Meson Spectrum in Strong Magnetic Fields

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We study the relativistic quark-antiquark system embedded in magnetic field (MF). The Hamiltonian containing confinement, one gluon exchange and spin-spin interaction is derived. We analytically follow the evolution of the lowest meson states as functions of MF strength. Calculating the one gluon exchange interaction energy $\langle V_{OGE} \rangle$ and spin-spin contribution $\langle a_{SS} \rangle$ we have observed, that these corrections remain finite at large MF, preventing the vanishing of the total $\rho$ meson mass at some $B_{\text{crit}}$, as previously thought. We display the $\rho$ masses as functions of MF in comparison with recent lattice data.

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I. INTRODUCTION

During the last years we have witnessed an impressive progress of the fundamental physics in ultra-intense magnetic field (MF) reaching the strength up to $eB \sim 10^{19} G \sim m_e^2$ \cite{1}. Until recently magnetars \cite{2} were the only physical objects, where such, or somewhat weaker MF could be realized. Now MF of the above strength and even stronger is within reach in peripheral heavy ion collisions at RHIC and LHC \cite{3}. High intensity lasers is another perspective tool to achieve MF beyond the Schwinger limit \cite{4}. On the theoretical side a striking progress has been achieved along several lines. It is beyond our scope to discuss these works or even present a list of corresponding references. We mention only two lines of research which have a certain overlap with our work. The first one \cite{5,6} is the behavior of the hydrogen atom and positronium in very strong MF. The second one \cite{7} is the conjecture of the vacuum reconstruction due to vector meson condensation in large MF. The relation between the above studies and our work will be clarified in what follows.

Our goal is to study from the first principles the spectrum of a meson composed of quark-antiquark embedded in MF. Use will be made of Fock-Feynman-Schwinger representation (see \cite{8} for review and references) of the quark Green’s function with strong (QCD) interaction and MF included. An alternative approach could have been Bethe-Salpeter type formalism. However, for the confinement originating from the area law of the Wilson loop, the use of the gluon propagator is inadequate. Numerous attempts in this direction failed because of gauge dependence and the vector character of the gluon propagator, while confinement is scalar and gauge invariant. Therefore it is sensible to use the path integral technique for QCD+QED Green’s functions. This method based on the proper-time formalism allows to represent the quark-antiquark Green’s function via the Hamiltonian (see \cite{9} for a new derivation), and it was used in \cite{10} to construct explicit expressions for meson Hamiltonians without MF. In this way spectra of light-light, light-heavy and heavy-heavy mesons were computed with a good accuracy, using the string tension $\sigma$, strong coupling constant $\alpha_S$ and quark current masses as an input \cite{11,12}.

In what follows we expand this technique to incorporate the effects of MF on mesons. The latter contains: 1) direct influence of MF on quark and antiquark, and 2) the influence on gluonic fields, e.g., on the gluon propagator via $q\bar{q}$ loops and on the gluon field correlators determining the string tension $\sigma$. Since MF acts on charged objects, its influence on the gluonic degrees of freedom enters only via $(N_c)^{-k}, \ k=1, 2, ..., $ however, corrections of the second type can be important, as shown in \cite{13}. 3) As was shown recently in the framework of our method, MF also changes quark condensate $\langle \bar{q}q \rangle$ and quark decay constants $\bar{f}_q$ etc., and in this way strongly influences chiral dynamics \cite{14}.

The important step in our relativistic formalism is the implementation of the pseudomomentum notion and center-of-mass (c.m.) factorization in MF, suggested in the nonrelativistic case in \cite{15} for neutral two particle systems. Recently the c.m. factorization was proved for the neutral 3-body system in \cite{16}, the situation with charged 2-body system was clarified and an approximation scheme was suggested in \cite{17}.

The plan of the paper is the following. Section 2 con-
tains a brief pedagogical reminder of how the two-body problem in MF is solved in quantum mechanics. The central point here is the integral of motion (“pseudomomentum”) which allows the separation of the center of mass. Here we also show how to diagonalize the spin-dependent interaction. In section 3 we formulate the path integral for quark-antiquark system with QCD+QED interaction. Then from Green’s function the relativistic Hamiltonian is obtained. Section 4 is devoted to the treatment of confining and color Coulomb terms. Here we also present the derivation of the eigenvalue equations for the relativistic Coulomb problem. In section 5 we discuss the spectrum of the system focusing on the regime of ultra-strong MF. Section 6 contains the discussion of the results, comparison with lattice calculations, drawing further perspectives and intersections of our results with those of other authors [5–7].

II. PSEUDOMOMENTUM AND WAVEFUNCTION FACTORIZATION

The total momentum of \(N\) mutually interacting particles with translation invariant interaction is a constant of motion and the center of mass motions can be separated in Schrödinger equation. It was shown [15] that a system embedded in a constant MF also possesses a constant of motion—“pseudomomentum”. As a result for the case of zero total electric charge \(Q = 0\) the c.m. motion can be removed from the total Hamiltonian. The simplest example is a two-particle system with equal masses \(m_1 = m_2 = m\) and electric charges \(e_1 = -e_2 = e\). We define

\[
R = \frac{r_1 + r_2}{2}, \quad \eta = r_1 - r_2, \quad P = p_1 + p_2. \tag{1}
\]

Straightforward calculation in the London gauge \(A = \frac{1}{2}(B \times r)\) yields

\[
\hat{H} = \frac{1}{4m} \left( P - \frac{e}{2} (B \times \eta) \right)^2 + \frac{1}{m} \left( -i \frac{\partial}{\partial \eta} - \frac{e}{2} (B \times R) \right)^2 + V(\eta). \tag{2}
\]

One can verify that the following “pseudomomentum” operator \(F\) commutes with the Hamiltonian [2]

\[
\hat{F} = P + \frac{e}{2} (B \times \eta). \tag{3}
\]

This immediately leads to the following factorization of the wave function (WF)

\[
\Psi(R, \eta) = \varphi(\eta) \exp \left\{ iPR - \frac{e}{2} (B \times \eta) R \right\}. \tag{4}
\]

For the oscillator-type potential \(V(\eta)\) the problem reduces to a set of three oscillators, two of them are in a plane perpendicular to the magnetic field and their frequencies are degenerate, while the third one is connected solely with \(V(\eta)\).

Next we briefly elucidate the spin interaction in presence of MF. The corresponding part of the Hamiltonian may be written as

\[
\hat{H}_s = a_{hf}(\sigma_1 \sigma_2) - \mu B(\sigma_1 - \sigma_2), \tag{5}
\]

where \(e_1 = -e_2 = e > 0\) and \(\mu > 0\). Diagonalization of \(H_s\) yields the following four eigenvalues e.g. for \(u\bar{u}\) system, comprising both \(\rho\) and \(\pi\) levels.

\[
E_{1,2}^{(s)} = a_{hf}, \quad E_{3,4}^{(s)} = \pm a_{hf} \left( 2\sqrt{1 + \left( \frac{\mu B}{a_{hf}} \right)^2} \mp 1 \right), \tag{6}
\]

where we assume that \(B\) is aligned along the positive \(z\)-axis and \(B = |B|\). In a strong MF when \(\mu B > a_{hf}\) spin-spin interaction becomes unimportant and \(E_{3,4}^{(s)} \approx \pm 2 \mu B\).

For the lowest level \(E_{1}^{(s)}\) this corresponds to a configuration \((+ -)\) when the spin of negatively charged particle is aligned antiparallel to \(B\), and the spin of the positively charged one – parallel to \(B\). This means that the spin (and isospin) are no more good quantum numbers and eigenvalues \([6]\) correspond to the mixture of spin 1 and spin 0 states. As a result the \(q\bar{q}\) state will split into 4 states (two of them coinciding \(E_{1}^{(s)} = E_{2}^{(s)}\)). Till now we treated a nonrelativistic system, to incorporate relativistic effects we shall exploit the path integral form of relativistic Green’s functions [8, 9].

III. RELATIVISTIC \(q\bar{q}\) GREEN’S FUNCTION AND EFFECTIVE HAMILTONIAN

The derivation of the relativistic Hamiltonian of the \(q\bar{q}\) system in MF consist of several steps. The first one is the 4d relativistic path integral for the \(q\bar{q}\) Green’s function. The starting point is the Fock-Feynmann-Schwinger (world-line) representation of the quark Green’s function [8]. The role of the “time” parameter along the path \(z_{\mu}^{(i)}(s_i)\) of the \(i\)-th quark is played by the Fock-Schwinger proper time \(s_i, i = 1, 2\). Consider a quark with a charge \(e_i\) in a gluonic field \(A_{\mu}\) and the electromagnetic vector potential \(A^{(e)}_{\mu}\), corresponding to a constant magnetic field \(B\). Then the quark propagator in the Euclidean spacetime is

\[
S_i(x, y) = (m_i + \hat{\partial} - ig \hat{A} - ie_i \hat{A}_{\mu}^{(e)}(x, y)^{-1})^{-1} \equiv (m_i + \hat{D}^{(i)})^{-1}. \tag{7}
\]

The path-integral representation for \(S_i\) [8] is

\[
S_i(x, y) = (m_i - \hat{D}^{(i)}) \int_0^\infty ds_i (D^{4})_{xy} e^{-K_i \Phi^{(i)}(x, y)} \equiv (m_i - \hat{D}^{(i)})G_i(x, y), \tag{8}
\]
Note, that operator $\hat{\varepsilon}$ replacement is valid:

\[
K_i = m_i^2 s_i + \frac{1}{4} \int_0^{s_i} d\tau_i \left( \frac{d\phi^{(i)}_\mu}{d\tau_i} \right)^2,
\]

\[
\Phi^{(i)}_\sigma(x, y) = P_A P_F \exp \left( ig \int_y^x A_\mu d\phi^{(i)}_\mu + ie_i \int_y^x A_\mu d\phi^{(i)}_\mu \right) \exp \left( \int_0^{\tau_i} d\tau_i \sigma_{\mu\nu}(gF_{\mu\nu} + e_i B_{\mu\nu}) \right).
\]

Here $F_{\mu\nu}$ and $B_{\mu\nu}$ are correspondingly gluon and MF tensors, $P_A, P_F$ are ordering operators, $\sigma_{\mu\nu} = \frac{1}{4}(\gamma_\mu \gamma_\nu - \gamma_\nu \gamma_\mu)$. Eqs. (11-12) hold for the quark, $i = 1$, while for the antiquark one should reverse the signs of $e_i$ and $g$. In explicit form one writes

\[
\sigma_{\mu\nu} F_{\mu\nu} = \begin{pmatrix} \sigma H & \sigma E \\ \sigma E & \sigma H \end{pmatrix}, \quad \sigma_{\mu\nu} B_{\mu\nu} = \begin{pmatrix} \sigma B & 0 \\ 0 & \sigma B \end{pmatrix}.
\]

Next we consider $q_1 \bar{q}_2$ system born at the point $x$ with the current $j_{\Gamma_1}(x) = \bar{q}_1(x) \Gamma_1 q_2(x)$ and annihilated at the point $y$ with the current $j_{\Gamma_2}(y)$. Here $x$ and $y$ denote the sets of initial and final coordinates of quark and antiquark. Using the nonabelian Stokes theorem and cluster expansion for the gluon field (see [11] for reviews) and leaving the MF term intact, we can write

\[
G_{q_1 \bar{q}_2}(x, y) = \int_0^\infty ds_1 \int_0^\infty ds_2 (D^4 z^{(1)})(x) (D^4 z^{(2)})(x) e^{-K_1-K_2 t \sigma_T} (\hat{T} W_\sigma(A))_A \times \\
\times \exp \left( ie_1 \int_y^x A_\mu^{(c)} d\phi^{(1)}_\mu - ie_2 \int_y^x A_\mu^{(c)} d\phi^{(2)}_\mu + e_1 \int_0^{\tau_1} d\tau_1 (\sigma B) - e_2 \int_0^{\tau_2} d\tau_2 (\sigma B) \right),
\]

where

\[
\hat{T} = \Gamma_1 (m_1 - \hat{D}_1) \Gamma_2 (m_2 - \hat{D}_2),
\]

and $\Gamma_1 = \gamma_\mu$, $\Gamma_2 = \gamma_\nu$ for vector currents, $\Gamma_i = \gamma_5$ for pseudoscalar currents, while

\[
\langle W_\sigma(A)_A = \exp \left( -\frac{g^2}{2} \int d\pi_{\mu\nu}(1) d\pi_{\lambda\sigma}(2) (F_{\mu\nu}(1) F_{\lambda\sigma}(2)) + O(\langle F F F \rangle) \right),
\]

where $d\pi_{\mu\nu} = d\pi_{\mu\nu} + \sigma^{(1)}_\nu d\tau_1 - \sigma^{(2)}_\nu d\tau_2$, and $d\pi_{\mu\nu}$ is an area element of the minimal surface, which can be constructed using straight lines, connecting the points $z^{(1)}_j(t)$ and $z^{(2)}_j(t)$ on the paths of $q_1$ and $\bar{q}_2$ at the same time $t \equiv t_\tau$. Note, that operator $\hat{T}$ actually do not participate in field averaging procedure: as was shown in [18], the following replacement is valid: $m - \hat{D} \rightarrow m - i \hat{p}$, $p_\mu = \frac{1}{2} \left( \frac{dz^{(1)}_\mu}{dt} \right)_T = s$.

As a result of the first step the $q \bar{q}$ Green’s function is represented as a 4d path integral (including Euclidean time paths) in addition also integrals over proper times $s_1, s_2$. In the second step one introduces monotonic Euclidean time $t_E(\tau) = x_4 + \frac{T}{2} \tau$, where $T \equiv |x_4 - y_4|$, so that $z_4(\tau) = t_E(\tau) + \Delta z_4(\tau)$, where $\Delta z_4(\tau)$ is fluctuation of time trajectory around $t_E(\tau)$. This new variable $t_E$ is an ordering parameter for trajectories $z^{(1)}(t_E)$, $z^{(2)}(t_E)$, and proper times transform into physical parameters – virtual $q$ and $\bar{q}$ energies $\omega_1 \equiv \frac{\omega_1}{2 \omega_1}$, so that $ds_i = -\frac{1}{2 \omega_1} d\omega_i$.

Combining for simplicity all fields into one Wilson loop $W(A, A^{(c)})$, one can rewrite the Green’s function in new variables as

\[
G_{q_1 \bar{q}_2}(x, y) = \frac{T}{8 \pi} \int_0^\infty \frac{d\omega_1}{\omega_1^{3/2}} \frac{d\omega_2}{\omega_2^{3/2}} (D^4 z^{(1)})(D^4 z^{(2)})_{xy} \times \\
\times e^{-K_1(\omega_1) - K_2(\omega_2)} (\hat{T} W_\sigma)_{\Delta z_4},
\]

(see [3] for details of derivation). Here $K_1(\omega_1), K_2(\omega_2)$ are obtained from $K_i$ in [11] by the same replacement $\frac{ds}{dt} = 2 \omega_1 \frac{dx_4}{dt_E}$,

\[
K_1(\omega_1) + K_2(\omega_2) = \left( \frac{m_1^2 + \omega_1^2}{2 \omega_1} + \frac{m_2^2 + \omega_2^2}{2 \omega_2} \right) T + \\
+ \int_0^T dt_E \left[ \frac{\omega_1}{2} \left( \frac{d\omega_1}{dt_E} \right)^2 + \frac{\omega_2}{2} \left( \frac{d\omega_2}{dt_E} \right)^2 \right].
\]

The final step is the use of the Wilson loop dynamics to express all dynamics in terms of instantaneous interaction. Indeed, the quadratic field correlator in (14) is represented through two scalar functions $D(z)$ and $D_1(z)$ (see, e.g., [11, 19] for details), first of them is responsible for confinement, while the second one gives one
gluon exchange (OGE) potential. So, for the case of zero quark orbital momenta with the minimal surface, discussed above, integrating over relative time \( \nu = t^E_1 - t^E_2 \) in \( D(\nu, z_1 - z_2), D_1(\nu, z_1 - z_2) \) one obtains a simple instantaneous answer for spin-independent (SI) part of \( \langle W_\nu(A) \rangle_A \),

\[
\langle W_\nu(A) \rangle_A^{SI} = \exp \left( - \int_0^T dt^E \left[ \sigma |z^{(1)} - z^{(2)}| - \frac{4}{3} \frac{\alpha_s}{|z^{(1)} - z^{(2)}|} \right] \right),
\]

containing \( V_{conf}(r) = \sigma r \) and \( V_{OGE}(r) = -\frac{4\alpha_s}{3r} \). Here \( \sigma \) is the QCD string tension, \( \sigma = 0.18 \text{ GeV}^2 \) in our calculations.

First we need to find the Hamiltonian \( H_{q\bar{q}} \) of the system at \( t^E_1 = t^E_2 = t^E \). To this end we define the Euclidean Lagrangian \( L^{E}_{q\bar{q}}. \) We write \( \frac{d\xi^{(i)}}{dt^E} = 2\omega_k \frac{dz_k^{(i)}}{dt^E} = 2\omega_k \dot{z}_k^i, k = 1, 2, 3. \) Then all terms in the exponents in (12), (13) and (17) can be represented as \( \exp(-\int dt^E L^{E}_{q\bar{q}}) \) and thus we arrive at the following representation:

\[
G_{q\bar{q}}(x, y) = \frac{T}{8\pi} \int_0^\infty \frac{d\omega_1}{\omega_1^{3/2}} \frac{d\omega_2}{\omega_2^{3/2}} (D_3z^{(1)} D_3z^{(2)})_{xy} \exp \left( e^{-S^E_{q\bar{q}}/T} \right)
\]

with the action

\[
S^E_{q\bar{q}} = \int_0^T dt^E \left[ \sum_i \left( \frac{\omega_i}{2} \dot{z}_k^{(i)} - i e_i A^{(e)}_k \dot{z}_k^{(i)} \right)^2 + \frac{\omega_i + \omega_2}{2} + \frac{m_1^2}{2\omega_1} + \frac{m_2^2}{2\omega_2} + e_1 \frac{\sigma_1 B}{2\omega_1} + e_2 \frac{\sigma_2 B}{2\omega_2} + \sigma |z^{(1)} - z^{(2)}| - \frac{4}{3} \frac{\alpha_s}{|z^{(1)} - z^{(2)}|} \right].
\]

Here \( A^{(e)}_k \) is the \( k \)-th component of the QED vector potential. The next step is the transition to the Minkowski metric. This is easy, since confinement is already expressed in terms of string tension. We have \( \exp(-\int L^{E} dt^E) \rightarrow \exp(i \int L^{M} dt^M), \quad t^E \rightarrow it^M, \) and

\[
p^{(i)}_k = \frac{\partial L^M}{\partial \dot{z}_k^{(i)}} = \omega_k \dot{z}_k^{(i)} + e_i A^{(e)}_k \dot{z}_k^{(i)}, \quad H_{q\bar{q}} = \sum_i \dot{z}_k^{(i)} p^{(i)}_k - L^M
\]

Explicit expression for Hamiltonian without spin-dependent terms is

\[
H_{q\bar{q}} = \sum_{i=1,2} \frac{(p^{(i)} - e_i A(z^{(i)})^2 + m_i^2 + \omega_i^2 - e_i \sigma^{(i)} B)}{2\omega_i} + \sigma |z^{(1)} - z^{(2)}| - \frac{4}{3} \frac{\alpha_s}{|z^{(1)} - z^{(2)}|}
\]

The \( q\bar{q} \) Green’s function (15) takes the “heat–kernel” form, when going back to Euclidean time with Hamiltonian (21)

\[
G_{q\bar{q}}(x, y) = \frac{T}{8\pi} \int_0^\infty \frac{d\omega_1}{\omega_1^{3/2}} \frac{d\omega_2}{\omega_2^{3/2}} \left( \sum_s \text{tr}(e^{-H_{q\bar{q}} T}) \right) \left| y \right|
\]

The c.m. projection of the Green’s function yields

\[
\int G_{q\bar{q}}(x, y) d^3(x - y) = \frac{T}{8\pi} \int_0^\infty \frac{d\omega_1}{\omega_1^{3/2}} \frac{d\omega_2}{\omega_2^{3/2}} \sum_{n=0}^\infty \phi^2_m(0) (\text{tr}(\hat{T})) e^{-M_n(\omega_1, \omega_2) T},
\]

where \( \varphi_n \) and \( M_n \) are eigenfunctions and eigenvalues of Hamiltonian \( H_{q\bar{q}} \). At large \( T \) the integral over \( \omega_1, \omega_2 \) can be taken by the stationary point method, and hence the effective energies \( \omega_i \) are to be found from the minimum of the total mass \( M_n(\omega_1, \omega_2) \), as it was suggested in (10). To introduce the minimization procedure and to
check its accuracy we shall begin by the calculation of the eigenvalues of one and two quarks in MF, and the energy of the ground state of a relativistic charge in the atom in the next section, reproducing the known exact results.

We have the following equations defining $\omega_i$ from the total mass $M(\omega_i)$

$$\hat{H}\psi = M_n(\omega_1, \omega_2)\psi, \quad \frac{\partial M_n(\omega_1, \omega_2)}{\partial \omega_i} = 0. \quad (24)$$

For a single quark in MF the first of the above equations gives

$$M_n(\omega) = \frac{p_z^2 + m_q^2 + |eB|(2n+1) - eB\sigma_z}{2\omega} + \frac{\omega}{2}. \quad (25)$$

Then the minimization over $\omega$ yields the correct answer

$$M_n = (p_z^2 + m_q^2 + |eB|(2n+1) - eB\sigma_z)^{1/2}. \quad (26)$$

Now we turn to the case of $q_1\bar{q}_2$ system and introduce the coordinates which are the generalization of (1):

$$R = \frac{\omega_1 z^{(1)} + \omega_2 z^{(2)}}{\omega_1 + \omega_2}, \quad \eta = z^{(1)} - z^{(2)}, \quad (27)$$

$$P = -i\frac{\partial}{\partial R}, \quad \pi = -i\frac{\partial}{\partial \eta}. \quad (28)$$

It is convenient to introduce the following two additional parameters

$$\tilde{\omega} = \frac{\omega_1 \omega_2}{\omega_1 + \omega_2}, \quad s = \frac{\omega_1 - \omega_2}{\omega_1 + \omega_2}. \quad (29)$$

Let us consider the case of neutral meson, so that $e_1 = -e_2 = e$. Then the total Hamiltonian may be written as

$$H_{q_1\bar{q}_2} = H_B + H_\sigma + W, \quad (30)$$

where

$$H_B = \frac{1}{2\omega_1} \left[ \frac{\tilde{\omega} P + \pi - eB}{2} \times (R + \frac{\tilde{\omega}}{\omega_1} \eta) \right]^2 + \frac{1}{2\omega_2} \left[ \frac{\tilde{\omega}}{\omega_1} P - \pi + eB \times \left( R - \frac{\tilde{\omega}}{\omega_2} \eta \right) \right]^2 = \frac{1}{2\omega_1} \left( \pi - \frac{eB}{2} \times R + s \frac{eB}{2} \times \eta \right)^2 + \frac{1}{2\omega_2} \left( P - \frac{eB}{2} \times \eta \right)^2. \quad (31)$$

Equation (31) is an obvious generalization of (2). The two other terms in (30) read

$$H_\sigma = \frac{m_1^2 + \omega_1^2 - e\sigma_1 B}{2\omega_1} + \frac{m_2^2 + \omega_2^2 + e\sigma_2 B}{2\omega_2}, \quad (32)$$

$$W = V_{\text{conf}} + V_{\text{OGE}} + \Delta W = \sigma \eta - \frac{4}{3} \frac{\alpha_s(\eta)}{\eta} + \Delta W, \quad (33)$$

and $\Delta W$ contains self-energy and spin–spin contributions, which come from unaccounted spin-dependent terms of $\langle W_s(A) \rangle$. One can verify, that the “pseudo-momentum” operator in (3), introduced in section II, commutes with $H_B$ and hence we can again separate the c.m. motion according to the ansatz (4):

$$H_B \Psi(\mathbf{R}, \eta) = \exp \left\{ i \mathbf{P} \mathbf{R} - \frac{e}{2}(\mathbf{B} \times \eta) \mathbf{R} \right\} \tilde{H}_B \varphi(\eta). \quad (34)$$

Then the problem reduces to the eigenvalue problem for $\varphi(\eta)$ with the Hamiltonian $\tilde{H}_B$ having the following form:

$$\tilde{H}_B = \frac{1}{\omega_2} \left( -i \frac{\partial}{\partial \eta} + s \frac{eB}{2} \times \eta \right)^2 + \frac{1}{2(\omega_1 + \omega_2)} \left( \mathbf{P} - e\mathbf{B} \times \eta \right)^2 \quad (35)$$

For $\mathbf{P} \times \mathbf{B} = 0$ the system has a rotational symmetry and the c.m. is freely moving along the $z$-axis. Here we shall consider a state with zero orbital momentum $(L_\eta)_z = [\eta \times \frac{\partial}{\partial \eta}_z] = 0$. As a result $\tilde{H}_B$ is replaced by a purely internal space operator

$$H_0 = \frac{1}{2\omega} \left( -\frac{\partial^2}{\partial \eta^2} + \frac{e^2}{4}(\mathbf{B} \times \eta)^2 \right). \quad (36)$$

To test our method we put $W = 0$ and arrive at the equation

$$(H_0 + h_\sigma) \varphi = M(\omega_1, \omega_2) \varphi. \quad (37)$$

Consequent minimization of $M(\omega_1, \omega_2)$ in $\omega_1, \omega_2$, as in (26), yields the expected answer for the two independent quarks,

$$M = \sqrt{m_1^2 + eB(2n_1 + 1) - e\sigma_1 B + m_2^2 + eB(2n_2 + 1) + e\sigma_2 B \mathbf{B}}. \quad (38)$$

We turn now to the particular case of charged two-body system in MF, $e_1 = e_2 = e$ and also $m_1 = m_2$, when exact factorization of $\mathbf{R}$ and $\eta$ can be done. In this case, for $\omega_1 = \omega_2 = \omega$ and $\mathbf{P} \times \mathbf{B} = 0$, the Hamiltonian has the following form (3):

$$H_{q_1\bar{q}_2} = \frac{P^2}{4\omega} + \frac{e^2}{4\omega}(\mathbf{B} \times \mathbf{R})^2 + \frac{\pi^2}{\omega} + \frac{e^2}{16\omega}(\mathbf{B} \times \eta)^2 + \frac{2m_1^2 + 2m_2^2 - e(\sigma_1 + \sigma_2)B}{2\omega} + \frac{\sigma}{2} (\frac{\eta^2}{\gamma} + \gamma) + V_{\text{OGE}} + V_{SS} + \Delta M_{SE}. \quad (39)$$

IV. TREATING CONFINEMENT AND GLUON EXCHANGE TERMS. THE ABSENCE OF THE MAGNETIC QCD COLLAPSE

From (33), (36) it is clear, that inclusion of $V_{\text{conf}}$ and $V_{\text{OGE}}$ in $H_0 + W$ leads to a differential equation in variables $\eta_1, \eta_2$, which can be solved numerically. However,
in order to obtain a clear physical picture, we shall represent \( V_{\text{conf}} \) in a quadratic form. This will allow to get an exact analytic solution in terms of oscillator functions with eigenvalue accuracy of the order of 5\%. The OGE contribution will be estimated as an average \( \langle \varphi | V_{\text{OGE}} | \varphi \rangle \), thus yielding an upper limit for the total mass.

For \( V_{\text{conf}} \) we choose the form

\[
V_{\text{conf}} \rightarrow \bar{V}_{\text{conf}} = \frac{\alpha}{2} \left( \frac{\eta^2}{\gamma} + \gamma \right)
\]

Here \( \gamma \) is a positive variational parameter; minimizing \( \bar{V}_{\text{conf}} \) w.r.t. \( \gamma \), one returns to \( V_{\text{conf}} \). We shall determine \( M(\omega_1, \omega_2, \gamma) \) corresponding to \( \bar{V}_{\text{conf}} \), and to define \( \gamma \) an additional condition

\[
\frac{\partial M(\omega_1, \omega_2, \gamma)}{\partial \gamma} |_{\gamma = \gamma_0} = 0
\]

will be added to (24). As a result \( M(\omega_1^{(0)}, \omega_2^{(0)}, \gamma_0) \) will be the final answer for the mass of the system, neglecting the \( \Delta W \) contribution. The difference of the exact numerical solution from that obtained with the genuine potential \( V_{\text{conf}} \) does not exceed 5\%. The solution of the equation \( (H_0 + \bar{V}_{\text{conf}}) \varphi = M(\omega_1, \omega_2, \gamma_0) \varphi \) for the ground state is

\[
\psi(\eta) = \frac{1}{\sqrt{\pi^{3/2} \gamma^3 r_0}} \exp \left( -\frac{\eta^2}{2r_\perp^2} - \frac{\eta^2}{2r_0^2} \right),
\]

where \( r_\perp = \sqrt{\frac{2}{eB}} \left( 1 + \frac{4\sigma}{\gamma^2 eB^2} \right)^{-1/4} \), \( r_0 = (\frac{\gamma}{\gamma_0})^{1/4} \). As we shall see below, for the lowest mass eigenvalue with \( eB \gg \sigma \), one has \( r_\perp \approx \sqrt{\frac{2}{eB}} \), \( r_0 \approx \frac{1}{\sqrt{\gamma}} \) and the \((q_1 \bar{q}_2)\) system acquires the form of an elongated ellipsoid. Similar quasi-one-dimensional picture was observed before for the hydrogen–like atoms in strong MF [5, 6]. In such geometrical configuration \( V_{\text{OGE}} \) manifests itself in a peculiar way, again similar to what happens in hydrogen, or positronium atoms, and as was shown in [13] in QCD the outcome is also similar to the case of QED, with the screening of the diverging effects.

We turn now to the OGE term to treat it in our formalism. As a starting point we present another check of our approach, namely we shall obtain the ground state energy of two relativistic particles with opposite charges without MF interacting via the Coulomb potential. The corresponding Hamiltonian reads \( H = H_0 + H_\sigma - \frac{\alpha}{\gamma} \), then \( H\phi = M\phi \), and for \( eB = 0 \) we have

\[
M = -\frac{\omega_0^2 \alpha^2}{2} + \frac{m_1^2 + \omega_1^2}{2\omega_1} + \frac{m_2^2 + \omega_2^2}{2\omega_2}.
\]

Minimizing in \( \omega_1 \) in the limit \( m_2 \gg m_1 \) (the hydrogen atom), one obtains

\[
M = m_1 \sqrt{1 - \alpha^2} + m_2,
\]

which coincides with the known eigenvalue of the Dirac equation.

In our \((q_1 \bar{q}_2)\) case one can calculate the expectation value of \( V_{\text{OGE}} = -\frac{\alpha \sigma(\gamma)}{\eta} \), with the asymptotic freedom and IR saturation behaviour in \( p \)-space (see [22] for a derivation and a short review)

\[
\alpha_s(q) = \frac{4\pi}{\beta_0 \ln \left( \frac{q^2 + M_B^2}{\Lambda_{QCD}^2} \right)},
\]

where \( \beta_0 = \frac{11}{3} N_c - \frac{2}{3} n_f, M_B \) is proportional to \( \sqrt{\sigma} \), \( M_B \approx 1 \text{ GeV} \). With the wavefunction (42) the average value of \( V_{\text{OGE}} \) takes form

\[
\Delta M_{\text{OGE}} = \int V_{\text{OGE}}(\phi) \psi^2(q) \frac{d^3 q}{(2\pi)^3} = -\frac{4}{3\pi} \int_0^\infty \alpha_s(q) dq e^{-\frac{\pi^2}{4} I \left[ \frac{q^2(r_0^2 - r_\perp^2)}{4} \right]},
\]

where \( \psi^2(q) \) is the Fourier transform of squared wave function \( \psi^2(\eta) \) and \( I(a^2) = \int_{-1}^{+1} dxe^{-a^2 x^2} \). Estimating the integral in (46), for \( eB \gg \sigma \), i.e. for \( r_0 \gg r_\perp \) one obtains for massless quarks

\[
\Delta M_{\text{OGE}} \approx -\frac{16\sqrt{\pi}}{3 r_0 \beta_0} \ln \frac{r_0^2}{r_\perp^2} \approx -\sqrt{\sigma} \ln \frac{eB}{\sigma}.
\]

With \( eB \) increasing the upper bound for the \( q\bar{q} \) mass is boundlessly decreasing. The exact eigenvalue should lie even lower.

This situation is similar to the hydrogen atom case, where \( \Delta M_{\text{ Coul}} \) diverges as \( -\ln \epsilon eB \), and in this case \( e^\epsilon \) loop contribution to the photon line stabilizes the result (the “screening effect” [4, 7]). In our case the \( q\bar{q} \) loop contribution to the OGE term can be written in a similar way, adding to the gluon loop also the Lowest Landau Level (LLL) of the \( q\bar{q} \) in the MF,

\[
\tilde{\psi}_{\text{OGE}}(Q) = -\frac{16\pi \alpha_s^{(0)}}{3} \left[ Q^2 \left( 1 + \frac{\alpha_s^{(0)}}{4\pi} \frac{11}{3} N_c \ln \frac{Q^2 + M_B^2}{\mu^2} \right) + \frac{\alpha_s^{(0)} n_f |e_B|}{\pi} \ln \left( \frac{-q^2}{2 |e_B|} \right) \right],
\]
FIG. 1. One gluon exchange correction to the meson mass in GeV as a function of magnetic field with (solid line) and without (broken line) account of quark loops contributions.

where $T(z) = \frac{\ln(\sqrt{1 + z^2} + \sqrt{z(z + 1)})}{\sqrt{z(z + 1)}} + 1$. Calculating now the average value of $\langle V_{\text{OGE}} \rangle$, one obtains saturation of $\Delta M_{\text{OGE}}$ at large $eB$, as shown in Fig. 1, eliminating in this way the possible “Color Coulomb catastrophe”, discussed in the first version of this paper [23].

V. MESON MASSES IN MAGNETIC FIELD

Our next task is to calculate analytically the mass $M_n(\omega_1, \omega_2, \gamma)$ of a $(q_1\bar{q}_2)$ meson. We have to solve the equation

$$(H_0 + H_\sigma + W)\Psi_n(\eta) = M_n(\omega_1, \omega_2, \gamma)\Psi_n(\eta), \quad (50)$$

where $H_0, H_\sigma, W$ are given in (32), (33), (36), the total Hamiltonian for charged meson is given in (39).

The resulting mass for neutral meson without spin-dependent contribution from $\Delta W$ is

$$M_n(\omega_1, \omega_2, \gamma) = \varepsilon_{n_1, n_2} + \Delta M_{\text{OGE}} + \frac{m_1^2 + \omega_1^2}{2\omega_1} eB\sigma_1 + \frac{m_2^2 + \omega_2^2 + eB\sigma_2}{2\omega_2} \equiv M_n(\omega_1, \omega_2, \gamma) - \frac{eB\sigma_1}{2\omega_1} + \frac{eB\sigma_2}{2\omega_2}, \quad (51)$$

where

$$\varepsilon_{n_1, n_2} = \frac{1}{2\omega} \left[ \sqrt{e^2 B^2 + \frac{4\sigma\omega}{\gamma}(2\omega_1 + 1) + \frac{4\sigma\omega}{\gamma}(n_2 + 1)} \right] + \frac{\gamma\sigma}{2}, \quad (52)$$

$\Delta M_{\text{OGE}}$ is given by (48) and (49). So, for fixed $n$ we have four states for different quark spin orientations, $|+\rangle, |\gamma\rangle, |\pm\rangle, |\mp\rangle$ and $|\pm\rangle$, where $+/-$ are up/down directions of individual quark spins, with corresponding masses

$$M_n^{++} = \bar{M}_n - eB \left( \frac{1}{2\omega_1} - \frac{1}{2\omega_2} \right), \quad (53)$$

$$M_n^{--} = \bar{M}_n + eB \left( \frac{1}{2\omega_1} - \frac{1}{2\omega_2} \right), \quad (54)$$

$$M_n^{+-} = \bar{M}_n + eB \left( \frac{1}{2\omega_1} + \frac{1}{2\omega_2} \right), \quad (55)$$

$$M_n^{-+} = \bar{M}_n - eB \left( \frac{1}{2\omega_1} + \frac{1}{2\omega_2} \right). \quad (56)$$

The spin-dependent part $\Delta W$ contains self-energy $V_{SE}$ and spin-spin $V_{SS}$ contributions. As was shown in [24], the mass correction, corresponding to $V_{SE}$, is given by

$$\Delta M_{SE} = -\frac{3\omega}{4\pi\omega_1} \left[ 1 + \eta \left( \frac{1}{\sqrt{2eB + m_1^2}} \right) - \frac{3\omega}{4\pi\omega_2} \left[ 1 + \eta \left( \frac{1}{\sqrt{2eB + m_2^2}} \right) \right], \quad (57)$$

where $\eta(t) = t \int_0^\infty z^2 K_1(tz)e^{-z}dz$ and $\lambda \sim 1$ GeV$^{-1}$ is vacuum correlation lengths.

Let us introduce now the spin-spin interaction. It has nondiagonal structure

$$V_{SS} = \frac{8\omega_1\omega_2}{9\omega_1\omega_2} \delta(3)(r)\sigma_1 \sigma_2 \equiv a_{SS} \sigma_1 \sigma_2, \quad (58)$$

so we should diagonalize the Hamiltonian with respect to spin variables. This results in new four states, two of them are mixture of $|+\rangle$ and $|-\rangle$ states, corresponding to $\pi^0$ and $\rho^0$ with zero spin projection $s_z = 0$, the other two states $|+\rangle$ and $|-\rangle$ correspond to $\rho^0$ states with $s_z = 1$ and $s_z = -1$ (we consider ground state $n = 0$).

We note that both $V_{SS}$ and $\Delta M_{SE}$ are to be considered as corrections and contain $\omega_1(0), \omega_2(0)$, obtained from minimization of the remaining part of the meson mass. Note also, that masses $M_n^{++}$ and $M_n^{-+}$ are symmetric with respect to $\omega_1 \leftrightarrow \omega_2$ for equal quark masses, so in this case we have in fact only two variables in minimization procedure $\omega$ and $\gamma$.

The masses of first two states are:

$$E_{1,2} = \frac{1}{2}(M_{11} + M_{22}) \pm \sqrt{\left( \frac{M_{22} - M_{11}}{2} \right)^2 + 4a_{12}a_{21}}, \quad (59)$$
where

\[ M_{11} = (M_0^0 + \Delta M_{SE} - \langle a_{SS} \rangle) \mid _{\omega_1^0 = \omega_2^0 = \omega_3^0}, \]
\[ M_{22} = (M_0^0 + \Delta M_{SE} - \langle a_{SS} \rangle) \mid _{\omega_1^0 = \omega_2^0 = \omega_3^0}, \]  

(60)

\[ a_{12} = a_{21} = \langle a_{SS} \rangle \mid _{\omega_1^0 = \omega_2^0 = \omega_3^0} \] and \( \langle a_{SS} \rangle \) is the averaging with the wave function (42) (see [24] for derivation).

The parameters \( \omega_1^0, \omega_2^0, \) and \( \omega_3^0 \) are obtained by minimizing of corresponding diagonal eigenvalues \( M_0^0 \) and \( M_0^0 \), the parameter \( \gamma_0 \) (see (35)) for the ground state is defined from the condition

\[ \frac{\partial M_0(\omega_1, \omega_2, \gamma)}{\partial \gamma} \bigg|_{\gamma = \gamma_0} = \frac{\partial E_{00}}{\partial \gamma} \bigg|_{\gamma = \gamma_0} = 0. \]  

(61)

It is easy to see, that at large \( eB \) masses \( E_1, E_2 \) tend to diagonal values

\[ E_1(eB \to \infty) \to M_{22}, \quad E_2(eB \to \infty) \to M_{11}. \]  

(62)

The remaining two states have masses

\[ E_3 = M_0^0 + \Delta M_{SE} + \langle a_{SS} \rangle, \quad E_4 = M_0^0 - \Delta M_{SE} + \langle a_{SS} \rangle, \]  

(63)

taken in point \( (\omega_1^0, \omega_2^0, \gamma^0) \) in accordance with minimization conditions [24] and (61).

It should be noted, that actually we have eight states instead of four, since \( \bar{q}q \) systems with different quark charges behave differently in MF, as we see from our Hamiltonians. Isospin is not conserved now and each neutral state splits into two states with different quark content \( u \bar{u} \) and \( d \bar{d} \).

Let us consider also the particular case of charged meson with Hamiltonian (33) in state with \( s_z = 1 \) (\( \{| + \rangle \} \) state, corresponding to \( \rho^+ \)). The eigenvalue, corresponding to this state, is given by the following expression

\[ M_n(\omega, \gamma) = \frac{eB}{2 \omega} (2N_\perp + 1) + \sqrt{\frac{eB^2}{2 \omega^2} + \frac{2 \sigma}{\omega \gamma} (2n_\perp + 1) + \frac{\sqrt{\frac{2 \sigma}{\omega \gamma}}}{n_\parallel + 1} - \frac{eB}{\omega} - \frac{\sigma}{\omega} \gamma^2 + \frac{n^2 + \omega^2}{\omega} - \frac{\Delta M_{OGE} + \Delta M_{SE} + \langle a_{SS} \rangle}{\omega}. \]  

(64)

Among considered states, the mass of charged meson ground state \( (\rho^+ \) with \( s_z = 1 \) and \( E_2 \), corresponding to \( \pi^0 \), tend to finite value at large MF due to cancellation of linearly growing terms in \( e_{n_\perp,n_z} \) and \( H_s \), while other masses grow with \( eB \). This is true, provided that the spin-spin contribution \( \langle a_{SS} \rangle \) remains finite at large MF. However, it contains the factor \( \psi^2(0) \sim eB \), which leads to unbounded decrease of \( E_2 \). As was shown in [24], this situation is not physical, the total mass eigenvalues should be positive, and the reason of this decrease is the unlawful use of the perturbation theory for the potential \( c \delta^{(3)}(r) \). One should replace \( a_{SS} \) by a smeared out version, e.g.,

\[ \delta^{(3)}(r) \to \tilde{\delta}^{(3)}(r) = \left( \frac{1}{\lambda \sqrt{\pi}} \right)^3 e^{-r^2/\lambda^2}, \quad \lambda \sim 1 \text{ GeV}^{-1}. \]  

(65)

Using the wave function (42), one obtains for \( \langle a_{SS} \rangle \)

\[ \langle a_{SS} \rangle = \frac{c}{\pi^{3/2} \sqrt{\lambda^2 + r_0^2}} \left( \lambda^2 + r_0^2 \right), \quad c = \frac{8 \pi \alpha_s}{9 \omega_1 \omega_2}. \]  

(66)

The smearing length \( \lambda \) on the lattice corresponds to the lattice unit \( \lambda (a \sim a) \), in physical situation the relativistic smearing is connected with the gluon mass parameters in \( D(z) \) and \( D_1(z) \), see [19] for details.

![FIG. 2. The masses of the systems in GeV as a functions of \( eB \). See the text for explanations.](image)

In Fig.2 we plot the masses of some selected systems as a functions of \( eB \) (\( e \) is the \( \rho^+ \) charge, not the charge of individual quarks). Calculations were performed according to (59), (63), (64) and the minimization procedure. The dashed curves correspond to the \( \rho^0 \) state with \( s_z = 0 \) (eigenvalue \( E_1 \)), the solid-symbol lines describe \( \rho^0 \) state with \( s_z = 1 \), the lower solid curve refers to the state of charged meson \( \rho^+ \) with \( s_z = 1 \). The black triangles are from lattice calculations [23]. One can see that the masses of first states are increasing, while the last one tends to finite limit in accordance with discussion above (note, that the results plotted in Fig.2 were obtained for massless quarks).

### VI. DISCUSSION AND CONCLUSIONS

In our treatment of relativistic quark–aniquark system embedded in MF we relied on pseudomomentum factorization of the wave function and relativistic Hamiltonian technique. The Hamiltonian for mesons in MF, containing confinement, one gluon exchange and spin interaction was derived. Using a suitable approximation for confining force we were able to calculate analytically meson masses as functions of the MF. In this paper to simplify...
things we started with $\rho^0$ meson states at $B = 0$ taking $\gamma_i$ in place of $\Gamma_1$ and $\Gamma_2$ in [13]. In this way we essentially left aside the complicated problem of chiral dynamics and pseudo-Goldstone spectrum. In this oversimplified picture the lowest neutral state with $s_z = 0$ is a mixture of the $\rho^0$ and $\pi^0$, as can be seen from its spin and isospin structure. Indeed, $uu$ or $dd$ system under consideration is a mixture of isospin $I = 0$ and $I = 1$ states, and at large MF it has a spin structure $|u \uparrow, \bar{u} \downarrow\rangle$, which is a mixture of $S = 0$ and $S = 1$ states. We have calculated mass of the higher state of this mixture, which we call $\rho(eB)$, while the lower state, associated with $\pi^0(eB)$, can be subject to chiral corrections. These states have negative corrections from one-gluon exchange and spin-spin interactions. As was shown, these corrections (and the total mass) stay finite at large $B$, preventing the so-called “magnetic collapse in QCD”, discussed earlier in [23].

As shown in Fig. 2, our analytical results are in agreement with lattice calculations [25] both for $\rho^0$ and $\rho^+$ states.

Another system which can be treated using the same technique is the neutral 3–body system, like neutron. The results might be important for the neutron stars physics.

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