Projective Curvature Tensors of Second Type Almost Geodesic Mappings

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Abstract

We consider equitorsion second type almost geodesic mappings of a non-symmetric affine connection space in this article. Using different computational methods, we obtained some invariants of these mappings. Last generalized Thomas projective parameter and Weyl projective tensor as invariants of a second type almost geodesic mapping of a non-symmetric affine connection space are further generalized here.

1 Introduction

A lot of research papers and monographs are dedicated to developments of the theory of differential geometry [1,2,6,8–21] and its applications [3–5,7]. Einstein (see [3–5]) concluded the symmetric affine connection theory covers researches about a gravitation. The theory electromagnetism is covered by anti-symmetric parts of affine connections. The research about non-symmetric affine connected spaces is started by L. P. Eisenhart [6].

An \(N\)-dimensional manifold \(\mathcal{M}_N\) endowed with a non-symmetric affine connection \(\nabla\) (affine connection coefficients \(L^i_{jk}\) and \(L^i_{kj}\) are different) is said to be the non-symmetric affine connection space \(\mathcal{GA}_N\). Because of the previous mentioned non-symmetry of affine connection coefficients it exists symmetric and anti-symmetric part of these coefficients respectively defined as:

\[
\tilde{S}^i_{jk} = \frac{1}{2}(L^i_{jk} + L^i_{kj}) \quad \text{and} \quad \tilde{T}^i_{jk} = \frac{1}{2}(L^i_{jk} - L^i_{kj}).
\]
A symmetrization and an anti-symmetrization without division by indices \( i \) and \( j \) will be denoted as \((i \ldots j)\) and \([i \ldots j]\) respectively.

The magnitude \( \tilde{T}_{jk}^i \) is a torsion tensor of the space \( \mathcal{G}_A N \). An affine connection space \( A_N \) endowed with an affine connection \( S \) which coefficients coincide with the symmetric part \( \tilde{S}_{jk}^i \) of the affine connection coefficients \( L_{jk}^i \) of the space \( \mathcal{G}_A N \) is said to be the associated space of the space \( \mathcal{G}_A N \).

There are a lot of researchers interested for a development of the non-symmetric affine connection space theory. Some significant results in this subject are obtained into the papers \([10, 11, 14–19, 21]\).

Four kinds of covariant differentiation (see \([10]\)) with regard to an affine connection of a non-symmetric affine connection space \( \mathcal{G}_A N \) are defined as:

\[
\begin{align*}
  a_{j|k}^{i_1} &= a_{j,k}^i + L^i_{\alpha k} a^\alpha_j - L^\alpha_{jk} a^i_j, \\
  a_{j|k}^{i_2} &= a_{j,k}^i + L^i_{k\alpha} a^\alpha_j - L^\alpha_{jk} a^i_j, \\
  a_{j|k}^{i_3} &= a_{j,k}^i + L^i_{\alpha k} a^\alpha_j - L^\alpha_{kj} a^i_j, \\
  a_{j|k}^{i_4} &= a_{j,k}^i + L^i_{k\alpha} a^\alpha_j - L^\alpha_{kj} a^i_j,
\end{align*}
\]

(1.2)

for a partial derivative denoted by comma and an indexed magnitude \( a_j^i \).

All of these covariant derivatives become restricted to a covariant derivative

\[
a_{j;k}^i = a_{j,k}^i + \tilde{S}_{\alpha k}^i a^\alpha_j - \tilde{S}_{jk}^\alpha a^i_j,
\]

(1.4)

of the magnitude \( a_j^i \) with regard to an affine connection of the associated space \( A_N \) of the space \( \mathcal{G}_A N \).

For this reason, it exists only one curvature tensor

\[
R_{jm;n}^i = \tilde{S}_{jm;n}^i - \tilde{S}_{jn;m}^i,
\]

(1.5)

of the associated space \( A_N \).

1.1 Almost geodesic mappings of a space \( \mathcal{G}_A N \)

In an attempt to generalize the term of geodesics N. S. Sinyukov (see \([12]\)) defined an almost geodesic line of a symmetric affine connection space \( A_N \). Consequently, he defined a term of an almost geodesic mapping \( f \) between symmetric affine connection spaces \( A_N \) and \( \overline{A}_N \). Sinyukov noticed three types \( \pi_1, \pi_2, \pi_3 \) of almost geodesic mappings between symmetric affine connection spaces. His research has been directly developed by many authors in a lot of papers \([1, 2, 8, 9, 13, 20]\).
The Sinyukov’s generalization of geodesics is primarily developed for the case of a generalized affine connection space $GA_N$ in [14–16]. In this space it exists four kinds of covariant differentiation but these covariant derivatives are reduced onto first two ones (1.2) for the case of any contra-variant tensor.

For this reason, there are two kinds of almost geodesic lines of the space $GA_N$ [14–19, 21] defined as a curve $\ell = \ell(t)$ which tangential vector $\lambda^i = d\ell^i/dt \neq 0$ satisfies the following equations:

\begin{align}
\lambda^i_\theta(2) &= \bar{a}(t)\lambda^i + \bar{b}(t)\lambda^i_\theta(1), \quad \lambda^i_\theta(1) = \lambda^i_\theta \parallel^\alpha \lambda^\alpha, \quad \lambda^i_\theta(2) = \lambda^i_\theta(1) \parallel^\alpha \lambda^\alpha, \\
\theta &= 1, 2, \text{ for covariant differentiation of the } \theta\text{-th kind with regard to affine connection of the space } GA_N \text{ denoted by } \parallel_\theta.
\end{align}

Because of two kinds of almost geodesic lines of this space a mapping $f : GA_N \to GA_N$ is the almost geodesic mapping of a $\theta$-th kind, $\theta = 1, 2$, if any geodesic line of the space $GA_N$ it turns into an almost geodesic line of the $\theta$-th kind of the space $GA_N$. For this reason, there are three types of almost geodesic mappings of the space $GA_N$ and any of these three types have two kinds. A class of almost geodesic mappings of a $\tau$-th type, $\tau = 1, 2, 3$, and of a $\theta$-th kind $\theta = 1, 2$ of the space $GA_N$ is denoted as $\pi_{\theta \tau}$.

Basic equations of a second type almost geodesic mapping $f : GA_N \to GA_N$ of a $\theta$-th kind, $\theta = 1, 2$, are [15]:

\begin{align}
\bar{L}^i_{jk} &= L^i_{jk} + \psi_j \delta^i_k + \psi_k \delta^i_j + \sigma_j F^i_k + \sigma_k F^i_j + \xi^i_{jk}, \\
F^i_{\theta|k} + F^i_{\theta|j} + F^\alpha_{\theta} F^i_j \sigma_k + F^\alpha_{\theta} F^i_k \sigma_j + (-1)^{\theta} (\xi^i_{j\alpha} F^\alpha_k + \xi^i_{k\alpha} F^\alpha_j) \\
&= \mu_j F^i_k + \mu_k F^i_j + \nu_j \delta^i_k + \nu_k \delta^i_j,
\end{align}

for covariant vectors $\mu_j, \nu_j$, an affinor $F^i_j$ and an anti-symmetric tensor $\xi^i_{jk}$.

A second type almost geodesic mapping $f : GA_N \to GA_N$ of a $\theta$-th kind, $\theta = 1, 2$, satisfies the property of reciprocity (it is an element of the class $\pi_{\theta}(e)$) if it saves the affinor $F^i_j$ and its inverse mapping is a second type almost geodesic mapping of the $\theta$-th kind. An almost geodesic mapping $f$ of the space $GA_N$ satisfies the property of reciprocity (see [15]) if and only if the affinor $F^i_j$ satisfies a relation

\begin{equation}
F^i_{\alpha} F^\alpha_j = \epsilon \delta^i_j, \quad \epsilon = 0, \pm 1.
\end{equation}
2 Invariants of second type almost geodesic mappings

The aim of this paper is to find some new invariants of almost geodesic mappings of a second type which satisfy the property of reciprocity. The results in this subject obtained until now are about the theories of special subclasses of the classes $\pi_\theta^2, \theta = 1, 2$.

Motivated by Sinyukov’s results, it is obtained (see [15]) magnitudes

\[
T_{1}^{i}j_k = \tilde{S}_{j_k}^i - \frac{1}{e - F^2} \left( (F \tilde{S}_{\alpha k}^\alpha - F_k^\alpha \tilde{S}_{\alpha \beta}^\beta) F_j^i + (F \tilde{S}_{j \alpha}^\alpha - F_j^\alpha \tilde{S}_{\alpha \beta}^\beta) F_k^i \right),
\]

\[
\hat{T}_{2}^{i}j_k = T_{j_k}^i + eF_a^i \left( F_{(j|k)}^\alpha - \tilde{T}_{\beta(k)j}^\alpha \right) - \frac{e}{1 + N} F_{\alpha}^\beta \left( (F_{\beta j}^\alpha - \tilde{T}_{\gamma(j)F_{\gamma}^\gamma}^\alpha) \delta_{k}^i + (F_{\beta j}^\alpha - \tilde{T}_{\gamma(k)F_{\gamma}^\gamma}^\alpha) \delta_{j}^i \right),
\]

for $F = F_a^\alpha, e - F^2 \neq 0$ and Thomas projective parameter $T_{1}^{i}j_k$ of the associated space $A_N$ in the expression of the invariant $T_{1}^{i}j_k$, are invariants of a canonical second type almost geodesic mapping of the first kind of the space $GA_N$.

Moreover, Weyl projective tensor of the space $GA_N$ which affine connection coefficients are $\hat{L}_{j_k}^i = L_{j_k}^i + eF_a^i F_{(j|k)}^\alpha - eT_{\beta(j)F_{k}^\gamma}^\alpha F_{\alpha}^i$ is an invariant of the canonical second type almost geodesic mapping $f$ of the first kind. The aim of our following research is to find some other more general invariants of special second type almost geodesic mappings of the space $GA_N$.

Let a mapping $f : GA_N \to GA_N$ be an equitorsion second type almost geodesic mapping of a $\theta$-th kind, $\theta = 1, 2$, which satisfies the property of reciprocity. The composition (1.9) involved into the basic equation (1.8) together with using of the fact the mapping $f$ is an equitorsion one ($\xi_{j_k}^i = 0$) involved into the both of basic equations (1.7, 1.8) proves it is satisfied relations

\[
\hat{T}_{j_k}^i = L_{j_k}^i + \psi_j^i \delta_{k}^i + \psi_k^i \delta_{j}^i + \sigma_j F_{k}^i + \sigma_k F_{j}^i,
\]

\[
F_{j_k}^i + F_{k_j}^i = \mu_j F_{k}^i + \mu_k F_{j}^i + (\nu_j - e\sigma_j) \delta_{k}^i + (\nu_k - e\sigma_k) \delta_{j}^i.
\]

It is proved a following proposition is satisfied in this way.
Proposition 2.1 Let $f : \mathbb{G}A_N \rightarrow \overline{\mathbb{G}A_N}$ be an equitorsion second type almost geodesic mapping of a $\theta$-th kind, $\theta = 1, 2$, which satisfies the property of reciprocity. The equations (2.12, 2.13, 1.9) are basic equations of this mapping.

Based on the fact the second type almost geodesic mapping $f$ satisfies the property of reciprocity the corresponding magnitudes $\overline{\psi}_i, \sigma_i, \overline{F}_j^i$ which determine an inverse mapping $f^{-1}$ of the mapping $f$ are

$$\overline{\psi}_i = -\psi_i, \quad \sigma_i = -\sigma_i, \quad \overline{F}_j^i = F_j^i.$$ 

After contracting the basic equation (2.12) by indices $i$ and $k$ and using the fact it is satisfied a relation $\sigma_j = \frac{1}{2}(\sigma_j - \overline{\sigma}_j)$ we obtain it is satisfied an equation

$$\psi_j = \frac{1}{N + 1}(L^\alpha_{j\alpha} - L^\alpha_{j\alpha}) + \frac{1}{2(N + 1)} \left[ (\sigma_j \overline{F} + \sigma_\alpha \overline{F}_j^\alpha) - (\sigma_j F + \sigma_\alpha F_j^\alpha) \right], \quad (2.14)$$

for $F = F_\alpha^\alpha$ as above.

Using the previous expression of the magnitude $\psi_j$ we conclude the basic equation (2.12) has a form

$$\overline{L}^i_{jk} = L^i_{jk} + \omega^i_{jk} - \omega^i_{jk}, \quad (2.15)$$

for

$$\omega^i_{jk} = -\frac{1}{2}(\sigma_j F^i_k + \sigma_k F_j^i) + \frac{1}{N + 1} \left[ (L^\alpha_{j\alpha} \delta^i_k + L^\alpha_{k\alpha} \delta^i_j) \right] + \frac{1}{2(N + 1)} \left[ (\sigma_j F + \sigma_\alpha F_j^\alpha) \delta^i_k + (\sigma_k F + \sigma_\alpha F_k^\alpha) \delta^i_j \right], \quad (2.16)$$

and the magnitude $\overline{\psi}_j$ defined in the same manner as a function of the corresponding elements of the space $\mathbb{G}A_N$.

The equation (2.15) proves it is satisfied an equality

$$\overline{T}^i_{2jk} = \overline{T}^i_{2jk},$$

for

$$\overline{T}^i_{2jk} = L^i_{jk} - \omega^i_{jk} \quad \text{and} \quad \overline{T}^i_{2jk} = \overline{L}^i_{jk} - \overline{\omega}^i_{jk}. \quad (2.17)$$

It is proved a following lemma in this way.
Lemma 2.1 Let \( f : \mathbb{G}A_N \rightarrow \mathbb{GA}_N \) be an equitorsion almost geodesic mapping of a second type which satisfies the property of reciprocity. A magnitude \( T_{2jk}^i \) defined in the first of the expressions (2.17) is an invariant of the mapping \( f \).

The invariant \( T_{2jk}^i \) of a second type almost geodesic mapping \( f : \mathbb{G}A_N \rightarrow \mathbb{GA}_N \) which satisfies the property of reciprocity is said to be the \( \pi_2 \)-generalized Thomas projective parameter.

Let us generalize Weyl projective tensor of an equitorsion almost geodesic mapping \( f : \mathbb{G}A_N \rightarrow \mathbb{GA}_N \) of the first type which satisfies the property of reciprocity. This generalization will be realized just for first type almost geodesic mappings. A result for the case of second type almost geodesic mappings may be obtained in the same manner.

First of all, we can observe the magnitude \( \omega_{jk}^i \) defined into the equation (2.16) is symmetric by indices \( j \) and \( k \). For this reason, symmetric parts \( \tilde{S}_{jk}^i \) and \( \tilde{S}_{ij}^k \) of affine connection coefficients \( L_{jk}^i \) and \( \bar{L}_{jk}^i \) of the spaces \( \mathbb{G}A_N \) and \( \mathbb{GA}_N \) satisfy a relation

\[
\tilde{S}_{jk}^i = \tilde{S}_{ij}^k + \omega_{jk}^i - \omega_{kj}^i,
\]

for the above defined magnitudes \( \omega_{jk}^i \) and \( \omega_{kj}^i \).

Using the covariant derivative of the first kind (1.2) we obtain a curvature tensor \( R_{jmn}^i \) of the associated space \( \mathbb{A}_N \) has a form:

\[
R_{jmn}^i = \tilde{S}_{jm|n}^i - \tilde{S}_{jn|m}^i - \tilde{T}_{\alpha n}^i \tilde{S}_{jm}^\alpha - \tilde{T}_{jm}^\alpha \tilde{S}_{\alpha n}^i + \tilde{T}_{jn}^\alpha \tilde{S}_{\alpha m}^i + \tilde{T}_{am}^\alpha \tilde{S}_{\alpha j}^n + 2 \tilde{T}_{mn}^\alpha \tilde{S}_{j\alpha}^i.
\]

Motivated by this result we are going to find a rule of change of the curvature tensor \( R_{jmn}^i \) bellow. Let us involve following substitutions:

\[
U_{2jk}^i = \frac{1}{2} (T_{2jk}^i + T_{2kj}^i) \quad \text{and} \quad \bar{U}_{2jk}^i = \frac{1}{2} (\bar{T}_{2jk}^i + \bar{T}_{2kj}^i),
\]

for the above obtained invariant \( T_{2jk}^i = \bar{T}_{2jk}^i \) of the mapping \( f \).

From the equations (2.17) and (2.20) together with the above mentioned symmetry of the magnitude \( \omega_{jk}^i \) from the equation (2.16) by indices \( j \) and \( k \) we conclude it is satisfied equalities.
\( U^i_{jk} = \tilde{S}^i_{jk} - \omega^i_{jk} \) and \( \overline{U}^i_{jk} = \tilde{S}^i_{jk} - \overline{\omega}^i_{jk} \). \hfill (2.21)

It is easy to be obtained covariant derivatives

\[
U^i_{jm|n} = \frac{1}{2} (T^i_{jm|1} + T^i_{mj|1}) \quad \text{and} \quad \overline{U}^i_{jm||n} = \frac{1}{2} (\overline{T}^i_{jm||1} + \overline{T}^i_{mj||1}),
\]

of the previous defined magnitudes \( U^i_{jm} \) and \( \overline{U}^i_{jm} \) satisfy a relation

\[
\overline{U}^i_{jm||n} = U^i_{jm|n} + \tilde{S}^i_{an} \overline{U}^a_{jm} - \tilde{S}^a_{jn} \overline{U}^i_{2am} - \tilde{S}^a_{mn} \overline{U}^i_{2j\alpha} - \tilde{S}^i_{an} U^a_{jm} + \tilde{\omega}^i_{jn} U^i_{2am} + \tilde{\omega}^a_{mn} U^i_{2j\alpha}.
\hfill (2.23)

The equations (2.21, 2.23) prove it is satisfied a following proposition.

**Proposition 2.2** Let \( f : \mathcal{A}_N \to \mathcal{A}_N \) be an equitorsion second type almost geodesic mapping of the first kind between non-symmetric affine connection spaces \( \mathcal{A}_N \) and \( \overline{\mathcal{A}}_N \). Covariant derivatives \( \tilde{S}^i_{jm|n} \) and \( \overline{T}^i_{jm||n} \) of symmetric parts \( \tilde{S}^i_{jm} \) and \( \overline{T}^i_{jm} \) of the corresponding affine connection coefficients \( L^i_{jm} \) and \( \overline{L}^i_{jm} \) satisfy a relation

\[
\tilde{S}^i_{jm|n} = \tilde{S}^i_{jm|1} + \omega^i_{jm|n} - \omega^i_{jm|1} + \tilde{S}^i_{an} \overline{U}^a_{jm} - \tilde{S}^a_{jn} \overline{U}^i_{2am} - \tilde{S}^a_{mn} \overline{U}^i_{2j\alpha} - \tilde{S}^i_{an} U^a_{jm} + \tilde{\omega}^i_{jn} U^i_{2am} + \tilde{\omega}^a_{mn} U^i_{2j\alpha},
\hfill (2.24)

for the magnitudes \( U^i_{jk} \) and \( \overline{U}^i_{jk} \) defined above. \( \square \)

Using the invariance \( \tilde{T}^i_{jk} = \tilde{\tilde{T}}^i_{jk} \) and the consequent invariances \( \tilde{T}^\alpha_{jm} \tilde{\tilde{T}}^i_{\alpha n} = \tilde{T}^\alpha_{jm} \tilde{T}^i_{\alpha n} \) such as \( \tilde{T}^i_{jm} \tilde{\tilde{T}}^\alpha_{mn} = \tilde{T}^i_{jm} \tilde{T}^\alpha_{mn} \) we obtain it is satisfied a following proposition.

**Proposition 2.3** Let \( f : \mathcal{A}_N \to \mathcal{A}_N \) be an equitorsion second type almost geodesic mapping of the first kind between non-symmetric affine connection spaces \( \mathcal{A}_N \) and \( \overline{\mathcal{A}}_N \). Magnitudes \( \tilde{S}^\alpha_{jm} \tilde{T}^i_{\alpha n}, \tilde{S}^\alpha_{aj} \tilde{T}^\alpha_{mn} \) and its deformations \( \tilde{\tilde{S}}^\alpha_{jm} \tilde{\tilde{T}}^i_{\alpha n}, \tilde{\tilde{S}}^\alpha_{aj} \tilde{\tilde{T}}^\alpha_{mn} \) under the mapping \( f \) satisfy equations...
\[ \tilde{S}_jm \tilde{T}_\alpha = \tilde{S}_jm \tilde{T}_\alpha + \tilde{\omega}_jm \tilde{T}_\alpha - \omega_{jm} \tilde{T}_\alpha, \]  
\[ \tilde{S}_ja \tilde{T}_{mn} = \tilde{S}_ja \tilde{T}_{mn} + \tilde{\omega}_ja \tilde{T}_{mn} - \omega_{ja} \tilde{T}_{mn}, \]

for the magnitude \( \omega_{jk} \) defined into the equation (2.16) and the corresponding one \( \tilde{\omega}_{jk} \).

If we contract the basic equation (2.13) by the indices \( i \) and \( k \) we conclude it is satisfied a relation

\[ F_{\theta j} = \mu_j F + \mu_\alpha F_\alpha^j + (N + 1)(\nu_j - e\sigma_j) - F_{\alpha j}^\alpha, \]

\( \theta = 1, 2 \). This equation proves it is satisfied a following proposition.

**Proposition 2.4** Let \( f : \mathbb{G}_A^N \rightarrow \mathbb{G}_A^\alpha_N \) be an equitorsion second type almost geodesic mapping of the first kind between non-symmetric affine connection spaces \( \mathbb{G}_A^N \) and \( \mathbb{G}_A^\alpha_N \) which satisfies the property of reciprocity. A covariant derivative \( \omega_{jm|n}^i \) of the magnitude \( \omega_{jk} \) defined into the equation (2.16) satisfies an equality

\[ \omega_{jm|n}^i = \frac{1}{N + 1} (L_{\alpha |n}^\alpha \delta_m^i + L_{\alpha m|n}^\alpha \delta_j^i) + \frac{1}{2}(\nu_n - e\sigma_n)(\sigma_j \delta_m^i + \sigma_m \delta_j^i) \\
- \frac{1}{2}(\sigma_j \delta_m^i F_m^i + \sigma_m \delta_j^i F_j^i + \sigma_j F_m^i \delta_j^i + \sigma_m F_j^i \delta_j^i) \\
+ \frac{1}{2(N + 1)} (\sigma_j \delta_m^i F_\alpha^\alpha_j + \sigma_\alpha \delta_m^i F_j^\alpha \sigma_\alpha + \sigma_\alpha \delta_m^i F_j^\alpha) \delta_j^i \\
+ \frac{1}{2(N + 1)} (\sigma_m \delta_j^i F_\alpha^\alpha_m + \sigma_\alpha \delta_m^i F_m^\alpha \sigma_\alpha + \sigma_\alpha \delta_m^i F_m^\alpha) \delta_j^i \\
+ \frac{1}{2(N + 1)} (\mu_n F + \mu_\alpha F_\alpha^\alpha_n - F_{\alpha n}^\alpha (\sigma_j \delta_m^i + \sigma_m \delta_j^i)), \]

for magnitudes \( \mu_i, \nu_i \) used into the basic equation (2.13).

A difference \( \Delta_{jmn}^i = \tilde{\omega}_{jm||n}^i - \omega_{jm|n}^i \), of the magnitudes \( \tilde{\omega}_{jm||n}^i \) and \( \omega_{jm|n}^i \) satisfies a relation

\[ \Delta_{jmn}^i = \frac{1}{N + 1} \left( (\tilde{S}_ja||n - \tilde{S}_ja|n) \delta_m^i + (\tilde{S}_\alpha ma|n - \tilde{S}_\alpha ma|n) \delta_j^i \right) \\
+ \tilde{\rho}_{jmn}^i - \tilde{\rho}_{jmn}^i, \]
such that the equation (2.24) has a form

\[
2\hat{\rho}_{jmn} = -2L^\beta_{jn} T_{i\beta\alpha} \delta^i_{m} - 2L^\beta_{mn} T_{i\beta\alpha} \delta^i_{j} + (\nu_n - e\sigma_n)(\sigma_j \delta^i_m + \sigma_m \delta^i_j) \\
- (\sigma_i F^i_m + \sigma_m F^i_j + \sigma_j F^i_m + \sigma_m F^i_j) \\
+ \frac{1}{N+1} (\sigma_j F^\alpha_i + \sigma_m F^\alpha_j) \delta^i_m \\
+ \frac{1}{N+1} (\sigma_m F^\alpha_m + \sigma_F^\alpha_j) \delta^i_j \\
+ \frac{1}{N+1} (\mu_n F_m + \mu_m F_j) (\sigma_j \delta^i_m + \sigma_m \delta^i_j),
\]

(2.30)

From the equations (2.29) and (2.30) we conclude it exists a magnitude \(\hat{\rho}_{jmn}\) from the space \(\mathbb{G}_N\) analogue to the magnitude \(\hat{\rho}_{jmn}\), the magnitudes \(\mu_i, \nu_i\) from the equation (2.13) and the corresponding ones \(\overline{\mu}_i, \overline{\nu}_i\).

such that the equation (2.24) has a form

\[
\tilde{S}_{jm\parallel n} = \tilde{S}_{jm\parallel n} - \delta^i_m \hat{\nu}_{jn} - \delta^i_j \hat{\nu}_{mn} \\
- \frac{1}{2} (\sigma_j F^i_m + \sigma_m F^i_j + \sigma_j F^i_m + \sigma_m F^i_j) \\
+ \frac{1}{2} (\sigma_j F^i_m + \sigma_m F^i_j + \sigma_j F^i_m + \sigma_m F^i_j) \\
+ \tilde{S}_{an} F^\alpha_{2jm} - \tilde{S}_{jn} U^\alpha_{2am} - \tilde{S}_{mn} F^\alpha_{2ja} - \tilde{S}_{an} U^\alpha_{2jm} + \tilde{S}_{jn} U^\alpha_{2am} + \tilde{S}_{mn} U^\alpha_{2ja},
\]

(2.32)
Using the equations \(2.20, 2.25, 2.26, 2.32\) and the expression \(2.19\) of the curvature tensors \(R^i_{jmn}\) and \(\overline{R}^i_{jmn}\) of the associated spaces \(\mathcal{A}_N\) and \(\overline{\mathcal{A}}_N\) we obtain it is satisfied the following equation

\[
\overline{R}^i_{jmn} = R^i_{jmn} - \delta^i_m \hat{\nu}^j_{1j} + \delta^i_n \hat{\nu}^j_{1jm} - \delta^i_j \hat{\nu}^m_{1mn} \\
- \frac{1}{2} (\sigma^i_m F^i_m + \sigma^m_i F^i_j + \sigma^i_j F^i_m + \sigma^i_j F^i_j)
\]

\[
+ \frac{1}{2} (\sigma^i_m F^i_m + \sigma^m_i F^i_j + \sigma^i_j F^i_m + \sigma^i_j F^i_j)
\]

\[
+ \frac{1}{2} (\sigma^i_j F^i_m + \sigma^m_i F^i_j + \sigma^i_j F^i_m + \sigma^i_j F^i_j)
\]

\[
- \frac{1}{2} (\sigma^i_m F^i_n + \sigma^m_i F^i_j + \sigma^i_j F^i_m + \sigma^i_j F^i_j)
\]

\[
+ 2 \tilde{S}^i_{\alpha j} - 2 \tilde{S}^i_{\alpha j} - 2 \tilde{S}^i_{\alpha m} - 2 \tilde{S}^i_{\alpha m} - 2 \tilde{S}^i_{\alpha m} - 2 \tilde{S}^i_{\alpha m}
\]

\[
- \tilde{F}^a_{m j} \alpha i + \omega^a_{m j} L^i_{\alpha m} - \tilde{F}^a_{m j} \alpha i + \omega^a_{m j} L^i_{\alpha m} + 2 \omega^a_{m j} L^i_{\alpha m} - 2 \omega^a_{m j} L^i_{\alpha m}
\]

\[
= R^i_{jmn} - \delta^i_m \hat{\nu}^j_{1j} + \delta^i_n \hat{\nu}^j_{1jm} - \delta^i_j \hat{\nu}^m_{1mn} + F^i_{1jmn} - \overline{F}^i_{1jmn},
\]

for

\[
F^i_{1jmn} = \frac{1}{2} (\sigma^i_m F^i_m + \sigma^m_i F^i_j + \sigma^i_j F^i_m + \sigma^i_j F^i_j)
\]

\[
- \frac{1}{2} (\sigma^i_m F^i_n + \sigma^m_i F^i_j + \sigma^i_j F^i_m + \sigma^i_j F^i_j)
\]

the magnitude \(\omega^i_{jk}\) defined into the equation \(2.16\) and the corresponding magnitude \(\overline{F}^i_{1jmn}\).

After contracting the equation \(2.33\) by indices \(i\) and \(n\) we conclude Ricci tensors \(R_{jm}\) and \(\overline{R}_{jm}\) of the associated spaces \(\mathcal{A}_N\) and \(\overline{\mathcal{A}}_N\) satisfy a relation

\[
\overline{R}_{jm} = R_{jm} + (N - 1) \hat{\nu}^i_{1jm} + \hat{\nu}^i_{1jm} + F^i_{1j} - \overline{F}^i_{1j},
\]

for \(F^i_{1j} = F^i_{1j\alpha}\) and \(\overline{F}^i_{1j} = \overline{F}^i_{1j\alpha}\).
After alternating the equation (2.35) by indices \( j \) and \( m \) we conclude it is satisfied a relation

\[
(N + 1)\dot{v}_{[jm]} = \overline{R}_{[jm]} - R_{[jm]} - \overline{F}_{[jm]} + \overline{F}_{[jm]},
\]

(2.36)

From the equations (2.35) and (2.36) we conclude it is satisfied an expression

\[
(N^2 - 1)\dot{v}_{jm} = (NR_{jm} + R_{mj}) - (NR_{jm} + R_{mj})
+ (N\overline{F}_{jm} + \overline{F}_{mj}) - (N\overline{F}_{jm} + \overline{F}_{mj}).
\]

(2.37)

After involving the results (2.36, 2.37) in the equation (2.33) we obtain it is satisfied an equality

\[
\overline{W}^i_{2 jmn} = W^i_{2 jmn},
\]

where we denoted by

\[
\begin{align*}
W^i_{2 jmn} &= R^i_{jmn} + \frac{1}{N + 1} \delta^i_j R_{[mn]} + \frac{N}{N^2 - 1} \delta^i_m R_{jn} + \frac{1}{N^2 - 1} \delta^i_n R_{jm} \\
&+ \frac{1}{N + 1} \delta^i_j F_{[mn]} + \frac{N}{N^2 - 1} \delta^i_m F_{jn} + \frac{1}{N^2 - 1} \delta^i_n F_{jm},
\end{align*}
\]

(2.38)

a geometric object of the space \( \mathcal{G} \mathcal{A}_N \), where the magnitude \( F^i_{1 jmn} \) is defined into the equation (2.34) and for the corresponding magnitude \( F^i_{1 ij} = F^i_{1 j} \). The corresponding magnitude \( \overline{W}^i_{2 jmn} \) of the space \( \mathcal{G} \mathcal{A}_N \) is defined in the same manner.

It is proved a following theorem is satisfied in this way.

**Theorem 2.1** Let \( f : \mathcal{G} \mathcal{A}_N \rightarrow \mathcal{G} \mathcal{A}_N \) be an equitorsion second type almost geodesic mapping of the first kind which satisfies the property of reciprocity. The magnitude \( W^i_{2 jmn} \) defined into the equation (2.38) is an invariant of this mapping. \( \square \)

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