Mirror Nesting of the Fermi Contour and Superconducting Pairing from the Repulsive Interaction

V. I. Belyavsky‡, Yu. V. Kopaev‡ and S. V. Shevtsov‡

Lebedev Physical Institute, Russian Academy of Sciences, Moscow, 119991, Russia

We consider the necessary conditions of superconducting pairing at repulsive interaction between particles composing a pair with large total momentum: (1) the existence of, at least, one negative eigenvalue of the repulsion potential and (2) mirror nesting of the Fermi contour. Under these conditions, we represent the solution of the self-consistency equation continuously depending on the momentum of the relative motion of the pair. The corresponding superconducting order parameter changes its sign on a line crossing the Fermi contour inside the domain of definition of the relative motion pair momentum. We argue that repulsive-induced superconducting pairing with large total pair momentum may be just the case relating to high-temperature superconducting cuprates.

PACS numbers: 78.47.+p, 78.66.-w
Keywords: superconductivity, repulsion interaction, nesting, large pair momentum, cuprates

I. INTRODUCTION

A conventional point of view relating to the mechanism of high-temperature superconductivity of cuprates is absent up to this point. Nevertheless, there is an experimental evidence that there is the singlet pairing of current carriers below the superconducting (SC) transition temperature in these compounds although the question is what interaction dominates pairing. In this connection, the models of the superconductivity both with dominating attraction and repulsion are considered. Another open question is what contribution into the pair binding energy (due to potential energy as it is in the conventional theory by Bardeen, Cooper and Schrieffer (BCS) or, maybe, due to kinetic energy) may be considered as a driven force of the SC transition in the cuprates.

In contrast to the conventional paradigm that the SC transition is a consequence of an instability of the Fermi liquid with respect to arbitrarily weak attraction between particles the question relating to the character of the ground state of normal (N) phase of cuprate compounds remains under discussion. The manifestation of the pseudogap in underdoped cuprates is a reason to suppose that the N phase may be something different from the normal Fermi liquid. Assuming that cuprates are strong correlated electron systems one can, in principle, connect rather wide spectrum of physical properties of these compounds which include, besides typical of cuprates superconductivity and antiferromagnetism (AF), some other ordered states arising due to a competition of SC and AF orders. In this connection one can raise the question of the applicability of the mean-field theory (like the BCS theory which describes conventional superconductors successfully) to cuprate superconductors.

Pairing with zero total pair momentum does not impose any kinematical constraint on the momenta of the relative motion coinciding, in this case, with the momenta of the particles composing the pair. In the BCS model with an attractive interaction between particles, however, there is a “dynamical” constraint connected with the fact that the domain of the effective attraction is a narrow layer enveloping the Fermi surface. The volume of this layer is proportional to the statistical weight of the pair which is a number of one-particle states which contribute to the state of the relative motion of the pair.

On the contrary, at non-zero total pair momentum there is an essential kinematical constraint resulting in a finite domain of the momentum space. The question of the relative motion of the pair should be defined. A supplementary dynamical constraint arising from an attractive interaction, decreases such a domain abruptly resulting in a considerable decrease in the statistical weight of the pair. It should be emphasized that, in the case of Coulomb repulsion, any dynamical constraint is absent and the statistical weight of the pair is fully determined by the volume of kinematically allowed domain of the momentum space.

The Fermi surface of quasi-two-dimensional (2D) electron system of copper–oxygen planes of a cuprate compound is degenerated into a line which we call the Fermi contour (FC). In cuprates, the FC is situated in an extended vicinity of a saddle point of the electron dispersion. As a consequence, one may expect a FC with strong nesting feature and, correspondingly, a considerable difference in the transversal (with respect to the FC) and the longitudinal Fermi velocities. In addition, one can assume that the corresponding effective masses should be of opposite signs and differ in absolute values considerably. Thus, one can expect that, for some definite (antinodal) directions coinciding with the sides of 2D Brillouin zone, there exists rather large area of 2D momentum space defining the statistical weight of the pair with certain non-zero total momentum. One can create a current-less state corresponding to such a pair as a definite linear combination of the states relating to crystal equivalent momenta.

The SC instability of 2D electron system arises due to the fact that the boundary separating occupied and unoccupied one-particle states is a line. In the case of zero total pair momentum, this line is the full FC whereas at non-zero total pair momentum such a boundary is,
generally speaking, a set of points. Therefore, in such a case, there is no logarithmic singularity in the self-consistency equation and, consequently, the SC pairing is impossible at arbitrarily weak interaction strength.

However, the extended saddle point may result in a rise of such a topology of isolines in a vicinity of the FC that the boundary between occupied and unoccupied states of the pair relative motion turns out to be a set of lines of finite length. These lines form a “pair” Fermi contour (PFC) for the relative motion of the pair with definite total momentum. Such a feature (mirror nesting) of the FC is held in a finite doping interval and may warrant a logarithmic singularity in the self-consistency equation resulting in the SC instability at arbitrarily interaction strength.

The channel of the SC pairing with large total pair momentum results quite naturally from a consideration of the competition of Cooper (at zero pair momentum) SC pairing and AF ordering in the framework both a band model and the models which concern strong electron correlations. Also, such a channel arises with necessity in rather general phenomenology based on symmetry considerations.

Recent experimental results relating to neutron scattering in cuprates (the so-called 41 meV peak) may be considered as an indirect evidence in favor of the SC pairing with large total momentum16. Another two fundamental experimental results, the so-called “peak–dip–hump structure” of angle-resolved photoemission spectrum (ARPES)17,18 and relatively small superfluid phase stiffness typical of cuprates19, can be explained qualitatively in the framework of the conception of mirror nesting and repulsion-induced SC pairing with large total pair momentum as well.

The goal of this paper is an investigation of the necessary conditions of the repulsion-induced SC pairing with large total pair momentum and a solution of the corresponding self-consistency equation arising within the mean-field scheme.

II. MIRROR NESTING

The kinematical constraint due to the presence of the FC results in the fact that both momenta \( k_\pm \) of like-charged particles composing a pair with total momentum \( K = k_+ + k_- \) have to be situated either inside or outside the FC. Thus they belong to dependent on \( K \) domain of the momentum space which may be considered as a statistical weight of the pair. This domain \( \Xi_K \) is symmetrical with respect to the inversion transformation \( k \leftrightarrow -k \) of the momentum of the relative motion of the pair, \( k = (k_+ - k_-)/2 \), and, generally speaking, consists of the two parts \( \Xi_K^(-) \) and \( \Xi_K^+(+) \), inside and outside the FC, respectively. An excitation energy of the pair with respect to the chemical potential \( \mu \) that is kinetic energy of the particles composing the pair,

\[
2\xi_K = \varepsilon(K/2 + k) + \varepsilon(K/2 - k) - 2\mu, \tag{1}
\]

equals zero on the boundary separating occupied (\( \Xi_K^(-) \)) and unoccupied (\( \Xi_K^+(+) \)) parts of the domain \( \Xi_K \). Hereafter, for the sake of simplicity, we restrict ourselves to a consideration of hole-doped cuprate superconductors within a simple single-band scheme; therefore, \( \varepsilon(k_\pm) \) makes sense of a hole dispersion.

As it is mentioned above, at \( K = 0 \), the momenta of the relative motion \( \pm k \) coincide with the momenta of the holes composing the pair and therefore the boundary separating the occupied and unoccupied parts of the domain \( \Xi_K \) is the full FC whereas at any \( K \neq 0 \) the boundary between \( \Xi_K^(-) \) and \( \Xi_K^+(+) \), generally speaking, turns out to be a set of isolated points. However, due to a special hole dispersion, such a boundary, at definite total pair momentum, may coincide with finite-length pieces of the FC as it is shown schematically in Fig.1. In such a case the boundary between \( \Xi_K^(-) \) and \( \Xi_K^+(+) \) plays role of a peculiar FC with respect to the relative motion of the pair and may be called a pair Fermi contour (PFC).

Using the symmetry property, \( k \leftrightarrow -k \), one can write the condition defining the PFC in the form

\[
\varepsilon(K/2 + k) - \mu = \varepsilon(K/2 - k) - \mu, \tag{2}
\]

under the additional condition that both \( k \) and \( -k \) belong to the domain \( \Xi_K \). It is obvious that, to obey the condition \( \Xi_K \), the vector \( K \) should be directed along one of the symmetrical (antinodal or nodal) direction of 2D Brillouin zone. Hence the boundary separating occupied and vacant states has to obey a mirror symmetry with respect to the pair momentum \( K \) direction and so Eq. (2) may be called a mirror nesting condition. It should be also noted that, under mirror nesting condition, the inversion transformation \( k \leftrightarrow -k \) superposes an occupied state with an occupied one and a vacant state with a vacant one, respectively.

In the framework of a single-band scheme relating to cuprates, one can give some rather obvious examples of the mirror nesting condition resulting in a rise of the PFC20,21. In a case when the FC of a hole-doped cuprate compound is simply connected and appears as a square with rounded corners, one can expect that a change of FC curvature sign, which can be easily realized because of the existence of long, almost rectilinear, pieces of the FC, may provide a realization of the mirror nesting condition for certain chosen pair momenta. One more single-band example represents an underdoped cuprate compound considering as doped Mott insulator in which the FC encloses some symmetrically disposed hole pockets around the points belonging to symmetrical directions in 2D Brillouin zone. In such a case, the mirror nesting condition is perfectly satisfied when a half of the total pair momentum corresponds to the center of a pocket and the whole of the line enclosing the pocket turns out to be...
be the PFC. The stripe structure arising in underdoped cuprates gives one more possibility resulting in a rise of PFC.

III. THE SELF–CONSISTENCY EQUATION

To consider the problem of pairing with non-zero total pair momentum in the mean-field framework let us write the Hamiltonian corresponding to pair relative motion in the form

$$\hat{H}_{K} = \sum_{k} \left[ (\varepsilon_{k_{+}} - \mu) \hat{a}_{k_{+},\uparrow} \hat{a}_{k_{+},\uparrow}^\dagger + (\varepsilon_{k_{-}} - \mu) \hat{a}_{k_{-},\downarrow} \hat{a}_{k_{-},\downarrow}^\dagger \right] + \frac{1}{S} \sum_{k,k'} U(k-k') \hat{a}_{k_{+},\uparrow}^\dagger \hat{a}_{k_{-},\downarrow}^\dagger \hat{a}_{k_{+},\downarrow}^\dagger \hat{a}_{k_{-},\uparrow} \tag{3}$$

where $\varepsilon_{k_{\pm}} = \varepsilon(k_{\pm})$, $U(k-k')$ is the Fourier transform of the interaction energy, $S$ is a normalizing area, $\hat{a}_{k_{+},\sigma}^\dagger$ ($\hat{a}_{k_{-},\sigma}$) creates (annihilates) a hole with the momentum $k_{\pm}$ and the spin quantum number $\sigma = \uparrow, \downarrow$. The summation in Eq. (3) is taken over all range of the allowed values of the momentum of the relative motion and thus is restricted by the domain $\Xi_{K}$. Note that the summation in the Hamiltonian Eq. (3) should be taken over only two (instead of three in a general case) variables, $k$ and $k'$, just as in the case of BCS Hamiltonian, taking into account the only interaction between particles composing the pairs with total pair momentum $K$.

As usual, to diagonalize the Hamiltonian Eq. (3) approximately one can introduce creation and annihilation operators of new quasiparticles using the well-known Bogoliubov–Valatin transformation:

$$\hat{a}_{k_{+},\uparrow} = u_{Kk} \hat{b}_{k_{+},1} + v_{Kk} \hat{b}_{k_{+},-1},$$
$$\hat{a}_{k_{-},\downarrow} = u_{Kk} \hat{b}_{k_{-},-1} - v_{Kk} \hat{b}_{k_{-},1}. \tag{4}$$

The Hamiltonian, up to the terms of the order of $\hat{b}^2$, can be written as

$$\hat{H}_{K} = E_{K0} + \hat{H}_{K}^{(0)} + \hat{H}_{K}^{(1)}. \tag{5}$$

The ground state energy has the form

$$E_{K0} = 2 \sum_{k} \xi_{KK} u_{KK} v_{KK} + \sum_{k} \Delta_{Kk} u_{KK} v_{KK}, \tag{6}$$

where, related to the value of the chemical potential, an energy of the relative motion of the pair with the total momentum $K$ is defined by the Eq. (1) and the order parameter is defined as

$$\Delta_{Kk} = \frac{1}{S} \sum_{k'} U(k-k') u_{Kk'} v_{Kk'} \tag{7}$$

Diagonal and nondiagonal, with respect to quasiparticle operators, parts of the Hamiltonian can be written as

$$\hat{H}_{K}^{(0)} = \sum_{k,\beta = \pm 1} \eta_{K\beta}(k) \hat{b}_{k,\beta}^\dagger \hat{b}_{k,\beta} \tag{8}$$

and

$$\hat{H}_{K}^{(1)} = \sum_{k} \left[ 2 \xi_{KK} u_{KK} v_{KK} - (v_{KK}^2 - u_{KK}^2) \Delta_{Kk} \right] \times \left( \hat{b}_{k_{+},1}^\dagger \hat{b}_{k_{-},-1} + \hat{b}_{k_{-},1} \hat{b}_{k_{+},-1}^\dagger \right), \tag{9}$$

respectively. Here the energies corresponding to two branches ($\beta = \pm 1$) of the one-particle excitation spectrum are equal to each other,

$$\eta_{K\beta}(k) = \sqrt{\xi_{KK}^2 + \Delta_{Kk}^2}. \tag{10}$$

A choice of the amplitudes, $u_{KK}$ and $v_{KK}$, in Bogoliubov–Valatin transformation Eq. (4) is determined in the zero-temperature limit by the conditions that (i) all of the states inside the subdomain $\Xi_{K}^{(+)}$, in which the kinetic energy of the relative motion of the pair is negative, $2\xi_{KK} < 0$, must be occupied and (ii) the nondiagonal part, Eq. (9), of the Hamiltonian vanishes. In addition, the condition $v_{KK}^2 + u_{KK}^2 = 1$ preserving Fermi’s commutation relations for quasiparticle operators must be fulfilled. These conditions yield

$$v_{KK}^2 = \frac{1}{2} \left( 1 - \frac{\xi_{KK}}{\sqrt{\xi_{KK}^2 + \Delta_{Kk}^2}} \right),$$
$$u_{KK} v_{KK} = -\frac{1}{2} \frac{\Delta_{Kk}}{\sqrt{\xi_{KK}^2 + \Delta_{Kk}^2}} \tag{11}$$
It should be noted that the Bogoliubov–Valatin amplitudes \( \epsilon(k) - \mu \) are determined not by the one-particle energy \( \epsilon(k) - \mu \) as it were at \( K = 0 \) but by the relative motion energy \( \mu \) which transforms into \( \epsilon(k) - \mu \) just in the case \( K = 0 \).

In the zero-temperature limit, one can obtain the self-consistency equation determining the SC order parameter,

\[
\Delta_{KK} = -\frac{1}{2\pi} \sum_{k' \neq 0} \frac{U(k - k') \Delta_{kk'}}{\sqrt{\xi_{kk'}} + \Delta_{kk'}},
\]

(12)

where \( k \) and \( k' \) are momenta of the relative motion of pairs with one and the same total momentum \( K \), \( U(k) \) a matrix element of the effective interaction potential in the Hamiltonian Eq. (3) depending on the momentum transfer \( \kappa = k - k' \) due to a scattering from an initial \( (k) \) to a finite \( (k') \) state of the relative motion of the pair being also a difference, \( k_\perp - k'_\perp \), between the corresponding momenta of the particles composing pairs with the total momentum \( K \). Summation over momenta of the relative motion in the equation (12) is performed within the whole of their domain of definition \( \Xi_K \). Thus, the SC order parameter is defined inside the domain \( \Xi_K \) as well.

To solve the equation (12) it is convenient to reduce it to the corresponding integral equation considering the quantities \( \Delta_{KK} = \Delta(k) \) and \( \xi_{kk} = \xi(k) \) as continuous functions of the relative motion momentum. Hereafter, for a simplification of all of the notations, we omit the label \( K \). Using the notations

\[
\eta(k) = \sqrt{\xi^2(k) + \Delta^2(k)}, \quad f(k) = 1/4\pi\eta(k),
\]

(13)

one can rewrite (12) in the form

\[
\Delta(k) = -\frac{1}{2\pi} \int_{\Xi} U(|k - k'|) f(k') \Delta(k') d^2k',
\]

(14)

where \( U(|k - k'|) \) may be called a kernel of the non-linear integral equation (14).

IV. NECESSARY CONDITIONS OF REPULSION–INDUCED SC PAIRING

In this section, we discuss the necessary conditions of the repulsion-induced SC pairing with large total pair momentum and put forward a procedure of reducing of the mean-field SC gap equation (14) to an approximate non-linear singular integral equation with a degenerate kernel.

The matrix element \( U(k - k') \) is connected with the effective interaction potential \( U(r) \) in the real space by the Fourier transform

\[
U(|k - k'|) = \int U(r) \exp[i(k - k')r] d^2r.
\]

(15)

One can deduce a simple criterion of the absence of non-trivial solutions of the equation (14). Let us multiply this equation and the function \( f(k)\Delta(k) \) together and integrate the obtained relation over \( k \) inside the domain \( \Xi \). Expressing \( U(k - k') \) in accordance with (14) and changing the order of integration over \( k, k' \) and \( r \) one can obtain

\[
\int_{\Xi} \Delta^2(k) f(k) d^2k = -\frac{1}{2\pi} \int_{\Xi} U(r) L(r) d^2r,
\]

(16)

where

\[
L(r) = \left| \int_{\Xi} \Delta(k) f(k) \exp(iKr) d^2k \right|^2
\]

(17)

is a nonnegative function of \( r \). The integration over \( r \) in (17) is performed over the whole of 2D real space. As far as \( f(k) \geq 0 \), the left side of (16) is nonnegative as well. If the potential \( U(r) \) were positive at any \( r \), the right-hand side of the Eq. (16), on the contrary, turns out to be negative and the equality (16) may be satisfied only in the trivial case when \( \Delta(k) \equiv 0 \) in the whole of the domain \( \Xi \). Thus, under the condition that the interaction is purely repulsive in the real space that is \( U(r) > 0 \) at any \( r \), the self-consistency equation leads only to the trivial solution.

Therefore, a rise of repulsion-induced SC order is possible only in the systems where the interaction potential is a function of \( r \) with alternating signs. As an example of such a potential with alternating signs, one can consider screened repulsive Coulomb interaction in a degenerated Fermi system which exhibits the Friedel oscillations. Further, we restrict ourselves to a consideration of such a repulsive potential \( U(r) \).

One can see that the maximal (positive) value of the function \( rU(r) \), arising when the integral in the right-hand side of (16) is calculated using the polar coordinates, corresponds to \( r = 0 \). This function with alternating signs is multiplied by a nonnegative function \( L(r) \) defined by Eq. (17). The governing contribution into the product \( \Delta(k) f(k) \) determining the function \( L(r) \) is due to a small vicinity of the PFC because of the equality \( \xi(k) = 0 \) which takes place just on the PFC.

As it will be shown below, the gap function \( \Delta(k) \) resulting from the repulsion interaction changes sign on a line crossing the PFC. Therefore, the absolute value of the integral in (17) represents a function of \( r \) appearing as a series of peaks with decreasing magnitude and a character separation between the neighboring peaks of about \( \pi/k_p \) where \( k_p \) is of about the length of the PFC. The first (and the largest) peak results in the fact that the product of the factor \( \Delta(k) f(k) \) with alternating signs and exponential function in the integral (17) turns out to be a function with the values mainly of constant sign.

The value \( L(0) \) is, generally speaking, a small quantity because of a partial (or even full) compensation of the contributions of the parts of the domain \( \Xi \) in which \( \Delta(k) \) has opposite signs. For example, in the case of the full compensation, \( L(r) \sim r^2 \), therefore the main positive extremum of the function \( rU(r) \) turns out to be
trivial solution only. In the opposite case of large \( r \), the function \( L(r) \) is also small due to rapidly oscillating factor \( \exp(ikr) \). As far as \( k_F \) is pronouncedly less than \( k_p \), the position of the first maximum of the function \( L(r) \), \( r_1 \sim \pi/k_p > \pi/k_F \), corresponds quite naturally to the region of the Friedel oscillations of the repulsive Coulomb potential. There is a significant region of the \( r \)-space in which the integrand in the right-hand side of the Eq. (16) is negative whereas the contributions of the regions with positive integrand are comparatively small as it is illustrated schematically in Fig. 2. Therefore the integral in Eq. (16) may turn out to be negative.

In such a case, the equality (16) is provided due to the fact that the function \( f(k) \) is singular on the PFC and may be arbitrarily large at \( \Delta(k) \to 0 \). The magnitude of the oscillations of the function \( L(r) \) depends on \( \Delta(k) \) because an effective width of the vicinity of the PFC leading to the main contribution into the integral (16) is determined by just the function \( \Delta(k) \). Thus, the condition (16) may be fulfilled at sufficiently small \( \Delta(k) \neq 0 \).

Therefore the presence of the PFC which results in the singularity of the Eq. (16) is the first necessary condition of the repulsion-induced SC pairing.

The equation (14) is a non-linear integral Hammerstein equation with a symmetric kernel \( U(|k-k'|) \). All of the eigenvalues of such a nondegenerate kernel are real and belong to a discrete spectrum \( \lambda_n \) (where \( n = 1, 2, \ldots \)) with the condensation point \( |\lambda_n| \to \infty \) at \( n \to \infty \). If a kernel \( U(|k-k'|) \) is the Fourier transform (15) of everywhere positive effective potential \( U(r) \) (a positive defined kernel) all its eigenvalues are positive as well. In such a case, as it follows from Eq. (16), the equation (14) has the trivial solution only.

Therefore, the presence of at least one negative eigenvalue in the spectrum of the kernel \( U(|k-k'|) \) is the second necessary condition of the existence of a non-trivial solution of the equation (14).

It should be noted that, as it follows from Eq. (14), a non-trivial solution \( \Delta(k) \neq 0 \) of the self-consistency equation at \( U(k-k') > 0 \), if such a solution exists, must be a function with alternating signs inside its domain of definition \( \Xi \).

In the case of the non-degenerate kernel \( U(|k-k'|) \), the existence of such a non-trivial solution of the self-consistency equation at the repulsive interaction can be demonstrated if one represents this kernel as an expansion over its eigenfunctions \( \varphi_n(k) \) which are the solutions of the linear integral equation

\[
\varphi_n(k) = \lambda_n \int U(|k-k'|) \varphi_n(k') d^2 k'.
\]

Let us write this expansion in the form

\[
U(k-k') = \sum_n \frac{\varphi_n(k) \varphi_n(k')}{\lambda_n}
\]

and substitute it into the self-consistency equation (14). Integrating the right-hand side of this equation term-by-term, one can see that a non-trivial solution, if it exists at all, may be represented in the form of an expansion over the complete system of the eigenfunctions \( \varphi_n(k) \),

\[
\Delta(k) = \sum_{n'} \Delta_{n'} \varphi_n(k'),
\]

where \( \Delta_n \) are the expansion coefficients which can be determined as the solutions of the infinite equation system,

\[
\Delta_n = -\frac{1}{2\pi \lambda_n} \sum_{n'} f_{nn'} \Delta_{n'}.
\]

Here, \( f_{nn'} \) are the matrix elements of the form

\[
f_{nn'} = \int \varphi_n^*(k) f(k) \varphi_n(k)
\]

where the function \( f(k) \) is determined by Eq. (13).

Multiplying Eq. (21) and \( \lambda_n \Delta_n \) together, one can, after a summation over \( n \), obtain the relation

\[
\sum_n \lambda_n \Delta_n^2 = -\frac{1}{2\pi} \sum_{n,n'} \Delta_n f_{nn'} \Delta_{n'}
\]

which is analogous to the relation (16). As it is follows from Eq. (20), there is only the trivial solution of this equation in the case when all of the eigenvalues are positive. Therefore, a non-trivial solution may, in principle, exist only in the case when at least one of the eigenvalues turns out to be negative. In fact, such a condition turns out to be not only necessary but the sufficient one.

To prove this statement, one can take advantage of the obvious analogy between the spectral expansion of the order parameter, Eq. (20), and its expansion into a series of
spherical harmonics. As it is well known, the SC pairing arises when the negative scattering length corresponds at least to one of the values of orbital angular momentum. For this reason, taking account of such an analogy, we do not present a direct proof of such a statement within the framework of the spectral expansion.

V. SIMPLE DEGENERATE KERNEL

The function defining a non-linear operator in Eq. (11) has a special form $\Delta f(\Delta)$ where $f(\Delta)$ may be called a nonlinearity factor. In this sense, the equation (11) may be referred to the class of quasi-linear integral equations. In the case of a degenerate kernel, the significant feature of such equations is that a form of a non-trivial solution (if it exists at all) reproduces the kernel structure. The kernel $U(|k - k'|)$ can be approximately reduced to a degenerate one if one takes into account that both variables, $k$ and $k'$, are defined in a relatively small domain $\Xi$ of the momentum space. Therefore, if the function $U$ can be expanded into the Taylor series, one can restrict himself to some lowest powers of the argument $\kappa = |k - k'|$. As it follows from the definition, the expansion of $U(\kappa)$ includes only even powers of $\kappa$ and the Fourier transform of the interaction potential can be written as

$$U(\kappa) = 2\pi \left[ u_0 - \frac{1}{2} u_2 \kappa^2 + \frac{3}{8} u_4 \kappa^4 - \ldots \right],$$

where

$$u_n = \frac{1}{n!} \int_0^\infty U(r) r^{n+1} dr.$$

A degenerate kernel corresponds to a finite number of the terms in the expansion and may be considered as a good approximation of the true nondegenerate kernel in the case of a small domain $\Xi$. It should be noted that the Fourier transform $U(\kappa)$ corresponding to a repulsive interaction is, at small $\kappa$, a decreasing function of its argument resulting in $u_0 > 0$ and $u_2 > 0$.

To solve the equation (11) it is convenient to transform it into a dimensionless form. Using two constants $U_0$ and $r_0$, being characteristic energy and length scales, respectively, one can define them by the relations $U_0r_0^2 = u_0$ and $U_0r_4^4 = u_2$. In the case of screened Coulomb potential the parameters $r_0$ and $U_0 = e^2/r_0$ make sense of a screening length and a characteristic Coulomb energy respectively. Thus, the energy $U_0$ determines the scale of the quantities $\Delta(k)$ and $\xi(k)$ whereas a momentum is measured by the units of $r_0^{-1}$. The equation (11) remains invariant with respect to such a scaling if one reduces the domain of integration $\Xi$ to dimensionless variables (the components $k_1$ and $k_2$ of the momentum of the relative motion of the pair).

If the domain $\Xi$ is small enough one may keep only two terms of the lowest order in the expansion and obtain a degenerate kernel in the form

$$U_d(\kappa) = 2\pi [1 - \kappa^2/2].$$

One can make sure that the obtained degenerate kernel has four eigenvalues three of which are positive whereas the fourth one is negative.

To show this one has to write down the linear integral equation determining the eigenvalues $\lambda$ and the eigenfunctions $\varphi_\lambda(k)$ corresponding to the degenerate kernel Eq. (26),

$$\varphi_\lambda(k) = \lambda \int_\Xi U_d(|k - k'|)\varphi_\lambda(k') dk'.$$

It follows immediately from the Eq. (27) and the form of the kernel that eigenfunctions must be taken in the form

$$\varphi_\lambda(k) = a + (\chi k) + bk^2$$

thus reproducing the form of the degenerate kernel. Here, $a$ and $b$ are scalar coefficients and $\chi$ is a constant vector being subjects to be determined for each of the eigenvalues. To obtain the parameters $a$, $b$ and $\chi$ of the solution one has to substitute this solution unto the Eq. (27) and, after integration, set the coefficients corresponding to one and the same power of the vector $k$ equal to each other.

It is obvious that the approximation is sufficient if $(k + k')^2 < 2$. Therefore, one has to assume that the domain $\Xi$ is such that the condition

$$k^2 < 1/2.$$  

is fulfilled for any momentum of the relative motion. If one takes into account that a screening length $r_0$ is of about a few interatomic distances, the restriction results in the fact that a characteristic size of the domain $\Xi$ should be much less as compared with a characteristic Brillouin zone size.

Due to a symmetry of the domain of integration $\Xi$ with respect to inversion transformation $\mathbf{k} \leftrightarrow -\mathbf{k}$ eigenfunctions $\varphi_\lambda(k)$ must be either even or odd. Odd eigenfunctions are determined by single unknown vector $\chi$ being the solution of the equation

$$\chi = 2\pi \lambda \int_\Xi (\chi k')k'^2 dk'.$$

This equation has two non-trivial solutions directed along the symmetry axes of the domain $\Xi$. Corresponding eigenvalues,

$$\lambda_i = \left[ 2\pi \lambda \int_\Xi k_i^2 k'^2 \right]^{-1}, \quad i = 1, 2,$$

are positive. Here, $k_i$ are the components of the vector $\mathbf{k}$ along the symmetry axes of the domain $\Xi$.

In the case of the even eigenfunctions, one can obtain the system of linear homogeneous equations

$$a = 2\pi \lambda \int_\Xi (1 - k'^2/2)(a + bk'^2)dk',$n

$$b = \pi \lambda \int_\Xi (a + bk'^2)dk'.$$
determining two unknown quantities, $a$ and $b$. The condition of the non-trivial compatibility of this system results in another two eigenvalues of the degenerate kernel Eq. (33),
\[ \lambda_\pm = \frac{1}{\pi} \left( (K_0 - K_1) \pm \sqrt{(K_0 - K_1)^2 + (K_0 K_2 - K_1^2)} \right)^{-1}. \]
(33)

Here we use the notation
\[ K_n = \int_{\Xi} k^{2n} d^2 k, \quad n = 0, 1, 2. \]
(34)

As it follows from the Cauchy–Schwarz–Bunyakovsky inequality for the integrals [24], $K_0 K_2 \geq K_1^2$, therefore one of the eigenvalues [23] is positive, $\lambda_+ > 0$, whereas the second one is negative, $\lambda_- < 0$. Here we take into account the fact that, due to the inequality [29], $2K_1 < K_0$.

It should be noted that two of the positive eigenvalues $\lambda_{1,2} \sim u_0^{-1}$, another one positive eigenvalue $\lambda_+ \sim u_2^{-1}$, and the negative eigenvalue $\lambda_- \sim -u_0/u_2^2$.

One may neglect the next terms of the expansion [21] only under the condition that, inside the domain $\Xi$, the mean square of the neglected terms is much less in comparison with the absolute value of any eigenvalue of the degenerate kernel.

\[ 2|u_4| \sqrt{\int_{\Xi} d^2 k \int_{\Xi} d^2 k' (|k - k'|^8 \ll |\lambda|^{-1}. \]
(35)

In the case of the positive eigenvalues, this inequality is satisfied under rather natural condition that the third term in the expansion [24] is small in comparison with the second one. In the case of the negative eigenvalue, the inequality [25] reduces to the condition that $u_0 |u_4| < c u_2^2$ where $c \ll 1$. In the case of everywhere positive potential $U(r)$, this relation does not fulfilled definitely since, for such a potential, the quantities $u_n$ defined in [24] should be connected by a reciprocal inequality following from the Cauchy–Schwarz–Bunyakovsky inequality for the integrals [25]. Thus, using the degenerate kernel [26] instead of the true kernel $U(k - k')$, one must assume that the conditions [21] and [25] are fulfilled.

VI. GENERAL SOLUTION OF THE SELF-CONSISTENCY EQUATION

Substituting the kernel [24] into the equation [14] and grouping the terms independent of the momentum and also linear and squared terms one can see that the dependence of the order parameter on the momentum of the relative motion of the pair has the form similar to the Eq. [28]

\[ \Delta(k) = a + (\chi k) - bk^2, \]
(36)

where $a$ and $b$ are unknown parameters and $\chi$ is an unknown constant vector being subjects to be determined. Thus, one can see that the solution of the equation [14] reproduces the momentum dependence of the degenerate kernel. Therefore, to solve this equation, one has to determine the parameters $a$, $b$ and $\chi$. To find these parameters one has to substitute the kernel [26] and an explicit form of the solution [36] into the equation [14].

After that, one obtains a system of integral equations determining the parameters $a$, $b$ and $\chi$. One of these equations, following from the comparison of the coefficients at the first power of $k$ in the left and right sides of the equation resulting from the Eq. [14], has the form

\[ \chi = -\int_{\Xi} f(k) \Delta(k) k d^2 k. \]
(37)

The equation (37) determines the vector $\chi$ implicitly because the right-hand side of this equation depends on $\Delta$ which itself depends on $\chi$. It is not difficult to see that there is at most one solution of the equation (37). To prove this proposition let us introduce an auxiliary function $H(\Delta) \equiv \Delta \cdot f(\eta(\Delta))$ which is a monotonically increasing function of its argument. Writing down the equation (37) in the component-wise form one can see that its right-hand side is a decreasing function of a component $\chi_i$ of the vector $\chi$ ($i = 1, 2$) whereas the left-hand side increases with $\chi_i$. Therefore the equality (37) turns out to be possible at a single value of the vector parameter $\chi$.

The obvious solution of (37) is $\chi = 0$. Indeed, due to central symmetry of the domain $\Xi$, the right-hand side of (37) vanishes at $\chi = 0$ as a result of an integration of an odd function over a symmetric domain. It should be noted that the order parameter $\Delta(k)$ and the excitation energy $\xi(-k) = \xi(k)$ are even functions of $k$ at $\chi = 0$. There are no other solutions of the equation (37). Thus, in the case of a degenerate kernel (26), the equation (14) may presuppose a simple solution of the form

\[ \Delta(k) = a - bk^2, \]
(38)

determining the SC gap as a parabolic function of the momentum of the relative motion of the pair.

Taking into account that $\chi = 0$ one can write down two equations determining the parameters $a$ and $b$:

\[ \left( J_0 - \frac{1}{2} J_1 + 1 \right) a - \left( J_1 - \frac{1}{2} J_2 \right) b = 0, \]

\[ \frac{1}{2} J_0 a - \left( \frac{1}{2} J_1 - 1 \right) b = 0, \]
(39)

where

\[ J_n \equiv \int_{\Xi} f(k) k^{2n} d^2 k, \]
(40)

where $n = 0, 1, 2$. Thus, we have a quasi-linear system of the equations (39) which contains three positive integrals (40) depending on two unknown parameters $a$ and $b$.

First of all, let us multiply the inequality $k^2 < 1/2$ by the function $f(\eta(k))$ and then by the another function,
\( k^2 f(\eta(k)) \). Integrating the obtained inequalities over the domain \( \Xi \) one can deduce two new inequalities, \( J_1 < J_0/2 \) and \( J_2 < J_1/2 \), which allow to conclude that both coefficients in the first equation are positive. Therefore, the solutions of the equation system have one and the same sign (positive, for example). As a result, the SC order parameter as a function of the momentum of the relative motion changes its sign at a certain \( k = k_0 \) and can be written as

\[
\Delta(k) = b(k_0^2 - k^2)
\]

(41)

where \( k_0^2 = a/b \). The parameter \( b \) determines an energy scale of the SC gap and \( k_0 \) is a radius of the circle on which the SC gap vanishes. In the case of repulsive interaction considered here, it is obvious that the SC gap has to vanish inside the domain \( \Xi \) because at \( U(|k - k'|) > 0 \) there exists no constant-sign non-trivial solution of the equation (14).

The conditions that \( a > 0 \) and \( b > 0 \), as it follows from the second equation, lead to a restrictive inequality \( J_1 > 2 \) which is a consequence of more strong inequality following from the condition of the non-trivial consistency of the system. Calculating the determinant of the system one can write down this condition in the form

\[
(J_1/2 - 1)^2 = J_0(J_1/2 - 4 - 1).
\]

(42)

It is seen that it must be \( J_2 > 4 \). Therefore, taking into account that \( J_1 > 2J_2 \), one can obtain \( J_1 > 8 \).

Instead of the pair parameters \( a \) and \( b \) characterizing the SC gap it is convenient to consider another pair, \( b \) and \( k_0 \). To determine these two parameters one can use any pair of the equations and (12). As one of them it is convenient to use the second equation of the system writing it in the form

\[
J_1 - k_0^2 J_0 = 2.
\]

(43)

Then, expressing the left-hand side of the equation with the help of the second of the equations, the Eq. (12) can be rewritten as

\[
J_2 - k_0^2 J_0 = 4.
\]

(44)

The system of equations is fully equivalent to initial system and can be used to determine the parameters \( b \) and \( k_0 \) of the SC gap. In this connection, it should be noted that the integrals \( J_0, J_1 \) and \( J_2 \), because of their dependence on \( \Delta \), depend themselves on \( b \) and \( k_0 \).

VII. ORDER PARAMETER

Let us rewrite the equations and in the explicit form

\[
\frac{1}{4\pi} \int_{\Xi} \frac{(k^{2n} - k_0^{2n})d^2k}{\sqrt{\xi^2(k) + b^2(k^2 - k_0^2)^2}} = 2^n,
\]

(45)

where \( n = 1, 2 \). Is is impossible to solve the equation system analytically in the whole of the domain of definition of unknown quantities (\( b > 0 \), \( 0 < k_0^2 < 1/2 \)) because the integrals in depend on the form of the domain of integration. However, it is easy to determine these integrals in the two limiting cases of large (\( b \to \infty \)) and small (\( b \to +0 \)) values of the gap energy scale \( b \). Indeed, at large \( b \) one can neglect the argument \( \xi(k) \) in the radicand of each of the integrals. After that, these integrals can be calculated easily and turn out to be proportional to \( 1/b \) therefore one can obtain the solution of in a trivial way.

Considering the integrals in in the most interesting case of small \( b \) one has to take into account the fact that pair excitation energy vanishes on the line separating occupied, \( \Xi^{(-)} \), and unoccupied, \( \Xi^{(+)} \), parts of the domain \( \Xi \) as far as this line (PFC) represents a piece of the FC. This results in a divergence of the integrals at small \( b \) and, if one takes into account that the main contribution into the integrals is from a narrow strip along the line \( \xi(k) = 0 \), it is this divergence that results in the existence of the solution of the system. Let us denote a width of the strip as \( 2\Delta k \) and, taking account of the equation of the line \( \xi(k) = 0 \) in polar coordinates, \( k = k(\varphi) \), consider an integral of a general form

\[
J = \frac{1}{4\pi} \int_{\Xi} \frac{F(k) dk d\varphi}{\sqrt{\xi^2(k, \varphi) + b^2(k_0^2 - k^2)^2}},
\]

(46)

where \( F(k) \) is a continuous function without any singularity. Let us consider a behavior of this integral at \( b \to +0 \) using a strip of the width \( 2\Delta k \) as an integration domain. First of all, one can perform an integration over a polar angle in the limits from \( \varphi \) to \( \varphi_2 \) corresponding to the endpoints of the line \( \xi(k, \varphi) = 0 \). Then, introducing a new variable of integration \( \xi \), one can perform an integration over \( k \) in the limits from \( k(\varphi) - \Delta k \) up to \( k(\varphi) + \Delta k \). On account of the smallness of \( \Delta k \), one can assume that \( k(k(\varphi) - \Delta k) \) to \( k(\varphi) + \Delta k \) and \( k(k(\varphi) + \Delta k) \) to \( k(\varphi) - \Delta k \). In addition, the argument \( k \) in the integrand can be replaced by its constant (at a given \( \varphi \)) value corresponding to the PFC: \( k = k(\varphi) \). Thus, after integration over \( \xi \), one can obtain a singular (logarithmic in \( b \)) contribution into the integral.

Besides the singular contribution there is also a regular one which depends on one-particle dispersion and on the form and size of the domain \( \Xi \). This contribution depends on \( k_0^2 \) as on a parameter. Thus, the integral can be represented as

\[
J = A \ln(1/b) + C,
\]

(47)

where the coefficient in front of the logarithm has the form

\[
A = \frac{1}{2\pi} \int_{\varphi_1}^{\varphi_2} \frac{kF(k)}{|\xi(k)|^2} dk,
\]

(48)

where an integration over the angle variable should be performed along the PFC (see Fig. 1). The regular part
of the integral denoted as $C$ can be taken equal to its value at $b = 0$.

Thus, at small $b$, the equation system (45) can be written in the form

$$ (A_n - k_0^{2n}A_0) \ln(1/b) = P_n, $$  \hspace{1cm} (49)

where the coefficients $A_n$ are independent of $k_0^2$ and can be determined by the expression (48) in which one has to put $F(k) = k^{2n}$ whereas the right-hand sides of the equations (49) depend on $k_0^2$ and can be written as

$$ P_n = 2^n - C_n + k_0^{2n}C_0. $$  \hspace{1cm} (50)

Here $C_n$ are the regular parts of the integrals (40).

As it follows from the Eqs. (49), the equation system (45) has a non-trivial solution with necessity. To prove such a statement let us consider each of the equations (45) as an equation of a line on the plane $b$, $k_0^2$: $b_n = b_n(k_0^2)$. Then the point of the intersection of these two lines corresponds to the solution of the equation system (45). Putting $k_0^2 = 0$ in these equations one can obtain the left-hand sides of each of them as functions of $b$ (Fig. 3). These functions monotonically decrease from $+\infty$ at $b \to +0$ to zero at $b \to +\infty$. Due to the inequality (29), the second of these functions, corresponding to $n = 2$, with necessity is less than the first one ($n = 1$) at any $b$. Therefore, at $k_0^2 = 0$, the solution $b_1(0)$ of the first equation turns out to be more than the solution $b_2(0)$ of the second equation because of the fact that the right-hand side of the second equation is always more as compared to the right-hand side of the first one. In the another limiting case when $b \to +0$, one can use an asymptotic representation of the equations (45) in the form (46). As far as the right-hand sides of the equations are finite and $\ln(1/b) \to +\infty$ at $b \to +0$ the values of $k_0^2$ corresponding to $b \to 0$ can be written as $k_0^{2n} = A_1/A_0$ at $n = 1$ and $k_0^{2n} = \sqrt{A_2/A_1}$ at $n = 2$.

Taking into account the condition (29) one can see that $k_0^{2n} < 1/2$. There is Cauchy–Schwarz–Bunyakovsky inequality, $A_1^2 \leq A_0A_2$, for the quantities $A_0$, $A_1$ and $A_2$ defining by such integrals as (45) therefore $k_0^{21} \leq k_0^{22}$ and, with necessity, there exists a point of the intersection of the lines $b_n = b_n(k_0^2)$.

To obtain the non-trivial solution in the case of small $b$ one has to perform term-by-term division of the equations (45). This leads to a closed equation determining the parameter $k_0^2$. As a result, a characteristic energy scale of the SC gap can be found from any equation of the system (45) and, for example, can be written in the form

$$ b = \exp \left( -\frac{P_1}{A_1 - A_0k_0^2} \right). $$  \hspace{1cm} (51)

This relation explicitly determines the dependence of SC order parameter on the momentum of the relative motion of the pair. Due to the fact that $b << 1$, it is obvious that the exponent in (51) is negative.

One can see that, in the case of small $b$, the line $k = k_0$ intersects the PFC. Indeed, let us suppose that $b$ and $k_0$ are the solutions of the equation system (45). At $b \to +0$ the integrals in (45) are determined by their singular parts following from the integration over a small vicinity of the PFC. An assumption that the circle $k = k_0$ does not intersect the PFC means that both the PFC and its small vicinity are situated either inside or outside this circle. Therefore in the whole of this vicinity one may expect either the inequality $k < k_0$ or the opposite inequality $k > k_0$. In such a case, the integrand in (45) must be of constant sign and the integral corresponding to $n = 2$ has to be less than the integral with $n = 1$ because of the inequality $k^2 + k_0^2 < 1$ following from (29). Thus, if one consider that the circle $k = k_0$ and the PFC does not intersect each other the equations in the system (45) turn out to be incompatible with each other. Consequently, at $b \to +0$, the parameter $k_0$ turns out to be such that the intersection of the PFC and the circle $k = k_0$ arises with necessity. In such a case, the quantity $k^2 - k_0^2$ has different signs on different pieces of the PFC interior and exterior the circle $k = k_0$. Such a conclusion proved under the condition that $b \to +0$ valid also at rather small but finite values of the parameter $b$.

As far as the circle $k = k_0$ intersects the PFC, one can assume that there exist the minimal, $k_{m1}$, and maximal, $k_{m2}$, values of $k_0$ corresponding to the endpoints of the PFC. Taking into account the explicit expressions of the integrals (45) at $F(k) = k^{2n}$, one can obtain an estimation

$$ |A_1 - k_0^{2n}A_0| \leq \frac{\Delta \varphi k_{m1}^3 - k_m^3}{2\pi \langle v_F \rangle} $$  \hspace{1cm} (52)

where $\Delta \varphi = \varphi_2 - \varphi_1$ is an angular size of the PFC and

$$ \frac{1}{v_F} = \frac{1}{\Delta \varphi} \int_{\varphi_1}^{\varphi_2} \frac{d\varphi}{|\xi_k|} $$  \hspace{1cm} (53)
is an average (over the PFC) value of the dimensionless Fermi velocity. One can see from Eq.(12) that the absolute value of the SC order parameter decreases exponentially with a decrease both of the angular size and the anisotropy (characterizing by the difference \( k_M - k_m \)) of the PFC. In particular, in the case of repulsion-induced pairing with zero total pair momentum (when the PFC coincides with the FC), the non-trivial solution similar to Eq. (21) is absent.

Thus, using conventional energy and momentum dimensions, one can write down the SC order parameter, being the non-trivial solution of the self-consistency equation (14), in the form

\[
\Delta(k) = \Delta_0(k) \exp\left(-\frac{1}{w}\right)
\]  

where

\[
w = (A_1 - A_0 k_0^2)/P_1
\]

can be treated as an effective interaction strength and

\[
\Delta_0(k) = U_0 \exp\left(k_0^2 - k^2\right)
\]

is a pre-exponential factor dependent on the momentum of the relative motion of the pair.

VIII. STEP-WISE APPROXIMATION OF THE ORDER PARAMETER

The continuous solution Eq. (21) of the self-consistency equation can be approximately presented as a step-wise function of the momentum of the relative motion. Here, we briefly discuss the simplest form of such an averaged solution assuming that the true solution, Eq. (21), is replaced by an approximate one which has the form

\[
\Delta(k) = \Delta_p, \quad p = 1, 2
\]

where

\[
\Delta_p = \frac{1}{\Xi_p} \int_{\Xi_p} \Delta(k) d^2k.
\]

The parameters \( \Delta_1 \) and \( \Delta_2 \) defined by the Eq. (57) may be considered as average values of the order parameter inside the domain \( \Xi \) at \( k < k_0 \) and \( k > k_0 \) respectively.

Performing the averaging of the left- and right-hand sides of the self-consistency equation (14) one can obtain the equation system:

\[
\begin{align*}
2\Delta_1 &= -\alpha \Xi U_{11} \Delta_1 f_1 - (1 - \alpha) \Xi U_{12} \Delta_2 f_2, \\
2\Delta_2 &= -\alpha \Xi U_{21} \Delta_1 f_1 - (1 - \alpha) \Xi U_{22} \Delta_2 f_2
\end{align*}
\]

where \( \alpha = \Xi_1/\Xi, \quad 1 - \alpha = \Xi_2/\Xi \), and

\[
f_p \equiv f_p(\Delta_p) = \frac{1}{(2\pi)^2 \Xi_p} \int_{\Xi_p} \frac{d^2k}{\sqrt{\xi^2(k) + \Delta_p^2}}
\]

is the value of the functional,

\[
f_p\{\Delta(k)\} = \frac{1}{(2\pi)^2 \Xi_p} \int_{\Xi_p} \frac{\Delta(k) d^2k}{\sqrt{\xi^2(k) + \Delta_p^2}}, \quad (60)
\]

corresponding to the average value \( \Delta_p \) of the order parameter inside the subdomain \( \Xi_p \). The average value of the interaction matrix element \( U/(k' - k') \) corresponding to a momentum transfer \( \kappa = k - k' \) is defined as

\[
U_{pp'} = U_{p'p} = \frac{1}{\Xi_p^{*} \Xi_{p'}} \int_{\Xi_p^{*} \Xi_{p'}} U(\kappa) d^2\kappa
\]

where \( \Xi_p^{*} \), is a domain of definition of the momentum transfer \( \kappa \) when an initial momentum, \( k \), belongs to the subdomain \( \Xi_p \) whereas a final one, \( k' \), belongs to \( \Xi_{p'} \).

If one represents the functions (59) in the explicit form, the equations (58) lead to a closed system of two transcendental equations. The integrals (59) over the subdomains \( \Xi_p \) are similar to the integrals (40). Therefore, there exists a logarithmic singularity of such an integral when the domain of integration includes the PFC. The analysis of the non-trivial consistency of the Eqs.(58) shows that the non-trivial solution exists under the condition that

\[
U_{13} U_{21} - U_{11} U_{22} > 0.
\]

This condition coincides with the well-known Suhl–Matthias–Walker condition and is definitely fulfilled in the case of repulsive potentials which have at least one negative eigenvalue.

The approximate approach developed here to determine the order parameter arising at singlet repulsion-induced pairing and corresponding to extended s-wave symmetry can be used in the case of d-wave symmetry as well. In this connection, it should be noted that a simple replacement of the true kernel \( U(k - k') \) of the self-consistency equation by a constant

\[
U(k - k') \to \langle \Delta(k) U(k - k') \Delta(k') \rangle / \langle \Delta^2(k) \rangle
\]

(63)

treated as an average over the FC hardly ever leads to a non-trivial solution because of the fact that such an averaging procedure, similar to performed in Eqs.(16) and (29), means the summation over both \( k \) and \( k' \) resulting in a non-negative value of the constant, in the case of repulsive interaction. Thus, strictly speaking, considering the scattering due to a repulsive interaction \( U(k - k') \), one has to take into account not only a difference between the areas of the momentum space where the order parameter has different signs but also the dependence of the interaction matrix element on the momentum transfer \( k - k' \).

IX. CONCLUSION

In this paper, we have shown that there must be at least one negative eigenvalue of the interaction matrix
to ensure the possibility of a repulsion-induced non-trivial solution of the self-consistency equation. A rise of the pair Fermi contour results in the fact that such a necessary condition of repulsion-induced superconducting pairing becomes the sufficient one at arbitrarily weak interaction strength.

It should be emphasized that negative eigenvalues have to arise with necessity in real Fermi systems, in particular, due to Friedel oscillations of screened Coulomb potential. Quasi-two-dimensional electron system typical of cuprate superconductors may be considered as an intermediate case between comparatively weak Friedel oscillations in three-dimensional system and one-dimensional charge density wave behavior. The pair Fermi contour arises when, at certain large total pair momentum, there exists mirror nesting of the Fermi contour.

Repulsion-induced singlet pairing results in an extended s-wave symmetry of the superconducting order parameter. This parameter is defined inside the domain of definition of pair relative motion momentum and changes its sign on a line intersecting the pair Fermi contour. An anisotropy is a necessary feature of one-electron dispersion which leads to the repulsion-induced non-trivial solution of the self-consistency equation: the corresponding superconducting order is absent in the framework of any isotropic model.

Acknowledgments

The work was supported, in part, by the Russian scientific-educational programme “Integration” (project B0049), the Department of Education of Russian Federation (Grant E02-3.4-147), the Russian Foundation for Basic Research (grant 02-02-17133) and Federal purpose-technical-scientific programme (State contracts N 40.072.1.1.1173, N 40.012.1.1.1357).

* State Pedagogical University, Voronezh, 394043, Russia
† Electronic address: kopaev@sci.lebedev.ru
1 D. J. Scalapino, *Physics Reports* **250**, 329 (1995).
2 H. Aoki, cond-mat/0305490 (and references therein).
3 J. Bardeen, L. N. Cooper, and J. R. Schrieffer, *Phys. Rev.* **108**, 1175 (1957).
4 J. E. Hirsch, *Physica* C **341-348**, 213 (2000).
5 M. R. Norman, M. Randeria, B. Jankó, and J. C. Campuzano, *Phys. Rev.* B **61**, 14742 (2000).
6 J. Orenstein and A. J. Millis, *Science* **288**, 468 (2000).
7 J. Schmalian, D. Pines, and B. Stojković, *Phys. Rev. Lett.* **86**, 9253 (1995);
8 S. Sachdev, *Science* **288**, 475 (2000).
9 J. R. Schrieffer, *Theory of Superconductivity* (Benjamin, New York, 1983).
10 P. Fulde and R. A. Ferrel, *Phys. Rev.* **135**, A550 (1964); A. I. Larkin and Yu. N. Ovchinnikov, *Sov. Phys. JETP* **20**, 762 (1964).
11 Z. - X. Shen, W. E. Spicer, D. M. King, D. S. Dessau, and B. O. Wells, *Science* **267**, 343 (1995).
12 M. Chiao, R. W. Hill, C. Lupien, L. Taillefer, P. Lampert, R. Gagnon, and P. Fournier, *Phys. Rev. B* **62**, 3554 (2000).
13 V. I. Belyavsky, V. V. Kapayev, and Yu. V. Kopaev, *JETP Letters* **60**, 667 (1999).
14 V. I. Belyavsky, Yu. V. Kopaev, V. M. Sofronov and S. V. Shevtsov, Zh. Eksp. Teor. Fiz. **124**, 1149 (2003) [JETP **97** (2003)].
15 V. I. Belyavsky, Yu. V. Kopaev, V. M. Sofronov and S. V. Shevtsov, Zh. Eksp. Teor. Fiz. **124**, 1149 (2003).
16 S.-C. Zhang, *Science* **275**, 1089 (1997).
17 M. Guidry, L.-A. Wu, Y. Sun and C.-L. Wu, *Phys.Rev. B* **63**, 134516 (2001).
18 J. C. Campuzano, M. R. Norman, and M. Randeria, in *Physics of Conventional and Unconventional Superconductors*, K. H. Bennemann and J. B. Ketterson, eds. (Springer - Verlag, 2002).
19 A. Damascelli, Z. Hussain, and Z.-X. Chen, *Rev. Mod. Phys.* **75**, 473 (2003).
20 V. I. Belyavsky and Yu. V. Kopaev, *Phys. Lett. A* **287**, 152 (2001).
21 V. I. Belyavsky, V. V. Kapayev, and Yu. V. Kopaev, *JETP Letters* **60**, 44 (2002).
22 R. Hübina and T. M. Rice, *Phys. Rev. B* **51**, 9253 (1995); J. Schmalian, D. Pines, and B. Stojković, *Phys. Rev. B* **60**, 667 (1999).
23 V. I. Belyavsky, Yu. V. Kopaev, V. M. Sofronov and S. V. Shevtsov, Zh. Eksp. Teor. Fiz. **124**, 1149 (2003) [JETP **97** (2003)].
24 C. Kittel, *Quantum Theory of Solids* (John Wiley and Sons, inc., New York–London, 1963).
25 S. G. Mikhlin, *Lections on integral equations* (FM, Moscow, 1959).
26 E. M. Lifshitz and L. P. Pitaevskii, *Statistical Physics*, Part 2 (Pergamon, New York, 1980).
27 S. V. Shevtsov, to be published.
28 H. Suhl, B. T. Matthias, L. R. Walker. *Phys. Rev. Lett.* **3**, 552 (1959).