An explicit model for the adiabatic evolution of quantum observables driven by 1D shape resonances

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Abstract

This paper is concerned with a linearized version of the quantum transport problem where the Schrödinger–Poisson operator is replaced by a non-autonomous Hamiltonian, slowly varying in time. We consider an explicitly solvable system where a semiclassical island is described by a flat potential barrier, while a time-dependent ‘delta’ interaction is used as a model for a single quantum well. Introducing, in addition to the complex deformation, a further modification formed by artificial interface conditions, we give a reduced equation for the adiabatic evolution of the sheet density of charges accumulating around the interaction point.

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1. Introduction

The derivation of reduced models for the dynamics of transverse quantum transport with concentrated nonlinearities plays a central role in the mathematical analysis of semiconductor heterostructures like tunnelling diodes or possibly more complex structures. The conduction band edge-profile of such systems has been described using Schrödinger–Poisson Hamiltonians with quantum wells in a semiclassical island, where a nonlinear potential term, depending on the local charge density, approximates in the mean-field limit the repulsive interaction between the charge carriers. A functional framework for such a model is proposed in [17], based on Mourre’s theory and Sigal–Soffer propagation estimates, and implements a dynamical nonlinear version of the Landauer–Büttiker approach. The analysis of the related

* This work is dedicated to the memory of Pierre Duclos and to our friend Naoufel Ben Abdallah who recently passed away.
steady-state problem, developed in [6, 7, 18] on the basis of the Helffer–Sjöstrand approach to resonances [14], has provided with an asymptotic reduced equation for the nonlinear potential, which elucidates the influence of the geometry of the potential on the feasibility of hysteresis phenomena, already studied in [15, 19], and confirms the general belief arising in the physical literature: the nonlinear phenomena are governed by a finite number of resonant states.

For the dynamical problem, we conjecture that the nonlinear dynamics follows the time evolution of those resonant states corresponding to shape resonances which are asymptotically embedded in some relevant energy interval when the quantum scale of the problem, parametrized by \( h \), goes to zero. It is known, at least in the linear case, that this evolution shows an exponential decay behaviour having physical interpretation in terms of truncated resonant states (lying in \( L^2 \)). The quasi-resonant states concentrate their mass inside the quantum well’s support—the classical region of motion of our model—on a long time scale given by the inverse of the imaginary part of the resonant energies \( E_{\text{res}} \). In this connection, the Poisson potential as well as the charge density for the nonlinear modelling are expected to evolve slowly in time, with an adiabatic parameter \( \varepsilon \) which is related to the quantum scale of the system according to \( \varepsilon = O(\text{Im} E_{\text{res}}) \sim e^{-\tau h} \), for some \( \tau > 0 \).

This paper is concerned with a linearized version of the transport problem where the Schrödinger–Poisson operator is replaced by a non-autonomous Hamiltonian, slowly varying in time, and whose time profile takes into account the evolution of the nonlinear potential. This allows us to separate the adiabatic evolution generated by the double scale Hamiltonian, from questions concerned with the nonlinear nature of the original problem. In particular, we consider an explicitly solvable model where a semiclassical island with a single quantum well is described by the superposition of a flat potential barrier plus an attractive time-dependent ‘delta’ interaction whose strength is proportional to a scale parameter \( h \). Our approach consists in introducing, in addition to the complex deformation, a further modification formed by artificial interface conditions. According to the results obtained in [12], an adiabatic theorem holds for this modified system (see theorem 7.1 in [12]), while small perturbations are produced on the relevant spectral quantities (actually the same remains true under more general assumptions). In this simplified framework, we give a reduced equation for the adiabatic evolution of the sheet density of charges accumulating around the interaction point. This result is coherent with the reduced model predicted in [19, 20]. Moreover, some corrections arise, depending on the time profile of the perturbation, which can be relevant in realistic physical situations.

2. The model

We consider the time evolution of a quantum observable for a family of non-self-adjoint Hamiltonians adiabatically depending on the time. Our model is defined by the Schrödinger operators \( H_{\Delta_{\theta},\alpha(t)}^{h} \):

\[
H_{\Delta_{\theta},\alpha(t)}^{h} = -\hbar^2 \Delta_{\theta} + 1_{(a,b)}V_0 + h\alpha(t)\delta_c,
\]

(2.1)

where \( \Delta_{\theta} \) is a singularly perturbed Laplacian with artificial interface conditions on the boundary of \( \mathbb{R}\setminus\{a,b\} \):

\[
\begin{cases}
D(\Delta_{\theta}) = \left\{ u \in H^2(\mathbb{R}\setminus\{a,b\}) : \begin{array}{ll}
\hbar^2 u(b^+) = u(b^-) ; & e^{-\hbar^2/2} u(b^+) = u(b^-) \\
\hbar^2 u(a^-) = u(a^+) ; & e^{-\hbar^2/2} u(a^-) = u(a^+) 
\end{array} \right\}, \\
\Delta_{\theta} u = \partial_x^2 u.
\end{cases}
\]

(2.2)
Meanwhile, $1_{(a,b)} V_0 + h \alpha \delta_c$ is a self-adjoint time-dependent point interaction defined with $V_0 > 0, c \in (a, b), \alpha \in C^\infty(0, T)$ and requiring the condition

$$u \in H^2((a, b) \setminus \{c\}) \cap H^1(a, b), \quad h[u'(c^+) - u'(c^-)] = \alpha(t) u(c)$$

for all $u \in D(H_{h\theta_0,\alpha(t)})$ (we refer to [1] for the definition of delta interaction Hamiltonians).

An accurate analysis of this class of operators has been given in [12]. It is shown that the interface conditions introduce small errors, controlled by $\theta_0$, with respect to the original self-adjoint model (defined by $\theta_0 = 0$). The main interest in introducing the artificial perturbation $\Delta_{\theta_0}$ rests upon the fact that the corresponding Hamiltonian defines, under complex deformation, a dynamical systems of contractions. This provides us with an alternative approach to the adiabatic evolution of the shape resonances possibly associated with our model, which can be treated in terms of (adiabatic evolution of) spectral projectors for the non-self-adjoint deformed operator (for this point, we refer to theorem 7.1 in [12]; see also the work of Joye [16] for the adiabatic evolution of dynamical systems without uniform time estimates on the semigroup).

Let us consider a positive smooth function $\chi$, supp $\chi \subset (a, b)$; in our framework $\chi$ is the quantum observable associated with the charge density accumulated in a small neighbourhood of the quantum well. The expected value of this density sheet is associated with

$$A_{\theta_0}(t) = \text{Tr}\{\chi \rho^h_t\}, \quad (2.4)$$

where $\rho^h_t$ is the time evolution of the density operator. The initial state of the system,

$$\rho^h_0 = \int \frac{dk}{2\pi \hbar} g(k) \langle \psi_{-}(k, \cdot, \alpha_0) | \psi_{-}(k, \cdot, \alpha_0) \rangle,$$

is defined by a superposition of incoming scattering states solving

$$(H_{h\theta_0,\alpha(t)}^h - k^2) \psi_{-}(k, \cdot, \alpha) = 0,$$

according to the out-of-equilibrium assumption $g = 1_{\mathbb{R}_+} g$. Using an adiabatic approximation for the time variations of the coupling parameter $\alpha$, $\rho^h_t$ is written as

$$\rho^h_t = \int \frac{dk}{2\pi \hbar} g(k) \langle u(k, \cdot, t) | u(k, \cdot, t) \rangle,$$

with

$$i \varepsilon \partial_t u(k, \cdot, t) = H_{h\theta_0,\alpha(t)}^h u(k, \cdot, t),$$

$$u_{t=0} = \psi_{-}(k, \cdot, \alpha_0).$$

Adiabatic dynamics have already been considered within the modelling of out-of-equilibrium quantum transport in [3, 4, 10], playing with the continuous spectrum with self-adjoint techniques. For energies close to the shape resonances, the relevant observable of this problem follow the adiabatic evolution of resonant states. Then, a different approach consists in using complex deformations, originally introduced in [2, 5]. In [12], we define a family of exterior complex deformations $U_\theta$ for Hamiltonians with compactly supported potentials in $(a, b)$

$$U_\theta u(x) = \begin{cases} e^{\frac{i}{\hbar} \theta u(x)} (x - b) + b, & x > b, \\ u(x), & x \in (a, b), \\ e^{\frac{i}{\hbar} \theta u(x)} (x - a) + a, & x < a. \end{cases}$$

The corresponding deformed operator is obtained by conjugation: $H_{h,\theta_0,\alpha(t)}^h(\theta) = U_\theta H_{h_0,\alpha(t)}^h U_\theta^{-1}$. It is explicitly written as

$$H_{h,\theta_0,\alpha(t)}^h(\theta) = -\hbar^2 \partial_{x^2} + \Delta_{\theta_0,\theta} + 1_{(a,b)} V_0 + h \alpha(t) \delta_c.$$

$$3$$
The initial state is defined with a smooth and compactly supported partition function $g$. The deformation and the interface conditions parameters are equal and the adiabatic parameter is fixed to the exponential scale defined by $\alpha_t$. Denoting with $S_{\theta,c}(t,s)$ the time propagator related to $\frac{1}{\hbar}H^b_{\theta,c}(t)$, we get

$$A_{\theta}(t) = \text{Tr}[\chi U_\theta S_{\theta,c}(t,0)\rho_{\theta} U_\theta^*]$$

where $(U_\theta^{-1})^* = (U_\theta^*)^{-1}$ is used. The conjugation $U_\theta S_{\theta,c}(t,0)U_\theta^{-1}$ defines the propagator associated with the deformed Hamiltonian $\frac{1}{\hbar}H^b_{\theta,c}(t)$. Thus, (2.4) reformulates as follows:

$$A_{\theta}(t) = \text{Tr}[\chi \rho_{\theta}^b(\theta)]$$

$$\rho_{\theta}^b(\theta) = \int \frac{dk}{2\pi\hbar} g(k)|u_\theta(k,\cdot,t)\rangle\langle u_\theta(k,\cdot,t)|$$

with

$$\left\{ \begin{array}{l}
    i\hbar \partial_t u_\theta(k,\cdot,t) = H^b_{\theta,c}(t)u_\theta(k,\cdot,t) \\
    u_{\theta,0} = U_\theta \psi_{-}(k,\cdot,0) \end{array} \right.$$  

We will consider this evolution problem under the following assumptions.

(h1) The deformation and the interface conditions parameters are equal and

$$\theta = \theta_0 = i\hbar\psi_{0}, \quad N_0 > 2.$$  

(h2) The time-dependent coupling parameter $\alpha_t$ is a $C^\infty(0,T)$ real-valued function with compact range in $(-2V_0^2,0)$ and such that

(i) its first variations have size $\hbar$, i.e.

$$\forall s, t \in [0,T] \Rightarrow |\alpha_t - \alpha_s| \leq \frac{2\hbar}{\sqrt{V_0d_0}}$$  

where $d_0 > 0$ is specified further;

(ii) there exists a positive integer $J$ such that the vector $\{\alpha_j(t)\}_{j=1}^J$ is not null for all $t$.

(h3) The initial state is defined with a smooth and compactly supported partition function $g$ such that

$$\text{supp} \ g(k) = \left\{ k > 0, |k^2 - \lambda_0| < \frac{2h}{d_0} \right\}$$

where $\lambda_0$ denotes some asymptotic energy: $\lambda_0 \in (0, V_0)$, while $d_0$ and $h_0$ are such that $\text{supp} \ g \subset (0, V_0)$ uniformly w.r.t. $h \in (0, h_0)$. Furthermore, we assume that $g(E^2)$ extends to a holomorphic function of $E$ in the complex neighbourhood of $\lambda_0$ of radius $\frac{h}{\pi}$. The function $\chi \in C^\infty_0(a,b)$ is real valued and such that

$$\text{supp} \ \chi = (c - 2\eta, c + 2\eta)$$

$$\chi(x)|_{x \in (-c, c + c\eta)} = 1,$$  

$$2\eta < d(c, [a,b]),$$  

with $d(c, [a,b])$ denotes the distance of $c$ from the the boundary of the interval $(a,b)$.

(h4) The adiabatic parameter is fixed to the exponential scale defined by

$$\varepsilon = e^{-\frac{2\pi}{\hbar}d(c,[a,b])}.$$
The scaling assumptions adopted in (2.1) are motivated by the fact that the heterostructures present a finite number of resonant states in the relevant energy interval. The particular shape featured by our potential—and, more generally, by quantum wells in semiclassical islands—permits us to keep these constraints even in the limit \( h \to 0 \). According to this point, the spectral asymptotics concerned with the model at issue is not semiclassical, and the parameter \( h \) can be rather considered as the quantum scale of the system. In applications to transport problems in semiconductor devices, the parameter \( h \) is given by a rescaled Fermi length depending on the Fermi level of the system. In particular, its size is fixed by the length of the barrier’s support and the donor density outside the device (we refer to sections 2 and 3 in [8] for this point). As remarked in [8], realistic situations for GaAs and Si devices are provided with \( h \sim 0.1, 0.3 \).

The explicit character of our model and the adiabatic theorem, obtained in [12] for this class of non-self-adjoint Hamiltonians, allow us to obtain a complete description of the asymptotic behaviour of \( \Lambda_0(t) \) as \( h \to 0 \), in particular concern with the position of the delta-shaped potential well.

To formulate our results, we adopt the following notation.

Notation

(a) The resonance at time \( t \) and the related resonant state are respectively denoted with \( E(t) \) and \( G(t) \).

(b) The expression \( X_\epsilon = \tilde{O}(\epsilon^h) \) is used for the following condition: \( \forall \delta \in (0, 1) \); there exists \( C_{X, \delta} \) such that

\[
|X_\epsilon| \leq C_{X, \delta} \epsilon^{\alpha - \delta}. \tag{2.19}
\]

**Theorem 2.1.** Let \( \lambda_\epsilon = V_0 - \frac{\alpha_\epsilon^2}{r^2}, \ t \in [0, T] \) and assume conditions (h1)–(h4) to hold with \( h \in (0, h_0), h_0 \) small, \( \lambda_0 = V_0 - \frac{\alpha_0^2}{r^2} \) and \( d_0 > 0 \) such that \( \lambda_\epsilon^u \in \text{supp} \ g \subset (0, V_0) \) for all \( t \). The following conditions hold.

(i) For any \( t \in [0, T] \), there exists a single resonance, \( E(t) \), of \( H_{h_0, a(t)}^h \) such that \( \text{Re} \, E(t) \in (0, V_0) \). With the notation \( E(t) = E_R(t) - i\Gamma_t \), the real and the imaginary parts of \( E(t) \) fulfill the conditions

\[
E_R(t) = \lambda_\epsilon + \tilde{O}(e^{-\frac{\alpha_\epsilon}{r^2}d(c, [a, b])}),
\]

\[
\Gamma_t = \tilde{O}(e^{-\frac{\alpha_\epsilon}{r^2}d(c, [a, b])}),
\]

with \( d(c, [a, b]) \) denoting the distance from the boundary points. The related resonant state, \( G(t) \), is locally defined as the solution of

\[
(H_{h_0,0}^h - E(t))u = \delta_c, \quad \text{in} \quad L^2(a, b). \tag{2.22}
\]

Both \( E(t) \) and \( G(t) \) are holomorphic w.r.t. \( \alpha \), and \( C^\infty \) in time.

(ii) There exists \( \tau_{\epsilon, J} > 0 \), depending on \( \chi, J \), such that the solution of (2.11)–(2.13) is

\[
\Lambda_\epsilon(t) = a(t) + J(t) + \tilde{O}(|\theta_0|) + \tilde{O}(e^{-\frac{\alpha_\epsilon}{r^2}}). \tag{2.23}
\]

The main contribution, \( a(t) \), is described by the equation

\[
\begin{align*}
\notag \frac{d}{dt}a(t) &= \left(-2 \frac{\beta^2}{\alpha_\epsilon^2} \right) \left(a(t) - \frac{\alpha_\epsilon}{\alpha_0} \right) g \left(\lambda_\epsilon^u\right), \quad \text{for} \quad d(c, [a, b]) = c - a, \\
&= g(\lambda_\epsilon^u), \quad \text{for} \quad d(c, [a, b]) = b - c.
\end{align*} \tag{2.24}
\]

or by \( a(t) = \tilde{O}(e^{-\frac{\beta_\epsilon}{2}}) \), \( \beta = \frac{\beta_\epsilon}{2}(c - a - (b - c)) \) if \( d(c, [a, b]) = b - c \).
(iii) When \( d(c, \{a, b\}) = c - a \), the remainder is \( \mathcal{J} = \mathcal{J}_1 + \mathcal{J}_2 + \mathcal{O}(e^{-\frac{c-a}{h}}) \), with
\[
\mathcal{J}_1(t) = \left| 1 - \frac{\alpha t}{\alpha_0} \right|^2 g(\lambda_1^2) = \mathcal{O}(h^2),
\]
and
\[
\mathcal{J}_2(t) = \mathcal{O}(\lambda^2(t - \lambda_0)^2 e^{\frac{1}{2}}).
\]

For \( d(c, \{a, b\}) = b - c \), the correction \( \mathcal{J} = \mathcal{O}(e^{-\frac{c-a}{h}}) \) is exponentially small.

The above result is concerned with situations where the two barriers composing our potential have different opacity w.r.t. the electron tunnelling. When the interaction point \( c \) is closer to the left boundary of the barrier, i.e. \( d(c, \{a, b\}) = c - a \), a macroscopic variation of the charges accumulating around \( c \) is observed and a reduced equation is given. This corresponds to the appearance of macroscopic hysteresis phenomena in the nonlinear modelling where a similar simplified equation was predicted [19]. On the opposite, for \( d(c, \{a, b\}) = b - c \), only exponentially small contributions to \( A_\theta \) appears as \( h \to 0 \). Although the critical case, given by \( b - c = c - a + \mathcal{O}(h) \), is not explicitly considered here; most of the computations developed in this work can be adapted to study this particular problem.

The reduced model of theorem 2.1 follows from explicit computations, which are made possible by our simplified setting. The coefficient \( \left| \frac{\alpha t}{\alpha_0} \right|^3 \), appearing in this formulation, arises from the ratio of the \( L^2 \) square norms of resonant functions:
\[
\frac{\|G(0)\|^2_{L^2(\mathbb{R})}}{\|G(t)\|^2_{L^2(\mathbb{R})}}.
\]
This provides a possible link to extend the analysis to more realistic situations.

Some of the assumptions in (h1)–(h4) can be relaxed according to the following points:

(I) If we limit to the point (ii) of the theorem, the condition \( \alpha t \in C^2(0, T) \) is sufficient for the derivation of the reduced model, once that a suitable adaptation of the proof of lemma 5.2 is provided. Nevertheless, the conditions \( \alpha t \in C^\infty \) and (h2)-(ii) play a central role when the asymptotics of the remainder terms is considered.

(II) Relation (2.16) fixes the out-of-equilibrium condition \( k > 0 \) for the density matrix. The constraint \( |k^2 - \lambda_0| < 2\frac{h}{\alpha_0} \) selects the leading term of the density kernel \( \rho_{11}(\theta, x, y) \); if additional contributions to this kernel were considered, with \( |k^2 - \lambda_0| \geq 2\frac{h}{\alpha_0} \), they would generate contributions to \( A_\theta \), allowing exponentially small bounds.

Developing effective models for Schrödinger–Poisson Hamiltonians with quantum wells and shape resonances would provide a deeper understanding of the complex physical systems related to—namely the quantum transport in tunnelling heterostructures—and, possibly, new tools for the numerical investigation of this problem. At present, this analysis has been completely carried out in the 1D nonlinear steady-state case in [6, 7]: the corresponding reduced equations have provided an effective reduction of the numerical complexity in the computation of the nonlinear eigenvalues, charge density, currents and potential, as functions of an external bias, in the multiple well case [8, 9]. Our work is a first attempt to extend this analysis to the non-stationary problem. The result of theorem 2.1 confirms in a simplified linear setting the general idea that the quantum transport in resonant heterostructures is driven by a finite number of resonant states. In the application perspective, a numerical check of the agreement between the solution to (2.24) and the dynamics provided by the full system (refer equations (2.11)–(2.13)) in realistic situations would test the relevance of our reduced model and, possibly, the related computational assets.
The exterior part of the solution is

\[ \psi_+(k, \cdot) = \begin{cases} e^{\frac{i}{k} + R(k) e^{-\frac{i}{k}}}, & x < a \\ T(k) e^{\frac{i}{k} x}, & x > b, \end{cases} \]

while according to the boundary conditions in \( D(\Delta_{0b}) \), the interior part of the solution is

\[
\begin{align*}
(\text{in } (a, b)) & \\
\left\{ \begin{array}{l}
(-h^2 \partial_x^2 + V_0 - k^2) \tilde{\psi}_-(k, \cdot) = 0 \\
(h \partial_x + i k e^{-\theta_0}) \tilde{\psi}_-(k, a^+) = 2 i k e^{\frac{i}{k} a} e^{-\frac{i}{k} a} \\
(h \partial_x - i k e^{-\theta_0}) \tilde{\psi}_-(k, b^-) = 0.
\end{array} \right.
\end{align*}
\]
It follows from a direct computation that
\[
1_{(a,b)} \tilde{\psi}_{-}(k, \cdot) = - \frac{2 \sin \gamma z e^{i \varphi} e^{-\frac{\theta}{z}}}{\sin \left( \frac{\Delta z}{\hbar} (b - a) + 2 \gamma z \right)} \cos \left( \frac{\Lambda_{z}}{\hbar} (x - b) - \gamma z \right),
\]
with
\[
\Lambda_{z} = (z - V_{0})^{rac{1}{2}}, \quad e^{2i\gamma z} = \frac{\Lambda_{z} - z^{rac{1}{2}} e^{-\delta c}}{\Lambda_{z} + z^{rac{1}{2}} e^{-\delta c}}.
\]

The generalized eigenstates of \( H_{\theta,0}(\theta_{0}) \) are obtained by transformation through the deformation map \( U_{\theta} \); in particular, the interior part of these functions is not affected by the deformation and one has \( 1_{(a,b)} U_{\theta} \tilde{\psi}_{-}(k, \cdot) = 1_{(a,b)} \tilde{\psi}_{-}(k, \cdot) \).

### 3.1. The Green’s functions of \( H_{\theta,0}(\theta_{0}) \)

Assume \( z \) to be close to some limit energy \( \lambda_{0} \) in the interval \((0, V_{0}): \ z \in \mathcal{G}_{\theta}(\lambda_{0}) \). In this set, we use the square root’s branch-cut fixed along the positive imaginary axis (corresponding to \( \arg z \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \)). The integral kernel of \( (H_{\theta,0}(\theta_{0}) - z)^{-1} \) is defined by
\[
(H_{\theta,0}(\theta_{0}) - z) G^{\pm}(\cdot, c) = \delta_{c}.
\]

Focusing our attention on the case \( c \in (a, b) \), \( G^{\pm}(\cdot, c) \) is written as follows:
\[
G^{\pm}(x, c) = \begin{cases} 
  u_{x} e^{i \int_{a}^{x} e^{-\delta} \, dx}, & x > b, \\
  u_{x} e^{-i \int_{a}^{x} e^{-\delta} \, dx}, & x < a,
\end{cases}
\]

while the inner problem can be rephrased as
\[
\begin{cases}
  \left(-\hbar^{2} \frac{d^{2}}{dx^{2}} + V_{0} - z\right) G^{\pm}(\cdot, c) = 0, & \text{for } x \in (a, b) \setminus \{c\}, \\
  G^{\pm}(\cdot, c) \in H^{1}(a, b); & \\
  \hbar^{2} \frac{d}{dx} G^{\pm}(\cdot, c) - \partial_{c} G^{\pm}(\cdot, c) = -1, \\
  (\hbar \partial_{c} + i \sqrt{z} e^{-\delta}) G^{\pm}(\cdot, c) = 0, \\
  (\hbar \partial_{c} - i \sqrt{z} e^{-\delta}) G^{\pm}(\cdot, c) = 0.
\end{cases}
\]

With the notation adopted in (3.11), the solution is
\[
G^{\pm}(x, c) = \frac{1}{\hbar \Lambda_{c}} \tan \left( \frac{\lambda_{0}}{\hbar} (c - a) + \varphi_{c} \right) - \tan \left( \frac{\lambda_{0}}{\hbar} (c - b) - \gamma_{c} \right) \times \\
\frac{\cos \left( \frac{\Delta_{c}}{\hbar} (x - a) + \gamma_{c} \right)}{\cos \left( \frac{\Delta_{c}}{\hbar} (c - a) + \varphi_{c} \right)}, & x \in (a, c), \\
\frac{\cos \left( \frac{\Delta_{c}}{\hbar} (x - b) - \gamma_{c} \right)}{\cos \left( \frac{\Delta_{c}}{\hbar} (c - b) - \varphi_{c} \right)}, & x \in (c, b).
\]

It follows from the definition of \( \mathcal{G}_{\theta}(\lambda_{0}) \) and our choice of the branch-cut, that \( \Im \sqrt{z} e^{\delta} > 0 \). Thus, (3.13) and (3.15) properly defines \( L^{2} \)-functions. When \( \Re z \in (0, V_{0}) \)—as is the case for \( z \in \mathcal{G}_{\theta}(\lambda_{0}) \)—the point \( z - V_{0} \) has a negative real part and, according to the definition of the square root, one has \( \Re \Lambda_{c} < 0 \). Therefore, the terms of the type \( e^{i \Delta z / \hbar} d \) appearing in (3.15) are exponentially increasing or decreasing functions of \( \frac{1}{\hbar} \) depending on the sign of \( d \). The asymptotic behaviour of the value \( G^{\pm}(\cdot, c) \) as \( \hbar \to 0 \) will be considered by using the formula
\[
l = (b - a), \quad p_{n_{0}}(z) = e^{-2i\gamma z};
\]
\[
G^{\pm}(\cdot, c) = \frac{i}{2\hbar \Lambda_{c}} \frac{1}{1 - e^{-2i\Delta_{c} / \hbar} p_{n_{0}}(z)} \times \\
\left[ 1 + e^{-2i\varphi_{c} (c - a)} p_{n_{0}}(z) + e^{-2i\varphi_{c} (b - c)} p_{n_{0}}(z) + e^{-2i\varphi_{c} / \hbar} p_{n_{0}}^{2}(z) \right].
\]

which is a rewriting of (3.15).
Remark 3.1. Although $G^z$ is properly defined for $z \in \mathbb{C} \setminus \mathbb{R}_+ e^{-2 \theta_0} \cup \left\{ \frac{-\pi}{2} \right\}$, and in particular for $z \in G_0(\lambda_0)$, relation (3.16) makes sense in $\mathbb{C} \setminus \left\{ z_0^0(\theta_0) \right\}$. Thus, considering the small-$h$ expansions of $G^z(c, c)$, a larger neighbourhood of $\lambda_0$ can be used, such that $\text{Im} \Lambda_z < 0$.

Next we give accurate upper and lower bounds and exponential estimates, for $G^z(\cdot, c)c \in (a, b)$. Using (3.13)–(3.15), allows us to show that this function is exponentially decaying outside a small neighbourhood of $x = c$, where all its mass concentrates as $h \to 0$.

Lemma 3.2. Let $z \in G_0(\lambda_0)$, $\theta_0 = i h \theta_0$ and $h \in (0, h_0)$ with $h_0$ small. For $c \in (a, b)$, the following estimates holds:

$$\left\| \frac{c_0}{h} \right\| L^2_{(a,b)} \leq \frac{c_1}{h},$$

(3.17)

$$\sup_{[a,b]} |e^{x} G^z(\cdot, c)| + \left\| e^{x} h \partial G^z(\cdot, c) \right\| L^2_{(a,b)} + \left\| e^{x} G^z(\cdot, c) \right\| L^2_{(a,b)} \leq \frac{C_{a,b}}{h},$$

(3.18)

$$\left\| G^z(\cdot, c) \right\| L^2_{(R, (a,b))} \leq \frac{C_{a,b}}{h} e^{-\frac{\beta(\lambda_0)}{h}},$$

(3.19)

$$\left\| G^z \right\| L^2_{(\mathbb{R}, (a,b))} \leq \frac{C_{a,b}}{h} e^{-\frac{\beta(\lambda_0)}{h}},$$

(3.20)

with $\varphi = (V_0 - \lambda_0)^2 \cdot (-e^c), \beta(\lambda_0) = (V_0 - \lambda_0)^2 \cdot d(c, (a, b)), N_1 = 2 N_0 + 5$ and constants depending on the data.

Proof. To simplify the notation, we use $G^z$ instead of $G^z(\cdot, c)$. Let us start considering an upper bound of $\left\| G^z \right\| L^2_{(a,b)}$ in the relevant energy range $\text{Re} z \in (0, V_0)$. Owing to (3.15), we have

$$G^z|_{x \in (a,b), c} = \frac{1}{i h \Lambda_z} e^{-\frac{1}{2} h \lambda (x-c)} \left[ 1 + O(e^{-\frac{1}{2} h \lambda d(x,(a,b))}) \right].$$

Since $\text{Im} \Lambda_z < 0$ when $\text{Re} z \in (0, V_0)$, the quantities $e^{-\frac{1}{2} h \lambda (x-c)}, e^{-\frac{1}{2} h \lambda d(x,(a,b))}$ are exponentially small as $h \to 0$, and an explicit computation yields

$$\left\| G^z(\cdot, c) \right\| L^2_{(a,b)} \leq \frac{C_1}{h |\text{Im} \Lambda_z|} \leq \frac{\tilde{C}_1}{h}.$$ (3.21)

For the lower bound, we refer to (3.15), with $x \in (a, c)$, to get

$$G^z(\cdot, c) \geq \frac{C_0}{h} \cos \left( \frac{\frac{1}{2} h \lambda (x-a) + \gamma_z}{h} \right) \geq \frac{\tilde{C}_0}{h} |e^{-\frac{1}{2} h \lambda (x-c)}|$$

for $\delta$ and $h$ small. It follows that

$$\left\| G^z \right\| L^2_{(a,b)} \geq \left\| G^z \right\| L^2_{(c-a, c)} \geq \frac{\tilde{C}_0}{4 h |\text{Im} \Lambda_z|}.$$ (3.22)

Exponential estimates are usually obtained from Agmon identities using as exponential weight the distance from the classical region of motion (we refer to [13]). However, in this particular case, the explicit formula (3.15) shows that the leading factors in $G^z$ and $h \partial G^z$ are controlled by $e^{-\frac{1}{2} h \lambda (x-c)} \leq e^{-\frac{\beta(\lambda_0)}{h} \cdot |x-c|}$; this gives (3.18). In the exterior domain, a direct computation yields

$$\left\| G^z \right\| L^2_{(\mathbb{R}, (a,b))} = \frac{h}{2 |\text{Im} (z e^\theta_0)|^2} \left( |G^z(a^-)|^2 + |G^z(b^-)|^2 \right).$$ (3.23)
For \( z \in \mathcal{G}_h(\lambda_0) \), \( \Im(z e^{2ih}) \frac{1}{z} \sim \mathcal{O}(hN_0) \), and we get
\[
\|G^z\|_{L^2(\mathbb{R}(a,b))} \leq \frac{1}{2hN_0-1} (|G^z(a^+)|^2 + |G^z(b^-)|^2).
\] (3.24)

According to (3.18), the boundary values of \( G^z \) are estimated by \( \frac{\mathcal{O}(h\delta c)}{h} \) and the inequality (3.19) follows.

From definitions (3.13)–(3.15), \( z \to G^z \) is an \( L^2 \)-valued holomorphic map in \( \mathcal{G}_h(\lambda_0) \). For \( z_1, z_2 \in \mathcal{G}_h(\lambda_0) \), we have
\[
G^{z_1} - G^{z_2} = \left[ (H_{h,0}^h(\theta_0) - z_1)^{-1} - (H_{h,0}^h(\theta_0) - z_2)^{-1} \right] \delta_c
= (z_1 - z_2) (H_{h,0}^h(\theta_0) - z_1)^{-1} (H_{h,0}^h(\theta_0) - z_2)^{-1} \delta_c
= (z_1 - z_2) (H_{h,0}^h(\theta_0) - z_1)^{-1} G^{z_2}.
\] (3.25)

Using (3.5), the first of (3.20) follows. The derivative \( \partial_z G^z \) is expressed by
\[
\partial_z G^z = (H_{h,0}^h(\theta_0) - z)^{-1} G^z,
\] (3.26)
which implies \( \partial_z G^z = (H_{h,0}^h(\theta_0) - z)^{-2} \delta_c \). Then, using (3.5) and (3.6) completes the proof of (3.20).

\section{4. A Krein’s resolvent formula and spectral expansions for small \( h \)}

Let consider the spectral problem for the deformed Hamiltonian with \( \theta = \theta_0 = ihN_0 \),
\[
H_{h,0}^h(\theta_0) = -h^2 e^{-2ih^{1/2}z^2} \Delta z + 1_{(a,b)}V_0 + h \alpha \delta_c, \quad c \in (a, b).
\] (4.1)

The related resolvent operator can be expressed as a finite rank perturbation of \( (H_{h,0}^h(\theta_0) - z)^{-1} \):
\[
(H_{h,0}^h(\theta_0) - z)^{-1} = (H_{h,0}^h(\theta_0) - z)^{-1} - \frac{h \alpha \overline{G}(\cdot, c) G(z, \cdot)}{1 + h \alpha |G(z, \cdot)|^2} G(z, \cdot).
\] (4.2)

This will provide an accurate description of the resonant energy as \( h \to 0 \).

\textbf{Proposition 4.1.} Let \( h \in (0, h_0) \), with \( h_0 \) small, \( \theta_0 = ihN_0 \) and \( \alpha \in (-2V_0^{1/2}, 0) \). The spectrum of \( H_{h,0}^h(\theta_0) \) is characterized by the following conditions.

(i) The essential spectrum is \( \sigma_{ess}(H_{h,0}^h(\theta_0)) = e^{-2hN_0} \mathbb{R}_+ \).

(ii) There exists a unique non-degenerate spectral point of \( H_{h,0}^h(\theta_0) \) in \( \{ \Re z \in (0, V_0) \} \), \( \arg z \in (-2\theta_0, 0) \), admitting the small-\( h \) expansion
\[
E_{res} = V_0 - \frac{\alpha^2}{4} - \frac{\alpha^2}{2} p_0(E^0) e^{-\frac{\alpha^2}{2} d(c, [a,b])} + \mathcal{O}(\theta_0 e^{-\frac{\alpha^2}{2} d(c, [a,b])}) + o(e^{-\frac{\alpha^2}{2} d(c, [a,b])}),
\] (4.3)
with \( p_0(E^0) = \frac{1}{2} \left[ \frac{1}{2} \frac{\alpha^2}{2} \right] \). The corresponding eigenvector is given by the Green’s function \( G_{E_{res}}(\cdot, c) \).

(iii) Both \( E_{res} \) and \( G_{E_{res}}(\cdot, c) \) are holomorphic w.r.t. \( \alpha \).

\textbf{Proof.}

(i) The first statement is a consequence of corollary 3.4 in [12] (holding for generic \( \mathcal{M}_{h} \)-perturbation of \( H_{h,0}^h(\theta_0) \) supported in \( (a, b) \)).
(ii) According to proposition 5.5 in [12] (partly relying on Helffer–Sjöstrand techniques in [14]), the points in \( \{ \Re z \in (0, V_0), \arg z \in (-2\theta_0, 0) \cap \sigma(H^{h\theta}_0(\theta_0)) \} \) are localized around the eigenvalues of the Dirichlet Hamiltonian \( H^D_D = -\Delta^D_{(a,b)} + 1_{(a,b)}V_0 + \alpha \delta_z \), with a one-to-one correspondence and an exponentially small bound. Using the dilution \( x \to \frac{x^2}{h} \), we can refer to the spectral problem for \( H^D_d = -\Delta^D_{(\frac{x^2}{h}, \frac{1}{h})} + 1_{(\frac{x^2}{h}, \frac{1}{h})}V_0 + \alpha \delta_z \). When \( h \to 0 \), the spectral subset \( (0, V_0) \cap \sigma(H^D_d) \) converges to \((0, V_0) \cap \sigma(H_0^d)\), with \( H_0^d = -\Delta + V_0 + \alpha \delta_z \), preserving the dimension of the respective subspaces (the proof of this point is based on standard convergence estimates in semiclassical analysis; a guide line for it can be recovered from the strategy used in [11] lemma 4.5). The point spectrum of \( H^d \) is explicitly computable for \( \alpha \in (-2V_0^\frac{1}{2}, 0) \); it is composed of a unique point of multiplicity 1 and equal to \( \lambda = V_0 - \frac{|\alpha|^2}{4} \). Therefore, there exists a unique non-degenerate eigenvalue, \( E^{h}_{\text{res}} \), of \( H^{h\theta}_{0,a}(\theta_0) \) in the prescribed region, converging to \( \lambda \) as \( h \to 0 \). The result of proposition 5.5 in [12] is written in this case as \( E^{h}_{\text{res}} = V_0 - \frac{|\alpha|^2}{4} + O(h^{-3} e^{-\frac{|\alpha|^2}{2d(c,a,b)}}) \).

A more refined asymptotic expression is obtained by using the explicit resolvent’s formula. Since the poles of \( (H^{h\theta}_{0,a}(\theta_0) - z)^{-1} \) are confined in \( \{ \Re z \geq V_0 \} \) (we refer to (3.3)), relation (4.2) leads to an equation for \( E^{h}_{\text{res}} \):

\[
1 + h\alpha G^E_{c,c}(\theta) = 0. \tag{4.4}
\]

In the strip \( \Re E \in (0, V_0) \), where \( \Im(E - V_0)^2 < 0 \) due to the determination \( \arg z \in (-\frac{1}{2}\pi, \frac{\pi}{2}) \), the above equation is rephrased as

\[
2(E - V_0)^2 = \frac{i\alpha}{1 - e^{2i(E-V_0)^2/p_{h\theta}(E)}} \left[ 1 + e^{2i(E-V_0)^2/p_{h\theta}(E)} p_{h\theta}(E) + e^{2i(E-V_0)^2/p_{h\theta}(E)} p_{h\theta}(E) \right] = 0, \quad (4.5)
\]

according to (3.16). Let \( E = V_0 - \frac{|\alpha|^2}{4} + \delta E e^{-\frac{|\alpha|^2}{2d(c,a,b)}} \) + \( O(e^{-\frac{|\alpha|^2}{2d(c,a,b)}}) \); an approximation of the first order in \( e^{-\frac{|\alpha|^2}{2d(c,a,b)}} \) of (4.5) yields

\[
\delta E = -\frac{\alpha^2}{2} p_{h\theta}(E_0^0), \tag{4.6}
\]

where \( p_{h\theta}(E) \) is holomorphic w.r.t. both the variables provided that \( E \sim V_0 - \frac{|\alpha|^2}{4} \) and \( |\theta_0| \ll 1 \). The value \( p_{h\theta}(E_0^0) \) is approximated by \( p_{h\theta}(E_0^0) = p_0(E_0^0) \) + \( O(\theta_0) \):

\[
p_0(E_0^0) = \frac{1}{V_0} \left[ \frac{1}{|\alpha|} \left( V_0 - \frac{|\alpha|^2}{4} \right)^{\frac{1}{2}} - \left( V_0 - \frac{|\alpha|^2}{4} \right) \right];
\]

this leads to expansion (4.3). Finally, relation (4.4) allows us to verify that \( (H^{h\theta}_{0,a}(\theta_0) - E^{h}_{\text{res}}) G^D_{\tilde{\theta}_{\tilde{c},\tilde{c}}}(c,c) = 0 \).

(iii) Consider the quadratic form associated with the operator \( H^{h\theta}_{0,a}(\theta_0) \). This is an accretive form (due to the choice \( \theta = \theta_0 \)) and its domain, \( Q(H^{h\theta}_{0,a}(\theta_0)) = H^1((\mathbb{R}\backslash\{a,c,b\}) \cup H^1((\mathbb{R}\backslash\{a,b\}), its action

\[
\langle u, H^{h\theta}_{0,a}(\theta_0) u \rangle_{L^2(\mathbb{R})} = h^2 \sin 2\theta_0 \int_{(\mathbb{R}\backslash\{a,b\})} |u'|^2 + V_0 \int_{\mathbb{R}} |u|^2 + h\alpha |u(c)|^2
\]

\[
11
\]
defines a holomorphic function of \( \alpha \). Thus, \( H_{\theta_0,\alpha}^h \) is an analytic family-type \( B \) w.r.t. \( \alpha \) and the Kato–Rellich theorem applies to the non-degenerate discrete eigenvalue \( E_{\text{res}}^h \). \( \square \)

**Remark 4.2.** The solution \( E_{\text{res}}^h \) is a singularity of the resolvent embedded in the second Riemann sheet and corresponds to the shape resonance produced by the attractive part of the interaction. It defines an eigenvalue of the deformed operator \( H^h_{\theta_0,\alpha}(\theta_0) \) provided that \( |\arg E_{\text{res}}^h| \) is lower than the deformation angle, given by \( h^{N_0} \) in our assumption. Since \( \text{Im } E_{\text{res}}^h \) is exponentially small w.r.t. \( h \) this condition definitively holds as \( h \to 0 \).

Next, we consider the expansions of relevant quantities involved in the computation of \( A_{\theta_0}(t) \) for energies close to the resonance. In what follows, we assume the results of proposition 4.1 to hold, and take \( |E - E_{\text{res}}^h| < \frac{\hbar}{2} \), with \( \hbar \) being a small constant which fixes a complex neighbourhood of \( E_{\text{res}}^0 \) of size \( h \). In such a domain, relation (3.16) can be used to write expansions of \( G^E(c, c) \) as \( h \to 0 \) (see remark 3.1). Recalling that \( 1 + h\alpha G^E_{\text{res}}(c, c) = 0 \), the function \( (1 + h\alpha G^E(c, c))^{-1} \) can be written in the form

\[
(1 + h\alpha G^E(c, c))^{-1} = \frac{M(E, E_{\text{res}}^h)}{E - E_{\text{res}}^h},
\]

\[ M(E, E_{\text{res}}^h) = \frac{E - V_0 + (E_{\text{res}}^h - V_0)^\frac{1}{2}(E - V_0)^\frac{1}{2}}{1 + h\alpha[E - V_0]^\frac{1}{2} + (E_{\text{res}}^h - V_0)^\frac{1}{2}]E - E_{\text{res}}^h, \quad (4.7)
\]

with the branch-cut fixed along \( i\mathbb{R}_+ \). The incremental ratio at the denominator in (4.8) is controlled by the derivative of \( (E - V_0)^\frac{1}{2} G^E(c) \) evaluated in a neighbourhood of \( E_{\text{res}}^h \). According to (3.16), this is written as

\[
(E - V_0)^\frac{1}{2} G^E(c) = -\frac{i}{2\hbar} \frac{1}{1 - e^{-2i(E-V_0)^\frac{1}{2}} p_{\theta_0}(E)} \left[ 1 + e^{-2i(E-V_0)^\frac{1}{2}} p_{\theta_0}(E) \right]
\]

\[
+ e^{-2i(E-V_0)^\frac{1}{2}} p_{\theta_0}(E) + e^{-2i(E-V_0)^\frac{1}{2}} p_{\theta_0}^2(E). \]

Using the holomorphicity of \( p_{\theta_0}(E) \) and \( (E - V_0)^\frac{1}{2} \) in \( |E - E_{\text{res}}^h| < \frac{\hbar}{2} \), and the asymptotic characterization (4.3), we get

\[
\partial_E(E - V_0)\frac{1}{2} G^E(c) = h^{-2} e^{-\frac{i}{\hbar} d(c, [a, b])} R(E). \quad (4.9)
\]

This yields the representation

\[
M(E, E_{\text{res}}^h) = \left[ E - V_0 + (E_{\text{res}}^h - V_0)^\frac{1}{2}(E - V_0)^\frac{1}{2} \right] + h^{-1} e^{-\frac{i}{\hbar} d(c, [a, b])} R(E), \quad (4.10)
\]

holding for \( |E - E_{\text{res}}^h| < \frac{\hbar}{2} \). In a closer neighbourhood of the resonance, the function \( M(E, E_{\text{res}}^h) \) is connected with scalar products of the Green’s functions.

**Lemma 4.3.** In the assumptions of proposition 4.1, let \( E \in \mathcal{U}_{\theta_0}(E_{\text{res}}^h) \), \( c \in (a, b) \), \( S = |a| d(c, [a, b]) \). The relations

\[
\text{Im } G^{E}(c, c) L^2(\mathbb{R}) = \frac{1}{M(E, E_{\text{res}}^h)} + h^{-N} e^{-\frac{i}{\hbar} R_0(E, h)}, \quad (4.11)
\]

\[
\text{Im } \partial_z G^{E}(c, c) L^2(\mathbb{R})|_{z = E_{\text{res}}} = \frac{1}{M(E, E_{\text{res}}^h)} + h^{-N} e^{-\frac{i}{\hbar} R_1(E, h)}, \quad (4.12)
\]
hold with $N, N_1$ suitable positive integers, while $\mathcal{R}(E, h)$ are the holomorphic functions of $E$ uniformly bounded w.r.t. $h$.

**Proof.** From the relations $\hbar a G^{E_h}(c, c) = -1$ and (3.25), it follows that

\[
1 + h\alpha G^E(c, c) = h\alpha(G^E(c, c) - G^{E_h}(c, c))
\]

\[
= h\alpha(E - E_h^\theta)(H^h_{\theta_0,0}(\theta_0) - E)^{-1}G^{E_h}(c, c),
\]

(4.13)

which is also written as

\[
(1 + h\alpha G^E(c, c)) = h\alpha(E - E_h^\theta) \int_{\mathbb{R}} G^E(c, x)G^{E_h}(x, c) \, dx.
\]

(4.14)

For $x \in (a, b)$, $G^E(c, x)$ is obtained from (3.15) by interchanging the variables $x$ and $c$; from a direct check on this formula it follows $G^E(x, c) = G^E(c, x)$. For any $x \in \mathbb{R}\setminus(a, b)$, the map $c \rightarrow G^E(c, x)$ is the solution of $(H^h_{\theta_0,0}(\theta_0) - \varepsilon)G^\varepsilon(\cdot, x) = \delta_x$ in $(a, b)$. According to the boundary conditions in (2.2), this problem is explicitly written as

\[
\begin{cases}
-h^2 \delta^2 c + V_0 - E \quad G^E(c, x) = 0 \quad \text{for } c \in (a, b), \quad x > b \quad \text{for} \\
(h\partial_c + i\sqrt{E} e^{-\theta_0})G^E(a^+, x) = 0 \quad -\frac{e^{-\theta_0}}{h} e^{i\frac{\pi}{4}E_x^\theta(x-b)} , \\
(h\partial_c - i\sqrt{E} e^{-\theta_0})G^E(b^-, x) = \frac{e^{-\theta_0}}{h} e^{i\frac{\pi}{4}E_x^\theta(x-b)} 
\end{cases}
\]

(4.15)

For $E \in \mathcal{O}(E_h^\theta)$, the lemma 4.3 in [12] applies, and a pointwise exponential estimate for $1_{(a,b)}G^E(\cdot, x)$ holds, depending on $x$: \[
\sup_{c \in (a, b)} |e^{\varphi} G^E(c, x)| \leq \frac{C_{a,b}}{h^2} (|1_{x>b} e^{i\frac{\pi}{4}E_x^\theta(x-b)}| + |1_{x<a} e^{i\frac{\pi}{4}E_x^\theta(a-x)}|), \]

(4.16)

with $\varphi = \frac{\omega}{2} d(\cdot, [a, b])$. From (i), the integral on the r.h.s. of (4.14) is written as

\[
\int_{\mathbb{R}} G^E(c, x)G^{E_h}(x, c) \, dx = \int_a^b G^E(c, x)G^{E_h}(x, c) \, dx + \int_{\mathbb{R}\setminus(a,b)} G^E(c, x)G^{E_h}(x, c) \, dx
\]

\[
= \int_{\mathbb{R}} G^E(c, x)G^{E_h}(x, c) \, dx - \int_{\mathbb{R}\setminus(a,b)} [G^E(c, x)] \, dx.
\]

The exterior contribution defines a holomorphic function of $E \in \mathcal{O}(E_h^\theta)$. From (4.16) the Cauchy–Schwarz inequality and estimate (3.19), applied with $\lambda_0 = V_0 - \frac{\omega^2}{4}$, this is bounded by $O(h^{-N} e^{-\varepsilon^2})$ for a suitable large $N$, uniformly w.r.t. $E$. Then (4.11) is deduced from (4.7). The second relation (4.12) similarly follows by using (3.20).

5. Adiabatic evolution of $A_{\theta_0}(t)$

We consider the asymptotic behaviour of the dynamical system (2.11)–(2.13) as $h \to 0$ goes to zero. Assumptions (h1)–(h4) fix the physical data of the problem, including: (1) the quantum observable $\chi$, corresponding to the charge accumulating around the interaction point $c$; (2) the time profile of the interaction, $a(t)$, which determines the resonant energy level at time $t$; (3) the energy partition function $g$, defining an out-of-equilibrium initial state; (4) the long time scale of the problem, corresponding to the inverse of the adiabatic parameter $\varepsilon$ defined in (2.18).
In particular, the constraint $\alpha_t \in (-2V_{d1}, 0)$ implies that the attractive part of the interaction generates, for each $t$, a single resonance whose small-$h$ expansion is given in (4.3). With the notation introduced in section 2, this corresponds to $E(t) = E_K(t) - i\Gamma_t$:

$$E_K(t) = V_0 - \frac{\alpha_t^2}{4} + O(e^{-\frac{|c|}{2} d(c, \{a, b\})})$$  \hfill (5.1)

$$\Gamma_t = O(e^{-\frac{|c|}{2} d(c, \{a, b\})})$$  \hfill (5.2)

Due to (2.15), exponentially small terms $O(e^{-\frac{|c|}{2} d(c, \{a, b\})})$ can be replaced by $O(e^{-\frac{|c|}{2} d(c, \{a, b\})})$; this leads to

$$\Gamma_t = O(e^{-\frac{|c|}{2} d(c, \{a, b\})}) \quad \forall t,$$  \hfill (5.3)

where definition (2.18) is taken into account. The corresponding resonant state, given by the Green’s function $G^{E(t)}(\cdot, c)$, will be simply denoted by $G(t)$.

The condition $\theta_0 = \bar{h} N_0$ in (h1) allows us to control the perturbation introduced by the interface conditions; namely, the distance between $E(t)$ and the corresponding resonant level for the unperturbed model is bounded by $O(\theta_0 e^{-\frac{|c|}{2} d(c, \{a, b\})})$, according to (4.3). A suitable choice of the parameter $d_0$ in (h2) and (h3) ensures that $E_K^2(t) \subset supp g \subset (0, V_0)$ definitely holds as $h \to 0$.

Conditions (h1)–(h4) also provide with a well-posed functional analytical framework for the study of the adiabatic problem. According to the result of proposition 3.7(d) in [12], the Hamiltonian $-\frac{\bar{h}}{\theta_0}(\theta_0)$, with $\alpha_t \in C^2((0, T), \mathbb{R})$, generates a dynamical system of contractions, $S^t(t, s), t \geq s$, defined by the equation

$$i\varepsilon \partial_t S^t(t, s) = H_{\theta_0,\alpha(t)}(\theta_0) S^t(t, s), \quad S^t(s, s) = Id,$$  \hfill (5.4)

which preserves the domains, $S^t(t, s) D(H_{\theta_0,\alpha(t)}(\theta_0)) \subset D(H_{\theta_0,\alpha(t)}(\theta_0))$ for $t \geq s$. An adiabatic theorem, for arbitrarily large time scales $\varepsilon = e^{-\frac{t}{\varepsilon}}$, has been proved to hold for a wide class of Hamiltonians with interface conditions and exterior complex dilations, including the case of $H_{\theta_0,\alpha(t)}(\theta_0)$ (theorem 7.1 in [12]). To fix this point consider the adiabatic evolution of the initial resonant state $G(0)$; our problem is

$$i\varepsilon \partial_t u = H_{\theta_0,\alpha(t)}(\theta_0) u, \quad u_{t=0} = G(0),$$  \hfill (5.5)

where $\varepsilon$ is fixed to the exponentially small scale $\varepsilon = e^{-\frac{|c|}{2} d(c, \{a, b\})}$. Let $G_{\theta}(E(t))$ denote the set

$$G_{\varepsilon}(E(t)) = \tilde{G}_{\varepsilon}(E(t)) \setminus \left\{ z \in \mathbb{C}, d(z, E(t)) \geq \frac{\bar{h} N_0}{C} \right\},$$

with $\tilde{G}_{\varepsilon}(\cdot)$ defined by (3.4). For $C$ suitably large, this forms a non-empty subset of $G_{\theta}(E(t))$ where we can define the normalized non-self-adjoint projector on $G(t)$ as

$$P(t) = \frac{1}{2\pi i} \int_{\gamma^t(t)} (z - H_{\theta_0,\alpha(t)}(\theta_0))^{-1} dz,$$  \hfill (5.6)

with $\gamma^t(t)$ being a smooth curve in $G_{\varepsilon}(E(t))$ simply connected to $E(t)$. With this notation, the result of [12] rephrases as follows:

$$\sup_{t \in [0, T]} |S^t(t, s)G(0) - \phi_t|_{L^2(\mathbb{R})} \leq \tilde{O}(\varepsilon) |G(0)|_{L^2(\mathbb{R})}$$  \hfill (5.7)

$$\phi_t = \mu(t) e^{-\frac{t}{\varepsilon} \int_0^t E(s) \, ds} G(t), \quad \mu(t) |G(t)|_{L^2(\mathbb{R})}^2 = -\mu(t) \langle G(t), P(t) \bar{\partial}_t G(t) \rangle$$

$$\mu(0) = 1.$$  \hfill (5.8)
The coefficient on the r.h.s. of the equation can be made explicit according to $P(t) = (G^s(t), G(t))$, where $G^s(t) = G^s(t')$ is the anti-resonant function. Using the inequalities (3.17) and (3.20), we have $G(t) - G^s(t) = \tilde{O}(E(t) - E^s(t)) = \tilde{O}(\varepsilon)$ in $L^2$, and

$$\langle G(t), P(t) \partial_t G(t) \rangle_{L^2(\mathbb{R})} = \frac{|G(t)|^2_{L^2(\mathbb{R})}}{G^s(t), G(t))_{L^2(\mathbb{R})}} \langle G^s(t), \partial_t G(t) \rangle_{L^2(\mathbb{R})}$$

$$= \frac{|G(t)|^2_{L^2(\mathbb{R})}}{|G(t)|^2_{L^2(\mathbb{R})} + \tilde{O}(\varepsilon)} \langle (G(t), \partial_t G(t))_{L^2(\mathbb{R})} + \tilde{O}(\varepsilon) \rangle$$

This provides with an expansion for $\mu(t)$:

$$\mu(t) = e^{-\int_{t_0}^{t_0} \frac{|G(t)|^2_{L^2(\mathbb{R})}}{|G(t)|^2_{L^2(\mathbb{R})} + \tilde{O}(\varepsilon)}} + \tilde{O}(\varepsilon).$$  

(5.9)

### 5.1. A decomposition of $A_{\theta_0}(t)$

We shall use a decomposition in the same spirit of the one proposed in [15] and [19] with additional specific information given by our specific model. For $\theta = \theta_0$, the Cauchy problem (2.13) is written as

$$\begin{cases}
  i \varepsilon \partial_t u_{\theta_0}(k, \cdot, t) = H^h_{\theta_0, \alpha(t)}(\theta_0) u_{\theta_0}(k, \cdot, t), \\
  u_{\theta_0, \alpha} = U_{\theta_0} \psi_-(k, \cdot, \alpha_0),
\end{cases}$$  

(5.10)

where, for a fixed $\alpha$, $U_{\theta_0} \psi_-(k, \cdot, \alpha)$ solves the equation

$$H^h_{\theta_0, \alpha}(\theta_0) \psi = -k^2 U_{\theta_0} \psi_-(k, \cdot, \alpha) = 0, \quad H^h_{\theta_0, \alpha}(\theta_0) u \in L^2_{\text{loc}}.$$

(5.11)

For time-dependent $\alpha$, the following representation holds:

$$U_{\theta_0} \psi_-(k, \cdot, \alpha(t)) = U_{\theta_0} \psi_-(k, \cdot, \alpha(t)) + C(k, t) G^{k^2},$$

(5.12)

where $\psi_-(k, \cdot)$ are the incoming scattering states of the unperturbed Hamiltonian (solving (3.7)), the coefficient $C(k, t)$ is defined according to

$$C(k, t) = -\frac{h \alpha_0 \psi_-(k, c)}{1 + h \alpha_0 G^{k^2}(c)},$$

(5.13)

while $G^{k^2}$ is explicitly given in (3.13)–(3.15). A possible decomposition of the solution of (5.10) is

$$u_{\theta_0}(k, \cdot, t) = e^{-i k^2 t} U_{\theta_0} \psi_-(k, \cdot, \alpha(t)) + R(t).$$

(5.14)

Denoting $\psi_t = e^{-i k^2 t} U_{\theta_0} \psi_-(k, \cdot, \alpha(t))$ we have

$$\begin{cases}
  i \varepsilon \partial_t \psi_t = H^h_{\theta_0, \alpha(t)}(\theta_0) \psi_t + i \varepsilon e^{-i k^2 t} \tilde{C}(k, t) G^{k^2}, \\
  \psi_0 = U_{\theta_0} \psi_-(k, \cdot, \alpha_0).
\end{cases}$$

(5.15)

As follows from (2.13) and (5.15), the remainder $R(t) = u_\theta(k, \cdot, t) - \psi_t$ solves the Cauchy problem

$$\begin{cases}
  i \varepsilon \partial_t R(t) = H^h_{\theta_0, \alpha(t)}(\theta_0) R(t) + i \varepsilon e^{-i k^2 t} \tilde{C}(k, t) G^{k^2}, \\
  R(0) = 0,
\end{cases}$$

(5.16)

and its explicit form is

$$R(t) = -\int_0^t S^h(t, s) e^{-i k^2 s} \tilde{C}(k, s) G^{k^2} \, ds,$$

(5.17)
where $S^\epsilon(t, s)$ is the dynamical system associated with $-\frac{i}{\hbar}H^{\hbar}_{\lambda_0, \alpha(t)}(\theta_0)$. Making use of (5.14) and (5.17), the time evolution $u_{\theta_0}(k, \cdot, t)$ further decomposes into the sum

$$u_{\theta_0}(k, \cdot, t) = \sum_{j=1}^{4} \psi_j(k, \cdot, t),$$

with

$$\psi_1(k, \cdot, t) = e^{-i\hat{k}^2} \left[ U_{\theta_0} \hat{\psi}_-(k, \cdot) + C(k, t)(G^\hbar - G(t)) \right],$$

$$\psi_2(k, \cdot, t) = -\int_0^t S^\epsilon(t, s) \hat{C}(k, s) e^{-i\hat{k}^2} \left( G^\hbar - G(s) \right) ds,$$

$$\psi_3(k, \cdot, t) = -\int_0^t \hat{C}(k, s) e^{-i\hat{k}^2} \left( S^\epsilon(t, s) G(s) - \mu(t) e^{-\frac{i}{2\hbar}E(\sigma)ds} G(t) \right) ds,$$

$$\psi_4(k, \cdot, t) = e^{-i\hat{k}^2} \left[ C(k, t) - \int_0^t \hat{C}(k, s) e^{-\frac{i}{2\hbar}E(\sigma)ds} \mu(t) G(t) + e^{-i\hat{k}^2} (1 - \mu(t)) C(k, t) G(t) \right].$$

The variable $A_{\theta_0}(t), \text{associated with} u_{\theta_0}(k, \cdot, t), \text{is now written as}$

$$A_{\theta_0}(t) = \sum_{j=1}^{4} \int \frac{dk}{2\pi \hbar} g(k) \langle \chi_j(k, \cdot, t), \psi_j(k, \cdot, t) \rangle_{L^2(\mathbb{R})}.\tag{5.23}$$

In order to get adiabatic estimates for the contributions to (5.23), we need accurate asymptotic expansions for the quantities involved in these computations, including: $\hat{\psi}_-(k, \cdot)$, and the integrals of $|C(k, t)|^2$. To this aim we introduce the following technical lemma.

**Lemma 5.1.** In assumptions (h1)–(h4), let $E \in \mathbb{R}$, $E^\epsilon \in \text{suppg}$ and $\lambda_\epsilon = \lim_{\epsilon \to 0} E(t)$; the solutions to (3.7) for $k^2 = E$ fulfil the conditions

$$|\hat{\psi}_-(E^\epsilon, c)|^2 = e^{-\frac{i}{2\hbar}E(c-a)} \mathcal{O}(1), \quad |\hat{\psi}_-(E^\epsilon, c)|^2 = e^{-\frac{i}{2\hbar}(b-c)} \mathcal{O}(1)$$

for all $c \in (a, b)$. In particular, for $|E - \lambda_\epsilon| < C_\epsilon$ and $d(c, [a, b]) = c - a$, the first of (5.24) is explicitly

$$|\hat{\psi}_-(E^\epsilon, c)|^2 = 2\lambda_\epsilon \frac{c}{|\alpha_\epsilon|} \frac{\Gamma_i}{M(\lambda_\epsilon, \lambda_\epsilon)} (1 + \mathcal{O}(|\theta_0|)) + o(\epsilon),$$

with $M(E_1, E_2)$, $\Gamma_i$, and $\epsilon$ being defined according to (4.8), (5.2) and (2.18), respectively. The function $\hat{\psi}_-(E^\epsilon, c)$ holomorphically extends to the complex neighbourhood $\mathbb{C} \cap \{|z - E^\epsilon_{\text{res}}| \leq \frac{\hbar}{d_0}\}$, where the representation

$$\hat{\psi}_-(E^\epsilon, c) \frac{\psi^\ast((E^\epsilon)^2, c)}{c} = e^{-\frac{i}{2\hbar}(c-a)} \mathcal{F}^{\hbar, \theta_0}(E)$$

holds, with $\mathcal{F}^{\hbar, \theta_0}(\cdot)$ being a holomorphic family uniformly bounded w.r.t. $h$ and $\theta_0$.

**Proof.** In (3.10) the explicit form of $\hat{\psi}_-(k, \cdot)$, $k > 0$, is given. For energies $k^2$ placed below the barrier level $V_0$, the decreasing behaviour of the terms $e^{-i\frac{k^2}{2\hbar}}$, $e^{-i\frac{(c-a)}{2\hbar}}$ w.r.t. $h$ allows us to write

$$\hat{\psi}_-(k, c) = e^{-i\frac{k^2}{2\hbar}(c-a)} \mathcal{F}^{\hbar}(k^2, \theta_0),$$

with $\Lambda_\epsilon$ defined as in (3.11), $\mathcal{F}^{\hbar}(z, \theta_0)$ a holomorphic map w.r.t. both variables, provided that $|z - E^\epsilon_{\text{res}}| < \frac{\hbar}{d_0}$ and $|\theta_0|, h$ are small enough. The first part of (5.24) follows by
using (5.27) with \( k^2 = E = \lambda_t + \mathcal{O}(\hbar) \). The second part of (5.24) can be carried out by a similar direct computation. For \( \hbar \to 0 \), the asymptotic behaviour of \( \tilde{\psi}_-(k, c) \) is determined by the factor \( e^{-i \int_0^t \Lambda(z) (c-a(z))} \). In the complex neighbourhood \( |z - E^{\hbar}_{res}| < \frac{\hbar}{2M} \), where \( E^{\hbar}_{res} = V_0 - \frac{\omega^2}{4} + \mathcal{O}(\hbar^{\frac{1}{2}} |d(c, a(z))|) \), the function \( \Lambda \) is analytic and the relation \( e^{-i \int_0^t \Lambda(z) (c-a(z))} = e^{-i \int_0^t \Lambda(z) h(z)} \) holds, with \( R(z) \) being an analytic family uniformly bounded w.r.t. \( \hbar \). A similar identity holds for \( \tilde{\psi}_-(k, c) \), once (5.27) is taken into account. Representation (5.26) is a direct consequence of this relation.

Next we use the notation \( \tilde{\psi}_{-0}(k, c) \) and \( G^{\hbar}_{0}(\cdot, c) \) to point out the dependence of scattering states and Green’s function on the interface conditions of the Hamiltonian. When \( \theta_0 = 0 \), \( H^{\hbar}_{0,0}(0) \) is a self-adjoint operator with a purely absolutely continuous spectrum. Stone’s formula yields in this case:

\[
\int_0^\infty \frac{dk}{2\pi \hbar} f(k^2)[|\tilde{\psi}_{-0}(k, c)|^2 + |\tilde{\psi}_{-0}(-k, c)|^2] = \frac{1}{2\pi i} \lim_{\delta \to 0} \int_0^{\infty} dE f(E) \delta \epsilon \left( (H^{\hbar}_{0,0}(0) - E + i\delta)^{-1} - (H^{\hbar}_{0,0}(0) - E - i\delta)^{-1} \right) 
\]

\[
\quad = \frac{1}{\pi} \lim_{\delta \to 0} \int dE f(E) \text{Im} G^{E-i0}_{0}(c, c) = \frac{1}{\pi} \lim_{\delta \to 0} \int dk 2|k| f(k^2) \text{Im} G^{k^2-i0}_{0}(c, c) 
\]

for continuous \( f \). This leads to

\[
|\tilde{\psi}_{-0}(k^2, c)|^2 + |\tilde{\psi}_{-0}(-k^2, c)|^2 = 4h|k| \text{Im} G^{k^2-i0}_{0}(c, c). 
\]

For \(|E - \lambda_t| < C\epsilon\), (5.24) implies that \(|\tilde{\psi}_{-0}(E^2, c)|^2 = \mathcal{O}(e^{-\frac{\omega}{\hbar} |d(c, b)|})\), and due to the assumption \( d(c, [a, b]) = c - a \), we get

\[
|\tilde{\psi}_{-0}(E^2, c)|^2 = 4h E^2 \text{Im} G^{E-i0}_{0}(c, c) + o(\epsilon). 
\]

Relation (5.25), for \( \theta_0 = 0 \), follows by using (4.7) and (4.10) to express \( \text{Im} G^{E-i0}_{0}(c, c) \) as a function of \( M(E, E(t)) \), and expanding for \( E = \lambda_t - i\Gamma_t + o(\epsilon) \). The general case is recovered by noting that (5.27) and the correspondent expression for \( \tilde{\psi}_{-0}(\cdot, k, c) \) imply

\[
|\tilde{\psi}_{-0}(k, c)|^2 + |\tilde{\psi}_{-0}(-k, c)|^2 = (|\tilde{\psi}_{-0}(k, c)|^2 + |\tilde{\psi}_{-0}(-k, c)|^2) (1 + \mathcal{O}(|\theta_0|)).
\]

(Actually (5.30) could also be recovered, with the less efficient bound \( \mathcal{O}(h^{-1}|\theta_0|) \), from propositions 4.5 in [12].) Thus, for \(|\theta_0| \ll 1\) (we refer to the assumption (h1)), (5.28) is written as

\[
|\tilde{\psi}_{-0}(k, c)|^2 + |\tilde{\psi}_{-0}(-k, c)|^2 = 4h|k| \text{Im} G^{k^2-i0}_{0}(c, c)(1 + \mathcal{O}(|\theta_0|)).
\]

Proceeding as before, we obtain (5.25).

Next computations involve the use of small-\( h \) expansions of the coefficients \( C(k, t) \) and \( \dot{C}(k, t) \). Using definition (5.13) and relation (4.7) leads to

\[
C(k, t) = -\frac{\hbar \alpha}{h} \tilde{\psi}_{-0}(k, c) M(k^2, E(t)) \frac{1}{k^2 - E(t)}. 
\]

The derivative \( \dot{C}(k, s) \) is explicitly given by

\[
\dot{C}(k, t) = \frac{\hbar \alpha}{(1 + \hbar \alpha) G^2(c) \frac{2\hbar \alpha}{G^2(c) - 1}} \frac{h \alpha}{(2h \alpha, G^2(c) - 1)} \frac{h \alpha}{(k^2 - E(t))} \frac{2h \alpha}{G^2(c) - 1} \mathcal{O}(1). 
\]
for all $k \in \text{supp} \ g$, and $t \in [0, T]$. Relations (5.24) and (5.25) allow us to identify $|C(k, t)|^2$ with a Lorentzian function on $\text{supp} \ g$, with the scale parameter given by $\Gamma$. In particular, for $d(c, \{a, b\}) = c - a$, it holds that

$$\int \frac{dk}{2\pi h} g(k)|C(k, t)|^2 = \frac{h|a|}{2} g(k^2/\Gamma) (1 + O(|\theta|)) + o(\varepsilon), \quad (5.34)$$

while for $d(c, \{a, b\}) = b - c$ we have

$$\int \frac{dk}{2\pi h} g(k)|C(k, t)|^2 = O(e^{-\frac{\varepsilon}{c}}), \quad (5.35)$$

with positive $\beta = \frac{|a|}{h}(c - a - (b - c))$. Both the above expansions follow by using the dilation $y = \frac{k^2 - E(t)}{\hbar^2}$ and taking the limit of the resulting integral as $h \to 0$. With a similar computation we also have

$$\int dk g(k) \frac{1}{|k^2 - E(t)|} = O\left(\frac{1}{h}\right). \quad (5.36)$$

**Lemma 5.2.** In assumptions (h1)–(h4), the estimates

$$\left| \int \frac{dk}{2\pi h} g(k)(\chi\psi_j(k, \cdot, t), \psi_j(k, \cdot, t))_{L^2(\mathbb{R})} \right| = \tilde{O}(\varepsilon^\frac{1}{4}) \quad (5.37)$$

hold with $j = 1, 2, 3$.

**Proof.** For $j = 1$, this product develops in the sum

$$\int \frac{dk}{2\pi h} g(k)(\chi U_{\theta_0} \tilde{\psi}_-(k, \cdot), U_{\theta_0} \tilde{\psi}_-(k, \cdot))_{L^1(\mathbb{R})}$$

$$+ \int \frac{dk}{2\pi h} g(k)(\chi C(k, t)(G^{k^2} - G(t)), C(k, t)(G^{k^2} - G(t)))_{L^2(\mathbb{R})}$$

$$+ 2\text{Re} \int \frac{dk}{2\pi h} g(k)(\chi U_{\theta_0} \tilde{\psi}_-(k, \cdot), C(k, t)(G^{k^2} - G(t)))_{L^2(\mathbb{R})}. \quad (5.38)$$

Using the exponential decreasing behaviour of $\tilde{\psi}_-(k, \cdot)$ on the supp $\chi$ (see lemma 5.1), the first contribution is estimated by

$$\left| \int \frac{dk}{2\pi h} g(k)(\chi U_{\theta_0} \tilde{\psi}_-(k, \cdot), U_{\theta_0} \tilde{\psi}_-(k, \cdot))_{L^1(\mathbb{R})} \right| = \tilde{O}(e^{-\frac{\varepsilon}{4h}(c-a)}). \quad (5.39)$$

For the second term, the definition of $C(k,t)$ (see (5.32)), equivalence (5.24) and the first inequality in (3.20) lead to

$$\left| \int \frac{dk}{2\pi h} g(k)|C(k, t)|^2\langle \chi(G^{k^2} - G(t)), (G^{k^2} - G(t)) \rangle_{L^2(\mathbb{R})} \right| \leq C \int \frac{dk}{2\pi h} |g(k)||C(k, t)|^2\|G^{k^2} - G(t)\|_{L^2(\mathbb{R})} = \tilde{O}(e^{-\frac{\varepsilon}{4h}(c-a)}). \quad (5.40)$$

The last contribution is a crossing term; it is estimated in terms of the previous ones by using the Hölder inequality in $L^2(\mathbb{R}^2)$:

$$\left| \int \frac{dk}{2\pi h} g(k)(\chi U_{\theta_0} \tilde{\psi}_-(k, \cdot), C(k, t)(G^{k^2} - G(t)))_{L^2(\mathbb{R})} \right| \leq \tilde{O}(\varepsilon). \quad (5.41)$$

For $j = 2$, the integral is

$$\int \frac{dk}{2\pi h} g(k)(\chi \psi_2(k, \cdot, t), \psi_2(k, \cdot, t))_{L^2(\mathbb{R})} = \int \frac{dk}{2\pi h} \int_0^t ds_1 \int_0^s ds_2 f(k, s_1, s_2). \quad (5.42)$$
with
\[ f(k, s_1, s_2) = g(k) \hat{C}(k, s_1) \hat{C}^*(k, s_2) e^{-i \frac{2\pi}{\hbar} k^2} \times \left\{ \chi S^\prime(t, s_1)(G^k - G(s_1)), S^\prime(t, s_2)(G^k - G(s_2)) \right\}_{L^1(\mathbb{R})}. \]
(5.42)

Since \( f(k, s_1, s_2) = f^*(k, s_2, s_1) \), this integral is written as
\[ \int \frac{dk}{2\pi \hbar} g(k) \left\{ \chi \psi_2(k, \cdot, t), \psi_2(k, \cdot, t) \right\}_{L^1(\mathbb{R})} = 2 \text{Re} \int \frac{dk}{2\pi \hbar} \int_0^t ds_1 \int_0^{s_1} ds_2 f(k, s_1, s_2). \]
(5.43)

The first inequality of (3.20) leads to \( \|S(t, s)(G^k - G(s))\|_{L^1(\mathbb{R})} \leq |k^2 - E(s)| \hat{O}(e^0) \). Then, according to definitions (5.32) and (5.33), we find
\[ |f(k, s_1, s_2)| \leq \hat{O}(e^0)|g(k)||C(k, s_1)C^*(k, s_2)|, \]
and
\[ \left| \int \frac{dk}{2\pi \hbar} \int_0^t ds_1 \int_0^{s_1} ds_2 f(k, s_1, s_2) \right| \leq \hat{O}(e^0) \int_0^t ds_1 \int_0^{s_1} ds_2 \int \frac{dk}{2\pi \hbar} |C(k, s_1)C^*(k, s_2)|. \]
(5.44)

The integral over \( k \) admits two independent estimates.

(1) Use the Cauchy–Schwarz inequality to write
\[ \int \frac{dk}{2\pi \hbar} \left| \frac{g(k)}{2\pi \hbar} \right| (C(k, s_1)C^*(k, s_2)) \]
\[ \leq \left( \int \frac{dk}{2\pi \hbar} |g(k)||C(k, s_1)|^2 \right)^{\frac{1}{2}} \left( \int \frac{dk}{2\pi \hbar} |g(k)||C(k, s_2)|^2 \right)^{\frac{1}{2}} = \mathcal{O}(1). \]
(5.45)

(2) Use (5.32) and the relation
\[ \frac{|E(s_1) - E(s_2)|}{|k^2 - E(s_1)||k^2 - E(s_2)|} = \left| \frac{1}{k^2 - E(s_1)} - \frac{1}{k^2 - E(s_2)} \right| \]
to write
\[ \int \frac{dk}{2\pi \hbar} \left| \frac{g(k)}{2\pi \hbar} \right| (C(k, s_1)C^*(k, s_2)) \]
\[ \leq \hat{O}(e^0) \left( \int_{\text{supp } g} \left| \frac{\hat{\psi}(-k, c)}{k^2 - E(s_1)} \right|^2 \right) + \int_{\text{supp } g} \left| \frac{\hat{\psi}(-k, c)}{k^2 - E(s_2)} \right|^2. \]
(5.46)

From (5.36) and (5.24), it follows that
\[ \int \frac{dk}{2\pi \hbar} \left| \frac{g(k)}{2\pi \hbar} \right| (C(k, s_1)C^*(k, s_2)) \leq \hat{O}(e^{-\frac{\alpha + (c-a)}{2\pi\hbar}}) \left| \frac{E(s_1) - E(s_2)}{E(s_1) - E(s_2)} \right|. \]
Interpolating between (5.45) and (5.46) yields
\[ \int \frac{dk}{2\pi \hbar} \left| \frac{g(k)}{2\pi \hbar} \right| (C(k, s_1)C^*(k, s_2)) \leq \hat{O}(e^{-\frac{\alpha + (c-a)}{2\pi\hbar}}) \left| \frac{E(s_1) - E(s_2)}{E(s_1) - E(s_2)} \right|^2 \leq \hat{O}(e^{-\frac{\alpha + (c-a)}{2\pi\hbar}}), \]
where we use the lower bound \( |E(s_1) - E(s_2)| \geq c_0 |\alpha(s_1) - \alpha(s_2)| \) following from (2.19) and (2.20) and the analyticity of the map \( \alpha \to E^\text{res}_\alpha \) (see proposition 4.1). Due to assumption (h2),
\[|\alpha(s_1) - \alpha(s_2)|^{-\frac{1}{2}} \text{ is integrable on the triangle } \{s_1 \in [0, T], s_2 \leq s_1 \} \text{ provided that } n \geq J + \delta.\]

This leads to
\[\left| \int \frac{dk}{2\pi \hbar} g(k) \langle \chi \psi_2(k, \cdot, t), \psi_2(k, \cdot, t) \rangle_{L^2(\mathbb{R})} \right| = \tilde{O}(e^{\frac{n}{2}(e-a)}).\]

For \(j = 3\), the integral is
\[\int \frac{dk}{2\pi \hbar} g(k) \langle \chi \psi_3(k, \cdot, t), \psi_3(k, \cdot, t) \rangle_{L^2(\mathbb{R})} = \int_0^t ds_1 \int_0^t ds_2 \int \frac{dk}{2\pi \hbar} g(k) \tilde{C}(k, s_1) \tilde{C}^*(k, s_2) \langle \chi \psi(t, s_1), \psi(t, s_2) \rangle_{L^2(\mathbb{R})},\]

where
\[\varphi(t, s) = S'(t, s) G(s) - \mu(t) e^{-\frac{i}{\hbar} \int_{E(s)}^t E(\sigma) d\sigma} G(t).\]

Using (5.7), (5.8) and (5.9), it follows \(|\varphi(t, s)|_{L^2(\mathbb{R})} \leq \tilde{O}(\varepsilon)\) uniformly w.r.t. \(t\) and \(s\); then, proceeding as above, we get
\[\left| \int \frac{dk}{2\pi \hbar} g(k) \langle \chi \psi_3(k, \cdot, t), \psi_3(k, \cdot, t) \rangle_{L^2(\mathbb{R})} \right| \leq \tilde{O}(\varepsilon^2) \int_0^t ds_1 \int_0^t ds_2 \int \frac{dk}{2\pi \hbar} g(k) |C(k, s_1) C^*(k, s_2)| |k^2 - E(s_1)| |k^2 - E(s_2)|.\]

Since \(|k^2 - E(s)|^{-1} \leq \frac{1}{2}\) on \(\text{supp } g\), a similar inequality to the one considered in (5.44) follows. We obtain
\[\left| \int \frac{dk}{2\pi \hbar} g(k) \langle \chi \psi_3(k, \cdot, t), \psi_3(k, \cdot, t) \rangle_{L^2(\mathbb{R})} \right| \leq \tilde{O}(e^{\frac{n}{2}(e-a)}).\]

\[\square\]

5.2. The reduced equation

We consider the term
\[\int \frac{dk}{2\pi \hbar} g(k) \langle \chi \psi_4(k, \cdot, t), \psi_4(k, \cdot, t) \rangle.\]  

Setting \(\psi_4(k, \cdot, t) = \varphi_1(k, \cdot, t) + \varphi_2(k, \cdot, t)\),

\[\varphi_1(k, \cdot, t) = e^{-i k^2} \left[ C(k, t) - \int_0^t \tilde{C}(k, s) e^{-\frac{i}{\hbar} \int_{E(s)}^t E(\sigma) - k^2} d\sigma ds \right] \mu(t) G(t),\]

\[\varphi_2(k, \cdot, t) = e^{-i k^2} (1 - \mu(t)) C(k, t) G(t),\]

and introducing the variables
\[a(t) = \int \frac{dk}{2\pi \hbar} g(k) \langle \chi \varphi_1(k, \cdot, t), \varphi_1(k, \cdot, t) \rangle,\]

\[\mathcal{J}_1(t) = \int \frac{dk}{2\pi \hbar} g(k) \langle \chi \varphi_2(k, \cdot, t), \varphi_2(k, \cdot, t) \rangle,\]

\[\mathcal{J}_2(t) = 2\text{Re } \int \frac{dk}{2\pi \hbar} g(k) \langle \chi \varphi_1(k, \cdot, t), \varphi_2(k, \cdot, t) \rangle,\]

it becomes
\[\int \frac{dk}{2\pi \hbar} g(k) \langle \chi \psi_4(k, \cdot, t), \psi_4(k, \cdot, t) \rangle = a(t) + \mathcal{J}_1(t) + \mathcal{J}_2(t).\]
In what follows the asymptotic analysis of these contributions as $h \to 0$ is developed. Let us start with $a(t)$; it can be rephrased as
\begin{equation}
 a(t) = \int \frac{d k}{2 \pi h} g(k) |\beta(k, t)|^2 |\mu(t)|^2 \langle \chi G(t), G(t) \rangle_{L^2(\mathbb{R})},
\tag{5.54}
\end{equation}
\begin{equation}
 \beta(k, t) = C(k, t) - \int_0^\ell \hat{C}(k, s) e^{-\frac{i}{\hbar} \int s (E(\sigma) - k^2) d \sigma} d s.
\tag{5.55}
\end{equation}

According to the definitions of $\mu$ (see (5.9)) and $\beta$, we have
\begin{equation}
 \partial_x \mu(t) = -\frac{1}{\hbar} \Big( 2 \Re \{ \mu(t) \} \Big),
\tag{5.56}
\end{equation}
\begin{equation}
 \partial_x \beta(k, t) = -\frac{i}{\hbar} (E(t) - k^2) (\beta(k, t) - C(k, t)).
\tag{5.57}
\end{equation}

Out of exponentially small terms, this leads to the differential relation
\begin{equation}
 \partial_t a(t) = \left[ -2 \Re \{ \langle \chi G(t), \partial_x G(t) \rangle \} + \mu(t) \partial_t \langle \chi G(t), G(t) \rangle_{L^2(\mathbb{R})} \right] a(t)
\end{equation}
\begin{equation}
 - 2 \Re \frac{i}{\hbar} \int \frac{d k}{2 \pi h} g(k)(E(t) - k^2) |\beta(k, t)|^2 |\mu(t)|^2 \langle \chi G(t), G(t) \rangle_{L^2(\mathbb{R})}
\end{equation}
\begin{equation}
 + 2 \Re \frac{i}{\hbar} \int \frac{d k}{2 \pi h} g(k)(E(t) - k^2) |\mu(t)|^2 \beta(k, t) C(k, t) \langle \chi G(t), G(t) \rangle_{L^2(\mathbb{R})}.
\tag{5.58}
\end{equation}

Using $E(t) - k^2 = (E_k(t) - k^2) - i \Gamma_k$, it follows that
\begin{equation}
 \partial_t a(t) = \left[ -2 \Re \{ \langle \chi G(t), \partial_x G(t) \rangle \} + \mu(t) \partial_t \langle \chi G(t), G(t) \rangle_{L^2(\mathbb{R})} - 2 \frac{\Gamma_k}{\hbar} \right] a(t) + S^h(t),
\tag{5.59}
\end{equation}
\begin{equation}
 S^h(t) = 2 \Re \frac{i}{\hbar} \int \frac{d k}{2 \pi h} g(k) (E(t) - k^2) |\mu(t)|^2 \beta(k, t) C(k, t) \langle \chi G(t), G(t) \rangle_{L^2(\mathbb{R})}.
\tag{5.60}
\end{equation}

The derivative on the l.h.s. is explicitly given by
\begin{equation}
 \partial_t \langle \chi G(t), G(t) \rangle_{L^2(\mathbb{R})} = 2 \Re \{ \langle \chi G(t), \partial_x G(t) \rangle \} \langle \chi G(t), G(t) \rangle_{L^2(\mathbb{R})},
\tag{5.61}
\end{equation}
so we get
\begin{equation}
 \partial_t a(t) = \left[ - \frac{2}{|G(t)|_{L^2(\mathbb{R})}^2} \Re \{ \langle \chi - 1, G(t), \partial_x G(t) \rangle \} - 2 \frac{\Gamma_k}{\hbar} \right] a(t) + S^h(t).
\tag{5.62}
\end{equation}

We next discuss the small-$h$ behaviour of the source term. This can further be developed as $S^h = S^h_1 + S^h_2$, with
\begin{equation}
 S^h_1(t) = 2 \int \frac{d k}{\pi h} g(k) |C(k, t)|^2,
\tag{5.63}
\end{equation}
\begin{equation}
 S^h_2(t) = 2 \Re \{ \chi G(t), G(t) \} \langle \chi G(t), G(t) \rangle_{L^2(\mathbb{R})} \int \frac{d k}{\pi h} h \frac{i}{\hbar} (k^2 - E(t)) C(k, t) \int \frac{d \sigma}{\pi h} \hat{C}^\ast(k, s) e^{-\frac{i}{\hbar} \int s (E(\sigma) - k^2) d \sigma} d s,
\tag{5.64}
\end{equation}
and $W(t) = 2 |\mu(t)|^2 \langle \chi G(t), G(t) \rangle_{L^2(\mathbb{R})}$.

Since
\begin{equation}
 |\mu(t)| = \frac{\| G(0) \|_{L^2(\mathbb{R})}}{\| G(t) \|_{L^2(\mathbb{R})}} + O(\epsilon),
\tag{5.65}
\end{equation}

the exponentially decreasing character of the Green’s functions outside supp $\chi$ (see lemma 3.2) and relation (4.11) lead to
\[ W(t) = 2\| \chi G(t), G(t) \|_{L^2(\mathbb{R})}^2 + \tilde{O}(\epsilon) = 2\| G(0) \|_{L^2(\mathbb{R})}^2 + \tilde{O}(\epsilon). \]

Thus, $\frac{h\alpha_0}{\epsilon} S^h_1(t)$ expands as (5.34) or (5.35), depending on the value of $d(c, \{a, b\})$. For the first contribution to $S^h$ we get
\[ S^h_1(t) = 2\left| \frac{\alpha}{\alpha_0} \right|^3 \frac{\Gamma}{\epsilon} g(\lambda^2_0)(1 + \tilde{O}(\mathbb{R})) + o(\epsilon), \quad \text{for } d(c, \{a, b\}) = c - a \] (5.67)
\[ S^h_1(t) = O(e^{-\frac{\beta t}{\epsilon}}) \quad \text{for } d(c, \{a, b\}) = b - c, \] (5.68)

with $\beta = \frac{\alpha}{\alpha_0}(c - a - (b - c))$. After changing the variable $E = k^2$, the second contribution is written as
\[ S^h_2(t) = W(t) \text{Re} \int F(E, t) dE. \] (5.69)
\[ F(E, t) = \frac{1}{2E^\frac{1}{2}} g(E^\frac{1}{2}) \left| E - (E - t) \right| C(E^\frac{1}{2}, t) \int_0^t \hat{C}^\ast((E^\ast)^\frac{1}{2}, s) e^{-\frac{\beta}{\epsilon}(E - E^\ast(s))} ds. \] (5.70)

According to assumption (h3), $F(\cdot, t)$ extends to a holomorphic function of $E \in \mathbb{C} \setminus \{ z \} \setminus \{ E \mid \sup g(E^\ast) \setminus \{ |E - \lambda_0| < \frac{\hbar}{\epsilon} \} \}$, while for $E \in \sup g(E^\ast) \setminus \{ |E - \lambda_0| < \frac{\hbar}{\epsilon} \}$ definitions (5.32), (5.33) and the exponential bounds (5.24) imply $|F(E, t)| = \tilde{O}(\epsilon)$. In particular, the term $(E - E(t))C(E, t)$ is analytic in a complex neighbourhood of $\lambda_0 = \lim_{h \to 0} E(0)$, while $\hat{C}^\ast((E^\ast)^\frac{1}{2}, s)$ is meromorphic with a double pole at $E = E^\ast(s)$, placed in the upper-half plane. Our strategy is to use a complex integration path formed by the semi-circumference $\tilde{C}_z(\lambda_0)$ of centre $\lambda_0$, radius $\frac{\hbar}{\epsilon}$ in the lower-half plane $\text{Im} C_z(\lambda_0) \leq 0$. Let us consider a holomorphic extension of $F(\cdot, t)$ to the half-disc whose boundary is determined by $\{ E \in \mathbb{R}, |E - \lambda_0| < \frac{1}{\epsilon} \} \cup C_z(\lambda_0)$. Using (5.32), (5.33), (5.26) and the function $\hat{\vartheta}(\cdot)$
\[ \hat{\vartheta}(z) = |\text{Im } z|; \] (5.71)
the restriction of $F(\cdot, t)$ to $C_z(\lambda_0)$ is bounded by
\[ F(\cdot, t)\big|_{C_z(\lambda_0)} \leq \frac{C}{\epsilon} e^{-\frac{\vartheta}{3}(\cdot - a)} \int_0^t e^{-\frac{\vartheta}{3}(\cdot - s)} ds \leq C \int_0^t e^{-\frac{\vartheta}{3}(\cdot - s)} ds \] (5.72)
for a suitable positive $C$. According to (5.72), the following estimates hold:

1. $F(\cdot, t)\big|_{C_z(\lambda_0)} \leq C \frac{e^{-\frac{\vartheta}{3}}}{\vartheta(E)} [1 - e^{-\frac{\vartheta}{3}(\cdot - a)}] \leq C \frac{e}{\vartheta(E)}$.
2. $F(\cdot, t)\big|_{C_z(\lambda_0)} \leq Ct$.

and by interpolation we obtain
\[ F(\cdot, t)\big|_{C_z(\lambda_0)} \leq \frac{C e^\frac{1}{2}}{\vartheta^\frac{1}{2}(E)}. \] (5.73)
By computing the residue, $S^b(t)$ is written as

$$S^b(t) = -N(t) \int \frac{dE}{\tilde{\theta}^{\tau}(E)} F(E, t) \, dE + \tilde{O}(\varepsilon).$$  \hspace{1cm} (5.74)

Denoting $z \in \mathcal{C}_z(\lambda_0)$ as $z = \lambda_0 + \frac{\eta}{\alpha} e^{i\psi} \omega \in (-\pi, 0)$, the previous inequality implies

$$\sup_t |S^b(t)| \leq C \varepsilon \frac{1}{d} \int_{\mathcal{C}_z(\lambda_0)} |dE| = C \varepsilon \frac{1}{d} \int_{-\pi}^{0} \frac{d\omega}{\sin^2(\alpha)} = O(h^{\frac{1}{2}}).$$  \hspace{1cm} (5.75)

Estimates (5.67) and (5.68) and (5.75) allow us to use $S^b = S^b_1 + O(|\theta_0|) + O(h \varepsilon^{\frac{1}{2}})$, with

$$S^b_1 = O(e^{-\frac{t}{\varepsilon^2}}) \quad \text{for} \quad d(c, [a, b]) = c - a,$$

$$S^b_1 = O(e^{-\frac{t}{\varepsilon^2}}) \quad \text{for} \quad d(c, [a, b]) = b - c,$$

with $\beta = \frac{|\omega|}{\varepsilon} (c - a - (b - c))$. Owing to the estimates in the lemma 3.2, the term

$$\frac{1}{|G(t)|^2} \left( |(\chi - 1) G(t), \tilde{\alpha}_1^i G(t)\right) \leq C |(\chi - 1) G(t)|_{L^2(\mathbb{R})} ||\tilde{\alpha}_1^i G(t)||_{L^2(\mathbb{R})}$$

$$= \tilde{O} \left( \inf_{\text{supp}(1-x)} e^{-\frac{|x|}{\varepsilon^2}} \right).$$  \hspace{1cm} (5.78)

When the interaction point 'c' is on the left side of the barrier's support and the condition $d(c, [a, b]) = c - a$ is fulfilled, the limit condition (5.76) and the estimates (5.75) and (5.78) allow us to write (5.62) as follows:

$$\tilde{\alpha}_1^i a(t) = \left( -2 \frac{\Gamma_1}{\varepsilon} \right) \left( a(t) - \frac{\alpha_1}{\alpha_0} g(\lambda_1^1) \right) + O(|\theta_0|) + \tilde{O}(e^{-\frac{t}{\varepsilon^2}}),$$

where $\tau_\chi > 0$ is defined according to the remainders in (5.67), (5.75) and (5.78). The initial datum for this equation is deduced by evaluating (5.54) at $t = 0$. With the above expansions, we obtain

$$a(0) = g(\lambda_0^1) (1 + O(|\theta_0|)) + o(\varepsilon), \quad \text{for} \quad d(c, [a, b]) = c - a.$$  \hspace{1cm} (5.80)

When $d(c, [a, b]) = c - a$, the solution $a(t)$ is

$$a(t) = a(0) e^{-\frac{t}{\varepsilon^2}} \tilde{\alpha}_1^i + \int_0^t e^{-\frac{t-s}{\varepsilon^2}} ||\tilde{\alpha}_1^i|| ds + O(|\theta_0|) + \tilde{O}(e^{-\frac{t}{\varepsilon^2}}),$$

$$a(0) = g(\lambda_0^1), \quad S^b_1(0) = 2 \frac{|\alpha_1|}{\alpha_0} \frac{\Gamma_1}{\varepsilon} g(\lambda_1^1).$$  \hspace{1cm} (5.81)

In the other case, when $d(c, [a, b]) = b - c$, the initial value of $a(t)$ is $a(0) = O(e^{-\frac{t}{\varepsilon^2}})$ which coincides with the size of the source term given in (5.77). This leads to

$$a(t) = O(e^{-\frac{t}{\varepsilon^2}}) \quad \text{for} \quad d(c, [a, b]) = b - c,$$

with $\beta = \frac{|\omega|}{\varepsilon} (c - a - (b - c)).$

To complete the proof of the second point of theorem 2.1, we need the following lemma.
Lemma 5.3. In assumptions (h1)–(h4), the relations
\[
\int \frac{dk}{2\pi \hbar} g(k) \langle \chi \psi_j(k, \cdot, t), \psi_j'(k, \cdot, t) \rangle_{L^2(\mathbb{R})} = O(\epsilon^{\frac{1}{\hbar^2}})
\]  
(5.83)
hold with \( j, j' = 1, 2, 3, 4 \) and \( j \neq j' \).

Proof. Let us consider the contributions \( J_{i=1,2}(t) \) to (5.47). The first term is explicitly written as
\[
J_i(t) = |1 - \mu(t)|^2 \langle \chi G(t), G(t) \rangle_{L^2(\mathbb{R})} \int \frac{dk}{2\pi \hbar} g(k)|C(k, t)|^2.
\]  
(5.84)
Then, estimate (3.17), and relations (5.34) and (5.65) yield \( J_i(t) = O(\epsilon^0) \), holding for any choice of \( \chi, g \) fulfilling the assumptions. For the second term, let us take \( \tilde{g}, \tilde{\chi} \) a couple of positive functions fulfilling (h3), and such that \(|g| < \tilde{g}, |\chi| < \tilde{\chi}|. With this conditions, a straightforward application of the Cauchy–Schwartz inequality gives
\[
\left| \frac{J_2(t)}{2} \right| \leq \left| \int \frac{dk}{2\pi \hbar} g(k) \langle \chi \psi_1(k, \cdot), \psi_2(k, \cdot, t) \rangle \right|
\leq \int \frac{dk}{2\pi \hbar} \frac{dx}{2\pi \hbar} \tilde{g}(k) \tilde{\chi}(x) |\psi_1(k, x, t), \psi_2(k, x, t)|
\leq \left( \int \frac{dk}{2\pi \hbar} \tilde{g}(k) \langle \tilde{\chi} \psi_1(k, \cdot), \psi_1(k, \cdot, t) \rangle \right)^{\frac{1}{2}}
\times \left( \int \frac{dk}{2\pi \hbar} \tilde{g}(k) \langle \tilde{\chi} \psi_2(k, \cdot), \psi_2(k, \cdot, t) \rangle \right)^{\frac{1}{2}}
= \tilde{a}^\dagger(t) \tilde{J}_i^\dagger(1),
\]  
with \( \tilde{a} \) and \( \tilde{J}_i \) denoting the principal contribution and the first remainder arising from the auxiliary data \( \tilde{g}, \tilde{\chi} \). Since \( \tilde{a}(t) = O(1) \) (as follows from (5.81)) and \( \tilde{J}_i(t) = O(\epsilon^0) \), we obtain \( J_2(t) = O(\epsilon^0) \). This leads to
\[
\int \frac{dk}{2\pi \hbar} g(k) \langle \chi \psi_4(k, \cdot, t), \psi_4(k, \cdot, t) \rangle_{L^2(\mathbb{R})} = O(\epsilon^0),
\]  
(5.85)
while the results of lemma 5.2 gives
\[
\int \frac{dk}{2\pi \hbar} g(k) \langle \chi \psi_j(k, \cdot, t), \psi_j(k, \cdot, t) \rangle_{L^2(\mathbb{R})} = O(\epsilon^{\frac{1}{\hbar}}), \quad \text{with} \quad j = 1, 2, 3.
\]  
(5.86)
Once more, we remark that these estimates hold for all choice of \( g, \chi \) fulfilling the conditions (h3). Let \( g_m \) and \( \chi_m \) be positively defined, verifying the required hypothesis and such that \( g_m > |g| \) and \( \chi_m > |\chi| \). For \( j \neq j' \), the Cauchy–Schwarz inequality implies
\[
\left| \int \frac{dk}{2\pi \hbar} g(k) \langle \chi \psi_j(k, \cdot, t), \psi_j'(k, \cdot, t) \rangle_{L^2(\mathbb{R})} \right|
\leq \int \int \frac{dk \, dx}{2\pi \hbar} g_m(k) \chi_m(x) |\psi_j(k, x, t), \psi_j'(k, x, t)|
\leq \left( \int \frac{dk}{2\pi \hbar} g_m(k) \langle \chi_m \psi_j(k, \cdot, t), \psi_j(k, \cdot, t) \rangle_{L^2(\mathbb{R})} \right)^{\frac{1}{2}}
\times \left( \int \frac{dk}{2\pi \hbar} g_m(k) \langle \chi_m \psi_j(k, \cdot, t), \psi_j'(k, \cdot, t) \rangle_{L^2(\mathbb{R})} \right)^{\frac{1}{2}}
\leq O(\epsilon^{\frac{1}{\hbar^2}}).
\]  
□
5.3. Remainder terms and proof of theorem 2.1

We next consider the terms \( \mathcal{J}_1(t) \) and \( \mathcal{J}_2(t) \) in (5.53) in the limit \( h \to 0 \). To this aim, an asymptotic formula for the difference \( 1 - \mu(t) \) is needed.

**Lemma 5.4.** With assumptions (h1)–(h4), the function \( \mu(t) \), defined in (5.9) is such that

\[
\mu(t) = \left| \frac{\alpha_t}{\alpha_0} \right|^2 (1 + \tilde{O}(\varepsilon)) = 1 + O(h). \tag{5.87}
\]

**Proof.** From (5.9) and (5.65), our function is written as

\[
\mu(t) = \frac{\|G(0)\|_{L^2(\mathbb{R})}}{\|G(t)\|_{L^2(\mathbb{R})}} e^{-i \int_0^t \frac{\lambda}{2} |\dot{\chi}(\mathbf{k})|^2 \mathop{d\mathbf{k}} \mathop{d\tau}} + \tilde{O}(\varepsilon). \tag{5.88}
\]

As \( h \to 0 \), an approximation of \( \text{Im}(G(s), \partial_\mu G(s)) \) is computable starting from relation (4.12) taken with \( E = E_{\text{res}} = E(s) \) and \( \alpha = \alpha_t \); this gives

\[
\text{Im}(G(s), \partial_\mu G(s)) = -\frac{1}{h \alpha_t} \text{Im} \frac{\dot{E}(s) \partial_\mu M(E(s), E(s))}{M^2(E(s), E(s))} + \tilde{O}(\varepsilon),
\]

where \( \partial_\mu \) denotes the derivative w.r.t. the second variable. A relation for \( \dot{E}(t) \) follows by taking the time derivative of (4.4):

\[
\dot{E}(t) = \frac{\alpha_t}{\alpha_0} \frac{G^{(0)}(c, c)}{\partial_\mu G(c, c)} |_{E(t)}.
\]

The r.h.s. of (5.89) is further developed by using (4.9); this leads to \( \dot{E}(t) = \frac{\alpha_t |\alpha_0|}{2} + \tilde{O}(e^{-\frac{\pi h}{2} d(c, \{a, b\})}) \). Thus, \( \dot{E}(t) \) is real, out of exponentially small terms, and the size of \( \text{Im}(G(s), \partial_\mu G(s)) \) is determined by the imaginary part of \( M^{-1}(E(s), E(s)) \partial_\mu M(E(s), E(s)) \).

According to (4.10), this quantity is expressed as

\[
\frac{\partial_\mu M(E(s), E(s))}{M^2(E(s), E(s))} = \frac{1}{2} + \tilde{O}(e^{-\frac{\pi h}{2} d(c, \{a, b\})}) = \frac{1}{2} + \tilde{O}(e^{-\frac{\pi h}{2} d(c, \{a, b\})}).
\]

We finally get \( \text{Im}(G(s), \partial_\mu G(s)) = \tilde{O}(e^{-\frac{\pi h}{2} d(c, \{a, b\})}) \). It follows that

\[
\mu(t) = \frac{\|G(0)\|_{L^2(\mathbb{R})}}{\|G(t)\|_{L^2(\mathbb{R})}} (1 + \tilde{O}(\varepsilon)). \tag{5.90}
\]

Since the Green’s function norms can be expressed in terms of \( (h \alpha_t M(E(t), E(t)))^{-1} \) (we refer to (4.11)), the above ratio further expands as

\[
\frac{\|G(0)\|_{L^2(\mathbb{R})}}{\|G(t)\|_{L^2(\mathbb{R})}} = \left| \frac{\alpha_t}{\alpha_0} \right|^2 + \tilde{O}(\varepsilon). \tag{5.91}
\]

This result, together with assumption (2.15), leads to (5.87). \( \square \)

The integral \( \mathcal{J}_1(t) \) has the form

\[
\mathcal{J}_1(t) = |1 - \mu(t)|^2 \langle \chi G(t), G(t) \rangle_{L^2(\mathbb{R})} \int \frac{dk}{2\pi h} g(k) |C(k, t)|^2. \tag{5.92}
\]

According to (5.34), (5.35), (4.11) and (5.90), and using the exponential estimates for \( G(t) \) outside \( \text{supp} \chi \), this can be rephrased as

\[
\mathcal{J}_1(t) = \left| 1 - \left( \frac{\alpha_t}{\alpha_0} \right)^2 \right|^2 g(\lambda_1^2) (1 + O(|\theta_0|)) + \tilde{O}(\varepsilon). \tag{5.93}
\]
for a suitable $\tau > 0$ and $d(c, \{a, b\}) = c - a$; otherwise we have $J_1 = \mathcal{O}(e^{-\frac{\tau}{2}})$. The second remainder is a crossing term (see definition (5.52)); in lemma 5.3 it has been shown that $J_2 = \mathcal{O}(\tilde{a}^\frac{1}{2} \tilde{J}_1)$, where the variables $\tilde{a}$ and $\tilde{J}_1$ are the principal contribution and the first remainder associated with a suitable couple of auxiliary data $\tilde{g}$, $\tilde{\chi}$. If we assume $d(c, \{a, b\}) = b - c$, we have $J_2 \sim \tilde{a} \cdot \tilde{J}_1 = \mathcal{O}(e^{-\frac{\tau}{2}})$. When $d(c, \{a, b\}) = c - a$, this term is explicitly given by

$$J_2(t) = 2\text{Re} \mu(t)(1 - \mu^*(t))\langle \chi G(t), G(t) \rangle_{L^2(\mathbb{R})} \int \frac{dk}{2\pi h} g(k)\beta(k, t)C^*(k, t).$$

(5.94)

After an integration by part, we get

$$J_2(t) = 2\text{Re} \mu(t)(1 - \mu^*(t))\langle \chi G(t), G(t) \rangle_{L^2(\mathbb{R})}[I + II],$$

(5.95)

$$I = \int \frac{dk}{2\pi h} g(k)C(k, 0)C^*(k, t) e^{-i \frac{1}{\hbar} (E(\sigma) - k^2)} d\sigma,$$

(5.96)

$$II = \frac{i}{\varepsilon} \int \frac{dk}{2\pi h} g(k) \int_0^\varepsilon C(k, s)C^*(k, t)(E(s) - k^2) e^{-i \frac{1}{\hbar} (E(\sigma) - k^2)} d\sigma ds.$$  

(5.97)

The small-$\hbar$ behaviour of $I$ and $II$ is investigated using a path-deformation argument and following the same line as in (5.64). As before, $C^*_\pi(\lambda_0)$ denotes the semicircle of centre $\lambda_0$, radius $\frac{h}{\pi}$, but now we fix Im $C^*_\pi(\lambda_0) > 0$. Replacing $C(k, 0)C^*(k, t)$ with $C(k, 0)C^*(k^*, t)$, we define a meromorphic function in a neighbourhood of $\lambda_0$ with simple poles at $k^2 = E(0), E^*(t)$. Thus, the first integral is

$$I = -\int_{C^*_\pi(\lambda_0)} \frac{dE}{4\pi h E^2} g(E^2)K(E, 0, t) e^{-i \frac{1}{\hbar} (E(\sigma) - E)} dE + 2\pi i \text{Res}_1(E^*(t)) + \mathcal{O}(\varepsilon),$$

(5.98)

where $\text{Res}_1(E^*(t))$ is the residue at $E^*(t)$, while $K(E, s, t)$ denotes

$$K(E, s, t) = C(E^2, s)C^*(E^*(t), t).$$

Since Im$(E(\sigma) - k^2) < 0$ and $K(E, s, t)$, $\mathcal{O}(\varepsilon)$ for $E \in C^*_\pi(\lambda_0)$ (according to (5.26)), we have

$$I = 2\pi i \text{Res}_1(E^*(t)) + \mathcal{O}(\varepsilon).$$

(5.99)

Computing the residue when $\hbar \to 0$, $E(t)$ can be replaced with its limit value $\lambda_t$, excepting those parts of the function where the difference $E^*(t) - E(0)$ appears. In this case we use $E(t) = \lambda_t - i\Gamma_t/2$. Out of exponentially small terms, the result is

$$\text{Res}_1(E^*(t)) = \frac{\hbar\alpha_0}{4\pi \lambda_t^2} g(\lambda_t^2)M(\lambda_t, \lambda_0)M^*(\lambda_t, \lambda_t) \left[ \psi_0(\lambda_t^2) \cdot e^{-\frac{1}{\hbar} (E(\sigma) - E^*(t)) d\sigma} \right]^2.$$  

(5.100)

Using (4.10) and (5.25), it follows that

$$2\pi i \text{Res}_1(E^*(t)) = i(\alpha_0)(\alpha_0^2 + \alpha_0 \alpha_t) g(\lambda_t^2) e^{-\frac{1}{\hbar} (E^*(t) - E(0))} \frac{\Gamma_t}{2} (\hbar + \mathcal{O}(\hbar \theta_0))).$$

(5.101)

Adopting the same notation, the second contribution is written as

$$II = -\frac{i}{\varepsilon} \int_{C^*_\pi(\lambda_0)} \frac{dE}{4\pi h E^2} g(E^2) \int_0^\varepsilon K(E, s, t)(E(s) - E) e^{-i \frac{1}{\hbar} (E(\sigma) - E)} d\sigma ds + 2\pi i \text{Res}_2(E^*(t)) + \mathcal{O}(\varepsilon).$$

(5.102)
Proceeding as the previous section (see the estimate of $S_2^0(t)$), the integral over $C^t_\pi (\lambda_0)$ is bounded as $O(\varepsilon^{\frac{1}{2}})$, while the residue in $E^*(t)$ is given, out of exponentially small terms, by

$$\text{Res}_2(E^*(t)) = -i \frac{h}{4\pi} g\left(\lambda^2_0\right) \frac{\Gamma_I}{\lambda}(1 + O(|\theta_0|)) \int_0^t f(s,t) e^{-\frac{i}{h} \varphi(s,t)} ds,$$

$$f(s,t) = \left(\alpha_s \alpha^2_t + \alpha^2_0 \alpha_t\right) e^{-\frac{1}{h} \int \psi(s,t) ds}, \quad \varphi(s,t) = \int_s^t (\lambda_\sigma - \lambda_t) d\sigma.$$  

If $d(c, \{a, b\}) = c - a$, the factor $\frac{\lambda}{\pi}$ is $O(1)$, and the size of $II$ is determined by the oscillatory integral. To this concern, we note that $\partial_s \varphi(s,t) = \lambda_t - \lambda_s$; according to the definition of $\lambda_t$, the stationary points of $\varphi(s,t)$ are defined by the equation

$$\alpha_s - \alpha_t = 0. \quad (5.103)$$

It follows from (h2) that the set of the 's' fulfilling condition (5.103) does not have accumulation points in $[0, t]$, forming a subset of finite cardinality. It means that $s \to \varphi(s,t)$ have a finite number of stationary points $\{s_j(t)\}_{1}^{\infty} \subset [0, t]$, depending on $t$. Since $s \to f(s,t)$ is a regular function (with $\alpha_s, \Gamma_s \in C^{\infty}$) the stationary phase method applies with $|\partial^{j+1}_s \varphi(s,t)| = |\partial^j_s E_k(s)| \gtrsim |\partial^j_s \alpha(t)| > 0$ for some $j \in \{1, \ldots, J\}$. This yields

$$\int_0^t f(s,t) e^{-\frac{i}{h} \varphi(s,t)} ds = O(\varepsilon^{\frac{1}{2}})$$

and uniformly w.r.t. the time. According to definition (5.95) and the expansions (5.99), (5.101) and (5.104), we get

$$\mathcal{J}_2(t) = 2 \text{Re} \left[ i\mu(t)(1 + \mu^*(t)) \langle \chi G(t), G(t) \rangle_{L^2(R)} g\left(\lambda^2_0\right) \frac{\Gamma_I}{2} \times |\alpha_0| \left|\left(\alpha_s^2 + \alpha_0 \alpha_t\right)\right| e^{-\frac{1}{h} \int_0^t (E(s) - E(t)) ds} e^{\frac{1}{h} \int_0^t E(s) ds} \right].$$

Expanding $\langle \chi G(t), G(t) \rangle_{L^2(R)}$ and $\mu(t)$ with (4.11) and (5.87), and using $E(t) = \lambda_t - i\Gamma_t + O(\varepsilon)$ leads to

$$\mathcal{J}_2(t) = \text{Re} 2i \left(1 - \left|\frac{\alpha_t}{\alpha_0}\right|^2\right) \frac{T(t) \Gamma_I}{\lambda_t - \lambda_0 - i(\Gamma_t + \Gamma_0)}.$$

$$T(t) = \left|\frac{\alpha_0 \alpha_s^2 + \alpha_0^2 \alpha_t}{(\alpha_0 \alpha_t)^2}\right| e^{-\frac{1}{h} \int_0^t (\Gamma_s + \Gamma_t) ds} e^{-\frac{1}{h} \int_0^t (\lambda_t - \lambda_s) ds}. \quad (5.106)$$

When $d(c, \{a, b\}) = c - a$, we have $\Gamma_t = O(\varepsilon)$, $T(t) = O(1)$ for all $t$ and the small-$h$ behaviour of this quantity is determined as follows: setting $D = \lambda_t - \lambda_0$, one has

$$\left|\frac{T(t) \Gamma_I}{\lambda_t - \lambda_0 - i(\Gamma_t + \Gamma_0)}\right|^2 = \frac{O(\varepsilon^2)}{D^2 + (\Gamma_t + \Gamma_0)^2} \leq \frac{O(\varepsilon^2)}{2(D^2 + (\Gamma_t + \Gamma_0)^2)} = O(D^{-1} \varepsilon). \quad (5.107)$$

Since $\lambda_t = O(\varepsilon)$, we get

$$\mathcal{J}_2(t) = O(D^{\frac{1}{2}} \varepsilon^{\frac{1}{2}}). \quad (5.108)$$
Proof of theorem 2.1.

(i) This first point is a rewriting of the result of proposition 4.1.
(ii) The second point comes from decomposition (5.23) and the results of lemmas 5.2 and 5.3. The reduced equation for the main contribution \( a(t) \) is obtained in (5.79) for \( d(c, [a, b]) = c - a \), while this variable is exponentially small, according to estimate (5.82), when \( d(c, [a, b]) = b - c \).
(iii) Once the small-\( h \) behaviour of the factors \( \mu(t), (1-\mu^*(t)) \) and \( \langle \chi G(t), G(t) \rangle \) is taken into account, the last point is a consequence of (5.93), (5.95), (5.99) and (5.105)–(5.108).

□

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