SEMICLASSICAL SPECTRAL ANALYSIS OF TOEPLITZ OPERATORS ON SYMPLECTIC MANIFOLDS: THE CASE OF DISCRETE WELLS

YURI A. KORDYUKOV

Abstract. We consider Toeplitz operators associated with the renormalized Bochner Laplacian on high tensor powers of a positive line bundle on a compact symplectic manifold. We study the asymptotic behavior, in the semiclassical limit, of low-lying eigenvalues and the corresponding eigensections of a self-adjoint Toeplitz operator under assumption that its principal symbol has a non-degenerate minimum with discrete wells. As an application, we prove upper bounds for low-lying eigenvalues of the Bochner Laplacian in the semiclassical limit.

1. Preliminaries and main results

1.1. Introduction. The Berezin-Toeplitz operator quantization is a quantization method for a compact quantizable symplectic manifold, which is a particularly effective version of geometric quantization. It was Berezin who recognized the importance of Toeplitz operators for quantization of Kähler manifolds in his pioneering work [4]. There are several approaches to Berezin-Toeplitz and geometric quantization (see, for instance, survey papers [1, 16, 33, 41]). For a general compact Kähler manifold, the Berezin-Toeplitz quantization was constructed by Bordemann-Meinrenken-Schlichenmaier [5], using the theory of Toeplitz structures of Boutet de Monvel and Guillemin [8]. In this case, the quantum space is the space of holomorphic sections of tensor powers of the prequantum line bundle over the Kähler manifold. In order to generalize the Berezin-Toeplitz quantization to arbitrary symplectic manifolds, one has to find a substitute for this quantum space. A natural candidate suggested by Guillemin and Vergne is the kernel of the spin$^c$ Dirac operator. The Berezin-Toeplitz quantization with such a quantum space was developed by Ma-Marinescu [35, 37]. It is based on the asymptotic expansion of the Bergman kernel outside the diagonal obtained by Dai-Liu-Ma [13]. Another candidate suggested by Guillemin-Uribe [17] is the space of eigensections of the renormalized Bochner Laplacian corresponding to eigenvalues localized near the origin. In this case, the Berezin-Toeplitz quantization was recently constructed in [26, 28], based on

2000 Mathematics Subject Classification. Primary 58J50; Secondary 53D50, 58J37.

Key words and phrases. Bochner Laplacian, symplectic manifolds, semiclassical analysis, Berezin-Toeplitz quantization, eigenvalue asymptotics.

Supported by the Russian Science Foundation, project no. 17-11-01004.
Ma-Marinescu work: the Bergman kernel expansion from [36] and Toeplitz calculus developed in [37] for spin$^c$ Dirac operator and Kähler case (also with an auxiliary bundle). We note also that Charles [12] proposed recently another approach to quantization of symplectic manifolds and Hsiao-Marinescu [25] constructed a Berezin-Toeplitz quantization for eigensections of small eigenvalues in the case of complex manifolds.

The simplest example of the Berezin-Toeplitz quantization is given by the Toeplitz operators on the Fock space. This quantization is related with the standard pseudodifferential calculus in the Euclidean space via the Bargmann transform. Therefore, for a general quantizable symplectic manifold, the Berezin-Toeplitz quantization can be considered as a kind of operator calculus on the manifold, similar to the semiclassical pseudodifferential calculus in the Euclidean space. Several basic notions and results of the semiclassical pseudodifferential calculus in the Euclidean space were extended to Berezin-Toeplitz operators on Kähler manifolds by Charles [9].

In this paper, we are interested in asymptotic spectral properties of self-adjoint Toeplitz operators in semiclassical limit. Several aspects of the semiclassical spectral analysis were studied for Toeplitz operators on compact Kähler manifolds: for instance, quantum ergodicity [13], the Gutzwiller trace formula [6], the Bohr-Sommerfeld conditions (both regular [10, 11] and singular [30, 31]). We consider Toeplitz operators associated with the renormalized Bochner Laplacian on an arbitrary quantizable compact symplectic manifold. We assume that the principal symbol of a self-adjoint Toeplitz operator has a non-degenerate minimum with discrete wells and study the asymptotic behavior, in the semiclassical limit, of its eigenvalues at the bottom of the spectrum (low-lying eigenvalues) and of the corresponding eigensections. For semiclassical pseudodifferential operators (in particular, for the Schrödinger operator $-h^2 \Delta + V$ in $\mathbb{R}^n$ with potential wells), the study of similar problems goes back to the papers of Helffer-Sjöstrand [24], Helffer-Robert [23] and Simon [42]. For compact Kähler manifolds, such problems were recently studied by Deleporte in [14, 15]. For compact Kähler surfaces, a full asymptotic expansion for the first eigenvalues, valid on a fixed interval, was obtained in [30].

As an immediate application of our results, we obtain asymptotic upper bounds, in the semiclassical limit, for low-lying eigenvalues of the Bochner Laplacian with discrete wells, thus extending to the case of arbitrary even dimension the results of [22, 18] on eigenvalue asymptotics for the two-dimensional magnetic Schrödinger operator with non-vanishing magnetic field and discrete wells.

1.2. Preliminaries on Toeplitz operators. Let $(X, \omega)$ be a compact symplectic manifold of dimension $2n$ and $(L, h^L)$ be a Hermitian line bundle on $X$ with a Hermitian connection

$$\nabla^L : C^\infty(X, L) \to C^\infty(X, T^*X \otimes L).$$
The curvature of this connection is given by $R^L = (\nabla^L)^2$. We will assume that $L$ satisfies the pre-quantization condition:

(1.1) \( \frac{i}{2\pi} R^L = \omega. \)

Thus, $[\omega] \in H^2(X, \mathbb{Z})$.

Let $g$ be a Riemannian metric on $X$. Let $J_0 : TX \to TX$ be a skew-adjoint operator such that

\[
\omega(u, v) = g(J_0 u, v), \quad u, v \in TX.
\]

Consider the operator $J : TX \to TX$ given by

(1.2) \( J = J_0 (-J_0^{-2})^{-1/2}. \)

Then $J$ is an almost complex structure compatible with $\omega$ and $g$, that is,

\[
g(Ju, Jv) = g(u, v), \quad \omega(Ju, Jv) = \omega(u, v) \quad \text{and} \quad \omega(u, Ju) \geq 0 \quad \text{for any} \quad u, v \in TX.
\]

For any $p \in \mathbb{N}$, let $L^p := L^{\otimes p}$ be the $p$-th tensor power of $L$. Let $\nabla^{L^p} : C^\infty(X, L^p) \to C^\infty(X, T^* X \otimes L^p)$ be the connection on $L^p$ induced by $\nabla^L$. Denote by $\Delta^{L^p}$ the induced Bochner Laplacian acting on $C^\infty(X, L^p)$ by

(1.3) \( \Delta^{L^p} = (\nabla^{L^p})^* \nabla^{L^p}, \)

where $(\nabla^{L^p})^* : C^\infty(X, T^* X \otimes L^p) \to C^\infty(X, L^p)$ denotes the formal adjoint of the operator $\nabla^{L^p}$. If $\{e_j\}_{j=1, \ldots, 2n}$ is a local orthonormal frame of $TX$, then $\Delta^{L^p}$ is given by

(1.4) \( \Delta^{L^p} = -\sum_{j=1}^{2n} \left[ (\nabla^{L^p}_{e_j})^2 - \nabla^{L^p}_{\nabla^{TX}_{e_j} e_j} \right], \)

where $\nabla^{TX}$ is the Levi-Civita connection of the metric $g$.

The renormalized Bochner Laplacian $\Delta_p$ is a second order differential operator acting on $C^\infty(X, L^p)$ by

\[
\Delta_p = \Delta^{L^p} - p\tau,
\]

where $\tau$ is a smooth function on $X$ given by

(1.5) \( \tau(x) = -\pi \text{Tr}[J_0(x)J(x)], \quad x \in X. \)

This operator was introduced by Guillemin-Uribe in [17]. When $(X, \omega)$ is a Kähler manifold, it is twice the corresponding Kodaira Laplacian on functions $\Box^{L^p} = \bar{\partial}^{L^p} \partial^{L^p}$.

Denote by $\sigma(\Delta_p)$ the spectrum of $\Delta_p$ in $L^2(X, L^p)$. Put

(1.6) \( \mu_0 = \inf_{u \in T^* X, x \in X} \frac{i R_x^L(u, J(x)u)}{|u|^2}. \)

By [33, Cor. 1.2], there exists a constant $C_L > 0$ such that for any $p$

\[
\sigma(\Delta_p) \subset [-C_L, C_L] \cup [2p\mu_0 - C_L, +\infty).
\]
Consider the finite-dimensional vector subspace \( \mathcal{H}_p \subset L^2(X, L^p) \) spanned by the eigensections of \( \Delta_p \) corresponding to eigenvalues in \( [-CL, CL] \). By [34 Cor. 1.2] (also cf. [7, 17] in the case \( J_0 = J \)), its dimension is given, for \( p \) large enough, by the Riemann-Roch-Hirzebruch formula

\[
d_p := \dim \mathcal{H}_p = \langle \text{ch}(L^p) \, \text{Td}(TX), [X] \rangle.
\]

Here \( \text{ch}(L^p) \) is the Chern character of \( L^p \) and \( \text{Td}(TX) \) is the Todd class of the tangent bundle \( TX \) considered as a complex vector bundle with complex structure \( J \). In particular, we have \( d_p \sim p^n \int_X \frac{\omega^n}{n!} \) as \( p \to \infty \).

Let \( P_{\mathcal{H}_p} \) be the orthogonal projection from \( L^2(X, L^p) \) onto \( \mathcal{H}_p \).

**Definition 1.1.** A Toeplitz operator is a sequence \( \{T_p\} = \{T_p\}_{p \in \mathbb{N}} \) of bounded linear operators \( T_p : L^2(X, L^p) \to L^2(X, L^p) \), satisfying the following conditions:

(i): For any \( p \in \mathbb{N} \), we have

\[
T_p = P_{\mathcal{H}_p} T P_{\mathcal{H}_p}.
\]

(ii): There exists a sequence \( g_l \in C^\infty(X) \) such that

\[
T_p = P_{\mathcal{H}_p} \left( \sum_{l=0}^{\infty} p^{-l} g_l \right) P_{\mathcal{H}_p} + O(p^{-\infty}),
\]

i.e. for any natural \( k \) there exists \( C_k > 0 \) such that, for any \( p \in \mathbb{N} \),

\[
\left\| T_p - P_{\mathcal{H}_p} \left( \sum_{l=0}^{k} p^{-l} g_l \right) P_{\mathcal{H}_p} \right\| \leq C_k p^{-k-1}.
\]

Here we follow the approach to Toeplitz operator calculus introduced in [35, 37] for spin\(^c\) Dirac operator and Kähler case (also with an auxiliary bundle) and extended to the case under consideration in [26, 28].

The full symbol of \( \{T_p\} \) is the formal series \( \sum_{l=0}^{\infty} h^l g_l \in C^\infty(X)[[h]] \) and the principal symbol of \( \{T_p\} \) is \( g_0 \). In the particular case when \( g_l = 0 \) for \( l \geq 1 \) and \( g_0 = f \), we get the operator \( T_{f,p} = P_{\mathcal{H}_p} T_{f,p} : L^2(X, L^p) \to L^2(X, L^p) \).

If each \( T_p \) is self-adjoint, then \( \{T_p\} \) is called self-adjoint. For real-valued \( f \), the operator \( T_{f,p} \) is obviously self-adjoint. Conversely, if \( \{T_p\} \) is self-adjoint, then its principal symbol is real-valued.

As shown in [26, 28], the set of Toeplitz operators forms an algebra. This is based on the Bergman kernel expansion from [36] and Toeplitz operator calculus developed in [37] for spin\(^c\) Dirac operator and Kähler case (also with an auxiliary bundle).

**Theorem 1.2.** Let \( f, g \in C^\infty(X) \). Then the product of the Toeplitz operators \( T_{f,p} \) and \( T_{g,p} \) is a Toeplitz operator. More precisely, it admits the asymptotic expansion

\[
T_{f,p} T_{g,p} = \sum_{r=0}^{\infty} p^{-r} T_{C_r(f,g),p} + O(p^{-\infty}),
\]

with some $C_r(f, g) \in C^\infty(X)$, where $C_r$ are bidifferential operators. In particular,

$$C_0(f, g) = fg, \quad C_1(f, g) - C_1(g, f) = i\{f, g\},$$

where $\{f, g\}$ is the Poisson bracket on $(X, 2\pi\omega)$.

Thus, the Toeplitz operators provide a Berezin-Toeplitz quantization for the compact symplectic manifold $(X, 2\pi\omega)$. The limit $p \to +\infty$ for Toeplitz operators can be thought of as a semiclassical limit, with semiclassical parameter $\hbar = \frac{1}{p} \to 0$. Theorem 1.2 shows that this quantization has a correct semiclassical limit.

Throughout in the paper, we will consider a self-adjoint Toeplitz operator $T_p$ with principal symbol $h$:

$$T_p = P_{\mathcal{H}_p} \left( \sum_{l=0}^{\infty} p^{-l} g_l \right) P_{\mathcal{H}_p} + \mathcal{O}(p^{-\infty}), \quad g_0 = h.$$  

(1.9)

Without loss of generality, we will assume that the principal symbol $h$ satisfies the condition:

$$\min_{x \in X} h(x) = 0. \tag{1.10}$$

We will suppose that the operator $T_p$ acts on the finite-dimensional space $\mathcal{H}_p$. Thus, the spectrum of $T_p$ consists of a finite number of eigenvalues $\lambda_0^p \leq \lambda_1^p \leq \ldots \leq \lambda_{d_p}^p$. We will study the asymptotic properties of eigenvalues and eigensections of $T_p$ in the semiclassical limit $p \to \infty$.

1.3. **Asymptotic decay of eigensections.** First, we study localization properties of eigensections of Toeplitz operators. We start with a result on exponential decay of the eigensections of $T_p$ in the classically forbidden region. For any $h_0 \geq 0$, denote

$$U_{h_0} = \{ x \in X : h(x) \leq h_0 \}.$$

**Theorem 1.3.** Let $\{T_p\}$ be a self-adjoint Toeplitz operator (1.9), satisfying the condition (1.10). Assume that there exist $C > 0$ and $a > 0$ such that for any $p \in \mathbb{N}$ and $(x, y) \in X \times X$,

$$|(T_p - P_{\mathcal{H}_p} h P_{\mathcal{H}_p})(x, y)| < C p^{-1} e^{-a \sqrt{d}(x, y)} \tag{1.11}$$

where $(T_p - P_{\mathcal{H}_p} h P_{\mathcal{H}_p})(x, y)$ denotes the Schwartz kernel of the operator $T_p - P_{\mathcal{H}_p} h P_{\mathcal{H}_p}$ with respect to the Riemannian volume form $dv_X$ on $X$ and $d$ is the geodesic distance function of $(X, g)$.

Suppose that, for any $p \in \mathbb{N}$, $\lambda_p$ is an eigenvalue of $T_p$ such that $\lambda_p \leq h_1$ with some $h_1 > 0$ independent of $p$ and $u_p \in \mathcal{H}_p$ is the corresponding normalized eigensection:

$$T_p u_p = \lambda_p u_p, \quad \|u_p\| = 1.$$
Then, for any $h_0 > h_1$, there exist $\alpha > 0$ and $C_1 > 0$ such that, for any $p \in \mathbb{N},$

$$\int_X e^{2\alpha \sqrt{d}d(x,U_{h_0})} |u_p(x)|^2 dv_X(x) < C_1,$$

where $d(x,U_{h_0})$ denotes the distance from $x$ to $U_{h_0}$.

The proof uses an approach to the study of Bergman kernels, developed in [28, 29]. It combines the methods of [13, 35, 36, 37] and weighted estimates with appropriate exponential weights as in [27].

Next, we consider the eigensections, corresponding to low-lying eigenvalues of a Toeplitz operator, under the assumption that the minimum of its principal symbol is non-degenerate. One can use the methods of the proof of Theorem 1.3 in this setting and get some exponential decay estimates (see Theorem 2.8). This result turns out to be insufficient for our applications. So instead we extend a general localization result proved by Deleporte [14] in the case when $X$ is a Kähler manifold.

So let $\{T_p\}$ be a self-adjoint Toeplitz operator (1.9), satisfying the condition (1.10). We assume that $h$ is non-degenerate on the zero set $U_0 = \{x \in X : h(x) = 0\}$: there exists $\alpha > 0$ such that

$$h(x) \geq \alpha d(x,U_0)^2, \quad x \in X.$$

For any $\delta > 0$, put

$$V_\delta = \{x \in X : d(x,U_0) < \delta\}.$$

**Theorem 1.4.** For any $c > 0$, $k \in \mathbb{N}$ and $\delta > 0$, there exists $C > 0$ such that for any $p \in \mathbb{N}$ and $u_p \in \mathcal{H}_p$ such that

$$T_p u_p = \lambda_p u_p, \quad \|u_p\| = 1.$$

with $\lambda_p$, satisfying the estimate

$$\lambda_p < \frac{c}{p},$$

we have

$$\int_{X \setminus V_\delta} |u_p(x)|^2 dv_X(x) < C p^{-k}, \quad p \in \mathbb{N}.$$

### 1.4. Asymptotic expansions of low-lying eigenvalues.

Let $T_p$ be a self-adjoint Toeplitz operator (1.9) with the principal symbol $h$, satisfying (1.10). For each non-degenerate minimum $x_0$ of $h$:

$$h(x_0) = 0, \quad \text{Hess } h(x_0) > 0,$$

one can define the model operator for $\{T_p\}$ at $x_0$ in the following way.

The second order term in the Taylor expansion of $h$ at $x_0$ (in normal coordinates near $x_0$):

$$q_{x_0}(Z) = \left( \frac{1}{2} \text{Hess } h(x_0)Z, Z \right)$$

is a positive quadratic form on $T_{x_0}X \cong \mathbb{R}^{2n}$. 

Consider a second order differential operator $L_{x_0}$ in $C^\infty(T_{x_0}X)$ given by

\begin{equation}
L_{x_0} = -\sum_{j=1}^{2n} \left( \nabla e_j + \frac{1}{2} R^L_{x_0}(Z, e_j) \right)^2 - \tau(x_0),
\end{equation}

where $\{e_j\}_{j=1,\ldots,2n}$ is an orthonormal base in $T_{x_0}X$. Here, for $U \in T_{x_0}X$, we denote by $\nabla_U$ the ordinary operator of differentiation in the direction $U$ on $C^\infty(T_{x_0}X)$. Let $P_{x_0}$ be the orthogonal projection in $L^2(T_{x_0}X)$ to the kernel of $L_{x_0}$ (see Section 3.1 for more details).

The model operator for $\{T_p\}$ at $x_0$ is the Toeplitz operator $T_{x_0}$ in $L^2(T_{x_0}X)$ given by

\[ T_{x_0} = P_{x_0}(q_{x_0}(Z) + g_1(x_0))P_{x_0}, \]

where $g_1$ is the second coefficient in the asymptotic expansion (1.9) for the operator $\{T_p\}$. It is an unbounded self-adjoint operator in $L^2(T_{x_0}X)$ with discrete spectrum (see also below). The eigenvalues of $T_{x_0}$ do not depend on the choice of normal coordinates, and the lowest eigenvalue is simple.

Now suppose that each minimum of $h$ is non-degenerate. Then the zero set $U_0$ is a finite set of points:

\[ U_0 = \{x_1, \ldots, x_N\}. \]

Let $T$ be the self-adjoint operator on $L^2(T_{x_1}X) \oplus \ldots \oplus L^2(T_{x_N}X)$ defined by

\begin{equation}
T = T_{x_1} \oplus \ldots \oplus T_{x_N}.
\end{equation}

**Theorem 1.5.** Let $\{\lambda^m_p\}$ be the non-decreasing sequence of the eigenvalues of $T_p$ on $\mathcal{H}_p$ (counted with multiplicities) and let $\{\mu_m\}$ be the non-decreasing sequence of the eigenvalues of $T$ (counted with multiplicities). Then, for any fixed $m$, $\lambda^m_p$ has an asymptotic expansion, when $p \to \infty$, of the form

\begin{equation}
\lambda^m_p = p^{-1}\mu_m + p^{-3/2}\phi_m + O(p^{-2})
\end{equation}

with some $\phi_m \in \mathbb{R}$.

**Theorem 1.6.** If $\mu$ is a simple eigenvalue of $T_{x_j}$ for some $j = 1, \ldots, N$, then there exists a sequence $\lambda_p$ of eigenvalues of $T_p$ on $\mathcal{H}_p$ which admits a complete asymptotic expansion of the form

\[ \lambda_p \sim p^{-1} \sum_{k=0}^{+\infty} a_k p^{-k}, \quad a_0 = \mu. \]

In the case when $(X, \omega)$ is a Kähler manifold, that is, $J = J_0$ and $J$ is a complex structure, these results are slight refinements of the main results of [14].
1.5. Applications to the Bochner Laplacian. We assume that the function $\tau \in C^\infty(X)$ satisfies the following conditions:

- $\min_{x \in X} \tau(x) = \tau_0$;
- There exists a unique $x_0 \in X$ such that $\tau(x) = \tau_0$, which is non-degenerate:

  $$\text{Hess } \tau(x_0) > 0.$$  

Denote by $q_{x_0}(Z) = \left( \frac{1}{2} \text{Hess } \tau(x_0) Z, Z \right)$, $Z \in T_{x_0}X$.

We will also need a smooth function $J_{1,2}$ on $X$, which appears in the leading coefficient in the asymptotic expansion of the restriction to the diagonal of the generalized Bergman kernel $P_{1,p}$ of $\Delta_p$, which is the smooth kernel of the operator $\Delta_p P_{H_p}$ with respect to $dv_X$. We refer the reader to Section 4.1 for more information on this function.

Consider the Toeplitz operator $T_{x_0}$ in $L^2(T_{x_0}X)$ defined by

$$T_{x_0} = P_{x_0}(q_{x_0}(Z) + J_{1,2}(x_0))P_{x_0}.$$  

Denote by $\{\mu_j\}$ the non-decreasing sequence of the eigenvalues of $T_{x_0}$ (counted with multiplicities).

Let $\{\lambda_j(\Delta^{L_p})\}$ be the non-decreasing sequence of the eigenvalues of the operator $\Delta^{L_p}$ (counted with multiplicities). As a consequence of Theorem 1.5 and variational technique, we obtain upper estimates for $\lambda_j(\Delta^{L_p})$.

Theorem 1.7. For any $j \in \mathbb{N}$, there exists $\phi_j \in \mathbb{R}$ such that

$$\lambda_j(\Delta^{L_p}) \leq p\tau_0 + \mu_j + p^{-1/2}\phi_j + O(p^{-1}), \quad p \to \infty.$$  

In the case when the line bundle $L$ is trivial, the corresponding Bochner Laplacian $\Delta^{L_p}$ is closely related with the magnetic Schrödinger operator $H^h$ with the semiclassical parameter $h = p^{-1}$ (see Section 4 for more details). The case when the form $\omega$ is symplectic corresponds to the case when the magnetic field is non-vanishing (of full rank). In the two-dimensional case, spectral properties of the magnetic Schrödinger operator with non-vanishing magnetic field were studied in [18, 22, 19, 21, 38] (see also the survey paper [20] and the book [39]). As shown in Section 4.4, for a two-dimensional magnetic Laplacian, the upper bounds of Theorem 1.7 are sharp and agree with the asymptotic expansions of [18, 22].

The paper is organized as follows. In Section 2, we study localization properties of eigensections of Toeplitz operators and prove Theorem 1.3 and 1.4. Section 3 is devoted to asymptotic expansions of low-lying eigenvalues. Here we prove Theorems 1.5 and 1.6. Finally, Section 4 contains applications to the Bochner Laplacian.
2. Asymptotic decay of eigensections

2.1. Differential operators and Sobolev spaces. We will need to consider families of differential operators $A_p$ acting in $C^\infty(X, L^p)$ with $p \in \mathbb{N}$. In this section, we collect some necessary results (see, for instance, [29]).

As usual, we introduce the $L^2$-norm on $L^2(X, L^p)$ by

$$
\|u\|_{p,0}^2 = \int_X |u(x)|^2 dv_X(x), \quad u \in L^2(X, L^p).
$$

For any integer $m > 0$, we introduce the norm $\| \cdot \|_{p,m}$ on the Sobolev space $H^m(X, L^p)$ of order $m$ by the formula

$$
\|u\|_{p,m}^2 = \sum_{\ell=0}^m \int_X \left( \frac{1}{\sqrt{p}} \nabla^L p \right)^\ell u(x)^2 dv_X(x), \quad u \in H^m(X, L^p).
$$

Denote by $\langle \cdot, \cdot \rangle_{p,m}$ the corresponding inner product on $H^m(X, L^p)$. For any integer $m < 0$, we define the norm in the Sobolev space $H^m(X, L^p)$ by duality. For any bounded linear operator $A : H^m(X, L^p) \to H^{m'}(X, L^p)$, $m, m' \in \mathbb{Z}$, we will denote its norm by $\|A\|_{p,m,m'}$.

Let $E$ be a vector bundle over $X$. Suppose that $E$ is Euclidean or Hermitian depending on whether it is real or complex and equipped with a metric connection $\nabla^E$. The Levi-Civita connection $\nabla^T X$ on $(X, g^T X)$ and the connection $\nabla^E$ define a metric connection $\nabla^E : C^\infty(X, (T^* X)^{\otimes j} \otimes E) \to C^\infty(X, (T^* X)^{\otimes (j+1)} \otimes E)$ on each vector bundle $(T^* X)^{\otimes j} \otimes E$ for $j \in \mathbb{N}$, that allows us to introduce the operator

$$
(\nabla^E)^\ell : C^\infty(X, E) \to C^\infty(X, (T^* X)^{\otimes \ell} \otimes E)
$$

for every $\ell \in \mathbb{N}$. Any differential operator $A$ of order $q$ acting in $C^\infty(X, E)$ can be written as

$$
(2.1) \quad A = \sum_{\ell=0}^q a_\ell \cdot (\nabla^E)^\ell,
$$

where $a_\ell \in C^\infty(X, (T^* X)^{\otimes \ell})$ and the endomorphism $\cdot : (T^* X)^{\otimes \ell} \otimes ((T^* X)^{\otimes \ell} \otimes E) \to E$ is given by the contraction.

Denote by $D^q(X, L^p)$ the space of differential operators of order $q$, acting on $C^\infty(X, L^p)$. We say that a family $\{A_p \in D^q(X, L^p), p \in \mathbb{N}\}$ is bounded in $p$, if

$$
A_p = \sum_{\ell=0}^q a_{p,\ell} \cdot \left( \frac{1}{\sqrt{p}} \nabla^L p \right)^\ell, \quad a_{p,\ell} \in C^\infty(X, (T^* X)^{\otimes \ell}),
$$

and, for any $\ell = 0, 1, \ldots, q$, the family $\{a_{p,\ell}, p \in \mathbb{N}\}$ is bounded in the Frechet space $C^\infty(X, (T^* X)^{\otimes \ell})$. An example of a bounded in $p$ family of differential operators is given by $\{\frac{1}{p} \Delta_p, p \in \mathbb{N}\}$.

Recall that any operator $A \in D^q(X, L^p)$ defines a bounded operator from $H^{m+q}(X, L^p)$ to $H^m(X, L^p)$ for any $m \in \mathbb{Z}$. The following proposition
provides a control on the operator norms of a family of differential operators as above.

**Proposition 2.1.** If a family \( \{ A_p \in D^q(X, L^p) \}, p \in \mathbb{N} \) is bounded in \( p \), then for any \( m \in \mathbb{Z} \), there exists \( C_m > 0 \) such that, for all \( p \in \mathbb{N} \),

\[
\| A_p u \|_{p,m} \leq C_m \| u \|_{p,m+q}, \quad u \in H^{m+q}(X, L^p).
\]

2.2. Weighted estimates for eigensections. Here we prove weighted estimates for eigensections of Toeplitz operators, satisfying (1.10) and (1.11).

Let \( \rho \) be a Lipschitz function on \( X \), that is, there exists \( L > 0 \) such that

\[
|\rho(x) - \rho(y)| \leq L d(x, y), \quad x, y \in X.
\]

**Proposition 2.2.** Assume that \( \{ T_p \} \) is a self-adjoint Toeplitz operator (1.9), satisfying the conditions (1.10) and (1.11). Let \( u_p \) be a sequence of eigensections of \( \{ T_p \} \):

\[
T_p u_p = \lambda_p u_p, \quad u_p \in \mathcal{H}_p.
\]

There exist \( C_1, C_2, C_3 > 0 \) and \( \alpha_0 > 0 \) such that for any \( \alpha \in \mathbb{R} \) such that \( |\alpha| < \alpha_0 \) and \( p \in \mathbb{N} \),

\[
\int_X e^{2\alpha \sqrt{p}\rho(x)} \left[ h(x) - (1 + C_1|\alpha|) \lambda_p - \frac{C_2|\alpha|}{\sqrt{p}} - \frac{C_3}{p} \right] |u_p(x)|^2 \, dv_X(x) \leq 0.
\]

The rest of this subsection is devoted to the proof of Proposition 2.2. By a standard averaging procedure [27, Proposition 4.1] (see also [29, Section 3.1]), one can construct a sequence \( \tilde{\rho}_p \in C^\infty(X), p \in \mathbb{N} \), satisfying the following conditions:

(1) there exists \( c_0 > 0 \) such that

\[
|\tilde{\rho}_p(x) - \rho(x)| < \frac{c_0}{\sqrt{p}}, \quad x \in X, p \in \mathbb{N};
\]

(2) for any \( k > 0 \), there exists \( c_k > 0 \) such that

\[
\| \tilde{\rho}_p \|_{C^k(X)} < c_k \left( \frac{1}{\sqrt{p}} \right)^{1-k}, \quad p \in \mathbb{N}.
\]

For any \( \alpha \in \mathbb{R} \) and \( p \in \mathbb{N} \), consider the function \( f_{\alpha,p} \in C^\infty(X) \) given by

\[
f_{\alpha,p}(x) = e^{\alpha \sqrt{p}\tilde{\rho}_p(x)}, \quad x \in X.
\]

These functions play role of weighted functions in our estimates. We will use the same notation \( f_{\alpha,p} \) for the multiplication operator by \( f_{\alpha,p} \) in \( C^\infty(X, L^p) \). Instead of working directly with the associated weighted spaces, we will consider operators of the form \( \{ f_{\alpha,p} A f_{-\alpha,p} \} \) and only at the very end switch to weighted estimates.

Observe that, for \( v \in C^\infty(X, TX) \),

\[
\nabla_{L^p}^{\alpha,v} := f_{\alpha,p} \nabla_v^{L^p} f_{-\alpha,p}^{-1} = \nabla_v^{L^p} - \alpha \sqrt{p} d\tilde{\rho}_p(v).
\]
Therefore, for any $a > 0$ and $v \in C^\infty(X, TX)$, the family $\{1 \big/ \sqrt{p} \nabla^{L^p}_{\alpha,v} : |\alpha| < a\}$ is a family of operators from $D^1(X, L^p)$, uniformly bounded in $p$. The operator $\Delta_{p,\alpha} := f_{\alpha,p} \Delta_p f_{\alpha,p}^{-1}$ has the form

$$\Delta_{p,\alpha} = \Delta_p + \alpha A_p + \alpha^2 B_p,$$

where $A_p \in D^1(X, L^p)$ and $B_p \in D^0(X, L^p)$. Moreover, the families $\{1 \big/ p A_p : p \in \mathbb{N}\}$ and $\{1 \big/ p B_p : p \in \mathbb{N}\}$ are uniformly bounded in $p$. Indeed, using (1.4), one can check that, if $\{e_j\}_{j=1}^{2n}$ is a local orthonormal frame of $TX$, then

$$A_p = \sqrt{p} \sum_{j=1}^{2n} \bigg[ \nabla^{L^p}_{e_j} \circ d\tilde{\rho}_p(e_j) + d\tilde{\rho}_p(e_j) \circ \nabla^{L^p}_{e_j} + d\tilde{\rho}_p(\nabla^{TX}_{e_j} e_j) \bigg],$$

$$B_p = -p \sum_{j=1}^{2n} (d\tilde{\rho}_p(e_j))^2 = -p \|d\tilde{\rho}_p\|^2.$$

**Lemma 2.3.** There exist $\alpha_1 > 0$ and $C > 0$ such that, for any $p \in \mathbb{N}$ and $|\alpha| < \alpha_1$,

$$\|f_{-\alpha,p}[P_{H_p}, f_{2\alpha,p}] f_{-\alpha,p}\| < C|\alpha|.$$

**Proof.** Observe that

$$f_{-\alpha,p}[P_{H_p}, f_{2\alpha,p}] f_{-\alpha,p} = f_{\alpha,p}^{-1} P_{H_p} f_{\alpha,p} - f_{\alpha,p} P_{H_p} f_{\alpha,p}^{-1}.$$

Let $\delta$ be the counterclockwise oriented circle in $\mathbb{C}$ centered at 0 of radius $\mu_0$. We will use the formula

$$P_{H_p} = \frac{1}{2\pi i} \int_\delta \left( \lambda - \frac{1}{p} \Delta_p \right)^{-1} d\lambda.$$

By [29, Theorem 2.3] (see also [37, Theorem 1.7]), we have the following theorem.

**Theorem 2.4.** For any $\lambda \in \delta$ and $p \in \mathbb{N}$, the operator $\lambda - \frac{1}{p} \Delta_p$ is invertible in $L^2(X, L^p)$, and there exists $C > 0$ such that for all $\lambda \in \delta$ and $p \in \mathbb{N}$,

$$\left\| \left( \lambda - \frac{1}{p} \Delta_p \right)^{-1} \right\|_{0,0}^0 \leq C,$$

$$\left\| \left( \lambda - \frac{1}{p} \Delta_p \right)^{-1} \right\|_{1,1}^p \leq C.$$

As in [29, Theorem 3.2] (see also [28]), we can derive the corresponding weighted estimates.

**Theorem 2.5.** There exist $\alpha_1 > 0$ and $C > 0$ such that, for all $\lambda \in \delta$, $p \in \mathbb{N}$, $|\alpha| < \alpha_1$, the operator $\lambda - \frac{1}{p} \Delta_{p,\alpha}$ is invertible in $L^2(X, L^p)$, and, for the inverse operator $(\lambda - \frac{1}{p} \Delta_{p,\alpha})^{-1}$, we have

$$\left\| \left( \lambda - \frac{1}{p} \Delta_{p,\alpha} \right)^{-1} \right\|_{0,0}^0 \leq C,$$

$$\left\| \left( \lambda - \frac{1}{p} \Delta_{p,\alpha} \right)^{-1} \right\|_{1,1}^p \leq C.$$
It is easy to see that the inverse operators \((\lambda - \frac{1}{p} \Delta_{p,\alpha})^{-1}\) and \((\lambda - \frac{1}{p} \Delta_{\alpha})^{-1}\) are related by the formula
\[
(2.12) \quad \left(\lambda - \frac{1}{p} \Delta_{p,\alpha}\right)^{-1} = f_{\alpha,p} \left(\lambda - \frac{1}{p} \Delta_{\alpha}\right)^{-1} f_{\alpha,p}^{-1}.
\]
By (2.10), it follows that
\[
(2.13) \quad f_{\alpha,p} P_{H_p} f_{\alpha,p}^{-1} = \frac{1}{2\pi i} \int_\delta f_{\alpha,p} \left(\lambda - \frac{1}{p} \Delta_{p}\right)^{-1} f_{\alpha,p}^{-1} d\lambda.
\]
Using the resolvent identity and (2.6), we get:
\[
(2.14) \quad f_{\alpha,p} \left(\lambda - \frac{1}{p} \Delta_{p}\right)^{-1} f_{\alpha,p}^{-1} - f_{\alpha,p}^{-1} f_{\alpha,p} \left(\lambda - \frac{1}{p} \Delta_{p}\right)^{-1} f_{\alpha,p}
= \left(\lambda - \frac{1}{p} \Delta_{p,\alpha}\right)^{-1} \left(\frac{1}{p} \Delta_{p,\alpha} - \frac{1}{p} \Delta_{p,-\alpha}\right) \left(\lambda - \frac{1}{p} \Delta_{p,-\alpha}\right)^{-1}
= \frac{2\alpha}{p} \left(\lambda - \frac{1}{p} \Delta_{p,\alpha}\right)^{-1} A_p \left(\lambda - \frac{1}{p} \Delta_{p,-\alpha}\right)^{-1}.
\]
Using (2.11) and (2.14), we proceed as follows:
\[
\left\| f_{\alpha,p} \left(\lambda - \frac{1}{p} \Delta_{p}\right)^{-1} f_{\alpha,p}^{-1} - f_{\alpha,p}^{-1} f_{\alpha,p} \left(\lambda - \frac{1}{p} \Delta_{p}\right)^{-1} f_{\alpha,p} \right\|_p^{0,0}
\leq \frac{2|\alpha|}{p} \left\| \left(\lambda - \frac{1}{p} \Delta_{p,\alpha}\right)^{-1} \right\|_p^{0,0} \left\| A_p \left(\lambda - \frac{1}{p} \Delta_{p,-\alpha}\right)^{-1} \right\|_p^{0,0}
\leq C|\alpha| \left\| \frac{1}{p} A_p \right\|_p^{1,0} \leq C_1 |\alpha|.
\]
Taking into account (2.9) and (2.13), this completes the proof. \(\square\)

**Corollary 2.6.** There exists \(C > 0\) such that, for any \(p \in \mathbb{N}\) and \(|\alpha| < \alpha_1\),
\[
\left\| f_{\alpha,p} P_{H_p} f_{\alpha,p}^{-1} \right\| < C.
\]

**Lemma 2.7.** There exists \(C > 0\) such that, for any \(p \in \mathbb{N}\) and \(\alpha\) such that \(|\alpha| < \alpha_1\) with \(\alpha_1\) as in Lemma 2.3, we have
\[
\left\| f_{\alpha,p}[h, P_{H_p}] f_{\alpha,p}^{-1} \right\| \leq \frac{C}{\sqrt{p}}.
\]

**Proof.** By (2.12) and (2.13), we can write
\[
(2.15) \quad f_{\alpha,p}[h, P_{H_p}] f_{\alpha,p}^{-1} = [h, f_{\alpha,p} P_{H_p} f_{\alpha,p}^{-1}]
= \frac{1}{2\pi i} \int_\delta \left[ h \left(\lambda - \frac{1}{p} \Delta_{p,\alpha}\right)^{-1} \right] d\lambda.
\]
We clearly have
\[ (2.16) \quad \left[ h, \left( \lambda - \frac{1}{p} \Delta_{\rho,\alpha} \right)^{-1} \right] \]
\[ = \left( \lambda - \frac{1}{p} \Delta_{\rho,\alpha} \right)^{-1} \left[ h, \frac{1}{p} \Delta_{\rho,\alpha} \right] \left( \lambda - \frac{1}{p} \Delta_{\rho,\alpha} \right)^{-1}. \]

We use the formula (2.6) to compute the commutator \( [h, 1_p \Delta_{\rho,\alpha}] \):
\[ [h, 1_p \Delta_{\rho,\alpha}] = [h, 1_p \Delta_{\rho}] + \alpha [h, 1_p A_p]. \]

Using the formulas (1.4) and (2.7), one can compute that, if \( \{e_j\}_{j=1}^{2n} \) is a local orthonormal frame of \( TX \), then
\[ (2.17) \quad [h, 1_p \Delta_{\rho}] = \frac{2}{\sqrt{p}} \sum_{j=1}^{2n} dh(e_j) \circ \nabla_{e_j} L \circ \nabla_{e_j} L \circ dh(e_j) - dh(\nabla_{e_j} T X e_j). \]

By (2.17) and (2.18), we see that, for any \( \alpha \), the family \( \sqrt{p} [h, 1_p \Delta_{\rho,\alpha}] \) is a bounded family in \( D^1(X, L^p) \). By Proposition 2.1, it follows that
\[ \left\| [h, 1_p \Delta_{\rho,\alpha}] \right\|_{p}^{1,0} \leq C_1 \frac{1}{\sqrt{p}}. \]

By this estimate, (2.15), (2.16) and Theorem 2.5, we immediately complete the proof. \( \square \)

Now we are ready to prove Proposition 2.2.

Proof of Proposition 2.2. Using (2.3), we get
\[ (f_{\alpha,p} h u_p, f_{\alpha,p} u_p) \]
\[ = (P_{H_p} f_{2\alpha,p} h u_p, u_p) \]
\[ = (f_{\alpha,p} P_{H_p} h u_p, f_{\alpha,p} u_p) + ([P_{H_p}, f_{2\alpha,p}] h u_p, u_p) \]
\[ = \lambda_p \| f_{\alpha,p} u_p \|^2 - (f_{\alpha,p} (T_p - P_{H_p} h P_{H_p}) u_p, f_{\alpha,p} u_p) \]
\[ + ([P_{H_p}, f_{2\alpha,p}] h u_p, u_p). \]

The kernel \( K_{\alpha,p} \in C^\infty(X \times X) \) of the operator \( f_{\alpha,p} (T_p - P_{H_p} h P_{H_p}) f_{-\alpha,p} \) is given by
\[ K_{\alpha,p}(x, y) = e^{\alpha \sqrt{p}\rho_p(x)} (T_p - P_{H_p} h P_{H_p})(x, y)e^{-\alpha \sqrt{p}\rho_p(y)}. \]

Using (2.4), (2.2) and (1.11), one can show that there exists \( C > 0 \) such that, for any \( p \in \mathbb{N} \) and \( (x, y) \in X \times X \),
\[ |K_{\alpha,p}(x, y)| \leq C p^{-1} e^{(L|\alpha|-a) \sqrt{\mathcal{P}}(x, y)}. \]
Therefore, if we take $|\alpha| < a/L$, then, for any $p \in \mathbb{N}$ and $(x, y) \in X \times X$, 
\[ |K_{\alpha,p}(x, y)| \leq C p^{-1}, \]
and, by Schur’s lemma, there exists $C_3 > 0$ such that, for any $p \in \mathbb{N}$,
\[ \|f_{\alpha,p}(T_p - P_{H_p}hP_{H_p})u_p, f_{\alpha,p}u_p\| \leq C_3 p^{-1}, \quad p \in \mathbb{N}. \]
By (2.20), we get the bound for the second term in the right-hand side of (2.22):
\[ |(f_{\alpha,p}(T_p - P_{H_p}hP_{H_p})u_p, f_{\alpha,p}u_p)| \leq \frac{C_3}{p} \|f_{\alpha,p}u_p\|^2, \quad p \in \mathbb{N}. \]

Put $\alpha_0 = \min(a/L, \alpha_1)$, where $\alpha_1$ is given by Lemma [2.3] and assume that $|\alpha| < \alpha_0$. For the third term in the right-hand side of (2.19), we proceed as follows:
\[
([P_{H_p}, f_{2\alpha,p}]hu_p, u_p) = ([P_{H_p}, f_{2\alpha,p}]P_{H_p}hu_p, u_p) + ([P_{H_p}, f_{2\alpha,p}](I - P_{H_p})hu_p, u_p) \\
= \lambda_p([P_{H_p}, f_{2\alpha,p}]u_p) + ([P_{H_p}, f_{2\alpha,p}](T_p - P_{H_p}hP_{H_p})u_p, u_p) \\
\quad + ([P_{H_p}, f_{2\alpha,p}][h, P_{H_p}]u_p, u_p).
\]

Using Lemma [2.3], Lemma [2.7] and [2.20], for any $p \in \mathbb{N}$, we get the estimate
\[ |([P_{H_p}, f_{2\alpha,p}]hu_p, u_p)| \leq C |\alpha| |\lambda_p||f_{\alpha,p}u_p||^2 + \frac{C_2 |\alpha|}{\sqrt{p}} \|f_{\alpha,p}u_p\|^2, \]
that immediately completes the proof.

\[ \Box \]

2.3. Tunneling estimates. Here we prove Theorem [1.3] the result on exponential decay of the eigensections in the classically forbidden region.

Let $\rho(x) = d(x, U_{h_0})$. This is a Lipschitz function, so we can construct a sequence $\rho_p \in C^{\infty}(X), p \in \mathbb{N}$, satisfying (2.4) and (2.5). By Proposition [2.2] it follows that, for any $M \in (h_1, h_0)$, there exists $\alpha > 0$ such that for any $p \in \mathbb{N}$,
\[ \int_X e^{2\alpha \sqrt{\rho_p}(x)} (h(x) - M) |u_p(x)|^2 dv_X(x) \leq 0. \]
Now the proof is completed as in [24]. For convenience of the reader, we recall the arguments.

We rewrite (2.21) in the form
\[ \int_{X \setminus U_{h_0}} e^{2\alpha \sqrt{\rho_p}(x)} (h(x) - M) |u_p(x)|^2 dv_X(x) \leq \int_{U_{h_0}} e^{2\alpha \sqrt{\rho_p}(x)} (M - h(x)) |u_p(x)|^2 dv_X(x). \]

For the left-hand side of (2.22), since $h(x) \geq h_0$ on $X \setminus U_{h_0}$, we get
\[ \int_{X \setminus U_{h_0}} e^{2\alpha \sqrt{\rho_p}(x)} (h(x) - M) |u_p(x)|^2 dv_X(x) \]
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≥ (h_0 - M) \int_{X \setminus U_{h_0}} e^{2\alpha \sqrt{\tilde{\rho}_p(x)}} |u_p(x)|^2 dv_X(x).

If \( x \in U_{h_0} \), then \( \rho(x) = 0 \) and, by (2.4), \( \tilde{\rho}_p(x) < c_0 / \sqrt{p} \). Hence, we have

(2.24) \( \int_{U_{h_0}} e^{2\alpha \sqrt{\tilde{\rho}_p(x)}} |u_p(x)|^2 dv_X(x) \)

≤ \( e^{2\alpha} \int_{U_{h_0}} |u_p(x)|^2 dv_X(x) \) ≤ \( e^{2\alpha} \int_{X} |u_p(x)|^2 dv_X(x) \).

Using (2.24) and the fact that \( h \) is bounded, for the right-hand side of (2.22), we get the estimate

(2.25) \( \int_{U_{h_0}} e^{2\alpha \sqrt{\tilde{\rho}_p(x)}} (M - h(x)) |u_p(x)|^2 dv_X(x) \)

≤ \( C \int_{U_{h_0}} e^{2\alpha \sqrt{\tilde{\rho}_p(x)}} |u_p(x)|^2 dv_X(x) \) ≤ \( C_1 \int_{X} |u_p(x)|^2 dv_X(x) \).

By (2.23), (2.22) and (2.25), it follows that

(2.26) \( \int_{X \setminus U_{h_0}} e^{2\alpha \sqrt{\tilde{\rho}_p(x)}} |u_p(x)|^2 dv_X(x) \) ≤ \( \int_{X} |u_p(x)|^2 dv_X(x) \).

Finally, by (2.24) and (2.26), we infer that, for \( p > p_0 \),

\( \int_{X} e^{2\alpha \sqrt{\tilde{\rho}_p(x)}} |u_p(x)|^2 dv_X(x) \) ≤ \( \int_{X} |u_p(x)|^2 dv_X(x) \),

that completes the proof of Theorem 1.3.

2.4. Low-lying eigenvalues. In this section, we study localization properties of eigensections, corresponding to low-lying eigenvalues, under the assumption that the minimum of the principal symbol is non-degenerate.

First, we prove Theorem 1.4.

Proof of Theorem 1.4. The proof is obtained by a slight modification of the proof of [13, Proposition 3.1]. So we will be sketchy.

First, we show that, for any \( k \in \mathbb{N} \), the operator \( T^k_{f_1, p} \) has the form

(2.27) \( T^k_{f_1, p} = P_{H_P} \left( h^k + \sum_{j=1}^{k-1} p^{-j} f_{j,k} + O(p^{-k}) \right) P_{H_P} \),

where, for any \( k \) and \( j = 1, \ldots, k-1 \), \( f_{j,k} \in C^\infty(X) \) and there exists \( C_{jk} > 0 \) such that

(2.28) \( |f_{j,k}(x)| < C_{jk} h(x)^{k-j}, \quad x \in X \).

Iterating (1.8), one can see that, for any \( f_1, \ldots, f_k \in C^\infty(X) \), the product of the Toeplitz operators \( T_{f_1, p} \ldots T_{f_k, p} \) is a Toeplitz operator, which admits
the asymptotic expansion

\[(2.29) \ T_{f_1,p} \ldots T_{f_k,p} = P_{\mathcal{H}_p} \left( \sum_{r=0}^{k-1} p^{-r} C_{r,k}(f_1, \ldots, f_k) + O(p^{-k}) \right) P_{\mathcal{H}_p}, \]

with some \( C_{r,k}(f_1, \ldots, f_k) \in C^\infty(X) \), where each \( C_{r,k} \), \( r = 0, \ldots, k-1 \), is a \( k \)-multilinear differential operator of order at most \( 2r \) and \( C_{0,k}(f_1, \ldots, f_k) = f_1 \ldots f_k \). Using (1.12), one can show (cf. [14]) that, if at least \( l \) of \( f_1, \ldots, f_k \) are equal to \( h \), then

\[(2.30) \quad |C_{r,k}(f_1, \ldots, f_k)| \leq Ch^{l-r}. \]

On the other hand, using the asymptotic expansion (1.9) for \( T_p \), one can see that each coefficient \( f_{j,k} \) in (2.27) is a linear combination of terms of the form \( C_{r,k}(f_1, \ldots, f_k) \), \( r \leq j \), where each \( f_m \), \( m = 1, \ldots, k \), equals \( g_r \) with some \( r = r_m \in \mathbb{N} \) and at least \( k - j + r \) of \( f_1, \ldots, f_k \) are equal to \( g_0 = h \). By (2.30), this implies (2.28).

Next, we prove by induction on \( k \) that, for any \( c > 0 \) and \( k \in \mathbb{N} \), there exists \( C > 0 \) such that for any \( p \in \mathbb{N} \) and \( u_p \in \mathcal{H}_p \), satisfying (1.13) and (1.14),

\[(2.31) \quad |(u_p, h^k u_p)| \leq C p^{-k}. \]

For \( k = 1 \), we have

\[(u_p, h u_p) = (u_p, T_p u_p) - (u_p, (T_p - P_{\mathcal{H}_p} h P_{\mathcal{H}_p}) u_p) = \lambda_p + (u_p, O(p^{-1}) u_p) = O(p^{-1}). \]

The induction step goes exactly as in the proof of [14, Proposition 3.1], using (2.27) and (2.28), that proves (2.31).

Finally, if \( x \notin V_\delta \), then \( d(x, U_0) > \delta \) and, by (1.12), \( h(x) \geq \alpha \delta^2 \). Therefore, we have

\[(2.32) \quad (u_p, h^k u_p) = \int_X h(x)^k |u_p(x)|^2 dv_X(x) \geq \int_{X \setminus V_\delta} h(x)^k |u_p(x)|^2 dv_X(x) \geq \alpha^k \delta^{2k} \int_{X \setminus V_\delta} |u_p(x)|^2 dv_X(x). \]

From (2.31) and (2.32), we get (1.15).

In the end of this section, we establish some exponential decay estimates for eigensections, corresponding to low-lying eigenvalues, using the methods of Section 2.3. As mentioned above, this result turns out to be insufficient for our applications.

So let \( \{T_p\} \) be a self-adjoint Toeplitz operator (1.9), satisfying the condition (1.10). Here we assume more generally that, for some \( k \geq 1 \), there exists \( C > 0 \) such that

\[(2.33) \quad h(x) \geq Cd(x, U_0)^{2k}, \quad x \in X. \]
Theorem 2.8. Suppose that \( \lambda_p \) is a sequence of eigenvalues of \( T_p \), satisfying the estimate
\[
\lambda_p < \frac{C_0}{p^{2k/(2k+1)}}, \quad p \in \mathbb{N},
\]
with some \( C_0 > 0 \), independent of \( p \), and \( u_p \in \mathcal{H}_p \) is the corresponding normalized eigensection:
\[
T_p u_p = \lambda_p u_p, \quad \|u_p\| = 1.
\]
For any \( c < \alpha_0 \), there exists \( C_1 > 0 \) such that, for any \( p \in \mathbb{N} \),
\[
\int_X e^{2cp^{1/(2k+1)}d(x, U_0)}|u_p(x)|^2 dv_X(x) < C_1.
\]

Proof. Let \( \rho(x) = d(x, U_0) \). Take a sequence \( \tilde{\rho}_p \in C^\infty(X), p \in \mathbb{N} \), satisfying \( \tilde{\rho}_p \leq \rho(x) \) and \( \tilde{\rho}_p \rightarrow 0 \) as \( p \rightarrow \infty \). We apply Proposition 2.2 with \( \alpha = cp^{-(2k-1)/(4k+2)} \) for the given \( c > 0 \). We conclude that there exists \( M > 0 \) such that, for any \( p \in \mathbb{N} \),
\[
\int_X e^{2cp^{1/(2k+1)}\tilde{\rho}_p(x)} \left( h(x) - \frac{M}{p^{2k/(2k+1)}} \right) |u_p(x)|^2 dv_X(x) \leq 0.
\]

Now the proof is completed similar to the proof of Theorem 1.5 and we omit it. \( \square \)

3. Asymptotic expansions of low-lying eigenvalues

This section is devoted to the proofs of Theorems 1.5 and 1.6.

3.1. Characterization of Toeplitz operators. In this section, we recall the description of Toeplitz operators in terms of their Schwartz kernels introduced in [35, 37].

First, we introduce normal coordinates near an arbitrary point \( x_0 \in X \), which we fix. Let \( a^X \) be the injectivity radius of \((X, g)\). We will identify \( B^{T_{x_0}X}(0, a^X) \) with \( B^X(x_0, a^X) \) by the exponential map \( \exp^X : T_{x_0}X \to X \). For \( Z \in B^{T_{x_0}X}(0, a^X) \) we identify \( L_Z \) to \( L_{x_0} \) by parallel transport with respect to the connection \( \nabla^L \) along the curve \( \gamma_Z : [0, 1] \ni u \to \exp^X_{x_0}(u Z) \). Consider the line bundle \( L_0 \) with fibers \( L_{x_0} \) on \( T_{x_0}X \). Denote by \( \nabla^L, h^L \) the connection and the metric on the restriction of \( L_0 \) to \( B^{T_{x_0}X}(0, a^X) \) induced by the identification \( B^{T_{x_0}X}(0, a^X) \cong B^X(x_0, a^X) \) and the trivialization of \( L \) over \( B^{T_{x_0}X}(0, a^X) \).

Let \( dv_{TX} \) be the Riemannian volume form of \((T_{x_0}X, g^{T_{x_0}X})\) and \( dv_X \) the volume form on \( B^{T_{x_0}X}(0, a^X) \), corresponding to the Riemannian volume form \( dv_X \) on \( B^X(x_0, a^X) \) under identification \( B^{T_{x_0}X}(0, a^X) \cong B^X(x_0, a^X) \). Let \( \kappa_{x_0} \) be the smooth positive function on \( B^{T_{x_0}X}(0, a^X) \) defined by the equation
\[
dv_X(Z) = \kappa_{x_0}(Z) dv_{TX}(Z), \quad Z \in B^{T_{x_0}X}(0, a^X).
\]

The almost complex structure \( J_{x_0} \) defined in (1.2) induces a splitting \( T_{x_0}X \otimes \mathbb{R}^2 = T^{(1,0)}_{x_0}X \oplus T^{(0,1)}_{x_0}X \), where \( T^{(1,0)}_{x_0}X \) and \( T^{(0,1)}_{x_0}X \) are the eigenspaces
of $J_{x_0}$ corresponding to eigenvalues $i$ and $-i$ respectively. Denote by $\det_C$ the determinant function of the complex space $T^{(1,0)}_{x_0}X$. Put 

$$J_{x_0} = -2\pi i J_0.$$ 

Then $J_{x_0} : T^{(1,0)}_{x_0}X \to T^{(1,0)}_{x_0}X$ is positive, and $J_{x_0} : T_{x_0}X \to T_{x_0}X$ is skew-adjoint. Put (see $[35, 37]$)

$$z \in l \text{[35, 37]}$$

Then define the coordinates $Z$ on $x, x$ with smooth kernel $\Xi$ with some $L$ or $J$ of $Y$. A. KORDYUKOV

Definition 3.1

$$J_{x_0} w_j = a_j w_j, \quad j = 1, \ldots, n,$$

with some $a_j > 0$. Then $e_{2j-1} = \frac{1}{\sqrt{2}}(w_j + \bar{w}_j)$ and $e_{2j} = \frac{1}{\sqrt{2}}(w_j - \bar{w}_j)$, $j = 1, \ldots, n$, form an orthonormal basis of $T_{x_0}X$. We use this basis to define the coordinates $Z$ on $T_{x_0}X \cong \mathbb{R}^{2n}$ as well as the complex coordinates $z$ on $\mathbb{C}^n \cong \mathbb{R}^{2n}$, $z_j = Z_{2j-1} + i Z_{2j}, j = 1, \ldots, n$. In this coordinates, we get

$$\mathcal{P}(Z, Z') = \frac{1}{(2\pi)^n} \prod_{j=1}^n a_j \exp \left( -\frac{1}{4} \sum_{k=1}^n a_k (|z_k|^2 + |z'_k|^2 - 2z_k \bar{z}_k) \right).$$

Let $\{\Xi_p\}_{p \in \mathbb{N}}$ be a sequence of linear operators $\Xi_p : L^2(X, L^p) \to L^2(X, L^p)$ with smooth kernel $\Xi_p(x, x')$ with respect to $dv_X$. Under our trivialization, $\Xi_p(x, x')$ induces a smooth function $\Xi_{p, x_0}(Z, Z')$ on the set of all $Z, Z' \in T_{x_0}X$ with $x_0 \in X$ and $|Z|, |Z'| < a_X$.

Definition 3.1 ($[35, 37]$). We say that 

$$p^{-n} \Xi_{p, x_0}(Z, Z') \equiv \sum_{r=0}^k (Q_{r, x_0} P_{x_0}) (\sqrt{p} Z, \sqrt{p} Z') p^{-\frac{r}{2}} + O(p^{-\frac{k+1}{2}})$$

with some $Q_{r, x_0} \in \mathbb{C}[Z, Z']$, $0 \leq r \leq k$, depending smoothly on the parameter $x_0 \in X$, if there exist $\varepsilon \in (0, a_X]$ and $C_0 > 0$ with the following property: for any $l \in \mathbb{N}$, there exist $C > 0$ and $M > 0$ such that for any $x_0 \in X$, $p \geq 1$ and $Z, Z' \in T_{x_0}X$, $|Z|, |Z'| < \varepsilon$, we have

$$\left| p^{-n} \Xi_{p, x_0}(Z, Z') \kappa_{x_0}^{\frac{1}{2}}(Z) \kappa_{x_0}^{\frac{1}{2}}(Z') - \sum_{r=0}^k (Q_{r, x_0} P_{x_0}) (\sqrt{p} Z, \sqrt{p} Z') p^{-\frac{r}{2}} \right|_{C^l(X)} 

\leq C p^{-\frac{k+1}{2}} \left(1 + \sqrt{p} |Z| + \sqrt{p} |Z'|\right)^M \exp(-\sqrt{C_0 p} |Z - Z'|) + O(p^{-\infty}).$$
Observe that if $P_{p,x_0}(Z, Z')$ denotes the Schwartz kernel of the Bergman projection $P_{H_p}$ in the normal coordinates near $x_0 \in X$, then, by \cite{28} Theorem 1.1, for any $k \in \mathbb{N},$

$$p^{-n}P_{p,x_0}(Z, Z') \cong \sum_{r=0}^{k} (J_{r,x_0}P_{x_0})(\sqrt{pZ}, \sqrt{pZ'})p^{-\frac{r}{2}} + O(p^{-\frac{k+1}{2}}),$$

$J_{r,x_0}(Z, Z')$ are polynomials in $Z, Z'$, depending smoothly on $x_0$, with the same parity as $r$ and $\deg J_{r,x_0} \leq 3r$. This estimate was introduced in \cite{10} for the spin$^c$ Dirac operator, see also \cite{35, 37} for the Kähler case and \cite{32} for the renormalized Bochner Laplacian.

Let $f \in C^\infty(X)$. The Schwartz kernel of $T_{f,p}$ is given by

$$T_{f,p}(x, x') = \int_X P_p(x, x'')f(x'')P_{p}(x'', x')dv_X(x'').$$

Therefore, for any $x_0 \in X$ and $k \in \mathbb{N}$, we have \cite{28} Lemma 6.4 (see also \cite{35, 37} for the spin$^c$ Dirac operator and the Kähler case and \cite{20} for the renormalized Bochner Laplacian)

$$p^{-n}T_{f,p,x_0}(Z, Z') \cong \sum_{r=0}^{k} (Q_{r,x_0}(f)P_{x_0})(\sqrt{pZ}, \sqrt{pZ'})p^{-\frac{r}{2}} + O(p^{-\frac{k+1}{2}}),$$

where the polynomials $Q_{r,x_0}(f)(Z, Z')$ have the same parity as $r$.

The coefficients $Q_{r,x_0}(f)$ are computed explicitly in \cite{35, 37} as follows. For any polynomial $F \in \mathbb{C}[Z, Z']$, denote by $FP_{x_0}$ the operator in $L^2(T_{x_0}X)$ with smooth kernel $(FP_{x_0})(Z, Z')$. For polynomials $F, G \in \mathbb{C}[Z, Z']$, define the polynomial $\mathcal{K}[F, G] \in \mathbb{C}[Z, Z']$ by the condition

$$((FP_{x_0}) \circ (GP_{x_0}))(Z, Z') = (\mathcal{K}[F, G]P_{x_0})(Z, Z'),$$

where $((FP_{x_0}) \circ (GP_{x_0}))(Z, Z')$ is the smooth kernel of the composition $(FP_{x_0}) \circ (GP_{x_0})$ of the operators $FP_{x_0}$ and $GP_{x_0}$ in $L^2(T_{x_0}X)$.

Then we have

$$Q_{r,x_0}(f) = \sum_{r_1+r_2+|\alpha|=r} \mathcal{K} \left[ J_{r_1,x_0}, \frac{\partial^\alpha f_{x_0}}{\partial Z^\alpha} \frac{Z^\alpha}{\alpha!} J_{r_2,x_0} \right],$$

where $f_{x_0}$ is the smooth function on $B^{T_{x_0}X}(0, a^X)$, corresponding to $f$ by the identification $B^{T_{x_0}X}(0, a^X) \cong B^X(x_0, a^X)$.

For the first three coefficients, we get the following expressions:

\begin{align*}
(3.3) \quad & Q_{0,x_0}(f) = f(x_0), \\
(3.4) \quad & Q_{1,x_0}(f) = f(x_0)J_{1,x_0} + \mathcal{K} \left[ J_{0,x_0}, \sum_{j=1}^{2n} \frac{\partial f_{x_0}}{\partial Z_j} (0) Z_j J_{0,x_0} \right], \\
\text{and} \\
(3.5) \quad & Q_{2,x_0}(f)
\end{align*}
is the smooth kernel of the operator $P$

Since $J$

Observe that

Theorem 3.3. A family $(\text{see also [26]})$. This type of criterion was introduced in [37, Theorem 4.9].

By definition of $K$

First of all, by (3.3) and (3.4), we have

Proof. First of all, by (3.3) and (3.3), we have

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(iii): There exists a family of polynomials $Q_{r,x_0} \in \mathbb{C}[Z, Z']$, $r \in \mathbb{N}$, depending smoothly on $x_0$, of the same parity as $r$ such that, for any $k \in \mathbb{N}$ and $x_0 \in X$,

Lemma 3.2. If $f(x_0) = 0$ and $df(x_0) = 0$, then

$p^{-n}T_{f,p,x_0}(Z, Z')$

\[ \approx p^{-1} \left[ \mathcal{P}_{x_0} \left( \frac{1}{2} \text{Hess } f(x_0) Z, Z \right) \mathcal{P}_{x_0} \right] \sqrt{pZ, \sqrt{p}Z'} + O(p^{-\frac{3}{2}}). \]

Proof. First of all, by (3.3) and (3.3), we have

Observe that

Since $J_{0,x_0}(Z, Z') = 1$, by (3.3), we get

By definition of $K$, the function

is the smooth kernel of the operator $\mathcal{P}_{x_0} \left( \frac{1}{2} \text{Hess } f(x_0) Z, Z \right) \mathcal{P}_{x_0}$.

We have the following criterion for Toeplitz operators [28, Theorem 6.5] (see also [26]). This type of criterion was introduced in [37, Theorem 4.9].

Theorem 3.3. A family $\{T_p : L^2(X, L^p) \rightarrow L^2(X, L^p)\}$ of bounded linear operators is a Toeplitz operator if and only if it satisfies the following three conditions:

(i): For any $p \in \mathbb{N}$,

(ii): For any $\varepsilon_0 > 0$, there exist $C > 0$ and $c > 0$ such that for any $p \geq 1$ and $(x, x') \in X \times X$ with $d(x, x') > \varepsilon_0$,

(iii): There exists a family of polynomials $Q_{r,x_0} \in \mathbb{C}[Z, Z']$, $r \in \mathbb{N}$, depending smoothly on $x_0$, of the same parity as $r$ such that, for any $k \in \mathbb{N}$ and $x_0 \in X$,

\[ p^{-n}T_{p,x_0}(Z, Z') \approx \sum_{r=0}^{k} \left( Q_{r,x_0} \mathcal{P}_{x_0} \right) \left( \sqrt{pZ}, \sqrt{p}Z' \right) p^{-\frac{3}{2}} + O(p^{-\frac{k+1}{2}}). \]
Moreover, it is shown in the proof of this theorem (see Proposition 4.11 and (4.30) in \[37\]) that, in this case, for all \(x_0 \in X\) and \(Z, Z' \in T_{x_0} X\),
\[
Q_{0,x_0}(Z, Z') = Q_{0,x_0}(0, 0),
\]
and the principal symbol \(g_0\) of \(T_p\) is given by
\[
g_0(x_0) = Q_{0,x_0}(0, 0).
\]

### 3.2. Construction of approximate eigensections.

We start the proofs of Theorems 1.5 and 1.6 with a construction of approximate eigensections of the operator \(T_p\). It follows closely the construction given in \[14, Proposition 4.2 and Proposition 5.1\] in the case of Kähler manifolds and heavily relies on the asymptotic expansions of kernels of Toeplitz operators. So we just give a sketch of this construction and state the main results. We will use notation introduced in Section 1.4.

Let \(T_p\) be a self-adjoint Toeplitz operator \((1.9)\), satisfying \((1.10)\) and \((1.11)\). Suppose that \(x_0 \in U_0\) is a non-degenerate minimum of \(h\). The approximate eigensections, which we are going to construct, will be supported in a small neighborhood of \(x_0\). So we will use the normal coordinates near \(x_0\) and the trivialization of the line bundle \(L\) constructed in Section 3.1. By Theorem 3.3 and Lemma 3.2, the smooth kernel \(T_{p,x_0}(Z, Z')\) of \(T_p\) in these coordinates admits the following asymptotic expansion: for any \(k \geq 2\),
\[
p^{-n}T_{p,x_0}(Z, Z') \sim \sum_{r=2}^{k} (Q_{r,x_0} P_{x_0})(\sqrt{p}Z, \sqrt{p}Z') p^{-r/2} + O(p^{-k+1}),
\]
where, for any \(r\), \(Q_{r,x_0} \in \mathbb{C}[Z, Z']\) is a polynomial of the same parity as \(r\) and
\[
(Q_{2,x_0} P_{x_0})(Z, Z') = [P_{x_0}(q_{x_0} + g_1(x_0)) P_{x_0}](Z, Z') = T_{x_0}(Z, Z').
\]
In particular, when \(k = 2\), we have
\[
p^{-n}T_{p,x_0}(Z, Z') \sim T_{x_0}(\sqrt{p}Z, \sqrt{p}Z') p^{-1} + O(p^{-3}),
\]

First, we construct a formal eigensection of the operator \(T_p\). We write formal asymptotic expansions in powers of \(p^{-1/2}\):
\[
u_p(Z) = \sum_{j=0}^{+\infty} p^{j/2} u^{(j)}(\sqrt{p}Z) p^{-j/2}, \quad \lambda_p = p^{-1} \sum_{j=0}^{+\infty} \lambda^{(j)} p^{-j/2}
\]
and express the cancellation of the coefficients of \(p^{-j/2}\) in the formal expansion for \((T_p - \lambda_p)u_p\) step by step, using the asymptotic expansions \((3.7)\) and \((3.8)\).

At the first step, we get:
\[
T_{x_0} u^{(0)} = \lambda^{(0)} u^{(0)}.
\]
Thus, \(\lambda^{(0)} = \lambda\) is an eigenvalue of \(T_{x_0}\) and \(u^{(0)}\) is an associated eigenfunction.
At the second step, we get:
\[ T_{x_0}u^{(1)} + Q_{1,x_0}P_{x_0}u^{(0)} = \lambda^{(0)}u^{(1)} + \lambda^{(1)}u^{(0)}. \]

One can show that this equation has a solution \((u^{(1)}, \lambda^{(1)})\).

At the third step, we get an equation, which can’t be solved in general. One can solve it in the case when \(\lambda\) is a simple eigenvalue. In this case, we can proceed further and obtain a solution as a formal asymptotic series in powers of \(p^{-1/2}\). To obtain approximate eigensections, we multiple the formal eigenfunctions by appropriate cut-off functions.

We arrived at the following statements.

**Proposition 3.4.** Let \(x_0 \in U_0\) be a non-degenerate minimum of \(h\) and \(\lambda\) be an eigenvalue of \(T_{x_0}\) of multiplicity \(m\). There exist an orthonormal system \(u^{(1)}, \ldots, u^{(m)}\) from \(S(T_{x_0}X)\) of eigenfunctions of \(T_{x_0}\) with eigenvalue \(\lambda\), functions \(v^{(1)}, \ldots, v^{(m)}\) from \(S(T_{x_0}X)\) and real numbers \(\mu^{(1)}, \ldots, \mu^{(m)}\) such that if for each \(p\) and \(j = 1, \ldots, m\) we define \(u^{(j)}_p \in L^2(X,L^p)\) and \(\lambda^{(j)}_p \in \mathbb{R}\) by
\[
u^{(j)}_p(Z) = p^n\chi(Z)u^{(j)}(\sqrt{p}Z) + p^{-1/2}v^{(j)}(\sqrt{p}Z), \quad \lambda^{(j)}_p = p^{-1}\lambda + p^{-1/2}\mu^{(j)},
\]
\(\chi\) is a cut-off function from \(C^\infty_c(B^{T_{x_0}}(0,\varepsilon)) \cong C^\infty_c(B^X(x_0,\varepsilon))\) with some \(\varepsilon \in (0,aX/4)\), then
\[
\|T_p u^{(j)}_p - \lambda^{(j)}_p u^{(j)}_p\|_{L^2(X,L^p)} = O(p^{-2}).
\]

**Proposition 3.5.** Let \(x_0 \in U_0\) be a non-degenerate minimum of \(h\), \(\lambda\) be a simple eigenvalue of \(T_{x_0}\) and \(u_0\) be a normalized eigenfunction of \(T_{x_0}\). There exist a sequence \((u_k)_{k \geq 1}\) of functions from \(S(T_{x_0}X)\) with \((u_0,u_k) = \delta_{0k}\) and a sequence \((\lambda_k)_{k \geq 0}\) of real numbers, with \(\lambda_0 = \lambda\) and \(\lambda_k = 0\) for \(k\) odd, such that if for each \(N\) and \(p\) we define \(u_{N,p} \in L^2(X,L^p)\) and \(\lambda_{N,p} \in \mathbb{R}\) by
\[
u_{N,p}(Z) = \chi(Z)p^n \sum_{k=0}^{N} p^{-k/2}u_k(\sqrt{p}Z), \quad \lambda_{N,p} = p^{-1} \sum_{k=0}^{N} p^{-k/2}\lambda_k,
\]
\(\chi\) is a cut-off function from \(C^\infty_c(B^{T_{x_0}}(0,\varepsilon)) \cong C^\infty_c(B^X(x_0,\varepsilon))\) with some \(\varepsilon \in (0,aX/4)\), then
\[
\|T_p u_{N,p} - \lambda_{N,p} u_{N,p}\|_{L^2(X,L^p)} = O(p^{-(N+3)/2}).
\]

### 3.3. Proofs of Theorems 1.5 and 1.6

Suppose that \(T_p\) is a self-adjoint Toeplitz operator with the principal symbol \(h\), satisfying 1.10, such that each minimum is non-degenerate. In this case, \(U_0 = \{x_1, \ldots, x_N\}\) and the model operator \(T\) associated with \(T_p\) is a self-adjoint operator on \(L^2(T_{x_1}X) \oplus \ldots \oplus L^2(T_{x_N}X)\) defined by 1.17.

**Lemma 3.6.** For any \(c > 0\) and \(\gamma \in (0,1/2)\), there exists \(C > 0\) such that for any sequence \(u_p \in \mathcal{H}_p\) of eigensections of \(T_p\):
\[
T_p u_p = \lambda_p u_p, \quad \|u_p\| = 1,
\]
with $\lambda_p \in (0, cp^{-1})$, we have for $p \in \mathbb{N}$,

$$(p\lambda_p - C p^{-1/2+\gamma}, p\lambda_p + C p^{-1/2+\gamma}) \cap \sigma(T) \neq \emptyset.$$  

**Proof.** Fix $c > 0$. By Theorem 1.4 for any $\delta > 0$ and $k \in \mathbb{N}$, there exists $C_1 > 0$ such that, for any sequence $u_p \in \mathcal{H}_p$ of normalized eigensections of $T_p$ with $\lambda_p \in (0, cp^{-1})$ and for any $p \in \mathbb{N}$,

$$(3.9) \quad \int_{X \setminus V_\delta} |u_p(x)|^2 dv_X(x) \leq C_1 p^{-k}.$$  

Using a partition of unity subordinate to the open covering

$$X = \bigcup_{j=1}^N B^X(x_j, \varepsilon_0) \cup (X \setminus U_0)$$  

with some $\varepsilon_0 \in (0, a_X)$ small enough, we can write

$$u_p = \sum_{j=1}^N u_p^{(j)} + u_p^{(0)},$$  

where each $u_p^{(j)}$ is supported in $B^X(x_j, \varepsilon_0) \cong B^{T_{x_j}X}(0, \varepsilon_0)$ and satisfies (3.9), $u_p^{(0)}$ is supported in $X \setminus U_0$ and, by (3.9), $\|u_p^{(0)}\| = \mathcal{O}(p^{-\infty})$. We will assume that the balls $B^X(x_j, \varepsilon_0)$ are pairwise disjoint.

We will use the same notation $u_p^{(j)}$ for the corresponding function on $B^{T_{x_j}X}(0, \varepsilon_0)$ and define a function $\tilde{u}_p^{(j)} \in C_c^\infty(B^{T_{x_j}X}(0, \varepsilon_0))$ by

$$\tilde{u}_p^{(j)}(Z) = \kappa_{x_j}^{1/2}(Z) u_p^{(j)}(Z), \quad Z \in B^{T_{x_j}X}(0, \varepsilon_0).$$

Put

$$(3.10) \quad \tilde{u}_p = \bigoplus_{j=1}^N \tilde{u}_p^{(j)} \in \bigoplus_{j=1}^N L^2(T_{x_j}X).$$

Observe that

$$\|\tilde{u}_p^{(j)}\|_{L^2(T_{x_j}X)} = \|\tilde{u}_p^{(j)}\|_{L^2(B^{T_{x_j}X}(0, \varepsilon_0))} = \|u_p^{(j)}\|_{L^2(B^X(x_j, \varepsilon_0))} = \|u_p^{(j)}\|,$$

$$\|\tilde{u}_p\| = 1 + \mathcal{O}(p^{-\infty}).$$

For any $j = 1, \ldots, N$ and $p \in \mathbb{N}$, let $T_{x_j,p}$ be the operator in $L^2(T_{x_j}X)$ with the Schwartz kernel

$$T_{x_j,p}(Z, Z') = p^a T_{x_j}(\sqrt{p}Z, \sqrt{p}Z'), \quad Z, Z' \in T_{x_j}X,$$

where $T_{x_j}(Z, Z')$ is the Schwartz kernel of the operator $T_{x_j}$. It is clear that the operator $T_{x_j,p}$ is obtained from the operator $T_{x_j}$ by an obvious scaling. So these operators are unitarily equivalent and, in particular, have the same spectrum. Let $T_p$ be the self-adjoint operator on $L^2(T_{x_1}X) \oplus \ldots \oplus L^2(T_{x_N}X)$ defined by

$$T_p = T_{x_1,p} \oplus \ldots \oplus T_{x_N,p}.$$
Fix some $\gamma \in (0, 1/2)$. Using (3.3) and (3.9), one can show (cf. [14, Proposition 2.7]) that there exists $C_2 > 0$ such that for any $p \in \mathbb{N}$ and $j = 1, \ldots, N$,

$$(3.11) \quad \|\kappa_{x,j}^{1/2} T_p u_p^{(j)} - p^{-1} T_{x,j} p \tilde{u}_p^{(j)}\|_{L^2(B^{T_{x,j} X}(0, \varepsilon_0 + \varepsilon))} \leq C_2 p^{-3/2+\gamma} \|\tilde{u}_p^{(j)}\|_{L^2(T_{x,j} X)}.$$  

Here $\varepsilon > 0$ is given by the asymptotic expansion (3.8). We choose $\varepsilon_0 > 0$ in such a way that $\varepsilon_0 + \varepsilon \in (0, a_X)$. The constant $C_2 > 0$ depends only on the constant $C_1 > 0$ in (3.9) and, therefore, on $c > 0$, but not on the particular sequence $u_p$.

Since each $\tilde{u}_p^{(j)}$ is supported in $B^{T_{x,j} X}(0, \varepsilon_0)$ and the kernel of $T_{x,j} p$ is rapidly decaying outside the diagonal as $p \to \infty$, we have

$$\|T_{x,j} p \tilde{u}_p^{(j)}\|_{L^2(T_{x,j} X \setminus B^{T_{x,j} X}(0, \varepsilon_0 + \varepsilon))} = O(p^{-\infty}) \|\tilde{u}_p^{(j)}\|,$$

and, therefore,

$$\|p^{-1} T_p \tilde{u}_p - \lambda_p \tilde{u}_p\|^2 = \sum_{j=1}^{N} \|p^{-1} T_{x,j} p \tilde{u}_p^{(j)} - \lambda_p \tilde{u}_p^{(j)}\|_{L^2(T_{x,j} X)}^2$$

$$= \sum_{j=1}^{N} \|p^{-1} T_{x,j} p \tilde{u}_p^{(j)} - \lambda_p \tilde{u}_p^{(j)}\|_{L^2(B^{T_{x,j} X}(0, \varepsilon_0 + \varepsilon))}^2 + O(p^{-\infty}) \|\tilde{u}_p^{(j)}\|^2.$$  

By (3.11), it follows that

$$(3.12) \quad \|p^{-1} T_p \tilde{u}_p - \lambda_p \tilde{u}_p\|^2$$

$$= \sum_{j=1}^{N} \|\kappa_{x,j}^{1/2} T_p u_p^{(j)} - \lambda_p u_p^{(j)}\|^2_{L^2(B^{T_{x,j} X}(0, \varepsilon_0 + \varepsilon))} + C_2^2 p^{-3+2\gamma} \|\tilde{u}_p\|^2$$

$$= \sum_{j=1}^{N} \|T_p u_p^{(j)} - \lambda_p u_p^{(j)}\|^2_{L^2(B^X(x_j, \varepsilon_0 + \varepsilon))} + C_2^2 p^{-3+2\gamma}.$$  

Since each $u_p^{(j)}$ is supported in $B^X(x_j, \varepsilon_0)$, we get

$$(3.13) \quad \|T_p u_p^{(j)} - \lambda_p u_p^{(j)}\|^2 = \|T_p u_p^{(j)} - \lambda_p u_p^{(j)}\|^2_{L^2(B^X(x_j, \varepsilon_0 + \varepsilon))} + O(p^{-\infty}).$$  

Moreover, since $B^X(x_j, \varepsilon_0)$ are pairwise disjoint, $(u_p^{(j)}, u_p^{(k)})_{L^2(X)} = 0$ for $j \neq k$. By Theorem 3.3 this implies that, for $j \neq k$,

$$(T_p u_p^{(j)} - \lambda_p u_p^{(j)}, T_p u_p^{(k)} - \lambda_p u_p^{(k)}) = O(p^{-\infty}).$$  

Using this almost orthogonality property and the fact that $u_p$ is an eigensection of $T_p$ with the eigenvalue $\lambda_p$, we get

$$(3.14) \quad \sum_{j=1}^{N} \|T_p u_p^{(j)} - \lambda_p u_p^{(j)}\|^2 = \|T_p u_p - \lambda_p u_p\|^2 + O(p^{-\infty}) = O(p^{-\infty}).$$
Combining (3.12), (3.13) and (3.14), we conclude that
\[
\|p^{-1}T_p \tilde{u}_p - \lambda_p \tilde{u}_p\| = C_3p^{-3/2+\gamma}\|\tilde{u}_p\|
\]
that completes the proof of the lemma. \(\square\)

Fix \(\gamma \in (0, 1/2)\) and \(c > 0\). By Lemma 3.6, there exists \(C > 0\) such that, for all \(p \in \mathbb{N}\),
\[
\sigma(T_p) \cap (0, cp^{-1}) \subset \bigcup_{\mu \in \sigma(T) \cap (0,c)} (\mu p^{-1} - C p^{-3/2+\gamma}, \mu p^{-1} + C p^{-3/2+\gamma}).
\]

Let us show that, for any \(\mu \in \sigma(T)\), the number \(m_p\) of eigenvalues of \(T_p\) in the interval \((\mu p^{-1} - C p^{-3/2+\gamma}, \mu p^{-1} + C p^{-3/2+\gamma})\) is independent of \(p\) for large \(p\) and equals the multiplicity \(m\) of \(\mu\):
\[
(3.16) \quad m_p = m, \quad p \gg 1.
\]

Using Proposition 3.4, one can easily show that, if \(\mu\) is an eigenvalue of the operator \(T\) of multiplicity \(m\), then there exists at least \(m\) eigenvalues of the operator \(T_p\) in the interval \((\mu p^{-1} - C p^{-3/2+\gamma}, \mu p^{-1} + C p^{-3/2+\gamma})\) for some \(C > 0\). Therefore, \(m_p \geq m\) for any \(p\) large enough.

On the other hand, let \(u_{1,p}, \ldots, u_{m_p,p}\) be an orthonormal basis of eigensections of \(T_p\) with the corresponding eigenvalues \(\lambda_{1,p}, \ldots, \lambda_{m_p,p}\) in the interval \((\lambda p^{-1} - C p^{-3/2+\gamma}, \lambda p^{-1} + C p^{-3/2+\gamma})\). For the functions \(\tilde{u}_{1,p}, \ldots, \tilde{u}_{m_p,p}\) defined by (3.10), we get
\[
(\tilde{u}_{j,p}, \tilde{u}_{k,p}) = \delta_{jk} + O(p^{-\infty})
\]
and, by (3.15),
\[
\|p^{-1}T_p \tilde{u}_{j,p} - \lambda_{j,p} \tilde{u}_{j,p}\| = C p^{-3/2+\gamma}\|\tilde{u}_p\|.
\]
It follows that there are at least \(m_p\) eigenvalues of \(T_p\) in the interval \((\mu - C p^{-1/2+\gamma}, \mu + C p^{-1/2+\gamma})\). But, for large \(p\), the only eigenvalue \(T_p\) in this interval is \(\mu\). Therefore, \(m_p \leq m\), that completes the proof of (3.16).

This shows that each eigenvalue \(\lambda_{p}^m\) has an asymptotic expansion, as \(p \to \infty\), of the form
\[
\lambda_{p}^m = p^{-1} \mu_m + O(p^{-3/2}).
\]
Now the asymptotic expansions (1.5) in Theorem 1.5 and the complete asymptotic expansions in Theorem 1.6 can be proved in a standard way by means of Propositions 3.4 and 3.5.

4. Applications to the Bochner Laplacian

4.1. Upper bounds for eigenvalues of the Bochner Laplacian.

Proof of Theorem 1.7. The Bochner Laplacian \(\Delta^L_p\) can be written in the following Schrödinger operator type form
\[
\Delta^L_p = \Delta_p + p\tau.
\]
So we have
\[ p^{-1} P_{\mathcal{H}_p} \Delta^{L^p} P_{\mathcal{H}_p} = p^{-1} P_{\mathcal{H}_p} \Delta P_{\mathcal{H}_p} + P_{\mathcal{H}_p} \tau P_{\mathcal{H}_p}. \]
It is well-known (see, for instance, [40] Theorem XIII.3), the Rayleigh-Ritz technique) that
\[ \lambda_j(\Delta^{L^p}) \leq \lambda_j(P_{\mathcal{H}_p} \Delta^{L^p} P_{\mathcal{H}_p}), \quad j \in \mathbb{N}, \]
It remains to get an upper bound for \( \lambda_j(P_{\mathcal{H}_p} \Delta^{L^p} P_{\mathcal{H}_p}) \).

Here the crucial fact is that the operator \( P_{\mathcal{H}_p} \Delta P_{\mathcal{H}_p} = \Delta P_{\mathcal{H}_p} \) is a Toeplitz operator. This was proved in [28], extending previous results of [36]. Denote by \( P_{\mathcal{H}_p}(x, x') \) the smooth kernel of the operator \( \Delta P_{\mathcal{H}_p} \) with respect to the Riemannian volume form \( dv \). It is called a generalized Bergman kernel of \( \Delta P \). For any \( k \in \mathbb{N} \) and \( x_0 \in X \), we have
\[ (4.1) \quad p^{-n} P_{1, p, x_0}(Z, Z') \cong \sum_{r=2}^{j} J_{1, r, x_0}(\sqrt{p}Z, \sqrt{p}Z') \kappa^{-\frac{r}{2}}(Z) \kappa^{-\frac{r}{2}}(Z') p^{-\frac{r}{2}+1}, \]
where
\[ (4.2) \quad F_{1, r, x_0}(Z, Z') = J_{1, r, x_0}(Z, Z') P_{x_0}(Z, Z'), \]
\( J_{1, r, x_0}(Z, Z') \) are polynomials in \( Z, Z' \), depending smoothly on \( x_0 \), with the same parity as \( r \) and \( \deg J_{1, r, x_0} \leq 3r \).

So the operator \( p^{-1} P_{\mathcal{H}_p} \Delta^{L^p} P_{\mathcal{H}_p} \) is a Toeplitz operator:
\[ p^{-1} P_{\mathcal{H}_p} \Delta^{L^p} P_{\mathcal{H}_p} = P_{\mathcal{H}_p} \left( \sum_{l=0}^{\infty} p^{-l} g_l \right) P_{\mathcal{H}_p} + O(p^{-\infty}), \]
where \( g_0 = \tau \) and \( g_1 \) is the principal symbol of \( P_{\mathcal{H}_p} \Delta P_{\mathcal{H}_p} \). By (3.6) and (4.2), it is given by
\[ (4.3) \quad g_1(x_0) = J_{1, 2}(x_0) \quad J_{1, 2, x_0}(0, 0) = \frac{F_{1, 2, x_0}(0, 0)}{P_{x_0}(0, 0)}. \]

By Theorem 1.5 we have
\[ p^{-1} \lambda_j(P_{\mathcal{H}_p} \Delta^{L^p} P_{\mathcal{H}_p}) = \tau_0 + p^{-1} \mu_j + p^{-3/2} \phi_j + O(p^{-2}), \]
with some \( \phi_j \in \mathbb{R} \), that immediately completes the proof of Theorem 1.7 \( \Box \)

The coefficient \( F_{1, 2, x_0} \) (and, therefore, the function \( J_{1, 2} \)) can be computed explicitly (see Subsection 2.1, in particular, the formula (2.12) in [36]). Let us recall this formula. We will use notation of Section 3.1. Put
\[ \frac{\partial}{\partial z_j} = \frac{1}{2} \left( \frac{\partial}{\partial Z_{2j-1}} - i \frac{\partial}{\partial Z_{2j}} \right), \quad \frac{\partial}{\partial \bar{z}_j} = \frac{1}{2} \left( \frac{\partial}{\partial Z_{2j-1}} + i \frac{\partial}{\partial Z_{2j}} \right). \]
Let \( \mathcal{R}(Z) = \sum_{j=1}^{2n} z_j \xi_j = Z \) denote the radial vector field on \( T_{x_0}X \). Define first order differential operators \( b_j, b^\perp_j, j = 1, \ldots, n, \) on \( T_{x_0}X \) by
\[ b_j = -2 \nabla_{\xi_j} \rho - R_{x_0}^L(\mathcal{R}, \frac{\partial}{\partial z_j}) = -2 \frac{\partial}{\partial z_j} + \frac{1}{2} \frac{\partial}{\partial \bar{z}_j}, \]
section, we compute the spectrum of the Toeplitz operator \( T \) in \( L^2 \) where

\[
\phi_j = 2 \nabla \frac{\partial}{\partial x_j} + R_{x_0}^L (R, \frac{\partial}{\partial x_j}) = 2 \frac{\partial}{\partial x_j} + \frac{1}{2} a_j z_j.
\]

So we can write

\[
L_{x_0} = \sum_{j=1}^n b_j b_j^+, \quad \tau(x_0) = \sum_{j=1}^n a_j.
\]

Then

\[
F_{1,2,x_0}(Z, Z') = [P_{x_0} F_{1,2,x_0} P_{x_0}](Z, Z'),
\]

where \( F_{1,2,x_0} \) is an unbounded linear operator in \( L^2(T_{x_0}X) \) given by

\[
F_{1,2,x_0} = 4 \left( R_{x_0}^T \left( \frac{\partial}{\partial x_j} \cdot \frac{\partial}{\partial z_k} \right) \frac{\partial}{\partial x_j} \cdot \frac{\partial}{\partial z_k} \right)
\]

\[
+ \left( (\nabla^X \nabla^X J(R,R)) \frac{\partial}{\partial x_j} \cdot \frac{\partial}{\partial z_k} \right) + \frac{i}{4} \text{tr}_{TX} \left( \nabla^X \nabla^X (J J) \right) \left( R, R \right)
\]

\[
+ \frac{1}{9} |(\nabla^X R J) R|^2 + \frac{4}{9} \sum_{j,k=1}^n \left( (\nabla^X R J) R, \frac{\partial}{\partial z_j} \right) b_j^+ L_{x_0}^{-1} b_k \left( (\nabla^X R J) R, \frac{\partial}{\partial z_k} \right).
\]

The formulas (4.3), (4.4) and (4.5) allow one to compute the function \( J_{1,2} \). For instance, as shown in [36, (2.30)], if \( J_0 = J \), then

\[
J_{1,2}(x_0) = \frac{1}{24} |\nabla^X J|^2_{x_0}.
\]

Here if \( \{e_j\}_{j=1,\ldots,2n} \) is a local orthonormal frame of \( (TX, g^{TX}) \), then

\[
|\nabla^X J|^2 = \sum_{i,j=1}^{2n} |(\nabla^X J) e_i|^2.
\]

4.2. Computation of the spectrum of the model operator. In this section, we compute the spectrum of the Toeplitz operator \( T(Q) \) in \( L^2(\mathbb{C}^n) \) given by

\[
T(Q) = PQ : \ker L_{x_0} \subset L^2(\mathbb{C}^n) \to \ker L_{x_0} \subset L^2(\mathbb{C}^n),
\]

where \( Q = Q(z, \bar{z}) \) is a polynomial in \( \mathbb{C}^n \) and \( P \) is the orthogonal projection in \( L^2(\mathbb{C}^n) \cong L^2(T_{x_0}X) \) to the kernel of \( L_{x_0} \) (see [32]). We will keep notation of Section 3.1.

Recall that the Fock space \( \mathcal{F}_n \) is the space of holomorphic functions \( F \) in \( \mathbb{C}^n \) such that \( e^{-\frac{1}{2}|z|^2} F \in L^2(\mathbb{C}^n) \). It is a closed subspace in \( L^2(\mathbb{C}^n; e^{-\frac{1}{2}|z|^2} dz) \) and the orthogonal projection \( \Pi : L^2(\mathbb{C}^n; e^{-\frac{1}{2}|z|^2} dz) \to \mathcal{F}_n \) is given by

\[
\Pi F(z) = \frac{1}{\pi^n} \int_{\mathbb{C}^n} \exp \left(-|z'|^2 + z \cdot z' \right) F(z') dz' d\bar{z}'.
\]

Consider the isometry \( S : L^2(\mathbb{C}^n; e^{-\frac{1}{2}|z|^2} dz) \to L^2(\mathbb{C}^n) \) given, for \( u \in L^2(\mathbb{C}^n; e^{-\frac{1}{2}|z|^2} dz), \) by

\[
Su(z) = \frac{\prod_{j=1}^n a_j}{2^n} e^{-\frac{1}{2} \sum_j a_j |z_j|^2} u(\phi(z)),
\]
where $\phi : \mathbb{C}^n \to \mathbb{C}^n$ is a linear isomorphism given by

$$\phi(z) = \left( \frac{\sqrt{a_1}}{\sqrt{2}}z_1, \ldots, \frac{\sqrt{a_n}}{\sqrt{2}}z_n \right), \quad z \in \mathbb{C}^n.$$  

It is easy to see that $S \Pi S^{-1} = P$. It follows that $S(F_n) = \ker L_{x_0}$ and

$$T(Q) = ST^0(Q \circ \phi^{-1})S^{-1}.$$  

where $T^0(Q)$ is a Toeplitz operator in the Fock space defined by

$$T^0(Q) = \Pi Q : F_n \to F_n.$$

In particular, the spectrum of $T(Q)$ in $\ker L_{x_0}$ coincides with the spectrum of $T^0(Q \circ \phi^{-1})$ in $F_n$.

To compute the spectrum of $T^0(P)$ in $F_n$ for a positive definite quadratic form $P$, we will use the well-known relation between the Bargmann-Fock and the Schrödinger representations of the canonical commutation relations via the Bargmann transform. Recall that the Bargmann transform [2] is an isometry $B : L^2(\mathbb{R}^n) \to F_n$ defined by

$$Bf(z) = \pi^{-n/4} \int_{\mathbb{R}^n} \exp \left[ -\left( \frac{1}{2} z \cdot z + \frac{1}{2} x \cdot x - \sqrt{2} z \cdot x \right) \right] f(x)dx, \quad z \in \mathbb{C}^n.$$  

It is easy to check (see, for instance, [2, (3.15b)]) that, for the standard position and momentum operators

$$\hat{q}_k = x_k, \quad \hat{p}_k = \frac{1}{i} \frac{\partial}{\partial x_k}, \quad k = 1, \ldots, n,$$

we have

$$B\hat{q}_kB^{-1} = \frac{1}{\sqrt{2}} \left( z_k + \frac{\partial}{\partial z_k} \right), \quad B\hat{p}_kB^{-1} = \frac{1}{\sqrt{2}i} \left( z_k - \frac{\partial}{\partial z_k} \right).$$

Accordingly, for the creation and annihilation operators

$$\hat{a}^*_k = \frac{1}{\sqrt{2}}(\hat{q}_k - i\hat{p}_k), \quad \hat{a}_k = \frac{1}{\sqrt{2}}(\hat{q}_k + i\hat{p}_k),$$

we get

$$B\hat{a}^*_kB^{-1} = z_k, \quad B\hat{a}_kB^{-1} = \frac{\partial}{\partial z_k}.$$  

Observe, for any $F \in L^2(\mathbb{C}^n; e^{-\frac{1}{2}|z|^2}dz)$,

$$\frac{\partial}{\partial z_k} \Pi F = \Pi \bar{z}_k F, \quad k = 1, \ldots, n,$$

and, for any $F \in F_n$,

$$z_k \Pi F = \Pi (z_k F) \quad k = 1, \ldots, n.$$

Let $P$ be a polynomial in $\mathbb{C}^n$. If we write $P$ as

$$P(\bar{z}, z) = \sum_{k,l \in \mathbb{Z}^n} A_{k,l} \bar{z}_k \cdots \bar{z}_1 z_l \cdots z_1,$$
then, using (4.8) and (4.9), one can easily see that, for any $F \in \mathcal{F}_n$,
\begin{equation}
(4.10)
P(\partial_z, z) F = \Pi(P(\bar{z}, z) F) = \mathcal{T}^0(P) F,
\end{equation}
where $P(\partial_z, z)$ is the operator in $\mathcal{F}_n$ given by
\[
P(\partial_z, z) = \sum_{k,l \in \mathbb{Z}_n^+} A_{k,l} \frac{\partial^{k_1}}{\partial z_1} \ldots \frac{\partial^{k_n}}{\partial z_n} z_1^{l_1} \ldots z_n^{l_n}.
\]
By (4.7), it follows that, under the Bargmann transform $B$, the operator $T^0(P)$ in $\mathcal{F}_n$ corresponds to the operator $B^{-1} T^0(P) B$ in $L^2(\mathbb{R}^n)$ given by
\[
B^{-1} T^0(P) B = \sum_{k,l \in \mathbb{Z}_n^+} A_{k,l} \hat{a}^{k_1} \ldots \hat{a}^{k_n} (\hat{a}_1)^{l_1} \ldots (\hat{a}_n)^{l_n}.
\]
In the terminology of [3], the operator $B^{-1} T^0(P) B$ is the differential operator with polynomial coefficients in $\mathbb{R}^n$ with anti-Wick symbol $P(\bar{z}, z)$. Here, by (4.6) and (4.7), the phase space $\mathbb{R}^{2n} \cong T^* \mathbb{R}^n$ is identified to $\mathbb{C}^n$ by the linear isomorphism
\begin{equation}
(4.11)
z_k = \frac{1}{\sqrt{2}} (x_k - i \xi_k), \quad k = 1, \ldots, n.
\end{equation}
One can compute the Weyl symbol of this operator by a well-known formula (see, for instance, [3]). The paper [3] also contains some sufficient self-adjointness conditions for the operator $B^{-1} P(\partial_z, z) B$. In particular, if $P$ is a positive definite quadratic form, then (see, for instance, [14, Proposition 3.6]),
\[
B^{-1} T^0(P) B = Op_w(\tilde{P}) + \frac{\text{tr}(\tilde{P})}{2},
\]
where $\tilde{P}$ is a quadratic form on $\mathbb{R}^{2n}$, corresponding to $P$ under the linear isomorphism (4.11), $Op_w(\tilde{P})$ is the pseudodifferential operator in $\mathbb{R}^n$ with Weyl symbol $\tilde{P}$.

Consider the case $n = 1$ and $P$ is a positive definite quadratic form on $\mathbb{C} \cong \mathbb{R}^2$. Using rotations, we can assume that
\[
P(Z_1, Z_2) = \alpha Z_1^2 + \beta Z_2^2.
\]
The corresponding form $\tilde{P}$ on $\mathbb{R}^2$ is given by
\[
\tilde{P}(x, \xi) = \frac{1}{2}(\alpha x^2 + \beta \xi^2).
\]
We have
\[
Op_w(\tilde{P}) = \frac{1}{2}(\alpha x^2 + \beta D_x^2), \quad \text{tr}(\tilde{Q}) = \frac{\alpha + \beta}{2},
\]
and the eigenvalues of $B T^0(P) B^{-1}$ are
\[
\lambda_j = \frac{\sqrt{\alpha \beta}}{2} (2j + 1) + \frac{\alpha + \beta}{4} = \sqrt{\alpha \beta j} + \frac{(\sqrt{\alpha} + \sqrt{\beta})^2}{4}, \quad j = 0, 1, \ldots.
\]
So, for a general positive definite quadratic form $P$, the eigenvalues of $B^T P B^{-1}$ are

$$
\lambda_j = \sqrt{Dj + \frac{A^2}{4}}, \quad j = 0, 1, \ldots,
$$

where $D = \det P$, $A = \text{tr} P^{1/2}$.

Coming back to the initial problem and applying this result when $P(z) = (Q \circ \phi^{-1})(z) = \frac{2}{a_1} Q(z)$, $z \in \mathbb{C} \cong \mathbb{R}^2$, we find the eigenvalues of $T(Q)$:

$$
(4.12) \quad \lambda_j = 2\sqrt{D} \frac{a_1}{j} + \frac{A^2}{2a_1}, \quad j = 0, 1, \ldots,
$$

where $D = \det Q$, $A = \text{tr} Q^{1/2}$.

4.3. Relation with the magnetic Schrödinger operator. In this section, we recall the relationship of the Bochner Laplacian with the standard magnetic Schrödinger operator.

Assume that the Hermitian line bundle $(L, h^L)$ on $X$ is trivial, that is, $L = X \times \mathbb{C}$ and $|z|^2_{h^L} = |z|^2$ for $(x, z) \in X \times \mathbb{C}$. Then the Hermitian connection $\nabla^L$ can be written as $\nabla^L = d - iA$ with some real-valued 1-form $A$ (a magnetic potential). Its curvature $R^L$ is given by

$$
(4.13) \quad R^L = -iB,
$$

where $B = dA$ is a real-valued 2-form (a magnetic field). For the form $\omega$ defined by (1.1), we have

$$
\omega = \frac{1}{2\pi} B.
$$

Thus, $\omega$ is symplectic if and only if $B$ is non-degenerate (of full rank).

The associated Bochner Laplacian $\Delta^{L_p}$ is related with the magnetic Laplacian

$$
H^h = (ihd + A)^* (ihd + A), \quad h > 0
$$

by the formula

$$
\Delta^{L_p} = h^{-2} H^h, \quad h = \frac{1}{p}, \quad p \in \mathbb{N}.
$$

Let $B : TX \to TX$ be a skew-adjoint endomorphism such that

$$
(4.14) \quad B(u, v) = g(Bu, v), \quad u, v \in TX.
$$

Then we have

$$
(4.15) \quad J_0 = \frac{1}{2\pi} B, \quad J = B(B^* B)^{-1/2}, \quad J = -iB.
$$

Finally, the function $\tau$ and the constant $\mu_0$ are given by

$$
(4.16) \quad \tau(x) = \frac{1}{2} \text{Tr}(B^* B)^{1/2} = \text{Tr}^+(B).
$$
\[
\mu_0 = \inf_{u \in TX} \frac{|(B^*B)^{1/4}u|^2}{g} = \inf_{x \in X} |B(x)|.
\]

4.4. The 2D case. In this subsection, we compute explicitly the upper bounds of Theorem 1.7 for a two-dimensional magnetic Laplacian and show that, in this case, they are sharp and agree with the asymptotic expansions of [22, 18].

Thus, we assume that \(X\) is a Riemann surface and \(L\) is trivial. Then one can write
\[
(4.17) \quad B = b(x)dv_X,
\]
where \(dv_X = \sqrt{g}dx_1 \wedge dx_2\) is the Riemannian volume form. If \(B\) is of full rank, then \(b(x) \neq 0\) for all \(x \in X\). We will assume \(b(x) > 0\) for any \(x \in X\) and there exists a unique \(x_0\) such that
\[
(4.18) \quad b(x_0) = b_0 := \min_{x \in X} b(x),
\]
First, we observe that
\[
\tau(x) = b(x), \quad x \in X.
\]
Indeed, choose an orthonormal basis \(\{w_1\}\) of \(T^{(1,0)}_X\), consisting of eigenvectors of \(J_x\):
\[
(4.19) \quad J_xw_1 = a_1w_1,
\]
with some \(a_1 > 0\). Then \(e_1 = \frac{1}{\sqrt{2}}(w_1 + \bar{w}_1)\) and \(e_2 = \frac{i}{\sqrt{2}}(w_1 - \bar{w}_1)\) form an orthonormal basis of \(T_xX\) such that
\[
(4.20) \quad b(x) = B_x(e_1, e_2) = g(B_xe_1, e_2) = a_1.
\]
Finally, by (4.16), we complete the proof of (4.18):
\[
\tau(x) = \frac{1}{2} \text{Tr}(B_x^*B_x)^{1/2} = b(x).
\]
Next, we show that the operator \(\mathcal{F}_{1,2,x_0}\) given by (4.5) vanishes.

Lemma 4.1. If \(x_0\) is a minimum point of \(b\), then \(\mathcal{F}_{1,2,x_0} = 0\).

Proof. First, observe that, in the two-dimensional case, the formula (4.5) takes the form
\[
(4.21) \quad \mathcal{F}_{1,2,x_0} = 4 \left\langle R'^X_{x_0} \left( \frac{\partial}{\partial \bar{z}_1}, \frac{\partial}{\partial \bar{z}_1} \right) \frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_1} \right\rangle
+ \left\langle (\nabla X \nabla X J)_{(\mathbb{R}, \mathbb{R})} \frac{\partial}{\partial \bar{z}_1}, \frac{\partial}{\partial \bar{z}_1} \right\rangle + \frac{\sqrt{-1}}{4} \text{tr}_{TX} \left( \nabla X \nabla X (J^*J) \right)_{(\mathbb{R}, \mathbb{R})}
\]
we get first term in (4.21) vanishes: identities (4.19) hold at \( x \) where (4.23)

So, for the first term in (4.22), we get

By (4.13) and (4.17), we have

Recall that we fix an oriented orthonormal basis \( \{ e_1, e_2 \} \) of \( T_{x_0}X \) and identify \( B^{T_{x_0}X}(0, a_X) \) with \( B^X(x_0, a_X) \) by the exponential map \( \exp^X : T_{x_0}X \to X \), getting a local frame \( \{ e_1, e_2 \} \) on \( B^X(x_0, a_X) \). We also assume that the identities (4.19) hold at \( x_0 \).

First, observe that, since the curvature \( R^{T_{x_0}X}(u, v) \) is skew-symmetric, the first term in (4.21) vanishes:

By (2.6), since \( (\nabla X \nabla X J)_{(R, R)} \) is skew-adjoint, for the first term in the second line of (4.21), we get

By (1.13) and (4.17), we have

where \( g_{jk}(Z) = \langle e_j(Z), e_k(Z) \rangle \), \( j, k = 1, 2 \).

Since \( (\partial^a b)_{x_0} = 0 \) for \( |a| = 1 \) and \( g_{ij}(0) = \delta_{ij} \), for the first term in (4.22), we get

By (1.31)], we have

So, for the first term in (4.22), we get

(4.23)  

\[
\int \frac{1}{2} (\nabla^X \nabla X J)^2 \frac{\partial}{\partial z_1} + \frac{1}{2} \left( \nabla^X \nabla X J, \frac{\partial}{\partial z_1} \right) b_1^L \nabla_{x_0} \frac{\partial}{\partial z_1} \right). 
\]
For the second term in (4.22), by (4.19) and (4.20), we get

\begin{equation}
(4.24) \quad \frac{i}{6} \langle R^T X (R, e_1) R, J e_2 \rangle - \langle R^T X (R, e_2) R, J e_1 \rangle = - \frac{1}{6} b_0 [\langle R^T X (R, e_1) R, e_1 \rangle + \langle R^T X (R, e_2) R, e_2 \rangle]
\end{equation}

Combining (4.23) and (4.24), for the first term in the second line of (4.21), we get

\begin{equation}
(4.25) \quad \langle (\nabla^X \nabla^X J)(R, R) \frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_1} \rangle = \sum_{|\alpha|=2} (\partial^\alpha b) x_0 Z^\alpha \alpha !.
\end{equation}

By (1.5) (see also [36, (1.94)]), for the second term in the second line of (4.21), we have

\begin{equation}
(4.26) \quad \frac{i}{4} \text{tr}_{T X} \left( \nabla^X \nabla^X (J J) \right)_{(R, R)} = - \sum_{|\alpha|=2} (\partial^\alpha b) x_0 Z^\alpha \alpha !.
\end{equation}

Combining (4.25) and (4.26), we infer that the second line in (4.21) vanishes:

\begin{equation}
\langle (\nabla^X \nabla^X J)(R, R) \frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_1} \rangle + \frac{\sqrt{-1}}{4} \text{tr}_{T X} \left( \nabla^X \nabla^X (J J) \right)_{(R, R)} = 0.
\end{equation}

Finally, the third line in (4.21) vanishes, since for any $U \in T_{x_0} X$,

\begin{equation}
\nabla^X_U J = 0.
\end{equation}

Indeed, observe that $\nabla^X_U J$ is skew-symmetric, and

\begin{equation}
(4.27) \quad \langle (\nabla^X_U J) e_1, e_2 \rangle = U \langle J e_1, e_2 \rangle - \langle J \nabla^X_U e_1, e_2 \rangle - \langle J e_1, \nabla^X_U e_2 \rangle.
\end{equation}

Using (4.15), (4.14) and (4.17), we show that the first term in (4.27) vanishes:

$U \langle J e_1, e_2 \rangle_{x_0} = -iU \langle [B e_1, e_2] \rangle_{x_0} = -iU \langle B(e_1, e_2) \rangle_{x_0} = -iU \langle b \sqrt{\det g} \rangle_{x_0} = 0$.

The last equality holds, because $Ub(x_0) = 0$ since $x_0$ is a minimum, and $U \langle \sqrt{\det g} \rangle_{x_0} = 0$ by properties of normal coordinates.

By properties of normal coordinates, we have

$\langle \nabla^X_U e_1 \rangle_{x_0} = \langle \nabla^X_U e_2 \rangle_{x_0} = 0$.

Therefore, the last two terms in (4.27) vanish. This completes the proof of the lemma.

Thus, $J_{1,2,x_0}(Z, Z') = 0$ and the model Toeplitz operator $\mathcal{T}_{x_0}$ in $L^2(T_{x_0} X)$ has the form

$\mathcal{T}_{x_0} = \mathcal{P}_{x_0} q_{x_0}(Z) \mathcal{P}_{x_0}$,

where

$q_{x_0}(Z) = \left( \frac{1}{2} \text{Hess} b(x_0) Z, Z \right), \quad Z \in T_{x_0} X$.

The spectrum of $\mathcal{T}_{x_0}$ is computed in (4.12). Therefore, if we denote

$a = \text{Tr} \left( \frac{1}{2} \text{Hess} b(x_0) \right)^{1/2}, \quad d = \det \left( \frac{1}{2} \text{Hess} b(x_0) \right)$,
by Theorem 1.7 we obtain the estimate
\[ \lambda_j(\Delta^{L^p}) \leq pb_0 + \left[ \frac{2d^{1/2}}{b_0} j + \frac{a^2}{2b_0} \right] + C_j p^{-1/2} \]
with some \( C_j > 0 \), which is sharp and agrees with the asymptotic expansions of [22] [18].

REFERENCES

[1] Ali, S. T., Englis, M.: Quantization methods: a guide for physicists and analysts. Rev. Math. Phys. 17, 391–490 (2005)
[2] Bargmann, V.: On a Hilbert space of analytic functions and an associated integral transform. Comm. Pure Appl. Math. 14, 187–214 (1961)
[3] Berezin, F.A.: Wick and anti-Wick symbols of operators. Mat. Sb. (N.S.) 86(128), 578–610 (1971)
[4] Berezin, F.A.: General concept of quantization. Commun. Math. Phys. 40, 153174 (1975)
[5] Bordemann, M., Meinrenken, E., Schlichenmaier, M.: Toeplitz quantization of Kähler manifolds and \( \mathfrak{g}(N), \ N \to \infty \) limits. Comm. Math. Phys. 165, 281–296 (1994)
[6] Borthwick, D., Paul, T., Uribe, A.: Semiclassical spectral estimates for Toeplitz operators. Ann. Inst. Fourier (Grenoble), 48, 1189–1229 (1998)
[7] Borthwick, D., Uribe, A.: Almost complex structures and geometric quantization. Math. Res. Lett. 3, 845–861 (1996)
[8] Boutet de Monvel, L., Guillemin, V.: The spectral theory of Toeplitz operators. Ann. Math. Studies, Nr. 99, Princeton University Press, Princeton, NJ (1981)
[9] Charles, L.: Berezin-Toeplitz operators, a semi-classical approach. Comm. Math. Phys., 239, 1–28 (2003)
[10] Charles, L.: Quasimodes and Bohr-Sommerfeld conditions for the Toeplitz operators. Comm. Partial Differential Equations 28, 1527–1566 (2003)
[11] Charles, L.: Symbolic calculus for Toeplitz operators with half-forms Journal of Symplectic Geometry 4, 171–198 (2006)
[12] Charles, L.: Quantization of compact symplectic manifolds. J. Geom. Anal. 26, 2664–2710 (2016)
[13] Dai, X., Liu, K., Ma, X.: On the asymptotic expansion of Bergman kernel. J. Differential Geom. 72, 1–41 (2006)
[14] Deleporte, A.: Low-energy spectrum of Toeplitz operators: the case of wells. J. Spectr. Theory 9, 79–125 (2019)
[15] Deleporte, A.: Low-energy spectrum of Toeplitz operators with a miniwell, Preprint arXiv:1610.05902 (2016)
[16] Engliš, M.: An excursion into Berezin-Toeplitz quantization and related topics. In: Quantization, PDEs, and geometry, Oper. Theory Adv. Appl., 251, Adv. Partial Differ. Equ. (Basel), pp. 69–115, Birkhäuser/Springer, Cham (2016)
[17] Guillemin, V., Uribe A.: The Laplace operator on the nth tensor power of a line bundle: eigenvalues which are uniformly bounded in n. Asymptotic Anal. 1, 105–113 (1988)
[18] Helffer, B., Kordyukov, Yu. A.: Semiclassical spectral asymptotics for a two-dimensional magnetic Schrödinger operator: The case of discrete wells. In: Spectral Theory and Geometric Analysis; Contemp. Math. 535, pp. 55–78; AMS, Providence, RI (2011)
[19] Helffer, B., Kordyukov, Yu. A.: Semiclassical spectral asymptotics for a two-dimensional magnetic Schrödinger operator II: The case of degenerate wells. Comm. Partial Differential Equations 37, 1057–1095 (2012)
[20] Helffer, B., Kordyukov, Yu. A.: Semiclassical spectral asymptotics for a magnetic Schrödinger operator with non-vanishing magnetic field. In: Geometric Methods in Physics (Bialowieza, Poland, 2013), pp. 259–278, Birkhäuser, Basel (2014)
[21] Helffer, B., Kordyukov, Yu. A.: Accurate semiclassical spectral asymptotics for a two-dimensional magnetic Schrödinger operator. Ann. Henri Poincaré 16, 1651–1688 (2015)
[22] Helffer, B., Morame, A.: Magnetic bottles in connection with superconductivity. J. Funct. Anal. 185, 604–680 (2001)
[23] Helffer, B., Robert, D.: Puits de potentiel généralisés et asymptotique semi-classique. Ann. Inst. H. Poincaré Phys. Théor. 41, 291–331 (1984)
[24] Helffer, B., Sjöstrand, J.: Multiple wells in the semiclassical limit. I. Comm. Partial Differential Equations 9, 337–408 (1984)
[25] Hsiao, C.-Y., Marinescu G.: Berezin-Toeplitz quantization for lower energy forms, Comm. Partial Differential Equations. 42, 895–942 (2017)
[26] Ioos, L., Lu, W., Ma, X., Marinescu, G.: Berezin-Toeplitz quantization for eigenstates of the Bochner-Laplacian on symplectic manifolds, J Geom Anal (2018). https://doi.org/10.1007/s12220-017-9977-y.
[27] Kordyukov, Yu. A.: $L^p$-theory of elliptic differential operators on manifolds of bounded geometry. Acta Appl. Math. 23, 223–260 (1991)
[28] Kordyukov, Yu. A.: On asymptotic expansions of generalized Bergman kernels on symplectic manifolds. (Russian) Algebra i Analiz 30, no. 2, 163–187 (2018); translation in St. Petersburg Math. J. 30, no. 2, 267–283 (2019)
[29] Kordyukov, Yu. A., Ma, X., Marinescu, G.: Generalized Bergman kernels on symplectic manifolds of bounded geometry. Comm. Partial Differential Equations 44, 1037–1071 (2019)
[30] Le Floch, Y.: Singular Bohr-Sommerfeld conditions for 1D Toeplitz operators: elliptic case. Communications in Partial Differential Equations 39, 213–243 (2014)
[31] Le Floch, Y.: Singular Bohr-Sommerfeld conditions for 1D Toeplitz operators: hyperbolic case. Anal. PDE 7, 1595–1637 (2014)
[32] Lu, W., Ma, X., Marinescu, G.: Donaldson’s $Q$-operators for symplectic manifolds. Sci. China Math. 60, 1047–1056 (2017)
[33] Ma, X.: Geometric quantization on Kähler and symplectic manifolds. In: Proceedings of the International Congress of Mathematicians. Volume II, pp. 785–810, Hindustan Book Agency, New Delhi (2010)
[34] Ma, X., Marinescu, G.: The Spin$^c$ Dirac operator on high tensor powers of a line bundle. Math. Z. 240, 651–664 (2002)
[35] Ma, X., Marinescu, G.: Holomorphic Morse inequalities and Bergman kernels. Progress in Mathematics, 254. Birkhäuser Verlag, Basel (2007)
[36] Ma, X., Marinescu, G.: Generalized Bergman kernels on symplectic manifolds. Adv. Math. 217, 1756–1815 (2008)
[37] Ma, X., Marinescu, G.: Toeplitz operators on symplectic manifolds. J. Geom. Anal. 18, 565–611 (2008)
[38] Raymond, N., Vũ Ngọc, S.: Geometry and spectrum in 2D magnetic wells. Ann. Inst. Fourier 65, 137–169 (2015)
[39] Raymond, N.: Bound states of the magnetic Schrödinger operator. EMS Tracts in Mathematics, 27. European Mathematical Society (EMS), Zürich (2017)
[40] Reed, M., Simon, B.: Methods of modern mathematical physics. IV. Analysis of operators. Academic Press [Harcourt Brace Jovanovich, Publishers], New York-London (1978)
[41] Schlichenmaier, M.: Berezin-Toeplitz quantization for compact Kähler manifolds. A review of results. Adv. Math. Phys., Art. ID 927280, 38 pp. (2010)
[42] Simon, B.: Semiclassical analysis of low lying eigenvalues. I. Nondegenerate minima: asymptotic expansions. Ann. Inst. H. Poincaré Sect. A (N.S.) 38, 295–308 (1983)
[43] Zelditch, S.: Index and dynamics of quantized contact transformations. Ann. Inst. Fourier (Grenoble) 47, 305–363 (1997)

Institute of Mathematics with Computing Centre, Ufa Federal Research Centre of Russian Academy of Sciences, 112 Chernyshevsky str., 450008 Ufa, Russia, ORCID: 0000-0003-2957-2873

E-mail address: yurikor@matem.anrb.ru