Oscillations of the variable cross beams under seismic and technogenic influences (part I)

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Abstract. Free compressed beam transverse vibrations of a variable cross section carrying a distributed mass are considered. The mathematical model of oscillations is presented as a boundary value problem from the basic partial differential equation of the fourth order hyperbolic type in spatial coordinate, the second order in time and boundary conditions. The technical theory of the rods’ bending vibrations, based on the Bernoulli hypothesis about the invariance of flat beam cross-sections, has been used. The task is to determine the eigenvalues, eigenmodes of oscillation and the attenuation coefficient (extended Sturm-Liouville problem). It has been argued that solving the problem by the analytical methods is impractical due to the large transformations and calculations volume. The variables separation methods, finite differences and computer technology have been used. An algorithm for solving the problem, implemented in the Matlab software environment in the form of high-precision graphical and analytical calculations, has been developed. On a concrete example of a beam, verification of the proposed mathematical model has been demonstrated. The practical conclusions have been made.

Introduction

To date, transverse and longitudinal vibrations of beams and, in general, rods have an extensive bibliography [2-8]. But the new tasks dictated by the requirements of modern technology and the latest technology constantly appear.

The horizontal beams’ transverse vibrations are the most dangerous during the earthquakes caused by seismic and man-made causes. The existing calculation and design rules in this case are of little use, since they take into account only horizontal displacements of the base, while the perturbing process is vector and random.

The influence of the horizontal component along the axis of the beam in this case is insignificant, the vertical and angular (rotary) component and their degree of correlation are more significant. Such a problem statement is very relevant for the structures located near the earthquakes’ epicenter, where the vertical component of the base movement significantly exceeds the horizontal component. Therefore, this work is a continuation and deepening of the article’s authors [1] and the first part of a more detailed study devoted to the beams’ forced bending vibrations during the anthropogenic and seismic impacts. The second part of the article, devoted to the forced harmonic and stochastic oscillations, is placed in this collection of the conference reports.
Mathematical Model of Oscillations
We compose a mathematical model with a focus on use in both parts of the article. Let us consider the formulation of the problem as an example of a single-span horizontal beam of a variable section, shown in Figure 1.

Figure 1. Single span horizontal beam of a variable section

The left end of the beam is sealed, the right one is supported on the elastic spring with a stiffness coefficient \( c \) and is compressed by the longitudinal force \( P \). The beam, in addition to its own distributed mass, carries an attached uniformly distributed mass \( q \). Its consideration is caused by the need to bring the oscillation model closer to real situations. It will affect the frequencies and waveforms.

The end sections of the beam are subjected to kinematic seismic or technogenic impacts in the form of a vector process \( F(t) = [f_1(t), f_2(t), f_3(t)] \). Here \( f_1, f_2, f_3 \) are the centered impact components. They can be in-phase, antiphase coupled or completely independent in the deterministic case, uncorrelated or correlated with the correlation coefficients in the range \( k \in [-1, 1] \) with random exposure. Two-pointed arrows in Fig. 1 correspond to the effects’ oscillatory nature.

Let us consider the oscillations mathematical model definition. With this purpose we use the technical theory of the rods’ bending vibrations. It is assumed that the Bernoulli hypothesis about the invariance of flat sections is true, the deflections and rotation angles are small, the longitudinal fibers do not press against each other.

We use the mathematical oscillations model for the deterministic formulation of the problem with its subsequent adaptation to the random oscillations. Then the main equation has the form of a homogeneous differential equation of hyperbolic type in partial derivatives with variable coefficients [4]

\[
(b(x)u^{\prime\prime})'' + Pu^{\prime\prime} + r(x)u^{\prime} + \mu m(x)u = 0, \quad x \in (0, l), \quad t > -\infty, \quad b(x) = EJ(x), \quad r(x) = m(x) + q.
\]

Here \( u(x,t) \) - is the desired function of the deflection of the beam during movement, \( b(x) \) - denotes bending stiffness, variable along the beam’s length, \( J(x) \) - is an axial moment of the cross section inertia, \( r(x) \) - is the total linear beam mass, \( P \) – is the longitudinal axial compressive force, \( \mu \) - is a specific coefficient of linearly viscous internal friction, \( m(x) \) - is the linear beam mass. The dashes in the upper index indicate the differentiation of the deflection function with respect to the \( x \) coordinate, the dots above the symbols indicate the differentiation in time \( t \).

The boundary conditions discussed above are added to the main equation (1).

\[
u(0,t) = f_1(t), \quad [b(l)u^{\prime\prime}(l,t)]' - cu(l,t) = f_2(t), \quad u'(0,t) = f_3(t), \quad u^{\prime\prime}(l,t) = 0, \quad t > -\infty
\]

\( c \) – is the stiffness coefficient of elastic support.
The equation (1) and boundary conditions (2) form a mathematical model that makes it possible to determine the function $u(x,t)$ for the free and forced vibrations. It will be further refined depending on the types of tasks under consideration. In this part of the article, only free oscillations are considered.

**Free vibrations**

The problem of free oscillations usually precedes the study of the forced oscillations and includes the determination of natural frequencies and forms, which significantly affect the quantitative and qualitative parameters of the forced oscillations. For this purpose, we turn to the work [1] with refinements of the mathematical model caused by the payload $q$ addition, using an easier method for calculating the desired characteristics of free oscillations with the recommendations for introducing the obtained results into design practice.

In the problem of free vibrations, the boundary conditions (2) will be homogeneous:

$$u(0,t) = 0, \quad (b(l)u'(l,t))' - cu(l,t) = 0, \quad u'(0,t) = 0, \quad u''(l,t) = 0, \quad t > -\infty.$$  

(3)

The main equation (1) and boundary conditions (3) form the spectral problem of determining the eigenvalues and eigenfunctions.

We use the variables separation method and present the desired function as a bundle:

$$u(x,t) = X(x)T(t).$$  

(4)

It can be rewritten more exact in this problem

$$u(x,t) = X(x)e^{\lambda t}.$$  

(5)

Here $\lambda = -\mu + j\omega$ is a characteristic indicator, $\omega$ - is the angular frequency of free oscillations to be determined, $j$ – defines the imaginary unit. The logic of such a refinement is the obviousness of the fact that the model must correspond to vibrations that are damped due to the presence of resistance forces. Using (5) instead of (4) will reduce the amount of computation.

Further continuation of the problem’s solution by the analytical methods will lead to greater difficulties due to its bulkiness and the presence of variable coefficients $b(x)$, $r(x)$, $m(x)$ before the derivatives in the main equation (1). The solution of this problem is the use of numerical methods and computer technology. For this purpose, we consider the finite difference method (FDM) [9, 10] and the Matlab computing complex [11].

In the FDM, the scope of continuous argument $0 \leq x \leq l$ is replaced by a discrete set of points (grid) $l_h = \{ x_i = (i-1)h, \quad i = 1, 2, ..., n \}$ with a pitch $h = l / (n - 1)$. Turning to its application, we carry out the necessary procedures and rewrite the equation (1) in the form:

$$v_i y_{i-2} + \alpha_i y_{i-1} + \beta_i y_i + \gamma_i y_{i+1} + \xi_i y_{i+2} = 0, \quad i = 3, 4, ..., n-2,$$  

(6)

where the notation is introduced:

$\alpha_i = -4 + 2b_0 h / b_i, \quad \beta_i = 6 - 2Ph^2 / b_i + \epsilon_i h^4 / b_i, \quad \gamma_i = -4 + 2b_0 h / b_i + Ph^2 / b_i, \quad \xi_i = 1 + b_0 h / b_i, \quad \epsilon_i = -r_i \omega^2 + j\omega \mu_i, \quad b_i = b(x_i), \quad b_0 = b_0'$. 

Boundary conditions (3) are also brought into correspondence with the finite difference method and we obtain:

On the left end:

$$y_1 = 0, \quad -3y_1 + 4y_2 - y_3 = 0.$$  

(7)

On the right end:

$$3y_{n-4} + \delta y_{n-3} + \sigma y_{n-2} + \rho y_{n-1} + \kappa y_n = 0, \quad -y_{n-3} + 4y_{n-2} - 5y_{n-1} + 2y_n = 0, \quad \delta = -2hb_0 / b_n - 14, \quad b_0 = -211 \cdot 10^8 E, \quad \sigma = 8hb_0 / b_n + 24, \quad \rho = -10hb_0 / b_n - 18, \quad \kappa = (4hb_0 - 2h^3 c) / b_n + 5.$$  

(8)
The equations (6) - (8) form a homogeneous algebraic system, which can be rewritten in the matrix-vector form:

\[ A(\lambda) Y = 0, \]

where \( A(\lambda) \) is the coefficient matrix, \( Y \) is the discrete argument vector \( Y = \{ y_1, y_2, \ldots, y_n \} \), \( y_i \approx X(x_i) \), \( i \) is the finite difference mesh node number. In this case, the replacement of analytical derivatives by the numerical ones is performed with accuracy \( O(h^2) \).

After carrying out the necessary procedures, we obtain a square five-diagonal ribbon matrix \( A(\lambda) \) of the \( n \) order.

\[
A(\lambda) = \begin{pmatrix}
1 & -3 & 4 & -1 & v_3 & \alpha_3 & \beta_3 & \gamma_3 & \xi_3 \\
-3 & v_4 & \alpha_4 & \beta_4 & \gamma_4 & \xi_4 & & & \\
& & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \ \\
& & & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \ \\
& & & & \ddots & \ddots & \ddots & \ddots & \ddots \ \\
& & & & & 3 & \delta & \sigma & \rho & \kappa \\
& & & & & & -1 & 4 & -5 & 2
\end{pmatrix}
\]

Here the null elements are not written out. The elements of the main diagonal \( \beta_i \) are the functions of a characteristic indicator \( \lambda \) and through it the attenuation coefficient can have the form \( \mu \) and the oscillation frequencies \( \omega \) in accordance with (5).

The condition for the nontrivial solution existence to the system of equations (9) gives the frequency equation

\[ \det A(\lambda) = 0, \]

from which the eigenvalues are determined \( \{ \lambda_1, \lambda_2, \ldots \} \).

The equation (10) is algebraic of the form \( f(\lambda) = 0 \). For the large values \( n \), its writing in expanded form is a cumbersome procedure possible though. In addition, its analytical solutions can only be obtained in the simplest cases. Therefore, we use the numerical methods and computers. In this case, we will focus on the algorithmic languages and software systems that allow to directly use the functions of a complex variable and carry out the algebraic and other actions on them (for example, in Matlab, etc.).

The equation (10), taking into account the fact that its left side represents a complex function, can be rewritten as a division into the real and imaginary parts:

\[ f_1(\omega) + f_2(j\omega) = 0 \]

or as a system of two nonlinear equations:

\[ f_1(\omega) = 0, \quad f_2(j\omega) = 0. \]

The first equation which roots determine the eigenvalues is of interest. It is not possible to find its solution by the analytical methods because of the large matrix \( A \) order. Considering this, we use the highly accurate graph-analytical methods implemented in the Matlab computing complex environment. Solving the nonlinear system of equations (11), in the previous work [1], the authors used the coordinate descent method, since there were two unknown quantities: natural frequency \( \omega \) and a specific linear viscous internal friction coefficient \( \mu \), which led to unreasonably complex algorithms and computer programs. Meanwhile, there is an opportunity for a significant reduction in the procedures. For this purpose, it is possible to use the bibliographic sources on the specific coefficient of linearly viscous internal friction \( \mu \). In this case, we can neglect the second equation and solve the first equation of system (11) autonomously, which is implemented below to determine the frequencies \( \omega_k \). The function
\( f_i(\omega) \) contains frequency \( \omega \) to very high degrees, since it is formed as a result of the large order \( n \) square matrix \( A \) elements’ multiplication (see the example below). The practical need for only the first few roots comes of the resulting polynomial roots. Therefore, without invoking the methods of linear algebra to determine the eigenvalues of square matrices, in this case we can solve the nonlinear equation \( f_i(\omega) = 0 \) graphically, determining the necessary roots on the computer using simple programs.

**Example 1.** Initial data:
\[ l = 6 \, \text{m}, \quad n = 601, \quad m(x) = 21 - 1.2x \, \text{kg/m}, \quad q = 300 \, \text{kg/m}, \quad P = 50 \, \text{kN}, \]
\[ J(x) = (1840-211x)10^{-8} \, \text{m}^4, \quad c = 10^4 \, \text{N/m}, \quad \mu = 0.01 \, \text{s}^{-1}. \]

According to the algorithm described above for system (11), three elements of the natural frequencies spectrum are determined \( \omega_k \), presented in Table 1.

| \( k \) | 1    | 2    | 3    |
|------|------|------|------|
| \( \omega_k \), \( \text{s}^{-1} \) | 11.01 | 54.57 | 147.52 |

The graph obtained on the computer monitor according to the first equation (11) has the form Fig. 2. The abscissas highlighted with the dots correspond to the frequencies in the table above. These results are highly accurate, since the monitor screen makes it possible to increase the picture size and consider the fragments and neighborhoods of the points in great detail.

**Figure 2.** Free frequencies

**Figure 3.** The dependence of the dynamism coefficient on the oscillation frequencies:
1. for the soils of I, II categories; 2. for the soils of III, IV category.

The first frequency of free vibrations \( \omega_1 = 11.01 \, \text{c}^{-1} \) represents significant danger to the beam. Its value according to the current building standards [12] and bibliographic the sources [13] gives the
greatest coefficient of dynamism $\beta = 2.5$ seismic load for the structures on the soils of III, IV category and $\beta \approx 2$ on the soils of I, II category. It is advisable to reduce it by changing the oscillatory system’s parameters. A qualified solution to such a problem should be based on the methods of dynamic damping of oscillations, methods of the dynamic mechanical systems’ optimal design [15, 16]. The arsenal of the techniques and special tools used in such cases is very extensive and the choice substantially depends on the specific circumstances. The solution to this problem is not described here, but some recommendations for reducing the natural frequencies in order to get out of the danger zone can be offered:

- to reduce the rigidity of the beam and elastic support,
- to increase the attached mass,
- to increase the value of the longitudinal compressive force,
- to attach the special dynamic vibration dampers to the beam, etc.

It should be noted that these proposals apply not only to the first natural frequency, but also to higher ones.

The oscillation modes normalized to unity and corresponding to the eigenvalues are represented by the graphs in Fig. 4. It is seen that with an increase in the eigenvalues, the vibration forms’ waviness increase, and they still correspond to the boundary conditions: the left end is pinched, the right support is articulated and elastic.

![Figure 4. Free oscillation forms](image)

**Summary**

1. Mathematical models are created for the kinematically excited free and forced vibrations of a complex beam structure under seismic and man-made impacts.
2. The algorithms and computer programs for determining the eigenvalues, eigenfunctions and damping coefficient of oscillations are developed.
3. A simple and effective method for determining the natural frequencies and damping coefficients of free vibrations is proposed.
4. The effectiveness of the mathematical model is confirmed by the numerical examples.
5. The recommendations on reducing the dynamic coefficient of the beam under seismic loading are given.
6. High-frequency forced oscillations will occur in the forms of great curvature and therefore will create the internal forces and stresses of high intensity.

**References**

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