Wilson–like fermions and the static $B_B$ parameter with no chirality breaking mixings 

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I consider the recent proposal by R. Frezzotti and G. Rossi to chirally improve Wilson fermions in such a way that mixings among operators of different chirality can be excluded. The method, which is based on the use of twisted mass QCD with several replica of valence quarks, is extended here to static-light systems. The operators relevant for the computation of the $B_B$ parameter (in the static approximation) are discussed. In this case the same renormalization pattern as for Ginsparg-Wilson fermions is obtained by a simple modification of the discretization of the action for valence quarks.

1. INTRODUCTION

The $B_B$ parameter describes $B$ → $\overline{B}$ oscillations and it is an important quantity in the analyses of the CKM unitarity triangle. It is defined through the matrix element (between the states $B$ and $\overline{B}$) of the $\Delta b = 2$ effective weak Hamiltonian operator $O_{VV+AA}$. Here I adopt the notation

$$O_{\gamma Z} = \langle \overline{b} \gamma \gamma Z q \rangle \langle b \Gamma q \rangle ,$$

where $q$ denotes the light (down or strange) quark and $\Gamma_X = 1, \gamma_5, \gamma_\mu, \gamma_\mu \gamma_5$ for $X = S, P, V, A$ respectively. A precise theoretical prediction for $B_B$ would provide a stringent test of the Standard Model. Given the definition in terms of an hadronic matrix element, this prediction has to be non-perturbative, e.g. from the lattice.

Heavy-light systems intrinsically involve scales differing by several orders of magnitude. This problem hampers direct simulations on the lattice. An alternative approach consists in the use of an effective theory like HQET. It is derived through a formal expansion of the QCD Lagrangian in powers of $1/m_b$. The $\mu$ denotes the renormalization scale in the HQET. The functions $C_{L}(m_b, \mu)$ and $C_{S}(m_b, \mu)$ have been computed at NLO in the MS scheme in [1].

2. LATTICE ACTIONS

The lattice discretizations of the static quark action are all derived from the Eichten-Hill action [2]

$$S_{\text{stat}} = \sum_{x} \left[ \bar{h}^{(+)}(x) \nabla_{\hat{0}}^{+} h^{(+)}(x) + \bar{h}^{(-)}(x) \nabla_{\hat{0}}^{-} h^{(-)}(x) \right] ,$$

where $\nabla_{\hat{0}}^{+}$ and $\nabla_{\hat{0}}$ are the covariant backward and forward derivatives respectively. The field $h^{(+)}$ annihilates a static quark, whereas $h^{(-)}$ creates a static anti-quark. They satisfy the constraints

$$\frac{1 + \gamma_0}{2} h^{(+)} = h^{(+)} , \quad \frac{1 - \gamma_0}{2} h^{(-)} = h^{(-)} .$$

For the action in eq. (3) the heavy quark spin symmetry (HQS) and the local conservation of heavy quark flavor number are realized at finite lattice spacing. HQS in particular played an important rôle in discussing the mixing pattern of the operator $O_{VV+AA}$ on the lattice [3]. It is the invariance of the action under the SU(2) rotations

$$h^{(\pm)} \rightarrow V(\phi^{(\pm)}) h^{(\pm)} , \quad \bar{h}^{(\pm)} \rightarrow \bar{h}^{(\pm)} V(\phi^{(\pm)}) \ ,$$

with $V = \exp(-i\phi \varepsilon_{ijk} \sigma_{jk})$, and transformation parameters $\phi_i$. Concerning rotational invariance, only discrete spatial rotations remain symmetries.
of the static action. The same set of symmetries is preserved by the statistically improved static actions proposed in \(\text{[1]}\). The following discussion goes through unchanged if those actions are used.

Moving to the action for the light quarks, the renormalization of the operator \(O_{VV+AA}\) has been discussed in \(\text{[3]}\) for the Wilson action and for Overlap fermions \(\text{[5]}\). The latter fulfill the Ginsparg-Wilson relation and they therefore exhibit an exact chiral symmetry on the lattice. To fix the notation, correlation functions of the type

\[
C_{\chi\psi}(x,y) = \langle \bar{\psi} \gamma_5 \sigma^{\mu\nu} T^a \psi \rangle (x) \langle \bar{\psi} \gamma_5 \sigma^{\mu\nu} T^a \psi \rangle (y),
\]

with \(\gamma_5 = (\gamma^1 \gamma^2 \gamma^3 \gamma^0)\), will be considered, as they provide the relevant matrix elements to compute \(B_\chi\) in the static approximation.

Considering a basis of parity even \(\Delta\)-operators: \(\{O_{VV+AA},O_{SS+PP},O_{VV-AA},O_{SS-PP}\}\), the main result in \(\text{[3]}\) is that for Wilson fermions HQS and \(O(3)\) symmetries constrain the mixings under renormalization in this basis to be described by the matrix \(Z\)

\[
Z = \begin{pmatrix}
Z_{11} & 0 & Z_{13} & 2Z_{13} \\
Z_{13}^{-1}Z_{22} & Z_{22} & Z_{23} & -Z_{13} - 2Z_{23} \\
Z_{31} & Z_{32} & Z_{33} & Z_{34} \\
Z_{23}^{-1}Z_{34} & Z_{34} & Z_{33} & Z_{33}
\end{pmatrix},
\]

whereas for Overlap fermions chiral symmetry rules out the mixings among operators of different chirality, yielding

\[
Z = \begin{pmatrix}
Z_{11} & 0 & 0 & 0 \\
Z_{13}^{-1}Z_{22} & Z_{22} & 0 & 0 \\
0 & 0 & Z_{33} & Z_{34} \\
0 & 0 & Z_{34}^{-1} & Z_{33}
\end{pmatrix}.
\]

3. tmQCD

Twisted mass QCD (tmQCD) has been introduced in \(\text{[6]}\), where it has been proven to be a legal regularization for QCD with two degenerate flavors. Choosing the twisting angle \(\omega = \pi/2\), in the physical basis the fermionic action for a doublet of quarks \(\psi = \begin{pmatrix} q \\ q' \end{pmatrix}\) reads

\[
S_{tm} = \sum_i \bar{\psi} i\gamma_5 \gamma_\tau \left[ \frac{1}{2} \gamma_\mu (\nabla^*_\mu + \nabla_\mu) + \frac{r}{2} \nabla_\mu \nabla_\mu - M_{ct}(r) \right] + m_q \psi(x),
\]

and it looks like a simple modification of the Wilson action, it just amounts to chirally twisting the Wilson term. In eq. \(\text{[9]}\) \(M_{ct}\) is the usual counter-term due to the Wilson term while \(m_q\) is the bare, multiplicatively renormalizable, quark mass. It is easy to see that in the case \(m_q = 0\) the action in eq. \(\text{[9]}\) is invariant under axial transformations with generators \(\tau_1\) and \(\tau_2\), while axial rotations generated by \(\tau_3\) change the action by cutoff effects. In particular the massless action is invariant under the finite chiral rotations

\[
\psi \rightarrow i\gamma_5 \gamma_\tau \psi \quad \text{or} \quad \tau_1 \rightarrow \tau_2 ,
\]

\[
\bar{\psi} \rightarrow i\bar{\psi} \gamma_5 \gamma_\tau \quad \text{or} \quad \tau_1 \rightarrow \tau_2 .
\]

On the other hand vector-flavor symmetry is in general broken, only \(\tau_3\)-vector rotations are exactly conserved. Thus this simple modification of Wilson regularization doesn’t change the number of exactly conserved vector/axial transformations, which is \(3\) in both cases. The consequences of flavor symmetry breaking have been theoretically investigated in \(\chi\)PT in \(\text{[7,8,9]}\) while a numerical study for example of the splitting between \(\pi^0\) and \(\pi^\pm\) is still missing.

Concerning renormalization, having a subset of the chiral symmetry exactly preserved should simplify the mixings. To show that for \(B_\chi\) in the static approximation this is really the case I closely follow \(\text{[10]}\). There it has been shown that using tmQCD with several replica of the valence quarks the chirality breaking mixings for a large set of 4-fermion operators can be ruled out. The exact number of replica to be introduced depends in general on the quantity of interest. We will see that the case I’m discussing here turns out to be among the simplest ones, as only the action for one doublet as in eq. \(\text{[9]}\) needs to be considered.

As rotational \(O(3)\) invariance and HQS are preserved by the twisting, the starting point is the matrix \(Z\) in eq. \(\text{[6]}\). For the moment I focus on the matrix element of the operator \(O_{VV+AA}\). Making use of Wick’s theorem one can show that the same (up to cutoff effects) matrix element can be extracted from the correlation function

\[
C_{\chi\chi}(x,y) = \langle \bar{\psi} \gamma_5 \sigma^{\mu\nu} T^a \psi \rangle (x) \langle \bar{\psi} \gamma_5 \sigma^{\mu\nu} T^a \psi \rangle (y),
\]

\[
\psi \rightarrow \bar{\psi}(r)(\bar{\psi} \gamma_5 \sigma^{\mu\nu} T^a \psi)(y) ,
\]

\[
\bar{\psi} \rightarrow \bar{\psi}(r)(\bar{\psi} \gamma_5 \sigma^{\mu\nu} T^a \psi)(x) ,
\]
with $Q_{YZ}$ symmetrised under $q \leftrightarrow q'$

$$Q_{YZ} = (\hat{h}^{(+)} \Gamma_y q)(\hat{h}^{(-)} \Gamma_z q') + (\hat{h}^{(+)} \Gamma_y q')(\hat{h}^{(-)} \Gamma_z q).$$

(12)

In addition to the mixings with $Q_{VV-\AA}$ and $Q_{SS-\PP}$, the operators of opposite parity $Q_{VA \pm AV}$ and $Q_{SP \pm PS}$ need to be considered due to the parity breaking induced by the tmQCD action. Let's introduce the transformations:

- $Ex_5$, which already appeared in [10]

\[ q \to -i\gamma_5 q', \quad \bar{q} \to -i\bar{q}' \gamma_5, \quad q' \to +i\gamma_5 q, \quad \bar{q}' \to +i\bar{q} \gamma_5. \quad (13) \]

It maps $S_{tm}(m_q)$ onto $S_{tm}(-m_q)$.

- $\mathcal{P}_{\pi/2}(x P = (-\vec{P}, x_0))$

\[ U_0(x) \rightarrow U_0(x P), \]

\[ U_k(x) \rightarrow U_k(x P - \hat{k}), \]

\[ q(x) \rightarrow i\gamma_5 \bar{q}(x P), \]

\[ \bar{q}(x) \rightarrow i\bar{q}(x P) \gamma_5, \]

\[ h^{(\pm)}(x) \rightarrow \gamma_5 h^{(\pm)}(x P) \gamma_5, \]

\[ \bar{h}^{(\pm)}(x) \rightarrow \bar{h}^{(\pm)}(x P) \gamma_5. \quad (14) \]

and analogously for $q'$ and $\bar{q}'$. Again its effect on $S_{tm}$ is a change in the sign of $m_q$.

- $\mathcal{P}_{\pi/2}'$, same as $\mathcal{P}_{\pi/2}$ except for

\[ h^{(\pm)}(x) \rightarrow \gamma_5 h^{(\pm)}(x P), \]

\[ \bar{h}^{(\pm)}(x) \rightarrow \bar{h}^{(\pm)}(x P) \gamma_5. \quad (15) \]

The $Q$-operators have been constructed to have a definite parity under these transformations, indeed $Ex_5^2 = \mathcal{P}_{\pi/2}^2 = \mathcal{P}_{\pi/2}'^2 = 1$. Parities are summarised in table 1. The action in eq. (9) on the other hand is invariant under $Ex_5 \times \mathcal{P}_{\pi/2}$ and $Ex_5 \times \mathcal{P}_{\pi/2}'$. Thus $Ex_5 \times \mathcal{P}_{\pi/2}$ can be used to exclude mixings of $Q_{VV+\AA}$ with $Q_{VV-\AA}$, $Q_{SS-\PP}$, $Q_{AV+\VA}$ and $Q_{SP+PS}$, while $Ex_5 \times \mathcal{P}_{\pi/2}'$ rules out $Q_{AV-\VA}$ and $Q_{SP-PS}$. The arguments can be repeated for the operator $Q_{SS+PP}$, again the result is that its renormalization pattern is the same as for Overlap fermions, i.e. described by the matrix in eq. (8).

Finally, the renormalization factors of the operators can be computed non-perturbatively in the

| $E_{x_5}$ | $\mathcal{P}_{\pi/2}$ | $\mathcal{P}_{\pi/2}'$ | $E_{x_5} \times \mathcal{P}_{\pi/2}$ | $E_{x_5} \times \mathcal{P}_{\pi/2}'$ |
|----------|-------------------|-------------------|-------------------|-------------------|
| $Q_{VV+\AA}$ | even | even | odd | even | odd |
| $Q_{VV-\AA}$ | odd | even | even | odd | odd |
| $Q_{SS-\PP}$ | odd | even | odd | odd | odd |
| $Q_{AV+\VA}$ | even | odd | even | odd | even |
| $Q_{AV-\VA}$ | odd | odd | odd | even | even |
| $Q_{SP+PS}$ | even | even | even | odd | odd |
| $Q_{SP-PS}$ | odd | odd | even | even | even |

Table 1. Parities of $Q$-operators.

Schrödinger functional (SF) scheme. To this purpose it is convenient to perform the change of variables

$$\psi = e^{i\frac{\tau}{2} \gamma_5 \chi}, \quad \bar{\psi} = \chi e^{i\frac{\tau}{2} \gamma_5 \frac{\gamma_0}{2}},$$

(16)

the twisting then completely moves to the mass term and the action in terms of $\chi$ and $\chi$ is consistent with SF boundary conditions. At the same time the operators need to be rotated, in particular $Q_{VV+\AA}(h^{(\pm)}, \psi) \to Q_{VV+\AA}(h^{(\pm)}, \chi)$. The approach is very similar to the one used in [11,12] to compute $B_K$ and its renormalization factor.

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REFERENCES

1. M. Ciuchini, E. Franco and V. Giménez, Phys. Lett. B 388 (1996) 167.
2. E. Eichten and B. Hill, Phys. Lett. B 240 (1990) 193.
3. D. Becirevic and J. Reyes, Nucl. Phys. Proc. Suppl. 129 (2004) 435.
4. M. Della Morte et al. [ALPHA Collaboration], Phys. Lett. B 581 (2004) 93.
5. H. Neuberger, Phys. Lett. B 417 (1998) 141.
6. R. Frezzotti et al. [ALPHA Collaboration], JHEP 0108 (2001) 058.
7. G. Münster and C. Schmidt, Europhys. Lett. 66 (2004) 652.
8. L. Scorzato, hep-lat/0407023.
9. S. Sharpe and J. Wu, hep-lat/0407035.
10. R. Frezzotti and G. Rossi, hep-lat/0407002.
11. M. Guagnelli et al., Nucl. Phys. Proc. Suppl. 106 (2002) 320.
12. M. Guagnelli et al. [ALPHA Collaboration], Nucl. Phys. Proc. Suppl. 119 (2003) 436.