Abstract

We obtain new lower bounds for the independence number of \( K_r \)-free graphs and linear \( k \)-uniform hypergraphs in terms of the degree sequence. This answers some old questions raised by Caro and Tuza \cite{8}. Our proof technique is an extension of a method of Caro and Wei \cite{7, 20}, and we also give a new short proof of the main result of \cite{8} using this approach. As byproducts, we also obtain some non-trivial identities involving binomial coefficients.

1 Introduction

For \( k \geq 2 \), a \( k \)-uniform hypergraph \( H \) is a pair \( (V(H), E(H)) \) where \( E \subseteq \binom{V(H)}{k} \). A set \( I \subseteq V(H) \) is an independent set of \( H \) if \( e \not\subseteq I \) for every \( e \in E(H) \), or equivalently, \( \binom{I}{k} \cap E(H) = \emptyset \). The independence number of \( H \), denoted by \( \alpha(H) \), is the maximum size of an independent set in \( H \). For \( u \in V(H) \), its degree in \( H \), denoted by \( d_H(u) \), is defined to be \( |\{e \in E(H) : u \in e\}| \) (we omit the subscript if it is obvious from context). Throughout this paper, we use \( t \) to denote \( k-1 \) except in some places where it stands for some real value (the correct meaning can be easily inferred from the context). Also, we use the term graph whenever \( k \) happens to be 2.

A \( k \)-uniform hypergraph is linear if it has no 2-cycles where a 2-cycle is a set of 2 hyperedges containing at most 2 \( t \) vertices. The dual of the above definition says that a linear hypergraph is one in which every pair of vertices is contained in at most one hyperedge.

In \cite{19}, Turán proved a theorem giving a tight bound on the maximum number of edges that a \( K_r \)-free graph can have, which has since become the cornerstone theorem of extremal graph theory. Turán’s theorem, when applied to the complement \( \overline{G} \) of a graph \( G \), yields a lower bound \( \alpha(G) \geq \frac{n}{d+1} \) where \( d \) denotes the average degree in \( G \) of its vertices.
Caro \[7\] and Wei \[20\] independently proved that \(\alpha(G) \geq \sum_v \frac{1}{d(v)+1}\) which is at least \(\frac{n}{d^2}\). The probabilistic proof of their result later appeared in \[3\]. One natural extension of Turán’s theorem to \(k\)-uniform hypergraphs \(H\) is the bound \(\alpha(H) \geq c_k \frac{n}{d} + 1\), and this was shown via an easy probabilistic argument by Spencer \[15\]. Caro and Tuza \[8\] improved this bound for irregular \(k\)-uniform hypergraphs by proving that

\[
\alpha(H) \geq \sum_{v \in V(H)} \left( \frac{1}{d(v)+1/t} \right). \tag{1}
\]

Indeed, an easy consequence of (1) is the following result.

**Theorem 1.1 (Caro-Tuza \[8\])**  For every \(k \geq 3\), there exists \(d_k > 0\) such that every \(k\)-uniform hypergraph \(H\) has

\[
\alpha(H) \geq d_k \sum_{v \in V(H)} \left( \frac{1}{d(v)+1} \right)^{1/t}.
\]

As a corollary, one infers the bound of Spencer above. Later, ThieLe \[18\] provided a lower bound on the independence number of non-uniform hypergraphs, based on the degree rank (a generalization of degree sequence).

In this paper, we prove new lower bounds for the independence number of locally sparse graphs and linear \(k\)-uniform hypergraphs. The starting point of our approach is the probabilistic proof of Boppana-Caro-Wei. This approach, together with some additional simple ideas, quickly yields a new short proof of Theorem 1.1 (see Section 2 for the detailed proof).

### 1.1 \(K_r\)-free graphs

For certain classes of sparse graphs, improvements of the Caro-Wei bound (in terms of average degree \(d\)) are known. Ajtai, Komlós and Szemerédi \[11\] proved a lower bound of \(\Omega \left( \frac{n \log d}{d} \right)\) for the independence number of triangle-free graphs. An elegant and simpler proof was later given by Shearer \[13\], who also improved the constant involved. Later Shearer \[14\] also proved a bound of \(\Omega \left( \frac{n \log d}{d \log \log d} \right)\) for \(K_r\)-free graphs when \(r > 3\).

Caro and Tuza \[8\] raised the following question in their 1991 paper :

(i) Can the lower bounds of Ajtai et al \[11\] and Shearer \[13, 14\] be generalized in terms of degree sequences?

We answer this question via the following two theorems.

**Theorem 1.2**  For every \(\epsilon \in [0,1)\) there exists \(c > 0\) such that the following holds: Every triangle-free graph \(G\) with average degree \(D\) has independence number at least

\[
c(\log D) \sum_{v \in V(G)} \frac{1}{\max \{D', d(v)\}}.
\]

---

\(^1\)According to R. Bopanna \[10\], the probabilistic argument in \[3\] was obtained by him, although it is possible that it was known earlier.
**Theorem 1.3** For every $\epsilon \in [0, 1)$ and $r \geq 4$, there exists $c > 0$ such that the following holds: Every $K_r$-free graph $G$ with average degree $D$ has independence number at least

$$c \frac{\log D}{\log \log D} \sum_{v \in V(G)} \frac{1}{\max \{D^r, d(v)\}}.$$  

1.2 Linear Hypergraphs

As mentioned earlier, a lower bound of $\Omega \left( \frac{n}{d^{1/t}} \right)$ for an $n$ vertex $k$-uniform hypergraph with average degree $d$ can be inferred from Theorem 1.1. Caro and Tuza [8] also raised the following question:

(ii) How can one extend the lower bounds of Ajtai et al [1] and Shearer [13, 14] to hypergraphs?

As it turns out, such extensions were known for the class of linear $k$-uniform hypergraphs. Indeed, the lower bound

$$\alpha(H) = \Omega \left( n \left( \frac{\log d}{d} \right)^{1/t} \right),$$

where $H$ is a linear $k$-uniform hypergraph with average degree $d$ was proved by Duke-Lefmann-Rödl [9], using the results of [2]. Our final result generalizes [2] in terms of the degree sequence of the hypergraph.

**Theorem 1.4** For every $k \geq 3$ and $\epsilon \in [0, 1)$, there exists $c > 0$ such that the following holds: Every linear $k$-uniform hypergraph $H$ with average degree $D$ has independence number at least

$$c (\log D)^{1/t} \sum_{v \in V(H)} \frac{1}{\max \{D^{r/t}, (d(v))^{1/t}\}}.$$  

We also describe an infinite family of $k$-uniform linear hypergraphs to illustrate that the ratio between the bounds of Theorem 1.4 and [2] can be unbounded in terms of the number of vertices.

The remainder of this paper is organized as follows. In Section 2, we give a new short proof of Theorem 1.1. In Section 3 we apply the analysis in Section 2 to the special case of linear hypergraphs, and obtain a “warm-up” result - Theorem 3.1 which will be helpful in proving the main technical result, Theorem 4.1 proved in Section 4. The expression obtained in Theorem 4.1 plays a crucial role in the proofs of Theorems 1.2, 1.3 and 1.4 these are provided in Section 5. In Section 6, we give infinite families of $K_r$-free graphs and $k$-uniform linear hypergraphs which illustrate that the bounds in Theorems 1.2, 1.3 and 1.4 can be bigger than the corresponding bounds in [1, 2, 9, 13, 14] by arbitrarily large multiplicative factors. Finally, in section 7 we state several combinatorial identities which follow as simple corollaries of Theorem 4.1.

2 A new proof of Theorem 1.1

In this section we obtain a new short proof of Theorem 1.1. First we obtain the following theorem which is later used to prove Theorem 1.1.
Theorem 2.1 For every $k \geq 2$, there exists a constant $c = c_k$ such that any $k$-uniform hypergraph $H$ on $n$ vertices and $m \geq 1$ hyperedges satisfies

$$\sum_{J \subseteq V(H)} \frac{1}{{n \choose |J|}} > c \frac{n}{m^{1/k}} \quad \ldots \quad (A)$$

where we sum over all independent sets $J$.

Proof Let $t_k(n, m)$ denote the LHS of $(A)$. Consider any edge $e \in E(H)$. $e$ can belong to at most $\binom{n-k}{j-k}$ non-independent sets of size $j$. Since there are $m$ edges there are at most $m \binom{n-k}{j-k}$ sets of size $j$ that are not independent. Thus, at least $\binom{n}{j} - m \binom{n-k}{j-k}$ sets of size $j$ are independent. Hence we have

$$t_k(n, m) \geq \sum_{j=1}^{n} \left(1 - m \frac{\binom{n-k}{j-k}}{\binom{n}{j}}\right) = \sum_{j=1}^{n} \left(1 - m \frac{\binom{j}{k}}{\binom{n}{k}}\right)$$

$$> \sum_{j=1}^{\lceil n/(2m)^{1/k} \rceil} \left(1 - m \frac{j}{n}\right) \geq \sum_{j=1}^{\lceil n/(2m)^{1/k} \rceil} \left(1 - m \frac{1}{2m}\right)$$

$$\geq \frac{1}{2} \frac{n}{(2m)^{1/k}} \geq c_k \frac{n}{m^{1/k}}$$

for some suitably chosen $c_k$ which is close to $2^{-(k+1)/k}$. \hfill \Box

Let $H = (V, E)$ be a $k$-uniform hypergraph. For $k \geq 3$ and for $u \in V$ with $d_H(u) \geq 1$, the link graph associated with $u$ in $H$ is the $t$-uniform hypergraph $L_u = (U, F)$ where $U := \{v \neq u : \exists e \in E : \{u, v\} \subseteq e\}$ and $F = \{e \setminus u : u \in e \in E\}$. Let $I(H)$ denote the collection of independent sets of $H$.

Proof of Theorem 1.1. Let $H = (V, E)$ be an arbitrary $k$-uniform hypergraph. Choose uniformly at random a total ordering $<$ on $V$. Define an edge $e \in E$ to be backward for a vertex $v \in e$ if $u < v$ for every $u \in e \setminus \{v\}$. Define a random subset $I$ to be the set of those vertices $v$ such that no edge $e$ incident at $v$ is backward for $v$ with respect to $<$. Clearly, $I$ is independent in $H$. We have $E[I] = \sum_v Pr[v \in I]$. If $d_v = 0$, then $v \in I$ with probability 1. Hence, we assume that $d(v) \geq 1$. From the definition of $I$, it follows that $v \in I$ if and only if for every $e$ incident at $v$, $e \setminus \{v\} \not\subseteq S_v = \{u \in V(L_v) : u < v\}$. In other words, $S_v$ is an independent set in $L_v$. Let $l_v = |V(L_v)|$. Then

$$Pr[v \in I] = \sum_{J \in \mathcal{I}(L_v)} \frac{|J|!(l_v - |J|)!}{l_v + 1} \leq \frac{1}{l_v + 1} \sum_{J \in \mathcal{I}(L_v)} \frac{1}{|J|}$$

Applying Theorem 2.1 to the $t$-uniform link graph $L_v$ (with $c = c_{k-1}$), we get

$$Pr[v \in I] \geq \frac{c}{l_v + 1} \frac{l_v}{d(v)^{1/(k-1)}} \geq \frac{cl_v}{l_v + 1} \frac{1}{(d(v) + 1)^{1/(k-1)}}$$

Since $l_v \geq k - 1$, we get $Pr[v \in I] \geq \frac{(k-1)c/k}{(d(v) + 1)^{1/(k-1)}}$. By choosing $d_k = (k-1)c/k$, we get the lower bound of the theorem. \hfill \Box
In this section, we state and prove a warm-up result on the probability of having no backward edges incident at a vertex for a randomly chosen linear ordering (Theorem 3.1 below). The problem is the same as in the previous section, only, now the hypergraph under consideration is assumed to be linear and we get an explicit closed-form expression for this probability. This result will be helpful for the proof of the main technical theorem, given in the next section. In order to state the lower bound, we need the following definition (of fractional binomial coefficients) from [11].

**Definition** For $t > 0$, $a \geq 0$, $d \in \mathbb{N}$

$$\binom{d + 1/t}{a} := \frac{(td + 1)(t(d - 1) + 1)...(t(d - a + 1) + 1)}{a!t^a}$$

**Theorem 3.1** Let $H$ be a linear $k$-uniform hypergraph and let $v$ be an arbitrary vertex having degree $d$. For a uniformly chosen total ordering $<$ on $V$, the probability $P_v(0)$ that $v$ has no backward edge incident at it, is given by

$$P_v(0) = \frac{1}{\binom{d+1/t}{d}}$$

**Remark.** It is interesting to note that the above expression when summed over all vertices, is the same bound which Caro and Tuza obtain in [8] (using very different methods), although their bound holds for independent sets in general $k$-uniform hypergraphs.

We prove the theorem using the well-known Principle of Inclusion and Exclusion (PIE). First we state an identity involving binomial coefficients.

**Lemma 3.2** Given non-negative integers $d$ and $t$,

$$\sum_{r=0}^{d} (-1)^r \binom{d}{r} \frac{1}{tr+1} = \frac{1}{\binom{d+1/t}{d}}$$

This identity is already known (see [11], Equation 5.41). However, we give an alternate proof (using hypergeometric series) in the Appendix.

**Proof of Theorem 3.1** Firstly, observe that since $H$ is linear, the number of vertices that are neighbors of $v$ is exactly $(k-1)d = td$. Next, notice that since the random ordering is uniformly chosen, only the relative arrangement of these $td$ neighbors and the vertex $v$, i.e. $td + 1$ vertices in all, will determine the required probability. Hence the total number of orderings under consideration is $(td + 1)!$.

Label the hyperedges incident at $v$ with 1, ..., $d$ arbitrarily. For a permutation $\pi$, we say that $\pi$ has the property $T \geq S$ if the edges with labels in $S$, $S \subseteq [d]$ are backward. Also, say $\pi$ has the property $T = S$ if the edges with labels in $S$ are backward and no other edges are backward. For a set $S$ of hyperedges incident at $v$, let $N(T \geq S)$ denote the number of orderings having the property $T \geq S$, that is, the number of permutations such that the hyperedges in $S$ will all be
backward edges. $N(T=S)$ is similarly defined. $N(T \geq S)$ is determined as follows:
Suppose $S$ has $r$ hyperedges incident at $v$. For a fixed arrangement of the vertices belonging to
dges in $S$, the number of permutations of the remaining vertices is $(td + 1)!/(tr + 1)!$. In each
allowed permutation, the vertex $v$ must occur only after the vertices of $S$ (i.e. the rightmost
position). However the remaining $tr$ vertices can be arranged among themselves in $(tr)!$ ways.
Thus we have

$$N(T \geq S) = (td + 1)!\frac{(tr)!}{(tr + 1)!} = \frac{(td + 1)!}{(tr + 1)!}.$$  

Clearly, if a permutation has the property $T \geq S$, it has the property $T = S'$ for some $S' \supseteq S$.
Hence for every $S \subset [d]$,

$$N(T \geq S) = \sum_{S' \supset S} N(T = S').$$

Therefore, by PIE (see [16], Chapter 2),

$$N(T = \emptyset) = \sum_{S} (-1)^{|S|} N(T \geq S)$$

$$\sum_{|S| = r} N(T \geq S) = \binom{d}{r} N(T \geq |r|) = \binom{d}{r} \frac{(td + 1)!}{tr + 1}$$

Hence we get the required probability to be

$$P_v(0) = \left( \sum_{r=0}^{d} \binom{d}{r} (-1)^r \frac{(td + 1)!}{tr + 1} \right) \times \frac{1}{(td + 1)!}$$

$$= \sum_{r=0}^{d} \frac{d!}{r!} (-1)^r \frac{1}{tr + 1}$$

By Lemma [3,2]

$$P_v(0) = \frac{1}{\binom{d + 1}{d}}$$

and this completes the proof.

\[\square\]

4 Linearity: Probability of having few backward edges

Now, we consider the more general case when at most $A - 1$ backward edges are allowed. In
this section, we get an exact expression for the corresponding probability. This estimate plays
an important role later in getting new and improved lower bounds on $\alpha(H)$ for locally sparse
graphs and linear hypergraphs. Our goal in this section is to prove the following result.

**Theorem 4.1** For a $k$-uniform linear hypergraph $H$, a vertex $v$ having degree $d$, a uniformly
chosen permutation $\pi$ induces at most $A - 1$ backward edges with probability $P_v(A - 1)$ given by

$$P_v(A - 1) = \begin{cases} 
\frac{1}{\binom{d}{r}} \frac{1}{tA + 1} \frac{tA}{tA + 1} \binom{d}{r-d-A} & \text{if } d \leq A - 1; \\
\frac{1}{\binom{d}{r}} \frac{1}{tA + 1} \frac{tA}{tA + 1} \binom{d}{r-d-A} & \text{if } d \geq A.
\end{cases}$$
Corollary 4.2 As \( d \to \infty \), the asymptotic expression for the probability \( P_v(A-1) \) is given by

\[
P_v(A-1) \sim \frac{1}{1 + (1/(tA))} \left( \frac{A}{d} \right)^{1/t} = \Omega((A/d)^{1/t})
\]

Proof The asymptotics are w.r.t. \( d \to \infty \), \( d \geq A \). The expression for having at most \( A-1 \) backward edges is

\[
P_v(A-1) = \frac{1}{1 + (tA)^{-1} (d - A)! (d + 1/t)(d - 1 + 1/t)...(d - A)}
\]

Now, for \( 0 < x \), we have \((1 + x)^{-1} > e^{-x}\). So we get

\[
P_v(A-1) > (1 + (tA)^{-1})^{-1} e^{-(1/(t)) \sum_{r=1}^{d-A} (1/(1/r))}
\]

The above expression therefore becomes \( \Omega((A/d)^{1/t}) \). \( \square \)

The version of PIE used most commonly deals with \( N(T_\emptyset) \), i.e. the number of elements in the set of interest - in this case, permutations of \( \left\lceil td \right\rceil \) which do not have any of the properties under consideration (in this case, backward edges with respect to \( v \)). However we need something slightly different - an expression for the number of permutations which have at least \( A \) backward edges. Clearly, the remaining permutations are those which have at most \( A-1 \) backward edges.

Therefore, we use a slightly modified version of PIE, which is stated below in Theorem 4.5. This form is well-known (see e.g. [16], Chapter 2, Exercise 1), although it seems to be used less frequently. For the sake of completeness, we provide a simple proof. First we state two identities involving binomial coefficients that we will prove in the Appendix.

Lemma 4.3 For \( a, b \) nonnegative integers,

\[
\sum_{i=0}^{b} (-1)^i \binom{a+b}{a+i} \binom{a+i-1}{i} = 1
\]

Lemma 4.4 Given non-negative integers \( d, A \), \( d \geq A \) and a positive integer \( t \),

\[
\sum_{r=0}^{d-A} (-1)^r \binom{d}{r+A} \binom{A+r-1}{r} \frac{1}{t(r+A)+1} = 1 - \left( \frac{At}{tA+1} \right) \frac{\binom{d}{A}}{\binom{d+1/t}{d-A}}
\]

We now present the generalized PIE and its well-known proof.
Theorem 4.5 Let $S$ be an $n$-set and $E_1, E_2, \ldots E_d$ not necessarily distinct subsets of $S$. For any subset $M$ of $[d]$, define $N(M)$ to be the number of elements of $S$ in $\cap_{i \in M} E_i$ and for $0 \leq j \leq d$, define $N_j := \sum_{|M|=j} N(M)$. Then the number $N_{\geq a}$ of elements of $S$ in at least $a$, $0 \leq a \leq d$ of the sets $E_i$, $1 \leq i \leq d$, is

$$N_{\geq a} = \sum_{i=0}^{d-a} (-1)^i \binom{a+i-1}{i} N_{i+a} \quad \text{(MPIE)}$$

Proof Take an element $e \in S$.

(i) Suppose $e$ is in no intersection of at least $a$ $E_i$'s. Then $e$ does not contribute to any of the summands in the RHS of the expression (MPIE), and hence, its net contribution to the RHS is zero.

(ii) Suppose $e$ belongs to exactly $a+j$ of the $E_i$'s, $0 \leq j \leq d-a$. Then its contribution to the RHS of (MPIE) is

$$\sum_{l=0}^{j} (-1)^l \binom{a+j}{a+l} \binom{a+l-1}{l}$$

and by Lemma 4.3 this is equal to 1. 

Proof of Theorem 4.1 If $d \leq A - 1$, then $P_v(A - 1) = 1$ obviously. The proof is similar to the proof of Theorem 3.1 except that in place of the PIE, we use Theorem 4.5. The set under consideration is the set of permutations of $[td+1]$, the subsets $E_i$ correspond to the permutations for which the $i$-th edge is backward. It is easy to see that $N(M) = N(T_{\geq M})$ under the notation used in Theorem 3.1 and hence $N(M) = \frac{(td+1)!}{(i+1)!}$. Therefore we have $N_j = \binom{d}{j} \frac{(td+1)!}{(tj+1)}$ as before. Hence the expression for the probability $Q_v(A)$ that at least $A$ edges are backward under a uniformly random permutation $\pi$, becomes:

$$Q_v(A) = \sum_{i=0}^{d-A} (-1)^i \binom{d}{i+1} \binom{A+i-1}{i} \frac{1}{t(i+A)+1}.$$ 

By Lemma 4.4 the RHS of the above expression is

$$Q_v(A) = 1 - \frac{1}{1 + (tA)^{-1}} \frac{\binom{d}{A}}{\binom{d+1/t}{d-A}}.$$ 

Hence the probability of having at most $A - 1$ backward edges is given by

$$P_v(A - 1) = \frac{1}{1 + (tA)^{-1}} \frac{\binom{d}{A}}{\binom{d+1/t}{d-A}}$$

and the proof is complete. \hfill \square

5 Lower bounds for linear hypergraphs and $K_r$-free graphs

In this section we prove Theorems 1.2, 1.3, and 1.4. These follow by a simple application of Corollary 4.2. Since the proofs follow the same outline, we prove them simultaneously, highlighting only the differences as and when they occur.
Proofs of Theorems 1.2, 1.3 and 1.4. Consider a uniformly chosen random permutation of the vertices of the graph/hypergraph under consideration. Let \( D \) be the average degree of the graph or hypergraph and \( A = D' \). Let \( I \) be the set of those vertices each having at most \( A - 1 \) backward edges incident on it. Clearly, the expected size of \( I \) is
\[
E[|I|] \geq c \sum_{v \in V} \left( \frac{A}{\max\{A, d(v)\}} \right)^{1/t} = cA^{1/t} \sum_{v \in V} \left( \frac{1}{\max\{A, d(v)\}} \right)^{1/t}
\]
for some constant \( c = c(k, \epsilon) \). (For a graph, \( k = 2 \) and hence \( t = 1 \).) Also, by construction, the average degree of the sub(hyper)graph induced by \( I \) is at most \( k(A - 1) \). Therefore, there exists an independent set \( I' \) of size at least as follows

(i) Case \( t = 1 \), graph is \( K_3 \)-free: By [13], \( \alpha(G) \) is at least
\[
\Omega \left( \log(2(A - 1)) \frac{|I|}{2(A - 1)} \right) = \Omega \left( \log D \sum_{v \in V} \frac{1}{\max\{A, d(v)\}} \right)
\]

(ii) Case \( t = 1 \), graph is \( K_r \)-free \( (r > 3) \): By [14], \( \alpha(G) \) is at least
\[
\Omega \left( \frac{\log(2(A - 1))}{\log \log(2(A - 1))} \frac{|I|}{2(A - 1)} \right) = \Omega \left( \frac{\log D}{\log \log D} \sum_{v \in V} \frac{1}{\max\{A, d(v)\}} \right)
\]

(iii) Case \( t > 1 \), hypergraph is linear: By [9], \( \alpha(H) \) is at least
\[
\Omega \left( \left( \log k(A - 1) \right)^{1/t} \frac{|I|}{(k(A - 1))^{1/t}} \right) = \Omega \left( \frac{(\log D)^{1/t}}{\max\{A, d(v)\}^{1/t}} \right)
\]

The above three cases prove Theorems 1.2, 1.3 and 1.4 respectively.

Note: An inspection of the proofs above show why we need \( \epsilon \) to be a fixed constant. It is because all three expressions above essentially have \( \log A \) i.e. \( \epsilon \log D \) in the numerator. So, if \( \epsilon = o(1) \), then \( \log A = o(\log D) \), and we would get asymptotically weaker results. \( \square \)

6 Construction comparing average degree vs. degree sequence based bounds

A degree sequence-based bound obviously reduces to a bound based on average degree, when the (hyper)graph is regular. However, the convexity of the function \( x^{-1/t} \), \( x \geq 1 \) and \( t \in \mathbb{N} \), shows that the bounds in Theorems 1.2, 1.3 and 1.4 are better than the corresponding average degree-based bounds proved in [2], [13] and [14] respectively provided the minimum degree is at least \( A \), although it is not clear \( a \ priori \) if the improvement can become significantly larger. Also, at least half the vertices will have degree at most \( 2D \), so even in the general case (no restriction on the minimum degree) our bounds are no worse than the average degree based bounds (ignoring the constant factors). However, they can be much larger than the latter bounds. We now give infinite families of \( K_r \)-free graphs and linear \( k \)-uniform hypergraphs which show that

(i) The bounds given by Theorem 1.2, 1.3 can be better than the bounds in [1] [13] [14] respectively by a multiplicative factor of \( \log(|V(G)|) \).
(ii) The bound in Theorem 1.2 can be better than the bound in [2] by a multiplicative factor of 

\((\log |V(H)|) / (\log \log |V(H)|)^{(1-\epsilon)/\epsilon}\), where \(\epsilon\) is the constant mentioned in Theorem 1.2.

Case (i) Take a set of \(n\) disjoint graphs, \(K_{1,1}, K_{2,2}, K_{4,4}, \ldots, K_{2^n-1,2^n-1}\). The total number of vertices is \(2^{n+1} - 2\), whereas the average degree is \(d_{av} = (2^n + 1) / 3\). Hence, the average degree based bound gives \(\Theta(|V(G)| \log d_{av} / d_{av}) = \Theta(d_{av})\). Denote by \(l\) the maximum \(j\) such that \(2^j \leq A = d_{av}^\epsilon\). It follows that \(A < 2^{l+1}\). Theorem 1.2 gives

\[
\alpha(H) = \Omega \left( mn^{1+1/t} (\log d_{av})^{1/t} / 2^{n/t} \right) \quad (A)
\]

On the other hand, the bound in Theorem 1.2 gives

\[
\alpha(H) = \Omega \left( m (\log d_{av})^{1/t} \left[ \sum_{j = 0}^{l} \frac{m}{2^{n/j}} + \sum_{j = l+1}^{n-1} \frac{m}{2^{j/t}} \right] \right)
\]

\[
= \Omega \left( m (\log d_{av})^{1/t} \left[ 2^{-en/t} n^{1+\epsilon/t} + 2^{-en/t} n^{\epsilon/t} (1 - 2^{-n-l-1/t}) / (1 - 2^{-1/t}) \right] \right)
\]

\[
= \Omega \left( m (\log d_{av})^{1/t} \times 2^{-en/t} \left[ \epsilon n^{1+\epsilon/t} + n^{\epsilon/t} (1 - 2^{-n-l-1/t}) / (1 - 2^{-1/t}) \right] \right)
\]

\[
= \Omega \left( m (\log d_{av})^{1/t} \times 2^{-en/t} (\epsilon n^{1+\epsilon/t} + \Theta(n^{\epsilon/t})) \right) \quad (B)
\]

The ratio of the bound in (B) to the one in (A) can be seen to be \(\Omega((2^n / n)^{(1-\epsilon)/t})\), which is \(\Omega((\log |V| / \log \log |V|)^{(1-\epsilon)/t})\) for an appropriately slow-growing function \(w\).

### 7 Concluding Remarks

In the course of this paper, some semi-combinatorial proofs of certain non-trivial identities involving binomial coefficients were also obtained. These are described below:
\[
\sum_{a=0}^{A} \sum_{i=0}^{d-a} \binom{d}{a+i} \binom{a+i}{i} 2^i (2d-2a-i)! (2a+i)! = (d!)^2 4^{d-A} (A+1) \binom{2A+1}{A}
\] (3)

The LHS (when divided by \((2d+1)\)!\) amounts to the expression for \(P_v(A)\) when \(k = 3\): choose \(a+i\) hyperedges from the \(d\) hyperedges incident on \(v\), of these \(a\) hyperedges are backward, while \(i\) hyperedges each have one vertex occurring prior to \(v\) in the random permutation. These \(i\) vertices can be chosen from \(i\) pairs in \(2^i\) ways. The \((2a+i)\) vertices before \(v\) can be arranged amongst themselves in \((2a+i)!\) ways. The \((2d-2a-i)\) vertices occur after \(v\) and can be arranged amongst themselves in \((2d-2a-i)!\) ways. The RHS is easily obtained from Theorem 4.1 by taking \(t = 2\).

Taking \(A = 0\) in the above expression gives us the simpler identity:

\[
\sum_{i=0}^{d} \binom{d}{i} 2^i (2d-i)! = (d!)^2 2^d
\] (4)

Dividing by \((d!)^2 2^d\) and changing the order of summation, we get

\[
\sum_{i=0}^{d} \binom{d+i}{d} 2^{-i} = 2^d
\] (5)

The above expression is discussed in some detail in [11] (Chapter 5, eqs. 5.20, 5.135-8); a nice combinatorial proof of it is provided in [17].

The next expression (for the more general case \(k \geq 3\)) is much more complicated:

\[
\sum_{a=0}^{A} \sum_{i=0}^{d-a} \sum_{\sum_{j=1}^{t-1} i_j = i} \binom{d}{a+i} \binom{a+i}{a,i_1,\ldots,i_{t-1}} \binom{t}{1}^{i_1} \binom{t}{2}^{i_2} \cdots \binom{t}{t-1}^{i_{t-1}} \times (ta+i_1+2i_2+\ldots(t-1)i_{t-1})!(td-ta-i_1-\ldots-(t-1)i_{t-1})! \times (td+1)!(1+(tA+t)^{-1})^{-1} \binom{d}{d+A+t} \binom{d+A+t}{d-A-1}
\] (6)

The LHS again follows by similar arguments as in the previous case, this time for general \(t\). There are \(a\) backward edges, \(i_1\) edges which have one vertex before \(v\), \(i_2\) edges with 2 vertices before \(v\), and so on. The RHS follows from Theorem 4.1.

It was in fact the non-triviality of the above LHS expressions that led to our use of the PIE and its variant (Theorem 4.5) in order to obtain closed-form expressions for Theorems 3.1 and 4.1.

With regard to the tightness of our results and the weakening parameter \(A\), firstly, from the proof of Theorems 1.2, 1.3 it is clear that \(\epsilon = \log A/\log D\) has to be at least a constant. Ideally, we may want to replace \(A\) by 1 in the bounds of Theorems 1.2, 1.3 and 1.4. This corresponds
to the case $\epsilon = 0$. The following example, however, shows that it is possible to construct a triangle-free graph for which the bound in say, Theorem 1.2 would give a value more than the number of vertices: Take a disjoint union of $A = K_{n/3, n/3}$ and $B = \overline{K}_{n/3}$, and introduce a perfect matching between $B$ and one of the parts of $A$. Now, $|V| = n$, $D \sim 2n/9$, and hence if $A = 1$, Theorem 1.2 would give a lower bound of $\Omega(n \log n)$, which is asymptotically larger than $|V|$. Similar examples can be constructed with linear hypergraphs also.

**Acknowledgement** The first and third authors would like to thank N.R. Aravind for some initial helpful discussions.

**References**

[1] M. Ajtai, J. Komlós and E. Szemerédi: A Note on Ramsey Numbers. *J. Comb. Theory Ser. A* 29 (1980) 354-360.

[2] M. Ajtai, J. Komlós, J. Pintz, J.H. Spencer and E. Szemerédi: Extremal uncrowded hypergraphs, *J. Combinatorial Theory Ser. A* 32, 321-335 (1982).

[3] N. Alon and J.H. Spencer (2008). *The Probabilistic Method*, Wiley International.

[4] George E. Andrews, Richard Askey and Ranjan Roy: Special Functions. Encyclopedia of Mathematics and its Applications (1999) 71 Cambridge University Press.

[5] W. N. Bailey: Generalized Hypergeometric Series (1935) Cambridge.

[6] Frits Beukers: Gauss’ Hypergeometric Function (2002) [http://www.math.uu.nl/people/beukers/MRIcourse93.ps](http://www.math.uu.nl/people/beukers/MRIcourse93.ps)

[7] Y. Caro: New results on the independence number. Technical Report, Tel Aviv University, 1979.

[8] Yair Caro and Zsolt Tuza: Improved Lower Bounds on $k$- Independence, *J. Graph Theory* 1991, Vol. 15, 99-107.

[9] R. Duke, H. Lefmann and V. Rödl: On uncrowded hypergraphs, *Random Structures and Algorithms*, 6, 209-212, 1995.

[10] R. Bopanna: [Comment on Lance Fortnow’s blog, article by Bill Gasarch] [http://blog.computationalcomplexity.org/2010/08/today-is-paul-turans-100th-birthday.html](http://blog.computationalcomplexity.org/2010/08/today-is-paul-turans-100th-birthday.html) comment #1 (2010).

[11] Ronald L. Graham, Donald E. Knuth, and Oren Patashnik: Concrete Mathematics (Reading, Massachusetts: Addison-Wesley, 1994)

[12] J. Komlós, J. Pintz and E. Szemerédi: A lower bound for Heilbronn’s problem, *J. London Math. Soc. (2)* 25 (1982), no. 1, 13-24.

[13] J.B. Shearer: A note on the independence number of triangle-free graphs. *Discrete Math.* 46 (1983) 83-87.

[14] J.B. Shearer: On the independence number of sparse graphs. *Random Structures Algorithms* 7 (1995) 269-271.

[15] J. Spencer, Turán’s theorem for $k$-graphs, Discrete Mathematics 2 (1972) 183–186.
Appendix

Proof of Lemma 3.2

Write the LHS as $\sum_{r \geq 0} t_r$, since $(-d)^r = 0$ for $r > d$. Now,

$$\frac{t_{r+1}}{t_r} = \frac{(-1)(d-r)(tr+1)}{(r+1)(tr+t+1)} = \frac{(r-d)(r+1/t)}{(r+1)(r+1+1/t)}$$

Also, notice that $t_0 = 1$. Therefore, the LHS can be written as the generalised hypergeometric function $F(1/t, -d; 1+1/t; 1)$, where the generalised hypergeometric function $F(a_1, ..., a_m; b_1, ..., b_n; z)$ is given by

$$F(a_1, ..., a_m; b_1, ..., b_n; z) = \sum_{r=0}^{\infty} \frac{(a_1)^{(r)}(a_2)^{(r)}...(a_m)^{(r)}(r)}{(b_1)^{(r)}...(b_n)^{(r)}(r)} \frac{z^r}{r!}$$

where $p^{(q)} = p(p+1)...(p+q-1)$ is the rising factorial. Next, we use the general version of Vandermonde convolution - also known as Chu-Vandermonde identity (a special case of Gauss’s Hypergeometric Theorem, see e.g. [11], Chapter 5, equation 5.93, also [4, 5, 6, 21])

$$F(a, -n; c; 1) = (c-a)^{(n)}_{c}$$

The above is true whenever $a, c$ are complex numbers and $n$ is a natural number, such that $\Re(a) - n < \Re(c)$. In our case, $a = 1/t, n = d$ and $c = 1+1/t$. Hence we get $(c-a)^{(n)} = dl$, and $c^{(n)} = (1+1/t)^{(d)} = (1+1/t)(2+1/t)...(d+1/t)$. Therefore, the LHS of 3 becomes

$$F(1/t, -d; 1+1/t; 1) = \frac{d!}{(1+1/t)(2+1/t)...(d+1/t)} = \frac{1}{(d+1/t)^d}$$

Proof of Lemma 4.3

The proof is by induction on $b$. For $b = 0$, the LHS reduces to

$$\sum_{i=0}^{0} (-1)^i \binom{a+0}{a+i} \binom{a+i-1}{i}$$

$^2$A proof based on $n$-th order difference operators follows from [11] (Chapter 5, eqn. 5.41). However, we were not aware of this at the time of solving. The hypergeometric proof is equally simple and the approach more general.
which is clearly 1. Assume the lemma to be true for \( b = c \) and consider the LHS when \( b = c + 1 \):

\[
\sum_{i=0}^{c+1} (-1)^i \binom{a + 1 + c}{a + i} \left( a + i - 1 \right)
\]

\[
= \sum_{i=0}^{c+1} (-1)^i \left[ \binom{a + c}{a + i} + \binom{a + c}{a + i - 1} \right] \left( a + i - 1 \right)
\]

\[
= \sum_{i=0}^{c+1} (-1)^i \left[ \binom{a + c}{a + i} \left( a + i - 1 \right) + \binom{a + c}{a + i - 1} \left( a + i - 1 \right) \right]
\]

\[
= 1 + \sum_{i=0}^{c+1} (-1)^i \binom{a + c}{a + i - 1} \left( a + i - 1 \right)
\]

by the induction hypothesis, since \( \binom{a+c}{a+c+1} = 0 \). Now, the second sum is

\[
\sum_{i=0}^{c+1} (-1)^i \binom{a + c}{a + i - 1} \left( a + i - 1 \right)
\]

\[
= \frac{(a + c)!}{(a - 1)!(c + 1)!} \sum_{i=0}^{c+1} (-1)^i \binom{c + 1}{i}
\]

\[
= 0
\]

\[
\square
\]

**Proof of Lemma 4.4** Let the LHS be denoted by \( S_d \). Then, using the identity \( \binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1} \), we have

\[
S_d = \sum_{r=0}^{d-A} (-1)^r \left[ \binom{d - 1}{r + A} + \binom{d - 1}{r + A - 1} \right] \left( A + r - 1 \right) \frac{1}{tr + tA + 1}
\]

\[
= S_{d-1} + \sum_{r=0}^{d-A} (-1)^r \binom{d - 1}{r + A - 1} \left( A + r - 1 \right) \frac{1}{tr + tA + 1}
\]

since \( \binom{d-1}{d} = 0 \). Now the second sum can be simplified as

\[
T_d = \sum_{r=0}^{d-A} (-1)^r \binom{d - 1}{r + A - 1} \left( A + r - 1 \right) \frac{1}{tr + tA + 1}
\]

\[
= \left( \frac{(d - 1)!}{(d - A)!(A - 1)!} \right) \sum_{r=0}^{d-A} (-1)^r \binom{d - A}{r} \frac{1}{tr + tA + 1}
\]

\[
= \left( \frac{d - 1}{A - 1} \right) \frac{1}{tA + 1} \sum_{r=0}^{d-A} (-1)^r \binom{d - A}{r} \frac{1}{(t/(tA + 1))r + 1}
\]

By Lemma 5.2 we get

\[
T_d = \frac{1}{tA + 1} \frac{(d-1)^{A-1}}{(d-A+(tA+1)/t)}
\]

14
Therefore,

\[ S_d = S_{d-1} + \frac{1}{tA+1} \left( \frac{d-1}{d-A} \right) \]

Unraveling the recursion and noticing that \( S_A = 1/(tA+1) \), we get that

\[ S_d = \left( 1/(tA+1) \right) \sum_{r=0}^{d-A} \left( \frac{A-1+r}{A-1} \right) \left( \frac{d}{d-A-r} \right) \]

by reversing the order of summation. Finally, the following claim completes the proof.

**Claim.** For \( d \geq A, t \geq 0, \)

\[ \frac{1}{tA+1} \sum_{r=0}^{d-A} \left( \frac{A-1+r}{A-1} \right) = 1 - \frac{tA}{tA+1} \left( \frac{d}{d-A} \right) \]

**Proof of Claim.** We use induction on \( d \). When \( d = A \), the LHS is \((tA+1)^{-1}\), while the RHS is \( 1 - \frac{tA}{tA+1} \), so we have equality. Now assume equality for \( d \) and consider the LHS for \( d + 1 \):

\[
\frac{1}{tA+1} \sum_{r=0}^{d-A+1} \left( \frac{A-1+r}{A-1} \right) = 1 - \frac{At}{(tA+1)^{-1} + \frac{d}{d-A+1}}
\]

which is the required expression on the RHS. \( \square \)