QUASI-EINSTEIN SHEARFREE SPACETIMES LIFTED FROM SASAKIAN MANIFOLDS

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ABSTRACT. In this article we prove that a certain class of smooth Sasakian manifolds admits lifts to 4-dimensional quasi-Einstein shearfree spacetimes of Petrov type II or D. This is related to an analogous result by Hill, Lewandowski and Nurowski [12] for general real-analytic CR manifolds. In particular, our result holds for smooth tubular CR manifolds. Furthermore, we show that any Sasakian manifold with underlying Kähler-Einstein manifold with non-zero Einstein constant has a lift to a shearfree Einstein metric of Petrov type II or D.

1. Introduction

A foliation by integral curves of a non-vanishing null vector field $k$ on a Lorentzian manifold $(M, g)$ is called a shearfree congruence if

$$\mathcal{L}_kg(v, w) = \rho g(v, w),$$

for some function $\rho$ and for all vector fields $v, w \in k^\perp$. These integral curves are automatically geodesics and in the case of a 4-dimensional Lorentzian manifold they can be interpreted as light rays. The 4-dimensional case directly applies to General Relativity and has been intensively studied by many authors, see e.g. [4, 7–9, 15, 17, 18, 19, 21–27].

The famous Goldberg-Sachs Theorem [10] (See also [11] for a generalisation of the Goldberg-Sachs theorem.) relates the existence of shearfree congruences to algebraically special solutions of the Einstein field equations for which the Weyl tensor satisfies certain degeneracy conditions, expressed through its so-called Petrov type. Of particular interest are the shearfree congruences with underlying Lorentzian metrics $g$ that satisfy the Einstein condition

$$\text{Ric}(g) = \Lambda g,$$

or the relaxed quasi-Einstein condition

$$\text{Ric}(g) = \Lambda g + \Phi \chi^2,$$  \hspace{1cm} (1.1)

where $\text{Ric}(g)$ is the Ricci curvature of $g$, $\Lambda$ is the so-called cosmological constant, $\chi = g(k, \cdot)$ and $\Phi$ is a smooth function corresponding to the energy momentum tensor of pure radiation.

Robinson and Trautman [18] described the relation between shearfree congruences in 4-dimensional spacetimes and the CR structure of their 3-dimensional leaf spaces. A CR structure on a 3-dimensional manifold is a distribution $D$ of rank 2 equipped with a field of endomorphisms $J: D \to D$ with $J^2 = -\text{Id}$. The distribution $D$ on the leaf space $M$ is induced by the distribution $k^\perp$ on $M$, and $J$ is induced by the conformal structure on $k^\perp \setminus k$. The CR structure of $M$ can be encoded as a pair $(\mu, \lambda)$ comprised of a complex 1-form $\mu$ and a real 1-form $\lambda$, such that $D = \ker \lambda$ and $\mu$ is a complex coordinate on $D$,
that is $\mu \wedge \bar{\mu} \wedge \lambda \neq 0$. By $[(\mu, \lambda)]$ we denote the equivalence class of pairs $(\mu, \lambda)$ that produce the same CR structure on $M$.

Locally, any shearfree Lorentzian metric associated with a given CR manifold $(M, [(\mu, \lambda)])$ can be written as

$$g = 2P^2 (\mu \bar{\mu} + \lambda (dr + W\mu + \bar{W}\bar{\mu} + 3\lambda)),$$

where $P \neq 0, H, r$ are real functions, and $W$ is a complex function on $M$ (see e.g. [12,18]). Any metric from this family is called a lift of the underlying CR structure.

As one of the main results in the paper [12], Hill, Lewandowski and Nurowski proved that for any real-analytic CR manifold, there exists a lift that renders the Lorentzian manifold $(M, g)$ quasi-Einstein, that is, (1.1) is satisfied. Moreover, $g$ is of Petrov type II or D.

In this paper, we focus on the case when the underlying CR manifold has a Sasakian structure, i.e., it is strictly pseudoconvex and it admits an infinitesimal CR automorphism transversal to the distinguished distribution $D$ (see Section 2 for precise definitions). The main result of this article is Theorem 1.1, which is an analog to the theorem by Hill, Lewandowski and Nurowski, cited above. We remove the assumption of real analyticity and prove that a particular class of smooth 3-dimensional Sasakian manifolds admit lifts to quasi-Einstein Lorentzian manifolds with positive cosmological constant $\Lambda$ and of Petrov type II or D. Petrov type II or D means that the Weyl tensor is to some degree degenerate, that is certain components vanish and another certain component is different from 0. In what follows, we use $(z = x + iy, w = u + iv)$ as the local coordinates of $\mathbb{C}^2$.

**Theorem 1.1.** Let $(M, D, J)$ be a 3-dimensional Sasakian manifold given by a defining equation $v = F(z, \bar{z})$, where $F: \mathbb{C} \to \mathbb{R}$ is such that the limits

$$\lim_{|z| \to \infty} F_{zz} > 0 \quad \text{and} \quad \lim_{|z| \to \infty} \partial_{z\bar{z}} \log(F_{zz}) < 0$$

exist, and let $\Lambda$ be an arbitrary positive constant. Then, there exists a 4-dimensional Lorentzian manifold $(M = M \times \mathbb{R}, g)$, which is a lift of $M$, such that $g$ is of Petrov type II or D, and satisfies the quasi-Einstein equation

$$\text{Ric}(g) = \Lambda g + \Phi \lambda^2,$$

where $\Phi$ is some real function.

We illustrate this result in the special case of tubular hypersurfaces. In this case the cosmological constant can be arbitrary.

**Theorem 1.2.** Let $(M, D, J)$ be a 3-dimensional tubular CR manifold with defining function $v = F(y)$, and let $\Lambda$ be an arbitrary constant. Then, there exists a 4-dimensional Lorentzian manifold $(M = M \times \mathbb{R}, g)$, which is a lift of $M$, such that $g$ is of Petrov type II or D, and satisfies the quasi-Einstein equation

$$\text{Ric}(g) = \Lambda g + \Phi \lambda^2,$$

where $\Phi$ is some real function.

In Theorem 1.3 we consider Sasakian manifolds for which the associated Kähler manifold of integral curves of the Reeb vector field is Einstein with non-zero Einstein
constant. We show that such Sasakian manifolds admit a lift to an Einstein Lorentzian manifold.

**Theorem 1.3.** Let \((M, D, J)\) be a 3-dimensional Sasakian manifold with defining function \(v = F(z, \bar{z})\), and assume the underlying Kähler manifold is Einstein with non-zero Einstein constant \(\Lambda_0\) and let the cosmological constant, \(\Lambda\), be any real value such that \(\Lambda \Lambda_0 > 0\). Then, there exists a 4-dimensional Lorentzian manifold \((M = M \times \mathbb{R}, g)\), which is a lift of \(M\), such that \(g\) is of Petrov type II or D, and satisfies the Einstein equation

\[ \text{Ric}(g) = \Lambda g. \]

### 2. CR manifolds and Sasakian manifolds

In this section, we recall some facts regarding CR manifolds and Sasakian manifolds, and fix our notational conventions.

**Definition 2.1.** A 3-dimensional CR manifold is a triple \((M, D, J)\) where \(M\) is a 3-dimensional manifold, \(D\) a codimension 1 distribution and \(J\) a smooth field of endomorphisms with \(J^2 = -\text{Id}\) on \(D\).

We assume that \((M, D, J)\) is strictly pseudoconvex, i.e. for any (local) non-vanishing section \(X\) of \(D\), \([X, JX] \not\in D\) at any point. For any choice of \(X\) we have an adapted complex frame \((\partial, \bar{\partial}, \partial_o)\), where

\[ \partial = X - iJX, \quad \partial_o = i[\partial, \bar{\partial}] = -2[X, JX]. \]

The complex vector field \(\partial = X - iJX\) spans the +i-eigen-distribution \(D^{1,0}\) of \(J\) in \(D \otimes \mathbb{C}\). We denote the corresponding dual coframe by \((\mu, \bar{\mu}, \lambda)\). Since \(\bar{\partial}\) and \(\bar{\mu}\) are just the conjugates of \(\partial\) and \(\mu\), we will also refer to \((\partial, \partial_o)\) and \((\mu, \lambda)\) as a frame and coframe, respectively. The real 1-form \(\lambda\) and the complex 1-form \(\mu\) completely determine the CR structure on \(M\), with \(D = \ker \lambda\) and \(\mu|_D: D \to \mathbb{C}\) as a complex coordinate on \(D\). Strict pseudoconvexity of \(M\) translates to

\[ d\lambda \wedge \lambda \neq 0. \]

Our choice implies the structure equations

\[ d\lambda = i\mu \wedge \bar{\mu} + c\mu \wedge \lambda + \bar{\mu} \wedge \lambda, \]
\[ d\mu = \alpha \mu \wedge \lambda + \beta \bar{\mu} \wedge \lambda. \quad (2.1) \]

where \(c, \alpha, \beta\) are complex valued functions on \(M\).

Notice that a strictly pseudoconvex CR manifold carries the structure of a contact manifold with contact form \(\lambda\). For any choice of a contact form \(\lambda\), one defines the uniquely determined *Reeb vector field* \(Z\) that satisfies \(Z \cdot \lambda = 1\) and \(Z \cdot d\lambda = 0\).

Any other adapted frame \((\partial', \bar{\partial}', \partial'_o)\) and coframe \((\mu', \bar{\mu}', \lambda')\) express through the original frame and coframe by

\[ \partial' = \frac{1}{f} \partial, \quad \partial'_o = \frac{1}{|f|^2} (\partial_o - \ell \partial - \bar{\ell} \bar{\partial}), \]
\[ \mu' = f(\mu + \ell \lambda), \quad \lambda' = |f|^2 \lambda. \quad (2.2) \]

(continues)
where \( f \neq 0 \) and \( \ell \) are complex valued functions, and

\[
\ell = -i \bar{\partial} (\log f) , \quad \alpha' = \frac{1}{|f|^2} (\alpha - \partial_o (\log f) + \ell \partial (\log f) + \partial \ell + \ell c) , \\
c' = \frac{1}{f} (c - 2i \bar{\ell} + \partial (\log f)) , \quad \beta' = \frac{1}{f^2} (\beta + i \ell^2 + \bar{\partial} \ell + \bar{c} \ell) . \tag{2.4}
\]

**Definition 2.2.** A CR function on a CR manifold \((M, D, J)\) is a complex-valued \(C^1\)-function \(\zeta\), such that

\[ \bar{\partial} (\zeta) = 0. \]

This definition does not depend on the choice of the adapted frame.

A (local) realisation or embedding into \(\mathbb{C}^2\) of a CR manifold \((M, D, J)\) is a (local) \(C^1\)-embedding \(M \to \mathbb{C}^2\) such that both components of the mapping are CR functions. Such embeddability is equivalent to the existence of two functionally independent CR functions \(z\) and \(w\), i.e.

\[ dz \wedge dw \neq 0. \]

If the CR manifold \((M, D, J)\) is (locally) embedded as a real hypersurface

\[ v = F(z, \bar{z}, u) \]

in \(\mathbb{C}^2\) with local coordinates \((z, w = u + iv)\), we can choose an adapted frame and coframe as in the Lemma below.

**Lemma 2.3.** Let \((M, D, J)\) be a CR manifold locally embeddable as a real hypersurface in \(\mathbb{C}^2 = \{ z = x + iy, w = u + iv \} \) as \(v = F(z, \bar{z}, u)\) for some smooth real function \(F\). Set

\[ L = \frac{F_z}{F_u + i} . \]

Then there exists a CR coframe \((\mu, \bar{\mu}, \lambda)\) and dual frame \((\partial, \bar{\partial}, \partial_o)\) where

\[
\mu = dz , \quad \lambda = \frac{du + L dz + \bar{L} d\bar{z}}{i (\partial L - \partial \bar{L})} , \\
\partial = \partial_z - L \partial_u , \quad \partial_o = i (\bar{\partial} L - \partial L \bar{L}) \partial_u .
\]

The coframe satisfies the structure equations:

\[ d\mu = 0 , \quad d\lambda = i \mu \wedge \bar{\mu} + c \mu \wedge \lambda + \bar{c} \bar{\mu} \wedge \lambda , \]

where

\[ c = -\partial \log (i (\bar{\partial} L - \partial L)) - L_u . \]

In particular,

\[ \partial \bar{c} = \bar{\partial} c . \]

**Proof.** Let \( \mu = dz \), and choose a contact form

\[ \lambda = J(dv - dF) = (1 + F_u^2) du + (-iF_z + F_u F_z) dz + (iF_{\bar{z}} + F_u F_{\bar{z}}) d\bar{z} = (1 + F_u^2) (du + L dz + \bar{L} d\bar{z}) \]

with \(L\) defined as above. We need to find \(\psi\) such that

\[ \lambda = \frac{\psi}{(1 + F_u^2)} \lambda' . \]
satisfies (2.1), i.e.
\[d\lambda = \psi \left( L \, dz + \bar{L} \, d\bar{z} \right) + d\psi \wedge \frac{1}{\psi} \lambda \equiv i \, dz \wedge d\bar{z} \mod \lambda.\]

Notice that the dual adapted frame to \((\mu, \bar{\mu}, \lambda)\) becomes
\[\partial = \partial_z - L \partial_u, \quad \bar{\partial} = \partial_{\bar{z}} - \bar{L} \partial_u, \quad \partial_\theta = \frac{1}{\psi} \partial_u.\] (2.5)

It follows
\[\frac{1}{\psi} = -i(\partial L - \bar{\partial} \bar{L}) = -i(\bar{L}z - L\bar{z} + Lu\bar{L} - \bar{L}u L),\] (2.6)

and therefore
\[\lambda = \frac{du + L \, dz + \bar{L} \, d\bar{z}}{i(\partial L - \bar{\partial} \bar{L})}, \quad \partial_\theta = i(\partial L - \bar{\partial} \bar{L}) \partial_u.\]

The structure function \(c\), in (2.1), now takes the form
\[c = \frac{\partial \psi}{\psi} - \psi \partial_\theta L = -\partial \log \left( i (\partial L - \bar{\partial} \bar{L}) \right) - L u.\] (2.7)

Moreover, it follows from \(\mu = dz\) and \(d^2\lambda = 0\) that
\[\partial \bar{c} = \bar{\partial} c.\] (2.8)

In the remainder of the paper, we use Cartan’s description of a CR manifold \(M\), defining the CR structure by the equivalence classes \([\mu, \lambda]\) of adapted coframes, defined as above in (2.3). The definition of Sasakian manifolds given below suits the purpose of this paper best. For other definitions and their equivalence see [1] or [9].

**Definition 2.4.** A Sasakian manifold (of dimension 3) is a strictly pseudoconvex CR manifold \((M, D, J)\) endowed with a vector field \(Z\) transversal to \(D\), which is an infinitesimal automorphism, i.e.
\[\mathcal{L}_Z D \subseteq D, \quad \mathcal{L}_Z J = 0.\] (2.9)

Notice that the first condition in (2.9) is equivalent to \(Z\) being the Reeb vector field for some contact form \(\lambda\). In terms of a chosen adapted frame the second condition in (2.9) is equivalent to
\[\partial_\theta \mathcal{L}_Z \mu = 0.\]

It is well known, see e.g. [3, 14, 28], and easy to see that (3-dimensional) Sasakian manifolds are locally realisable as submanifolds of \(\mathbb{C}^2\). Indeed, consider the manifold \(M \times \mathbb{R}_r\). The complex vector fields \(\partial\) and \(Z + i\partial\theta\) define an integrable complex structure on \(M \times \mathbb{R}\) in which \(M\) is embedded as \(r = 0\). In suitable local coordinates \(z, w = u + iv\) on the complex manifold \(M \times \mathbb{R}\), \(M\) can be expressed as
\[w = F(z, \bar{z}).\] (2.10)

The leaf space of the integral curves of the Reeb vector field \(Z = \partial_u\) is in this case simply \(\mathbb{C}_z\). It has a natural Kähler structure with Kähler potential \(F(z, \bar{z})\) and Kähler metric \(h = 2F_{\bar{z}z} dz d\bar{z}\) in \(\mathbb{C}_z\). On the other hand, a Kähler manifold with Kähler potential \(F\) also completely determines the geometry of the Sasakian manifold \(M\). We refer to the
metric $h$ as the underlying Kähler metric of the Sasakian manifold. The Ricci form of the underlying Kähler metric is given by

$$R = -\partial_{zz} \log (F_{zz}),$$

(2.11)

From Lemma 2.3 for the embedding (2.10) the expressions for the frame (2.5) simplify to

$$\partial = \partial_z + iF_z \partial_u, \quad \partial_o = 2F_{zz} \partial_u$$

(2.12)

and the function $c$ takes the form

$$c = -\partial_z \log (F_{zz}).$$

(2.13)

We now show that 3-dimensional Sasakian manifolds can also be characterised through the structure function $c$ defined by (2.7).

**Proposition 2.5.** Let $(M, D, J)$ be a 3-dimensional strictly pseudoconvex CR manifold. Suppose it admits a non-constant CR function $z$. Let $(\mu, \lambda)$ be an adapted coframe for $M$, where $\mu = dz \neq 0$, and let $(\partial, \partial_o)$ be the dual frame. Let $c$ be the structure function defined by (2.7). Then $M$ is Sasakian if and only if

$$\partial_o c = 0.$$

**Proof.** If $(M, D, J)$ is Sasakian then with respect to the coordinates $(z, u + iv)$ from the embedding (2.10)

$$\partial_o = \frac{1}{\psi} \partial_u.$$

Since the function $c$ from (2.13) does not depend on $u$, it follows that

$$\partial_o c = 0.$$

Proving now the converse, we assume $\partial_o c = 0$. We show that there exists a real function $A$, such that the vector field

$$Z = e^\varphi \partial_o,$$

is an infinitesimal CR automorphism transversal to $D$, i.e. it is the Reeb vector field for the contact form

$$e^{-\varphi} \lambda,$$

and it preserves the complex structure on the CR distribution, i.e.

$$\bar{\partial} \lrcorner \mathcal{L}_Z \mu = \bar{\partial} \lrcorner (d(Z \lrcorner \mu) + Z \lrcorner d\mu) = 0.$$

The condition

$$e^\varphi \partial_o \lrcorner d(e^{-\varphi} \lambda) = 0,$$

is equivalent to

$$(\partial \varphi - c) \mu + (\bar{\partial} \varphi - \bar{c}) \bar{\mu} = 0.$$  

(2.14)

Therefore, it remains to show that the equation

$$\partial \varphi = c,$$  

(2.15)

has a real solution. Let $(z = x + iy, u)$ be local coordinates as above, such that $\partial_o = \frac{1}{\psi} \partial_u$. Since $\partial_o c = 0$, it follows from (2.8) that

$$\partial_z \bar{c} = \partial_{\bar{z}} c.$$
Substituting \( c(x,y) = a(x,y) + ib(x,y) \) and \( \partial_z = \frac{1}{2}(\partial_x - i\partial_y) \) into the above equation gives us

\[
b_x = -a_y. \tag{2.16}
\]

For the real function \( \varphi \), the equation (2.15) is equivalent to

\[
\begin{aligned}
\varphi_x &= 2a \\
\varphi_y &= -2b.
\end{aligned} \tag{2.17}
\]

Now, the condition (2.16) guarantees the existence of a local solution \( \varphi(x,y) \) with \( \partial_o \varphi = 0 \). It follows,

\[
\partial \varphi = \partial_z \varphi - L \partial_u \varphi = \partial_z \varphi = c. \tag{2.18}
\]

As a special case we consider the so-called tubular hypersurfaces \( \nu = F(y) \), which have an underlying Kähler potential \( F(y) \) depending only on one real coordinate. In this case, from the Lemma 2.3, the expressions for the frame (2.5) and structure function simplify to

\[
\begin{aligned}
\partial &= \partial_z + \frac{1}{2} F_y \partial_u, \\
\partial_o &= \frac{1}{2} F_{yy} \partial_u, \\
c &= \frac{iF_{yyy}}{2F_{yy}}.
\end{aligned} \tag{2.18}
\]

3. Shearfree congruences of null geodesics and CR geometry

In this section, we recall some notions related to shearfree congruences of null geodesics on Lorentzian manifolds following the paper by Hill, Lewandowski and Nurowski [12].

**Definition 3.1.** [2] A shearfree congruence of null geodesics on a 4-dimensional Lorentzian manifold \( (\mathcal{M},g) \) is a foliation by integral curves of a nowhere vanishing vector field \( \mathbf{k} \), such that

(i) The vector field \( \mathbf{k} \) is null, i.e. \( g(\mathbf{k},\mathbf{k}) = 0 \).

(ii) \( \mathcal{L}_k g = \rho g + \theta \chi \), where \( \theta = g(\mathbf{k},\cdot) \), \( \rho \) is a real function on \( \mathcal{M} \) and \( \chi \) is a 1-form.

For the sake of brevity we will call the vector field \( \mathbf{k} \) shearfree for the Lorentzian metric \( g \).

In the definition above, and throughout this paper, we use the standard notation of the symmetric tensor product of two 1-forms \( \omega \) and \( \gamma \),

\[
\omega \gamma = \frac{1}{2}(\omega \otimes \gamma + \gamma \otimes \omega).
\]

Notice that conditions (i) and (ii) in the definition above mean that the metric \( g \) changes conformally under the flow of \( \mathbf{k} \), when restricted to the distribution

\[
k^\perp = \{ X \in T\mathcal{M} \mid g(X, \mathbf{k}) = 0 \}.
\]

In this sense a shearfree vector field can be considered as a generalisation of a conformal Killing vector field.

Following [12], we define, for a given 3-dimensional CR manifold \( (\mathcal{M},[(\mu,\lambda)]) \), a class of Lorentzian metrics on the line bundle \( \mathcal{M} = M \times \mathbb{R} \)

\[
g = 2\mathcal{P}^2 (\mu \bar{\mu} + \lambda (dr + W\mu + \bar{W}\bar{\mu} + \mathcal{H})).
\]

Here \( \mathcal{P} \) is a non-zero real function, \( r, \mathcal{H} \) are real functions and \( W \) is a complex function on \( \mathcal{M} \). We have used the notation \( \lambda \) and \( \mu \) for the forms on \( M \) as well as for their pullbacks to \( M \). We will call \((\mathcal{M},g)\) a lift of the CR manifold \( M \).
The vector field \( k = \frac{\partial}{\partial r} \) is shearfree for all metrics of the family (3.1). This family of metrics is independent of the choice of coframe \((\mu, \lambda)\), depending only on the CR structure of \( \mathcal{M} \). With respect to the choice of coframe on \( \mathcal{M} \)

\[
\theta^1 = \mathcal{P}_\mu, \quad \theta^2 = \mathcal{P}_{\bar{\mu}}, \quad \theta^3 = \mathcal{P}_\lambda, \quad \theta^4 = \mathcal{P} (dr + W_\mu + \bar{W}_{\bar{\mu}} + \mathcal{H}(\lambda)),
\]

(3.2)

and corresponding frame

\[
e_1 = \frac{1}{\mathcal{P}}(\partial - W\partial r), \quad e_2 = \frac{1}{\mathcal{P}}(\partial - \bar{W}\partial r), \quad e_3 = \frac{1}{\mathcal{P}}(\partial_{\bar{\nu}} - \mathcal{H}\partial_r), \quad e_4 = \frac{1}{\mathcal{P}}\partial_r,
\]

(3.3)

the metric (3.1) takes the form

\[
g = 2(\theta^1\theta^2 + \theta^3\theta^4).
\]

(3.4)

In the Theorem 3.2 below, proved in [12], we summarise a list of consequences for the functions \( \mathcal{P}, W \) and \( \mathcal{H} \) resulting from the vanishing of the complexified Ricci curvature restricted to the so-called \( \alpha \)-planes, that is the following components with respect to the frame (3.3) chosen above

\[
\text{Ric}_{44} = \text{Ric}_{22} = \text{Ric}_{24} = 0.
\]

Then the Goldberg-Sachs theorem [10, 11] implies that the following components of the Weyl tensor vanish

\[
C_{4141} = C_{4341} = 0.
\]

Here, again, the subscripts indicate the components of the tensor with respect to the frame (3.3). Together with

\[
C_{4132} \neq 0
\]

this establishes that the Lorentzian metric is of Petrov type II or D.

**Theorem 3.2.** [12] 1. Assume that for the metric (3.1) the components of the Ricci curvature \( \text{Ric}_{44}, \text{Ric}_{22}, \text{Ric}_{24} \) with respect to the frame (3.3) vanish. Then

\[
\mathcal{W} = i\mathcal{X}e^{-iR} + \mathcal{Y},
\]

(3.5)

where \( \mathcal{X} \) and \( \mathcal{Y} \) are complex-valued functions, which do not depend on \( r \) and the following statements hold:

\[
\mathcal{P} = \frac{p}{\cos\left(\frac{r - s}{2}\right)},
\]

(3.6)

\[
\partial t + (c - t)t = 0, \quad t = c + 2\partial \log p - e^{is}\mathcal{X},
\]

(3.7)

\[
\mathcal{Y} = ic + 2i\partial \log p + \partial s - 2it,
\]

(3.8)

where \( p \) and \( s \) are real-valued functions with \( p_r = s_r = 0 \), and the complex functions \( c, \mathcal{X}, \) and \( \mathcal{Y} \) are defined by (2.1) and (3.5), respectively.

2. Vice versa, if a metric (3.1) satisfies (3.5), (3.6), (3.7), (3.8) then

\[
\text{Ric}_{44} = \text{Ric}_{22} = \text{Ric}_{24} = 0.
\]

3. Furthermore, assuming \( \text{Ric}_{44} = \text{Ric}_{22} = \text{Ric}_{24} = 0 \),

\[
\text{Ric}_{12} = \text{Ric}_{34} = \Lambda \iff \mathcal{B} = 0,
\]

(3.9)

with
\[ B = \left( \partial \bar{\partial} + \partial \bar{c} + \bar{c} \partial + c \bar{\partial} + \frac{1}{2} |c|^2 + \frac{3}{4} (\partial \bar{c} + \partial c) - \frac{3}{2} (\partial \bar{t} + \partial t)^2 \right) p - \frac{m + \bar{m}}{p^3} - \frac{2}{3} \Lambda p^3, \]

where \( m \) is a complex function satisfying \( m_r = 0 \).

4. Assuming \( \text{Ric}_{44} = \text{Ric}_{22} = \text{Ric}_{24} = 0 \) and \( \text{Ric}_{12} = \text{Ric}_{34} = \Lambda \), we have
\[ \text{Ric}_{13} = 0 \iff \partial m + 3(c - t)m = 0. \quad (3.10) \]

5. Finally, assuming \( \text{Ric}_{44} = \text{Ric}_{22} = \text{Ric}_{24} = 0 \), \( \text{Ric}_{12} = \text{Ric}_{34} = \Lambda \) and \( \text{Ric}_{13} = 0 \), we have
\[ C_{4132} = \frac{(1 + e^{i(r+s)})^3}{2p^6}m. \quad (3.11) \]

We will use the results of the theorem above in the case of Sasakian manifolds, given by a defining equation (2.10) in coordinates \((z, w = u + iv)\). In this setting, the identity (2.8), \( \partial \bar{c} = \bar{\partial} c \), allows us to simplify some of the expressions. Moreover, one may notice that \( t = 0 \) is a solution of the equation (3.7). For the remainder of the paper, we proceed under the ansatz \( t \equiv 0 \), with the CR structure on \( M \) given by the adapted frame and coframe defined in (2.12). In addition, we assume \( s = 0 \) and \( p_u = 0 \). As a consequence of these assumptions and Theorem 3.2, the function \( H \) takes the form
\[ H = m p e^{2i r} + \bar{m} p e^{-2i r} + Q e^{i r} + \bar{Q} e^{-i r} + T, \quad (3.12) \]

where
\[ Q = \frac{3m + \bar{m}}{p^4} + \frac{2}{3} \Lambda p^2 + \frac{2\partial p \partial \bar{p} - p (\partial \bar{p} \partial p + \bar{\partial} \partial p)}{2p^2} - \partial c, \quad (3.13) \]
and
\[ T = \frac{3m + 3\bar{m}}{p^4} + 2\Lambda p^2 + \frac{2\partial p \partial \bar{p} - p (\partial \bar{p} \partial p + \bar{\partial} \partial p)}{p^2} - 2\bar{c}, \quad (3.14) \]

and
\[ \text{Ric}_{33} = \left( \frac{8}{p^4} (\partial + 2c) \left[ p^2 (\partial \bar{\partial} - 2\Lambda (2\partial \log p + c) p^2) \right] + \frac{16\Lambda}{p} \mathcal{B}_o + \frac{16i}{p^3} \partial \left( \frac{m}{p^3} \right) \right) \cos \left( \frac{r}{2} \right), \quad (3.15) \]

where the function \( J \) is defined by
\[ J = \partial (\partial \log p + c) + (\partial \log p + c)^2, \quad (3.16) \]
and
\[ \mathcal{B}_o = \mathcal{B}|_{t=0} = \left( \partial \bar{\partial} + \partial \bar{c} + \bar{c} \partial + c \bar{\partial} + \frac{1}{2} |c|^2 + \frac{3}{4} (\partial \bar{c} + \partial c) \right) p - \frac{m + \bar{m}}{p^3} - \frac{2}{3} \Lambda p^3. \quad (3.17) \]

4. Quasi-Einstein Lorentzian manifolds

Hill, Nurowski and Lewandowski proved in [12] that for any real analytic CR manifold there exists a representative in the family of metrics (3.1) of Petrov type II or D, which is quasi-Einstein, that is, it satisfies the equations
\[ \text{Ric}(g) = \Lambda g + \Phi \lambda^2, \]

where \( \Phi \) is some real function. We prove an analogous result for Sasakian manifolds without the assumption of real analyticity.
Prior to presenting our main result, we state a theorem proved by Du and Ma in [6], (see also [5] by Dong and Du) which gives sufficient conditions for the existence of a solution for the so-called logistic elliptic equation. The existence of such solution is crucial in the proof of our theorem.

**Theorem 4.1.** [5, 6] Consider the logistic elliptic equation

\[-\Delta q = a(x)q - b(x)q^\sigma, \quad x \in \mathbb{R}^n, \quad (4.1)\]

where \(\sigma > 1\), and \(a(x), b(x)\) are continuous real-valued functions with \(b(x) > 0\), for all \(x \in \mathbb{R}^n\). Then, the equation (4.1) has a unique positive solution \(q\), if the following limits exist and are positive

\[\lim_{|x| \to \infty} a(x) > 0 \quad \text{and} \quad \lim_{|x| \to \infty} b(x) > 0.\]

In the following subsections, we prove Theorems 1.1, 1.2, 1.3.

4.1. **Proof of Theorem 1.1.** Suppose that \((M, D, J)\) is a 3-dimensional Sasakian manifold given by an equation \(v = F(z, \bar{z})\), such that the limits

\[\lim_{|z| \to \infty} F(z, \bar{z}) > 0 \quad \text{and} \quad \lim_{|z| \to \infty} \partial_z \bar{z} \log (F(z, \bar{z})) < 0 \quad (4.2)\]

exist, and let \(\Lambda\) be an arbitrary positive constant. Consider the metric (3.1) with \(\mu = dz\), \(\lambda = \frac{1}{2F_{z\bar{z}}} (du - iF_z + iF_{\bar{z}})\).

We determine the functions \(\mathcal{P}, \mathcal{X}, \mathcal{Y}\) and \(\mathcal{H}\) using Theorem 3.2. From (3.7) and (3.8) the functions \(\mathcal{X}, \mathcal{Y}\) and \(\mathcal{P}\) are

\[\mathcal{X} = c + 2\partial \log p, \quad \mathcal{Y} = i (c + 2\partial \log p), \quad \mathcal{P} = \frac{p}{\cos(\frac{t}{2})},\]

where \(c\) is given by (2.13). We now determine the function \(p\). Note that the equation \(B = 0\) from (3.9) with our ansatz \(t = 0\) becomes \(B_0 = 0\), where \(B_0\) is given by (3.17). Assuming that \(m\) is purely imaginary and that \(p\) does not depend on \(u\) this simplifies further to

\[p_{zz} + \frac{\bar{c}}{2} p_z + c + 2\partial \bar{c} p + \left(\frac{|c|^2}{4} + \frac{3}{4} \partial_z \bar{c}\right) p = \frac{\Lambda}{3} p^3. \quad (4.3)\]

Let \(q\) be the function defined by

\[p = f q, \quad \text{where} \quad f = \sqrt{F_{z\bar{z}}}. \quad (4.4)\]

In terms of \(q\) the equation (4.3) becomes the logistic elliptic equation

\[q_{z\bar{z}} + a(z, \bar{z}) q = b(z, \bar{z}) q^3, \quad (4.5)\]

with

\[a(z, \bar{z}) = \frac{f_{z\bar{z}}}{f} + \frac{f_z}{2f} \bar{c} + \frac{f_{\bar{z}}}{2f} c + \frac{|c|^2}{4} + \frac{3}{4} \partial_z \bar{c}. \quad (4.6)\]
and
\[ b(z, \bar{z}) = \frac{\Lambda}{3} f^2 = \frac{\Lambda}{3} F_{zz}. \]

The expression (4.6) for the function \( a(z, \bar{z}) \) simplifies to
\[ a(z, \bar{z}) = -\frac{1}{4} \partial_{zz} \log (F_{zz}) = \frac{1}{4} R, \]

where \( R \) is given by (2.11). By our assumptions the limits (4.2) of both \( a \) and \( b \) exist and are positive real numbers, and, by Theorem 4.1, the logistic elliptic equation (4.5) has a unique positive solution. This determines the function \( p \).

It remains to choose the purely imaginary function \( m \) such that \( R_{13} = 0 \) and \( C_{1324} \neq 0 \). Let \( m = i(\partial_o w)^3 \), where \( w = u + i F(z, \bar{z}) \) is a CR function, that is \( \bar{\partial}w = (\partial \bar{z} - iF\partial_u)w = 0 \). From (2.12) it follows that
\[ \partial_o w = 2F_{zz} \neq 0, \]

is a real function, hence \( m \) is purely imaginary. Now the non-zero function \( m \) satisfies
\[ \partial m + 3cm = 3i(\partial_o w)^2 \partial \partial_o w + 3ic(\partial_o \bar{w})^3 = 3i(\partial_o w)^2(\partial_o \partial \bar{w} - c\partial_o \bar{w}) + 3ic(\partial_o \bar{w})^3 = 0, \]

since \( \partial \bar{w} = 0 \) and \( \partial \partial_o - \partial \partial_o = c\partial_o \). Then, by (3.10), \( \text{Ric}_{13} = 0 \) and the function \( H \) defined by (3.12) is determined. Since, \( m \neq 0 \), the Weyl component
\[ C_{1324} = \frac{(1 + e^{3r})^3}{2p^6} m \]

is non-zero, and therefore, the metric \( g \) is of Petrov type II or D.

4.2. Proof of Theorem 1.2. Let \((M, D, J)\) be a 3-dimensional tubular CR manifold with defining function \( v = F(y) \). The proof is the same as the proof of Theorem 1.1 with one exception. The equation \( B_o = 0 \) becomes the ODE
\[ p_{yy} - ic p_y + ic p_y + \left( |c|^2 - \frac{3}{2} i \bar{c}_y \right) p = \frac{16}{3} A p^3, \]  

which, by the Picard–Lindelöf theorem has a two-parametric family of solutions for any \( \Lambda \). We remark that by the same substitution as in the proof of Theorem 1.1 the ODE 4.7 simplifies to
\[ q_{yy} + R q = \frac{16}{3} A f^2 q^3. \]  

\[ \Box \]
4.3. Proof of Theorem 1.3. Let \((M, D, J)\) be a 3-dimensional Sasakian manifold with defining equation \(v = F(z, \bar{z})\), and assume that the underlying Kähler manifold with Kähler metric

\[ h = 2F_{\bar{z}z}dz\,d\bar{z}, \]

is Einstein with non-zero Einstein constant \(\Lambda_o\) i.e.,

\[ \text{Ric}(h) = R = \Lambda_o F_{\bar{z}z}. \]

We follow again the proof of Theorem 1.1. The equation (4.5) takes now the form

\[ q_{\bar{z}z} + \left(\frac{\Lambda_o}{4} F_{\bar{z}z}\right) q = \left(\frac{\Lambda}{3} F_{\bar{z}z}\right) q^3. \tag{4.9} \]

For any real constant \(\Lambda\), such that \(\Lambda \Lambda_o > 0\) the constant function

\[ q = \pm \sqrt{\frac{3\Lambda_o}{4\Lambda}}, \]

is a solution of the above equation (4.9).

It remains to show that in this case \(R_{33} = 0\), hence the metric (3.1) becomes Einstein. Notice that the function \(I\) from (3.16) now becomes

\[ I = \frac{1}{2} \partial c + \frac{1}{4} c^2, \]

and consequently,

\[ \partial_z \bar{J} = \frac{1}{2} \partial_z (\partial_z \bar{c}) + \frac{1}{2} \bar{c} \partial_z \bar{c} = \frac{\Lambda_o}{2} \partial_z F_{\bar{z}z} - \frac{\Lambda_o}{2} \partial_z F_{\bar{z}z} = 0. \]

This reduces (3.15) to

\[ \text{Ric}_{33} = \frac{8 \cos^4 \left(\frac{\xi}{2}\right)}{p^4} (\partial + 2c) \left(\frac{16i}{p^3} \partial_o \left(\frac{m}{p^4}\right)\right). \]

Since

\[ \partial_o m = 8i \partial_o (F_{\bar{z}z}) = 0, \]}

and \(\partial_o p = 0\), this implies \(\text{Ric}_{33} = 0\). \(\square\)

5. Examples

Let \((M, D, J)\) be a 3-dimensional Sasakian manifold with defining function

\[ v = F(z, \bar{z}) = z\varphi(z) + \bar{z}\varphi(z), \tag{5.1} \]

where \(\varphi\) is a holomorphic function, so that

\[ c = \frac{-\varphi''(z)}{\varphi(z) + \varphi'(z)}. \]

Then, there exists a 4-dimensional lift with Ricci flat Lorentzian metric \(g\) of Petrov type II or D. The metric \(g\) is given by

\[ g = \frac{2 \Re \varphi'}{\cos^2 \left(\frac{\xi}{2}\right)} \left( h + \lambda' \left( dr + Wdz + \bar{W}d\bar{z} + \frac{3\lambda}{4 \Re \varphi'}\lambda'\right) \right), \]

where

\[ h = 4 \Re \varphi' dz\,d\bar{z}, \quad \lambda' = \psi \lambda, \quad W = -i c(e^{-ir} + 1), \]
and the function $\mathcal{H}$ is defined by (3.12) with $m = i\psi^{-3}$.

Note that for the defining function (5.1)

$$\left( F_{\bar{z}z} \right)_{\bar{z}z} = 0,$$

Replacing this condition by

$$\left( F_{\bar{z}z} \right)^2 = 0,$$

yields another Ricci flat 4-dimensional lift of Petrov type II or D. The choices

$$p = \left( F_{\bar{z}z} \right)^2, \quad m = i\psi^{-3},$$

together with $\Lambda = 0$, solve all differential equations of Theorem 3.2. The resulting metric becomes

$$g = \left( F_{\bar{z}z} \right)^4 \left( h + X' \left( dr + Wdz + \bar{W}d\bar{z} + \frac{3\mathcal{H}}{2F_{\bar{z}z}}X' \right) \right), \quad (5.2)$$

with

$$W = \frac{i}{\sqrt{3}}e^{-i\pi/2} \left( e^{-i\pi/2} + 1 \right),$$

and the function $\mathcal{H}$ defined by (3.12) with $m = i\psi^{-3}$. The metric (5.2) is an example of the Fefferman-Robinson-Trautman metric introduced in [20] in relation with the embedding problem of a 3-dimensional CR manifold. For more details see [20].

**References**

[1] D. V. Alekseevsky, V. Cortés, K. Hasegawa, and Y. Kamishima, *Homogeneous locally conformally Kähler and Sasakian manifolds*, Internat. J. Math. 26 (2015), no. 6, 1541001, 29, DOI 10.1142/S0129167X15410013. MR3356872

[2] Dmitri V. Alekseevsky, Masoud Ganji, and Gerd Schmalz, *CR-geometry and shearfree Lorentzian geometry*, Geometric complex analysis, Springer Proc. Math. Stat., vol. 246, Springer, Singapore, 2018, pp. 11–22. MR3923214

[3] M. S. Baouendi, Linda Preiss Rothschild, and F. Trèves, *CR structures with group action and extendability of CR functions*, Invent. Math. 82 (1985), no. 2, 359–396, DOI 10.1007/BF01388808. MR809720

[4] G. C. Debney, R. P. Kerr, and A. Schild, *Solutions of the Einstein and Einstein-Maxwell equations*, J. Mathematical Phys. 10 (1969), 1842–1854, DOI 10.1063/1.1664769. MR250641

[5] Wei Dong and Yihong Du, *Unbounded principal eigenfunctions and the logistic equation on $\mathbb{R}^N$*, Bull. Austral. Math. Soc. 67 (2003), no. 3, 413–427, DOI 10.1017/S0004972700037229. MR1983874

[6] Yihong Du and Li Ma, *Logistic type equations on $\mathbb{R}^N$ by a squeezing method involving boundary blow-up solutions*, J. London Math. Soc. (2) 64 (2001), no. 1, 107–124, DOI 10.1017/S0024610701002289. MR1840774

[7] Anna Fino, Thomas Leistner, and Arman Taghavi-Chabert, *Almost Robinson geometries*. https://arxiv.org/abs/2102.05634

[8] Anna Fino, Thomas Leistner, and Arman Taghavi-Chabert, *Optical geometries*. https://arxiv.org/abs/2009.10012

[9] Masoud Ganji, *Shearfree Lorentzian geometry and CR geometry*, PhD thesis, University of New England, 2019.

[10] J. N. Goldberg and R. K. Sachs, *A theorem on Petrov types*, Acta Phys. Polon. 22 (1962), no. suppl., 13–23. MR156679

[11] A. Rod Gover, C. Denson Hill, and Paweł Nurowski, *Sharp version of the Goldberg-Sachs theorem*, Ann. Mat. Pura Appl. (4) 190 (2011), no. 2, 295–340, DOI 10.1007/s10231-010-0151-4. MR2786175
[12] C. Denson Hill, Jerzy Lewandowski, and Paweł Nurowski, *Einstein’s equations and the embedding of 3-dimensional CR manifolds*, Indiana Univ. Math. J. **57** (2008), no. 7, 3131–3176, DOI 10.1512/iumj.2008.57.3473. MR2492229

[13] Howard Jacobowitz, *An introduction to CR structures*, Mathematical Surveys and Monographs, vol. 32, American Mathematical Society, Providence, RI, 1990. MR1067341

[14] Jerzy Lewandowski, Paweł Nurowski, and Jacek Tafel, *Einstein’s equations and realizability of CR manifolds*, Classical Quantum Gravity **7** (1990), no. 11, L241–L246. MR1078890

[15] Andrei D.Polyanin and Valentin F. Zaitsev, *Handbook of nonlinear partial differential equations*, 2nd ed., CRC Press, Boca Raton, FL, 2012. MR2865542

[16] Jerzy Lewandowski, Paweł Nurowski, and Jacek Tafel, *Einstein’s equations and realizability of CR manifolds*, Classical Quantum Gravity **7** (1990), no. 11, L241–L246. MR1078890

[17] Andrzej Trautman, *Robinson manifolds and Cauchy-Riemann spaces*, Classical Quantum Gravity **19** (2002), no. 2, R1–R10, DOI 10.1088/0264-9381/19/2/201. MR1885472

[18] Xuefeng Zhang and Daniel Finley, *CR structures and twisting vacuum spacetimes with two Killing vectors and cosmological constant: type II and more special*, Classical Quantum Gravity **30** (2013), no. 11, 115006, 20, DOI 10.1088/0264-9381/30/11/115006. MR3055095

[19] Vladimir Ezhov, Martin Kolář, and Gerd Schmalz, *Rigid embeddings of Sasakian hyperquadrics in \( C^{n+1} \),* J. Geom. Anal. **28** (2018), no. 3, 2185–2205, DOI 10.1007/s12220-017-9900-6. MR3833790

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