Kondo-based Qubits for Topological Quantum Computation

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We propose to use residual parafermions of the overscreened Kondo effect for topological quantum computation. A superconducting proximity gap of $\Delta < T_K$ can be utilized to isolate the parafermion from the continuum of excitations and stabilize the non-trivial fixed point. We use weak-coupling renormalization group, dynamical large-N technique and bosonization to show that the residual entropy of multichannel Kondo impurities survives in a superconductor. We find that while (in agreement with recent numerical studies) the non-trivial fixed point is unstable against intra-channel pairing, it is robust in presence of a finite inter-channel pairing. Based on this observation, we suggest a superconducting charge Kondo setup for isolating and detecting the Majorana fermion in the two-channel Kondo system.

**Introduction** - The quest for realizing non-Abelian anyons, like Majorana zero modes (MZMs) and parafermions, has propelled an extensive research due to their application in topological quantum computation [1, 2]. After a decade of active research, an unequivocal demonstration of MZMs is yet to be seen. Moreover, the currently pursued Ising anyons are insufficient for an all-topological quantum computation, which requires Fibonacci anyons. The main option for the latter is the edge state of fractional quantum Hall states in proximity to a superconductor (SC) [3, 4]. Even so, an elaborate technique is required to isolate the Fibonacci sector [5]. Here, we propose an alternative route of using the fractionalization inherent to the Kondo effect, to realize MZMs and Fibonacci anyons.

The Kondo effect arises when the electrons in a metal screen a magnetic impurity spin (Fig. 1a), so that the spin effectively disappears at low temperatures [6]. When various channels compete in screening a magnetic impurity, the spin is overscreened and this typically leads to a fractionalization of the spin and a residual degree of freedom at low temperatures [7, 8]. In the simplest case of two-channel Kondo (2CK) model, the infrared (IR) fixed point (FP) contains a decoupled MZM with ground state degeneracy of $g_{2CK} = \sqrt{2}$, similar to the edge mode of the 1D Kitaev model. The three channel Kondo (3CK) has $g_{3CK} = (1 + \sqrt{5})/2$, corresponding to a Fibonacci anyon. Can these anyons, e.g. the 2CK MZM, be utilized for quantum computation?

Nowadays, multichannel Kondo systems are not as out-of-reach as before. Experiments on semiconducting quantum dots [9, 10] and charge Kondo effect [11, 12] in quantum Hall regime [13, 14] have demonstrated 2CK and 3CK physics. There are also proposals based on Majorana boxes [15], Floquet-driven Anderson impurity [16], or magnetically frustrated Kondo systems [17, 18]. Even though a MZM can be achieved using simpler non-interacting setups, a 2CK realization paves the way for producing Fibonacci anyons using 3CK model. However to this end, there are various conceptual challenges:

i) It is unclear if the residual MZM in the 2CK model is localized at the position of the impurity spin, or delocalized throughout the system.

ii) The gapless spectrum of the conduction band prohibits singling out the topological sector and braiding.

iii) The coupling between two spin-impurities mediated by the conduction band destabilize individual 2CK FPs, driving them to a Fermi liquid at the IR.

A possible solution to these problems is to gap out the conduction band at low-energies by a superconductor with a gap $\Delta < T_K$. The low-energy effective theory of such a system is only in terms of fractionalized degrees of freedom $\gamma_j$ localized at the position of spin impurities, suitable for braiding. For example in the 2CK case we have $H_{eff} = i \sum_{m<n} M_{mn} \gamma_m \gamma_n + O(\gamma^4)$. The spin impurities do not need to move in real space and a braiding in configuration space is sufficient for computation. This is feasible, e.g. by moving a potential scatterings at $\vec{x}_j$ induced by an scanning tip to control $M_{mn}(\{\vec{x}_j\})$.

For simplicity and practicality we consider spin-singlet s-wave pairing. A 2D problem can be mapped to a semi-infinite 1D wire with the Kondo impurity at the
Pauli matrices. We consider a singlet proximity pairing $K$ with $\tau|$, Here, $a$ the MZM. inter-channel pairing, and we propose a setup to isolate gain insight about the more relevant SU(2) spin, we use gap in the spectrum, suggestive of topological order. To trivial ground state degeneracy can coexist with a finite problem here. After a brief discussion of weak-coupling regime, we use dynamical large-N technique to show 2CK FP is unstable against pairing [27].

Motivated by the potential application, we revisit the Model - The model consists of $K$ channels of non-interacting spinful electrons, proximity paired to a SC and Kondo coupled to an impurity spin. The Hamiltonian is $H = H_0 + H_\Delta + H_K$, where

$$H_0 = \sum_{k,a,\alpha} \epsilon_k c_{k\alpha a}^\dagger c_{k\alpha a}, \quad H_K = J_K \sum_{a,\alpha,\beta} S^z_{k\alpha a} \sigma_{\alpha\beta} c_{k\alpha a} c_{k\beta a}$$

Here, $a = 1...K$ is the channel index. In the SU(2) case $\alpha,\beta = \uparrow, \downarrow$ and $S^z$ is a $S = 1/2$ spin operator and $\sigma$ are the Pauli matrices. We consider a singlet proximity pairing

$$H_\Delta = \Delta \sum_a [c_{k,\alpha,\uparrow}^\dagger c_{-k,\alpha,\downarrow}^\dagger + \text{h.c.}],$$

which for a wide conduction band of electrons, $|\epsilon_k| < D_0$, results in the local Green’s function

$$g_c(z) = -2\rho \frac{\pi \rho' + \Delta \phi'}{\sqrt{\Delta^2 - \Delta'^2}} \log \frac{D_0 - \sqrt{z^2 - \Delta^2} - \Delta'}{D_0 - \sqrt{z^2 - \Delta^2}}$$

with $\rho' = x_{0/z}$ being Pauli matrices in the Nambu space.

Weak-coupling - At weak-coupling (small $J_K/D_0$) for any $K$, the Kondo coupling evolves as $dJ_K/d\ell = \rho(D)F_K^2$, where $d\ell = -dD/D$ and $\rho(e) = g_{c,e}^\dagger(e - i0+)$. This density of states is shown in the inset of Fig. 2a. As the cutoff $D$ is reduced $J_K(D)$ increases. $T_K(\Delta)$ is defined as the $D$ at which $J_K$ diverges. If this happens before $D$ is reduced to $\Delta$, the Kondo coupling has already renormalized to strong coupling and the moment is screened. Otherwise, the ground state is an unscreened local moment, separated from the screened phase with a first-order transition. $T_K(\Delta)$ as function of $T_K = D_0 e^{-2D_0/J_K}$ is shown in Fig. 1b. The two phases are separated by $T_K^0/\Delta = 1/2$. To study the IR FP, we next turn to the large-N limit.

Dynamical large-N - We can use Schwinger bosons [28] to represent SU(N) spins as $S_{\alpha\beta} = b_{\alpha}^\dagger b_{\beta}$ with $\alpha, \beta = 1...N$. The bosons have larger Hilbert space than the original spin and a constraint $b_{\alpha}^\dagger b_{\alpha} = 2S$ has to imposed to stay within the physical subspace. Plugging this into $H_K$ and using a Hubbard-Stratonovich transformation, $H_K$ reduces to

$$H_K = \sum_a \frac{\chi_a^\dagger \chi_a}{J_K} + \sum_{aa'} \left[ \chi_a^\dagger b_{a} c_{a'}^\dagger + \text{h.c.} \right],$$

where Grassmannian “holons” $\chi$ are introduced. At $N, K, 2S \to \infty$ but finite ratios, the Green’s function and self-energies of bosons and holons obey simple forms:

$$\Sigma_B(\tau) = -\frac{K}{N} g_c(\tau) G_\chi(\tau), \quad \Sigma_\chi(\tau) = g_c(-\tau) G_B(\tau),$$

which can be solved self-consistently together with the Dyson equations $G_{\chi}^{-1}(z) = z - \lambda - \Sigma_\chi(z)$ and $G_B^{-1}(z) = -J_K - \Sigma_B(z)$. See [28] [31] for details.

We first study the fully screened case $K = 2S$. The phase shift $\Delta \delta_c$, the residual entropy $S$, and the effective moment $T_\chi$ are shown in Fig. 2a as a function of $T/T_K$. The local moment phase at $T_K^0/\Delta < 1$ (red) and the screened phase at $T_K^0/\Delta > 1$ (blue) are clearly visible. Fig. 2b shows the same information in the overscreened case. While for $T_K^0/\Delta > 1$ (blue), the moment disappears at low temperature, the residual entropy (predicted for the gapless system [8] [10]) survives. In order to better
understand this for the case of SU(2) spins, we will use field-theory techniques.

_Bosonization_ - The Hamiltonian $H$ in 2D can be reduced to a sum of 1D wires (in radial direction, see appendix) terminated at the position of the impurity. We first linearize the spectrum using $\epsilon_{aa} = e^{i k_F x} \psi_{Laa} + e^{-i k_F x} \psi_{Raa}$ and use the recipe $\psi_{L/R,aa} = \Gamma_a \exp(i \sqrt{\pi}(\theta_{aa} \pm \theta_{aR}))$ to express the left/right-moving fermions in terms of chiral right-movers $\theta$. The open boundary condition at ultraviolet corresponds to $\theta_{aa}(0) = 0$.

1CK - In the single-channel case ($K = 1$) we define charge/spin bosons according to $\phi_{c/s} = (\phi_1 \pm \phi_2)/\sqrt{2}$ (and same for $\theta_{c/s}$) in terms of which

$$ H_0 = \frac{v_F}{2} \sum_{\mu = c/s} \int_0^\infty dx [ (\partial_x \phi_\mu)^2 + (\partial_x \theta_\mu)^2 ], \quad (6) $$

and the proximity pairing becomes

$$ H_\Delta = \frac{2 \Delta}{\pi} \int_0^\infty dx \cos \sqrt{2 \pi} \phi_s(x) \cos \sqrt{2 \pi} \theta_s(x). \quad (7) $$

This tends to pin $\phi_c = 0, \theta_s = \sqrt{\pi/2}$ or vice versa at low energies. An anisotropic Kondo interaction bosonizes to

$$ H_K = \frac{J_K}{2} [ S^z e^{-i \sqrt{2 \pi} \phi_s(0)} + \text{h.c.}] - \frac{J_z}{\sqrt{2 \pi}} S^z \partial_x \theta_s(0). \quad (8) $$

We use the unitary transformation $U = \exp[i \mu S^z \phi_s(0)]$ in order to tune the system $H \to U^\dagger H U$ to the Toulouse II or _decoupling point_. This is best achieved by unfolding the system in terms of chiral right-movers $\phi, \theta(x) = [\varphi(-x) \pm \varphi(x)]/\sqrt{2}$. For any $K$, the decoupling point is defined as the choice of $\mu$ which maximizes the scaling dimension for the transverse Kondo coupling $J_K^{1/2}$. For the single channel case, the bosonic factor $e^{i \sqrt{2 \pi} \phi_s}$ can be eliminated by $\mu = \sqrt{2 \pi}$. Moreover, $\phi(x) = \phi(x)$ but

$$ \hat{\theta}(x) = \theta(x) + \mu \sqrt{2} S^z \text{sign}(x), \quad (9) $$

and the Hamiltonian becomes

$$ U^\dagger (H_K + H_0) U = H_0 + J_K^{1/2} S^z - \frac{J_z}{\sqrt{2 \pi}} \partial_x \theta_s(0) S^z. \quad (10) $$

At the decoupling point, $J_K^{1/2}$ is highly relevant whereas $J_K^{1/2} = J_K - 2 \mu \sqrt{2 \pi} v_F$ term is (marginally) irrelevant. Via this term the conduction electrons ‘observe’ the $S^z$ state of the spin and entangle-to (decohere) it. The strong coupling FP corresponds to $J_K \to \infty$ and $J_K \to 0$, i.e. the dressed spin polarized along the $x$-direction.

Now, we turn to coexisting Kondo and pairing terms. Due to Eq. (6) the pairing term is modified by the unitary transformation $U^\dagger H_\Delta U$, and the pinning of $\theta_s(x)$ depends on the state of $S$. An infinitesimal $J_K^{1/2} S^z$ cannot flip the spin. However, a large $J_K^{1/2}$ can ‘melt’ the bosonic solid in an area $\xi_K$. Near the impurity, $J_K$ is large and the spin precession is negligible. But at long distances $J_K^{1/2} \to 0$ and the spin-precession is set by $J_z$ which grows at low energies, i.e. $S^z(\tau) = e^{i J_z S^z \tau} e^{-i J_z S^z \tau}$ is fluctuating in time, with a rate given by $J_z$. For $\theta_s(x)$ to follow this evolution, it costs an energy $\frac{\pi}{2 \sqrt{2}} |\theta_s(0)|^2 = \frac{\pi}{2 \sqrt{2}} |\theta_s|^2 \propto |\partial_x S^z(\tau)|^2$ per unit length. Hence beyond certain distance, $\theta_s(x)$ can no longer follow the spin and it is pinned to $\langle S^z \rangle$. This defines a characteristic distance $\xi_K \sim v_F/T_K(\Delta)$.

2CK - In the two-channel case ($K = 2$), we follow and define collective bosons according to

$$ \left( \begin{array}{cc} \phi_c \\ \phi_s \\ \phi_f \\ \phi_{sf} \end{array} \right) = \frac{1}{2} \left( \begin{array}{cccc} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \end{array} \right) \left( \begin{array}{c} \phi^\dagger \\ \phi_{1+} \\ \phi_{1-} \\ \phi_{2+} \end{array} \right), \quad (11) $$

and similar relation for $\theta_{1,2}$ with $\mu = c,s,f,sf$. Again $U = \exp[i \mu S^2 \phi_s(0)]$ can be used with $\mu = \sqrt{\pi}$ to tune the system to the decoupling point:

$$ U^\dagger H_K U = 2J_K^{1/2} S^z \cos \sqrt{\pi} \phi_{sf}(0) - \frac{J_z}{\sqrt{\pi}} S^z \partial_x \theta_s(0). \quad (12) $$

The ground state of $J_K^{1/2}$ term is a Schrödinger’s cat state between the spin and the pinning of the boson $\phi_{sf}(0) = 0, \sqrt{\pi}$. This means the difference of the spin in the two channels $\theta_s(0)$, which is the only non-conserved charge, strongly fluctuates. Representing the spin with three Majorana fermions $\gamma, \gamma'$ and $\Gamma$ so that $S^z = i \gamma \gamma'$ and $S^z = i \gamma \Gamma$ and refermionizing the dim-1/2, $\cos \sqrt{\pi} \phi_{sf}$ as $\eta_{sf}(x) \equiv \Gamma \cos \sqrt{\pi} \phi_{sf}(x)$ the Hamiltonian becomes a resonant Andreev scattering

$$ U^\dagger H_K U \to 2iJ_K^{1/2} \gamma \eta_{sf}(0) - \frac{J_z}{\sqrt{\pi}} i \gamma \gamma' \partial_x \theta_s(0). \quad (13) $$

The first term hybridizes $\gamma'$ with the conduction electrons $\eta_{sf}(0)$, and provides it with the scaling dimension $1/2$. Near this IR FP, the originally marginal interaction $J_K^{1/2}$, coupling $\gamma$ and $\gamma'$, becomes a dim-3/2 irrelevant interaction (the leading $\xi^2$). At the IR FP, $\gamma$ is entirely decoupled consistent with the $S = 1/2$ residual entropy, realizing a MZM.

Can this MZM survive in a SC? The simplest form of SC is an induced intra-channel pairing, expressed as

$$ H_{\Delta}^{\text{intra}} = \frac{2 \Delta}{\pi} \int_0^\infty dx [ \cos \sqrt{2 \pi} \phi_{c1} \cos \sqrt{2 \pi} \theta_{s1} + \cos \sqrt{2 \pi} \phi_{c2} \cos \sqrt{2 \pi} \theta_{s2} ]. \quad (14) $$

$U^\dagger H_{\Delta}^{\text{intra}} U$ has the effect of $\theta_s \to \hat{\theta}_s$ as in Eq. (9). Far from the impurity the two lines can be minimized independently. Since both $\theta_{s1}$ and $\theta_{s2}$ are pinned, $\theta_{s1} = \theta_{s2}$. Therefore, $1CK$ and $2CK$ apply separately.
leads to purely inter-channel pairing. This means that $\phi_{sf}$ is strongly fluctuating and the term $J_{2s}^S S^z \cos \sqrt{\pi} \phi_{sf}(0)$ becomes highly irrelevant. Thus inclusion of a small $\Delta$ destabilizes the 2CK FP, in agreement with NRG results [27]. On the other hand, a singlet/triplet inter-channel pairing has the form

$$H^\text{inter}_\Delta = \Delta \int_0^\infty dx[(c_{1\uparrow}^\dagger c_{2\downarrow}^\dagger + c_{2\uparrow}^\dagger c_{1\downarrow}^\dagger) + h.c.]$$

$$\sim \frac{2\Delta}{\pi} \int_0^\infty dx[\cos \sqrt{\pi}(\phi_c + \phi_{sf}) \cos \sqrt{\pi}(\theta_c + \theta_f)]$$

Note that there is no $\theta_{sf}$ here. Again $U^\dagger H^\text{inter}_\Delta U$ has the effect of $\theta_c \to \theta_s$ as in Eq. (9). This interaction tends to pin $\phi_{sf}$ and is expected to be benign to the 2CK FP. This can be seen in a two-site problem, where the addition/removal of a pair of inter-channel electrons maps the strong coupling to weak-coupling or vice versa without affecting the channel isotropy. The possibility of coexistence of interchannel pairing and 2CK FP seen here, can be verified in future NRG studies.

**Multiple impurities** - The case of two 2CK impurities coupled to the same bath is discussed before [38]. The double 2CK FP is transformed to a line of FPs by the RKKY interaction $J_H \vec{S}_1 \cdot \vec{S}_2$ which becomes a marginal interaction $iJ_H' \gamma_1 \gamma_2 \partial_x \Phi(0)$ at the non-trivial FP. Here, $J_H'$ is the renormalized RKKY interaction at the IR FP and $\Phi$ is a linear combination of the spin bosons of each impurity [38]; the two decoupled Majoranas form a non-local charge qubit whose state is dynamically measured (/decohered) by the gapless $\partial_x \Phi$ mode. Presence of a gap in the spectrum has the additional feature of suppressing such decoherence effects and reducing it to a static $iM_{12} \gamma_1 \gamma_2$ interaction discussed before.

**Experimental realization** - Based on above discussion, we propose a modified version of the charge Kondo setup [18, 19] at zero magnetic field to isolate the MZM in the 2CK case. In the simplest charge Kondo effect, a spinless single electron transistor (SET) with large charging energy is coupled to a spinless conduction bath. The SET is tuned to a charge degeneracy point $\Delta Q = 0$ so that the charge parity plays the role of the pseudospin. The location of the electron, either in SET or the conduction bath, plays the role of conduction electron pseudo-spin [15]. The spinful case provides the simplest realization of a two-channel charge Kondo effect. Incidentally, this is ideally suitable for combining with previous discussion. A proximity pairing of the SET (or SET made of SC) and conduction bath with singlet SC leads to purely inter-channel pairing. Such Coulomb}

1 Since the role of the spin and channel are reversed, a singlet pairing in the lead and the SET correspond to $T_\perp$ inter-channel blockaded superconducting islands are common in topological Kondo effect [20, 21]. However, the MZM here is produced by the 2CK, rather than by the band topology.

Realization of the charge Kondo effect requires $\delta E \ll k_B T \ll T_K \ll E_C$ where $\delta E$ is the mean-level spacing, $T_K \sim E_C e^{-\pi^2/2T}$ is the Kondo temperature expressed in terms of transmission $T$, and $E_C = e^2/C$ is the charging energy. This condition can be met in small metallic grains, e.g. the hybrid metal-semiconductor setup of Iftikhar et al. [18, 19]. Alternatively, a purely proximity-induced superconductivity in semiconductor heterostructures with large carrier mass can be used. Since carrier mobility is unimportant, one possible option is a dot with a large charging energy defined using in-plane gates in a shallow 2D valence band hole gas [30, 41].

**Detection** - The presence of the MZM has to be inferred indirectly; we consider an additional normal lead (e.g. a scanning tunnelling microscope) weakly coupled to the SET to measure the conductance between the two leads. At the 2CK FP, the coupling of the probe channel is irrelevant [16, 17] and a conductance of $G \propto T$ is expected on resonance at $T > \Delta$ [16, 17]. For $T < \Delta$, the capacitive coupling to the SET, with another scanning SET seem to be the ideal probe. Alternatively, entropy measurements along [12] can be envisioned.

**Conclusion** - We have proposed to use Kondo-based parafermions in proximity with superconductivity for quantum computation. We found that the residual entropy survives a gap in the spectrum, in particular if the gap is due to an inter-channel proximity pairing. We have suggested a superconducting version of the charge Kondo setup for isolating the MZM in the 2CK model.

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spin-triplet pairing in the original basis, whereas Eq. (10) showed that a $T_0$ triplet pairing is benign to 2CK physics. Considering $c_{1\uparrow}^\dagger c_{2\downarrow}^\dagger - c_{1\downarrow}^\dagger c_{2\uparrow}^\dagger = c_{1\uparrow}^\dagger c_{2\downarrow}^\dagger - c_{1\downarrow}^\dagger c_{2\uparrow}^\dagger$, we expect the latter to hold for all inter-channel triplet pairings albeit with a π phase difference, although this cannot be seen in abelian bosonization.
Appendix: Dimensional reduction

For completeness, here we show how a problem of $M$ impurities in two dimension can be reduced to a sum of 1D metallic systems terminated at the spin impurities, appropriate for field theory analysis. This is a generalization of $M = 2$ from \cite{22}. We start from a two-dimensional system described by $H = H_0 + H_K + H_\Delta$ defined in Eqs. 1 and 2 and generalize it to the case that there are $M$ Kondo impurities located at positions $\vec{d}_n$ coupled to the conduction band.

![Diagram](image)

FIG. 3: (a) The problem of three (multichannel) Kondo impurities in 2D can be mapped to (b) the problem of three Kondo impurities at the end of semi-infinite wires. This is so even in presence of proximity pairing of the host metal. Also various potential scatterings (e.g. a scanning gate microscope) only modify the mutual Kondo coupling and RKKY interactions among the impurities.

Kondo coupling

We consider $M$ impurities located on a 2D plane at positions $\vec{d}_n$. We have

$$H_K = \int d^2 k d^2 k' \psi_{n}^{\dagger} \vec{\sigma} \cdot \vec{d} \psi_{n} = \sum_{n=1}^{M} J_n \vec{S}_n \cdot \vec{d}_n. \quad (16)$$

We start by defining the fermions

$$\psi_{nE} = \int \frac{dk}{(2\pi)^2} \epsilon_k e^{i\vec{k}\cdot\vec{d}_n} \psi_{n}^{\dagger} \psi_{n}^{\dagger} \cdot \vec{d}_n. \quad (17)$$

in terms of which the Kondo Hamiltonian is

$$H_K = \int dE dE' \sum_{n=1}^{M} J_n \psi_{nE}^{\dagger} \vec{\sigma} \cdot \vec{d}_n \psi_{nE'}. \quad (18)$$

The problem is that $\psi_{nE}$ do not obey standard anticommutation relations. Rather

$$\{ \psi_{nE}, \psi_{mE'}^{\dagger} \} = g_{nm}(E) \delta(E - E'), \quad (19)$$

where

$$g_{nm}(E) = \int \frac{d^2 k}{(2\pi)^2} \delta(\epsilon_k - E) e^{i\vec{k}\cdot\vec{d}_n} \delta(\vec{d}_n - \vec{d}_m). \quad (20)$$

Representing the vector $\vec{d}_{nm} = \vec{d}_n - \vec{d}_m$ in polar coordinates by $\vec{d}_{nm} = \vec{d}_n (\cos \phi_{nm}, \sin \phi_{nm})$ we can write

$$g_{nm}(E) = \frac{k_E}{2\pi \partial E} \int_0^{2\pi} d\phi e^{i\vec{k}\cdot\vec{d}_n \cos(\phi - \phi_{nm})} \quad (21)$$

$$= \frac{k_E}{2\pi \partial E} \sum_p e^{-i\phi_{nm}} J_p(k_E d_{nm}) \int_0^{2\pi} d\phi e^{i\phi} \quad (22)$$

$$= g_{E} J_0(k_E d_{nm}), \quad (23)$$

where

$$g_E = \frac{k_E}{2\pi \partial E}. \quad (24)$$

Remarkably, $g_{nm}$ depends only on mutual distances of the impurities, measured by the corresponding wavelength $k_E d_{nm}$. The matrix $g_{nm}$ is real and symmetric and it has real eigenvalues $\lambda_n$ and can be diagonalized by orthogonal eigenvectors $\vec{u}_n$ where $g_{\vec{u}_n} = \lambda_n \vec{u}_n$. So we can write $g_{nm} = \sum_{p} u_{np} \lambda_p u^{*}_{mp}$ and the orthogonality is $\sum_{p} u_{np} u^{*}_{mp} = \delta_{nm}$. Defining new operators with

$$\tilde{\psi}_{nE} = \sum_{j} u_{nj}(E) \sqrt{\lambda_j(E)} \tilde{\psi}_{jE}, \quad (25)$$

we find that they are orthonormal

$$\{ \tilde{\psi}_{nE}, \tilde{\psi}_{jE'} \} = \delta_{ij} \delta(E - E'), \quad (26)$$

and the Kondo Hamiltonian becomes

$$H_K = \sum_{n} \tilde{\psi}_{nE}^{\dagger} \vec{\sigma} \cdot \vec{d}_n \tilde{\psi}_{nE} \sqrt{\lambda_n(E)}. \quad (27)$$

Although the $J_{nij}$ couplings are complex in general, the relation $J_{nij}(E, E') = J_{nij}(E', E)$ ensures the hermiticity of the Hamiltonian. Whether or not a spin $n$ couples the channels $i$ and $j$ depend on the product of wavefunctions of $i$ and $j$ at the site $n$, which can be tuned by moving potential scatterings induced by scanning tips. Next, we do a Taylor expansion of $J_{nij}(E, E')$ function around Fermi energy and keep only the leading term. $H_K$ can be written in the matrix form $[J_{n}]_{ij} = J_{nij},$

$$H_K = \sum_{n} \tilde{\psi}_{nE}^{\dagger} [J_{n}]_{ij} \tilde{\psi}_{jE} \sqrt{\lambda_j(E)} \tilde{\psi}_{nE}. \quad (28)$$

where $\Psi$ is a vector in $m = 1...M$ index and the spin is implicit. As we see the impurities talk to all the channels and scatter electron between all the channels. A result of the Taylor expansion of $J_{nij}(E, E')$ is that RKKY interactions will be induced between the spins

$$H_K \rightarrow H_K + \sum_{ij} J_{H}^{ij} \vec{S}_i \cdot \vec{S}_j. \quad (29)$$
Mapping to 1D

The mapping to (unfolded) left-movers follows \cite{22}

\[ \psi_L(x,t) = \frac{1}{\sqrt{v}} \int_{-D}^{D} dE e^{-iE(t+x/v)} \psi_E, \]

(30)

for \( x \in (-\infty, +\infty) \) with commutation relations

\[ \{ \psi_L^\dagger(x), \psi_L(y) \} = 2\pi \delta(x-y). \]

(31)

Alternatively, we can work in the folded space of left and right-movers defined as

\[ \psi_R(x,t) = \psi_L(-x,t) \quad (x > 0). \]

(32)

Kinetic term

As we saw above, the Kondo interaction involves \( M \) 1D conduction bands pulled out of the 2D conduction band. The natural guess for their kinetic energy is

\[ H_0(\vec{d}) \equiv \int dE E^2 \psi_{nE}^\dagger \psi_{nE} \]

(33)

\[ = \int \frac{d^2k d^2k'}{(2\pi)^4} \epsilon_k \delta(\epsilon_k - \epsilon_{k'}) e^{i(\vec{k}-\vec{k}') \cdot \vec{d}} \psi_k^\dagger \psi_{k'}. \]

Summing over all the positions we have

\[ \int d^2r H_0(\vec{r}) = \delta(0) \int \frac{d^2k d^2k'}{(2\pi)^4} \epsilon_k \psi_k^\dagger \psi_{k'}. \]

(34)

which is the total Hamiltonian up to a \( \delta(0) \), i.e. the total volume. Transformation from \( \psi_{nE} \) to \( \psi_{lE} \) is a unitary transformation at each energy and it doesn’t change the kinetic part. Therefore, we have

\[ H_0 \rightarrow M \sum_{n=1}^{M} \int dE \psi_{nE}^\dagger \psi_{lE} \]

(35)

Pairing term

Without loss of generality, we consider a channel-diagonal s-wave singlet pairing

\[ \Delta \]

(36)

Let us look at

\[ H_\Delta(\vec{d}_n) \equiv \Delta \int dE \psi_{nE}^\dagger \psi_{nE} \]

\[ = \Delta \int dE \int \frac{d^2k d^2k'}{(2\pi)^4} \delta(\epsilon_k - E) \delta(\epsilon_{k'} - E) \psi_k^\dagger \psi_{k'} e^{-i(\vec{k} + \vec{k}') \cdot \vec{d}_n}. \]

Summing over all the points gives a \( (2\pi)^2 \delta(\vec{k} + \vec{k}') \) term that gives

\[ \int d^2r H_\Delta(\vec{r}) = \delta(0) \Delta \int \frac{d^2k}{(2\pi)^2} \psi_k^\dagger \psi_{-k}, \]

(37)

which is our initial Hamiltonian up to the system volume. Therefore,

\[ H_\Delta \rightarrow \Delta \sum_{n=1}^{M} \int dE (\psi_{nE}^\dagger \psi_{lE} + h.c.) \]

\[ = \Delta \sum_{ij} \int dE (\psi_{iE}^\dagger \psi_{jE} + h.c.). \]

(38)