Partial regularity for the Navier-Stokes-Fourier system

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Abstract

This paper addresses a nonstationary flow of heat-conductive incompressible Newtonian fluid with temperature-dependent viscosity coupled with linear heat transfer with advection and a viscous heat source term, under homogeneous Dirichlet boundary conditions. The partial regularity for the velocity of the fluid is proved to each proper weak solution, that is, for such weak solutions which satisfy some local energy estimates in a similar way to the suitable weak solutions of the Navier-Stokes system. Finally, we study the nature of the set of points in space and time upon which proper weak solutions could be singular.

Keywords: Navier-Stokes-Fourier system, Joule effect, suitable weak solutions
MSC2000: 76D03, 35Q30, 80A20

1 Introduction

Due to diverse applications, coupled systems governing incompressible flows are subject of intensive analytical and numerical investigation (see for instance [6, 7, 9, 11, 17, 19, 20, 21, 25] and the references therein). Here we deal with a nonstationary flow of heat-conductive incompressible Newtonian fluid with temperature-dependent viscosity coupled with linear heat transfer
with advection and a viscous heat source term, under homogeneous Dirichlet boundary conditions, and we study the partial regularity for proper weak solutions to the coupled system under study. The proper weak solution is each weak solution that satisfies some local energy estimates in a similar way to the suitable weak solutions of the Navier-Stokes system \[^8\]. Although the techniques used in the present work could be considered standard, the result is new because the mentioned techniques can not be directly applied. Under isothermal effects, the viscosity is constant and the problem is described by the Navier-Stokes equations. We refer to \[^22, 23, 24\] where the partial regularity theory is studied: at the first instant of time when a viscous incompressible fluid flow with finite kinetic energy in three space becomes singular, the singularities in space are concentrated on a closed set whose one dimensional Hausdorff measure is finite. This investigation characterizes some geometric properties and measures theoretic properties of the sets of points in space and time upon which weak solutions could be singular \[^3, 16\]. In such conditions, it is known that the concept of weak solutions in the sense given by Leray and Hopf is not sufficient to establish their partial regularity. Different regularity criteria for suitable weak solutions to the N-S system has been introduced in terms of the smallness of functionals that are invariant with respect to the natural scaling either the velocity or its gradient \[^10, 13, 18\]. We prove the existence of regular points in the sense due to \[^15\], that is, the function is Hölder continuous on a parabolic cylinder centred at such point. We remark that in the popular definition given at \[^3\] the Hölder space is replaced by the space of essentially bounded functions.

We recall that the following continuous inclusion

$$W^{2,1}_q(Q_T) := \{ v \in L^q(0,T; W^{2,q}(\Omega)) : \partial_t v \in L^q(Q_T) \} \hookrightarrow C^{k,\alpha}(\bar{Q}_T)$$

only occurs if \( q > (n+2)/(2-k) \). This means for \( k = 0 \) that \( q > n/2 + 1 \geq 2 \) (\( n = 2, 3 \)), i.e., the Banach space \( W^{2,1}_2(Q_T) \) is not embedded in the Banach space of Hölder continuous functions with exponent \( \alpha \) in the \( x \)-variables and \( \alpha/2 \) in the \( t \)-variable. It is known that the boundedness of the velocity of the fluid, when a weak solution of N-S equations is care of, requires the so-called Ladyzhenskaya-Prodi-Serrin sufficient condition of the norm of the mixed Lebesgue space \( L^{s,r}(Q_T) \) with \( 2/r + n/s = 1 \) if \( s > n \). Likewise recently a supplementary criterion concerning the N-S equations ensures the boundedness of solutions \[^2\], representing the pressure by a relation on the velocity which involves its spatial derivatives of second order. We emphasize that even in 2D, the regularity theory for the second derivative in space
or the derivative with respect to the time variable is not available to N-S-F system because the viscosity is a non-Hölderian function. It remains to know whether or not it is possible to find a $L^\infty$-bound of the velocity of heat-conducting fluids.

The partial regularity up to the boundary for the N-S-F system is still another open problem, since the study for N-S equations is provided by a constant viscosity in order to request that the spatial derivative of second order, $\nabla^2 u$, belongs to a convenient mixed Lebesgue space, namely $L^{\frac{9}{8},\frac{3}{2}}(Q_T)$ [26, 27] what in the present study does not happen. Even in the study of parabolic equations the partial regularity up to the boundary is proved by means of the existence of $\partial_t u \in L^2(Q_T)$ [1]. Once more such regularity does not occur in the present study.

Let $\Omega \subset \mathbb{R}^n$ be a bounded open domain, $\partial \Omega$ its boundary, and $T > 0$. Let us consider the initial boundary value problem of the N-S-F system:

\begin{align*}
\partial_t u - \text{div}(\mu(\theta)D u) + (u \cdot \nabla)u &= f - \nabla p \quad \text{in } Q_T := \Omega \times ]0, T[; \\
\text{div } u &= 0 \quad \text{in } Q_T; \\
\partial_t \theta - k \Delta \theta + u \cdot \nabla \theta &= \mu(\theta)||D u||^2 \quad \text{in } Q_T; \\
u|_{t=0} &= u_0, \quad \theta|_{t=0} = \theta_0 \quad \text{in } \Omega; \\
u = 0, \quad \theta = 0 \quad \text{on } \partial \Omega \times ]0, T[,
\end{align*}

where $\theta$ represents the temperature, $u$ the velocity of the fluid and $D u = \frac{1}{2}(\nabla u + \nabla u^T)$, $p$ denotes the pressure, $\mu$ the viscosity, $f$ denotes the given external body forces, $k$ denotes the conductivity assumed to be a fixed positive constant, and $u_0$ and $\theta_0$ are some given functions. For simplicity, the density and the heat capacity are constants assumed equal to one and we do not consider the existence of the external heat source, since the heating dissipative term is the main mathematical difficulty. The product of two tensors is given by $D : \tau = D_{ij}\tau_{ij}$ (in Einstein’s convention) and the norm by $|D|^2 = D : D$.

The outline of the paper is as follows. Next section is concerned to the presentation of the appropriate functional framework and the main results. The Section 3 is devoted to the proof of some auxiliary results introduced at Section 2. In Section 4 we prove the partial regularity result (Theorem 2.2). Finally an estimate to the parabolic Hausdorff dimension on the set of the singularities for the fluid velocity is provided in Section 5.
2 Assumptions and main results

Here we assume that \( \Omega \subset \mathbb{R}^n \) \((n = 2, 3)\) is a bounded open domain sufficiently regular, e.g. of class \( C^2 \). For \( 1 \leq q \leq \infty \), we introduce the Sobolev space of divergence-free fields

\[
J_0^{1,q}(\Omega) = \{ u \in W_0^{1,q}(\Omega) : \nabla \cdot u = 0 \text{ in } \Omega \}
\]

with norm

\[
\| \cdot \|_{1,q,\Omega} = \| \nabla \cdot \|_{q,\Omega}.
\]

We denote by bold the vector spaces of vector-valued or tensor-valued functions. For any set \( A \), we write \((u,v)_A := \int_A uv\) whenever \( u \in L^q(A) \) and \( v \in L^{q'}(A) \), where \( q' = q/(q - 1) \) is the conjugate exponent to \( q \), or simply \((\cdot,\cdot)\) whenever there exists no confusing at all, and we use the symbol \((\cdot,\cdot)\) to denote a generic duality pairing, not distinguished between scalar and vector fields.

**Definition 2.1** We say that the triple \((u,p,\theta)\) is a weak solution to the Navier-Stokes-Fourier (N-S-F) problem (1)-(4) in \( Q_T \) if,

\[
\begin{align*}
\theta &\in \mathcal{E} := L^\infty(0,T;L^1(\Omega)) \cap L^q(0,T;W_0^{1,q}(\Omega)), \\
\frac{1}{\ell} &= \frac{n}{q(n+1)} + \frac{n}{2(n+2)}, \\
\partial_t \theta &\in L^1(0,T;W^{-1,\ell}(\Omega)), \\
\partial_t u &\in \mathcal{X} := L^2(0,T;W^{-1,2}(\Omega)) \cap L^{(n+2)/n}(0,T;W^{-1,(n+2)/n}(\Omega)), \\
\partial_t u &\in U := L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;J_0^{1,2}(\Omega)), \\
\partial_t \theta &\in L^1(0,T;W^{-1,\ell}(\Omega)), \\
\partial_t u &\in X := L^2(0,T;W^{-1,2}(\Omega)) \cap L^{(n+2)/n}(0,T;W^{-1,(n+2)/n}(\Omega)), \\
\partial_t \theta &\in L^1(0,T;W^{-1,\ell}(\Omega)),
\end{align*}
\]

and satisfies the variational formulation

\[
\langle \partial_t u, v \rangle + \int_{Q_T} \left( \mu(\theta) D u : D v + (u \cdot \nabla) u \cdot v \right) dx dt = \int_{Q_T} (f \cdot v + p \text{ div } v) dx dt \\
(5)
\]

\[
\forall v \in L^\infty(0,T;W_0^{1,\infty}(\Omega)), \quad u|_{t=0} = u_0 \text{ in } \Omega;
\]

\[
\langle \partial_t \theta, \phi \rangle + \int_{Q_T} \left( k \nabla \theta - \theta u \right) : \nabla \phi dx dt = \int_{Q_T} \mu(\theta)|D u|^2 \phi dx dt \\
(6)
\]

\[
\forall \phi \in L^\infty(0,T;W_0^{1,\infty}(\Omega)), \quad \theta|_{t=0} = \theta_0 \text{ in } \Omega.
\]
In (5) the convective term verifies $(u \cdot \nabla)u \in L^{(n+2)/(n+1)}(Q_T)$, for every $u \in U \hookrightarrow L^{2(n+2)/n}(Q_T)$. In (6) the advection term $\theta u \in L^\ell(Q_T)$ if $\theta \in \mathcal{E} \hookrightarrow L^{q(n+1)/n}(Q_T)$ for $\ell \geq 1$, i.e. $q > 1$ if $n = 2$ and $q \geq 15/14$ if $n = 3$ $(q \geq 2n(n+2)/((n+4)(n+1)))$. For $n = 2, 3$, we remark that the embedding holds $\mathcal{X} \hookrightarrow L^{(n+2)/n}(0,T;W^{-1,(n+2)/n}(\Omega))$. In conclusion, all terms in the above equalities are meaningful.

We begin by recalling the existence of weak solutions such that verify some local energy inequalities (the proof of the existence result can be found in [8]). We assume that the following hypotheses hold

(A1) $f : Q_T \to \mathbb{R}^n$ is given such that $f \in L^{2(1+\epsilon_0)}(Q_T)$ with $\epsilon_0 > 0$;

(A2) $\mu : \mathbb{R} \to \mathbb{R}$ is a continuous function such that
$$0 < \mu_\# \leq \mu(s) \leq \mu_\#, \quad \forall s \in \mathbb{R};$$

(A3) $u_0 \in L^2(\Omega)$, $\theta_0 \in L^1(\Omega)$ such that
$$\text{div } u_0 = 0 \quad \text{in } \Omega; \quad \text{ess inf}_{x \in \Omega} \theta_0(x) \geq 0.$$ 

**Theorem 2.1** Under the assumptions (A1)-(A3), the N-S-F problem defined by (5)-(6) has proper weak solutions in the following sense. The weak solution in accordance to Definition 2.1 satisfies the following local energy inequalities

$$\int_{\Omega} \frac{|u(x,t) - a|^2}{2} \varphi^2(x,t)dx + \int_{Q_t=\Omega \times ]0,t[} \mu(\theta)|Du|^2 \varphi^2 dx d\tau \leq$$

$$\leq 2 \int_{Q_t} \mu(\theta)\varphi Du : ((u - a) \otimes \nabla \varphi) + \int_{Q_t} |u - a|^2 \varphi(\partial_t \varphi + u \cdot \nabla \varphi) +$$

$$+ 2 \int_{Q_t} p(x)u \cdot \nabla \varphi dx d\tau + \int_{Q_t} f \cdot (u - a)\varphi^2 dx d\tau, \quad (9)$$

$$\int_{\Omega} \sqrt{\zeta + \theta^2} \psi(x,t)dx + \zeta \int_{Q_t} \frac{\nabla \theta^2}{(\zeta + \theta^2)^{3/2}} \psi dx d\tau \leq$$

$$\leq \int_{Q_t} \sqrt{\zeta + \theta^2} (\partial_t \psi + k\Delta \psi + u \cdot \nabla \psi) + \int_{Q_t} \mu(\theta)|Du|^2 \frac{\theta \psi}{(\zeta + \theta^2)^{1/2}}, \quad (10)$$

for all $\varphi \in C_0^\infty(Q_T)$, for any $a \in \mathbb{R}^n$, for any $\zeta > 0$, and for all $\psi \in C_0^\infty(Q_T)$ such that $\psi \geq 0$, a.e $t \in ]0,T[$. Moreover, $\theta \geq 0$ in $Q_T$. 

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Remark 2.1 The local energy inequality for the temperature, (10), plays an essential role in the proof of the decay lemma (cf. Theorem 3.1) because if we take $\zeta \to 0^+$ we obtain $\theta \in L^\infty(0,T;L^1_{\text{loc}}(\Omega))$.

In order to establish the partial regularity, let us introduce some additional notations.

Definition 2.2 Given a positive number $\lambda$, we set

$$c_\lambda(f,\omega) := \sup \left\{ \frac{1}{R^{2-n}} \left( \int_{Q(z,R)} |f|^2 \right)^{1/2} : R > 0, Q(z,R) \subset \omega \right\} \quad (11)$$

for all $\omega \subset Q_T$. In particular, we set $c_\lambda(f) = c_\lambda(f,Q_T)$.

Given a positive number $\lambda$ we introduce the “parabolic” variant of the Morrey spaces \([15]\)

$$M_{2,\lambda}(Q_T) = \{ f \in L^2_{\text{loc}}(Q_T) : c_\lambda(f,Q_T) < \infty \}.$$

Let $z_0 = (x_0,t_0) \in Q_T$ and $R > 0$ be such that $Q(z_0,R) \equiv B(x_0,R) \times ]t_0-R^2,t_0[ \subset Q_T$. For any set $A$, we denote by $|A|$ the usual Lebesgue measure of the set, and we write $\int_A v := \frac{1}{|A|} \int_A v$ whenever $v \in L^1(A)$, or simply $(v)_A = \int_A v$. Observing that $2(n+2)/(n+4) < a' < q(n+1)/n$ is equivalent to

$$\frac{q(n+1)}{q(n+1)-n} < a < \frac{2(n+2)}{n} = \begin{cases} 4 & \text{if } n = 2 \\
10/3 & \text{if } n = 3 \end{cases}$$

we can choose $1 < q < (n+2)/(n+1)$ such that $q > 3n/(2(n+1))$, i.e. $q(n+1)/(q(n+1)-n) < 3$. So we set

$$3 \leq a < 2(n+2)/n. \quad (12)$$

Let us introduce the following scaling invariant functional:

$$\bar{Y}_R(z_0;u,p,\theta) \equiv \left( \int_{Q(z_0,R)} |u|^a \, dz \right)^{1/a} + R \left( \int_{Q(z_0,R)} |p|^{\frac{n+2}{2}} \, dz \right)^{\frac{n}{n+2}} + R \left( \int_{Q(z_0,R)} |	heta|^{a'} \, dz \right)^{1/a'}.$$

The choice of the above exponents is consequence of $u \in L^{2(n+2)/n}(Q_T)$, $p \in L^{(n+2)/n}(Q_T)$, $\theta \in L^{q(n+1)/n}(Q_T)$ and $u\theta \in L^1(Q_T)$. We refer \([15]\) in order to compare to the those exponents in the analysis for the N-S system into the three-dimensional space $(n = 3)$. 

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Theorem 2.2 Suppose \( \lambda \) and \( \alpha \) are positive numbers such that (12) is verified. Let \( f \in M_{2,\lambda}(Q_T) \) and (A1)-(A3) be fulfilled. Assume that \((u,p,\theta)\) is a proper weak solution of the N-S-F system in \( Q_T \). Then, there exists a positive constant \( \Lambda \) depending only on \( \mu#, \mu#', \lambda \) and \( n \). Moreover, if for any \( z_0 \in Q_T \)
\[
\limsup_{R \to 0^+} R\widetilde{Y}_R(z_0; u, p, \theta) \leq \Lambda,
\]
then \( z_0 \) is a regular point in the following sense: the function \( z \mapsto u(z) \) is Hölder continuous in some neighbourhood of the point \( z_0 \).

The local energy inequality for the temperature has a nonstandard format. It is still an open problem the proof that the pointwise criterion
\[
\limsup_{R \to 0} \frac{1}{R} \left( \int_{Q(z_0, R)} |\nabla u|^2 dz \right)^{1/2} + \left( \int_{Q(z_0, R)} |\nabla \theta|^q dz \right)^{1/q} \leq \gamma^*_a
\]
implies the local condition for the regularity (13), for some constant \( \gamma^*_a > 0 \).

Remark 2.2 Under all conditions of Theorem 2.2, we denote by \( Q_0 \) the set of all regular points such that satisfy (13). By definition, \( Q_0 \) is an open set. The set \( S = Q_T \setminus Q_0 \), known as the singular set, reads
\[
S = \{ z \in Q_T : \limsup_{R \to 0^+} R\widetilde{Y}_R(z; u, p, \theta) > \Lambda \}.
\]
Moreover, it can be proved that it is of Lebesgue measure zero.

The parabolic Hausdorff dimension on the singular set can be estimated. Indeed, we state the following result.

Theorem 2.3 Let \( a \) be given as in (12). Then \( \mathcal{H}^{n-(a-2)/(a-1)}(S) = 0 \), where \( \mathcal{H}^d \) is the d-dimensional Hausdorff measure with respect to the standard parabolic metric of a set \( S \) defined as follows
\[
\mathcal{H}^d(S) := \liminf_{\delta \to 0} \{ \sum_{i=1}^{\infty} R_i^d : S \subset \bigcup_{i=1}^{\infty} Q(z_i, R_i), \ 0 < R_i \leq \delta \}.
\]
Therefore, the parabolic Hausdorff dimension of \( S \) does not exceed \( n - (a - 2)/(a - 1) \), i.e. \( \dim_{\mathcal{H}}(S) := \inf\{ d \geq 0 : \mathcal{H}^d(S) = 0 \} \leq n - (a - 2)/(a - 1) \).
3 Auxiliary results

The main tool for the partial regularity analysis is the decay property of the scaled Lebesgue norms of the triple velocity-pressure-temperature which is based on a standard "blow up" method and the decomposition of the pressure. Let us introduce the following scaling invariant functional:

\[
Y_R(z_0; u, p, \theta) \equiv \left( \int_{Q(z_0, R)} |u(x, t) - \langle u \rangle_{Q(z_0, R)}|^a \, dz \right)^{1/a} + \\
R \left( \int_{Q(z_0, R)} |p(x, t) - \langle p \rangle_{B(z_0, R)}(t)|^{(n+2)/n} \, dz \right)^{n/(n+2)} + \\
R \left( \int_{Q(z_0, R)} |\theta(x, t) - \langle \theta \rangle_{Q(z_0, R)}|^{a'} \, dz \right)^{1/a'} .
\]

**Theorem 3.1 (Decay estimate)** Suppose that \( 0 < \varsigma < 1 \) and \( 0 < \beta < \lambda \) are fixed numbers. For each function \( \mu \) satisfying (7) there exist positive constants \( \gamma \) and \( \varepsilon > 0 \) depending only on \( \mu^\#, \mu^\#, \varsigma, \beta \) and \( \lambda \) such that for any proper weak solution \((u, p, \theta)\) of the \( N \)-\( S \)-\( F \) problem in \( Q_T \) verifying

\[
\forall Q(z, R) \subset \subset Q_T : 0 < R < \varepsilon, \\
R|\langle u \rangle_{Q(z, R)}| < 1, R|\langle \theta \rangle_{Q(z, R)}| < \gamma/4 \\
Y_R(z; u, p, \theta) + c_\lambda(f) R^\beta < \gamma
\]

we have the decay estimate

\[
Y_{\varsigma R}(z; u, p, \theta) \leq C_* \varsigma^\alpha (Y_R(z; u, p, \theta) + c_\lambda(f) R^\beta)
\]

with \( 0 < \alpha < 2 - n/2 \) and some absolute constant \( C_* \) depending only on \( \mu^\# \) and \( \mu^\# \).

To prove Theorem 2.2 some iterative estimates are required [15, Lemmas 2.5, 3.1]. Let us state an iterative result, in which a careful choice of \( \varsigma \) significantly simplifies the iterative formula as well as its proof (compare to [15, Lemmas 2.5, 3.1]).

**Lemma 3.1** Under all conditions of Theorem 3.1, let \( 0 < \delta < \alpha \). If additionally \( \varsigma \) is such that

\[
C_* \varsigma^{\alpha - \delta} \leq 1/2 \quad \text{and} \quad \varsigma^\delta + \varsigma^\beta \leq 1,
\]

then for any proper weak solution \((u, p, \theta)\) of the \( N \)-\( S \)-\( F \) system in \( Q_T \) satisfying (14) verify
1. for any $m \in \mathbb{N}$,
\[
Y_{\varsigma^m}(z; u, p, \theta) \leq \varsigma^m(Y_R(z; u, p, \theta) + c_\lambda(f) R^\beta); \quad (16)
\]

2. for all $\rho \in [0, \varsigma R]$,
\[
Y_\rho(z; u, p, \theta) \leq C \left( \frac{\rho}{R} \right)^\delta (Y_R(z; u, p, \theta) + c_\lambda(f) R^\beta), \quad (17)
\]

where $C = C(\varsigma, \delta)$ denotes a positive constant.

Finally, we state the additional technical result.

**Lemma 3.2** Assume that $(u, p, \theta)$ satisfies the system \([1] - [2]\) in the sense of distributions. For every $R, r > 0$, $e_0 = (y_0, s_0)$ and $z_0 = (x_0, t_0)$ such that $Q(z_0, R) \subset Q_T$,

1. if we introduce the functions
\[
\begin{align*}
\bar{u}^R(y, s) &\equiv \frac{R}{r} u(x_0 + \frac{R}{r}(y - y_0), t_0 + \left( \frac{R}{r} \right)^2 (s - s_0)), \\
\bar{p}^R(y, s) &\equiv \left( \frac{R}{r} \right)^2 p(x_0 + \frac{R}{r}(y - y_0), t_0 + \left( \frac{R}{r} \right)^2 (s - s_0)), \\
\bar{\theta}^R(y, s) &\equiv \left( \frac{R}{r} \right)^2 \theta(x_0 + \frac{R}{r}(y - y_0), t_0 + \left( \frac{R}{r} \right)^2 (s - s_0)),
\end{align*}
\]

then $(\bar{u}^R, \bar{p}^R, \bar{\theta}^R)$ satisfy the transported system in $Q(e_0, r)$
\[
\begin{align*}
\partial_s \bar{u}^R - \text{div}_y(\mu^R(\bar{\theta}^R) D_y \bar{u}^R) + (\bar{u}^R \cdot \nabla_y) \bar{u}^R &= f^R - \nabla_y \bar{p}^R; \quad (18) \\
\text{div}_y \bar{u}^R &= 0; \quad (19) \\
\partial_s \bar{\theta}^R - k \Delta_y \bar{\theta}^R + \bar{u}^R \cdot \nabla_y \bar{\theta}^R &= \mu^R(\bar{\theta}^R) |D_y \bar{u}^R|^2; \quad (20)
\end{align*}
\]

with the function $\mu^R(\vartheta) \equiv \mu(r^2 \vartheta / R^2)$ and
\[
\bar{f}^R(y, s) = \left( \frac{R}{r} \right)^3 f(x_0 + \frac{R}{r}(y - y_0), t_0 + \left( \frac{R}{r} \right)^2 (s - s_0));
\]

2. then
\[
r \bar{Y}_\rho(e_0; \bar{u}^R, \bar{p}^R, \bar{\theta}^R) = R \bar{Y}_R(z_0; u, p, \theta). \quad (21)
\]

Moreover
\[
c_\lambda(f^R, Q(e_0, r)) \leq \left( \frac{R}{r} \right)^{\lambda+1} c_\lambda(f, Q(z_0, R)). \quad (22)
\]

**Remark 3.1** The function $\mu^R$ satisfies \([7]\) with the same constants $\mu^\#, \mu^\#.$

Henceforth the symbol $C$ will denote a positive, finite constant that may vary from line to line; the relevant dependencies on the data will be specified whenever it will be required. Notice that the dependence never occurs on the unknown functions $u, p$ or $\theta.$

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4 Proof of the auxiliary results

4.1 Proof of Lemma 3.2

1) Considering the change of variables

\[ Q(e_0, r) \equiv B(y_0, r) \times (s_0 - r^2, s_0) \rightarrow Q(z_0, R) \equiv B(x_0, R) \times (t_0 - R^2, t_0) \]

\( (y, s) \mapsto \left( x_0 + \frac{R}{r}(y - y_0), t_0 + \left( \frac{R}{r} \right)^2 (s - s_0) \right) \)

we get (18)-(20), observing that

\[
\begin{align*}
\partial_s u^R + (u^R \cdot \nabla_y) u^R + \nabla_y p^R &= \left( \frac{R}{r} \right)^3 (\partial_s u + (u \cdot \nabla)u + \nabla p); \\
div_y (\mu^R(\theta^R) D_y u^R) &= \left( \frac{R}{r} \right)^2 \div_y (\mu(\theta) D_y u) \\
\partial_s \theta^R + u^R \cdot \nabla_y \theta^R - \Delta_y \theta^R &= \left( \frac{R}{r} \right)^4 (\partial_t \theta + u \cdot \nabla \theta + \Delta \theta); \\
\mu^R(\theta^R)|D_y u^R|^2 &= \left( \frac{R}{r} \right)^4 \mu(\theta)|Du|^2.
\end{align*}
\]

2) Considering the change of variables

\[ Q(z_0, R) \rightarrow Q(e_0, r) \]

\( (x, t) \mapsto \left( y_0 + \frac{r}{R}(x - x_0), s_0 + \left( \frac{r}{R} \right)^2 (t - t_0) \right) \)

de the Jacobian is \((r/R)^{n+2}\). Then, for every \(m \geq 1\) and \(u \in \mathbb{L}^m(Q_T),\)

\[
\int_{Q(e_0, r)} |u(x_0 + \frac{R}{r}(y - y_0), t_0 + \left( \frac{R}{r} \right)^2 (s - s_0))|^m \, de = \int_{Q(z_0, R)} |u|^m \, dz.
\]

In particular, it follows

\[
\begin{align*}
R \left( \int_{Q(e_0, r)} |u^R|^a \right)^{1/a} &= \left( \int_{Q(z_0, R)} |u|^a \right)^{1/a} \\
R^2 \left( \int_{Q(e_0, r)} |p^R|^{(n+2)/n} \right)^{n/(n+2)} &= \left( \int_{Q(z_0, R)} |p|^{(n+2)/n} \right)^{n/(n+2)} \\
R^2 \left( \int_{Q(e_0, r)} |\theta^R|^a' \right)^{1/a'} &= \left( \int_{Q(z_0, R)} |\theta|^a' \right)^{1/a'}.
\end{align*}
\]
Consequently, (21) holds. Also we have
\[
\frac{1}{r^{\lambda-2}} \left( \int_{Q(e_0,r)} |R|^2 \, dc \right)^{1/2} = \left( \frac{R}{r} \right)^{\lambda+1} \frac{1}{R^{\lambda-2}} \left( \int_{Q(z_0,R)} |f|^2 \, dz \right)^{1/2}.
\]

Thus we obtain (22) and Lemma 3.2 is proved.

4.2 Proof of the decay estimate (Theorem 3.1)
Suppose the opposite, i.e., there exist a sequence \( \{ (\varepsilon_m, \gamma_m) \} \) and solutions \( (u_m, p_m, \theta_m) \) of (1)-(2) in \( \Omega \times ]0, T_m[ \) and \( Q(z_m, R_m) \subset \subset \Omega \times ]0, T_m[ \) such that
\[
R_m |(u_m)_{Q(z_m, R_m)}| < 1, \quad R_m |(\theta_m)_{Q(z_m, R_m)}| < \gamma_m / 4 \tag{23}
\]
\[
\varepsilon_m \to 0, \quad Y_{R_m}(z_m; u_m, p_m, \theta_m) + d_m R^\beta_m = \gamma_m \to 0, \quad (m \to \infty) \tag{24}
\]
\[
Y_{< R_m}(z_m; u_m, p_m, \theta_m) \geq C \gamma_m^a \tag{25}
\]
where \( d_m = c_\lambda(f_m; \Omega \times ]0, T_m[) \). Applying the change of variables
\[
Q \equiv Q(0,1) \leftrightarrow Q_m \equiv Q(z_m, R_m),
\]
\[
e = (y, s) \leftrightarrow z = (x, t)
\]
we introduce the transported functions:
\[
w_m(e) := \frac{1}{\gamma_m} (u_m(z) - (u_m)_{Q_m}),
\]
\[
\pi_m(e) := \frac{R_m}{\gamma_m} (p_m(z) - (p_m)_{B_m}), \quad B_m = B(x_m, R_m),
\]
\[
\kappa_m(e) := \frac{R_m}{\gamma_m} (\theta_m(z) - (\theta_m)_{Q_m}).
\]

These functions satisfy the system (in the sense of distributions) in \( Q \)
\[
\begin{aligned}
\partial_s w_m + R_m ((\gamma_m w_m + a_m) \cdot \nabla_y) w_m - \text{div}_y (\mu(\gamma_m \kappa_m + b_m) D_y w_m) &= \\
= -\nabla_y \pi_m + g_m, \quad \text{div}_y w_m = 0, \\
\partial_s \kappa_m + R_m (\gamma_m w_m + a_m) \cdot \nabla_y \kappa_m - k \Delta_y \kappa_m &= \\
= R_m \gamma_m \mu(\gamma_m \kappa_m + b_m) |D_y w_m|^2.
\end{aligned}
\tag{26}
\]
Here \( g_m(e) = \frac{R^2_m}{\gamma_m} f_m(z) \), \( a_m := (u_m)_{Q_m} \) and \( b_m := (\theta_m)_{Q_m} \).
From (24) we can extract subsequences, still denoted by the same symbols, such that

\[ \mathbf{w}_m \rightharpoonup \mathbf{w} \text{ in } L^a(Q), \quad \pi_m \rightharpoonup \pi \text{ in } L^{(n+2)/n}(Q), \quad \varkappa_m \rightharpoonup \varkappa \text{ in } L^{a'}(Q) \]

and taking into account that by definition \((\mathbf{w}_m)_Q = (\varkappa_m)_Q = 0\) and \((\pi_m)_B(s) = 0, s \in (-1, 0)\) and also the mean integrals remain zero for the weak limits, it follows

\[ 1 \geq \liminf_{m \to \infty} Y_1(0; \mathbf{w}_m, \pi_m, \varkappa_m) \geq Y_1(0; \mathbf{w}, \pi, \varkappa); \quad (27) \]

\[ \liminf_{m \to \infty} Y_\varsigma(0; \mathbf{w}_m, \pi_m, \varkappa_m) \geq C_\varsigma \varsigma^a. \quad (28) \]

In order to obtain more information about \((\mathbf{w}, \pi, \varkappa)\) we wish to pass to the limit the system satisfied by \((\mathbf{w}_m, \pi_m, \varkappa_m)\) considering the weak formulation

\[ -\int_Q \mathbf{w}_m \cdot \partial_s \mathbf{v} \, de + \int_Q \mu \left( \frac{\gamma_m}{R_m} \varkappa_m + b_m \right) D_y \mathbf{w}_m : D_y \mathbf{v} \, de = 
\]

\[ = R_m \int_Q (\gamma_m \mathbf{w}_m + a_m) \otimes \mathbf{w}_m : \nabla_y \mathbf{v} \, de + \int_Q \pi_m \div_y \mathbf{v} \, de + \int_Q g_m \cdot \mathbf{v} \, de, \quad \text{for all } \mathbf{v} \in C^\infty_0(Q); \quad (29) \]

\[ -\int_Q \varkappa_m \partial_s \phi \, de - k \int_Q \varkappa_m \Delta_y \phi \, de = R_m \int_Q \varkappa_m (\gamma_m \mathbf{w}_m + a_m) \cdot \nabla_y \phi \, de + 
\]

\[ + R_m \gamma_m \int_Q \mu \left( \frac{\gamma_m}{R_m} \varkappa_m + b_m \right) |D_y \mathbf{w}_m|^2 \phi \, de, \quad \text{for all } \phi \in C^\infty_0(Q). \quad (30) \]

The dependence of the viscosity on the temperature does not allow to proceed as in the N-S system (under isothermal behaviour). In the following we use the local energy inequalities (9-10) for \((\mathbf{u}_m, p_m, \theta_m)\) in order to obtain some indispensable estimates on the required solutions under some standard conditions on the functions involved therein.

### 4.2.1 \( L^\infty(-\varsigma^2, 0; L^2(B(0, \varsigma))) \cap L^2(-\varsigma^2, 0; H^1(B(0, \varsigma))) \) estimate for \( \mathbf{w}_m \)

The local energy inequality (9) for \((\mathbf{u}_m, p_m, \theta_m)\) can be rewritten for \((\mathbf{w}_m, \pi_m, \varkappa_m)\), for a.e. \( s \in \mathbb{R} \),

\[ \int_B \frac{\mathbf{w}_m(y, s)^2}{2} \phi^2(y, s) \, dy + \int_{-1}^s \int_B \mu \left( \frac{\gamma_m}{R_m} \varkappa_m + b_m \right) |D_y \mathbf{w}_m|^2 \phi^2 \, dy \, d\tau \leq \]

\[ \int_B \frac{\mathbf{w}_m(y, 0)^2}{2} \phi^2(y, 0) \, dy. \]
\[
\leq R_m \int_{-1}^{s} \int_B \mu \left( \frac{\gamma_m}{R_m} x_m + b_m \right) \varphi D_y w_m : (w_m \otimes \nabla y \varphi) dyd\tau + \\
+ \int_{-1}^{s} \int_B |w_m|^2 \varphi (\partial_s \varphi + R_m (\gamma_m w_m + a_m) \cdot \nabla y \varphi) dyd\tau + \\
+ 2 \int_{-1}^{s} \int_B \pi_m \varphi w_m \cdot \nabla y \varphi dyd\tau + \int_{-1}^{s} \int_B g_m \cdot w_m \varphi^2 dyd\tau, \quad \forall \varphi \in C_0^\infty(Q), (31)
\]

where \( B \equiv B(0,1) \subset \mathbb{R}^n \). Using the Cauchy-Schwarz and Young inequalities we deduce

\[
\int_B \frac{|w_m(y,s)|^2}{2} \varphi^2(y,s) dy \leq \frac{1}{2} \int_{-1}^{s} \int_B \mu \left( \frac{\gamma_m}{R_m} x_m + b_m \right) |D_y w_m|^2 \varphi^2 dyd\tau \leq \\
\leq \frac{1}{2} \int_{-1}^{s} \int_B \mu \left( \frac{\gamma_m}{R_m} x_m + b_m \right) |w_m \otimes \nabla y \varphi|^2 dyd\tau + \\
+ \int_{-1}^{s} \int_B |w_m|^2 \varphi (\partial_s \varphi + R_m (\gamma_m w_m + a_m) \cdot \nabla y \varphi) dyd\tau + \\
+ 2 \int_{-1}^{s} \int_B \pi_m \varphi w_m \cdot \nabla y \varphi dyd\tau + \int_{-1}^{s} \int_B g_m \cdot w_m \varphi^2 dyd\tau. (32)
\]

Thus, taking a conveniently chosen \( \varphi \) and using (27) under (12) and also (23) we obtain

\[
\operatorname{ess \ sup}_{s \in [-\varsigma^2,0[} \|w_m(s)\|_{2,B(0,\varsigma)} + \|\nabla y w_m\|_{2,Q(0,\varsigma)} \leq C + \|g_m\|_{2,Q}, \quad (33)
\]

with \( C = C(\mu_\#, \mu^\#) \).

### 4.2.2 \( L^\infty(-\varsigma^2,0;L^1(B(0,\varsigma))) \) estimate for \( x_m \)

After letting \( \varsigma \rightarrow 0^+ \), the local energy inequality (10) for \((u_m, p_m, \theta_m)\) can be rewritten for \((w_m, \pi_m, x_m)\), for a.e. \( s \in [-1,0[ \),

\[
\int_B \left( \frac{\gamma_m}{R_m} x_m + b_m \right) \psi(y,s) dy \leq \gamma_m \int_{-1}^{s} \int_B (\gamma_m x_m + b_m) w_m : \nabla y \psi dyd\tau + \\
+ \int_{-1}^{s} \int_B \left( \frac{\gamma_m}{R_m} x_m + b_m \right) (\partial_s \psi + k \Delta y \psi + a_m \cdot \nabla y \psi) dyd\tau + \\
+ \gamma_m^2 \int_{-1}^{s} \int_B \mu \left( \frac{\gamma_m}{R_m} x_m + b_m \right) |D_y w_m|^2 \psi dyd\tau.
\]

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Thus, taking a conveniently chosen $\psi$ and using the Gronwall Lemma, (33), (27), (23) and (7), we obtain
\[
\text{ess sup}_{s \in -\varsigma^2,0} \left\| \frac{\gamma_m}{R_m} \kappa_m + b_m \right\|_{1,B(0,\varsigma)} \leq \gamma_m^2 N(1 + \|g_m\|_{2,Q}) \exp[C + R_m \gamma_m \|g_m\|_{2,Q}].
\] (34)

By definition (11) we have
\[
\left( \int_Q |g_m|^2 \, de \right)^{1/2} = \frac{R_m^2}{\gamma_m} \left( \int_{Q(z_m,R_m)} |f_m|^2 \, dz \right)^{1/2}
\leq \frac{R_m^2}{\gamma_m} d_m R_m^{\lambda-2} = \frac{R_m^3}{\gamma_m} d_m R_m^{\lambda-\beta} \leq R_m^{\lambda-\beta} \to 0, \quad \text{as } m \to 0 \quad (\beta < \lambda). \quad (35)
\]

Finally, since $b_m \geq 0$ and using (35), from (34) we find
\[
\text{ess sup}_{s \in -\tau^2,0} \left\| \kappa_m \right\|_{1,B(0,\tau)} \leq CR_m \gamma_m.
\] (36)

### 4.2.3 Passage to the limit as $m \to \infty$

In order to be in conditions to pass to the limit in (29)-(30), let us state the following convergences. From (23) we get $R_m a_m \to a$ in $\mathbb{R}^n$. Moreover, $|a| < 1$ arises. Inserting (35) into (33) we get
\[
D_y w_m \to D_y w \quad \text{in } L^2(Q(0,\varsigma)),
\]
and consequently
\[
R_m \gamma_m \int_Q \mu(\gamma_m \kappa_m + b_m) |D_y w_m|^2 \phi \, de \leq R_m \gamma_m \mu \|D_y w_m\|_{2,Q}^2 \|\phi\|_{\infty,Q}
\leq CR_m \gamma_m \to 0.
\]

Inserting (35) in (34) and (36) as tending $m$ to infinity, we obtain
\[
\frac{\gamma_m}{R_m} \kappa_m + b_m \to 0, \quad \kappa_m \to 0 \quad \text{in } L^1(Q(0,\varsigma)) \quad (37)
\]
and consequently $\mu(\gamma_m \kappa_m + b_m) \to \mu(0)$ in $\mathbb{R}$.

Then we can pass to the limit in (29)-(30) with $\kappa = 0$ and $(w, \pi)$ is a solution to the time dependent Stokes system (in the sense of distributions)
\[
\begin{align*}
\partial_t w - \mu(0) \Delta_y w + \nabla_y \pi &= -(a \cdot \nabla_y) w = -\nabla_y \cdot (a \otimes w) \\
\text{div}_y w &= 0
\end{align*}
\] in $Q$. 14
The classical theory for the nonstationary Stokes equations \([29]\) claims from \((a \cdot \nabla_y) w \in L^2(Q)\) that \(w \in W^{2,1}_2(Q) \hookrightarrow L^{2(n+2)/n}(-1,0;\mathbb{W}^{1,2(n+2)/n}(B))\). Applying the bootstrap argument from \((a \cdot \nabla_y) w \in L^{2(n+2)/n}(Q)\) it follows that \(w \in W^{2,1}_{2(n+2)/n}(Q) \hookrightarrow C^{0,\alpha}(Q)\) for \(0 \leq \alpha < 2 - n/2\). Then \(w\) is Hölder continuous in the closure of the cylinder \(Q(0,\varsigma/2)\) and the following estimate holds

\[
Y_\varsigma(0;w,0,0) \leq C_{**} \varsigma^\alpha
\]

(38)

with some constant \(C_{**}\) depending only on \(\mu_\#\) and \(\mu^\#\).

**Remark 4.1** The above bootstrap argument can still be applied if we use first the embedding \(\mathcal{U} \hookrightarrow L^{2(n+1)/n}(Q)\) and next the \(L^p\)-theory for the Stokes equation with RHS in the divergence form \([14]\), that is, from \(w \in L^{2(n+1)/n}(Q)\) it follows that \(w \in L^{2(n+1)/n}(-1,0;\mathbb{W}^{1,2(n+1)/n}(B))\).

On the other hand, we need to extract subsequences which converge strongly in order to pass to the limit in the integral \(Y_\varsigma(0;w_m,\pi_m,\kappa_m)\). From \([29]\) and using \([7], (23), (33), (35), (27)\), we derive that \(\{\partial_s w_m\}_{m \in \mathbb{N}}\) is bounded in \(L^{(n+2)/2}(-1,0;\mathbb{W}^{-1,(n+2)/2}(B))\) since the following estimate holds

\[
\int_Q w_m \cdot \partial_s v \, dx \leq (\mu_\# \|D_y w_m\|_{2,Q} + R_m \gamma_m \|w_m\|^2_{2(n+2)/n,Q} + R_m |a_m| \|w_m\|_{2(n+2)/n,Q} + \|\pi_m\|_{(n+2)/n,Q} \|\nabla_y v\|_{(n+2)/2,Q} + \|g_m\|_{2,Q} \|v\|_{2,Q}),
\]

for every \(v \in L^{(n+2)/2}(-1,0;\mathbb{W}^0_{0,(n+2)/2}(B))\). Thanks to \([33]\) and \([35]\), the sequence \(\{w_m\}\) is bounded in \(L^\infty(-\varsigma^2,0;L^2(B(0,\varsigma))) \cap L^2(-\varsigma^2,0;H^1(B(0,\varsigma))) \hookrightarrow L^{2(n+2)/n}(Q(0,\varsigma))\). Using \(H^1(B) \hookrightarrow L^5(B)\) and a compactness result \([28]\) we obtain

\[
w_m \rightarrow w \quad \text{in} \quad L^a(Q(0,\varsigma)).
\]

(39)

Therefore, using \([35], (38)\) and \([36]\) we obtain

\[
\limsup_{m \to \infty} Y_\varsigma(0;w_m,\pi_m,\kappa_m) \leq \limsup_{m \to \infty} \varsigma \left( f_{Q(0,\varsigma)} |\pi_m - (\pi_m)_{B(0,\varsigma)}|^{(n+2)/n} \, dx \right)^{n/(n+2)} \leq C_{**} \varsigma^\alpha + 4 \limsup_{m \to \infty} \varsigma \left( f_{Q(0,\varsigma)} |\pi_m|^{(n+2)/n} \, dx \right)^{n/(n+2)}.
\]

In order to estimate the pressure, the idea is to decompose the pressure into two parts, which one part is harmonic and hence is smooth (see \([15]\).
Choosing \( v(y, s) = \chi(s)\nabla \varphi(y) \), where \( \chi \in C_0^\infty(0, 1) \) and \( \varphi \in C_0^\infty(B) \), as a test function in (29), remarking that \( \text{div} \ w_m = 0 \) and

\[- \int_Q w_m \cdot \nabla \varphi \frac{d\chi}{ds} dy ds = \int_Q \nabla \cdot w_m \varphi \frac{d\chi}{ds} dy ds = 0,\]

we obtain, for a.e. \( s \in ]-1, 0[ \),

\[ \int_B \mu \left( \frac{\gamma_m}{R_m} \varphi_m + b_m \right) D_y w_m : D_y \nabla \varphi dy - R_m \gamma_m \int_B w_m \otimes w_m : \nabla^2 \varphi dy = \int_B \pi_m \Delta_y \varphi dy + \int_B g_m \cdot \nabla \varphi dy. \]

Let \( q_m \) be defined by

\[ \int_B \mu \left( \frac{\gamma_m}{R_m} \varphi_m + b_m \right) D_y w_m : D_y \nabla \varphi dy - R_m \gamma_m \int_B w_m \otimes w_m : \nabla^2 \varphi dy = \int_B q_m \Delta_y \varphi dy + \int_B g_m \cdot \nabla \varphi dy, \quad (40) \]

which is valid for a.e. \( s \in ]-1, 0[ \) and for all \( \varphi \in W_0^{2,(n+2)/2} (B) \), and

\[ (\pi_m - q_m, \Delta_y \varphi) = 0. \quad (41) \]

In (41), if we consider \( \varphi \) as a solution of the Laplace problem

\[ \Delta_y \varphi(s) = |(\pi_m - q_m)(s)|^{2/n} \text{sign}((\pi_m - q_m)(s)) \quad \text{in} \quad B, \]

we obtain \( \pi_m \equiv q_m \) a.e. in \( Q \). In (40), if we consider \( \varphi \) as the unique solution of the Dirichlet-Laplace problem

\[ \Delta_y \varphi(s) = |q_m(s)|^{2/n} \text{sign}(q_m(s)) \quad \text{in} \quad B(0, \varsigma); \]
\[ \varphi(s) = 0 \quad \text{on} \quad \bar{B} \setminus B(0, \varsigma), \]

then using (7) and the Hölder inequality, it results

\[ \int_{B(0, \varsigma)} |q_m|^{(n+2)/n} \leq \|g_m\|_{2,B(0,\varsigma)} \|\nabla \varphi\|_{2,B(0,\varsigma)} + \left( \mu^\# \|\nabla_y w_m\|_{(n+2)/n,B(0,\varsigma)} + R_m \gamma_m \|w_m\|_{2(n+2)/n,B(0,\varsigma)}^2 \right) \|\nabla^2 \varphi\|_{(n+2)/2,B(0,\varsigma)}. \]
It is known that the following estimate holds
\[ \|\nabla y \varphi(s)\|_{(n+2)/2, B(0, \varsigma)} + \|\nabla^2 y \varphi(s)\|_{(n+2)/2, B(0, \varsigma)} \leq C \|q_m(s)\|_{(n+2)/n, B(0, \varsigma)}^{2/n}, \]
for some constant \( C > 0 \). So it follows
\[ \|\pi_m\|_{(n+2)/n, B(0, \varsigma)} \leq C (R_m^{\lambda - \beta} + \mu^\# \|\nabla_y w_m\|_{(n+2)/n, B(0, \varsigma)} + R_m \gamma_m), \]
taking into account (35) and (24). Applying the Hölder inequality, we conclude that
\[ \limsup_{m \to \infty} \int_{Q(0, \varsigma)} |\pi_m|^{(n+2)/n} \, dx \leq C_1 \limsup_{m \to \infty} \left( \int_{Q(0, \varsigma)} |\nabla_y w_m|^2 \, dx \right)^{\frac{n+2}{2n}}, \quad (42) \]
with \( C_1 = C(\mu^\#) \).

Taking in (32) \( \varphi \in C^\infty_0(Q(0, 2\varsigma)) \) the cut-off function such that \( \varphi \equiv 1 \) in \( Q(0, \varsigma) \) and \( |\nabla \varphi| \leq C/\varsigma^{2\alpha/(n+2)}, |\partial_t \varphi| \leq C/\varsigma^{4\alpha/(n+2)} \) in \( Q(0, 2\varsigma) \), it results
\[ \mu^\# \int_{Q(0, \varsigma)} |\nabla_y w_m|^2 \, dx \leq \frac{C}{\varsigma^{4\alpha/(n+2)}} \int_{Q(0, 2\varsigma)} |w_m|^2 \, dx \, ds + \]
\[ + \frac{C}{\varsigma^{2\alpha/(n+2)}} \int_{Q(0, 2\varsigma)} |\pi_m w_m| \, dx \, ds + \int_{Q(0, 2\varsigma)} |g_m|^2 \, dx \, ds. \]

Using (36), the Hölder inequality and observing that
\[ \left( \frac{1}{\mu^\#} \frac{C}{\varsigma^{2\alpha/(n+2)}} \int_{Q(0, 2\varsigma)} |\pi_m w_m| \, dx \right)^{\frac{n+2}{2n}} \leq \frac{1}{2C_1} \int_{Q(0, 2\varsigma)} |\pi_m|^{(n+2)/n} \, dx + \]
\[ + \frac{C}{\varsigma^{2\alpha/n}} \left( \int_{Q(0, 2\varsigma)} |w_m|^{\alpha} \, dx \right)^{\frac{n+2}{\alpha n}}, \]
we conclude
\[ \limsup_{m \to \infty} \left( \int_{Q(0, \varsigma)} |\nabla_y w_m|^2 \, dx \right)^{\frac{n+2}{2n}} \leq \frac{C_2}{\varsigma^{2\alpha/n}} \limsup_{m \to \infty} \left( \int_{Q(0, 2\varsigma)} |w_m|^{\alpha} \, dx \right)^{\frac{n+2}{\alpha n}} + \]
\[ + \frac{1}{2C_1} \limsup_{m \to \infty} \int_{Q(0, 2\varsigma)} |\pi_m|^{\frac{n+2}{n}} \, dx, \quad (43) \]
with \( C_2 = C(\mu^\#, \mu^\#) \).
Introducing (43) into (42) and applying (38), we obtain
\[
\limsup_{m \to \infty} \int_{Q(0,c)} |\pi_m|^{(n+2)/n} \, de \leq C_3(2\varsigma)^\alpha + \frac{1}{2} \limsup_{m \to \infty} \int_{Q(0,2c)} |\pi_m|^{(n+2)/n} \, de,
\]
with \( C_3 = C_1C_2 2^{2\alpha/n} C_{**}^{n+2}/n \).

Iterating over \( i = 1, \cdots, k \), where \( k \) is such that \( 1/2 < 2^k \varsigma < 1 \) we derive
\[
\limsup_{m \to \infty} \int_{Q(0,c)} |\pi_m|^{n+2}/n \, de \leq 4C_3 \sum_{i=1}^{\lfloor \log_2 \varsigma \rfloor} 2^i(\alpha-2) \varsigma^\alpha + \limsup_{m \to \infty} \int_{Q(0,2^k c)} |\pi_m|^{n+2}/n \, de.
\]
Considering that
\[
\int_{Q(0,2^k c)} |\pi_m|^{n+2}/n \, de \leq C2^{n+2} \int_{Q} |\pi_m|^{n+2}/n \, de \leq C2^{n+2}
\]
it results
\[
\limsup_{m \to \infty} \int_{Q(0,c)} |\pi_m|^{n+2}/n \, de \leq \frac{4C_3}{2^{2-\alpha}-1} \varsigma^\alpha + C \leq C_{**}^{n+2}/n.
\]
Hence we obtain
\[
\limsup_{m \to \infty} Y_\varsigma(0; w_m, \pi_m, \kappa_m) \leq C_{**} \varsigma^\alpha + C_{**} \varsigma \leq (C_{**} + C_{**}) \varsigma^\alpha.
\]
This is a contradiction with (28) if we set \( C_* > C_{**} + C_{**} \). Theorem 3.1 is proved.

4.3 Proof of Lemma 3.1

(i) We prove (16) by induction on \( m \). The proof of (16) when \( m = 1 \) follows from Theorem 3.1, observing that (14) holds and
\[
C_* \varsigma^\alpha = C \varsigma^{\alpha-\delta} \varsigma^\delta \leq \varsigma^\delta.
\]
Now we suppose that (14) holds and, for \( i = 1, \cdots, m \), we have
\[
Y_{\varsigma R}(z; u, p, \theta) \leq \varsigma^{i\delta}(Y_{R}(z; u, p, \theta) + c\lambda(f)R^\beta).
\]
Thus (14) and (14) imply that
\[
Y_{\varsigma R}(z; u, p, \theta) + c\lambda(f)\varsigma^{i} R^\beta \leq \varsigma^{i\delta}(Y_{R}(z; u, p, \theta) + c\lambda(f)R^\beta) + c\lambda(f)\varsigma^{i\beta} R^\beta \leq \varsigma^{i\delta} Y_{R}(z; u, p, \theta) + c\lambda(f)R^\beta (\varsigma^{i\delta} + \varsigma^{i\beta}) < \gamma,
\]
considering that (15) holds.

We can apply Theorem 3.1 and subsequently (44) resulting

\[
Y_{\varsigma + 1}(z; u, p, \theta) = Y_{\varsigma R}(z; u, p, \theta) \leq C_{\varsigma} (Y_{\varsigma R}(z; u, p, \theta) + c_{\lambda}(f) R^{\beta}) \\
\leq \frac{1}{2} \varsigma \delta (Y_{\varsigma R}(z; u, p, \theta) + 2c_{\lambda}(f) R^{\beta}) \\
\leq \varsigma^{(i+1)}(Y_{\varsigma R}(z; u, p, \theta) + c_{\lambda}(f) R^{\beta}),
\]

which completes the proof of (i) in Lemma 3.1.

(ii) Let \( \rho \in [0, \varsigma R] \) be arbitrary and \( m \in \mathbb{N} \) be such that

\[
\varsigma^{m+1} < \frac{\rho}{R} \leq \varsigma^m.
\]

Proceeding as in [15] we have

\[
Y_{\rho}(z; u, p, \theta) \leq C(\varsigma) Y_{\varsigma R}(z; u, p, \theta) \leq (\text{see (44)}) \\
\leq C(\varsigma) \left( \frac{\varsigma^{m+1}}{\varsigma} \right)^{\delta} (Y_{\varsigma R}(z; u, p, \theta) + c_{\lambda}(f) R^{\beta}) \\
\leq C(\varsigma) \left( \frac{1}{\varsigma R} \right)^{\delta} (Y_{\varsigma R}(z; u, p, \theta) + c_{\lambda}(f) R^{\beta}),
\]

which concludes the proof of Lemma 3.1.

5 Proof of Theorem 2.2

Choose \( \varsigma \) such that (15) is fulfilled. For instance, taking \( \delta = \alpha/2 \) and \( \beta = \lambda/2 \) it follows

\[
\varsigma^{\alpha/2} \leq 1/C_{\varsigma} \quad \text{and} \quad \varsigma^{\alpha/2} + \varsigma^{\lambda/2} \leq 1.
\]

Thanks to Theorem 3.1 with the above chosen \( \varsigma \) and \( \beta \), there exist \( \bar{\varepsilon} = \varepsilon(C_{\varsigma}, \alpha, \lambda) \) and \( \bar{\gamma} = \gamma(C_{\varsigma}, \alpha, \lambda) \) such that for any proper weak solution \((u, p, \theta)\) of the N-S-F problem in \( Q_T \) satisfying

\[
Q(z, R) \subset Q_T, \quad 0 < R < \bar{\varepsilon}, \\
R|(|u|_{Q(z, R)}| < 1, \quad R|(|\theta|_{Q(z, R)}| < \bar{\gamma}/4 \\
Y_{R}(z; u, p, \theta) + c_{\lambda}(f) R^{\lambda/2} < \bar{\gamma}),
\]

(45)
and Lemma 3.1 can be applied. Indeed, for all $0 < \rho \leq \varsigma R$ we get

$$Y_\rho (z; u, p, \theta) \leq C \left( \frac{\rho}{R} \right)^{\alpha/2} (Y_R (z; u, p, \theta) + c_\lambda(f) R^{\lambda/2}). \tag{46}$$

Let $\Lambda = \bar{\varepsilon}\bar{\gamma}/8$ be the desired constant. By hypothesis (13) for each $z_0 \in Q_T$ there exists a constant $R_0 > 0$ that can be chosen such that $R_0 < \bar{\varepsilon}/2$ and for all $R < R_0$ we have

$$Q(z_0, R) \subset \subset Q_T; \quad R(\bar{Y}_R(z_0; u, p, \theta) + c_\lambda(f) R^{\lambda/2}) < \frac{\bar{\varepsilon}\bar{\gamma}}{8} + \frac{\bar{\varepsilon}\bar{\gamma}}{8} = \frac{\bar{\varepsilon}\bar{\gamma}}{4}. \quad (47)$$

By the continuity of $z \mapsto \bar{Y}_R(z; u, p, \theta)$ at $z_0$, there exists a neighbourhood of $z_0$, $O(z_0)$, such that for all $z \in O(z_0)$ we have

$$R(\bar{Y}_R(z; u, p, \theta) + c_\lambda(f) R^{\lambda/2}) < \frac{\bar{\varepsilon}\bar{\gamma}}{4}. \quad (47)$$

Indeed, there exists $0 < R_1 \leq R_0$ such that for all $R < R_1$ and $Q(z, R) \subset O(z_0)$ the relation (47) is satisfied. Applying Lemma 3.2 with $e = (y, s)$ and $r = \bar{\varepsilon}$ in (21)-(22) and using (47) we obtain

$$\bar{Y}_r(e; u^R, p^R, \theta^R) + c_\lambda(f^R; Q_r) r^{\lambda/2} \leq \frac{\bar{\varepsilon}\bar{\gamma}}{4} \quad (R < r),$$

with $Q_r$ denoting the transported domain. Let us consider the transported solution $(u^R, p^R, \theta^R)$ and $0 < r < \min\{\bar{\varepsilon}, 4/\bar{\gamma}\}$. Then the assumption (45) is fulfilled regarding that $r < \bar{\varepsilon}$, and

$$r \left| (u^R)_{Q(e, r)} \right| \leq r \bar{Y}_r(e; u^R, p^R, \theta^R) < 1,$$

$$r \left| (\theta)_{Q(e, r)} \right| \leq \bar{Y}_r(e; u^R, p^R, \theta^R) < \frac{\bar{\gamma}}{4}.$$

$$Y_r(e; u^R, p^R, \theta^R) + c_\lambda(f^R; Q_r) r^{\lambda/2} \leq 4 \bar{Y}_r(e; u^R, p^R, \theta^R) + c_\lambda(f^R; Q_r) r^{\lambda/2} < \bar{\gamma}.$$

Now using (46) it results

$$Y_\rho (e; u^R, p^R, \theta^R) \leq C \left( \frac{\rho}{r} \right)^{\alpha/2} \left( Y_r(e; u^R, p^R, \theta^R) + c_\lambda(f^R; Q_r) r^{\lambda/2} \right) < \bar{\gamma} C \left( \frac{\rho}{r} \right)^{\alpha/2}.$$

Thus $u$ is Hölder continuous with exponent $\alpha/4$ in a neighbourhood of $e$, taking into account the parabolic version of the Campanato criterion [4, Theorem I.2]. Theorem 2.2 is proved.
6 Proof of Theorem 2.3

We begin by recalling the following result which plays a central role in the proof.

**Lemma 6.1 ([12, Lemma 11])** Let $f \in L^1_{\text{loc}}(Q_T)$ and, for $0 < d < n + 2$, let

$$F = \{ z \in Q_T : \limsup_{R \to 0^+} R^{-d} \int_{Q(z,R)} |f| dx dt > 0 \}.$$  

Then, we have $\mathcal{H}^d(F) = 0$.

Let $z \in S$ be arbitrary and, for $R > 0$, denote

$$A = \frac{1}{\omega_n R^n} \int_{Q(z,R)} |u|^a, \quad B = \frac{1}{\omega_n R^n} \int_{Q(z,R)} |p|^{\frac{n+2}{a}}, \quad D = \frac{1}{\omega_n R^n} \int_{Q(z,R)} |\theta|^{a'},$$

with $\omega_n$ denoting the measure of the unit $n$-dimensional ball.

First, we observe that $S \subset \bigcup_{i=1}^5 F_i$, where each $F_i$ is the set of points of $S$ that verify the following $i$-case.

1. $[A \leq 1, \ B \leq 1, \ D \leq 1]$ This case is impossible due to

$$0 < \Lambda < \limsup_{R \to 0^+} R(R^{-2/a} + R(R^{-2n/(n+2)} + R^{-2/a'}))$$

$$= \limsup_{R \to 0^+} (R^{1-2/a} + R^{4/(n+2)} + R^{2/a}) = 0,$$

taking $a > 2$ into account (cf. [12]). Then we find that $F_1 = \emptyset$.

2. $[A \leq 1, \ B \leq 1, \ D \geq 1]$ Since $A^{a'/a} \leq 1$ and $B^{a'/(n+2)} \leq 1$, we get

$$0 < \Lambda^a < 4 \limsup_{R \to 0^+} \left(R^{(1-2/a)a'} + R^{4a'/(n+2)} + R^{2a'/a} D \right)$$

$$\leq 4 \limsup_{R \to 0^+} \frac{R^{2a'/a}}{\omega_n R^n} \int_{Q(z,R)} |\theta|^{a'} dx dt.$$  

Then from Lemma 6.1 it follows $\mathcal{H}^{n-2/(a-1)}(F_2) = 0$.  

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3. \([A \leq 1, \ B \geq 1, \ \forall D]\) Considering that
\[
1 < \frac{2(n + 2)}{n + 4} < a' \leq \frac{3}{2} < \frac{n + 2}{n} < 3 < a < \frac{2(n + 2)}{n}
\]
implies that \(B^{a'n/(n+2)} \leq B\), we get
\[
0 < \Lambda^{a'} < 4 \limsup_{R \to 0^+} \left( R^{(1-2/a)a'} + R^{4a'/n(n+2)}B + R^{2a'/a}D \right)
\]
\[
\leq 4 \limsup_{R \to 0^+} \frac{R^{2a'/a}}{\omega_n R^n} \int_{Q(z,R)} |p|^{\frac{n+2}{n}} + |\theta|^{a'} dxdt.
\]
Then from Lemma 6.1 it follows \(\mathcal{H}^{n-2/(a-1)}(F_3) = 0\).

4. \([A \geq 1, \ B \leq 1, \ \forall D]\) Arguing as in the above two cases, now considering that \(A^{a'/a} \leq A\), we get
\[
0 < \Lambda^{a'} < 4 \limsup_{R \to 0^+} \left( R^{(1-2/a)a'} A + R^{4a'/n(n+2)} + R^{2a'/a}D \right)
\]
\[
\leq 4 \limsup_{R \to 0^+} \frac{R^{(1-2/a)a'}}{\omega_n R^n} \int_{Q(z,R)} |u|^a + |\theta|^{a'} dxdt.
\]
Then from Lemma 6.1 it follows \(\mathcal{H}^{n-(a-2)/(a-1)}(F_4) = 0\).

5. \([A \geq 1, \ B \geq 1, \ \forall D]\) Analogously to the above cases, now with \(A^{a'/a} \leq A\) and \(B^{a'n/(n+2)} \leq B\), we get
\[
0 < \Lambda^{a'} < 4 \limsup_{R \to 0^+} \left( R^{(1-2/a)a'} A + R^{4a'/n(n+2)}B + R^{2a'/a}D \right)
\]
\[
\leq 4 \limsup_{R \to 0^+} \frac{R^{(1-2/a)a'}}{\omega_n R^n} \int_{Q(z,R)} (|u|^a + |p|^{\frac{n+2}{n}} + |\theta|^{a'}) dxdt.
\]
Then from Lemma 6.1 it follows \(\mathcal{H}^{n-(a-2)/(a-1)}(F_5) = 0\).

Next, from the properties of measure, we have
\[
\mathcal{H}^d(S) \leq \sum_{i=2}^{5} \mathcal{H}^d(F_i) \leq \sum_{i=2}^{5} \mathcal{H}^{d_i}(F_i),
\]
supposing that \(d = \max\{d_i, \ i = 2, \cdots, 5\}\).

Then, by the assumption (12) we have \(a \leq 4 (n = 2, 3)\) and we conclude that \(\mathcal{H}^{n-(a-2)/(a-1)}(S) = 0\). Therefore by definition of parabolic Hausdorff dimension the proof of Theorem 2.3 is finished.

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