**Research article**

**Plenty of wave solutions to the ill-posed Boussinesq dynamic wave equation under shallow water beneath gravity**

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**Abstract:** This paper applies two computational techniques for constructing novel solitary wave solutions of the ill-posed Boussinesq dynamic wave (IPB) equation. Jacques Hadamard has formulated this model for studying the dynamic behavior of waves in shallow water under gravity. Extended simple equation (ESE) method and novel Riccati expansion (NRE) method have been applied to the investigated model’s converted nonlinear ordinary differential equation through the wave transformation. As a result of this research, many solitary wave solutions have been obtained and represented in different figures in two-dimensional, three-dimensional, and density plots. The explanation of the methods used shows their dynamics and effectiveness in dealing with certain nonlinear evolution equations.

**Keywords:** ill-posed Boussinesq dynamical wave; extended simple equation (ESE) method; novel Riccati expansion (NRE) method; nonlinear soliton lattice wave solutions

**Mathematics Subject Classification:** 35C08, 34A25, 49M05
1. Introduction

Beginning from the mid of 18th century, Euler, Cauchy, d’Alembert, Hamilton, Jacobi, Lagrange, Laplace, Monge, and many others have started formulating some complex nonlinear phenomena through nonlinear partial differential equations [1, 2]. Many complex phenomena, such as solid-state physics, mechanics, fluid mechanics, surface plasma physics, quantum mechanics, civic engineering, population climate, neural networks, epidemiology for infectious disorders, plasma wave, thermodynamics, physics of condensed matter, nonlinear optics, etc., are extracted in distinct mathematical formulas [3–5]. Many computational schemes have been formulated for constructing novel solitary wave solutions that describe the physical and dynamical behavior of these phenomena [6–8]. In 1834 John Russell pointed out the lonely solutions of moving waves by adding a particular parameter value in the closed shape [9,10]. There are several common types of solitary wave solutions. Such solitary wave solutions explain much about the physical properties of these models [11–13].

Here, this article studies the well-known nonlinear IPB model through the ESE and NRE methods for constructing novel lattice soliton solutions and investigating the physical characterizes of the along wave in shallow water beneath gravity [14, 15]. The mathematical model of the considered model is given by [16, 17]

\[ U_{tt} = U_{xx} + (U^2)_{xx} + U_{xxx}, \]  

(1.1)

where \( U = U(x,t) \) describes promulgation of small amplitude long waves (long compared to the amplitude of the wave) in sundry physical contexts inclusive shallow water under gravity. Implementing the next wave transformation \( U = \mathcal{V}(3), \mathcal{V} = x - ct, \) where \( c \) is arbitrary constant to be evaluated later, converts the nonlinear partial differential equation into the following the ordinary differential equation

\[ (c^2 - 1) \mathcal{V} - \mathcal{V}^2 - \mathcal{V}' = 0. \]  

(1.2)

Applying the homogeneous balance principles to the highest order derivative term and nonlinear term of Eq (1.2), get \( N = 2 \). Using the general formula of the suggested computational schemes [3–7], get the traveling solutions of the IPB model in the next formula

\[
\begin{align*}
\mathcal{V}(3) = & \left\{ \sum_{j=-N}^{N} a_j \phi^j(3) = \frac{a_{-2}}{\phi(3)^2} + \frac{a_{-1}}{\phi(3)} + a_0 + a_1 \phi(3) + a_2 \phi^2(3), \\
& \sum_{j=-N}^{N} a_j (d + \psi(3))^j = \frac{a_{-2}}{(d + \psi(3))^2} + \frac{a_{-1}}{(d + \psi(3))} + a_0 + a_1 (d + \psi(3)) + a_2 (d + \psi(3))^2,
\end{align*}
\]  

(1.3)

where \( a_j, j = -2, \cdots, 2, \) are arbitrary constants to be determined later. While \( \phi(3), \psi(3) \) satisfy the following auxiliary equations

\[
\begin{align*}
\phi'(3) = & \mathcal{G}_1 + \mathcal{G}_2 \phi(3) + \mathcal{G}_3 \phi^2(3), \\
\psi'(3) = & \left( d^2 \mathcal{E}_3 - d^2 - d \mathcal{E}_2 + \mathcal{E}_1 \right) + \left( -2 d \mathcal{E}_3 + 2d + \mathcal{E}_2 \right) \psi(3) + \left( \mathcal{E}_3 - 1 \right) \psi(3)^2,
\end{align*}
\]

(1.4)

where \( d, \mathcal{G}_i, \mathcal{E}_i, (i = 1, 2, 3) \) are arbitrary constants to be determined through the suggested analytical schemes’ framework.
The paper’s rest sections are organized as follows: Section 2 handles the considered model through the ESE and NRE methods [18–24]. Additionally, the solitary wave solutions are explained through two, three-dimensional, and density plots to illustrate the dynamical behavior of shallow water waves beneath gravity. Section 4 discusses the obtained solitary wave solutions and their novelty. Section 5 gives the conclusion of the whole paper.

2. Nonlinear soliton lattice wave solutions

Here, we use the suggested computational schemes’ framework to determine the above-shown parameters.

2.1. ESE method’s solutions

Handling the considered model along with the ESE method gives the following sets of parameters’ value.

Set A

\[ G_3 = \frac{1 - c^2}{16 G_1}, \quad G_2 = 0, \quad a_{-2} = -6 G_1^2, \quad a_{-1} = a_1 = 0, \quad a_0 = \frac{3}{4} (c^2 - 1), \quad a_2 = \frac{-3 (1 - 2 c^2 + c^4)}{128 G_1^2}. \]

Thus, the model’s soliton solutions are given by

**When \( G_2 = 0 \), we acquire:**

**Case 1.** When \( G_1 G_3 > 0 \)

\[ U(Z) = -6 G_1 G_3 \cot^2 \left( \sqrt{G_1 G_3} (3 + C) \right) + \frac{3}{4} \left( c^2 - 1 \right) - \frac{3 (1 - 2 c^2 + c^4)}{128 G_1 G_3} \]

\[ \times \tan^2 \left( \sqrt{G_1 G_3} (3 + C) \right), \quad (2.1) \]

**Case 2.** When \( G_1 G_3 < 0 \)

\[ U(Z) = 6 G_1 G_3 \coth^2 \left( \sqrt{-G_1 G_3} (3 + C) \right) + \frac{3}{4} \left( c^2 - 1 \right) + \frac{3 (1 - 2 c^2 + c^4)}{128 G_1 G_3} \]

\[ \times \tanh^2 \left( \sqrt{-G_1 G_3} (3 + C) \right), \quad (2.3) \]

**When \( \pm G_1 G_3 > 0 \)**

\[ U(Z) = 6 G_1 G_3 \tan^2 \left( \sqrt{G_1 G_3} (3 + C) \right) + \frac{3}{4} \left( c^2 - 1 \right) + \frac{3 (1 - 2 c^2 + c^4)}{128 G_1 G_3} \]

\[ \times \coth^2 \left( \sqrt{G_1 G_3} (3 + C) \right). \quad (2.4) \]
Set B
\[
G_3 = \frac{c^2 - 1}{16 G_1}, \ G_2 = 0, \ a_{-2} = -6 G_1^2, \ a_{-1} = a_1 = 0, \ a_0 = \frac{1}{4} (c^2 - 1), \ a_2 = \frac{-3 \left(1 - 2 c^2 + c^4\right)}{128 G_1^2}.
\]

Thus, the model’s soliton solutions are given by

Case 1. When \( G_1 G_3 > 0 \)
\[
U(3) = -6 G_1 G_3 \cos^2 \left( \sqrt{G_1 G_3} (3 + C) \right) + \frac{1}{4} \left( \frac{c^2 - 1}{c} \right) - \frac{3 \left(1 - 2 c^2 + c^4\right)}{128 G_1 G_3} \tan^2 \left( \sqrt{G_1 G_3} (3 + C) \right).
\]

Case 2. When \( G_1 G_3 < 0 \)
\[
U(3) = -6 G_1 G_3 \cosh^2 \left( \sqrt{-G_1 G_3} (3 + C) \right) + \frac{1}{4} \left( \frac{c^2 - 1}{c} \right) + \frac{3 \left(1 - 2 c^2 + c^4\right)}{128 G_1 G_3} \coth^2 \left( \sqrt{-G_1 G_3} (3 + C) \right).
\]

Set C
\[
G_3 = \frac{c^2 - 1 + G_2^2}{4 G_1}, \ a_{-2} = -6 G_1^2, \ a_{-1} = -6 G_1 G_2, \ a_0 = \frac{1}{2} \left(3 G_2^2 - 1 + c^2\right), \ a_1 = a_2 = 0.
\]

Thus, the model’s soliton solutions are given by

When \( G_2 = 0 \), we acquire:

Case 1. When \( G_1 G_3 > 0 \)
\[
U(3) = -6 G_1 G_3 \cos^2 \left( \sqrt{G_1 G_3} (3 + C) \right) - \frac{1}{2} \left(1 - c^2\right).
\]

Case 2. When \( G_1 G_3 < 0 \)
\[
U(3) = -6 G_1 G_3 \cosh^2 \left( \sqrt{-G_1 G_3} (3 + C) \right) - \frac{1}{2} \left(1 - c^2\right).
\]
When $G_3 > G_2$ and $G_3 > 0$

$$U(3) = 6 G_1 G_3 \tanh^2 \left( \sqrt{-G_1 G_3} \frac{\ln(C)}{2} \right) - \frac{1}{2} (-1 + c^2). \tag{2.13}$$

Additionally, the general soliton solutions are given by

Case 1. When $4 G_1 G_3 > G_2^2$ and $G_3 > 0$

$$U(3) = \frac{-24 G_1^2 G_3^2}{\left( \sqrt{4 G_1 G_3 - G_2^2} \tan \left( \frac{\sqrt{4 G_1 G_3 - G_2^2}}{2} (3 + C) \right) - G_2 \right)^2} - \frac{12 G_1 G_3}{\sqrt{4 G_1 G_3 - G_2^2} \tan \left( \frac{\sqrt{4 G_1 G_3 - G_2^2}}{2} (3 + C) \right) - G_2} \frac{1}{2} \left( 3 G_2^2 - 1 + c^2 \right). \tag{2.14}$$

Case 2. When $4 G_1 G_3 > G_2^2$ and $G_3 < 0$

$$U(3) = \frac{-24 G_1^2 G_3^2}{\left( \sqrt{4 G_1 G_3 - G_2^2} \cot \left( \frac{\sqrt{4 G_1 G_3 - G_2^2}}{2} (3 + C) \right) + G_2 \right)^2} - \frac{12 G_1 G_3}{\sqrt{4 G_1 G_3 - G_2^2} \cot \left( \frac{\sqrt{4 G_1 G_3 - G_2^2}}{2} (3 + C) \right) + G_2} \frac{1}{2} \left( 3 G_2^2 - 1 + c^2 \right). \tag{2.15}$$

Set $D$

$$G_3 = -\frac{c^2 - 1 - G_2^2}{4 G_1}, \quad a_2 = -6 G_1^2, \quad a_1 = -6 G_1 G_2, \quad a_0 = -\frac{3}{2} \left( G_2^2 - 1 + c^2 \right), \quad a_1 = a_2 = 0.$$

Thus, the model’s soliton solutions are given by

When $G_2 = 0$, we acquire:

Case 1. When $G_1 G_3 > 0$

$$U(3) = -6 G_1 G_3 c \cot^2 \left( \sqrt{G_1 G_3} (3 + C) \right) - \frac{3}{2} \left( -1 + c^2 \right). \tag{2.18}$$
\[ U(3) = -6G_1G_3 \tan^2 \left( \sqrt{G_1G_3} (3 + C) \right) - \frac{3}{2} \left( -1 + c^2 \right). \] (2.19)

**Case 2.** When \( G_1G_3 < 0 \)

\[ U(3) = 6G_1G_3 \coth^2 \left( \sqrt{-G_1G_3} \frac{3 + \ln(C)}{2} \right) - \frac{3}{2} \left( -1 + c^2 \right), \] (2.20)

\[ U(3) = 6G_1G_3 \tanh^2 \left( \sqrt{-G_1G_3} \frac{3 + \ln(C)}{2} \right) - \frac{3}{2} \left( -1 + c^2 \right). \] (2.21)

Additionally, the general soliton solutions are given by

**Case 1.** When \( 4G_1G_3 > G_2^2 \) and \( G_3 > 0 \)

\[ U(3) = \frac{-24G_1^2G_3^2}{\sqrt{4G_1G_3 - G_2^2} \tan \left( \sqrt{\frac{4G_1G_3 - G_2^2}{2}} (3 + C) \right) - G_2^2} - \frac{12G_1G_2G_3}{\sqrt{4G_1G_3 - G_2^2} \tan \left( \sqrt{\frac{4G_1G_3 - G_2^2}{2}} (3 + C) \right) - G_2^2} \] (2.22)

\[- \frac{3}{2} \left( G_2^2 - 1 + c^2 \right), \]

**Case 2.** When \( 4G_1G_3 > G_2^2 \) and \( G_3 < 0 \)

\[ U(3) = \frac{-24G_1^2G_3^2}{\sqrt{4G_1G_3 - G_2^2} \cot \left( \sqrt{\frac{4G_1G_3 - G_2^2}{2}} (3 + C) \right) + G_2^2} - \frac{12G_1G_2G_3}{\sqrt{4G_1G_3 - G_2^2} \cot \left( \sqrt{\frac{4G_1G_3 - G_2^2}{2}} (3 + C) \right) + G_2^2} \] (2.24)

\[- \frac{3}{2} \left( G_2^2 - 1 + c^2 \right), \]

\[ U(3) = \frac{-24G_1^2G_3^2}{\sqrt{4G_1G_3 - G_2^2} \cot \left( \sqrt{\frac{4G_1G_3 - G_2^2}{2}} (3 + C) \right) + G_2^2} - \frac{12G_1G_2G_3}{\sqrt{4G_1G_3 - G_2^2} \cot \left( \sqrt{\frac{4G_1G_3 - G_2^2}{2}} (3 + C) \right) + G_2^2} \] (2.25)

\[- \frac{3}{2} \left( G_2^2 - 1 + c^2 \right). \]
Set \( E \)
\[
G_1 = \frac{-36 G_3^2 + 36 c^2 G_3^2 + a_1^2}{144 G_3^2}, \quad G_2 = \frac{-a_1}{6 G_3}, \quad a_2 = a = 0, \quad a_0 = \frac{-a_1 - 12 G_3^2 + 12 c^2 G_3^2}{24 G_3^2}, \quad a_2 = -6 G_3^2.
\]

Thus, the model’s soliton solutions are given by

When \( G_2 = 0, \) we acquire:

Case 1. When \( G_1 G_3 > 0 \)
\[
U(3) = -\frac{a_1^2 - 12 G_3^2 + 12 c^2 G_3^2}{24 G_3^2} - 6 G_3 G_1 \tan^2 \left( \sqrt{G_1 G_3} (3 + C) \right), \tag{2.26}
\]
\[
U(3) = -\frac{a_1^2 - 12 G_3^2 + 12 c^2 G_3^2}{24 G_3^2} - 6 G_3 G_1 \cot^2 \left( \sqrt{G_1 G_3} (3 + C) \right). \tag{2.27}
\]

Case 2. When \( G_1 G_3 < 0 \)
\[
U(3) = -\frac{a_1^2 - 12 G_3^2 + 12 c^2 G_3^2}{24 G_3^2} + 6 G_3 G_1 \tanh^2 \left( \sqrt{-G_1 G_3} 3 \pm \frac{\ln(C)}{2} \right), \tag{2.28}
\]
\[
U(3) = -\frac{a_1^2 - 12 G_3^2 + 12 c^2 G_3^2}{24 G_3^2} + 6 G_3 G_1 \coth^2 \left( \sqrt{-G_1 G_3} 3 \pm \frac{\ln(C)}{2} \right). \tag{2.29}
\]

When \( G_1 = 0, \) we acquire

Case 1. When \( G_2 > 0 \)
\[
U(3) = -\frac{a_1^2 - 12 G_3^2 + 12 c^2 G_3^2}{24 G_3^2} + \frac{a_1 G_2 e^{G_2(3+C)}}{1 - G_3 e^{G_2(3+C)}} - 6 G_3^2 \left( \frac{G_2 e^{G_2(3+C)}}{1 - G_3 e^{G_2(3+C)}} \right)^2, \tag{2.30}
\]

Case 2. When \( G_2 < 0 \)
\[
U(3) = -\frac{a_1^2 - 12 G_3^2 + 12 c^2 G_3^2}{24 G_3^2} - \frac{a_1 G_2 e^{G_2(3+C)}}{1 + G_3 e^{G_2(3+C)}} - 6 G_3^2 \left( \frac{G_3 e^{G_2(3+C)}}{1 + G_3 e^{G_2(3+C)}} \right)^2. \tag{2.31}
\]

Additionally, the general soliton solutions are given by

Case 1. When \( 4 G_1 G_3 > G_2^2 \) and \( G_3 > 0 \)
\[
U(3) = -\frac{a_1^2 - 12 G_3^2 + 12 c^2 G_3^2}{24 G_3^2} + \frac{a_1}{2 G_3} \left( \sqrt{4 G_1 G_3 - G_2^2} \tan \left( \frac{\sqrt{4 G_1 G_3 - G_2^2}}{2} (3 + C) - G_2 \right) \right) - \frac{3}{2} \left( \sqrt{4 G_1 G_3 - G_2^2} \tan \left( \frac{\sqrt{4 G_1 G_3 - G_2^2}}{2} (3 + C) - G_2 \right) \right)^2, \tag{2.32}
\]

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Thus, the model’s soliton solutions are given by

\[
U(3) = - \frac{a_1^2 - 12 G_3^2 + 12 c^2 G_3^2}{24 G_3^2} + \frac{a_1}{2 G_3} \left( \sqrt{4 G_1 G_3 - G_2^2} \cot \left( \frac{\sqrt{4 G_1 G_3 - G_2^2}}{2} (3 + C) \right) - G_2 \right)
\]
\[
- \frac{3}{2} \left( \sqrt{4 G_1 G_3 - G_2^2} \cot \left( \frac{\sqrt{4 G_1 G_3 - G_2^2}}{2} (3 + C) \right) - G_2 \right)^2.
\] (2.33)

Case 2. When \(4 G_1 G_3 > G_2^2\) and \(G_3 < 0\)

\[
U(3) = - \frac{a_1^2 - 12 G_3^2 + 12 c^2 G_3^2}{24 G_3^2} + \frac{a_1}{2 G_3} \left( \sqrt{4 G_1 G_3 - G_2^2} \tan \left( \frac{\sqrt{4 G_1 G_3 - G_2^2}}{2} (3 + C) \right) + G_2 \right)
\]
\[
- \frac{3}{2} \left( \sqrt{4 G_1 G_3 - G_2^2} \tan \left( \frac{\sqrt{4 G_1 G_3 - G_2^2}}{2} (3 + C) \right) + G_2 \right)^2.
\] (2.34)

Case 2. When \(4 G_1 G_3 > G_2^2\) and \(G_3 < 0\)

\[
U(3) = - \frac{a_1^2 - 12 G_3^2 + 12 c^2 G_3^2}{24 G_3^2} + \frac{a_1}{2 G_3} \left( \sqrt{4 G_1 G_3 - G_2^2} \cot \left( \frac{\sqrt{4 G_1 G_3 - G_2^2}}{2} (3 + C) \right) + G_2 \right)
\]
\[
- \frac{3}{2} \left( \sqrt{4 G_1 G_3 - G_2^2} \cot \left( \frac{\sqrt{4 G_1 G_3 - G_2^2}}{2} (3 + C) \right) + G_2 \right)^2.
\] (2.35)

Set \(F\)

\[
G_1 = - \frac{36 G_1^3 + 36 c^2 G_3^2 - a_1^2}{144 G_3^3}, \quad G_2 = -\frac{a_1}{6 G_3}, \quad a_{-2} = a_{-1} = 0, \quad a_0 = - \frac{a_1^2 + 36 G_1^3 - 36 c^2 G_3^2}{24 G_3^2}, \quad a_2 = -6 G_3^2.
\]

Thus, the model’s soliton solutions are given by

When \(G_3 = 0\), we acquire:

Case 1. When \(G_1 G_3 > 0\)

\[
U(3) = - \frac{a_1^2 + 36 G_1^3 - 36 c^2 G_3^2}{24 G_3^2} - 6 G_3 G_1 \tan^2 \left( \sqrt{G_1 G_3} (3 + C) \right).
\] (2.36)

Case 2. When \(G_1 G_3 < 0\)

\[
U(3) = - \frac{a_1^2 + 36 G_1^3 - 36 c^2 G_3^2}{24 G_3^2} - 6 G_3 G_1 \cot^2 \left( \sqrt{G_1 G_3} (3 + C) \right).
\] (2.37)

Case 2. When \(G_1 G_3 < 0\)

\[
U(3) = - \frac{a_1^2 + 36 G_1^3 - 36 c^2 G_3^2}{24 G_3^2} + 6 G_3 G_1 \tanh^2 \left( \sqrt{-G_1 G_3} 3 \pm \frac{\ln(C)}{2} \right).
\] (2.38)
When $G_1 = 0$, we acquire

**Case 1. When $G_2 > 0$**

\[
U(3) = -\frac{a_1^2 + 36 G_3^2 - 36 c^2 G_3^2}{24 G_3^2} + \frac{a_1 G_2}{2 G_3} \left( \frac{4 G_1 G_3 - G_2^2}{2} (3 + C) - G_2 \right) - \frac{3}{2} \left( \sqrt{4 G_1 G_3 - G_2^2} \tan \left( \frac{\sqrt{4 G_1 G_3 - G_2^2}}{2} (3 + C) - G_2 \right) \right)^2,
\]

**Case 2. When $G_2 < 0$**

\[
U(3) = -\frac{a_1^2 + 36 G_3^2 - 36 c^2 G_3^2}{24 G_3^2} - \frac{a_1 G_2}{2 G_3} \left( \frac{4 G_1 G_3 - G_2^2}{2} (3 + C) - G_2 \right) - \frac{3}{2} \left( \sqrt{4 G_1 G_3 - G_2^2} \cot \left( \frac{\sqrt{4 G_1 G_3 - G_2^2}}{2} (3 + C) - G_2 \right) \right)^2.
\]

Additionally, the general soliton solutions are given by

**Case 1. When $4 G_1 G_3 > G_2^2$ and $G_3 > 0$**

\[
U(3) = -\frac{a_1^2 + 36 G_3^2 - 36 c^2 G_3^2}{24 G_3^2} + \frac{a_1}{2 G_3} \left( \sqrt{4 G_1 G_3 - G_2^2} \tan \left( \frac{\sqrt{4 G_1 G_3 - G_2^2}}{2} (3 + C) - G_2 \right) \right)
\]

\[
- \frac{3}{2} \left( \sqrt{4 G_1 G_3 - G_2^2} \tan \left( \frac{\sqrt{4 G_1 G_3 - G_2^2}}{2} (3 + C) - G_2 \right) \right)^2
\]

**Case 2. When $4 G_1 G_3 > G_2^2$ and $G_3 < 0$**

\[
U(3) = -\frac{a_1^2 + 36 G_3^2 - 36 c^2 G_3^2}{24 G_3^2} + \frac{a_1}{2 G_3} \left( \sqrt{4 G_1 G_3 - G_2^2} \cot \left( \frac{\sqrt{4 G_1 G_3 - G_2^2}}{2} (3 + C) - G_2 \right) \right)
\]

\[
- \frac{3}{2} \left( \sqrt{4 G_1 G_3 - G_2^2} \cot \left( \frac{\sqrt{4 G_1 G_3 - G_2^2}}{2} (3 + C) - G_2 \right) \right)^2.
\]
Thus, the model’s soliton solutions are given by

\[ U(\lambda) = -\frac{a_1^2 + 36 G_3^2 - 36 c^2 G_3^2}{24 G_3^2} + \frac{a_1}{2 G_3} \left( \sqrt{4 G_1 G_3 - G_2^2} \cot \left( \frac{\sqrt{4 G_1 G_3 - G_2^2}}{2} (3 + C) \right) + G_2 \right) \]

\[ \frac{3}{2} \left( \sqrt{4 G_1 G_3 - G_2^2} \cot \left( \frac{\sqrt{4 G_1 G_3 - G_2^2}}{2} (3 + C) \right) + G_2 \right)^2. \]  

(2.45)

2.2. NRE method’s solutions

Handling the considered model along with the NRE method gives the following sets of parameters’ value.

Set A

\[ c = \sqrt{4 E_1 E_3 - 4 E_1 + 1 - E_3^2}, \quad a_2 = a_{-1} = 0, \quad a_0 = - (2 E_3 - 2) E_1 - E_2^2 - (6 E_3 d + 6 d) E_2 - 6 E_3^2 d^2 + 12 E_3 d^2 - 6 d^2, \quad a_1 = - E_2 (6 E_3 - 6) + 12 E_3^2 d - 24 E_3 d + 12 d, \quad a_2 = -6 E_3^2 - 6 + 12 E_3. \]

Thus, the model’s soliton solutions are given by

When \((C = E_2^2 - 4 E_2 E_1 + 4 E_1 > 0)\) and \((E_2(E_3 - 1) \neq 0)\) or \((E_1(E_3 - 1) \neq 0)\):

\[ U(\lambda) = - (2 E_3 - 2) E_1 - E_2^2 - (6 E_3 d + 6 d) E_2 - 6 E_3^2 d^2 + 12 E_3 d^2 - 6 d^2 + (-E_2 \times (6 E_3 - 6) + 12 E_3^2 d - 24 E_3 d + 12 d)(d - \frac{1}{2(E_3 - 1)} (E_2 + \sqrt{\infty} \tanh(\frac{\sqrt{\infty}}{2} 3))) \]

\[ + (6 E_3^2 - 6 + 12 E_3) \left( d - \frac{1}{2(E_3 - 1)} (E_2 + \sqrt{\infty} \tanh(\frac{\sqrt{\infty}}{2} 3)) \right)^2, \]  

(2.46)

\[ U(\lambda) = - (2 E_3 - 2) E_1 - E_2^2 - (6 E_3 d + 6 d) E_2 - 6 E_3^2 d^2 + 12 E_3 d^2 - 6 d^2 + (-E_2 (6 E_3 - 6) + 12 E_3^2 d - 24 E_3 d + 12 d)(d - \frac{1}{2(E_3 - 1)} (E_2 + \sqrt{\infty} \coth(\frac{\sqrt{\infty}}{2} 3))) + (-6 E_3^2 - 6) + 12 E_3 \left( d - \frac{1}{2(E_3 - 1)} (E_2 + \sqrt{\infty} \coth(\frac{\sqrt{\infty}}{2} 3)) \right)^2, \]  

(2.47)

\[ U(\lambda) = - (2 E_3 - 2) E_1 - E_2^2 - (6 E_3 d + 6 d) E_2 - 6 E_3^2 d^2 + 12 E_3 d^2 - 6 d^2 + (-E_2 (6 E_3 - 6) + 12 E_3^2 d - 24 E_3 d + 12 d)(d - \frac{1}{2(E_3 - 1)} (E_2 + \sqrt{\infty} (\tanh(\frac{\sqrt{\infty}}{3}) + i \text{sech}(\frac{\sqrt{\infty}}{3})))) \]

\[ + (6 E_3^2 - 6 + 12 E_3) \left( d - \frac{1}{2(E_3 - 1)} (E_2 + \sqrt{\infty} (\tanh(\frac{\sqrt{\infty}}{3}) + i \text{sech}(\frac{\sqrt{\infty}}{3}))) \right)^2, \]  

(2.48)

\[ U(\lambda) = - (2 E_3 - 2) E_1 - E_2^2 - (6 E_3 d + 6 d) E_2 - 6 E_3^2 d^2 + 12 E_3 d^2 - 6 d^2 + (-E_2 (6 E_3 - 6) + 12 E_3^2 d - 24 E_3 d + 12 d)(d - \frac{1}{2(E_3 - 1)} (E_2 + \sqrt{\infty} (\coth(\frac{\sqrt{\infty}}{3}) \pm \text{csch}(\frac{\sqrt{\infty}}{3})))) \]

\[ + (6 E_3^2 - 6 + 12 E_3) \left( d - \frac{1}{2(E_3 - 1)} (E_2 + \sqrt{\infty} (\coth(\frac{\sqrt{\infty}}{3}) \pm \text{csch}(\frac{\sqrt{\infty}}{3}))) \right)^2. \]  

(2.49)
\[ U(3) = - (2 \mathcal{E}_3 - 2) \mathcal{E}_1 - \mathcal{E}_2^2 - (-6 \mathcal{E}_3 d + 6 d) \mathcal{E}_2 - 6 \mathcal{E}_3^2 d^2 + 12 \mathcal{E}_3 d^2 - 6 d^2 + \left( - \mathcal{E}_2 (6 \mathcal{E}_3 - 6) \right. \\
+ 12 \mathcal{E}_3^2 d^2 - 24 \mathcal{E}_3 d + 12 d) \left( d - \frac{1}{4(\mathcal{E}_3 - 1)} \left( 2 \mathcal{E}_2 + \sqrt{\mathcal{C}} \left( \sqrt{\frac{\mathcal{C}}{4}} - 3 \pm \coth(\sqrt{\frac{\mathcal{C}}{4}}) \right) \right) \right) \right) \\
\left. (2.50) \right) \\
+ (-6 \mathcal{E}_3^2 - 6 + 12 \mathcal{E}_3) \left( d - \frac{1}{4(\mathcal{E}_3 - 1)} \left( 2 \mathcal{E}_2 + \sqrt{\mathcal{C}} \left( \sqrt{\frac{\mathcal{C}}{4}} - 3 \pm \coth(\sqrt{\frac{\mathcal{C}}{4}}) \right) \right) \right)^2, \\
\]

\[ U(3) = - (2 \mathcal{E}_3 - 2) \mathcal{E}_1 - \mathcal{E}_2^2 - (-6 \mathcal{E}_3 d + 6 d) \mathcal{E}_2 - 6 \mathcal{E}_3^2 d^2 + 12 \mathcal{E}_3 d^2 - 6 d^2 + (- \mathcal{E}_2 (6 \mathcal{E}_3 - 6) \right. \\
+ 12 \mathcal{E}_3^2 d^2 - 24 \mathcal{E}_3 d + 12 d) \left( d + \frac{1}{2(\mathcal{E}_3 - 1)} \left( \mathcal{E}_2 + \frac{\pm \sqrt{\mathcal{C}} (A^2 + B^2) - A \sqrt{\mathcal{C}} \cosh(\sqrt{\frac{\mathcal{C}}{3}})}{A \sinh(\sqrt{\frac{\mathcal{C}}{3}}) + B} \right) \right) \\
\left. (2.51) \right) \\
+ \left( -6 \mathcal{E}_3^2 - 6 + 12 \mathcal{E}_3 \right) \left( d + \frac{1}{2(\mathcal{E}_3 - 1)} \left( \mathcal{E}_2 + \frac{\pm \sqrt{\mathcal{C}} (A^2 + B^2) - A \sqrt{\mathcal{C}} \cosh(\sqrt{\frac{\mathcal{C}}{3}})}{A \sinh(\sqrt{\frac{\mathcal{C}}{3}}) + B} \right) \right)^2, \\
\]

\[ U(3) = - (2 \mathcal{E}_3 - 2) \mathcal{E}_1 - \mathcal{E}_2^2 - (-6 \mathcal{E}_3 d + 6 d) \mathcal{E}_2 - 6 \mathcal{E}_3^2 d^2 + 12 \mathcal{E}_3 d^2 - 6 d^2 + (- \mathcal{E}_2 (6 \mathcal{E}_3 - 6) + 12 \mathcal{E}_3^2 d \\
- 24 \mathcal{E}_3 d + 12 d) \left( d + \frac{2 \mathcal{E}_1 \cosh(\sqrt{\frac{\mathcal{C}}{3}})}{\sqrt{\mathcal{C}} \sinh(\sqrt{\frac{\mathcal{C}}{3}}) - \mathcal{E}_2 \cosh(\sqrt{\frac{\mathcal{C}}{3}})} \right) \right) \left( -6 \mathcal{E}_3^2 - 6 + 12 \mathcal{E}_3 \right) \left( d \\
(2.53) \right) \\
+ \left( \frac{2 \mathcal{E}_1 \cosh(\sqrt{\frac{\mathcal{C}}{3}})}{\sqrt{\mathcal{C}} \sinh(\sqrt{\frac{\mathcal{C}}{3}}) - \mathcal{E}_2 \cosh(\sqrt{\frac{\mathcal{C}}{3}})} \right)^2, \\
\]

\[ U(3) = - (2 \mathcal{E}_3 - 2) \mathcal{E}_1 - \mathcal{E}_2^2 - (-6 \mathcal{E}_3 d + 6 d) \mathcal{E}_2 - 6 \mathcal{E}_3^2 d^2 + 12 \mathcal{E}_3 d^2 - 6 d^2 + (- \mathcal{E}_2 (6 \mathcal{E}_3 - 6) + 12 \mathcal{E}_3^2 d \\
- 24 \mathcal{E}_3 d + 12 d) \left( d + \frac{2 \mathcal{E}_1 \sinh(\sqrt{\frac{\mathcal{C}}{3}})}{\sqrt{\mathcal{C}} \cosh(\sqrt{\frac{\mathcal{C}}{3}}) - \mathcal{E}_2 \sinh(\sqrt{\frac{\mathcal{C}}{3}})} \right) \right) \left( -6 \mathcal{E}_3^2 - 6 + 12 \mathcal{E}_3 \right) \left( d \\
(2.54) \right) \\
+ \left( \frac{2 \mathcal{E}_1 \sinh(\sqrt{\frac{\mathcal{C}}{3}})}{\sqrt{\mathcal{C}} \cosh(\sqrt{\frac{\mathcal{C}}{3}}) - \mathcal{E}_2 \sinh(\sqrt{\frac{\mathcal{C}}{3}})} \right)^2, \\
\]

\[ U(3) = - (2 \mathcal{E}_3 - 2) \mathcal{E}_1 - \mathcal{E}_2^2 - (-6 \mathcal{E}_3 d + 6 d) \mathcal{E}_2 - 6 \mathcal{E}_3^2 d^2 + 12 \mathcal{E}_3 d^2 - 6 d^2 + (- \mathcal{E}_2 (6 \mathcal{E}_3 - 6) + 12 \mathcal{E}_3^2 d \\
- 24 \mathcal{E}_3 d + 12 d) \left( d + \frac{2 \mathcal{E}_1 \cosh(\sqrt{\mathcal{C}})}{\sqrt{\mathcal{C}} \sinh(\sqrt{\mathcal{C}}) - \mathcal{E}_2 \cosh(\sqrt{\mathcal{C}}) + i \sqrt{\mathcal{C}}} \right) \right) \left( -6 \mathcal{E}_3^2 - 6 + 12 \mathcal{E}_3 \right) \left( d \\
(2.55) \right) \\
+ \left( \frac{2 \mathcal{E}_1 \cosh(\sqrt{\mathcal{C}})}{\sqrt{\mathcal{C}} \sinh(\sqrt{\mathcal{C}}) - \mathcal{E}_2 \cosh(\sqrt{\mathcal{C}}) + i \sqrt{\mathcal{C}}} \right)^2, \\
\]

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\[ U(3) = -(2 \mathcal{E}_3 - 2) \mathcal{E}_1 - \mathcal{E}_2^2 - (-6 \mathcal{E}_3 d + 6 d) \mathcal{E}_2 - 6 \mathcal{E}_3^2 d^2 + 12 \mathcal{E}_3 d^2 - 6 d^2 + (-\mathcal{E}_2 (6 \mathcal{E}_3 - 6) + 12 \mathcal{E}_3^2 d - 24 \mathcal{E}_3 d + 12 d)(d + \frac{2 \mathcal{E}_1 \sinh(\sqrt{\mathcal{E}})}{\sqrt{\mathcal{E}} \cosh(\sqrt{\mathcal{E}}) - \mathcal{E}_2 \sinh(\sqrt{\mathcal{E}} \mp i \sqrt{\mathcal{E}})})(-6 \mathcal{E}_1^2 - 6 + 12 \mathcal{E}_3)(d) \] (2.56)

While \( A, B \) are arbitrary real constants and \( A^2 + B^2 > 0 \).

When \( \mathcal{E} = 2 - 4 \mathcal{E}_2 \mathcal{E}_1 + 4 \mathcal{E}_1 < 0 \) and \( (\mathcal{E}_2(\mathcal{E}_3 - 1) \neq 0) \) or \( (\mathcal{E}_2(\mathcal{E}_3 - 1) \neq 0) \):

\[ U(3) = -(2 \mathcal{E}_3 - 2) \mathcal{E}_1 - \mathcal{E}_2^2 - (-6 \mathcal{E}_3 d + 6 d) \mathcal{E}_2 - 6 \mathcal{E}_3^2 d^2 + 12 \mathcal{E}_3 d^2 - 6 d^2 + (-\mathcal{E}_2 (6 \mathcal{E}_3 - 6) + 12 \mathcal{E}_3^2 d - 24 \mathcal{E}_3 d + 12 d)(d + \frac{1}{2(\mathcal{E}_3 - 1)}(-\mathcal{E}_2 + \sqrt{-\mathcal{E}} \tanh(\frac{\sqrt{-\mathcal{E}}}{2}))(\mathcal{E}_3))(2.57) \]

\[ U(3) = -(2 \mathcal{E}_3 - 2) \mathcal{E}_1 - \mathcal{E}_2^2 - (-6 \mathcal{E}_3 d + 6 d) \mathcal{E}_2 - 6 \mathcal{E}_3^2 d^2 + 12 \mathcal{E}_3 d^2 - 6 d^2 + (-\mathcal{E}_2 (6 \mathcal{E}_3 - 6) + 12 \mathcal{E}_3^2 d - 24 \mathcal{E}_3 d + 12 d)(d + \frac{1}{2(\mathcal{E}_3 - 1)}(-\mathcal{E}_2 + \sqrt{-\mathcal{E}} \coth(\frac{\sqrt{-\mathcal{E}}}{2}))(\mathcal{E}_3))(2.58) \]

\[ U(3) = -(2 \mathcal{E}_3 - 2) \mathcal{E}_1 - \mathcal{E}_2^2 - (-6 \mathcal{E}_3 d + 6 d) \mathcal{E}_2 - 6 \mathcal{E}_3^2 d^2 + 12 \mathcal{E}_3 d^2 - 6 d^2 + (-\mathcal{E}_2 (6 \mathcal{E}_3 - 6) + 12 \mathcal{E}_3^2 d - 24 \mathcal{E}_3 d + 12 d)(d + \frac{1}{2(\mathcal{E}_3 - 1)}(-\mathcal{E}_2 + \sqrt{-\mathcal{E}} \csc(\sqrt{-\mathcal{E}}))(\mathcal{E}_3))(2.59) \]

\[ U(3) = -(2 \mathcal{E}_3 - 2) \mathcal{E}_1 - \mathcal{E}_2^2 - (-6 \mathcal{E}_3 d + 6 d) \mathcal{E}_2 - 6 \mathcal{E}_3^2 d^2 + 12 \mathcal{E}_3 d^2 - 6 d^2 + (-\mathcal{E}_2 (6 \mathcal{E}_3 - 6) + 12 \mathcal{E}_3^2 d - 24 \mathcal{E}_3 d + 12 d)(d + \frac{1}{2(\mathcal{E}_3 - 1)}(-\mathcal{E}_2 + \sqrt{-\mathcal{E}} (\tan(\sqrt{-\mathcal{E}}))(\mathcal{E}_3))(2.60) \]

\[ U(3) = -(2 \mathcal{E}_3 - 2) \mathcal{E}_1 - \mathcal{E}_2^2 - (-6 \mathcal{E}_3 d + 6 d) \mathcal{E}_2 - 6 \mathcal{E}_3^2 d^2 + 12 \mathcal{E}_3 d^2 - 6 d^2 + (-\mathcal{E}_2 (6 \mathcal{E}_3 - 6) + 12 \mathcal{E}_3^2 d - 24 \mathcal{E}_3 d + 12 d)(d + \frac{1}{2(\mathcal{E}_3 - 1)}(-\mathcal{E}_2 + \sqrt{-\mathcal{E}} (\cot(\sqrt{-\mathcal{E}}))(\mathcal{E}_3))(2.61) \]
\[ U(3) = -(2E_3 - 2)E_1 - E_2^2 - (6E_3 + 6d)E_2 - 6E_3^2d^2 + 12E_3 d^2 - 6d^2 + (-E_2 (6E_3 - 6) + 12E_3^2d \\
+ 12E_3^2d - 24E_3 d + 12d) \left( d + \frac{1}{4(\overline{E_3}-1)}(-2E_2 + \sqrt{-c} \tan(\frac{\sqrt{-c}}{4} 3)) \right) \right) \cos(\frac{\sqrt{-c}}{4} 3) + \cot(\frac{\sqrt{-c}}{4} 3)) \right) + (-6E_3^2 - 6 + 12E_3) \right) \left( d + \frac{1}{4(\overline{E_3}-1)}(-2E_2 + \sqrt{-c} \tan(\frac{\sqrt{-c}}{4} 3)) \right) \right) \right) \right) \right)^2. \]
While \( A, B \) are arbitrary real constants and \( A^2 - B^2 > 0 \).

**When** \( \mathcal{E}_1 = 0 \) **and** \( \mathcal{E}_2(\mathcal{E}_3 - 1) \neq 0 \), **we have:**

\[
\mathcal{U}(3) = - (2\mathcal{E}_3 - 2)\mathcal{E}_1 - \mathcal{E}_2^2 - (-6\mathcal{E}_3 d + 6 d)\mathcal{E}_2 - 6\mathcal{E}_3^2 d^2 + 12\mathcal{E}_3 d^2 - 6 d^2 + (-\mathcal{E}_2 (6\mathcal{E}_3 - 6) + 12\mathcal{E}_3^2 d
- 24\mathcal{E}_3 d + 12 d)(d - \frac{\mathcal{E}_2 k}{(\mathcal{E}_3 - 1)(k + \coth(\mathcal{E}_2 3) - \sinh(\mathcal{E}_2 3))) + (-6\mathcal{E}_3^2 - 6 + 12\mathcal{E}_3)
\times (d - \frac{\mathcal{E}_2 k}{(\mathcal{E}_3 - 1)(k + \coth(\mathcal{E}_2 3) - \sinh(\mathcal{E}_2 3)))^2 ,
(2.68)
\]

\[
\mathcal{U}(3) = - (2\mathcal{E}_3 - 2)\mathcal{E}_1 - \mathcal{E}_2^2 - (-6\mathcal{E}_3 d + 6 d)\mathcal{E}_2 - 6\mathcal{E}_3^2 d^2 + 12\mathcal{E}_3 d^2 - 6 d^2 + (-\mathcal{E}_2 (6\mathcal{E}_3 - 6) + 12\mathcal{E}_3^2 d
- 24\mathcal{E}_3 d + 12 d)(d - \frac{\mathcal{E}_2 (\coth(\mathcal{E}_2 3) + \cosh(\mathcal{E}_2 3)))}{(\mathcal{E}_3 - 1)(k + \cosh(\mathcal{E}_2 3) + \sinh(\mathcal{E}_2 3))) + (-6\mathcal{E}_3^2 - 6 + 12\mathcal{E}_3)
\times (d - \frac{\mathcal{E}_2 (\coth(\mathcal{E}_2 3) + \sinh(\mathcal{E}_2 3)))}{(\mathcal{E}_3 - 1)(k + \cosh(\mathcal{E}_2 3) + \sinh(\mathcal{E}_2 3)))^2 ,
(2.69)
\]

\[
\mathcal{U}(3) = - (2\mathcal{E}_3 - 2)\mathcal{E}_1 - \mathcal{E}_2^2 - (-6\mathcal{E}_3 d + 6 d)\mathcal{E}_2 - 6\mathcal{E}_3^2 d^2 + 12\mathcal{E}_3 d^2 - 6 d^2 + (-\mathcal{E}_2 (6\mathcal{E}_3 - 6) + 12\mathcal{E}_3^2 d
- 24\mathcal{E}_3 d + 12 d)(d - \frac{1}{(\mathcal{E}_3 - 1)3 + C}) + (-6\mathcal{E}_3^2 - 6 + 12\mathcal{E}_3)
\times (d - \frac{1}{(\mathcal{E}_3 - 1)3 + C})^2 .
(2.70)
\]

Set \( B \)

\[
c = \sqrt{-4\mathcal{E}_1 \mathcal{E}_3 + 4\mathcal{E}_1 + 1 - \mathcal{E}_2^2 , a_{-2} = a_{-1} = 0, a_0 = -(6\mathcal{E}_3 - 6)\mathcal{E}_1 - \mathcal{E}_2^2 - (-6\mathcal{E}_3 d + 6 d)\mathcal{E}_2 - 6\mathcal{E}_3^2 d^2 + 12\mathcal{E}_3 d^2 - 6 d^2 + (-\mathcal{E}_2 (6\mathcal{E}_3 - 6) + 12\mathcal{E}_3^2 d}
+ 12\mathcal{E}_3 d^2 - 6 d^2 , a_1 = -\mathcal{E}_2 (6\mathcal{E}_3 - 6) + 12\mathcal{E}_3^2 d - 24\mathcal{E}_3 d + 12 d , a_2 = -6\mathcal{E}_3^2 - 6 + 12\mathcal{E}_3
(2.71)
\]

Thus, the model’s soliton solutions are given by **when** \( \mathcal{C} = \mathcal{E}_2^2 - 4\mathcal{E}_2 \mathcal{E}_1 + 4\mathcal{E}_1 > 0 \) **and** \( (\mathcal{E}_2(\mathcal{E}_3 - 1) \neq 0) \) or \( (\mathcal{E}_1(\mathcal{E}_3 - 1) \neq 0) \):

\[
\mathcal{U}(3) = - (6\mathcal{E}_3 - 6)\mathcal{E}_1 - \mathcal{E}_2^2 - (-6\mathcal{E}_3 d + 6 d)\mathcal{E}_2 - 6\mathcal{E}_3^2 d^2 + 12\mathcal{E}_3 d^2 - 6 d^2 + (-\mathcal{E}_2 (6\mathcal{E}_3 - 6) + 12\mathcal{E}_3^2 d
- 24\mathcal{E}_3 d + 12 d)(d - \frac{1}{2(\mathcal{E}_3 - 1)}(\mathcal{E}_2 + \sqrt{\mathcal{C}} \tanh(\sqrt{\frac{\mathcal{C}}{2}))) + (-6\mathcal{E}_3^2 - 6 + 12\mathcal{E}_3)
\times (d - \frac{1}{2(\mathcal{E}_3 - 1)}(\mathcal{E}_2 + \sqrt{\mathcal{C}} \tanh(\sqrt{\frac{\mathcal{C}}{2}})))^2 ,
(2.72)
\]

\[
\mathcal{U}(3) = - (6\mathcal{E}_3 - 6)\mathcal{E}_1 - \mathcal{E}_2^2 - (-6\mathcal{E}_3 d + 6 d)\mathcal{E}_2 - 6\mathcal{E}_3^2 d^2 + 12\mathcal{E}_3 d^2 - 6 d^2 + (-\mathcal{E}_2 (6\mathcal{E}_3 - 6) + 12\mathcal{E}_3^2 d
- 24\mathcal{E}_3 d + 12 d)(d - \frac{1}{2(\mathcal{E}_3 - 1)}(\mathcal{E}_2 + \sqrt{\mathcal{C}} \coth(\sqrt{\frac{\mathcal{C}}{2}))) + (-6\mathcal{E}_3^2 - 6 + 12\mathcal{E}_3)
\times (d - \frac{1}{2(\mathcal{E}_3 - 1)}(\mathcal{E}_2 + \sqrt{\mathcal{C}} \coth(\sqrt{\frac{\mathcal{C}}{2}})))^2 .
(2.73)
\]
\[ U(3) = -(6E_3 - 6)E_1 - E_2^2 - (-6E_3 d + 6 d)E_2 - 6E_3^2 d^2 + 12E_3 d^2 - 6 d^2 + (-E_2 (6E_3 - 6) + 12E_3^2 d - 24E_3 d + 12 d), \]
\[ \times \left( d - \frac{1}{2(E_3 - 1)} \right) (E_2 + \sqrt{E} \left( \tanh \left( \frac{\sqrt{E}}{3} \right) \pm \sech(\sqrt{E} \frac{\sqrt{E}}{3}) \right)) + (-6E_3^2 - 6 + 12E_3) \]
\times \left( d - \frac{1}{2(E_3 - 1)} \right) (E_2 + \sqrt{E} \left( \tanh \left( \frac{\sqrt{E}}{3} \right) \pm \sech(\sqrt{E} \frac{\sqrt{E}}{3}) \right))^2. \]
\[ \text{(2.74)} \]

\[ U(3) = -(6E_3 - 6)E_1 - E_2^2 - (-6E_3 d + 6 d)E_2 - 6E_3^2 d^2 + 12E_3 d^2 - 6 d^2 + (-E_2 (6E_3 - 6) + 12E_3^2 d - 24E_3 d + 12 d), \]
\[ \times \left( d - \frac{1}{2(E_3 - 1)} \right) (E_2 + \sqrt{E} \left( \coth \left( \frac{\sqrt{E}}{3} \right) \pm \csch(\sqrt{E} \frac{\sqrt{E}}{3}) \right)) + (-6E_3^2 - 6 + 12E_3) \]
\times \left( d - \frac{1}{2(E_3 - 1)} \right) (E_2 + \sqrt{E} \left( \coth \left( \frac{\sqrt{E}}{3} \right) \pm \csch(\sqrt{E} \frac{\sqrt{E}}{3}) \right))^2. \]
\[ \text{(2.75)} \]

\[ U(3) = -(6E_3 - 6)E_1 - E_2^2 - (-6E_3 d + 6 d)E_2 - 6E_3^2 d^2 + 12E_3 d^2 - 6 d^2 + (-E_2 (6E_3 - 6) + 12E_3^2 d - 24E_3 d + 12 d), \]
\[ \times \left( d - \frac{1}{4(E_3 - 1)} \right) (2E_2 + \sqrt{E} \left( \tanh \left( \frac{\sqrt{E}}{4} \frac{\sqrt{E}}{3} \right) \pm \coth(\sqrt{E} \frac{\sqrt{E}}{4} \frac{\sqrt{E}}{3}) \right)) + (-6E_3^2 - 6) \]
\[ + 12E_3) \left( d - \frac{1}{4(E_3 - 1)} \right) (2E_2 + \sqrt{E} \left( \tanh \left( \frac{\sqrt{E}}{4} \frac{\sqrt{E}}{3} \right) \pm \coth(\sqrt{E} \frac{\sqrt{E}}{4} \frac{\sqrt{E}}{3}) \right))^2. \]
\[ \text{(2.76)} \]

\[ U(3) = -(6E_3 - 6)E_1 - E_2^2 - (-6E_3 d + 6 d)E_2 - 6E_3^2 d^2 + 12E_3 d^2 - 6 d^2 + (-E_2 (6E_3 - 6) + 12E_3^2 d - 24E_3 d + 12 d), \]
\[ \times \left( d + \frac{1}{2(E_3 - 1)} \right) \left( E_2 + \frac{\pm \sqrt{E} (A^2 + B^2) - A \sqrt{E} \cosh(\sqrt{E} \frac{\sqrt{E}}{3})}{A \sinh(\sqrt{E} \frac{\sqrt{E}}{3}) + B} \right) + (-6E_3^2 - 6) \]
\[ + 12E_3 \right) \left( d + \frac{1}{2(E_3 - 1)} \right) \left( E_2 + \frac{\pm \sqrt{E} (A^2 + B^2) - A \sqrt{E} \cosh(\sqrt{E} \frac{\sqrt{E}}{3})}{A \sinh(\sqrt{E} \frac{\sqrt{E}}{3}) + B} \right)^2. \]
\[ \text{(2.77)} \]

\[ U(3) = -(6E_3 - 6)E_1 - E_2^2 - (-6E_3 d + 6 d)E_2 - 6E_3^2 d^2 + 12E_3 d^2 - 6 d^2 + (-E_2 (6E_3 - 6) + 12E_3^2 d - 24E_3 d + 12 d), \]
\[ \times \left( d + \frac{1}{2(E_3 - 1)} \right) \left( E_2 + \frac{\pm \sqrt{E} (A^2 + B^2) + A \sqrt{E} \cosh(\sqrt{E} \frac{\sqrt{E}}{3})}{A \sinh(\sqrt{E} \frac{\sqrt{E}}{3}) + B} \right)^2. \]
\[ \text{(2.78)} \]

\[ U(3) = -(6E_3 - 6)E_1 - E_2^2 - (-6E_3 d + 6 d)E_2 - 6E_3^2 d^2 + 12E_3 d^2 - 6 d^2 + (-E_2 (6E_3 - 6) + 12E_3^2 d - 24E_3 d + 12 d), \]
\[ \times \left( d + \frac{2E_1 \cosh(\sqrt{E} \frac{\sqrt{E}}{3})}{\sqrt{E} \sinh(\sqrt{E} \frac{\sqrt{E}}{3}) - E_2 \cosh(\sqrt{E} \frac{\sqrt{E}}{3})} \right) + (-6E_3^2 - 6 + 12E_3) \]
\[ \times \left( d + \frac{2E_1 \cosh(\sqrt{E} \frac{\sqrt{E}}{3})}{\sqrt{E} \sinh(\sqrt{E} \frac{\sqrt{E}}{3}) - E_2 \cosh(\sqrt{E} \frac{\sqrt{E}}{3})} \right)^2. \]
\[ \text{(2.79)} \]
\[ U(3) = - (6 \mathcal{E}_3 - 6) \mathcal{E}_1 - \mathcal{E}_2^2 - (-6 \mathcal{E}_3 d + 6 d) \mathcal{E}_2 - 6 \mathcal{E}_3^2 d^2 + 12 \mathcal{E}_3 d^2 - 6 d^2 + (-\mathcal{E}_2 (6 \mathcal{E}_3 - 6) + 12 \mathcal{E}_3^2 d \\
- 24 \mathcal{E}_3 d + 12 d) \left( d + \frac{2 \mathcal{E}_1 \sinh(\sqrt{\mathcal{E}}/2)}{\sqrt{\mathcal{E}} \cosh(\sqrt{\mathcal{E}}/2) - \mathcal{E}_2 \sinh(\sqrt{\mathcal{E}}/2)} \right) \right) + (-6 \mathcal{E}_3^2 - 6 + 12 \mathcal{E}_3) \\
\times \left( d + \frac{2 \mathcal{E}_1 \sinh(\sqrt{\mathcal{E}}/2)}{\sqrt{\mathcal{E}} \cosh(\sqrt{\mathcal{E}}/2) - \mathcal{E}_2 \sinh(\sqrt{\mathcal{E}}/2)} \right)^2, \]

(2.80)

\[ U(3) = - (6 \mathcal{E}_3 - 6) \mathcal{E}_1 - \mathcal{E}_2^2 - (-6 \mathcal{E}_3 d + 6 d) \mathcal{E}_2 - 6 \mathcal{E}_3^2 d^2 + 12 \mathcal{E}_3 d^2 - 6 d^2 + (-\mathcal{E}_2 (6 \mathcal{E}_3 - 6) + 12 \mathcal{E}_3^2 d \\
- 24 \mathcal{E}_3 d + 12 d) \left( d + \frac{2 \mathcal{E}_1 \cosh(\sqrt{\mathcal{E}}/2)}{\sqrt{\mathcal{E}} \sinh(\sqrt{\mathcal{E}}/2) - \mathcal{E}_2 \cosh(\sqrt{\mathcal{E}}/2) \pm i \sqrt{\mathcal{E}}} \right) \right) + (-6 \mathcal{E}_3^2 - 6 + 12 \mathcal{E}_3) (d \\
+ \frac{2 \mathcal{E}_1 \cosh(\sqrt{\mathcal{E}}/2)}{\sqrt{\mathcal{E}} \sinh(\sqrt{\mathcal{E}}/2) - \mathcal{E}_2 \cosh(\sqrt{\mathcal{E}}/2) \pm i \sqrt{\mathcal{E}}} )^2, \]

(2.81)

\[ U(3) = - (6 \mathcal{E}_3 - 6) \mathcal{E}_1 - \mathcal{E}_2^2 - (-6 \mathcal{E}_3 d + 6 d) \mathcal{E}_2 - 6 \mathcal{E}_3^2 d^2 + 12 \mathcal{E}_3 d^2 - 6 d^2 + (-\mathcal{E}_2 (6 \mathcal{E}_3 - 6) + 12 \mathcal{E}_3^2 d \\
- 24 \mathcal{E}_3 d + 12 d) \left( d + \frac{2 \mathcal{E}_1 \sinh(\sqrt{\mathcal{E}}/2)}{\sqrt{\mathcal{E}} \cosh(\sqrt{\mathcal{E}}/2) - \mathcal{E}_2 \sinh(\sqrt{\mathcal{E}}/2) \pm i \sqrt{\mathcal{E}}} \right) + (-6 \mathcal{E}_3^2 - 6 + 12 \mathcal{E}_3) (d \\
+ \frac{2 \mathcal{E}_1 \sinh(\sqrt{\mathcal{E}}/2)}{\sqrt{\mathcal{E}} \cosh(\sqrt{\mathcal{E}}/2) - \mathcal{E}_2 \sinh(\sqrt{\mathcal{E}}/2) \pm i \sqrt{\mathcal{E}}} )^2, \]

(2.82)

While \( A, B \) are arbitrary real constants and \( A^2 + B^2 > 0 \).

When \( (\mathcal{C} = \mathcal{E}_2^2 - 4 \mathcal{E}_2 \mathcal{E}_1 + 4 \mathcal{E}_1 < 0) \) and \( (\mathcal{E}_2(\mathcal{E}_3 - 1) \neq 0) \) or \( (\mathcal{E}_1(\mathcal{E}_3 - 1) \neq 0) \):

\[ U(3) = - (6 \mathcal{E}_3 - 6) \mathcal{E}_1 - \mathcal{E}_2^2 - (-6 \mathcal{E}_3 d + 6 d) \mathcal{E}_2 - 6 \mathcal{E}_3^2 d^2 + 12 \mathcal{E}_3 d^2 - 6 d^2 + (-\mathcal{E}_2 (6 \mathcal{E}_3 - 6) + 12 \mathcal{E}_3^2 d \\
- 24 \mathcal{E}_3 d + 12 d) \left( d + \frac{1}{2(\mathcal{E}_3 - 1)}(-\mathcal{E}_2 + \sqrt{-\mathcal{C}} \tanh(\sqrt{-\mathcal{C}}/2))) \right) + (-6 \mathcal{E}_3^2 - 6 + 12 \mathcal{E}_3) \\
\left( d + \frac{1}{2(\mathcal{E}_3 - 1)}(-\mathcal{E}_2 + \sqrt{-\mathcal{C}} \tanh(\sqrt{-\mathcal{C}}/2))) \right)^2, \]

(2.83)

\[ U(3) = - (6 \mathcal{E}_3 - 6) \mathcal{E}_1 - \mathcal{E}_2^2 - (-6 \mathcal{E}_3 d + 6 d) \mathcal{E}_2 - 6 \mathcal{E}_3^2 d^2 + 12 \mathcal{E}_3 d^2 - 6 d^2 + (-\mathcal{E}_2 (6 \mathcal{E}_3 - 6) + 12 \mathcal{E}_3^2 d \\
- 24 \mathcal{E}_3 d + 12 d) \left( d - \frac{1}{2(\mathcal{E}_3 - 1)}(\mathcal{E}_2 + \sqrt{-\mathcal{C}} \coth(\sqrt{-\mathcal{C}}/2))) \right) + (-6 \mathcal{E}_3^2 - 6 + 12 \mathcal{E}_3) \\
\times \left( d - \frac{1}{2(\mathcal{E}_3 - 1)}(\mathcal{E}_2 + \sqrt{-\mathcal{C}} \coth(\sqrt{-\mathcal{C}}/2))) \right)^2, \]

(2.84)
\[ \mathcal{U}(3) = - (6 \mathcal{E}_3 - 6) \mathcal{E}_1 - \mathcal{E}_2^2 - (-6 \mathcal{E}_3 d + 6 d) \mathcal{E}_2 - 6 \mathcal{E}_3 \mathcal{E}_2 d^2 + 12 \mathcal{E}_3 d^2 - 6 d^2 + (-\mathcal{E}_2) (6 \mathcal{E}_3 - 6) + 12 \mathcal{E}_3 d d \\
\]
\[ - 24 \mathcal{E}_3 d + 12 d) (d + \frac{1}{2(E_3 - 1)} (\mathcal{E}_2 + \sqrt{-\mathcal{E}} \tan(\sqrt{-\mathcal{E}} 3) \pm \sec(\sqrt{-\mathcal{E}} 3))) + (-6 \mathcal{E}_3^2 - 6 \mathcal{E}_3 d + 12 d) (d + \frac{1}{2(E_3 - 1)} (\mathcal{E}_2 + \sqrt{-\mathcal{E}} \tan(\sqrt{-\mathcal{E}} 3) \pm \sec(\sqrt{-\mathcal{E}} 3)))^2, \]
(2.85)
\[ \mathcal{U}(3) = - (6 \mathcal{E}_3 - 6) \mathcal{E}_1 - \mathcal{E}_2^2 - (-6 \mathcal{E}_3 d + 6 d) \mathcal{E}_2 - 6 \mathcal{E}_3 \mathcal{E}_2 d^2 + 12 \mathcal{E}_3 d^2 - 6 d^2 + (-\mathcal{E}_2) (6 \mathcal{E}_3 - 6) + 12 \mathcal{E}_3 d d \\
\]
\[ - 6 + 12 \mathcal{E}_3) (d + \frac{1}{2(E_3 - 1)} (\mathcal{E}_2 + \sqrt{-\mathcal{E}} \tan(\sqrt{-\mathcal{E}} 3) \pm \sec(\sqrt{-\mathcal{E}} 3)))^2, \]
(2.86)
\[ \mathcal{U}(3) = - (6 \mathcal{E}_3 - 6) \mathcal{E}_1 - \mathcal{E}_2^2 - (-6 \mathcal{E}_3 d + 6 d) \mathcal{E}_2 - 6 \mathcal{E}_3 \mathcal{E}_2 d^2 + 12 \mathcal{E}_3 d^2 - 6 d^2 + (-\mathcal{E}_2) (6 \mathcal{E}_3 - 6) + 12 \mathcal{E}_3 d d \\
\]
\[ - 6 + 12 \mathcal{E}_3) (d + \frac{1}{2(E_3 - 1)} (\mathcal{E}_2 + \sqrt{-\mathcal{E}} \tan(\sqrt{-\mathcal{E}} 3) \pm \sec(\sqrt{-\mathcal{E}} 3)))^2, \]
(2.87)
\[ \mathcal{U}(3) = - (6 \mathcal{E}_3 - 6) \mathcal{E}_1 - \mathcal{E}_2^2 - (-6 \mathcal{E}_3 d + 6 d) \mathcal{E}_2 - 6 \mathcal{E}_3 \mathcal{E}_2 d^2 + 12 \mathcal{E}_3 d^2 - 6 d^2 + (-\mathcal{E}_2) (6 \mathcal{E}_3 - 6) + 12 \mathcal{E}_3 d d \\
\]
\[ - 6 + 12 \mathcal{E}_3) (d + \frac{1}{4(E_3 - 1)} (-2 \mathcal{E}_2 + \sqrt{-\mathcal{E}} \tan(\sqrt{-\mathcal{E}} 3) \pm \cot(\sqrt{-\mathcal{E}} 3))) + (-6 \mathcal{E}_3^2 \\
\]
\[ - 6 + 12 \mathcal{E}_3) (d + \frac{1}{2(E_3 - 1)} (-2 \mathcal{E}_2 + \sqrt{-\mathcal{E}} \tan(\sqrt{-\mathcal{E}} 3) \pm \cot(\sqrt{-\mathcal{E}} 3)))^2, \]
(2.88)
\[ \mathcal{U}(3) = - (6 \mathcal{E}_3 - 6) \mathcal{E}_1 - \mathcal{E}_2^2 - (-6 \mathcal{E}_3 d + 6 d) \mathcal{E}_2 - 6 \mathcal{E}_3 \mathcal{E}_2 d^2 + 12 \mathcal{E}_3 d^2 - 6 d^2 + (-\mathcal{E}_2) (6 \mathcal{E}_3 - 6) + 12 \mathcal{E}_3 d d \\
\]
\[ - 24 \mathcal{E}_3 d + 12 d) (d + \frac{1}{2(E_3 - 1)} (\mathcal{E}_2 + \pm \sqrt{-\mathcal{E}} (A^2 - B^2) - A \sqrt{-\mathcal{E}} \cos(\sqrt{-\mathcal{E}} 3)) \pm \sec(\sqrt{-\mathcal{E}} 3) \\
\]
\[ + \left(-6 \mathcal{E}_3^2 - 6 + 12 \mathcal{E}_3 \right) \left( d + \frac{1}{2(E_3 - 1)} (\mathcal{E}_2 + \pm \sqrt{-\mathcal{E}} (A^2 - B^2) - A \sqrt{-\mathcal{E}} \cos(\sqrt{-\mathcal{E}} 3)) \right)^2, \]
(2.89)
\[ \mathcal{U}(3) = - (6 \mathcal{E}_3 - 6) \mathcal{E}_1 - \mathcal{E}_2^2 - (-6 \mathcal{E}_3 d + 6 d) \mathcal{E}_2 - 6 \mathcal{E}_3 \mathcal{E}_2 d^2 + 12 \mathcal{E}_3 d^2 - 6 d^2 + (-\mathcal{E}_2) (6 \mathcal{E}_3 - 6) + 12 \mathcal{E}_3 d d \\
\]
\[ - 24 \mathcal{E}_3 d + 12 d) (d - \frac{2\mathcal{E}_1 \cos(\frac{\sqrt{-\mathcal{E}} 3}{2})}{\sqrt{-\mathcal{E}} \sin(\frac{\sqrt{-\mathcal{E}} 3}{2}) + \mathcal{E}_2 \cos(\frac{\sqrt{-\mathcal{E}} 3}{2})} + (-6 \mathcal{E}_3^2 - 6 + 12 \mathcal{E}_3 \right) \\
\times (d - \frac{2\mathcal{E}_1 \cos(\frac{\sqrt{-\mathcal{E}} 3}{2})}{\sqrt{-\mathcal{E}} \sin(\frac{\sqrt{-\mathcal{E}} 3}{2}) + \mathcal{E}_2 \cos(\frac{\sqrt{-\mathcal{E}} 3}{2})})^2, \]
(2.90)
\[ \mathcal{U}(3) = -(6 \mathcal{E}_3 - 6) \mathcal{E}_1 - \mathcal{E}_2^2 - (-6 \mathcal{E}_3 d + 6 d) \mathcal{E}_2 - 6 \mathcal{E}_3^2 d^2 + 12 \mathcal{E}_3 d^2 - 6 d^2 + (-\mathcal{E}_2 (6 \mathcal{E}_3 - 6) + 12 \mathcal{E}_3^2 d \\
- 24 \mathcal{E}_3 d + 12 d)(d + \frac{2 \mathcal{E}_1 \sin(\sqrt{-\mathcal{E}_3} \frac{\theta}{2})}{\sqrt{-\mathcal{E}_3} \cos(\sqrt{-\mathcal{E}_3} \frac{\theta}{2}) - \mathcal{E}_2 \sin(\sqrt{-\mathcal{E}_3} \frac{\theta}{2})}) + (-6 \mathcal{E}_3^2 - 6 + 12 \mathcal{E}_3) \\
\times \left(d + \frac{2 \mathcal{E}_1 \sin(\sqrt{-\mathcal{E}_3} \frac{\theta}{2})}{\sqrt{-\mathcal{E}_3} \cos(\sqrt{-\mathcal{E}_3} \frac{\theta}{2}) - \mathcal{E}_2 \sin(\sqrt{-\mathcal{E}_3} \frac{\theta}{2})}\right)^2, \] (2.91)

While \( A, B \) are arbitrary real constants and \( A^2 - B^2 > 0. \)

When \( \mathcal{E}_1 = 0 \) and \( \mathcal{E}_2(\mathcal{E}_3 - 1) \neq 0, \) we have:

\[ \mathcal{U}(3) = -(6 \mathcal{E}_3 - 6) \mathcal{E}_1 - \mathcal{E}_2^2 - (-6 \mathcal{E}_3 d + 6 d) \mathcal{E}_2 - 6 \mathcal{E}_3^2 d^2 + 12 \mathcal{E}_3 d^2 - 6 d^2 + (-\mathcal{E}_2 (6 \mathcal{E}_3 - 6) + 12 \mathcal{E}_3^2 d \\
- 24 \mathcal{E}_3 d + 12 d)(d - \frac{\mathcal{E}_2 k}{(\mathcal{E}_3 - 1)(k + \cosh(\mathcal{E}_2 \frac{\theta}{2}) - \sinh(\mathcal{E}_2 \frac{\theta}{2}))}) + (-6 \mathcal{E}_3^2 - 6 + 12 \mathcal{E}_3) \\
\times \left(d - \frac{\mathcal{E}_2 k}{(\mathcal{E}_3 - 1)(k + \cosh(\mathcal{E}_2 \frac{\theta}{2}) - \sinh(\mathcal{E}_2 \frac{\theta}{2}))}\right)^2, \] (2.93)

\[ \mathcal{U}(3) = -(6 \mathcal{E}_3 - 6) \mathcal{E}_1 - \mathcal{E}_2^2 - (-6 \mathcal{E}_3 d + 6 d) \mathcal{E}_2 - 6 \mathcal{E}_3^2 d^2 + 12 \mathcal{E}_3 d^2 - 6 d^2 + (-\mathcal{E}_2 (6 \mathcal{E}_3 - 6) + 12 \mathcal{E}_3^2 d \\
- 24 \mathcal{E}_3 d + 12 d)(d - \frac{\mathcal{E}_2 (\cosh(\mathcal{E}_2 \frac{\theta}{2}) + \sinh(\mathcal{E}_2 \frac{\theta}{2}))}{(\mathcal{E}_3 - 1)(k + \cosh(\mathcal{E}_2 \frac{\theta}{2}) + \sinh(\mathcal{E}_2 \frac{\theta}{2}))}) + (-6 \mathcal{E}_3^2 - 6 + 12 \mathcal{E}_3) \\
\times \left(d - \frac{\mathcal{E}_2 (\cosh(\mathcal{E}_2 \frac{\theta}{2}) + \sinh(\mathcal{E}_2 \frac{\theta}{2}))}{(\mathcal{E}_3 - 1)(k + \cosh(\mathcal{E}_2 \frac{\theta}{2}) + \sinh(\mathcal{E}_2 \frac{\theta}{2}))}\right)^2, \] (2.94)

\[ \mathcal{U}(3) = -(6 \mathcal{E}_3 - 6) \mathcal{E}_1 - \mathcal{E}_2^2 - (-6 \mathcal{E}_3 d + 6 d) \mathcal{E}_2 - 6 \mathcal{E}_3^2 d^2 + 12 \mathcal{E}_3 d^2 - 6 d^2 + (-\mathcal{E}_2 (6 \mathcal{E}_3 - 6) + 12 \mathcal{E}_3^2 d \\
- 24 \mathcal{E}_3 d + 12 d)(d - \frac{1}{(\mathcal{E}_3 - 1)\frac{\theta}{3} + C}) + (-6 \mathcal{E}_3^2 - 6 + 12 \mathcal{E}_3) \left(d - \frac{1}{(\mathcal{E}_3 - 1)\frac{\theta}{3} + C}\right)^2, \] (2.95)

Set \( C \)

\[ c = \sqrt{4 \mathcal{E}_1 \mathcal{E}_3 - 4 \mathcal{E}_1 + 1 - \mathcal{E}_2^2}, \ a_{-2} = -6 \mathcal{E}_1^2 - (-12 \mathcal{E}_2 d - 12 d^2 + 12 \mathcal{E}_3 d^2) \mathcal{E}_1 - 6 \mathcal{E}_2^2 d^2 - (-12 \mathcal{E}_3 d^3 \\
+ 12 d^3) \mathcal{E}_2 - 6 d^4 - 6 \mathcal{E}_3^2 d^4 + 12 \mathcal{E}_3 d^4, \ a_{-1} = (-12 \mathcal{E}_3 d + 12 d + 6 \mathcal{E}_2) \mathcal{E}_1 + 6 \mathcal{E}_2^2 d - (18 \mathcal{E}_3^2 d^2 \\
- 18 d^2) \mathcal{E}_2 + 12 \mathcal{E}_3^2 d^2 - 24 \mathcal{E}_3 d^3 + 12 d^3, \ a_1 = a_2 = 0, \ a_0 = (2 \mathcal{E}_3 - 2) \mathcal{E}_1 - \mathcal{E}_2^2 - (-6 \mathcal{E}_3 d + 6 d) \mathcal{E}_2 \\
- 6 \mathcal{E}_3^2 d^2 + 12 \mathcal{E}_3 d^2 - 6 d^2. \]
Thus, the model’s soliton solutions are given by

When \((E = E_1^2 - 4 E_2 E_1 + 4 E_1 > 0)\) and \((E_2(E_3 - 1) \neq 0)\) or \((E_1(E_3 - 1) \neq 0)\):

\[
\mathcal{U}(3) = (-6 E_1^2 - (-12 E_2 d - 12 d^2 + 12 E_3 d^2) E_1 - 6 E_2^2 d^2 - (-12 E_3 d^3 + 12 d^3) E_2 - 6 d^4 - 6 E_3^2 d^4
+ 12 E_3 d^4)(d - \frac{1}{2(E_3 - 1)}(E_2 + \sqrt{E} \tan(rac{\sqrt{E}}{2}))^{-2})^2 + (-(-12 E_3 d + 12 d + 6 E_2) E_1 + 6 E_2^2 d
- (18 E_3 d^2 - 18 d^2) E_2 + 12 E_3^2 d^3 - 24 E_2 d^3 + 12 d^3)(d - \frac{1}{2(E_3 - 1)}(E_2 + \sqrt{E} \tan(rac{\sqrt{E}}{2}))^{-1})^2
- (2 E_3 - 2) E_1 - E_2^2 - (-6 E_3 d + 6 d) E_2 - 6 E_3^2 d^2 + 12 E_3 d^2 - 6 d^2),
\]

\[
\mathcal{U}(3) = (-6 E_1^2 - (-12 E_2 d - 12 d^2 + 12 E_3 d^2) E_1 - 6 E_2^2 d^2 - (-12 E_3 d^3 + 12 d^3) E_2 - 6 d^4 - 6 E_3^2 d^4
+ 12 E_3 d^4)(d - \frac{1}{2(E_3 - 1)}(E_2 + \sqrt{E} \coth(rac{\sqrt{E}}{2}))^{-2})^2 + (-(-12 E_3 d + 12 d + 6 E_2) E_1 + 6 E_2^2 d
- (18 E_3 d^2 - 18 d^2) E_2 + 12 E_3^2 d^3 - 24 E_2 d^3 + 12 d^3)(d - \frac{1}{2(E_3 - 1)}(E_2 + \sqrt{E} \coth(rac{\sqrt{E}}{2}))^{-1})^2
- (2 E_3 - 2) E_1 - E_2^2 - (-6 E_3 d + 6 d) E_2 - 6 E_3^2 d^2 + 12 E_3 d^2 - 6 d^2),
\]

\[
\mathcal{U}(3) = (-6 E_1^2 - (-12 E_2 d - 12 d^2 + 12 E_3 d^2) E_1 - 6 E_2^2 d^2 - (-12 E_3 d^3 + 12 d^3) E_2 - 6 d^4
+ 6 E_3^2 d^4 + 12 E_3 d^4)(d - \frac{1}{2(E_3 - 1)}(E_2 + \sqrt{E} \coth(rac{\sqrt{E}}{2}) \pm csch(\frac{\sqrt{E}}{2}))^{-2})^2
+ (-(-12 E_3 d + 12 d + 6 E_2) E_1 + 6 E_2^2 d - (18 E_3 d^2 - 18 d^2) E_2 + 12 E_3^2 d^3 - 24 E_3 d^3)(d - \frac{1}{2(E_3 - 1)}(E_2 + \sqrt{E} \coth(rac{\sqrt{E}}{2}) \pm csch(\frac{\sqrt{E}}{2})))^{-1} - (2 E_3 - 2) E_1 - E_2^2
- (-6 E_3 d + 6 d) E_2 - 6 E_3^2 d^2 + 12 E_3 d^2 - 6 d^2),
\]

\[
\mathcal{U}(3) = (-6 E_1^2 - (-12 E_2 d - 12 d^2 + 12 E_3 d^2) E_1 - 6 E_2^2 d^2 - (-12 E_3 d^3 + 12 d^3) E_2 - 6 d^4
+ 6 E_3^2 d^4 + 12 E_3 d^4)(d - \frac{1}{4(E_3 - 1)}(2 E_2 + \sqrt{E} \tan(rac{\sqrt{E}}{4}))^{-2})^2
+ (-(-12 E_3 d + 12 d + 6 E_2) E_1 + 6 E_2^2 d - (18 E_3 d^2 - 18 d^2) E_2 + 12 E_3^2 d^3)(d - \frac{1}{4(E_3 - 1)}(2 E_2 + \sqrt{E} \tan(rac{\sqrt{E}}{4}))^{-1})^2
- (2 E_3 - 2) E_1 - E_2^2 - (-6 E_3 d + 6 d) E_2 - 6 E_3^2 d^2 + 12 E_3 d^2 - 6 d^2),
\]
\[ U(3) = \begin{aligned} &(-6 E_1^2 - (-12 E_2 d - 12 d^2 + 12 E_3 d^3) E_1 - 6 E_2^2 d^2 - (-12 E_3 d^3 + 12 d^3) E_2 - 6 d^4 \\
&- 6 E_3^2 d^4 + 12 E_3 d^4)(d + \frac{1}{2(E_3 - 1)}(-E_2 + \frac{\pm \sqrt{E}(A^2 + B^2)}{A \sinh(\sqrt{E} 3) + B}) - 2 d^2 + 12 E_3^2 d^3 \end{aligned} \]

\[ U(3) = (-6 E_1^2 - (-12 E_2 d - 12 d^2 + 12 E_3 d^3) E_1 - 6 E_2^2 d^2 - (-12 E_3 d^3 + 12 d^3) E_2 - 6 d^4 \\
- 6 E_3^2 d^4 + 12 E_3 d^4)(d + \frac{1}{2(E_3 - 1)}(-E_2 + \frac{\pm \sqrt{E}(A^2 + B^2)}{A \sinh(\sqrt{E} 3) + B}) - 2 d^2 + 12 E_3^2 d^3 \]

\[ U(3) = \begin{aligned} &(-6 E_1^2 - (-12 E_2 d - 12 d^2 + 12 E_3 d^3) E_1 - 6 E_2^2 d^2 - (-12 E_3 d^3 + 12 d^3) E_2 - 6 d^4 - 6 E_3^2 d^4 \\
&+ 12 E_3 d^4)(d + \frac{2 E_1 \cosh(\sqrt{E} 3)}{\sqrt{E} \sinh(\sqrt{E} 3) - E_2 \cosh(\sqrt{E} 3)}) - (18 E_3 d^2 - 18 d^2) E_2 + 12 E_3^2 d^3 \end{aligned} \]

\[ U(3) = \begin{aligned} &(-6 E_1^2 - (-12 E_2 d - 12 d^2 + 12 E_3 d^3) E_1 - 6 E_2^2 d^2 - (-12 E_3 d^3 + 12 d^3) E_2 - 6 d^4 - 6 E_3^2 d^4 \\
&+ 12 E_3 d^4)(d + \frac{2 E_1 \sinh(\sqrt{E} 3)}{\sqrt{E} \cosh(\sqrt{E} 3) - E_2 \sinh(\sqrt{E} 3)}) - (18 E_3 d^2 - 18 d^2) E_2 + 12 E_3^2 d^3 \end{aligned} \]
\[ U(3) = (-6E_1^2 - (-12E_2 d - 12 d^2 + 12 E_3 d^3)E_1 - 6E_2^2 d^2 - (-12 E_3 d^3 + 12 d^3)E_2 - 6E_3^2 d^4 \]
\[ + 12 E_3 d^4)(d + \frac{2E_1 \cosh(\sqrt{E} 3)}{\sqrt{E} \sinh(\sqrt{E} 3) - E_2 \cosh(\sqrt{E} 3) \pm i \sqrt{E}})^{-2} + (-(-12 E_3 d + 12 d \]
\[ + 6 E_2)E_1 + 6 E_2^2 d - (18 E_3 d^2 - 18 d^2)E_2 + 12 E_3^2 d^3 - 24 E_3 d^3 + 12 d^3) \]
\[ \times (d + \frac{2E_1 \cosh(\sqrt{E} 3)}{\sqrt{E} \sinh(\sqrt{E} 3) - E_2 \cosh(\sqrt{E} 3) \pm i \sqrt{E}})^{-1} - (2 \frac{E_3 - 2)E_1 - E_2^2 - (-6 E_3 d + 6 d)E_2}{2} - 6 E_3^2 d^2 + 12 E_3 d^2 - 6 d^2, \]
\[ (2.105) \]

\[ U(3) = (-6E_1^2 - (-12E_2 d - 12 d^2 + 12 E_3 d^3)E_1 - 6E_2^2 d^2 - (-12 E_3 d^3 + 12 d^3)E_2 - 6E_3^2 d^4 \]
\[ + 12 E_3 d^4)(d + \frac{2E_1 \sinh(\sqrt{E} 3)}{\sqrt{E} \cosh(\sqrt{E} 3) - E_2 \sinh(\sqrt{E} 3) \pm i \sqrt{E}})^{-2} + (-(-12 E_3 d + 12 d + 6 E_2)E_1 \]
\[ + 6 E_2^2 d - (18 E_3 d^2 - 18 d^2)E_2 + 12 E_3^2 d^3 - 24 E_3 d^3 + 12 d^3) \]
\[ \times \frac{2E_1 \sinh(\sqrt{E} 3)}{\sqrt{E} \cosh(\sqrt{E} 3) - E_2 \sinh(\sqrt{E} 3) \pm i \sqrt{E}})^{-1} - (2 \frac{E_3 - 2)E_1 - E_2^2 - (-6 E_3 d + 6 d)E_2}{2} - 6 E_3^2 d^2 + 12 E_3 d^2 - 6 d^2, \]
\[ (2.106) \]

While \( A, B \) are arbitrary real constants and \( A^2 + B^2 > 0 \).

When \( (E = E_2 - (E_2 E_1 + 4 E_1 < 0) \) and \( (E_2 E_3 - 1) \neq 0 \) or \( (E_1 E_3 - 1) \neq 0 \):

\[ U(3) = (-6E_1^2 - (-12E_2 d - 12 d^2 + 12 E_3 d^3)E_1 - 6E_2^2 d^2 - (-12 E_3 d^3 + 12 d^3)E_2 - 6E_3^2 d^4 \]
\[ + 12 E_3 d^4)(d + \frac{1}{2(E_3 - 1)}(-E_2 + \sqrt{-E} \tanh(\frac{-E}{2} )))^{-2} + (-(-12 E_3 d + 12 d + 6 E_2)E_1 \]
\[ + 6 E_2^2 d - (18 E_3 d^2 - 18 d^2)E_2 + 12 E_3^2 d^3 - 24 E_3 d^3 + 12 d^3) \]
\[ \times \frac{1}{2(E_3 - 1)}(-E_2 + \sqrt{-E} \tanh(\frac{-E}{2} )))^{-1} - (2 \frac{E_3 - 2)E_1 - E_2^2 - (-6 E_3 d + 6 d)E_2 - 6 E_3^2 d^2 + 12 E_3 d^2 - 6 d^2, \]
\[ (2.107) \]

\[ U(3) = (-6E_1^2 - (-12E_2 d - 12 d^2 + 12 E_3 d^3)E_1 - 6E_2^2 d^2 - (-12 E_3 d^3 + 12 d^3)E_2 - 6E_3^2 d^4 \]
\[ + 12 E_3 d^4)(d + \frac{1}{2(E_3 - 1)}(-E_2 + \sqrt{-E} \coth(\frac{-E}{2} )))^{-2} + (-(-12 E_3 d + 12 d + 6 E_2)E_1 + 6 E_2^2 d \]
\[ - (18 E_3 d^2 - 18 d^2)E_2 + 12 E_3^2 d^3 - 24 E_3 d^3 + 12 d^3) \]
\[ \times \frac{1}{2(E_3 - 1)}(-E_2 + \sqrt{-E} \coth(\frac{-E}{2} )))^{-1} - (2 \frac{E_3 - 2)E_1 - E_2^2 - (-6 E_3 d + 6 d)E_2 - 6 E_3^2 d^2 + 12 E_3 d^2 - 6 d^2, \]
\[ (2.108) \]

\[ U(3) = (-6E_1^2 - (-12E_2 d - 12 d^2 + 12 E_3 d^3)E_1 - 6E_2^2 d^2 - (-12 E_3 d^3 + 12 d^3)E_2 - 6E_3^2 d^4 \]
\[ + 12 E_3 d^4)(d + \frac{1}{2(E_3 - 1)}(-E_2 + \sqrt{-E} (\tan(\sqrt{-E} 3)) \pm \sec(\sqrt{-E} 3)))^{-2} + (-(-12 E_3 d \]
\[ + 12 d + 6 E_2)E_1 + 6 E_2^2 d - (18 E_3 d^2 - 18 d^2)E_2 + 12 E_3^2 d^3 - 24 E_3 d^3 + 12 d^3) \]
\[ \times(-E_2 + \sqrt{-E} (\tan(\sqrt{-E} 3)) \pm \sec(\sqrt{-E} 3)))^{-1} - (2 \frac{E_3 - 2)E_1 - E_2^2}{2} - 6 E_3 d + 6 d)E_2 - 6 E_3^2 d^2 + 12 E_3 d^2 - 6 d^2, \]
\[ (2.109) \]
\[
\mathcal{U}(3) = (-6E_1^2 - (-12E_2 d - 12d^2 + 12E_3 d^3)E_1 - 6E_2^2 d^2 - (-12E_3 d^3 + 12d^3)E_2 - 6d^4 - 6E_3^2 d^4 \\
+ 12E_3 d^4)(d - \frac{1}{2(E_3 - 1)}(E_2 + \sqrt{-C} (\cot(\sqrt{-C} 3) \pm \csc(\sqrt{-C} 3))))^{-2} + (-12E_3 d + 12d \\
+ 6E_2)E_1 + 6E_2^2 d - (18E_3 d^2 - 18d^2)E_2 + 12E_3^2 d^3 - 24E_3 d^3 + 12d^3)(d - \frac{1}{2(E_3 - 1)}(E_2(2.110) \\
+ \sqrt{-C} (\cot(\sqrt{-C} 3) \pm \csc(\sqrt{-C} 3))))^{-1} - (2E_3 - 2)E_1 - E_2^2 - (-6E_3 d + 6d)E_2 \\
- 6E_3^2 d^2 + 12E_3 d^2 - 6d^2.
\]

\[
\mathcal{U}(3) = (-6E_1^2 - (-12E_2 d - 12d^2 + 12E_3 d^3)E_1 - 6E_2^2 d^2 - (-12E_3 d^3 + 12d^3)E_2 - 6d^4 - 6E_3^2 d^4 \\
+ 12E_3 d^4)(d + \frac{1}{4(E_3 - 1)}(-2E_2 + \sqrt{-C} (\tan(\frac{-\sqrt{-C} 4}{3}) - \cot(\frac{-\sqrt{-C} 4}{3}))))^{-2} + (-12E_3 d \\
+ 12d + 6E_2)E_1 + 6E_2^2 d - (18E_3 d^2 - 18d^2)E_2 + 12E_3^2 d^3 - 24E_3 d^3 + 12d^3)(d + \frac{1}{4(E_3 - 1)} \\
\times (-2E_2 + \sqrt{-C} (\tan(\frac{-\sqrt{-C} 4}{3}) - \cot(\frac{-\sqrt{-C} 4}{3}))))^{-1} - (2E_3 - 2)E_1 - E_2^2 \\
- (-6E_3 d + 6d)E_2 - 6E_3^2 d^2 + 12E_3 d^2 - 6d^2.
\]

\[
\mathcal{U}(3) = (-6E_1^2 - (-12E_2 d - 12d^2 + 12E_3 d^3)E_1 - 6E_2^2 d^2 - (-12E_3 d^3 + 12d^3)E_2 - 6d^4 - 6E_3^2 d^4 \\
+ 12E_3 d^4)(d + \frac{1}{2(E_3 - 1)}(-E_2 + \frac{\pm \sqrt{-C} (A^2 - B^2) - A \sqrt{-C} \cos(\sqrt{-C} 3))}{A \sin(\sqrt{-C} 3) + B})^{-2} \\
+ (-12E_3 d + 12d + 6E_2)E_1 + 6E_2^2 d - (18E_3 d^2 - 18d^2)E_2 + 12E_3^2 d^3 - 24E_3 d^3 + 12d^3 \\
\times (d + \frac{1}{2(E_3 - 1)}(-E_2 + \frac{\pm \sqrt{-C} (A^2 - B^2) - A \sqrt{-C} \cos(\sqrt{-C} 3))}{A \sin(\sqrt{-C} 3) + B})^{-1} - (2E_3 - 2)E_1 - E_2^2 \\
- (-6E_3 d + 6d)E_2 - 6E_3^2 d^2 + 12E_3 d^2 - 6d^2.
\]

\[
\mathcal{U}(3) = (-6E_1^2 - (-12E_2 d - 12d^2 + 12E_3 d^3)E_1 - 6E_2^2 d^2 - (-12E_3 d^3 + 12d^3)E_2 - 6d^4 - 6E_3^2 d^4 \\
+ 12E_3 d^4)(d + \frac{1}{2(E_3 - 1)}(-E_2 + \frac{\pm \sqrt{-C} (A^2 - B^2) + A \sqrt{-C} \cos(\sqrt{-C} 3))}{A \sin(\sqrt{-C} 3) + B})^{-2} \\
+ (-12E_3 d + 12d + 6E_2)E_1 + 6E_2^2 d - (18E_3 d^2 - 18d^2)E_2 + 12E_3^2 d^3 - 24E_3 d^3 + 12d^3 \\
\times (d + \frac{1}{2(E_3 - 1)}(-E_2 + \frac{\pm \sqrt{-C} (A^2 - B^2) + A \sqrt{-C} \cos(\sqrt{-C} 3))}{A \sin(\sqrt{-C} 3) + B})^{-1} - (2E_3 - 2)E_1 - E_2^2 \\
- (-6E_3 d + 6d)E_2 - 6E_3^2 d^2 + 12E_3 d^2 - 6d^2.
\]

\[
\mathcal{U}(3) = (-6E_1^2 - (-12E_2 d - 12d^2 + 12E_3 d^3)E_1 - 6E_2^2 d^2 - (-12E_3 d^3 + 12d^3)E_2 - 6d^4 - 6E_3^2 d^4 \\
+ 12E_3 d^4)(d + \frac{1}{2(E_3 - 1)}(-E_2 + \frac{\pm \sqrt{-C} (A^2 - B^2) + A \sqrt{-C} \cos(\sqrt{-C} 3))}{A \sin(\sqrt{-C} 3) + B})^{-2} \\
+ (-12E_3 d + 12d + 6E_2)E_1 + 6E_2^2 d - (18E_3 d^2 - 18d^2)E_2 + 12E_3^2 d^3 - 24E_3 d^3 + 12d^3 \\
\times (d + \frac{1}{2(E_3 - 1)}(-E_2 + \frac{\pm \sqrt{-C} (A^2 - B^2) + A \sqrt{-C} \cos(\sqrt{-C} 3))}{A \sin(\sqrt{-C} 3) + B})^{-1} - (2E_3 - 2)E_1 - E_2^2 \\
- (-6E_3 d + 6d)E_2 - 6E_3^2 d^2 + 12E_3 d^2 - 6d^2.
\]
\[
\mathcal{U}(3) =\left(-6 E_1^2 - (12 E_2 d - 12 d^2 + 12 E_3 d^3)E_1 - 6 E_2^2 d^2 - (12 E_3 d^3 + 12 d^3)E_2 - 6 d^4 - 6 E_3^2 d^4
\right.

\[\left.+ 12 E_3 d^4\right)\left(d - \frac{2 E_1 \cos\left(\frac{\sqrt{-\mathcal{C}}}{2} 3\right)}{\sqrt{-\mathcal{C}} \sin\left(\frac{\sqrt{-\mathcal{C}}}{2} 3\right) + E_2 \cos\left(\frac{\sqrt{-\mathcal{C}}}{2} 3\right)}\right)^2 + \left(-(-12 E_3 d + 12 d + 6 E_2)E_1 + 6 E_2^2 dight.

\[\left.- (18 E_3 d^2 - 18 d^2)E_2 + 12 E_3^2 d^3 - 24 E_3 d^3 + 12 d^3\right)\left(d - \frac{2 E_1 \cos\left(\frac{\sqrt{-\mathcal{C}}}{2} 3\right)}{\sqrt{-\mathcal{C}} \sin\left(\frac{\sqrt{-\mathcal{C}}}{2} 3\right) + E_2 \cos\left(\frac{\sqrt{-\mathcal{C}}}{2} 3\right)}\right)^{-1}

\left.- (2 E_3 - 2)E_1 - E_2^2 - (-6 E_3 d + 6 d)E_2 - 6 E_3^2 d^2 + 12 E_3 d^2 - 6 d^2\right],
\]

(2.114)

\[
\mathcal{U}(3) =\left(-6 E_1^2 - (12 E_2 d - 12 d^2 + 12 E_3 d^3)E_1 - 6 E_2^2 d^2 - (12 E_3 d^3 + 12 d^3)E_2 - 6 d^4 - 6 E_3^2 d^4
\right.

\[\left.+ 12 E_3 d^4\right)\left(d + \frac{2 E_1 \sin\left(\frac{\sqrt{-\mathcal{C}}}{2} 3\right)}{\sqrt{-\mathcal{C}} \cos\left(\sqrt{-\mathcal{C}} 2 3\right) - E_2 \sin\left(\frac{\sqrt{-\mathcal{C}}}{2} 3\right)}\right)^2 + \left(-(-12 E_3 d + 12 d + 6 E_2)E_1 + 6 E_2^2 dight.

\[\left.- (18 E_3 d^2 - 18 d^2)E_2 + 12 E_3^2 d^3 - 24 E_3 d^3 + 12 d^3\right)\left(d + \frac{2 E_1 \sin\left(\frac{\sqrt{-\mathcal{C}}}{2} 3\right)}{\sqrt{-\mathcal{C}} \cos\left(\sqrt{-\mathcal{C}} 2 3\right) - E_2 \sin\left(\frac{\sqrt{-\mathcal{C}}}{2} 3\right)}\right)^{-1}

\left.- (2 E_3 - 2)E_1 - E_2^2 - (-6 E_3 d + 6 d)E_2 - 6 E_3^2 d^2 + 12 E_3 d^2 - 6 d^2\right],
\]

(2.115)

\[
\mathcal{U}(3) =\left(-6 E_1^2 - (12 E_2 d - 12 d^2 + 12 E_3 d^3)E_1 - 6 E_2^2 d^2 - (12 E_3 d^3 + 12 d^3)E_2 - 6 d^4 - 6 E_3^2 d^4
\right.

\[\left.+ 12 E_3 d^4\right)\left(d + \frac{2 E_1 \sin\left(\frac{\sqrt{-\mathcal{C}}}{2} 3\right)}{\sqrt{-\mathcal{C}} \cos\left(\sqrt{-\mathcal{C}} 3\right) - E_2 \sin\left(\frac{\sqrt{-\mathcal{C}}}{2} 3\right) \pm \sqrt{-\mathcal{C}}\right)^2 + \left(-(-12 E_3 d + 12 d + 6 E_2)E_1

\[+ 6 E_2^2 d - (18 E_3 d^2 - 18 d^2)E_2 + 12 E_3^2 d^3 - 24 E_3 d^3 + 12 d^3\right)\left(d
\right.

\[+ \frac{2 E_1 \sin\left(\frac{\sqrt{-\mathcal{C}}}{2} 3\right)}{\sqrt{-\mathcal{C}} \cos\left(\sqrt{-\mathcal{C}} 3\right) - E_2 \sin\left(\frac{\sqrt{-\mathcal{C}}}{2} 3\right) \pm \sqrt{-\mathcal{C}}\right)^{-1} - (2 E_3 - 2)E_1 - E_2^2 - (-6 E_3 d + 6 d)E_2

\[+ 6 E_3^2 d^2 + 12 E_3 d^2 - 6 d^2\right],
\]

(2.116)

while \(A, B\) are arbitrary real constants and \(A^2 - B^2 > 0\).

**When \(E_1 = 0\) and \(E_2(E_3 - 1) \neq 0\), we have:**

\[
\mathcal{U}(3) =\left(-6 E_1^2 - (12 E_2 d - 12 d^2 + 12 E_3 d^3)E_1 - 6 E_2^2 d^2 - (12 E_3 d^3 + 12 d^3)E_2 - 6 d^4 - 6 E_3^2 d^4
\right.

\[\left.+ 12 E_3 d^4\right)\left(d - \frac{E_2 k}{E_3 - 1)(k + \cosh(E_2 3)) - \sinh(E_2 3))\right)^2 + \left(-(-12 E_3 d + 12 d + 6 E_2)E_1

\[+ 6 E_2^2 d - (18 E_3 d^2 - 18 d^2)E_2 + 12 E_3^2 d^3 - 24 E_3 d^3 + 12 d^3\right)\left(d
\right.

\[\times \frac{E_2 k}{(E_3 - 1)(k + \cosh(E_2 3)) - \sinh(E_2 3))^{-1} - (2 E_3 - 2)E_1 - E_2^2 - (-6 E_3 d + 6 d)E_2

\[+ 6 E_3^2 d^2 + 12 E_3 d^2 - 6 d^2\right],
\]

(2.117)
\[ U(3) = (-6E_1^2 - (-12E_2d - 12d^2 + 12E_3d^2)E_1 - 6E_2^2d^2 - (-12E_3d^3 + 12d^3)E_2 - 6d^4 - 6E_3^2d^4) \]
\[
+ 12E_3d^4)(d \left( \frac{E_2(cosh(2E_2Z) + sinh(2E_2Z))}{(E_3 - 1)k + 3} \right) \right)^2 + (-(-12E_3d + 12d + 6E_2)dE_1 + 6E_2^2d

- (18E_3d^2 - 12E_3d^3 - 24E_3d^3 + 12d^2) d \left( \frac{E_2(cosh(2E_2Z) + sinh(2E_2Z))}{(E_3 - 1)k + 3} \right) \right)^{-1}

- (2E_3 - 2)E_1 - E_2^2 - (6E_3d + 6d)E_2 - 6E_3^2d^2 + 12E_3d^2 - 6d^2,
\]

(2.118)

\[ U(3) = (-6E_1^2 - (-12E_2d - 12d^2 + 12E_3d^2)E_1 - 6E_2^2d^2 - (-12E_3d^3 + 12d^3)E_2 - 6d^4 - 6E_3^2d^4) \]
\[
+ 12E_3d^4)(d \left( \frac{1}{(E_3 - 1)3 + C} \right) \right)^2 + (-(-12E_3d + 12d + 6E_2)dE_1 + 6E_2^2d - (18E_3d^2

- 18d^2)E_2 + 12E_3^2d^3 - 24E_3d^3 + 12d^2) d \left( \frac{1}{(E_3 - 1)3 + C} \right) \right)^{-1} - (2E_3 - 2)E_1 - E_2^2

- (6E_3d + 6d)E_2 - 6E_3^2d^2 + 12E_3d^2 - 6d^2,
\]

(2.119)

3. Solutions’ demonstrating

This section explains the constructed solutions through some distinct plots in two, three, and contour graphs under some special values of above-mentioned parameters. The following Figures 1–5 show cone, dark, solitary, bright, and singular dark waves respectively.

**Figure 1.** Cone wave graphs of Eq (2.3) represented by three, two-dimensional and contour plot three dimensional.

**Figure 2.** Dark wave graphs of Eq (2.12) represented by three, two-dimensional and contour plot three dimensional.
Figure 3. Solitary wave graphs of Eq (2.30) represented by three, two-dimensional and contour plot three dimensional.

Figure 4. Bright wave graphs of Eq (2.46) represented by three, two-dimensional and contour plot three dimensional.

Figure 5. Singular dark wave graphs of Eq (2.50) represented by three, two-dimensional and contour plot three dimensional.

4. Results and discussion

Here, the solutions’ physical interpretation and novelty are shown for demonstrating the research paper’s contribution. This process is given in the following items:

- **Employed schemes:**
  Two computational schemes (ESE and NRE methods) have been employed in the nonlinear IPB
model for constructing novel nonlinear soliton lattice wave solutions. These methods have not been applied to this model before. Both schemes depend on different forms of well-known Riccati equation that helps formulate their solutions in various forms such as hyperbolic, trigonometric, exponential and rational forms. Additionally, Their obtained solutions show the dynamical and physical characterizations of the shallow water waves under gravity.

- **Obtained results:**
  The obtained schemes have obtained many different forms of solutions that cover many previous published solutions through using some different schemes. All our solutions are completely different and novel that have been obtained in [25–30].

- **Shown figures:**
  Some solutions have been explained through some different figures in various forms such as contour, two, three-dimensional plots. These solutions show many novel properties of the shallow water waves under gravity, such as cone, solitary, dark, bright and periodic features. These figures have been plotted using a particular value of each show-parameters in the solutions.

5. Conclusions

This article has studied the nonlinear IPB model along with two recent analytical schemes. Many novel solutions have been obtained and demonstrated through some magnificent figures to show many undiscovered features of the considered model. The novelty and paper’s contribution are investigated. The used schemes’ performance shows their effectiveness and the ability to handle many nonlinear evolution equations.

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Conflicts of interest

There is no conflict of interest.

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