Global Estimates for Generalized Forchheimer Flows of Slightly Compressible Fluids

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Abstract

This paper is focused on the generalized Forchheimer flows of slightly compressible fluids in porous media. They are reformulated as a degenerate parabolic equation for the pressure. The initial boundary value problem is studied with time-dependent Dirichlet boundary data. The estimates up to the boundary and for all time are derived for the $L^\infty$-norm of the pressure, its gradient and time derivative. Large-time estimates are established to be independent of the initial data. Particularly, thanks to the special structure of the pressure’s nonlinear equation, the global gradient estimates are obtained in a relatively simple way, avoiding complicated calculations and a prior requirement of Hölder estimates.

1 Introduction

In studies of fluid dynamics in porous media, Darcy’s law is ubiquitously used. However this linear relation between the velocity and pressure gradient is not always valid. The deviation from Darcy’s law is well-known when the Reynolds number increases \cite{2,16}. Such a deviation was even noticed in early works by Darcy \cite{3} and Dupuit \cite{6}. Nonlinear alternatives were formulated by Forchheimer \cite{7,8}, and were studied extensively afterward in physics and engineering, see \cite{2,16,17,21,26} and references therein. In contrast to the vast mathematical research of Darcy’s flows, see e.g. \cite{25}, existing mathematical papers on Forchheimer flows are much fewer and came much later. Even rarer are the ones for compressible fluids. (See, e.g., \cite{1,9,10} for more introduction to Forchheimer flows.) Generalized Forchheimer equations were proposed \cite{1,9,11,13} in order to cover a general class of fluid flows in porous media formulated from experiments. In our previous paper \cite{12}, we derive interior estimates for generalized Forchheimer flows of slightly compressible fluids. In this article, we focus on the spatially global estimates, i.e., on the entire domain. Since the equation for pressure is of degenerate parabolic type, finding $L^\infty$-bounds for its gradient, in general, is a difficult task and requires much work, see, for e.g., \cite{3,18,19,24}. We will show that the gradient estimates for the particular flows under the current study can be obtained in a relatively simple way thanks to their equations’ specific structure.

In this paper, we consider the initial boundary value problem for the pressure in a bounded domain with time-dependent Dirichlet boundary data. Our goal is to obtain bounds for the pressure’s gradient and time derivative in terms of initial and boundary data. Aiming at studying the
long-term dynamics of the system, we also emphasize the bounds for large time. The standard technique from [14], see also [4, 15], requires a Hölder estimate first, which, by itself, is not a small task. Moreover, the estimates obtained in [14] for degenerate equations depend on the initial value of the gradient. This will result in long-time estimates for the gradient that are dependent on the initial data. That means that the long-time dynamics cannot be reduced to a much smaller set, say, the global attractor (c.f. [20, 23]). In this paper, we will demonstrate that these two setbacks can be overcome without overcomplicated calculations. First, we extend the techniques for parabolic equations in [14] to our degenerate one in a suitable way. Such extension is possible due to the equation’s special structure. Second, by doing analysis on the entire domain, we avoid the spatially local consideration, and hence, the requirement for the Hölder estimates. Finally, localizing the estimates in time and utilizing uniform Gronwall-type inequalities remove the estimates’ dependence on the initial data.

The paper is organized as follows. In section 2 we recall basic facts about generalized Forchheimer equations, and prove a global embedding of Ladyzhenskaya-Uraltseva type. It is different from the versions in [12, 14] which are localizations at an interior or a boundary point. This will contribute to simpler proofs for the global estimates in section 6. In section 3 we review some relevant estimates from our previous works [11, 12] which will be used repeatedly. In section 4, we use De Giorgi’s iteration to obtain (spatially) global estimates. Time-local inequality (4.2) is “quasi-homogeneous” in terms of $L_\alpha^{x, t}$-norm of the solution. This improves estimates in our previous works, and is similar to a recent improvement on interior estimates in [22]. Explicit estimates in terms of initial and boundary data are established in Theorem 4.4. Section 5 is devoted to estimating the maximum of the gradient’s modulus on the boundary. Unlike the results in [14], we are able to localize the estimate in time in Theorem 5.2 hence, it is independent of the initial gradient. This is essential to finding the gradient bounds for large time, by combining it with certain uniform Gronwall-type estimates. This is demonstrated in Corollary 5.3. In section 6 we establish global $L^s$-estimates for the gradient for all $s > 0$ and all positive time. This requires only moderate regularity for the initial data, say, $L^\alpha$ and $W^{1,2-a}$ with an appropriate $\alpha > 0$. Section 7 contains $L^\infty$-estimates for the gradient. Thanks to the special form of the resulting equations (7.2) for the gradient, it is possible to apply the De Giorgi technique, see Theorem 7.1. Section 8 has the same results as section 7 but for the time derivative. The main estimates are in Theorems 8.2 and 8.3.

2 Preliminaries

Consider a fluid in a porous medium in space $\mathbb{R}^n$. For physics problem $n = 3$, but here we consider any $n \geq 2$. Let $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$ be the spatial and time variables. The fluid flow has velocity $v(x, t) \in \mathbb{R}^n$, pressure $p(x, t) \in \mathbb{R}$ and density $\rho(x, t) \in [0, \infty)$.

The generalized Forchheimer equations studied in [1, 9–11, 13] are of the the form:

$$g(|v|)v = -\nabla p,$$

(2.1)

where $g(s) \geq 0$ is a function defined on $[0, \infty)$. When $g(s) = \alpha + \alpha + \beta s + \gamma s^2 + \alpha + \gamma m s^{m-1}$, where $\alpha, \beta, \gamma, m, \gamma_m$ are empirical constants, we have Darcy’s law, Forchheimer’s two-term, three-term and power laws, respectively.

In this paper, we study the case when the function $g$ in (2.1) is a generalized polynomial with positive coefficients, that is,

$$g(s) = a_0 s^{\alpha_0} + a_1 s^{\alpha_1} + \ldots + a_N s^{\alpha_N} \quad \text{for} \quad s \geq 0,$$

(2.2)
where \( N \geq 1 \), the powers \( \alpha_0 = 0 < \alpha_1 < \ldots < \alpha_N \) are fixed real numbers (not necessarily integers), the coefficients \( a_0, a_1, \ldots, a_N \) are positive. This function \( g(s) \) is referred to as the Forchheimer polynomial in equation (2.2).

From (2.1) one can solve \( v \) implicitly in terms of \( \nabla p \) and derives a nonlinear version of Darcy’s equation:

\[
v = -K(|\nabla p|)\nabla p,
\]

where the function \( K : [0, \infty) \to [0, \infty) \) is defined by

\[
K(\xi) = \frac{1}{g(s(\xi))}, \quad \text{with } s = s(\xi) \geq 0 \text{ satisfying } sg(s) = \xi, \quad \text{for } \xi \geq 0.
\]

In addition to (2.1), we have the continuity equation

\[
\phi \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho v) = 0,
\]

where number \( \phi \in (0, 1) \) is the constant porosity. Also, for slightly compressible fluids, the equation of state is

\[
\frac{d\rho}{dp} = \frac{\rho}{\kappa}, \quad \text{with } \kappa = \text{const.} > 0.
\]

From (2.3), (2.5) and (2.6) one derives a scalar equation for the pressure:

\[
\phi \frac{\partial p}{\partial t} = \kappa \nabla \cdot (K(|\nabla p|)\nabla p) + K(|\nabla p|)|\nabla p|^2.
\]

On the right-hand side of (2.7), the constant \( \kappa \) is very large for most slightly compressible fluids in porous media [16], hence we neglect its second term and by scaling the time variable, we study the reduced equation

\[
\frac{\partial p}{\partial t} = \nabla \cdot (K(|\nabla p|)\nabla p).
\]

Note that this reduction is commonly used in engineering.

Our aim is to study the initial boundary value problem for equation (2.8) in a bounded domain. Here afterward \( U \) is a bounded, open, connected subset of \( \mathbb{R}^n, n = 2, 3, \ldots \) with \( C^2 \) boundary \( \Gamma = \partial U \). Throughout, the Forchheimer polynomial \( g(s) \) in (2.2) is fixed. The following number is frequently used in our calculations:

\[
a = \frac{\alpha_N}{1 + \alpha_N} \in (0, 1).
\]

The function \( K(\xi) \) in (2.3) has the following properties (c.f. [1][9]): it is decreasing in \( \xi \) mapping \( \xi \in [0, \infty) \) onto \( (0, 1/a_0) \), and

\[
\frac{d_1}{(1 + \xi)^a} \leq K(\xi) \leq \frac{d_2}{(1 + \xi)^{a}},
\]

\[
d_3(\xi^{2-a} - 1) \leq K(\xi)^2 \leq d_2\xi^{2-a},
\]

\[
-aK(\xi) \leq K'(\xi)\xi \leq 0,
\]

where \( d_1, d_2, d_3 \) are positive constants depending on \( g \). As in previous works, we use the function \( H(\xi) \) defined by

\[
H(\xi) = \int_0^{\xi^2} K(\sqrt{s}) \\mathrm{d}s \quad \text{for } \xi \geq 0.
\]
It satisfies $K(\xi)\xi^2 \leq H(\xi) \leq 2K(\xi)\xi^2$, thus, by (2.11),
\[ d_3(\xi^{2-a} - 1) \leq H(\xi) \leq 2d_2\xi^{2-a}. \] (2.13)

The following parabolic Poincaré-Sobolev inequalities are needed for our study. For each $T > 0$, denote $Q_T = U \times (0, T)$. We define a threshold exponent
\[ \alpha_* = \frac{an}{2-a}. \] (2.14)

**Lemma 2.1** (cf. [12], Lemma 2.1). Assume
\[ \alpha \geq 2 \quad \text{and} \quad \alpha > \alpha_. \] (2.15)

Let
\[ p = \alpha \left( 1 + \frac{2-a}{n} \right) - a. \] (2.16)

Then
\[ \|u\|_{L^p(Q_T)} \leq C(1 + \delta T)^{1/p}[[u]], \] (2.17)
where $\delta = 1$ in general, $\delta = 0$ in case $u$ vanishes on the boundary $\partial U$, and
\[ [[u]] = \esssup_{[0,T]} \|u(\cdot, t)\|_{L^\alpha(U)} + \left( \int_0^T \int_U |u(x, t)|^{\alpha-2} |\nabla u(x, t)|^{2-a} dxdt \right)^{\frac{1}{\alpha-a}}. \] (2.18)

The next is a particular embedding with spatial weights from Lemma 2.4 of [13] (see inequality (2.28) with $m = 2$ there).

**Lemma 2.2** (cf. [13], Lemma 2.4). Given $W(x, t) > 0$ on $Q_T$. Let $r$ be a number that satisfies
\[ \frac{2n}{n+2} < r < 2. \] (2.19)

Set
\[ q = q(r) = 4(1 - 1/r^*). \] (2.20)

Then
\[ \|u\|_{L^q(Q_T)} \leq C[[u]]_{2, W; T} \left\{ \delta T^\frac{1}{q} + \esssup_{t\in[0,T]} \left( \int_U W(x, t)^{-\frac{n}{2-r}} \chi_{\supp u}(x, t) dx \right)^{\frac{2-r}{q-r}} \right\}, \] (2.21)
where $\delta = 1$ in general, $\delta = 0$ in case $u$ vanishes on the boundary $\partial U$, and
\[ [[u]]_{2, W; T} = \esssup_{[0,T]} \|u(\cdot, t)\|_{L^2(U)} + \left( \int_0^T \int_U W(x, t)|\nabla u(x, t)|^2 dxdt \right)^{\frac{1}{2}}. \] (2.22)

Above, $\supp f$ denotes the support set of a function $f$, and $\chi_A$ denotes the characteristic functions of a set $A$.

The following embedding is a global version of Lemma 3.7 from [13] and Lemma 5.4 on page 93 in [14].

**Lemma 2.3.** Suppose $w$ is a function on $\bar{U}$ that satisfies $|\nabla w| \leq M$ on $\Gamma$.

Let $v = \max\{|\nabla w|^2 - M^2, 0\}$. For each $s \geq 1$, there exists a constant $C > 0$ depending on $s$ such that for any $k \in \mathbb{R}$,
\[ \int_U K(|\nabla w|)v^{s+1} dx \leq C \max |w - k|^2 \int_U K(|\nabla w|)|\nabla^2 w|^2 v^{s-1} dx + CM^4 \int_U K(|\nabla w|)v^s dx. \] (2.23)
Proof. Let \( I = \int_U K(|\nabla w|)v^{s+1} \, dx \). Note that \( v = 0 \) on \( \Gamma \). First, we see that

\[
I \leq \int_U K(|\nabla w|)v^s|\nabla w|^2 \, dx = \sum_{i=1}^{n} \int_U K(|\nabla w|)v^s \partial_i w \partial_i (w - k) \, dx.
\]

By integration by parts,

\[
I = -\sum_{i=1}^{n} \int_U \partial_i (K(|\nabla w|)v^s \partial_i w) \cdot (w - k) \, dx
\]

\[
= -\sum_{i,j=1}^{n} \int_U \left( K'(|\nabla w|) \frac{\partial_i \partial_j w \partial_j w}{|\nabla w|} \right) v^s \partial_i w \cdot (w - k) \, dx - \int_U K(|\nabla w|)v^s \Delta w \cdot (w - k) \, dx
\]

\[
- s \sum_{i,j=1}^{n} \int_U K(|\nabla w|) \left( v^{s-1} \partial_i \partial_j w \partial_j w \right) \cdot \partial_i w \cdot (w - k) \chi_{\{v > 0\}} \, dx.
\]

From this and (2.12), it follows that

\[
I \leq a \int_U K(|\nabla w|)|\nabla^2 w|v^s|w - k| \, dx + \int_U K(|\nabla w|)v^s|\Delta w||w - k| \, dx
\]

\[
+ s \int_U K(|\nabla w|)v^{s-1}|\nabla^2 w||\nabla w|^2|w - k| \chi_{\{v > 0\}} \, dx
\]

\[
\leq C \int_U K(|\nabla w|)|\nabla^2 w||w - k| \, dx + C \int_U K(|\nabla w|)|\nabla^2 w|v^{s-1}(v + M^2)|w - k| \, dx.
\]

Hence,

\[
I \leq C \int_U K(|\nabla w|)v^s|\nabla^2 w||w - k| \, dx + CM^2 \int_U K(|\nabla w|)|\nabla^2 w|v^{s-1}|w - k| \, dx
\]

\[
= C \int_U K(|\nabla w|)v^{s-1} \cdot |\nabla^2 w|v^s |w - k| \, dx + C \int_U K(|\nabla w|)v^{s-1}|\nabla^2 w||w - k| \cdot M^2 \, dx.
\]

This last inequality and Cauchy’s inequality imply that

\[
I \leq \int_U K(|\nabla w|)\left( \frac{v^{s+1}}{2} + C|\nabla^2 w|^2|v^{s-1}|w - k|^2 \right) \, dx + C \int_U K(|\nabla w|)v^{s-1}(|\nabla^2 w|^2|w - k|^2 + M^4) \, dx
\]

\[
= \frac{1}{2} I + C \int_U K(|\nabla w|)v^{s-1}|\nabla^2 w|^2|w - k|^2 \, dx + CM^4 \int_U K(|\nabla w|)v^{s-1} \, dx.
\]

Therefore, we obtain (2.23).

The following is a generalization of the convergence of fast decay geometry sequences in Lemma 5.6, Chapter II of [14]. It will be used in the De Giorgi iterations.

Lemma 2.4 (cf. [13], Lemma A.2). Let \( \{Y_i\}_{i=0}^{\infty} \) be a sequence of non-negative numbers satisfying

\[
Y_{i+1} \leq \sum_{k=1}^{m} A_k B^{i} Y_{i}^{1+\mu_k}, \quad i = 0, 1, 2, \ldots,
\]

where \( B > 1, A_k > 0 \) and \( \mu_k > 0 \) for \( k = 1, 2, \ldots, m \). If \( Y_0 \leq \min\{(m^{-1}A_k^{-1}B^{-\frac{1}{\mu_k}})^{1/\mu_k} : 1 \leq k \leq m\} \) then \( \lim_{i \to \infty} Y_i = 0 \).
3 Previous results

We study the following initial boundary value problem (IBVP) for \( p(x,t) \):

\[
\begin{aligned}
\frac{\partial p}{\partial t} &= \nabla \cdot (K(|\nabla p|) \nabla p) \quad \text{in } U \times (0, \infty), \\
p(x,0) &= p_0(x) \quad \text{in } U, \\
p(x,t) &= \psi(x,t) \quad \text{on } \Gamma \times (0, \infty).
\end{aligned}
\]  

(3.1)

In order to deal with the non-homogeneous boundary condition, the data \( \psi(x,t) \) with \( x \in \Gamma \) and \( t > 0 \) is extended to a function \( \Psi(x,t) \) with \( x \in \bar{U} \) and \( t \geq 0 \). Throughout, our results are stated in terms of \( \Psi \) instead of \( \psi \). Nonetheless, corresponding results in terms of \( \psi \) can be retrieved as performed in [9]. The function \( \Psi \) is always assumed to have adequate regularities for all calculations in this paper.

**Solutions.** It is proved in section 3 of \([11]\) that (3.1) possesses a weak solution \( p(x,t) \) for all \( t > 0 \). It, in fact, has more regularity in spatial and time variables, see \([11]\). For the current study, we assume that solution \( p(x,t) \) has sufficient regularities both in \( x \) and \( t \) variables such that our calculations hereafter can be performed legitimately. Specifically, we assume \( p, \nabla p, p_t \in C(\bar{U} \times (0, \infty)) \), the Hessian matrix of second spatial derivatives \( \nabla^2 p \in C(U \times (0, \infty)) \), and the function \( t \to p(\cdot, t) \) is continuous from \([0, \infty) \) to \( W^{1,2-a}(U) \cap L^a(U) \), for an appropriate \( a > 0 \) which will be determined for each type of estimate. Some particular estimates may require much less regularity such as the \( L^\infty \)-estimates in section \([11]\). The obtained estimates also hold without the \( C^2 \)-requirement by using an approximation process, see \([14]\). However, to avoid further complications, we do not perform such approximation here.

**Generic constants notation.** Hereafter, the symbols \( C \) and \( C' \) are used to denote positive numbers independent of the initial and boundary data, and the time variables \( t, T_0, T \); it may depend on many parameters, namely, exponents and coefficients of polynomial \( g \), the spatial dimension \( n \) and domain \( U \), other involved exponents \( \alpha, s \), etc. in calculations. The values of \( C \) and \( C' \) may vary from place to place, even on the same line.

**Functions and Norms.** Throughout, whenever unspecified, the norm Lebesgue or Sobolev norms mean for the whole domain \( U \). Also, for a function \( f(x,t) \), we use \( t(f) \) to denote the functions \( t \to f(\cdot, t) \). For example, \( \|p(t)\|_{L^1} = \|p(\cdot, t)\|_{L^1(U)} \), \( \max_{0 \leq t \leq T} \|p(t)\|_{L^\infty} = \max_{t \in [0,T]} \|p(\cdot, t)\|_{L^\infty(U)} \).

We recall some relevant results from \([11]\). For \( \alpha \geq 1 \), we define

\[
A(\alpha, t) = \left[ \int_U |\nabla \Psi(x,t)|^{\frac{\alpha(2-a)}{2}} dx \right]^{\frac{2(\alpha-a)}{\alpha(2-a)}} + \left[ \int_U |\Psi_t(x,t)|^{\alpha} dx \right]^{\frac{\alpha-a}{\alpha(1-a)}}
\]  

(3.2)

for \( t \geq 0 \), and

\[
A(\alpha) = \limsup_{t \to \infty} A(\alpha, t) \quad \text{and} \quad \beta(\alpha) = \limsup_{t \to \infty}[A'(\alpha, t)]^{-}.
\]  

(3.3)

Also, define for \( \alpha > 0 \) the number

\[
\hat{\alpha} = \max \{ \alpha, 2, \alpha_s \}.
\]  

(3.4)

Whenever \( \beta(\alpha) \) is in use, it is understood that the function \( t \to A(\alpha, t) \) belongs to \( C^1((0, \infty)) \).

For a function \( f : [0, \infty) \to \mathbb{R} \), we denote by \( Envf \) a continuous and increasing function \( F : [0, \infty) \to \mathbb{R} \) such that \( F(t) \geq f(t) \) for all \( t \geq 0 \).

Let \( p(x,t) \) be a solution to IBVP (3.1), and denote \( \overline{p} = p - \Psi \).

**Theorem 3.1** (cf. \([11]\), Theorem 4.3). Let \( \alpha > 0 \).
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Theorem 3.2

(i) For all $t \geq 0$,
$$
\int_{U} |\bar{p}(x,t)|^{\alpha} dx \leq C \left( 1 + \int_{U} |\bar{p}(x,0)|^{\alpha} dx + [EnvA(\bar{\alpha},t)]^{\frac{\alpha}{\alpha - \alpha\beta}} \right). \tag{3.5}
$$

(ii) If $A(\bar{\alpha}) < \infty$ then
$$
\limsup_{t \to \infty} \int_{U} |\bar{p}(x,t)|^{\alpha} dx \leq C \left( 1 + A(\bar{\alpha})^{\frac{\alpha}{\alpha\beta}} \right). \tag{3.6}
$$

(iii) If $\beta(\bar{\alpha}) < \infty$ then there is $T > 0$ such that
$$
\int_{U} |\bar{p}(x,t)|^{\alpha} dx \leq C \left( 1 + \beta(\bar{\alpha})^{\frac{\alpha}{\alpha - \alpha\beta}} + A(\bar{\alpha},t)^{\frac{\alpha}{\alpha - \alpha\beta}} \right) \text{ for all } t \geq T. \tag{3.7}
$$

For gradient and time derivative estimates, we denote
$$
G_1(t) = \int_{U} \left| \nabla \Psi(x,t) \right|^2 dx + \left[ \int_{U} \left| \Psi_t(x,t) \right|^{\alpha_0} dx \right]^{\frac{2 - \alpha}{\alpha_0 (1 - \alpha)}}, \quad G_2(t) = \int_{U} \left| \nabla \Psi_t(x,t) \right|^2 dx + \int_{U} \left| \Psi_t(x,t) \right|^2 dx, \quad G_3(t) = G_1(t) + G_2(t).
$$

with $\alpha_0 = \frac{n(2 - \alpha)}{(2 - \alpha)(\alpha + 1) - \alpha}$. For $t \geq 0$, recall from (4.20) in [11] and from (3.25) in [9] that
$$
\int_{0}^{t} \int_{U} H(|\nabla p|)(x,t) dx d\tau \leq C \int_{U} \bar{p}^2(x,0) dx + C \int_{0}^{t} G_1(\tau) d\tau, \tag{3.8}
$$
$$
\int_{U} H(|\nabla p|)(x,t) dx + \int_{0}^{t} \int_{U} \bar{p}_t(x,\tau)^2 dx d\tau \leq C \int_{U} H(|\nabla p(x,0)|) + \bar{p}^2(x,0) dx + C \int_{0}^{t} G_3(\tau) d\tau. \tag{3.9}
$$

Below are estimates for gradient and time derivative when time is large.

Theorem 3.2 (cf. [12], Corollary 3.3). Let $\alpha \geq 2$.

(i) For $t \geq 1$,
$$
\int_{t-1}^{t} \int_{U} H(|\nabla p(x,\tau)|) dx d\tau \leq C \left( 1 + \int_{U} |\bar{p}_0(x)|^{\alpha} dx + [EnvA(\alpha, t)]^{\frac{\alpha}{\alpha - \alpha\beta}} + \int_{t-1}^{t} G_1(\tau) d\tau \right), \tag{3.10}
$$
$$
\int_{t-1/2}^{t} \int_{U} \bar{p}_t^2(x,\tau) dx d\tau \leq C \left( 1 + \int_{U} |\bar{p}_0(x)|^{\alpha} dx + [EnvA(\alpha, t)]^{\frac{\alpha}{\alpha - \alpha\beta}} + \int_{t-1}^{t} G_3(\tau) d\tau \right). \tag{3.11}
$$

(ii) If $A(\alpha) < \infty$ then
$$
\limsup_{t \to \infty} \int_{t-1}^{t} \int_{U} H(|\nabla p(x,\tau)|) dx d\tau \leq C \left( 1 + A(\alpha)^{\frac{\alpha}{\alpha\beta}} + \limsup_{t \to \infty} \int_{t-1}^{t} G_1(\tau) d\tau \right), \tag{3.12}
$$
$$
\limsup_{t \to \infty} \int_{t-1/2}^{t} \int_{U} \bar{p}_t^2(x,\tau) dx d\tau \leq C \left( 1 + A(\alpha)^{\frac{\alpha}{\alpha\beta}} + \limsup_{t \to \infty} \int_{t-1}^{t} G_3(\tau) d\tau \right). \tag{3.13}
$$

(iii) If $\beta(\alpha) < \infty$ then there is $T > 0$ such that one has for all $t \geq T$ that
$$
\int_{t-1}^{t} \int_{U} H(|\nabla p(x,\tau)|) dx d\tau \leq C \left( 1 + \beta(\alpha)^{\frac{\alpha}{\alpha\beta}} + A(\alpha, t-1)^{\frac{\alpha}{\alpha\beta}} + \int_{t-1}^{t} G_1(\tau) d\tau \right), \tag{3.14}
$$
$$
\int_{t-1/2}^{t} \int_{U} \bar{p}_t^2(x,\tau) dx d\tau \leq C \left( 1 + \beta(\alpha)^{\frac{\alpha}{\alpha\beta}} + A(\alpha, t-1)^{\frac{\alpha}{\alpha\beta}} + \int_{t-1}^{t} G_3(\tau) d\tau \right). \tag{3.15}
$$
4 \(L^\infty\)-estimates

In this section we estimate the \(L^\infty\)-norm of the pressure. We will focus on estimates for \(\bar{p}\) instead of \(p\). First, we give a local (in time) \(L^\infty\)-estimate which does not depend on the initial data’s \(L^\infty\)-norm.

Let \(p(x,t)\) be a solution to IBVP \(
\bar{p}(x,t) = \nabla \cdot (K(|\nabla p|)\nabla p) - \Psi_t \quad \text{in} \ U \times (0,\infty),
\)
\(\bar{p}(x,t) = 0 \quad \text{on} \ \Gamma \times (0,\infty).\) \hfill (4.1)

**Theorem 4.1.** Let \(\alpha\) satisfy \((2.12)\). If \(T_0 \geq 0, T > 0\) and \(\theta \in (0,1)\) then
\[
\sup_{[T_0 + \theta T, T_0 + T]} \|\bar{p}(t)\|_{L^\infty(U)} \leq C \left\{ \left(\theta T\right)^{-1/\delta_1} \|\bar{p}\|_{L^\infty(U \times (T_0,T_0+T))} + T^{\alpha-2} \|\bar{p}\|_{L^\infty(U \times (T_0,T_0+T))} \right. \\
+ T^{\alpha-1} \left\| |\Psi_t| \right\|_{L^{p_1/(1+\delta_1)}} \left\| \bar{p}\right\|_{L^\infty(U \times (T_0,T_0+T))} \right. \\
+ T^{\alpha-2} \left\| |\Psi_t| \right\|_{L^{p_1/(1+\delta_1)}} \left\| \bar{p}\right\|_{L^\infty(U \times (T_0,T_0+T))} \right. \\
\left. + \left(\theta T\right)^{-1/\delta_2} \left\| |\Psi_t| \right\|_{L^{p_1/(1+\delta_2)}} \left\| \bar{p}\right\|_{L^\infty(U \times (T_0,T_0+T))} \right. \\
\left. \right\}.
\] \hfill (4.2)

where \(r_1 = \alpha(1 + (2 - \alpha)/n) - \alpha\), numbers \(p_1\) and \(q_1\) are conjugates of each other that satisfy
\[
1 \leq p_1 < r_1/\alpha, \text{ or equivalently, } 1 + \frac{1}{\Delta(1/\Delta - 1/\alpha)} < q_1 = \frac{p_1}{p_1 - 1} \leq \infty,
\]
and \(\delta_1, \delta_2, \delta_3, \delta_4\) are positive numbers defined by
\[
\delta_1 = 1 - \alpha/r_1, \quad \delta_2 = \alpha/(\alpha - \alpha/r_1), \quad \delta_3 = \alpha/(\alpha - \alpha/r_1), \quad \delta_4 = \alpha/(\alpha - \alpha/r_1).
\] \hfill (4.3)

**Proof.** Without loss of generality, we assume \(T_0 = 0\). Let \(k \geq 0, \) define \(\bar{p}^{(k)} = \max\{\bar{p} - k, 0\}\), and denote by \(\chi_k\) the characteristic function on the set \(\text{supp} \ \bar{p}^{(k)}\) - the support set of \(\bar{p}^{(k)}\).

Let \(\zeta = \zeta(t)\) be a smooth function on \(\mathbb{R}\) satisfying \(0 \leq \zeta \leq 1\) and \(\zeta_t \geq 0\) on \(\mathbb{R}\), and \(\zeta = 0\) on \((-\infty,0]\).

Multiplying the partial differential equation (PDE) in \((3.1)\) by \(|\bar{p}^{(k)}|^{\alpha-1}\zeta\), integrating over \(U\) and using integration by parts, we have
\[
\alpha^{-1} \int_U \frac{\partial |\bar{p}^{(k)}|^{\alpha}}{} \zeta dx + (\alpha - 1) \int_U K(|\nabla \bar{p}^{(k)}|)|\nabla \bar{p}^{(k)}|^{2} |\bar{p}^{(k)}|^{\alpha-2} \zeta dx = - \int_U |\Psi_t| |\bar{p}^{(k)}|^{\alpha-1} \zeta dx.
\]

Using \((2.11)\) to estimate the second integral on the left-hand side, we have
\[
\frac{d}{dt} \int_U |\bar{p}^{(k)}|^{\alpha} \zeta dx + \int_U |\nabla \bar{p}^{(k)}|^{2} |\bar{p}^{(k)}|^{\alpha-2} \zeta dx \\
\leq C \int_U |\bar{p}^{(k)}|^{\alpha} \zeta_t dx + C \int_U (|\bar{p}^{(k)}|^{\alpha-2} \zeta + |\Psi_t| |\bar{p}^{(k)}|^{\alpha-1} \zeta) dx.
\]

Applying Young’s inequality to the integrands of the last integral yields
\[
\frac{d}{dt} \int_U |\bar{p}^{(k)}|^{\alpha} \zeta dx + \int_U |\nabla \bar{p}^{(k)}|^{2} |\bar{p}^{(k)}|^{\alpha-2} \zeta dx \leq C \int_U |\bar{p}^{(k)}|^{\alpha} \zeta_t dx \\
+ C \int_U |\bar{p}^{(k)}|^{\alpha} \zeta dx + C \varepsilon^{1-\alpha/2} \int_U \chi_k \zeta dx dt + C \varepsilon^{1-\alpha} \int_U |\Psi_t|^{\alpha} \chi_k \zeta dx dt.
\]
Integrating form 0 to $t$ for $t \in [0, T]$ and taking the supremum give

$$\sup_{[0,T]} \int_U |\bar{p}^{(k)}|^{\alpha} \zeta dx + \int_0^T \int_U |\nabla \bar{p}^{(k)}|^{2-\alpha} |\bar{p}^{(k)}|^{\alpha-2} \zeta dx dt \leq C \int_0^T \int_U |\bar{p}^{(k)}|^{\alpha} \zeta dx dt$$

$$+ \varepsilon T \sup_{[0,T]} \int_U |\bar{p}^{(k)}|^{\alpha} \zeta dx + C \varepsilon^{1-\alpha/2} \int_0^T \int_U \chi_k \zeta dx dt + C \varepsilon^{1-\alpha} \int_0^T \int_U |\Psi_t|^{\alpha} \chi_k \zeta dx dt.$$

Selecting $\varepsilon = 1/(2T)$, we obtain

$$\sup_{[0,T]} \int_U |\bar{p}^{(k)}|^{\alpha} \zeta dx + \int_0^T \int_U |\nabla \bar{p}^{(k)}|^{2-\alpha} |\bar{p}^{(k)}|^{\alpha-2} \zeta dx dt \leq C \int_0^T \int_U |\bar{p}^{(k)}|^{\alpha} \zeta dx dt$$

$$+ CT^{\alpha/2-1} \int_0^T \int_U \chi_k \zeta dx dt + CT^{\alpha-1} \int_0^T \int_U |\Psi_t|^{\alpha} \chi_k \zeta dx dt.$$ 

Applying Hölder’s inequality to the last integral gives

$$\sup_{[0,T]} \int_U |\bar{p}^{(k)}|^{\alpha} \zeta dx + \int_0^T \int_U |\nabla \bar{p}^{(k)}|^{2-\alpha} |\bar{p}^{(k)}|^{\alpha-2} \zeta dx dt \leq C \int_0^T \int_U |\bar{p}^{(k)}|^{\alpha} \zeta dx dt$$

$$+ CT^{\alpha/2-1} \int_0^T \int_U \chi_k \zeta dx dt + CT^{\alpha-1} \|\Psi_t\|^{\alpha} \|L^{q_1}(U \times (0,T))\left( \int_0^T \int_U \chi_k \zeta dx dt \right)^{1/p_1}.$$ \quad (4.4)

Let $M_0 > 0$ be fixed which will be determined later. For $i \geq 0$, define

$$k_i = M_0(1 - 2^{-i}), \quad t_i = \theta T(1 - 2^{-i}).$$ \quad (4.5)

Then $t_0 = 0 < t_1 < \ldots < \theta T$ and $\lim_{i \to \infty} t_i = \theta T$; $k_0 = 0 < k_1 < k_2 < \ldots$ and $\lim_{i \to \infty} k_i = M_0$.

For $i, j \geq 0$, we denote

$$Q_i = U \times (t_i, T) \quad \text{and} \quad A_{i,j} = \{(x, t) \in Q_j : p(x, t) > k_i\}, \quad A_i = A_{i,i}.$$ \quad (4.6)

For each $i$, we use a cut-off function $\zeta_i(t)$ with $\zeta_i \equiv 1$ in $[t_{i+1}, T]$ and $\zeta_i \equiv 0$ on $[0, t_i]$, and

$$|\zeta_i(t)| \leq \frac{C}{t_{i+1} - t_i} = \frac{C2^{i+1}}{\theta T}.$$ \quad (4.7)

for some $C > 0$. Applying (4.4) with $k = k_{i+1}$ and $\zeta = \zeta_i$ gives

$$\sup_{[0,T]} \int_U |\bar{p}^{(k_{i+1})}|^{\alpha} \zeta dx + \int_0^T \int_U |\nabla \bar{p}^{(k_{i+1})}|^{2-\alpha} |\bar{p}^{(k_{i+1})}|^{\alpha-2} \zeta dx dt$$

$$\leq \int_0^T \int_U |\bar{p}^{(k_{i+1})}|^{\alpha} \zeta dx dt + CT^{\alpha/2-1} \int_0^T \int_U \chi_{k_{i+1}} \zeta dx dt$$

$$+ CT^{\alpha-1} \|\Psi_t\|^{\alpha} L^{q_1}(U \times (0,T)) \left( \int_0^T \int_U \chi_{k_{i+1}} \zeta dx dt \right)^{1/p_1}.$$ \quad (4.8)

Define

$$F_i \overset{\text{def}}{=} \sup_{[t_{i+1}, T]} \int_U |\bar{p}^{(k_{i+1})}|^{\alpha} dx + \int_{t_i}^T \int_U |\nabla \bar{p}^{(k_{i+1})}|^{2-\alpha} |\bar{p}^{(k_{i+1})}|^{\alpha-2} dx dt.$$
Let $\mathcal{E}_1 = T^{\alpha/2-1}$ and $\mathcal{E}_2 = T^{\alpha-1} \|\Psi_t\|^2_{L^{\alpha,q_1}(U \times (0,T))}$. Then (4.8) yields

$$F_i \leq \int_{t_i}^{t_{i+1}} \int_{U} |\tilde{p}^{(k+1)}|^{\alpha} \zeta dx dt + C\mathcal{E}_1 \int_{t_{i+1}}^{T} \int_{U} \chi_{k+1} dx dt + C\mathcal{E}_2 \left( \int_{t_{i+1}}^{T} \int_{U} \chi_{k+1} dx dt \right)^{1/p_1}.$$  

(4.9) 

$$\leq C2^i (\theta T)^{-1} ||\tilde{p}^{(k+1)}||_{L^\alpha(A_{i+1,i+1})}^\alpha + C\mathcal{E}_1 |A_{i+1,i}| + C\mathcal{E}_2 |A_{i+1,i}|^{1/p_1}.$$  

(4.10) 

Since $||\tilde{p}^{(k)}||_{L^\alpha(A_i)} \geq ||\tilde{p}^{(k)}||_{L^\alpha(A_{i+1,i+1})} \geq (k_{i+1} - k_i) |A_{i+1,i}|^{1/\alpha}$, thus

$$|A_{i+1,i}| \leq (k_{i+1} - k_i)^{-\alpha} ||\tilde{p}^{(k)}||_{L^\alpha(A_i)}^\alpha \leq C2^\alpha M_0^{-\alpha} ||\tilde{p}^{(k)}||_{L^\alpha(A_i)}^\alpha.$$  

(4.11) 

This and (4.9) imply

$$F_i \leq C2^i (\theta T)^{-1} ||\tilde{p}^{(k+1)}||_{L^\alpha(A_{i+1,i+1})}^\alpha + C\mathcal{E}_1 2^\alpha M_0^{-\alpha} ||\tilde{p}^{(k)}||_{L^\alpha(A_i)}^\alpha + C\mathcal{E}_2 C^1/p_1 M_0^{-\alpha/p_1} ||\tilde{p}^{(k)}||_{L^\alpha(A_i)}^{\alpha/p_1}.$$  

(4.12) 

Note that $r_1$ is the exponent defined in (2.16). By Lemma 2.1

$$||\tilde{p}^{(k+1)}||_{L^{\alpha} (A_{i+1,i+1})} \leq C (F_{i}^{1/\alpha} + F_{i}^{1/(\alpha-a)}).$$  

(4.13) 

Hölder’s inequality gives

$$||\tilde{p}^{(k+1)}||_{L^\alpha(A_{i+1,i+1})} \leq ||\tilde{p}^{(k+1)}||_{L^{\alpha} (A_{i+1,i+1})} |A_{i+1,i+1}|^{1/\alpha-1/r_1} \leq ||\tilde{p}^{(k+1)}||_{L^{\alpha} (A_{i+1,i+1})} |A_{i+1,i}|^{1/\alpha-1/r_1}.$$  

(4.14) 

Combining (4.13) with (4.10), (4.11) and (4.12) yields

$$||\tilde{p}^{(k+1)}||_{L^\alpha(A_{i+1,i+1})} \leq CB^i \left( (\theta T)^{-1} + \mathcal{E}_1 M_0^{-\alpha} \right)^{1/\alpha} ||\tilde{p}^{(k)}||_{L^\alpha(A_i)} + \mathcal{E}_2 C^{1/(\alpha p_1)} M_0^{-1/p_1} ||\tilde{p}^{(k)}||_{L^\alpha(A_i)}^{1/p_1}$$

$$+ \left( (\theta T)^{-1} + \mathcal{E}_1 M_0^{-\alpha} \right)^{1/(\alpha-a)} ||\tilde{p}^{(k)}||_{L^\alpha(A_i)}^{\alpha/(\alpha-a)} + \mathcal{E}_2 C^{1/(\alpha-a)p_1} M_0^{-\alpha/(\alpha-a)p_1} ||\tilde{p}^{(k)}||_{L^\alpha(A_i)}^{\alpha/(\alpha-a)p_1} \right).$$

(4.15) 

where $B = 2^\alpha$. Let $Y_i = ||\tilde{p}^{(k)}||_{L^\alpha(A_i)}$. Then

$$Y_{i+1} \leq CB^i \left(D_1 Y_i^{1+\delta_1} + D_2 Y_i^{1+\delta_2} + D_3 Y_i^{1+\delta_3} + D_4 Y_i^{1+\delta_4} \right),$$

where

$$D_1 = \left( (\theta T)^{-1/\alpha} + \mathcal{E}_1 1/\alpha M_0^{-1} \right) M_0^{-1+\alpha/r_1} = (\theta T)^{-1} M_0^{-\delta_1} + T^{1/2-1/\alpha} M_0^{-1-\delta_1};$$

$$D_2 = \mathcal{E}_2 C^{1/(\alpha p_1)} M_0^{-1/p_1} \cdot M_0^{-1+\alpha/r_1} = T^{\alpha-1} ||\Psi_t||_{L^{\alpha,q_1}(U \times (0,T))} M_0^{-1/p_1-\delta_1};$$

$$D_3 = \left( (\theta T)^{-1/(\alpha-a)} + \mathcal{E}_1 1/(\alpha-a) M_0^{-\alpha/(\alpha-a)} \right) M_0^{-1+\alpha/r_1} = (\theta T)^{-\alpha/(\alpha-a)} M_0^{-\delta_1} + T^{\alpha-2/(\alpha-a)} M_0^{-\alpha/(\alpha-a)-\delta_1};$$

$$D_4 = \mathcal{E}_2 C^{1/(\alpha-a)p_1} M_0^{-\alpha/(\alpha-a)p_1} \cdot M_0^{-1+\alpha/r_1} = T^{\alpha/(\alpha-a)p_1} ||\Psi_t||_{L^{\alpha,q_1}(U \times (0,T))} M_0^{-\alpha/(\alpha-a)p_1-\delta_1}.$$ 

(4.16) 

Take $M_0$ sufficiently large such that

$$Y_0 \leq C \min \{ D_1^{-1/\delta_1}, D_2^{-1/\delta_2}, D_3^{-1/\delta_3}, D_4^{-1/\delta_4} \}.$$  

(4.17)
or equivalently,

\[ D_j \leq CY_0^{-\delta_j}, \quad j = 1, 2, 3, 4. \]  \hfill (4.15)

Then by Lemma 2.4 \( \lim_{i \to \infty} Y_i = 0 \), consequently, \( \int_0^T \int_U |\bar{p}(M_0)|^{\alpha} dx \leq 0 \), that is

\[ \bar{p}(x, t) \leq M_0 \quad \text{in} \ U \times [\theta T, T]. \]

Repeating the argument above for \(-p, -\psi\) instead of \(p, \psi\), we obtain

\[ |\bar{p}(x, t)| \leq M_0 \quad \text{in} \ U \times [\theta T, T]. \]  \hfill (4.16)

It remains to determine \( M_0 \). Since \( Y_0 \leq \|\bar{p}\|_{L^\alpha(U \times (0, T))} \), we have sufficient conditions for (4.15):

\[ \begin{align*}
(\theta T)^{-1} M_0^{-\delta_1} &\leq C \|\bar{p}\|_{L^\alpha(U \times (0, T))}^{\delta_1}, \\
T^{1/2 - 1/\alpha} M_0^{-1/\alpha} &\leq C \|\bar{p}\|_{L^\alpha(U \times (0, T))}^{-\delta_2}, \\
(\theta T)^{-\frac{1}{\alpha - a} M_0^{-\delta_3}} &\leq C \|\bar{p}\|_{L^\alpha(U \times (0, T))}^{-\delta_3}, \\
T^{\frac{1}{\alpha - a} M_0^{-\delta_4}} &\leq C \|\bar{p}\|_{L^\alpha(U \times (0, T))}^{\delta_4}.
\end{align*} \]

Solving these inequalities gives

\[ M_0 \geq C \max \left\{ \left(\theta T\right)^{-1/\delta_1} \|\bar{p}\|_{L^\alpha(U \times (0, T))}^{\delta_1}, \ T^{\frac{1}{2 - 1/\alpha}} \|\bar{p}\|_{L^\alpha(U \times (0, T))}^{\delta_2}, \ (\theta T)^{-\frac{1}{\alpha - a}} \|\bar{p}\|_{L^\alpha(U \times (0, T))}^{\delta_3}, \ T^{\frac{1}{\alpha - a}} \|\bar{p}\|_{L^\alpha(U \times (0, T))}^{\delta_4} \right\}. \]  \hfill (4.17)

with an appropriate positive constant \( C \). Choosing \( M_0 \) to be the right-hand side of (4.17) with the sum replacing the maximum, we obtain (4.16) that

\[ \sup_{[\theta T, T, t_0 + T]} \|\bar{p}(t)|_{L^\infty(U)} \leq C \left\{ \left(\theta T\right)^{-1/\delta_1} \|\bar{p}\|_{L^\alpha(U \times (0, T))}^{\delta_1} + \left(T^{\frac{1}{2 - 1/\alpha}} \|\bar{p}\|_{L^\alpha(U \times (T_0, T_0 + T))}^{\delta_2} + (\theta T)^{-\frac{1}{\alpha - a}} \|\bar{p}\|_{L^\alpha(U \times (T_0, T_0 + T))}^{\delta_3} + T^\frac{1}{\alpha - a} \|\bar{p}\|_{L^\alpha(U \times (T_0, T_0 + T))}^{\delta_4} \right) \right\} \leq C \left\{ \left(\theta T\right)^{-1/\delta_1} \|\bar{p}\|_{L^\alpha(U \times (0, T))}^{\delta_1} + \left(T^{\frac{1}{2 - 1/\alpha}} \|\bar{p}\|_{L^\alpha(U \times (T_0, T_0 + T))}^{\delta_2} + (\theta T)^{-\frac{1}{\alpha - a}} \|\bar{p}\|_{L^\alpha(U \times (T_0, T_0 + T))}^{\delta_3} + T^\frac{1}{\alpha - a} \|\bar{p}\|_{L^\alpha(U \times (T_0, T_0 + T))}^{\delta_4} \right) \right\}. \]  \hfill (4.18)

Rewriting the powers in (4.18), noticing that

\[ \frac{1}{p_1} + \delta_1 = 1 + \delta_3, \quad \frac{\alpha}{\alpha - a} + \delta_1 = 1 + \delta_2, \quad \frac{\alpha}{(\alpha - a)p_1} + \delta_1 = 1 + \delta_4, \]

we obtain [122].

**Remark 4.2.** In [122], the norm \( \|\bar{p}\|_{L^\infty(U \times (T_0, T_0 + T))} \) is estimated by a sum of homogeneous terms in \( \|\bar{p}\|_{L^\alpha(U \times (T_0, T_0 + T))} \). Therefore it is appropriate for both small and large \( \|\bar{p}\|_{L^\alpha(U \times (T_0, T_0 + T))} \). This improves our previous interior versions in Theorem 4.1 of [12] and Proposition 3.2 of [13], where the bounds contain an additive constant, which makes them more suitable for large value \( \|\bar{p}\|_{L^\alpha(U \times (T_0, T_0 + T))} \). For doubly nonlinear parabolic equations of \( p \)-Laplacian type, the interior estimate of this kind was obtained in [22] using Moser’s iteration.
To simplify our future estimates, we use the following weaker version of Theorem 4.1.

**Corollary 4.3.** Let $\alpha, T_0, T, \theta, r_1, p_1, q_1, \delta_1, \delta_2, \delta_3, \delta_4$ be as in Theorem 4.1. Then

\[
\sup_{[T_0+\theta T, T_0+T]} \| \bar{p}(t) \|_{L^\infty(U)} \leq C \left( 1 + (\theta T)^{-1/\delta_1} + T^{z_1} \right) \left( 1 + \| \Psi_t \|_{L^{\alpha q_1}(U \times (T_0, T_0+T))} \right)^{z_2} \cdot \left( 1 + \| \bar{p} \|_{L^\alpha(U \times (T_0, T_0+T))} \right)^{z_3},
\]

where

\[
z_1 = \max \left\{ \frac{\alpha - 2}{2(\alpha - a)(1 + \delta_1)}, \frac{\alpha - 1}{\alpha p_1(1 + \delta_3)} \right\},
\]

(4.20)

\[
z_2 = \frac{\alpha}{(\alpha - a)p_1(1 + \delta_4)}.
\]

(4.21)

\[
z_3 = \max \left\{ 1, \frac{\delta_2}{1 + \delta_3} \right\}.
\]

(4.22)

**Proof.** We apply Young’s inequality in (4.2) to the terms involving $(\theta T)^{-1}$, $T$, $\| \Psi_t \|_{L^{\alpha q_1}}$, $\| \bar{p} \|_{L^\alpha}$. First, we note that $\delta_2 > \delta_1 \geq \delta_3$ and $\delta_2 \geq \delta_4 > \delta_3$.

- For the power of $(\theta T)^{-1}$, we have $1/\delta_1 > 1/(\alpha - a)\delta_1$.
- The largest power of $T$ is

\[
z_1 = \max \left\{ \frac{\alpha - 2}{2(\alpha - a)(1 + \delta_1)}, \frac{\alpha - 1}{\alpha p_1(1 + \delta_3)} \right\}.
\]

Note that

\[
\frac{\alpha - 2}{2(\alpha - a)(1 + \delta_1)\alpha} = \frac{\alpha - 2}{2(\alpha - a)(1 + \delta_2)},
\]

\[
\frac{\alpha - 1}{\alpha p_1(1 + \delta_3)} < \frac{\alpha - 1}{\alpha + \delta_1(\alpha - a)p_1} = \frac{\alpha - 1}{(\alpha - a)p_1(1 + \delta_4)}.
\]

Hence $z_1$ is as in (4.20).

- The largest power of $\| \Psi_t \|_{L^{\alpha q_1}}$ is

\[
z_2 = \max \left\{ \frac{1}{p_1(1 + \delta_3)}, \frac{\alpha}{(\alpha - a)p_1(1 + \delta_4)} \right\}.
\]

Since $\frac{1}{p_1(1 + \delta_3)} = \frac{1}{\alpha + (\alpha - a)p_1} = \frac{\alpha}{(\alpha - a)p_1(1 + \delta_4)}$, we have $z_2$ as in (4.21).

- The largest power of $\| \bar{p} \|_{L^\alpha}$ is

\[
z_3 = \left\{ 1, \frac{\delta_1}{1 + \delta_1}, \frac{\delta_2}{1 + \delta_3}, \frac{\delta_3}{1 + \delta_2}, \frac{\delta_4}{1 + \delta_4} \right\}.
\]

Note that $\frac{\delta_1}{1 + \delta_1}, \frac{\delta_2}{1 + \delta_2}, \frac{\delta_3}{1 + \delta_3}, \frac{\delta_4}{1 + \delta_4} < 1$. Hence $z_3$ is as in (4.22). Therefore, we obtain (4.19) from (4.2).

We then derive the $L^\infty$-estimates in terms of the problem’s data.

**Theorem 4.4.** Assume $\alpha$ satisfies (2.17). Let $q_1, \delta_1$ be defined as in Theorem 4.1 and $z_1, z_2, z_3$ be defined as in Corollary 4.3.
Global Estimates for Generalized Forchheimer Flows

(i) If \( 0 < t \leq 3 \) then
\[
\|\bar{p}(t)\|_{L^\infty(U)} \leq Ct^{-\frac{1}{\alpha_1}}(1 + \|\bar{p}_0\|_{L^\alpha(U)} + [EnvA(\alpha, t)]^{\frac{1}{\alpha-a}})^{23}(1 + \|\Psi_t\|_{L^{\alpha_q}(U \times (0, t))})^{22}. \tag{4.23}
\]

If \( t \geq 1 \) then
\[
\|\bar{p}(t)\|_{L^\infty(U)} \leq C(1 + \|\bar{p}_0\|_{L^\alpha(U)} + [EnvA(\alpha, t)]^{\frac{1}{\alpha-a}})^{23}(1 + \|\Psi_t\|_{L^{\alpha_q}(U \times (0, t))})^{22}. \tag{4.24}
\]

(ii) If \( A(\alpha) < \infty \) then
\[
\limsup_{t \to \infty} \|\bar{p}(t)\|_{L^\infty(U)} \leq C(1 + A(\alpha)^{\frac{1}{\alpha-a}})^{23}(1 + \limsup_{t \to \infty} \|\Psi_t\|_{L^{\alpha_q}(U \times (0, t))})^{22}. \tag{4.25}
\]

(iii) If \( \beta(\alpha) < \infty \) then there is \( T > 0 \) such that
\[
\|\bar{p}(t)\|_{L^\infty(U)} \leq C(1 + \beta(\alpha)^{\frac{1}{\alpha-a}} + \|A(\alpha, \cdot)\|_{L^{\alpha-a}(U \times (0, t))})^{23}(1 + \|\Psi_t\|_{L^{\alpha_q}(U \times (t-1, t))})^{22} \tag{4.26}
\]
for all \( t \geq T \).

Proof. (i) Note that \( \alpha = \bar{\alpha} \). Let \( t \in (0, 3] \). Applying (1.19) for \( T = t \) and \( \theta = 1/2 \), we have
\[
\|\bar{p}(t)\|_{L^\infty} \leq C(1 + t^{-\frac{1}{\alpha_1}})(1 + \|\Psi_t\|_{L^{\alpha_q}(U \times (0, t))})^{22}(1 + \|\bar{p}\|_{L^\alpha(U \times (0, t))})^{23}. \tag{4.27}
\]

Using (3.5) to estimate \( \|\bar{p}\|_{L^\alpha(U \times (0, t))} \) in (4.27), we obtain (4.23).

For \( t \geq 1 \), applying (1.19) with \( T_0 = t - 1, T = 1 \) and \( \theta = 1/2 \) we obtain
\[
\|\bar{p}(t)\|_{L^\infty} \leq C(1 + \|\Psi_t\|_{L^{\alpha_q}(U \times (t-1, t))})^{22}(1 + \|\bar{p}\|_{L^\alpha(U \times (t-1, t))})^{23}. \tag{4.28}
\]
Again using (3.5) with noticing that the function \( EnvA(\alpha, t) \) increasing in \( t \), then we obtain (4.24) from (4.28).

(ii) From (4.28) we have
\[
\limsup_{t \to \infty} \|\bar{p}(t)\|_{L^\infty} \leq C(1 + \limsup_{t \to \infty} \|\Psi_t\|_{L^{\alpha_q}(U \times (t-1, t))})^{22}(1 + \limsup_{t \to \infty} \|\bar{p}\|_{L^\alpha(U \times (t-1, t))})^{23}. \tag{4.29}
\]

By (3.6),
\[
\limsup_{t \to \infty} \|\bar{p}\|_{L^\alpha(U \times (t-1, t))} \leq \limsup_{t \to \infty} \int_U |\bar{p}(x, t)|^{\alpha} dx \leq C(1 + A(\alpha))^{\alpha/(\alpha-a)}. \tag{4.29}
\]

Thus (4.25) follows.

(iii) Using (3.7) we have for large \( t \) that
\[
\int_{t-1}^t \int_U |\bar{p}(x, \tau)|^{\alpha} dx d\tau \leq C \left( 1 + \beta(\alpha)^{\frac{1}{\alpha-a}} + \int_{t-1}^t A(\alpha, \tau)^{\frac{\alpha}{\alpha-a}} d\tau \right). \tag{4.30}
\]
Therefore, (4.26) follows from (4.28) and (4.30).
5 Gradient estimates on the boundary

In this section, we estimate the maximum of $|\nabla p|$ on the boundary $\Gamma \times (0, \infty)$. It will be used in estimates of $L^s$-norms for the gradient in section 6 and its $L^\infty$-norm in section 7. We extend Ladyzhenskaya-Ural'tseva’s technique [14] in order to have better estimates for large time.

We rewrite the PDE in (3.1) for $p$ in the non-divergence form as

$$p_t - \sum_{i,j=1}^{n} A_{ij} p_{x_i x_j} = 0,$$

(5.1)

where

$$A_{ij} = A_{ij}(x, t) = K(|\nabla p(x, t)|) \delta_{ij} + \frac{K'(|\nabla p(x, t)|)}{|\nabla p(x, t)|} p_{x_i}(x, t) p_{x_j}(x, t), \quad i, j = 1, 2, \ldots, n.$$ 

Thanks to (2.10)–(2.12) we find that

$$|A_{ij}| \leq (1 + a) K(|\nabla p|),$$

(1−a)K(|\nabla p|) |y|^2 \leq \sum_{i,j=1}^{n} A_{ij} y_i y_j \leq K(|\nabla p|) |y|^2, \quad \forall y \in \mathbb{R}^n.$$

By (2.10), we have more explicit relations:

$$|A_{ij}| \leq (1 + a) d_2 (1 + |\nabla p|)^{-a},$$

(5.2)

$$(1−a)d_1(1 + |\nabla p|)^{-a} |y|^2 \leq \sum_{i,j=1}^{n} A_{ij} y_i y_j \leq d_2 (1 + |\nabla p|)^{-a} |y|^2, \quad \forall y \in \mathbb{R}^n.$$ 

(5.3)

We will establish boundary estimates in case the boundary is flat first. For general boundary, we will flatten it out and hence transform equation (5.1) to a different, but similar, one. To prepare for this transformation, we consider a more general PDE:

$$\mathcal{L} p \overset{\text{def}}{=} p_t - \sum_{i,j=1}^{n} \tilde{A}_{ij} p_{x_i x_j} + \sum_{i=1}^{n} \tilde{b}_i p_{x_i} = 0,$$

(5.4)

where $\tilde{A}_{ij} = \tilde{A}_{ij}(x, t)$, $\tilde{b}_i = \tilde{b}_i(x, t)$ for $i, j = 1, 2, \ldots, n$.

Define for $\xi \geq 0$ the function

$$\tilde{K}(\xi) = \frac{1}{(1 + \xi)^a}.$$

Similar to (5.2) and (5.3), we assume that $\tilde{A}_{ij}$ satisfy

$$(\sum_{i,j=1}^{n} |\tilde{A}_{ij}|^2)^{1/2} \leq c_1 \tilde{K}(|\nabla p|),$$

(5.5)

$$c_2 \tilde{K}(|\nabla p|) |y|^2 \leq \sum_{i,j=1}^{n} \tilde{A}_{ij} y_i y_j \leq c_3 \tilde{K}(|\nabla p|) |y|^2, \quad \forall y \in \mathbb{R}^n,$$

(5.6)

where $c_1, c_2, c_3$ are positive constants. In addition, we also assume that

$$|\tilde{b}(x, t)| \leq c_4 \tilde{K}(|\nabla p|) \text{ with } \tilde{b} = (\tilde{b}_1, \ldots, \tilde{b}_n).$$

(5.7)
for some positive constant $c_4$. Note, particularly when $y = e_k$, a unit vector in the standard basis of $\mathbb{R}^n$, we have from (5.6) that

$$\tilde{A}_{kk} \geq c_2 \tilde{K}(|\nabla p|), \quad k = 1, 2, \ldots, n. \quad (5.8)$$

Denote the ball in $\mathbb{R}^n$ centered at the origin with radius $R$ by $B_R$, and the upper half ball by $B_R^+$, i.e., $B_R^+ = B_R \cap \{x = (x', x_n) : x' \in \mathbb{R}^{n-1}, \ x_n > 0\}$. Also, denote $\Gamma_R = \partial B_R^+ \cap \{x_n = 0\}$-the flat portion of the boundary of $B_R^+$.

We start with estimates on a flat boundary.

**Proposition 5.1.** Let $T > 0$ and $R > 0$. Let $p(x, t) \in C^{1,0}(\bar{B}_R^+ \times [0, T]) \cap C^{2,1}(B_R^+ \times (0, T))$ be a solution of (5.4) in $B_R^+ \times (0, T)$, with $\bar{A}_{ij}$ and $\bar{b}_i$ satisfying (5.5)-(5.7) in $\bar{B}_R^+ \times [0, T]$. Then

$$\max_{\Gamma_R/2 \times [\theta T, T]} |\nabla p(x, t)| \leq C(1 + R^{-2})e^{C' R(1 + (\theta T)^{-1})^{\mu_0}} \exp\left(C' \max_{\bar{B}_R^+ \times [\theta T/2, T]} |\bar{p}|\right) \cdot (1 + \max_{\bar{B}_R^+ \times [0, T]} |\Psi_t|^{\mu_0} + \max_{\bar{B}_R^+ \times [0, T]} |\nabla \Psi|^2 + \max_{\bar{B}_R^+ \times [0, T]} |\nabla^2 \Psi|), \quad (5.9)$$

for any $\theta \in (0, 1)$, where $C, C'$ are positive numbers independent of $R, T, \theta$, and

$$\mu_0 = \frac{2}{2 - a}. \quad (5.10)$$

**Proof.** (a) Let $v = -1 + e^{\kappa_0 \bar{p}}$ with the constant $\kappa_0 > 0$ chosen later. Then

$$v = 0 \text{ on } \partial B_R^+ \cap \{x_n = 0\}. \quad (5.11)$$

For $i, j = 1, 2, \ldots, n$, we have

$$v_t = \kappa_0 e^{\kappa_0 \bar{p}} \bar{p}_t, \quad v_{x_i} = \kappa_0 e^{\kappa_0 \bar{p}} \bar{p}_{x_i}, \quad v_{x_i x_j} = \kappa_0 e^{\kappa_0 \bar{p}} \bar{p}_{x_i x_j} + \kappa_0^2 e^{\kappa_0 \bar{p}} \bar{p}_{x_i} \bar{p}_{x_j}. \quad (5.12)$$

Substituting (5.12) into (5.4) gives

$$\kappa_0^{-1} e^{-\kappa_0 \bar{p}} v_t + \Psi_t - \sum_{i, j = 1}^n \bar{A}_{ij} \left\{ \kappa_0^{-1} e^{-\kappa_0 \bar{p}} v_{x_i x_j} + \Psi_{x_i x_j} - \kappa_0 \bar{p}_{x_i} \bar{p}_{x_j} \right\} + \bar{b} \cdot \left( \kappa_0^{-1} e^{-\kappa_0 \bar{p}} \nabla v + \nabla \Psi \right) = 0.$$

Multiplying by $\kappa_0 e^{\kappa_0 \bar{p}}$, it shows that $v(x, t)$ solves the equation

$$v_t - \sum_{i, j = 1}^n \bar{A}_{ij} v_{x_i x_j} + \bar{b} \cdot \nabla v = - \sum_{i, j = 1}^n \bar{A}_{ij} \kappa_0^2 e^{\kappa_0 \bar{p}} \bar{p}_{x_i} \bar{p}_{x_j} + \kappa_0 e^{\kappa_0 \bar{p}} \left( \sum_{i, j = 1}^n \bar{A}_{ij} \Psi_{x_i x_j} - \Psi_t - \bar{b} \cdot \nabla \Psi \right).$$

Thus, by (5.9) and Cauchy-Schwarz inequality, we have

$$\mathcal{L} v \leq -c_2 \kappa_0^2 e^{\kappa_0 \bar{p}} \tilde{K}(|\nabla p|)|\nabla \bar{p}|^2 + \kappa_0 e^{\kappa_0 \bar{p}} \left( \sum_{i, j = 1}^n |\bar{A}_{ij}|^2 \right)^{1/2} \left( \sum_{i, j = 1}^n |\Psi_{x_i x_j}|^2 \right)^{1/2} + |\bar{b}||\nabla \Psi| + |\Psi_t|. \quad (5.13)$$

Using inequality $|\nabla \bar{p}|^2 = |\nabla p - \nabla \Psi|^2 \geq |\nabla p|^2/2 - |\nabla \Psi|^2$, and inequalities (5.5), (5.7), we get

$$\mathcal{L} v \leq -c_2 \kappa_0^2 e^{\kappa_0 \bar{p}} \tilde{K}(|\nabla p|)|\nabla \bar{p}|^2 + c_2 \kappa_0^2 e^{\kappa_0 \bar{p}} \tilde{K}(|\nabla p|) |\nabla \Psi|^2 + \kappa_0 e^{\kappa_0 \bar{p}} \tilde{K}(|\nabla p|) (c_1 |\nabla^2 \Psi| + c_4 |\nabla \Psi| + |\Psi_t|).$$
Let

\[ M_1 = \max_{B_R \times [0,T]} (c_1|\nabla^2 \Psi| + c_4|\nabla \Psi|), \quad M_2 = \max_{B_R \times [0,T]} |\Psi_t|, \quad M_3 = c_2 \max_{B_R \times [0,T]} |\nabla \Psi|^2. \]

Then

\[
\mathcal{L}v \leq -\frac{c_2}{2}\kappa_0^2 e^{\kappa_0 \bar{p}} \tilde{K}(|\nabla p|)|\nabla p|^2 + \kappa_0 (M_3 \kappa_0 + M_1) e^{\kappa_0 \bar{p}} \tilde{K}(|\nabla p|) + \kappa_0 e^{\kappa_0 \bar{p}} M_2. \tag{5.13}
\]

Let \( \varepsilon > 0 \). We write \( M_2 = \tilde{K}(|\nabla p|) \cdot M_2(1 + |\nabla p|)^a \) and estimate

\[
M_2(1 + |\nabla p|)^a \leq M_2(1 + |\nabla p|^a) \leq M_2 + \varepsilon^{-a/(2-a)} M_2^{2/(2-a)} + \varepsilon |\nabla p|^2.
\]

From this and (5.13), we find that

\[
\mathcal{L}v \leq -\left(\frac{c_2}{2}\kappa_0 - \varepsilon\right) \kappa_0 e^{\kappa_0 \bar{p}} \tilde{K}(|\nabla p|)|\nabla p|^2 + \kappa_0 e^{\kappa_0 \bar{p}} \tilde{K}(|\nabla p|)(\kappa_0 M_3 + M_1 + M_2 + \varepsilon^{-a/(2-a)} M_2^{2/(2-a)}).
\tag{5.14}
\]

(b) We localize the above calculations. Let \( \zeta = \zeta(x,t) \in [0,1] \) be a cut-off function on \( B_R \times (0,T) \) with \( \zeta = 0 \) on \( (B_R \times [\theta T/2,T]) \cup (\partial B_R \times [0,T]) \), and \( \zeta(x,t) = 1 \) on \( B_{R/2} \times (\theta T,T) \), and satisfy

\[
|\nabla \zeta| \leq \frac{C_0}{R}, \quad |\nabla^2 \zeta| \leq \frac{C_0}{R^2}, \quad |\zeta_t| \leq \frac{C_0}{\theta T},
\]

where \( C_0 > 0 \) is independent of \( R, \theta, \) and \( T \). Denote by \( \chi = \chi(t) \) the characteristic function of \([\theta T/T, T]\).

Let \( w = v\zeta^2 \). Then

\[
\mathcal{L}w = \zeta^2 \mathcal{L}v + v \mathcal{L}(\zeta^2) - 2\zeta(\mathcal{A} \nabla v) \cdot \nabla \zeta. \tag{5.15}
\]

For the last term, by (5.5) and (5.12):

\[
|2\zeta(\mathcal{A} \nabla v) \cdot \nabla \zeta| \leq 2c_1 \zeta \tilde{K}(|\nabla p|)|\nabla v||\nabla \zeta| \leq C_1 \zeta_0 e^{\kappa_0 \bar{p}} \tilde{K}(|\nabla p|)|\nabla p|
\leq \kappa_0 e^{\kappa_0 \bar{p}} \tilde{K}(|\nabla p|) \left[C_1 \zeta_0 |\nabla p| + C_1 \zeta \nabla \Psi \right],
\]

where \( C_1 = 2c_1 \). Thus

\[
|2\zeta(\mathcal{A} \nabla v) \cdot \nabla \zeta| \leq \kappa_0 e^{\kappa_0 \bar{p}} \tilde{K}(|\nabla p|)(\varepsilon \zeta^2|\nabla p|^2 + \varepsilon^{-1} C_1^2 \chi + C_1 \zeta |\nabla \Psi|). \tag{5.16}
\]

For the second term on the right-hand side of (5.15),

\[
|v \mathcal{L}(\zeta^2)| \leq |v|2\zeta|\zeta_t| + |v| \left(\sum_{i,j=1}^n |\hat{A}_{ij}|(\zeta^2)_{x_i x_j} + 2|\hat{b}||\nabla \zeta|\right) \leq |v|2\zeta|\zeta_t| + C_2 |v| \chi \tilde{K}(|\nabla p|), \tag{5.17}
\]

where \( C_2 = C_3(1 + R^{-2}) \), with \( C_3 > 0 \) independent of \( R, \theta, T \). Treating the term \( 2|v|\zeta|\zeta_t| \) in (5.17) the same way as we did for \( M_2 \) in (5.13):

\[
2|v|\zeta|\zeta_t| \leq C_4 |v|\zeta \leq C_4 |v|\zeta \tilde{K}(|\nabla p|)(1 + |\nabla p|^a) \leq \tilde{K}(|\nabla p|)(C_4|v|\zeta + C_4 |v|\zeta |\nabla p|^a),
\]

where \( C_4 = 2C_0(\theta T)^{-1} \). Note that

\[
C_4 |v|\zeta |\nabla p|^a = C_4 |v|\zeta^1-a [\varepsilon \kappa_0 e^{\kappa_0 \bar{p}}]^{-a/2} \cdot [\varepsilon^{1/2} \kappa_0^{1/2} e^{\kappa_0 \bar{p}}/2] |\nabla p|^a.
\]
Applying Young’s inequality with power $2/(2 - a)$ and $2/a$ yields

$$2|v|\xi|\xi| \leq \tilde{K}(|\nabla p|)\left[C_4|v|\xi + (C_4|v|\xi)^{2/(2-a)}(\varepsilon\kappa_0 e^{\kappa_0 p})^{-a/(2-a)}\right] + \varepsilon\kappa_0 e^{\kappa_0 p} \tilde{K}(|\nabla p|)\xi^2|\nabla p|^2. \tag{5.18}$$

It follows from (5.17) and (5.18) that

$$|v|\mathcal{L}(\xi^2) \leq \varepsilon\kappa_0 e^{\kappa_0 p} \tilde{K}(|\nabla p|)\xi^2|\nabla p|^2$$

$$+ \tilde{K}(|\nabla p|)\left[C_4|v|\xi + (C_4|v|\xi)^{2/(2-a)}(\varepsilon\kappa_0 e^{\kappa_0 p})^{-a/(2-a)}e^{\kappa_0 p}a/(2-a) + C_2|v|\xi\right]. \tag{5.19}$$

Using (5.11), (5.19) and (5.10) in (5.13), we obtain

$$\mathcal{L} w \leq -\xi^2\left(\frac{C_2}{2}\kappa_0 - 3\varepsilon\right)\kappa_0 e^{\kappa_0 p} \tilde{K}(|\nabla p|)\xi^2$$

$$+ \kappa_0 e^{\kappa_0 p} \tilde{K}(|\nabla p|)\left[\xi^2(\kappa_0 M_3 + M_1 + M_2 + M_2^{2/(2-a)}e^{-a/(2-a)}) + C_1^2\varepsilon - 1 \chi + C_1\xi|\nabla \Psi|\right]$$

$$+ \tilde{K}(|\nabla p|)[C_2\xi|v| + C_4|v|\xi + (C_4|v|\chi)^{2/(2-a)}(\kappa_0 e^{\kappa_0 p})^{-a/(2-a)}e^{\kappa_0 p}a/(2-a)]. \tag{5.20}$$

Choose $\varepsilon = 1$ and $\kappa_0 = 8/c_2$. Then

$$\mathcal{L} w \leq \kappa_0 \chi e^{\kappa_0 p} \tilde{K}(|\nabla p|)\left[C_0 M_3 + M_1 + M_2 + M_2^{2/(2-a)} + C_1^2 + C_1|\nabla \Psi|\right]$$

$$+ \tilde{K}(|\nabla p|)[C_2\chi|v| + C_4|v|\chi + (C_4|v|\chi)^{2/(2-a)}(\kappa_0 e^{\kappa_0 p})^{-a/(2-a)}e^{\kappa_0 p}a/(2-a)]. \tag{5.21}$$

Let $M_4 = \max_{B_R^+ \times [0,T]}|\bar{p}|$. Note that $\chi e^{\kappa_0 p}, \chi|v| \leq e^{\kappa_0 M_4}$. Define

$$M_5 = \kappa_0 e^{\kappa_0 M_4}(\kappa_0 M_3 + M_1 + c_0 M_2 + M_2^{2/(2-a)} + C_1^2 + C_2 + \frac{\max_{B_\bar{R}^+ \times [0,T]}|\nabla \Psi|}{B_\bar{R}^+ \times [0,T]} \max_{B_\bar{R}^+ \times [0,T]}|\nabla \Psi|^2 + \frac{\max_{B_\bar{R}^+ \times [0,T]}|\nabla \Psi|^2}{B_\bar{R}^+ \times [0,T]} \max_{B_\bar{R}^+ \times [0,T]}|\nabla \Psi|^2) + e^{\kappa_0 M_4(2+a)/(2-a)}(1 + \kappa_0^{-a/(2-a)})(C_2 + C_4 + C_4^{2-a/2-a})).$$

We obtain $\mathcal{L} w \leq M_5 \tilde{K}(|\nabla p|)$. Note that

$$M_5 \leq \kappa_0 e^{\kappa_0 M_4}(\kappa_0 M_3 + M_1 + c_0 M_2 + M_2^{2/(2-a)} + C_1^2 + C_2 + \frac{\max_{B_\bar{R}^+ \times [0,T]}|\nabla \Psi|}{B_\bar{R}^+ \times [0,T]} \max_{B_\bar{R}^+ \times [0,T]}|\nabla \Psi|^2 + \frac{\max_{B_\bar{R}^+ \times [0,T]}|\nabla \Psi|^2}{B_\bar{R}^+ \times [0,T]} \max_{B_\bar{R}^+ \times [0,T]}|\nabla \Psi|^2) + e^{\kappa_0 M_4(2+a)/(2-a)}(1 + \kappa_0^{-a/(2-a)})(1 + C_2)2(1 + C_4^{2-a/2-a}) \leq M_6,$$
Choose \( \mu = (1 + c_4)/c_2 \) and
\[
\lambda = M_0 e^{\mu R}/\mu. \tag{5.24}
\]
Then on \( B^*_R \times (0, T] \), \( \mathcal{L} \bar{\omega} \leq 0. \) Since \( p(x,t) \in C^{1,0}(\bar{U} \times [0, T]) \), the function \( \tilde{K}(|\nabla p|) \) is bounded below by a positive number. Then by (5.6), \( \mathcal{L} \) is a parabolic operator. Therefore, the maximum principle for operator \( \mathcal{L} \) implies
\[
\max_{B^*_R \times [0, T]} \bar{w} = \max_{\partial_p(\partial_p^* \times (0, T))} \bar{w}.
\]
Here \( \partial_p \) denotes the parabolic boundary.

When \( (x, t) \in (\mathcal{B}^*_R \times \{0\}) \cup ((\partial \mathcal{B}^*_R \setminus \Gamma) \times (0, T]) \), we have \( \zeta(x, t) = 0 \) hence
\[
\tilde{w}(x, t) = 0 + \lambda e^{-\mu x_n} \leq \lambda.
\]

When \( (x, t) \in \Gamma_R \times (0, T] : v(x, t) = 0 \) hence \( \tilde{w}(x, t) = \lambda \).

Thus,
\[
\max_{B^*_R \times [0, T]} \bar{w} = \max_{\partial_p(\partial_p^* \times (0, T))} \bar{w} \leq \lambda = \tilde{w}(x, t), \quad \forall (x, t) \in \Gamma_R \times (0, T].
\]
Hence, we have \( \bar{w}_{x_n} \leq 0 \) on \( \Gamma_R \times (0, T] \), equivalently, \( w_{x_n} \leq \mu \lambda \) on \( \Gamma_R \times (0, T] \). Note that \( w = v \) on \( \Gamma_R/2 \times [\theta T, T] \). Thus, on \( \Gamma_R/2 \times [\theta T, T] \) we have
\[
\kappa_0 (p_{x_n} - \Psi_{x_n}) = v_{x_n} = w_{x_n} \leq \mu \lambda,
\]
which implies
\[
p_{x_n} \leq \Psi_{x_n} + \kappa_0^{-1} \mu \lambda. \tag{5.25}
\]
Replacing \( p \) by \( -p \), and \( \Psi \) by \( -\Psi \), we obtain again (5.25) with \(-p_{x_n}\) and \(-\Psi_{x_n}\) in place of \(p_{x_n}\) and \(\Psi_{x_n}\). Therefore, we have on \( \Gamma_{R/2} \times [\theta T, T] \) that
\[
|p_{x_n}| \leq C(|\Psi_{x_n}| + \kappa_0^{-1} \mu \lambda).
\]
Combining with the fact \( p_{x_i} = \Psi_{x_i} \) on \( \Gamma_R \times (0, T] \) for \( i < n \), we assert that
\[
|\nabla p| \leq C(|\nabla \Psi| + \kappa_0^{-1} \mu \lambda) \quad \text{on} \quad \Gamma_{R/2} \times [\theta T, T]. \tag{5.26}
\]
Using (5.24) and (5.22) in (5.26), we obtain
\[
\max_{\Gamma_{R/2} \times [\theta T, T]} |\nabla p(x, t)| \leq C(1 + R^{-2} + e^{C'R} (1 + (\theta T)^{-1})^{\mu_0} \exp \left( C' \max_{\partial \mathcal{B}^*_R \times [\theta T/2, T]} |\bar{p}| \right)
\]
\[
\cdot (1 + \max_{\partial \mathcal{B}^*_R \times [0, T]} |\Psi| \frac{2}{2 - n} + \max_{\partial \mathcal{B}^*_R \times [0, T]} |\nabla \Psi|^{2} + \max_{\partial \mathcal{B}^*_R \times [0, T]} |\nabla^2 \Psi|), \tag{5.27}
\]
thus proving (5.9). \( \square \)

The general domain and boundary are treated in the next theorem.

**Theorem 5.2.** Let \( p(x, t) \) be a solution of (5.1). Then for any \( T_0 > 0, T > 0, \) and \( \theta \in (0, 1), \)
\[
\max_{\Gamma \times [T_0 + \theta T, T_0 + T]} |\nabla p(x, t)| \leq C(1 + (\theta T)^{-1})^{\mu_0} \exp \left( C' \max_{U \times [T_0 + \theta T/2, T_0 + T]} |\bar{p}| \right)
\]
\[
\cdot (1 + \max_{U \times [T_0, T_0 + T]} |\Psi| \frac{2}{2 - n} + \max_{U \times [T_0, T_0 + T]} |\nabla \Psi|^{2} + \max_{U \times [T_0, T_0 + T]} |\nabla^2 \Psi|), \tag{5.28}
\]
where \( C, C' \) are positive numbers independent of \( T_0, T, \) and \( \theta. \)
Proof. By replacing $p(x, t)$ with $p(x, T_0 + t)$, we can assume, without loss of generality, that $T_0 = 0$ and $p \in C^{2,1}(U \times [0, T])$. Let $x_0 \in \partial U$. There exist an open neighborhood $V$ of $x_0$, a radius $R > 0$ and $C^2$-bijections $y = \Phi(x) : U \cap V \to B_{2R}^+$ and $x = \Upsilon(y) : B_{2R}^+ \to U \cap V$ such that $\Phi = \Upsilon^{-1}$,

$$
\Phi(\Gamma \cap V) = B_{2R} \cap \{y_n = 0\},
$$

and

$$
\|\Phi\|_{C^2(U \cap V)}, \|\Upsilon\|_{C^2(B_{2R}^+)} \leq c_0 \text{ for some } c_0 > 0.
$$

Define $\tilde{p}(y) = p(\Upsilon(y))$ and $\tilde{\Psi}(y) = \Psi(\Upsilon(y))$. We use the fact that $p(x, t)$ is a solution of (5.31). Simple calculations give

$$
p_{x_i} = \sum_{k=1}^{n} \tilde{p}_{y_k} \Phi^k_{x_i}, \quad p_{x_ix_j} = \sum_{k,l=1}^{n} \tilde{p}_{y_ky_l} \Phi^k_{x_i} \Phi^l_{x_j} + \sum_{k=1}^{n} \tilde{p}_{y_k} \Phi^k_{x_ix_j}
$$

with $\Phi = (\Phi^k)_{k=1,...,n}$. Thus,

$$
p_t - \sum_{i,j=1}^{n} A_{ij} p_{x_ix_j} = \tilde{p}_t - \sum_{k,l=1}^{n} \tilde{A}_{kl} \tilde{p}_{y_ky_l} + \sum_{k=1}^{n} \tilde{b}_k \tilde{p}_{y_k} \overset{\text{def}}{=} \mathcal{L}\tilde{p},
$$

where

$$
\tilde{A}_{kl} = \sum_{i,j=1}^{n} A_{ij} \Phi^k_{x_i} \Phi^l_{x_j}, \quad \tilde{b}_k = -\sum_{i,j=1}^{n} A_{ij} \Phi^k_{x_ix_j}.
$$

Therefore, $\tilde{p}(y, t)$ satisfies the equation

$$
\mathcal{L}\tilde{p}(y, t) = 0 \text{ on } B_{2R}^+ \times (0, T), \quad \tilde{p}(y, t) = \tilde{\Psi}(y, t) \text{ for } y_n = 0.
$$

Now we check the conditions (5.5), (5.6) and (5.7). By chain rule and (5.30),

$$
|\nabla\tilde{p}(y, t)| \leq \tilde{c}_0 |\nabla p(x, t)|, \quad |\nabla p(x, t)| \leq \tilde{c}_0 |\nabla \tilde{p}(y, t)|, \quad |\nabla^2 \tilde{\Psi}| \leq \tilde{c}_0 (|\nabla^2 \tilde{\Psi}| + |\nabla \tilde{\Psi}|), \quad |\nabla^2 \tilde{\Psi}| \leq \tilde{c}_0 (|\nabla^2 \tilde{\Psi}| + |\nabla \tilde{\Psi}|),
$$

for some $\tilde{c}_0 \geq 1$. We have for any $k, l = 1, 2, \ldots, n$ that

$$
|\tilde{A}_{kl}| = |\sum_{i,j=1}^{n} A_{ij} \Phi^k_{x_i} \Phi^l_{x_j}| \leq n^2 \tilde{c}_0^2 \max_{1 \leq i,j \leq n} |A_{ij}| \leq n^2 \tilde{c}_0^2 (1 + a) d_2 (1 + |\nabla p|)^{-a}
\leq n^2 \tilde{c}_0^2 (1 + a) d_2 (1 + \tilde{c}_0 |\nabla \tilde{p}|)^{-a} \leq n^2 \tilde{c}_0^2 (1 + a) d_2 \tilde{c}_0^a (1 + |\nabla \tilde{p}|)^{-a}.
$$

This yields (5.5). For all $\xi \in \mathbb{R}^n$,

$$
\sum_{k,l=1}^{n} \tilde{A}_{kl} \xi_k \xi_l = \sum_{k,l=1}^{n} \sum_{i,j=1}^{n} A_{ij} \Phi^k_{x_i} \Phi^l_{x_j} \xi_k \xi_l = \sum_{i,j=1}^{n} A_{ij} \eta_i \eta_j
$$

with $\eta = (D\Phi)\xi$, i.e., $\xi = (D\Upsilon)\eta$. By (5.33) we have

$$
(1 - a) d_1 (1 + |\nabla p|)^{-a} |\eta|^2 \leq \sum_{i,j=1}^{n} A_{ij} \eta_i \eta_j \leq d_2 (1 + |\nabla p|)^{-a} |\eta|^2.
$$
Note that $c_0^{-1}|\xi| \leq |\eta| \leq c_0|\xi|$. It follows from (5.34) that
\[
\sum_{k,l=1}^{n} \mathcal{A}_{kl} \xi_k \xi_l \leq c_0^2 d_2 (1 + c_0^{-1} |\nabla y \hat{p}|)^{-a} |\xi|^2 \leq c_0^2 d_2 c_0^a (1 + |\nabla y \hat{p}|)^{-a} |\xi|^2,
\]
and
\[
\sum_{k,l=1}^{n} \mathcal{A}_{kl} \xi_k \xi_l \geq (1 - a) c_0^{-2} d_1 (1 + c_0 |\nabla y \hat{p}|)^{-a} |\xi|^2 \geq (1 - a) c_0^{-2} d_1 c_0^{-a} (1 + |\nabla y \hat{p}|)^{-a} |\xi|^2,
\]
hence (5.6) is satisfied. Next, we bound $|\hat{b}_k|$ for each $k = 1, 2, \ldots, n$ by
\[
|\hat{b}_k| \leq \sum_{i,j=1}^{n} |A_{ij} \Psi^k_{x,x,j}| \leq c_0 n^2 (1 + a) d_2 (1 + |\nabla p|)^{-a} \leq c_0 n^2 (1 + a) d_2 c_0^a (1 + |\nabla y \hat{p}|)^{-a}.
\]
Hence we have (5.7).

Applying estimate (5.9) from Proposition 5.1 to $\hat{p}$, $\hat{\Psi}$ and equation (5.31), we obtain
\[
\max_{\Gamma \cap B_{\rho}(x_0) \times [\theta T, T]} |\nabla \hat{y} \hat{p}(y, t)| \leq \hat{C}_R (1 + (\theta T)^{-1})^{\mu_0} \exp \left( C' \max_{B_R^+ \times [\theta T/2, T]} |\hat{p} - \hat{\Psi}| \right) \cdot \left( 1 + \max_{B_R^+ \times [0, T]} |\hat{\Psi}|^{\mu_0} + \max_{B_R^+ \times [0, T]} |\nabla \hat{\Psi}|^{2} + \max_{B_R^+ \times [0, T]} |\nabla^2 \hat{\Psi}| \right). \tag{5.35}
\]
Let $\rho > 0$ sufficiently small such that $U \cap B_\rho(x_0) \subset \Upsilon(B_R^+)$, we can use (5.29), (5.35) and (5.32), it follows that
\[
\max_{(\Gamma \cap B_\rho(x_0) \times [\theta T, T]} |\nabla \hat{p}(x, t)| \leq C_R (1 + (\theta T)^{-1})^{\mu_0} \exp \left( C' \max_{U \times [\theta T/2, T]} |\hat{p}| \right) \cdot \left( 1 + \max_{U \times [0, T]} |\Psi_t|^{\mu_0} + \max_{U \times [0, T]} |\nabla \Psi|^{2} + \max_{U \times [0, T]} |\nabla^2 \Psi| \right). \tag{5.36}
\]
By Cauchy’s inequality for the last $|\nabla \Psi|$, we obtain
\[
\max_{(\Gamma \cap B_\rho(x_0) \times [\theta T, T]} |\nabla \hat{p}(x, t)| \leq C_R (1 + (\theta T)^{-1})^{\mu_0} \exp \left( C' \max_{U \times [\theta T/2, T]} |\hat{p}| \right) \cdot \left( 1 + \max_{U \times [0, T]} |\Psi_t|^{\mu_0} + \max_{U \times [0, T]} |\nabla \Psi|^{2} + \max_{U \times [0, T]} |\nabla^2 \Psi| \right). \tag{5.37}
\]
By using a finite open covering of $\Gamma$, we obtain the desired estimate (5.28) from (5.37). The proof is complete. $\square$

The bounds of $\|\nabla p(t)\|_{L^\infty(\Gamma)}$, in fact, can be expressed in terms of the initial and boundary data as follows.

**Corollary 5.3.** Let $p(x, t)$ be a solution of (3.7), and let $\alpha$ satisfy (2.13).

(i) If $0 < t \leq 3$ then
\[
\|\nabla p(t)\|_{L^\infty(\Gamma)} \leq C t^{-\mu_0} (1 + \max_{t/4, t} (\|\Psi_t\|_{L^\infty(U)}^{\mu_0} + \|\nabla \Psi\|_{L^\infty(U)}^2 + \|\nabla^2 \Psi\|_{L^\infty(U)})) \cdot \exp \left( C t^{-\frac{1}{3}} (1 + \|\tilde{p}_0\|_{L^\infty(U)})^{2} (1 + [EnvA(\alpha, t)]^{-\frac{1}{2}})^{2} (1 + \|\Psi_t\|_{L^\infty(U \times (0, t))})^{2} \right). \tag{5.38}
\]
If \( t > 1 \) then
\[
\|\nabla p(t)\|_{L^\infty(V)} \leq C(1 + \max_{[t-1,t]}(\|\Psi_t\|_{L^\infty} + \|\nabla \Psi\|_{L^\infty}^2 + \|\nabla^2 \Psi\|_{L^\infty}))
\]
\[
\cdot \exp \left\{ C' \left(1 + \|\bar{p}_0\|_{L^\infty(U)} \right)^{2\beta} (1 + [\text{Env}A(\alpha, t)]^{-\frac{1}{\alpha}})^{2\beta} (1 + \|\Psi_t\|_{L^{2q}(U \times (t-1,t))})^{2\beta} \right\}. \tag{5.39}
\]

(ii) If \( A(\alpha) < \infty \) then
\[
\lim_{t \to \infty} \|\nabla p(t)\|_{L^\infty(V)} \leq C(1 + \max_{[t-1,t]}(\|\Psi_t\|_{L^\infty} + \|\nabla \Psi\|_{L^\infty}^2 + \|\nabla^2 \Psi\|_{L^\infty}))
\]
\[
\cdot \exp \left\{ C' \left(1 + A(\alpha)^{-\frac{1}{\alpha}} \right)^{2\beta} (1 + \lim_{t \to \infty} \|\Psi_t\|_{L^{2q}(U \times (t-1,t))})^{2\beta} \right\}. \tag{5.40}
\]

(iii) If \( \beta(\alpha) < \infty \) then there is \( T > 0 \) such that for all \( t \geq T \)
\[
\|\nabla p(t)\|_{L^\infty(V)} \leq C(1 + \max_{[t-1,t]}(\|\Psi_t\|_{L^\infty} + \|\nabla \Psi\|_{L^\infty}^2 + \|\nabla^2 \Psi\|_{L^\infty}))
\]
\[
\cdot \exp \left\{ C' \left(1 + \beta(\alpha)^{-\frac{1}{\alpha}} \right)^{2\beta} (1 + \lim_{t \to \infty} \|\Psi_t\|_{L^{2q}(U \times (t-1,t))})^{2\beta} \right\}. \tag{5.41}
\]

Proof. (i) For \( 0 < t \leq 3 \), applying Theorem 5.2 to \( T_0 = t/4 \), \( T = 3t/4 \), \( \theta = 1/3 \) and using estimate (4.24) we have
\[
\|\nabla p(t)\|_{L^\infty(V)} \leq C t^{-\mu_0} \left(1 + \max_{[t/4,t]}(\|\Psi_t\|_{L^\infty} + \|\nabla \Psi\|_{L^\infty}^2 + \|\nabla^2 \Psi\|_{L^\infty})
\]
\[
\cdot \exp \left\{ C' \left(1 + \frac{1}{3t} \right)^{2\beta} (1 + \|\bar{p}_0\|_{L^\infty(U)} + [\text{Env}A(\alpha, t)]^{-\frac{1}{\alpha}})^{2\beta} (1 + \|\Psi_t\|_{L^{2q}(U \times (0,t))})^{2\beta} \right\}. \tag{5.35}
\]

Then (5.35) follows.

When \( t > 1 \), applying Theorem 5.2 to \( T_0 = t-1 \), \( T = 1 \), \( \theta = 1/2 \) gives
\[
\|\nabla p(t)\|_{L^\infty(V)} \leq C(1 + \max_{[t-1,t]}(\|\Psi_t\|_{L^\infty} + \|\nabla \Psi\|_{L^\infty}^2 + \|\nabla^2 \Psi\|_{L^\infty})) \cdot \exp \left\{ C' \max_{[t-3/4,t]} \|\bar{p}\|_{L^\infty}\right\}. \tag{5.42}
\]

Using estimate (4.24) we have
\[
\|\nabla p(t)\|_{L^\infty(V)} \leq C(1 + \max_{[t-1,t]}(\|\Psi_t\|_{L^\infty} + \|\nabla \Psi\|_{L^\infty}^2 + \|\nabla^2 \Psi\|_{L^\infty}))
\]
\[
\cdot \exp \left\{ C' \left(1 + \|\bar{p}_0\|_{L^\infty(U)} + [\text{Env}A(\alpha, t)]^{-\frac{1}{\alpha}}\right)^{2\beta} (1 + \|\Psi_t\|_{L^{2q}(U \times (t-1,t))})^{2\beta} \right\}. \tag{5.39}
\]

Then (5.39) follows.

(ii) If \( A(\alpha) < \infty \) then taking limit superior of (5.42) and using (4.25) yield
\[
\lim_{t \to \infty} \|\nabla p(t)\|_{L^\infty(V)} \leq C(1 + \lim_{t \to \infty} \max_{[t-1,t]}(\|\Psi_t\|_{L^\infty} + \|\nabla \Psi\|_{L^\infty}^2 + \|\nabla^2 \Psi\|_{L^\infty}))
\]
\[
\cdot \exp \left\{ C' \left(1 + A(\alpha)^{-\frac{1}{\alpha}}\right)^{2\beta} (1 + \lim_{t \to \infty} \|\Psi_t\|_{L^{2q}(U \times (t-1,t))})^{2\beta} \right\}. \tag{5.40}
\]

Thus we obtain (5.40).

(iii) Using (4.26) in (5.42) gives
\[
\|\nabla p(t)\|_{L^\infty(V)} \leq C(1 + \max_{[t-1,t]}(\|\Psi_t\|_{L^\infty} + \|\nabla \Psi\|_{L^\infty}^2 + \|\nabla^2 \Psi\|_{L^\infty}))
\]
\[
\cdot \exp \left\{ C' \left(1 + \beta(\alpha)^{-\frac{1}{\alpha}} \right)^{2\beta} (1 + \lim_{t \to \infty} \|\Psi_t\|_{L^{2q}(U \times (t-1,t))})^{2\beta} \right\}. \tag{5.41}
\]

for all \( t \geq T \) with some \( T > 0 \). This proves (5.41). \( \square \)
6 \textit{L}^s\textit{-estimates for the gradient}

In this section, we estimate the pressure gradient in \textit{L}^s\textit{-norm for any } 0 < s < \infty. Throughout this section, \( p(x,t) \) is a solution of \((3.1)\). First, we establish the basic step for the Ladyzhenskaya-Ural'tseva iteration.

**Lemma 6.1.** Let \( s > 0 \), \( T_0 \geq 0 \), and \( T > T' \geq 0 \). Define
\[
M_b = \max_{\Gamma \times [T_0 + T', T_0 + T]} |\nabla p| \quad \text{and} \quad v = \max\{|\nabla p|^2 - M_b^2, 0\}.
\]

Let \( \zeta(t) \) be a smooth cut-off function on \([T_0, T_0 + T]\) with \( \zeta = 0 \) on \([T_0, T_0 + T']\). Then
\[
\sup_{[T_0, T_0 + T]} \left| \int_U v^{s+1}(x,t) \zeta \, dx + \int_{T_0}^{T} K(|\nabla p|)|\nabla^2 p|^2 v^s \zeta \, dx \, dt \right| \leq C \int_{T_0}^{T} \int_U v^{s+1}|\zeta| \, dx \, dt. \tag{6.1}
\]

**Proof.** Without loss of generality, assume \( T_0 = 0 \). Note that \( v\zeta = 0 \) on \( \Gamma \times [0, T] \). Denote \( \chi = \chi_{\{v > 0\}} \). Multiplying the equation \((6.1)\) by \( -\nabla \cdot (v^s \zeta \nabla p) \), integrating the resultant over \( U \) and using integration by parts, we obtain
\[
\frac{1}{2s + 2} \frac{d}{dt} \int_U v^{s+1} \zeta \, dx = -\sum_{i,j=1}^{n} \int_U \partial_j (K(|\nabla p|)\partial_i p) \partial_i (v^s \partial_j p \zeta) \, dx + \frac{1}{2s + 2} \int_U v^{s+1} \zeta \, dx. \tag{6.2}
\]
Using the product rule for the first term in the right-hand side of \((6.2)\) (see Lemma 3.6 of [13]) we rewrite above equation as
\[
\frac{1}{2s + 2} \frac{d}{dt} \int_U v^{s+1} \zeta \, dx = -\sum_{i,j=1}^{n} \int_U \left[ \partial_j (K(|\nabla p|)y_i) \right]_{y=\nabla p} \partial_j \partial_i p \zeta \, dx \\
- s \sum_{i,j=1}^{n} \sum_{l,m=1}^{n} \int_U \left[ \partial_j (K(|\nabla p|)y_i) \right]_{y=\nabla p} \partial_j \partial_i p \zeta \, dx + \frac{1}{2s + 2} \int_U v^{s+1} \zeta \, dx.
\]

We denote the three terms on the right-hand side by \( I_1 \), \( I_2 \), and \( I_3 \). It follows from the calculations in Lemma 3.6 of [13] that
\[
I_1 \leq -(1 - a) \sum_{j=1}^{n} \int_U K(|\nabla p|)|\nabla (\partial_j p)|^2 v^s \zeta \, dx,
\]
\[
I_2 \leq -(1 - a) s \int_U K(|\nabla p|) \left| \nabla \left( \frac{1}{2} |\nabla p|^2 \right) \right|^2 v^{s-1} \chi \zeta \, dx \leq 0.
\]
Combining these estimates, we find that
\[
\frac{1}{2s + 2} \frac{d}{dt} \int_U v^{s+1} \zeta \, dx + (1 - a) \int_U K(|\nabla p|)|\nabla^2 p|^2 v^s \zeta \, dx \leq \frac{1}{2s + 2} \int_U v^{s+1} \zeta \, dx. \tag{6.3}
\]
Inequality \((6.1)\) follows directly by integrating \((6.3)\) from 0 to \( T \). \( \square \)

In the following proposition, we iterate the inequality in Lemma 6.1 in order to estimate \( W^{1,s} \)-norm of \( p \) in term of its \( W^{1,2-a} \) and \( L^\infty \) norms.
Lemma 6.2. Let $s \geq 1$, $T_0 \geq 0$, $T > 0$, and $0 < \theta' < \theta < 1$. Define
\[
M_b = \max_{[T_0+\theta'T, T_0+T]} \|\nabla p\|_{L^\infty(\Gamma)} \quad \text{and} \quad v = \max\{\|\nabla p\|^2 - M_b^2, 0\}.
\]
Then
\[
\int_{T_0+T}^{T_0+T} \int_{U} K(|\nabla p|)v^s \, dx\, dt \leq C T U(s) + C d_0 \frac{s-1}{a} \int_{T_0+T}^{T_0+T} \int_{U} (1 + |\nabla p|^{2-a}) \, dx\, dt,
\]
where constant $C > 0$ is independent of $T_0$, $T$, $\theta$, and $\theta'$,
\[
U(s) = \begin{cases} 0, & \text{if } 1 \leq s \leq 3 - a, \\ d_0 + d_0 \frac{s-1}{a} + \frac{1}{a}, & \text{if } s > 3 - a, \end{cases}
\]
\[
d_0 = N_0^4 ((\theta' - \theta) T)^{-2} + M_b^4, \quad \text{with} \quad N_0 = \sup_{[T_0+\theta'T, T_0+T]} \|p\|_{L^\infty(U)}.
\]

Proof. Without loss of generality, assume $T_0 = 0$. The proof consists of three steps.

Step 1. Let $\zeta(t)$ be the cut-off function with $\zeta = 0$ for $t \leq \theta'T$ and $\zeta = 1$ for $t \geq \theta'T$. For $s \geq 0$, by applying Lemma 2.3 with $k = 0$, and $s + 1$ in place of $s$, multiplying (6.1) by $\zeta^2(t)$ and integrating from 0 to $T$, we have
\[
\int_{0}^{T} \int_{U} K(|\nabla p|)v^{s+2} \zeta^2 \, dx\, dt \leq C \max |p|^2 \int_{0}^{T} \int_{U} K(|\nabla p|)|\nabla^2 p|^2 v^s \zeta^2 \, dx\, dt + CM_b^4 \int_{0}^{T} \int_{U} K(|\nabla p|)v^s \zeta^2 \, dx\, dt.
\]

To estimate the first integral on the right-hand side, we use (6.1) with $\zeta^2$ in place of $\zeta$, and find that
\[
\int_{0}^{T} \int_{U} K(|\nabla p|)v^{s+2} \zeta^2 \, dx\, dt \leq CN_0^2 \int_{0}^{T} \int_{U} v^{s+1} \zeta_t \, dx\, dt + CM_b^4 \int_{0}^{T} \int_{U} K(|\nabla p|)v^s \zeta^2 \, dx\, dt.
\]

Now using Young’s inequality we have
\[
\int_{0}^{T} \int_{U} K(|\nabla p|)v^{s+2} \zeta^2 \, dx\, dt \leq \frac{1}{2} \int_{0}^{T} \int_{U} K(|\nabla p|)v^{s+2} \zeta^2 \, dx\, dt + CN_0^4 \int_{0}^{T} \int_{U} K^{-1}(|\nabla p|)v^s \zeta_t^2 \, dx\, dt + CM_b^4 \int_{\theta T}^{T} \int_{U} K(|\nabla p|)v^s \zeta^2 \, dx\, dt.
\]

Thus
\[
\int_{0}^{T} \int_{U} K(|\nabla p|)v^{s+2} \zeta^2 \, dx\, dt \leq CN_0^4 \int_{0}^{T} \int_{U} K^{-1}(|\nabla p|)v^s \zeta_t^2 \, dx\, dt + CM_b^4 \int_{\theta T}^{T} \int_{U} K(|\nabla p|)v^s \zeta^2 \, dx\, dt.
\]

Note that
\[
K^{-1}(|\nabla p|)v^s \leq CK(|\nabla p|)(1 + |\nabla p|)^{2a} v^s \leq CK(|\nabla p|)(1 + (v + M_b^2) v^s) v^s \leq CK(|\nabla p|)(1 + v^{s+a} + M_b^2 v^s) \leq CK(|\nabla p|)(v^{s+a} + 1 + M_b^{2(s+a)}),
\]
and
\[
K(|\nabla p|)v^s \leq CK(|\nabla p|)(1 + v^{s+a}).
\]
Thus (6.8) implies
\[
\int_0^T \int_U K(|\nabla p|)v^{s+2}\zeta dxdt \leq C[N_0^4((\theta - \theta')T)^{-2} + M_b^4] \int_0^T \int_U K(|\nabla p|)v^{s+a} \chi_{\zeta > 0} dxdt
\]
\[+ C \int_0^T \int_U [N_0^4((\theta - \theta')T)^{-2}(1 + M_b^{2(s+a)}) + M_b^4] \chi_{\zeta > 0} dxdt.
\]
Therefore,
\[
\int_{\theta T}^T \int_U K(|\nabla p|)v^{s+2}\zeta dxdt \leq C[N_0^4((\theta - \theta')T)^{-2} + M_b^4] \int_{\theta T}^T \int_U K(|\nabla p|)v^{s+a}dxdt
\]
\[+ CT[N_0^4((\theta - \theta')T)^{-2}(1 + M_b^{2(s+a)}) + M_b^4].
\]
Let \( N_1 = TN_0^4((\theta - \theta')T)^{-2} \) and \( N_2 = Td_0 \). Then
\[
\int_{\theta T}^T \int_U K(|\nabla p|)v^{s+2}\zeta dxdt \leq C d_0 \int_{\theta T}^T \int_U K(|\nabla p|)v^{s+a} dxdt + CN_1M_b^{2(s+a)} + CN_2. \tag{6.9}
\]
Equivalently, for \( s \geq 2 \)
\[
\int_{\theta T}^T \int_U K(|\nabla p|)v^{s}\zeta dxdt \leq C d_0 \int_{\theta T}^T \int_U K(|\nabla p|)v^{s-(2-a)}dxdt + CN_1(1 + M_b^{2})^{s-(2-a)} + CN_2. \tag{6.10}
\]

**Step 2.** We prove (6.3) for \( s \in [1, 3-a] \). First,
\[
\int_{\theta T}^T \int_U K(|\nabla p|)v^{3-a}\zeta dxdt \leq \int_{\theta T}^T \int_U K(|\nabla p|)|\nabla p|^2 dxdt \leq \int_{\theta T}^T \int_U (1 + |\nabla p|^{2-a}) dxdt. \tag{6.11}
\]
This yields (6.3) for \( s = 1 \).

Second, let \( s = 1 - a \) in (6.8),
\[
\int_{\theta T}^T \int_U K(|\nabla p|)v^{3-a} dxdt \leq CN_0^4((\theta - \theta')T)^{-2} \int_{\theta T}^T \int_U (1 + |\nabla p|)^a v^{1-a} dxdt + CM_b^4 \int_{\theta T}^T \int_U v^{1-a} dxdt
\]
\[\leq CN_0^4((\theta - \theta')T)^{-2} \int_0^T \int_U (1 + |\nabla p|)^a |\nabla p|^{2(1-a)} dxdt + CM_b^4 \int_0^T \int_U |\nabla p|^{2(1-a)} dxdt.
\]
Therefore,
\[
\int_{\theta T}^T \int_U K(|\nabla p|)v^{3-a}\zeta dxdt \leq C d_0 \int_0^T \int_U (1 + |\nabla p|^{2-a}) dxdt \tag{6.12}
\]
This implies (6.4) when \( s = 3-a \).

Third, when \( s \in (1, 3-a) \), let \( \beta \) be the number in (0, 1) such that \( \frac{1}{s} = \frac{1-\beta}{1} + \frac{\beta}{s-a} \). Then, by interpolation inequality:
\[
\left( \int_{\theta T}^T \int_U K(|\nabla p|)v^{s}\zeta dxdt \right)^\frac{1}{s} \leq \left( \int_{\theta T}^T \int_U K(|\nabla p|)v^{\frac{\beta a}{s-a}} dxdt \right)^\frac{1}{\frac{\beta a}{s-a}} \left( \int_{\theta T}^T \int_U K(|\nabla p|)v^{3-a}\zeta dxdt \right)^\frac{\beta a}{3-a}.
\]
Using (6.12) to estimate the last double integral, we obtain
\[
\int_{\theta T}^T \int_U K(|\nabla p|)v^{s}\zeta dxdt \leq C d_0^{\frac{\beta a}{s-a}} \left( \int_0^T \int_U (1 + |\nabla p|^{2-a}) dxdt \right)^{s(1-\beta + \frac{\beta a}{s-a})}.
\]
Note that $\beta s/(3-a) = (s-1)/(2-a)$. Then

$$\int_{\Theta T} \int_U K(\nabla p)|\nabla u|^s\,dx \, dt \leq C d_0^m \int_{\Theta T} \int_U (1 + |\nabla p|^{2-a})\,dx \, dt. \quad (6.13)$$

Therefore, (6.11) and (6.13) imply (6.4) for $s \in [1, 3-a]$.

**Step 3.** When $s > 3-a$, let $m \in \mathbb{N}$ such that

$$\frac{s - (3-a)}{2-a} \leq m < \frac{s - (3-a)}{2-a} + 1,$$

then $s-m(2-a) \in (1, 3-a]$. For each $k = 1, 2, \ldots, m$, let $\theta_k = \theta - (\theta-\theta')k/m$ and $Q_k = U \times (\theta_k T, T)$. Also, let $\zeta_k(t)$ be a smooth cut-off function which is equal to one on $Q_k$ and zero on $Q_T \setminus Q_k$. There is a positive constant $c > 0$ depending on $U$, such that $|\zeta_k| \leq c[(\theta - \theta') T]^{-1}$, for all $k = 1, 2, \ldots, m$.

From (6.10) we have

$$\int_{\Theta T} \int_U K(\nabla p)|\nabla u|^s\,dx \, dt \leq C d_0^m \int_{\Theta T} \int_U K(\nabla p)|\nabla u|^{s-m(2-a)}\,dx \, dt$$

$$+ C N_1 \left[M_b^{2(s-(2-a))} + d_0 M_b^{2(s-2(2-a))} + d_0^2 M_b^{2(s-3(2-a))} + \ldots + d_0^{m-1} M_b^{2(s-m(2-a))}\right]$$

$$+ C N_2 (1 + d_0 + d_0^2 + \ldots + d_0^{m-1}).$$

Using inequality $\sum_{i=0}^{m-1} z^i \leq (1 + z)^{m-1}$, where $z > 0$, we infer

$$\int_{\Theta T} \int_U K(\nabla p)|\nabla u|^s\,dx \, dt \leq C d_0^m \int_{\Theta T} \int_U K(\nabla p)|\nabla u|^{s-m(2-a)}\,dx \, dt$$

$$+ C N_1 M_b^{2(s-(2-a))}[1 + d_0 M_b^{2(2-a)}]^{m-1} + CT d_0(1 + d_0)^{m-1}.$$

Therefore we obtain

$$\int_{\Theta T} \int_U K(\nabla p)|\nabla u|^s\,dx \, dt \leq C d_0^m \int_{\Theta T} \int_U K(\nabla p)|\nabla u|^{s-m(2-a)}\,dx \, dt$$

$$+ C N_1 M_b^{2(s-(2-a))}[1 + d_0 M_b^{-2(2-a)}]^{m-1} + CT d_0(1 + d_0)^{m-1}.$$

Since $s-m(2-a) \in (1-a, 3-a]$, estimating the double integral on the right-hand side by (6.13) gives

$$\int_{\Theta T} \int_U K(\nabla p)|\nabla u|^s\,dx \, dt \leq C N_1 M_b^{2(s-(2-a))}[1 + d_0 M_b^{-2(2-a)}]^{m-1} + CT d_0(1 + d_0)^{m-1}$$

$$+ C d_0^{\frac{s-m(2-a)-1}{2-a}} \int_{\Theta T} \int_U (1 + |\nabla p|^{2-a})\,dx \, dt.$$

Since $m \leq \frac{s-1}{2-a}$ and $\frac{s-m(2-a)-1}{2-a} + m = \frac{s-1}{2-a}$, then

$$\int_{\Theta T} \int_U K(\nabla p)|\nabla u|^s\,dx \, dt \leq C N_1 M_b^{2(s-(2-a))}[1 + d_0 M_b^{-2(2-a)}]^{\frac{s-1}{2-a}} + CT d_0(1 + d_0)^{\frac{s-1}{2-a}}$$

$$+ C d_0^{\frac{s-1}{2-a}} \int_{\Theta T} \int_U (1 + |\nabla p|^{2-a})\,dx \, dt \leq C N_1 (M_b^{2(s-(2-a))} + d_0^{\frac{s-1}{2-a}} M_b^2) + CT(d_0 + d_0^{\frac{s-1}{2-a}})$$

$$+ C d_0^{\frac{s-1}{2-a}} \int_{\Theta T} \int_U (1 + |\nabla p|^{2-a})\,dx \, dt.$$
Note that
\[ N_1(M_b^{2s-(2-a)} + d_0^{\frac{s-a}{2}} - 1 M_b^2) + T(d_0 + d_0^{\frac{s-a}{2}}) \leq T(d_0 d_0^{2s-(2-a)/4} + d_0 d_0^{\frac{s-a}{2}} + d_0^{1/2} + d_0 + d_0^{\frac{s-a}{2}}) \]
\[ = T(d_0^{\frac{s-a}{2}} + d_0^{\frac{s-a}{2}} + d_0 + d_0^{\frac{s-a}{2}}) \leq CT(d_0^{\frac{s-a}{2}} + d_0^{\frac{s-a}{2}} + d_0) \leq CT(d_0 + d_0^{\frac{s-a}{2}} + d_0) \]
since \( 1 < \frac{s-a}{2} < \frac{s-a}{2} + \frac{1}{2} \) for \( s > 3 - a \). So,
\[ \int_0^T \int_U K(|\nabla p|) v^s dx dt \leq CT U(s) + C d_0^{\frac{s-a}{2}} \int_0^T (1 + |\nabla p|^{2-a}) dx dt. \quad (6.14) \]
This completes the proof of (6.14) for all \( s \geq 1 \).

From the estimates of \( K(|\nabla p|) v^s \) is Lemma 6.2, we derive direct estimates for \(|\nabla p|^s\).

**Proposition 6.3.** Let \( T_0 \geq 0, \ T > 0 \) and \( \theta \in (0, 1) \). If \( s \geq 2 - a \) then
\[ \int_{T_0+\theta T}^{T_0+T} \int_U |\nabla p|^s dx dt \leq CT(1 + D^s) + CD^{\frac{s}{2-a} - 1} \int_{T_0}^{T_0+T} \int_U |\nabla p|^{2-a} dx dt, \quad (6.15) \]
and if \( s > 2 \) then
\[ \sup_{[T_0+\theta T,T_0+T]} \int_U |\nabla p|^s dx \leq C \theta^{-1}(1 + D^s) + C(\theta T)^{-1} D^{\frac{s}{2-a} - 1} \int_{T_0}^{T_0+T} \int_U |\nabla p|^{2-a} dx dt, \quad (6.16) \]
where constant \( C > 0 \) is independent of \( T_0, T, \) and \( \theta \),
\[ \tilde{s} = \begin{cases} \max\{\frac{s+a}{2}, \frac{s-a}{2} - 1\}, & \text{if } 2 - a \leq s \leq 3(2 - a), \\ \frac{s}{2-a}, & \text{if } s > 3(2 - a), \end{cases} \quad (6.17) \]
and
\[ D = (\theta T)^{-1} \sup_{[T_0+\theta T/2,T_0+T]} ||p||_{L^\infty(U)} + \sup_{[T_0+\theta T/2,T_0+T]} |\nabla p|_{L^\infty(\Gamma)}. \quad (6.18) \]

**Proof.** Without loss of generality, assume \( T_0 = 0 \). Let \( M_0, v \) and \( d_0 \) be defined as in Lemma 6.2.

**Proof of (6.15).** From the definition of the function \( v \) and inequality \(|a - b| \geq 2^{-\gamma} a^\gamma - b^\gamma\) for \( a, b \geq 0, \gamma > 0 \), we have
\[ \int_0^T \int_U K(|\nabla p|)|\nabla p|^{2s} dx dt \leq \int_0^T \int_U K(|\nabla p|) v^s dx dt + CT M_0^{2s}. \]
We apply (6.4) with \( \theta' = \theta/2 \) to estimate the integral on the right-hand side. It results in
\[ \int_0^T \int_U K\langle |\nabla p| \rangle |\nabla p|^{2s} dx dt \leq CT(1 + D^{\frac{s}{2-a}}) + C d_0^{\frac{s-a}{2-a}} \int_0^T \int_U (1 + |\nabla p|^{2-a}) dx dt. \quad (6.19) \]
For \( s > 3 - a \), then by definition (6.5) and then Young’s inequality, it follows
\[ \int_0^T \int_U K\langle |\nabla p| \rangle |\nabla p|^{2s} dx dt \leq CT(d_0 + d_0^{\frac{s-a}{2}} + d_0^{s/2} + d_0^{\frac{s-a}{2}}) + C d_0^{\frac{s-a}{2}} \int_0^T \int_U |\nabla p|^{2-a} dx \]
\[ \leq CT(1 + d_0^{\frac{s-a}{2}}) + C d_0^{\frac{s-a}{2}} \int_0^T \int_U |\nabla p|^{2-a} dx dt. \]
Estimating \( K(|\nabla p|) \) from below by \((2.11)\), we get
\[
\int_{\theta T}^{T} \int_{U} |\nabla p|^{2s-a} dx dt \leq CT(1 + d_0^{\frac{s}{2-a}} + \frac{\theta}{4}) + C d_0^{\frac{s}{2-a}} \int_{0}^{T} \int_{U} |\nabla p|^{2-a} dx dt. \tag{6.20}
\]
Replacing \( 2s - a \) by \( s > 2(3 - a) - a = 3(2 - a) \) in \((6.20)\) we obtain
\[
\int_{\theta T}^{T} |\nabla p|^{s} dx dt \leq CT(1 + d_0^{\frac{s}{2-a}} + d_0^{\frac{s}{2-a}}) + C d_0^{\frac{s}{2-a}} \int_{0}^{T} \int_{U} |\nabla p|^{2-a} dx dt. \tag{6.21}
\]

For \( 1 \leq s \leq 3 - a \), we have from \((6.19)\) that
\[
\int_{\theta T}^{T} \int_{U} K(|\nabla p|)|\nabla p|^{2s} dx dt \leq CT(d_0^{s/2} + d_0^{\frac{s}{2-a}}) + C d_0^{\frac{s}{2-a}} \int_{0}^{T} \int_{U} |\nabla p|^{2-a} dx dt.
\]
Using property \((2.11)\) again yields
\[
\int_{\theta T}^{T} \int_{U} |\nabla p|^{2s-a} dx dt \leq CT(1 + d_0^{\frac{s}{2-a}} + d_0^{\frac{s}{2-a}}) + C d_0^{\frac{s}{2-a}} \int_{0}^{T} \int_{U} |\nabla p|^{2-a} dx dt.
\]
Replacing \( 2s - a \) by \( s \in [2 - a, 3(2 - a)] \) gives
\[
\int_{\theta T}^{T} \int_{U} |\nabla p|^{s} dx dt \leq CT(1 + C D^{s/2} + d_0^{\frac{s}{2-a}}) + C d_0^{\frac{s}{2-a}} \int_{0}^{T} \int_{U} |\nabla p|^{2-a} dx dt \tag{6.22}
\]
Note that \( d_0^{1/2} \leq CD \). Then combining \((6.21)\), for the case \( s > 3(2 - a) \), with \((6.22)\), for the case \( s \in [2 - a, 3(2 - a)] \), we obtain \((6.15)\) for all \( s \geq 2 - a \).

**Proof of \((6.16)\).** For \( s > 2 \), we have from \((6.1)\) with \( \theta' = \theta/2 \) that
\[
\sup_{[0,T]} \int_{U} |\nabla p|^{s} dx \leq C \sup_{[0,T]} \int_{U} (v^{s/2} + M_\theta^s) \zeta dx \leq CM_\theta^s + \int_{0}^{T} \int_{U} v^{s/2} \zeta dx dt
\]
\[
\leq CM_\theta^s + C \int_{0}^{T} \int_{U} |\nabla p|^{s/2} \zeta dx dt.
\]
Choose \( \zeta(t) \) such that \( \zeta = 0 \) for \( t \leq 3\theta T/4 \), and \( \zeta = 1 \) for \( t \geq \theta T \). Then
\[
\sup_{[\theta T,T]} \int_{U} |\nabla p|^{s} dx \leq Cd_0^{s/4} + C(\theta T)^{-1} \int_{\theta T}^{T} \int_{U} |\nabla p|^{s} dx dt, \tag{6.23}
\]
with \( d_0 \) being defined by using, again, \( \theta' = \theta/2 \). We apply \((6.15)\) with \( \theta \) being replaced by \( \theta_1 = 3\theta/4 \) and \( d_0 \), hence \( D \), defined by using \( \theta' = 2\theta_1/3 \). We obtain
\[
\sup_{[\theta T,T]} \int_{U} |\nabla p|^{s} dx \leq CD^{s/2} + C \theta^{-1}(1 + D^s) + C(\theta T)^{-1}D^{\frac{s}{2-a}} \int_{0}^{T} \int_{U} |\nabla p|^{2-a} dx dt \leq C \theta^{-1}(1 + D^s) + C(\theta T)^{-1}D^{\frac{s}{2-a}} \int_{0}^{T} \int_{U} |\nabla p|^{2-a} dx dt.
\]
Thus, we obtain \((6.16)\). This completes the proof of the theorem.
Now, we combine Proposition 6.3 with estimates in section 3 to express the bounds for the gradient’s $L^s$-norms in terms of the initial and boundary data.

In the following $\alpha$ always satisfies (2.14). Let $p_1, q_1$ be fixed and $\delta_1$ be defined as in Theorem 4.1. Also, let $z_1, z_2, z_3$ be defined as in Corollary 4.3 and $\mu_0$ be defined by (5.10). We define a number of constants and quantities that will be used for the rest of the paper:

$$\kappa_1 = \max\{2\mu_0, 1 + 2/\delta_1\}, \quad \kappa_1(t) = 1 + \left[\text{Env}A(\alpha, t)\right]^{\frac{1}{\alpha-2}}, \quad \tilde{\kappa}_1(t) = 1 + \beta(\alpha)^{\frac{1}{\alpha-2}} + \sup_{[t-2,t]} A(\alpha, \cdot)^{\frac{1}{\alpha-2}},$$

$$K_1(t) = 1 + \sup_{[0,t]} (\|\Psi\|_{L^\infty} + \|\Psi_t\|_{L^\infty} + \|\nabla\Psi\|_{L^\infty} + \|\nabla^2\Psi\|_{L^\infty}) + \|\Psi_t\|_{L^{q_1}(U \times (0,t))},$$

$$\tilde{K}_2(t) = 1 + \sup_{[t-2,t]} (\|\Psi\|_{L^\infty} + \|\Psi_t\|_{L^\infty} + \|\nabla\Psi\|_{L^\infty} + \|\nabla^2\Psi\|_{L^\infty}) + \|\Psi_t\|_{L^{q_1}(U \times (t-2,t))},$$

**Theorem 6.4.** Let $s > 2$. \hspace{1cm} (i) If $0 < t \leq 3$ then

$$\int_U |\nabla p(x, t)|^s dx \leq Ct^{-1-\tilde{s}\kappa_1} (1 + \|\tilde{p}_0\|_{L^\infty(U)})^{2z_3+2} \kappa_1(t)^{2z_3} \tilde{K}_2(t)^{2s} \left(1 + \int_0^t G_1(\tau) d\tau\right) \cdot \exp\left\{C't^{-1/\delta_1} (1 + \|\tilde{p}_0\|_{L^\infty(U)})^{\tilde{s}\kappa_1(t)^{\tilde{s}}}(1 + \|\Psi_t\|_{L^{q_1}(U \times (0,t))})^{2}\right\}. \quad (6.28)$$

(ii) If $t > 2$ then

$$\int_U |\nabla p(x, t)|^s dx \leq C (1 + \|\tilde{p}_0\|_{L^\infty(U)})^{2z_3+\alpha} \kappa_1(t)^{2z_3+\alpha} \tilde{K}_2(t)^{2s} \left(1 + \int_{t-1}^t G_1(\tau) d\tau\right) \cdot \exp\left\{C'(1 + \|\tilde{p}_0\|_{L^\infty(U)}) \kappa_1(t)^{\tilde{s}}(1 + \|\Psi_t\|_{L^{q_1}(U \times (t-2,t))})^{2}\right\}. \quad (6.29)$$

**Proof.** (i) Let $t \in (0,\frac{1}{2})$. Applying (6.16) to $T_0 = 0, T = t$ and $\theta = 1/2$ gives

$$\int_U |\nabla p(x, t)|^s dx \leq C \left(1 + D^s + Ct^{-1} D^{\frac{s}{\alpha-2}} \int_0^t \int_U |\nabla p|^{2-\alpha} dxd\tau\right),$$

where $D$ is defined by (6.18). Since $\tilde{s} \geq \frac{s}{\alpha-2} - 1$, then

$$\int_U |\nabla p(x, t)|^s dx \leq Ct^{1-1} (1 + D) \tilde{s} \left(1 + \int_0^t \int_U |\nabla p|^{2-\alpha} dxd\tau\right), \quad (6.30)$$

Note that

$$\mathcal{D} \leq Ct^{-1} \sup_{[t/4,t]} \|p\|_{L^\infty}^2 + \sup_{[t/4,t]} \|\nabla p\|_{L^\infty(\Gamma)}^2 \leq Ct^{-1} \sup_{[t/4,t]} \|\tilde{p}\|_{L^\infty}^2 + Ct^{-1} \sup_{[t/4,t]} \|\Psi\|_{L^\infty}^2 + \sup_{[t/4,t]} \|\nabla p\|_{L^\infty(\Gamma)}^2.$$
where $R_2 = \sup_{[0,t]} \|\Psi\|_{L^\infty}$, $R_3 = 1 + \sup_{[0,t]} (\|\Psi_t\|_{L^\infty} + \|\nabla \Psi\|_{L^\infty}^2 + \|\nabla^2 \Psi\|_{L^\infty})$.

By relation (2.13), we have $|\nabla p|^{2-a} \leq C(1 + H(|\nabla p|))$, hence we can use (3.8) to estimate the last integral in (6.30). Substituting these estimates into (6.30) gives

$$\int_U |\nabla p(x,t)|^s dx \leq C t^{-1} \bar{D} \left( 1 + \|\check{p}_0\|_{L^2}^2 + \int_0^t G_1(\tau) d\tau \right),$$  

(6.31)

hence

$$\int_U |\nabla p(x,t)|^s dx \leq C t^{-1-\delta_1} (R_1 + R_2 + R_3) \bar{D}^2 R_4 e^{C t^{-\delta_1} R_1},$$  

(6.32)

where $R_4 = (1 + \|\check{p}_0\|_{L^2})^2 (1 + \int_0^t G_1(\tau) d\tau)$. For the sum $R_1 + R_2 + R_3$, we clearly see that $R_2, R_3 \leq \mathcal{K}_2(t)$, and

$$R_1 \leq C (1 + \|\check{p}_0\|_{L^\alpha(U)})^2 \mathcal{K}_1(t)^2 \mathcal{K}_2(t).$$

Therefore

$$\int_U |\nabla p(x,t)|^s dx \leq C t^{-1-\delta_1} (1 + \|\check{p}_0\|_{L^2})^2 (1 + \|\check{p}_0\|_{L^\alpha})^{2(\frac{2}{\alpha} + 1)} \mathcal{K}_1(t)^{2} \mathcal{K}_2(t)^2 \left( 1 + \int_0^t G_1(\tau) d\tau \right) \exp \left\{ C t^{-1/\delta_1} R_1 \right\}.$$

(6.33)

Then by Hölder’s inequality, we can combine the powers of $L^2$ and $L^\alpha$ norms of $\check{p}_0$, and obtain (6.28).

(ii) Let $t > 2$. Applying (6.16) with $T_0 = t - 1, T = 1, \theta = 1/2$,

$$\int_U |\nabla p(x,t)|^s dx \leq C (1 + \bar{D}^3) + C \bar{D}^{\frac{2}{\alpha} - 1} \int_{t-1}^t \int_U |\nabla p|^{2-a} dx d\tau \leq C (1 + \bar{D})^3 \left[ 1 + \int_{t-1}^t \int_U |\nabla p|^{2-a} dx d\tau \right],$$

(6.34)

where $\bar{D}$ is defined in (6.18). We have

$$\bar{D} \leq C \sup_{[t-1,t]} \left[ \|p\|_{L^\infty} + \|\nabla p\|_{L^\infty(\Gamma)} \right]^2 \leq C \sup_{[t-1,t]} \left[ \|\check{p}\|_{L^\infty} + \|\Psi\|_{L^\infty} + \|\nabla p\|_{L^\infty(\Gamma)} \right]^2.$$  

(6.35)

Similar to part (i), but using (4.23) and (5.39) instead of (4.23) and (5.38), we have

$$\bar{D} \leq C (R_5^2 + R_6^2 e^{C t R_5}),$$

(6.36)

where

$$R_5 = (1 + \|\check{p}_0\|_{L^\alpha})^{2} \mathcal{K}_1(t)^3 (1 + \|\Psi_t\|_{L^\infty(U \times (t-2,t))})^{\frac{2}{\alpha}},$$

$$R_6 = \sup_{[t-1,t]} \|\Psi\|_{L^\infty}, \quad R_7 = 1 + \max_{[t-2,t]} (\|\Psi_t\|_{L^\infty} + \|\nabla \Psi\|_{L^\infty}^2 + \|\nabla^2 \Psi\|_{L^\infty}).$$
Combining (6.34), (6.36) and (3.10) gives

\[ \int_U |\nabla p(x,t)|^s \, dx \leq C(R_2^2 + R_6^2 + R_7^2 e^{C'R_5})^2 \left( 1 + \|\tilde{p}_0\|_{L^\alpha}^n + Env A(\alpha,t) \frac{\alpha^n}{\alpha} + \int_{t-1}^t G_1(\tau) \, d\tau \right) \]

\[ \leq C(R_5 + R_6 + R_7 e^{C'R_5}) e^{2\beta} \left( 1 + \|\tilde{p}_0\|_{L^\alpha}^n \right)^\alpha \mathcal{K}_1(t) \left( 1 + \int_{t-1}^t G_1(\tau) \, d\tau \right). \]  

(6.37)

Note that \( R_6, R_7 \leq \tilde{K}_2(t) \) and

\[ R_5 \leq C \left( 1 + \|\tilde{p}_0\|_{L^\alpha} \right)^{2s} \mathcal{K}_1(t)^s \tilde{K}_2(t)^{2s}. \]

Therefore, similar to part (i), we obtain (6.29) from (6.37).

Remark 6.5. (a) Regarding the regularity requirement for the initial data \( p_0(x) \) in estimates of \( \|\nabla p\|_{L^s} \) in later time \( t > 0 \), the right-hand side of (6.28) and (6.29) only need \( \|p_0\|_{L^\alpha} \). This shows the regularity gain of the solution when \( t > 0 \).

(b) The \( W^{1,s} \)-estimates established above hold for \( s > n \), hence the Hölder estimates follow thanks to Morrey’s imbedding. This is opposite to Ladyzhenskaya-Ural’tseva’s proofs in [14] which establish Hölder continuity first and use it to obtain \( W^{1,s} \)-estimates, but not for all \( s \). The reason for this simplification is our equation’s special structure, see (2.8)–(2.13).

For large time estimates, the bounds in Theorem 6.4 can be established to be independent of the initial data as in the following.

Theorem 6.6. (i) If \( A(\alpha) < \infty \) then

\[ \limsup_{t \to \infty} \int_U |\nabla p(x,t)|^s \, dx \]

\[ \leq C A_1^{2s_3+\alpha} A_2^{2s} G_1 \cdot \exp \left\{ C' A_1^{2s_3} (1 + \limsup_{t \to \infty} \|\Psi(t)\|_{L^{a_1}(U \times (t-1,t))}^{2s_2}) \right\}, \]  

(6.38)

where

\[ A_1 = 1 + A(\alpha)^{\frac{1}{\alpha-n}}, \quad G_1 = 1 + \limsup_{t \to \infty} \int_{t-1}^t G_1(\tau) \, d\tau, \]  

(6.39)

\[ A_2 = 1 + \frac{\limsup_{t \to \infty} (\|\Psi(t)\|_{L^\infty} + \|\Psi_1(t)\|_{L^{\alpha} (U \times (t-2,t))}^{2s_2})}{\limsup_{t \to \infty} \|\Psi(t)\|_{L^\alpha (U \times (t-2,t))}}. \]  

(6.40)

(ii) If \( \beta(\alpha) < \infty \) then there is \( T > 0 \) such that for all \( t > T \),

\[ \int_U |\nabla p(x,t)|^s \, dx \leq C \bar{K}_1(t)^{2s_3+\alpha} \bar{K}_2(t)^{2s} \left( 1 + \int_{t-1}^t G_1(\tau) \, d\tau \right) \]

\[ \cdot \exp \left\{ C' \bar{K}_1(t)^{2s_3} (1 + \|\Psi(t)\|_{L^{a_1}(U \times (t-2,t))}^{2s_2}) \right\}. \]

(6.41)

Proof. (i) Taking the limit superior as \( t \to \infty \) in (6.34) and (6.35) gives

\[ \limsup_{t \to \infty} \int_U |\nabla p(x,t)|^s \, dx \, dt \]

\[ \leq C \left( 1 + \limsup_{t \to \infty} (\|p\|_{L^\infty(U)} + \|\Psi\|_{L^\infty(U)} + \|\nabla p\|_{L^\infty(U)}) \right)^{2s} \cdot (1 + \limsup_{t \to \infty} \int_{t-1}^t \int_U |\nabla p|^{2-a} \, dx \, d\tau). \]

(6.42)
Applying \(^{(3.12)}\), \(^{(5.40)}\) and \(^{(3.44)}\) gives

\[
\limsup_{t \to \infty} \int_U |\nabla p(x, t)|^s dx \leq C \left[ \mathcal{A}_1^{\frac{s}{2}} \left( 1 + \limsup_{t \to \infty} \|\Psi_t(t)\|_{L^{q_1}(U \times (t-1, t))} \right)^{\frac{s}{2}} + \limsup_{t \to \infty} \|\Psi\|_{L^\infty(U)} \right] \\
+ \limsup_{t \to \infty} (1 + \|\Psi_t\|_{L^\infty(U)}^2 + \|\nabla \Psi\|_{L^\infty}^2 + \|\nabla^2 \Psi\|_{L^\infty}^2) \\
\cdot \exp \{ C' \mathcal{A}_1^{\frac{s}{2}} (1 + \limsup_{t \to \infty} \|\Psi_t(t)\|_{L^{q_1}(U \times (t-1, t))}^{\frac{s}{2}}) \}^{\frac{2s}{s-2}} \times \left( 1 + \mathcal{A}_{1^0} \right) + \limsup_{t \to \infty} \int_{t-1}^t G_1(\tau) d\tau.
\]

Then estimate \(^{(6.38)}\) follows.

(ii) By combining \(^{(6.34)}\), \(^{(6.36)}\) with \(^{(4.26)}\), \(^{(5.41)}\) and \(^{(3.14)}\), we have

\[
\int_U |\nabla p(x, t)|^s dx \leq C \left\{ [1 + \beta(\alpha)1^{-\frac{2s}{2a}} + \|A(\alpha, \cdot)\|_{L^{\frac{2s}{2a}}(t-2, t)}^2] \right\}^{\frac{s}{2}} \left[ 1 + \|\Psi_t\|_{L^{q_1}(U \times (t-2, t))}^2 \right] \\
+ \sup_{[t-1, t]} \|\Psi\|_{L^\infty(U)} + \sup_{[t-1, t]} (\|\Psi_t\|_{L^\infty(U)}^2 + \|\nabla \Psi\|_{L^\infty}^2 + \|\nabla^2 \Psi\|_{L^\infty}^2) \\
\cdot \exp \left\{ C'[1 + \beta(\alpha)1^{-\frac{2s}{2a}} + \|A(\alpha, \cdot)\|_{L^{\frac{2s}{2a}}(t-2, t)}^2] \right\}^{\frac{s}{2}} \left[ 1 + \|\Psi_t\|_{L^{q_1}(U \times (t-2, t))}^2 \right] \\
\cdot \left( 1 + \beta(\alpha)1^{-\frac{2s}{2a}} + A(\alpha, t-1)1^{-\frac{2s}{2a}} + \int_{t-1}^t G_1(\tau) d\tau \right),
\]

for all \(t \geq T\) with some \(T > 2\). Note that \(\|A(\alpha, \cdot)\|_{L^{\frac{2s}{2a}}(t-2, t)} \leq 2^{1/a} \sup_{[t-2, t]} A(\alpha, \tau)^{1/(a-a)}\). Then

\[
\int_U |\nabla p(x, t)|^s dx \leq C \left\{ \tilde{K}_1(t)^{\frac{s}{2}} \tilde{K}_2(t) + \tilde{K}_2(t) \exp \left\{ C' \tilde{K}_1(t)^{\frac{s}{2}} \left[ 1 + \|\Psi_t\|_{L^{q_1}(U \times (t-2, t))}^2 \right] \right\} \right\}^{\frac{2s}{s-2}} \\
\cdot \tilde{K}_1(t)^{\alpha} \left( 1 + \int_{t-1}^t G_1(\tau) d\tau \right),
\]

Therefore, inequality \(^{(6.41)}\) follows. \(\square\)

**Remark 6.7.** In case \(\Psi = \Psi(x)\), we want to investigate the long-time dynamics by using the notion of global attractors, see, e.g., \(^{[20],[23]}\). However, the independence of estimates \(^{(6.38)}\) and \(^{(6.41)}\) on the initial data does not yet prove the existence of a absorbing ball. Therefore the existence of the global attractor is still an open problem.

### 7 \(L^\infty\)-estimates for the gradient

In this section, we obtain global \(L^\infty\)-estimates for the gradient of pressure. Let \(p(x, t)\) be a solution of IBVP \(^{(3.11)}\). For each \(m = 1, 2, \ldots, n\), denote \(u_m = p_{x_m}\) and \(u = (u_1, u_2, \ldots, u_n) = \nabla p\). We have

\[
\frac{\partial u_m}{\partial t} = \partial_m (\nabla \cdot (K(|u|) u)) = \nabla \cdot (K(|u|) \partial_m u) + \nabla \cdot \left[ K'(|u|) \sum_i u_i \partial_m u_i \right]. \quad (7.1)
\]

Since \(\partial_i u_m = \partial_m u_i\), we have \(\partial_m u = (\partial_m u_1, \ldots, \partial_m u_n) = (\partial_1 u_m, \ldots, \partial_n u_m) = \nabla u_m\), and \(\sum_i u_i \partial_m u_i = \sum_i u_i \partial_i u_m = u \cdot \nabla u_m\). Therefore, we rewrite \((7.1)\) as

\[
\frac{\partial u_m}{\partial t} = \nabla \cdot (K(|u|) \nabla u_m) + \nabla \cdot \left[ K'(|u|) \frac{u \cdot \nabla u_m}{|u|} u \right]. \quad (7.2)
\]
We consider this as a linear equation for \( u_m \) with variable coefficients, and write
\[
\frac{\partial u_m}{\partial t} = \nabla \cdot A(x, t, \nabla u_m),
\] (7.3)
where
\[
A(x, t, \xi) = K(|u(x, t)|)\xi + K'(|u(x, t)|) \frac{u(x, t) \cdot \xi}{|u|} u(x, t).
\]
Using properties (2.11) and (2.12), one can verify that
\[
|A(x, t, \xi)| \leq (1 + a)K(|u(x, t)|)|\xi|,
\] (7.4)
\[
|A(x, t, \xi) \cdot \xi| \geq (1 - a)K(|u(x, t)|)|\xi|^2.
\] (7.5)

We will apply De Giorgi’s technique to equation (7.2). In the following, we fix a number \( s_0 \) such that \( r = s_0 \) satisfies (2.19). Note that \( s_0^2 > 2 \). We will also use \( s_j \) for \( j \geq 1 \) to denote some exponents that depend on \( s_0 \) but are independent of \( \alpha \). Let
\[
s_1 = (1 - 2/s_0^2)^{-1} > 1.
\] (7.6)

**Theorem 7.1.** For any \( T_0 \geq 0, T > 0, \) and \( \theta \in (0, 1) \), if \( t \in [T_0 + \theta T, T_0 + T] \) then
\[
\|\nabla p(t)\|_{L^\infty(U)} \leq C(1 + (\theta T)^{-1})^{\frac{n+1}{2}}\lambda^{\frac{n}{2}} \|\nabla p\|_{L^2(U \times (T_0 + \theta T/2, T_0 + T))} + C \sup_{[T_0 + \theta T/2, T_0 + T]} \|\nabla p\|_{L^\infty(\Gamma)},
\] (7.7)
where
\[
\lambda = \lambda(T_0, T, \theta) = \left( \int_{T_0 + T}^{T_0 + T/2} \int_U (1 + |\nabla p|)^{\frac{s_0}{2} \frac{u_{s_0}}{s_0}} dx dt \right)^{\frac{2-s_0}{s_0}},
\] (7.8)
and constant \( C > 0 \) is independent of \( T_0, T, \theta \).

**Proof.** Without loss of generality, assume \( T_0 = 0 \). Denote \( M_b = \sup_{\Gamma \times [\theta T/2, T]} |\nabla p| \).

Fix \( m \in \{1, 2, \ldots, n\} \). We will show for \( t \in [\theta T, T] \) that
\[
\|p_{x_m}(t)\|_{L^\infty(U)} \leq C(1 + (\theta T)^{-1})^{\frac{n+1}{2}}\lambda^{\frac{n}{2}} \|p_{x_m}\|_{L^2(V \times (T_0 + \theta T/2, T_0 + T))} + 2M_b.
\] (7.9)

Let \( \zeta(t) \) be a cut-off function with \( \zeta(t) = 0 \) for \( t \leq \theta T/2 \). We define for \( k \geq M_b \), \( u_m^{(k)} = \max\{u_m - k, 0\} \), then \( u_m^{(k)} = 0 \) on \( \Gamma \).

Multiplying (7.3) by \( u_m^{(k)} \zeta \) and integrating over \( U \), using properties (7.3) and (7.5), we obtain
\[
\frac{1}{2} \frac{d}{dt} \int_U |u_m^{(k)} \zeta|^2 dx \leq \int_U |u_m^{(k)}|^2 \zeta \zeta_t dx - (1 - a) \int_U K(|u|)|\nabla u_m^{(k)}|^2 \zeta_t dx.
\] (7.10)
Integrating (7.10) from 0 to \( t \) for \( t \in [0, T] \), and then taking supremum in \( t \) give
\[
\max_{[0, T]} \int_U |u_m^{(k)} \zeta|^2 dx + C \int_0^T \int_U K(|u|)|\nabla u_m^{(k)}|^2 \zeta_t dx dt \leq \int_0^T \int_U |u_m^{(k)}|^2 \zeta_t dx.
\] (7.11)
Let
\[
\nu_0 = 4(1 - 1/s_0^2) > 2.
\] (7.12)
Applying Lemma 2.2 to function \( u_m^{(k)} \) which vanishes on the boundary, weight \( W = K(|u|) \) and exponents \( r = s_0, q = q(s_0) = \nu_0 \), we have

\[
\| u_m^{(k)} \|_{L^q(T)} \leq C \left[ \text{ess sup}_{t \in [0,T]} \| u_m^{(k)} \|_{L^q(U)} + \left( \int_0^T \int_U K(|u|) |\nabla (u_m^{(k)} \zeta)|^2 \, dx \, dt \right)^{\frac{1}{2}} \right] 
\cdot \left( \int_0^T \int_{\text{supp}\zeta} K(|u|)^{-\frac{a_0}{q}} dx \, dt \right)^{\frac{2-a_0}{a_0}}.
\]

Using (2.11), we have \( K(|u|)^{-\frac{a_0}{q}} \leq C(1 + |u|)^{a_0}, \) hence

\[
\| u_m^{(k)} \zeta \|_{L^q(T)} \leq C \lambda^{1/\nu_0} \left[ \max_{t \in [0,T]} \int_U |u_m^{(k)} \zeta|^2 \, dx + \int_0^T \int_U K(|u|) |\nabla (u_m^{(k)} \zeta)|^2 \, dx \, dt \right]^{\frac{1}{2}}.
\]

(7.13)

By (7.14), (7.15) and the boundedness of function \( K(\cdot) \) we find that

\[
\| u_m^{(k)} \zeta \|_{L^q(T)} \leq C \lambda^{1/\nu_0} \left( \int_0^T \int_U |u_m^{(k)} \zeta|^2 \, dx \right)^{\frac{1}{2}}.
\]

(7.14)

Let \( M_0 \geq M_b \). For \( i \geq 0 \), let \( t_i = \theta T(1 - 1/2^{i+1}), \) \( k_i = 2M_0(1 - 2^{i+1}), \) and let \( Q_i, \zeta_i(t) \) be the same as in the proof of Theorem 4.1. For \( i, j \geq 0 \), define

\[
A_{i,j} = \{ (x, t) : u_m(x, t) > k_i, t \in (t_j, T) \}, \quad A_i = A_{i,i}.
\]

Let \( Y_i = \| u_m^{(k_i)} \|_{L^q(A_{i+1})} \). Applying (7.14) with \( k = k_{i+1} \) and \( \zeta = \zeta_i \) and then using the same arguments as in the proof of Theorem 4.1 (see detailed calculations in Theorem 5.6 of [12]), we obtain

\[
Y_{i+1} \leq DB^3 Y_i^{1+\nu_3} \quad \text{for all } i \geq 0,
\]

where \( \nu_3 = 1 - 2/\nu_0, \) \( B = 4 \) and \( D = C(1 + \frac{1}{\theta T})^{1/2} \lambda^{1/\nu_0} M_0^{-\nu_3}. \)

We now determine \( M_0 \) so that \( Y_0 \leq D^{-1/\nu_3} B^{-1/\nu_3} \). This condition is met if

\[
M_0 \geq C \left[ \lambda^{1/\nu_0} \left( 1 + \frac{1}{\theta T} \right)^{1/2} \right]^{1/\nu_3} Y_0 = C \lambda^{\nu_1/2} \left( 1 + \frac{1}{\theta T} \right)^{\frac{1+\nu_3}{2}} Y_0.
\]

Since \( Y_0 = \| u_m^{(k_0)} \|_{L^q(A_{0,0})} \leq \| u_m \|_{L^2(U \times (\theta T/2, T))} \) and \( M_0 \geq M_b \), it suffices to choose \( M_0 \) as

\[
M_0 = C \lambda^{\nu_1/2} \left( 1 + \frac{1}{\theta T} \right)^{\frac{1+\nu_3}{2}} \| u_m \|_{L^2(U \times (\theta T/2, T))} + M_b.
\]

(7.15)

Then Lemma 2.3 gives \( \lim_{i \to \infty} Y_i = 0. \) Hence, \( \int_{\theta T}^T \int_U |u_m^{(2M_0)}|^2 \, dx \, dt = 0. \) Thus, \( u_m(x, t) \leq M_0 \) a.e. in \( U \times (\theta T, T). \) Replace \( u_m, u \) by \( -u_m, -u \) and use the same argument we obtain

\[
|u_m(x, t)| \leq 2M_0 \quad \text{a.e. in } U \times (\theta T, T).
\]

(7.16)

By the choice of \( M_0 \) we obtain from (7.16) that

\[
|u_m(x, t)| \leq C \left( 1 + \frac{1}{\theta T} \right)^{\frac{1+\nu_3}{2}} \lambda^{\nu_1/2} \| u_m \|_{L^2(U \times (\theta T/2, T))} + 2M_b
\]

for all \( m = 1, \ldots, n. \) Then (7.9) follows, and the proof is complete. \( \square \)
We will combine Theorem 7.1 with the high integrability of \( \nabla p \) in section 6 to obtain the \( L^\infty \)-estimates. For the rest of this paper, we fix the following constants

\[
s_2 = \max \left\{ \frac{2a s_0}{2 - s_0}, \frac{a s_0}{2 - s_0} \right\} \quad \text{and} \quad s_3 = s_1 (2 - s_0) / s_0 + 1, \tag{7.17}\]

where \( s_0 \) and \( s_1 \) are as in (7.6). We use the same notation as in (6.24)-(6.27), in Theorem 6.6 and also denote

\[
\kappa_2 = 1 + s_1 + \bar{s}_2 \kappa_3, \quad \kappa_3 = \bar{s}_2 z_3 + \alpha / 2, \tag{7.18}\]

\[
\tilde{K}_1(t) = 1 + \beta(\alpha) \frac{1}{\alpha - a} + \sup_{[t - 3, t]} A(\alpha, \cdot) \frac{1}{\alpha - a}, \tag{7.19}\]

\[
\tilde{K}_2(t) = 1 + \sup_{[t - 3, t]} \left( ||\Psi(t)||_{L^\infty} + ||\Psi_t(t)||_{L^\infty}^\mu + \frac{1}{\alpha - a} \cdot \exp \left[ C t^{-1} (1 + ||\Psi(t)||_{L^\infty}^\mu) \right] \right), \tag{7.20}\]

We recall that the definition of \( \tilde{s} \) is given by (6.17).

\textbf{Theorem 7.2.} (i) If \( 0 < t \leq 3 \) then

\[
||\nabla p(t)||_{L^\infty(U)} \leq C t^{-\kappa_2/2} (1 + ||\bar{p}_0||_{L^\infty(U)}) (\bar{s}_2 z_3 + 1) s_3 K_1(t) \bar{s}_2 z_3 s_3 \bar{s}_2 s_3 \left( 1 + \int_0^t G_1(\tau) d\tau \right)^{s_3/2}
\times \exp \left[ C t^{-1/\beta_1} (1 + ||\bar{p}_0||_{L^\infty(U)}) K_1(t) (1 + ||\Psi_t||_{L^\infty(U \times (0, t))})^{2s} \right]. \tag{7.21}\]

(ii) If \( t > 3 \) then

\[
||\nabla p(t)||_{L^\infty(U)} \leq C (1 + ||\bar{p}_0||_{L^\infty(U)}) \kappa_3 s_3 K_1(t) \kappa_3 s_3 \tilde{K}_2(t) \bar{s}_2 s_3 \left( 1 + \int_{t-2}^t G_1(\tau) d\tau \right)^{s_3/2}
\times \exp \left\{ C (1 + ||\bar{p}_0||_{L^\infty(U)}) K_1(t) (1 + ||\Psi_t||_{L^\infty(U \times (t-2, t))})^{2s} \right\}. \tag{7.22}\]

\textbf{Proof.} First, we note in (7.7) that

\[
\lambda \leq C \left( \int_{T_0 + \theta T/2}^{T_0 + T} \int_U (1 + |\nabla p|)^{s_2} dxd\tau \right)^{2 - s_0},
\]

\[
||\nabla p||_{L^2(U \times (T_0 + \theta T/2, T_0 + T))} \leq C \left( \int_{T_0 + \theta T/2}^{T_0 + T} \int_U (1 + |\nabla p|)^{s_2} dxd\tau \right)^{1/2}.
\]

Hence, we have from (7.7) that

\[
\sup_{[T_0 + \theta T, T_0 + T]} ||\nabla p(t)||_{L^\infty(U)} \leq C (1 + (\theta T)^{-1}) \frac{1}{4} \left( \int_{T_0 + \theta T/2}^{T_0 + T} \int_U (1 + |\nabla p|)^{s_2} dxd\tau \right)^{s_3/2}
\times \sup_{[T_0 + \theta T/2, T_0 + T]} ||\nabla p||_{L^\infty(\Gamma)}. \tag{7.23}\]

(i) Let \( t \in (0, 3) \), applying (7.23) with \( T_0 = 0 \), \( T = t \), and \( \theta = 1/2 \), we obtain

\[
||\nabla p(t)||_{L^\infty(U)} \leq C t^{-1/2} \left( \int_{t/4}^t \int_U (1 + |\nabla p|)^{s_2} dxd\tau \right)^{s_3/2} + C \sup_{[t/4, t]} ||\nabla p||_{L^\infty(\Gamma)}. \tag{7.24}\]
Using (6.28) with $s = s_2$ and (5.38) we obtain

$$
\|\nabla p(t)\|_{L^\infty(U)} \leq C t^{\frac{1+\kappa s}{2}} \left\{ t \cdot t^{-1-\tilde{s}_2} (1 + \| \tilde{p}_0 \|_{L^\infty(U)})^{2(\tilde{s}_2 s_3 + 1)} K_1(t) K_2(t)^{2s_2} \left( 1 + \int_0^t G_1(\tau) d\tau \right) \cdot \exp \left[ C' t^{1-\delta_1} (1 + \| \tilde{p}_0 \|_{L^\infty(U)})^{s_3} K_1(t)^{s_3} \left( 1 + \| \Psi_t \|_{L^{\alpha q_1}(U \times (0,t))} \right)^{s_2} \right] \right\}^{s_3/2}
+ C t^{-\mu_0} K_2(t) \exp \left[ C' t^{1-\delta_1} (1 + \| \tilde{p}_0 \|_{L^\infty(U)})^{s_3} K_1(t)^{s_3} \left( 1 + \| \Psi_t \|_{L^{\alpha q_1}(U \times (0,t))} \right)^{s_2} \right].
$$

Hence,

$$
\|\nabla p(t)\|_{L^\infty(U)} \leq C \left( t^{\frac{1+\kappa s}{2}} + t^{-\mu_0} \right) \left( 1 + \| \tilde{p}_0 \|_{L^\infty(U)} \right)^{2\tilde{s}_2 s_3} K_2(t)^{s_2} \left( 1 + \int_0^t G_1(\tau) d\tau \right)^{s_3/2} \cdot \exp \left[ C' t^{1-\delta_1} (1 + \| \tilde{p}_0 \|_{L^\infty(U)})^{s_3} K_1(t)^{s_3} \left( 1 + \| \Psi_t \|_{L^{\alpha q_1}(U \times (0,t))} \right)^{s_2} \right].
$$

(7.25)

Concerning the power of $t^{-1}$, we note that $s_3, \tilde{s}_2 \geq 1$ and $\frac{\tilde{s}_2 s_3}{2} \geq \frac{2\tilde{s}_2 \mu_0 s_3}{2} = \tilde{s}_2 \mu_0 s_3 \geq \mu_0$. Then (7.21) follows (7.25).

(ii) Consider $t > 3$. Applying (7.23) with $T_0 = t - 1, T = 1, \theta = 1/2$ yields

$$
\|\nabla p(t)\|_{L^\infty(U)} \leq C \left( 1 + \int_{t-3/4}^t \int_U |\nabla p|^{s_2} dx \, dt \right)^{s_3/2} + C \sup_{[t-3/4,t]} \|\nabla p\|_{L^\infty(U)}. \tag{7.26}
$$

Thanks to (6.29) with $s = s_2$, and (5.39), and also noting that $\sup_{[t-3/4,t]} \tilde{K}_2(\cdot) \leq \tilde{K}_2(t)$, we obtain

$$
\|\nabla p(t)\|_{L^\infty(U)} \leq C \left[ (1 + \| \tilde{p}_0 \|_{L^\infty(U)}^{2s_3} K_1(t)^{2s_3} \tilde{K}_2(t)^{2\tilde{s}_2} \left( 1 + \int_{t-2}^t G_1(\tau) d\tau \right) \right)^{s_3/2} \cdot \exp \left\{ C (1 + \| \tilde{p}_0 \|_{L^\infty(U)}^{s_3} K_1(t)^{s_3} \left( 1 + \| \Psi_t \|_{L^{\alpha q_1}(U \times (t-2,t))} \right)^{s_2} \right\}]^{s_3/2}
+ C \tilde{K}_2(t) \exp \left\{ C' (1 + \| \tilde{p}_0 \|_{L^\infty(U)}^{s_3} K_1(t)^{s_3} \left( 1 + \| \Psi_t \|_{L^{\alpha q_1}(U \times (t-2,t))} \right)^{s_2} \right\}.
$$

Then (7.22) follows. □

Combining (7.26) with Theorem 6.6 we have the following asymptotic estimates.

**Theorem 7.3.** (i) If $A(\alpha) < \infty$ then

$$
\limsup_{t \to \infty} \|\nabla p(t)\|_{L^\infty(U)} \leq C A_1^{\alpha s_3} A_2^{\tilde{s}_2 s_3} G_1^{s_3/2} \cdot \exp \left\{ C' A_1^{\alpha s_3} (1 + \limsup_{t \to \infty} \| \Psi_t \|_{L^{\alpha q_1}(U \times (t-1,t))} \right)^{s_2} \}. \tag{7.27}
$$

(ii) If $\beta(\alpha) < \infty$ then there is $T > 0$ such that for all $t > T$,

$$
\|\nabla p(t)\|_{L^\infty(U)} \leq C \tilde{K}_1(t)^{s_3} \tilde{K}_2(t)^{s_2} \left( 1 + \int_{t-3/2}^t G_1(\tau) d\tau \right)^{s_3/2} \cdot \exp \left\{ C' \tilde{K}_1(t)^{s_3} \left( 1 + \| \Psi_t(t) \|_{L^{\alpha q_1}(U \times (t-3,t))} \right)^{s_2} \right\}. \tag{7.28}
$$
Proof. Taking the limit superior as \( t \to \infty \) of (7.26), and using estimates (6.38) and (5.40) give:

\[
\limsup_{t \to \infty} \|\nabla p(t)\|_{L^\infty(U)} \leq C \left[ A_1^{2s_3} A_2^{2s_2} G_1 \cdot \exp \left\{ C' A_1^{s_3} \left( 1 + \limsup_{t \to \infty} \|\Psi(t)\|_{L^{\alpha q_1}(U \times (t-1,t))}^2 \right) \right\} \right]^{s_3/2} + C A_2 \exp \left\{ C' A_1^{s_3} \left( 1 + \limsup_{t \to \infty} \|\Psi(t)\|_{L^{\alpha q_1}(U \times (t-1,t))}^2 \right) \right\}.
\]

Then (7.27) follows.

(ii) Combining (7.26) with estimate (6.41) for \( s = s_2 \), and with estimate (5.41) we have:

\[
\|\nabla p(t)\|_{L^\infty(U)} \leq C \left\{ \sup_{[t-3/4,t]} K_1(\cdot)^{2s_3} \sup_{[t-3/4,t]} K_2(\cdot)^{2s_2} \left( 1 + \int_{t-7/4}^{t} G_1(\tau) d\tau \right) \right\} \cdot \exp \left\{ C' \sup_{[t-3/4,t]} K_1(\cdot)^{s_3} \left( 1 + \|\Psi(t)\|_{L^\alpha q_1(U \times (t-11/4,t))}^2 \right) \right\} \left\{ 1 + \sup_{[t-11/4,t]} \left( \|\Psi(t)\|_{L^\infty(U \times (t-7/4,t))}^2 + \|\nabla^2 \Psi\|_{L^\infty(U \times (t-7/4,t))} \right) \right\},
\]

for all \( t \geq T \) with some \( T > 3 \). This leads to (7.28). \( \square \)

8 \( L^\infty \)-estimates for the time derivative

We now estimate the \( L^\infty \)-norm of the pressure’s time derivative. Let \( p(x,t) \) be a solution of IBVP (3.1), and let \( q = p_t \). Then

\[
\frac{\partial q}{\partial t} = \nabla \cdot (K(\nabla p)\nabla p)_t.
\]

Proposition 8.1. If \( T_0 \geq 0 \), \( T > 0 \) and \( \theta \in (0,1) \), then

\[
\sup_{[T_0+\theta T,T_0+T]} \|p_t\|_{L^\infty(U)} \leq C L^\frac{s_3}{2} (1 + (\theta T)^{-1}) L^\frac{s_3}{2} \|p_t\|_{L^2(U \times (T_0+\theta T/2,T_0+T))} + \max_{\Gamma \times (T_0+\theta T))} \|\psi_t\|,
\]

where \( s_3 \) and \( \lambda = \lambda(T_0,T,\theta) \) are defined in Theorem 7.1 and constant \( C > 0 \) is independent of \( T_0 \), \( T \), and \( \theta \).

Proof. Without loss of generality, assume \( T_0 = 0 \). For \( k \geq \max_{\Gamma \times [0,T]} |p_t| = \max_{\Gamma \times [0,T]} |\psi_t| \), let \( q^{(k)} = \max\{q - k, 0\} \) and \( S_k(t) = \{ x \in U : q(x,t) > k \} \), and \( \chi_k(x,t) \) be the characteristic function of \( S_k(t) \). On \( S_k(t) \), we have \( (\nabla p)_t = \nabla q = \nabla q^{(k)} \).

Let \( \zeta = (\zeta(t)) \) be the cut-off function satisfying \( \zeta(0) = 0 \). We will use test function \( q^{(k)} \zeta^2 \), noting that \( \nabla (q^{(k)} \zeta^2) = \zeta^2 \nabla q^{(k)} \). Multiplying (8.1) by \( q^{(k)} \zeta^2 \) and integrating the resultant on \( U \), we get

\[
\frac{1}{2} \frac{d}{dt} \int_U |q^{(k)} \zeta^2|^2 dx = \int_U |q^{(k)} \zeta^2| \zeta_t dx - \int_U (K(|\nabla p|))_t \nabla p \cdot \nabla q^{(k)} \zeta^2 dx - \int_U K(|\nabla p|)(\nabla p)_t \cdot \nabla q^{(k)} \zeta^2 dx.
\]

Note that \( (\nabla p)_t \cdot \nabla q^{(k)} \zeta^2 = |\nabla (q^{(k)} \zeta)|^2 \). Taking into account (2.12),

\[
|(K(|\nabla p|))_t \nabla p \cdot \nabla q^{(k)} \zeta^2| = |K'(|\nabla p|)|\frac{\nabla p \cdot \nabla p_t}{|\nabla p|}|\nabla p \cdot \nabla q^{(k)} \zeta^2| \leq a K(|\nabla p|)|\nabla q||\nabla q^{(k)} \zeta^2|.
\]
Note that $|\nabla q||\nabla q^k|^2 = |\nabla (q^k)\zeta|^2$. It follows that

$$\frac{d}{dt} \int_U |q^k\zeta|^2 dx + (1 - a) \int_U K(|\nabla p|)|\nabla (q^k)\zeta|^2 dx \leq 2 \int_U |q^k|^2 |\zeta_t| dx.$$

Integrating this inequality from 0 to $T$, we obtain

$$\max_{[0,T]} \int_U |q^k\zeta|^2 dx + \int_0^T \int_U K(|\nabla p|)|\nabla (q^k)\zeta|^2 dx dt \leq C \int_0^T \int_U |q^k|^2 |\zeta_t| dx dt.$$

The last inequality uses the fact that function $K(\cdot)$ is bounded above. Applying Lemma 2.2 to $q^k\zeta$ with $W = K(|\nabla p|)$ and $q = \nu_0$ defined by (7.12), we have

$$\|q^k\zeta\|_{L^0(\Omega_T)} \leq C \lambda^{1/\nu_0} \left( \max_{[0,T]} \int_U |q^k\zeta|^2 dx + \int_0^T \int_U K(|\nabla p|)|\nabla (q^k)\zeta|^2 dx dt \right)^{1/2}. $$

and hence,

$$\|q^k\zeta\|_{L^0(\Omega_T)} \leq C \lambda^{1/\nu_0} \left( \int_0^T \int_U |q^k|^2 |\zeta_t| dx dt \right)^{1/2}. $$

(8.3)

This is similar to inequality (7.14). Then by following arguments of Theorem 7.1 applied for $q^k$ instead of $u_0^k$, we obtain the estimate (8.2). We omit the details.

In the following estimates of $p_t$ we use the same notation as in sections 6 and 7, particularly, (6.24)–(6.27), (6.39), (6.40), (7.17)–(7.20), and also define new numbers

$$\kappa_4 = 1 + s_1 + s_2\kappa_1(s_3 - 1), \quad \kappa_5 = (s_2z_3 + 1)(s_3 - 1) + 1,$$

$$\kappa_6 = (s_2z_3 + \alpha/2)(s_3 - 1) + \alpha/2 = \kappa_3(s_3 - 1) + \alpha/2.$$

**Theorem 8.2.** (i) If $0 < t \leq 3$ then

$$\|p_t(t)\|_{L^\infty(U)} \leq C t^{-\kappa_4/2} \left( 1 + \|p_0\|_{L^\nu} \right)^{\kappa_5} \left( 1 + \int_U H(|\nabla p_0(x)|) dx \right)^{1/2}$$

$$\times \mathcal{K}_1(t)^{s_2(s_3 - 1)z_3}\mathcal{K}_2(t)^{s_2(s_3 - 1)} \left( 1 + \int_0^t G_3(\tau) d\tau \right)^{s_3/2}$$

$$\times \exp \left\{ Ct^{-1/\delta_1} \left( 1 + \|p_0\|_{L^\nu(U)} \right)^{\frac{3}{2}} \mathcal{K}_1(t)^{z_3} \left( 1 + \|\Psi_t\|_{L^{\nu_1(U \times (0,t))}} \right)^{\frac{3}{2}}\right\} + \sup_{[0,t]} \|\psi_t\|_{L^\infty(\Gamma)}. $$

(8.4)

(ii) If $t > 3$ then

$$\|p_t(t)\|_{L^\infty(U)} \leq C \left( 1 + \|p_0\|_{L^\nu} \right)^{\kappa_6} \mathcal{K}_1(t)^{\kappa_6} \mathcal{K}_2(t)^{s_2(s_3 - 1)} \left( 1 + \int_{t-2}^t G_3(\tau) d\tau \right)^{s_3/2}$$

$$\times \exp \left\{ C' \left( 1 + \|p_0\|_{L^\nu(U)}^3 \right) \mathcal{K}_1(t)^{z_3} \left( 1 + \|\Psi_t\|_{L^{\nu_1(U \times (t-2,t))}} \right)^{z_2} \right\} + \sup_{[t-1,t]} \|\psi_t\|_{L^\infty(\Gamma)}.$$
where \( \lambda \) is defined by (8.8). By Young’s inequality, we have
\[
\lambda^{s_1/2} = \left( \int_U \int t \left( 1 + |\nabla p| \right)^{\alpha q_0} \, dx \, d\tau \right)^{\nu_1/2} \leq \left( \int_U \int t \left( 1 + |\nabla p| s_2 \right)^{s_2} \, dx \, d\tau \right)^{\nu_1/2},
\]
where \( \nu_1 = (2 - s_0)s_1/s_0 = s_3 - 1 \). Combining with (6.28) for \( s = \tilde{s}_2 \) yields
\[
\lambda^{s_1/2} \leq C \left[ t \cdot t^{-1-\tilde{s}_2} \left( 1 + \|\tilde{p}\|_{L^\infty(U)} \right)^{2\tilde{s}_2} \right]^{s_2} \left( 1 + \int_0^t G_1(\tau) \, d\tau \right)^{\nu_1/2} \cdot \exp \left\{ C' t^{-1/\tilde{s}_2} \left( 1 + \|\tilde{p}\|_{L^\infty(U)} \right)^{\tilde{s}_2} \left( 1 + \|\Psi_t\|_{L^{\infty}(U \times (0,t))} \right)^{\tilde{s}_2} \right\} \nu_1/2.
\]
Combining this with (8.6) and (8.9) gives
\[
\| p_t(t) \|_{L^\infty} \leq C t^{-\tilde{s}_2} \left( 1 + \|\tilde{p}\|_{L^\infty(U)} \right)^{\tilde{s}_2} \left( 1 + \int_0^t \left( \int_U H(|\nabla p_0(x)|) \, dx \right) + \int_0^t G_3(\tau) \, d\tau \right)^{\nu_1/2} \nonumber
\]
\[
\cdot \exp \left\{ C' t^{-1/\tilde{s}_2} \left( 1 + \|\tilde{p}\|_{L^\infty(U)} \right)^{\tilde{s}_2} \left( 1 + \|\Psi_t\|_{L^{\infty}(U \times (0,t))} \right)^{\tilde{s}_2} \right\} \left( 1 + \int_0^t \left( \int_U |\psi_t(x, \tau)|^2 \, dx \, d\tau \right)^{1/2} + \sup_{[0,t]} \|\psi_t\|_{L^\infty} \right).
\]
Therefore,
\[
\| p_t(t) \|_{L^\infty} \leq C t^{-\kappa_4/2} \left( 1 + \|\tilde{p}\|_{L^\infty(U)} \right)^{\tilde{s}_2} \left( 1 + \int_0^t H(|\nabla p_0(x)|) \, dx \right)^{1/2}
\]
\[
\cdot \left( 1 + \int_0^t G_3(\tau) \, d\tau \right)^{\nu_1/2} \nonumber
\]
\[
\cdot \exp \left\{ C' t^{-1/\tilde{s}_2} \left( 1 + \|\tilde{p}\|_{L^\infty(U)} \right)^{\tilde{s}_2} \left( 1 + \|\Psi_t\|_{L^{\infty}(U \times (0,t))} \right)^{\tilde{s}_2} \right\} \left( 1 + \int_0^t \left( \int_U |\psi_t(x, \tau)|^2 \, dx \, d\tau \right)^{1/2} + \sup_{[0,t]} \|\psi_t\|_{L^\infty} \right).
\]
Thus, we obtain (8.7).

(ii) Now consider \( t > 3 \). Applying (8.2) with \( T_0 = t - 1, T = 1 \) and \( \theta = 1/2 \), we obtain
\[
\| p_t(t) \|_{L^\infty(U \times (t-3/4,t))} \leq C \lambda^{s_1/2} \| p_t \|_{L^2(U \times (t-3/4,t))} + \sup_{[t-1,t]} \|\psi_t\|_{L^\infty},
\]
where \( \lambda = \left( \int_{t-3/4}^t \int_U |\nabla p|^{\alpha q_0} \, dx \, dt \right)^{\alpha q_0 / \alpha q_0} \). Then by Young’s inequality and (6.29) for \( s = \tilde{s}_2 \),
\[
\lambda^{s_1/2} \leq C \left( \int_{t-1}^t \int_U \left( 1 + |\nabla p| \right)^{s_2} \, dx \, d\tau \right)^{\nu_1/2}
\]
\[
\leq C \left( 1 + \|\tilde{p}\|_{L^\infty(U)} \right)^{s_2} \left( 1 + \int_{t-2}^t G_1(\tau) \, d\tau \right)^{\nu_1/2}
\]
\[
\cdot \exp \left\{ C' \left( 1 + \|\tilde{p}\|_{L^\infty(U)} \right)^{s_2} \left( 1 + \|\Psi_t\|_{L^{\infty}(U \times (t-3,t))} \right)^{s_2} \right\}.
\]
Combining this with (8.7) and (8.11) gives
\[
\| p_t(t) \|_{L^\infty(U)} \leq C \left( 1 + \|\tilde{p}\|_{L^\infty(U)} \right)^{s_2} \left( 1 + \int_{t-2}^t G_1(\tau) \, d\tau \right)^{\nu_1/2}
\]
\[
\cdot \exp \left\{ C' \left( 1 + \|\tilde{p}\|_{L^\infty(U)} \right)^{s_2} \left( 1 + \|\Psi_t\|_{L^{\infty}(U \times (t-3,t))} \right)^{s_2} \right\}
\]
\[
\cdot \left( 1 + \int_U |\tilde{p}(x)|^2 \, dx + \lim_{\alpha \to 0} \int_{t-1}^t \int_U \left( 1 + \int_U |\psi_t(x, \tau)|^2 \, dx \, d\tau \right)^{1/2} + \sup_{[t-1,t]} \|\psi_t\|_{L^\infty} \right).
\]
Using the facts $G_1(t) \leq G_3(t)$ and $\int_U |\Psi_t(x,t)|^2 \, dx \leq G_3(t)$, we infer that

$$\|p_t(t)\|_{L^\infty(U)} \leq C(1 + \|\tilde{p}_0\|_{L^\infty}^{\frac{\nu_1}{s_3 - 1}})\|\tilde{K}_2(t)\|^{\frac{s_3}{2}} \left(1 + \int_{t-2}^{t} G_3(\tau) \, d\tau\right)^{\frac{s_3}{2}}$$

and obtain (8.5). The proof is complete.

For large time or asymptotic estimates, we have the following.

**Theorem 8.3.** (i) If $A(\alpha) < \infty$ then

$$\limsup_{t \to \infty} \|p_t(t)\|_{L^\infty(U)} \leq C A_1^{G_1(\alpha)} A_2^{\frac{s_3 - 1}{s_3}} G_2 \left(1 + \int_{t-2}^{t} G_3(\tau) \, d\tau\right)^{\frac{s_3}{2}} + \sup_{[t-1,t]} \|\psi_t\|_{L^\infty}$$

where $G_2 = 1 + \limsup_{t \to \infty} \int_{t-1}^{t} G_3(\tau) \, d\tau$.

(ii) If $\beta(\alpha) < \infty$ then there is $T > 0$ such that for all $t > T$,

$$\|p_t(t)\|_{L^\infty(U)} \leq C \left(1 + \int_{t-2}^{t} G_3(\tau) \, d\tau\right)^{\frac{s_3}{2}} + \sup_{[t-1,t]} \|\psi_t\|_{L^\infty}$$

Theorem 8.3. (ii) Combining inequality (8.7) with estimate (6.41) for $s = s_2$ and estimate (3.15), we obtain

$$\limsup_{t \to \infty} \|p_t(t)\|_{L^\infty(U)} \leq C A_1^{G_1(\alpha)} A_2^{\frac{s_3 - 1}{s_3}} G_1 \left(1 + \int_{t-2}^{t} G_3(\tau) \, d\tau\right)^{\frac{s_3}{2}}$$

and estimate (3.15), we obtain (8.9).

Proof. Again, let $\nu_1 = s_3 - 1$.

(i) Taking the limit superior of (8.7) as $t \to \infty$, and using (6.33) with $s = s_2$, and (3.15), we obtain

$$\limsup_{t \to \infty} \|p_t(t)\|_{L^\infty(U)} \leq C A_1^{G_1(\alpha)} A_2^{\frac{s_3 - 1}{s_3}} G_1 \left(1 + \int_{t-2}^{t} G_3(\tau) \, d\tau\right)^{\frac{s_3}{2}}$$

Simple manipulations yield

$$\limsup_{t \to \infty} \|p_t(t)\|_{L^\infty(U)} \leq C A_1^{G_1(\alpha)} A_2^{\frac{s_3 - 1}{s_3}} G_1 \left(1 + \sup_{[t-1,t]} \|\psi_t\|_{L^\infty} \right)$$

Since $G_1 \leq G_2$, we have $G_1^{s_3/2} \left(1 + \sup_{[t-1,t]} G_3(\tau) \right)^{s_3/2} \leq G_2^{(s_3 - 1)/2} G_2^{s_3/2}$. Thus, inequality (8.8) follows (8.10).

(ii) Combining inequality (8.7) with estimate (6.41) for $s = s_2$ and estimate (3.15), we obtain for large $t$ that

$$\|p_t(\tau)\|_{L^\infty(U)} \leq C \left(1 + \int_{t-2}^{t} G_3(\tau) \, d\tau\right)^{\frac{s_3}{2}}$$

Using similar calculations as in part (i) and the fact $G_1 \leq G_3$, we obtain (8.9).
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