Integers represented by positive-definite quadratic forms and Petersson inner products

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Abstract: We give a survey of results about the problem of determining which integers are represented by a given quaternary quadratic form $Q$. A necessary condition for $Q(x_1, x_2, x_3, x_4)$ to represent $n$ is for the equation $Q(x_1, x_2, x_3, x_4) = n$ to have a solution with $x_1, x_2, x_3, x_4 \in \mathbb{Z}_p$ for all $p$. But even when $n$ is sufficiently large, this is not sufficient for $Q$ to represent $n$. The form $Q$ is anisotropic at the prime $p$ if for $x_1, x_2, x_3, x_4 \in \mathbb{Z}_p$, $Q(x_1, x_2, x_3, x_4) = 0$ implies that $x_1 = x_2 = x_3 = x_4 = 0$. Suppose that $A$ is the Gram matrix for $Q$ and $D(Q) = \det(A)$. We show that if $n >> D(Q)^{6+\epsilon}$, $n$ is locally represented by $Q$, but $Q$ fails to represent $n$, then there is an anisotropic prime $p$ so that $p^2 | n$ and $np^{2k}$ is not represented by $Q$ for any $k \geq 1$. We give sharper results when $D(Q)$ is a fundamental discriminant and discuss applications to universality theorems like the 15 and 290 theorems of Bhargava and Hanke.