MODERATE DEVIATIONS FOR A FRACTIONAL STOCHASTIC HEAT EQUATION WITH SPATIALLY CORRELATED NOISE

YUMENG LI, RAN WANG, NIAN YAO, AND SHUGUANG ZHANG

Abstract: In this paper, we study the Moderate Deviation Principle for a perturbed stochastic heat equation in the whole space \( \mathbb{R}^d, d \geq 1 \). This equation is driven by a Gaussian noise, white in time and correlated in space, and the differential operator is a fractional derivative operator. The weak convergence method plays an important role.

Keyword: Fractional derivative operator \cdot stochastic heat equation \cdot moderate deviation principle \cdot weak convergence method.

Mathematics Subject Classification (2000) 60H15 \cdot 60F05 \cdot 60F10

1. INTRODUCTION

Since the work of Freidlin and Wentzell [16], the theory of small perturbation large deviations for stochastic (partial) differential equation has been extensively developed (see [2, 12]). The large deviation principle (LDP) for stochastic reaction-diffusion equations driven by the space-time white noise was first obtained by Freidlin [15] and later by Sowers [25], Chenal and Millet [8], Carrai and Röckner [6] and other authors. An LDP for a stochastic heat equation driven by a Gaussian noise, white in time and correlated in space was proved by Márquez-Carreras and Sarrà [23]. Recently, El Mellali and Mellouk proved an LDP for a fractional stochastic heat equation driven by a spatially correlated noise in [7].

Like the large deviations, the moderate deviation problems arise in the theory of statistical inference quite naturally. The moderate deviation principle (MDP) can provide us with the rate of convergence and a useful method for constructing asymptotic confidence intervals, see [14, 17, 19, 24, 28] and references therein. Results on the MDP for processes with independent increments were obtained in De Acosta [1], Ledoux [22] and so on. The study of the MDP estimates for other processes has been carried out as well, e.g., Wu [29] for Markov processes, Guillin and Liptser [18] for diffusion processes, Wang and Zhang [27] for stochastic reaction-diffusion equations in \( \mathbb{R} \), Budhiraja et al. [5] for stochastic differential equations with jumps, and references therein.

In this paper, we study the MDP for the fractional stochastic heat equation in spatial dimension \( \mathbb{R}^d \) driven by a spatially correlated noise. In [21], we studied the MDP for a perturbed stochastic heat equations defined on \([0, T] \times [0, 1]^d\), driven by a spatially correlated noise. In that paper, the method is the exponential approximation theorem (see [12 Theorem 4.2.13]), which needs some exponential estimates. However, due to the lack of good regularity properties of the Green function for the fractional heat
equation, it is difficult to get those exponential estimates. Instead of proving exponential estimates, we will use the weak convergence approach (see [4]) in this paper.

Now, let us give the fractional stochastic heat equation

\[
\begin{cases}
\frac{\partial u^\varepsilon}{\partial t}(t, x) = D_\alpha^{\varepsilon} u^\varepsilon(t, x) + b(u^\varepsilon(t, x)) + \sqrt{\varepsilon} \sigma(u^\varepsilon(t, x)) \dot{F}(t, x), \\
u^\varepsilon(0, x) = 0,
\end{cases}
\]

where \( \varepsilon > 0, (t, x) \in [0, T] \times \mathbb{R}^d, d \geq 1, \alpha = (\alpha_1, \ldots, \alpha_d), \delta = (\delta_1, \ldots, \delta_d) \) and we will assume that \( \alpha_i \in [0, 2] \setminus \{1\} \) and \( |\delta_i| \leq \min\{\alpha_i, 2 - \delta_i\}, i = 1, \ldots, d, \) \( F \) is the “formal” derivative of the Gaussian perturbation and \( D_\alpha^{\varepsilon} \) denotes a non-local fractional differential operator on \( \mathbb{R}^d \) defined by

\[
D_\alpha^{\varepsilon} := \sum_{i=1}^d D_{\delta_i}^{\alpha_i}.
\]

Here \( D_{\delta_i}^{\alpha_i} \) denotes the fractional differential derivative with respect to (w.r.t.) the \( i \)-th coordinate defined via its Fourier transform \( \mathcal{F} \) by

\[
\mathcal{F}(D_{\delta_i}^{\alpha_i} \phi)(\xi) = -|\xi|^{\alpha_i} \exp \left(-i\delta_i \frac{\pi}{2} \text{sgn} \xi \right) \mathcal{F}(\phi)(\xi),
\]

with \( i^2 + 1 = 0 \). The noise \( F(t, x) \) is a martingale measure in the sense of Walsh [26] and Dalang [10], which will be defined with details in the sequel. The coefficients \( b \) and \( \sigma : \mathbb{R} \to \mathbb{R} \) are given functions. From now on, we shall refer to Eq. (1) as \( E_{D_\alpha^{\varepsilon}}(d, b, \sigma) \). See Section 2 for details.

As the parameter \( \varepsilon \) tends to zero, the solutions \( u^\varepsilon \) of (1) will tend to the solution of the deterministic equation defined by

\[
\begin{cases}
\frac{\partial u^0}{\partial t}(t, x) = D_\alpha^{0} u^0(t, x) + b(u^0(t, x)), \\
u^0(0, x) = 0.
\end{cases}
\]

In this paper we shall investigate deviations of \( u^\varepsilon \) from the deterministic solution \( u^0 \), as \( \varepsilon \) decreases to 0, that is, the asymptotic behavior of the trajectories,

\[
\frac{1}{\sqrt{\varepsilon} \lambda(\varepsilon)} (u^\varepsilon - u^0)(t, x), \quad (t, x) \in [0, T] \times \mathbb{R}^d,
\]

where \( \lambda(\varepsilon) \) is some deviation scale, which strongly influences the asymptotic behavior.

The case \( \lambda(\varepsilon) = 1/\sqrt{\varepsilon} \) provides some large deviations estimates. Under suitable assumptions, El Mellali and Mellouk proved that the law of the solution \( u^\varepsilon \) satisfies a large deviation principle on the Hölder space in [7].

If \( \lambda(\varepsilon) \) is identically equal to 1, we are in the domain of the central limit theorem. To fill in the gap between the central limit theorem scale \([\lambda(\varepsilon) = 1]\) and the large deviations scale \([\lambda(\varepsilon) = 1/\sqrt{\varepsilon}]\), we will study moderate deviations, that is when the deviation scale satisfies

\[
\lambda(\varepsilon) \to +\infty, \quad \sqrt{\varepsilon} \lambda(\varepsilon) \to 0 \quad \text{as} \ \varepsilon \to 0.
\]
The moderate deviation principle enables us to refine the estimates obtained through the central limit theorem. It provides the asymptotic behavior for \( \mathbb{P} \left( \left\| u^\varepsilon - u^0 \right\| \geq \delta \sqrt{\varepsilon \lambda(\varepsilon)} \right) \) while the central limit theorem gives asymptotic bounds for \( \mathbb{P} \left( \left\| u^\varepsilon - u^0 \right\| \geq \delta \sqrt{\varepsilon} \right) \). Throughout this paper, we assume (3) is in place.

The rest of this paper is organized as follows. In Section 2, the precise framework is stated. In Section 3, the skeleton equation is studied. It is proved that the solution is a continuous map from the level set into the Hölder space. Section 4 is devoted to the proof of the moderate deviation principle by the weak convergence approach. We give some precise estimates of the fundamental solution \( G \) in the appendix.

Throughout the paper, \( C_p \) is a positive constant depending on the parameter \( p \), and \( C, C_1, \cdots \) are constants depending on no specific parameter (except \( T \) and the Lipschitz constants), whose value may be different from line to line by convention.

For any \( T > 0, K \subset \mathbb{R}^d, \beta = (\beta_1, \beta_2) \), let \( C^{\beta,0}(0, T] \times K; \mathbb{R}^d) \) be the Hölder space equipped with the norm defined by

\[
\| f \|_{\beta,K} := \sup_{(t,x)\in[0,T] \times K} |f(t,x)| + \sup_{s \neq t \in [0,T]} \sup_{x \neq y \in K} \frac{|f(t,x) - f(s,y)|}{|t-s|^{\beta_1} + |x-y|^{\beta_2}}.
\]

Since \( C^{\beta,0}(0, T] \times K; \mathbb{R}^d) \) is not separable, we consider the space \( C^{\beta,0}(0, T] \times K; \mathbb{R}^d) \) of Hölder continuous \( f \) with the degree \( \beta_i < \beta_i', i = 1, 2 \) such that

\[
\lim_{\delta \to 0^+} \left( \sup_{|t-s| + |x-y| < \delta} \frac{|f(t,x) - f(s,y)|}{|t-s|^{\beta_i} + |x-y|^{\beta_2}} \right) = 0
\]

and \( C^{\beta,0}(0, T] \times K; \mathbb{R}^d) \) is a Polish space containing \( C^{\beta}(0, T] \times K; \mathbb{R}^d) \). From now on, let \( \mathcal{E}^{\beta}(0, T] \times K; \mathbb{R}^d) := C^{\beta,0}(0, T] \times K; \mathbb{R}^d \), where \( \beta = (\beta_1, \beta_2) \).

2. Framework

In this section, let us give the framework taken from Boulanba et al. [3] and El Mellali and Mellouk [7].

2.1. The operator \( D^\alpha_\delta \). In one dimension space, the operator \( D^\alpha_\delta \) is a closed, densely defined operator on \( L^2(\mathbb{R}) \) and it is the infinitesimal generator of a semigroup which is in general not symmetric and not a contraction. It is self-adjoint only when \( \delta = 0 \) and in this case, it coincides with the fractional power of the Laplacian.

According to [13], [20], \( D^\alpha_\delta \) can be represented for \( 1 < \alpha < 2 \), by

\[
D^\alpha_\delta = \int_{-\infty}^{+\infty} \frac{\phi(x+y) - \phi(x) - y\phi'(x)}{|y|^{1+\alpha}} (\kappa_-^\delta 1_{(0,\infty)}(y) + \kappa_+^\delta 1_{(0,\infty)}(y))dy,
\]

and for \( 0 < \alpha < 1 \), by

\[
D^\alpha_\delta = \int_{-\infty}^{+\infty} \frac{\phi(x+y) - \phi(x)}{|y|^{1+\alpha}} (\kappa_-^\delta 1_{(0,\infty)}(y) + \kappa_+^\delta 1_{(0,\infty)}(y))dy,
\]

where \( \kappa_-^\delta \) and \( \kappa_+^\delta \) are two non-negative constants satisfying \( \kappa_-^\delta + \kappa_+^\delta > 0 \) and \( \phi \) is a smooth function for which the integral exists, and \( \phi' \) stands for its derivative. This
representation identifies it as the infinitesimal generator for a non-symmetric $\alpha$-stable Lévy process.

Let $G_{\alpha,\delta}(t,x)$ denotes the fundamental solution of the equation $Eq_{\delta,1}(1,0,0)$, that is, the unique solution of the Cauchy problem

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) = D_{\alpha}^\gamma u^\gamma(t,x), \\
u(0,x) = \delta_0(x), \quad t > 0, x \in \mathbb{R}, \end{cases}$$

where $\delta_0$ is the Dirac distribution. Using Fourier’s calculus one gets

$$G_{\alpha,\delta}(t,x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp \left( -ix - t|z|^\alpha \exp \left( -i\frac{\pi}{2} \text{sgn}(z) \right) \right) dz. \quad (5)$$

The relevant parameters, $\alpha$ called the index of stability and $\delta$ called the skewness, are real numbers satisfying $\alpha \in [0,2]$ and $|\delta| \leq \min\{\alpha, 2 - \alpha\}$.

Now, for higher dimension $d \geq 1$ and any multi index $\underline{\alpha} = (\alpha_1, \cdots, \alpha_d)$ and $\underline{\delta} = (\delta_1, \cdots, \delta_d)$, let $G_{\underline{\alpha},\underline{\delta}}(t,x)$ be the Green function of the deterministic equation $Eq_{\underline{\alpha},\underline{\delta}}(d,0,0)$. Clearly,

$$G_{\underline{\alpha},\underline{\delta}}(t,x) = \prod_{i=1}^{d} G_{\alpha_i,\delta_i}(t,x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{\alpha_1 d_1 \pi |\xi_1|^{\alpha_1} \exp \left( -i\frac{\pi}{2} \text{sgn}(\xi_1) \right)} d\xi, \quad (6)$$

where $\langle \cdot, \cdot \rangle$ stands for the inner product in $\mathbb{R}^d$.

The properties of the Green function $G_{\underline{\alpha},\underline{\delta}}(t,x)$ will be given in the appendix.

2.2. The driving noise $F$. Let us explicitly describe here the spatially homogeneous noise, see Dalang [10].

Let $S(\mathbb{R}^{d+1})$ be the space of Schwartz test functions. On a complete probability space $(\Omega, \mathcal{G}, \mathbb{P})$, the noise $F = \{F(\phi), \phi \in S(\mathbb{R}^{d+1})\}$ is assumed to be an $L^2(\Omega, \mathcal{G}, \mathbb{P})$-valued Gaussian process with mean zero and covariance functional given by

$$J(\varphi, \psi) := \mathbb{E} [F(\varphi)F(\psi)] = \int_{\mathbb{R}_+} ds \int_{\mathbb{R}^d} \left( \phi(s, \ast) * \tilde{\psi}(s, \ast) \right)(x) \Gamma(dx) ds, \quad \varphi, \psi \in S(\mathbb{R}^{d+1}),$$

where $\tilde{\psi}(s,x) := \psi(s,-x)$ and $\Gamma$ is a non-negative and non-negative definite tempered measure, therefore symmetric. The symbol $\ast$ denotes the convolution product and $\ast$ stands for the spatial variable.

Let $\mu$ be the spectral measure of $\Gamma$, which is also a trivial tempered measure, that is $\mu = F^{-1}(\Gamma)$ and this gives

$$J(\phi, \psi) = \int_{\mathbb{R}_+} ds \int_{\mathbb{R}^d} \mu(d\xi) \mathcal{F} \phi(s, \ast)(\xi) \mathcal{F} \tilde{\psi}(s, \ast)(\xi), \quad (7)$$

where $\bar{z}$ is the complex conjugate of $z$.

As in Dalang [10], the Gaussian process $F$ can be extended to a worthy martingale measure, in the sense of Walsh [26],

$$M := \{M_t(A), \ t \in \mathbb{R}_+, A \in \mathcal{B}_0(\mathbb{R}^d)\},$$
where $\mathcal{B}_b(\mathbb{R}^d)$ denotes the collection of all bounded Borel measurable sets in $\mathbb{R}^d$. Let $\mathcal{G}_t$ be the completion of the $\sigma$-field generated by the random variables $\{F(s, A); 0 \leq s \leq t, A \in \mathcal{B}_b(\mathbb{R}^d)\}$.

Then, Boulanba et al. \[3\] gave a rigorous meaning to the solution of equation $E_{\mathcal{F}_t}(d, b, \sigma)$ by means of a joint measurable and $\mathcal{G}_t$-adapted process $\{u^\varepsilon(t, x); (t, x) \in \mathbb{R}_+ \times \mathbb{R}^d\}$ satisfying, for each $t \geq 0$ and for almost all $x \in \mathbb{R}^d$ the following evolution equation

$$u^\varepsilon(t, x) = \sqrt{\varepsilon} \int_0^t \int_{\mathbb{R}^d} G_{\mathcal{F}_s}(t-s, x-y) \sigma(u^\varepsilon(s, y)) F(dsdy) + \int_0^t ds \int_{\mathbb{R}^d} G_{\mathcal{F}_s}(t-s, x-y) b(u^\varepsilon(s, y)) dy.$$  \hspace{1cm} (8)

In order to prove our main result, we are going to give other equivalent approach to the solution of $E_{\mathcal{F}_t}(d, b, \sigma)$, see [11]. To start with, let us denote by $H$ the Hilbert space obtained by the completion of $S(\mathbb{R}^d)$ with the inner production 

$$\langle \phi, \psi \rangle_H := \int_{\mathbb{R}^d} \Gamma(dx)(\phi \ast \tilde{\psi})(x) = \int_{\mathbb{R}^d} \mu(d\xi) F(\phi(\xi) \overline{\psi}(\xi)), \phi, \psi \in S(\mathbb{R}^d).$$

The norm induced by $\langle \cdot, \cdot \rangle_H$ is denoted by $\| \cdot \|_H$.

By the Walsh’s theory of the martingale measures [26], for $t \geq 0$ and $h \in H$ the stochastic integral

$$B_t(h) = \int_0^t \int_{\mathbb{R}^d} h(y) F(ds, dy),$$

is well-defined and the process $\{B_t(h); t \geq 0, h \in H\}$ is a cylindrical Wiener process on $H$, that is:

(a) for every $h \in H$ with $\|h\|_H = 1$, $\{B_t(h)\}_{t \geq 0}$ is a standard Wiener process,

(b) for every $t \geq 0$, $a, b \in \mathbb{R}$ and $f, g \in H$,

$$B_t(af + bg) = aB_t(f) + bB_t(g) \quad \text{almost surely.}$$

Let $\{e_k\}_{k \geq 1}$ be a complete orthonormal system (CONS) of the Hilbert space $H$, then

$$\left\{ B^k_t := \int_0^t \int_{\mathbb{R}^d} e_k(y) F(ds, dy); k \geq 1 \right\}$$

defines a sequence of independent standard Wiener processes and we have the following representation

$$B_t := \sum_{k \geq 1} B^k_t e_k.$$  \hspace{1cm} (9)

Let $\{\mathcal{F}_s\}_{s \in [0, T]}$ be the $\sigma$-field generated by the random variables $\{B^k_s; s \in [0, t], k \geq 1\}$. We define the predictable $\sigma$-field in $\Omega \times [0, T]$ generated by the sets $\{(s, t) \times A; A \in \mathcal{F}_s, 0 \leq s \leq t \leq T\}$. In the following, we can define the stochastic integral with
respect to cylindrical Wiener process \((B_t(h))_{t \geq 0}\) (see e.g. [9] or [11]) of any predictable square-integrable process with values in \(\mathcal{H}\) as follows

\[
\int_0^t \int_{\mathbb{R}^d} g \cdot dB := \sum_{k \geq 1} \int_0^t \langle g(s), e_k \rangle_{\mathcal{H}} dB^k_s.
\]

Note that the above series converges in \(L^2(\Omega, \mathcal{F}, \mathbb{P})\) and the sum does not depend on the selected CONS. Moreover, each summand, in the above series, is a classical Itô integral with respect to a standard Brownian motion, and the resulting stochastic integral is a real-valued random variable.

In the sequel, we shall consider the mild solution to equation \(Eq_{\alpha, \delta}(d, b, \sigma)\) given by

\[
u^\varepsilon(t, x) = \varepsilon \sum_{k \geq 1} \int_0^t \langle G_{\alpha, \delta}(t-s, x-s) \sigma(u^\varepsilon(s, *)) e_k \rangle_{\mathcal{H}} dB^k_s + \int_0^t \left[ G_{\alpha, \delta}(t-s) \ast b(u^\varepsilon(s, *)) \right](x) ds,
\]

for any \(t \in [0, T], x \in \mathbb{R}^d\).

2.3. Existence, uniqueness and Hölder regularity to equation. For a multi-index \(\underline{\alpha} = (\alpha_1, \cdots, \alpha_d)\) such that \(\alpha_i \in [0, 2] \setminus \{1\}, i = 1, \cdots, d\) and any \(\xi \in \mathbb{R}^d\), let

\[
S_{\underline{\alpha}}(\xi) = \sum_{i=1}^d |\xi_i|^{\alpha_i}.
\]

Assume the following assumptions on the functions \(\sigma, b\) and the measure \(\mu\):

(C): The functions \(\sigma\) and \(b\) are Lipschitz, that is there exists some constant \(L\) such that

\[
|\sigma(x) - \sigma(y)| \leq L|x - y|, \quad |b(x) - b(y)| \leq L|x - y|\]

for all \(x, y \in \mathbb{R}^d\).

(H\(\alpha, \eta\)): Let \(\underline{\alpha}\) as defined above and \(\eta \in [0, 1]\), it holds that

\[
\int_{\mathbb{R}^d} \frac{\mu(d\xi)}{(1 + S_{\underline{\alpha}}(\xi))^\eta} < +\infty.
\]

The last assumption stands for an integrability condition w.r.t. the spectral measure \(\mu\). Indeed, the following stochastic integral

\[
\int_0^T \int_{\mathbb{R}^d} G_{\alpha, \delta}(T-s, x-y) F(ds, dy)
\]

is well-defined if and only if

\[
\int_0^T ds \int_{\mathbb{R}^d} \mu(d\xi) |F G_{\alpha, \delta}(s, \ast)(\xi)|^2 < +\infty.
\]

More precisely, by [3, Lemma 1.2], there exist two positive constants \(c_1, c_2\) such that

\[
c_1 \int_{\mathbb{R}^d} \frac{\mu(d\xi)}{1 + S_{\underline{\alpha}}(\xi)} \leq \int_0^T ds \int_{\mathbb{R}^d} \mu(d\xi) |F G_{\alpha, \delta}(s, \ast)(\xi)|^2 \leq c_2 \int_{\mathbb{R}^d} \frac{\mu(d\xi)}{1 + S_{\underline{\alpha}}(\xi)}.
\]
Under the assumptions \((C)\) and \(\left(H^+_\eta\right)\), Boulanba et al. proved that Eq. (10) admits a unique solution \(u^\varepsilon\) such that
\[
\sup_{t \in [0,T]} \sup_{x \in \mathbb{R}^d} \mathbb{E}|u^\varepsilon(t, x)|^p < +\infty, \quad \forall T > 0, p \geq 2.
\]
See [3, Theorem 2.1]. Moreover, Theorem 3.1 in [3] tells us that the trajectories of the solution \(u^\varepsilon(t, x) : (t, x) \in \mathbb{R}_+ \times \mathbb{R}^d\) to Eq. (10) are \(\beta = (\beta_1, \beta_2)\)-Hölder continuous in \((t, x) \in [0, T] \times K\) for every \(K\) compact subset of \(\mathbb{R}^d\) and every \(\beta_1 \in (0, 1 - \eta/2), \beta_2 \in (0, \min\{\alpha_0(1 - \eta), 1/2\}\), where \(\alpha_0 := \min_{1 \leq i \leq d}\alpha_i\).

Consequently, the random field solution \(t \mapsto u^\varepsilon(t, x)\) to Eq. (10) lives in the Hölder space \(C^{\beta_1, \beta_2}_{\varepsilon, \eta} K\) for every \(K\) compact subset of \(\mathbb{R}^d\) and every \(\beta_1 \in (0, \min\{\alpha_0(1 - \eta), 1/2\}\), where \(\alpha_0 := \min_{1 \leq i \leq d}\alpha_i\).

Particularly, taking \(\varepsilon = 0\), the deterministic solution \(u^0\) to (2) has the following estimates
\[
\sup_{t \in [0,T]} \sup_{x \in \mathbb{R}^d} \mathbb{E}|u^0(t, x)|^p < +\infty, \quad \forall T > 0, p \geq 2.
\]
and \(\|u^0\|_{\beta, K} < \infty\) for any compact set \(K \subset \mathbb{R}^d\).

3. Skeleton equations

The purpose of this section is to study the skeleton equation, which will be used in the weak convergence approach.

From now on, we furthermore suppose that

\(\textbf{(D): The function } b \text{ is differentiable, and its derivative } b' \text{ is Lipschitz. More precisely, there exists a positive constant } L' \text{ such that } \left|b'(y) - b'(z)\right| \leq L'|y - z| \text{ for all } y, z \in \mathbb{R}.\)

(15)

Combined with the Lipschitz continuity of \(b\), we conclude that
\[
\|b'(z)\| \leq L, \quad \forall z \in \mathbb{R}.
\]

(16)

For \(T > 0\), let \(\mathcal{H}_T := L^2([0, T]; \mathcal{H})\), which is a real separable Hilbert space such that, if \(\varphi, \psi \in \mathcal{H}_T\),
\[
\langle \varphi, \psi \rangle_{\mathcal{H}_T} := \int_0^T \langle \varphi(s, \cdot), \psi(s, \cdot) \rangle_{\mathcal{H}} ds.
\]

Denote \(\| \cdot \|_{\mathcal{H}_T}\) the norm induced by \(\langle \cdot, \cdot \rangle_{\mathcal{H}_T}\). For any \(N > 0\), define
\[
\mathcal{H}_T^N := \{ h \in \mathcal{H}_T; \| h \|_{\mathcal{H}_T} \leq N\},
\]
and we consider that \(\mathcal{H}_T^N\) is endowed with the weak topology of \(\mathcal{H}_T\).

For any \(h \in \mathcal{H}_T\), consider the deterministic evolution equation (called Skeleton equation)
\[
Z^h(t, x) = \int_0^t \langle G_{\alpha, \delta}(t - s, x - \bullet) \sigma(u^0(s, \bullet)), h(s, \bullet) \rangle_{\mathcal{H}} ds
+ \int_0^t \left[ G_{\alpha, \delta}(t - s) \ast (b'(u^0(s, \bullet)Z^h(s, \bullet)) \right] (x) ds,
\]

(17)
where the first term on the right-hand side of the above equation can be written as

$$\sum_{k \geq 1} \int_0^t \langle G_{\alpha \delta}(t - s, x - \cdot)\sigma(u^0(s, \cdot)), e_k \rangle_H h_k(s) ds,$$

with $h_k(t) := \langle h(t), e_k \rangle_H, t \in [0, T], k \geq 1$.

Using the strategy in the proof of Proposition 2.7 in [7], one can obtain the following result. Here we omit its proof.

Proposition 3.1. Assuming conditions $(C), (H_{\alpha}^0)$ and $(D)$, there exists a unique solution $Z^h$ to Eq. (17), which satisfies

$$\sup_{h \in \mathcal{H}_T^N} \sup_{(t,x) \in [0,T] \times \mathbb{R}^d} |Z^h(t,x)| < +\infty. \quad (18)$$

Theorem 3.2. Assuming conditions $(C), (H_{\alpha}^0)$ and $(D)$, the mapping $h : \mathcal{H}_T^N \rightarrow Z^h \in \mathcal{E}^\beta([0,T] \times K; \mathbb{R}^d)$ is continuous with respect to the weak topology, where $K$ is a compact set in $\mathbb{R}^d$, $\beta = (\beta_1, \beta_2)$ satisfies that $0 < \beta_1 < \alpha_0(1-\eta)/2, 0 < \beta_2 < 1-\eta$ and $\alpha_0 = \min_{1 \leq i \leq d}(\alpha_i)$.

Proof. Let $0 < \beta_1 < \alpha_0(1-\eta)/2, 0 < \beta_2 < 1-\eta$ and $\{h, (h_n)_{n \geq 1}\} \subset \mathcal{H}_T^N$ such that for any $g \in \mathcal{H}_T$,

$$\lim_{n \to \infty} \langle h_n - h, g \rangle_{\mathcal{H}_T} = 0.$$

We need to prove that

$$\lim_{n \to \infty} \|Z^{h_n} - Z^h\|_{\beta, K} = 0. \quad (19)$$

According to Lemma 5.4 below (a particular case of Lemma A.1 in [2]), the proof of (19) can be divided into two steps:

1. **Pointwise convergence**: for any $(t,x) \in [0,T] \times K$,

$$\lim_{n \to \infty} |Z^{h_n}(t,x) - Z^h(t,x)| = 0. \quad (20)$$

2. **Estimation of the increments**: for any $(t,x), (s,y) \in [0,T] \times K$,

$$\sup_{n \geq 1} \left| (Z^{h_n}(t,x) - Z^h(t,x)) - (Z^{h_n}(s,y) - Z^h(s,y)) \right| \leq C \left( |t-s|^{\beta_1} + |x-y|^{\beta_2} \right). \quad (21)$$

We will prove those two estimates in the following two steps.

**Step 1.** **Pointwise convergence.** For any $(t,x) \in [0,T] \times K$,

$$Z^{h_n}(t,x) - Z^h(t,x)$$

$$= \int_0^t \langle G_{\alpha \delta}(t - s, x - \cdot)\sigma(u^0(s, \cdot)), h_n(s, \cdot) - h(s, \cdot) \rangle_H ds$$

$$+ \int_0^t \{G_{\alpha \delta}(t - s) \ast [b'(u^0(s, \cdot) (Z^{h_n}(s, \cdot) - Z^h(s, \cdot)))] \} (x) ds$$

$$=: I^1_1(t,x) + I^1_2(t,x). \quad (22)$$
Since $h_n, h \in \mathcal{H}^N_T$ and $u^0$ is bounded in $[0, T] \times \mathbb{R}^d$, by Cauchy-Schwarz's inequality on the Hilbert space $\mathcal{H}_T$, (C) and (12), we have

$$|I^n_1(t, x)|^2 \leq \int_0^t \|G_{\Delta}(t-s, x)\|_{\mathcal{H}^N}^2 \cdot \int_0^t \|h_n(s) - h(s)\|_{\mathcal{H}^N}^2 ds$$
$$\leq 4N^2 C \int_0^t \|G_{\Delta}(t-s, x)\|_{\mathcal{H}^N}^2 ds$$
$$\leq C(N) < +\infty,$$

here $C(N)$ is independent of $n, t, x$. Since $h_n \to h$ weakly in $\mathcal{H}^N_T$, we know that $I^n_1 \to 0$ in $C([0, T] \times \mathbb{R}^d; \mathbb{R})$ by Arzela-Ascoli Theorem. This implies that

$$\lim_{n \to \infty} \sup_{t \in [0, T], x \in \mathbb{R}^d} |I^n_1(t, x)| = 0. \quad (23)$$

Set $\zeta^n(t) := \sup_{0 \leq s \leq t, x \in \mathbb{R}^d} |Z^{h_n}(s, x) - Z^h(s, x)|$. By Lemma 5.1 and (16), we have

$$|I_2(t, x)| \leq \int_0^t \int_{\mathbb{R}^d} G_{\Delta}(t-s, x-y) |b'(u^0(s, y))(Z^{h_n}(s, y) - Z^h(s, y))| dy ds$$
$$\leq L \int_0^t \int_{\mathbb{R}^d} G_{\Delta}(t-s, x-y) \sup_{0 \leq s \leq t, y \in \mathbb{R}^d} |Z^{h_n}(s, y) - Z^h(s, y)| dy ds$$
$$\leq L \int_0^t \zeta^n(s) ds. \quad (24)$$

By (22) and (24), we have

$$\zeta^n(t) \leq L \int_0^t \zeta^n(s) ds + \sup_{t \in [0, T], x \in \mathbb{R}^d} |I^n_1(t, x)|.$$

Hence, by the Gronwall’s lemma and (23), we obtain that

$$\zeta^n(T) \leq e^{LT} \lim_{n \to \infty} \sup_{t \in [0, T], x \in \mathbb{R}^d} |I^n_1(t, x)| \to 0, \quad n \to \infty,$$

which is stronger than (20).
Step 2. Estimation of the increments. For any $0 \leq t \leq T$, $s > 0$, $x \in \mathbb{R}^d$, $y \in K$,

$$\left[ Z_{h,n}(t+s, x+y) - Z^h(t+s, x+y) \right] - \left[ Z_{h,n}(t, x) - Z^h(t, x) \right]$$

$$= \int_0^{t+s} \left\langle G_{\alpha \delta}(t+s-l, x+y-*) \sigma(u^0(l,*)), h_n(l,*) - h(l,*) \right\rangle_{\mathcal{H}} dl$$

$$+ \int_0^{t+s} \left\{ G_{\alpha \delta}(t+s-l) * \left[ b'(u^0(l,*)) \left( Z_{h,n}(l,*) - Z^h(l,*) \right) \right] \right\} (x+y) dl$$

$$- \int_0^{t} \left\langle G_{\alpha \delta}(t-l, x-*) \sigma(u^0(l,*)), h_n(l,*) - h(l,*) \right\rangle_{\mathcal{H}} dl$$

$$- \int_0^{t} \left\{ G_{\alpha \delta}(t-l) * \left[ b'(u^0(l,*)) \left( Z_{h,n}(l,*) - Z^h(l,*) \right) \right] \right\} (x) dl$$

$$= A^n_1 + A^n_2 + A^n_3 + A^n_4. \quad (25)$$

Since $u^0$ is bounded, $h_n, h \in \mathcal{H}_T^N$, we know that

$$\sup_{n \geq 1} \int_0^T \| u^0(s,*) (h^n - h)(s,*) \|^2_{\mathcal{H}} ds < \infty. \quad (26)$$

This, together with Cauchy-Schwarz’s inequality, (25) and Lemma 5.2 implies that for each $0 < \beta_1 < (1 - \eta)/2, 0 < \beta_2 < \min\{(1 - \eta)\alpha_0/2, 1/2\}$, there exists a constant $C$ independent of $n$ such that

$$|A^n_1| \leq C |y|^{\beta_2}, \quad |A^n_2| \leq C s^{\beta_1}, \quad |A^n_3| \leq C s^{\beta_1}. \quad (27)$$

Let us now give the estimate of $A^n_4$. Denote

$$V_n(t, x) := b'(u^0(t,x)) \left( Z_{h,n}(t,x) - Z^h(t,x) \right).$$

By the estimates in Step 1, (16) and (18), we know that

$$\sup_{n \geq 1} \sup_{t \in [0, T], x \in \mathbb{R}^d} |V_n(t, x)| < \infty. \quad (28)$$
After a change of variable, we have
\[ A_n^4 = \int_0^s \int_{\mathbb{R}^d} G_{\alpha,\beta}(t - s - l, x + y - z) V(l, z) dldz \]
\[ + \int_0^t \int_{\mathbb{R}^d} G_{\alpha,\beta}(t - l, x - z) [V(s + l, y + z) - V(l, z)] dldz. \]

By (i) of Lemma 5.1 and (28), we know that
\[ \sup_{n \geq 1} \int_0^s \int_{\mathbb{R}^d} G_{\alpha,\beta}(t - s - l, x + y - z) |V_n(l, z)| dldz \leq c_1 s. \]

By the Lipschitz continuity of \( b' \), the Hölder continuity of \( u^0 \) and the boundness of \( Z^{h_n} \) and \( Z^h \), we have
\[ |V(s + l, y + z) - V(l, z)| \]
\[ \leq |b'(u^0(s + l, y + z)) - b'(u^0(l, z))| \cdot |Z^{h_n}(s + l, y + z) - Z^h(s + l, y + z)| \]
\[ + |b'(u^0(l, z))| \cdot |Z^{h_n}(s + l, y + z) - Z^h(s + l, y + z)| - [Z^{h_n}(l, z) - Z^h(l, z)]| \]
\[ \leq C(L') |u^0(s + l, y + z) - u^0(l, z)| \]
\[ + |b'(u^0(l, z))| \cdot |Z^{h_n}(s + l, y + z) - Z^h(s + l, y + z)| - [Z^{h_n}(l, z) - Z^h(l, z)]| \]
\[ \leq c_1 (s^{\beta_1} + |y|^{\beta_2}) + c_2 \sup_{z \in \mathbb{R}^d} |Z^{h_n}(s + l, y + z) - Z^h(s + l, y + z)| - [Z^{h_n}(l, z) - Z^h(l, z)]|. \]

Putting
\[ \phi(l, s, y) := \sup_{z \in \mathbb{R}^d} \left\{ [Z^{h_n}(s + l, y + z) - Z^h(s + l, y + z)] - [Z^{h_n}(l, z) - Z^h(l, z)] \right\}. \]

Hence, by (25), (27) and (29), we have
\[ \phi(t, s, y) \leq c_3 (s + s^{\beta_1} + |y|^{\beta_2}) + c_2 \int_0^t \phi(l, s, y) dl. \]

Therefore by the Gronwall’s inequality,
\[ \sup_{0 \leq t \leq T} \sup_{x \in \mathbb{R}^d} |Z^{h_n}(t + s, x + y) - Z^h(t + s, x + y)| - [Z^{h_n}(t, x) - Z^h(t, x)]| \]
\[ \leq c(s + s^{\beta_1} + |y|^{\beta_2}). \]

The proof is complete. \( \square \)

4. Moderate deviation principle

The main aim of this paper is to prove that \( \lambda(\varepsilon) \) satisfies an LDP on the Hölder space, where \( \lambda(\varepsilon) \) satisfies (3). This special type of LDP is usually called the moderate deviation principle of \( u^\varepsilon \) (cf. [12]).
4.1. **Large Deviation Principle.** First, recall the definition of large deviation principle. See [12].

Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space with an increasing family \(\{\mathcal{F}_t\}_{0 \leq t \leq T}\) of the sub-\(\sigma\)-fields of \(\mathcal{F}\) satisfying the usual conditions. Let \(\mathcal{E}\) be a Polish space with the Borel \(\sigma\)-field \(\mathcal{B}(\mathcal{E})\).

**Definition 4.1.** A function \(I : \mathcal{E} \to \mathbb{R}_+\) is called a rate function on \(\mathcal{E}\), if for each \(M \leq \infty\), the level set \(\{x \in \mathcal{E} : I(x) \leq M\}\) is a compact subset of \(\mathcal{E}\). A family of positive numbers \(\{\lambda(\varepsilon)\}_{\varepsilon > 0}\) is called a speed function if \(\lambda(\varepsilon) \to +\infty\) as \(\varepsilon \to 0\).

**Definition 4.2.** A family \(\{X^\varepsilon\}\) of \(\mathcal{E}\)-valued random elements is said to satisfy the large deviation principle on \(\mathcal{E}\) with rate function \(I\) and with speed function \(\{\lambda(\varepsilon)\}_{\varepsilon > 0}\), if the following two conditions hold.

\(a\) (Large deviation upper bound) For each closed subset \(F\) of \(\mathcal{E}\),

\[
\limsup_{\varepsilon \to 0} \frac{1}{\lambda(\varepsilon)} \log \mathbb{P}(X^\varepsilon \in F) \leq - \inf_{x \in F} I(x).
\]

\(b\) (Large deviation lower bound) For each open subset \(G\) of \(\mathcal{E}\),

\[
\liminf_{\varepsilon \to 0} \frac{1}{\lambda(\varepsilon)} \log \mathbb{P}(X^\varepsilon \in G) \geq - \inf_{x \in G} I(x).
\]

4.2. **The main result.** From now on, we always assume that \(K\) is a compact set in \(\mathbb{R}^d\), \(\alpha = (\alpha_1, \cdots, \alpha_d), \alpha_i \in [0, 2] \setminus \{1\}\) for \(i = 1, \cdots, d\), \(\beta = (\beta_1, \beta_2)\) satisfies that \(0 < \beta_1 < \alpha_0(1 - \eta)/2, 0 < \beta_2 < 1 - \eta\) where \(\alpha_0 = \min_{1 \leq i \leq d} \{\alpha_i\}, \eta \in ]0, 1]\).

The main result of this paper is the following theorem.

**Theorem 4.1.** Assuming conditions \((C), (H_{\alpha}^0)\) and \((D)\) for \(\eta \in ]0, 1]\). Let \(u^\varepsilon\) be the solution of Eq. (10). Then the law of \((u^\varepsilon - u^h) / (\sqrt{\varepsilon} \lambda(\varepsilon))\) obeys an LDP on the space \(\mathcal{E}^\beta([0, T] \times K; \mathbb{R}^d)\) with speed \(\lambda^2(\varepsilon)\) and with rate function

\[
I(f) = \inf_{h \in \mathcal{H}_T; Z^h = f} \left\{ \frac{1}{2} \|h\|_{\mathcal{H}_T}^2 \right\}, \tag{30}
\]

where \(Z^h\) is defined by (17).

4.3. **Proof of Theorem 4.1.**

We shall apply the weak convergence approach to establish moderate deviation principle.

Denote by \(\mathcal{A}_\mathcal{H}\) the set of predictable process which belongs to \(L^2(\Omega \times [0, T]; \mathcal{H})\). For any \(N > 0\), let

\[
\mathcal{A}_\mathcal{H}^N := \{ h \in \mathcal{H}_T; \|h\|_{\mathcal{H}_T} \leq N \}.
\]
For any $v \in \mathcal{A}_H^N$ and $\varepsilon \in (0, 1]$, define the controlled equation $Z^{\varepsilon, v}$ by

$$Z^{\varepsilon, v}(t, x) = \frac{1}{\lambda(\varepsilon)} \sum_{k \geq 1} \int_0^t \langle G_{\vartheta}(t-s, x-*) \sigma(u^0(s, *) + \sqrt{\varepsilon} \lambda(\varepsilon) Z^{\varepsilon, v}(s, *)), e_k \rangle dF^k_s$$

$$+ \int_0^t \langle G_{\vartheta}(t-s, x-*) \sigma(u^0(s, *) + \sqrt{\varepsilon} \lambda(\varepsilon) Z^{\varepsilon, v}(s, *)), v(s, *) \rangle dF^k_s$$

$$+ \int_0^t \left\{ G_{\vartheta}(t-s) \ast \frac{b(u^0(s, *)) + \sqrt{\varepsilon} \lambda(\varepsilon) Z^{\varepsilon, v}(s, *) - b(u^0(s, *))}{\sqrt{\varepsilon} \lambda(\varepsilon)} \right\} ds.$$

(31)

Following the proof of Theorem 2.1 in [3] and Proposition 2.7 in [7], one can show the existence and uniqueness of the stochastic controlled equation given by (31).

**Lemma 4.2.** Assuming conditions (C), $(H^N_H)$ and (D) for $\eta \in [0, 1]$, there exists a unique random field solution to Eq. (31), $\{Z^{\varepsilon, v}(t, x); (t, x) \in [0, T] \times \mathbb{R}^d\}$, which satisfies that for any $p \geq 1$,

$$\sup_{\varepsilon \in [0, 1]} \sup_{v \in \mathcal{A}_H^N} \sup_{(t, x) \in [0, T] \times \mathbb{R}^d} \mathbb{E}[|Z^{\varepsilon, v}(t, x)|^p] < \infty.$$

(32)

Inspiring by [4], let us consider the following two conditions which correspond on the weak convergence approach frame in our setting. Also refer to the weak convergence approach used in [7].

(a) The set $\{Z^h; h \in \mathcal{H}_T^N\}$ is a compact set of $\mathcal{E}_\beta$, where $Z^h$ is the solution of Eq. (17).

(b) For any family $\{v^\varepsilon; \varepsilon > 0\} \subset \mathcal{A}_H^N$ which converges in distribution as $\varepsilon \to 0$ to $v \in \mathcal{A}_H^N$, as $\mathcal{H}_T^N$-valued random variables, we have

$$\lim_{\varepsilon \to 0} Z^{\varepsilon, v^\varepsilon} = Z^v$$

in distribution, as $\mathcal{E}_\beta$-valued random variables, where $Z^v$ denotes the solution of Eq. (17) corresponding to the $\mathcal{H}_T^N$-valued random variable $v$ (instead of a deterministic function $h$).

**The proof of Theorem 4.2** Applying to Theorem 6 in [4], a verification of conditions (a) and (b) implies the validity of Theorem 4.1. Condition (a) follows from the continuity of the mapping $h : \mathcal{H}_T^N \to Z^h \in \mathcal{E}_\beta([0, T] \times K; \mathbb{R}^d)$, which has been established in Theorem 3.2. Next, we verify Condition (b).

By the Skorokhod representation theorem, there exist a probability $(\bar{\Omega}, \bar{\mathcal{F}}, (\bar{\mathcal{F}}_t), \bar{\mathbb{P}})$, and, on this basis, a sequence of independent Brownian motions $\bar{B} = (\bar{B}_k)_{k \geq 1}$ and also a family of $\mathcal{F}_t$-predictable processes $\{\bar{v}^\varepsilon; \varepsilon > 0\}$, $\bar{v}$ belonging to $L^2(\bar{\Omega} \times [0, T]; \mathcal{H})$ taking values on $\mathcal{H}_T^N$, $\bar{\mathbb{P}}$-a.s., such that the joint law of $(v^\varepsilon, v, B)$ under $\bar{\mathbb{P}}$ coincides with that of $(\bar{v}^\varepsilon, \bar{v}, \bar{B})$ under $\bar{\mathbb{P}}$ and

$$\lim_{\varepsilon \to 0} \langle \bar{v}^\varepsilon - \bar{v}, g \rangle_{\mathcal{H}_T} = 0, \ \forall g \in \mathcal{H}_T, \bar{\mathbb{P}}$ - a.s..
Let $\tilde{Z}^{\varepsilon, \bar{v}}$ be the solution to a similar equation as (31) replacing $v$ by $\bar{v}$ and $B$ by $\bar{B}$. Thus, to verify the condition (b), it is sufficient to prove that

$$\lim_{\varepsilon \to 0} \| \tilde{Z}^{\varepsilon, \bar{v}} - \tilde{Z}^{\bar{v}} \|_{\beta, K} = 0 \text{ in probability.} \quad (33)$$

From now on, we drop the bars in the notation for the sake of simplicity, and we denote

$$Y^{\varepsilon, \bar{v}, v} := Z^{\varepsilon, \bar{v}} - Z^v.$$

According to Lemma 5.4, the proof of (33) can be divided into two steps: for any $(t, x), (s, y) \in [0, T] \times K$ with $K$ being compact in $\mathbb{R}^d$,

1. **Pointwise convergence:**

$$\lim_{\varepsilon \to 0} |Y^{\varepsilon, \bar{v}, v}(t, x)| = 0 \text{ in probability.} \quad (34)$$

2. **Estimation of the increments:** there exists a constant $C$ satisfied that

$$\sup_{\varepsilon \in \varepsilon_1} \mathbb{E} \left| Y^{\varepsilon, \bar{v}, v}(t, x) - Y^{\varepsilon, \bar{v}, v}(s, y) \right|^2 \leq C \left[ |t - s|^{\beta_1} + |x - y|^{\beta_2} \right] \quad (35)$$

Those two estimates will be established in the following steps.

**Step 1. Pointwise convergence.** For any $(t, x) \in [0, T] \times K$,

$$Z^{\varepsilon, \bar{v}}(t, x) - Z^v(t, x)$$

$$= \frac{1}{\lambda(\varepsilon)} \sum_{k \geq 1} \int_0^t \left\{ \langle G_{\alpha, \delta}(t - s, x - \star) \sigma(u^0(s, \star) + \sqrt{\varepsilon} \lambda(\varepsilon) Z^{\varepsilon, \bar{v}}(s, \star)), e_k \rangle_H dB^k_s \right. 

+ \left\{ \int_0^t \langle G_{\alpha, \delta}(t - s, x - \star) \sigma(u^0(s, \star) + \sqrt{\varepsilon} \lambda(\varepsilon) Z^{\varepsilon, \bar{v}}(s, \star)), v^\varepsilon(s, \star) \rangle_H ds 

- \int_0^t \langle G_{\alpha, \delta}(t - s, x - \star) \sigma(u^0(s, \star)), v(s, \star) \rangle_H ds \right\} 

+ \left\{ \int_0^t \left\{ G_{\alpha, \delta}(t - s) \ast \frac{b(u^0(s, \star) + \sqrt{\varepsilon} \lambda(\varepsilon) Z^{\varepsilon, \bar{v}}(s, \star)) - b(u^0(s, \star))}{\sqrt{\varepsilon} \lambda(\varepsilon)} \right\} (x) ds 

- \int_0^t \left\{ G_{\alpha, \delta}(t - s) \ast [b'(u^0(s, \star))] Z^v(s, \star) \right\} (x) ds \right\} 

= A^\varepsilon_1(t, x) + A^\varepsilon_2(t, x) + A^\varepsilon_3(t, x). \quad (36)$$

Let

$$\mathcal{J}(t) := \int_{\mathbb{R}^d} \mu(d\xi) |\mathcal{F}G_{\alpha, \delta}(t)(\xi)|^2. \quad (37)$$
For the first term \( A_1^\varepsilon \), by Burkholder’s inequality, the Lipschitz property of \( \sigma \), Eq. (14) and (32), we have that
\[
\mathbb{E}\left[|A_1^\varepsilon(t, x)|^2\right] = \lambda^{-2}(\varepsilon)\mathbb{E}\left[\int_0^t \|G_{\alpha, \delta}(t-s, x-\ast)\sigma(u^0(s, \ast) + \sqrt{\varepsilon}\lambda(\varepsilon)Z^{\varepsilon,v}(s, \ast))\|_{\mathcal{H}}^2 ds\right]
\leq \lambda^{-2}(\varepsilon) c_1 \int_0^t ds \left(1 + \sup_{(r, y) \in [0, s] \times \mathbb{R}^d} \mathbb{E}[u^0(r, y) + \sqrt{\varepsilon}\lambda(\varepsilon)Z^{\varepsilon,v}(s, y)]^2\right) \times \mathcal{J}(t-s)
\leq c_2 \lambda^{-2}(\varepsilon).
\]

The second term is further divided into two terms:
\[
|A_2^\varepsilon| \leq \int_0^t |\langle G_{\alpha, \delta}(t-s, x-\ast) [\sigma(u^0(s, \ast) + \sqrt{\varepsilon}\lambda(\varepsilon)Z^{\varepsilon,v}(s, \ast)) - \sigma(u^0(s, \ast))], v^\varepsilon(s, \ast) \rangle_{\mathcal{H}}| \; ds
\]
\[
+ \int_0^t |\langle G_{\alpha, \delta}(t-s, x-\ast) \sigma(u^0(s, \ast)), v^\varepsilon(s, \ast) - v(s, \ast) \rangle_{\mathcal{H}}| \; ds,
\]

By the Lipschitz condition of \( \sigma \), (32) and Cauchy-Schwarz’s inequality, we have
\[
\mathbb{E}[A_{2,1}^\varepsilon(t, x)]^2 \leq \varepsilon \lambda^2(\varepsilon)L^2 \left(\int_0^t \sup_{(r, y) \in [0, s] \times \mathbb{R}^d} \mathbb{E}[|Z^{\varepsilon,v}(r, y)|^2] \times \mathcal{J}(t-s)ds\right) \times \left(\int_0^t \|v^\varepsilon(s, \ast)\|_{\mathcal{H}}^2 ds\right) \frac{1}{2}
\leq \varepsilon \lambda^2(\varepsilon)C(L, N).
\]

By the similar arguments as in the proof of (23), we can show that
\[
\lim_{\varepsilon \to 0} \sup_{(t, x) \in [0, T] \times \mathbb{R}^d} |A_{2,2}^\varepsilon(t, x)| \to 0 \text{ in Probability.}
\]

For the third term \( A_3^\varepsilon \), by the Taylor’s formula, the Lipschitz continuity of \( b' \) and (16), we have
\[
|A_3^\varepsilon(t, x)| \leq \int_0^t \left\{ G_{\alpha, \delta}(t-s) \ast \left[ \frac{b(u^0(s, \ast) + \sqrt{\varepsilon}\lambda(\varepsilon)Z^{\varepsilon,v}(s, \ast)) - b(u^0(s, \ast))}{\sqrt{\varepsilon}\lambda(\varepsilon)} \right] \right\} ds
\]
\[
+ \int_0^t \left\{ G_{\alpha, \delta}(t-s) \ast \left[ b'(u^0(s, \ast))(Z^{\varepsilon,v}(s, \ast) - Z^v(s, \ast)) - Z^v(s, \ast) \right] \right\} (x) ds
\]
\[
\leq L' \sqrt{\varepsilon}\lambda(\varepsilon) \int_0^t G_{\alpha, \delta}(t-s) \ast |Z^{\varepsilon,v}(s, \ast)| (x) ds
\]
\[
+ L \int_0^t G_{\alpha, \delta}(t-s) \ast |Z^{\varepsilon,v}(s, \ast) - Z^v(s, \ast)| (x) ds.
\]
This inequality, together with (42), implies that

\[ 
|A_{\varepsilon}^{3}(s, x)|^2 \leq 2L^2 \varepsilon \lambda^2(\varepsilon) \int_{0}^{t} \int_{\mathbb{R}^d} \mathbf{G}_{\varepsilon}(t - s, x - y) \sup_{(r, z) \in [0, s] \times \mathbb{R}^d} \mathbb{E} \left[ |Z^{\varepsilon, \varepsilon}(r, z)|^2 \right] ds dy 
+ 2L^2 \int_{0}^{t} \int_{\mathbb{R}^d} \mathbf{G}_{\varepsilon}(t - s, y) \sup_{(r, z) \in [0, s] \times \mathbb{R}^d} \mathbb{E} \left[ |Z^{\varepsilon, \varepsilon}(r, z) - Z^{\varepsilon}(r, z)|^2 \right] ds dy 
\leq C(L') \varepsilon \lambda^2(\varepsilon) + 2L^2 \int_{0}^{t} \sup_{(r, z) \in [0, s] \times \mathbb{R}^d} \mathbb{E} \left[ |Y^{\varepsilon, \varepsilon, \varepsilon}(r, z)|^2 \right] ds. \tag{41} 
\]

Define the stopping time

\[ 
\tau_{\varepsilon} := \inf \left\{ t \leq T; \sup_{(s, x) \in [0, t] \times \mathbb{R}^d} |A_{\varepsilon}^3(s, x)| \geq M \right\}, 
\]

where \(M\) is some constant large enough.

Putting (36), (38)-(41) together, we have

\[ 
\sup_{(s, x) \in [0, t] \times \mathbb{R}^d} \mathbb{E} \left[ |Y^{\varepsilon, \varepsilon, \varepsilon}(s \wedge \tau_{\varepsilon}^M, x)|^2 \right] 
\leq C \left( \sup_{(s, x) \in [0, t] \times \mathbb{R}^d} \mathbb{E} \left[ |A_{1}^\varepsilon(s \wedge \tau_{\varepsilon}^M, x)|^2 \right] + \sup_{(s, x) \in [0, t] \times \mathbb{R}^d} \mathbb{E} \left[ |A_{2,1}^\varepsilon(s \wedge \tau_{\varepsilon}^M, x)|^2 \right] 
+ \sup_{(s, x) \in [0, t] \times \mathbb{R}^d} \mathbb{E} \left[ |A_{2,2}^\varepsilon(s \wedge \tau_{\varepsilon}^M, x)|^2 \right] + \mathbb{E} \left[ |A_{3}^\varepsilon(t \wedge \tau_{\varepsilon}^M, x)|^2 \right] 
+ \int_{0}^{t} \sup_{(r, z) \in [0, s] \times \mathbb{R}^d} \mathbb{E} \left[ |Y^{\varepsilon, \varepsilon, \varepsilon}(r \wedge \tau_{\varepsilon}^M, z)|^2 \right] ds \right). \]

By Gronwall’s lemma, we obtain that

\[ 
\sup_{(s, x) \in [0, T] \times \mathbb{R}^d} \mathbb{E} \left[ |Y^{\varepsilon, \varepsilon, \varepsilon}(s \wedge \tau_{\varepsilon}^M, x)|^2 \right] 
\leq C(T, L, L') \left( \lambda^{-2}(\varepsilon) + 2\varepsilon \lambda^2(\varepsilon) + \sup_{(s, x) \in [0, T] \times \mathbb{R}^d} \mathbb{E} \left[ |A_{2,2}^\varepsilon(s \wedge \tau_{\varepsilon}^M, x)|^2 \right] \right). \tag{42} 
\]

Since \(|A_{2,2}^\varepsilon(s \wedge \tau_{\varepsilon}^M, x)| \leq M\) for any \(s \in [0, T]\), by (40) and the dominated convergence theorem, we know that

\[ 
\sup_{(s, x) \in [0, T] \times \mathbb{R}^d} \mathbb{E} \left[ |A_{2,2}^\varepsilon(s \wedge \tau_{\varepsilon}^M, x)|^2 \right] \longrightarrow 0, \quad \text{as } \varepsilon \to 0. \]

This inequality, together with (42), implies that

\[ 
\sup_{(s, x) \in [0, T] \times \mathbb{R}^d} \mathbb{E} \left[ |Y^{\varepsilon, \varepsilon, \varepsilon}(s \wedge \tau_{\varepsilon}^M, x)|^2 \right] \longrightarrow 0, \quad \text{as } \varepsilon \to 0. \tag{43} 
\]
Letting $M \to \infty$, we obtain that for any $(t, x) \in [0, T] \times \mathbb{R}^d$,

$$\lim_{\varepsilon \to 0} |Y^{\varepsilon, v}(t, x)| = 0 \quad \text{in probability.}$$

**Step 2. Estimation of the increments.**
For any $(t, x), (s, y) \in [0, T] \times K$ with $K$ being compact in $\mathbb{R}^d$ and $t \geq s$,

$$Y^{\varepsilon, v}(t, x) - Y^{\varepsilon, v}(s, y)$$

$$= \frac{1}{\lambda(\varepsilon)} \sum_{k \geq 1} \left\{ \int_0^t \langle G_{\alpha, \delta}(t - r, x - *) \sigma(u^0(r, *) + \sqrt{\varepsilon} \lambda(\varepsilon)Z^{\varepsilon, v}(r, *)), e_k \rangle \mathcal{H} dB_r^k ight. $$

$$- \int_0^s \langle G_{\alpha, \delta}(s - r, y - *) \sigma(u^0(r, *) + \sqrt{\varepsilon} \lambda(\varepsilon)Z^{\varepsilon, v}(r, *)), e_k \rangle \mathcal{H} dB_r^k \}
$$

$$+ \left\{ \int_0^t \langle G_{\alpha, \delta}(t - r, x - *) \sigma(u^0(r, *) + \sqrt{\varepsilon} \lambda(\varepsilon)Z^{\varepsilon, v}(r, *)), v^\varepsilon(r, *) \rangle \mathcal{H} \, dr ight. $$

$$- \int_0^s \langle G_{\alpha, \delta}(s - r, y - *) \sigma(u^0(r, *) + \sqrt{\varepsilon} \lambda(\varepsilon)Z^{\varepsilon, v}(r, *)), v^\varepsilon(r, *) \rangle \mathcal{H} \, dr \}
$$

$$- \left\{ \int_0^t \langle G_{\alpha, \delta}(t - r, x - *) \sigma(u^0(r, *)) \, v(r, *) \rangle \mathcal{H} \, dr ight. $$

$$- \int_0^s \langle G_{\alpha, \delta}(s - r, y - *) \sigma(u^0(r, *)) \, v(r, *) \rangle \mathcal{H} \, ds \}
$$

$$+ \left\{ \int_0^t G_{\alpha, \delta}(t - r) \left[ b(u^0(r, *) + \sqrt{\varepsilon} \lambda(\varepsilon)Z^{\varepsilon, v}(r, *)) - b(u^0(r, *)) \right] (x) \, dr ight. $$

$$- \int_0^s G_{\alpha, \delta}(s - r) \left[ b(u^0(r, *) + \sqrt{\varepsilon} \lambda(\varepsilon)Z^{\varepsilon, v}(r, *)) - b(u^0(r, *)) \right] (y) \, dr \}
$$

$$- \left\{ \int_0^t G_{\alpha, \delta}(t - r) \left[ b'(u^0(r, *))Z^v(r, *) \right] (x) \, dr ight. $$

$$- \int_0^s G_{\alpha, \delta}(s - r) \left[ b'(u^0(r, *))Z^v(r, *) \right] (y) \, dr \}
$$

$$=: \frac{1}{\lambda(\varepsilon)} B_1^{\varepsilon}(t, s, x, y) + B_2^{\varepsilon}(t, s, x, y) + B_3^{\varepsilon}(t, s, x, y) + B_4^{\varepsilon}(t, s, x, y) + B_5^{\varepsilon}(t, s, x, y).$$
For the first term, by the Lipschitz continuity of $\sigma$, (32) and Lemma 5.2, we obtain that for any $\beta_1 \in \mathbb{R}$, $\beta_2 < \frac{1}{2}$

$$E \left[ |B^\varepsilon(t, s, y)|^2 \right] \leq C_1 E \left[ \int_0^t \|G_{\Delta\delta}(t - r, r, \star) - G_{\Delta\delta}(s - r, r, \star) \|^2 d\mathcal{H} dr \right]$$

where $\Delta$ is independent of $\varepsilon$, $\varepsilon > 0$.

Using the similar argument in (44) and noticing the fact $v^\varepsilon, v \in \mathcal{A}^N$, we give the following estimates without the proof

$$E \left[ B^\varepsilon_i(t, s, y) \right] \leq C(t - s)^{2\beta_1} + |x - y|^{2\beta_2}, \quad i = 2, 3,$$

where $C$ is independent of $\varepsilon > 0$.

where in the last inequality we have used the estimate in the proof of Theorem 3.1 in [3], $C$ is independent of $\varepsilon$.

Putting together all the estimates, we get (35). The proof is complete. $\Box$
5. Appendix

To make reading easier, we present here some results on the kernel $G$ from Boulanba et al. \cite{3}.

**Lemma 5.1.** \cite{3, Lemma 1.1} For $\alpha \in [0, 2] \setminus \{1\}$ such that $|\delta| \leq \min\{\alpha, 2 - \alpha\}$, the following statements hold.

(i) The function $G_{\alpha, \delta}(t, x)$ is the density of a Lévy $\alpha$-stable process in time $t$.
(ii) Semigroup property: $G_{\alpha, \delta}(t, x)$ satisfies the Chapman-Kolmogorov equation, i.e., for $0 < s < t$,

$$G_{\alpha, \delta}(t + s, x) = \int_{\mathbb{R}} G_{\alpha, \delta}(t, y) G_{\alpha, \delta}(s, y - x) dy.$$  

(iii) Scaling property: $G_{\alpha, \delta}(t, x) = t^{-1/\alpha} G_{\alpha, \delta}(1, t^{-1/\alpha} x)$.
(iv) There exists a constant $c_\alpha$ such that $0 \leq G_{\alpha, \delta}(1, x) \leq c_\alpha/(1 + |x|^{1+\alpha})$, for all $x \in \mathbb{R}$.

The next proposition studies the Hölder regularity of the Green function, whose proof is contained in the proof of Proposition 3.2 in \cite{3}.

**Lemma 5.2.** Under $(H_{\alpha}^\eta)$, it holds that

(i) For each $0 \leq s < t \leq T, \beta_1 \in [0, (1 - \eta)/2]$, there exists a constant $C > 0$ such that

$$\int_0^s \|G_{\alpha, \delta}(t - r, \star) - G_{\alpha, \delta}(s - r, \star)\|_{\mathcal{H}}^2 dr \leq C |t - s|^{2\beta_1},$$

and

$$\int_s^t \|G_{\alpha, \delta}(t - r, \star)\|_{\mathcal{H}}^2 dr \leq C |t - s|^{2\beta_1}.$$  

(ii) For each $0 < \beta_2 < \min\{(1 - \eta)\alpha_0/2, 1/2\}$, there exists a constant $C > 0$ for any $x, y \in \mathbb{R}^d$,

$$\int_0^T \|G_{\alpha, \delta}(T - s, x - \star) - G_{\alpha, \delta}(T - s, y - \star)\|_{\mathcal{H}}^2 ds \leq C |x - y|^{2\beta_2}.$$

The next lemma is about the Hölder regularity of the stochastic integral. See \cite{3} Proposition 3.2]. For a given predictable random field $V$, we set

$$U(t, x) := \sum_{k \geq 1} \int_0^t \langle G_{\alpha, \delta}(t - s, x - \star) V(s, \star), e_k \rangle_{\mathcal{H}} dB_s^k.$$  

**Lemma 5.3.** Assume that $\sup_{0 \leq t \leq T} \sup_{x \in \mathbb{R}^d} \mathbb{E}(|V(t, x)|^p)$ is finite for some $p$ large enough. Then under $(H_{\alpha}^\eta)$, we have

(i) For each $x \in \mathbb{R}^d$ a.s. the mapping $t \to U(t, x)$ is $\beta_1$-Hölder continuous for $0 < \beta_1 < (1 - \eta)/2$.
(ii) For each $t \in [0, T]$ a.s. the mapping $x \to U(t, x)$ is $\beta_2$-Hölder continuous for $0 < \beta_2 < \min\{\alpha_0(1 - \eta)/2, 1/2\}$.

The following result is a consequence of Lemma A.1 in \cite{2}.
Lemma 5.4. Let $K$ be a compact set in $\mathbb{R}^d$ and let $\{V^\varepsilon(t, x) : (t, x) \in [0, T] \times K\}$ be a family of real-valued functions. Assume

(A1). For any $(t, x) \in [0, T] \times K$,
\[ \lim_{\varepsilon \to 0} |V^\varepsilon(t, x)| = 0. \]

(A2). There exist $\beta_1, \beta_2 > 0$ satisfied that for any $(t, x), (t', x') \in [0, T] \times K$,
\[ |V^\varepsilon(t, x) - V^\varepsilon(t', x')| \leq C(|t - t'|^{\beta_1} + |x - x'|^{\beta_2}), \]
where $C$ is a constant independent of $\varepsilon$.

Then for any $\theta \in (0, 1)$, we have
\[ \lim_{\varepsilon \to 0} \|V^\varepsilon\|_{\theta \beta_1, \theta \beta_2} = 0. \]

Proof. The stochastic version of this lemma in [2, Lemma A1] is proved by the Garsia-Rodemich-Rumsey’s lemma. Here we give a direct proof for the deterministic case.

In view of Arzelà-Ascoli theorem, a sequence in $C([0, T] \times K; \mathbb{R})$ converges uniformly if and only if it is equicontinuous and converges pointwise. Thence, under conditions (A.1) and (A.2), we know that
\[ \lim_{\varepsilon \to 0} \sup_{(t, x) \in [0, T] \times K} |V^\varepsilon(t, x)| \to 0. \]

Thus, for any $\theta \in (0, 1), (t, x) \neq (t', x') \in [0, T] \times K$, we have
\[ \frac{|V^\varepsilon(t, x) - V^\varepsilon(t', x')|}{(|t - t'|^{\beta_1} + |x - x'|^{\beta_2})^\theta} = \frac{|V^\varepsilon(t, x) - V^\varepsilon(t', x')|^\theta |V^\varepsilon(t, x) - V^\varepsilon(t', x')|^{1-\theta}}{(|t - t'|^{\beta_1} + |x - x'|^{\beta_2})^\theta} \leq C(|V^\varepsilon(t, x)| + |V^\varepsilon(t', x')|)^{1-\theta} \to 0 \quad \text{uniformly on } ([0, T] \times K)^2. \]

The proof is complete. \qed

Acknowledgements: Y. Li and S. Zhang were supported by Natural Science Foundation of China (11471304, 11401556). R. Wang was supported by Natural Science Foundation of China (11301498, 11431014) and the Fundamental Research Funds for the Central Universities (WK 0010000048). N. Yao was supported by Natural Science Foundation of China (11101313).

References

[1] De Acosta, A.: Moderate deviations and associated Laplace approximations for sums of independent random vectors. Trans. Amer. Math. Soc. 329, 357-375 (1992)
[2] Bally, V., Millet, A., Sanz-Solé, M.: Approximation and support theorem in Hölder norm for parabolic stochastic partial differential equations. Ann. Probab. 23, 178-222 (1995)
[3] Boulanba, L., Eddahbi, M., Mellouk, M.: Fractional spdes driven by spatially correlated noise: Existence of the solution and smoothness of its density. Osaka J. Math. 47, 41-65 (2010)
[4] Budhiraja, A., Dupuis, P., Maroulas, V.: Large deviations for infinite dimensional stochastic dynamical systems. Ann. Probab. 36, 1390-1420 (2008).
[5] Budhiraja, A., Dupuis, P., Ganguly, A.: Moderate deviation principles for stochastic differential equations with jumps. To appear in Ann. Probab. (2015)
[6] Cerrai, S., Röckner, M.: Large deviations for stochastic reaction-diffusion systems with multiplicative noise and non-Lipschitz reaction term. Ann. Probab. 32, 1100-1139 (2004)

[7] El Mellali, T., Mellouk, M.: Large deviations for a fractional stochastic heat equation in spatial dimension $\mathbb{R}^d$ driven by a spatially correlated noise. Stoch. Dyn. 16, 1650001 (2015).

[8] Chenal, F., Millet, A.: Uniform large deviations for parabolic SPDEs and applications. Stochastic Process. Appl. 72, 161-186 (1997)

[9] Da Prato, G., Zabczyk, J.: Stochastic equations in infinite dimensions. Cambridge Univ. Press, Cambridge (1992)

[10] Dalang, R.: Extending the martingale measure stochastic integral with applications to spatially homogeneous s.p.d.e’s. Electron. J. Probab. 4, 1-29 (1999)

[11] Dalang, R., Quer-Sardanyons, L.: Stochastic integrals for spde’s: a comparison. Expo. Math. 29(1), 67-109 (2011)

[12] Dembo, A., Zeitouni, O.: Large deviations techniques and applications. Second edition. Applications of Mathematics 38, Springer-Verlag (1998)

[13] Droniou, J., Imbert, C.: Fractal first-order partial differential equations. Arch. Ration. Mech. Anal. 182(2), 299-331 (2006).

[14] Ermakov, M.: The sharp lower bound of asymptotic efficiency of estimators in the zone of moderate deviation probabilities. Electron. J. Stat. 6, 2150-2184 (2012)

[15] Freidlin, M.I.: Random perturbations of reaction-diffusion equations: The quasi-deterministic approach. Trans. Amer. Math. Soc. 305, 665-697 (1988)

[16] Freidlin, M.I., Wentzell, A.D.: Random perturbation of dynamical systems. Translated by Szuc, J., Springer, Berlin (1984)

[17] Gao, F.Q., Zhao, X.Q.: Delta method in large deviations and moderate deviations for estimators. Ann. Statist. 39, 1211-1240 (2011)

[18] Guillen, A., Liptser, R.: Examples of moderate deviation principle for diffusion processes. Discrete Contin. Dyn. Syst. Ser. B 6, 803-828 (2006)

[19] Inglot, T., Kallenberg, W.: Moderate deviations of minimum contrast estimators under contamination. Ann. Statist. 31, 852-879 (2003)

[20] Komatsu, T.: On the martingale problem for generators of stable processes with perturbations. Osaka J. Math. 21(1), 113-132 (1984)

[21] Li, Y., Wang, R., Zhang, S.: Moderate Deviations for a Stochastic Heat Equation with Spatially Correlated Noise. To appear in Acta Appl. Math.

[22] Ledoux, M.: Sur les deviations moderées des sommes de variables aléatoires vectorielles indépendantes de même loi. Ann. H. Poincaré 28, 267-280 (1992)

[23] Márquez-Carreras, D., Sarrà, M.: Large deviation principle for a stochastic heat equation with spatially correlated noise. Electron. J. Probab. 8, 1-39 (2003)

[24] Miao, Y., Shen S.: Moderate deviation principle for autoregressive processes. J. Multivariate Anal. 100, 1952-1961 (2009).

[25] Sowers, R.: Large deviations for a reaction-diffusion equation with non-Gaussian perturbation. Ann. Probab. 20, 504-537 (1992)

[26] Walsh, J.: An introduction to stochastic partial differential equations. École d’été de Probabilités St Flour XIV, Lect Notes Math, vol 1180, Springer, Berlin, Heidelberg, New York (1986)

[27] Wang R., Zhang, T.: Moderate deviations for stochastic reaction-diffusion equations with multiplicative noise. Potential Anal., 42, 99-113 (2015)

[28] Williams, N.: Small noise asymptotics for a stochastic growth model. J. Economic Theory, 119(2), 271-298 (2004).

[29] Wu, L.: Moderate deviations of dependent random variables related to CLT. Ann. Probab. 23, 420-445 (1995)
Yumeng Li, Department of Statistics and Finance, University of Science and Technology of China, Hefei, Anhui Province, 230026, China.
E-mail address: liyumeng@mail.ustc.edu.cn

Ran Wang, School of Mathematical sciences, University of Science and Technology of China, Hefei, Anhui Province, 230026, China.
E-mail address: wangran@ustc.edu.cn

Nian Yao, College of Mathematics and Computational Science, Shenzhen University, 518060, Shenzhen, Guangdong Province, China.
E-mail address: yaonian@szu.edu.cn

Shuguang Zhang, Department of Statistics and Finance, University of Science and Technology of China, Hefei, Anhui Province, 230026, China.
E-mail address: sgzhang@ustc.edu.cn