On Root Multiplicities of Some
Hyperbolic Kac-Moody Algebras

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Abstract.
Using the coset construction, we compute the root multiplicities at level three for some hyperbolic Kac-Moody algebras including the basic hyperbolic extension of $A_1^{(1)}$ and $E_{10}$.
Hyperbolic or generalized Kac-Moody algebras re-appear periodically in string theory. There have been recent indications that they should play a role in duality properties of supersymmetric gauge theories. It is indeed quite striking that the modular group is often a subgroup of the Weyl group of hyperbolic Kac-Moody algebras. They also seem to have a role to play in two dimensional field theory since some affine Toda field theories are known to be invariant under quantum affine algebras but with a central charge acting as a topological charge. This suggests that hyperbolic algebras could be related to the spectrum generating algebra of these models.

However little is known about hyperbolic Kac-Moody algebras, (but more is known concerning generalized Kac-Moody algebras). In particular the root multiplicities are known only for a small subset of roots, the so-called level two roots. The aim of this short note is to slightly extend this knowledge by determining the root multiplicities for the level three roots.

1 Hyperbolic extension of affine algebras.

1.1 Presentation.

We are going to consider Kac-Moody algebras which are extensions of affine algebras by a basic representation. We mainly use notation from the book [1]. We denote by $G$ the underlying affine algebra, which we assume to be untwisted, and by $\hat{G}$ its extension.

Let $(a_{ij})$, $i, j = 0, 1, \cdots, r$, be the Cartan matrix of $G$, and $e_i, f_i, h_i$ be the generators associated to the simple root $\alpha_i$. By convention the root $\alpha_0$ is the extended root of the affine algebra $G_0$ and the remaining roots $\alpha_j$, $j = 1, \cdots, r$, are those of a finite simple Lie algebra, which we denote by $\mathcal{G}$. As usual, we extend $G_0$ by adding a derivation $d$ commuting with the Cartan generators $h_i$ and such that $[d, e_i] = -\delta_{i,0} e_i$, and $[d, f_i] = \delta_{i,0} f_i$. In the following $G_0$ will refer to the affine algebra extended by the derivation. It will be useful to introduce a basis $\{d; k; J^a_n, n \in \mathbb{Z}, a = 1, \cdots, dim \mathcal{G}\}$ of $G_0$ with Lie brackets:

$$[J^a_n, J^b_m] = f_c^{ab} J^c_{n+m} + nkq^{ab} \delta_{n+m,0}$$
$$[d, J^a_n] = -nJ^a_n$$

where $f_c^{ab}$ and $q^{ab}$ are the structure constants and the Killing form of $\mathcal{G}$, and $k$ is central. The invariant bilinear form of $G_0$ is defined by $(J^a_n, J^b_m) = q^{ab} \delta_{n+m,0}$ and $(d, k) = (-1, -1)$. As is well known, the derivation $d$ can be realized on any integrable highest weight $G_0$-module as $d = L_0 + C^1 \cdot \delta_{i,0}$ where $L_0$ is the zero mode of the Sugawara Virasoro generators:

$$L_0 = \frac{1}{2(k+h^*)} \sum_{a,b} (q^{ab} J^a_0 J^b_0 + 2 \sum_{n>0} q^{ab} J^a_{-n} J^b_{n})$$

In the above formula, $h^*$ is the dual Coxeter number of $\mathcal{G}$ and $q^{ab}$ the inverse Killing form, $q^{ab} q_{bc} = \delta^a_c$.

The Cartan matrix $\bar{a}$ of the extension $\hat{G}$ of $G_0$ is defined by:

$$\bar{a}_{ij} = a_{ij}, \quad \text{for} \quad i, j = 0, 1, \cdots, r$$
$$\bar{a}_{i,-1} = \bar{a}_{-1,i} = -\delta_{i,0}$$

(1)

The algebra $\hat{G}$ is thus of rank $(r + 1)$. The root $\alpha_{-1}$ is usually referred as the overextended root. Although this Cartan matrix always define a Kac-Moody algebra, it does not always define a hyperbolic algebra. For instance it does not if the rank of $G_0$ is too big. See ref. [2] for a complete classification of hyperbolic algebras. The algebra $\hat{G}$ can be defined as the Lie algebra generated by the elements $\{e_i, f_i, h_i, i = -1, 0, 1, \cdots, r\}$ with relations:

$$[h_i, h_j] = 0 \quad , \quad [e_i, f_j] = \delta_{ij} h_i$$
$$[h_i, e_j] = a_{ij} e_j \quad , \quad [h_i, f_j] = -a_{ij} f_j$$
$$(ade_i)^1-a_{ij} e_j \quad = \quad (adf_i)^1-a_{ij} f_j = 0 \quad i \neq j$$

Notice that these relations in particular imply that we may identify $d$ with $h_{-1}$ up to a multiple of $k$. Consistency of the following constructions enforces the choice $d = h_{-1} + k$. 

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We will use an alternative presentation of $\hat{G}$ due to Feingold and Frenkel \[8\], see also \[8\] or \[8\]. It corresponds to decompose $\hat{G}$ with respect to its affine subalgebra $G_0$. This yields a graded decomposition of $\hat{G}$ as:

$$\hat{G} = \cdots + G_{-2} + G_{-1} + G_0 + G_1 + G_2 + \cdots$$

where each subspaces $G_n$ are $G_0$-modules of level $n$. So this decomposition is graded by $k$ the central element of $G_0$.

By definition the modules $G_1$ (resp. $G_{-1}$) are the highest (resp. lowest) $G_0$-modules generated by $f_{-1}$ (resp. $e_{-1}$). They are isomorphic to the basic module $V(\Lambda_0 + \delta)$ and its dual $V^*(\Lambda_0 + \delta)$:

$$G_1 \simeq V(\Lambda_0 + \delta) , \quad G_{-1} \simeq V^*(\Lambda_0 + \delta)$$

The $d$-grading of the highest vector $|\Lambda_0 + \delta\rangle$, which is identified with $f_{-1}$, is fixed by the identification $d = h_{-1} + k$. It is such that $d|\Lambda_0 + \delta\rangle = -|\Lambda_0 + \delta\rangle$. We denote by $(v^*, v)$ for $v \in G_1$ and $v^* \in G_{-1}$ the pairing between $G_1$ and $G_{-1}$.

We first need to describe the Lie brackets between elements in $G_0$ and $G_{\pm 1}$. Let $x \in G_0$, $v \in G_1$ and $v^* \in G_{-1}$, then by construction \[3\]:

$$[x, v] = x \cdot v, \quad [x, v^*] = x \cdot v^*, \quad [v^*, v] = \sum_I (v^*, X^I \cdot v) X^I$$

where $x \cdot v$ and $x \cdot v^*$ refer to the action of $x \in G_0$ on $v \in G_1$ or on $v^* \in G_{-1}$, and where $(X^I)$ form an orthonormal basis of $G_0$. It is easy to check that it satisfies the Jacobi identity:

$$[x, [v^*, v]] = [[x, v^*], v] + [v^*, [x, v]]$$

The Lie subalgebras $G_{\pm} = \bigoplus_{n>0} G_{\pm n}$ are defined as certain quotients of the free Lie algebra $F_+ = F(V)$ generated by $V$, or $F_- = F(V^*)$ generated by $V^*$. A few basic facts concerning free Lie algebras are recalled in the next subsection. Let us introduce $\hat{F} = F_+ + G_0 + F_-$. As shown in \[8\] we can endow $\hat{F}$ with a Lie bracket which reduces to the commutator \[8\] for the elements in $G_0$, $G_{\pm 1}$. Let now $J_{\pm} = \bigoplus_{n>0} J_{\pm n}$ be the ideals in $\hat{F}$ generated respectively by $J_{\pm 2}$ which are the subsets of $F_{\pm 2}$ defined by:

$$J_{\pm 2} = \{ z \in F_{\pm 2}; \quad [y, z] = 0, \forall y \in G_{\mp 1} \}$$

Let $J = J_+ + J_-$. It is an ideal in $\hat{F}$. As proved in \[8\], taking the quotient of $\hat{F}$ ensures that the Serre relations are fulfilled, and the resulting algebra is isomorphic to the hyperbolic extension $\hat{G}$:

$$\hat{G} \simeq \hat{F}/J \simeq F_-/J_- + G_0 + F_+/J_+$$

Ie. $G_{\pm} = F_{\pm}/J_{\pm}$. This is the definition of $\hat{G}$ we will use to compute the root multiplicities.

### 1.2 Details on the free Lie algebra $F(V)$.

We need a finer description of the free Lie algebra $F_+ = F(V)$. Let $(T(V), \odot)$ be the tensor algebra of $V$. This is a graded associative algebra, so we can endow it with a graded Lie algebra structure by taking the usual commutator as Lie bracket. We identify $V$ with the subspace of $T(V)$ consisting of elements of degree 1 and write $(T(V), [, ,])$ when we want to stress that we view $T(V)$ as a Lie algebra. Let $F(V)$ be the smallest Lie subalgebra of $(T(V), [, ,])$ containing $V$. As $F(V)$ is generated by homogeneous elements of degree 1, it is a graded Lie algebra, and we write $F(V) = \bigoplus_{n=1}^\infty F_n$, with $F_1 = V$. By definition, $V$ generates $F(V)$. Let $\mathcal{U}(F(V))$ be the universal enveloping algebra of $F(V)$. As usual we view $F(V)$ as a subspace of $\mathcal{U}(F(V))$. Then it can be shown that \[8\]

**Proposition.** i) The associative algebras $T(V)$ and $\mathcal{U}(F(V))$ are canonically isomorphic.

ii) Any linear map from $V$ to a Lie algebra $L$ extends uniquely to a Lie algebra homomorphism from $F(V)$ to $L$. This explains why $F(V)$ is called a free Lie algebra.

**Proof.**

The first part is a consequence of the universal properties of tensor algebras and universal enveloping algebras. The identity map from $V \subset T(V)$ to $V \subset \mathcal{U}(F(V))$ extends (in a unique way) to a homomorphism
of associative algebras. The identity map from $F(V) \subset \mathcal{U}(F(V))$ to $F(V) \subset T(V)$, a Lie algebra homomorphism, extends (in a unique way) to a homomorphism of associative algebras. The composition in any order of those two homomorphisms is the identity so they have to be isomorphisms.

The second part goes as follows: The linear map from $V$ to $L$ extends (in a unique way) to a homomorphism of associative algebras from $T(V)$ to $\mathcal{U}(L)$. So by the first part there is a homomorphism of associative algebras from $\mathcal{U}(F(V))$ to $\mathcal{U}(L)$. This leads to a Lie algebra homomorphisms of the underlying Lie algebras mapping $V$, hence $F(V)$ to $L$.

Further information comes from the Poincaré-Birkhoff-Witt (or PBW) theorem which states that for a Lie algebra $L$, the graded algebra associated to the filtered algebra $\mathcal{U}(L) = \bigcup_{n=0}^{\infty} S_n(L)$ is the symmetric algebra $S(L) = \bigoplus_{n=0}^{\infty} S_n(L)$. In particular $\mathcal{U}_n(L)$ and $\bigoplus_{n=0}^{\infty} S_n(L)$ carry isomorphic representations of $GL(L)$. For the free Lie algebra $F(V) = \bigoplus_n F_n$ we see that $S(F(V)) \simeq \bigotimes_n S(F_n)$ and $T(V)$ are isomorphic as graded representations of $GL(V)$. This isomorphism has several useful manifestations, involving Poincaré series.

We introduce a formal variable $t$ to keep track of degrees. Then we can write $T(V) = \frac{1}{1-tV}$ as a formal identity whose meaning is that the expansion in $t$ on the right-hand side is:

$$T(V) = \frac{1}{1-tV} = 1 + tV + t^2V \oplus 2 + t^3V \oplus 3 + \cdots$$

(4)

The corresponding formal expansion for $S(F_n)$ is $1 + t^n F_n + t^{2n} S^2(F_n) + t^{3n} S^3(F_n) + \cdots$, which by a well-known boson-fermion reciprocity can be rewritten as

$$S(F_n) = \frac{1}{1 - t^n F_n + t^{2n} \Lambda^2(F_n) - t^{3n} \Lambda^3(F_n) + \cdots}$$

(5)

where $\Lambda^p(F_n)$ denotes the wedge product of $p$ copies of $F_n$. So we have

$$\frac{1}{1-tV} = \bigotimes_{n>0} \left( \frac{1}{1-t^n F_n + t^{2n}(\Lambda^2 F_n) - t^{3n}(\Lambda^3 F_n) + \cdots} \right)$$

(6)

which we shall use by taking its inverse.

We also introduce a formal character. Let $A \in \text{End}(V)$ be diagonalizable, with eigenvalues $a_\alpha$, treated as formal variables of degree 1. It multiplicatively induces a degree preserving endomorphism on $T(V)$, which we again denote by $A$. Let $a_{\alpha n}$ be the eigenvalues of the restriction of $A$ to $F_n$. Then the character of $T(V)$ is $(1 - t \sum \alpha a_\alpha)^{-1}$ and the character of $S(F_n)$ is $\prod_{\alpha n} (1 - t^n a_{\alpha n})^{-1}$. So we have

$$\left( \frac{1}{1-t\sum \alpha a_\alpha} \right) = \prod_{n>0} \prod_{\alpha n} \left( \frac{1}{1-t^n a_{\alpha n}} \right)$$

(7)

which we shall use by taking its logarithm. To summarize:

**Proposition.** Let $F(V) = \bigoplus_{n>0} F_n$ be the free Lie algebra generated by $V$, then:

i) The spaces $F_n$ can be computed as subspaces of the tensor algebra $T(V)$ using the following generating function:

$$1-tV = \bigotimes_{n>0} (1-t^n F_n + t^{2n}(\Lambda^2 F_n) - t^{3n}(\Lambda^3 F_n) + \cdots)$$

(8)

ii) The characters are given by:

$$\log \left( 1 - t \sum \alpha a_\alpha \right) = \sum_{n>0} \sum_{\alpha n} \log (1 - t^n a_{\alpha n})$$

(9)

The first formula (8) gives for the first few spaces $F_n$:

$$
F_2 = \Lambda^2 V = V \wedge V \\
F_3 = V \otimes (V \wedge V) - \Lambda^3 V \\
F_4 = V \otimes F_3 - [S^2(V \wedge V) - \Lambda^4 V]
$$

(10)

These formulas have a simple and nice interpretation: all states in $F_n$ can be represented as nested commutators between elements in $V$, and the relations (10) describe the linear combinations between these states.
due to the Jacobi identity and the antisymmetry of the Lie bracket. We can formulate this interpretation more precisely. Consider the linear maps \( I_{n+1} \) from \( V \otimes F_n \) to \( F_{n+1} \) defined by:

\[
I_{n+1}(v \otimes f) = [v, f]
\]

(11)

for any \( v \in V \) and \( f \in F_n \). By definition we have the isomorphism:

\[
F_{n+1} \cong (V \otimes F_n)/Ker I_{n+1}
\]

(12)

The relations (11) tell us how to determine the kernel of \( I_{n+1} \) using the following sequences:

\[
0 \rightarrow \Lambda^3 V \rightarrow V \otimes (V \wedge V) \overset{I_3}{\rightarrow} F_3 \rightarrow 0
\]

\[
0 \rightarrow \Lambda^4 V \rightarrow S^2(V \wedge V) \overset{id \otimes I_3}{\rightarrow} V \otimes F_3 \overset{I_4}{\rightarrow} F_4 \rightarrow 0
\]

It is easy to see that these sequences define complexes. The relations (11) tell us that these complexes have no cohomology, i.e., the sequences are exact. Thus:

\[
Ker I_3 = \Lambda^3 V
\]

\[
Ker I_4 = (id \otimes I_3)S^2(\Lambda^2 V) \simeq S^2(\Lambda^2 V) - \Lambda^4 V
\]

(13)

This construction can clearly be generalized to any level.

The second formula (11) leads to

\[
\left( \sum_{\alpha} a_\alpha \right)^n = \sum_{p+q=n} p \left( \sum_{\alpha_n} a_{\alpha_n}^2 \right)
\]

(14)

For dimensions, this could be inverted using the Möbius function. Anyway, the term \( p = n \) involves the character \( ChF_n = \sum_{\alpha_n} a_{\alpha_n} \) of the space \( F_n \). For instance

\[
ChF_2 = \frac{1}{2} \left( \sum_{\alpha} a_\alpha \right)^2 - \frac{1}{2} \sum_{\alpha} a_\alpha^2
\]

(15)

\[
ChF_3 = \frac{1}{3} \left( \sum_{\alpha} a_\alpha \right)^3 - \frac{1}{3} \sum_{\alpha} a_\alpha^3
\]

(16)

\[
ChF_4 = \frac{1}{4} \left( \sum_{\alpha} a_\alpha \right)^4 - \frac{1}{4} \sum_{\alpha} a_\alpha^4 - \frac{1}{2} \sum_{\alpha_2} a_{\alpha_2}^2
\]

(17)

Finally let us point out that the algebra \( \hat{F} = F_- + G_0 + F_+ \) is not a generalized Kac-Moody algebra [3]. In particular, taking the quotient of \( \hat{F} \) by the ideal generated by \( J_{\pm 2} \) is required to have a contravariant form which is not degenerate outside the Cartan subalgebra.

1.3 Details on the ideal \( J_2 \)

To compute the roots multiplicities we need a finer identification of \( J_2 \) and the ideal it generates. We describe how the Feingold-Frenkel presentation of the hyperbolic algebras is related to the coset construction [3]. The fact that the coset construction should play a role is clear from the formula (12) which identifies \( F_{n+1} \) as a \( G_0 \)-submodule of the tensor product of \( V_0 \), a level one module, with \( F_n \), a level \( n \)-module.

Let us introduce the tensor Casimir operator \( \mathcal{C} \) which is the element of \( G_0 \otimes G_0 \) defined by \( \mathcal{C} = \sum_{x} X^I \otimes X^I \), with \( \{X^I\} \) an orthonormalized basis of \( G_0 \). In the basis \( \{d; k; J_a^\pm, a = 1, \ldots, dim \mathcal{G}, n \in \mathbb{Z}\} \) it reads:

\[
\mathcal{C} = -d \otimes k - k \otimes d + \sum_{n \in \mathbb{Z}} q_{ab} J_n^a \otimes J_n^b
\]

(18)

Clearly, when acting on tensor product of two representations \( W_1 \) and \( W_2 \) of \( G_0 \), the operator \( \mathcal{C} \) commutes with the diagonal action of \( G_0 \). It is actually related to the coset Virasoro generators.
Proposition. Consider highest weight $G_0$-modules $W_1$ and $W_2$. Suppose that on these modules we identify the derivation $d$ with $L_0 + \eta_1, \eta_2$ where $\eta_1$ and $\eta_2$ are constants. Then, on $W_1 \otimes W_2$ we have:

$$C = -k \otimes \eta_2 - \eta_1 \otimes k - (\Delta(k) + h^*)_L^0 \coset$$

where $L^0_0 \coset$ is the zero mode of the coset Virasoro generators:

$$L^0_0 \coset = L_0 \otimes 1 + 1 \otimes L_0 - \frac{1}{2(\Delta(k) + h^*)} \sum_{a,b} \left( q_{ab} \Delta J^a_0 \Delta J^b_0 + 2 \sum_{n>0} q_{ab} \Delta J^a_{-n} \Delta J^b_n \right)$$

where $\Delta(X) = X \otimes 1 + 1 \otimes X$ for all $X \in G_0$.

As a consequence, one gets the following description of $J_2$.

$$J_2 = \text{Ker} \mathcal{C}_{|V \wedge V} = \{ |\omega\rangle \in V \wedge V; (L^0_0 \coset|_{V \wedge V}) |\omega\rangle = \frac{2}{h^* + 2} |\omega\rangle \}$$

where $L^0_0 \coset|_{V \wedge V}$ is the coset Virasoro generator as defined in eq. (21) acting on $V \wedge V$. The first equality, which follows from the definitions, was proved in [3]. The second equality follows from the relation between $\mathcal{C}$ and the coset Virasoro generators, from the identification $d = L_0 - 1$ when acting on $V$, and from the fact that $V \simeq V(\Lambda_0 + \delta)$ is a level one module.

Notice that the value $(\frac{2}{h^* + 2})$ for $L^0_0 \coset$ identifies the vectors of $J_2$ as highest weight vectors of the coset Virasoro generators.

1.4 Roots multiplicities at level 2 and 3.

In order to compute the roots multiplicities of $\hat{G}$ for the very first few levels, we extend the relation between $J_2$ and the coset construction to other components $J_n$ of the ideal generated by $J_2$.

The definition of $\hat{G} = \bigoplus_{n \in \mathbb{Z}} G_n$ as the quotient of $\hat{F}$ by the ideal $J = \bigoplus_{n \in \mathbb{Z}} J_n$ generated by $J_{\pm 2}$ gives us the following description of $\hat{G}_n$ for $n > 0$ as:

$$G_{n+1} \simeq F_{n+1} \cap J_{n+1} \simeq V \otimes F_n - \text{Ker} I_{n+1} - J_{n+1}$$

Using the formula (13) for the kernel of the maps $I_{n+1}$, we obtain:

Proposition. As subspaces of $V \otimes J_2$ we have the following inclusion,

$$(V \otimes J_2) \cap \Lambda^3 V \subset \{ |\omega\rangle \in V \otimes J_2; \ (L^0_0 \coset|_{V \otimes J_2}) |\omega\rangle = \frac{4h^* + 6}{(h^* + 3)(h^* + 2)} |\omega\rangle \}$$

where $L^0_0 \coset|_{V \otimes J_2}$ is the coset Virasoro generator defined in eq. (21) acting on $V \otimes J_2$.

Proof.

The space $V \otimes J_2$ is embedded into $V \otimes V \otimes V$. Let $P_{ij}$ be the operator permuting the $i^{th}$ and $j^{th}$ copies of $V$ in $V \otimes^3$. Let us denote by $C_{ij}$ the tensor Casimir $C$ acting on the $i^{th}$ and $j^{th}$ copies of $V$ in $V \otimes^3$. If $|\omega\rangle \in (V \otimes J_2) \cap \Lambda^3 V$ then $C_{23}|\omega\rangle = 0$, by definition of $J_2$, and $P_{ij}|\omega\rangle = -|\omega\rangle$, by definition of $\Lambda^3 V$. Therefore,

$$|\omega\rangle \in (V \otimes J_2) \cap \Lambda^3 V \Rightarrow C_{ij}|\omega\rangle = 0 \ \forall i,j = 1,2,3$$

since $P_{12}C_{23} = C_{13}P_{12}$, etc... Now, taking into account the identification of $d = L_0 - 1$ in each copy of $V$ in $V \otimes^3$, we can express $C_{12} + C_{13}$ in terms of the coset Virasoro generators:

$$C_{12} + C_{13} = (h^* + 3) L^0_0 \coset|_{V \otimes J_2} + \left( \frac{C_{23}}{h^* + 2} \right) + \left( \frac{4h^* + 6}{h^* + 2} \right)$$
This proves the result.

Remark 1: this identifies a finite set of possible $G_0$-modules, which are made of highest weight vectors for $L^{\text{coset}}$ and which can be analysed case by case.

Remark 2: if $(V \otimes J_2) \cap \Lambda^3 V$ is the trivial module $\{ |\omega \rangle = 0 \}$, then $J_3 \simeq V \otimes J_2$, and the subspace of level three $G_3$ is simply

$$G_3 \simeq V \otimes (V \wedge V) - \Lambda^3 V - (V \otimes J_2)$$  \hspace{1cm} (24)

It is then a simple exercise in affine algebras to decompose this space into irreducible $G_0$-modules, exercise that we will describe in two cases in the next sections. This covers the case of $\hat{A}_1^{(1)}$ but needs a slight generalization for $E_{10}$.

Remark 3: to compute the root multiplicities at level 4 one would need to compute the intersection $(V \otimes J_3) \cap \text{Ker} I_4$. This is not so easy because the permutation group does not act in a simple way on this space.

2 Example 1: $\hat{A}_1^{(1)}$.

In this section, we expand the formula (24) in the case of $\hat{A}_1^{(1)}$ which is the hyperbolic extension of the affine algebra $A_1^{(1)}$ by its basic representation. Its Dynkin diagram is:

It represents the Cartan matrix:

$$\tilde{a} = \begin{pmatrix} 2 & -2 & 0 \\ -2 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

We label the roots by $\alpha_1$, $\alpha_0$ and $\alpha_{-1}$ as indicated on the Dynkin diagram. The dual Coxeter number is $h^* = 2$.

The module $V$ is the basic level one module $V(\Lambda_0 + \delta)$. We recall that its character can be expressed in terms the Theta and Dedekind functions:

$$Ch V \equiv ChV(\Lambda_0 + \delta) = \frac{q^{-1}}{\eta(q)} \Theta_{\Lambda_0}$$  \hspace{1cm} (25)

Since $V$ has level one, all the coset constructions associated to the identifications (12) will be related to the minimal representations of the Virasoro algebra.

Consider first roots at level two. This space was already described in [3, 8] but we need to recall it for computing the level three root multiplicities. The coset Virasoro associated to $V \otimes V$ has central charge $c = \frac{1}{2}$. Therefore the decomposition of $V \otimes V$ with respect to the diagonal action of $A_1^{(1)}$ can be expressed in terms of the $c = \frac{1}{2}$ minimal Virasoro characters. As is well known one has:

$$V \wedge V = V(\Lambda_0 + \delta) \wedge V(\Lambda_0 + \delta) \simeq Vir_{c=1/2}^{h=1/2} (q) \otimes V(2\Lambda_1 + 2\delta)$$

where $Vir_{c}^{h}(q)$ denotes the Virasoro characters of the irreducible highest weight vector representation of central charge $c$ and conformal weight $h$. Since $h^* = 2$, the ideal at level 2 is

$$J_2 \simeq |h = \frac{1}{2} \rangle \otimes V(2\Lambda_1 + 2\delta)$$

Therefore, we have [3, 8]:

$$G_2 = \left( Vir_{c=1/2}^{h=1/2} (q) - q^{1/2} \right) \otimes V(2\Lambda_1 + 2\delta)$$  \hspace{1cm} (26)

Consider now the root multiplicities at level three. The Virasoro generators representing the coset construction associated to the isomorphism $F_3 \simeq (V \otimes F_2)/\text{Ker} I_3$ have central charge $c = 7/10$. We need the
decomposition of the tensor product $V(\Lambda_0 + \delta) \otimes V(2\Lambda_1 + 2\delta)$ with respect to the diagonal action of $A_1^{(1)}$. It is given by:

$$V(\Lambda_0 + \delta) \otimes V(2\Lambda_1 + 2\delta) \simeq \text{Vir}_{c=7/10} \otimes V(\Lambda_0 + 2\Lambda_1 + 3\delta) + \text{Vir}_{c=7/10} \otimes V(3\Lambda_0 + 3\delta)$$

Comparing this formula and the inclusion (23) we learn that $(V \otimes J_2) \cap \Lambda^3 V = 0$ reduces to the trivial module: $\frac{4h^c + 4}{(h^c + 3)(h^c + 2)} = \frac{3}{10}$ and there is no state $|\omega\rangle \in (V \otimes J_2)$ such that $(L_0^\text{ext} |\omega\rangle) \equiv \frac{3}{10}|\omega\rangle$. Therefore

$$J_3 = V \otimes J_2 \quad \text{for} \quad A_1^{(1)}$$

As a consequence we have:

$$G_3 = \left[ \text{Vir}_{c=1/2} - q^{1/2} \right] \otimes \left[ \text{Vir}_{c=7/10} \otimes V(\Lambda_0 + 2\Lambda_1 + 3\delta) + \text{Vir}_{c=7/10} \otimes V(3\Lambda_0 + 3\delta) \right]$$

The last step consists in finding an explicit expression for the character of $\Lambda^3 V(\Lambda_0 + \delta) \equiv \Lambda^3 V$. This can be done using the formula (16) and (25). It gives:

$$\text{Ch}(\Lambda^3 V) = \frac{1}{3} (\text{ChV})^3 - \frac{1}{3} \frac{q^{-3}}{\eta(q^3)} \Theta_{3\Lambda_0}$$

But $(\text{ChV})^3$ is the character of $V^{3 \otimes 3}$ and therefore can be expressed in terms of the Virasoro characters as follows:

$$(\text{ChV})^3 = \left[ \text{Vir}_{c=1/2} \otimes \text{Vir}_{c=7/10} \right] \text{ChV}(\Lambda_0 + 2\Lambda_1 + 3\delta) + \left[ \text{Vir}_{c=1/2} \otimes \text{Vir}_{c=7/10} \right] \text{ChV}(3\Lambda_0 + 3\delta)$$

The second term in (28) is explicit but we can also reexpress it in terms of the $A_1^{(1)}$ characters of representations of level 3. To do it we use the fact that the Theta functions of level three are linearly related to the characters at level three with coefficients given by the so-called string functions:

$$\text{ChV}(i\Lambda_0 + j\Lambda_1) = \sum_{n,m} C_{ij}^{nm}(q) \Theta_{n\alpha + m\lambda}$$

At level three the string functions can again be expressed in terms of the Virasoro characters but with central charge $c = 4/5$. Namely:

$$C_{30}^{30}(q) = \frac{1}{\eta(q)} \left[ \text{Vir}_{c=4/5}^{h=0} + \text{Vir}_{c=4/5}^{h=3} \right]$$

$$C_{12}^{30}(q) = \frac{1}{\eta(q)} \left[ \text{Vir}_{c=4/5}^{h=2/3} \right]$$

$$C_{12}^{12}(q) = \frac{1}{\eta(q)} \left[ \text{Vir}_{c=4/5}^{h=1/5} \right]$$

$$C_{30}^{12}(q) = \frac{1}{\eta(q)} \left[ \text{Vir}_{c=4/5}^{h=2/5} + \text{Vir}_{c=4/5}^{h=7/5} \right]$$

Inverting the linear system relating the characters and the string functions gives the expression of $\Theta_{3\Lambda_0}$:

$$\left( C_{12}^{30} C_{30}^{30} - C_{12}^{30} C_{30}^{12} \right) \Theta_{3\Lambda_0} = C_{12}^{12}(q) \text{ChV}(3\Lambda_0) - C_{30}^{12}(q) \text{ChV}(\Lambda_0 + 2\Lambda_1)$$

Gathering all the formula gives an explicit formula for the decomposition of the level three space $G_3$ as a $G_0$-module. So it gives an explicit formula for the root multiplicities at level three. Explicit but ugly!

### 3 Example 2: $E_{10}$.

In this section we present the root multiplicities at level three for $E_{10}$. Once again it consists in making formula (23) explicit. As we explained the $A_1^{(1)}$ case in detail we will be more sketchy for $E_{10}$.
The Dynkin diagram of $E_{10}$ is

![Dynkin Diagram](image)

The indices refer to the root labelling. The dual Coxeter number is $h^* = 30$. Let $\Lambda_1$ be the weight dual to the affine root $\alpha_i$. The weight $\Lambda_0$ is the only integrable weight at level one for the affine subalgebra $E_8^{(1)}$. The integrable weights at level two are $2\Lambda_0$, $\Lambda_8$ and $\Lambda_1$. Those integrable at level three are $3\Lambda_0$, $\Lambda_8 + \Lambda_0$, $\Lambda_1 + \Lambda_0$, $\Lambda_2$ and $\Lambda_7$.

The basic module which generates the hyperbolic extension of $E_8^{(1)}$ is $V \simeq V(\Lambda_0 + \delta)$. As explained previously, the root multiplicities at level two are related to the coset construction $(E_8^{(1)};E_8^{(1)};E_8^{(1)})/E_8^{(1)}$.

This is described by the $c = \frac{1}{2}$ minimal Virasoro representation. Explicitly,

$$V(\Lambda_0 + \delta) \wedge V(\Lambda_0 + \delta) \simeq V^{h=1/16}(q) \otimes V(\Lambda_8 + 2\delta)$$

(30)

Comparing with (21) gives:

$$J_2 \simeq |h = \frac{1}{16}| \otimes V(\Lambda_8 + 2\delta)$$

(31)

Thus the root multiplicities at level two are:

$$G_2 \simeq \left[ \left. V(\Lambda_0 + \delta) \right|_{c=1/2}^{h=1/16} \right] \otimes V(\Lambda_8 + 2\delta)$$

(32)

For future convenience we need the expression of the highest weight vector $|\Lambda_8\rangle$ in $J_2$:

$$|\Lambda_8\rangle = (J_{-1}^\theta \otimes 1 - 1 \otimes J_{-1}^\theta) |\Lambda_0\rangle \otimes |\Lambda_0\rangle$$

where $\theta$ is the highest root of $E_8$.

To compute the root multiplicities at level three we have to evaluate $J_3$. The Virasoro generators representing the coset construction associated to the isomorphism $F_3 \simeq (V \otimes F_2)/\text{Ker}I_3$ have central charge $c = 1 - \frac{6}{11.16}$, again a minimal model. One computes that $\frac{(h^* + 3)(h^* + 2)}{11.16} = \frac{21}{11.16}$. As explained in the previous section, we have to look for the state $|\omega\rangle \in (V \otimes J_2)$ such that $(L_0^{\coset}|_{V \otimes J_2}) |\omega\rangle = \frac{21}{11.16} |\omega\rangle$.

Contrary to the $A_{10}^{(1)}$ case, for $E_{10}$ there is a candidate for such a state. It occurs in the coset decomposition $V(\Lambda_0) \otimes V(\Lambda_8)/V(\Lambda_7)$. More precisely:

**Proposition.** For $E_{10}$ we have:

$$(V \otimes J_2) \cap \Lambda^3 V \simeq |h = \frac{21}{11.16}| \otimes V(\Lambda_7 + 3\delta)$$

(33)

Here $|h = \frac{21}{11.16}|$ denotes the highest weight vector of the Virasoro coset algebra with conformal weight $\frac{21}{11.16}$.

Thus, we have: $J_3 \simeq (V \otimes J_2) - |h = \frac{21}{11.16}| \otimes V(\Lambda_7 + 3\delta)$.

Proof.

The states $|\omega\rangle$ which are highest weight vectors for the affine algebra with weight $\Lambda_7$ and for the Virasoro coset with conformal weight $\frac{(h^* + 3)(h^* + 2)}{11.16} = \frac{21}{11.16}$ occur at level one in the tensor product $V(\Lambda_0) \otimes V(\Lambda_8)$.

Therefore, it has to be a linear combination of the four following vectors:

$$|\Lambda_0\rangle \otimes J_{-1}^\theta |\Lambda_0\rangle \otimes J_{-1}^\theta |\Lambda_0\rangle$$

$$|\Lambda_0\rangle \otimes J_{-1}^\theta |\Lambda_0\rangle \otimes J_{-1}^\theta |\Lambda_0\rangle$$

But recall that we have to view $|\Lambda_8\rangle$ as the element $(J_{-1}^\theta \otimes 1 - 1 \otimes J_{-1}^\theta) |\Lambda_0\rangle \otimes |\Lambda_0\rangle$ in $V(\Lambda_0) \otimes V(\Lambda_0)$. Therefore, these four states can written in terms of the action of $J_{-1}^\theta$ and $J_{-1}^\theta$ on one of the three copies of $|\Lambda_0\rangle$ in $|\Lambda_0\rangle \otimes |\Lambda_0\rangle \otimes |\Lambda_0\rangle$. The only state of this form which is in $\Lambda^3 V$ is:

$$|\omega\rangle = (J_{-1}^\theta \otimes 1 \otimes J_{-1}^\theta - J_{-1}^\theta \otimes J_{-1}^\theta \otimes J_{-1}^\theta) |\Lambda_0\rangle \otimes |\Lambda_0\rangle \otimes |\Lambda_0\rangle$$
It is easy to check that this is a highest weight vector for the affine algebra. One may also verify that $|\omega\rangle \in (J_2 \otimes V)$, since it can written alternatively as follows:

$$|\omega\rangle = (\Delta J_0^{-\alpha_8}|\Lambda_8\rangle) \otimes (J_{-1}^0|\Lambda_0\rangle) + (\Delta J_{-1}^0|\Lambda_8\rangle) \otimes |\Lambda_0\rangle - |\Lambda_8\rangle \otimes (J_{-1}^0|\Lambda_0\rangle)$$

The fact that the multiplicity in eq. (33) is one follows from the fact that the states $|\omega\rangle$ defined above is unique. This proves the result (33). □

As a corollary we obtain the root multiplicities at level three for $E_{10}$:

$$G_3 \simeq V \otimes (V \wedge V) - \Lambda^3 V - (V \otimes J_2) + |h = \frac{21}{11.16}\rangle \otimes V(\Lambda_7 + 3\delta)$$

(34)

with $V = V(\Lambda_0 + \delta)$ and $J_2 \simeq |h = \frac{1}{2}\rangle \otimes V(\Lambda_8 + 2\delta)$. It is a simple exercise, which we leave to the reader, to expand this formula in terms of affine $E_8^{(1)}$ characters at level three times branching functions. Since the coset construction $(E_{8,k=1}^{(1)} \otimes E_{8,k=2}^{(1)})/E_{8,k=3}$ is a minimal model, the necessary branching functions can be written in terms of characters of $c = 1 - \frac{6}{11.16}$ Virasoro characters.

Clearly the same method can be applied to any extensions of an affine algebra by one of its basic representations.

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