ON IWASE’S MANIFOLDS

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Abstract. In [Iw2] Iwase has constructed two 16-dimensional manifolds $M_2$ and $M_3$ with LS-category 3 which are counter-examples to Ganea’s conjecture: $\text{cat}_{LS}(M \times S^n) = \text{cat}_{LS} M + 1$. We show that the manifold $M_3$ is a counter-example to the logarithmic law for the LS-category of the square of a manifold: $\text{cat}_{LS}(M \times M) = 2 \text{cat}_{LS} M$. Also we construct a map of degree one

$$f : 2(M_3 \times S^2 \times S^{14}) \# -(M_2 \times S^2 \times S^{14}) \rightarrow M_2 \times M_3$$

which reduces Rudyak’s conjecture to the question whether $\text{cat}_{LS}(M_2 \times M_3) \geq 5$ and show that $\text{cat}_{LS}(M_2 \times M_3) \geq 4$.

1. Introduction

The Lusternik-Schnirelmann category $\text{cat}_{LS} X$ is a celebrated numerical invariant of topological spaces $X$ which was introduced in the late 30s of the last century. By definition $\text{cat}_{LS} X$ is the minimal number $k$ such that $X$ can be covered by $k+1$ open sets $U_0, \ldots, U_k$ such that each $U_i$ is contractible in $X$. This invariant is of special importance when $X$ is a closed manifold, since it brings a lower bound on the number of critical points of smooth functions on $X$ [CLOT] Section 1.3.

It turns out that $\text{cat}_{LS}$ behaves differently with respect to two basic operations on manifolds, the connected sum and the cartesian product. In the case of connected sum there is a natural formula [DS1], [DS2]:

$$(*) \quad \text{cat}_{LS}(M \# N) = \max\{\text{cat}_{LS} M, \text{cat}_{LS} N\}.$$ 

In the case of the product the LS-category behavior is weird. There was the upper bound formula

$$\text{cat}_{LS}(X \times Y) \leq \text{cat}_{LS} X + \text{cat}_{LS} Y$$

since the late 30s [Ba], [F]. Since then there was a natural question for which spaces $X$ and $Y$ this upper bound is attained. In particular, there was a longstanding conjecture of Ganea that

$$\text{cat}_{LS}(X \times S^n) = \text{cat}_{LS} X + 1$$

Date: August 5, 2021.

2000 Mathematics Subject Classification. Primary 55M30; Secondary 53C23, 57N65.
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for all $X$. The Ganea Conjecture was verified for many classes of spaces $X$, yet it turned out to be false. At the end of the last century Noiro Iwase had constructed counterexamples to Ganea’s conjecture, first when $X$ is a finite complex [Iw] and then when $X$ is a closed manifold [Iw1], [Iw2]. He constructed two 16-dimensional manifolds denoted by $M_2$ and $M_3$ satisfying

$$\text{cat}_{LS}(M_i \times S^n) = \text{cat}_{LS} M_i = 3$$

for sufficiently large $n$, $i = 2, 3$. Also Iwase proved that

$$\text{cat}_{LS}(M \times S^n) = \text{cat}_{LS} M$$

for all $n$ either with $M = M_3$ or with $M = M_3 \times S^1$. The reason that Iwase manifolds have indexes 2 and 3 is that their constructions are related to the 2-primary and the 3-primary components of the group $\pi_{13}(S^3) = \mathbb{Z}_2 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_3$ respectively.

In this paper we exhibit a relation of Iwase’s examples to the following two intriguing problems in the LS-category. The first is a problem from [Ru1] which is also known as the Rudyak Conjecture:

For a degree one map $f : M \to N$ between closed manifolds, $\text{cat}_{LS} M \geq \text{cat}_{LS} N$.

The second is the question (which I also attribute to Yu. Rudyak):

Does the equality $\text{cat}_{LS}(M \times M) = 2 \text{cat}_{LS} M$ hold true for closed manifolds $M$?

There were partial results on both problems. Thus, the Rudyak conjecture was proven in some special cases [Ru1], [Ru2], [DSc]. A finite 2-dimensional complex that does not satisfy the equality $\text{cat}_{LS}(X \times X) = 2 \text{cat}_{LS} X$ were constructed in [H], [St2].

In this paper we connect both problems to Iwase’s examples. In particular, we prove the following

1.1. Theorem.

$$\text{cat}_{LS}(M_3 \times M_3) \leq 5 < 2 \text{cat}_{LS} M_3.$$

We connect the Rudyak Conjecture with computation of $\text{cat}_{LS}(M_2 \times M_3)$. Namely, we show that if $\text{cat}_{LS}(M_2 \times M_3) \geq 5$, then there is a counterexample to the Rudyak Conjecture. In this paper we managed to prove only the inequality $\text{cat}_{LS}(M_2 \times M_3) \geq 4$.

2. Preliminaries

2.1. Ganea-Schwarz approach to the LS-category. An element of an iterated join $X_0 \ast X_1 \ast \cdots \ast X_n$ of topological spaces is a formal linear combination $t_0 x_0 + \cdots + t_n x_n$ of points $x_i \in X_i$ with $\sum t_i = 1$, $t_i \geq 0$, in which all terms of the form $0 x_i$ are dropped. Given fibrations
The fiberwise join of fibrations $f_0, \ldots, f_n$ is the fibration

$$f_0 * \cdots * f_n : X_0 * \cdots * X_n \to Y$$

defined by taking a point $t_0x_0 + \cdots + t_nx_n$ to $f_i(x_i)$ for any $i$. As the name ‘fiberwise join’ suggests, the fiber of the fiberwise join of fibrations is given by the join of fibers of fibrations.

When $X_i = X$ and $f_i = f : X \to Y$ for all $i$ the fiberwise join of spaces is denoted by $s^{-1}_Y \cdot X$ and the fiberwise join of fibrations is denoted by $s^{-1}_Y \cdot f$. For a path connected topological space $X$, we turn an inclusion of a point $* \to X$ into a fibration $p_0^Y : G_0X \to X$. The $n$-th Ganea space of $X$ is defined to be the space $G_nX = s^{-1}_X \cdot G_0X$, while the $n$-th Ganea fibration $p_n^X : G_nX \to X$ is the fiberwise join of fibrations $p_0^X : G_0X \to X$. Thus, the fiber $F_0X$ of $p_0^X$ is the iterated join product $F_nX = s^{-1}_X \cdot \Omega X$ of the loop space of $X$.

The following theorem was proven by D. Stanley [St] in greater generality.

2.1. Theorem. Let $X$ be a connected CW-complex. Then $\text{cat}_{\text{LS}}(X) \leq n$ if and only if the fibration $p_n^X : G_nX \to X$ admits a section.

There is a chain of inclusions $G_0X \subset G_1X \subset G_2X \subset \ldots$ such that each $p_n^X$ is a restriction of $p_{n+1}^X$ to $G_nX$.

We recall that a map $f : X \to Y$ is called an $n$-equivalence if it induces isomorphisms of homotopy groups $f_* : \pi_i(X) \to \pi_i(Y)$ for $i < n$ and an epimorphism for $i = n$.

2.2. Proposition ([DS2]). Let $f : X \to Y$ be an $s$-equivalence of pointed $r$-connected CW-complexes with $r \geq 0$. Then the induced map $f_k : F_kX \to F_kY$ of the fibers of the $k$-th Ganea spaces is a $(k(r + 1) + s - 1)$-equivalence.

The following theorem was proven by D. Stanley [St] Theorem 3.5].

2.3. Theorem. Let $X$ be a connected CW-complex with $\text{cat}_{\text{LS}} X = k > 0$. Then $\text{cat}_{\text{LS}} X^{(s)} \leq k$ for any $s$, where $X^{(s)}$ is the $s$-skeleton of $X$.

Proof. Since the inclusion $j : X^{(s)} \to X$ is an $s$-equivalence, by Proposition 2.2 the induced map $j_k : F_kX^{(s)} \to F_kX$ of the fibers of the $k$-th Ganea fibrations is a $(k + s - 1)$-equivalence. Since $s \leq k + s - 1$, this implies that a section of $p_k^X$ defines a section of $p_k^{X^{(s)}}$. \hfill $\square$
2.2. **Berstein-Hilton invariant.** The Ganea spaces \( G_nX \) admit a reasonable homotopy theoretic description \([CLOT]\). We mention that \( G_1X \) is homotopy equivalent to \( \Sigma \Omega X \) where \( \Omega \) denote the loop space and \( \Sigma \) the reduced suspension. Moreover, \( p_1^X \) is homotopic to the composition \( e \circ h \) where \( h : G_1X \to \Sigma \Omega X \) is a homotopy equivalence and \( e : \Sigma \Omega X \to X \) is the evaluation map, \( e(\phi, t) = \phi(t) \).

There is a natural inclusion \( i_Y : Y \to \Omega \Sigma Y \) which takes \( y \in Y \) to the meridian loop \( \psi_y \) through \( y \). Note that \( i_\Omega X \) defines a section of the map \( \Omega(e) : \Omega \Sigma \Omega X \to \Omega X \). This proves the following

2.4. **Proposition.** The loop fibration \( \Omega p_n^X : \Omega G_nX \to \Omega X \) admits a canonical section \( s : \Omega p_n^X \to \Omega G_nX \).

2.5. **Corollary.** The induced map \( (p_n^X)_* : \pi_k(G_nX) \to \pi_k(X) \) has a natural splitting for all \( n \geq 2 \).

We denote by \( r_* : \pi_k(G_nX) \to \pi_k(F_nX) \) the projection defined by the above splitting.

2.6. **Definition.** \([BH]\) Let \( \alpha \in \pi_k(X) \). We define the Berstein-Hilton invariant of \( \alpha \) as the family

\[
H_k(\alpha) = \{ r_* \sigma_*(\alpha) \in \pi_k(F_nX) \mid \sigma : X \to G_kX \text{ is a section}. \}
\]

We use notation \( H^p_k(\alpha) \in H_k(\alpha) \) for a representative of the homotopy class \([r_* \sigma_*(\alpha)] \in \pi_k(F_nX)\).

Since \( e : \Sigma \Omega S^n \to S^n \) has a unique homotopy section, \( H_1([f]) \) consists of one element for any \( f : S^k \to S^n \).

By the James decomposition formula \( \Sigma \Omega S^2 \) is homotopy equivalent to the wedge \( \bigvee_{i=1}^{\infty} S^{i+1} \) and for any \( \alpha \in \pi_k(S^2) \) the Berstein-Hilton invariant \( H_1(\alpha) \) is the collection of the \( j \)-th James-Hopf invariants (see \([CLOT]\), Section 6.2) \( h_j(\alpha) \in \pi_k(S^{j+1}) \), \( j \geq 2 \). Thus, \( H_1(\eta) = 1 \in \pi_3(S^3) = \mathbb{Z} \) for the Hopf bundle \( \eta : S^3 \to S^2 \).

The Berstein-Hilton invariant can be helpful in determining whether

\[
\text{cat}_{LS}(X \cup_{\alpha} D^{n+1}) \leq \text{cat}_{LS} X
\]

where \( \alpha : \partial D^{n+1} \to Y \) is the attaching map \([BH],[lw],[CLOT]\):

2.7. **Theorem.** Let \( \text{cat}_{LS} X = k \). If \( H_k(\alpha) \) contains 0, then

\[
\text{cat}_{LS}(X \cup_{\alpha} D^{n+1}) \leq k.
\]

The converse holds true whenever \( \dim X \leq k + n - 2 \).
2.3. **Fibration-Cofibration Lifting problem.** Let

\[
\begin{array}{ccc}
FY & \xrightarrow{i} & \bar{Y} \\
& s & \downarrow g \\
S^n & \xrightarrow{f} & X & \xrightarrow{j} & C(f)
\end{array}
\]

be a homotopy commutative diagram where the top row is a fibration \( p : \bar{Y} \to Y \) with the fiber \( FY \) and the bottom row is a cofibration sequence. The Fibration-Cofibration Lifting problem is to find a homotopy lift \( \bar{s} : C(f) \to \bar{Y} \) of \( g \) extending \( s \).

2.8. **Proposition.** Suppose that the inclusion homomorphism \( i_* : \pi_n(FY) \to \pi_n(\bar{Y}) \) is injective. Then there is unique up to homotopy map \( \phi : S^n \to FY \) that makes the diagram

\[
\begin{array}{ccc}
FY & \xrightarrow{i} & \bar{Y} \\
\phi & \downarrow s & \downarrow g \\
S^n & \xrightarrow{f} & X & \xrightarrow{j} & C(f)
\end{array}
\]

homotopy commutative.

The Fibration-Cofibration Lifting problem has a solution if and only if \( \phi \) is null-homotopic.

**Proof.** A canonical homotopy of \( jf \) to a constant map defines a homotopy \( H : S^n \times I \to Y \) of \( gjf \) to a constant map. There is a lift \( \bar{H} : S^n \times I \to \bar{Y} \) with \( \bar{H}|_{S^n \times 0} \) homotopic to \( sf \). Then \( \phi = \bar{H}|_{S^n \times 1} \). The injectivity condition implies the uniqueness of \( \phi \).

Clearly, a lift \( \bar{s} : C(f) \to \bar{Y} \) defines the lift \( \bar{H} \) such that \( \phi = \bar{H}|_{S^n \times 1} \) is a constant map to \( FY \).

Now assume that \( \phi = \bar{H}|_{S^n \times 1} \) is null-homotopic. Then the homotopy \( \bar{H} \), and a contraction of \( \bar{H}|_{S^n \times 1} \) to a point in \( FY \) define \( \bar{s} \). \( \square \)

2.9. **Remark.** When \( \text{cat}_{LS} X \leq k \) we fix a section \( \sigma : X \to G_kX \) of \( p_k^X \) and consider the following Fibration-Cofubration Lifting problem.

\[
\begin{array}{ccc}
F_kY & \xrightarrow{i} & G_kY \\
\phi & \downarrow s & \downarrow = \\
S^n & \xrightarrow{f} & X & \xrightarrow{j} & C(f)
\end{array}
\]

where \( s = G_k(j) \circ \sigma \) and \( j : X \to Y = X \cup_\alpha D^{n+1} \) is the inclusion. Since \( p_k \) is a split surjection for homotopy groups, the condition of
Proposition 2.8 is satisfied. Moreover, \( \phi \) is homotopic to \( F_k(j) \circ H_k(\alpha) \). This explains the first part of Theorem 2.7. By Proposition 2.2, \( F_\alpha \) is the attaching map. 2.10. Theorem category of the product \( B \) of a map \( f \) be the quotient map. Then \( M \) is the image under the pull-back map \( \bar{g} : D^{t+1} \times S^r \to M \). Let \( \Psi : S^t \times S^r \to S^t \) be the restriction of \( \bar{g} \) to \( \partial D^{t+1} \times S^r \). Then \( M = S^r \cup \Psi (D^{t+1} \times S^r) \). We consider the standard CW complex structures on \( D^{t+1} = S^t \cup e^{t+1} \) and \( S^r = \ast \cup e^r \). Then \( D^{t+1} \times \ast = (S^t \times S^r) \cup (e^{t+1} \times \ast) \cup (e^{t+1} \times e^r) \). Thus, we obtain

\[
M = S^r \cup \alpha e^{t+1} \cup \psi e^{r+t+1}
\]

with \( \alpha = \Psi |_{S^t \times \ast} \) where the sphere \( S^r \) is identified with \( q^{-1}(x_0) \).

If \( q' : M' \to S^{t+1} \) is the pull-back of \( q \) with respect to the suspension of a map \( f : S^r \to S^t \), then \( \Psi' \) factors through \( \Psi \) and \( \alpha' = \alpha \circ f \).

Clearly, \( \text{cat}_{LS} M \leq 3 \). The category of \( M \) and in some cases the category of the product \( M \times S^n \) can be computed in terms of the attaching map \( \alpha \) in view of the following.

2.10. Theorem (LW2). Let \( t > r > 1 \) and \( H_1(\alpha) \neq 0 \). Then

- \( \text{cat}_{LS} M = 3 \) if and only if \( \Sigma^r H_1(\alpha) \neq 0 \).
- \( \Sigma^{n+r} H_1(\alpha) = 0 \) implies \( \text{cat}_{LS} (M \times S^n) = 3 \).
- \( \Sigma^{n+r+1} H_2(\alpha) \neq 0 \) implies \( \text{cat}_{LS} (M \times S^n) = 4 \).

2.5. Homotopy groups of spheres. We follow the notations from Toda’s book [1]. The Hopf bundles \( \eta : S^3 \to S^2 \), \( \nu : S^7 \to S^4 \), and \( \sigma : S^{15} \to S^8 \) produce by suspensions the elements \( \eta_n \in \pi_{n+1}(S^n) \), \( \nu_n \in \pi_{n+3}(S^n) \), and \( \sigma_n \in \pi_{n+7}(S^n) \). We use the notation \( \eta^2_n \) for the composition \( \eta_n \circ \eta_{n+1} : S^{n+2} \to S^n \) as well as for the generator of \( \pi_{n+2}(S^n) = \mathbb{Z}_2 \). The generator \( \epsilon_3 \) of the \( \mathbb{Z}_2 \) summand of \( \pi_{11}(S^3) \) and its suspensions produce generators \( \epsilon_n \in \pi_{n+8}(S^n) \).

2.11. Proposition. Let \( \phi = \eta^2_3 \circ \epsilon_5 \in \pi_{13}(S^2) \). Then \( \Sigma^5 \phi \neq 0 \) and \( \Sigma^6 \phi = 0 \).

Proof. For \( n \geq 2 \), \( \Sigma^n \phi = \eta^2_{n+3} \circ \epsilon_{n+5} = 4(\nu_{n+3} \circ \sigma_{n+6}) \) by (7.10) of [1]. By Theorem 7.3 (2) [1], \( \nu_{n+3} \circ \sigma_{n+6} \) generates the subgroup \( \mathbb{Z}_8 \subset \pi_{n+13}(S^{n+3}) \) for \( n = 2, 3, 4, 5 \). Hence, \( \Sigma^5 \phi \neq 0 \). By (7.20) in [1] \( \Sigma^6 \phi = 4(\nu_9 \circ \sigma_{12}) = 0 \).

We recall some facts about primary \( p \)-components \( \pi_i(S^n; p) \) of homotopy groups \( \pi_i(S^n) \) for odd prime \( p \). Namely, for \( i \in \{1, 2, \ldots, p - 1\}, m \geq 1 \),

\[
\pi_{2m+1+2i(p-1)-1}(S^{2m+1}; p) = \mathbb{Z}_p
\]
with the generators $\alpha_i(2m + 1)$ satisfying the condition $\Sigma^2\alpha_i(2m - 1) = \alpha_i(2m + 1)$. Using suspension we define $\alpha_i(n)$ for even $n$ as well. Then the group $\pi_{2(p-1)+1}(S^3; p) \cong \mathbb{Z}_p$ (see [T] Proposition 13.6.) for $2 \leq i \leq p$ is generated by $\alpha_1(p) \circ \alpha_{i-1}(2p)$. There is Serre isomorphism

$$\pi_i(S^{2m}; p) \cong \pi_{i-1}(S^{2m-1}; p) \oplus \pi_i(S^{4m-1}; p)$$

such that the suspension $\Sigma : \pi_{i-1}(S^{2m-1}; p) \to \pi_i(S^{2m}; p)$ defines the embedding of the first summand.

2.12. Proposition. Let $\psi = \alpha_1(3) \circ \alpha_2(6)$. Then $\Sigma^3\psi \neq 0$ and $\Sigma^4\psi = 0$.

Proof. By (13.6)' in [T] the group $\pi_{13}(S^3; 3) = \mathbb{Z}_3$ is generated by $\psi$. By Theorem 13.9 [T], $\pi_{14}(S^3; 3) = \mathbb{Z}_3$, $\pi_{16}(S^5; 3) = \mathbb{Z}_9$, and $\pi_{15}(S^5; 3) = \mathbb{Z}_9$. The exact sequence (13.2) from [T] in view of (13.6) produces the exact sequence

$$0 \to \pi_{14}(S^3; 3) \xrightarrow{\Sigma^2} \pi_{16}(S^5; 3) \to \mathbb{Z}_3 \to \pi_{13}(S^3; 3) \xrightarrow{\Sigma^2} \pi_{15}(S^5; 3) \to \mathbb{Z}_3$$

which implies that $\pi_{13}(S^3; 3) \xrightarrow{\Sigma^2} \pi_{15}(S^5; 3)$ is injective. Therefore, $\Sigma^2\psi$ generates a subgroup $\mathbb{Z}_3 \subset \mathbb{Z}_9 = \pi_{15}(S^5; 3)$. In particular, $\Sigma^2\psi \neq 0$.

By the Serre isomorphism, the suspension homomorphism $\pi_{15}(S^5; 3) \to \pi_{16}(S^6; 3)$ is a monomorphism. Hence, $\Sigma^3\psi \neq 0$.

The exact sequence (13.2) from [T] implies that the following sequence

$$\mathbb{Z}_3 \to \pi_{15}(S^5; 3) \xrightarrow{\Sigma^2} \pi_{17}(S^7; 3) \to 0$$

is exact. Since $\pi_{17}(S^7; 3) = \mathbb{Z}_3$ (Theorem 3.19 [T]), this implies that $\Sigma^4\psi = 0$. \qed

3. Iwase’s Examples

3.1. Manifold $M_2$. The $S^1$-action on $S^7$ defines a factorization of the Hopf bundle $\nu_4 : S^7 \to S^4$ through the $S^2$-bundle $h : \mathbb{C}P^3 \to S^4$. Iwase defined $M_2$ as the total space of the $S^2$-bundle $q_2 : M_2 \to S^{14}$ induced from $h$ by means of the suspension map $f_2 = \Sigma f_2'$ where $f_2'$ represents $\eta_3^2 \circ \epsilon_5 \in \pi_{13}(S^3)$. Then the gluing map $\Psi : S^{13} \times S^2 \to S^2$ for $M_2$ is the composition $\Psi_0 \circ f_2'$ where $\Psi_0 : S^3 \times S^2 \to S^2$ is the gluing map for $h$. Then the attaching map for the 14-cell in $M_2$ is the composition $\alpha = \alpha_0 \circ f_2'$ where $\alpha_0$ is the attaching map of the 4-cell in $\mathbb{C}P^2$. Thus, $\alpha$ represents $\eta \circ \eta_3^2 \circ \epsilon_5$:

$$S^{13} \xrightarrow{\epsilon_5} S^5 \xrightarrow{\eta_3} S^4 \xrightarrow{\eta_3} S^3 \xrightarrow{\eta} S^2.$$

The following is a minor refinement of Iwase’s theorem [Iw2].
3.1. **Proposition.** The manifold $M_2$ has the following properties:

1. $\text{cat}_{LS}(M_2 \times S^n) = 3$ for $n \geq 4$;
2. $\text{cat}_{LS} M_2 = 3$;
3. $\text{cat}_{LS}(M_2 \times S^1) = \text{cat}_{LS}(M_2 \times S^2) = 4$;
4. There is a map $f : S^{14} \times S^2 \to M_2$ of degree 2.

**Proof.** To prove (1)-(3) we show that $H_1(\alpha) = h_2(\alpha) = \phi$. Then by Proposition 2.11, $\Sigma^2 H_1(\alpha) \neq 0$, $\Sigma^6 H_1(\alpha) = 0$, and $\Sigma^5 h_2(\alpha) \neq 0$. Theorem 2.10 implies that $\text{cat}_{LS} M_2 = 3$, $\text{cat}_{LS}(M_2 \times S^n) = 3$ for $n \geq 4$ and $\text{cat}_{LS}(M_2 \times S^1) = \text{cat}_{LS}(M \times S^2) = 4$.

Let $i_X \to \Omega \Sigma X$ denote the natural inclusion. If $\beta = \eta \circ \gamma$ were $\gamma = \Sigma \gamma'$ is a suspension, $\gamma' : S^k \to S^r$, then the commutativity of diagram

\[
\begin{array}{ccc}
\Sigma \Omega \Sigma S^k \ & \xrightarrow{\eta_1} \ & \Sigma \Omega \Sigma S^r \\
\Sigma i_{S^k} \uparrow & & \downarrow \Sigma i_{S^r}
\end{array}
\]

implies that $H_1(\beta) = H_1(\eta) \circ \gamma$ and $h_j(\beta) = h_j(\eta) \circ \gamma$. If $\eta : S^3 \to S^2$, then $h_j(\eta) = 0$ for $j \geq 3$, $h_2(\eta) = H_1(\eta)$, and $h_2(\beta) = H_1(\beta)$.

Note that $\eta^2_3 \circ \epsilon_5$ is a suspension since by definition $\eta^2_3$ and $\epsilon_m$ are suspensions for $n > 2$ and $m > 3$, and for the Hopf map, $H_1(\eta) = 1$. Then, $h_2(\alpha) = H_1(\alpha) = H_1(\eta) \circ (\eta^2_3 \circ \epsilon_5) = \eta^2_3 \circ \epsilon_5 = \phi \in \pi_{13}(S^3)$ for the map $\phi$ defined in Proposition 2.11.

Proof of (4). Let $2 : S^{14} \to S^{14}$ be a map of degree 2. It induces the map of the pull-back manifold $f : M' \to M_2$ of degree 2. Note that $M'$ is the pull-back of $\mathbb{C}P^3$ with respect to the map $f_2 \circ q$ which defines zero element of $\pi_{14}(S^4)$, since $\Sigma(\eta^2_3 \circ \epsilon_5) \circ 2 = \eta^2_3 \circ \epsilon_6 \circ 2 = (2 \eta^2_3) \circ \epsilon_6 = 0$ in view of the equality $2 \eta_n = 0$ for $n > 2$. Therefore, $M'$ is homeomorphic to $S^{14} \times S^2$. \hfill \square

3.2. **Manifold** $M_3$. The manifold $M_3$ is defined as the total space of the $S^2$-bundle $q_3 : M_3 \to S^{14}$ induced from $h : \mathbb{C}P^3 \to S^4$ by means of the suspension map $f_3 = \Sigma f_3^4$ where $f_3^4 : S^{13} \to S^3$ is a map representing $\alpha_1(3) \circ \alpha_2(6)$. Then in the construction of $M_2$ the attaching map $\alpha = \alpha_0 \circ f_3$ where $\alpha_0$ is the attaching map of the 4-cell in $\mathbb{C}P^2$. Thus, $\alpha$ represents $\eta_2 \circ \alpha_1(3) \circ \alpha_2(6)$:

\[
\begin{array}{cccc}
S^{13} \xrightarrow{\Sigma \alpha_2(5)} S^6 & \xrightarrow{\alpha_1(3)} & S^3 & \xrightarrow{\eta_2} S^2.
\end{array}
\]

3.2. **Proposition.** The manifold $M_3$ has the following properties:

1. $\text{cat}_{LS}(M_3 \times S^n) = 3$ for $n \geq 2$;
(2) \( \text{cat}_{LS} M_3 = 3 \);
(3) There is a map \( f : S^{14} \times S^2 \to M_3 \) of degree 3.

Proof. Properties (1) and (2) were proven in [Iw1] by application of Theorem 2.10. Namely, it was shown that \( H_1(\alpha) = h_2(\alpha) = \psi \) and then Proposition 2.12 was applied.

The argument for this is similar to the argument in the proof of Proposition 3.1 with the difference is that \( \alpha_1 p_3 q : S^6 \to S^3 \) is a suspension. It turns out that \( \alpha_1 p_3 q \) is a co-H map and this is sufficient to get the equality \( H_1(\alpha) = H_1(\eta) \circ (\alpha_1(3) \circ \Sigma \alpha_2(5)) \).

Proof of (3) is similar to the proof of (4) in Proposition 3.1 and it is based on the fact that \( \alpha_1(3) \) has the order 3. \( \square \)

4. Category of the product

It is known [Iw],[SS] that in Ganea’s definition of the category of the product of two complexes \( X \times Y \) instead of the Ganea fibrations \( p_k : G_k(X \times Y) \to X \times Y \) one can take a map with the smaller domain

\[
\hat{G}_k(X \times Y) = \bigcup_{i+j=k} G_iX \times G_jY \subset G_kX \times G_kY.
\]

There is the natural projection \( \hat{p}_k : \hat{G}_k(X \times Y) \to X \times Y \) with fibers

\[
\hat{F}_k(X \times Y) = \bigcup_{i+j=k} F_iX \times F_jY \subset F_kX \times F_kY.
\]

It is known that \( \text{cat}_{LS}(X \times Y) \leq k \) if and only if \( \hat{p}_k \) admits a homotopy section [Iw]. Though it is not important for our main result, we give a sketch of proof that in the case of CW complexes \( X \) and \( Y \) a homotopy section of \( \hat{p}_k \) can be replaced by a section.

4.1. Proposition. For CW complexes \( X \) and \( Y \) the map

\[
\hat{p}_k : \hat{G}_k(X \times Y) \to X \times Y
\]

is a Serre fibration.

We recall that a map is called a Serre fibration if it satisfies the homotopy lifting property for CW complexes. This is equivalent to have the homotopy lifting property for \( n \)-cubes for all \( n \).

The proof of Proposition 4.1 is based on the following two facts.

We say that a map \( p : E \to B \) satisfies the Homotopy Lifting Property for a pair \( (X, A) \) if for any homotopy \( H : X \times I \to B \) with a lift \( H' : A \times I \to E \) of the restriction \( H|_{A \times I} \) and a lift \( H_0 \) of \( H|_{X \times 0} \) which agrees with \( H' \), there is a lift \( \tilde{H} : X \times I \to E \) of \( H \) which agrees with \( H_0 \) and \( H' \). The following is well-known [Ha]:
4.2. Theorem. Any Serre fibration $p : E \to B$ satisfies the Homotopy Lifting Property for CW complex pairs $(X, \Lambda)$.

We use the abbreviation ANE for absolute neighborhood extensors for the class of finite dimensional spaces. Such spaces can be characterized as those which are locally $n$-connected for all $n$.

4.3. Lemma (Pasting Lemma for Fibrations). Let $p : E \to B$ be a map between ANE spaces with closed subsets $E_1, E_2 \subset E$ such that all spaces $E_1, E_2, E_1 \cap E_2$ are ANE. Suppose that the restrictions $p_1 = p|_{E_1} : E_1 \to B$, $p_2 = p|_{E_2} : E_2 \to B$, and $p_0 = p|_{E_1 \cap E_2} : E_1 \cap E_2 \to B$, are Serre fibrations. Then the restriction $\hat{p} = p|_{E_1 \cup E_2} : E_1 \cup E_2 \to B$ is a Serre fibration.

Proof. Let $H : Z \times I \to B$ be a homotopy of a finite CW complex $Z$ with a fixed lift $\tilde{h} : Z \to E_1 \cup E_2$ of $H|_{Z \times \{0\}}$. Let $Z_1 = \tilde{h}^{-1}(E_1)$, $Z_2 = \tilde{h}^{-1}(E_2)$, and $Z_0 = \tilde{h}^{-1}(E_1 \cap E_2)$. If it happens to be that $Z_0$ is a CW complex we apply homotopy lifting property of $p_0$ to $H|_{Z_0 \times I}$. Then we apply Theorem 4.2 twice, first for to $p_1$ and then for $\hat{p}$ and we are done.

For general $Z_0$ we apply the standard trick sketched below. We use the ANE property to find a CW complex neighborhood $A$ of $Z_0$ and a map $\hat{h} : Z \to E_1 \cup E_2$ with $\hat{h}(A) \subset E_1 \cap E_2$, and with a small homotopy $\hat{h}_t : Z \to E_1 \cup E_2$ joining $\tilde{h}$ and $\hat{h}$ and stationary on $Z_0$. Using local contractibility property of $B$ we may achieve that the homotopy $\hat{h}_t$ is fiberwise. Thus, we obtain a lift of $H$ if our homotopy $H$ is stationary on a small neighborhood of $Z \times \{0\}$. Generally, we change the fiberwise homotopy $\hat{h}_t$ by pushing it along the paths of $H$ with such control that $\hat{h}_1(A) \subset E_1 \cap E_2$. \hfill \Box

Now the proof of Proposition 4.1 can be given by induction using the Pasting Lemma and the fact that for a CW-complex $X$ all the spaces $G_k X$ are ANE.

There are two important facts about $\hat{p}_k$ [16]: There is a lift $\mu : G_k(X \times Y) \to \hat{G}_k(X \times Y)$ of $p_k$ with respect to $\hat{p}_k$ and there is the inequality $\text{cat}_{LS} \hat{G}_k(X \times Y) \leq k$. The latter can be proven by the cone length estimate (see [SS]). The inequality $\text{cat}_{LS} \hat{G}_k(X \times Y) \leq k$ implies that there is a lift

$$\lambda = \lambda_{k,X,Y} : \hat{G}_k(X \times Y) \to G_k(X \times Y)$$

of $\hat{p}_k$ with respect to $p^{X \times Y}_k : G_k(X \times Y) \to X \times Y$. 
Since $p_k$ is a split surjection for homotopy groups, the lift $\mu$ ensures that $\hat{p}_k$ is also a split surjection for homotopy groups. Hence the inclusion homomorphism $\pi_n(\hat{F}_k(X \times Y)) \to \pi_n(\hat{G}_k(X \times Y))$ is injective for all $n$.

Since the inclusions $F_r Z \to F_{r+1} Z$ are null-homotopic for all $Z$ and $k$, there are natural maps
\[
\eta_{i,j} : F_i X \times F_j Y \to (F_{i+1} X \times F_j Y) \cup (F_i X \times F_{j+1} Y) \subset \hat{F}_{i+j+1}.
\]

Let $Q = S^2 \cup_{\alpha} D^{14}$ with $\alpha : S^{13} \to S^2$ from the construction of $M_3$.

4.4. **Theorem.** Iwase’s manifold $M_3$ satisfies the inequality
\[
\text{cat}_{LS}(M_3) \leq 5 < 2 \text{cat}_{LS} M_3.
\]

**Proof.** We recall that $M_3 = S^2 \cup_{\alpha} e^{14} \cup_{\psi} e^{16} = Q \cup e^{16}$. Consider the product CW-complex structure on $M_3 \times M_3$. Define a sequence of CW subspaces
\[
X_1 \subset X_2 \subset X_3 \subset X_4 = M_3 \times M_3
\]
as follows
\[
X_1 = (Q \times S^2) \cup (S^2 \times Q),
X_2 = X_1 \cup (M_3 \times *) \cup (*) \times M_3 = X_1 \cup (e^{16} \times *) \cup (\ast \times e^{16}),
X_3 = (M_3 \times S^2) \cup (S^2 \times M_3) \cup (Q \times Q) = X_2 \cup (e^{16} \times e^2) \cup (e^2 \times e^{16} \cup (e^{14} \times e^{14}),
X_4 = (M_3 \times Q) \cup (Q \times M_3) = X_2 \cup (e^{16} \times e^{14}) \cup (e^{14} \times e^{16}),
X_5 = M_3 \times M_3 = X_3 \cup (e^{16} \times e^{16}).
\]

It suffices to prove the inequality $\text{cat}_{LS} X_3 \leq 3$. Then since $X_4$ is obtained from $X_3$ by attaching cells to $X_3$, we get $\text{cat}_{LS} X_4 \leq 4$. Finally, $\text{cat}_{LS} X_4 \leq 5$. $\Box$

4.5. **Proposition.** $\text{cat}_{LS} X_3 \leq 3$.

**Proof.** Note that $X_3$ is obtained by attaching three cells to $X_2$, two cells of dimension 18 and one of dimension 28. The attaching map $\alpha$ defines the attaching map $\bar{\alpha}$ in
\[
Q \times Q = X_1 \cup_{\bar{\alpha}} e^{28}.
\]

We denote by $\bar{\alpha}$ the attaching map of the 28-cell to $X_2$. Thus $\bar{\alpha}$ is the composition $\bar{\alpha}$ and the inclusion $X_1 \subset X_2$. Let
\[
\bar{\psi} : S^{17} \to (M_3 \times *) \cup (Q \times S^2)
\]
denote the attaching map of the top cell in $M_3 \times S^2$. Note that
\[
X_2 = (((M_3 \times *) \cup (Q \times S^2)) \cup ((S^2 \times Q) \cup (* \times M_3))).
\]
Then the attaching maps of the 18-cells in $X_3$ can be presented as $ψ_− = i_− \circ ψ$ and $ψ_+ = i_+ \circ ψ$ where $i_±$ are two symmetric inclusions of $(M_3 \times *) \cup (Q \times S^2)$ into $X_2$.

We consider the embeddings $G_3M_3 \times * \to G_3M_3 \times G_0M_3$ and $G_2Q \times G_1S^2 \to G_2M_3 \times G_1M_3$ generated by the inclusions $*, S^2, Q \subset M_3$. Then

$\hat{X}_2 = (G_3M_3 \times *) \cup (G_2Q \times G_1S^2) \cup (G_1S^2 \times G_2Q) \cup (* \times M_3)$

is embedded in $\hat{G}_3(M_3 \times M_3)$. Let $\bar{i}$ denotes the embedding. There is a natural projection $p'$ of $\hat{X}_2$ onto $X_2 = (M_3 \times *) \cup (Q \times S^2) \cup (S^2 \times Q) \cup (* \times M_3)$ that makes a commutative diagram

\[
\begin{array}{ccc}
\hat{X}_2 & \xrightarrow{\bar{i}} & \hat{G}_3(M_3 \times M_3) \\
p' \downarrow & & \downarrow \bar{p}_3 \\
X_2 & \xrightarrow{\bar{e}} & M_3 \times M_3.
\end{array}
\]

We define a section $\hat{s} : X_2 \to \hat{X}$ of $p'$ as follows.

Let $ψ : S^{15} \to Q$ be the attaching map in the construction of $M_3 = S^2 \cup_α e^{14} \cup_ψ e^{16}$. It was proven in [Iw1] and explicitly exhibited in [Iw2] that there is a section $σ : Q \to G_2Q$ such that $H^2_2(ψ)$ is homotopic to the composition $β \circ Σ^2H_1(α)$ for some $β$. The section $σ$ extends to a section $σ' : M_3 \to G_3M_3$ (called a standard section in [Iw1]). Then we define

$\hat{σ} : X_2 \to X_2' = (G_2Q \times S^2) \cup (S^2 \times G_2Q) \cup (G_3M_3 \times pt) \cup (pt \times G_3M_3)$

to be the restriction of $σ' \times 1_{S^2} \cup 1_{S^2} \times σ'$ to $Y$. The space $X_2'$ has a natural inclusion $ξ : X_2' \to X_2$. We define

$s = \bar{i} \circ ξ \circ \hat{σ} : X_2 \to \hat{G}_3(M_3 \times M_3)$.

For each attaching map $f \in \{ψ_−, ψ_+, \bar{α}\}$ we consider the Fibration-\(\text{Cofibration Lifting problem}\)

\[
\begin{array}{ccc}
\hat{F}_3(M_3 \times M_3) & \xrightarrow{i} & \hat{G}_3(M_3 \times M_3) & \xrightarrow{\bar{p}_3} & M_3 \times M_3 \\
S^n & \xrightarrow{f} & X_2 & \xrightarrow{j} & C(f)
\end{array}
\]

In Lemma 4.6 and Lemma 4.7 we show that each of the three lifting problems has a solution. This defines a lift $\hat{s} : X_3 \to \hat{G}_3(M_3 \times M_3)$ of the inclusion $X_3 \subset M_3 \times M_3$ extending $s$.

Let $\hat{s} : X_3 \to \hat{G}_3(M_3 \times M_3)$ be such a lift. Then $λ \circ \hat{s} : X_3 \to G_3(M_3 \times M_3)$ is a lift of the inclusion $X_3 \subset M_3 \times M_3$ with respect to
$p_3 : G_3(M_3 \times M_3) \to M_3 \times M_3$ where $\lambda : \hat{G}_3(M_3 \times M_3) \to G_3(M_3 \times M_3)$ is a lift of $\hat{p}_3$ with respect to $p_3$. Since the inclusion $X_3 \to M_3 \times M_3$ is a $29$-equivalence, by Proposition 2.2 the inclusion of fibers $F_3X_3 \to F_3(M_3 \times M_3)$ is a $(2 \times 3 + 29 - 1)$-equivalence. In the pull-back diagram

$\begin{array}{ccc}
G_3X_3 & \to^q & Z \\
p' \downarrow & & p_3 \downarrow \\
X_3 & \subseteq & M_3 \times M_3.
\end{array}$

the lift $\lambda \tilde{s}$ defines a section $\nu' : X_3 \to Z$. The map $p^{X_3}$ factors as $p' \circ q$ where $q$ is a $34$-equivalence. Since $\dim X_3 = 28$, there is a lift of $\nu'$ to a section of $G_3X_3$. Therefore, $\text{cat}_{LS} X_3 \leq 3$.

4.6. Lemma. There are lifts $s_{1, 2} : C(\psi_{\pm}) \to \hat{G}_3(M_3 \times M_3)$ with respect to $\hat{p}_3 : \hat{G}_3(M_3 \times M_3) \to M_3 \times M_3$ of the inclusion $C(\psi_{\pm}) \subset M_3 \times M_3$ that extend $s$.

Proof. Since the inclusion $i : \hat{F}_3(M_3 \times M_3) \to \hat{G}_3(M_3 \times M_3)$ is injective on the homotopy groups, in view of Proposition 2.3 it suffices to define a null-homotopic map $\phi_{\pm} : S^{17} \to \hat{F}_3(M_3 \times M_3)$ such that $i \circ \phi_{\pm}$ is homotopic to $s \circ \psi_{\pm}$.

The construction of $\phi_{\pm}$ is based on Iwase’s proof of the inequality $\text{cat}_{LS}(M_3 \times S^2) \leq 3$ [Iw1]. We consider the homotopy commutative diagram from Proposition 3.7 in [Iw1] amended by the right column

$\begin{array}{ccc}
F_2M_3 \ast F_0S^2 & \longrightarrow & G_3M_3 \times pt \cup G_2M_3 \times G_1S^2 \\
\downarrow j & & \downarrow \xi \\
F_2Q \ast S^1 & \longrightarrow & G_3M_3 \times pt \cup G_2Q \times S^2 \\
\downarrow H_2^\sigma(\psi) \ast 1_{S^1} & & \downarrow \sigma_1 \\
S^{15} \ast S^1 & \longrightarrow & M_3 \times pt \cup Q \times S^2 \\
\bar{\psi} & & \bar{\sigma} \\
& & \bar{\sigma} \\
& & \bar{\sigma} \\
\end{array}$

where $j$ is generated by the inclusions $Q \to M_3$ and $S^1 \to \Omega \Sigma S^1 = F_0S^2$, $\sigma_1$ is the restriction of $\bar{\sigma}$, and $\psi$ is taken from Proposition 3.2.

By Proposition 2.12 $\Sigma^4 \psi = 0$. Therefore,

$H_2^\sigma(\psi) \ast 1_{S^1} = \Sigma^2H_2^\sigma(\psi) = \Sigma^2\beta \circ \Sigma^4H_1(\alpha) = 0$. 


Therefore, $\theta = \tilde{j} \circ (H_\tau^s(\psi) \ast 1_{s_1})$ is null-homotopic. Thus, we have a homotopy commutative diagram

$$
\begin{array}{c}
\hat{F}_3(M_3 \times M_3) \\
\phi \downarrow \\
S^{17} \\
\psi \downarrow \\
X_2
\end{array} \xrightarrow{i} \begin{array}{c}
\hat{G}_3(M_3 \times M_3) \\
\phi \downarrow \\
S^{17} \\
\psi \downarrow \\
X_2
\end{array}
$$

with null-homotopic $\phi = j_\pm \circ \eta_{2,0} \circ \theta$. Here $\eta_{2,0} : F_2M_3 \ast F_0S^2 \rightarrow \hat{F}_3(M_3 \times S^2)$ and $j_\pm : \hat{F}_3(M_3 \times S^2) \rightarrow \hat{F}_3(M_3 \times M_3)$ are the inclusions induced by $i_\pm$. By Proposition 2.8 there are the required lifts $s_1$ and $s_2$. □

4.7. Lemma. There is lifts $s_3 : C(\hat{\alpha}) \rightarrow \hat{G}_3(M_3 \times M_3)$ with respect to $\hat{p}_3 : \hat{G}_3(M_3 \times M_3) \rightarrow M_3 \times M_3$ of the inclusion $C(\hat{\alpha}) \subset M_3 \times M_3$ that extends $s$.

Proof. The construction of $s_3$ is based on Harper’s proof \[H\] of the inequality $\text{cat}_{LS}(Q \times Q) \leq 3$. We consider the homotopy commutative diagram from \[H\] completed by the right commutative square

$$
\begin{array}{ccc}
F_1Q \ast F_1Q & \longrightarrow & G_2Q \times G_1Q \cup G_1Q \times G_2Q \\
\downarrow H^s_1(\alpha) \ast H^s_1(\alpha) & & \downarrow s' \\
S^{13} \ast S^{13} & \longrightarrow & Q \times S^2 \cup S^2 \times Q \\
\alpha \downarrow & & \downarrow s \\
& & X_2
\end{array}
$$

where $s'$ is the restriction of $s$. By the Barratt-Hilton formula (\[T\], Proposition 3.1) we obtain

$$H^s_1(\alpha) \ast H^s_1(\alpha) = \Sigma(H_1(\alpha) \wedge H_1(\alpha)) = \Sigma(\Sigma^2H_1(\alpha) \circ \Sigma^3H_1(\alpha)).$$

Since $\Sigma^4H_1(\alpha) = 0$, we obtain that the map $\theta = H^s_1(\alpha) \ast H^s_1(\alpha)$ is null-homotopic.

Again, we apply Proposition 2.8 to the homotopy commutative diagram

$$
\begin{array}{c}
\hat{F}_3(M_3 \times M_3) \\
\phi \downarrow \\
S^{28} \\
\alpha \downarrow \\
X_2
\end{array} \xrightarrow{} \begin{array}{c}
\hat{G}_3(M_3 \times M_3) \\
\phi \downarrow \\
S^{28} \\
\alpha \downarrow \\
X_2
\end{array}
$$

with null-homotopic $\phi = \tilde{j} \circ \eta_{1,1} \circ \theta$ to obtain a lift $s_3$. Here $\eta_{1,1} : F_1Q \ast F_1Q \rightarrow \hat{F}_3(Q \times Q)$ and $\tilde{j} : \hat{F}_3(Q \times Q) \rightarrow \hat{F}_3(M \times M_3)$ is the inclusion induced by the inclusion $Q \times Q \subset M_3 \times M_3$. □

4.8. Proposition. cat$_{LS}(M_2 \times M_3) \geq 4$. 
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Proof. We recall that both $M_2$ and $M_3$ have CW complex structures with one cell in each of the dimensions 0, 2, 14, and 16. Consider the product CW-complex structure on $X = M_2 \times M_3$. We show that $\text{cat}_{\text{LS}} X^{(18)} = 4$. Then by Theorem 2.3 $\text{cat}_{\text{LS}} X \geq 4$. Assume the contrary: $\text{cat}_{\text{LS}} X^{(18)} \leq 3$. Consider the pull-back diagram generated by the 3-rd Ganea fibrations and the inclusion $j : M_2 \times S^2 \to X^{(18)}$

$$
\begin{array}{cccc}
G_3(M_2 \times S^2) & \xrightarrow{\xi} & Z & \xrightarrow{p} \to G_3(X^{(18)}) \\
\downarrow & & \downarrow & \downarrow \text{p}_3 \\
M_2 \times S^2 & \xrightarrow{j} & X^{(18)} \\
\end{array}
$$

where $p' \circ \xi = p_3^{M_2 \times S^2}$. By the assumption there is a section of $p_3$ which induces a section of $p'$. Since the inclusion $j : M_2 \times S^2 \to X^{(18)}$ is a 13-equivalence and the spaces are 1-connected, by Proposition 2.2 the mapping between fibers of $F_3(M_2 \times S^2) \to F_3X^{(18)}$ is a $(2 \times 3 + 13 - 1)$-equivalence. Hence, the homotopy fiber of $\xi$ is 18-connected. Since $\dim(M_2 \times S^2) = 18$, the section of $p'$ can be lifted to a section of $p_3^{M_2 \times S^2}$. This implies a contradiction: $\text{cat}_{\text{LS}}(M_2 \times S^2) \leq 3$. □

4.9. Remark. By Theorem 2.3 the proper filtration $X^{(18)} \subset X^{(28)} \subset X^{(30)} \subset X$ defines a chain of inequalities $\text{cat}_{\text{LS}} X^{(18)} \leq \text{cat}_{\text{LS}} X^{(28)} \leq \text{cat}_{\text{LS}} X^{(30)} \leq \text{cat}_{\text{LS}} X$. In view of Theorem 2.7, whether any of the above inequalities is strict can be determined by Berstein-Hilton invariants. If one of this three opportunities for $\text{cat}_{\text{LS}}$ to jump up is realized, then, as we show in the next section, there is a counter-example to the Rudyak conjecture.

5. Rudyak Conjecture

For a closed oriented manifold $M$ and $k \in \mathbb{N}$ by $kM$ we denote the connected sum of $k$ copies off $M$ and by $-kM$ we denote the connected sum $|k|M$ where $\bar{M}$ is $M$ taken with the opposite orientation. The following is obvious

5.1. Lemma. If $M_1, \ldots, M_r$ are connected $n$-manifold, then there is a cofibration sequence

$$
\bigvee_{i=1}^{r-1} S^{n-1} \longrightarrow \#_{j=1}^{r} M_j \longrightarrow \bigvee_{j=1}^{r} M_j.
$$

A special case of the following proposition was proven in [Dr].

5.2. Proposition. Suppose that $g : N_1 \to M_1$ and $h : N_2 \to M_2$ are maps between closed manifolds of degree $p$ and $q$ for mutually prime $p$ and $q$. Then there are $k, \ell \in \mathbb{Z}$ and a degree one map

$$
f : k(M_1 \times N_2) \# \ell(N_1 \times M_2) \to M_1 \times M_2.
$$
Proof. Let \( \dim N_1 = n_1 \) and \( \dim N_2 = n_2 \). Take \( k \) and \( \ell \) such that \( \ell p + k q = 1 \). We may assume that the above connected sum is obtained by taking the wedge of \((|k| + |\ell| - 1)\) copies of \((n_1 + n_2 - 1)\)-spheres embedded in one of the summands and gluing all other summands along those spheres. Consider the cofibration map from Lemma 5.1

\[
\psi : k(M_1 \times N_2) \# \ell(N_1 \times M_2) \to \bigvee_k (M_1 \times N_2) \vee \bigvee_\ell (N_1 \times M_2).
\]

Let the map

\[
\phi : \bigvee_k (M_1 \times N_2) \vee \bigvee_\ell (N_1 \times M_2) \to M_1 \times M_2
\]

be defined as the union

\[
\phi = \bigcup_k (1 \times g) \cup \bigcup_\ell (h \times 1).
\]

The degree of \( h \times 1 \) is \( p \) and the degree of \( 1 \times g \) is \( q \). Hence the degree of \( f = \phi \circ \psi \) is \( \ell p + k q = 1 \). \( \square \)

5.3. Theorem. Suppose that \( g : N_1 \to M_2 \) and \( h : N_2 \to M_2 \) are maps between closed manifolds of degree \( p \) and \( q \) for mutually prime \( p \) and \( q \) and

\[
\max\{\text{cat}_{LS}(M_1 \times N_2), \text{cat}_{LS}(N_1 \times M_2)\} < \text{cat}_{LS}(M_1 \times M_2).
\]

Then there is a counter-example to Rudyak’s Conjecture.

Proof. By Proposition 5.2 there is a degree one map

\[
f : k(M_1 \times N_2) \# \ell(N_1 \times M_2) \to M_1 \times M_2.
\]

By the connected sum formula (\(*\)),

\[
\text{cat}_{LS}(k(M_1 \times N_2) \# \ell(N_1 \times M_2)) \leq \max\{\text{cat}_{LS}(M_1 \times N_2), \text{cat}_{LS}(N_1 \times M_2)\}.
\]

\( \square \)

5.4. Corollary. If \( \text{cat}_{LS}(M_2 \times M_3) \geq 5 \), then there is a counter-example to Rudyak’s conjecture.

Proof. By Proposition 3.1 and Proposition 3.2 there are maps of degree two, \( g : S^{14} \times S^2 \to M_2 \), and of degree three, \( h : S^{14} \times S^2 \to M_3 \). Then the map

\[
f : -(M_2 \times S^{14} \times S^2) \# 2(S^{14} \times S^2 \times M_3) \to M_2 \times M_3
\]

of Proposition 5.2 has degree one. We note that

\[
\text{cat}_{LS}(M_2 \times S^{14} \times S^2) \leq \text{cat}_{LS}(M_2 \times S^{14}) + \text{cat}_{LS} S^2 = 3 + 1 = 4
\]

and

\[
\text{cat}_{LS}(S^{14} \times S^2 \times M_3) \leq \text{cat}_{LS} S^2 + \text{cat}_{LS}(S^{14} \times M_3) = 4.
\]

\( \square \)
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