Noise reduction in Laguerre-domain discrete delay estimation

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Abstract—This paper introduces a stochastic framework for a recently proposed discrete-time delay estimation method in Laguerre-domain, i.e. with the delay block input and output signals being represented by the corresponding Laguerre series. A novel Laguerre-domain disturbance model allowing the involved signals to be square-summable sequences is devised. The relation to two commonly used time-domain disturbance models is clarified. Furthermore, by forming the input signal in a certain way, the signal shape of an additive output disturbance can be estimated and utilized for noise reduction. It is demonstrated that a significant improvement in the delay estimation error is achieved when the noise sequence is correlated. The noise reduction approach is applicable to other Laguerre-domain problems than pure delay estimation.

I. INTRODUCTION

Delays, also termed as dead time, time lag, latency, etc., are ubiquitous in real world and have to be quantified to be properly taken into account in control [1] or estimation applications. Pure delay estimation is instrumental in the remote sensing technology such as radar, sonar, ultrasound and lidar [2]. These methods are based on time-of-arrival estimation and make use of emitted pulses with finite support.

Laguerre functions are traditionally used for representing both dynamical systems and signals. In the former case, they are used to approximate the input-output mapping of the system, and, in the latter, to capture the signal forms of the involved inputs and outputs. In time-domain, the Laguerre functions are essentially exponentials with polynomial coefficients and, therefore, are highly suitable for describing solutions of linear time-invariant systems [3].

The idea of using Laguerre functions to estimate delay in continuous and discrete time has been investigated before (see, e.g. [4], [5]) but the discrete-time case has drawn less attention until recently. An extensive comparison of delay estimation approaches in simulation experiments was performed in [6] and highlighted the robustness of the methods based on the use of Laguerre functions.

The contribution of the present work is threefold: First, a novel disturbance model constituting of a linear combination of a finite number of Laguerre functions with random weights is introduced. Second, an approach to reconstructing the signal shape of additive disturbances through shaping the excitation in Laguerre-domain is proposed. Third, the performance of a time-delay estimation algorithm is improved by applying Laguerre-domain noise reduction making use of the signal shape reconstruction.

The rest of the paper is organized as follows. After summarizing the necessary background on Laguerre-domain system representation, three stochastic disturbance models are formulated. Further, making use of the considered models, the impact of disturbance on the estimated Laguerre spectrum of the measured output is analyzed. An approach to reconstructing the signal shape of an additive measurement disturbance realization is described and analyzed for the considered noise models. The disturbance estimate is shown to be instrumental in noise reduction of Laguerre-domain estimation algorithms. Finally, the efficacy of the proposed noise reduction method is demonstrated on a Laguerre-domain time-delay estimation algorithm via numerical experiments.

II. SYSTEM DESCRIPTION

Consider the model
\[ y(t) = u(t - \tau) + e(t), \quad t \in \mathbb{N}_0, \] (1)
where \( u(t), y(t) \in \mathbb{R}, \tau \in \mathbb{N}_+ \) is a constant delay, and \( e(t) \) represents unknown noise.

A. Laguerre spectrum

Let \( \mathbb{H}_2^2 \) be the Hardy space of analytic functions on the complement of the unit disc that are square-integrable on the unit circle and equipped with the inner product
\[ \langle W, V \rangle = \frac{1}{2\pi i} \int_D W(z) V(z^{-1}) \frac{dz}{z}, \] (2)
where \( D \) is the unit circle. An orthonormal complete basis in \( \mathbb{H}_2^2 \) is given by the Laguerre functions specified in \( \mathbb{Z} \)-domain
\[ L_j(z; p) = \frac{\sqrt{1-p} \sqrt{z}}{\sqrt{z-\sqrt{p}}} T^j(z; p), \quad T(z; p) \triangleq \frac{1 - \sqrt{pz}}{z - \sqrt{p}}, \] (3)
for all \( j \in \mathbb{N} \), where the constant \( 0 < p < 1 \) is the discrete Laguerre parameter. Then, any function \( W \in \mathbb{H}_2^2 \) can be represented as an infinite series
\[ W(z) = \sum_{k=0}^{\infty} w_k L_k(z; p), \quad w_j = \langle W, L_j \rangle, \] (4)
and the set \( \{ w_j \} \) is called the Laguerre spectrum of \( W \).

A system is said to be considered in Laguerre-domain when its inputs and outputs are given by their Laguerre spectra.

The time-domain representations of the Laguerre functions
\[ \ell_j(t;p) = \mathbb{Z}^{-1}\{ L_j(z; p) \} \quad (j \in \mathbb{N}) \] yield an orthonormal basis in \( \ell_2^2(0, \infty) \), the space of square-summable sequences defined for non-negative integer arguments, where \( \mathbb{Z}^{-1} \) denotes the inverse \( \mathbb{Z} \)-transform.

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In this paper we use “noise” and “disturbance” interchangeably.
B. Linear time-invariant system in Laguerre-domain

Consider the linear time-invariant (LTI) system with
\[ x(t + 1) = Ax(t) + Bu(t), \quad \forall t \in \mathbb{N}_0, \]
\[ y(t) = Cx(t), \]
where \( x : \mathbb{R} \to \mathbb{R}^n \), \( A, B, C \) are real matrices of suitable dimensions, \( x(0) = 0 \).

**Proposition 1 ([7]):** Let the input signal \( u(t) \in \mathcal{E}^2[0, \infty) \) to system (5) be defined by its Laguerre spectrum \( \{u_j\}_{j \in \mathbb{N}} \). Then the output Laguerre spectrum \( \{y_j\}_{j \in \mathbb{N}} \) is given by the output of the system
\[ x_{j+1} = Fx_j + Gu_j, \]
\[ y_j = Hx_j + Ju_j, \] \( j = 0, 1, \ldots \) for \( \tau \) operator (7).

Then the operation of the delay block in Laguerre-domain is directly relating the input coefficient \( u_j \) to the output coefficient \( y_j \) is always present but the coefficients of higher order, i.e. \( u_k, j < k \) do not contribute to the value of \( y_j \). This is despite the fact that each Laguerre coefficient is, according to (4), evaluated from the whole signal sequence defined on \( \mathbb{N}_0 \). This property of the Laguerre-domain description will be exploited in Section V.

C. Discrete delay in Laguerre-domain

Consider now a noise-free case of (1), i.e. let \( e(t) \equiv 0 \),
\[ y(t) = u(t - \tau). \] \( \tau \)

Then the operation of the delay block in Laguerre-domain is readily described by the following result.

**Proposition 2 ([9]):** Let the input and output of (7) be
\[ U(z) = \sum_{k=0}^{\infty} u_k L_k(z), \quad Y(z) = \sum_{j=0}^{\infty} y_j L_j(z). \]

Then the Laguerre spectrum of the output is related to that of the input by
\[ y_j = (1 - p) \sum_{k=0}^{j-1} L_{j-k}(\sqrt{p})u_k + \sqrt{p}^j u_j, \] \( j = 1, 2, \ldots \)

where
\[ L_{m}(\sqrt{p}) = (-\sqrt{p})^{m-\tau} \sum_{n=0}^{\tau-1} \binom{m+n}{n} \binom{m-1}{\tau-n-1} (-p)^n, \]
and it is agreed that \( (\sqrt{p})^k = 0 \) for \( k > n \) by definition.

Naturally, the discrete delay operator is an LTI system and can be written in state-space form (5). Then the delay length \( \tau \) becomes the order of the state-space representation. By transforming (7) to Laguerre-domain, the delay estimation problem can be formulated as a parameter estimation problem and solved in a closed-form, see [10] for details.

The role of the polynomials \( L_m(\sqrt{p}) \) in the Laguerre-domain description of the delay operator is revealed by the convolution operator in (8). Indeed, the following relationship holds
\[ (1 - p)L_j(\sqrt{p}) = h_j \triangleq H_j F_{j-1} G, \quad j = 1, 2, \ldots, \] \( \tau \)

where \( H_j, F_j, G \) are the matrices given by (6) and evaluated for delay operator (7).

III. MEASUREMENT NOISE IN LAGUERRE-DOMAIN

There is no established noise model in Laguerre-domain. Below, three models are analyzed. The first two are conventional and defined in time-domain, while the third one is novel and introduced directly in Laguerre-domain.

Define the time-domain noise vector
\[ E_{\text{time}} \triangleq \begin{bmatrix} e(0) & e(1) & \cdots & e(T-1) \end{bmatrix}^T, \] \( T \)

and denote its covariance matrix as \( \Sigma_{\text{time}} \). Then the Laguerre spectrum, which is also further on referred to as (spectrum) distortion, is given by
\[ E_{\text{lag}} \triangleq \begin{bmatrix} e(0) & e(1) & \cdots & e_L-1 \end{bmatrix}^T \Psi_p E_{\text{time}}, \]
where
\[ \Psi_p = (\Phi_1(p) \Phi_L(p))^{-1} \Phi_1(p) \]
is the projection matrix onto the space spanned by the first \( L \) Laguerre functions, and the matrix \( \Phi_k(p) \in \mathbb{R}^{T \times L} \) contains the first \( T \) instants of the Laguerre functions in time domain.

A. White noise

The white noise model is typically used in communication systems to represent channel noise, in radar/sonar to describe the ambient noise, and, in solid-state electronics [11], to capture electronic noise.

**Proposition 3:** Suppose that the time-domain noise sequence \( \{e(t), t = 0, \ldots, T-1\} \) is uncorrelated with zero mean and variance \( \lambda \). Then, for a sufficiently large \( T \), the sequence of the Laguerre-domain coefficients \( \{e_k, k = 0, \ldots, L-1\} \) has arbitrary small correlations and, for each \( k \), the variance \( \mathbb{E}[e_k^2] = \lambda_L \approx \lambda \).

**Proof:** Omitted.

Noise stationarity is significant here: When the time-domain noise is uncorrelated but non-stationary, i.e. has time-varying variances \( \lambda_t \), the sequence of the Laguerre-domain coefficients \( e_0, \ldots, e_{L-1} \) is correlated. A main conclusion of Proposition 3 is that, for a white \( e(t) \) and sufficiently large \( T \), the second-order properties of the Laguerre-domain distortion are independent of the Laguerre parameter \( p \).

B. Colored noise

In most real-life applications, disturbances can be correlated over time; i.e., colored. Stationary colored disturbances are routinely modeled as filtered white noise. A variety of filter structures ranging from autoregressive filters to rational transfer operators can be employed in the modeling [12]. These are usually estimated from experimental data, where the signal is set to zero so that data contains only noise. Regardless of the used model structure, and under zero mean assumption, the second-order properties of colored noise are given by its correlation function. It can be used to construct a full covariance matrix \( \Sigma_{\text{time}} \) of the noise vector \( E_{\text{time}} \).

The Cholesky factorization of a known positive-definite covariance \( \Sigma_{\text{time}} \) is \( \Sigma_{\text{time}} = \lambda SS^\top \), where \( \lambda > 0 \) is the variance
of \( e(t) \) and \( S \) is a lower-triangular matrix. The columns of \( S \) are given by the impulse response of the spectral factor of the noise process. Then it holds that
\[
\Sigma_{\text{lag}}(p) = \lambda \Psi_p S^T \Psi_p = \lambda \Psi_p \Psi_p^T,
\]
and a convolution between the basis functions and the impulse response of the spectral factor of the noise takes place. In general, \( \Psi_p \Psi_p^T \) is a full matrix and does not converge to the identity matrix as \( T \to \infty \). Thus, in contrast with the case of white noise, the second-order properties of \( E_{\text{lag}} \) depend on \( p \), and, in general, the entries of \( E_{\text{lag}} \) are correlated.

C. Random combination of Laguerre functions

When no assumptions are made regarding \( e(t) \), the measured signal \( \{y(t), t \in [0,T-1]\} \), \( T \to \infty \), may not be in \( \ell^2 \). It will be almost surely in \( \ell^2 \) only if the realizations of \( e(t) \) are almost surely in \( \ell^2 \). This property is guaranteed for \( e(t) \) defined as a random combination of a finite number of Laguerre functions
\[
e(t) \triangleq \sum_{k=0}^{K} \epsilon_k \psi_k(t; p_e),
\]
where \( K \in \mathbb{N}_+ \), \( 0 < p_e < 1 \), and \( \epsilon_0, \ldots, \epsilon_K \) are random variables with zero mean and finite variances. This is in contrast with (4), where the Laguerre coefficients are constant. Model (11) is related to what is employed in the Karhunen–Loève expansion ([13, Ch.3, Sec. 4]). Both models decouple the probabilistic behavior of the signal from its behavior in time. Yet, the models differ in character: The Karhunen–Loève expansion appears in a representation theorem applicable to an infinite sum and requires finite support of the orthonormal functions, while (11) is a defining model. Another important difference is that the random variables \( \epsilon_k \) in (11) are not necessarily uncorrelated.

Noise model (11) produces a non-stationary second-order process, in contrast with the standard stationary noise models used in time-domain. Its correlation function is given by
\[
R_e(t,s) = \mathbb{E}[e(t)e(s)] = \sum_{m=0}^{K} \sum_{n=0}^{K} \sigma_{mn} \epsilon_m(s; p_e) \epsilon_n(t; p_e),
\]
where \( \sigma_{mn} \) is the correlation between \( \epsilon_m \) and \( \epsilon_n \).

Notably, model (11) captures projections of both white and colored time-domain finite noise subsequences onto the space spanned by the first \( K+1 \) Laguerre functions. In this subspace and with the freedom to model the correlation matrix of \( E_{\text{lag}} \), it is sufficient to assume model (11) irrespective of the actual time-domain correlation properties. The noise models are illustrated in Fig. 1, see also Section VI.

IV. NOISE-CORRUPTED LAGUERRE SPECTRUM

Introduce the vectors
\[
Y_{\text{time}} \triangleq [y(0) \ y(1) \ \ldots \ y(T-1)]^T,
Y_{\text{lag}} \triangleq [y_0 \ y_1 \ \ldots \ y_{L-1}]^T.
\]

Then, in terms of the noise-free output \( Y_{\text{time}} \), an approximation of the first \( L \) Laguerre spectrum coefficients \( Y_{\text{lag}} \) from the data in the interval \([0,T-1]\) is given by
\[
\hat{Y}_{\text{lag}} = \Psi_p Y_{\text{time}}.
\]

The errors \( \epsilon = Y_{\text{meas.}} - \hat{Y}_{\text{lag}} \) are truncation errors (residuals), due to the contribution from the Laguerre coefficients of orders higher than \( L \). Notice that the dependence of this error on \( p \) is implicit in the notation.

When the measurement is noisy, only \( Y_{\text{meas.}} = Y_{\text{time}} + E_{\text{time}} \) is available, where \( E_{\text{time}} \) is defined in (10) and \( Y_{\text{meas}} \) is a vector stacking the measured outputs. This introduces further errors, and the approximation becomes
\[
\hat{Y}_{\text{lag}} = \Psi_p Y_{\text{meas.}} = \Psi_p (Y_{\text{time}} + E_{\text{time}}) = \hat{Y}_{\text{lag}} + \Psi_p E_{\text{time}}
= \hat{Y}_{\text{lag}} + E_{\text{lag}} = Y_{\text{lag}} + \epsilon + E_{\text{lag}}.
\]
where \( E_{\text{lag}} \) represents the distortion in the first \( L \) Laguerre spectrum coefficients of \( y(t) \) due to the noise, and its covariance matrix is given as
\[
\Sigma_{\text{lag}} = \mathbb{E}[E_{\text{lag}} E_{\text{lag}}^T] = \Psi_p \Sigma \Psi_p^T.
\]

Then, from (8), the following relation holds
\[
\hat{Y}_{\text{lag}} = M_{\tau}(p) U_{\text{lag}} + \epsilon + E_{\text{lag}},
\]
where \( U_{\text{lag}} \) is a column vector stacking the Laguerre coefficients of the input \( u(t) \) (usually known by design), and \( M_{\tau}(p) \) is the matrix of Markov parameters. In other words,
\[
\hat{y}_j = (1 - p) \sum_{k=0}^{j-1} L_{j-k}^{[\tau]} \sqrt{p} u_k + \sqrt{p} \hat{y}_j + \epsilon_j + \hat{e}_j,
\]
where \( j = 0, \ldots, L-1 \). By the completeness of the Laguerre basis in \( \ell^2 \), the norm of the truncation error goes to zero as \( L \to \infty \). Notice that the truncation is not a problem when the input spectrum is finite. Even with a significant “tail” in the noise-free output, all the relations hold if the output spectrum is evaluated from a complete realization. However, unlike the truncation error, the distortion due to the noise persists even when \( L \) and \( T \) are large. In what follows, we will assume that \( \epsilon_j = 0 \) for all \( j \).
V. SIGNAL ESTIMATION IN LAGUERRE-DOMAIN

Recall that causality in Laguerre-domain holds in the sense that the Laguerre coefficients \( y_j \) in (6) are independent of \( u_i, i > j \). Consider now (5) with output measurements corrupted by additive noise. Assuming that the input signal is formed so that \( u_t = 0, i = 0, \ldots, m-1 \), for some \( m \in \mathbb{N}_+ \), the first \( m \) coefficients of the output, i.e. \( y_j, i = 0, \ldots, m-1 \), are independent of the input and constitute instead the first \( m \) coefficients of the Laguerre spectrum of the noise realization. Thus, the signal shape of the realization can be reconstructed and utilized for noise reduction. The accuracy of this reconstruction depends on how complex the signal shape of the disturbance realization is, that is how many Laguerre coefficients in a truncated Laguerre series of the signal it takes to achieve the desired result. Further, when the Laguerre noise spectrum is random and constitutes a correlated sequence, the noise realization coefficients of higher order than \( m \) can be predicted and enhance noise reduction even more.

A. Noise vector estimation

Consider a particular case of (14) where the input is delayed in Laguerre-domain. That is, the input signal is designed in Laguerre-domain so that, for some \( m \in \mathbb{N}_+ \), it holds that \( u_n = 0 \) for all \( n < m \)

\[
\hat{y}_j = \begin{cases} 
  e_j, & j < m, \\
  \sqrt{p} u_j + e_j, & j = m, \\
  (1 - p) \sum_{j = m}^{T} t_j (\sqrt{p}) u_j + \sqrt{p} u_j + e_j, & j > m.
\end{cases}
\]  

(15)

Then \( \hat{y}_0, \ldots, \hat{y}_{m-1} \) are equal to the distortion coefficients \( e_0, \ldots, e_{m-1} \), respectively.

Let the matrix \( \Xi_p \) be the submatrix of \( \Psi_p \) given by its first \( m \) rows. Then,

\[
\begin{bmatrix}
  e_0 & \ldots & e_{m-1}
\end{bmatrix}^T = \Xi_p Y_{\text{meas}}.
\]

An approximation of the whole time-domain noise vector can then be obtained as

\[
E_{\text{time}} = \begin{bmatrix} e(0) & \ldots & e(T-1) \end{bmatrix}^T \triangleq \Phi_L(p) \Xi_p Y_{\text{meas}}.
\]

(16)

In other words, for \( t = 0, \ldots, T-1 \),

\[
\hat{e}(t) = \sum_{k=0}^{m-1} \Psi_p \tilde{\ell}_k(t; p) = \sum_{k=0}^{m-1} \Xi_p \tilde{\ell}_k(t; p),
\]

where \( \Xi_p \) denotes the \( k \)-th row of \( \Xi_p \).

B. Best linear estimate

When the covariance matrix of the distortion vector \( E_{\text{lag}} \) is non-diagonal and known, the best linear estimator (BLE) of \( e_{m}, \ldots, e_{L-1} \) in terms of \( e_0, \ldots, e_{m-1} \) can be readily computed. If, in addition, the measurement noise is Gaussian, the estimator obtained is in fact mean squared error optimal.

Partition the vector \( \hat{Y}_{\text{lag}} \) and the covariance matrix \( \Sigma_{\text{lag}} \) as

\[
\hat{Y}_{\text{lag}} = \begin{bmatrix} \hat{Y}_{\text{lag}}^{(1)} \\ \hat{Y}_{\text{lag}}^{(2)} \end{bmatrix}, \quad E_{\text{lag}} = \begin{bmatrix} E_{\text{lag}}^{(1)} \\ E_{\text{lag}}^{(2)} \end{bmatrix}, \quad \Sigma_{\text{lag}} = \begin{bmatrix} \Sigma_{\text{lag}}^{(11)} & \Sigma_{\text{lag}}^{(12)} \\ \Sigma_{\text{lag}}^{(21)} & \Sigma_{\text{lag}}^{(22)} \end{bmatrix},
\]

such that \( \hat{Y}_{\text{lag}}^{(1)} \) and \( E_{\text{lag}}^{(1)} \) contain the first \( m \) entries of their corresponding vectors, and \( \Sigma_{\text{lag}}^{(11)} \) is the covariance matrix of \( E_{\text{lag}}^{(1)} \) that is square and of size \( m \). From (15), \( \hat{Y}_{\text{lag}}^{(1)} = E_{\text{lag}}^{(1)} \), and the BLE of \( E_{\text{lag}}^{(2)} \) is given as

\[
\hat{E}_{\text{lag}}^{(2)} = \begin{bmatrix} \hat{e}_0 & \ldots & \hat{e}_{L-1} \end{bmatrix}^T = (\Sigma_{\text{lag}}^{(11)})^{-1} \hat{Y}_{\text{lag}}^{(1)}\end{bmatrix}^T.
\]

(17)

It has a covariance

\[
\text{cov}\left( \hat{E}_{\text{lag}}^{(2)} - E_{\text{lag}}^{(2)} \right) = (\Sigma_{\text{lag}}^{(11)})^{-1} \Sigma_{\text{lag}}^{(12)}.(12)
\]

and the covariance of the error is

\[
\text{cov}\left( \hat{E}_{\text{lag}}^{(2)} - \hat{Y}_{\text{lag}}^{(2)} \right) = (\Sigma_{\text{lag}}^{(11)})^{-1} \Sigma_{\text{lag}}^{(12)}.(12)
\]

Then an estimate of the time-domain noise is given as

\[
\hat{e}(t) = \sum_{k=0}^{m-1} e_k \tilde{\ell}_k(t; p) + \sum_{k=m}^{L-1} \hat{e}_k \tilde{\ell}_k(t; p), \quad t = 0, \ldots, T-1,
\]

or, equivalently,

\[
\hat{E}_{\text{time}} = \Phi_L(p) \hat{Y}_{\text{lag}} = \Phi_L(p) \begin{bmatrix} E_{\text{lag}}^{(1)} \\ E_{\text{lag}}^{(2)} \end{bmatrix}^T.
\]

(18)

where \( \Theta_p = \Phi_{m+1:L}(p) \Sigma_{\text{lag}}^{(21)} \Sigma_{\text{lag}}^{(11)}^{-1} \Xi_p \) and \( \Phi_{m+1:L}(p) \) denotes the last \( L-m \) columns of \( \Phi_L(p) \). This is to be compared to (16) where the coefficients \( e_m, \ldots, e_{L-1} \) are estimated using their (unconditional) expected value, namely zero.

C. Noise reduction

Suppose that an estimate \( \hat{Y}_{\text{lag}} \) is obtained according to (16) and subtract the noise estimate from the measurement vector; viz. \( \tilde{\hat{Y}}_{\text{lag}} = Y_{\text{meas}} - \hat{E}_{\text{lag}} \). Using (16) and recalling that \( Y_{\text{meas}} = Y_{\text{time}} + E_{\text{time}} \),

\[
\tilde{\hat{Y}}_{\text{lag}} = \Phi_L(p) Y_{\text{meas}} - \hat{E}_{\text{lag}} = \hat{Y}_{\text{lag}} - \Psi_p \hat{E}_{\text{time}} \]

\[
= \Psi_p Y_{\text{meas}} + \Phi_L(p) \Xi_p E_{\text{time}} = Y_{\text{lag}} + \tilde{\hat{Y}}_{\text{lag}},
\]

where the equality before the last one holds because \( \Xi_p Y_{\text{time}} = 0_{m \times 1} \). Here, the truncation errors are ignored, i.e., it is assumed that \( Y_{\text{lag}} = \hat{Y}_{\text{lag}} \). Then it holds, \( \Psi_p (I_T - \Phi_m(p) \Xi_p) E_{\text{time}} = \begin{bmatrix} 0 & E_{\text{lag}}^{(2)} \end{bmatrix}^T \). Comparing to (12), where the distortion due to noise has a covariance as in (13), the covariance matrix is

\[
\Psi_p (I_T - \Phi_m(p) \Xi_p) \Sigma_{\text{time}} (I_T - \Phi_m(p) \Xi_p)^T \Psi_p.
\]

It is not difficult to see that this matrix is exactly equal to the submatrix of \( \Sigma_{\text{lag}} \), in (13), given by the last \( L-m \) rows and columns

\[
\begin{bmatrix}
  0 & 0
\end{bmatrix} \Sigma_{\text{lag}} - \begin{bmatrix}
  0 & 0
\end{bmatrix} L_{m-m},
\]

where 0 denotes zero matrices, and \( L_{m-m} \) is the identity matrix with dimension \( L-m \). Thus, no improvement in the signal-to-noise ratio in Laguerre-domain is obtained (the signal, i.e., non-zero spectrum, starts at \( m \)).
D. Noise reduction using BLE in Laguerre-domain

Now suppose that the BLE \( \hat{E}_{time} \) (18) is used instead. Then

\[
\hat{Y}_{\text{lag.}}^* = \Psi_p \Delta P \Theta_p \hat{E}_{time}
\]

In the light of (17), a reduction in the noise variance of \( \hat{Y}_{\text{lag.}} \) compared to \( \hat{Y}_{\text{lag.}} \) is guaranteed.

To recapitulate the above results, no noise reduction can be achieved in the Laguerre-domain if the distortion \( E_{lag.} \) is white. When, however, the distortion is correlated with a known correlation function, then the BLE can be used to improve the signal-to-noise ratio in the Laguerre-domain.

VI. DELAY ESTIMATION

In the rest of the paper, the results of Section V are applied to the problem of delay estimation in Laguerre-domain. Whereas a discrete-time delay estimation problem is essentially system order estimation in time-domain, it can be formulated as a parameter estimation problem in Laguerre-domain. An additional argument for the use of Laguerre-domain is that the input in delay estimation applications is typically a finite pulse of a certain signal shape and readily lends itself to a Laguerre series representation.

A. Algorithm

A result from an earlier contribution [10] showed that the value of \( \tau \) can be computed from three subsequent Markov parameters, \( h_{m-1}, h_m, \) and \( h_{m-1} \) in (9) for any value of \( m \geq n+1 \) using the formula

\[
\tau = -\left( \frac{(m+1)h_{m+1} + (m-1)h_{m-1}}{\beta h_m} \right) - \frac{ma}{\beta},
\]

where \( n \) is the index of the first non-zero Markov parameter, \( \alpha = (\sqrt{p} + \sqrt{p^{-1}}), \beta = (\sqrt{p} - \sqrt{p^{-1}}), h_k = \sum_{j=1}^{\infty} g_{k-j} y_j, \) and

\[
g_n = \frac{1}{u_n}, \quad g_k = \frac{1}{u_n} \sum_{j=0}^{k-1} u_k g_{j}, \quad k \geq n+1, \quad u_n \neq 0.
\]

From this, and using the first \( M \) non-zero Markov parameters, it is straightforward to see that the following equality holds

\[
a - b \tau = 0,
\]

where the column vectors \( a \) and \( b \) are defined as

\[
a \triangleq \Omega(\alpha) \begin{bmatrix} h_{n+1} \\ h_{n+1} \\ \vdots \\ h_{M+n} \\ h_{M+n} \end{bmatrix} + (M+n) \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} \triangleq \beta \begin{bmatrix} h_{n+1} \\ \vdots \\ h_{M+n} \end{bmatrix},
\]

and \( \Omega(\alpha) \) is the tridiagonal matrix

\[
\begin{bmatrix}
\alpha & 2 & \alpha & \cdots & \alpha \\
1 & 2\alpha & 3 & \cdots & 4 \\
0 & 2 & 3\alpha & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & \cdots & (M-1)\alpha
\end{bmatrix}
\]

From this, the delay is given by the closed-form formula for

\[
\tau = -\frac{b^T a}{b^T b}.
\]

Hence, the delay is given by the closed-form formula for

When the Markov parameters are estimated \( \{ \hat{h}_k \} \) from noisy data, e.g., using (14) and ordinary least-squares, the following estimate of the time delay is obtained

\[
\hat{\tau} = -\frac{\hat{b}^T \hat{a}}{\hat{b}^T \hat{b}}.
\]

B. Numerical Experiment

A Monte-Carlo numerical experiment is presented below to illustrate the performance of estimate (19). Three data sets are considered that correspond to the three noise models detailed in Section III.

Let the true delay value in (1) be \( \tau = 4 \), and consider the following three noise models (NM)

- NM1: \( e_1(t) \) is a stationary Gaussian white noise with variance \( \lambda = 0.3 \);
- NM2: \( e_2(t) \) is a stationary colored noise defined as

\[
e_2(t) = \frac{1}{q^2 - 0.9464q + 0.7408} \nu(t),
\]

where \( q \) is the time-domain shift operator, and \( \nu(t) \) is a Gaussian white noise whose variance is such that the variance of \( e_2(t) \) is, as \( e_1(t) \), equal to \( \lambda = 0.3 \);
- NM3: \( e_3(t) \) is given by (11) with \( p_e = 0.5, K = 19 \), such that \( \text{cov}([e_3(0) \ldots e_3(T-1)]) = \text{cov}([e_2(0) \ldots e_2(T-1)]) \).

In Laguerre-domain and, for sufficiently large \( T \), all the three models have the same marginal second-order properties, and the last two produce Laguerre-domain noise vectors \( E_{lag.} \) with the same covariance matrix. Yet the time-domain properties of NM2 and NM3 are different: NM2 is an autoregressive model with stable complex-conjugate poles (fixed to arbitrary values for this experiment), and thus stationary, while NM3 is non-stationary.

We ran a Monte-Carlo simulation experiment for the three noise models and the total number of simulations was \( 15e^3 \). The time-domain data set length was 300 samples, and all computations were made using the first \( L = 20 \) Laguerre functions with a Laguerre parameter \( p = 0.5 \). The input was designed to possess the Laguerre spectrum

\[
U_{lag.} = \begin{bmatrix} 0 & \cdots & 0 & 3.1 & 3 & 0 & 0 & 0 \end{bmatrix}^T,
\]

and kept fixed during all simulations. Thus, the first 15 Laguerre coefficients of the noise-free outputs are identically zero. Only the coefficients from \( y_{15} \) to \( y_{19} \) were used to compute the least-squares estimate of the first 5 Markov parameters yielding a delay estimate according to (19).

The results of the experiment for each case is summarized in terms of the mean and the variance of the estimator in Table I. The variances for the colored noise cases are larger than those for the white noise case. This is natural as all the noise models have the same marginal variance.

Table II shows the results obtained when the BLE is used to reduce the noise, as pointed out in Section V-C, using the
TABLE I
MEAN AND VARIANCE OF $\hat{\tau}$ FOR THE THREE NOISE MODELS.

|       | NM1   | NM2   | NM3   | τ  |
|-------|-------|-------|-------|----|
| Mean  | 3.3807| 3.2229| 3.2234| 4  |
| Var   | 0.8904| 1.0827| 1.0839|    |

TABLE II
MEAN AND VARIANCE OF $\hat{\tau}$ FOR THE THREE NOISE MODEL AFTER NOISE REDUCTION VIA THE BLE. BOTH THE BIAS AND THE VARIANCE ARE IMPROVED COMPARED TO THE RESULTS IN TABLE I.

|       | NM1   | NM2   | NM3   | τ  |
|-------|-------|-------|-------|----|
| Mean  | 3.3807| 3.6920| 3.6925| 4  |
| Var   | 0.8904| 0.5918| 0.5919|    |

Fig. 2. A realization of the BLE of the Laguerre distortion for the three noise models. Top panel: NM1, Middle panel: NM2, and Lower panel: NM3. In the three cases, the “true” and “estimated” first 15 coefficients coincide. For the case of NM1, the BLE of the last five coefficients is zero. The Laguerre-domain estimation errors ($\|E^{M}_{\text{true}} - E^{M}_{\text{est}}\|^2$) for these three realizations are: 0.7834 (NM1), 0.2009 (NM2), and 0.3993 (NM3). The corresponding noise vectors in time-domain are shown in Fig. VI-B.

The same exact data sets. As expected, there is no improvement in the case of the white noise model. However, the improvement in the mean value as well as the variance of $\hat{\tau}$ is clear in case of NM2 and NM3: the bias is reduced from 0.7771 to 0.3080 (about 60% drop), and the variance is reduced from 1.0827 to 0.5918 (about 45% drop). Interestingly, in Table II, the mean value rounds up to the true delay value. We also note that the results obtained using NM2 and NM3 are almost identical, despite NM3 being non-stationary. As discussed earlier, this is expected. Fig. 2 and Fig. VI-B show a realization of the BLE of the Laguerre distortion vector, and the reconstructed time-domain noise, respectively. Clearly, the signal shape of the noise realization is reconstructed closely only in case of NM3, for which Parseval’s identity holds.

Fig. 3. Examples of reconstructed time-domain noise vectors. Top panel: NM1, Middle panel: NM2, and Lower panel: NM3. The time-domain estimation errors ($\|E_{\text{true}} - E_{\text{est}}\|^2$) for these three realizations are: 80.0112 (NM1), 92.6170 (NM2) and 0.3993 (NM3).

VII. CONCLUSIONS
The implications of stochastic additive noise on the accuracy of Laguerre-domain estimation are studied both analytically and via Monte-Carlo simulations. It is shown that, by selecting the input signal as a linear combination of higher-order Laguerre functions, the signal shape of the actual noise realization can be reconstructed from the spectrum of the output in the case of strongly correlated noise sequence. The reconstructed signal can be then used for noise reduction. The efficacy of the proposed approach is demonstrated with respect to a Laguerre-domain time delay estimation algorithm.

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