1. Introduction

1.1. A dimension for a triangulated category has been introduced by Rouquier in [Ro], which gives a new invariant for algebras and algebraic varieties under derived equivalences. For related topics see also [BV], and [Hap], p.70.

Let $\mathcal{C}$ be a triangulated category with shift functor $[1]$, $\mathcal{I}$ and $\mathcal{J}$ full subcategories of $\mathcal{C}$. Denote by $\langle \mathcal{I} \rangle$ the smallest full subcategory of $\mathcal{C}$ containing $\mathcal{I}$ and closed under isomorphisms, finite direct sums, direct summands, and shifts. Any object of $\langle \mathcal{I} \rangle$ is isomorphic to a direct summand of a finite direct sum $\bigoplus I_i[n_i]$ with each $I_i \in \mathcal{I}$ and $n_i \in \mathbb{Z}$. Define $\mathcal{I} \ast \mathcal{J}$ to be the full subcategory of $\mathcal{C}$ consisting of the objects $M$, for which there is a distinguished triangle $I \to M \to J \to I[1]$ with $I \in \mathcal{I}$ and $J \in \mathcal{J}$. Now define $\langle \mathcal{I} \rangle_n := \{0\}$, and $\langle \mathcal{I} \rangle_n := (\langle \mathcal{I} \rangle_{n-1} \ast \langle \mathcal{I} \rangle)$ for $n \geq 1$. Then $\langle \mathcal{I} \rangle_1 = \langle \mathcal{I} \rangle$, and $\langle \mathcal{I} \rangle_n = (\langle \mathcal{I} \rangle \ast \cdots \ast \langle \mathcal{I} \rangle)$, by the associativity of $\ast$ (see [BV]). Note that $\langle \mathcal{I} \rangle_\infty := \bigcup_{n=0}^{\infty} \langle \mathcal{I} \rangle_n$ is the smallest thick triangulated subcategory of $\mathcal{C}$ containing $\mathcal{I}$.

By definition, the dimension of $\mathcal{C}$, denoted by $\dim(\mathcal{C})$, is the minimal integer $d \geq 0$ such that there exists an object $M \in \mathcal{C}$ with $\mathcal{C} = \langle M \rangle_{d+1}$, or $\infty$ when there is no such an object $M$. See [Ro].

Let $A$ be a finite-dimensional algebra over a field $k$. Denote by $A\text{-mod}$ the category of finite-dimensional left $A$-modules, and by $D^b(A\text{-mod})$ the bounded derived category. Define the derived dimension of $A$, denoted by $\text{der.dim}(A)$, to be the dimension of the triangulated category $D^b(A\text{-mod})$. By [Ro] and [KK] one has

$$\text{der.dim}(A) \leq \min\{l(A), \, \text{gl.dim}(A), \, \text{rep.dim}(A)\}$$

where $l(A)$ is the smallest integer $l \geq 0$ such that $\text{rad}^{l+1}(A) = 0$, $\text{gl.dim}(A)$ and $\text{rep.dim}(A)$ are the global dimension and the representation dimension of $A$ (for the definition of $\text{rep.dim}(A)$ see [Au]), respectively. In particular, we have $\text{der.dim}(A) < \infty$. 

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Our main result is

**Theorem** Let $A$ be a finite-dimensional algebra over an algebraically closed field $k$. Then $\text{der.dim}(A) = 0$ if and only if $A$ is an iterated tilted algebra of Dynkin type.

1.2. Let us fix some notation. For an additive category $\mathcal{A}$, denote by $C^*(\mathcal{A})$ the category of complexes of $\mathcal{A}$, where $* \in \{-, +, b\}$ means bounded-above, bounded-below, and bounded, respectively; and by $C(\mathcal{A})$ the category of unbounded complexes. Denote by $K^*(\mathcal{A})$ the corresponding homotopy category. If $\mathcal{A}$ is abelian, we have derived category $D^*(\mathcal{A})$.

For a finite-dimensional algebra $A$, denote by $A$-mod, $A$-proj and $A$-inj the category of finite-dimensional left $A$-modules, projective $A$-modules and injective $A$-modules, respectively.

For triangulated categories and derived categories we refer to [V], [Har], and [Hap]; for representation theory of algebras we refer to [ARS] and [Ri]; and for tilting theory we refer to [Ri] and [Hap], in particular, for iterated tilted algebras we refer to [Hap], p.171.

2. **Proof of Theorem**

Before giving the proof of Theorem, we make some preparations.

2.1. Let $A = \bigoplus_{j \geq 0} A(j)$ be a finite-dimensional positively-graded algebra over $k$, and $A$-gr the category of finite-dimensional left $\mathbb{Z}$-graded $A$-modules with morphisms of degree zero. An object in $A$-gr is written as $M = \bigoplus_{j \in \mathbb{Z}} M(j)$. For each $i \in \mathbb{Z}$, we have the degree-shift functor $(i) : A$-gr $\rightarrow$ $A$-gr, defined by $M(i)(j) = M(i+j), \forall j \in \mathbb{Z}$. Let $U : A$-gr $\rightarrow$ $A$-mod be the degree-forgetful functor. Then $U(M(i)) = U(M), \forall i \in \mathbb{Z}$. Clearly, $A$-gr is a Hom-finite abelian category, and hence by Remark A.2 in Appendix it is Krull-Schmidt. An indecomposable in $A$-gr is called a gr-indecomposable module. The category $A$-gr has projective covers and injective hulls. Assume that $\{e_1, e_2, \cdots, e_n\}$ is a set of orthogonal primitive idempotents of $A(0)$, such that $\{P_i := Ae_i = \bigoplus_{j \geq 0} A(j)e_i \mid 1 \leq i \leq n\}$ is a complete set of pairwise non-isomorphic indecomposable projective $A$-modules. Then $P_i$ (resp. $I_i := D(e_i, A) = \bigoplus_{j \leq 0} D(e_i, A_{i-j})$) is a projective (resp. an injective) object in $A$-gr. One deduces that $\{P_i(j) \mid 1 \leq i \leq n, j \in \mathbb{Z}\}$ is a complete set of pairwise non-isomorphic indecomposable projective objects in $A$-gr, and $\{I_i(j) \mid 1 \leq i \leq n, j \in \mathbb{Z}\}$ is a complete set of pairwise non-isomorphic indecomposable injective objects in $A$-gr.

Let $0 \neq M \in A$-gr. Define $t(M) := \max\{i \in \mathbb{Z} \mid M(i) \neq 0\}$ and $b(M) := \min\{i \in \mathbb{Z} \mid M(i) \neq 0\}$. For a graded $A$-module $M = \bigoplus_{i \in \mathbb{Z}} M(i)$, set $\text{top}(M) := M(\text{top}(M))$, and $\text{bot}(M) := M(\text{bot}(M))$, both of which are viewed as $A(0)$-modules. Denote by $\Omega^n$ (resp. $\Omega_{A(0)}$) the $n$-th syzygy functor on $A$-gr (resp. $A(0)$-mod), $n \geq 1$. Similarly we have $\Omega^{-n}$ and $\Omega_{A(0)}^{-n}$.

We need the following observation.

**Lemma 2.1.** Let $M$ be a non-zero non-projective and non-injective graded $A$-module. With notation above we have
(i) Either $b(\Omega(M)) = b(M)$ and $\bot(\Omega(M)) = \Omega_{A(0)}(\bot(M))$, or $b(\Omega(M)) > b(M)$.

(i') Either $t(\Omega^{-1}(M)) = t(M)$ and $\top(\Omega^{-1}(M)) = \Omega_{A(0)}^{-1}(\top(M))$, or $t(\Omega^{-1}(M)) < t(M)$.

**Proof.** We only justify (i). Note that $\rad(A) = \rad(A(0)) \oplus A(1) \oplus \cdots$, and that for a graded $A$-module $M$, the projective cover $P$ of $M/\rad(A)M$ in $A$-mod is graded. It follows that it gives the projective cover of $M$ in $A$-gr. Since $A$ is positively-graded, it follows that $b(P) = b(M)$, and that $\bot(P)$ is the projective cover of $\bot(M)$ as $A(0)$-modules. If $\bot(P) = \bot(M)$, then $b(\Omega(M)) > b(M)$. Otherwise, $b(\Omega(M)) = b(M)$ and $\bot(\Omega(M)) = \Omega_{A(0)}(\bot(M))$.

2.2. Let $A = \bigoplus_{j \geq 0} A_{(j)}$ be a finite-dimensional positively-graded algebra over $k$. The category $A$-gr is said to be locally representation-finite, provided that for each $i \in \mathbb{Z}$, the set

$$\{ [M] \mid M \text{ is gr-indecomposable such that } M_{(i)} \neq 0 \}$$

is finite, where $[M]$ denote the isoclass in $A$-gr of the graded module $M$. By degree-shifts, one sees that $A$-gr is locally representation-finite if and only if the set

$$\{ [M] \mid M \text{ is gr-indecomposable such that } M_{(0)} \neq 0 \}$$

is finite, if and only if $A$-gr has only finitely many indecomposable objects up to degree-shifts.

If $A$ is in addition self-injective, then $A$-gr is a Frobenius category. In fact, we already know that $A$-gr has enough projective objects and injective objects, and each indecomposable projective object is of the form $P_{(j)}$; since $A$ is self-injective, it follows that $P_{(j)}$ is injective in $A$-mod, so is $P_{(j)}$ in $A$-gr; similarly, each $I_{(j)}$ is a projective object in $A$-gr.

Note that the stable category $A$-gr is triangulated (see [Hap, Chap. 1, Sec. 2]), with shift functor induced by $\Omega^{-1}$.

**Proposition 2.2.** Let $A = \bigoplus_{j \geq 0} A_{(i)}$ be a finite-dimensional positively-graded algebra which is self-injective. Assume that $\dim(A$-gr) = 0 and $\gl\dim(A_{(0)}) < \infty$. Then $A$-gr is locally representation-finite.

**Proof.** Since $\dim(A$-gr) = 0, it follows that $A$-gr = $\langle X \rangle$ for some graded module $X$. Without loss of generality, we may assume that $X = \bigoplus_{l=1}^{r} M_{(i)}$, where $M_{(i)}$'s are pairwise non-isomorphic non-projective gr-indecomposable modules. It follows that every gr-indecomposable $A$-module is in the set $\{ \Omega_{(j)}^{l}(M_{(i)}) \mid i \in \mathbb{Z}, 1 \leq l \leq r, 1 \leq j \leq n \}$. Therefore, it suffices to prove that for each $1 \leq l \leq r$, the set

$$\{ j \in \mathbb{Z} \mid \Omega_{(l)}^{l}(M_{(i)}) \neq 0 \}$$

is finite.

For this, assume that $\gl\dim(A_{(0)}) = N$, $b(M_{(i)}) = j_{0}$, and $t(M_{(i)}) = i_{0}$. Since $\gl\dim(A_{(0)}) < \infty$, it follows from Lemma 2.1(i) that if $b(\Omega(M)) = b(M)$ then $p.d(\bot(\Omega(M))) = p.d(\bot(M)) - 1$ as $A_{(0)}$-modules, and otherwise $b(\Omega(M)) > b(M)$. By using Lemma 2.1(i) repeatedly we have

$$\text{if } j \geq \max\{ 1, -j_{0}N \}, \text{ then } b(\Omega^{l}(M_{(i)})) > 0.$$
Dually, if \( j \geq \max\{1, i_0 \} \), then \( t(\Omega^{-j}(M^i)) < 0 \). Note that \( b(\Omega^j(M^i)) > 0 \) (resp. \( t(\Omega^{-j}(M^i)) < 0 \)) implies that \( \Omega^j(M^i)(0) = 0 \) (resp. \( \Omega^{-j}(M^i)(0) = 0 \)). It follows that the set considered above is finite.

2.3. Let us recall some related notion in [BG] and [G]. Let \( A \) and \( \{e_1, e_2, \ldots, e_n\} \) be the same as in 2.1, and \( \mathbf{M} \) the full subcategory of \( A\text{-gr} \) consisting of objects \( \{P_j(i) \mid 1 \leq j \leq n, \ i \in \mathbb{Z} \} \). Then \( \mathbf{M} \) is a locally finite-dimensional in the sense of [BG]. One may identify \( A\text{-gr} \) with \( \text{mod}(\mathbf{M}) \) such that a graded \( A \)-module \( M \) is identified with a contravariant functor sending \( P_j(i) \) to \( e_j M_{(j-i)} \). Now it is direct to see that \( A\text{-gr} \) is locally representation-finite if and only if the category \( \mathbf{M} \) is locally representation-finite in the sense of [BG], p.337.

Let us follow [G], p.85-93. Let \( G \) be the group \( \mathbb{Z} \). Then \( G \) acts freely on \( \mathbf{M} \) by degree-shifts. Moreover, the orbit category \( \mathbf{M}/G \) can be identified with the full subcategory of \( A\text{-mod} \) consisting of \( \{P_j \mid 1 \leq j \leq n\} \). Hence we may identify \( \text{mod}(\mathbf{M}/G) \) with \( A\text{-mod} \). With these two identifications, the push-down functor \( F_\lambda : \text{mod}(\mathbf{M}) \rightarrow \text{mod}(\mathbf{M}/G) \) is nothing but the degree-forgetful functor \( U : A\text{-gr} \rightarrow A\text{-mod} \). The following is just a restatement of Theorem d) in 3.6 of [G].

**Lemma 2.3.** Let \( k \) be algebraically closed, and \( A \) be a finite-dimensional positively-graded \( k \)-algebra. Assume that \( A\text{-gr} \) is locally representation-finite. Then the degree-forgetful functor \( U \) is dense, and hence \( A \) is of finite representation type.

2.4. **Proof of Theorem:** If \( A \) is an iterated tilted algebra of Dynkin type, then by Theorem 2.10 in [Hap], p.109, we have a triangle-equivalence \( D^b(A\text{-mod}) \simeq D^b(kQ\text{-mod}) \) for some Dynkin quiver \( Q \). Note that \( kQ \) is of finite representation type, and that \( D^b(kQ\text{-mod}) = \langle M(0) \rangle \), where \( M \) is the direct sum of all the (finitely many) indecomposable \( kQ \)-modules. It follows that \( \text{der.dim}(A) = \text{der.dim}(kQ) = 0 \).

Conversely, if \( \text{dim}D^b(A\text{-mod}) = 0 \), it follows from the fact that \( D^b(A\text{-mod}) \) is Krull-Schmidt (see e.g. Theorem B.2 in Appendix) that \( D^b(A\text{-mod}) \) has only finitely many indecomposable objects up to shifts. Since \( K^b(A\text{-proj}) \) is a thick subcategory of \( D^b(A\text{-mod}) \), it follows that \( K^b(A\text{-proj}) \) has finitely many indecomposable objects up to shifts. Consequently, \( \text{s.gl.dim}(A) < \infty \) (for the definition of \( \text{s.gl.dim}(A) \) see B.3 in Appendix).

By Theorem 4.9 in [Hap], p.88, and Lemma 2.4 in [Hap], p.64, we have an exact embedding
\[
F : D^b(A\text{-mod}) \rightarrow T(A)\text{-gr},
\]
where \( T(A) = A \oplus DA \) is the trivial extension algebra of \( A \), which is graded with \( \text{deg}A = 0 \) and \( \text{deg}DA = 1 \). Since \( \text{gl.dim}A \leq \text{s.gl.dim}(A) - 1 < \infty \) (see Corollary B.3 in Appendix), it follows from Theorem 4.9 in [Hap] that the embedding \( F \) is an equivalence. Now by applying Proposition 2.2 to the graded algebra \( T(A) \) we know that \( T(A)\text{-gr} \) is locally representation-finite. It follows from Lemma 2.3 that \( T(A) \) is of finite representation type, and then the assertion follows from a theorem of Assem, Happel, and Roldán in [AHR], which says the trivial extension algebra \( T(A) \) is of finite representation type if and only if \( A \) is an iterated tilted algebra of Dynkin type (see also Theorem 2.1 in [Hap], p.199, and [HW]).

Appendix

This appendix includes an exposition on some material we used. They are well-known, however their proofs seem to be scattered in various literature.

A. Krull-Schmidt categories

This part includes a review of Krull-Schmidt categories.

A.1. An additive category \( C \) is **Krull-Schmidt** if any object \( X \) has a decomposition \( X = X_1 \oplus \cdots \oplus X_n \), such that each \( X_i \) is indecomposable with local endomorphism ring (see [Ri], p.52).

Directly by definition, a factor category (see [ARS], p.101) of a Krull-Schmidt category is Krull-Schmidt.

Let \( C \) be an additive category. An idempotent \( e = e^2 \in \text{End}_C(X) \) splits, if there are morphisms \( u : X \to Y \) and \( v : Y \to X \) such that \( e = vu \) and \( \text{Id}_Y = uv \). In this case, \( u \) (resp. \( v \)) is the cokernel (resp. kernel) of \( \text{Id}_X - e \); and \( \text{End}_C(Y) \simeq e \text{End}_C(X)e \) by sending \( f \in \text{End}_C(Y) \) to \( vf u \). If in addition \( \text{Id}_X - e \) splits via \( X \xrightarrow{\psi} Y' \xrightarrow{\psi'} X \), then \( \psi' X \simeq Y \oplus Y' \). One can prove directly that an idempotent \( e \) splits if and only if the kernel of \( \text{Id}_X - e \) exists, if and only if the kernel of \( \text{Id}_X - e \) exists. It follows that if \( C \) has cokernels (or kernels) then each idempotent in \( C \) splits; and that if each idempotent in \( C \) splits, then each idempotent in a full subcategory \( D \) splits if and only if \( D \) is closed under direct summands.

A ring \( R \) is **semiperfect** if \( R/\text{rad}(R) \) is semisimple and any idempotent in \( R/\text{rad}(R) \) can be lifted to \( R \), where \( \text{rad}(R) \) is the Jacobson radical.

**Theorem A.1.** An additive category \( C \) is Krull-Schmidt if and only if any idempotent in \( C \) splits, and \( \text{End}_C(X) \) is semiperfect for any \( X \in C \).

In this case, any object has a unique (up to order) direct decomposition into indecomposables.

**Proof.** For \( X \in C \) denote by \( \text{add}X \) the full subcategory of the direct summands of finite direct sums of copies of \( X \), and set \( R := \text{End}_C(X)^{op} \). Let \( \text{R-proj} \) denote the category of finitely-generated projective left \( R \)-modules. Consider the fully-faithful functor

\[
\Phi_X := \text{Hom}_C(X, \_): \text{add}X \to \text{R-proj}.
\]

Assume that \( C \) is Krull-Schmidt. Then \( X = X_1 \oplus \cdots \oplus X_n \) with each \( X_i \) indecomposable and \( \text{End}_C(X_i) \) local. Set \( P_i := \Phi_X(X_i) \). Then \( _RR = P_1 \oplus \cdots \oplus P_n \) with \( \text{End}_R(P_i) \simeq \text{End}_C(X_i) \) local. Thus \( R \) is semiperfect by Theorem 27.6(b) in [AF], and so is \( \text{End}_C(X) \simeq R^{op} \). Note that every object \( P \in \text{R-proj} \) is a direct sum of finitely many \( P_i \)'s: in fact, note that \( \{S_i := P_i/\text{rad}(P_i)\}_{1 \leq i \leq n} \) is the set of pairwise non-isomorphic simple \( R \)-modules and that the projection \( P \to \overline{P} := P/\text{rad}(P) = \bigoplus_i S_i^{m_i} \) is a projective cover, thus \( P \simeq \bigoplus_i P_i^{m_i} \). It follows that \( P \) is essentially contained in the image of \( \Phi_X \), and hence \( \Phi_X \) is an equivalence. Consider \( \text{R-Mod} \), the category of left \( R \)-modules. Since \( \text{R-Mod} \) is abelian, it follows that any idempotent in \( \text{R-Mod} \) splits. Since \( \text{R-proj} \) is a full subcategory of \( \text{R-Mod} \)
closed under direct summands, it follows that any idempotent in $R$-proj splits. So each idempotent in \( \text{add}(X) \) splits. This proves that any idempotent in $C$ splits.

Conversely, assume that each idempotent in $C$ splits and \( R^{\text{op}} = \text{End}_C(X) \) is semiperfect for each $X$. Then again by Theorem 27.6(b) in [AF] we have $R = Re_1 \oplus \cdots \oplus Re_n$ where each $e_i$ is idempotent such that $e_iRe_i$ is local. Since $1 = e_1 + \cdots + e_n$ and $e_i$ splits in $C$ as $X \xrightarrow{u_i} Y_i \xrightarrow{v_i} X$, it follows that $X \simeq Y_1 \oplus \cdots \oplus Y_n$ via the morphism $(u_1, \cdots, u_n)^t$ with inverse $(v_1, \cdots, v_n)$. Note that $\text{End}_C(Y_i) \simeq e_i\text{End}_C(X)e_i = (e_iRe_i)^{\text{op}}$ is local. This proves that $C$ is Krull-Schmidt.

For the last statement, it suffices to show the uniqueness of decomposition in $\text{add}X$ for each $X$. This follows from the fact that $\Phi_X$ is an equivalence, since the uniqueness of decomposition in $R$-proj is well known by Azumaya’s theorem (see e.g. Theorem 12.6(2) in [AF]). This completes the proof. $lacksquare$

A.2. Let $k$ be a field. An additive category $C$ is a Hom-finite $k$-category if $\text{Hom}_C(X,Y)$ is finite-dimensional $k$-space for any $X,Y \in C$, or equivalently, $\text{End}_C(X)$ is a finite-dimensional $k$-algebra for any object $X$.

**Corollary A.2.** Let $C$ be a Hom-finite $k$-category. Then the following are equivalent.

(i) $C$ is Krull-Schmidt.

(ii) Each idempotent in $C$ splits.

(iii) For any indecomposable $X \in C$, $\text{End}_C(X)$ has no non-trivial idempotents.

**Remark A.2.** By Corollary A.2 (ii), a Hom-finite abelian $k$-category is Krull-Schmidt.

In particular, the category of coherent sheaves on a complete variety is Krull-Schmidt (see [At], Theorem 2(i)).

**B. Homotopically-minimal complexes**

In this part $A$ is a finite-dimensional algebra over a field $k$.

**B.1.** A complex $P^\bullet = (P^n, d^n) \in C(A\text{-proj})$ is called homotopically-minimal provided that a chain map $\phi^\bullet : P^\bullet \rightarrow P^\bullet$ is an isomorphism if and only if it is an isomorphism in $K(A\text{-proj})$ (see [K]).

Applying Lemma B.1 and Proposition B.2 in [K], and duality, we have

**Proposition B.1.** (Krause) Let $P^\bullet = (P^n, d^n) \in C(A\text{-proj})$. The following statements are equivalent.

(i) The complex $P^\bullet$ is homotopically-minimal.

(ii) Each differential $d^n$ factors through $\text{rad}(P^{n+1})$.

(iii) The complex $P^\bullet$ has no non-zero direct summands in $C(A\text{-proj})$ which are null-homotopic.

Moreover, in $C(A\text{-proj})$ every complex $P^\bullet$ has a decomposition $P^\bullet = P'^\bullet \oplus P''^\bullet$ such that $P'^\bullet$ is homotopically-minimal and $P''^\bullet$ is null-homotopic.
B.2. For $P^\bullet \in C(A\text{-proj})$, consider the ideal of $\End_{C(A\text{-proj})}(P^\bullet)$:
\[ \Htp(P^\bullet) = \{ \phi^\bullet : P^\bullet \to P^\bullet \mid \phi^\bullet \text{ is homotopic to zero} \} . \]

Lemma B.2. Assume $\rad^1(A) = 0$. Let $P^\bullet$ be homotopically-minimal. Then $\Htp(P^\bullet)^k = 0$. 

Proof. Let $\phi^\bullet \in \Htp(P^\bullet)$ with homotopy $\{ h^n \}$. Then $\phi^n = d^{n-1}h^n + h^{n+1}d^n$. Since by assumption both $d^{n-1}$ and $d^n$ factor through radicals, it follows that $\phi^n$ factors through $\rad P^n$. Therefore, for $k \geq 1$ morphisms in $\Htp(P^\bullet)^k$ factor through the $k$-th radicals. So the assertion follows from $\rad^1(A) = 0$. □

Denote by $C^{-,b}(A\text{-proj})$ the category of bounded above complexes of projective modules with finitely many non-zero cohomologies, and by $K^{-,b}(A\text{-proj})$ its homotopy category. It is well-known that there is a triangle-equivalence $\mathbf{p} : D^b(A\text{-mod}) \simeq K^{-,b}(A\text{-proj})$.

The following result can be deduced from Corollary 2.10 in [BS]. See also [BD].

Theorem B.2. The bounded derived category $D^b(A\text{-mod})$ is Krull-Schmidt.

Proof. Clearly $D^b(A\text{-mod})$ is Hom-finite. By Corollary A.2 it suffices to show that $\End_{D^b(A\text{-mod})}(X^\bullet)$ has no non-trivial idempotents, for any indecomposable $X^\bullet$.

By Proposition B.1 we may assume that $P^\bullet := pX^\bullet$ is homotopically-minimal. Since $P^\bullet$ is indecomposable in $K^{-,b}(A\text{-proj})$, it follows from Proposition B.1(iii) that $P^\bullet$ is indecomposable in $C(A\text{-proj})$. Since idempotents in $C(A\text{-proj})$ split, it follows that $\End_{C(A\text{-proj})}(P^\bullet)$ has no non-trivial idempotents. Note that
\[ \End_{D^b(A\text{-mod})}(X^\bullet) = \End_{K^{-,b}(A\text{-proj})}(P^\bullet) = \End_{C(A\text{-proj})}(P^\bullet)/\Htp(P^\bullet). \]

Since by Lemma B.2 $\Htp(P^\bullet)$ is a nilpotent ideal, it follows that any idempotent in the quotient algebra $\End_{C(A\text{-proj})}(P^\bullet)/\Htp(P^\bullet)$ lifts to $\End_{C(A\text{-proj})}(P^\bullet)$. Therefore $\End_{C(A\text{-proj})}(P^\bullet)/\Htp(P^\bullet)$ has no non-trivial idempotents. □

B.3. For $X^\bullet = (X^n, d^n)$ in $C^b(A\text{-mod})$, define the width $w(X^\bullet)$ of $X^\bullet$ to be the cardinality of $\{ n \in \mathbb{Z} \mid X^n \neq 0 \}$. The strong global dimension $\sgl \dim(A)$ of $A$ is defined by (see [S])
\[ \sgl \dim(A) := \sup \{ w(X^\bullet) \mid X^\bullet \text{ is indecomposable in } C^b(A\text{-proj}) \} . \]

By Proposition B.1 an indecomposable $X^\bullet$ in $C^b(A\text{-proj})$ is either homotopically-minimal, or null-homotopic (thus it is of the form $\cdots \to 0 \to P \overset{\text{id}}{\to} P \to 0 \to \cdots$, for some indecomposable projective $A$-module $P$). So we have
\[ \sgl \dim(A) = \sup \{ 2, w(P^\bullet) \mid P^\bullet \text{ is homotopically-minimal and indecomposable in } C^b(A\text{-proj}) \} . \]

Let $M$ be an indecomposable $A$-module with minimal projective resolution $P^\bullet \overset{\epsilon}{\to} M$. Denote by $\tau_{\leq -m}P^\bullet$ the brutal truncation of $P^\bullet$, $m \geq 1$. By Proposition B.1(ii) $\tau_{\leq -m}P^\bullet$ is homotopically-minimal.

If $\tau_{\leq -m}P^\bullet = P'^\bullet \oplus Q'^\bullet$ in $C^b(A\text{-proj})$ with $P'^\bullet = (P^n, d^n)$, $P'^\bullet = (P'^n, \delta'^n)$, and $Q'^\bullet = (Q^n, \partial'^n)$, then both $P'^\bullet$ and $Q'^\bullet$ are homotopically-minimal. Assume that
$P^{t_0} \neq 0$, and set $t_0 := \max\{t \in \mathbb{Z} \mid Q^t \neq 0\}$. Then $-m \leq t_0 \leq 0$. Since $M$ is indecomposable and both $P^\bullet$ and $Q^\bullet$ are homotopically-minimal, it follows that $t_0 \neq 0$, and hence $Q^{t_0} \subseteq \text{Ker}d^{t_0} \subseteq \text{rad}(P^{t_0}) = \text{rad}(P^{t_0} \oplus Q^{t_0})$, a contradiction. This proves

Lemma B.3. The complex $\tau_{\geq -m} P^\bullet$ is homotopically-minimal and indecomposable in $C^b(A\text{-proj})$.

As a consequence we have

Corollary B.3. ([S], p.541) Let $A$ be a finite-dimensional algebra. Then

$$\text{s.gl.dim}(A) \geq \max(2, 1 + \text{gl.dim}(A)).$$

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