Anomalous diffusion and generalized Sparre Andersen scaling

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Abstract – We are discussing the long-time scaling limit for the anomalous diffusion composed of the subordinated Lévy-Wiener process. The limiting anomalous diffusion is in general non-Markov, even in the regime, where ensemble averages of a mean-square displacement or quantiles representing the group spread of the distribution follow the scaling characteristic for an ordinary stochastic diffusion. To discriminate between a truly memory-less process and the non-Markov one, we are analyzing the deviation of the survival probability from the (standard) Sparre Andersen scaling.

Introduction. – The question of the time that it takes for stochastic process to reach a specific point or state by the first time is central in many applications of stochastic modeling in physics (Kramers problem \cite{1}), chemistry (reaction kinetics \cite{2}), biology (neural activity models \cite{3}) and economics (estimation of the ruin time \cite{4}). For a random walk sequence, a nontrivial theorem due to Sparre Andersen \cite{5,6} states that the asymptotic form of the probability of motion from a position \(x\) to \(0\) during its motion.

\[
\lim_{n \to \infty} P(x_n \leq \Delta x) = \frac{1}{\sqrt{4\pi D\Delta t}} \exp\left(-\frac{(x-\Delta x)^2}{4D\Delta t}\right)
\]

where the prefactor \(c(x_0)\) depends on the initial position. The result can be easily generalized for the continuous-time version of the process. For unbiased, continuous Gaussian random walk its first passage time density (FPTD) from \(x_0\) to \(x\) can be easily calculated explicitly \cite{2}: Let us denote \(p(x_2,t|x_0,0) = p(x_2 - x_0)\) the probability of motion from a position \(x_0\) to \(x_2\) in time \(t\) with \(x\) denoting the position on the way from \(x_0\) to \(x_2\), i.e. \(x_0 < x < x_2\). By taking \(f(x,t|x_0,0) = f(x-x_0,\tau)\) as the probability of arriving for the first time at \(x\) at time \(\tau\), the equation for \(p(x_2 - x_0, t)\) reads

\[
p(x_2 - x_0, t) = \int_0^t p(x_2 - x, t - \tau) f(x, x_0, \tau) d\tau
\]

with its Laplace transform given by

\[
\tilde{p}(x_2 - x_0, s) = \tilde{p}(x_2 - x, s) \tilde{f}(x - x_0, s).
\]

For unbiased Gaussian random walk we have \(p(x, t|0,0) = p(x, t) = (2\pi\sigma^2 t)^{-1/2} \exp\left(-\frac{x^2}{2\sigma^2 t}\right)\). With the Laplace transform

\[
\tilde{p}(x, s) = \int_0^\infty p(x, t)e^{-st} dt = (2s\sigma^2)^{-1/2} \exp\left(-\sqrt{2x^2 s/\sigma^2}\right)
\]

one obtains

\[
\tilde{f}(x - x_0, s) = \exp(-\sqrt{2(x - x_0)^2 s/\sigma^2})
\]

which, by the inverse Laplace transform, yields the Lévy-Smirnov distribution

\[
f(x, t|x_0, 0) = f(x - x_0, t) = \frac{1}{t} \sqrt{\frac{(x-x_0)^2}{2\pi\sigma^2 t}} e^{-\frac{(x-x_0)^2}{2\pi\sigma^2 t}},
\]

where \(x_0\) represents the initial condition. This “inverse Gaussian distribution” decays for long times as \(f(x, t|x_0, 0) \propto t^{-3/2}\) and does not have a first moment, i.e. the mean first passage time from \(x_0\) to \(x\) diverges. On the other hand, since \(\int_0^\infty f(x, t|x_0, 0) dt = 1\), the particle performing the one dimensional Gaussian random walk will certainly hit any point \(x\) during its motion.

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Assuming the absorbing boundary located at the origin, i.e., at \( x = 0 \), formula (4) with \( x = 0 \) gives the probability density of the first passage time from the positive semi-axis for a Gaussian random walk. It should be stressed, however that for generally non-Gaussian noises, the knowledge of the boundary location may be insufficient to specify in full the corresponding conditions for absorption or reflection [7–10]. In particular, trajectories of Lévy walks may exhibit discontinuous jumps and in a consequence, the location of the boundary itself is not hit by the majority of sample trajectories. In order to properly take care of possible excursion of the trajectories beyond the location of the boundary (at, say \( x = 0 \)) with subsequent re-crossings into the interval \( (x > 0) \), the whole semi-line \( (x \leq 0) \) has to be assumed “absorbing”. This nonlocal definition of the boundary conditions secures proper evaluation of the first passage time distribution and survival probability [7–10], see below.

The very same scenario, see eq. (4), dictated by the Sparre Andersen theorem holds also true for “paradoxical” diffusion-like processes studied in terms of CTRW (continuous-time random walks) where kinetics of the walker is determined by the distribution of jump lengths and distribution of waiting times before a next jump to occur [11]. If the process is regular in time but with nontrivial jump distribution following the (symmetric) Lévy law of stability (so-called symmetric Lévy flight), the first passage time density (FPTD) follows the Sparre Andersen universality [7,8,12]. Notably, however, if the first passage timedensity (FPTD) follows the SparreAndersen universality [7,8,12]. Notably, however, if the first passage timedensity (FPTD) follows the Sparre Andersen scaling [7,8,12–14] the universal behaviour can be observed [11]. In particular trajectories of Lévy walks may exhibit discontinuous jumps and in a consequence, the location of the boundary itself is not hit by the majority of sample trajectories. In order to properly take care of possible excursion of the trajectories beyond the location of the boundary (at, say \( x = 0 \)) with subsequent re-crossings into the interval \( (x > 0) \), the whole semi-line \( (x \leq 0) \) has to be assumed “absorbing”. This nonlocal definition of the boundary conditions secures proper evaluation of the first passage time distribution and survival probability [7–10], see below.

To further elucidate the nature of deviation from the “standard” Sparre Andersen scaling in subordinated scenarios, we consider the process \( X(t) = \tilde{X}(S_t) \), for which the parental process \( \tilde{X}(\tau) \) is described by a Langevin equation [13]

\[
dX(\tau) = \sigma dL_\alpha(\tau)
\]

driven by a symmetric \( \alpha \)-stable Lévy motion \( L_\alpha(\tau) \) with the Fourier transform \( \langle e^{ikL_\alpha(\tau)} \rangle = e^{-|k|^\alpha} \). Here \( \tau \) stands for the operational time scale which is changed to the physical time scale \( t \) by subordination \( \tilde{X}(S_t) \). The subordinator \( S_t \) is defined as \( S_t = \inf \{ \tau : U(\tau) > t \} \) with \( U(\tau) \) denoting a strictly increasing \( \nu \)-stable Lévy motion \( (0 < \nu < 1) \) and is assumed to be independent from the noise term \( L_\alpha(\tau) \).

The above setup has been recently proved [13–15] to give a proper stochastic realization of the random process described otherwise by a fractional diffusion equation [8,16–20]

\[
\frac{\partial p(x,t)}{\partial t} = \alpha D_t^{1-\nu} \left[ \sigma^\alpha \frac{\partial^\alpha}{\partial |x|^\alpha} \right] p(x,t).
\]

Here \( \alpha D_t^{1-\nu} \) denotes the Riemann-Liouville fractional derivative defined as \( \alpha D_t^{1-\nu} f(t) = \Gamma(\nu)^{-1} \int_0^t (t-\tau)^{\nu-1} f(\tau) d\tau \) with \( 0 < \nu < 1 \) and \( \frac{\partial^\alpha}{\partial |x|^\alpha} \) with \( 0 < \alpha \leq 2 \) stands for the Riesz fractional derivative with the Fourier transform \( \mathcal{F}\{ \partial^\alpha_{|x|^\alpha} f(x) \} = -i|k|^\alpha \hat{f}(k) \) [21]. Occurrence of the operator \( \alpha D_t^{1-\nu} \) is due to the heavy-tailed waiting times between successive jumps and presence of the Riesz fractional derivative \( \frac{\partial^\alpha}{\partial |x|^\alpha} \) is a consequence of the Lévy-flight character of the jumps.

In this paper, instead of seeking an analytical solution to eq. (6), we switch to a Monte Carlo method [13–15,22,23] which allowed generating trajectories of the subordinated process \( X(t) \) with the parent process \( \tilde{X}(\tau) \). Furthermore, we study the potential free case, see eq. (6), i.e., we assume \( V(x) = 0 \). The assumed algorithm provides means to investigate the competition between subdiffusion (controlled by a \( \nu \)-parameter) and Lévy flights characterized by a stability index \( \alpha \).

For Markov processes, the Sparre Andersen scaling [5,6] presents a universal law which is independent of detailed properties of the jump length distribution (if it is only continuous and symmetric). In particular, for continuous times, the scaling predicts the \( \tau^{-1/2} \) decay of the survival probability, independently of whether the moments of the underlying jump process exist or not. For example, for \( \alpha < 2 \), the moments of the process \( X(t) \) (cf. eq. (5)) exist only for \( \delta < \alpha \) with obvious divergence of moments of order \( \delta > \alpha \), i.e.,

\[
\langle |X|^\delta \rangle = \int_{-\infty}^{\infty} |x|^\delta p(x,t) dx = \infty.
\]

This divergence can be easily demonstrated in the case of (pure) Lévy flights described by eq. (5), for which the operational time \( \tau \) and physical time \( t \) is equivalent, i.e. \( S_t = t \) and consequently \( \tau = \tau \), see below. In such a case, the \( p(x,t) \) is a Lévy stable density (whose width is growing with time) and \( \langle |X|^\delta \rangle \) stays infinite for \( \delta > \alpha \). Clearly, for finite time \( t \) and finite number of representative trajectories \( N \) (otherwise called realizations of the process \( X(t) \)), variance \( \langle (X - \bar{X})^2 \rangle \) of (symmetric) Lévy flights stays finite, see [24], eq. (1.19), and [25]. In fact, finite number of realizations (to be distinguished from the number of steps \( n \) used in simulation of a single trajectory of time duration \( t = n\Delta t \)) and finite time introduce an effective cutoff to the jump length distribution. In contrast, for any fixed time the variance diverges with increasing number of simulated trajectories \( N \). Analogously, for any fixed \( N \), the variance diverges with increasing time (scaling like \( t^{1/\alpha} \), see [24], eq. (1.19), and [25]). The problem of mathematical divergence can be resolved either by introducing spatiotemporal coupling (typical for so called Lévy walks [26,27]) or by proper truncation of the jump length distribution [28–31]. Several truncation methods have been proposed [28–31] to retain the finite second moment. In particular, paralleling the simulation studies of Mantegna and Stanley [28], a smooth exponential cutoff has been introduced by Koponen [29]. Instead of
truncating tails of a distribution, this approach is based upon the exponential tempering of the Lévy density and preserves the infinite divisibility [4] of the distribution. The classical tempered stable distribution has been further generalized by Rosiński (for more detailed discussion see [32], Chap. 5.7, and [29,33,34]).

**Methods and Results.** – For systems driven by symmetric processes the generalized Sparre Andersen scaling [2,4–6,35] can be used to discriminate between Markovian and non-Markovian situations. More precisely, according to the Sparre Andersen theorem for a stochastic processes driven by any symmetric white noises, the first passage time densities, \( f(t) = \frac{dF}{dt} \), from the real half line asymptotically behave like

\[
f(t) \propto t^{-3/2}.
\]

Consequently the survival probability \( S(t) \), i.e. the probability of finding a particle starting its motion at \( x_0 > 0 \) in the real (positive) half line, scales like

\[
S(t) = 1 - F(t) \propto t^{-1/2}.
\]

Therefore, any deviation of the survival probability from \( t^{-1/2} \)-dependence indicates violation of assumptions assuring the proof of the theorem. It can mean either that a system is driven by nonsymmetric or not “memory-less” driving. In consequence, for symmetric drivings, analysis of data based on (assumed *a priori*) Sparre Andersen scaling may reveal deviations from Markovianity.

We study statistical properties of a symmetric free Lévy motion eq. (5) constrained to the initial position \( x(0) = 1 \). To achieve the goals, we use the scheme of stochastic subordination [13–15,25], i.e. we obtain the process of primary interest \( X(t) \) as a function \( X(t) = \hat{X}(S_t) \) by randomizing the time clock of the process \( X(\tau) \) using a different clock \( S_t \). The parent process \( \hat{X}(\tau) \) is composed of increments of symmetric \( \alpha \)-stable motion described in an operational time \( \tau \) and in every jump moment the relation \( U(S_t) = t \) is fulfilled. The (inverse-time) subordinator \( S_t \) is (in general) non-Markovian hence, as it will be shown, the diffusion process \( \hat{X}(S_t) \) possesses also some degree of memory.

The survival probability, see eq. (9), was estimated from ensemble of trajectories of the process \( X(t) \) starting at \( x_0 \) \( (x_0 > 0) \). For \( \alpha < 2 \), in order to correctly account for non-local boundary conditions [7–9] we have excluded multiple recrossing events, i.e. every time the particle reached any point \( x \) beyond the boundary it was removed from the system.

In figs. 1, 2 the survival probability \( S(t) = 1 - F(t) \) is depicted for various stability indices \( \alpha \) and various subdiffusion parameters \( \nu \). It is clearly visible that the survival probability \( S(t) \) behaves like a power law for all considered values of the subdiffusion parameter \( \nu \) and the stability index \( \alpha \). However, the exponent characterizing the power law dependence is equal to \(-1/2\), as predicted by the (standard) Sparre Andersen theorem, only for the Markovian case \( (\nu = 1) \). In more general case the power law is characterized by the exponent \( b \)

\[
S(t) = 1 - F(t) \propto t^b
\]

(10)
which differs from

\[ b = -\frac{1}{2} \]  

(11)

For \( \alpha = 2 \) with any \( \nu \) (0 < \( \nu \) < 1), the first passage time distribution is one sided Lévy distribution characterized by the stability index \( \nu/2 \) [36–38], i.e.

\[ b = -\frac{\nu}{2} \]  

(12)

Furthermore, in the general case, the value of the exponent \( b \) does not depend on the stability index \( \alpha \) of the jump length distribution [35].

Figures 1, 2 confirm that the value of the exponent \( b \) depends on the subdiffusion parameter, \( \nu \), only. Figure 2 shows results for \( \alpha = 1.1 \). Results for others values of the stability index \( \alpha \) are the same as those one for \( \alpha = 1.1 \). Finally, fig. 3 presents value of the exponent \( b \), see eq. (10), as the function of the subdiffusion parameter \( \nu \) and the stability index \( \alpha \). Figure 3 confirms that the exponent \( b \) depends on the subdiffusion parameter \( \nu \) and the influence of the stability index \( \alpha \) is negligible. Furthermore, \( b \) depends linearly on \( \nu \): \( b = -(0.54 \pm 0.01)\nu + (0.03 \pm 0.03) \), what agrees with earlier findings [35,36,38], see fig. 3. Value of the exponent \( b \) is the decreasing function of the subdiffusion parameter \( \nu \) leading to the slowest decay of the survival probability for small values of the \( \nu \) parameter, i.e. when the exponent \( \nu \) deviates the most from its Markovian—“memory-less” value—1. The deviation of the exponent \( b \) from \(-1/2\) clearly indicates a typical slowing down of the subdiffusive process in comparison to its (Markov) regular diffusion analogue. The \( \alpha \)-independence of the survival probability \( S(t) \) in this case shows that the properties of the decay kinetics are determined by the subdiffusive part of the process only. This observation is different from the results obtained by Sokolov and Metzler for a class of Lévy random processes subordinated (via the relation connecting distribution of number of jumps \( n \) in physical time \( t \)) to Lévy flights or to Brownian random walks. In particular, in their derivation of subordination, the authors are using the Markovian Lévy-flight process \( X(t) \) transformed to the process \( X(T(t)) \) by use of the operational time \( T \) which, by itself, is called the directing process \( T(t) \). The density for the process \( X(T(t)) \) assumes the form \( P(x,t) = \int_0^\infty p(x,\tau)\nu(\tau,t)d\tau \) with \( p(x,\tau) \), \( \nu(\tau,t) \) representing densities of a Lévy-flight process and the density of the directing process, respectively. If \( X(t) \) is a stable process with a stability parameter \( \alpha \) and \( T(t) \) is a one-sided stable process with an exponent \( \nu \), the subordinated process \( X(T(t)) \) becomes a stable process with the stability index \( \nu/\alpha \). In contrast, in more general terms of the CTRW scenario, after waiving the assumption about independent increments of the \( T(t) \) process, the asymptotic form of the distribution \( P(x,t) \) can be derived by use of Tauberian theorems [16] and is known to be \( \nu/\alpha \) self-similar, i.e. \( P(x,t) = t^{-\nu/\alpha}P(xt^{-\nu/\alpha},1) \) [15,39].

**Summary and conclusions.** – We have discussed effects of the subordination scheme leading to the fractional diffusion equation, eq. (6). By use of the Monte Carlo method we have created trajectories of the process \( X(t) = \bar{X}(S_t) \) with \( S_t \) being the inverse time \( \alpha \)-stable subordinator. Since the \( S_t \) process appears as an asymptotic one in the CTRW scheme with heavy-tailed waiting time distribution between successive jumps and the parent process \( X(\tau) \) is assumed symmetric \( \alpha \)-stable, the proposed subordination [15] leads to \( \nu/\alpha \) self-similar process whose survival probabilities are governed by the stability exponent \( \nu \). Information gained from the analysis of generated trajectories brings around further confirmation of non-Markov property of the motion [25]. Moreover, due to the interplay between the subdiffusion in time and superdiffusion in step lengths, the resulting process violates the ergodicity (in the weak sense) so that the long-time average is different from the average taken over the ensemble of trajectories [40–42]. This issue is of special interest in the context of single-particle measurements [43] which require analysis of time series representative for the motion. In this work we demonstrate that subdiffusive and non-Markovian character of the motion can be grasped by analyzing survival probabilities which deviate from the (standard) Sparre Andersen scaling also
in those cases when the ensemble averages suggest a Brownian diffusion with $\nu/\alpha = 1/2$ [25].

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