Noncommutative Differential Geometry with Higher Order Derivatives

Andrzej Sitarz \(^*\)\(^\dagger\)
Department of Field Theory
Institute of Physics
Jagiellonian University
Reymonta 4, 30-059 Kraków, Poland

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Abstract

We build a toy model of differential geometry on the real line, which includes derivatives of the second order. Such construction is possible only within the framework of noncommutative geometry. We introduce the metric and briefly discuss two simple physical models of scalar field theory and gauge theory in this geometry.

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\(^*\)Partially supported by KBN grant 2 P302 168 4
\(^\dagger\)E-mail: sitarz@if.uj.edu.pl
1 Introduction

The noncommutative differential geometry has proved to be a very useful
generalisation of the standard differential geometry \([1]-[4]\). It enables us to
construct and study models, which allows description only in the language
of algebras and modules. However, often even in the classical well-known
models we are able to find new features or noncommutative extensions \([5, 6]\).

This paper is devoted to the presentation of a toy model of the diffe-
rential algebra and sample physical models on the real line (or circle), which, how-
ever, involve the higher order derivatives. It is interesting that such extension
of the ordinary differential algebra, demonstrates the noncommutative prop-
erties. We find it similar to the situation we have discovered in the case of
discrete spaces, where the differential algebra was noncommutative, thought
the algebra of functions was the commutative one.

In the paper we construct the differential algebra, then we briefly discuss
its metric properties and finally we present a simple model of scalar field
theory based on the constructed geometry.

2 The Differential Algebra

Let \( \mathcal{A} \) be the algebra of smooth functions on the real line \( C^\infty(\mathbb{R}) \) with the
pointwise multiplication and addition. In the first step we shall construct a
two-dimensional bimodule over \( \mathcal{A} \):

Observation 2.1 Let \( dx \) and \( \eta \) be the generators of the right module \( \mathcal{M} \)
over \( \mathcal{A} \). Then, if we define the left multiplication rules

\[
fdx = dx f + 2\eta f', \quad (1)
\]
\[
f\eta = \eta f. \quad (2)
\]

where by \( f' \) we denote the derivative of \( f \), the module \( \mathcal{M} \) shall be equipped
with the structure of a bimodule.

Now let introduce the operator \( d \) acting on the algebra \( \mathcal{A} \), taking values in
\( \mathcal{M} \):

\[
df = df' + \eta f''. \quad (3)
\]

The operator \( d \) has all properties of the external derivative, as we shall
demonstrate in the following lemma:
Lemma 2.1 The operator \( d \) defined as above obeys the Leibniz rule:

\[
 d(fg) = (df)g + f(dg),
\]

(4)

and \( \text{Ker}(d) = \mathbb{C} \).

The last property follows directly from the definition (3), therefore we shall concentrate only on proving (4). By definition we have:

\[
 d(fg) = dx(f'g + fg') + \eta(f''g + 2f'g' + fg''),
\]

(5)

where we have used the Leibniz rule for differentiating the product of functions. Now, if we use the rules of the left multiplication for the module \( M \) we shall obtain:

\[
 (df)g + f(dg) = dx(f'g + fg') + \eta(f''g + 2f'g' + fg''),
\]

(6)

which is precisely the right-hand side of (5) and so it ends the proof.

The bimodule \( M \) could be now seen as the module of one-forms, and in order to construct the differential algebra we need to introduce the product of one forms. First let us denote by \( \Omega^n_0 \) the following direct sum:

\[
 \Omega^n_0 = \bigoplus_{k=0}^n M \otimes_A \cdots \otimes_A M, \quad \text{n times}
\]

(7)

where the first term (for \( k = 0 \) is, of course, the algebra \( A \). We may now denote the limit as \( n \to \infty \) by \( \Omega_0 \). By definition, \( \Omega_0 \) is an algebra, with the standard addition and the tensor product defining the multiplication of its elements.

Now, let us define the ideal \( J \subset \Omega_0 \) as the smallest ideal generated by the elements \( \eta \otimes \eta \) and \( \eta \otimes dx + dx \otimes \eta \). Now, we shall define our differential algebra as the quotient \( \Omega = \Omega_0/J \) and the product of forms, which we shall denote by \( \bullet \), as the product in this quotient. In particular we have:

\[
 \eta \bullet \eta = 0, \quad \eta \bullet dx = -dx \bullet \eta.
\]

(8)

(9)

Our next task would be the extension of the operator \( d \) to the algebra \( \Omega \) in such a way that it obeys the graded Leibniz rule.

\[2\]
Lemma 2.2 Let us define the action of $d$ on $\mathcal{M}$ in the following way:

$$d(fdx + g\eta) = df \cdot dx + dg \cdot \eta + gdx \cdot dx,$$

then $d$ has the following properties:

$$d^2 f = 0, \forall f \in \mathcal{A},$$

$$d(f\omega) = df \cdot \omega + f(d\omega),$$

$$d(\omega f) = (d\omega)f - \omega \cdot df.$$  

The proof of the properties (11-13) follows directly from the definitions (3),(10) and the rules of left multiplications (1,2).

Since we have already successfully extended $d$ to $\mathcal{M}$ let us now define $d$ on an arbitrary element of $\Omega_0$. We take:

$$d(\omega_1 \otimes \ldots \otimes \omega_n) = \pi \left( \sum_{i=1}^{n} (-1)^{i+1} \omega_1 \otimes \ldots \otimes (d\omega_i) \otimes \ldots \omega_n \right).$$

where $\pi$ is the quotient projection $\pi : \Omega_0 \to \Omega$. In the next lemma we shall demonstrate that $d$ can be, in fact, restricted to the algebra $\Omega$, thus the latter will be our differential algebra.

Lemma 2.3 The ideal $\mathcal{J}$ is a differential ideal, i.e. the action of $d$ as defined in (14) annihilates it:

$$d\mathcal{J} = 0,$$

so that $d$ may be defined on $\Omega$ and then it obeys the graded Leibniz rule:

$$d(\omega \cdot \rho) = (d\omega) \cdot \rho + (-1)^{\deg \omega} \omega \cdot (d\rho),$$

for any $\omega, \rho \in \Omega$.

From the definition (14) it is clear that it is sufficient to verify that $d\mathcal{J}_2 = 0$, where $\mathcal{J}_2$ is the restriction of $\mathcal{J}$ to $\Omega_0^2$ and further we may restrict only to its generators. From the definitions (14) and (10) we obtain:

$$d(dx \otimes \eta + \eta \otimes dx) = \pi (-2dx \cdot dx \cdot dx + 2dx \cdot dx \cdot dx) = 0,$$

$$d(\eta \otimes \eta) = (2dx \cdot dx \cdot \eta - 2\eta \cdot dx \cdot dx) = 0,$$

which ends the proof of this part of the lemma. Now, since $d$ is defined on $\Omega$, the graded Leibniz rule property for the higher order forms follows directly.
from the definition (14) and we already know that it is satisfied for \( \mathcal{A} \) and \( \Omega^1 = \mathcal{M} \), as demonstrated in (4) and (12,13). Of course, it is nilpotent, i.e. \( d^2 = 0 \), which is the consequence of (11) and the graded Leibniz rule.

The above constructed differential algebra is infinite-dimensional and each its component (the module of \( n \)-forms for any fixed \( n > 0 \)) is a two-dimensional free bimodule over \( \mathcal{A} \). The standard, finite differential algebra on \( \mathbb{R} \) is, of course, its subalgebra.

The diffeomorphism \( x \to y(x) \) of \( \mathbb{R} \) generates the following automorphism of \( \mathcal{M} \):

\[
\begin{align*}
\eta & \to \eta \frac{dy}{dx}, \\
dx & \to dx \frac{dy}{dx} + \eta \frac{d^2 y}{dx^2}.
\end{align*}
\]  

(19) (20)

3 Metric

Before we proceed with the introduction of the analogue of metric we shall prove the following simple lemma:

**Lemma 3.1** There exists a conjugation \( * \) on the algebra \( \Omega \), which after restriction to \( \mathcal{A} \) is just the complex conjugation and it graded commutes with \( d \):

\[
d(\omega^*) = (-1)^{\deg \omega} (d\omega)^*.
\]

(21)

The proof is straightforward if we take \( f^* = \bar{f} \) for \( f \in \mathcal{A} \) and \( dx^* = dx \) and \( \eta^* = -\eta \).

We define the metric as a middle linear functional [5]-[7]:

\[
G : \mathcal{M} \otimes_\mathcal{A} \mathcal{M} \to \mathcal{A},
\]

(22)

\[
G(a\omega b, \rho c) = aG(\omega, b\rho) c.
\]

(23)

The condition (23) is very strong and in particular, it results in the following constraints:

\[
\begin{align*}
G(\eta, \eta) &= 0, \\
G(dx, \eta) &= -G(\eta, dx).
\end{align*}
\]

(24) (25)
We may also require that the metric is hermitian, i.e.:

\[ G(\omega, \rho) = (G(\rho^*, \omega^*))^*, \]  

(26)

and then we obtain that both \( G(dx, dx) \) and \( G(\eta, dx) \) must be real.

As the trace on the algebra \( A \), we take the diffeomorphism invariant integral:

\[ \text{Tr} f = \int_{\mathbb{R}} dx \frac{1}{\sqrt{G(dx, \eta)}} f(x), \]  

(27)

with the diffeomorphism invariant measure \( dx \frac{1}{\sqrt{G(dx, \eta)}} \).

4 Field Theory

Let us construct now two simple models of field theory in the considered geometrical setup. We shall begin here with the simplest one, the scalar field theory.

4.1 Scalar Field Theory

Let \( \phi \) be the scalar field, which is just the element of \( A \). We define the action \( S(\phi) \) in the simplest way:

\[ S(\phi) = \text{Tr} G(d\phi^*, d\phi). \]  

(28)

Now, if we write the right-hand side of (28) explicitly we obtain:

\[ S(\phi) = \int_{\mathbb{R}} dx \frac{1}{\sqrt{G(dx, \eta)}} \left( G(x, x)\phi'\bar{\phi}' + G(dx, \eta)(\bar{\phi}''\phi' + \bar{\phi}'\phi'') \right). \]  

(29)

Now, if denote by \( F(x) \) such function that \( F' = \sqrt{G(dx, \eta)} \) we may rewrite (29) as:

\[ S(\phi) = \int_{\mathbb{R}} dx \frac{1}{\sqrt{G(dx, \eta)}} \left( G(x, x) - F(x)\sqrt{G(dx, \eta)} \right) (\bar{\phi}'\phi'). \]  

(30)

where we have eliminated the boundary terms (full derivative of \( \sqrt{G(dx, \eta)\bar{\phi}'\phi'} \)).
Now, this does not differ from the classical action of the scalar field in one-dimension (up to the boundary terms, of course), with the metric $E$:

$$E(dx, dx) = \frac{G(x, x) - F(x)\sqrt{G(dx, \eta)}}{\sqrt{G(dx, \eta)}}, \quad (31)$$

so the effective theory would be the same, the only possible difference arising from the above mentioned boundary terms.

4.2 Gauge Theory

Here, we would like to present the possibility of construction of a simple $U(1)$ gauge theory in the considered differential geometry. This is a completely new feature, as in the standard differential geometry one cannot build gauge theory in less than two dimensions.

The gauge group $G$ shall be in our case the unitary group of $\mathcal{A}$ consisting of all $U(1)$ valued functions on $\mathbb{R}$, and the hermitian connection will be the one-form $A$:

$$A = dx A_x + \eta A_\eta, \quad (32)$$

such that $A^* = -A$, which is satisfied only if

$$\bar{A}_x = -\bar{A}_x, \quad (33)$$

$$\bar{A}_\eta = A_\eta - 2A_x'. \quad (34)$$

Their gauge transformations rules are as follow:

$$A_x \to A_x + i\phi', \quad (35)$$

$$A_\eta \to A_\eta - 2iA_x\phi' + i\phi'' + (\phi')^2, \quad (36)$$

where $\phi$ is a real field defining $e^{i\phi} \in U(\mathcal{A})$. From the conditions (33,34) we see that in fact only the imaginary part of $A_x$ and real part of $A_\eta$ make independent real variables. If we call them $\Phi$ and $\Psi$, respectively, we may rewrite the connection as:

$$A = dx(i\Phi) + \eta(\Psi + i\Phi'), \quad (37)$$

with the tranformation rules:

$$\Phi \to \Phi + \phi', \quad (38)$$

$$\Psi \to \Psi + 2\Phi\phi' + (\phi')^2. \quad (39)$$
Let us notice that by gauge transformation we can eliminate $\Phi$ or in some cases $\Psi$ but not both of them at the same time.

Finally, let us calculate the curvature $F = dA + A \cdot A$:

$$F = dx \bullet dx \left( \Psi - \Phi^2 \right) \quad (40)$$

$$+ \; dx \bullet \eta \left( \Psi' - 2\Phi \Phi' \right). \quad (41)$$

The curvature is, of course, selfadjoint, i.e. $F^* = F$ and additionally we have $dF = 0$.

There exist two possible gauge invariant lagrangian terms, one linear in $F$:

$$L \sim \left( \Psi - \Phi^2 \right), \quad (42)$$

since by the similar argument as in the case of a scalar field we can eliminate the second component of $F$, which effectively gives contribution to some boundary terms. The other one shall be simply its square:

$$L \sim \left( \Psi - \Phi^2 \right)^2, \quad (43)$$

and it appears that it is equal to the Yang-Mills lagrangian in our case.

Both of the derived candidates for the Lagrange function contain no dynamical dependance on the fields $\Phi$ and $\Psi$, therefore the resulting gauge theory is trivial. Let us notice, however, that in combination with the standard gauge theory on a smooth manifold it could provide the potential term for two real fields, which then could be treated also as fields on the manifold. The potential $L$ is rather similar to the Higgs potential, so the above construction may provide a similar symmetry-breaking mechanism as one obtains in the case of discrete geometry.

5 Conclusions

As we have demonstrated in this paper one could extend the differential calculus on the real line to include derivatives of second order, the construction appears, however, to be a noncommutative one. Of course, following the same procedure as described in the second section we may generalise the idea to higher order derivatives and by taking tensor products of the obtained differential algebras we may also build such calculus for $\mathbb{R}^n$, however, further generalisation to arbitrary smooth manifolds would be more difficult.
What is a remarkable picture of this geometry, is that due to its non-commutative properties we obtain no new physics if we attempt to construct physical models. Naively, one would expect that such models would include higher order terms like $\phi'' \bar{\phi}''$. In our case, the only difference with the classical geometry appears in the existence of possible topological (or boundary) terms. The gauge theory, though possible to construct, has no dynamical degrees of freedom and may only contribute to some potential terms in the products with some standard gauge theory.

We may believe that this could be a profound geometrical reason that no higher order derivatives are necessary in the description of fundamental physical theories.

Although we treat the above construction mainly as a toy model there are still some interesting problems in it. First, one could study such differential algebras on smooth manifolds in general. From the physical point of view, it would be interesting to analyse the gauge theories on the product of standard geometry and the one discussed above.

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