3-systems which do not represent a real number and its square

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Abstract

In 2013 Schmidt and Summerer showed that the parametric successive minima function $L_u$ of a given vector $u \in \mathbb{R}^n$ can be approximated up to a bounded difference by a function from a certain class. Roy recently proved that the same is true within a smaller class of functions called $n$-systems. Conversely, given an $n$-system, Roy also showed that there exists a point $u \in \mathbb{R}^n$ whose associated function $L_u$ is approximated by this $n$-system up to a bounded difference. In this paper we study the case $n = 3$ and we construct 3-systems such that there is no vector $u$ of the form $(1, \xi, \xi^2)$ whose associated function $L_u$ may be approximated by these $n$-systems up to a bounded difference.

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1 Introduction

Let $n \geq 2$ be an integer and let $u = (1, \xi_1, \ldots, \xi_{n-1}) \in \mathbb{R}^n$ be a point whose coordinates are linearly independent over $\mathbb{Q}$ (in the following we will have $n = 3$ and $u = (1, \xi, \xi^2)$). Details about parametric geometry of numbers are gathered in Section 3. For $x \in \mathbb{R}^n$ we define its norm $\|x\|$ and the quantity $M(x)$ by

$$\|x\| = \max_{0 \leq i < n} |x_i| \quad \text{and} \quad M(x) = \max_{0 \leq i < n} |x_0 \xi_i - x_i|.$$

One of the founding ideas of the parametric geometry of numbers is to consider a family of convex bodies parameterized by a positive real number $q$ and to study the successive minima associated to this family. The choice of the family may differ according to the context. In this paper we choose to consider the following convex bodies family (which, up to a bounded multiplicative constant, is that of [14]). We set

$$C_u^*(e^q) := \{x \in \mathbb{R}^n ; \|x\| \leq e^q, M(x) \leq 1\}.$$

For $j = 1, \ldots, n$, $\lambda_j^*(q)$ denotes the $j$-th successive minimum of the convex body $C_u^*(e^q)$ with respect to the lattice $\mathbb{Z}^n$. We also define $L_j^*(q) = \log \lambda_j^*(q)$ and we group these successive minima $L_j^*$ into a single map $L_u^* = (L_1^*, \ldots, L_n^*)$. All functions $L_j^*$ are continuous piecewise linear with slopes $0$ and $-1$, and satisfy $L_1^*(q) + \cdots + L_n^*(q) = -q + O(1)$. We set

$$\bar{\psi}_j = \limsup_{q \to \infty} \frac{L_j^*(q)}{q} \quad \text{and} \quad \underline{\psi}_j = \liminf_{q \to \infty} \frac{L_j^*(q)}{q}.$$

Schmidt and Summerer describe precisely the behavior of components $L_j$ by introducing the model of $(n, \gamma)$-systems in [17]. In [14] Roy gives the definition of an $n$-system which is equivalent to the
Definition 1.1. He shows that one may only consider this smaller and simpler class of functions to describe the behavior of the previous successive minima functions \( L_j \) (see Theorem 1.1 below). In this paper we work with dual \( n \)-systems which are more adapted to the parametrization of \( C_*^n(q) \). A dual \( n \)-system is a map \( P = (P_1, \ldots, P_n) \) such that \(-P_n, \ldots, -P_1\) is an \( n \)-system (see Definition 3.1).

Theorem 1.1 (Roy, 2015). For each non-zero point \( u \in \mathbb{R}^n \), there exist \( q_0 > 0 \) and a dual \( n \)-system \( P \) on \([q_0, +\infty)\) such that \( \|L_u^* - P\|_\infty \) is bounded over \([q_0, +\infty)\). Conversely, for each dual \( n \)-system \( P \) on an interval \([q_0, +\infty)\), there exists a non-zero point \( u \in \mathbb{R}^n \) such that \( \|L_u^* - P\|_\infty \) is bounded on \([q_0, +\infty)\).

Note that Schmidt and Summerer first gave a weaker version of the first part of Theorem 1.1 in [17]. Precisely, they showed that for each non-zero point \( u \in \mathbb{R}^n \), there exist \( q_0 > 0 \) and a \((n, \gamma)\)-system \( P \) on \([q_0, +\infty)\) such that \( \|L_u^* - P\|_\infty \) is bounded over \([q_0, +\infty)\). See [17] for the definition of an \((n, \gamma)\)-system. An \( n \)-system is an \((n, 0)\)-system for Schmidt and Summerer.  

Note that if \( \|L_u^* - P\|_\infty \) is bounded on \([q_0, +\infty)\) and if we write \( P = (P_1, \ldots, P_n) \), then

\[
\bar{\psi}_j = \limsup_{q \to \infty} \frac{P_j(q)}{q} \quad \text{and} \quad \underline{\psi}_j = \liminf_{q \to \infty} \frac{P_j(q)}{q},
\]

(1.1)

Definition 1.1. Let \( \xi \) be a real number and write \( u = (1, \xi, \ldots, \xi^n) \). We say that a dual \( n \)-system \( P \) on \([q_0, +\infty)\) represents \( \xi \) if \( \|L_u^* - P\|_\infty \) is bounded on \([q_0, +\infty)\).

Theorem 1.1 of Roy ensures that there always exists a dual \( n \)-system which represents a real number \( \xi \). Thus the following question can naturally be raised: can we describe (even partially) the set of dual \( n \)-systems which represent a real number? A very satisfying answer would be to find a “simple” subfamily of dual \( n \)-systems such that for each real number \( \xi \) there would be a dual \( n \)-system of this subfamily which represents \( \xi \) and, conversely, each dual \( n \)-system of this subfamily would represent some real number.

Presently, even for \( n = 3 \) there are not so many families of real numbers for which we can give an almost complete description of associated 3-systems. If \( \tilde{\omega}_3(\xi) > 2 \) there are essentially Roy’s extremal numbers [9] and numbers of Sturmian type (see Proposition 7.20 and Figure 2 of [7]) which generalize both Roy’s Fibonacci type numbers [12] and the Sturmian continued fractions constructed by Bugeaud and Laurent [2].

The main goal of this article is to construct a family of dual 3-systems which do not represent any real number. All of our constructions of such examples are based on properties of some diophantine exponents. The simplest way to construct dual 3-systems which do not represent any real number is to use the following property. Let us consider \( u = (1, \xi, \xi^2) \) and its parametric exponents. If \( \bar{\psi}_1 < -\frac{1}{3} \), then \( \bar{\psi}_1 \geq -\frac{1}{2} \) (note that for a general dual 3-system we only have estimates \(-1 \leq \bar{\psi}_1 \leq \bar{\psi}_1 \leq \frac{1}{2} \), see Section 3). This property is equivalent to the implication \( \tilde{\lambda}_2(\xi) > \frac{1}{3} \Rightarrow \lambda_2(\xi) \leq 1 \), where \( \lambda_2 \) are classical diophantine exponents of simultaneous approximation, see Propositions 2.1 and 2.2 of Section 2. Using (1.1) we can therefore deduce the following result:

Proposition 1.2. If \( P = (P_1, P_2, P_3) \) is a dual 3-system such that

\[
\limsup \frac{P_j(q)}{q} < -\frac{1}{3} \quad \text{and} \quad \liminf \frac{P_1(q)}{q} < -\frac{1}{2},
\]

then \( P \) does not represent any real number.
In Section 4 we define the exponent \( \kappa \). Classical properties coming from the parametric geometry of numbers are gathered in Section 3.

This paper is organized as follows. In Section 2 we recall definitions of classical exponents \( \beta_0(\xi) \) (see \[6\]) defined as follows. Fix \( \xi \) a real number which is algebraic of degree \( \leq 2 \). For \( 0 < \varepsilon \leq 1 \) the exponent \( \beta_{\varepsilon}(\xi) \) is the infimum of the set of all \( \beta \) such that for any sufficiently large \( B > 0 \) there exists \( x = (x_0, x_1, x_2) \in \mathbb{Z}^3 \) such that

\[
1 \leq ||x|| \leq B \quad \text{and} \quad M(x) \leq \min(B^{-1/\beta}, ||x||^{-1+\varepsilon}).
\]

If this set of \( \beta \) is empty, we set \( \beta_{\varepsilon}(\xi) = +\infty \). For \( \varepsilon = 0 \), we set

\[
\beta_0(\xi) = \sup_{0 < \varepsilon \leq 1} \beta_{\varepsilon}(\xi) = \lim_{\varepsilon \to 0^+} \beta_{\varepsilon}(\xi)
\]

since for any given \( \xi \) the map \( \varepsilon \mapsto \beta_{\varepsilon}(\xi) \) is non-increasing. Note that for \( \varepsilon = 1 \) we have \( \beta_1(\xi) = 1/\lambda_2(\xi) \), where \( \lambda_2(\xi) \) is the classical uniform exponent of simultaneous approximation (see Section \[2\]). Also note that \( \beta_0(\xi) < 2 \) implies \( \lambda_2(\xi) = 1 \). Fischler studied the spectrum of \( \beta_0 \) and he showed the following result:

**Theorem 1.2** (Fischler, 2007). Let us set \( S_0 = \{ \beta_0(\xi) \mid \xi \in \mathbb{R}, \text{irrational, non-quadratic} \} \) and \( S' = \{ \limsup|1; 1, s_k, \ldots, s_1| \mid (s_i)_i \text{bounded sequence of positive integers} \} \), where \( [a_0; a_1, a_2, \ldots] \) denotes the continued fraction whose partial quotients are \( a_0, a_1, a_2, \ldots \). Then we have

\[
S_0 \cap (1, \sqrt{3}) = S' \cap (1, \sqrt{3}).
\]

The smallest element of \( S' \) is \( \gamma = \frac{1 + \sqrt{5}}{2} \) (which is obtained when 1 is the only integer appearing infinitely many times in the sequence \( (s_i)_i \)). The values immediately superior were found by Cassaigne [3]. They constitute an increasing sequence of quadratic numbers converging to the smallest accumulation point of \( S' \). Elements of this set also appear in the description of the classical exponents of Sturmian continued fractions (see Bugeaud and Laurent’s article \[2\] on this topic) and of Sturmian type numbers (see \[7\]). In this paper we define a new parametric exponent \( \kappa^* \) (see Section \[1\]). For a real number \( \xi \) (not algebraic of degree \( \leq 2 \)) such that \( \lambda_2(\xi) = 1 \) we have

\[
\kappa^*(\xi) = -\frac{1}{\beta_0(\xi) + 1}
\]

(see Proposition \[1.3\] in Section \[4\]). For any number \( \alpha \in (-\frac{1}{2}, -\frac{1}{3}) \) we will construct a dual 3-system \( P \) whose associated exponent \( \kappa^* \) is equal to \( \alpha \) and such that \( \psi_0^* = -\frac{1}{2} \). To conclude the proof of Theorem \[1.2\] it will suffice to choose \( \alpha \) not of the form \( -\frac{1}{p+1} \) with \( \beta \in S_0 \).

This paper is organized as follows. In Section \[2\] we recall definitions of classical exponents \( \lambda_2, \tilde{\lambda}_2 \) and the result which implies Proposition \[1.2\] (namely, Proposition \[2.2\]). Definitions and classical properties coming from the parametric geometry of numbers are gathered in Section \[3\]. In Section \[4\] we define the exponent \( \kappa^* \) and we establish the relation \( (1.2) \) (valid for any real number \( \xi \) satisfying \( \lambda_2(\xi) = 1 \)). Finally, we prove Theorem \[1.2\] in Section \[5\].
2 Classical exponents $\lambda_2$ and $\hat{\lambda}_2$

In this section we give definitions of the classical Diophantine exponents $\lambda_2$ and $\hat{\lambda}_2$. For our purpose we only need Proposition 2.1 and its parametric version (namely, Proposition 2.2) given at the end of this section.

If $\xi \in \mathbb{R}$ is not an algebraic number of degree $\leq 2$ we may study the following standard problem of simultaneous approximation:

**Problem $E'_{\lambda,X}$**: We search for non-zero integer points $x = (x_0, x_1, x_2) \in \mathbb{Z}^3$ solutions of the system

$$
\begin{align*}
\|x\| \leq X \\
M(x) &= \max(|x_0\xi - x_1|, |x_0\xi^2 - x_2|) \leq X^{-\lambda}.
\end{align*}
$$

We denote by $\lambda_2(\xi)$ (resp. $\hat{\lambda}_2(\xi)$) the supremum of real numbers $\lambda$ for which the problem $E'_{\lambda,X}$ admits a non-zero integer solution for arbitrarily large values of $X$ (resp. for each sufficiently large value of $X$).

The previous exponents have been studied a lot during the last decades. The reader may refer to [1] which contains a well supplied summary of the subject and to [2] for a generalization to the approximation of a vector $(1, \xi, \xi_2) \in \mathbb{R}^3$ (in this paper we study the special case $u = (1, \xi, \xi_2)$). Equation (2.1) below states a classical result in parametric geometry of numbers and establishes a relation between standard diophantine exponents attached to a real number $\xi$ and the parametric exponents $\psi^*_i, \bar{\psi}_i$ (attached to the vector $u = (1, \xi, \xi_2)$). See [16] and [13]; note that $\lambda_2(\xi), \hat{\lambda}_2(\xi)$ are respectively denoted by $\lambda(u), \hat{\lambda}(u)$ in [13]. We have

$$
(\bar{\psi}_1, \psi^*_1) = \left(-\frac{\hat{\lambda}_2(\xi)}{\lambda_2(\xi) + 1}, -\frac{\lambda_2(\xi)}{\hat{\lambda}_2(\xi) + 1}\right).
$$

(2.1)

The following result can be obtained from Davenport and Schmidt’s work by generalizing Lemmas 2 and 6 of [5] (see for example [8, Corollaire 6.2.7]). It is also a corollary of a recent result due to Schleischitz [15, Theorem 1.6]:

**Proposition 2.1.** Suppose that $\hat{\lambda}_2(\xi) > \frac{1}{2}$. Then

$$
\lambda_2(\xi) \leq 1.
$$

(More generally [15, Theorem 1.6] implies that if $\hat{\lambda}_n(\xi) > \frac{1}{n}$ then $\lambda_n(\xi) \leq 1$ (but here we have restricted our study to the case $n = 2$). Recall that $\bar{\psi}_i$ and $\psi^*_i$ ($i = 1, 2, 3$) denote the parametric exponents attached to a vector $u = (1, \xi, \xi_2)$, then thanks to (2.1) Proposition 2.1 may be rewritten as follows

**Proposition 2.2.** Suppose that $\bar{\psi}_1(\xi) < -\frac{1}{3}$. Then

$$
\psi^*_1(\xi) \geq -\frac{1}{2}.
$$

3 Parametric geometry of numbers

In this section we quickly present Schmidt and Summerer’s tools from the parametric geometry of numbers (cf. [16] and [17]). For our purpose the most important notion is the definition of a
For and we set

Definition 3.1. Fix a real number \( q \) and we set

\[ \lambda_j(q) = \log(\lambda_j^*(q)) \]

where \( \lambda_j^*(q) \) denotes the \( j \)-th successive minimum of the convex body \( C_u(e^q) \) with respect to the lattice \( \mathbb{Z}^n \) and

\[ \overline{\psi}_j = \limsup_{q \to \infty} \frac{L_j^*(q)}{q} \quad \text{and} \quad \underline{\psi}_j = \liminf_{q \to \infty} \frac{L_j^*(q)}{q}. \]

We set \( L_n^* = (L_1^*, \ldots, L_n^*) \). All functions \( L_j^* \) are continuous, piecewise linear with slopes 0 and \(-1\), and satisfy for any \( q \geq 0 \)

(a) \( L_1^*(q) \leq \cdots \leq L_n^*(q) \).

(b) \( L_1^*(q) + \cdots + L_n^*(q) = -q + O(1) \).

The following definition is that of an \( n \)-system (see [14] Definition 4.1; this is a \((n, 0)\)-system for Schmidt and Summerer [17]).

Definition 3.1. Fix a real number \( q_0 \geq 0 \). An \( n \)-system on \([q_0, +\infty)\) is a continuous piecewise linear map \( P = (P_1, \ldots, P_n) : [q_0, +\infty) \to \mathbb{R}^n \) with the following properties:

(a) For each \( q \geq q_0 \), we have \( 0 \leq P_1(q) \leq \cdots \leq P_n(q) \) and \( P_1(q) + \cdots + P_n(q) = q \).

(b) If \( H \) is a non-empty open subinterval of \([q_0, +\infty)\) on which \( P \) is differentiable, then there is an integer \( r \) \((1 \leq r \leq n)\), such that \( P_r \) has slope 1 on \( H \) while the other components \( P_j \) of \( P \) \((j \neq r)\) are constant on \( H \).

(c) If \( q > q_0 \) is a point at which \( P \) is not differentiable and if the integers \( r \) and \( s \), for which \( P_r \) has slope 1 on \((q - \varepsilon, q)\), and \( P_s \) has slope 1 on \((q, q + \varepsilon)\) \((\varepsilon > 0 \text{ small enough})\), satisfy \( r < s \), then we have \( P_r(q) = P_{r+1}(q) = \cdots = P_s(q) \).

Given a subset \( A \) of \( \mathbb{R} \), we call interval of \( A \) any interval of \( \mathbb{R} \) included in \( A \). Here, the condition \( "P \) is piecewise linear" means that for all bounded intervals \( J \subset [q_0, +\infty) \), the intersection of \( J \) with the set \( D \) of points in \([q_0, +\infty)\) at which \( P \) is not differentiable is finite, and that the derivative of \( P \) is locally constant on \([q_0, +\infty) \setminus D \). The slope of a component \( P_j \) of \( P \) on a non-empty open interval \( H \) of \([q_0, +\infty) \setminus D \) is the constant value of its derivative on \( H \), or equivalently the slope of its graph over \( H \).

Recall that in this paper we work with dual \( n \)-systems, defined as follows:

Definition 3.2. Fix a real number \( q_0 \geq 0 \). A dual \( n \)-system on \([q_0, +\infty)\) is a map \( P = (P_1, \ldots, P_n) : [q_0, +\infty) \to \mathbb{R}^n \) such that \((-P_n, \ldots, -P_1)\) is an \( n \)-system on \([q_0, +\infty)\).

We follow [17] §3 and we define the combined graph of a set of real valued functions defined on an interval \( I \) to be the union of their graphs in \( I \times \mathbb{R} \). For a map \( P : [c, +\infty) \to \mathbb{R}^n \) and an interval \( I \subset [c, +\infty) \), we also define the combined graph of \( P \) on \( I \) to be the combined graph of its components \( P_1, \ldots, P_n \) restricted to \( I \).
**Definition 3.3.** In order to draw the combined graph of the map \( L_u^* \), it is useful to define for each point \( x \in \mathbb{R}^n \setminus \{0\} \) the quantity \( \lambda_1(x) \) to be the smallest real number \( \lambda > 0 \) such that \( x \in \lambda C_u^*(e^\gamma) \). Then we set
\[
L_u^*(x) = \log(\lambda_1(x)).
\]
Roy calls the graph of \( L_u^* \) the trajectory of \( x \). Locally, the combined graph of \( L_u^* \) is included in the combined graph of a finite set of \( L_{\lambda, u}^* \), and for each \( x \neq 0 \) we have
\[
L_u^*(x) = \max \{ \log(M(x)), \log(||x||) - q \}.
\]
Note that
\[
L_1^*(x) = \min_{x \neq 0} L_u^*(x). \quad (3.1)
\]

**Case n = 3 and u = (1, ξ, ξ^2)**

The following properties are only informative and are not required for the proof of Theorem 4.1.2. Fix \( \xi \) a real number not algebraic of degree \( \leq 2 \). We set \( u = (1, \xi, \xi^2) \) (whose coordinates are linearly independent over \( \mathbb{Q} \)). Recall that classical Diophantine exponents and parametric exponents are related by the formula (2.1)
\[
(\overline{\psi}_1, \overline{\psi}_2^*) = \left( -\frac{\lambda_2(\xi)}{\lambda_2(\xi) + 1}, \frac{\lambda_2(\xi)}{\lambda_2(\xi) + 1} \right).
\]

In general exponents \( \psi_i^* \) and \( \overline{\psi}_i \) satisfy
\[
-1 \leq \psi_i^* \leq \frac{1}{\gamma} \leq -\frac{1}{2} \quad \text{(it is a translation of the classical estimates)}
\]
\[
\frac{1}{2} \leq \lambda_2(\xi) \leq \lambda_2(\xi) \quad \text{(implied by Dirichlet’s box principle). By using Proposition 2.2 and}
\]
\[
\text{the classical estimate } \lambda_2(\xi) \leq \frac{1}{\gamma} \text{ due to Davenport and Schmidt [5] Theorem 1a} \quad \text{where } \gamma \text{ denotes}
\]
\[
\text{the golden ratio } \frac{1 + \sqrt{5}}{2}, \text{we have in our context the following property. Suppose } \overline{\psi}_1 < -\frac{1}{3}. \text{ Then}
\]
\[
-1 \leq \psi_i^* \leq -\frac{1}{3} \quad \text{and} \quad -\frac{1}{\gamma} \leq \overline{\psi}_1 \leq -\frac{1}{3}.
\]

It is an open problem to give a complete description of the joint spectrum of the six parametric exponents when \( u \) is of the form \( (1, \xi, \xi^2) \) (we actually do not even know an explicit description of the spectrum of \( \overline{\psi}_1^* \)).

## 4 Exponents ψ*

In Definition 4.1 we define the exponent \( \kappa^*(\xi) \) and in Proposition 4.1.3 we give the relation between \( \kappa^*(\xi) \) and Fischler’s exponents \( \beta_i(\xi) \). Let \( \xi \) be a real number not algebraic of degree \( \leq 2 \) and let \( u \) denote the vector \((1, \xi, \xi^2)\). We consider the function \( L_u^* \) and exponents \( \overline{\psi}_i, \psi_i^* \) of Section 3. In [6], Fischler considers real numbers \( \xi \) satisfying \( \lambda_2(\xi) = 1 \) (if \( \lambda_2(\xi) < 1 \) then we have \( \beta_0(\xi) = +\infty \) by definition of \( \beta_0(\xi) \)). In order to include numbers \( \xi \) such that \( \lambda_2(\xi) < 1 \) it is convenient to consider \( \lim_{x \rightarrow \alpha^+} \beta_i(\xi) \) with \( \varepsilon_0 = 1 - \lambda_2(\xi) \) rather than \( \beta_0(\xi) \). Exponent \( \kappa^* \) is a parametric version of this new exponent.

For all non-zero \( x, y \in \mathbb{R}^n \) such that \( ||x|| < ||y|| \) and \( M(x) > M(y) \) we set
\[
\alpha(x) = \frac{\log(M(x))}{\log(||x||) - \log(M(x))} \quad \text{and} \quad \alpha(x, y) = \frac{\log(M(x))}{\log(||y||) - \log(M(x))}. \quad (4.1)
\]
The situation is illustrated on Figure 1 below. Geometrically, $\alpha(x,y)$ is the slope of the line passing through origin 0 and the intersection point of the trajectories of $x$ and $y$. The quantity $\alpha(x)$ is the slope of the line passing through the origin 0 and the point $(q^*_x, L^*_x(q^*_x))$ (where $q^*_x$ is the abscissa at which $L^*_x$ changes slope). We have

$$\alpha(x) = \frac{L^*_x(q^*_x)}{q^*_x}. $$

Figure 1: Trajectories of points $x$ and $y$

For each $\alpha > \psi^*_1$ we set

$$A^*_\alpha = \{ x \in \mathbb{Z}^3 \setminus \{0\} \mid \alpha(x) \leq \alpha \}. $$

(4.2)

Geometrically, $A^*_\alpha$ is the set of integer points $x$ whose trajectory goes under the line of slope $\alpha$ passing through the origin 0, more precisely we have the equivalence:

$$\text{There exists } q > 0 \text{ such that } \frac{L^*_x(q^*_x)}{q^*_x} \leq \alpha \iff x \in A^*_\alpha. $$

(4.3)

Fix $\alpha > \psi^*_1$. By definition of $\psi^*_1$ the set $A^*_\alpha$ is infinite. We define the function $L^*_1,\alpha$ by

$$L^*_1,\alpha(q) = \min_{x \in A^*_\alpha} L_x(q). $$

Definition 4.1. For each $\alpha > \psi^*_1$ we set $\kappa^*_\alpha(\xi) = \kappa^*_\alpha = \limsup_{q \to +\infty} \frac{L^*_1,\alpha(q)}{q}$ and we define

$$\kappa^*(\xi) = \sup_{\alpha > \psi^*_1} \kappa^*_\alpha = \lim_{\alpha \to \psi^*_1} \kappa^*_\alpha. $$

Remark 4.2. Note that (3.1) implies that for each $\alpha > \psi^*_1$ we have $\kappa^*_\alpha \geq \psi^*_1$, with equality if $\alpha > \psi^*_1$. Roughly speaking, $\kappa^*$ corresponds to $\psi^*_1$ when we ignore all the “bad” approximations of $u = (1, \xi, \xi^2)$, i.e. integer points $x \in \mathbb{Z}^3$ such that $\alpha(x)$ is not “close” to $\psi^*_1$.

If we define $\bar{\varepsilon}$ and $\tilde{\alpha}$ (which are reciprocal functions) by $\bar{\varepsilon}(\alpha) = \frac{1+2\alpha}{1+\alpha}$ and $\tilde{\alpha}(\varepsilon) = -\frac{1-\varepsilon}{2-\varepsilon}$, then we have for each $\alpha > \psi^*_1$

$$A^*_\varepsilon = \{ x \in \mathbb{Z}^3 \setminus \{0\} \mid \|x\|^{-1-\bar{\varepsilon}(\alpha)} \}, $$

which is the set $A_\varepsilon$ used by Fischler in [6] (with $\varepsilon = \bar{\varepsilon}(\alpha)$). Note that $\alpha = \tilde{\alpha}(\varepsilon)$. 

7
Sequence of minimal points

Now let \((a_i)_i\) be a sequence of minimal points, i.e. a sequence of integers points satisfying

\[
\begin{cases}
1 < \|a_1\| < \|a_2\| < \ldots \\
1 > M(a_1) > M(a_2) > \ldots,
\end{cases}
\]

and such that for any non-zero integer point \(z\), if \(|z| < \|a_{k+1}\|\) then \(M(z) \geq M(a_k)\). The notion of minimal point first appears in Davenport and Schmidt articles [4], [5]; it is also used by Roy [10] and [11], and it plays a crucial role in Fischler’s article [6] (note that the notion of minimal point may change according to the context). If \(q^*_i\) denotes the abscissa at which \(L^*_a\) changes slope and \(q'_i\) absissa of the intersection point of the trajectories of \(a_i\) and \(a_{i+1}\), we have

\[
\psi^*_i(\xi) = \liminf_{i \to +\infty} \frac{L^*_i(q^*_i)}{q^*_i} \quad \text{and} \quad \varphi^*_i(\xi) = \limsup_{i \to +\infty} \frac{L^*_i(q'_i)}{q'_i}.
\]

This can be rewritten with our notations:

\[
\psi^*_{i\downarrow}(\xi) = \liminf_{i \to +\infty} \alpha_{a_i} \quad \text{and} \quad \varphi^*_{i\downarrow}(\xi) = \limsup_{i \to +\infty} \alpha_{a_i, a_{i+1}}.
\]

Fix \(\alpha > \psi^*_{i\downarrow}\). If \((u_i)_i\) denotes the subsequence of \((a_i)_i\) consisting in those \(a_i\) which are in \(A^*_a\) (ordered according to their norm), then we have \(L^*_{i\alpha}(q) = \min_i L_{u_i}(q)\) and

\[
\kappa^*_{a} = \limsup_{i \to +\infty} \alpha_{u_i, u_{i+1}} = \limsup_{i \to +\infty} \frac{\log(M(u_i))}{\log(\|u_{i+1}\|) - \log(M(u_i))}. \tag{4.5}
\]

Moreover, setting \(\varepsilon = \varepsilon(\alpha)\), Equation (11) of [6] gives

\[
\beta_\varepsilon(\xi) = \limsup_{i \to +\infty} \frac{\log(u_{i+1})}{-\log(M(u_i))}. \tag{4.6}
\]

Equations (4.5) and (4.6) give the following result.

**Proposition 4.3.** For each \(\alpha > \psi^*_{i\downarrow}\) we have

\[
\kappa^*_{a}(\xi) = -\frac{1}{\beta_\varepsilon(\xi) + 1}, \tag{4.7}
\]

where \(\varepsilon = \varepsilon(\alpha) = \frac{1 + 2\alpha}{1 + \alpha}\). In particular if we set \(\varepsilon_0 = \varepsilon(\psi^*_{i\downarrow}) = 1 - \lambda_2(\xi)\), then we have \(\kappa^*(\xi) = \lim_{\varepsilon \to \varepsilon^*_{0\downarrow}} -\frac{1}{\beta_\varepsilon(\xi) + 1}\). If \(\lambda_2(\xi) = 1\) (which is equivalent to \(\psi^*_{i\downarrow} = -\frac{1}{2}\)), then \(\varepsilon_0 = 0\) and we have

\[
\kappa^*(\xi) = -\frac{1}{\beta_0(\xi) + 1}. \tag{4.8}
\]

## 5 Proof of Theorem 1.2

Using the exponent \(\kappa^*\) we construct an uncountable set of dual 3-systems which do not represent any real number. The construction stated below is inspired by the construction of a rigid \(n\)-system and a canvas with mesh \(\delta\) from Roy (see [13, Sect. 1]).

Let \(\alpha\) be a real number such that \(-\frac{1}{\alpha} < \alpha < -\frac{1}{3}\). We have \(-\frac{1}{2} < -(1 + \alpha)^2 < \alpha\). For \(k \in \mathbb{N}\) we set

\[
q_{2k} = \left(\frac{1}{\alpha} - 1\right)^k \quad \text{and} \quad q_{2k+1} = -\frac{q_{2k}}{2\alpha}.
\]
The sequence \((q_k)_{k \geq 0}\) is increasing and tends to infinity. For each \(k \geq 0\) we define a point \(a^{(k)} = (a_1^{(k)}, a_2^{(k)}, a_3^{(k)}) \in \mathbb{R}^3\) by

\[
a^{(2k)} = \left(-\frac{1}{2}q_{2k}, -\frac{1}{2}q_{2k}, \frac{\alpha}{2}q_{2k}\right) \quad \text{and} \quad a^{(2k+1)} = \left(\frac{(1 + \alpha)^2}{2\alpha}q_{2k}, a_1^{(2k)}, a_2^{(2k)}\right).
\]

By hypothesis on \(\alpha\) we have for each \(k\)

\[
a_1^{(k)} < a_2^{(k)} < a_3^{(k)} \quad \text{and} \quad a_1^{(k)} + a_2^{(k)} + a_3^{(k)} = -q_k.
\]

Let us set \(\Delta = \{(x_1, x_2, x_3) \in \mathbb{R}^3; x_1 \leq x_2 \leq x_3\}\) and \(\Phi : \mathbb{R}^3 \to \Delta\) the continuous map which lists the coordinates of a point in monotone non-decreasing order. We define the dual 3-system \(P_\alpha = (P_1, P_2, P_3)\) by

\[
P_\alpha(q) = \begin{cases} 
\Phi(a_1^{(2k)}, a_2^{(2k)}, a_3^{(2k)} - q + q_{2k}) & \text{if } q_{2k} \leq q < q_{2k+1} \\
\Phi(a_1^{(2k+1)}, a_2^{(2k+1)} - q + q_{2k+1}, a_3^{(2k+1)}) & \text{if } q_{2k+1} \leq q < q_{2k+2}.
\end{cases}
\]

The function \(P_\alpha\) is a dual 3-system whose combined graph is represented on Figure 2. By decreasing slope order, the first solid line passes through the origin 0 and has slope \(\alpha\); the second dotted line passes through 0 and has slope \(-(1 + \alpha)^2\); the last thin line passes through the origin and has slope \(-\frac{1}{2}\).

![Figure 2: combined graph of the dual-system \(P_\alpha\)](image)

We can immediately deduce that

\[
\liminf_{q \to +\infty} \frac{P_1(q)}{q} = -\frac{1}{2}.
\]

Let us prove that if \(P_\alpha\) represents some real number \(\xi\), then we have

\[\kappa^*(\xi) = \alpha.\]

If \(P_\alpha\) represents indeed a real number \(\xi\), then for all \(\alpha' < -(1 + \alpha)^2\) (where \(-(1 + \alpha)^2\) is the slope of the dotted line), up to a bounded difference, the graph of function \(L_{t,\alpha'}^*\) associated to \(u = (1, \xi, \xi^2)\) is the one represented on Figure 3 (see section 4 for the definition of \(L_{t,\alpha'}^*\)).
We have therefore $\kappa^*_\alpha(x) = \alpha$ for all $\alpha' < -(1 + \alpha)^2$ and thus $\kappa^*(x) = \alpha$. According to Proposition 4.3 this means that $\beta_0(x) = -1 - \frac{1}{\alpha}$, i.e. $\alpha = -\frac{1}{\beta_0(x)+1}$.

Recall that in Theorem 1.3 we set

$$S_0 = \{\beta_0(x) \mid x \text{ not algebraic of degree } \leq 2\}.$$  

Theorem 1.3 gives

$$S_0 \cap [\gamma, \sqrt{3}) = S' \cap [\gamma, \sqrt{3}),$$

where $S_0 = \text{spec}(\beta_0)$ and $S'$ is related to Sturmian sequences. Cassaigne proved [3] that the set $S'$ has an empty interior (see also the last remark of [2]). In particular, the set $[\gamma, \sqrt{3}) \setminus S_0$ forms a dense subset of $[\gamma, \sqrt{3})$. This remark and the following Proposition (which is a corollary of Theorem 1.3) imply Theorem 1.2.

**Proposition 5.1.** For each $\alpha = -\frac{1}{1+\beta}$ with $\beta \in [\gamma, \sqrt{3}) \setminus S_0$ the dual 3-system $P_\alpha$ does not represent any real number $\xi$.

**Remark 5.2.** Let us set $\Delta = \left\{-\frac{1}{1+\beta} \mid \beta \in [\gamma, \sqrt{3}) \setminus S_0\right\}$. Note that $S' \cap (\gamma, 1+\frac{\sqrt{3}}{2}) = \emptyset$, so that for each $\beta \in (\gamma, 1+\frac{\sqrt{3}}{2})$ we have $-\frac{1}{1+\beta} \in \Delta$. In particular, $\Delta$ contains an interval (not reduced to a point); it has thus a non-empty interior.

Note that it is an open problem to describe precisely the set $S_0 \cap [\sqrt{3}, +\infty)$.

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