The distribution function (df) of a random variable $X$ is defined as:
\[ F_X(x) = \Pr[ X \leq x ] \]

The generalized inverse for a df, the quantile function $q_X(p)$, is defined as:
\[ F_X^{-1}(p) = \inf \{ \text{real } x \mid F_X(x) \geq p \} \]
\[ = \sup \{ \text{real } x \mid F_X(x) < p \} \]

The ‘check’ function is defined as:
\[ \rho_q(r) = qr - r1_{\{r < 0\}} \quad \text{for } 0 \leq q \leq 1, \]
e.g. $\rho_{0.5}(r) = 0.5 |r|$
Curious folk result

For an integrable rv $X$, the minimizer of $E [\rho_q ( X - x ) ]$ with respect to $x$, is the $q$-quantile of $X$.

For an elementary proof, see [Hunter, Lange 1998, Appendix]
An even more curious result

Suppose rv $X$ has $E[ |X| ] < \infty$. Then the Fenchel-Legendre transform of the convex function $\Psi(x) = E[ (x - X)^+ ]$ is given by

$$\Psi^*(y) = \sup_{x \in \text{Real}} (xy - \Psi(x))$$

= Integral from 0 to $y$ of $q_X$, if $0 \leq y \leq 1$, and $+\infty$ otherwise.

Moreover, for $0 < y < 1$, the supremum above is attained in $x$ if and only if $x$ is a $y$-quantile of $X$, that is $x = q_X(y)$.

[Follmer, Schied, 2004, Lemma A.22]
Their connection!

\[
\begin{align*}
\arg \min_{x \in \mathbb{R}} \{ E [ \rho_q (X - x) ] \} \\
= \arg \min_{x \in \mathbb{R}} \{ qE[X] - qx - E[ (X - x)1_{\{X-x < 0\}} ] \} \\
= \arg \max_{x \in \mathbb{R}} \{ qx + E[ (X - x)1_{\{X-x < 0\}} ] \} \\
= \arg \max_{x \in \mathbb{R}} \{ xq - E[ (x - X)^+ ] \} \\
= \arg \max_{x \in \mathbb{R}} \{ xq - \Psi(x) \} \\
= q_X(q), \text{ for } 0 < q < 1. 
\end{align*}
\]

So we have an elegant proof of the folk result via functional analysis!
Applications

Quantile regression is a statistical technique used to estimate and make inference about conditional quantile functions [Koenker, Bassett, 1978]. Financial applications of quantile functions include asset pricing [Follmer, Schied, 2004], portfolio construction [Ma, Pohlman, 2005], [Bassett, Koenker, Kordas, 2004], risk management [Chernozhukov, 2002], [McNeil, Frey, Embrechts, 2005], and insurance [Denuit, Dhaene, Goovaerts, Kaas, 2005].
Some quantile function properties

First order quantile ODE:
\[
dq/dp = 1/f(q) \text{ where } q \text{ is the quantile function, } 0 \leq p \leq 1, \text{ and } f \text{ is the pdf.}
\]

Second order non-linear ODE:
\[
d^2q/dp^2 = H(q) \left( dq/dp \right)^2
\text{ where } H(q) = -d/dq \ln\{ f(q) \}.
\]
For power series solutions, see [Steinbrecher, Shaw, 2007].

The quantile-characteristic function connection:
\[
\phi_X(t) := E\left[ \exp(itX) \right] = \text{Integral from 0 to 1 of } \exp(itq_X)
\text{ is explored via differentiation in [Shaw, McCabe, 2009].}
\]

May you discover more curious quantile properties!
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