BLUE SCHEMES AS RELATIVE SCHEMES AFTER TOÈN AND VAQUIÉ

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ABSTRACT. The Yoneda embedding establishes an equivalence between schemes and schemes relative to abelian groups (in the sense of Toën and Vaquié). Similarly, it establishes an equivalence between monoidal schemes (or $\mathbb{F}_1$-schemes in the sense of Deitmar) and schemes relative to sets. In this text, we show that these equivalences extend to an equivalence of a certain subcategory of blue schemes with a certain subcategory of schemes relative to blue $\mathbb{F}_1$-modules. This includes, next to usual schemes and monoidal schemes, all blue schemes of finite type over $\mathbb{F}_1$. Due to different Grothendieck topologies, however, the Yoneda embedding does not extend to an equivalence between blue schemes with schemes relative to blue $\mathbb{F}_1$-modules.

INTRODUCTION

The usual definition of a scheme is as a locally ringed space that is locally isomorphic to spectra of rings, and the topology on the spectrum of a ring is defined in terms of localizations of this ring. Alternatively, the Yoneda embedding identifies a scheme $X$ with its functor of points $h_X = \text{Hom} (\text{Spec}(-), X)$, which is a sheaf on the category of rings endowed with the Zariski topology. Open immersions can be characterized in terms of flat epimorphisms of finite presentation of rings. This makes it possible to describe schemes in a completely functorial language as sheaves on the Zariski site of rings with an open covering by affine schemes $\text{Hom} (B_i, -)$ where $B_i$ are rings, see Demazure and Gabriel’s book [3].

Toën and Vaquié generalize in [8] this functorial viewpoint from rings, which are commutative monoids in $\mathbb{Z}$-modules, to commutative monoids in any complete and cocomplete closed symmetric monoidal category $C$. This yields the notion of a scheme relative to $C$ as a sheaf on the Zariski site of commutative monoids in $C$ that can be covered by affines.

This yields, in particular, the notion of an $\mathbb{F}_1$-scheme as a scheme relative to sets together with the Cartesian product. Vezzani shows in [9] that taking the functor of points establishes an equivalence between monoidal schemes, aka, $\mathbb{F}_1$-schemes in the sense of Deitmar [2], and $\mathbb{F}_1$-schemes in Toën and Vaquié’s sense.

The purpose of this text is to extend these equivalences of usual and monoidal schemes with schemes relative to $\mathbb{Z}$-modules and sets $[1]$ respectively, to a comparison of the category $\text{Sch}_{\mathbb{F}_1}$ of blue schemes and the category $\text{Sch}_{\mathbb{F}_1}^{\text{rel}}$ of schemes relative to blue $\mathbb{F}_1$-modules, or, for short, relative schemes. This is no longer an equivalence of categories, but it turns out that $\text{Sch}_{\mathbb{F}_1}$ and $\text{Sch}_{\mathbb{F}_1}^{\text{rel}}$ are different, and that the functor of points $h_X$ of a blue scheme $X$ is in general not a relative scheme.

The reason for this digression lies in the fact that the geometrical approach via prime spectra and topological coverings endows the category $\mathcal{Bpr}$ of blueprints with a finer Grothendieck topology than Toën and Vaquié’s functorial approach via flat descend does. While for every blueprint $B$, the functor $h_B = \text{Hom} (B, -)$ is a sheaf for the latter site, it is, in general, not a sheaf for the former site. The sheafification of $h_B$ leads to an idempotent endofunctor $\Gamma : \mathcal{Bpr} \to \mathcal{Bpr}$ together with a functorial blueprint morphisms $\sigma : B \to \Gamma B$, which is called the globalization. This means that the restriction of $\Gamma$ to its image $\Gamma \mathcal{Bpr}$, the subcategory of global blueprints, is isomorphic to the identity functor of $\Gamma \mathcal{Bpr}$, and $[1]$ To be precise, we consider pointed sets in this text, cf. Remark [12].
that the category of affine blue schemes is anti-equivalent to $\Gamma_{\text{BlPr}}$. Therefore, it is more accurate to view blue schemes as functors on $\Gamma_{\text{BlPr}}$.

If we consider the category $\mathcal{A}^{\text{rel}} = \text{BlPr}^{\text{op}}$ of affine relative schemes as a site with respect to flat descend and the category $\mathcal{A}^{\text{blue}} = \Gamma_{\text{BlPr}}^{\text{op}}$ of affine blue schemes as a site with respect to topological coverings, then $\mathcal{G} = \Gamma_{\text{BlPr}} : \mathcal{A}^{\text{rel}} \to \mathcal{A}^{\text{blue}}$ is a morphism of sites. This extends naturally to a functor $\mathcal{G} : \text{Sch}_{\mathbb{F}_1}^{\text{rel}} \to \text{Sch}_{\mathbb{F}_1}$ from relative schemes to blue schemes. We make this precise by introducing affine presentations, which are certain commutative diagrams of affines whose colimit is a (blue or relative) scheme.

However, the inclusion $\iota : \Gamma_{\text{BlPr}} \to \text{BlPr}$ as a subcategory is not a morphism of sites. This makes it more difficult to associate a relative scheme with a blue scheme in a meaningful way. In full generality, this does not seem to be possible. But if the blue scheme $X$ has an affine presentation $U$ such that $\mathcal{F}(U)$ is an affine presentation for some relative scheme $Y$ and if there are “enough” refinements $\mathcal{V}$ of $U$ such that $\mathcal{F}(V)$ is an affine presentation for the same relative scheme $Y$, then the definition $\mathcal{F}(X) = Y$ extends naturally to a functor. We make this idea rigorous with the notion of an algebraic presentation for $X$. Examples of algebraically presented schemes are usual schemes, monoidal schemes and all schemes that are of finite type over $\mathbb{F}_1$.

Thus we obtain a functor $\mathcal{F} : \text{Sch}_{\mathbb{F}_1}^{\text{alg}} \to \text{Sch}_{\mathbb{F}_1}^{\text{rel}}$ where $\text{Sch}_{\mathbb{F}_1}^{\text{alg}}$ is the full subcategory of algebraically presented blue schemes in $\text{Sch}_{\mathbb{F}_1}$. The main result of this note is the following (see Theorem 11.1).

**Theorem.** The functor $\mathcal{F}$ is a faithfully flat embedding of categories, and the composition $\mathcal{G} \circ \mathcal{F} : \text{Sch}_{\mathbb{F}_1}^{\text{alg}} \to \text{Sch}_{\mathbb{F}_1}$ is isomorphic to the embedding of $\text{Sch}_{\mathbb{F}_1}^{\text{alg}}$ as subcategory.

This means that the subcategory $\text{Sch}_{\mathbb{F}_1}^{\text{alg}}$ can be identified with a full subcategory of relative schemes such that $\mathcal{F}$ and $\mathcal{G}$ restrict to mutual inverse equivalences between these subcategories. These equivalences extend the equivalence of usual schemes with schemes relative to $\mathbb{Z}$-modules and the equivalence of monoidal schemes with schemes relative to pointed sets.

**Content overview.** In Section 1 we introduce the concept of an affine presentation, which allows us to talk about scheme-like objects in terms of affine objects. We state certain properties (1–6), which are typically satisfied for theories that generalize schemes. In Section 2 we review Toën and Vaquié’s definition of schemes relative to a complete and cocomplete closed symmetric monoidal category $\mathcal{C}$. We show that properties (1–6) hold in this context.

In Section 3 we review the definition of a blueprint. In Section 4 we set up the notion of a blue module over a blueprint $B$ and reason that blue $B$-modules form a complete and cocomplete closed symmetric monoidal category with respect to the tensor product. We reinterpret blueprints as commutative monoids in the category of blue $\mathbb{F}_1$-modules.

In Section 5 we give an algebraic characterization of morphisms of blueprints that are of finite presentation. In Section 6 we show that a morphism of blueprints is flat if and only if it is a localization. This leads to the characterization of flat epimorphisms of finite presentation as localizations at finitely generated multiplicative subsets.

In Section 7 we recall the definition of a blue scheme, and, in particular, of the globalization of a blueprint. We show that properties (1–6) hold in this context.

In Section 8 we construct the functor $\mathcal{G}$ from relative schemes, i.e. schemes relative to the category of $\mathbb{F}_1$-modules, to blue schemes that extends the globalization functor $\Gamma : \text{BlPr} \to \Gamma_{\text{BlPr}}$. In Section 9 we introduce the notion of an algebraic presentation and show that algebraic presentations are stable under common refinements in a certain sense, which is needed in the construction of the functor $\mathcal{F}$. In Section 10 we construct the functor $\mathcal{F}$ from algebraically presented blue schemes to relative schemes.
In Section 11 we prove the main result, namely, that \( \mathcal{F} \) is fully faithful and that \( \mathcal{G} \circ \mathcal{F} \) is isomorphic to the embedding of algebraically presented blue schemes into blue schemes. We conclude this text with several remarks in Section 12.

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1. Affine presentations

Typically, a theory of scheme-like objects looks as follows: to a category \( \mathcal{B} \) of algebraic objects, one associates a category \( \mathcal{A} \) of affine schemes that is anti-equivalent to \( \mathcal{B} \). This category \( \mathcal{A} \) of affine schemes carries a Grothendieck topology and is embedded as a full subcategory in a category \( \mathcal{F} \), which allows to glue affine schemes in some way. The purpose of this section is to abstract this mechanism of gluing as taking colimits over certain commutative diagrams in \( \mathcal{A} \), which we call affine presentations. At the end of this section, we state certain properties (i)–(vi) that are typically satisfied in scheme-like theory.

Let \( \mathcal{A} \) be a category with finite limits, i.e. for all morphisms \( U \to V \) and \( U' \to V \) the fibre product \( U \times_V U' \) exist. We endow \( \mathcal{A} \) with a Grothendieck pretopology, i.e. with a distinguished class of open morphisms \( U \to V \) (or, for short, opens) and for every \( U \) a collection of covering families \( \{W_i \to U\} \), subject to the usual axioms. We assume that opens are stable under base change along arbitrary morphisms.

Let \( \mathcal{V} \) be a commutative diagram in \( \mathcal{A} \). The category spanned by \( \mathcal{V} \) is the category \( \langle \mathcal{V} \rangle \) whose objects are the objects of \( \mathcal{V} \) and whose morphisms are all possible compositions of morphisms of \( \mathcal{V} \) in \( \mathcal{A} \), including the identity morphisms as compositions. Since \( \mathcal{V} \) is commutative, the morphism sets \( \text{Hom}(U,V) \) of \( \mathcal{C}(\mathcal{V}) \) have at most one element, i.e. \( \langle \mathcal{V} \rangle \) is a thin category. The commutative diagram \( \mathcal{V} \) can be considered as a subdiagram of \( \langle \mathcal{V} \rangle \) and there is a natural functor \( \langle \mathcal{V} \rangle \to \mathcal{A} \).

Definition 1.1. An affine presentation (in \( \mathcal{A} \)) is a commutative diagram \( \mathcal{V} \) in \( \mathcal{A} \) of open morphisms that satisfies the following two properties.

(i) Existence of maximal elements:
For all objects \( W \) in \( \mathcal{V} \) there is a morphism \( W \to U \) in \( \mathcal{V} \) such that \( U \) is a maximal element of \( \mathcal{V} \), i.e. the only possible morphism \( \varphi : U \to U' \) in \( \mathcal{V} \) with domain \( U \) is the identity of \( U \).

(ii) Cocycle condition:
For all diagrams

\[
\begin{array}{ccc}
W & \xrightarrow{\psi_{12}} & U_{12} \\
\downarrow{\psi_{13}} & & \downarrow{\varphi_{12}^{12}} \\
U_{13} & \xrightarrow{\psi_{13}} & U_1 \\
\downarrow{\psi_{23}} & & \downarrow{\varphi_{23}^{13}} \\
U_{23} & \xrightarrow{\psi_{12}} & U_{23} \\
\downarrow{\psi_{13}} & & \downarrow{\varphi_{13}^{12}} \\
U_2 & \xrightarrow{\varphi_{12}^{12}} & U_1 \\
\downarrow{\varphi_{23}^{12}} & & \downarrow{\psi_{13}} \\
U_3 & \xrightarrow{\psi_{23}} & U_3 \\
\end{array}
\]

such that for all \( i,j \in \{1,2,3\} \) and \( k \in \{i,j\} \), the morphisms \( \psi_{ij} : W \to U_{ij} \) are in \( \mathcal{A} \), the morphisms \( \varphi_{k}^{ij} : U_{ij} \to U_k \) are in \( \mathcal{V} \) and the objects \( U_k \) are maximal elements of \( \mathcal{V} \). we have that if two squares of the diagram commute, then the third square commutes, i.e. if \( \varphi_{12}^{12} \circ \psi_{12} = \varphi_{13}^{12} \circ \psi_{13} \) and \( \varphi_{12}^{12} \circ \psi_{12} = \varphi_{23}^{12} \circ \psi_{23} \), then \( \varphi_{13}^{12} \circ \psi_{13} = \varphi_{23}^{12} \circ \psi_{23} \).

A morphism \( \Phi : \mathcal{V} \to \mathcal{V}' \) is a functor \( \Phi : \langle \mathcal{V} \rangle \to \langle \mathcal{V}' \rangle \) between the respective spanned categories, which we denote by the same symbol \( \Phi \) by abuse of language, together with a
family of morphisms $\Phi_U : U \to \Phi(U)$ for each object $U$ of $\mathcal{U}$ such that

\[
\begin{array}{cccc}
U_1 & \xrightarrow{\Phi_{U_1}} & \Phi(U_1) \\
\downarrow & \swarrow_{\psi} & \downarrow \\
U_2 & \xrightarrow{\Phi_{U_2}} & \Phi(U_2)
\end{array}
\]

commutes for every morphism $\psi : U_1 \to U_2$ in $\mathcal{U}$.

Note that if $\mathcal{U}$ is an affine presentation, then $\langle \mathcal{U} \rangle$ is also one, and the natural embedding $\mathcal{U} \to \langle \mathcal{U} \rangle$ is a morphism of affine presentations. We denote by $\mathcal{U}_{\text{max}}$ the family of maximal elements of $\mathcal{U}$. For $U, V$ in $\mathcal{U}_{\text{max}}$, we denote by $\mathcal{U}_{U,V}$ the full subdiagram of all objects $W$ of $\mathcal{U}$ such that there are morphisms $W \to U$ and $W \to V$ in $\mathcal{U}$.

We lend and bend terminology from the theory of manifolds. The atlas of $\mathcal{U}$ is the diagram $\text{Atlas}(\mathcal{U})$ whose objects are $\mathcal{U}_{\text{max}}$ and the disjoint union $\bigsqcup \{ \mathcal{U}_{U,V} \}$ where $U$ and $V$ range through $\mathcal{U}_{\text{max}}$, together with the morphisms $W \to U$ and $W \to V$ for $W$ in $\mathcal{U}_{U,V}$. Note that the atlas of $\mathcal{U}$ is itself an affine presentation. It comes together with a morphism $\text{Atlas}(\mathcal{U}) \to \mathcal{U}$, and a morphism of affine presentations induces a morphism of their atlases in a natural way. The atlas of $\text{Atlas}(\mathcal{U})$ is isomorphic to $\text{Atlas}(\langle \mathcal{U} \rangle)$. We call an affine presentation $\mathcal{U}$ an affine atlas if it is isomorphic to its atlas.

A refinement of $\mathcal{V}$ is a morphism $\Phi : \mathcal{U} \to \mathcal{V}$ of affine presentations such that all morphisms $\Phi_U : U \to \Phi(U)$ are opens and such that for all $V$ in $\text{Atlas}(\mathcal{V})$, the family $\{ \Phi_U : U \to \Phi(U) \}$ with $\Phi(U) = V$ is a covering family of $V$. Note that the natural morphisms $\text{Atlas}(\mathcal{U}) \to \mathcal{U}$ and $\mathcal{U} \to \langle \mathcal{U} \rangle$ are refinements.

**Lemma 1.2.** The category of affine presentations in $\mathcal{A}$ contains fibre products and the base change of a refinement is a refinement.

**Proof.** Let $\Phi : \mathcal{U} \to \mathcal{V}$ and $\Psi : \mathcal{V}' \to \mathcal{V}$ be two morphisms of affine presentations. We construct the fibre product $\mathcal{U}' = \mathcal{U} \times_{\mathcal{V}} \mathcal{V}'$ as follows. The objects of $\mathcal{U}'$ are the fibre products $U \times_{\mathcal{V}} V'$ for every pair of morphisms $U \to V$ in $\Phi$ and $V' \to V$ in $\Psi$. The morphisms of $\mathcal{U}'$ are the morphisms $U \times_{\mathcal{V}} V' \to U' \times_{\mathcal{V}} V'$ that are induced by morphisms $U \to U'$ in $\mathcal{V}$ and the morphisms $U \times_{\mathcal{V}} V' \to U \times_{\mathcal{V}} V'$ that are induced by morphisms $V' \to V'$ in $\mathcal{V}'$. It is clear that $\mathcal{U}'$ together with the canonical morphism $\Phi' : \mathcal{U}' \to \mathcal{V}'$ and $\Psi' : \mathcal{V}' \to \mathcal{V}$ satisfies the universal property of the fibre product $\mathcal{U} \times_{\mathcal{V}} \mathcal{V}'$.

Assume that $\Psi' : \mathcal{U}' \to \mathcal{V}'$ is an refinement. For an open $V \to V'$ in $\Psi'$, the morphism $U \times_{\mathcal{V}} V' \to U$ in $\Psi'$ is open by our assumption that opens are stable under base change. The covering property of an refinement follows for $\Psi' : \mathcal{U}' \to \mathcal{V}'$ from the stability under base change of a Grothendieck pretopology. This shows that refinements are stable under base changes. \hfill \Box

In theories of scheme-like objects one faces typically the following situation. There is a fully faithful embedding $\mathcal{A} \to \mathcal{F}$ of the category of affine schemes into some larger category $\mathcal{F}$, which allows to define a full subcategory $\text{Sch}_{\mathcal{A}}$ of schemes in $\mathcal{F}$ that contains $\mathcal{A}$. These categories satisfy the following properties.

(i) The categories $\mathcal{A}$, $\text{Sch}_{\mathcal{A}}$ and $\mathcal{F}$ contain finite limits and the functors $\mathcal{A} \to \text{Sch}_{\mathcal{A}}$ and $\text{Sch}_{\mathcal{A}} \to \mathcal{F}$ commute with finite limits.

(ii) The category $\mathcal{C}$ contains colimits of affine presentations in $\mathcal{A}$, and these colimits coincide with the class of schemes in $\mathcal{C}$.

(iii) For every scheme $X$, the functor $\text{Hom}(-, X)$ is a sheaf on $\mathcal{A}$.

(iv) Refinements $\mathcal{U} \to \mathcal{V}$ induce isomorphisms $\text{colim} \mathcal{U} \to \text{colim} \mathcal{V}$ of schemes.

(v) Affine presentations $\mathcal{U}$ and $\mathcal{V}$ whose colimits are isomorphic have a common refinement $\mathcal{U} \to \mathcal{W}$ and $\mathcal{V} \to \mathcal{W}$.

(vi) Every morphism of schemes is induced by a morphism of affine presentations.
2. Relative schemes after Toën and Vaquié

We recall the definition of relative schemes from Toën and Vaquié’s paper [8]. Let \( \mathcal{C} \) be a closed symmetric monoidal category that is complete and cocomplete. We denote by Comm(\( \mathcal{C} \)) the category of commutative, associative and unital semigroups in \( \mathcal{C} \). We call an object \( B \) of Comm(\( \mathcal{C} \)) for short a commutative monoid if the context is clear. A B-algebra is a morphism \( B \to C \) of commutative monoids and a B-algebra morphism is a morphism \( C \to C' \) that commutes with the morphisms \( B \to C \) and \( B \to C' \).

A B-module is an object \( M \) together with a morphism \( B \times M \to M \) in \( \mathcal{C} \) that satisfies usual axioms “\( (ab, m) = a \cdot (b \cdot m) \)” and “\( 1 \cdot m = m \)” of a monoid action. A morphism \( M \to N \) of B-modules is a morphism in \( \mathcal{C} \) that commutes with the actions of \( B \) on \( M \) and \( N \), respectively. This defines the (complete and cocomplete) category \( \mathcal{Mod}_B \) of B-modules.

Let \( f : B \to C \) be a morphism of commutative monoids in \( \mathcal{C} \). The morphism \( f \) is flat if \( - \otimes_B C : \mathcal{Mod}_B \to \mathcal{Mod}_C \) commutes with finite limits and colimits. The morphism \( f : B \to C \) is of finite representation if for all directed systems \( \mathcal{D} \), the canonical map

\[
\Psi_{\mathcal{D}} : \text{colim} \, \text{Hom}_B(C, \mathcal{D}) \to \text{Hom}_B(C, \text{colim} \, \mathcal{D})
\]

is bijective.

An affine scheme relative to \( \mathcal{C} \) is an object of the dual category of Comm(\( \mathcal{C} \)), which we denote by Aff_{\mathcal{C}}. Let spec : Comm(\( \mathcal{C} \)) \to Aff_{\mathcal{C}} be the anti-equivalence of dual categories. Let \( f : B \to C \) be a morphism of commutative monoids in \( \mathcal{C} \). Then \( f^\ast : \text{spec} C \to \text{spec} B \) is called a Zariski open immersion if \( f : B \to C \) is a flat epimorphism of finite representation.

A family \( \{ \varphi_i : \text{spec} B_i \to \text{spec} B \}_{i \in I} \) is a covering family if all the morphisms \( \varphi_i \) are Zariski open immersions and if there is a finite subset \( J \subset I \) such that the functor

\[
\Phi = \prod_{j \in J} \otimes_B B_j : \mathcal{Mod}_B \to \prod_{j \in J} \mathcal{Mod}_B
\]

is conservative (i.e. \( f : M \to N \) is an isomorphism if \( \Phi(f) \) is an isomorphism). This endows the category Aff_{\mathcal{C}} of affine schemes with a Grothendieck pretopology, called the Zariski topology of Aff_{\mathcal{C}}. This defines the full subcategory \( \text{Sh}(\text{Aff}_{\mathcal{C}}) \) of sheaves in the category \( \text{Pr}(\text{Aff}_{\mathcal{C}}) \) of pre-sheaves.

We use the characterization of relative schemes as a quotient of a disjoint union of affine schemes by a suitable equivalence relation in order to bypass some notions that are needed in the original definition of a scheme relative to \( \mathcal{C} \) from [8]. Namely, we define a scheme relative to \( \mathcal{C} \) in terms of the following proposition.

**Proposition 2.1.** A sheaf \( F \) on Aff_{\mathcal{C}} is a scheme relative to \( \mathcal{C} \) if and only if it is the colimit (in Sh(Aff_{\mathcal{C}})) of an affine presentation in Aff_{\mathcal{C}}.

**Proof.** Without recalling all definitions from [8], we outline how the proposition follows from [8] Prop. 2.18. If a sheaf \( F \) on Aff_{\mathcal{C}} has an affine presentation \( \mathcal{U} \), then we define an equivalence relation \( R \) on \( X = \coprod \mathcal{U} \) as follows. For \( U, V \in \mathcal{U} \), we define \( R_{U,V} \) as the subsheaf of \( U \times V \) that is generated by the image of \( \coprod W \) with \( W \in \mathcal{U}_{U,V} \). By the definition of a Zariski open subsheaf of an affine scheme, \( R_{U,V} \) is an open subsheaf of both \( U \) and \( V \). We define \( R \) as the disjoint union \( \coprod R_{U,V} \) over all \( U, V \in \mathcal{U} \), which is a subsheaf of \( X \times X \).

In the case \( U = V \), we obtain that \( R_{U,U} \) is the image of the diagonal \( U \to U \times U \). Therefore, \( R \) is reflexive on \( X \). It is symmetric since \( \mathcal{U}_{U,V} = \mathcal{U}_{V,U} \). The cocycle condition of an affine presentation is equivalent to the transitivity of \( R \). By definition of the quotient \( X/R \), it is the same as the colimit of \( \mathcal{U} \).

Given a representation of a scheme \( F \) as \( X/R \), we define an affine presentation \( \mathcal{U} \) as follows. By definition \( X \) is the disjoint union of affine schemes. We define \( \mathcal{U}_{\text{max}} \) as the family of all these affine schemes. The equivalence relation \( R \) is a disjoint union of Zariski open subsheaves \( R_{U,V} \) of \( U \) resp. \( V \). By definition, \( R_{U,V} \) is a (sheaf theoretic) union of affine Zariski opens \( W_k \) of \( U \) and it is a union of affine Zariski opens \( W_l \) of \( V \). Then the
fibre products $Z_{k,j} = W_k \times_{R_{i,j}} W_j$ are affine Zariski opens of both $U$ and $V$, and $R_{i,j}$ is the union of all $Z_{k,j}$. If we define $\mathcal{U}_{U,V}$ as the diagram of these $Z_{k,j}$ together with the inclusions into $U$ and $V$, and $\mathcal{W}$ as the diagram that is the union of $\mathcal{U}_{\max}$ with all $\mathcal{U}_{U,V}$ for $U,V \in \mathcal{U}_{\max}$, together with the morphisms $Z \to U$ and $Z \to V$ for $Z \in \mathcal{U}_{U,V}$.

Then $\mathcal{W}$ is a commutative diagram of Zariski opens and it has maximal elements, namely, $\mathcal{U}_{\max}$. The cocycle condition follows by the transitivity of the equivalence relation $R$. This means that $\mathcal{W}$ is an affine presentation, which is actually an affine atlas, whose colimit is the scheme $F$. □

**Theorem 2.2.** Properties (i)–(vi) from the Section 2 are satisfied for the embedding of $\text{Aff}_\emptyset$ into $\text{Sh} (\text{Aff}_\emptyset)$.

**Proof.** Property (i) is satisfied since $\text{lim} \text{Hom} (-, B_i) = \text{Hom} (-, \lim B_i)$ by the universal property of limits and since schemes are stable under products of sheaves, see [8, Prop. 2.18]. Property (ii) is established by Proposition 2.1. Property (iii) is shown in [8, Cor. 2.11] for affine schemes. Since $X$ and $Y$ are sheaves on $\text{Aff}_\emptyset$, a morphism $W \to Y$ is represented by an affine presentation $\mathcal{W}$ of $W$ and a morphism $\mathcal{W} \to Y$. We can assume that $\mathcal{W} = \text{Atlas}(\mathcal{W})$ since the colimit $X$ of an affine presentation $\mathcal{W}$ only depends on the maximal elements and a covering of their pairwise intersections $U \times_X V$, which are given by the elements of $\mathcal{U}_{U,V}$. By the base change property of Grothendieck pretopologies, the refinement $\mathcal{W} \to \mathcal{W}$ defines a refinement $\mathcal{W} \to \mathcal{W}$ for $\mathcal{W} = \mathcal{W} \times \mathcal{W}$ and a morphism $\mathcal{W} \to \mathcal{W}$. By the local character of Grothendieck pretopologies, colim $\mathcal{W} = W$. This with, it is easy to verify that the induced morphism colim $\mathcal{W} \to \mathcal{W}$ is an isomorphism. This establishes property (iv).

Let $\mathcal{W}$ and $\mathcal{Y}$ be affine presentations such that colim $\mathcal{W} \simeq X \simeq \text{colim} \mathcal{Y}$. We define a common refinement $\mathcal{W}$ of $\mathcal{W}$ and $\mathcal{Y}$ as follows. By (iv), we can assume that $\mathcal{W}$ and $\mathcal{Y}$ are affine atlases. For every $U \in \mathcal{W}$ and $V \in \mathcal{Y}$, the common open subscheme $W = U \times_X V$ of $U$ and $V$ can be covered by affine open subschemes $W_i$. If $U \to U'$ is a morphism in $\mathcal{W}$, $V \in \mathcal{Y}$ and $\{W_i\}$ and $\{W'_i\}$ are the coverings of $U \times_X V$ and $U' \times_X V$, respectively, then we can refine the covering $\{W_i\}$ such that the induced morphism $U \times_X V \to U' \times_X V$ restricts to morphisms $W_i \to W'_i(\iota)$ for all $i$. The same argument holds for morphisms $V \to V'$. All the $W_i$ together with the morphisms $W_i \to W'_i(\iota)$ yield a diagram $\mathcal{W}$, which is commutative since there are no morphisms to compare. Since open immersions are stable under base change ([8, Lemme 2.13]) all morphisms of $\mathcal{W}$ are open immersions. The maximal elements of $\mathcal{W}$ are the $W_i$ that cover $U \times_X V$ for some $U \in \mathcal{W}_{\max}$ and $V \in \mathcal{Y}_{\max}$. The cocycle condition follows easily from the cocycle condition for $U$ and $V$. Therefore $\mathcal{W}$ is an affine presentation, and indeed an affine atlas, that is a common refinement of $\mathcal{W}$ and $\mathcal{Y}$ with respect to the canonical morphisms $\mathcal{W} \to \mathcal{W}$ and $\mathcal{W} \to \mathcal{Y}$. This establishes property (vi).

By similar arguments, we find for a given morphism colim $\mathcal{W} \to \mathcal{W}$ of schemes an refinement $\mathcal{W}' \to \mathcal{W}$ and a morphism $\mathcal{W}' \to \mathcal{Y}$ that induces this morphism of schemes. This is property (vi). □

3. **Blueprints**

We recall the definition of a blueprint. Note that we follow the convention of [6], i.e. all blueprints are proper and with zero, according to the terminology in [5].

By a monoid with zero, we mean a multiplicatively written commutative semigroup $A$ with a neutral element 1 and an absorbing element 0, which are characterized by the properties $1 \cdot a = a$ and $0 \cdot a = 0$ for all $a \in A$. A morphism of monoids with zero is a multiplicative map $f : A_1 \to A_2$ that maps 1 to 1 and 0 to 0.

A blueprint $B$ is a monoid $A$ with zero together with a pre-addition $\mathcal{R}$, i.e. $\mathcal{R}$ is an equivalence relation on the semiring $\mathbb{N}[A] = \{ \sum a_i | a_i \in A \}$ of finite formal sums of elements...
of $A$ that satisfies the following axioms (where we write $\sum a_i \equiv \sum b_j$ whenever $(\sum a_i, \sum b_j) \in \mathcal{R}$):

(i) $\sum a_i \equiv \sum b_j$ and $\sum c_k \equiv \sum d_l$ implies $\sum a_i \pm \sum c_k \equiv \sum b_j \pm \sum d_l$ and $\sum a_i c_k \equiv \sum b_j d_l$,

(ii) $0 \equiv$ (empty sum), and

(iii) if $a \equiv b$, then $a \equiv b$ (as elements in $A$).

A morphism $f : B_1 \rightarrow B_2$ of blueprints is a multiplicative map $f : A_1 \rightarrow A_2$ between the underlying monoids of $B_1$ and $B_2$, respectively, with $f(0) = 0$ and $f(1) = 1$ such that for every relation $\sum a_i \equiv \sum b_j$ in the pre-addition $\mathcal{R}_1$ of $B_1$, the pre-addition $\mathcal{R}_2$ of $B_2$ contains the relation $\sum f(a_i) \equiv \sum f(b_j)$. Let $\mathcal{B}^{pr}$ be the category of blueprints.

In the following, we write $B = A \sslash R$ for a blueprint $B$ with underlying monoid $A$ and pre-addition $\mathcal{R}$. We adopt the conventions used for rings: we identify $B$ with the underlying monoid $A$ and write $a \in B$ or $S \subset B$ when we mean $a \in A$ or $S \subset A$, respectively. Further, we think of a relation $\sum a_i \equiv \sum b_j$ as an equality that holds in $B$ (without the elements $\sum a_i$ and $\sum b_j$ being defined, in general).

Given a set $S$ of relations, there is a smallest equivalence relation $\mathcal{R}$ on $\mathbb{N}[A]$ that contains $S$ and satisfies Axioms (i) and (ii). If $\mathcal{R}$ satisfies also Axiom (iii), then we say that $\mathcal{R}$ is the pre-addition generated by $S$, and we write $\mathcal{R} = \langle S \rangle$. In particular, every monoid $A$ with zero has a smallest pre-addition $\mathcal{R} = \langle \emptyset \rangle$.

More generally, let $A$ be a monoid with zero and $\mathcal{R}$ an equivalence relation on $\mathbb{N}[A]$ that satisfies Axioms (i) and (ii). We can form the quotient set $A' = A/\sim$ where $a \sim b$ whenever $a \equiv b$. Then $A'$ inherits the structure of a monoid by the multiplicity of $\mathcal{R}$, and the image $\mathcal{R}'$ of $\mathcal{R}$ in $\mathbb{N}[A'] \times \mathbb{N}[A']$ is a pre-addition on $A'$ satisfying Axiom (iii) (see Lemma 1.6 in [2] for more details on the construction of the proper quotient). We say that $A/A'$ is a representation of the blueprint $A' \sslash \mathcal{R}'$, and we say that the representation $A/A'$ of $B = A/\mathcal{R}$ is proper if $A = A'$.

4. BLUE $B$-MODULES

In this section, we introduce the notion of a blue $B$-model for a blueprint $B$.

Let $M$ be a pointed set. We denote the base point of $M$ by $\ast$. A pre-addition on $M$ is an equivalence relation $\mathcal{P}$ on the semigroup $\mathbb{N}[M] = \{\sum a_i | a_i \in M\}$ of finite formal sums in $M$ with the following properties (as usual, we write $\sum m_i \equiv \sum n_j$ if $\sum m_i$ stays in relation to $\sum n_j$):

(i) $\sum m_i \equiv \sum n_j$ and $\sum p_k \equiv \sum q_l$ implies $\sum m_i + \sum p_k \equiv \sum n_j + \sum q_l$,

(ii) $\ast \equiv$ (empty sum), and

(iii) if $m \equiv n$, then $m = n$ (in $M$).

Let $B = A \sslash \mathcal{R}$ be a blueprint. A blue $B$-module is a set $M$ together with a pre-addition $\mathcal{P}$ and a $B$-action $B \times M \rightarrow M$, which is a map $(b, m) \mapsto bm$ that satisfies the following properties:

(i) $1 \cdot m = m$, $0 \cdot m = \ast$ and $a \cdot \ast = \ast$,

(ii) $(ab) \cdot m = a \cdot (b \cdot m)$, and

(iii) $\sum a_i \equiv \sum b_j$ and $\sum m_k \equiv \sum n_l$ implies $\sum a_i \cdot m_k \equiv \sum b_j \cdot n_l$.

A morphism of blue $B$-modules $M$ and $N$ is a map $f : M \rightarrow N$ such that

(i) $f(a \cdot m) = a \cdot f(m)$ for all $a \in B$ and $m \in M$ and

(ii) whenever $\sum m_i \equiv \sum n_j$ in $M$, then $\sum f(m_i) \equiv \sum f(n_j)$ in $N$.

This implies in particular that $f(\ast) = \ast$. We denote the category of blue $B$-modules by $\mathcal{M}_{od}B$. Note that in case of a ring $B$, every $B$-module is a blue $B$-module, but not vice versa.

Lemma 4.1. The category $\mathcal{M}_{od}B$ is closed, complete and cocomplete. The trivial blue module $0 = \{\ast\}$ is an initial and terminal object of $\mathcal{M}_{od}B$. 

Proof. All arguments are essentially the same as in the case of $A$-sets. We refer to [1] Section 2.2.1] for the facts that $\mathcal{M}odB$ is closed and $0$ is initial and terminal. The construction of limits and colimits can be found in [1 Prop. 2.13]. □

Lemma 4.2. The category $\mathcal{M}odB$ has tensor products $M \otimes_B N$, which are characterized by the universal property that every bi-$B$-linear morphism $M \times N \to P$ factors through a unique $B$-linear map $M \otimes_B N \to P$. The canonical map $M \times N \to M \otimes_B N$ is surjective. The functor $- \otimes_B M$ is left-adjoint to $Hom_B(M, \cdot)$. Together with the tensor product, $\mathcal{M}odB$ is a symmetric monoidal category.

Proof. The blue $B$-module $M \otimes_B N$ can be constructed as the quotient of $M \times N$ by the equivalence relation that is generated by the relations of the form $(b,m,n) \sim (m,b,n)$ for all combinations of $b \in B$, $m \in M$ and $n \in N$. It is easily verified that this defines a blue $B$-module that satisfies the desired properties; cf. Section 2.2.3 of [1] for the analogous case of $A$-sets where $A$ is a monoid. □

Let $B$ be a blueprint. We denote the category of $B$-algebras by $B\text{-}pr_B$ and its morphism sets by $Hom_B(C, C')$. Let $\mathbb{F}_1$ be the blueprint $\{0, 1\}/\{0\}$, which is an initial object in $B\text{-}pr$. Then the association $(\mathbb{F}_1 \to B) \mapsto B$ establishes an equivalence between $B\text{-}pr_{\mathbb{F}_1}$ and $B\text{-}pr$.

Lemma 4.3. Let $B$ be a blueprint. Then the category $B\text{-}pr_B$ is equivalent to the category of commutative monoids in $\mathcal{M}odB$.

Proof. A $B$-algebra $f : B \to C$ is a blue $B$-module w.r.t. the multiplication defined by $b.c = f(b)c$ for $b \in B$ and $c \in C$. The multiplication of $C$ turns $C$ into a commutative monoid in $\mathcal{M}odB$. A morphism $C \to C'$ of $B$-algebras induces naturally a morphism of the associated commutative monoids in $\mathcal{M}odB$. It is immediately verified that this functor is an equivalence of categories. □

5. BLUEPRINT MORPHISMS OF FINITE PRESENTATION

In this section, we characterize morphisms $f : B \to C$ of finite presentation in terms of the finiteness of certain sets of generators.

Let $B = A \underset{\mathcal{B}}{\rightarrow}$ be a blueprint. The we denote by $B[T_{i \in I}]$ the free blueprint over $B$ in the indeterminants $T_i$, cf. 1.12 of [3]. Its elements are $\{0\}$ and all monomials $b \prod_{i \in I}T_i^{n_i}$ with coefficients $b \in B - \{0\}$ where $n_i \geq 0$ with $n_i = 0$ for almost but finitely many $i \in I$. The blueprint $B$ can be seen as the subset of all constants $b \prod_{i \in I}T_i^0$, and the pre-addition of $B[T]$ is generated by the image of $\mathcal{B}$ in $B[T_i]$.

A $B$-algebra $f : B \to C$ is generated by a subset $\{b_i\}_{i \in I}$ of $C$ if there are for every element $c \in C$ finitely many $a_i \in B$ and not necessarily different indices $i \in I$ such that $c \equiv \sum_i f(a_i)b_i$. A presentation of a $B$-algebra $f : B \to C$ is pair $(b, S)$ where $b = \{b_i\}_{i \in I}$ generates $f : B \to C$ and $S$ is a set of relations on the free $B$-algebra $B[T_{i \in I}]$ that satisfies the following property: for $B = B[T_i]/\langle S \rangle$, there is a monomorphism $\tilde{f} : B \to C$ of $B$-algebras that sends $T_{i}$ to $b_i$ and the pre-addition of $C$ is generated by the image of the pre-addition of $\tilde{B}$. Note that the underlying monoid of $\tilde{B}$ is in general a proper quotient of $B[T_i]$.

A $B$-algebra $f : B \to C$ is algebraically of finite presentation if there is a presentation $(b, S)$ of $f : B \to C$ with finite $b$ and finite $S$. We say that such a pair $(b, S)$ is a finite presentation for $f : B \to C$.

Proposition 5.1. A morphism $f : B \to C$ of blueprints is of finite presentation if and only if it is algebraically of finite presentation.

Proof. We unfold the definition of a finitely presented morphism of blueprints. Consider a directed system $\mathcal{D}$ of $B$-algebras $D_i$ (where $i$ ranges through an index set $I$) and morphisms $f_{i,j} : D_i \to D_j$ (for a directed subset of indices $(i, j) \in I \times I$). Define $J(i) = \{j \in I | f_{i,j} \}$
and presentation of $B$.

In this section, we show that a flat morphisms of blueprints coincide with localizations.

Assume that there are a finite subset $J$ of $I$ such that

$$\psi_k = \sum_{\psi \in \Psi_D} f_{j,k} \circ \psi$$

for all $j,k \in J(i)$ where $i$ is some element of $I$. The canonical map

$$\Psi_D : \text{colim} \Hom_B(C, D) \to \Hom_B(C, \text{colim} \mathcal{D})$$

send the class of a tuple $(\varphi_j : B \to D)$ to the morphism $\varphi = \psi \circ \varphi_j : C \to \text{colim} \mathcal{D}$.

Then $F : B \to C$ is of finite presentation if $\Psi_D$ is a bijection for all directed systems $\mathcal{D}$.

Assume $(\{b_1, \ldots, b_n\}, S)$ is a finite presentation for $f : B \to C$. Let $\mathcal{D}$ be a directed system.

We show that $\Psi_D$ is injective. Let $(\varphi_j)_{j \in J(i)}$ and $(\psi_j)_{j \in J(i)}$ be two elements of $\text{colim} \Hom_B(C, D)$ such that $\varphi = \psi \circ \varphi_j : C \to \text{colim} \mathcal{D}$. This means that for every $l = 1, \ldots, n$, there is an $j_l \in J(I(i))$ such that $\varphi_{j_l}(b_l) = \psi_{j_l}(b_l)$ in $D_{j_l}$. If $j \in J(I(i))$, then $\varphi_j(b_l) = \psi_j(b_l)$ in $D_{j_l}$ for all $l = 1, \ldots, n$. Therefore, we obtain for an arbitrary element $c \equiv \psi(c)$, this shows the injectivity of $\Psi_D$.

We show that $\Psi_D$ is surjective. Let $\varphi : C \to \text{colim} \mathcal{D}$ be a morphism of $B$-algebras. Then for every relation $R$ in $f(S)$, there is an $i_R \in I$ and $c_l \in D_{i_R}$ such that $\varphi(c_l) = \psi(c_l)$ in $D_{i_R}$ for $l = 1, \ldots, n$ and such that the $c_l$ satisfy the relation $\varphi_{i_R}(R)$. Since $S$ is finite, we can replace the $i_R$ by an $i$ in $\bigcup J(i)$ and can assume that there are elements $c_l \in D_{i}$ that satisfy all relations in $\varphi_i(f(S))$ and $\varphi_{i}(c_l) = b_l$. This means that $\varphi : C \to \text{colim} \mathcal{D}$ factors into a morphism $\varphi_i : C \to D_i$, defined by $\varphi_i(b_l) = c_l$, followed by $i : D_i \to \text{colim} \mathcal{D}$. This establishes the surjectivity of $\Psi_D$ and shows that $f : B \to C$ is of finite presentation, which is one direction of the proposition.

Assume that $\Psi_D$ is a bijection for every directed system $\mathcal{D}$. Let $(\{b_1, \ldots, b_n\}, S)$ be a presentation of $B \to C$ such that the cardinality of $I \cup S$ is minimal. We show that both $I$ and $S$ are finite.

Define for every pair of finite subsets $J \subset I$ and $T \subset S$ such that all relations in $T$ involve only elements of $J$ the blueprint $D_{J,T} = B[b_{j \in J}] / (\mathcal{D}_{J,T})$ where $\mathcal{D}_{J,T}$ is the pre-addition that is generated by $T$ and $\mathcal{D}_B$. Then every $D_{I,T}$ is naturally a $B$-algebra and a pair of inclusions $J_1 \subset J_2$ and $T_1 \subset T_2$ yields a morphism $D_{J_1, T_1} \to D_{J_2, T_2}$ of $B$-algebras. This defines a directed system $\mathcal{D}$ whose colimit $\text{colim} \mathcal{D}$ is $C$. Since $\Psi_D$ is bijective, the identity morphism $\text{id} : C \to C = \text{colim} \mathcal{D}$ comes from an element $(\varphi_{J,T})$ of $\text{colim} \Hom_B(C, \mathcal{D})$. This means that there are a finite subset $J$ of $I$ and a finite subset $T$ of $S$ such that $\text{id} : C \to C$ factors into a morphism $\varphi_{J,T} : C \to D_{J,T}$, followed by $\psi_{J,T} : D_{J,T} \to C$. By the minimality of $I$ and $S$, this can only be the case if both $J = I$ and $T = S$, i.e. $I$ and $S$ are finite. This finishes the proof of the proposition.

6. Flat morphisms

In this section, we show that a flat morphisms of blueprints coincide with localizations.

We recall the definition of a localization of a blueprint at a multiplicative subset. Let $B = A / \mathcal{R}$ be a blueprint. Let $S$ be a multiplicative set in $B$, i.e. a subset of $B$ that contains 1 and $ab$ for all $a, b \in S$. We define $\mathcal{R}^{-1}A$ as the quotient of $A \times S$ by the equivalence relation
~ given by \((a, s) \sim (a', s')\) if and only if there is a \(t \in S\) such that \(tsa' = ts'a\). We write \(a:t\) for the equivalence class of \((a, s)\) in \(S^{-1}A\). We define \(S^{-1}\mathcal{R}\) as the set

\[
S^{-1}\mathcal{R} = \left\{ \sum a_i s_i = \sum b_j r_j \mid \exists t \in S \text{ such that } \sum ts_i a_i = \sum tr_i b_j \right\}
\]

where

\[
s^d = \prod_{k \neq i} s_k \cdot \prod_j r_j \quad \text{and} \quad r^d = \prod_i s_i \cdot \prod_{i \neq j} r_i.
\]

Then \(S^{-1}A\) is a monoid (with the multiplication inherited from \(A \times S\)) and that \(S^{-1}\mathcal{R}\) is a pre-addition for \(S^{-1}A\). We define the localization of \(B\) at \(S\) as the blueprint \(S^{-1}B = S^{-1}A/\mathcal{R}\).

The association \(a \mapsto \frac{a}{S}\) defines an epimorphism \(B \to S^{-1}B\). It satisfies the universal property that every morphism \(f : B \to C\) that maps \(S\) to the units of \(C\) factors uniquely through \(B \to S^{-1}B\). If \(S = \{h'\}_{\geq 0}\) is generated by some \(h \in B\), then we denote \(S^{-1}B\) by \(B[h^{-1}]\).

Given a blue \(B\)-module \(M\) and a multiplicative subset \(S\) of \(B\), we define \(S^{-1}M\) as the following blue \(B\)-module. Its underlying set is the quotient of \(M \times S\) by the equivalence relation \(\sim\) defined by \((m, s) \sim (m', s')\) if and only if there is a \(t \in S\) such that \(ts.m' = ts'.m\). We denote by \(\frac{m}{S}\) the equivalence class of \((m, s)\) and denote by \(f : M \to S^{-1}M\) the canonical map that sends \(m\) to \(\frac{m}{S}\). The pre-addition of \(S^{-1}M\) is generated by \(f(P)\) where \(P\) is the pre-addition of \(M\). With this \(f : M \to S^{-1}M\) is a morphism of blue \(B\)-modules, and \(S^{-1}M\) is naturally a blue \(S^{-1}B\)-module.

We say that a morphism \(f : B \to C\) of blueprints is a localization if there is a multiplicative set \(S\) in \(B\) and an isomorphism \(g : C \to S^{-1}B\) such that \(g \circ f : B \to S^{-1}B\) equals the localization map \(a \mapsto \frac{a}{S}\). Recall that a morphism \(f : B \to C\) is flat if \(- \otimes_B C : \mathcal{M}odB \to \mathcal{M}odC\) commutes with finite limits and colimits.

**Proposition 6.1.** A morphism \(f : B \to C\) of blueprints is flat if and only if it is a localization.

**Proof.** Since it is not surprising that localizations are flat, we restrict ourselves to an outline of this direction of the proof. Let \(S\) be a multiplicative subset of \(B\). Since \(- \otimes_B S^{-1}B\) is left-adjoint to \(\text{Hom}(S^{-1}B, -)\), it commutes with colimits. It is easily verified that \(- \otimes_B S^{-1}B\) commutes with finite limits (cf. [11], Prop. 2.24) for the case of a monoid \(B\). Therefore \(B \to S^{-1}B\) is flat.

For two blue \(B\)-modules \(M\) and \(N\), the universal morphism \(M \times N \to M \otimes_B N\) is a surjection between the underlying sets, see Lemma 4.2. This means that every element of \(M \otimes_B N\) can be represented as \(m \otimes n\) with \(m \in M\) and \(n \in N\). This is the basic property that allows us to show that every flat morphism is a localization.

Assume that \(f : B \to C\) is flat. Define \(S = f^{-1}(C^\times)\), which is a multiplicative subset of \(B\). By the universal property of the localization, the morphism \(f : B \to C\) factors into \(B \to S^{-1}B\) and a unique morphism \(f_S : S^{-1}B \to C\). The proof is completed once we have shown that \(f_S\) is an isomorphism.

We show that \(f_S\) is surjective. For \(b \in B\) and \(c \in C\), we write \(b.c = f(b)c\). By the flatness of \(f : B \to C\), the morphism

\[
\Phi : (B \times B) \otimes_B C \longrightarrow (B \otimes_B C) \times (B \otimes_B C) = C \times C
\]

\(\Phi(b_1, b_2) \otimes c \mapsto (b_1 \otimes c, b_2 \otimes c) = (b_1, c, b_2, c)\)

is an isomorphism. This means that we find for every \(d \in C\) elements \(b_1, b_2 \in B\) and \(c \in C\) such that \((d, 1) = (b_1, c, b_2, c)\). Therefore \(b_2.d = b_1 b_2.c = b_1, 1\) is in \(f(B)\) and \(f(b_2) \in C^\times\).

This means that \(d = f_S(b_2)\) is in the image of \(f_S\), which shows that \(f_S\) is surjective.

If \(A\) is the underlying monoid of \(S^{-1}B\) and \(\mathcal{R}\) is its pre-addition, then we can represent \(C\) as \(A/\mathcal{R}C\), which is not necessarily a proper representation. If we show that \(f_S\) induces a bijection between \(\mathcal{R}_B\) and \(\mathcal{R}_C\), then it follows that \(f_S\) is an isomorphism.
To do so, consider a relation $\sum a_i \equiv \sum b_j$ between elements $a_i, b_j \in A$ in $\mathcal{R}_C$. Define $a = \sum a_i$ and $b = \sum b_j$ in the semiring $S^{-1}B^+$ and consider the two morphisms

$$S^{-1}B^+ \xrightarrow{f_a} S^{-1}B^+ \xrightarrow{f_b} S^{-1}B^+$$

of blue $B$-modules given by $f_a(c) = ac$ and $f_b(c) = bc$. Since $B \to S^{-1}B$ is an epimorphism and $S^{-1}B \to C$ is surjective, we have $S^{-1}B^+ \otimes_B C = S^{-1}B^+ \otimes_{S^{-1}B} C = C^+$. Therefore the base change $- \otimes_B C$ yields the morphisms

$$C^+ \xrightarrow{f_a \otimes_B C} C^+ \xrightarrow{f_b \otimes_B C} C^+,$$

which are the same since $a = \sum a_i = \sum b_j = b$ in $C^+$. Since $f : B \to C$ is flat, we have $\text{eq}(f_a, f_b) \otimes_B C = \text{eq}(f_a \otimes_B C, f_b \otimes_B C) = C^+$. Therefore, there exists a $c \in \text{eq}(f_a, f_b)$ such that $f(c) \in (C^+)^\times$ and $c \otimes f(c)^{-1} = 1$ in $C^+$. By the definition of $S$, $c$ is invertible in $S^{-1}B^+$. Therefore the equality $ac = f_a(c) = f_b(c) = bc$ implies

$$a = acc^{-1} = c^{-1}f_a(c) = c^{-1}f_b(c) = bcc^{-1} = b$$
in $S^{-1}B^+$. This means that $\sum a_i = \sum b_j$ in $S^{-1}B$, which was to be shown. □

**Corollary 6.2.** Every flat morphism of blueprints is an epimorphism. □

We say that a morphism $f : B \to C$ is a finite localization if it is isomorphic to a localization $B \to S^{-1}B$ at a finitely generated multiplicative subset $S = \{h_1^{e_1} \cdots h_n^{e_n} | e_i \geq 0\}$ of $B$. Note that in this case, $S^{-1}B = B[h_1^{-1}, \ldots, h_n^{-1}]$.

**Corollary 6.3.** A morphism of blueprints is a flat epimorphism of finite presentation if and only if it is a finite localization.

**Proof.** We know that a flat epimorphism is the same as a localization. Let $B \to S^{-1}B$ be a finite localization, i.e. $S$ is generated by finitely many elements $h_1, \ldots, h_n$. Then

$$\{(T_1, \ldots, T_n), \{T_i h_1 \equiv 1, \ldots, T_i h_n \equiv 1\}\}$$
is a finite presentation for $B \to S^{-1}B$. Conversely, if $B \to S^{-1}B$ is a localization with a finite presentation $(\{T_1, \ldots, T_n\}, R)$, then $S$ is finitely generated by those $T_i$ that are invertible in $S^{-1}B$. □

## 7. Blue Schemes

In this section, we recall the definition of a blue scheme as a locally blueprinted space that is locally isomorphic to the spectra of blueprints. More details can be found in Section 3 of [5].

A **locally blueprinted space** is a topological space $X$ together with a structure sheaf $\mathcal{O}_X$ into the category of blueprints such that the stalk $\mathcal{O}_{X,x}$ at every point $x$ of $X$ is a local blueprint, i.e. $\mathcal{O}_{X,x}$ has a unique maximal ideal $m_x$. The residue field at $x$ is the quotient $\kappa_x = \mathcal{O}_{X,x}/m_x$. A local morphism of locally blueprinted spaces $X$ and $Y$ consists of a continuous map $\varphi : X \to Y$ between the underlying topological spaces and a morphism $\varphi^*: \mathcal{O}_Y \to \varphi_* \mathcal{O}_X$ of sheaves such that the induced morphisms $\varphi^*_x : \mathcal{O}_{Y, \varphi(x)} \to \mathcal{O}_{X,x}$ of stalks is a local morphism, i.e. it maps the maximal ideal of $\mathcal{O}_{Y, \varphi(x)}$ to the maximal ideal of $\mathcal{O}_{X,x}$. This induces, in particular, a morphism $\kappa(\varphi(x)) \to \kappa(x)$ of the residue fields.

A **prime ideal** of a blueprint $B$ is a subset $p$ such that $pB = p$, such that $S = B - p$ is a multiplicative subset of $B$ and such that every relation $\sum a_i + c \equiv \sum b_j$ in $B$ with $a_i, b_j \in p$ implies $c \in p$. We endow the set $X$ of all prime ideals of $B$ with the topology that is generated by the subsets $U_h = \{p \in X | h \not\in p\}$. Note that $U_{gh} \cap U_h = U_{gh}$, which implies that $\{U_h | h \in B\}$ forms a basis for the topology of $X$. A **covering family** for $X$ is an collection of open subsets whose union equals $X$. The **structure sheaf** $\mathcal{O}_X$ is defined as the sheaf that
associates with each open subset $U$ of $X$ the set of locally representable sections $s : U \rightarrow \coprod_{p \in U} B_p$, i.e. there is a covering family $\{U_h\}$ of $X$ and elements $s_i \in B[h_i^{-1}]$ such that $s(p) = s_i$ in $B_p$ for all $i$ and $p \in U_h$. Note that each set $\mathcal{O}_X(U)$ of local sections comes with the natural structure of a blueprint. The spectrum $\text{Spec } B$ of $B$ is the topological space $X$ together with the structure sheaf $\mathcal{O}_X$.

The spectrum of a blueprint is a locally blueprinted space. A morphism $f : B \rightarrow C$ of blueprints defines naturally a local morphism $f^* : \text{Spec } C \rightarrow \text{Spec } B$ between the spectra of the blueprints. Thus Spec defines a functor from $\mathcal{B}pr$ to the category $\mathcal{L}oc\mathcal{B}pr\mathcal{F}p$ of locally blueprinted spaces. Conversely, taking global sections $\Gamma(X, \mathcal{O}_X)$ defines a functor from the category of locally blueprinted spaces to $\mathcal{B}pr$. If $X = \text{Spec } B$, we obtain an endofunctor on blueprints that sends $B$ to $B\Gamma = \Gamma(X, \mathcal{O}_X)$.

The difficulty in comparing blue schemes with schemes relative to the category of blue $\mathbb{F}_1$-modules lies in the fact that the functor $\text{Spec} : \mathcal{B}pr \rightarrow \mathcal{L}oc\mathcal{B}pr\mathcal{F}p$ is not fully faithful, and that the canonical morphism $\sigma : B \rightarrow B\Gamma$, called the globalization of $B$, is in general not an isomorphism. We call $B$ global if $\sigma$ is an isomorphism. We summarize some results from Section 3 of [5], in particular cf. [5 Thm. 3.12].

**Theorem 7.1.** For every blueprint $B$, $\sigma : B \rightarrow B\Gamma$ defines an isomorphism $\sigma^* : \text{Spec } B\Gamma \rightarrow \text{Spec } B$. Consequently, $B\Gamma$ is a global blueprint and every morphism $f : B \rightarrow C$ into a global blueprint $C = \Gamma C$ factors uniquely through $\sigma : B \rightarrow B\Gamma$.

If we denote the full subcategory of global blueprints in $\mathcal{B}pr$ by $\mathcal{G}\mathcal{B}pr$, then the restriction of Spec to $\mathcal{G}\mathcal{B}pr$ is a fully faithful embedding into $\mathcal{L}oc\mathcal{B}pr\mathcal{F}p$, and $\Gamma \circ \text{Spec}$ is isomorphic to the identity functor of $\mathcal{G}\mathcal{B}pr$.

Let $\mathcal{A}$ be the essential image of Spec, which is a full subcategory of $\mathcal{L}oc\mathcal{B}pr\mathcal{F}p$. We endow $\mathcal{A}$ with the Grothendieck pretopology whose opens are morphisms $W \rightarrow U$ that are induced by a finite localization $B \rightarrow S^{-1}B$ (in $\mathcal{B}pr$), and whose covering families are families $\{W_i \rightarrow U\}$ of opens such that the set-theoretic union $\bigcup W_i$ covers the underlying set of $U$. Then we can define blue schemes in terms of the following characterization.

**Proposition 7.2.** A locally blueprinted space $X$ is a blue scheme if and only if there is an affine presentation $\mathcal{U}$ in $\mathcal{A}$ such that $X \simeq \text{colim } \mathcal{U}$ in $\mathcal{L}oc\mathcal{B}pr\mathcal{F}p$.

**Proof.** Without recalling all notions from [5], we sketch the proof of this fact. Let $X$ be a blue scheme and $\mathcal{U}$ an affine open covering. We define $\mathcal{V}$ as follows. The maximal elements are $\mathcal{U}_{\text{max}} = \mathcal{U}$. For every $U$ and $V$ in $\mathcal{U}_{\text{max}}$, we let $\mathcal{U}_{U,V}$ be the set of all affine open subschemes of $U \cap V$. We define $\mathcal{U}$ as the disjoint union of $\mathcal{U}_{\text{max}}$ with the sets $\mathcal{U}_{U,V}$ where $U$ and $V$ range through $\mathcal{U}_{\text{max}}$, and the morphisms of $\mathcal{U}$ are the inclusions $W \rightarrow U$ and $W \rightarrow V$ for $U, V \in \mathcal{U}_{\text{max}}$ and $W \in \mathcal{U}_{U,V}$. Then $\mathcal{U}$ is a commutative diagram of open immersions such that $X = \text{colim } \mathcal{U}$. Its maximal elements are $\mathcal{U}_{\text{max}}$, and the cocycle condition is equivalent to the fact that the maximal elements of $\mathcal{U}$ are open subschemes of $X$. Thus $\mathcal{U}$ is an affine presentation and, indeed, an affine atlas.

Conversely, if $X$ is the colimit of an affine presentation $\mathcal{U}$, then $X$ is covered by the images of the maximal elements in $\mathcal{U}$. Due to the cocycle condition, the maximal elements of $\mathcal{U}$ are isomorphic to their image in $X$. Therefore $X$ is a blue scheme. \hfill \Box

We denote the full subcategory of $\mathcal{L}oc\mathcal{B}pr\mathcal{F}p$ whose objects are blue schemes by $\text{Sch}\mathcal{B}1$.

**Theorem 7.3.** Properties (i)–(vi) from the Section 7 are satisfied for the embedding of $\mathcal{A}$ into $\mathcal{L}oc\mathcal{B}pr\mathcal{F}p$.

**Proof.** Property (i) follows from the construction of the fibre product of blue schemes, cf. [5 Prop. 3.27]. Property (ii) is established by Proposition 7.2. Properties (iv) and (v) follow easily from the facts that open subschemes are completely determined by their underlying topological space and that taking colimits of affine presentations commutes
with the forgetful functor to topological spaces. Property \([3]\) is \([5]\) Thm. 3.23, and \([11]\) follows from \([10]\).

8. The Blue Scheme Associated with a Relative Scheme

Let \(\text{Sch}^{\text{bl}}_{\text{rel}}\) be the category of schemes relative to \(\text{Mod} F_1\), which we also call relative schemes if there is no risk of confusion. In this section, we construct a functor \(G : \text{Sch}^{\text{bl}}_{\text{rel}} \to \text{Sch}_{\text{rel}}\) from relative schemes to blue schemes that extends the dual of the globalization functor \(\Gamma : \text{Blpr} \to \Gamma \text{Blpr}\).

In the definition of \(G\), we make use of the properties \([4]–[7]\) from Section \([1]\) see Theorems \([2.2]\) and \([7.3]\) for their validity for relative schemes and blue schemes, respectively. We will refer to the corresponding property in brackets where we make use of it.

**Objects.** Let \(X\) be a scheme relative to \(\text{Mod} F_1\). Then \(X\) is the colimit of an affine presentation \(\mathcal{U}\) (property \([11]\)). The category \(\text{Sch}^\text{bl}_{\text{rel}}\) is dual to the category \(\text{Blpr}\) of blueprints, so we can apply the functor \(\text{Spec}\) to the dual diagram \(\mathcal{U}^{\text{op}}\), which yields a diagram \(\mathcal{V} = \text{Spec}(\mathcal{U}^{\text{op}})\) in the category of affine blue schemes \(\mathcal{A}\). We define \(G(X)\) as the colimit of \(\mathcal{V}\) in \(\text{Sch}^\text{bl}_{\text{rel}}\) (property \([11]\)).

This definition is independent (up to canonical isomorphism) from the choice of affine presentation for the following reason. Since two affine presentation have a common refinement (property \([5]\)), it is enough to show that a refinement \(\mathcal{U} \to \mathcal{U}'\) of an affine presentation \(\mathcal{U}\) of \(X\) induces an isomorphism \(\text{colim} \text{Spec} \mathcal{U}^{\text{op}} \to \text{colim} \text{Spec} \mathcal{U}'^{\text{op}}\). Since all morphisms \(W \to U\) that are part of \(\mathcal{U} \to \mathcal{U}'\) are open immersions, the corresponding blueprint morphisms \(\mathcal{U}^{\text{op}} \to \mathcal{W}^{\text{op}}\) are flat epimorphisms of finite presentation, i.e. finite localizations by Corollary \([6.3]\). Therefore, \(\text{Spec} \mathcal{U}^{\text{op}} \to \text{Spec} \mathcal{U}'^{\text{op}}\) is an open immersion of affine blue schemes. Assume that a finite family of blueprint morphisms \(B \to B_i\) defines a conservative functor \(\text{Mod} F_i \to \prod \text{Mod} B_i\) and let \(x\) be a topological point of \(\text{Spec} B\). Since a non-trivial torsion module with support \(\{x\}\) is not isomorphic to the trivial \(B\)-module, \(x\) must lie in the image of one of the morphisms \(\text{Spec} B_i \to \text{Spec} B\). This shows that a covering \(\{W \to U\}\) of an affine scheme relative to \(B\) yields a covering \(\{W_i^{\text{op}} \to \text{Spec} U_i^{\text{op}}\}\) of affine blue schemes. It follows that \(\text{Spec} \mathcal{U}^{\text{op}} \to \text{Spec} \mathcal{U}'^{\text{op}}\) is a refinement of affine presentations in \(\mathcal{A}\), which yields an isomorphism \(\text{colim} \text{Spec} \mathcal{U}^{\text{op}} \to \text{colim} \text{Spec} \mathcal{U}'^{\text{op}}\) of blue schemes.

**Morphisms.** Let \(\varphi : X \to Y\) be a morphism in \(\text{Sch}^\text{bl}_{\text{rel}}\). Then there is a morphism \(\mathcal{U} \to \mathcal{V}\) of affine presentations of \(X\) and \(Y\), respectively, that induces \(\varphi\) (property \([11]\)). We define \(G(\varphi)\) as the colimit over the induced morphism \(\text{Spec} \mathcal{U}^{\text{op}} \to \text{Spec} \mathcal{V}^{\text{op}}\) of affine presentations in the category of affine blue schemes.

This definition is independent from the choice of affine presentations for the following reason. As explained before, \(G(X)\) and \(G(Y)\) are independent from the choice of affine presentation. If \(\mathcal{U} \to \mathcal{V}\) is a morphism that induces \(\varphi : X \to Y\), then there are common refinements \(\mathcal{U}''\) of \(\mathcal{U}\) and \(\mathcal{V}'\) and \(\mathcal{V}''\) of \(\mathcal{V}\) and \(\mathcal{V}'\). It follows from Lemma \([1.2]\) that, after refining \(\mathcal{V}''\), there is a morphism \(\varphi'' : \mathcal{U}'' \to \mathcal{V}''\) that is compatible with \(\varphi : \mathcal{U} \to \mathcal{V}\) and thus induces \(\varphi : X \to Y\). Therefore \(\varphi''\) is also compatible with \(\varphi'\), which shows that there are canonical identifications \(G(\varphi') = G(\varphi'') = G(\varphi)\).

The independence from the affine presentation also shows that \(G\) commutes with the composition of morphisms.

9. Algebraically Presented Blue Schemes

When we want to associate a relative scheme with a blue scheme, we face two differences between the categories of affine schemes in the context of schemes relative to \(\text{Mod} F_1\) and in the context of blue schemes. Namely, an open immersion \(X \to Y\) of affine blue schemes does in general not yield a finite localization \(\Gamma Y \to \Gamma X\) of coordinate blueprints.
Furthermore, an open affine cover $\mathcal{V}$ of an affine scheme $X = \text{Spec} B$ does not imply that the functor
\[
\prod_{U \in \mathcal{V}} - \otimes_B \Gamma U \rightarrow \prod_{U \in \mathcal{V}} \mathcal{M} \odot B \rightarrow \prod_{U \in \mathcal{V}} \mathcal{M} \odot \Gamma U
\]
is conservative (see Example 9.2). If it is so, we say that $\mathcal{V}$ is a conservative cover of $X$.

In this section, we will introduce the class of algebraically presented schemes, that allows us to bridge the gap between blue schemes and schemes relative to $\mathcal{M} \odot \Gamma F_1$ by using an affine presentation that is fine enough. We say that a morphism $X \rightarrow Y$ of affine blue schemes is a finite localization if the induced blueprint morphism $\Gamma Y \rightarrow \Gamma X$ is a finite localization. Note that finite localizations are composable.

**Definition 9.1.** An affine scheme $X = \text{Spec} B$ is with an algebraic basis if every covering of $X$ is conservative and if the affine open subsets $U$ of $X$ that are finite localizations form a basis of the topology of $X$. A blue scheme is with an algebraic basis if every affine open subscheme is with an algebraic basis.

A blue scheme $X$ is algebraically presented if it has an affine presentation $\mathcal{V}$ such that all $U$ in $\mathcal{V}$ are with an algebraic basis and if all morphisms of $\mathcal{V}$ are finite localizations. We call such an affine presentation $\mathcal{V}$ an algebraic presentation of $X$. We denote the full subcategory of $\text{Sch}_{\mathcal{V}}$ whose objects are algebraically presented blue schemes by $\text{Sch}^{\text{alg}}_{\mathcal{V}}$.

**Example 9.2.** The usual technique of proof applies to show that schemes (in the usual sense) are with an algebraic basis, e.g. see [4, Prop. II.5.1]. Since monoids are local, affine monoidal schemes have a unique maximal point. Therefore also monoidal schemes are with an algebraic basis.

Note that a blue scheme with an algebraic basis is algebraically presented, but not vice versa. An example of a blue scheme that is algebraically presented, but not with an algebraic basis is $X = \text{Spec} B$ with $B = F_1[a,b,g,h]/(ah \equiv bg, g + h \equiv 1)$, which is not global since $\Gamma B$ contains a global section $s$ that equals $a/g$ on $U_g = \text{Spec} B[g^{-1}]$ and $b/h$ on $U_h = \text{Spec} B[h^{-1}]$, cf. [5, Ex. 3.8]. It is not with an algebraic basis because the cover $\{U_g, U_h\}$ is not conservative. But the affine presentation that consists of $U_g, U_h, U_{gh}$ and the open immersions $U_{gh} \hookrightarrow U_g$ and $U_{gh} \hookrightarrow U_h$ is an algebraic presentation of $X$. An example of a blue scheme that is not algebraically presented is $X = \text{Spec} B$ with $B = F_1[a_i, b_i, g_i, h_i]_{i \in \mathbb{N}}/(a_i h_i \equiv b_i g_i, g_i + h_i \equiv 1)_{i \in \mathbb{N}}$ since every open subset has a covering that is not conservative.

Examples of algebraically presented blue schemes are locally finite blue schemes, i.e. blue schemes that can be covered by open subschemes with only finitely many points. Indeed, a locally finite blue scheme $X$ has the following canonical algebraic presentation $\mathcal{V}$. Since $X$ is locally finite, for every point $x$ of $X$ has a minimal open neighbourhood $U_x$, which is the intersection of all open subsets of $X$ that contain $x$. Explicitly, $U_x$ consists of all points $z$ of $X$ such that $x$ is contained in the closure of $z$. In particular, the only open subset of $U_x$ that contains $x$ is $U_x$ itself. Therefore $U_x$ is conservative, and if $U_x = \text{Spec} B$ for some blueprint $B$, then $B$ is a global blueprint. This implies that every inclusion $U_x \hookrightarrow U_y$ (for $y$ in the closure of $x$) is a finite localization. Note that the family of the $U_x$ where $x$ ranges through all points of $X$ forms a basis for the topology of $X$. This reasons that the diagram $\mathcal{V}$ of all $U_x$ together with all inclusions is an algebraic presentation of $X$.

We will need the following property of schemes with localization bases.

**Proposition 9.3.** Let $\mathcal{W}$ and $\mathcal{V}$ be two algebraic presentations of a blue scheme $X$ and $\mathcal{W}'$ a common refinement of $\mathcal{W}$ and $\mathcal{V}$. Then there exists a refinement $\mathcal{W}$ of $\mathcal{W}'$ such that all morphisms of $\mathcal{W}$, all morphisms $\Phi_W : W \rightarrow \Phi(W)$ in $\Phi : \mathcal{W} \rightarrow \mathcal{V}$ and all morphisms $\Psi_V : W \rightarrow \Psi(W)$ in $\Psi : \mathcal{W} \rightarrow \mathcal{V}$ are finite localizations.

We will need some preliminary statements before we can turn to the proof of the proposition.
**Lemma 9.4.** Let \( B \) be a global blueprint and \( f : B \rightarrow C \) a morphism of blueprints such that \( f^* : \text{Spec}C \rightarrow \text{Spec}B \) is an isomorphism of affine blue schemes. Then \( f \) is an isomorphism of blueprints.

**Proof.** Note that \( B \cong \Gamma C \). After applying a suitable automorphism, we can assume that \( f^* \) is the inverse to \( \sigma_C : \text{Spec} \Gamma C \rightarrow \text{Spec} C \). In particular, this means that the pre-addition of \( B \cong \Gamma C \) is defined by the pre-addition of \( C \). Therefore it follows that \( f \) is an isomorphism if we can show that \( \sigma_C \) is a bijection between the underlying sets of \( C \) and \( \Gamma C \).

We show that \( \sigma_C \) is surjective. Consider a section \( s : \text{Spec} C \rightarrow \coprod C_p \) where \( p \) varies through all prime ideals of \( C \). Then we have a section \( \tilde{s} = s \circ (f^*)^{-1} : \text{Spec} B \rightarrow \coprod B_p \). Since the globalization \( \sigma_B : B \rightarrow \Gamma B \) is an isomorphism, there is a \( b \in B \) with \( \sigma_B(b) = \tilde{s} \). Therefore \( s = \sigma_C(f(b)) \) is the image of \( f(b) \in C \).

We show that \( \sigma_C \) is injective. Consider \( c \) and \( c' \) in \( C \) with \( \sigma_C(c) = \sigma_C(c') \). Then
\[
\tilde{s} = \sigma(c) \circ (f^*)^{-1} = \sigma(c') \circ (f^*)^{-1} : \text{Spec} B \longrightarrow \coprod C_p = \coprod B_p
\]
is a global section of \( B \). Since \( B \) is global, there is a unique \( b \in B \) such that \( \tilde{s} = \sigma_B(b) \). Therefore we have \( c = f(b) = c' \).

**Lemma 9.5.** Let \( S \subset B \) be a finitely generated multiplicative subset and \( f : B \rightarrow C \) and \( g : C \rightarrow S^{-1}B \) blueprints such that \( g \circ f \) equals the canonical morphism \( B \rightarrow S^{-1}B \). Let \( T = f(S) \). If \( f^* : \text{Spec} C \rightarrow \text{Spec} B \) is an open immersion and \( S^{-1}B \) is global, then \( f \) induces an isomorphism \( f_\ast : S^{-1}B \rightarrow T^{-1}C \) of blueprints. In particular, \( T^{-1}C \) is a global blueprint.

**Proof.** By the universal property of localizations, \( f \) and \( g \) induce morphisms \( f_\ast : S^{-1}B \rightarrow T^{-1}C \) and \( g_\ast : T^{-1}C \rightarrow S^{-1}B \), respectively. Let \( U = \text{Spec} B \), \( U_S = \text{Spec} S^{-1}B \), \( V = \text{Spec} C \) and \( V_T = \text{Spec} T^{-1}C \). Then we obtain a commutative diagram

\[
\begin{array}{ccc}
V_T & \longrightarrow & V \\
\downarrow f_\ast & & \downarrow f^* \\
U_S & \longrightarrow & U
\end{array}
\]

where the solid arrows are open immersions. This means that \( U_S, V \) and \( V_T \) are open subsets of \( U \) and that the respective structure sheaves are restrictions of the structure sheaf of \( U \). Consequently, the dashed arrows are open immersions as well. In particular \( f_\ast \) and \( g_\ast \) must be mutual inverse isomorphisms. Therefore, we can apply Lemma 9.4 to \( f_\ast : S^{-1}B \rightarrow T^{-1}C \), which says that \( f_\ast \) is an isomorphism.

**Corollary 9.6.** Let \( \varphi : V \rightarrow U \) be an open immersion and \( \psi : W \rightarrow V \) a morphism such that \( \psi \circ \varphi : W \rightarrow U \) is a finite localization. Then \( \psi \) is a finite localization.

**Proof.** The statement follows from applying Lemma 9.5 to \( B = \Gamma U \), \( C = \Gamma V \), \( S^{-1}B = \Gamma W \), \( f = \Gamma \varphi \) and \( g = \Gamma \psi \).

**Proof of Proposition 9.3.** Let \( \Psi : \mathcal{W}' \rightarrow \mathcal{W} \) and \( \Phi' : \mathcal{W}' \rightarrow \mathcal{Y}' \) be refinements and \( \mathcal{W} \) and \( \mathcal{Y} \) algebraic presentations. We construct \( \mathcal{W}' \) as follows. Since \( \mathcal{W} \) is an algebraic presentation, we can cover each \( W' \in \mathcal{W}_\text{max} \) with finite localizations \( W_i \) of \( U = \Phi(W') \in \mathcal{W}_\text{max} \). Since \( \mathcal{Y} \) is an algebraic presentation, we can cover each of the \( W_i \) with finite localizations \( W_{i,j} \) of \( V = \Psi(W') \). By Corollary 9.6, each \( W_{i,j} \) is a finite localization of \( W_i \), and thus of \( U \).

We define \( \mathcal{W}_\text{max} \) as the collection of all \( W_{i,j} \) (for varying \( W' \)), which will be the maximal elements of an affine presentation \( \mathcal{W}' \). These sets \( W_{i,j} \) come with an open immersions \( \xi_{W_{i,j}} : W_{i,j} \rightarrow W_i \), which will be part of a refinement \( \Xi : \mathcal{W}' \rightarrow \mathcal{W}' \).

For \( W'_1, W'_2 \in \mathcal{W}_\text{max}' \) and \( W \in \mathcal{W}' \), \( W_1, W_2 \in \mathcal{W}_\text{max} \) with \( \Xi_1(W_1) = W'_1 \) and \( \Xi_2(W_2) = W'_2 \), define \( W_{1,2} = W_1 \times_{W'_1} W'_2 \times_{W'_2} W_2 \), which comes together with open immersions \( W_{1,2} \rightarrow W_1 \) and \( W_{1,2} \rightarrow W_2 \) and \( \Xi_{W_{1,2}} : W_{1,2} \rightarrow W' \). Note that the family of all \( W_{1,2} \) for \( W_1 \) and \( W_2 \) varying...
through the open subschemes of the covering of \( W'_1 \) and \( W'_2 \), respectively, cover \( W' \) by the stability of coverings under base change. Let \( U = \Phi'(W') \) and \( V = \Psi'(W') \). Then we have the following commutative diagram of open immersions.

\[
\begin{array}{ccc}
W' & \xrightarrow{\Xi_{W_1}} & W'_1 \\
\downarrow{\Xi_{W_{1,2}}} & & \downarrow{\Phi_{W'}} \\
W'_2 & \xrightarrow{\Psi_{W'}} & V
\end{array}
\]

Since \( \Phi'_{W'} \circ \Xi_{W_i} : W_i \to W'_i \to U_i \) for \( U_i = \Phi'(W'_i) \) in \( \mathcal{U} \) and \( i = 1, 2 \) are open immersions, Corollary 9.6 implies that the finite localizations of \( W_i \) form a basis of its topology. Therefore, we can cover \( W_{1,2} \) with finite localizations \( W_i \) of \( W_1 \). We can cover each \( W_i \) with finite localizations \( W_{i,j} \) of \( W_2 \), which are finite localizations of \( W_i \) by Corollary 9.6 and thus of \( W_1 \). We can cover each \( W_i \) with finite localizations \( W_{i,j,k} \) of \( U_i \), and each \( W_{i,j,k} \) with finite localizations \( W_{i,j,k,l} \) of \( V \). By the same argument as before, Corollary 9.7 implies that each \( W_{i,j,k,l} \) is a common finite localization of \( W_1, W_2, U \) and \( V \). Note that the family of all \( W_{i,j,k,l} \) covers \( W_{1,2} \).

We define \( \mathcal{W} \) as the union of \( \mathcal{W}_{\text{max}} \) together with all sets \( W_{i,j,k,l} \), together with the finite localizations \( W_{i,j,k,l} \to W_1 \) and \( W_{i,j,k,l} \to W_2 \), where \( W'_1 \) and \( W'_2 \) vary through \( \mathcal{W}'_{\text{max}} \). \( W' \) varies through \( \mathcal{W}'_{W_1'}, W'_2 \), and \( W_1 \) and \( W_2 \) vary through \( \mathcal{W}_{\text{max}} \), together with the finite \( W_i \) and \( W_{i,j,k,l} \) as well as all finite localizations in \( \mathcal{W}'_{\text{max}} \). By the same argument as before, Corollary 9.6 implies that each \( W_i \) and \( W_{i,j,k,l} \) is a common finite localization of \( W_1, W_2, U \) and \( V \). This establishes Proposition 9.3.

**Corollary 9.7.** Let \( \varphi : X \to Y \) be a morphism between two blue schemes with affine presentations \( \mathcal{U} \) and \( \mathcal{V} \). Then there exists refinements \( \mathcal{U}' \to \mathcal{U} \) and \( \mathcal{V}' \to \mathcal{V} \) with the following properties: there is a morphism \( \Phi : \mathcal{U}' \to \mathcal{V}' \) of affine presentations that induces \( \varphi \); all morphisms of \( \mathcal{U}' \) and \( \mathcal{V}' \) are finite localizations; and the refinements \( \mathcal{U}' \to \mathcal{U} \) and \( \mathcal{V}' \to \mathcal{V} \) consist of finite localizations.

**Proof.** Let \( \Phi' : \widehat{\mathcal{U}}' \to \widehat{\mathcal{V}}' \) be a morphism of affine presentations that induces \( \varphi \). Choose a common refinement \( \widehat{\mathcal{V}}'' \) of \( \mathcal{V}' \) and \( \widehat{\mathcal{V}}'' \). By Proposition 9.3, we find a refinement \( \mathcal{V}''' \) of \( \mathcal{V}' '' \) whose morphisms are finite localizations and such that \( \mathcal{V}''' \to \mathcal{V} \) consists of finite localizations. Then \( \widehat{\mathcal{V}}'' = \mathcal{V}''' \times_{\widehat{\mathcal{V}}'} \mathcal{V}' \) is a refinement of \( \widehat{\mathcal{V}}' \) and it comes with a morphism \( \Phi'' : \widehat{\mathcal{V}}'' \to \mathcal{V}' '' \) that induces \( \varphi \). Let \( \mathcal{U}' '' \) be a common refinement of \( \mathcal{U}' \) and \( \widehat{\mathcal{V}}'' \). By Proposition 9.3, there is a refinement \( \mathcal{U}' \) of \( \mathcal{U}' '' \) whose morphisms are all finite localization and such that the refinement \( \mathcal{U}' \to \mathcal{U} \) is finite localization. It is clear that the composition \( \Phi : \widehat{\mathcal{U}}' \to \widehat{\mathcal{V}}'' \to \mathcal{V}' '' \) induces \( \varphi \), which concludes the proof of the corollary.

10. **Algebraically Presented Blue Schemes as Relative Schemes**

For the construction of a functor from blue schemes to relative schemes, we need the requirement that there is a class of affine presentations \( \mathcal{U} \) of a blue scheme such that the corresponding diagram \( \text{spec}\Gamma \mathcal{U} \) in the category of affine relative schemes is an affine presentation as well and that all affine presentations in this class lead to the same relative scheme. Such a class is provided by algebraic presentations.
Objects. Let $X$ be a blue scheme with an algebraic presentation $\mathcal{U}$. Then all morphisms of the dual diagram $\Gamma_{\mathcal{U}}$ in the category of blueprints are finite localizations, and therefore $\text{spec} \Gamma_{\mathcal{U}}$ is a diagram of Zariski opens in $\text{Sch}^\text{rel}_{F_1}$ and $\text{spec} \Gamma_{\mathcal{U}}$ is an affine presentation. We define $\mathcal{F}(X) = \text{colim} \text{spec} \Gamma_{\mathcal{U}}$.

This definition is independent (up to canonical isomorphism) from the choice of affine presentation for the following reason. Let $\mathcal{U}$ and $\mathcal{V}$ be two algebraic presentations of $X$, and $\mathcal{U}'$ and $\mathcal{V}'$ a common refinement of $\mathcal{U}$ and $\mathcal{V}$. By Proposition 9.3 there is a refinement $\mathcal{U}'$ of $\mathcal{U}$, all morphisms $W \to \Phi(W)$ in $\Phi : \mathcal{U} \to \mathcal{U}$ and all morphisms $W \to \Psi(W)$ in $\Psi : \mathcal{V} \to \mathcal{V}$ are finite localizations. Therefore, $\text{spec} \Gamma_{\mathcal{U}'}$ is an affine presentation in the category of affine relative schemes. The induced morphisms $\text{spec} \Gamma_{\mathcal{U}} : \text{spec} \Gamma_{\mathcal{U}'} \to \text{spec} \Gamma_{\mathcal{U}'}$ and $\text{spec} \Gamma_{\mathcal{V}} : \text{spec} \Gamma_{\mathcal{V}'} \to \text{spec} \Gamma_{\mathcal{V}'}$ are refinements since all morphisms $\Phi_W$ and $\Psi_W$ are finite localizations and since all coverings of an object $U \in \mathcal{U}$ (resp. $V \in \mathcal{V}$) are conservative by the definition of an algebraic presentation.

Morphisms. Let $\varphi : X \to Y$ be morphism of schemes with algebraic presentations $\mathcal{U}$ and $\mathcal{V}$. By Corollary 9.7 there are refinements $\mathcal{U}' \to \mathcal{U}$ and $\mathcal{V}' \to \mathcal{V}$ such that all morphisms involved are finite localizations and such that $\varphi$ is induced by a morphism $\Phi : \mathcal{U}' \to \mathcal{V}'$ of affine presentations. This means that $\text{spec} \Gamma_{\mathcal{U}'}$ and $\text{spec} \Gamma_{\mathcal{V}'}$ are affine presentations in the category of affine relative schemes. Since $\mathcal{U}$ and $\mathcal{V}$ are algebraic presentations, $\text{spec} \Gamma_{\mathcal{U}'} \to \text{spec} \Gamma_{\mathcal{U}}$ and $\text{spec} \Gamma_{\mathcal{V}'} \to \text{spec} \Gamma_{\mathcal{V}}$ are refinements. Thus $\mathcal{F}(X)$ can be presented as the colimit of colimspec $\Gamma_{\mathcal{U}'}$ and $\mathcal{F}(Y)$ can be presented as the colimit of colimspec $\Gamma_{\mathcal{V}'}$. We define $\mathcal{F}(\varphi) : \mathcal{F}(X) \to \mathcal{F}(Y)$ as the colimit of the induced morphism $\text{spec} \Gamma_{\Phi} : \text{spec} \Gamma_{\mathcal{U}'} \to \text{spec} \Gamma_{\mathcal{V}'}$ of affine presentations.

The independence of this definition from the chosen affine presentations $\mathcal{U}$, $\mathcal{V}$, $\mathcal{U}'$, and $\mathcal{V}'$ can be seen by considering suitable common refinements. We omit the arguments, which are similar to the ones that we used before. Since the definition of $\mathcal{F}$ is stable under refinements, it follows that $\mathcal{F}(\varphi \circ \psi) = \mathcal{F}(\varphi) \circ \mathcal{F}(\psi)$.

11. The comparison theorem

Theorem 11.1. The functor $\mathcal{G} \circ \mathcal{F} : \text{Sch}^\text{alg}_{F_1} \to \text{Sch}_{\mathcal{U}_1}$ is isomorphic to the embedding of $\text{Sch}^\text{alg}_{F_1}$ as a subcategory of $\text{Sch}_{\mathcal{U}_1}$. The functor $\mathcal{F}$ is fully faithful.

Proof. Let $X$ be a blue scheme with algebraic presentation $\mathcal{V}$. Then $\mathcal{F}(X)$ is isomorphic to the colimit of the affine presentation $\text{spec} \Gamma_{\mathcal{U}}$. By the definition of $\mathcal{G}$, $\mathcal{G}(\mathcal{F}(X))$ is isomorphic to the colimit of the affine presentation $\text{Spec}(\text{spec} \Gamma_{\mathcal{U}})$, which is canonically isomorphic to $\mathcal{U}$ itself. This means that $\mathcal{G} \circ \mathcal{F}(X)$ is canonically isomorphic to $X$.

Let $\varphi : X \to Y$ be a morphism of blue schemes with algebraic presentations $\mathcal{U}$ and $\mathcal{V}$, respectively. Then $\varphi$ is induced by a morphism $\Phi : \mathcal{U}' \to \mathcal{V}'$ of refinements of $\mathcal{U}$ and $\mathcal{V}$, respectively, which induce isomorphisms $\text{colim} \text{spec} \Gamma_{\mathcal{U}'} \to \mathcal{F}(X)$ and $\text{colim} \text{spec} \Gamma_{\mathcal{V}'} \to \mathcal{F}(Y)$. This means that $\mathcal{F}(\varphi)$ is represented by $\text{spec} \Gamma_{\Phi} : \text{spec} \Gamma_{\mathcal{U}'} \to \text{spec} \Gamma_{\mathcal{V}'}$. This induces an isomorphism of $\mathcal{G}(\mathcal{F}(\varphi))$ with $\text{Spec}(\text{spec} \Gamma_{\Phi})$, which is canonically isomorphic to $\varphi$ itself. This shows the former claim of the theorem.

If $X$ and $Y$ are in the essential image of $\mathcal{F}$ and $\varphi : X \to Y$ is a morphism of relative schemes, then it is clear from the local nature of $\mathcal{F}$ and $\mathcal{G}$ that $\mathcal{G}(\mathcal{F}(\varphi))$ is naturally identified with $\varphi$. This shows that $\mathcal{F}$ is fully faithful.

12. Concluding remarks

12.1. Let $X$ be an algebraically presented blue scheme. Then the functors $\mathcal{F}(X)$ and $\text{Hom}(\text{Spec}(\cdot), X)$ from $\text{Bl}pr$ to sets are not isomorphic, basically, because they are sheaves on different sites. Let $\text{Gamma}pr^\text{alg}$ be the full subcategory of $\text{Bl}pr$ whose objects are global blueprints $B$ such that $\text{Spec} B$ is algebraically presented. Then Theorem 11.1 implies that
the restrictions of \( \mathcal{F}(X) \) and \( \text{Hom}(\text{Spec}(-), X) \) to functors from \( \mathcal{B}lpr^{alg} \) to sets are isomorphic.

12.2. The functors \( \mathcal{F} \) and \( \mathcal{G} \) restrict to mutual inverse equivalences between usual schemes and schemes relative to abelian groups. The restriction of \( \mathcal{F} \) to schemes is nothing else than assigning the functor of points \( \text{Hom}(\text{Spec}(-), X) \) (evaluated on rings) to a scheme \( X \), cf. Demazure and Gabriel’s book [3].

Similarly, the functors \( \mathcal{F} \) and \( \mathcal{G} \) restrict to mutual inverse equivalences between monoidal schemes and schemes relative to pointed sets. Also in this case, \( \mathcal{F} \) associates with a monoidal scheme its functor of points (restricted to monoids). Up to the technical variance of the base point, this is Vezzani’s result in [9]. I expect that Vezzani’s method’s transfer without problems to the case of pointed sets, and I expect also that the methods of this text extend to the setting of general blueprints, which include commutative monoids (without a fixed zero).

12.3. While the functor \( \mathcal{G} \) is an extension of the dual \( \Gamma^{op} : \mathcal{B}lpr^{op} \to \Gamma \mathcal{B}lpr \) of the globalization functor \( \Gamma : \mathcal{B}lpr \to \Gamma \mathcal{B}lpr \), which maps affine relative schemes to affine blue schemes, the functor \( \mathcal{F} \) does not restrict to a functor from affine blue schemes to affine relative schemes. In fact, I am not aware of a meaningful way to extend the dual \( \iota^{op} \) of the inclusion functor \( \iota : \Gamma \mathcal{B}lpr \to \mathcal{B}lpr \) to a functor from blue schemes to relative schemes.

12.4. The functors \((-)^{+}\) from blueprints to semirings and \((-)_2^{+}\) from blueprints to rings extend to functors on blue schemes as well as functors on relative schemes, which we denote by the same symbols \((-)^{+}\) and \((-)_2^{+}\), respectively. Since \( \mathcal{G} \) is independent from the chosen affine presentation, \( \mathcal{F} \) commutes with both \((-)^{+}\) and \((-)_2^{+}\). Since every affine presentation of a scheme is an algebraic presentation, \( \mathcal{F} \) commutes with \((-)_2^{+}\).

It is, however, not true that \( \mathcal{F} \) commutes with \((-)^{+}\). This happens not to be the case already for the most elementary example \( \text{Spec} \mathbb{F}_1 \). The relative scheme \( \mathcal{F}(\text{Spec} \mathbb{F}_1) \) is obviously \( \text{spec} \mathbb{F}_1 \) and thus \( \mathcal{F}(\text{Spec} \mathbb{F}_1)^{+} = \text{spec} \mathbb{N} \). It turns out that \( \text{Spec} \mathbb{N} = (\text{Spec} \mathbb{F}_1)^{+} \) is algebraically presented, but that \( \mathcal{F}(\text{Spec} \mathbb{N}) \) is not even affine. This can be seen as follows.

First, we show that the relative scheme \( \text{spec} \mathbb{N} \) has no non-trivial cover, or, what amounts to the same, that all conservative covers of the blue scheme \( \text{Spec} \mathbb{N} \) contain \( \text{Spec} \mathbb{N} \) itself. Note that the inclusion \( \mathbb{N} \to \mathbb{Z} \) induces a homeomorphism \( \text{Spec} \mathbb{Z} \to \text{Spec} \mathbb{N} \), i.e. \( \text{Spec} \mathbb{N} \) has the same open subsets as \( \text{Spec} \mathbb{Z} \). In particular, \( \mathbb{N} \) is a global blueprint and all affine opens are of the form \( U_h = \text{Spec} \mathbb{N}[h^{-1}] \), which are finite localizations of \( \text{Spec} \mathbb{N} \). Let \( \{ U_i \} \) be an affine open cover of \( \text{Spec} \mathbb{N} \). Then \( U_i \) is of the form \( \text{Spec} \mathbb{N}[h_i^{-1}] \) for some \( h_i \in \mathbb{N} \). Consider the non-isomorphic blue \( \mathbb{N} \)-algebras \( M = \mathbb{N}[X]/(hX + Xh) \) and \( M' = \mathbb{N}[s]/(\emptyset) \) and the morphism \( f : M \to M' \) that sends \( X \) to \( h s \). Then \( s = X/h \) on \( U_i \), which means that the base extension functor \( \prod - \otimes \mathbb{N}[h_i^{-1}] \) is not conservative unless \( 1 \) is an \( \mathbb{N} \)-linear combination of the \( h_i \). But this is only possible if one of the \( h_i \) equals 1, which means that \( U_1 = \text{Spec} \mathbb{N} \).

Note that this also shows that \( \text{Spec} \mathbb{N} \) is not an algebraic presentation of itself. However, any non-trivial presentation of \( \text{Spec} \mathbb{N} \) is an algebraic presentation. This can be seen as follows. All morphisms between affine opens of \( \text{Spec} \mathbb{N} \) are finite localizations. Therefore the finite localizations of every open subset form a basis for its topology. Every cover \( \{ U_i \} \) with \( U_i = \text{Spec} \mathbb{N}[h_i^{-1}] \) of a subset \( U_h \) for \( h > 1 \) is conservative since some power of \( h \) can be written as an \( \mathbb{N} \)-linear combination \( h^a = \sum a_i h_i \) and with this, it is possible to adopt the usual proof from scheme theory.

Therefore \( \mathcal{F}(\text{Spec} \mathbb{N}) \) is the colimit of \( \text{spec} \mathcal{W} \) for some non-trivial affine presentation \( \mathcal{W} \) of \( \text{Spec} \mathbb{N} \). Since \( \text{spec} \mathbb{N} \) has no non-trivial covering, \( \mathcal{F}(\text{Spec} \mathbb{N}) \) cannot be \( \text{spec} \mathbb{N} \). It is easy to see that there is no other semiring \( B \) than \( \mathbb{N} \) such that all \( U \) in \( \mathcal{W} \) are finite localizations of \( \text{Spec} B \). Therefore \( \mathcal{F}(\text{Spec} \mathbb{N}) \) is not affine.
12.5. The previous example shows that the digression between blue schemes and relative schemes is already present in the respective subcategories of semiring schemes. Since the tensor product in the category of semirings does not coincide with the tensor product of semirings in the category of blueprints, the situation might look different if we compare semiring schemes in the subcategory $\text{Sch}_{\mathbb{N}}$ of semiring schemes with schemes relative to $\mathbb{N}$-modules, i.e. commutative semigroups with a neutral element. In particular, it is not true in the category of semirings that all flat morphisms of semirings are localizations, cf. Proposition 6.1.

12.6. Finally, we pose the following question. Is there a complete and cocomplete closed symmetric monoidal category $\mathcal{C}$ such that the category $\Gamma \mathcal{B}_{\text{pr}}$ of global blueprints is equivalent to the category $\text{Comm}(\mathcal{C})$ of commutative monoids in $\mathcal{C}$? If so, are $\Gamma \mathcal{B}_{\text{pr}}$ and $\text{Comm}(\mathcal{C})$ isomorphic as sites? A positive answer to these questions would imply that the category of blue schemes is equivalent to the category of schemes relative to $\mathcal{C}$.

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