Stronger Lower Bounds for Polynomial Time Problems

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Abstract

We reinvestigate the classical topic of limited nondeterminism, to prove stronger conditional lower bounds for polynomial time problems. In particular, we show that CircuitSAT for circuits with $m$ gates and $\log(m)$ inputs (denoted by log-CircuitSAT) is not solvable in quasilinear time unless the exponential time hypothesis (ETH) is false. In other words, a polynomial time improvement for log-CircuitSAT would lead to an exponential time improvement for NP-complete problems.

1 Introduction

Developing a deeper understanding of polynomial time problems is essential to the fields of algorithm design and computational complexity theory. In this work, we build on prior concepts from the topic of limited nondeterminism to show a new kind of conditional lower bound for polynomial time problems where a polynomial time improvement for one problem leads to an exponential time improvement for another. We proceed by introducing basic notions and explaining how they relate to existing research work.

A polynomial time problem is a decision problem that can be decided in $O(n^k)$ time for some constant $k$, where $n$ denotes the input length. As usual, $\mathbf{P}$ denotes the class of polynomial time problems. A decision problem has an unconditional time complexity lower bound $t(n)$ if it cannot be decided in less than $t(n)$ time. Polynomial time problems with non-trivial unconditional time complexity lower bounds do not commonly appear in complexity theory research (aside from lower bounds for one-tape Turing machines [21,28]). Such problems are known to exist by the time hierarchy theorem [20], but to the best of our knowledge, there are few examples that appear in the literature. Most of the
known examples are related to pebbling games \[4\] or intersection non-emptiness for automata \[33, 34\]. For these examples, the unconditional lower bounds are proven by combining Turing machine simulations with classical diagonalization arguments.

Although unconditional lower bounds are rare, many polynomial time problems have been shown to have conditional lower bounds in recent works on fine-grained complexity theory (see surveys \[38, 7\]). The primary goal of this work is to introduce stronger conditional lower bounds by applying new relationships between deterministic and nondeterministic computations.

2 Background

2.1 Conditional Lower Bounds

Fine-grained complexity theory is a subject focused on exact runtime bounds and conditional lower bounds. A conditional lower bound for a polynomial time problem typically takes the following form. Polynomial time problem $A$ is not solvable in $O(n^{\alpha - \varepsilon})$ time for all $\varepsilon > 0$ assuming that problem $B$ is not solvable in $O(t(n)^{\beta - \varepsilon})$ time for all $\varepsilon > 0$ where $\alpha$ and $\beta$ are constants and $t(n)$ is a function (typically $t(n)$ is either polynomial or exponential). This is referred to as a conditional lower bound because problem $A$ has a lower bound under the assumption that $B$ has a lower bound. Conditional lower bounds are known for many polynomial time problems including Triangle Finding, Orthogonal Vectors Problem (OVP), and 3SUM \[38, 7\].

There are some known cases where conditional lower bounds can be strengthened. In particular, it was shown that some polynomial problems have lower bounds conditioned on a disjunction of assumptions \[2\]. A conditional lower bound of this type takes the following form. Polynomial time problem $A$ is not solvable in $O(n^{\alpha - \varepsilon})$ time for all $\varepsilon > 0$ assuming that a family of problems $\{B_i\}_{i \in [k]}$ satisfies the property that for some $i \in [k]$, $B_i$ is not solvable in $O(t_i(n)^{\beta - \varepsilon})$ time for all $\varepsilon > 0$. In other words, problem $A$ has a lower bound under the assumption that at least one of the $B_i$ problems has a lower bound.

In this paper, we demonstrate a much stronger conditional lower bound. A quasilinear time problem is typically defined as a decision problem that can be decided in $O(n \cdot \text{poly}(\log(n)))$ time \[32\]. In this work, we relax this definition to include problems that can be decided in $O(n^{1+\varepsilon})$ time for all $\varepsilon > 0$. We provide an example of a polynomial time problem $A$ such that $A$ is not solvable in quasilinear time assuming that problem $B$ is not solvable in $O(t(n)^{\varepsilon})$ time for some $\varepsilon > 0$. In other words, a polynomial improvement for the runtime of $A$ leads to an exponential improvement\[4\] for the runtime of $B$. In order to prove this stronger lower bound, we must carefully explore relationships between deterministic and nondeterministic computations.

\[1\] By exponential improvement, we mean that when $t(n) = 2^{O(n)}$, the runtime improves from exponential time $2^{O(n)}$ to subexponential time $2^{o(n)}$. In other words, the improvement breaks the barrier between a linear and sublinear exponent.
2.2 Exponential Time Hypothesis

Many classic problems related to Boolean formulas are well known to be computationally hard [13, 27]. As a result, satisfiability of Boolean formulas (SAT) is a natural candidate for lower bound assumptions. In particular, it is common to focus on satisfiability of Boolean formulas in conjunctive normal form with clause width at most \( k \) (\( k \)-CNF-SAT) for a fixed \( k \).

The exponential time hypothesis (ETH) states that for some \( \varepsilon > 0 \), \( 3 \)-CNF-SAT cannot be solved in \( O(\text{poly}(n) \cdot 2^{\varepsilon \cdot v}) \) time, where \( n \) denotes the input size and \( v \) denotes the number of variables [22]. The strong exponential time hypothesis (SETH) states that for every \( \varepsilon > 0 \), there is a sufficiently large \( k \) such that \( k \)-CNF-SAT cannot be solved in \( O(\text{poly}(n) \cdot 2^{(1-\varepsilon) \cdot v}) \) time [22, 23, 11].

Conditional lower bounds are frequently shown relative to the \( k \)-CNF-SAT problems. For example, it is well known that the Orthogonal Vectors Problem (OVP) on polylogarithmic length vectors is not solvable in \( O(n^2 - \varepsilon) \) time for all \( \varepsilon > 0 \) assuming that for all \( \varepsilon > 0 \), there exists \( k \) sufficiently large such that \( k \)-CNF-SAT is not solvable in \( O(\text{poly}(n) \cdot 2^{(1-\varepsilon) \cdot v}) \) time [35, 36, 7]. Moreover, Orthogonal Vectors Problem (OVP) is not solvable in \( O(n^{2-\varepsilon}) \) time for all \( \varepsilon > 0 \) unless SETH is false.

Remark 1. As far as we know, the current best reduction shows that an \( O(n^a) \) time algorithm for OVP leads to a roughly \( O(\text{poly}(n) \cdot 2^{a \cdot v^2}) \) time algorithm for \( k \)-CNF-SAT for all \( k \) [25, 28, 7]. Furthermore, we do not yet know whether the existence of a quasilinear time algorithm for OVP would imply ETH is false.

We also consider at a generalization of SAT called the satisfiability of Boolean circuits (CircuitSAT) decision problem. In particular, we focus on the satisfiability of bounded fan-in Boolean circuits with \( m \) gates and \( \log(m) \) inputs (log-CircuitSAT) which is also considered in [8, 3].

2.3 Limited Nondeterminism

A nondeterministic polynomial time problem is a decision problem that can be decided in polynomial time by a nondeterministic algorithm or machine. There are multiple ways to introduce nondeterminism into a computation. For example, a machine could have nondeterministic bits written on a tape in advance or it could make nondeterministic guesses as the computation is carried out. Such variations don’t appear to make much difference when defining nondeterministic polynomial time (NP). However, when considering limited nondeterminism, the definitions will require special care and attention.

Limited nondeterminism is simply the restriction or bounding of the amount of nondeterminism in a computation. A survey by Goldsmith, Levy, and Mundhenk [18] provides a useful summary on this topic. We proceed by discussing some key notions and results from prior works.

Kintala and Fischer [25, 26] defined \( P_{f(n)} \) as the class of languages that can be decided by a polynomial time bounded machine \( M \) such that \( M \) scans at most \( f(n) \) cells of a nondeterministic guess tape for each input of size \( n \).
Álvarez, Díaz and Torán [5, 15], making explicit a concept of Xu, Doner, and Book [29], then defined $\beta_k$ as the class of languages that can be decided by a polynomial-time bounded machine $M$ such that $M$ uses at most $O((\log(n))^k)$ bits of nondeterminism. Farr took a similar approach [16], defining $f(n)$-NP as the languages that can be decided by a polynomial-time bounded machine $M$ with a binary guess tape alphabet such that $M$ scans at most $f(q(n))$ cells of the guess tape for each input of size $n$. Here $q$ is a polynomial that depends only on $M$. Note that $f(n)$-NP is the union over all $k$ of the classes $P_{f(n^k)}$. Another related approach was taken by Buss and Goldsmith [8] where $N^{mP_l}$ is defined as the class of languages decided by nondeterministic machines in quasi-$n^l$ time making at most $m \cdot \log(n)$ nondeterministic guesses.

Cai and Chen [10] then focused on machines that partition access to nondeterminism, by first creating the contents of the guess tape, and then using a deterministic machine to check the guess. In this terminology, $GC(s(n), C)$ is the class of languages that can be decided by a machine that guesses $O(s(n))$ bits and then uses the power of class $C$ to verify. A related definition followed in Santhanam [31] where $NTIGU(t(n), g(n))$ is the class of languages that can be decided by a machine that makes $g(n)$ guesses and runs for $t(n)$ time.

It follows from the definitions that

$$ P = P_{\log(n)} = NTIGU(P, \log(n)) = N^{O(1)}P_{O(1)} = GC(O(1), P) = \beta_1 $$

and

$$ NP = n^*\cdot NP = NTIGU(P, P) = GC(n^{O(1)}, P) = P_{n^{O(1)}}. $$

Furthermore, the $\beta_k$ classes are meant to capture classes between $P$ and $NP$.

In this work, we focus on the log-CircuitSAT problem and the levels within $P = \beta_1$. It’s worth noting that a loosely related work [17] investigated the log-Clique problem which is in $\beta_2$.

## 3 Trade-offs Between Time and Nondeterminism

### 3.1 Motivation

By introducing a new notion of limited nondeterminism, we are able to prove new relationships between deterministic and nondeterministic computations. In particular, if more efficient deterministic algorithms exist, then there are elegant trade-offs between time and nondeterminism.

In order to prove these relationships we need to pay close attention to the details of our model for limited nondeterminism combining various elements from the existing models from the literature and introducing new elements. In particular, for our model (like others) nondeterminism will be measured in bits and the coefficients on how many bits we are using will be very important. Unlike existing models, the nondeterministic guesses will be preallocated as placeholders within an input string. This means that we can only fill in
placeholders with nondeterministic bits. This property is essential for proving structural properties (see translation and padding lemmas in Subsection 3.3). With other models, proofs of structural properties are much more complex and lead to various overheads from tape reduction theorems.

### 3.2 A Model for Limited Nondeterminism

Consider strings over a ternary alphabet $\Sigma = \{0, 1, p\}$ where $p$ is referred to as the placeholder character. For any string $x \in \Sigma^*$, we let $|x|$ denote the length of $x$ and $\#_p(x)$ denote the number of placeholder character occurrences in $x$.

**Definition 1.** Let a string $r \in \{0, 1\}^*$ be given. Define a function

$$\text{sub}_r : \Sigma^* \to \Sigma^*$$

such that for each string $x \in \Sigma^*$, $\text{sub}_r(x)$ is obtained from $x$ by replacing placeholder characters with bits from $r$ so that the $i$th placeholder character from $x$ is replaced by the $i$th bit of $r$ for all $i$ satisfying $0 \leq i < \min\{|r|, \#_p(x)\}$. Also, define $\text{SUB}(n)$ such that $\text{SUB}(n) := \{\text{sub}_r \mid |r| \leq n\}$. We call $\text{sub}_r$ a substitution operator and $\text{SUB}(n)$ a set of substitution operators.

**Example 1.** Consider strings $x = 11p01p0p$ and $r = 0110$. By applying the preceding definition, we have that $\text{sub}_r(x) = 11001101$.

A substitution operator $\text{sub}_r$ replaces the first $|r|$ placeholder characters with the bits of $r$ in order. We will also consider the notion of partial filling, which is any replacement of placeholders, without restricting replacements to a particular order.

**Definition 2.** Let strings $x$ and $y \in \Sigma^*$ be given. We write $x \preceq y$ if $x$ can be obtained from $y$ by replacing any number of placeholders in $y$ with 0 or 1. Let a language $L \subseteq \Sigma^*$ be given. We define $\text{Clo}(L)$ such that

$$\text{Clo}(L) := \{x \in \Sigma^* \mid (\exists y \in L) x \preceq y\}.$$ 

Intuitively, $\text{Clo}(L)$ is the closure of $L$ under partial fillings.

**Example 2.** Consider a language $L = \{0p1p\}$. By applying the preceding definition, we have that $\text{Clo}(L) = \{0p1p, 0p10, 0p11, 001p, 011p, 0001, 0110, 0111\}$.

We now proceed by defining a complexity class for limited nondeterminism $\text{DTIWI}(t(n), w(n))$ where intuitively $t(n)$ represents a time bound and $w(n)$ represents a bound on nondeterministic bits, in other words the witness length.

**Definition 3.** Consider a language $L \subseteq \Sigma^*$. We write $L \in \text{DTIWI}(t(n), w(n))$ if there exist languages $U$ and $V \subseteq \Sigma^*$ satisfying the following properties.

- $U \in \text{DTIME}(O(n))$,
- $\text{Clo}(U) \in \text{DTIME}(O(n))$,
• \( V \in \text{DTIME}(O(t(n))) \), and

• for all \( x \in \Sigma^* \), \( x \in L \) if and only if \( x \in U \) and there exists \( s \in \text{SUB}(w(|x|)) \) such that \( s(x) \in V \).

We refer to \( V \) as a verification language for \( L \) with input string universe \( U \).

**Remark 2.** The language \( U \) represents the set of input strings that will be considered. The language \( \text{Clo}(U) \) represents any partial filling of a string in \( U \). We generally won’t pay too close attention to languages \( U \) and \( \text{Clo}(U) \), but they are important for restricting input strings to a specific format or encoding.

**Remark 3.** We offer a generic definition for \( \text{DTIWI} \) that can be adapted for different machine models. Later it will be important to specify a particular machine model. At that time, we will write \( \text{Turing-\text{DTIWI}} \) (and \( \text{Turing-\text{DTIME}} \)) to specify that the machines are multitape Turing machines that allow for read and write access to all tapes (including the input tape).

There are many different ways to encode mathematical structures as input strings over a fixed size alphabet. Therefore, common decision problems can take many different forms as formal languages. Furthermore, in order to put a problem within \( \text{DTIWI}(t(n), w(n)) \), we need to provide an encoding for its input strings where we put placeholder characters at the appropriate positions. Consider the following examples.

**Example 3.** Sumset Sum is represented such that each input is a sequence of placeholder and binary number pairs. When a placeholder is nondeterministically filled with 0 that means the number is not to be included in the sum and 1 means that the number is to be included.

**Example 4.** SAT is represented such that each input has placeholder characters out front followed by an encoding of a Boolean formula. Each variable is represented as a binary number representing an index to a specific placeholder. The placeholders will be nondeterministically filled to create a variable assignment.

Notice that in Example 3 the placeholders occur throughout an input while in Example 4 the placeholders appear as a prefix at the beginning of an input.

### 3.3 Structural Properties of Limited Nondeterminism

The following two lemmas demonstrate elegant structural properties relating time and nondeterminism. These structural properties will be essential to proving speed-up theorems in Subsection 3.4 that reveal new relationships between deterministic and nondeterministic computations.

**Lemma 4** (Translation Lemma). If \( \text{DTIWI}(t(n), w(n)) \subseteq \text{DTIME}(t'(n)) \), then for all efficiently computable functions \( w' \),

\[
\text{DTIWI}(t(n), w(n) + w'(n)) \subseteq \text{DTIWI}(t'(n), w'(n)).
\]
Proof. Suppose that DTIWI(t(n), w(n)) ⊆ DTIME(t’(n)).

Let a function w’ be given. Let L ∈ DTIWI(t(n), w(n) + w’(n)) be given. By definition, there exist an input string universe U and a verification language V ∈ DTIME(O(t(n))) satisfying for all x ∈ Σ*, x ∈ L if and only if x ∈ U and there exists s ∈ SUB(w(|x|) + w’(|x|)) such that s(x) ∈ V. Consider a new language

\[ L' := \{ x ∈ \text{Clo}(U) \mid (\exists s ∈ \text{SUB}(w(|x|))) s(x) ∈ V \}. \]

By interpreting V as a verification language for L’ with input string universe Clo(U), we get L’ ∈ DTIWI(t(n), w(n)). By assumption, it follows that

\[ L' ∈ DTIME(t'(n)). \]

Finally, by interpreting L’ as a verification language for L with input string universe U, we get L ∈ DTIWI(t’(n), w’(n)). □

Lemma 5 (Padding Lemma). If DTIWI(t(n), w(n)) ⊆ DTIME(t’(n)), then for all efficiently computable functions f,

\[ DTIWI(t(f(n)), w(f(n))) ⊆ DTIME(t'(f(n))). \]

Proof. Suppose that DTIWI(t(n), w(n)) ⊆ DTIME(t’(n)).

Let a function f be given. Let L ∈ DTIWI(t(f(n)), w(f(n))) be given. By definition, there exist an input string universe U and a verification language

\[ V ∈ DTIME(O(t(f(n)))) \]

satisfying for all x ∈ Σ*, x ∈ L if and only if x ∈ U and there exists

\[ s ∈ \text{SUB}(w(f(|x|))) \]

such that s(x) ∈ V. Consider new languages L’, V’, and U’ such that

\[ L' := \{ 1^{k-1}\cdot 0 \cdot x \mid k + |x| = f(|x|) \land x ∈ L \} \]
\[ V' := \{ 1^{k-1}\cdot 0 \cdot x \mid k + |x| = f(|x|) \land x ∈ V \} \]
\[ U' := \{ 1^{k-1}\cdot 0 \cdot x \mid k ≥ 1 \land x ∈ U \}. \]

Since V ∈ DTIME(O(t(f(n)))), assuming that f can be computed efficiently and that the machine model is capable of ignoring the input string’s prefix of 1’s, we have that V’ ∈ DTIME(O(t(n))). By interpreting V’ as a verification language for L’ with input string universe U’, we get L’ ∈ DTIWI(t(n), w(n)). By assumption, it follows that L’ ∈ DTIME(t’(n)). Again, assuming that f can be computed efficiently and that the machine model can efficiently prepend a padding prefix to the input string, we get L ∈ DTIME(t’(f(n))). □

Remark 4. Initially, we tried to use existing notions of limited nondeterminism to prove the preceding lemmas. However, the proofs were messy with input size and time complexity blow-ups. In contrast, our model for limited nondeterminism (DTIWI) offers straightforward proofs with no input size blow-ups.
3.4 Speed-up Theorems

Unlike prior works that investigated speed-up theorems for time and space trade-offs [36, 9], we focus specifically on speed-up results for time and nondeterminism trade-offs. Our speed-up results are also distinct from recent hardness magnification results from [12] which amplify circuit lower bounds rather than speed-up computations.

We prove the first speed-up theorem by repeatedly applying the structural properties of limited nondeterminism from the preceding subsection.

**Theorem 6 (First Speed-up Theorem).** Let \( \alpha \) such that \( 1 \leq \alpha < 2 \) be given. If

\[
\text{DTIWI}(n, \log(n)) \subseteq \text{DTIME}(n^\alpha),
\]

then for all \( k \in \mathbb{N} \),

\[
\text{DTIWI}(n, (\sum_{i=0}^{k} \alpha^i) \log(n)) \subseteq \text{DTIME}(n^{\alpha^{k+1}}).
\]

**Proof.** Let \( \alpha \) such that \( 1 \leq \alpha < 2 \) be given. (Note that when \( \alpha = 1 \) some of the following formulas can be simplified, but the proof still holds for this case.)

Suppose that \( \text{DTIWI}(n, \log(n)) \subseteq \text{DTIME}(n^\alpha) \). We prove by induction on \( k \) that for all \( k \in \mathbb{N} \),

\[
\text{DTIWI}(n, (\sum_{i=0}^{k} \alpha^i) \log(n)) \subseteq \text{DTIME}(n^{\alpha^{k+1}}).
\]

The base case \((k = 0)\) is true by assumption. Now, let’s consider the induction step. Suppose that

\[
\text{DTIWI}(n, (\sum_{i=0}^{k} \alpha^i) \log(n)) \subseteq \text{DTIME}(n^{\alpha^{k+1}}).
\]

By applying this assumption with Lemma 4, we get that

\[
\text{DTIWI}(n, (\sum_{i=0}^{k+1} \alpha^i) \log(n)) \subseteq \text{DTIME}(n^{\alpha^{k+1}}, \alpha^{k+1} \cdot \log(n)).
\]

Next, we apply our initial assumption and Lemma 5 with \( f(n) = n^{\alpha^{k+1}} \), \( w(n) = \log(n) \), and \( t(n) = n \). Therefore,

\[
\text{DTIWI}(n^{\alpha^{k+1}}, \alpha^{k+1} \cdot \log(n)) \subseteq \text{DTIME}(n^{\alpha^{k+2}}).
\]

It follows that

\[
\text{DTIWI}(n, (\sum_{i=0}^{k+1} \alpha^i) \log(n)) \subseteq \text{DTIME}(n^{\alpha^{k+2}}). \quad \square
\]

**Remark 5.** Theorem 6 is a speed-up result because when \( 1 \leq \alpha < 2 \), the exponent from the runtime divided by the constant factor for the witness string length decreases as \( k \) increases. In particular, we have

\[
\lim_{k \to \infty} \frac{\alpha^{k+1}}{\sum_{i=0}^{k} \alpha^i} = (\alpha - 1) \cdot \lim_{k \to \infty} \frac{\alpha^{k+1}}{\alpha^{k+1} - 1} = \alpha - 1 < 1.
\]

\(^2\)This repeated application of structural properties leads to a kind of iterated simulation.
Remark 6. The value $k$ in Theorem 6 is a constant. Replacing $k$ by a function of $n$ may result in several difficulties including compounding overheads on time complexity, formalizing machine encodings, and resolving how to algorithmically apply the translation and padding lemmas on arbitrary machine encodings.

We prove the second speed-up theorem by combining the first speed-up theorem with the padding lemma.

**Theorem 7** (Second Speed-up Theorem). Suppose that $g$ is an efficiently computable non-decreasing function such that $(\forall n \in \mathbb{N}) g(n) \leq n$ and $g(n)$ is $\omega(\log(n))$. Let $\alpha$ such that $1 < \alpha < 2$ be given. If

$$\text{DTIWI}(n, \log(n)) \subseteq \text{DTIME}(n^\alpha),$$

then

$$(\forall \varepsilon > 0) \text{DTIWI}(\text{poly}(n), g(n)) \subseteq \text{DTIME}(2^{(1+\varepsilon)\cdot(\alpha-1)\cdot g(n)}).$$

*Proof.* Let $\alpha$ such that $1 < \alpha < 2$ be given. Let $z(\alpha, k) = \sum_{i=0}^{k} \alpha^i$. Note that

$$z(\alpha, k) = \frac{\alpha^{k+1} - 1}{\alpha - 1}.$$ 

Suppose that $\text{DTIWI}(n, \log(n)) \subseteq \text{DTIME}(n^\alpha)$. Let $\varepsilon > 0$ be given. By Theorem 6 we have that for all $k \in \mathbb{N}$,

$$\text{DTIWI}(n, z(\alpha, k) \log(n)) \subseteq \text{DTIME}(n^\alpha k^{k+1}).$$

Next, we apply Lemma 5 with $f(n) = 2^{g(n)/z(\alpha, k)}$, $w(n) = z(\alpha, k) \log(n)$, and $t(n) = n$. Therefore,

$$\text{DTIWI}(2^{g(n)/z(\alpha, k)}, g(n)) \subseteq \text{DTIME}(2^{(\alpha^{k+1}) \cdot g(n)/z(\alpha, k)}).$$

Since $g(n)$ is $\omega(\log(n))$ and $z(\alpha, k) > 0$ is a constant, $2^{g(n)/z(\alpha, k)}$ is super-polynomial. Hence,

$$\text{DTIWI}(\text{poly}(n), g(n)) \subseteq \text{DTIME}(2^{(\alpha^{k+1}) \cdot g(n)/z(\alpha, k)}).$$

Since

$$\lim_{k \to \infty} \frac{\alpha^{k+1}}{\alpha^{k+1} - 1} = 1,$$

there exists $k$ sufficiently large such that

$$\frac{\alpha^{k+1}}{\alpha^{k+1} - 1} \leq 1 + \varepsilon.$$

Therefore, by choosing sufficiently large $k$, we have

$$\text{DTIWI}(\text{poly}(n), g(n)) \subseteq \text{DTIME}(2^{\alpha^{k+1} \cdot g(n)/z(\alpha, k)}).$$  \qed
Remark 7. In Theorem 7, the exponent from the first runtime is $\alpha$ and the exponent from the second runtime is a factor of $\alpha - 1$. This is a speed-up result because as $\alpha$ approaches 1, the second runtime improves exponentially while the first runtime only improves polynomially.

Corollary 8. Suppose that $g$ is an efficiently computable non-decreasing function such that $(\forall n \in \mathbb{N}) g(n) \leq n$ and $g(n)$ is $\omega(\log(n))$. If for all $\alpha > 1$,

$$\text{DTIWI}(n, \log(n)) \subseteq \text{DTIME}(n^\alpha),$$

then

$$(\forall \varepsilon > 0) \text{DTIWI}(\text{poly}(n), g(n)) \subseteq \text{DTIME}(2^{\varepsilon g(n)}).$$

Proof. Follows directly from Theorem 7. \qed

4 Stronger Conditional Lower Bounds

4.1 log-CircuitSAT Decision Problem

Satisfiability of bounded fan-in Boolean circuits with $m$ gates and $\log(m)$ inputs is denoted by log-CircuitSAT. We encode this problem so that the placeholder characters are out front followed by a conventional encoding of a bounded fan-in Boolean circuit. Such an encoding can be naturally carried out so that if $n$ denotes the total input length and $m$ denotes the number of gates, then $n$ is $\Theta(m \cdot \log(m))$. Using brute force search, a robust random access machine model can be used to solve this problem in roughly $O(n^2)$ time by preprocessing the circuit into a graph data structure for efficient evaluation and then evaluating the circuit on every possible input assignment.

Whether or not we can solve log-CircuitSAT in $O(n^{2-\varepsilon})$ time for some $\varepsilon > 0$ is an open problem. Furthermore, as far as we know, no unconditional super-linear lower bounds are known for log-CircuitSAT. Later in this section, we prove a strong conditional lower bound for log-CircuitSAT. In particular, we show that if log-CircuitSAT is solvable in quasilinear time, then ETH is false (Corollary 14) meaning that a polynomial time improvement for log-CircuitSAT leads to an exponential time improvement for $\text{NP}$-complete problems.

4.2 Simulating Turing Machines Using Boolean Circuits

It is a well known result of Pippenger and Fischer (1979) that any $O(t(n))$ time bounded Turing machine can be simulated by an $O(t(n) \cdot \log(t(n)))$ time oblivious Turing machine \[^{[30]}\]. Moreover, any $O(t(n))$ time bounded Turing machine can be simulated by $O(t(n) \cdot \log(t(n)))$ size Boolean circuits and such circuits can be computed efficiently\[^{[4]}\].

\[^{[4]}\]Such circuits can even be computed efficiently by a Turing machine.
Theorem 9 ([30] [8] [24]). If \( L \in \text{Turing-\text{DTIME}}(t(n)) \), then in 
\[ O(t(n) \cdot \text{poly}(\log(t(n)))) \]
time, we can compute Boolean circuits for \( L \) of size at most 
\[ O(t(n) \cdot \log(t(n))) \].

Next, we apply Theorem 9 to show that any problem in Turing-\text{DTIWI}(n, \log(n)) is efficiently reducible to log-CircuitSAT.

Theorem 10. For all \( L \in \text{Turing-\text{DTIWI}}(n, \log(n)) \), \( L \) is reducible to logarithmically many instances of log-CircuitSAT in quasilinear time by a Turing machine.

Proof. Let \( L \in \text{Turing-\text{DTIWI}}(n, \log(n)) \) be given. Let \( V \in \text{Turing-\text{DTIME}}(n) \) denote a verification language for \( L \) with input string universe \( U \in \text{Turing-\text{DTIME}}(n) \).

Let an input string \( x \in U \) of length \( n \) be given. By Theorem 9, we can compute an \( n \) input Boolean circuit \( C \) for \( V \) with at most 
\[ O(n \cdot \log(n)) \] gates in quasilinear time on a Turing machine. Now, we construct a family of circuits \( \{C_i\}_{\log(n)} \) such that for each \( i \in [\log(n)] \), \( C_i \) is obtained by fixing the characters of \( x \) into the circuit \( C \) so that only \( i \) input bits remain where these input bits are associated with the first \( i \) placeholders within \( x \). Therefore, the circuit \( C_i \) has at most \( \log(n) \) inputs and at most 
\[ O(n \cdot \log(n)) \] gates. It follows that \( x \in L \) if and only if there exists \( i \in [\log(n)] \) such that \( C_i \) is satisfiable. \( \square \)

Remark 8. To the best of our knowledge, it’s an open problem to determine whether \( O(t(n)) \) time bounded random access machines can be simulated by \( O(t(n) \cdot \log(t(n))) \) size Boolean circuits. As a result, Theorem 10 is currently limited to Turing machines.

Corollary 11. If \((\forall \alpha > 1) \text{log-CircuitSAT} \in \text{Turing-\text{DTIME}}(n^\alpha)\), then 
\[ (\forall \alpha > 1) \text{Turing-\text{DTIWI}}(n, \log(n)) \subseteq \text{Turing-\text{DTIME}}(n^\alpha). \]

Proof. Follows directly from Theorem 10. \( \square \)

Corollary 12. Suppose that \( g \) is an efficiently computable non-decreasing function such that \((\forall n \in \mathbb{N}) g(n) \leq n \) and \( g(n) \) is \( \omega(\log(n)) \). If 
\[ (\forall \alpha > 1) \text{log-CircuitSAT} \in \text{Turing-\text{DTIME}}(n^\alpha), \]
then 
\[ (\forall \varepsilon > 0) \text{Turing-\text{DTIWI}}(\text{poly}(n), g(n)) \subseteq \text{Turing-\text{DTIME}}(2^{\varepsilon \cdot g(n)}). \]

Proof. Follows by combining Corollary 11 with Corollary 8. \( \square \)

\footnote{Since \( L \) and \( V \) are over a ternary alphabet, the input strings are encoded into binary before being fed into the Boolean circuits. Therefore, \( C \) will actually have slightly more than \( n \) input bits to account for encoding \( x \) in binary.}
4.3 ETH-hardness

We apply results from the preceding subsection to prove strong conditional lower bounds for log-CircuitSAT. In particular, we show that the existence of quasilinear time algorithms for log-CircuitSAT would imply that ETH is false and that NP-complete problems would have exponentially faster algorithms.

**Theorem 13.** If \((\forall \alpha > 1) \log\text{-}\text{CircuitSAT} \in \text{Turing-DTIME}(n^\alpha)\), then

\[(\forall \varepsilon > 0) \text{CircuitSAT} \in \text{Turing-DTIME}(\text{poly}(n) \cdot 2^{\varepsilon \cdot m})\]

where \(m\) is the number of gates.

**Proof.** Suppose that for all \(\alpha > 1\), \(\log\text{-}\text{CircuitSAT} \in \text{Turing-DTIME}(n^\alpha)\). By applying Corollary 12 with 
\[g(n) = \frac{n}{\log(n)},\]
we get that

\[(\forall \varepsilon > 0) \text{Turing-DTIWI}(\text{poly}(n), \frac{n}{\log(n)}) \subseteq \text{Turing-DTIME}(2^{\varepsilon \cdot \frac{n}{\log(n)}}).\]

Recall that we encode Boolean circuits so that \(n = \Theta(m \cdot \log(m))\) where \(m\) is the number of gates. In fact, under reasonable encoding conventions, \(\frac{n}{\log(n)}\) will actually be larger than the number of gates and inputs. Hence,

\[\text{CircuitSAT} \in \text{Turing-DTIWI}(\text{poly}(n), \frac{n}{\log(n)}).\]

Therefore, \((\forall \varepsilon > 0) \text{CircuitSAT} \in \text{Turing-DTIME}(2^{\varepsilon \cdot \frac{n}{\log(n)}}).\) It follows that

\[(\forall \varepsilon > 0) \text{CircuitSAT} \in \text{Turing-DTIME}(\text{poly}(n) \cdot 2^{\varepsilon \cdot m}).\]

\[\square\]

**Corollary 14.** If \((\forall \alpha > 1) \log\text{-}\text{CircuitSAT} \in \text{Turing-DTIME}(n^\alpha)\), then ETH is false.

**Proof.** Because 3-CNF-SAT is a special case of CircuitSAT, Theorem 13 implies that

\[(\forall \varepsilon > 0) \text{3-CNF-SAT} \in \text{Turing-DTIME}(\text{poly}(n) \cdot 2^{\varepsilon \cdot m})\]

where \(m\) is the number of bounded AND, OR, and NOT gates (which is approximately three times the number of clauses). By applying the Sparsification Lemma [23], we get that

\[(\forall \varepsilon > 0) \text{3-CNF-SAT} \in \text{Turing-DTIME}(\text{poly}(n) \cdot 2^{\varepsilon \cdot v})\]

where \(v\) is the number of variables. It follows that ETH is false. \[\square\]

4.4 Reductions from Other Polynomial Time Problems

In the preceding subsection, we proved strong conditional lower bounds for the log-CircuitSAT problem. In particular, we showed that quasilinear time algorithms for log-CircuitSAT would imply the existence of exponentially more efficient algorithms for NP-complete problems. Now, we consider how quasilinear...
time algorithms for log-CircuitSAT would imply the existence of quasilinear time algorithms for polynomial time problems such as OVP, 3SUM, Triangle Finding, and even $k$-Clique for all $k$.

Before proving these results, we briefly state the definitions for the OVP, 3SUM, Triangle Finding, and $k$-Clique decision problems.

- **Orthogonal vectors decision problem (OVP)** is defined as follows. Given a list of binary vectors, do there exist two vectors that are orthogonal?

- **3SUM decision problem** is defined as follows. Given a list of integers, do there exist three integers that sum to 0?

- **Triangle Finding decision problem** is defined as follows. Given an undirected graph, do there exist three vertices that form a triangle?

- **$k$-Clique decision problem** is defined as follows for any fixed number $k$. Given an undirected graph, do there exist $k$ vertices that form a clique (i.e., a graph where every pair of vertices is adjacent)?

**Corollary 15.** OVP, 3SUM, and Triangle Finding are reducible to log-CircuitSAT in quasilinear time. Moreover, log-CircuitSAT is not solvable in quasilinear time under the assumption that quasilinear time algorithms do not exist for at least one of the problems OVP, 3SUM, or Triangle Finding.

**Proof.** We observe that OVP, 3SUM, and Triangle Finding (under appropriate encodings) are all in Turing-DTIWI($n \cdot \text{poly}(\log(n)), \log(n)$) where $n$ denotes the total input string length.

- For OVP, we are given a list of bit vectors. We use nondeterministic bits to guess one of the bit vectors. Then, we compute the dot product between it and all other bit vectors.

- For 3SUM, we are given a list of binary numbers (appropriately padded so that they all have the same length). We use nondeterministic bits to guess one of the binary numbers $x$. Then, for all other binary numbers, we multiply by 2 and add $x$. Finally, we solve the 2SUM problem on the resulting list of binary numbers.

- For Triangle Finding, we are given a list of edges represented as a pair of vertices. We use nondeterministic bits to guess one of the edges. Then, we check if this edge combined with an additional vertex forms a triangle.

By applying a slight variation to Theorem 10 for quasilinear time verifiers with log($n$) length witnesses, we get that OVP, 3SUM, and Triangle Finding are all quasilinear time reducible to log-CircuitSAT. Therefore, quasilinear time algorithms for log-CircuitSAT would imply the existence of quasilinear time algorithms for OVP, 3SUM, and Triangle Finding which is currently not known.

\[\Box\]

\footnote{Notice that Triangle Finding is equivalent to 3-Clique.}
Theorem 16. If for all $\alpha > 1$, $\text{DTIWI}(n, \log(n)) \subseteq \text{DTIME}(n^{\alpha})$, then

$$(\forall k) \ (\forall \alpha > 1) \ \text{DTIWI}(n, k \cdot \log(n)) \subseteq \text{DTIME}(n^{\alpha}).$$

Proof. Suppose that for all $\alpha > 1$, $\text{DTIWI}(n, \log(n)) \subseteq \text{DTIME}(n^{\alpha})$. By Theorem 6 for all $\alpha$ such that $1 \leq \alpha < 2$ and for all $k \in \mathbb{N}$,

$$\text{DTIWI}(n, (\Sigma_{i=0}^{k} \alpha^i) \log(n)) \subseteq \text{DTIME}(n^{\alpha_{k+1}}).$$

Notice that when $\alpha > 1$, we have $k \leq (\Sigma_{i=0}^{k} \alpha^i)$. Therefore, for all $\alpha$ such that $1 \leq \alpha < 2$ and for all $k \in \mathbb{N},$

$$\text{DTIWI}(n, k \cdot \log(n)) \subseteq \text{DTIME}(n^{\alpha_{k+1}}).$$

Let $k \in \mathbb{N}$ and $\alpha_1 > 1$ be given. We can choose $\alpha_2 > 1$ sufficiently close to $1$ so that $\alpha_{2}^{k+1} \leq \alpha_1$ (just take $\alpha_2 = \alpha_1^{1/(k+1)}$). It follows that

$$\text{DTIWI}(n, k \cdot \log(n)) \subseteq \text{DTIME}(n^{\alpha_{k+1}}) \subseteq \text{DTIME}(n^{\alpha_{1}}).$$

Corollary 17. If $(\forall \alpha > 1) \ \log\text{-CircuitSAT} \in \text{Turing-DTIME}(n^{\alpha})$, then

$$(\forall k) \ (\forall \alpha > 1) \ \text{Turing-DTIWI}(n, k \cdot \log(n)) \subseteq \text{Turing-DTIME}(n^{\alpha}).$$

Proof. Follows by combining Corollary 11 with Theorem 16.

Corollary 18. $\log\text{-CircuitSAT}$ is not solvable in quasilinear time under the assumption that quasilinear time algorithms do not exist for $k\text{-Clique}$ for some sufficiently large fixed $k$.

Proof. Follows by combining Corollary 17 with the observation that $k\text{-Clique} \in \text{Turing-DTIWI}(n, k \cdot \log(n))$ for all fixed $k$.

Remark 9. Although we do not focus on parameterized complexity theory here, the preceding arguments can also be used to show that if $\log\text{-CircuitSAT}$ is solvable in quasilinear time, then $W[1] \subset \text{non-uniform-FPT}$. Moreover, we suggest that this implication could be extended to $W[P] \subset \text{non-uniform-FPT}$. We refer the reader to [3] for background information on $W[P]$ and the $W$ hierarchy.

\[\text{[6] Although it may be more common for } n \text{ to denote the number of vertices, we instead use } n \text{ to denote the total length of the input string which encodes the graph.}\]
5 Conclusion

In this work, we have demonstrated strong conditional lower bounds for the log-CircuitSAT decision problem by carefully investigating properties of limited nondeterminism. In particular, in Corollary 14 we showed that the existence of quasilinear time Turing machines for log-CircuitSAT would imply that ETH is false. This means that a polynomial time improvement for log-CircuitSAT would lead to an exponential time improvement for NP-complete problems. Through this investigation we revealed new relationships between deterministic and nondeterministic computations.

The main two open problems of this work are as follows.

- Does Corollary 14 hold for random access machine models? This question is related to the well known question of whether linear time for random access machines can be simulated in subquadratic time by multitape Turing machines [14]. It is also related to whether random access machines can be made oblivious [19].

- Can the construction from the first speed up theorem (Theorem 6) be carried out for a non-constant number of iterations $k$. We speculate that if it can, then $\text{DTIW}(n, \log(n)) \subseteq \text{DTIME}(n \cdot \log(n))$ would imply that $\text{NTIME}(n) \subseteq \text{DTIME}(2^{\sqrt{n}})$.

Also, although this work doesn’t focus on circuit lower bounds, we suggest that recent results connecting the existence of faster algorithms with circuit lower bounds [1, 37, 36, 6] could be applied to show that the existence of faster algorithms for log-CircuitSAT would imply new circuit lower bounds for $E^{NP}$ as well as other complexity classes.

Finally, we leave the reader with the thought that the speed-up theorems for limited nondeterminism (Theorems 6 and 7) might be a special case of a more general speed-up result connecting nondeterminism, alternation, and time.

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References

[1] Amir Abboud, Thomas Dueholm Hansen, Virginia Vassilevska Williams, and Ryan Williams. Simulating branching programs with edit distance and friends: Or: a polylog shaved is a lower bound made. In Proc. of the 48th Annual ACM Symposium on Theory of Computing, STOC ’16, pages 375–388. Association for Computing Machinery, 2016. doi:10.1145/2897518.2897653

[2] Amir Abboud, Virginia Vassilevska Williams, and Huacheng Yu. Matching triangles and basing hardness on an extremely popular conjecture. In Proceedings of the Forty-Seventh Annual ACM Symposium on Theory of Computing, STOC ’15, page 41–50. Association for Computing Machinery, 2015. doi:10.1145/2746539.2746594

[3] Karl A. Abrahamson, Rodney G. Downey, and Michael R. Fellows. Fixed-parameter tractability and completeness IV: On completeness for W[P] and PSPACE analogues. Annals of Pure and Applied Logic, 73(3):235–276, 1995. doi:10.1016/0168-0072(94)00034-Z

[4] Akeo Adachi, Shigeki Iwata, and Takumi Kasai. Some combinatorial game problems require Ω(n^k) time. J. ACM, 31(2):361–376, March 1984. doi:10.1145/62.322433

[5] Carme Álvarez, Josep Díaz, and Jacobo Torán. Complexity classes with complete problems between P and NP-C. In J. Csirik, J. Demetrovics, and F. Gécseg, editors, FCT 1989: Proceedings of the 7th International Conference on Fundamentals of Computation Theory, volume 380 of LNCS, pages 13–24. Springer, 1989. doi:10.1007/3-540-51498-8_2

[6] Eli Ben-Sasson and Emanuele Viola. Short PCPs with projection queries. In Javier Esparza, Pierre Fraigniaud, Thore Husfeldt, and Elias Koutsoupias, editors, ICALP 2014: International Colloquium on Automata, Languages, and Programming, volume 8572 of LNCS, pages 163–173. Springer, 2014. doi:10.1007/978-3-662-43948-7_14

[7] Karl Bringmann. Fine-grained complexity theory (tutorial). In Rolf Niedermeier and Christophe Paul, editors, STACS 2019: 36th International Symposium on Theoretical Aspects of Computer Science, volume 126 of LIPIcs, pages 4:1–4:7. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2019. doi:10.4230/LIPIcs.STACS.2019.4

[8] Jonathan Buss and Judy Goldsmith. Nondeterminism within P. SIAM J. Comput., 22(3):560–572, 1993. doi:10.1137/0222038

[9] Jonathan Buss and Kenneth Regan. Simultaneous bounds on time and space. Manuscript, 2014.
[10] Liming Cai and Jianer Chen. On the amount of nondeterminism and the power of verifying. *SIAM J. Comput.*, 26(3):733–750, 1997. doi:10.1137/S0097539793258295

[11] Chris Calabro, Russell Impagliazzo, and Ramamohan Paturi. The complexity of satisfiability of small depth circuits. In Jianer Chen and Fedor V. Fomin, editors, *IWPEC 2009: Parameterized and Exact Computation*, volume 5917 of *LNCS*, pages 75–85. Springer, 2009. doi:10.1007/978-3-642-11269-0_6

[12] Lijie Chen, Ce Jin, and R. Ryan Williams. Sharp threshold results for computational complexity. In *Proceedings of the 52nd Annual ACM SIGACT Symposium on Theory of Computing*, STOC 2020, page 1335–1348. Association for Computing Machinery, 2020. doi:10.1145/3357713.3384283

[13] Stephen A. Cook. The complexity of theorem-proving procedures. In *Proceedings of the Third Annual ACM Symposium on Theory of Computing*, STOC ’71, page 151–158. Association for Computing Machinery, 1971. doi:10.1145/800157.805047

[14] Stephen A. Cook and Robert A. Reckhow. Time bounded random access machines. *Journal of Computer and System Sciences*, 7(4):354–375, 1973. doi:10.1016/S0022-0000(73)80029-7

[15] J. Díaz and J. Torán. Classes of bounded nondeterminism. *Mathematical Systems Theory*, 23(1):21–32, 1990. doi:10.1007/BF02090764

[16] Graham E. Farr. *Topics in computational complexity*. PhD thesis, University of Oxford, 1986. URL: https://ora.ox.ac.uk/objects/uuid:ad3ed1a4-fea4-4b46-8e7a-a0c6a3451325/

[17] Uriel Feige and Joe Kilian. On Limited versus Polynomial Nondeterminism. *Chicago Journal of Theoretical Computer Science*, 1997(1), March 1997. URL: http://cjtcs.cs.uchicago.edu/articles/1997/1/cj97-01.pdf

[18] Judy Goldsmith, Matthew A. Levy, and Martin Mundhenk. Limited nondeterminism. *SIGACT News*, 27(2):20–29, 1996. doi:10.1145/235767.235769

[19] Yuri Gurevich and Saharon Shelah. Nearly linear time. In Albert R. Meyer and Michael A. Taitslin, editors, *Logic at Botik 1989*, volume 363 of *LNCS*, pages 108–118. Springer, 1989. doi:10.1007/3-540-51237-3_10

[20] J. Hartmanis and R. E. Stearns. On the computational complexity of algorithms. *Transactions of the AMS*, 117:285–306, 1965. URL: http://www.jstor.org/stable/1994208

[21] F.C. Hennie. One-tape, off-line Turing machine computations. *Information and Control*, 8(6):553–578, 1965. doi:10.1016/S0019-9958(65)90399-2
Russell Impagliazzo and Ramamohan Paturi. On the complexity of k-SAT. *Journal of Computer and System Sciences*, 62(2):367–375, 2001. doi:10.1006/jcss.2000.1727

Russell Impagliazzo, Ramamohan Paturi, and Francis Zane. Which problems have strongly exponential complexity? *Journal of Computer and System Sciences*, 63(4):512–530, 2001. doi:10.1006/jcss.2001.1774

Richard J. Lipton and Richard Williams. Amplifying circuit lower bounds against polynomial time with applications. In *Proceedings of the Annual IEEE Conference on Computational Complexity*, CCC 2012, pages 1–9, 2012. doi:10.1109/CCC.2012.44

Chandra M. R. Kintala and Patrick C. Fischer. Computations with a restricted number of nondeterministic steps (extended abstract). In *Proceedings of the ninth annual ACM symposium on Theory of computing*, STOC ’77, pages 178–185. Association for Computing Machinery, 1977. doi:10.1145/800105.803407

Chandra M. R. Kintala and Patrick C. Fischer. Refining nondeterminism in relativized polynomial-time bounded computations. *SIAM Journal on Computing*, 9(1):46–53, 1980. doi:10.1137/0209003

Leonid Anatolevich Levin. Universal sequential search problems. *Problemy peredachi informatsii*, 9(3):115–116, 1973.

Wolfgang Maass. Quadratic lower bounds for deterministic and nondeterministic one-tape turing machines. In *Proceedings of the Sixteenth Annual ACM Symposium on Theory of Computing*, STOC ’84, pages 401–408. Association for Computing Machinery, 1984. doi:10.1145/800057.808706

Xu Mei-rui, John E. Doner, and Ronald V. Book. Refining nondeterminism in relativizations of complexity classes. *J. ACM*, 30(3):677–685, 1983. doi:10.1145/2402.322399

Nicholas Pippenger and Michael J. Fischer. Relations among complexity measures. *J. ACM*, 26(2):361–381, 1979. doi:10.1145/322123.322138

Rahul Santhanam. On separators, segregators and time versus space. In *Proceedings of the Sixteenth Annual Conference on Computational Complexity*, CCC ’01, pages 286–294, 2001. doi:10.1109/CCC.2001.933895

C. P. Schnorr. Satisfiability is quasilinear complete in NQL. *J. ACM*, 25(1):136–145, 1978. doi:10.1145/322047.322060

Joseph Swernofsky and Michael Wehar. On the complexity of intersecting regular, context-free, and tree languages. In Magnús M. Halldórsson, Kazuo Iwama, Naoki Kobayashi, and Bettina Speckmann, editors, *ICALP 2015: Automata, Languages, and Programming - 42nd International Colloquium, Proceedings, Part II*, volume 9135 of LNCS, pages 414–426. Springer, 2015. doi:10.1007/978-3-662-47666-6_33
[34] Michael Wehar. *On the Complexity of Intersection Non-Emptiness Problems*. PhD thesis, State University of New York at Buffalo, 2016. URL: [http://michaelwehar.com/documents/mwehar_dissertation.pdf](http://michaelwehar.com/documents/mwehar_dissertation.pdf)

[35] Ryan Williams. A new algorithm for optimal 2-constraint satisfaction and its implications. *Theoretical Computer Science*, 348(2):357–365, 2005. doi: [10.1016/j.tcs.2005.09.023](http://dx.doi.org/10.1016/j.tcs.2005.09.023)

[36] Ryan Williams. Improving exhaustive search implies superpolynomial lower bounds. *SIAM J. Comput.*, 42(3):1218–1244, 2013. doi: [10.1137/10080703X](http://dx.doi.org/10.1137/10080703X)

[37] Ryan Williams. Non-uniform ACC circuit lower bounds. *J. ACM*, 61(1):2:1–2:32, 2014. doi: [10.1145/2559903](http://dx.doi.org/10.1145/2559903)

[38] Virginia Vassilevska Williams. Hardness of easy problems: Basing hardness on popular conjectures such as the strong exponential time hypothesis (invited talk). In *IPEC 2015: Proc. of the 10th International Symposium on Parameterized and Exact Computation*, volume 43 of *LIPIcs*, pages 17–29, 2015. doi: [10.4230/LIPIcs.IPEC.2015.17](http://dx.doi.org/10.4230/LIPIcs.IPEC.2015.17)