Abstract

We study the problem of privacy-preserving collaborative filtering where the objective is to reconstruct the entire users-items preference matrix using a few observed preferences of users for some of the items. Furthermore, the collaborative filtering algorithm should reconstruct the preference matrix while preserving the privacy of each user. We study this problem in the setting of joint differential privacy where each user computes her own preferences for all the items, without violating privacy of other users’ preferences.

We provide the first provably differentially private algorithm with formal utility guarantees for this problem. Our algorithm is based on the Frank-Wolfe (FW) method, and consistently estimates the underlying preference matrix as long as the number of users \( m \) is \( \omega(n^{5/4}) \), where \( n \) is the number of items, and each user provides her preference for at least \( \sqrt{n} \) randomly selected items. We also empirically evaluate our FW-based algorithm on a suite of datasets, and show that our method provides nearly same accuracy as the state-of-the-art non-private algorithm, and outperforms the state-of-the-art private algorithm by as much as 30%.

1 Introduction

Collaborative filtering (or matrix completion) is a popular approach for modeling the recommendation system problem, where the goal is to provide personalized recommendations about certain items to a user [32]. In other words, the objective of a personalized recommendation system is to learn the entire users-items preference matrix \( Y^* \in \mathbb{R}^{m \times n} \) using a small number of user-item preferences \( Y_{ij}^*, (i, j) \in [m] \times [n] \), where \( m \) is the number of users and \( n \) is the number of items. Naturally, in absence of any structure in \( Y^* \), the problem is ill-defined as the unknown entries of \( Y^* \) can be arbitrary. Hence, a popular modeling hypothesis is that the underlying preference matrix \( Y^* \) is low-rank, and thus, the collaborative filtering problem reduces to that of low-rank matrix completion [8, 38]. One can also enhance this formulation using side-information like user-features or item-features [47].

Naturally, personalization problems require collecting and analyzing sensitive customer data like their preferences for various items, which can lead to serious privacy breaches [7, 33, 37]. In this work, we attempt to address this problem of privacy-preserving recommendations using collaborative filtering [35, 36]. In particular, we answer the following question in the affirmative:
Can we design a matrix completion algorithm which keeps all the ratings of a user private, i.e., guaranteeing user-level privacy while still providing accurate recommendations? In particular, we provide the first differentially private [12] matrix completion algorithms with provable accuracy guarantees. Differential privacy is a rigorous privacy notion which formally protects the privacy of any user participating in a statistical computation by controlling her influence to the final output.

Most of the prior works on differentially private matrix completion (and low-rank approximation) [3, 9, 15, 19, 20, 28] have provided guarantees which are non-trivial only in the entry-level privacy setting, i.e., they preserve privacy of only a single rating of a user. Hence, they are not suitable for preserving a user’s privacy in practical recommendation systems. In fact, their trivial extension to user-level privacy leads to vacuous bounds (see Table 1). Some works [35, 36] do serve as an exception, and directly address the user-level privacy problem. However, these results only show empirical evidences of their effectiveness; they do not provide formal error bounds. In case of [35], the differential privacy guarantee itself might require exponential amount of computation. In contrast, we provide an efficient algorithm based on the classic Frank-Wolfe procedure [17], and show that it can provide strong utility guarantees while preserving user-level privacy. Furthermore, we empirically demonstrate the effectiveness of this algorithm on various standard benchmark datasets.

**Notion of privacy:** To measure privacy, we select differential privacy (DP) [12] which is a de-facto privacy notion for large-scale learning systems, and has been widely adopted by the academic community as well as big corporations like Apple [1], Google [16], etc. The underlying principle of differential privacy is that the output of the algorithm should not change significantly due to presence or absence of a user. In the context of matrix completion, where the goal is to release the entire preference matrix \( Y \) while preserving privacy, this implies that the computed ratings/preferences of any particular user \( Y_i \) cannot depend strongly on her own personal preferences. Naturally, the resulting preference computation is going to be trivial and inaccurate [19].

To alleviate this concern, we consider a relaxed but natural differential privacy notion (for recommendation systems) called joint differential privacy [30]. Consider an algorithm \( A \) that produces individual outputs \( Y_i \) for each user \( i \), i.e., the \( i \)-th row of preference matrix \( Y \). For each user \( i \), joint differential privacy ensures that the output of \( A \) for all other users (denoted by \( Y_{-i} \)) does not reveal “much” about the preference data of user \( i \). That is, recommendations made to all the users other than \( i \)-th user do not depend significantly upon \( i \)-th user’s preferences. Although not mentioned explicitly, previous works on differentially private matrix completion [35, 36] do strive to ensure joint differential privacy. Formal definitions are provided in Section 2.

**Granularity of privacy:** Differential privacy protects the information about a user in the context of presence or absence of her data record. Prior works on differentially private matrix completion [35, 36], and its close analogue, low-rank approximation [3, 9, 14, 19, 20], have considered different variants of the notion of presence of a data record. Some have considered a single entry in the matrix \( Y^* \) as a data record (which corresponds to entry-level privacy), whereas others have considered a more practical setting where the complete row is a data record (which corresponds to user-level privacy). In this work, we present all our results in the strictly harder user-level privacy setting. To ensure a fair comparison, we present the results of prior work in the same setting.
1.1 Problem definition: Matrix completion

The goal of a low-rank matrix completion problem is to estimate a low-rank (or a convex relaxation of bounded nuclear norm) matrix $Y^* \in \mathbb{R}^{m \times n}$ having seen only a small number of entries from it. Let $\Omega = \{(i, j) \subseteq [m] \times [n]\}$ be the index set of the observed entries from $Y^*$, and let $P_\Omega : \mathbb{R}^{m \times n} \rightarrow \mathbb{R}^{m \times n}$ be a matrix operator s.t. $P_\Omega(Y)_{ij} = Y_{ij}$ if $(i, j) \in \Omega$, and 0 otherwise. Given, $P_\Omega(Y^*)$, the objective is to output a matrix $Y$ such that the following generalization error, i.e., the error in approximating a uniformly random entry from the matrix $Y^*$, is minimized:

$$F(Y) = \mathbb{E}_{(i,j) \sim \text{unif}[m] \times [n]} \left[ (Y_{ij} - Y^*_{ij})^2 \right].$$

Generalization error captures the predictive ability of the algorithm on unseen data samples from the matrix $Y^*$. We would want the generalization error to be $o(1)$ in terms of the problem parameters. Throughout the paper, we will assume that the number of rows ($m$) of $Y^*$, i.e., the number of users is larger than the number of columns ($n$), i.e., the number of items.

1.1.1 Our contributions

In this work, we provide the first joint differentially private algorithm for low-rank matrix completion with formal non-trivial error bounds. The error bounds for our method are summarized in Tables 1 and 2. At a high level, our key result can be summarized as follows:

**Informal Theorem 1.1** (Corresponds to Corollary 3.1). Assume that each entry of a hidden matrix $Y^* \in \mathbb{R}^{m \times n}$ is in $[-1, 1]$, and there are $\sqrt{n}$ observed entries per user. Also, assume that the nuclear norm of $Y^*$ is bounded by $O(\sqrt{mn})$, i.e., $Y^*$ is nearly constant rank. Then, there exist $(\epsilon, \delta)$-joint differentially private algorithms that have $o(1)$ generalization error, as long as the number of rows $m = \omega(n^{5/4})$.

That is, even with $\sqrt{n}$ observed ratings per user, on an average we obtain asymptotically the correct estimation of each entry of $Y^*$ as long as the number of users $m$ is large enough. The sample complexity bound dependency on the number of users ($m$) can be strengthened if we make additional assumptions on the hidden matrix such as incoherence. See Appendix B for details.

Our algorithm is based on two important ideas: a) using local and global computation, b) using the Frank-Wolfe method as a base optimization technique.

**Local and global computation:** The key idea that defines our algorithm and allows us to get strong error bounds under joint differential privacy is that of splitting the overall algorithm into two components: global and local. Recall that each row of the hidden matrix $Y^*$ belongs to an individual user. The global component of our algorithm computes statistics that are aggregate in nature (e.g., computing the correlation across columns of the revealed matrix $P_\Omega(Y^*)$). On the other hand, the local component of the algorithm independently fine tunes the statistics computed by the global component to generate accurate predictions for each user. Since the global component depends on the data of all users, adding noise to it (in order to protect privacy) does not significantly affect the accuracy of the predictions. [35] [36] also exploit a similar idea of segregating the computation, but they do not utilize it formally to provide non-trivial error bounds.

**Frank-Wolfe based method:** We use standard nuclear norm formulation [8, 38, 39, 44] for the matrix completion problem:

$$\min_{\|Y\|_{\text{nuc}} \leq k} \frac{1}{2|\Omega|} \|P_\Omega(Y - Y^*)\|^2_F,$$ (1)
where $\|Y\|_{\text{nuc}}$ is the sum of singular values of $Y$ and the underlying hidden matrix $Y^*$ is assumed to have nuclear norm of at most $k$. We use the popular Frank-Wolfe algorithm [17, 24] as our algorithmic building block to solve (1). At a high-level, Frank-Wolfe computes the solution to (1) as a convex combination of rank one matrices, each with nuclear norm at most $k$. These rank one matrices are added iteratively to the solution.

Our main contribution is to design a version of the Frank-Wolfe method which preserves joint differential privacy. That is, if the standard Frank-Wolfe algorithm decides to add matrix $u \cdot v^T$ during an iteration, then our global component computes a noisy version of $v \in \mathbb{R}^n$ via our global component. Then each user computes the respective row of $u \in \mathbb{R}^m$ to obtain the desired update matrix. The noisy version of $v$ suffices for the joint differential privacy guarantee, and allows us to provide the strong error bound outlined in Theorem 1.1 above.

We want to emphasize that the choice of Frank-Wolfe as the underlying matrix completion algorithm is critical for our system. Frank-Wolfe algorithm updates via rank one matrices in each step, hence the error incurred due to addition of noise in each step is small (i.e., proportional to the rank) and allows for an easy decomposition into the local-global computation model. Other standard techniques like proximal gradient descent based techniques [5, 34] can involve nearly full-rank updates in an iteration and hence might incur large error and lead to arbitrary inaccurate solutions. Also note that a prior work [42] also proposed a differentially private Frank-Wolfe (FW) algorithm for high dimensional regression but it was for a completely different problem in a different setting where the segregation of computation into global and local components was not required.

**SVD-based method:** In Appendix B, we also extend our technique to a singular value decomposition (SVD) based method for matrix completion/factorization. Our utility analysis shows that there are certain settings where this method outperforms our Frank-Wolfe based method, but in general can provide significantly worse solution. The main purpose of this algorithm is to study the power of the simple SVD-based method which is still a popular method for collaborative filtering.

**Empirical results:** Finally, we show that our Frank-Wolfe based algorithm not only have strong analytical guarantees, it also scores well empirically. In particular, we show the efficacy of our algorithms on benchmark collaborative filtering datasets like Jester [18], MovieLens [22], the Netflix prize dataset, and the Yahoo! Music recommender dataset [46]. We observe that our algorithm’s accuracy nearly matches that of the standard non-private nuclear norm minimization methods, while significantly outperforming (in terms of accuracy) the existing state-of-the-art differentially private matrix completion method by up to 30%.

**1.2 Comparison to prior work**

As discussed earlier, our results are the first to provide non-trivial error bounds for matrix completion with differential privacy. In Table 1 we place our sample complexity bounds, i.e., the number of users $m$, and the total number of samples $|\Omega|$ (i.e., number of observed preferences) required to attain an $o(1)$ generalization error. For comparison of different results, we consider the following setting of the hidden matrix $Y^* \in \mathbb{R}^{m \times n}$ and the set of released entries $\Omega$: i) $|\Omega| \approx m\sqrt{n}$, ii) each row of $Y^*$ has an $\ell_2$ norm of $\sqrt{n}$, and iii) each row of $P_{\Omega}(Y^*)$ has an $\ell_2$-norm of at most $n^{1/4}$, i.e., roughly around $\sqrt{n}$ random entries are revealed for each row. Furthermore, we assume the spectral norm of $Y^*$ is at most $O(\sqrt{mn})$, and $Y^*$ is rank one. Note that these conditions are satisfied by a matrix $Y^* = u \cdot v^T$ where $u_i, v_j \in [-1, 1] \forall i, j$, and random $\sqrt{n}$ entries are observed per user.
We compare our results with the best non-private algorithm for matrix completion based on nuclear norm minimization [39], and the prior work on differentially private matrix completion [35, 36]. The comparison is based on the sample complexity, i.e., the number of users (m) and the number observed entries |Ω| needed to get a generalization error of o(1). The comparison is tabulated in Table 1. We see that the sample complexity on the number of users m increases from ω(n) to ω(n^{5/4}) for our Frank-Wolfe based algorithm, while keeping the sample complexity |Ω| the same. While [35, 36] work under the same notion of joint differential privacy as us, but does not provide any formal accuracy guarantees.

| Algorithm                      | Bound on m | Bound on |Ω||
|-------------------------------|------------|----------|
| Nuclear norm min. (non-private) [39] | ω(n)       | ω(m/√n) |
| Noisy SVD + kNN [36]          | –          | –        |
| Noisy SGLD [35]               | –          | –        |
| Noisy Frank-Wolfe (This work) | ω(n^{5/4}) | ω(m/√n) |

Table 1: Sample complexity bounds for matrix completion. n is the number of items and m is the number of users. The bounds hide privacy parameters ϵ and log(1/δ), and polylog factors in m, n.

Interlude: Low-rank approximation. We compare our results with the prior work on a related problem of differentially private low-rank approximation as well. The problem is given a matrix Y* ∈ ℜ_{m×n}, compute a differentially private low-rank approximation Y_{priv}, such that it is close to Y* either in the spectral or Frobenius norm. Notice that this is very similar to matrix completion if the set of the revealed entries Ω is the complete matrix and hence our methods can be applied directly. In order to be consistent with the existing literature, we assume that each row of Y* has ℓ2-norm of at most one (i.e., ∥Y_i*∥_2 ≤ 1 for all i ∈ [n]), and Y* is rank one matrix. Table 2 compares various results; the “Rank” column in Table 2 is the rank of the output matrix Y_{priv}. Notice that all the prior works provide trivial error bounds (in both Frobenius and spectral norm), since ∥Y*∥_2 = ∥Y*∥_F ≤ √m, while our methods provide non-trivial bounds. The key difference is that we ensure joint differential privacy, while existing methods ensure stricter standard differential privacy (Definition 2.1).

2 Background: Notions of privacy

Let D = {d_1, · · · , d_m} be a dataset of m entries. Each entry d_i lies in a fixed domain T, and belongs to an individual i, whom we refer to as an agent in this paper. Furthermore, d_i encodes potentially sensitive information about agent i. Let A be an algorithm that operates on dataset D, and produces a vector of m outputs, one for each agent i and from a set of possible outputs S. Formally, let A : T^m → S^m. Let D−i denote the data set D without the entry of the i-th agent, and similarly A−i(D) be the set of outputs without the output for the i-th agent. In the following, we define both standard differential privacy (Definition 2.1) and joint differential privacy (Definition 2.2), and contrast them.

Definition 2.1 (Standard differential privacy [11, 12]). An algorithm A satisfies (ϵ, δ)-differential privacy if for any agent i, any two possible values of data entry d_i, d'_i ∈ T for agent i, any tuple of
Table 2: Approximation error bounds ($\|Y-Y^*\|_F$) for low-rank approximation. $m, n$ is the number of users and items, respectively. $\mu \in [0, m]$ is the incoherence parameter (Definition B.2). The bounds hide privacy parameters $\epsilon$ and $\log(1/\delta)$, and polylog factors in $m$ an $n$. The privacy notion for the algorithm in this work is Joint DP, whereas it is Standard DP for all the other algorithms.

| Algorithm                                      | Error                                   | Rank                      |
|------------------------------------------------|-----------------------------------------|---------------------------|
| Randomized response                            | $O(\sqrt{m + n})$                       | $O(1)$                    |
| Gaussian measurement                           | $O \left( \sqrt{m + \frac{mn}{m}} \right)$ | $O(1)$                    |
| Noisy power method                             | $O(\sqrt{\mu})$                        | $O(1)$                    |
| Exponential mechanism                          | $O(m + n)$                              | $O(1)$                    |
| Noisy Frank-Wolfe (This work)                  | $O \left( \frac{10}{\sqrt{m^2/n}} \right)$ | $O \left( \frac{m^{3/5}}{n^{1/5}} \right)$ |
| Noisy SVD (This work)                          | $O \left( \sqrt{\mu \left( \frac{n^2}{m} + \frac{m}{n} \right)} \right)$ | $O(1)$                    |

data entries for all other agents, $D_{-i} \in \mathcal{T}^{m-1}$, and any output $S \in \mathcal{S}^m$,

$$\Pr[A(d_i; D_{-i}) \in S] \leq e^\epsilon \Pr[A(d'_i; D_{-i}) \in S] + \delta.$$  

Here the notation $(d_i; D_{-i})$ corresponds to a data set obtained by adding data entry $d_i$ to the data set $D_{-i}$. 

At a high-level, an algorithm is $(\epsilon, \delta)$-standard differentially private if for every data set $D$, the output $A(D)$ and $D_{-i}$ does not reveal “much” about the data entry $(d_i)$ for agent $i$. In this paper, we will consider the privacy parameter $\epsilon$ to be a small constant ($\approx 0.1$), and $\delta = O(1/m^2)$. There are semantic reasons for such choice of parameters [29], but that is beyond the scope of the current work.

For reasons mentioned in Section 1, our matrix completion algorithms provide privacy guarantee based on a relaxed notion of differential privacy, called joint differential privacy, which was initially proposed in [30]. At a high-level, an algorithm preserves $(\epsilon, \delta)$-joint differential privacy if for each agent $i$, knowledge of the output of Algorithm $A$ for the other $(m-1)$ agents (denoted by $A_{-i}(D)$) and their data entries $D_{-i}$ does not reveal “much” about the data data entry of agent $i$. For matrix completion, such a relaxation is necessary because an accurate completion of the row of an agent can reveal a lot of information about her observed entries. However, it is still a very strong privacy guarantee for an agent, even if every other agent colludes against her, as long as she does not make the predictions made to her public.

**Definition 2.2 (Joint differential privacy [30]).** An algorithm $A$ satisfies $(\epsilon, \delta)$-joint differential privacy if for any agent $i$, any two possible values of data entry $d_i, d'_i \in \mathcal{T}$ for agent $i$, any tuple of data entries for all other agents, $D_{-i} \in \mathcal{T}^{m-1}$, and any output $S \in \mathcal{S}^{m-1}$,

$$\Pr[A_{-i}(d_i; D_{-i}) \in S] \leq e^\epsilon \Pr[A_{-i}(d'_i; D_{-i}) \in S] + \delta.$$  

Here the notation $(d_i; D_{-i})$ corresponds to a data set obtained by adding data entry $d_i$ to the data set $D_{-i}$. 

6
3 Private matrix completion via Frank-Wolfe

Recall that the objective is to solve the matrix completion problem (defined in Section 1.1) under joint differential privacy. A standard modeling assumption is that \( Y^* \) is nearly low-rank, leading to the following empirical risk minimization problem \([26, 27, 31]\): \[
\min_{Y} \frac{1}{m} \sum_{(i,j) \in \Omega} (Y_{ij} - \Omega_{ij})^2 \quad \text{s.t.} \quad \text{rank}(Y) \leq k, \quad \text{where} \quad k \ll \min(m, n). \]

As the above mentioned problem is a challenging non-convex optimization problem, a popular approach is to relax the rank constraint to a nuclear-norm constraint \([8, 38, 39, 44]\), i.e.,

\[
\min_{Y} \frac{1}{2|\Omega|} \|P_{\Omega}(Y - Y^*)\|^2_F, \quad \text{s.t.} \quad \|Y\|_{\text{nuc}} \leq k,
\]

where \( \|Y\|_{\text{nuc}} = \sum_i \sigma_i(Y) \) is the nuclear norm of \( Y \), and \( \sigma_i(Y) \) is the \( i \)-th singular value of \( Y \).

To this end, we use the Frank-Wolfe \([17]\) algorithm as our building block (see Appendix A for a detailed description). Frank-Wolfe is a popular conditional gradient algorithm where the current iterate is updated by:

\[
Y(t) \leftarrow (1 - \eta)Y(t-1) + \eta \cdot G, \text{ where } \eta \text{ is the step size, and } G \text{ is given by:}
\]

\[
\min_{G} \langle G, \nabla_{Y(t-1)} F(Y) \rangle, \text{s.t. } \|G\|_{\text{nuc}} \leq k, \text{ and } F(Y) = \frac{\|P_{\Omega}(Y) - P_{\Omega}(Y^*)\|^2_F}{2|\Omega|}.
\]

Note that the optimal solution to the above problem is: \( G = -kuv^\top \), where \( (\lambda, u, v) \) are the top singular components of \( A(t-1) = P_{\Omega}(Y(t-1) - Y^*) \). Also, the optimal \( G \) is a rank one matrix.

**Algorithmic ideas:** In order ensure joint differential privacy and still have strong error guarantees, we develop the following ideas. These ideas have been formally compiled into Algorithm 1. Notice that both the functions \( A_\text{global} \) and \( A_\text{local} \) are parts of the Private Frank-Wolfe technique, where \( A_\text{global} \) consists of the global component, and each user runs \( A_\text{local} \) at her end to carry out a local update. Throughout this discussion, we assume that \( \max_{i \in [m]} \|P_{\Omega}(Y_i^*)\|_2 \leq L \).

**Splitting the update into global and local components:** One can equivalently write the Frank-Wolfe update as follows:

\[
Y(t) \leftarrow (1 - \eta)Y(t-1) - \eta \cdot \frac{k}{\lambda} A(t-1)vv^\top, \quad \text{where } A(t-1), v, \text{ and } \lambda \text{ are defined as above. Note that } v \text{ and } \lambda^2 \text{ can also be obtained as the top right eigenvector and eigenvalue of } A(t-1)^\top A(t-1) = \sum_{i=1}^m A_i(t-1)^\top A_i(t-1), \text{ where } A_i(t-1) = P_{\Omega}(Y_i(t-1) - Y_i^*) \text{ is the } i\text{-th row of } A(t-1).
\]

We will use the **global component** \( A_\text{global} \) in Algorithm 1 to compute \( v \) and \( \lambda \). Using the output of \( A_\text{global} \), each user (row) \( i \in [m] \) can compute her **local update** (using \( A_\text{local} \)) as follows:

\[
Y_i(t) = (1 - \eta)Y_i(t-1) - \frac{\eta k}{\lambda} P_{\Omega}(Y(t-1) - Y_i^*)vv^\top.
\]

A block schematic of this idea is presented in Figure 3.

**Noisy rank one update:** Observe that \( v \) and \( \lambda \), the statistics that are computed in each iteration via \( A_\text{global} \), are aggregate statistics that use information from all the rows of \( Y^* \). This ensures that they are noise tolerant, and hence adding sufficient noise can ensure standard differential privacy (Definition 2.1) for \( A_\text{global} \). Since the final objective is to satisfy joint differential privacy (Definition 2.2), the local update procedure \( A_\text{local} \) can compute the update for each user (corresponding to (2)) without adding any noise.

**Controlling norm via projection:** In order to control the amount of noise needed to ensure differential privacy, it is important that any individual data entry (here, any row of \( Y^* \)) has bounded
effect on the aggregate statistic computed by $A_{\text{global}}$. One challenge we face with the design is that each intermediate computation $Y_i(t)$ in (2) can have high $\ell_2$-norm even if $\|P_\Omega(Y_i^*)\|_2 \leq L$. We address this challenge by applying the projection $\Pi_{L,\Omega}$ operator (defined below) to $Y_i(t)$, and compute the local update as $\Pi_{L,\Omega}(Y_i(t))$ in place of (2). $\Pi_{L,\Omega}$ is defined as follows: For any matrix $M$, the projector $\Pi_{L,\Omega}(M)$ ensures that any row of the “zeroed out” matrix $P_\Omega(M)$ does not have $\ell_2$-norm higher than $L$. Formally, $\Pi_{L,\Omega}(M)_{i,j} = L \cdot \frac{M_{i,j}}{\|P_\Omega(M_i)\|_2}$ for all entries $(i,j)$ of the matrix $M$ where $\|P_\Omega(M_i)\|_2 \geq L$. In our analysis, we will show that this projection operation does not increase the error.

3.1 Privacy and utility analysis

**Theorem 3.1.** Algorithm 1 satisfies $(\epsilon, \delta)$-joint differential privacy if $\epsilon \leq 2 \log \left(\frac{1}{\delta}\right)$.

For a proof of Theorem 3.1 see Appendix C.1.1. The proof uses standard differential privacy properties of Gaussian noise addition from [4]. We now show that the empirical risk of our algorithm is close to the optimal as long as the number of users $m$ is “large”.

**Theorem 3.2 (Excess empirical risk guarantee).** Let $Y^*$ be a matrix with $\|Y^*\|_{\text{nuc}} \leq k$, and $\max_{i \in [m]} \|P_\Omega(Y^*)_i\|_2 \leq L$. Let $Y^{(T)}$ be a matrix, with its rows being $Y_i^{(T)}$ for all $i \in [m]$, computed by function $A_{\text{local}}$ in Algorithm 1 after $T$ iterations. Then, with probability at least $\frac{2}{3}$ over the outcomes
Algorithm 1: The Private Frank-Wolfe algorithm

1 Function Global Component $A_{\text{global}}$ [Input: privacy parameters, $(\epsilon, \delta)$, total number of iterations $T$, bound on $\|P_O(Y^*_i)\|_2 \leq L$, failure probability $\beta$, number of users $m$, number of items $n$]

2 $\sigma \leftarrow L^2 \sqrt{64 \cdot T \log(1/\delta) / \epsilon}$

3 Initialize: $\hat{\nu} \leftarrow \{0\}^n$, $\hat{\lambda} \leftarrow 0$

4 for $t \in T$ do

5 $W(t) \leftarrow \{0\}^{n \times n}$

6 $\hat{\lambda}_{\text{biased}} \leftarrow \hat{\lambda} + \sqrt{\sigma \log(n/\beta)} n^{1/4}$

7 for $i \in [m]$ do

8 $W(t) \leftarrow W(t) + A_{\text{local}}(i, \hat{\nu}, \hat{\lambda}_{\text{biased}}, T, t, L)$

9 end

10 $W(t) \leftarrow W(t) + N(t)$, where $N(t) \in \mathbb{R}^{n \times n}$ corresponds to a matrix with i.i.d. entries from $\mathcal{N}(0, \sigma^2)$.

11 $\hat{\nu} \leftarrow \text{Top right eigenvector of } W(t), \hat{\lambda} \leftarrow \text{Top eigenvalue of } W(t)$.

end

13 Function Local Update $A_{\text{local}}$ [Input: user number $i$, top right singular vector $\hat{\nu}$, top singular value $\hat{\lambda}_{\text{biased}}$, total number of iterations $T$, current iteration $t$, bound on $\|P_O(Y^*_i)\|_2$ as $L$, private true matrix row $P_O(Y^*_i)$]

14 $Y_i(0) \leftarrow \{0\}^n, A_i(t-1) \leftarrow P_O(Y_i^{(t-1)} - Y^*_i)$

15 $\hat{u}_i \leftarrow (A_i^{(t-1)} \cdot \hat{\nu}) / \hat{\lambda}_{\text{biased}}$

16 Define an operator $\Pi_{L, O}(M)_{i,j} = L \cdot \frac{M_{i,j}}{\|P_O(M_{i,j})\|_2}$ if $\|P_O(M_{i,j})\|_2 \geq L$

17 $Y_i(t) \leftarrow \Pi_{L, O} \left( (1 - \frac{1}{T}) Y_i^{(t-1)} - \frac{k}{T} \hat{u}_i(\hat{\nu})^T \right)$

18 $A_i(t) \leftarrow P_O\left(Y_i(t) - Y^*_i\right)$

19 if $t = T$ then

20 Output $Y_i(T)$ as prediction to user $i$ and stop

21 else

22 Return $A_i^{(t)} A_i^{(t)}$ to $A_{\text{global}}$

23 end

9
of the algorithm, the following is true:

\[
\frac{1}{2|\Omega|} \left\| P_\Omega \left( Y^{(T)} - Y^* \right) \right\|_F^2 = O \left( \frac{k^2}{|\Omega|T} + \frac{kT^{1/4}L \sqrt{n^{1/2} \log^{1/2}(1/\delta) \log n}}{|\Omega| \sqrt{\epsilon}} \right). \tag{3}
\]

Furthermore, the following is true after setting \( T = \tilde{O} \left( \frac{k^{4/5} \epsilon^{2/5}}{n^{1/5} L^{4/5}} \right) \) and hiding poly-logarithmic terms:

\[
\frac{1}{2|\Omega|} \left\| P_\Omega \left( Y^{(T)} - Y^* \right) \right\|_F^2 = \tilde{O} \left( \frac{k^{6/5} n^{1/5} L^{4/5}}{|\Omega| \epsilon^{2/5}} \right).
\]

The \( O(\cdot) \) hides only universal constants.

See Appendix C.1 for a detailed proof of the theorem. At a high-level, our proof combines the noisy eigenvector estimation error for Algorithm 1 with a noisy-gradient analysis of the Frank-Wolfe algorithm. Also, note that the first term in (3) corresponds to standard Frank-Wolfe convergence error, while the second term can be attributed to the noise added to preserve privacy, which directly depends on the number of iterations \( T \). We also compute the optimal number of iterations required to minimize the empirical risk. Finally, the rank of \( Y^{(T)} \) is at most \( T \), but its nuclear-norm is bounded by \( k \). As a result, \( Y^{(T)} \) has low generalization error (see Section 3.1.1).

Remark 1. We further illustrate our empirical risk bound by considering a simple setting: let \( Y^* \) be a rank one matrix with \( Y^*_{ij} \in [-1, 1] \) and let \( |\Omega| = m \sqrt{n} \). Then \( k = O(\sqrt{mn}) \), and \( L = O(n^{1/4}) \), implying an error of \( \tilde{O}(\frac{n}{m^{2/5}}) \) with hiding the privacy parameter \( \epsilon \); in contrast, a trivial solution like \( Y = 0 \) leads to \( O(1) \) error. Naturally, the error increases with \( n \) as there is more information to be protected. However, it decreases with a larger number of users \( m \) as the presence/absence of a user has lesser effect on the solution with increasing \( m \). We leave further investigation of the dependency of the error on \( m \) for future work.

Remark 2. Our analysis does not require an upper bound on the nuclear norm of \( Y^* \) (as stated in Theorem 3.2); instead, we would incur an additional error of \( \min \| Y \|_{nuc} \leq k \frac{1}{|\Omega|} \| P_\Omega (Y^* - Y) \|_F^2 \). Additionally, consider a similar scenario as in Remark 1 but \( \Omega = [m] \times [n] \), i.e., all the entries of \( Y^* \) are revealed. In such a case, \( L = O(\sqrt{n}) \), and the problem reduces to that of standard low-rank matrix approximation of \( Y^* \). Note that our result here leads to an error bound of \( \tilde{O} \left( \frac{n^{1/5}}{m^{2/5}} \right) \), while the state-of-the-art result by [20] leads to an error bound of \( O(1) \) due to their much stricter privacy model.

3.1.1 Generalization error guarantee

We now present a generalization error bound which shows that our approach provides accurate prediction over unknown entries. Formally, the generalization error (or true risk) and the empirical risk for a solution \( Y \) are defined as:

\[
F(Y) = E_{(i,j) \sim \text{unif} [m] \times [n]} \left[ (Y_{ij} - Y^*_{ij})^2 \right], \quad \hat{F}(Y) = \frac{1}{|\Omega|} \sum_{i,j \in \Omega} (Y_{ij} - Y^*_{ij})^2. \tag{4}
\]

Now, for our bound, we use the following result by [41]:

\[
\]
Theorem 3.3 (From [41]). Let $Y^*$ be a hidden matrix, and the data samples in $\Omega$ be drawn uniformly at random from $[m] \times [n]$. Let $A \in \mathbb{R}^{m \times n}$ be a matrix with $\text{rank}(A) \leq r$, and let each entry of $A$ be bounded by a constant. Then, the following holds with probability at least $2/3$:

$$|F(A) - \hat{F}(A)| = \tilde{O}\left(\sqrt{\frac{r \cdot (m + n)}{|\Omega|}}\right).$$

The $\tilde{O}(\cdot)$ hides only universal constants and poly-logarithmic terms in $m$ and $n$.

Also, the output of our Private Frank-Wolfe algorithm (Algorithm 1) has rank at most $T$, where $T$ is the number of iterations. Therefore, by replacing $T$ from Theorem 3.2, we have the following corollary:

Corollary 3.1 (Generalization Error). Let $\|Y^*\|_{\text{nuc}} \leq k$ for a hidden matrix $Y^*$, and for every row $i$ of $Y^*$, let $\|P_{\Omega}(Y^*)_i\|_2 \leq L$. If we choose the number of rounds in Algorithm 1 to be $O\left(\frac{k^{4/3}}{\sqrt[4]{|\Omega|}} (m + n)^{1/3}\right)$, and the data samples in $\Omega$ are drawn uniformly at random from $[m] \times [n]$, then with probability at least $2/3$ over the outcomes of the algorithm, the following is true for the final completed matrix $Y$:

$$F(Y) = \tilde{O}\left(\frac{k^{4/3} L n^{1/4}}{\sqrt{\epsilon |\Omega|^{13/6}} (m + n)^{1/6}} + \left(\frac{k \sqrt{m + n}}{|\Omega|}\right)^{2/3}\right).$$

The $\tilde{O}(\cdot)$ hides only universal constants and poly-logarithmic terms in $m, n, |\Omega|$ and $\delta$.

Remark 3. We further illustrate our bound using a setting similar to the one considered in Remark 1. Let $Y^*$ be a rank one matrix with $Y^*_{ij} \in [-1, 1]$ for all $i, j$; let $|\Omega| \geq m \sqrt{n \cdot \text{polylog}(n)}$, i.e., the fraction of movies rated by each user is arbitrarily small for larger $n$. For this setting, our generalization error bound is $o(1)$ for $m = \omega(n^{5/4})$. This is slightly higher than the bound by [39], where $m = \omega(n)$ is sufficient in the non-private setting to get a generalization error bound of $o(1)$. Also, the first term in the error bound pertains to differential privacy. Hence, it decreases with a larger number of users $m$, and increases with $n$ as it has to preserve privacy of a larger number of items. In contrast, the second term is the matrix completion error, which does not depend on $m$, and decreases with $n$. This is intuitive, as a larger number of movies enables more sharing of information between users, thus allowing a better estimation of preferences $Y^*$. However, just increasing $m$ may not always lead to a more accurate solution (for example, consider the case of $n = 1$).

Remark 4. The generalization guarantee in Corollary 3.1 is for uniformly random $\Omega$, but using results of [40], it is straightforward to extend our results to any general i.i.d. distribution over $\Omega$. Moreover, we can extend our results to handle strongly convex and smooth loss functions $\ell(Y_{ij}; Y^*_{ij})$ instead of the squared loss considered in this paper.

4 Experimental evaluation

We now present empirical results for our private Frank-Wolfe based algorithm (Algorithm 1) on a few benchmark datasets and compare our methods to existing methods.
**Baselines:** To the best of our knowledge, only [35] and [36] address the user-privacy preserving matrix completion problem. While we present empirical evaluation of [36], we refrain from comparing our method to [35], as the exact privacy parameters (\(\epsilon\) and \(\delta\)) for their Langevin Dynamics based algorithm are unclear. In particular, [35] uses a Markov chain based sampling method and for such a method to satisfy differential privacy, we require the sampled distribution to converge (non-asymptotically) to a differential privacy preserving distribution in \(L_1\) distance. The final privacy parameters depend on this final \(L_1\) distance, for which we are not aware of any analysis.

**Datasets:** As we want to preserve privacy of every user, and the output for each user itself is \(n\)-dimensional, we can expect the private recommendations to be accurate only when \(m \gg n\) (See Theorem 3.1). Due to this constraint, we conduct experiments on the below given datasets where the number of users is significantly higher than the number of movies:

- **Synthetic dataset:** We generate a random rank-one matrix \(Y^* = uv^T\) with unit \(\ell_\infty\)-norm, \(m = 500\text{K (users)}\) and \(n = 400\) (items), and sample 80 ratings per user.

- **Jester dataset:** This dataset contains \(n = 100\) jokes, and \(m \approx 73\text{K users}\). We rescale the ratings to be from 0 to 5.

- **MovieLens10M dataset (Top 400):** We pick the \(n = 400\) most rated movies from the original Movielens10M dataset, which has \(m \approx 70\text{K users}\) of the \(\approx 71\text{K users}\) in the complete dataset.

- **Netflix prize dataset (Top 400):** We pick the \(n = 400\) most rated movies from the original Netflix prize dataset, which has \(m \approx 474\text{K users}\) of the \(\approx 480\text{K users}\) in the complete dataset.

- **Yahoo! Music dataset (Top 400):** We pick the \(n = 400\) most rated songs from the original Yahoo! music dataset, which has \(m \approx 995\text{K users}\) of the \(\approx 1\text{M users}\) in the complete dataset. We rescale the ratings to be from 0 to 5.

For all the datasets, we randomly sample 1% of the given ratings for measuring the test error; we use root mean squared error (RMSE) to measure the performance of a method. For the experiments with privacy, we randomly select at most \(\xi = 80\) ratings per user to construct \(P_{\Omega}(Y^*)\) for all datasets except Jester. We vary the privacy parameter \(\epsilon\), but keep \(\delta = 10^{-6}\), thus ensuring that \(\delta < \frac{1}{m}\) for all the considered datasets. Moreover, we report results averaged over 10 independent runs. The requirement of Theorem 3.1 that \(\epsilon \leq 2 \log (1/\delta)\) is satisfied by all the values of \(\epsilon \in [0.1, 20]\) considered for the experiments. We note that the privacy guarantee is per-user, i.e., it corresponds to the privacy of all the ratings given by a user. This effectively translates to a per-rating privacy guarantee of \(\epsilon_{\text{rating}} = \frac{\epsilon_{\text{user}}}{\xi}\), which results in \(\epsilon_{\text{rating}} \in [0.00125, 0.25]\) as \(\epsilon_{\text{user}} \in [0.1, 20]\) in all our experiments. As we haven’t performed any sampling for the experiments with privacy on the Jester dataset, the range for \(\epsilon_{\text{rating}}\) is even smaller.

To get the non-private baseline, we normalize the training data by removing the per-user and per-movie averages (as in [24]), and run our algorithm for at least 400 iterations on the training data. For all the experiments with private Frank-Wolfe, we normalize the training data as \(\hat{r}_{i,j} = r_{i,j} - u_i\) for all \(i \in [m], j \in [n]\), where \(r_{i,j}\) is user \(i\)’s rating for item \(j\), whereas \(u_i\) is the average rating of user \(i\). Note that each user can safely perform such a normalization at her end without incurring any privacy cost. For faster training, we calibrate the standard deviation of the noise in every iteration according to the number of iterations that the algorithm has completed, while still ensuring the overall differential privacy guarantee. For each dataset, we cross-validate over the nuclear norm bound \(k\), and the total number of iterations \(T\). For \(k\), we set it to the actual nuclear norm for
Figure 2: Root mean squared error (RMSE) versus $\epsilon$, on (a) synthetic, (b) Jester, (c) MovieLens10M, (d) Netflix, and (e) Yahoo! Music datasets, for $\delta = 10^{-6}$. A legend for all the plots is provided in (f).

For a private baseline, we implement the ‘SVD after cleansing method’ of [36], setting $\delta = 10^{-6}$, and selecting $\epsilon$ appropriately to ensure a fair comparison with our setting. For this algorithm, we normalize the data by removing the private versions of the global average rating and the per-movie averages, as suggested in [36]. Moreover, we set the values of all the parameters as per [36]. We choose this algorithm as our private baseline because other methods such as the private SGLD method [45], and the private SGD technique [2] suffer from a $O\left(\frac{1}{\alpha^2}\right)$ convergence rate, where $\alpha$ is the empirical risk, whereas private Frank-Wolfe’s rate of convergence in this setting is $O\left(\frac{1}{\alpha}\right)$ steps. This may imply addition of a much larger amount of noise for such SGD methods, thus overall poorer performance.

In Figure 2 we show the results of our experiments on the synthetic dataset in plot (a), Jester dataset in plot (b), MovieLens10M dataset (Top 400) in plot (c), the Netflix dataset (Top 400) in plot (d), and Yahoo! Music dataset (Top 400) in plot (e). In all the plots, we see that the test RMSE for private Frank-Wolfe is close to the non-private baseline even for very small values of $\epsilon$, and it goes down significantly as we increase $\epsilon$. Moreover, our Frank-Wolfe based method incurs a significantly lower error than the method of [36].
5 Future directions

For future work, it would be interesting to understand the optimal dependence of the generalization error for private matrix completion w.r.t. the number of users and the number of items. Moreover, extending our techniques to other popular matrix completion methods, like alternating minimization, projected gradient descent, etc., could be another promising direction.

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A Frank-Wolfe algorithm

We use the classic Frank-Wolfe algorithm [17] as one of the optimization building blocks for our differentially private algorithms. In Algorithm 2 we state the Frank-Wolfe method to solve the following convex optimization problem:

$$\hat{Y} = \arg \min_{\|Y\|_{\text{nuc}} \leq k} 1 \over 2|\Omega| \|P_{\Omega} (Y - Y^\ast)\|_F^2. \tag{5}$$

In this paper, we use the approximate version of the algorithm from [23]. The only difference is that, instead of using an exact minimizer to the linear optimization problem, Line 4 of Algorithm 2 uses an oracle that minimizes the problem up to a slack of $\gamma$. In the following, we provide the convergence guarantee for Algorithm 2.

Note: Observe that the algorithm converges at the rate of $O(1/T)$ even with an error slack of $\gamma$. While such a convergence rate is sufficient for us to prove our utility guarantees, we observe that this rate is rather slow in practice.

**Algorithm 2: Approximate Frank-Wolfe algorithm**

**Input:** Set of revealed entries: $\Omega$, operator: $P_{\Omega}$, matrix: $P_{\Omega}(Y^\ast) \in \mathbb{R}^{m \times n}$, nuclear norm constraint: $k$, time bound: $T$, slack parameter: $\gamma$

1. $Y^{(0)} \leftarrow \{0\}^{m \times n}$
2. for $t \in [T]$ do
3.  $W^{(t-1)} \leftarrow \frac{\perp_{\Omega}}{\|W^{(t-1)}\|_{\text{nuc}} \leq k}$ s.t. $\langle W^{(t-1)}, Z^{(t-1)} \rangle - \min_{\|\Theta\|_{\text{nuc}} \leq k} \langle W^{(t-1)}, \Theta \rangle \leq \gamma$
4.  Obtain $Z^{(t-1)}$ with $\|Z^{(t-1)}\|_{\text{nuc}} \leq k$ s.t. $\langle W^{(t-1)}, Z^{(t-1)} \rangle - \min_{\|\Theta\|_{\text{nuc}} \leq k} \langle W^{(t-1)}, \Theta \rangle \leq \gamma$
5.  $Y^{(t)} \leftarrow (1 - \frac{1}{T}) Y^{(t-1)} + Z^{(t-1)}$
6. end
7. Return $Y^{(T)}$

**Theorem A.1 (Utility guarantee).** Let $\gamma$ be the slack in the linear optimization oracle in Line 4 of Algorithm 2. Then, following is true for $Y^{(T)}$:

$$\frac{1}{2|\Omega|} \|P_{\Omega} (Y^{(T)} - Y^\ast)\|_F^2 - \min_{\|Y\|_{\text{nuc}} \leq k} \frac{1}{2|\Omega|} \|P_{\Omega} (Y - Y^\ast)\|_F^2 \leq \frac{k^2}{2|\Omega|T} + \gamma.$$
Proof (Adapted from [23]). Let \( D \in \mathbb{R}^{m \times n} \) some fixed domain. We will define the curvature parameter \( C_f \) of any differentiable function \( f : D \to \mathbb{R} \) to be the following:

\[
C_f = \max_{x, s \in D, \mu \in [0, 1]} \frac{2}{\mu^2} \left( f(y) - f(x) - \langle y - x, \nabla f(x) \rangle \right).
\]

In the optimization problem in (5), let \( f(Y) = \frac{1}{2|\Omega|} \| P_\Omega (Y - Y^*) \|^2_F \), and \( G(t-1) = \arg \min_{\| \Theta \|_\infty \leq k} \langle W(t-1), \Theta \rangle \), where \( W(t-1) \) is as defined in Line 3 of Algorithm 2. We now have the following due to smoothness:

\[
f(Y(t)) = f(Y(t-1) + \frac{1}{T} (Z(t-1) - Y(t-1))) \\
\leq f(Y(t-1)) + \frac{1}{2T} C_f \left( \frac{1}{T} \langle Z(t-1) - Y(t-1), \nabla f(Y(t-1)) \rangle \right). \tag{6}
\]

Now, by the \( \gamma \)-approximation property in Line 4 of Algorithm 2, we have:

\[
\langle Z(t-1) - Y(t-1), \nabla f(Y(t-1)) \rangle \leq \langle G(t-1) - Y(t-1), \nabla f(Y(t-1)) \rangle + \gamma.
\]

Therefore, we have the following from (6):

\[
f(Y(t)) \leq f(Y(t-1)) + \frac{C_f}{2T^2} \left( 1 + \frac{2T\gamma}{C_f} \right) + \frac{1}{T} \langle G(t-1) - Y(t-1), \nabla f(Y(t-1)) \rangle. \tag{7}
\]

Recall the definition of \( \hat{Y} \) from [5], and let \( h(\Theta) = f(\Theta) - f(\hat{Y}) \). By convexity, we have the following (also called the duality gap):

\[
\langle Y(t) - G(t), \nabla f(Y(t)) \rangle \geq h(Y(t)). \tag{8}
\]

Therefore, from (7) and (8), we have the following:

\[
h(Y(T)) \leq h(Y(T-1)) - \frac{h(Y(T-1))}{T} + \frac{C_f}{2T^2} \left( 1 + \frac{2T\gamma}{C_f} \right) \\
\leq \left( 1 - \frac{1}{T} \right) h(Y(T-1)) + \frac{C_f}{2T^2} \left( 1 + \frac{2T\gamma}{C_f} \right) \\
\leq \frac{C_f}{2T^2} \left( 1 + \frac{2T\gamma}{C_f} \right) \cdot \left( 1 + \left( 1 - \frac{1}{T} \right) + \left( 1 - \frac{1}{T} \right)^2 + \cdots \right) \\
\leq \frac{C_f}{2T} \left( 1 + \frac{2T\gamma}{C_f} \right) = \frac{C_f}{2T} + \gamma
\]

\[
\iff f(Y(T)) - f(\hat{Y}) \leq \frac{C_f}{2T} + \gamma. \tag{9}
\]

With the above equation in hand, we bound the term \( C_f \) for the stated \( f(\Theta) \) to complete the proof. Notice that \( \frac{2k^2}{|\Omega|} \) is an upper bound on the curvature constant \( C_f \) (See Lemma 1 from [39], or Section 2 of [10], for a proof). Therefore, from (9), we get:

\[
f(Y(T)) - f(\hat{Y}) \leq \frac{k^2}{2|\Omega|T} + \gamma,
\]

which completes the proof. \( \square \)
B Private matrix completion via singular value decomposition (SVD)

In this section, we study a simple SVD-based algorithm for differentially private matrix completion. Our SVD-based algorithm for matrix completion just computes a low-rank approximation of $P_\Omega(Y^*)$, but still provides reasonable error guarantees [31]. Moreover, the algorithm forms a foundation for more sophisticated algorithms like alternating minimization [21], singular value projection [25] and singular value thresholding [6]. Thus, similar ideas may be used to extend our approach.

Algorithmic idea: At a high level, given rank $r$, Algorithm 3 first computes a differentially private version of the top-$r$ right singular subspace of $P_\Omega(Y^*)$, denoted by $V_r$. Each user projects her data record onto $V_r$ (after appropriate scaling) to complete her row of the matrix. Since each user’s completed row depends on the other users via the global computation which is performed under differential privacy, the overall algorithm satisfies joint differential privacy. In principle, this is the same as in Section 3, except now it is a direct rank-$r$ decomposition instead of an iterative rank-1 decomposition. Also, our overall approach is similar to that of [36], except that each user in [36] uses a nearest neighbor algorithm in the local computation phase (see Algorithm 3). Additionally, in contrast to [36], we provide a formal generalization guarantee.

Algorithm 3: Private Matrix Completion via SVD

Input: Privacy parameters: $(\epsilon, \delta)$, matrix dimensions: $(m, n)$, uniform $\ell_2$-bound on the rows of $P_\Omega(Y^*)$: $L$, and rank bound: $r$

1 Global computation: Compute the top-$r$ subspace $V_r$ for the matrix $W \leftarrow \sum_{i=1}^{m} W_i + N$,

where $W_i = \Pi_L (P_\Omega(Y_i^*))^\top \Pi_L (P_\Omega(Y_i^*))$, $\Pi_L$ is the projection onto the $\ell_2$-ball of radius $L$, $N \in \mathbb{R}^{n \times n}$ corresponds to a matrix with i.i.d. entries from $\mathcal{N}(0, \sigma^2)$, and $\sigma \leftarrow L^2 \sqrt{64 \log(1/\delta)/\epsilon}$

2 Local computation: Each user $i$ computes the $i$-th row of the private approximation $\hat{Y}$:

$\hat{Y}_i \leftarrow \frac{mn}{|\Omega|} P_\Omega(Y_i^*) V_r V_r^\top$

B.1 Privacy and utility analysis

We now present the privacy and generalization guarantees for the above algorithm.

Theorem B.1. Algorithm 3 satisfies $(\epsilon, \delta)$-joint differential privacy.

The proof of privacy for Algorithm 3 follows immediately from the proof of Theorem 3.1 as the key step of computing the top singular vectors of the $W$ matrix remains the same.

For the generalization error bound for Algorithm 3, we use the standard low-rank matrix completion setting, i.e., entries are sampled i.i.d., and the underlying matrix $Y^*$ is incoherent [B.2]. Intuitively, incoherence ensures that the left and right singular subspaces of a matrix have a low correlation with the standard basis. The scale of $\mu$ is $[0, \max\{m, n\}]$. Since, we are assuming $m \geq n$ throughout the paper, $\mu \in [0, m]$. 


Definition B.2 \((\mu\text{-incoherence})\). Let \(Y \in \mathbb{R}^{m \times n}\) be a matrix of rank at most \(r\), and let \(U \in \mathbb{R}^{m \times r}\) and \(V \in \mathbb{R}^{r \times n}\) be the left and right singular subspaces of \(Y\). Then, the incoherence \(\mu\) is the following:

\[
\mu = \max \left\{ \frac{m}{r} \max_{1 \leq i \leq m} \|UU^Te_i\|_2, \frac{n}{r} \max_{1 \leq i \leq n} \|VV^Tf_i\|_2 \right\}.
\]

Here, \(e_i \in \mathbb{R}^m\) and \(f_i \in \mathbb{R}^n\) are the \(i\)-th standard basis vectors in \(m\) and \(n\) dimensions, respectively.

Under the above set of assumptions, we get:

**Theorem B.3.** Let \(Y^* \in \mathbb{R}^{m \times n}\) be a rank-\(r\), \(\mu\)-incoherent matrix with condition number \(\kappa = \|Y^*\|_2 / \lambda_r(Y^*)\), where \(\lambda_r(\cdot)\) corresponds to the \(r\)-th largest singular value. Also, let the set of known entries \(\Omega\) be sampled uniformly at random s.t. \(|\Omega| \geq c_0 \kappa^2 \mu m n \log m\) for a large constant \(c_0 > 0\). Let \(\|P_\Omega(Y^*)_i\|_2 \leq L\) for every row \(i\) of \(Y^*\). Then, with probability at least \(2/3\) over the outcomes of the algorithm, the following holds for \(\hat{Y}\) estimated by Algorithm 3:

\[
F(\hat{Y}) = O\left( \frac{L^4 \kappa^4 m^3 n^4 \cdot r \cdot \Delta_{\epsilon, \delta}^2}{|\Omega|^4 \|Y^*\|_2^2} + \frac{\mu \|Y^*\|_2^2 \cdot r^2 \log m}{n \cdot |\Omega|} \right),
\]

where \(F(Y) = \frac{1}{|\Omega|} \|P_\Omega(Y - Y^*)\|_F^2\), and the privacy parameter is \(\Delta_{\epsilon, \delta} = \sqrt{64 \log(1/\delta) / \epsilon}\).

Using \(L \leq \|Y^*\|_2\), we get:

\[
F(\hat{Y}) = O\left( \frac{\min\left( L^2, \frac{n^2 \mu}{m} \right) \kappa^4 m^3 n^4 \cdot r \cdot \Delta_{\epsilon, \delta}^2}{|\Omega|^4} + \frac{\mu \|Y^*\|_2^2 \cdot r^2 \log m}{n \cdot |\Omega|} \right).
\]

The \(O(\cdot)\) hides only universal constants.

For a proof of this theorem, see Section B.1.1.

**Remark 5.** Let \(Y^*\) be a rank one incoherent matrix with \(Y^*_i = \Theta(1), |\Omega| = m \sqrt{n}, L = O(n^{1/4})\), and \(\mu = O(1)\). Notice that the spectral norm \(\|Y^*\|_2 \approx \sqrt{mn}\). Hence, the first term in the bound reduces to \(O\left( \frac{n^2}{m}\right)\) and the second error term is \(O\left( \frac{1}{\sqrt{n}} \right)\), whereas a trivial solution of \(Y = 0\) leads to \(O(1)\) error. Similar to the behavior in Remark 3, the first term above increases with \(n\), and decreases with increasing \(m\) due to the noise added, while the second term decreases with increasing \(n\) due to more sharing between users.

**Remark 6.** Under the assumptions of Theorem B.3, the second term can be arbitrarily small for other standard matrix completion methods like the FW-based method (Algorithm 1) studied in Section 3 above. However, the first error term for such methods can be significantly larger. For example, the error of Algorithm 1 in the setting of Remark 5 is \(O\left( \frac{n^{13/24}}{m^{7/12}} \right)\) as the second term in Corollary B.1 vanishes in this setting; in contrast, the error of the SVD-based method (Algorithm 3) is \(O\left( \frac{n^2}{mn} + \frac{1}{\sqrt{n}} \right)\). On the other hand, if the data does not satisfy the assumptions of Theorem B.3, then the error incurred by Algorithm 3 can be significantly larger (or even trivial) when compared to that of Algorithm 1.
B.1.1 Proof of Theorem \[B.3\]

Proof. Let \( B = \frac{1}{p} P_{\Omega}(Y^*) \) where \( p = |\Omega|/m \cdot n \) and let \( V_r \) be the top-\( r \) right singular subspace of \( B \). Suppose \( \Pi_r = V_r V_r^T \) be the projector onto that subspace. Recall that \( \hat{V}_r \) is the right singular subspace defined in Algorithm 3 and let \( \hat{\Pi}_r = \hat{V}_r \hat{V}_r^T \) be the corresponding projection matrix.

Then, using the triangular inequality, we have:

\[
\|B \hat{\Pi}_r - Y^*\|_2 \leq \|B \hat{\Pi}_r - B \Pi_r\|_2 + \|B \Pi_r - Y^*\|_2 \leq \|B \hat{\Pi}_r - B \Pi_r\|_2 + \epsilon \|Y^*\|_2 \sqrt{\frac{\mu m r \log m}{|\Omega|}},
\]

where the second inequality follows from the following standard result (Lemma \[B.4\]) from the matrix completion literature, and holds w.p. \( \geq 1 - 1/m^{10} \).

Lemma \[B.4\] (Follows from Lemma A.3 in [27]). Let \( M \) be an \( m \times n \) matrix with \( m \geq n \), rank \( r \), and incoherence \( \mu \), and \( \Omega \) be a subset of i.i.d. samples from \( M \). There exists universal constants \( c_1 \) and \( c_0 \) such that if \( |\Omega| \geq c_0 \mu m r \log m \), then with probability at least \( 1 - 1/m^{10} \), we have:

\[
\left\| M - \frac{mn}{|\Omega|} P_{\Omega}(M) \right\|_2 \leq c_1 \|M\|_2 \sqrt{\frac{\mu m r \log m}{|\Omega|}}.
\]

Using Theorem 6 of [15], the following holds with probability at least 2/3,

\[
\left\| \hat{\Pi}_r - \Pi_r \right\|_2 = O \left( \frac{L^2 \sqrt{n} \Delta_{r, \delta}}{\alpha_r^2 \alpha_{r+1}^2} \right),
\]

where \( \alpha_i \) is the \( i \)-th singular value of \( P_{\Omega}(Y^*) = p \cdot B \).

Recall that \( \kappa = \|Y^*\|_2 / \lambda_r(Y^*) \), where \( \lambda_r \) is the \( r \)-th singular value of \( Y^* \). Let \( |\Omega| \geq c_0 \kappa^2 \mu m r \log m \) with a large constant \( c_0 > 0 \). Then, using Lemma \[B.4\] and Weyl’s inequality, we have (w.p. \( \geq 1 - 1/m^{10} \)):

\[
\alpha_r \geq 0.9 \cdot p \cdot \frac{1}{\kappa} \|Y^*\|_2, \quad \alpha_{r+1} \leq c_1 p \cdot \|Y^*\|_2 \sqrt{\frac{\mu m r \log m}{|\Omega|}} \leq 0.1 \cdot \alpha_r.
\]

Similarly,

\[
\|B\|_2 \leq 2 \|Y^*\|_2, \text{w.p.} \geq 1 - 1/m^{10}.
\]

Using \( (10), (11), (12) \), and \( (13) \), we have w.p. \( \geq 2/3 - 5/m^{10} \):

\[
\|B \hat{\Pi}_r - Y^*\|_2 \leq 8 \|Y^*\|_2 \cdot \frac{L^2 \kappa^2 \sqrt{n} \Delta_{r, \delta}}{p^2 \|Y^*\|_2^2} + c_1 \|Y^*\|_2 \sqrt{\frac{\mu m r \log m}{|\Omega|}}.
\]

Recall that \( \hat{Y} = \frac{1}{p} P_{\Omega}(Y^*) \hat{\Pi}_r = B \hat{\Pi}_r \). Hence:

\[
\frac{\|B \hat{\Pi}_r - Y^*\|_2^2}{mn} \leq O \left( \frac{L^4 \kappa^4 n \Delta_{r, \delta}^2}{mn \cdot p^4 \|Y^*\|_2^2} \right) + c_1 \|Y^*\|_2^2 \frac{\mu m r \log m}{mn \cdot |\Omega|}.
\]

The theorem now follows by using \( \|A\|_F^2 \leq r \|A\|_2^2 \), where \( r \) is the rank of \( A \). □
C Omitted Proofs

In this section, we provide detailed proofs that have been omitted from the main body of the paper.

C.1 Proofs of privacy and utility for Private Frank-Wolfe (Algorithm 1)

C.1.1 Proof of privacy

Proof of Theorem 3.1. Consider the sequence of matrices $W^{(1)}, \ldots, W^{(T)}$ produced by function $A_{\text{global}}$. Notice that, if every user $i \in [m]$ knows this sequence, then she can construct her updates $Y_i^{(1)}, \ldots, Y_i^{(T)}$ by herself independent of any other user’s data. Therefore, by the post-processing property of differential privacy [12, 13], it follows that as long as function $A_{\text{global}}$ satisfies $(\epsilon, \delta)$-differential privacy, one can ensure $(\epsilon, \delta)$-joint differential privacy for Algorithm 1, i.e., the combined pair of functions $A_{\text{global}}$ and $A_{\text{local}}$. (Recall that the post-processing property of differential privacy states that any operation performed on the output of a differentially private algorithm, without accessing the raw data, remains differentially private with the same level of privacy.) Hence, Lemma C.1 completes the proof of privacy.

Lemma C.1. Let $W^{(t)}$ be the output in every iteration $t \in [T]$ of function $A_{\text{global}}$ in Algorithm 1. Then, $A_{\text{global}}$ is $(\epsilon, \delta)$-differentially private.

Proof. We are interested in the function $\text{Cov}(A^{(t)}) = A^{(t)\top}A^{(t)}$, where $A^{(t)} = P_{\Omega}(Y^{(t)} - Y^*)$. Since $\|P_{\Omega}(Y^{(t)})\|_2 \leq L$ and $\|P_{\Omega}(Y^*)\|_2 \leq L$ for all rows $i \in [m]$, we have that the $\ell_2$-sensitivity of $\text{Cov}(A^{(t)})$ is $4L^2$. Recall that the $\ell_2$-sensitivity of $\text{Cov}$ corresponds to the maximum value of $\|\text{Cov}(A) - \text{Cov}(A')\|_F$ for any two matrices $A, A'$ in the domain, and differing in exactly one row. Using the Gaussian mechanism from [4], and using Propositions 3, 6 and Lemma 7 (composition property) from [4], it follows that adding Gaussian noise with standard deviation $\sigma = \frac{L^2\sqrt{64T \log(1/\delta)}}{\epsilon}$ in each iteration of the global component of private Frank-Wolfe (function $A_{\text{global}}$) ensures $(\epsilon, \delta)$-differential privacy for $\epsilon \leq 2 \log (1/\delta)$.

C.2 Proof of utility

Proof of Theorem 3.2. Recall that in function $A_{\text{global}}$ of Algorithm 1, the matrix $W^{(t)}$ captures the total error covariance corresponding to all the users at a given step $t$, i.e., $A^{(t)\top}A^{(t)} = \sum_{i \in [m]} A_i^{(t)\top}A_i^{(t)}$. Spherical Gaussian noise of appropriate scale is added to ensure that $W^{(t)}$ is computed under the constraint of differential privacy. Let $\hat{\nu}$ be the top right singular vector of $W^{(t)}$, and let $\hat{\lambda}$ be the corresponding singular value. In Lemma C.2, we first show that $\hat{\lambda}$ is a reasonable approximation to the energy of $A^{(t)}$ captured by $\hat{\nu}$, i.e., $\|A^{(t)}\hat{\nu}\|_2^2$. Furthermore, in Lemma C.3 we show that $\hat{\nu}$ captures sufficient energy of the matrix $A^{(t)}$. Hence, we can conclude that one can use $\hat{\nu}$ as a proxy for the top right singular vector of $A^{(t)}$.

Lemma C.2. With probability at least $1 - \beta$, the following is true:

$$\|A^{(t)}\hat{\nu}\|_2^2 \leq \hat{\lambda} + O\left(\sqrt{\sigma \log(n/\beta)} \sqrt{n}\right).$$
Proof. Let $E = W^{(t)} - A^{(t)\top} A^{(t)}$, where the matrix $W^{(t)}$ is the output of function $\mathcal{A}_{\text{global}}$ for iteration $t$. We have,

$$
\left\| A^{(t)} \hat{v} \right\|_2^2 = \hat{v}^\top A^{(t)\top} A^{(t)} \hat{v} = \hat{v}^\top \left( A^{(t)\top} A^{(t)} + E \right) \hat{v} - \hat{v}^\top E \hat{v} \leq \hat{\lambda}^2 + \| E \|_2
$$

\[ \leq \hat{\lambda}^2 + O \left( \sigma \log(n/\beta) \sqrt{n} \right) \text{ w.p. } \geq 1 - \beta. \]  
\[ \Rightarrow \left\| A^{(t)} \hat{v} \right\|_2 \leq \hat{\lambda} + O \left( \sqrt{\sigma \log(n/\beta) \sqrt{n}} \right) \text{ w.p. } \geq 1 - \beta. \]  

(14)

(15)

The inequality in (14) follows from the spectral norm bound on the Gaussian matrix $E$ drawn i.i.d. from $\mathcal{N}(0, \sigma^2)$. (See Corollary 2.3.5 in [43] for a proof). Inequality (15) completes the proof. \qed

Lemma C.3 (Follows from Theorem 3 of [15]). Let $A \in \mathbb{R}^{m \times p}$ be a matrix and let $W = A^\top A + E$, where $E \sim \mathcal{N}(0, I_{p \times p} \sigma^2)$. Let $\mathbf{v}$ be the top right singular vector of $A$, and let $\hat{\mathbf{v}}$ be the top right singular vector of $W$. The following is true with probability at least $1 - \beta$:

$$
\| A \hat{\mathbf{v}} \|_2^2 \geq \| A \mathbf{v} \|_2^2 - O \left( \sigma \log(n/\beta) \sqrt{n} \right).
$$

Now, one can compactly write the update equation of $Y^{(t)}$ in function $\mathcal{A}_{\text{local}}$ of Algorithm 1 for all the users as:

$$
Y^{(t)} \leftarrow \Pi_{L, \Omega} \left( \left( 1 - \frac{1}{T} \right) Y^{(t-1)} - \frac{k}{T} \hat{\mathbf{u}} \hat{\mathbf{v}}^\top \right),
$$

(16)

where $\hat{\mathbf{u}}$ corresponds to the set of entries $\hat{u}_i$ in function $\mathcal{A}_{\text{local}}$ represented as a vector. Also, by Lemma C.2, we can conclude that $\| \hat{\mathbf{u}} \|_2 \leq 1$. Hence, $Y^{(t)}$ is in the set $\{ Y : \| Y \|_{\text{nuc}} \leq k \}$ for all $t \in [T]$.

In the following, we incorporate the noisy estimation in the analysis of original Frank-Wolfe (stated in Section A). In order to do so, we need to ensure a couple of properties: i) We need to obtain an appropriate bound on the slack parameter $\gamma$ in Algorithm 2 and ii) we need to ensure that the projection operator $\Pi_{L, \Omega}$ in function $\mathcal{A}_{\text{local}}$ does not introduce additional error. We do this via Lemma C.4 and C.5 respectively.

Lemma C.4. For the noise variance $\sigma$ used in function $\mathcal{A}_{\text{global}}$ of Algorithm 1, w.p. at least $1 - \beta$, the slack parameter $\gamma$ in the linear optimization step of Frank-Wolfe algorithm is at most $O \left( \frac{k}{\| \hat{\mathbf{u}} \|_{\text{nuc}} \sqrt{\sigma \log(n/\beta) \sqrt{n}}} \right)$.

Proof. Recall that $\hat{\lambda}^2$ corresponds to the maximum eigenvalue of $W^{(t)}$, and notice that $A^{(t)}$ is the scaled gradient of the loss function $\frac{1}{\| \hat{\mathbf{u}} \|_{\text{nuc}}} \| P_{\Omega} (\Theta - Y^*) \|_F^2$ at $\Theta = \Pi_{L, \Omega} (Y^{(t)})$. Essentially, we need to compute the difference between $\left( \frac{1}{\| \hat{\mathbf{u}} \|_{\text{nuc}}} A^{(t)}, k \hat{\mathbf{u}} \mathbf{v}^\top \right)$ and $\left( \frac{1}{\| \hat{\mathbf{u}} \|_{\text{nuc}}} A^{(t)}, k \hat{\mathbf{u}} \hat{\mathbf{v}}^\top \right)$. Let $\alpha = \left( \frac{1}{\| \hat{\mathbf{u}} \|_{\text{nuc}}} A^{(t)}, k \hat{\mathbf{u}} \mathbf{v}^\top \right)$,
Lemma C.5. \( (18) \). Since \( \hat{\alpha} \) can conclude from (19) that 
\[
\hat{\alpha} = \frac{k \hat{\nu}^T A(t)^T \hat{\mathbf{u}}}{|\Omega|} \lambda \geq \frac{k \hat{\nu}^T A(t)^T A(t) \hat{\nu}}{|\Omega| \left( \hat{\lambda} + \Theta \left( \sqrt{\sigma \log(n/\beta) \sqrt{n}} \right) \right)} = \frac{k \| A(t) \hat{\nu} \|_2^2}{|\Omega| \left( \hat{\lambda} + \Theta \left( \sqrt{\sigma \log(n/\beta) \sqrt{n}} \right) \right)}
\]
where \( \lambda^2 \) is the maximum eigenvalue of \( A(t)^T A(t) \), the second equality follows from the definition of \( \hat{\mathbf{u}} \), and the inequality follows from Lemma C.3. One can rewrite (17) as:
\[
\alpha - \hat{\alpha} \leq \left( 1 - \frac{\lambda}{\left( \hat{\lambda} + \Theta \left( \sqrt{\sigma \log(n/\beta) \sqrt{n}} \right) \right)} \right) \alpha + O \left( \frac{k \sigma (n/\beta) \sqrt{n}}{|\Omega| \left( \hat{\lambda} + \Theta \left( \sqrt{\sigma \log(n/\beta) \sqrt{n}} \right) \right)} \right). \tag{18}
\]
We will analyze \( E_1 \) and \( E_2 \) in (18) separately. One can write \( E_1 \) in (18) as follows:
\[
E_1 = \left( \frac{\hat{\lambda} + O \left( \sqrt{\sigma \log(n/\beta) \sqrt{n}} \right)}{\left( \hat{\lambda} + \Theta \left( \sqrt{\sigma \log(n/\beta) \sqrt{n}} \right) \right)} - \lambda \right) = k \left( \frac{\hat{\lambda} + O \left( \sqrt{\sigma \log(n/\beta) \sqrt{n}} \right)}{\left( \hat{\lambda} + \Theta \left( \sqrt{\sigma \log(n/\beta) \sqrt{n}} \right) \right)} - \lambda \right) \lambda. \tag{19}
\]
By Weyl’s inequality for eigenvalues, and the fact that w.p. at least \( 1 - \beta \), we have 
\[
\| W(t) - A(t)^T A(t) \|_2 = O \left( \sigma \log(n/\beta) \sqrt{n} \right)
\]
because of spectral properties of random Gaussian matrices (Corollary 2.3.5 in [43]), it follows that \( \hat{\lambda} - \lambda = O \left( \sqrt{\sigma \log(n/\beta) \sqrt{n}} \right) \). Therefore, one can conclude from (19) that \( E_1 = O \left( k \frac{\sigma}{n} \sqrt{\sigma \log(n/\beta) \sqrt{n}} \right) \). Now, we will bound the term \( E_2 \) in (18). Since \( \hat{\lambda} \geq 0 \), it follows that \( E_2 = O \left( k \frac{\sigma}{n} \sqrt{\sigma \log(n/\beta) \sqrt{n}} \right) \). Therefore, the slack parameter \( \alpha - \hat{\alpha} = E_1 + E_2 = O \left( k \frac{\sigma}{n} \sqrt{\sigma \log(n/\beta) \sqrt{n}} \right) \).

Lemma C.5. Define the operators \( \mathcal{P}_\Omega \) and \( \Pi_{L, \Omega} \) as described in function \( \mathcal{A}_{\text{local}} \) in Section 3. Let \( f(Y) = \frac{1}{2 \| \mathcal{P}_\Omega (Y - Y^*) \|_F^2} \) for any matrix \( Y \in \mathbb{R}^{m \times n} \). The following is true for all \( Y \in \mathbb{R}^{m \times n} \): 
\[
f (\Pi_{L, \Omega} (Y)) \leq f (\mathcal{P}_\Omega (Y)).
\]

Proof. First, notice that for any matrix \( M = [m_1^T, \cdots, m_m^T] \) (where \( m_i^T \) corresponds to the \( i \)-th row of \( M \)), \( \| M \|_F^2 = \sum_i \| m_i \|_2^2 \). Let \( \Pi_L \) be the \( \ell_2 \) projector onto a ball of radius \( L \), and \( \mathbb{B}_L^n \) be a ball of radius \( L \) in \( n \)-dimensions, centered at the origin. Then, for any pair of vectors, \( v_1 \in \mathbb{R}^n \) and \( v_2 \in \mathbb{B}_L^n \), \( \| \Pi_L (v_1) - v_2 \|_2 \leq \| v_1 - v_2 \|_2 \). This follows from the contraction property of \( \ell_2 \)-projection. Hence, by the above two properties, and the fact that each row of the matrix \( \mathcal{P}_\Omega (Y^*) \in \mathbb{B}_L^n \), we can conclude \( f (\Pi_L (\mathcal{P}_\Omega (Y))) \leq f (\mathcal{P}_\Omega (Y)) \) for any \( Y \in \mathbb{R}^{m \times n} \). This concludes the proof. \( \square \)
This means we can still use Theorem A.1. Hence, we can conclude that:

\[
\frac{1}{2|\Omega|} \left\| P_{\Omega} \left( Y^{(T)} - Y^* \right) \right\|_F^2 = O \left( \frac{k^2}{|\Omega|T} + \frac{k}{|\Omega|} \sqrt{\sigma \log(n/\beta)\sqrt{n}} \right) \text{ w.p. } \geq 1 - \beta.
\]

(Here we used the fact that the curvature parameter \(C_f\) from Theorem A.1 is at most \(k^2/|\Omega|\). See [24] for a proof.) Setting \(\beta = 1/3\) completes the proof. \(\square\)