THE FIRST INITIAL BOUNDARY VALUE PROBLEM FOR HESSIAN EQUATIONS OF PARABOLIC TYPE ON RIEMANNIAN MANIFOLDS

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Abstract. In this paper, we are concerned with the first initial boundary value problem for a class of fully nonlinear parabolic equations on Riemannian manifolds. As usual, the establishment of the a priori \( C^2 \) estimates is our main part. Based on these estimates, the existence of classical solutions is proved under conditions which are nearly optimal.

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1. Introduction

In this paper, we study the Hessian equations of parabolic type of the form

\[
f(\lambda(\nabla^2 u + \chi), -u_t) = \psi(x, t)
\]

in \( M_T = M \times (0, T] \subset M \times \mathbb{R} \) satisfying the boundary condition

\[
u = \varphi \text{ on } \mathcal{P}M_T,
\]

where \((M, g)\) is a compact Riemannian manifold of dimension \( n \geq 2 \) with smooth boundary \( \partial M \) and \( \bar{M} := M \cup \partial M \), \( \mathcal{P}M_T = BM_T \cup SM_T \) is the parabolic boundary of \( M_T \) with \( BM_T = M \times \{0\} \) and \( SM_T = \partial M \times [0, T] \), \( f \) is a symmetric smooth function of \( n + 1 \) variables defined in an open convex symmetric cone \( \Gamma \subset \mathbb{R}^{n+1} \) with vertex at the origin and

\[
\Gamma_{n+1} = \{ \lambda \in \mathbb{R}^{n+1} : \text{each component } \lambda_i > 0, \ 1 \leq i \leq n + 1 \} \subseteq \Gamma,
\]

\( \nabla^2 u \) denotes the Hessian of \( u(x, t) \) with respect to \( x \in M \), \( u_t = \frac{\partial u}{\partial t} \) is the derivative of \( u(x, t) \) with respect to \( t \in [0, T] \), \( \chi \) is a smooth \((0, 2)\) tensor on \( \bar{M} \) and \( \lambda(\nabla^2 u + \chi) = (\lambda_1, \ldots, \lambda_n) \) denotes the eigenvalues of \( \nabla^2 u + \chi \) with respect to the metric \( g \).

As in [4] (see [9] also), we assume that \( f \) satisfies the following structural conditions:

(1.3) \( f_i = \frac{\partial f}{\partial \lambda_i} > 0 \text{ in } \Gamma, \ 1 \leq i \leq n + 1, \)

(1.4) \( f \) is concave in \( \Gamma \)

and

(1.5) \( \delta_{\psi, f} \equiv \inf_{\lambda \in \Gamma} \psi - \sup_{\lambda \in \Gamma} f(\lambda) \geq 0 \), where \( \sup_{\lambda \in \Gamma} f \equiv \lim_{\lambda_0 \to \lambda_0} \sup_{\lambda \in \Gamma} f(\lambda). \)

We mean an admissible function by \( u \in C^2(M_T) \) satisfying \( (\lambda(\nabla^2 u + \chi), -u_t) \in \Gamma \) in \( M_T \), where \( C^k(M_T) \) denotes the space of functions defined on \( M_T \) which are
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$k$-times continuously differentiable with respect to $x \in M$ and $[k/2]$-times continuously differentiable with respect to $t \in (0, T]$ and $[k/2]$ is the largest integer not greater than $k/2$. We note that (1.1) is parabolic for admissible solutions (see [4]).

We first recall the following notations

$$|u|_{C^k(M_T)} = \sum_{|\beta| + 2r \leq k} \sup_{M_T} |\nabla^\beta D_t^r u|,$$

$$|u|_{C^{k+\alpha}(M_T)} = |u|_{C^k(M_T)} + \sup_{|\beta|+2r=k} \sup_{(x,s),(y,t) \in M_T \setminus \{(x,s) \neq (y,t)\}} \frac{|\nabla^\beta D_t^r u(x,s) - \nabla^\beta D_t^r u(y,t)|}{(|x-y| + |s-t|^{1/2})^{\alpha}}$$

and $C^{k+\alpha}(M_T)$ denotes the subspace of $C^k(M_T)$ defined by

$$C^{k+\alpha}(M_T) := \{ u \in C^k(M_T) : |u|_{C^{k+\alpha}(M_T)} < \infty \}.$$

In the current paper, we are interested in the existence of admissible solutions to (1.1)-(1.2). The key step is to establish the a priori $C^2$ estimates. Using the methods from [10], where Guan studied the elliptic counterpart of (1.1):

$$f(\lambda(\nabla^2 u + \chi)) = \psi(x)$$

in $M$ satisfying the Dirichlet boundary condition, we are able to obtain these estimates under nearly minimal restrictions on $f$.

Our main results are stated in the following theorem.

**Theorem 1.1.** Suppose that $\psi \in C^\infty(M_T)$, $\varphi \in C^\infty(PM_T)$ for $0 < T \leq \infty$, 

(1.7) 

$$(\lambda(\nabla^2 \varphi(x,0) + \chi(x)), -\varphi_t(x,0)) \in \Gamma \text{ for all } x \in \overline{M}$$

and

(1.8) 

$$f(\lambda(\nabla^2 \varphi(x,0) + \chi(x)), -\varphi_t(x,0)) = \psi(x,0) \text{ for all } x \in \partial M.$$

In addition to (1.7)-(1.8), assume that

(1.9) 

$$f_j(\lambda) \geq \nu_0 \left( 1 + \sum_{i=1}^{n+1} f_i(\lambda) \right) \text{ for any } \lambda \in \Gamma \text{ with } \lambda_j < 0,$$

for some positive constant $\nu_0$,

(1.10) 

$$\sum_{i=1}^{n+1} f_i \lambda_i \geq -K_0 \sum_{i=1}^{n+1} f_i, \quad \forall \lambda \in \Gamma$$

for some $K_0 \geq 0$ and that there exists an admissible subsolution $u \in C^2(M_T)$ satisfying

(1.11) 

$$\left\{ \begin{array}{ll}
    f(\lambda(\nabla^2 u + \chi), -u_t) \geq \psi(x,t) & \text{in } M_T, \\
    u = \varphi & \text{on } SM_T, \\
    u \leq \varphi & \text{on } BM_T.
\end{array} \right.$$
Remark 1.2. Condition (1.9) is only used to derive the gradient estimates as many authors, see [21], [14], [8], [20], [23] and [27] for examples.

Condition (1.10) is used in the estimates for both $|\nabla u|$ and $|u_t|$. We will see that in the gradient estimates, condition (1.10) can be weakened by

$$\sum_{i=1}^{n+1} f_i \lambda_i \geq -K_0 \left( 1 + \sum_{i=1}^{n+1} f_i \right), \forall \lambda \in \Gamma.$$  

As in [10], the existence of $u$ is useful to construct some barrier functions which are crucial to our estimates.

The most typical examples of $f$ satisfying the conditions in Theorem 1.1 are $f = \sigma_{1/k}$ and $f = (\sigma_k/\sigma_l)^{1/(k-l)}$, $1 \leq l < k \leq n+1$, defined in the Gårding cone

$$\Gamma_k = \{ \lambda \in \mathbb{R}^{n+1} : \sigma_j(\lambda) > 0, j = 1, \ldots, k \},$$

where $\sigma_k$ are the elementary symmetric functions

$$\sigma_k(\lambda) = \sum_{i_1 < \ldots < i_k} \lambda_{i_1} \ldots \lambda_{i_k}, \quad k = 1, \ldots, n+1.$$ 

When $f = \sigma_{1/(n+1)}$, equation (1.1) can be written as the parabolic Monge-Ampère equation:

$$-u_t \det(\nabla^2 u + \chi) = \psi^{n+1},$$

which was introduced by Krylov in [17] when $\chi = 0$ in Euclidean space. Our motivation to study (1.1) is from their natural connection to the deformation of surfaces by some curvature functions. For example, equation (1.1) plays a key role in the study of contraction of surfaces by Gauss-Kronecker curvature (see Firey [6] and Tso [25]). For the study of more general curvature flows, the reader is referred to [1], [2], [15], [22] and their references. (1.1) is also relevant to a maximum principle for parabolic equations (see Tso [26]).

In [21], Lieberman studied the first initial-boundary value problem of equation (1.1) when $\chi \equiv 0$ and $\psi$ may depend on $u$ and $\nabla u$ in a bounded domain $\Omega \subset \mathbb{R}^{n+1}$ under various conditions. Jiao and Sui [16] considered the parabolic Hessian equation of the form

$$f(\lambda(\nabla^2 u + \chi)) - u_t = \psi(x,t)$$

on Riemannian manifolds using techniques from [10] and [11] where the authors studied the corresponding elliptic equations. Guan, Shi and Sui [13] extended the work of [16] using the idea of [10]; they also treated the parabolic equation of the form

$$f(\lambda(\nabla^2 u + \chi)) = e^{u_t+\psi}.$$ 

Applying the methods of [9], Bao and Dong solved (1.1)-(1.2) under an additional condition which is introduced in [9] (see [11] also)

$$T_\lambda \cap \partial \Gamma^\sigma \text{ is a nonempty compact set, } \forall \lambda \in \Gamma \text{ and } \sup_{\partial \Gamma^\sigma} f < \sigma < f(\lambda),$$

where $\partial \Gamma^\sigma = \{ \lambda \in \Gamma : f(\lambda) = \sigma \}$ is the boundary of $\Gamma^\sigma = \{ \lambda \in \Gamma : f(\lambda) > \sigma \}$ and $T_\lambda$ denote the tangent plane at $\lambda$ of $\partial \Gamma f(\lambda)$, for $\sigma > \sup_{\partial \Gamma^\sigma} f$ and $\lambda \in \Gamma$. The reader is referred to [19], [27], [8], [9], [10], [11], [12] and their references for the study of elliptic Hessian equations on manifolds.
We can prove the short time existence as Theorem 15.9 in [21]. So without loss of generality, we may assume
\[(1.17) \quad f(\lambda(\nabla^2 \varphi(x,0) + \chi(x)), -\varphi_t(x,0)) = \psi(x,0) \quad \text{for all} \ x \in \overline{M}.
\]

As usual, the main part of this paper is to derive the \textit{a priori} $C^2$ estimates. We see that (1.13) is uniformly parabolic after establishing the $C^2$ estimates by (1.3) and (1.5). The $C^{2,\alpha}$ estimates can be obtained by applying Evans-Krylov theorem (see [3] and [18]). Finally Theorem 1.1 can be proved as Theorem 15.9 of [21].

The rest of this paper is organized as follows. In section 2, we introduce some notations and useful lemmas. $C^1$ estimates are derived in Section 3. An \textit{a priori} bound for $|u_t|$ is obtained in Section 4. Section 5 and Section 6 are devoted to the global and boundary estimates for second order derivatives respectively.

2. Preliminaries

Let $F$ be the function defined by $F(A, \tau) = f(\lambda(A), \tau)$ for $A \in S^n$, $\tau \in \mathbb{R}$ with $(\lambda(A), \tau) \in \Gamma$, where $S^n$ is the set of $n \times n$ symmetric matrices. It was shown in [4] that $F$ is concave from (1.3). For simplicity we shall use the notations $U = \nabla^2 u + \chi$, $\bar{U} = \nabla^2 \bar{u} + \chi$ and under an orthonormal local frame $e_1, \ldots, e_n$,
\[
U_{ij} \equiv U(e_i, e_j) = \nabla_{ij} u + \chi_{ij}, \quad \bar{U}_{ij} \equiv \bar{U}(e_i, e_j) = \nabla_{ij} \bar{u} + \chi_{ij}.
\]
Thus, (1.1) can be written in the form locally
\[(2.1) \quad F(U, -u_t) = f(\lambda(U_{ij}), -u_t) = \psi.
\]
Let
\[
F^{ij} = \frac{\partial F}{\partial A_{ij}}(U, -u_t), \quad F^\tau = \frac{\partial F}{\partial \tau}(U, -u_t)
\]
\[
F^{ij, kl} = \frac{\partial^2 F}{\partial A_{ij} \partial A_{kl}}(U, -u_t), \quad F^{ij, \tau} = \frac{\partial^2 F}{\partial A_{ij} \partial \tau}(U, -u_t).
\]
By (1.3) we see that $F^\tau > 0$ and \{F^{ij}\} is positive definite. We shall also denote the eigenvalues of \{F^{ij}\} by $f_1, \ldots, f_n$ when there is no possible confusion. We note that \{U_{ij}\} and \{F^{ij}\} can be diagonalized simultaneously and that
\[
F^{ij} U_{ij} = \sum f_i \tilde{\lambda}_i, \quad F^{ij} U_{ik} U_{kj} = \sum f_i \tilde{\lambda}_i^2,
\]
where $\lambda(\{U_{ij}\}) = (\tilde{\lambda}_1, \ldots, \tilde{\lambda}_n)$.

Similarly to [10], we write
\[
\mu(x, t) = (\lambda(U(x, t)), -\nu(x, t)),
\]
\[
\lambda(x, t) = (\lambda(U(x, t)), -u_t(x, t))
\]
and $\nu_\lambda \equiv DF(\lambda)/|DF(\lambda)|$ is the unit normal vector to the level hypersurface $\partial \Gamma^f(\lambda)$ for $\lambda \in \Gamma$. Since $K \equiv \{\mu(x, t) : (x, t) \in \overline{M_T}\}$ is a compact subset of $\Gamma$, there exist uniform constants $\beta \in (0, \frac{1}{2\sqrt{n}+1})$ such that
\[(2.2) \quad \nu_\mu(x, t) - 2\beta 1 \in \Gamma_{n+1}, \forall (x, t) \in \overline{M_T}.
\]
where $1 = (1, \ldots, 1) \in \mathbb{R}^{n+1}$ (see [10]).

We need the following Lemma which is proved in [10].
Lemma 2.1. Suppose that $|\nu_\mu - \nu_\lambda| \geq \beta$. Then there exists a uniform constant $\varepsilon > 0$ such that

$$
\sum_{i=1}^{n+1} f_i(\lambda)(\mu_i - \lambda_i) \geq \varepsilon \left(1 + \sum_{i=1}^{n+1} f_i(\lambda)\right).
$$

Define the linear operator $L$ locally by

$$
L v = F^{ij} \nabla_{ij} v - \tau v_t, \quad v \in C^2(M_T).
$$

From Lemma 2.1 and Lemma 6.2 of [4] it is easy to derive that when $|\nu_\mu(x,t) - \nu_\lambda(x,t)| \geq \beta$,

$$
L(u - u) \geq \varepsilon \left(1 + \sum_{i=1}^{n+1} F^{ii} + \tau\right).
$$

If $|\nu_\mu - \nu_\lambda| < \beta$, we have $\nu_\lambda - \beta 1 \in \Gamma_{n+1}$. It follows that

$$
f_i \geq \beta \frac{\sqrt{n+1}}{n+1} \sum_{j=1}^{n+1} f_j, \quad \forall 1 \leq i \leq n+1.
$$

3. The $C^1$ estimates

Since $u$ is admissible and $\Gamma \subset \{\lambda \in \mathbb{R}^{n+1} : \sum_{i=1}^{n+1} \lambda_i > 0\}$, we see that $u$ is a subsolution of

$$
\begin{cases}
\triangle h - h_t + \text{tr}(\chi) = 0, & \text{in } M_T, \\
h = \varphi, & \text{on } \partial M_T.
\end{cases}
$$

Let $h$ be the solution of (3.1). It follows from the maximum principle that $u \leq u \leq h$ on $\overline{M_T}$. Therefore, we have

$$
\sup_{\overline{M_T}} |\nabla u| + \sup_{\partial M_T} |\nabla u| \leq C.
$$

For the global gradient estimates, we can prove the following theorem.

Theorem 3.1. Suppose that (1.3), (1.4), (1.9) and (1.12) hold. Let $u \in C^3(\overline{M_T})$ be an admissible solution of (1.7) in $M_T$. Then

$$
\sup_{M_T} |\nabla u| \leq C(1 + \sup_{\partial M_T} |\nabla u|),
$$

where $C$ depends on $|\psi|_{C^1(\overline{M_T})}$, $|u|_{C^0(\overline{M_T})}$ and other known data.

Proof. Set

$$
W = \sup_{(x,t) \in \overline{M_T}} w^\phi,
$$

where $w = \frac{|\nabla w|^2}{2}$ and $\phi$ is a function to be determined. It suffices to estimate $W$ and we may assume that $W$ is achieved at $(x_0, t_0) \in \overline{M_T} - \partial M_T$. Choose a smooth orthonormal local frame $e_1, \ldots, e_n$ about $x_0$ such that $\nabla_i e_j = 0$ at $x_0$ and $U(x_0, t_0)$ is diagonal. We see that the function $\log w + \phi$ attains its maximum at $(x_0, t_0)$. Therefore, at $(x_0, t_0)$, we have

$$
\frac{\nabla w_i}{w} + \nabla_i \phi = 0, \quad \text{for each } i = 1, \ldots, n,
$$

$$
\frac{w_t}{w} + \phi_t \geq 0
$$
and
\[ \frac{\nabla_{ii}w}{w} - \left( \frac{\nabla w}{w} \right)^2 + \nabla_{ii}\phi \leq 0. \]

Differentiating the equation (1.1), we get
\[ F^{ii}\nabla_k U_{ii} - F^\tau \nabla_k u_t = \nabla_k \psi \text{ for } k = 1, \ldots, n \]
and
\[ F^{ii}(U_{ii})_t - F^\tau u_{tt} = \psi_t. \]

Note that
\[ \nabla_{i}w = \nabla_{k}u \nabla_{ik}u, \quad w_t = \nabla_{k}(u(\nabla_{k}u))_t, \quad \nabla_{ii}w = (\nabla_{ik}u)^2 + \nabla_{k}u \nabla_{iik}u \]
and that
\[ \nabla_{ijk}u - \nabla_{jik}u = R^l_{kij} \nabla_{l}u. \]

We have, by (3.5), (3.7), (3.9) and (3.10),
\[ F^{ii}\nabla_{ii}w \geq \nabla_{k}u F^{ii}\nabla_{iik}u \]
\[ \geq -C|\nabla u| - C|\nabla u|^2 \sum F^{ii} + F^\tau \nabla_{k}u \nabla_{k}u_t \]
\[ \geq -C|\nabla u| - C|\nabla u|^2 \sum F^{ii} - wF^\tau \phi_t, \]
provided \( |\nabla u| \) is sufficiently large. Combining (3.12), (3.3), we obtain
\[ 0 \geq -C\frac{\nabla_{i}w}{|\nabla u|} - C\sum F^{ii} - F^{ii}(\nabla_{i}\phi)^2 + \mathcal{L}\phi. \]

Let \( \phi = \delta v^2 \), where \( v = u + \sup E \sum \nabla u | + 1 \) and \( \delta \) is a small positive constant to be chosen. Thus, choosing \( \delta \) sufficiently small such that \( 2\delta - 4\delta^2 v^2 \geq c_0 > 0 \) for some uniform constant \( c_0 \), by (1.12),
\[ \mathcal{L}\phi - F^{ii}(\nabla_{i}\phi)^2 = 2\delta v (F^{ii}\nabla_{ii}u - F^\tau u_t) + (2\delta - 4\delta^2 v^2) F^{ii}(\nabla_{i}u)^2 \]
\[ \geq -C\delta (1 + \sum F^{ii} + F^\tau) + c_0 F^{ii}(\nabla_{i}u)^2. \]

It follows from (3.12) and (3.14) that
\[ c_0 F^{ii}(\nabla_{i}u)^2 \leq C \left( 1 + \sum F^{ii} + F^\tau \right), \]
provided \( |\nabla u| \) is sufficiently large. We may assume \( |\nabla u(x_0, t_0)| \leq n\nabla_1 u(x_0, t_0) \) and by (3.3),
\[ U_{11} = -2\delta vw + \frac{\nabla_{k}u \chi_{1k}}{\nabla_{1k}u} < 0 \]
provided \( w \) is sufficiently large. Then we can derive from (1.9) that
\[ F^{11} \geq \nu_0 \left( 1 + \sum F^{ii} + F^\tau \right). \]

Therefore, we obtain a bound \( |\nabla u(x_0, t_0)| \leq Cn^2/c_0\nu_0 \) by (3.14) and (3.3) is proved. \( \square \)

**Remark 3.2.** We see that in the proof of Theorem 3.1 we do not need the existence of \( u_1 \).

By (3.2) and (3.3), the \( C^1 \) estimates are established.
4. ESTIMATE FOR $|u_t|$ 

In this section, we derive the estimate for $|u_t|$. 

**Theorem 4.1.** Suppose that (4.3), (4.4), (4.5) and (4.6) hold. Let $u \in C^3(M_T)$ be an admissible solution of (1.1) in $M_T$. Then there exists a positive constant $C$ depending on $|u|_{C^3(M_T)}$, $|\psi|_{C^2(M_T)}$ and other known data such that

$$
\sup_{M_T} |u_t| \leq C(1 + \sup_{\mathcal{P}M_T} |u_t|).
$$

**Proof.** We first show that

$$
\sup_{M_T} (-u_t) \leq C(1 + \sup_{\mathcal{P}M_T} (-u_t))
$$

for which we set

$$W = \sup_{M_T} (-u_t)e^\phi,$$

where $\phi$ is a function to be chosen. We may assume that $W$ is attained at $(x_0, t_0) \in M_T - \mathcal{P}M_T$. As in the proof of Theorem 5.1, we choose an orthonormal local frame $e_1, \ldots, e_n$ about $x_0$ such that $\nabla e_i e_j = 0$ and $\{U_{ij}(x_0, t_0)\}$ is diagonal. We may assume $-u_t(x_0, t_0) > 0$. At $(x_0, t_0)$ where the function $\log(-u_t) + \phi$ achieves its maximum, we have

$$
\nabla_i u_t + \nabla_i \phi = 0, \text{ for each } i = 1, \ldots, n,
$$

$$
\frac{u_{tt}}{u_t} + \phi_t \geq 0,
$$

and

$$
0 \geq F^{ii} \left\{ \frac{\nabla_i u_t}{u_t} - \left( \frac{\nabla_i u_t}{u_t} \right)^2 + \nabla_i \phi \right\}.
$$

Combining (4.3), (4.4) and (4.5), we find

$$
0 \geq \frac{1}{u_t} \left( F^{ii} \nabla_i u_t - F^r u_{tt} - F^{ii}(\nabla_i \phi)^2 + \mathcal{L} \phi \right).
$$

By (4.8) and (4.9),

$$
\mathcal{L} \phi \leq -\frac{\psi_t}{u_t} + F^{ii}(\nabla_i \phi)^2.
$$

Let $\phi = \frac{\delta^2}{2} |\nabla u|^2 + \delta u + b(u - u)$, where $\delta \ll b \ll 1$ are positive constants to be determined. By straightforward calculations, we see

$$
\nabla_i \phi = \delta^2 \nabla_k u \nabla_i u_{k} + \delta \nabla_i u + b \nabla_i (u - u),
$$

$$
\phi_t = \delta^2 \nabla_k u (\nabla_k u)_t + \delta u_t + b(u - u)_t,
$$

$$
\nabla_i u_\phi \phi = \delta^2 (\nabla_i u)^2 + \delta^2 \nabla_k u \nabla_i u + \delta \nabla_i u + b \nabla_i (u - u).
$$

It follows that, in view of (4.7) and (4.10),

$$
\mathcal{L} \phi \geq \delta^2 \nabla_k u (F^{ii} \nabla_i u_{k} - F^r (\nabla_k u)_t) + \frac{\delta^2}{2} F^{ii} u^2_{ii} + \delta F^{ii} u_{ii} - \delta F^r u_t - C\delta^2 \sum F^{ii} + b \mathcal{L} (u - u)
$$

$$
\geq - C\delta^2 \left( 1 + \sum F^{ii} \right) + \frac{\delta^2}{2} F^{ii} u^2_{ii} + \delta \mathcal{L} u + b \mathcal{L} (u - u).
$$

Next,
\[(4.9) \quad (\nabla_i \phi)^2 \leq C \delta^4 U_{ii}^2 + C b^2.\]
Thus, we can derive from \((4.7), (4.8)\) and \((4.9)\) that
\[(4.10) \quad b \mathcal{L}(u - u) + \frac{\delta^2}{4} F_{ii} U_{ii}^2 + \delta \mathcal{L} u \leq -\frac{C}{u_t} + C \delta^2 \left(1 + \sum F_{ii}\right) + C b^2 \sum F_{ii},\]
when \(\delta\) is small enough. Now we use the idea of \([10]\) to consider two cases: (i) \(|\nu_{\mu_0} - \nu_{\lambda_0}| \geq \beta\) and (ii) \(|\nu_{\mu_0} - \nu_{\lambda_0}| < \beta\), where \(\mu_0 = \mu(x_0, t_0)\) and \(\lambda_0 = \lambda(x_0, t_0)\).
In case (i), by Lemma 2.1, we see that \((2.4)\) holds. By \((1.10)\), we have
\[(4.11) \quad b \mathcal{L}(u - u) \geq C \sum F_{ii} \geq -K_0 (\sum F_{ii} + F_{\tau}) - C \sum F_{ii}.\]
Combining with \((4.11)\) and \((4.10)\), we have
\[(4.12) \quad b \mathcal{L}(u - u) \leq -\frac{C}{u_t} + C \delta^2 \left(1 + \sum F_{ii} + F_{\tau}\right) + C b^2 \sum F_{ii}.\]
Now using \((2.4)\) we can choose \(\delta \ll b \ll 1\) to obtain a bound
\[-u_t(x_0, t_0) \leq C b \varepsilon.\]
In case (ii), we see that \((2.5)\) holds. By \((4.10)\), we have
\[(4.13) \quad b \mathcal{L}(u - u) + \frac{\delta^2}{4} F_{ii} U_{ii}^2 + \delta (F_{ii} U_{ii} - F_{\tau} u_t) \leq -\frac{C}{u_t} + C \delta^2 \left(1 + \sum F_{ii}\right) + C (\delta + b^2) \sum F_{ii}.\]
Note that
\[(4.14) \quad \frac{\delta^2}{4} F_{ii} U_{ii}^2 \geq \delta F_{ii} |U_{ii}| - \sum F_{ii}\]
and
\[(4.15) \quad \mathcal{L}(u - u) \geq 0\]
by the concavity of \(F\). Therefore, by \((4.13), (4.14)\) and \((4.15)\), we have
\[(4.16) \quad -\delta F_{\tau} u_t \leq -\frac{C}{u_t} + C \delta^2 + C \sum F_{ii}.\]
By \((1.10)\), similar to \([10]\),
\[(4.17) \quad -F_{\tau} u_t \geq \frac{-\beta u_t}{\sqrt{n+1}} (\sum F_{ii} + F_{\tau}) \geq \frac{-\beta u_t}{2 \sqrt{n+1}} (\sum F_{ii} + F_{\tau}) + \frac{\beta b_0}{2 \sqrt{n+1}}.\]
Choose \(\delta\) sufficiently small such that
\[\frac{\beta b_0 \delta}{2 \sqrt{n+1}} - C \delta^2 \geq c_1 > 0\]
for some constant $c_1$. Therefore, we can derive from (4.16) that

$$-u_t(x_0, t_0) \leq \max \left\{ \frac{C\sqrt{n+1}}{\beta \delta}, \frac{C}{c_1} \right\}.$$

So (4.2) holds.

Similarly, we can show (4.18) by setting

$$W = \sup_{M_T} u_t e^\phi$$

and $\phi = \frac{\delta^2}{2} |\nabla u|^2 - \delta u + b(u - u)$. Combining (4.2) and (4.18), we can see that (4.1) holds.

Since $u_t = \varphi_t$ on $SM_T$ and (1.17), we can derive the estimate

(4.19) $\sup_{M_T} |u_t| \leq C$.

5. Global estimates for second order derivatives

In this section, we derive the global estimates for the second order derivatives. We prove the following maximum principle.

**Theorem 5.1.** Let $u \in C^1(\overline{M_T})$ be an admissible solution of (1.1) in $M_T$. Suppose that (1.3), (1.4), and (1.11) hold. Then

(5.1) $\sup_{M_T} |\nabla^2 u| \leq C(1 + \sup_{P M_T} |\nabla^2 u|),$  

where $C > 0$ depends on $|u|_{C^1(\overline{M_T})}$, $|u_t|_{C^0(\overline{M_T})}$, $|\psi|_{C^2(\overline{M_T})}$ and other known data.

**Proof.** Set

$$W = \max_{(x, t) \in \overline{M_T}} \max_{\xi \in T_x M, \xi = 1} (\nabla_{\xi \xi} u + \chi(\xi, \xi)) e^\phi,$$

where $\phi$ is a function to be determined. We may assume $W$ is achieved at $(x_0, t_0) \in \overline{M_T} - P M_T$ and $\xi_0 \in T_{x_0} M$. Choose a smooth orthonormal local frame $e_1, \ldots, e_n$ about $x_0$ as before such that $\xi_0 = e_1$, $\nabla_{e_i} e_j = 0$, and $\{U_{ij}(x_0, t_0)\}$ is diagonal. We see that $W = U_{11}(x_0, t_0) e^\phi(x_0, t_0)$. We may also assume that $U_{11} \geq \ldots \geq U_{nn}$ at $(x_0, t_0)$.

Since the function $\log(U_{11}) + \phi$ attains its maximum at $(x_0, t_0)$, we have, at $(x_0, t_0)$,

(5.2) $\frac{\nabla_i U_{11}}{U_{11}} + \nabla_i \phi = 0$ for each $i = 1, \ldots, n$,

(5.3) $\frac{(\nabla_{11} u)_t}{U_{11}} + \phi_t \geq 0,$

and

(5.4) $0 \geq \sum_i F^{ii} \left\{ \frac{\nabla_i U_{11}}{U_{11}} - \left( \frac{\nabla_i U_{11}}{U_{11}} \right)^2 + \nabla_i \phi \right\}.$
Therefore, by (5.3) and (5.4), we find
\[ \mathcal{L}\phi \leq -\frac{1}{U_{11}}(F^{ii}\nabla_{ii}U_{11} - F^{\tau}(\nabla_{11}u)_{\tau}) + F^{ii}\left(\frac{\nabla_i U_{11}}{U_{11}}\right)^2. \]

By the formula
\[ \nabla_{ijk}\nabla - \nabla_{klj}\nabla = R_{tjk} \nabla_{im}v + \nabla_i R^{m}_{tjk} \nabla_{m}v + R^{m}_{tik} \nabla_{jm}v + R^{m}_{tjk} \nabla_{lm}v + R^{m}_{tjk} \nabla_{km}v + \nabla_k R^{m}_{tik} \nabla_{m}v \]
we have
\[ \nabla_i U_{11} \geq \nabla_{11}U_{ii} - C U_{11}. \]

Differentiating equation (1.1) twice, we have
\[ F^{ij}\nabla_{11}U_{ij} - F^{i}\nabla_{11}u_i + F^{ij,kl}\nabla_{1}U_{ij} \nabla_{1}U_{kl} \]
\[ + F^{i}\nabla_{1}U_{ij} \nabla_{1}u_i = \nabla_{11}\psi \geq -C. \]

It follows from (5.5), (5.7) and (5.8) that
\[ \mathcal{L}\phi \leq \frac{C}{U_{11}} + C \sum F^{ii} + E, \]
where
\[ E = \frac{1}{U_{11}}(F^{ij,kl}\nabla_{1}U_{ij} \nabla_{1}U_{kl} - 2F^{ij,\tau}\nabla_{1}U_{ij} \nabla_{1}u_i + F^{i}\nabla_{1}U_{ij} \nabla_{1}u_i)^2 + F^{ii}\left(\frac{\nabla_i U_{11}}{U_{11}}\right)^2. \]

E can be estimated as in [9] using an idea of Urbas [27] to which the following inequality proved by Andrews [1] and Gerhardt [7] is crucial.

**Lemma 5.2.** For any symmetric matrix \( \eta = \{\eta_{ij}\} \) we have
\[ F^{ij,kl}\eta_{ij}\eta_{kl} = \sum_{i,j} \frac{\partial^2 f}{\partial x_i \partial x_j} \eta_{ii} \eta_{jj} + \sum_{i \neq j} \frac{f_i - f_j}{\lambda_i - \lambda_j} \eta_{ij}^2. \]
The second term on the right hand side is nonpositive if \( f \) is concave, and is interpreted as a limit if \( \lambda_i = \lambda_j \).

Similar to [9], we can derive (see [3] also)
\[ E \leq \sum_{i \in J} F^{ii}(\nabla_i\phi)^2 + C \sum_{i \in K} F^{ii} + CF^{11} \sum_{i \in K} (\nabla_i\phi)^2, \]
where \( J = \{i : 3U_{ii} \leq -U_{11}\} \) and \( K = \{i : 3U_{ii} > -U_{11}\} \).

Let
\[ \phi = \frac{\delta |\nabla u|^2}{2} + b(u - u), \]
where \( \delta \) and \( b \) are positive constants to be determined. Thus, we can derive from (5.10) that
\[ E \leq Cb^2 \sum_{i \in J} F^{ii} + C\delta^2 F^{ii}U_{11}^2 + C \sum_{i \in K} F^{ii} + C(\delta^2 U_{11}^2 + b^2)F^{11}. \]

On the other hand, by (5.7) and (5.10),
\[ \mathcal{L}\phi = \delta F^{ii} \sum_k (\nabla_{ik}u)^2 + \delta \nabla_k u F^{ii} \nabla_{ik}u - \delta \nabla_k u F^{\tau}(\nabla_{k}u)_t + b\mathcal{L}(u - u) \]
\[ \geq \delta F^{ii}U_{11}^2 + b\mathcal{L}(u - u) - C\delta\left(1 + \sum F^{ii}\right) \]
Combining (5.9), (5.11) and (5.12), we obtain
\[
(5.13) \quad \frac{\delta}{2} F_{ii} U_{ii}^2 + b \mathcal{L}(\mathbf{u} - u) \leq \frac{C}{U_{11}} + C b^2 \sum_{i \neq j} F_{ii} + C b^2 F^{11} + C \left(1 + \sum F_{ii}\right)
\]
provided \( \delta \) is sufficiently small. Note that \( |U_{jj}| \geq \frac{1}{4} U_{11} \), for \( j \in J \). Therefore, by (5.13), we have
\[
(5.14) \quad \frac{\delta}{4} F_{ii} U_{ii}^2 + b \mathcal{L}(\mathbf{u} - u) \leq C \left(1 + \sum F_{ii}\right)
\]
when \( U_{11}^2 \geq \max\{C b^2/\delta, 1\} \).

Now let \( \mu_0 = \mu(x_0, t_0) \) and \( \lambda_0 = \lambda(x_0, t_0) \). If \( |\lambda_0 - \mu_0| \geq \beta \), we can obtain a bound of \( U_{11}(x_0, t_0) \) by (2.5) as in [9].

If \( |\lambda_0 - \mu_0| < \beta \), we see that (2.5) holds. Let \( \hat{\lambda} = \lambda(U(x_0, t_0)) \). We may assume \( |\hat{\lambda}| \geq |u_t(x_0, t_0)| \). Similar to (10), by the concavity of \( f \),
\[
(5.15) \quad |\hat{\lambda}| \left(\sum F_{ii} + F^r\right) \geq f(\hat{\lambda}|1) - f(\lambda(U), -u_t) + \sum F_{ii} U_{ii} - F^r u_t
\]
\[
\geq f(\hat{\lambda}|1) - f(\lambda(U), -u_t) - |\hat{\lambda}| \left(\sum F_{ii} + F^r\right)
\]
\[
\geq 2b_0 - |\hat{\lambda}| \left(\sum F_{ii} + F^r\right)
\]
for some uniform positive constant \( b_0 \), provided \( |\hat{\lambda}| \) is sufficiently large. By (2.5), (4.16) and (5.14), we see that
\[
(5.16) \quad 2c_0 |\hat{\lambda}|^2 \left(\sum F_{ii} + F^r\right) \leq C \left(1 + \sum F_{ii}\right),
\]
where
\[
c_0 := \frac{\delta \beta}{8 \sqrt{n} + 1}.
\]
Then we can derive a bound of \( |\hat{\lambda}| \) from (5.15). \( \square \)

6. Boundary estimates for second order derivatives

In this section, we consider the estimates of second order derivatives on \( SM_T \). We may assume \( \varphi \in C^4(M_T) \). For simplicity we shall make use of the condition (1.12) though stronger results may be proved (see [9], [10] and [12]).

The pure tangential second derivatives are easy to estimate from the boundary condition \( u = \varphi \) on \( \mathcal{P}M_T \). So we are focused on the estimates for mixed tangential-normal and pure normal second derivatives.

Fix a point \( (x_0, t_0) \in SM_T \). We shall choose smooth orthonormal local frames \( e_1, \ldots, e_n \) around \( x_0 \) such that when restricted to \( \partial M \), \( e_n \) is normal to \( \partial M \).

Let \( \rho(x) \) and \( d(x) \) denote the distance from \( x \in M \) to \( x_0 \) and \( \partial M \) respectively and set
\[
M_T^\delta = \{X = (x, t) \in M \times (0, T) : \rho(x) < \delta\}.
\]
We shall use the following barrier function as in [9].
\[
(6.1) \quad \Psi = A_1 v + A_2 \rho^2 - A_3 \sum_{\gamma < n} |\nabla_\gamma (u - \varphi)|^2,
\]
where
\[
v = (u - \varphi) + ad - \frac{N d^2}{2}.
\]
Now we show the following lemma which is useful to construct barrier functions (see Lemma 6.2).

**Lemma 6.1.** Suppose (1.3) and (1.4) hold. Then for any \( σ > 0 \) and any index \( r \),

\[
\sum f_i |\hat{\lambda}_i| \leq σ \sum f_i \hat{\lambda}_i^2 + \frac{C}{σ} \left( \sum f_i + F^\tau \right) + C
\]

**Proof.** If \( \hat{\lambda}_r < 0 \), by (1.2), we see

\[
\sum f_i |\hat{\lambda}_i| = 2 \sum f_i \hat{\lambda}_i - \sum f_i \hat{\lambda}_i + F^\tau u_t - F^\tau u_t
\]

\[
\leq σ \sum f_i \hat{\lambda}_i^2 + \frac{1}{σ} \sum f_i + K_0 \left( 1 + \sum f_i + F^\tau \right) + CF^\tau
\]

and (6.2) follows.

If \( \hat{\lambda}_r \geq 0 \), by the concavity of \( f \),

\[
\sum f_i |\hat{\lambda}_i| = \sum f_i \hat{\lambda}_i - 2 \sum f_i \hat{\lambda}_i
\]

\[
\leq σ \sum f_i \hat{\lambda}_i^2 + \frac{1}{σ} \sum f_i + \sum f_i \hat{\lambda}_i - F^\tau (\hat{\mu} - u_t)
\]

\[
\leq σ \sum f_i \hat{\lambda}_i^2 + \frac{1}{σ} \sum f_i + C \left( \sum f_i + F^\tau \right).
\]

Then (6.2) is proved. \( \square \)

The following Lemma is crucial to our estimates and the idea is mainly from [10] and [11] (see [13] also).

**Lemma 6.2.** Suppose that (1.3), (1.4) and (1.11) hold. Then for any constant \( K > 0 \), there exist uniform positive constants \( a, δ \) sufficiently small, and \( A_1, A_2, A_3, N \) sufficiently large such that \( \Psi \geq K(d + ρ^2) \) in \( \overline{M}_T^\delta \) and

\[
\mathcal{L} \Psi \leq -K \left( 1 + \sum_{i=1}^n f_i |\hat{\lambda}_i| + \sum_{i=1}^n f_i + F^\tau \right) \text{ in } \overline{M}_T^\delta.
\]

**Proof.** For any fixed \((x, t) \in M_T^\delta\), we may assume that \( U_{ij} \) and \( F_{ij} \) are both diagonal at \((x, t)\). Firstly, we have (see [9] for details),

\[
\mathcal{L} (\nabla_k (u - \varphi)) \leq C \left( 1 + \sum f_i |\hat{\lambda}_i| \right) \text{ for } 1 \leq k \leq n.
\]

Therefore,

\[
\sum_{k<n} \mathcal{L} (|\nabla_k (u - \varphi)|^2) \geq \sum_{k<n} F_{ij} U_{il} U_{lj} - C \left( 1 + \sum f_i |\hat{\lambda}_i| \right) \text{ for } 1 \leq k \leq n.
\]

Using the same proof of Proposition 2.19 in [9], we can show

\[
\sum_{l<n} F_{il} U_{il} \geq \frac{1}{2} \sum_{i \neq r} f_i \hat{\lambda}_i^2,
\]

for some index \( r \). Write \( \mu = \mu(x, t) \) and \( \lambda = \lambda(x, t) \) and note that \( \mu = (\hat{\mu}, -\hat{u}_t) \) and \( \lambda = (\hat{\lambda}, -u_t) \), where \( \hat{\mu} = \lambda(U) \).
We shall consider two cases as before: (a) $|\nu_\mu - \nu_\lambda| < \beta$ and (b) $|\nu_\mu - \nu_\lambda| \geq \beta$.

Case (a). By (2.5), we have

$$f_i \geq \frac{\beta}{\sqrt{n+1}} \left( \sum_{i' \neq i} f_{i'} + F^r \right), \quad \forall 1 \leq i \leq n.$$  

(6.7)

Now we make a little modification of the proof of Lemma 3.1 in [10] to show the following inequality

$$\sum_{i \neq r} f_i \hat{\lambda}_i^2 \geq c_0 \sum_{i \neq r} f_i \hat{\lambda}_i^2 - C_0 \left( \sum_i f_i + F^r \right)$$  

for some $c_0, C_0 > 0$. If $\hat{\lambda}_r < 0$, we have

$$\hat{\lambda}_r^2 \leq n \sum_{i \neq r} \hat{\lambda}_i^2 + C,$$

where $C$ depends on the bound of $u_t$ since $\sum \hat{\lambda}_i - u_t > 0$.

Therefore, by (6.7) and (6.9), we have

$$f_r \hat{\lambda}_r^2 \leq n \sum_{i \neq r} \hat{\lambda}_i^2 + C f_r \leq \frac{n \sqrt{n+1}}{\beta} \sum_{i \neq r} f_i \hat{\lambda}_i^2 + C \sum f_i$$

(6.1) holds.

Now suppose $\hat{\lambda}_r \geq 0$. By the concavity of $f$, 

$$f_r \hat{\lambda}_r \leq f_r \tilde{\mu}_r - F^r (u_t - u_i) + \sum_{i \neq r} f_i (\tilde{\mu}_i - \tilde{\lambda}_i).$$

(6.11)

Thus, by (6.7) and Schwarz inequality, we have

$$\frac{\beta f_r \hat{\lambda}_r^2}{\sqrt{n+1} \left( \sum_i f_i + F^r \right)} \leq f_r^2 \tilde{\lambda}_r^2 \leq C \left( f_r^2 \tilde{\mu}_r^2 + \sum_{k \neq r} f_k \sum_{i \neq r} f_i (\tilde{\mu}_i^2 + \hat{\lambda}_i^2) + (F^r)^2 \right)$$

$$\leq C \left( \sum_i f_i + F^r \right) \left( \sum_{i \neq r} \left( f_i + F^r \right) + \sum_{i \neq r} f_i \hat{\lambda}_i^2 \right),$$

(6.12)

where $C$ may depend on the bound of $|u_i|$. It follows that

$$f_r \hat{\lambda}_r^2 \leq C \sum_{i \neq r} f_i \hat{\lambda}_i^2 + C \left( \sum_i f_i + F^r \right)$$

(6.13)

and (6.8) holds.

We first suppose $|\lambda| \geq R$ for $R$ sufficiently large. By (5.15), we see

$$\sum_{i \neq r} f_i \hat{\lambda}_i^2 \geq b_0 |\lambda|$$

(6.14)

when $R$ is sufficiently large. Since $|\nabla d| \equiv 1$, when $a$ and $\delta$ are sufficiently small, by (6.7), we have,

$$\mathcal{L} v \leq \left( \mathcal{L} (u - u) + C_0 (a + N d) \sum f_i - NF^{ij} \nabla_i d \nabla_j d \right)$$

$$\leq - \frac{\beta N}{2 \sqrt{n+1}} \left( \sum f_k + F^r \right).$$

(6.15)
Note that for any $\sigma > 0$,

\[
\sum f_i |\tilde{\lambda}_i| \leq \sigma \sum f_i \tilde{\lambda}_i^2 + \frac{1}{\sigma} \sum f_i.
\]

(6.16)

Therefore, it follows from (6.8), (6.15) and (6.16) that for any $\sigma > 0$,

\[
\mathcal{L}\Psi \leq -\frac{\beta A_1 N}{2\sqrt{n} + 1} \left( \sum f_k + F^r \right) + CA_2 \sum f_i
\]

\[
- \frac{A_3}{2} \sum_{i \neq r} f_i \tilde{\lambda}_i^2 + CA_3 \left( 1 + \sum f_i |\tilde{\lambda}_i| + \sum f_i + F^r \right)
\]

(6.17)

\[
\leq -\frac{\beta A_1 N}{2\sqrt{n} + 1} \left( \sum f_k + F^r \right) - \frac{A_3 c_0}{4} \sum f_i \tilde{\lambda}_i^2 + CA_2 \sum f_i
\]

\[
+ CA_3 \left( 1 + \sum f_i |\tilde{\lambda}_i| + \sum f_i + F^r \right)
\]

\[
\leq -\frac{\beta A_1 N}{2\sqrt{n} + 1} \left( \sum f_k + F^r \right) + \left( A_3 \sigma - \frac{A_3 c_0}{2} \right) \sum f_i \tilde{\lambda}_i^2
\]

\[
+ C \left( A_2 + \frac{A_3}{\sigma} \right) \sum f_i + CA_3 \left( 1 + F^r \right).
\]

Let $\sigma = c_0/4$, we find

\[
\mathcal{L}\Psi \leq -\frac{\beta A_1 N}{2\sqrt{n} + 1} \left( \sum f_k + F^r \right) - \frac{A_3 c_0}{4} \sum f_i \tilde{\lambda}_i^2
\]

\[
+ C (A_2 + A_3) \left( \sum f_i + F^r \right) + CA_3
\]

\[
\leq -\frac{\beta A_1 N}{2\sqrt{n} + 1} \left( \sum f_k + F^r \right) - \frac{A_3 c_0 b_0}{8} |\tilde{\lambda}| - A_3 \sum f_i |\tilde{\lambda}_i|
\]

\[
+ C (A_2 + A_3) \left( \sum f_i + F^r \right) + CA_3
\]

\[
\leq -\frac{\beta A_1 N}{2\sqrt{n} + 1} \left( \sum f_k + F^r \right) - A_3 \sum f_i |\tilde{\lambda}_i|
\]

\[
+ C (A_2 + A_3) \left( \sum f_i + F^r \right)
\]

\[
\text{by choosing } R \geq 8C/c_0b_0 + 1.
\]

If $|\tilde{\lambda}| \leq R$, by (1.3) and (1.5), we have

\[
c_1 I \leq \{F^U\} \leq C_1, \quad c_1 \leq F^r \leq C_1
\]

for some uniform positive constants $c_1$, $C_1$ which may depend on $R$. Therefore, we have

\[
\mathcal{L}\Psi \leq C (-A_1 + A_2 + A_3) \left( 1 + \sum f_i + \sum f_i |\tilde{\lambda}_i| + F^r \right)
\]

(6.19)

where $C$ depends on $c_1$ and $C_1$.

Case (b). By Lemma 2.4, we may fix $a$ and $\delta$ sufficiently small such that $v \geq 0$ in $M^\delta_T$ and

\[
\mathcal{L}v \leq \frac{\varepsilon}{2} \left( 1 + \sum f_i + F^r \right) \quad \text{in } M^\delta_T.
\]

(6.20)
Thus, by Lemma [6.1] we have
\[ L\Psi \leq -\frac{\varepsilon A_1}{2} \left( 1 + \sum f_i + F^r \right) + CA_2 \sum f_i - \frac{A_3}{2} \sum_{i \neq r} f_i \hat{\lambda}_i^2 \]
\[ + CA_3 \left( 1 + \sum f_i + \sum f_i |\hat{\lambda}_i| \right) \]
\[ \leq \left( -\frac{\varepsilon A_1}{2} + CA_2 + CA_3 \right) \left( 1 + \sum f_i + F^r \right) - \frac{A_3}{4} \sum_{i \neq r} f_i \hat{\lambda}_i^2 \]
\[ \leq \left( -\frac{\varepsilon A_1}{2} + CA_2 + CA_3 \right) \left( 1 + \sum f_i + F^r \right) - A_3 \sum f_i |\hat{\lambda}_i|, \]
Checking (6.18), (6.19) and (6.21), we can choose \( A_1 \gg A_2 \gg A_3 \gg 1 \) such that (6.3) holds and \( \Psi \geq K(d + \rho^2) \) in \( M_\delta^T \). Therefore, Lemma 6.2 is proved. \( \square \)

The estimates for mixed tangential-normal second derivatives can be established immediately using \( \Psi \) as a barrier function by (6.4) and the maximum principle (see [3]).

The pure normal second derivatives can be derived as [9] using an idea of Trudinger [24]. The reader is referred to [3] for details.

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