ON MONOMIAL IDEAL RINGS AND A THEOREM OF TREVISAN

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Abstract. A direct proof is presented of a form of Alvise Trevisan’s result [7], that every monomial ideal ring is represented by the cohomology of a topological space. Certain of these rings are shown to be realized by polyhedral products indexed by simplicial complexes.

1. Introduction

In the paper [7], Alvise Trevisan showed that every ring which is a quotient of an integral polynomial ring by an ideal of monomial relations, can be realized as the integral cohomology ring of a topological space. Moreover, he showed that the rings could be all realized with spaces which are generalized Davis-Januszkiewicz spaces. These spaces are colimits over multicomplexes which are generalizations of simplicial complexes.

Here is presented a direct proof of the “realization” part of Trevisan’s theorem. It uses a result of Fröberg from [2] which asserts that a map known as “polarization” produces in a natural way, a regular sequence of degree-two elements. This allows for the realization of any monomial ideal ring by a certain pullback.

It is noted also that certain families of monomial ideal rings, beyond Stanley-Reisner rings, can be realized as generalized Davis-Januszkiewicz spaces based on ordinary simplicial complexes. Of course, as Trevisan shows, multicomplexes are needed in general.

Through the paper, all cohomology is taken with integral coefficients.

2. The main result

Let \( \mathbb{Z}[x_1, \ldots, x_n] \) be a polynomial ring and

\[
M = \{m_j\}_{j=1}^r, \quad m_j = x_1^{t_{1j}} x_2^{t_{2j}} \cdots x_n^{t_{nj}}
\]

be a set of minimal monomials, that is, no monomial divides another. Here, the exponent \( t_{ij} \) might be equal to zero but every \( x_i \) must appear in some \( m_j \). Notice that the set \( M \) is determined by the \( n \times r \) matrix \( (t_{ij}) \). Denote by \( I(M) \) the ideal in \( \mathbb{Z}[x_1, \ldots, x_n] \) generated by the minimal monomials \( m_j \) and set

\[
A = A(M) = \mathbb{Z}[x_1, \ldots, x_n]/I(M)
\]

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a monomial ideal ring. From this is defined a second monomial ideal ring \( A(M) \) with monomial ideal generated by square free monomials. For each \( i = 1, 2, \ldots, n \) set
\[
t_i = \max\{t_{i1}, t_{i2}, \ldots, t_{ir}\}
\]
the largest entry in the \( i \)-th row of \((t_{ij})\). Next, introduce new variables \( y_{i1}, y_{i2}, \ldots, y_{it_i} \) for each \( i = 1, 2, \ldots, n \). For each monomial \( m_j = x_1^{t_{1j}} x_2^{t_{2j}} \cdots x_n^{t_{nj}} \), set
\[
\overline{m}_j = (y_{11}y_{12} \cdots y_{it_1})(y_{21}y_{22} \cdots y_{it_2}) \cdots (y_{n1}y_{n2} \cdots y_{nt_n}).
\]
Let \( M = \{m_j\}_{j=1}^r \) and define an algebra \( B \) by
\[
B = B(M) = \mathbb{Z}[y_{11}, y_{12}, \ldots, y_{it_1}, y_{21}, y_{22}, \ldots, y_{it_2}, \ldots, y_{n1}, y_{n2}, \ldots, y_{nt_n}] / I(M).
\]
The monomials here are square-free so \( B \) is a Stanley-Reisner algebra which determines a simplicial complex \( K(M) \). (This process which constructs \( B \) from \( A \) is known in the literature as polarization.) Associated to this simplicial complex is a fibration
\[
Z(K(M); (D^2, S^1)) \longrightarrow \mathcal{D}\mathcal{J}(K(M)) \longrightarrow BT^{d(M)}
\]
where \( d(M) = \sum_{i=1}^n t_i \), with \( t_i \) as in (2.3), \( \mathcal{D}\mathcal{J}(K(M)) \) is the Davis-Januszkiewicz space of the simplicial complex \( K(M) \) and \( Z(K(M); (D^2, S^1)) \) is the moment-angle complex corresponding to \( K(M) \), (3). Recall that the Davis-Januszkiewicz space has the property that
\[
H^*(\mathcal{D}\mathcal{J}(K(M))) \cong B.
\]
Define next a diagonal map \( \Delta : T^n \longrightarrow T^{d(M)} \) by
\[
\Delta(x_1, x_2, \ldots, x_l) = (\Delta_{t_1}(x_1), \Delta_{t_2}(x_2), \ldots, \Delta_{t_n}(x_l))
\]
where \( \Delta_{t_i}(x_i) = (x_i, x_i, \ldots, x_i) \in T^{t_i} \). In the diagram below, let \( W(A) \) be defined as the pullback of the fibration.
\[
\begin{array}{ccc}
Z(K(M); (D^2, S^1)) & \longrightarrow & Z(K(M); (D^2, S^1)) \\
\downarrow & & \downarrow \\
W(A) & \longrightarrow & \mathcal{D}\mathcal{J}(K(M)) \\
\downarrow & & \downarrow \\
BT^n & \longrightarrow & BT^{d(M)}
\end{array}
\]
The diagram extends to to the right and produces the fibration
\[
W(A) \overset{\Delta}{\longrightarrow} \mathcal{D}\mathcal{J}(K(M)) \overset{p}{\longrightarrow} BT^{d(M)} - n.
\]
Recall that \( d(\overline{M}) = \sum_{i=1}^{n} t_i \) and choose generators
\[
H^*(BT^{d(\overline{M})-n}) \cong \mathbb{Z}[u_{12}, \ldots, u_{1t_1}, u_{22}, \ldots, u_{2t_2}, \ldots, u_{nt_n}],
\]
so that
\[
p^*(u_{ik}) = y_{i1} - y_{ik}, \quad i = 1, 2, \ldots, n, \quad k = 2, 3, \ldots, t_i.
\]

The fact that \( p^* \) is determined by the diagonal \( \Delta \) in diagram \((2.8)\), allows this choice. Set \( \theta_{ik} := p^*(u_{ik}) \). The proposition following is a basic result about the diagonal map \( \Delta \), (the polarization map); a proof may be found in [5, page 30].

**Proposition 2.1** (Fröberg). Over any field \( k \), the sequence \( \{\theta_{ik}\} \) is a regular sequence of degree-two elements in the ring \( H^*(\mathcal{D}\mathcal{J}(K(\overline{M}));k) \).

This result allows for a direct proof of the realization theorem.

**Theorem 2.2.** There is an isomorphism of rings
\[
H^*(W(A);Z) \rightarrow A(M).
\]

**Proof.** Working over a field \( k \) and following Masuda-Panov, [6, Lemma 2.1], we use the Eilenberg-Moore spectral sequence associated to the fibration \((2.9)\). It has
\[
E_2^{*,*} = \text{Tor}_{H^*(BT^{d(\overline{M})-n})}^*(H^*(\mathcal{D}\mathcal{J}(K(\overline{M}))),k).
\]

Now \( H^*(\mathcal{D}\mathcal{J}(K(\overline{M}))) \) is free as an \( H^*(BT^{d(\overline{M})-n}) \)-module by Proposition 2.1, so
\[
\text{Tor}_{H^*(BT^{d(\overline{M})-n})}^*(H^*(\mathcal{D}\mathcal{J}(K(\overline{M}))),k) = \text{Tor}_{H^*(BT^{d(\overline{M})-n})}^0(H^*(\mathcal{D}\mathcal{J}(K(\overline{M}))),k)
\]
\[
= H^*(\mathcal{D}\mathcal{J}(K(\overline{M}))) \otimes_{H^*(BT^{d(\overline{M})-n})} k
\]
\[
= H^*(\mathcal{D}\mathcal{J}(K(\overline{M}))) / p^*(H^{>0}(BT^{d(\overline{M})-n})).
\]

It follows that the Eilenberg-Moore spectral sequence collapses at the \( E_2 \) term and hence, as groups,
\[
H^*(W(A)) = H^*(\mathcal{D}\mathcal{J}(K(\overline{M}))) / p^*(H^{>0}(BT^{d(\overline{M})-n}))
\]
from which we conclude that \( H^*(W(A);k) \) is concentrated in even degree. Taking \( k = \mathbb{Q} \) gives the result that in odd degree, \( H^*(W(A);\mathbb{Z}) \) consists of torsion only. Unless this torsion is zero, the argument above with \( k = \mathbb{F}_p \) for an appropriate \( p \), implies a contradiction. It follows that \( H^*(W(A);\mathbb{Z}) \) is concentrated in even degree.

**Lemma 2.3.** The integral Serre spectral sequence of the fibration \((2.9)\) collapses.

**Proof.** The spaces in the fibration have integral cohomology concentrated in even degrees. \( \square \)
The $E_2$-term of the Serre spectral sequence is

$$H^s(W(A); \mathbb{Z}) \otimes H^s(BT^d(M) - n; \mathbb{Z}).$$

It follows that, as a ring, $H^s(W(A); \mathbb{Z})$ is the quotient of $H^s(DJ(K(M)))$ by the two-sided ideal $L$ generated by the image of $p^*$. So there is an isomorphism of graded rings,

$$H^s(W(A); \mathbb{Z}) \to H^s(DJ(K(M)))/L \cong A(M)/L \cong A(M)$$

completing the proof of Theorem 2.2.

3. On the geometric realization of certain monomial ideal rings by ordinary polyhedral products

In this section, polyhedral products, [1], involving finite and infinite complex projective spaces are used to realize certain classes of monomial ideal rings. As noted earlier, generalizations of the Davis-Januszkiewicz spaces to the realm of multicomplexes are required in order to realize all monomial ideal rings, see Trevisan [7].

The class which can be realized by ordinary polyhedral products is restricted to those monomials

$$M = \{m_j\}_{j=1}^r, \quad m_j = x_1^{t_{j1}} x_2^{t_{j2}} \cdots x_n^{t_{jn}}$$

of (2.1), which satisfy the condition:

$\ast$ $t_{ij}$ is constant over all monomials $m_j$ which have $t_{ij}$ and at least one other exponent both non-zero.

In particular, a monomial ring of the form

$$(3.1) \quad \mathbb{Z}[x_1, x_2, x_3]/\langle x_1^2 x_2, x_2^2 x_3, x_3^5 \rangle$$

can be realized by an ordinary polyhedral product. As usual, let $(X, A)$ denote a family of CW pairs

$$\{(X_1, A_1), (X_2, A_2), \ldots, (X_n, A_n)\}.$$ 

Given a monomial ring $A(M)$ of the form (2.2), satisfying the condition $\ast$ above, a simplicial complex $K$ and a family of pairs $(X, A)$ will be specified so that

$$H^*(Z(K; (X, A)); \mathbb{Z}) = A(M)$$

where $Z(K; (X, A))$ represents a polyhedral product as defined in [1].

**Construction 3.1.** Let $K$ be the simplicial complex on $n$ vertices $\{v_1, v_2, \ldots, v_n\}$ which has a minimal non-face corresponding to each $m_i$ having at least two non-zero exponents. If $m_i$ has non-zero exponents

$$t_{j_{i1}}, t_{j_{i2}}, \ldots, t_{j_{in}}$$

...
then \( K \) will have a corresponding minimal non-face \( \{v_{j_1}, v_{j_2}, \ldots, v_{j_t}\} \). Moreover, these will be the only minimal non-faces of \( K \).

For example, the ring \((3.1)\) above will have associated to it, the simplicial complex \( K \) on vertices \( \{v_1, v_2, v_3\} \) and will have minimal non-faces \( \{v_1, v_2\} \) and \( \{v_1, v_3\} \). So, \( K \) will be the disjoint union of a point and a one-simplex.

For the set of monomials \( M \) satisfying condition \( \star \), the cases following are distinguished in terms of \((2.1)\) for fixed \( i \in \{1, 2, \ldots, n\} \).

(1) For certain \( j \), \( t_{ij} = 1 \), \( t_{i'}j \neq 0 \) for some \( i' \neq i \) and \( t_{ik} = 0 \) otherwise.
(2) For certain \( j \), \( t_{ij} = q_i > 1 \), \( t_{i'}j \neq 0 \) for some \( i' \neq i \) and \( t_{ik} = 0 \) otherwise.
(3) \( m_j = x_i^{s_i} \) for some \( j \) and \( t_{ik} = 0 \) for \( k \neq j \).
(4) \( m_j = x_i^{s_i} \) for some \( j \) and if \( t_{ik} \neq 0 \) for \( k \neq j \), then \( t_{ik} = q_i < s_i \).

With this classification in mind, define a family of CW-pairs

\[
(X, A) = \{(X_i, A_i) : i = 1, \ldots, n\}
\]

by

\[
(X_i, A_i) = \begin{cases} 
(CP_\infty, *) & \text{if } i \text{ satisfies (1)}, \\
(CP_\infty, CP_{q_i-1}) & \text{if } i \text{ satisfies (2)}, \\
(CP_{s_i-1}, *) & \text{if } i \text{ satisfies (3)}, \\
(CP_{s_i-1}, CP_{q_i-1}) & \text{if } i \text{ satisfies (4)}. 
\end{cases}
\]

(3.2)

The next theorem describes the polyhedral products which have cohomology realizing the monomial ideal rings satisfying condition \( \star \).

**Theorem 3.2.** Let \( A(M) \) be a monomial ring of the form \((2.2)\), satisfying the condition \( \star \) and \( K \), the simplicial complex defined by Construction \((3.1)\) then

\[
H^*(Z(K; (X, A)); \mathbb{Z}) = A(M)
\]

where \((X, A)\) is the pair specified by \((3.2)\).

**Remark.** The improvement here over \([2, \text{ Theorem 10.5}]\) consists of the inclusion of cases (3) and (4) above. The polyhedral products which realize the monomial ideal rings discussed in \([1]\) have \( X_i = CP_\infty \) for all \( i = 1, 2, \ldots, n \).

**Proof of Theorem 3.2.** Set \( Q = (q_1, q_2, \ldots, q_n) \) with \( q_i \geq 1 \) for all \( i \) and write the spaces \( A_i \) of \((3.2)\) as \( CP_{q_i-1} \) where \( q_i = 1 \) if \( A_i = * \), a point. Write

\[
(X, A) = (X, CP^Q) = \{(X_i, CP^{q_i-1}) : i = 1, 2, \ldots, n\}
\]
and consider the commutative diagram

\[
\begin{array}{ccc}
H^*(\prod_{i=1}^n X_i) & \xleftarrow{p^*} & H^*(\prod_{i=1}^n \mathbb{C}P^\infty) \\
\downarrow i^* & & \downarrow k^* \\
H^*(Z(K; (X, \mathbb{C}P^{Q-1}))) & \xleftarrow{b^*} & H^*(Z(K; (\mathbb{C}P^\infty, \mathbb{C}P^{Q-1})))
\end{array}
\]  

(3.3)

induced by the various inclusion maps. According to [2, Theorem 10.5], there is an isomorphism of rings

\[
H^*(Z(K; (\mathbb{C}P^\infty, \mathbb{C}P^{Q-1}))) \rightarrow \mathbb{Z}[x_1, \ldots, x_n]/I(M^Q)
\]

where \(I(M^Q)\) is the ideal generated by all monomials \(x_{i_1}^{q_{i_1}}, x_{i_2}^{q_{i_2}}, \ldots, x_{i_k}^{q_{i_k}}\) corresponding to the minimal non-faces \(\{v_{i_1}, v_{i_2}, \ldots, v_{i_k}\}\) of \(K\). Moreover, the proof of [2, Lemma 10.3] shows that the composition \(i^*p^*\) is a surjection. The commutativity of diagram (3.3) implies that these relations all hold in \(H^*(Z(K; (X, \mathbb{C}P^{Q-1})))\). In addition to these, the relation \(x_i^{s_i} = 0\) is included for each \(i\) satisfying \(X_i = \mathbb{C}P^{q_i-1}\). These relations account for all the relations determined by \(I(M)\). The remainder of the argument shows that \(I(M)\) determines all relations in \(H^*(Z(K; (X, A)); \mathbb{Z})\). Consider now the space

\[
W_k = \mathbb{C}P^{q_1-1} \times \cdots \times \mathbb{C}P^{q_k-1} \times X_k \times \mathbb{C}P^{q_{k+1}-1} \times \cdots \times \mathbb{C}P^{q_n-1}
\]

corresponding to the simplex \(\{v_k\} \in K\), consisting of a single vertex. The composition

\[
W_k \rightarrow Z(K; (X, \mathbb{C}P^{Q-1})) \rightarrow \prod_{i=1}^n X_i
\]

factors the natural inclusion \(W_k \rightarrow \prod_{i=1}^n X_i\). From this observation follows the fact that no other monomial relations occur in \(H^*(Z(K; (X, \mathbb{C}P^{Q-1})))\) other than those determined by \(I(M)\). Suppose next that there is a linear relationship of the form

\[
a \omega = \sum_{i=1}^k a_i \omega_i
\]

(3.4)

where \(a, a_i \in \mathbb{Z}\) and \(\omega, \omega_i\) are monomials in the \(x_i, i = 1, 2, \ldots, n\). Without loss of generality, \(\omega\) and \(\omega_i\) can be assumed to be not divisible by any of the monomials in \(M\). Suppose \(\omega = x_{j_1}^{\lambda_1} x_{j_2}^{\lambda_2} \cdots x_{j_l}^{\lambda_l}\), then \(\sigma = \{v_{j_1}, v_{j_2}, \ldots, v_{j_l}\} \in K\) is a simplex and so is a full subcomplex of \(K\). (The corresponding polyhedral product \(Z(\sigma; (X, \mathbb{C}P^{Q-1}))\) is a product of finite and infinite complex projective spaces.) This implies, by [4, Lemma 2.2.3], that \(H^*(Z(\sigma; (X, \mathbb{C}P^{Q-1})))\) must be a direct summand in \(H^*(Z(K; (X, \mathbb{C}P^{Q-1})))\) contradicting the relation (3.4). \(\square\)
REFERENCES

[1] A. Bahri, M. Bendersky, F. Cohen and S. Gitler, The Polyhedral Product Functor: a method of computation for moment-angle complexes, arrangements and related spaces. Advances in Mathematics, 225 (2010), 1634–1668.

[2] A. Bahri, M. Bendersky, F. Cohen and S. Gitler, A New Topological Construction of Infinite Families of Toric Manifolds Implying Fan Reduction. Online at: http://arxiv.org/abs/1011.0094

[3] V. Buchstaber and T. Panov, Torus actions and their applications in topology and combinatorics, AMS University Lecture Series, 24, (2002).

[4] G. Denham and A. Suciu, Moment-angle complexes, monomial ideals and Massey products, Pure and Applied Mathematics Quarterly 3, no. 1, (2007), 25–60.

[5] R. Fröberg, A study of graded extremal rings and of monomial rings, Math. Scand. 51 (1982), 22–34.

[6] M. Masuda and T. Panov On the cohomology of torus manifolds. Available at: http://arxiv.org/abs/math/0306100

[7] A. J. Trevisan, Generalized Davis-Januszkiewicz spaces, multicomplexes and monomial rings, Homology, Homotopy and Applications, 13, no. 1, (2011), 205–221.

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