Automated Analysis of MUTEX Algorithms with \textsc{FASE} *

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In this paper we study the liveness of several MUTEX solutions by representing them as processes in PAFAS, a CCS-like process algebra with a specific operator for modelling non-blocking reading behaviours. Verification is carried out using the tool \textsc{FASE}, exploiting a correspondence between violations of the liveness property and a special kind of cycles (called \textit{catastrophic cycles}) in some transition system. We also compare our approach with others in the literature. The aim of this paper is twofold: on the one hand, we want to demonstrate the applicability of \textsc{FASE} to some concrete, meaningful examples; on the other hand, we want to study the impact of introducing non-blocking behaviours in modelling concurrent systems.

1 \textbf{Introduction}

MUTEX algorithms can exhibit an intricate behaviour and their correctness can be hard to establish, because our intuitive notion of the program flow can be misled by the fact that a shared variable may change from one statement to the other, even if the process we are tracing does not modify it. There are two kinds of properties to verify: the \textit{safety property} that two competing processes are never in their critical sections at the same time, and the \textit{liveness property} that a requesting process will always enter its critical section. The first kind of property can be proven fairly easily because only the static configuration of the system at any time must be taken into account. The liveness property is much more difficult to prove since it usually requires some fairness assumption.

In [8], we have developed the process description language PAFAS, a CCS-like [11] process algebra originally introduced as a tool for evaluating the worst-case efficiency of asynchronous systems. Processes are compared via a variant of the testing approach of De Nicola and Hennessy [12] where tests are test environments (or user behaviours) together with a time bound. A process is embedded into the environment (via parallel composition) and satisfies a (timed) test, if success is reached before the time bound in \textit{every} run of the composed system, i.e. even in the worst case. This gives rise to a \textit{faster-than} preorder relation over processes that is naturally an \textit{efficiency preorder}. In [7] it has been shown that the test-based preorder in [8] can equivalently be defined on the basis of a performance function that gives the worst-case time needed to satisfy any test environment. Whenever the above testing scenario is adapted to a setting where tests belong to a very specific, but often occurring, class of \textit{request-response} user behaviours (processes serving these users receive requests via an \textit{in}-action and provide responses via an \textit{out}-action) this performance function is \textit{asymptotically linear}. This provides us with a quantitative measure of systems performance that measures how fast the system under consideration responds to requests from the environment. In [7] we have also shown how to determine this performance measure.

\*This work was supported by the PRIN Project ‘Paco:Performability-Aware Computing: Logics, Models, and Languages’.

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Giovanna D’Agostino, Salvatore La Torre (Eds.):
Proceedings of the Second International Symposium on
“Games, Automata, Logics and Formal Verification” (GandALF 2011)
EPTCS 54, 2011, pp. 45-59 doi:10.4204/EPTCS.54.4
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for finite-state processes. This result only holds for request-response processes (i.e. processes that can only perform in and out as visible actions) that pass certain sanity checks: they must not produce more responses than requests, and they must allow requests and provide responses in finite time. While the first requirement can easily be read off from the transition system, violation of the latter one is characterised as the existence of a special kind of cycles (called catastrophic cycles) in a reduced transition system (we remind the reader to [7] for the complete description of such a reduction). Finally, a corresponding tool FASE that allows the automated evaluation of systems performance function has been developed; see [2] for a first informal account.

The notion of timing in PAFAS is strongly related to (weak) fairness of actions which requires that an action must be performed whenever it is enabled continuously in a run. We have shown that each everlasting (or non-Zeno) timed process execution is fair and vice versa, where fairness is defined in an intuitive but complicated way in the spirit of [9, 10]. In fact, we have proven this correspondence for fairness of actions and, with a modified notion of timing, for fairness of components. These characterisations have been used in [5] to prove that Dekker’s algorithm is live under the assumption of fairness of components but not under the assumption of fairness of actions. This result can be improved by means of suitable assumptions about the hardware, namely we must assume that reading a value from a storage cell is non-blocking; to model this we have introduced specific reading prefixes for PAFAS in [6].

Here, we add reading in the form of a read-set prefix \{a_1,\ldots,a_n\} \triangleright P (the new process description language is called PAFAS_s) which behaves as P but, like a variable or a more complex data structure, can also be read with actions in the set \{a_1,\ldots,a_n\}. Since being read does not change the state, each action \(a_i (i = 1,\ldots,n)\) can be performed repeatedly until the execution of some ordinary action of \(P\).

A first key property of non-blocking actions is that they have a direct impact on timed behaviour of concurrent systems (see the examples at the end of Section 2). They are also an important feature for proving the liveness of MUTEX solutions under the assumption of weak fairness of actions. Indeed, one result in [6] shows that Dekker’s algorithm is live when assuming fairness of actions, provided we regard as non-blocking the reading of a variable as well as its writing in the case that the written value equals the current one. It had long been an open problem how to achieve such a result in a process algebra (see e.g. [13]). In [6] we have also discovered an interesting connection between liveness of MUTEX algorithms and catastrophic cycles; we have shown that violations of the liveness property can be traced back to catastrophic cycles of a suitably modified process (cf. Section 3). Even though FASE was originally developed for automatically checking whether a process of (original) PAFAS has a catastrophic cycle, it has been recently adapted to a setting with reading actions. This has opened the way to check automatically the liveness property for MUTEX algorithms.

In this paper we use FASE to study the liveness of four MUTEX solutions–Peterson’s, Lamport’s, Dijkstra’s and Knuth’s algorithms (see [13] and references therein)–under the assumption of fairness of actions. Our aim is twofold: we want to show the applicability of FASE to concrete, meaningful examples, but also to stress the impact of introducing non-blocking actions in PAFAS (and in general in modelling concurrent systems). We prove that Peterson is live provided we regard the reading of a variable as a non-blocking action. We also show that the liveness of Dijkstra and Knuth cannot be ensured even if (as in [6]) we consider as non-blocking the reading of a variable and its writing in the case the written value equals the current one. With the same assumption on program variables, we finally prove that Lamport (which is not symmetric) is live for just one of the two competing processes, i.e. it is not live.

To even more emphasize the role of non-blocking reading in proving liveness property, we have implemented some ideas taken from [13] that describe how fairness can be assumed in a CCS setting in order to enable a proof of liveness. At the time of writing, these ideas could not be expressed for the use of the Concurrency Workbench [4], but this is now possible within newer tools like the Concurrency Workbench. 
Workbench of the New Century [3]. A comparison of the results provided by the two approaches shows that the liveness of Dekker’s and Peterson’s algorithms strongly depends on the liveness of the hardware. This is exactly the sort of consideration for which non-blocking actions provide a formal treatment.

We proceed as follows: In Section 2 we recall PAFAS’, its timed operational semantics and the correspondence between fair traces and everlasting timed computations. In Section 3 we introduce the four algorithms and provide our results. Finally, in Section 4 we compare our approach with that in [13].

2 A process algebra for describing reading behaviours

PAFAS [8] is a CCS-like process description language [11] (with a TCSP-like parallel composition), where actions are atomic and instantaneous, but have associated an upper time bound (either 0 or 1, for simplicity) as a maximal delay for their execution. As shown in [8], these upper time bounds can be used to evaluate efficiency, but they do not influence functionality (which actions are performed); so compared with CCS also PAFAS treats the full functionality of asynchronous systems. In [6], PAFAS has been extended with a new operator ▷ to represent non-blocking behaviour of processes. Intuitively, \{α₁, ..., αₙ\} ▷ P models a process like a variable or a more complex data structure that behaves as P but can additionally be read with α₁, ..., αₙ: since being read does not change the state, each action αᵢ can be performed repeatedly without blocking a synchronization partner as described below.

We use the following notation: \( A \) is an infinite set of visible actions; the additional action τ represents a real-time activity, unobservable for other components, and \( A_τ = A \cup \{τ\} \). Elements of \( A \) are denoted by \( a, b, c, \ldots \) and those of \( A_τ \) by \( α, β, \ldots \) . Actions in \( A_τ \) can ‘let time’ pass before their execution, i.e. 1 is their maximal delay. After that time, they become urgent actions written \( a \) or \( τ \); these have maximal delay 0.

The set of urgent actions is denoted by \( A_{τ} = \{a \mid a ∈ A\} \cup \{τ\} \) and is ranged over by \( α, β, \ldots \).

Elements of \( A_{τ} ∪ A_τ \) are ranged over by ρ and v. We also assume that, for any \( α ∈ A_{τ} \), when time elapses \( α = α. \)

\( \mathcal{X} \) (ranged over by \( x, y, z, \ldots \)) is the set of process variables, used for recursive definitions. \( Φ : A_{τ} → A_{τ} \) is a general relabelling function if the set \( \{α ∈ A_{τ} \mid Φ(α) = α\} \) is finite and \( Φ(τ) = τ \). Such a function can also be used to define hiding: \( P/A \), where the actions in \( A \) are made internal, is the same as \( P[Φ_A] \), where the relabelling function \( Φ_A \) is defined by \( Φ_A(α) = τ \) if \( α ∈ A \) and \( Φ_A(α) = α \) if \( α ⊈ A \).

Below, initial processes are just processes of a standard process algebra extended with ▷, while general processes are those reachable from the initial ones according to the operational semantics. The set \( \mathcal{S}_i \) of initial (timed) process terms \( P \) and \( \mathcal{S}_i \) of (general) (timed) process terms \( Q \) are generated by:

\[
P := \text{nil} \mid x \mid α.P \mid \{α₁, ..., αₙ\} ▷ P \mid P + P \mid P ⊥ A P \mid P[Φ] \mid \text{rec x}.P
\]

\[
Q := P \mid α.P \mid \{μ₁, ..., μₙ\} ▷ Q \mid Q + Q \mid Q ⊥ A Q \mid Q[Φ]
\]

where nil is a constant, \( x ∈ \mathcal{X} \), \( α ∈ A_{τ} \), \( Φ \) is a general relabelling function and \( A ⊆ A \) possibly infinite; \{α₁, ..., αₙ\} and \{μ₁, ..., μₙ\} are (finite and nonempty) subsets of \( A_{τ} \) and \( A_{τ} ∪ A_τ \), resp.

We assume that the latter kind of read-sets can only contain a copy (either lazy or urgent) of each action α, i.e. \{μ₁, ..., μₙ\} cannot contain both \( α \) and \( α \) for any \( α ∈ A_{τ} \). By the operational semantics, terms not satisfying this property are not reachable from initial ones anyway. A process term is closed if every variable \( x \) is bound by the corresponding \text{rec x}-operator; the set of closed timed process terms in \( \mathcal{P}_1 \) and \( \mathcal{P}_1 \), simply called processes and initial processes resp., is denoted by \( \mathcal{P} \) and \( \mathcal{P}_1 \) resp.

\( \text{nil} \) is the Nil-process: it cannot perform any action but can let time pass without limits. \( α.P \) and \( α.P \) is action-prefixing known from CCS. Process \( α.P \) performs α within time 1; i.e. it can perform α immediately and evolve to \( P \) (as usual in CCS), or let one time unit pass and become \( α.P \). In this latter case, \( α \) cannot be further delayed (i.e. it must occur or be deactivated) unless \( α.P \) has to wait for a synchronisation on \( α \neq τ \). Our processes are patient: as a stand-alone process \( α.P \) has no reason to
wait, but as a component of a larger system, e.g. \(a.P\|_a a.\text{nil}\), it can wait for a synchronisation on \(a\); this can take up to time 1 since component \(a.\text{nil}\) can idle so long. \(\{\mu_1, \ldots, \mu_n\} \triangleright Q\) can perform actions from \(\{\mu_1, \ldots, \mu_n\}\) without changing state (including urgencies and, hence, the syntax of the term itself), and the actions of \(Q\) in the same way as \(Q\), i.e. the read-set is removed after such an action. \(Q_1 + Q_2\) is a non-deterministic choice between two conflicting processes \(Q_1\) and \(Q_2\). \(Q_1\) and \(Q_2\) run in parallel in \(Q_1\|\lambda Q_2\) and have to synchronize on all actions from \(A\). \(P[\Phi]\) behaves as \(P\) but with actions changed according to \(\Phi\). rec.\(x.P\) models a recursive definition; we often use equations to define recursive processes.

**Functional and temporal behaviour of PAFAS\(_s\) processes.** We first introduce the transitional semantics describing the functional behaviour of PAFAS\(_s\) processes, i.e. which actions they can perform.

**Definition 2.1** (functional operational semantics) Let \(Q \in \bar{S}\) and \(\alpha \in \bar{A}\). The SOS-rules defining the transition relation \(\alpha \subseteq (\bar{S} \times \bar{S})\) (the action transitions) are given in Table 1. As usual, we write \(Q \xrightarrow{\alpha} Q'\) if \((Q, Q') \in \alpha\) and \(Q \xrightarrow{\alpha}\) if there exists a \(Q' \in \bar{S}\) such that \((Q, Q') \in \alpha\). Similar conventions will apply later on. We also define the set of the activated or enabled actions to be the set of all \(\alpha\) such that \(Q \xrightarrow{\alpha}\).

\[
\begin{align*}
\text{PREF}_s & \quad \mu \in \{\alpha, \bar{\alpha}\} \quad \mu.P \xrightarrow{\alpha} P \\
\text{SUM_s} & \quad Q_1 \xrightarrow{\alpha} Q', Q_2 \xrightarrow{\alpha} Q' \quad \frac{Q_1 + Q_2 \xrightarrow{\alpha} Q'}{Q_1 \| \lambda Q_2 \xrightarrow{\alpha} Q_1 \| \lambda Q_2} \\
\text{READ_1} & \quad \mu_i \in \{\alpha, \bar{\alpha}\} \quad \{\mu_1, \ldots, \mu_n\} \triangleright Q \xrightarrow{\alpha} \quad \frac{\{\mu_1, \ldots, \mu_n\} \xrightarrow{\alpha} Q \xrightarrow{\alpha}}{\{\mu_1, \ldots, \mu_n\} \triangleright Q} \\
\text{READ_2} & \quad Q \xrightarrow{\alpha} Q' \quad \begin{cases} \alpha \notin \bar{A} & \text{PAR_1} \quad Q_1 \| \lambda Q_2 \xrightarrow{\alpha} Q_1 \| \lambda Q_2 \\
\alpha \in \bar{A} & \text{PAR_2} \quad \{Q_1, Q_2\} \xrightarrow{\alpha} Q_1 \| \lambda Q_2 \xrightarrow{\alpha} Q_2 \| \lambda Q_2 \\
\end{cases} \\
\text{REL_3} & \quad Q[\Phi] \xrightarrow{\Phi(a)} Q'[\Phi] \\
\text{REC_4} & \quad Q[\Phi.x/P] \xrightarrow{\Phi.x/P} Q' \xrightarrow{\alpha} Q' \\
\end{align*}
\]

Table 1: Functional behaviour of PAFAS\(_s\) processes

Rules in Table 1 are quite standard. Timing can be disregarded in \(\text{PREF}_s\); when an action is performed, one cannot see whether it was urgent or not, and thus \(\alpha.P \xrightarrow{\alpha} P\); furthermore, component \(\alpha.P\) has to act within time 1, i.e. it can also act immediately, giving \(\alpha.P \xrightarrow{\alpha} P\). Rules \(\text{READ}_1\) and \(\text{READ}_2\) say that \(\{\mu_1, \ldots, \mu_n\} \triangleright Q\) can either repeatedly perform one of its non-blocking actions or evolve as \(Q\). Other rules are as expected; symmetric rules have been omitted. Actually, the above SOS-rules describe reading in a sensible way only under some syntactic restrictions, cf. [6]. All the example processes we consider here meet these restrictions.

We now define the refusal traces of a term \(Q \in \bar{S}\). Intuitively a refusal trace records, along a computation, which actions process \(Q\) can perform (\(Q \xrightarrow{\alpha} Q', \alpha \in \bar{A}\)) and which actions \(Q\) can refuse to perform when time elapses (\(Q \xrightarrow{X} Q', X \subseteq \bar{A}\)). \(Q \xrightarrow{X} Q'\) is called a (partial) time-step. The actions listed in \(X\) are not urgent; hence \(Q\) is justified in not performing them, but performing a time step instead. This time step is partial because it can occur only in contexts that can refuse the actions not in \(X\). If \(X = \bar{A}\) then \(Q\) is fully justified in performing this time-step; i.e., \(Q\) can perform it independently of the environment. In such a case, we say that \(Q\) performs a \(1\)-step written \(Q \xrightarrow{1} Q'\); moreover we often write \(Q\) (the urgent version of \(Q\)) instead of \(Q'\). To provide the reader with a better intuition we observe that any \(Q\) can perform a \(1\)-step whenever it can refuse to perform, because not urgent, all its activated actions. In the next definition, \(\mathcal{U}(\{\mu_1, \ldots, \mu_n\}) = \{\alpha | \mu_i = \alpha\ \text{for some} \ i \in [1,n]\}\) is the set of urgent actions in \(\{\mu_1, \ldots, \mu_n\}\).

\[\text{1We do here without clean and unmark, used e.g. in [5] to get a closer relationship between states of untimed fair runs and timed non-Zeno runs. They do not change the behaviour (up to an injective bisimulation) and would complicate the setting.}\]
**Definition 2.2 (refusal transitional semantics)** The SOS-rules in Table 2 define $X_r \subseteq (\mathbb{S} \times \mathbb{S})$ where $X \subseteq A$.

\[
\begin{array}{c}
\text{NIL}_r \xrightarrow{\alpha} \text{NIL} \\
\text{PREF}_r \xrightarrow{\alpha.P \not\in X} \alpha.P \\
\text{PREF}_2 \xrightarrow{\alpha.P \not\in X} \alpha.P \\
\text{SUM}_r \xrightarrow{Q_1 \xrightarrow{\alpha} Q_i \text{ for } i = 1, 2} \frac{Q_1 + Q_2 \xrightarrow{\alpha} Q_i + Q_2}{Q_1 \cup Q_2 \xrightarrow{\alpha} Q_i + Q_2} \\
\text{REL}_r \xrightarrow{Q_i \xrightarrow{\alpha} Q' \text{ for } i = 1, 2, X \subseteq (A \cap (X_1 \cup X_2)) \cup ((X_1 \cap X_2) \backslash A)} \frac{Q_1 || Q_2 \xrightarrow{\alpha} Q_i || Q_2}{Q_1 || Q_2 \xrightarrow{\alpha} Q_i || Q_2} \\
\text{REC}_r \xrightarrow{\text{rec.x} \cdot Q / x} \frac{Q \xrightarrow{\alpha} Q'}{\text{rec.x} \cdot Q \xrightarrow{\alpha} Q'} \\
\end{array}
\]

Table 2: Refusal transitional semantics of PAFAS$_s$ processes

Rule **PREF$_1$** says that a process $\alpha.P$ can let time pass and can refuse to perform any action, while rule **PREF$_2$** says that a process $\alpha.P$, can let time pass but action $\alpha$ cannot be refused. Process $\tau.P$ cannot let time pass and cannot refuse any action; in any context, $\tau.P$ has to perform $\tau$ before time can pass further. Rule **PAR$_i$** defines which actions a parallel composition can refuse during a time-step. The intuition is that $Q_1||Q_2$ can refuse an action $\alpha$ if either $\alpha \not\in A$ ($Q_1$ and $Q_2$ can do $\alpha$ independently) and both $Q_1$ and $Q_2$ can refuse $\alpha$, or $\alpha \in A$ ($Q_1$ and $Q_2$ are forced to synchronise on $\alpha$) and at least one of them can refuse the action, i.e. can delay it. Thus, an action in a parallel composition is urgent (cannot be further delayed) only when all synchronising 'local' actions are urgent. Rule **READ$_i$** says that $\{\mu_1, \ldots, \mu_n\} \triangleright Q$ can refuse the same actions as $Q$ provided these are not urgent in $\{\mu_1, \ldots, \mu_n\}$; moreover, as for the action-prefixing, process $\{\mu_1, \ldots, \mu_n\} \triangleright Q$ cannot let time pass and cannot refuse any action, whenever one of the urgent actions in $\{\mu_1, \ldots, \mu_n\}$ is a $\tau$. Other rules are as expected. Again symmetric rules have been omitted.

In [8], it is shown that inclusion of refusal traces characterises a testing-based faster-than relation that compares processes w.r.t. their worst-case efficiency. In this sense, e.g. $P = \{a\} \triangleright b.nil$ is faster than the functionally equivalent $P' = \text{rec.x}.a.x + b.nil$, since only the latter has the refusal traces $1a(1a)^*$. After $1a$, $P'$ returns to itself (recursion unfolding creates fresh $a$ and $b$); intuitively, $b$ is disabled during the occurrence of $a$, so $a$ and also $b$ can be delayed again. In contrast, after a 1-step and any number of $a$'s, $P$ turns into $\{a\} \triangleright b.nil$ and no further 1-step is possible; read actions do not block or delay other activities, they make processes faster. If $a$ models the reading of a value stored by $P$ or $P'$ and two parallel processes want to read it, this should take at most time 1 in a setting with non-blocking reads. And indeed, whereas $P' || \{a\} (a.nil || \neg a.nil)$ has the refusal trace $1a(1a)$, this behaviour is not possible for $P || \{a\} (a.nil || \neg a.nil)$ since, when performing $1a$, this evolves into e.g. $\{a\} \triangleright b.nil || \{a\} (nil || \neg a.nil)$, and then 1 is not possible.

Another application of refusal traces is the modelling of weak fairness of actions. Weak fairness requires that an action must be performed whenever continuously enabled in a run. Thus, a run from $P$ above with infinitely many $a$'s is not fair; the read action does not block $b$ or change the state, so the same $b$ is always enabled but never performed. In contrast, if $P'$ performs $a$, a fresh $b$ is created; in conformance to [9], a run from $P'$ with infinitely many $a$'s is fair. In [6], generalising [5], fair traces for PAFASs are first defined in an intuitive, but very complex fashion in the spirit of [9, 10] and then characterised: they are the sequences of visible actions occurring in transition sequences with infinitely many 1-steps. Due to lack of space, we cannot properly formulate this as a theorem, but take it as a definition of fair traces instead. With this, infinitely many $a$'s are a fair trace of $P'$ since it can repeat $1a$ indefinitely, but the fair traces of finite-state $P$ are those that end with $b$. We use this definition of fair
traces to study liveness property of MUTEX solutions we consider in the next section.

For request-response processes the transition system (built according to Def. 2.1 and 2.2) must be reduced as described in [7]; a cycle in the resulting system is catastrophic if it contains (at least) one time step but no in- or out-transition.

3 Liveness property of MUTEX algorithms

In this section we use the approach of [6] to study the liveness of four different MUTEX solutions: Peterson’s, Lamport’s, Dijkstra’s and Knuth’s algorithm. We first translate the algorithms into PAFAS, and then use FASE to automatically decide whether each of them is live or not. Negative results are discussed by means of counterexamples, i.e. fair violating traces which are built from catastrophic cycles detected with FASE. The results of this section are collected in Table 3.

**Peterson’s algorithm** There are two processes P₁ and P₂, two Boolean-valued variables b₁ and b₂, whose initial value is false, and a variable k, which takes values in {1, 2} and whose initial value is arbitrary. The b₁ variables are “request” variables and k is a “turn” variable: b₁ is true if P₁ is requesting entry to its critical section and k is i if it is P₁’s turn to enter its critical section. Only P₁ writes b₁, but both processes read it. Process Pᵢ (with i = 1, 2) is described as follows; j is the index of the other process:

```
PETERSON()
1   while true
2     do  (non-critical section);
3       bᵢ ← true;   k ← j;
4     while bᵢ and k = j do skip;
5     (critical section);
6       bᵢ ← false;
```

In our translation of the algorithm into PAFAS, we use essentially the same coding as Walker in [13]. Each variable is represented as a family of processes. For example, the process B₁(f) denotes the variable b₁ with value false. The sort of B₁(f) (i.e. the set of actions it can ever perform) is {b₁rf, b₁rt, b₁wf, b₁wt} Unlike [13], we model the actions that correspond to the reading of a variable (e.g. b₁rt and b₂rt) as non-blocking. Below, we let $\mathbb{B} = \{ f, t \}$ and $\mathbb{K} = \{ 1, 2 \}$.

**Definition 3.1 (Peterson’s algorithm)** Let $i \in \{ 1, 2 \}$. Program variables are represented as follows:

- $B_i(f) = \{ b_{1rf}, b_{1rt}, b_{1wf}, b_{1wt} \}$
- $B_i(t) = \{ b_{2rf}, b_{2rt}, b_{2wf}, b_{2wt} \}$

Given $b₁, b₂ \in \mathbb{B}, k \in \mathbb{K}$, we define $PV(b₁, b₂, k) = B₁(b₁) \parallel B₂(b₂) \parallel K(k)$. Processes P₁ and P₂ are represented by the following PAFAS processes: the actions reqᵢ and csᵢ indicate the request to enter and the execution of the critical section by the process Pᵢ:

$$
P₁ = req₁.b₁wt.kw2.P₁₁ + τ.P₁  \quad P₂ = req₂.b₂wt.kw1.P₂₁ + τ.P₂
$$

$$
P₁₁ = b₁rf.P₁₁ + b₁rt.P₁₂  \quad P₂₁ = b₁rf.P₂₁ + b₁rt.P₂₂
$$

$$
P₁₂ = kr2.P₁₁ + kr1.P₁₃  \quad P₂₂ = kr1.P₂₁ + kr2.P₂₃
$$

$$
P₁₃ = cs₁.b₁wf.P₁  \quad P₂₃ = cs₂.b₂wf.P₂
$$

Since no process should be forced to request by the fairness assumption, P₁ has the alternative of an internal move, i.e. staying in its non-critical section. Peterson’s algorithm is defined to be the PAFAS process $\text{Peterson} = (P₁ \parallel P₂) \parallel PV(f, f, 1) / B$; here (and in the following) B is the set of all actions except reqᵢ and csᵢ (i = 1, 2). A MUTEX algorithm like Peterson’s satisfies liveness if, in every fair trace, each reqᵢ is eventually followed by the respective csᵢ.
We now show how to modify the process Peterson such that it is live under the assumption of fairness of actions iff the modified process, that we call Petersonio, does not have catastrophic cycles. Observe that FASE only accepts request-response behaviours (having only in and out as visible actions) as input and, hence, it cannot be applied directly. Moreover, Peterson can perform a 1-step followed by the two internal actions of P₁ and P₂ (see Def. 3.1) giving a catastrophic cycle which is not relevant for the liveness property. So, we modify Peterson as follows: we first change req₁ and cs₁ into τ’s and req₂ and cs₂ into in and out, resp.; we finally delete the τ summand of P₂. As in [6] (see Theorem 8.22), we can prove that Petersonio does not have catastrophic cycles iff each request from process P₂ will eventually be satisfied along fair traces, i.e. iff Peterson is live for process P₂ under the assumption of fairness of actions. The liveness of Peterson follows by the symmetry of the algorithm. In case of non-symmetric algorithms, as e.g. Lamport, the liveness for processes P₁ and P₂ must be proven separately. Since FASE has shown that Petersonio does not have catastrophic cycles, our first result is:

**Proposition 3.2** Peterson is live under the assumption of fairness of actions.

We now consider Peterson’, a slightly different specification of Peterson where all actions – including the reading of program variables – are ordinary actions. E.g., in this version, we define Bᵢ(f) = bᵢrf.Bᵢ(f) + bᵢwf.Bᵢ(f) + bᵢwt.Bᵢ(t). Then, Peterson’ can be defined as in Def. 3.1

**Proposition 3.3** Peterson’ is not live under the assumption of fairness of actions.

**Proof:** FASE shows that Petersonio has catastrophic cycles as, e.g., those in the next examples. □

The following example shows a timed computation along which both processes P₁ and P₂ get stuck after a request. To ease understanding, we leave the actions on program variables visible, i.e. we consider a timed computation of \( P = (P₁ || P₂) || PV(f,f,1) \). Indeed, by the operational semantics, we know that P behaves as Peterson’ as long as we rename actions in B with τ. We will proceed in this fashion later on in this section. Furthermore, we write PV(t₁,t₁) and PV(t₂,t₂) to abbreviate \((B₁(t₁) || B₂(t₂)) || K(1)\) and \((B₁(t₁) || B₂(t₂)) || K(1)\), resp. In general, we underline a value to denote the urgent version of the PAFASᵢ process that represents the corresponding variable.

**Example 3.4** Consider the following timed computation from P.

\[
P = (P₁ || P₂) || PV(f,f,1) \quad \xrightarrow{req₁ bᵢwt kw₁ req₂ bᵢwt kw₁} \quad P₁ || P₂ || PV(t,t,1) \quad \xrightarrow{Bᵢrf Bᵢwt} \quad R₁ \quad \xrightarrow{Bᵢrf} \quad R₂ \quad \xrightarrow{Bᵢrf} \quad R₃ \quad \xrightarrow{Bᵢrf} \quad R
\]

Process R can only perform kr₁ as a synchronisation between K(1) and either P₁₁ or P₂₂; after the first 1-step, this action becomes urgent. Once in R, we perform kr₁ and K(1) evolves into K(1) which can delay kr₁. As a consequence, Q can refuse to perform kr₁ and, since this is its only activated action, Q \xrightarrow{kr₁} R. The execution sequence Peterson’ = \( P/B \xrightarrow{τᵢq₁ τᵢq₂ τᵢ} R/B \xrightarrow{τᵢ} R/B \ldots \) is fair but not live since

\[\text{The proof of Theorem 8.2 we provide in [6] is partly independent from the specific algorithm we were analysing, i.e. Dekker’s algorithm, and it can be easily adapted to all the algorithms we consider in this paper. From now on, we freely use the correspondence between liveness and catastrophic cycles without explicitly proving it. In the following, if P is a PAFASᵢ process that models a given MUTEX solution, we write P₀ to denote the process we obtain by changing P as Peterson.} \]
no process will ever enter the critical section: \( R/B \xrightarrow{\tau} Q/B \xrightarrow{1} R/B \) corresponds to a catastrophic cycle in the reduced transition system of Peterson’s.

This example describes a scenario where process \( P_1 \) will never move because process \( P_2 \) repeatedly reads variables \( k \) and \( b_1 \). There is another fair run where \( P_1 \), reading variable \( b_2 \), can repeatedly delay and, thus, indefinitely block \( P_2 \) that wants to write it. On the contrary, the representation of program variables we use in Def. 3.1 ensures the liveness of the hardware under the assumption of fairness of actions; namely, it ensures that no process can be indefinitely blocked by infinite reading.

**Lamport’s algorithm** There are \( n \geq 2 \) processes and \( n \) Boolean-valued variables \( b_i \) \((i = 1 \ldots n)\), each with initial value false; only \( P_i \) writes \( b_i \), but all the processes can read it. The \( i \)-th process is described below:

```
LAMPORT()
1 var j : integer;
2 while true
3 do {non-critical section};
4 b_i \leftarrow true;
5 for j \leftarrow 1 \text{ to } i - 1
6 do if b_j
7 then b_i \leftarrow false;
8 do while b_j do skip;
9 goto 4;
10 for j \leftarrow i + 1 \text{ to } n
11 do while b_j do skip;
12 {critical section};
13 b_i \leftarrow false;
```

Now we provide the PAFAS\(_s\) specification in case of \( n = 2 \) processes.

**Definition 3.5** (Lamport’s algorithm) Again we first define the family of PAFAS\(_s\) processes representing the program variables. Let \( B_i(f) = \{ b_{i,rf}, b_{i,wf} \} > b_{i, wf}. B_i(t) \) and \( B_i(t) = \{ b_{i,rt}, b_{i,wt} \} > b_{i, wf}. B_i(f) \) where \( i \in \{ 1, 2 \} \). We also define \( PV(b_1, b_2) = B_1(b_1) \parallel B_2(b_2) \) where \( b_1, b_2 \in \mathbb{B} \).

Processes \( P_1 \) and \( P_2 \) are represented by:

\[
\begin{align*}
P_1 &= \text{req}_1.b_{1,wt}.P_1 + \tau.P_1 \\
P_1 &= b_{2,rf}.P_2 + b_{2,rt}.P_1 \\
P_2 &= \text{req}_2.b_{2,wt}.P_2 + \tau.P_2 \\
P_2 &= b_{1,rf}.P_2 + b_{1,rt}.P_2 + b_{1,wt}.P_2 \\
P_2 &= b_{1,rf}.b_{2,wt}.P_2 + b_{1,rt}.P_2 \\
P_2 &= \text{cs}_{1}.b_{1, wf}.P_1 \\
P_2 &= \text{cs}_{2}.b_{2, wf}.P_2
\end{align*}
\]

Finally Lamport = \(((P_1 \parallel P_2) \parallel PV(f,f))/B\).

Note that now we regard as non-blocking not only the reading of a variable but also its writing in case that the written value equals the current one. This kind of re-write does not change the state of the variable and can be thought of as a non-destructive or non-consuming operation (allowing potential concurrent behaviour). This way of accessing a variable is not new. It has been implemented e.g. in area of database. Unlike in Peterson’s specification, we make this assumption on the hardware to show that Lamport’s algorithm is not live with respect to \( P_2 \):

**Proposition 3.6** If we assume fairness of actions, Lamport is live for process \( P_1 \) but not for process \( P_2 \).

**Proof:** Lamport is not live for \( P_2 \) because Lamport\(_{io}\) has catastrophic cycles. To prove the other statement, we need symmetric changes; namely, we rename actions \( \text{req}_1 \) and \( \text{cs}_1 \) into \textit{in} and \textit{out} resp. and
actions \( \tau \textnormal{req}_2 \) and \( cs_2 \) into \( \tau \); we also delete the \( \tau \)-summand of process \( P_1 \). Since this modified process does not have catastrophic cycles, we conclude that Lamport is live for process \( P_1 \).

Prop. 3.6 still holds if we use the same representation of program variables as in Def. 3.1 while we lose liveness for \( P_1 \) whenever processes representing program variables are those used for Peterson’. Then, while reading variable \( b_1 \), process \( P_2 \) can forever block the other process that wants to write it. The next example explains why Lamport is not live for process \( P_2 \).

**Example 3.7** The following timed computation corresponds to an execution sequence from Lamport = \( L/B \) which is fair but not live since process \( P_2 \) never enters its critical section.

\[
\begin{align*}
L &= (P_1 || b P_2) || B \text{PV}(f,f) \\
&= (b_1 \text{wt.P}_1 || b_2 \text{wt.P}_2) || B \text{PV}(f,f) \\
&= (P_2 || b_2 \text{wt.P}_2) || B \text{PV}(t,f)
\end{align*}
\]

\[
\begin{align*}
R &= (P_1 || b_2 || P_2) || B \text{PV}(t,f) \\
&\rightarrow R
\end{align*}
\]

\[
\begin{align*}
Q &= (P_2 || b_2 || P_2) || B \text{PV}(t,f)
\end{align*}
\]

\( R \) can do either a \( cs_1 \)- or a \( b_1 \text{rt} \)-action (due to a synchronisation between \( P_2 \) and \( B_1(t) \)); both actions become urgent after the first \( 1 \)-step. Later, we perform \( cs_1 \) followed by \( b_1 \text{wt} \) (and, hence, \( B_1(t) \) evolves into \( B_1(f) \)) which, in turn, is followed by \( \text{req}_1 \) and \( b_1 \text{wt} \). At this stage, \( B_1(f) \) becomes \( B_1(t) \) and \( Q \) can refuse to perform its activated actions, again \( cs_1 \) and \( b_1 \text{rt} \), and evolve into \( R \). Finally, \( R/B \rightarrow Q/B \) corresponds to a catastrophic cycle in the reduced transition system of Lamport\(_{io} \). A key observation here is that process \( P_1 \), along this cycle, continuously changes the value of \( b_1 \) from \( true \) to \( false \) and vice versa. Consequently, the PAFAS process representing this variable always offers a new instance of \( b_1 f \) and \( b_1 rt \) to its synchronisation partners, and in particular to \( P_2 \). So, any possible move of process \( P_2 \) can be arbitrarily delayed (and, hence, this process can indefinitely be blocked) even in fair traces. No reasonable assumption about program variables can prevent this unwanted behaviour under weak fairness.

**Dijkstra’s algorithm** This algorithm considers \( n \geq 2 \) processes that share two Boolean-valued arrays \( b \) and \( c \) (whose components are initialised to \( true \)) and a turn variable \( k \) initially chosen in \( \{1, 2, \ldots, n\} \). The \( i \)-th process is described below:

Dijkstra()

1. \textbf{var} \( j \) : integer;
2. \textbf{while} true
3. \textbf{do} (non-critical section);
4. \( b[i] \leftarrow false \);
5. \textbf{if} \( k \neq i \)
6. \textbf{then} \( c[i] \leftarrow true \);
7. \textbf{if} \( b[k] \) \textbf{then} \( k \leftarrow i \);
8. \textbf{goto} 5;
9. \textbf{else} \( c[i] \leftarrow false \);
10. \textbf{for} \( j \leftarrow 1 \) to \( n \)
11. \textbf{do} if \( j \neq i \) and \( \neg c[j] \) \textbf{then} \textbf{goto} 5;
Peterson and Knuth are live, but Dijkstra is not.

Dijkstra’s algorithm is defined as

\[ P = \text{get}((\text{kr1}.P_{13} + \text{kr2}.P_{14}) \; \text{rt} \; P_{12} ) \]

\[ P_{13} = b_1 \text{rt}.\text{put} \; \text{kwl}.P_{11} + b_1 \text{rf}.\text{put}.P_{11} \]

\[ P_{14} = b_2 \text{rt}.\text{put} \; \text{kwl}.P_{11} + b_2 \text{rf}.\text{put}.P_{11} \]

\[ P_{15} = c_1 \text{wf}.(c_2 \text{rf}.P_{11} + c_2 \text{rt}.P_{16}) \]

\[ P_{16} = c_1 \text{wt}.b_1 \text{wt}.P_{1} \]

Dijkstra’s algorithm is defined as

\[ \text{Dijkstra} = ((P_1 \parallel P_2) \parallel (\text{get.PUT} \text{BK}) \parallel B \text{PV}(t,t,t,t,1)) / (B \cup \{\text{get.put}\}) \]

where \( \text{BK} = \text{get.PUT.BK} \).

As in [13] we must ensure that whenever, during the execution of the statement “if \( b[k] \) then \( k \leftarrow i \)”, process \( P_i \) has read variable \( k \) but not yet \( b[k] \), the other process cannot change the value of the former variable. Note that \( \text{BK} \) locks the variable \( k \) in writing mode when evaluating \( b[k] \). Indeed, after a get-action, \( k \) can be written only after a subsequent put-action, i.e. once \( b[k] \) has been read.

As other papers in the literature (see e.g. [11]), we cannot prove the liveness of the algorithm\(^3\). In case \( k = 1 \), process \( P_1 \) can immediately enter its critical section (after setting \( b[1] \) to false, both conditions \( k \neq 1 \) and \( \neg c[2] \) are false), while process \( P_2 \) has to wait until \( b[1] \) becomes true (i.e. until \( P_1 \) ends its critical section) and, hence, it can change \( k \). If \( P_1 \) is fast enough to perform its critical section, reset variables \( c[1] \) and \( b[1] \), and submit a further request (again, by setting \( b[1] \) to false) before \( P_2 \) can actually read \( b[1] \), the latter process can never enter its critical section. This scenario is fair and, hence, admissible; see e.g. in [11] where Dijkstra is analysed by exploiting the model checker SMV ([11] and references therein). The fairness notion assumed in [11] ensures that each process executes infinitely often and that no process can stay in its critical or non-critical section forever. The next example shows that the above scenario is also admissible if one assumes fairness of actions and introduces reasonable non-blocking behaviours.

**Example 3.9** Let us consider the following timed computation:

---

\(^3\)Paper [11] studies the liveness of the same algorithms we consider here except for Lamport. In [1] it has been proven that Peterson and Knuth are live, but Dijkstra is not.
There are two processes Knuth’s algorithm Dijkstra variables. Moreover, processes c true 0, and a turn variable k whose initial value is 0, and a turn variable k that takes values in {1, 2} and whose initial value is arbitrary. Process \( P_i \) \((i = 1, 2)\) is described as follows, where \( j \) is the index of the other process:

Knuth’s algorithm There are two processes \( P_1 \) and \( P_2 \), two variables \( c_1 \) and \( c_2 \) that take values in \( \{0, 1, 2\} \) and whose initial value is 0, and a turn variable \( k \) that takes values in \( \{1, 2\} \) and whose initial value is arbitrary. Process \( P_i \) \((i = 1, 2)\) is described as follows, where \( j \) is the index of the other process:

**Example 3.7** This means that it can block forever process \( P_2 \).

**Proposition 3.10** Dijkstra is not live under the assumption of fairness of actions.

**Definition 3.11** (Knuth’s algorithm) The turn variable \( k \) is given in Def. [3.4] and modelled according to Def. [3.8] Variables \( c_1 \) and \( c_2 \) are represented as follows, where \( i = 1, 2 \):

\[
\begin{align*}
C_i(0) &= \{c_ir0, c_jw0\} \triangleright (c_iw1.C_i(1) + c_jw2.C_i(2)) \\
C_i(1) &= \{c_ir1, c_jw1\} \triangleright (c_iw0.C_i(0) + c_jw2.C_i(2)) \\
C_i(2) &= \{c_ir2, c_jw2\} \triangleright (c_iw0.C_i(0) + c_jw1.C_i(1))
\end{align*}
\]

Let \( c_1, c_2 \in \{0, 1, 2\} \) and \( k \in \mathbb{K} \). We let \( PV(c_1, c_2, k) \) to be the parallel composition of all program variables. Moreover, processes \( P_1 \) and \( P_2 \) are defined as follows:

\[
\begin{align*}
P_1 &= \text{req} r c_1w1.P_{11} + \tau.P_1 \\
P_1 &= \text{req} r c_2w1.P_{11} + \tau.P_1 \\
P_{11} &= kr1.P_{13} + kr2.P_{12} \\
P_{12} &= c_2r0.P_{13} + c_2r1.P_{11} + c_2r2.P_{11} \\
P_{13} &= c_1w2.P_{14} \\
P_2 &= \text{req} r c_2w1.P_{21} + \tau.P_2 \\
P_2 &= \text{req} r c_2w1.P_{21} + \tau.P_2 \\
P_{21} &= kr2.P_{23} + kr1.P_{22} \\
P_{22} &= c_1r0.P_{23} + c_1r1.P_{21} + c_1r2.P_{21} \\
P_{23} &= c_2w2.P_{24}
\end{align*}
\]
\[P_{14} = c_2 r_0.P_{15} + c_2 r_1.P_{15} + c_2 r_2.P_{16}\]
\[P_{15} = kw_1.cs_1.kw_2.c_1 w_0.P_1\]
\[P_{16} = c_1 w_1.P_{11}\]
\[P_{24} = c_1 r_0.P_{25} + c_1 r_1.P_{25} + c_1 r_2.P_{26}\]
\[P_{25} = kw_2.cs_2.kw_1.c_2 w_0.P_2\]
\[P_{26} = c_2 w_1.P_{21}\]

We define Knuth = \((P_1 \parallel P_2) \parallel_B PV(0,0,1))/B\).

We now provide an example that shows the existence of a catastrophic cycle in the reduced transition system of the modified Knuth. This example also implies Prop. 3.13.

**Example 3.12** Let us consider the following timed computation:

\[K = (P_1 \parallel P_2) \parallel_B PV(0,0,1) \xrightarrow{\text{req}_2 c_2 w_1} (P_1 \parallel_0 P_{21}) \parallel_B PV(0,1,1) \xrightarrow{\text{kr}_1 c_1 r_0 c_2 w_2} (P_1 \parallel_0 P_{24}) \parallel_B PV(0,2,1) \xrightarrow{\text{req}_1 c_1 w_1} (P_1 \parallel_0 P_{24}) \parallel_B PV(1,2,1) \xrightarrow{\text{kr}_1 c_1 w_2} (P_1 \parallel_0 P_{24}) \parallel_B PV(2,2,1) \xrightarrow{\text{c}_2 r_2} (P_1 \parallel_0 P_{24}) \parallel_B PV(2,2,1) \xrightarrow{\text{c}_1 w_1 k r_1 c_1 w_2} (P_1 \parallel_0 P_{24}) \parallel_B PV(2,2,1) \xrightarrow{\text{c}_2 r_2} (P_1 \parallel_0 P_{24}) \parallel_B PV(2,2,1) \xrightarrow{\text{c}_1 w_1 k r_1 c_1 w_2} (P_1 \parallel_0 P_{24}) \parallel_B PV(2,2,1) \xrightarrow{\text{c}_2 r_2} \cdots \]

Once in \(R\), process \(P_1\) cannot enter its critical section because \(c_2\) is 2; but, the value of this variable will never change because \(P_2\) is blocked. Moreover, as in Examples 3.7 and 3.9, repeated changes of variable \(c_1\) (from 2 to 1 and vice versa) allows a further I-step in \(Q\). The execution sequence Knuth = \(K/B \xrightarrow{\text{req}_2 c_2 w_1} R/B \xrightarrow{\tau} R/B \cdots\) is fair but not live since process \(P_2\) never enters its critical section.

Let us finally notice that Knuth is live e.g. in [1] since the above execution sequence is not fair as defined there, and hence not admissible, because process \(P_2\) does not execute infinitely often.

**Proposition 3.13** Knuth is not live under the assumption of fairness of actions.

### 4 Related works and Conclusion

This work partly originates from [13] where Walker aimed at verifying six MUTEX algorithms with the Concurrency Workbench [4] (CWB, for short). Walker translated the algorithms into CCS and then verified the safety property that the two competing processes are never in their critical sections at the same time. Regarding the liveness property, Walker first considered the following interpretation – which could be expressed as a modal mu-calculus formula and then checked with the CWB:

An algorithm is live if whenever at some point in a computation the process \(P_i\) requests the execution of its critical section, then in any continuation from that point in which between them the processes execute an infinite number of critical sections, \(P_i\) performs its critical section at least once.

The fairness (or progress) assumption assumed here is that infinitely often a critical section is entered. This assumption allows a run where one process enters its critical section repeatedly, while the other one requests the execution of its critical section, but then – for no good reason at all – refuses to take the necessary steps to actually enter it. So, it may be no surprising that four of the six algorithms (Dekker, Dijkstra, Lamport and Hyman) fail to satisfy this property. Moreover, in order to economize on computational effort, the six algorithms in [13] have been minimized w.r.t. weak bisimulation. This allowed Walker to ignore some \(\tau\)-loops that could invalidate the liveness property. And, indeed, all of them are not live whenever the formula expressing the first interpretation of liveness is evaluated over
the transition system that does not abstract from \( \tau \)'s. By examining process \( P_i \), it is clear that these \( \tau \)-loops arise, e.g. in Peterson, from repeated reading and writing of variables by the same process. This is common to all the algorithms and it is not introduced by the translation into CCS (or in PAFAS). Rather its presence reflects the faithfulness of the translation itself.

Then, Walker considered the same liveness property we study in Section 3. To establish that any of the algorithms is live under this second interpretation, Walker added some assumption. Indeed, one characteristic of the \( \tau \)-loops arising from repeated reading and writing of variables by one process is that the other one is excluded from an infinite computation of the system. It is natural to ask if only the presence of such ‘unfair’ loops prevents any of the algorithms from being live. So, Walker proposed to use enriched formulas of the form \( F \Rightarrow P \) where \( P \) is the property of interest (i.e. liveness) and \( F \) is a fairness assumption that assumes as admissible only those paths to which each process contributes infinitely often. Even if at the time of writing no automated analysis was possible, Walker discussed how fairness could be assumed. The basic idea is to tag each action with a unique probe or label; then, we can say the \( i \)-th process \( P_i \) contributes infinitely often to a computation whenever none of its probes is continuously possible from a certain point on. Finally, the liveness under this fairness assumption is expressed by letting \( K_i \) be the set of all probes of \( P_i \) and defining \( \text{FairLive} = \text{FairLive}_1 \land \text{FairLive}_2 \) where \( \text{FairLive}_i = (\forall a \in K_i. GF[a].false) \Rightarrow G(\langle req \rangle .true \Rightarrow F(\langle cs \rangle .true), \) and the operators \( G \) (always), \( F \) (future), \( \langle \rangle \) (possibly) and \( [] \) (necessarily) are standard modal logics operators.

This fairness induced with probes is closely related to fairness of actions as it has been defined in [9][10]. W.r.t. our characterisation (cf. Section 2), the main difference is that, instead of time and time passing, probes are used to decide whenever an action is continuously enabled along a computation and, hence, must be performed eventually. To allow a comparison, we have implemented these ideas within the Concurrency Workbench of the New Century [3] (CWBNBC, for short) that, unlike CWB, can handle modal formulas with fairness constrains. To be able to attach a probe to each process action, the algorithms have been translated into Timed CCS (this is not possible by using the standard CCS language); probes are introduced by annotating synchronisation actions or \( \tau \)'s. For instance, the \( i \)-th processes of Peterson can be defined as follows:

\[
P_i = b_{wt}(req_i).kwj(a_i).P_{i1} \quad P_{i2} = krj(a_i).P_{i1} + kri(a_i).P_{i3}
\]

\[
P_{i1} = b_{rf}(b_i).P_{i3} + b_{rt}(b_i).P_{i2} \quad P_{i3} = cs_i(cs_i).b_{wf}(a_i).P_1
\]

Note that two consecutive actions (as, e.g., \( b_{wt} \) and \( kwj \) in \( P_1 \)) never have the same label. Moreover, since the overall number of labels impacts on the computational effort (see below), we also try to reduce the number of labels we use. For example, we can reuse \( a_i \) to label the actions of \( P_{i2} \) because none of its actions is adjacent to \( kwj \) and this action has already been executed once \( P_{i2} \) is reached.

Whenever an action is performed, the corresponding label becomes visible\(^4\) and can be used as a probe in FairLive. Table 3 shows that all the algorithms we consider are not live according to this second liveness interpretation (also in this setting, Lamport is live for process \( P_1 \) but not for \( P_2 \)). As an example, consider a path from Peterson along which the first process reaches \( P_{11} = b_{rf}(b_1).P_{i3} + b_{rt}(b_1).P_{i2} \),

\(^4\)E.g., if \( P_i \) synchronises with \( B_i(f) \) on the execution of \( b_{wt} \), the label \( @req_i \) becomes visible; similarly, whenever process \( P_i \) executes \( cs_i \) we get the label \( @cs_i \).
Table 4: Execution time (expressed in milliseconds): CWBNC vs. FASE

|       | CWBNC | FASE | CWBNC | FASE | CWBNC | FASE |
|-------|-------|------|-------|------|-------|------|
| **Dekker** | 103125 | 119 | 4844  | 34  | Knuth | 110391 | 166 |
| **Dijkstra** | 110797 | 647 | Lamport | 1734 | 22  |       |     |

*b*₂ is true and *k* is 2. Once in such a state, process *P*₁ can read *b*₂ and *k* and come back to *P*₁₁. Along this cycle, no probe of *P*₁ is continuously possible (probes *b*₁ and *a*₁ are alternately possible) but *c*₁ will never be performed. So, *FairLive*₁ is false and Peterson is not live. As in Example 3.4, there is a path along which a process can be indefinitely blocked by repeated reading. Also in this setting, the liveness of the algorithm strongly depends on the liveness of the hardware, i.e. on the the possibility of making some behaviours non-blocking.

As a further counter-check, we again consider Peterson but now we tag its actions in such a way that the same probe is associated to all the actions that appear along consecutive reading (trying to simulate the intuition behind non-blocking behaviours). So, let us replace *P*₂ with *P*₂ = *kri*(*b*₁)·*P*₁ + *kri*(*b*₁)·*P*₃. Now, whenever in *P*₁₁ and assuming *b*₂ and *k* equal to true and 2, the process *P*₁ can still repeatedly read variables *b*₂ and *k*, but the corresponding path is not fair because probe *b*₁ is continuously possible. With these probes, Peterson and Dekker turn out to be live. So, probes can be used to somehow simulate non-blocking actions. But they must be added and (whenever necessary) tuned by the user by hand. This task is subject to errors and wrong assumptions that would give erroneous results. On the contrary, FASE can be more easily used by also a non-expert user that has only to decide whether (and, in case, which) non-blocking behaviours are necessary. In our opinion, the use of probes requires a deeper knowledge of the problem and much more attention in both modelling and analysis phases.

Another difference between the two approaches deals with performance issues. In Table 4 we report the execution time of both FASE and CWB-NC to perform the analysis on the algorithms discussed in this paper. In particular, in [2] an efficient algorithm for detecting catastrophic cycles has been proposed and implemented. This works in time $O(N + E)$ where *N* and *E* are, resp., the number of nodes and edges of the state space of the process. On the contrary, CWBNC uses an on-the-fly model checking algorithm whose complexity is exponential in the size of the formula (see [3]); in our case, this size strongly depends on the number of probes.

FASE is a good first step towards the creation of an integrated framework for the analysis of concurrent systems. The improvements introduced by the tool (and, in particular, the possibility to easily check non-functional properties such as liveness) allows us to derive results – as those in this paper – very hard to prove by hand. Since these results are very promising, we are currently planning to extend FASE in order to improve the analysis of more complex systems with a larger state space.

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