A SHARPENED RIESZ-SOBOLEV INEQUALITY

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ABSTRACT. The Riesz-Sobolev inequality provides an upper bound, in integral form, for the convolution of indicator functions of subsets of Euclidean space. We formulate and prove a sharper form of the inequality. This can be equivalently phrased as a stability result, quantifying an inverse theorem of Burchard that characterizes cases of equality.

1. Introduction

1.1. The Riesz-Sobolev inequality. Let $E = (E_1, E_2, E_3)$ be an ordered triple of Lebesgue measurable subsets of $\mathbb{R}^d$ with finite Lebesgue measures. The Riesz-Sobolev inequality \cite{12,13} states that

$$\int_{E_3} 1_{E_1} * 1_{E_2} \leq \int_{E_3} 1_{E_1^*} * 1_{E_2^*}$$

where $E^*$ denotes the closed ball, centered at the origin, that satisfies $|E^*| = |E|$, $*$ denotes convolution of functions, and $1_E$ denotes the indicator function $1_E(x) = 1$ if $x \in E$ and $= 0$ if $x \notin E$. This can be read both as an upper bound for $\int_{E_3} 1_{E_1} * 1_{E_2}$ as $E_j$ vary over all sets of prescribed measures, and as a statement that this upper bound is attained by $E^* = (E_1^*, E_2^*, E_3^*)$.

Burchard \cite{2} characterized those triples $E$ that realize equality in (1). Such a characterization must take into account two features of the inequality, namely affine invariance and the concept of admissibility. Affine invariance holds in the sense that if $\psi : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is a measure-preserving linear transformation and $v = (v_1, v_2, v_3) \in (\mathbb{R}^d)^3$ satisfies $v_3 = v_1 + v_2$ then the sets $\tilde{E}_j = \psi(E_j) + v_j$ satisfy $\int_{E_3} 1_{\tilde{E}_1} * 1_{\tilde{E}_2} = \int_{E_3} 1_{E_1} * 1_{E_2}$ and $|\tilde{E}_j| = |E_j|$. For arbitrary $\psi \in \text{Gl}(d)$, $\int_{E_3} 1_{E_1} * 1_{E_2} = |\det(\psi)|^2 \int_{E_3} 1_{E_1^*} * 1_{E_2^*}$.

An ordered triple $r = (r_1, r_2, r_3)$ of positive real numbers is said to be admissible if $r_k \leq r_i + r_j$ for all permutations $(i, j, k)$ of $(1, 2, 3)$, and to be strictly admissible if $r_k < r_i + r_j$ for all permutations. An ordered triple $E$ of measurable subsets of $\mathbb{R}^d$ is said to be admissible (respectively strictly admissible) if $(|E_j|^{1/d} : 1 \leq j \leq 3)$ is admissible (respectively strictly admissible). Burchard’s theorem states that if $E$ is strictly admissible and realizes equality in the Riesz-Sobolev inequality, then there exist a measure-preserving linear transformation $\psi$ and $v \in (\mathbb{R}^d)^3$ satisfying $v_3 = v_1 + v_2$ such that $E_j = \psi(E_j^*) + v_j$ for all $j \in \{1, 2, 3\}$. In particular, the sets $E_j$ are mutually homothetic ellipsoids. Here, and throughout this paper, two sets are regarded as equivalent if their symmetric difference is a Lebesgue null set.

In the borderline admissible but not strictly admissible case, equality holds if and only if the sets $E_j$ are (equivalent to) suitably translated mutually homothetic convex sets \cite{2}. This is equivalent to the well-known characterization of equality in the Brunn-Minkowski inequality. This borderline case will not be discussed in the present paper.

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In order to state our main result we need the following notion of distance from $E$ to the best approximating ordered triple of compatibly translated homothetic ellipsoids of appropriate Lebesgue measures.

**Definition 1.** Let $E = (E_1, E_2, E_3)$ and $F = (F_1, F_2, F_3)$ be ordered triples of Lebesgue measurable subsets of $\mathbb{R}^d$ with $|E_j|, |F_j| < \infty$ for each index $j$. The distance from $E$ to the orbit of $F$ is

$$
\text{Distance}(E, \mathcal{O}(F)) = \inf_{\psi, v \in \{1, 2, 3\}} \max_{j \in \{1, 2, 3\}} |E_j \Delta (\psi(F_j) + v_j)|
$$

where the infimum is taken over all $v = (v_1, v_2, v_3) \in (\mathbb{R}^d)^3$ satisfying $v_3 = v_1 + v_2$ and over all Lebesgue measure–preserving invertible linear automorphisms $\psi$ of $\mathbb{R}^d$.

We will be especially interested in $\text{Distance}(E, \mathcal{O}(E^*))$. It is elementary that this quantity vanishes if and only if there exist $\psi, v$, with $\psi$ measure-preserving and $v$ satisfying $v_3 = v_1 + v_2$, such that $E_j = \psi(E_j^*) + v_j$ for all $j \in \{1, 2, 3\}$.

We also require a quantitative concept of strict admissibility.

**Definition 2.** Let $\rho > 0$. An ordered triple $r$ of positive real numbers is $\rho$–strictly admissible if $r_k \leq (1 - \rho)(r_i + r_j)$ for all permutations $(i, j, k)$ of $(1, 2, 3)$, and $\min(r_1, r_2, r_3) \geq \rho \max(r_1, r_2, r_3)$.

An ordered triple $E$ of Lebesgue measurable subsets of $\mathbb{R}^d$ with positive, finite Lebesgue measures is $\rho$–strictly admissible if $(|E_j|^{1/d} : 1 \leq j \leq 3)$ is a $\rho$–strictly admissible triple of positive real numbers.

Our main result is:

**Theorem 1.** For each $d \geq 1$ and each $\rho > 0$ there exists $c > 0$ such that for each $\rho$–strictly admissible ordered triple $E$ of Lebesgue measurable subsets of $\mathbb{R}^d$,

$$
\int_{E_3} 1_{E_1} * 1_{E_2} \leq \int_{E_3^*} 1_{E_1^*} * 1_{E_2^*} - c \text{Distance}(E, \mathcal{O}(E^*))^2.
$$

The exponent 2 is optimal. This bound does not hold in the borderline admissible case.

Theorem 1 can also be read as a characterization of those triples $E$ that nearly extremize the Riesz-Sobolev functional.

**Theorem 2.** For each $d \geq 1$ and each $\rho > 0$ there exists $c > 0$ such that for any $\delta > 0$ and any $\rho$–strictly admissible ordered triple $E$ of Lebesgue measurable subsets of $\mathbb{R}^d$, if

$$
\int_{E_3} 1_{E_1} * 1_{E_2} \geq (1 - \delta) \int_{E_3^*} 1_{E_1^*} * 1_{E_2^*},
$$

then

$$
\text{Distance}(E, \mathcal{O}(E^*)) \leq C\delta^{1/2} \max_j |E_j|.
$$

The exponent $\frac{1}{2}$ is optimal. Burchard’s theorem, in the strictly admissible range, is a corollary.

A sharper version, treating the dependence on $\rho$ more quantitatively, was established for $d = 1$ in [8]. This improved on a weaker version [6], whose main hypothesis was that $\int_{E_3} 1_{E_1} * 1_{E_2}$ should be nearly maximal for sets $E_{3,1}$ and $E_{3,2}$ with $|E_{3,2}|/|E_{3,1}|$ nearly equal to an odd integer. This weaker result was still sufficient to serve as the key ingredient in a characterization of near-maximizers for Young’s convolution inequality in [7].

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[1][7] was subsequently revised to incorporate the simpler formulation in [8].
A SHARPENED RIESZ-SOBOLEV INEQUALITY

was quite different from the method developed below, relying on a result from additive combinatorics concerning sets whose sumsets have nearly minimal size, adapted from the discrete case to the continuum context. For \( d > 1 \), a less quantitative form

\[
\int_{E_3} 1_{E_1} \ast 1_{E_2} \leq \int_{E_3} 1_{E_1'} \ast 1_{E_2} - \Theta \left( \frac{\text{Distance}(E, O(E'))}{\max_j |E_j|} \right) \cdot \max_j |E_j|^2
\]

of (3) was established in [9], with an unspecified function \( \Theta \) that vanishes only at 0.

1.2. A variant inequality. The following inequality for subsets of \( \mathbb{R}^d \) is closely related to the Riesz-Sobolev inequality. For \( d = 1 \) it is discussed in [8]. A proof is included below.

**Theorem 3.** For any \( d \geq 1 \) and any Lebesgue measurable sets \( E_j \subset \mathbb{R}^d \) with finite Lebesgue measures, for any \( \tau > 0 \),

\[
\int_{\mathbb{R}^d} \min(1_{E_1} \ast 1_{E_2}, \tau) \geq \int_{\mathbb{R}^d} \min(1_{E_1'} \ast 1_{E_2'}, \tau).
\]

A sharpened form, parallel to Theorem 1, is as follows. Define \( \text{Distance}((A, B), O(A'^*, B'^*)) \) to be the infimum, over all Lebesgue measure-preserving linear transformations of \( \mathbb{R}^d \) and over all \( (u, v) \in (\mathbb{R}^d)^2 \), of \( \max( |A \Delta (\psi(A'^*)) + u|, |B \Delta (\psi(B'^*) + v)|) \).

**Theorem 4.** For any \( d \geq 1 \), any compact set \( \Lambda \subset (0, 1) \), and any \( \rho > 0 \) there exists \( c > 0 \) such that for any Lebesgue measurable sets \( E_j \subset \mathbb{R}^d \) with finite, positive Lebesgue measures satisfying \( \min(|E_1|, |E_2|) \geq \rho \max(|E_1|, |E_2|) \) and any \( \tau \in \mathbb{R}^+ \) such that \( \tau / \min(|E_1|, |E_2|) \in \Lambda \),

\[
\int_{\mathbb{R}^d} \min(1_{E_1} \ast 1_{E_2}, \tau) \geq \int_{\mathbb{R}^d} \min(1_{E_1'} \ast 1_{E_2'}, \tau) + c \text{Distance}((E_1, E_2), O(E_1'^*, E_2'^*))^2.
\]

In particular, for \( 0 < \tau < \min(|E_1|, |E_2|) \), equality holds in (4) only if \( (E_1, E_2) \) is a pair of homothetic ellipsoids. For \( d = 1 \) a slightly more quantitative result is proved in [8].

A formally sharper variant holds. Given \( E_1, E_2, \tau \), define

\[
S_{\tau} = \{ x \in \mathbb{R}^d : (1_{E_1} \ast 1_{E_2})(x) > \tau \}
\]

\[
S_{\tau}' = \{ x \in \mathbb{R}^d : (1_{E_1'} \ast 1_{E_2'})(x) > \tau \}.
\]

**Theorem 5.** For any \( d \geq 1 \), any compact set \( \Lambda \subset (0, 1) \), and any \( \rho > 0 \) there exists \( c > 0 \) such that for any Lebesgue measurable sets \( E_j \subset \mathbb{R}^d \) with finite, positive Lebesgue measures satisfying \( \min(|E_1|, |E_2|) \geq \rho \max(|E_1|, |E_2|) \) and any \( \tau \in \mathbb{R}^+ \) such that \( \tau / \min(|E_1|, |E_2|) \in \Lambda \),

\[
\int_{\mathbb{R}^d} \min(1_{E_1} \ast 1_{E_2}, \tau) \geq \int_{\mathbb{R}^d} \min(1_{E_1'} \ast 1_{E_2'}, \tau) + c \text{Distance}((E_1, E_2), O(E_1'^*, E_2'^*))^2 + c(|S_{\tau} - |S_{\tau}'|)^2.
\]

A corresponding improvement of Theorem 1 holds. See [8] for the case \( d = 1 \).

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2. Notation and reformulation

\( 1_E \) denotes the indicator function of a set \( E \), and \( |E| \) denotes its Lebesgue measure. All functions in this paper are real-valued. \( \langle f, g \rangle = \int fg \); the integral is understood to be taken over \( \mathbb{R}^d \) with respect to Lebesgue measure unless the contrary is explicitly indicated.
$A \Delta B = (A \setminus B) \cup (B \setminus A)$ denotes the symmetric difference between two sets. Since $|A \Delta B| = \|1_A - 1_B\|_{L^1}$, one has the triangle inequality $|A \Delta C| \leq |A \Delta B| + |B \Delta C|$ for any three measurable sets.

It will be convenient to reformulate the Riesz-Sobolev inequality in more symmetric form. Let $\lambda$ be the natural Lebesgue measure on $\Sigma = \{x = (x_1, x_2, x_3) \in (\mathbb{R}^d)^3 : x_1 + x_2 + x_3 = 0\}$.

\begin{equation}
\int_{\Sigma} f(x) \, d\lambda(x) = \int_{\mathbb{R}^d \times \mathbb{R}^d} f(x_1, x_2, -x_1 - x_2) \, dx_1 \, dx_2.
\end{equation}

Define

\begin{equation}
\mathcal{T}(E) = \int_{\Sigma} \prod_{j=1}^{3} 1_{E_j}(x_j) \, d\lambda(x).
\end{equation}

This is equal to $\int_{-E_3} 1_{E_1} * 1_{E_2}$, where $-E_3 = \{ -x : x \in E_3 \}$. Since $(-E)^* \equiv E^*$, the Riesz-Sobolev inequality is equivalent to

$$\mathcal{T}(E) \leq \mathcal{T}(E^*) \quad \text{for all } E.$$

**Definition 3.** The distance from $E$ to the orbit of $F$ is

\begin{equation}
\text{dist}(E, \mathcal{O}(F)) = \inf_{\psi} \max_{j \in \{1, 2, 3\}} |E_j \Delta (\psi(F_j) + v_j)|
\end{equation}

where the infimum is taken over all $\mathbf{v} = (v_1, v_2, v_3) \in (\mathbb{R}^d)^3$ satisfying $v_1 + v_2 + v_3 = 0$ and over all Lebesgue measure-preserving invertible linear automorphisms $\psi$ of $\mathbb{R}^d$.

Our main result can equivalently be formulated as follows.

**Theorem 6.** For each $d \geq 1$ and each $\rho > 0$ there exists $c > 0$ such that for each $\rho$–strictly admissible ordered triple $E$ of Lebesgue measurable subsets of $\mathbb{R}^d$,

\begin{equation}
\mathcal{T}(E) \leq \mathcal{T}(E^*) - c \text{Distance}(E, \mathcal{O}(E^*))^2.
\end{equation}

The Steiner symmetrization $E^\dagger$ of a Lebesgue measurable set $E \subset \mathbb{R}^d$, satisfying $|E| < \infty$, is defined as follows. Regard $\mathbb{R}^d$ as $(\mathbb{R}^{d-1} \times \mathbb{R})^1$ with coordinates $(x', t)$. Define the vertical slices $E_{x'} = \{ t \in \mathbb{R} : (x', t) \in E \}$. Denote by $O(d)$ the group of all orthogonal linear transformations of $\mathbb{R}^d$.

**Definition 4.**

\begin{equation}
E^\dagger = \{(x', t) : |t| \leq \frac{1}{2} |E_{x'}| \}.
\end{equation}

For $O \in O(d)$,

\begin{equation}
E^\dagger_O = O^{-1}[(O(E))^\dagger].
\end{equation}

If $|E_{x'}| < \infty$ for every $x'$ then $E^\dagger = \{(x', t) : t \in (E_{x'})^* \}$, where $E_{x'}^*$ denotes the symmetrization of $E_{x'} \subset \mathbb{R}^1$. $E^\dagger_O$ is the Steiner symmetrization of $E$ in the direction $O^{-1}(0, 0, \ldots, 0, 1) \in \mathbb{R}^d$.

3. A Flow of Sets

For general sets, the following result is perhaps best described as folklore. It was known long ago to Burchard [4], and a version of it is mentioned in her 2009 lecture notes [3]. It appears in a recent work of Carillo, Hittmeir, Volzone, and Yao [5]. While it is not essential to our analysis, it does simplify one step, and deserves to be more widely known.
Proposition 7. There exists a flow \((t, E) \mapsto E(t)\) of equivalence classes of Lebesgue measurable subsets of \(\mathbb{R}^1\) with finite measures, defined for \(t \in [0, 1]\), having the following properties for all sets \(E\):

1. \(E(0) = E\) and \(E(1) = E^*\).
2. Measure preserving: \(|E(t)| = |E|\) for all \(t \in [0, 1]\).
3. Continuity: \(|E(s) \Delta E(t)| \to 0\) as \(s \to t\).
4. Inclusion monotonicity: If \(E \subset \tilde{E}\) then \(E(s) \subset \tilde{E}(s)\) for all \(s \in [0, 1]\).
5. Contractivity: \(|E_1(t) \Delta E_2(t)| \leq |E_1 \Delta E_2|\) for all sets \(E_1, E_2\).
6. Independence of past history: If \(0 \leq s \leq t \leq 1\) then \(E(t)\) depends only on \(E(s), s, t\). Moreover, \(E(t) = (E(s))(\tau)\) where \(1 - \tau = \frac{t-s}{t}\).
7. Functional monotonicity and continuity: The function \(t \mapsto \mathcal{T}(E_1(t), E_2(t), E_3(t))\) is continuous and nondecreasing on \([0, 1]\).

All of these statements are to be interpreted in terms of equivalence classes of measurable sets, with \(E\) equivalent to \(E^*\) whenever \(|E \Delta E^*| = 0\). In the case in which the initial set \(E\) is a finite union of pairwise disjoint closed intervals, this flow is a well known device [1], [11]. From its construction it is clear that for sets that are finite unions of closed intervals this flow satisfies \(|E(t)| = |E|\), and if \(E_1 \subset E_2\) then \(E_1(t) \subset E_2(t)\). We now sketch a proof of the extension of the flow to arbitrary sets.

Lemma 8. For \(j = 1, 2, 3\) let \(E_j\) be a finite union of pairwise disjoint closed bounded intervals. Let \(t \mapsto E_j(t)\) be as defined in [1], [11]. Then \(|E_1(t) \Delta E_2(t)| \leq |E_1 \Delta E_2|\).

Proof. Consider the flows of \(A = E_1 \cap E_2\) and \(B = E_1 \cup E_2\). Both of these sets are finite unions of closed bounded intervals, so their flows are defined. Since \(A \subset B\), \(A(t) \subset B(t)\) for all \(t\). Therefore

\[
|A(t) \Delta B(t)| = |B(t) \setminus A(t)| = |B(t)| - |A(t)| = |B| - |A| = |(E_1 \cup E_2) \setminus (E_1 \cap E_2)| = |E_1 \Delta E_2|.
\]

Since \(A \subset E_j \subset B\), \(A(t) \subset E_j(t) \subset B(t)\) for all \(t\). Therefore

\[
A(t) \subset E_1(t) \cap E_2(t) \subset E_1(t) \cup E_2(t) \subset B(t)
\]

and consequently

\[
E_1(t) \Delta E_2(t) \subset A(t) \Delta B(t).
\]

\[\Box\]

To define the flow for a general Lebesgue measurable set \(E \subset \mathbb{R}^1\) satisfying \(|E| < \infty\), consider any approximating sequence of sets \(E_n\) satisfying \(|E_n \Delta E| \to 0\) as \(n \to \infty\). Since \(|E_m \Delta E_n| \leq |E_m \Delta E| + |E \Delta E_n|\), for each \(t\) we have \(\lim_{n \to \infty} |E_m(t) \Delta E_n(t)| = 0\). Therefore (since \(L^1\) is complete) there exists a set \(\tilde{E}(t)\) such that \(\lim_{n \to \infty} |E_n(t) \Delta \tilde{E}(t)| = 0\). Moreover, this set \(\tilde{E}(t)\) is independent of the choice of approximating sequence \((E_n)\). Define the flow by setting \(E(t) = \tilde{E}(t)\).

It is clear that if \(A\) is a finite union of bounded closed intervals then \(t \mapsto A(t)\) is continuous at \(t = 0\) in the sense that \(|A(t) \Delta A| \to 0\) as \(t \to 0\). From this and the contraction property \(|A(t) \Delta B(t)| \leq |A \Delta B|\) it follows immediately that \(t \mapsto E(t)\) is continuous at \(t = 0\) for any set \(E\) of finite Lebesgue measure. Continuity of \(t \mapsto E(t)\) at an arbitrary \(s \in [0, 1]\) follows from this together with the independence of past history.

It is straightforward to verify the other conclusions of Proposition 7. \[\Box\]
An auxiliary property of this flow is useful: \(|E(t) \Delta E^*|\) is a nonincreasing function of \(t\). This holds, by inspection, for finite unions of intervals, and follows for general sets by continuity of \(t \mapsto E(t)\).

We record in passing a smoothing property of this flow, which is not used in the proofs of our main results. It is established in [12].

**Proposition 9.** Let \(E \subset \mathbb{R}^1\) be a Lebesgue measurable set with finite measure. For each \(t > 0\), \(E(t)\) equals a union of intervals.

That is, there exists a countable family of intervals \(I_n\) such that \(|E(t) \Delta \cup_n I_n| = 0\).

A less canonical, but still useful, higher-dimensional analogue of Proposition 9 can be constructed by combining this flow with iterated Steiner symmetrization. The next lemma, used in this construction, is proved in [1] and in [11].

**Lemma 10.** Let \(d \geq 1\). Let \(E\) be an ordered triple of bounded Lebesgue measurable subsets of \(\mathbb{R}^d\), each with positive, finite measure. There exists a sequence \(O_n \in O(d)\) such that the sequence of iterated Steiner symmetrizations defined recursively by \(E_{0,j} = E_j\) and \(E_{n,j} = (E_{n-1,j})^\star_{O_n}\) satisfies

\[
\lim_{n \to \infty} |E_{n,j} \Delta E_j^*| = 0 \quad \text{for each } j \in \{1, 2, 3\}.
\]

In higher dimensions there exists a natural flow \(t \mapsto E(t)\) satisfying \(E(0) = E\) and \(E(1) = E^\star\), the Steiner symmetrization of \(E\). For each \(x' \in \mathbb{R}^d\), define \(E_j(t) \subset \mathbb{R}^d\) by setting the fiber \(\{x_d \in \mathbb{R} : (x', x_d) \in E_j(t)\}\) equal to the flow defined above, at time \(t\), of the fiber \(\{x_d \in \mathbb{R} : (x', x_d) \in E_j\}\). We call this the Steiner flow in the proof of Proposition 11.

A property of Steiner flow is that

\[
|E(t) \Delta E^\star| = \text{a nonincreasing function of } t.
\]

This property, for \(d > 1\), is an immediate consequence of the case \(d = 1\).

The next result asserts the existence of a flow with corresponding properties for subsets of \(\mathbb{R}^d\), for arbitrary \(d > 1\). This higher-dimensional analogue is not canonical; its construction involves certain choices; but it is sufficient for our purpose.

**Proposition 11.** Let \(d \geq 1\). For \(j \in \{1, 2, 3\}\) let \(E_j \subset \mathbb{R}^d\) be a bounded Lebesgue measurable set. There exist mappings \([0, 1] \ni t \mapsto E_j(t)\) of equivalence classes of Lebesgue measurable subsets of \(\mathbb{R}^d\), with the following properties:

1. \(E_j(0) = E_j\) and \(E_j(1) = E_j^\star\).
2. \(|E_j(t)| = |E_j|\) for all \(t \in [0, 1]\).
3. \(|E_j(s) \Delta E_j(t)| \to 0\) as \(s \to t\).
4. The function \(t \mapsto \mathcal{T}(E(t))\) is continuous and nondecreasing on \([0, 1]\).

**Proof.** Let \(E\) be given. Let \((O_n : n \in \mathbb{N})\) be as in Lemma 10. Define the flow \(t \mapsto E(t)\) for \(t \in [0, \frac{1}{2}]\) to be the Steiner flow of \(E\), conjugated with the rotation \(O_1\), with the time rescaled so that \(E_1\) is reached at \(t = \frac{1}{2}\) rather than at \(t = 1\). Next, define the flow for \(t \in [\frac{1}{2}, \frac{3}{4}]\) so that \(\mathcal{E}(\frac{3}{4}) = E_2\), by using the same construction, conjugated by \(O_2\). Use the time interval \([1 - 2^{-k}, 1 - 2^{-k-1}]\) in the same way to make a continuous deformation from \(E_k\) to \(E_{k+1}\) for each \(k \in \mathbb{N}\). Define \(E(1) = E^\star\). The flow thus defined is clearly continuous on \([0, 1]\), and \(\mathcal{T}(E(t))\) is a nondecreasing function of \(t\).

From (13) and the property that \(|E_{n,j} \Delta E_j^*| \to 0\) as \(n \to \infty\) it follows that \(|E_j(t) \Delta E_j^*| \to 0\) as \(t \to 1\) from below. Thus defining \(E_j(1) = E_j^\star\), \(t \mapsto \mathcal{T}(E(t))\) becomes a continuous function on the closed interval \([0, 1]\). 

\(\square\)
4. Reduction to small measure perturbations

In order to establish Theorem 6 it suffices to prove it in the following perturbative regime.

**Proposition 12.** For each \(d \geq 1\) and each \(\rho > 0\) there exist \(\delta_0 > 0\) and \(c > 0\) such that the inequality

\[
\mathcal{T}(E) \leq \mathcal{T}(E^*) - c \text{Distance}(E, O(E^*))^2
\]

holds for each \(\rho\)-strictly admissible ordered triple \(E\) of Lebesgue measurable subsets of \(\mathbb{R}^d\) satisfying

\[
\text{dist}(E, O(E^*)) \leq \delta_0 \max_{1 \leq j \leq 3} |E_j|.
\]

In this section, we show how Theorem 6 is a consequence of Proposition 12 which in turn will be proved below. We may suppose without loss of generality that \(\max_j |E_j| = 1\). For if \(rE_j = \{rx : x \in E_j\}\) and \(rE = (rE_1, rE_2, rE_3)\) then

\[
\frac{\mathcal{T}(rE)}{\max_j |rE_j|^2} = \frac{\mathcal{T}(E)}{\max_j |E_j|^2}
\]

and \(\text{dist}((rE_1, rE_2, rE_3), O((rE_1)^*, (rE_2)^*, (rE_3)^*)) = r^d \text{dist}(E, O(E^*))\). Likewise, \(E\) is \(\rho\)-strictly admissible if and only if \(E\) is so. Thus the conclusion of Theorem 6 holds for \(E\) with a given constant \(c\), if and only if it holds for \(rE\), with the same constant \(c\). Choosing \(r = \max_j |E_j|^{1/d}\) reduces matters to the case in which \(\max_j |E_j| = 1\).

Let \(\delta_0 > 0\) and suppose (10) to be known for all \(\rho\)-strictly admissible \(E\) satisfying \(\max_j |E_j| = 1\) and \(\text{dist}(E, O(E^*)) \leq \delta_0\). Let \(E\) be a \(\rho\)-strictly admissible ordered triple of bounded sets satisfying \(\max_j |E_j| = 1\) and \(\text{dist}(E, O(E^*)) > \delta_0 \max_j |E_j| = \delta_0\). Consider \(E(t) = (E_1(t), E_2(t), E_3(t))\), where \([0, 1] \ni t \mapsto E(t)\) is the flow of Proposition 11.

The function \([0, 1] \ni t \mapsto \text{dist}(E(t), O(E^*))\) is continuous, and it tends to 0 as \(t \to 1\) since \(|E_j(t) \Delta E^*_j| \to 0\). Therefore there exists a smallest \(t_0 \in [0, 1]\) for which \(\text{dist}(E(t_0), O(E^*)) = \delta_0\). By monotonicity of \(\mathcal{T}\) under the threefold flow,

\[
\mathcal{T}(E) \leq \mathcal{T}(E(t_0)) \leq \mathcal{T}(E^*) - c \text{dist}(E(t_0), O(E^*))^2 = \mathcal{T}(E^*) - c \delta_0^2.
\]

The maximum possible value of \(\text{dist}(A, O(A^*))\), as \(A\) ranges over all ordered triples of sets satisfying \(\max_j |A_j| = 1\), is equal to 2. Therefore \(\mathcal{T}(E^*) - c \delta_0^2 \leq \mathcal{T}(E^*) - c' \text{dist}(E, O(E^*))^2\), yielding the desired inequality by transitivity.

5. Reduction to perturbations near the boundary

Let \(e = (e_1, e_2, e_3) \in (\mathbb{R}^+)^3\) be given, and suppose that \((e_j^{1/d} : 1 \leq j \leq 3)\) is \(\rho\)-strictly admissible. Let \(E_j \subset \mathbb{R}^d\) be Lebesgue measurable sets satisfying \(|E_j| = e_j\). Define

\[
\delta = \text{dist}(E, O(E^*)�).
\]

Suppose that \(\max_j |E_j| = 1\), and that \(\delta\) is small. Choose \(\psi, v\) so that \(\tilde{E}_j = \psi(E_j) + v_j\) satisfy

\[
\max_j |\tilde{E}_j \Delta E^*_j| \leq 2 \text{dist}(E, O(E^*))�,
\]

and replace \(E\) by \((\tilde{E}_j : 1 \leq j \leq 3)\), as we may do without affecting the inequality in question.

Let \(B_j = E^*_j\), and let \(r_j > 0\) be the radius of \(B_j\). Define functions \(f_j\) by

\[
f_j = 1_{E_j} = 1_{E^*_j} + f_j = 1_{B_j} + f_j.
\]
Thus $f_j$ takes values in $\{-1,0,1\}$, $\int f_j = 0$, and the essential support of $f_j$ has measure $\leq 2|E_j \Delta E_j^*| \leq 4\delta$.

One has

$$T(E) = T(\mathbf{1}_{B_1} + f_1, \mathbf{1}_{B_2} + f_2, \mathbf{1}_{B_3} + f_3) = T(E^*) + \sum_{k=1}^{3} \langle K_k, f_k \rangle + O(\delta^2)$$

where the kernels $K_k$ are defined for $k \in \{1, 2, 3\}$ by

(19) 

$$K_k = \mathbf{1}_{B_i} \ast \mathbf{1}_{B_j}$$

with the notation $\{1, 2, 3\} = \{i, j, k\}$. Each of the functions $K_k$ is radial, nonnegative, and nonincreasing along each ray emanating from the origin. Moreover, if $r_i \leq r_j$ then $\nabla K_k(x) \neq 0$ whenever $r_j - r_i < |x| < r_j + r_i$. Strict admissibility of $E$ is equivalent to the assertion that $r_j - r_i < r_k < r_i + r_j$ for all permutations $(i, j, k)$ of $(1, 2, 3)$. Therefore $\nabla K_k(x)$ is nonzero when $|x| = r_k$. Define $\gamma_k$ by

(20) 

$$|\nabla K_k(x)| = \gamma_k \text{ when } |x| = r_k.$$ 

We will abuse notation mildly by writing $K_k(s)$ for $K_k(x)$ where $|x| = s$.

Since $|E_k \setminus B_k| = |B_k \setminus E_k|$, $\int f_k = 0$ and consequently

$$\langle K_k, f_k \rangle = \int (K_k(x) - K_k(r_k)) f_k(x) \, dx$$

$$= -\int |K_k(x) - K_k(r_k)| \cdot |f_k(x)| \, dx$$

since the functions $x \mapsto K_k(x) - K_k(r_k)$ and $-f_k$ are both nonnegative on $B_k$ and nonpositive on its complement.

Let $\lambda$ be a large positive constant, which is to be independent of $\delta$ and is to be chosen below. If $\lambda \delta$ is bounded above by a small constant depending only on $e$ then it follows from the nonvanishing of $\nabla K_k$ in a neighborhood of $|x| = r_k$ that

$$\langle K_k, f_k \rangle \leq -c\lambda \delta \int_{|x| - r_k \geq \lambda \delta} |f_k(x)| \, dx$$

$$\leq -c\lambda \delta \left| \{ x \in E_k \Delta B_k : |x| - r_k \geq \lambda \delta \} \right|.$$

We aim to reduce to the case in which $E_k \Delta B_k$ is entirely contained in $\{ x : |x| - r_k \leq \lambda \delta \}$ for each index $k$. To accomplish this, for each index $j \in \{1, 2, 3\}$ choose a set $E_j^\perp$ so that

$$|E_j^\perp| = |E_j|,$$

$E_j \Delta B_j$ is the disjoint union of $E_j^\perp \Delta B_j$ and $E_j \Delta E_j^\perp$

$$\{ x \in E_j \Delta B_j : |x| - r_j \geq \lambda \delta \} \subset E_j^\perp \Delta E_j$$

$$|E_j^\perp \Delta E_j| \leq 2 \left| \{ x \in E_j \Delta B_j : |x| - r_j > \lambda \delta \} \right|.$$ 

To construct such a set, define $S = \{ x \in E_j \Delta B_j : |x| - r_j \leq \lambda \delta \}$, $S_+ = S \cap (E_j \setminus B_j)$, and $S_- = S \cap (B_j \setminus E_j)$. If $|S_+| \geq |S_-|$ then choose $\tilde{S}_+ \subset S_+$ to be a measurable set satisfying $|\tilde{S}_+| = |S_-|$, and define $E_j^\perp$ by

(21) 

$$E_j^\perp = (B_j \cup \tilde{S}_+) \setminus S_-.$$
If on the other hand $|S_+| < |S_-|$ then choose $\tilde{S}_- \subset S_-$ to be a measurable set satisfying $|\tilde{S}_-| = |S_+|$, and define $E_j^\perp$ by

$$E_j^\perp = (B_j \cup S_+) \setminus \tilde{S}_-.$$ \hfill (22)

Then $E_j \Delta B_j$ is the disjoint union of $E_j^\perp \Delta B_j$ and $E_j \Delta E_j^\perp$, so

$$|E_j \Delta B_j| = |E_j \Delta E_j^\perp| + |E_j^\perp \Delta B_j|.$$ \hfill (23)

Moreover, $E_j^\perp \Delta B_j \subset \{x : |x| - r_j \leq \lambda \delta\}$.

Set $f_j^\perp = 1_{E_j^\perp} - 1_{B_j}$ and expand

$$1_{E_j} = 1_{B_j} + f_j^\perp + (f_j - f_j^\perp) = 1_{E_j^\perp} + \tilde{f}_j$$

where $\tilde{f}_j = f_j - f_j^\perp = 1_{E_j} - 1_{E_j^\perp}$. Thus $\tilde{f}_j$ takes values in $\{-1, 0, 1\}$ and has essential support equal to $E_j \Delta E_j^\perp$. Write $E_j^\perp = (E_j^\perp, E_j^\perp, E_j^\perp)$.

**Lemma 13.** Let $d \geq 1$ and $\rho > 0$. There exist $\lambda < \infty$, and $\delta_0, c > 0$, with the following property. Let $E$ be a $\rho$-strictly admissible ordered triple of subsets of $\mathbb{R}^d$ satisfying $\max_j |E_j| = 1$ and $\max_j |E_j \Delta E_j^*| \leq \delta_0$. Define $E_j^\perp$ as above. Then

$$\mathcal{T}(E) \leq \mathcal{T}(E^*) - c\lambda \sum_{i=1}^3 |E_i \Delta E_i^*| \cdot \sum_{j=1}^3 |E_j \Delta E_j^\perp|.$$ \hfill (24)

In the following proof, $c$ denotes a small strictly positive constant, whose value is permitted to change from one occurrence to the next.

**Proof.** Set $\delta = \max_{i=1}^2 |E_i \Delta E_i^*| \leq \delta_0$. Write $1_{E_k} = 1_{E_k^\perp} + f_k^\perp + \tilde{f}_k$ for each index, and expand $\mathcal{T}(E)$ accordingly to obtain 27 terms. Eight of those terms do not involve the functions $\tilde{f}_j$; these eight recombine to give $\mathcal{T}(E_j^\perp)$. Three terms are of the form $\{K_k, \tilde{f}_k\}$; their sum is less than or equal to $-c\lambda \delta \sum_k |E_k \Delta E_k^\perp|$, as discussed above. The remaining terms involve either two or more $\tilde{f}_j$, or one $\tilde{f}_j$ and at least one $f_j^\perp$. By the elementary inequality

$$\mathcal{T}(E_1, E_2, E_3) \leq \min_{i \neq j \in \{1, 2, 3\}} |E_i| \cdot |E_j|,$$

each of these terms is

$$O\left(\max_j |E_j \Delta E_j^*| \cdot \max_k |E_k^\perp \Delta E_k|\right) = O(\delta \max_k |E_k^\perp \Delta E_k|),$$

uniformly in $\lambda$. Thus

$$\mathcal{T}(E) \leq \mathcal{T}(E_j^\perp) - c\lambda \delta \sum_j |E_j \Delta E_j^\perp| + O(\delta \sum_j |E_j \Delta E_j^\perp|)$$ \hfill (26)

where the constant factor implicit in the $O$ notation is independent of $\lambda$.

Choose $\lambda$ to be a sufficiently large constant to ensure that the remainder term in (26) can be absorbed, yielding

$$\mathcal{T}(E) \leq \mathcal{T}(E_j^\perp) - c\lambda \sum_{i=1}^3 |E_i \Delta E_i^*| \cdot \sum_{j=1}^3 |E_j \Delta E_j^\perp|$$ \hfill (27)

with a smaller but still positive value of $c$. Since $|E_j^\perp| = |E_j|$, $(E_j^\perp)^* = E_j^\perp$. Therefore by the Riesz-Sobolev inequality, $\mathcal{T}(E_j^\perp) \leq \mathcal{T}(E^*)$. Inserting this into (27) yields (24). \hfill \Box
If \( \max_{1 \leq j \leq 3} |E_j \Delta E_j^1| \geq \frac{1}{10} \max_{1 \leq j \leq 3} |E_j \Delta E_j^*| \) we conclude immediately from Lemma 13 that

\[
\mathcal{T}(E) \leq \mathcal{T}(E^*) - c \sum_{j=1}^{3} |E_j \Delta E_j^1|^2 \leq \mathcal{T}(E^*) - c \text{dist}(E, O(E^*))^2.
\]

There remains the main case, in which \( \max_j |E_j \Delta E_j^1| \leq \frac{1}{10} \max_j |E_j \Delta E_j^*| \). In this case the nonpositive term \(-c\lambda \sum_{i=1}^{3} |E_i \Delta E_i^*| \cdot \sum_{j=1}^{3} |E_j \Delta E_j^1| \) in (24) may not be useful. However, (27) still gives

\[
(28) \quad \mathcal{T}(E) \leq \mathcal{T}(E^1)
\]

and it suffices to prove that \( \mathcal{T}(E^1) \leq \mathcal{T}(E^*) - c \text{dist}(E, O(E^*))^2 \). Indeed,

\[
\text{dist}(E^1, O(E^*)) \geq \text{dist}(E, O(E^*)) - \max_j |E_j \Delta E_j^1| \geq \frac{1}{2} \text{dist}(E, O(E^*)�)
\]

6. Reduction to the boundary(ies)

We have shown that \( E \) may be replaced by \( E^1 \). Thus our assumptions henceforth are that \( E \) is \( \rho^- \)-strictly admissible, that \( \max_j |E_j| = 1 \), that \( \text{dist}(E, O(E^*)) \leq \delta_0 \), that \( \delta = \max_{1 \leq j \leq 3} |E_j \Delta E_j^1| \) satisfies \( \delta \leq 4 \text{dist}(E, O(E^*)) \), and that

\[
(29) \quad E_j \Delta B_j \subset \{ x : |x| - r_j \leq \lambda \delta \}.
\]

We aim to prove that for any \( \lambda, \rho \in (0, \infty) \) there exists \( \delta_0 > 0 \) such that \( \mathcal{T}(E) \) satisfies the desired inequality whenever all of these assumptions are satisfied.

We continue to write \( B_j = E_j^* \) and to denote by \( r_j \) the radius of \( B_j \). The kernels \( K_j \) and positive constants \( \gamma_j \) are as defined above. It is elementary that \( K_k \) is twice continuously differentiable in a neighborhood of the boundary of \( B_k \). We write \( e = (e_j : 1 \leq j \leq 3) = (|E_j| : 1 \leq j \leq 3) \).

Continue to represent \( 1_{E_j} = 1_{B_j} + f_j \). Each \( f_j \) is supported in \( \{ x : |x| - r_j \leq \lambda \delta \} \), and satisfies \( \int f_j = 0 \) and \( \| f_j \|_{L^\infty} \leq 1 \). Refine this representation by defining functions \( f_j^\pm \), taking values in \( \{0,1\} \), by \( 1_{E_j \setminus B_j} = f_j^+ \) and \( 1_{B_j \setminus E_j} = f_j^- \), so that \( f_j = f_j^+ - f_j^- \). Introduce polar coordinates \( (r, \theta) \) in \( \mathbb{R}^d \) and define functions \( F_j^\pm \in L^2(S^{d-1}) \) by

\[
F_j^+ (\theta) = \int_{\mathbb{R}^+} f_j^+ (t\theta) t^{d-1} dt
\]

where \( \theta \) is regarded as a unit vector, so that \( t\theta \) is the point with polar coordinates \( (t, \theta) \). Define \( F_j \in L^2(S^{d-1}) \) by \( F_j = F_j^+ - F_j^- \).

Under the hypothesis that \( E_j \Delta B_j \) is contained in a small neighborhood of the boundary of \( B_j \),

\[
(30) \quad |E_j \Delta B_j|^2 \lesssim \| F_j^+ \|^2_{L^2(S^{d-1})} + \| F_j^- \|^2_{L^2(S^{d-1})},
\]

where \( u \preceq v \) means that \( u \leq Cv \) and \( v \leq Cu \) with a positive, finite constant \( C \) that depends on \( d, \rho \) but not otherwise on \( E \). Thus it suffices to establish an upper bound of the form

\[
\mathcal{T}(E) \leq \mathcal{T}(E^*) - c \sum_{j=1}^{3} \left( \| F_j^+ \|^2_{L^2(S^{d-1})} + \| F_j^- \|^2_{L^2(S^{d-1})} \right).
\]
Under the hypotheses introduced at the beginning of Proposition 14.

The expression \( T \) metric, and the compact linear operator \( Q \) of Lemma 15.

Denote by \( \sigma \) the rotation-invariant measure on \( S^{d-1} \), normalized so that Lebesgue measure in \( \mathbb{R}^d \) is equal to \( r^{d-1} dr d\sigma(\theta) \) in polar coordinates. For each \( k \in \{1, 2, 3\} \) define the quadratic form \( Q_k \) on \( L^2(S^{d-1}) \) by

\[
Q_k(F, G) = \int_{S^{d-1} \times S^{d-1}} F(x)G(y)1_{[r_x, y] \leq r_k} d\sigma(x) d\sigma(y).
\]

Define also

\[
Q(F_1, F_2, F_3) = Q_1(F_2, F_3) + Q_2(F_3, F_1) + Q_3(F_1, F_2).
\]

Whenever \( |x| = |y| = 1, |r_x + r_y|^2 = r_x^2 + r_y^2 + 2r_x r_y x \cdot y \). Therefore \( Q_k \) is symmetric, and the compact linear operator \( T : L^2(S^{d-1}) \to L^2(S^{d-1}) \) defined by \( TF(x) = \int_{S^{d-1}} F(y)1_{[r_x, y] \leq r_k} d\sigma(y) \) is selfadjoint, and commutes with rotations.

The goal of this section is the following second order expansion, in whose formulation \( L^2 \) denotes \( L^2(S^{d-1}, \sigma) \).

**Proposition 14.** Under the hypotheses introduced at the beginning of \( \Xi \)

\[
T(E) \leq T(E^*) - \frac{3}{2} \sum_{k=1}^{3} \gamma_k r_k^{(d-1)} \left( \|F_k^+\|^2_{L^2} + \|F_k^-\|^2_{L^2} \right) + Q(F_1, F_2, F_3) + O(\delta^3).
\]

To begin the proof, substitute \( 1_{B_1} + f_3 \) for \( 1_E \) for each index, and expand \( T(E) \) as a sum of the resulting eight terms. The main term is \( T(B_1, B_2, B_3) = T(E^*) \). There are three other types of terms, which are analyzed in the next three lemmas.

**Lemma 15.** \( |T(f_1, f_2, f_3)| = O(\delta^3) \).

**Proof.** If \( x_1 \in \mathbb{R}^d \) satisfies \( |x_1 - r_1| \leq C\delta \) then the \( \sigma \)-measure of the set of all \( x_2 \in \mathbb{R}^d \) satisfying both \( |x_2 - r_2| \leq C\delta \) and \( |x_1 + x_2 - r_3| \leq C\delta \) is \( O(\delta^2) \) under the hypothesis of \( \rho \)-strict admissibility, provided that \( C\delta \) is sufficiently small relative to \( \rho \). \( \square \)

**Lemma 16.** For each index \( j \in \{1, 2, 3\} \),

\[
\langle K_j, f_j \rangle \leq -\frac{3}{2} \gamma_j r_j^{1-d} \left( \|F_+\|^2_{L^2} + \|F_-\|^2_{L^2} \right) + O(\delta^3).
\]

**Proof.** It is elementary that for each index \( k \in \{1, 2, 3\} \), \( K_k \) is twice continuously differentiable in a neighborhood of the support of \( f_k \). Therefore since \( f_k(t\theta) = 0 \) unless \( |t - r_k| \leq C\delta \),

\[
\langle K_k, f_k \rangle = \int_{S^{d-1}} \int_{0}^{\infty} K_k(t\theta) f_k(t\theta) t^{d-1} dt d\sigma(\theta)
\]

\[
= \int_{S^{d-1}} \int_{0}^{\infty} (K_k(r_k) - \gamma_k(t - r_k) + O(\delta^2)) f_k(t\theta) t^{d-1} dt d\sigma(\theta).
\]

The expression \( K_k(r_k) - \gamma_k(t - r_k) + O(\delta^2) \) leads to three terms. The first of these is \( K_k(r_k) \int_{\mathbb{R}^d} f_k = 0 \). The second is \( O(\delta^2) \|f_k\|_{L^1} = O(\delta^3) \). The second is \( -\gamma_k \int_{S^{d-1}} \int_{0}^{\infty} (t - r_k)(f_k^+ - f_k^-)(t\theta) t^{d-1} dt d\sigma(\theta) \). The integrand is nonnegative, since \( f^+_k - f^-_k \) has the same sign as \( t - r_k \).

Let us momentarily imagine that \( F_k^+ \) is given, and that \( f^+_k \) varies among those functions supported in \( \{t\theta : t \geq r_k\} \), taking values in \( \{0, 1\} \), satisfying \( \int_{0}^{\infty} f^+_k(t\theta) t^{d-1} dt = F^+_k(\theta) \).

\( ^2 \)For \( d = 1 \) this set is empty.
Among all such functions $f_k$, $\int_{\mathbb{R}_+} f_k^+ (t \theta) t^{d-1} dt$ is minimized if $t \mapsto f_k(t \theta)$ is the indicator function of an interval $[r_k, r_k + h(\theta)]$ where $h(\theta) \geq 0$ is defined by the relation $\int_{r_k}^{r_k + h(\theta)} t^{d-1} dt = F_k^+(\theta)$. That is, $(r_k + h(\theta))^d - r_k^d = dF_k^+(\theta)$. Therefore, since $h(\theta) = O(\delta)$,

\[
\tag{35}
 h(\theta) = r_k^{-(d-1)} F_k^+(\theta) + O(\delta F_k^+(\theta)).
\]

This gives

\[
\tag{36}
 \int_{\mathbb{R}_+} (t - r_k) f_k^+ (t \theta) t^{d-1} dt \geq \int_{r_k}^{r_k + h(\theta)} (t - r_k) t^{d-1} dt = \frac{1}{2} r_k^d - \frac{1}{2} h(\theta)^2 - O(h(\theta)^3)
\]

where the constants implicit in the $O(\cdot)$ notation depend only on $d, r_k, \lambda, \delta_0$. The remainder term $O(h(\theta)^3)$ is $O(\delta^3)$ according to (35), since $\|F_k^+\|_{L^\infty} = O(\delta)$. Moreover, $\frac{1}{2} r_k^{d-1} h(\theta)^2 = \frac{1}{2} r_k^{-(d-1)} F_k^+(\theta)^2$, also by (35).

The same analysis may be applied to $f_k^+, f_k^-$, yielding (36).

Lemma 17.

\[
\mathcal{T}(f_i, f_j, 1_{B_k}) = Q_k(F_i, F_j) + O(\delta^3).
\]

That is,

\[
\tag{37}
 \int_{\mathbb{R}^d \times \mathbb{R}^d} f_i(x) f_j(y) 1_{B_k} (x + y) \, dx \, dy = \int_{S^{d-1} \times S^{d-1}} F_i(x) F_j(y) 1_{|r_i x + r_j y| \leq r_k} \, d\sigma(x) \, d\sigma(y) + O(\delta^3).
\]

Proof. For any $\theta_i, \theta_j \in S^{d-1}$,

\[
\int_{\mathbb{R}^d \times \mathbb{R}^d} f_i(t_i, \theta_i) f_j(t_j, \theta_j) t_i^{d-1} t_j^{d-1} 1_{|t_i \theta_i + t_j \theta_j| \leq r_k} (t_i, t_j) \, dt_i \, dt_j = F_i(\theta_i) F_j(\theta_j) 1_{|r_i \theta_i + r_j \theta_j| \leq r_k}
\]

unless $|r_i \theta_i + r_j \theta_j| - r_k | \leq C \delta$. The $\sigma \times \sigma$ measure of the set of all ordered pairs $(\theta_i, \theta_j) \in S^{d-1} \times S^{d-1}$ satisfying $|r_i \theta_i + r_j \theta_j| - r_k | \leq C \delta$ is $O(\delta)$ by the $\rho$-strict admissibility hypothesis. Since $\int |f_i(t_i, \theta_i)| t_i^{d-1} dt_i = O(\delta)$, $F_i = O(\delta)$, and the same bounds hold with $i$ replaced by $j$, the total contribution of all such exceptional pairs $(\theta_i, \theta_j)$ to either side of (37) is $O(\delta^3)$.

Combining the last three lemmas establishes Proposition 14.

7. Spectral analysis

A natural question, in light of what has been shown thus far, is what is the value of the optimal constant $A$ in the inequality

\[
\tag{38}
 Q(F_1, F_2, F_3) \leq A \sum_{k=1}^3 \gamma_k r_k^{1-d} \|F_k\|_{L^2}^2 \quad \forall F_j \in L^2(S^{d-1}) \text{ satisfying } \int_{S^{d-1}} F_j \, d\sigma = 0.
\]

This is potentially relevant because of the inequality

\[
\tag{39}
 \|F_k\|_{L^2}^2 = \|F_k^+\|_{L^2}^2 + 2 \langle F_k^+, F_k^- \rangle \leq \|F_k^+\|_{L^2}^2 + \|F_k^-\|_{L^2}^2,
\]

which is valid since $F_k^\pm$ are nonnegative and $F_k = F_k^+ - F_k^-$. The optimal constant $A$ in (38) cannot be strictly less than $\frac{1}{2}$. For if (38) were to hold for some $A < \frac{1}{2}$ then it would be a direct consequence of the foregoing analysis that for any $\rho$-strictly admissible $E$ satisfying $\max_j |E_j| = 1$ with $\max_j |E_j \Delta E_j^*|$ sufficiently small and
\[ E_j \Delta E_j^* \subset \{ x : |x| - r_j \leq C \max_i |E_i \Delta E_i^*| \text{ for all } j, T(E) \leq T(E^*) - c \max_j |E_j \Delta E_j^*|^2. \]

But this conclusion is false; by virtue of affine invariance of the functional \( T \), it fails for every admissible ordered triple \( E \) of homothetic ellipsoids with centers \( v \) with \( \sum_j v_j = 0 \). Thus the analysis must be refined, to exploit the full strength of the assumption that \( \max_j |E_j \Delta E_j^*| \) has the same order of magnitude as \( \text{dist}(E, O(E^*)) \).

For this purpose, we recast \( Q \) in terms of spherical harmonics. For each nonnegative integer \( n \), denote by \( H_n \subset L^2(S^{d-1}) \) the finite-dimensional subspace of all spherical harmonics of degree \( n \). Then \( L^2(S^{d-1}) = \bigoplus_{n=0}^\infty H_n \). Denote by \( \pi_n \) the orthogonal projection from \( L^2(S^{d-1}) \) onto \( H_n \). This decomposition diagonalizes each of the quadratic forms \( Q_k \), in the sense that there exist compact selfadjoint operators \( T_k \) on \( L^2(S^{d-1}) \) such that \( Q_k(F, G) = \langle T_k(F), G \rangle \), \( T_k : H_n \to H_n \) for all \( n \), and \( T_k \) agrees with a scalar multiple \( \lambda(n, r_1, r_2, r_3) \) on \( H_n \).

Because \( |E_j| = |E_j^*| \), \( \int_{\mathbb{R}^d} f_j = 0 \) for each index \( j \) and consequently \( \int_{S^{d-1}} F_j \, d\sigma = 0 \); that is, \( \pi_0(F_j) = 0 \). Therefore for each \( d \geq 2 \),

\[ Q(F_1, F_2, F_3) = \sum_{n=1}^\infty Q(\pi_n(F_1), \pi_n(F_2), \pi_n(F_3)). \]

The compactness of the linear operators \( T_k \) has the following consequence.

**Lemma 18.** For each \( d \geq 2 \) there exists a sequence \( \Lambda_n = \Lambda_n(\mathbf{r}) \) satisfying \( \lim_{n \to \infty} \Lambda_n = 0 \) such that for each \( n \geq 0 \) and all \( (G_1, G_2, G_3) \in H_3^n \),

\[ |Q(G_1, G_2, G_3)| \leq \Lambda_n \sum_{j=1}^3 \|G_j\|_{L^2(S^{d-1})}^2. \]

Denote by \( I \) the identity mapping \( I : \mathbb{R}^d \to \mathbb{R}^d \). The next two results will be proved in (C) below.

**Lemma 19.** Let \( d \geq 2 \). Let \( E \) be as above, and let \( \delta = \text{dist}(E, O(E^*)) \). There exist \( v, \psi \) satisfying \( |v| = O(\delta) \) and \( \|\psi - I\| = O(\delta) \) such that the functions \( \tilde{F}_j \) associated to the sets \( \tilde{E}_j = \psi(E_j^*) + v_j \) satisfy

\[ \begin{cases} 
\pi_1(\tilde{F}_j) = 0 \text{ for } j = 1 \text{ and } j = 2 \\
\pi_2(\tilde{F}_1) = 0.
\end{cases} \]

Here \( v \in (\mathbb{R}^d)^3 \) satisfies \( v_1 + v_2 + v_3 = 0 \), and \( \psi \) is a Lebesgue measure-preserving linear automorphism of \( \mathbb{R}^d \).

“Associated”, throughout the discussion, means that \( \tilde{F}_j(\theta) = \int_0^\infty (1_{E_j} - 1_{B_j})(t\theta) t^{d-1} \, dt \).

The norm \( \|\psi - I\| \) is defined by choosing any fixed norm on the vector space of all \( d \times d \) real matrices, expressing the elements \( \psi, I \) of the general linear group as such matrices, and taking the norm of the difference of the two associated matrices.

There are no spherical harmonics of degrees \( > 1 \) for \( d = 1 \), and the group of measure-preserving linear automorphisms is trivial, so the situation is simpler.

**Lemma 20.** Let \( d = 1 \). Let \( E \) be as above, and let \( \delta = \text{dist}(E, O(E^*)) \). There exists \( v \in \mathbb{R}^3 \) satisfying \( |v| = O(\delta) \) and \( \sum_{j=1}^3 v_j = 0 \) such that the functions \( \tilde{F}_j \) associated to the sets \( \tilde{E}_j = E_j + v_j \) vanish identically on \( S^0 \).

The symmetry group of the functional \( T \) is not sufficiently large to enable further reductions of these types, as a simple dimension count demonstrates.
For $d = 1$, $F_j = \pi_1(F_j)$ for each index $j$, because there are no spherical harmonics of higher degrees. Therefore Lemma 20 suffices to complete the proof of Proposition 12 hence the proof of Theorem 6 for $d = 1$.

For $d \geq 2$, Lemma 19 eliminates crucial terms from (40). In particular, spherical harmonics of degree 2 are eliminated; if two of the three functions $F_m$ vanish then $Q_k(F_1, F_j) = 0$ for all $(i,j,k)$. Some analysis will be required to show that elimination of these terms suffices to make the optimal constant $A$ strictly less than $\frac{1}{2}$.

The significance of the conclusions $|v| = O(\delta)$ and $\|\phi - I\| = O(\delta)$ is that these ensure that the sets $\tilde{E}_j = \phi(E_j) + v_j$ continue to satisfy

$$\tilde{E}_j \Delta E_j^* \subset \{x : |x| - r_j \leq C\delta\}$$

for a certain finite constant $C$. Therefore the above analysis applies equally well to these sets, and we can simply replace $E$ by $\tilde{E}$ henceforth.

For $d \geq 2$, replace $E_j$ by $\tilde{E}_j$ for all three indices. In order to prove Proposition 12 and hence Theorem 6 for arbitrary dimensions, it now suffices to prove the following two results.

**Lemma 21.** Let $d \geq 2$. Let $(r_1, r_2, r_3)$ be strictly admissible. For each $n \geq 3$ there exists $A < \frac{1}{2}$ such that for all ordered triples $(G_1, G_2, G_3)$ of spherical harmonics $G_j : S^{d-1} \to \mathbb{R}$ of degree $n$,

$$Q(G_1, G_2, G_3) \leq A \sum_{k=1}^{3} \gamma_k r_k^{1-d} \|G_k\|_{L^2(S^{d-1})}^2. \tag{43}$$

**Lemma 22.** Let $d \geq 2$. Let $(r_1, r_2, r_3)$ be strictly admissible. There exists $A < \frac{1}{2}$ such that for all spherical harmonics $G_2, G_3 : S^{d-1} \to \mathbb{R}$ of degree 2,

$$Q_1(G_2, G_3) \leq A \sum_{k=2}^{3} \gamma_k r_k^{1-d} \|G_k\|_{L^2(S^{d-1})}^2. \tag{44}$$

These two results will be proved in 110. Together with Lemma 18, Lemma 22 gives this corollary:

**Corollary 23.** For each $d \geq 2$ there exists $A < \frac{1}{2}$ with the following property. Let $F_j \in L^2(S^{d-1})$. Suppose that $\pi_0(F_j) = 0$ for all $j \in \{1, 2, 3\}$, that $\pi_1(F_j) = 0$ for $j = 1, 2$, and that $\pi_2(F_1) = 0$. Then

$$Q(F_1, F_2, F_3) \leq A \sum_{k=1}^{3} \gamma_k r_k^{1-d} \|F_k\|_{L^2}^2. \tag{45}$$

Rather than calculating $\gamma_j$ and the eigenvalues of the operators associated to the quadratic forms $Q_j$ on $\mathcal{H}_n$ (all of which are functions of $r = (r_1, r_2, r_3)$), we will carry out a more conceptual analysis of the difference $Q(G_1, G_2, G_3) - \frac{1}{2} \sum_{k=1}^{3} \gamma_k r_k^{1-d} \|G_k\|_{L^2(S^{d-1})}^2$ for ordered triples $(G_1, G_2, G_3)$ of spherical harmonics of degree $n$.

8. **Interlude**

We digress to explain why we are not able to analyze $Q_k$ by means of explicit formulas for spherical harmonics. $Q_k(G_i, G_j)$ takes the form $(S_{\rho}(G_i), G_j)$ where the inner product is that of $L^2(S^{d-1})$, and $S_{\rho}$ is the linear operator on $L^2(S^{d-1})$ defined by

$$S_{\rho}G(x) = \int_{S^{d-1}} G(y) 1_{x \cdot y \leq \rho} \, d\sigma(y)$$
where \( \rho = (2r_i r_j)^{-1}(r_k^2 - r_i^2 - r_j^2) \). All \( \rho \) in a certain open interval arise from admissible ordered triples \((r_1, r_2, r_3)\).

Acting on spherical harmonics of degree \( k \), \( S_\rho \) is a scalar multiple \( \lambda_k(\rho) \) of the identity. Let \( P_k \) be the Gegenbauer polynomials. These can be defined by the generating function expansion

\[
(1 + s^2 - 2st)^{-(d-2)/2} = \sum_{k=0}^{\infty} P_k(t)s^k.
\]

Then \( Z_k(x) = P_k(x, \rho) \) is a spherical harmonic of degree \( k \); these are the zonal harmonics, up to scalar factors which are of no consequence here.

The value of \( S_\rho(Z_k) \) at the point \( N = (0, 0, \ldots, 0, 1) \) is the integral of \( Z_k \) over a spherical cap centered at \( N \), whose radius varies with \( \rho \). Thus a calculation of \( \lambda_k(\rho) \) for all \( \rho \) equivalent to a calculation of the ratio of \( S_\rho(Z_k)(N) \) to \( Z_k(N) \). This amounts to a calculation of the indefinite integral \( \int P_k(t)(1-t^2)^{(d-3)/2} dt \). An explicit formula for the indefinite integral would give an explicit formula for \( P_k(t) \), after differentiation and division by \( (1-t^2)^{(d-3)/2} \).

9. Balancing via affine automorphisms

Let \( \mathbb{B} \) be the closed ball of radius 1, centered at the origin, in \( \mathbb{R}^d \). Consider bounded Lebesgue measurable sets \( E \subset \mathbb{R}^d \) that satisfy \( |E| = |\mathbb{B}| \). To \( E \) is associated the function \( F = F_E : S^{d-1} \rightarrow \mathbb{R} \) defined by

\[
F_E(\theta) = \int_0^\infty (1_E - 1_\mathbb{B})(t\theta) t^{d-1} \, dt.
\]

**Definition 5.** Let \( D \in \mathbb{N} \). A bounded Lebesgue measurable set \( E \subset \mathbb{R}^d \) satisfying \( |E| = |\mathbb{B}| \) is balanced up to degree \( D \) if the function \( F_E : S^{d-1} \rightarrow \mathbb{R} \) associated to \( E \) by \([17]\) satisfies

\[
\int_{S^{d-1}} F_E(y)P(y) \, d\sigma(y) = 0
\]

for every polynomial \( P : \mathbb{R}^d \rightarrow \mathbb{R} \) of degree less than or equal to \( D \).

For \( D = 0 \), \([17]\) asserts that \( \int (1_E - 1_\mathbb{B}) = 0 \), which is simply a restatement of the hypothesis \( |E| = |\mathbb{B}| \).

Denote by \( \text{Aff}(d) \) the group of all affine automorphisms of \( \mathbb{R}^d \). Denote by \( \mathcal{M}_d \) the vector space of all \( d \times d \) square matrices with real entries, and by \( \mathcal{M}_d \oplus \mathbb{R}^d \) the set of all ordered pairs \((T, v)\) where \( T \in \mathcal{M}_d \) and \( v \in \mathbb{R}^d \), with the natural vector space structure. Identify elements of \( \mathcal{M}_d \) with linear endomorphisms of \( \mathbb{R}^d \) in the usual way; \( S = (T, v) \) acts by \( S(x) = T(x) + v \). Fix any norm \( \| \cdot \|_{\mathcal{M}_d} \) on \( \mathcal{M}_d \).

Elements \( \phi \in \text{Aff}(d) \) take the form \( \phi(x) = T(x) + v \) where \( T : \mathbb{R}^d \rightarrow \mathbb{R}^d \) is an invertible linear transformation, and \( v \in \mathbb{R}^d \). \( T \) can be identified with an element of \( \mathcal{M}_d \), and \( (T, v) \) is thus identified with a unique element of \( \mathcal{M}_d \oplus \mathbb{R}^d \). Define \( \|\phi\|_{\text{Aff}(d)} = \|T\|_{\mathcal{M}_d} + \|v\|_{\mathbb{R}^d} \). We abuse notation by writing \( \det(\phi) \) for the determinant of the unique \( T \in \mathcal{M}_d \) thus associated to \( \phi \), and likewise \( \text{trace}(\phi) = \text{trace}(T) \).

**Lemma 24.** Let \( d \geq 1 \). There exists \( c > 0 \) such that for every \( \lambda \geq 1 \) there exists \( C_\lambda < \infty \) with the following property. For any Lebesgue measurable set \( E \subset \mathbb{R}^d \) satisfying \( |E| = |\mathbb{B}| \), \( \lambda |E \Delta \mathbb{B}| \leq c \), and \( E \Delta \mathbb{B} \subset \{ x : |x| - 1 \leq \lambda |E \Delta \mathbb{B}| \} \), there exists a measure-preserving
affine transformation \( \phi \in \text{Aff}(d) \) such that

\[
\phi(E) \text{ is balanced up to degree 2,}
\]

\[
\|\phi - I\|_{\text{Aff}(d)} \leq C_\lambda |E| \Delta B,
\]

\[
\phi(E) \Delta B \subset \{x : |1 - |x|| \leq C_\lambda |E| \Delta B\}.
\]

Denote by \( W_2 \) the real vector space of all polynomials \( P : \mathbb{R}^d \to \mathbb{R} \) that are finite linear combinations of homogeneous harmonic polynomials of degrees \( \leq 2 \). Denote by \( V_2 \) the real vector space of all restrictions to \( S^{d-1} \) of real-valued polynomials of degrees \( \leq 2 \). The natural linear mapping from \( W_2 \) to \( V_2 \) induced by restriction from \( \mathbb{R}^d \) to \( S^{d-1} \) is a bijection \([13]\).

Regard \( V_2 \) as a real inner product space, with the \( L^2(S^{d-1}, \sigma) \) inner product. Denote by \( \Pi \) the orthogonal projection of \( L^2(S^{d-1}) \) onto its subspace \( V_2 \). Define \( \mathfrak{A} : \mathcal{M}_d \oplus \mathbb{R}^d \to V_2 \) by

\[
(48) \quad \mathfrak{A}(S)(\alpha) = \Pi(\alpha \cdot S(\alpha)),
\]

that is, the right-hand side equals the restriction to \( S^{d-1} \) of the quadratic polynomial \( \mathbb{R}^d \ni x \mapsto x \cdot S(x) \).

**Lemma 25.** \( \mathfrak{A} : \mathcal{M}_d \oplus \mathbb{R}^d \to V_2 \) is surjective.

**Proof.** The range of \( \mathfrak{A} \) is the collection of all functions \( S^{d-1} \ni \alpha \mapsto S(\alpha) \cdot \alpha \), as the function \( S \) varies over all affine mappings from \( \mathbb{R}^d \) to \( \mathbb{R}^d \). Because \( S \mapsto \mathfrak{A}(S) \) is linear, this range is a subspace of \( V_2 \).

Firstly, the constant function \( \alpha \mapsto 1 \) equals \( \mathfrak{A}(S) \) when \( S(x) \equiv x \), since \( S(\alpha) \cdot \alpha = \alpha \cdot \alpha \equiv 1 \) for \( \alpha \in S^{d-1} \). Secondly, a linear monomial \( \alpha = (\alpha_1, \ldots, \alpha_d) \mapsto \alpha_k \) is expressed by choosing \( S(x) \equiv e_k \), the \( k \)-th unit coordinate vector. Thirdly, to express a monomial \( \alpha \mapsto \alpha_j \alpha_k \) in the form \( S(\alpha) \cdot \alpha \), define \( S(x) = (S_1(x), \ldots, S_d(x)) \) by \( S_i(x) \equiv 0 \) for all \( i \neq j \), and \( S_j(x) = x_k \). Then \( \alpha_j \alpha_k = S(\alpha) \cdot \alpha \). Functions of these three types span \( V_2 \), so \( \mathfrak{A} \) is indeed surjective. \( \square \)

**Proof of Lemma 24.** If \( c \leq \frac{1}{2} \) then \( E \) contains the ball of radius \( \frac{1}{2} r \) centered at 0, so if \( \phi \in \text{Aff}(d) \) is sufficiently close to the identity then \( \phi(E) \) contains the ball of radius \( \frac{1}{4} r \) centered at 0.

Let \( k \in \{0, 1, 2\} \). Let \( P : \mathbb{R}^d \to \mathbb{R} \) be a homogeneous harmonic polynomial of degree \( k \). Let \( g(x) \) be a smooth function that agrees with \( |x|^{-k} P(x) \) in \( \{x : ||x|| - 1| \leq \frac{3}{4}\} \). Moreover, choose \( g \) so that the map \( P \mapsto g \) is linear over \( \mathbb{R} \).

For \( \phi \in \text{Aff}(d) \) let \( f_{\phi(E)} = 1_{\phi(E)} - 1_B \) and \( F_{\phi(E)} \) be the functions associated to \( \phi(E) \) in the same way that \( f = 1_E - 1_B \) and \( F \) are associated to \( E \). Then if \( \phi \) is sufficiently close to the identity,

\[
(49) \quad \int_{S^{d-1}} F_{\phi(E)}(y) P(y) d\sigma(y) = \int_{S^{d-1}} \int_{0}^{\infty} (1_{\phi(E)} - 1_B)(ry) r^{d-1} dr P(y) d\sigma(y)
\]

\[
= \int_{\mathbb{R}^d} (1_{\phi(E)} - 1_B)(x)|x|^{-k} P(x) dx
\]

\[
= \int_{\mathbb{R}^d} (1_E \circ \phi^{-1} - 1_B) g
\]

\[
= \int_{\mathbb{R}^d} (f \circ \phi^{-1}) g + \int_{\mathbb{R}^d} (1_B \circ \phi^{-1} - 1_B) g.
\]
The second to last equation holds because both \( \mathbb{B} \) and \( \phi(E) \) contain the ball of radius \( \frac{1}{4} \) centered at 0, and \( g(x) \equiv |x|^{-k} P(x) \) for all \( x \) in the complement of this ball. All of these quantities depend linearly on \( g \).

We seek the desired \( \phi \in \text{Aff}(d) \) in the form \( \phi = I + S \), where \( \|S\|_{\text{Aff}(d)} \) is small and \( I \) is the identity matrix; that is, \( \phi(x) = x + S(x) \) where \( S \) is an affine mapping. The second term on the right-hand side of (19) is independent of \( P \).

Moreover,

\[
\int_{\mathbb{R}^d} (1_\mathbb{B} \circ \phi^{-1}) g = |\det(\phi)| \int_{\mathbb{B}} g \circ \phi
= (1 + \text{trace}(S)) \int_{\mathbb{B}} g \circ \phi + O_P(\|S\|^2_{\text{Aff}(d)})
= (1 + \text{trace}(S)) \int_{\mathbb{B}} g(x + S(x)) \, dx + O_P(\|S\|^2_{\text{Aff}(d)}).
\]

Here and below, \( O_P(\|S\|^2_{\text{Aff}(d)}) \) denotes a quantity that depends linearly on \( P \), whose norm or absolute value, as appropriate, is bounded above by a constant multiple of the norm of \( P \) times the \( \text{Aff}(d) \) norm squared of \( S \).

Invoking the Taylor expansion of \( g \) about \( x \) gives

\[
\int_{\mathbb{R}^d} (1_\mathbb{B} \circ \phi^{-1}) g = (1 + \text{trace}(S)) \int_{\mathbb{B}} g + \int_{\mathbb{B}} \nabla g \cdot S + O_P(\|S\|^2_{\text{Aff}(d)})
= \int_{\mathbb{B}} g + \int_{\mathbb{B}} (g \text{trace}(S) + \nabla g \cdot S) + O_P(\|S\|^2_{\text{Aff}(d)})
= \int_{\mathbb{B}} g + \int_{\mathbb{B}} \text{div}(gS) + O_P(\|S\|^2_{\text{Aff}(d)})
= \int_{\mathbb{B}} g + \int_{S^{d-1}} g(\alpha)S(\alpha) \cdot \alpha \, d\sigma(\alpha) + O_P(\|S\|^2_{\text{Aff}(d)})
= \int_{\mathbb{B}} g + \int_{S^{d-1}} P(\alpha)S(\alpha) \cdot \alpha \, d\sigma(\alpha) + O_P(\|S\|^2_{\text{Aff}(d)}).
\]

The second to last equality is justified by the divergence theorem, and the last by the identity \( g \equiv P \) on \( S^{d-1} \). Thus

\[
\int_{\mathbb{R}^d} (1_\mathbb{B} \circ \phi^{-1} - 1_\mathbb{B}) g = \int_{S^{d-1}} P(\alpha)S(\alpha) \cdot \alpha \, d\sigma(\alpha) + O_P(\|S\|^2_{\text{Aff}(d)}).
\]

Since \( \int_{\mathbb{R}^d} (f \circ \phi^{-1}) g = \int_{\mathbb{R}^d} (f \circ \phi^{-1}) P(x) |x|^{-k} \), by returning to (19) we find that the equation \( \int_{S^{d-1}} F_{\phi(E)} P \, d\sigma = 0 \) for all \( P \in V_2 \), for an unknown \( S \in \text{Aff}(d) \), takes the form

\[
(50) \quad \int_{S^{d-1}} P(\alpha)S(\alpha) \cdot \alpha \, d\sigma(\alpha) = - \int_{\mathbb{R}^d} (f \circ \phi^{-1}) P(x) |x|^{-k} \, dx + O_P(\|S\|^2_{\text{Aff}(d)}) \quad \forall \, P \in V_2.
\]

All three terms in this equation depend linearly on \( P \), so by interpreting each term as the inner product of \( P \) with an element of \( V_2^* \) we may regard this as an equation in \( V_2^* \), thus eliminating \( P \). Equivalently, it will be regarded as an equation in the Hilbert space \( V_2 \).

Write this equation as

\[
(51) \quad \mathfrak{A}(S) = \mathcal{N}_f + \mathcal{R}(S)
\]

where \( \mathfrak{A} \) is defined above, \( \mathcal{N}_f \) is the mapping \( P \mapsto - \int_{\mathbb{R}^d} (f \circ \phi^{-1}) P(x) |x|^{-k} \, dx \), and \( \mathcal{R} \) represents the term \( P \mapsto O_P(\|S\|^2_{\text{Aff}(d)}) \). Both \( \mathfrak{A} \) and \( \mathcal{R} \) are twice continuously differentiable.
functions of $S \in \mathcal{M}_d \oplus \mathbb{R}^d$. Moreover,
\begin{equation}
\|N_f\|((S)v_2) \leq C|E \Delta B|
\end{equation}

simply because $|f| \leq 1_{|E \Delta B|}$ and $f$ is supported where $\frac{1}{2} \leq |x| \leq \frac{3}{2}$. Since $\mathfrak{A} : \mathcal{M}_d \oplus \mathbb{R}^d \rightarrow V_2$ is surjective, the Implicit Function Theorem guarantees that the equation $\mathfrak{A}(S) = N_f + \mathcal{R}(S)$ admits a solution $S \in \mathcal{M}_d \oplus \mathbb{R}^d$ satisfying $\|S\|_{\mathcal{M}_d \oplus \mathbb{R}^d} \leq C|E \Delta B|$. 

**Lemma 26.** Let $d \geq 1$. There exists $c > 0$ such that for every positive constant $\lambda < \infty$ there exists $C_\lambda < \infty$ with the following property. For any Lebesgue measurable set $E \subset \mathbb{R}^d$ satisfying $|E| = |B|$, $\lambda|E \Delta B| \leq c$, and $E \Delta B \subset \{x : |x| - 1| \leq \lambda|E \Delta B|\}$, there exists $v \in \mathbb{R}^d$ such that

$$E + v \text{ is balanced up to degree } 1,$$

$$|v| \leq C_\lambda|E \Delta B|,$$

$$(E + v) \Delta B \subset \{x : |1 - |x|| \leq C_\lambda|E \Delta B|\}.$$

Again, the constant $C_\lambda$ depends on the constant $\lambda$ and on the dimension $d$, but not on the set $E$.

The proof of Lemma 26 is a simplified variant of the proof of Lemma 24. No additional ideas are required. Details are omitted. 

**Lemma 19** is a direct application of Lemmas 24 and 26 together with dilation. Choose $\psi$ and $v_1$ so that $\tilde{E}_1 = \psi(E_1) + v_1$ satisfies the desired conclusion; $\pi_n(\tilde{E}_1) = 0$ for $n = 1$ and $n = 2$. Define $v_2 = 0$ and $v_3 = -v_1$, and define $\tilde{E}_j = \psi(E_j) + v_j$ for $j = 2, 3$. Rename these new sets to be $E_j$, and begin again. Now choose $\psi$ to be the identity, and define $\tilde{E}_2 = E_2 + v_2$ where the new vector $v_2$ is chosen so that $\pi_1(\tilde{E}_2) = 0$. Define new vectors $v_1, v_3$ by $v_1 = 0$ and $v_3 = -v_2$, and define $\tilde{E}_j = E_j + v_j$ for $j = 1, 3$. The resulting doubly modified ordered triple of sets satisfies the conclusions of Lemma 19.

Likewise, Lemma 20 follows directly from Lemma 26.

10. **Final steps**

Let $n \geq 3$, and let $G = (G_1, G_2, G_3)$ be an ordered triple of spherical harmonics of common degree $n$ on $S^{d-1}$, satisfying $\|G\|^2 = \sum_{j=1}^3 \|G_j\|_{L^2}^2 = 1$. For each $j \in \{1, 2, 3\}$ define $\varphi_j : S^{d-1} \times (-\frac{1}{2}, \frac{1}{2}) \rightarrow \mathbb{R}^+$ as follows. If $sG_j(\theta) \geq 0$ then $\varphi_j(\theta, s) \geq 0$, and

$$\int_{r_j + \varphi_j(\theta, s)}^{r_j} t^{d-1} dt = sG_j(\theta).$$

If $sG_j(\theta) \leq 0$ then $\varphi_j(\theta, s) \leq 0$, and

$$\int_{r_j + \varphi_j(\theta, s)}^{r_j} t^{d-1} dt = -sG_j(\theta).$$

Equivalently, for either sign,

\begin{equation}
(r_j + \varphi_j(\theta, s))^d - r_j^d = dsG_j(\theta)
\end{equation}

Thus

\begin{equation}
r_j^{d-1}\varphi_j(\theta, s) = sG_j(\theta) + O(s^2),
\end{equation}

and in a neighborhood of $s = 0$, $(\theta, s) \mapsto \varphi_j(\theta, s)$ is a $C^\infty$ function specified by (52).

For $s \in \mathbb{R}$ with $|s|$ small define sets $E_j(s) \subset \mathbb{R}^d$ by

\begin{equation}
E_j(s) = \{t\theta : t \leq r_j + \varphi_j(\theta, s)\}.
\end{equation}

Since $\int_{S^{d-1}} G d\sigma = 0$, $|E_j(s)| = |E_j|$ for all $s$ in a neighborhood of 0. The function $F_{j,s}$ associated to $E_j(s)$ depends smoothly on $(\theta, s)$ and satisfies $F_{j,s} \equiv sG_j + O(s^2)$. Define

\begin{equation}
E(s) = E_G(s) = (E_j(s) : 1 \leq j \leq 3).
\end{equation}
Lemma 27. For any \( d \geq 2, \rho > 0 \), and \( n \in \mathbb{N} \) there exists \( c > 0 \) such that uniformly for each \( \rho \)-strictly admissible \( r \) satisfying \( \max_j r_j = 1 \) and for all 3-tuples \( G \) of spherical harmonics of degree \( n \) satisfying \( \|G\| = 1 \), there exists \( \eta > 0 \) such that

\[
T(E(s)) = T(E^*) - \frac{1}{2} s^2 \sum_{k=1}^{3} \frac{\gamma_k r_k^{1-d}}{c} ||G_k||^2_{L^2} + s^2 \mathcal{Q}(G) + O(|s|^3)
\]

whenever \( |s| \leq \eta \).

Lemma 27 and Proposition 14 are closely related, but differ in essential ways. The lemma has the stronger conclusion; in the lemma the two sides of the equation are equal, whereas in the proposition, the left-hand side is less than or equal to the right-hand side. The stronger conclusion does not hold under the hypotheses of the proposition. On the other hand, the lemma applies only to a very special class of sets. Two properties of these sets make possible a more detailed analysis of the terms \( \langle K_k, f_k \rangle \) for \( E(s) \), which leads to the stronger conclusion. Firstly, \( F_k^\pm \) have disjoint supports, so that \( ||F_k^\pm||^2_{L^2} = ||F_k^+||^2_{L^2} + ||F_k^-||^2_{L^2} \). Secondly, for each \( \theta \in S^{d-1} \), \( \{ t \in \mathbb{R}^+ : \theta \in E_k \setminus B_k \} \) is an interval whose left endpoint equals \( r_k \), and likewise \( \{ t \in \mathbb{R}^+ : \theta \in B_k \setminus E_k \} \) is an interval whose right endpoint equals \( r_k \). Combining these two facts with the proof of Proposition 14 establishes Lemma 27.

The next two lemmas, 28 and 29, will be proved below.

Lemma 28. Let \( n \geq 3, d \geq 2, \) and \( \rho > 0 \). There exists \( c > 0 \), depending on \( n, d, \rho \) such that for each \( \rho \)-strictly admissible \( r \) satisfying \( \max_j r_j = 1 \), for all 3-tuples \( G \) of spherical harmonics of degree \( n \) satisfying \( \|G\| = 1 \),

\[
T(E(s)) \leq T(E^*) - cs^2
\]

for all \( s \in \mathbb{R} \) sufficiently close to 0.

The conclusion holds uniformly for all \( s \) in a neighborhood of 0 that is independent of \( G \). This neighborhood, and the constant \( c \), are permitted to depend on \( n, d, \rho \). What is essential for the application is that \( c \) is independent of \( s, G \) for all \( s \) sufficiently close to 0.

Lemma 29. Let \( d \geq 2 \) and \( \rho > 0 \). There exists \( c > 0 \), depending on \( d, \rho \) such that for each \( \rho \)-strictly admissible \( r \) satisfying \( \max_j r_j = 1 \), for all 3-tuples \( G \) of spherical harmonics of degree 2 satisfying \( \|G\| = 1 \) and \( G_1 = 0 \),

\[
T(E(s)) \leq T(E^*) - cs^2
\]

for all \( s \in \mathbb{R} \) sufficiently close to 0.

Proof of Lemma 27. Upon dividing by \( s^2 \) in (57) and extracting the limit as \( s \to 0 \), recalling the normalization \( \|G\| = 1 \), we conclude from (58) that there exists \( c' = c'(n, d, \rho) > 0 \) such that for all ordered triples of spherical harmonics of common degree \( n \),

\[
-\frac{1}{2} \sum_{k=1}^{3} \frac{\gamma_k r_k^{1-d}}{c} ||G_k||^2_{L^2} + \mathcal{Q}(G) \leq -c' \|G\|^2.
\]

As noted in the above discussion of the lack of need for bounds uniform in \( n \geq 3 \), this is equivalent to the conclusion of Lemma 27.

In the same way, Lemma 22 is a direct consequence of Lemma 29.

The remainder of this section is devoted to the proofs of Lemmas 28 and 29. To begin the proof of Lemma 28 let \( n, d, \rho, r, G \) be given. For any fixed degree \( n \), the hypothesis \( \|G\| = 1 \) implies upper bounds on each \( G_j \) in \( C^\infty(S^{d-1}) \). For each \( s \in \mathbb{R} \) with small
absolute value define $E(s)$ as above. We now proceed to analyze $T(E(s))$ directly, without using the reduction to $S^{d-1}$ developed earlier in the analysis. Via (57), this will give the desired control on the optimal constant $A$ in (38).

Define $\Sigma$ to be the set of all $x' = (x'_1, x'_2, x'_3) \in (\mathbb{R}^{d-1})^3$ that satisfy $x'_1 + x'_2 + x'_3 = 0$. For each index $j$, define

$$I_j(x', s) = \{ t \in \mathbb{R} : (x', t) \in E_j(s) \}.$$  

From the uniform upper bounds for $G$ and all of its derivatives, and from the $\rho$-strict admissibility hypothesis, it follows that there exist a neighborhood $V$ of $(0, 0, 0) \in (\mathbb{R}^{d-1})^3$ and $\eta > 0$ such that for all $x' \in V \cap \Sigma$ and all $s \in [-\eta, \eta]$, each set $I_j(x', s) \subset \mathbb{R}$ is an interval, and $\left( |I_j(x', s)| : 1 \leq j \leq 3 \right)$ is a $2\rho$–strictly admissible ordered triple of positive real numbers close to $(r_1, r_2, r_3)$.

Let $c_j(x', s)$ be the center of the interval $I_j(x', s)$. For $x' \in \mathbb{R}^{d-1}$ in a small neighborhood of 0 and for $|s|$ small, the upper endpoint, $t_+$, of $I_j(x', s)$ is the unique solution $t$ of

$$|x'|^2 + t^2 = (r_j + \varphi_j(\theta, s))^2$$

where $S^{d-1} \ni \theta = (|x'|^2 + t^2)^{-1/2}(x', t)$. Write $t_0 = t_0(x')$ for the positive solution of $|x'|^2 + t_0^2 = r_j^2$. Thus by (54),

$$t_+ = t_0(1 + 2t_0^{-2}r_j^{-2-d}G_j(\theta) + O(s^2))^{1/2} = t_0 + sr_j^{2-d}t_0^{-1}G_j(\theta) + O(s^2).$$

$G_j$ is equal to the restriction to $S^{d-1}$ of a (unique) homogeneous harmonic polynomial of degree $n$, also denoted by $G_j$, defined on $\mathbb{R}^d$. Writing $G_j(\theta) = (|x'|^2 + t_+^2)^{-n/2}G_j(x', t_+)$ and noting that $t_+ = t_0 + O(s)$ gives

$$t_+ = t_0 + sr_j^{2-d-n}t_0^{-1}G_j(x', t_0) + O(s^2),$$

bearing in mind that $t_0$ is a function of $x'$. In the same way, the lower endpoint, $t_-$, of $I_j(x', s)$ is

$$t_- = -t_0 - sr_j^{2-d-n}t_0^{-1}G_j(x', -t_0) + O(s^2).$$

Therefore

$$c_j(x', s) = \frac{1}{2}sr_j^{2-d-n}t_0^{-1}G_j(x', t_0) + O(s^2).$$

Write $G_j = G_{j,e} + G_{j,o}$ by expanding $G_j(x', x_d)$ (regarded as a function of $(x', x_d) \in \mathbb{R}^d$) as a linear combination of monomials in $x' = (x', x_d)$ and defining $G_{j,e}(x', x_d)$ to be the contribution of all monomials having even degrees with respect to $x_d$, and $G_{j,o}(x', x_d)$ to be the contribution of all monomials having odd degrees with respect to $x_d$. Then

$$c_j(x', s) = sr_j^{2-d-n}x_d^{-1}G_{j,o}(x', x_d) + O(s^2).$$

The quantity $r_j^{2-d-n}x_d^{-1}G_{j,o}(x', x_d)$ is a sum of monomials, in each of which $x_d = (r_j^2 - |x'|^2)^{1/2}$ is raised to an even power, because of the factor of $x_d^{-1}$. Therefore we may rewrite this last identity in the form

$$c_j(x', s) = sP_j(x') + O(s^2)$$

where

$$P_j(x') = \frac{1}{2}sr_j^{2-d-n}G_{j,e}(x', x_d).$$
where \( P_j : \mathbb{R}^{d-1} \to \mathbb{R} \) is a polynomial of degree at most \( n - 1 \), defined by
\[
P_j(x') = r_j^{2-d-n}x_d^{-1}G_{j,0}(x', x_d) \quad \text{with} \quad x_d = (r_j^2 - |x'|^2)^{1/2}.
\] (64)

The coefficients of \( P_j \) are bounded above, uniformly in all ordered triples \( G \) of spherical harmonics of degree \( n \) satisfying \( \|G\| = 1 \).

Write
\[
\mathcal{T}(E(s)) = \int_{x_1' + x_2' + x_3' = 0} \mathcal{T}_1(I_1(x_1', s), I_2(x_2', s), I_3(x_3', s)) \, d\lambda(x').
\] (65)

For any \( x' \),
\[
\mathcal{T}_1(I_1(x_1', s), I_2(x_2', s), I_3(x_3', s)) \leq \mathcal{T}_1(I_1(x_1', s)^*, I_2(x_2', s)^*, I_3(x_3', s)^*)
\] (66)

by the one-dimensional Riesz-Sobolev inequality. Crucially, there is an improvement in the case in which the intervals \( \mathcal{T}_1(I_j(x_j', s)) \) do not have compatible centers.

**Lemma 30.** For each \( \rho > 0 \) there exists \( a_\rho > 0 \) with the following property. For \( j \in \{1, 2, 3\} \) let \( I_j \subset \mathbb{R} \) be closed bounded intervals with centers \( c_j \). Suppose that \( (|I_j| : 1 \leq j \leq 3) \) is \( \rho \)-strictly admissible. Then
\[
\mathcal{T}_1(I_1, I_2, I_3) \leq \mathcal{T}_1(I_1^*, I_2^*, I_3^*) - a_\rho|c_1 + c_2 + c_3|^2.
\] (67)

The proof of this lemma is straightforward, and is omitted. As an alternative, one could invoke Theorem 1 for \( d = 1 \); but the case of intervals is much simpler than that of general sets.

Applying this lemma yields
\[
\mathcal{T}_1(I_1(x_1', s), I_2(x_2', s), I_3(x_3', s)) \leq \mathcal{T}_1(I_1(x_1', s)^*, I_2(x_2', s)^*, I_3(x_3', s)^*) - a|x_1'| + |x_2'| + |x_3'|
\] (68)

for a certain constant \( a > 0 \), for all \( x' \) in a sufficiently small neighborhood of the origin in \((\mathbb{R}^{d-1})^3\), uniformly for all sufficiently small \( s \). Therefore by the relation (63) between \( c_j(x', s), s \), and \( P_j(x') \), for all \( x' \in \Sigma \) sufficiently close to \((0, 0, 0)\),
\[
\mathcal{T}_1(I_1(x_1', s), I_2(x_2', s), I_3(x_3', s)) \leq \mathcal{T}_1(I_1(x_1', s)^*, I_2(x_2', s)^*, I_3(x_3', s)^*) - as^2P^2(G)(x')^2 + O(s^3)
\] (69)

uniformly in \( x', s, G \) for fixed \( d, n, \rho \), where \( P^2(G) \) is defined on \( \Sigma \) by
\[
P^2(G)(x') = \sum_{j=1}^3 P_j(x_j'),
\] (70)

with the polynomial \( P_j \) defined in terms of \( G_j \) as above.

By a polynomial \( P \) of degree \( D \) with domain \( \Sigma \) we mean any function with domain such that \( (x_1, x_2) \mapsto P(x_1, x_2, -x_1 - x_2) \) is a polynomial of degree \( D \). Introduce any norm on the vector space of all polynomials \( P : \Sigma \to \mathbb{R} \) of degrees \( \leq n - 1 \). Combining (65) and (66), we have established the following lemma.

**Lemma 31.** With the above hypotheses and notations,
\[
\mathcal{T}(E_G(s)) \leq \mathcal{T}(E^*) - cs^2\|P^2(G)\|^2 + O(s^3).
\] (71)
Lemma 33. Let \( \alpha \) be a rotation in the \((1,2,3)\) plane; \( G \) is a nonzero ordered triple of spherical harmonics of degree \( n \) then there exists \( \mathcal{O} \in O(d) \) such that \( P^2(\mathcal{O}(\mathcal{G})) \neq 0 \).

It follows immediately from a simple compactness argument that for each \( n, d, \rho \), the infimum over all \( \mathcal{G} \) of \( \max_{\mathcal{O} \in O(d)} \| P^2(\mathcal{O}(\mathcal{G})) \| \) is strictly positive, where \( \mathcal{G} \) ranges over the set of all 3-tuples of spherical harmonics of common degree \( n \) satisfying \( \| \mathcal{G} \| = 1 \).

Proof of Lemma 32. If \( \sum_{j=1}^{3} P_j(x_j') \) vanishes identically in a neighborhood in \( \Sigma \) of \((0,0,0)\) then \( P_j(x') \) must be an affine function of \( x' \in \mathbb{R}^{d-1} \) for each index \( j \). Therefore it suffices to show that for any \( k \in \{1,2,3\} \) for which \( G_k \) does not vanish identically on \( \mathbb{R}^d \), there exists \( \mathcal{O} \in O(d) \) such that the polynomial \( x' \mapsto P_k(x') \) associated to \( G_k \circ \mathcal{O} \) via \((64)\) fails to be affine.

Fix such an index \( k \). \( G_k \) has degree equal to \( n \). It is well-known that any measurable solutions \( \varphi_j \) of the functional equation \( \sum_{j=1}^{3} \varphi_j(x_j') = 0 \) on \( \Sigma \cap (I_1 \times I_2 \times I_3) \) must be affine functions in a neighborhood of any point of the intersection of \( \Sigma \) with the interior of \( I_1 \times I_2 \times I_3 \). Therefore if the associated polynomial \( P_k : \mathbb{R}^{d-1} \to \mathbb{R} \) is not affine, then the proof is complete.

Suppose instead that \( P_k \) is affine. By exploiting the identity \( |x'|^2 + x_d^2 = 1 \) for \( x', x_d \in S^{d-1} \) to eliminate powers of \( x_d \), one can express \( G_k(x', x_d) \), as a function of \((x', x_d) \in S^{d-1}, \) in the form \( p_1(x') + x_d p_2(x') \) where \( p_1, p_2 \) are uniquely determined polynomials of degrees \( \leq n \) and \( \leq n - 1 \), respectively. Now according to \((64)\), \( P_k(x') = r_2^{2-d-n} p_2(x') \). Since \( P_k \) is affine, this representation can be simplified to

\[
G_k(x) = G_k(x', x_d) = p(x') + (x' \cdot v)x_d + bx_d \quad \text{for} \ x \in S^{d-1}
\]

where \( p \) is a real-valued polynomial, \( v \in \mathbb{R}^{d-1} \), and \( b \in \mathbb{R} \). Since \( G_k \) has degree equal to \( n > 2 \), \( p \) must have degree equal to \( n \).

Consider \( \tilde{G}(x', x_d) = G_k(Tx', x_d) = p(Tx') + (Tx' \cdot v)x_d + bx_d \) where \( T \in O(d - 1) \) is chosen so that the coefficient \( b \) of \( x_d^n \) for \( p(Tx') \) is nonzero. Consider \( \tilde{G}(S(x', x_d)) \) where \( S \) is a rotation in the \((x_1, x_d)\) plane; \( S \) preserves the coordinates \( x_i \) for \( 2 \leq i < d \), and maps \((x_1, x_d)\) to

\[
(\cos(\alpha)x_1 + \sin(\alpha)x_d, -\sin(\alpha)x_1 + \cos(\alpha)x_d),
\]

where \( \alpha \in \mathbb{R} \) is a free parameter. Expanding \( \tilde{G}(S(x', x_d)) \) in the canonical form \( p(x') + x_d q(x') \), the monomial \( x_1^{n-1} x_d \) occurs with coefficient equal to \( nb \alpha + O(\alpha^2) \). Indeed, the term \( (Tx' \cdot v)x_d + bx_d \) has degree less than or equal to \( 2 < n \), and this upper bound is preserved by the rotation \( S \). Therefore for all sufficiently small nonzero \( \alpha \), this coefficient is nonzero. For any such \( \alpha \), the associated polynomial \( P(x') \) fails to be affine. \( \square \)

While Lemma 32 does not hold for spherical harmonics of degree \( n = 2 \), there is a satisfactory substitute, which yields Lemma 33 in the same way that Lemma 32 established Lemma 28. Let \( P_j, P^2 \) continue to be defined by \((64)\) and by \((70)\), respectively.

Lemma 33. Let \( d \geq 2 \). If \( \mathcal{G} \) is a nonzero ordered triple of spherical harmonics of degree 2, and if \( G_1 \equiv 0 \), then there exists \( \mathcal{O} \in O(d) \) such that \( P^2(\mathcal{O}(\mathcal{G})) \neq 0 \).
Proof. We follow the reasoning in the proof of Lemma 52. If \( \sum_{j=1}^{3} P_j(x'_j) \) vanishes identically in a neighborhood in \( \Sigma \) of \((0, 0, 0)\), and if \( P_1 \equiv 0 \), then \( P_2, P_3 \) must be constant functions. Therefore for \( k = 2, 3 \), \( G_k \) takes the form \( p(x') + bx_d \) for some constant \( b \), where \( p \) is a polynomial of degree \( \leq 2 \). The coefficient \( b \) must vanish, for otherwise the term \( bx_d \) would be a spherical harmonic of degree 1. Thus \( G_k \) is a function of \( x' \) alone.

This can only hold for the composition of \( G_k \) with an arbitrary rotation if \( G_k \equiv 0 \). □

11. Variant Inequality

Inequality 41 and the Riesz-Sobolev inequality are quite closely related, as will be seen in the proof of Theorem 44 below, but are not merely restatements of one another. For \( t \geq 0 \) define

\[
S_t(A, B) = \{ x \in \mathbb{R}^d : 1_A \ast 1_B(x) > t \}.
\]

Then for any Lebesgue measurable \( A, B \subset \mathbb{R}^d \) with \( |A|, |B| < \infty \),

\[
|A| \cdot |B| = \int_{\mathbb{R}^d} 1_A \ast 1_B = \int_0^\infty |S_t(A, B)| \, dt
\]

and

\[
\int_{S_t(A, B)} 1_A \ast 1_B = \tau |S_t(A, B)| + \int_\tau^\infty |S_t(A, B)| \, dt.
\]

Therefore

\[
\int_{\mathbb{R}^d} \min(1_A \ast 1_B, \tau) = \int_{\mathbb{R}^d} 1_A \ast 1_B - \int_{1_A \ast 1_B(x) > \tau} ((1_A \ast 1_B)(x) - \tau) \, dx
\]

\[
= |A| \cdot |B| - \int_{S_t(A, B)} 1_A \ast 1_B + \tau |S_t(A, B)|.
\]

Thus 41 can be equivalently restated as \( \Psi(A, B, \tau) \leq \Psi(A^*, B^*, \tau) \), where

\[
\Psi(A, B, \tau) = \int_{S_t(A, B)} 1_A \ast 1_B - \tau |S_t(A, B)|.
\]

That is,

\[
\int_{S_t(A, B)} 1_A \ast 1_B - \tau |S_t(A, B)| \leq \int_{S_t(A^*, B^*)} 1_{A^*} \ast 1_{B^*} - \tau |S_t(A^*, B^*)|.
\]

Compare this with the Riesz-Sobolev inequality, with \( E_1 = A, E_2 = B, \) and \( E_3 = S_t(A, B) \), which states that

\[
\int_{S_t(A, B)} 1_A \ast 1_B \leq \int_{S_t(A^*, B^*)} 1_{A^*} \ast 1_{B^*}.
\]

There are two differences in comparison to the inequality \( \Psi(A, B, \tau) \leq \Psi(A^*, B^*, \tau) \): There are no negative terms \(-\tau |S_t(\cdot, \cdot)|\), and the domain of integration \( S_t(A^*, B^*) \) is changed to \( [S_t(A, B)]^* \). If \( |S_t(A, B)| = |S_t(A^*, B^*)| \) then \( (S_t(A, B))^* = S_t(A^*, B^*) \) and the two inequalities become direct restatements of one another. The relation \( \int 1_A \ast 1_B = \int 1_{A^*} \ast 1_{B^*} \) is valid for all sets \( A, B \), and can be rewritten as \( \int_0^\infty |S_t(A, B)| \, dt = \int_0^\infty |S_t(A^*, B^*)| \, dt \), but there is no pointwise inequality relating the two quantities \( |S_t(E, E_2)| \) and \( |S_t(E^*_1, E^*_2)| \), in general.
Lemma 34. Let \( F_j \in L^1(\mathbb{R}^+) \) be nonincreasing, nonnegative functions satisfying \( \int_y^\infty F_0(x) \, dx \geq \int_y^\infty F_1(x) \, dx \) for all \( y \in \mathbb{R}^+ \). Then for each \( \tau \in \mathbb{R}^+ \),

\[
(78) \quad \int_0^\infty \min(F_0(x), \tau) \, dx \geq \int_0^\infty \min(F_1(x), \tau) \, dx.
\]

Proof. Via simple approximation and limiting arguments we can reduce to the case in which \( F_0, F_1 \) vanish outside of some bounded interval, belong to \( C^1([0, \infty)) \), are strictly decreasing with strictly negative derivatives where they are nonzero, and satisfy \( \sup_x F_j(x) > \tau > 0 \). For \( t \in [0, 1] \) consider \( F(x, t) = tF_1(x) + (1 - t)F_0(x) \). It suffices to show that \( \int_0^\infty \min(F(x, t), \tau) \, dx \) is a nonincreasing function of \( t \).

Our hypotheses on \( F_0, F_1 \) guarantee that for each \( t \) there exists a unique \( a(t) \in \mathbb{R}^+ \) satisfying \( F(a(t), t) = \tau \), and that \( a \) is a differentiable function of \( t \). Then

\[
\int_0^\infty \min(F(x, t), \tau) \, dx = \int_0^{a(t)} \tau \, dx + \int_{a(t)}^\infty F(x, t) \, dx
\]

and consequently

\[
\frac{d}{dt} \int_0^\infty \min(F(x, t), \tau) \, dx = \tau a'(t) - \tau a'(t) + \int_{a(t)}^\infty \frac{\partial F(x, t)}{\partial t} \, dx
\]

\[
= \int_{a(t)}^\infty (F_1(x) - F_0(x)) \, dx \leq 0.
\]

□

Proof of Theorem 3. Let \( F_0 : \mathbb{R}^+ \to [0, \infty) \) be right continuous, nonincreasing, and satisfy \( |\{y \in \mathbb{R}^+ : F_0(y) > t\}| = |\{x \in \mathbb{R}^d : 1_A \ast 1_B > t\}| \) for all \( t \in [0, \infty) \). Let \( F_1 \) be associated to \( 1_A \ast 1_B \) in the same way. The Riesz-Sobolev inequality states that \( \int_0^x F_0 \leq \int_0^x F_1 \) for all \( x \in [0, \infty) \). Since \( \int_0^\infty F_0 = |A| \cdot |B| = |A^*| \cdot |B^*| = \int_0^\infty F_1 \), this can be equivalently restated as \( \int_x^\infty F_0 \geq \int_x^\infty F_1 \) for all \( x \in \mathbb{R}^+ \). Moreover, \( \int_0^\infty \min(F_0, \tau) = \int_{\mathbb{R}^d} \min(1_A \ast 1_B, \tau) \), with a corresponding identity for \( 1_B \). Therefore an application of Lemma 34 yields the conclusion of the theorem.

□

Lemma 35. Let \( A, B \subset \mathbb{R}^d \) be Lebesgue measurable sets with finite Lebesgue measures. Let \( A(s), B(s) \) be their flows, as described in Proposition 4. For any \( \tau \in \mathbb{R}^+ \), \( \int_\mathbb{R} \min(1_A(s) \ast 1_B(s), \tau) \) is a nonincreasing continuous function of \( s \in [0, 1] \).

Proof. Continuity is easy, since \( t \mapsto \min(t, \tau) \) is a Lipschitz function and \( s \mapsto 1_A(s) \ast 1_B(s) \) is a continuous mapping from \([0, 1]\) to \( L^1(\mathbb{R}^d) \) by Proposition 4.

Define \( F_s : (0, \infty) \to [0, \infty) \) to be the unique nonincreasing right continuous function that satisfies

\[
|\{x \in \mathbb{R}^+ : F_s(x) > u\}| = |\{y \in \mathbb{R} : (1_A(s) \ast 1_B(s))(y) > u\}| \text{ for almost every } u \in \mathbb{R}^+.
\]

We claim that whenever \( s_0 \leq s_1 \), \( \int_0^{s_1} F_{s_0}(y) \, dy \leq \int_0^{s_1} F_{s_1}(y) \, dy \). It suffices to prove this for \( s_0 = 0 \). Observe that for any \( s \in [0, 1] \) and any \( x \in \mathbb{R}^+ \), \( \int_0^x F_s(y) \, dy \) is equal to the supremum of \( \int_E 1_A(s) \ast 1_B(s) \), with the supremum taken over all \( E \subset \mathbb{R} \) satisfying \( |E| = x \). This supremum is attained. Choose \( E \) so that \( \int_E 1_A \ast 1_B = \int_0^x F_0(y) \, dy \), and consider
the flow \( s \mapsto E(s) \) and the associated expression \( \Psi(s) = \int_{E(s)} 1_{A(s)} \ast 1_{B(s)} \). According to Proposition 7, \( \Psi \) is a nondecreasing function. But

\[
\int_0^x F_0(y) \, dy = \int_{E(0)} 1_{A(0)} \ast 1_B \leq \int_{E(s)} 1_{A(s)} \ast 1_{B(s)}
\]

\[
\leq \sup_{|E| = |E(s)|} \int_E 1_{A(s)} \ast 1_{B(s)} = \int_0^x F_s(y) \, dy
\]

since \( |E(s)| \equiv |E| \).

For any \( s, \int_{E^+} F_s = |A(s)| \cdot |B(s)| = |A| \cdot |B| \). Therefore since \( |A(s_0)| \cdot |B(s_0)| = |A(s_1)| \cdot |B(s_1)| \), the inequality \( \int_0^x F_{s_0} \leq \int_0^x F_{s_1} \) for all \( x \), can be rewritten as \( \int_0^x F_{s_0} \geq \int_0^x F_{s_1} \) for all \( x \). Therefore an application of Lemma 34 completes the proof. \( \square \)

**Proof of Theorem 4.** By virtue of the continuity and monotonicity of the functional \( \int \min(1_{A(s)} \ast 1_{B(s)}, \tau) \) discussed in Lemma 35, together with its affine invariance, we may reduce matters, as in the proof of Theorem 6, to the small perturbation case, in which Distance\((A, B, O(A^*, B^*))\) is much less than \( \max(|A|, |B|) \). By making a suitable measure-preserving affine change of variables we may reduce to the case in which

\[ \max(|A \Delta A^*|, |B \Delta B^*|) \leq 2 \text{Distance}((A, B, O(A^*, B^*))). \]

Write

\[ \int \min(1_A \ast 1_B, \tau) = |A| \cdot |B| + \tau |S_r(A, B)| - \int_{S_r(A, B)} 1_A \ast 1_B. \]

There is a corresponding identity for \( \int \min(1_A \ast 1_B, \tau) \), and \( |A| \cdot |B| = |A^*| \cdot |B^*| \). Let \( S = S_r(A, B) \) and \( S^2 = S_r(A^*, B^*) \). Therefore we seek to bound \( \int_{S^2} 1_A \ast 1_B - \tau |S^2| \), minus a suitable nonnegative term.

One has

\[
\int_{S^2} 1_A \ast 1_B^* \leq \int_{S^2} 1_A \ast 1_B^* - \tau(|S^2| - |S|)
\]

in general, and

\[
\int_{S^2} 1_A \ast 1_B^* \leq \int_{S^2} 1_A \ast 1_B^* - \tau(|S^2| - |S|) - c(|S^2| - |S|)^2
\]

for ordered triples \((A, B, S)\) in the strictly admissible range.

From the elementary uniform bound

\[ \|1_A \ast 1_B - 1_A \ast 1_B^*\|_{L^\infty} \leq |A \Delta A^*| \cdot |B| + |A| \cdot |B \Delta B^*| \]

and the assumption that \( \max(|A \Delta A^*|, |B \Delta B^*|) \ll \max(|A|, |B|) \) it follows that

\[ |S_r(A, B) \Delta S_r(A^*, B^*)| \ll \max(|A|, |B|). \]

Therefore the ordered triple \((A, B, S_r(A, B))\) is \( \rho \)-strictly admissible, where \( \rho > 0 \) depends only on the parameters in the hypotheses of Theorem 4.
Therefore the Riesz-Sobolev inequality in the form (10) can be invoked to obtain
\[
\int_S 1_A * 1_B - \tau |S| \leq \int_{S^c} 1_{A^c} * 1_{B^c} - \tau |S| - c \text{ Distance}((A, B), O(A^*, B^*))^2
\]
\[
\leq \int_{S^c} 1_{A^c} * 1_{B^c} - \tau(|S|^2 - |S|) - c(|S|^2 - |S|)^2 - \tau |S|
\]
\[-c \text{ Distance}((A, B), O(A^*, B^*))^2
\]
\[
= \int_{S^c} 1_{A^c} * 1_{B^c} - \tau(|S|^2 - c(|S|^2 - |S|)^2 - c \text{ Distance}((A, B), O(A^*, B^*))^2.
\]
This completes the proof of Theorem 4, as well as of the formally sharper form, Theorem 5.
\[\square\]

12. A PROPERTY OF THE FLOW

Here we prove Proposition 9, which states that for any \( t > 0 \), \( E(t) \) equals a union of intervals, up to a Lebesgue null set.

The flow of \( E \) can be regarded as a flow \( (t, x) \mapsto x(t) \in E(t) \) of the points \( x \in E \), in the following natural way. Firstly, define \( \phi_E : E \to E^* \) by
\[
\phi_E(x) = |E \cap (x, \infty)| - \frac{1}{2} |E|.
\]
\( \phi_E \) is a nondecreasing function, and \( |\phi_E(E \cap I)| = |E \cap I| \) for every interval \( I \). Secondly, define \( \bar{\phi}_E : (-\frac{1}{2} |E|, \frac{1}{2} |E|) \to \mathbb{R} \) by
\[
\bar{\phi}(x) = y \in \mathbb{R}
\]
where \( y \) is the smallest element of \( \mathbb{R} \) satisfying \( |E \cap (\infty, y]| = y + \frac{1}{2} |E| \). \( \bar{\phi}_E \) is a nondecreasing Lebesgue measure-preserving function. It is a consequence of the Lebesgue density theorem that for almost every \( x \in E \), the only point \( y \in \mathbb{R} \) satisfying \( |E \cap (\infty, y]| = |E \cap (\infty, x]| \) is \( y = x \) itself. Therefore \( \bar{\phi}_E(\phi_E(x)) = x \) for almost every \( x \in E \), and \( |E \Delta \phi_E(E^*)| = 0 \).

For each \( t \in [0, 1] \) let \( \phi_{E(t)} : E(t) \to E(t)^* = E^* \) be defined in this way. Set \( \psi_E(t) = \bar{\phi}_{E(t)} \circ \phi_E : E \to E(t) \). This is a well-defined nondecreasing function, which preserves Lebesgue measure of Borel sets. The mapping \( E \ni x \mapsto \psi_E(t)(x) \) defines the desired flow on the underlying points of \( E \).

The next lemma states that if \( I \) is a bounded interval, and if \( E \cap I \) is sufficiently dense in \( I \), then \( \Psi_t(E) \) contains an interval of length comparable to \( I \), for all \( t > 0 \) that are not too small.

**Lemma 36.** Let \( [a, b] \subset [0, \infty) \) be a closed bounded interval of positive length. Let \( E \subset \mathbb{R} \) be a Lebesgue measurable set satisfying \( 0 < |E| < \infty \). Let \( \delta \in [0, \frac{1}{2}] \). Suppose that \( |E \cap I| \geq (1 - \delta)|I| \). Then for every \( T > 2\delta(1 - 2\delta)^{-1} \), the set \( \psi_I(E \cap I) \) is an interval.

We will prove Lemma 36 in the special case in which \( E \) is a finite union of closed intervals. In that case, the mapping \( t \mapsto x(t) \) is continuous and is almost everywhere differentiable for almost every \( x \in E \). Then for each \( x \in E \), for each \( t \) let \( c(t) \) be the center of the largest interval that is contained in \( E(t) \), and contains \( x(t) \). Then for almost every \( t \), \( dx(t)/dt = -c(t) \).

**Proof of Lemma 36.** Write \( I = [a^-, a^+] \) and \( |I| = a^+ - a^- \). Consider \( a^-(t), a^+(t) \in E(t) \). Define \( \eta^-(t) \) to be the supremum of all \( \eta \geq 0 \) such that
\[
[a^-(t), a^-(t) + 2\eta|I|] \subset E(t).
\]
Likewise define $\eta^+(t)$ to be the supremum of all $\eta \geq 0$ such that
\[
[a^+(t) - 2\eta|I|, a^+(t)] \subset E(t).
\]
$\eta^\pm(t)$ are nondecreasing functions of $t \in [0, 1]$.

Let $c^\pm(t)$ be the centers of the largest intervals contained in $E(t)$ that contain $a^\pm(t)$, respectively. Provided that these two intervals are disjoint,
\[
c^-(t) \leq a^-(t) + \eta^-(t) \quad \text{and} \quad c^+(t) \geq a^+(t) - \eta^+(t).
\]
Moreover, for almost every $t$, $da^\pm(t)/dt$ exists and satisfies
\[
\frac{d}{dt}a^\pm(t) = -c^\pm(t).
\]

Let $T > 0$ and suppose that $\eta^+(T) + \eta^-(T) < \frac{1}{2}$, and consequently the two intervals $[a^-(t), a^-(t) + \eta^-(t)|I|]$ and $[a^+(t), a^+(t) - \eta^+(t)|I|]$ are disjoint for each $t \in [0, T]$.

Since $a^+(t) - a^-(t) \geq |E \cap I| \geq (1 - \delta)|I|,$
\[
\frac{d}{dt}(a^+(t) - a^-(t)) = c^+(t) - c^-(t)
\leq [a^-(t) + \eta^-(t)|I|] - [a^+(t) - \eta^+(t)|I|]
\leq [-(1 - \delta) + (\eta^+(t) + \eta^-(t))|I|]
\leq (-\frac{1}{2} + \delta)|I|
\]
for almost every $t \in [0, T]$. Integrating over $t$ and using the initial condition $a^+(0) - a^-(0) = a^+ - a^- = |I|$ gives
\[
a^+(T) - a^-(T) \leq [1 + (-\frac{1}{2} + \delta)T]|I|.
\]
Combining this inequality with the constraint $a^+(T) - a^-(T) \geq (1 - \delta)|I|$ gives
\[
1 - \delta \leq 1 - T(\frac{1}{2} - \delta),
\]
that is, $T \leq 2\delta(1 - 2\delta)^{-1}$.

Now consider any $\tau$ strictly greater than $2\delta(1 - 2\delta)^{-1}$. The hypothesis $\eta^+(\tau) + \eta^-(\tau) < \frac{1}{2}$ underlying the above reasoning cannot hold, since the conclusion does not. Thus $\eta^+(\tau) + \eta^-(\tau) \geq \frac{1}{2}$. Therefore the interval $[a^-(\tau), a^+(\tau)]$ is contained in $E(\tau)$, up to a Lebesgue null set.

Proposition 36 is an immediate corollary of the next lemma.

Lemma 37. Let $E \subset \mathbb{R}$ be a Lebesgue measurable set satisfying $0 < |E| < \infty$. Let $\varepsilon > 0$. There exists $t < \varepsilon$ such that $E(t)$ can be expressed as a countable union of intervals, together with a set of Lebesgue measure less than $\varepsilon$.

Proof. If $E(\tau)$ can be expressed as a union of countably many intervals together with a set of Lebesgue measure less than $\varepsilon$ for some $\tau > 0$, then the same holds for $E(t)$, for every $t > \tau$.

According to the Lebesgue density theorem, there exist a collection of pairwise disjoint bounded intervals $I_j$ and a subset $E' \subset E$ such that $|E'| < \varepsilon$, $E' \cap I_j = \emptyset$ for each index $j$, and $|E \cap I_j| \geq (1 - \varepsilon)|I_j|$ for every $j$. According to Lemma 36, $\psi_{E,t}(E \cap I_j)$ is an interval for each index $j$, for every $t > 2\varepsilon(1 - 2\varepsilon)^{-1}$. Moreover, $|\psi_{E,t}(E')| = |E'| < \varepsilon$. Therefore $\psi_{E,t}(E \setminus E')$ can be expressed as a countable union of intervals. \qed
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