Lattice and q-difference Darboux-Zakharov-Manakov systems via \( \tilde{\partial} \)-dressing method

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Abstract

A general scheme is proposed for introduction of lattice and q-difference variables to integrable hierarchies in frame of \( \tilde{\partial} \)-dressing method. Using this scheme, lattice and q-difference Darboux-Zakharov-Manakov systems of equations are derived. Darboux, Bäcklund and Combescure transformations and exact solutions for these systems are studied.

1 Introduction

In the present paper we will discuss the lattice and q-difference integrable versions of the well-known Darboux-Zakharov-Manakov (DZM) system

\[
\partial_i \partial_j H_k = (\partial_j H_i) H_i^{-1} \partial_i H_k + (\partial_i H_j) H_j^{-1} \partial_j H_k
\]

where \( i, j, k = 1, 2, 3 \) and \( i \neq j \neq k \neq i \). The system (1) is about one hundred years old. It has been discovered by G. Darboux within his study of the triply conjugate systems of surfaces and then has been studied intensively by geometrists (see e.g. [1]-[4]).

The system (1) has been rediscovered 10 years ago by Zakharov and Manakov within the framework of dressing method. But only few years ago the interrelation has been revealed between the new results of Zakharov and Manakov and the old geometrical constructions of Darboux. During the last years the DZM system has been studied in detail: wide classes of exact solutions have been constructed, different transformations and reductions have been analyzed.

It is remarkable that the applications of the DZM system are not exhausted only by the differential geometry. It was discovered recently that the system (1) plays a key role in the theory of the Hamiltonian and semi-Hamiltonian systems of hydrodynamical type and in two-dimensional...
topological field theories \[13\]. The DZM system \([1]\) arises also as the universal equations for the certain hierarchies of integrable equations \([14]\), \([15]\).

Our study of the difference and q-difference versions of the DZM system is motivated by the well established understanding that the difference versions of integrable systems reveal the deeper nature and algebraic structure of the corresponding nonlinear integrable PDEs.

In the present paper we construct the difference and q-difference DZM systems using the $\bar{\partial}$-dressing method \([5]\), \([15]\), \([5]\) (see also \([7]\)). We discuss the first order form of the difference and q-difference DZM system and its properties. Darboux, Bäcklund and Combescure transformations for the DZM system are derived via $\bar{\partial}$-dressing. Exact solution are found.

We derive also the analogue of the Hirota bilinear identity for the difference and q-difference DZM systems (see \([17]\)) via the $\bar{\partial}$-dressing method. Such bilinear identities can be used as the starting point of the Hirota-Sato approach to the DZM hierarchies.

## 2 Lattice and q-difference variables in $\bar{\partial}$-dressing formalism

The scheme of the $\bar{\partial}$-dressing method uses the nonlocal $\bar{\partial}$-problem with the special dependence of the kernel on additional variables

$$
\bar{\partial}(\chi(x, \lambda) - \eta(x, \lambda)) = \int\int_{\mathbb{C}} d\mu \wedge d\bar{\mu} \chi(\mu) g^{-1}(\mu) R(\mu, \lambda) g(\lambda),
$$

where $\lambda \in \mathbb{C}$, $\bar{\partial} = \partial/\partial \lambda$, $\eta(x, \lambda)$ is a rational function of $\lambda$ (normalization). In this work we treat non-commutative case, so the function $\chi(\lambda)$ and the kernel $R(\lambda, \mu)$ are matrix-valued functions.

A dependence of the solution $\chi(\lambda)$ of the problem \((2)\) on extra variables is hidden in the function $g(\lambda)$. Usually these variables are continuous space and time variables, but it is possible also to introduce discrete (lattice) and q-difference variables into $\bar{\partial}$-dressing formalism. We will consider the following functions $g(\lambda)$

$$
g_{i}^{-1} = \exp(K_{i}x_{i}); \quad \frac{\partial}{\partial x_{i}} g^{-1} = K_{i}g^{-1},
$$

$$
g_{i}^{-1} = (1 + K_{i}n_{i})^{n_{i}}; \quad \Delta_{i}g^{-1} = \frac{g^{-1}(n_{i} + 1) - g^{-1}(n_{i})}{l_{i}} = K_{i}g^{-1},
$$

$$
g_{i}^{-1} = e_{q}(K_{i}y_{i}); \quad \delta_{i}^{q}g^{-1} = \frac{g^{-1}(qy_{i}) - g^{-1}(y_{i})}{(q - 1)y_{i}} = K_{i}g^{-1}.
$$

Here $K_{i}(\lambda)$ are meromorphic matrix functions commuting for different values of $i$. The function \((3)\) introduces a dependence on continuous variable $x_{i}$, the function \((4)\) – on discrete variable $n_{i}$ and the function \((5)\) defines a dependence of $\chi(\lambda)$ on the variable $y_{i}$ (we will call it a q-difference variable). To introduce a dependence on several variables (may be of different type), one should consider a product of corresponding functions $g(\lambda)$ (all of them
commute). Equations in the right part of (3-5) and the boundary condition \( g(0) = 1 \) characterize the corresponding functions (and give a definition of \( e_q(y) \)). These equations play a crucial role in the algebraic scheme of constructing integrable equations in frame of \( \partial \) dressing method. This scheme is based on the assumption of unique solvability of the problem (2), and on the existence of special operators, which transform solutions of the problem (2) into the solutions of the same problem with other normalization.

We suppose that the kernel \( R(\lambda, \mu) \) equals to zero in some open subset \( G \) of the complex plane with respect to \( \lambda \) and to \( \mu \). This subset should typically include all zeroes and poles of the considered class of functions \( g(\lambda) \) and a neighborhood of infinity.

In this case the solution of the problem (1) normalized by \( \eta \) is the function

\[
\chi(\lambda) = \eta(x, \lambda) + \varphi(x, \lambda),
\]

where \( \eta(\lambda) \) is a rational function of \( \lambda \) (normalization), all poles of \( \eta(\lambda) \) belong to \( G \), \( \varphi(\lambda) \) decreases as \( \lambda \to \infty \) and is analytic in \( G \).

The solutions of the problem (2) with a rational normalization form a linear space, let us denote this space \( W \). This space depends on corresponding extra variables (in fact it if a functional of the function \( g \)). It is easy to check that

\[
W(g) = gW(1) \quad (6)
\]

The \( \partial \)-problem (2) implies the difference and q-difference extensions of the famous Hirota bilinear identity. Indeed, let us consider the problem (2) and its formally adjoint for the function normalized by \( (\lambda - \mu)^{-1} \) with different functions \( g \) (i.e. with different values of coordinates)

\[
\frac{\partial}{\partial \lambda} \chi(\lambda, \mu) = 2\pi i \delta(\lambda - \mu) + \iint_{C} d\nu \wedge d\bar{\nu} \chi(\nu, \mu) g_1(\nu)^{-1} R(\nu, \lambda) g_1(\lambda),
\]

\[
\frac{\partial}{\partial \lambda} \chi^*(\lambda, \mu) = -2\pi i \delta(\lambda - \mu) - \iint_{C} d\nu \wedge d\bar{\nu} g_2(\mu)^{-1} R(\lambda, \nu) g_2(\nu) \chi^*(\nu, \mu). \quad (7)
\]

After simple calculations (in the case of continuous variables see [18]) we obtain

\[
\int_{\gamma} \chi(\nu, \lambda; g_1) g_1^{-1}(\nu) g_2(\nu) \chi^*(\nu, \mu; g_2) d\nu = 0, \quad (8)
\]

where \( \gamma \) is the boundary of \( G \). It follows from (8) that in \( \tilde{G} \) the function \( \chi(\lambda, \mu) \) is equal to \( \chi^*(\mu, \lambda) \), so in fact this identity should be written for one function. It is possible to take identity (8) instead of (2) as a starting point for the algebraic scheme of constructing equations.

3 Derivation of DZM equations

The algebraic scheme of constructing equations is based on the following property of the problem (2) with the dressing functions (3-5): if \( \chi(x, n, y, \lambda) \in W(x, n, y) \), then the functions

\[
D^c_i \chi = \frac{\partial}{\partial x_i} \chi + \chi K_i(\lambda)
\]

\[
D^d_i \chi = \Delta \chi + T_i \chi K_i(\lambda)
\]

\[
D^q_i \chi = \delta^q_i \chi + T^q_i \chi K_i(\lambda)
\]

(9)
also belong to \( W \), where \( Tf(n) = f(n + 1), T^qf(y) = f(qy) \). We can multiply the solution from the left by the arbitrary matrix function of additional variables, \( u(x, n, y) \in W \). So the operators (9) are the generators of Zakharov-Manakov ring of operators, that transform \( W \) into itself.

Combining this property with the unique solvability of the problem (1), one obtains the differential relations between the coefficients of expansion of functions \( \chi(x, n, y, \lambda) \) into powers of \( (\lambda - \lambda_p) \) at the poles of \( K_i(\lambda) \) [15].

The derivation of equations in this case is completely analogous to the continuous case [3]. First we choose three functions \( K_i(\lambda), K_j(\lambda), K_k(\lambda) \) in the form

\[
K_i(\lambda) = \frac{A_i}{\lambda - \lambda_i} \tag{10}
\]

where \( A_i, A_j, A_k \) are commuting matrices, \( \lambda_i \neq \lambda_j \neq \lambda_k \neq \lambda_i \). Then we introduce the solution of the problem (2) \( \chi(\lambda) \) with the canonical normalization \( \eta(\lambda) = 1 \). The following derivation will be conducted for q-difference case, to get the difference case you should just change \( \delta^q_i \) for \( \Delta_i \), and \( T_i \) for \( T_i^q \).

The function \( \chi \) satisfies the linear equations

\[
D_i^qD_j^q\chi = T_i^q((D_j^q(\lambda_i)\chi_i)\chi_i^{-1})D_i^q(\lambda_k)\chi + T_j^q((D_j^q(\lambda_j)\chi_j)\chi_j^{-1})D_i^q(\lambda_k)\chi \tag{11}
\]

where \( \chi_i = \chi(\lambda_i) \). Evaluating the equation (11) at the point \( \lambda_k \), we obtain the closed system of equations for the functions \( \chi_i \)

\[
D_i^q(\lambda_k)D_j^q(\lambda_k)\chi_k = T_i^q((D_j^q(\lambda_i)\chi_i)\chi_i^{-1})D_i^q(\lambda_k)\chi_k + T_j^q((D_j^q(\lambda_j)\chi_j)\chi_j^{-1})D_i^q(\lambda_k)\chi_k. \tag{12}
\]

It is possible to transform the operators \( D \) to usual derivatives or difference operators by the substitution

\[
\chi_k = H_kg_i(\lambda_k)g_j(\lambda_k); \quad \psi = \chi g_i(\lambda)g_j(\lambda)g_k(\lambda) \tag{13}
\]

(this substitution works for all cases - continuous, difference and q-difference, you should only take the corresponding function (3) [3]). Then the linear equations (11) and the equations for the functions \( H_i \) read

\[
\delta_i^q\delta_i^q\psi = T_i^q((\delta_i^q H_i)H_i^{-1})\delta_i^q\psi + T_j^q((\delta_i^q H_j)H_j^{-1})\delta_i^q\psi, \tag{14}
\]

\[
\delta_i^q\delta_j^qH_k = T_i^q((\delta^q H_i)H_i^{-1})\delta_j^qH_k + T_j^q((\delta^q H_j)H_j^{-1})\delta_j^qH_k. \tag{15}
\]

The equations (13) represent a q-difference integrable deformation of Zakharov-Manakov system.

The system (13) can be rewritten in the first order form by the substitution

\[
\beta_{ij} = (T_i^q H_i)^{-1}\delta_i^q H_j. \tag{16}
\]

Using the identity \( \delta^q(g^{-1}) = -(T^q g)^{-1}(\delta^q g)g^{-1} \) we obtain the equations

\[
\delta_k^q\beta_{ij} = (T_k^q \beta_{ik})\beta_{kj}. \tag{17}
\]

Correspondingly the linear system (14) becomes

\[
\delta_k^q\psi_i = (T_k^q \beta_{ik})\psi_k \tag{18}
\]
where $\psi_i = (T^q_i H_i)^{-1}\delta^q_i \psi$. Note that the equations (17) and (18) can be derived directly from the bilinear identity (8). In this case the subset $G$ consists of the neighborhoods of the points $\lambda_i, \lambda_j, \lambda_k$ and $\lambda = \infty$.

The continuous version of the system (17) has important applications in the theory of systems of hydrodynamical type and topological field theory [13]. We hope that similar applications will be found for equations (17) too.

4 Darboux transformation

Now we will demonstrate how certain symmetry transformations for the DZM system can be derived via $\bar{\partial}$-dressing method.

Let us introduce the function $\tilde{g}(\lambda)$ in addition to the functions $g_i, g_j, g_k$

$$\tilde{g}(\lambda) = \frac{\lambda - \lambda_i}{\lambda - \bar{\lambda}}$$  (19)

where $\bar{\lambda} \in G$. It follows from (8) that

$$\tilde{W}(y_i, y_j, y_k) = \tilde{g}W(y_i, y_j, y_k)$$  (20)

Let the canonically normalized function $\chi$ in the space $W$ be given; it satisfies linear equations (11). The property (20) gives an opportunity to calculate canonically normalized function $\tilde{\chi}$ in the space $\tilde{W}$ in terms of $\chi$, this function also satisfies equations (11) (with other potentials). It implies that

$$\tilde{\chi} = u(y)\tilde{g}\chi + v(y)\tilde{g}D^q_i \chi$$  (21)

with the properly chosen functions $u$ and $v$. Using two conditions: the absence of poles and and unit asymptotics at infinity, we get

$$\tilde{\chi} = \tilde{g}(\chi - \chi(\bar{\lambda})(D^q_i \chi)^{-1}(\bar{\lambda})D^q_i \chi)$$  (22)

If one transforms operators $D^q_i$ into q-difference operators $\delta^q_i$ by the substitution $\chi = \psi g_i g_j g_k$, one obtains

$$\tilde{\psi}(\lambda) = \tilde{g}(1 - \psi(\bar{\lambda})(\delta^q_i \psi)^{-1}(\bar{\lambda})\delta^q_i \psi(\lambda))$$  (23)

that is the well-known Darboux transformation for the DZM system (see [1], [8], [9])

5 Bäcklund transformation

Introduction of group element (19) may be treated in a different manner, namely, as introduction of extra discrete variable with

$$\tilde{D} = \Delta + \tilde{K}T; \quad \tilde{K} = \frac{\lambda_i - \bar{\lambda}}{\lambda - \lambda_i}$$  (24)
In these notations $\tilde{\chi} = \tilde{T}\chi$. Using the formulae (15) for two q-difference variables $y_i, y_j$ and discrete variable $\tilde{n}$, we obtain the following transformation for the q-deformation of DZM system (13)

$$\Delta \delta^q_i H_i = \tilde{T}((\delta^q_i H_k)H_k^{-1})\Delta H_i + T^q_i((\Delta H_j)H_j^{-1})\delta^q_i H_i$$

$$\delta^q_i \Delta H_j = T^q_i((\Delta H_i)H_i^{-1})\delta^q_i H_j + \tilde{T}((\delta^q_i H_k)H_k^{-1})\Delta H_j$$

(25)

(26)

This transformation establishes a connection between two solutions of the system (13): $H_i, H_j, H_k \rightarrow \tilde{T}H_i, \tilde{T}H_j, \tilde{T}H_k$ and it is nothing but the Bäcklund transformation.

6 Combescure transformation

To derive Combescure transformation in frame of $\tilde{\partial}$-dressing method, it is necessary to use freedom to choose a normalization of the problem (2) in quite a nontrivial way. In this section we consider the commutative case of $\tilde{\partial}$-dressing method, so all functions take their values in $C$. Let us introduce solution of the problem (2) $\chi(\lambda, \mu)$ normalized by $(\lambda - \mu)^{-1}$, where $\mu$ is a parameter, $\mu \in G$, and let us modify operators $D$ just adding constants $c_i = \frac{A_i}{(\lambda_i - \mu)}$ to them

$$D'^q_i = D^q_i + c_i = D^q_i + \frac{A_i}{(\lambda_i - \mu)} = \delta^q_i = \frac{A_i}{(\lambda_i - \mu)}(T^q_i - 1) + A_i \frac{\lambda - \mu}{(\lambda - \lambda_i)(\lambda_i - \mu)} T^q_i$$

(27)

We would like to emphasize the kernel of the problem (2) remains the same. Then the function $\chi(\lambda, \mu)$ satisfies the equation (24) with the modified operators $D'$. To transform operators $D'$ to q-difference operators in this case one should use a substitution

$$\chi_k = H_kg_{\lambda}(\lambda_k)\epsilon^1_q(c_{ij}y_i)g_j(\lambda_k)\epsilon^1_q(c_{ij}y_j),$$

$$\psi = \chi(g_1(\lambda)g_2(\lambda)g_k(\lambda)\epsilon^1_q(c_{ij}y_i)\epsilon^1_q(c_{ij}y_j)e^1_q(c_{k}y_k).$$

(28)

So using different normalizations we obtain different solutions of the system (13). It happens that the connection between these solutions is given by the Combescure transformation. Indeed, unique solvability of the problem (2) implies that $D'^q_i\chi(\lambda, \mu) = u_i(y)D^q_i\chi(\lambda)$ or, after substitution (28)

$$\delta^q_i \psi(\lambda, \mu) = U_i(y)\delta^q_i \psi(\lambda).$$

(29)

The compatibility conditions for equations (24) give the q-deformation of equations of Combescure transformation [1]

$$\delta^q_i U_i = (T_i U_j - T_j U_i)T^q_i((\delta^q_i H_i)H_i^{-1}).$$

(30)

These equations in the continuous case are important for the connection with the systems of hydrodynamical type [10]-[12].
7 Exact solutions

The procedure of getting exact solutions in frame of $\bar{\partial}$-dressing method is based on the fact that for degenerate kernel $R(\lambda, \mu)$ the problem (3) is explicitly solvable. This property keeps for the case of lattice and $q$-difference variables too. Let us treat a simple example.

There is one important special case of nonlocal $\bar{\partial}$-problem which is exactly solvable, which corresponds to plane soliton solutions. This is a case of $\delta$-functional kernels

$$R(\lambda, \mu) = 2\pi i \sum_{\alpha=1}^{N} R_{\alpha} \delta(\lambda - \lambda_{\alpha}) \delta(\mu - \mu_{\alpha}),$$

(31)

where $\lambda_{\alpha}$, $\mu_{\alpha}$ is a set of points in the complex plane, $\lambda_{\alpha} \neq \mu_{\alpha^{'}}$.

In this case the solution of the problem (1) is a rational function, and the problem (1) reduces to the system of linear equations. As a result the canonically normalized function $\chi$ is given by

$$\chi(\lambda) = 1 - \sum_{a,a^{'}} ((A')^{-1})_{aa^{'}} \frac{1}{(\lambda - \lambda_{a})},$$

(32)

$$A'_{aa^{'}} = R^{-1}_{\alpha} \delta_{aa^{'}} - \frac{1}{\mu_{\alpha} - \lambda_{a^{'}}}.$$ 

To get solutions of the DZM equations (15) from this formula, one should use relation $\chi_i = \chi(\lambda_i)$, substitution (13) and corresponding explicit expressions for the functions $g_i$.

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References
[1] G. Darboux, *Lecons sur les systèmes orthogonaux et les coordonneés curvilignes*, Paris (1910).
[2] L. Bianchi, *Lezioni di geometria differentiale*, (2nd ed.), Piza, Spoerri (1902)
[3] L.P. Eisenhart, *A treatise on the differential geometry of curves and surfaces*, Dover Publ., New York (1909)
[4] A.R. Forsyth, *Lectures on the differential geometry of curves and surfaces*, Cambridge Univ. Press (1920).
[5] V.E. Zakharov and S.V. Manakov, 1984 Zap. N. S. LOMI 133 77
Zakharov V E and Manakov S V 1985 Funk. Anal. Ego Prilozh. 19 11

[6] V.S. Dryuma, Proc. of NEEDS Workshop, Dubna 1990, World Scientific, Singapore (1991), 94;
V.S. Dryuma, Math. Studies, vol. 124, Kishinev (1992), 56.

[7] B.G. Konopelchenko, Solitons in multidimensions, World Scientific, Singapore (1993)

[8] B.G. Konopelchenko and W.K. Schief, Lamé and Zakharov-Manakov systems: Combesure, Darboux and Bäklund transformations, preprint AM 93/9, UNSW, Sydney (1993)

[9] W.K. Schief, Inverse Problems, 10, 1185 (1994).

[10] S.P. Tsarev, Math. in the USSR Izvestiya, 37(2), 347 (1991)

[11] B.A. Dubrovin, S.P. Novikov, Usp. Mat. Nauk, 44, 29 (1989)

[12] S.P. Tsarev, Classical differential geometry and integrability of systems of hydrodynamical type, Exeter Workshop (1992), preprint hep-th 9303092

[13] B. Dubrovin, Nucl. Phys., B379, 627(1992);
B. Dubrovin, Commun. Math. Phys., 152, 539 (1993);
B. Dubrovin, Geometry of 2D topological field theories, preprint SISSA-89/94 /FM (1994).

[14] F. Nijhoff and J.-M. Maillet in Nonlinear evolution, Proc. of IV NEEDS 1987, J.J.P. Léon(Ed.), World Scientific, Singapore (1988), 281.

[15] L.V. Bogdanov and S.V. Manakov, 1988 J. Phys. A.:Math. Gen. 21 L537

[16] V.E. Zakharov, 1990 On the dressing method Inverse Methods in Action ed P C Sabatier (Berlin: Springer)

[17] L.V. Bogdanov, hep-th 9401080, Teor. i Mat. Fiz., 99, 177 (1994);
L.V. Bogdanov, talk at NLS conference, Chernogolovka 1994, to be published.

[18] R. Carroll and B. Konopelchenko, Lett. Math. Phys., 28, 307 (1993)