QUADRIC SURFACE BUNDLES OVER SURFACES

ASHER AUEL, R. PARIMALA, AND V. SURESH

Abstract. Let \( f : T \to S \) be a finite flat morphism of degree 2 of regular integral schemes of dimension \( \leq 2 \) (with 2 invertible), having regular branch divisor \( D \subset S \). We establish a bijection between Azumaya quaternion algebras on \( T \) and quadric surface bundles with simple degeneration along \( D \). This is a manifestation of the exceptional isomorphism \( \tilde{\mathbb{A}}_1 = \mathbb{D}_2 \) degenerating to the exceptional isomorphism \( \mathbb{A}_1 = \mathbb{B}_1 \). In one direction, the even Clifford algebra yields the map. In the other direction, we show that the classical algebra norm functor can be uniquely extended over the discriminant divisor. Along the way, we study the orthogonal group schemes, which are smooth yet nonreductive, of quadratic forms with simple degeneration. Finally, we provide two applications: constructing counter-examples to the local-global principle for isotropy, with respect to discrete valuations, of quadratic forms over surfaces; and a new proof of the global Torelli theorem for very general cubic fourfolds containing a plane.

Introduction

A quadric surface bundle \( \pi : Q \to S \) over a scheme \( S \) is the flat fibration in quadrics associated to a line bundle-valued quadratic form \( q : E \to \mathcal{L} \) of rank 4 over \( S \). A natural class of quadric surface bundles over \( \mathbb{P}^2 \) appearing in algebraic geometry arise from cubic fourfolds \( Y \subset \mathbb{P}^5 \) containing a plane. Projection from the plane \( \pi : \tilde{Y} \to \mathbb{P}^2 \) defines a quadric surface bundle, where \( \tilde{Y} \) is the blow-up of \( Y \) along the plane. Such quadric bundles have degeneration along a sextic curve \( D \subset \mathbb{P}^2 \). If \( Y \) is sufficiently general then \( D \) is smooth and the double cover \( T \to \mathbb{P}^2 \) branched along \( D \) is a K3 surface of degree 2. Over the surface \( T \), the even Clifford algebra \( C_0 \) associated to \( \pi \) becomes an Azumaya quaternion algebra representing a Brauer class \( \beta \in \text{Br}(T) \). For \( T \) sufficiently general, the association \( Y \mapsto (T, \beta) \) is injective: smooth cubic fourfolds \( Y \) and \( Y' \) giving rise to isomorphic data \( (T, \beta) \cong (T', \beta') \) are linearly isomorphic. This result was originally obtained via Hodge theory by Voisin [52] in her celebrated proof of the global Torelli theorem for cubic fourfolds.

In this work, we provide an algebraic generalization of this result to any regular integral scheme \( T \) of dimension \( \leq 2 \) with 2 invertible, which is a finite flat double cover of a regular scheme \( S \) with regular branch divisor \( D \subset S \). Let \( \text{Quad}_2(T/S) \) denote the set of \( S \)-isomorphism classes of quadric surface bundles over \( S \) with simple degeneration along \( D \) and discriminant cover \( T \to S \) (see §1 for details). Let \( \text{Az}_2(T/S) \) denote the set of \( \mathcal{O}_T \)-isomorphism classes of Azumaya algebras of degree 2 over \( T \) with generically trivial corestriction to \( S \) (see §5 for details). The even Clifford algebra functor yields a map \( C_0 : \text{Quad}_2(T/S) \to \text{Az}_2(T/S) \). In §5, we define a generalization \( N_{T/S} \) of the algebra norm functor, which gives a map in the reverse direction. Our main result is the following.

Theorem 1. Let \( S \) be a regular integral scheme of dimension \( \leq 2 \) with 2 invertible and \( T \to S \) a finite flat morphism of degree 2 with regular branch divisor \( D \subset S \). Then the even Clifford algebra and norm functors

\[
\begin{align*}
\text{Quad}_2(T/S) & \xrightarrow{C_0} \text{Az}_2(T/S) \\
\text{Az}_2(T/S) & \xleftarrow{N_{T/S}} \text{Quad}_2(T/S)
\end{align*}
\]

give rise to mutually inverse bijections.

Date: May 2, 2014.
This result can be viewed as a significant generalization of the exceptional isomorphism \( 2A_1 = D_2 \) correspondence over fields and rings (cf. [33, IV.15.B] and [35, §10]) to the setting of line bundle-valued quadratic forms with simple degeneration over schemes. Most of our work goes toward establishing fundamental local results concerning quadratic forms with simple degeneration (see §3) and the structure of their orthogonal group schemes, which are nonreductive (see §2). In particular, we prove that these group schemes are smooth (see Proposition 2.3) and realize a degeneration of exceptional isomorphisms \( 2A_1 = D_2 \) to \( A_1 = B_1 \). We also establish some structural results concerning quadric surface bundles over schemes (see §1) and the formalism of gluing tensors over surfaces (see §4).

Also, we give two surprisingly different applications of our results. In §6, we provide a class of quadratic forms that are counter-examples to the local-global principle for isotropy, with respect to discrete valuations, over function fields of surfaces over algebraically closed fields. Moreover, such forms exist even over rational function fields, where regular quadratic forms fail to provide such counter-examples. In §7, using tools from the theory of moduli of twisted sheaves, we are able to provide a new proof of the global Torelli theorem for very general cubic fourfolds containing a plane, a result originally obtained by Voisin [52] using Hodge theory.

Our perspective comes from the algebraic theory of quadratic forms. We employ the even Clifford algebra of a line bundle-valued quadratic form constructed by Bichsel [11]. Bichsel–Knus [12], Caenepeel–van Oystaeyen [13] and Parimala–Sridharan [43, §4] give alternate constructions, which are all detailed in [2, §1.8]. In a similar vein, Kapranov [31, §4.1] (with further developments by Kuznetsov [36, §3]) considered the homogeneous Clifford algebra of a quadratic form—the same as the generalized Clifford algebra of [12] or the graded Clifford algebra of [13]—to study the derived category of projective quadrics and quadric bundles. We focus on the even Clifford algebra as a sheaf of algebras, ignoring its geometric manifestation via the relative Hilbert scheme of lines in the quadric bundle, as in [52, §1] and [29, §5]. In this context, we refer to Hassett–Tschinkel [28, §3] for a version of our result in the case of smooth projective curves over an algebraically closed field.

Finally, our work on degenerate quadratic forms may also be of independent interest. There has been much recent focus on classification of degenerate (quadratic) forms from various number theoretic directions. An approach to Bhargava’s [10] seminal construction of moduli spaces of “rings of low rank” over arbitrary base schemes is developed by Wood [54] where line bundle-valued degenerate forms (of higher degree) are crucial ingredients. In related developments, building on the work of Delone–Faddeev [18] over \( \mathbb{Z} \) and Gross–Lucianovic [23] over local rings, Venkata Balaji [8], and independently Voight [51], used Clifford algebras of degenerate ternary quadratic forms to classify degenerations of quaternion algebras over arbitrary bases. In this context, our main result can be viewed as a classification of quaternionic quadratic forms with squarefree discriminant in terms of their even Clifford algebras.

Acknowledgements. The first author benefited greatly from a visit at ETH Zürich and is partially supported by National Science Foundation grant MSPRF DMS-0903039. The second author is partially supported by National Science Foundation grant DMS-1001872. The authors would specifically like to thank M. Bernardara, J.-L. Colliot-Thélène, B. Conrad, M.-A. Knus, E. Macrì, and M. Ojanguren for many helpful discussions.

1. Reflections on simple degeneration

Let \( S \) be a noetherian separated integral scheme. A (line bundle-valued) quadratic form on \( S \) is a triple \( (\mathcal{E}, q, \mathcal{L}) \), where \( \mathcal{E} \) is a locally free \( \mathcal{O}_S \)-module and \( q : \mathcal{E} \to \mathcal{L} \) is a quadratic morphism of sheaves such that the associated morphism of sheaves \( b_q : S^2\mathcal{E} \to \mathcal{L} \), defined on sections by \( b_q(v, w) = q(v + w) - q(v) - q(w) \), is an \( \mathcal{O}_S \)-module morphism. Equivalently, a quadratic form is an \( \mathcal{O}_S \)-module morphism \( q : S_2\mathcal{E} 

\to \mathcal{L} \), see [49, Lemma 2.1] or [2, Lemma 1.1]. Here, \( S^2\mathcal{E} \) and \( S_2\mathcal{E} \) denote the second symmetric power
and the submodule of symmetric second tensors of \( E \), respectively. There is a canonical isomorphism \( S^2(E^\vee) \otimes \mathcal{L} \cong \mathcal{H}om(S_2E, \mathcal{L}) \). A line bundle-valued quadratic form then corresponds to a global section

\[
q \in \Gamma(S, \mathcal{H}om(S_2E, \mathcal{L})) \cong \Gamma(S, S^2(E^\vee) \otimes \mathcal{L}) \cong \Gamma_S(\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathbb{P}(\mathcal{E})}/S(2) \otimes p^*\mathcal{L}),
\]

where \( p : \mathbb{P}(\mathcal{E}) = \text{Proj} S^*(E^\vee) \to S \) and \( \Gamma_S \) denotes sections over \( S \). There is a canonical \( \mathcal{O}_S \)-module morphism

\[
\psi_q : \mathcal{E} \to \mathcal{H}om(E, \mathcal{L}) \text{ associated to } b_q.
\]

A line bundle-valued quadratic form \((\mathcal{E}, q, \mathcal{L})\) is \( \text{regular} \) if \( \psi_q \) is an \( \mathcal{O}_S \)-module isomorphism. Otherwise, the radical \( \text{rad}(q) \) is the sheaf kernel of \( \psi_q \), which is a torsion-free subsheaf of \( \mathcal{E} \). We will mostly dispense with the adjective “line bundle-valued.” We define the rank of a quadratic form to be the rank of the underlying module.

A similarity \((\varphi, \lambda_\varphi) : (\mathcal{E}, q, \mathcal{L}) \to (\mathcal{E}', q', \mathcal{L}')\) consists of \( \mathcal{O}_S \)-module isomorphisms \( \varphi : \mathcal{E} \to \mathcal{E}' \) and \( \lambda_\varphi : \mathcal{L} \to \mathcal{L}' \) such that \( q'(\varphi(v)) = \lambda_\varphi \circ q(v) \) on sections. A similarity \((\varphi, \lambda_\varphi)\) is an isometry if \( \mathcal{L} = \mathcal{L}' \) and \( \lambda_\varphi \) is the identity map. We write \( \simeq \) for similarities and \( \cong \) for isometries. Denote by \( \text{GO}(\mathcal{E}, q, \mathcal{L}) \) and \( \text{O}(\mathcal{E}, q, \mathcal{L}) \) the presheaves, on the fppf site \( S_{\text{fppf}} \), of similitudes and isometries of a quadratic form \((\mathcal{E}, q, \mathcal{L})\), respectively. These are sheaves and are representable by affine group schemes of finite presentation over \( S \), indeed closed subgroupschemes of \( \text{GL}(\mathcal{E}) \). The similarity factor defines a homomorphism

\[
\lambda : \text{GO}(\mathcal{E}, q, \mathcal{L}) \to \text{G}_m \text{ with kernel } \text{O}(\mathcal{E}, q, \mathcal{L}).
\]

If \((\mathcal{E}, q, \mathcal{L})\) has even rank \( n = 2m \), then there is a homomorphism \( \det/\lambda^m : \text{GO}(\mathcal{E}, q, \mathcal{L}) \to \mu_2 \), whose kernel is denoted by \( \text{GO}^+(\mathcal{E}, q, \mathcal{L}) \) (this definition of \( \text{GO}^+ \) assumes 2 is invertible on \( S \); in general it is defined as the kernel of the Dickson invariant). The similarity factor \( \lambda : \text{GO}^+(\mathcal{E}, q, \mathcal{L}) \to \text{G}_m \) has kernel denoted by \( \text{O}^+(\mathcal{E}, q, \mathcal{L}) \). Denote by \( \text{PGO}(\mathcal{E}, q, \mathcal{L}) \) the sheaf cokernel of the central subgroupschemes \( \text{G}_m \to \text{GO}(\mathcal{E}, q, \mathcal{L}) \) of homotheties; similarly denote \( \text{PGO}^+(\mathcal{E}, q, \mathcal{L}) \).

At every point where \((\mathcal{E}, q, \mathcal{L})\) is \( \text{regular} \), these group schemes are smooth and reductive (see [19, II.1.2.6, III.5.2.3]) though not necessarily connected. In §2, we will study their structure over points where the form is not regular.

The quadric bundle \( \pi : Q \to S \) associated to a nonzero quadratic form \((\mathcal{E}, q, \mathcal{L})\) of rank \( n \geq 2 \) is the restriction of \( p : \mathbb{P}(\mathcal{E}) \to S \) via the closed embedding \( j : Q \to \mathbb{P}(\mathcal{E}) \) defined by the vanishing of the global section \( q \in \Gamma_S(\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathbb{P}(\mathcal{E})}/S(2) \otimes p^*\mathcal{L}) \). Write \( \mathcal{O}_{Q/S}(1) = j^*\mathcal{O}_{\mathbb{P}(\mathcal{E})}/S(1) \). We say that \((\mathcal{E}, q, \mathcal{L})\) is primitive if \( q_x \neq 0 \) at every point \( x \) of \( S \), i.e., if \( q : \mathcal{E} \to \mathcal{L} \) is an epimorphism. If \( q \) is primitive then \( Q \to \mathbb{P}(\mathcal{E}) \) has relative codimension 1 over \( S \) and \( \pi : Q \to S \) is flat of relative dimension \( n-2 \), cf. [38, 8 Thm. 22.6]. We say that \((\mathcal{E}, q, \mathcal{L})\) is \( \text{generically regular} \) if \( q \) is regular over the generic point of \( S \).

Define the projective similarity class of a quadratic form \((\mathcal{E}, q, \mathcal{L})\) to be the set of similarity classes of quadratic forms \((\mathcal{N} \otimes \mathcal{E}, \text{id}_{\mathcal{N} \otimes \mathcal{E}} \otimes q, \mathcal{N}^\otimes \otimes \mathcal{L})\) ranging over all line bundles \( \mathcal{N} \) on \( S \). Equivalently, this is the set of isometry classes \((\mathcal{N} \otimes \mathcal{E}, \phi \circ (\text{id}_{\mathcal{N} \otimes \mathcal{E}} \otimes q), \mathcal{L}')\) ranging over all isomorphisms \( \phi : \mathcal{N}^\otimes \otimes \mathcal{L} \to \mathcal{L}' \) of line bundles on \( S \). This is referred to as a lax-similarity class in [9]. The main result of this section shows that projectively similar quadratic forms yield isomorphic quadric bundles, while the converse holds under further hypotheses.

Let \( \eta \) be the generic point of \( S \) and \( \pi : Q \to S \) a quadric bundle. Restriction to the generic fiber of \( \pi \) gives rise to a complex

\[
0 \to \text{Pic}(S) \xrightarrow{\pi_*} \text{Pic}(Q) \to \text{Pic}(Q_\eta) \to 0
\]

whose exactness we will study in Proposition 1.6 below.

**Proposition 1.1.** Let \( \pi : Q \to S \) and \( \pi' : Q' \to S \) be quadric bundles associated to quadratic forms \((\mathcal{E}, q, \mathcal{L})\) and \((\mathcal{E}', q', \mathcal{L}')\). If \((\mathcal{E}, q, \mathcal{L})\) and \((\mathcal{E}', q', \mathcal{L}')\) are in the same projective similarity class then \( Q \) and \( Q' \) are \( S \)-isomorphic. The converse holds if \( q \) is assumed to be \( \text{generically regular} \) and (1) is assumed to be exact in the middle.
Proof. Let \((\mathcal{E}, q, \mathcal{L})\) and \((\mathcal{E}', q', \mathcal{L}')\) be projectively similar with respect to an invertible \(\mathcal{O}_S\)-module \(\mathcal{N}\) and \(\mathcal{O}_S\)-module isomorphisms \(\varphi : \mathcal{N} \otimes \mathcal{E} \to \mathcal{E}'\) and \(\lambda : \mathcal{N}^{\otimes 2} \otimes \mathcal{L} \to \mathcal{L}'\) preserving quadratic forms. Let \(p : \mathbb{P}(\mathcal{E}) \to S\) and \(p' : \mathbb{P}(\mathcal{E}') \to S\) be the associated projective bundles and \(h : \mathbb{P}(\mathcal{E}') \to \mathbb{P}(\mathcal{N} \otimes \mathcal{E})\) the \(\mathcal{S}\)-isomorphism associated to \(\varphi^\vee\). There is a natural \(\mathcal{S}\)-isomorphism \(g : \mathbb{P}(\mathcal{N} \otimes \mathcal{E}) \to \mathbb{P}(\mathcal{E})\) satisfying \(g^* \mathcal{O}_{\mathbb{P}(\mathcal{E})/S}(1) \cong \mathcal{O}_{\mathbb{P}(\mathcal{E} \otimes \mathcal{N})/S}(1) \otimes p^* \mathcal{N}\), see [26, II Lemma 7.9].

Denote by \(f = g \circ h : \mathbb{P}(\mathcal{E}') \to \mathbb{P}(\mathcal{E})\) the composition. Then we have an equality of global sections \(f^* q = q'\) in

\[\Gamma_S(\mathbb{P}(\mathcal{E}'), f^* (\mathcal{O}_{\mathbb{P}(\mathcal{E})/S}(2) \otimes p^* \mathcal{L}')) \cong \Gamma_S(\mathbb{P}(\mathcal{E}'), \mathcal{O}_{\mathbb{P}(\mathcal{E}')/S}(2) \otimes p^* \mathcal{L}').\]

and so \(f\) induces a \(\mathcal{S}\)-isomorphism \(Q' \to Q\) upon restriction.

Now suppose that \(q\) is generically regular and that \((1)\) is exact in the middle. Let \(f : Q \to Q'\) be an \(\mathcal{S}\)-isomorphism. First, we have a canonical \(\mathcal{O}_S\)-module isomorphism \(\mathcal{E}^\vee \cong \pi_* \mathcal{O}_{Q/S}(1)\). Indeed, considering the long exact sequence associated to \(p_*\) applied to the short exact sequence

\[0 \to \mathcal{O}_{\mathbb{P}(\mathcal{E})/S}(1) \otimes p^* \mathcal{L}^\vee \to \mathcal{O}_{\mathbb{P}(\mathcal{E})/S}(1) \to j_* \mathcal{O}_{Q/S}(1) \to 0\]

and noting that \(R^ip_* \mathcal{O}_{\mathbb{P}(\mathcal{E})/S}(-1) = 0\) for \(i = 0, 1\), we have

\[\mathcal{E}^\vee = p_* \mathcal{O}_{\mathbb{P}(\mathcal{E})/S}(1) = p_* j_* \mathcal{O}_{Q/S}(1) = \pi_* \mathcal{O}_{Q/S}(1)\]

Next, we claim that \(f^* \mathcal{O}_{Q/S}(1) \cong \mathcal{O}_{Q/S}(1) \otimes \pi^* \mathcal{N}\) for some line bundle \(\mathcal{N}\) on \(S\). Indeed, over the generic fiber, we have \(f^* \mathcal{O}_{Q/S}(1) \cong \mathcal{O}_{Q'/S}(1)\) by the case of smooth quadrics (since \(q\) is generically regular) over fields, cf. [21, Lemma 69.2]. The exactness of \((1)\) in the middle finishes the proof of the claim.

Then, note that we have the following chain of \(\mathcal{O}_S\)-module isomorphisms

\[\mathcal{E}^\vee \cong \pi_* \mathcal{O}_{Q/S}(1) \cong \pi_* f_* (f^* \mathcal{O}_{Q'/S}(1) \otimes \pi^* \mathcal{N}^\vee) \cong \pi_* \mathcal{O}_{Q'/S}(1) \otimes \pi_* \pi^* \mathcal{N}^\vee \cong \mathcal{E'}^\vee \otimes \mathcal{N}^\vee\]

(again we need \(n \geq 1\)). Finally, one is left to check that the induced dual \(\mathcal{O}_S\)-module isomorphism \(\mathcal{N} \otimes \mathcal{E}' \to \mathcal{E}'\) preserves the quadratic forms. \(\square\)

**Definition 1.2.** The determinant \(\psi_q : \det \mathcal{E} \to \det \mathcal{E}^\vee \otimes \mathcal{L}^{\otimes n}\) gives rise to a global section of \((\det \mathcal{E}^\vee)^{\otimes 2} \otimes \mathcal{L}^{\otimes n}\), whose divisor of zeros is called the discriminant divisor \(D\).

The reduced subscheme associated to \(D\) is precisely the locus of points where the radical of \(q\) is nontrivial. If \(q\) is generically regular, then \(D \subset S\) is closed of codimension one.

**Definition 1.3.** We say that a quadratic form \((\mathcal{E}, q, \mathcal{L})\) has simple degeneration if

\[\text{rk}_{\kappa(x)} \text{rad}(\mathcal{E}_x, q_x, \mathcal{L}_x) \leq 1\]

for every closed point \(x\) of \(S\), where \(\kappa(x)\) is the residue field of \(\mathcal{O}_{S,x}\).

Our first lemma concerns the local structure of simple degeneration.

**Lemma 1.4.** Let \((\mathcal{E}, q)\) be a quadratic form with simple degeneration over the spectrum of a local ring \(R\) with \(2\) invertible. Then \((\mathcal{E}, q) \cong (\mathcal{E}_1, q_1)\) \(\perp (R, < \pi >)\) where \((\mathcal{E}_1, q_1)\) is regular and \(\pi \in R\).

*Proof.* Over the residue field \(k\), the form \((\mathcal{E}, q)\) has a regular subform \((\mathcal{E}_1, q_1)\) of corank one, which can be lifted to a regular orthogonal direct summand \((\mathcal{E}_1, q_1)\) of corank 1 of \((\mathcal{E}, q)\), cf. [7, Cor. 3.4]. This gives the required decomposition. Moreover, we can lift a diagonalization \(q_1 \cong < \pi_1, \ldots, \pi_{n-1}, \pi >\) with \(\pi_i \in k^\times\), to a diagonalization

\[q \cong < u_1, \ldots, u_{n-1}, \pi >\]

with \(u_i \in R^\times\) and \(\pi \in R\). \(\square\)

Let \(D \subset S\) be a regular divisor. Since \(S\) is normal, the local ring \(\mathcal{O}_{S,D'}\) at the generic point of a component \(D'\) of \(D\) is a discrete valuation ring. When \(2\) is invertible on \(S\),
Lemma 1.4 shows that a quadratic form \((\mathcal{E}, q, L)\) with simple degeneration along \(D\) can be diagonalized over \(O_{S,D'}\) as
\[ q \cong < u_1, \ldots, u_{r-1}, u_r \pi^e > \]
where \(u_i\) are units and \(\pi\) is a parameter of \(O_{S,D'}\). We call \(e \geq 1\) the multiplicity of the simple degeneration along \(D\). If \(e\) is even for every component of \(D\), then there is a birational morphism \(g : S' \to S\) such that the pullback of \((\mathcal{E}, q, L)\) to \(S'\) is regular. We will focus on quadratic forms with simple degeneration of multiplicity one along (all components of) \(D\).

We can give a geometric interpretation of having simple degenerate.

Proposition 1.5. Let \(\pi : Q \to S\) be the quadric bundle associated to a generically regular quadratic form \((\mathcal{E}, q, L)\) over \(S\) and \(D \subset S\) its discriminant divisor. Then:

a) \(q\) has simple degeneration if and only if the fiber \(Q_x\) of its associated quadric bundle has at worst isolated singularities for each closed point \(x\) of \(S\);

b) if 2 is invertible on \(S\) and \(D\) is reduced, then any simple degeneration along \(D\) has multiplicity one;

c) if 2 is invertible on \(S\) and \(D\) is regular, then any degeneration along \(D\) is simple of multiplicity one;

d) if \(S\) is regular and \(q\) has simple degeneration, then \(D\) is regular if and only if \(Q\) is regular.

Proof. The first claim follows from the classical geometry of quadrics over a field: the quadric of a nondegenerate form is smooth while the quadric of a form with nontrivial radical has isolated singularity if and only if the radical has rank one. As for the second claim, the multiplicity of the simple degeneration is exactly the scheme-theoretic multiplicity of the divisor \(D\). For the third claim, see [16, §3], [29, Rem. 7.1], or [3, Rem. 2.6]. The final claim is standard, cf. [29, Lemma 5.2]. □

We do not need the full flexibility of the following general result, but we include it for completeness.

Proposition 1.6. Let \(\pi : Q \to S\) be a flat morphism of noetherian integral separated normal schemes and \(\eta\) the generic point of \(S\). Then the complex (1) is:

a) exact at right if \(Q\) is locally factorial;

b) exact in the middle if \(S\) is locally factorial;

c) exact at left if \(\pi : Q \to S\) is proper with geometrically integral fibers.

Proof. First, note that flat pullback and restriction to the generic fiber give rise to an exact sequence of Weil divisor groups
\[ 0 \to \text{Div}(S) \xrightarrow{\pi^*} \text{Div}(Q) \to \text{Div}(Q_{\eta}) \to 0. \]
Indeed, as \(\text{Div}(Q_{\eta}) = \lim_{\leftarrow} \text{Div}(Q_U)\), where the limit is taken over all dense open sets \(U \subset S\) and we write \(Q_U = Q \times_S U\), the exactness at right and center of sequence (2) then follows from usual exactness of the excision sequence
\[ Z^0(\pi^{-1}(S \setminus U)) \to \text{Div}(Q) \to \text{Div}(Q_U) \to 0 \]
cf. [22, 1 Prop. 1.8]. Finally, sequence (2) is exact at left since \(\pi\) is surjective on codimension 1 points.

Since \(\pi\) is dominant, the sequence of Weil divisor groups induces an exact sequence of Weil divisor class groups, which is the bottom row of the following commutative diagram
\[ \begin{array}{cccc}
\text{Pic}(S) & \xrightarrow{\pi^*} & \text{Pic}(Q) & \xrightarrow{\pi^*} \text{Pic}(Q_{\eta}) \xrightarrow{} 0 \\
\text{Cl}(S) & \xrightarrow{\pi^*} & \text{Cl}(Q) & \xrightarrow{\pi^*} \text{Cl}(Q_{\eta}) \xrightarrow{} 0
\end{array} \]
of abelian groups. The vertical inclusions become equalities under a locally factorial hypothesis, cf. [24, Cor. 21.6.10]. Thus a) and b) are immediate consequences of diagram chases.

To prove c), assume \( \pi^* [\mathcal{L}] = [\pi^* \mathcal{L}] = 0 \) in \( \text{Cl}(Q) \) for the class \( [\mathcal{L}] \in \text{Cl}(S) \) of some line bundle \( \mathcal{L} \in \text{Pic}(Q) \). Then \( [\pi^* \mathcal{L}] = \text{div}_Y(f) \) in \( \text{Div}(Q) \) for some \( f \in K^X_Q \). But as \( [\pi^* \mathcal{L}]_{\eta} = 0 \) in \( \text{Div}(Q_\eta) \), we have that \( \text{div}_{Q_\eta}(f) = 0 \) (since \( K_Q = K_{Q_\eta} \)) so that \( f \in \Gamma(Q_\eta, \mathcal{O}_{Q_\eta}^\times) \), i.e., \( f \) has neither zeros nor poles. By the hypothesis on \( \pi \), we have that \( Q_\eta \) is a proper geometrically integral \( K_S \)-scheme, so that \( \Gamma(Q_\eta, \mathcal{O}_{Q_\eta}^\times) = K_S \). In particular, \( f \in K_S \) via the inclusion \( K_S \hookrightarrow K_Q \). Hence \( \pi^*([\mathcal{L}] - \text{div}_S(f)) = 0 \) in \( \text{Div}(Q) \), thus \( [\mathcal{L}] = \text{div}_S(f) \) in \( \text{Div}(S) \), and so \( \mathcal{L} \) is trivial in \( \text{Pic}(S) \).

\[ \square \]

**Corollary 1.7.** Let \( S \) be a regular integral scheme with 2 invertible and \((\mathcal{E}, q, \mathcal{L})\) a quadratic form on \( S \) of rank \( \geq 4 \) having at most simple degeneration along a regular divisor \( D \subset S \). Let \( \pi : Q \to S \) be the associated quadric bundle. Then the complex (1) is exact.

**Proof.** First, recall that a quadratic form over a field contains a nondegenerate subform of rank \( \geq 3 \) if and only if its associated quadric is irreducible, cf. [26, I Ex. 5.12]. Hence the fibers of \( \pi \) are geometrically irreducible. By Proposition 1.5, \( Q \) is regular. Quadratic forms with simple degeneration are primitive, hence \( \pi \) is flat. Thus we can apply all the parts of Proposition 1.6. \[ \square \]

We will define \( \text{Quad}_D^n(S) \) to be the set of projective similarity classes of line bundle-valued quadratic forms of rank \( n + 2 \) on \( S \) with simple degeneration of multiplicity one along an effective Cartier divisor \( D \). An immediate consequence of Propositions 1.1 and 1.5 and Corollary 1.7 is the following.

**Corollary 1.8.** For \( n \geq 2 \) and \( D \) reduced, the set \( \text{Quad}_D^n(S) \) is in bijection with the set of \( S \)-isomorphism classes of quadric bundles of relative dimension \( n \) with isolated singularities in the fibers above \( D \).

**Definition 1.9.** Now let \((\mathcal{E}, q, \mathcal{L})\) be a quadratic form of rank \( n \), \( \mathcal{E}_0 = \mathcal{E}_0(\mathcal{E}, q, \mathcal{L}) \) its even Clifford algebra (see [12] or [2, §1.8]), and \( \mathcal{Z} = \mathcal{Z}(\mathcal{E}, q, \mathcal{L}) \) its center. Then \( \mathcal{E}_0 \) is a locally free \( \mathcal{O}_S \)-algebra of rank \( 2^{n-1} \), cf. [32, IV.1.6]. If \( q \) is generically regular of even rank (we are still assuming that \( S \) is integral and regular) then \( \mathcal{Z} \) is a locally free \( \mathcal{O}_S \)-algebra of rank two, see [32, IV Prop. 4.8.3]. The associated finite flat morphism \( f : T \to S \) of degree two is called the discriminant cover.

**Lemma 1.10** ([3, App. B]). Let \((\mathcal{E}, q, \mathcal{L})\) be a quadratic form of even rank with simple degeneration of multiplicity one along \( D \subset S \) and \( f : T \to S \) its discriminant cover. Then \( f^* \mathcal{O}(D) \) is a square in \( \text{Pic}(T) \) and the branch divisor of \( f \) is precisely \( D \).

By abuse of notation, we also denote by \( \mathcal{E}_0 = \mathcal{E}_0(\mathcal{E}, q, \mathcal{L}) \) the \( \mathcal{O}_T \)-algebra associated to the \( \mathcal{Z} \)-algebra \( \mathcal{E}_0 = \mathcal{E}(\mathcal{E}, q, \mathcal{L}) \). The center \( \mathcal{Z} \) is an étale algebra over every point of \( S \) where \((\mathcal{E}, q, \mathcal{L})\) is regular and \( \mathcal{E}_0 \) is an Azumaya algebra over every point of \( T \) lying over a point of \( S \) where \((\mathcal{E}, q, \mathcal{L})\) is regular.

**Lemma 1.11.** Let \((\mathcal{E}, q, \mathcal{L})\) be a quadratic form with simple degeneration over an integral scheme \( S \) with 2 invertible and let \( T \to S \) be the discriminant cover. Then \( \mathcal{E}_0 \) is a locally free \( \mathcal{O}_T \)-algebra.

**Proof.** The question is local, so we can assume that \( S = \text{Spec} \, R \) for a local domain \( R \) with 2 invertible. We fix a trivialization of \( \mathcal{L} \) and let \( n \) be the rank of \( q \). By Lemma 1.4, we write \((\mathcal{E}, q) = (\mathcal{E}_1, q_1) \perp (R, < \pi >)\) for \( q_1 \) regular and \( \pi \in R \). Consider the canonically induced homomorphism \( \mathcal{E}_0(\mathcal{E}_1, q_1) \to \mathcal{E}_0(\mathcal{E}, q) \). We claim that the map

\begin{equation}
\mathcal{E}_0(\mathcal{E}_1, q_1) \otimes_{\mathcal{O}_S} \mathcal{Z}(\mathcal{E}, q) \to \mathcal{E}_0(\mathcal{E}, q)
\end{equation}


induced from multiplication in \( \mathcal{O}_0(\mathcal{E}, q) \), is a \( \mathcal{Z}(\mathcal{E}, q) \)-algebra isomorphism. Indeed, since \( \mathcal{O}_0(\mathcal{E}_1, q_1) \) is an Azumaya \( \mathcal{O}_S \)-algebra, then \( \mathcal{O}_0(\mathcal{E}_1, q_1) \otimes_{\mathcal{O}_S} \mathcal{Z}(\mathcal{E}, q) \) is an Azumaya \( \mathcal{Z}(\mathcal{E}, q) \)-algebra. In particular, the map is injective. If \( q \) is generically regular (i.e., \( \pi \neq 0 \)), then then this map is generically an isomorphism, hence an isomorphism. Otherwise \( \pi = 0 \), in which case \( \mathcal{Z}(\mathcal{E}, q) \cong \mathcal{O}_S[\varepsilon]/(\varepsilon^2) \) and we can argue directly using the exact sequence

\[
0 \to \varepsilon \mathcal{O}_1(\mathcal{E}_1, q_1) \to \mathcal{O}_0(\mathcal{E}, q) \to \mathcal{O}_0(\mathcal{E}_1, q_1) \to 0
\]

where \( \varepsilon \in \mathcal{E} \) generates the radical, and the fact that \( \varepsilon \mathcal{O}_1(\mathcal{E}_1, q_1) \cong \varepsilon \mathcal{O}_0(\mathcal{E}_1, q_1) \). \( \square \)

Finally, we prove a strengthened version of \([36, \text{Prop. 3.13}]\).

**Lemma 1.12.** Let \((V, q)\) be a quadratic form of even rank \( n = 2m > 2 \) over a field \( k \). Then the following are equivalent:

a) The radical of \( q \) has rank at most 1.

b) The center \( Z(q) \subset C_0(q) \) is a \( k \)-algebra of rank 2.

c) The algebra \( C_0(q) \) is \( Z(q) \)-Azumaya of degree \( 2^{m-1} \).

If \( n = 2 \), then \( C_0(q) \) is always commutative.

**Proof.** We will prove that \( a) \Rightarrow c) \Rightarrow b) \Rightarrow a) \). If \( q \) is nondegenerate (i.e., has trivial radical), then it is classical that \( Z(q) \) is an étale quadratic algebra and \( C_0(q) \) is an Azumaya \( Z(q) \)-algebra. If \( \text{rad}(q) \) has rank 1, generated by \( v \in V \), then a straightforward computation shows that \( Z(q) \cong k[\varepsilon]/(\varepsilon^2) \), where \( \varepsilon \in vC_1(q) \cap Z(q) \subset k \). Furthermore, we have that \( C_0(q) \otimes_{k[\varepsilon]/(\varepsilon^2)} k \cong C_0(q)/vC_1(q) \cong C_0(q/\text{rad}(q)) \) where \( q/\text{rad}(q) \) is nondegenerate of rank \( n-1 \), cf. \([21, \text{II \S11, p. 58}]\). Since \( C_0(q) \) is finitely generated and free as a \( Z(q) \)-module by Lemma 1.11, and has special fiber a central simple algebra \( C_0(q/\text{rad}(q)) \) of degree \( 2^{m-1} \), it is \( Z(q) \)-Azumaya of degree \( 2^{m-1} \). Hence we’ve proved that \( a) \Rightarrow c) \)

The fact that \( c) \Rightarrow b) \chi \) clear from a dimension count. To prove \( b) \Rightarrow a) \), suppose that \( \text{rk}_k \text{rad}(q) \geq 2 \). Then the embedding \( \bigwedge^2 \text{rad}(q) \subset C_0(q) \) is central (and does not contain the central subalgebra generated by \( V \otimes \varepsilon \), as \( q \) has rank \( > 2 \)). More explicitly, if \( e_1, e_2, \ldots, e_n \) is a block diagonalization (into 1 and 2 dimensional spaces), then \( k \otimes ke_1 \cdots e_n \otimes \bigwedge^2 \text{rad}(q) \subset Z(q) \). Thus \( Z(q) \) has \( k \)-rank at least \( 2 + \text{rk}_k \bigwedge^2 \text{rad}(q) \geq 3 \). \( \square \)

**Proposition 1.13.** Let \((\mathcal{E}, q, \mathcal{L})\) be a generically regular quadratic form of even rank on \( S \) with discriminant cover \( f : T \to S \). Then \( \mathcal{O}_0(\mathcal{E}, q, \mathcal{L}) \) is an Azumaya \( \mathcal{O}_T \)-algebra if and only if \((\mathcal{E}, q, \mathcal{L})\) has simple degeneration or has rank 2 (and any degeneration).

**Proof.** Since \( \mathcal{O}_0 \) is a locally free \( \mathcal{O}_S \)-module it is a locally free \( \mathcal{O}_T \)-module (as \( \mathcal{O}_T \) is locally free of rank 2). Thus \( \mathcal{O}_0 \) is an \( \mathcal{O}_T \)-Azumaya algebra if and only if its fiber at every closed point \( y \) of \( T \) is a central simple \( \kappa(y) \)-algebra. If \( f(y) = x \), then \( \kappa(y) \) is a \( \kappa(x) \)-algebra of rank 2. Then we apply Lemma 1.12 to the fiber of \( \mathcal{O}_0 \) at \( x \). \( \square \)

## 2. Orthogonal groups with simple degeneration

The main results of this section concern the special (projective) orthogonal group schemes of quadratic forms with simple degeneration over semilocal principal ideal domains. Let \( S \) be a regular integral scheme. Recall, from Proposition 1.13, that if \((\mathcal{E}, q, \mathcal{L})\) is a line bundle-valued quadratic form on \( S \) with simple degeneration along a closed subscheme \( D \) of codimension 1, then the even Clifford algebra \( \mathcal{O}_0(q) \) is an Azumaya algebra over the discriminant cover \( T \to S \).

**Theorem 2.1.** Let \( S \) be a regular scheme with 2 invertible, \( D \) a regular divisor, \((\mathcal{E}, q, \mathcal{L})\) a quadratic form of rank 4 on \( S \) with simple degeneration along \( D \), \( T \to S \) its discriminant cover, and \( \mathcal{O}_0(q) \) its even Clifford algebra over \( T \). The canonical homomorphism

\[
c : \text{PO}_0^+(q) \to R_{T/S} \text{PGL}(\mathcal{O}_0(q))
\]

induced from the functor \( \mathcal{O}_0 \), is an isomorphism of \( S \)-group schemes.
The proof involves several preliminary general results concerning orthogonal groups of quadratic forms with simple degeneration and will occupy the remainder of this section.

Let \( S = \text{Spec} R \) be an affine scheme with 2 invertible, \( D \subset S \) be the closed scheme defined by an element \( \pi \) in the Jacobson radical of \( R \), and let \( (V,q) = (V_1,q_1) \perp (R,<n>) \) be a quadratic form of rank \( n \) over \( S \) with \( q_1 \) regular and \( V_1 \) free. Let \( Q_1 \) be a Gram matrix of \( q_1 \). Then as an \( S \)-group scheme, \( \mathcal{O}(q) \) is the subvariety of the affine space of block matrices

\[
\begin{pmatrix}
A & v \\
 w & u
\end{pmatrix}
\]

(4) satisfying

\[
A^t Q_1 A + \pi w^t w = Q_1 \\
v^t Q_1 v = (1 - u^t u) \pi
\]

where \( A \) is an invertible \((n-1) \times (n-1)\) matrix, \( v \) is an \( n \times 1 \) column vector, \( w \) is a \( 1 \times n \) row vector, and \( u \) a unit. Note that since \( A \) and \( Q_1 \) are invertible, the second relation in (4) implies that \( v \) is determined by \( w \) and \( u \) and that \( v \) is invertible for \( R/\pi \), and hence the third relation implies that \( \pi^2 = 1 \) in \( R/\pi \). Define \( \mathcal{O}^+(q) = \ker(\det : \mathcal{O}(q) \to \mathbb{G}_m) \). If \( R \) is an integral domain then \( \det \) factors through \( \mu_2 \) and \( \mathcal{O}^+(q) \) is the irreducible component of the identity.

**Proposition 2.2.** Let \( R \) be a regular local ring with 2 invertible, \( \pi \in m \) in the maximal ideal, and \( (V,q) = (V_1,q_1) \perp (R,<n>) \) a quadratic form with \( q_1 \) regular of rank \( n - 1 \) of \( R \). Then \( \mathcal{O}(q) \) and \( \mathcal{O}^+(q) \) are smooth \( R \)-group schemes.

**Proof.** Let \( K \) be the fraction field of \( R \) and \( k \) its residue field. First, we’ll show that the equations in (4) define a local complete intersection morphism in the affine space \( A_R^2 \) of \( n \times n \) matrices over \( R \). Indeed, the condition that the generic \( n \times n \) matrix \( M \) over \( R[x_1, \ldots, x_n] \) is orthogonal with respect to a given symmetric \( n \times n \) matrix \( Q \) over \( R \) can be written as the equality of symmetric matrices \( M^t Q M = Q \) over \( R[x_1, \ldots, x_n][\{\det M^{-1}\}] \), hence giving \( n(n+1)/2 \) equations. Hence, the orthogonal group is the scheme defined by these \( n(n+1)/2 \) equations in the Zariski open of \( A_R^2 \) defined by \( \det M \).

Since \( q \) is generically regular of rank \( n \), the generic fiber of \( \mathcal{O}(q) \) has dimension \( n(n-1)/2 \). By (4), the special fiber of \( \mathcal{O}^+(q) \) is isomorphic to the group scheme of euclidean transformations of the regular quadratic space \((V_1,q_1)\), which is the semidirect product

\[
\mathcal{O}^+(q) \times_R k \cong G_a^{n-1} \rtimes \mathcal{O}(q_{1,k})
\]

where \( G_a^{n-1} \) acts in \( V_1 \) by translation and \( \mathcal{O}(q_{1,k}) \) acts on \( G_a^{n-1} \) by conjugation. In particular, the special fiber of \( \mathcal{O}^+(q) \) has dimension \( (n-1)(n-2)/2 + (n-1) = n(n-1)/2 \), and similarly with \( \mathcal{O}(q) \).

In particular, \( \mathcal{O}(q) \) is a local complete intersection morphism. Since \( R \) is Cohen–Macaulay (being regular local) then \( R[x_1, \ldots, x_n][\{\det M^{-1}\}] \) is Cohen–Macaulay, and thus \( \mathcal{O}(q) \) is Cohen–Macaulay. By the “miracle flatness” theorem, equidimensional and Cohen–Macaulay over a regular base implies that \( \mathcal{O}(q) \to \text{Spec} R \) is flat, cf. [24, Prop. 15.4.2] or [38, 8 Thm. 23.1]. Thus \( \mathcal{O}^+(q) \to \text{Spec} R \) is also flat. The generic fiber of \( \mathcal{O}^+(q) \) is smooth since \( q \) is generically regular while the special fiber is smooth since it is a (semi)direct product of smooth schemes (recall that \( \mathcal{O}(q_1) \) is smooth since \( 2 \) is invertible). Hence \( \mathcal{O}^+(q) \to \text{Spec} R \) flat and has geometrically smooth fibers, hence is smooth. \( \square \)

**Proposition 2.3.** Let \( S \) be a regular scheme with 2 invertible and \((\mathcal{E},q,\mathcal{L})\) a quadratic form of even rank on \( S \) with simple degeneration. Then the group schemes \( \mathcal{O}(q), \mathcal{O}^+(q), \mathcal{G}O(q), \mathcal{G}O^+(q), \mathcal{P}GO(q), \) and \( \mathcal{P}GO^+(q) \) are \( S \)-smooth. If \( T \to S \) is the discriminant cover and \( \mathcal{C}_0(q) \) is the even Clifford algebra of \((\mathcal{E},q,\mathcal{L}) \) over \( T \), then \( R_{T/S} \mathcal{G}L_1(\mathcal{C}_0(q)), R_{T/S} \mathcal{S}L_1(\mathcal{C}_0(q)), \) and \( R_{T/S} \mathcal{P}GL_1(\mathcal{C}_0(q)) \) are smooth \( S \)-schemes.

**Proof.** The \( S \)-smoothness of \( \mathcal{O}(q) \) and \( \mathcal{O}^+(q) \) follows from the fibral criterion for smoothness, with Proposition 2.2 handling points of \( S \) contained in the discriminant divisor. As \( \mathcal{G}O \cong (\mathcal{O}(q) \times \mathbb{G}_m)/\mu_2, \mathcal{G}O^+(q) \cong (\mathcal{O}^+(q) \times \mathbb{G}_m)/\mu_2, \mathcal{P}GO(q) \cong \mathcal{G}O(q)/\mathbb{G}_m, \)
PGO\(^+\)(q) \cong GO\(^+\)(q)/G_m are quotients of S-smooth group schemes by flat closed subgroups, they are S-smooth. Finally, \(\mathcal{O}(q)\) is an Azumaya \(\mathcal{O}\)-algebra by Proposition 1.13, hence \(\text{GL}_1(\mathcal{O}(q)), \text{SL}_1(\mathcal{O}(q))\), and \(\text{PGL}_1(\mathcal{O}(q))\) are smooth \(T\)-schemes, hence their Weil restrictions via the finite flat map \(T \to S\) are \(S\)-smooth by [17, App. A.5, Prop. A.5.2]. □

**Remark 2.4.** If the radical of \(q_s\) has rank \(\geq 2\) at a point \(s\) of \(S\), a calculation shows that the fiber of \(O(q) \to S\) over \(s\) has dimension \(> n(n-1)/2\). In particular, if \(q\) is generically regular over \(S\) then \(O(q) \to S\) is not flat. The smoothness of \(O(q)\) is a special feature of quadratic forms with simple degeneration.

We will also make frequent reference to the classical version of Theorem 2.1 in the regular case, when the discriminant cover is étale.

**Theorem 2.5.** Let \(S\) be a scheme and \((\mathcal{S}, q, \mathcal{L})\) a regular quadratic form of rank 4 with discriminant cover \(T \to S\) and even Clifford algebra \(\mathcal{O}(q)\) over \(T\). The canonical homomorphism
\[
c : PGO^+(q) \to R_{T/S}PGL(\mathcal{O}(q)),
\]
induced from the functor \(\mathcal{O}\), is an isomorphism of \(S\)-group schemes.

**Proof.** The proof over affine schemes \(S\) in [35, §10] carries over immediately. See [33, IV.15.B] for the particular case of \(S\) the spectrum of a field. Also see [2, §5.3]. □

Finally, we come to the proof of the main result of this section.

**Proof of Theorem 2.1.** We will use the following fibral criteria for relative isomorphisms (cf. [24, IV.4 Cor. 17.9.5]): let \(g : X \to Y\) be a morphism of \(S\)-schemes locally of finite presentation over a scheme \(S\) and assume \(X\) is \(S\)-flat, then \(g\) is an \(S\)-isomorphism if and only if its fiber \(g_s : X_s \to Y_s\) is an isomorphism over each geometric point \(s\) of \(S\).

For each \(s\) in \(S\setminus D\), the fiber \(g_s\) is a regular quadratic form over \(\kappa(s)\), hence the fiber \(c_s : PGO^+(q_s) \to R_{T/S}PGL(\mathcal{O}(q_s))\) is an isomorphism by Theorem 2.5. We are thus reduced to considering the geometric fibers over points in \(D\). Let \(s = \text{Spec } k\) be a geometric point of \(D\). By Lemma 1.12, there is a natural identification of the fiber \(T_s = \text{Spec } k_s\), where \(k_s = k[\epsilon]/(\epsilon^2)\).

We use the following criteria for isomorphisms of group schemes (cf. [33, VI Prop. 22.5]): let \(g : X \to Y\) be a homomorphism of affine \(k\)-group schemes of finite type over an algebraically closed field \(k\) and assume that \(Y\) is smooth, then \(g\) is a \(k\)-isomorphism if and only if \(g : X(k) \to Y(k)\) is an isomorphism on \(k\)-points and the Lie algebra map \(dg : \text{Lie}(X) \to \text{Lie}(Y)\) is an injective map of \(k\)-vector spaces.

First, we shall prove that \(c\) is an isomorphism on \(k\) points. Applying cohomology to the exact sequence
\[
1 \to \mu_2 \to O^+(q) \to PGO^+(q) \to 1,
\]
we see that the corresponding sequence of \(k\)-points is exact since \(k\) is algebraically closed. Hence it suffices to show that \(O^+(q)(k) \to PGL_1(\mathcal{O}(q))(k)\) is surjective with kernel \(\mu_2(k)\).

Write \(q = q_1 \perp <0>\), where \(q_1\) is regular over \(k\). Denote by \(E\) the unipotent radical of \(O^+(q)\). We will now proceed to define the following diagram
\[
\begin{array}{ccccccccc}
1 & \rightarrow & \mu_2 & \rightarrow & O^+(q) & \rightarrow & O(q_1) & \rightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \rightarrow & E & \rightarrow & PGL_1(\mathcal{O}(q)) & \rightarrow & PGL_1(\mathcal{O}(q_1)) & \rightarrow & 1 \\
\downarrow & & & & & & & & \\
1 & \rightarrow & I + \epsilon \mathcal{O}(q) & \rightarrow & PGL_1(\mathcal{O}(q)) & \rightarrow & PGL_1(\mathcal{O}(q_1)) & \rightarrow & 1 \\
\end{array}
\]
of groups schemes over $k$, and verify that it is commutative with exact rows and columns. This will finish the proof of the statement concerning $c$ being an isomorphism on $k$-points. We have $H^1_{\text{et}}(k, E) = 0$ and also $H^1_{\text{et}}(k, \mu_2) = 0$, as $k$ is algebraically closed. Hence it suffices to argue after taking $k$-points in the diagram.

The central and right most vertical columns are induced by the standard action of the (special) orthogonal group on the even Clifford algebra. The right most column is an exact sequence

$$1 \to \mu_2 \to O(q_1) \cong \mu_2 \times O^+(q_1) \to \text{PGL}_1(C_0(q_1)) \to 1$$

arising from the split isogeny of type $A_1 = B_1$, cf. [33, IV.15.A]. The central row is defined by the map $O^+(q)(k) \to O(q)(k)$ defined by

$$ \begin{pmatrix} A & v \\ w & u \end{pmatrix} \mapsto A$$

in the notation of (4). In particular, the group $E(k)$ consists of block matrices of the form

$$ \begin{pmatrix} I & 0 \\ w & 1 \end{pmatrix}$$

for $w \in k^3$. Since $O(q_1)$ is semisimple, the kernel contains the unipotent radical $E$, so coincides with it by a dimension count. The bottom row is defined as follows. By (3), we have $C_0(q) \cong C_0(q_1) \otimes_k Z(q) \cong C_0(q_1) \otimes_k k$. The map $\text{PGL}_1(C_0(q)) \to \text{PGL}_1(C_0(q_1))$ is thus defined by the reduction $k_c \to k$. This also identifies the kernel as $I + \epsilon c_0(q)$, where $c_0(q)$ is the affine scheme of reduced trace zero elements of $C_0(q)$, which is identified with the Lie algebra of $\text{PGL}_1(C_0(q))$ in the usual way. The only thing to check is that the bottom left square commutes (since by (5), the central row is split). By the five lemma, it will then suffice to show that $E(k) \to 1 + \epsilon c_0(q)(k)$ is an isomorphism.

To this end, we can diagonalize $q = < 1, -1, 1, 0 >$, since $k$ is algebraically closed of characteristic $\neq 2$. Let $e_1, \ldots, e_4$ be the corresponding orthogonal basis. Then $C_0(q_1)(k)$ is generated over $k$ by $1, e_1 e_2, e_2 e_3$, and $e_1 e_3$ and we have an identification $\phi : C_0(q_1)(k) \to M_2(k)$ given by

$$1 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad e_1 e_2 \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad e_2 e_3 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e_1 e_3 \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Similarly, $C_0(q)$ is generated over $Z(q) = k_c$ by $1, e_1 e_2, e_2 e_3$, and $e_1 e_3$, since we have

$$e_1 e_4 = \epsilon e_2 e_3, \quad e_2 e_4 = \epsilon e_1 e_3, \quad e_3 e_4 = \epsilon e_1 e_2, \quad e_1 e_2 e_3 e_4 = \epsilon.$$

and we have a identification $\psi : C_0(q) \to M_2(k_c)$ extending $\phi$. With respect to this $k_c$-algebra isomorphism, we have a group isomorphism $\text{PGL}_1(C_0(q))(k) = \text{PGL}_2(k_c)$ and a Lie algebra isomorphism $c_0(q)(k) \cong \mathfrak{sl}_2(k_c)$, where $\mathfrak{sl}_2$ is the scheme of traceless $2 \times 2$ matrices. We claim that the map $E(k) \to I + \epsilon \mathfrak{sl}_2(k)$ is explicitly given by

$$ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mapsto I - \frac{\epsilon}{2} \begin{pmatrix} a & -b + c \\ b + c & -a \end{pmatrix}. $$

Indeed, let $\phi_{a,b,c} \in E(S_0)$ be the orthogonal transformation whose matrix displayed in (6), and $\sigma_{a,b,c}$ its image in $I + \epsilon \mathfrak{sl}_2(k)$, thought of as an automorphism of $C_0(q)(k_c)$. Then we have

$$\sigma_{a,b,c}(e_1 e_2) = e_1 e_2 + b e_2 e_3 - a e_1 e_3$$
$$\sigma_{a,b,c}(e_2 e_3) = e_2 e_3 + a e_1 e_3 - b e_1 e_2$$
$$\sigma_{a,b,c}(e_1 e_3) = e_1 e_3 + a e_2 e_3 - c e_1 e_2$$
and $\sigma_{a,b,c}(\epsilon) = \epsilon$. It is then a straightforward calculation to see that

$$\sigma_{a,b,c} = \text{ad}(1 - \frac{1}{2}\epsilon(ce_1e_2 + ae_2e_3 - be_1e_3)),$$

where $\text{ad}$ is conjugation in the Clifford algebra, and furthermore, that $\psi$ takes $ce_1e_2 + ae_2e_3 - be_1e_3$ to the $2 \times 2$ matrix displayed in (6). Thus the map $E(k) \to I + \epsilon\mathfrak{sl}_2(k)$ is as stated, and in particular, is an isomorphism. Thus the diagram is commutative with exact rows and columns, and in particular, $c : \text{PGO}^+(q) \to \text{PGL}_1(\mathcal{E}_0(q))$ is an isomorphism on $k$-points.

Now we prove that the Lie algebra map $dc$ is injective. Consider the commutative diagram

$$
\begin{array}{cccc}
1 & \longrightarrow & I + x\mathfrak{so}(q)(k) & \longrightarrow & \mathcal{O}^+(q)(k[x]/(x^2)) & \longrightarrow & \mathcal{O}^+(q)(k) & \longrightarrow & 1 \\
1 + x\text{dc} & \downarrow & \downarrow & & \downarrow & & \downarrow & \\
1 & \longrightarrow & I + x\mathfrak{g}(k) & \longrightarrow & \text{PGL}_1(\mathcal{E}_0(q))(k[x]/(\epsilon^2, x^2)) & \longrightarrow & \text{PGL}_1(\mathcal{E}_0(q))(k) & \longrightarrow & 1
\end{array}
$$

where $\mathfrak{so}(q)$ and $\mathfrak{g}$ are the Lie algebras of $\mathcal{O}^+(q)$ and $R_{k_c/k}\text{PGL}_1(\mathcal{E}_0(q))$, respectively.

The Lie algebra $\mathfrak{so}(q_1)$ of $\mathcal{O}(q_1)$ is identified with the scheme of $3 \times 3$ matrices $A$ such that $AQ_1$ is skew-symmetric, where $Q_1 = \text{diag}(1, -1, 1)$. It is then a consequence of (4) that $I + x\mathfrak{so}(q_1)(k)$ consists of block matrices of the form

$$\begin{pmatrix}
I + xA & 0 \\
xw & 1
\end{pmatrix}
$$

for $w \in \mathbb{A}^3(k)$ and $A \in \mathfrak{so}(q_1)(k)$. Since

$$\begin{pmatrix}
I + xA & 0 \\
xw & 1
\end{pmatrix} = \begin{pmatrix}
I + xA & 0 \\
0 & 1
\end{pmatrix} \begin{pmatrix}
I & 0 \\
xw & 1
\end{pmatrix} = \begin{pmatrix}
I & 0 \\
xw & 1
\end{pmatrix} \begin{pmatrix}
I + xA & 0 \\
0 & 1
\end{pmatrix},$$

we see that $I + x\mathfrak{so}(q)$ has a direct product decomposition $E \times (I + x\mathfrak{so}(q_1))$. We claim that the map $\mathfrak{h} \to \mathfrak{g}$ is explicitly given by the product map

$$\begin{pmatrix}
I + xA & 0 \\
xw & 1
\end{pmatrix} \mapsto (I - \epsilon\beta(xw))(I - \alpha(xA)) = I - x(\alpha(A) + \epsilon\beta(w))$$

where $\alpha : \mathfrak{so}(q_1) \to \mathfrak{sl}_2$ is the Lie algebra isomorphism

$$\begin{pmatrix}
0 & a & -b \\
a & 0 & c \\
b & c & 0
\end{pmatrix} \mapsto \frac{1}{2} \begin{pmatrix}
a & -b + c \\
b + c & -a
\end{pmatrix},$$

induced from the isomorphism $\text{PSO}(q_1) \cong \text{PGL}_2$ and $\beta : \mathbb{A}^3 \to \mathfrak{sl}_2$ is the Lie algebra isomorphism

$$\begin{pmatrix}
a & b & c
\end{pmatrix} \mapsto \frac{1}{2} \begin{pmatrix}
a & -b + c \\
b + c & -a
\end{pmatrix}$$

as above. Thus $dc : \mathfrak{so}(q) \to \mathfrak{g}$ is an isomorphism. \hfill $\Box$

**Remark 2.6.** The isomorphism of algebraic groups in the proof of Theorem 2.1 can be viewed as a degeneration of an isomorphism of semisimple groups of type $^2A_1 = D_2$ (on the generic fiber) to an isomorphism of nonreductive groups whose semisimplification has type $A_1 = B_1 = C_1$ (on the special fiber).
3. Simple degeneration over semi-local rings

The semilocal ring \( R \) of a normal scheme at a finite set of points of codimension 1 is a semilocal Dedekind domain, hence a principal ideal domain. Let \( R_i \) denote the (finitely many) discrete valuation overrings of \( R \) contained in the fraction field \( K \) (the localizations at the height one prime ideals), \( \hat{R_i} \) their completions, and \( \hat{K}_i \) their fraction fields. If \( \hat{R} \) is the completion of \( R \) at its Jacobson radical \( \text{rad}(R) \) and \( \hat{K} \) the total ring of fractions, then \( \hat{R} \cong \prod \hat{R}_i \) and \( \hat{K} \cong \prod \hat{K}_i \). We call an element \( \pi \in \hat{R} \) a parameter if \( \pi = \prod_i \pi_i \) is a product of parameters \( \pi_i \) of \( R_i \).

We first recall a well-known result, cf. [14, §2.3.1].

**Lemma 3.1.** Let \( R \) be a semilocal principal ideal domain and \( K \) its field of fractions. Let \( q \) be a regular quadratic form over \( R \) and \( u \in R^\times \) a unit. If \( u \) is represented by \( q \) over \( K \) then it is represented by \( q \) over \( R \).

We now provide a generalization of Lemma 3.1 to the case of simple degeneration.

**Proposition 3.2.** Let \( R \) be a semilocal principal ideal domain with 2 invertible and \( K \) its field of fractions. Let \( q \) be a quadratic form over \( R \) with simple degeneration of multiplicity one and let \( u \in R^\times \) be a unit. If \( u \) is represented by \( q \) over \( K \) then it is represented by \( q \) over \( R \).

For the proof, we’ll first need to generalize, to the degenerate case, some standard results concerning regular forms. If \((V,q)\) is a quadratic form over a ring \( R \) and \( v \in V \) is such that \( q(v) = u \in R^\times \), then the reflection \( r_v : V \to V \) through \( v \) given by

\[
r_v(w) = w - u^{-1}b_q(v,w)v
\]

is an isometry over \( R \) satisfying \( r_v(v) = -v \) and \( r_v(w) = w \) if \( w \in v^\perp \).

**Lemma 3.3.** Let \( R \) be a semilocal ring with 2 invertible. Let \((V,q)\) be a quadratic form over \( R \) and \( u \in R^\times \). Then \( \text{O}(V,q)(R) \) acts transitively on the set of vectors \( v \in V \) such that \( q(v) = u \).

**Proof.** Let \( v, w \in V \) be such that \( q(v) = q(w) = u \). We first prove the lemma over any local ring with 2 invertible. Without loss of generality, we can assume that \( q(v-w) \in R^\times \). Indeed, \( q(v+w) + q(v-w) = 4u \in R^\times \) so that, since \( \hat{R} \) is local, either \( q(v+w) \) or \( q(v-w) \) is a unit. If \( q(v-w) \) is not a unit, then \( q(v+w) \) is and we can replace \( w \) by \(-w \) using the reflection \( r_w \). Finally, by a standard computation, we have \( r_{v-w}(v) = w \). Thus any two vectors representing \( u \) are related by a product of at most two reflections.

For a general semilocal ring, the quotient \( R/\text{rad}(R) \) is a product of fields. By the above argument, \( \overline{\pi} \) can be transported to \( -\overline{\pi} \) in each component by a product \( \overline{\pi} \) of at most two reflections. By the Chinese remainder theorem, we can lift \( \overline{\pi} \) to a product of at most two reflections \( \pi \) of \((V,q)\) transporting \( v \) to \(-w + z \) for some \( z \in \text{rad}(R) \otimes_R V \). Replacing \( v \) by \(-w + z \), we can assume that \( v + w = z \in \text{rad}(R) \otimes_R V \). Finally, \( q(v+w) + q(v-w) = 4u \) and \( q(v+w) \in \text{rad}(R) \), thus \( q(v-w) \) is a unit. As before, \( r_{v-w}(v) = w \). \( \square \)

**Corollary 3.4.** Let \( R \) be a semilocal ring with 2 invertible. Then regular forms can be cancelled, i.e. if \( q_1 \) and \( q_2 \) are quadratic forms and \( q \) a regular quadratic form over \( R \) with \( q_1 \perp q \cong q_2 \perp q \), then \( q_1 \cong q_2 \).

**Proof.** Regular quadratic forms over a semilocal ring with 2 invertible are diagonalizable. Hence we can reduce to the case of rank one form \( q = (\hat{R}, <u>) \) for \( u \in R^\times \). Let \( \varphi : q_1 \perp (\hat{R}w_1, <u>) \cong q_2 \perp (\hat{R}w_2, <u>) \) be an isometry. By Lemma 3.3, there is an isometry \( \psi \) of \( q_2 \perp (\hat{R}w_2, <u>) \) taking \( \varphi(w_1) \) to \( w_2 \), so that \( \psi \circ \varphi \) takes \( w_1 \) to \( w_2 \). By taking orthogonal complements, \( \varphi \) induces an isometry \( q_1 \cong q_2 \). \( \square \)

**Lemma 3.5.** Let \( R \) be a complete discrete valuation ring with 2 invertible and \( K \) its fraction field. Let \( q \) be a quadratic form with simple degeneration of multiplicity one and let \( u \in R^\times \). If \( u \) is represented by \( q \) over \( K \) then it is represented by \( q \) over \( R \).
Thus and elementary hyperbolic isometries \( \hat{\gamma} \) for quadratic forms that are not necessarily regular. Let \( \phi \) be a ring with \( 2 \) invertible, \( (K, \phi) \) a quadratic form over \( K \) that is nondegenerate over \( \phi \in O(q \perp h)(R) \) by

\[
E_v(w) = w + b(v, w)e, \quad E_v^*(w) = w + b(v, w)f, \quad \text{for } w \in V \\
E_v(e) = e, \quad E_v^*(e) = -v - 2^{-1}q(v)f + e \\
E_v(f) = -v - 2^{-1}q(v)e + f, \quad E_v^*(f) = f.
\]

Define the group of elementary hyperbolic isometries \( EO(q, h)(R) \) to be the subgroup of \( O(q \perp h)(R) \) generated by \( E_v \) for \( v \in V \).

For \( u \in R^\times \), define \( \alpha_u \in O(h)(R) \) by

\[
\alpha_u(e) = ue, \quad \alpha_u(f) = u^{-1}f
\]

and \( \beta_u \in O(h)(R) \) by

\[
\beta_u(e) = u^{-1}f, \quad \beta_u(f) = ue.
\]

Then \( O(h)(R) = \{ \alpha_u : u \in R^\times \} \cup \{ \beta_u : u \in R^\times \} \). One can verify the following identities:

\[
\alpha_u^{-1}E_v\alpha_u = E_{u^{-1}v}, \quad \beta_u^{-1}E_v\beta_u = E_v^* \\
\alpha_u^{-1}E_v^*\alpha_u = E_{u^{-1}v}, \quad \alpha_u^{-1}E_v\alpha_u = E_v^*.
\]

Thus \( O(h)(R) \) normalizes \( EO(q, h)(R) \).

If \( R = K \) is a field and \( q \) is nondegenerate, then \( EO(q, h)(K) \) and \( O(h)(K) \) generate \( O(q \perp h)(K) \) (see [20, ch. 1]) so that

\[
O(q \perp h)(K) = EO(q, h)(K) \times O(h)(K). \tag{7}
\]

**Proposition 3.6.** Let \( R \) be a semilocal principal ideal domain with \( 2 \) invertible and \( K \) its fraction field. Let \( \hat{R} \) be the completion of \( R \) at the radical and \( \hat{K} \) its fraction field. Let \( (V, q) \) be a quadratic form over \( R \) that is nondegenerate over \( K \). Then every element \( \varphi \in O(q \perp h)(\hat{K}) \) is a product \( \varphi_1\varphi_2 \), where \( \varphi_1 \in O(q \perp h)(\hat{K}) \) and \( O(q \perp h)(\hat{K}) \).

**Proof.** We follow portions of the proof in [42, Prop. 3.1]. As topological rings \( \hat{R} \) is open in \( \hat{K} \), and hence as topological groups \( O(q \perp h)(\hat{R}) \) is open in \( O(q \perp h)(\hat{K}) \). In particular, \( O(q \perp h)(\hat{R}) \cap EO(q, h)(\hat{K}) \) is open in \( EO(q, h)(\hat{K}) \). Since \( R \) is dense in \( \hat{R} \), \( K \) is dense in \( \hat{K} \), \( V \otimes_R K \) is dense in \( V \otimes_R \hat{K} \), and hence \( EO(q, h)(\hat{K}) \) is dense in \( EO(q, h)(\hat{K}) \).

Thus, by topological considerations, every element \( \varphi' \) of \( EO(q \perp h)(\hat{K}) \) is a product \( \varphi'_1\varphi'_2 \), where \( \varphi'_1 \in EO(q, h)(K) \) and \( \varphi'_2 \in EO(q, h)(\hat{K}) \cap O(q \perp h)(\hat{R}) \). Clearly, every element \( \gamma \) of \( O(h)(\hat{K}) \) is a product \( \gamma_1\gamma_2 \), where \( \gamma_1 \in O(h)(\hat{K}) \) and \( \gamma_2 \in O(h)(\hat{R}) \).

The form \( q \perp h \) is nondegenerate over \( \hat{K} \), so by \( (7) \), every \( \varphi' \in O(q \perp h)(\hat{K}) \) is a product \( \varphi'\gamma \), where \( \varphi' \in EO(q, h)(\hat{K}) \) and \( \gamma \in O(h)(\hat{K}) \). As above, we can write

\[
\varphi = \varphi'\gamma = \varphi'_1\varphi'_2\gamma_1\gamma_2 = \varphi'_1\gamma_1^{-1}\varphi'_2\gamma_1\gamma_2.
\]
Since $EO(q, h)(\hat{K})$ is a normal subgroup, $\gamma_1^{-1}\varphi_2'\gamma_1 \in EO(q, h)(\hat{K})$ and is thus a product $\psi_1\psi_2$, where $\psi_1 \in EO(q, h)(K)$ and $\psi_2 \in EO(q, h)(\hat{K}) \cap O(q \perp h)(\hat{R})$. Finally, $\varphi$ is a product $(\varphi'\gamma_1\psi_1)(\psi_2\gamma_2)$ of the desired form.

**Proof of Proposition 3.2.** Let $\hat{R}$ be the completion of $R$ at the radical and $\hat{K}$ the total ring of fractions. As $q|_K$ represents $u$, we have a splitting $q|_K \cong q_1 \perp u$. We have that $q|_{\hat{R}} = \prod_i q|_{\hat{R}_i}$ represents $u$ over $\hat{R} = \prod_i \hat{R}_i$, by Lemma 3.5, since $u$ is represented over $\hat{K} = \prod_i \hat{K}_i$. We thus have a splitting $q|_{\hat{R}} \cong q_2 \perp u$. By Witt cancellation over $\hat{K}$, we have an isometry $\varphi : q_1|_{\hat{K}} \cong q_2|_{\hat{R}}$, which by patching defines a quadratic form $\hat{q}$ over $\hat{R}$ such that $\hat{q}|_{\hat{K}} \cong q_1$ and $\hat{q}|_{\hat{R}} \cong q_2$.

We claim that $q \perp -u \cong \hat{q} \perp h$. Indeed, as $h \cong -u, -u$, we have isometries

$$\psi^K : (q \perp -u)|_K \cong (\hat{q} \perp h)|_K, \quad \psi^R : (q \perp -u)|_R \cong (\hat{q} \perp h)|_R.$$

By Proposition 3.6, there exists $\theta_1 \in O(\hat{q} \perp h)(\hat{R})$ and $\theta_2 \in O(\hat{q} \perp h)(K)$ such that $\psi^R(\psi^K)^{-1} = \theta_1^{-1}\theta_2$. The isometries $\theta_1\psi^R$ and $\theta_2\psi^K$ then agree over $\hat{K}$ and so patch to yield an isometry $\psi : q \perp -u \cong \hat{q} \perp h$.

As $h \cong -u, -u$, we have $q \perp -u \cong \hat{q} \perp -u, -u$. By Corollary 3.4, we can cancel the regular form $-u$, so that $q \cong \hat{q} \perp u$. Thus $q$ represents $u$ over $\hat{R}$.

**Lemma 3.7.** Let $R$ be a discrete valuation ring and $(E, q)$ a quadratic form of rank $n$ over $R$ with simple degeneration. If $q$ represents $u \in R^\times$ then it can be diagonalized as $q \cong <u, u_2, \ldots, u_n, \pi >$ for $u_i \in R^\times$ and some parameter $\pi$.

**Proof.** If $q(v) = u$ for some $v \in E$, then $q$ restricted to the submodule $Rv \subset E$ is regular, hence $(E, q)$ splits as $(R, <u>) \perp (Rv^\perp, q|_{Rv^\perp})$. Since $(Rv^\perp, q|_{Rv^\perp})$ has simple degeneration, we are done by induction. 

**Corollary 3.8.** Let $R$ be a semilocal principal ideal domain with $2$ invertible and $K$ its fraction field. If quadratic forms $q$ and $q'$ with simple degeneration and multiplicity one over $R$ are isometric over $K$, then they are isometric over $R$.

**Proof.** Any quadratic form $q$ with simple degeneration and multiplicity one has discriminant $\pi \in R/R^\times 2$ given by a parameter. Since $R^\times/ R^\times 2 \to K^\times/K^\times 2$ is injective, if $q'$ is another quadratic form with simple degeneration and multiplicity one, such that $q|_K$ is isomorphic to $q'|_K$, then $q$ and $q'$ have the same discriminant.

Over each discrete valuation overring $R_i$ of $R$, we thus have diagonalizations,

$$q|_{R_i} \cong <u_1, \ldots, u_{r_i-1}, u_1 \cdots u_{r_i-1} \pi_i >, \quad q'|_{R_i} \cong <u_1', \ldots, u_{r_i'-1}, u_1' \cdots u_{r_i'-1} \pi_i >,$$

for a suitable parameter $\pi_i$ of $R_i$, where $u_j, u_j' \in R_i^\times$. Now, since $q|_K$ and $q'|_K$ are isometric, $q|_K$ represents $u_i$ over $K$, hence by Proposition 3.2, $q'|_{R_i}$ represents $u_1$ over $R_i$. Hence by Lemma 3.7, we have a further diagonalization

$$q'|_{R_i} \cong <u_1, u_2', \ldots, u_{r_i'-1}, u_1 u_2' \cdots u_{r_i'-1} \pi_i >$$

with possibly different units $u_i'$. By cancellation over $K$, we have $<u_2, \ldots, u_{r_i-1}, u_1 \cdots u_{r_i-1} \pi_i >|_K \cong <u_2', \ldots, u_{r_i'-1}, u_1' \cdots u_{r_i'-1} \pi_i >|_K$. By an induction hypothesis over the rank of $q$, we have that $<u_2, \ldots, u_{r_i-1}, u_1 \cdots u_{r_i-1} \pi_i > \cong <u_2', \ldots, u_{r_i'-1}, u_1' \cdots u_{r_i'-1} \pi_i >$ over $R$. By induction, we have the result over each $R_i$.

Thus $q|_{\hat{R}} \cong q'|_{\hat{R}}$ over $\hat{R} = \prod_i \hat{R}_i$. Consider the induced isometry $\psi^R : (q \perp h)|_{\hat{R}} \cong (q' \perp h)|_{\hat{R}}$ as well as the isometry $\psi^K : (q \perp h)|_K \cong (q' \perp h)|_K$ induced from the given one. By Proposition 3.6, there exists $\theta \in O(q \perp h)(\hat{R})$ and $\theta^K \in O(q \perp h)(K)$ such that $\psi^R \circ \theta^K = \theta^R \circ \psi^K$. The isometries $\psi^R \theta^\hat{R}$ and $\psi^K \theta^K$ then agree over $\hat{K}$ and so patch to yield an isometry $\psi : q \perp h \cong q' \perp h$ over $R$. By Corollary 3.4, we then have an isometry $q \cong q'$. 

Remark 3.9. Let $R$ be a semilocal principal ideal domain with 2 invertible, closed fiber $D$, and fraction field $K$. Let $\text{QF}^D(R)$ be the set of isometry classes of quadratic forms on $R$ with simple degeneration of multiplicity one along $D$. Corollary 3.8 says that $\text{QF}^D(R) \rightarrow \text{QF}(K)$ is injective, which can be viewed as an analogue of the Grothendieck–Serre conjecture for the (nonreductive) orthogonal group of a quadratic form with simple degeneration of multiplicity one over a discrete valuation ring.

Corollary 3.10. Let $R$ be a complete discrete valuation ring with 2 invertible and $K$ its fraction field. If quadratic forms $q$ and $q'$ of even rank $n = 2m \geq 4$ with simple degeneration and multiplicity one over $R$ are similar over $K$, then they are similar.

Proof. Let $\psi : q|_K \simeq q'|_K$ be a similarity with factor $\lambda = u\pi^e$ where $u \in R^\times$ and $\pi$ is a parameter whose square class can assume is the discriminant of $q$ and $q'$. If $e$ is even, then $\pi^{e/2}\psi : q|_K \simeq q'|_K$ has factor $u$, so defines an isometry $q|_K \simeq uq'|_K$. Hence by Corollary 3.8, there is an isometry $q \simeq uq'$, hence a similarity $q \simeq q'$. If $e$ is odd, then $\pi^{(e-1)/2}\psi^K$ defines an isometry $q|_K \simeq u\pi^{|_K}$. Writing $q \simeq q_1 \perp <\alpha\pi>$ and $q' = q'_1 \perp <\beta\pi>$ for regular quadratic forms $q_1$ and $q'_1$ over $R$, then $\pi q|_K \simeq u\pi q'_1 \perp <\alpha\pi>$. Comparing first residues, we have that $q_1$ and $<\beta u>$ are equal in $W(k)$, where $k$ is the residue field of $R$. Since $R$ is complete, $q_1$ splits off the requisite number of hyperbolic planes, and so $q_1 \simeq h^m \perp <(-1)^m\alpha>$. Now note that $(-1)^{m-1}\pi$ is a similarity factor of the form $q|_K$. Finally, we have $( -1)^{m-1}\pi q|_K \simeq q|_K \simeq u\pi q'_1$, so that $q|_K \simeq (-1)^{m-1}uq'_1|_K$. Thus by Corollary 3.8, $q \simeq (-1)^{m}uq'$ over $R$, so that there is a similarity $q \simeq q'$ over $R$. □

We need the following relative version of Theorem 2.1.

Proposition 3.11. Let $R$ be a semilocal principal ideal domain with 2 invertible and $K$ its fraction field. Let $q$ and $q'$ be quadratic forms of rank 4 over $R$ with simple degeneration and multiplicity one. Given any $R$-algebra isomorphism $\varphi : \mathcal{O}(q) \cong \mathcal{O}(q')$ there exists a similarity $\psi : q \simeq q'$ such that $\mathcal{O}(\psi) = \varphi$.

Proof. By Theorem 2.5, there exists a similarity $\psi^K : q \simeq q'$ such that $\mathcal{O}(\psi^K) = \varphi|_K$. Thus over $\tilde{R} = \prod_i \tilde{R}_i$, Corollary 3.10 applied to each component provides a similarity $\rho : q|_{\tilde{R}} \simeq q'|_{\tilde{R}}$. Now $\mathcal{O}(\rho)^{-1} \circ \varphi : \mathcal{O}(q)|_{\tilde{R}} \cong \mathcal{O}(q)|_{\tilde{R}}$ is a $\tilde{R}$-algebra isomorphism, hence by Theorem 2.1, is equal to $\mathcal{O}(\sigma)$ for some similarity $\sigma : q|_{\tilde{R}} \simeq q'|_{\tilde{R}}$. Then $\psi_{\tilde{R}} = \rho \circ \sigma : q|_{\tilde{R}} \simeq q'|_{\tilde{R}}$ satisfies $\mathcal{O}(\psi_{\tilde{R}}) = \varphi|_{\tilde{R}}$.

Let $\lambda \in K^\times$ and $u \in \tilde{R}^\times$ be the factor of $\psi^K$ and $\psi_{\tilde{R}}$, respectively. Then $\psi_{\tilde{R}}^{-1} \circ \psi_{\tilde{R}} : q|_{\tilde{R}} \simeq q'|_{\tilde{R}}$ has factor $u^{-1}\lambda \in \tilde{R}^\times$. But since $\mathcal{O}(\psi_{\tilde{R}})^{-1} \circ \psi_{\tilde{R}} : q|_{\tilde{R}} \simeq q'|_{\tilde{R}}$ is given by multiplication by $\mu \in \tilde{K}^\times$. In particular, $u^{-1}\lambda = \mu^2$ and thus the valuation of $\lambda \in K^\times$ is even in every $R_i$. Thus $\lambda = v^2$ with $v \in R^\times$ and so $\psi$ defines an isometry $q|_K \simeq vq'|_K$. By Corollary 3.8, there’s an isometry $\alpha : q \simeq q'$, i.e., a similarity $\alpha : q \simeq q'$.

Finally, we need the following generalization of [15, Prop. 2.3] to the setting of quadratic forms with simple degeneration.

Proposition 3.12. Let $S$ be the spectrum of a regular local ring $(R, m)$ of dimension $\geq 2$ with 2 invertible and $D \subset S$ a regular divisor. Let $(V, q)$ be a quadratic form over $S$ such that $(V, q)|_{S \setminus \{m\}}$ has simple degeneration of multiplicity one along $D \setminus \{m\}$. Then $(V, q)$ has simple degeneration along $D$ of multiplicity one.

Proof. First note that the discriminant of $(V, q)$ (hence the subscheme $D$) is represented by a regular element $\pi \in m \setminus m^2$. Now assume, to get a contradiction, that the radical of
\((V,q)_{\kappa(m)}\), where \(\kappa(m)\) is the residue field at \(m\), has dimension \(r > 1\) and let \(e_1, \ldots, e_r\) be a \(\kappa(m)\)-basis of the radical. Lifting to unimodular elements \(e_1, \ldots, e_r\) of \(V\), we can complete to a basis \(e_1, \ldots, e_n\). Since \(b_q(e_i, e_j) \in m\) for all \(1 \leq i \leq r\) and \(1 \leq j \leq n\), inspecting the Gram matrix \(M_q\) of \(b_q\) with respect to this basis, we find that \(\det M_q \in m^r\), contradicting the description of the discriminant above. Thus the radical of \((V,q)\) has rank 1 at \(m\) and \((V,q)\) has simple degeneration along \(D\). Similarly, \((V,q)\) also has multiplicity one at \(m\), hence on \(S\) by hypothesis.

\[\square\]

**Corollary 3.13.** Let \(S\) be a regular integral scheme of dimension \(\leq 2\) with \(2\) invertible and \(D\) a regular divisor. Let \((\mathcal{E},q,\mathcal{L})\) be a quadratic form over \(S\) that is regular over every codimension 1 point of \(S \setminus D\) and has simple degeneration of multiplicity one over every codimension one point of \(D\). Then over \(S\), \(q\) has simple degeneration along \(D\) of multiplicity one.

**Proof.** Let \(U = S \setminus D\). The quadratic form \(q|_U\) is regular except possibly at finitely many closed points. But regular quadratic forms over the complement of finitely many closed points of a regular surface extend uniquely by [15, Prop. 2.3]. Hence \(q|_U\) is regular. The restriction \(q|_D\) has simple degeneration at the generic point of \(D\), hence along the complement of finitely many closed points of \(D\). At each of these closed points, \(q\) has simple degeneration by Proposition 3.12. Thus \(q\) has simple degeneration along \(D\).

\[\square\]

4. Gluing tensors

In this section, we reproduce some results on gluing (or patching) tensor structures on vector bundles communicated to us by M. Ojanguren and inspired by Colliot-Thélène–Sansuc [15, §2, §6]. As usual, any scheme \(S\) is assumed to be noetherian.

**Lemma 4.1.** Let \(S\) be a scheme of dimension \(n\), \(U \subset S\) a dense open subset, \(x \in S \setminus U\) a point of codimension \(1\) of \(S\), \(V \subset S\) an open neighborhood of \(x\), and \(W \subset U \cap V\) a dense open subset of \(S\). Then there exists an open neighborhood \(V'\) of \(x\) such that \(V' \cap U \subset W\).

**Proof.** The closed set \(Z = S \setminus W\) is of dimension \(n - 1\) and contains \(x\), hence has a decomposition \(Z = Z_1 \cup Z_2\), where

\[Z_1 = \{x\} \cup \{x_1\} \cup \cdots \cup \{x_r\}, \quad Z_2 = \{y_1\} \cup \cdots \cup \{y_s\}\]

into closed irreducible sets with distinct generic points, where \(x, x^1, \ldots, x^r \notin U\) and \(y_1, \ldots, y_s \in U\). Let \(F = \{y_1\} \cup \cdots \cup \{y_s\}\) and \(V'' = S \setminus F\). Note that \(W \subset V''\).

Since \(Z_1 \cap U = \emptyset\), we have that \(U \cap F = U \cap Z\). This shows that \(V'' \cap U = U \setminus (U \cap Z) = U \cap (X \setminus Z) = U \cap W = W\). Thus we can take \(V' = V \cap V''\).

\[\square\]

Let \(\mathcal{V}\) be a locally free \(\mathcal{O}_S\)-module (of finite rank). A tensorial construction \(t(\mathcal{V})\) in \(\mathcal{V}\) is any locally free \(\mathcal{O}_S\)-module that is a tensor product of modules \(\mathcal{L}(\mathcal{V}), \mathcal{L}(\mathcal{V}^\vee), \mathcal{S}(\mathcal{V}), \mathcal{S}(\mathcal{V}^\vee)\). Let \(\mathcal{L}\) be a line bundle on \(S\). An \(\mathcal{L}\)-valued tensor \((\mathcal{V}, q, \mathcal{L})\) of type \(t(\mathcal{V})\) on \(S\) is a global section \(q \in \Gamma(S, t(\mathcal{V}) \otimes \mathcal{L})\) for some tensorial construction \(t(\mathcal{V})\) in \(\mathcal{V}\).

For example, an \(\mathcal{L}\)-valued quadratic form is an \(\mathcal{L}\)-valued tensor of type \(t(\mathcal{V}) = \mathcal{S}(\mathcal{V})\); an \(\mathcal{O}_S\)-algebra structure on \(\mathcal{V}\) is an \(\mathcal{O}_S\)-valued tensor of type \(t(\mathcal{V}) = \mathcal{V}^\vee \otimes \mathcal{V} \otimes \mathcal{V}\). If \(U \subset S\) is an open set, denote by \((\mathcal{V}, q, \mathcal{L})|_U = (\mathcal{V}|_U, q|_U, \mathcal{L}|_U)\) the restricted tensor over \(U\). If \(D \subset S\) is a closed subscheme, let \(\mathcal{O}_{S,D}\) denote the semilocal ring at the generic points of \(D\) and \((\mathcal{V}, q, \mathcal{L})|_D = (\mathcal{V}, q, \mathcal{L}) \otimes_{\mathcal{O}_S} \mathcal{O}_{S,D}\) the associated tensor over \(\mathcal{O}_{S,D}\). If \(S\) is integral and \(K\) its function field, we write \((\mathcal{V}, q, \mathcal{L})|_K\) for the stalk at the generic point.

A similarity between line bundle-valued tensors \((\mathcal{V}, q, \mathcal{L})\) and \((\mathcal{V}', q', \mathcal{L}')\) consists of a pair \((\varphi, \lambda)\) where \(\varphi : \mathcal{V} \cong \mathcal{V}'\) and \(\lambda : \mathcal{L} \cong \mathcal{L}'\) are \(\mathcal{O}_S\)-module isomorphisms such that \(t(\varphi) \otimes \lambda : t(\mathcal{V}) \otimes \mathcal{L} \cong t(\mathcal{V}') \otimes \mathcal{L}'\) takes \(q\) to \(q'\). A similarity is an isomorphism if \(\mathcal{L} = \mathcal{L}'\) and \(\lambda = \text{id}\).

**Proposition 4.2.** Let \(S\) be an integral scheme, \(K\) its function field, \(U \subset S\) a dense open subscheme, and \(D \subset S \setminus U\) a closed subscheme of codimension 1. Let \((\mathcal{V}^U, q^U, \mathcal{L}^U)\)
be a tensor over $U$, $(\mathcal{V}^D, q^D, \mathcal{L}^D)$ a tensor over $\mathcal{O}_{S,D}$, and $(\varphi, \lambda) : (\mathcal{V}^U, q^U, \mathcal{L}^U)|_K \cong (\mathcal{V}^D, q^D, \mathcal{L}^D)|_K$ a similarity of tensors over $K$. Then there exists a dense open set $U' \subset S$ containing $U$ and the generic points of $D$ and a tensor $(\mathcal{V}^U, q^U, \mathcal{L}^U)$ over $U'$ together with similarities $(\mathcal{V}^U, q^U, \mathcal{L}^U) \cong (\mathcal{V}^U, q^U, \mathcal{L}^U)|_U$ and $(\mathcal{V}^D, q^D, \mathcal{L}^D) \cong (\mathcal{V}^U, q^U, \mathcal{L}^U)|_D$. A corresponding statement holds for isomorphisms of tensors.

**Proof.** By induction on the number of irreducible components of $D$, gluing over one at a time, we can assume that $D$ is irreducible. Choose an extension $(\mathcal{V}^A, q^A, \mathcal{L}^A)$ of $(\mathcal{V}^D, q^D, \mathcal{L}^D)$ to some open neighborhood $V$ of $D$ in $S$. Since $(\mathcal{V}^V, q^V, \mathcal{L}^V)|_K \cong (\mathcal{V}^U, q^U, \mathcal{L}^U)|_K$, there exists an open subscheme $W \subset U \cap V$ over which $(\mathcal{V}^V, q^V, \mathcal{L}^V)|_W \cong (\mathcal{V}^U, q^U, \mathcal{L}^U)|_W$. By Lemma 4.1, there exists an open neighborhood $V' \subset S$ of $D$ such that $V' \cap U \subset W$. We can glue $(\mathcal{V}^V, q^V, \mathcal{L}^V)$ and $(\mathcal{V}^V, q^V, \mathcal{L}^V)$ over $U \cap V'$ to get a tensor $(\mathcal{V}^U, q^U, \mathcal{L}^U)$ over $U'$ extending $(\mathcal{V}^U, q^U, \mathcal{L}^U)$, where $U' = U \cap V'$. But $U'$ contains the generic points of $D$ and we are done. \hfill $\square$

For an open subscheme $U \subset S$, a closed subscheme $D \subset S \setminus U$ of codimension 1, a similarity gluing datum (resp. gluing datum) is a triple $((\mathcal{V}^U, q^U, \mathcal{L}^U), (\mathcal{V}^U, q^U, \mathcal{L}^U), (\varphi, \lambda))$ consisting of a tensor over $U$, a tensor over $\mathcal{O}_{S,D}$, and a similarity (resp. an isomorphism) of tensors $(\varphi, \lambda) : (\mathcal{V}^U, q^U, \mathcal{L}^U)|_K \cong (\mathcal{V}^D, q^D, \mathcal{L}^D)|_K$ over $K$. There is an evident notion of isomorphism between two (similarity) gluing data. Two isomorphic gluing data yield, by Proposition 4.2, tensors $(\mathcal{V}^U, q^U, \mathcal{L}^U)$ and $(\mathcal{V}^U, q^U, \mathcal{L}^U)$ over open dense subsets $U', U'' \subset S$ containing $U$ and the generic points of $D$ such that there is an open dense refinement $U'' \subset U \cap U'$ over which we have $(\mathcal{V}^U, q^U, \mathcal{L}^U)|_{U''} \cong (\mathcal{V}^U, q^U, \mathcal{L}^U)|_{U'}$.

Together with results of [15], we get a well-known result—purity for division algebras over surfaces—which we state in a precise way, due to Ojanguren, that is conducive to our usage. If $K$ is the function field of a regular scheme $S$, we say that $\beta \in Br(K)$ is unramified if its in the image of the injection $Br(S) \to Br(K)$.

**Theorem 4.3.** Let $S$ be a regular integral scheme of dimension $\leq 2$, $K$ its function field, $D \subset S$ a closed subscheme of codimension 1, and $U = S \setminus D$.

a) If $\mathcal{A}^U$ is an Azumaya $\mathcal{O}_U$-algebra such that $\mathcal{A}^U|_K$ is unramified along $D$ then there exists an Azumaya $\mathcal{O}_S$-algebra $\mathcal{A}$ such that $\mathcal{A}|_U \cong \mathcal{A}^U$.

b) If a central simple $K$-algebra $A$ has Brauer class unramified over $S$, then there exists an Azumaya $\mathcal{O}_S$-algebra $\mathcal{A}$ such that $\mathcal{A}|_K \cong A$.

**Proof.** For a), since $\mathcal{A}^U|_K$ is unramified along $D$, there exists an Azumaya $\mathcal{O}_{S,D}$-algebra $\mathcal{B}^D$ with $\mathcal{B}^D|_K$ Brauer equivalent to $A$.

We argue that we can choose $\mathcal{B}^D$ such that $\mathcal{B}^D|_K = A$. Indeed, writing $\mathcal{B}^D|_K = M_m(\Delta)$ for a division $K$-algebra $\Delta$ and choosing a maximal $\mathcal{O}_{S,D}$-order $\mathcal{D}^D$ of $\Delta$, then $M_m(\mathcal{D}^D)$ is a maximal order of $\mathcal{B}^D|_K$. Any two maximal orders are isomorphic by [6, Prop. 3.5], hence $M_m(\mathcal{D}^D) \cong \mathcal{B}^D$. In particular, $\mathcal{D}^D$ is an Azumaya $\mathcal{O}_{S,D}$-algebra. Finally writing $A = M_m(\Delta)$, then $M_m(\mathcal{D}^D)$ is an Azumaya $\mathcal{O}_{S,D}$-algebra and is our new choice for $\mathcal{B}^D$.

Applying Proposition 4.2 to $\mathcal{A}^U$ and $\mathcal{B}^D$, we get an Azumaya $\mathcal{O}_U$-algebra $\mathcal{A}^U'$ extending $\mathcal{A}^U$, where $U'$ contains all points of $S$ of codimension 1. Finally, by [15, Thm. 6.13] applied to the group $\text{PGL}_n$ (where $n$ is the degree of $A$), $\mathcal{A}^U'$ extends to an Azumaya $\mathcal{O}_S$-algebra $\mathcal{A}$ such that $\mathcal{A}|_U = \mathcal{A}^U$.

For b), the $K$-algebra $A$ extends, over some open subscheme $U \subset S$, to an Azumaya $\mathcal{O}_U$-algebra $\mathcal{A}^U$. If $U$ contains all codimension 1 points, then we apply [15, Thm. 6.13] as above. Otherwise, $D = S \setminus U$ has codimension 1 and we apply part (1). \hfill $\square$

Finally, we note that isomorphic Azumaya algebra gluing data on a regular integral scheme $S$ of dimension $\leq 2$ yield, by [15, Thm. 6.13], isomorphic Azumaya algebras on $S$. 

5. The norm form $N_{T/S}$ for ramified covers

Let $S$ be a regular integral scheme, $D \subset S$ a regular divisor, and $f : T \to S$ a ramified cover of degree 2 branched along $D$. Then $T$ is a regular integral scheme. Let $L/K$ be the corresponding quadratic extension of function fields. Let $U = S \setminus D$, and for $E = f^{-1}(D)$, let $V = T \setminus E$. Then $f|_V : V \to U$ is étale of degree 2. Let $\iota$ be the nontrivial Galois automorphism of $T/S$.

The following lemma is not strictly used in our construction but we need it for the applications in §6.

**Lemma 5.1.** Let $S$ be a regular integral scheme and $f : T \to S$ a finite flat cover of prime degree $\ell$ with regular branch divisor $D \subset S$ on which $\ell$ is invertible. Let $L/K$ be the corresponding extension of function fields.

a) The corestriction map $N_{L/K} : \text{Br}(L) \to \text{Br}(K)$ restricts to a well-defined map $N_{T/S} : \text{Br}(T) \to \text{Br}(S)$.

b) If $S$ has dimension $\leq 2$ and $\mathcal{B}$ is an Azumaya $\mathcal{O}_T$-algebra of degree $d$ representing $\beta \in \text{Br}(T)$ then there exists an Azumaya $\mathcal{O}_S$-algebra of degree $\ell d$ representing $N_{T/S}(\beta)$ whose restriction to $U$ coincides with the classical étale norm algebra $N_{V/U}\mathcal{B}|_V$.

**Proof.** The hypotheses imply that $T$ is regular integral and so by [5], we can consider $\text{Br}(S) \subset \text{Br}(K)$ and $\text{Br}(T) \subset \text{Br}(L)$. Let $L/K$ be an Azumaya $\mathcal{O}_T$-algebra of degree $d$ representing $\beta \in \text{Br}(T)$. As $V/U$ is étale, the classical norm algebra $N_{V/U}\mathcal{B}|_V$ is an Azumaya $\mathcal{O}_U$-algebra of degree $\ell d$ representing the class of $N_{L/K}(\beta) \in \text{Br}(K)$. In particular, $N_{L/K}(\beta)$ is unramified at every point (of codimension 1) in $U$. As $D$ is regular, it is a disjoint union of irreducible divisors and let $D'$ be one such irreducible component. If $E' = f^*D'$, then $\mathcal{O}_{T,E'}$ is totally ramified over $\mathcal{O}_{S,D'}$ (since it is ramified of prime degree).

In particular, $E' \subset T$ is an irreducible component of $E = f^*D$. The commutative diagram

$$
\begin{array}{ccc}
\text{Br}(\mathcal{O}_{T,E'}) & \longrightarrow & \text{Br}(L) \\
\downarrow & & \downarrow \text{Br}(\kappa(E')) \\
\text{Br}(\mathcal{O}_{S,D'}) & \longrightarrow & \text{Br}(K) \\
\end{array}
$$

of residue homomorphisms implies, since $\beta$ is unramified along $E'$, that $N_{L/K}(\beta)$ is unramified along $D'$. Thus $N_{L/K}(\beta)$ is an unramified class in $\text{Br}(K)$, hence is contained in $\text{Br}(S)$ by purity for the Brauer group (cf. [25, Cor. 1.10]). This proves $a$).

By Theorem 4.3, $N_{V/U}(\mathcal{B}|_V)$ extends (since by $a$), it is unramified along $D$) to an Azumaya $\mathcal{O}_S$-algebra of degree $\ell d$, whose generic fiber is $N_{L/K}(\beta)$. $\square$

Suppose that $S$ has dimension $\leq 2$. We are interested in finding a good extension of $N_{V/U}(\mathcal{B}|_V)$ to $S$. We note that if $\mathcal{B}$ has an involution of the first kind $\tau$, then the corestriction involution $N_{V/U}(\tau|_V)$, given by the restriction of $\iota_*\tau|_V \otimes \tau|_V$ to $N_{V/U}(\mathcal{A}|_V)$, is of orthogonal type. If $N_{V/U}(\mathcal{B}|_V) \cong \text{End}(\mathcal{E}^U)$ is split, then $N_{V/U}(\tau|_V)$ is adjoint to a regular line bundle-valued quadratic form $(\mathcal{E}^U, q^U, \mathcal{L}^U)$ on $U$ unique up to projective similarity. The main result of this section is that this extends to a line bundle-valued quadratic form $(\mathcal{E}, q, \mathcal{L})$ on $S$ with simple degeneration along a regular divisor $D$ satisfying $\mathcal{C}_0(\mathcal{E}, q, \mathcal{L}) \cong \mathcal{B}$.

**Theorem 5.2.** Let $S$ be a regular integral scheme of dimension $\leq 2$ with $2$ invertible and $f : T \to S$ a finite flat cover of degree $2$ with regular branch divisor $D$. Let $\mathcal{B}$ be an Azumaya quaternion $\mathcal{O}_T$-algebra with standard involution $\tau$. Suppose that $N_{V/U}(\mathcal{B}|_V)$ is split and $N_{V/U}(\tau|_V)$ is adjoint to a regular line bundle-valued quadratic form $(\mathcal{E}^U, q^U, \mathcal{L}^U)$ on $U$. There exists a line bundle-valued quadratic form $(\mathcal{E}, q, \mathcal{L})$ on $S$ with simple degeneration along $D$ with multiplicity one, which restricts to $(\mathcal{E}^U, q^U, \mathcal{L}^U)$ on $U$ and such that $\mathcal{C}_0(\mathcal{E}, q, \mathcal{L}) \cong \mathcal{B}$. 


First we need the following lemma. Let \( S \) be a normal integral scheme, \( K \) its function field, \( D \subset S \) a regular divisor, and \( \mathcal{O}_{S,D} \) the semilocal ring at the generic points of \( D \).

**Lemma 5.3.** Let \( S \) be a normal integral scheme with 2 invertible, \( T \to S \) a finite flat cover of degree 2 with regular branch divisor \( D \subset S \), and \( L/K \) the corresponding extension of function fields. Under the restriction map \( H^1_{\text{ét}}(U, \mathbb{Z}/2\mathbb{Z}) \to H^2_{\text{ét}}(K, \mathbb{Z}/2\mathbb{Z}) = K^\times/K^\times^2 \), the class of the étale quadratic extension \( [V/U] \) maps to a square class represented by a parameter \( \pi \in K^\times \) of the semilocal ring \( \mathcal{O}_{S,D} \).

**Proof.** Consider any \( \pi \in K^\times \) with \( L = K(\sqrt{\pi}) \). For any irreducible component \( D' \) of \( D \), if \( v_D(\pi) \) is even, then we can modify \( \pi \) up to squares in \( K \) so that \( v_D(\pi) = 0 \). But then \( T/S \) would be étale at the generic point of \( D' \), which is impossible. Hence, \( v_D(\pi) \) is odd for every irreducible component \( D' \) of \( D \). Since \( \mathcal{O}_{S,D} \) is a principal ideal domain, we can modify \( \pi \) up to squares in \( K \) so that \( v_D(\pi) = 1 \) for every component \( D' \) of \( D \). Under \( H^1_{\text{ét}}(U, \mathbb{Z}/2\mathbb{Z}) \to H^2_{\text{ét}}(K, \mathbb{Z}/2\mathbb{Z}) = K^\times/K^\times^2 \), the class of \([V/U]\) is mapped to the class of \([L/K]\), which corresponds via Kummer theory to the square class \((\pi)\).

**Proof of Theorem 5.2.** If \( D = \cup D_i \) is the irreducible decomposition of \( D \), then \( \pi = \prod_i \pi_i \) is a parameter of \( \mathcal{O}_{S,D_i} \). Define \( \mathcal{O}_{S,D} \), a parameter of \( \mathcal{O}_{S,D} \), and choose a regular quadratic form \((\mathcal{O}_{S,D}, q_D^U, L^U)\) on \( U \) adjoint to \( N_{V/S}(\mathcal{O}_V) \). Since \( \mathcal{O}_{S,D} \) is a principal ideal domain, modifying by squares over \( K \), the form \( q_D^U \) has a diagonalization \( <a_1^2, a_2^2, a_3^2, a_4^2> \), where \( a_i \in \mathcal{O}_{S,D} \) are squarefree. By Lemma 5.3, we can choose \( \pi \in K^\times \) so that \([V/U] \in H^1_{\text{ét}}(U, \mathbb{Z}/2\mathbb{Z}) \) maps to \([V/U]\). Since \( \mathcal{O}_{S,D} \) is a principal ideal domain, \( a_1 \cdots a_4 \equiv \mu^2 \pi \), for some \( \mu \in \mathcal{O}_{S,D} \). If \( \pi_i \) divides \( \mu < a_1, a_2, a_3, a_4 > \) yields a form \((\mathcal{O}_{S,D}, q_D^U, L^U)\) on \( U \) over \( \mathcal{O}_{S,D} \) with simple degeneration along \( D \), which over \( K \), is isometric to \( \mu q_D^U \). Define \((\mathcal{O}_{S,D}, q_D^U, L^U) = (\mathcal{O}_{S,D}, q_D^U, <a_1^2, a_2^2, a_3^2, a_4^2>, \mathcal{O}_{S,D}) \).

By definition, the identity map is a similarity \( q_D^U | K = \beta \) with similarity factor \( \mu \) (up to \( K^\times^2 \)). Our aim is to find a good similarity enabling a gluing to a quadratic form over \( S \) with simple degeneration along \( D \) and the correct even Clifford algebra.

First note that by the classical theory of \( \mathbb{A}_1^2 = \mathbb{D}_2 \) over \( V/U \) (cf. Theorem 2.5), we can choose an \( \mathcal{O}_T \)-algebra isomorphism \( \varphi^U: \mathcal{O}_T(q_D^U, L^U) \to \mathcal{B}_V \). Second, we can pick an \( \mathcal{O}_T \)-algebra isomorphism \( \varphi^D: \mathcal{O}_T(q_D^D, L^D) \to \mathcal{B}_V \), where \( D = f^{-1}D \). Indeed, by the classical theory of \( \mathbb{A}_2 = \mathbb{D}_2 \) over \( L/K \) (cf. Theorem 2.5), the central simple algebras \( \mathcal{O}_T(q_D^D) \) and \( \mathcal{B}_L \) are isomorphic over \( L \), hence they are isomorphic over the semilocal principal ideal domain \( \mathcal{O}_{T,E} \). Now consider the \( L \)-algebra isomorphism \( \varphi^L = (\varphi^U | L)^{-1} \circ \varphi^D | L : \mathcal{O}_T(q_D^D) | L \to \mathcal{O}_T(q_D^D) | L \). Again by the classical theory of \( \mathbb{A}_1^2 = \mathbb{D}_2 \) over \( L/K \) (cf. Theorem 2.5), this is induced by a similarity \( \psi^K: q_D^D | K \to q_D^U | K \), unique up to multiplication by scalars. By Proposition 4.2, the quadratic forms \((\mathcal{O}_T(q_D^D, L^D)\) (\( \mathcal{O}_T(q_D^D, L^D)\) glue, via the similarity \( \psi^K \), to a quadratic form \((\mathcal{O}_T(q_D^D, L^D)\) on a dense open subscheme \( U \subset S \) containing \( U \) and the generic points of \( D \), hence all points of codimension 1. By [9, Prop. 2.3], the quadratic form \((\mathcal{O}_T(q_D^D, L^D)\) extends uniquely to a quadratic form \((\mathcal{O}_T(q_D^D, L^D)\) on \( S \) since the underlying vector bundle \( \mathcal{O}_U^D \) extends to a vector bundle \( \mathcal{O}_S \) on \( S \) (because \( S \) is a regular integral schemes of dimension \( \leq 2 \)). By Corollary 3.13, this extension has simple degeneration along \( D \).

Finally, we argue that \( \mathcal{O}_T(q) \cong \mathcal{B}_V \). We know that \( q|U = q_D^U \) and \( q|D = q_D^D \) and we have algebra isomorphisms \( \varphi^U: \mathcal{O}_T(q)_U \cong \mathcal{B}_U \) and \( \varphi^D: \mathcal{O}_T(q)_D \cong \mathcal{B}_D \) such that \( \varphi^L = (\varphi^U | L)^{-1} \circ \varphi^D | L \). Hence the gluing data \((\mathcal{O}_T(q)_U, \mathcal{O}_T(q)_D, \varphi^L)\) is isomorphic to the gluing data \((\mathcal{B}_U, \mathcal{B}_D, \text{id})\). Thus \( \mathcal{O}_T(q) \) and \( \mathcal{B} \) are isomorphic over an open subset \( U \subset S \) containing all codimension 1 points of \( S \). Hence by [15, Thm. 6.13], these Azumaya algebras are isomorphic over \( S \).
Finally, we give the proof of our main result.

Proof of Theorem 1. Theorem 5.2 implies that $\mathcal{C}_0 : \text{Quad}_2(T/S) \to \Lambda_2(T/S)$ is surjective. To prove the injectivity, let $(\mathcal{E}_1, q_1, \mathcal{L}_1)$ and $(\mathcal{E}_2, q_2, \mathcal{L}_2)$ be line bundle-valued quadratic forms of rank 4 on $S$ with simple degeneration along $D$ of multiplicity one such that there is an $O_T$-algebra isomorphism $\varphi : \mathcal{C}_0(q_1) \cong \mathcal{C}_0(q_2)$. By the classical theory of $2A_1 = D_2$ over $V/U$ (cf. Theorem 2.5), we know that $\varphi|_U : \mathcal{C}_0(q_1)|_U \cong \mathcal{C}_0(q_2)|_U$ is induced by a similarity transformation $\psi^U : q_1|_U \cong q_2|_U \otimes O_U$, for some line bundle $O_U$ on $U$, which we can assume is the restriction of a line bundle $\mathcal{N}$ on $S$. Replacing $(\mathcal{E}_2, q_2, \mathcal{L}_2)$ by $(\mathcal{E}_2 \otimes \mathcal{M}_2 \otimes \angle_1 >, \mathcal{L}_2 \otimes \mathcal{N} \otimes \mathcal{M}_2)$, which is in the same projective similarity class, we can assume that $\psi^U : q_1|_U \cong q_2|_U$. In particular, $\mathcal{L}_1|_U \cong \mathcal{L}_2|_U$ so that we have $\mathcal{L}_1 \cong \mathcal{L}_2 \otimes \mathcal{M}$ for some $\mathcal{M}|_U \cong O_U$ by the exact excision sequence

$$A^0(D) \to \text{Pic}(S) \to \text{Pic}(U) \to 0$$

of Picard groups (really Weil divisor class groups), cf. [24, Cor. 21.6.10], [22, 1 Prop. 1.8].

By Theorem 3.11, we know that $\varphi|_D : \mathcal{C}_0(q_1)|_D \cong \mathcal{C}_0(q_2)|_D$ is induced by some similarity transformation $\psi^D : q_1|_D \cong q_2|_D$. Thus $\psi^K = (\psi^D|^K)^{-1} \circ \psi^U|_K \in \text{GO}(q_1|_K)$. Since $\mathcal{C}_0(\psi^U|_K) = \mathcal{C}_0(\psi^D|^K) = \varphi|_K$, we have that $\psi^K \in \text{GO}(q_1|_K)$ is a homothety, multiplication by $\lambda \in K^\times$. As in §4, define a line bundle $\mathcal{P}$ on $S$ by the gluing datum $(\mathcal{E}_U, \mathcal{E}_D, \lambda^{-1} : \mathcal{E}_U|_K \cong \mathcal{E}_D|_K)$. Then $\mathcal{P}$ comes equipped with isomorphisms $\rho^U : \mathcal{E}_U \cong \mathcal{P}|_U$ and $\rho^D : \mathcal{E}_D \cong \mathcal{P}|_D$ with $(\rho^D|^K)^{-1} \circ \rho^U|_K = \lambda^{-1}$. Then we have similarities $\psi^U \otimes \rho^U : q_1|_U \cong q_2|_D \otimes \rho^D : q_1|_D \cong q_2|_D \otimes \mathcal{P}|_D$ such that

$$(\psi^D \otimes \rho^D)|_K^{-1} \circ (\psi^U \otimes \rho^U)|_K = (\psi^D|^K \psi^U|_K)(\rho^D|^K \rho^U|_K) = \psi^K \lambda^{-1} = \text{id}$$

in $\text{GO}(q_1|_K)$. Hence, as in §4, $\psi^U \otimes \rho^U$ and $\psi^D \otimes \rho^D$ glue to a similarity $(\mathcal{E}_1, q_1, \mathcal{L}_1) \cong (\mathcal{E}_2 \otimes \mathcal{P}, q_2 \otimes 1 >, \mathcal{L}_2 \otimes \mathcal{P} \otimes 2)$. Thus $(\mathcal{E}_1, q_1, \mathcal{L}_1)$ and $(\mathcal{E}_2, q_2, \mathcal{L}_2)$ define the same element of $\text{Quad}_2(T/S)$.

6. Failure of the local-global principle for isotropy of quadratic forms

In this section, we mention one application of the theory of quadratic forms with simple degeneration over surfaces. Let $S$ be a regular proper integral scheme of dimension $d$ over an algebraically closed field $k$ of characteristic $\neq 2$. For a point $x$ of $X$, denote by $K_x$ the fraction field of the completion $\hat{O}_{S,x}$ of $O_{S,x}$ at its maximal ideal.

Lemma 6.1. Let $S$ be a regular integral scheme of dimension $d$ over an algebraically closed field $k$ of characteristic $\neq 2$ and let $D \subset S$ be a divisor. Fix $i > 0$. If $(\mathcal{E}, q, \mathcal{L})$ is a quadratic form of rank $> 2^{d-1} + 1$ over $S$ with simple degeneration along $D$ then $q$ is isotropic over $K_x$ for all points $x$ of $S$ of codimension $\geq i$.

Proof. The residue field $k(x)$ of $K_x$ has transcendence degree $\leq d - i$ over $k$ and is hence a $C_{d-i}$-field. By hypothesis, $q$ has, over $K_x$, a subform $q_1$ of rank $> 2^{d-i}$ that is regular over $\hat{O}_{S,x}$. Hence $q_1$ is isotropic over $\kappa(x)$, thus $q$ is isotropic over the complete field $K_x$.

As usual, denote by $K = k(S)$ the function field. We say that a quadratic form $q$ over $K$ is locally isotropic if $q$ is isotropic over $K_x$ for all points $x$ of codimension one.

Corollary 6.2. Let $S$ be a proper regular integral surface over an algebraically closed field $k$ of characteristic $\neq 2$ and let $D \subset S$ be a regular divisor. If $(\mathcal{E}, q, \mathcal{L})$ is a quadratic form of rank $\geq 4$ over $S$ with simple degeneration along $D$ then $q$ over $K$ is locally isotropic.

For a different proof of this corollary, see [41, §3]. However, quadratic forms with simple degeneration are mostly anisotropic.

Theorem 6.3. Let $S$ be a proper regular integral surface with 2 invertible and $2\text{Br}(S)$ trivial. Let $T \to S$ be a finite flat morphism of degree 2 and $D \subset S$ a smooth divisor. Each nontrivial class in $2\text{Br}(T)$ gives rise to an anisotropic quadratic form over $K$, unique up to similarity.
Proof. Let \( L = k(T) \) and \( K = k(S) \). Let \( \beta \in 2\text{Br}(T) \) be nontrivial. Then by [1], \( \beta_L \in 2\text{Br}(L) \) has index 2 and by purity for division algebras over regular surfaces (Theorem 4.3), there exists an Azumaya quaternion algebra \( \mathcal{B} \) over \( T \) whose Brauer class is \( \beta \). Since \( N_{L/K}(\beta_L) \) is unramified on \( S \), by Lemma 5.1, it extends to an element of \( 2\text{Br}(S) \), which is assumed to be trivial. By the classical theory of \( 2\mathbb{A}_1 = D_2 \) over \( L/K \) (cf. Theorem 2.5), the quaternion algebra \( \mathcal{B}_L \) corresponds to a unique similarity class of quadratic form \( q^K \) of rank 4 on \( K \). By Theorem 1, \( \mathcal{B} \) corresponds to a unique projective similarity class of quadratic form \((\mathcal{E}, q, \mathcal{L})\) of rank 4 with simple degeneration along \( D \) and such that \( q|_K = q^K \).

A classical result in the theory of quadratic forms of rank 4 is that \( q^K \) is isotropic over \( K \) if and only if \( \mathcal{E}_0(q^K) \) splits over \( L \) (here \( L/K \) is the discriminant extension of \( q^K \)), see [34, Thm. 6.3], [47, 2 Thm. 14.1, Lemma 14.2], or [7, II Prop. 5.3] in characteristic 2. Hence \( q^K \) is anisotropic since \( \mathcal{E}_0(q^K) = \mathcal{B}_L \) has nontrivial Brauer class \( \beta \) by assumption. \( \square \)

We make Theorem 6.3 explicit as follows. Write \( L = K(\sqrt{d}) \). Let \( \mathcal{B} \) be an Azumaya quaternion algebra over \( T \) with \( \mathcal{B}_L \) given by the quaternion symbol \((a,b)\) over \( L \). Since \( N_{L/K}(\mathcal{B}_L) \) is trivial, the restriction-corestriction sequence shows that \( \mathcal{B}_L \) is the restriction of a class from \( 2\text{Br}(K) \), so we can choose \( a,b \in K^\times \). The corresponding quadratic form over \( K \) given by Theorem 1 is then given, up to similarity, by \( <1,a,b,abd> \), since it has discriminant \( d \) and even Clifford invariant \((a,b)\) over \( L \), see [34].

Example 6.4. Let \( T \to \mathbb{P}^2 \) be a branched double cover over a smooth sextic curve over an algebraically closed field of characteristic zero. Then \( T \) is a smooth projective K3 surface of degree 2. The Picard rank \( \rho \) of \( T \) can take values \( 1 \leq \rho \leq 20 \), with the “generic” such surface having \( \rho = 1 \). In particular, \( 2\text{Br}(T) \cong (\mathbb{Z}/2\mathbb{Z})^{22-\rho} \neq 0 \), so that \( T \) gives rise to \( 2^{22-\rho} - 1 \) similarity classes of locally isotropic yet anisotropic quadratic forms of rank 4 over \( K = k(\mathbb{P}^2) \). Explicit examples of such a quadratic form are given in [29] and [4].

7. The Torelli theorem for cubic fourfolds containing a plane

Let \( Y \) be a cubic fourfold, i.e., a smooth projective cubic hypersurface of \( \mathbb{P}^5 = \mathbb{P}(V) \) over \( \mathbb{C} \). Let \( W \subset V \) be a subspace of rank three, \( P = \mathbb{P}(W) \subset \mathbb{P}(V) \) the associated plane, and \( P' = \mathbb{P}(V/W) \). If \( Y \) contains \( P \), let \( \tilde{Y} \) be the blow-up of \( Y \) along \( P \) and \( \pi : \tilde{Y} \to P' \) the projection from \( P \). The blow-up of \( \mathbb{P}^5 \) along \( P \) is isomorphic to the total space of the projective bundle \( p : \mathbb{P}(\mathcal{E}) \to P' \), where \( \mathcal{E} = W \otimes \mathcal{O}_{P'} \oplus \mathcal{O}_{P'}(-1) \), and in which \( \pi : \tilde{Y} \to P' \) embeds as a quadric surface bundle. The degeneration divisor of \( \pi \) is a sextic curve \( D \subset P' \).

It is known that \( D \) is smooth and \( \pi \) has simple degeneration along \( D \) if and only if \( Y \) does not contain any other plane meeting \( P \), cf. [52, §1, Lemme 2]. In this case, the discriminant cover \( T \to P' \) is a K3 surface of degree 2. All K3 surfaces considered will be smooth and projective.

We choose an identification \( P' = \mathbb{P}^2 \) and suppose, for the rest of this section, that \( \pi : \tilde{Y} \to P' = \mathbb{P}^2 \) has simple degeneration. If \( Y \) contains another plane \( R \) disjoint from \( P \), then \( R \subset \tilde{Y} \) is the image of a section of \( \pi \), hence \( \mathcal{E}_0(\pi) \) has trivial Brauer class over \( T \) by a classical result concerning quadratic forms of rank 4, cf. proof of Theorem 6.3. Thus if \( \mathcal{E}_0(\pi) \) has nontrivial Brauer class \( \beta \in 2\text{Br}(T) \), then \( P \) is the unique plane contained in \( Y \).

Given a scheme \( T \) with \( 2 \) invertible and an Azumaya quaternion algebra \( \mathcal{B} \) on \( T \), there is a canonically lift \([\mathcal{B}] \in H^2_{\text{et}}(T, \mu_2)\) of the Brauer class of \( \mathcal{B} \), defined in [44] by taking into account the standard symplectic involution on \( \mathcal{B} \). Denote by \( c_1 : \text{Pic}(T) \to H^2_{\text{et}}(T, \mu_2) \) the mod 2 cycle class map arising from the Kummer sequence.

Definition 7.1. Let \( T \) be a K3 surface of degree 2 over \( k \) together with a polarization \( \mathcal{F} \), i.e., an ample line bundle of self-intersection 2. For \( \beta \in H^2_{\text{et}}(T, \mu_2)/[c_1(\mathcal{F})] \), we say that a cubic fourfold \( Y \) represents \( \beta \) if \( Y \) contains a plane whose associated quadric bundle \( \pi : \tilde{Y} \to \mathbb{P}^2 \) has simple degeneration and discriminant cover \( f : T \to \mathbb{P}^2 \) satisfying \( f^* \mathcal{O}_{\mathbb{P}^2}(1) \cong \mathcal{F} \) and \([\mathcal{E}_0(\pi)] = \beta \).
Remark 7.2. For a K3 surface $T$ of degree 2 with a polarization $\mathcal{F}$, not every class in $H^2_\text{et}(T, \mu_2)/\langle c_1(\mathcal{F}) \rangle$ is represented by a cubic fourfold, though one can characterize such classes. Consider the cup product mapping $H^2_\text{et}(T, \mu_2) \times H^2_\text{et}(T, \mu_2) \to H^4_\text{et}(T, \mu_2^2) \cong \mathbb{Z}/2\mathbb{Z}$. Define
\[ B(T, \mathcal{F}) = \{ \tau \in H^2_\text{et}(T, \mu_2)/\langle c_1(\mathcal{F}) \rangle \mid x \cup c_1(\mathcal{F}) \neq 0 \}. \]

Note that the natural map $B(T, \mathcal{F}) \to 2\text{Br}(T)$ is injective if and only if Pic$(T)$ is generated by $\mathcal{F}$. A consequence of the global description of the period domain for cubic fourfolds containing a plane is that for a K3 surface $T$ of degree 2 with polarization $\mathcal{F}$, the subset of $H^2_\text{et}(T, \mu_2)/\langle c_1(\mathcal{F}) \rangle$ represented by a cubic fourfolds containing a plane coincides with $B(T, f) \cup \{0\}$, cf. [50, §9.7] and [29, Prop. 2.1].

We can now state the main result of this section. Using Theorem 1 and results on twisted sheaves described below, we provide an algebraic proof of the following result, which is due to Voisin [52] (cf. [50, §9.7] and [29, Prop. 2.1]).

**Theorem 7.3.** Let $T$ be a generic K3 surface of degree 2 with a polarization $\mathcal{F}$. Then each element of $B(T, \mathcal{F})$ is represented by a single cubic fourfold containing a plane up to isomorphism.

We now explain the interest in this statement. The global Torelli theorem for cubic fourfolds states that a cubic fourfold $Y$ is determined up to isomorphic by the polarized Hodge structure on $H^4(Y, \mathbb{Z})$. Here polarization means a class $h^2 \in H^4(Y, \mathbb{Z})$ of self-intersection 3. Voisin’s approach [52] is to deal first with cubic fourfolds containing a plane, then apply a deformation argument to handle the general case. For cubic fourfolds containing a plane, we can give an alternate formulation, assuming the global Torelli theorem for K3 surfaces of degree 2, which is a celebrated result of Piatetsky-Shapiro and Shafarevich [45].

**Proposition 7.4.** Assume the global Torelli theorem holds for a K3 surface $T$ of degree 2. If the statement of Theorem 7.3 holds for $T$ then the global Torelli theorem holds for all cubic fourfolds containing a plane with $T$ as associated discriminant cover.

**Proof.** Let $Y$ be a cubic fourfold containing a plane $P$ with discriminant cover $f : T \to \mathbb{P}^2$ and even Clifford algebra $\mathcal{C}_0$. Consider the cycle class of $P$ in $H^4(Y, \mathbb{Z})$. Then $\mathcal{F} = f^*\mathcal{C}_{\mathbb{P}^2}(1)$ is a polarization on $T$, which together with $[\mathcal{C}_0] \in H^2_\text{et}(T, \mu_2)$, determines the sublattice $(h^2, P) \perp \subseteq H^4(Y, \mathbb{Z})$. The key lattice-theoretic result we use is [52, §1, Prop. 3], which can be stated as follows: the polarized Hodge structure $H^2(T, \mathbb{Z})$ and the class $[\mathcal{C}_0] \in H^2_\text{et}(T, \mu_2)$ determines the Hodge structure of $Y$; conversely, the polarized Hodge structure $H^4(Y, \mathbb{Z})$ and the sublattice $(h^2, P)$ determines the primitive Hodge structure of $T$, hence $T$ itself by the global Torelli theorem for K3 surfaces of degree 2.

Now let $Y$ and $Y'$ be cubic fourfolds containing a plane $P$ with associated discriminant covers $T$ and $T'$ and even Clifford algebras $\mathcal{C}_0$ and $\mathcal{C}'_0$. Assume that $\Psi : H^4(Y, \mathbb{Z}) \cong H^4(Y', \mathbb{Z})$ is an isomorphism of Hodge structures preserving the polarization $h^2$. By [27, Prop. 3.2.4], we can assume (by composing $\Psi$ with a Hodge automorphism fixing $h^2$) that $\Psi$ preserves the sublattice $(h^2, P)$. By [52, §1, Prop. 3], $\Psi$ induces an isomorphism $T \cong T'$, with respect to which $[\mathcal{C}_0] = [\mathcal{C}'_0] = \beta \in H^2_\text{et}(T, \mu_2) \cong H^2_\text{et}(T', \mu_2)$. Hence if there is at most a single cubic fourfold representing $\beta$ up to isomorphism then $Y \cong Y'$.

The following lemma, whose proof we could not find in the literature, holds for smooth cubic hypersurfaces $Y \subset \mathbb{P}^{2r+1}_k$ containing a linear subspace of dimension $r$ over any field $k$. Since $\text{Aut}(\mathbb{P}^{2r+1}_k) \cong \text{PGL}_{2r+2}(k)$ acts transitively on the set of linear subspaces in $\mathbb{P}^{2r+1}_k$ of dimension $r$, any two cubic hypersurfaces containing linear subspaces of dimension $r$ have isomorphic representatives containing a common such linear subspace.

**Lemma 7.5.** Let $Y_1$ and $Y_2$ be smooth cubic hypersurfaces in $\mathbb{P}^{2r+1}_k$ containing a linear space $P$ of dimension $r$. The associated quadric bundles $\pi_1 : \tilde{Y}_1 \to \mathbb{P}^r_k$ and $\pi_2 : \tilde{Y}_2 \to \mathbb{P}^r_k$ are $\mathbb{P}^r_k$-isomorphic if and only if the there is a linear isomorphism $Y_1 \cong Y_2$ fixing $P$. 


Proof. Any linear isomorphism $Y_1 \cong Y_2$ fixing $P$ will induce an isomorphism of blow-ups $\tilde{Y}_1 \cong \tilde{Y}_2$ commuting with the projections from $P$. Conversely, assume that $\tilde{Y}_1$ and $\tilde{Y}_2$ are $\mathbb{P}^r_k$-isomorphic. Since $\text{PGL}_{2r+2}(k)$ acts transitively on the set of linear subspaces of dimension $r$, without loss of generality, we can assume that $P = \{x_0 = \cdots = x_r = 0\}$ where $(x_0 : \cdots : x_r : y_0 : \cdots : y_r)$ are homogeneous coordinates on $\mathbb{P}^{2r+1}_k$. For $l = 1, 2$, write $Y_l$ as

$$\sum_{0 \leq m \leq n \leq r} a^l_{mn} y_m y_n + \sum_{p=0}^r b^l_p y_p + c^l = 0$$

for homogeneous linear forms $a^l_{mn}$, quadratic forms $b^l_p$, and cubic forms $c^l$ in $k[x_0, \ldots, x_r]$. The blow-up of $\mathbb{P}^{2r+1}_k$ along $P$ is identified with the total space of the projective bundle $\pi : \mathbb{P}(\mathcal{E}) \to \mathbb{P}^r_k$, where $\mathcal{E} = \mathcal{O}^{r+1}_P \oplus \mathcal{O}_{\mathbb{P}^r_k}(-1)$. The homogeneous coordinates $y_0, \ldots, y_r$ correspond, in the blow-up, to a basis of global sections of $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$.

Let $z$ be a nonzero global section of $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1) \otimes \pi^* \mathcal{O}_{\mathbb{P}^r_k}(-1)$. Then $z$ is unique up to scaling, as we have

$$\Gamma(\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1) \otimes \pi^* \mathcal{O}_{\mathbb{P}^r_k}(-1)) \cong \Gamma(\mathbb{P}^r_k, \pi^* \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1) \otimes \mathcal{O}_{\mathbb{P}^r_k}(-1)) = \Gamma(\mathbb{P}^r_k, \mathcal{E}^\vee \otimes \mathcal{O}_{\mathbb{P}^r_k}(-1)) = k$$

by the projection formula. Thus $(y_0 : \cdots : y_r : z)$ forms a relative system of homogeneous coordinates on $\mathbb{P}(\mathcal{E})$ over $\mathbb{P}^r_k$. Then $\tilde{Y}_l$ can be identified with the subscheme of $\mathbb{P}(\mathcal{E})$ defined by the global section

$$q_l(y_0, \ldots, y_r, z) = \sum_{0 \leq m \leq n \leq r} a^l_{mn} y_m y_n + \sum_{0 \leq p \leq r} b^l_p y_p z + c^l z^2 = 0$$

of $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(2) \otimes \pi^* \mathcal{O}_{\mathbb{P}^r_k}(1)$. Under these identifications, $\pi_l : \tilde{Y}_l \to \mathbb{P}^r_k$ can be identified with the restriction of $\pi$ to $\tilde{Y}_l$, hence with the quadric bundle associated to the line bundle-valued quadratic form $(\mathcal{E}, q_l, \mathcal{O}_{\mathbb{P}^r_k}(1))$. Since $Y_l$ and $P$ are smooth, so is $\tilde{Y}_l$. Thus $\pi_l : \tilde{Y}_l \to \mathbb{P}^r_k$ is flat, being a morphism from a Cohen–Macaulay scheme to a regular scheme. Thus by Propositions 1.1 and 1.6, the $\mathbb{P}^r_k$-isomorphism $\tilde{Y}_1 \cong \tilde{Y}_2$ induces a projective similarity $\psi$ between $q_1$ and $q_2$. But as $\mathcal{E} \otimes \mathcal{N} \cong \mathcal{E}$ implies $\mathcal{N}$ is trivial in $\text{Pic}(\mathbb{P}^r_k)$, we have that $\psi : q_1 \simeq q_2$ is, in fact, a similarity. In particular, $\psi \in \text{GL}(\mathcal{E})(\mathbb{P}^r_k)$, hence consists of a block matrix of the form

$$\begin{pmatrix} H & v \\ 0 & u \end{pmatrix}$$

where $H \in \text{GL}(\mathcal{O}_{\mathbb{P}^r_k}(1)) = \text{GL}_{r+1}(k)$ and $u \in \text{GL}(\mathcal{O}_{\mathbb{P}^r_k}(-1)) = \text{GL}_m(\mathbb{P}^r_k) = k^\times$, while $v \in \text{Hom}_{\mathcal{O}_{\mathbb{P}^r_k}}(\mathcal{O}_{\mathbb{P}^r_k}(-1), \mathcal{O}_{\mathbb{P}^r_k}(1)) = \text{Hom}(\mathbb{P}^{r+1}_k, \mathcal{O}_{\mathbb{P}^r_k}(1))$. Let $v = G \cdot (x_0, \ldots, x_r)^t$ for a matrix $G \in M_{r+1}(k)$. Then writing $H = (h_{ij})$ and $G = (g_{ij})$, we have that $\psi$ acts as

$$x_i \mapsto x_i, \quad y_i \mapsto \sum_{0 \leq j \leq r} (h_{ij} y_j + g_{ij} x_j z), \quad z \mapsto uz$$

and satisfies $q_2(\psi(y_0), \ldots, \psi(y_r), \psi(z)) = \lambda q_1(y_0, \ldots, y_r, z)$ for some $\lambda \in k^\times$. Considering the matrix $J \in M_{2r+2}(k)$ with $(r + 1) \times (r + 1)$ blocks

$$J = \begin{pmatrix} uI & 0 \\ G & H \end{pmatrix}$$

as a linear automorphism of $\mathbb{P}^{2r+1}_k$, then $J$ acts on $(x_0 : \cdots : x_r : y_0 : \cdots : y_r)$ as

$$x_i \mapsto ux_i, \quad y_i \mapsto \sum_{0 \leq j \leq r} (h_{ij} y_j + g_{ij} x_j),$$

and hence satisfies $q_2(J(y_0), \ldots, J(y_r), 1) = u \lambda q_1(y_0, \ldots, y_r, 1)$ due to the homogeneity properties of $x_i$ and $z$. Thus $J$ is a linear automorphism taking $Y_1$ to $Y_2$ and fixes $P$. \(\square\)
Let $T$ be a K3 surface. We shall freely use the notions of $\beta$-twisted sheaves, $B$-fields associated to $\beta$, the $\beta$-twisted Chern character, and $\beta$-twisted Mukai vectors from [30]. For a Brauer class $\beta \in \frac{1}{2} \text{Br}(T)$ we choose the rational $B$-field $\beta/2 \in H^{2}(T, \mathbb{Q})$. The $\beta$-twisted Mukai vector of a $\beta$-twisted sheaf $\mathcal{V}$ is
\[
v^B(\mathcal{V}) = c_B(\mathcal{V}) \sqrt{\text{Td}_T} = (\text{rk } \mathcal{V}, c_1^B(\mathcal{V}), \text{rk } \mathcal{V} + \frac{1}{2} c_2^B(\mathcal{V}) - c_2^B(\mathcal{V})) \in H^*(T, \mathbb{Q})\]
where $H^*(T, \mathbb{Q}) = \bigoplus_{i=0} H^{2i}(T, \mathbb{Q})$. As in [39], one introduces the Mukai pairing
\[
(v, w) = v_2 \cup w_2 - v_0 \cup w_0 - v_4 \cup w_4 \in H^4(T, \mathbb{Q}) \cong \mathbb{Q}
\]
for Mukai vectors $v = (v_0, v_2, v_4)$ and $w = (w_0, w_2, w_4)$.

By [55, Thm. 3.16], the moduli space of stable $\beta$-twisted sheaves $\mathcal{V}$ with Mukai vector $v = v^B(\mathcal{V})$ satisfying $(v, v) = 2n$ is isomorphic to the Hilbert scheme $\text{Hilb}_{2n+1}$. In particular, when $(v, v) = -2$, this moduli space consists of one point; we give a direct proof of this fact inspired by [39, Cor. 3.6].

**Lemma 7.6.** Let $T$ be a K3 surface and $\beta \in \frac{1}{2} \text{Br}(T)$ with chosen $B$-field. Let $v \in H^*(T, \mathbb{Q})$ with $(v, v) = -2$. If $\mathcal{V}$ and $\mathcal{V}'$ are stable $\beta$-twisted sheaves with $v^B(\mathcal{V}) = v^B(\mathcal{V}') = v$ then $\mathcal{V} \cong \mathcal{V}'$.

**Proof.** Assume that $\beta$-twisted sheaves $\mathcal{V}$ and $\mathcal{V}'$ have the same Mukai vector $v \in H^2(T, \mathbb{Q})$.

Since $-2 = (v, v) = \chi(\mathcal{V}, \mathcal{V}) = \chi(\mathcal{V}, \mathcal{V}')$, a Riemann–Roch calculation shows that either $\text{Hom}(\mathcal{V}, \mathcal{V}') \neq 0$ or $\text{Hom}(\mathcal{V}, \mathcal{V}') \neq 0$. Without loss of generality, assume $\text{Hom}(\mathcal{V}, \mathcal{V}') \neq 0$. Then since $\mathcal{V}$ is stable, a nonzero map $\mathcal{V} \to \mathcal{V}'$ must be injective. Since $\mathcal{V}'$ is stable, the map is an isomorphism. □

**Lemma 7.7.** Let $T$ be a K3 surface of degree 2 and $\beta \in \frac{1}{2} \text{Br}(T)$ with chosen $B$-field. Let $Y$ be a smooth cubic fourfold containing a plane whose even Clifford algebra $\mathcal{C}_0$ represents $\beta \in \frac{1}{2} \text{Br}(T)$. If $\mathcal{V}_0$ is a $\beta$-twisted sheaf associated to $\mathcal{C}_0$ then $(v^B(\mathcal{V}_0), v^B(\mathcal{V}_0)) = -2$. Furthermore, if $T$ is generic then $\mathcal{V}_0$ is stable.

**Proof.** By the $\beta$-twisted Riemann–Roch theorem, we have
\[
-(v^B(\mathcal{V}_0), v^B(\mathcal{V}_0)) = \chi(\mathcal{V}_0, \mathcal{V}_0) = \sum_{i=0}^2 \text{Ext}^i_T(\mathcal{V}_0, \mathcal{V}_0).
\]
Then $v^B(\mathcal{V}_0) = 2$ results from the fact that $\mathcal{V}_0$ is a spherical object, i.e., $\text{Ext}^i_T(\mathcal{V}_0, \mathcal{V}_0) = \mathbb{C}$ for $i = 0, 2$ and $\text{Ext}^1(\mathcal{V}_0, \mathcal{V}_0) = 0$. Indeed, as in [37, Rem. 2.1], we have $\text{Ext}^i_T(\mathcal{V}_0, \mathcal{V}_0) = H^i(\mathbb{P}^2, \mathcal{E}_0)$, which can be calculated directly using the fact that, as $\mathcal{O}_{\mathbb{P}^2}$-algebras,
\[
\mathcal{E}_0 \cong \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-3) \oplus \mathcal{O}_{\mathbb{P}^2}(-1)^3 \oplus \mathcal{O}_{\mathbb{P}^2}(-2)^3
\]
If $T$ is generic, stability follows from [39, Prop. 3.14], see also [55, Prop. 3.12]. □

**Lemma 7.8.** Let $T$ be a K3 surface of degree 2. Let $Y$ and $Y'$ be smooth cubic fourfolds containing a plane whose respective even Clifford algebras $\mathcal{C}_0$ and $\mathcal{C}_0'$ represent the same $\beta \in \frac{1}{2} \text{Br}(T)$. If $T$ is generic then $\mathcal{C}_0 \cong \mathcal{C}_0'$.

**Proof.** Let $\mathcal{V}_0$ and $\mathcal{V}_0'$ be $\beta$-twisted sheaves associated to $\mathcal{C}_0$ and $\mathcal{C}_0'$, respectively. A consequence of [37, Lemma 3.1] and (8) is that $v = v^B(\mathcal{V}_0) = v^B(\mathcal{V}_0' \otimes \mathcal{N})$ for some line bundle $\mathcal{N}$ on $T$. Replacing $\mathcal{V}_0'$ by $\mathcal{V}_0' \otimes \mathcal{N}^{\vee}$, we can assume that $v^B(\mathcal{V}_0) = v^B(\mathcal{V}_0')$. By Lemma 7.7, we have $(v, v) = -2$ and that $\mathcal{V}_0$ and $\mathcal{V}_0'$ are stable. Hence by Lemma 7.6, we have $\mathcal{V}_0 \cong \mathcal{V}_0'$ as $\beta$-twisted sheaves, hence $\mathcal{C}_0 \cong \mathcal{E}nd(\mathcal{V}_0) \cong \mathcal{E}nd(\mathcal{V}_0') \cong \mathcal{C}_0'$. □

**Proof of Theorem 7.3.** Suppose that $Y$ and $Y'$ are smooth cubic fourfolds containing a plane whose associated even Clifford algebras $\mathcal{C}_0$ and $\mathcal{C}_0'$ represent the same class $\beta \in B(T, \mathcal{F}) \subset H^{2}(T, \mathbb{Q}^2)/(c_1(\mathcal{F})) \cong \frac{1}{2} \text{Br}(T)$. By Lemma 7.8, we have $\mathcal{C}_0 \cong \mathcal{C}_0'$. By Theorem 1, the quadric surface bundles $\pi : \tilde{Y} \to \mathbb{P}^2$ and $\pi' : \tilde{Y}' \to \mathbb{P}^2$ are $\mathbb{P}^2$-isomorphic. Finally, by Lemma 7.5, we have $Y \cong Y'$. □
References

[1] M. Artin, *Brauer-Severi varieties*, Brauer groups in ring theory and algebraic geometry (Wilrijk, 1981), Lecture Notes in Math., vol. 917, Springer, Berlin, 1982, pp. 194–210.

[2] A. Auel, *Clifford invariants of line bundle-valued quadratic forms*, MPIM preprint series 2011-33, 2011.

[3] A. Auel, M. Bernardara, and M. Bolognesi, *Fibrations in complete intersections of quadrics, Clifford algebras, derived categories, and rationality problems*, preprint arXiv:1109.6938, 2011.

[4] A. Auel, M. Bernardara, M. Bolognesi, and A. Várilly-Alvarado, *Rational cubic fourfolds containing a plane with nontrivial clifford invariant*, preprint arXiv:1205.0237, 2012.

[5] M. Auslander and O. Goldman, *The Brauer group of a commutative ring*, Trans. Amer. Math. Soc. 97 (1960), 367–409.

[6] *Maximal orders*, Trans. Amer. Math. Soc. 97 (1960), 1–24.

[7] R. Baeza, *Quadratic forms over semilocal rings*, Lecture Notes in Mathematics, Vol. 655, Springer-Verlag, Berlin, 1978.

[8] V. Balaji, *Line-bundle valued ternary quadratic forms over schemes*, J. Pure Appl. Algebra 208 (2007), 237–259.

[9] P. Balmer and B. Calmès, *Bases of total Witt groups and lax-similitude*, J. Algebra Appl. 11, no. 3 (2012), 24 pages.

[10] A. Auel and M. Bernardara, *Fibrations in complete intersections of quadrics, Clifford algebras, derived categories, and rationality problems*, preprint arXiv:1109.6938, 2011.

[11] A. Auel, M. Bernardara, and M. Bolognesi, *Clifford invariants of line bundle-valued quadratic forms*, MPIM preprint series 2011-33, 2011.

[12] A. Auel, M. Bernardara, M. Bolognesi, and A. Várilly-Alvarado, *Rational cubic fourfolds containing a plane with nontrivial clifford invariant*, preprint arXiv:1205.0237, 2012.

[13] M. Auslander and O. Goldman, *The Brauer group of a commutative ring*, Trans. Amer. Math. Soc. 97 (1960), 367–409.

[14] *Maximal orders*, Trans. Amer. Math. Soc. 97 (1960), 1–24.

[15] R. Baeza, *Quadratic forms over semilocal rings*, Lecture Notes in Mathematics, Vol. 655, Springer-Verlag, Berlin, 1978.

[16] V. Balaji, *Line-bundle valued ternary quadratic forms over schemes*, J. Pure Appl. Algebra 208 (2007), 237–259.

[17] P. Balmer and B. Calmès, *Bases of total Witt groups and lax-similitude*, J. Algebra Appl. 11, no. 3 (2012), 24 pages.

[18] A. Auel and M. Bernardara, *Fibrations in complete intersections of quadrics, Clifford algebras, derived categories, and rationality problems*, preprint arXiv:1109.6938, 2011.

[19] A. Auel, M. Bernardara, and M. Bolognesi, *Clifford invariants of line bundle-valued quadratic forms*, MPIM preprint series 2011-33, 2011.

[20] A. Auel and M. Bernardara, *Fibrations in complete intersections of quadrics, Clifford algebras, derived categories, and rationality problems*, preprint arXiv:1109.6938, 2011.

[21] A. Auel and M. Bernardara, *Fibrations in complete intersections of quadrics, Clifford algebras, derived categories, and rationality problems*, preprint arXiv:1109.6938, 2011.
[30] D. Huybrechts and P. Stellari, *Equivalences of twisted K3 surfaces*, Math. Ann. 332 (2005), no. 4, 901–936.

[31] M. M. Kapranov, *On the derived categories of coherent sheaves on some homogeneous spaces*, Invent. Math. 92 (1988), 479–508.

[32] M.-A. Knus, *Quadratic and hermitian forms over rings*, Springer-Verlag, Berlin, 1991.

[33] M.-A. Knus, A. Merkurjev, M. Rost, and J.-P. Tignol, *The book of involutions*, Colloquium Publications, vol. 44, AMS, 1998.

[34] M.-A. Knus, R. Parimala, and R. Sridharan, *On rank 4 quadratic spaces with given Arf and Witt invariants*, Math. Ann. 274 (1986), no. 2, 181–198.

[35] A classification of rank 6 quadratic spaces via pfaffians, J. reine angew. Math. 398 (1989), 187–218.

[36] A. Kuznetsov, *Derived categories of quadric fibrations and intersections of quadrics*, Adv. Math. 218 (2008), no. 5, 1340–1369.

[37] E. Macrì and P. Stellari, *Fano varieties of cubic fourfolds containing a plane*, Math. Ann., to appear.

[38] H. Matsumura, *Commutative ring theory*, second ed., Cambridge Studies in Advanced Mathematics, vol. 8, Cambridge University Press, Cambridge, 1989, Translated from the Japanese by M. Reid.

[39] S. Mukai, *On the moduli space of bundles on K3 surfaces. I*, Vector bundles on algebraic varieties (Bombay, 1984), Tata Inst. Fund. Res. Stud. Math., vol. 11, Tata Inst. Fund. Res., Bombay, 1987, pp. 341–413.

[40] M. Ojanguren, *Formes quadratiques sur les algèbres de polynomes*, C. R. Acad. Sci. Paris Sér. A-B 287 (1978), no. 9, A695–A698.

[41] M. Ojanguren and R. Parimala, *Quadratic forms over complete local rings*, Advances in algebra and geometry (Hyderabad, 2001), Hindustan Book Agency, New Delhi, 2003, pp. 53–55.

[42] R. Parimala, *Quadratic forms over polynomial rings over Dedekind domains*, Amer. J. Math. 103 (1981), no. 2, 289–296.

[43] R. Parimala and R. Sridharan, *Reduced norms and pfaffians via Brauer-Severi schemes*, Contemp. Math. 155 (1994), 351–363.

[44] R. Parimala and V. Srinivas, *Analogues of the Brauer group for algebras with involution*, Duke Math. J. 66 (1992), no. 2, 207–237.

[45] I. I. Pjatecki-ˇSapiro and I. R. ˇSafareviˇc, *Torelli’s theorem for algebraic surfaces of type K3*, Izv. Akad. Nauk SSSR Ser. Mat. 35 (1971), 530–572.

[46] A. Roy, *Cancellation of quadratic form over commutative rings*, J. Algebra 10 (1968), 286–298.

[47] W. Scharlau, *Quadratic and Hermitian forms*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 270, Springer-Verlag, Berlin, 1985.

[48] V. Suresh, *Linear relations in Eichler orthogonal transformations*, J. Algebra 168 (1994), no. 3, 804–809.

[49] R. G. Swan, *K-theory of quadric hypersurfaces*, Ann. of Math. (2) 122 (1985), no. 1, 113–153.

[50] B. van Geemen, *Some remarks on Brauer groups of K3 surfaces*, Adv. Math. 197 (2005), no. 1, 222–247.

[51] J. Voight, *Characterizing quaternion rings over an arbitrary base*, J. Reine Angew. Math. 657 (2011), 113–134.

[52] C. Voisin, *Théorème de Torelli pour les cubiques de P5*, Invent. Math. 86 (1986), no. 3, 577–601.

[53] C. T. C. Wall, *On the orthogonal groups of unimodular quadratic forms. II*, J. Reine Angew. Math. 213 (1963/1964), 122–136.

[54] M. M. Wood, *Gauss composition over an arbitrary base*, Adv. Math. 226 (2011), no. 2, 1756–1771.

[55] K. Yoshioka, *Moduli spaces of twisted sheaves on a projective variety*, Moduli spaces and arithmetic geometry, Adv. Stud. Pure Math., vol. 45, Math. Soc. Japan, Tokyo, 2006, pp. 1–30.