Odd harmonious labeling on squid graph and double squid graph

F Febriana and K A Sugeng
Mathematics Department, Faculty of Mathematics and Natural Sciences, Universitas Indonesia, Indonesia
E-mail: moh.faqih@sci.ui.ac.id, kiki@sci.ui.ac.id

Abstract. An injective function $f$ from set of vertices in graph $G$ to a set of $\{0,1,...,|E|-1\}$ is called an odd harmonious labeling if the function $f$ induced the edge function $f^{*}$ from the set of edges of $G$ to a set of odd positive integer number $\{1,3,5,...,2|E|-1\}$ with $f^{*}(xy) = f(x) + f(y)$ for every edge $xy$ in $E$. Graph that has an odd harmonious labeling is called odd harmonious graph. The squid graph $T_{n,k}$ is a graph which is obtained from a cycle $C_{n}$ and we add $k$ pendant to one vertex of the cycle. It is known that $C_{n}$ is an odd harmonious graph if and only if $n \equiv 0 \mod 4$. However, by adding at least one pendant in the cycle graph, we can label the new graph odd harmoniously for all even number of vertices. In this paper, we showed that the graph $T_{n,k}$ and $T_{2n,k}$ are an odd harmonious graph, for $n \equiv 0 \mod 2$, $n \geq 4$ and $k \geq 1$. The construction of the odd harmonious labeling of the graph $T_{n,k}$ and $T_{2n,k}$ are inspired by the odd harmonious labeling of $C_{n}$ for $n \equiv 0 \mod 4$.

1. Introduction
Let $G = (V,E)$ be a graph with $p = |E|$ edges and $q = |V|$ vertices. A graph $G$ is said to be odd harmonious if there exists an injection $f : V \rightarrow \{0,1,2,...,2q-1\}$ such that induced function $f^{*} : E \rightarrow \{1,3,5,...,2q-1\}$ where $f^{*}(xy) = f(x) + f(y), \forall xy \in E(G)$ is a bijection function. The labeling $f$ is said to be odd harmonious labeling of $G$ [1]. A graph which has an odd harmonious labeling is called odd harmonious graph. Odd harmonious labeling is a variation of harmonious graph which is introduced by Graham and Sloane [2].

Liang and Bai [1] proved some necessary conditions for the existence of odd harmonious labeling of graphs:
1. If $G$ is an odd harmonious graph, then the $G$ is a bipartite graph.
2. If $(p,q)$-graph $G$ is odd harmonious, then $2\sqrt{q} \leq p \leq 2q - 1$.

A cycle is a 2-regular connected graph, and a cycle with $n$ vertices, $n \geq 3$ is denoted by $C_{n}$. For $C_{n}$ we have $|V(C_{n})| = |E(C_{n})|$. A path graph is a connected graph which two vertices of the graph have degree 1, while the rest vertices have degree 2. Many results regarding the odd harmonious labeling have been proved (see [1], [3], [4], and [5]). For a dynamic survey of various graph labeling results, the reader can find it in Gallian [6].

It is known that Cycle graph $C_{n}$, is odd harmonious if and only if $n \equiv 0$(mod 4) [1]. However, if we have cycle with $n \equiv 0$(mod 2) and add at least one pendant, then the resulting graph is called a squid graph $T_{n,k}$, and we can proved that the new graph is odd harmonious for all $n$ even.

In this paper, we proved the existence of odd harmonious labeling for squid graph $T_{n,k}$ and $T_{2n,k}$ when $n \equiv 0 \mod 2$, $n \geq 4$ and $k \geq 1$. 


2. Main results

A squid graph $T_{n,k}$ is a graph which is constructed from a cycle $C_n$ and adding $k$ pendants at one of the cycle vertex. The sets of vertices and edges of a squid graph $T_{n,k}$ are given by vertex set $\mathcal{V}(T_{n,k}) = \{v_i|i = 0,\ldots,n-1\} \cup \{x_i|i = 1,2,\ldots,k\}$ and for edge set $\mathcal{E}(T_{n,k}) = \{v_{n-1}v_0\} \cup \{v_iv_{i+1}|i = 0,\ldots,n-2\} \cup \{v_0x_j|j = 1,\ldots,k\}$.

A double squid graph $T_{2n,k}$ is a graph which is constructed from two cycles $C_n$ which are have one common vertex $v_0$ and adding $k$ pendants at vertex $v_0$. The sets of vertices and edges of a double squid graph $T_{2n,k}$ are given by vertex set $\mathcal{V}(T_{2n,k}) = \{v_i|i = 0,\ldots,2n-1\} \cup \{x_i|i = 1,2,\ldots,k\}$ and for edge set $\mathcal{E}(T_{2n,k}) = \{v_iv_{i+1}|i = 0,\ldots,n-2\} \cup \{v_{n-1}v_0\} \cup \{v_iv_{i+1}|i = n,\ldots,2n-2\} \cup \{v_{2n-1}v_0\} \cup \{v_0x_j|j = 1,\ldots,k\}$.

**Theorem 2.1** If $G$ is an odd harmonious graph, then the $G$ is a bipartite graph, and if $(p,q)$-graph $G$ is odd harmonious, then $2\sqrt{q} \leq p \leq 2q - 1$. [1]

**Theorem 2.2** Cycle graph $C_n$ is odd harmonious if and only if $n \equiv 0 \pmod{4}$. [1]

From Theorem 2.1, consequently, a squid graph $T_{n,k}$ with $n$ odd are not odd harmonious graphs, since the graph is not a bipartite graph. However, from Theorem 2.1, the cycle $C_n$ is not odd harmonious for $n \equiv 2 \pmod{4}$, but we proved that squid graph $T_{n,k}$, for all $n$ even.

**Theorem 2.3** The squid graph $T_{n,k}$, $n \geq 4$, $k \geq 1$ is odd harmonious if and only if $n \equiv 0 \pmod{2}$.

**Proof.** Let the squid graph $T_{n,k}$, $n \geq 4$, $k \geq 1$ is odd harmonious. Since the squid graph has $C_n$ as a subgraph, then $n$ cannot be odd. Suppose that $n$ is odd, then there is two adjacent vertices that both of them have odd labels or even labels. It makes the label of edge is not odd (contradiction). Let $n \equiv 0 \pmod{2}$. Defined labeling of vertices and edges squid graph $T_{n,k}$ as follows:

Label the vertices in cycle:

$$f(v_i) = \begin{cases} 
  i, & i = 0,\ldots,\frac{n}{2}, \\
  i + 2, & i = \frac{n}{2} + 1,\ldots,n-1.
\end{cases}$$

Label the vertices of the pendant:

$$f(x_i) = \begin{cases} 
  n + 5, & i = 1, \\
  2n - 1 + 2i, & i = 2,\ldots,k.
\end{cases}$$

The labels for all vertices are as follows:

$$f:\left(\mathcal{V}(T_{n,k})\right) = \left\{0,\ldots,\frac{n}{2}\right\} \cup \left\{\frac{n}{2} + 3,\ldots,n + 1\right\} \cup \left\{n + 5\right\} \cup \left\{2n + 3, 2n + 5,\ldots,2n - 1 + 2k\right\}$$

We can see that all vertex labels are all different and the largest vertex labels is $2n - 3 + 2k < 2q - 1$, then $f$ is an injective function.
Next we check the edge function $f^*$. Label the edges as follows:

$$f^*(v_i v_{i+1}) = \begin{cases} 2i + 1, & i = 0, ..., \frac{n}{2} - 1, \\ n + 3, & i = \frac{n}{2}, \\ 2i + 5, & i = \frac{n}{2} + 1, ..., n - 2, \\ n + 1, & i = n - 1. \end{cases}$$

Label the edge of the pendant

$$f^*(v_0 v_j) = \begin{cases} n + 5, & j = 1, \\ 2n - 1 + 2j, & j = 2, ..., k'. \end{cases}$$

From the formula, we can see that the edge labels are all different and

$$f:\left(E\left(T_{n,k}\right)\right) = \{1, ..., n - 1\} \cup \{n + 1\} \cup \{n + 3\} \cup \{n + 7, ..., 2n + 1\} \cup \{n + 5\} \cup \{2n + 3, ..., 2n - 1 + 2k\} = \{1,3,5, ..., 2n - 1 + 2k\}.\text{Thus, } f^* \text{ is a bijective function.}$$

Since $f$ is injective and $f^*$ is bijective, the we can conclude that $T_{n,k}$, for $n \geq 4$, $n$ even and $k \geq 1$ is odd harmonious.

In Figure 1, we can see an example of odd harmonious labeling for $T_{8,3}$.

![Figure 1. Example of odd harmonious labeling for $T_{8,3}$](image)

**Theorem 2.3** The Double Squid graph $T_{2n,k}$, $n \geq 4, k \geq 1$ is odd harmonious if and only if $n \equiv 0 \pmod{2}$.

**Proof.** Let the double squid graph $T_{2n,k}$, $n \geq 4, k \geq 1$ is odd harmonious. Since $C_n$ is a subgraph of the squid graph, then $n$ cannot be odd. Suppose that $n$ is odd, then there are two adjacent vertices that both odd labels or even labels. It makes the label of edge is even (contradiction).
Let \( n \equiv 0 \pmod{2} \). Defined labeling of vertices and edges of double squid graph \( T_{2n,k} \) as follows:

Label the vertices on both cycles:

\[
f(v_i) = \begin{cases} 
    i, & i = 0, \ldots, \frac{n}{2}, \\
    i + 2, & i = \frac{n}{2} + 1, \ldots, n - 1, \\
    i + 5, & i \text{ even}, n \leq i \leq 2n - 2, \\
    i + 1, & i \text{ odd}, n + 1 \leq i \leq 2n - 3.
\end{cases}
\]

Label the vertices of the pendant:

\[
f(x_i) = \begin{cases} 
    2n + 5, & i = 1, \\
    4n - 1 + 2i, & i = 2, \ldots, k.
\end{cases}
\]

The labels for all vertices are as follows:

\[
f: \left( V(T_{2n,k}) \right) = \left\{ 0, \ldots, \frac{n}{2} \right\} \cup \left\{ \frac{n}{2} + 3, \ldots, n + 1 \right\} \cup \{ n + 5, n + 7, \ldots, 2n + 3 \} \\
\cup \{ n + 2, n + 4, \ldots, 2n - 2 \} \cup \{ 2n + 5 \} \cup \{ 4n + 3, 4n + 5, \ldots, 4n + 2k - 1 \}.
\]

We see that all vertex labels are all different and the largest vertex labels is \( 4n + 2k - 1 \), where \( 4n + 2k - 1 < 2q - 1 \), then \( f \) is an injective function.

Next we consider the edge function \( f^* \). Label the edges on both cycles as follows:

\[
f^*(v_i, v_{i+1}) = \begin{cases} 
    2i + 1, & i = 0, \ldots, \frac{n}{2} - 1, \\
    n + 3, & i = \frac{n}{2}, \\
    2i + 5, & i = \frac{n}{2} + 1, \ldots, n - 2, \\
    n + 1, & i = n - 1, \\
    n + 5, & i = n, \\
    n + 2i - 3, & i = n + 1, \ldots, 2n - 2, \\
    2n + 3, & i = 2n - 1.
\end{cases}
\]

Label the edge of the pendants as follows:

\[
f^*(v_0 v_j) = \begin{cases} 
    2n + 5, & j = 1, \\
    4n - 1 + 2j, & j = 2, \ldots, k.
\end{cases}
\]

From the formula, we can see that the edge labels are all different and

\[
f: \left( E(T_{2n,k}) \right) = \{ 1, \ldots, n - 1 \} \cup \{ n + 3 \} \cup \{ n + 7, \ldots, 2n + 1 \} \cup \{ n + 1 \} \cup \{ n + 5 \} \cup \{ 3n - 1, \ldots, 3n - 7 \} \cup \{ 2n + 3 \} \cup \{ 2n + 5 \} \cup \{ 4n + 3, \ldots, 4n + 2k - 1 \} = \{ 1, \ldots, 4n + 2k - 1 \}.
\]

Thus, \( f^* \) is a bijective.

Since \( f \) is an injective and \( f^* \) is a bijective, then we proved that \( T_{2n,k} \), for \( n \geq 4, n \text{ even and } k \geq 1 \) is odd harmonious \( \blacksquare \)
In Figure 2, we can see an example of odd harmonious labeling for $T_{2(8),3}$.

![Figure 2. Odd harmonious labeling for $T_{2(8),3}$.](image)

### 3. Conclusions

In this paper, we have discussed the odd harmonious graph labeling. It is known that the cycle graph $C_n$ is odd harmonious only for $n \equiv 0 \pmod{4}$. However, by adding at least one pendant at one cycle vertex of $C_n$ then we have a new graph that we called a squid graph $T_{n,k}$, and it is proved that the squid graph $T_{n,k}$ is odd harmonious for $n \equiv 0 \pmod{2}$ and $k \geq 1$. Moreover, we proved that double squid graph $T_{2n,k}$ is also odd harmonious for $n \equiv 0 \pmod{2}$ and $k \geq 1$. There are still many families of graphs which are not known odd harmonious or not. This opportunity can be used for the open problem in this topic.

### Acknowledgment

The authors supported by Hibah Pitta A-UI 2019 No NKB-0440/UN2.R3.1/HKP.05.00/2019.

### References

[1] Liang A and Bai Z 2009 On the harmonious graphs with applications *J. Appl. Math Comput.* 29 105-116

[2] Graham R L and Sloane N J A 1980 On additive bases and harmonious graph *SIAM J. Alg. Discrete Methods* 1 382-404

[3] Jeyanthi P, Philo S and Sugeng K A 2015 Odd harmonious labeling of some new families of graphs *SUT Journal of Mathematics* 51(2) 181–193

[4] Saputri G A, Sugeng K A and Froncek D 2013 The odd harmonious labeling of dumbbell and generalized prism graphs *AKCE: Int. J. Graphs Combin.* 10(2) 221-228

[5] Vadya S K and Shah N H 2012 Some new odd harmonious graphs *International J. Math. Combin.* 3 105-112

[6] Gallian J A 2017 Dynamic survey of graph labeling *The Electronic Journal of Combinatorics* #DS6.