Proper size of the visible Universe in FRW metrics with a constant spacetime curvature

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Abstract
In this paper, we continue to examine the fundamental basis for the Friedmann–Robertson–Walker (FRW) metric and its application to cosmology, specifically addressing the question: What is the proper size of the visible universe? There are several ways of answering the question of size, though often with an incomplete understanding of how far light has actually traveled in reaching us today from the most remote sources. The difficulty usually arises from an inconsistent use of the coordinates or an over-interpretation of the physical meaning of quantities such as the so-called proper distance \( R(t) = a(t) r \), written in terms of the (unchanging) co-moving radius \( r \) and the universal expansion factor \( a(t) \). In this paper, we prove for the five non-trivial FRW metrics with a constant spacetime curvature that, when the expansion began from an initial singularity, the visible universe today has a proper size equal to \( R_h(t_0/2) \), i.e., the gravitational horizon at half its current age. The exceptions are de Sitter and Lanczos, whose contents had pre-existing positions away from the origin. In so doing, we confirm earlier results showing the same phenomenon in a broad range of cosmologies, including \( \Lambda \)CDM, based on the numerical integration of null geodesic equations through an FRW metric.

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1. Introduction
Recent efforts aimed at providing a better understanding of the fundamental basis for the Friedmann–Robertson–Walker (FRW) metric and its application to cosmology have uncovered several previously unrecognized properties relevant to the interpretation of cosmological data. The standard model of cosmology (\( \Lambda \)CDM) is only marginally consistent with these developing theoretical considerations, reflected in the growing tension between its predictions and what is actually observed, both in the cosmic microwave background (CMB) and the
unexpected early appearance of quasars and galaxies at high redshift, and in the matter
distribution, gamma-ray burst rate and Type Ia supernovae in the nearby Universe.

For example, the use of Birkhoff’s theorem and its corollary [1] has shown that the
Universe possesses a gravitational horizon (with radius \( R_h \)) coincident with the better known
Hubble sphere emerging empirically from the observed universal expansion [2]. This new
insight has allowed us to consider the impact of strictly adhering to the requirements of both
the Cosmological principle and Weyl’s postulate [3], which together force \( R_h \) to always be
equal to \( ct \), the distance light could have traveled during a time \( t \) since the big bang [4]. \( \Lambda \)CDM
agrees with this constraint only partially, oddly very early in the universal expansion close to
the Planck time, and more recently, where the various observations tell us that \( R_h(t_0) \approx ct_0 \)
today—but not in between. For a summary of how the current cosmological data compare
with the condition \( R_h = ct \) and the predictions of the standard model, see [5–10].

This fundamental approach to the study of the cosmological spacetime has also allowed
us to examine the nature of cosmological redshift \( z \) in FRW metrics with a constant spacetime
curvature. We recently showed that the interpretation of \( z \) as due to the ‘stretching’ of space
is coordinate dependent [11]. An equally important outcome of this study has been a greatly
improved understanding of how null geodesics behave in FRW, allowing us to better appreciate
which sources are actually observable today. We recently confirmed the importance of \( R_h \)
in delimiting the size of the observable universe [12, 13] by proving that in all cosmologies with
an equation-of-state parameter \( w \geq -1 \), where the pressure \( p \) and density \( \rho \) are related by the
expression \( p = w\rho \), no light rays reaching us today could have ever attained a proper distance
\( R(t) \) greater than \( R_h(t_0) \).

A principal motivation for this paper is actually another interesting result that emerged
from the numerical integration of the null geodesics in [12]. There, we showed that for a
broad range of cosmologies, including \( \Lambda \)CDM, no null geodesics reaching us today (at time
\( t_0 \)) could have ever started from, or reached, a proper distance greater than \( \sim c t_0/2 \) away
from us. Our purpose here is to examine the fundamental basis for this constraint, and we will
prove that in FRW metrics with a constant spacetime curvature, the most distant sources we
see today—particularly the CMB—emitted their light at time \( (1/2)t_0 \) from a proper distance
\( R_h(t_0/2) \) away, which therefore defines the size of the visible universe today.

Applied to the CMB, this result may seem paradoxical because the time \( t_r \) at recombination
was presumably much earlier than \( (1/2)t_0 \). Needless to say, this issue has itself caused
confusion over the years, with some workers believing that light must have therefore traveled
a proper distance \( c(t_0 - t_r) \) in reaching us. For example, a recent recalibration (by \( \Delta t \sim 2 \) Gyr)
of the age of extragalactic eclipsing binaries was used to stretch the cosmic distance ladder
by \( \sim c \Delta t \) [14]. Similarly, conclusions concerning the Universe’s topology are often based on
how far light has traveled since the big bang [15, 16]. And an older (often cited) publication
on distance measures makes several incorrect associations between how far light could have
traveled and the inferred distance to horizons [17]. But it is easy to demonstrate that the proper
distance to a source is not equal to the light-travel distance, and that the difference is merely
due to the time dilation between frames moving at relative speeds close to \( c \). In other words,
we shall see that whereas \( t_r \) may be close to \( 0 \) (for, say, the CMB), the corresponding time on
clocks at rest with respect to us was dilated significantly to a value \( \sim (1/2)t_0 (\gg t_r) \).

2. The FRW metrics with constant spacetime curvature

The high degree of symmetry afforded by the FRW metric is a direct consequence of the
cosmological principle and Weyl’s postulate, which together require that any distance in the
cosmos be expressible as the product of an unchanging co-moving radius \( r \) and a universal
expansion factor \( a(t) \) depending only on the cosmic time \( t \) (see [18] for a pedagogical description). The FRW metric for a spatially homogeneous and isotropic three-dimensional space may be written in the general form,
\[
\text{d}s^2 = c^2 \text{d}t^2 - a^2(t) \left[ \text{d}r^2 (1 - kr^2)^{-1} + r^2 (\text{d}\theta^2 + \sin^2 \theta \text{d}\phi^2) \right],
\]
where \( \theta \) and \( \phi \) are the corresponding angular coordinates in the co-moving frame. The spatial curvature constant \( k \) is \(+1\) for a closed universe, \( 0 \) for a flat, open universe, or \(-1\) for an open universe.

But one must be careful in using this simplification, because the so-called proper distance \( R(t) = a(t) \int_0^r \text{d}r (1 - kr^2)^{-1/2} \) in these coordinates is not measured using rulers and clocks at rest with respect to an individual observer; instead, \( R(t) \) represents a community distance, compiled from the infinitesimal contributions of myriads of observers lined up between the endpoints, all at the same time \( t \) [19]. Of course, there is nothing intrinsically wrong with the usage of \( R(t) \) as a measure of distance—but only so long as one does not over-interpret its physical meaning. For example, a source at \( R \) with \( \text{d}R/\text{d}t > c \) is not receding ‘superluminally,’ because although \( c \) is measured with rulers and clocks at rest with respect to an individual observer, \( \text{d}R/\text{d}t \) is not (we will return to this shortly).

In previous applications [11], we had demonstrated that a single observer can assess the speed of expansion relative to \( c \) only in terms of his proper distance and proper time, both measured on rulers and clocks at rest with respect to himself (see equations (3) and (4) below).

Only then is the speed of light invariant—and always equal to \( c \)—and a true upper limit to the speed of any object in the cosmos. It is in this context, therefore, that to meaningfully address the question of how big the visible universe is, the most straightforward way is to first find an alternative set of coordinates \( \text{d}x^\mu \) to rewrite the FRW metric in its static form,
\[
\text{d}s^2 = g_{\mu\nu} \text{d}x^\mu \text{d}x^\nu,
\]
where the metric coefficients \( g_{\mu\nu} (\mu, \nu = 0, 1, 2, 3) \) are independent of time \( x^0 \), because only then can one claim that the distance and time are being measured at rest with respect to the observer. The form of the metric in equation (1) clearly does not satisfy this condition because \( g_{rr}, g_{\theta\theta} \) and \( g_{\phi\phi} \) are all functions of \( t \) through the expansion factor \( a(t) \). The FRW metrics that can be transformed in this fashion are those with a constant spacetime curvature [11]. Throughout this paper, we will write \( x^\mu = (cT, R, \theta, \phi) \) for the coordinates that render the FRW metric static.

As it turns out, there are exactly six such metrics [20], though one of these—the Minkowski spacetime—is highly trivial; in each of the five remaining cases, a transformation of coordinates permits us to write these solutions in a static form [21]. In the following sections, we will consider each of these in turn, the Milne Universe, de Sitter space, anti-de Sitter space, an open Lanczos-like Universe and the Lanczos Universe itself. But we shall also learn that de Sitter and Lanczos [22] are quite different from the rest because they do not begin their expansion from a singularity at time \( t = 0 \). The size of the visible universe for these two cases is therefore revealingly different from that of all the others.

It is important to stress as we proceed through this exercise that although the spacetime curvature is constant in the cases we consider here, it is generally nonzero. This is a crucial point because the result we obtain is not just an artifact of a cosmology without any spacetime curvature (as in the Milne Universe); it is actually independent of what the spacetime curvature happens to be. In other words, static FRW metrics do not simplify the expansion by eliminating the effects of gravity (or dark energy, for that matter). Gravitational effects are present even when the FRW metric is time independent, as is well known from the Schwarzschild and Kerr spacetimes.
3. The ‘earliest’ visible light

Our detailed proof will be presented in section 4, but before we begin that treatment, it will be helpful for us to consider the essential elements and ideas of this procedure using the following simple motivational argument. Quite generally, an observer’s proper length is the spacelike separation given in terms of equation (2):

$$dL \equiv \sqrt{-ds^2} = \sqrt{-g_{ij} \, dx^i \, dx^j},$$

where, following convention, the Latin indices $i$ and $j$ run from 1 to 3, representing the spatial coordinates only. Similarly, the proper time is the timelike separation

$$d\tau \equiv \frac{1}{c} \sqrt{-g_{00} \, dx^0 \, dx^0}.$$  (4)

These are the spatial and temporal elements the observer must use in order to claim that the proper speed of light $V_\gamma$ is $c$. One can show this trivially by using the null condition in equation (2), for then

$$\sqrt{g_{00}} \, c \, dT = \sqrt{-g_{ij} \, dx^i \, dx^j}.$$  (5)

(This condition implies that we are only considering metrics with a zero velocity shift, but for the cases we include here, this subclass is sufficient. See [23] for further details.)

Thus, defining

$$\mathcal{V} \equiv \frac{dL}{d\tau} = \sqrt{-\frac{g_{ij} \, dx^i \, dx^j}{g_{00} \, dT \, dT}},$$

one can see that the proper speed for light, $\mathcal{V}_\gamma$, is always equal to $c$, irrespective of how curved the spacetime happens to be.

The proper speed $\mathcal{V}$ should not be confused with the so-called co-ordinate speed

$$\nu \equiv \sqrt{-\frac{n_{ij} \, dx^i \, dx^j}{n_{00} \, dT \, dT}},$$

where $n_{\alpha\beta}$ is the Minkowski metric tensor. The coordinate speed can take on any value, even much greater than $c$. However, the proper speed for particles and objects with mass must always be less than $c$ because $ds^2 > 0$. That is, since

$$c \, d\tau > dL,$$

we must have $\mathcal{V} < c$. For example, in the widely known Schwarzschild metric for a central mass $M$, we have $g_{00} = (1 - 2GM/c^2r)$ and $g_{rr} = (1 - 2GM/c^2r)^{-1}$. Therefore, a static observer will see a photon approaching the event horizon located at $R_S = 2GM/c^2$ with a proper speed $\mathcal{V}_\gamma = c$, whereas its coordinate speed $\nu_\gamma = \mathcal{V}_\gamma (1 - 2GM/c^2r)$ actually goes to zero.

Let us now consider sources of light expanding radially away from the observer in an FRW spacetime. (A more formal background to the discussion in this section may be found, e.g., in [24, 23].) With reference to equation (2) written in static form, the proper time and proper distance in this frame are, respectively,

$$d\tau = \sqrt{g_{TT}} \, dT,$$

and

$$dL = \sqrt{-g_{RR}} \, dR.$$

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Thus, the proper speed of light in this frame is

\[ V' = \sqrt{-\frac{g_{RR}}{g_{TT}}} \frac{dR'}{dT} \]  

(11)

Reference to ‘superluminal’ motion in cosmology is often based on the coordinate speed,

\[ v' = \frac{dR'}{dT} = c\sqrt{-\frac{g_{TT}}{g_{RR}}}, \]

which, as we have said, is not the speed of light measured on rulers and clocks at rest with respect to the observer. This coordinate speed diverges for sources beyond the Hubble radius.

We can now ask the question: ‘What was the earliest time \( T_e \) at which light we are receiving right now could have been emitted?’ As shown in [12], the earliest cosmic time \( t_e \) at which this light could have been emitted was 0, because all null geodesics linking us to our past actually started at the origin of the coordinates for \( t \to 0 \). But this is not true of the time \( T_e \) on the individual observer’s clocks, because when viewed in terms of the observer’s coordinates, the sources were not at the origin when they emitted the light we see today.

Instead, the sources had to first travel out to a proper distance equal to that traversed by light once it was emitted back toward us. Designating the proper speed of a source as \( V_S \), we therefore see that

\[ \int_{T_e}^{T_0} V_S (\sqrt{g_{TT}} \, dT) = \int_{T_e}^{T_0} V' (\sqrt{g_{TT}} \, dT), \]

(12)

where \( T_0 \) is the present time. But since the proper lightspeed is always \( c \), we can write the somewhat simpler expression

\[ \int_{T_0}^{T_e} V_S (\sqrt{g_{TT}} \, dT) = \int_{T_0}^{T_0} c (\sqrt{g_{TT}} \, dT). \]

(13)

However, we have just argued that the maximum proper speed of any particle or object is \( V_S \to c \). The earliest time \( T_e \) that a source could have emitted light just reaching us today corresponds to the fastest among them—those that reached the greatest proper distance in the shortest time. In other words, the earliest time \( T_e \) corresponds to \( V_S = c \), and therefore the condition on \( T_e \) becomes

\[ \int_{T_0}^{T_e} \sqrt{g_{TT}} \, dT \to \int_{T_0}^{T_0} \sqrt{g_{TT}} \, dT, \]

(14)

which has the obvious solution \( T_e = (1/2)T_0 \). But \( T_0 = t_0 \), since \( t \) is actually the proper time on the clock at rest with respect to us at our location, and we thus infer that the earliest time light visible to the observer that could have been emitted must have been \( T_e = (1/2)t_0 \). In the following section, we will prove this result for each of the FRW metrics with a constant spacetime curvature that begin their expansion from an initial singularity, and then we will discuss what it means to have \( T_e \gg t_e \) (i.e., \( T_e \gg 0 \)).

4. Detailed proof

4.1. The Milne Universe

The Milne Universe [25] has no density \( (\rho = 0) \) and is characterized by a spatial curvature \( k = -1 \). It therefore corresponds to a simple solution of Einstein’s equations with

\[ a(t) = ct, \]

(15)

in which the scale factor grows linearly with time at a rate equal to the speed of light \( c \). Since the acceleration \( \ddot{a}(t) \) is zero in this cosmology, one might expect such a universe to have a zero spacetime curvature and be a mere re-parametrization of Minkowski space. Indeed, Milne
built this type of expansion based solely on special relativity, without any constraints imposed by the more general theory.

To cast the FRW metric for the Milne Universe in its static form [26, 27], we first introduce the co-moving distance variable $\chi$, defined in terms of $r$, according to

$$r = \sinh \chi,$$

which allows us to write

$$ds^2 = c^2 dt^2 - c^2 t^2 [d\chi^2 + \sinh^2 \chi \, d\Omega^2],$$

where, for simplicity, we have also introduced the notation $d\Omega^2 = d\theta^2 + \sin^2 \theta \, d\phi^2$. The transformation that brings equation (10) into a static form is

$$T = t \cosh \chi,$$

$$R = ct \sinh \chi,$$

for then

$$ds^2 = c^2 dT^2 - dR^2 - R^2 d\Omega^2.$$  

Clearly, a null geodesic in this frame has

$$cdT = -dR$$

(for light approaching us along a radius), and therefore

$$R_e = c(T_0 - T_e),$$

where the proper distance $L_e$ to the source in the static coordinate system is here identical to $R_e$ because $g_{TT} = g_{RR} = 1$, and $T_e$ and $T_0$ are the emission and observation times, respectively, in this frame. To be clear, since the metric coefficients $g_{\mu\nu}$ in equation (12) are independent of $T$, the quantities defined in equation (11) are now the coordinates for an individual observer, because time $T$ is being measured on clocks at rest in his frame.

Suppose now that at cosmic time $t_e$, a source at $\chi_e$ emits the light reaching the observer today. The corresponding proper time $T_e$ on the observer’s clock at the location of the source will be greater than $t_e$ due to the effects of relativistic time dilation. The most distant sources were moving at proper speeds close to $c$ in this frame when they emitted their light at $t_e$ [11], so the time dilation between $T_e$ and $t_e$ for them approaches infinity. For this exercise, we may even allow $t_e \to 0$, in order to examine the time at which the CMB was produced. From equation (11), we see that when $t_e \to 0$,

$$\frac{T_e}{t_e} = \cosh \chi_e \gg 1.$$  

In this limit, therefore, $\chi_e \to \infty$, and so we may write

$$\frac{T_e}{t_e} \to \frac{1}{2} e^{\nu_e}.$$  

But we also know from equation (10) that a null geodesic satisfies

$$\int_0^{\nu_e} d\nu' = \int_{t_e}^0 \frac{dt'}{t'},$$

and so

$$\frac{\nu_0}{t_e} = e^{\nu_e}.$$  

A comparison between equations (16) and (18) immediately tells us that, in the limit $t_e \to 0$,

$$T_e \to \frac{1}{2} t_0.$$  

a simple and beautiful result that confirms our supposition from the previous section, and one that will be repeated with each subsequent cosmology we consider (except for de Sitter and Lanczos, as we have already anticipated).

Thus, in the Milne Universe, the greatest proper distance (defined in equation (14)) traveled by light reaching the observer today is only \((1/2)c t_0\), though the sources there look very young due to the effects of time dilation between the source frame and that of the observer. In other words, though the CMB may have been produced at a cosmic time \(t_e \rightarrow 0\) in the co-moving frame, that event occurred at time \(T_e = (1/2)t_0\) in the observer’s frame, and though the light signal carries information pertaining to those earliest moments in the Universe’s expansion history, according to equation (14) it has therefore only traveled a proper distance \((1/2)c t_0\) in reaching the observer.

Later in the discussion section, we will describe in greater detail what is actually happening, but the basic principle is rather simple—since in Milne the expansion began from a singularity at time \(t = 0\), and it took the most distant sources a time \((1/2)t_0\) to reach a proper distance \((R_e = c t_0/2)\) from which the most distant light signal could then have reached the observer by time \(t_0\). This feature is illustrated schematically in figure 1. Note that formally, \(R_h = c/H\), and since here \(H \equiv \dot{a}/a = 1/t\), we may also write this result in the form \(R_e = R_h(t_0/2)\). Observationally, we recognize \(R_h\) as the Hubble radius.
4.2. Anti-de Sitter space

A universe with negative mass density and spatial curvature \( k = -1 \) is known as anti-de Sitter space, due to its negative spacetime curvature. This metric is given by

\[
\begin{align*}
\text{d}s^2 &= c^2 \text{d}t^2 - (cb)^2 \sin^2(t/b) \left[ \frac{\text{d}r^2}{1 + r^2} + r^2 \Omega_1^2 \right],
\end{align*}
\]

where clearly the expansion factor is now \( a(t) = cb \sin(t/b) \). The coordinate transformation

\[
R = cbr \sin(t/b),
\]

and

\[
\tan(T/b) = (1 + r^2)^{1/2} \tan(t/b),
\]

produces the static form of the metric,

\[
\begin{align*}
\text{d}s^2 &= \left[ 1 + \left( \frac{R}{cb} \right)^2 \right] c^2 \text{d}T^2 - \left[ 1 + \left( \frac{R}{cb} \right)^2 \right]^{-1} \text{d}R^2 - R^2 \Omega_1^2.
\end{align*}
\]

Now, along a geodesic connecting the emission point \( r_e \) at time \( t_e \) with the observer at \( r = 0 \) and time \( t_0 \),

\[
\int_0^{t_e} \frac{\text{d}r}{\sqrt{1 + r^2}} = \int_{t_0/b}^{t_e/b} \frac{\text{d}u}{\sin(u)},
\]

which has the solution [11]

\[
r_e = \frac{1}{2} \left( \frac{\tan(t_0/2b)}{\tan(t_e/2b)} - \frac{\tan(t_e/2b)}{\tan(t_0/2b)} \right).
\]

Thus, for a source emitting light very early in the Universe’s history, i.e., for \( t_e \to 0 \),

\[
r_e \to \frac{1}{2} \frac{\tan(t_0/2b)}{t_e/2b}.
\]

In this limit, \( r_e \) clearly diverges, so the factor \( \sqrt{1 + r_e^2} \) in the equation for \( T_e \),

\[
\tan(T_e/b) = (1 + r_e^2)^{1/2} \tan(t_e/b),
\]

simply becomes \( r_e \). In addition, \( \tan(t_e/b) \) in this limit reduces to \( t_e/b \), and so

\[
\tan(T_e/b) \to \frac{1}{2} \frac{\tan(t_0/2b)}{t_e/2b} \frac{t_e}{b} = \tan(t_0/2b),
\]

which again leads to the beautiful result that

\[
T_e \to \frac{1}{4} t_0.
\]

This is the observer’s coordinate time at which the most distant sources visible to him at \( t_0 \) produced their light in anti-de Sitter space.

4.3. A Lanczos-like Universe with \( k = -1 \)

In co-moving coordinates, the third FRW metric with a constant spacetime curvature (and \( k = -1 \)) may be written [20, 21]

\[
\begin{align*}
\text{d}s^2 &= c^2 \text{d}t^2 - (cb)^2 \sinh^2(t/b) \left[ \frac{\text{d}r^2}{1 + r^2} + r^2 \Omega_1^2 \right],
\end{align*}
\]

where \( a(t) = (cb) \sinh(t/b) \) and \( b \) is a constant (though clearly not the Hubble constant \( H \equiv \dot{a}/a \)). The metric may also be written in static form with the transformation

\[
R = cbr \sinh(t/b),
\]
and
\[ \tanh(T/b) = (1 + r^2)^{1/2} \tanh(t/b), \] (39)
which together allow us to write the interval in the form
\[ ds^2 = \left[ 1 - \left( \frac{R}{cb} \right)^2 \right] c^2 dT^2 - \left[ 1 - \left( \frac{R}{cb} \right)^2 \right]^{-1} dR^2 - R^2 d\Omega^2. \] (40)
identical (in terms of \( R \) and \( T \)) to the actual Lanczos metric we shall examine below.

As before, we calculate the co-moving radius \( r_e \) to a source at time \( t_e \) using the geodesic equation
\[ \int_{r_0}^{r_e} \frac{dr}{\sqrt{1 + r^2}} = \int_{t_0/b}^{t_e/b} \frac{du}{\sinh(u)}, \] (41)
whose solution is
\[ \sinh^{-1}(r_e) = \ln(\tanh(t_0/2b)) - \ln(\tanh(t_e/2b)). \] (42)
That is,
\[ r_e = \frac{1}{2} \left( \frac{\tanh(t_0/2b)}{\tanh(t_e/2b)} - \frac{\tanh(t_e/2b)}{\tanh(t_0/2b)} \right). \] (43)

The most distant sources visible by the observer at time \( t_0 \) emitted their light at time \( t_e \to 0 \), so their co-moving radius is evidently
\[ r_e \to \frac{1}{2} \frac{\tanh(t_0/2b)}{t_e/2b}. \] (44)
As was true for anti-de Sitter space, this radius diverges for the earliest times, and therefore \( \sqrt{1 + r_e^2} \to r_e \). Thus,
\[ \tanh(T_e/b) \to \frac{1}{2} \frac{\tanh(t_0/2b)}{t_e/2b} \frac{t_e}{b} = \tanh(t_0/2b), \] (45)
so once again
\[ T_e \to \frac{1}{2} t_0. \] (46)

### 4.4. de Sitter space

We next consider the situation with de Sitter which, unlike Milne, is a cosmology in which objects not only recede from each other, but also accelerate under the influence of gravity. However, unlike anti-de Sitter space and the open Lanczos-like universe we have just considered that de Sitter space does not begin its expansion from a singularity.

The de Sitter cosmology [28] corresponds to a universe devoid of matter and radiation, but filled with a cosmological constant whose principal property is the equation of state \( p = -\rho \). The FRW metric in this case may be written as
\[ ds^2 = c^2 dt^2 - e^{2Ht} [dr^2 + r^2 d\Omega^2], \] (47)
where \( k = 0 \) and the expansion factor has the specific form
\[ a(t) = e^{Ht}, \] (48)
in terms of the Hubble constant \( H \).
Unlike the Milne model, de Sitter space contains mass–energy (in the form of a cosmological constant). The transformation of coordinates that brings the metric (equation (40)) into its stationary form is as follows:

\[ R = a(t) r \]
\[ T = t - \frac{1}{2H} \ln \Phi, \]

(49)

where

\[ \Phi \equiv 1 - \left( \frac{R}{R_h} \right)^2, \]

(50)

and

\[ R_h \equiv \frac{c}{H} \]

(51)

is the gravitational (or Hubble) radius. With these, the de Sitter metric becomes

\[ ds^2 = c^2 \Phi \, dT^2 - \Phi^{-1} \, dR^2 - R^2 \, d\Omega^2, \]

(52)

and note that all the metric coefficients are now independent of time \( T \). Of course, the chief difference between this case and that exhibited in equation (12) is that the de Sitter spacetime is curved, and therefore both \( g_{TT} \) and \( g_{RR} \) depend on the spatial coordinates \([2]\). The form of the metric in equation (45) is how de Sitter himself first presented his now famous solution.

To be precise, de Sitter does not actually have an initial singularity because \( a(t) \to 1 \) as \( t \to 0 \). Rather, the transformation exhibited in equations (42)–(44) demonstrates the dependence of \( R \) and \( T \) only on the time difference between \( t \) and an initial time \( t_i \). The form of these equations corresponds to the choice \( t_i = 0 \). In talking about the proper size of the visible universe in de Sitter (see section 5), we therefore necessarily refer to how far sources and light have moved during a time \( t_0 = t_i \). Again, in the expressions that follow, we will adhere to the convention that \( t_i = 0 \).

As we did with Milne, let us now consider the time \( T_e \) at which the source located at \( R_e(t_e) \) emits the light we see today, corresponding to the proper time \( t_e \) in its own rest frame. It will be useful for us to borrow a result we derived earlier \([11]\), allowing us to express the redshift in the form

\[ 1 + z = \frac{1}{1 - R_e(t_e)/R_h}. \]

(53)

This follows very easily from the well-known formulation \([19]\)

\[ 1 + z = \frac{a(t_0)}{a(t_e)}, \]

(54)

or

\[ 1 + z = \exp[H(t_0 - t_e)]. \]

(55)

According to equations (42) and (43),

\[ T_e = t_e - \frac{1}{2H} \ln \left( 1 - \left( \frac{R_e(t_e)}{R_h} \right)^2 \right), \]

(56)

so that

\[ T_e = t_e - \frac{1}{2H} \ln \left( 1 - \left( \frac{R_e(t_e)}{R_h} \right)^2 \right) - \frac{1}{2H} \ln \left( 1 + \left( \frac{R_e(t_e)}{R_h} \right)^2 \right). \]

(57)

This equation may be manipulated further, yielding

\[ T_e = t_e - \frac{1}{2H} \ln e^{-H(t_0-t_e)} - \frac{1}{2H} \ln \left( \frac{1 + 2z}{1 + z} \right). \]

(58)
And this leads to the result we were seeking,
\[ T_e = \frac{1}{2} t_0 + \frac{1}{2} t_e - \frac{1}{2H} \ln \left( \frac{1 + 2z}{1 + \frac{1}{2} z} \right). \]  
(59)

To find the time \( T_e \) in the observer’s frame corresponding to the earliest emission of observable light in the Universe, we put \( t_e \to 0 \) and \( z \to \infty \), so that
\[ T_e \to \frac{1}{2} t_0 - \frac{1}{2H} \ln 2. \]  
(60)

Therefore, when \( t_0 \gg 1/H \), we obtain the rather remarkable result that, even in this kind of curved spacetime,
\[ T_e \to \frac{1}{2} t_0. \]  
(61)

This is what one would expect on the basis of our discussion in the previous section. As we shall see shortly, however, the proper distance associated with this emission time is not the same as that for the previous three cases.

4.5. The Lanczos (closed) Universe

The fifth, and final, FRW metric with a constant spacetime curvature (other than Minkowski) is known as the Lanczos Universe, described by the metric
\[ ds^2 = c^2 dt^2 - (cb)^2 \cosh^2(t/b) \left[ \frac{dr^2}{1 - r^2} + r^2 d\Omega^2 \right]. \]  
(62)

where \( k = +1 \), and the expansion factor is now \( a(t) = (cb) \cosh(t/b) \). We use the following transformation to render this metric in static form:
\[ R = cb \cosh(t/b), \]  
(63)

and
\[ \tanh(T/b) = (1 - r^2)^{-1/2} \tanh(t/b), \]  
(64)

which together allow us to write the interval as
\[ ds^2 = \left[ 1 - \left( \frac{R}{cb} \right)^2 \right] c^2 dT^2 - \left[ 1 - \left( \frac{R}{cb} \right)^2 \right]^{-1} dR^2 - R^2 d\Omega^2. \]  
(65)

But it does not take much to realize that this universe is quite different from the others. For one thing, it is closed \( (k = +1) \); the others are all open, with \( k = 0 \) or \( k = -1 \). Moreover, as in de Sitter, the expansion factor \( a(t) \) does not vanish at \( t = 0 \). Instead, \( a \to cb \), so this universe does not begin its expansion from a singularity, and because the sources all occupied pre-existing positions at \( t = 0 \), one should therefore expect that an observer can detect the light they emitted all the way back to \( T_e = 0 \), which we now demonstrate formally.

As shown in [11], the co-moving distance to a source emitting light at \( t_e \) is here given by the expression
\[ r_e = 2(e^{h/b} - e^{t_e/b})(1 + e^{(h+t_e)/b})(1 + e^{2t_e/b})^{-1}(1 + e^{2h/b})^{-1}, \]  
(66)

and it is not difficult to show that as \( t_e \to 0 \),
\[ r_e \to \tanh(t_0/b). \]  
(67)

Therefore, as \( t_e \to 0 \),
\[ \tanh(T_e/b) \to \cosh(t_0/b) \tanh(t_e/b) \to 0, \]  
(68)

for which \( T_e \to 0 \).
5. Proper size of an FRW Universe

In all the cases we have considered, the coordinate distance \( R \) in the frame where all the metric coefficients are independent of time actually coincides with the definition of proper radius \( R = a(t) r \) in the co-moving frame. We may therefore determine the ‘proper’ size of the visible universe by merely calculating \( R_e(T_e) \) at the time of emission \( T_e \).

The Milne Universe is unique among the other FRW metrics because it has a zero spacetime curvature. Therefore, \( g_{RR} = g_{TT} = 1 \), and \( R \) is exactly equal to the light-travel distance \( cT \), as indicated in equation (14). Thus, using equation (19), we see that for Milne, \( R_e = (1/2)c t_0 = R_h(t_0/2) \).

For anti-de Sitter space, \( R_e = cb \sinh(t_e/b) \), and introducing the limiting form of \( r_e \) from equation (26), we find that
\[
R_e \to cb \tanh(t_0/2b), \tag{70}
\]
which therefore gives, again, \( R_e = R_h(t_0/2) \).

A similar result follows for the third universe we considered, since in this case \( R_e = cb \sin(t_e/b) \), and therefore using the limiting form of \( r_e \) from equation (37), we find that
\[
R_e \to cb \tan(t_0/2b), \tag{72}
\]
Since in this case \( H(t) = 1/b \tanh(t/b) \), we find that \( R_e = R_h(t_0/2) \), as was the case for Milne and anti-de Sitter.

These three independent cases all demonstrate the principal result of this paper—that in a FRW metric with a constant spacetime curvature expanding from a singularity, the earliest signal we can see today was produced when the Universe was half its current age, \( t_0 \) (as measured on clocks at rest with respect to us), from a proper distance equal to the size of the gravitational horizon at that time. As such, this formal derivation is fully consistent with the numerical calculations reported in [12], and helps to explain the conclusions in that paper—that light reaching us today, even in the case of \( \Lambda \)CDM, never attained a proper distance greater than \( R_h(t_0/2) \).

Let us now compare this fundamental result with de Sitter and Lanczos, for which \( a(0) \neq 0 \).

It is straightforward to see from equation (42) that
\[
-t_0 H = \ln \left( 1 - \frac{R_e}{R_h} \right)^2, \tag{73}
\]
and therefore \( R_e \to R_h \) for \( t_0 H \gg 1 \). Even though \( T_e = t_0/2 \), the proper size of the visible de Sitter universe is nonetheless \( R_h \), which is a constant. The fact that \( T_e \neq 0 \) is entirely due to our choice of following the expansion from a defined initial time \( t_i = 0 \). But the structure of this universe is independent of time because its expansion is eternally exponentiated. So the most distant sources we can see are always at the gravitational horizon.

There are strong similarities between de Sitter and the final cosmology we have considered—the Lanczos Universe—though the gravitational horizon in this case is not constant. We see from equation (56) that
\[
R_e = cb \cosh(t_e/b), \tag{74}
\]
so evidently
\[
R_e(t_e) \to cb \tanh(t_0/b) = \frac{c}{H(t_0)} = R_h(t_0). \tag{75}
\]
This is a very interesting and important result in itself, because it demonstrates that, regardless of whether or not the gravitational horizon is moving, the most distant sources we see today in a universe without an initial singularity coincide with the location of this horizon today.

Furthermore, note that even though $R_h$ is here not constant, we find that $R_e \rightarrow cb$ for all $t_0/b \gg 1$, which mirrors the situation with de Sitter. The principal difference between these last two cosmologies and all the others is that whereas the sources in the previous cases first had to travel a proper distance $R_h(t_0/2)$ to reach the edge of the visible universe, the most distant sources in de Sitter and Lanczos were already situated at $R_e = R_h(\infty)$ from the very beginning, and therefore the observer can see their light emitted from that maximal distance at arbitrarily early times.

6. Discussion and conclusions

To fully understand and appreciate the results we have presented in this paper, one must acknowledge the critical role played by the choice of coordinates in describing the expansion of the Universe. We had already seen an example of this, based on how the choice of frames impacts our interpretation of the cosmological redshift $z$ [11]. We proved earlier that, although $z$ is conventionally calculated directly from the expansion factor $a(t)$, its origin cannot be attributed to an expansion of space when viewed in terms of the FRW metric written in a stationary form. We found that $z$ is actually the cosmological version of a lapse function encountered more typically in the context of the Schwarzschild and Kerr metrics. That is, $z$ is simply due to the combined effects of the kinematic expansion and the gravitational acceleration—but only in terms of the proper velocity, calculated using the proper distance and proper time for an individual observer.

In this paper, we have expanded our study of the fundamental aspects of the cosmic spacetime by using these alternative sets of coordinates to address another issue that sometimes gives rise to confusion and ambiguity: What is the true size of the visible universe? The question itself is fraught with ambiguity because it goes without saying that to measure a size, one must have a precise definition of distance. General relativity is founded on the basic principle that $c$ is invariant and is measured to have the same value for all observers. But what is often overlooked or forgotten is that in order to make the measurements consistent with this tenet, distances and times must be determined with devices at rest with respect to the observer. Only then can he claim that $c$ is an upper limit to all speeds and that light travels at speed $c$ under all circumstances and at all times.

These notions are particularly important to the question we have addressed in this paper, especially for cosmologies that begin their expansion from a singularity at time $t = 0$. The reason for this is rather straightforward. In these cosmologies, all the worldlines of sources we see today started from the same location—very near the same co-moving point we ourselves are now occupying. Clearly, to suggest that the light they emitted has traveled a distance $c(t_0 - t_e) \rightarrow ct_0$ since the big bang is quite non-sensical. The correct statement is that the most distant sources we see today are precisely those moving at close to proper lightspeed, which reached a proper distance $R_h(t_0/2)$ before emitting the light that is just now reaching us at time $t_0$.

It is remarkable—though obvious in retrospect—how elegantly and beautifully this simple result emerges from the properties of the FRW metric itself written in a stationary form, when we take the limit $t_e \rightarrow 0$ for the time at which the light from the most distant sources was emitted. One of the principal results of our analysis has been the demonstration that even though $t_e \approx 0$ for these sources, the time measured on our clocks at rest with respect to us
was actually $T_e = (1/2)t_0$. And now we understand that this effect is entirely due to the time dilation between us and sources receding at a proper speed $c$ when they emitted this light.

These conclusions do come with a caveat, however, because most of these results are based on the use of FRW metrics with a constant spacetime curvature, allowing us to find an alternative set of coordinates to write them in stationary form. Without this option, we would not yet know how to evaluate distances and times in such a way as to demonstrate without any doubt how far sources could have traveled before emitting the light we see today. One ought to expect the proper size of the visible universe to be measurable against the gravitational horizon even in cases where the spacetime curvature is not constant, but only future work can establish this result conclusively.

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