty \), we have \( \lim_{x \to \infty} |x| E_x^V[\xi] \leq \varepsilon \). Since \( \varepsilon \) is chosen arbitrarily, the condition II is satisfied.

**Example 4.7.** Let \( \alpha \in (0, 2] \) and \( d > \alpha \), and \( X = (\{X_t\}_{t \geq 0}, \{P_x\}_{x \in \mathbb{R}^d}, \zeta) \) be the rotationally symmetric \( \alpha \)-stable process on \( \mathbb{R}^d \). We note that \( X \) is transient. Let us consider the additive functional \( \{A_t\}_{t \geq 0} \) of \( X \) defined by

\[
A_t = \int_0^t W(X_s)^{-1} ds, \quad t \geq 0.
\]

Here \( W \) is a Borel measurable function on \( \mathbb{R}^d \) with the condition:

\[
1 + |x|^\beta \leq W(x) < \infty, \quad x \in \mathbb{R}^d,
\]

where \( \beta \geq 0 \) is a constant. The Revuz measure of \( \{A_t\}_{t \geq 0} \) is identified with \( W^{-1} \). Denote \( \mu = W^{-1} \). \( \mu \) is not necessary a finite measure on \( \mathbb{R}^d \). Noting that \( A_t \) is continuous and strictly increasing in \( t \), we define \( X^\mu = (\{X^\mu_t\}_{t \geq 0}, \{P_x\}_{x \in \mathbb{R}^d}, \zeta^\mu) \) by

\[
X^\mu_t = X_t, \quad t \geq 0, \quad \tau = A^{-1}, \quad \zeta^\mu = \zeta^\infty.
\]

Then, \( X^\mu \) becomes a \( \mu \)-symmetric Hunt process on \( \mathbb{R}^d \), \( X^\mu \) is transient because the transience is preserved by time-changed transform ([7] Theorem 6.2.3). The semigroup and the resolvent of \( X^\mu \) are denoted by \( \{p^\mu_t\}_{t \geq 0}, \{R^\mu_t\}_{\alpha \geq 0} \), respectively.

**Theorem 4.8.** If \( \beta > \alpha \), \( X^\mu \) satisfies the conditions from I to III.

Before proving Theorem [L3], we give some notions and lemmas. Let \((\mathcal{E}, \mathcal{F})\) be the Dirichlet form of \( X \). \((\mathcal{E}, \mathcal{F})\) is identified with

\[
\mathcal{E}(f, g) = \frac{K(d, \alpha)}{2} \int_{\mathbb{R}^d} \hat{f}(\xi) \hat{g}(\xi) |\xi|^\alpha d\xi,
\]

\[
f, g \in \mathcal{F} = \left\{ f \in L^2(\mathbb{R}^d, m) \middle| \int_{\mathbb{R}^d} |\hat{f}(\xi)|^2 |\xi|^\alpha d\xi < \infty \right\}.
\]

Here \( \hat{f} \) denotes the Fourier transform of \( f \) and \( K(d, \alpha) \) is a positive constant. Recall that \( m \) is the Lebesgue measure on \( \mathbb{R}^d \), \( m \) is also denoted by \( dx \). Let \((\mathcal{E}, \mathcal{F}_c)\) denotes the extended Dirichlet space of \((\mathcal{E}, \mathcal{F})\). Namely, \( \mathcal{F}_c \) is the family of Lebesgue measurable functions \( f \) on \( \mathbb{R}^d \) such that \( |f| < \infty \) \( m \)-a.e. and there exists a sequence \( \{f_n\}_{n=1}^\infty \) of functions in \( \mathcal{F} \) such that \( \lim_{n \to \infty} f_n = f \) \( m \)-a.e. and \( \lim_{n, k \to \infty} \mathcal{E}(f_n - f_k, f_n - f_k) = 0 \). \( \{f_n\}_{n=1}^\infty \) as above called an approximating sequence for \( f \in \mathcal{F}_c \) and \( \mathcal{E}(f, f) \) is defined by \( \mathcal{E}(f, f) = \lim_{n \to \infty} \mathcal{E}(f_n, f_n) \). Since the quasi support of \( \mu \) is identified with \( \mathbb{R}^d \), the Dirichlet form \((\mathcal{E}^\mu, \mathcal{F}^\mu)\) of \( X^\mu \) is described as follows (see [7] Theorem 6.2.1, (6.2.22)] for details).

\[
\mathcal{E}^\mu(f, g) = \mathcal{E}(f, g), \quad \mathcal{F}^\mu = \mathcal{F}_c \cap L^2(\mathbb{R}^d, \mu).
\]

By identifying the Dirichlet form of \( X^\mu \), we see that the semigroup of \( X^\mu \) is ultracontractive.

**Lemma 4.9.** For any \( f \in L^1(\mathbb{R}^d, \mu) \) and \( t > 0 \), \( p^\mu_t f \in L^\infty(\mathbb{R}^d, \mu) \).

**Proof.** By [3] Theorem 1, p138 and [5] Theorem 6.5] for \( \alpha \in (0, 2) \), there exist positive constants \( C > 0 \) and \( q \in (2, \infty) \) such that

\[
\left\{ \int_{\mathbb{R}^d} |f|^q d\mu \right\}^{2/q} \leq \left\{ \int_{\mathbb{R}^d} |f|^q dm \right\}^{2/q} \leq C \mathcal{E}(f, f), \quad f \in \mathcal{F}.
\]
Let \( \{f_n\}_{n=1}^\infty \subset \mathcal{F} \) be an approximating sequence of \( f \in \mathcal{F}^\mu = \mathcal{F}_\epsilon \cap L^2(\mathbb{R}^d, \mu) \). By using Fatou’s lemma and (4.2), we have

\[
\left\{ \int_{\mathbb{R}^d} |f|^q \, d\mu \right\}^{2/q} \leq \lim_{n \to \infty} \left\{ \int_{\mathbb{R}^d} |f_n|^q \, d\mu \right\}^{2/q} \leq C \lim_{n \to \infty} \mathcal{E}(f_n, f_n) = C \mathcal{E}(f, f).
\]

The proof is complete by [2]. See also [7, Theorem 4.2.7].

Let \( U \) be an open subset of \( \mathbb{R}^d \) and \( X^{\mu, U} \) be the part of \( X^\mu \) on \( U \):

\[
X^{\mu, U}_t = \begin{cases} \mu_t, & t < T_U := \inf \{ t > 0 \mid X^\mu_t \notin U \} \\ \partial, & t \geq T_U. \end{cases}
\]

The semigroup and the resolvent are denoted by \( \{p^{\mu, U}_t\}_{t>0} \) and \( \{R^{\mu, U}_\gamma\}_{\gamma>0} \), respectively.

**Lemma 4.10.** Let \( f \in \mathcal{B}_0(U) \), \( \gamma > 0 \), and \( U \subset \mathbb{R}^d \) be an open subset. Then, \( R^{\mu, U}_\gamma f \in C_0(\mathbb{R}^d) \). In particular, for each \( \gamma > 0 \) and \( x \in U \), the kernel \( R^{\mu, U}_\gamma(x, \cdot) \) is absolutely continuous with respect to \( \mu|_U \).

**Proof.** It is easy to see that \( \lim_{\epsilon \to 0} \sup_{x \in \mathbb{R}^d} E_x[A_\epsilon] = 0 \). This means that \( \mu \) is in the Kato class of \( X \) in the sense of [8]. Since the resolvent of \( X \) is doubly Feller in the sense of [8], by [8, Theorem 7.1], the resolvent of \( X^\mu \) is also doubly Feller. By using [8, Theorem 3.1], we complete the proof. “In particular” part follows from the same argument as in [7, Exercise 4.2.1].

Following the arguments in [1, Theorem 5.1], we strengthen Lemma 4.10 as follows.

**Proposition 4.11.** Let \( f \in \mathcal{B}_0(U) \), \( t > 0 \), and \( U \subset \mathbb{R}^d \) be a bounded open subset. Then, \( p^{\mu, U}_t f \in C_b(U) \).

**Proof.** Step 1: We denote by \( (\mathcal{L}_U, D(\mathcal{L}_U)) \) the non-positive generator of \( \{p^{\mu, U}_t\} \) on \( L^2(U, \mu) \). By Lemma 4.9, \( -\mathcal{L}_U \) has only discrete spectrum. Let \( \{\lambda_n\}_{n=1}^\infty \subset [0, \infty) \) be the eigenvalues of \( -\mathcal{L}_U \) written in increasing order repeated according to multiplicity, and let \( \{\varphi_n\}_{n=1}^\infty \subset D(\mathcal{L}_U) \) be the corresponding eigenfunctions:

\[
-\mathcal{L}_U \varphi_n = \lambda_n \varphi_n.
\]

Then, \( \varphi_n = e^{\lambda_n t} p^{\mu, U}_t \varphi_n \in L^\infty(\mathbb{R}^d, \mu) \) by Lemma 4.9. Hence, for each \( n \in \mathbb{N} \), there exists a bounded measurable version of \( \varphi_n \) (still denoted as \( \varphi_n \)). By Lemma 4.10, for each \( \gamma > 0 \) and \( n \in \mathbb{N} \), \( R^{\mu, U}_\gamma \varphi_n \) is continuous on \( U \).

Furthermore, we see from [7, Theorem 4.2.3] that

\[
R^{\mu, U}_\gamma \varphi_n = (\gamma - \mathcal{L}_U)^{-1} \varphi_n = (\gamma + \lambda_n)^{-1} \varphi_n \quad \mu\text{-a.e. on } U.
\]

Therefore, there exists a (unique) bounded continuous version of \( \varphi_n \) (still denoted as \( \varphi_n \)). By [4, Theorem 2.1.4], the series

\[
p^{\mu, U}_t(x, y) := \sum_{n=1}^\infty e^{-\lambda_n t} \varphi_n(x) \varphi_n(y)
\]

absolutely converges uniformly on \( [\epsilon, \infty) \times U \times U \) for any \( \epsilon > 0 \). Since \( \{\varphi_n\}_{n=1}^\infty \) are bounded continuous on \( U \), \( p^{\mu, U}_t(x, y) \) is also continuous on \( (0, \infty) \times U \times U \) and defines an integral kernel of \( \{p^{\mu, U}_t\}_{t>0} \). Namely, for each \( t > 0 \) and \( f \in L^2(U, \mu) \),

\[
p^{\mu, U}_t f(x) = \int_U p^{\mu, U}_t(x, y) f(y) \, d\mu(y) \quad \text{for } \mu\text{-a.e. } x \in U.
\]
The uniform convergence of the series (4.4) imply the boundedness of \( p_t^{\mu,U}(x, y) \) on \( [\varepsilon, \infty) \times U \times U \) for each \( \varepsilon > 0 \). We also note that \( p_t^{\mu,U}(x, y) \geq 0 \) by (4.5) and the fact that \( p_t^{\mu,U} f \geq 0 \) \( \mu \)-a.e. for any \( f \in L^2(U, \mu) \) with \( f \geq 0 \).

Step 2: In this step, we show that for each \( x \in U, \gamma > 0, \) and \( f \in \mathcal{B}_b(\mathbb{R}^d), \)

\[
\int_0^\infty e^{-\gamma t} E_x[f(X_t^{\mu,U})] \, dt = \int_0^\infty e^{-\gamma t} \left( \int_U p_t^{\mu,U}(x, y) f(y) \, d\mu(y) \right) \, dt.
\]

By the absolute continuity of \( R_\gamma^{\mu,U} \) (Lemma 4.10), for any \( \varepsilon > 0, \)

\[
\int_\varepsilon^\infty e^{-\gamma t} E_x[f(X_t^{\mu,U})] \, dt = e^{-\gamma \varepsilon} \int_\varepsilon^\infty R_\gamma^{\mu,U}(p_\varepsilon^{\mu,U} f)(x) \, dt
\]

\[
= e^{-\gamma \varepsilon} \int_\varepsilon^\infty \left( \sum_{n=1}^{\infty} e^{-\lambda_n \varepsilon} \left( \int_U \varphi_n(y) f(y) \, d\mu(y) \right) \varphi_n(x) \right) \, dt
\]

\[
= \sum_{n=1}^{\infty} \int_\varepsilon^\infty \int_U e^{-\lambda_n \varepsilon} \varphi_n(y) \varphi_n(x) f(y) \, d\mu(y) \, e^{-\gamma \varepsilon} \, dt
\]

\[
= \sum_{n=1}^{\infty} \int_U p_\varepsilon^{\mu,U}(x, y) f(y) \, d\mu(y) \, e^{-\gamma \varepsilon} \, dt.
\]

Here, we used the identity (4.3) and the uniform convergence of the series (4.4). Set

\[ a_\varepsilon = e^{-(\gamma + \lambda_n) \varepsilon} (\gamma + \lambda_n)^{-1} = \int_\varepsilon^\infty e^{-(\gamma + \lambda_n) t} \, dt. \]

Since the series (4.4) uniformly converges on \( [\varepsilon, \infty) \times U \times U \) for each \( \varepsilon > 0, \)

\[
\int_\varepsilon^\infty e^{-\gamma t} E_x[f(X_t^{\mu,U})] \, dt = \sum_{n=1}^{\infty} a_\varepsilon \left( \int_U \varphi_n(y) f(y) \, d\mu(y) \right) \varphi_n(x) \]

\[
= \sum_{n=1}^{\infty} \int_\varepsilon^\infty \int_U e^{-\lambda_n \varepsilon} \varphi_n(y) \varphi_n(x) f(y) \, d\mu(y) \, e^{-\gamma \varepsilon} \, dt
\]

\[
= \sum_{n=1}^{\infty} \int_U p_\varepsilon^{\mu,U}(x, y) f(y) \, d\mu(y) \, e^{-\gamma \varepsilon} \, dt.
\]

By letting \( \varepsilon \to 0 \) in (4.7), we obtain (4.6).

Step 3: By (4.6) and the uniqueness of Laplace transforms, it holds that

\[
E_x[f(X_t^{\mu,U})] = \int_U p_t^{\mu,U}(x, y) f(y) \, d\mu(y) \quad \text{dt-a.e. } t \in (0, \infty)
\]

for any \( x \in E \) and \( f \in \mathcal{B}_b(\mathbb{R}^d) \). If \( f \) is bounded continuous on \( U \), by the continuity of \( X_t^\mu \) and \( p_t^{\mu,U}(x, y) \), (4.8) holds for any \( t \in (0, \infty) \). By using a monotone class argument, we have

\[
E_x[f(X_t^{\mu,U})] = \int_U p_t^{\mu,U}(x, y) f(y) \, d\mu(y)
\]

for any \( x \in E \) and \( f \in \mathcal{B}_b(\mathbb{R}^d) \), and \( t > 0 \). By Step 1, for each \( t > 0 \), \( p_t^{\mu,U}(x, y) \) is bounded continuous on \( U \times U \). Since \( \mu(U) < \infty \), the proof is complete by dominated convergence theorem.

\[ \square \]

**Corollary 4.12.** For any \( f \in \mathcal{B}_b(\mathbb{R}^d) \) and \( t > 0 \), \( p_t^\mu f \in C_b(\mathbb{R}^d) \).
Proof. Let $K$ be a compact subset of $\mathbb{R}^d$. For any bounded open subset $U \subset \mathbb{R}^d$ with $K \subset U$, 

$$\sup_{x \in K} |p_t^U f(x) - p_t f(x)| \leq \|f\|_{L^\infty(E,\mu)} \times \sup_{x \in K} P_x [t \geq TU].$$

By Proposition 4.11, $p_t^U f$ is continuous on $K$. By Lemma 3.1 and Dini’s theorem, 

$$\lim_{U \nearrow \mathbb{R}^d} \sup_{x \in K} P_x [t \geq TU] = 0,$$

which complete the proof. \qed

Proof of Theorem 4.8. By Lemma 4.9 and Corollary 4.12, the conditions I and III are satisfied. We shall prove the condition II. Let $\gamma_1, \gamma_2 > 0$ such that $\gamma_1 < d$ and $\gamma_1 + \gamma_2 > d$. Setting 

$$J_{\gamma_1,\gamma_2}(x) = \int_{\mathbb{R}^d} \frac{dy}{|x - y|^\gamma_1 (1 + |y|^\gamma_2)} \quad x \in \mathbb{R}^d,$$

$J_{\gamma_1,\gamma_2}$ is bounded on $\mathbb{R}^d$ and there exist positive constants $c_1, c_2, c_3$ such that 

$$(4.9) \quad J_{\gamma_1,\gamma_2}(x) \leq \begin{cases} 
  c_1 |x|^{d-(\gamma_1+\gamma_2)}, & \text{if } \gamma_2 < d, \\
  c_2 (1 + |x|)^{-\gamma_1} \log |x| & \text{if } \gamma_2 = d, \\
  c_3 (1 + |x|)^{-\gamma_1} & \text{if } \gamma_2 > d
\end{cases}$$

for any $x \in \mathbb{R}^d$. See [10, Lemma 6.1] for the bounds (4.9).

We denote by $G(x,y)$ the Green function of $X$. It is known that 

$$G(x, y) = c(d, \alpha)|x - y|^\alpha - d.$$

Here $c(d, \alpha) = 2^{1-\alpha} \pi^{-d/2} \Gamma((d - \alpha)/2) \Gamma(\alpha/2)^{-1}$ and $\Gamma$ is the gamma function: 

$$\Gamma(s) = \int_0^\infty x^{s-1} \exp(-x) \, dx.$$ 

Recall that $\beta > \alpha$. Since 

$\begin{align*}
  R_0^\mu 1_{\mathbb{R}^d}(x) &= \int_{\mathbb{R}^d} G(x, y) \, d\mu(y) \\
  &\leq c(d, \alpha) \int_{\mathbb{R}^d} \frac{dy}{|x - y|^{d-\alpha} W(y)} \\
  &\leq c(d, \alpha) \int_{\mathbb{R}^d} \frac{dy}{|x - y|^{d-\alpha} (1 + |y|^\beta)} \\
  &= c(d, \alpha) J_{d-\alpha, \beta}(x),
\end{align*}$

$R_0^\mu 1_{\mathbb{R}^d}$ is bounded on $\mathbb{R}^d$ and $\lim_{x \in \mathbb{R}^d, |x| \to \infty} R_0^\mu 1_{\mathbb{R}^d}(x) = 0$. \qed

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