Limit Distribution of Eigenvalues for Random Hankel and Toeplitz Band Matrices

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Abstract

Consider real symmetric, complex Hermitian Toeplitz and real symmetric Hankel band matrix models, where the bandwidth $b_N \to \infty$ but $b_N/N \to b$, $b \in [0, 1]$ as $N \to \infty$. We prove that the distributions of eigenvalues converge weakly to universal, symmetric distributions $\gamma_T(b)$ and $\gamma_H(b)$. In the case $b > 0$ or $b = 0$ but with the addition of $b_N \geq C N^{\frac{1}{2}} + \epsilon_0$ for some positive constants $\epsilon_0$ and $C$, we prove almost sure convergence. The even moments of these distributions are the sum of some integrals related to certain pair partitions. In particular, when the bandwidth grows slowly, i.e. $b = 0$, $\gamma_T(0)$ is the standard Gaussian distribution and $\gamma_H(0)$ is the distribution $|x| \exp(-x^2)$. In addition, from the fourth moments we know that the $\gamma_T(b)$’s are different for different $b$’s, the $\gamma_H(b)$’s different for different $b \in [0, \frac{1}{2}]$ and the $\gamma_H(b)$’s different for different $b \in [\frac{1}{2}, 1]$.

1 Introduction

In Random Matrix Theory, the most important information is contained in the eigenvalues of matrices and the most prominent analytical object is the distribution of eigenvalues. That is, for a real symmetric or complex Hermitian $N \times N$ matrix $A$ with eigenvalues $\lambda_1, \cdots, \lambda_N$, the distribution of

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its eigenvalues is the normalized probability measure

$$
\mu_A := \frac{1}{N} \sum_{j=1}^{N} \delta_{\lambda_j}.
$$

(1.1)

If $A$ is a random matrix, then $\mu_A$ is a random measure.

The limit distribution of (1.1) is of much interest in Random Matrix Theory. In [18], Wigner found his famous semicircular law and then in [19] pointed out that the semicircular law was valid for a wide class of real symmetric random matrices. Since then much more has been done on various random matrix models, a standard reference is Metha’s book [16]. Recently in his review paper [1], Bai proposes the study of random matrix models with certain additional linear structure. In particular, the properties of the distributions of eigenvalues for random Hankel and Toeplitz matrices with independent entries are listed among the unsolved random matrix problems posed in [1]. Section 6. Bryc, Dembo and Jiang [7] proved the existence of limit distributions $\gamma_H$ and $\gamma_T$ for real symmetric Hankel and Toeplitz matrices. The moments of $\gamma_H$ and $\gamma_T$ are the sum of volumes of solids, from which we can see that $\gamma_H$ and $\gamma_T$ are symmetric and of unbounded support. At the same time Hammond and Miller [12] also independently proved the existence of limit distribution for symmetric Toeplitz matrices. In the present paper we shall prove the existence of limit distribution of eigenvalues for real symmetric and complex Hermitian Toeplitz band matrices, and also for real symmetric Hankel band matrices. Random band matrices have arisen in connection with the theory of quantum chaos [8,11] and the limit distribution of eigenvalues of band matrices has been studied in [3,17].

Note that Toeplitz matrices emerge in many aspects of mathematics and physics and also in plenty of applications, e.g. [6], especially [2,9] for connections with random matrices. Hankel matrices arise naturally in problems involving power moments, and are closely related to Toeplitz matrices. Explicitly, for a Toeplitz matrix of the form $T_N = (a_{i-j})_{i,j=1}^{N}$ and a Hankel matrix of the form $H_N = (a_{i+j-2})_{i,j=1}^{N}$, if $P_N = (\delta_{i-1,N-j})_{i,j=1}^{N}$ the “backward identity” permutation, then $P_N T_N$ is a Hankel matrix and $P_N H_N$ is a Toeplitz matrix. In this paper we always write a Hankel matrix $H_N = P_N T_N$ where we assume the matrix entries $a_{-N+1}, \cdots, a_0, \cdots, a_{N-1}$ are real-valued, thus $H_N$ is a real symmetric matrix. In addition, if we introduce the Toeplitz or Jordan matrices $B = (\delta_{i+1,j})_{i,j=1}^{N}$ and $F = (\delta_{i,j+1})_{i,j=1}^{N}$, called the “backward shift” and “forward shift” because of their effect on the elements of the standard basis $\{e_1, \cdots, e_N\}$ of $\mathbb{R}^N$, then an $N \times N$ matrix
A can be written in the form

$$A = \sum_{j=0}^{N-1} a_{-j} B^j + \sum_{j=1}^{N-1} a_j F^j$$

(1.2)

if and only if $A$ is a Toeplitz matrix where $a_{-N+1}, \cdots, a_0, \cdots, a_{N-1}$ are complex numbers [13]. It is worth emphasizing that this representation of a Toeplitz matrix is of vital importance and the starting point of our method. The “shift” matrices $B$ and $F$ exactly present the information of the traces.

Consider a Toeplitz band matrix as follows. Given a bandwidth $b_N < N$, let

$$\eta_{ij} = \begin{cases} 1, & |i - j| \leq b_N; \\ 0, & \text{otherwise}. \end{cases}$$

(1.3)

Then a Toeplitz band matrix is

$$T_N = (\eta_{ij} a_{i-j})_{i,j=1}^N.$$

(1.4)

Moreover, the Toeplitz band matrix $T_N$ can be also written in the form

$$T_N = \sum_{j=0}^{b_N} a_{-j} B^j + \sum_{j=1}^{b_N} a_j F^j.$$

(1.5)

Obviously, a Toeplitz matrix can be considered as a band matrix with the bandwidth $b_N = N - 1$. Note that when referring to a Hankel band matrix $H_N$, we always mean $H_N = P_N T_N$ where $T_N$ is a Toeplitz band matrix.

In this paper the three models under consideration are:

(i) Hermitian Toeplitz band matrices. The model consists of $N$-dimensional random Hermitian matrices $T_N = (\eta_{ij} a_{i-j})_{i,j=1}^N$ in Eq. (1.4). We assume that $\Re a_j = \Re a_{-j}$ and $\Im a_j = -\Im a_{-j}$ for $j = 1, 2, \cdots$, and \{a_0\} $\cup \{\Re a_j, \Im a_j\}_{j \in \mathbb{N}}$ is a sequence of independent real random variables such that

$$\mathbb{E}[a_j] = 0, \quad \mathbb{E}[|a_j|^2] = 1 \quad \text{for} \quad j \in \mathbb{Z}$$

(1.6)

and further

$$\sup_{j \in \mathbb{Z}} \mathbb{E}[|a_j|^k] = C_k < \infty \quad \text{for} \quad k \in \mathbb{N}.$$  

(1.7)

Notice that

$$\sup_{j \in \mathbb{Z}} \mathbb{E}[|a_j|^k] < \infty \iff \sup_{j \in \mathbb{Z}} \{\mathbb{E}[|\Re a_j|^k], \mathbb{E}[|\Im a_j|^k]\} < \infty,$$
and the assumption (1.7) shows that when fixing $k$, these moments of the independent random variables whose orders are not larger than $k$ can be controlled by some constant only depending on $k$.

The case of real symmetric Toeplitz band matrices is very similar to the Hermitian case, except that we now consider $N$-dimensional real symmetric matrices $T_N = (\eta_{ij} a_{i-j})_{i,j=1}^N$.

(ii) Symmetric Toeplitz band matrices. We assume that $a_j = a_{-j}$ for $j = 0, 1, \ldots$, and $\{a_j\}_{j=0}^\infty$ is a sequence of independent real random variables such that

$$E[a_j] = 0, \ E[|a_j|^2] = 1 \quad \text{for} \quad j = 0, 1, \ldots, \tag{1.8}$$

and further

$$\sup_{j \geq 0} E[|a_j|^k] = C_k < \infty \quad \text{for} \quad k \in \mathbb{N}. \tag{1.9}$$

(iii) Symmetric Hankel band matrices. Let $T_N = (\eta_{ij} a_{i-j})_{i,j=1}^N$ be a Toeplitz band matrix and $H_N = P_N T_N$ be the corresponding Hankel band matrix. We assume that $\{a_j\}_{j \in \mathbb{Z}}$ is a sequence of independent real random variables such that

$$E[a_j] = 0, \ E[|a_j|^2] = 1 \quad \text{for} \quad j \in \mathbb{Z}, \tag{1.10}$$

and further

$$\sup_{j \in \mathbb{Z}} E[|a_j|^k] = C_k < \infty \quad \text{for} \quad k \in \mathbb{N}. \tag{1.11}$$

We assume that $b_N$ grows either as 1) $b_N/N \to b$ as $N \to \infty$, $b \in (0, 1]$ (proportional growth), or as 2) $b_N \to \infty$ as $N \to \infty$, $b_N = o(N)$ (slow growth). Under these assumptions we will establish the limit distributions $\gamma_T(b)$ for Toeplitz band matrices and $\gamma_H(b)$ for Hankel band matrices. In the case $b > 0$, we prove that the distribution of eigenvalues for Toeplitz (Hankel) band matrices converges almost surely to $\gamma_T(b)$ ($\gamma_H(b)$), the moments of which are the sum of some integrals depending on $b$. For $b = 0$, we obtain that $\gamma_T(0)$ is the standard Gaussian distribution and $\gamma_H(0)$ is the distribution $|x| \exp(-x^2)$. In the case $b = 0$, with the addition of $b_N \geq C N^{\frac{1}{2} + \epsilon_0}$ for some positive constants $\epsilon_0$ and $C$, we also prove almost sure convergence.

The plan of the remaining part of our paper is the following. Sections 2 and 3 are devoted to the statements and proofs of the main theorems for band matrices with proportionally growing and slowly growing bandwidths $b_N$, respectively. In Section 4, the fourth moments of the limit distributions $\gamma_T(b)$ and $\gamma_H(b)$ are calculated, which shows their difference for different $b$'s.
2 Proportional Growth of $b_N$

In order to calculate the moments of the limit distribution, we review some basic combinatorial concepts.

**Definition 2.1.** Let the set $[n] = \{1, 2, \ldots, n\}$.

1. We call $\pi = \{V_1, \ldots, V_r\}$ a partition of $[n]$ if the blocks $V_j (1 \leq j \leq r)$ are pairwise disjoint, non-empty subsets of $[n]$ such that $[n] = V_1 \cup \cdots \cup V_r$. The number of blocks of $\pi$ is denoted by $|\pi|$, and the number of the elements of $V_j$ is denoted by $|V_j|$. 

2. Without loss of generalization, we assume that $V_1, \ldots, V_r$ have been arranged such that $s_1 < s_2 < \cdots < s_r$, where $s_j$ is the smallest number of $V_j$. Therefore we can define the projection $\pi(i) = j$ if $i$ belongs to the block $V_j$; furthermore for two elements $p, q$ of $[n]$ we write $p \sim \pi q$ if $\pi(p) = \pi(q)$. 

3. The set of all partitions of $[n]$ is denoted by $\mathcal{P}(n)$, and the subset consisting of all pair partitions, i.e. all $|V_j| = 2, 1 \leq j \leq r$, is denoted by $\mathcal{P}_2(n)$. The subset of $\mathcal{P}_2(n)$ consisting of such pair partitions that each contains exactly one even number and one odd number is denoted by $\mathcal{P}_1^2(n)$. Note that $\mathcal{P}_2(n)$ is an empty set if $n$ is odd.

We can formulate our results for Hankel and Toeplitz band matrices with the proportional growth of $b_N$ as follows. By convention, $I_B$ represents the characteristic function of a set $B$.

**Theorem 2.2.** Let $T_N$ be either a Hermitian ((1.6)–(1.7)) or real symmetric ((1.8)–(1.9)) Toeplitz random band matrix, where $b_N/N \to b$ as $N \to \infty$, $b \in (0, 1]$. Take the normalization $X_N = T_N/\sqrt{(2 - b)bN}$, then $\mu_{X_N}$ converges almost surely to a symmetric probability distribution $\gamma_T(b)$ which is determined by its even moments

$$m_{2k}(\gamma_T(b)) = \frac{1}{(2 - b)^k} \sum_{\pi \in \mathcal{P}_2(2k)} \int_{[0,1] \times [-1,1]^k} \prod_{j=1}^{2k} I_{[0,1]}(x_0 + b \sum_{i=1}^{j} \epsilon_{\pi}(i) x_{\pi(i)}) \prod_{l=0}^{k} dx_l$$

where $\epsilon_{\pi}(i) = 1$ if $i$ is the smallest number of $\pi^{-1}(\pi(i))$; otherwise, $\epsilon_{\pi}(i) = -1$.

**Theorem 2.3.** Let $H_N$ be a real symmetric ((1.10)–(1.11)) Hankel random band matrix, where $b_N/N \to b$ as $N \to \infty$, $b \in (0, 1]$. Take the normalization $Y_N = H_N/\sqrt{(2 - b)bN}$, then $\mu_{Y_N}$ converges almost surely to a symmetric
probability distribution \( \gamma_H(b) \) which is determined by its even moments

\[
m_{2k}(\gamma_H(b)) = \frac{1}{(2-b)^k} \sum_{\pi \in P_k(2k)} \int_{[0,1] \times [-1,1]^k} \prod_{j=1}^{2k} I_{[0,1]}(x_0 - b \sum_{i=1}^{j} (-1)^i x_{\pi(i)}) \prod_{l=0}^{k} d x_l.
\]

(2.2)

Let us first give two basic lemmas about traces of Toeplitz and Hankel band matrices. Although their proofs are simple, they are very useful in treating random matrix models closely related to Toeplitz matrices.

**Lemma 2.4.** For a Toeplitz band matrix \( T = (\eta_{ij} \ a_{i-j})_{i,j=1}^N \) with the band-width \( b_N \) where \( a_{-N+1}, \ldots, a_{N-1} \) are complex numbers, we have the trace formula

\[
\text{tr}(T^k) = \sum_{i=1}^{N} \sum_{j_1, \ldots, j_k = -b_N}^{b_N} \prod_{l=1}^{k} a_{j_l} \prod_{q=1}^{l} I_{[1, N]}(i + \sum_{q=1}^{l} j_q) \delta_{0, \sum_{q=1}^{l} j_q}, \quad k \in \mathbb{N}.
\]

(2.3)

**Proof.** For the standard basis \( \{e_1, \ldots, e_N\} \) of the Euclidean space \( \mathbb{R}^N \), we have

\[
T e_i = \sum_{j=0}^{b_N} a_{-j} B^j e_i + \sum_{j=1}^{b_N} a_j F^j e_i = \sum_{j=-b_N}^{b_N} a_j I_{[1, N]}(i + j) e_{i+j}.
\]

Repeating \( T \)'s effect on the basis, we have

\[
T^k e_i = \sum_{j_1, \ldots, j_k = -b_N}^{b_N} \prod_{l=1}^{k} a_{j_l} \prod_{q=1}^{l} I_{[1, N]}(i + \sum_{q=1}^{l} j_q) e_{i + \sum_{q=1}^{l} j_q}.
\]

By \( \text{tr}(T^k) = \sum_{i=1}^{k} e_i^t T^k e_i \), we complete the proof.

**Lemma 2.5.** Given a Toeplitz band matrix \( T = (\eta_{ij} a_{i-j})_{i,j=1}^N \) with the bandwidth \( b_N \) where \( a_{-N+1}, \ldots, a_{N-1} \) are real numbers, let \( H = PT \) be the corresponding Hankel band matrix where \( P = (\delta_{i-1,N-j})_{i,j=1}^N \). We have the trace formula
\[
\text{tr}(T^k) = \left\{
\begin{array}{ll}
\sum_{i=1}^{N} \frac{b_N}{j_1, \ldots, j_k = -b_N} \sum_{i=1}^{N} a_{j_1} \prod_{l=1}^{k} I_{[1,N]}(i - \sum_{q=1}^{l} (-1)^{q-j_q}) \delta_{0, \sum_{q=1}^{k} (-1)^{q-j_q}}, & k \text{ even}, \\
\sum_{i=1}^{N} \frac{b_N}{j_1, \ldots, j_k = -b_N} \sum_{i=1}^{N} a_{j_1} \prod_{l=1}^{k} I_{[1,N]}(i - \sum_{q=1}^{l} (-1)^{q-j_q}) \delta_{2i-1-N, \sum_{q=1}^{k} (-1)^{q-j_q}}, & k \text{ odd}.
\end{array}
\right.
\]

(2.4)

Proof. Follow a similar procedure as in the proof of Lemma 2.4.

We are now ready to prove the main results of this section.

Proof of Theorem 2.2. We prove the theorem by the following steps:

Step 1. Calculation of the Moments

Let

\[
m_{k,N} = \mathbb{E}[\int x^k \mu_X(d.x)] = \frac{1}{N} \mathbb{E}[\text{tr}(X_N^k)].
\]

Using Lemma 2.4, we have

\[
m_{k,N} = \frac{N^{-\frac{k}{2}}-1}{(2-b)^{\frac{k}{2}} b^\frac{k}{2}} \sum_{i=1}^{N} \sum_{j_1, \ldots, j_k = -b_N} a_{j_1} \prod_{l=1}^{k} I_{[1,N]}(i + \sum_{q=1}^{l} j_q) \delta_{0, \sum_{q=1}^{k} j_q} \prod_{l=1}^{k} a_{j_l}.
\]

For \(j = (j_1, \ldots, j_k)\), we construct a set of numbers \(S_j = \{|j_1|, \ldots, |j_k|\}\) with multiplicities. Note that the random variables whose subscripts have different absolute values are independent. If \(S_j\) has one number with multiplicity 1, by independence of the random variables, we have \(\mathbb{E}[\prod_{l=1}^{k} a_{j_l}] = 0\). Thus the only contribution to the \(k\)-th moment comes when each of \(S_j\) has multiplicity at least 2, which implies that \(S_j\) has at most \(\left[\frac{k}{2}\right]\) distinct numbers. Once we have specified the distinct numbers of \(S_j\), the subscripts \(j_1, \ldots, j_k\) are determined in at most \(2^k(\left[\frac{k}{2}\right])^k\) ways. By independence and the assumptions (1.7) and (1.9), we find

\[
m_{k,N} = O((b_N)^{-\frac{k}{2}} + |\frac{k}{2}|) = O(N^{-\frac{k}{2}} + |\frac{k}{2}|).
\]

Therefore, for odd \(k\)

\[
\lim_{N \to \infty} m_{k,N} = 0.
\]

It suffices to deal with \(m_{2k,N}\). If each of \(S_j = \{|j_1|, \ldots, |j_{2k}|\}\) has multiplicity at least 2 but one of which at least 3, then \(S_j\) has at most \(k - 1\) distinct numbers. Thus, the contribution of such terms to \(m_{2k,N}\) is
That is, for \( \pi \in \mathcal{P}_2(2k) \), if \( p \sim \pi q \), then it is always the case that \( j_p = j_q \) or \( j_p = -j_q \). Under the condition
\[
2k \sum_{q=1}^{2k} j_q = 0
\]
according to (2.3), considering the main contribution to the trace, we should take \( j_p = -j_q \); otherwise, there exists \( p_0, q_0 \in [2k] \) such that
\[
j_{p_0} = j_{q_0} = 1 - 2k \sum_{q=1}^{2k} j_q.
\]
We can choose other \( k - 1 \) distinct numbers, which determine \( j_{p_0} = j_{q_0} \). This shows that there is a loss of at least one degree of freedom and the contribution of such terms is \( O(N^{-1}) \). Thus \( \gamma_{\pi}(b) \) and \( \gamma_{\bar{\pi}}(b) \) are conjugate. So we can write
\[
m^{2k}_{2k,N} = o(1) + \frac{N-k-1}{(2-b)kbk} \sum_{\pi \in \mathcal{P}_2(2k)} \sum_{j_1, \ldots, j_k = -b_N} b_N \prod_{l=1}^{2k} I_{[1,N]}(i + \sum_{q=1}^{l} \epsilon_\pi(q) j_{\pi(q)}).
\]
(2.5)

Follow a similar argument from [7, 3]: for fixed \( \pi \in \mathcal{P}_2(2k) \),
\[
\frac{1}{N^{k+1}} \sum_{i=1}^{N} \sum_{j_1, \ldots, j_k = -b_N} b_N \prod_{l=1}^{2k} I_{[1,N]}(i + \sum_{q=1}^{l} \epsilon_\pi(q) j_{\pi(q)}),
\]
i.e.
\[
\frac{1}{N^{k+1}} \sum_{i=1}^{N} \sum_{j_1, \ldots, j_k = -b_N} b_N \prod_{l=1}^{2k} I_{[1,N]}(i + \sum_{q=1}^{l} \epsilon_\pi(q) j_{\pi(q)} N),
\]
can be considered as a Riemann sum of the definite integral
\[
\int_{[0,1] \times [-b,b]^k} I_{[0,1]}(x_0 + \sum_{q=1}^{l} \epsilon_\pi(q)x_{\pi(q)}) \prod_{i=0}^{k} dx_i.
\]
Given \( \pi \in \mathcal{P}_2(2k) \), as \( N \to \infty \), each term in (2.5) can be treated as an integral. Thus we have \( m^{2k}_{2k}(\gamma_{\pi}(b)) = \lim_{N \to \infty} m^{2k}_{2k,N} \) is the representation of the form (2.1).

**Step 2. Carleman’s Condition**
Obviously, from (2.1) we have
\[ m_2k(\gamma_{T}(b)) \leq \frac{1}{(2-b)^k} \sum_{i \in \mathcal{P}_2(2k)} \int_{[0,1] \times [-1,1]^k} 1 \prod_{l=0}^{k} dx_l = \frac{2^k}{(2-b)^k(2k-1)!!}. \]

By Carleman’s theorem (see [10]), the limit distribution \( \gamma_{T}(b) \) is uniquely determined by the moments.

**Step 3. Almost Sure Convergence**

It suffices to show that
\[
\sum_{N=1}^{\infty} \frac{1}{N^4} \mathbb{E}[(\text{tr}(X_N^k) - \mathbb{E}[\text{tr}(X_N^k)])^4] < \infty \quad (2.6)
\]
for each fixed \( k \). Indeed, by Lemma 2.4, we have
\[
\text{tr}(X_N^k) = \frac{1}{N^{2k}} \sum_{p_0, p} A[p_0; p]
\]
where
\[
A[p_0; p] = \prod_{l=1}^{k} a_{p_l} \prod_{l=1}^{k} [1,N](p_0 + \sum_{q=1}^{l} p_q)\delta_{0, \sum_{q=1}^{l} p_q},
\]
p = \((p_1, \ldots, p_k)\), and the summation \( \sum_{p_0,p} \) runs over all possibilities that \( p \in \{-b_N, \ldots, b_N\}^k \) and \( p_0 \in \{1, \ldots, N\} \). Thus,
\[
\frac{1}{N^4} \mathbb{E}[(\text{tr}(X_N^k) - \mathbb{E}[\text{tr}(X_N^k)])^4] = \frac{(2-b)^{-2k}b^{-2k}}{N^{4+2k}} \sum_{i_0,j_0,s_0,t_0} \mathbb{E}[(A[p_0; p] - \mathbb{E}[A[p_0; p]])] \quad (2.7)
\]
where the summation \( \sum_{i_0,j_0,s_0,t_0} \) runs over all possibilities that \( i,j,s,t \in \{-b_N, \ldots, b_N\} \) and \( i_0,j_0,s_0,t_0 \in \{1, \ldots, N\} \).

For \( i = (i_1, \ldots, i_k) \), as in Step 1 we still construct a set of numbers \( S_i = \{|i_1|, \ldots, |i_k|\} \) with multiplicities. Obviously, if one of \( S_1, \ldots, S_t \), for example, \( S_t \) does not have any number coincident with numbers of the other three sets, then the term in (2.7) equals 0 by independence. Also, if the union \( S = S_1 \cup \cdots \cup S_t \) has one number with multiplicity 1, the term in (2.7) equals 0.

Now, let us estimate the non-zero term in (2.7). Assume that \( S \) has \( p \) distinct numbers with multiplicities \( \nu_1, \ldots, \nu_p \), subject to the constraint
$\nu_1 + \ldots + \nu_p = 4k$. If $p \leq 2k - 2$, by (1.6)–(1.9), the contribution of the terms corresponding to $S$ is bounded by

$$C_k \frac{(2 - b)^{-2k}b^{-2k}}{N^{4+2k}} N^4 b_N^{2k-2} \leq C_{k,b} N^{-2}$$

for some constants $C_k$ and $C_{k,b}$.

For the case of $p = 2k - 1$, we have two numbers shared by three of four sets $S_1, \ldots, S_t$ each or one number shared by the four sets. It is not hard to see that there always exists one of $S_1, \ldots, S_t$, for example, $S_i$, which has one number with multiplicity 1, denoted by $|i_{q_0}|$ for some $q_0 \in \{1, \ldots, k\}$. Consequently, under the constraint

$$\sum_{q=1}^{k} i_q = 0$$

according to (2.3), we have $i_{q_0} = i_{q_0} - \sum_{q=1}^{k} i_q$. Hence we can choose the other $2k - 2$ distinct numbers of $S$, which determine the subscript $i_{q_0}$. This shows that there is a loss of at least one degree of freedom and the contribution of the terms corresponding to $S$ is bounded by $C_{k,b} N^{-2}$.

The case of $p = 2k$ implies each number in $S$ occurs exactly two times. Thus there exist two of $S_1, \ldots, S_t$, for example, $S_i$ and $S_j$, which share one number with one of $S_s$ and $S_t$ respectively. Consequently, under the constraints

$$\sum_{q=1}^{k} i_q = 0 \quad \text{and} \quad \sum_{q=1}^{k} j_q = 0$$

according to (2.3), there is a loss of at least two degrees of freedom and the contribution of the terms corresponding to $S$ is bounded by $C_{k,b} N^{-2}$.

Therefore,

$$\frac{1}{N^4} \mathbb{E}[(\text{tr}(X_N^k) - \mathbb{E}[\text{tr}(X_N^k)])^4] \leq C_{k,b} N^{-2}$$

and almost sure convergence is proved. Consequently, the proof of Theorem 2.2 is complete.

\begin{proof}[Proof of Theorem 2.3] Using Lemma 2.5 and our assumptions, the proof is very similar to that of Theorem 2.2. Here we don’t repeat the process. \end{proof}
Note that in the representation of even moments in Eq. (2.2) the pair partition $\pi \in \mathcal{P}_2(2k)$ comes from the constraint

$$\sum_{q=1}^{k} (-1)^q j_q = 0$$  \hspace{1cm} (2.8)$$

in the form of the trace for Hankel matrices in Lemma 2.5.

3 Slow Growth of $b_N$

We give our results for Hankel and Toeplitz band matrices with slow growth of the bandwidth as follows.

**Theorem 3.1.** Let $T_N$ be either a Hermitian ((1.6)–(1.7)) or real symmetric ((1.8)–(1.9)) Toeplitz random band matrix, where $b_N \to \infty$ but $b_N/N \to 0$ as $N \to \infty$. Take the normalization $X_N = T_N/\sqrt{2b_N}$, then $\mu_{X_N}$ converges weakly to the standard Gaussian distribution. In addition, if there exist some positive constants $\epsilon_0$ and $C$ such that $b_N \geq C N^{1/2 + \epsilon_0}$, then $\mu_{X_N}$ converges almost surely to the standard Gaussian distribution.

**Theorem 3.2.** Let $H_N$ be a real symmetric ((1.10)–(1.11)) Hankel random band matrix, where $b_N \to \infty$ but $b_N/N \to 0$ as $N \to \infty$. Take the normalization $Y_N = H_N/\sqrt{2b_N}$, then $\mu_{Y_N}$ converges weakly to the distribution $f(x) = |x| \exp(-x^2)$. In addition, if there exist positive constants $\epsilon_0$ and $C$ such that $b_N \geq C N^{1/2 + \epsilon_0}$, then $\mu_{Y_N}$ converges almost surely to the distribution $f(x) = |x| \exp(-x^2)$.

**Remark 3.3.** For real symmetric palindromic Toeplitz Matrices, i.e. real symmetric Toeplitz matrices under extra conditions: $a_{-j} = a_{N-j}, 0 < j < N$, Massey, Miller and Sinsheimer [15] have obtained the Gaussian normal distribution for eigenvalues. And for random reverse circulant matrices, i.e., Hankel matrices under extra conditions: $a_{-j} = a_{N-j}, 1 < j < N$, Bose and Mitra [5] obtained the same distribution of eigenvalues $f(x) = |x| \exp(-x^2)$ as in Theorem 3.2.

Since proofs of both theorems are very similar, we prove only the Toeplitz case.

**Proof of Theorem 3.1.** The proof is quite similar to that of Theorem 2.2 and some details will be omitted. We will lay a strong emphasis on the derivation of the Gaussian distribution. Here the slow growth of $b_N$ leads to an easy
calculation of the complicated integrals. We now complete the proof of the Gaussian law by showing

(1) $m_{k,N} = \frac{1}{N} E[\text{tr}(X_N^k)]$ converges to the $k$-th moment of the standard Gaussian distribution.

(2) Assume that $b_N \geq C N^{-\frac{1}{2}+\epsilon_0}$ for some positive constants $\epsilon_0$ and $C$.

For each fixed $k$,

$$\sum_{N=1}^{\infty} \frac{1}{N^4} E[(\text{tr}(X_N^k) - E[\text{tr}(X_N^k)])^4] < \infty. \quad (3.1)$$

By Lemma 2.4 and our assumptions (1.6)–(1.9), it follows that

$$m_{k,N} = O((b_N)^{-\frac{k}{2}+\frac{1}{2}}).$$

Since $b_N \to \infty$ as $N \to \infty$, for odd $k$

$$\lim_{N \to \infty} m_{k,N} = 0.$$

It suffices to deal with $m_{2k,N}$. However, again by Lemma 2.4 and our assumptions, the contribution with the exception of all pair partitions to $m_{2k,N}$ is $O(b_N^{-1})$. So it suffices to consider all pair partitions of $[2k] = \{1, 2, \ldots, 2k\}$. That is, for $\pi \in \mathcal{P}_2(2k)$, if $p \sim_\pi q$, then it is always the case that $j_p = j_q$ or $j_p = -j_q$. Under the condition

$$\sum_{q=1}^{2k} j_q = 0$$

according to (2.3), considering the main contribution to the trace, we should take $j_p = -j_q$ (otherwise, there is a loss of at least one degree of freedom).

Thus $a_{j_p}$ and $a_{j_q}$ are conjugate. So we can write

$$m_{2k,N} = o(1) + \frac{N^{-1}}{2k b_N^k} \sum_{\pi \in \mathcal{P}_2(2k)} \sum_{i=1}^{N} \sum_{j_i, \ldots, j_{2k} = -b_N} \prod_{l=1}^{2k} I_{[1,N]}(i + \sum_{q=1}^{l} \epsilon_{\pi}(q) j_{\pi(q)}). \quad (3.2)$$

Note that the constraints

$$1 \leq i + \sum_{q=1}^{l} \epsilon_{\pi}(q) j_{\pi(q)} \leq N, \quad 1 \leq l \leq 2k,$$

i.e.

$$\frac{1}{N} \leq \frac{i}{N} + \sum_{q=1}^{l} \epsilon_{\pi}(q) \frac{j_{\pi(q)}}{N} \leq 1, \quad 1 \leq l \leq 2k.$$
Since
\[ \left| \sum_{q=1}^{l} \epsilon_\pi(q) \frac{j_\pi(q)}{N} \right| \leq \frac{b_N}{N} \rightarrow 0 \]
as \( N \rightarrow \infty \), we have
\[ I_{[1,N]}(i + \sum_{q=1}^{l} \epsilon_\pi(q) j_\pi(q)) = 1 + o(1). \]

Hence
\[ m_{2k,N} = o(1) + (2k - 1)!! \left( \frac{2b_N + 1}{2b_N} \right)^k (1 + o(1))^{2k} \]
converges to \((2k - 1)!!\) as \( N \rightarrow \infty \). Assertion (1) is then proved.

We now prove (2). The same argument as in the proof of almost sure convergence in Theorem 2.2 shows that
\[ \frac{1}{N^4} \mathbb{E}[\left( \text{tr}(X_N^k) - \mathbb{E}[\text{tr}(X_N^k)] \right)^4] = O(\frac{1}{b_N^2}). \]

Since \( b_N \geq C N^{\frac{1}{2} + \epsilon_0} \), we have
\[ \frac{1}{N^4} \mathbb{E}[\left( \text{tr}(X_N^k) - \mathbb{E}[\text{tr}(X_N^k)] \right)^4] \leq C_k \frac{1}{N^{1 + 2\epsilon_0}}, \]
where \( C_k \) is a constant depending on \( k \) only. This completes the proof of assertion (2).

The proof of Theorem 3.1 is then complete. \( \square \)

4 First Four Moments of \( \gamma_T(b) \) and \( \gamma_H(b) \)

We will compute the second and fourth moments of \( \gamma_T(b) \) and \( \gamma_H(b) \). From the fourth moment we can read that the \( \gamma_T(b) \)'s are different for different \( b \)'s, the \( \gamma_H(b) \)'s different for different \( b \in [0, \frac{1}{2}] \) and the \( \gamma_H(b) \)'s different for different \( b \in [\frac{1}{2}, 1] \). However, for the \( 2k \)-th moments \( (k \geq 3) \), the integrals on the right-hand sides of (2.1) and (2.2) are in general different for different pair partitions, which makes difficult the explicit calculations of the higher moments.

Observe that for \( \pi \in P_{2}^{1}(2k) \), the corresponding integrals on the right-hand sides of (2.1) and (2.2) are in fact the same. Therefore, it is sufficient
to calculate the integral in the Toeplitz case for every pair partition. For the pair partition $\pi \in P_2(2k)$, we introduce the symbol

$$p_\pi(b) = \int_{[0,1]\times[-1,1]^k} \prod_{j=1}^{2k} f_{[0,1]}(x_0 + b \sum_{i=1}^{j} \epsilon_\pi(i) x_{\pi(i)}) \prod_{l=0}^{k} d x_l.$$

For $k=1$ it is easy to obtain the second moments

$$m_2(\gamma_T(b)) = 1, \quad m_2(\gamma_H(b)) = 1.$$ 

Thus the fourth moment is the first “free” moment in seeing the shape of the distribution. When $k = 2$, a direct calculation (this can be checked using Mathematica) for all pair partitions

$$\pi_1 = \{\{1,2\}, \{3,4\}\}, \pi_2 = \{\{1,4\}, \{2,3\}\}, \pi_3 = \{\{1,3\}, \{2,4\}\}$$

of $P_2(4)$ shows

$$p_{\pi_1}(b) = p_{\pi_2}(b) = \begin{cases} \frac{2}{3}(6 - 5b), \ b \in [0, \frac{1}{2}]; \\ \frac{-1 + 6b - 2b^3}{3b^2}, \ b \in (\frac{1}{2}, 1] \end{cases} \quad (4.1)$$

and

$$p_{\pi_3}(b) = \begin{cases} 4(1 - b), \ b \in [0, \frac{1}{2}]; \\ \frac{2(-1 + 6b - 6b^2 + 3b^3)}{3b^2}, \ b \in (\frac{1}{2}, 1]. \end{cases} \quad (4.2)$$

Thus, for $b \in [0, \frac{1}{2}]$,

$$m_4(\gamma_T(b)) = \frac{4(9 - 8b)}{3(2 - b)^2}$$

which strictly decreases on $[0, \frac{1}{2}]$. When $b \in (\frac{1}{2}, 1]$,

$$m_4(\gamma_T(b)) = \frac{4(-1 + 6b - 3b^2)}{3b^2(2 - b)^2}$$

which also strictly decreases on $(\frac{1}{2}, 1]$.

Therefore, one knows that $m_4(\gamma_T(b))$ strictly decreases on $[0, 1]$. Further, the distributions $\gamma_T(b)$’s are different. In particular, $\gamma_T(b)$ ($0 < b \leq 1$) is indeed different from the normal distribution. Note that for $b = 1$ the fact that the limit distribution is not Gaussian has been observed in [4, 7, 12].

We turn to the Hankel-type distribution $\gamma_H(b)$. From (4.1), one obtains that for $b \in [0, \frac{1}{2}]$

$$m_4(\gamma_H(b)) = \frac{4(6 - 5b)}{3(2 - b)^2}$$

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which strictly decreases on $[0, \frac{1}{2}]$ while for $b \in (\frac{1}{2}, 1]$,
\[ m_4(\gamma_H(b)) = \frac{2(-1 + 6b - 2b^2)}{3b^2(2-b)^2} \]
which strictly increases on $(\frac{1}{2}, 1]$. Thus, according to the fourth moments, we only know that the $\gamma_H(b)$’s ($0 \leq b \leq \frac{1}{2}$) are different and the $\gamma_H(b)$’s ($\frac{1}{2} \leq b \leq 1$) are different.

**An added note.** After the paper as submitted we learned from the Associate Editor and the referee about a related preprint “Limiting Spectral Distribution of Some Band Matrices” by Basak and Bose at the site http://www.isical.ac.in/~statmath/html/publication/techreport.html. Although their paper and ours contain the same results for Hankel and Toeplitz band matrices, the former assumes less integrability on the entries of a matrix, allows more general “rates” for the bandwidth, and also covers more ensembles of “structured matrices” related to Toeplitz matrices. On the other hand, our paper covers Hermitian Toeplitz matrices and gives low order moment calculations. Besides, our paper gives a different method for analyzing Toeplitz matrices by treating them as a linear combination of deterministic matrices with independent coefficients. This method can be used to derive other results, including those that deal with semicircle law.

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