Fusion Rules and $\mathcal{R}$-Matrices
For Representations of $SL(2)_q$ at Roots of Unity*

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We recall the classification of the irreducible representations of $SL(2)_q$, and then give fusion rules for these representations. We also consider the problem of $\mathcal{R}$-matrices, intertwiners of the differently ordered tensor products of these representations, and satisfying altogether Yang–Baxter equations.

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1. Introduction

Representations of $SL(2)_q$ at roots of unity \([1,2]\) play several roles in physics. The irreducible representations corresponding to deformations of classical ones are used in conformal theory \([3,4]\) and in statistical theory \([5,6]\), whereas the new periodic representations (which exist only when $q$ is a root of unity) appear in relation with statistical models \([1,7–12]\).

We present here fusion rules for both types of representations, and $\mathcal{R}$-matrices that intertwine $\Delta$ and $\Delta'$ on tensor products.

The end of the introduction is devoted to definitions and we will also recall the classification of the irreducible representations (irreps) of $SL(2)_q$. In section 2, we recall the fusion rules of $q$-deformed irreps. In sections 3 and 4, we consider the fusion rules involving the representations that exist only when $q$ is a root of 1. Section 5 is devoted to $\mathcal{R}$-matrices.

$SL(2)_q$ is defined by the generators $k$, $k^{-1}$, $e$, $f$, and the relations

\[
\begin{align*}
kk^{-1} &= k^{-1}k = 1, \\
kek^{-1} &= q^2e, \\
kf k^{-1} &= q^{-2}f, \\
[e, f] &= \frac{k - k^{-1}}{q - q^{-1}}. 
\end{align*}
\]

The coproduct $\Delta$ is given by

\[
\begin{align*}
\Delta(k) &= k \otimes k \\
\Delta(e) &= e \otimes 1 + k \otimes e \\
\Delta(f) &= f \otimes k^{-1} + 1 \otimes f,
\end{align*}
\]

while the opposite coproduct $\Delta'$ is $\Delta' = P\Delta P$ where $P$ is the permutation map $Px \otimes y = y \otimes x$.

The result of the composition of two representations $\rho_1$ and $\rho_2$ of $SL(2)_q$ is the representation $\rho = (\rho_1 \otimes \rho_2) \circ \Delta$, whereas the composition in the reverse order is equivalent to $\rho' = (\rho_1 \otimes \rho_2) \circ \Delta'$.

When $q$ is not a root of unity, the representation theory is similar to the classical one \([13]\).
In the following, the parameter $q$ will be a root of unity. Let $m'$ be the smallest integer such that $q^{m'} = 1$. Let $m$ be equal to $m'$ if $m'$ is odd, and to $m'/2$ otherwise.

In addition to the usual quadratic Casimir

$$C = fe + (q - q^{-1})^{-2} (qk + q^{-1}k^{-1})$$

the centre of $SL(2)_q$ contains also $e^m, f^m,$ and $k^±m$. Following ref. [14], we will denote by $x, y, z^±1,$ and $c$ the values of $e^m, f^m, k^±m,$ and $C$ on irreducible representations.

We now recall the classification [2] of the irreducible representations of $SL(2)_q$. The new facts are that the dimension of the finite dimensional irreps are bounded by $m$, and that the irreps of dimension $m$ depend on three complex continuous parameters. In the following, we will call type A irreps those that have a classical analogue and type B irreps the others. We will mostly use a module notation in the following.

The $q$-deformed classical irreps (type A) are labelled by their half-integer spin $j$, such that $1 ≤ 2j + 1 ≤ m$, and by another discrete parameter $ω = ±1$. They are given [2] by the basis \{w_0, ..., w_{2j}\} and, in a notation of module,

$$\begin{align*}
k w_p &= ωq^{2j−2p}w_p \\
f w_p &= w_{p+1} \\
f w_{2j} &= 0 \\
e w_p &= ω[p][2j − p + 1]w_{p−1} \text{ for } 1 ≤ p ≤ 2j \\
ev_0 &= 0
\end{align*}$$

where as usual

$$[x] \equiv \frac{q^x - q^{-x}}{q - q^{-1}}.$$ 

We denote this representation by Spin ($j, ω$). On it, the central elements $e^m, f^m, k^m$, and $C$ take the values $x = y = 0, z = (ωq^{2j})^m = ±1,$ and $c = ω(q - q^{-1})^{-2} (q^{2j+1} + q^{-2j−1})$ respectively.

Note that the representation Spin ($j, ω = −1$) can be obtained as the tensor product of Spin ($j, 1$) by the one-dimensional representation Spin ($j = 0, ω$).

A type B irrep is an irreducible representation that has no finite dimensional analogue when $q$ is equal to one. It has dimension $m$ and is characterized by three complex parameters $β, y,$ and $λ$. This representation is given in the basis \{v_0, ..., v_{m−1}\} by

$$\begin{align*}
k v_p &= λq^{−2p}v_p \\
f v_p &= v_{p+1} \\
f v_{m−1} &= yv_0 \\
e v_p &= (p)[μ − p + 1] + yβ) v_{p−1} \text{ for } 1 ≤ p ≤ m−1 \\
ev_0 &= βv_{m−1}
\end{align*}$$

(1.6)
with the definition \( q^\mu \equiv \lambda \).

The central elements \( e^m, f^m, k^m, \) and \( C \) take the values
\[
x = \beta \prod_{p=1}^{m-1} ([p][\mu - p + 1] + y\beta),
\]
where \( y, z = \lambda^m, \) and \( c = y\beta + (q - q^{-1})^{-2} (q\lambda + q^{-1}\lambda^{-1}) \) respectively. The numbers \( x, y \) and \( z \) actually almost characterize the irreps, up to a discrete choice for the value of the quadratic Casimir \( C \) \(^{(1.3)}\), which satisfies an \( m \)-th-degree polynomial equation with coefficients depending on \( x, y \) and \( z \) \(^{(14)}\).

We will denote the representation given by \((1.6)\) either by \( B(\beta, y, \lambda) \) or equivalently by \( B'(x, y, z, c) \). The first notation turns out to be of much simpler use when there is a highest- or lowest-weight vector. The second one is directly related to the spectrum of the centre.

The representation \((1.6)\) is actually irreducible iff one of the three following conditions is satisfied:

a. \( x \neq 0 \),

b. \( y \neq 0 \),

c. \( \beta = 0 \) and \( \lambda^2 \in C \setminus \{1, q^2, ..., q^{2(m-2)}\} \).

(Note that \( B(0, 0, \pm q^{m-1}) = \text{Spin}((m-1)/2, \pm 1) \) is actually of type A.)

The representation \((1.6)\) will be called periodic if \( xy \neq 0 \). In this case it is irreducible and has no highest-weight and no lowest-weight vector.

A semi-periodic representation is a representation for which one only of the parameters \( x \) or \( y \) vanishes. It is then also irreducible.

### 2. Composition of type A representations

This section will be a brief review of the results of Pasquier and Saleur \(^{(6)}\), and of Keller \(^{(15)}\).

The tensor product of two representations \( \text{Spin}(j_1, \omega_1) \) and \( \text{Spin}(j_2, \omega_2) \) decomposes into irreducible representations of the same type and also, if \( 2(j_1 + j_2) + 1 \) is greater than \( m \), into some indecomposable spin representations \(^{(3,13)}\).
An indecomposable spin representation \( \text{Ind} (j, \omega) \) has dimension \( 2m \). It is characterized by a half-integer \( j \) such that \( 1 \leq 2j + 1 < m \) and by \( \omega = \pm 1 \). On a basis \( \{ w_0, \ldots, w_m, x_0, \ldots, x_m \} \) the generators of \( SL(2)_q \) act as follows:

\[
\begin{align*}
kw_p &= \omega q^{-2j-2-2p} w_p \\
fw_p &= w_{p+1} \\
fw_{m-1} &= 0 \\
ev_p &= \omega[p][-2j - p - 1] w_{p+1} \\
kx_p &= \omega q^{2j-2p} x_p \\
x_{p+1} &= x_p = x_{m-1} = 0 \\
ex_p &= f^{p+m-2j-2} w_0 + \omega[p][2j - p + 1] x_{p-1} \\
&\quad \text{for } 0 \leq p \leq m-1
\end{align*}
\]

(\text{In particular, } ex_0 = w_{m-2j+2} \text{ and } ex_{2j+1} = w_{m-1}, \text{and } e^m, f^m \text{ are 0 on such a module.})

This indecomposable representation contains the sub-representation \( \text{Spin} (j, \omega) \). It is a deformation of the sum of the classical \( \text{Spin} (j) \) and \( \text{Spin} (m/2 - j - 1) \) representations.

The fusion rules are

\[
\text{Spin} (j_1, \omega_1) \otimes \text{Spin} (j_2, \omega_2) = \left( \bigoplus_{j = [j_1-j_2]} \text{Spin} (j, \omega_1\omega_2) \right) \oplus \left( \bigoplus_{j = [m-j_1-j_2-1]} \text{Ind} (j, \omega_1\omega_2) \right),
\]

(2.2)

where the sums are limited to integer values of \( j \) if \( j_1 + j_2 \) is integer, and to half-(odd)-integer values if \( j_1 + j_2 \) is half-(odd)-integer. In conformal field theories, the fusion rules (2.2) are truncated to the first parenthesis, keeping only the representations of \( q \)-dimension different from 0 in the result.

The fusion rules of type A representations close with

\[
\text{Spin} (j_1, \omega_1) \otimes \text{Ind} (j_2, \omega_2) = \bigoplus_{\text{some } j, \omega} \text{Ind} (j, \omega)
\]

\[
\text{Ind} (j_1, \omega_1) \otimes \text{Ind} (j_2, \omega_2) = \bigoplus_{\text{some } j, \omega} \text{Ind} (j, \omega).
\]

(2.3)

3. Fusion rules mixing type A and type B representations

\textbf{Proposition 1:} The tensor product of a semi-periodic representation with a spin \( j \) representation is completely reducible. More precisely,

\[
B (0, y, \lambda) \otimes \text{Spin} (j, \omega) = \bigoplus_{l=0}^{2j} B (0, (\omega(q)^{2j})^m y, q^{2(j-l)} \lambda\omega).
\]

(3.1)
Proof: All the weight spaces of the tensor product have the same dimension $2j + 1$. In each of them there is a singular vector, i.e., a vector in the kernel of $\rho(e)$. Since $\rho(f)$ is injective, these vectors generate semi-periodic sub-representations which do not mix, because the quadratic Casimir takes different values on each of them.

Remark 1: We stated the proposition with a highest-weight semi-periodic representation. A similar decomposition holds if $y = 0$ and $x \neq 0$.

Remark 2: The decomposition (3.1) also holds with $y = 0$, as soon as $\lambda^2$ is not a power of $q$.

Proposition 2: The tensor product of a semi-periodic representation with the indecomposable spin representation $\text{Ind} (j, \omega)$ is completely reducible. Moreover,

$$B(0, y, \lambda) \otimes \text{Ind} (j, \omega) = \bigoplus_{l=0}^{m-1} 2B(0, (\omega(q)^{2j})^m y, q^{2l} \lambda \omega).$$  \hspace{1cm} (3.2)

Proof: We can find $2m$ highest weight vectors in the tensor product, or use the previous proposition and the coassociativity of $\Delta$.

Proposition 3: The tensor product of a periodic representation $B'(x, y, z, c)$ with the spin representation $\text{Spin} (j, \omega)$ is reducible for generic values of the parameters defining the periodic representation, and

$$B'(x, y, z, c) \otimes \text{Spin} (j, \omega) = \bigoplus_{l=0}^{2j} B'(x, (\omega(q)^{2j})^m y, (\omega(q)^j)^m z, c_l).$$  \hspace{1cm} (3.3)

Proof: One can first consider $j = 1/2$. The quadratic Casimir $C$ (1.3) is diagonalizable on the tensor product iff $c \neq \pm 2/(q - q^{-1})^2$. (This defines the generic case for $j = 1/2$.) So we obtain in this case the direct sum (3.3), where $c_0$ and $c_1$ are the (different) eigenvalues found for $C$ on the tensor product. In the non-generic case (i.e., if $c$ takes one of the two values $\pm 2/(q - q^{-1})^2$) the result is a (periodic) indecomposable representation of dimension $2m$ with a quite simple structure (two copies of (1.4) with the same parameters, plus a branching from one to the other). We then go to $j > 1/2$ using the coassociativity of $\Delta$ or by a direct analogous proof.
4. Fusion of type B representations

Consider two irreps of type B: $\rho_1 = B'(x_1, y_1, z_1, c_1)$ and $\rho_2 = B'(x_2, y_2, z_2, c_2)$.

Then the central elements $e^m, f^m, k^m$ are scalar on the tensor product $\rho = (\rho_1 \otimes \rho_2) \circ \Delta$ and take the values

$$
\begin{align*}
    x &= x_1 + z_1 x_2, \\
    y &= y_1 z_2^{-1} + y_2, \\
    z &= z_1 z_2.
\end{align*}
$$

(4.1)

They are also scalar on $\rho' = (\rho_1 \otimes \rho_2) \circ \Delta'$ and take the values $(x' = x_2 + z_2 x_1, y' = y_2 z_1^{-1} + y_1, z' = z_1 z_2)$.

We see that $\rho$ and $\rho'$ can be equivalent only if their parameters belong to the same algebraic curve \[8\]

$$
\frac{x_1}{1 - z_1} = \frac{x_2}{1 - z_2}, \quad \frac{y_1}{1 - z_1^{-1}} = \frac{y_2}{1 - z_2^{-1}} \quad (4.2)
$$

and that in this case $x = x', y = y', z = z'$ also satisfy these relations.

Until the end of this section we consider the composition of $\rho_1$ and $\rho_2$ with $\Delta$, without imposing the condition \((1.2)\).

Each weight space of $B'(x_1, y_1, z_1, c_1) \otimes B'(x_2, y_2, z_2, c_2)$ has dimension $m$. The weights are all the $m^{th}$ roots of $z = z_1 z_2$.

The rank of $\Delta(e)$ restricted to a weight space is either $m$ or $m - 1$. It does not depend on the weight. It is equal to $m$ if $x \neq 0$ and to $m - 1$ if $x = 0$.

**Proposition 4:** For generic values of the parameters $(x_1, y_1, z_1, c_1)$ and $(x_2, y_2, z_2, c_2)$, the tensor product is reducible and

$$
B'(x_1, y_1, z_1, c_1) \otimes B'(x_2, y_2, z_2, c_2) = \bigoplus_{l=0}^{m-1} B'(x, y, z, c_l), \quad (4.3)
$$

where $x, y$ and $z$ are given by \((1.1)\).

**Proof:** Consider the characteristic polynomial of the quadratic Casimir $C$ \((1.3)\) on one of the weight spaces of the tensor product. The parameters $x_1, y_1, x_2$ and $y_2$ always enter in the coefficient through the products $x_1 y_1$ and $x_2 y_2$, except in the constant term where a non-trivial linear combination of the products $x_1 y_2$ and $x_2 y_1$ appears. So this polynomial has $m$ distinct roots for generic values of the parameters. These roots are then all the allowed values for $c$ with a given $(x, y, z)$. Since the characteristic polynomial of $C$ is continuous in the parameters, it is then proportional, for all the values of the parameters
of the representations, to the polynomial of ref. [14], the roots of which are the possible values of \( c \) for given \((x, y, z)\). Our non-generic case happens when this polynomial has not only simple zeroes.

In the generic case, the eigenvectors of \( C \) generate the \( m \) periodic representations \( B'(x, y, z, c_l) \).

**Remark:** in ref. [8], the underlying quantum Lie algebra is the affine \( \hat{SL}(N)_q \). Analogous tensor products are in this case irreducible, in contrast with the present results. Remember that in our case the dimension of irreps is bounded by \( m \).

**Proposition 5:** Consider values of the parameters \((x_1, y_1, z_1, c_1)\) and \((x_2, y_2, z_2, c_2)\) such that on the tensor product \( xy = 0 \) and \( z^2 \neq 1 \). Let us choose \( x = 0 \). Then the tensor product is reducible and

\[
B'(x_1, y_1, z_1, c_1) \otimes B'(x_2, y_2, z_2, c_2) = \bigoplus_{l=0}^{m-1} B'(0, y, z, c_l).
\]

**Proof:** Since \( x = 0 \), \( \Delta(e) \) has rank \( m - 1 \) on each weight space. So there is one highest-weight vector in each weight space. Since \( z^2 \neq 1 \), each of them generates an \( m \)-dimensional representation with no singular vector. The values of \( C \) are distinct on these representations.

**Remarks:**

1) Proposition 5 includes the case of the composition of semi-periodic representations, except when \( z_1z_2 = \pm 1 \). This last case is more subtle. We will discuss it after the second remark.

2) Two tensor products of type B irreps giving the same \((x, y, z)\) are generically equivalent: according to proposition 4 they have the same decomposition. This is also true when the parameters satisfy the assumptions of proposition 5. However, this is not true for all the values of the parameters, as we will see.

Consider now the tensor product \( B'(x_1, y_1, z_1, c_1) \otimes B'(x_2, y_2, z_2, c_2) \) leading to \( x = y = 0, z = \pm 1 \). Note that, according to (4.1), there is still some freedom for the choice of the parameters (on a three-dimensional manifold). We claim that this tensor product is equivalent, for generic values of the remaining parameters, to the tensor product

\[
B(\beta_1 = 0, y_1 = 0, \lambda_1) \otimes B(\beta_2 = 0, y_2 = 0, \lambda_2), \quad \text{with} \quad (\lambda_1 \lambda_2)^m = z,
\]

which is a sum of indecomposable representations \( \text{Ind}\ (j, \omega) \) (plus some \( \text{Spin}\ ((m-1)/2, \omega) \)).
However, there are values of the remaining parameters for which the decomposition is not equivalent to (4.3). For values of the parameters lying on a submanifold (of the three-dimensional manifold leading to \(x = y = 0, z = \pm 1\)), the invariant subspace which generically leads to a sub-representation equivalent to \(\text{Ind}(j, \omega)\) can now be split into the terms of the sum \(B(\beta \neq 0, y = 0, \lambda = q^{2j}) \oplus B(\beta \neq 0, y = 0, q^{-2j-2})\). These representations are indecomposable and they never appear in the fusion rules of type A irreps. They are not periodic in the sense that they correspond to \(x = y = 0\), but they share with periodic representations the fact that \(e^p\) and \(f^m - p\) can have non-vanishing matrix elements between the same vectors, in the basis of (1.6) which diagonalizes \(k\). They contain as irreducible sub-representation the Spin \((m/2 - j - 1, -\omega)\) and Spin \((j, \omega)\) irreps, respectively. So the signal of this non-generic event in the already non-generic case \(x \neq y \neq 0, z = \pm 1\) is the appearance of Spin \((m/2 - j - 1, -\omega)\) as an irreducible sub-representation. (Note that when \(m\) is odd, \(m/2 - j - 1\) is non-integer when \(j\) is integer, and vice versa). As an example when \(m\) is even, the splitting of the part \(\text{Ind}(j = m/2 - 1, \omega)\) occurs when \(c_1 = c_2 = 0\) and Spin \((0, -\omega)\) appears in the spectrum.

We end this section with a remark on the regular representation of \(SL(2)_q\). It is defined on the vector space \(U_q(SL(2))\) itself with the further relations \(e^m = f^m = 0, k^m = 1\), and has dimension \(m^3\). It is equivalent to \(\bigoplus_{p=0}^m B(0, 0, \lambda) \otimes B(0, 0, \lambda^{-1} q^{2p})\).

5. \(R\)-matrices

When \(q\) is a root of unity, there is no universal \(R\)-matrix intertwining \(\Delta\) and \(\Delta'\) at the level of the algebra. When the representations \((\rho_1 \otimes \rho_2) \circ \Delta\) and \((\rho_1 \otimes \rho_2) \circ \Delta'\) are equivalent, there exist \(R(1, 2)\) such that

\[
\forall X \in SL(2)_q \quad R(1, 2)(\rho_1 \otimes \rho_2) \circ \Delta(X) = (\rho_1 \otimes \rho_2) \circ \Delta'(X)R(1, 2) .
\]

(5.1)

The truncation of the formal universal \(R\)-matrix

\[
R_u = q^{-\frac{1}{2} h \otimes h} \sum_{n=0}^{m-1} q^n \frac{(1 - q^2)^n}{[n]!} q^{-n(n-1)/2(k-1)e^n} \otimes (kf)^n
\]

(5.2)

(where \(k \equiv q^h\)) provides intertwiners for \(\Delta\) and \(\Delta'\) when evaluated on tensor product of type A representations. This is also true with the truncation of the inverse of the permuted universal \(R\)-matrix

\[
\tilde{R}_u = q^{\frac{1}{2} h \otimes h} \sum_{n=0}^{m-1} \frac{(g - q^{-1})^n}{[n]!} q^{-n(n-1)/2} f^n \otimes e^n .
\]

(5.3)
These intertwiners satisfy Yang–Baxter equations altogether.

Tensor products with \( \Delta \) and \( \Delta' \) of type B representations are not always equivalent \([8,10]\). When the parameters of the representations lie on the same algebraic curve (4.2), intertwiners for these tensor products have been found in refs. \([8,10]\), in relation with the Boltzmann weight of some statistical model. They also satisfy the Yang–Baxter equation.

We now recall results of refs. \([16,17]\) on \( \mathcal{R} \)-matrices for tensor products involving both types (A and B) of representations. Let us call \( \mathcal{R}_{+}(1,2) \) (resp. \( \mathcal{R}_{-}(1,2) \)) the evaluation of \( \mathcal{R}_u \) (resp. \( \tilde{\mathcal{R}}_u \)) on the tensor product of representations 1 and 2.

**Proposition 6:** Let \( B' (x, y, z, c) \) and \( B' (x', y', z', c') \) be two representations for which there exists an intertwiner \( \mathcal{R}(x, x') \) (\( x \) and \( x' \) will refer in the following to the whole sets of parameters \( (x, y, z, c) \) and \( (x', y', z', c') \)). Let Spin \((J, \omega)\) be a type A irrep. We denote by \( V_x, V_{x'} \) and \( V_J \) the vectors spaces on which these representations act. Then the following Yang–Baxter equations are satisfied,

\[ \mathcal{R}_{12}(x, x') \mathcal{R}_{13}^+(x, J) \mathcal{R}_{23}^+(x', J) = \mathcal{R}_{23}^+(x', J) \mathcal{R}_{13}^+(x, J) \mathcal{R}_{12}(x, x') . \]  

(5.4)

b) On \( V_x \otimes V_J \otimes V_{x'} \),

\[ \mathcal{R}_{12}^+(x, J) \mathcal{R}_{13}(x, x') \mathcal{R}_{23}^-(J, x') = \mathcal{R}_{23}^-(J, x') \mathcal{R}_{13}(x, x') \mathcal{R}_{12}^+(x, J) . \]  

(5.5)

c) On \( V_J \otimes V_x \otimes V_{x'} \),

\[ \mathcal{R}_{12}^-(J, x) \mathcal{R}_{13}^-(J, x') \mathcal{R}_{23}(x, x') = \mathcal{R}_{23}(x, x') \mathcal{R}_{13}^-(J, x') \mathcal{R}_{12}^-(J, x) . \]  

(5.6)

d) One can replace in a), b) and c) above one or both of the type B representations \( B' (x, y, z, c) \) and \( B' (x', y', z', c') \) by type A irreps, changing \( \mathcal{R}(x, x') \) to the corresponding \( \mathcal{R}_{+} \) (or also \( \mathcal{R}_{-} \)), and eqs. (5.4), (5.5), (5.6) are still valid. Furthermore, all the type A irreps can also be replaced by indecomposable representations occurring in the fusion rules of type A irreps. Finally, \( \mathcal{R}_{+} \) and \( \mathcal{R}_{-} \) can be exchanged globally in each equation.

However,
e) the Yang–Baxter equation

\[ R_{12}^+(x, J) R_{13}^+(x, x') R_{23}^+(J, x') = R_{23}^+(J, x') R_{13}^+(x, x') R_{12}^+(x, J) \] (5.7)

cannot be satisfied on \( V_x \otimes V_J \otimes V_{x'} \) for generic \( x \) and \( x' \).

The pairs \((R^+(x, J), R^-(J, x))\) and \((R^-(x, J), R^+(J, x))\) are actually the only solutions for intertwiners satisfying Yang–Baxter equations altogether and with \( R(x, x') \) when periodic representations are involved.

In ref. [17], a quantum chain is presented as an example of a new physical model involving both type A and type B representations.

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