Complete complementarity relations in curved spacetimes

Marcos L. W. Basso$^{1,*}$ and Jonas Maziero$^{1,†}$

$^1$Departamento de Física, Centro de Ciências Naturais e Exatas, Universidade Federal de Santa Maria, Avenida Roraima 1000, Santa Maria, RS, 97105-900, Brazil

We extend complete complementarity relations to curved spacetimes by considering a succession of infinitesimal local Lorentz transformations, which implies that complementarity remains valid locally. This result allows the study of these different complementary aspects of a quantum system as it travels through spacetime. In particular, we study the behavior of these different complementary properties of massive spin-1/2 particles in the Schwarzschild spacetime. For geodetic circular orbits, we find that the spin state of one particle oscillates between a separable and an entangled state. For non-geodetic circular orbits, we notice that the frequency of these oscillations gets bigger as the orbit gets nearer to the Schwarzschild radius $r_s$.

Keywords: Complete complementarity relation; Curved spacetimes; Schwarzschild spacetime

I. INTRODUCTION

According to Schrödinger, entanglement is the characteristic feature of quantum mechanics, the one that imposes its total departure from the classical lines of thought [1]. Its central importance in quantum foundations [2, 3], as well as its main role in the fields of quantum information and quantum computation [4, 5], has made entanglement theory achieve great progress in recent decades. Perhaps, the most astonishing application of this unique feature is quantum teleportation, where two observers use two quantum systems in an entangled state to transmit information about the state of a third system [6]. Moreover, concern about how entanglement behaves in relativistic scenarios has grown more and more [7]. In the end of the last century, Czachor considered the relativistic version of the famous Einstein-Podolsky-Rosen (EPR) experiment with massive spin-1/2 particles [8]. While, in the beginning of this century, the authors of Refs. [9, 10] showed that the entanglement of Bell states depends on the velocity of the observer. On the other hand, the authors in Ref. [11] argued that the entanglement fidelity of a Bell state remains invariant for a Lorentz boosted observer. However, in the same year, it was demonstrated by Peres et al. [12] that the entropy of a single massive spin-1/2 particle does not remain invariant under Lorentz boosts. These apparently conflicting results involve systems containing different particle states and boost geometries [13]. Therefore, entanglement under Lorentz boosts is highly dependent on the boost scenario in question [14], which led to a rich variety of works exploring these different scenarios by several researchers [15–22]. More generally, the entanglement for observers constantly accelerated in a flat space-time was considered in [23–25]. A step forward in the investigations of these relativistic scenarios was taken by Terashima and Ueda [26], who studied EPR correlation and the violation of Bell’s inequalities in curved spacetimes. In addition, the same authors, in Ref. [27], studied the decoherence of spin states due to the presence of a gravitational field, by considering a succession of infinitesimal Lorentz transformations. It turns out that decoherence is quite general for a particle in a gravitational field [28, 29].

However, entanglement is not the only quantum feature that occupies a central position in the world of quantum weirdness. The other feature, known as wave-particle duality, also turns apart the quantum world from the classical world. It is usually considered the main example of Bohr’s complementarity principle, which states that quantum systems, or quantons [30], may possess properties that are equally real but mutually exclusive [31]. Attempts have been made to formalize the wave-particle duality in a quantitative way [32–34]. In these efforts, quantitative measures of wave and particle properties were constructed and constrained in a complementarity inequality

$$P^2 + V^2 \leq 1, \quad (1)$$

where $P$ is the predictability and $V$ is the visibility of the interference pattern. Together with the quantitative formulation of the wave-particle duality, it was noticed that not only extreme cases of full wave and particle natures existing in mutual exclusion is possible, but also intermediate cases of partial wave and particle natures coexisting in a compatibility relation. Until now, many approaches were taken for quantifying the wave-particle properties of a quantum system [35–39], as well, with the development of the field of quantum information, it was suggested that the quantum coherence [40] would be a good generalization of the visibility measure [41–43]. However, as pointed out by Qian et al. [44], complementarity relation like Eq. (1) does not really capture a balanced exchange between $P$ and $V$ because the inequality permits, for instance, that $V$ decreases due to the interaction of the system with its environment while $P$ can remain unchanged, or even worse, it can decrease together with the visibility of system. It even allows the extreme case $P = V = 0$. Thus, something must be missing from Eq. (1). As noticed by Jakob and Bergou

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$^*$Electronic address: marcoslw.basso@mail.ufsm.br

$^†$Electronic address: jonas.maziero@ufsm.br
[45], this lack of knowledge about the system is due to entanglement. This means that the information is being shared with another system and this kind of quantum correlation can be seen as responsible for the loss of purity of each subsystem such that, for pure maximally entangled states, it is not possible to obtain information about the local properties of the subsystems, since we can always purify our system and think of it as part of a multipartite pure quantum system. Even though entanglement entropy does not remain invariant under Lorentz boosts, and neither do the measures of predictability and coherence, in [46] we showed that these three measures taken together, what is known as a complete complementarity relation (CCR), are Lorentz invariant. Hence, in this work, we extend this result to curved spacetimes by considering a succession of infinitesimal Lorentz transformations, allowing us to study the different complementary aspects of a quantum, as it moves through spacetime. In particular, we study the behavior of these different complementary aspects of massive spin-1/2 particles (or qubits) in the Schwarzschild spacetime. For geodetic circular orbits, we find that the spin-state of one particle oscillates between a separable and entangled state. For circular orbits, we note that the frequency of these oscillations gets bigger as the orbit gets near the Schwarzschild radius, which agrees with the fact that the spin precession near rs is very rapid, as reported in Ref. [26]. Hence, our work contributes for a better understanding of how the spacetime curvature affects the behavior of these complementary properties of a quantum system.

The organization of this article is as follows. In Sec. II, we discuss the spin dynamics in curved spacetimes, by focusing in spin-1/2 massive particles. In Sec. III, we extend complete complementarity relations for curved spacetimes. Thereafter, in Sec. IV, we turn to the study of the behavior of CCR in the Schwarzschild spacetime, by exploring two types of circular orbits. Lastly, in Sec. V, we give our conclusions.

II. SPIN DYNAMICS IN CURVED SPACETIMES

A. Spin States in Local Inertial Frames

To study the dynamics of spin-1/2 particles in gravitational fields, the use of local inertial frames of reference, which can be defined at each point of spacetime, is required. These local inertial frames are defined through a tetrad field (or vielbein), which is a set of four linearly independent four-vector fields [47]. In general relativity, the gravitational field is encoded in the metric components of a curved spacetime, which is a differential manifold $\mathcal{M}$ [48]. A manifold is simultaneously a very flexible and powerful structure, since it comes equipped naturally with a tangent (or contravariant) and a cotangent (or covariant) vector spaces in each point $p \in \mathcal{M}$, denoted by $T_p(\mathcal{M})$ and $T^*_p(\mathcal{M})$, respectively. Then, tensor fields of arbitrary rank can be constructed from elements of $T_p(\mathcal{M})$ and $T^*_p(\mathcal{M})$ using the tensor product $\otimes$. The differential structure of $\mathcal{M}$ provides, in each point $p$, a coordinate basis for the vector spaces $T_p(\mathcal{M})$ and $T^*_p(\mathcal{M})$ given by $\{\partial_{\mu}\}$ and $\{dx^\nu\}$, respectively, such that $dx^\nu(\partial_{\mu}) := \partial_{\mu}x^\nu = \delta^\nu_{\mu}$. We can proceed by defining a metric $g$ in $\mathcal{M}$, which gives us a Riemannian (or pseudo-Riemannian) manifold. The metric is a covariant tensor field of rank 2, which defines, in each point $p \in \mathcal{M}$, a inner product in $T_p(\mathcal{M})$ that, in turn, allows us to compute lengths, volumes, angles, time intervals, and so on. Given the basis in $T^*_p(\mathcal{M})$, we can express the metric as $g = g_{\mu \nu}(x)dx^\mu \otimes dx^\nu$, and the elements of the metric which encode the gravitational field are given by $g_{\mu \nu}(x) = g(\partial_{\mu}, \partial_{\nu})$ [49]. However, the natural basis $\{\partial_{\mu}\} \subset T_p(\mathcal{M})$ and $\{dx^\nu\} \subset T^*_p(\mathcal{M})$ are not necessarily orthonormal. But we can set up any basis as we like. In particular, we can form an orthonormal basis with respect to the pseudo-Riemannian manifold (spacetime) on which we are working. Following [49], let us consider the linear combination

$$e_a = e^\mu_a(x)\partial_{\mu}, \quad e^a = e^a_{\mu}(x)dx^\mu,$$

(2)

$$\partial_{\mu} = e^a_{\mu}(x)e_a, \quad dx^\mu = e^a_{\mu}(x)e^a.$$

(3)

To define a local inertial frame at each point $p \in \mathcal{M}$, we require $\{e_a\}$ to be orthonormal in the following sense

$$g(e_a, e_b) := \eta_{ab}, \quad g := \eta_{ab}e^a \otimes e^b,$$

(4)

where $\eta_{ab} = diag(-1, 1, 1, 1)$ is the Minkowski metric. Equivalently, we can define the tetrad field in terms of its components

$$g_{\mu \nu}(x)e^\mu_a(x)e^\nu_b(x) = \eta_{ab},$$

(5)

$$\eta_{ab}e^\mu_a(x)e^\nu_b(x) = g_{\mu \nu}(x),$$

(6)

with

$$e^\mu_a(x)e^\mu_b(x) = \delta^b_a, \quad e^\mu_a(x)e^\nu_b(x) = \delta^\nu_{\mu}.$$  

(7)

Here, and from now on, we assumed that Latin letters $a, b, c, d, \cdots$ refers to coordinates in the local inertial frame; Greek indices $\mu, \nu, \cdots$ runs over the four general-coordinate labels; and repeated indices are to be summed over. Furthermore, for general coordinate indices, the lowering and raising of indices is done with the metric $g_{\mu \nu}(x)$ and its inverse $g^{\mu \nu}(x)$, respectively. The indices in the local inertial frame are lowered by $\eta_{ab}$ and raised by its inverse $\eta^{ab}$. The components of the tetrad field and its inverse transforms a tensor in the general coordinate system into one in the local inertial frame, and vice versa. Therefore it can be used to shift the dependence of spacetime curvature of the vector fields to the tetrad fields. Indeed, instead of working with $A^\mu$ defined in the general coordinate system, it is possible to work with $e^\mu_a(x)A^a$. As $A^a$ is a set of four Lorentz scalar fields, then all the information about the spacetime curvature is encoded in

$$A^a = \eta_{ab}e^\mu_b(x)\partial_{\mu}A^a.$$  

(8)
the tetrad field \(e_a{}^\mu(x)\) [50]. In addition, Eq.(6) tells us that the tetrad field encodes all the information about the spacetime curvature hidden in the metric, which allowed an equivalent formulation of General Relativity in terms of the tetrad fields [51]. Besides, it is worth pointing out that the tetrad field \(\{e_a{}^\mu(x), a = 0, 1, 2, 3\}\) is a set of four contravariant vector fields, and not a single second-rank tensor of indices \(a\) and \(\mu\). Therefore, the tetrad field transforms as
\[
e_a{}^\mu(x) \rightarrow e_a{}^\mu'(x') = \frac{\partial x'^\mu}{\partial x^\nu} e_a{}^\nu(x) \tag{8}
\]
under general coordinate transformation, and as
\[
e_a{}^\mu(x) \rightarrow e_a{}^{\mu'}(x') = \Lambda_a{}^b(x)e_b{}^{\mu'}(x) \tag{9}
\]
in the local inertial system, which is a local Lorentz transformation. Since the inertial frame remains inertial under Lorentz transformations, the choice of the local inertial frame is not unique. Therefore, a tetrad representation of a particular metric is not uniquely defined, and different tetrad fields will provide the same metric tensor, as long as they are related by local Lorentz transformations [52].

By using the set of orthonormal four-vectors \(e_a{}^\mu(x)\), the observer succeed in making the metric components of his laboratory locally flat, \(g(e_a, e_b) = \eta_{ab}\). The observer can go even further, constructing coordinates in his laboratory such that the derivative of the metric components \(g_{\mu\nu}(x)\) vanishes along the geodetic trajectory of its world-line. Coordinates constructed in this way are known as Riemann normal coordinates, which provide a realization of the locally inertial frames (or freely falling frames). A way to accomplish this is by the exponential map [48]. By constructing the local Lorentz transformation, we can define a particle with spin-1/2 in curved spacetime as a particle whose one-particle states furnish the spin-1/2 representation of the local Lorentz transformation [26]. Thus, let’s consider a massive spin-1/2 particle moving with four-momentum \(p^\mu(x) = m u^\mu(x)\) with \(p^\mu(x)\) \(p_\mu(x) = -m^2\), where \(m\) is the mass of the quanton, \(u^\mu(x)\) is the four-velocity in the general coordinate system, and we already putted \(c = 1\). Now, we can use the tetrad field \(e_a{}^\mu(x)\) to project the four-momentum \(p^\mu(x)\) into the local inertial frame, i.e., \(p^\mu(x) = e_a{}^\mu(x)p^\mu(x)\). Thus, in the local inertial frame at point \(p \in \mathcal{M}\) with coordinates \(x^a = e_a{}^\mu(x) x^\mu\), a momentum eigenstate of a Dirac particle in a curved spacetime is given by [50]
\[
|p^\mu(x), \sigma; x\rangle := |p^\mu(x), \sigma; x^a, e_a{}^\mu(x), g_{\mu\nu}(x)\rangle, \tag{10}
\]
and represents the state with spin \(\sigma\) and momentum \(p^\mu(x)\) as observed from the position \(x^a = e_a{}^\mu(x) x^\mu\) of the local inertial frame defined by \(e_a{}^\mu(x)\) in the spacetime \(\mathcal{M}\) with metric \(g_{\mu\nu}(x)\). The description of a Dirac particle state can only be provided regarding the tetrad field and the local inertial structure that it describes. By definition, the state \(|p^\mu(x), \sigma; x\rangle\) transforms as the spin-1/2 representation under the local Lorentz transformation.

In the case of special relativity, a one-particle spin-1/2 state \(|p^\mu, \sigma\rangle\) transforms under a Lorentz transformation \(\Lambda^\mu_\nu\) as [53]
\[
U(\Lambda)|p^\mu, \sigma\rangle = \sum_\lambda D_{\sigma\lambda}(W(\Lambda, p))|\Lambda p^\mu, \lambda\rangle, \tag{11}
\]
where \(D_{\sigma\lambda}(W(\Lambda, p))\) is a unitary representation of the Wigner’s little group, whose elements are Wigner rotations \(W^\sigma_\lambda(\Lambda, p)\) [54]. It’s worth pointing out that the superscripts in \(D_{\sigma\lambda}(W(\Lambda, p))\) are just to emphasize that in general \(U(\Lambda)\) generates superposition in the spin-states. We could very well suppress the superscripts and write \(U(\Lambda)|p^\mu, \sigma\rangle = |\Lambda p^\mu\rangle \otimes D(W(\Lambda, p))|\sigma\rangle\), as sometimes we’ll do. In other words, under a Lorentz transformation \(\Lambda\), the momenta \(p^\mu\) goes to \(\Lambda p^\mu\), and the spin transforms under the representation \(D_{\sigma\lambda}(\Lambda, p)\) of the Wigner’s little group [55]. Meanwhile, in a curved spacetime everything above remains essentially the same, except by the fact that single-particle states now form a local representation of the inhomogeneous Lorentz group at each point \(p \in \mathcal{M}\), i.e.,
\[
U(\Lambda(x))|p^\mu(x), \sigma; x\rangle = \sum_\lambda D_{\sigma\lambda}(W(x))|\Lambda p^\mu(x), \lambda; x\rangle, \tag{12}
\]
where \(W(x) := W(\Lambda(x), p(x))\) is a local Wigner rotation.

### B. Spin Dynamics

Following Terashima and Ueda [26], let us consider how the spin changes when the quanton moves from one point to another in curved spacetime. In the local inertial frame at point \(p\) with coordinates \(x^a = e_a{}^\mu(x)x^\mu\), the momentum of the particle is given by \(p^\mu(x) = e_a{}^\mu(x)p^\mu(x)\). After an infinitesimal proper time \(d\tau\), the quanton moves to a new point with general coordinates \(x'^\mu = x^\mu + u^\mu d\tau\). Then, the momentum of the particle in the local inertial frame at the new point becomes \(p'^\mu(x') = p^\mu(x) + \delta p^\mu(x)\), where the variation of the momentum in the local inertial frame can be described by the combination of changes due to non-gravitational external forces \(\delta p^\mu(x)\), and spacetime geometry effects \(\delta e_a{}^\mu(x)\):
\[
\delta p^\mu(x) = e_a{}^\mu(x)\delta p^\mu(x) + \delta e_a{}^\mu(x)p^\mu(x). \tag{13}
\]
The variation \(\delta p^\mu(x)\) in the first term on the right hand side of the last equation is simply given by
\[
\delta p^\mu(x) = u^\nu(x)\nabla_\nu p^\mu(x)d\tau = ma^\mu(x)d\tau, \tag{14}
\]
where \(\nabla_\nu\) is the covariant derivative and \(a^\mu(x) := u^\nu(x)\nabla_\nu u^\mu(x)\) is the acceleration due to a non-gravitational force. Once \(p^\mu(x)p_\mu(x) = -m^2\) and \(p^\mu(x)a_\mu(x) = 0\), Eq.(14) can be rewritten as
\[
\delta p^\mu(x) = -\frac{1}{m}(a^\mu(x)p_\nu(x) - p^\nu(x)a_\nu(x))p^\nu(x)d\tau. \tag{15}
\]
Meanwhile, the variation of the tetrad field is given by
\[
\delta e^a_\mu(x) = u^\nu(x) \nabla_\nu e^a_\mu(x) d\tau
= -u^\nu(x) \omega^a_\nu b(x) e^b_\mu(x) d\tau,
\] (16)
where \( \omega^a_\nu b := e^a_\lambda \nabla_\nu e^\lambda_b \) is the connection 1-form (or spin connection) [56]. Collecting these results and substituting in Eq. (14), we obtain
\[
\delta \rho^a(x) = \lambda^a_b(x) p^b(x) d\tau,
\] (17)
where
\[
\lambda^a_b(x) = -\frac{1}{m} (u^\mu(x) p_\mu(x) - p^\mu(x) a_\mu(x)) p^a(x) + \chi^a_b
\] (18)
with \( \chi^a_b := -u^\nu(x) \omega^a_\nu b(x) \). It can be shown that Eqs. (17) and (18) constitute an infinitesimal local Lorentz transformation since, as the particle moves in spacetime, the momentum in the inertial frame will transform under an infinitesimal local Lorentz transformation by using a unitary representation of the local Lorentz group. This is the basis for the Wigner little group. However, the result obtained doing this, we are considering a particular representation of the Wigner group. In general, the Wigner rotation varies at different points along the trajectory.

### III. Complementarity Relations in Curved Spacetimes

In Ref. [58], we developed a general framework to obtain a complete complementarity relation (CCR) for a subsystem that belongs to an arbitrary multipartite pure quantum system, by exploring the purity of the multipartite quantum system. While, in Ref. [46], we demonstrated that this procedure turns out to be useful to prove that the CCR obtained is invariant under Lorentz boosts. In this section, we extend this result to curved spacetimes by considering a succession of infinitesimal Lorentz transformations, as discussed in the previous section. We will restrict ourselves to discrete momentum states, as in Refs. [13, 18, 20], which correspond to plane waves solutions of the Dirac equation. Besides, this can be justified once we can consider narrow distributions by composing different plane waves solutions such that the momentum states are centered around different momentum values, what makes possible representing them by orthogonal state vectors, i.e., \( \langle p_a | p_b \rangle = \delta_{a,b} \). Although narrow momenta are an idealization, it’s a system worth studying, since it helps understand more realistic systems. Also, it’s worth pointing out that throughout this article we consider only massive particles of spin-1/2. By doing this, we are considering a particular representation of the Wigner little group. However, the result obtained in this section will not depend on the particular choice of representation, given that the representation remains unitary.

So, let’s consider \( n \) massive quanta with spin-1/2 in a pure state described by \( |\Psi\rangle_{A_1,\ldots,A_{2n}} \in \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_{2n}, \) with dimension \( d = d_{A_1} d_{A_2} \cdots d_{A_{2n}}, \) in the local inertial frame defined by the tetrad field in the point \( p \) of spacetime represented by the coordinates \( x^a = e^a_\mu(x) x^\mu. \) For instance, \( A_1, A_2 \) are referred as the momentum and spin of the first quanta, and so on. By defining a local orthormal basis for each degree of freedom (DOF) \( A_m, \) \( \{ |i_m \rangle_{A_m} \}_{m=0}^{d_m-1}, m = 1, \ldots, 2n, \) the state of the multipartite quantum system can be written as [59]

\[
\rho = |\Psi\rangle_{A_1,\ldots,A_{2n}} \langle \Psi |
= \sum_{i_{1,\ldots,i_{2n}},j_{1,\ldots,j_{2n}}} \rho_{i_{1},\ldots,i_{2n},j_{1},\ldots,j_{2n}} |i_{1},\ldots,i_{2n}\rangle_{A_1,\ldots,A_{2n}} \langle j_{1},\ldots,j_{2n}|.
\] (23)

Without loss of generality, let’s consider the state of the DOF \( A_1, \) which is obtained by tracing over the other
The linear entropy of the subsystem $A_1$, measures the quantum correlations of $A_1$ with rest of the system. Identifying the predictability, visibility/coherence, and quantum correlations measures within Eq. (27), we can write down the following CCR:

$$P_l(\rho_{A_1}) + C_{hs}(\rho_{A_1}) + S_l(\rho_{A_1}) = \frac{d_{A_1} - 1}{d_{A_1}}.$$  

(31)

The proof of this result can be found in Ref. [46]. It is worthwhile mentioning the CCR given by Eq. (31) is a natural generalization of the complementarity relation obtained by Jakob and Bergou [61, 62] for bipartite pure quantum systems. More generally, $E = \sqrt{2S_l(\rho_{A_1})}$, where $E$ is the generalized concurrence obtained in [63] for multi-particle pure states.

Now, since the dynamics of the quantum system through spacetime can be described by successive local Lorentz transformations, the multipartite quantum system is described by $|\Psi\rangle_{A_1,\ldots,A_{2n}} = U(\Lambda(x))|\Psi\rangle_{A_1,\ldots,A_{2n}}$ at the point $x^\alpha = e^\mu_\alpha(x')x'^\mu$, and the density matrix of the multipartite pure quantum system can be written as [64, 65]

$$\rho_{\Lambda} = |\Psi\rangle_{A_1,\ldots,A_{2n}}\langle\Psi\rangle = U(\Lambda(x))\rho U^\dagger(\Lambda(x)),$$  

(32)

implying that $\text{Tr}\rho_{\Lambda}^2 = \text{Tr}\rho^2$, and the whole system remains pure as the quantum system moves along its trajectory in spacetime. As we used the purity of the density matrix to obtain the complete complementarity relation for $A_1$, then, from $1 - \text{Tr}\rho_{\Lambda}^2 = 0$, the following CCR for $A_1$ remains valid throughout the world-line of the multipartite quantum system

$$P_l(\rho_{A_1}^\Lambda) + C_{hs}(\rho_{A_1}^\Lambda) + S_l(\rho_{A_1}^\Lambda) = \frac{d_{A_1} - 1}{d_{A_1}}.$$  

(33)

This proves our claim that this complete complementarity relation can be extended to curved spacetimes, allowing us to quantify the different complementary aspects of the subsystems as they move through spacetime.

IV. QUBITS IN THE SCHWARZSCHILD SPACETIME

In this section, we’ll study the behavior of the different complementary aspects of a spin-1/2 quanton (or a qubit), which is in motion in the Schwarzschild spacetime. Because we are interested in qubits, it’s worth pointing out that the motion of spinning particles, either classical or quantum, does not follow geodesics because the spin and curvature couple in a non-trivial manner [66]. However, the deviation from geodesic motion is very small, of order $\hbar$, and it can be safely ignored except for the case of supermassive compact objects and/or ultra-relativistic test particles [50, 67, 68]. The Schwarzschild solution was the first exact solution to Einstein’s field equation, and it describes the spacetime outside of a static and spherically symmetric body.
of mass $M$, which constitutes a vacuum solution. Because of its symmetries, the Schwarzschild metric describes a static and spherically symmetric gravitational field [69]. In the spherical coordinates system $(t, r, \theta, \phi)$, the line element of the Schwarzschild metric is given by

$$ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu = -f(r)dt^2 + f^{-1}(r)dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2),$$

where $f(r) = 1 - r_s/r$, with $r_s = 2GM$ being the Schwarzschild radius. It’s straightforward to observe that the metric diverges in two distinct points, at $r = r_s$ and at $r = 0$. However, it is important to distinguish the different nature of both singularities. Since all the information about the physics and the spacetime curvature is contained in the curvature tensor $R^\alpha_{\beta\mu\nu}$ and its contractions, to establish when a metric has a singularity with some physical meaning, it is necessary to search for nontrivial scalars that can be constructed from the curvature tensor, which are independent from coordinate systems. For instance, $R^\alpha_{\beta\mu\nu}R_\alpha^\mu^\nu = 12r_s^2/r^6$ tell us that there exists a singular point in $r = 0$ [48]. This suggests that the singularity at $r = r_s$ is not an intrinsic singularity, since it can be shown that all curvature scalars are finite at $r = r_s$. This type of singularity is called apparent singularity (or coordinate singularity) and it is related to our specific choice of coordinates. Therefore, it can be removed by changing the coordinate system.

To make the Schwarzschild metric reduce to the Minkowski metric, it is possible to choose the following tetrad field

$$e^0_t(x) = \sqrt{f(r)}, \quad e^1_r(x) = \frac{1}{\sqrt{f(r)}},$$
$$e^2_\theta(x) = r, \quad e^3_\phi(x) = r \sin \theta,$$

and all the other components are zero. Also, only nonzero components will be shown from now on. The inverse of these elements are given by

$$e_0^t(x) = \frac{1}{\sqrt{f(r)}}, \quad e_1^t(x) = \sqrt{f(r)},$$
$$e_2^\theta(x) = \frac{1}{r}, \quad e_3^\phi(x) = \frac{1}{r \sin \theta}.$$  

Thus, we can write the line element as

$$ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu = g_{\mu\nu}(x)e^\mu_a(x)e^\nu_b(x)e^a e^b = \eta_{ab}e^a e^b.$$  

This vierbein represents a static local inertial frame at each point. Therefore it can used to represent an observer in the associated local inertial frame [26]. In addition, at each point, the $0-, 1-, 2-$, and $3-$axes are parallel to the $t, r, \theta,$ and $\phi$ directions, respectively. A straightforward calculation shows that the non-zero components of the connection 1-form $\omega^a_{\phantom{a}b} := e^a \nabla_\mu e^b \gamma^{\mu}$ are given by

$$\omega^1_{\phantom{1}0}(x) = \omega^3_{\phantom{3}0}(x) = \frac{r_s}{2r^2},$$
$$\omega^3_{\phantom{3}2}(x) = -\omega^3_{\phantom{3}2}(x) = -\cos \theta.$$  

Now that we have the ingredients required to study the behavior of the Wigner rotation and the different aspects of a qubit in motion in the Schwarzschild spacetime, in the following subsections we’ll consider two examples: (1) an equatorial circular geodesic and (2) a non-geodetic equatorial circular orbit.

### A. Equatorial Circular Geodesics

Following [50, 69], let’s consider the case of a free-falling test particle moving around the source of the gravitational field in a geodetic circular orbit, which can be obtained by solving the geodesic equation. The four-velocity of these circular geodesics in the equatorial plane, $\theta = \pi/2$, are given by:

$$u^t = \frac{K}{f(r)}, \quad u^r = 0,$$
$$u^\theta = 0, \quad u^\phi = \frac{J}{r^2},$$

where $K, J$ are integration constants related to the energy and angular momentum of the required orbit, respectively, and are given by

$$K = \frac{1 - r_s/r}{\sqrt{1 - \frac{3r_s}{2r}}}, \quad J^2 = \frac{1}{2}\frac{r_s}{1 - \frac{3r_s}{2r}}.$$  

The energy of the spin-1/2 quanton of rest mass $m$ in a circular orbit of radius $r$ is then given by $E = Km$. Furthermore, the value of $J$ implies that the angular velocity is given by

$$u^\phi = \pm \sqrt{\frac{r_s}{2r^3(1 - \frac{3r_s}{2r})}},$$

which means that stable circular geodesic orbits are only possible when $r > \frac{3}{2}r_s$. The non-zero infinitesimal Lorentz transformations in the local inertial frame defined by the tetrad field are given by

$$\lambda^0_1 = -\frac{K r_s}{2r^2 f(r)},$$
$$\lambda^3_1 = -\lambda^3_1 = -\frac{J \sqrt{f(r)}}{r^2},$$

which corresponds to a boost in the direction of the 1-axes and a rotation over the 2-axis, respectively. While, the four-velocity in the local inertial frame is found to be

$$u^a = e^a_\mu(x) = \left(\frac{K}{\sqrt{f(r)}}, 0, 0, \frac{J}{r}\right).$$
Therefore, the Wigner angle that corresponds to the rotation over the 2-axis is given by:

\[
\vartheta^3(x) = \gamma^3(x) + \lambda^3_0(x)p_s(x) - \lambda^3_0(x)p^1(x) \]

\[
= J\sqrt{f(r)}f(r) (1 - Ks + \frac{1}{2rf(r)K + \sqrt{f(r)}}). \tag{50}
\]

After the test particle has moved in the circular orbit across some proper time \(\tau\), the total angle is given by

\[
\Theta = \int \vartheta^3(x) d\tau = \int \vartheta^3(x) \frac{d\tau}{d\phi} \tag{51}
\]

\[
= \frac{\vartheta^3(x) r^2}{J} \Phi, \tag{52}
\]

since, for a circular orbit, \(r\) is fixed and \(\vartheta^3(x)\), \(K\), and \(J\) are constants. The angle \(\Phi\) is the angle traversed by the particle during the proper time \(\tau\). It is noteworthy that the angle \(\Theta\) reflects all the rotation suffered by the spin of the qubit as it moves in the circular orbit, which means that there are two contributions: The “trivial rotation” \(\Phi\) and the rotation due to gravity [26]. Therefore, to obtain the Wigner rotation angle that is produced solely by spacetime effects, it’s necessary to compensate the trivial rotation angle \(\Phi\), i.e., \(\Omega := \Theta - \Phi\) is the total Wigner rotation of the spin exclusively due to the spacetime curvature, which only depends on the radius of the circular geodesic \(r\) and the mass of the source of the gravitational field expressed by \(r_s\).

Similarly to Terashima and Ueda [26], let’s consider a pair of entangled spin-1/2 particles emitted at given a point on a geodesic equatorial circle with the local quantization axis along the 1-axis, as one of the particles of the bipartite state circulates the orbit clockwise, the other circulates it counterclockwise. In other words, we have a pair of entangled particles moving in opposite directions with constant four-velocity \(u^\mu_{\pm} = (K/\sqrt{f(r)}, 0, \pm J/r)\) and in the following initial state

\[
|\Psi\rangle_{A,B} = \frac{1}{\sqrt{2}} \left( |p^a_-, \uparrow; 0\rangle_A \otimes |p^a_-, \downarrow; 0\rangle_B + |p^a_+, \uparrow; 0\rangle_A \otimes |p^a_+, \downarrow; 0\rangle_B \right), \tag{53}
\]

where \(\phi = 0\) is the coordinate of the point where the quanta were emitted. After some proper time \(\tau = r^2\Phi/J\), the particles travelled along its circular paths and the spin representation of the finite Wigner rotation due only to gravitation effects is given by

\[
D(W(\pm\Phi)) = e^{\mp \frac{i}{2} \sigma_3 \Omega}, \tag{54}
\]

since \(\vartheta^3(x)\) is constant along the path, the time-ordering operator is not necessary. Therefore, the state of the bipartite system in the local inertial frame at points \(\phi = ±\Phi\) is given

\[
|\Psi\rangle_{A,B} = \frac{1}{\sqrt{2}} \left( |p^a_+, \uparrow; 0\rangle_A \otimes |p^a_+, \downarrow; 0\rangle_B + |p^a_-, \uparrow; 0\rangle_A \otimes |p^a_-, \downarrow; 0\rangle_B \right) \]

\[
= \frac{1}{\sqrt{2}} \left( |p^a_+, \uparrow; 0\rangle_A \otimes |p^a_+, \downarrow; 0\rangle_B \right) \otimes (|\uparrow, \uparrow\rangle_{A,B} + |\downarrow, \downarrow\rangle_{A,B}) \]

\[
+ \frac{1}{\sqrt{2}} |p^a_-, \uparrow; 0\rangle_A \otimes |p^a_-, \downarrow; 0\rangle_B \otimes (\cos \frac{\Omega}{2} |\uparrow, \uparrow\rangle_{A,B} + |\downarrow, \downarrow\rangle_{A,B}) \]

\[
+ \sin \frac{\Omega}{2} |\uparrow, \downarrow\rangle_{A,B} + \frac{1}{\sqrt{2}} |p^a_+, \downarrow; 0\rangle_A \otimes |p^a_-, \uparrow; 0\rangle_B \otimes (\sin \frac{\Omega}{2} |\uparrow, \downarrow\rangle_{A,B} + \cos \frac{\Omega}{2} |\downarrow, \uparrow\rangle_{A,B}), \tag{55}
\]

where \(|p^a_+, \downarrow; 0\rangle_A \otimes |p^a_-, \uparrow; 0\rangle_B := |p^a_+, \uparrow; 0\rangle_A \otimes |p^a_-, \downarrow; 0\rangle_B\). Whereas the reduced spin density matrices of each particle are given by

\[
\rho^A_{\Lambda_s} = \rho^B_{\Lambda_s} = \left( \cos \frac{\Omega}{2} \sin \frac{\Omega}{2} \cos \frac{\Omega}{2} \sin \frac{\Omega}{2} \right), \tag{56}
\]

and \(\rho^A_{\Lambda_p} = \rho^B_{\Lambda_p} = \frac{1}{2} I_{2 \times 2}\). By inspecting Eq. (56), we can see that part of the entanglement between the spins were turned into quantum coherence of each spin state.

In Figs. 1(a) and 1(b), we plotted \(S_l(\rho^A_{\Lambda_s})\) and \(C_{hs}(\rho^A_{\Lambda_s})\) as a function of \(\Phi\) for different circular orbits. As \(\Phi \propto \tau\), these figures shows the behavior of \(S_l(\rho^A_{\Lambda_s})\) and \(C_{hs}(\rho^A_{\Lambda_s})\) as the particle travels along its circular orbit. It’s interesting to notes that for \(r = 2r_s\), the spin-state of the quantum A oscillates between a separable and entangled state with the spin-state of the particle B, if both particles completes a circular orbit. This behavior is due to the fact that as \(r \rightarrow \frac{3}{2}r_s\), the Wigner rotation \(\Omega\) varies more rapidly. While, in Figs. 1(c) and 1(d), we plotted \(S_l(\rho^A_{\Lambda_s})\) and \(C_{hs}(\rho^A_{\Lambda_s})\) as a function of \(r_s/r\) for different values of \(\Phi\).

On the other hand, if we consider a one-particle state in a separable state between spin and momentum with maximally coherent momentum-state in the clock and counterclockwise direction and the spin-state also maximally coherent, then \(C_{hs}(\rho_{\Lambda_s})\) would start in its maximum value and decrease, while \(S_l(\rho_{\Lambda_s})\) would start in its minimum value and increase. Similarly, if we consider a one-particle state in a separable state with a momentum state maximally coherent in the clock and counterclockwise direction and a spin-state completely predictable, for instance \(|\uparrow\rangle\), as the particle travels along its superposition paths, the states of the momenta will become entangled with the spin-states, and there will be a interchange between predictability \(P_l(\rho_{\Lambda_s})\), and the entanglement entropy \(S_l(\rho_{\Lambda_s})\) of the spin states. This effect of spacetime curvature in the complementary behavior of these quantum states is analogous to the effect reported in [70], since each clockwise and counterclockwise circular path can be taken as the different path of a Mach-Zehnder interferometer.
is given by
\[
    u^t = E, \quad u^r = 0, \\
    u^\theta = 0, \quad u^\phi = \omega E,
\]
such that the standard angular velocity is \(d\phi/dt = u^\phi/u^t = \omega\). Since \(u^\mu u_\mu = -1\), or equivalently
\[
    g_{tt}(u^t)^2 + g_{\phi\phi}(u^\phi)^2 = -f(r)E^2 + r^2\omega^2E^2 = -1,
\]
which implies \(E = 1/\sqrt{f(r)} - r^2\omega^2\). Besides, Eq.(58) suggests a familiar parametrization: \(f(r)E^2 = \cosh^2\xi\) and \(r^2\omega^2E^2 = \sinh^2\xi\) such that
\[
    u^t = \frac{\cosh \xi}{\sqrt{f(r)}}, \quad u^\phi = \frac{\sinh \xi}{r}.
\]
Therefore, the non-zero elements of the four-velocity in the local inertial frame defined by the tetrad field are expressed by
\[
    u^0 = e^0(x)u^t = \cosh \xi, \quad u^3 = e^3(x)u^\phi = \sinh \xi,
\]
with the speed of the particle in this frame being \(v = dx^3/dx^0 = u^3/u^0 = \tanh \xi\), which implies that \(r\omega = \sqrt{f(r)}v\) and the familiar expressions \(\sinh \xi = v\gamma\) and \(\cosh \xi = \gamma\), where \(\gamma = (1 - u^2)^{-1/2}\). In order for the particle to maintain such non-geodetic circular orbit, it’s necessary to apply an external radial force against gravity and the centrifugal force \(^1\), allowing the quanton to travel in the circular orbit with the specific angular velocity \(\omega\) at a given distance \(r\) from the source. Therefore the non-zero component of the acceleration due to non-gravitational external forces is given by:
\[
    a^r = u^\nu \nabla_\nu u^r = -\frac{\sinh^2 \xi}{r} \left(1 - \frac{r_s}{2rf(r)} \coth^2 \xi\right) f(r).
\]
For instance, in the specific case where \(u^\phi = J/r^2\), which corresponds to a geodetic circular orbit, then \(a^r = 0\). The non-zero infinitesimal local Lorentz transformations, defined by Eq. (18), are
\[
    \lambda^1_{0}(x) = -\frac{\cosh \xi \sinh^2 \xi}{r} \left(1 - \frac{r_s}{2rf(r)}\right) \sqrt{f(r)},
\]
\[
    \lambda^1_{3}(x) = \frac{\cosh^2 \xi \sinh \xi}{r} \left(1 - \frac{r_s}{2rf(r)}\right) \sqrt{f(r)},
\]
which also corresponds to a boost along the 1-axis and a rotation about the 2-axis. The infinitesimal Wigner

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\(^1\) In General Relativity, gravity and the centrifugal force are just manifestations of the spacetime curvature.
The reduced spin density matrices of each particle are given as a function of

\[ \rho^A = \rho^B = \left( \cos \frac{\Omega}{2} \sin \frac{\Omega}{2} \right) \left( \cos \frac{\Omega}{2} \sin \frac{\Omega}{2} \right), \]  

and \( \rho^A_{sp} = \rho^B_{sp} = \frac{1}{2} I_{2 \times 2} \), with \( \Omega \) being expressed by Eq. (67). In Figs. 2(b) and 2(c), we plotted the behavior \( S_1(\rho_{\Lambda s}) \) and \( C_{hs}(\rho_{\Lambda s}) \) as a function of \( r_s/r \), for \( \Phi = \pi/8 \) and \( v/c = 0.1 \).

This rapid oscillation near \( r_s \) persists for any value of \( \Phi \) and \( v \), and it is due to the fact that the Wigner angle varies very rapidly near the Schwarzschild radius. Whereas, in Figs. 3(a) and 3(b), we plotted \( S_1(\rho^A_{\Lambda s}) \) and \( C_{hs}(\rho^A_{\Lambda s}) \) as a function of \( \Phi \) for different circular orbits. As \( \Phi \propto \tau \), these figures express the behavior of these density matrices along circular paths such that the spinor representation of the finite Wigner rotation due only to gravitational effects is given by \( D(W(\pm \Phi)) = e^{ \mp \xi \hat{\sigma}_2 \Omega} \). Therefore, the spin correlation, and their complementarity, for the state in Eq. (55) with \( \Omega \) given by Eq. (67).

In this article, we extended complete complementarity relations to curved spacetimes by considering a succession of infinitesimal local Lorentz transformations, which implies that complementarity remains valid locally. This result allowed us to study these different complementary aspects of a quantum system as it travels through spacetime. In particular, we studied the behavior of these different complementary properties of massive spin-1/2 particles in the Schwarzschild spacetime. For geodetic circu-
given by Eq. (67).

Figure 3: (Color online) Quantum coherence and correlation, and their complementarity, for the state in Eq. (55) with \( \Omega \) given by Eq. (67).

(a) \( S_l(\rho_{\Lambda s}) \) as a function of \( \Phi \) for different values of \( r_s/r \).

(b) \( S_{h,s}(\rho_{\Lambda s}) \) as a function of \( \Phi \) for different values of \( r_s/r \).

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