Deformations of vector bundles over Lie groupoids

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Abstract
VB-groupoids are vector bundles in the category of Lie groupoids. They encompass several classical objects, including Lie group representations and 2-vector spaces. Moreover, they provide geometric pictures for 2-term representations up to homotopy of Lie groupoids. We attach to every VB-groupoid a cochain complex controlling its deformations and discuss its fundamental features, such as Morita invariance and a van Est theorem. Several examples and applications are given.

Keywords Lie groupoids · Lie algebroids · VB-groupoids · Deformations · Differentiable stacks

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Disclaimer: I cannot verify the accuracy of the mathematical content in the document. It is recommended to consult with a qualified expert to verify the correctness of the mathematical statements and proofs.
Introduction

Lie groupoids are unifying structures in differential geometry. Their theory generalizes that of Lie groups: they have an infinitesimal counterpart, Lie algebroids, and there exists a Lie functor differentiating Lie groupoids to Lie algebroids. Moreover, Lie groupoids provide a general framework to deal with many geometric situations such as orbifolds, foliations, Lie group actions and the geometry of PDEs. In all these cases, the main difficulty lies in studying spaces that are obtained as quotients of smooth manifolds, but are not smooth: they are rather *differentiable stacks* [4] and it is well-known that the latter are morally just Lie groupoids up to Morita equivalence.

This leads to the idea that the category of Lie groupoids should be understood as a more general setting for differential geometry, and indeed several kinds of geometric structures on Lie groupoids (symplectic structures, complex structures, Riemannian metrics and so on) have been considered in the last years. In particular, vector bundles in the category of Lie groupoids are known as *VB-groupoids*. They first appeared in [21] in connection to symplectic groupoids. Later, in [15], it was shown that they are geometric and more intrinsic models for 2-term representations up to homotopy of Lie groupoids [2]. For example, the tangent and the cotangent bundle of a Lie groupoid are VB-groupoids: they correspond to the adjoint and the coadjoint representations.

This paper is part of a bigger project devoted to the study of deformations of geometric structures on Lie groupoids (and on differentiable stacks). Many such structures can be understood as vector bundle maps, so it is important to understand deformations of VB-groupoids first: we address this problem in the present paper concentrating on cohomological aspects. Our starting point is the paper [7] by Crainic, Mestre and Struchiner, where deformations of Lie groupoids are studied. We will also partly follow [16] where we discussed deformations of VB-algebroids, the infinitesimal version of VB-groupoids.

The paper is divided in two main sections: the first one presents the relevant deformation cohomology and its main properties, the second one discusses examples and applications. In its turn, the first section is divided in five subsections. In Sect. 1.1 we recall from [7, 9, 16] the necessary background on deformations of Lie groupoids, Lie algebroids and VB-algebroids. We also review therein some cohomological aspects of VB-groupoids. In Sect. 1.2 we discuss the deformation theory of VB-groupoids. If $(\mathcal{V} \rightrightarrows E; \mathcal{G} \rightrightarrows M)$ is a VB-groupoid, the top groupoid $\mathcal{V} \rightrightarrows E$ has an associated deformation complex $C_{\text{def}}(\mathcal{V})$. We show that (infinitesimal) deformations of the VB-groupoid structure are controlled by a subcomplex $C_{\text{def, lin}}(\mathcal{V})$ of $C_{\text{def}}(\mathcal{V})$, the *linear deformation complex* of $\mathcal{V}$, originally introduced in [12]. We compute the first cohomology groups and we show that dual VB-groupoids have the same linear deformation cohomology. In Sect. 1.3 we introduce the *linearization map*, an important technical tool, adapting ideas from [6] and [16]. The linearization map is then used to prove that the linear deformation cohomology of a VB-groupoid is embedded in the deformation
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cohomology of the top groupoid. The linearization map also has applications in the subsequent subsections.

Section 1.4 is dedicated to the proof of a van Est theorem for the linear deformation cohomology of VB-groupoids. We show that the linear deformation complex of a VB-groupoid and that of the associated VB-algebroid are intertwined by a van Est map, which is a quasi-isomorphism under certain connectedness conditions. Finally, in Sect. 1.5 we show that the linear deformation cohomology is Morita invariant. This is particularly important, because it means that this cohomology is really an invariant of the associated vector bundle in differentiable stacks.

In the second section we deal with examples. Section 2.1 is about vector bundles in the category of Lie groups. The latter are equivalent to Lie group representations and we show that, in this case, the linear deformation cohomology and the classical cohomology controlling deformations of Lie group representations [20] fit into a long exact sequence. In Sect. 2.2 we study deformations of 2-vector spaces, i.e. Lie groupoids in the category of vector spaces. In Sect. 2.3, we discuss deformations of the tangent and the cotangent VB-groupoids of a Lie groupoid. Representations of foliation groupoids and Lie group actions on vector bundles can be also encoded by VB-groupoids: we study the associated deformation complexes in Sects. 2.4 and 2.5 respectively.

We assume that the reader is familiar with the basic notions on Lie groupoids and algebroids. Despite they are not as standard objects as the latter, we will also assume familiarity with VB-algebroids and VB-groupoids. The unfamiliar reader may refer e.g. to [5, 14, 15, 17] for the details, and [12, 16] for our notation/conventions about VB-algebroids/groupoids.

Before we start, we recall from [13] the concept of homogeneity structure of a vector bundle, a basic tool that will be used throughout the paper. Let \( E \rightarrow M \) be a vector bundle. The monoid \( \mathbb{R}_{\geq 0} \) of non-negative real numbers \( \lambda \) acts on \( E \) by fiber-wise scalar multiplications \( h_{\lambda} : E \rightarrow E, e \mapsto \lambda \cdot e, \lambda \in \mathbb{R}_{\geq 0} \). The action \( h : \mathbb{R}_{\geq 0} \times E \rightarrow E, e \mapsto h_\lambda(e) \), is called the homogeneity structure of \( E \). Together with the smooth structure, it fully characterizes the vector bundle structure. This implies that every notion that involves the linear structure of \( E \) can be expressed in terms of \( h \) only: for example, a smooth map between the total spaces of two vector bundles is a vector bundle map if and only if it commutes with the homogeneity structures.

1 Deformations of VB-groupoids

1.1 Preliminaries

In this section, we briefly recall the deformation theory of Lie groupoids, Lie algebroids and VB-algebroids, originally presented in [7, 9] and [16] respectively. We will also fix here our notation.
1.1.1 Deformations of Lie groupoids

Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid. We denote by $s$, $t$, $1$, $m$, $i$ its structure maps (source, target, unit, multiplication, inversion respectively), by $\bar{m}$ the division map and by $\mathcal{G}^{(k)}$ the manifold of $k$-tuples of composable arrows of $\mathcal{G}$.

The deformation complex $(C_{\text{def}}(\mathcal{G}), \delta)$ of $\mathcal{G}$ is defined as follows. For $k \geq 0$, $C_{\text{def}}^k(\mathcal{G})$ is the set of smooth maps $c : \mathcal{G}^{(k+1)} \to T\mathcal{G}$ such that:

1. $c(g_0, \ldots, g_k) \in T_{g_0}\mathcal{G}$;
2. $(T\bar{s} \circ c)(g_0, \ldots, g_k)$ does not depend on $g_0$ for any $(g_0, \ldots, g_k) \in \mathcal{G}^{(k+1)}$.

Thus one can define the $s$-projection of $c$ to be

$$s_c : \mathcal{G}^{(k)} \to TM, \quad s_c(g_1, \ldots, g_k) := (T\bar{s} \circ c)(g_0, \ldots, g_k).$$

The differential of $c \in C_{\text{def}}^k(\mathcal{G})$ is defined by

$$\delta c(g_0, \ldots, g_{k+1}) = -T\bar{m}(c(g_0g_1, \ldots, g_{k+1}), c(g_1, \ldots, g_{k+1}))$$
$$+ \sum_{i=1}^{k} (-1)^{i-1} c(g_0, \ldots, g_1g_{i+1}, \ldots, g_{k+1}) + (-1)^k c(g_0, \ldots, g_k).$$

Moreover, $C_{\text{def}}^{-1}(\mathcal{G}) := \Gamma(A)$, where $A \rightarrow M$ is the Lie algebroid of $\mathcal{G}$, and $\delta \alpha = \overrightarrow{\alpha} + \overleftarrow{\alpha}$ for each $\alpha \in \Gamma(A)$, where $\overrightarrow{\alpha}$ and $\overleftarrow{\alpha}$ are the left-invariant and right-invariant vector fields corresponding to $\alpha$. Notice that we adopt a different grading convention from [7], to be coherent with [16], where deformations of VB-algebroids were studied.

Remark 1.1.1 We observe, for later use, that conditions (1) and (2) can be expressed in the following way. For every $k \geq 1$, define the surjective submersions

$$p_k : \mathcal{G}^{(k)} \rightarrow \mathcal{G}, \quad (g_1, \ldots, g_k) \mapsto g_1,$$
$$q_k : \mathcal{G}^{(k)} \rightarrow M, \quad (g_1, \ldots, g_k) \mapsto s(g_1).$$

Then an element $c \in C_{\text{def}}^k(\mathcal{G})$ is simply a section of the pull-back bundle $p_{k+1}^*T\mathcal{G} \rightarrow \mathcal{G}^{(k+1)}$ that is $s$-projectable, i.e. such that there exists a section $s_c$ of $q_k^*TM \rightarrow \mathcal{G}^{(k)}$ fitting into the following commutative diagram

$$
\begin{array}{ccc}
p_{k+1}^*T\mathcal{G} & \rightarrow & \mathcal{G}^{(k+1)} \\
\downarrow T\bar{s} & & \downarrow s_c \\
q_k^*TM & \rightarrow & \mathcal{G}^{(k)}
\end{array}
$$

where the right arrow is the projection onto the last $k$ elements.
The normalized deformation complex \( \hat{C}^k_{\text{def}}(G) \) is defined as follows. For \( k \geq 1 \), \( \hat{C}^k_{\text{def}}(G) \) consists of all cochains \( c \in C^k_{\text{def}}(G) \) such that

\[
c(1_x, g_1, \ldots, g_k) \in T_x M \subset T_{1_x} G, \quad \text{and} \quad c(g_0, \ldots, 1_x, \ldots, g_k) = 0, \tag{1.3}
\]

while \( \hat{C}^0_{\text{def}}(G) \) consists of 0-cochains \( c \) such that \( c(1_x) = s_c(x) \). Finally, \( \hat{C}^{-1}_{\text{def}}(G) := \Gamma_1(\mathcal{A}) \).

Notice that the first condition in (1.3) implies that \( c(1_x, g_1, \ldots, g_k) \) can be identified with \( (Ts \circ c)(1_x, g_1, \ldots, g_k) = s_c(g_1, \ldots, g_k) \), so we recover the definition in [7].

Clearly, the group \( \text{Aut}(G) \) of automorphisms of \( G \) acts naturally on \( C_{\text{def}}(G) \) by cochain maps, via pull-backs.

Now we recall from [7] the deformation cohomology of a Lie groupoid in degrees \(-1\) and \(0\). In degree \(-1\) we have

\[
H^{-1}_{\text{def}}(G) = \Gamma(i)^{\text{inv}}
\]

where the right hand side \( \Gamma(i)^{\text{inv}} \) consists of invariant sections of the isotropy bundle, i.e. sections \( \alpha \) of \( A \) in the kernel of the anchor map \( \rho : \Gamma(A) \to \mathcal{X}(M) \) which are invariant under the set-theoretic action of \( G \) on isotropies, the latter being well defined even if \( i = \ker \rho \) is not a subbundle in \( A \). Such action does even allow to define \( H(G, i) \): differentiable cohomologies of \( G \) with coefficients in \( i \). A cochain \( u : G^{(k)} \to i \) is smooth if it is smooth as an \( A \)-valued map, and there is an inclusion of cochain complexes

\[
r : C(G, i) \hookrightarrow C_{\text{def}}(G)[-1], \quad u \mapsto c_u \tag{1.4}
\]

given by

\[
c_u(g_1, \ldots, g_k) = TR g_1(u(g_1, \ldots, g_k))
\]

where \( R_g \) denotes right translation by \( g \in G \) (see [7] for more details).

In degree 0 we have

\[
H^0_{\text{def}}(G) = \frac{\text{multiplicative vector fields on } G}{\text{inner multiplicative vector fields on } G} \tag{1.5}
\]

and there is a linear map

\[
\pi : H^0_{\text{def}} \to \Gamma(v)^{\text{inv}}
\]

where \( \Gamma(v)^{\text{inv}} \) consists of invariant sections of the normal bundle, i.e. elements \([V]\) in the quotient module

\[
\Gamma(v) = \frac{\mathcal{X}(M)}{\text{im}(\rho : \Gamma(A) \to \mathcal{X}(M))} \tag{1.6}
\]
which are invariant under the set-theoretic action of \( \mathcal{G} \) on the normal bundle \( \nu = TM / \text{im} \rho \) (the latter being well-defined even when \( \nu \) is not of constant rank). This means that \( \nu \) possesses an \((s, t)\)-lift, i.e. a vector field \( X \in \mathcal{X}(\mathcal{G}) \) projecting onto \( \nu \) via both \( s \) and \( t \).

Finally, there is an exact sequence

\[
0 \longrightarrow H^1(\mathcal{G}, i) \overset{r}{\longrightarrow} H^0_{\text{def}}(\mathcal{G}) \overset{\pi}{\longrightarrow} \Gamma(\nu)^{\text{inv}} \overset{K}{\longrightarrow} H^2(\mathcal{G}, i) \overset{r}{\longrightarrow} H^1_{\text{def}}(\mathcal{G}).
\]

where \( K \) maps \([\nu] \in \Gamma(\nu)^{\text{inv}}\), with \( \nu \in \mathcal{X}(\mathcal{M}) \), to the cohomology class of \( \delta X \in C^2(\mathcal{G}, i) \), where \( X \in \mathcal{X}(\mathcal{G}) \) is an \((s, t)\)-lift of \( \nu \).

The group \( H^1_{\text{def}}(\mathcal{G}) \) is directly related to (infinitesimal) deformations of \( \mathcal{G} \). Recall that a family of Lie groupoids

\[
\tilde{\mathcal{G}} \Rightarrow \tilde{\mathcal{M}} \overset{\pi}{\longrightarrow} \mathcal{B},
\]

consists of a Lie groupoid \( \tilde{\mathcal{G}} \Rightarrow \tilde{\mathcal{M}} \) and a surjective submersion \( \pi : \tilde{\mathcal{M}} \rightarrow \mathcal{B} \) such that \( \pi \circ \tilde{s} = \pi \circ \tilde{t} \). In particular, for every \( b \in \mathcal{B} \), \( \tilde{\mathcal{G}}_b := (\pi \circ \tilde{s})^{-1}(b) \) is a Lie groupoid over \( M_b := \pi^{-1}(b) \).

This definition encodes the idea of a “smoothly varying” Lie groupoid. If \( \mathcal{B} \) is an open interval \( I \) containing \( 0 \), we say that \( \tilde{\mathcal{G}} \) is a deformation of \( \mathcal{G}_0 \Rightarrow M_0 \) and we denote the latter by \( \mathcal{G} \Rightarrow \tilde{\mathcal{M}} \). We will often denote by \( \epsilon \) the canonical coordinate on \( I \). Accordingly, a deformation of \( \mathcal{G} \) is also denoted by \( (\mathcal{G}_\epsilon) \). The structure maps of \( \mathcal{G}_\epsilon \) are denoted \( s_\epsilon, \, t_\epsilon, \, \iota_\epsilon, \, m_\epsilon, \, i_\epsilon \). The division map is denoted \( m_\epsilon \). The constant deformation is the one with \( \mathcal{G}_\epsilon = \mathcal{G} \) as groupoids for all \( \epsilon \).

Two deformations \( (\mathcal{G}_\epsilon) \) and \( (\mathcal{G}_\epsilon') \) of \( \mathcal{G} \) are said to be equivalent if there exists a smooth family of groupoid isomorphisms \( \Psi_\epsilon : \mathcal{G}_\epsilon \rightarrow \mathcal{G}_\epsilon' \) such that \( \Psi_0 = \text{id}_{\mathcal{G}_0} \). A deformation \( (\mathcal{G}_\epsilon) \) is trivial if it is equivalent to the constant deformation.

Let \( \tilde{\mathcal{G}} \) be a deformation of \( \mathcal{G} \). A transverse vector field for \( \tilde{\mathcal{G}} \) is a vector field \( X \in \mathcal{X}(\tilde{\mathcal{G}}) \) which is \( s \)-projectable to a vector field on \( \tilde{\mathcal{M}} \) which is, in turn, \( \pi \)-projectable to \( \frac{d}{d\epsilon} \). Transverse vector fields \( \tilde{X} \in \mathcal{X}(\tilde{\mathcal{G}}) \) always exist. Additionally

1. \( \delta \tilde{X} \) restricts to \( \mathcal{G} \) and \( \xi_0 := \delta \tilde{X} \in C^1_{\text{def}}(\mathcal{G}) \) is a cocycle;
2. the cohomology class of \( \xi_0 \) does not depend on the choice of \( \tilde{X} \).

The resulting cohomology class in \( H^1_{\text{def}}(\mathcal{G}) \) is called the deformation class associated to the deformation \( \tilde{\mathcal{G}} \). The deformation class is invariant under equivalence of deformations.

We conclude this short review of deformations of Lie groupoids recalling a result about general families of Lie groupoids. Let \( \tilde{\mathcal{G}} \Rightarrow \tilde{\mathcal{M}} \overset{\pi}{\longrightarrow} \mathcal{B} \) be a family of Lie groupoids. Then any curve \( \gamma : I \rightarrow \mathcal{B} \) induces a deformation \( \gamma^* \tilde{\mathcal{G}} \) of \( \tilde{\mathcal{G}}_{\gamma(0)} \). Now, let \( b \in \mathcal{B} \). Then, for any curve \( \gamma : I \rightarrow \mathcal{B} \) with \( \gamma(0) = b \), the deformation class of \( \gamma^* \tilde{\mathcal{G}} \) at time 0 does only depend on \( \gamma(0) \). This defines a linear map

\[
\text{Var}_b^\tilde{\mathcal{G}} : T_b \mathcal{B} \rightarrow H^1_{\text{def}}(\tilde{\mathcal{G}}_b),
\]

called the variation map of \( \tilde{\mathcal{G}} \) at \( b \).
1.1.2 Deformations of Lie algebroids and VB-algebroids

Let \( E \to M \) be a vector bundle. A \textit{multiderivation with \( k \) entries} of \( E \) (and \( C^\infty(M) \)-multilinear symbol), also called a \( k \)-\textit{derivation}, is a skew-symmetric, \( \mathbb{R}^k \)-linear map
\[
c : \Gamma(E) \times \cdots \times \Gamma(E) \to \Gamma(E)
\]
such that there exists a bundle map \( \sigma_c : \bigwedge^{k-1} E \to TM \), the \textit{symbol} of \( c \), satisfying the following Leibniz rule:
\[
c(\varepsilon_1, \ldots, \varepsilon_{k-1}, f \varepsilon_k) = \sigma_c(\varepsilon_1, \ldots, \varepsilon_{k-1})(f)\varepsilon_k + fc(\varepsilon_1, \ldots, \varepsilon_k),
\]
for all \( \varepsilon_1, \ldots, \varepsilon_k \in \Gamma(E) \), \( f \in C^\infty(M) \). The space of \( k \)-derivations of \( E \to M \) is denoted \( D_k(E) \) and we set \( D^\cdot(E) := \bigoplus_k D_k(E) \).

1-derivations are simply derivations and they are of a particular interest. The space of derivations is denoted by \( D(E) \). Recall that derivations of \( E \) are sections of a Lie algebroid \( DE \Rightarrow M \), that sits in the following short exact sequence (the Spencer sequence):
\[
0 \to \text{End } E \to DE \overset{\sigma}{\to} TM \to 0,
\]
where \( \sigma : DE \to TM \) is the \textit{symbol map} and plays the role of the anchor. The Lie algebroid \( DE \) is sometimes called the \textit{gauge algebroid} of \( E \). Before going on, we recall its main properties. First of all, there is a canonical isomorphism of Lie algebroids
\[
DE \cong DE^*, \quad \delta \mapsto \delta^*,
\]
defined as follows: if \( \delta \in D_x E \), then \( \delta^* : \Gamma(E^*) \to E^*_x \) is uniquely determined by the condition
\[
\sigma_\delta(\varphi, \varepsilon) = \langle \delta^* \varphi, \varepsilon \rangle + \langle \varphi, \delta \varepsilon \rangle,
\]
for all \( \varepsilon \in \Gamma(E) \), \( \varphi \in \Gamma(E^*) \).

There is an alternative description of derivations of a vector bundle \( \pi : E \to M \) that will be useful in what follows. Namely, consider the vector bundle \( T\pi : TE \to TM \). We denote by \( TE|_v \) the fiber \( T\pi^{-1}(v) \) over a tangent vector \( v \in TM \). Then a derivation \( \delta \in D_x E \) (at a point \( x \in M \)) with symbol \( \sigma_\delta \) determines a linear map \( \hat{\delta} : E_x \to TE|_{\sigma_\delta} \) via
\[
\hat{\delta}(e)(\varphi) = \langle \delta^* \varphi, e \rangle,
\]
for all \( e \in E_x \) and \( \varphi \in \Gamma(E^*) \), where, in the lhs, \( \varphi \) is also interpreted as a fiber-wise linear function on \( E \). The assignment \( \delta \mapsto (\sigma_\delta, \hat{\delta}) \) establishes a one-to-one correspondence between the fiber \( D_x E \) of the gauge algebroid over \( x \), and the space
of pairs \((v, h)\) where \(v \in T_x M\) and \(h : E_x \to T E|_v\) is a right inverse of the projection \(T E|_v \to E_x\). In other words, \(D E\) is the \textit{fat algebroid} \([14]\) of the VB-algebroid \((T E \Rightarrow E; TM \Rightarrow M)\), and derivations are the same as linear sections of \((T E \Rightarrow E; TM \Rightarrow M)\), or, which is the same, linear vector fields \(\mathfrak{x}_{\text{lin}}(E)\) on \(E\): \(\mathcal{D}(E) \cong \mathfrak{x}_{\text{lin}}(E)\) \([17]\). We will sometimes identify \(\delta\) and the pair \((\sigma_{\delta}, \widehat{\delta})\).

The assignment \(E \mapsto D E\) is functorial in the following sense. Let

\[
\begin{array}{ccc}
E_N & \xrightarrow{\phi} & E \\
\downarrow & & \downarrow \\
N & \xrightarrow{f} & M
\end{array}
\]

be a \textit{regular} vector bundle morphism, i.e. \(\phi\) is an isomorphism on each fiber, so that \(E_N\) is canonically isomorphic to the pullback bundle \(f^*E\). Then there is a pullback map \(\phi^* : \Gamma(E) \to \Gamma(E_N)\) defined by

\[(\phi^*\varepsilon)_y = \phi^{-1}_y(\varepsilon_{f(y)})\]

for all \(\varepsilon \in \Gamma(E)\), and all \(y \in N\). One can use this pull-back to define a Lie algebroid morphism \(D\phi : D E_N \to D E\). Specifically, for all \(\delta \in D_y E_N\) we define \(D\phi(\delta) : \Gamma(E) \to E_{f(y)}\) by

\[D\phi(\delta) = \phi \circ \delta \circ \phi^*.\]

It is then easy to see that

\[\widehat{D\phi(\delta)} = T\phi \circ \widehat{\delta} \circ \phi^{-1}_x : E_x \to T E|_{T f(\sigma_{\delta})}.\]

Finally, the diagram

\[
\begin{array}{ccc}
D E_N & \xrightarrow{D\phi} & D E \\
\downarrow & & \downarrow \\
TN & \xrightarrow{T f} & TM
\end{array}
\]

is a pull-back diagram, and this induces an isomorphism \(D E_N \cong T N \times_{T M} D E\) which is sometimes useful. From now on, we will often identify \(E_N\) with the pull-back \(f^*E\). For more details about the gauge algebroid we refer to \([17]\) (see also \([12]\)).

Now, let us go back to the main topic of this subsection and take a Lie algebroid \(A \Rightarrow M\). The graded vector space \(C_{\text{def}}(A) := D^*(A)[1]\) can be turned into a cochain complex, the \textit{deformation complex} of \(A\), using the Lie bracket \([-,-]\) on sections of \(A \to M\). The differential \(\delta : C_{\text{def}}^k(A) \to C_{\text{def}}^{k+1}(A)\) is defined by (see \([9]\))
\[ \delta c(\alpha_0, \ldots, \alpha_{k+1}) = \sum_i (-1)^i [\alpha_i, c(\alpha_0, \ldots, \hat{\alpha}_i, \ldots, \alpha_{k+1})] + \sum_{i<j} (-1)^{i+j} c([\alpha_i, \alpha_j], \alpha_0, \ldots, \hat{\alpha}_i, \ldots, \hat{\alpha}_j, \ldots, \alpha_{k+1}). \]

for all \( \alpha_0, \ldots, \alpha_{k+1} \in \Gamma(A) \).

Clearly, the group \( \text{Aut}(A) \) of Lie algebroid automorphisms of \( A \) acts naturally on \( \mathcal{C}_\text{def}(A) \) by cochain maps, via pullback.

Finally, take a VB-algebroid \( (W \Rightarrow E; A \Rightarrow M) \). Deformations of the VB-algebroid structure are controlled by a subcomplex \( \mathcal{C}_\text{def, lin}(W) \) of \( \mathcal{C}_\text{def}(W) \), the \textit{linear deformation complex} of \( W \), defined as follows \cite{12, 16}. Let \( h_\lambda \) be the homogeneity structure of the vector bundle \( W \Rightarrow A \). We say that a deformation cochain \( \tilde{c} \in \mathcal{C}_\text{def}(W) \) is \textit{linear} if

\[ h_\lambda \tilde{c} = \tilde{c} \]

for every \( \lambda > 0 \). The linear deformation complex consists, by definition, of linear deformation cochains.

\subsection{Cohomology of VB-groupoids}

In this subsection we recall from \cite{15} the main cohomological aspects of VB-groupoids. Let \((\mathcal{V} \Rightarrow E; \mathcal{G} \Rightarrow M)\) be a VB-groupoid. We denote \( \mathring{s}, \mathring{t}, \mathring{\iota}, \mathring{m}, \mathring{i} \) the structure maps of \( \mathcal{V} \), \( s, t, 1, m, i \) the structure maps of \( \mathcal{G} \), \( \mathring{\pi} : \mathcal{V} \to \mathcal{G}, \pi : E \to M \) the vector bundle projections and \( \mathring{\mathcal{G}} : \mathcal{G} \to \mathcal{V} \), \( 0 : M \to E \) the zero sections. The total groupoid \( \mathcal{V} \) comes with its Lie groupoid complex \( (C(\mathcal{V}), \delta) \). There is an induced vector bundle structure on \( \mathcal{V}^{(k)} \to \mathcal{G}^{(k)} \), so there is a natural subcomplex \( \mathcal{C}_\text{lin}(\mathcal{V}) \) of \( C(\mathcal{V}) \), whose \( k \)-cochains are functions on \( \mathcal{V}^{(k)} \) that are linear over \( \mathcal{G}^{(k)} \). Inside \( \mathcal{C}_\text{lin}(\mathcal{V}) \), there is a distinguished subcomplex \( \mathcal{C}_\text{proj}(\mathcal{V}) \) consisting of \textit{left-projectable linear cochains} \cite{15}. By definition, a linear cochain \( f \in C^k_{\text{lin}}(\mathcal{V}) \) is left-projectable if

1. \( f((\mathring{0}_g, v_2, \ldots, v_k)) = 0 \) for every \( (\mathring{0}_g, v_2, \ldots, v_k) \in \mathcal{V}^{(k)} \);
2. \( f((\mathring{0}_g \cdot v_1, \ldots, v_k)) = f(v_1, \ldots, v_k) \) for \( (v_1, \ldots, v_k) \in \mathcal{V}^{(k)} \) and \( g \in \mathcal{G} \) such that \( \mathring{\iota}(v_1) = 0_{s_1(g)} \).

The complexes \( \mathcal{C}_\text{lin}(\mathcal{V}) \) and \( \mathcal{C}_\text{proj}(\mathcal{V}) \) are called the \textit{linear complex} and the \textit{VB-complex} of \( \mathcal{V} \), respectively. Their cohomologies are denoted \( H_{\text{lin}}(\mathcal{V}) \) and \( H_{\text{proj}}(\mathcal{V}) \) and called the \textit{linear cohomology} and the \textit{VB-cohomology} of \( \mathcal{V} \).

\textbf{Remark 1.1.2} We are adopting the terminology of \cite{10}. In \cite{15} and \cite{7}, instead, the VB-complex and the VB-cohomology of \( \mathcal{V} \) are defined to be \( \mathcal{C}_{\text{proj}}(\mathcal{V}^*) \) and \( H_{\text{proj}}(\mathcal{V}^*) \), respectively.

Notice that \( \mathcal{C}_{\text{proj}}(\mathcal{V}) \) is a \( (\mathcal{G}) \)-DG-module in the obvious way and it turns out that the linear cohomology and the VB-cohomology are isomorphic. More precisely, the inclusion \( \mathcal{C}_{\text{proj}}(\mathcal{V}) \hookrightarrow \mathcal{C}_{\text{lin}}(\mathcal{V}) \) induces an isomorphism of \( H(\mathcal{G}) \)-modules in cohomology \cite[Lemma 3.1]{6}.
For our purposes, the \( \text{VB-complex} \) is particularly important because it gives another description of the deformation complex of a Lie groupoid \( \mathcal{G} \). Indeed, there is an isomorphism of \( C(\mathcal{G}) \)-DG-modules

\[
\phi : C_{\text{def}}(\mathcal{G}) \to C_{\text{proj}}(T^*\mathcal{G})[1]
\]
given by

\[
\phi(c)(\theta_0, \ldots, \theta_k) = (\theta_0, c(g_0, \ldots, g_k))
\]
for all \( c \in C^k_{\text{def}}(\mathcal{G}) \) and \((\theta_0, \ldots, \theta_k) \in (T^*\mathcal{G})^{(k+1)}\) such that \( \theta_i \in T^*_{g_i}\mathcal{G} [7, \text{Proposition 3.9}].\)

### 1.2 The linear deformation complex of a VB-groupoid

In this subsection we introduce the main object of this paper: the \textit{linear deformation complex of a VB-groupoid}, first introduced in [12] (for different purposes from the present ones). The definition is entirely analogous to the one recalled in Sect. 1.1 for VB-algebroids. Here and in what follows we will need the notion of a \textit{double vector bundle} (DVB for short): we refer to [17] for definitions and basic properties and to our previous paper [16] for notations.

Let \((V \rightrightarrows E; \mathcal{G} \rightrightarrows M)\) be a VB-groupoid, and let \((W \rightrightarrows E; A \rightrightarrows M)\) be its VB-algebroid. The fiber-wise scalar multiplication \(h_\lambda\) is a Lie groupoid automorphism for every \(\lambda > 0\), so it acts on the deformation complex \(C_{\text{def}}(V)\) of \(V \rightrightarrows E\). We say that a deformation cochain \(\tilde{c}\) is \textit{linear} if

\[
h_\lambda^*\tilde{c} = \tilde{c}
\]
for every \(\lambda > 0\). Hence, linear cochains are those which are invariant under the homogeneity structure. Equivalently, a deformation cochain \(c : \mathcal{V}^{(k+1)} \to TV\) is linear if it is a vector bundle map between the vector bundles \(\mathcal{V}^{(k+1)} \to \mathcal{G}^{(k+1)}\) and \(TV \to T\mathcal{G}\) (covering some smooth map \(c : \mathcal{G}^{(k+1)} \to TG\), necessarily a deformation cochain of \(\mathcal{G}\), see also the discussion below). This explains the terminology.

\textit{Linear deformation cochains form a subcomplex of} \(C_{\text{def}}(\mathcal{V})\). We denote the latter by \(C_{\text{def,lin}}(\mathcal{V})\) and we call it the \textit{linear deformation complex} of \(\mathcal{V}\). Its cohomology is called the \textit{linear deformation cohomology} of \(\mathcal{V}\) and denoted \(H_{\text{def,lin}}(\mathcal{V})\). It is easy to see that \(C_{\text{def,lin}}(\mathcal{V})\) is a \(C(\mathcal{G})\)-DG-module (more precisely a DG-submodule of \(C_{\text{def}}(\mathcal{V})\)).

The action of \(h_\lambda\) on \(C^{-1}_{\text{def}}(\mathcal{V})\) coincides, by definition, with the action induced by the homogeneity structure of \(W \rightrightarrows A\) on \(\Gamma(W, E)\), so \(C^{-1}_{\text{def,lin}}(\mathcal{V})\) is simply the space \(\Gamma_{\text{lin}}(W, E)\) of linear sections of \(W \rightrightarrows E\). For \(k \geq 0\), Eq. (1.7) is equivalent to saying that \(\tilde{c} : \mathcal{V}^{(k+1)} \to TV\) intertwines the homogeneity structures of \(\mathcal{V}^{(k+1)} \to \mathcal{G}^{(k+1)}\) and \(TV \to T\mathcal{G}\). This also means that \(\tilde{c}\) is a vector bundle map over some map \(c : \mathcal{G}^{(k+1)} \to T\mathcal{G}\). In this way we recover the definition in [12].

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For \( k \geq 0 \), a linear \( k \)-cochain \( \tilde{c} \) can also be seen as an \( \tilde{s} \)-projectable, linear section of the DVB \( (p_{k+1}^*T\mathcal{V} \to \mathcal{V}^{(k+1)}; p_{k+1}^*T\mathcal{G} \to \mathcal{G}^{(k+1)}) \):

\[
\begin{array}{ccc}
\mathcal{V}^{(k+1)} & \xrightarrow{\tilde{c}} & \mathcal{V}^{(k)} \\
p_{k+1}^*T\mathcal{V} & \searrow & \downarrow \\
\mathcal{V}^{(k+1)} & \left\downarrow \right\searrow & \mathcal{G}^{(k+1)}
\end{array}
\]

where we denote \( \tilde{p}_k : \mathcal{V}^{(k)} \to \mathcal{V} \) and \( \tilde{q}_k : \mathcal{V}^{(k)} \to E \) the maps (1.2) for \( \mathcal{V} \).

There is another way to describe the linear deformation complex of a VB-groupoid. Let \( (\mathcal{V} \Rightarrow E; \mathcal{G} \Rightarrow M) \) be a VB-groupoid, let \( C \) be its core, and let \( (W \Rightarrow E; A \Rightarrow M) \) be its VB-algebroid. Consider the cotangent VB-groupoid of \( \mathcal{V} \Rightarrow E \), \( (T^*\mathcal{V} \Rightarrow W^*E; \mathcal{V} \Rightarrow E) \) (here we denote by \( W^*_E \to E \) the dual of a vector bundle \( W \to E \)).

Actually, \( T^*\mathcal{V} \to W^*_E \) is the top groupoid of another VB-groupoid. To see this, first take the dual of \( \mathcal{V} \), that is \( (\mathcal{V}^* \Rightarrow C^*; \mathcal{G} \Rightarrow M) \). Then, the cotangent VB-groupoid of \( \mathcal{V}^* \Rightarrow C^* \) is \( (T^*\mathcal{V}^* \Rightarrow (W^*_A)^+_{C^*}; \mathcal{V}^* \Rightarrow C^*) \). Finally, recall from [17] that there is a canonical isomorphism

\[
B : T^*\mathcal{V} \to T^*\mathcal{V}^*
\]

of both DVBs and Lie groupoids, covering an isomorphism

\[
\beta : W^*_E \to (W^*_A)^+_{C^*}.
\]

of DVBs. Combining all these maps, we obtain a diagram:

\[
\begin{array}{ccc}
T^*\mathcal{V} & \xrightarrow{\cong} & W^*_E \\
\downarrow B & \downarrow \cong & \downarrow \beta \\
(W^*_A)^+_{C^*} & \xrightarrow{\cong} & C^*
\end{array}
\]

(1.8)

The maps \( B \) and \( \beta \) are isomorphisms of vector bundles over the identity, so the back face in the diagram (1.8) is also a VB-groupoid.

It follows that inside \( C(T^*\mathcal{V}) \) there are two distinguished subcomplexes, those of cochains that are linear over \( \mathcal{V} \) and over \( \mathcal{V}^* \): we denote them by \( C_{\text{lin}}(T^*\mathcal{V}) \) and \( C_{\text{lin}}(T^*\mathcal{V}) \) respectively. Moreover, denote \( C_{\text{proj}}(T^*\mathcal{V}) \) the subcomplex of left-projectable linear cochains over \( \mathcal{V} \) and define \( C_{\text{lin,lin}}(T^*\mathcal{V}) := \text{lin}(T^*\mathcal{V}) \cap \text{lin}(T^*\mathcal{V}) \), \( C_{\text{proj,lin}}(T^*\mathcal{V}) := \text{proj}(T^*\mathcal{V}) \cap \text{lin}(T^*\mathcal{V}) \), \( C_{\text{lin,proj}}(T^*\mathcal{V}) := \text{lin}(T^*\mathcal{V}) \cap \text{proj}(T^*\mathcal{V}) \), \( C_{\text{proj,proj}}(T^*\mathcal{V}) := \text{proj}(T^*\mathcal{V}) \cap \text{proj}(T^*\mathcal{V}) \). We denote their cohomologies \( H_{\text{lin,lin}}(\mathcal{V}) \) and \( H_{\text{proj,lin}}(\mathcal{V}) \) respectively.
From [7, Proposition 3.9], there is an isomorphism of $C(\mathcal{V})$-DG-module

$$C_{\text{def}}(\mathcal{V}) \cong C_{\text{proj}, \ast}(T^*\mathcal{V})[1]. \quad (1.9)$$

It is easy to check that this isomorphism takes linear deformation cochains to cochains on $T^*\mathcal{V}$ that are linear over $\mathcal{V}^*$. So we get the following

**Proposition 1.2.1** There is an isomorphism of $C(\mathcal{G})$-DG-module

$$C_{\text{def, lin}}(\mathcal{V}) \cong C_{\text{proj, lin}}(T^*\mathcal{V})[1]. \quad (1.10)$$

For later use, we notice that a “linear version” of [6, Lemma 3.1] holds. Namely, we have

**Lemma 1.2.2** The inclusion $C_{\text{proj, lin}}(T^*\mathcal{V}) \hookrightarrow C_{\text{lin, lin}}(T^*\mathcal{V})$ induces an isomorphism in cohomology.

**Proof** The proof of [6, Lemma 3.1] works identically in our setting without significant modifications. \qed

### 1.2.1 Deformations of $\mathcal{G}$ from linear deformations of $\mathcal{V}$

Recall from [12, Lemma 2.27] that, if $\tilde{c} : \mathcal{V}^{(k+1)} \to T\mathcal{V}$ belongs to $C^{k}_{\text{def, lin}}(\mathcal{V})$, then its projection $c : \mathcal{G}^{(k+1)} \to T\mathcal{G}$ belongs to $C^k_{\text{def}}(\mathcal{G})$ and $\delta \tilde{c}$ projects to $\delta c$. It follows that there exists a natural cochain map:

$$C_{\text{def, lin}}(\mathcal{V}) \to C_{\text{def}}(\mathcal{G}). \quad (1.11)$$

In degree $k = -1$, this is simply the projection $\Gamma_{\text{lin}}(W, E) \to \Gamma(A)$ and we have the well-known short exact sequence:

$$0 \to \mathfrak{Hom}(E, C) \to \Gamma_{\text{lin}}(W, E) \to \Gamma(A) \to 0,$$

where $\mathfrak{Hom}(E, C)$ is the $C^\infty(M)$-module of vector bundle morphisms $E \to C$.

We now show that the map (1.11) is surjective for all $k \geq 0$. We will need a technical Lemma

**Lemma 1.2.3** Consider a DVB morphism:
such that all the horizontal maps are surjective submersions. Let \( \tilde{\alpha}_2 \) be a linear section of \( W_2 \to E_2 \), and let \( \alpha_1 \) be a section of \( A_1 \) that projects on the same section \( \alpha_2 \) of \( A_2 \). Then there exists a linear section of \( W_1 \to E_1 \) that projects simultaneously on \( \tilde{\alpha}_2 \) and \( \alpha_1 \).

**Proof** We can decompose the morphism \( \phi \) in the following way:

\[
\begin{array}{ccc}
W_1 & \xrightarrow{\hat{\phi}} & E_1 \times_{E_2} W_2 \times_{A_2} A_1 \\
\downarrow \tilde{\alpha}_1 & & \downarrow \phi_E \\
E_1 & \xrightarrow{\alpha} & E_2 \\
\downarrow \pi_{E_1} & & \downarrow \alpha_2 \\
A_1 & \xrightarrow{\alpha_1} & A_2 \\
\downarrow \alpha_{e} & & \downarrow \alpha_{e} \\
M_1 & \xrightarrow{\alpha_{e}} & M_2
\end{array}
\]

Define

\[
\hat{\alpha} : E_1 \to E_1 \times_{E_2} W_2 \times_{A_2} A_1, \quad \hat{\alpha}_e = (e, \tilde{\alpha}_2|_{\phi_E(e)}, \alpha_1|_{\pi_{E_1}(e)}).
\]

It is clear that \( \hat{\alpha} \) is a well-defined linear section of \( E_1 \times_{E_2} W_2 \times_{A_2} A_1 \to E_1 \), that projects simultaneously on \( \tilde{\alpha}_2 \) and \( \alpha_1 \), so the problem is reduced to find a splitting of \( \hat{\phi} \). This can be done first locally, and then globally via the choice of a partition of unity on \( M_1 \).

Now, to see that the map (1.11) is surjective for all \( k \geq 0 \), take \( c \in C^k_{\text{def}}(G) \). By Remark 1.1.1, we have a diagram:

\[
\begin{array}{ccc}
G^{(k+1)} & \xrightarrow{c} & p_{k+1}^*T^*G \\
\downarrow s_c & & \downarrow T^*s \\
G^{(k)} & \xrightarrow{s_c} & q_k^*TM
\end{array}
\]

We can lift \( s_c \) to a linear section \( s_c \) of \( \tilde{q}_k^*TE \to \gamma^{(k)} \), so we obtain the following diagram:

\[
\begin{array}{ccc}
p_{k+1}^*TV & \xrightarrow{\tilde{p}_{k+1}^*} & \tilde{q}_k^*TE \\
\gamma^{(k+1)} & \xrightarrow{s_c} & \gamma^{(k)} \\
\downarrow & & \downarrow \\
p_{k+1}^*TG & \xrightarrow{s_c} & q_k^*TM
\end{array}
\]
where the left and the right faces are DVBs and the horizontal arrows form a surjective DVB morphism. Thus we are in the situation of Lemma 1.2.3 and we conclude that there exists a linear section \( \tilde{c} : \mathcal{V}^{(k+1)} \to \tilde{p}_{k+1}^* T\mathcal{V} \) that projects on \( c \), as desired. Summarizing, there is a canonical short exact sequence of cochain maps:

\[
0 \rightarrow \ker \Pi \rightarrow C_{\text{def, lin}}(\mathcal{V}) \xrightarrow{\Pi} C_{\text{def}}(G) \rightarrow 0.
\] (1.12)

Now we describe \( \ker \Pi \). By definition, a \( k \)-cochain \( \tilde{c} : \mathcal{V}^{(k+1)} \to T\mathcal{V} \) is killed by \( \Pi \) if and only if it takes values in the vertical bundle \( T\tilde{s}_\mathcal{V} \) of \( \tilde{s}_\mathcal{V} : \mathcal{V} \rightarrow G \). It is easy to check that \( T\tilde{s}_\mathcal{V} \Rightarrow T\pi E \) is a subgroupoid of \( T\mathcal{V} \Rightarrow TE \). Moreover, \( T\tilde{s}_\mathcal{V} \cong \mathcal{V} \times G \mathcal{V} \) and \( T\pi E \cong E \times_M E \) canonically as vector bundles. Under these isomorphisms, \( T\tilde{s}_\mathcal{V} \) is identified with the groupoid \( \mathcal{V} \times G \mathcal{V} \Rightarrow E \times_M E \) with the component-wise structure maps. We will understand this identification.

It is clear that \( \tilde{c}(v_0, \ldots, v_k) \) has \( v_0 \) as first component, so one can think of \( \tilde{c} \) as a vector bundle map \( \mathcal{V}^{(k+1)} \rightarrow \mathcal{V} \) covering \( p_{k+1} : G^{(k+1)} \rightarrow G \). Then \( \ker \Pi \) is given by elements \( \tilde{c} \in \mathcal{H}om(\mathcal{V}^{(k+1)}, p_{k+1}^* \mathcal{V}) \) such that \( \tilde{s}(c(v_0, \ldots, v_k)) \) does not depend on \( v_0 \) for any \( (v_0, \ldots, v_k) \in \mathcal{V}^{(k+1)} \). Finally, we observe that, on \( \ker \Pi \), the differential is simply given by

\[
\delta \tilde{c}(v_0, \ldots, v_{k+1}) = -\tilde{m}(\tilde{c}(v_0v_1, \ldots, v_{k+1}), \tilde{c}(v_1, \ldots, v_{k+1})) + \sum_{i=1}^k (-1)^{i-1} \tilde{c}(v_0, \ldots, v_i v_{i+1}, \ldots, v_{k+1}) + (-1)^k \tilde{c}(v_0, \ldots, v_k).
\]

### 1.2.2 Trivial-core VB-groupoids

Let \( (\mathcal{V} \Rightarrow E : G \Rightarrow M) \) be a trivial-core VB-groupoid. We want to show that its linear deformation complex has a particularly simple shape. First recall that the total groupoid of a trivial-core VB-groupoid is canonically isomorphic to the action groupoid \( G \ltimes E \Rightarrow E \) associated to a representation of the base groupoid \( G \Rightarrow M \) on the side bundle \( E \). Therefore, without loss of generality, we will assume that \( \mathcal{V} = G \ltimes E \). As a vector bundle over \( G \), it is the pull-back \( s^* E \), and we denote it also by \( E_G \).

Consider a linear cochain \( \tilde{c} \in C^k_{\text{def, lin}}(E_G) \). By definition, it gives a commutative diagram:

\[
\begin{array}{ccc}
E_G^{(k+1)} & \xrightarrow{\tilde{c}} & TE_G \\
\downarrow s_G & & \downarrow s_E \\
E_G^{(k)} & \xrightarrow{s} & TE \\
\downarrow s_G & & \downarrow s_E \\
G^{(k+1)} & \xrightarrow{c} & T\tilde{G} \\
\downarrow s_G & & \downarrow s_E \\
G^{(k)} & \xrightarrow{s} & TM
\end{array}
\] (1.13)
But, for every $k$, there is a canonical isomorphism

$$E_G^{(k)} \cong \mathcal{G}^{(k)} \times_p E, \quad ((g_1, e_1), \ldots, (g_k, e_k)) \mapsto ((g_1, \ldots, g_k); e_k)$$

(1.14)

where $s$ is the projection $\mathcal{G}^{(k)} \to M$, $(g_1, \ldots, g_k) \mapsto s(g_k)$. We will often understand this isomorphism. We also have that $T E_G \cong T \mathcal{G} \times_{T p} T E$. So, we get the following alternative description of (1.13):

$$\mathcal{G}^{(k+1)} \times_p E \xrightarrow{\bar{c}} T \mathcal{G} \mathcal{G}^{(k)} \xrightarrow{\bar{c}} T E \xrightarrow{c} T \mathcal{G} \xrightarrow{s_c} T M$$

(1.15)

where the vertical arrows, except for the front right one, are projections onto the first factor. In particular, $\bar{c}$ is fully determined by $c$ and $s_c$. Set $\tilde{c}_1 := c$ and observe that, for every $(g_1, \ldots, g_k) \in \mathcal{G}^{(k)}$, $s_c((g_1, \ldots, g_k); -)$ is a linear map $E_{s(g_k)} \to T E|_{s_c(g_1, \ldots, g_k)}$. Consider the linear map

$$\tilde{c}_2(g_1, \ldots, g_k): E_{t(g_1)} \to T E|_{s_c(g_1, \ldots, g_k)}, \quad e \mapsto s_c((g_1, \ldots, g_k); (g_1 \cdots g_k)^{-1} \cdot e).$$

(1.16)

It is easy to see that $\tilde{c}_2(g_1, \ldots, g_k)$ splits the projection $T E|_{s_c(g_1, \ldots, g_k)} \to E_{t(g_1)}$. Hence, the pair $(s_c(g_1, \ldots, g_k), \tilde{c}_2(g_1, \ldots, g_k))$ corresponds to a derivation in $D_{t(g_1)} E$ (via the inverse to the correspondence $\delta \mapsto (\sigma_\delta, \hat{\delta})$ discussed in Sect. 1.1.2). We denote by $\tilde{c}_2(g_1, \ldots, g_k)$ again the latter derivation. In this way, we have defined a map $\tilde{c}_2: \mathcal{G}^{(k)} \to DE$ such that

(TC1) $\tilde{c}_2(g_1, \ldots, g_k) \in D_{t(g_1)} E$ for every $(g_1, \ldots, g_k) \in \mathcal{G}^{(k)}$;

(TC2) $\sigma \circ \tilde{c}_2 = s_c \tilde{c}_1$.

Conversely, from a pair $(\tilde{c}_1, \tilde{c}_2)$ with $\tilde{c}_1 \in C^k_{\text{def}}(\mathcal{G})$ and $\tilde{c}_2 : \mathcal{G}^{(k)} \to DE$ satisfying (TC1) and (TC2) above, we can reconstruct a linear deformation cochain $\tilde{c} \in C^k_{\text{def,lin}}(E_G)$ in the obvious way. Finally, a direct computation exploiting [7, Formula (2)] shows that, for every $\tilde{c} \in C^k_{\text{def,lin}}(E_G)$,

$$(\delta \tilde{c})_1 = \delta(\tilde{c}_1)$$

(1.17)
and \((\delta \tilde{c})_2\) is given by the following formula

\[
(\delta \tilde{c})_2(g_1, \ldots, g_{k+1})(e) = -\tilde{c}_1(g_1, \ldots, g_{k+1}) \cdot_T \left( \tilde{c}_2(g_2, \ldots, g_{k+1})(g_1^{-1} e) \right) + \sum_{i=1}^k (-1)^{i-1} \tilde{c}_2(g_1, \ldots, g_i g_{i+1}, \ldots, g_{k+1})(e) + (-1)^k \tilde{c}_2(g_1, \ldots, g_k)(e),
\]

where the \(\cdot_T\) is the tangent map \(\cdot_T : TG \times TM TE \to TE\) to the action \(\tilde{t} : G \times E = \tilde{G} \times_M E \to E\).

The above discussion proves the following

**Lemma 1.2.4** Let \((G \ltimes E = \tilde{E}; G \rightrightarrows M)\) be a trivial-core VB-groupoid and let \(k \geq 0\).

The assignment \(\tilde{c} \mapsto (\tilde{c}_1, \tilde{c}_2)\) establishes an isomorphism between \(C^k_{\text{def,lin}}(E_G)\) and the space of pairs \((\tilde{c}_1, \tilde{c}_2)\) with \(\tilde{c}_1 \in C^k_{\text{def}}(G)\), and \(\tilde{c}_2 : G^{(k)} \to DE\) satisfying (TC1) and (TC2) above. Under this isomorphism the differential \(\delta \tilde{c}\) corresponds to the pair \(((\delta \tilde{c})_1, (\delta \tilde{c})_2)\) given by Formulas (1.17) and (1.18).

Now, take \(\tilde{c} \in C^k_{\text{def,lin}}(E_G)\). Using that \(DE_G \cong TG \times TM DE\) define a map \(\widehat{\tilde{c}} : G^{(k+1)} \to DE_G\) by putting

\[
\widehat{\tilde{c}}(g_0, \ldots, g_k) := (\tilde{c}_1(g_0, \ldots, g_k), \tilde{c}_2(g_1, \ldots, g_k)),
\]

and observe that

- (TC3) \(\widehat{\tilde{c}}(g_0, \ldots, g_k) \in D g_0 E_G\), for all \((g_0, \ldots, g_k) \in G^{(k+1)}\),
- (TC4) there exists a (necessarily unique) smooth map \(G^{(k)} \to DE\) making the following diagram commutative:

\[
\begin{array}{ccc}
G^{(k+1)} & \longrightarrow & DE_G \\
\downarrow & & \downarrow D \delta \\
\tilde{G}^{(k)} & \longrightarrow & DE
\end{array}
\]

where the map on the left is the projection onto the last \(k\) arrows.

Conversely, given a map \(\widehat{\tilde{c}} : G^{(k+1)} \to DE_G\) satisfying (TC3) and (TC4) we can reconstruct \(\tilde{c}_1, \tilde{c}_2\) and hence \(\tilde{c}\). A direct computation exploiting Formulas (1.17) and (1.18) shows that \(\widehat{\delta \tilde{c}} : G^{(k+2)} \to DE_G\) is given by

\[
\widehat{\delta \tilde{c}}(g_0, \ldots, g_{k+1}) = -D \widehat{\delta}(\widehat{\tilde{c}}(g_0 g_1, \ldots, g_{k+1}), \widehat{\tilde{c}}(g_1, \ldots, g_{k+1}))
+ \sum_{i=1}^k (-1)^{i-1} \widehat{\tilde{c}}(g_0, \ldots, g_i g_{i+1}, \ldots, g_{k+1}) + (-1)^k \widehat{\tilde{c}}(g_0, \ldots, g_k),
\]

(1.19)
where we used that \( \tilde{m} : E_{G} s \times s E_{G} \to E_{G} \) is a regular vector bundle map (covering \( \tilde{m} : G s \times s G \to G \)) and \( D(E_{G} s \times s E_{G}) \cong DE_{G} Ds \times Ds DE_{G} \) (we leave the easy details to the reader). Summarizing, we have the following

**Corollary 1.2.5** Let \((G \ltimes E \rightrightarrows E; G \rightrightarrows M)\) be a trivial-core VB-groupoid and let \(k \geq 0\). The assignment \( \tilde{c} \mapsto \hat{\tilde{c}} \) establishes an isomorphism between \( C_{k}^{\text{def,lin}}(G) \) and the space of maps \( \hat{\tilde{c}} : G^{(k+1)} \to DE_{G} \) satisfying (TC3) and (TC4). Under this isomorphism the differential \( \delta \tilde{c} \) corresponds to the map \( \hat{\delta} \hat{\tilde{c}} : G^{(k+2)} \to DE_{G} \) given by Formula (1.19).

Notice the analogy between (1.19) and (1.1).

Finally, in this case, the projection (1.11) is the map \( \tilde{c} \mapsto \tilde{c} \). From the condition (TC2), a cochain in the kernel of this projection is equivalent to a map \( \tilde{c} : G^{(k)} \to DE \) such that \( \sigma \circ \tilde{c} = 0 \), so \( \tilde{c} \) takes values in \( \text{End} E \), and it is easy to see that the sequence (1.12) takes the form:

\[
0 \longrightarrow C(G, \text{End} E) \longrightarrow C_{\text{def,lin}}(G \ltimes E) \longrightarrow C_{\text{def}}(G) \longrightarrow 0,
\]

where \( C(G, \text{End} E) \) is the Lie groupoid complex of \( G \) with coefficients in the representation of \( G \) on \( \text{End} E \) induced by that on \( E \). Therefore, there is a long exact sequence in cohomology:

\[
\cdots \longrightarrow H^{k}(G, \text{End} E) \longrightarrow H_{\text{def,lin}}^{k}(G \ltimes E) \\
\longrightarrow H_{\text{def}}^{k}(G) \longrightarrow H^{k+1}(G, \text{End} E) \longrightarrow \cdots.
\]

We conclude this subsection proving a rigidity result. Recall that a Lie groupoid \( G \rightrightarrows M \) is proper if the map \((s, t) : G \to M \times M \) is proper.

**Proposition 1.2.6** Let \((G \ltimes E \rightrightarrows E; G \rightrightarrows M)\) be a trivial-core VB-groupoid, and suppose that \( G \) is proper. Then \( H_{\text{def,lin}}^{k}(G \ltimes E) = 0 \) for \( k \geq 1 \).

**Proof** Consider the long exact sequence (1.21). In degrees \( k \geq 1 \), both the Lie groupoid cohomology with coefficients [8] and the deformation cohomology [7] of a proper Lie groupoid vanish. It follows that \( H_{\text{def,lin}}^{k}(G \ltimes E) = 0 \) for \( k \geq 1 \). \( \square \)

At the end of Sect. 1.2, we will prove an analogous result for full-core VB-groupoids. (Corollary 1.2.18).

### 1.2.3 Low-degree cohomology groups

Next we describe low-degree cohomology groups. The entire discussion of Sect. 1.1 goes through, with minor changes. We report it here for completeness.

Let \((\mathcal{V} \rightrightarrows E; \mathcal{G} \rightrightarrows M)\) be a VB-groupoid, and let \((W \rightrightarrows E; A \rightrightarrows M)\) be its VB-algebroid. Consider the isotropy bundle \( i \) of \( \mathcal{V} \) and its sections. It is natural to define linear sections of \( i \):

\[
\Gamma_{\text{lin}}(i) := \Gamma(i) \cap \Gamma_{\text{lin}}(W, E)
\]
and linear invariant sections:

\[ H^0_{\text{lin}}(V, i) = \Gamma_{\text{lin}}(i)^{\text{inv}} := \Gamma(i)^{\text{inv}} \cap \Gamma_{\text{lin}}(W, E). \]

The following proposition is easily proved as in [7, Proposition 4.1].

**Proposition 1.2.7** Let \((V \Rightarrow E; G \Rightarrow M)\) be a VB-groupoid. Then \(H^{-1}_{\text{def,lin}}(V) \cong H^0_{\text{lin}}(V, i) = \Gamma_{\text{lin}}(i)^{\text{inv}}\).

Now we consider the complex \(C(V, i)\). It is clear that there is a distinguished subcomplex \(C_{\text{lin}}(V, i)\), the one of cochains \(V^{(k)} \to W\) that are linear over some map \(G^{(k)} \to A\). From a direct computation it follows that the map (1.4) takes \(C_{\text{lin}}(V, i)\) to \(C_{\text{def,lin}}(V)\), so we have a map

\[ r : C_{\text{lin}}(V, i) \hookrightarrow C_{\text{def,lin}}(V). \]

**Proposition 1.2.8** (see also [12, Sect. 2.3])

\[ H^0_{\text{def,lin}}(V) = \frac{\text{linear multiplicative vector fields on } V}{\text{inner linear multiplicative vector fields on } V}. \]

**Proof** Let \(X \in C^0_{\text{def,lin}}(V)\). Then \(X\) is a linear vector field, and from (1.5) it follows that \(X\) is closed if and only if it is multiplicative, and is exact if and only if it is inner multiplicative, as desired. \(\square\)

Recall that also the normal bundle \(\nu\) of \(V\) is defined. Observe that the anchor \(\rho : W \to TE\) of the Lie algebroid \(W\) is a morphism of DVBs, hence it takes linear sections to linear vector fields \(\mathcal{X}_{\text{lin}}(E)\). We set

\[ \Gamma_{\text{lin}}(\nu) := \mathcal{X}_{\text{lin}}(E) / \rho(\Gamma_{\text{lin}}(W, E)). \]

Following the discussion in Sect. 1.1, we declare that a section \([V] \in \Gamma_{\text{lin}}(V)\) is invariant if it possesses an \((s, t)\)-lift \(X \in \mathcal{X}_{\text{lin}}(V)\). The space of invariant linear sections is denoted \(H^0_{\text{lin}}(V, \nu)\) or \(\Gamma_{\text{lin}}(\nu)^{\text{inv}}\). Observing that the projection on \(E\) of a linear multiplicative vector field is linear, we obtain a linear map

\[ \pi : H^0_{\text{def,lin}}(V) \to \Gamma_{\text{lin}}(\nu)^{\text{inv}}. \]

From [7, Lemma 4.9], [7, Proposition 4.11] and their proofs, the “linear versions” follow immediately.

**Lemma 1.2.9** Let \([V] \in \Gamma_{\text{lin}}(\nu)^{\text{inv}}\) and \(X \in \mathcal{X}_{\text{lin}}(V)\) an \((s, t)\)-lift of \(V\). Then \(\delta X \in C^2_{\text{lin}}(V, i)\) and its cohomology class does not depend on the choice of \(X\), hence there is an induced linear map

\[ K : \Gamma_{\text{lin}}(\nu)^{\text{inv}} \to H^2_{\text{lin}}(V, i). \]
Proposition 1.2.10 There is an exact sequence
\[ 0 \to H_{\text{lin}}^1(\mathcal{V}, i) \overset{r}{\to} H_{\text{def, lin}}^0(\mathcal{V}) \overset{\pi}{\to} \Gamma_{\text{lin}}(\nu)^{\text{inv}} \overset{K}{\to} H_{\text{lin}}^2(\mathcal{V}, i) \overset{r}{\to} H_{\text{def, lin}}^1(\mathcal{V}). \]

1.2.4 Deformations

Here we describe deformations of a VB-groupoid \((\mathcal{V} \rightrightarrows E; \mathcal{G} \rightrightarrows M)\) and their relation with the 1-cohomology \(H_{\text{def, lin}}^1(\mathcal{V})\). Let \(B\) be a smooth manifold.

Definition 1.2.11 A family of VB-groupoids over \(B\) is a diagram

\[
\begin{array}{ccc}
\tilde{\mathcal{V}} & \xrightarrow{\sim} & \tilde{\mathcal{E}} \\
\downarrow & & \downarrow \\
\tilde{\mathcal{G}} & \xrightarrow{\sim} & \tilde{\mathcal{M}} \\
\end{array} \xrightarrow{\sim} \begin{array}{ccc} E_b & \xrightarrow{\sim} & B \\
\downarrow & & \downarrow \\
M_b & \xrightarrow{\sim} & B \\
\end{array}
\]

such that the first square is a VB-groupoid and the rows are families of Lie groupoids. In particular, for every \(b \in B\),

\[
\begin{array}{ccc}
\mathcal{V}_b & \xrightarrow{\sim} & E_b \\
\downarrow & & \downarrow \\
\mathcal{G}_b & \xrightarrow{\sim} & M_b \\
\end{array}
\]

is a VB-groupoid.

If \(B\) is an open interval \(I\) containing 0, the family is said to be a deformation of \(\mathcal{V}_0\) and the latter is denoted simply by \((\mathcal{V} \rightrightarrows E; \mathcal{G} \rightrightarrows M)\). A deformation of \(\mathcal{V}\) is also denoted \((\mathcal{V}_\epsilon)\).

Constant deformations are defined as in the plain groupoid case. Two deformations \((\mathcal{V}_\epsilon)\) and \((\mathcal{V}_\epsilon')\) of \(\mathcal{V}\) are called equivalent if there exists a smooth family of VB-groupoid isomorphisms \(\Psi_\epsilon : \mathcal{V}_\epsilon \to \mathcal{V}_\epsilon'\) such that \(\Psi_0 = \text{id}\). We say that \((\mathcal{V}_\epsilon)\) is trivial if it is equivalent to the constant deformation.

We now define a linear version of the deformation class. Recall from [2] that an Ehresmann connection on a Lie groupoid \(\mathcal{G} \rightrightarrows M\) is a splitting of the short exact sequence

\[ 0 \to T^*\mathcal{G} \to T\mathcal{G} \overset{T_\mathcal{S}}{\to} s^*TM \to 0 \]

that restricts to the canonical splitting

\[ 0 \to A \to 1^*T\mathcal{G} \xrightarrow{\text{inv}} TM \to 0 \]
over $M$. Such a connection always exists. Let $(\mathcal{V} \rightrightarrows E, \mathcal{G} \rightrightarrows M)$ be a VB-groupoid. Then there is a diagram of DVB morphisms

\[
\begin{array}{ccccccccc}
0 & \rightarrow & T\tilde{s}\mathcal{V} & \rightarrow & T\mathcal{V} & \rightarrow & T\mathcal{G} & \rightarrow & s^*TM & \rightarrow & 0 \\
0 & \rightarrow & T\tilde{s}\mathcal{G} & \rightarrow & T\mathcal{G} & \rightarrow & s^*TM & \rightarrow & 0 \\
\tilde{G} & \rightarrow & \mathcal{V} & \rightarrow & \mathcal{G} & \rightarrow & \mathcal{V} & \rightarrow & \mathcal{G}
\end{array}
\]

where the top rows are short exact sequences of vector bundles.

**Definition 1.2.12** A linear Ehresmann connection on $\mathcal{V}$ is a morphism of DVBs

\[
(\tilde{s}^*TE \rightarrow s^*TM; \mathcal{V} \rightarrow \mathcal{G}) \rightarrow (T\mathcal{V} \rightarrow T\mathcal{G}; \mathcal{V} \rightarrow \mathcal{G})
\]
such that the maps $\tilde{s}^*TE \rightarrow T\mathcal{V}$ and $s^*TM \rightarrow T\mathcal{G}$ are Ehresmann connections on $\mathcal{V}$ and $\mathcal{G}$, respectively.

**Lemma 1.2.13** On every VB-groupoid $(\mathcal{V} \rightrightarrows E; \mathcal{G} \rightrightarrows M)$ there exists a linear Ehresmann connection.

**Proof** First of all, choose local coordinates on $\mathcal{G}$ adapted to the submersion $s : \mathcal{G} \rightarrow M$. Up to a translation, one can also assume that they are adapted to the immersion $1 : M \rightarrow \mathcal{G}$. Using a right-decomposition $\mathcal{V} \cong s^*E \oplus t^*C$, we find fiber coordinates on $\mathcal{V}$ with analogous properties. Now it is easy to see that linear Ehresmann connections exist locally, and one can conclude with a partition of unity argument.

**Proposition 1.2.14** Let $\tilde{\mathcal{V}}$ be a deformation of $\mathcal{V}$. Then transverse linear vector fields $\tilde{X}$ always exist. Additionally

1. $\delta \tilde{X}$ restricts to $\mathcal{V}$ and the cochain $\xi_0 := \delta \tilde{X} \in C^1_{\text{def, lin}}(\mathcal{V})$ is a cocycle;
2. the cohomology class of $\xi_0$ does not depend on the choice of $\tilde{X}$.

**Proof**

(1) Take a vector field $Y$ on $\tilde{M}$ that projects on $\frac{d}{d\epsilon}$. Choosing a linear connection on $\tilde{E} \rightarrow \tilde{M}$, one can lift it to a linear vector field $\tilde{Y}$ on $\tilde{E}$, that obviously projects on $\frac{d}{d\epsilon}$. Now the choice of a linear Ehresmann connection on $\mathcal{V}$ gives a linear vector field $\tilde{X}$ on $\mathcal{V}$ that projects on $\tilde{Y}$, as desired.

(2) Let $\tilde{X}$ be a transverse linear vector field. Then $\delta \tilde{X} \in C^1_{\text{def, lin}}(\tilde{\mathcal{V}})$ and it restricts to $\mathcal{V}$ as in the plain groupoid case. The restriction $\xi_0 := \delta \tilde{X}|_\mathcal{G}$ clearly belongs to $C^1_{\text{def, lin}}(\mathcal{V})$.

(3) This can be proved as in [7, Proposition 5.12].

The cohomology class $[\xi_0] \in H^1_{\text{def, lin}}(\mathcal{V})$ is called the linear deformation class associated to the deformation $\tilde{\mathcal{V}}$. From the last proposition, it follows directly that this
class is also independent of the equivalence class of the deformation. In particular, the linear deformation class of a trivial deformation is trivial. Hence, from Proposition 1.2.6, the linear deformation class of every deformation of a trivial-core VB-groupoid with proper base is trivial.

Remark 1.2.15 Clearly, a deformation $\tilde{V}$ of the VB-groupoid $V$ induces a deformation $\tilde{G}$ of the base groupoid $G$. Now, it is easy to check that the projection (1.11) sends the linear deformation class of $\tilde{V}$ to the deformation class of $\tilde{G}$.

Finally, we discuss the variation map associated to a family of VB-groupoids. Let $(\tilde{V} \Rightarrow \tilde{E}; \tilde{G} \Rightarrow \tilde{M})$ be a family of VB-groupoids over a smooth manifold $B$. Then any curve $\gamma : I \to B$ induces a deformation $\gamma^* \tilde{V}$ of $\tilde{V}_{\gamma(0)}$, and we have:

Proposition 1.2.16 (The linear variation map) Let $b \in B$. For any curve $\gamma : I \to B$ with $\gamma(0) = b$, the deformation class of $\gamma^* \tilde{V}$ at time 0 does only depend on $\dot{\gamma}(0)$. This defines a linear map

$$\text{Var}_{lin,b}^{\tilde{V}} : T_b B \to H^1_{\text{def}, \text{lin}}((\tilde{V})_b),$$

called the linear variation map of $\tilde{V}$ at $b$, that makes the following diagram commutative:

$$\begin{array}{ccc}
T_b B & \xrightarrow{\text{Var}_{lin,b}^{\tilde{V}}} & H^1_{\text{def}, \text{lin}}((\tilde{V})_b) \\
\downarrow & & \downarrow \\
H^1_{\text{def}}(\tilde{G}_b) & \xleftarrow{\text{Var}_b^G} & H^1_{\text{def}}((\tilde{V})_b)
\end{array}$$

Proof The first statement is proved as in [7, Proposition 5.15], while the second statement trivially follows from Remark 1.2.15.

1.2.5 Deformations of the dual VB-groupoid

We conclude this section noticing that the linear deformation cohomology of a VB-groupoid is canonically isomorphic to that of its dual.

Theorem 1.2.17 Let $(\mathcal{V} \Rightarrow E; \mathcal{G} \Rightarrow M)$ be a VB-groupoid. Then there is a canonical isomorphism

$$H_{\text{def}, \text{lin}}(\mathcal{V}) \cong H_{\text{def}, \text{lin}}(\mathcal{V}^*)$$

(1.22)

defined $H(\mathcal{G})$-modules.

Proof Using Proposition 1.2.1 and Lemma 1.2.2, we get:

$$H_{\text{def}, \text{lin}}(\mathcal{V}) \cong H_{\text{proj}, \text{lin}}(T^*\mathcal{V})[1] \cong H_{\text{lin}, \text{lin}}(T^*\mathcal{V})[1].$$

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For the same reason, $H_{\text{def,lin}}(V^*) \cong H_{\text{lin,lin}}(T^*V^*)[1]$. But we have already noticed that $T^*V \cong T^*V^*$ as double vector bundles and Lie groupoids, so we obtain (1.22). □

As the dual of a full-core VB-groupoid is a trivial-core VB-groupoid, from Proposition 1.2.6 we immediately get:

**Corollary 1.2.18** Let $(V \rightrightarrows 0_M; G \rightrightarrows M)$ be a full-core VB-groupoid with proper base. Then $H_{\text{def,lin}}^k(V) = 0$ for $k \geq 1$. In particular, the linear deformation class of every deformation of $V$ is trivial.

### 1.3 The linearization map

Let $(V \rightrightarrows E; G \rightrightarrows M)$ be a VB-groupoid. We have shown that deformations of the VB-groupoid structure are controlled by a subcomplex $C_{\text{def,lin}}(V)$ of the deformation complex $C_{\text{def}}(V)$ of the top Lie groupoid $V \rightrightarrows E$. In this section, we prove that there is a canonical splitting of the inclusion $C_{\text{def,lin}}(V) \hookrightarrow C_{\text{def}}(V)$ in the category of cochain complexes, the *linearization map*. This will imply, in particular, that the inclusion induces an injection in cohomology $H_{\text{def,lin}}(V) \hookrightarrow H_{\text{def}}(V)$.

The procedure we are going to describe is analogous to the one we used in [16] to define the linearization of sections of a DVB. As usual, denote by $h$ the homogeneity structure of $V \rightrightarrows G$. For every $\lambda > 0$, $h_\lambda$ is a groupoid automorphism of $V \rightrightarrows E$. By definition the action of $h_\lambda$ on $C_{\text{def,lin}}^{-1}(V) = \Gamma(W, E)$ coincides with that induced by the homogeneity structure of $W \rightrightarrows A$. For $k \geq 0$, $C_{\text{def}}^k(V) \subset \Gamma(p_{k+1}^*T^*V, \gamma^{(k+1)})$, and a direct computation shows that the action of $h_\lambda$ on $C_{\text{def}}^k(V)$ coincides with that induced by the homogeneity structure of the vector bundle $p_{k+1}^*T^*V \rightrightarrows p_{k+1}^*T^*G$ on sections of $\tilde{p}_{k+1}^*T^*V \rightrightarrows \gamma^{(k+1)}$. Therefore, by [16, Propositions 1.4.1, 1.4.2], for every $\tilde{c} \in C_{\text{def}}(V)$, the limits

$$\tilde{c}_{\text{core}} := \lim_{\lambda \to 0} (\lambda \cdot h_\lambda^*\tilde{c})$$

$$\tilde{c}_{\text{lin}} := \lim_{\lambda \to 0} (h_\lambda^*\tilde{c} - \lambda^{-1} \cdot \tilde{c}_{\text{core}})$$

are well-defined. The latter equation defines a linear map

$$\text{lin} : C_{\text{def}}(V) \to C_{\text{def,lin}}(V), \quad \tilde{c} \mapsto \tilde{c}_{\text{lin}},$$

which we call the *linearization map*.

**Theorem 1.3.1** The map (1.23) is a cochain map that splits the inclusion $C_{\text{def,lin}}(V) \hookrightarrow C_{\text{def}}(V)$. Hence $C_{\text{def,lin}}(V)$ is a direct summand of $C_{\text{def}}(V)$ and the inclusion induces an injection in cohomology:

$$H_{\text{def,lin}}(V) \hookrightarrow H_{\text{def}}(V).$$

**Proof** We only need to prove that the linearization map respects the differential. As $h_\lambda$ is an automorphism of $V \rightrightarrows E$ for every $\lambda$, we have that $h_\lambda^*$ commutes with $\delta$. It is

\[\square\]
also clear that $\delta$ preserves limits, so we compute
\[(\delta\tilde{c})_{\text{core}} = \lim_{\lambda \to 0} (\lambda \cdot h_\lambda^*(\delta\tilde{c})) = \lim_{\lambda \to 0} (\lambda \cdot \delta(h_\lambda^*\tilde{c})) = \delta(\lim_{\lambda \to 0} \lambda \cdot \delta(h_\lambda^*\tilde{c})) = \delta\tilde{c}_{\text{core}},\]
\[(\delta\tilde{c})_{\text{lin}} = \lim_{\lambda \to 0} (h_\lambda^*(\delta\tilde{c}) - \lambda^{-1}(\delta\tilde{c})_{\text{core}}) = \delta\left(\lim_{\lambda \to 0} (h_\lambda^*\tilde{c} - \lambda^{-1}\delta\tilde{c}_{\text{core}})\right) = \delta\tilde{c}_{\text{lin}}\]
and we are done. \(\square\)

**Remark 1.3.2** Applying the isomorphisms (1.9) and (1.10), we obtain that $H_{\text{proj, lin}}(T^*\mathcal{V})$ is a direct summand of $H_{\text{proj, lin}}(T^*\mathcal{V})$: it identifies with classes in $H_{\text{proj, lin}}(T^*\mathcal{V})$ which can be represented by cochains that are linear over $\mathcal{V}^*$.

Finally, we discuss a first consequence of Theorem 1.3.1. We call $\widehat{C}_{\text{def, lin}}(\mathcal{V}) := C_{\text{def, lin}}(\mathcal{V}) \cap \widehat{C}_{\text{def}}(\mathcal{V})$ the linear normalized deformation complex of $\mathcal{V}$.

**Proposition 1.3.3** The inclusion $\widehat{C}_{\text{def, lin}}(\mathcal{V}) \hookrightarrow C_{\text{def, lin}}(\mathcal{V})$ is a quasi-isomorphism.

**Proof** Take $c \in C^k_{\text{def, lin}}(\mathcal{V})$. As, by [7, Proposition 11.8], the inclusion $\widehat{C}_{\text{def}}(\mathcal{V}) \hookrightarrow C_{\text{def}}(\mathcal{V})$ is a quasi-isomorphism, there exist $\tilde{c} \in \widehat{C}_{\text{def}}(\mathcal{V})$ and $c' \in C^{k-1}_{\text{def}}(\mathcal{V})$ such that $c - \tilde{c} = \delta c'$. Applying the linearization map, we get $c - \tilde{c}_{\text{lin}} = \delta c'_{\text{lin}}$ and $\tilde{c}_{\text{lin}} \in \widehat{C}^{k}_{\text{def, lin}}(\mathcal{V})$, as desired. \(\square\)

Other applications of the linearization map will be considered in the next sections.

### 1.4 The van Est map

The van Est theorem is a classical result relating the differentiable cohomology of a Lie group and the Chevalley–Eilenberg cohomology of its Lie algebra [23, 24]. It was later extended to differentiable cohomology [8, 25] and deformation cohomology [7] of a Lie groupoid, and to the VB-cohomology of a VB-groupoid [6]. In this subsection, we want to prove an analogous theorem for the linear deformation cohomology of a VB-groupoid.

Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid, and let $A \Rightarrow M$ be its Lie algebroid. The normalized deformation complex of $\mathcal{G}$ and the deformation complex of $A$ are intertwined by the *van Est map*, defined as follows. Given a section $\alpha \in \Gamma(A)$, recall from [7] the map $R_\alpha : \widehat{C}^k(\mathcal{G}) \to \widehat{C}^{k-1}(\mathcal{G})$ given by
\[R_\alpha(c) = [c, \overset{\alpha}{\overrightarrow{\delta}}]|_M\] (1.24)
if $k = 0$, and
\[R_\alpha(c)(g_1, \ldots, g_k) = (-1)^k \frac{d}{d\epsilon}|_{\epsilon=0} c(g_1, \ldots, g_k, \Phi_\epsilon^\alpha(s(g_k))^{-1})\] (1.25)
if $k > 0$, where $\Phi_\epsilon^\alpha(x) = \Phi_{\epsilon, \alpha}^\alpha(1_x)$ for every $x \in M$: the image of $1_x$ under the flow $\{\Phi_{\epsilon, \alpha}^\alpha\}$ of the right invariant vector field $\overset{\alpha}{\overrightarrow{\delta}}$ associated to $\alpha$. Then the van Est map $\text{VE} : \widehat{C}_{\text{def}}(\mathcal{G}) \to C_{\text{def}}(A)$ (1.26)
is given by:

\[
\text{VE}(c)(\alpha_0, \ldots, \alpha_k) = \sum_{\tau \in S_{k+1}} (-1)^\tau (R_{\alpha_{\tau(k)}} \circ \cdots \circ R_{\alpha_{\tau(0)}})(c).
\]  

The van Est map \( \text{VE} \) is a cochain map [7]. Moreover, if \( \mathcal{G} \) has \( k \)-connected \( s \)-fibers, it induces an isomorphism in cohomology in all degrees \( p < k \) [7, Theorem 10.1]. We are going to prove an analogous result for the linear deformation complex of a VB-groupoid. To do this, we need a simple preliminary lemma. Before the statement, notice that, for a Lie groupoid \( G \Rightarrow M \), with Lie algebroid \( A \Rightarrow M \), the group \( \text{Aut}(G) \) of automorphisms of \( G \) acts both on \( \hat{C}^{k}_{\text{def}}(G) \) and on \( C^{k}_{\text{def}}(A) \), via the Lie functor. As it might be expected, we have the following

**Lemma 1.4.1** The van Est map (1.26) is equivariant with respect to the action of \( \text{Aut}(G) \).

**Proof** For a Lie groupoid automorphism \( \Psi \in \text{Aut}(G) \) we denote by \( \psi \in \text{Aut}(A) \) the Lie algebroid automorphism corresponding to \( \Psi \), via the Lie functor. We will prove that

\[
\Psi^*(R_\alpha(c)) = R_{\psi^*\alpha}(\Psi^*c)
\]  

for all \( c \in \hat{C}^k_{\text{def}}(G) \), \( \alpha \in \Gamma(A) \), \( \Psi \in \text{Aut}(G) \). If \( k = 0 \), we have

\[
\Psi^*(R_\alpha(c)) = \Psi^*((c, \overrightarrow{\alpha})|_M) = (\Psi^*[c, \overrightarrow{\alpha}])|_M = [\Psi^*c, \Psi^*\overrightarrow{\alpha}]|_M
\]

\[
= [\Psi^*c, \psi^*\alpha]|_M = R_{\psi^*\alpha}(\Psi^*c).
\]

Now, let \( k > 0 \). Then

\[
\Psi^*(R_\alpha(c))(g_1, \ldots, g_k)
\]

\[
= T \Psi^{-1}(R_\alpha(c)(\Psi(g_1), \ldots, \Psi(g_k)))
\]

\[
= T \Psi^{-1}\left( (-1)^k \frac{d}{d\epsilon} \bigg|_{\epsilon=0} c(\Psi(g_1), \ldots, \Psi(g_k), \Phi_\epsilon^\alpha(s(\Psi(g_k)))^{-1}) \right)
\]  

(1.29)

Using that \( \Psi^*\overrightarrow{\alpha} = \psi^*\alpha \), we compute:

\[
\Phi_\epsilon^\alpha(s(\Psi(g_k))) = \Phi_\epsilon^\alpha(1_{s(\Psi(g_k))}) = \Phi_\epsilon^\alpha(\Psi(1_{s(g_k)}))
\]

\[
= \Psi(\Phi_\epsilon^{\psi^*\alpha}(1_{s(g_k)})) = \Psi(\Phi_\epsilon^{\psi^*\alpha}(s(g_k)))
\]  

\[\square\] Springer
So, from (1.29) we have:

\[
\Psi^*(R_\alpha(c))(g_1, \ldots, g_k) = (-1)^k \frac{d}{d\epsilon} \bigg|_{\epsilon=0} \Psi^{-1}(c(\Psi(g_1), \ldots, \Psi(g_k), \Psi(\Phi_\epsilon^\ast \alpha(s(g_k))))^{-1}) \\
= (-1)^k \frac{d}{d\epsilon} \bigg|_{\epsilon=0} (\Psi^*c)(g_1, \ldots, g_k, \Phi_\epsilon^\ast \alpha(s(g_k)))^{-1} \\
= R_{\psi^\ast \alpha}(\Psi^*c)(g_1, \ldots, g_k).
\]

Finally, by applying repeatedly Formula (1.28) in (1.27), we obtain

\[
\Psi^*(VE(c)) = VE(\Psi^*c)
\]

for every \( c \in \hat{C}_{\text{def}}(G) \), as desired.

Now we are ready for the main theorem of this section. Notice that the first part of the following statement has been already proved in [12] in the special case where the rank of \( E \) is greater than zero. Here we provide an alternative proof, which is valid in all cases, exploiting Lemma 1.4.1.

**Theorem 1.4.2** (Linear van Est map) Let \((\mathcal{V} \rightrightarrows E; \mathcal{G} \rightrightarrows M)\) be a VB-groupoid, and let \((W \rightrightarrows E; A \rightrightarrows M)\) be its VB-algebroid. Then the van Est map for the Lie groupoid \(\mathcal{V} \rightrightarrows E\) restricts to a cochain map

\[
VE : \hat{C}_{\text{def, lin}}(\mathcal{V}) \to C_{\text{def, lin}}(W),
\]

which we call the linear van Est map. If \(\mathcal{G}\) has \(k\)-connected \(s\)-fibers, this map induces an isomorphism in cohomology in all degrees \(p < k\).

**Proof** As before, we denote by \(h\) the homogeneity structure of \(\mathcal{V} \rightrightarrows \mathcal{G}\). Then the last proposition shows that

\[
h^\ast_\lambda(VE(\tilde{c})) = VE(h^\ast_\lambda \tilde{c})
\]

for every \(\tilde{c} \in \hat{C}_{\text{def}}(\mathcal{V}), \lambda > 0\). But the VB-algebroid automorphism corresponding to \(h_\lambda\) is exactly the one induced by the homogeneity structure of \(W \rightrightarrows A\), so the last equation implies that the van Est map preserves linear cochains.

Now we prove the second part of the theorem. First, we have to observe that the van Est map commutes with linearization, i.e.

\[
VE(\tilde{c})_{\text{lin}} = VE(\tilde{c}_{\text{lin}}).
\]

To see this, take \(\tilde{c} \in C^k_{\text{def, lin}}(\mathcal{V})\) and \(w_0, \ldots, w_k \in \Gamma(W, E)\) and compute

\[
VE(\tilde{c})_{\text{lin}}(w_0, \ldots, w_k) = \lim_{\lambda \to 0} \bigg( h^\ast_\lambda(VE(\tilde{c})) - \lambda^{-1}VE(\tilde{c}_{\text{core}}) \bigg)(w_0, \ldots, w_k) \\
= \lim_{\lambda \to 0} \bigg( h^\ast_\lambda \tilde{c} - \lambda^{-1}\tilde{c}_{\text{core}} \bigg)(w_0, \ldots, w_k) \\
= \lim_{\lambda \to 0} \sum_{\tau \in S_{k+1}} (-1)^\tau (R_{w_{\tau(k)}} \circ \cdots \circ R_{w_{\tau(0)}})(h^\ast_\lambda \tilde{c} - \lambda^{-1}\tilde{c}_{\text{core}}).
\]

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We would like to swap the limit and the derivatives that appear in definitions (1.24) and (1.25). This is ultimately possible because of smoothness, and we get

\[ \text{VE}(\tilde{c})_{\text{lin}}(w_0, \ldots, w_k) = \sum_{\tau \in S_{k+1}} (-1)^{\tau} (R_{w_{\tau(k)}} \circ \cdots \circ R_{w_{\tau(0)}}) \left( \lim_{\lambda \to 0} (h_{\lambda}^* \tilde{c} - \lambda^{-1} \tilde{c}_{\text{core}}) \right) \]

\[ = \sum_{\tau \in S_{k+1}} (-1)^{\tau} (R_{w_{\tau(k)}} \circ \cdots \circ R_{w_{\tau(0)}})(\tilde{c}_{\text{lin}}) \]

\[ = \text{VE}(\tilde{c}_{\text{lin}})(w_0, \ldots, w_k) \]

as desired.

Finally, suppose that \( \mathcal{G} \) has \( k \)-connected \( s \)-fibers. Then \( \mathcal{V} \) has \( k \)-connected \( \tilde{s} \)-fibers (they are vector bundles over the \( s \)-fibers of \( \mathcal{G} \)). Take \( p < k \). We want to prove that the induced map

\[ \text{VE} : H_{\text{def,lin}}^p(\mathcal{V}) \to H_{\text{def,lin}}^p(\mathcal{W}) \]

is an isomorphism. If \( \tilde{c} \in C_{\text{def,lin}}^p(\mathcal{V}) \) is closed, we denote \([\tilde{c}]\) its class in \( H_{\text{def}}^p(\mathcal{V}) \) and \([\tilde{c}]_{\text{lin}} \) its class in \( H_{\text{def,lin}}^p(\mathcal{V}) \); we use an analogous notation for \( \mathcal{W} \).

First, suppose that \([\text{VE}(\tilde{c})]_{\text{lin}} = 0\). Then \([\text{VE}(\tilde{c})] = 0\) and, by [7, Theorem 10.1], \([\tilde{c}] = 0\), i.e. \( \tilde{c} = \delta \tilde{\gamma} \) for some \( \tilde{\gamma} \in C_{\text{def}}^{p-1}(\mathcal{V}) \). Applying the linearization map, we get \( \tilde{c} = \delta \gamma_{\text{lin}} \) and \( \gamma_{\text{lin}} \in C_{\text{def}}^{p-1}(\mathcal{V}) \), i.e. VE is injective in degree \( p \). To conclude, take \([c]_{\text{lin}} \in H_{\text{def,lin}}^p(\mathcal{W}) \). Then \([c] \in H_{\text{def}}^p(\mathcal{W}) \), so \([c] = [\text{VE}(\tilde{c})] \), i.e. \( c - \text{VE}(\tilde{c}) = \delta \gamma \) for some \( \tilde{c} \in \tilde{C}_{\text{def}}^p(\mathcal{V}) \), \( \gamma \in C_{\text{def}}^{p-1}(\mathcal{W}) \). Applying again the linearization map and using (1.30), we get \( c - \text{VE}(\tilde{c}_{\text{lin}}) = \delta \gamma_{\text{lin}} \), i.e. VE is also surjective in degree \( p \), as desired.

\( \square \)

1.5 Morita invariance

In this subsection we assume the reader is familiar with Morita equivalence of Lie groupoids. For details we refer to [4, 11]. The notion of Morita equivalence of VB-groupoids first appears in [10]. In that reference, the authors prove that the VB-cohomologies of Morita equivalent VB-groupoids are isomorphic. As a corollary, they give a conceptual and very simple proof of the fact, first appeared in [7], that Morita equivalent Lie groupoids have isomorphic deformation cohomologies. This second result means that the deformation cohomology of a Lie groupoid is in fact an invariant of the associated differentiable stack. In this subsection, we want to prove an analogous result for the linear deformation cohomology of a VB-groupoid.

Let \( (\mathcal{V}_1 \rightrightarrows E_1; \mathcal{G}_1 \rightrightarrows M_1) \) and \( (\mathcal{V}_2 \rightrightarrows E_2; \mathcal{G}_2 \rightrightarrows M_2) \) be VB-groupoids. A VB-groupoid morphism \( \Psi : \mathcal{V}_1 \to \mathcal{V}_2 \) is a VB-Morita map [10] if the Lie groupoid morphism \( \Psi \) is a Morita map. The VB-groupoids \( \mathcal{V}_1 \) and \( \mathcal{V}_2 \) are Morita equivalent if there exist a VB-groupoid \( \mathcal{W} \) and VB-Morita maps \( \mathcal{W} \to \mathcal{V}_1, \mathcal{W} \to \mathcal{V}_2 \). VB-cohomology is Morita invariant in the sense that a VB Morita map \( \Psi : \mathcal{V}_1 \to \mathcal{V}_2 \) induces an isomorphism \( \Psi^* : H_{\text{proj}}(\mathcal{V}_2) \to H_{\text{proj}}(\mathcal{V}_1) \) [10, Theorem 4.2].
**Theorem 1.5.1** Let \( \Psi : \mathcal{V}_1 \to \mathcal{V}_2 \) be a VB-Morita map. Then \( H_{\text{def, lin}}(\mathcal{V}_1) \cong H_{\text{def, lin}}(\mathcal{V}_2) \).

**Proof** It is enough to show that \( H_{\text{proj, lin}}(T^*\mathcal{V}_1) \cong H_{\text{proj, lin}}(T^*\mathcal{V}_2) \). To do this we will use linearization together with some properties of VB Morita maps from [10].

Recall that both \( T^*\mathcal{V}_1 \) and \( T^*\mathcal{V}_2 \) have two VB-groupoid structures, as discussed in Sect. 1.2, and observe that \( \Psi_{\ast}(T^*\mathcal{V}_2) \) possesses also two VB-groupoid structures, that fit in the following commuting diagram:

\[
\begin{array}{ccc}
\Psi^\ast(T^*\mathcal{V}_2) & \longrightarrow & \Psi^\ast(W_2^\ast|_{E_2}) \\
\downarrow & & \downarrow \\
\mathcal{V}_1 & \longrightarrow & E_1 \\
\downarrow & & \downarrow \\
\mathcal{V}_2^\ast & \longrightarrow & \Psi^\ast C_2^* \\
\mathcal{G}_1 & \longrightarrow & M_1
\end{array}
\]

We denote by \( C_{\text{proj, lin}}(\Psi^\ast(T^*\mathcal{V}_2)) \) the VB-complex of the VB-groupoid upstairs and by \( C_{\text{proj, lin}}(\Psi^\ast(T^*\mathcal{V}_2)) \) its subcomplex of cochains that are linear with respect to the vertical projections. As usual, we denote their cohomologies by \( H_{\text{proj, lin}}(\Psi^\ast(T^*\mathcal{V}_2)) \) and \( H_{\text{proj, lin}}(\Psi^\ast(T^*\mathcal{V}_2)) \) respectively. Now, from [6] we know that there is a linearization map

\[
\text{lin} : C(\Psi^\ast(T^*\mathcal{V}_2)) \to C_{\text{lin}}(\Psi^\ast(T^*\mathcal{V}_2)), \quad c \mapsto c_{\text{lin}},
\]

that is a cochain map splitting the inclusion \( C_{\text{lin}}(\Psi^\ast(T^*\mathcal{V}_2)) \hookrightarrow C(\Psi^\ast(T^*\mathcal{V}_2)) \). If \( \cdot \Psi_{\ast}\mathcal{V}_2^\ast \) denotes the scalar multiplication in the fibers of \( \Psi^\ast(T^*\mathcal{V}_2) \to \Psi^\ast C_2^* \), \( c \in C^k(\Psi^\ast(T^*\mathcal{V}_2)) \), then \( c_{\text{lin}} \) is simply given by

\[
c_{\text{lin}}(\theta_1, \ldots, \theta_k) = \left. \frac{d}{d\lambda} \right|_{\lambda=0} c(\lambda \cdot \Psi_{\ast}\mathcal{V}_2^\ast(\theta_1, \ldots, \theta_k)).
\]

From the definition, it is clear that \( \text{lin} \) restricts to a cochain map

\[
\text{lin} : C_{\text{proj, lin}}(\Psi^\ast(T^*\mathcal{V}_2)) \to C_{\text{proj, lin}}(\Psi^\ast(T^*\mathcal{V}_2)),
\]

so \( H_{\text{proj, lin}}(\Psi^\ast(T^*\mathcal{V}_2)) \) embeds in \( H_{\text{proj, lin}}(\Psi^\ast(T^*\mathcal{V}_2)) \) as a direct summand.

The map \( \Psi \) is VB-Morita, so, from [10, Corollary 3.8], \( T\Psi : T\mathcal{V}_1 \to T\mathcal{V}_2 \) is VB-Morita again. It follows from [10, Corollary 3.7] and [10, Corollary 3.9] that the dual map \( T\Psi^\ast : \Psi^\ast(T^*\mathcal{V}_2) \to T^*\mathcal{V}_1 \) is VB-Morita as well. Remember that also the canonical map \( \Psi^\ast(T^*\mathcal{V}_2) \to T^*\mathcal{V}_2 \) is VB-Morita ( [10, Corollary 3.7] again). As a result, we get isomorphisms in VB-cohomology:

\[
H_{\text{proj, lin}}(T^*\mathcal{V}_1) \xrightarrow{\cong} H_{\text{proj, lin}}(\Psi^\ast(T^*\mathcal{V}_2)) \xleftarrow{\cong} H_{\text{proj, lin}}(T^*\mathcal{V}_2).
\]
Now, the maps
\[ T^*\mathcal{V}_1 \xleftarrow{\Psi} \Psi^*(T^*\mathcal{V}_2) \rightarrow T^*\mathcal{V}_2 \]
are also DVB morphisms. This implies, on one hand, that the maps (1.31) preserve linear cohomologies, on the other hand that they commute with the respective linearization maps. From this last property we deduce, as in Theorem 1.4.2, that the maps (1.31) induce isomorphisms on linear cohomologies, so
\[ H_{\text{proj},\text{lin}}(T^*\mathcal{V}_1) \cong H_{\text{proj},\text{lin}}(\Psi^*(T^*\mathcal{V}_2)) \cong H_{\text{proj},\text{lin}}(T^*\mathcal{V}_2) \]
as desired.

Remark 1.5.2 The deformation cohomology of a Lie groupoid \( \mathcal{G} \) is Morita invariant [7] (see also [10]). This means that the deformation cohomology is an invariant of the differentiable stack represented by \( \mathcal{G} \). Similarly, according to Theorem 1.5.1, the linear deformation cohomology of a VB-groupoid \( \mathcal{V} \rightarrow \mathcal{G} \) is VB-Morita invariant. In this sense, the linear deformation cohomology is an invariant of the vector bundle in differentiable stacks represented by \( \mathcal{V} \). This suggests a relationship to deformation theory of (vector bundles in) differentiable stacks. Namely, it is clear that a deformation of \( \mathcal{G} \) (resp., of \( \mathcal{V} \)) should induce a, generically non-trivial, deformation of the differentiable stack (resp., the vector bundle in differentiable stacks). So the following questions, raised by one of the referees, that we thank, is very natural:

1. which deformation cohomology classes in \( H^1_{\text{def}}(\mathcal{G}) \) do correspond to (non-trivial deformations of \( \mathcal{G} \) but) trivial deformations of the differentiable stack?
2. which linear deformation cohomology classes in \( H^1_{\text{def},\text{lin}}(\mathcal{V}) \) do correspond to (non-trivial deformations of \( \mathcal{V} \) but) trivial deformations of the vector bundle in differentiable stacks?
3. which linear deformation cohomology classes in \( H^1_{\text{def},\text{lin}}(\mathcal{V}) \) do correspond to trivial deformations of the base differentiable stack?

Answering these questions goes far beyond the aims of the present paper, and we believe it is actually a new, and extremely interesting, research line. The main issue is the lack in the literature of a deformation theory even for plain differentiable stacks, not to speak of vector bundles in differentiable stacks. At this stage, we don’t even have yet in the literature a definition of what a deformation of a differentiable stack is. We were actually informed by João Nuno Mestre that he is currently working on this and related problems but his research is still at the very early stages. The following observations, including the cited literature, were communicated to us by him [18]. In principle, one could define deformations of differentiable stacks paralleling what is done for Artin stacks [1], which can be considered the Algebraic Geometry counterparts of the former (see also [19, 22]). But notice that this might pose some difficulties due to the fact that the usual differential geometric language and the usual algebraic geometric language for stacks do not match exactly and one usually needs to adopt an appropriate dictionary to pass from one to the other [4]. The same dictionary should be probably used to relate deformations of Lie groupoids (and VB groupoids) and deformations of...
the associated differentiable stacks (and vector bundles in differentiable stacks). See [1] again for a relation between deformations of algebraic groupoids and deformations of the associated algebraic stack.

2 Examples and applications

In this section we provide several examples. Examples in Sects. 2.1, 2.4 and 2.5 parallel the analogous examples in [7], connecting our linear deformation cohomology to known cohomologies, while examples in Sects. 2.2 and 2.3 are specific to VB-groupoids. The infinitesimal counterparts of all these examples were discussed in our previous paper [16].

2.1 VB-groups and their duals

A VB-group is a vector bundle object in the category of Lie groups. In other words, it is a VB-groupoid of the form

$$\begin{array}{ccc}
K & \longrightarrow & 0 \\
\downarrow & & \downarrow \\
G & \longrightarrow & * \\
\end{array}$$

In particular, $K$ and $G$ are Lie groups. Let $C := \ker p$ be the core of $(K \rightrightarrows 0; G \rightrightarrows *)$. The map $p$ has a canonical section in the category of Lie groups, the zero section $0 : G \to K$. By standard Lie group theory, $G$ acts on the core $C$ via

$$g \cdot c = 0_g \cdot c \cdot 0_g^{-1}$$

and there is a canonical isomorphism of Lie groups

$$K \cong G \ltimes C, \quad k \mapsto (p(k), k \cdot 0_{p(k)}^{-1}),$$

where $G \ltimes C$ is the semidirect product Lie group. Thus VB-groups are equivalent to Lie group representations.

It is then natural to study the relationship between the linear deformation complex of $K$ and the classical complex $C(G, \text{End} C)$ (of the Lie group $G$ with coefficients in the representation $\text{End} C$) that controls deformations of the $G$-module $C$ [20]. To do this, we notice that the dual of $K$ is the VB-groupoid

$$\begin{array}{ccc}
G \ltimes C^* & \longrightarrow & C^* \\
\downarrow & & \downarrow \\
G & \longrightarrow & * \\
\end{array}$$
i.e. it is the action VB-groupoid associated to the dual representation of $G$ on $C^*$. In particular, it is a trivial-core VB-groupoid, so there is a short exact sequence of cochain complexes:

$$0 \longrightarrow C(G, \text{End } C^*) \longrightarrow C_{\text{def, lin}}(G \ltimes C^*) \longrightarrow C_{\text{def}}(G) \longrightarrow 0.$$  

But $C(G, \text{End } C^*) \cong C(G, \text{End } C)$ canonically, so the latter is recovered as the sub-complex of $C_{\text{def, lin}}(G \ltimes C^*)$ controlling deformations of the representation $C^*$ that fix the Lie group structure on $G$. Moreover, $H_{\text{def, lin}}(G \ltimes C^*) \cong H_{\text{def, lin}}(K)$, so there is a long exact sequence in cohomology:

$$\cdots \longrightarrow H^k(G, \text{End } C) \longrightarrow H^k_{\text{def, lin}}(K) \longrightarrow H^k_{\text{def}}(G) \longrightarrow H^{k+1}(G, \text{End } C) \longrightarrow \cdots.$$  

Finally, suppose that $G$ is compact. Then Proposition 1.2.6 applies, and we get the following

**Corollary 2.1.1** Let $(K \Rightarrow 0; G \Rightarrow *)$ be a VB-group with compact base. Then $H^k_{\text{def, lin}}(K) = 0$ for $k \geq 1$.

### 2.2 2-Vector spaces

A 2-vector space is a (Lie) groupoid object in the category of vector spaces. In other words, it is a VB-groupoid of the form

$$\begin{array}{c}
V_1 \\
\downarrow \quad \downarrow \\
\ast \\
V_0 \\
\ast
\end{array}$$

In [3] it is proved that, if $s$ and $t$ are the source and the target maps of $V_1 \Rightarrow V_0$, $C = \ker s$, $\partial = t|_C : C \to V_0$, then $V_1 \Rightarrow V_0$ is canonically isomorphic to the action groupoid $C \ltimes V_0 \Rightarrow V_0$, where $C$ acts on $V_0$ by

$$c \cdot v = \partial c + v.$$  

Notice that $C$ does not act by linear isomorphisms, but by translations. We will identify $V_1$ with $C \ltimes V_0$.

Now we compute the linear deformation complex of $(V_1 \Rightarrow V_0; \ast \Rightarrow \ast)$. First of all, as $V_1 \Rightarrow V_0$ is an action groupoid, Equation (1.14) yields an isomorphism $(C \ltimes V_0)^{(k)} \cong C^k \oplus V_0$. We will understand this isomorphism. Recall also that $s : C \times V_0 \to V_0$ is just the projection onto the second factor. It follows that, for $k \geq 0$, $C_{\text{def, lin}}^{V_1}$ is the set of linear maps $C^{k+1} \oplus V_0 \to C \oplus V_0$ such that the second component does not depend on the first arrow, hence these maps are equivalent to pairs of linear maps $C^{k+1} \oplus V_0 \to C$ and $C^k \oplus V_0 \to V_0$. So

$$C_{\text{def, lin}}^{V_1} \cong \text{Hom}(C^{k+1} \oplus V_0, C) \oplus \text{Hom}(C^k \oplus V_0, V_0).$$  

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We will identify a deformation cochain $\gamma$ with the corresponding pair of linear maps $(\gamma_1, \gamma_2)$. A direct computation shows that the differential is given by the following formulas. For $\gamma \in C_{\text{def},\text{lin}}^{-1}(V_1)$

$$\delta \gamma = (\gamma \circ \partial, \partial \circ \gamma),$$

and, for $\gamma \in C_{\text{def},\text{lin}}^k(V_1), k \geq 0$,

$$\delta \gamma = (\Gamma_1, \Gamma_2),$$

where

$$\Gamma_1(c_0, \ldots, c_{k+1}, v)$$

$$= -\gamma_1(c_0, 0, \ldots, 0) + \sum_{i=1}^{k} (-1)^{i-1} \gamma_1(c_0, \ldots, c_i + c_{i+1}, \ldots, c_{k+1}, v)$$

$$+ (-1)^k \gamma_1(c_0, \ldots, c_k, c_{k+1} \cdot v),$$

and

$$\Gamma_2(c_1, \ldots, c_{k+1}, v)$$

$$= -\gamma_1(c_1, \ldots, c_{k+1}, v) \cdot \gamma_2(c_2, \ldots, c_{k+1}, v)$$

$$+ \sum_{i=1}^{k} (-1)^{i-1} \gamma_2(c_1, \ldots, c_i + c_{i+1}, \ldots, c_{k+1}, v)$$

$$+ (-1)^k \gamma_2(c_1, \ldots, c_k, c_{k+1} \cdot v).$$

We further notice that, if $\gamma \in C_{\text{def},\text{lin}}^k(V_1), k \geq 0$, then $\gamma = (\gamma_1, \gamma_2)$ belongs to the normalized deformations subcomplex $\widehat{C}_{\text{def},\text{lin}}(V_1)$ if and only if

$$\gamma_1(c_0, \ldots, c_i, \ldots, c_k, v) = 0 \text{ for every } i \geq 0,$$

$$\gamma_2(c_1, \ldots, c_i, \ldots, c_k, v) = 0 \text{ for every } i \geq 1.$$

It follows that $\widehat{C}_{\text{def},\text{lin}}(V_1)$ reduces to

$$0 \longrightarrow \text{Hom}(V_0, C)[1] \xrightarrow{\delta_0} \text{End} C \oplus \text{End} V_0 \xrightarrow{\delta_1} \text{Hom}(C, V_0)[-1] \longrightarrow 0 \quad (2.1)$$

with

$$\delta_0 \gamma = (\gamma \circ \partial, \partial \circ \gamma),$$

$$\delta_1 (\gamma_1, \gamma_2) = \gamma_2 \circ \partial - \partial \circ \gamma_1.$$
So the linear deformation cohomology of $V_1$ is:

\[ H_{\text{def}, \text{lin}}^{-1}(V_1) = \text{Hom}(\ker \partial, \text{coker} \partial), \]
\[ H_{\text{def}, \text{lin}}^{0}(V_1) = \text{End}(\text{coker} \partial) \oplus \text{End}(\ker \partial), \]
\[ H_{\text{def}, \text{lin}}^{1}(V_1) = \text{Hom}(\ker \partial, \text{coker} \partial). \]

Finally, notice that the VB-algebroid of $V_1$ is an $\text{LA-vector space}$ [16] of the form $(V_1 \Rightarrow V_0; 0 \Rightarrow *)$. In [16] we showed that the linear deformation complex of $(V_1 \Rightarrow V_0; 0 \Rightarrow *)$ is again (2.1), and it is easy to show, for example in coordinates, that the van Est map

\[ \text{VE} : \widehat{C}_{\text{def}, \text{lin}}(V_1 \Rightarrow V_0) \to C_{\text{def}, \text{lin}}(V_1 \Rightarrow V_0) \]

is simply the identity.

### 2.3 Tangent and cotangent VB-groupoids

Let $G \rightrightarrows M$ be a Lie groupoid. We want to relate the linear deformation cohomology of $T \overline{G}$ with the deformation cohomology of $\overline{G}$. First recall that there is a projection

\[ \Pi : C_{\text{def}, \text{lin}}(T \overline{G}) \to C_{\text{def}}(\overline{G}). \]  

(2.2)

Now define an inclusion

\[ \iota : C_{\text{def}}(\overline{G}) \to C_{\text{def}, \text{lin}}(T \overline{G}), \quad c \mapsto \iota_c := J \circ Tc \]

where $J : TT \overline{G} \to TT \overline{G}$ is the canonical involution of the DVB $TT \overline{G}$ (see, e.g., [17] for definition and basic properties). For later use, we recall that, if $\pi : T \overline{G} \to \overline{G}$ is the canonical projection, the canonical involution is an automorphism of the DVB $TT \overline{G}$ that inverts the two projections $T \pi : TT \overline{G} \to T \overline{G}$ and $\pi_{T \overline{G}} : TT \overline{G} \to T \overline{G}$:

\[ J \circ T \pi = \pi_{T \overline{G}} \circ J. \]  

(2.3)

The inclusion $\iota$ is well defined, i.e., if $c \in C_{\text{def}}^k(\overline{G})$, then $\iota_c \in C_{\text{def}, \text{lin}}^k(T \overline{G})$. To see this we show that properties def-(1) and def-(2) in the definition of a deformation cochain at the beginning of Sect. 1.1 (see p. 3) hold. So, let $\epsilon \mapsto (g_0(\epsilon), \ldots, g_k(\epsilon))$ be a curve in $\overline{G}^{(k+1)}$ defined around 0. Then $(\pi \circ \iota)(g_0(\epsilon), \ldots, g_k(\epsilon)) = g_1(\epsilon)$. Differentiating at $\epsilon = 0 = 0$ we get

\[ (T \pi \circ Tc)(\dot{g}_0(0), \ldots, \dot{g}_k(0)) = \ddot{g}_0(0). \]

Applying $J$ and remembering Eq. (2.3) we get

\[ \pi_{T \overline{G}}(\iota_c(\dot{g}_0(0), \ldots, \dot{g}_k(0))) = \ddot{g}_0(0), \]
i.e. \( \iota_c(\dot{g}_0(0), \ldots, \dot{g}_k(0)) \in T_{\dot{g}_0(0)}T\mathcal{G} \) as desired.

Now notice that property def-(2) can be expressed in the following way: if \( p : \mathcal{G}^{(k+1)} \to \mathcal{G}^{(k)} \) is the map that forgets the first arrow, then \( T \circ c \) descends to a map \( \mathcal{G}^{(k)} \to T\mathcal{M} \):

\[
\begin{array}{ccc}
\mathcal{G}^{(k+1)} & \xrightarrow{T\circ c} & T\mathcal{M} \\
\downarrow{\rho} & & \downarrow{}
\end{array}
\]

Differentiating, we obtain that also \( TT\circ c : T\mathcal{G}^{(k+1)} \to TT\mathcal{M} \) descends to a map \( T\mathcal{G}^{(k)} \to TT\mathcal{M} \). Applying \( J \) and remembering that it commutes with \( TT\circ c \), we obtain the same statement for \( TT\circ \iota_c \), as desired. Moreover, \( \iota_c \) is obviously linear.

A direct computation shows that \( \iota \) is a cochain map. Finally, we observe that the following diagram commutes:

\[
\begin{array}{ccc}
T\mathcal{G}^{(k+1)} & \xrightarrow{T\circ c} & TT\mathcal{G} \\
\downarrow{\pi_{T\mathcal{G}}} & & \downarrow{TT\pi} \\
\mathcal{G}^{(k+1)} & \xrightarrow{c} & T\mathcal{G}
\end{array}
\]

This shows that \( \iota \) inverts the projection (2.2). It follows that

\[
C_{\text{def, lin}}(T\mathcal{G}) \cong C_{\text{def}}(\mathcal{G}) \oplus \ker \Pi
\]

as cochain complexes, hence

\[
H_{\text{def, lin}}(T\mathcal{G}) = H_{\text{def, lin}}(T^*\mathcal{G}) = H_{\text{def}}(\mathcal{G}) \oplus H(\ker \Pi).
\]

### 2.4 Representations of foliation groupoids

A foliation groupoid is a Lie groupoid whose anchor map is injective. This condition ensures that the connected components of the orbits of the groupoid are the leaves of a regular foliation of the base manifold, whence the name. On the other hand, foliation groupoids encompass several classical groupoids associated to a foliated manifold, such as the holonomy and the monodromy groupoids.

Representations of a foliation groupoid \( \mathcal{G} \) are equivalent to trivial-core VB-groupoids over \( \mathcal{G} \). Here we want to study the linear deformation cohomology of such VB-groupoids. First of all, let \( \mathcal{G} \Rightarrow M \) be a foliation groupoid, let \( A \Rightarrow M \) be its Lie algebroid, \( \rho : A \to T\mathcal{M} \) the (injective) anchor map, \( \nu = T\mathcal{M} / \text{im} \rho \) the normal bundle and let \( \pi : T\mathcal{M} \to \nu \) be the projection. In this case, \( \nu \) has constant rank and the normal representation is a plain representation of \( \mathcal{G} \) on \( \nu \). Moreover, we recall from...
Consider a representation $E \to M$ of $G$ and construct the associated trivial-core VB-groupoid $(\mathcal{G} \times E \rightrightarrows E; \mathcal{G} \rightrightarrows M)$. At the infinitesimal level, there is an induced representation of $A$ on $E$, i.e. an $A$-flat connection $\nabla : A \to DE$, and $\nabla$ is injective because $\rho$ is so. Therefore, the cokernel $\tilde{\nu} = DE / \text{im } \nabla$ is a vector bundle over $M$. Denote by

$$\tilde{\pi} : DE \to \tilde{\nu}, \quad \delta \mapsto \bar{\delta}$$

the projection.

We want to show that, in this situation, $G$ acts on $\tilde{\nu}$. To see this, recall that the group $\text{Bis}(\mathcal{G})$ of bisections of $\mathcal{G}$ acts on $\Gamma_1(E)$ via

$$(\beta \star \epsilon)_x = \beta((t \circ \beta)^{-1}(x)) \cdot \epsilon((t \circ \beta)^{-1}(x)), \quad (2.4)$$

so it also acts on derivations of $E$ by

$$(\beta \bullet \Delta)(\epsilon) = \beta \star (\Delta(\beta^{-1} \star \epsilon)). \quad (2.5)$$

If $\beta$ is a local bisection around $x$ and $\beta(x) = g : x \to y$, these formulas still make sense: Eq. (2.4) shows that the action of $\beta$ takes local sections around $y$ to local sections around $x$, Eq. (2.5) shows that $\beta$ acts on derivations locally defined around $x$ (to give a derivation locally defined around $y$).

Now, let $g : x \to y$ be an arrow in $\mathcal{G}$ and $\delta \in D_x E$. Choose a local bisection $\beta$ of $\mathcal{G}$ passing through $g$ and a derivation $\Delta \in \mathcal{D}(E)$ such that $\Delta_x = \delta$. Our action is then defined by

$$g \cdot \bar{\delta} = \bar{\beta \bullet \Delta}|_y.$$  

A routine computation shows that the definition does not depend on the choice of $\Delta$. Let us prove that this definition is also independent of the choice of $\beta$. This is equivalent to prove that, if $\beta \in \text{Bis}(\mathcal{G})$, $\beta_z = 1_z$ for some $z \in M$, then there exists $\alpha \in \Gamma(A)$ such that

$$(\Delta - \beta \bullet \Delta - \nabla\alpha)_z = 0.$$  

Consider the vector field $\sigma_{\Delta - \beta \star \Delta}$. We have

$$\xi := (\sigma_{\Delta - \beta \star \Delta})_z = (\sigma_\Delta - (t \circ \beta)_*(\sigma_\Delta))_z = \sigma_{\Delta_z} - T(t \circ \beta)(\sigma_{\Delta_z}). \quad (2.6)$$

But $t \circ \beta$ preserves the orbits of $\mathcal{G}$, hence it maps a sufficiently small neighborhood of $z$ in the leaf $\mathcal{L}_z$ of $\text{im } \rho$ through $z$ to itself. It then follows from (2.6) that $\xi$ kills all the functions that are constant along $\mathcal{L}_z$, hence it belongs to $\text{im } \rho$.  

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Now, let \( \alpha \in \Gamma(A) \) be any section such that \( \rho(\alpha z) = \xi \) and put \( D := \Delta - \beta \cdot \Delta - \nabla_{\alpha} \).
By construction, \( \sigma_{Dz} = 0 \), so it suffices to show that \( Dz \) vanishes on \( \nabla \)-flat sections. If \( \varepsilon \) is such a section, then
\[
Dz \varepsilon = (\Delta - \beta \cdot \Delta)z \varepsilon = \Delta_z (\varepsilon - \beta^{-1} \ast \varepsilon).
\] (2.7)

But the hypothesis \( \nabla \varepsilon = 0 \) implies that \( \varepsilon \) is invariant under the action of \( \text{Bis}(\mathcal{G}) \), at least locally around \( z \), and the claim follows from (2.7).

Notice that the symbol map \( \sigma : DE \to TM \) descends to a \( \mathcal{G} \)-equivariant map \( \tilde{\nu} \to \nu \), and the fact that \( \text{End} E \cap \text{im} \nabla = 0 \) implies that its kernel is again \( \text{End} E \). Hence we have a short exact sequence of vector bundles with \( \mathcal{G} \)-action:
\[
0 \longrightarrow \text{End} E \longrightarrow \tilde{\nu} \longrightarrow \nu \longrightarrow 0.
\]

In turn, this induces a short exact sequence of DG-modules:
\[
0 \longrightarrow C(\mathcal{G}, \text{End} E) \longrightarrow C(\mathcal{G}, \tilde{\nu}) \longrightarrow C(\mathcal{G}, \nu) \longrightarrow 0.
\]

But \( \mathcal{G} \ltimes E \) is also a trivial-core VB-groupoid, so we also have the sequence (1.20):
\[
0 \longrightarrow C(\mathcal{G}, \text{End} E) \longrightarrow C(\text{def}, \text{lin}(\mathcal{G} \ltimes E)) \longrightarrow C(\text{def}(\mathcal{G})) \longrightarrow 0.
\]

We are looking for a map relating the two sequences. If \( \tilde{c} \in C^k(\text{def}, \text{lin}(\mathcal{G} \ltimes E)) \) and \( \tilde{c}_2 : \mathcal{G}^{(k)} \to DE \) is the map (1.16), we can simply define:
\[
\tilde{p} : C(\text{def}, \text{lin}(\mathcal{G} \ltimes E)) \to C(\mathcal{G}, \tilde{\nu}), \quad \tilde{c} \mapsto \tilde{\pi} \circ \tilde{c}_2.
\]

This is a cochain map and we obtain the following commutative diagram:
\[
\begin{array}{cccccc}
0 & \longrightarrow & C(\mathcal{G}, \text{End} E) & \longrightarrow & C(\text{def}, \text{lin}(\mathcal{G} \ltimes E)) & \longrightarrow & C(\text{def}(\mathcal{G})) & \longrightarrow & 0 \\
& & \| & \downarrow \tilde{p} & & \downarrow p & & \| & & \\
0 & \longrightarrow & C(\mathcal{G}, \text{End} E) & \longrightarrow & C(\mathcal{G}, \tilde{\nu}) & \longrightarrow & C(\mathcal{G}, \nu) & \longrightarrow & 0
\end{array}
\]

The rows are short exact sequences of DG-modules and the vertical arrows are DG-module surjections; additionally, \( p \) is a quasi-isomorphism. Hence, it immediately follows from the Snake Lemma and the Five Lemma that \( \tilde{p} \) is a quasi-isomorphism as well. We have thus proved the following

**Proposition 2.4.1** There is a canonical isomorphism between the linear deformation cohomology of the VB-groupoid \( (\mathcal{G} \ltimes E \Rightarrow E; \mathcal{G} \Rightarrow M) \) and the Lie groupoid cohomology with coefficients in \( \tilde{\nu} \):
\[
H_{\text{def}, \text{lin}}(\mathcal{G} \ltimes E) = H(\mathcal{G}, \tilde{\nu}).
\]
2.5 Lie group actions on vector bundles

Let $G$ be a Lie group with Lie algebra $\mathfrak{g}$. Assume that $G$ acts on a vector bundle $E \rightarrow M$ by vector bundle automorphisms. Then $G$ acts also on $M$ and $(G \ltimes E \rightrightarrows E; G \ltimes M \rightrightarrows M)$ is a trivial-core VB-groupoid. We want to discuss its linear deformation cohomology.

Of course the action of $G$ on $M$ induces an infinitesimal action of $\mathfrak{g}$ on $M$, and the Lie algebroid of $G \ltimes M$ is the action algebroid $\mathfrak{g} \ltimes M$. We recall from [7] that $G \ltimes M$ acts naturally on $\mathfrak{g} \ltimes M$, by extending the adjoint action of $G$ on $\mathfrak{g}$, and on $TM$ by differentiating the action of $G$ on $M$. Moreover, there is a short exact sequence of complexes:

$$0 \rightarrow C(G \ltimes M, TM) \rightarrow C_{\text{def}}(G \ltimes M) \rightarrow C(G \ltimes M, \mathfrak{g} \ltimes M)[1] \rightarrow 0.$$  

The projection $p$ is defined as follows. Take a $k$-cochain $c : (G \ltimes M)^{(k+1)} \rightarrow T(G \ltimes M)$ in the deformation complex of $G \ltimes M$ and let $(h_0, \ldots, h_k) \in (G \ltimes M)^{(k+1)}$, $h_0 = (g, x)$. Then

$$c(h_0, \ldots, h_k) \in T_{(g_0, x_0)}(G \ltimes M) \cong T_{g_0}G \times T_{x_0}M \cong \mathfrak{g} \times T_{x_0}M$$

via right translations, and we compose with the projection $TM \rightarrow M$ to get an element in $\mathfrak{g} \ltimes M$. The kernel of $p$ is given by $TM$-valued cochains. Since the $TM$-component is the projection by the source map, it does not depend on the first component, and we conclude that the kernel is $C(G \ltimes M, TM)$.

We want to construct a similar sequence for $C_{\text{def, lin}}(G \ltimes E)$ taking into account the linear nature of the action. First of all, there is an obvious induced action of $G$ on $DE$ and the symbol map $\sigma : DE \rightarrow TM$ is $G$-equivariant. Hence there is a short exact sequence of cochain complexes:

$$0 \rightarrow C(G \ltimes M, \text{End } E) \rightarrow C(G \ltimes M, DE) \rightarrow C(G \ltimes M, TM) \rightarrow 0.$$  

In this case, the sequence (1.20) reads

$$0 \rightarrow C(G \ltimes M, \text{End } E) \rightarrow C_{\text{def, lin}}(G \ltimes E) \rightarrow C_{\text{def}}(G \ltimes M) \rightarrow 0$$

and, composing $\Pi$ with $p$, we get a cochain map

$$C_{\text{def, lin}}(G \ltimes E) \rightarrow C(G \ltimes M, \mathfrak{g} \ltimes M)[1] \rightarrow 0. \quad (2.8)$$

Applying the isomorphism (1.14), we get $(G \ltimes E)^{(k)} \cong G^k \ltimes E$, and similarly $(G \ltimes M)^{(k)} \cong G^k \ltimes M$. Then if $\tilde{c} \in C_{\text{def, lin}}^k(G \ltimes E)$ is a linear cochain, the diagram (1.15) takes the following form:

\[ \]
Moreover, $\tilde{c}$ is in the kernel of $\text{(2.8)}$ if and only if its $TG$-component is 0. In this case, it is clear that $\tilde{c}$ is determined by $s_\tilde{c}$, that is in turn equivalent to the map $\tilde{c}_2 : G^k \times M \to DE$ defined by (1.16). Therefore, the kernel of $\text{(2.8)}$ is $C(G \times M, DE)$.

Summarizing there is an exact diagram of cochain complexes

$$
\begin{array}{ccccccccc}
0 & & 0 & & 0 & & 0 & & 0 & & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & C(G \times M, \text{End } E) & \to & C(G \times M, \text{End } E) & \to & 0 & & 0 & & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & C(G \times M, DE) & \to & C_{\text{def,lin}}(G \times E) & \to & C(G \times M, g \times M)[1] & \to & 0 & & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & C(G \times M, TM) & \to & C_{\text{def}}(G \times M) & \to & C(G \times M, g \times M)[1] & \to & 0 & & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0 & & 0 & & 0 & & 0
\end{array}
$$

This proves the following

**Proposition 2.5.1** Let $G$ be a Lie group acting on a vector bundle $E \to M$ by vector bundle automorphisms. The linear deformation cohomology of the VB-groupoid $(G \times E \Rightarrow E, G \times M \Rightarrow M)$ fits in the exact diagram:

$$
\begin{array}{ccccccccc}
\cdots & \to & H^k(G \times M, \text{End } E) & \to & H^k(G \times M, \text{End } E) & \to & 0 & \to & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\cdots & \to & H^k(G \times M, DE) & \to & H^k_{\text{def,lin}}(G \times E) & \to & H^{k+1}(G \times M, g \times M) & \to & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\cdots & \to & H^k(G \times M, TM) & \to & H^k_{\text{def}}(G \times M) & \to & H^{k+1}(G \times M, g \times M) & \to & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\cdots & \to & H^{k+1}(G \times M, \text{End } E) & \to & H^{k+1}(G \times M, \text{End } E) & \to & 0 & \to & \cdots \\
\end{array}
$$

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References

1. Aoki, M.: Deformation theory of algebraic stacks. Compos. Math. 141, 19–34 (2005)
2. Arias Abad, C., Crainic, M.: Representations up to homotopy and Bott’s spectral sequence for Lie
groupoids. Adv. Math. 248, 416–452 (2013)
3. Baez, J.C., Crans, A.S.: Higher-dimensional algebra VI: Lie 2-algebras. Theory Appl. Categ. 12,
492–528 (2004)
4. Behrend, K., Xu, P.: Differentiable stacks and gerbes. J. Sympl. Geom. 9, 285–341 (2011)
5. Bursztyn, H., Cabrera, A., del Hoyo, M.: Vector bundles over Lie groupoids and algebroids. Adv. Math.
290, 163–207 (2016)
6. Cabrera, A., Drummond, T.: Van Est isomorphism for homogeneous cochains. Pacific J. Math. 287,
297–336 (2017)
7. Crainic, M., Mestre, J. N., Struchinerm I.: Deformations of Lie groupoids. Int. Math. Res. Not. IMRN
7662–7746 (2020)
8. Crainic, M.: Differentiable and algebroid cohomology, van Est isomorphisms, and characteristic
classes. Commun. Math. Helv. 78(4), 681–721 (2003)
9. Crainic, M., Moerdijk, I.: Deformations of Lie brackets: cohomological aspects. J. Eur. Math. Soc.
287, 1037–1059 (2008)
10. del Hoyo, M., Ortiz, C.: Morita equivalences of vector bundles, Int. Math. Res. Not. IMRN 2020,
4395–4432
11. del Hoyo, M.: Lie groupoids and their oribispaces. Port. Math. 70, 161–209 (2013)
12. Esposito, C., Tortorella, A.G., Vitagliano, L.: Infinitesimal automorphisms of VB-groupoids and alge-
broids. Q. J. Math. 70, 1039–1089 (2019)
13. Grabowski, J., Rotkiewicz, M.: Higher vector bundles and multi-graded symplectic manifolds. J. Geom.
Phys. 59, 1285–1305 (2009)
14. Gracia-Saz, A., Mehta, R.A.: Lie algebroid structures on DVBs and representation theory of Lie
algebroids. Adv. Math. 223, 1236–1275 (2010)
15. Gracia-Saz, A., Mehta, R.A.: VB-groupoids and representation theory of Lie groupoids. J. Sympl.
Geom. 15, 741–783 (2017)
16. La Pastina, P.P., Vitagliano, L.: Deformations of linear Lie brackets. Pacific J. Math. 303, 265–298
(2019)
17. Mackenzie, K.C.H.: General Theory of Lie Groupoids and Algebroids. Cambridge Univ. Press, Cam-
bridge (2005)
18. Mestre, J. N.: Private communication
19. Nagai, Y., Sato, F.: Deformation of a smooth Delign–Mumford stack via differential graded Lie algebra.
J. Algebra 320, 3481–3492 (2008)
20. Nijenhuis, A., Richardson, R.W.: Deformations of homomorphisms of Lie groups and Lie algebras.
Bull. Am. Math. Soc. 73, 175–179 (1967)
21. Pradines, J.: Remarque sur le groupoide cotangent de Weinstein-Dazord. C.R. Acad. Sci. Paris Sér. I Math. 306, 557–560 (1988)
22. Pridham, J.P.: Derived deformations of Artin stacks. Commun. Anal. Geom. 23, 419–477 (2015)
23. van Est, W.T.: Group cohomology and Lie algebra cohomology in Lie groups I, II. Proc. Kon. Ned. Akad. 56, 484–504 (1953)
24. van Est, W.T.: On the algebraic cohomology concepts in Lie groups I, II. Proc. Kon. Ned. Akad. 58(225–233), 286–294 (1955)
25. Weinstein, A., Xu, P.: Extensions of symplectic groupoids and quantization. J. Reine Angew. Math. 417, 159–189 (1991)

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