SELF-REGULATION IN INFINITE POPULATIONS WITH 
FISSION-DEATH DYNAMICS

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Abstract. The evolution of an infinite population of interacting point entities placed in $\mathbb{R}^d$ is studied. The elementary evolutionary acts are death of an entity with rate that includes a competition term and independent fission into two entities. The population states are probability measures on the corresponding configuration space and the result is the construction of the evolution of states in the class of sub-Poissonian measures, that corresponds to the lack of clusters in such states. This is considered as a self-regulation in the population due to competition.

1. Introduction

1.1. Regulating population dynamics. Simple population dynamics models are mostly based on two evolutionary acts: disappearance (death) of an entity and procreation, in the course of which new entities join the population. A commonly accepted viewpoint on the evolution of a finite population of this kind is that it either dies out or grows ad infinitum being unhampered by habitat restrictions. Clearly, such restrictions can only be ignored if the population size is small, i.e., at the early stage of its development. In developed populations, environmental restrictions force the entities to compete with each other – a crowding effect. In the mentioned models, this effect manifests itself in a state-dependent increment of the death toll. In Verhulst’s phenomenological theory based on the equation $\frac{d}{dt}N = \lambda N - (\mu + \alpha N)N$, such an increment is $\alpha N$. Here $N = N(t)$ is the (expected) number of entities at time $t$, and positive $\lambda$ and $\mu$ are the intrinsic procreation and death rates, respectively. Later on, Pearl and Reed rewrote this in the form of the logistic growth equation $\frac{d}{dt}N = rN(1 - N/K)$ with $r = \lambda - \mu$ and $K = r/\alpha$. The latter parameter gives rise to the notion of carrying capacity as the solution $N \equiv K$ is a stationary one, to which $N(t)$ tends in the limit $t \to +\infty$. Since then, this notion is used in the theory of biological populations, see Introduction in [1], and not only in the context of the competition caused crowding effect. For instance, in the Galton-Watson model with binary fission considered in [2], the probability of fission of a member of generation $n$ consisting of $Z_n$ entities was taken to be $K/(K + Z_n)$. Thereby, the constructed process gets super- or subcritical under or over the level $K$, respectively. This aspect of the theory may be viewed as a phenomenological (mean-field-like) way of regulating the population dynamics. Here regulating means preventing the population from infinite growth and mean-field corresponds to imitating interactions as state-dependent external actions (fields), cf. [3, Sect. 13].

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In the theory of populations with interactions explicitly taken into account, a usual assumption is that each entity interacts mostly (or even entirely) with the subpopulation located in a compact subset of the habitat. Then the local structure of the population is determined by the network of such interactions. Since a finite population occupies a compact set, it is always local as each of its members has a compact neighborhood containing the whole remaining population. Thus, in order to understand the global behavior of populations of this kind, one should take them infinite. In the statistical mechanics of interacting physical particles developed from phenomenological thermodynamics, this conclusion had led to the concept of the infinite-volume limit, see, e.g., [6, pp. 5,6]. In this note, and in the accompanying paper [4] where all the technical details are presented, we introduce an individual-based model of an infinite population of point entities placed in \( \mathbb{R}^d \) which undergo binary fission and death caused also by crowding (local competition). Its aim is to demonstrate that the local competition – interaction explicitly taken into account – can produce a global regulating effect. Here, however, one has to make precise the very notion of regulation as the considered population is already infinite. Instead of characterizing it by the number of constituents, we will look at the spatial distribution of the population by comparing it with the distribution governed by a Poisson law.

1.2. Presenting the result. Similarly as in [3], we deal with the phase space \( \Gamma \) consisting of all locally finite subsets \( \gamma \subset \mathbb{R}^d \), called configurations. Local finiteness means that \( \gamma_\Lambda := \gamma \cap \Lambda \) is finite whenever \( \Lambda \subset \mathbb{R}^d \) is compact. For compact \( \Lambda \) and \( n \in \mathbb{N}_0 \), we then set \( \Gamma^{\Lambda,n} = \{ \gamma \in \Gamma : |\gamma_\Lambda| = n \} \), where \( | \cdot | \) denotes cardinality, and equip \( \Gamma \) with the \( \sigma \)-field \( \mathcal{B}(\Gamma) \) generated by all such \( \Gamma^{\Lambda,n} \). This allows one to consider probability measures on \( \Gamma \) as states of the system. In a Poisson state, the entities are independently distributed over \( \mathbb{R}^d \). A homogeneous Poisson measure \( \pi_\kappa \) with intensity \( \kappa > 0 \) is characterized by its values on \( \Gamma^{\Lambda,n} \) given by the following expression

\[
\pi_\kappa(\Gamma^{\Lambda,n}) = \left( \frac{\kappa|\Lambda|}{n!} \right)^n \exp \left( -\kappa|\Lambda| \right),
\]

where \( |\Lambda| \) stands for the Lebesgue measure of \( \Lambda \). Note that \( \pi_\kappa(\Gamma_0) = 0 \), for all \( \kappa > 0 \), where \( \Gamma_0 \) is the set of all finite configurations. Let \( \mathcal{P}(\Gamma) \) be the set of all probability measures on \( \Gamma \). We say that a given \( \mu \in \mathcal{P}(\Gamma) \) is sub-Poissonian if, for each compact \( \Lambda \), all \( n \in \mathbb{N}_0 \) and some \( \kappa > 0 \), the following holds

\[
\mu(\Gamma^{\Lambda,n}) \leq \pi_\kappa(\Gamma^{\Lambda,n}).
\]

It is believed that sub-Poissonian states are characterized by the lack of clustering, typical to procreating populations with noninteracting (noncompeting) constituents, see the corresponding discussion in [3].

In dealing with states on \( \Gamma \), one employs observables – appropriate functions \( F : \Gamma \to \mathbb{R} \). Their evolution is obtained by solving the Kolmogorov equation

\[
\frac{d}{dt} F_t = LF_t, \quad F_t|_{t=0} = F_0, \quad t > 0,
\]

in which the operator \( L \) specifies the model. The model which we introduce here is based on the following evolutionary acts: (a) an entity located at \( x \) dies with rate (probability per unit time) \( m(x) + \sum_{y \in \gamma \setminus x} a(x-y) \), where \( m(x) \geq 0 \) corresponds to a per se mortality and \( a \geq 0 \) is the competition kernel; (b) an entity located at
real-valued functions with compact support. Then the map by setting

$$b$$

Note also that we do not exclude the case where possible. The version studied in [4] is characterized by less restrictive conditions.

We proceed as follows. Let

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Then the translation invariance and the finite-range property are imposed here only to make the presentation of the model and the results as simple as possible. As in the Bolker-Pacala model, here the following situations may occur:

- Short dispersal: there exists $$\omega > 0$$ such that $$a(x) \geq \omega \beta(x)$$ for all $$x \in \mathbb{R}^d$$; corresponds to $$R \leq r$$.
- Long dispersal: for each $$\omega > 0$$, there exists $$x \in \mathbb{R}^d$$ such that $$a(x) < \omega \beta(x)$$; corresponds to $$R > r$$.

The direct use of $$L$$ as a linear operator in an appropriate Banach space is possible only if one restricts the consideration to states on $$\Gamma_0$$, see [4, Sect. 3]. Otherwise, the sums in (1.3) – taken over infinite configurations – may not exist. In view of this, we proceed as follows. Let $$C_0(\mathbb{R}^d)$$ stand for the set of all continuous real-valued functions with compact support. Then the map

$$\Gamma \ni \gamma \mapsto F^\theta(\gamma) := \prod_{x \in \gamma}(1 + \theta(x)),$$

$$\Theta := \{\theta \in C_0(\mathbb{R}^d) : \theta(x) \in (-1, 0]\},$$
Hence, the unbounded linear operator \(\|\) and with the usual point-wise linear operations. Clearly, the action of \(L\) of such functions equipped with the norm states of thermal equilibrium of systems of interacting physical particles satisfy \((2.2)\), see \([5]\). By means of \(\kappa\) holding with some \(\alpha\), one can define the function \(k_\mu: \Gamma_0 \to \mathbb{R}\) by setting \(k_\mu(\{x_1, \ldots, x_n\}) = k_\mu^{(n)}(x_1, \ldots, x_n), n \in \mathbb{N}\). Let us consider the Banach space \(\mathcal{K}_\alpha\) of such functions equipped with the norm

\[
\|k\|_\alpha = \sup_{n \geq 0} \|k^{(n)}\|_{L^\infty(\mathbb{R}^d)^n} \exp(-\alpha n), \quad \alpha \in \mathbb{R},
\]

and with the usual point-wise linear operations. Clearly, \(\|k\|_{\alpha'} \leq \|k\|_\alpha\) whenever \(\alpha' > \alpha\), which yields that

\[
\mathcal{K}_\alpha \hookrightarrow \mathcal{K}_{\alpha'}, \quad \alpha < \alpha'.
\]

Hence, \(\{\mathcal{K}_\alpha\}_{\alpha \in \mathbb{R}}\) form an ascending scale of Banach spaces. In each \(\mathcal{K}_\alpha\), one defines the unbounded linear operator \((L^\Delta, \mathcal{D}_\alpha)\) by setting \(\mathcal{D}_\alpha = \{k \in \mathcal{K}_\alpha : L^\Delta k \in \mathcal{K}_\alpha\}\), where the action of \(L^\Delta\) on \(k_\mu\) is calculated from the formula, cf. \((2.1)\),

\[
\frac{d}{dt}\mu_t(F^\theta) = \mu_t(LF^\theta).
\]

Here \(\mathcal{P}_{\exp}(\Gamma)\) is a class of measures each element of which is sub-Poissonian, see below, and such that \(\mu(LF^\theta) < \infty\).

2. The Result

For the Poisson measure as in \((1.1)\), it follows that

\[
\pi_\nu(F^\theta) = \exp \left( - \int_{\mathbb{R}^d} \theta(x) dx \right).
\]

Having this in mind we introduce the class of measures \(\mathcal{P}_{\exp}(\Gamma)\) by the condition that, for each \(\mu \in \mathcal{P}_{\exp}(\Gamma), \mu(F^\theta)\) can be continued to an exponential type entire function of \(\theta \in L^1(\mathbb{R}^d)\). It can be shown that \(\mu \in \mathcal{P}_{\exp}(\Gamma)\) if and only if \(\mu(F^\theta)\) is written in the form

\[
\mu(F^\theta) = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \int_{(\mathbb{R}^d)^n} k^{(n)}_{\mu}(x_1, \ldots, x_n) \theta(x_1) \cdots \theta(x_n) dx_1 \cdots dx_n,
\]

where \(k^{(n)}_{\mu}\) is the \(n\)-th order correlation function of \(\mu\). Each \(k^{(n)}_{\mu}\) is a symmetric positive element of \(L^\infty((\mathbb{R}^d)^n)\) satisfying the Ruelle bound, cf. \([5]\),

\[
k^{(n)}_{\mu}(x_1, \ldots, x_n) \leq \varkappa^n, \quad n \in \mathbb{N},
\]

holding with some \(\varkappa > 0\). Note that \((2.2)\) readily yields \((1.2)\). Note also that states of thermal equilibrium of systems of interacting physical particles satisfy \((2.2)\), see \([5]\). By means of \(k^{(n)}_{\mu}\) one can define the function \(k_\mu: \Gamma_0 \to \mathbb{R}\) by setting \(k_\mu(\{x_1, \ldots, x_n\}) = k^{(n)}_{\mu}(x_1, \ldots, x_n), n \in \mathbb{N}\). Let us consider the Banach space \(\mathcal{K}_\alpha\) of such functions equipped with the norm

\[
\|k\|_\alpha = \sup_{n \geq 0} \|k^{(n)}\|_{L^\infty(\mathbb{R}^d)^n} \exp(-\alpha n), \quad \alpha \in \mathbb{R},
\]

and with the usual point-wise linear operations. Clearly, \(\|k\|_{\alpha'} \leq \|k\|_\alpha\) whenever \(\alpha' > \alpha\), which yields that

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\[
\mu(LF^\theta) = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \int_{(\mathbb{R}^d)^n} (L^\Delta k_\mu)^{(n)}(x_1, \ldots, x_n) \theta(x_1) \cdots \theta(x_n) dx_1 \cdots dx_n,
\]
Then with the help of (2.1) and (2.3) the evolution \( \mu_0 \to \mu_t \) is obtained by employing the correlation functions in the following three steps:

(a) Constructing \( k_0 \to k_t \) for \( t < T < \infty \) by solving the corresponding evolution equation

\[
\frac{d}{dt} k_t = L^\Delta k_t, \quad k_t|_{t=0} = k_{\mu_0}.
\]  

(b) Proving that \( k_t \) is the correlation function of a unique \( \mu_t \in \mathcal{P}_{\mathbb{R}}(\Gamma) \)

(c) Continuing \( k_t \) to all \( t > 0 \).

To perform step (a), for each \( \alpha_0 \in \mathbb{R} \) and \( \alpha_1 > \alpha_0 \), we construct a family of operators \( Q_{\alpha_1,\alpha_0}(t), t \in [0, T(\alpha_1,\alpha_0)) \). Here \( T(\alpha_1,\alpha_0) = (\alpha_1 - \alpha_0)/\tau(\alpha_1) \) with certain (dependent on the model parameters and explicitly found) function \( \tau(\alpha) \). Each \( Q_{\alpha_1,\alpha_0}(t) \) acts as a bounded operator from \( \mathcal{K}_{\alpha_0} \) to \( \mathcal{K}_{\alpha_1} \), cf. (2.3). Then the (classical) solution of (2.5) with \( k_0 \in \mathcal{K}_{\alpha_0} \) is obtained in the form \( k_t = \mu_{\alpha_1,\alpha_0}(t)k_0, t < T(\alpha_1,\alpha_0) \). The important peculiarities of this solution are: (i) the function \( \tau(\alpha) \) is rapidly increasing, which means that the time interval shrinks to zero as \( \alpha \to +\infty \); (ii) as \( t \) increases, \( k_t \) passes to an ever-larger space, cf. (2.3); (iii) the solution \( k_t \) need not be a correlation function of any state. In view of (i) and (ii), the direct continuation of \( k_t \) to all \( t > 0 \) is impossible.

To perform step (b) we use a special cone \( \mathcal{K}_\alpha^* \subset \mathcal{K}_\alpha \) (explicitly constructed, see eq. (4.11) in [4]) such that \( k \in \mathcal{K}_\alpha^* \) is the correlation function of a unique \( \mu \in \mathcal{P}_{\mathbb{R}}(\Gamma) \) if and only if \( k \in \mathcal{K}_\alpha^* \). Then we prove that the solution mentioned above lies in \( \mathcal{K}_\alpha^* \) for all \( t < T(\alpha_1,\alpha_0)/3 \). Along with the identification of \( k_t \) as a correlation function, this yields also that \( k_t \in \mathcal{K}_{\alpha_t} \) with \( \alpha_t = \alpha_0 + ct \). Here \( \alpha_0 \) is chosen to be such that \( k_0 \in \mathcal{K}_{\alpha_0} \) and \( \alpha_0 > -\log \omega \) with \( \omega \) as in Remark 1.1. One can take \( c = 0 \) if \( s := \inf_{x \in \mathbb{R}_d} m(x) > \langle b \rangle \), where the latter is the same as in (1.5). In the short dispersal case, one can take \( c = 0 \) already for \( s = \langle b \rangle \). For \( c = 0 \), the solution stays in the same space and hence can be continued to all \( t > 0 \) by repeating the above construction. This is not the case if \( c > 0 \). Then the solution passes to an ever-larger space, but with a much slower increase than in the construction made in step (a). This allows one to prove that \( k_t \in \mathcal{K}_{\alpha_t} \) for all \( t > 0 \) also for positive \( c \), and hence to perform step (c). Note that in the essentially different cases of short and long dispersal the qualitative difference of the corresponding dynamics appear only at the borderline case of \( s = \langle b \rangle \). This may mean that the dispersal range affects finer properties of the corresponding system.

As the result, under the assumptions on \( a, m \) and \( b \) made above we prove the following statement, see [4] Theorem 4.1 and Corollary 4.2.

**Theorem 2.1.** There exist \( c \in \mathbb{R} \) and \( \omega > 0 \) such that, for each \( \mu_0 \in \mathcal{P}_{\mathbb{R}}(\Gamma_0) \), there exists a unique map \( [0, +\infty) \ni t \mapsto k_t \in \mathcal{K}_\alpha^* \) with \( \alpha_t = \alpha_0 + ct \) and \( \alpha_0 > -\log \omega \) such that \( k_0 = k_{\mu_0} \in \mathcal{K}_{\alpha_0}^* \), which has the following properties:

(i) For each \( T > 0 \) and all \( t \in [0, T) \), the map

\[
[0, T) \ni t \mapsto k_t \in \mathcal{K}_{\alpha_t} \subset \mathcal{D}_{\alpha_t} \subset \mathcal{K}_{\alpha_T}
\]

is continuous on \( [0, T) \) and continuously differentiable on \( (0, T) \) in \( \mathcal{K}_{\alpha_T} \).

(ii) For all \( t \in (0, T) \), it satisfies \( \frac{d}{dt} k_t = L^\Delta k_t \).

By Theorem 2.1 the evolution \( \mu_0 \to \mu_t \) in question is obtained by identifying \( \mu_t \) by its values on \( \mathcal{F}^d \) with the help of (2.1) and by the evolution \( k_0 \to k_t \) constructed.
therein. Then the validity of (1.10) follows by (2.20). Since $\mu_t \in \mathcal{P}_{\exp}(\Gamma)$ for all $t > 0$, this evolution preserves the sub-Poissonicity of the states and hence the self-regulation – in the above-mentioned sense – takes place. Like in the Bolker-Pacala model, see the corresponding discussion in [3], in our case it can be shown that the self-regulation of this kind does not hold for $a \equiv 0$.

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