Corrigendum: Incorporating gravity into trace dynamics: the induced gravitational action

2013 Class. Quantum Grav. 30 195015

Stephen L Adler

Institute for Advanced Study, Einstein Drive, Princeton, NJ 08540, USA
E-mail: adler@ias.edu

Received 13 September 2013
Published 7 October 2013
Online at stacks.iop.org/CQG/30/239501

Equations (60) and (61) (which were not used elsewhere in the paper) are not correct. The Bianchi identity should read

$$G'_{rr} - \frac{2A}{r^3} G_{\theta\theta} + \left( \frac{B'}{2B} + \frac{2}{r} - \frac{A'}{A} \right) G_{rr} + \frac{AB'}{2B^2} G_{rr} = 0,$$

and equation (61) should read

$$\Delta T_{tt} = \frac{N}{D},$$

$$N = \Delta T'_{rr} - \frac{2A}{r^3} \Delta T_{\theta\theta} + \left( \frac{B'}{2B} + \frac{2}{r} - \frac{A'}{A} \right) \Delta T_{rr},$$

$$D = -\frac{AB'}{2B^2}.\quad (2)$$

Using $\Delta T_{rr}$ and $\Delta T_{\theta\theta}$ from equation (58), this gives as the modified equation for $G_{tt}$

$$G_{tt} - \frac{3A}{B} = 0.\quad (3)$$
Incorporating gravity into trace dynamics: the induced gravitational action

Stephen L. Adler

Institute for Advanced Study, Einstein Drive, Princeton, NJ 08540, USA
E-mail: adler@ias.edu

Received 14 June 2013, in final form 16 August 2013
Published 11 September 2013
Online at stacks.iop.org/CQG/30/195015

Abstract
We study the incorporation of gravity into the trace dynamics framework for classical matrix-valued fields, from which we have proposed that quantum field theory is the emergent thermodynamics, with state vector reduction arising from fluctuation corrections to this thermodynamics. We show that the metric must be incorporated as a classical, not a matrix-valued, field, with the source for gravity the exactly covariantly conserved trace stress–energy tensor of the matter fields. We then study corrections to the classical gravitational action induced by the dynamics of the matrix-valued matter fields, by examining the average over the trace dynamics canonical ensemble of the matter field action, in the presence of a general background metric. Using constraints from global Weyl scaling and three-space general coordinate transformations, we show that to zeroth order in derivatives of the metric, the induced gravitational action in the preferred rest frame of the trace dynamics canonical ensemble must have the form

\[
\Delta S = \int d^4x \left( \frac{(4)}{g_{00}} \right)^{1/2} A (g_{00} g_{0i} g_{ij})^{1/2} g_{00} D_i D_j / g_{00} g_{00} D_i / g_{00},
\]

with \( D_i \) defined through the co-factor expansion of \( (4) g \) by \( (3) g = g_{00} + \sum D_i \), and with \( A(x, y, z) \) a general function of its three arguments. This action has ‘chameleon-like’ properties: for the Robertson–Walker cosmological metric, it exactly reduces to a cosmological constant, but for the Schwarzschild metric it diverges as \( (1 - 2M/r)^{-2} \) near the Schwarzschild radius, indicating that it may substantially affect the horizon structure.

PACS numbers: 03.50.Kk, 03.65.-w, 04.20.Cv, 04.60.Bc

1. Introduction

In papers culminating in a book [1–3], we proposed ‘trace dynamics’ as the fundamental pre-quantum dynamics of matter degrees of freedom. In this dynamics, the matter fields are non-commuting matrix-valued fields, with cyclic permutation under a trace action resolving factor-ordering problems. We identified globally conserved quantities and used them to
construct a canonical ensemble for the statistical dynamics of trace dynamics. We then argued, with approximations that remain to be justified, that the statistical thermodynamics of trace dynamics gives rise to quantum field theory, with fluctuation corrections to this thermodynamics leading to state vector reduction in measurements.

We did not, however, address the issue of incorporating gravity into trace dynamics, and that is the purpose of this paper. We shall see that without knowing the precise underlying trace dynamics action, restrictive qualitative statements can be made. The organization of this paper is as follows. In section 2 we give a very brief survey of trace dynamics. In section 3 we present arguments indicating that gravity must be incorporated into trace dynamics as a classical (a diagonal matrix) field, as opposed to a general matrix-valued field. We show that this leads to a consistent dynamics for classical gravity coupled to matrix-valued matter, with the source term for the Einstein equations the exactly covariantly conserved matter trace stress–energy tensor. In section 4 we define a matter induced effective action as the canonical ensemble average of the matter trace action. In section 5, we use global Weyl scaling invariance and three-space general coordinate invariance to derive a general functional form for the structure of the induced effective action to leading orders in derivatives of the metric. In section 6 we deduce the rules for using this frame dependent effective action as a source for the Einstein equations. In section 7, we show that although this effective action does not have the structure of a cosmological constant action, on a Robertson–Walker space-time it exactly reduces to a cosmological term. In section 8 we take a first look at the implications of the effective action for a time-independent, spherically symmetric metric. In section 9 we make further remarks about the structure and implications of our results. In appendix A we state our notational conventions and give formulas for matter field actions and conserved quantities derived from them. In appendix B we discuss the construction and properties of the mixed index gravitational pseudotensor that is referenced in the course of our argument.

2. Brief overview of trace dynamics

Trace dynamics [1–3] is a new kinematic framework for pre-quantum dynamics, in which the dynamical variables are pairs of operator-valued variables \( \{q_r\}, \{p_r\} \), with no assumed a priori commutativity properties, acting on an underlying complex Hilbert space. A theory of dynamical flows can be set up by starting from an operator Hamiltonian \( H[\{q_r\}, \{p_r\}] \) and forming the trace Hamiltonian \( H \equiv \text{Tr} \, H \). Although noncommutativity of the operator variables prevents one from differentiating \( H \) with respect to them, we can use the cyclic property of the trace to define derivatives of a general trace functional \( A \) by forming \( \delta A / \delta q_r \) and cyclically reordering the operator variations \( \delta q_r, \delta p_r \) to the right. This gives the fundamental definition

\[
\delta A = \text{Tr} \sum_r \left( \frac{\delta A}{\delta q_r} \delta q_r + \frac{\delta A}{\delta p_r} \delta p_r \right),
\]

in which \( \delta A / \delta q_r \) and \( \delta A / \delta p_r \) are operators. Applying this definition to the trace Hamiltonian, a symplectic dynamics of the operator phase space variables is generated by the operator Hamilton equations

\[
\frac{\delta H}{\delta q_r} = -\dot{p}_r, \quad \frac{\delta H}{\delta p_r} = \epsilon_r \dot{q}_r,
\]

with \( \epsilon_r = 1 (-1) \) according to whether \( q_r, p_r \) are bosonic (fermionic).

Substituting equation (3) into equation (2), we see that \( H \) is a constant of motion in trace dynamics. Another conserved trace quantity is the trace fermion number \( N \). An essential
feature of trace dynamics is that there are two other conserved quantities. The first is the traceless anti-self-adjoint operator $\tilde{C}$ defined by

$$\tilde{C} = \sum_{r,B} [q_r, p_i] - \sum_{r,F} [q_r, p_i],$$

(4)

with the subscripts $B, F$ denoting respectively sums over bosonic and fermionic degrees of freedom. When the trace Hamiltonian is constructed using only non-operator numerical coefficients, there is a global unitary invariance for which $\tilde{C}$ is the conserved Noether charge [2].

A second important conserved quantity is the natural integration measure $d\mu$ for the underlying operator phase space. Conservation of $d\mu$ gives a trace dynamics analog of Liouville’s theorem, and permits the use of statistical mechanics methods. Specifically, the canonical ensemble is given by

$$d\mu = d\mu(\tilde{C}, \tilde{\lambda}; H, \tau) = \frac{d\mu \exp[-Tr(\tilde{\lambda} \tilde{C}) - \tau H - \eta N]}{\int d\mu \exp[-Tr(\tilde{\lambda} \tilde{C}) - \tau H - \eta N]},$$

(5)

with the denominator enforcing the normalization condition $\int d\mu = 1$. The ensemble parameters (generalized temperatures) are the real numbers $\tau$ and $\eta$, and the anti-self-adjoint operator $\tilde{\lambda}$, chosen so that the ensemble averages

$$\langle H \rangle_{AV} = \int d\mu \rho H, \quad \langle \tilde{C} \rangle_{AV} = \int d\mu \rho \tilde{C}, \quad \langle N \rangle_{AV} = \int d\mu \rho N$$

(6)

have specified values. Since $\langle \tilde{C} \rangle_{AV}$ is itself an anti-self-adjoint operator, it can be brought to the canonical form

$$\langle \tilde{C} \rangle_{AV} = i_{eff} D_{eff}, \quad Tr(i_{eff} D_{eff}) = 0, \quad i_{eff} = -i_{eff}^1, \quad i_{eff}^2 = -1, \quad [i_{eff}, D_{eff}] = 0,$$

(7)

with $D_{eff}$ a real diagonal and non-negative operator.

The simplest case corresponds to an ensemble that does not favor any state in the underlying Hilbert space over any other, in which case $D_{eff}$ is a real constant multiple of the unit operator. This real constant has the dimensions of action, and plays the role of Planck’s constant in the emergent quantum mechanics derived from the canonical ensemble, so we shall denote it by $\hbar$, giving

$$\langle \tilde{C} \rangle_{AV} = i_{eff} \hbar, \quad Tr i_{eff} = 0.$$  

(8)

Since the relations $i_{eff} = -i_{eff}$ and $i_{eff}^2 = -1$ imply that $i_{eff}$ can be diagonalized to the form $i \text{ diag}(\pm 1, \pm 1, \ldots, \pm 1)$, the condition $\text{Tr} i_{eff} = 0$ requires that the positive and negative eigenvalues must be paired so as to give a vanishing trace. Therefore the dimension $N$ of the underlying Hilbert space must be even, say $N = 2K$, and $i_{eff}$ diagonalizes to the form

$$i_{eff} = i \text{ diag}(1, -1, 1, -1, \ldots, 1, -1),$$

(9)

with equal numbers of eigenvalues 1, −1 along the principal diagonal.

We remark now that the connection between trace dynamics and an emergent quantum theory leads to two copies of the quantum theory, one with a $K$ dimensional Hilbert space on which the effective imaginary unit is $i$, and the other with a $K$ dimensional Hilbert space on which the effective imaginary unit is $-i$, corresponding to the two ways in which $i_{eff}$ can act. This dichotomy is borne out by calculations [3] showing that, under suitable approximations, canonical ensemble averages of products of dynamical variables in trace dynamics can be put into correspondence with Wightman functions of an emergent quantum theory. For a general $N \times N$ matrix $M$, let us denote by $M_{eff}$ the part that commutes with $i_{eff}$, that is $M_{eff} = \frac{1}{2}(M - i_{eff} M i_{eff})$. Then the emergent quantum equations take the following form: for
time evolution of effective quantum operators \( x_{r \text{eff}} \) with \( x_r \) a \( q_r \) or a \( p_r \), we find the effective Heisenberg equation of motion

\[
\dot{x}_{r \text{eff}} = \frac{i \epsilon_{\text{eff}}}{\hbar} [H_{\text{eff}}, x_{r \text{eff}}].
\] (10)

For the non-vanishing effective canonical commutators of bosonic degrees of freedom, we find

\[
[q_{u \text{eff}}, p_{v \text{eff}}] = i \epsilon_{\text{eff}} \hbar \delta_{uv},
\] (11)

and for the non-vanishing effective canonical anticommutators of fermionic degrees of freedom, we find

\[
\{q_{u \text{eff}}, p_{v \text{eff}}\} = i \epsilon_{\text{eff}} \hbar \delta_{uv}.
\] (12)

On the sector where \( \epsilon_{\text{eff}} = i \), we get the usual quantum mechanical relations, while on the sector where \( \epsilon_{\text{eff}} = -i \), we get quantum mechanics with \( i \) replaced by \(-i\) in the Heisenberg equations of motion and the canonical commutation/anticommutation relations. In the concluding chapter of [3], and as elaborated in [4], we have suggested that the \(-i\) sector is a candidate for the dark matter sector of the universe.

In applications of trace dynamics to field theory, the discrete index \( r \) labeling degrees of freedom becomes a spatial coordinate label \( \vec{x} \). We show in [3] that for Lorentz invariant Lagrangians, the operator \( \hat{C} \) is a Lorentz scalar, and that there is a conserved trace stress–energy tensor of the usual form. We also show [5] that the rigid supersymmetry theories of spin 0, 1/2, and 1 fields, as well as the ‘matrix model for M-theory’, have trace dynamics extensions, whereas [3] supergravity does not have a trace dynamics extension.

3. Arguments for the metric being c-number valued in trace dynamics

We show now that to incorporate gravity into trace dynamics, the metric must be introduced as a c-number or classical field, that is, as a purely diagonal matrix. There are a number of independent arguments for this.

**Invariant volume.** Rewriting a flat spacetime theory in curved coordinates requires a spacetime volume element \( dV \) that is invariant under general coordinate transformations. The usual recipe is \( dV = d^4x(\sqrt{\det g})^{1/2} \), where the scalar density \( \sqrt{\det g} \) is given by \( \sqrt{|\det g_{\mu\nu}|} \). Under a change of coordinates \( x_\mu \rightarrow x_\mu(x') \), the scalar density transforms as \( \sqrt{\det g_{\mu\nu}} \rightarrow \sqrt{|\det J|} \sqrt{\det g_{\mu'\nu'}} \), with \( J \) the Jacobian of the transformation obeying \( |J|d^4x = d^4x' \). However, for general operator-valued metric components \( g_{\mu\nu} \), the product property of the determinant is lost. That is, the determinant of a matrix, whose elements are the matrix product of an operator-valued \( g_{\mu\nu} \) with a c-number matrix \( \partial x^\sigma / \partial x'^\nu \) is not the product of the respective determinants of the matrices. Thus, one cannot construct the appropriate invariant volume element if \( g_{\mu\nu} \) is operator-valued.

**Obstacle to trace dynamics extension of rigid supersymmetry theories.** If \( g_{\mu\nu} \) and \( (\sqrt{\det g})^{1/2} \) are operator-valued, the constructions of [5] for trace dynamics extensions of rigid supersymmetry theories fail in curved spacetime, because the metric factors do not commute with the matter fields, and prevent the cyclic permutation of matter fields inside the trace needed to verify supersymmetry.

**Supergravity.** As already mentioned, for reasons explained in section 3.4 of [3], supergravity does not admit a trace dynamics extension in which the metric and Rarita–Schwinger spinor are operator-valued quantities.
For these reasons, we are led to introduce the metric into trace dynamics as a c-number field. There has been considerable discussion in the literature of whether gravity has to be quantized. Dyson [6] argues that the Bohr–Rosenfeld argument for quantization of the electromagnetic field does not apply to gravity, and moreover, by a number of examples, shows that it is hard (perhaps not possible) to formulate an experiment that can detect a graviton. Dyson also notes that the papers of Page and Geilker [7] and Eppley and Hannah [8], which have been cited to argue that gravity must be quantized, really only show that a particular model for classical gravity coupled to quantized matter is inconsistent. Specifically, these papers consider the Møller and Rosenfeld proposal for constructing a semi-classical Einstein equation by writing

$$G_{\mu\nu} = -\frac{8\pi G}{\text{Tr}} \langle \psi | T_{\mu\nu} | \psi \rangle,$$

and argue that this construction has insurmountable problems when confronted with measurements giving rise to state vector reduction. But, as Page and Geilker note in their conclusion, while this rules out the semi-classical source postulate, it does not rule out more complicated forms of classical gravity theories.

To incorporate classical gravity into trace dynamics we proceed as follows. We start from a flat spacetime trace matter action

$$S_m = \int dt L = \int d^4x \text{Tr} L(x),$$

with $L$ an operator Lagrangian density. We then generalize this to curved spacetime in the usual fashion, by introducing a classical metric $g_{\mu\nu}$ and writing

$$S_m[g] = \int dt L = \int d^4x (\sqrt{g})^{1/2} \text{Tr} L(x; g),$$

with $L(x; g)$ the operator Lagrangian density with the classical metric $g_{\mu\nu}$ used to form covariant derivatives and to contract indices to form scalars. The total action will now be

$$S_{\text{tot}} = S_m[g] + S_g,$$

with the gravitational action $S_g$ given by

$$S_g = \frac{1}{16\pi G} \text{Tr} \int d^4x (\sqrt{g})^{1/2} R = \frac{\text{Tr}(1)}{16\pi G} \int d^4x (\sqrt{g})^{1/2} R,$$

where $G$ is the gravitational constant and $R$ is the curvature scalar. In the second line we have used the fact that since the metric is a c-number, $R$ is also a c-number, so the trace just gives a numerical factor $\text{Tr}(1)$, which is the dimension of the underlying Hilbert space. It is convenient now to divide out this factor, by writing $S_g = S_g/\text{Tr}(1)$ and $S_m = S_m/\text{Tr}(1)$. Varying the metric, we get

$$\delta S_g = -\frac{1}{16\pi G} \int d^4x (\sqrt{g})^{1/2} G_{\mu\nu} \delta g_{\mu\nu},$$

and

$$\delta S_m = -\frac{1}{2} \int d^4x (\sqrt{g})^{1/2} [T_{\mu\nu}/\text{Tr}(1)] \delta g_{\mu\nu},$$

with $T_{\mu\nu}$ the trace stress–energy tensor. Equating the metric variation of the total action to zero, we get as the trace dynamics gravitational field equations

$$G_{\mu\nu} + \frac{8\pi G}{\text{Tr}(1)} T_{\mu\nu} = 0.$$
which defining
\[ T^{\mu\nu} = \frac{T^{\mu\nu}}{\text{Tr}(1)} \] (20)
takes the usual form
\[ G^{\mu\nu} + 8\pi G T^{\mu\nu} = 0. \] (21)
Since \( T^{\mu\nu} \) obeys the covariant conservation condition
\[ \nabla_\mu T^{\mu\nu} = 0, \] (22)
equations (19)–(21) are fully consistent with the gravitational Bianchi identities
\[ \nabla_\mu G^{\mu\nu} = 0. \] (23)
Thus, if our conjecture that the underlying equations of trace dynamics give rise, at the level of thermodynamics and statistical mechanics, to both quantum theory and state vector reduction, the consistency problems that afflict the Møller–Rosenfeld semi-classical gravity theory are absent in the trace dynamics framework.

Additionally, we note that convergence of the partition function \( Z \) for the canonical ensemble requires \( H \geq 0 \) over phase space, and if we were to consider an ensemble translating with velocity \( v_i \), with \( v_i v_i/c^2 \leq 1 \), convergence of the analogous ensemble would require positivity of \( H + v_i P_i \), with \( P_i \) the trace momentum. This positivity requirement is guaranteed if the trace Hamiltonian and trace three momentum satisfy the ‘dominant energy’ condition, which is also the condition needed to prove the positive energy theorems in relativity. In the conventional approach to quantum gravity, with a quantum stress–energy tensor as the source of gravity, it has never been clear why the dominant energy condition should hold after stress–energy tensor regularization.

4. The matter-induced effective action for gravity

Even when no particulate matter sources are present, the averaged pre-quantum matter field motions can influence gravitational dynamics. This is taken into account by defining an induced gravitational action as the action calculated from the average of the matter field Lagrangian density over the canonical ensemble,
\[ S_{\text{g, induced}} = \int d^4x (^{(4)}g)^{1/2} \left[ \text{Tr}(L(x))_{AV}/\text{Tr}(1) \right], \] (24)
where \( \langle L(x) \rangle_{AV} \) denotes an average over the trace dynamics canonical ensemble \( \rho \) of equation (5),
\[ \langle L(x) \rangle_{AV} = \int d\mu \rho L(x). \] (25)
In more detail, this average is computed as follows. Writing \( x = (x^0, \vec{x}) = (t, \vec{x}) \), the Lagrangian density \( L \) at time \( t \) is defined as a function of the matter fields and their time derivatives. Labeling the matter fields, which can be bosonic or fermionic, by an index \( a \), the set of fields are \( q_a(t, \vec{x}) \) and their time derivatives are \( \dot{q}_a(t, \vec{x}) \). We can now rewrite the matter field time derivatives in terms of the corresponding canonical momenta \( p_a(t, \vec{x}) \) (for matter gauge fields, this will involve a gauge fixing), so that the Lagrangian density becomes a function of the fields and momenta. Thus at each fixed time \( t \) we can write
\[ L(x) = L(q_a(t, \vec{x}), p_a(t, \vec{x})). \] (26)
Recall that the canonical ensemble \( \rho \) and the phase space measure \( d\mu = \prod_a \prod_{\vec{x}} dq_a(t, \vec{x}) dp_a(t, \vec{x}) \) are time-independent (with \( dq \) for a complex matrix \( q \) defined
in the usual way [3] as the product of the differentials of the real and imaginary parts of the matrix elements of $q$). So at fixed time $t$, the average required by equation (25) with the Lagrangian density rewritten in the form of equation (26) is now explicitly defined. Note that the metric $g_{\mu\nu}$ is held fixed in this averaging, which leads to a functional of the metric as stated in equation (24).

The assertion that the canonical ensemble $\rho$ is time-independent needs elaboration when in curved spacetime. It requires that the three quantities in the exponent of equation (5), $H$, $N$, and $\tilde{C}$, which are constants of the motion [3] in flat spacetime, remain constants of the motion in curved spacetime. For $N$ and $\tilde{C}$ this follows from the fact (shown explicitly in appendix A) that these are charges formed from conserved currents, which generalize to covariantly conserved currents in curved spacetime. That is,

$$\tilde{C} = \int d^3x (4g)^{1/2} \tilde{C}_0,$$

$$N = \int d^3x (4g)^{1/2} N_0,$$

(27)

with $\tilde{C}_0$ and $N_0$ covariantly conserved four vector currents obeying $\nabla_\mu \tilde{C}_\mu = \nabla_\mu N_\mu = 0$. The usual identity for any contravariant vector current $V_\mu$,

$$\nabla_\mu V_\mu = (4g)^{-1/2} \partial_\mu \left[ (4g)^{1/2} V_\mu \right]$$

(28)

then shows that in curved spacetime, $\tilde{C}$ and $N$ are time-independent.

For the trace Hamiltonian more explanation is needed. In flat spacetime the canonical matter field trace Hamiltonian is defined by

$$H_m = \int d^3x \text{Tr} \sum_a p_a(t, \vec{x}) q_a(t, \vec{x}) - L.$$  

(29)

Since $1 = \delta^0_0$, the natural generalization of this to curved spacetime is

$$H_m = \int d^3x (4g)^{1/2} T^0_0(t, \vec{x})$$

(30)

with $T^\nu_\mu$ the mixed index trace stress–energy tensor. The fact that we use the mixed tensor, and not the more customary $T^{00}$, will be crucial to the global Weyl scaling argument that follows. However, it is well known that neither of these tensors defines a conserved matter Hamiltonian, because the energy of the gravitational field must be taken into account. In both cases, it is also known that one can construct a gravitational stress–energy pseudotensor, the Einstein–Dirac [9] pseudotensor $t^\nu_\mu$ in the mixed index case, and the Landau–Lifshitz [10] pseudotensor $t^{0\mu}$ in the upper index case, that yield conserved quantities. Specifically, in the mixed index case needed here, $t^\nu_\mu$ is a function solely of the metric, constructed so that

$$\partial_\nu \left[ (4g)^{1/2} (T^\nu_\mu + (\text{Tr}(1)) t^\nu_\mu) \right] = 0.$$  

(31)

Thus, when we define

$$H = \int d^3x (4g)^{1/2} \left[ T^0_0(t, \vec{x}) + (\text{Tr}(1)) t^0_0(t, \vec{x}) \right],$$

(32)

we obtain a Hamiltonian function that is conserved in a general curved spacetime, which can then be used to construct the canonical ensemble.

We shall not actually need the detailed form of $t^0_0$, because since it does not depend on the matter fields, it cancels out of the definition of the canonical ensemble between the numerator in equation (5) and the normalizing denominator. So we can then simply use $H_m$ for the Hamiltonian in the canonical ensemble, and henceforth will drop the subscript $m$. A discussion of the construction of $t^\nu_\mu$ and its useful properties is given in appendix B.
5. Constraints on the form of the induced effective action

We next address constraints on the structural form of the induced effective action implied by the structure of the canonical ensemble. We begin by noting that although $\tilde{C}$ and $N$ are Lorentz scalars, the trace Hamiltonian $H$ is the time component of a four-vector, and so the canonical ensemble picks out a preferred frame. We shall make the natural assumption that this preferred frame is the rest frame of the cosmological background radiation. However, we shall also assume that there is no other Lorentz violation present, in particular, we assume that the matter field action in curved spacetime is the usual minimal transcription a Lorentz invariant flat spacetime action.

Let us consider now a purely spatial general coordinate transformation, which leaves $x^0$ invariant. Under this transformation, $g_{00}$ transforms as a three-space scalar, $g_{0a}$ as a three-space covariant vector, and $(^{(3)}g - \det g_{ij})$ as a three-space scalar density. Expanding $(^{(4)}g)$ in a cofactor expansion written in the form

\[(^{(4)}g')^{(3)}g = g_{00} + g_{0a}D^a,\]  

we see that $D^a$ transforms as a three-space contravariant vector. Since the canonical ensemble is a three-space scalar under purely spatial general coordinate transformations, the induced effective action must share this property, and so must be a function of the three-space scalars that we can construct from the above quantities and their derivatives, times the invariant volume element $dV$. For example, the leading order effective action in an expansion in powers of derivatives of the metric must have the form

\[\Delta S_{\text{g, induced}} = \int d^4x (^{(4)}g)^{1/2}A(g_{00}, g_{0a}g^{ab}, D^4g_{ij}, g_{0a}D^a).\]  

with $A(a, b, c, d)$ a general function of its four arguments.

Further restrictions on the form of the induced action come from considering global Weyl scaling invariance. A detailed study of the Weyl scaling invariance properties of classical fields in curved spacetime has been given in an important paper by Forger and Römer [11]. In $n$-dimensional spacetime, they define global Weyl scale transformations of the metric $g_{\mu\nu}$ and the n-bein $e^a_\mu$ by the substitutions

\[
g_{\mu\nu}(x) \rightarrow \lambda^2 g_{\mu\nu}(x), \quad g^{\mu\nu}(x) \rightarrow \lambda^{-2} g^{\mu\nu}(x), \quad e^a_\mu(x) \rightarrow \lambda e^a_\mu(x), \quad e^a_\mu(x) \rightarrow \lambda^{-1} e^a_\mu(x).\]  

For a generic matter field $q(x)$ with canonical momentum $p(x)$, the corresponding transformation is

\[q(x) \rightarrow \lambda^{-w_q}q(x), \quad p(x) \rightarrow \lambda^{-w_p}p(x),\]  

with

\[
w_q = \frac{1}{2}(n - 2), \quad w_p = w_q + 2, \quad q \text{ a scalar field } (w_q = 1, w_p = 3 \text{ for four dimensions}),
\]

\[
w_q = \frac{1}{2}(n - 1), \quad w_p = w_q + 1, \quad q \text{ a Dirac spinor field } (w_q = 3/2, w_p = 5/2 \text{ for four dimensions}),
\]

\[
w_q = w_p = 0, \quad q \text{ a Yang–Mills gauge field } (w_q = w_p = 0 \text{ for four dimensions}).\]  

Thus, in this scheme, the metric $g_{\mu\nu}$ has Weyl dimension $-2$, and the n-bein $e^a_\mu$ has Weyl dimension $-1$. The necessity for giving Weyl dimension zero to Yang–Mills fields arises from the fact that the Yang–Mills field strength $F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu + A_\mu \times A_\nu$ involves both
linear and quadratic terms in the gauge potential $A_\mu$. (Forger and Römer do not consider $U(1)$ gauge fields, for which one could consistently assign a scale dimension of $w_q = \frac{1}{2}(n - 4)$ in $n$ dimensions; however, from a grand unification point of view, $U(1)$ fields arise from Yang–Mills fields, and are not present in the original matter action.)

Forger and Römer study the global Weyl transformation properties of standard actions for massless spin 0, 1/2, and 1 matter fields in curved spacetime and find the following results (which will be derived in appendix A).

1. The massless spin 0 actions, both ‘improved’ with an additional term $b(n)R\phi^2$, and ‘minimal’ without this term, are globally Weyl invariant off-shell (without use of the equations of motion) in $n$ dimensions.
2. The massless Dirac spinor action is globally Weyl invariant off-shell in $n$ dimensions.
3. The Yang–Mills gauge field action is globally Weyl invariant off-shell only in $n = 4$ dimensions.

In order to study the Weyl scaling properties of the canonical ensemble, we must additionally know the Weyl properties of the trace stress–energy tensor $T_{\mu\nu}$, as well as of $\tilde{C}$ and $N$. In appendix A we also give the formulas for $T_{\mu\nu}$ given by Forger and Römer for classical fields. (These are converted to formulas for $T_{\mu\nu}$ by reinterpreting the fields as matrix valued and symmetrizing coupling terms where there are factor-ordering ambiguities. However, since the Weyl scaling factor $\lambda$ for a classical metric is necessarily classical, the Weyl invariance calculations for both the action and the stress–energy tensor are the same for both the classical models in appendix A and their trace dynamics transcriptions.) In $n = 4$ dimensions, we find the following Weyl scaling properties.

1. From the global Weyl invariance of the matter field actions, and the definition of equation (A.3), we deduce that $(^(4)g)^{1/2}T_{\mu\nu}^\phi$ is off-shell globally Weyl invariant for the massless scalar, massless Dirac, and Yang–Mills gauge fields.
2. For matrix-valued scalar, Yang–Mills, and Dirac spinor fields, as needed to construct the current $\tilde{C}^\mu$, we find off-shell that $(^(4)g)^{1/2}\tilde{C}^\mu$ is globally Weyl invariant.
3. For matrix-valued Dirac spinor fields, as needed to construct the trace fermion number current $N^\mu$, we find off-shell that $(^(4)g)^{1/2}N^\mu$ is globally Weyl invariant.

To summarize these results, all of the on-shell conserved quantities used to form the canonical ensemble are off-shell invariant under global Weyl scalings. This means that they are globally Weyl invariant over the entire phase space that is integrated over in the canonical ensemble. Since the Weyl scaling factors cancel between the phase space measure factors $d\mu$ in equation (5), and since the matter action is globally Weyl scaling invariant, we learn that the matter induced gravitational effective action defined in equation (24) must be globally Weyl invariant. Since $D^I$ has Weyl scaling weight 0, this allows us to further restrict the functional form given in equation (34) to read

$$\Delta S_g = \int d^4x(^{(4)}g)^{1/2}(g_{00})^{-2}A(g_{00}, g^I / g_{00}, D^ID^J / g_{00}, g_{00}D^J / g_{00})$$  (38)

with $A(x, y, z)$ a general function of its three arguments. We remark that this result excludes a cosmological constant term in the induced gravitational action, which would correspond to an action

$$S_{\text{cosmological constant}} \propto \int d^4x(^{(4)}g)^{1/2}$$  (39)
that is not globally Weyl scaling invariant. (Thus, we have given here a corrected version of the argument which we initially attempted in [12].) We also remark that in the important case of metrics for which $g_{0i} = g_{0j} = D_i = 0$, equation (38) greatly simplifies to read

$$\Delta S_g = A_0 \int d^4x \frac{1}{(g_{00})^{1/2}} (g_{00}^{-2})$$

(40)

where $A_0 = A(0, 0, 0)$ is a constant factor. Similarly, when $g_{0i}$ and $D_i$ are effectively small, as for the metrics for slowly rotating bodies, we can expand $A(x, y, z)$ to first order in its arguments, giving the effective action

$$\Delta S_g = \int d^4x \frac{1}{(g_{00})^{1/2}} \left[ A_0 + (g_{00})^{-1} (B_1 g_{0i} g_{0j} g_{ij} + B_2 D_i D_j g_{ij} + B_3 g_{0i} D_i) \right]$$

(41)

6. Rules for use of the frame dependent effective action

When particulate matter (baryonic matter, dark matter, and radiation) is present, with action $S_{pm}$, the total action that we have obtained is

$$S_{\text{total}} = S_g + \Delta S_g + S_{pm}. \quad (42)$$

The familiar actions $S_g$ and $S_{pm}$ are general coordinate transformation scalars, but the induced action $\Delta S_g$ is frame dependent, and as we have seen is only invariant under the subset of general coordinate transformations that act on the spatial coordinates $\vec{x}$, but leave the time coordinate $t$ invariant. As a result, the spacetime stress–energy tensor obtained by varying $\Delta S_g$ with respect to the full metric $g_{\mu\nu}$ will not satisfy the covariant conservation condition, and thus cannot be used as a source for the full spacetime Einstein equations. However, it is perfectly consistent to use $\Delta S_g$ as the source for the spatial components of the Einstein tensor $G^{ij}$ in the preferred rest frame of the canonical ensemble, which we have assumed to be the rest frame of the cosmological background radiation. Thus we get the following rules.

1. The spatial components $G^{ij}$ of the Einstein equations are obtained by varying $S_{\text{total}}$ of equation (42) with respect to the spatial components $g_{ij}$ of the metric tensor, giving the gravitational field equations

$$G^{ij} + 8\pi G (\Delta T^{ij} + T_{pm}^{ij}) = 0,$$

(43)

with $T_{pm}^{ij}$ the spatial components of the usual particulate matter stress–energy tensor $T_{\mu\nu}^{pm}$, which is covariantly conserved, and with $\Delta T^{ij}$ given by

$$\delta \Delta S_g = -\frac{1}{2} \int d^4x \frac{1}{(g_{00})^{1/2}} \Delta T^{ij} \delta g_{ij}.$$

(44)

2. The components of the Einstein tensor $G^{00} = G^{0i} = 0$ are obtained from the Bianchi identities, with $G^{ij}$ as input, and from them we can infer the conserving extensions $\Delta T^{00}$ and $\Delta T^{0i}$ of the induced gravitational stress–energy tensor. Equivalently, we can infer these by imposing the covariant conservation condition on the full induced tensor $\Delta T^{\mu\nu}$, with $\Delta T^{00}$ as input.

3. With this interpretation, comparing equation (43) with equation (21), we see that we have defined a splitting of the trace matter stress–energy tensor into a part $\Delta T^{\mu\nu}$ that arises from the hidden averaged motions of the pre-quantum matter fields, and a part $T_{pm}^{\mu\nu}$ that arises from the observable particulate matter,

$$T^{\mu\nu} = \frac{T^{\mu\nu}_{\text{Tr}}}{\text{Tr}(1)} = \Delta T^{\mu\nu} + T_{pm}^{\mu\nu}.$$ 

(45)
These rules have an analog in statistical mechanics and condensed matter theory, where there is a large literature showing how to obtain ‘conserving approximations’ when the full equations of motion are truncated or are averaged over certain dynamical variables. Fortunately, the general relativity case just described is simpler. We shall see that for certain metrics of particular interest, such as the Robertson–Walker cosmological metric, and the static spherically symmetric metric, it is easy to write down the conserving extension of $\Delta T^{ij}$.

7. Application to Robertson–Walker cosmology

The standard $\Lambda$CDM model of cosmology, which is in excellent agreement with observational data from the WMAP and Planck satellites, is based on the Robertson–Walker line element

$$ds^2 = dt^2 - a(t)^2 \left( \frac{dr^2}{1 - kr^2} + r^2 (d\theta^2 + \sin^2 \theta \, d\phi^2) \right),$$

(46)

corresponding to the metric components

$$g_{00} = 1, \quad g_{rr} = -a(t)^2 / (1 - kr^2), \quad g_{tt} = -a(t)^2 r^2, \quad g_{\theta\phi} = -a(t)^2 r^2 \sin^2 \theta.$$  

(47)

Since $g_{0i} = g_{i0} = 0, D^i = 0$, we can use the simplified form of the induced action given in equation (40). Substituting $g_{00} = 1$, we get

$$\Delta S_g = A_0 \int d^4x \frac{1}{2} (g^{ij})^{\frac{1}{2}}.$$  

(48)

Varying the spatial components $g_{ij}$ of the metric, while taking $\delta g_{00} = \delta g_{0i} = 0$, and using $\delta (g^{ij})^{\frac{1}{2}} = \frac{1}{4} (g^{ij})^{\frac{1}{2}} g^{\mu\nu} \delta g_{\mu\nu}$, we find from equation (44) that the spatial components of $\Delta T^{ij}$ are given by

$$\Delta T^{ij} = -A_0 g^{ij}.$$  

(49)

The conserving extension of the induced gravitational stress–energy tensor for this case is obviously given by

$$\Delta T^{\mu\nu} = -A_0 g^{\mu\nu},$$  

(50)

and we see that for a homogeneous, isotropic cosmological metric, the induced term has exactly the structure of a cosmological constant! Assuming that there is no ‘bare’ cosmological constant, the induced term is to be identified with the observed cosmological constant. In this interpretation the so-called ‘dark energy’ is the energy associated with the hidden motions of the pre–quantum matter fields, and is strictly constant over the course of cosmic evolution, even as the matter sector undergoes phase transitions associated with successive stages of spontaneous symmetry breaking. Comparing with the standard form of the Einstein equations in the presence of a cosmological constant $\Lambda$,

$$G^{\mu\nu} + \Lambda g^{\mu\nu} + 8\pi G T^{\mu\nu}_{pm} = 0,$$

(51)

we identify the constant $A_0$ in equation (40) as

$$A_0 = -\frac{\Lambda}{8\pi G}.$$  

(52)

Using the relation $\Lambda = 3H_0^2 \Omega_\Lambda$, between $\Lambda$, the Hubble constant $H_0$ and the cosmological fraction $\Omega_\Lambda$, we get the alternative expression

$$A_0 = -\frac{3H_0^2 \Omega_\Lambda}{8\pi G}.$$  

(53)

We emphasize that we are inferring the value of $A_0$ from the experimentally observed cosmological constant, and so have not given an explanation of why $A_0$ is so small compared to the scale set by the Planck mass.
8. A first look at the static, spherically symmetric metric

The standard form for the static, spherically symmetric line element is

$$\mathrm{d}s^2 = B(r) \, \mathrm{d}t^2 - A(r) \, \mathrm{d}r^2 - r^2 (\mathrm{d}\theta^2 + \sin^2 \theta \, \mathrm{d}\phi^2),$$  \hspace{1cm} (54)

corresponding to the metric components

$$g_{00} = B(r), \quad g_{rr} = -A(r), \quad g_{0\theta} = -r^2, \quad g_{\phi\phi} = -r^2 \sin^2 \theta.$$  \hspace{1cm} (55)

Again we have $g_0 = g_\theta = 0$, so we can again use the simplified form of the induced action given in equations (40) and (52). Substituting $g_{00} = B(r)$, we get

$$\Delta S_k = -\frac{\Lambda}{8\pi G} \int \mathrm{d}^4x (\sqrt{-g}) \, B(r)^{-2}.$$  \hspace{1cm} (56)

Again by varying the spatial components $g_{ij}$ of the metric, while taking $\delta g_{00} = \delta g_{0\theta} = 0$, we find from equations (44) and (56) that the spatial components $\Delta T^{ij}$ are given by

$$\Delta T^{ij} = \frac{\Lambda}{8\pi G} g^{ij} B(r)^2, \quad \Delta T_{ij} = \frac{\Lambda}{8\pi G} g_{ij} B(r)^2$$  \hspace{1cm} (57)

and the Einstein equations for $G_{rr}$ and $G_{\theta\theta}$ are modified to read

$$G_{rr} - \frac{\Lambda A(r)}{B(r)^2} = 0,$$

$$G_{\theta\theta} - \frac{\Lambda r^2}{B(r)^2} = 0,$$  \hspace{1cm} (58)

with the equation for $G_{\phi\phi}$ proportional to that for $G_{\theta\theta}$. Although $\Lambda$ is very small, we see that near the horizon of a Schwarzschild black hole, where the unperturbed solution is $A(r)^{-1} = B(r) = 1 - r_s/r$ (with $r_s$ the Schwarzschild radius), the induced term becomes infinite and so may have a significant effect on the horizon structure, which we plan to study.

From the expressions for $G_{tt}$, $G_{rr}$, and $G_{\theta\theta}$, with $\delta$ denoting $\mathrm{d}/\mathrm{d}r$, and with $A \equiv A(r)$ and $B \equiv B(r)$ in equations (59)–(61),

$$G_{tt} = \frac{B}{rA} \left[ \frac{-A'}{A} + \frac{1}{r} (1 - A) \right],$$

$$G_{rr} = -\frac{B'}{rB} + \frac{1}{r^2} (A - 1),$$

$$G_{\theta\theta} = -\frac{r^2}{2A} \left[ \frac{B'}{B} - \frac{B'}{2A} \left( \frac{A'}{A} + \frac{B'}{B} \right) + \frac{1}{r} \left( -\frac{A'}{A} + \frac{B'}{B} \right) \right],$$  \hspace{1cm} (59)

we find the linear relation (the Bianchi identity)

$$G_{rr} = \frac{2\Lambda}{r^3} G_{\theta\theta} + \left( \frac{B'}{2B} + \frac{1}{r} (3 - A) \right) G_{rr} + \left( \frac{A(1 - A)}{rB} - \frac{AB'}{2B^2} \right) G_{tt} = 0.$$  \hspace{1cm} (60)

When $G_{\mu\nu}$ is replaced in this equation by the covariantly conserved $\Delta T_{\mu\nu}$ it must also be satisfied, so for the conserving extension $\Delta T_{tt}$, $\Delta T_{rr}$, and $\Delta T_{\theta\theta}$ we find

$$\Delta T_{tt} = N/D,$$

$$N = \Delta T_{rr} - \frac{2\Lambda}{r^3} \Delta T_{\theta\theta} + \left( \frac{B'}{2B} + \frac{1}{r} (3 - A) \right) \Delta T_{rr},$$

$$D = \frac{AB'}{2B^2} - \frac{A(1 - A)}{rB}.$$  \hspace{1cm} (61)
9. Some remarks

In conclusion we make some remarks, first on subtleties of our derivations, and then on speculations and possible future directions.

9.1. Remarks on the derivations

(1) Our calculation is a form of a ‘fast-slow’ calculation, in which ‘fast’ degrees of freedom (in our case, the pre-quantum matter) are averaged to get effective equations for ‘slow’ ones, in our case, the metric. In usual applications of this method, one averages the Hamiltonian instead of the action. When the action and Hamiltonian have the form (using subscripts \(f\) and \(s\) to label the fast and slow degrees of freedom)

\[
S = T - V = T(q_f) + T(q_s) - V(q_f, q_s),
\]

\[
H = T + V = T(p_f) + T(p_s) + V(q_f, q_s),
\]

and one averages over a normalized weighting of either the form \(\rho(q_f, p_f)\), or the factorized form \(\rho(p_f)\eta(q_f, q_s)\) (which includes the thermal ensemble for this system), the averages of \(T(p_f)\) and \(T(q_f)\) are constants which do not contribute to the averaged Euler–Lagrange and Hamilton equations for the slow variables. The averaged action and the averaged Hamiltonian then give the same equations of motion. However, when the fast kinetic terms depend on slow variables, and so have the form \(T(q_f, q_s)\) or \(T(p_f, q_s)\), when averaged these give extra potential-like terms for the slow variables, and the two averaging procedures are not manifestly equivalent. This corresponds to the gravitational case that we have studied, where the matter kinetic terms depend on the metric. Since the action form of gravitation is much simpler than the Hamiltonian form, we have chosen to average the action.

(2) We have focused on global Weyl invariance properties, which are enough to restrict the leading terms in the induced gravitational effective action in an expansion in powers of derivatives of the metric. As discussed in appendix A, the ‘improved’ scalar, Dirac, and Yang–Mills actions and Hamiltonians in \(n = 4\) dimensions are also invariant under local Weyl transformations with \(\lambda = \lambda(x)\), and therefore under time-independent transformations with \(\lambda = \lambda(\vec{x})\), which can still be scaled out of the canonical ensemble integration measure \(d\mu\). For Dirac and Yang–Mills fields, \((4g)^{1/2}\mathcal{L}\) is also locally Weyl invariant, while in the scalar case, \((4g)^{1/2}\mathcal{L}\) is locally invariant up to a total derivative, which does not contribute to the action. However, in the scalar case, since \(p_\phi = \delta^{\mu \nu} \partial_\mu \phi = \delta^{00} \partial_0 \phi + \delta^{ij} \partial_i \phi\), the local scaling properties of \(p_\phi\) and \(\phi\) are consistent only when \(\lambda = \lambda(\vec{x})\) and when the metric is specialized to \(g^{00} = 0\). These results place no additional restrictions on the leading non-derivative effective action terms, but can be used to place scaling restrictions on the terms in the effective action that depend on derivatives of the metric.

9.2. Speculations and possible future directions

(1) In general there will be induced corrections to the \(R\) term in the gravitational action, with a coefficient \(C\) of dimension \([\text{mass}]^2\), whereas the coefficient \(A\) of the leading term in the derivative expansion has dimension \([\text{mass}]^4\). If we make the naive estimate \(C \sim A^{-1/2}\), then the size of the correction to the \(R\) action relative to the Einstein–Hilbert action will be of order \(GA^{1/2} \sim H_0 G^{1/2} \sim 10^{-60}\), that is the ‘induced gravitational’ action is much too small to serve as the gravitational action by itself. So a fundamental \(R\) action is needed.
This suggests studying trace dynamics generalizations of various extended supergravity theories as a possible way of unifying the matter and gravity sectors.

(2) Because the parameter $\tau$ with dimension of $[\text{mass}]^{-1}$ appears in the canonical ensemble, there are two constants with dimension of $[\text{mass}]^{-1}$ present, $G^{-1/2}$ and $\tau$. We suggest that these can be related by imposing an initial condition of zero total energy $\langle H \rangle_{\text{AV}} = 0$, with $H$ including the gravitational energy as in equation (32). It is well known [13] that in a Newtonian universe, a spatially flat universe with $\Omega = 1$ corresponds to zero total energy, with the kinetic energy of matter balanced by the negative potential energy of its gravitational attraction. We show in appendix B that this statement has a general relativistic analog: in a spatially flat universe (Robertson–Walker with $k = 0$), using Cartesian coordinates, the Einstein–Dirac pseudotensor takes the locally uniform value

$$\rho_{\text{tot}} = -\frac{3}{8\pi G} \left( \frac{\dot{a}}{a} \right)^2 = -\rho_{\text{tot}} \text{,}$$

with $\rho_{\text{tot}}$ the total matter contribution to the energy density coming from the sum of the induced gravitational term $\Delta T_{00}^0$ and the particulate matter term $T_{00}^0$ in the Einstein equations. Thus in a general relativistic sense as well, the observation of $\Omega = 1$ corresponds to a zero energy condition.

Acknowledgments

I wish to thank Gerhard Grössing for inviting me to be a keynote speaker at the second international conference on ‘Emergent Quantum Mechanics’, EmQM13, to be held in Vienna 4th–6th October 2013. This invitation prompted me to return to the subject addressed in my book [3], on which I had talked at the first conference in Vienna two years ago, to try to address some of the problems that I had left unresolved. I am grateful to Freeman Dyson for sending me a pre-publication copy of his Poincaré prize lecture [6], and for e-mail correspondence about the issues he discusses in it. I also wish to thank Angelo Bassi for reading the paper. Final revisions were supported in part by the National Science Foundation under grant no. PHYS-1066293 and the hospitality of the Aspen Center for Physics.

Appendix A. Notational conventions and formulas for matter field actions and conserved quantities derived from them

A.1. Notational conventions

Since the books on gravitation and cosmology that we have consulted use many different conventions, we summarize our notational conventions here. They follow the conventions of the book of Parker and Toms [14] and the paper of Forger and Römer [11].

(1) The Lagrangian in flat spacetime is $L = T - V$, with $T$ the kinetic energy and $V$ the potential energy, and the flat spacetime Hamiltonian is $H = T + V$.

(2) We use a $(1, -1, -1, -1)$ metric convention, so that in flat spacetime, where the metric is denoted by $\eta_{\mu\nu}$, the various $00$ components of the stress–energy tensor $T_{\mu\nu}$ are equal, $T_{00} = T_{0}^{0} = T^{00}$.

(3) The affine connection, curvature tensor, contracted curvatures, and the Einstein tensor, are given by

$$\Gamma^k_{\mu\nu} = \frac{1}{2} g^{k\sigma}(g_{\sigma\nu,\mu} + g_{\sigma\mu,\nu} - g_{\mu\nu,\sigma}) \text{.}$$
\( R_{\tau\nu}^{\lambda\kappa} = \Gamma_{\tau\nu}^{\lambda\kappa} - \Gamma_{\tau\kappa}^{\lambda\nu} + \text{quadratic terms in } \Gamma, \)

\( R_{\mu\nu} = R_{\mu\nu}^{\lambda\kappa} = -\Gamma_{\mu\nu\kappa}^{\lambda} + \text{other terms}, \)

\( R = g^{\mu\nu} R_{\mu\nu}, \)

\( G_{\mu\nu} = R_{\mu\nu} = -\frac{1}{2} g_{\mu\nu} R. \)  

(A.1)

(4) The gravitational action, with cosmological constant \( \Lambda_1 \), and its variation with respect to the metric \( g_{\mu\nu} \) are

\[ S_g = \frac{1}{16\pi G} \int d^4x (4) g^{1/2} (R - 2\Lambda), \]

\[ \delta S_g = -\frac{1}{16\pi G} \int d^4x (4) g^{1/2} (G_{\mu\nu} + \Lambda g_{\mu\nu}) \delta g_{\mu\nu}. \]  

(A.2)

(5) The matter action and its variation with respect to the metric \( g_{\mu\nu} \) are

\[ S_m = \int dt L = \int d^4x (4) g^{1/2} L(x), \]

\[ \delta S_m = -\frac{1}{2} \int d^4x (4) g^{1/2} T_{\mu\nu} \delta g_{\mu\nu}. \]  

(A.3)

(6) The Einstein equations are

\[ G_{\mu\nu} + \Lambda g_{\mu\nu} + 8\pi G T_{\mu\nu} = 0. \]  

(A.4)

(7) The gravitational covariant derivative that leaves the metric invariant is denoted by \( \nabla_{\mu} \), an ordinary partial derivative by \( \partial_{\mu} \), and a covariant derivative with respect to both the metric and gauge fields, by \( D_{\mu} \).

(8) For uniformly distributed matter in the rest frame of the Robertson–Walker metric, the stress–energy tensor is

\[ T_{\mu\nu} = (p + \rho) u^\mu u^\nu - pg_{\mu\nu}, \]  

with \( p \) the pressure, \( \rho \) the mass density, and \( u^0 = 1, u^i = 0. \)

A.2. Formulas for matter field actions and conserved quantities derived from them

To calculate local Weyl scaling properties, it is convenient to write \( \lambda(x) = \exp(\omega(x)) \) and to study the effect of an infinitesimal \( \omega \). The calculations given in [11] show that under Weyl scaling of the metric and the matter fields in \( n = 4 \) dimensions, the matter Lagrangian densities all obey \( \delta \omega L = -4\omega L \) off-shell, which since \( \delta \omega ((4) g)^{1/2} = 4\omega ((4) g)^{1/2} \), implies that

\[ \delta \omega \left[ \frac{1}{2} (4) g \right] = 0. \]  

(A.6)

off-shell. This in turn implies the invariance of the corresponding action integral

\[ \delta \omega \int d^4x ((4) g)^{1/2} L = 0. \]  

(A.7)

The matter Lagrangian densities used in [11], for which these properties hold, are as follows.

(1) The ‘improved’ or ‘modified’ scalar field Lagrangian density (with \( \Box = \nabla^\mu \nabla_{\mu} \)),

\[ L_{\text{scalar}} = -\frac{1}{2} \phi \Box \phi - K \phi^4 + \frac{1}{12} R \phi^2, \]  

(A.8)
(2) the Yang–Mills gauge field Lagrangian density (with , the internal index inner product),
\[ \mathcal{L}_{\text{gauge}} = -\frac{1}{4} g^{\mu \kappa} g^{\nu \lambda} (F_{\mu \kappa}, F_{\nu \lambda}), \] (A.9)

(3) the Dirac spinor field Lagrangian density (with \( \bar{\psi} = \psi \gamma_0 \) and \( \gamma_\mu = e_\mu \gamma_a \), with \( \gamma_a \) the flat spacetime gamma matrices),
\[ \mathcal{L}_{\text{spinor}} = \frac{i}{2} g^{\mu \nu} \bar{\psi} \gamma_{\mu} \nabla_{\nu} \psi, \] (A.10)

(4) the usual renormalizable interaction Lagrangian densities \( \mathcal{L}_{\text{interaction}} \) for gauge fields coupling to scalar and spinor fields (obtained by replacing \( \nabla_\mu \rightarrow D_\mu \)) and for Dirac spinors with Yukawa couplings to scalars.

We note that for the alternative form of the modified scalar Lagrangian density, which differs only by a total derivative that does not contribute to the action,
\[ \mathcal{L}'_{\text{scalar}} = \frac{1}{2} g^{\alpha \beta} \partial_\alpha \phi \partial_\beta \phi - K \phi^4 + \frac{1}{12} R \phi^2, \] (A.11)

By varying the above Lagrangians with respect to the metric, which again requires lengthy calculations, Forger and Römer [11] calculate formulas for the corresponding stress–energy tensors \( T_{\mu \nu} \). Raising the index \( \nu \), these become

(1) The ‘modified’ scalar field stress–energy tensor
\[ T_{\nu}^{\mu}_{\text{scalar}} = \partial_\mu \phi \partial_\nu \phi - \delta_\nu^{\mu} \mathcal{L}'_{\text{scalar}} + \frac{1}{\delta_\mu} \left( \delta_\nu^{\mu} \nabla_\alpha \nabla_\alpha + R^\nu_{\mu} \right) \phi^2, \] (A.13)

(2) the Yang–Mills gauge field stress–energy tensor
\[ T_{\nu}^{\mu}_{\text{gauge}} = -g^{\kappa \lambda} (F_{\mu \kappa}, F_{\nu \lambda}) - \delta_\nu^{\mu} \mathcal{L}_{\text{gauge}}, \] (A.14)

(3) the Dirac spinor stress–energy tensor
\[ T_{\nu}^{\mu}_{\text{spinor}} = \frac{i}{4} g^{\nu \alpha} (\bar{\psi} \gamma_\alpha \nabla_\mu \psi + \bar{\psi} \gamma_\alpha \nabla_\nu \gamma_\mu \psi) - \delta_\nu^{\mu} \mathcal{L}_{\text{spinor}}. \] (A.15)

(4) the usual contribution to the stress–energy tensor arising from the interaction Lagrangian densities
\[ T_{\nu}^{\mu}_{\text{interaction}} = -\delta_\nu^{\mu} \mathcal{L}_{\text{interaction}}. \] (A.16)

By direct calculation, we have verified that the mixed component tensors \( T_{\mu \nu} \) for the modified scalar, gauge field, and spinor cases, satisfy the satisfy the following local Weyl scaling conditions,
\[ \delta_\alpha \left( \left( \frac{\sqrt{-g}}{g} \right)^{1/2} T_{\mu \nu}^{\text{scalar}} \right) = \delta_\alpha \left( \left( \frac{\sqrt{-g}}{g} \right)^{1/2} T_{\mu \nu}^{\text{gauge}} \right) = \delta_\alpha \left( \left( \frac{\sqrt{-g}}{g} \right)^{1/2} T_{\mu \nu}^{\text{spinor}} \right) = 0. \] (A.17)

Thus, in all three cases, we learn that the three-space integral
\[ \int d^3x \left( \frac{\sqrt{-g}}{g} \right)^{1/2} T_{\mu \nu} \] (A.18)
is Weyl scale invariant for general time-independent but space dependent \( \omega(\vec{x}) \). The global Weyl invariance specialization of these results can again be read off from the expressions for the stress–energy tensors without detailed calculation.
These local invariance results for the stress–energy tensor can also be deduced from the local Weyl variation of the action by the following argument. Since the order of variations can be interchanged, we have

$$\delta \omega \delta g_{\mu \nu} = \delta g_{\mu \nu} \delta \omega.$$  \hspace{1cm} (A.19)

Applying the right-hand side to the product of \((4g)^{1/2}\) with the Lagrangian density, and using equation (A.6) we get

$$\delta g_{\mu \nu} \delta \omega \left[ \left( (4g)^{1/2} \right) L \right] = 0.$$  \hspace{1cm} (A.20)

Now in general, the metric variation of \(\left( (4g)^{1/2} \right) L \) has the form

$$\delta g_{\mu \nu} \left[ \left( (4g)^{1/2} \right) T^\alpha_{\mu} \right] \xi^\alpha = -2 \delta \omega \partial^\kappa \Sigma^\kappa (\xi, \eta).$$  \hspace{1cm} (A.24)

In the gauge field and Dirac spinor cases the total derivative term \(\Sigma^\kappa\) vanishes (for the spinor, this requires a lengthy calculation given in [11]), and so using the fact that \(\xi\) and \(\eta\) are arbitrary, we learn that

$$\delta \omega \left[ \left( (4g)^{1/2} T^\alpha_{\mu} \right) \right] = 0.$$  \hspace{1cm} (A.25)

which is the result obtained by direct calculation. In the scalar case, several integrations by parts are needed to get from the variation of the action to the stress–energy tensor, so \(\Sigma^\kappa\) is nonzero. To get the detailed form of \(\delta \omega \Sigma^\kappa\), one must calculate the surface term \(\Sigma^\kappa\), which we have done as an independent check on the scalar case results stated above, but which involved considerable effort.

We stress again that all of the above Weyl scaling results are valid ‘off-shell’, that is without use of the equations of motion. The main focus of Forger and Römer [11] was not on Weyl scaling of the stress–energy tensor, but rather on the connection between scale invariance and vanishing of the stress–energy tensor. Here the equations of motion are used in their theorem 5.1: “‘on-shell’, that is, assuming the matter fields to satisfy their equations of motion, the matter field action is locally Weyl invariant if and only if the corresponding energy–momentum tensor is traceless.”

We have stated the previous results in terms of classical Lagrangian densities. But as noted in the text, if these are generalized to trace dynamics Lagrangian densities by making the fields matrix valued, adding an overall trace over the underlying Hilbert space, and symmetrizing Yukawa coupling terms where needed, all of the manipulations described above go through for a classical metric \(g_{\mu \nu}\). The only change will be that \(T^\alpha_{\mu}\) becomes the trace Hamilton density \(H = Tr T^\alpha_{\mu}\), and so the trace Hamiltonian given by equation (30) is Weyl scale invariant.

The spinor number current \(N^\alpha = \bar{\psi} \gamma^\alpha \psi\), which obeys \(\nabla_\mu N^\alpha = 0\) on shell, clearly obeys \(\delta \omega N^\alpha = -4\omega N^\alpha\) off-shell, so the conserved fermion number \(N = \int d^4x (4g)^{1/2} N^\alpha\) is Weyl scale invariant off-shell. These statements immediately carry over to the trace dynamics generalization \(N\). We digress to remark that the corresponding scalar quantity \(M = \bar{\psi} \psi\) obeys \(\delta \omega M = -3\omega M\), and so the mass-like action term formed from this, \(\int d^4x (4g)^{1/2} M\) is not Weyl invariant.
scale invariant. Hence we expect that when spinor source terms are introduced, the induced effective action for the spinor sources will not acquire a mass term, in direct analogy with the exclusion of a true cosmological constant term in the gravitational effective action. We expect this to play an important role in the application of trace dynamics to building models unifying the standard model of particle physics with gravitation, since it will extend the class of models in which the generation of Planck scale masses is forbidden.

Returning to the conserved quantities appearing in the trace dynamics canonical ensemble, we consider finally the current \( \tilde{C}^\mu \) associated with the conserved operator \( \tilde{C}^\nu \). These have the interpretation as the on-shell conserved current and charge associated with global \( U(N) \) invariance of the trace dynamics action. The current \( \tilde{C}^\mu \) is easily calculated by replacing all matrix fields \( q \) by the commutator \( [/Lambda_1, q] \) and isolating the term \( \text{Tr} \partial_\mu /Lambda_1 C^\mu \). Applying this recipe to the trace dynamics generalizations of the actions given above, we find the following.

1. For the scalar field, we have
   \[
   \tilde{C}^\mu = g^\mu\nu [\phi, \partial_\nu \phi],
   \tilde{C}^0 = g^0\nu [\phi, \partial_\nu \phi] = [\phi, p_\phi].
   \] (A.26)

2. For the Yang–Mills field, we have
   \[
   \tilde{C}^\mu = - [A_\lambda, F^\mu\lambda],
   \tilde{C}^0 = - [A_\lambda, F^{0\lambda}] = [A_\lambda, p_{A_\lambda}].
   \] (A.27)

3. For the Dirac spinor field, we have
   \[
   \tilde{C}^\mu = - i \{ \bar{\psi} \gamma^\mu, \psi \},
   \tilde{C}^0 = - i \{ \bar{\psi} \gamma^0, \psi \} = \{ \psi, p_\psi \}.
   \] (A.28)

In all three cases we see that off-shell \( \delta_\omega \tilde{C}^\mu = -4\omega \tilde{C}^\mu \), and so
\[
\delta_\omega \left( (4g)^{1/2} \tilde{C}^\mu \right) = 0,
\] (A.29)
which implies the Weyl scaling invariance of the conserved charge \( \int d^3 x (4g)^{1/2} \tilde{C}^0 \) appearing in the canonical ensemble.

Appendix B. Construction and properties of the mixed index gravitational pseudotensor

We show here how to construct a mixed index gravitational pseudotensor with the following properties.

1. When the metric is written as \( g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \), with \( h_{\mu\nu} \) not necessarily small, the tensor is quadratic or higher order in \( h_{\mu\nu} \).

2. The pseudotensor obeys the conservation law \( \partial_\nu \left( \sqrt{g} (T^\nu_\mu + t^\nu_0) \right) = 0 \), where for purposes of this appendix we abbreviate \( \sqrt{g} \equiv (4g)^{1/2} \).

3. For an isolated system, the three-space integral \( \int d^3 x \sqrt{g} (T^0_\mu + t^0_0) \) gives the usual four-momentum \( P_\mu \) defined by the asymptotic solution.

4. The matter component of the total energy, and the gravitational component of the total energy \( \int d^3 x \sqrt{g} t^0_0 \), are both invariant under three-space coordinate transformations \( t \rightarrow t \), \( \vec{x} \rightarrow \vec{x}(\vec{x}) \).

5. The total gravitational energy for an isolated system is equal to that calculated from any pseudotensor obeying properties (1) and (2), such as the Einstein–Dirac pseudotensor \( t^\mu_{\mu, \text{ED}} \), that is, \( \int d^3 x \sqrt{g} t^0_0 = \int d^3 x \sqrt{g} t^0_0_{\text{ED}} \).
6. For linear coordinate transformations, the Einstein–Dirac pseudotensor transforms as a tensor.

7. For the special case of a spatially flat universe using Cartesian coordinates, the Einstein–Dirac pseudotensor is spatially uniform, taking the form of equation (63), and has a local physical significance.

To prove these statements, we follow a constructive procedure given by Weinberg [15], with modifications appropriate to the mixed index case and to include factors of \( \sqrt{g} \) where needed. We start from the mixed index form of the Einstein equations,

\[
G_{\mu\nu} = -8\pi G T_{\mu\nu}, \tag{B.1}
\]

and separate \( G_{\mu\nu} \) into a part \( G^{(1)}_{\mu\nu} \) and a remainder \( \Delta G_{\mu\nu} \),

\[
G_{\mu\nu} = G^{(1)}_{\mu\nu} + \Delta G_{\mu\nu}, \tag{B.2}
\]

so that \( \Delta G \) is quadratic (and higher) order in \( h_{\mu\nu} \). Adopting the convention that indices on first order quantities like \( h_{\mu\nu}, \partial_{\mu} \) and \( G^{(1)}_{\mu\nu} \) are raised and lowered with \( \eta_{\mu\nu} \), we have explicitly

\[
G^{(1)}_{\mu\nu} = \eta_{\nu\kappa} G^{(1)}_{\mu\kappa},
\]

\[
G^{(1)}_{\mu\kappa} = R^{(1)}_{\mu\kappa} - \frac{1}{2} \eta_{\mu\kappa} R^{(1)}_{\lambda\lambda}, \tag{B.3}
\]

and with (see equation (7.6.2) of [15])

\[
R^{(1)}_{\mu\kappa} = \frac{1}{2} \left( \frac{\partial^2 h^\lambda_{\mu}}{\partial x^\lambda \partial x^\nu} - \frac{\partial^2 h^\lambda_{\nu}}{\partial x^\lambda \partial x^\mu} - \frac{\partial^2 h^\lambda_{\kappa}}{\partial x^\lambda \partial x^\mu} + \frac{\partial^2 h_{\mu\kappa}}{\partial x^\mu \partial x^\kappa} \right). \tag{B.4}
\]

We make a similar splitting for \( \sqrt{g} \), by writing

\[
\sqrt{g} = 1 + \Delta \sqrt{g}, \tag{B.5}
\]

so that \( \Delta \sqrt{g} \) is at least linear in \( h_{\mu\nu} \).

Multiplying equation (B.1) by \( \sqrt{g} \) and doing some algebraic rearrangement, it can be rewritten in the form

\[
G^{(1)}_{\mu\nu} = -8\pi G \sqrt{g} (T^{\nu}_{\mu} + t^{\nu}_{\mu}), \tag{B.6}
\]

with

\[
t^{\nu}_{\mu} = \frac{1}{8\pi G} \left[ \Delta G^{\nu}_{\mu} + \frac{\sqrt{g}}{\sqrt{g}} G^{(1)}_{\mu\nu} \right]. \tag{B.7}
\]

By construction, \( t^{\nu}_{\mu} \) is at least quadratic in \( h^{\nu}_{\mu} \), and since \( G^{(1)}_{\mu\nu} \) obeys the linearized Bianchi identity

\[
\partial_{\nu} G^{(1)}_{\mu\nu} = 0, \tag{B.8}
\]

we have

\[
\partial_{\nu} \left[ \sqrt{g} (T^{\nu}_{\mu} + t^{\nu}_{\mu}) \right] = 0. \tag{B.9}
\]

This construction completes the demonstration of properties (1) and (2) listed above. We note, however, that we cannot rewrite \( t^{\nu}_{\mu} \) as a symmetric tensor by lowering the index \( \nu \) with \( \eta_{\nu\kappa} \), because while this turns \( G^{(1)}_{\mu\nu} \) into a symmetric tensor \( G^{(1)}_{\mu\kappa} \), the quantity \( \Delta G^{\nu}_{\mu} \) is the difference of a tensor \( G^{(1)}_{\mu\nu} \) that needs the full metric \( g_{\nu\kappa} \) to lower the index \( \nu \) to give a symmetric tensor, and of \( G^{(1)}_{\mu\nu} \).

Following the discussion in [15], we now use the fact that \( G^{(1)}_{\mu\nu} = \eta_{\mu\kappa} G^{(1)}_{\nu\kappa} \) can be written as a total divergence,

\[
G^{(1)}_{\mu\nu} = \partial_{\nu} (\eta_{\mu\kappa} Q^{\nu\kappa}), \tag{B.10}
\]
with \( Q^{\rho \nu \lambda} \) given by equation (7.6.19) of [15],
\[
Q^{\rho \nu \lambda} = \frac{1}{2} \left( \frac{\partial h^{\rho \nu}}{\partial x^\lambda} - \frac{\partial h^{\rho \lambda}}{\partial x^\nu} + \frac{\partial h^{\nu \lambda}}{\partial x^\rho} - (\nu \leftrightarrow \rho) \right). \tag{B.11}
\]

Since \( Q^{\rho \nu \lambda} \) is antisymmetric in \( \nu \) and \( \rho \), the contracted Bianchi identity of equation (B.8) is automatically satisfied. Let us now form the volume integral
\[
\int d^3x \sqrt{g}(T^{\mu}_\mu + i^{\mu}_\mu) = -\frac{1}{8\pi G} \int d^3x G^{(1)\nu}_\mu = -\frac{1}{8\pi G} \int d^3x \eta_{\mu\nu} \partial_\nu Q^{\rho \nu \lambda}. \tag{B.12}
\]

Defining the total four momentum by \( P_\mu = \eta_{\mu\lambda} P^\lambda \), we have
\[
P^\mu = \eta^{\lambda\mu} \int d^3x \sqrt{g}(T^{\mu}_\nu + i^{\mu}_\nu) = -\frac{1}{8\pi G} \int d^3x \partial_\nu Q^{\rho \nu \lambda} = -\frac{1}{8\pi G} \int d^3x \partial_{\nu}Q^{0\nu\dot{\rho}}, \tag{B.13}
\]

with the surface integral on the second line evaluated over the sphere at spatial infinity. This demonstrates property (3) listed above. Note that once we have identified \( \frac{1}{8\pi G} \partial_{\nu}Q^{0\nu\dot{\rho}} \) as an expression of energy–momentum density, we can similarly define a total angular momentum by
\[
J^{\rho \nu \lambda} = -\frac{1}{8\pi G} \int d^3x (x^\nu \partial_\lambda Q^{0\nu\dot{\rho}} - x^\lambda \partial_\nu Q^{0\nu\dot{\rho}}), \tag{B.14}
\]

and convert it to a surface integral over the sphere at infinity. But because \( i^{\mu}_\mu \) is not symmetric in its indices, the integrand in this equation cannot be rewritten in terms of a locally conserved angular momentum four vector current density constructed from \( Q^{\rho \nu \lambda} \).

Property (4) is a consequence of the facts that \( \int d^3x \sqrt{g}P^0_i \) is invariant under three-space coordinate transformations that keep the time \( t \) fixed, since \( d^3x \sqrt{g} \) and \( T^{\mu}_\mu \) both are invariant under these transformations, and that for an isolated system with an asymptotically flat metric, the total energy \( P^0 \) defined by the spatial integral in equation (B.13) is also invariant under spatial coordinate transformations in the interior (non-asymptotic) region. Hence \( \int d^3x \sqrt{g}P^0_i = P^0_i - \int d^3x \sqrt{g}T^{\mu}_\mu \) is invariant under such spatial coordinate transformations. This was Dirac’s [9] motivation for including the \( \sqrt{g} \) factor in his definition of the gravitational stress–energy tensor.

The construction we have given for \( i^{\mu}_\mu \) is not unique. Suppose there is another \( \tilde{i}^{\mu}_\mu \) that is at least quadratic in \( h_{\mu\nu} \) and obeys \( \partial_\nu[\sqrt{g}(T^{\mu}_\nu + \tilde{i}^{\mu}_\nu)] = 0 \). Forming the difference \( \Delta i^{\mu}_\mu = \tilde{i}^{\mu}_\mu - i^{\mu}_\mu \), we have \( \partial_\nu[\sqrt{g}\Delta i^{\mu}_\mu] = 0 \), which implies that
\[
\sqrt{g}\Delta i^{\mu}_\mu = \partial_\rho D^{\rho \mu}_\mu, \tag{B.15}
\]

with \( D^{\rho \mu}_\mu \) antisymmetric in \( \nu \) and \( \rho \). Then calculating the corresponding total gravitational energy difference, we have
\[
\Delta P_\mu \propto \int d^3x \sqrt{g}\Delta i^{\mu}_\mu = \int d^3x \partial_\mu D^{\rho \mu}_\mu = \int d^3x \partial_\mu D^{\rho \mu}_\mu = \int d\Sigma D^{\rho \mu}_\mu = 0, \tag{B.16}
\]

since the fact that \( \Delta i^{\mu}_\mu \) is of quadratic or higher order in \( h_{\mu\nu} \) implies that for an isolated system, the surface integral at spatial infinity vanishes. Hence one obtains the same total gravitational energy momentum from either \( i^{\mu}_\mu \) or \( \tilde{i}^{\mu}_\mu \), even though they define different local energy–momentum distributions. A particular elegant choice of \( \tilde{i}^{\mu}_\mu \) has been given by Einstein and
Dirac [9], and so we have demonstrated property (5) stated above. That is, the Einstein–Dirac pseudotensor $t^\nu_{\mu \text{ED}}$ given by

$$t^\nu_{\mu \text{ED}} = \frac{1}{16\pi G (^{(4)}g)^{1/2}} \left[ (^{(4)}g)^{1/2} \Gamma^\nu_{\mu \alpha} - \delta^\nu_\mu \Gamma^\alpha_{\nu \sigma} \right] - \delta^\nu_\mu (^{(4)}g)^{1/2} \left[ \Gamma^\alpha_{\nu \sigma} \Gamma^\rho_{\sigma \rho} - \Gamma^\sigma_{\nu \sigma} \Gamma^\rho_{\rho \rho} \right],$$

(B.17)

yields the same total $P_\mu$ for an isolated system as the $t^\nu_{\mu}$ constructed above, which we have shown gives the usual asymptotically defined energy–momentum for an isolated system. Subtracting the matter energy–momentum, it also yields the same total gravitational contribution to $P_\mu$ as any pseudotensor obeying properties (1) and (2).

Since the Einstein–Dirac pseudotensor is constructed in terms of the affine connection, it transforms as a tensor when the affine connection transforms as a tensor. Because the inhomogeneous terms in the transformation of the affine connection under a coordinate transformation arise from second derivatives of the coordinate transformation, in the special case of linear coordinate transformations, the Einstein–Dirac pseudotensor transforms as a tensor. This is property (6).

Let us now consider a spatially flat Robertson–Walker universe ($k = 0$), for which the line element in Cartesian coordinates is simply

$$ds^2 = dt^2 - a(t)^2 (dx^2 + dy^2 + dz^2),$$

(B.18)

corresponding to the metric components

$$g_{00} = 1, \quad g_{0i} = g_{i0} = 0, \quad g_{ij} = -a(t)^2 \delta_{ij}.$$  

(B.19)

An easy calculation shows that the only non-vanishing affine connection components are

$$\Gamma^i_{0j} = -\dot{a} \delta^i_j, \quad \Gamma^0_{ij} = \alpha \dot{a} \delta_{ij}.$$ 

(B.20)

A further easy calculation then shows that the Einstein–Dirac pseudotensor takes the spatially uniform value

$$t^0_{0 \text{ED}} = -\frac{3}{8\pi G} \left( \frac{\dot{a}}{a} \right)^2,$$

$$t^i_{0 \text{ED}} = 0,$$

(B.21)

which is property (7). The spatial uniformity of this result could have been anticipated from property (6), since the isometries of the metric of equation (B.19), which are spatial translations and rigid spatial rotations, are realized as linear coordinate transformations. Since the Friedmann equations tell us that

$$\left( \frac{\dot{a}}{a} \right)^2 + \frac{k}{a^2} = \frac{8\pi G}{3} \rho_{\text{tot}},$$

(B.22)

with $\rho_{\text{tot}}$ the total matter contribution to the energy density, we see that for $k = 0$ the sum of the gravitational and matter energy densities is zero. (A related, more complicated calculation has been given by Mitra [16] starting from the Einstein form of the pseudotensor. He concludes that a spatially flat universe has zero total energy when it is static, but we do not find this restriction. We have not analyzed the reason for the discrepancy between his result and ours.)

For further discussion and properties of the mixed index pseudotensor, see [17] and [18].

References

[1] Adler S L and Millard A C 1996 Nucl. Phys. B 473 199 (other papers are cited in [3])
[2] Adler S L and Kempf A 1998 J. Math. Phys. 39 5083
[3] Adler S L 2004 Quantum Theory as an Emergent Phenomenon: The Statistical Mechanics of Matrix Models as the Precursor of Quantum Field Theory (Cambridge: Cambridge University Press)

[4] Adler S L 2013 ‘Shadow dark matter as a manifestation of $i \leftrightarrow -i$ symmetry in pre-quantum trace dynamics’, honorable mention in the 2013 gravitation essay competition Int. J. Mod. Phys. D at press

[5] Adler S L 1997 Nucl. Phys. B 499 569

Adler S L 1997 Phys. Lett. B 407 229

[6] Dyson F J 2012 ‘Is a graviton detectable?’, Poincaré prize lecture at press

[7] Page D N and Geilker C D 1981 Phys. Rev. Lett. 47 979

[8] Eppley K and Hannah E 1977 Found. Phys. 7 51

[9] Dirac P A M 1996 General Theory of Relativity (Princeton, NJ: Princeton University Press) sections 31 and 32

[10] Landau L and Lifshitz E 1951 The Classical Theory of Fields (Reading, MA: Addison-Wesley) section 11–9

[11] Forger M and Römer H 2004 Ann. Phys. 309 306

[12] Adler S L 1997 Gen. Rel. Grav. 29 1357

[13] Mukhanov V 2005 Physical Foundations of Cosmology (Cambridge: Cambridge University Press) section 1.2

[14] Parker L E and Toms D J 2009 Quantum Field Theory in Curved Spacetime (Cambridge: Cambridge University Press)

[15] Weinberg S 1972 Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity (New York, NY: Wiley) pp 165–71

[16] Mitra S 2010 Gen. Rel. Grav. 42 443

[17] Goldberg J N 1958 Phys. Rev. 111 315

[18] Bergmann P G 1958 Phys. Rev. 112 287