Fractional G-White Noise Theory, Wavelet Decomposition for Fractional G-Brownian Motion, and Bid-Ask Pricing Application to Finance Under Uncertainty

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Abstract G-framework is presented by Peng [41] for measure risk under uncertainty. In this paper, we define fractional G-Brownian motion (fGBm). Fractional G-Brownian motion is a centered G-Gaussian process with zero mean and stationary increments in the sense of sub-linearity with Hurst index $H \in (0,1)$. This process has stationary increments, self-similarity, and long range dependence properties in the sense of sub-linearity. These properties make the fractional G-Brownian motion a suitable driven process in mathematical finance. We construct wavelet decomposition of the fGBm by wavelet with compactly support. We develop fractional G-white noise theory, define G-Itô-Wick stochastic integral, establish the fractional G-Itô formula and the fractional G-Clark-Ocone formula, and derive the G-Girsanov’s Theorem. For application the G-white noise theory, we consider the financial market modelled by G-Wick-Itô type of SDE driven by fGBm. The financial asset price modelled by fGBm has volatility uncertainty, using G-Girsanov’s Theorem and G-Clark-Ocone Theorem, we derive that sublinear expectation of the discounted European contingent claim is the bid-ask price of the claim.

Keywords Fractional G Brownian motion, G expectation, fractional G-noise, wavelet decomposition, volatility uncertainty, Wick product, G-Itô-Wick stochastic integral, fractional G-Black-Scholes market

MSC-classification: 60E05, 60H40,60K,60G18,60G22
JEL-classification: G10,G12,G13
The stochastic process $B^0_H(t)$, which is continuous Gaussian process with stationary increments

$$E[B^0_H(t)B^0_H(s)] = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H}), \quad H \in (0,1), \quad (E[\cdot] \text{ is some linear expectation})$$

was originally introduced by Kolmogorov [28] (1940) in study of turbulence under the name "Wiener Spiral", and the process has the self-similarity property: for $a > 0$

$$\text{Law}(B^0_H(at), \ t \geq 0) = \text{Law}(a^H B^0_H(t), \ t \geq 0).$$

Later, when the papers of Hurst [23] (1951) and Hurst, Black and Simaika [24] (1965) devoted to long-term storage capacity in Nile river, were published, the parameter $H$ got the name "Hurst parameter". The current name fractional Brownian motion (fBm) comes from the other pioneering paper by Mandelbrot and Van Ness [33] (1968), in which the stochastic calculus with respect to the fBm was considered. The fractional Brownian motion has similarity property and long range dependence property, which was leaded to describe a great variety of natural and physical phenomena, such as, hydrodynamics, natural images, traffic modelling in broadband networks, telecommunications, and fluctuations of the stock market.

The first continuous-time stochastic model for a financial asset appeared in the thesis of Bachelier [4] (1900). He proposed modelling the price of a stock with Brownian motion plus a linear drift. The drawbacks of this model are that the asset price could become negative and the relative returns are lower for higher stock prices. Samuelson [45] (1965) introduced the more realistic model

$$S_t = S_0 \exp((\mu - \frac{\sigma^2}{2})t + \sigma B^0_H(t)),$$

which have been the foundation of financial engineering. Black and Scholes [7] (1973) derived an explicit formula for the price of a European call option by using the Samuelson model with $S_0 = \exp(\mu t)$ through the continuous replicate trade. Such models exploded in popularity because of the successful option pricing theory, as well as the simplicity of the solution of associated optimal investment problems given by Merton [34] (1973).

However, the Samuelson model also has deficiencies and up to now there have been many efforts to build better models. Cutland et al. [11] (1995) discuss the empirical evidence that suggests that long-range dependence should be accounted for when modelling stock price movements and present a fractional version of the Samuelson model. For $H \in (\frac{1}{2},1)$ the fractional Gaussian noise $B^0_H(k+1) - B^0_H(k)$ exhibits long-range dependence, which is also called the Joseph effect in mandelbrot’s terminology [32] (1997), for $H = \frac{1}{2}$ the fBm is semimartingale and all correlations at non-zero lags are zero, and for $H \in (0,\frac{1}{2})$ the correlations sum up to zero which is less interesting for financial applications [11] (1995). However, empirical evidence is given of a Hurst parameter with values in $(0,\frac{1}{2})$ for foreign exchange rates [30].

Hu and Øksendal [25] (2003) develop fractional white noise theory in a white noise probability space $(\mathcal{S}(\mathcal{R}), \mathcal{F})$ with $\mathcal{F}$ the Borel field, modelled the financial market by Wick-Itô type of stochastic differential equations driven by fractional Brownian motions with Hurst index $H \in (\frac{1}{2},1)$, and compute explicit the price and replicating portfolio of a European option in this market. Elliott and Hoek [15] (2003) present an extension of the work of Hu and Øksendal [25] for fractional Brownian motion in which processes with all indices include $H \in (0,1)$ under the same probability measure, describe the financial market by a SDE driven by a sum of fractional Brownian motion with various Hurst indices and develop the European option pricing in such a market.

In an uncertainty financial market, the uncertainty of the fluctuation of the asset price comes from the drift uncertainty and the volatility uncertainty. For the drift uncertainty, in the probability
framework Chen and Epstein [9] (2002) propose to use g-expectation introduced by Peng in [36] (1997) for a robust valuation of stochastic utility. Karoui, Peng and Quenez in [27] and Peng in [37] (1997) propose to use time consistent condition g-expectation defined by the solution of a BSDE, as bid-ask dynamic pricing mechanism for the European contingent claim. Delbaen, Peng and Gianin ([12]) (2010) prove that any coherent and time consistent risk measure absolutely continuous with respect to the reference probability can be approximated by a g-expectation.

In the probability framework, the volatility uncertainty model was initially studied by Avellaneda, Levy and Paras [3] (1995) and Lyons [31] (1995) in the risk neutral probability measures, they intuitively give the bid-ask prices of the European contingent claims as superior and inferior expectations corresponding with a family of equivalent probability.

There is uncertainty in economics, and no one knows its probability distribution. Almost all the financial market fluctuations show volatility uncertainty (VIX, S&P 500, Nasdaq, Dow Jones, Eurodollar, and DAX, etc), and the volatility uncertainty is the most important, interesting and open problem in valuation (see [46] (2011)). Motivated by the problem of coherent risk measures under the volatility uncertainty (see [2] (1999)), Peng develops the process with volatility uncertainty, which is called G-Brownian motion in sublinear expectation space \((\Omega, \mathcal{H}, \hat{\mathbb{E}})\). He constructs the G-framework which is a very powerful and beautiful tool to analyse the uncertainty risk (see [39], [41], and [42]). In the sublinear expectation space the G-Brownian motion is a G-martingale under the G-expectation, the market modelled by the G-Brownian motion is incomplete. Using G-framework, Epstein and Ji [16] study the utility uncertainty application in economics, and Chen [9] gives a time consistent G-expectation bid-ask dynamic pricing mechanism for the European contingent claim in the uncertainty financial market modelled by SDE driven by the G-Brownian motion.

In this paper we consider to develop a fractional G-white noise theory under uncertainty. We define fractional G-Brownian motion \((f\text{GBm}) B_H(t)\) with Hurst index \(H \in (0, 1)\) in a G-white noise space, which is a centered G-Gaussian process (see Peng [43]) with stationary increment in the sense of sub-linearity, and it is more realistic to model the financial market by using the fGBm. Meanwhile, we construct wavelet decomposition of fGBm on the family of wavelet with compactly support. We develop a fractional G-white noise theory in a sublinear expectation space (or G-white noise space) \((S'(\mathbb{R}), S(\mathbb{R}), \hat{\mathbb{E}})\), consider fGBm on the G-white noise space, define fractional G-noise and set up fractional G-\text{Wi}ick stochastic integral with respect to fGBm. We derive the fractional G-\text{Itô} formula, define the fractional Malliavin differential derives, and prove the fractional G-Clark-Ocone formula. Furthermore, we present the G-Girsanov’s Theorem. Applying our theory in the financial market modelled by G-Wick-\text{Itô} type stochastic differential equations driven by fGBm \(B_H(t)\), we prove that the sublinear expectation of the discounted European contingent claim is the bid-ask price of the European claim.

Our paper is organized as follows: In Sec. 2 we define the fGBm with Hurst index \(H \in (0, 1)\) in the sublinear space. We prove that the fGBm is a continuous stochastic path with Hölder exponent in \([0, H]\), and has self-similarity property and long range dependence property in the sense of the sub-linearity. Furthermore, we establish the wavelet decomposition for the fGBm by using wavelets with compactly support. In Sec. 3 we present fractional G-white noise theory. In Sec. 4 we present the G-Girsanov’s Theorem. In Sec. 5, we apply our theory in the financial market modelled by G-Wick-\text{Itô} type stochastic differential equations driven by fGBm \(B_H(t)\), and derive the bid-ask price for the European contingent claim.
2 Fractional G-Brownian Motion

2.1 Sublinear Expectation and Fractional G-Brownian Motion

Let $\Omega$ be a given set and let $\mathcal{H}$ be a linear space of real valued functions defined on $\Omega$ containing constants. The space $\mathcal{H}$ is also called the space of random variables.

**Definition 1** A sublinear expectation $\hat{E}$ is a functional $\hat{E}: \mathcal{H} \rightarrow \mathbb{R}$ satisfying

(i) **Monotonicity:**
$$\hat{E}[X] \geq \hat{E}[Y] \text{ if } X \geq Y.$$  

(ii) **Constant preserving:**
$$\hat{E}[c] = c \text{ for } c \in \mathbb{R}.$$  

(iii) **Sub-additivity:** For each $X, Y \in \mathcal{H},$
$$\hat{E}[X + Y] \leq \hat{E}[X] + \hat{E}[Y].$$  

(iv) **Positive homogeneity:**
$$\hat{E}[\lambda X] = \lambda \hat{E}[X] \text{ for } \lambda \geq 0.$$  

The triple $(\Omega, \mathcal{H}, \hat{E})$ is called a sublinear expectation space.

In this paper, we mainly consider the following type of sublinear expectation spaces $(\Omega, \mathcal{H}, \hat{E})$: if $X_1, X_2, \ldots, X_n \in \mathcal{H}$ then $\varphi(X_1, X_2, \ldots, X_n) \in \mathcal{H}$ for $\varphi \in \operatorname{C}_{b,Lip}(\mathbb{R}^n)$, where $\operatorname{C}_{b,Lip}(\mathbb{R}^n)$ denotes the linear space of functions $\varphi$ satisfying

$$|\varphi(x) - \varphi(y)| \leq C(1 + |x|^m + |y|^m)|x - y| \text{ for } x, y \in \mathbb{R},$$  

some $C > 0, m \in \mathbb{N}$ is depending on $\varphi$.

For each fixed $p \geq 1$, we take $\mathcal{H}_0^p = \{X \in \mathcal{H}, \hat{E}[|X|^p] = 0\}$ as our null space, and denote $\mathcal{H} / \mathcal{H}_0^p$ as the quotient space. We set $\|X\|_p : = (\hat{E}[|X|^p])^{1/p}$, and extend $\mathcal{H} / \mathcal{H}_0^p$ to its completion $\hat{\mathcal{H}}_p$ under $\| \cdot \|_p$. Under $\| \cdot \|_p$, the sublinear expectation $\hat{E}$ can be continuously extended to the Banach space $(\hat{\mathcal{H}}_p, \| \cdot \|_p)$. Without loss generality, we denote the Banach space $(\hat{\mathcal{H}}_p, \| \cdot \|_p)$ as $L^p_\varnothing(\Omega, \mathcal{H}, \hat{E})$. For the G-framework of sublinear expectation space, we refer to [38], [39], [40], [41], [42] and [43]. In this paper we assume that $\mu, \overline{\mu}, \sigma$ and $\overline{\sigma}$ are nonnegative constants such that $\mu \leq \overline{\mu}$ and $\sigma \leq \overline{\sigma}$.

**Definition 2** Let $X_1$ and $X_2$ be two random variables in a sublinear expectation space $(\Omega, \mathcal{H}, \hat{E})$. $X_1$ and $X_2$ are called identically distributed, denoted by $X_1 \overset{d}{=} X_2$ if

$$\hat{E}[\varphi(X_1)] = \hat{E}[\varphi(X_2)] \text{ for } \forall \varphi \in \operatorname{C}_{b,Lip}(\mathbb{R}^n).$$

**Definition 3** In a sublinear expectation space $(\Omega, \mathcal{H}, \hat{E})$, a random variable $Y$ is said to be independent of another random variable $X$, if

$$\hat{E}[\varphi(X, Y)] = \hat{E}[\hat{E}[\varphi(x, Y)]|_{x = X}].$$

**Definition 4** (G-normal distribution) A random variable $X$ on a sublinear expectation space $(\Omega, \mathcal{H}, \hat{E})$ is called G-normal distributed if

$$aX + b\tilde{X} = \sqrt{a^2 + b^2}X \quad \text{for } a, b \geq 0,$$

where $\tilde{X}$ is an independent copy of $X$. 


Remark 1 For a random variable $X$ on the sublinear space $(\Omega, \mathcal{H}, \hat{E})$, there are four typical parameters to characterize $X$:

$$
\begin{align*}
\overline{\mu} &= \hat{E}X, & \mu &= -\hat{E}[-X], \\
\overline{\sigma}^2 &= \hat{E}X^2, & \sigma^2 &= -\hat{E}[-X^2],
\end{align*}
$$

where $[\mu, \overline{\mu}]$ and $[\sigma^2, \overline{\sigma}^2]$ describe the uncertainty of the mean and the variance of $X$, respectively.

It is easy to check that if $X$ is G-normal distributed, then

$$
\begin{align*}
\overline{\mu} &= \hat{E}X = \mu = -\hat{E}[-X] = 0, \\
\overline{\sigma}^2 &= \hat{E}X^2 = \sigma^2 = -\hat{E}[-X^2] = 0,
\end{align*}
$$

and we denote the G-normal distribution as $N(\{0\}, [\sigma^2, \overline{\sigma}^2])$. If $X$ is maximal distributed, then

$$
\begin{align*}
\overline{\mu} &= \hat{E}X = \mu = -\hat{E}[-X] = 0, \\
\overline{\sigma}^2 &= \hat{E}X^2 = \sigma^2 = -\hat{E}[-X^2] = 0,
\end{align*}
$$

and we denote the maximal distribution as $N([\mu, \overline{\mu}], \{0\})$.

Definition 5 We call $(X_t)_{t \in \mathbb{R}}$ a $d$-dimensional stochastic process on a sublinear expectation space $(\Omega, \mathcal{H}, \hat{E})$, if for each $t \in \mathbb{R}$, $X_t$ is a $d$-dimensional random vector in $\mathcal{H}$.

Definition 6 Let $(X_t)_{t \in \mathbb{R}}$ and $(Y_t)_{t \in \mathbb{R}}$ be $d$-dimensional stochastic processes defined on a sublinear expectation space $(\Omega, \mathcal{H}, \hat{E})$, for each $\mathcal{L} = (t_1, t_2, \ldots, t_n) \in \mathcal{T}$,

$$
F^X_{\mathcal{L}}[\varphi] := \hat{E}[\varphi(X_t)], \quad \forall \varphi \in C_{lip}(\mathbb{R}^{n \times d})
$$

is called the finite dimensional distribution of $X_t$. $X_t$ and $Y_t$ are said to be identically distributed, i.e., $X_t \overset{d}{=} Y_t$, if

$$
F^X_{\mathcal{L}}[\varphi] = F^Y_{\mathcal{L}}[\varphi], \quad \forall \mathcal{L} \in \mathcal{T} \quad \text{and} \quad \forall \varphi \in C_{lip}(\mathbb{R}^{n \times d})
$$

where $\mathcal{T} := \{\mathcal{L} = (t_1, t_2, \ldots, t_n) : \forall n \in \mathbb{N}, t_i \in \mathbb{R}, t_i \neq t_j, 0 \leq i, j \leq n, i \neq j\}$.

Definition 7 A process $(B_t)_{t \geq 0}$ on the sublinear expectation space $(\Omega, \mathcal{H}, \hat{E})$ is called a G-Brownian motion if the following properties are satisfied:

(i) $B_0(\omega) = 0$;

(ii) For each $t, s > 0$, the increment $B_{t+s} - B_t$ is G-normal distributed by $N(\{0\}, [\sigma^2, \overline{\sigma}^2])$ and is independent of $(B_{t_1}, B_{t_2}, \ldots, B_{t_n})$, for each $n \in \mathbb{N}$ and $t_1, t_2, \ldots, t_n \in (0, t]$.

Definition 8 A process $(X_t)_{t \in \mathbb{R}}$ on a sublinear expectation space $(\Omega, \mathcal{H}, \hat{E})$ is called a centered G-Gaussian process if for each fixed $t \in \mathbb{R}$, $X_t$ is G-normal distributed $N(\{0\}, [\sigma^2, \overline{\sigma}^2])$, where $0 \leq \sigma_t \leq \overline{\sigma}_t$.

Remark 2 Peng in [41] constructs G-framework, which is a powerful and beautiful analysis tool for risk measure and pricing under uncertainty. In [33], Peng defines G-Gaussian processes in a nonlinear expectation space, $q$-Brownian motion under a complex-valued nonlinear expectation space, and presents a new type of Feynman-Kac formula as the solution of a Schrödinger equation.

From now on, in this section we start to define a two-sided G-Brownian motion and a fractional G-Brownian motion, furthermore we construct the fractional G-Brownian motion and present the similarity property and long range dependent property for the fractional G-Brownian motion in the sense of linearity.
Definition 9 A process \((B_{1/2}(t))_{t \in R} \in \Omega\) on the sublinear expectation space \((\Omega, \mathcal{H}, \hat{E})\) is called a two-sided G-Brownian motion if for two independent G-Brownian motions \((B_1^{(1)})_{t \geq 0}\) and \((B_1^{(2)})_{t \geq 0}\) we denote the fractional G-Brownian motion as \(f_{GBm}\).

We can easily check that \(f_{GBm}\) is called fractional G-Brownian motion with Hurst index \(H\) if

\[
\begin{align*}
\hat{E}[B_H(s)B_H(t)] &= \frac{\pi^2}{\Gamma^2(H+1/2)} \left( |s|^{2H} + |t|^{2H} - |s-t|^{2H} \right), \\
\hat{E}[-B_H(s)B_H(t)] &= \frac{\pi^2}{\Gamma^2(H+1/2)} \left( |s|^{2H} + |t|^{2H} - |s-t|^{2H} \right),
\end{align*}
\]

We denote the fractional G-Brownian motion as \(f_{GBm}\).

2.2 Moving Average Representation

Similar with the Mandelbrot-Van Ness representation of fBm, we give the moving average representation of \(f_{GBm}\) with respect to the G-Brownian motion as follows.

Theorem 1 Let \(H \in (0, 1)\), for \(t \in R\) the fractional G-Brownian Motion with Hurst index \(H\) is represented as

\[
B_H(t, \omega) = C_H^H \int_R [(t-s)^{H-1/2} - (-s)^{H-1/2}] dB(s, \omega),
\]

where \(C_H^H = \frac{(2H \sin \pi H \Gamma(2H))^{1/2}}{\Gamma(H+1/2)}\) and \((B_t)_{t \in R}\) is a two-sided G-Brownian motion.

Proof. It is clear that \(B_H(0) = \hat{E}[B_H(t)] = 0\), and it is trivial to prove the equations in (2) for \(s = t\).

From the Definition 7 and 9 and by using G-Itô stochastic integral (11) and the integral transform we have that, for \(s > t\)

\[
\begin{align*}
\hat{E}[B_H(s)B_H(t)] &= \frac{2\pi \sin \pi H \Gamma(2H)}{\Gamma^2(H+1/2)} \sigma^2 \left\{ \int_{-\infty}^0 [(s-u)^{H-1/2} - (-u)^{H-1/2}][(t-u)^{H-1/2} - (-u)^{H-1/2}] du \\
&\quad + \int_0^t (s-u)^{H-1/2}(t-u)^{H-1/2} du \right\} \\
&= \sigma^2 \left\{ |s|^{2H} + |t|^{2H} - |s-t|^{2H} \right\} - \frac{2\pi \sin \pi H \Gamma(2H)}{\Gamma^2(H+1/2)} \sigma^2 \left\{ \int_{-\infty}^0 [(s-u)^{H-1/2} - (-u)^{H-1/2}][(t-u)^{H-1/2} - (-u)^{H-1/2}] du \\
&\quad + \int_0^t (s-u)^{H-1/2}(t-u)^{H-1/2} du \right\},
\end{align*}
\]

thus we prove the first equation in (2), and other cases can be proved in a similar way.

We can prove the second equation in (2) with \(\hat{E}[-\cdot] \) replaced by \(-\hat{E}[-\cdot] \) in the above equation, hence we prove (2). □
2.3 Properties of Fractional G-Brownian Motion

**Definition 11** In the sublinear expectation space \((\Omega, \mathcal{H}, \hat{E})\), a process \((X_t)_{t \in \mathbb{R}}\) is called to have \(H\)-self-similarity property, if

\[ X(at) \overset{d}{=} a^H X(t) \quad \text{for } a > 0. \]  

(4)

**Theorem 2** A fGBm \(B_H(t)\) with Hurst index \(H \in (0, 1)\) in \((\Omega, L^p_{\mathcal{G}}(\Omega), \hat{E})\) has the following properties

(i) \(H\)-self-similar property

\[ B_H(at) \overset{d}{=} a^H B_H(t) \quad \text{for } a > 0. \]

(ii) The fGBm \((B_H(t))_{t \in \mathbb{R}}\) is a continuous path with stationary increment, with \(B \in [0, H]\) order Hölder continuous and almost nowhere Hölder continuous with order \(\gamma > H\), i.e., for \(\alpha \geq 0\),

\[ \hat{E}|B_H(s) - B_H(t)|^\alpha = \hat{E}|B_H(1)|^\alpha |t-s|^{\alpha H}, \]

\[-\hat{E}[-|B_H(s) - B_H(t)|^\alpha] = -\hat{E}[-|B_H(1)|^\alpha |t-s|^{\alpha H}]. \]

**Proof.**

(i) From the Definition \([10]\), the fGBm is a centered G-Gaussian process, and from \(\hat{E}[B_H^2(at)] = a^{2H} \hat{E}[B_H^2(t)]\) and \(-\hat{E}[-B_H^2(at)] = -a^{2H} \hat{E}[-B_H^2(t)]\) we prove the H-self-similar property.

(ii) It is easy to check that \(B_H(s) - B_H(t)\) and \(B_H(s-t)\) is identity distributed with G-normal distribution \(N(0, \sigma^2 (t-s)^{2H})\), the fGBm \((B_H(t))_{t \geq 0}\) has the self similarity property, therefore, we derive that

\[ \hat{E}|B_H(s) - B_H(t)|^\alpha = \hat{E}|B_H(s-t)|^\alpha \]

\[ = \hat{E}|B_H(1)|^\alpha |s-t|^{\alpha H}, \]

with the similar argument for \(-\hat{E}[-\cdot]\), we prove the theorem. \(\square\)

**Theorem 3** (Long range dependence) For the fGBm \((B_H(t))_{t \in \mathbb{R}}\) with Hurst index \(H\) in sublinear expectation space \((\Omega, \mathcal{H}, \hat{E})\)

(i) For \(H \in (0, 1)\)

\[ \hat{E}[(B_H(n+1) - B_H(n))B_H(1)] = \frac{1}{2} \sigma^2 [(n+1)^{2H} - 2n^{2H} + (n-1)^{2H}] \]

\[ -\hat{E}[-(B_H(n+1) - B_H(n))B_H(1)] = \frac{1}{2} \sigma^2 [(n+1)^{2H} - 2n^{2H} + (n-1)^{2H}] \]  

(5)

(6)

(ii) If \(H \in (1/2, 1)\), there exhibits long range dependence, i.e.,

\[ 0 < \underline{r}(n) < \overline{r}(n), \quad \forall n \in \mathbb{N}, \]

if \(H = 1/2\) there exhibits uncorrelated, i.e.,

\[ \overline{r}(n) = \underline{r}(n) = 0, \]

and if \(H \in (0, 1/2)\)

\[ \lim_{n \to \infty} \underline{r}(n) = \lim_{n \to \infty} \overline{r}(n) = 0, \]

where

\[ \overline{r}(n) = \hat{E}[(B_H(n+1) - B_H(n))B_H(1)] \]

and

\[ \underline{r}(n) = -\hat{E}[-(B_H(n+1) - B_H(n))B_H(1)] \]

are upper and lower auto-correlation function of \(B_H(n+1) - B_H(n)\), respectively.
Proof. (i) From the construction of fGBm in the next section (see (28) in the next section) we have
\[ \hat{\mathbf{E}}[(B_H(n+1) - B_H(n))B_H(1)] = \sigma^2 \int_{R} [C_{H}(n+1)](x)M_{H}(0,1)(x)dx, \] (7)
\[ - \hat{\mathbf{E}}[(B_H(n+1) - B_H(n))B_H(1)] = \sigma^2 \int_{R} [C_{H}(n,1)](x)M_{H}(0,1)(x)dx. \] (8)
Define\[ C'_H = \frac{[\sin(\pi H)\Gamma(2H + 1)]^{-1/2}C_H}{}, \]
where \[ C_H = [2\Gamma(H - \frac{1}{2})\cos(\frac{1}{2}\pi(H - \frac{1}{2}))]^{-1}[\sin(\pi H)\Gamma(2H + 1)]^{1/2}, \]
similar with the Definition(12) in the next section for the operator \( M_H \), we denote \( M'_H \) as the operator with replace \( C_H \) by \( C'_H \) in the definition \( M_H \). For \( 0 \leq a < b \) and \( 0 < H < 1 \), by Parseval’s Theorem
\[ \int_{R} [M'_{H}]_{[a,b]}(x)^2 dx = \frac{1}{2\pi} \int_{R} [M_{H}']_{[a,b]}(\xi)^2 d\xi \]
\[ = \frac{1}{2\pi} \int_{R} |\xi|^{-2H}[\hat{f}_{[a,b]}(\xi)]^2 d\xi \]
\[ = \frac{1}{2\pi} \int_{R} |\xi|^{-2H} \frac{e^{-ib\xi} - e^{-ia\xi}}{-i\xi}^2 d\xi \]
\[ = \frac{1}{\sin \pi H \Gamma(2H + 1)}(b-a)^{2H}, \]
from which and notice (7) and (8) we can prove (i).
(ii) From (5) we have that
\[ \tau_n \sim H(2H - 1)n^{2H-2}E[B_H^2(1)], \quad n \to \infty, \quad H \neq \frac{1}{2}, \]
\[ \tau_n = 0, \quad H = \frac{1}{2}. \]
and
\[ \sum_{n=1}^{\infty} \tau_n = \hat{\mathbf{E}}[B_H^2(1)] \lim_{n \to \infty} (n + 1)^{2H} - n^{2H} - 1 \left\{ \begin{array}{l} < \infty, \quad H \in (0,1/2); \\ = 0; \quad H = 1/2; \\ = \infty, \quad H \in (1/2,1), \end{array} \right. \]
we can also derive the similar expressions for \( L_n \), thus we finish the proof. □

2.4 Wavelet Decomposition of Fractional G-Brownian Motion
We consider to expand the fGBm on the periodic compactly supported wavelet family (see (14) and (19)):
\[ \{ \psi_{j,k} : x \to \sum_{l \in Z} \psi(2^j(x-l) - k), \quad j \geq 0, \quad 0 \leq k \leq 2^j - 1 \} \] (9)
where \( \psi \) is a mother wavelet, and we denote \( \phi(x) \) as its periodic scaling function. We assume that
- the wavelet \( \psi \) belongs to the Schwartz class \( S(R) \);
• the $\psi$ has $N(\geq 2)$ vanishing moments, i.e.,

$$\int_{-\infty}^{\infty} t^N \psi(t) dt = 0.$$ 

By convention, if $j = -1$, $0 \leq k \leq 2^{-1} - 1$ means $k = 0$, we denote

$$2^{-\frac{1}{2}j} \psi_{-1,k}(t) = \phi(x-k), 0 \leq k \leq 2^{-1} - 1.$$ 

Then the periodized wavelet family

$$\{2^{j/2} \psi_{j,k}(t), j \geq -1, 0 \leq k \leq 2^j - 1\}$$ 

form an orthonormal basis of $L^2(T)$, where $T := R/Z$ (1-period). Without loss of generality, we consider $T = [0, 1]$.

For $\alpha > 0$, we denote Liouville fractional integral as

$$(L^\alpha f)(x) := \frac{1}{\Gamma(\alpha)} \int_0^x (t-x)^{\alpha-1} f(x) dx,$$ 

and define Riemann-Liouville fractional integral coincide with the Marchaud fractional integral as follows

$$(I^\alpha_M f)(x) := \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{+\infty} [(t-x)_+^{\alpha-1} - (-x)_+^{\alpha-1}] f(x) dx,$$ 

Theorem 4 There exists a wavelet expansion for a fGBm process $B_H(t)$, i.e., for $H \in (0, 1)$

$$B_H(t) = C_H^w \sum_{j=-1}^{\infty} \sum_{k=0}^{2^j - 1} \mu_{j,k} (I^\alpha_M \psi_{j,k})(t)$$ 

where

$$\alpha = H + \frac{1}{2},$$

$$C_H^w = \frac{(2H \sin \pi H \Gamma(2H))^{1/2}}{\Gamma(H + 1/2)},$$

$$\mu_{j,k} = 2^{-(H-\frac{1}{2})j} \epsilon_{j,k}$$

and $\epsilon_{j,k}$ are i.i.d. $G$-normal distributed with $B_H(1) \sim N(\{0\}, [\sigma^2, \sigma^2])$.

Proof. (i) We denote the right hand side of (13) as

$$F(t) = C_H^w \sum_{j=-1}^{\infty} \sum_{k=0}^{2^j - 1} \mu_{j,k} (I^\alpha_M \psi_{j,k})(t)$$

Without loss generality, we can rewrite $F(t)$ as follows

$$F(t) = C_H^w \sum_{n=-1}^{\infty} f_n(t) \epsilon_n,$$ 

where $\{f_n\}_{n=-1}^{\infty}$ denotes the countable Riesz basis $\{2^{-(H+1)j} (I^\alpha_M \psi_{j,k})(x)\}_{j=-1, k=0, \ldots, 2^j-1}$ of $L^2(R)$ (see [47]).
For proving that the right-hand side of above equation defines a generalized process, i.e., as a linear functional

$$F(u) = \int_{-\infty}^{\infty} F(t)u(t)dt, \text{ for } \forall u \in S(R),$$  \hfill (15)

we only need to prove

$$\|F\|_{H^{-1}} = \|C_H^n \sum_{n=-1}^{\infty} f_n(t)\varepsilon_n\|_{H^{-1}} < \infty.$$  \hfill (16)

By the representation theorem of a sublinear expectation (see [41]), there exists a family of linear expectations \(\{E_\theta : \theta \in \Theta\}\) such that

$$\hat{E}[X] = \sup_{\theta \in \Theta} E_\theta[X], \text{ for } X \in \mathcal{H}. \hfill (17)$$

Thus, by Kolmogorov’s convergence critera we conclude that

$$\|C_H^n \sum_{n=-1}^{\infty} f_n(t)\varepsilon_n\|_{H^{-1}} < \infty.$$  \hfill (18)

Consequently, by Plancherel theorem

$$| < F, u > | = \frac{1}{2\pi} \left| \int_{-\infty}^{\infty} \hat{F}(\xi) \hat{u}(\xi) d\xi \right| = \frac{1}{2\pi} \left| \int_{-\infty}^{\infty} \hat{F}(\xi)(1 + \xi^2)^{-1/2} \hat{u}(\xi)(1 + \xi^2)^{1/2} d\xi \right| \leq \frac{1}{2\pi \infty} \|F\|_{H^{-1}} \|u\|_{H^{-1}} < \infty,$$

where \(\hat{u}(\xi) = \int_{-\infty}^{\infty} u(t)e^{-\xi t}dt\) is Fourier transform of \(u\).

(ii) We prove that \(\{B_H(t), t \in R\}\) is a centered generalized G-Gaussian process with stationary increment, i.e., with zero mean and

$$\hat{EB}_H(u)\overline{B}_H(v) = \frac{\sigma}{2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \vert t \vert^{2H} + \vert s \vert^{2H} - \vert t-s \vert^{2H} u(t)\overline{v(s)}dt \, ds,$$

$$-\hat{E}[-B_H(u)\overline{B}_H(v)] = \frac{\sigma}{2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \vert t \vert^{2H} + \vert s \vert^{2H} - \vert t-s \vert^{2H} u(t)\overline{v(s)}dt \, ds.$$

(19)

From the definition of the fractional integral [12], we have

$$B_H(u) = C_H^n \int_{-\infty}^{\infty} \sum_{j=-1}^{\infty} \sum_{k=0}^{2^{-1}} 2^{-(H+\frac{1}{2})j} \varepsilon_{j,k}(M_{\psi,j,k}(t)\overline{\mu}(t))dt$$

$$= C_H^n \int_{-\infty}^{\infty} \sum_{j=-1}^{\infty} \sum_{k=0}^{2^{-1}} 2^{-(H+\frac{1}{2})j} \varepsilon_{j,k} \int_{-\infty}^{\infty} \mu(t) \int_{-\infty}^{\infty} \left[ ([I^n\delta](t-x))_+ - ([I^n\delta](x-t))_+ \right] \psi_{j,k}(x)dx \, dt,$$

where \((I^n\delta)(t-s) = \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\).
G-normal distributed \(e_{j,k} \ (j = -1, 0, 1, \ldots; k = 0, \ldots, 2^j - 1)\) are independent, we derive

\[
\mathbb{E}_{H}(u)\mathbb{E}_{\hat{H}}(v)
\]

\[
= \mathbb{E}_{H}(1)[C_{H}^{w}] \equiv \sum_{j=1}^{\infty} \sum_{k=0}^{2^j-1} 2^{-(2H+1)j} \int_{R} \left[ (\psi^\alpha(t-s))_+ - (\psi^\alpha(-s))_+ \right] u(t) \psi_{j,k}(s) ds dt
\]

\[
\int_{R} \left[ (\psi^\alpha(t-s))_+ - (\psi^\alpha(-s))_+ \right] u(t) dt
\]

\[
= \mathbb{E}_{H}(C_{H}^{w}) \int_{R} \left[ (\psi^\alpha(t-s))_+ - (\psi^\alpha(-s))_+ \right] u(t) dt \left[ (\psi^\alpha(t-s))_+ - (\psi^\alpha(-s))_+ \right] ds
\]

\[
\int_{R} \psi(t) \sum_{j=1}^{\infty} \sum_{k=0}^{2^j-1} \left[ (\psi^\alpha(t-s'))_+ - (\psi^\alpha(-s'))_+ \right] 2^{-(H+\frac{1}{2}) |s|} \psi_{j,k}(s) ds dt
\]

Following from the proof of Theorem 1 we have

\[
\mathbb{E}_{H}(C_{H}^{w}) \int_{R} \left[ (\psi^\alpha(t-s'))_+ - (\psi^\alpha(-s'))_+ \right] \left[ (\psi^\alpha(t'-s'))_+ - (\psi^\alpha(-t'))_+ \right] dt'
\]

\[
= \mathbb{E}_{H}(C_{H}^{w}) \int_{R} \left[ (\psi^\alpha(t-s'))_+ - (\psi^\alpha(-s'))_+ \right] \left[ (\psi^\alpha(t'-s'))_+ - (\psi^\alpha(-t'))_+ \right] dt'
\]

\[
= \frac{1}{2} [t^H + s^H - |t-s|^H],
\]

we finish the first equation in 19. With the similar argument, we can prove the second part in 19. Thus, we prove that the right hand of 13 is a generalized fBm, we finish the proof of the theorem.

\[
\square
\]

Remark 3 For construct the G-normal distributed random vector, for example \(B_{H}(1)\), Peng in [41] proposed the central limit theorem with zero-mean.

Let \(\{X_i\}_{i=1}^{n}\) be a sequence of \(\mathbb{R}^{d}\) valued random variables on a sublinear expectation space \((\Omega, \mathcal{F}, \mathbb{E})\), \(\mathbb{E}[X_1] = -\mathbb{E}[X_1] = 0\), and assume that \(X_{t+1} \overset{d}{=} X_t\) and \(X_{t+1}\) is independence from \(\{X_1, \ldots, X_t\}\). Then

\[
S_n \overset{la}{\rightarrow} X,
\]

where

\[
S_n := \frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_i
\]

and \(X\) is G-normal distributed.

3 Fractional G-Noise and Fractional G-Itô Formula

3.1 Fractional G-Brownian Motion on the G-White Noise Space

Let \(S(R)\) denotes the Schwartz space of rapidly decreasing infinitely differentiable real valued functions, let \(S'(R)\) be the dual space of \(S(R)\), and \(< \cdot, \cdot >\) denotes the dual operation, for \(f \in L^{2}(R)\) by approximating by step functions

\[
<f, \omega> := \int f dB(\omega),
\]

(20)
where $B(\omega) = B(\cdot, \omega)$ is the two-sided G-Brownian motion with $B(1) \sim N(\{0\}, [\sigma^2, \sigma^2])$. Then

$$(S'(R), S(R), \hat{E})$$

is a sublinear expectation space.

**Remark 4** Concerning the G-framework and G-Itô stochastic integral theorem, we refer to Peng’s paper [40], book [42] and references therein.

Denote $I_{[0,t]}(s)$ as the indicator function

$$I_{[0,t]}(s) = \begin{cases} 
1 & \text{if } 0 \leq s \leq t \\
-1 & \text{if } t \leq s \leq 0 \\
0 & \text{otherwise}
\end{cases} \quad (21)$$

Define the following process

$$\tilde{B}_t(\omega) := \langle I_{[0,t]}(\cdot), \omega \rangle \quad (22)$$

then $\tilde{B}_t$ is two-sided G-Brownian motion with $\tilde{B}_t \sim N(\{0\}, [\sigma^2|t|, \sigma^2|t|])$. Without loss generality, for $t \in R$ we denote $B_t$ as two-sided G-Brownian motion $\tilde{B}_t$.

For $H \in (0, 1)$, we define the following operator $M_H$

**Definition 12** The operator $M_H$ is defined on functions $f \in S(R)$ by

$$\hat{M}_H f(y) = |y|^{1/2-H} \hat{f}(y), \quad y \in R, \quad (23)$$

where

$$\hat{g} := \int_R e^{-ixy} g(x) dx$$

denotes the Fourier transform.

For $0 < H < \frac{1}{2}$ we have

$$M_H f(x) = C_H \int_R \frac{f(x-t) - f(x)}{|t|^{3/2-H}} dt, \quad (24)$$

where

$$C_H = [2\Gamma(H - \frac{1}{2}) \cos(\frac{1}{2} \pi(H - \frac{1}{2}))]^{-1} [\sin(\pi H) \Gamma(2H + 1)]^{1/2}.$$

For $H = \frac{1}{2}$ we have

$$M_H f(x) = f(x). \quad (25)$$

For $\frac{1}{2} < H < 1$ we have

$$M_H f(x) = C_H \int_R \frac{f(t)}{|t-x|^{3/2-H}} dt. \quad (26)$$

We define

$$L^2_H(R) := \{ f : M_H f \in L^2(R) \}$$

$$= \{ f : |y|^{1/2-H} \hat{f}(y) \in L^2(R) \}$$

$$= \{ f : \| f \|_{L^2_H(R)} < \infty \}, \quad \text{where } \| f \|_{L^2_H(R)} = \| M_H f \|_{L^2(R)},$$

then the operator $M_H$ can be extended from $S(R)$ to $L^2_H(R)$. 

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For $H \in (0, 1)$, consider the following process

$$
\tilde{B}_H(t, \omega) := \langle M_H I_{(0,t)}(\cdot), \omega \rangle 
$$

(28)

then, for $t \in \mathbb{R}$ it is a centered G-Gaussian process (see Peng (2011) [43]) with $\tilde{B}_H(0) = \tilde{E}[B_H(t)] = 0$, and

$$
\tilde{E}[\tilde{B}_H(s)\tilde{B}_H(t)] = \frac{1}{2} \sigma^2 \left[ \int_{\mathbb{R}} M_{H} I_{(0,s)}(x) M_{H} I_{(0,t)}(x) \, dx \right] 
$$

$$
= \frac{1}{2} \sigma^2 [t^{2H} + |s|^{2H} - |s-t|^{2H}],
$$

$$
-\tilde{E}[-\tilde{B}_H(s)\tilde{B}_H(t)] = \frac{1}{2} \sigma^2 \left[ \int_{\mathbb{R}} M_{H} I_{(0,s)}(x) M_{H} I_{(0,t)}(x) \, dx \right] 
$$

$$
= \frac{1}{2} \sigma^2 [t^{2H} + |s|^{2H} - |s-t|^{2H}],
$$

Then the continuous process $\tilde{B}_H(t)$ is a fGBm with Hurst index $H$, we denote $\tilde{B}_H(t)$ as $B_H(t)$.

Let $f(x) = \sum_{j} a_{j} I_{(\frac{j}{n}, \frac{j+1}{n})}(x)$ be a step function, then

$$
< M_{H} f, \omega > = \int_{\mathbb{R}} f(t) \, dB_H(t), \quad (29)
$$

and can be extended to all $f \in L_{H}^{2}(\mathbb{R})$. And we also have

$$
\int_{\mathbb{R}} f(t) \, dB_H(t) = \int_{\mathbb{R}} M_{H} f(t) \, dB(t), \quad f \in L_{H}^{2}(\mathbb{R}). \quad (30)
$$

### 3.2 Fractional G-Noise

Recall the Hermite polynomials

$$
h_n(x) = (-1)^n e^{-\frac{x^2}{2}} \frac{d^n}{d x^n} e^{-\frac{x^2}{2}}, \quad n = 0, 1, 2, \ldots
$$

We denote the Hermite functions as follows:

$$
\tilde{h}_n(x) = \pi^{-\frac{1}{4}} ((n-1)!)^{-\frac{1}{2}} h_{n-1}(\sqrt{2}x) e^{-\frac{x^2}{2}}, \quad n = 1, 2, \ldots
$$

Then $\{\tilde{h}_n, n = 1, 2, \cdots\}$ is an orthonormal basis of $L_{2}(\mathbb{R})$ and

$$
|\tilde{h}_n(x)| \leq \left\{ \begin{array}{ll}
C n^{-\frac{1}{4}} & \text{if } |x| \leq 2\sqrt{n} \\
C e^{-\gamma^2} & \text{if } |x| > 2\sqrt{n},
\end{array} \right.
$$

where $C$ and $\gamma$ are constants independent of $n$. Define

$$
e_i(x) := M_{H}^{-1} \tilde{h}_i(x), \quad i = 1, 2, \ldots
$$

Then $\{e_i, i = 1, 2, \cdots\}$ is an orthonormal basis of $L_{H}^{2}(\mathbb{R})$.

We denote $J$ as the set of all finite multi-indices $\alpha = (\alpha_1, \cdots, \alpha_n)$ for some $n \geq 1$ and $\alpha_i \in \mathbb{N}_0 = \{0, 1, 2, \cdots\}$, and for $\alpha \in J$

$$
H_{\alpha}(\omega) = \Pi_{i=1}^{n} h_{\alpha_i}(< \tilde{h}_i, \omega >) 
$$

$$
= \Pi_{i=1}^{n} h_{\alpha_i}(\int_{\mathbb{R}} \tilde{h}_i dB(\omega)),
$$
where \((B(t, \omega))_{t \in \mathbb{R}}\) is the two-sided G-Brownian motion.

Denote \(\tilde{h}^{\otimes \alpha}\) as the symmetric product with factors \(\tilde{h}_1, \ldots, \tilde{h}_n\) with each \(\tilde{h}_i\) being taken \(\alpha_i\) times, similar with the statement for the fundamental result of Itô [26] (1951) we denote

\[
\int_{\mathbb{R}^n} \tilde{h}^{\otimes \alpha} dB^{\otimes \alpha} := H_\alpha(\omega).
\] (31)

**Definition 13** We define space \(\hat{L}^2_{\tilde{H}}(\mathbb{R}^n)\) as follows

\[
\hat{L}^2_{\tilde{H}}(\mathbb{R}^n) := \{ f(x_1, \ldots, x_n) \text{ be symmetric function of } (x_1, \ldots, x_n) \mid M^n_H f \in L^2(\mathbb{R}^n) \},
\]

where \(M^n_H f\) means the operator \(M_H\) is applied to each variable of \(f\), and we denote

\[
\|f\|_{\hat{L}^2_{\tilde{H}}(\mathbb{R}^n)}^2 := \int_{\mathbb{R}^n} (M^n_H f)^2 ds.
\] (32)

For \(f \in \hat{L}^2_{\tilde{H}}(\mathbb{R}^n)\), we define

\[
\int_{\mathbb{R}^n} f dB^{\otimes n} := \int_{\mathbb{R}^n} (M^n_H f) dB^{\otimes n}.
\]

**Definition 14** The random variable

\[
F \in L^2_{G,H}(S'(\mathbb{R}), S(\mathbb{R}), \hat{\mathbb{E}}), \text{ if and only if } F \circ M_H \in \hat{L}^2_{\tilde{H}}(S'(\mathbb{R}), S(\mathbb{R}), \hat{\mathbb{E}}).
\]

The expansion of \(F \circ M_H\) in terms of the Hermite functions:

\[
F(M_H \omega) = \sum_\alpha c_\alpha H_\alpha(\omega) = \sum_\alpha c_\alpha h_\alpha(\langle M_H e_1, \omega \rangle) \ldots h_\alpha(\langle M_H e_n, \omega \rangle)
\]

Consequently,

\[
F(\omega) = \sum_\alpha c_\alpha h_\alpha(\langle e_1, \omega \rangle) \ldots h_\alpha(\langle e_n, \omega \rangle).
\]

For giving \(F \in \hat{L}^2_{\tilde{H}}(S'(\mathbb{R}), S(\mathbb{R}), \hat{\mathbb{E}})\), we have

\[
F(\omega) = \sum_\alpha c_\alpha H_\alpha(\omega) = \sum_\alpha \sum_{|\alpha| = n} c_\alpha \int_{\mathbb{R}^n} \tilde{h}^{\otimes \alpha} dB^{\otimes n} = \sum_\alpha \sum_{|\alpha| = n} c_\alpha \int_{\mathbb{R}^n} e^{\otimes \alpha} dB_{M^H}^{\otimes n}
\]

Aase et al. [11] (2000), Holden et al. [21], and Elliott & Hoek [15] (2003) gave the definitions of Hida space (S) and \((S)^*\) under the probability framework. Here we define G-Hida space (S) and \((S)^*\) as follows
Definition 15  (i) We define the G-Hida space \((S)\) to be all functions \(\psi\) with the following expansion
\[
\psi(\omega) = \sum_{\alpha \in J} a_\alpha \mathcal{H}_\alpha(\omega)
\] (33)
satisfies
\[
\|\psi\|_{k,(S)} := \sum_{\alpha \in J} a_\alpha^2 \alpha!(2N)^{k\alpha} < \infty \text{ for all } k = 1, 2, \ldots ,
\]
where
\[
(2N)^\gamma := (2 \cdot 1)^\gamma (2 \cdot 2)^\gamma \cdots (2 \cdot m)^\gamma \text{ if } \gamma = (\gamma_1, \gamma_2, \cdots, \gamma_m) \in J.
\]
(ii) We define the G-Hida space \((S)^*\) to be the set of the following expansions
\[
\Phi(\omega) = \sum_{\alpha \in J} b_\alpha \mathcal{H}_\alpha(\omega)
\] (34)
such that
\[
\|\Phi\|_{-q,(S)^*} := \sum_{\alpha \in J} b_\alpha^2 \alpha!(2N)^{-q\alpha} < \infty \text{ for some integer } q \in (0, +\infty).
\]
The duality between \((S)\) and \((S)^*\) is given as follows: for
\[
\Psi(\omega) = \sum_{\alpha \in J} a_\alpha \mathcal{H}_\alpha(\omega) \in (S),
\]
\[
\Phi(\omega) = \sum_{\alpha \in J} b_\alpha \mathcal{H}_\alpha(\omega) \in (S)^*
\]
\[
\langle \langle \Psi, \Phi \rangle \rangle := \sum_{\alpha \in J} \alpha! a_\alpha b_\alpha.
\]
The space \((S)^*\) is convenient for the Wick product

**Definition 16** If for \(F_i(\cdot) \in (S)^*\)
\[
F_i(\omega) = \sum_{\alpha \in J} c^{(i)}_\alpha H_\alpha(\omega), \ i = 1, 2,
\]
we define their Wick product \((F_1 \circ F_2)(\omega)\) by
\[
(F_1 \circ F_2)(\omega) = \sum_{\alpha, \beta \in J} c^{(1)}_\alpha c^{(2)}_\beta H_{\alpha + \beta}(\omega) = \sum_{\gamma \in J} \left( \sum_{\alpha + \beta = \gamma} c^{(1)}_\alpha c^{(2)}_\beta \right) H_{\gamma}(\omega).
\] (35)

From the chaos expansion of the fGBm \(B_H(t)\), we have
\[
B_H(t) = \langle M_H I_{[0,t]}, \omega \rangle = \langle I_{[0,t]}, M_H \omega \rangle
\]
\[
= \langle \sum_{k=1}^{\infty} (I_{[0,t]}, e_k) L^2(R) e_k, M_H \omega \rangle
\]
\[
= \langle \sum_{k=1}^{\infty} (M_H I_{[0,t]}, M_H e_k) L^2(R) e_k, M_H \omega \rangle
\]
\[
= \sum_{k=1}^{\infty} (M_H I_{[0,t]}, \tilde{H}_k) L^2(R) e_k, M_H \omega \rangle
\]
\[
= \sum_{k=1}^{\infty} \int_{0}^{t} M_H \tilde{H}_k(s) dsH_{e_k}(\omega),
\]

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where $\varepsilon^{(k)} = (0, 0, \cdots, 0, 1, 0, \cdots, 0)$ with 1 on the $k$th entry, 0 otherwise, and $k = 1, 2, \cdots$. We define the $H$-fractional G-noise $W_H(t)$ by the expansion as follows

**Definition 17** $H$-fractional G-noise with respect to the fGBm $B_H(t)$ is defined as following

$$W_H(t) = \sum_{k=1}^{\infty} M_{H} \hat{h}_k(t) H_{e^{(i)k}}(\omega).$$

For $H = \frac{1}{2}$, we call $\frac{1}{2}$-fractional G-noise $W_{\frac{1}{2}}(t)$ with respect to the G-Brownian motion $B(t)$ as G-white noise.

For $\frac{1}{2} < H < 1$, $H$-fractional G-noise $W_H(t)$ is called fractional G-black noise.

For $0 < H < \frac{1}{2}$, $H$-fractional G-noise $W_H(t)$ is called fractional G-pink noise.

Then it can be shown that $W_H(t)$ for all $t$ as

$$\frac{d B_H(t)}{dt} = W_H(t).$$

**Remark 5** Long Range Dependence Theorem characterizes the $H$-fractional G-noise, especially, its uncertainty. The fractional G-black noise has persistence long memory, the fractional G-pink noise has negative correlations in the sense of the sub-linearity and quickly alternatively change its uncertainty. The fractional G-black noise has persistence long memory, the fractional G-pink noise characterizes the $H$-fractional G-noise, especially,

$$H_{-\frac{1}{2}} + \sum_{k=1}^{\infty} \left| \hat{h}_k(y) \right| = \sqrt{2\pi}.$$
Definition 18 Suppose \( Y \in (S)^* \) is such that \( Y(t) \circ W_H(t) \) is integrable in \((S)^*\), we say that \( Y \) is dB\(_H\)-integrable and we define the fractional G-Itô-Wick integral of \( Y(t) = Y(t, \omega) \) with respect to \( B_H(t) \) by

\[
\int_Y Y(t) dB_H(t) := \int_Y Y(t) \circ W_H(t) dt.
\]

Using Wick calculus we derive

\[
\int_0^T B_H(t) dB_H(t) = \frac{1}{2} (B_H(T))^2 - \frac{1}{2} T^{2H}.
\]

3.3 Fractional G-Itô Formula

Consider the fractional stochastic differential equation driven by fGBm \( B_H(t) \)

\[
\begin{align*}
\left\{ \begin{array}{l}
\frac{dX(t)}{dt} = X(t) \circ (\alpha(t) dt + \beta(t) dB_H(t)), \quad t \geq 0, \\
X(0) = x.
\end{array} \right.
\end{align*}
\]

We write the above fractional stochastic differential equation in \((S)^*\)

\[
\frac{dX(t)}{dt} = X(t) \circ [\alpha(t) + \beta(t) W_H(t)] dt,
\]

by Wick product formula we have

\[
X(t) = X(0) \circ \exp \left( \int_0^t \alpha(s) ds + \int_0^t \beta(s) dB_H(s) \right),
\]

i.e.

\[
X(t) = x \exp \left( \int_0^t \beta(s) dB_H(s) + \int_0^t \alpha(s) ds - \frac{1}{2} \int_0^t (M_H(\beta(s) I_{[0,t]}))^2 ds \right).
\]

is the solution of the fractional SDE (40).

Theorem 5 Assume that \( f(s, x) \in C^{1,2}(R \times R) \), the fractional G-Itô formula is as follows:

\[
f(t, B_H(t)) = f(0, 0) + \int_0^t \frac{\partial f}{\partial s}(s, B_H(s)) ds + \int_0^t \frac{\partial f}{\partial x}(s, B_H(s)) dB_H(s) + H \int_0^t \frac{\partial^2 f}{\partial x^2}(s, B_H(s)) \frac{d^2 H - 1}{ds^2} ds.
\]

Proof. Let

\[
g(t, x) = \exp(\alpha x + \beta(t))
\]

where \( \alpha \in R \) be a constant and \( \beta : R \rightarrow R \) be a deterministic differentiable function. Set

\[
Y(t) = g(t, B_H(t)).
\]

Then

\[
Y(t) = \exp \left( \beta(t) + \alpha B_H(t) + \frac{1}{2} \alpha^2 t^{2H} \right),
\]
by Wick calculus in $(S)^*$, $Y(t)$ is the solution of the following fractional SDE
\[
\left\{ \begin{array}{l}
\frac{d}{dt}Y(t) = Y(t)\beta'(t) + Y(t)\diamond (\omega W_H(t)) + Y(t)H\alpha^2 t^{2H-1}, \\
Y(0) = \exp(\beta(0)).
\end{array} \right.
\]
Hence
\[
Y(t) = Y(0) + \int_0^t Y(s)\beta'(s)ds + \int_0^t Y(s)\alpha dB_H(s) + H\int_0^t Y(s)\alpha^2 s^{2H-1}ds,
\]
which means that
\[
g(t,B_H(t)) = g(0,0) + \int_0^t \frac{\partial g}{\partial s}(s,B_H(s))ds + \int_0^t \frac{\partial g}{\partial s}(s,B_H(s))dB_H(s) + H\int_0^t \frac{\partial^2 g}{\partial s^2}(s,B_H(s))s^{2H-1}ds. \tag{46}
\]
For $f(t,x) \in C^{1,2}(R \times R)$, we can find a sequence $f_n(t,x)$ of a linear combinations of function $g(t,x)$ in $(\ref{E})$ such that $f_n(t,x), \frac{\partial f_n(t,x)}{\partial t}, \frac{\partial f_n(t,x)}{\partial x},$ and $\frac{\partial^2 f_n(t,x)}{\partial x^2}$ pointwisely convergence to $f(t,x), \frac{\partial f(t,x)}{\partial t}, \frac{\partial f(t,x)}{\partial x},$ and $\frac{\partial^2 f(t,x)}{\partial x^2}$, respectively, in $C^{1,2}(R \times R)$, then $f_n(t,x)$ satisfying (46) for all $n$. Since
\[
\int_0^t \frac{\partial f_n}{\partial x}(s,B_H(s))dB_H(s)
= \int_0^t \frac{\partial f_n}{\partial x}(s,B_H(s))\diamond W_H(s)ds,
\]
we can find a sequence $f_n(t,x)$ such that
\[
\int_0^t \frac{\partial f_n}{\partial x}(s,B_H(s))\diamond W_H(s)ds \rightarrow \int_0^t \frac{\partial f}{\partial x}(s,B_H(s))\diamond W_H(s)ds \quad \text{in} \quad (S)^*, \quad \text{as} \quad n \rightarrow \infty.
\]
We prove the theorem. \[\square\]

### 3.4 Fractional Differentiation

Similar with the approach to differentiation in Aase et al. \[1\] (2000), Hu & Øksendal \[25\] (2003), and Elliott & Van der Hoek \[15\] (2004), we define the fractional differentiation of the function defined on the white noise space $(S'(R),S(R),\hat{E})$.

**Definition 19** Suppose $F : S'(R) \rightarrow R$ and suppose $\gamma \in S(R)$. We say $F$ has a directional $M_H$-derivative in the direction $\gamma$ if
\[
D^{(H)}_\gamma F(\omega) := \lim_{\varepsilon \rightarrow 0} \frac{F(\omega + \varepsilon M_H\gamma) - F(\omega)}{\varepsilon}
\]
exists in $G$-Hida space $(S)^*$, and $D^{(H)}_\gamma F(\omega)$ is called the directional $M_H$-derivative of $F$ in the direction $\gamma$.

**Definition 20** We say that $F : S'(R) \rightarrow R$ is $M_H$-differentiable if there is a map
\[
\Psi : R \rightarrow (S)^*
\]
such that $(M_H\Psi(t)) \cdot (M_H\gamma)(t)$ is $(S)^*$ integrable and
\[
D^{(H)}_\gamma F(\omega) = \langle F, \gamma \rangle_{M_H}, \quad \text{for all} \quad \gamma \in L^2_{H}(R),
\]
where $\langle F, \gamma \rangle_{M_H} := \int_R (M_H\Psi(t)) \cdot (M_H\gamma)(t)dt$. We then define
\[
D^{(H)}_\gamma F(\omega) := \frac{\partial^H}{\partial \omega} F(t,\omega) = \Psi(t,\omega)
\]
and we call $D^{(H)}_\gamma F(\omega)$ the Malliavin derivative or stochastic $M_H$-gradient of $F$ at $t$. 

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Definition 21 Suppose \( k \in \{1, 2, \ldots \} \) and for \( n = 0, 1, 2, \ldots \), \( f_n \in L^2_H (\mathbb{R}^n) \). We say that
\[
\Psi = \sum_{n=0}^{\infty} \int_{\mathbb{R}^n} f_n dB_H^{\otimes n} (x)
\]
belongs to the space \( G_h (M_H) \) if
\[
\| \Psi \|_{G_h}^2 := \sum_{n=0}^{\infty} n! \| M_H^n f_n \|^2_{L^2 (\mathbb{R}^n)} e^{2kn} < \infty.
\]
Define \( G (M_H) = \bigcap_h G_h (M_H) \) and give \( G (M_H) \) the projective topology.

Definition 22 Suppose \( q \in \{1, 2, \ldots \} \). We say the formal expansion
\[
Q = \sum_{n=0}^{\infty} \int_{\mathbb{R}^n} g_n dB_H^{\otimes n} (x),
\]
with \( g_n \in L^2_H (\mathbb{R}^n), n = 0, 1, 2, \ldots, \) belongs to the space \( G_{-q} (M_H) \) if
\[
\| Q \|_{G_{-q}}^2 := \sum_{n=0}^{\infty} n! \| M_H^n g_n \|^2_{L^2 (\mathbb{R}^n)} e^{-2qn} < \infty.
\]
Define \( G^* (M_H) = \bigcup_{q=1}^{\infty} G_{-q} (M_H) \) and given \( G^* (M_H) \) the inductive topology. Then \( G^* (M_H) \) is the dual space of \( G (M_H) \). For \( Q \in G^* (M_H) \) and \( \Psi \in G (M_H) \) define
\[
<< Q, \Psi >>_G = \sum_{n=0}^{\infty} n! < M_H^n g_n, M_H^n f_n >_{L^2 (\mathbb{R}^n)}.
\]

Definition 23 Suppose
\[
Q = \sum_{n=0}^{\infty} \int_{\mathbb{R}^n} g_n (s) dB_H^{\otimes n} ,
\]
is in \( G^* (M_H) \). We define the quasi-G-conditional expectation of \( Q \) with respect to \( \mathcal{F}_t^{M_H} = \sigma \{ B_H (s), 0 \leq s \leq t \} \) as
\[
\tilde{E}_{M_H} [Q | \mathcal{F}_t^{M_H}] := \sum_{n=0}^{\infty} \int_{\mathbb{R}^n} g_n (s) I_n (0, 1) (s) dB_H^{\otimes n} (s),
\]
where \( I_n (0, 1) (s) = I (0, 1) (s_1) \cdots I (0, 1) (s_n) \). We say that \( Q \in G^* (M_H) \) is \( \mathcal{F}_t^{M_H} \) measurable if
\[
\tilde{E}_{M_H} [Q | \mathcal{F}_t^{M_H}] = Q
\]
Note that, if \( H \neq \frac{1}{2} \) the quasi-G-conditional expectation \( \tilde{E}_{M_H} [B_H | \mathcal{F}_t^{M_H}] = B_H (s) \neq \tilde{E} [B_H | \mathcal{F}_t^{M_H}] \).

Definition 24 Suppose \( F = \sum_{\alpha} c_\alpha H_\alpha \in G^* (M_H) \). Then the stochastic \( M_H \)-gradient of \( F \) at \( t \) is defined by:
\[
D_t^H F (\omega) = \sum_{\alpha} c_\alpha \sum_{i} \alpha_i H_{\alpha - e_i} (\omega) e_i (t) = \sum_{\beta} \sum_{i} c_{\beta + e_i} (\beta_i + 1) e_i (t) H_{\beta} (\omega).
\]
From now on we denote $\nabla_I^H Q := E_{M_H}[D_I^H Q|\mathcal{F}_{I}^{M_H}]$.

**Theorem 6 (Fractional G-Clark-Ocone formula for polynomials)** Suppose $P(x) = \sum_c c_\alpha x^\alpha, x = (x_1, \ldots, x_n)$, is a polynomial. Consider $f_i \in L^2_{M_H}(R), 1 \leq i \leq n$, and $X_i(t) = \int_0^t f_i dB_H(t)$. Then $D_x \geq \int_0^T E_{M_H}[D_x^H F|\mathcal{F}_{I}^{M_H} | dB(t).$ (47)

**Proof.** From the Definition of the quasi-G-conditional expectation, we see that $F$ is $\mathcal{F}_T^{M_H}$ measurable, so

$$F(\xi) = E_{M_H}[F|\mathcal{F}_T^{M_H}] = \sum_\alpha c_\alpha E_{M_H}[X^\alpha|\mathcal{F}_T^{M_H}] = P^\alpha(X_T).$$

Thus we have

$$\int_0^T E_{M_H}[F|\mathcal{F}_T^{M_H}] | dB_H(t) = \int_0^T E_{M_H} \left[ \left( \sum_\alpha c_\alpha \frac{\partial P}{\partial \xi} \right) X^\alpha f_i(t) \right] | dB_H(t)$$

$$= \int_0^T \frac{d}{dt} P^\alpha(X(t)) | dt$$

$$= P^\alpha(X(T)) - P^\alpha(X(0))$$

$$= F - E[F]. \quad \square$$

Using the similar argument in Aase et al. [11] (2000), Hu & Øksendal [25] (2003), and Elliott & Van der Hoek [15] (2004), we can establish the following fractional G-Clark-Ocone Theorem

**Theorem 7 (Fractional G-Clark-Ocone Theorem)** (a) Suppose $Q \in G^*(M_H)$ is $\mathcal{F}_T^{M_H}$ measurable. Then $D_t^H Q \in G^*(M_H)$ and $\hat{E}_{M_H}[D_t^H Q|\mathcal{F}_{I}^{M_H}] \in G^*(M_H)$ for almost all $t$. $\hat{E}_{M_H}[D_t^H Q|\mathcal{F}_{I}^{M_H}] \circ W_H(t)$ is integrable in $S^*$ and

$$Q = \hat{E}[Q] + \int_0^T \hat{E}_{M_H}[D_t^H Q|\mathcal{F}_{I}^{M_H}] \circ W_H(t) dt.$$ (48)

(b) Suppose $Q \in L^2_{M_H}$ is $\mathcal{F}_T^{M_H}$ measurable. Then $\hat{E}_{M_H}[D_t^H Q|\mathcal{F}_{I}^{M_H}](\omega) \in L^1_{M_H}(R)$ (the definition of $L^1_{M_H}(R)$ is the analogue in Elliott & Hoek [15]) and

$$Q = \hat{E}[Q] + \int_0^T \hat{E}_{M_H}[D_t^H Q|\mathcal{F}_{I}^{M_H}] | dB_H(t).$$ (49)
4 G-Girsanov’s Theorem

In this section we will construct G-Girsanov’s Theorem, a analogue in a special situation was proposed in [9] defined from PDE and [25] in a view of SDE. For \( \phi \in L^2(\mathbb{R}) \), setting

\[
B^G(t) := B(t) - \int_0^t \phi(s) ds,
\]

(50)

where \( B(t) \) is a G-Brownian motion in sublinear space \((\Omega, \mathcal{H}, \mathcal{F}, \hat{E})\), \( \mathcal{F}_t = \sigma\{B(s), s \leq t\} \). We will construct a time consistent G expectation \( E^G \), and transfer the sublinear expectation space \((\Omega, \mathcal{H}, \hat{E})\) to the sublinear expectation space \((\Omega, \mathcal{H}, E^G)\) such that \( B^G(t) \) is a G-Brownian motion on \((\Omega, \mathcal{H}, E^G)\).

Define a sublinear function \( G(\cdot) \) as follows

\[
G(\alpha) = \frac{1}{2}(\overline{\alpha}^2 - \overline{\alpha}^-), \quad \forall \alpha \in R.
\]

(51)

For given \( \varphi \in C_{b, lip}(R) \), we denote \( u(t,x) \) as the viscosity solution of the following G-heat equation

\[
\partial_t u - G(\partial_x u) = 0, \quad (t,x) \in (0,\infty) \times R,
\]

\[
u(0,x) = \varphi(x).
\]

(52)

Remark 6 The G-heat equation \([52]\) is a special kind of Hamilton-Jacobi-Bellman equation, also the Barenblatt equation except the case \( \sigma = 0 \) (see \([53]\) and \([6]\)). The existence and uniqueness of \([52]\) in the sense of viscosity solution can be found in, for example \([17], [10], \) and \([40]\) for \( C^{1,2} \)-solution if \( \sigma > 0 \).

The stochastic path information of \( B^G(t) \) up to \( t \) is the same as \( B(t) \), without loss of generality we still denote \( \mathcal{F} \) as the path information of \( B^G(t) \) up to \( t \). Consider the process \( B^G(t) \), we define \( E^G[\cdot]: \mathcal{H} \longrightarrow R \) as

\[
E^G[\varphi(B^G(t))] = u(t,0), t \in (0,+\infty)
\]

and for each \( t, s \geq 0 \) and \( 0 < t_1 < \cdots < t_N \leq t \)

\[
E^G[\varphi(B^G(t_1), \ldots , B^G(t_N), B^G(t+s) - B^G(t))] := E^G[\varphi(B^G(t_1), \ldots , B^G(t_N))]
\]

where \( \psi(x_1, \ldots , x_N) = E^G[\varphi(x_1, \ldots , x_N, B^G(s))] \).

For \( 0 < t_1 < t_2 < \cdots < t_i < t_{i+1} < \cdots < t_N < +\infty \), we define G conditional expectation with respect to \( \mathcal{F}_t \) as

\[
E^G[\psi(B^G(t_1), B^G(t_2) - B^G(t_1), \ldots , B^G(t_{i+1}) - B^G(t_i), \ldots , B^G(t_N) - B^G(t_{N-1})|\mathcal{F}_t] = E^G[\psi(B^G(t_1), B^G(t_2) - B^G(t_1), \ldots , B^G(t_i) - B^G(t_{i-1})]
\]

where \( \psi(x_1, \ldots , x_i) = E^G[\varphi(x_1, \ldots , x_i, B^G(t_{i+1}) - B^G(t_i), \ldots , B^G(t_N) - B^G(t_{N-1})] \).

By the comparison theorem of the G-heat equation \([52]\) and the sublinear property of the function \( G(\cdot) \), we consistently define a sublinear expectation \( E^G \) on \( \mathcal{H} \). Under sublinear expectation \( E^G \) we define above \( B^G(t) \) is a G-Brownian motion and \( B^G(t) \) is \( N(\{0\}, \alpha^2, \sigma^2) \) distributed. We call \( E^G[\cdot] \) as G-expectation on \((\Omega, \mathcal{H})\). Without loss generality, we denote \( B^G(t) \) as two-sided G-Brownian motion.
With a similar argument used in Section 2, we denote the Banach space \((\hat{\mathcal{H}}_p, \| \cdot \|_p)\) as \(L^p_G(\Omega, \hat{\mathcal{H}}_p, E^G)\), and \(B^G(t)\) is a G-Brownian motion in \(L^p_G(\Omega, \hat{\mathcal{H}}_p, E^G)\), consequently,

\[
B^G_H(t) := \int_R M_H I^{(0,J)}(s) dB^G(s) = \int_R M_H I^{(0,J)}(s) dB(s) - \int_R M_H I^{(0,J)}(s) \phi(s) ds
\]

is a fGBm under G-expectation \(E^G\).

To eliminate a drift in the fGBm we need to solve equations of the form

\[
\int_R M_H I^{(0,J)}(s) \phi(s) ds = g(t).
\]

That is

\[
\int_0^t M_H \phi(s) ds = g(t).
\]

Consequently, with \((M_H \phi)(t) = g'(t)\)

\[
\phi(t) = (M_H^{-1} g')(t).
\]

If \(g'(t) = A\) on \([0, T]\)

\[
\phi(t) = AM_{1-H} I^{(0,T)}(t) = \frac{A(\sin(\pi(1-H)) \Gamma(3-2H))^{\frac{1}{2}}}{2 \Gamma(\frac{1}{2}-H) \cos(\frac{\pi}{2}(\frac{1}{2}-H))} \left[ \frac{T-t}{|T-t|^\frac{1}{2} + t} \right],
\]

and for \(0 \leq t \leq T\)

\[
\phi(t) = \frac{A(\sin(\pi(1-H)) \Gamma(3-2H))^{\frac{1}{2}}}{2 \Gamma(\frac{1}{2}-H) \cos(\frac{\pi}{2}(\frac{1}{2}-H))} \left[ (T-t)^{\frac{1}{2}-H} + (t)^{\frac{1}{2}-H} \right].
\]

**Theorem 8** (G-Girsanov Theorem) Assume that in the sublinear expectation space \((\Omega, \mathcal{H}, \hat{\mathcal{E}})\) the stochastic process \(B_H(t)\) is a fGBm with Hurst index \(H \in (0, 1)\), then there exists G-expectation \(E^G\), such that, in the G-expectation space \((\Omega, \mathcal{H}, \hat{\mathcal{E}})\)

\[
B^G(t) := B(t) - \int_0^t \phi(s) ds,
\]

is a G-Brownian motion with notation \(B(t) = B^G(t)\), and

\[
B^G_H(t) := \int_R M_H I^{(0,J)}(s) dB(s) - \int_R M_H I^{(0,J)}(s) \phi(s) ds
\]

is a fGBm under G-expectation \(E^G\) with drift \(\int_R M_H I^{(0,J)}(s) \phi(s) ds = g(t)\), and \(\phi(t) = (M_{1-H} g')(t)\).

## 5 Financial Application

We start from a family of probability space with equivalent probability measure \(\{(S^0(R), S(R), \mathcal{F}, P^\theta) : \theta \in \Theta\}\) where \(P^\theta\) is an induced probability by \(\int_0^t \sigma^\theta dB^\theta(t)\) with \(B^\theta(t)\) be a standard Brownian motion in reference space \((S^0(R), S(R), \mathcal{F}, P_0)\), \(\mathcal{F}\) is the augment filter constructed from the Brownian motion \(B^0(t)\), and \((\sigma^\theta)_{\theta \in \Theta, \theta_0 \geq 0}\) are unknown processes parameterized by \(\theta \in \Theta\), and \(\Theta\) is a nonempty convex set. Here \(\sigma^\theta_0\) means the uncertainty of the volatility of the process \(\sigma^\theta_0 B^0(t)\), we assume that
Assumption 1 (H1) Assume that \((\sigma_t^0)_{t \geq 0}\) is an adapt process for \(\mathcal{F}_t\), and satisfying
\[
\sigma_t^0 \in [\underline{\sigma}, \overline{\sigma}], \text{for all } \theta \in \Theta.
\]
We set
\[
\hat{E}[X] = \sup_{\theta \in \Theta} E_\theta[X],
\]
where \(E_\theta\) is the correspond linear expectation of probability \(P^\theta\). Thus \(((S'(R), S(R), \mathcal{F}, \hat{E}))\) is a sublinear expectation space, Denis, Hu and Peng [13] prove that \(\sigma_t^0B^t(\theta)\) is a \(G\)-Brownian motion in sublinear expectation space \(((S'(R), S(R), \mathcal{F}, \hat{E}))\), we denote \(B^t_\theta := \sigma_t^0B^t(\theta)\). Set \(B^t_H(t)\) is fBm with Hurst index \(H \in (0, 1)\) in \(((S'(R), S(R), \mathcal{F}, P_0))\), then it is easy to prove that \(\sigma_t^0B^t_H(\theta)\) is a fGBm with Hurst index \(H\) in \((\Omega, \mathcal{F}, \hat{E})\), we denote it as \(B^t_H(t)\).

We consider an incompleted financial market which contains a bond \(P(t)\) with
\[
dP(t) = rP(t)dt, \quad 0 \leq t \leq T,
\]
and a stock whose price \(S(t)\) with uncertain volatility and satisfying the following SDE driven by a fGBm:
\[
dS(t) = S(t) \circ [\mu dt + dB^t_H(t)]
= S(t) \circ [\mu + W^t_H(t)]dt, \quad 0 \leq t \leq T,
\]
\[
S(0) = x,
\]
which equivalent to a family of SDE with \(\theta \in \Theta\)
\[
dS(t) = S(t) \circ [\mu dt + \sigma^0_t dB^t_H(\theta)]
= S(t) \circ [\mu + \sigma^0_t W^t_H(\theta)]dt, \quad 0 \leq t \leq T,
\]
\[
S(0) = x,
\]
where \(W^t_H(\theta)\) is fractional noise with respect to fBm \(B^t_H(\theta)\).

By G-Girsanov’s Theorem, there exists G-expectation \(E^G[\cdot]\) and G-Brownian motion \(B^G(t)\) in the sublinear expectation space \(\mathcal{L}_p^G(S'(R), S(R), E^G)\) \((p \geq 1)\), such that
\[
B^G(t) = (\mu - r)t + B(t).
\]
Under \(E^G\)
\[
B^G_H(t) := (\mu - r)t + B_H(t), \quad \text{or } W^G_H := (\mu - r) + W_H(t),
\]
is a fGBm with Hurst index \(H\), and
\[
\phi(t) = (r - \mu)M_{1-H_I(0, T)}(t).
\]
Then we have
\[
dS(t) = S(t) \circ [r dt + dB^G_H(t)]
= S(t) \circ [r + W^G_H(t)]dt, \quad 0 \leq t \leq T.
\]
By fractional G-Itô formula
\[
S(t) = x \exp\left(rt + B^G_H(t) - \frac{1}{2}t^{2H}\right)
\]
is the solution to the SDE \[ 58 \]

Suppose that a portfolio \( \pi(t) = (u(t), v(t)) \) be a pair of \( \mathcal{F}_t^H \) adapted processes, where \( \mathcal{F}_t^H \) is the augment filter of fBm \( B_H^t(t) \). The corresponding wealth process is

\[
W^\pi(t) = u(t)P(t) + v(t)S(t),
\]

where \( \pi \) is called admissible if \( W^\pi(t) \) is bounded below for all \( t \in [0, T] \) and \( \pi \) is self-financing if

\[
dW^\pi(t) = u(t)dP(t) + v(t)S(t) \circ [r dt + dB_H^t(t)] = rW^\pi(t) dt + v(t)S(t) \circ W_H^t(s) ds.
\]

\[ e^{-rt}W^\pi(t) = W^\pi(0) + \int_0^t e^{-rs}v(s)S(s) \circ W_H^t(s) ds. \]

Suppose there is a negative and \( \mathcal{F}_t^H \) measurable contingent claim \( \xi \in \mathcal{L}_G^2(S'(R), S(R), E^G) \) with maturity \( T > 0 \) in the market, applying our fractional G-Clark-Ocone Theorem to \( e^{-rt}\xi(\omega) \), we have

\[ e^{-rT}\xi = E^G[e^{-rT}\xi + \int_0^T \hat{E}_{\mathcal{M}_t}[e^{-rT}D^H_t \xi | \mathcal{F}_t^H] \circ W_H^t(t) dt], \tag{59} \]

where \( \hat{E}_{\mathcal{M}_t} \) is the quasi-G-conditional expectation defined in Definition \( 23 \). \( D^H_t \) is the G-fractional Malliavin differential operator.

On the Probability framework, by Girsanov’s Theorem given in \( 15 \)

\[ \hat{B}_H(t) := \frac{\mu - r}{\sigma_H^0} t + B_H^0(t), \quad (\text{or } \hat{W}_H(t) := \frac{\mu - r}{\sigma_H^0} t + W_H^0(t)) \tag{60} \]

is a fBm with respect to the measure \( \hat{P}^0 \) by

\[ \frac{d\hat{P}^0}{P^0}(\omega) = \exp[<\phi^0, \omega> - \frac{1}{2}\|\phi^0\|^2] \]

where \( \phi^0(t) = \phi(t)/\sigma_H^0 \). We can rewrite the equation (60) as

\[ \sigma_H^0 \hat{B}_H(t) := (\mu - r)t + \sigma_H^0 B_H^0(t), \quad (\text{or } \sigma_H^0 \hat{W}_H(t) := (\mu - r)t + \sigma_H^0 W_H^0(t)). \]

Define

\[ E[X] := \sup_{\theta \in \Theta} E_{\hat{P}^0}[X], \quad X \in \mathcal{H}, \]

where \( E_{\hat{P}^0}[\cdot] \) is the linear expectation corresponding to the probability measure \( \hat{P}^0 \). Note that under the sublinear expectation \( E[\cdot] \), \( \sigma_H^0 \hat{B}_{1/2}(t) \) is a G-Brownian motion on the sublinear expectation space \( (S'(R), S(R), E^G) \). Notice that

\[ B^G(t) \equiv \sigma_H^0 \hat{B}_{1/2}(t) \]

and \( B^G(t) \) is G-Brownian motion on the sublinear expectation space \( (S'(R), S(R), E^G) \), thus

\[ E^G[\cdot] = E[\cdot], \]

i.e.,

\[ E^G[X] = \sup_{\theta \in \Theta} E_{\hat{P}^0}[X], \quad X \in \mathcal{H}. \tag{61} \]
It is easy to proof that \( \sigma^0 \hat{B}_H(t) \) is fGBm on the sublinear expectation space \((S(R), S(R), E^G)\).

By using the Girsanov’s transform \([55]\), the stock price process \([55]\) can be rewritten as
\[
\begin{align*}
    dS(t) &= S(t) \circ [rdt + \sigma^0 \hat{W}_H(t)] \\
    S(0) &= x.
\end{align*}
\]

Elliott and Hoek \([15]\), and Hu and Øksendal \([25]\) prove that
\[
E^\theta [e^{-rT} \xi] = \sup_{\theta \in \Theta} E^{\hat{P}}_{\theta} [e^{-rT} \xi] + \int_0^T \hat{E}^\theta_H [e^{-rT} D^H_t \xi | F^H_t] \circ W^G_H(t) dt,
\]
where \( \hat{E}^\theta_H \) is the quasi-conditional expectation, and \( D^H_t \) is the fractional Malliavin differential operator defined in \([15]\).

From \([61]\), we have
\[
E^G [e^{-rT} \xi] = \sup_{\theta \in \Theta} E^{\hat{P}}_{\theta} [e^{-rT} \xi],
\]
thus, notice \([59]\), \( E^G [e^{-rT} \xi] \) is the bid price of the claim \( \xi \) with
\[
v(t) = S(t)^{-1} e^{-r(T-t)} \hat{E}^\theta_H \xi | F^H_t \]
determines the portfolio in the super hedging. Similarly, we can derive that
\[
- E^G [-e^{-rT} \xi] = \inf_{\theta \in \Theta} E^{\hat{P}}_{\theta} [e^{-rT} \xi],
\]
is the ask price of the claim.

References

[1] Aase, K., Øksendal, B., Privault, N., and Ubøe, J. (2000) White noise generalizations of the Clark-haussmann-Ocone theorem with application to mathematical finance, Finance stoch. 4, 465-496.

[2] Artzner, Ph., Delbaen F., Eber J. M. (1999) Coherent measures of risk, Mathematical Finance. 9, 73-88.

[3] Avellaneda, M., Levy, A. and Parás, A. (1995) Pricing and Hedging Derivative Securities in Markets With Uncertain Volatilities, Applied Mathematical Finance. 2, 73-88.

[4] Bachelier, L. (1900). Theorie de la speculation. Ann. Sci. Ecole Norm. Sup. 17, 21-86.

[5] Barenblatt, G.I. (1978) Similarity, self-similarity and intermediate asymptotics, Consultants Bureau, New York (there exists a revised second Russian edition, Leningrad Gidrometeoizdat, 1982).
[6] Barenblatt, G.I. and Sivashinski, G.I. (1969) Self-similar solutions of the second kind in non-linear filtration, Applied Math. Mech. 33, 836-845 (translated from Russian PMM, pages 861-870).

[7] Black, F. and Scholes, M. (1973) The pricing of options and corporate liabilities, J. Political Economy, 81, 673-659.

[8] Chen, W. (2011) Time consistent G-expectation and bid-ask dynamic pricing mechanisms for contingent claims under uncertainty, Preprint arXiv:1111.4298v1.

[9] Chen, Z., Epstein, L. (2002) Ambiguity, risk and asset returns in continuous time, Econometrica, 70(4), 1403-1443.

[10] Crandall, M.G., Ishii, H., Lions, P.L. (1992) User’s guide to viscosity solutions of second order partial differential equations, Bull. Amer. Math. Soc. 27(1), 1-67.

[11] Cutland, N.J., Kopp P.E. and Willinger, W. (1995). Stock price returns and the Joseph effect: a fractional version of the Black-Scholes model. Progress in Probability 36, 327-351.

[12] Delbaen, F., Peng, S., Rosazza Gianin, E. (2010) Representation of the penalty term of dynamic concave utilities, Finance and Stochastic, 14(3), 449-472.

[13] Denis, L., Hu, M. and Peng, S. (2010) Function spaces and capacity related to a Sublinear Expectation: application to G-Brownian Motion Paths, Potential Analysis, 34, 139-161.

[14] Daubechies, I. (1992) Ten lectures on wavelets, S.I.A.M., Philadelphia.

[15] Elliott, R.J., Hoek, J.V. (2003) A general fractional white noise thoery and applications to finance, Mathematical Finance. 13, 301-330.

[16] Epstein, L., Ji, Shaolin. (2011) Ambiguous volatility, possibility and utility in continuous time, arXiv:1103.1652v4.

[17] Fleming, W., Soner, M. (1992) Controlled markov processes and viscosity solutions. Spring Verlag, New York.

[18] Ghashghaie, S., Breymann, W., Peinke, J., Talkner, P. and Dodge, Y. (1996) Turbulent cascades in foreign exchange markets, Nature, 381, 27 June, 767-770.

[19] Hem` andex, E. and Weiss, G. (1996) A first course on wavelets, CRC Press, Boca Raton, FL.

[20] Hille, E. (1958) A class of reciprocal functions, Ann. math. (2nd series), 27, 427-464.

[21] Holden, H., Øksendal, B., Ubøe, and Zhang T. (1996) Stochastic partial differential equations. Basel: Birkhauser.

[22] Hu, M., Ji, S., Peng, S., Song, Y. (2012) Comparison Theorem, Feynman-Kac Formula and Girsanov Transformation for BSDEs Driven by G-Brownian Motion, Preprint arXiv:1212.5403v1.

[23] Hurst, H.E. (1951) Long-term storage capacity in reservoirs. Trans. Amer Soc. Civil. Eng., 116, 400C410.

[24] Hurst, H.E., Black, R.P., Simaika, Y.M. (1965) Long Term Storage in Reservoirs. An Experimental Study. Constable, London.

[25] Hu, Y., Øksendal, B. (2003) Fractional white noise calculus and applications to finance. Infinite Dimensional Analysis, Quantum Probability and Related Topics, 6, 1-32.
[26] Itô, K. (1951) Multiple Wiener integral, J. Math. Soc. Japan 3, 157-169.

[27] El Karoui, Peng, S., Quenez, M.-C. (1997) Backward stochastic differential equations in finance, Mathematical Finance. 7, 1-71.

[28] Kolmogorov, A.N. (1940) The Wiener spiral and some other interesting curves in Hilbert space. Dokl. Akad. Nauk SSSR, 26, 115C118.

[29] Kolmogorov, A.N. (1941) The local structures of turbulence in incompressible viscous fluid for very large Reynold’s numbers, Comptes Rendus (Dokl) de l’Académie des Sciences de l’URSS, 30, 301-305.

[30] Los, C., Karuppiah, J. (1997) Wavelet multiresolution analysis of high frequency Asian exchange rates: working paper Dept. of Finance, Kent State University, OH.

[31] Lyons, T. J. (1995) Uncertain volatility and the risk-free synthesis of derivatives, Applied mathematical Finance, 2, 117-133.

[32] Mandelbrot, Benoit B., 1997. Fractals and Scaling in Finance, Springer New York.

[33] Mandelbrot, B.B., van Ness J.W.: Fractional Brownian motions, fractional noises and applications. SIAM Review, 10, 422C437 (1968)

[34] Merton, R.C. (1973). Theory of rational option pricing. Bell J. Econom. Management Sci. 4, 141-183.

[35] Thangavelu, S. (1993) Lectures on Hermite and laguerre expansions. Princeton, NJ: Princeton University Press.

[36] Peng, S. (1997) Backward SDE and related g-expectations. Backward stochastic differential equations, in El N. Karoui and L. Mazliak, eds, Pitman Res. notes Math. Ser. Longman Harlow, vol. 364, 141-159.

[37] Peng, S. (2006) Modelling derivatives pricing mechanisms with their generationg functions, in arXiv:math/0605599.

[38] Peng, S., (2004) Filtration Consistent Nonlinear Expectations and Evaluations of Contingent Claims. Acta Mathematicae Applicatae Sinica, English Series, 20(2), 1-24.

[39] Peng, S. (2005) G-Expectation, G-Brownian Motion and Related Stochastic Calculus of Ito Type, Stochastic Analysis and Applications, The Abel Symposium, 541-567.

[40] Peng, S. (2008) Multi-dimensional G-Brownian Motion and Related stochastic Calculus under G-Expectation, Stochastic Processes and their Applications, 118, 2223-2253.

[41] Peng, S. (2010) Nonlinear expectations and stochastic calculus under uncertainty - with robust central limit theorem and G-Brownian Motion, Preprint arXiv:1002.4546v1.

[42] Peng, S. (2009) Survey on normal distributions, central limit theorem, Brownian motion and the related stochastic calculus under sublinear expectations, Science in China Series A: Mathematics, 52(7), 1391-1411.

[43] Peng, S. (2011) G-Gaussian processes under sublinear expectation and q-Brownian motion in quantum mechanics, Preprint arXiv: 1105.1055v1.
[44] Peters, E. E. (1991) Chaos and order in the capital markets: a new view of cycles prices, and market volatility, Wiley, New York.

[45] Samuelson, P. A. (1965). Rational theory of warrant pricing. Industrial Management Review Vol. 6, No. 2, 13-31.

[46] Schwartz, R. A., Byrne, J. A., Colaninno A. (2011) Volatility risk and uncertainty in financial markets, Springer.

[47] Unser, M. A., Blu, T. (2003) Fractional wavelets, derivatives, and Besov spaces. Proc. SPIE 5207, Wavelets: Applications in Signal and Image Processing X, 147 (November 14, 2003).