Abstract. We are checking the closed categories beginning with the category of all sets $\text{SET}$ and ending with the category of all categories $\text{CAT}$. The novelty is a generalizing the adjoint functors to the category of directed graphs $\text{GRPH}$. We have described for what condition we get bijective name mapping for graphs transports $\text{GRPH}(X \times T; Y) \rightarrow \text{GRPH}(X; Y^T)$. As an example of future applications we introduce a notion of relator to some tens or product.

Key words: exponential functor, name mapping, directed graphs, relator.

The scalar space $V = [0; \infty]$ was the first example of closed monoidal category. It was values space for metric spaces. Its objects were called values and arrows $r \rightarrow s$ were inequality between values $r \geq s$.

The monoidal category has a tensor product $r \otimes s$ defined with some bi-functor. For arbitrary binary word of objects we define the tensor product of these objects uniquely up to unique canonical isomorphism. The monoidal category is closed if its tensor product defines functor $r \otimes s$ having coadjoint functor $t^* = \exp t_s$. It will be called an exponential functor. Otherwise for exponential functor $t^*$ tensor product $r \otimes s$ will be an adjoint functor. We shall say that both functors provide a joint pair of functors, The first functor is adjoint to the second, and the second is coadjoint to the first. For truly joint pair of functors we have natural bijection between arrows sets

$$V(r \otimes s; t) \rightarrow V(r; t^*) .$$

For an arrow $f : r \otimes s \rightarrow t$ this bijection appoints its name $[f] : x \rightarrow t^x$. The inverse bijection will be called a cobijection of taken joint pair of functors

$$V(r; t^*) \rightarrow V(r \otimes s; t) .$$

It for the arrow $g : x \rightarrow t^x$ appoints its realization $[g] : r \times s \rightarrow t$.

We extend the notion of joint pair of functors for more general case. The joint pair of functors from category $X$ to category $Y$ will be arbitrary pair of functors

$$\langle F, G \rangle : X \leftrightarrow Y .$$

One $F : X \Rightarrow Y$ is called an adjoint functor, another $G : Y \Rightarrow X$ is called a coadjoint functor. A unit transform $i : 1d_X \rightarrow FG$ is defined by collection of arrows $i_x : x \rightarrow x^{FG}$ for every point in the source category $x \in X$. Sometimes such arrows haven’t any property needed in source category $X$, so
unit transform for joint pair of functors is only auxiliary. The existing unit transform defines a name mapping between arrows sets

\[ Y(x^F; y) \to X(x; y^G) . \]

For an arrow \( f \in Y(x^F; y) \) it appoints the composition of unit arrow with image of taken arrow for the adjoint functor \( i_x \circ f^G \in X(x; y^G) \). For a natural unit transform we get the natural name mappings with respect to arrows in each category \( u \in X(x'; x) \) and \( v \in Y(y; y') \).

\[
\begin{align*}
Y((x')^F; y') & \to X(x'; (y')^G) \\
(u^F)^* \times v_s & \to u^* \times (v^G)^*_s \\
Y(x^F; y) & \to Y(x; y^G)
\end{align*}
\]

It can be expressed as commuting diagrams in the category \( X \)

\[
\begin{array}{ccc}
x' & \xrightarrow{i_{x'}} & (x')^F \to (y')^G \\
u & \downarrow & u^F \downarrow \\
x & \xrightarrow{i_x} & x^F \to y^G
\end{array}
\]

A counit transform \( e : GF \to \text{Id}Y \) is defined by collection of arrows \( e_y : y^GF \to y \) for every point in the target category \( y \in Y \). The counit transform for joint pair of functors is also only auxiliary. The existing counit transform defines a realization mapping between former arrows sets in opposite direction

\[ X(x; y^G) \to Y(x^F; y) . \]

For an arrow \( g \in X(x; y^G) \) it appoints the composition of image for the coadjoint functor of taken arrow with counit arrow \( g^F \circ e_y \in Y(x^F; y) \). For a natural counit transform we get the natural realization mapping according to the arrows in each category \( u \in X(x'; x) \) and \( v \in Y(y; y') \).

\[
\begin{align*}
X(x'; (y')^G) & \to Y((x')^F; y') \\
u^* \times (v^G)^*_s & \to (u^F)^* \times v_s \\
X(x; g^G) & \to Y(x^F; y)
\end{align*}
\]

It can be expressed as commuting diagrams in category \( Y \)

\[
\begin{array}{ccc}
(x')^F & \xrightarrow{(g')^p} & (y')^GF \\
u^F & \downarrow & u^GF \downarrow \\
x^F & \xrightarrow{g^p} & y^GF \to (v^GF)^* \to v \\
& & e_y \to y
\end{array}
\]

For truly joint pair of functors we demand that realization mapping would be inverse to the name mapping, i.e. both mappings would be bijections. We shall say that a realization mapping reverses the name mapping, and a name mapping reverses the realization mapping.
Taking an arrow \( f = 1_x \) we construct its name’s realization. It coincides with arrows composition

\[
(i_x)^F \circ \epsilon(xF) : x^F \to x^F.
\]

So if the realization mapping reverses name mapping we get the first triangular equality for unit and counit transforms

\[
i \times F \circ F \times e = \text{Id}_F.
\]

Otherwise, having the first triangular equality the realization of name maps the same arrow \( f \in Y(x^F; y) \) exactly when we have commuting quadrat begun with an arrow \((i_x)^F : x^F \to x^{FGF}\) for the unit transform ended with the adjoint functor

\[
\begin{array}{ccc}
X^F & \xrightarrow{i \times F} & x^{FGF} & \xrightarrow{F \times e} & y^{GF} \\
\downarrow & \downarrow & \downarrow e & \downarrow \quad f & \downarrow y \\
X^F & \xrightarrow{F \times e} & y \\
\end{array}
\]

This can be proved by completing the first triangular diagram with the unit arrow of identity transform \( \text{Id}_F : F \to F \)

\[
\begin{array}{ccc}
X^F & \xrightarrow{i \times F} & x^{FGF} & \xrightarrow{F \times e} & y^{GF} \\
\downarrow & \downarrow & \downarrow e & \downarrow \quad f & \downarrow y \\
X^F & \xrightarrow{F \times e} & y \\
\end{array}
\]

Taking an arrow \( g = 1_y \) we construct its realization’s name. It coincides with arrows composition

\[
(i_y)G \circ (e_y)^G : y^G \to y^G.
\]

So if the name mapping reverses realization mapping we get the second triangular equality for unit and counit transforms

\[
(G \times i) \circ (e \times G) = 1_G.
\]

Otherwise, having the second triangular equality the name of realization maps the same arrow \( g \in X(x; y^G) \) exactly when we have commuting quadrat ended with an arrow \((e_y)^G : y^{FG} \to x^G\) for the counit transform ended with the coadjoint functor

\[
\begin{array}{ccc}
y^G & \xleftarrow{\epsilon \times G} & y^{FG} & \xleftarrow{F} & x^G \\
\uparrow G \times i & \uparrow i & \uparrow \quad x \\
y^G & \xleftarrow{\epsilon \times G} & y^{FG} \\
\end{array}
\]

This can be proved by completing the second triangular diagram with the unit arrow of identity transform \( \text{Id}_G : G \to G \)

\[
\begin{array}{ccc}
y^G & \xleftarrow{\epsilon \times G} & y^{FG} & \xleftarrow{F} & x^G \\
\uparrow G \times i & \uparrow i & \uparrow \quad x \\
y^G & \xleftarrow{\epsilon \times G} & y^{FG} \\
\end{array}
\]
For natural unit and counit transforms the first and second triangular equalities provides that the name mapping and the realization mapping are inverse each to other natural bijections

\[ Y(x^F; y) = \to X(x; y^G). \]

In monoidal category \( V \) else we have a final object \( * \in V \), i.e. for every object \( r \in V \) we always have exactly one arrow \( ! : r \to * \). We demand that final object \( * \in V \) would be also the unit for tensor product, i.e. we should have canonical isomorphisms

\[ * \otimes r = r = r \otimes *. \]

We shall look for exponential functor in various categories. At first we shall probe to define the exponential functor in the category of all sets \( \text{SET} \) or in the category of small sets \( \text{Set} \). Then the most familiar will be the case of the category \( \text{CAT} \) of all possible categories. The arrows set \( \text{CAT}(X; Y) \) will be compounded of functors \( f : X \Longrightarrow Y \) between categories \( X \) and \( Y \). The exponential functor will be coadjoint for the Carte tensor product. It for a category \( X \) appoints the category \( X^A \) with arrows sets \( X^A(f; g) \) compounded of all natural transforms \( \phi : f \to g \) between functors \( f, g : A \Longrightarrow X \).

In next part of this work the construction of such arrows sets will be rewritten for the category \( \text{Cat}_{\text{Set}} \) of \( \text{Set} \)-enriched categories with values category \( \text{Set} \) of small sets. It will be a sample for a construction of exponential functor in the category \( \text{Cat}_V \) of \( V \)-enriched categories for arbitrary values category \( V \). So we acquire the construction of exponential functor also in the category of metric spaces \( \text{Cat}_{[0, \infty]} \). Finally we shall construct exponential functor in the category \( \text{Cat}_V \) with another tensor product in values category \( V \) which will be called a relator to the original tensor product of closed monoidal category \( V \). Then we provide the proving of Yoneda lemma for \( \text{Set} \)-enriched categories. The exponential functor is needed for the \( V \)-enriched category \( X \) to define the second dual space \( \hat{X} = V^{X^{op}} \) with Yoneda imbedding \( X \subset \hat{X} \).

We could work with the category of all sets \( \text{SET} \). We shall define exponential functor in such tremendous category. The objects of category \( \text{SET} \) will be arbitrary sets \( X \) as entity of points \( x \in X \). They will be called members of taken set. The arrows between two sets will be arbitrary appointment \( f : X \to Y \). It for each member in the source set \( x \in X \) appoints the member in the target set \( f(x) \in Y \). For appointments \( f : X \to Y \) and \( g : Y \to Z \) we have usual composition \( f \circ g : X \to Z \) with

\[ f \circ g(x) = g(f(x)), \]

and we can choose identity appointments \( \text{Id}_X : X \to X \) with \( \text{Id}_X(x) = x \) for each point \( x \in X \) as unit arrows for such composition

\[ \text{Id}_X \circ f = f = f \circ \text{Id}_Y. \]
In the category \textbf{SET} we shall define the tensor product of two sets \(X \times Y\). It will be a set of all pairs
\[
X \times Y = \{(x, y) : x \in X, y \in Y\}
\]
We take two appointments \(p_1 : X \times Y \to X\) and \(p_2 : X \times Y \to Y\)
\[
p_1((x, y)) = x, \quad p_2((x, y)) = y.
\]
as projections of taken tensor product.

It will provide one possible realization of the Cartesian product in the category \textbf{SET}, i.e. we can check the universal property of such product. For arbitrary two appointments \(f : Z \to X\) and \(g : Z \to Y\) we have exactly one appointment to the product set \(h : Z \to X \times Y\) with projections equal to taken appointments
\[
h \circ p_1 = f, \quad h \circ p_2 = g.
\]
It can be taken
\[
h(z) = (f(z), g(z)) \in X \times Y,
\]
and if we have some appointment \(h' : Z \to X \times Y\) with taken projections, then for the point \(z \in Z\) appointed pair \(h'(z) = (x, y) \in X \times Y\) will have the same projections
\[
x = h' \circ p_1(z) = f(z), \quad y = h' \circ p_2(z) = g(z),
\]
therefore we have equality of appointed pairs \((x, y) = (f(z), g(z))\) and equality of both appointments
\[
h' = h.
\]

Using universal property of the Cartesian product we get unique isomorphisms
\[
S_{X,Y,Z} : X \times (Y \times Z) \to (X \times Y) \times Z.
\]
These isomorphisms are natural for possible appointments
\[
f : X \to X', \quad g : Y \to Y', \quad h : Z \to Z'.
\]

They admit the pentagon diagram, so we can get coherence property from \textbf{MacLane 1971} part VII. Monoids to get tensor product for arbitrary binary word of sets uniquely up to unique canonical isomorphism. In our case the canonical isomorphisms are chosen next appointments and their inverse together with all possible compositions of various tensor products. First we take identity appointment of any taken set \(1_X : X \to X\), secondly we take former isomorphism of associativity \(S_{X,Y,Z}\).

Any onepoint set \(* = \{o\}\) provides final object in the category \textbf{Set}. It becomes unit for the tensor product with chosen canonical isomorphisms
\[
R_X : X \times * \to X
\]
which for a couple $\langle x, o \rangle \in X \times \ast$ appoints the point $x \in X$, and

$$L_X : \ast \times X \to X$$

which for a couple $\langle o, x \rangle \in \ast \times X$ appoints the point $x \in X$.

Every onepoint set $\ast = \{ o' \}$ correspondingly will provide another possible realization of the final object in the category Set. It can be taken as another unit for taken product with another natural isomorphisms for the projections

$$R_X : X \times \ast \to X, \quad L_X : \ast \times X \to X,$$

These projections are natural transforms. It rests to check that such projections are isomorphisms.

For the projection $R_X : X \times \ast \to X$ we take an appointment to the product $\theta : X \to X \times \ast$ defined with the pair of appointments $1_X : X \to X$ and unique appointment to the final object $! : X \to \ast$, i. e. we have composition equal to the identic appointment

$$\theta \circ R_X = 1_X.$$

Otherwise the composition $R_X \circ \theta : X \times \ast \to X \times \ast$ is unanimously defined by the projections. One projection is

$$(R_X \circ \theta) \circ R_X = R_X \circ (\theta \circ R_X) = R_X,$$

and another coincides with unique arrow to the final object

$$(R_X \circ \theta) \circ ! = !.$$

Therefore we get another equality

$$R_X \circ \theta = 1_X \times 1_\ast = 1_{X \times \ast}.$$

These projections admit the triangular cancellation diagrams. Such diagrams were drown in G. Valiukevičius 1992 p. 144.

And finally we get equal projections $R_\ast = L_\ast$, as the product of onepoint set is contained in the diagonal of product

$$\triangle = \ast \times \ast.$$

So we have got a tensor product for monoidal category of Bénabou 1963. We shall call it a Carte tensor product. Usually we understand the Carte product more freely as a final object in the category of cones over discrete diagrams of factors. In G. Valiukevičius 2009 p. 13 we have discussed abstract objects investigated in the theory of categories. Such theory was announced by one of its exploiter Norman Steenrod as abstract nonsense. The Carte product is an example of such abstract objects. Otherwise the topology in general sense by G. Valiukevičius deals with mappings of concrete objects. Functors will be examples of continuous mappings in some topological spaces and tensor products in category is defined as a family of concrete objects with
unique canonical isomorphisms between different realizations of some finite tensor product.

The importance of abstract objects has become evident in the science of program writing for any computer. The programs are written in abstract way, without trouble about concrete processes fulfilling such calculations.

We wish that the category of all sets SET would be a closed monoidal category.

The Carte tensor product of two sets $X \times Y$ defines bifunctor

$$\text{SET} \times \text{SET} \Rightarrow \text{SET}.$$  

For a pair of sets $(X,Y)$ we appoint the new set $X \times Y$, and for a pair of appointments $f : X \to X'$ and $g : Y \to Y'$ we get the product of appointments

$$f \times g : X \times Y \to X' \times Y'$$

which for a pair of points $(x,y) \in X \times Y$ appoints the pair of appointed points $(f(x),g(y)) \in X' \times Y'$.

Additionally we have maintenance of arrows composition

$$(f \times g) \circ (f' \times g') = (f \circ f') \times (g \circ g'),$$

and maintenance of unit arrows

$$\text{Id}_X \times \text{Id}_Y = \text{Id}_{X \times Y}.$$  

The chosen one-point set $* = \{o\}$ with correspondent isomorphic projections

$$R_X : X \times * \to X , \quad L_X : * \times X \to X$$

becomes a neutral set for taken Carte tensor product.

For arbitrary assistant set $T$ the Carte tensor product defines a functor denoted $X \times T$

$$\text{SET} \Rightarrow \text{SET}$$

which for arbitrary set $X$ appoints the Carte tensor product $X \times T$ and for arbitrary appointment $f : X \to X'$ appoints another appointment provided as Carte product of appointments $f \times \text{Id}_T : X \times T \to X' \times T$.

Now we shall construct a coadjoint exponential functor $Y^T$. For arbitrary set $Y$ we appoint the functional space $Y^T$ compounded of all possible appointments over assistant set $\phi : T \to Y$.

For an appointment $f : Y \to Y'$ this functor appoints the appointment of functional spaces $f^T : Y^T \to (Y')^T$, which for an appointment $\phi \in Y^T$ provides the composed appointment $\phi \circ f \in (Y')^T$. We shall call such composition as changing of target space $Y$.

We shall check that we have got a truly joint pair of functors

$$\langle F,G \rangle : \text{SET} \leftrightarrow \text{SET}$$
defined with Carte tensor product \( X^F = X \times T \) and functional space \( Y^G = Y^T \).

The \textit{unit transform} of joint pair \( i : \mathbb{I}_\text{SET} \to FG \) will be defined with collection of \textit{sections appointments}

\[
\lambda_X : X \to (X \times T)^T
\]

which for a point \( x \) appoints the \textit{section} \( \lambda_X(x) \in (X \times T)^T \), i.e. a graph of constant appointment \( \lambda_X(x)_t = (x, t) \).

We get a \textit{natural transform} of functors as for arbitrary appointment \( f : X \to X' \) we have commuting diagram:

\[
\begin{array}{ccc}
X & \xrightarrow{f} & X' \\
\downarrow \lambda_X & & \downarrow \lambda_X' \\
(X \times T)^T & \xrightarrow{(f \times 1)^T} & (X' \times T)^T
\end{array}
\]

Indeed. The section \( \lambda_X(x) \in (X \times T)^T \) which for the argument \( t \in T \) appoints a couple \( (x, t) \in X \times T \) now will be transformed to a new section which for the argument \( t \in T \) will appoint a couple \( (f(x), t) \in X' \times T \), therefore this new section is get by composition \( f \circ \lambda_X \).

The \textit{counit transform} of joint pair \( e : GF \to \mathbb{I}_\text{SET} \) will be defined with collection of \textit{evaluation appointments}

\[
ev_Y : Y^T \times T \to Y
\]

which for an appointment \( f \in Y^T \) and a point \( t \in T \) appoints the value of taken appointment over taken point \( f(t) \in Y \). These arrows define a natural transform of functors as for arbitrary mapping \( g : Y \to Y' \) we get commuting diagram

\[
\begin{array}{ccc}
(Y^T) \times T & \xrightarrow{s^T \times 1} & (Y')^T \times T \\
\downarrow \text{ev}_Y & & \downarrow \text{ev}_Y' \\
Y & \xrightarrow{g} & Y'
\end{array}
\]

Indeed. For a mapping \( g : Y \to Y' \) the composition of functors \( GF \) appoints the mapping \( Y^T \times T \to (Y')^T \times T \) which for an appointment \( \phi \in Y^T \) and a point \( t \in T \) provides the composed function \( \phi \circ g \in (Y')^T \) and the same argument \( t \in T \), therefore the new evaluation \( \text{ev}_{Y'} \) will provide the value \( g(\phi(t)) \) which coincides with value appointed by composition \( \text{ev}_X \circ g \).

For the \textit{truly joint pair of functors} we additionally must check both \textit{triangular equalities}. At first I shall explain the notation for \textit{horizontal composition of natural transforms} \( s : F \to G \) and \( t : K \to L \) between functors \( F, G : A \Rightarrow B \) and \( K, L : B \Rightarrow C \) with \( s \times t : FK \to GL \) defined with composition of arrows

\[
s^K_a : a^{FG} \to a^{GK}, \quad t_a^G : a^{GK} \to a^{GL},
\]

or another equal composition

\[
t_{a^F} : a^{FK} \to a^{FL}, \quad s^L_a : a^{FL} \to a^{GL}.
\]
So we have an equality
\[ s^K_a \circ t_a \circ t_a \circ s^L_a. \]

For a functor \( F : A \Rightarrow B \) we take the identity transform \( \text{Id}_F : F \rightarrow F \)
and define the horizontal composition
\[ F \times t = \text{Id}_F \times t \]
which is defined with collection of arrows \( t_{aF} : a^{FK} \rightarrow a^{FL} \).

And for a functor \( K \) we can use the horizontal composition
\[ s \times K = s \times \text{Id}_K, \]
it is defined with new arrows \( s^K_a : a^{FK} \rightarrow a^{GK} \). More about horizontal composition can be seen in G. Valiukevičius 1992 p. 98.

The first triangular equality is for the unit transform ended with the adjoint functor and composed with the counit transform begun with the same adjoint functor
\[ i \times F \circ F \times e = 1_F. \]
For a set \( X \) we take the Carte tensor product \( X \times T \) and we must check that we shall get the composition of two natural mappings equal to the identity mapping. The first natural transform \( i \times T \) for a couple \( \langle x, t \rangle \in X \times T \) will appoint the couple of section \( \lambda_X(x) \in (X \times T)_T \) and the same point \( t \in T \)
\[ \langle \lambda_X(x), t \rangle \in (X \times T)_T \times T. \]

The second transform is an evaluation for the space \( X \times T \). It for a section \( \lambda_X(x) \) appoints its value over the point \( t \in T \), therefore we get the same value \( \lambda_X(x)_t = \langle x, t \rangle \in X \times T \).

The second triangular equality is defined for the unit transform begun with the coadjoint functor and composed with the counit transform ended with the same coadjoint functor
\[ G \times i \circ e \times G = 1_G. \]
This composition must be the identity transform over functional space \( Y^T \) which is appointed by coadjoint functor \( G \) for taken set \( Y \).

The unit transform is defined with sections appointments
\[ \lambda_{Y^T} : Y^T \rightarrow (Y^T 	imes T)_T. \]
They for an appointment \( \phi \in X^T \) will appoint the section \( \lambda_{Y^T}(\phi) \) with values
\[ \lambda_{Y^T}(\phi)_t = \langle \phi, t \rangle. \]
Such section will be a point from functional space appointed by coadjoint functor \( G \) for the space \( Y^T \times T \)
\[ \lambda_{Y^T}(\phi) \in (Y^T \times T)_T, \]
therefore the counit appointment first will be applied as evaluation

\[ \text{ev}_Y : Y^T \times T \to Y \]

and then will be changed by coadjoint functor, i.e. we transform the function \( \psi \in Y^T \) by changing target space with evaluation arrow. For taken function it will be appointed composed function \( \psi \circ \text{ev}_Y \).

The section \( \lambda_{Y^T}(\phi) : T \to Y^T \times T \) will be appointed to appointment which for the point \( t \in T \) appoints evaluation of taken appointment over taken point \( t \). Therefore we get the same appointment \( \phi \).

So we have checked that category \( \text{SET} \) is a monoidal closed category.

In some cases we need to work with small sets by means of S. MacLane 1971 part 1 §6. It means the members from some universe \( U \). This was a problem of logic to say what is a set of all sets which aren’t members of itself. Therefore it was proposed to work only with members of some large set \( U \), which is called a universe. It was demanded that every member \( x \in y \) of any small set \( y \in U \) would be again a small set \( x \in U \),

- the product of small sets \( u \in U, v \in U \) remains small set \( u \times v \in U \),
- for small set \( u \in U \) the union of members \( \bigcup u \) remains small set
- and the set of all subsets \( 2^U \) remains a small set,
- the set of all finite ordinals \( N \) is a small set \( N \in U \),
- for arbitrary function \( f : X \to Y \) the image of small set \( u \in U \) remains small set \( f(u) \in U \).

Also we use the principle of comprehension that the members of small set with some logical property \( \phi \)

\[ \{ x : x \in u, \phi(x) = \text{true} \} \]

remains small set.

The small sets were applied by N. Bourbaki Théorie des ensembles 1957 to construct a universal arrow. Later we shall use small topological space to construct strongest Hausdorff simplification.

We shall work with a category of small sets. It again will be a closed monoidal category with Carte tensor product and coadjoint exponential functor.

The objects of category Set will be arbitrary small sets \( X \), and the arrows will be arbitrary mappings between small sets \( f : X \to Y \). It can be identify with graphics as partial set \( f \subset X \times Y \), therefore the whole arrows set \( \text{Set}(X;Y) \subset 2^{X \times Y} \) remains small set.

Obviously we have an imbedding of categories \( \text{Set} \subset \text{SET} \).

The Carte tensor product of small sets

\[ X \times Y = \{ (x, y) : x \in X, y \in Y \} \]

remains a small set. So we have got a tensor product in the category Set with the same neutral onepoint set \( * = \{ o \} \).
We shall check that the category of small sets $\text{Set}$ again is a \textit{closed monoidal category}, i.e. for the functor provided with Carte tensor product $X \otimes T$ we get a coadjoint exponential functor $Y^T$.

For small assistant set $T$ and another small set $Y$ we appoint the \textit{functional space} $Y^T$ compounded of all possible mappings $\phi : T \to Y$, i.e. it is contained in the small set $2^{T \times Y}$.

For the mapping $g : Y \to Y'$ this functor appoints a mapping of functional spaces provided by changing target space $g^T : Y^T \to (Y')^T$, which for a function $\phi \in Y^T$ provides the composed function $\phi \circ g \in (Y')^T$.

In imbedded smaller category $\text{Set} \subset \text{SET}$ we get again a truly joint pair of induced functors

$$\langle F_0, G_0 \rangle : \text{Set} \rightleftharpoons \text{Set} ,$$

as we have the unit transform $i : \text{Id}_{\text{Set}} \to F_0G_0$ and the counit transform $e : G_0F_0 \to \text{Id}_{\text{Set}}$ defined with arrows from smaller category $\text{Set}$, i.e. these arrows are mappings between small sets

$$\lambda_X : X \to (X \times T)^T , \quad \text{ev}_Y : Y \times T \to Y .$$

Next we shall construct an exponential functor in the category $\text{CAT}$ of all categories. But first we shall stop a little with a category of all directed graphs $\text{GRPH}$.

The graph is defined as a set of edges $X$ between vertexes from set $X_0$. We have two projections $p_1, p_2 : X \to X_0$. One $p_1$ is for the source of edges, and another $p_2$ for the target of edges. Between arbitrary two vertexes $a, b \in X$ we define an edges set as the inverse image of both projections

$$X(a;b) = (p_1 \times p_2)^{-1}((a,b)) \uparrow (a,b) \in X_0 \times X_0 .$$

The transport of graphs $f : X \Rightarrow Y$ will be defined by a mapping of edges $f : X \to Y$ and a mapping for vertexes $f_0 : X_0 \to Y_0$. These two mappings must commute with source and target projections

$$\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
p_1 \downarrow & & \downarrow p_1 \\
X_0 & \xrightarrow{f_0} & Y_0 \\
p_2 \downarrow & & \downarrow p_2 \\
X_0 & \xrightarrow{f_0} & Y_0
\end{array}$$

A transport of graphs can be defined with collection of mappings over edges sets

$$f_{a,b} : X(a;b) \to X(f_0(a); f_0(t)) .$$

If the projections of arrow's source or target would be understood as an instance of convergence in the set $X_0$ of vertexes for the sequences defined with one edge, then transport of mappings can be viewed as continuous mappings $f : X_0 \to Y_0$ for such convergences in the spaces of vertexes.

Arbitrary category $X$ is an example of graph. The edges will be called as arrows, and the vertexes will be called as points. Additionally we define
composition of arrows. It is functional mapping defined over the \textit{pullback of projections}

\[ X \times_{X_0} X \rightarrow X . \]

Such mapping can be defined with collection of mappings over arrows sets

\[ X_{a,b,c} : X(a;b) \times X(b;c) \rightarrow X(a;c) . \]

We can denote this operation by its image set

\[ X(a;b) \times X(b;c) \rightarrow X(a;b) \circ X(b;c) \subset X(a;c) . \]

We demand associativity for mappings of arrows composition. It can be expressed as equalities

\[ X(a;b) \circ (X(b;c) \circ X(c;d)) = (X(a;b) \circ X(b;c)) \circ X(c;d) . \]

For the \textit{true category} we also need an appointment of unit arrows

\[ 1_a \in X(a,a) . \]

We demand that for an arrow \( u \in X(a;b) \) the appointed arrows would be neutral for \textit{composition before arrow}

\[ 1_a \circ u = u , \]

and \textit{composition after arrow}

\[ u \circ 1_b = u . \]

The categories without unit arrows can be called as \textit{semicategories}, together with \textit{semimonoids} or \textit{semigroups}. The structure of such algebraic objects would become much more delicate.

In G. Valiukevičius 2009 I proposed to call algebraic operation of composition as a convergence for finite sequences of two members, and the appointment of unit arrows as a convergence for the sequence of one member. Then the category becomes convergence space over the set of arrows. Such way unifies the nomenclature of different mathematical regions - algebra and topology.

A functor between two categories \( f : X \Rightarrow Y \) is defined as a transport of underlying graphs maintaining the composition of arrows

\[ f(u \circ v) = f(u) \circ f(v) \]

and the appointment of unit arrows

\[ f(1_a) = 1_{f(a)} . \]

Arbitrary points in a category \( X \) can be identified with appointed unit arrow. So the category entirely is defined by its arrows, and functors between them coincide with continuous mappings for defined convergence of arrow’s sequences.
We can define a category of graphs \( \text{GRPH} \) compounded of all graphs transports. The points will be arbitrary graph and arrows will be transports between such graphs. Such transport \( f : X \Rightarrow Y \) are defined as an appointment of vertexes \( f_0 : X_0 \to Y_0 \) and appointments of edges sets

\[
f_{x,y} : X(x;y) \to Y(f(x);f(y)).
\]

It can be concerned that the edges defines two different convergences for sequences of one member, and mapping of vertexes are continuous mappings for both convergences.

The composition of graph transports are defined in usual way

\[
f \circ g(x) = g(f(x)), \quad f_{x,y} \circ g_{f(x),f(y)} = (f \circ g)_{x,y}(g(f(x));g(f(y))).
\]

The identity transport \( \text{Id}_X : X \Rightarrow X \) is defined by identity appointments

\[
\text{Id}_X(x) = x, \quad (\text{Id}_X)_{x,y} = \text{Id}_{X(x,y)}.
\]

It will be units for chosen composition of transports.

A Carte tensor product of two graphs \( X \times Y \) in the category of graphs \( \text{GRPH} \) is defined by Carte tensor product for the sets of vertexes \( X_0 \times Y_0 \) and the sets of edges \( X \times Y \). The source and target projections are also provided by Carte tensor products

\[
p_1 \times p_1 : X \times Y \to X_0 \times Y_0, \quad p_2 \times p_2 : X \times Y \to X_0 \times Y_0.
\]

So for the Carte tensor product of graphs \( X \times Y \) we have edges sets

\[
X \times Y(\langle x,y \rangle;\langle x',y' \rangle) = X(x;x') \times Y(y;y').
\]

It provides a bifunctor over the category of graphs \( \text{GRPH} \), but we shall use only the functor \( F : \text{GRPH} \Rightarrow \text{GRPH} \) which for the graph \( X \) appoints tensor product \( X \times T \) with chosen assistant graph \( T \).

For a graphs transport \( f : X \Rightarrow Y \) we appoint another graphs transport

\[
f \times \text{Id}_T : X \times T \Rightarrow Y \times T
\]

taking the appointment for edges \( f \times \text{Id}_T : X \times T \to Y \times Y \) and the appointment for vertexes \( f_0 \times \text{Id}_{X_0} : X_0 \times T_0 \to Y_0 \times T_0 \). We have checked in the category \( \text{SET} \) that such appointments maintains the arrows composition and unit arrows appointment.

We wish to define a joint pair of functors

\[
\langle F,G \rangle : \text{GRPH} \leftrightarrow \text{GRPH}.
\]

We need to construct an coadjoint exponential functor \( G \). For the graph \( Y \) we provide the set \( Y^T \) of all graphs transports \( \phi : T \Rightarrow Y \). It will be a set of vertexes for a new graph. The edges will be defined by transforms \( \alpha : \phi \to \psi \)
between graphs transports $\phi, \psi : T \Rightarrow Y$. It is understood as a collection of edges in target graph $Y$

$$\alpha_t \in Y(\phi_0(t); \psi_0(t)) \uparrow t \in T_0 .$$

We shall work only with full collections of edges, but we also could work with partial collection and get a more large exponential graph $X^T$.

The unit transform $i : \text{Id}_{\text{GRAPH}} \rightarrow FG$ will be defined only for graphs $X$ with chosen unit arrows $1_a \in X(a; a)$. Also for assistant graph $T$ we demand to have chosen unit arrows $1_s \in T(s; s)$.

We choose arbitrary edge $1_a \in X(a; a)$ which will be called a unit arrow. Such choosing provides a section $\lambda_X(x) : T \Rightarrow X \times T$ with unanimously defined edges appointment

$$\lambda_X(x)_u = (1_x, u) \in X \times T \uparrow u \in T(s; t) .$$

Choosing an edge $1_t \in T(t; t)$ provides a graphs transport

$$\lambda_X : X \Rightarrow (X \times T)^T$$

with unanimously defined edges appointment. It for an edge $\alpha \in X(a; b)$ will appoint the transform between two sections

$$\prod_{s \in T} \langle \alpha, 1_s \rangle : \lambda_X(a) \rightarrow \lambda_X(t) .$$

A unit transform will be defined with collection of graphs transports

$$\lambda_X : X \Rightarrow (X \times T)^T .$$

It for a vertex $a \in X_0$ appoints the section $\lambda_X(a) : T \Rightarrow X \times T$ with mapping of vertexes $\lambda_X(a) : T_0 \rightarrow X_0 \times T_0$

$$\lambda_X(a)_t = (a, t) \in X_0 \times T_0$$

and mapping of edges which for an edge $u \in T(s; t)$ appoints the edge in the product of graphs

$$\lambda_X(a)_u = (1_a, u) \in X \times T .$$

For an edge $\alpha \in X(a, b)$ sections transport appoints the transform between sections

$$\prod_{s \in T} \langle \alpha, 1_s \rangle : \lambda_X(a) \rightarrow \lambda_X(b)$$

defined by collection of edges

$$\langle \alpha, 1_s \rangle \in X \times T((a, s); (b, s)) = X(a; b) \times T(s; s) .$$

A counit transform for taken joint pair will be defined only for a graph $Y$ with chosen mappings of diagonal arrows

$$Y(c; d) \times Y(d; e) \rightarrow Y(c; d) \circ Y(d; e) \subset Y(c; e) .$$
This transform will be defined by collection of evaluation transports 

\[ \text{ev}_Y : Y^T \times T \rightarrow Y . \]

For the graphs transport \( \phi : T \rightarrow Y \) and vertex \( s \in T_0 \) it appoints the vertex \( \phi(s) \in Y_0 \). For the transform \( \beta : \phi \rightarrow \psi \) and an edge \( u \in T(s; t) \) it appoints the prediagonal arrow

\[
\alpha_s \circ \psi(u) \in Y(\phi_0(s); \psi_0(t)).
\]

\[
\begin{array}{ccc}
\phi_0(s) & \downarrow \alpha_s & \psi_0(s) \\
\psi_0(s) \rightarrow & \psi_0(t) & \\
\end{array}
\]

So we have got a joint pair having both unit transform and counit transform. Nevertheless these transforms are only partial transforms, without maintaining of arrows associative composition the evaluation graphs transports defined counit transform can’t be natural. So we can’t get natural bijection between sets of graphs transports

\[ \text{GRPH}(X \times T; Y) \rightarrow \text{GRPH}(X; Y^T) . \]

We shall prove the bijection only between the sets of special graphs transports.

**Proposition 1.** The choosing of unit arrows in the source graph \( 1_a \in X(a; a) \) and in an assistant graph \( 1_t \in T(t; t) \) allows us to define a name mapping between the sets of graphs transports 

\[ \text{GRPH}(X \times T; Y) \rightarrow \text{GRPH}(X; Y^T) . \]

The choosing of diagonal arrows in the target graph \( Y \) with composition 

\[ \circ : Y(c; d) \times Y(d; e) \rightarrow Y(c; e) \]

allows us to define a realization mapping

\[ \text{GRPH}(X; Y^T) \rightarrow \text{GRPH}(X \times T; Y) . \]

The name mapping is injective over graphs transports \( \Phi : X \times T \rightarrow Y \) which are decomposable with appointment of diagonal arrows in the target graph \( Y \)

\[ \Phi(\alpha, u) = \Phi(\alpha, 1_s) \circ \Phi(1_b, u) \uparrow \alpha \in X(a; b), u \in T(s; t)) . \]

The realization mapping is injective over graphs transports \( \Psi : X \rightarrow Y^T \) which are neutral for choosing of unit arrows in the source graph \( X \) and assistant graph \( T \)

\[ \Psi(1_a)_s \circ \Psi_0(a)_u = \Psi_0(a)_u , \]
\[ \Psi(\alpha)_s \circ \Psi_0(b)_{1_s} = \Psi(\alpha)_s . \]

**Proof:** For a graphs transport \( \Phi : X \times T \Rightarrow Y \) we shall construct its name \( \Psi : X \Rightarrow Y^T \). For a vertex \( a \in X_0 \) it appoints graphs transport

\[ \Psi(a) : T \Rightarrow Y \]

which for a vertex \( t \in T_0 \) appoints the vertex in the target graph

\[ \Psi_0(a)_t = \Phi_0(a, t) \in Y_0 \]

and for an edge \( u \in T(s; t) \) it appoints the edge in the target graph

\[ \Psi_0(a)_u = \Phi(1_a, u) \in Y_0(\Phi_0(a, s); \Phi_0(a, t)) . \]

For an edge \( \alpha \in X(a; b) \) the new graphs transport \( \Psi : X \Rightarrow Y^T \) appoints the transform of graphs transports \( \Psi(\alpha) : \Psi_0(a) \Rightarrow \Psi_0(b) \) defined with collection of edges

\[ \Psi(\alpha)_t = \Phi(1_a, t) \in Y_0(\Phi_0(a, s); \Phi_0(a, t)) . \]

Now for a graph transport \( \Psi : X \Rightarrow Y^T \) we shall construct its realization \( \Phi : X \times T \Rightarrow Y \). For a vertexes \( a \in X_0 \) and \( s \in T_0 \) we appoint the vertex in the target graph \( Y \) which is got as value \( \Psi_0(a)_s \in Y_0 \) for the graphs transport \( \Psi_0(a) : T \Rightarrow Y \).

For an edge in the Carte tensor product \( \langle \alpha, u \rangle \in X(a; b) \times T(s; t) \) we appoint the diagonal edge for graphs transform \( \Psi(\alpha)_s \in Y(\Psi_0(b)_s; \Psi_0(t)_s) \) and the edge appointed by graphs transport \( \Psi_0(b)_u \in Y_0(\Psi_0(b)_s; \Psi_0(b)_t) \)

\[ \Phi(\alpha, u) = \Psi(\alpha)_s \circ \Psi_0(b)_u \in Y_0(\Psi_0(a)_s; \Psi_0(b)_t) . \]

In the Carte tensor product we have a natural appointment of diagonal arrow

\[ \langle \alpha, u \rangle = \langle \alpha, 1_s \rangle \circ \langle 1_b, u \rangle \uparrow \alpha \in X(a; b), u \in T(s; t) . \]

Let we have decomposable graphs transform \( \Phi : X \times T \Rightarrow Y \) maintaining such diagonal arrow

\[ \Phi(\langle \alpha, u \rangle) = \Phi(\langle \alpha, 1_s \rangle) \circ \Phi(1_b, u) \uparrow \alpha \in X(a; b), u \in T(s; t) . \]

Then for a name \( \Psi : X \Rightarrow Y^T \) we get the realization \( \Phi' : X \times T \Rightarrow Y \) equal to the same graphs transform. It for the arrows \( \alpha \in X(a; b) \) and \( u \in T(s; t) \) appoints

\[ \Psi(\alpha)_s \circ \Psi_0(b)_u = \Phi(\langle \alpha, 1_s \rangle) \circ \Phi(1_b, u) = \Phi(\langle \alpha, u \rangle) . \]

If we have equalities for the diagonal edges appointment

\[ \Psi(\alpha)_s \circ \Psi_0(b)_{1_s} = \Psi(\alpha)_s , \]
\[ \Psi(1^a) \circ \Psi_0(a)u = \Psi_0(a)u , \]

then for its realization

\[ \Phi(\langle \alpha, u \rangle) = \Psi(\alpha) \circ \Psi_0(b)u \]

we get the same name \( \Psi' : X \implies Y^T \)

\[ \Psi'_0(a)u = \Phi(\langle 1^a, u \rangle) = \Psi(1^a) \circ \Psi_0(a)u = \Psi_0(a)u \]

and

\[ \Psi'(\alpha)u = \Phi(\langle \alpha, 1_s \rangle) = \Psi(\alpha) \circ \Psi_0(b)1_s = \Psi(\alpha)u . \]

\[ \square \]

We shall provide another proving of this proposition with standard tools in general categories. First we shall check that realization will reverse the name mapping. For this we shall construct commuting diagram

\[
\begin{array}{ccc}
X \times T & \xrightarrow{\lambda_x \times 1_T} & (X \times T)^T \times T \\
\downarrow & & \downarrow \text{ev}_{X \times T} \times T \\
X \times T & \xrightarrow{\text{ev}_X} & Y \times T
\end{array}
\]

We get the evaluation in the space \( X \times T \) by choosing composition

\[ \langle \alpha, 1_s \rangle \circ \langle 1_b, u \rangle = \langle \alpha, u \rangle \in X(a;b) \times T(s;t) \]

for arbitrary \( \alpha \in X(a;b) \) and \( u \in T(s;t) \).

This is natural choice, but we need to check that with such choice we have got a commuting diagram.

At first we shall check the first triangular equality for arbitrary graphs \( X \) and \( T \) with chosen unit arrows \( 1_a \in X(a;a) \) and \( 1_t \in T(t;t) \). We must prove the identical transport for composition of graphs transports

\[ X \times T \xrightarrow{\lambda_x \times 1_T} (X \times T)^T \times T \xrightarrow{\text{ev}_{X \times T} \times T} X \times T . \]

We have projection of categories \( \text{GRPH} \implies \text{SET} \) which for a graph \( X \) appoints its vertexes set \( X_0 \), and for a graphs transport \( f : X \implies Y \) appoints the vertexes appointment \( f_0 : X_0 \to Y_0 \).

The constructed Carte tensor product of graphs \( X \times Y \) is projected to the Carte tensor product of sets \( X_0 \times Y_0 \), i.e. we have commuting diagram of functors

\[
\begin{array}{ccc}
\text{GRPH} \times \text{GRPH} & \xrightarrow{X \times Y} & \text{GRPH} \\
\downarrow & & \downarrow \\
\text{SET} \times \text{SET} & \xrightarrow{X_0 \times Y_0} & \text{SET}
\end{array}
\]

For an exponential graph \( Y^T \) such projections appoints the set of all graphs transports \( \phi : T \implies Y \). It will be a primary projection. So we get the composition of vertexes appointments

\[
X_0 \times T_0 \xrightarrow{(\lambda_x)_0 \times 1_{T_0}} ((X \times T)^T)_0 \times T_0 \xrightarrow{\text{ev}_{(X \times T)_0}} X_0 \times T_0 .
\]
The secondary projection is defined with deformation of each graphs trans-
port \( \phi : T \Rightarrow Y \) to its vertexes appointment \( \phi_0 : T_0 \rightarrow Y_0 \). We shall need to
calculate these projections only for some concrete graphs transports. For the
secondary projection we get composition of appointments
\[
X_0 \times T_0 \xrightarrow{\lambda_{X_0 \times T_0}} (X_0 \times T_0)^{T_0} \xrightarrow{\text{ev}_{(X_0 \times T_0)}} X_0 \times T_0 ,
\]
which is equal to identity appointment by the first triangular equality in the
category of sets \( \text{SET} \).

The evaluation appointment for a couple \( \langle \phi, s \rangle \in Y^T_0 \times T_0 \) appoints the
value \( \phi_0(s) \in Y_0 \), therefore for arbitrary transport \( \phi : T \Rightarrow X \times T \) it depends
only from its projection \( \phi_0 : T_0 \rightarrow X_0 \times T_0 \). So we have get the first triangular
equality also for the primary projection.

It rests to check the first triangular equality only for edges appoin-
tments. For a couple of edges \( \alpha \in X(a; b) \) and \( u \in T(s; t) \) the first transport
\[
\lambda_X \times \text{Id}_T : X \times T \Rightarrow (X \times T)^T \times T
\]
will appoint the couple of transform between two sections
\[
\prod_{s' \in T} \langle \alpha, 1_{s'} \rangle : \lambda_X(a) \rightarrow \lambda_X(b)
\]
and the same edge \( u \in T(s; t) \). Further the evaluation transport in the space
\( X \times T \)
\[
\text{ev}_{X \times T} : (X \times T)^T \times T \Rightarrow X \times T
\]
will appoint the diagonal arrow
\[
\langle \alpha, 1_s \rangle \circ \langle 1_b, u \rangle = \langle \alpha, u \rangle \in X(a; b) \times T(s; t) .
\]
So we get the same couple of edges \( \alpha \in X(a; b) \) and \( u \in T(s; u) \), and we end
the checking of the first triangular equality for graphs.

Now we shall check for what graphs transport \( f : X \times T \Rightarrow Y \) we have
commuting quadrat with a transform
\[
\prod_{s' \in T} \langle \alpha, 1_{s'} \rangle : \lambda_X(a) \rightarrow \lambda_X(b)
\]
between two sections \( \lambda_X(a), \lambda_X(b) \in (X \times T)^T \)
\[
(X \times T)^T \times T \xrightarrow{f^T \times 1_T} Y^T \times T \xrightarrow{\text{ev}_Y} Y
\]

Let we have decomposable transport \( f : X \times T \Rightarrow Y \)
\[
f(\alpha, u) = f(\alpha, 1_s) \circ f(1_b, u) \in Y(f_0(a, s); f_0(1_b, u)) .
\]
Then the first evaluation $\mathbf{ev}_{(X \times T)}$ for the couple of transform between sections $\prod_{s' \in T}(\alpha, 1_{s'}) : \lambda_X(a) \rightarrow \lambda_X(b)$ and the edge $u \in T(s; t)$ will appoint the diagonal arrow

$$
(\alpha, 1_s) \circ \lambda_X(b)_u = (\alpha, 1_s) \circ (1_b, u) = (\alpha, u)
$$

and the transport $f : X \times T \rightarrow Y$ will appoint the value

$$f(\alpha, u) \in Y(f_0(a, s); f_0(b, t)).$$

Changing of target space with decomposable transport $f : X \times T \rightarrow Y$ for the transform between sections $\prod_{s' \in T}(\alpha, 1_{s'}) : \lambda_X(a) \rightarrow \lambda_X(b)$ will appoint the transform between changed sections

$$\prod_{s' \in T} f(\alpha, 1_{s'}) : \lambda_X(a) \circ f \rightarrow \lambda_X(b) \circ f,$$

so the evaluation transport for the couple of such transform and the same edge $u \in T(s; t)$ will appoint the same diagonal arrow

$$f(\alpha, 1_s) \circ f(\lambda_Y(b)_u) = f(\alpha, 1_s) \circ f(1_b, u) = f(\alpha, u).$$

Now we shall check that the name mapping reverses the realization. For this we shall construct another commuting diagram

$$Y^T \xleftarrow{\mathbf{ev}_{Y}} (Y^T \times T)^T \xleftarrow{(g \times 1_T)^T} (X \times T)^T \xleftarrow{\lambda_X} X.$$

In the graph $Y$ we have diagonal arrows mapping

$$\circ : Y(c; d) \times Y(d; e) \rightarrow Y(c; e),$$

but for a sections transport $\lambda_{Y^T}$ we need to choose the unit arrows in exponential graph $Y^T$. It will be defined with collection of unit arrows images only for image vertexes

$$g_{a,a}(1_a) \in Y^T(g_0(a); g_0(a)).$$

We shall check the second triangular equality as an identity transform by composition

$$Y^T \xrightarrow{\lambda_{Y^T}} (Y^T \times T)^T \xrightarrow{\mathbf{ev}_{Y}} Y^T,$$

begun with taken transport $g : X \rightarrow Y^T$ having the neutral properties for images of unit arrows from the source graph $1_a \in X(a; a)$ and assistant graph $1_s \in T(s; s)$

$$g(1_a)_u \circ g_0(a)_u = g_0(a)_u \in Y(g_0(a); g_0(a)) \uparrow u \in T(s; t),$$

$$g(\alpha)_s \circ g_0(b)_1 = g(\alpha)_s \in Y(g_0(a); g_0(b)) \uparrow \alpha \in X(a; b).$$

19
The sections transport \( \lambda_{Y^T} : Y^T \rightarrow (Y^T \times T)^T \) will be defined using the images of chosen unit arrows \( g_0(1_a) : g_0(a) \rightarrow g_0(a) \). We use a notation for edges in assistant graph \( u \in T(s; t) \). But for the same repeated graph \( T \) we apply a new notation \( u' \in T(s'; t') \).

First we shall check the second triangular equality with the primary projection for the set of all graph transports \( (Y^T)_0 = \{ \phi : T \Rightarrow Y \} \) in the category of sets \( \text{SET} \)

\[
(Y^T)_0 \xrightarrow{(\lambda_{Y^T})_0} ((Y^T \times T)^T)_0 \xrightarrow{(\text{ev}_{Y^T})_0} (Y^T)_0
\]

began with the appointment of taken graphs transport

\( g_0 : X_0 \rightarrow (Y^T)_0 \).

For sections transport \( \lambda_{Y^T} : Y^T \Rightarrow (Y^T \times T)^T \) we take vertexes appointment \( \lambda_{Y^T} : (Y^T)_0 \rightarrow ((Y^T \times T)^T)_0 \) which for a graphs transport \( \phi : T \Rightarrow Y \) appoints another graphs transport

\( \lambda_{(Y^T)}(\phi) : T \Rightarrow Y^T \times T \).

We can apply the secondary projection to get a new triangular equality for the same graphs transform \( \phi : T \Rightarrow Y \), but with a new sections appointment in the category of sets \( \text{SET} \)

\[
(Y^T)_0 \xrightarrow{\lambda_{(Y^T)}(\phi)} ((Y^T)_0 \times T)_0 \xrightarrow{(\text{ev}_{(Y^T)}(\phi))_0} (Y^T)_0
\]

It for a graphs transport \( \phi : T \Rightarrow Y \) will appoint the section in the category of sets

\( \lambda_{(Y^T)}(\phi) : T_0 \rightarrow (Y^T)_0 \times T_0 \)

which for a vertex \( t \in T_0 \) appoints the couple \( \langle \phi, t \rangle \in (Y^T)_0 \times T_0 \).

For the set of all appointments \( \phi : T_0 \rightarrow Y_0 \) we have already the second triangular equality

\[
(Y_0)_{T_0} \xrightarrow{\lambda_{(Y^T)}(\phi)} ((Y_0')_{T_0} \times T_0)_{T_0} \xrightarrow{(\text{ev}_{(Y^T)}(\phi))_{T_0}} (Y_0)_{T_0}
\]

The first arrow is a sections appointment defined for arbitrary appointment \( \phi : T_0 \rightarrow Y_0 \)

\( \lambda_{(Y^T)}(\phi) : T_0 \rightarrow (Y^T)_0 \times T_0 \)

So the triangular equality remains valid also for an imbedded smaller set \( (Y^T)_0 \subset (Y_0)_{T_0} \) of all appointments get from graphs transports \( \phi : T \Rightarrow Y \).

This new triangular equality will provides also the triangular equality for the primary projection. It rests to check the changing of target space with evaluation transport

\( (\text{ev}_Y)^T : (Y^T \times T)^T \Rightarrow Y^T \)

only for edges \( u' \in T(s'; t') \). The section \( \lambda_{Y^T}(g_0(a)) : T \Rightarrow Y^T \times T \) for arbitrary edge \( u' \in T(s'; t') \) will appoint the edge between couples

\( \langle g_{a,a}(1_a), u' \rangle : \langle g(a), s' \rangle \rightarrow \langle g(a), t' \rangle \)
defined with the image of unit arrows in source graph $1_a \in X(a; a)$. Further evaluation transport $ev_Y : Y^T \times T \Rightarrow Y$ will provide the diagonal arrow
\[ g_a, a(1_a)_{s'} \circ g_0(a)_{u'} \in Y(g_0(a)_{s'}; g_0(a)_{u'}) \]
equal by the neutral property to the same edge appointed by taken transport $g(a) \in Y^T$
\[ g_a, a(1_a)_{s'} \circ g_0(a)_{u'} = g_0(a)_{u'} \in Y(g_0(a)_{s'}; g_0(a)_{u'}) . \]
So we have got the same graphs transport $g(a)$.

Finally we need to check the second triangular equality for the edges appointments. By the graphs transport $g : X \Rightarrow Y^T$ the edge $\alpha \in X(a; b)$ will be appointed to the transform
\[ g(\alpha) : g_0(a) \rightarrow g_0(b) \]
declared with collection of edges $g(\alpha)_s \in Y(g_0(a)_s; g_0(b)_s)$. The sections transport $\lambda_{Y^T} : Y^T \Rightarrow (Y^T \times T)^T$ for such transform will be fully defined with chosen unit arrows in assistant graph $1_{s'} \in T(s'; s')$. The transform between sections
\[ \lambda_{Y^T}(g(\alpha)) : \lambda_{Y^T}(g_0(a)) \rightarrow \lambda_{Y^T}(g_0(b)) \]
will be defined with collection of edges in the product of graphs
\[ \lambda_{Y^T}(g(\alpha))_{s'} = \langle g(\alpha), 1_{s'} \rangle : \langle g_0(a), s' \rangle \rightarrow \langle g_0(b), s' \rangle . \]

Further we shall check the changing of target space with evaluation transport
\[ (ev_Y)^T : (Y^T \times T)^T \Rightarrow Y^T . \]
The evaluation transport $ev_Y : Y^T \Rightarrow Y$ for the couple $\langle g(\alpha), 1_{s'} \rangle$ will appoint the diagonal arrow
\[ g(\alpha)_{s'} \circ g_0(b)_{1_s} \in Y(g_0(a)_s; g_0(b)_s) \]
By the neutral property of chosen unit arrows in assistant graph $1_{s'} \in T(s'; s')$ we have equality
\[ g(\alpha)_{s'} \circ g_0(b)_{1_s} = g(\alpha)_{s'} . \]
So we have got the same transform between taken graphs transports
\[ g(\alpha) = \prod_{s' \in T_0} g(\alpha)_{s'} \in Y^T(g_0(a); g_0(b)) . \]
We have finally shown the second triangular equality begun with taken graphs transport $g : X \Rightarrow Y^T$.

Now we shall check the commuting quadrat between defined sections mappings
\[
\begin{array}{ccc}
(Y^T \times T)^T & (g \times 1_T)^T & (X \times T)^T \\
\uparrow \alpha_{Y^T} & \uparrow \lambda_X & \uparrow g \\
Y^T & X &
\end{array}
\]
For a vertex \( a \in X_0 \) we appoint the graphs transport \( g_0(a) \in Y^T \) which provides the section \( \lambda_{Y^T}(g_0(a)) : T \rightarrow Y^T \times T \) which for a vertex \( s' \in T_0 \) appoints the couple \( \langle g_0(a), s' \rangle \) and for an edge \( u' \in T(s'; t') \) appoints the transform between sections

\[
\langle g(1_a), u' \rangle : \langle g_0(a), s' \rangle \rightarrow \langle g_0(a), t' \rangle .
\]

We get the same sections and their transforms also by graphs transport \( g \times \text{Id}_T : (X 	imes T)^T \rightarrow (Y 	imes T)^T \).

For an edge \( \alpha \in X(a, b) \) taken graphs transport will appoint the transform between two graphs transports \( g(\alpha) : g_0(a) \rightarrow g_0(b) \), and sections transport provides the transform between two sections defined with edges in the product of graphs \( Y^T \times T \)

\[
\langle g(\alpha), 1_{s'} \rangle : \langle g_0(a), s' \rangle \rightarrow \langle g_0(b), s' \rangle .
\]

This ends another more categorical proving of former proposition.

Choosing of diagonal arrows mappings in the target graph \( Y \) provides a possibility to have another exponential superfunctor. For two graphs transports \( \phi, \psi : T \rightarrow Y \) we appoint the set of natural transforms \( \alpha : \phi \rightarrow \psi \). We will adapt a new notation for such graph \( Y^{(T)} \).

Evaluation transport can be defined in two different ways. For the couple of a transform \( \alpha : \phi \rightarrow \psi \) between two graphs transports \( \phi, \psi : T \rightarrow Y \) and an edge between two vertexes in source graph \( u \in T(s, t) \) we have defined 

\text{prediagonal arrow}

\[
\text{ev}^+_Y(\langle \alpha, u \rangle) = \alpha_s \circ \psi_s(u),
\]

and now we can define another 

\text{postdiagonal arrow}

\[
\text{ev}^-_Y(\langle \alpha, u \rangle) = \phi_u \circ \alpha_t.
\]

I haven’t find another names in English. It may be similar to using of names prefix and suffix. A transform \( \alpha : \phi \rightarrow \psi \) is called 

\text{natural} if we get commuting quadrats from equality of prediagonal and postdiagonal arrows

\[
\alpha_s \circ \psi_0(u) = \phi_0(u) \circ \alpha_t.
\]

We wish to define the new exponential space \( Y^{(T)} \) in more categorical way.

We have two evaluation arrows. One is for 

\text{preevaluation}

\[
\text{ev}^+_Y : \prod_{s \in T_0} Y(\phi_0(s); \psi_0(s)) \times T(s, t) \rightarrow
\]

\( 22 \)
\[ Y(\phi_0(s); \psi_0(s)) \times T(s; t) \rightarrow Y(\phi_0(s); \psi_0(t)) \]

and another is for postevaluation
\[ \text{ev}_Y^\alpha : \prod_{s \in T_0} Y(\phi_0(s); \psi_0(s)) \times T(s; t) \rightarrow Y(\phi_0(t); \psi_0(t)) \times T(s; t) \rightarrow Y(\phi_0(s); \psi_0(t)) . \]

Using the names of such mappings we get new mappings to the functional space defined in closed category \( \text{SET} \)
\[ \prod_{s \in T_0} Y(\phi_0(s); \psi_0(s)) \Rightarrow Y(\phi_0(s); \psi_0(t))^{T(s; t)} . \]

Their product for all couples \( s, t \in T_0 \) provides equalizer coinciding with new space of natural transforms
\[ Y(T) \rightarrow \prod_{s \in T_0} Y(\phi_0(s); \psi_0(t)) \Rightarrow \prod_{s, t \in T_0} Y(\phi_0(s); \psi(t))^{T(s; t)} . \]

For graphs transports \( g : X \Rightarrow Y \) we get different realization
\[ f = (g \times \text{Id}_T) \circ \text{ev}_Y^\alpha : X \times T \Rightarrow Y . \]

Preevaluation arrows define \( \text{prerealization} \), and postevaluation arrows provide \( \text{postrealization} \). The name mapping
\[ g = \lambda_X \circ f^T : X \Rightarrow Y^T \]
remains the same for both evaluation arrows.

We repeat former proposition for the exponential set of natural transforms.
We shall say that graphs transport \( \Phi : X \times T \Rightarrow Y \) to the target space \( Y \) with diagonal arrows mappings
\[ \circ : Y(a; b) \times Y(b; c) \rightarrow Y(a; c) \]
is \( \text{predecomposable} \) if we have an equality
\[ \Phi(\alpha, 1_s) \circ \Phi(1_b, u) = \Phi(\alpha, u) \uparrow \alpha \in X(a; b), u \in T(s; t) \]
and \( \text{postdecomposable} \) if we have another equality
\[ \Phi(\alpha, u) = \Phi(1_a, u) \circ \Phi(\alpha, 1_t) \uparrow \alpha \in X(a; b), u \in T(s; t) \]
The graphs transport will be called \( \text{decomposable} \) if it is predecomposable and postdecomposable.

We shall say that graphs transport \( \Psi X \Rightarrow Y^T \) is \( \text{preneutral} \) for chosen unit arrows in source graph \( 1_a \in X(a; a) \) and assistant graph \( 1_s \in T(s; s) \) if we have equalities
\[ \Psi(\alpha)_a \circ \Psi_0(b)_1 = \Psi(\alpha)_a , \]
\[ \Psi(1_a)_a \circ \Psi_0(a)_u = \Psi_0(a)_u . \]

And we shall say that such graphs transport is \( \text{postneutral} \) if we have another equalities
\[ \Psi(\alpha)_a = \Psi_0(a_1_a) \circ \psi(\alpha)_s , \]
\[ \Psi_0(a)_u = \Psi_0(a)_u \circ \Psi(1_a)_t . \]
The graphs transport will be called \( \text{neutral} \) for chosen unit arrows in source graph and assistant graph if it is preneutral and postneutral.
Proposition 2. The choosing of unit arrows in a source graph $1_a \in X(a;a)$ and in an assistant graph $1_s \in T(s;s)$ allows us to define the name mapping between the sets of graphs transports

$$\text{GRPH}(X \times T; Y) \to \text{GRPH}(X; Y^T).$$

For a diagonal edges mapping in a target graph $Y$

$$\circ : Y(c;d) \times Y(d;e) \to Y(c;e)$$

the choosing of evaluation transport defined with prediagonal or postdiagonal arrow allows us to define the realization mapping

$$\text{GRPH}(X; Y^T) \to \text{GRPH}(X \times T; Y).$$

The name of graphs transport $\Phi : X \times T \Rightarrow Y$ which is predecomposable and postdecomposable for diagonal arrows mappings in the target graph $Y$

$$\Phi(\alpha, 1_s) \circ \Phi(1_b,u) = \Phi(\alpha,u) = \Phi(1_a,u) \circ \Phi(\alpha, 1_t) \uparrow \alpha \in X(a;b), u \in T(s;t)$$

is corestrained in exponential graph of natural transforms $Y(T)$.

The name mapping is injective over such graphs transports

$$\text{GRPH}(X \times T; Y) \to \text{GRPH}(X; Y^{(T)}).$$

The realization mapping coincides over graphs transports corestrained in the exponential graph of natural transforms $\Psi : X \Rightarrow Y^{(T)}$ for both pre-diagonal or postdiagonal arrows and it is injective if taken graphs transport $\Psi : X \Rightarrow Y^{(T)}$ is preneutral and postneutral for choosing of unit arrows in the source graph $X$ and the assistant graph $T$

$$\Psi(\alpha)_s \circ \Psi_0(b)_{1_s} = \Psi(\alpha)_s = \Psi_0(a)_{1_s} \circ \Psi(\alpha)_s,$$

$$\Psi(1_a)_s \circ \Psi_0(a)_{u} = \Psi_0(a)_u = \Psi_0(a)_u \circ \Psi(1_a)_t.$$

Proof: We present the more detailed categorical proving of such proposition.

First we notice that the name of graphs transport $f : X \times T \Rightarrow Y$ is defined using sections transport $\lambda_X : X \Rightarrow (X \times T)^T$

$$g = \lambda_X \circ f^T : X \Rightarrow Y^T.$$

For predecomposable graphs transport $f : X \times T \Rightarrow Y$ the prerealization reverses its name. So the name mapping will be injective over the set of predecomposable graphs transports and prerealization mapping covers this set.

The same is true for the set of postdecomposable graphs transports. The name mapping is injective and postrealization mapping covers this set.

The name of decomposable graphs transport $f : X \times T \Rightarrow Y$ is included in the exponential space of natural transforms $Y^{(T)}$. 24
Having commuting diagrams with preevaluation or postevaluation transports

\[
\begin{array}{ccc}
X \times T & \xrightarrow{\lambda_X \times 1_T} & (X \times T)^T \times T \\
\searrow & \downarrow \text{ev}^\pm_{X \times T} & \downarrow \text{ev}^\pm_Y \\
X \times T & \xrightarrow{f^T \times 1_T} & Y^T \times T \\
\end{array}
\]

we shall check the corestriction of name \(g\):

\[
g : X \xrightarrow{\lambda_X} (X \times T)^T \xrightarrow{f^T} Y^T.
\]

in exponential space of natural transforms \(Y^{(T)}\).

For a vertex \(a \in X_0\) it appoints the section \(\lambda_X(a) : T \implies X \times T\) which for an edge \(u \in T(s; t)\) appoints the couple of edges

\[
\langle 1_a, u \rangle : \langle a, s \rangle \to \langle a, t \rangle
\]

between the couples of vertexes \(\langle a, s \rangle \in X_0 \times T_0\) and \(\langle a, t \rangle \in X_0 \times T_0\). Further the changing of target space by graphs transport \(f : X \times T \implies Y\) will appoint a graphs transport \(T \implies Y\) which for the edge \(u \in T(s; t)\) will appoint the edge between vertexes in target space \(Y\)

\[
f(1_a, u) : f_0(a, s) \to f_0(a, t).
\]

For an edge in source space \(\alpha \in X(a; b)\) we get the transform between sections \(\lambda_X(\alpha) : \lambda_X(a) \to \lambda_X(b)\) defined with collection of couples of edges

\[
\langle \alpha, 1_s \rangle : \langle a, s \rangle \to \langle b, s \rangle \uparrow s \in T_0.
\]

Further by changing of target space with graphs transport \(f : X \times T \implies Y\) we get transform between earlier graphs transports \(T \implies Y\) defined with collection of edges

\[
f(\alpha, 1_s) : f_0(a, s) \to f_0(b, t).
\]

Such transform will be neutral if we should have commuting diagrams

\[
\begin{array}{ccc}
f_0(a, s) & \xrightarrow{f(1_a, u)} & f_0(a, t) \\
f(\alpha, 1_s) \downarrow & \downarrow f(\alpha, 1_t) & \\
f_0(b, s) & \xrightarrow{f(1_b, u)} & f_0(b, 1_t)
\end{array}
\]

This is proved by diagonal arrows mappings

\[
\begin{array}{ccc}
f_0(a, s) & \xrightarrow{f(1_a, u)} & f_0(a, t) \\
f(\alpha, 1_s) \downarrow & \downarrow f(\alpha, 1_t) & \\
f_0(b, s) & \xrightarrow{f(1_b, u)} & f_0(b, 1_t)
\end{array}
\]

Such diagonal arrow is provided by \(f(\alpha, u) : f_0(a, s) \to f_0(b, t)\) from commuting diagrams drown for both evaluation transports \(\text{ev}_{Y}^\pm\), i. e. we have equalities for prediagonal and postdiagonal arrows

\[
f(\alpha, 1_s) \circ f(1_b, u) = f(\alpha, u) = f(1_a, u) \circ f(\alpha, 1_t).
\]
Finally we need to check that for a graphs transport $g : X \Longrightarrow Y^T$ with neutral properties we get injective realization mapping

$$\text{GRPH}(X; Y^{(T)}) \to \text{GRPH}(X \times T; Y) .$$

First we can notice that both prerealization and postrealization coincides over graphs transports corestricted in exponential space of natural transforms. Such realizations are defined by composition of graphs transports $f : X \times T \rightarrow Y^T \times T \xrightarrow{\text{ev}_Y} Y$. For the natural transform both evaluation transports appoints the same diagonal arrow as for the graphs transports corestricted in exponential space of natural transforms $g : X \Longrightarrow Y^{(T)}$ both preneutral and postneutral properties coincide.

Taking one of neutral properties, for example preneutral one, we get injective realization mapping over exponential space of natural transforms.

Later we could want to work with categories having weakly associative composition of arrows, i. e. with natural isomorphisms $S_{\alpha, \beta, \gamma} : \alpha \circ (\beta \circ \gamma) \rightarrow (\alpha \circ \beta) \circ \gamma$.

If we should work with graphs transforms $\Phi : X \times T \Longrightarrow Y$ which maintain both arrow composition

$$\Phi(1_a, u) \circ \Phi(\alpha, 1_t) = \Phi(\alpha, u) = \Phi(\alpha, 1_s) \circ \Phi(1_b, u) ,$$

then we could work only with natural transforms between graphs transforms $\phi, \psi : T \Longrightarrow Y$ to get a smaller exponential graph $Y^{(T)}$. The evaluation transport for such nonassociative arrows composition doesn’t maintain the arrows composition, but it is not difficult to calculate the isomorphism for evaluation transport $Y^T \times T \Longrightarrow Y$

$$\text{ev}_Y(\alpha \circ \beta, u \circ v) \rightarrow \text{ev}_Y(\alpha, u) \circ \text{ev}_Y(\beta, v) .$$

So we can apply former proposition to get bijection between arrows sets in such categories

$$\text{GRPH}(X \times T; Y) = \rightarrow \text{GRPH}(X; Y^{(T)}) .$$

Next we shall define an exponential superfunctor in the category of all categories $\text{CAT}$. The points in this category will be all possible categories. We define a category $X$ as the graph with the set $X$ of edges coinciding with
the set of category’s arrows and the set $X_0$ of vertexes coinciding with the set of category’s points. We have arrows composition

$$\circ : X(a; b) \times X(b; c) \to X(a; c)$$

with associativity equality

$$f \circ (g \circ h) = (f \circ g) \circ h \uparrow f \in X(a; b), g \in X(b; c), h \in X(c; d) .$$

Also we have unit arrows $1_a \in X(a; a)$ which are preneutral and postneutral for arrows composition

$$1_a \circ f = f = f \circ 1_b \uparrow f \in X(a; b) .$$

The arrows sets in the category of categories $\text{CAT}(X; Y)$ will be compounded of all possible functors $f : X \Rightarrow Y$ between two categories $X$ and $Y$. They will be defined by transport of graphs with two appointments. One for the set of edges $f : X \to Y$, and another for the set of vertexes $f_0 : X_0 \to Y_0$. The arrows appointment is defined with collection of mappings for arrows sets between two points

$$f_{a,b} : X(a; b) \to Y(f(a); f(B)) .$$

The functor must maintain the composition of arrows

$$f_{a,c}(\alpha \circ \beta) = f_{a,b}(\alpha) \circ f_{b,c}(\beta) \uparrow \alpha \in X(a; b), \beta \in X(b; c) .$$

Also it must maintain unit arrows

$$f_{a,a}(1_a) = 1_{f(a)} .$$

Functors between the categories can be understood as continuous transports between graphs with convergence defined by composition of arrows and appointment of unit arrows. It also can be considered as continuous mapping between the sets of points with additional convergences defined by edges between such points. It will be helpful when we want to imagine more tremendous construction such as superfunctors.

The superfunctors will be usual functors, only defined on larger category. It can be imagined as a functor over the category of convergence spaces compounded with all continuous mappings. A superfunctor $F$ for the category $X$ will appoint another category $X^F$, and it defines mappings over the sets of functors

$$F_{X,Y} : \text{CAT}(X; Y) \to \text{CAT}(X^F; Y^F) .$$

It must maintain the composition of functors

$$(f \circ g)^F = f^F \circ g^F$$
and the appointment of identity functors $\text{Id}_X : X \rightarrow X$

$$(\text{Id}_X)^F = \text{Id}_{(X^F)}.$$  

We shall be interested in some concrete superfunctors

$$F : \text{CAT} = \rightarrow \text{CAT}. $$

The first example of superfunctor is provided with a Carte tensor product of two categories. A Carte tensor product of two categories $X \times Y$ is defined on the Carte tensor product of graphs, i.e. we have the Carte tensor product for the sets of vertexes $X_0 \times Y_0$ and the Carte tensor product for the sets of arrows $X \times Y$. Source and target projections are defined with Carte product of correspondent projections

$$p_1 \times p_1 : X \times Y \rightarrow X_0 \times Y_0, \quad p_2 \times p_2 : X \times Y \rightarrow X_0 \times Y_0.$$  

The arrows composition and appointment of unit arrows are also defined as Carte products

$$(X \times Y(\langle x, y; \langle x', y' \rangle \rangle) \times (X \times Y(\langle x', y'; \langle x'', y'' \rangle \rangle)) \rightarrow X \times Y(\langle x, y; \langle x'', y'' \rangle \rangle),$$

$$1_{\langle x, y \rangle} = 1_x \times 1_y.$$  

This is an instance of general construction of Carte tensor product in convergence spaces.

For chosen assistant category $T$ we get a superfunctor in the category $\text{CAT}$ of categories. For arbitrary category $X$ we appoint the Carte tensor product $X \times A$ and for arbitrary functor $f : X = \rightarrow X'$ we appoint the Carte tensor product with identity functor of assistant category $f \times \text{Id}_T : X \times A = \rightarrow X' \times T$.

We shall try to define a coadjoint exponential superfunctor. For a category $Y^T$ we take a set of all possible functors $\phi : T = \rightarrow Y$ as a set of points, and a set of all possible natural transformations $\phi \rightarrow \psi$ as a set of arrows between two functors $\phi, \psi : T = \rightarrow Y$. The natural transform is defined with collection of arrows in the target category

$$\phi(s) \rightarrow \psi(s) \in Y$$

which meets the coherence conditions of commuting diagrams for arbitrary arrow in source category $u \in T(s; t)$

$$\phi(s) \xrightarrow{\phi(u)} \phi(t) \quad \downarrow \quad \downarrow \quad \psi(s) \xrightarrow{\psi(u)} \psi(t).$$

The composition of transforms $\phi \rightarrow \psi$ and $\psi \rightarrow \xi$ is defined over each point $s \in T$

$$\phi(s) \rightarrow \psi(s) \rightarrow \xi(s).$$
Easy to see that such composition maintains the property of commuting
diagrams. The most easy way it to do is a *chasing correspondent diagrams*. This
can be also interpreted as a property of continuous transforms.

Having commuting quadrats for arbitrary edge in assistant category \( u \in T(s;t) \)

\[
\phi(u) \circ \alpha_t = \alpha_s \circ \psi(u) ,
\]

\[
\psi(u) \circ \beta_t = \beta_s \circ \xi(u) ,
\]

we get a new commuting quadrat

\[
\begin{array}{c}
\phi(u) \circ (\alpha_t \circ \beta_t) = (\ast \alpha_s \circ \beta_s) \circ \xi(u) .
\end{array}
\]

\[
\begin{array}{ccc}
\phi_0(s) & \xrightarrow{\alpha_s} & \psi_0(s) \\
\phi(u) & \downarrow & \psi(u) \\
\phi_0(t) & \xrightarrow{\beta_t} & \phi_0(t)
\end{array} \quad
\begin{array}{ccc}
& \xrightarrow{\xi_0(s)} & \\
& \downarrow & \\
& \xi(u) & \\
& \beta_t & \\
& \xi_0(t) &
\end{array}
\]

The proving uses three ways from a vertex \( \phi_0(s) \) to the vertex \( \psi_0(t) \)

\[
(\alpha_s \circ \beta_s) \circ \xi(u) = \alpha_s \circ (\beta_s \circ \xi(u)) = \alpha_s \circ (\psi(u) \circ \beta_t) =
\]

\[
(\alpha_s \circ \psi(u)) \circ \beta_t = (\phi(u) \circ \alpha_t) \circ \beta_t = \phi(u) \circ (\alpha_t \circ \beta_t) .
\]

The unit transform \( 1_\phi : \phi \to \phi \) is defined with collection of unit arrows \( 1_{\phi_0(s)} : \phi_0(s) \to \phi_0(s) \). It obviously also meets the property of commuting
diagrams.

So for arbitrary category \( Y \) we have constructed the category \( Y(T) \) which
will be called an *exponential category*. It defines an appointment of objects for
an *exponential superfunctor*.

Functors between the categories will be arrows in category \( \text{CAT} \) and will
be also mapped by superfunctor. For the functor \( g : Y \Rightarrow Y' \) we define
a functor between exponential categories \( g(T) : Y(T) \Rightarrow (Y')(T) \). For each
functor \( \phi : T \Rightarrow Y \) we appoint a new functor provided with changing of target
space \( \phi \circ g : T \Rightarrow Y' \), and for each natural transform \( \alpha : \phi \Rightarrow \psi \) we get a new
natural transform \( \alpha \times g : \phi \circ g \Rightarrow \psi \circ g \).

We must check that for an identity functor \( \text{Id}_Y : Y \Rightarrow Y \) we get again
the identity functor over exponential category

\[
(\text{Id}_Y)^{(T)} = \text{Id}_{Y(T)}(()) ,
\]

and for a composition of functors \( g \circ k \) we get the composition of functors between exponential categories

\[
(g \circ k)^{(T)} = g^{(T)} \circ k^{(T)} .
\]

Such both superfunctors we have defined even for the category of graphs. Now the associative composition of arrows allows us to have truly joint pair of
superfunctors with counit arrows defined by collection of *evaluation functors*

\[
ev_Y : Y(T) \times T \Rightarrow Y .
\]
Possibility to use natural transforms together with associative arrows composition is essential.

For some functor $\phi : T \Rightarrow Y$ and a point in source category $s \in T$ it appoints the point in the target category $\phi(s) \in Y$, and for a natural transform $\alpha : \phi \rightarrow \psi$ and an arrow in source category $u : s \rightarrow t$ it appoints the composition of arrows

$$\phi(u) \circ \alpha_b = \alpha_a \circ \psi(u) : \phi(s) \rightarrow \psi(t).$$

First we need to check that such application maintains the composition of arrows. Let we have another natural transform $\beta : \psi \rightarrow \xi$ and arrow in the source category $\ell : t \rightarrow r$. Then

$$\phi(s)$$

$$\psi(s) \downarrow \nu$$

$$\psi(t)$$

$$\xi(s) \downarrow \nu$$

$$\xi(t) \downarrow \nu$$

$$\xi(r)$$

and we need to check the equalities

$$(\alpha_s \circ \psi(u)) \circ (\beta_t \circ \xi(\ell)) =$$

$$\alpha_s \circ (\psi(u) \circ \beta_t) \circ \xi(\ell) =$$

$$\alpha_s \circ (\beta_a \circ \xi(u)) \circ \xi(\ell) =$$

$$(\alpha_s \circ \beta_a) \circ (\xi(u) \circ \xi(\ell)).$$

For a transform with unit arrows $\alpha_s = 1_{\phi(s)}$ and a unit arrow in the source category $1_a : a \rightarrow a$ we also get a unit arrow in the target category

$$1_{\phi(s)} \circ \phi(1_a) = 1_{\phi(s)}.$$  

This completes the checking that we indeed have got a functor $Y^{(T)} \times T \Rightarrow Y$.

We have projection of category of categories $\text{CAT}$ to the category of all sets $\text{SET}$. This is also an instance of superfunctor $\text{CAT} \Rightarrow \text{SET}$. For each category $X$ we appoint its points set $X_0$, and for a functor $f : X \Rightarrow Y$ we have the appointment of points sets $f : X_0 \rightarrow Y_0$. Obviously we have maintenance of composition for functors and for the identity functor $\text{Id}_X : X \Rightarrow X$ it appoints the identity appointment of points set $\text{Id}_X : X \rightarrow X$. This projection will be called a primary projection.

The commuting diagrams from the category $\text{CAT}$ rest commuting in the category of sets $\text{SET}$. Each joint pair of superfunctors in the category $\text{CAT}$

$$\langle F, G \rangle : \text{CAT} \Leftrightarrow \text{CAT}$$

provides the joint pair of projected functors in the category $\text{SET}$

$$\langle F_0, G_0 \rangle : \text{SET} \Leftrightarrow \text{SET}.$$
The study of unit and counit transforms must be begun by investigation of their projections.

The constructed Carte tensor product of categories $X \times Y$ is projected to the Carte tensor product of sets $X_0 \times Y_0$, i.e., we have commuting diagram of superfunctors

\[
\begin{array}{ccc}
\text{CAT} \times \text{CAT} & \xrightarrow{X \times Y} & \text{CAT} \\
\downarrow & & \downarrow \\
\text{SET} \times \text{SET} & \xrightarrow{X_0 \times Y_0} & \text{SET}
\end{array}
\]

For the exponential functor $Y^{(T)}$ we additionally apply the deformation. We change the set of graph transforms $\phi : T \Rightarrow Y$ by the set of points appointments $\phi_0 : T_0 \to Y_0$ and we get another projection $\text{CAT} \Rightarrow \text{SET}$, which will be called a secondary projection.

The equality of two arrows $f, g : X \Rightarrow Y$ in category $\text{CAT}$ is defined as equality for primary projections $f_0, g_0 : X_0 \to Y_0$ in the category $\text{SET}$ and equality of arrows appointments

\[f_{x,x'} = g_{x,x'} : X(x;x') \to Y(f_0(x); f_0(x')) = Y(g_0(x); g_0(x')) .\]

For the joint pair of functors $X \times A$ and $Y^{(T)}$ in the category of all categories $\text{CAT}$ we shall construct unit and counit transforms. The unit transform for joint pair of superfunctors

\[\text{Id}_{\text{CAT}} \to FG\]

will be defined with the collection of sections functors

\[\lambda_X : X \Rightarrow (X \times T)^{(T)} .\]

It for a point $x \in X$ appoints the section

\[\lambda_X(x) : T \Rightarrow X \times T\]

which for a point $s \in T$ appoints the couple $(x, s) \in X_0 \times T_0$ and for an arrow in assistant category $u \in T(s;t)$ appoints the arrow in the product of categories

\[(1_x, u) \in X \times T .\]

For an arrow $\theta \in X(x;y)$ it appoints the natural transform between two sections defined with the collection of arrows

\[\theta \times 1_s \in X \times T((x,s);(y,s)) .\]

We can see that the composition of two arrows $\theta : x \to y$ and $\rho : y \to z$ is mapped to the composition of natural transforms between section functors

\[(\theta \circ \rho) \times 1_s = (\theta \times 1_s) \circ (\rho \times 1_s) .\]
and a unit arrows $1_x : x \to x$ will provide the identity transform between the
same section functor
$$1_x \times 1_s : \lambda_X(x) \to \lambda_X(x).$$
So we indeed have defined functors.

Such sections functors $\lambda_X : X \Rightarrow (X \times T)^{(T)}$ defines natural transform
between correspondent superfuntors. For arbitrary functor $f : X \Rightarrow X'$ the
superfunctor appoints another functor $(f \times 1_{1T}) : (X \times T)^{(T)} \Rightarrow (X' \times T)^{(T)}$
with commuting quadrat

$$\begin{array}{ccc}
X & \overset{f}{\to} & X' \\
\lambda_X \downarrow^{(X \times T)^{(T)}} & \downarrow & \downarrow_{\lambda_{X'}}^{(X' \times T)^{(T)}} \\
(X \times T)^{(T)} & \overset{(f \times 1_{1T})^{(T)}}{\to} & (X' \times T)^{(T)}
\end{array}$$

At first we wish to check that the primary projection provides commuting
quadrats in the category $\text{SET}$. With secondary projection we get $\text{sections appointments}$ in the category $\text{SET}$

$$\lambda_{X_0} : X_0 \to (X_0 \times T_0)^{T_0},$$
therefore they will be arrows for natural transforms between superfuntors in
the category $\text{SET}$.

To get commuting diagram for the primary projection we need to check
that the section $\lambda_X(x) \in (X \times T)^{(T)}$ is mapped by functor $(f \times 1_{1T})^{(T)}$ to
another section $\lambda_{X'}(f_0(x)) \in (X' \times T)^{(T)}$. This is not obvious only for an
appointment of arrows.

For arbitrary arrow $u \in T(s; t)$ the first section $\lambda_X(x)$ appoints an arrow
$$\langle 1_x, u \rangle \in X \times T((x, s); (x, t))$$
and the functor $(f \times 1_{1T})^{(T)}$ appoints another section $\lambda_{X'}(f_0(x))$, which for
arbitrary arrow $u \in T$ provides the arrow
$$\langle 1_{f_0(x)}, u \rangle \in X' \times T((f_0(x), s); (f_0(x), t)).$$
This value will be the same for the section get in category $X'$ with the changed
point $f_0(x) \in X'_0$.

It rests to check commuting diagram for arrows appointments. For an
arrow $\theta \in X(x; y)$ we appoint a natural transform between section

$$\lambda_X(x) \to \lambda_X(y)$$
defined with the collection of arrows
$$\theta \times 1_s \in X \times T((x, a); (y, a)) \uparrow s \in T$$

The functor $(f \times 1_{1T})^{(T)} : (X \times T)^{(T)} \Rightarrow (X' \times T)^{(T)}$ appoints the natural
transform defined with a new collection of arrows
$$\langle f(\theta), 1_s \rangle \in X' \times T((f_0(x), s); (f_0(y), s)) \uparrow s \in T_0.$$
This indeed coincides with the natural transform appointed by the functor \( \lambda_{X'} : X' \to (X' \times T)^{(T)} \) to an arrow \( f(\theta) \in X'(f_0(x); f_0(y)) \).

The counit transform

\[
GF \to \text{Id}_{\text{CAT}}
\]

will be defined by collection of evaluation functors

\[
ev_Y : Y^{(T)} \times T \to Y
\]

This transform is natural. We need to check commuting diagrams for arbitrary functor \( g : Y \to Y' \)

\[
\begin{array}{ccc}
Y^{(T)} \times T & \xrightarrow{g^{(T)} \times 1_T} & (Y')^{(T)} \times T \\
ev_Y & \downarrow & \downarrow \ev_{Y'} \\
Y & \xrightarrow{g} & Y'
\end{array}
\]

Their secondary projection will be the evaluation appointment for functional set in the category \( \text{SET} \)

\[
ev_{Y_0} : (Y_0)^{T_0} \times T_0 \to Y_0
\]

therefore in category \( \text{SET} \) we get arrows for natural transform.

We can see that the primary projection also defines natural transform between superfunctors in the category \( \text{SET} \).

The primary projection of evaluation functor \( \text{ev}_Y : Y^{(T)} \times T \to Y \) for a functor \( \phi : T \to Y \) and a point \( s \in T_0 \) appoints the value \( \phi_0(s) \in Y_0 \) of points appointment \( \phi_0 : T_0 \to Y_0 \), i.e. depends only from second projection of exponential category \( Y^{(T)} \).

It rests to check commuting diagram for the arrows appointment. Let we have a natural transform \( \alpha : \phi \to \psi \) between two functors \( \phi, \psi \in Y^{(T)} \) defined by collection of arrows \( \alpha_s \in Y(\phi_0(s); \psi_0(s)) \uparrow s \in T \) and an arrow in an assistant category \( u \in T(s; t) \). Then functor \( g^{(T)} \times 1_T : Y^{(T)} \times T \to (Y')^{(T)} \times T \) appoints the new natural transform between new functors with changed target space

\[
\alpha \times g : \phi \circ g \to \psi \circ g
\]

defined by collection of arrows \( g(\alpha_s) \in Y'(g_0(\phi_0(s)); g_0(\psi_0(s))) \uparrow s \in T \) and the same arrow in assistant category \( u \in T(s; t) \).

Otherwise the evaluation functor \( \text{ev}_{Y'} \) will appoint the diagonal arrow

\[
g(\alpha_s) \circ \psi(u) \in Y'(g_0(\phi_0(s)); g_0(\psi_0(t)))
\]

the same as the image of arrow \( \alpha_s \circ \psi(u) \in Y(\phi_0(s); \psi_0(t)) \)

\[
g(\alpha_s \circ h) = g(\alpha) \circ g(u).
\]

For the truly joint pair of superfunctors \( X \times T \) and \( Y^T \) we must check both triangular equalities for unit and counit transforms. It can be seen by earlier proved propositions for graphs transports.
We can apply the proposition about bijective name mapping between sets of functors
\[ \text{CAT}(X \times T; Y) = \rightarrow \text{CAT}(X; Y^{(T)}) \, . \]
The unit arrows \( 1_x \in X(x;x) \) are maintained by any functor, and they are neutral for arrows composition. The functor \( f : X \times T \implies Y \) is decomposable for the arrows composition in target category \( Y \), as we have equality in product category
\[
(\alpha, 1_s) \circ (1_b \circ u) = (\alpha \circ 1_b, 1_s \circ u) = (\alpha, u) \uparrow \alpha \in X(a;b), u \in T(s;t) \, .
\]
So we get injective name mapping to the set of functors corestricted in the exponential space of natural transforms. The realization mapping is also injective over such set of functors. Therefore both name mapping and realization mapping are bijective.

The bijective name mapping provides both triangular equalities. The first triangular equality is the property for realization mapping to reverse the name of identity functor
\[
\text{Id}_{X \otimes T} : X \otimes T \implies X \otimes T \, .
\]
The second triangular equality is the property for name mapping to reverse the realization of identity functor
\[
\text{Id}_{Y^T} : Y^T \implies Y^T \, .
\]
So we have proved already these properties. Nevertheless we shall check these triangular equalities once more without applying the proposition on bijective name mapping in category of graphs. However we shall only repeat the earlier proving for this special case.

The first triangular equality is for the unit transform ended with the adjoint superfunctor and composed with the counit transform begun with the adjoint superfunctor
\[
(i \times F) \circ (F \times e) = 1_F \, .
\]
The first natural transform is defined by collection of sections functors
\[
\lambda_X \times \text{Id}_T : X \times T \implies (X \times T)^{(T)} \times T \, ,
\]
the second natural transform is defined by collection of evaluation functors
\[
\text{ev}_{X \times T} : (X \times T)^{(T)} \times T \implies X \times T \, .
\]
The triangular equality in the category \( \text{SET} \) we have for the secondary projection. It is also valid for the primary projection, as evaluation functors depends on secondary projection only.

So it rests to check the triangular equality only for arrows appointments.

For a couple of arrows \( \langle \theta, u \rangle \in X \times T(\langle x, s \rangle; \langle y, t \rangle) \) the sections functor appoints the natural transform between two sections \( \lambda_X(x), \lambda_X(y) \in Y^{(T)} \) defined with collection of couples compounded by arrows
\[
\langle \theta, 1_s \rangle \in X \times T(\langle x, s \rangle; \langle y, s \rangle)
\]
and the identity functor $\text{Id}_T : T \Rightarrow T$ appoints the same arrow in assistant category $u \in T$.

Further the evaluation functor in the category $X \times T$ appoints the diagonal arrows

$$\langle \theta, 1_s \rangle \circ \lambda_X(y)(u) = \langle \theta, 1_s \rangle \circ (1_y, u) = \langle \theta, u \rangle \in X \times T(\langle x, s \rangle; \langle y, t \rangle).$$

So we have got the same couple of arrows.

The second triangular equality is for the unit arrow begun with the coadjoint superfunctor $G$ and composed with the counit arrow ended with the coadjoint superfunctor $G$

$$(G \times i) \circ (e \times G) = 1_G.$$

This composition must be identity transform defined by collection of the identity functors over exponential categories $Y^{(T)}$.

The first natural transform is defined by collection of sections functors

$$\lambda_{Y^{(T)}} : Y^{(T)} \Rightarrow (Y^{(T)} \times T)^{(T)}$$

and the second natural transform is defined with collection of changing target space by evaluation functor

$$(\text{ev}_Y)^{(T)} : (Y^{(T)} \times T)^{(T)} \Rightarrow Y^{(T)}.$$

For the secondary projection we have the triangular equality in the category $\text{SET}$ We need to check such triangular equality also for the primary projection.

For a functor $\phi : T \Rightarrow Y$ we must check that for an arrow $u' \in T(s'; t')$ finally we appoint the value $\phi(u') \in Y(\phi_0(s'); \phi_0(t'))$ of the same functor $\phi$.

The first sections functor will appoint the section $\lambda_{Y^{(T)}}(\phi) : T \Rightarrow Y^{(T)} \times T$ which for a point $s' \in T_0$ will appoint the couple of points $\langle \phi, s' \rangle \in Y^{(T)} \times T$ and for an arrow $u' \in T(s'; t')$ it appoints the couple of arrows

$$\langle \phi, u' \rangle \in Y^{(T)} \times T(\langle \phi, s' \rangle; \langle \phi, t' \rangle).$$

Changing the target space with evaluation functor $\text{ev}_Y : Y^T \times T \Rightarrow Y$ provides the diagonal arrow

$$1_{\phi_0(s') \circ \phi(u')} = \phi(u') \in Y(\phi_0(s'); \phi_0(t')).$$

It rests to check identity appointment for the composition of arrows appointments. For the natural transform $\alpha : \phi \rightarrow \psi$ between functors $\phi : T \Rightarrow Y$ and $\psi : T \Rightarrow Y$ the first sections functor will appoint a natural transform between two sections

$$\lambda_{Y^{(T)}}(\phi) \rightarrow \lambda_{Y^{(T)}}(\psi)$$

defined with the collection of arrows

$$\langle \alpha, 1_{s'} \rangle : \langle \phi, s' \rangle \rightarrow \langle \psi, s' \rangle \uparrow s' \in T_0.$$
which are natural transforms between functors \( T \Rightarrow Y(T) \times T \), defined with collections of arrows \( \langle \alpha_s, 1_{s'} \rangle \in Y \times T((\phi_0(s), s'); (\psi_0(s), s')) \). The changing target space with the evaluation functor \( \text{ev}_Y : Y(T) \times T \Rightarrow Y \) for every couple \( \langle \alpha, 1_{s'} \rangle \in Y(T) \times T(\langle \phi, s' \rangle; \langle \psi, s' \rangle) \) will appoint the diagonal arrow

\[
\alpha_{s'} \circ 1_{s'} = \alpha_{s'} \in T(\phi_0(s'); \psi_0(s')) .
\]

So we have shown completely the second triangular equality.

The bijective realization mapping

\[
f : X \times T \xrightarrow{g} Y(T) \times T \xrightarrow{\text{ev}_Y} Y
\]

can be interpreted as arrow of unit transform \( \lambda_X : X \Rightarrow (X \times T)(T) \) to be initial in comma category

\[(X \downarrow G)\]

for exponential superfunctor of natural transforms \( Y^G = Y(T) \). The functors \( X \Rightarrow Y^G \) are taken as points and commuting functors between target spaces \( h : Y \Rightarrow Y \) are taken as arrows of comma category \( (X \downarrow G) \). The bijectivity of realization mapping provides that for each functor \( m : X \Rightarrow Y^G \) unanimously exists an arrow \( n : F(X) \Rightarrow Y \) which realization coincides with taken arrow, i. e.

\[
\lambda_X \circ n^G = m .
\]

The initial arrow is unique up the unique isomorphism, therefore we can construct unique isomorphism between two possible exponential functors. However it is hard imagine what can be another exponential functor aside the exponential functor of natural transforms \( Y(T) \). This will be useful when we shall apply the notion of equalizer to define the exponential space of natural transforms \( Y(T) \). Equalizer is also defined unanimously up unanimously defined isomorphism.

Interesting to apply such constructions even in cases when exponential space of natural transforms doesn’t maintain composition of arrows, i. e. isn’t superfunctor.

We present once more another closed monoidal category of Banach spaces \( \text{Ban} \). Such spaces were introduced by S. Banach 1922. Banach space \( X \) can be presented as complete metric space with linear structure. In Banach space \( X \) we have additive Abel group with addition of vectors

\[
+: X \times X \rightarrow X
\]

and additive homothety by scalar \( r \in R \)

\[
r : X \rightarrow X .
\]

Such linear structure we can understood as a convergence for finite sequences. Additionally we introduce a metric with norm

\[
X(x; y) = \|y - x\| \in \mathbb{R}_+ .
\]
We demand that taken vector space \( X \) would be complete for Cauchy sequences of such metric. The morphism between two Banach spaces will be taken bounded linear mappings \( u : X \to X \) with bounded norm

\[
\|u\| = \sup_{\|x\| \leq 1} \|u(x)\| < +\infty.
\]

The \textit{algebraic tensor product} is defined as an initial object in the bilinear mappings \( U \times V \to W \) with commuting triangular diagram defined by linear mappings of target spaces \( W \to W' \). Therefore for existing of tensor product \( U \otimes V \) with bilinear mapping \( U \times V \to U \otimes V \) and arbitrary other bilinear mapping \( U \times V \to W \) we have exactly one linear mapping \( U \otimes U \to W \) providing commuting diagram

\[
\begin{array}{ccc}
U \times V & \rightarrow & U \otimes V \\
\downarrow & & \downarrow \\
W & & W' \\
\end{array}
\]

Constructing of such algebraic tensor product of linear spaces can be seen in \textbf{N. Bourbaki Algèbre}, chp. 3. Algèbre multilineaire.

The \textit{projective tensor product} of Banach spaces \( U \) is defined as completion of metric space on algebraic tensor product \( U \otimes V \) with projective product of norms

\[
p_{1} \square p_{2}(u) = \inf \{ \sum_{i} p_{1}(x_{i})p_{2}(y_{i}) : u = \sum_{i} x_{i} \otimes y_{i} \}.
\]

cl. Topological tensor products in \textbf{H. Schaefer 1966} part III. Linear mappings §6. For arbitrary Banach space \( Z \) we have isometrical bijection between the space of bilinear mappings \( f : X \times Y \to Z \) with norm

\[
\|f\| = \sup_{\|x\| \leq 1, \|y\| \leq 1} \|f(x, y)\|
\]

and correspondent linear mappings \( u : X \boxdot Y \to Z \) with norm

\[
\|u\| = \sup_{\|x\| \leq 1, \|y\| \leq 1} \|u(x \otimes y)\|.
\]

The correspondent linear mapping is produced as continuous extension of linear mapping defined by bilinear mapping over algebraic tensor product.

The points of projective tensor product \( u \in U \boxdot V \) are expressed as absolutely convergent sum

\[
u = \sum_{i \geq 1} \lambda_{i} x_{i} \otimes y_{i}
\]

with \( \sum_{i \geq 1} |\lambda_{i}| < +\infty \) and \( x_{i} \in U, y_{i} \in V \) are sequences converging to zero point. This characterization is proposed in \textbf{A. Pietsch 1963}.

We shall check again that the category of Banach spaces \textbf{Ban} with projective tensor product \( U \boxdot V \) is closed monoidal category.
For arbitrary assistant Banach space $A$ we define adjoint functor which for the Banach space $U$ appoints a new Banach space $U \boxtimes A$ and for bounded linear mapping $u : U \to U'$ it appoints a new bounded linear mapping $u \times 1_A : U \boxtimes A \to U' \boxtimes A$.

The coadjoint functor for the arbitrary Banach space $V$ appoints a new Banach space $V^A$ compounded of all bounded linear mappings $\phi : A \to V$. For bounded linear mapping $v : V \to V'$ it appoints changing of target space $g_* : V^A \to (V')^A$ which for a mapping $\phi \in V^A$ appoints the composed mapping $v \circ \phi : A \to (V')^A$.

The unit transform for joint pair of functors will be defined by collection of sections mappings $\lambda_U : U \to (U \boxtimes A)^A$. We have section $\lambda_U(x) : A \to U \boxtimes A$ which for the point $s \in T$ appoints a tensor $x \otimes a \in U \boxtimes A$ with norm

$$\|x \otimes a\| = \|x\| \cdot \|a\|,$$

so we get section’s norm

$$\|\lambda_U(x)\| = \sup_{\|a\| \leq 1} \|x \otimes a\| = \|x\|,$$

and the norm of sections mapping

$$\|\lambda_U\| = \sup_{\|x\| \leq 1} \|x\| = 1.$$

The counit transform will be defined by collection of evaluation mappings $\ev_V : V^A \boxtimes A \to V$. At first it is defined as a bilinear continuous mapping

$$\ev_V((\phi, s)) = \phi(s)$$

with inequality for norm

$$\|\phi(s)\| \leq \|\Phi u\| \cdot \|a\|.$$

Then we take its continuous extension over projective tensor product.

Both triangular equalities for truly joint pair of functors

$$\langle F, G \rangle : \text{Ban} \leftrightarrow \text{Ban}$$

is provided by such equalities in larger category $\text{Set}$, or smaller category of convergence spaces with continuous mappings.
Instead we shall show with some difficulties that we have natural bijection between sets of bounded linear mappings

\[ \text{Ban}(U \Box A; V) \to \text{Ban}(U; V^A) . \]

For a bounded linear mapping \( \Phi : U \Box A \to V \) we get a continuous bilinear mapping \( U \times U \to V \), therefore it will be partially continuous, so we get continuous linear mapping \( \Phi_1 : U \to V^T \). Otherwise for arbitrary continuous mapping \( \Psi : U \to V^T \) we get partially continuous bilinear mapping \( U \times A \to V \), which for Banach space will be continuous, so it has an extension over the whole projective tensor product \( \Phi(x \otimes y) = \Psi(x)_y \).

In the category \( \text{Ban} \) we can take another tensor product. For two Banach space \( U \) and \( V \) we appoint \( \text{biequicontinuous tensor product} \ U \bar{\Box} V \). On algebraic tensor product \( U \otimes V \) we take a norm of uniform convergence over tensor product of equicontinuous sets. For tensor \( u = \sum_{i \geq 1} x_i \otimes y_i \) we take a norm defined with continuous forms \( f \in U' \) and \( g \in V' \):

\[ [u] = \sup_{\|f\| \leq 1, \|g\| \leq 1} \sum_i \langle x_i, f \rangle \cdot \langle y_i, g \rangle . \]

So we get a smaller norm

\[ [u] \leq \sum_{i \geq 1} p_1(x_i) \cdot p_2(y_i) , \]

therefore \([u] \leq p_1 \bar{\Box} p_2(u)\) and we have got a bounded imbedding

\[ U \bar{\Box} V \subset U \Box V . \]

In H. Schaefer 1966 part IV. Duality §9.2 we can see the dual space for biequicontinuous tensor product \( X \bar{\Box} Y \). It coincides with the space of integrable mappings \( f : X \bar{\Box} Y \to R \) defined with Radon measure over weakly compact space \( \mu \uparrow S \times T \) with \( S = \{ x' : \|x'\| \leq 1 \} \subset X' \) and \( T = \{ y' : \|y'\| \leq 1 \} \subset Y' \).

\[ f(u) = \int \mu(x', y') \cdot \langle u, x' \otimes y' \rangle . \]

For Banach space \( Z \) the continuous linear mappings \( f : X \bar{\Box} Y \to Z \) is defined by bounded vector measure \( \mu \uparrow S \times T \) with the equalities for scalar forms

\[ \langle f(u), z' \rangle = \int \langle \mu, z' \rangle \, d(x', y') \cdot \langle u, x' \otimes y' \rangle . \]

We demand that scalar measures would have uniformly bounded variations

\[ \sup_{\|z'\| \leq 1} \|\langle \mu, z' \rangle\| \leq C . \]

Once again we want to get a joint pair of functors. We take adjoint functor defined by new tensor product \( U \bar{\Box} V \). The coadjoint functor would be defined
with the same exponential space $V^A$ compounded of continuous linear mappings with the norm of uniform convergence over the unit ball in the space $A$

$$\|u\| = \sup_{\|a\| \leq 1} \|u(s)\| .$$

However we must take a smaller category to get again truly joint pair of functors. Difficulties arises with evaluation mapping. It must be continuous for weaker topology of biequicontinuous tensor product

$$\text{ev}_Y : Y^A \boxtimes U \rightarrow Y .$$

This problem is solved taking integrable linear mappings.

The integrable mappings between two Banach spaces $u : U \rightarrow V$ is defined by integral of bounded vector measure $\mu \, d\, x'$ over weakly compact set in dual source space $U'$

$$B^\circ = \{ x' \in U' : \langle x, x' \rangle \leq 1 \uparrow \|x\| \leq 1 \} .$$

We shall denote

$$u = \int \mu \, d\, x' \cdot x'$$

and its values are unanimously defined by scalar integrals

$$\langle u(s), y' \rangle = \int \langle \mu, y' \rangle \, d\, x' \cdot \langle a, x' \rangle \in R \uparrow y' \in Y' .$$

For bounded vector measure we shall have an inequality

$$\sup_{|y'| \leq 1} \langle u(s), y' \rangle \leq \sup_{\|y'\| \leq 1} \int |\langle \mu, y' \rangle| \, d\, x' \cdot \|a\| \leq C \cdot \|a\| .$$

We shall define a smaller category of Banach spaces $\text{SBan} \subset \text{Ban}$ leaving as arrows only the integrable linear mappings. First we must check is the identity mapping integrable. For this we need construct a weak base in Banach spaces. J. Schauder 1927 for some Banach spaces constructed topological base $x_i \in X \uparrow \, i \in I$ with dual collection of linear forms $f_j \in X' \uparrow \, j \in X'$ permitting the orthogonal relations

$$\langle x_i, f_j \rangle = \delta_{i,j} \uparrow \, i, j \in I$$

and providing sums unconditionally converging in initial topology of Banach space

$$\sum_{i \in I} x_i f_i(x) = x .$$

S. Karlin 1948 has shown that there isn’t any such base even in a space of continuous scalar functions over compact interval $C \uparrow [0, 1]$. However for weak
topology such bases exist always for every Banach space \( X \). More exactly we have continuous imbeddings
\[
\ell^1 \subset X \subset \ell^\infty, \quad \ell^\infty \supset X' \supset \ell^1
\]
for some collection of vectors \( x_i \in E \cup i \in I \) and forms \( f_i \in E' \cup i \in I \) with orthogonality condition
\[
\langle x_i, f_j \rangle = \delta_{i,j}, \quad \|x_i\| = 1, \quad \|f_j\| = 1.
\]

**Proposition 3.** In Banach space \( X \) we can find a weak topological basis \( x_i \in B, \|x_i\| = 1 \) with dual collection of continuous forms \( f_j \in X' \)
\[
\langle x_i, f_j \rangle = \delta_{i,j}.
\]

**Proof:** In H. Schaefer 1966 part II. Locally convex spaces §4.2 Corollary 1 of theorem we can see construction of such collection of forms \( f_j \in X' \) for arbitrary finite collection of independent vectors \( x_i \in X \). We extend this construction by infinite induction for arbitrary Banach space,

Let we have a collections of independent vectors \( x_i \in X, \|x_i\| = 1 \) and continuous forms \( f_j \in X', \|f_j\| = 1 \) with orthogonality relation
\[
\langle x_i, f_j \rangle = \delta_{i,j} \uparrow i, j \in I.
\]

Let these vectors \( x_i \in X \) generate closed linear subspace \( E \subset X \). By Hahn 1928 and Banach 1929 theorem generated closed linear space is characterized by property for arbitrary continuous form \( f \in X' \) to vanish if it vanishes over each basic vector
\[
f(x_i) = 0 \uparrow i \in I \implies f(E) = 0.
\]

If \( E \neq X \), we can find a form vanishing for basic vectors \( f(x_i) = 0 \uparrow i \in I \) and having not zero value \( f(x) \neq 0 \) over some vector \( \|x\| = 1 \). We want find such point in intersection of zero spaces of all basic forms
\[
x \in \bigcap_{j \in I} f_j^{-1}(0).
\]

For arbitrary summable series of scalars \( \sum_j |\alpha_j| < +\infty \) we get a continuous form again
\[
\sum_j \alpha_j f_j \in X'.
\]

and for arbitrary \( x \in X \) we get a bounded series
\[
\sup_j f_j (x) < +\infty.
\]

So we get continuous imbeddings \( \ell^1 \subset X' \subset \ell^\infty \). The similar dual imbeddings we have for the space \( X \) itself \( \ell^\infty \supset X \supset \ell^1 \) as for the summable series of scalar
\( \beta_j \uparrow j \in I \) we get a point \( x = \sum_j \beta_j x_j \), and for each point \( x \in X \) we appoint a bounded sequence of scalars \( f_i(x) \uparrow i \in I \). We have equalities of dual mappings

\[
\langle x, i(\alpha_i) \rangle_{X \times X'} = \langle (f_i(x)), (\alpha_i) \rangle_{\ell^\infty \times \ell^1},
\]

\[
\langle \sum \beta_j x_j, f \rangle_{X \times X'} = \langle (\beta_j), (f(x_j)) \rangle_{\ell^1 \times \ell^\infty}.
\]

By H. Schaefer 1966 Part IV. Duality §2.3 for dual mappings we have equality of polars \( u(s)^o = (u')^{-1}(A^*) \).

So for the image of the whole space \( u(\ell^1) \subset X \) we get a polar \( (u')^{-1}(0) \subset X' \), i.e. for all basic mappings \( f_i \) we have \( f_i(x) = 0 \).

Otherwise for some summable series

\[
x = \sum_i \beta_i x_i \in X
\]

with \( f(x) \neq 0 \) we can correct this point

\[
x' = x - \sum_j \beta_j x_j \in X
\]

to get \( f(x') = f(x) \neq 0 \) and \( f_i(x') = 0 \) for all \( i \in I \).

We take such \( x \in X \) with unit norm \( \|x\| = 1 \) as next point in the collection of basic points \( (x_i \uparrow i \in I) \) and define next basic form with \( f(x) = 1 \). So we have got for all \( i \in I \)

\[
f_i(x) = 0, \quad f(x_i) = 0, \quad f(x) = 1.
\]

After such construction we can apply infinite induction to provide a maximal collection of basic points \( x_i \in X \) and dual basic forms \( f_i \in X' \). For it we shall have injective dual linear mappings

\[
\ell^1 \subset X \subset \ell^\infty, \quad \ell^\infty \supset X' \supset \ell^1.
\]

Weak base in Banach space \( x_i \in X \) with dual collection of basic forms \( f_i \in X' \) provides integrable presentation of identity mapping \( \text{Id}_X : X \to X \)

\[
x = \sum_{i \in I} x_i \otimes f_i(x)
\]

with vector measure

\[
\langle \mu, y' \rangle \, d\, x' = \sum_{i \in I} \langle x_i, y' \rangle f_i.
\]
Without such representation we could work only with semicategory of integrable mappings. Such situation will be inevitable with the semicategory of compact linear mappings.

The composition of two integrable linear mappings rests integrable. If we have integrals of vector measures
\[
\langle u(x), y' \rangle = \int \langle \mu, y' \rangle \, d x' \cdot \langle x, x' \rangle, \quad \langle v(y), z' \rangle = \int \langle \nu, z' \rangle \, d y' \cdot \langle y, y' \rangle,
\]
then
\[
\langle v(u(x)), z' \rangle = \int \langle \nu, z' \rangle \cdot \langle u(x), y' \rangle = \int \langle \nu, z' \rangle \, d y' \cdot \int \langle \mu, y' \rangle \, d x' \cdot \langle x, x' \rangle.
\]
So the composition is get by bounded vector measure
\[
\langle \kappa, z' \rangle = \int \langle \nu, z' \rangle \, d y' \cdot \langle \mu, y' \rangle.
\]

We have get a category $\text{SBan}$ of integrable linear mappings between Banach spaces.

Exponential space $V^A$ will be compound by all integrable linear mappings $u : A \to V$ with the same norm of uniform convergence over unit ball in source space
\[
\|u\| = \sup_{\|a\| \leq 1} \|u(s)\|.
\]

The unit transform for joint pair of functors will be defined by collection of sections mappings
\[
\lambda_U : U \to (U \square A)^A.
\]
For a point $u \in U$ it appoints the section $\lambda_U(u) : A \to U \square A$ with values
\[
\lambda_U(u)_s = u \otimes a.
\]
Such section will have summable expression get from weak base in assistant Banach space $T$
\[
u \otimes a = u \otimes \left( \sum_{i \in I} a_i g_i(s) \right).
\]

The integral expression of sections mapping needs also weak base in Banach space $U$
\[
\lambda_U(u) = \left( \sum_{j \in J} u_j f_j(u) \right) \otimes \left( \sum_i a_i \otimes g_i \right).
\]

The counit transform will be defined by collection of evaluation mappings
\[
ev_V : V^A \square A \to V.
\]
It will be integrable linear mapping for integrable linear mappings $\phi : A \to V$. Let we have integrable expression
\[
\phi(s) = \int \mu \, d x' \cdot \langle a, x' \rangle \in Y.
\]
Then evaluation mapping will be continuous over biequicontinuous tensor product. For weakly continuous function $\phi$ its values can be checked by Dirac measures. So we get inequality

$$\phi(s) \leq \sup_{\nu \cdot x' \cdot \phi(s) \leq 1} \int_0^\infty a^* \cdot \phi(s)$$

with upper boundary is taken over all Radon measures on the ball in assistant space $T$

$$\{ s \in T : \| a \| \leq 1 \}$$

with weakly topology. Also the polar of unit ball $B \subset V$ will be compound of bounded Radon measures $\nu \cdot x'$ over the ball in assistant space

$$\{ s \in T : \| a \| \leq 1 \} .$$

So we get an estimate

$$|\phi(s)| \leq \sup_{\| \nu \| \leq 1} |\int_0^\infty a^* \cdot \phi(s)| =$$

$$\sup_{\| \nu \| \leq 1, \| x' \| \leq 1} |\int_0^\infty a^* \cdot \phi(s)| \cdot |\langle a, x' \rangle| = [\phi \bowtie a] .$$

Both triangular equalities are the same as in larger category $\text{Ban}$. So we get truly joint functors $X \bowtie A$ and $Y^T$ in the category $\text{SBan}$ and a natural bijections between sets of integrable linear mappings

$$\text{SBan}(X \bowtie A; Y) \rightarrow \text{SBan}(X; Y^A) .$$

This can be deduced also from description of continuous linear mappings over biequicontinuous tensor product in H. Schaefer 1966 part IV. Duality §9.2. Continuous linear mappings over biequicontinuous tensor product of Banach spaces $f : X \bowtie T \rightarrow Y$ coincide with integrable linear mappings get with Radon measures over polars of unit balls $B = B_1(0) \subset X$ and $D = B_1(0) \subset A$

$$f(u) = \int_{B^* \times D^*} \mu \cdot d(x', s') \cdot u(x', s') .$$

This can be expressed as integrable linear mappings to the space $Y^A$

$$X \rightarrow Y^A .$$

We need only to apply integration of double integral

$$\int_{B^* \times D^*} \mu \cdot d(x', s') \cdot u(x', s') = \int_{B^*} \nu \cdot d x' \cdot \int_{D^*} \mu(x', s') \cdot u(x', s') .$$

We can show that the biequicontinuous tensor product is a relator for the projective tensor product, i. e. we have bounded linear mapping

$$(X \bowtie Y) \bowtie (Z \bowtie W) \subset (X \bowtie Y) \bowtie (Y \bowtie W) .$$
It is enough to show continuity of correspondent bilinear mapping
\[(X \square Y) \times (Z \square W) \rightarrow (X \square Y) \square (Y \square W)\,.
\]
This bilinear mapping is constructed by pairs of projections
\[p_1: X \square Y \rightarrow X, \quad q_1: Z \square W \rightarrow Z\]
and
\[p_2: X \square Y \rightarrow Y, \quad q_2: Z \square W \rightarrow W\,.
\]
Finally we get two continuous bilinear mappings
\[p_1 \square q_1: (X \square Y) \times (Z \square W) \rightarrow (X \square Z)\]
and
\[p_2 \square q_2: (X \square Y) \times (Z \square W) \rightarrow (Y \square W)\,.
\]
So we get also continuous biequicontinuous product
\[(p_1 \square q_1) \square (p_2 \square q_2): (X \square Y) \times (Z \square W) \rightarrow (X \square Z) \square (Y \square W)\,.
\]

Let in arbitrary monoidal closed category \(V\) with tensor product
\[r \otimes s \in V \uparrow r \in V, s \in V\,.
\]
we have another tensor product \(r \square s\) with a property
\[(r \square r') \otimes (s \square s') \rightarrow (r \otimes s) \square (r' \otimes s')\]
and the same neutral object
\[\ast \square r = r = r \square \ast\,.
\]
We demand that such arrows would define natural transform between two functors.
It will be called \textit{relator} to the previous tensor product. Similar notion perhaps we can see in \textbf{R. Backhouse, P. Hoogendijk 2000}. As a consequence we get an arrow
\[r \otimes s = (r \square \ast) \otimes (s \square \ast) \rightarrow (r \square s) \otimes (\ast \square \ast) = r \square s\,.
\]
They also define a natural transform between two functor, in this case we have natural transform between two tensor products.

REFERENCES

1. Stefan Banach 1922, Sur les opération dans les ensembles abstraits et leurs applications aux équations intégrales, 7-33, Fund. Math. \textbf{3}. 45
2. J. Schauder 1927. Zur Theorie stetiger Abbildungen in Funktionräumen, 47-65, 417-431, Math. Z. 26.
3. S. Karlin 1948, Bases in Banach spaces, 971-985, Duke Math. J. 15.
4. Nicolas Bourbaki, Théorie des ensembles ch. 4, Hermann: Paris 1957, Mir: Moskva 1965.
5. Nicolas Bourbaki, Algèbre chap 1-3, Masson: Paris 1970, Chap 4-6, Hermann: Paris, Nauka: Moskva 1965.
6. Jean Benabou 1963, Catégories avec multiplication, 1887-1890, C. R. Paris 256.
7. A. Pietsch 1963, Zur Theorie der topologischen Tensorprodukte, 19-30, Math. Nachr. 25.
8. Helmut Schaefer 1966, Topological vector spaces, MacMillan: N. Y., London, 1971 Mir: Moskva.
9. Saunders MacLane 1971, Categories for the working mathematician, Springer: New York/ Berlin/ Heidelberg, 1997 second edition.
10. G. Valiukevičius 1992, Integration in ordered spaces, I, Veja: Vilnius.
11. R. Backhouse, P. Hoogedijk 2000, Elements of a relational theory of datatypes.
12. G. Valiukevičius 2009, United sight to an algebraic operations and convergence, arXiv.org.

21 Septembre 2022.