Dynamical Generation of Linear $\sigma$
Model SU(3) Lagrangian and Meson Nonet Mixing

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This paper is the SU(3) extension of the dynamically generated SU(2) linear $\sigma$ model Lagrangian worked out previously using dimensional regularization. After discussing the quark-level Goldberger-Treiman relations for SU(3) and the related gap equations, we dynamically generate the meson cubic and quartic couplings. This also constrains the meson-quark coupling constant to $g = 2\pi/\sqrt{3}$ and determines the SU(3) scalar meson masses in a Nambu-Jona-Lasinio fashion. Finally we dynamically induce the U(3) pseudoscalar and scalar mixing angles in a manner compatible with data.

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1. Introduction

In a recent paper ref. [1], we have extended the original spontaneously broken SU(2) linear $\sigma$ model (L$\sigma$M) to the quark-level dynamically generated L$\sigma$M. The latter L$\sigma$M is close in spirit to the four-fermion theory of NJL, only the tight-binding bound states with chiral-limiting mass $m_\pi = 0$ and $m_\sigma = 2m_q$ in the NJL approach become elementary particle states in the L$\sigma$M scheme. Dimensional regularization and $Z = 0$ compositeness conditions are the key ingredients making the SU(2) theory extremely predictive. In this paper we generalize the dynamically generated L$\sigma$M to SU(3) symmetry and also discuss meson nonet mixing.

Experimental signatures [2–4] of the elusive nonstrange isoscalar and strange isospinor scalar resonances $\sigma(600−700)$ and $\kappa(800−900)$ combined with recent theoretical observations on scalar mesons [5,6] make the original SU(2) and SU(3) linear sigma model (L$\sigma$M) field theories [7,8] of interest once again. Specifically, a broad nonstrange scalar $\sigma$ (400–900) was extracted in the last reference in [2] and supported in the 1996 PDG tables [3] (with an upper limit mass scale 300 MeV higher).

Such a scalar $\sigma$ (400–900) has a mean value of $m_\sigma = 650$ MeV, which is in agreement with the prediction of the dynamically generated L$\sigma$M [1]. This SU(2) L$\sigma$M computed in one-loop order reproduces [1,9] many satisfying chiral-limiting results: $m_\pi = 0$, $m_\sigma = 2m_q$ (the latter two of course are true in the four-fermion NJL model [10]), vector meson dominance (VMD) universality [11] $g_{\rho\pi\pi} = g_\rho$, the dynamically generated scale $g_{\rho\sigma\rho} = 2\pi$, the KSRF [12] rho mass $m_\rho = \sqrt{2}g_\rho f_\pi$, and Weinberg’s [5] mended chiral symmetry decay width relation $\Gamma_\sigma = (9/2)\Gamma_\rho$. Moreover the semileptonic $\pi \rightarrow e\nu\gamma$ empirical [3] structure-dependent form factors are approximately recovered [13] from the SU(2) L$\sigma$M quark and meson loops. Finally the observed [14] $a_0(984) \rightarrow \gamma\gamma$ radiative decay width has been obtained [15] using SU(3) quark and meson loops in the L$\sigma$M.
Very recently, the SU(2) $L\sigma M$ Lagrangian density (shifted around the stable vacuum with $\langle \sigma \rangle = \langle \pi \rangle = 0$ and quarks now with mass $m_q$) having interacting part for elementary quarks and $\pi$ and $\sigma$ mesons,

$$\mathcal{L}_{int} = g\bar{\psi}(\sigma + i\gamma_5 \vec{\tau} \cdot \vec{\pi})\psi + g'\sigma(\sigma^2 + \vec{\pi}^2) - \lambda(\sigma^2 + \vec{\pi}^2)^2/4 \quad (1a)$$

with chiral-limiting meson-quark and meson-meson couplings [7]

$$g = m_q/f_\pi, \quad g' = m^2_\sigma/2f_\pi = \lambda f_\pi \quad (1b)$$

(for $f_\pi \approx 90$ MeV), has been dynamically generated [1]. Such dynamical generation is driven by the meson-quark interaction $g\bar{\psi}(\sigma + i\gamma_5 \vec{\tau} \cdot \vec{\pi})\psi$ alone. This leads to the chiral-limiting meson masses $m_\pi = 0$, $m_\sigma = 2m_q$, meson-meson cubic and quartic couplings $g'$, $\lambda = g'/f_\pi$ and also constrains the fundamental meson-quark coupling to $g = 2\pi/\sqrt{N_c}$. The latter coupling together with the NJL scalar $\sigma$ mass follow from dimensional regularization considerations. However, these results are regularization independent as shown in the second reference in [1].

Since the analogous (but much more complex) SU(3) $L\sigma M$ Lagrangian [8] has only been considered in its (unshifted) spontaneously generated form (but also giving rise to interesting physics [16,17]), in this paper we try to dynamically generate the SU(3) $L\sigma M$ Lagrangian, the U(3) meson masses, couplings, and in addition, dynamically induce the empirical meson mixing angles.

In Section 2 we focus on the quark-level SU(3) Goldberger-Treiman (GT) relations and corresponding SU(3) “gap equations.” Then in Section 3 we dynamically generate the nonstrange (NS) $\sigma$ meson–$\pi\pi$ and $K\bar{K}$ couplings $g'_{\sigma\text{NS}\pi\pi}$, $g'_{\sigma\text{NS}KK}$ obtained from the vanishing chiral-limiting pseudoscalar meson masses $m_\pi = 0$, $m_K = 0$. The latter also gives rise to the strange (S) $\sigma$ meson–$K\bar{K}$ coupling $g'_{\sigma\text{S}KK}$. Next in Section 4 we dynamically generate the SU(3)-broken scalar meson masses (but with $m_\pi = m_K = 0$)

$$m_{\sigma\text{NS}} = 2\hat{m}, \quad m_\kappa = 2\sqrt{m_\sigma\bar{m}}, \quad m_{\sigma\text{S}} = 2m_s. \quad (2)$$
Here the nonstrange, kappa and strange scalar meson masses are $m_{\sigma NS}$, $m_{\kappa}$, $m_{\sigma S}$ and the nonstrange and strange constituent quark masses are $\hat{m}$, $m_s$, respectively.

In Section 5 we comment on “bootstrapping” the cubic and quartic meson-meson couplings from one-loop order to tree order based on the gap equations discussed in Section 2. Finally, in Section 6 we dynamically induce the U(3) quark-annihilation graphs in the SU(3) $\sigma M$. They simulate (but do not double count) the effects of nonperturbative QCD by predicting $\eta - \eta'$ and $\sigma - f_0$ mixing angles that in fact are compatible with data. The latter mixing approach, while fitted self-consistently, bypasses a direct nonperturbative calculation of the singlet U(3) meson masses. We summarize our dynamically generated findings in Section 7 and list in the Appendix the needed nonstrange and strange (quark basis) U(3) structure constants.

\section{Quark Level GT Relations And Gap Equations}

Using only constituent quark masses already induced through vacuum expectation values of scalar fields along with the meson-quark (chiral) coupling $g\bar{\psi}\left(\sigma_{NS} + i\gamma^5 \vec{\lambda} \cdot \vec{\pi}\right) + \lambda^a (\kappa^a + i\gamma^5 K^a) + \cdots |\psi$, the quark loop pion and kaon decay constants depicted in Figs. 1 are in the chiral limit ($q_\pi \to 0$, $q_K \to 0$ but $m_s \neq \hat{m}$) with $d^4 p = d^4 p (2\pi)^{-4}$,

\begin{align}
if_\pi &= 4N_c g \int \frac{d^4 p \hat{m}}{(p^2 - \hat{m}^2)^2}, \quad if_K = 4N_c g \int \frac{d^4 p \frac{1}{2}(m_s + \hat{m})}{(p^2 - \hat{m}^2)(p^2 - m_s^2)}. \tag{3a}
\end{align}

Then invoking the quark-level pion GT relation in (1b) and its natural kaon generalization [18],

\begin{align}
f_\pi g &= \hat{m}, \quad f_K g = \frac{1}{2}(m_s + \hat{m}), \tag{3b}
\end{align}

eqs. (3a) lead to the (log-divergent) CL gap equations

\begin{align}
1 &= -i4N_c g^2 \int_{\Lambda^2} \frac{d^4 p}{(p^2 - \hat{m}^2)^2}, \tag{4a}
1 &= -i4N_c g^2 \int_{\Lambda^2} \frac{d^4 p}{(p^2 - m_s^2)(p^2 - \hat{m}^2)}. \tag{4b}
\end{align}
Since in the SU(3) LσM, the meson-quark coupling constant \( g \) is the same for pions and for kaons (in or away from the CL), the knowledge of \( g \) in (4a) and in (4b) fixes the log-divergent scales \( \Lambda \) and \( \Lambda' \). In fact it has been shown [1] in the dynamically generated SU(2) LσM that for \( N_c = 3 \),

\[
g = \frac{2\pi}{\sqrt{N_c}} \approx 3.6276 ,
\]

compatible with the nonstrange GT estimate in (3b) away from the CL \( g = \hat{m}/f_\pi \approx 340 \text{ MeV}/93 \text{ MeV} \approx 3.66 \). (We shall return to the derivation of (5) in Section 4.) Accordingly, the gap equations in (4) give the Euclidean integrals

\[
1 = \int_0^{\Lambda^2/\hat{m}^2} \frac{(dq^2/\hat{m}^2)(q^2/\hat{m}^2)}{(1 + q^2/\hat{m}^2)^2} , \quad 1 \approx \int_0^{\Lambda'^2/m_s\hat{m}} \frac{(dq^2/m_s\hat{m})(q^2/m_s\hat{m})}{(1 + q^2/m_s\hat{m})^2} .
\]

The denominator in the second integral of (6) is the geometric average of the exact product \((1 + q^2/\hat{m}^2)(1 + q^2/m_s^2)\). Then both integrals in (6) being unity in turn leads to

\[
\Lambda^2/\hat{m}^2 \approx \Lambda'^2/m_s\hat{m} \approx 5.3 .
\]

To verify that the dimensionless cutoffs in (7) make physical sense, we must first introduce a dimensionful scale. Returning to the CL nonstrange GT relation in (3b), we invoke the CL pion decay constant scale [19] \( f_\pi^{CL} \approx 90 \text{ MeV} \), so that the CL nonstrange quark mass is (with divergenceless axial current \( \partial A^\pi = 0 \) generating the GT relation for pions)

\[
f_\pi^{CL} g = \hat{m}^{CL} \approx (90 \text{ MeV})(3.6276) \approx 326 \text{ MeV} .
\]

Away from the CL the ratio of the two GT relations in (3b) is fixed to the empirical value [12]

\[
\frac{f_K}{f_\pi} = \frac{1}{2} \left( \frac{m_s}{\hat{m}} + 1 \right) \approx 1.22 \quad \text{or} \quad \frac{m_s}{\hat{m}} \approx 1.44 .
\]

In fact this constituent quark mass ratio of about 1.4 is also known to hold for baryon magnetic dipole moments [20], meson charge radii [21] and \( K^* \to K\gamma \) decays [22]. However
in the chiral limit we might expect \( m_s/\hat{m} \leq 1.44 \), say 4/3. Finally then the cutoff scales in (7) become

\[
\Lambda \sim \sqrt{5.3 \hat{m}^{CL}} \sim 750\text{MeV}, \quad \Lambda' \sim \sqrt{5.3 m_s^{CL} \hat{m}^{CL}} \sim 860\text{MeV}.
\] (10)

The above nonstrange cutoff scale of 750 MeV separates SU(2) LσM elementary particles (the \( u, d \) quarks, \( \bar{\pi} \) and \( \sigma \) mesons, the latter taken as \( \sigma \) (650) as justified from [2,3] and the discussion in the introduction) from the \( \bar{q}q \) bound states with mass \( > 750 \) MeV (\( \rho(770), \omega(783), A_1(1260), f_2(1275), A_2(1320) \)). Likewise the qualitative isospinor cutoff scale in (10) of 860 MeV separates elementary \( K(495), \kappa(820) \) mesons and \( m_s \approx 480 \) MeV quarks (as we shall see later) from bound state \( K^*(895), K^{**}(1350) \) mesons in the SU(3) LσM. In field theory language, the merging of the elementary particle and bound state cutoff scales inferred from the gap equations (4) correspond to \( Z = 0 \) compositeness conditions [23], whereby the scalar mesons \( \sigma(650) \) and \( \kappa(820) \) can be consistently treated either as elementary (in the LσM) or as bound states (in the NJL picture).

3. Dynamical Generation Of The Cubic Meson Couplings

Consider now the Nambu-Goldstone massless pion and kaon in the chiral limit (CL). Starting only with the quark-meson coupling used in Section 2, the quark one-loop order pion self-energies are depicted in Figs. 2. In the CL the “vacuum polarization (VP)”-bubble-type amplitude of Fig. 2a is (displaying the two quark propagators in the denominator to parallel eqs. (3) and (4))

\[
M_{VP}^{\pi} = -i4N_c 2g^2 \int \frac{d^4 p(p^2 - \hat{m}^2)}{(p^2 - \hat{m}^2)^2}, \quad (11a)
\]

where the factor of 2 in (11a) arises from \( u \) and \( d \) quark loops for \( \pi^0qq \) couplings, or \( (\sqrt{2})^2 \) for \( \pi^+ud \) couplings. Likewise the “quark tadpole” amplitude of Fig. 2b is in the CL

\[
M_{qktad}^{\pi} = i4N_c 2g 2g' \frac{\hat{m}}{m_s^{\sigma NS}} \int \frac{d^4 p \hat{m}}{p^2 - \hat{m}^2}. \quad (11b)
\]
The “new” $\sigma\pi\pi$ coupling $2g'$ (the factor 2 is from Bose symmetry) in Fig. 2b is dynamically generated so that the CL pion mass remains zero in one-loop order in the CL:

$$m_{\pi}^2 = M_{V\pi}^2 + M_{qktad}^\pi = 0 \quad (12a)$$

$$\left(g - \frac{2g'}{m_{\sigma NS}^2} \hat{m}\right) \int \frac{d^4p}{(p^2 - \hat{m}^2)} = 0 \quad (12b)$$

$$g'_{\sigma NS\pi\pi} = m_{\sigma NS}^2/2f_{\pi}. \quad (12c)$$

Note that regardless of the two quadratic divergent integrals in Eqs. (11), the dynamically generated meson-meson tree-level coupling $g'$ in (12c) “conspires” to keep the CL pion massless in (12a) and (12b) [1,9].

This SU(2) $L\sigma M$ result (12c) can be extended to SU(3) by considering the bubble and quark tadpole graphs in Figs. 3 which contribute to the kaon mass. The bubble amplitude for Fig. 3a is in the CL ($p_K \to 0$, $m_s \neq \hat{m}$),

$$M_{V\pi}^K = -i4N_c2g^2 \int \frac{d^4p(p^2 - m_s\hat{m})}{(p^2 - m_s^2)(p^2 - \hat{m}^2)}, \quad (13a)$$

while the two nonstrange (NS) and strange (S) quark tadpole graphs of Fig. 3b and Fig. 3c have the respective CL amplitudes

$$M_{qktadNS}^K = i\frac{4N_cgg'_{NS}}{m_{\sigma NS}^2} \int \frac{d^4p2\hat{m}}{(p^2 - \hat{m}^2)}, \quad (13b)$$

$$M_{qktadS}^K = i\frac{4N_c\sqrt{2}gg'_{SS}}{m_{\sigma S}^2} \int \frac{d^4p m_s}{(p^2 - m_s^2)}. \quad (13c)$$

Note that the factor of $2g^2$ in (13a) is due to the two $K^0 sd$ vertices each with coupling $\sqrt{2}g$, while the factor of $2\hat{m}$ in (13b) counts 2 nonstrange quarks in the NS tadpole loop of Fig. 3b. Finally the factor of $\sqrt{2}g$ in (13c) is due to the $\sigma_s SS$ coupling in Fig. 3c. In order to combine Eqs. (13a), (13b) and (13c) so that the CL kaon mass remains zero,

$$m_K^2 = M_{V\pi}^K + M_{qktadNS}^K + M_{qktadS}^K = 0, \quad (14a)$$
we invoke the partial-fraction identity

\[
\frac{(m_s + \hat{m})(p^2 - m_s\hat{m})}{(p^2 - m_s^2)(p^2 - \hat{m}^2)} = \frac{\hat{m}}{p^2 - \hat{m}^2} + \frac{m_s}{p^2 - m_s^2} .
\]  

(14b)

Replacing the integrand of the kaon bubble amplitude in (13a) by the right-hand-side (RHS) of (14b), we see that the vanishing of \(m_K^2\) in (14a) requires the two coefficients of the nonstrange loop integral to cancel and also the two coefficients of the strange loop integral to cancel. Thus we have dynamically generated two more tree-level meson cubic couplings in the CL:

\[
g'_{\sigma NSKK} = \frac{m_{\sigma NS}^2}{2f_K}
\]  

(15a)

\[
g'_{\sigma SKK} = \frac{m_{\sigma S}^2}{\sqrt{2}f_K} .
\]  

(15b)

The respective Clebsch-Gordon coefficients of 1, 1/2, 1/√2 in (12c), (15a) and (15b) correspond to the SU(3) structure constants \(d_{NS33} = 1, d_{NSKK} = 1/2, d_{SKK} = 1/\sqrt{2}\), derived in the Appendix. Thus the dynamically generated cubic meson-meson couplings (12c), (15a) and (15b) indeed follow an SU(3) LσM pattern.

4. Scalar Cubic Couplings And Scalar Meson Masses

By chiral symmetry we expect the respective scalar-scalar-scalar meson couplings to be identical to the analog scalar-pseudoscalar-pseudoscalar couplings. For the SU(2) case, the two \(\gamma_5\) vertices in the \(\sigma_{NS}\pi\pi\) loop of Fig. 4a reduce the divergence to

\[
g'_{\sigma NS\pi\pi} = 2g\hat{m} \left[ -i4N_c g^2 \int \frac{d^4p(p^2 - \hat{m}^2)}{(p^2 - \hat{m}^2)^3} \right] = 2g\hat{m} ,
\]  

(16a)

by virtue of the gap equation (4a). But the two factors of unity replacing the \(\gamma_5\)'s for the analogue \(\sigma_{NS}\sigma_{NS}\sigma_{NS}\) loop of Fig. 4a mean expanding out the trace in the CL gives [1]

\[
g'_{\sigma NS\sigma NS\sigma NS} = 2g \left[ -iN_c g^2 \int \frac{d^4pTr(\phi + \hat{m})^3}{(p^2 - \hat{m}^2)^3} \right] = 6g\hat{m} ,
\]  

(16b)
where we keep only the log divergent piece in the integral of (16b) since it dominates the coupling constant $g'_{\sigma_{NS}\sigma_{NS}\sigma_{NS}}$. Only then is the tree-order chiral symmetry $g'_{\sigma_{NS}\pi\pi} = g'_{\sigma_{NS}\sigma_{NS}\sigma_{NS}}$ recovered in one-loop order.

Moreover both cubic loop couplings in (16) “bootstrap” the $g'$ tree coupling $g' = m_{\sigma_{NS}}^2/2f_\pi$ in the CL provided [1,9]

$$m_{\sigma_{NS}} = 2\hat{m},$$  \hspace{1cm} (17a)

found by setting $g' = 2g\hat{m} = m_{\sigma_{NS}}^2/2f_\pi$ and using the GTR $f_\pi g = \hat{m}$. This “shrinking” of quark loops to points in eqs. (16) again corresponds to a $Z = 0$ compositeness condition [23].

To dynamically generate (17a) we consider instead Figs. 5 representing $m_{\sigma_{NS}}^2$ in the CL $p \to 0$. Using dimensional regularization, these graphs sum to [1]

$$m_{\sigma_{NS}}^2 = 16iN_c g^2 \int d^4p \left[ \frac{\hat{m}^2}{(p^2 - \hat{m}^2)^2} - \frac{1}{p^2 - \hat{m}^2} \right] = \frac{N_c g^2 \hat{m}^2}{\pi^2},$$  \hspace{1cm} (17b)

using a $\Gamma$ function identity $\Gamma(2-l) + \Gamma(1-l) \to -1$ in $2l = 4$ dimensions. Combining (17a) and (17b) one obtains the meson-quark coupling $g = 2\pi/\sqrt{N_c}$. The latter and (17a) can also be dynamically generated via the quark mass gap tadpole combined with the above dimensional regularization identity again [1].

In a similar fashion, the scalar kappa meson self energies of Figs. 6 have the bubble (VP) amplitude in the CL

$$M_{VP}^\kappa = -i4N_c(\sqrt{2}g)^2 \int \frac{d^4p}{(p^2 - m_s^2)(p^2 - \hat{m}^2)} \left[ p^2 + m_s\hat{m} \right].$$  \hspace{1cm} (18a)

Again adding and subtracting $m_s\hat{m}$ terms to the numerator of (18a), cancelling the quadratic divergent VP part against the tadpole graphs of Fig. 6b,c and using the gap equation (4b), we find in the CL

$$m_\kappa^2 = 0 + 2\cdot 2m_s\hat{m} \quad \text{or} \quad m_\kappa = 2\sqrt{m_s\hat{m}}.$$  \hspace{1cm} (18b)
Finally for the purely strange meson self-energy graphs of Figs. 7, by analogy with the nonstrange scalar mass equation (17b), the graphs sum via the above dimensionless regularization identity to

\[ m_{\sigma_s}^2 = 8iN_c g_{S}^2 \int d^4p \left[ \frac{m_s^2}{(p^2 - m_s^2)^2} - \frac{1}{p^2 - m_s^2} \right] = \frac{N_c g_{S}^2 m_s^2}{2\pi^2}. \] (19a)

Invoking the U(3) coupling \( g_{S} = \sqrt{2} g \) (which can be generated via the strange quark mass gap), equation (19a) generates the strange scalar meson mass

\[ m_{\sigma_s} = 2m_s. \] (19b)

Thus we have dynamically generated the chiral-limiting (NJL-like) scalar masses (17a), (18b), (19b) in the SU(3) LσM as indicated in Eq. (2).

There are in fact two independent ways of extending these CL results away from the chiral limit so as to obtain the “physical” quark and scalar meson masses. More specifically with \( m_{\pi} \neq 0 \), the nonstrange CL relation (16) becomes

\[ m_{\sigma_{NS}}^2 - m_{\pi}^2 = (2\hat{m}^{CL})^2 = (653 \text{ MeV})^2 \quad \text{or} \quad m_{\sigma_{NS}} \approx 668 \text{MeV}. \] (20)

Alternatively with \( f_{\pi} \approx 93 \text{ MeV} \) away from the chiral limit, the quark-level GT relation (8) becomes (still with \( g = 2\pi/\sqrt{3} \)),

\[ \hat{m} = f_{\pi} g \approx (93 \text{MeV})(3.6276) \approx 337 \text{MeV}, \] (21a)

in close agreement with the \( u \) constituent quark mass found from magnetic dipole moments [20]. Then a NJL-type estimate of the chiral-broken nonstrange scalar \( \sigma \) mass is

\[ m_{\sigma_{NS}} = 2\hat{m} \approx 674 \text{MeV}. \] (21b)

Henceforth we will take \( m_{\sigma_{NS}} \approx 670 \text{ MeV} \) as the average between (20) and (21b).
The $I = 1/2$ scalar kappa meson with mass $m_\kappa \neq 0$ follows from an “equal-splitting-law” [24] compared to (20),

$$m_\kappa^2 - m_K^2 = m_{\sigma_{NS}}^2 - m_\pi^2 \approx 0.43 \text{GeV}^2 \quad \text{or} \quad m_\kappa \approx 820 \text{MeV} . \quad (22)$$

On the other hand, the chiral-broken strange constituent quark mass found from (21a) and the ratio $m_s/\hat{m} \approx 1.4$ from (9) is

$$m_s = \hat{m}(m_s/\hat{m}) \approx 475 \text{MeV} , \quad (23a)$$

also in reasonable agreement with the magnetic moment determination [20]. Then the NJL-type estimate of the chiral-broken kappa mass in Eq. (2) scaled to (21a) above is

$$m_\kappa = 2 \sqrt{m_s \hat{m}} \approx 2 \cdot 337 \sqrt{1.4} \text{MeV} \approx 805 \text{MeV} , \quad (23b)$$

with an average $m_\kappa \approx 810$ MeV, midway between (22) and (23b).

In the early 1970’s, the particle data group (PDG) suggested the ground state kappa mass is in the 800-900 MeV region. Since 1974, however, this $\kappa$ has been replaced by the $\kappa(1450)$. But scaled to the $\sigma(670)$ mass of (20) or (21), a $\kappa$ in the 800-900 MeV region as in (22) and (23) is unavoidable. Nevertheless it is worth commenting on why a (peripherally produced) $\kappa(810)$ has not been observed. We suggest it is because of the soft pion theorem [25] (SPT) suppressing the $A_1 \to \pi(\pi\pi)_{sw}$ decay rate due to the interfering $A_1 \to \sigma\pi$ amplitude. Likewise, a similar SPT suppresses the $\kappa(810) \to K\pi$ decay amplitude, explaining why the PDG tables no longer list the $\kappa(810)$. Specifically, the latter peripherally produced $\kappa(810)$ in $K^-p \to K^-\pi^n$ is suppressed by the quark (box plus triangle) SPT chiral cancellation as in ref. [25].

Henceforth we will take the ground state kappa mass at $m_\kappa \approx 810$ MeV as the average between (22) and (23b). It is satisfying that these elusive scalar $\sigma$ and $\kappa$ masses were recently seen in polarization measurements (Svec et al., Refs. [4]) at 750 MeV and 887 MeV respectively, which are unaffected by the above soft pion theorem of ref. [25].

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We can also estimate the pure strange $s\bar{s}$ scalar meson mass in two ways. The equal-splitting-law analogue of (20) and (22), is

$$m_{\sigma_s}^2 - m_\kappa^2 = m_\kappa^2 - m_{\sigma_{NS}}^2 \quad \text{or} \quad m_{\sigma_s} \approx 930\text{MeV} ,$$

(24)

while the NJL-like strange scalar mass from (2) using $m_s \approx 475$ MeV from (23a) is

$$m_{\sigma_s} = 2m_s \approx 950\text{MeV} ,$$

(25)

with average mass $m_{\sigma_s} \approx 940$ MeV. Tornqvist and Roos in ref. [2] claim the $f_0$ (980) is mostly an $s\bar{s}$ scalar meson. Accounting for the observed scalar mixing angle of 20° (scalar mixing is discussed in eqs. (37)–(39) of Section 6), this observed $f_0$ (980) is compatible with the above predicted $\sigma_s$ (930–950). The average “physical” chiral-broken scalar meson masses which we shall henceforth use in our dynamically generated SU(3) L\sigma M are then

$$m_{\sigma_{NS}} \approx 670\text{MeV} , \quad m_\kappa \approx 810\text{MeV} , \quad m_{\sigma_s} \approx 940\text{MeV} .$$

(26)
5. Bootstrapping the Quartic Meson Lagrangian

Once the $Z = 0$ compositeness conditions [23] (or the quark mass gap and meson mass equations) are known, via the SU(3) gap equations (4) and SU(3) NJL equations (2), to shrink quark loops to $L\sigma M$ tree graphs, one should study how to induce the SU(3) quartic Lagrangian density

$$L_{\text{quartic}}^{L\sigma M} = -\lambda [\sigma_{NS}^2 + \bar{\pi}^2 + \kappa^2 + K^2 + \sigma_S^2 + \eta_{S}^2]^2/4.$$  (27)

The U(2) nonstrange sector of (27) was investigated in ref.[1] via the $u,d$ quark box of Fig.8a, leading to the chiral limiting (CL) $\pi^o\pi^o \rightarrow \pi^o\pi^o$ amplitude

$$T = -i8N_c g^4 \int d^4p(p^2 - \hat{m}^2)^{-2} = 2g^2,$$  (28a)

by virtue of the log-divergent gap equation (4a). Similarly the $\pi^+K^o \rightarrow \pi^+K^o$ quark box of Fig.8b has CL amplitude

$$T = -i(\sqrt{2})^4N_c g^4 \int d^4p(p^2 - \hat{m}^2)^{-1}(p^2 - m_s^2)^{-1}[p^2 - m_s\hat{m}] = 2g^2,$$  (28b)

by virtue of the partial fraction identity (14b) and gap equation (4). Likewise the $\eta_S\eta_S \rightarrow \eta_S\eta_S$ strange quark box of Fig.8c has CL amplitude

$$T = -i4N_c g_S^4 \int d^4p(p^2 - m_s^2)^{-2} = g_S^2 = 2g^2,$$  (28c)

due to the strange quark gap equation analogous to (4a) together with $g_S = \sqrt{2}g$.

Thus all three quark box graphs of Figs.8 and equ.(28) have effective quartic (box) couplings in the chiral limit

$$\lambda_{\text{quartic box}} \rightarrow 2g^2,$$  (29a)

whereas the SU(3) $L\sigma M$ quartic lagrangian (27) has $L\sigma M$ tree strength

$$\lambda = g'/f_\pi = (2\hat{m}g)/f_\pi = 2g^2,$$  (29b)
by virtue of the quark-level GT relation $\hat{m}/f_\pi = g$. The fact that $\lambda = 2g^2$ means that, starting from the meson-quark interaction, the SU(3) $L\sigma M$ quark box graphs of Figs. 8 and equs. (28) bootstrap back to the $L\sigma M$ quartic Lagrangian (27). These are all further examples of the $Z = 0$ compositeness condition helping to dynamically generate the entire SU(3) $L\sigma M$ Lagrangian.

6. Dynamically Inducing Mixing Of Pseudoscalar And Scalar Meson States

Thus far, starting from the fundamental SU(3) meson-quark chiral interaction $g\bar{\psi}\lambda^i[S^i + i\gamma_5 P^i]\psi$, we have dynamically generated the $L\sigma M$ cubic and quartic meson-meson couplings, the chiral-limiting pseudoscalar and scalar SU(3) meson masses, and even the meson-quark couplings $g$ and $g_S$. Taken together this forms the interacting part of the SU(3) linear sigma model ($L\sigma M$) Lagrangian density

$$L_{int}^{L\sigma M} = L_{meson-quark} + L_{meson-meson}. \quad (30)$$

It is then natural to study the additional U(3) mixing Lagrangian $L_{mixing}$. In the spontaneously generated $L\sigma M$ scheme of Refs. [8], such an “input” $L$ mixing term introduces extra mixing parameters in (30) which are to be determined by experiment. Alternatively in our dynamically generated approach to the SU(3) $L\sigma M$, the predicted parameters in (30) already match observation without introducing new (arbitrary) parameters. That is, $L_{L\sigma M}$ in (30) is an “output” Lagrangian rather than an input, and there is no additional $L$ mixing Lagrangian.

In this dynamically generated $L\sigma M$ theory, the chiral-broken seven pseudoscalar (Nambu-Goldstone) meson masses $m_{\pi}^2$ and $m_K^2$ are inserted in the theory by hand and then the six chiral-broken scalar (NJL-like) masses of Eq. (26) will in turn dynamically generate (fit) the observed $\eta$ and $\eta'$ pseudoscalar along with the $\sigma$ and $f_\rho$ scalar meson masses.
More specifically, for the case of pseudoscalar (P) meson states, the U(3) meson \( \eta - \eta' \) mixing is generated by the quark annihilation amplitude \( \beta_P \) which turns a \( \bar{u}u \) or \( \bar{d}d \) meson \( P \) into a \( \bar{u}u, \bar{d}d \) or \( \bar{s}s \) meson \( P' \) state. This dynamical breaking of the OZI rule can be characterized in the language of QCD [20,26], in the model-independent mixing approach of Ref. [27], or in terms of the SU(3) L\( \sigma \)M.

To reach the same mixing conclusions (28)-(30) in the context of QCD, one observes that a singlet \( \eta_0 \) is twice formed via the “pinched” quark annihilation graph in Fig.8. But such quark triangle graphs “shrink” to points in the L\( \sigma \)M by the log-divergent gap equations (4), i.e. via \( Z = 0 \) conditions. Then \( I = 0 \) mesons take the place of QCD gluons in Fig.8 so that one must consider the L\( \sigma \)M meson loop graphs of Fig.9 as simulating \( \beta_P \) for \( \eta' - \eta \) mixing. In all of the above cases, one classifies the \( I = 0 \) nonstrange meson mass matrix as

\[
M_P^2 = \begin{pmatrix}
    m_{\pi}^2 + 2\beta_P & \sqrt{2}\beta_P \\
    \sqrt{2}\beta_P & 2m_K^2 - m_{\pi}^2 + \beta_P
\end{pmatrix} \rightarrow \begin{pmatrix}
    m_\eta^2 & 0 \\
    0 & m_{\eta'}^2
\end{pmatrix},
\]  

where the arrow indicates rediagonalization to the observed \( \eta \) and \( \eta' \) \( I = 0 \) states. Here nonstrange (NS) and strange (S) \( I = 0 \) meson states contribute to a unitary singlet state according to \( |0\rangle = \sqrt{2/3}|NS\rangle + \sqrt{1/3}|S\rangle \), where \( |NS\rangle = |\bar{u}u + \bar{d}d|/\sqrt{2} \) and \( |S\rangle = |\bar{s}s\rangle \).

Note that the one parameter \( \beta_P \) on the LHS of (31) determines the two measured masses \( m_\eta^2 \) and \( m_{\eta'}^2 \) on the RHS of (31). Specifically, the trace of (31) requires

\[
2m_K^2 + 3\beta_P = m_\eta^2 + m_{\eta'}^2 \quad \text{or} \quad \beta_P \sim 0.24\text{GeV}^2,
\]  

while the determinant of (31) gives

\[
m_{\pi}(2m_K^2 - m_{\pi}^2) + (4m_K^2 - m_{\pi}^2)\beta_P = m_\eta^2m_{\eta'}^2 \quad \text{or} \quad \beta_P \sim 0.28\text{GeV}^2.
\]

Rather than work with isospin L\( \sigma \)M intermediate states and dynamically generate \( \beta_P \), it will be more straightforward to consider intermediate QCD glue (which is automatically flavor blind and \( I = 0 \)) and “dynamically fit” \( \beta_P \) in (31) to the scale of (32) even in the context of the L\( \sigma \)M [28]. There is then no need to introduce an additional SU(3)
Lagrangian simulating this mixing; it is already built into the above quantum-mechanical picture.

Further fine tuning of (32) comes from accounting for the SU(3)-breaking ratio of the nonstrange and strange quark propagators in Fig. 9 via the constituent quark mass ratio $X = \hat{m}/m_s \approx 0.7$ from Eq. (9). In the latter case, keeping $X$ free and weighting the off-diagonal NS-$S$ $\beta_P$ in (31) by $X$ and the $S-S$ $\beta_P$ by $X^2$, the two parameters $\beta_P$ and $X$ on the LHS of this modified matrix (31) are uniquely determined by the $\eta$ and $\eta'$ masses to be [26]

$$\beta_P = \frac{(m_{\eta'}^2 - m_{\pi}^2)(m_{\eta}^2 - m_{\pi}^2)}{4(m_K^2 - m_{\pi}^2)} \approx 0.28 \text{ GeV}^2,$$

$$X \approx 0.78.$$  \hspace{1cm} (33a)

These latter two fitted parameters are compatible with (32) and with $X \approx 0.7$ from (9). Thus there is only the one nonperturbative parameter $\beta_P \approx 0.28 \text{ GeV}^2$ to be explained in our dynamically fitted scheme. In fact this $\beta_P$ can be partially understood from a perturbative 2-gluon anomaly-type of graph [29,30]. Since the SU(3) nonstrange and strange pseudoscalar and scalar meson masses have already been dynamically generated in (26), our dynamically fitted U(3) extension in (33) cannot alter the masses in (26), but instead rotates the states via an $\eta'-\eta$ mixing angle. In effect, $\beta_P = m_{\eta_o}^2/3$ in (33) for $m_{\eta_o} \approx 915$ MeV regardless of the mixing scheme: QCD in Fig.9 or the LoSM induced by Figs. 8.

To this end one can recast the nonperturbative fitted pseudoscalar mixing scale of $\beta_P$ in (33) in terms of the quark nonstrange (NS)-strange (S) basis pseudoscalar mixing angle $\phi_P$ with physical $\eta$ and $\eta'$ states defined by

$$|\eta\rangle = \cos \phi_P |\eta_{NS}\rangle - \sin \phi_P |\eta_S\rangle, \quad |\eta'\rangle = \sin \phi_P |\eta_{NS}\rangle + \cos \phi_P |\eta_S\rangle,$$  \hspace{1cm} (34a)

or equivalently in terms of the more familiar singlet-octet mixing angle

$$\theta_P = \phi_P - \tan^{-1} \sqrt{2}.$$  \hspace{1cm} (34b)
Given the dynamically generated structure of the mixing mass matrix (31), the pseudoscalar mixing angle in (34) is predicted (fitted) to be [26]

$$\phi_P = \tan^{-1} \left[ \frac{(m_{\eta'}^2 - 2m_K^2 + m_{\pi}^2)(m_{\eta}^2 - m_{\pi}^2)}{(2m_K^2 - m_{\pi}^2 - m_{\eta}^2)(m_{\eta}^2 - m_{\pi}^2)} \right]^{1/2} \approx 42^\circ, \quad (34c)$$
or $$\theta_P \approx -13^\circ$$ since $$\tan^{-1} \sqrt{2} \approx 55^\circ$$ in (34b). We believe it significant that the “world data” in 1990 pointed to [27] $$\theta_P = -14^\circ \pm 2^\circ$$ or $$\phi_P = 41^\circ \pm 2^\circ$$, in good agreement with (34c).

In order to reconfirm this $$\eta - \eta'$$ mixing angle prediction (34c) specifically in the context of the LoSM, we consider the radiative decays $$\pi^0 \rightarrow \gamma\gamma$$, $$\eta \rightarrow \gamma\gamma$$, $$\eta' \rightarrow \gamma\gamma$$. In the former case, the usual $$u$$ and $$d$$ constituent quark triangle graphs lead to the $$\pi^0\gamma\gamma$$ amplitude $$F_{\alpha\beta\gamma\delta}k^\alpha_\gamma k^\beta_\gamma \epsilon^*_\delta$$ with $$F = \alpha/\pi f_{\pi}$$. This of course is the ABJ [30] anomaly amplitude or the Steinberger [31] fermion loop result with $$g_A = 1$$ and $$N_c = 3$$ at the quark level. Moreover this $$\pi^0\gamma\gamma$$ amplitude is also the LoSM prediction since there can be no three-pion (loop) correction. The resulting (LoSM) decay rate is then

$$\Gamma(\pi^0\gamma\gamma) = m_{\pi}^3(-\alpha/\pi f_{\pi})^2/64\pi \approx 7.6\text{eV}, \quad (35a)$$

which is very close to experiment [12] $$7.74 \pm 0.55 \text{ eV}$$.

For $$\eta, \eta' \rightarrow \gamma\gamma$$ decays, however, an additional constituent strange quark loop must be folded into the $$\pi^0\gamma\gamma$$ (LoSM) rate prediction (35a). This leads to the decay rate ratios [26,27] constrained to recent data in [3,32],

$$\frac{\Gamma(\eta\gamma\gamma)}{\Gamma(\pi^0\gamma\gamma)} = \left( \frac{m_\eta}{m_{\pi}} \right)^3 \left( \frac{5}{3} \right)^2 \cos^2 \phi_P \left( 1 - \frac{\sqrt{2}}{5} \frac{\hat{m}}{m_s} \tan \phi_P \right)^2 = 60 \pm 7, \quad (35b)$$

$$\frac{\Gamma(\eta'\gamma\gamma)}{\Gamma(\pi^0\gamma\gamma)} = \left( \frac{m_{\eta'}}{m_{\pi}} \right)^3 \left( \frac{5}{3} \right)^2 \sin^2 \phi_P \left( 1 + \frac{\sqrt{2}}{5} \frac{\hat{m}}{m_s} \cot \phi_P \right)^2 = 550 \pm 68. \quad (35c)$$
Using the constituent quark mass ratio $m_s/\hat{m} \approx 1.4$ from (9) or from Refs. [21,22], the LσM rate ratios in (35b,c) respectively predict $\phi_P = 45^\circ \pm 2^\circ$ and $\phi_P = 38^\circ \pm 4^\circ$, which average to the pseudoscalar $\eta' - \eta$ mixing angle extracted from $\eta, \eta' \to \gamma\gamma$ observations,

$$\phi_P = 40.5^\circ \pm 3^\circ \quad \text{or} \quad \theta_P = -14^\circ \pm 3^\circ . \quad (36)$$

Again we believe it is significant that the phenomenological pseudoscalar mixing angle in (36) is compatible with the world average $-14^\circ \pm 2^\circ$ in [27] and with the dynamically fitted value $\phi_P \approx 42^\circ$ in (34c) obtained from quark-annihilation graphs for intermediate QCD states.

As regards the chiral analog scalar mixing angle $\phi_S$, the parallel to the $\eta - \eta'$ angle $\phi_P$ in (34a) in the nonstrange-strange quark basis is defined via the physical $\sigma - f_o$ states

$$|\sigma\rangle = \cos \phi_S |\sigma_{NS}\rangle - \sin \phi_S |\sigma_S\rangle , \quad |f_o\rangle = \sin \phi_S |\sigma_{NS}\rangle + \cos \phi_S |\sigma_S\rangle . \quad (37)$$

Instead of the dynamical fitted approach to $\phi_S$ obtained through the quark-annihilation amplitude $\beta_S$ of Ref. [26] (yielding $\phi_S \sim 17^\circ$), given the already determined $\sigma_{NS}$ and $\sigma_S$ LσM scalar masses in (26), we can find $\phi_S$ via $\langle \sigma | f_o \rangle = 0$ and (37):

$$m_{\sigma_{NS}}^2 = m_\sigma^2 \cos^2 \phi_S + m_{f_o}^2 \sin^2 \phi_S \quad (38a)$$

$$m_{\sigma_S}^2 = m_\sigma^2 \sin^2 \phi_S + m_{f_o}^2 \cos^2 \phi_S . \quad (38b)$$

These two equations (38) and the physical mass $m_{f_o} \approx 980$ MeV constrain $m_\sigma$ and $\phi_S$ to the fitted values (with $m_{\sigma_{NS}} \approx 670$MeV, $m_{\sigma_S} \approx 940$MeV)

$$m_\sigma = \left[ m_{\sigma_{NS}}^2 + m_{\sigma_S}^2 - m_{f_o}^2 \right]^{1/2} \approx 610 \text{ MeV} , \quad (39a)$$

$$\phi_S = \sin^{-1} \left[ m_{f_o}^2 - m_{\sigma_S}^2 \right]^{1/2} \approx 20^\circ . \quad (39b)$$

For the pseudoscalar U(3) nonet (\(\pi, K, \eta, \eta'\)), we have used the dynamically generated SU(3) LσM and have self-consistently computed (dynamically fitted) the $\eta - \eta'$
mixing angle $\phi_P \approx 42^\circ$ via the Nambu-Goldstone $\pi$ and $K$ masses leading to the nonperturbative mass matrix (31) and Eqs. (33) or mixing angles in (34). For the scalar $U(3)$ nonet, however, we started with the SU(2) dynamically generated NJL-L$\sigma$M $NS$ and $S$ scalar masses (26) and used equal-splitting laws to fit the $I = 1/2$ kappa mass (squared) half-way between as in the averages (26). Together with the observed $f_0(980)$ this led to the fitted scalar mixing angle $\phi_S \approx 20^\circ$ and a slightly mixed $I = 0$ $\sigma(610)$ mass in (39). Note that the nearness of the observed $f_0(980)$ to our dynamically generated pure $\bar{s}s$ scalar mass $\sigma_S(940)$ is what is forcing $\phi_S$ in (39b) to be small. This parallels the $\bar{q}q$ vector case where the (nearby) $\phi(1020)$ vector meson is known to be almost purely $\bar{s}s$ strange.

All that remains undetermined in the latter nonet is the $I = 1$ scalar $a_o$ mass. Again it is the chiral equal-splitting laws [ESL] that require [24,33] in analogy with (22),

$$m_{a_o}^2 - m_{\eta_{NS}}^2 = m_{\sigma_{NS}}^2 - m_{\pi}^2 = m_{\kappa}^2 - m_{K}^2 \approx 0.43 \text{ GeV}^2 . \quad (40a)$$

Then with $\phi_P \approx 42^\circ$ so that $m_{\eta_{NS}} \approx 760$ MeV by analogy with (38), Eq. (40a) predicts the fitted $I = 1$ $a_o$ mass to be for $m_{\sigma_{NS}} \approx 670$ MeV from (20) and (21),

$$m_{a_o} = \left[ m_{\sigma_{NS}}^2 - m_{\pi}^2 + m_{\eta_{NS}}^2 \right]^{1/2} \approx 1.00 \text{ GeV} . \quad (40b)$$

Of course this latter predicted mass is presumably the observed [12] $a_o(984)$. The fact that this $I = 1$ $a_o(984)$ is near the $I = 0$ $f_0(980)$ does not necessarily signal that both the $a_o$ and $f_0$ have the same (nonstrange) flavor quarks (as do the $\rho$ and $\omega$ and the $a_o$ and $f_0$ in the alternative $\bar{q}q\bar{q}q$ scheme [34]). Rather, our $\bar{q}q$ SU(3) L$\sigma$M picture of a mostly strange $f_0(980)$ and nonstrange $a_o(984)$ is based on the (standard) mixing equations of (38) and (39) and the (infinite momentum frame) ESLs of Eqs. (40). To support this latter $\bar{q}q$ L$\sigma$M picture is the known almost purely strange vector meson $\phi(1020)$ being near this mostly strange $f_0(980)$. Moreover, Figs. 13 in the DM2 Collab. in refs. [2] also suggests that the $f_0(980)$ is composed mostly of $\bar{s}s$ quarks.
In the context of this same LσM, the SU(2) meson-meson coupling \( g'_{\sigma NS\pi\pi} = (m_{\sigma NS}^2 - m_{\pi}^2)/2g_{\pi} \) can be replaced by the SU(3) ESL coupling \( g'_{\delta NS\pi\pi} = (m_{a_0}^2 - m_{\eta NS}^2)/2f_{\pi} \).

Continuing to invoke SU(3) symmetry, one can then compute the scalar decay rate ratio as

\[
\frac{\Gamma(f_o(980)\pi\pi)}{\Gamma(a_o(984)\eta\pi)} = \frac{3p_{f_o}}{2p_{a_o}} \left( \frac{\sin \phi_S}{\cos \phi_P} \right)^2 = \frac{37 \pm 7 \text{MeV}}{57 \pm 11 \text{MeV}} ,
\]

where the observed rates are taken from the 1992 PDG tables [32]. For \( \phi_P \approx 42^\circ \) from (32c), the above (39a) requires (with momentum \( p_{f_o} = 467 \text{ MeV}, p_{a_o} = 319 \text{ MeV} \))

\[
\phi_S = 23^\circ \pm 3^\circ ,
\]

compatible with (39b). The 1992 PDG tables [32] takes the \( a_o \rightarrow \eta\pi \) rate as 57 \pm 11 MeV, but the high-statistics \( a_o \rightarrow \eta\pi \) rate measured by Armstrong et al., [35] of 95 \pm 14 MeV predicts \( \phi_S \approx 18^\circ \pm 3^\circ \) from (41a), more in line with (39b).

The above ground state \( \bar{q}q \) scalar LSM nonet (\( \sigma(610), \kappa(810), f_o(980), a_o(984) \)) with dynamically fitted mixing angle \( \phi_S \approx 20^\circ \) is qualitatively different from a \( \bar{q}q\bar{q}q \) four-quark [34] or \( \bar{K}K \) molecule [36] scheme. However the latter objections to a \( \bar{q}q \) picture [37] based on the recent Crystal Ball [14] radiative decays \( a_o \rightarrow \gamma\gamma \) and \( f_o \rightarrow \gamma\gamma \) can also be understood in the SU(3) LσM (but not in a pure \( \bar{q}q \) quark model). These narrow scalar decays have \( \bar{q}q \) quark loops which interfere destructively [15] with SU(3) LσM meson loops and lead to rates \( \sim 0.5 \text{ keV} \) or smaller as measured [14,32].
7. Conclusion

In summary, we started in Sec. 2 only with the fundamental SU(3) meson-quark (chiral quark model) interaction

\[ L_{\text{meson-qk}} = g \bar{\psi} \lambda^i [S^i + i \gamma_5 P^i] \psi , \]  

(42)

(where \( i = 0, \ldots 8 \) and \( S_i, P_i \) are scalar and pseudoscalar elementary meson fields), with quark level SU(3) Goldberger-Treiman relations

\[ f_\pi g = \hat{m} , \quad f_K g = \frac{1}{2} (m_s + \hat{m}) , \]  

(3b)

ensuring \( \partial A^j = 0 \) for \( j = 1 \ldots 8 \). Then we dynamically generated the log-divergent chiral-limiting gap equations

\[ 1 = -i 4 N_c g^2 \int_{\Lambda}^\infty \frac{d^4 p}{(p^2 - \hat{m}^2)^2} , \quad 1 = -i 4 N_c g^2 \int_{\Lambda'}^\infty \frac{d^4 p}{(p^2 - \hat{m}^2)(p^2 - m_s^2)} , \]  

(4)

which self-consistently fixed the cutoffs to

\[ \frac{\Lambda^2}{\hat{m}^2} \approx \frac{\Lambda'^2}{m_s \hat{m}} \approx 5.3 , \]  

(7)

so that \( \Lambda \sim 750 \text{ MeV} \), or \( \Lambda' \sim 860 \text{ MeV} \) in (10). This required all \( I = 0, 1 \) or \( I = 1/2 \) masses less than 750 MeV, 860 MeV to be elementary, such as \( \hat{m}, m_s \) for quarks and \( m_\pi, m_K, m_{\sigma_{NS}} \sim 670 \text{ MeV}, m_{\kappa} \sim 810 \text{ MeV} \) for pseudoscalar and scalar mesons, respectively.

Next in Sec. 3 we dynamically generated the cubic meson SU(3) couplings in the CL,

\[ g'_{\sigma_{NS}\pi\pi} = m_{\sigma_{NS}}^2 / 2 f_\pi , \quad g'_{\sigma_{NS}KK} = m_{\sigma_{NS}}^2 / 2 f_K , \quad g'_{\sigma_SKK} = m_{\sigma_S}^2 / \sqrt{2} f_K , \]  

(43)

and analogously for \( \sigma\sigma\sigma \)-like scalar couplings in Sec. 4 with the additional meson-quark SU(3)-limiting coupling constraint for \( N_c = 3 \)

\[ g = 2\pi / \sqrt{N_c} \approx 3.6276 . \]  

(5)
Recall that the cubic part of the standard [8] spontaneously broken SU(3) LσM Lagrangian density has the SU(3)-limiting form

\[ \mathcal{L}_{\text{cubic}}^{L\sigma M} = g' d^{ijk} S^i (S^j S^k + P^j P^k) . \]  

(44a)

On the other hand, our dynamically generated SU(3) LσM Lagrangian also has the SU(3)-limiting structure of (44a), but away from the SU(3) limit it becomes

\[ \mathcal{L}_{\text{cubic}}^{L\sigma M} = g'_\pi \sigma_{\pi} \eta_{\pi} \sigma_{\pi} + g'_{\pi K} K \sigma_{\pi} + g' K \sigma_{\pi} K \sigma_{\pi} \]  

(44b)

Here we use the SU(3) partially-broken LσM meson-meson couplings with massive pseudoscalars [38]

\[ g'_{\pi \sigma_{\pi} \sigma_{\pi}} = (m_{\sigma_{\pi}}^2 - m_{\pi}^2) / 2 f_{\pi} \approx 2.3 \text{ GeV} , \]  

(45a)

\[ g'_{\pi \delta \eta_{\pi}} = (m_{\delta}^2 - m_{\eta_{\pi}}^2) / 2 f_{\pi} \approx 2.1 \text{ GeV} , \]  

(45b)

\[ g'_{\pi \kappa K} = (m_{\kappa}^2 - m_{K}^2) / 2 f_{\pi} \approx 2.2 \text{ GeV} , \]  

(45c)

\[ g'_{K \delta K} = (m_{\delta}^2 - m_{K}^2) / 2 f_{K} \approx 3.2 \text{ GeV} , \]  

(45d)

\[ g'_{K \sigma_{K} K} = (m_{\sigma_{K}}^2 - m_{K}^2) / 2 \sqrt{2} f_{K} \approx 2.1 \text{ GeV} , \]  

(45e)

and the more severely broken SU(3) LσM couplings

\[ g'_{K \kappa \eta_{\pi}} = (m_{\kappa}^2 - m_{\eta_{\pi}}^2) / \sqrt{2} f_{K} \approx 0.10 \text{ GeV} , \]  

(46a)

\[ g'_{K \sigma_{K} K} = (m_{\sigma_{K}}^2 - m_{K}^2) / 4 f_{K} \approx 0.45 \text{ GeV} , \]  

(46b)

\[ g'_{K \kappa \eta_{\pi} K} = (m_{\kappa}^2 - m_{\eta_{\pi}}^2) / 4 f_{K} \approx 0.20 \text{ GeV} . \]  

(46c)

In Section 4 we dynamically generated the NJL-LσM chiral-broken average scalar meson masses appearing in (44) and (45) as

\[ m_{\sigma_{\pi} \pi} = 2\hat{m} \sim 670 \text{ MeV} , \quad m_{\kappa} = 2\sqrt{\hat{m} m_{s}} \sim 810 \text{ MeV} , \quad m_{\sigma_{K}} = 2m_{s} \sim 940 \text{ MeV} . \]  

(47)
Also in Section 5 the bootstrapping of the SU(3) quartic meson couplings in the Lagrangian were discussed, giving the usual value $\lambda = g'/f_\pi \approx 26$ in the CL.

Finally in Section 6 we focused on U(3) $\eta - \eta'$ and $\sigma - f_o$ particle mixing. In the original spontaneous broken SU(3) LoSM [8], (undetermined) particle mixing parameters were introduced in the extended version of the LoSM Lagrangian (44a). However in our dynamically generated version of the SU(3) LoSM, no additional mixing parameters enter the Lagrangian (44b). Rather, OZI violations generate quantum-mechanical particle mixing via the diagonalization of the mass matrix (31), but the resulting mixing parameters do not enter the dynamically generated Lagrangian (44b). Instead one dynamically fits the latter $\eta - \eta'$ and $\sigma - f_o$ mixing angles, in agreement with empirical data. This gives

$$\phi_P \approx 42^\circ \quad \phi_S \sim 20^\circ ,$$

respectively for the U(3) nonets ($\pi(138), K(495), \eta(547), \eta'(958))_P$ and ($\sigma(610), \kappa(810), f_o(980), a_o(984))_S$. There is much recent data supporting this above SU(3) LoSM nonet picture [39].

In short, the theoretical dynamically generated and dynamically fitted SU(2) and SU(3) linear sigma model Lagrangians of ref.[14] and here appear to give a good description of low energy strong interaction physics. Moreover the above LoSM picture is a natural generalization of the four-quark Nambu-Jona-Lasinio dynamically generated scheme and is also compatible with low-energy QCD [40].

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Appendix

Here we translate standard symmetric SU(3) cartesian structure constants \( d_{oij} = \sqrt{2/3} \delta_{ij} \), \( d_{338} = -d_{888} = 1/\sqrt{3} \), \( d_{344} = d_{355} = 1/2 \), \( d_{366} = d_{377} = -1/2 \), \( d_{448} = d_{558} = d_{668} = d_{778} = -1/2\sqrt{3} \), to the strange (S)-nonstrange (NS) quark basis with

\[
|NS\rangle = \frac{|\bar{u}u + \bar{d}d\rangle}{\sqrt{2}} = \sqrt{\frac{2}{3}}|0\rangle + \sqrt{\frac{1}{3}}|8\rangle
\] (A1)

\[
|S\rangle = |\bar{s}s\rangle = \sqrt{\frac{1}{3}}|0\rangle - \sqrt{\frac{2}{3}}|8\rangle .
\] (A2)

Then one finds

\[
d_{3NS,S} = d_{3SS} = d_{33S} = 0 ,
\]

\[
d_{33NS} = d_{NS,NS,NS} = 1 , \quad d_{SSS} = \sqrt{2} ,
\]

\[
d_{0NS,NS} = d_{0SS} = \sqrt{\frac{2}{3}} , \quad d_{8NS,NS} = \frac{1}{\sqrt{3}}
\] (A3)

\[
d_{8SS} = -\frac{2}{\sqrt{3}} , \quad d_{NSKK} = \frac{1}{2} , \quad d_{SKK} = \frac{1}{\sqrt{2}} .
\]
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Figure Captions

Fig. 1. Pion (a) and kaon (b) decay constants generated by quark loops

Fig. 2. Pion bubble (a) and quark tadpole (b) graphs

Fig. 3. Kaon bubble (a) and quark tadpole (b,c) graphs

Fig. 4. Bootstrap of $g_{\sigma_{NS\pi\pi}}$ quark triangle (a) to $g'$ tree (b) graph

Fig. 5. Scalar $\sigma_{NS}$ bubble (a) and quark tadpole (b) graphs

Fig. 6. Scalar kappa bubble (a) and quark tadpole (b,c) graphs

Fig. 7. Scalar $\sigma_S$ bubble (a) and quark-tadpole (b) graphs

Fig. 8. Isoscalar gluon-mediated quark annihilation diagrams for intermediate QCD states

Fig. 9. Quark box graphs for $\pi^0\pi^0$ (a), $\pi^+\pi^0$ (b), $\eta_s\eta_s$ (c) scattering.