THE ERDÖS BIPARTIFICATION CONJECTURE IS TRUE IN
THE SPECIAL CASE OF ANDRÁSFAI GRAPHS

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Abstract. Let the Andrásfai graph $\text{And}_k$ be defined as the graph with vertex
set $\{v_0, v_1, \ldots, v_{3k-2}\}$ and two vertices $v_i$ and $v_j$ being adjacent iff $|i - j| \equiv 1 \mod 3$. The graphs $\text{And}_k$ are maximal triangle-free and play a role in
characterizing triangle-free graphs with large minimum degree as homomorphic
preimages. A minimal bipartification of a graph $G$ is defined as a set of edges
$F \subseteq E(G)$ having the property that the graph $(V(G), E(G) \backslash F)$ is bipartite and
for every $e \in F$ the graph $(V(G), E(G) \backslash \{F \backslash e\})$ is not bipartite. In this note
it is shown that there is a minimal bipartification $F_k$ of $\text{And}_k$ which consists
of exactly $\left\lfloor \frac{k^2}{4} \right\rfloor$ edges. This equals $\frac{1}{36}|\text{And}_k|^2$, where $|\cdot|$ denotes the
number of vertices of a graph. For all $k$ this is consistent with a conjecture of
Paul Erdös that every triangle-free graph $G$ can be made bipartite by deleting
at most $\frac{1}{25}|G|^2$ edges.

Bipartifications like $F_k$ may be useful for proving that arbitrary homomor-
phic preimages of an Andrásfai graph can be made bipartite by deleting at
most $\frac{1}{36}|G|^2$ edges.

1. Introduction

All notation is standard and follows [5]. In particular, if $G$ is a graph then $|G|$ denotes
the number of its vertices. A minimal bipartification of a graph $G$ is defined
as a set of edges $F \subseteq E(G)$ having the property that the graph $(V(G), E(G) \backslash F)$ is bipartite and
for every $e \in F$ the graph $(V(G), E(G) \backslash \{F \backslash e\})$ is not bipartite. A homomorphic preimage
of a graph $G$ is a preimage of $G$ under some graph homomorphism. The present short note is concerned with the following special
class of graphs.

Definition 1 (Andrásfai graphs). For every integer $k \geq 2$ the graph $\text{And}_k$ is
defined as the graph with vertex set $\{v_0, v_1, \ldots, v_{3k-2}\}$ and two vertices $v_i$ and $v_j$
being adjacent iff $|i - j| \equiv 1 \mod 3$.

By Lemma 6.10.1 in [6], every graph $\text{And}_k$ is a triangle-free graph of diameter
two, which is the same as saying that it is maximal triangle-free.

In the proof below, the following lemma will be used for the inductive step.

Lemma 2 (Inductive construction of Andrásfai graphs). Deleting from $\text{And}_k$ the
path $v_{3k-4}v_{3k-3}v_{3k-2}$ leaves the graph $\text{And}_{k-1}$.

Proof. This is stated above Lemma 6.11.2 in [6] and easy to see from the definition
of $\text{And}_k$.\qed
2. Main result

The following theorem exhibits a minimal bipartification for the graphs \( \text{And}_k \).

**Theorem 3.** For every integer \( k \geq 2 \) the set of edges \( F_k := U_k^{(1)} \cup U_k^{(2)} \), where

\[
U_k^{(1)} := \bigcup_{i=0}^{\left\lfloor \frac{k-1}{4} \right\rfloor} \bigcup_{j=1}^{\left\lfloor \frac{k-1}{2} \right\rfloor} \{ v(3(i-1)-3j), v(3k-5-3j) \},
\]

\[
U_k^{(2)} := \bigcup_{i=0}^{\left\lfloor \frac{k-1}{4} \right\rfloor} \bigcup_{j=0}^{\left\lfloor \frac{k-1}{2} \right\rfloor} \{ v_i(v, v_{(3i+1)+3j}) \},
\]

is a minimal bipartification of \( \text{And}_k \) in the stronger sense that omitting an element from it creates a 5-cycle. Moreover, the bipartite graph \( \text{And}_k - F_k \) admits the bipartition \( A_k \cup B_k \) where

\[
A_k := \{ v_{3i} : i = 0, \ldots, k-1 \} \cup \{ v_{3i+1} : i = 0, \ldots, (k-1)/2 \}, \quad \text{and}\quad
B_k := \{ v_{3i-1} : i = 0, \ldots, k-2 \} \cup \{ v_{3i+1} : (k-1)/2, \ldots, k-1 \}.
\]

Moreover, the set \( F_k \) consists of exactly \( \left\lfloor \frac{k^2}{4} \right\rfloor = \frac{1}{4}(\text{And}_k + 1)^2 \) edges.

**Proof.** This will be proved by induction on \( k \). For \( k = 2 \), the graph \( \text{And}_k \) is the 5-cycle \( v_0v_1v_2v_3v_4v_5v_0 \) and the lemma correctly states that the single edge \( \{ v_0, v_1 \} \) is a minimal bipartification in the stronger sense stated above and that \( v_0v_1v_2v_3v_4v_5v_0 - \{ v_0, v_1 \} \) admits the bipartition \( A_2 \cup B_2 \). The statement about the number of edges is correct, too.

Now suppose that \( k \geq 3 \) and that the statement is true for \( k-1 \). By 2, it is known that \( \text{And}_k - v_{3k-4}v_{3k-3}v_{3k-2} = \text{And}_{k-1} \). By induction, \( F_{k-1} \) is a minimal bipartification of \( \text{And}_{k-1} \) in the stronger, 5-cycle-sense, and \( A_{k-1} \cup B_{k-1} \) is a bipartition of \( \text{And}_{k-1} - F_{k-1} \).

To prove the statement about being a bipartification for \( k \), it suffices to show that in \( \text{And}_k \) every edge having at least one endvertex \( v \) which is either new (i.e. \( v \in (A_k \cup B_k) \setminus (A_{k-1} \cup B_{k-1}) \)), or has changed partition classes (i.e. \( v \in A_k \cap B_{k-1} \) or \( v \in B_k \cap A_{k-1} \)), lies in \( F_k \).

From the definition of \( A_k \) and \( B_k \) it is clear that \( (A_k \cup B_k) \setminus (A_{k-1} \cup B_{k-1}) = \{ v_{3k-4}, v_{3k-3}, v_{3k-2} \} \), and that \( v_{3k-4} \in B_k, v_{3k-3} \in A_k \) and \( v_{3k-2} \in B_k \).

As to \( v_{3k-4} \in B_k \), from the definition of \( \text{And}_k \) it is clear that this vertex is adjacent to exactly the \( k \) vertices in \( \{ v_{3k-3} \} \cup \{ v_{3k-5} : j = 0, \ldots, k-2 \} \). Of these, exactly those in \( \{ v_{3k-5} : j = 0, \ldots, \left\lfloor \frac{k-1}{2} \right\rfloor - 1 \} \) lie in \( B_k \). Therefore, it suffices to check that the edges in \( \{ v_{3k-4}, v_{3k-5} : j = 0, \ldots, \left\lfloor \frac{k-1}{2} \right\rfloor - 1 \} \) lie in \( F_k \). This becomes obvious by setting \( i = 0 \) in (1).

As to \( v_{3k-3} \in A_k \), from the definition of \( \text{And}_k \), it is clear that this vertex is adjacent to exactly the \( k \) vertices \( \{ v_{3i} : i = 0, \ldots, k-1 \} \), all of which lie in \( B_k \).

As to \( v_{3k-2} \in B_k \), from the definition of \( \text{And}_k \), it is clear that this vertex is adjacent to exactly the \( k \) vertices \( \{ v_{3i} : i = 0, \ldots, k-1 \} \), all of which lie in \( A_k \).

From the definition of \( A_k \) and \( B_k \), it is clear by divisibility that \( v \in B_k \cap A_{k-1} = \{ v_{3i+1} : \left( \frac{k-1}{2} \right), \ldots, k-1 \} \cap \{ v_{3i+1} : i = 0, \ldots, (k-2)/2 \} \) and this intersection is clearly empty for every integer \( k \geq 2 \). Thus, a vertex in class \( A \) never changes over to the class \( B \).
Analogously, \( v \in A_k \cap B_{k-1} = \{v_{3i+1} : i = 0, \ldots, [(k-1)/2] - 1\} \cap \{v_{3i+1} : [(k-2)/2], \ldots, k-1\} \), but now this is non-empty iff \( k \) is odd with the intersection being equal to \( \{v_{3i+1} : [(k-2)/2] + 1\} \) because the oddness of \( k \) implies \([(k-1)/2] - 1 = [(k-2)/2]\). From the definition of \( \text{And}_k \) it is easy to see that \( v_{3i+1} : [(k-2)/2] + 1 \in A_k \) is adjacent exactly to the \( k \) vertices

\[
\{v_{3i} : i = 0, \ldots, [(k-2)/2]\} \cup \{v_{3i+2} : i = [(k-2)/2], \ldots, k-2\}. \tag{5}
\]

Of these, exactly those in the first set lie in \( A_k \), so it remains to check that the set of edges

\[
\{v_{3i} : [(k-2)/2] + 1, v_{3i}\} : i = 0, \ldots, [(k-2)/2]\}
\]

is a subset of \( F_k \). To see this, fix \( j = \lfloor \frac{k-i}{3} \rfloor - 1 \) in (2). Using \([(k-1)/2] - 1 = [(k-2)/2]\), which implies \((3i + 1) + 3(\lfloor \frac{k-i}{3} \rfloor - 1 - i) = 3 \lfloor \frac{k-i}{3} \rfloor + 1\), it is clear that the subset of \( U_{2}^{(i)} \) thus obtained is equal to (6). This completes the induction as far as being a bipartification is concerned.

For proving the strong minimality of \( F_k \) using the strong minimality of \( F_{k-1} \) (which is known by induction), it suffices to show that for every edge in \( F_k \setminus F_{k-1} \), there is a 5-cycle in \( \text{And}_k \) which intersects \( F_k \) in this edge only and is disjoint from \( F_{k-1} \), which implies that the edge in question is indispensable.

To prepare for the determination of the set \( F_k \setminus F_{k-1} \), note that for every pair of integers \( k_1 \geq 2 \) and \( k_2 \geq 2 \), since none of the edges in \( U_{1}^{(k_1)} \) contains a vertex with an index divisible by 3 whereas every edge in \( U_{1}^{(k_2)} \) does, the intersections \( U_{1}^{(k_1)} \cap U_{2}^{(k_2)} \) and \( U_{1}^{(k_1)} \cap U_{2}^{(k_2)} \) are both empty. In particular, for every integer \( k \geq 2 \),

\[
U_{1}^{(k_1)} \cap U_{2}^{(k_2)} = \emptyset \quad \text{and} \quad U_{1}^{(k_1)} \cap U_{2}^{(k_2)} = \emptyset. \tag{7}
\]

Obviously, for sets \( S_1, S_2, S_3, S_4 \), the condition that \( S_1 \cap S_4 = \emptyset \) and \( S_2 \cap S_3 = \emptyset \) implies that \( (S_1 \cup S_2) \setminus (S_3 \cup S_4) = (S_1 \setminus S_3) \cup (S_2 \setminus S_4) \), hence

\[
F_{k} \setminus F_{k-1} = (U_{1}^{(k)} \setminus U_{2}^{(k-1)}) \cup (U_{2}^{(k)} \setminus U_{1}^{(k-1)}) \tag{8}
\]

As to \( U_{1}^{(k)} \setminus U_{2}^{(k-1)} \), let \( e_{k}^{i,j} := \{v_{(3i-4) - 3i}, v_{(3i-5) - 3i}\} \) and \( f_{k}^{i,j} := \{v_{3i}, v_{3i+1} + 3j\} \), note that \( e_{k}^{i,j} = e_{k-1}^{i-1,j-1} \), and that, due to the different parities of the indices, for all integers \( i_1, i_2, j_1, j_2 \geq 2 \), the two edges \( e_{k}^{i_1,j_1} \) and \( e_{k}^{i_2,j_2} \), and the two edges \( f_{i_1,j_1} \) and \( f_{i_2,j_2} \), are equal iff \( i_1 = i_2 \) and \( j_1 = j_2 \).

If \( k \) is odd, then \( \lfloor \frac{k-2}{3} \rfloor = \lfloor \frac{k-1}{3} \rfloor - 1 \), hence, by the criterion for equality of two edges \( f_{i_1,j_1} \) and \( f_{i_2,j_2} \),

\[
U_{1}^{(k)} \setminus U_{2}^{(k)} = \bigcup_{i=0}^{\lfloor \frac{k-2}{3} \rfloor - 1} \bigcup_{j=0}^{\lfloor \frac{k-1}{3} \rfloor - 1} \{f_{i,j}\} \cup \bigcup_{i=0}^{\lfloor \frac{k-2}{3} \rfloor - 1} \bigcup_{j=0}^{\lfloor \frac{k-1}{3} \rfloor - 1} \{f_{i,j}\} \tag{9}
\]

\[
= \bigcup_{i=0}^{\lfloor \frac{k-2}{3} \rfloor - 1} \{f_{i,\lfloor \frac{k-1}{3} \rfloor - 1-i}\} = \bigcup_{i=0}^{\lfloor \frac{k-2}{3} \rfloor - 1} \{v_{3i}, v_{3i+1} \rfloor - 2\}. \tag{10}
\]
Using (9), since the identity and note that this is a 5-cycle since four of the needed five adjacencies are obvious.

\[ U_{k-1}^{(1)} \setminus U_{k-1}^{(1)} = \bigcup_{i=0}^{\lfloor k/2 \rfloor - 1} \bigcup_{j=i}^{\lfloor k/2 \rfloor - 1} \{ e_{k-1}^{i,j} \} \setminus \bigcup_{i=0}^{\lfloor k/2 \rfloor - 1} \bigcup_{j=i}^{\lfloor k/2 \rfloor - 1} \{ e_{k-1}^{i,j} \} \]  

(11)

\[ = \bigcup_{i=0}^{\lfloor k/2 \rfloor - 1} \bigcup_{j=i}^{\lfloor k/2 \rfloor - 1} \{ e_{k-1}^{i,j} \} \setminus \bigcup_{i=0}^{\lfloor k/2 \rfloor - 1} \bigcup_{j=i}^{\lfloor k/2 \rfloor - 1} \{ e_{k-1}^{i,j} \} \]  

(12)

\[ = \bigcup_{i=0}^{\lfloor k/2 \rfloor - 1} \bigcup_{j=i}^{\lfloor k/2 \rfloor - 1} \{ e_{k-1}^{i,j} \} \setminus \bigcup_{i=0}^{\lfloor k/2 \rfloor - 1} \bigcup_{j=i}^{\lfloor k/2 \rfloor - 1} \{ e_{k-1}^{i,j} \} \]  

(13)

\[ \bigcup_{j=0}^{\lfloor k/2 \rfloor - 1} \{ e_{k-1}^{i,j-1} \} = \bigcup_{j=0}^{\lfloor k/2 \rfloor - 1} \{ \{ v_{3k-4}, v_{(3k-5)-3j} \} \} \]  

(14)

where the penultimate equality is true by the criterion for equality of two edges \( e_{k}^{i_1,j_1} \) and \( e_{k}^{i_2,j_2} \). Using (8) it follows that, if \( k \) is odd,

\[ F_{k} \setminus F_{k-1} = \bigcup_{j=0}^{\lfloor k/2 \rfloor - 1} \{ \{ v_{3k-4}, v_{(3k-5)-3j} \} \} \bigcup_{i=0}^{\lfloor k/2 \rfloor - 1} \{ \{ v_{3i}, v_{3j} \} \} \bigcup_{i=0}^{\lfloor k/2 \rfloor - 1} \{ \{ v_{3i}, v_{3j} \} \}. \]  

(15)

If \( k \) is even, then \( \lfloor k/2 \rfloor = \lfloor k/2 \rfloor - 1 \), hence \( U_{k}^{(2)} \setminus U_{k-1}^{(2)} = \emptyset \), and \( \lfloor k/2 \rfloor = \lfloor k/2 \rfloor - 1 \), hence

\[ U_{k}^{(1)} \setminus U_{k-1}^{(1)} = \bigcup_{i=0}^{\lfloor k/2 \rfloor - 1} \bigcup_{j=i}^{\lfloor k/2 \rfloor - 1} \{ e_{k-1}^{i,j-1} \} \setminus \bigcup_{i=0}^{\lfloor k/2 \rfloor - 1} \bigcup_{j=i}^{\lfloor k/2 \rfloor - 1} \{ e_{k-1}^{i,j-1} \} \]  

(16)

\[ \bigcup_{j=0}^{\lfloor k/2 \rfloor - 1} \{ e_{k-1}^{i,j-1} \} = \bigcup_{j=0}^{\lfloor k/2 \rfloor - 1} \{ \{ v_{3k-4}, v_{(3k-5)-3j} \} \} \]. \]  

(17)

showing that \( U_{k}^{(1)} \setminus U_{k-1}^{(1)} \) is given by the same formula regardless of the parity of \( k \). Using (8), it follows that, if \( k \) is even,

\[ F_{k} \setminus F_{k-1} = \bigcup_{j=0}^{\lfloor k/2 \rfloor - 1} \{ \{ v_{3k-4}, v_{(3k-5)-3j} \} \}. \]  

(18)

To prove the indispensability of each of the edges in \( \bigcup_{j=0}^{\lfloor k/2 \rfloor - 1} \{ \{ v_{3k-4}, v_{(3k-5)-3j} \} \} \), for every \( k \) and every \( j \in \{ 0, 1, \ldots, \lfloor k/2 \rfloor - 1 \} \) define

\[ C_{k}^{(j)} := v_{3k-4}v_{(3k-5)-3j}v_{3j}^{\lfloor k/2 \rfloor - 1}v_{3k-2}v_{3k-3}v_{3k-4}, \]  

(19)

and note that this is a 5-cycle since four of the needed five adjacencies are obvious and since the identity \( k = \lfloor (k-1)/2 \rfloor + \lfloor k/2 \rfloor + 1 \) implies that \( (3k-5)-3j \geq 3\lfloor (k-1)/2 \rfloor \) for every integer \( k \) and every \( j \in \{ 0, 1, \ldots, \lfloor k/2 \rfloor - 1 \} \), so \( 3\lfloor (k-1)/2 \rfloor - ((3k-5)-3j) = (3k-5)-3j-3\lfloor (k-1)/2 \rfloor = 3(k-\lfloor (k-1)/2 \rfloor - j-2)+1 \). Since the expression in the parentheses is nonnegative for every \( j \in \{ 0, 1, \ldots, \lfloor k/2 \rfloor - 1 \} \), this shows that the absolute value of the difference of the indices of the two vertices \( v_{(3k-5)-3j} \) and \( v_{3\lfloor (k-1)/2 \rfloor} \) is congruent to 1 modulo 3, hence the vertices are adjacent.
To see that for every edge in \( \bigcup_{j=0}^{\lfloor k/2 \rfloor - 1} \{v_{3k-4}, v_{3(3k-5)-3j}\} \), the cycle \( C_k^{(j)} \) intersects \( F_k = U_k^{(1)} \cup U_k^{(2)} \) precisely in the edge \( \{v_{3k-4}, v_{3(3k-5)-3j}\} \), take the indices in the edge sets \( U_k^{(1)} \), \( U_k^{(2)} \) and \( E(C_k^{(j)}) \) modulo 3. For \( E(C_k^{(j)}) \) this results in the ‘signature’ \( \{2, 1\} \) \( \{0, 1\} \), whereas every element of \( U_k^{(1)} \) has signature \( \{2, 1\} \) and every element of \( U_k^{(2)} \) has \( \{0, 1\} \). This shows that \( C_k^{(j)} \) can intersect \( U_k^{(1)} \) in at most the edge \( \{v_{3k-4}, v_{3(3k-5)-3j}\} \), which does (when \( i = 0 \) in \( U_k^{(1)} \), and \( U_k^{(2)} \) in at most the three edges \( \{v_{3k-4}, v_{3(3k-5)-3j}, v_3|\frac{k+1}{2}|\}, \{v_3|\frac{k-1}{2}|, v_{3k-2}\} \) \( \{v_{3k-2}, v_{3k-3}\} \), which it does not, since in \( U_k^{(2)} \) the index which is divisible by three rises only as high as \( 3\lfloor (k-1)/2 \rfloor - 3 \), which prevents an equality with any of the three edges.

Moreover, to see that for every \( j \in \{0, 1, \ldots, |k/2| - 1\} \), the cycle \( C_k^{(j)} \) is disjoint from \( F_{k-1} \), repeat this argument but note that now all of the candidate-edges arising from considering the indices modulo 3 fail to be actually contained in the intersection \( E(C_k^{(j)}) \cap F_{k-1} \) for reasons of magnitude of indices.

Since by (18), \( F_k \setminus F_{k-1} = \bigcup_{j=0}^{\lfloor k/2 \rfloor - 1} \{v_{3k-4}, v_{3(3k-5)-3j}\} \), this completes the induction and proves the minimality of \( F_k \) in the case of even \( k \). In the case of odd \( k \), by (15) one still has to prove the indispensability each of the edges \( \bigcup_{i=0}^{\lfloor k/2 \rfloor - 1} \{v_{3i}, v_3|\frac{k+1}{2}|\} \). To this end, for every \( i \in \{0, \ldots, |(k-1)/2|\} \) define

\[
D_k^{(i)} := v_{3i}v_3v_2^{-1}v_3^{-1}v_{3k-4}v_{3k-3}v_{3k-2}v_{3i}
\]

and note that this is a 5-cycle since all of the five needed adjacencies are obvious (in the sense that it is obvious how to compute each of the absolute values of the difference of indices).

To see that for every edge in \( \bigcup_{i=0}^{\lfloor k/2 \rfloor - 1} \{v_{3i}, v_3|\frac{k+1}{2}|\} \), the cycle \( D_k^{(i)} \) intersects \( F_k = U_k^{(1)} \cup U_k^{(2)} \) precisely in the edge \( \{v_{3i}, v_3|\frac{k+1}{2}|\} \), repeat the argument given three paragraphs earlier.

This time, the cycle has signature \( \{0, 1\} \{2, 0\} \{0, 1\} \{1, 0\} \{1, 0\} \). This shows that \( D_k^{(i)} \) can intersect \( U_k^{(1)} \) in at most the edge \( \{v_{3i}|\frac{k+1}{2}, v_{3k-4}\} \). Again by looking at the remainders modulo 3 it is clear that for this to happen it is necessary that \( i = 0 \) in \( U_k^{(1)} \), whereupon the \( j \) would have to satisfy \( 3\lfloor (k-1)/2 \rfloor - 2 = 3k-5-3j \) which is equivalent to \( j = |k/2| \) which contradicts \( j \in \{0, \ldots, |k/2| - 1\} \), hence \( D_k^{(i)} \) does not intersect \( U_k^{(1)} \). Furthermore, the signatures of the cycle show that \( D_k^{(i)} \) can intersect \( U_k^{(2)} \) in at most the three edges \( \{v_{3i}, v_3|\frac{k-1}{2}, v_{3k-3}\} \{v_{3k-3}, v_{3k-2}\} \) and \( \{v_{3k-2}, v_{3k}\} \). Setting \( j = \lfloor (k-1)/2 \rfloor - 1 - i \) in \( U_k^{(2)} \) shows that the first of these edges actually lies in the intersection, and considering the magnitude of the index which is divisible by three in the second edge shows that the second edge does not. As to the third candidate-edge, since \( j \leq \lfloor (k-1)/2 \rfloor - 1 - i \), hence \( i + j \leq \lfloor (k-1)/2 \rfloor - 1 \), implies that \( 3(i+1) - 3j = 2 \), which contradicts \( 3\lfloor (k-1)/2 \rfloor - 2 \), the index of the vertex which is not divisible by three cannot reach \( 3k-2 \), so the edge is not in the intersection.

Moreover, to see that for every \( i \in \{0, \ldots, |(k-1)/2|\} \) the cycle \( D_k^{(i)} \) is disjoint from \( F_{k-1} \), repeat the argument from above and note that all arguments for an
edge not lying in the intersection \( E(D_k^{(i)}) \cap F_k \) can be adapted to the intersection \( E(D_k^{(i)}) \cap F_{k-1} \) and that the edge \( \{v_{3i}, v_3, 3i - 2\} \), which was the only one to make it into the intersection before, does not lie in \( U_{k-1}^{(2)} \) since an analogous estimate as the one above now shows that \( (3i + 1) + 3j \leq 3\lfloor (k-2)/2 \rfloor - 2 \) and \( 3\lfloor (k-2)/2 \rfloor - 2 < 3\lfloor (k-1)/2 \rfloor - 2 \) since \( k \) is odd.

The statement about the cardinality of \( F_k \) needs no induction. It is obvious that \( |F_k| = \frac{1}{2} \left( \frac{k}{2} \right) \left( \frac{k}{2} \right) + 1 + \frac{1}{2} \left( \frac{k-1}{2} \right) \left( \frac{k-1}{2} \right) + 1 \), and by distinguishing between odd and even \( k \) it is easy to see that this is equal to \( \frac{k}{4} \left( |\text{And}_k| + 1 \right)^2 \). \( \square \)

Since it is equally easy to show that \( \frac{1}{36} \left( |\text{And}_k| + 1 \right)^2 \leq \frac{1}{36} |\text{And}_k|^2 \), with the inequality being strict for every \( k \geq 5 \), Theorem 3 is consistent with the following well-known conjecture of Paul Erdős.

**Conjecture 4** (Erdős bipartition conjecture; for more information see the introductions of [1] and [7] and the references therein). Every triangle-free graph \( G \) can be made bipartite by deleting at most \( \frac{1}{25} |G|^2 \) edges.

3. **Concluding remarks**

There are two interesting questions concerning Theorem 3.

3.1. **Is the bipartition \( F_k \) minimum?** It is easy to see that there exist minimal bipartitions of \( \text{And}_k \) in the stronger sense of Theorem 3, which nevertheless have almost twice as many edges. An example is the set of all edges running between the vertex set \( \{v_{2i+3} : i \in \{0, \ldots, k-2\} \} \) and the vertex set \( \{v_{3i} : i \in \{0, \ldots, k-1\} \} \). This set consists of exactly \( \frac{1}{2} (k-1)k \) edges, and it is easy to check that \( \frac{|F_k|}{\frac{1}{2} (k-1)k} \) is strictly less than \( \frac{1}{2} + \frac{k+1}{(k-1)k} \). Given so much variation in the cardinalities of minimal bipartitions, it is natural to wonder whether the minimal bipartition \( F_k \) from Theorem 3 is also minimum, i.e. has the smallest cardinality a bipartition of \( \text{And}_k \) can have. The author thinks it likely that this is the case.

**Conjecture 5.** The set \( F_k \) defined in Theorem 3 is a minimum bipartition for \( \text{And}_k \).

However, \( F_k \) is not unique in the sense that there are several other minimal bipartitions which are mutually edge-disjoint and have the same cardinality as \( F_k \), and still more such bipartitions when one does not require mutual edge-disjointness. This leads to a second question concerning Theorem 3.

3.2. **Can Theorem 3 be helpful for proving a somewhat less special special case of the Erdős bipartition conjecture?** The graphs \( \text{And}_k \) are important for characterizing triangle-free graphs with large minimum degree as homomorphic preimages. By Theorem 3.8 in [4], later given a simpler proof in [2], if a triangle-free graph \( G \) has minimum degree \( \delta(G) > \frac{1}{2} |G| \) and chromatic number \( \chi(G) \leq 3 \), then it is homorphic to an Andrásfai graph. This is why proving the following conjecture would be a little more than merely a drop in the ocean with regard to the Erdős bipartitifcation conjecture.
Conjecture 6. By considering several copies of the bipartification $F_k$, and then optimizing a system of quadratic inequalities, it is possible to prove that an arbitrary homomorphic preimage $H$ of $\text{And}_k$ can be made bipartite by deleting at most $\frac{1}{25}|H|^2$ edges.

Furthermore, by Corollary 4.2 in [3], every triangle-free graph $G$ with $\delta(G) > \frac{1}{3}|G|$ is four-colourable, and by Theorem 1.2, Corollary 1.3 and Theorem 1.4 in [2], if the chromatic number is indeed four, then such graphs must simultaneously contain a Petersen graph with one contracted edge, a Wagner graph (Moebius ladder with four rungs) and a Grötzsch graph as subgraphs.

Therefore, proving Conjecture 6 (and thereby settling the case of $\delta(G) > \frac{1}{3}|G|$ and $\chi(G) \leq 3$) would allow anyone interested in the Erdős bipartification conjecture to assume that one of the following holds:

1. The minimum degree of the graph $G$ is at most $\frac{1}{3}|G|$. If $\delta(G) < \frac{1}{3}|G|$, then the chromatic number $G$ can be arbitrarily high (see Section 6 in [3]).
2. The minimum degree of the graph $G$ is strictly larger than $\frac{1}{3}|G|$, the chromatic number of $G$ is exactly four, and the graph $G$ contains a Petersen graph with one edge contracted, a Wagner graph, and a Grötzsch graph as subgraphs.

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REFERENCES

1. Stephan Brandt, The local density of triangle-free graphs, Discrete Math. 183 (1998), 17–25.
2. , On the structure of dense triangle-free graphs, Combin. Probab. Comput. 8 (1999), 237–245.
3. Stephan Brandt and Stéphan Thomassé, Dense triangle-free graphs are four-colorable, To appear in J. Combin. Theory Ser. B.
4. C.C. Chen, G.P. Jin, and K.M. Koh, Triangle-free graphs with large degree, Combin. Probab. Comput. 6 (1997), 381–396.
5. Reinhard Diestel, Graph theory, third ed., Graduate Texts in Mathematics, vol. 173, Springer, 2005.
6. Chris Godsil and Gordon Royle, Algebraic graph theory, first ed., Graduate Texts in Mathematics, vol. 207, Springer, 2001.
7. Benjamin Sudakov, Making a $K_4$-free graph bipartite, Combinatorica 27 (2007), no. 4, 509–518.

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