ON THE CHARACTER DEGREES OF A SYLOW $p$-SUBGROUP OF A
FINITE CHEVALLEY GROUP $G(p^f)$ OVER A BAD PRIME

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ABSTRACT. Let $q$ be a power of a prime $p$ and let $U(q)$ be a Sylow $p$-subgroup of a finite
Chevalley group $G(q)$ defined over the field with $q$ elements. We first give a parametrization
of the set $\text{Irr}(U(q))$ of irreducible characters of $U(q)$ when $G(q)$ is of type $G_2$. This is uniform
for primes $p \geq 5$, while the bad primes $p = 2$ and $p = 3$ have to be considered separately.
We then use this result and the contribution of several authors to show a general result,
namely that if $G(q)$ is any finite Chevalley group with $p$ a bad prime, then there exists a
character $\chi \in \text{Irr}(U(q))$ such that $\chi(1) = q^n/p$ for some $n \in \mathbb{Z}_{\geq 0}$. In particular, for each
$G(q)$ and every bad prime $p$, we construct a family of characters of such degree as inflation
followed by an induction of linear characters of an abelian subquotient $V(q)$ of $U(q)$.

1. Introduction

Let $q$ be a power of a prime $p$, and let $\mathbb{F}_q$ denote the field with $q$ elements. A major research
problem in the representation theory of finite groups is to understand the characters of a
finite Chevalley group $G(q)$ defined over $\mathbb{F}_q$. Namely finite Chevalley groups contribute to a
large part of all finite nonabelian simple groups. The study of the set $\text{Irr}(G(q))$ of ordinary
irreducible characters of $G(q)$ has been carried out to an extensive progress, starting from
the groundbreaking work of Deligne and Lusztig [DL76], to developments which allowed to
compute and process the character table of $G(q)$ in [CHEVIE] when the rank of $G(q)$ is small.
In particular, the set $\text{cd}(G(q)) = \{\chi(1) \mid \chi \in \text{Irr}(G(q))\}$ of irreducible character degrees of
$G(q)$ is essentially known.

The situation is different when we consider characters of $G(q)$ over an algebraically closed
field of characteristic $\ell \neq p$, the so called cross-characteristics case. One has considerably
less amount of information for such characters. The problem of studying such characters is
closely related to the one of parametrizing the ordinary irreducible characters of a fixed
Sylow $p$-subgroup $U(q)$ of $G(q)$, which we also denote by $U(G(q))$. Namely the induction
of $\psi \in \text{Irr}(U(q))$ to $G(q)$ remains an $\ell$-projective character, as $\ell$ and $p$ are different. A
decomposition of such induced character can be provided, if the behavior of $\psi$ is known and
we have enough information about the fusion of the conjugacy classes of $U(q)$ to $G(q)$.

Even for groups of small rank, the set $\text{cd}(U(q))$ is more complicated to describe than the
set $\text{cd}(G(q))$, and is in general also much bigger. We summarize here some of the main known
results in this direction. If $G(q) = A_{n-1}(q)$ for $n \geq 2$, then every degree of a character in
$\text{Irr}(U(q))$ is a power of $q$ [Is95], and in fact $\text{cd}(U(q)) = \{1, q, \ldots, q^{\mu(n)}\}$, where $\mu(n) = m^2 - m$
if $n = 2m$ and $\mu(n) = m^2$ if $n = 2m + 1$, see [Hup] and [Is07]. If $G(q) = D_n(q)$ for $n \geq 4$
and $p$ is odd, then $\text{cd}(U(q)) = \{1, q, \ldots, q^{f(n)}\}$ with $f(n) = n(n-1)/2 - [n/2]$ [Mar99]. A similar
result holds for other $G(q)$ of classical type for odd $p$; in particular, $\text{cd}(U(q)) \subseteq \{q^n \mid n \in \mathbb{Z}_{\geq 0}\}$
for $G(q)$ classical if and only if $p$ is an odd prime, see [Sze03] and [San03]. Via the Kirillov
work is to show that \( \text{cd}(U) \subseteq \{q^n \mid n \in \mathbb{Z}_{\geq 0}\} \) for \( G(q) \) an arbitrary finite Chevalley group.

When \( p \) is a bad prime for \( G(q) \), the set \( \text{cd}(U) \) is often not known. The goal of this work is to show that \( \text{cd}(U) \subseteq \{q^n \mid n \in \mathbb{Z}_{\geq 0}\} \) does not occur in this case, by means of an explicit construction of a character of degree \( q^2/p \) for some positive integer \( n \).

We first determine a parametrization of \( \text{Irr}(U) \) when \( G(q) \) is of type \( G_2 \). We provide full details just for the primes \( p = 2, 3 \); the result is straightforward from [GLMP16, Algorithm 3.3] if \( p \geq 5 \). We use the subsequent Lemma 2.1 to parametrize certain characters of \( \text{Irr}(U) \), and a counting argument to see that these determine all of \( \text{Irr}(U) \). In particular, we find characters of degree \( q/p \) for \( p \in \{2, 3\} \).

**Theorem 1.1.** Let \( G(q) = G_2(3^f) \) or \( G(q) = G_2(2^f) \). The irreducible characters of \( U(q) \) are parametrized in Table 2. In particular, we have that

\[
|\text{Irr}(U(q))| = \begin{cases} 
q^3 + 2q^2 - q - 1, & \text{if } p \geq 5, \\
2q^3 + 5q^2 - 10q + 4, & \text{if } p = 3, \\
q^3 + 5q^2 - 7q + 2, & \text{if } p = 2.
\end{cases}
\]

It is not difficult to produce characters of degree \( q/2 \) in \( U(B_2(q)) \) when \( q \) is a power of 2, which we can inflate to \( U(B_3(q)) \), \( U(C_3(q)) \) and \( U(F_4(q)) \). We could similarly inflate characters of \( U(D_4(2^f)) \) of degree \( q^3/2 \), obtained in [HLM11], to \( U(D_n(2^f)) \) and \( U(E_k(2^f)) \), for \( k = 6, 7, 8 \). For \( q = 3^f \), we have characters of degree \( q^4/3 \) in type \( F_4(q) \) [GLMP16] and of degree \( q^7/3 \) in type \( E_6 \) [LM15] which we can inflate to \( U(E_k(2^f)) \) for \( k = 7, 8 \). The work [LM15] also gives an example of an irreducible character of \( U(E_8(5^f)) \) of degree \( q^{16}/5 \). Finally, the construction of characters of degrees \( q/2 \) in \( \text{Irr}(U(G_2(2^f))) \) and \( q/3 \) in \( \text{Irr}(U(G_2(3^f))) \) follows from Theorem 1.1.

This collection of results allows us to state the following.

**Theorem 1.2.** Let \( G(q) \) be a finite Chevalley group over \( \mathbb{F}_q \). If \( p \) is a bad prime for \( G(q) \), then there exist \( \chi \in \text{Irr}(U(q)) \) and some \( n \geq 1 \) such that \( \chi(1) = q^n/p \). In particular, families of characters of such degree are constructed as an inflation, followed by an induction of a linear character of an abelian subquotient \( V(q) \) of \( U(q) \), with labels as in Table 1.

The labels of the characters are given as in [GLMP16]. In general, a label of the form \( a_i \) (respectively \( b_j \)) of \( \chi \in \text{Irr}(U(q)) \) corresponds to an element of \( \mathbb{F}_q^\times \) (respectively \( \mathbb{F}_q \)), which is the value on \( x_i(1) \) (respectively \( x_j(1) \)) of the linear character that we inflate and induce to obtain \( \chi \). More details on these labels are given in the sequel for each case taken into consideration.

The importance of the construction of such characters lies in the fact that these could replace some classes of characters, defined just for good \( p \), helpful for investigating the cross-characteristics representations of \( G(q) \). Let us for instance take \( G(q) = D_4(q) \), and \( \ell \neq p \) with \( \ell \mid q + 1 \). The decomposition numbers are obtained in [GP92] in the case when \( p \) is an odd prime; this assumption is required for exploiting properties of the generalized Gelfand-Graev characters and the parametrization of Green functions in [LS90]. A calculation shows that in the case \( p = 2 \), by inducing to \( G(q) \) the four irreducible characters of \( U(q) \) corresponding to \( (d_{1,2,4}, d_3) \in \mathbb{F}_2 \times \mathbb{F}_2 \) in Proposition 4.3 we get characters that play the role of the \( \ell \)-projective characters \( \Phi_0, \ldots, \Phi_9 \) in [GP92, §5], which turn out to be of major importance to determine the unitriangular shape of the decomposition matrix of \( D_4(2^f) \).
We now examine the question of whether each of the families of irreducible characters of degree $q^n/p$ in Table 1 consists of all the characters of $\text{Irr}(U(q))$ of such degree. It turns out that the previously mentioned works also determine that if $G(q)$ is not $E_8(5^f)$, then the families in Table 1 that do not arise from an embedding of a root system of smaller rank give in fact all the characters of $\text{Irr}(U(q))$ of degree $q^n/p$ for the corresponding values of $n$. On the other hand, if a family $\mathcal{F}$ does arise from an embedding of a smaller root system, then it is straightforward to get other characters in $\text{Irr}(U(q))$ of the same degree which are not in $\mathcal{F}$ by tensoring the characters in $\mathcal{F}$ with linear characters of a certain root subgroup indexed by a root in the difference of the two root systems.

We then propose the following conjecture.

**Conjecture 1.3.** The family in Table 1 of irreducible characters of degree $q^{16}/5$ in $\text{Irr}(U(E_8(5^f)))$ consists of all irreducible characters of $U(E_8(5^f))$ whose degree is not a power of $q$.

Finally, we present further progress and a question on fractional degrees in $U(q)$. The work [GMP01] provides a construction of irreducible characters of fractional degrees with denominator of the form $p^t$ with $t \geq 2$. Namely if $G(q) = C_n(q)$ then there exist irreducible characters of $U(q)$ of degree $q^{n(n-1)/2}/2^t$ for every $0 \leq t \leq \lfloor n/2 \rfloor$, and if $G(q) = D_n$, then there exist irreducible characters of $U(q)$ of degree $q^{3n(n-1)/2}/2^t$ (respectively $q^{3n(n+1)/2}/2^t$) for every $1 \leq t \leq \lfloor r/2 \rfloor$ if $n = 2r$ (respectively if $n = 2r + 1$). On the one hand, such characters seem to maximize the power $m$ for character degrees of the form $q^m/p^t$. On the other hand, they seem not to maximize $t$, as in the case of $U(D_6(2^f))$ one just gets $t = 1$ by applying the above formula, while by [LMP17] we know that there also exist characters of the form $q^m/2^t$ with $t = 2$ in $U(D_6(2^f))$. It would be interesting to determine, in general, all powers $q^m/p^t$ that can occur as character degrees for $\text{Irr}(U(q))$, in particular the maximum value of $t$, for each finite Chevalley group $G(q)$.

**2. Preliminaries**

We first let $G$ be any finite group, $H$ be a subgroup of $G$, and $N$ be a normal subgroup of $G$. We recall some notation on characters of $G$ and its subgroups.

We let $\text{Irr}(G)$ be the set of irreducible characters of the group $G$. For a character $\chi \in \text{Irr}(G)$, we denote by $\ker(\chi)$ the kernel of $\chi$ and by $Z(\chi)$ its centre. We denote by $\chi|_H$ the restriction of $\chi$ to $H$. Let $\varphi \in \text{Irr}(G/N)$. Then we denote by $\text{Inf}_{G/N}^G \varphi \in \text{Irr}(G)$ the inflation of the character $\varphi$ to $G$. If $\psi$ is a character of $H$, then we denote by $\text{Ind}_H^G(\psi)$ the induction of the character $\psi$ to $G$. We denote by $\langle , \rangle$ the usual inner product defined on the characters of $G$. If $\eta \in \text{Irr}(H)$, then we denote

$$\text{Irr}(G | \eta) = \{\chi \in \text{Irr}(G) \mid \langle \chi|_H, \eta \rangle \neq 0\} = \{\chi \in \text{Irr}(G) \mid \langle \chi, \text{Ind}_H^G \eta \rangle \neq 0\}.$$  

We recall a result that we use several times in the sequel. The proof is a particular case of [HLM16, Lemma 2.1] detailed in [GLMP16, §4.1].

**Lemma 2.1.** Let $G$ be a finite group, let $H \leq G$ and let $X$ be a transversal of $H$ in $G$. Let $Y$ and $Z$ be subgroups of $H$, and $\lambda \in \text{Irr}(Z)$. Suppose that

(i) $Z \subseteq Z(G)$,
(ii) $X$ and $Y$ are elementary abelian groups with $|X| = |Y|$,
(iii) $Y \trianglelefteq H$,
(iv) $Z \cap Y = 1$, and
If we put\( G \) of simple roots. We fix an enumeration for \( \Phi \).

| Type | Bad primes | Character labels | Size of family | Degree | Unique |
|------|------------|-----------------|----------------|--------|--------|
| \( B_n \) | \( p = 2 \) | \( x_{d_n-1,d_n}^{n+1,n+2} \) | \( 4(q - 1)^2 \) | \( q/2 \) | iff \( n = 2 \) |
| \( C_n \) | \( p = 2 \) | \( x_{d_n,d_n}^{n+1,n+2} \) | \( 4(q - 1)^2 \) | \( q/2 \) | iff \( n = 2 \) |
| \( D_n \) | \( p = 2 \) | \( x_{d_2,d_2}^{n+1,n+2,n+3} \) | \( 4(q - 1)^4 \) | \( q^3/2 \) | iff \( n = 4 \) |
| \( G_2 \) | \( p = 2 \) | \( x_{d_2,d_2}^{n+1,n+2} \) | \( 4(q - 1)^2 \) | \( q/2 \) | yes |
| \( F_4 \) | \( p = 3 \) | \( x_{d_2,d_2}^{n+1,n+2} \) | \( 9(q - 1)^2/2 \) | \( q/3 \) | no |
| \( E_6 \) | \( p = 2 \) | \( x_{d_2,d_2}^{n+1,n+2} \) | \( 4(q - 1)^4 \) | \( q^3/2 \) | yes |
| \( E_7 \) | \( p = 3 \) | \( x_{d_2,d_2}^{n+1,n+2} \) | \( 9(q - 1)^6/2 \) | \( q^3/3 \) | no |
| \( E_8 \) | \( p = 2 \) | \( x_{d_2,d_2}^{n+1,n+2} \) | \( 9(q - 1)^6/2 \) | \( q^3/3 \) | no |

Table 1. Families of characters of \( U(q) \) of degree \( q^n/p \) for some \( n \in \mathbb{N} \), and their uniqueness of such degree, for each bad prime \( p \) and every Lie type.

(v) the commutator group \([X,Y]\) is contained in \( Z \).

If we put

\[ X' := \{ x \in X \mid \lambda([x,y]) = 1 \text{ for all } y \in Y \} \]

and

\[ Y' := \{ y \in Y \mid \lambda([x,y]) = 1 \text{ for all } x \in X \}, \]

and if \( \tilde{Y} \) is a complement of \( Y' \) in \( Y \), then the map

(2.1) \[ \text{Ind}^G_{H, X', \text{Int}_{H, X', \text{ker} \lambda}} : \text{Irr}(H X'/\tilde{Y} \text{ ker} \lambda \mid \lambda) \longrightarrow \text{Irr}(G \mid \lambda). \]

is a bijection.

We keep the notation for \( q, G(q), U(q) \) and \( \text{Irr}(U(q)) \) as in the Introduction. We briefly recall the notion of bad primes. Let \( \Phi \) be the root system associated with \( G(q) \), and let \( \Phi^+ \) be set of positive roots in \( \Phi \). We fix an enumeration \( \alpha_1, \ldots, \alpha_{|\Phi^+|} \) of the positive roots, with \( \alpha_1, \ldots, \alpha_n \) the simple roots of \( \Phi^+ \), and \( \alpha_0 := \alpha_{|\Phi^+|} \) the highest root in \( \Phi^+ \). We say that \( p \) is a bad prime for \( \Phi^+ \) if \( p \) divides one of the coefficients of \( \alpha_0 \) in its linear combination in terms of simple roots.

We recall that as \( G(q) \) is a split group, we have that

\[ U(q) = \prod_{\alpha \in \Phi^+} X_\alpha, \quad \text{with} \quad X_\alpha = \{ x_\alpha(t) \mid t \in \mathbb{F}_q \} \cong (\mathbb{F}_q, +), \]

where...
hence we have \(|U(q)| = q^{\Phi^+}\). The group \(X_\alpha\) is called the root subgroup of \(U(q)\) associated to \(\alpha \in \Phi^+\), and each element \(x_\alpha(t)\) is called the root element with respect to \(\alpha \in \Phi^+\) and \(t \in \mathbb{F}_q\).

We say \(\mathcal{P} \subseteq \Phi^+\) is a pattern in \(\Phi^+\) if for every \(\alpha, \beta \in \mathcal{P}\), either \(\alpha + \beta \in \mathcal{P}\) or \(\alpha + \beta \notin \Phi^+\). For a pattern \(\mathcal{P}\), we have that the product

\[
X_\mathcal{P} := \prod_{\alpha \in \mathcal{P}} X_\alpha
\]

is well defined, and it is a subgroup of \(U(q)\). We call \(X_\mathcal{P}\) the pattern group corresponding to \(\mathcal{P}\). If \(\mathcal{P} := \{\alpha_{i_1}, \ldots, \alpha_{i_m}\}\), then we also write \(X_{\{i_1, \ldots, i_m\}}\) for \(X_\mathcal{P}\); similarly we write \(x_{i}(t)\) for the root element \(x_{\alpha_{i}}(t)\), with \(\alpha_{i} \in \Phi^+\) and \(t \in \mathbb{F}_q\). A subset \(\mathcal{N}\) of a pattern \(\mathcal{P}\) is normal in \(\mathcal{P}\), or \(\mathcal{N} \trianglelefteq \mathcal{P}\), if for every \(\alpha \in \mathcal{N}\) and \(\beta \in \mathcal{P}\), one has either \(\alpha + \beta \in \mathcal{N}\) or \(\alpha + \beta \notin \Phi^+\). It is easy to check that if \(\mathcal{N} \trianglelefteq \mathcal{P}\), then \(X_\mathcal{N} \trianglelefteq X_\mathcal{P}\). For \(\chi \in \text{Irr}(U(q))\), we define the central root support \(rs(\chi) := \{\alpha \in \Phi^+ \mid X_\alpha \subseteq Z(\chi)\}\) and \(X_\alpha \notin \ker(\chi)\).

In order to construct the subquotients \(V(q)\) as in Theorem 1.2, and to parametrize the corresponding characters, we need to fix a nontrivial character of \((\mathbb{F}_q,+). Denote by \(\text{Tr} : \mathbb{F}_q \to \mathbb{F}_p\) the field trace map. We define \(\phi : \mathbb{F}_q \to \mathbb{C}^\times\) by \(\phi(t) = e^{\frac{2\pi i \text{Tr}(t)}{p}}\) for \(t \in \mathbb{F}_q\). Notice that

\[
(2.2) \quad \ker \phi = \{t^p - t \mid t \in \mathbb{F}_q\}.
\]

Remark 2.2. In the sequel, for each cyclic group \(C\) of order \(p\) we implicitly fix a morphism \(\varphi : \mathbb{Z}/p\mathbb{Z} \to C\), and for each \(d \in \mathbb{Z}/p\mathbb{Z}\) we denote by \(\mu_C^d \in \text{Irr}(C)\) the character such that \(\mu_C^d(\varphi(1)) = \zeta_p^d\), where \(\zeta_p\) is a fixed primitive \(p\)-th root of unity.

3. A parametrization of \(\text{Irr}(U(G_2(q)))\), when \(q = 2^f\) or \(q = 3^f\)

In this section we provide a parametrization of the irreducible characters of \(U(q)\) when \(G(q) = G_2(q)\) for every prime \(p\). This is done by parametrizing families of characters of certain subquotients of \(U(q)\), and checking by using the well-known formula

\[
(3.1) \quad |U(q)| = \sum_{\chi \in \text{Irr}(U(q))} \chi(1)^2
\]

that these families give in fact all of \(\text{Irr}(U(q))\). We will denote by \(T_1, T_2, \ldots\) such subquotients of \(\text{Irr}(U(q))\) in the sequel. Each of the labels \(\chi_{b_{j_1}, \ldots, b_{j_s}}^{a_{i_1}, \ldots, a_{i_r}}\) in Tables 1 and 2, with \(a_{i_1}, \ldots, a_{i_r} \in \mathbb{F}_q^\times\) and \(b_{j_1}, \ldots, b_{j_s} \in \mathbb{F}_q\), is obtained in a similar way as in [GLMP16], namely by inflation-induction process of the corresponding family of characters

\[
\chi_{a_{i_1}, \ldots, a_{i_r}} \otimes \mu_{j_1} \otimes \cdots \otimes \mu_{j_s} \in \text{Irr}(V(q)).
\]

The characters of \(\text{Irr}(U(G_2(q)))\) with labels of the form \(d \in \mathbb{F}_2\) or \(e \in \mathbb{F}_3\) are described in more detail in this section.

We denote by \(\alpha_1\) the long simple root in \(G_2\), hence \(\alpha_2\) is its short simple root. For every prime \(p\), and for every \(s, t \in \mathbb{F}_q\), the commutator relations among root elements are as follows,

\[
[x_1(s), x_2(t)] = x_3(-st)x_4(st^2)x_5(-st^3)x_6(-s^2t^3),
[x_2(s), x_3(t)] = x_4(2st)x_5(-3s^2t)x_6(3st^2),
[x_1(s), x_5(t)] = x_6(st),
[x_3(s), x_4(t)] = x_6(-3st),
\]
and \([x_i(s), x_j(t)] = 1\) in the remaining cases. Observe that the irreducible characters of \(U(G_2(q))\) when \(p \geq 5\) can be easily parametrized by [GLMP16, Algorithm 3.3].

**Proposition 3.1.** Let \(q = p^f\) with \(p \geq 5\). Then \(U(G_2(q))\) has exactly

(i) \(q(q - 1)\) irreducible characters of degree \(q^2\),

(ii) \(q^2(q - 1) + q(q - 1) + (q - 1)\) irreducible characters of degree \(q\), and

(iii) \(q^2\) linear characters.

The characters of degree \(q^2\) are precisely the ones with central root support \(\{\alpha_6\}\), and each of the summands \(q^2(q - 1)\) in the expression for the number of irreducible characters of degree \(q\) corresponds to the family of irreducible characters with central root support \(\{\alpha_{i+3}\}\) for \(i = 0, 1, 2\).

We examine next the case \(q = 2^f\).

**Proposition 3.2.** Let \(q = 2^f\) and \(G(q) = G_2(q)\). Then \(U(q)\) has exactly

(i) \(q(q - 1)\) irreducible characters of degree \(q^2\),

(ii) \((q - 1)q^2 + 2(q - 1)\) irreducible characters of degree \(q\),

(iii) \(4(q - 1)^2\) irreducible characters of degree \(q/2\), and

(iv) \(q^2\) linear characters.

**Proof.** Let \(T_1 := U(q)\), and let \(Z := Z(T_1) = X_6\). Let us define \(\lambda^{a_6} \in \text{Irr}(Z)\) in the usual way. By the commutator relations, it is an easy check to deduce that the assumptions of Lemma 2.1 are verified with \(X := X_1 X_3\), \(Y := X_4 X_5\) and \(H := X_1 Y Z\). We have that \(X' = Y' = 1\). Let \(V_1(q) := X_2 X_6\). Then the family

\[
\mathcal{F}_1 := \{\text{Ind}_{V_1(q)}^{U(q)} \text{Ind}_{V_1(q)}^{V_1(q)} (\lambda^{a_6} \otimes \mu_2) \mid a_6 \in \mathbb{F}_q^*, \mu_2 \in \text{Irr}(X_2)\}
\]

consists of \(q(q - 1)\) irreducible characters of \(U(q)\) of degree \(q^2\).

Let now \(T_2 := U(q)/X_6\). We have \(Z := Z(T_2) = X_5\). Again we apply Lemma 2.1; it is an easy check that its hypotheses are satisfied with \(X := X_2\), \(Y := X_4\) and \(H := X_1 X_3 Y Z\). We have \(X' = Y' = 1\) also in this case. If \(V_2(q) := X_1 X_3 X_5\), then we have that

\[
\mathcal{F}_2 := \{\text{Ind}_{V_2(q)}^{U(q)} \text{Ind}_{V_2(q)}^{V_2(q)} (\lambda^{a_5} \otimes \mu_1 \otimes \mu_3) \mid a_5 \in \mathbb{F}_q^*, \mu_1 \in \text{Irr}(X_1), \mu_3 \in \text{Irr}(X_3)\}
\]

is a family of \(q^2(q - 1)\) irreducible characters of degree \(q\).

We now notice that \(T_3 := U(q)/X_5 X_6\) is isomorphic to \(U(B_2(2^f))\) in the obvious way. By the subsequent Proposition 4.1, we get a family \(\mathcal{F}_3\) of \(2(q - 1)\) irreducible characters of degree \(q\), a family \(\mathcal{F}_4\) of \(4(q - 1)^2\) irreducible characters of degree \(q/2\), and the family \(\mathcal{F}_5\) of \(q^2\) linear characters.

Finally, notice that if \(\chi_i\) is one of the characters in \(\mathcal{F}_i\), for \(i = 1, \ldots, 5\), then we have

\[
\sum_{i=1}^{5} \chi_i(1)^2 |\mathcal{F}_i| = q^5(q - 1) + q^4(q - 1) + 2q^2(q - 1) + 4q^2(q - 1)^2/4 + q^2 = q^6 = |U(q)|,
\]

hence by Equation (3.1) we have \(\mathcal{F}_1 \cup \cdots \cup \mathcal{F}_5 = \text{Irr}(U(q))\). \(\square\)

We now determine the irreducible characters of \(U(G_2(q))\) when \(q = 3^f\).

**Proposition 3.3.** Let \(q = 3^f\) and \(G(q) = G_2(q)\). Then \(U(q)\) has

(i) \((q - 1)^2\) irreducible characters of degree \(q^2\),
Proof. Let \( T_1 := U(q) \). We have that \( Z(T_1) = X_4X_6 \). Let us put \( Z := X_6 \) and let us define \( \lambda^{a_6} \) as usual for \( a_6 \in \mathbb{F}_q^\times \). Then Lemma 2.1 applies with \( X := X_2, Y := X_5 \) and \( H := X_1X_3YZ \), and \( X' = Y' = 1 \). Let \( V_1(q) := X_1X_3X_4X_6 \). Then \( V_1(q) \cong (X_1X_3X_4) \times X_6 = T' \times \mathbb{F}_q \), where \( T' \) is a special group of the form \( q^{1+2} \) with \( Z(T') = X_4 \). One then has that \( T' \) has \( q - 1 \) irreducible characters of degree \( q \) and \( q^2 \) linear characters. Hence we get two families of characters, namely

\[
F_1 = \{ \text{Ind}_{X(1,3,4,6)}^{U(q)} \text{Ind}_{X(1,3,4,6)}^{X(1,3,4,6)}(\lambda^{a_4a_6}) \mid a_4, a_6 \in \mathbb{F}_q^\times \},
\]

which consists of \((q - 1)^2\) characters of \( U(q) \) of degree \( q^2 \), and

\[
F_2 = \{ \text{Ind}_{X(1,3,4,5,6)}^{U(q)} \text{Ind}_{X(1,3,4,5,6)}^{X(1,3,4,5,6)}(\lambda^{a_6} \otimes \mu_1 \otimes \mu_3) \mid a_6 \in \mathbb{F}_q^\times, \mu_1 \in \text{Irr}(X_1) \text{ and } \mu_3 \in \text{Irr}(X_3) \},
\]

which has \( q^2(q - 1) \) characters of \( U(q) \) of degree \( q \).

Let us now define \( T_2 := U(q)/X_6 \). Then \( X_4 \subseteq Z(T_2) \). We let \( Z := X_4 \) and \( \lambda^{a_4} \in \text{Irr}(Z) \). It is a straightforward check that \( X := X_2, Y := X_4 \) and \( H := X_1X_5YZ \) satisfy the assumptions of Lemma 2.1. Again we have that \( X' = Y' = 1 \). Notice that \( V_2(q) := X_1X_2X_5X_6/X_6 \) is an abelian group. Hence we get a family

\[
F_3 = \{ \text{Ind}_{X(1,3,4,5,6)}^{U(q)} \text{Ind}_{X(1,3,4,5,6)}^{X(1,3,4,5,6)}(\lambda^{a_4} \otimes \mu_1 \otimes \mu_5) \mid a_4 \in \mathbb{F}_q^\times, \mu_1 \in \text{Irr}(X_1) \text{ and } \mu_5 \in \text{Irr}(X_5) \}
\]

of \( q^2(q - 1) \) irreducible characters of \( U(q) \) of degree \( q \).

We now let \( T_3 := U(q)/X_{1,4,6} \). In this case, we have \( Z := Z(T_3) = X_{1,3,5} \), and we define \( \lambda^{a_3a_5} \in \text{Irr}(Z) \) for \( a_3, a_5 \in \mathbb{F}_q^\times \) in a similar way as in the case of \( T_1 \) and \( T_2 \). The groups \( X := X_1, Y := X_2 \) and \( H := X_{1,2,3,5} \) satisfy the hypotheses of Lemma 2.1. We now want to compute the sets \( X' \) and \( Y' \). We have that

\[
\lambda([x_2(t), x_1(s)]) = \lambda(x_3(st)x_5(st^3)) = \phi(st(a_3 + a_5t^2)).
\]

Let us first assume that \( a_3, a_5, a_3, a_5 \in \mathbb{F}_q^\times \), and that \(-a_3/a_5\) is a square. In this case, we write \( a_3^* \) for \( a_3 \). Notice that there are \((q - 1)^2/2\) such pairs of elements \( a_3^*, a_5 \in \mathbb{F}_q^\times \). Namely the set \( S \) of squares in \( \mathbb{F}_q^\times \) is a subgroup of \( \mathbb{F}_q^\times \) of order \((q - 1)/2\), and \(-a_3^*/a_5 \) is a square if and only if \( a_5 \in \mathbb{F}_q^\times \) and \( a_3^* = -a_5S \). Let \( \omega_{3,5} \) be a fixed square root of \(-a_3^*/a_5 \). By Equation (2.1), we have that

\[
X' := \{1, x_1(\pm 1/(a_3\omega_{3,5}))\} \quad \text{and} \quad Y' := \{1, x_2(\pm \omega_{3,5})\}.
\]

In this case we have \([G : HX'] = q/3\). Moreover, \( V_3(q) := ZX'Y' \cong HX'/Y \ker \lambda \text{ is abelian.} \)

By Lemma 2.1, we obtain a family

\[
F_4 := \{ \chi_{e_1,e_2}^{a_3^*, a_5} \mid e_1, e_2 \in \mathbb{F}_3, a_3 \in \mathbb{F}_q^\times \text{ and } a_3^* \in -a_5S \},
\]

where

\[
\chi_{e_1,e_2}^{a_3^*, a_5} := \text{Ind}_{HX}^{U(q)} \text{Ind}_{V_3(q)}^{HX}(\lambda^{a_3^*, a_5} \otimes \mu_1^{e_1} \otimes \mu_2^{e_2}),
\]

of \( 9(q - 1)^2/2 \) irreducible characters of \( U(q) \) of degree \( q/3 \).
We now suppose that $a_4, a_5 \in \mathbb{F}_q^\times$, and $a_5/a_3$ is not a square. We write $a'_3$ for $a_3$. In this case, we have that $X' = Y' = 1$. We put $V_4(q) := H/Y \cong X_3X_5$. We get a family
\[
\mathcal{F}_5 := \{ \operatorname{Ind}_{U}^{U(q)} \operatorname{Ind}_{V_4(q)}^{U(q)} (\lambda(a'_3,a_5) \mid a_5 \in \mathbb{F}_q^\times \text{ and } a'_3 \in \mathbb{F}_q^\times \setminus \{a_5S\}\}
\]
of $(q-1)^2/2$ irreducible characters of $U(q)$ of degree $q$.

If exactly one of $a_3$ or $a_5$ is nonzero, then we also get $X' = Y' = 1$. Let $V_5(q) := X_3X_5$. Then we get a family
\[
\mathcal{F}_6 := \{ \operatorname{Ind}_{U}^{U(q)} \operatorname{Ind}_{V_5(q)}^{U(q)} (\lambda(a_3,a_5) \mid (a_3,a_5) \in (\mathbb{F}_q^\times \times \{0\}) \cup (\{0\} \times \mathbb{F}_q^\times)\}
\]
of $2(q-1)$ irreducible characters of degree $q$ of $U(q)$. The choice $a_3 = a_5 = 0$ corresponds to the family $\mathcal{F}_7$ of $q^2$ linear characters of $U(q)$.

Finally, if $\chi_i$ is any character in $\mathcal{F}_i$, for $i = 1, \ldots, 7$, then we have
\[
\sum_{i=1}^{7} \chi_i(1)^2 |\mathcal{F}_i| = q^4(q-1)^2 + q^2(2q^2(q-1) + (q-1)(q+3)/2) + q^2(q-1)^2/2 + q^2 = q^6 = |U(q)|,
\]

hence $\mathcal{F}_1 \cup \cdots \cup \mathcal{F}_7 = \operatorname{Irr}(U(G_2(q)))$.

\[\square\]

4. Characters of fractional degree of $U(q)$ in classical type

We now focus on the characters of $U(B_2(q))$ when $q = 2^f$. The family of characters of degree $q/2$ in $U(B_2(q))$ was obtained in [Lus03, §7] and revisited in [BD06, §7] in the context of character sheaves. We construct it here as an inflation-induction process from some subquotient of $U(B_2(q))$.

**Proposition 4.1.** Let $q = 2^f$. Then there are exactly $4(q-1)^2$ irreducible characters of $U(B_2(q))$ of degree $q/2$.

**Proof.** Since $p = 2$, we have that $[x_3(s), x_3(t)] = 1$. Hence $Z := Z(U(q)) = X_{(3,4)}$. Let $X := X_1, Y := X_2$ and $H := X_{(2,3,4)}$. Then the assumptions of Lemma 2.1 are satisfied. Let
us fix $a_3, a_4 \in \mathbb{F}_q^\times$, and let $\lambda = \lambda^{a_3, a_4} \in \text{Irr}(Z)$ be such that $\lambda(x_i(t)) = \phi(a_it)$ for $i = 3, 4$. We have that
\[ \lambda([x_1(s), x_2(t)]) = \lambda(x_3(st)x_4(st^2)) = \phi(st(a_3 + a_4t)), \]
and in the notation of Lemma 2.1, by Equation (2.2) we have
\[ X' := \{1, x_1(a_4/a_3^2)\} \quad \text{and} \quad Y' := \{1, x_2(a_3/a_4)\}. \]
Notice that $[U(q) : HX'] = q/2$ and that $HX'/Y' \ker \lambda \cong ZX'Y'$. Let us define $V(q) := ZX'Y'$. By Equation (2.1) we have that
\[ \text{Ind}_{HX'}^{U(q)} \text{Ind}_{Y'}^{HX'} : \text{Irr}(V(q) \mid \lambda) \rightarrow \text{Irr}(G \mid \lambda) \]
is a bijection. Moreover, we have that $V(q)$ is abelian, hence
\[ \text{Irr}(V(q) \mid \lambda^{a_3, a_4}) = \{\psi_{d_1, d_2}^{a_3, a_4} \mid d_1, d_2 \in \mathbb{F}_2\}, \]
where
\[ \psi_{d_1, d_2}^{a_3, a_4} := \lambda^{a_3, a_4} \otimes \mu_{X'}^{d_1} \otimes \mu_{Y'}^{d_2}. \]
Notice that the sets $\text{Irr}(G \mid \lambda^{a_3, a_4})$ are disjoint for $a_3, a_4 \in \mathbb{F}_q^\times$.

Finally, recall by [Lus03, §7] that the other characters in $\text{Irr}(U(q))$ consist of two families of size $q - 1$ of characters of degree $q$, namely the irreducible characters with central root support $\{\alpha_3\}$ and $\{\alpha_4\}$ respectively, and the family of the $q^2$ linear characters.

---

**Figure 1.** The Dynkin diagrams of $B_n$ and $C_n$. Simple roots are labelled as in CHEVIE.

---

By inflation of the irreducible characters in Proposition 4.1 we obtain the following.

**Corollary 4.2.** Let $G(q) = B_n(q)$ or $G(q) = C_n(q)$. Then $U(q)$ has at least $4(q - 1)^2$ characters of degree $q/2$.

**Proof.** Let us define
\[ \mathcal{N} := \bigcup_{i=1}^{n-2}\{\alpha \in \Phi^+ \mid \alpha_i \leq \alpha\}. \]
We have that $\mathcal{N} \subseteq \Phi^+$, hence $X_\mathcal{N} \subseteq U(q)$, and $U(q)/X_\mathcal{N} \cong X_{\{n-1,n,2n-1,3n-2\}}$ is isomorphic to $U(B_2(q))$, with $\alpha_{n-1}$ long and $\alpha_n$ short in type $B_n$ and vice versa in type $C_n$, in the notation of Figure 1. If $a_{2n-1}, a_{3n-2} \in \mathbb{F}_q^\times$ and $d_{n-1}, d_n \in \mathbb{F}_2$, we then define
\[ \chi_{a_{2n-1}, a_{3n-2}}^{d_{n-1}, d_n} := \text{Ind}_{U(q)/X_\mathcal{N}}^{U(q)} \psi_{a_{2n-1}, a_{3n-2}}^{d_{n-1}, d_n} \]
if $U(q) = U(B_2(q))$, and
\[ \chi_{a_{2n-1}, a_{3n-2}}^{d_{n-1}, d_n} := \text{Ind}_{U(q)/X_\mathcal{N}}^{U(q)} \psi_{a_{2n-1}, a_{3n-2}}^{d_{n-1}, d_n} \]
if $U(q) = U(C_2(q))$, where $\psi_{d_{n-1}, d_n}^{a_{2n-1}, a_{3n-2}}$ and $\psi_{d_{n-1}, d_n}^{a_{2n-1}, a_{3n-2}}$ are defined as in Proposition 4.1. 

---
Let us now examine the groups of type $D_4$. The irreducible characters of $U(D_4(q))$ have been completely parametrized in [HLM11] for every prime $p$. Unlike the case of type $B_n$ and $C_n$, there are no characters of degree $q/2$ in type $D_4$. The statement and the construction below combine the study of the family $\mathcal{F}_{8,9,10}$ in [HLM11] and the approach of [GLMP16].

**Proposition 4.3.** Let $G(q) = D_4(q)$. Then $U(q)$ has exactly $4(q - 1)^4$ irreducible characters of degree $q^3/2$.

**Construction.** Let $\mathcal{N} := X_{\{11,12\}}$. Notice that $\mathcal{N} \subseteq \Phi^+$. For fixed $a_8, a_9, a_{10} \in \mathbb{F}_q^\times$, define

$$x_{1,2,4}(t) := x_1(a_1t)x_2(a_9t)x_4(a_8t) \quad \text{and} \quad x_{5,6,7}(s) := x_5(a_{10}s)x_6(a_9s)x_7(a_8s)$$

for every $s, t \in \mathbb{F}_q$. Let

$$X' := \{x_{1,2,4}(t) \mid t \in \mathbb{F}_q\} \quad \text{and} \quad Y' := \{x_{5,6,7}(s) \mid s \in \mathbb{F}_q\},$$

and let $\bar{Y} := X_5X_6$. Then $Y = Y' \times \bar{Y}$.

Let us also fix $a_{5,6,7} \in \mathbb{F}_q^\times$. We define $\lambda = \lambda_{a_{5,6,7},a_8,a_9,a_{10}}$ by $\lambda(x_{5,6,7}(s)) = \phi(a_{5,6,7}a_8a_9a_{10})$, and

$$W_1 := \{1, x_{1,2,4}(a_{5,6,7}/(a_8a_9a_{10})) \} \quad \text{and} \quad W_2 := \{1, x_3(a_8a_9a_{10}/a_{5,6,7})\}$$

and $V(q) := W_1W_2\mathbb{Y}Z/(\mathbb{Y} \ker \lambda)$. Then each character of

$$\mathcal{F} := \{\psi^{a_{5,6,7},a_8,a_9,a_{10}}_{d_1,d_2,d_3} \mid a_{5,6,7}, a_8, a_9, a_{10} \in \mathbb{F}_q^\times, d_1, d_2, d_3 \in \mathbb{F}_2\},$$

where

$$\psi^{a_{5,6,7},a_8,a_9,a_{10}}_{d_1,d_2,d_3} := \text{Ind}_{\mathcal{N}}^{U(q)}(\text{Ind}_{\mathcal{F}}^{XW_2YX_{\{8,9,10\}}}(\mathbb{Y}^{a_{5,6,7},a_8,a_9,a_{10}} \otimes \mu_{W_1} \otimes \mu_{W_2})), \quad \text{is irreducible in } U(q)$$

of degree $q^3/2$. The characters $\psi^{a_{5,6,7},a_8,a_9,a_{10}}_{d_1,d_2,d_3}$ are all distinct.

Finally, by [HLM11], there are no other characters in $\text{Irr}(U(q))$ of degree $q^n/2$ for any $n \geq 0$. \qed

As done in the case of type $B_n$ and $C_n$, we obtain by inflation characters of degree $q^3/2$ in type $D_n$ for every $n \geq 5$.

**Corollary 4.4.** For $n \geq 5$, the group $U(D_n(q))$ has $4(q - 1)^4$ irreducible characters of degree $q^3/2$.

**Proof.** In a similar way as in Corollary 4.2, we have that

$$\mathcal{N} := \bigcup_{i=5}^{n}\{\alpha \in \Phi^+ \mid \alpha_i \leq \alpha\} \subseteq \Phi^+.$$

Then we have that

$$U(q)/X_\mathcal{N} = X_\mathcal{S} \cong U(D_4(q)),$$

where $\mathcal{S} := \{1, 2, 3, 4, n + 1, n + 2, n + 3, 2n, 2n + 1, 2n + 2, 3n - 1, 4n - 3\}$. We then apply Proposition 4.3, namely if $a_{n+1,n+2,n+3}, a_{2n}, a_{2n+1}, a_{2n+2} \in \mathbb{F}_q^\times$ and $d_1, d_2, d_3 \in \mathbb{F}_2$, the characters as in the claim are given by

$$\psi^{a_{n+1,n+2,n+3},a_2,a_3,a_{2n+1},a_{2n+2}}_{d_1,d_2,d_3} := \text{Ind}_{U(q)/X_\mathcal{N}}^{U(q)/X_\mathcal{N}}(\psi^{a_{n+1,n+2,n+3},a_2,a_3,a_{2n+1},a_{2n+2}}_{d_1,d_2,d_3})$$

where each of the $\psi^{a_{n+1,n+2,n+3},a_2,a_3,a_{2n+1},a_{2n+2}}_{d_1,d_2,d_3}$ is defined as in Proposition 4.3. \qed
5. Characters of fractional degree of $U(q)$ in types $F_4$ and $E_k$

We now move on to character degrees of the form $q^n/p$ in type $F_4$. We first consider the case of the prime $p = 2$. We have
\[
\mathcal{N} := \{ \alpha \in \Phi^+ \mid \alpha_1 \leq \alpha \} \cup \{ \alpha \in \Phi^+ \mid \alpha_4 \leq \alpha \} \leq \Phi^+,
\]
and $U(q)/X_{\mathcal{N}} \simeq U(B_2(q))$. Hence we obtain characters of degrees $q/2$ in $U(F_4(q))$. For $p = 3$, we have the following explicit construction.

**Proposition 5.1** ([GLMP16], §4.3). Let $q = 3^l$, and let $G(q) = F_4(q)$. There exist exactly $9(q - 1)^4/2$ irreducible characters of $U(q)$ of degree $q^4/3$.

*Construction.* Let $\mathcal{N} := \{\alpha_1, \ldots, \alpha_{24}\}$. Then $\mathcal{N} \leq \Phi^+$. For $a_{11}, a_{12}, a_{13} \in \mathbb{F}_q^\times$, let
\[
X_{1,3,4,7}(t) := x_1(a_{13}t)x_1(13t)x_3(12t)x_4(11t)x_7(-a_{11}a_{12}t^2)
\]
for every $t \in \mathbb{F}_q$. Moreover, let $S$ be the set of squares in $\mathbb{F}_q^\times$, and for every $a_6^* \in \mathbb{F}_q^\times$ such that $a_6^*/a_{11}a_{12}a_{13} \in S$ let $e$ be a square root of $a_6^*/a_{11}a_{12}a_{13}$, and let
\[
X' := \{x_{1,3,4,7}(t) \mid t \in \mathbb{F}_q\}
\]
and
\[
W_1 := \{x_{1,3,4,7}(es) \mid s \in \mathbb{F}_3\}, \quad W_2 := \{x_2(t/(a_{11}a_{12}a_{13}e^3)) \mid t \in \mathbb{F}_3\}.
\]

Let $\lambda := \lambda^{a_{11},a_{12},a_{13},a_6^*}_{e_1,e_2,e_3,e_4}$ be defined as $\lambda(x_i(t)) = \phi(a_it)$ for $i = 11, 12, 13$ and $\lambda(x_6(t)) = \phi(a_6^*t)$. Let $Y := X_{6,9,10}$, and let $V(q) := W_1W_2X_6YZ/\ker(\lambda)$. Then we have that
\[
\mathcal{F} := \{\lambda^{a_{11},a_{12},a_{13},a_6^*}_{e_1,e_2,e_3,e_4} \mid a_{11}, a_{12}, a_{13} \in \mathbb{F}_q^\times, e_{1,3,4,7} \in \mathbb{F}_3 \text{ and } a_6^*/a_{11}a_{12}a_{13} \in S\},
\]
where
\[
\lambda^{a_{11},a_{12},a_{13},a_6^*}_{e_1,e_2,e_3,e_4} := \text{Ind}^{U(q)}_{X'W_2X_6YZX_N} \text{Ind}^{X'W_2X_6YZ}_V(\lambda^{a_{11},a_{12},a_{13},a_6^*} \otimes \mu_{W_1}^{e_1} \otimes \mu_{W_2}^{e_2}),
\]
is a family of $9(q - 1)^4/2$ irreducible characters of $U(q)$ of degree $q^4/3$.

By [GLMP16, Section 4], this family consists of all irreducible characters of $U(q)$ of degree $q^4/3$ for some $n \geq 0$.

\begin{figure}[h]
\centering
\begin{tikzpicture}
  \node (a1) at (0,0) {$\alpha_1$};
  \node (a2) at (1,1) {$\alpha_2$};
  \node (a3) at (2,0) {$\alpha_3$};
  \node (a4) at (3,0) {$\alpha_4$};
  \node (a5) at (4,0) {$\alpha_5$};
  \node (a6) at (5,0) {$\alpha_6$};
  \node (a7) at (6,0) {$\alpha_7$};
  \node (a8) at (7,0) {$\alpha_8$};
  \draw (a1) -- (a3) -- (a5) -- (a7) -- (a8);
  \draw (a2) -- (a3);
  \draw (a4) -- (a5);
  \draw (a6) -- (a7);
\end{tikzpicture}
\caption{The Dynkin diagram of $E_k$ for $k = 6, 7, 8$. Simple roots are labelled as in CHEVIE.}
\end{figure}

We are left with the exceptional groups of types $E_6$, $E_7$ and $E_8$. Let us first consider $p = 2$. We define
\[
\mathcal{N}_k := \{\alpha \in \Phi^+ \mid \alpha_1 \leq \alpha \} \cup \bigcup_{i=6}^{k}\{\alpha \in \Phi^+ \mid \alpha_k \leq \alpha\},
\]
for $k = 6, 7, 8$. If $\Phi^+$ is a root system in type $E_k$, then $\mathcal{N}_k \subseteq \Phi^+$, and $U(E_k(q))/X_{\mathcal{N}_k} \cong U(D_4(q))$. We obtain a family $\mathcal{F}_k$ of $4(q - 1)^4$ characters of degree $q^3/2$ of $U(E_k(q))$ by inflation from $U(E_k(q))/X_{\mathcal{N}_k}$, for $k = 6, 7, 8$.

We summarize in the following proposition the result obtained by the first and the second author in [LM15] for $p = 3$.

**Proposition 5.2 ([LM15], Section 3).** Let $q = 3^f$, and let $G(q) = E_6(q)$. There exist exactly $9(q - 1)^6/2$ irreducible characters of $U(q)$ of degree $q^7/3$.

**Construction.** Let $\mathcal{N} = \{\alpha_{22}, \ldots, \alpha_{36}\}$. Notice that $X_{\mathcal{N}} \subseteq U(q)$ and $Z := Z(U(q)/X_{\mathcal{N}}) \cong X_{\{17, \ldots, 21\}}$. For every $t, r, s \in \mathbb{F}_q$, we let

$$x_{8,9,10}(t) := x_8(-t)x_9(t)x_{10}(t),$$

$$x_{1,2,3,5,6,7,11}(t, r, s) := x_2(t)x_1(t)x_3(-t)x_5(t)x_6(-t)x_7(r)x_{11}(s),$$

and for every square $a_{8,9,10}^* \in \mathbb{F}_q^*$ and a fixed a square root $e$ of $a_{8,9,10}^*$, let

$$X_{8,9,10} := \{x_{8,9,10}(t) \mid t \in \mathbb{F}_q\}$$

and

$$F_2 := \{1, x_{1,2,3,5,6,7,11}(\pm e, 2e^2, 2e^2)\}, \quad F_4 := \{1, x_4(\pm 1/e^3)\}.$$ 

Let $V(q) := ZX_{8,9,10}F_2F_4$. For $e_1 = e_{1,2,3,5,6,7,11}$ and $e_2 = e_4$ in $\mathbb{F}_3$, we denote by $\lambda_{e_1,e_2}^{a_{8,9,10}^*} \in \text{Irr}(V(q))$ the character such that

$$\lambda_{e_1,e_2}^{a_{8,9,10}^*}(x_{8,9,10}(t)) = \phi(a_{8,9,10}^* t), \quad \lambda_{e_1,e_2}^{a_{8,9,10}^*}|_{F_2} = \mu_{e_2}^{e_1}, \quad \lambda_{e_1,e_2}^{a_{8,9,10}^*}|_{F_4} = \mu_{e_4},$$

and $\lambda(x_i(t)) = \phi(t)$ for $i = 17, \ldots, 21$, and we let $H = ZX_{\{17, \ldots, 21\}}X_{\mathcal{N}}$. Then we have that

$$\mathcal{F} := \{\lambda_{e_1,e_2}^{a_{8,9,10}^*} \mid e_1, e_2 \in \mathbb{F}_3 \text{ and } a_{8,9,10}^* \text{ is a square in } \mathbb{F}_q^*\},$$

with

$$\lambda_{e_1,e_2}^{a_{8,9,10}^*} := \text{Ind}_{HX_4F_2}^{U(q)} \text{Inf}_{V(q)}^{HX_4F_2}(\lambda_{e_1,e_2}^{a_{8,9,10}^*})$$

is a family of $9(q - 1)/2$ irreducible characters of degree $q^7/3$.

A split maximal torus of $G(q)$ acts transitively on $\text{Irr}(X_{17})^* \times \cdots \times \text{Irr}(X_{21})^*$; here $\text{Irr}(X_i)^*$ denotes $\text{Irr}(X_i) \setminus \{1_{X_i}\}$. In particular, if $\lambda_{e_1,e_2}^{a_{17,\ldots,21},a_{8,9,10}^*} \in \text{Irr}(V(q))$ is defined such that

$$\lambda_{e_1,e_2}^{a_{17,\ldots,21},a_{8,9,10}^*}|_{X_{8,9,10}F_2F_4} = \lambda_{e_1,e_2}^{a_{8,9,10}^*}|_{X_{8,9,10}F_2F_4},$$

and $\lambda(x_i(t)) = \phi(a_it)$ for $i = 17, \ldots, 21$, then we obtain the family

$$\mathcal{F}' := \{\lambda_{e_1,e_2}^{a_{17,\ldots,21},a_{8,9,10}^*} \mid e_1, e_2 \in \mathbb{F}_3, a_{17}, \ldots, a_{21} \in \mathbb{F}_q^*, a_{8,9,10}^* \text{ is a square in } \mathbb{F}_q^*\},$$

where

$$\lambda_{e_1,e_2}^{a_{17,\ldots,21},a_{8,9,10}^*} := \text{Ind}_{HX_4F_2}^{U(q)} \text{Inf}_{V(q)}^{HX_4F_2}(\lambda_{e_1,e_2}^{a_{17,\ldots,21},a_{8,9,10}^*}),$$

which consists of $9(q - 1)^6/2$ elements of $\text{Irr}(U(q))$ of degree $q^7/3$.

Finally, by [LMP17+] the family $\mathcal{F}'$ consists of all irreducible characters of $U(q)$ of degree $q^n/3$ for some $n \geq 0$. 

\[\square\]
The construction in Proposition 5.2 allows us to produce characters of degree $q^7/3$ also in $U(E_7(3^f))$ and $U(E_8(3^f))$. Let $k \in \{7, 8\}$, and let

$$\mathcal{N}_k := \bigcup_{i=7}^{k} \{\alpha \in \Phi^+ \mid \alpha_k \leq \alpha\}.$$  

Then $\mathcal{N}_k \subseteq \Phi^+$, and we inflate the family of $9(q-1)^6/2$ irreducible characters of $U(E_k(q))/X_{\mathcal{N}_k} \cong U(E_6(q))$ obtained in Proposition 5.2 to $U(E_k(q))$.

We finish with a construction of irreducible characters of degree $q^{16}/5$ in $U(E_8(5^f))$.

**Proposition 5.3** ([LM15], Section 4). Let $q = 5^f$, and let $G(q) = E_8(q)$. Then there exist at least $25(q-1)^8/4$ irreducible characters of $U(q)$ of degree $q^{16}/5$.

**Construction.** The set $\mathcal{N} := \{\alpha_{44}, \ldots, \alpha_{120}\}$ is a normal subset of $\Phi^+$, and $Z := U(q)/X_{\mathcal{N}} \cong X_{\{37, \ldots, 43\}}$. Fix $a^*_{37, \ldots, 43} \in \mathbb{F}_q^\times$ such that $a^*_{37, \ldots, 43} = e^4$ for some $e \in \mathbb{F}_q^\times$. Observe that such an element $a^*_{37, \ldots, 43} \in \mathbb{F}_q^\times$ can take $(q - 1)/4$ distinct values in $\mathbb{F}_q^\times$. For every $u_1, u_2, u_3 \in \mathbb{F}_q$, we let

$$l_1(u_1) := x_1(u_1)x_2(2u_1)x_3(-2u_1)x_4(u_1)x_5(u_1)x_7(2u_1)x_8(-u_1),$$

$$l_2(u_2) := l_1(u_2)x_9(u_2)x_{10}(-u_2^2)x_{11}(u_2)x_{12}(-u_2)x_{15}(2u_2),$$

$$l_3(u_3) := l_2(u_3)x_{16}(4u_3^3)x_{17}(2u_3^3)x_{22}(3u_3^3),$$

and we define $x_{12,13} := \{x_{12}(t)x_{13}(-t) \mid t \in \mathbb{F}_q\}$ and

$$F_4 := \{l_3(eu)x_{23}(3e^4u^4) \mid u \in \mathbb{F}_5\}, \quad F_5 := \{x_5(v/e^5) \mid e \in \mathbb{F}_5\}.$$  

We put $V(q) := ZX_{12,13}F_4F_5$, and for $f_1 = f_1, f_2, f_3, f_4, f_5, f_6$ and $f_7 = f_6$ in $\mathbb{F}_5$, we denote by $\lambda = \lambda^*_{f_1, f_2, f_3}$ the irreducible character of $V(q)$ such that

$$\lambda(x_{12}(t)x_{13}(-t)) = \phi(a^*_{37, \ldots, 43}t), \quad \lambda|_{F_4} = \mu_{f_1}, \quad \lambda|_{F_5} = \mu_{f_6},$$

and $\lambda(x_i(t)) = \phi(t)$ for $i = 37, \ldots, 43$.

Let $H := ZX_{\{44, \ldots, 21\}}X_{\{24, \ldots, 36\}}X_{\mathcal{N}}$. Then we have a family

$$\mathcal{F} := \{\chi_{f_1, f_2, f_3} \mid f_1, f_2 \in \mathbb{F}_5 \text{ and } a^*_{37, \ldots, 43} \text{ is a fourth power in } \mathbb{F}_q^\times\},$$

where

$$\chi_{f_1, f_2, f_3} := \text{Ind}^{U_q}_{H_XF_4} \text{Ind}^{H_XF_4}_{V(q)} \lambda^*_{f_1, f_2, f_3},$$

of $25(q - 1)/4$ irreducible characters of degree $q^{16}/5$.

In a similar way of Proposition 5.2, we observe that a split maximal torus of $G(q)$ acts transitively on $\text{Irr}(X_{37})^{\times} \times \cdots \times \text{Irr}(X_{43})^{\times}$. This gives $(q - 1)^7 \cdot 25(q - 1)/4 = 25(q - 1)^8/4$ irreducible characters of $U(q)$ of degree $q^{16}/5$.  

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