ONE-DIMENSIONAL SCHRÖDINGER OPERATORS WITH
SINGULAR POTENTIALS: A SCHWARTZ
DISTRIBUTIONAL FORMULATION

NUNO COSTA DIAS, CRISTINA JORGE, AND JOÃO NUNO PRATA

Abstract. Using an extension of the Hörmander product of distributions, we obtain an intrinsic formulation of one-dimensional Schrödinger operators with singular potentials. This formulation is entirely defined in terms of standard Schwartz distributions and does not require (as some previous approaches) the use of more general distributions or generalized functions. We determine, in the new formulation, the action and domain of the Schrödinger operators with arbitrary singular boundary potentials. We also consider the inverse problem, and obtain a general procedure for constructing the singular (pseudo) potential that imposes a specific set of (local) boundary conditions. This procedure is used to determine the boundary operators for the complete four-parameter family of one-dimensional Schrödinger operators with a point interaction. Finally, the δ and δ' potentials are studied in detail, and the corresponding Schrödinger operators are shown to coincide with the norm resolvent limit of specific sequences of Schrödinger operators with regular potentials.

Keywords. Schrödinger operators; singular potentials; point interactions; products of distributions; quantum systems with boundaries

AMS subject classifications. 34L40, 81Q10, 81Q80, 34B09

1. Introduction.

Let \( \hat{H}_0 = -D_x^2 \) be the free Schrödinger operator with domain in the Sobolev-Hilbert space \( \mathcal{H}^2(\mathbb{R}) \subset \mathcal{L}^2(\mathbb{R}) \), and let \( \hat{S} \) be its symmetric restriction to the set \( \mathcal{D}(\mathbb{R}\setminus\{0\}) \) of smooth functions with support on a compact subset of \( \mathbb{R}\setminus\{0\} \). The self-adjoint (s.a.) extensions of \( \hat{S} \) are usually called Schrödinger operators with point interactions \([2,4,11]\). We shall denote them generically by \( \hat{L} \). In the one-dimensional case they form a four-parameter family of operators, each of which is characterized by two boundary conditions at \( x = 0 \). These objects yield exactly solvable models \([2,4,5,6,8,11,12,26,38,45]\) and have been widely used in applications in quantum mechanics (e.g. in models of low-energy scattering \([3,13,14,35]\) and quantum systems with boundaries \([22,23,24,27,32]\)), condensed matter physics \([10,17,27]\) and, more recently, on the approximation of thin quantum waveguides by quantum graphs \([1,15,16,20]\).
An interesting topic is the relation between the operators \( \hat{L} \) and the additive perturbations of the operator \( \hat{H}_0 \) by sharply localized potentials. The books [2, 4, 37] and the papers [6, 7, 11, 26, 30, 39, 42, 44, 46, 48] provide a detailed discussion and an extensive list of references on this subject. In the present work we will address the problem, that emerges naturally in this context, of constructing a boundary potential formulation of the operators \( \hat{L} \) that is entirely defined in terms of standard Schwartz distributions.

Let us consider this problem in more detail. The aim is to determine, for each \( \hat{L} \), a boundary potential operator \( \hat{B} = \hat{B} \cdot \), acting by multiplication by a Schwartz distribution \( \hat{B} \) (called the boundary potential), and such that:

\[
\hat{L} = \hat{H}_0 + \hat{B}
\]

where \( \hat{H}_0 = -D^2_x \) is defined in the generalized sense. Since \( \hat{L}\psi = \hat{S}\psi \) for all \( \psi \in \mathcal{D}(\hat{S}) = \mathcal{D}(\mathbb{R}\setminus\{0\}) \), the distribution \( B \) has to be supported at \( x = 0 \) (assuming that the product \( \cdot \) is local). Hence, for \( \psi \in \mathcal{D}(\hat{L}) \), the term \( \hat{B}\psi \) will in general stand for the product of a singular distribution by a discontinuous function. Such a product is not well defined within the standard theory of Schwartz distributions and so, unless some additional structure is introduced, the r.h.s. of (1.1) has only a formal meaning.

A common interpretation of the formal operator \( \hat{H}_0 + \hat{B} \) is that it stands for the limit (e.g. in the norm resolvent sense) of sequences of operators of the form

\[
\hat{H}_n = \hat{H}_0 + \hat{B}_n, \quad \hat{B}_n = B_n.
\]

where \( B_n \) belongs to some space of regular functions and satisfies \( B_n \rightarrow B \) in \( \mathcal{D}' \). Sequences of this kind have been used in many applications, (see, for instance, [30] and the references therein), and their convergence properties were studied for particular cases [1, 15, 16, 18, 30, 44, 47, 48]. Unfortunately, the relation between the convergence of \( B_n \) in \( \mathcal{D}' \) and the convergence of the associated operators \( \hat{H}_n \) in the norm resolvent is not straightforward [12, 28, 29, 48, 49]. One finds that different sequences of regular potentials, converging to the same distribution, may lead to sequences of operators (1.2) converging to distinct operators. The case \( B = \delta' \) was studied in detail. It was found that the formal operator \( \hat{H}_0 + \delta' \) stands (in the sense above) for, at least, a one-parameter family of distinct s.a. Schrödinger operators with a point interaction [30, 44, 49].

An interesting problem is then whether the operators \( \hat{H}_0 + \hat{B} \) admit an intrinsic formulation in terms of distributions and also, whether such formulation coincides with the norm resolvent limit of suitable sequences of Schrödinger operators with regular potentials. The first of these problems was addressed by P. Kurasov [39] (and extended to higher order linear operators by Kurasov and Boman [40]) using a theory of distributions acting on discontinuous test functions (see also [chapter 3, [3] and [41, 42]). Let us denote by \( \tilde{K} \) the set of \( C^\infty(\mathbb{R}\setminus\{0\}) \)-functions that display (together with
all their derivatives) finite lateral limits at $x = 0$, and let $\mathcal{K}$ be the set of $\bar{\mathcal{K}}$-functions with bounded support. The set of Kurasov distributions is the dual of $\mathcal{K}$, denoted by $\mathcal{K}'$. As usual, one can introduce most of the algebraic operations in $\mathcal{K}'$. One can also introduce a distributional derivative $D_K$ and, more importantly, a product $*_{\mathcal{K}}$ of an element of $\mathcal{K}'$ by an element of $\mathcal{K}$. This product is defined for all $F \in \mathcal{K}'$ and $\psi \in \bar{\mathcal{K}}$ by:

$$\langle F *_{\mathcal{K}} \psi, \xi \rangle = \langle F, \psi \xi \rangle, \quad \forall \xi \in \mathcal{K}$$

where $\langle , \rangle$ denotes the distributional bracket. Using the product $*_{\mathcal{K}}$ one may define the Kurasov boundary operators $\hat{B}_K \psi = B_K *_{\mathcal{K}} \psi$ where $B_K$ is a Kurasov distribution. It turns out that by considering a simple extension of the product $*_{\mathcal{K}}$, the complete four parameter family of Schrödinger operators $\hat{L}$ can be written in the form (1.1) with $\hat{B} = B_K *_{\mathcal{K}}$ and $B_K$ a (pseudo) potential (since in general it also involves the operator $D_K$) from $\mathcal{K}'$ [4, 39].

Besides Kurasov’s product, the Colombeau formalism [19] has also been used [9] to obtain a precise definition of the boundary potential operators $\hat{B}$. One should notice, however, that both formalisms are defined outside from the space of Schwartz distributions, which renders the formulation of the Schrödinger operators considerably more intricate than in the standard Schwartz case. For instance, the Kurasov operators $\hat{H}_K = -D_K^2 + \hat{B}_K$ are written in terms of distributions in $\mathcal{K}'$ and the new derivative operator $D_K$.

It turns out that $D_K$ is a complicated object that does not satisfy the standard properties of a derivative operator (it does not satisfy the Leibnitz rule, it does not reproduce the derivative of functions for smooth regular distributions and the derivative of a constant is not zero [39]). Moreover, Kurasov’s formulation yields operators of the form $\hat{H}_K : \mathcal{K} \to \mathcal{K}'$ and so $\hat{H}_K \psi \notin \mathcal{D}'$, not even for $\psi \in \mathcal{D}$. If one needs to produce standard distributional results (like when modelling point interactions), in the end one has to project down to $\mathcal{D}'$ the results of the formulation in $\mathcal{K}'$.

An alternative approach would be to use intrinsic products of Schwartz distributions in the definition of $\hat{B}$. Such products have been used in the past to obtain consistent formulations of ordinary and partial differential equations with singular terms [34, 43]. However, and up to our knowledge, this approach has never been considered in the context of Schrödinger operators with singular potentials.

The main goal of this paper is then to follow the latter approach and show that a rigorous formulation of the operators $\hat{H}_0 + \hat{B}$ can be obtained strictly within the framework of the Schwartz distributional theory. Our approach rests upon the definition of an intrinsic multiplicative product of Schwartz distributions that we shall denote by $\ast$. This product was originally defined and studied in [21]. It constitutes a generalization (to the case of possibly intersecting singular supports) of the Hörmander product of distributions with non-intersecting singular supports (pag.55, [33]). The product $\ast$ is associative, distributive, non-commutative and local. It satisfies the Leibnitz
rule with respect to the Schwartz distributional derivative $D_x$ and reproduces the usual product of functions for regular distributions. Contrary to what happens in the Colombeau and Kurasov cases, the product $\ast$ is defined (and is an inner operation) in a subspace of $\mathcal{D}'$. This subspace, endowed with the product $\ast$ and the Schwartz distributional derivative $D_x$, becomes an associative differential algebra that contains Kurasov’s function space $\mathcal{K}$ and, more importantly, all the distributional derivatives of the elements of $\mathcal{K}$.

Using the product $\ast$ we can define the boundary potential operators
\begin{equation}
\hat{B}\psi = \psi \ast B_1 + B_2 \ast \psi
\end{equation}
for an arbitrary pair $B_1, B_2$ of Schwartz distributions with support on a finite set. The operators $\hat{B}$ are extensions of the operators $B$ where $B = B_1 + B_2$ and $\cdot$ is the standard product of a distribution by a test function. In particular, they are well-defined on the spaces of discontinuous functions that are important for studying point interactions. Let us denote the association between the operators $\hat{B}$ (1.3) and the corresponding boundary potentials $B = B_1 + B_2$ by $B \leftrightarrow \hat{B}$. Notice that different choices of the pair $B_1, B_2$ (satisfying $B_1 + B_2 = B$) yield different operators $\hat{B}$, and so the association $B \leftrightarrow \hat{B}$ is not one-to-one.

More generally, we can also define the boundary pseudo potential operators
\begin{equation}
\hat{B}\psi = \sum_{i,j=0}^{1} D^i_x \hat{B}_{ij} D^j_x \psi
\end{equation}
where $\hat{B}_{ij}$ are boundary potential operators of the form (1.3). The operators (1.4) will be used to construct a boundary pseudo potential formulation of all S.A. Schrödinger operators $\hat{L}$.

The operators of the form (1.3) and (1.4) are particular examples of boundary operators. We use this terminology to denote an arbitrary linear operator $\hat{B}$ for which exists a finite set $I \subset \mathbb{R}$, such that
\begin{equation}
\text{supp } \hat{B}\psi \subseteq I, \quad \forall \psi \in \mathcal{D}(\hat{B}) \subset \mathcal{D}'.
\end{equation}

In this paper we study several properties of the operators $\hat{H}_0 + \hat{B}$, when $\hat{B}$ is an operator of the form (1.3), (1.4) or (1.5). In the next section we review, for the convenience of the reader, the definition and the main properties of the distributional product proposed in [21]. In section 3, we prove that the operators $\hat{B}$, given by (1.3), can be equivalently defined as weak operator limits of particular sequences of operators of multiplication by smooth functions; these sequences converge in $\mathcal{D}'$ to the associated boundary potential $B$. In section 4, we prove some general results about the operators of the form $\hat{H}_0 + \hat{B}$, when $\hat{B}$ is an arbitrary boundary operator (1.5). In particular, we determine their action and domain, explicitly. In section 5, we develop a general method for finding a boundary pseudo potential operator (1.4) that imposes a specific given set of boundary conditions. This
method is then used to obtain a boundary pseudo potential formulation of all one-dimensional Schrödinger operators $\hat{L}$. We also show that such a representation is not possible (for all $\hat{L}$) if only boundary potential operators of the form (1.3) are used. Finally, in section 6, we study the operators (1.3) in detail. For an arbitrary boundary potential $B = a\delta(x) + b\delta'(x)$, we determine which operators $\hat{H}_0 + \hat{B}$, where $\hat{B} \leftrightarrow B$, are s.a. and conversely, which s.a. operators $\hat{L}$ admit a boundary potential representation. The particular cases $B = a\delta(x)$ and $B = b\delta'(x)$, $a, b \in \mathbb{R}$, are then studied in detail. We show that they yield families of Schrödinger operators that coincide (exactly in the first case, and to a large extend in the second case) with the families of norm resolvent limit operators that were obtained in [29, 30].

Notation. Operators are denoted by letters with a hat and distributions by capital roman letters ($F, G$...). The exception is the Dirac measure $\delta$. Generic functions are denoted by lower case Greek letters ($\psi, \phi, \xi$...). Lower case roman letters from the middle of the alphabet ($f, g, h$...) are reserved for continuous functions and those from the end of the alphabet ($s, t, u$...) for smooth functions of compact support. Capital Greek letters ($\Omega, \Xi$...) denote open subsets of $\mathbb{R}$. The functional spaces are denoted by calligraphic capital letters ($L^2(\Omega), H^2(\Omega), \mathcal{D}(\Omega)$...). When $\Omega = \mathbb{R}$ we usually write only $L^2$, $H^2$, $\mathcal{D}$, etc. The domain of an operator $\hat{A}$ is written $\mathcal{D}(\hat{A})$ and the statement $\hat{A} \subseteq \hat{B}$ means, as usual, that $\mathcal{D}(\hat{A}) \subseteq \mathcal{D}(\hat{B})$ and $\hat{A}\psi = \hat{B}\psi$ for all $\psi \in \mathcal{D}(\hat{A})$.

2. A multiplicative product of Schwartz distributions

In this section we review the main properties of the multiplicative product of distributions $*$ that was proposed in [21]. For details and proofs the reader should refer to [21].

We start by presenting some basic definitions. Let the $n$th order singular support of a distribution $F \in \mathcal{D}'$ (denoted $n$-sing supp $F$) be the closed set of points where $F$ is not a $C^n$-function. More precisely: let $\Omega \subseteq \mathbb{R}$ be the largest open set for which there is $f \in C^n(\Omega)$ such that $F|_\Omega = f$ (where $F|_\Omega$ denotes the restriction of $F$ to $\mathcal{D}(\Omega)$). Then $n$-sing supp $F = \mathbb{R}\setminus\Omega$. This definition generalizes the usual definition of singular support of a distribution. We have, of course, $m$-sing supp $F \subseteq n$-sing supp $F$ for all $m \leq n$ and $\infty$-sing supp $F = \text{sing supp } F$.

Another useful concept is the order of a distribution [36]: we say that $F \in \mathcal{D}'$ is of order $n$ (and write $n = \text{ord } F$) iff $F$ is the $n$th order distributional derivative (but not a lower order distributional derivative) of a regular distribution.

Let now $C^n_p$ be the space of piecewise $n$ times continuously differentiable functions: $\psi \in C^n_p$ iff there is a finite set $I \subset \mathbb{R}$ such that $\psi \in C^n(\mathbb{R}\setminus I)$ and the lateral limits $\lim_{x \to x_0^\pm} \psi^{(j)}(x)$ exist and are finite for all $x_0 \in I$ and all $j$-order derivatives of $\psi$, with $j = 0, .., n$. 
Finally, let $\mathcal{F}^n$ be the space of distributions $F \in \mathcal{D}'$ such that $\text{supp} \ F$ is a finite set and $\text{ord} \ F \leq n + 1$.

A distributional extension of the sets $\mathcal{C}^n_p$ is then given by:

**Definition 2.1.** Let $\mathcal{A}^n = \mathcal{C}^n_p \oplus \mathcal{F}^n$, where the elements of $\mathcal{C}^n_p$ are regarded as distributions. Moreover, the space of distributions of the form $F|_\Omega$, where $F \in \mathcal{A}^n$, is denoted by $\mathcal{A}^n(\Omega)$.

We remark that only the spaces $\mathcal{A}^0$ and $\mathcal{A}^1$ (and their restrictions $\mathcal{A}^0(\Omega)$ and $\mathcal{A}^1(\Omega)$) will be used in the rest of the paper (sections 3 to 6). Hence, for the remainder of this and the next sections, the reader may always assume that $n = 0, 1$.

Let us then consider the definition of $\mathcal{A}^n$. We have, of course, $\mathcal{C}^n_p \subset \mathcal{A}^n \subset \mathcal{D}'$. All the elements of $\mathcal{A}^n$ are distributions of order (at most) $n + 1$ and finite $n$th order singular support. They can be written in the form $F = \Delta F + f$, where $\Delta F \in \mathcal{F}^n$ and $f \in \mathcal{C}^n_p$. The next Theorem states this property in a more explicit form:

**Theorem 2.2.** $F \in \mathcal{A}^n$ iff there is a finite set $I = \{x_1, ..., x_m\}$ associated with a set of open intervals $\Omega_i = (x_i, x_{i+1})$, $i = 0, ..., m$ (where $x_0 = -\infty$ and $x_{m+1} = +\infty$) such that ($\chi_{\Omega_i}$ is the characteristic function of $\Omega_i$):

$$F = \sum_{i=1}^{m} \sum_{j=0}^{n} c_{ij} \delta^{(j)}(x - x_i) + \sum_{i=0}^{m} f_i \chi_{\Omega_i}$$

for some $c_{ij} \in \mathbb{C}$ and $f_i \in \mathcal{C}^n$. We have, necessarily, $n\text{-sing supp} \ F \subseteq I$.

The product $\ast$ will be defined in the spaces $\mathcal{A}^n$. First, we recall some basic definitions about products of distributions. Let $\Xi \subset \mathbb{R}$ be an open set. The standard product of $F \in \mathcal{D}'(\Xi)$ by $g \in \mathcal{C}^\infty(\Xi)$ is defined by

$$\langle Fg, t \rangle = \langle Ft, g \rangle, \quad \forall t \in \mathcal{D}(\Xi).$$

Moreover, this product is also well-defined for all pairs $F \in \mathcal{A}^n(\Xi)$, $g \in \mathcal{C}^n(\Xi)$, where $n \in \mathbb{N}_0$ (because $\text{ord} \ F \leq n + 1$ and $gt \in \mathcal{C}^n_0(\Xi)$).

The Hörmander product of distributions generalizes the previous product to the case of two distributions with finite and disjoint singular supports (pag.55, [33]). We present a slightly more general definition which we call the extended Hörmander product.

**Definition 2.3.** For $n \in \mathbb{N}_0$, let $F, G \in \mathcal{A}^n$ be two distributions such that $n\text{-sing supp} \ F$ and $n\text{-sing supp} \ G$ are finite disjoint sets. Then there exists a finite open cover of $\mathbb{R}$ (denote it by $\{\Xi_i \subset \mathbb{R}, i = 1, ..., d\}$) such that, on each open set $\Xi_i$, either $F$ or $G$ is a $\mathcal{C}^n(\Xi_i)$-function. Hence, on each $\Xi_i$, the two distributions can be multiplied using the previous product of a $\mathcal{A}^n$-distribution by a $\mathcal{C}^n$-function. The extended Hörmander product of $F$ by $G$ is then defined as the unique distribution $F \cdot G \in \mathcal{A}^n$ that satisfies:

$$F \cdot G|_{\Xi_i} = F|_{\Xi_i} G|_{\Xi_i}, \quad i = 1, ..., d.$$
Finally, the new product * generalizes the extended Hörmander product to the case of an arbitrary pair of distributions in \( \mathcal{A}^n \):

**Definition 2.4.** The multiplicative product * is defined for all \( F, G \in \mathcal{A}^n \) by:

\[
F * G = \lim_{\epsilon \downarrow 0} F(x) \cdot G(x + \epsilon),
\]

where the product in \( F(x) \cdot G(x + \epsilon) \) is the extended Hörmander product and the limit is taken in the distributional sense.

The explicit form of \( F * G \) is given in Theorem 2.5 below and the main properties of * are stated in Theorem 2.6. Let \( F, G \in \mathcal{A}^n \) and let \( I_F \) and \( I_G \) be the \( n \)-singular supports of \( F \) and \( G \), respectively. Let \( I = \{x_1, \ldots, x_m\} \) (assuming \( x_i < x_{i+1} \)). Define the open sets \( \Omega_i = (x_i, x_{i+1}) \), \( i = 0, \ldots, m \) (with \( x_0 = -\infty \) and \( x_{m+1} = +\infty \)). Then, in view of Theorem 2.2, \( F \) and \( G \) can be written in the form:

\[
F = \sum_{i=1}^m \sum_{j=0}^n a_{ij} \delta^{(j)}(x - x_i) + \sum_{i=0}^m f_i \chi_{\Omega_i},
\]

\[
G = \sum_{i=1}^m \sum_{j=0}^n b_{ij} \delta^{(j)}(x - x_i) + \sum_{i=0}^m g_i \chi_{\Omega_i},
\]

where \( f_i, g_i \in C^n(\mathbb{R}) \) and \( a_{ij} = 0 \) if \( x_i \notin I_F \) or if \( j \geq \text{ord} \, F \), and likewise for \( G \). Then we have:

**Theorem 2.5.** Let \( F, G \in \mathcal{A}^n \) be written in the form (2.2). Then \( F * G \) is given explicitly by

\[
F * G = \sum_{i=1}^m \sum_{j=0}^n [a_{ij}g_i(x) + b_{ij}f_{i-1}(x)] \cdot \delta^{(j)}(x - x_i) + \sum_{i=0}^m f_i g_i \chi_{\Omega_i},
\]

and \( F * G \in \mathcal{A}^n \).

The main properties of * are summarized in the following

**Theorem 2.6.** The product * is an inner operation in \( \mathcal{A}^n \), it is associative, distributive, non-commutative and it reproduces the extended Hörmander product of distributions if the \( n \)-singular supports of \( F \) and \( G \) do not intersect. Since \( \mathcal{A}^n \) is not closed with respect to the distributional derivative \( D_x \), the Leibnitz rule is valid only under the condition that \( D_x F, D_x G \in \mathcal{A}^n \).

### 3. The Shifting Delta operators

Using the spaces \( \mathcal{A}^n \) and the product *, we can now define the following sets of operators:

**Definition 3.1.** (1) Let \( n \in \mathbb{N}_0 \). The set of all boundary potential operators of the form (1.3):

\[
\hat{B} : \mathcal{D}(\hat{B}) \subset \mathcal{D}' \rightarrow \mathcal{D}'; \quad \hat{B} \psi = \psi * B_1 + B_2 * \psi
\]
where \( B_1, B_2 \in \mathcal{A}^n \) and \( B_1, B_2 \) have finite support, is denoted by \( \hat{\mathcal{A}}^n \). We notice that \( \cup_{i \geq n} \mathcal{A}^i \subseteq \mathcal{D}(\hat{B}) \). Moreover, \( \mathcal{A}^n \subset \hat{\mathcal{A}}^m \) for all \( n < m \).

(2) The set of all boundary pseudo potential operators \((1.4)\) of the form:

\[
\hat{B} : \mathcal{D}(\hat{B}) \subseteq \mathcal{D}'(\hat{B}) \rightarrow \mathcal{D}'(\hat{B}); \quad \hat{B} \psi = \hat{B}_1 \psi + \hat{B}_2 D_x \psi + D_x \hat{B}_3 D_x \psi
\]

where \( \hat{B}_1 \in \hat{\mathcal{A}}^1 \) and \( \hat{B}_2, \hat{B}_3 \in \hat{\mathcal{A}}^0 \), is denoted by \( \hat{\mathcal{P}} \). We notice that \( \cup_{i \geq 1} \mathcal{A}^i \subseteq \mathcal{D}(\hat{B}) \).

(3) The linear operators \( \hat{B} : \mathcal{D}(\hat{B}) \subseteq \mathcal{D}' \rightarrow \mathcal{D}' \) that satisfy \((1.4)\) are called boundary operators. The set of all boundary operators is denoted by \( \hat{\mathcal{B}} \).

We have, of course, \( \hat{\mathcal{A}}^0 \subset \hat{\mathcal{A}}^1 \subset \hat{\mathcal{P}} \subset \hat{\mathcal{B}} \). In this section, we will study two important families of boundary potential operators:

**Definition 3.2.** For \( n \in \mathbb{N}_0 \) and \( x_0 \in \mathbb{R} \) let:

\[
\hat{\delta}^\pm_+(x_0) : \mathcal{A}^n \rightarrow \mathcal{A}^n; \quad F \rightarrow \hat{\delta}^\pm_+(x_0) F = \delta^{(n)}(x-x_0) * F
\]

\[
\hat{\delta}^\pm_-(x_0) : \mathcal{A}^n \rightarrow \mathcal{A}^n; \quad F \rightarrow \hat{\delta}^\pm_-(x_0) F = F * \delta^{(n)}(x-x_0).
\]

We call \( \hat{\delta}^\pm_+(x_0) \) the \( n \)th-order right shifting delta, and \( \hat{\delta}^\pm_-(x_0) \) the \( n \)th-order left shifting delta. For \( n = 0 \), we simply write \( \hat{\delta}_+(x_0) \) and \( \hat{\delta}_-(x_0) \); for \( x_0 = 0 \) we write \( \hat{\delta}^\pm_0 \).

We have \( \hat{\delta}^\pm_0(x_0) \in \hat{\mathcal{A}}^n \). We also notice that, since \( \delta^{(n)}(x-x_0) \in \mathcal{A}^m \) for all \( m \geq n \), the operators \( \hat{\delta}^\pm_0 \) can be extended to \( \hat{\mathcal{A}}^m \), for all \( m \geq n \).

These operators satisfy the following basic properties:

(1) Let us define

\[
\hat{\delta}^{(n)}(x_0) : \mathcal{C}^n \rightarrow \mathcal{D}'; \quad f \rightarrow \hat{\delta}^{(n)}(x_0) f = \delta^{(n)}(x-x_0) \cdot f
\]

where \( \cdot \) is the extended Hörmander product. The operator \( \hat{\delta}^{(n)}(x_0) \) is just the standard operator of ”multiplication by the \( n \)th-order derivative of a Dirac delta”. We have explicitly

\[
\langle \delta^{(n)}(x-x_0) \cdot f, t \rangle = (-1)^n \frac{d^n}{dx^n} (f \ t)(x_0), \quad \forall t \in \mathcal{D}.
\]

Since, for all \( f \in \mathcal{C}^n \) (cf. Theorem 2.5),

\[
\delta^{(n)}(x-x_0) \ast f = f \ast \delta^{(n)}(x-x_0) = \delta^{(n)}(x-x_0) \cdot f
\]

we have

\[
\hat{\delta}^{(n)}_-(x_0) f = \hat{\delta}^{(n)}_+(x_0) f = \hat{\delta}^{(n)}(x_0) f, \quad \forall f \in \mathcal{C}^n
\]

and so the operators \( \hat{\delta}^{(n)}_+(x_0) \) and \( \hat{\delta}^{(n)}_-(x_0) \) are extensions of \( \hat{\delta}^{(n)}(x_0) \) to the space \( \mathcal{A}^n \).
(2) For all \( n \in \mathbb{N}_0 \), both \( \tilde{\delta}^{(n)}(x_0) \) and \( \tilde{\delta}^{(n)}_+(x_0) \) can be cast as weak operator limits of one-parameter families of (operators of multiplication by) smooth functions. Let us set \( x_0 = 0 \). For every \( \epsilon > 0 \), let \( v_\epsilon \in \mathcal{D} \) be a non-negative and even function with support on \([-\epsilon, \epsilon]\) and such that \( \int_{-\infty}^{+\infty} v_\epsilon(x) \, dx = 1 \). Since
\[
\lim_{\epsilon \downarrow 0} \langle v_\epsilon, t \rangle = t(0) \quad \forall t \in \mathcal{D}
\]
we have, in the sense of distributions, \( \lim_{\epsilon \downarrow 0} v_\epsilon(x) = \delta(x) \).

Now consider the operators \( (n \in \mathbb{N}_0, \epsilon > 0) \):
\[
(3.3) \quad \tilde{\delta}^{(n)}_\epsilon : \mathcal{A}^n \to \mathcal{A}^n; \quad F(x) \mapsto \tilde{\delta}^{(n)}_\epsilon F(x) = v^{(n)}_\epsilon(x - \epsilon) \cdot F(x)
\]
where \( \cdot \) is the extended Hörmander product. In the distributional sense, we have once again
\[
\lim_{\epsilon \downarrow 0} v^{(n)}_\epsilon(x - \epsilon) = \delta^{(n)}(x)
\]
for all \( n \in \mathbb{N}_0 \). On the other hand, in the operator sense, we get:

**Theorem 3.3.** For all \( n \in \mathbb{N}_0 \), the one parameter family \( v^{(n)}_\epsilon \) converges, in the weak operator sense, to the operator \( \tilde{\delta}^{(n)}_+ \), i.e.
\[
w - \lim_{\epsilon \downarrow 0} \tilde{\delta}^{(n)}_\epsilon = \tilde{\delta}^{(n)}_+.
\]

**Proof.** Let us start by considering the case \( n = 0 \). Let \( F \in \mathcal{A}^0 \) and \( t \in \mathcal{D} \). We have
\[
\lim_{\epsilon \downarrow 0} \langle \tilde{\delta}_\epsilon F, t \rangle = \lim_{\epsilon \downarrow 0} \langle v_\epsilon(x - \epsilon) \cdot F(x), t(x) \rangle
\]
\[
= \lim_{\epsilon \downarrow 0} \langle F(x), v_\epsilon(x - \epsilon)t(x) \rangle
\]
\[
= \lim_{\epsilon \downarrow 0} \int_{0}^{2\epsilon} v_\epsilon(x - \epsilon)f(x)t(x) \, dx
\]
where we used the fact that \( v_\epsilon(x - \epsilon) = 0 \) for all \( x \notin (0, 2\epsilon) \) and also that, for sufficiently small \( \epsilon > 0 \), there is a function \( f \in \mathcal{C}^0 \) such that \( F|_{(0,2\epsilon)} = f|_{(0,2\epsilon)} \) (cf. Theorem 2.2).

Setting \( g = ft \in \mathcal{C}^0 \), we get
\[
\lim_{\epsilon \downarrow 0} \int_{0}^{2\epsilon} v_\epsilon(x - \epsilon)g(x) \, dx
\]
\[
= \lim_{\epsilon \downarrow 0} \left[ \int_{0}^{2\epsilon} v_\epsilon(x - \epsilon)g(0) \, dx + \int_{0}^{2\epsilon} v_\epsilon(x - \epsilon)[g(x) - g(0)] \, dx \right]
\]
\[
= g(0) + \lim_{\epsilon \downarrow 0} \int_{0}^{2\epsilon} v_\epsilon(x - \epsilon)[g(x) - g(0)] \, dx.
\]
Now, for every \( \epsilon > 0 \), let
\[
M_\epsilon = \max_{x \in [0,2\epsilon]} \left[ g(x) - g(0) \right] \quad \text{and} \quad m_\epsilon = \min_{x \in [0,2\epsilon]} \left[ g(x) - g(0) \right].
\]
Since $v_\epsilon(x - \epsilon)$ is non-negative, we have for all $\epsilon > 0$
\[ m_\epsilon \leq \int_0^{2\epsilon} v_\epsilon(x - \epsilon) \left[ g(x) - g(0) \right] \, dx \leq M_\epsilon \]
and since $\lim_{\epsilon \downarrow 0} M_\epsilon = \lim_{\epsilon \downarrow 0} m_\epsilon = 0$, we get
\[ \lim_{\epsilon \downarrow 0} \int_0^{2\epsilon} v_\epsilon(x - \epsilon) \left[ g(x) - g(0) \right] \, dx = 0. \]
Hence
\[ \lim_{\epsilon \downarrow 0} (\hat{v}_\epsilon F, t) = g(0) = f(0)t(0). \]

The generalization of this result to $n \in \mathbb{N}$ is easily obtained. Notice first that the equation (3.4) is also valid if we replace $\hat{v}_\epsilon$ by $\hat{v}^{(n)}_\epsilon$ and require that $F \in \mathcal{A}^n$ (in which case $f, g \in C^n$). Integrating by parts, it follows that
\[ \lim_{\epsilon \downarrow 0} \int_0^{2\epsilon} v^{(n)}_\epsilon(x - \epsilon)g(x) \, dx = (-1)^n \lim_{\epsilon \downarrow 0} \int_0^{2\epsilon} v_\epsilon(x - \epsilon)g^{(n)}(x) \, dx = (-1)^n g^{(n)}(0). \]

Finally, from Definition 3.2 and Theorem 2.5, we also have:
\[ \hat{\delta}^{(n)}_+ F = \delta^{(n)}(x) \cdot f(x) \implies (\hat{\delta}^{(n)}_+) F, t = (-1)^n \frac{d^n}{dx^n}(ft)(0), \quad \forall t \in \mathcal{D} \]
and so $w - \lim_{\epsilon \downarrow 0} \hat{v}^{(n)}_\epsilon = \hat{\delta}^{(n)}_+$, for all $n \in \mathbb{N}_0$.

If we re-define
\[ \tilde{v}^{(n)}_\epsilon : \mathcal{A} \rightarrow \mathcal{A}^n; \quad F(x) \rightarrow \tilde{v}^{(n)}_\epsilon F = v^{(n)}_\epsilon(x + \epsilon) \cdot F(x) \]
then we also have
\[ w - \lim_{\epsilon \downarrow 0} \tilde{v}^{(n)}_\epsilon = \hat{\delta}^{(n)} \]
for all $n \in \mathbb{N}_0$.

Moreover, if we combine $v^{(n)}_\epsilon(x + \epsilon)$ and $v^{(n)}_\epsilon(x - \epsilon)$, we easily obtain one-parameter families of smooth functions that converge in $\mathcal{D}'$ to $\delta^{(n)}(x)$
\[ c v^{(n)}_\epsilon(x - \epsilon) + (1 - c)v^{(n)}_\epsilon(x + \epsilon) \xrightarrow{\mathcal{D}'} \delta^{(n)}(x), \quad c \in \mathbb{R} \]
and which in the weak operator sense converge to
\[ w - \lim_{\epsilon \downarrow 0} \left[ c v^{(n)}_\epsilon(x - \epsilon) \cdot + (1 - c)v^{(n)}_\epsilon(x + \epsilon) \cdot \right] = c\hat{\delta}^{(n)}_+ + (1 - c)\hat{\delta}^{(n)}_- . \]

Finally, we notice that every boundary potential operator $\hat{B} \in \hat{\mathcal{A}}^n$ is given by a linear combination of the operators $\hat{\delta}^{(m)}_+$ and $\hat{\delta}^{(m)}_-$, $m \leq n$. Hence, every $\hat{B} \in \hat{\mathcal{A}}^n$ can also be written as the weak operator limit of sequences of operators of multiplication by smooth functions. The associated sequence of smooth functions converges in $\mathcal{D}'$ to $B = B_1 + B_2 \leftrightarrow \hat{B}$.

(3) Since the Sobolev-Hilbert spaces $\mathcal{H}^2(\mathbb{R}_\pm)$ satisfy
\[ \mathcal{H}^2(\mathbb{R}_\pm) = \chi_{\mathbb{R}_\pm} [\mathcal{H}^2] \quad \text{and} \quad \mathcal{H}^2 \subset C^1, \]
we have $\mathcal{H}^2(\mathbb{R}_-) \oplus \mathcal{H}^2(\mathbb{R}_+) \subset C^1_p \subset A^1$ and $D_x[\mathcal{H}^2(\mathbb{R}_-) \oplus \mathcal{H}^2(\mathbb{R}_+)] \subset A^0$. Hence, the three operators $\delta_\pm(x)$, $\delta_\pm'(x)$ and $\delta_\pm(x)D_x$ are well defined on $\mathcal{H}^2(\mathbb{R}_-) \oplus \mathcal{H}^2(\mathbb{R}_+)$. The same is then true for all $\hat{B} \in \hat{\mathcal{P}}$.

A general element of $\mathcal{H}^2(\mathbb{R}_-) \oplus \mathcal{H}^2(\mathbb{R}_+)$ can be written as:

$$\psi = \chi_{\mathbb{R}_-} \psi_- + \chi_{\mathbb{R}_+} \psi_+$$

where $\psi_+ \in \mathcal{H}^2$. Then we have explicitly

$$\delta_\pm(x) \psi(x) = \delta(x) \cdot \psi_\pm(x) = \delta(x) \psi_\pm(0)$$

$$\delta_\pm'(x) \psi(x) = \delta'(x) \cdot \psi_\pm(x) = \delta'(x) \psi_\pm(0) - \delta(x) \psi_\pm'(0)$$

$$\delta_\pm D_x \psi(x) = \delta(x) \cdot \psi_\pm'(x) = \delta(x) \psi_\pm'(0) .$$

4. One-dimensional boundary operators: General results

Let $\hat{H}_0 = -D_x^2$ be the one-dimensional free Schrödinger operator with domain on the Sobolev-Hilbert space $\mathcal{H}^2$. As before, let $\hat{S}$ be its symmetric restriction to the domain $\mathcal{D}(\mathbb{R}\setminus\{0\})$ and let $\hat{S}^*$ be the adjoint of $\hat{S}$. In this section we prove some general results about one-dimensional Schrödinger operators with arbitrary boundary operators:

$$\hat{Z} : \mathcal{D}(\hat{H}) \subseteq \mathcal{L}^2 \longrightarrow \mathcal{L}^2; \quad \hat{H} = \hat{H}_0 + \hat{B}$$

where $\hat{B} \in \hat{\mathcal{B}}$. More precisely, we show that every $\hat{Z}$ of the previous form satisfies $\hat{Z} \subseteq \hat{S}^*$, and we determine the maximal domain of $\hat{Z}$ in terms of the operator $\hat{B}$, explicitly. Moreover, we show that $\hat{S}^*$ can be written in the previous form (1.1), and determine the associated boundary operator $\hat{B}$, explicitly.

Here, and from now on, $\hat{H}_0$ is defined in the generalized sense

$$\hat{H}_0 : \mathcal{L}^2 \longrightarrow \mathcal{D}' ; \quad \hat{H}_0 = -D_x^2.$$

Moreover, the maximal domain of an arbitrary operator $\hat{Z}$ is defined to be

$$\mathcal{D}_{\text{max}}(\hat{Z}) \equiv \{ \psi \in \mathcal{L}^2 : \hat{Z} \psi \in \mathcal{L}^2 \}$$

where the condition $\hat{Z} \psi \in \mathcal{L}^2$ means precisely that: i) $\hat{Z} \psi$ is well-defined as a distribution in $\mathcal{D}'$ and ii) there exists $\phi \in \mathcal{L}^2$ such that $\hat{Z} \psi = \phi$, weakly.

We start by proving the following general result:

**Lemma 4.1.** Let $\hat{Z} = \hat{H}_0 + \hat{B}$ where $\hat{B} \in \hat{\mathcal{B}}$ is an arbitrary boundary operator. Then the maximal domain of $\hat{Z}$ satisfies

$$\mathcal{D}_{\text{max}}(\hat{Z}) \subseteq \mathcal{H}^2(\mathbb{R}_-) \oplus \mathcal{H}^2(\mathbb{R}_+).$$

**Proof.** We have $\psi \in \mathcal{D}_{\text{max}}(\hat{Z})$ iff $\psi \in \mathcal{L}^2 \cap \mathcal{D}(\hat{B})$ and there exists $\phi \in \mathcal{L}^2$ such that $\hat{Z} \psi = \phi$ in the weak sense. This implies that

$$\langle \hat{Z} \psi, t \rangle = \langle \phi, t \rangle, \ \forall t \in \mathcal{D}(\mathbb{R} \setminus \{0\}) \iff \begin{cases} (-D_x^2 \psi)|_{\mathbb{R}_-} = \phi|_{\mathbb{R}_-} \\ (-D_x^2 \psi)|_{\mathbb{R}_+} = \phi|_{\mathbb{R}_+} \end{cases}$$

and
where the identities are in the distributional sense and we used the fact that \( \text{supp} \, \hat{B} \psi \subseteq \{0\} \). It follows from \((-D^2_x \psi)|_{\mathbb{R}_\pm} = -D^2_x (\psi|_{\mathbb{R}_\pm})\) and Grubb’s [Theorem 4.20, Remark 4.21 [31] that \( \psi|_{\mathbb{R}_+} \in \mathcal{H}^2(\mathbb{R}_+) \), and likewise \( \psi|_{\mathbb{R}_-} \in \mathcal{H}^2(\mathbb{R}_-) \). Since \( \psi \in \mathcal{L}^2 = \mathcal{L}^2(\mathbb{R}_-) \oplus \mathcal{L}^2(\mathbb{R}_+) \), we conclude that \( \psi \in \mathcal{H}^2(\mathbb{R}_-) \oplus \mathcal{H}^2(\mathbb{R}_+) \) and so:
\[
D_{\text{max}}(\tilde{Z}) \subseteq \mathcal{H}^2(\mathbb{R}_-) \oplus \mathcal{H}^2(\mathbb{R}_+) .
\]

\[\square\]

A natural question is then whether \( \tilde{Z} \subseteq \tilde{S}^* \). First we show that \( \tilde{S}^* \) is itself of the form (4.1). Recall that the domain of \( \tilde{S}^* \) is \( [2, 4, 6, 8] \):
\[
\mathcal{D}(\tilde{S}^*) = \mathcal{H}^2(\mathbb{R}_-) \oplus \mathcal{H}^2(\mathbb{R}_+) \subseteq \mathcal{L}^2
\]
and that all \( \psi \in \mathcal{H}^2(\mathbb{R}_-) \oplus \mathcal{H}^2(\mathbb{R}_+) \) can be written in the form:
\[
\psi = \chi_{\mathbb{R}_-} \psi_- + \chi_{\mathbb{R}_+} \psi_+ ,
\]
where \( \psi_- , \psi_+ \in \mathcal{H}^2 \). Moreover, \( \tilde{S}^* \) acts as:
\[
\tilde{S}^* [\chi_{\mathbb{R}_-} \psi_- + \chi_{\mathbb{R}_+} \psi_+] = -\chi_{\mathbb{R}_-} \psi''_+ - \chi_{\mathbb{R}_+} \psi''_+ ,
\]
where, as usual, \( \psi''_+ = D^2_x \psi \).

Let us define the operators
\[
\hat{\beta}^{(n)} : \mathcal{A} \rightarrow \mathcal{A} ; \quad F \rightarrow \left[ \hat{\delta}^{(n)} - \hat{\beta}^{(n)} \right] F
\]
and introduce the notation \( \hat{\beta} = \beta^{(0)} \) and \( \hat{\beta}' = \beta^{(1)} \). Then

**Theorem 4.2.** The adjoint of \( \tilde{S} \) is given by
\[
(4.2) \quad \tilde{S}^* = \tilde{H}_0 + 2\hat{\beta} D_x + \hat{\beta}'
\]
and the domain of \( \tilde{S}^* \) coincides with the maximal domain of the expression on the right hand side,
\[
\mathcal{D}(\tilde{S}^*) = \{ \psi \in \mathcal{L}^2 : (\tilde{H}_0 + 2\hat{\beta} D_x + \hat{\beta}') \psi \in \mathcal{L}^2 \} .
\]

**Proof.** Consider the action of \( D^2_x \) on \( \psi \in \mathcal{D}(\tilde{S}^*) :
\[
D^2_x [\chi_{\mathbb{R}_-} \psi_- + \chi_{\mathbb{R}_+} \psi_+ ] = \chi_{\mathbb{R}_-} \psi''_- + \chi_{\mathbb{R}_+} \psi''_+ + 2\delta(x) [\psi'_+ - \psi'_-] + \delta'(x) [\psi_+ - \psi_-]
\]
and let us re-express the r.h.s in terms of the operators \( \hat{\delta}_\pm \) and \( \hat{\delta}'_\pm \),
\[
-D^2_x \psi = \tilde{S}^* \psi - 2 \left[ \hat{\delta}_+ - \hat{\delta}_- \right] \psi' - \left[ \hat{\delta}'_+ - \hat{\delta}'_- \right] \psi .
\]
Using the operators \( \hat{\beta} \) and \( \hat{\beta}' \) we immediately obtain (4.2).

It remains to prove that the maximal domain of the r.h.s. of (4.2) is \( \mathcal{H}^2(\mathbb{R}_-) \oplus \mathcal{H}^2(\mathbb{R}_+) \). Let us set \( \tilde{B} = 2\hat{\beta} D_x + \hat{\beta}' \). Since \( \text{supp} \, \tilde{B} \psi \subseteq \{0\} \) for all \( \psi \in \mathcal{D}(\tilde{B}) \), it follows from Lemma 4.1 that \( D_{\text{max}}(\tilde{H}_0 + \tilde{B}) \subseteq \mathcal{H}^2(\mathbb{R}_-) \oplus \mathcal{H}^2(\mathbb{R}_+) \).
Moreover, if \( \psi \in \mathcal{H}^2(\mathbb{R}_-) \oplus \mathcal{H}^2(\mathbb{R}_+) \) then \( \psi = \chi_{\mathbb{R}_-} \psi_- + \chi_{\mathbb{R}_+} \psi_+ \) for some \( \psi_-, \psi_+ \in \mathcal{H}^2 \) and
\[
(\tilde{H}_0 + 2\tilde{\beta}D_x + \tilde{\beta}') \psi = -\chi_{\mathbb{R}_-} \psi''_- - \chi_{\mathbb{R}_+} \psi''_+ \in \mathcal{L}^2.
\]
Hence, \( \mathcal{D}_{\max}(\tilde{H}_0 + 2\tilde{\beta}D_x + \tilde{\beta}') = \mathcal{H}^2(\mathbb{R}_-) \oplus \mathcal{H}^2(\mathbb{R}_+) = \mathcal{D}(\tilde{S}^*) \). \( \square \)

We are now able to prove the main result of this section:

**Theorem 4.3.** Let \( \tilde{B} \in \tilde{B} \) be an arbitrary boundary operator and let
\[
\tilde{Z} : \mathcal{D}_{\max}(\tilde{Z}) \subset \mathcal{L}^2 \longrightarrow \mathcal{L}^2, \quad \tilde{Z} = \tilde{H}_0 + \tilde{B}.
\]
Then \( \tilde{Z} \subseteq \tilde{S}^* \), and
\[
\mathcal{D}_{\max}(\tilde{Z}) = \text{Ker} \, \tilde{F} \cap (\mathcal{H}^2(\mathbb{R}_-) \oplus \mathcal{H}^2(\mathbb{R}_+))
\]
where
\[
\tilde{F} = -2\tilde{\beta}D_x - \tilde{\beta}' + \tilde{B}.
\]

**Proof.** Since \( \text{supp} \, \tilde{B} \psi \subseteq \{0\} \) for all \( \psi \in \mathcal{D}(\tilde{B}) \), we have from Lemma 4.1
\[
\mathcal{D}_{\max}(\tilde{Z}) \subseteq \mathcal{H}^2(\mathbb{R}_-) \oplus \mathcal{H}^2(\mathbb{R}_+).
\]
It follows from Theorem 4.2 (and the definition of \( \tilde{Z} \)) that on \( \mathcal{D}_{\max}(\tilde{Z}) \)
\[
\tilde{Z} = \tilde{S}^* - 2\tilde{\beta}D_x - \tilde{\beta}' + \tilde{B} = \tilde{S}^* + \tilde{F}
\]
where \( \tilde{F} \) is given by (4.3). Since \( \text{supp} \, \tilde{F} \psi \subseteq \{0\} \), the term \( \tilde{F} \psi \) is a linear combination of a Dirac delta and its derivatives and so
\[
\psi \in \mathcal{D}_{\max}(\tilde{Z}) \implies (\tilde{S}^* + \tilde{F})\psi \in \mathcal{L}^2 \implies \tilde{F} \psi = 0.
\]
Hence, \( \mathcal{D}_{\max}(\tilde{Z}) \subseteq \text{Ker} \, \tilde{F} \). Conversely
\[
\psi \in \text{Ker} \, \tilde{F} \cap (\mathcal{H}^2(\mathbb{R}_-) \oplus \mathcal{H}^2(\mathbb{R}_+)) \implies \tilde{Z} \psi = \tilde{S}^* \psi \in \mathcal{L}^2.
\]
We conclude that \( \mathcal{D}_{\max}(\tilde{Z}) = \text{Ker} \, \tilde{F} \cap (\mathcal{H}^2(\mathbb{R}_-) \oplus \mathcal{H}^2(\mathbb{R}_+)) \) and \( \tilde{Z} \subseteq \tilde{S}^* \). \( \square \)

5. One-dimensional Schrödinger operators with point interactions

Let now \( \tilde{Z} \) denote an arbitrary restriction of \( \tilde{S}^* \) to a domain characterized by two local boundary conditions at \( x = 0 \):
\[
\tilde{Z} : \mathcal{D}(\tilde{Z}) \subseteq \mathcal{L}^2 \longrightarrow \mathcal{L}^2; \quad \tilde{Z} \psi = \tilde{S}^* \psi
\]
\[
\mathcal{D}(\tilde{Z}) = \{ \psi \in \mathcal{H}^2(\mathbb{R}_-) \oplus \mathcal{H}^2(\mathbb{R}_+) : f_i(\psi_{\pm}(0), \psi'_{\pm}(0)) = 0, i = 1, 2 \}
\]
where \( \psi_{\pm}(0) = (\psi_{+}(0), \psi_{-}(0)), \ \psi'_{\pm}(0) = (\psi'_{+}(0), \psi'_{-}(0)) \) and \( f_i : \mathbb{C}^4 \longrightarrow \mathbb{C} \), \( i = 1, 2 \), are linear functions.

In this section we show that every operator \( \tilde{Z} \) can be written in the form
\[
\tilde{Z} = \tilde{H}_0 + \tilde{\beta} \tilde{Z} \in \mathcal{D}_{\max}(\tilde{H}_0 + \tilde{\beta})
\]
where $\hat{B} \in \hat{P}$ is a boundary pseudo potential operator of the form (5.1).

The one-dimensional Schrödinger operators with point interactions $\hat{\mathcal{L}}$ are all of the form (5.1). For each $\hat{\mathcal{L}}$, we will calculate a boundary pseudo potential representation (5.2) explicitly (Corollaries 5.4, 5.5 and 5.6).

We remark that for each $\hat{B} \in \hat{P}$ there are, in general, several different operators $\hat{B} \in \hat{P}$ such that (5.2) is valid. This will be shown explicitly for the operator $\hat{\mathcal{L}}$ with Dirichlet boundary conditions at $x = 0$ (Corollaries 5.4 and 5.5). In the next section, the non-uniqueness of $\hat{B}$ will be studied in more detail.

A natural question is whether the operators $\hat{Z}$ (or at least the operators $\hat{\mathcal{L}}$) admit a (simpler) boundary potential representation (and not only a boundary pseudo potential representation), i.e. a representation of the form (5.2) with $\hat{B} \in \hat{A}$. While this is true for a large class of operators $\hat{\mathcal{L}}$ (see Theorems 6.2 and 6.4, in the next section) the following Theorem shows that it is not true for all $\hat{\mathcal{L}}$, not even if we only require $\hat{B}$ to be of the form (5.1) with $\hat{B}_3 = 0$.

**Theorem 5.1.** The set of operators $\hat{Z} = \hat{H}_0 + \hat{B}$ (5.2) where $\hat{B}$ is of the form (5.1) with $\hat{B}_3 = 0$, does not contain all s.a. extensions of $\hat{S}$.

**Proof.** If $\hat{B}_3 = 0$ then $\hat{B}\psi = \hat{B}_1\psi + \hat{B}_2\psi'$ and $\hat{F} = -2\beta D_x - \beta' + \hat{B}$, given by (4.3), is also of the form

$\hat{F}\psi = \hat{F}_1\psi + \hat{F}_2\psi'$

for some $\hat{F}_1 \in \hat{A}$ and $\hat{F}_2 \in \hat{A}^0$.

On the other hand, if $\hat{Z} \subset \hat{S}^*$ then supp $\hat{B}\psi \subset \{0\}$ for all $\psi \in D(\hat{B})$. The same is then true for $\hat{F}_1, \hat{F}_2$ and so

$\hat{F}_1 = a_1\delta_- + b_1\delta_+ + c_1\delta_- + d_1\delta_+$

for some $a_i, b_i, c_i, d_i \in \mathbb{C}$, $i = 1, 2$ and $c_2 = d_2 = 0$.

We have from Theorem 4.3 that $D(\hat{Z}) = \text{Ker } \hat{F} \cap (\mathcal{H}^2(\mathbb{R}_-) \oplus \mathcal{H}^2(\mathbb{R}_+))$ and so $\psi \in D(\hat{Z})$ satisfies:

$\hat{F}\psi = 0 \iff \delta(x)f_1(\psi_\pm(x), \psi'_\pm(x)) + \delta'(x)f_2(\psi_\pm(x)) = 0$

for some linear functions $f_1 : \mathbb{C}^4 \to \mathbb{C}$ and $f_2 : \mathbb{C}^2 \to \mathbb{C}$. This is equivalent to:

$\delta(x)\left[f_1(\psi_\pm(x), \psi'_\pm(x)) - f_2(\psi'_\pm(x))\right] + D_x[\delta(x)f_2(\psi_\pm(x))] = 0$.

The two terms on the l.h.s. are linear independent and so:

\[
\begin{cases}
    f_1(\psi_\pm(0), \psi'_\pm(0)) - f_2(\psi'_\pm(0)) = 0 \\
    f_2(\psi_\pm(0)) = 0
\end{cases}
\]

These conditions are unable to implement any two boundary conditions that involve two linear independent combinations of $\psi_\pm(0)$ and $\psi'_\pm(0)$. This is
the case, for instance, of the conditions:
\[
\psi'_-(0) = 0 \quad \text{and} \quad \psi'_+(0) = 0
\]
which correspond to the operator $\hat{L}$ with Neumann boundary conditions at both sides of the boundary at $x = 0$. Hence, it is not possible to construct a boundary potential formulation of all operators $\hat{L}$.

We remark that the more general possibility $\hat{B}_2 \in \hat{A}^1 \setminus \hat{A}^0$ cannot be considered, because then $\mathcal{H}^2(\mathbb{R}_-) \oplus \mathcal{H}^2(\mathbb{R}_+) \not\subseteq \mathcal{D}(\hat{B}_2 D_x)$. Instead, $\hat{B}_2 \in \hat{A}^0$, which implies $\hat{F}_2 \in \hat{A}^0$ and so $c_2 = d_2 = 0$.

Our approach to obtain the representation (5.2) is then based on the following general result, which is a Corollary of Theorem 4.3:

**Corollary 5.2.** Let $\hat{Z} \subseteq \hat{S}^*$. If exists
\[
\hat{F} : \mathcal{D}(\hat{F}) \subseteq \mathcal{D}' \rightarrow \mathcal{D}'
\]
such that:
(i) supp $\hat{F}\psi \subseteq \{0\}$ for all $\psi \in \mathcal{D}(\hat{F})$;
(ii) Ker $(\hat{F}) \cap (\mathcal{H}^2(\mathbb{R}_-) \oplus \mathcal{H}^2(\mathbb{R}_+)) = \mathcal{D}(\hat{Z})$,
then $\hat{Z}$ admits the representation
\[
(5.3) \quad \hat{Z} : \mathcal{D}(\hat{Z}) \subset \mathcal{L}^2 \rightarrow \mathcal{L}_2; \quad \hat{Z} = \hat{H}_0 + \hat{B}
\]
where
\[
\hat{B} = 2\hat{\beta} D_x + \hat{\beta}' + \hat{F}.
\]

**Proof.** It follows from the definition of $\hat{B}$ that if supp $\hat{F}\psi \subseteq \{0\}$ for all $\psi \in \mathcal{D}(\hat{F})$, then also supp $\hat{B}\psi \subseteq \{0\}$ for all $\psi \in \mathcal{D}(\hat{B})$.

Consider the operator $\hat{H}_0 + \hat{B}$. From Theorem 4.3,
\[
\mathcal{D}_{\text{max}}(\hat{H}_0 + \hat{B}) = \text{Ker} \left( \hat{B} - 2\hat{\beta} D_x - \hat{\beta}' \right) \cap (\mathcal{H}^2(\mathbb{R}_-) \oplus \mathcal{H}^2(\mathbb{R}_+))
\]
\[
= \text{Ker} \left( \hat{F} \right) \cap (\mathcal{H}^2(\mathbb{R}_-) \oplus \mathcal{H}^2(\mathbb{R}_+)) = \mathcal{D}(\hat{Z})
\]
which proves the second formula in (5.3).

Since, by assumption, $\hat{Z} \subseteq \hat{S}^*$, and from Theorem 4.3 also $\hat{H}_0 + \hat{B} \subseteq \hat{S}^*$, we conclude that $\hat{Z} = \hat{H}_0 + \hat{B}$.

The main point in determining a boundary operator representation for all operators $\hat{Z}$ (5.1) is then to determine, for each $\hat{Z}$, a suitable operator $\hat{F}$. In general, there are many possible choices of the operator $\hat{F}$ (and consequently of $\hat{B}$). Since every $\hat{Z}$ is characterized by two boundary conditions at $x = 0$, one natural possibility is the following:
Theorem 5.3. Let $\hat{Z}$ be the operator (5.1), and let

$$\hat{F} = \hat{F}_1 + D_x \hat{F}_2$$

where

$$\hat{F}_i : \mathcal{H}^2(\mathbb{R}_-) \oplus \mathcal{H}^2(\mathbb{R}_+) \subset \mathcal{A}^1 \rightarrow \mathcal{D}'$$, \hspace{1em} i = 1, 2$$

are the linear operators acting as $$(\delta_\pm = (\delta_+, \delta_-))$$:

$$\hat{F}_i \psi = f_i(\delta_\pm, \delta_\pm D_x) \psi$$

and $f_i : \mathbb{C}^4 \rightarrow \mathbb{C}$, $i = 1, 2$ are the linear functions that impose the boundary conditions of $\mathcal{D}(\hat{Z})$ (5.1).

Then $\hat{F}$ satisfies the conditions (i) and (ii) of Corollary 5.2:

(i) $\text{supp } \hat{F} \psi \subseteq \{0\}$ for all $\psi \in \mathcal{D}(\hat{F})$;

(ii) $\text{Ker } \hat{F} \cap (\mathcal{H}^2(\mathbb{R}_-) \oplus \mathcal{H}^2(\mathbb{R}_+)) = \mathcal{D}(\hat{Z})$.

Proof. We start by noticing that for all $\psi \in \mathcal{D}(\hat{F})$

$$\text{supp } (\hat{F}_i \psi) \subseteq \{0\}$$

and the condition (i) is satisfied. Moreover,

$$\psi \in \text{Ker } \hat{F} \iff \delta(x)f_1(\psi_\pm(0), \psi'_\pm(0)) + \delta'(x)f_2(\psi_\pm(0), \psi'_\pm(0)) = 0$$

$$\iff f_1(\psi_\pm(0), \psi'_\pm(0)) = f_2(\psi_\pm(0), \psi'_\pm(0)) = 0.$$ 

Hence, Ker $(\hat{F}) \cap (\mathcal{H}^2(\mathbb{R}_-) \oplus \mathcal{H}^2(\mathbb{R}_+)) = \mathcal{D}(\hat{Z})$ and so $\hat{F}$ also satisfies the condition (ii).

We conclude from Corollary 5.2 and the previous Theorem that each operator $\hat{Z}$ (5.1) admits a boundary pseudo potential representation of the form (5.2), with

$$\hat{B} = \beta' + 2\beta D_x + f_1(\delta_\pm, \delta_\pm D_x) + D_x f_2(\delta_\pm, \delta_\pm D_x).$$

Boundary pseudo potential representation of the operators $\hat{L}$. The previous results are now used to determine a boundary pseudo potential representation for each $\hat{L}$, explicitly. It is also shown, using a particular example, that each $\hat{L}$ may admit more than one representation of the form (5.2) and, in particular, it may admit a simpler boundary potential representation (this question will be studied in detail in the next section).

All operators $\hat{L}$ are s.a. restrictions of $\hat{S}^*$ of the form of $\hat{Z}$ (5.1). The two boundary conditions that characterize $\mathcal{D}(\hat{L})$ can be separating, in which case they can be written as [39, 45]:

$$\begin{align*}
(a_- \psi'(0^-) = b_- \psi(0^-) \\
(a_+ \psi'(0^+) = b_+ \psi(0^+) \\quad, \quad (a_\pm, b_\pm) \in \mathbb{R}^2 \setminus \{(0, 0)\}
\end{align*}$$

and lead to confining Schrödinger operators of the form $\hat{L} = \hat{L}_- \oplus \hat{L}_+$:

$$\hat{L}_- \oplus \hat{L}_+ : \mathcal{D}(\hat{L}_-) \oplus \mathcal{D}(\hat{L}_+) \rightarrow \mathcal{L}^2; \hspace{1em} \psi \rightarrow (\hat{L}_- \oplus \hat{L}_+)\psi = \hat{S}^*\psi$$
where
\[ D(\tilde{L}_\pm) = \{ \psi_\pm = \chi_{\mathbb{R}_\pm} \psi : \psi \in \mathcal{H}^2 \land a_\pm \psi'(0) = b_\pm \psi(0) \}. \]

Notice that \( \tilde{L}_\pm \) are s.a. extensions of the restrictions of \( \tilde{H}_0 \) to \( \mathcal{D}(\mathbb{R}_\pm) \), \[24\]. The operators \( \tilde{L} \) of the form \( \tilde{L}_- \oplus \tilde{L}_+ \) commute with the projection operators \( \chi_{\mathbb{R}_\pm} \), and provide a “global” description of quantum systems confined to either of the domains \( \mathbb{R}_- \) or \( \mathbb{R}_+ \) \[24\].

The other possibility is that the s.a. boundary conditions are interacting (\( a, c \in \mathbb{R}, b \in \mathbb{C} : (1 + b)(1 - b) - ac \neq 0 \), \[6\]):
\[
(5.5) \quad \begin{cases}
\psi(0^+) - \psi(0^-) = a (\psi'(0^+) + \psi'(0^-)) + b (\psi(0^+) + \psi(0^-)) \\
\psi'(0^+) - \psi'(0^-) = c (\psi(0^+) + \psi(0^-)) - b (\psi'(0^+) + \psi'(0^-))
\end{cases}
\]
in which case they relate the values of the wave function at the two sides of the boundary. The associated operator \( \tilde{L} \) cannot be written in the form \( \tilde{L}_- \oplus \tilde{L}_+ \). This kind of operators describe quantum systems formed by two sub-systems which are not isolated from each other (as in the case of separating boundary conditions) but instead display some sort of interaction at their common boundary.

In this section we construct a boundary pseudo potential formulation of both the separating and interacting operators \( \tilde{L} \). Using the prescription of Theorem \[5.3\] we get

**Corollary 5.4.** The separating Schrödinger operators can be written in the form:
\[ \tilde{L}^S : \mathcal{D}(\tilde{L}^S) \subseteq \mathcal{L}^2 \longrightarrow \mathcal{L}^2, \quad \tilde{L}^S = \tilde{H}_0 + \tilde{B}^S \]
\[ \mathcal{D}(\tilde{L}^S) = \mathcal{D}_\text{max}(\tilde{H}_0 + \tilde{B}^S) \]

where
\[ \tilde{B}^S : \mathcal{H}^2(\mathbb{R}_-) \oplus \mathcal{H}^2(\mathbb{R}_+) \longrightarrow \mathcal{D}' \]
\[ \tilde{B}^S = 2\tilde{\beta}D_x + \tilde{\beta}' + \tilde{F}^S \]
and
\[ \tilde{F}^S : \mathcal{H}^2(\mathbb{R}_-) \oplus \mathcal{H}^2(\mathbb{R}_+) \longrightarrow \mathcal{D}' \]
\[ \tilde{F}^S = \tilde{F}_+^{\pm} + D_x \tilde{F}_-^{\pm} \]
\[ \tilde{F}_\pm^S : \mathcal{H}^2(\mathbb{R}_-) \oplus \mathcal{H}^2(\mathbb{R}_+) \longrightarrow \mathcal{D}', \quad \tilde{F}_\pm^S = a_\pm \delta D_x - b_\pm \delta' \]

**Proof.** The proof follows from the Corollary \[5.2\] since \( \text{supp} \tilde{F}_\pm^S \psi \subseteq \{0\} \) and
\[ \tilde{F}_\pm^S \psi = 0 \iff \tilde{F}_\pm^S \psi = 0 \iff a_\pm \delta(x) \psi'_\pm(x) - b_\pm \delta(x) \psi_\pm(x) = 0 \]
\[ \iff a_\pm \psi'_\pm(0) - b_\pm \psi_\pm(0) = 0 \]
which shows that \( \text{Ker} \tilde{F}_\pm^S = \mathcal{D}(\tilde{L}^S) \).

The previous result is valid for general separating boundary conditions. For particular cases, the expression of \( \tilde{L}^S \) simplifies considerably. For instance, for Dirichlet boundary conditions (i.e. \( a_\pm = 0, b_\pm = 1 \)) the operator \( \tilde{L}^S \) becomes
\[ \tilde{L}^D = -D_x^2 + 2\tilde{\beta}D_x + \tilde{\beta}' + \tilde{F}_1^D \]
where
\[ \hat{F}^D_1 = -\hat{\delta}_+ - D_x\hat{\delta}_-. \]

As we have already pointed out, for each \( \hat{L} \), there are many possible choices of the operator \( \hat{F} \). To illustrate this let us introduce the operator \( \hat{\alpha} \) and its "derivatives" \( \hat{\alpha}^{(n)} \):
\[
\hat{\alpha}^{(n)} : \mathcal{A}^n \rightarrow \mathcal{A}^n, \quad \hat{\alpha} = \hat{\delta}^{(n)} + \hat{\delta}^{(n)}_-. \]

We then have, for instance

**Corollary 5.5.** The Schrödinger operator satisfying Dirichlet boundary conditions at both sides of the boundary at \( x = 0 \), i.e. \( \psi(0^\pm) = 0 \), can also be written as
\[
\hat{L}^D : \mathcal{D}(\hat{L}^D) \subseteq \mathcal{L}^2 \rightarrow \mathcal{L}^2, \quad \hat{L}^D = \hat{H}_0 + \hat{\alpha} - \hat{\beta}'
\]
\[ \mathcal{D}(\hat{L}^D) = \mathcal{D}_{\max}(\hat{H}_0 + \hat{\alpha} - \hat{\beta}') \]

which yields a boundary potential representation of \( \hat{L}^D \).

**Proof.** Let us define
\[
\hat{F}^D_2 : \mathcal{H}^2(\mathbb{R}_-) \oplus \mathcal{H}^2(\mathbb{R}_+) \rightarrow \mathcal{D}', \quad \hat{F}^D_2 = \hat{\alpha} - 2D_x\hat{\beta}.
\]

Then \( \text{supp} (\hat{F}^D_2 \psi) \subseteq \{0\}, \forall \psi \in \mathcal{D}(\hat{F}^D_2) \) and
\[
\hat{F}^D_2 \psi = 0 \iff \begin{cases} 
\delta(x)(\psi_+(0) + \psi_-(0)) = 0 \\
\delta'(x)(\psi_+(0) - \psi_-(0)) = 0
\end{cases} \iff \psi_+(0) = \psi_-(0) = 0.
\]

Hence, \( \text{Ker} \hat{F}^D_2 = \mathcal{D}(\hat{L}^D) \) and so \( \hat{F}^D_2 \) satisfies the two conditions of Corollary 5.2. The proof is concluded by
\[
\hat{H}_0 + 2\hat{\beta}D_x + \hat{\beta}' + \hat{F}^D_2 = \hat{H}_0 + \hat{\alpha} - \hat{\beta}'.
\]

\[ \square \]

Finally, let \( \hat{L}' \) be the Schrödinger operator satisfying the *interacting* boundary conditions at \( x = 0 \). A boundary potential representation of \( \hat{L}' \) can also be determined using the method of Theorem 5.3.

**Corollary 5.6.** The operators \( \hat{L}' \) admit the representation
\[
\hat{L}' : \mathcal{D}(\hat{L}') \subseteq \mathcal{L}^2 \rightarrow \mathcal{L}^2, \quad \hat{L}' = \hat{H}_0 + \hat{B}'
\]
\[ \mathcal{D}(\hat{L}') = \mathcal{D}_{\max}(\hat{H}_0 + \hat{B}') \]

where
\[
\hat{B}' : \mathcal{H}^2(\mathbb{R}_-) \oplus \mathcal{H}^2(\mathbb{R}_+) \rightarrow \mathcal{D}', \quad \hat{B}' = c\hat{\alpha} - b\hat{\alpha}D_x + aD_x\hat{\alpha}D_x + \tilde{b}D_x\hat{\alpha}.
\]

**Proof.** Setting \( \hat{F} = \hat{F}_1 + D_x\hat{F}_2 \) with
\[
\hat{F}_1 : \mathcal{H}^2(\mathbb{R}_-) \oplus \mathcal{H}^2(\mathbb{R}_+) \rightarrow \mathcal{D}', \quad \hat{F}_1 = c\hat{\alpha} - b\hat{\alpha}D_x - \tilde{b}D_x
\]
\[
\hat{F}_2 : \mathcal{H}^2(\mathbb{R}_-) \oplus \mathcal{H}^2(\mathbb{R}_+) \rightarrow \mathcal{D}', \quad \hat{F}_2 = a\hat{\alpha}D_x + \tilde{b}\hat{\alpha} - \tilde{b}
\]
we have supp $\hat{F}\psi \subseteq \{0\}$ for all $\psi \in \mathcal{D}(\hat{F})$, and
\[
\mathcal{D}(\hat{L}^I) = \text{Ker}(\hat{F}_1) \cap \text{Ker}(\hat{F}_2) = \text{Ker}(\hat{F}).
\]
It follows that
\[
\hat{L}^I = \hat{S}^* + \hat{F} = -D_x^2 + 2\hat{\beta}D_x + \hat{\beta} + c\hat{\alpha} - b\hat{\alpha}D_x - \hat{\beta}D_x + D_x(a\hat{\alpha}D_x + \hat{b}\hat{\alpha} - \hat{\beta})
= \hat{H}_0 + \hat{B}^I
\]
which concludes the proof. 

6. The $a\delta + b\delta'$ Potential

In this section we address a problem which, in some sense, is the inverse of the one studied in the previous section. We are given a singular boundary potential $B$ and the aim is to determine the explicit form of the operators $\hat{H}_0 + \hat{B}$, where $\hat{B}$ is a boundary operator associated with $B$.

The crucial point here is the definition of the association between $B$ and $\hat{B}$. Recall that the simplest definition $\hat{B} = B$ (where $\cdot$ is the standard product of a distribution by a test function; or some obvious extension of it) yields boundary operators with very restricted domains.

Other ways of implementing the boundary potential $B$ yield a richer structure. As we have already mentioned in the introduction, one possible interpretation of $\hat{H} = \hat{H}_0 + \hat{B}$ is that it stands for the norm resolvent limit of a sequence of operators $\hat{H}_n = \hat{H}_0 + \hat{B}_n$, where $\hat{B}_n = B_n$. and $B_n$ is a sequence of regular potentials such that $B_n \rightarrow B$ in $\mathscr{S}'$. The case $B = a\delta(x) + b\delta'(x)$, with $a, b \in \mathbb{R}$, has been extensively studied in the literature (see [30, 49] and the references therein). It turns out that for $B_n \rightarrow B = a\delta(x)$ in $\mathscr{S}'$, the norm resolvent limit of $\hat{H}_n$ is (for a large class of regular potentials $B_n$)

\[
\hat{H}_{a\delta} : \mathcal{D}(\hat{H}_{a\delta}) \subset \mathcal{L}^2 \rightarrow \mathcal{L}^2, \quad \psi \mapsto \hat{H}_{a\delta}\psi = \hat{S}^*\psi
\]
\[
\mathcal{D}(\hat{H}_{a\delta}) = \left\{ \psi \in \mathcal{H}^2(\mathbb{R}_-) \oplus \mathcal{H}^2(\mathbb{R}_+) : \begin{align*}
\psi(0^+) &= \psi(0^-) \\
\psi'(0^+) - \psi'(0^-) &= a\psi(0)
\end{align*} \right\}
\]
and is independent of the particular sequence $B_n$ such that $B_n \rightarrow a\delta(x)$.

The case is different if $B = b\delta'(x)$. Golovaty, Man’ko and Hryniv [28, 29, 30] and Zolotaryuk [49] determined families of sequences $B_n \rightarrow B$, displaying a single distributional limit $B = b\delta'(x)$ but yielding, in the norm resolvent sense, the family of limit operators:

\[
\hat{H}_{b\delta', \theta} : \mathcal{D}(\hat{H}_{b\delta', \theta}) \subset \mathcal{L}^2 \rightarrow \mathcal{L}^2, \quad \psi \mapsto \hat{H}_{b\delta', \theta}\psi = \hat{S}^*\psi
\]
\[
\mathcal{D}(\hat{H}_{b\delta', \theta}) = \{ \psi \in \mathcal{H}^2(\mathbb{R}_-) \oplus \mathcal{H}^2(\mathbb{R}_+) : \psi(0^+) = \theta\psi(0^-) \wedge \theta\psi'(0^+) = \psi'(0^-) \}
\]
where the parameter $\theta$ depends on the shape of the potentials $B_n$.

In this section we will study the Schrödinger operators $\widetilde{H} = \widetilde{H}_0 + \tilde{B}$, where $\tilde{B}$ is a boundary potential operator in $\mathcal{A}^1$:

\[
\tilde{B} = \psi * B_1 + B_2 * \psi
\]
with \(B_1, B_2 \in \mathcal{A}^1\). The operator \(\hat{B}\) provides a natural operator representation of the boundary potential \(B = B_1 + B_2 = a\delta(x) + b\delta'(x)\). Notice that \(\hat{B}\) is an extension of \(B\) to \(\mathcal{A}^1 \supset \mathcal{H}^2(\mathbb{R}_-) \oplus \mathcal{H}^2(\mathbb{R}_+)\).

In subsection 6.1, we obtain in Theorem 6.1 the explicit form of the operators \(\hat{H} = \hat{H}_0 + \hat{B}\). Then in Theorems 6.2, 6.4 and Corollary 6.3 we determine which operators \(\hat{H}\) are s.a. and conversely, which s.a. extensions of \(\hat{S}\) admit a boundary potential representation of the form \(\hat{H}\). We will see that the two sets of operators (the ones of the form \(\hat{H} = \hat{H}_0 + \hat{B}\), and the Schrödinger operators \(\hat{L}\)) have a large intersection, but do not coincide, nor one contains the other. In Theorems 6.2, 6.4 and Corollary 6.3 we also determine, for the s.a. case, the entire set of boundary potential operators \(\hat{B} \in \mathcal{A}^1\), that yield a single operator \(\hat{L}\). Finally, in Theorem 6.5, we calculate the sesquilinear form associated with \(\hat{H}\).

In subsection 6.2, we consider the particular cases \(B = a\delta(x)\) and \(B = b\delta'(x)\), and compare the results of the boundary operator formulations with the results of the norm resolvent approach (Corollaries 6.6, 6.7 and 6.8).

### 6.1. Schrödinger operators with boundary potentials

Let \(\hat{B} \in \mathcal{A}^1\) be of the form (1.3) with \(B_i = c_i\delta(x) + b_i\delta'(x), c_i, b_i \in \mathbb{C}, i = 1, 2\). Then:

\[
\hat{B} = c_1\delta_+ + c_2\delta_+ + b_1\delta_- + b_2\delta_+
\]

and \(\hat{B} \leftrightarrow B = c\delta(x) + b\delta'(x)\), where \(c = c_1 + c_2\) and \(b = b_1 + b_2\). The following Theorem completely characterizes the action and the domain of the operators \(\hat{H} = \hat{H}_0 + \hat{B}\).

**Theorem 6.1.** Let \(\hat{H} = \hat{H}_0 + \hat{B}\) where \(\hat{B}\) is given by (6.3). Then

(i) \(\hat{H} \subseteq \hat{S}^*\);

(ii) \(\psi \in \mathcal{D}_{\text{max}}(\hat{H}) \iff \psi \in \mathcal{H}^2(\mathbb{R}_-) \oplus \mathcal{H}^2(\mathbb{R}_+)\) and

\[
\begin{pmatrix}
-c_1 & -c_2 \\
(b_1 + 1) & (b_2 - 1)
\end{pmatrix}
\begin{pmatrix}
(b_1 - 1) \\
0
\end{pmatrix}
\begin{pmatrix}
\psi_-(0) \\
\psi^+_0
\end{pmatrix}
\begin{pmatrix}
\psi_+(0) \\
\psi^+_0
\end{pmatrix}
= 0
\]

where we wrote, as usual, \(\psi = \chi_{\mathbb{R}_-}\psi_+ + \chi_{\mathbb{R}_+}\psi_+, \psi_+ \in \mathcal{H}^2\).

**Proof.** Since \(\text{supp} \hat{B}\psi \subseteq \{0\}\) for all \(\psi \in \mathcal{D}(\hat{B}) = \mathcal{A}^1\), it follows from Theorem 6.3 that \(\hat{H} \subseteq \hat{S}^*\). Moreover, also from Theorem 6.3

\[
\mathcal{D}_{\text{max}}(\hat{H}) = \text{Ker} \hat{F} \cap (\mathcal{H}^2(\mathbb{R}_-) \oplus \mathcal{H}^2(\mathbb{R}_+))
\]

where \(\hat{F}\) is given by (4.3):

\[
\hat{F}\psi = -2\hat{D}_x\psi - \hat{\beta}\psi + \hat{B}\psi
\]

\[
= \delta(x) \left[ c_1\psi_+ + c_2\psi_+ + 2\psi_+ - 2\psi_+ \right] + \delta'(x) (b_1 + 1)\psi_+ + (b_2 - 1)\psi_+ .
\]
Since \( \delta'(x)\psi_\pm = D_x(\delta(x)\psi_\pm) - \delta(x)\psi'_\pm \), we easily get:

\[
\hat{F}\psi = \delta(x)\left[ c_1\psi_- + c_2\psi_+ - (b_1 - 1)\psi'_- - (b_2 + 1)\psi'_+ \right] + D_x\left[ \delta(x)((b_1 + 1)\psi_- + (b_2 - 1)\psi_+) \right]
\]

and so

\[
\hat{F}\psi = 0 \iff \begin{cases} c_1\psi_-(0) + c_2\psi_+(0) - (b_1 - 1)\psi'_-(0) - (b_2 + 1)\psi'_+(0) = 0 \\
(b_1 + 1)\psi_-(0) + (b_2 - 1)\psi_+(0) = 0
\end{cases}
\]

which is equivalent to the condition (6.3).

A natural question is then which operators \( \hat{H} = \hat{H}_0 + \hat{B} \) are s.a., and conversely, which s.a. restrictions of \( \hat{S}^* \) admit a boundary potential representation \( \hat{H}_0 + \hat{B} \) with \( \hat{B} \) of the form (6.3). We start by recalling that \( \hat{Z} \subseteq \hat{S}^* \) is s.a. iff \( D(\hat{Z}) \subseteq \mathcal{H}^2(\mathbb{R}_-) \oplus \mathcal{H}^2(\mathbb{R}_+) \) is characterized by two separating boundary conditions (6.4):

\[
(6.5) \begin{bmatrix} b_- & 0 & -a_- & 0 \\ 0 & b_+ & 0 & -a_+ \end{bmatrix} \begin{bmatrix} \psi_-(0) \\ \psi_+(0) \\ \psi'_-(0) \\ \psi'_+(0) \end{bmatrix} = 0 ; \quad (a_\pm, b_\pm) \in \mathbb{R}^2 \setminus \{(0, 0)\}
\]

in which case the operator is denoted by \( \hat{L}^S \) or, more explicitly, by \( \hat{L}^S(a_-, a_+, b_-, b_+) \).

Alternatively, \( \hat{Z} \) might satisfy two interacting boundary conditions of the form (6.5):

\[
(6.6) \begin{bmatrix} -c & -c & (b - 1) & (b + 1) \\ (\overline{b} + 1) & (\overline{b} - 1) & a & a \end{bmatrix} \begin{bmatrix} \psi_-(0) \\ \psi_+(0) \\ \psi'_-(0) \\ \psi'_+(0) \end{bmatrix} = 0
\]

\[a, c \in \mathbb{R} \quad b \in \mathbb{C} : (\overline{b} + 1)(1 - b) - ac \neq 0\]

in which case the operator \( \hat{Z} \) is denoted by \( \hat{L}^I \) or, more explicitly, by \( \hat{L}^I(a, b, c) \).

The two following theorems study the relation between the operators \( \hat{H} = \hat{H}_0 + \hat{B} \) and the s.a. operators \( \hat{L}^I \) and \( \hat{L}^S \).

**Theorem 6.2.** (1) Let \( \hat{H} = \hat{H}_0 + \hat{B} \) where \( \hat{B} \) is given by (6.3). For arbitrary \( (c_1, c_2, b_1, b_2) \in \mathbb{C}^4 \), the operator \( \hat{H} \) is an interacting s.a. Schrödinger operator \( \hat{L}^I(a, b, c) \subseteq \hat{S}^* \) iff either (1a) or (1b) holds true:

(1a) \( b_1 = \overline{b}_2, b_1 \neq \pm 1 \) and \( \text{Im} \left[ c_1(\overline{b}_1 - 1) - c_2(b_1 + 1) \right] = 0 \). In this case, \( \hat{H} = \hat{L}^I(a, b, c) \) with:

\[
(6.7) a = 0 \quad b = \frac{b_1 + \overline{b}_1}{b_1 - b_1 + 2} \quad c = \frac{2c_1(\overline{b}_1 - 1) - 2c_2(b_1 + 1)}{(\overline{b}_1 - b_1)^2 - 4}
\]

Alternatively:
(1b) \( b_1 + b_2 = 0, b_1 \neq \pm 1, \) and \( \text{Im} \left[ \frac{c_1 + c_2}{2(1 - b_1)} \right] = 0. \) In this case \( \hat{H} = \hat{L}^I_{(a,b,c)} \) with:

\[
(6.8) \quad a = 0 \quad , \quad b = 0 \quad , \quad c = \frac{c_1 + c_2}{2(1 - b_1)}. \]

(2) Conversely, \( \hat{L}^I_{(a,b,c)} \) admits a boundary potential representation of the form \( \hat{H}_0 + \hat{B} \), with \( \hat{B} \) given by (6.3), iff one of the following holds true:

(2a) \( a = 0, b + \bar{b} \neq 0 \) and \( b \neq \pm 1. \) Then \( \hat{B} \) has parameters satisfying the conditions:

\[
(6.9) \quad b_1 = \frac{2b + b - \bar{b}}{b + \bar{b}}, \quad b_2 = \bar{b}_1
\]

and

\[
(6.10) \quad (c_1, c_2) = (k_1/X_1, k_2/X_2) \quad , \quad k_1, k_2 \in \mathbb{C}: k_1 + k_2 = c
\]

where

\[
(6.11) \quad X_1 = -\frac{b + \bar{b}}{4} + \frac{b + \bar{b}}{4b}, \quad X_2 = \frac{b + \bar{b}}{4} + \frac{b + \bar{b}}{4b}.
\]

(2b) \( a, b = 0. \) In this case the parameters \( (c_1, c_2, b_1, b_2) \in \mathbb{C}^4 \) satisfy the conditions:

\[
(6.12) \quad b_1 \in \mathbb{C}\setminus\{-1, 1\} \quad , \quad b_2 = -b_1 \quad , \quad c_1 + c_2 = 2c(1 - b_1).
\]

Proof. The operators \( \hat{L}^I \) and \( \hat{H} = \hat{H}_0 + \hat{B} \) are both restrictions of \( \hat{S}^* \). Hence, \( \hat{L}^I = \hat{H} \) iff their domains are the same. This is true iff the boundary conditions (6.4) and (6.6) are equivalent.

Since the two equations in (6.6) are linearly independent, the two sets of boundary conditions (6.4) and (6.6) are equivalent iff exists \( \lambda_1, \lambda_2, \mu_1, \mu_2 \in \mathbb{C} \) such that

\[
(6.13) \quad \begin{cases} 
\lambda_1(-c_1, -c_2, b_1 - 1, b_2 + 1) + \mu_1(b_1 + 1, b_2 - 1, 0, 0) = (-c, -c, b - 1, b + 1) \\
\lambda_2(-c_1, -c_2, b_1 - 1, b_2 + 1) + \mu_2(b_1 + 1, b_2 - 1, 0, 0) = (\bar{b} + 1, \bar{b} - 1, a, a)
\end{cases}
\]

From the second equation, if \( a \neq 0 \) then \( b_1 - 1 = b_2 + 1. \) Using the first equation this implies that \( b - 1 = b + 1, \) which is not possible. Hence, \( a = 0 \) and consequently \( \lambda_2 = 0. \) From the condition on the parameters \( a, b, c \) (cf. eq.(6.6)), we also have:

\[
(1 + b)(1 - b) - ac \neq 0 \implies b \neq \pm 1.
\]

The system (6.12) then reduces to the simpler systems:

\[
(6.13) \quad \begin{cases} 
\lambda_1 c_1 - \mu_1 (b_1 + 1) = c \\
\lambda_1 c_2 - \mu_1 (b_2 - 1) = c
\end{cases}
\]
and
\[
\begin{align*}
\lambda_1(b_1 - 1) &= b - 1 \\
\lambda_1(b_2 + 1) &= b + 1
\end{align*}
\quad \wedge \quad
\begin{align*}
\mu_2(b_1 + 1) &= \overline{b} + 1 \\
\mu_2(b_2 - 1) &= \overline{b} - 1
\end{align*}
\]

Since \( b \neq \pm 1 \), the two latter systems imply
\[
\lambda_1, \mu_2 \neq 0, \quad b_1, b_2 \neq \pm 1.
\]

Adding and subtracting the two equations in the two latter systems:

(6.14)
\[
\begin{align*}
\frac{b}{\overline{b}} &= \frac{\frac{1}{2} \lambda_1(b_1 + b_2)}{\frac{1}{2} \mu_2(b_1 + b_2)} \\
\lambda_1(b_2 - b_1 + 2) &= 2 \\
\mu_2(b_2 - b_1 - 2) &= -2
\end{align*}
\]

Hence, \( b_2 - b_1 \neq \pm 2 \) and

(6.15)
\[
\begin{align*}
b &= \frac{b_1 + b_2}{b_2 - b_1 + 2}, \\
\overline{b} &= -\frac{b_1 + b_2}{b_2 - b_1 - 2}.
\end{align*}
\]

Then
\[
\frac{b_1 + b_2}{b_2 - b_1 + 2} = -\frac{\overline{b}_1 + \overline{b}_2}{\overline{b}_2 - \overline{b}_1 - 2} \iff \begin{cases} b_2 \overline{b}_2 = b_1 \overline{b}_1 \\ \overline{b}_1 + \overline{b}_2 = b_1 + b_2 \end{cases}
\]

and so:

(6.16)
\[
b_1 = \overline{b}_2 \quad \lor \quad b_1 = -b_2.
\]

In the first case, from (6.15) and (6.14):

(6.17)
\[
b = \frac{b_1 + \overline{b}_1}{b_1 - b_1 + 2}, \quad \lambda_1 = \frac{2}{\overline{b}_1 - b_1 + 2}, \quad \mu_2 = \frac{-2}{b_1 - b_1 - 2}
\]

and in the second:

(6.18)
\[
b = 0, \quad \lambda_1 = \frac{1}{b_1 - b_1 + 1}, \quad \mu_2 = \frac{1}{b_1 + 1}.
\]

The previous formulas (6.16), (6.17) and (6.18) almost complete the proof of the statements (1a) and (1b). It remains only to proof that, in the two cases (6.16), for arbitrary \((c_1, c_2)\) there exists \(\mu_1\) and \(c\) such that (6.13) holds.

We start by considering the first case \(b_1 = \overline{b}_2\). Subtracting the two equations in (6.13), and taking into account (6.17), we get:

\[
\mu_1 = \frac{2(c_2 - c_1)}{(b_1 - b_1)^2 - 4}.
\]

Replacing this result in (6.13), we find
\[
c = \frac{2c_1(\overline{b}_1 - 1) - 2c_2(b_1 + 1)}{(b_1 - b_1)^2 - 4}.
\]

Since \(c\) is real, the triple \((c_1, c_2, b_1)\) has to satisfy the condition
\[
Im \left[ c_1(\overline{b}_1 - 1) - c_2(b_1 + 1) \right] = 0.
\]

This concludes the proof of statement (1a) in the Theorem.
We proceed by considering the second case in (6.16): \( b_1 + b_2 = 0 \) and \( b_1 \neq \pm 1 \). Adding and subtracting the two equations in (6.13), and taking into account (6.18), we find

\[
\mu_1 = \frac{c_2 - c_1}{2(b_1^2 - 1)}, \quad c = \frac{c_1 + c_2}{2(1 - b_1)}
\]

and, since \( c \) is real, \( c_1, c_2 \) should satisfy the condition

\[
\text{Im} \left[ \frac{c_1 + c_2}{2(1 - b_1)} \right] = 0.
\]

This concludes the proof of the statement (1b) of the Theorem.

It remains to prove the statement (2). This basically amounts to invert the equations (6.7) and (6.8) (in the two cases (1a) and (1b), respectively) and determine for which values of \((a, b, c)\) is this possible.

We start by considering the case (1a): \( b_1 = \overline{b}_2, b_1 \neq \pm 1 \). It follows from (6.15) that

\[
(b_1 - b_1 + 2)b = (b_1 - \overline{b}_1 + 2)\overline{b} \iff (b_1 - b)(b + \overline{b}) = 2(b - \overline{b}).
\]

Then we have three possibilities:

(i) If \( b + \overline{b} = 0 \) then also \( \overline{b} - b = 0 \) and so \( b = 0 \). Hence, \( b \) cannot be pure imaginary number.

(ii) If \( b = 0 \) then, from (6.14), it follows that \( b_1 + b_1 = 0 \), and so \( b_1 \) is an arbitrary imaginary number. Hence, \( b_2 = \overline{b}_1 = -b_1 \), and this is the case (1a), which will be consider below.

(iii) Finally, if \( b + \overline{b} \neq 0 \) then \( b_1 - b_1 = 2\frac{b_2 - b}{b_2 + b} \). Substituting in (6.17) (which is valid in the first case \( b_1 = \overline{b}_2 \)), we get \( \overline{b}_1 + b_1 = 4\frac{b_2}{b_2 + b} \). Then

\[
b_1 = \frac{2b\overline{b} + b - \overline{b}}{b + \overline{b}}
\]

which proves (6.9).

We then consider the equation for \( c \) in (6.7). Using (6.19), we get

\[
c = \frac{b + \overline{b}}{2} \left[ \frac{c_2 - c_1}{b_2} + \frac{c_1 + c_2}{2b} \right] = c_1 X_1 + c_2 X_2
\]

where \( X_1, X_2 \) are given by (6.11). Since \( b + \overline{b} \neq 0 \) and \( b \neq \pm 1 \), then also \( X_1, X_2 \neq 0 \). Hence, given \( b \) and \( c \), we immediately realize that the solutions \((c_1, c_2)\) of (6.20) can be written in the form (6.10), which concludes the proof of (2a).

We proceed with the study of the case (1b): \( b_1 + b_2 = 0 \), \( b_1 \neq \pm 1 \). This always yields \( b = 0 \). It follows immediately from (6.8) that, for an arbitrary \( c \in \mathbb{R} \), the operator \( \tilde{L}_{(0,0,c)}^I \) admits a boundary potential representation with \( b_1 \) an arbitrary number in \( \mathbb{C} \setminus \{-1, 1\} \), \( b_2 = -b_1 \) and \( c_1, c_2 \in \mathbb{C} \) such that:

\[
c_1 + c_2 = 2c(1 - b_1).
\]
This proves (2b).

If the parameters of the boundary potential $\hat{B}$ are real, the previous result considerably simplifies:

**Corollary 6.3.** Let $\hat{H} = \hat{H}_0 + \hat{B}$ where $\hat{B}$ is given by (6.3) with $(c_1, c_2, b_1, b_2) \in \mathbb{R}^4$. Then $\hat{H} = \hat{L}^I(a,b,c)$ for some $a, c \in \mathbb{R}$ and $b \in \mathbb{C}$ iff:

$$b_1 = \pm b_2, \quad b_1 \neq \pm 1, \quad a = 0, \quad b = \frac{b_1 + b_2}{2}.$$  

In addition:

(i) If $b_1 = b_2$ then $c = \frac{c_1(1-b_1)+c_2(1+b_1)}{2}$. 

(ii) If $b_1 = -b_2$ then $c = \frac{c_1+c_2}{2(1-b_1)}$.

**Proof.** The proof is a direct consequence of (1a) and (1b) in Theorem 6.2.

We proceed with the separating case:

**Theorem 6.4.** Let $(a_-, b_-), (a_+, b_+) \in \mathbb{R}^2 \setminus \{(0,0)\}$. Then $L^S_{(a_-, a_+, b_-, b_+)} = \hat{H}_0 + \hat{B}$ for some boundary potential operator of the form (6.3) iff one of the following holds true:

(1a) $a_- = 0, a_+ \neq 0, b_- \neq 0$. In this case $b_1 = b_2 = 1$, $c_1$ is arbitrary and $c_2 = 2b_+/a_+$. This corresponds to Dirichlet boundary conditions at $x = 0^-$ and Robin (or Neumann, if $b_+ = 0$) boundary conditions at $x = 0^+$.

(1b) $a_- \neq 0, a_+ = 0, b_+ \neq 0$. In this case $b_1 = b_2 = -1$, $c_1 = -2b_-/a_-$. This corresponds to Dirichlet boundary conditions at $x = 0^+$ and Robin (or Neumann, if $b_- = 0$) boundary conditions at $x = 0^-$. 

(1c) $a_- = a_+ = 0$ and $b_-, b_+ \neq 0$. In this case $b_1 = 1, b_2 = -1$ and $c_1 + c_2 \neq 0$. This corresponds to Dirichlet boundary conditions at $x = 0^-$ and $x = 0^+$.

**Proof.** The two sets of boundary conditions (6.4) and (6.5) are equivalent iff exists $\lambda_1, \lambda_2, \mu_1, \mu_2 \in \mathbb{C}$ such that:

$$\begin{cases}
\lambda_1(-c_1, -c_2, b_1 - 1, b_2 + 1) + \mu_1(b_1 + 1, b_2, b_1 - 1, 0, 0) = (b_-, 0, -a_-, 0) \\
\lambda_2(-c_1, -c_2, b_1 - 1, b_2 + 1) + \mu_2(b_1 + 1, b_2, b_1 - 1, 0, 0) = (0, b_+, 0, -a_+)
\end{cases}$$

This implies:

$$\begin{cases}
\lambda_1(b_2 + 1) = 0 \\
\lambda_1(b_1 - 1) = -a_- \\
\lambda_2(b_1 - 1) = 0 \\
\lambda_2(b_2 + 1) = -a_+
\end{cases}$$

From these systems it follows that:

$$a_+ \neq 0 \implies b_2 + 1 \neq 0 \implies \lambda_1 = 0 \implies a_- = 0$$
and likewise \( a_- \neq 0 \Rightarrow a_+ = 0 \). Hence,

\[
a_+ = 0 \quad \vee \quad a_- = 0.
\]

We then have three distinct cases:

(i) \( a_- = 0 \wedge a_+ \neq 0 \). Then also \( b_- \neq 0 \). From (6.23), \( \lambda_2 \neq 0 \), \( b_2 + 1 \neq 0 \) and so \( \lambda_1 = 0 \) and \( b_1 = 1 \). Hence, from (6.22),

\[
\mu_1(b_1 + 1, b_2 - 1) = (b_-, 0)
\]

and since \( b_- \neq 0 \), we have \( \mu_1 \neq 0 \) and so \( b_2 = 1 \). Substituting in (6.23), we also find \( \lambda_2 = -a_+/2 \).

From (6.22), we then have

\[-\lambda_2 c_2 + \mu_2(b_2 - 1) = b_+ \Rightarrow \lambda_2 c_2 = -b_+ \Rightarrow c_2 = 2b_+/a_+.
\]

This concludes the proof of the case (1a).

(ii) \( a_- \neq 0 \wedge a_+ = 0 \). This case yields the conditions (1b). The proof follows exactly the same steps as in (i).

(iii) \( a_+ = a_- = 0 \) and \( b_-, b_+ \neq 0 \). From (6.23), we get \( \lambda_1 = \lambda_2 = 0 \) or \( (b_1, b_2) = (1, -1) \). If \( \lambda_1 = \lambda_2 = 0 \) then, from (6.22),

\[
\mu_1(b_1 + 1, b_2 - 1) = (b_-, 0) \quad \wedge \quad \mu_2(b_2 - 1) = b_+
\]

and the first equation implies \( \mu_1 \neq 0 \) and \( b_2 = 1 \), while the second yields \( b_2 \neq 1 \). Hence, \( \lambda_1, \lambda_2 \neq (0, 0) \) and so \( b_1 = 1 \) and \( b_2 = -1 \). Then, from (6.22)

\[
\begin{cases}
2\mu_1 - \lambda_1 c_1 = b_- \\
-2\mu_1 - \lambda_1 c_2 = 0
\end{cases} \quad \Rightarrow \quad c_1 + c_2 \neq 0
\]

which concludes the proof of the case (1c).

\[\square\]

We now determine the sesquilinear form associated with \( \hat{H} \).

**Theorem 6.5.** Let \( \hat{H} = \hat{H}_0 + \hat{B} \), where \( \hat{B} \) is the boundary potential given by (6.3). The sesquilinear form associated with \( \hat{H} \) is given in the domain \( \mathcal{D}(h) = \mathcal{H}^2(\mathbb{R}_-) \oplus \mathcal{H}^2(\mathbb{R}_+) \) by

\[
(6.24) \quad h(\psi, \phi) = (\psi_+', \phi_+')_{L^2(\mathbb{R}_-)} + (\psi_+', \phi_+')_{L^2(\mathbb{R}_+)} + b_B(\psi, \phi)
\]

where the boundary term is

\[
(6.25) \quad b_B(\psi, \phi) = \overline{\phi_+(0)} \psi_+(0) - \overline{\phi_-(0)} \psi_-(0).
\]

If \( \hat{H} \) is s.a. then \( h \) can be extended to \( \mathcal{D}(h) = \mathcal{H}^1(\mathbb{R}_-) \oplus \mathcal{H}^1(\mathbb{R}_+) \). In this case, \( h \) is also given by (6.24), but the boundary term \( b_B \) depends on the particular operator \( \hat{H} \):
(1) If $\tilde{H}$ is a s.a interacting Schrödinger operator then the parameters defining $\tilde{B}$ satisfy (cf. Theorem 6.2) $b_1 \neq \pm 1$ and $b_1 = \overline{b}_2$ or $b_1 + b_2 = 0$. In this case:

$$
(6.26) \quad b_B(\psi, \phi) = \begin{cases} 
\frac{c_1}{1-b_1} \overline{\phi}_-(0) \psi_-(0) + \frac{c_2}{1+b_1} \overline{\phi}_+(0) \psi_+(0), & b_1 = \overline{b}_2 \\
\frac{c_1}{1-b_1} \overline{\phi}_-(0) \psi_-(0) + \frac{c_2}{1+b_1} \overline{\phi}_+(0) \psi_+(0), & b_1 = -b_2
\end{cases}
$$

(2) If $\tilde{H}$ is a s.a. separating Schrödinger operator then the parameters of $\tilde{B}$ satisfy (cf. Theorem 6.4) $b_1 = b_2 = \pm 1$ or $b_1 = -b_2 = 1$ and $c_1 + c_2 \neq 0$. In this case:

$$
(6.27) \quad b_B(\psi, \phi) = \begin{cases} 
\frac{c_1}{2} \overline{\phi}_+(0) \psi_+(0), & b_1 = b_2 = 1 \\
0, & b_1 = -b_2 = 1, c_1 + c_2 \neq 0 \\
\frac{c_1}{2} \overline{\phi}_-(0) \psi_-(0), & b_1 = b_2 = -1
\end{cases}
$$

Proof. The sesquilinear form generated by $\tilde{H}$ is by definition

$$
h(\psi, \phi) = \left( \tilde{H} \psi, \phi \right)_{L^2}, \quad \mathcal{D}(h) = \mathcal{D}_{max}(\tilde{H})
$$

where $(\ , \ )_{L^2}$ is the standard inner product in $L^2$. Since $\tilde{H} \subseteq \hat{S}^*$

$$
h(\psi, \phi) = \left( \hat{S}^* \psi, \phi \right)_{L^2} = - (\psi''_+, \phi_-)_{L^2(\mathbb{R}_-)} - (\psi''_+, \phi_+)_{L^2(\mathbb{R}_+)}
$$

for all $\psi, \phi \in \mathcal{D}_{max}(\tilde{H})$. Integrating by parts, we immediately obtain (6.24) and (6.25). This expression is well-defined in $D(h) = \mathcal{H}^2(\mathbb{R}_-) \oplus \mathcal{H}^2(\mathbb{R}_+)$. We now consider the case where $\tilde{H}$ is a s.a interacting Schrödinger operator. In this case, the parameters in $\tilde{B}$ satisfy (cf. Theorem 6.2) $b \neq \pm 1$.

From (6.25) and using the boundary conditions (6.4), we get:

$$
(6.28) \quad b_B(\psi, \phi) = \frac{\overline{\phi}_-(0)}{1-b_2} \left( (1+b_1) \psi'_+(0) + (\overline{b}_2 - 1) \psi'_-(0) \right).
$$

We now set $b_1 = \overline{b}_2$ (the case (1a) in Theorem 6.2). Using again the boundary conditions (6.4), we obtain from (6.28)

$$
b_B(\psi, \phi) = \frac{\overline{\phi}_-(0)}{1-b_1} \left( c_1 \psi_-(0) + c_2 \psi_+(0) \right).
$$

Finally, using again (6.4), we obtain the first formula in (6.26).

The other case in Theorem 6.2 is $b_1 = -b_2$. Substituting in (6.28) and proceeding as in the first case, we obtain the second formula in (6.26).

Finally, we consider the case where $\tilde{H}$ is a separating Schrödinger operator. Then the parameters in $\tilde{B}$ satisfy (cf. Theorem 6.4) $b_1 = b_2 = \pm 1$ or $b_1 = -b_2 = 1$ and $c_1 + c_2 \neq 0$.

If $b_1 = b_2 = 1$ then from Theorem 6.4 we have Dirichlet boundary conditions at $x = 0^-$, and Robin boundary conditions at $x = 0^+$ (the latter are of the form $\psi'_+(0) = c_2 \psi_+(0)/2$). Substituting these conditions in (6.25), we
obtain the first formula in (6.27). The other two cases in (6.27) are proved exactly in the same way.

6.2. The \( \delta \) and \( \delta' \) potentials: Norm resolvent versus boundary operator formulations. We now consider the case of a \( \delta \) and a \( \delta' \) potential operator and a \( \delta' \) pseudo potential operator and determine the explicit form of the corresponding Schrödinger operators (Corollaries 6.6, 6.7 and 6.8). We will see that these Schrödinger operators are closely related to the norm resolvent limit operators (6.1) and (6.2).

Let \( B = b \delta' \). A family of boundary potential operators associated with \( B \) is

\[
(6.29) \quad \hat{B}_{\delta',1} \psi = c \delta'(x) \ast \psi + d \psi \ast \delta'(x) = (c \delta'_+ + d \delta'_-) \psi , \quad c + d = b
\]

which is well-defined for all \( \psi \in A^1 \) and satisfies

\[
\hat{B}_{\delta',1} \psi = b \delta'(x) \cdot \psi , \quad \psi \in C^1.
\]

Hence, the operators \( \hat{B}_{\delta',1} \) are extensions of \( b \delta'(x) \cdot \) to the space \( A^1 \supset C^1 \). As usual, there are many possible extensions. A more general family of boundary operators associated with \( B = (c + d) \delta'(x) \) is

\[
(6.30) \quad \hat{B}_{\delta',2} = c \delta'_+ + d \delta'_- + e D_x (\delta_+ - \delta_-) + f (\delta_+ - \delta_-) D_x , \quad c, d, e, f \in \mathbb{R}.
\]

These are boundary pseudo potential operators, all of which are also well-defined on \( A^1 \) and satisfy \( \hat{B}_{\delta',2} \psi = (c + d) \delta'(x) \cdot \psi \) for all \( \psi \in C^1 \).

Finally, let us introduce the operators

\[
(6.31) \quad \hat{B}_\delta = c \delta_+ + d \delta_- ,
\]

which are boundary potential operators associated with the \( \delta \)-potential \( B = (c + d) \delta(x) \).

The boundary operators \( \hat{B}_\delta = c \delta_+ + d \delta_- \) and \( \hat{B}_{\delta',1} = c \delta'_+ + d \delta'_- \) are particular examples of operators of the form (6.3). We then have

**Corollary 6.6.** The operator \( \hat{H} = -D_x^2 + \hat{B}_\delta \) is given explicitly by (6.1) with \( a = c + d \).

*Proof.* The boundary operator \( \hat{B}_\delta \) is of the form (6.3) with \( c_1 = d, c_2 = c \) and \( b_1 = b_2 = 0 \). Hence, from Theorem 6.1 \( \hat{H} \subseteq \hat{S}^* \) and \( D_{\max}(\hat{H}) \) is characterized by the two boundary conditions (6.4):

\[
\begin{aligned}
- d \psi_-(0) - c \psi_+(0) - \psi'_-(0) + \psi'_+(0) &= 0 \\
\psi_-(0) - \psi_+(0) &= 0
\end{aligned}
\]

We conclude that \( \hat{H} \) is exactly the operator (6.1) with \( a = c + d \). \( \square \)

The \( \delta' \)-potential is more subtle. We have
Corollary 6.7. Let \( \hat{H} = -D_x^2 + \hat{B}_{\delta'}, \) where \( \hat{B}_{\delta'} = c\delta'_+ + d\delta'_-, \) \( c, d \in \mathbb{R} \). Then

1. For \( c = d \neq 1 \) the operator \( \hat{H} \) is given explicitly by (6.2), with \( \theta = \frac{c+1}{1-c} \).
2. Conversely, all operators \( \hat{H}_{\delta', \theta} \), \( \theta \neq -1 \) given by (6.2) can be written in the form \( \hat{H} = -D_x^2 + \hat{B}_{\delta'}, \) with \( c = d = \frac{\theta-1}{\theta+1} \).
3. The operator \( \hat{H} \) is of the form (6.2) only for \( c = d \neq 1 \) and for \( c = -d \notin \{-1, 1\} \). The family \( c = -d \in \mathbb{R}\backslash\{-1, 1\} \) generates the single operator \( \hat{H}_{b_{\delta'}, \theta} \) with \( \theta = 1 \).

Proof. (1) Since \( \hat{B}_{\delta'} \) is of the form (6.3) with \( c_1 = c_2 = 0 \) and \( b_1 = d, \) \( b_2 = c \), it follows from Theorem 6.1 that \( \hat{H} \subseteq \hat{S}^* \) and that \( D_{\max}(\hat{H}) \) satisfies the boundary conditions (6.4):

\[
\begin{cases}
(d-1)\psi'_-(0) + (c+1)\psi'_+(0) = 0 \\
(d+1)\psi'_-(0) + (c-1)\psi'_+(0) = 0
\end{cases}
\]

For \( c = d \neq 1 \) we recover the boundary conditions that characterize the domain of the operators \( \hat{H}_{b_{\delta'}, \theta} \) (given by (6.2))

\[ \psi_+(0) = \theta \psi_-(0) \quad \text{and} \quad \theta \psi'_+(0) = \psi'_-(0), \quad \theta \in \mathbb{R}\backslash\{-1\} \]

with \( \theta = \frac{c+1}{1-c} \).

(2) All the values \( \theta \in \mathbb{R}\backslash\{-1\} \) can be obtained from suitable values of \( c \)

\[ \theta = \frac{c+1}{1-c} \iff c = \frac{\theta-1}{\theta+1}. \]

The case \( \theta = -1 \) cannot be generated by the family of boundary operators \( \hat{B}_{\delta'}, \) even if we consider the more general case \( c \neq d \). This can be easily realized from equation (6.32).

(3) For \( c = -d \in \mathbb{R}\backslash\{-1, 1\} \), eq.(6.32) yields the boundary conditions that correspond to the case \( \theta = 1 \). Finally, when \( c = 1 \) or \( d = 1 \), the boundary conditions (6.32) are obviously not of the form (6.2) and when \( c \neq \pm d \) (for \( c, d \neq 1 \)) the operators \( -D_x^2 + \hat{B}_{\delta'} \) are also not of the form (6.2) since

\[ \frac{d+1}{1-c} = \frac{1+c}{1-d} \iff c = \pm d. \]

We conclude that for each value of \( b = c + d \) the family of operators \( \hat{H} = -D_x^2 + \hat{B}_{\delta'} \) generates one, and only one, operator of the form (6.2). Hence, there is no hidden structure in the \( \delta' \) potential operators \( \hat{B}_{\delta'} \). This situation changes when we consider the more general family \( \hat{B}_{\delta,2} \). The extra terms, which yield a zero contribution when acting on smooth functions, generate for each value of \( c + d \) the entire family of operators (6.2). This is proved in the next corollary.
Corollary 6.8. Let \( \hat{H} = -D_x^2 + \hat{B}_{\theta,2} \) where \( \hat{B}_{\theta,2} \) is given by eq. (6.30)

\[
\hat{B}_{\theta,2} = c\delta_+ + d\delta_- + eD_x(\delta_+ - \delta_-) + f(\delta_+ - \delta_-)D_x
\]

We then have:

1. For \( c = d \) and \( e = f \neq 1 - c \), the operator \( \hat{H} \) is given explicitly by (6.2) with \( \theta = \frac{1 - c}{1 + e} \).

2. Each family of operators of the form \( \hat{H} = -D_x^2 + \hat{B}_{\theta,2} \), obtained by fixing the value of \( c = d \neq 0 \) and letting \( e = f \in \mathbb{R} \setminus \{1 - c\} \), contains all the operators (6.2) with \( \theta \in \mathbb{R} \setminus \{1\} \).

3. All the operators \( \hat{H} = -D_x^2 + \hat{B}_{\theta,2} \) with \( c = d = 0 \) and \( e = f \in \mathbb{R} \setminus \{1\} \) are given explicitly by the operator \( \hat{H}_{b\delta',\theta} \) with \( \theta = 1 \).

Proof. Since \( \text{supp} \hat{B}_{\theta,2} \psi \subseteq \{0\} \) for all \( \psi \in \mathcal{D}(\hat{B}_{\theta,2}) \), it follows from Theorem 4.3 that \( \hat{H} \subseteq \hat{S}^* \). Moreover, also from Theorem 4.3, \( \psi \in \mathcal{D}_{\max}(\hat{H}) \) iff 

\[
\psi \in \mathcal{H}^2(\mathbb{R}_-) \oplus \mathcal{H}^2(\mathbb{R}_+) \quad \text{and} \quad \hat{F}_{\theta,2} \psi = 0.
\]

The operator \( \hat{F}_{\theta,2} \) is

\[
\hat{F}_{\theta,2} = -2\beta D_x - \beta' + c\delta_+ + d\delta_- + eD_x\beta + f\beta D_x
\]

and so,

\[
\hat{F}_{\theta,2} \psi = 0 \iff D_x [\delta(x)(c\psi_+ + d\psi_- + (e - 1)(\psi_+ - \psi_-))] - \delta(x) [c\psi'_+ + d\psi'_- - (f - 1)(\psi'_+ - \psi'_-)] = 0
\]

\[
\iff \begin{cases}
(1 - c - e)\psi_+(0) = (1 + d - e)\psi_-(0) \\
(1 + c - f)\psi'_+(0) = (1 - d - f)\psi'_-(0)
\end{cases}
\]

For \( c = d \) and \( e = f \) we get

\[
\begin{cases}
(1 - e - c)\psi_+(0) = (1 - e + c)\psi_-(0) \\
(1 - e + c)\psi'_+(0) = (1 - e - c)\psi'_-(0)
\end{cases}
\]

For \( e \neq 1 - c \) we recover the boundary conditions for the operators \( \hat{H}_{b\delta',\theta} \) with \( \theta = \frac{1 - e + c}{1 - e - c} \). This proves (1).

2. This statement is easily proved by noticing that, for arbitrary fixed \( c \neq 0 \) and \( \theta \neq 1 \),

\[
\theta = \frac{1 - e + c}{1 - e - c} \iff e = \frac{\theta(c - 1) + 1 + c}{1 - \theta}
\]

and that the solution of the equation satisfies \( e \neq 1 - c \).

3. It follows directly from (6.33) that for \( c = d = 0 \) and \( e = f \), we have \( \theta = \frac{1 - e}{1 - e} = 1 \), for all values of \( e \neq 1 \). \( \square \)

Finally, we remark that a boundary pseudo potential operator formulation of the \( \delta \)-potential (of the form (6.30)) but for the \( \delta \)-potential yields exactly the Schrödinger operators that were determined in Corollary 6.6. This can
be easily realized by reproducing the calculations of Corollary 6.8 for the new boundary operators. Hence, this more general formulation of the $\delta$-potential does not display an "inner structure" as the one of the $\delta'$-potential. This agrees with the results of the norm resolvent limit formulation (6.1).

Acknowledgements. The authors would like to thank the anonymous referee for several insights and useful suggestions.

Nuno Costa Dias and João Nuno Prata have been supported by the research grant PTDC/MAT-CAL/4334/2014 of the Portuguese Science Foundation.

Cristina Jorge was supported by the PhD grant SFRH/BD/85839/2012 of the Portuguese Science Foundation.

References

[1] S. Albeverio, C. Cacciapuoti, D. Finco: Coupling in the singular limit of thin quantum waveguides, *J. Math. Phys.* **48** (3) (2007), 032103, 21pp.

[2] S. Albeverio, F. Gesztesy, R. Högh-Krohn, H. Holden: *Solvable Models in Quantum Mechanics*, 2nd ed., (AMS, Chelsea, 2005).

[3] S. Albeverio, F. Gesztesy, R. Högh-Krohn: The low energy expansion in nonrelativistic scattering theory, *Ann. Inst. H. P.* **A37** (1982) 1–28.

[4] S. Albeverio, P. Kurasov, *Singular perturbations of differential operators and solvable Schrödinger type operators*, (Cambridge University Press, 2000).

[5] S. Albeverio, V. Koshmanenko: Singular rank one perturbations of selfadjoint operators and Krein theory of selfadjoint extensions, *Potential Anal.* **11** (1999) 279-287.

[6] S. Albeverio, L. Nizhnik: Approximation of general zero-range potentials, *Ukrainian Math. J.* **52** no. 5 (2000) 664-672.

[7] S. Albeverio, R. Högh-Krohn: Point interactions as limits of short range interactions, *J. Operator Theory* **6** (1981) 313-339.

[8] S. Albeverio, F. Gesztesy, R. Högh-Krohn, W. Kirsch: On point interactions in one dimension, *J. Operator Theory* **12** (1984) 101-126.

[9] A. Antonevich: The Schrödinger equation with point interactions in an algebra of new generalized functions. In: *Nonlinear theory of generalized functions* Chapman and Hall, *Research notes in mathematics series*, **401** (1999).

[10] J. Avron, P. Exner, Y. Last: Periodic Schrödinger-operators with large gaps and Wannier-Stark ladders, *Phys. Rev. Lett.* **72** (1994) 896–899.

[11] F.A. Berezin, I.D. Fadeev: Remark on the Schrödinger equation with singular potential, *Dokl. Akad. Nauk. SSSR* **137** (1961) 1011.

[12] J.F. Brasche, R. Figari, A. Teta: Singular Schrödinger operators as limits of point interaction Hamiltonians, *Potential Analysis* **8** (1998) 163-178.

[13] D. Bollé, F. Gesztesy, M. Klaus: Scattering theory for one-dimensional systems with $\int dxV(x) = 0$, *J. Math. Anal. Appl.* **122** (1987) 496–518.

[14] D. Bollé, F. Gesztesy, S. Wilk: A complete treatment of low-energy scattering in one dimension, *J. Operator Theory* **13** (1985) 3–31.

[15] C. Cacciapuoti, P. Exner: Nontrivial edge coupling from a Dirichlet network squeezing: the case of a bent waveguide, *J. Phys. A: Math. Theor.* **40** (26) (2007) F511-F523.

[16] C. Cacciapuoti, D. Finco: Graph-like models for thin waveguides with robin boundary conditions, *Asymptotic Analysis* **70** (3-4) (2010) 199-230.

[17] T. Cheon, P. Exner, P. Seba: Wave function shredding by sparse quantum barriers, *Phys. Lett. A* **277** (2000) 1–6.
P. Christiansen, H. Arnbak, A. Zolotaryuk, V. Ermakov, Y. Gaididei: On the existence of resonances in the transmission probability for interactions arising from derivatives of the Dirac delta function, *J. Phys. A: Math. Gen* 36 (2003) 7589–7600.

J.F. Colombeau, *New generalized functions and multiplication of distributions*, (North Holland, 1989).

G. Dell’Antonio, G. Panati: The flux-across-surfaces theorem and zero-energy resonances, *J. Stat. Phys.* 116 (2004) 1161–1180.

N.C. Dias, J.N. Prata: A multiplicative product of distributions and a class of ordinary differential equations with distributional coefficients, *J. Math. Anal. Appl.* 359 (2009) 216-228.

N.C. Dias, J.N. Prata: Wigner functions with boundaries, *J. Math. Phys.* 43 (2002) 4602–4627.

N.C. Dias, J.N. Prata: Deformation quantization of confined systems, *Int. J. Quant. Inf.* 5 (2007) 257–263.

N.C. Dias, A. Posilicano, J.N. Prata: Self-adjoint, globally defined Hamiltonian operators for systems with boundaries, *Comm. Pure Appl. Anal.* 10, no.6 (2011) 1687-1706.

P. Exner: Lattice Kronig-Penney models, *Phys. Rev. Lett.* 74 (1995) 3503-3506.

P. Exner, H. Neidhardt, V. Zagrebnov: Potential approximations to $\delta'$: an inverse Klauder phenomenon with norm resolvent convergence, *Comm. Math. Phys.* 224 (2001) 593-612.

P. Garbaczewski, W. Karwowski: Impenetrable barriers and canonical quantization, *Am. J. Phys.* 72 (2004) 924–933.

Y.D. Golovaty, R.O. Hryniv: On norm resolvent convergence of Schrödinger operators with $\delta'$-like potentials, *J. Phys. A: Math. Theor.* 43 (2010) 155204 (14pp).

Y.D. Golovaty, S.S. Man'ko: Solvable models for the Schrödinger operators with $\delta'$-like potentials, *Ukr. Math. Bulletin* 6 (2009) 173-207.

Y.D. Golovaty, R.O. Hryniv: Norm resolvent convergence of singularly scaled Schrödinger operators and $\delta'$-potentials, *Proc. R. Soc. Edinb. A.* 143, no.4 (2013) 791–816.

G. Grubb: *Distributions and Operators*, Graduate Texts in Mathematics, 257 (Springer, 2009).

M. Hirokawa, T. Kosaka: One-dimensional tunnel-junction formula for the Schrödinger particle, *Siam J. Appl. Math* 73, no.6, (2013) 2247–2261.

L. Hörmander, *The analysis of linear partial differential operators I* (Springer-Verlag, 1983).

G. Hörmann, L. Oparnica: Distributional solution concepts for the Euler-Bernoulli beam equation with discontinuous coefficients, *Applicable Analysis* 86, no.11 (2007) 1347–1363.

A. Jensen, T. Kato: Spectral properties of Schrödinger operators and time-decay of the wave functions, *Duke Mathematical Journal* 46 (1979) 583–611.

R.P. Kanwal, *Generalized Functions: Theory and Technique*, Second Edition, (Birkhäuser, 1998).

V.D. Kosmanenko, *Singular quadratic forms in perturbation theory* (Kluwer Academic Publishers, 1999).

A. Kostenko, M. Malamud, 1-D Schrödinger operators with local point interactions on a discrete set, *J. Differential Equations* 249 (2010) 253-304.

P. Kurasov: Distribution theory for discontinuous test functions and differential operators with generalized coefficients, *J. Math. Anal. Appl.* 201 (1996) 297-323.

P. Kurasov, J. Boman: Finite rank singular perturbations and distributions with discontinuous test functions, *Proc. Amer. Math. Soc.* 126, no.6 (1998) 1673-1683.

P. Kurasov, A. Scrinzi, N. Elander: On the $\delta'$-interaction arising in the exterior complex scaling, *Phys. Rev. A* 49 (1994) 5095–5097.
[42] L.P. Nizhnik: A one-dimensional Schrödinger operator with point interactions on Sobolev spaces, *Funct. Anal. Appl.* **40** (2006), N 2, 143-147.

[43] C. Sarrico: Collision of delta-waves in a turbulent model studied via a distribution product, *Nonlinear Anal.* **73**, no.9 (2010) 2868-2875.

[44] P. Seba: Schrödinger particle on a half line, *Lett. Math. Phys.* **10** (1985) 21–27.

[45] P. Seba: The generalized point interaction in one dimension, *Czech. J. Phys. B* **36** (1986), 667-673.

[46] P. Seba: Some remarks on the $\delta'$-interaction in one dimension, *Rep. Math. Phys.* **24** (1986) N 1, 111-120.

[47] A.V. Zolotaryuk: Point interactions of the dipole type defined through a three-parametric power regularization, *J. Phys A: Math. Theor.* **43** (2010) 105302 (21 pp).

[48] A.V. Zolotaryuk: Boundary conditions for the states with resonant tunnelling across the $\delta'$-potential, *Phys. Lett. A* **374** (15-16) (2010) 1636–1641.

[49] A.V. Zolotaryuk, Y. Zolotaryuk: Controlling a resonant transmission across the $\delta'$-potential: the inverse problem, *J. Phys. A: Math and Theor.* **44** (37) (2011) 375305, 21pp.

*******************************************************************

**Author’s addresses:**

- **Nuno Costa Dias** and **Cristina Jorge**: Departamento de Matemática. Universidade Lusófona de Humanidades e Tecnologias. Av. Campo Grande, 376, 1749-024 Lisboa, Portugal and Grupo de Física Matemática, Universidade de Lisboa, Av. Prof. Gama Pinto 2, 1649-003 Lisboa, Portugal

- **João Nuno Prata**: Escola Superior Náutica Infante D. Henrique. Av. Eng. Bonneville Franco, 2770-058 Paço d’Arcos, Portugal and Grupo de Física Matemática, Universidade de Lisboa, Av. Prof. Gama Pinto 2, 1649-003 Lisboa, Portugal

*******************************************************************

*E-mail address: (NCD) ncdias@meo.pt*

*E-mail address: (CJ) cristina.goncalves.jorge@gmail.com*

*E-mail address: (JNP) joao.prata@mail.telepac.pt*