Discrete equations and the singular manifold method

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Abstract

The Painlevé expansion for the second Painlevé equation (PII) and fourth Painlevé equation (PIV) have two branches. The singular manifold method therefore requires two singular manifolds. The double singular manifold method is used to derive Miura transformations from PII and PIV to modified Painlevé type equations for which auto-Bäcklund transformations are obtained. These auto-Bäcklund transformations can be used to obtain discrete equations.

1 Introduction

An especially important subject related to the study of discrete equations is that concerning the discretization of Painlevé equations (cf. [2, 16, 27, 30]). Recently there has been substantial interest in the discrete Painlevé equations (dP1–dPVI), which in a variety of physical applications and indeed dP1 and dPII were first discovered in physical situations [5, 17, 28]. In the continuous limit, the discrete Painlevé equations yield their corresponding continuous Painlevé equation. This is not all as they have a variety of other properties in common with their continuous counterparts. For example have Lax pairs, bilinear representations, Bäcklund transformations and particular solutions for certain parameter values, expressible in terms of rational functions or discrete special functions (see, for example, [14] and the references therein).
The discrete equations, though, are much richer than the continuous equations. They have a host of properties which are lost in the continuous limit. Perhaps the most fundamental difference is that the continuous Painlevé equations have a unique canonical form whereas this is not true for the discrete equations: for any discrete Painlevé equation there exists a number of different forms.

It is well known that continuous Painlevé equations satisfy the Painlevé property (cf. [8, 22]) which means that all their solutions \( y(x) \) can be locally expressed as

\[
y(x) = \sum_{j=0}^{\infty} a_j (x - x_0)^j - \alpha,
\]

in a neighborhood of the movable singularity \( \phi \equiv x - x_0 = 0 \), where \( x_0 \) is an arbitrary constant, \( \alpha \) is a positive integer and \( a_j, j = 0, 1, \ldots \), are constants to be determined. More precisely, the singular manifold method (SMM) consists of truncation of the series (1.1) to the constant level

\[
y(x) = \sum_{j=0}^{\alpha} a_j (x - x_0)^j - \alpha,
\]

This condition seems to be very restrictive, nevertheless it provides much information about a given equation (cf. [6, 36]).

In the present paper we shall apply the SMM to the second and fourth Painlevé equations, namely PII and PIV, which are usually written as

\[
\begin{align*}
\text{PII} : & \quad y'' = 2y^3 + xy + \ell, \\
\text{PIV} : & \quad y'' = \frac{(y')^2}{2y} + \frac{3}{2}y^3 + 4xy^2 + 2(x^2 - 2\ell)y + \frac{\mu}{y},
\end{align*}
\]

where \( ' \equiv \frac{d}{dx} \) and \( \ell \) and \( \mu \) are constants (cf. [22]).

1.1 Two Painlevé branches and the singular manifold method

These equations (1.2, 1.3) have the property that the leading analysis of the expansion (1.1) provides \( \alpha = 1 \) and \( a_0 = \pm 1 \), where the \( \pm \) sign in \( a_0 \) indicates that there are two possible Painlevé branches.

Equations with two Painlevé branches and how to extend the SMM to them have been studied in previous papers (cf. [8, 10]). According to these, when an equation, such as (1.2) or (1.3), has two Painlevé branches, then the truncation of (1.1) should be made for both branches simultaneously. This means that we
should work with truncated solutions $\tilde{y}$ of the following form
\begin{equation}
\tilde{y} = y + \frac{g'}{g} - \frac{h'}{h},
\end{equation}
where $y(x)$ is also a solution of the equation, and $g(x) = 0$ is the singularity for the + expansion and $h(x) = 0$ the singularity for the − expansion. Henceforth we shall call $g$ and $h$ singular manifolds.

The SMM usually requires that each coefficient in the different powers of the singular manifold that arises from substitution of the truncated expansion in the differential equation be equal to zero. Due to the non-linearity of PII and PIV, the substitution of (1.4) in (1.2) and (1.3) respectively provides terms that mix the powers in $h$ and $g$. To solve this problem we use a decoupling ansatz given by
\begin{equation}
\frac{g' h'}{g'} = A \frac{g'}{g} + B \frac{h'}{h},
\end{equation}
where $A$ and $B$ are functions of $y$, and $g$ and $h$ to be determined from the truncation itself. Furthermore, taking the derivative of (1.5) with respect to $x$ we have
\begin{equation}
A' = A(r - A - B) \quad \quad B' = B(v - A - B),
\end{equation}
where $v = g''/g'$ and $r = h''/h'$.

Within this framework, the objective of this paper is to prove that for PII and PIV the improved version of the SMM including two singular manifolds provides the following results:

(i) Modified versions of PII and PIV (namely mPII and mPIV).

(ii) Miura transformations between PII and mPII and between PIV and mPIV.

(iii) Bäcklund transformations for mPII and mPIV.

(iv) By regarding the parameter $\ell$ as a discrete variable, discrete equations can be derived associated with mPII and mPIV.

(v) Solutions of these discrete equations allow us to construct solutions of PII and PIV as a “linear” superposition of solutions of the discrete equations.
2 Painlevé II

Substituting (1.4) into PII (1.2) and the use of (1.5) to decouple the crossed terms yields

\[ A = y + \frac{1}{2} v, \quad (2.1) \]
\[ B = -y + \frac{1}{2} r, \quad (2.2) \]
\[ 0 = v' + v^2 - 6y^2 - x + 6A(2y + B - A), \quad (2.3) \]
\[ 0 = r' + r^2 - 6y^2 - x - 6B(2y + B - A). \quad (2.4) \]

Using (1.6) in (2.1–2.2) gives

\[ AB = k, \]

where \( k \) is a constant. By substituting (2.1–2.2) into (2.3–2.4), the following singular manifold equations are obtained

\[ v' - \frac{1}{2} v^2 + 6k - x = 0, \quad r' - \frac{1}{2} r^2 + 6k - x = 0. \quad (2.5) \]

2.1 A Miura transformation for modified PII

If we return to equation (1.6), substituting \( A \) and \( B \) as given by (2.1) and (2.2), respectively, and using (2.5), we have

\[ y' - y^2 - \frac{1}{2} x + v' + 2k = 0, \quad -y' - y^2 - \frac{1}{2} x + r' + 2k = 0, \quad (2.6) \]

which means that the expression \( y' - y^2 - \frac{1}{2} x \) depends only on the singular manifold \( g \) whilst \( -y' - y^2 - \frac{1}{2} x \) depends only on \( h \). Hence it is useful to define the following functions \( m' \) and \( n' \) as follows

\[ 2m' = y' - y^2 - \frac{1}{2} x, \quad 2n' = -y' - y^2 - \frac{1}{2} x. \quad (2.7) \]

With the aid of (1.2), these equations (2.7) can be integrated to give

\[ 2m = (y')^2 - (y^2 + \frac{1}{2} x)^2 - (2\ell - 1)y, \quad 2n = (y')^2 - (y^2 + \frac{1}{2} x)^2 - (2\ell + 1)y, \quad (2.8) \]

respectively. In order to identify the equations that \( m \) and \( n \) satisfy, we take the derivative of (2.7), which gives

\[ 2m'' = -4ym' + \left( \ell - \frac{1}{2} \right), \quad 2n'' = 4yn' - \left( \ell - \frac{1}{2} \right). \quad (2.9) \]
Now if we take $y'$ from (2.7) and $y$ from (2.9) and substitute them in (2.8), then we obtain the following equation for $m$ and $n$

\[
\frac{(M''_f)^2}{M_f^4} + 4(M'_f)^2 + 2(xM'_f - M_f) - \frac{(2\ell - 1)^2}{16M'_f} = 0, \tag{2.10}
\]

where $m = M_f$ and $n = M_{f+1}$. It is easy to prove that this equation has the Painlevé property. In fact, equation (2.10) is the potential version of the equation 34 of the Gambier classification [13] (see also [11, 22, 30]), which is commonly referred to as P34. Indeed, if we set $M'_f = Q_f$, then (2.10) becomes

\[
Q''_f - \frac{(Q'_f)^2}{2Q_f} + 4Q^2_f + xQ_f + \frac{(\ell - \frac{1}{2})^2}{8Q_f} = 0, \tag{2.11}
\]

which is precisely P34. Equations (2.7) and (2.9) can be written as

\[
2Q_f = y' - y^2 - \frac{1}{2}x, \quad 2Q_{f+1} = -y' - y^2 - \frac{1}{2}x,
\]

\[
y = \frac{-2Q'_f + \ell - \frac{1}{2}}{4Q_f}, \quad y = \frac{-2Q'_{f+1} + \ell + \frac{1}{2}}{4Q_{f+1}}.
\]

These can be interpreted as a Miura transformation between PII (2.2) and P34 (2.11) [11]. According to Ramani and Grammaticos [30], P34 (2.11) may be also be thought of as a modified PII (mPII), since the relation between their solutions is analogous to that between solutions of the Korteweg-de Vries and modified Korteweg-de Vries equations (see also [11]).

### 2.2 Auto-Bäcklund transformations for modified PII

By subtracting the two equations of (2.8), we have

\[
y(x) = M_f(x) - M_{f+1}(x), \tag{2.12}
\]

which combined with (2.9) gives

\[
M_{f+1} = M_f + \frac{4M''_f - (2\ell - 1)}{8M'_f}, \quad M_f = M_{f+1} + \frac{4M''_{f+1} + (2\ell + 1)}{8M'_{f+1}}, \tag{2.13}
\]

which are auto-Bäcklund transformations for potential mPII.

### 2.3 Linear superposition for PII

Suppose we have two solutions $M_f$ and $M_{f+1}$ of potential mP34 (2.10), related by the Bäcklund transformations (2.13), then we can construct a solution of PII (1.2) by using (2.12).

5
2.4 Discrete equations for potential P34

If we let \( \ell \to \ell - 1 \) in the second of equations (2.13) then
\[
M_{\ell+1} = M_{\ell} + \frac{4M''_{\ell} - (2\ell - 1)}{8M'_{\ell}}, \quad M_{\ell-1} = M_{\ell} + \frac{4M''_{\ell} + (2\ell - 1)}{8M'_{\ell}}.
\] (2.14)

Adding and subtracting these two equations gives
\[
M'_{\ell} = -\frac{2\ell - 1}{4(M_{\ell+1} - M_{\ell-1})}, \quad M''_{\ell} = -\frac{(2\ell - 1)(M_{\ell+1} + M_{\ell-1} - 2M_{\ell})}{4(M_{\ell+1} - M_{\ell-1})}.
\] (2.15)

By substituting (2.15) in (2.10), the result is the nonautonomous discrete equation
\[
(M_{\ell-1} - M_{\ell+1}) \left\{ \left( \ell - \frac{1}{2} \right) \left( (M_{\ell-1} - M_{\ell})(M_{\ell+1} - M_{\ell}) + \frac{1}{2} x \right) 
- M_{\ell}(M_{\ell-1} - M_{\ell+1}) \right\} + \left( \ell - \frac{1}{2} \right)^2 = 0,
\]
where the parameter \( \ell \) can be interpreted as the discrete variable.

3 Painlevé IV

In an analogous way as we studied PII in the previous section, in this section we study PIV (1.3) by using the truncated expansion (1.4) together with the decoupling ansatz (1.5). The result is
\[
A = y + x + \frac{1}{2} v, \quad B = -y - x + \frac{1}{2} r,
\] (3.1)
\[
0 = (24x + 6v)(A - y) - 4x^2 + 8\ell + v^2 + 2(v' + y') - 18y^2 - 16A^2 + 36yA + 8AB - 2rA, \quad (3.3)
\]
\[
0 = (-24x + 6r)(B + y) - 4x^2 + 8\ell + r^2 + 2(r' - y') - 18y^2 - 16B^2 - 36yB + 8AB - 2rB. \quad (3.4)
\]

Equations (3.1–3.2) combined with (1.6) gives \( AB = k \), with \( k \) a constant. Then substituting (3.1–3.2) into (3.3–3.4), for the singular manifolds we have the equations
\[
v' - \frac{1}{2} v^2 + 6k + 2x^2 + 8\ell - 2 = 0, \quad r' - \frac{1}{2} r^2 + 6k + 2x^2 + 8\ell + 2 = 0. \quad (3.5)
\]
3.1 A Miura transformation for modified PIV

The substitution of (2.1-2.2) into (1.6) gives
\[ y' - y^2 - 2xy + v' + 2(k + 2\ell) = 0, \quad -y' - y^2 - 2xy + r' + 2(k + 2\ell) = 0. \] (3.6)

It is reasonable therefore to define new functions \( m \) and \( n \) in the following form
\[ 2m' = y' - y^2 - 2xy, \quad 2n' = -y' - y^2 - 2xy. \] (3.7)

These equations can be integrated, taking (1.3) into account, as
\[ 2m = \frac{(y')^2 + 2\mu - (y^2 + 2xy)^2}{4y} + (2\ell + 1)y, \]
\[ 2n = \frac{(y')^2 + 2\mu - (y^2 + 2xy)^2}{4y} + (2\ell - 1)y. \] (3.8)

Furthermore, the derivation of (3.7) provides
\[ 2m'' = \frac{2(m')^2 + \mu}{y} - 2(m' + 2\ell + 1)y, \]
\[ 2n'' = -\frac{2(n')^2 + \mu}{y} + 2(n' + 2\ell - 1)y. \] (3.9)

Taking \( y' \) as defined by (3.7) and substituting in (3.8) we have
\[ 4(m - xm') = \frac{2(m')^2 + \mu}{y} + 2(m' + 2\ell + 1)y, \]
\[ 4(n - xn') = \frac{2(n')^2 + \mu}{y} + 2(n' + 2\ell - 1)y. \] (3.10)

Adding and subtracting (3.9) and (3.10) the result is
\[ y = \frac{2(m - xm') - m''}{2(m' + 2\ell + 1)}, \quad \frac{1}{y} = \frac{2(m - xm') + m''}{2(m')^2 + \mu}, \]
\[ y = \frac{2(n - xn') + n''}{2(n' + 2\ell - 1)}, \quad \frac{1}{y} = \frac{2(n - xn') - n''}{2(n')^2 + \mu}. \] (3.11)

We can eliminated \( y \) by multiplication of the left equation by the right one. Thus we then have
\[ (M''_{\ell})^2 - 4(M_{\ell} - xM'_{\ell})^2 + 2(M'_{\ell} + 2\ell + 1) \left[ 2(M'_{\ell})^2 + \mu \right] = 0, \] (3.12)
where \( m = M_{\ell} \) and \( n = M_{\ell-1} \). Equation (3.12) is of Painlevé type, and it can be considered as the modified version of PIV (mPIV). The Miura transformation that relates PII and mPII are (3.9), which can now be written as
\[ 2M'_{\ell} = y' - y^2 - 2xy, \quad 2M'_{\ell-1} = -y' - y^2 - 2xy. \] (3.13)
3.2 Auto-Bäcklund transformations

Subtracting the two equations of (3.8), and with the aid of (3.13), we have

\[ y = M_\ell - M_{\ell-1}, \]  

then combined with the left part of (3.11) yields to

\[ M_{\ell-1} = M_\ell + \frac{M''_\ell - 2(M_\ell - xM'_\ell)}{2(M'_\ell + 2\ell + 1)}, \]
\[ M_\ell = M_{\ell-1} + \frac{M''_{\ell-1} + 2(M_{\ell-1} - xM'_{\ell-1})}{2(M'_{\ell-1} + 2\ell - 1)}, \]  

which are auto-Bäcklund transformation for mPIV (3.12).

3.3 Linear superposition for PIV

If we have two solutions \( M_\ell \) and \( M_{\ell+1} \) of potential mPIV (3.12), related by the Bäcklund transformations (3.15), then we can construct a solution of PIV (1.3) by using (3.14).

3.4 Discrete equations for mPIV

As we did in the previous section, the Bäcklund transformations (3.15) can be written as

\[ M_{\ell-1} = M_\ell + \frac{M''_\ell - 2(M_\ell - xM'_\ell)}{2(M'_\ell + 2\ell + 1)}, \]
\[ M_{\ell+1} = M_\ell + \frac{M''_{\ell-1} + 2(M_{\ell-1} - xM'_{\ell-1})}{2(M'_{\ell-1} + 2\ell - 1)}, \]  

Then by addition and subtraction, it is easy to show that

\[ M'_\ell = \frac{2M_\ell - (2\ell + 1)(M_{\ell+1} - M_{\ell-1})}{M_{\ell+1} - M_{\ell-1} + 2x}, \]
\[ M''_\ell = \frac{(M'_\ell + 2\ell + 1)(M_{\ell+1} + M_{\ell-1} - 2M_\ell)}{M_{\ell+1} - M_{\ell-1} + 2x}, \]  

whose substitution in (3.13) yields the discrete equation

\[ (M_{\ell+1} - M_{\ell-1} + 2x)(M_\ell - M_{\ell-1})(M_\ell - M_{\ell+1}) \left[ M_\ell + 2 \left( \ell + \frac{1}{2} \right) x \right] \]
\[ + \left\{ 2 \left[ M_\ell - \left( \ell + \frac{1}{2} \right) \right] (M_{\ell+1} - M_{\ell-1}) \right\}^2 + \frac{1}{2} x (M_{\ell+1} - M_{\ell-1} + 2x)^2 = 0. \]

where \( \ell \) is the discrete parameter.
4 Conclusions

In this paper we have derived Miura transformations, modified equations and associated discrete equations for the second and fourth Painlevé equations. Recently there have been several studies of the derivation of discrete equations, and in particular the discrete Painlevé equations, from Bäcklund transformations of the (continuous) Painlevé equations [12, 14, 15, 21, 27, 33, 34].

Hierarchies of solutions of PII and PIV are well-known (cf. [1, 3, 7, 18, 19, 25, 26, 29, 35]). Since there is an explicit relationship between PII and PIV and discrete equations, then these Hierarchies of solutions of PII and PIV also satisfy difference equations in addition to the ordinary differential equations. This is analogous to the situation for the classical special functions, such as Bessel, hypergeometric, Legendre, Weber-Hermite and Whittaker functions, which satisfy both an ordinary differential equation and a recurrence relation, which is a discrete equation. This provides further evidence that the Painlevé equations may be thought of as nonlinear special functions and that there is a deep relationship between the classical special functions, the Painlevé equations and the discrete Painlevé equations (see, for example, [32]).

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