Experimental determination of the parameters of the nonlinearly elastic Hencky model

Yuri Astapov, Dmitrii Khristich
Department of Computational Mechanics and Mathematics, Tula State University, 92, Prospect Lenina, Tula 300012, Russia
E-mail: ast3x3@gmail.com, dmitrykhristich@rambler.ru

Abstract. This work presents a variant of physically nonlinear constitutive elasticity relations derived using the Hencky strain tensor. A test plant for determining the model’s constants is designed. These constants for weakly compressible materials are derived by compression tests of prismatic elastomeric specimens. The results of solving the problem of indentation of a rigid spherical stamp into a cylindrical specimen are provided. The suggested constitutive relations were used with the constants found by the compression tests. In addition, calculations were performed according to the relations known for the Hencky material and to the hypoelasticity relations with the neutral derivative of the stress tensor. To stabilize the calculation procedures, the Hencky tensor and its derivative were expanded by the monotonous parameter in a series by the Cauchy strain tensor degrees without any restrictions for allowable strain values. A numerical algorithm was built for tracking the variable area of contact. The calculation procedures of discretizing the stated problem by the finite elements method and step-by-step loading were implemented as an applied software suite. One of the integral characteristics of the process considered was the indentation curve defined as the dependence of the stamp immersion effort on the stamp immersion depth. The curves derived by computation are compared with the data of the indentation test conducted on the designed plant. It has been shown that the calculations according to the Hencky model and hypoelasticity relations provide a fair description of the curve to strains of around 30%. The suggested physical nonlinear model allows describing the continued strain growth to a sufficient degree of accuracy.

1. Introduction
Structural components exposed to finite elastic strains when in service play a major role in considering such issues as, e.g., water seal and vibration safety problems. These component parts are made from rubber-like materials characterized from the standpoint of deformable solid mechanics as weakly compressible isotropic elastic media. To accurately describe the response of these materials in contact with rigid matrices is an actual problem that affects the fabrication quality and application of component parts.

This work is aimed at developing the mathematical model to describe heterogeneous finite strains of nonlinearly elastic weakly compressible bodies, identifying the parameters of this model, and evaluating the model for confidence by comparing the computation data with the experimental data.

These models are based on constitutive relations connecting strains and stresses in elastomers. The choice of a variant of constitutive relations that would adequately describe the elastic behavior of specimens in such processes remains an issue in abeyance.
According to [1, 2, 3, 4], elastomers exposed to slow quasi-static loading are usually treated as nonlinearly elastic weakly compressible materials. The models of the nonlinear elasticity theory can be tentatively divided in three groups: generalization of Hooke’s law for finite strains by means of various finite strain and stress measures (e.g., Seth’s body), models of hypoelastic and hyperelastic materials.

Modern hypoelastic body models have been known since the middle of the past century. They were also developed in works by Truesdell, Green, Bernstein and Ericsson, Prager as well as in modern studies by Satya N. Atluri, Paul Steinmann, H. Xiao, O.T. Bruhns, A. Meyers, G.L. Brovko, A.A. Adamov, S.N. Korobeinikov, A.A. Markin, etc.

A material is referred to as hypoelastic when stress rates are homogeneous linear functions of strain rates [1, 5, 6, 7].

Usually, hyperelastic materials comprise models in which the existence of elastic potential is postulated and stresses are found through partial derivatives of elastic potential by invariants of the Cauchy–Green strain tensor. The best known hyperelastic body models are the Moony–Rivlin, Murnaghan, Signorini, Bartenev–Khazanovich, Chernykh–Shubina materials, etc. [6, 8]. It is known that in an isotropic material the corotational stress tensor is energetically conjugate with the Hencky logarithmic measure of strains. Consequently, it is proposed to record the connection between these tensors in hyperelastic body models in linear and nonlinear forms. The linear variant of relations is similar to the model of Neo-Hookean material and contains two material constants for an isotropic body. The nonlinear variant of constitutive relations is derived by retaining third-order terms in expansion for the specific potential energy.

An issue important to the considered elastomeric models is the identification of constitutive relations. Elastic constants are usually found by monoaxial and biaxial extension, bending, compression or indentation tests [9, 10, 11, 12]. In this article the constants of the material are found by the compression tests of a supposedly incompressible specimen on the built kinematic test bench. The model is verified by the example of solving the problem of embedding a rigid spherical indenter in a nonlinearly elastic cylinder. These processes are widely used to determine both the elastic characteristics of the material [9, 10, 12, 13, 14] and its hardness. The results of the calculations according to various nonlinear elasticity models will be compared with the experimental curves derived from the tests on the kinematic bench. This will allow making the conclusion about the adequacy of the suggested models and the proximity of the computed data with the test data.

2. Statement of the problem

Homogeneous compression and indentation can be treated as quasi-static processes [15]. To model the quasi-static response of the material to an external force action, it is possible to use the condition for the balanced course of strain. The system of equations [8] resulting from the requirement that not only the main vector of the loads applied to the body be equal to zero but also the variation rate of the vector upon the application of Jourdain’s principle is

\[
\begin{align*}
\int_V \left( \nabla \cdot \mathbf{S} + \rho \mathbf{F} \right) \cdot \delta \mathbf{v} \, dV &= 0, \\
\int_V \left[ \frac{d}{dt} \left( \nabla \cdot \mathbf{S} + \rho \mathbf{F} \right) \cdot \delta \mathbf{v} + \dot{\theta} \left( \nabla \cdot \mathbf{S} + \rho \mathbf{F} \right) \cdot \delta \mathbf{v} \right] \, dV &= 0,
\end{align*}
\]

(1)

where \( \mathbf{S} \) is the true Cauchy stress tensor; \( \mathbf{F}^{(n)} \) and \( \mathbf{F} \) are the external fields of surface and mass forces, respectively; \( \rho \) is the density; \( \mathbf{v} \) is the field of velocities of points in the medium; \( V \) is the volume taken by the body.

In its variational form the condition for the balanced course of strain (1) derived upon transforming the mass forces and leaving them beyond consideration is recorded as
\[
\int_V \left( \mathbf{S} + \mathbf{S} \dot{\theta} - \dot{\mathbf{v}} \nabla \cdot \mathbf{S} \right) \cdot \delta \left( \dot{\mathbf{v}} \nabla \right) dV = \int_{\Sigma} \left( \dot{P}^{(n)} + \ddot{P}^{\text{int}}(\dot{\theta} - \ddot{n} \cdot \mathbf{W} \cdot \ddot{n}) \right) \cdot \delta \mathbf{v} d\Sigma. \tag{2}
\]

This equation must be supplemented by the relations that connect the rate of stress variations with the rate of strains as well as by the initial and boundary conditions.

Consider that there are no stresses in the body at the initial moment of strain. According to static boundary conditions, the law of external load change must be preset in each point on the surface \( \Sigma_P \) as the function of time and Eulerian coordinates:

\[
\mathbf{P} = \mathbf{P}_0(\mathbf{x}, t) \quad \forall \mathbf{x} \in \Sigma_P, \quad \forall t > t_0. \tag{3}
\]

The kinematic boundary conditions in each point on the surface \( \Sigma_u \) determine the law of changes in material points movement

\[
\mathbf{u} = \mathbf{u}_0(\mathbf{x}, t) \quad \forall \mathbf{x} \in \Sigma_u, \quad \forall t > t_0. \tag{4}
\]

3. Variants of linear constitutive relations

To record constitutive relations of finite elastic behavior of elastomers, one usually makes use of pairs of energetically conjugate stress and strain tensors included in the specific stress power equation [8]. If to use the true Cauchy stress tensor \( \mathbf{S} \) as the stress measure, the specific stress power can be recorded as

\[
N^{(i)} = -\frac{1}{\rho} \mathbf{S} \cdot \mathbf{W},
\]

where \( \rho \) is the density of the material, \( \mathbf{W} = \frac{1}{2} \left( \nabla \mathbf{v} + \mathbf{v} \nabla \right) \) is the strain rate tensor. As a general matter, the \( \mathbf{W} \) tensor is not the absolute time derivative of any of the known finite strain measures.

In works [16, 17, 18] the \( \mathbf{S} \) and \( \mathbf{W} \) tensors are used to build constitutive hypoelasticity relations linearly connecting stress and strain rates as

\[
\mathbf{S}^* = \mathbf{N} \cdot \mathbf{W}, \tag{5}
\]

where \( \mathbf{N} \) is the fourth-order tensor of elastic constants, \( \mathbf{S}^* \) is the derivative of the tensor stress measure by the process parameter. Relations (5) must meet the material objectivity principle, i.e., the components of this measure must not change at hard turns of the strained medium [5, 8, 19], for which purpose the objective (corotational) derivative of the stress tensor in these components is used. The problem is to choose the type of this derivative, which is determined by choosing a rotating orthogonal basis relative to which the stress tensor change is found. The generalized Jaumann derivative [6] determines the tensor derivative relative to the polar basis \( \mathbf{n}_k, k = 1, \ldots, 3 \); the basis rotates at a rate of \( \Omega = \mathbf{R}^{-1} \cdot \mathbf{R} \) (\( \mathbf{R} \) is the orthogonal tensor included in the polar expansion of the strain affinor \( \Phi = \mathbf{U} \cdot \mathbf{R}, \mathbf{U} = \mathbf{U}^T, \mathbf{R}^T = \mathbf{R}^{-1} \)):

\[
\mathbf{S}^\Delta = \dot{\mathbf{S}} + \Omega \cdot \mathbf{S} - \mathbf{S} \cdot \Omega, \tag{6}
\]

Relation (5) for isotropic materials is recorded as

\[
\mathbf{S}^\Delta = K\dot{\theta} \mathbf{E} + 2G \left( \mathbf{W} - \frac{1}{3} \bar{\dot{\theta}} \mathbf{E} \right). \tag{7}
\]

The invariant \( \dot{\theta} = I_1(\mathbf{W}) \) is the elementary volume change rate.

As shown in works [7, 20], the energetically conjugate tensors in an isotropic material are the generalized corotational stress tensor \( \Sigma_R = \frac{\partial N^{(i)}}{\partial \mathbf{R}} \cdot \mathbf{S} \cdot \mathbf{R}^{-1} \) and the Hencky logarithmic strain tensor \( \Gamma = \ln \mathbf{U} \), where \( \mathbf{U} \) is the left deviation measure included in the polar expansion of the strain affinor \( N^{(i)} = -\frac{1}{\rho_0} \Sigma_R \cdot \Gamma \). If to treat the potential strain energy as the analytical function of the Hencky strain tensor, one will derive the following expansion

\[
\int V \left( \mathbf{S} + \mathbf{S} \dot{\theta} - \dot{\mathbf{v}} \nabla \cdot \mathbf{S} \right) \cdot \delta \left( \dot{\mathbf{v}} \nabla \right) dV = \int \Sigma \left( \dot{P}^{(n)} + \ddot{P}^{\text{int}}(\dot{\theta} - \ddot{n} \cdot \mathbf{W} \cdot \ddot{n}) \right) \cdot \delta \mathbf{v} d\Sigma. \tag{2}
\]
where $W_0 = 0$, $A = 0$ for the initial unstressed state and $N$ and $L$ are the tensors of the elastic constants of the material. If to keep in expansion (8) only the second-order term relative to the logarithmic strain tensor, then, using the fact that $\Sigma_R = \frac{\partial W}{\partial \dot{\Gamma}}$, and taking into account the expression for the components of the $N$ tensor through the $K$ and $G$ constants in an isotropic material, the resulting linear constitutive relations will be recorded as

$$\Sigma_R = K \dot{\theta} E + 2G \left( \Gamma - \frac{1}{3} \dot{\theta} E \right),$$  \hspace{1cm} (9)

where $\theta = I_1(\Gamma) = \ln \frac{\partial W}{\partial \dot{\Gamma}}$ is the first invariant of the Hencky tensor.

To use relations (9) in variational relations (2), it is necessary to differentiate (9) by time on the assumption that the constants of the material do not change. The variant of these relations in velocities results from (9) and is recorded as

$$\dot{\Sigma}_R = K \dot{\theta} E + 2G \left( \dot{\Gamma} - \frac{1}{3} \dot{\theta} E \right).$$  \hspace{1cm} (10)

4. Variant of nonlinear constitutive relations using the Hencky measure

The retention in expansion of specific potential energy (8) of the first two non-zero terms results in the following kind of constitutive relations:

$$\Sigma_R = \sigma_0 E + \tau_e \hat{\Gamma} + \tau_q Q,$$ \hspace{1cm} (11)

where $\sigma_0$, $\tau_e$, $\tau_q$ are the functions of the invariants of the strain measure, including the relative change of volume $\theta$ and the forming intensity $e = \sqrt{\Gamma \cdot \dot{\Gamma}}$; these functions are determined as $\sigma_0 = K \theta + \frac{C_1}{6\sqrt{3}} \theta^2 + \frac{C_2}{6\sqrt{3}} e^2$, $\tau_e = 2G + \frac{C_2}{3\sqrt{3}} \theta$, $\tau_q = C_3$. The $\hat{\Gamma}$ is the deviator of the logarithmic strain tensor and the tensor $Q$ is defined as the deviator of the tensor $\hat{\Gamma}^2$.

Formal differentiation of (11) leads to the form of constitutive relations that is recorded in velocities as

$$\dot{\Sigma}_R = \left[ \left( K - \frac{2}{3} G \right) \dot{\theta} + \frac{C_1}{3\sqrt{3}} - \frac{2C_2}{9\sqrt{3}} + \frac{1}{3} C_3 \right] \dot{\theta} E + \left( \frac{C_1}{3\sqrt{3}} - \frac{2}{3} C_3 \right) \dot{e} \dot{\Gamma} + \left( \frac{C_1}{3\sqrt{3}} - \frac{2}{3} C_3 \right) \dot{\theta} \dot{\Gamma} + \frac{2G}{3\sqrt{3}} \dot{\Gamma} \left( \dot{\Gamma} \cdot \dot{\Gamma} + \Gamma \cdot \dot{\Gamma} \right).$$ \hspace{1cm} (12)

In case of incompressible or weakly compressible materials in relation (11) it should be assumed that $\theta = 0$; thus the connection between the stress deviator and the strain tensor for this class of materials is recorded as

$$\dot{\Sigma}_R = 2G \ddot{\Gamma} + C_3 \left( \dot{\Gamma} \cdot \dot{\Gamma} + \Gamma \cdot \dot{\Gamma} \right).$$ \hspace{1cm} (13)

In relations (11), (12) one must determine the material constants $K$, $G$, $C_1$, $C_2$, $C_3$. The first two of them are the constants of the linear elasticity theory, whereas $C_1$, $C_2$, $C_3$ are responsible for additional nonlinear terms in constitutive relations.

If to insert constitutive relations (7), (9), and (11) in the condition for the balanced course of strain, the resolvent equations will be recorded as
\[
\int_V \left[ K \dot{\theta} E + 2G \left( W - \frac{1}{3} \dot{\theta} E \right) - \Omega \cdot S + S \cdot \Omega + S \dot{\theta} - \vec{v} \nabla \cdot S \right] \cdot \delta \left( \vec{v} \nabla \right) dV = \\
= \int_\Sigma \left[ \dot{P}^{(n)} + \vec{P}^{(n)} \left( \dot{\theta} - \vec{n} \cdot W \cdot \vec{n} \right) \right] \cdot \delta \vec{v} d\Sigma,
\]

\[
\int_V \left( \frac{dV_0}{dV} R^{-1} \cdot \Sigma_R \cdot R - \Omega \cdot S + S \cdot \Omega - \vec{v} \nabla \cdot S \right) \cdot \delta \left( \vec{v} \nabla \right) dV = \\
= \int_\Sigma \left[ \dot{P}^{(n)} + \vec{P}^{(n)} \left( \dot{\theta} - \vec{n} \cdot W \cdot \vec{n} \right) \right] \cdot \delta \vec{v} d\Sigma,
\]

where relations (14) correspond to the hypoelastic material model with constitutive relations (7) and relations (15) correspond to the nonlinearly elastic material models with constitutive relations (9)–(11).

5. Determining constants of the material by compression test

To conduct the tests on full-scale specimens, a kinematic bench was constructed for compression and indentation tests of prismatic specimens [11]. For the outward appearance of the plant see figure 1a. The bench design consists of two guides with a rigidly fixed work face on one side and a traveling piston rod with changeable heads on the other. The piston rod is firmly connected to strain-gage sensors and an optic displacement sensor. For the main specifications of the bench see table 1.

| Piston rod operating stroke, mm | Displacement sensor resolution, un/mm | Maximum allowable force, N | Strain-gage sensor resolution, un/N |
|---------------------------------|--------------------------------------|-----------------------------|-----------------------------------|
| 78                              | 48                                   | 0.75                        | 100                               |

Table 1. Kinematic bench specifications.

The crude rubbers with a low volumetric impurity content considered in this work can be referred to incompressible materials. In this case the identification of a material confines to determining the \( G \) and \( C_3 \) constants in relations (13). A series of tests was conducted on 20 × 5 × 5 mm rectangular block specimens. The bearing surfaces of the slabs between which the compression occurred were lubricated with mineral oil to reduce friction and, therefore, to reduce the impact of shearing stresses in the boundary regions.

The stressed state was approximately considered homogeneous and described in the following stress and strain fields: \( \Sigma_R^1 = P, \Sigma_R^2 = \Sigma_R^3 = 0, \Gamma_1 = \ln(\lambda), \Gamma_2 = \Gamma_3 = -\frac{1}{2} \ln(\lambda). \)

The connection of the true stress \( S_{11} \) with the elongation ratio \( \lambda \) for the nonlinear Hencky model is found using relation (11) as

\[
S_{11} = 3G \ln(\lambda) + \frac{3}{4} C_3 \ln^2(\lambda).
\]

The results of the tests are presented as a fitted curve in figure 1b. The constant values are found as \( G = 2 \cdot 10^6 \) Pa and \( C_3 = -2 \cdot 10^6 \) Pa. The least square method was used to find the approximation to the experimental data using relations (13).
Figure 1. a) Bench for compression and indentation tests of soft rubber specimens; b) Axial force-to-elongation rate relation in the compression.

The red curve in the diagram corresponds to the specific case of (16) when $C_3 = 0$, i.e., to the linear Hencky model. The $G$ and $C_3$ are determined so that relations (16) approximate the measured curve in the best possible manner.

6. Expanding the logarithmic strain measure in a series by degrees of the Hencky strain tensor

In general, to calculate the components of the Hencky tensor, it is necessary to go over to the main axes of the strain measure $U$. However, even in case of relatively simple processes there is ambiguity in choosing the directions of the main tensor axes, which may cause uncertainties and even errors overcomable only under additional conditions. The Hencky measure is related to the other strain measures as $\Gamma = \ln U = \frac{1}{2} \ln G = \frac{1}{2} \ln (E + 2\varepsilon)$, where $\varepsilon = \frac{1}{2} (G - g)$ is the Cauchy strain tensor, $G = \Phi \cdot \Phi^T$ is the Cauchy strain measure, and $g$ is the metric tensor of the reference configuration. By using the MacLaurin series for the logarithmic function, one can represent the function $\Gamma(\varepsilon) = \frac{1}{2} \ln (E + 2\varepsilon)$ as a tensor series convergent at $\|\varepsilon\| < \frac{1}{2}$. However, this condition significantly restricts the class of considered problems. There is an approach known from [21] that allows expanding the series convergence region for the logarithmic function. In particular, at $\forall a > 0, z > 0$ the following convergent series is observed:

$$\ln (a + z) = \ln a + 2 \left( \frac{z}{2a + z} + \frac{1}{3} \left( \frac{z}{2a + z} \right)^3 + \frac{1}{5} \left( \frac{z}{2a + z} \right)^5 + \ldots \right).$$

The Hencky tensor is expanded in series (17) by the formal substitution recorded as
\[ \Gamma = \frac{1}{2} \ln (\mathbf{E} + 2\varepsilon) = \sum_{k=1}^{n} \frac{1}{2k-1} \mathbf{H}^{2k-1}, \]  

where \( \mathbf{H} = \varepsilon \cdot (\mathbf{E} + \varepsilon)^{-1} = (\mathbf{E} + \varepsilon)^{-1} \cdot \varepsilon \). The \( \mathbf{H} \) derivative by \( t \) is determined as

\[ \dot{\mathbf{H}} = (\mathbf{E} + \varepsilon)^{-1} \cdot \dot{\varepsilon} \cdot (\mathbf{E} + (\mathbf{E} + \varepsilon)^{-1} \cdot \varepsilon). \]

If to use the formula for differentiating the product of tensor functions \( \frac{d(\mathbf{H}^n)}{dt} = \sum_{i=1}^{n} \mathbf{H}^{i-1} \cdot \dot{\mathbf{H}} \cdot \mathbf{H}^{n-i} \), the logarithmic measure derivative will appear to have the following expansion:

\[ \dot{\Gamma} = \sum_{k=1}^{n} \frac{1}{2k-1} \left( \sum_{m=1}^{2k-1} \varepsilon^{m-1} \cdot (\mathbf{E} + \varepsilon)^{-m} \cdot \varepsilon \cdot (\mathbf{E} - (\mathbf{E} + \varepsilon)^{-1} \cdot \varepsilon) \cdot ((\mathbf{E} + \varepsilon)^{-1} \cdot \varepsilon)^{2k-m-1} \right) \]

Expansion (19) converges to values of the logarithmic strain measure derivative by parameter without any restrictions for strain values.

7. Numerical solution procedure

Boundary value problems (2)–(4), (12), (14)–(15) are numerically solved by the finite elements method. This method allows going over from variational problem (2)–(4), (12), (14)–(15) to solving the system of linear algebraic equations relative to unknown nodal velocities.

The discrete models of variational equations (2)–(4), (12), (14)–(15) are built using the direct stiffness approach [22]. A computational region is divided in a finite number of non-overlapping subregions. In each element the components of the velocity vector are determined in the finite-size space of the piecewise linear function. When considering elastomeric models, many researchers [10, 12] proceed to considering incompressible materials, which generally increases the number of resolvent equations of the system for grid methods and requires special methods of solving systems of equations with low conditioning numbers. This work considers the weakly compressible material with the Poisson ratio close to 0.5. It is undesirable to use triangular simplex elements for such materials because the numerical volume locking effect shows at some kinds of the stress-strain state. It is known from [23] that, with the Poisson ratio \( \nu \) coming close to 0.5, the increase in the rigidity of the axially symmetric quadrangular linear element with the interpolation polynomial \( v(r, z) = \alpha + \beta r + \gamma z + \Delta rz \) is by an order less than for the axially symmetric triangular linear element. The coefficients \( \alpha, \beta, \gamma, \) and \( \Delta \) are defined given that the velocity in the grid nodes is continuous. The distribution of velocities in the element found by setting as known the column vector of the nodal element velocities \( [V]_i = v_i, \)

\( i = 1, \ldots, 8, \) is \( v(r, z) = N_i(r, z)v_i, \) where \( N_i(r, z), i = 1, \ldots, 8 \) are the linear form functions of the finite element. The velocity field thus preassigned will be used to define in sequence the subintegral expressions in (14)–(15). The impact of volume locking was reduced by the reduced integration method, which corresponds to the integration by one central Gaussian point for the chosen quadrangular axially symmetric element.

The evolution of the process over time is approximated by step-by-step loading method that divides the process parameter in finite sections. In each time section nodal velocities are considered constant. There are no stresses at the initial time instant. Then the specimen is exposed to the axially symmetric action on the part of an absolutely rigid stamp the shape of which is defined by the piecewise smooth curve \( f(\xi), \) where \( \xi \) is the monotonous parameter (see figure 2). The change in the contact zone is tracked at each loading step together with fulfilling the inequation \( f(\bar{x}) > 0; \) this inequation is the condition for the reciprocal nonpenetration of the elastic specimen and the rigid stamp, where \( \bar{x} \in V. \) When a point of the boundary falls within an introduced slip band with a relatively small width \( \varepsilon, \) this point is considered sliding.
along the stamp generatrix given that the stamp yields a non-negative connection response. The displacements of contact nodal points are found by integrating velocities along the generatrix proceeding from a sufficiently small per step change in the curvature of the function $f(\xi)$ [24].

The solution convergence is attained by refining the finite-element grid near the region with a supposedly high strain intensity.

This work considers the penetration of a spherical stamp into a cylindrical body lying on a rigid plane. For the computational pattern see figure 3.

The convergence of the numerical solution was studied by the condensing grid method.

8. **Verifying the numerical model by indentation test**

The loading pattern in figure 3 corresponds to the axially symmetric impression of a flat steel spherical stamp into a rectangular plate [21]. The problem was numerically solved by constitutive relations (10) and (12) for a weakly compressible material and with the aid of the above described
computational procedures. The models made use of the elastic constant values found by the compression tests. For the stress intensity distributions see figures 4a and 4b.

In figure 5 the points denote the data obtained by a range of tests in which $50 \times 50 \times 12$ mm rubber specimens were indented with a spherical head with a radius $R$ of 3mm. These data are represented as the dependence of the active force $P$ applied to the stamp on the penetration depth $D$.

The red curve corresponds to numerical solution (10) for the Hencky material, whereas the black curve corresponds to the solution using constitutive relations (12). The blue curve corresponds to hypoelastic constitutive relations (7). According to the plots, the dependences attained in the article according to physically nonlinear statement (12) introduce significant amendments to the linear solution for large strains and better describe the experimental data.
9. Conclusion
The article suggests a variant of the physically nonlinear Hencky model and its specific case for incompressible and weakly compressible materials. A plant was designed and built and a number of experiments were conducted to compress prismatic rubber specimens to define the model constants. According to the comparison of the experimental data with the results of the numerical modeled indentation of rubber specimens with the spherical indenter, the models of this process that are built using Hencky relations describe the indentation curve fairly well to strains of around 30%. Solutions using linear Hencky relations and hypoelastic relations are close. In case of larger strains the experimental data are described in the best manner by the model using the suggested physically nonlinear relations. As compared with the contribution of nonlinear summands, the accounting of contact friction introduces minor inaccuracies in the calculation of elastic constants.

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