On $C^1$-approximability of functions by solutions of second order elliptic equations on plane compact sets and $C$-analytic capacity

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Abstract
Criteria for approximability of functions by solutions of homogeneous second order elliptic equations (with constant complex coefficients) in the norms of the Whitney $C^1$-spaces on compact sets in $\mathbb{R}^2$ are obtained in terms of the respective $C^1$-capacities. It is proved that the mentioned $C^1$-capacities are comparable to the classic $C$-analytic capacity, and so have a proper geometric measure characterization.

Keywords Second order homogeneous elliptic operator · $C^1$-approximation · Localization operator of Vitushkin type · $L$-oscillation · $LC^1$-capacity · $C$-analytic capacity · Curvature of measure
1 Introduction

For the history of the subject under consideration we refer to the survey [1]. Let

\[ L(x) = c_{11}x_1^2 + 2c_{12}x_1x_2 + c_{22}x_2^2, \quad x = (x_1, x_2) \in \mathbb{R}^2, \]

be any fixed homogeneous polynomial of second order in \( \mathbb{R}^2 \) (with constant complex coefficients \( c_{11}, c_{12}, c_{22} \)) that satisfies the ellipticity condition: \( L(x) \neq 0 \) for all \( x \neq 0 \). With the polynomial \( L(x) \) we associate the elliptic differential operator

\[ \mathcal{L} = c_{11} \frac{\partial^2}{\partial x_1^2} + 2c_{12} \frac{\partial^2}{\partial x_1 \partial x_2} + c_{22} \frac{\partial^2}{\partial x_2^2}. \]

The basic examples: the Laplacian \( \Delta \) and the Bitsadze operator \( \overline{\Delta} = \partial^2/\partial \overline{z}^2 \) in \( \mathbb{R}^2 \) (\( z = x_1 + ix_2 \) is a complex variable). For an open set \( U \) in \( \mathbb{R}^2 \) set \( \mathcal{A}_L(U) = \{ u \in C^2(U) \mid Lu = 0 \text{ in } U \} \). The functions of this class we call \( \mathcal{L} \)-analytic in \( U \). It is well known that \( \mathcal{A}_L(U) \subset C^\infty(U) \) (see [2, Theorem 4.4.1]).

Denote by \( BC^1(U) \) (\( U \) is open in \( \mathbb{R}^2 \)) the space of complex valued functions \( f \) of class \( C^1(U) \) with finite norm

\[ ||f||_U = \max\{||f||_E, ||\nabla f||_U\}, \]

where \( ||g||_E = \sup_{x \in E} |g(x)| \) is the uniform norm of the (vector-) function \( g \) on the set \( E \neq \emptyset \) (for \( U = \mathbb{R}^2 = E \) write \( BC^1, ||f||_1, ||g|| \) respectively).

Let \( X \neq \emptyset \) be a compact set in \( \mathbb{R}^2 \) and \( f \in BC^1 \). The main problem considered in this paper consists of the following.

To find conditions on \( L, X \) and \( f \) necessary and sufficient for existence of a sequence \( \{f_n\}_{n=1}^{+\infty} \subset BC^1 \) such that each \( f_n \) is \( \mathcal{L} \)-analytic in (its own) neighborhood of \( X \) and \( ||f - f_n||_1 \to 0 \) as \( n \to +\infty \).

The class of all functions \( f \in BC^1 \) with this approximation property is denoted by \( \mathcal{A}_L^1(X) \). It is not difficult to show that the following condition always takes place: \( \mathcal{A}_L^1(X) \subset C^1_L(X) = BC^1 \cap \mathcal{A}_L(X) \). Therefore, the following \( C^1 \)-approximation problem for classes of functions naturally appears:

For which compact sets \( X \) one has \( \mathcal{A}_L^1(X) = C^1_L(X) \)?

Recall that analogous approximation problems in the spaces \( BC^m \) (see [3] for the definitions) are solved for all \( m > 0, m \neq 1 \) (even in \( \mathbb{R}^N, N \geq 3 \), [3–5]).

An open disc with center \( a \) and of radius \( r > 0 \) is denoted by \( B(a, r) \); also, for \( B = B(a, r) \) and \( \lambda > 0 \) by \( \lambda B \) we mean the disc \( B(a, \lambda r) \). Positive parameters (constants), which can depend only on \( \mathcal{L} \), will be designated by \( A, A_1, A_2, \ldots \) (they may be different in different occurrences).

Let \( \Phi(x) = \Phi_L(x) \) be the standard fundamental solution for the equation \( \mathcal{L}u = 0 \) (see [5, p. 161] and Section 2 below). We shall basically use the so-called \( \mathcal{L}C^1 \)-capacity, which is connected to the operator \( \mathcal{L} \) and the space \( BC^1 \). Namely, for a
bounded nonempty set $E \subset \mathbb{R}^2$ we let

$$\alpha_1(E) = \alpha_{1L}(E) = \sup_{T} \{|\langle T, 1 \rangle| : \text{Supp}(T) \subset E, \Phi \ast T \in C^1(\mathbb{R}^2), ||\nabla \Phi \ast T|| \leq 1\},$$

(1.1)

where $\langle T, \varphi \rangle$ means the action of the distribution $T$ on the function $\varphi \in C^\infty$, $*$—convolution, and $\text{Supp}(T)$ is the support of the distribution (function, measure) $T$.

On the other hand, the $L$-Lipschitz capacity of $E$, denoted by $\gamma_1(E)$ or $\gamma_{1L}(E)$ is defined in the same way as $\alpha_1(E)$, but with the condition $\Phi \ast T \in C^1(\mathbb{R}^2)$ replaced by $\Phi \ast T \in \text{Lip}(\mathbb{R}^2)$.

Recall also the definition of the classic $C$-analytic capacity [6], basically used in the theory of uniform holomorphic approximations. For a bounded (nonempty) set $E \subset \mathbb{C}$,

$$\alpha(E) = \sup_{T} \{|\langle T, 1 \rangle| : \text{Supp}(T) \subset E, 1/z \ast T \in C(\mathbb{C}), \quad ||1/z \ast T|| \leq 1\}. \quad (1.2)$$

If in this supremum one replaces the condition $1/z \ast T \in C(\mathbb{C})$ by $1/z \ast T \in L^\infty(\mathbb{C})$, one gets the well known analytic capacity, $\gamma(E)$, which is specially useful in the study of removable singularities for bounded holomorphic functions [7].

Clearly, for a bounded open set $E$ one has $\alpha_1(E) = \gamma_1(E)$ and $\alpha(E) = \gamma(E)$.

Our first main result is the following.

**Theorem 1.1** There exist constants $A_1 \in (0, 1)$ and $A_2 \geq 1$ dependent only on $L$, such that

$$A_1 \alpha(E) \leq \alpha_1(E) \leq A_2 \alpha(E) \quad (1.3)$$

and

$$A_1 \gamma(E) \leq \gamma_1(E) \leq A_2 \gamma(E) \quad (1.4)$$

for any bounded set $E$.

Let us remark that the capacities $\alpha$ and $\gamma$ admit a characterization in terms of measures with linear growth and finite curvature, by [8,9] [see (4.4) and (4.3)]. This characterization extends to $\alpha_1$ and $\gamma_1$ by the last theorem. In particular, the capacities $\alpha_1$ and $\gamma_1$ are countably semiadditive.

For the above mentioned elliptic polynomial $L(x), \ f \in C(\mathbb{R}^2)$ and a disc $B = B(a, r)$ define the so-called $L$-oscillation of $f$ on $B$ (see [3]):

$$O^L_B(f) = \frac{1}{2\pi r} \int_{\partial B} f(x) \frac{L(x - a)}{r^2} d\ell_x - \frac{c_{11} + c_{12}}{2\pi r^2} \int_B f(x) dx,$$

where $\ell$ is the Lebesgue measure (the length) on $\partial B$.

For instance, when $L(x) = x_1^2 + x_2^2$ (that is $L = \Delta$), we have (first time in analogous context appeared in [10]):

$$O^L_B(f) = \frac{1}{2\pi r} \int_{\partial B} f(x) \ell_x dx - \frac{1}{\pi r^2} \int_B f(x) dx,$$
and for \( L(x) = 4^{-1}(x_1 + ix_2)^2 = z^2/4 \) (e.g. \( L = \partial^2/\partial z \partial \bar{z} \)):
\[
O_B^L(f) = \frac{1}{8\pi i r^2} \int_{\partial B} f(z)(z - a) \, dz.
\]

Now we formulate our second main result. Fix a compact set \( X \subset \mathbb{R}^2 \) and \( f \in BC^1 \).

Without loss of generality we shall suppose that \( \text{Supp}(f) \) is compact, e.g. \( f \in C^1_0(\mathbb{R}^2) \).

Let \( \omega(g, r) \) be the modulus of continuity of the (vector-)function \( g \) on \( \mathbb{R}^2 \).

**Theorem 1.2** The following conditions are equivalent:

(a) \( f \in A^1_{L}(X) \);

(b) there exist \( k \geq 1 \) and a function \( \omega(r) \to 0 \) as \( r \to 0^+ \) such that for each disk \( B = B(a, r) \) one has
\[
\left| O_{B(a, kr)}^L(f) \right| \leq \omega(r) \alpha_1(B(a, kr) \setminus X);
\]

(c) the property (b) holds for \( k = 1 \) and \( \omega(r) = A \omega(\nabla f, r) \).

The plan of the paper is the following: In Sect. 2 we give some preliminary results. In Sect. 3 we prove Theorem 3.1 (more general than Theorem 1.2). In Sect. 4 we prove Theorem 1.1 and recall the main properties of \( C \)-analytic capacity. In Sect. 5 we present some corollaries of our main results: the \( C^1 \)-approximation criteria for classes of functions and corresponding criteria for the \( C^1 \)-approximation by \( L \)-polynomials on plane compact sets. Theorems 1.2, 3.1, and 5.1 were obtained in the frameworks of the Project 17-11-01064 by the Russian Science Foundation.

## 2 Background

The next lemma is proved in [3].

**Lemma 2.1** For \( a \in \mathbb{R}^2 \) and \( r \in (0, +\infty) \) let \( \psi_r^a(x) = (r^2 - |x - a|^2)/(4\pi r^2) \) in \( B = B(a, r) \) and \( \psi_r^a(x) = 0 \) outside of \( B(a, r) \). Then for all \( \varphi \in C^\infty(\mathbb{R}^2) \) one has:
\[
\int_B \psi_r^a(x) L \varphi(x) \, dx = O_B^L(\varphi),
\]
that is, the action \( (L \psi_r^a, \varphi) \) of the distribution \( L \psi_r^a \) on the function \( \varphi \) is equal to \( O_B^L(\varphi) \) (and it can be continuously extended on all class of functions \( \varphi \in C(\mathbb{R}^2) \)).

For a given \( \varphi \in C_0^\infty(\mathbb{R}^2) \) define the Vitushkin type localization operator (see [5,6]) corresponding to the operator \( L \):
\[
f \mapsto \mathcal{V}_\varphi(f) = \Phi \ast (\varphi L f), \quad f \in C(\mathbb{R}^2).
\]

The basic property of this operator consists of the following simple fact: \( L(\mathcal{V}_\varphi(f)) = \varphi L f \), which means that \( L \)-singularities of \( \mathcal{V}_\varphi(f) \) are contained in the intersection of
the $L$-singularities of $f$ and $\text{Supp} \varphi$. We now present one new property of operator $V_\varphi$ connected to the possibility of its extension to some wider class of “indices” $\varphi$ whenever $f \in C^1$. For a compact set $X \subset \mathbb{R}^2$ put $C^1(X) = C^1(\mathbb{R}^2)|_X$.

**Lemma 2.2** Fix any function $\varphi \in C^1(\overline{B(a, r)})$, $\varphi = 0$ outside of $B(a, r)$. Then for each $f \in C^1(\mathbb{R}^2)$ the following properties hold:

(a) the function $V_\varphi(f) \in C^1(\mathbb{R}^2)$ is well defined and

$$||\nabla (V_\varphi(f))|| \leq A \omega(\nabla f, r) ||\nabla \varphi|| r;$$

(b) $\text{Supp}(L V_\varphi(f)) \subset \text{Supp}(L f) \cap \text{Supp}(\varphi)$;

(c) if $f \in C^2(\mathbb{R}^2)$ then $L(V_\varphi(f)) = \varphi L f$.

**Proof** We prove (a); (b) and (c) then follow from usual regularization arguments. From the last mentioned arguments we also can additionally suppose that $f \in C^\infty(\mathbb{R}^2)$. Then it can be easily seen that $V_\varphi(f) \in C^1(\mathbb{R}^2)$. Let $\omega_1(r) = \omega(\nabla f, r)$. Fix $x \in \mathbb{R}^2$ and for $y \in \mathbb{R}^2$ set

$$F(y) = f(y) - f(x) - \nabla f(x) \cdot (y - x), \quad \text{if } x \in B(a, 2r),$$

$$F(y) = f(y) - f(a) - \nabla f(a) \cdot (y - a), \quad \text{if } x \notin B(a, 2r).$$

Then $LF = L f$ and for all $y \in \overline{B(a, r)}$ we have:

$$|F(y)| \leq 3\omega_1(r)|y - x|, \quad |\nabla F(y)| \leq 3\omega_1(r), \quad \text{if } x \in B(a, 2r), \quad (2.2)$$

$$|F(y)| \leq r\omega_1(r), \quad |\nabla F(y)| \leq \omega_1(r), \quad \text{if } x \notin B(a, 2r). \quad (2.3)$$

Let $\partial_j g(y) = \partial g(y)/\partial y_j$ and $c_{21} = c_{12}$. Then

$$\varphi L F = L(\varphi F) - F L \varphi - 2 \sum_{i, j=1}^{2} c_{ij} \partial_i \varphi \partial_j F,$$

so, substituting the last equality to (2.1) and taking into account that $\varphi(x) F(x) = 0$, we obtain:

$$V_\varphi(f)(x) = \langle \Phi(x - y), \varphi(y) L F(y) \rangle - \langle \Phi(x - y), F(y) L \varphi(y) \rangle$$

$$- 2 \sum_{i, j=1}^{2} c_{ij} \langle \Phi(x - y), \partial_i \varphi(y) \partial_j F(y) \rangle.$$

From the equality $F \partial_i \partial_j \varphi = \partial_j (F \partial_i \varphi) - \partial_i \varphi \partial_j F$ we find that

$$- \langle \Phi(x - y), F(y) L \varphi(y) \rangle = \sum_{i, j=1}^{2} c_{ij} \left( \langle \partial_j \Phi(x - y), F(y) \partial_i \varphi(y) \rangle \right.$$

$$\left. + \langle \Phi(x - y), \partial_i \varphi(y) \partial_j F(y) \rangle \right),$$
which gives that
\[
V_\varphi(f)(x) = \sum_{i,j=1}^{2} c_{ij}(\langle \partial_j \Phi(x - y), F(y) \partial_i \varphi(y) \rangle - \langle \Phi(x - y), \partial_i \varphi(y) \partial_j F(y) \rangle),
\]
\[
\partial_k V_\varphi(f)(x) = \sum_{i,j=1}^{2} c_{ij}(\langle \partial_k \partial_j \Phi(x - y), F(y) \partial_i f(y) \rangle - \langle \partial_k \Phi(x - y), \partial_i \varphi(y) \partial_j F(y) \rangle).
\]

From the last equality it follows that it remains to estimate the following expressions:
\[
\langle \partial_k \partial_j \Phi(x - y), F(y) \partial_i \varphi(y) \rangle, \tag{2.4}
\]
and
\[
\langle \partial_k \Phi(x - y), \partial_i \varphi(y) \partial_j F(y) \rangle \tag{2.5}
\]
for all possible triples \(\{k, i, j\}\).

Since \(|\nabla \Phi(y)| \leq A/|y|\) (see Lemma 2.3 below), by (2.2) and (2.3) the absolute value in (2.5) can be easily estimated by the following convergent integral:
\[
A_1 ||\nabla \varphi|| \omega_1(r) \int_{B(a,r)} \frac{dy}{|x - y|} \leq A \omega_1(r) r ||\nabla \varphi||.
\]

It is not so directly simple to estimate (2.4), because the kernel
\[
|\partial_k \partial_j \Phi(x - y)| \simeq |x - y|^{-2}
\]
is not locally integrable with respect to the Lebesgue measure in \(\mathbb{R}^2\). Nevertheless, according to [5, Lemma 1.1], for the function \(\chi \in C^0(\mathbb{R}^2)\) (properly tending to 0 as \(y \to x\)) one has:
\[
\langle \partial_k \partial_j \Phi(x - y), \chi(y) \rangle = (v, p) \int K_{kj}(x - y) \chi(y) \, dy,
\]
where each \(K_{kj}\) is a standard (of class \(C^\infty\) outside 0, homogeneous of order \(-2\), with zero average over \(\partial B(0, 1)\)) Calderon-Zygmund kernel in \(\mathbb{R}^2\) with respect to Lebesgue measure. In our case, the function \(\chi(y) = F(y) \partial_i \varphi(y)\) tends to zero like \(|y - x|\) as \(y \to x\), because of (2.2) and since \(\varphi = 0\) outside of \(B(a, r)\).

Therefore, the last integral (in the principle value sense) as a matter of fact is absolutely convergent and can be estimated for \(x \in B(a, 2r)\) as follows (using (2.2)):
\[
\int |\partial_k \partial_j \Phi(x - y) F(y) \partial_i \varphi(y)| \, dy \leq A_1 \omega_1(r) ||\nabla \varphi|| \int_{B(a,r)} \frac{dy}{|x - y|} \leq A \omega_1(r) r ||\nabla \varphi||.
\]

For \(x \notin B(a, 2r)\) the corresponding estimate is trivial [by (2.3)]. Lemma 2.2 is proved.
Recall the basic properties for solutions of our main equation

\[ \mathcal{L}u = c_{11} \frac{\partial^2 u}{\partial x_1^2} + 2c_{12} \frac{\partial^2 u}{\partial x_1 \partial x_2} + c_{22} \frac{\partial^2 u}{\partial x_2^2} = 0. \]

Let \( \lambda_1, \lambda_2 \) be the roots of the characteristic equation \( c_{11} \lambda^2 + 2c_{12} \lambda + c_{22} = 0 \). It follows from the ellipticity condition that \( \lambda_1, \lambda_2 \not\in \mathbb{R} \). Define

\[ \partial_1 = \frac{\partial}{\partial x_1} - \lambda_1 \frac{\partial}{\partial x_2}, \quad \partial_2 = \frac{\partial}{\partial x_1} - \lambda_2 \frac{\partial}{\partial x_2} \quad \text{if} \quad \lambda_1 \neq \lambda_2, \]

or

\[ \partial_1 = \frac{\partial}{\partial x_1} - \lambda_1 \frac{\partial}{\partial x_2}, \quad \partial_2 = \frac{\partial}{\partial x_1} + \lambda_1 \frac{\partial}{\partial x_2} \quad \text{if} \quad \lambda_1 = \lambda_2. \]

We then have the following decomposition of \( \mathcal{L} \):

\[ \mathcal{L}u = \begin{cases} c_{11} \partial_1 (\partial_2 (u)), & \text{if } \lambda_1 \neq \lambda_2; \\ c_{11} \partial_1^2 (u), & \text{if } \lambda_1 = \lambda_2. \end{cases} \]

We also introduce the following new coordinates:

\[ z_1 = \frac{\lambda_2}{\lambda_2 - \lambda_1} \left( x_1 + \frac{1}{\lambda_2} x_2 \right), \quad z_2 = \frac{\lambda_1}{\lambda_1 - \lambda_2} \left( x_1 + \frac{1}{\lambda_1} x_2 \right) \quad \text{if } \lambda_1 \neq \lambda_2, \]

or

\[ z_1 = \frac{1}{2} \left( x_1 - \frac{1}{\lambda_1} x_2 \right), \quad z_2 = \frac{1}{2} \left( x_1 + \frac{1}{\lambda_1} x_2 \right) \quad \text{if } \lambda_1 = \lambda_2, \]

that satisfy the “orthogonality” relations:

\[ \partial_1 z_1 = 1, \quad \partial_1 z_2 = 0, \quad \partial_2 z_1 = 0, \quad \partial_2 z_2 = 1. \]

Finally, we “identify” \( z = x_1 + ix_2 \) in \( \mathbb{C} \) and \( x = (x_1, x_2) \) in \( \mathbb{R}^2 \) in the sense that \( f(x) \) or \( f(z) \) will mean the same for any function \( f \). Notice that for \( s = 1 \) and \( s = 2 \) the linear transformations \( \Lambda_s(z) = z_s \) of \( \mathbb{R}^2 \) are nondegenerate. Nevertheless, we shall never use \( x \) as a complex variable, as well as the symbol \( x_s \) (not the same as \( x_s \)) will not be used.

The following well known results take place [11, Chapter IV, §6, (4.77)] (see also [12] for a simple direct proof).

**Lemma 2.3** Let \( \mathcal{L} \) be as above. If \( \lambda_1 \neq \lambda_2 \) then there exist in \( \mathbb{C}\setminus\{0\} \) a fixed analytic branch \( \log(z_1 z_2^s) \) of the multivalued function \( \text{Log}(z_1 z_2^s) \) and a complex constant \( k_1 = k_1(\mathcal{L}) \neq 0 \) such that

\[ \Phi(z) = \Phi_\mathcal{L}(z) = k_1 \log(z_1 z_2^s) \]
is a fundamental solution of \( L \), where \( \nu = 1 \) if \( \text{sgn}(\text{Im}\lambda_1) \neq \text{sgn}(\text{Im}\lambda_2) \), and \( \nu = -1 \) otherwise.

If \( \lambda_1 = \lambda_2 \), then \( \Phi_L(z) = k_1 \frac{z_1}{z_2} \) is a fundamental solution of \( L \), where \( k_1 = k_1(L) \neq 0 \).

**Lemma 2.4** There is \( k_2 = k_2(L) > 1 \) with the following properties. Let \( T \) be a distribution with compact support in the disc \( B(a, r) \) and \( g = \Phi_L * T \).

If \( \lambda_1 \neq \lambda_2 \) then for \( |z - a| > k_2 r \) we have the expansion

\[
g(z) = c_0 \Phi(z - a) + \sum_{m=1}^{\infty} \frac{c_1}{(z - a)^m_1} + \sum_{m=1}^{\infty} \frac{c_2}{(z - a)^m_2},
\]

where \( c_0 = c_0(g) = \langle T, 1 \rangle \) and

\[
c_s = c_m(g, a) = -k_1 \frac{m^{s-1}}{m} \langle T, (w - a)^m_s \rangle, \quad s \in \{1, 2\}, \quad m = 1, 2, \ldots.
\]

If \( \lambda_1 = \lambda_2 \) then for \( |z - a| > k_2 r \) we have the expansion

\[
g(z) = c_0 \Phi(z - a) + \sum_{m=1}^{\infty} \frac{c_1}{(z - a)^m_1} + \sum_{m=1}^{\infty} \frac{c_2(z - a)_1}{(z - a)^m_2},
\]

where \( c_0 = c_0(g) = \langle T, 1 \rangle \) and

\[
c_1 = -k_1 \langle T(w), (w - a)_1(w - a)^{m-1}_2 \rangle,
\]

\[
c_2 = k_1 \langle T(w), (w - a)^m_2 \rangle, \quad m = 1, 2, \ldots.
\]

The series in (2.6) and (2.7) converge in \( C^\infty(\mathbb{C} \setminus B(a, k_2 r)) \).

**Example 2.5** For the Laplacian \( L = \Delta \), one has \( \lambda_1 = i, \lambda_2 = -i, z_1 = z/2, z_2 = \bar{z}/2 \) and

\[
\partial_1 = \frac{\partial}{\partial x_1} - i \frac{\partial}{\partial x_2} = 2 \frac{\partial}{\partial x_1}, \quad \partial_2 = \frac{\partial}{\partial x_1} + i \frac{\partial}{\partial x_2} = 2 \frac{\partial}{\partial x_1}, \quad \Phi_\Delta(z) = \frac{1}{4\pi} \log \left( \frac{z\bar{z}}{4} \right).
\]

For the Bitsadze operator \( L = \frac{\partial^2}{\partial z^2} = \frac{1}{4} \left( \frac{\partial^2}{\partial x_1^2} + 2i \frac{\partial^2}{\partial x_1 \partial x_2} - \frac{\partial^2}{\partial x_2^2} \right) \), one gets \( \lambda_1 = \lambda_2 = -i, z_1 = \bar{z}/2, z_2 = z/2 \) and

\[
\partial_1 = 2 \frac{\partial}{\partial z}, \quad \partial_2 = 2 \frac{\partial}{\partial z}, \quad \Phi_L(z) = \frac{1}{\pi} \frac{\bar{z}}{z}.
\]

For a class \( \mathcal{I} \) of functions and \( \tau \geq 0 \) we denote by \( \tau \mathcal{I} \) the class \( \{ \tau g : g \in \mathcal{I} \} \). Rewrite the definition of \( \alpha_{1\mathcal{L}}(E) \) for a nonempty bounded set \( E \):

\[
\alpha_1(E) = \alpha_{1\mathcal{L}}(E) = \sup \{ \| \langle L g, 1 \rangle \| : g \in \mathcal{I}_1(E) \},
\]
where
\[
\mathcal{I}_1(E) = \{ \Phi \mathcal{L} \ast T \mid \text{Supp}(T) \subset E, \Phi \mathcal{L} \ast T \in C^1(\mathbb{R}^2), \|\nabla \Phi \ast T\| \leq 1 \}.
\]

Clearly, \(\alpha_1(B(a, r)) \leq Ar\) for each disc \(B(a, r)\), and this is the only property of \(\alpha_1\) that we need in this and the next section.

For \(g \in C^1(\mathbb{R}^2)\) define \(\nabla_c g = (\partial_1 g, \partial_2 g)\). Then \(|\nabla_c g|\) is comparable to \(|\nabla g|\) and \(\omega(\nabla_c g, r)\) is comparable to \(\omega(\nabla g, r)\).

The following lemma (where we use the notations of Lemma 2.4 above) is analogous to Lemma 3.3 and Corollary 3.4 in [13].

**Lemma 2.6** Let \(E \subset B(a, r) \) and \(g \in \mathcal{I}_1(E)\). Then there are \(k_3 = k_3(\mathcal{L}) > 1\), \(k_4 = k_4(\mathcal{L}) > 1\) and \(A = A(\mathcal{L}) > 0\) such that
\[
|c_0(g)| \leq \alpha_1(E), \quad |c_m^s(g, a)| \leq A(k_3 r)^m \alpha_1(E), \quad s \in \{1, 2\}, \quad m = 1, 2, \ldots \tag{2.8}
\]
and for \(|z - a| > k_4r\) one has
\[
|\nabla_c g(z)| \leq \frac{A\alpha_1(E)}{|z - a|}; \tag{2.9}
\]
\[
|\nabla_c(g(z) - c_0 \Phi \mathcal{L}(z))| \leq \frac{Ar\alpha_1(E)}{|z - a|^2}; \tag{2.10}
\]
\[
|\nabla_c\left(g(z) - c_0 \Phi \mathcal{L}(z) - \frac{c_1^1}{(z - a)_1} - \frac{c_1^2}{(z - a)_2}\right)| \leq \frac{Ar^3}{|z - a|^3}, \quad \text{if } \lambda_1 \neq \lambda_2; \tag{2.11}
\]
\[
|\nabla_c\left(g(z) - c_0 \Phi \mathcal{L}(z) - \frac{c_1^1}{(z - a)_1} - \frac{c_2^2(z - a)_1}{(z - a)_2}\right)| \leq \frac{Ar^3}{|z - a|^3}, \quad \text{if } \lambda_1 = \lambda_2.
\]

**Proof** We give a short proof here for completeness, and only for the cases \(\lambda_1 \neq \lambda_2\). The cases \(\lambda_1 = \lambda_2\) can be done almost the same way. To check (2.8) fix \(m \geq 1\) and \(s\) and take \(g_m^s = \Phi \ast (T(w)(w - a)_s)^m\), where \(T = \mathcal{L} g\). Then \(|c_m^s| = |k_1|m^{-1}|c_0(g_m^s)|\). Take \(\chi \in C^1(B(a, 2r))\), \(\chi = 0\) outside of \(B(a, 2r)\), with \(\chi = 1\) in \(B(a, r)\) and \(|\nabla \chi| < 2/r\). Then \(g_m^s = \chi \nabla(\chi(w)(w - a)_s)^m g\). Since, clearly, \(|\nabla(\chi(w)(w - a)_s)^m|\| \leq A(k_3 r)^m\|\), it suffices to apply Lemma 2.2 and definition of \(\alpha_1(E)\). The remaining estimates in the last lemma can be now easily checked. In fact, let \(d_1 = \min\{|z_s| : |z| = 1, \ s \in \{1, 2\}\}\). One then can take \(k_4 = (k_3 + 1)/d_1\). \(\square\)

### 3 Proof of Theorem 1.2

We now formulate and prove some generalization of Theorem 1.2.
Fix any even function \( \varphi_1 \) in \( C(\mathbb{R}^2) \cap C^1(B(0,1)) \) with \( \text{Supp} \varphi_1 \) in \( \overline{B}(0,1) \) and with the property \( \int \varphi_1(x)dx = 1 \).

Set \( \varphi^a_r(x) = \varphi_1((x-a)/r^2) \) and \( \varphi_r = \varphi^0_r \). Clearly, \( \| \nabla \varphi^a_r \| = r^{-3} \| \nabla \varphi_1 \| \).

By analogy with [14, Theorem 2.2], the proof of Theorem 1.2 is based on the following result.

**Theorem 3.1** For a compact set \( X \) and \( f \in C^1_0(\mathbb{R}^2) \) the following are equivalent:

(a) \( f \in A^1_c(X) \);
(b) there exist \( k \geq 1 \) and a function \( \omega(r) \to 0 \) as \( r \to 0+ \) such that for each disc \( B = B(a,r) \) one has

\[
\left| \int_{B(a,r)} \partial_1 f(x) \partial_2 \varphi^a_r(x)dx \right| \leq \omega(r)r^{-2} \alpha_1(B,a,r) < \infty.
\]

(c) the property (b) holds for \( k = 1 \) and \( \omega(r) = A \omega(\nabla f, r) \).

In particular, for \( \varphi_1 = 8\psi^0_1 \) (see Lemma 2.1) this theorem coincides with Theorem 1.2.

**Proof of (a) \( \Rightarrow \) (c) in Theorem 3.1.** Let \( f \in A^1_c(X) \) and take a sequence \( \{f_n\}_{n=1}^{\infty} \subset BC^1 \) such that each \( f_n \) is \( \mathcal{L} \)-analytic in (its own) neighborhood \( U_n \) of \( X \) and \( \| f - f_n \|_1 \to 0 \) as \( n \to +\infty \). By regularization arguments we can additionally suppose that each \( f_n \in C^\infty(\mathbb{R}^2) \). Fix \( B = B(a,r) \) and \( \epsilon \in (0,r/2) \). Then there is \( n_\epsilon \in \mathbb{N} \) such that for all \( n \geq n_\epsilon \) one has \( \| f - f_n \|_1 < \epsilon \), and then also \( \omega((\nabla f - \nabla f_n), r) < 2\epsilon \). So it is enough to prove the estimate

\[
\left| \int_{B(a,r)} \partial_1 f_n(x) \partial_2 \varphi^a_r(x)dx \right| \leq A \omega_n(r)r^{-2} \alpha_1(B,a,r) < \infty
\]

with \( A = A(\mathcal{L}) \) and \( \omega_n(r) = \omega(\nabla f_n, r) \), and then tend \( \epsilon \) to 0. Let \( h_n = \mathcal{V}_\varphi f_n \), where \( \varphi(x) = \varphi^a_{r-\epsilon}(x) \). By Lemma 2.2, \( h_n \in BC^1 \), \( \| \nabla h_n \| \leq A \omega_n(r) \| \nabla \varphi \| \) and \( h_n \) is \( \mathcal{L} \)-analytic outside some compact set \( E \subset B \setminus X \). By (1.1) and Lemma 2.2 we have

\[
|\langle \mathcal{L} h_n, 1 \rangle | = |\langle \varphi, \mathcal{L} f_n \rangle | = |c_{11}| |\langle \partial_2 \varphi, \partial_1 f_n \rangle | \leq A \omega_n(r) \| \nabla \varphi \| \alpha_1(B,a,r) \]

which ends the proof of (a) \( \Rightarrow \) (c).

Since (c) \( \Rightarrow \) (b) is evident, we pass to the following more complicated part of the proof.

**Proof of (b) \( \Rightarrow \) (a) in Theorem 3.1.**

We can suppose that for some \( R > 0 \) we have \( X \subset B(0, R) \) and \( f(z) = 0 \) for \( |z| > R \). In (3.1) we also take \( \omega(\delta) \geq \omega(\nabla f, \delta) \).

Fix \( \delta > 0 \) and any standard \( \delta \)-partition of unity \( \{(\varphi_j, B_j) : j = (j_1, j_2) \in \mathbb{Z}^2 \} \) in \( \mathbb{C} \). This means that \( B_j = B(a_j, \delta) \), where \( a_j = j_1 \delta + j_2 \delta \in \mathbb{C} \), \( \varphi_j \in C^\infty_0(B_j) \), \( 0 \leq \varphi_j \leq 1 \), \( \| \nabla \varphi_j \| \leq A/\delta \), \( \sum_j \varphi_j = 1 \).

Now consider the new partition of unity \( \{(\psi_j, B'_j) \} \), where \( \psi_j = \varphi_\delta * \varphi_\delta * \varphi_j \), \( B'_j = B(a_j, 3\delta) \) (recall that \( \varphi_\delta = \varphi^0_\delta \)). Clearly, \( \psi_j \in C^\infty_0(B'_j) \) and \( \| \nabla \psi_j \| \leq A/\delta \). Define the so-called localized functions \( f_j = \Phi_L * (\psi_j \mathcal{L} f) \).
Lemma 3.2  The functions $f_j$ satisfy the following properties:

1. $f_j \in A\omega(\nabla f, \delta)I_1(B_j' \setminus X^0)$;
2. $f = \sum_j f_j$ and the sum is finite ($f_j = 0$ if $B_j' \cap B(0, R) = \emptyset$);
3. if $\lambda_1 \neq \lambda_2$ then for $|z - a_j| > 3k_2\delta$ we have the expansion

$$f_j(z) = c_{0j} \Phi(z - a_j) + \sum_{m=1}^{\infty} \frac{c_{mj}^1}{(z - a_j)_1^m} + \sum_{m=1}^{\infty} \frac{c_{mj}^2}{(z - a_j)_2^m},$$

or for $\lambda_1 = \lambda_2$:

$$f_j(z) = c_{0j} \Phi(z - a_j) + \sum_{m=1}^{\infty} \frac{c_{mj}^1}{(z - a_j)_1^m} + \sum_{m=1}^{\infty} \frac{c_{mj}^2}{(z - a_j)_2^{m+1}},$$

where

$$c_{0j} = \int f(x) \mathcal{C}\psi_j(x) dx = -c_{11} \int \partial_1 f(x) \partial_2 \psi_j(x) dx,$$

$$c_{1j}^s = c_{11} \kappa_1 \nu^{s-1} \int \partial_s f(x) \partial_{(3-s)}(\psi_j(x)(z - a_j)_s) dx, \ s \in \{1, 2\}$$

with $\nu = -1$ whenever $\lambda_1 = \lambda_2$. Define $G_j = B(a_j, (k + 2)\delta) \setminus X$. Then

$$|c_{0j}| \leq A\omega(\nabla f, \delta)\alpha_1(G_j),$$

$$|c_{1j}^s| \leq A\omega(\nabla f, \delta)\delta\alpha_1(G_j), \ s \in \{1, 2\}.\ (3.5)$$

Proof  Notice that the last two estimates are corollaries of (3.1), not only (2.8). We follow analogous proof for Lemma 2.5 in [14].

First we obtain (3.4) using (3.2), which follows from Lemma 2.4, the definition of $f_j$ and integration by parts. Set $\varphi_j^* = \varphi_\delta * \varphi_j$. Then

$$\varphi_j^* \in C_0^\infty(B(a_j, 2\delta)), \ 0 \leq \varphi_j^* \leq 1, \ \psi_j = \varphi_\delta * \varphi_j^*.$$

By (3.1) and Fubini’s theorem

$$|c_{0j}| = |c_{11}| \left| \int \partial_1 f(x) \partial_2 \left( \int \varphi_\delta(x - y) \varphi_j^*(y) dy \right) dx \right|$$

$$\leq A_1 \left| \int \varphi_j^*(y) \omega(\delta)\delta^{-2} \alpha_1(B(y, k\delta) \setminus X) dy \right| \leq A\omega(\delta)\alpha_1(G_j).$$

In order to estimate $|c_{1j}^s|$ we first need to check that in (3.3) the function $\psi_j(x)(z - a_j)_s$ has the form $\varphi_\delta * \chi_j$, where $\chi_j \in C_0^\infty(B(a_j, 2\delta))$ and $||\chi_j|| \leq A\delta$. It can be done the same way as in [14, p. 1331] or [15, Lemma 3.4] using the Fourier transform. Then we proceed as in the first part of the proof. \qed
We are ready to describe the scheme for approximation of the function \( f = \sum f_j \) following and [13,15, §6].

Put \( J = \{ j \in \mathbb{Z}^2 : B_j' \cap \delta X \neq \emptyset \} \). For \( j \notin J \) by Lemma 3.2 (1), clearly, \( f_j \in A^1_{\mathcal{L}}(X) \), so these \( f_j \) don’t need to be approximated. Let now \( j \in J \). By definition of \( \alpha_1(G_j) \) (recall that \( G_j = B(a_j, (k+2)\delta) \setminus X \)) and by (3.4) we can find functions \( f_j^* \in A(\delta)I_1(G_j) \subset A^1_{\mathcal{L}}(X) \) such that \( c_0(f_j^*) = c_0(f_j) \). Put \( g_j = f_j - f_j^* \) (\( f_j^* = f_j, g_j \equiv 0 \) for \( j \notin J \)). Then

\[
||\nabla g_j|| \leq A(\delta); \quad c_0(g_j) = 0. \tag{3.6}
\]

Therefore, by (2.8) (with \( m = 1 \)) for \( E = G_j \) and \( g = f_j^* \) and by (3.5) we can write (clearly, \( c_1^s(g_j, a) = c_1^s(g_j) \) do not depend on \( a \)):

\[
|c_1^s(g_j)| \leq A(\delta)\delta \alpha_1(G_j), \quad s \in \{1, 2\}. \tag{3.7}
\]

Using (2.9)–(2.11) for \( g = g_j \) and \( E = B(a_j, (k+2)\delta) = B_j^* \), we obtain for \( |z - a_j| > p\delta \) (here \( p = \max\{k_2, k_3, k_4, k + 2\} + 1 \)):

\[
|\nabla g_j(z)| \leq \frac{A(\delta)\delta \alpha_1(G_j)}{|z - a_j|^2} + \frac{A(\delta)\delta^3}{|z - a_j|^3}. \tag{3.8}
\]

We need to introduce the following abbreviate notations. Recall that \( \delta \) is fixed and small enough.

For \( j \in J \) set \( \alpha_j = \alpha_1(G_j) \), so that all \( \alpha_j > 0 \). For \( I \subset J \) and \( z \in \mathbb{C} \) put

\[
B_i^* = \bigcup_{j \in I} B_j^*, \quad G_I = \bigcup_{j \in I} G_j, \quad \alpha_I = \sum_{j \in I} \alpha_j, \quad g_I = \sum_{j \in I} g_j,
\]

\[
I'(z) = \{ j \in I : |z - a_j| > p\delta \}, \quad S_I'(z) = \sum_{j \in I'(z)} \left( \frac{\delta \alpha_j}{|z - a_j|^2} + \frac{\delta^3}{|z - a_j|^3} \right).
\]

Set also \( S_I(z) = S_I'(z) \) if \( I = I'(z) \) and \( S_I(z) = S_I'(z) + 1 \) if \( I \neq I'(z) \). For \( I \subset J, l \in I \) and \( s \in \{1, 2\} \) define \( P_s(I, l) = \{ j \in I : j_{s-1} = l_{s-1} \} \).

**Definition 3.3** Fix \( s \in \{1, 2\}, I \subset J \) and \( l \in I \). A subset \( L_s = L_s(l) \) of \( I \) is called a **complete s-chain in I with vertex l** if the following conditions are satisfied:

1. **L_s is s-directional and connected in I**; this means that \( L_s \subset P_s(I, l), j_s \geq j_s' \) for all \( j \in L_s, \) and for each \( j \in L_s \) and \( j' \in P_s(I, l) \) such that \( l_s \leq j' \leq j_s \) we have \( j' \in L_s \);
2. it is possible to represent \( L_s \) as \( L_s = L_s^1 \cup L_s^2 \cup L_s^3 \) with the following properties: for each \( j^0 \in L_s^\theta, \theta = 1, 2, 3, \) one has

\[
j_s^1 < j_s^2 < j_s^3 \quad \text{and} \quad |a_{j^1} - a_{j^3}| \geq q\delta,
\]

where \( q \geq 3p \) depending only on \( \mathcal{L} \) will be chosen later;
(3) for \( \theta = 1 \) and \( \theta = 3 \) we have \( \alpha_{L_1^3} \geq \delta \) and \( L_s \) is minimal with the properties above (then, clearly, \( \alpha_{L_s} \leq A\delta \)).

**Definition 3.4** Let \( l \in I \subset J \). A set \( \Gamma \subset I \) is called a complete group in \( I \) with vertex \( l \) if there exist complete 1- and 2-chains \( L_1 \) and \( L_2 \) in \( I \) with vertex \( l \) such that \( \Gamma = L_1 \cup L_2 \).

Now we divide the set of indices \( J \) into a finite number of nonintersecting groups \( \Gamma^n, n \in \{1, \ldots, N\} \) by induction as follows. First define a natural order in \( J \): for \( j \neq j' \) in \( J \) write \( j < j' \) if \( j_2 < j_2' \) or \( j_2 = j_2' \) but \( j_1 < j_1' \). Now choose the minimal \( l_1 \) in \( J \). If there exists a complete group \( \Gamma = \Gamma_1 \cup \Gamma_2 \) in \( J \) with vertex \( l_1 \) we define \( \Gamma_1 = \Gamma \). If such \( \Gamma \) does not exist, we put \( \Gamma_1 = \Gamma_2(J, l) \) if \( L_1 \) does not exist and call \( \Gamma_1 \) incomplete 1-group, otherwise put \( \Gamma_1 = \Gamma_2(J, l) \) (if \( L_1 \) exists, but \( L_2 \) does not exist in the above sense) and call \( \Gamma_1 \) incomplete 2-group. If \( \Gamma_1, \ldots, \Gamma_n \) are constructed, take \( J^n = J \setminus (\Gamma_1 \cup \ldots \cup \Gamma^n) \) and make the same procedure for \( J^{n+1} \) instead of \( J \) defining \( \Gamma^{n+1} \). Let \( N \) be the maximal number with the property \( J^N \neq \emptyset \). Now we fix this partition \( \{ \Gamma^n \} = \{ \Gamma^n \}_{n=1}^N \) of \( J \).

For each group \( \Gamma = \Gamma^n \) (complete or not) by (3.6)–(3.8) one has:

\[
\alpha_\Gamma \leq A\delta \cdot c_0(\alpha) = 0, |c_s(\alpha)| \leq A\omega(\delta)\delta (s \in \{1, 2\}); \]
\[
|\nabla g_\Gamma(z)| \leq A\omega(\delta)S_\Gamma(z), ||\nabla g_\Gamma|| \leq A\omega(\delta), \]
\[
||S_\Gamma|| \leq A, ||S_{P_\Gamma}|| \leq A (s \in \{1, 2\}). \tag{3.9}
\]

**Lemma 3.5** For each complete group \( \Gamma = \Gamma^n \) there exists \( h_\Gamma \in A\omega(\delta)I_1(G_\Gamma) \subset A^1_\mathcal{C}(X) \) such that

\[
c_0(h_\Gamma) = 0, c_s^1(h_\Gamma) = c_s^1(\alpha) (s \in \{1, 2\}).
\]

and for all \( z \in \mathbb{C} \)

\[
|\nabla h_\Gamma(z)| \leq A\omega(\delta)S_\Gamma(z).
\]

**Proof** We follow the idea in [14, Lemma 2.7]. Let \( \Gamma \) be a complete group in \( J \) with vertex \( l \) and complete 1- and 2-chains \( L_1 \) and \( L_2 \) respectively, and let \( L_1 = L_1^1 \cup L_2^1 \cup L_3^1 \) (like in definitions just above).

For each \( j \in \Gamma \) we can choose \( h_j \in 2I_1(G_j) \) with \( c_0(h_j) = \alpha_j = \alpha_1(G_j) \). Let \( T_j = \partial h_j \), so that \( \alpha_j = (T_j, 1) \). Fix \( j_1 \in L_1^1, j_2 \in L_2^1 \) and for \( \theta = 1 \) and \( \theta = 3 \) put:

\[
h^\theta = h_{j_\theta}, T^\theta = \partial h_{j_\theta}, a^\theta = a_{j_\theta}, G^\theta = G_{j_\theta}.
\]

Put \( M = |a_1^1 - a_3^1|/\delta \). Let \( \lambda_1 \in (0, 1) \) and \( \lambda_3 \in (0, 1) \) be such that \( \lambda^1 \alpha^1 = \lambda^3 \alpha^3 := \alpha \).

Define

\[
h^{13}(z) = h^{13}(j_1^1, j_3^1, \lambda^1, \lambda^3, z) = (\lambda^3 h^1(z) - \lambda^1 h^1(z))/M. \tag{3.10}
\]

Then, clearly, \( c_0(h^{13}) = 0 \) and by Lemma 2.4

\[
v^s k_1^{-1} M c_1^s(h^{13}) = (\lambda^3 T^3 - \lambda^1 T^1, z_s)
\]
Moreover, for $z \geq 146$ P. V. Paramonov, X. Tolsa

The last equality has to be checked: it is enough, instead of estimating $\langle \nabla z \rangle$, using finally, the following elementary trick:

$$\lambda(\theta)$$

between the sets of indices $\lambda(\theta)(\kappa)$, $\kappa \in \{1, \ldots, \kappa_j\}$, $\kappa_j \in \mathbb{N}$) with the following properties:

(a) $\lambda(j, \kappa) > 0$, $\sum_{\kappa=1}^{\kappa_j} \lambda(j, \kappa) \leq 1$ for each $j$;

(b) between the sets of indices

$$\Psi^{\theta} = \{(j, \kappa) : j \in L_1^{\theta}, 1 \leq \kappa \leq \kappa_j\}, \theta = 1 \text{ and } 3,$$

we have one to one correspondence

$$\psi^1 \ni (j^1, \kappa^1) \leftrightarrow (j^3, \kappa^3) \in \Psi^3,$$

for which $\lambda(j^1, \kappa^1) \alpha_{j^1} = \lambda(j^3, \kappa^3) \alpha_{j^3}$.

where $|R_{s^1}^{13}| \leq A\delta\alpha$, which follows from Lemma 2.4 and (2.8) with $m = 1$. Therefore,

$$\left(\frac{1}{z-a^1}, \frac{1}{z-a^3}\right) = \left(\frac{(a^3-a^1)_s}{(z-a^1)_s(z-a^3)_s}\right) \leq A|a^3-a^1|(|z-a^1|^{-2} + |z-a^3|^{-2}).$$

In fact the last can be done for the case $\lambda_1 = \lambda_2$ (see Lemma 2.3). The remaining case ($\lambda_1 = \lambda_2$) we propose to the reader’s control. Additionally notice that for the case when $|z-a^\theta| \leq p\delta, \theta = 1$ or $\theta = 3$, we have by (2.9) (since $|a^3-a^1| = M\delta \geq q\delta \geq 3p\delta$):

$$|\nabla h^{13}(z)| \leq \frac{\lambda^\theta}{p} + A\frac{\lambda^{4-\theta} \alpha^{4-\theta} \delta}{|z-a^{3-\theta}|^2}. \quad (3.13)$$

Now we construct a special linear combination (just a sum) of such functions $h^{13}(z) = h^{13}(j^1, j^3, \lambda^1, \lambda^3, z)$. It is easily seen that for each $j \in L_1^3 \cup L_1^3$ there exist $\lambda(j, \kappa)$ ($\kappa \in \{1, \ldots, \kappa_j\}, \kappa_j \in \mathbb{N}$) with the following properties:

$$= \lambda^3 \langle T^3, (z-a^3)_s \rangle + \lambda^3 \langle T^3, a^3_s \rangle$$

$$- \lambda^1 \langle T^1, (z-a^1)_s \rangle - \lambda^1 \langle T^1, a^1_s \rangle$$

$$= \alpha(a^3-a^1)_s + R_{s^1}^{13},$$

$$= k_1 M^{-1} \alpha(\nu(a^3-a^1)_1, (a^3-a^1)_2) + \delta\alpha O(1/M)$$

$$= k_1 \delta\alpha ((\nu(1+i0)_1, (1+i0)_2) + O(1/M)). \quad (3.11)$$

Moreover, for $z$ with $|z-a^1| > p\delta$ and $|z-a^3| > p\delta$ we have by (2.10):

$$\nabla h^{13}(z) = \alpha M^{-1}(\nabla \Phi(z-a^3) - \nabla \Phi(z-a^1)) + \alpha \delta(\nu(|z-a^1|^2) + O(|z-a^3|^2))$$

$$= \alpha \delta(\nu(|z-a^1|^2) + O(|z-a^3|^2)). \quad (3.12)$$

The last equality has to be checked: it is enough, instead of estimating $|\nabla \Phi(z-a^1) - \nabla \Phi(z-a^3)|$, to estimate $|\partial_z \Phi(z-a^1) - \partial_z \Phi(z-a^3)|, s \in \{1, 2\}$ (see Lemma 2.3), using, finally, the following elementary trick:

$$\left| \frac{1}{(z-a^1)_s} - \frac{1}{(z-a^3)_s} \right| = \left| \frac{(a^3-a^1)_s}{(z-a^1)_s(z-a^3)_s} \right| \leq A|a^3-a^1|(|z-a^1|^{-2} + |z-a^3|^{-2}).$$

$\left(\frac{1}{z-a^1}, \frac{1}{z-a^3}\right) = \left(\frac{(a^3-a^1)_s}{(z-a^1)_s(z-a^3)_s}\right) \leq A|a^3-a^1|(|z-a^1|^{-2} + |z-a^3|^{-2}).$
(c) for θ = 1 and θ = 3

\[ \sum_{(j, \kappa) \in \Psi^\theta} \lambda(j, \kappa) \alpha_j = \delta. \]

We define

\[ h_1(z) = h_1(L_1, z) = \sum_{(j, \kappa) \in \Psi^1} \delta \frac{\delta(g(j^3, \kappa^3)h_{j^3} - \lambda(j^1, \kappa^1)h_{j^1})}{|a_{j^3} - a_{j^1}|}, \]

where \((j^3, \kappa^3)\) corresponds to \((j^1, \kappa^1)\) in the above sense. Each member of the last sum is precisely of the form (3.10) with \(\lambda(\theta) = \lambda(j^\theta, \kappa^\theta)\). Clearly, \(c_0(h_1) = 0\) and by (3.11) we have

\[ \left( c_1(h_1), c_2(h_1) \right) = k_1 \delta^2 (\nu(1 + i0)_1, (1 + i0)_2 + O(1/M)). \quad (3.14) \]

Arguing the same way for the complete chain \(L_2\) of \(\Gamma\) we construct the function 
\(h_2(z) = h_2(L_2, z)\) with the same properties as for \(h_1\), but

\[ \left( c_1(h_2), c_2(h_2) \right) = k_1 \delta^2 (\nu(0 + i1)_1, (0 + i1)_2 + O(1/M)). \quad (3.15) \]

Now choose and fix \(q\) so large in Definition 3.3 that \(O(1/M) = O(1/q)\) does not “spoil” (in (3.14) and (3.15)) the linear independence of the vectors \(k_1 \delta^2((\nu(1 + i0)_1, (1 + i0)_2))\) and \(k_1 \delta^2((\nu(0 + i1)_1, (0 + i1)_2))\). Notice that (in \(\mathbb{C}^2\)) the vector \((\nu(1 + i0)_1, (1 + i0)_2)\) is collinear to \((\nu \lambda_2, -\lambda_1)\) (when \(\lambda_1 \neq \lambda_2\)) and to \((-1, 1)\) (if \(\lambda_1 = \lambda_2\)); the vector \((\nu(0 + i1)_1, (0 + i1)_2)\) is collinear to \((\nu, -1)\) (when \(\lambda_1 \neq \lambda_2\)) and to \((1, 1)\) (if \(\lambda_1 = \lambda_2\).

By (3.12), (3.13) and the property (c) just above, we have

\[ |\nabla h_s(z)| \leq AS_{L_s}(z). \quad (3.16) \]

Taking into account the estimate \(|(c_1^1(g^\Gamma), c_1^2(g^\Gamma))| \leq A \omega(\delta) \delta^2\) (see (3.7) and Definition 3.3), (3.16), (3.14) and (3.15), we clearly can find the required \(h_{\Gamma}^s\) as an appropriate linear combination of functions \(h_1\) and \(h_2\).

It remains to show that the function \(\nabla \sum_{j \in J} f_j\) is uniformly approximated on \(\mathbb{C}\) with accuracy \(A \omega(\delta)\) by the function \(\nabla F\), where

\[ F = \sum_n' \left( \sum_{j \in \Gamma^n} f_j + h_{\Gamma^n} \right) + \sum_n'' \sum_{j \in \Gamma^n} f_j^n, \]

where \(\sum_n'\) and \(\sum_n''\) are summations over all complete and incomplete groups respectively.
For the proof of this assertion it is sufficient to check that for each \( z \in \mathbb{C} \) we have
\[
|\nabla (F(z) - f(z))| \leq \sum_n |\nabla (g_{\Gamma^n}(z) - h_{\Gamma^n}(z))| + \sum_{n''} |\nabla g_{\Gamma^n}(z)| \leq A \omega(\delta).
\]

After that it will be enough let \( \delta \) tend to 0.

Now our situation is absolutely analogous to that of [13, pp. 200–203] (2-dimensional case); some simple details (of the following last part of the proof), dropped here, can be found there.

First, estimate the sum \( \sum_{n''} |g_{\Gamma^n}(z)|. \) This is very easy, because in each \( P_s(J, l) = \{ j \in J : j_{3-s} = l_{3-s} \} \) we can find at most one incomplete group \( \Gamma \) (\( s \)-incomplete chain \( L_s = \Gamma \)). Therefore, by (3.9) and (3) of Definition 3.3, we can majorize the considered sum by \( A \omega(\delta) \sum_{m=1}^{+\infty} m^{-2} \), and this is it.

The estimating of \( \sum_{n'} |\nabla (g_{\Gamma^n}(z) - h_{\Gamma^n}(z))| \) is more complicated. For each complete group \( \Gamma^n \) set \( \chi^n = g_{\Gamma^n} - h_{\Gamma^n}. \) Then we have by (3.9) and Lemma 3.5
\[
|\nabla \chi^n(z)| \leq A \omega(\delta) S_t^n(z), \quad c_0(\chi^n) = c^1_1(\chi^n) = c^2_1(\chi^n) = 0. \quad (3.17)
\]

It suffices to prove that
\[
\sum_n |\nabla \chi^n(z)| \leq A \omega(\delta)
\]
for each \( z \in \mathbb{C} \). From now on we fix \( z \in \mathbb{C} \); without loss of generality we can suppose that \( |z| < \delta \). All further constructions will be relative also to \( z \).

Let \( \Gamma^n = L^n_1 \cup L^n_2 \) (with vertex \( l^n \)) be a complete group. Put \( a^n = aL^n, M^n_s = \text{diam}(B_{L^n_s})/\delta, M^n = \max\{M^n_1, M^n_2\} \). Divide the collection of all complete groups into two classes.

**Class (1)** Here we take all complete groups \( \Gamma^n \) with \( M^n \leq |l^n|^{1/4} \).

Clearly, the latter is possible only if \( |z - a^n| \geq (|l^n| - 1)\delta \geq ((M^n)^4 - 1)\delta > 2pM^n\delta. \) Since \( \chi^n \in A\omega(\delta)\bar{\mathcal{I}}(B(a^n, M^n\delta)) \) and (3.17) holds, we have by (2.11):
\[
|\nabla \chi^n(z)| \leq A \omega(\delta) \left( \frac{M^n\delta}{|l^n|} \right)^3 \leq A \omega(\delta) \left( \frac{|l^n|^{-9/4}}{\delta} \right).
\]

Since in each annulus \( B(0, (m + 1)\delta) \setminus B(0, m\delta) \) \( (m > p) \) we can find at most \( Am \) vertices of groups, we can see that
\[
\sum_n \sum_{(1)} |\nabla \chi^n(z)| \leq A \omega(\delta) \sum_{m > p} \left( m^{-5/4} \right) \leq A_1 \omega(\delta),
\]
where the last sum corresponds to all complete groups of the Class (1), which now is well estimated.

**Class (2)** Here we place all complete groups \( \Gamma^n \) for which \( M^n > |l^n|^{1/4} \). Fix such a group \( \Gamma^n \). Then, for some \( s = s^n \in \{1, 2\} \) we have \( M^n > |l^n|^{1/4} \). First we consider the
case when \( s^n = 1 \) is just one with the last property for \( \Gamma^n \). Clearly, then \(|l^n| > 2p\delta\), \( M^*_n \leq |l^n|^{1/4} \), so that

\[
S_{L_2^n}(z) \leq \frac{A\omega(\delta)\delta^2}{|z - al|^2},
\]

and then

\[
|\nabla \chi^n(z)| \leq A\omega(\delta)\left(\frac{\delta^2}{|z - al|^2} + S_{L_1^n}(z)\right).
\]

The same way we argue when \( s^n = 2 \) is just one with the property \( M^n_* > |l^n|^{1/4} \).

In any case we have the following lemma.

**Lemma 3.6** Fix an integer \( m \), and let \( V_{ms} \) denote the collection of all complete groups \( \Gamma^n \) of the Class (2) with \( l_{3-s}^n = m \) and such that \( M^n_* > |l^n|^{1/4} \). Then

\[
\sum_{n \in V_{ms}} S_{L_2^n}(z) \leq A, \quad |m| < 2p,
\]

and

\[
\sum_{n \in V_{ms}} S_{L_2^n}(z) \leq Am^{-5/4}, \quad |m| \geq 2p.
\]

**Proof** For \(|m| < 2p\) this follows from the estimate \(|S_{P^*}(J, j)| \leq A\) for each \( j \in J \).

Let now \(|m| \geq 2p\). Since all \( L^n_s, n \in V_{ms} \), are pairwise “disjoint” and \( M^n_* > |l^n|^{1/4} \geq |m|^{1/4} \), we have

\[
\sum_{n \in V_{ms}} S_{L_2^n}(z) \leq A_1 \sum_{\tau \in \mathbb{Z}} \frac{\delta^2}{m^2\delta^2 + (|m|^{1/4}\tau)^2\delta^2} \\
\leq A_2|m|^{-1/2} \int_0^{+\infty} \frac{dt}{(|m|^{3/4})^2 + t^2} = A|m|^{-5/4}.
\]

\( \square \)

Summation by \( m \) and \( s \) now gives the desired estimate for \( \sum_n (2)|\nabla \chi^n(z)| \), corresponding to the Class (2). This ends the proof of Theorem 3.1.

## 4 Proof of Theorem 1.1

Observe that, by Lemma 2.3 we have \((\partial_1 \Phi(z), \partial_2 \Phi(z)) = k_1(1/z_1, \nu/z_2)\) if \( \lambda_1 \neq \lambda_2 \) or \((\partial_1 \Phi(z), \partial_2 \Phi(z)) = k_1(1/z_2, -z_1/z_2^2)\) otherwise. Define

\[
(K_1(z), K_2(z)) = \left(\frac{1}{z_1}, \frac{1}{z_2}\right) \quad \text{if} \ \lambda_1 \neq \lambda_2,
\]
and
\[
(K_1(z), K_2(z)) = \left( \frac{z_1}{z_2^2}, \frac{1}{z_2} \right) \quad \text{if } \lambda_1 = \lambda_2.
\]

Then, clearly, \(\alpha_1(E)\) is comparable to
\[
\alpha_{12}(E) = \alpha_{12L}(E) = \sup_{T} |\langle T, 1 \rangle| : \text{Supp}(T) \subset E, K_s * T \in C(\mathbb{R}^2),
\]
\[
||K_s * T|| \leq 1, s \in \{1, 2\},
\]
and \(\gamma_1(E)\) is comparable to
\[
\gamma_{12}(E) = \gamma_{12L}(E) = \sup_{T} |\langle T, 1 \rangle| : \text{Supp}(T) \subset E, K_s * T \in L_\infty(\mathbb{C}),
\]
\[
||K_s * T|| \leq 1, s \in \{1, 2\}.
\]

4.1 Preliminaries

We assume all measures to be positive, Borel and locally finite. A measure \(\mu\) in \(\mathbb{C}\) is said to have linear growth (or \(A_0\)-linear growth) if there exists some constant \(A_0 > 0\) such that
\[
\mu(B(z, r)) \leq A_0 r \quad \text{for all } z \in \mathbb{C}, r > 0.
\]

The maximal Hardy–Littlewood operator with respect to \(\mu\) applied to a signed measure \(\nu\) is defined by
\[
M_\mu \nu(z) = \sup_{r > 0} \frac{||\nu||(B(z, r))}{\mu(B(z, r))}.
\]

For a function \(f \in L^1_{loc}(\mu)\), we write
\[
M_\mu f(z) = \sup_{r > 0} \frac{1}{\mu(B(z, r))} \int_{B(z, r)} |f| d\mu.
\]

It is well known (see Chapter 2 of [16], for example) that \(M_\mu\) is bounded in \(L^p(\mu)\) for \(1 < p < \infty\) and also from the space of finite signed measures \(M(\mathbb{C})\) into \(L^{1,\infty}(\mu)\). The latter means that there exists some constant \(A\) such that
\[
\mu\left( \{ z \in \mathbb{C} : M_\mu \nu(z) > \lambda \} \right) \leq A \frac{||\nu||}{\lambda} \quad \text{for all } \lambda > 0 \text{ and all } \nu \in M(\mathbb{C}).
\]

Given a signed measure \(\nu\) and a kernel \(K(\cdot)\) which is \(C^1\) away from the origin and satisfies
\[
|K(z)| \leq \frac{A}{|z|}, \quad |\nabla K(z)| \leq \frac{A}{|z|^2} \quad \text{for } z \in \mathbb{C}\setminus\{0\},
\] (4.1)
we denote

\[ T_Kv(z) = \int K(z-w) \, dv(w) \]

whenever the integral makes sense. For \( \varepsilon > 0 \) we consider the truncated version of \( T_K \):

\[ T_{K,\varepsilon}v(z) = \int_{|z-w| > \varepsilon} K(z-w) \, dv(w), \]

and the maximal operator

\[ T_{K,\ast}v(z) = \sup_{\varepsilon > 0} |T_{K,\varepsilon}v(z)|. \]

For a fixed positive Borel measure \( \mu \) and \( f \in L^1_{\text{loc}}(\mu) \), we write \( T_{K,\mu}f = T_K(f \mu) \), \( T_{K,\mu,\varepsilon}f = T_{K,\varepsilon}(f \mu) \), and \( T_{K,\mu,\ast}f = T_{K,\ast}(f \mu) \). We say that \( T_{K,\mu} \) is bounded in \( L^p(\mu) \) if the operators \( T_{K,\mu,\varepsilon} \) are bounded uniformly on \( \varepsilon > 0 \) in \( L^p(\mu) \), and we set

\[ \|T_{K,\mu}\|_{L^p(\mu) \to L^p(\mu)} = \sup_{\varepsilon > 0} \|T_{K,\mu,\varepsilon}\|_{L^p(\mu) \to L^p(\mu)}. \]

Analogously, if

\[ \mu\left( \{ z \in \mathbb{C} : |T_{K,\varepsilon}v(z)| > \lambda \} \right) \leq A \frac{\|v\|}{\lambda} \text{ for all } \lambda > 0, \text{ all } v \in M(\mathbb{C}) \text{ and all } \varepsilon > 0, \]

we say that \( T_K \) is bounded from \( M(\mathbb{C}) \) into \( L^{1,\infty}(\mu) \), and we denote by \( \|T_K\|_{M(\mathbb{C}) \to L^{1,\infty}(\mu)} \) the optimal constant \( A \).

For technical reasons we need to consider also smoothly truncated operators. We fix a radial \( C^\infty \) function \( \varphi \) which vanishes in \( B(0, 1/2) \) and equals 1 in \( \mathbb{C} \setminus B(0, 1) \), and for \( \varepsilon > 0 \) we set \( \varphi_\varepsilon(z) = \varphi(\varepsilon^{-1}z) \). We write

\[ T_{K,(\varepsilon)}v(z) = \int \varphi \left( \frac{z-w}{\varepsilon} \right) K(z-w) \, dv(w) \]

and

\[ T_{K,(\ast)}v(z) = \sup_{\varepsilon > 0} |T_{K,(\varepsilon)}v(z)| \]

and also \( T_{K,\mu,(\varepsilon)}f = T_{K,(\varepsilon)}(f \mu) \), \( T_{K,\mu,(\ast)}f = T_{K,(\ast)}(f \mu) \). If \( \mu \) has linear growth, it is immediate to check that there exists some constant \( A \) depending only on the kernel \( K \) such that

\[ |T_{K,\mu,(\varepsilon)}f(z) - T_{K,\mu,\varepsilon}f(z)| \leq A M_\mu f(z) \text{ for all } z \in \mathbb{C} \text{ and } \varepsilon > 0. \] (4.2)
By the $L^p(\mu)$ boundedness of $M_\mu$ for $1 < p < \infty$, this implies that $T_{K,\mu}$ is bounded in $L^p(\mu)$ if and only if the operators $T_{K,\mu,(\varepsilon)}$ are bounded in $L^p(\mu)$ uniformly on $\varepsilon > 0$ (under the linear growth assumption for $\mu$).

When $K$ is the Cauchy kernel, that is, $K(z) = \frac{1}{z}$, we have that $T_K$ is the Cauchy integral operator (or Cauchy transform) and we denote $T_K = C$ and $T_{K,\mu} = C_\mu$. Notice also that the kernels $K_1$ and $K_2$ defined above satisfy (4.1).

Given three pairwise distinct points $z, w, \xi \in \mathbb{C}$, we denote by $R(z, w, \xi)$ the radius of the circumference passing through $z, w, \xi$, with $R(z, w, \xi) = \infty$ if these points are aligned. Their Menger curvature is $c(z, w, \xi) = \frac{1}{R(z, w, \xi)}$. If two among the points coincide or the points are aligned, we write $c(z, w, \xi) = 0$. The curvature of the measure $\mu$ is defined by

$$c^2(\mu) = \int\int\int c(z, w, \xi)^2 \, d\mu(z) \, d\mu(w) \, d\mu(\xi).$$

This notion was introduced by Mark Melnikov in [17] while studying a discrete version of analytic capacity.

For a given compact set $E \subset \mathbb{C}$, let $\Sigma(E)$ be the set of Borel measures supported on $E$ such that

$$\mu(B(z, r)) \leq r \quad \text{for all } z \in \mathbb{C}, \ r > 0.$$  

Also, let $\Sigma_0(E)$ be the set of Borel measures $\mu \in \Sigma(E)$ such that

$$\lim_{r \to 0} \frac{\mu(B(z, r))}{r} = 0 \quad \text{for all } x \in \text{Supp} \, \mu.$$  

In [9] it was shown

$$\gamma(E) \asymp \sup \{ \mu(E) : \mu \in \Sigma(E), \| C_\mu \|_{L^2(\mu) \to L^2(\mu)} \leq 1 \}$$

and in [8],

$$\alpha(E) \asymp \sup \{ \mu(E) : \mu \in \Sigma_0(E), \| C_\mu \|_{L^2(\mu) \to L^2(\mu)} \leq 1 \}$$

and

$$c_2(\mu) \leq \mu(E) \}.$$

Another result that will be needed for the proofs of $\gamma_1 \asymp \gamma$ and $\alpha_1 \asymp \alpha$ is the following:

**Theorem 4.1** [18] Let $\mu$ be a locally finite Borel measure without point masses in $\mathbb{C}$. If the Cauchy transform $C_\mu$ is bounded in $L^2(\mu)$, then any singular integral operator $T_{K,\mu}$ associated with an odd kernel $K \in C^\infty(\mathbb{C} \setminus \{0\})$ satisfying

$$|z|^{1+j} |\nabla^j K(z)| \in L^\infty(\mathbb{C}) \quad \text{for all } z \neq 0 \text{ and } j = 0, 1, 2, \ldots$$
is also bounded on $L^2(\mu)$. Further, the norm of the operator $T_{K,\mu}$ in $L^2(\mu)$ is bounded by some constant depending only on the one of $C_{\mu}$ as an operator in $L^2(\mu)$ and on the numbers $\sup_{z \neq 0} |z|^{1+j} |\nabla^j K(z)|$, $j = 0, 1, 2 \ldots$.

4.2 Proof of $\gamma \gtrsim \gamma_{12}$ and $\alpha \gtrsim \alpha_{12}$

Let $E \subset \mathbb{C}$ be compact. By the definition of $\gamma_{12}$, there exists a distribution $T$ supported on $E$ such that $\|K_s * T\|_{L^\infty(\mathbb{C})} \leq 1$ for $s = 1, 2$ and $\gamma_{12}(E) \leq 2|\langle T, 1 \rangle|$, with $K_1, K_2$ as above. In particular, we have $\|K_2 * T\|_{L^\infty(\mathbb{C})} \leq 1$.

Consider the non-degenerate linear map in $\mathbb{C}$ defined by $\Lambda_1(z) = z^2$. Let $\Lambda_1 T$ be the push-forward distribution defined by

$$\langle \Lambda_1 T, \varphi \rangle = \langle T, \varphi \circ \Lambda_1 \rangle \quad \text{for all } \varphi \in C^\infty(\mathbb{C}).$$

Notice that $\Lambda_1 T$ is supported on $\Lambda_1(E)$, and for the kernel

$$\tilde{K}_2(z) = K_2 \left( \Lambda^{-1}(z) \right) \quad \text{for all } z \in \mathbb{C} \setminus \{0\}$$

it is easy to check that

$$\left((\Lambda_1 T) * \tilde{K}_2\right)(z) = (T * K_2)(\Lambda^{-1}(z)).$$

Observe that $\tilde{K}_2(z)$ coincides with the Cauchy kernel $\frac{1}{z}$ and thus, by the definition of analytic capacity,

$$\gamma(\Lambda(E)) \geq |\langle (\Lambda_1 T), 1 \rangle| = |\langle T, 1 \rangle| \geq 2^{-1} \gamma_{12}(E).$$

Now we could use the fact that, by [19] for any bilipschitz mapping $f : \mathbb{C} \to \mathbb{C}$ one has $\gamma(f(F)) \asymp \gamma(F)$ for any compact set $F$, with the comparability constant depending just on the bilipschitz constant, and so

$$\gamma(\Lambda(E)) \asymp \gamma(E),$$

concluding the proof of $\gamma \gtrsim \gamma_{12}$.

An alternative argument which exploits the fact that $\Lambda$ is a linear map is the following: by (4.3) we know that there exists some measure $\mu \in \Sigma(\Lambda(E))$ such that $\mu(\Lambda(E)) \asymp \gamma(\Lambda(E))$ and $c_2^2(\mu) \leq \mu(\Lambda(E))$. From the fact that

$$R(z, w, \xi) \asymp R(\Lambda(z), \Lambda(w), \Lambda(\xi)) \quad \text{for all } z, w, \xi \in \mathbb{C},$$

with the comparability constant depending just on $\Lambda$, we infer that the push-forward measure $\sigma = (\Lambda^{-1})_\sharp \mu$ satisfies $c_2^2(\sigma) \asymp c_2^2(\mu)$. Further, it is also easy to check that

$$\sigma(B(z, r)) \leq A_1 r \quad \text{for all } z \in \mathbb{C}.$$
Thus, for a suitable constant $c_0 > 0$ depending just on $\Lambda$, $c_0\sigma \in \Sigma(E)$ and $c^2(c_0\sigma) \leq c_0\sigma(E)$. Hence applying again (4.3), we derive

$$\gamma(E) \geq \sigma(E) = \mu(E) \geq \gamma(\Lambda(E)),$$

which together with (4.5) yields $\gamma(E) \geq \gamma_12(E)$.

The proof of the fact that $\alpha \geq \alpha_12$ is almost the same. The only required change is that above we have to require the function $K_s^* T$ to be continuous, which in turn implies that $(\Lambda_2 T) * \tilde{K}_2$ is continuous, and thus (4.5) holds with $\gamma$ and $\gamma_12$ replaced by $\alpha$ and $\alpha_12$, respectively. Then one concludes in the same way applying either the fact that $\alpha(\Lambda(E))$ is comparable to $\alpha(E)$ because $\Lambda$ is bilipschitz (by applying [19] to $\alpha$), or by following the last alternative argument, using (4.4) instead of (4.3).

**4.3 Proof of $\gamma \succeq \gamma_12$**

We need some auxiliary lemmas. The first one is based on some work which goes back to Davie and Oksendal, and its proof can be found with minor modifications in [20] (see Lemma 4.2 there and the definitions of the standard notations $\mathcal{M}(X), C_0(X), T_j^*$ just before it).

**Lemma 4.2** Let $\mu$ be a Radon measure on a locally compact Hausdorff space $X$ and let $T_j : \mathcal{M}(X) \to C_0(X), \quad j = 1, \ldots, d$ be linear bounded operators. Suppose that each transpose $T_j^* : \mathcal{M}(X) \to C_0(X)$ is bounded from $\mathcal{M}(X)$ to $L^{1, \infty}(\mu)$, that is to say that there exists a constant $A$ such that

$$\mu\{x : |T_j^* \nu(x)| > \lambda\} \leq C \frac{\parallel \nu \parallel}{\lambda}$$

for $j = 1, \ldots, d$, $\lambda > 0$ and $\nu \in \mathcal{M}(X)$. Then, for each $\tau > 0$ and any Borel set $E \subset X$ with $0 < \mu(E) < \infty$, there exists $h : X \to [0, 1]$ in $L^\infty(\mu)$ satisfying $h(x) = 0$ for $x \in X \setminus E$,

$$\int_E h \, d\mu \geq \frac{1}{1 + \tau} \mu(E)$$

and

$$\parallel T_j(h \mu) \parallel_\infty \leq A(C, \tau, d), \quad \text{for} \quad j = 1, \ldots, d.$$

From this lemma we get the following.

**Lemma 4.3** Let $\mu$ be a measure in $\mathbb{C}$ with compact support which has $A_0$-linear growth. For $j = 1, \ldots, d$, let $K^j$ be a kernel satisfying the conditions in (4.1), and denote $T^j = T^j_{K^j, \mu}, \quad T^j_{(\varepsilon)} = T^j_{K^j, \mu, (\varepsilon)}$. Suppose that $T^j$ is bounded in $L^2(\mu)$, with norm at most $C$. Then for each $\tau > 0$ there exists some function $h : \text{Supp} \mu \to [0, 1]$
such that
\[ \int h \, d\mu \geq \frac{1}{1 + \tau} \|\mu\|, \]
\[ \|T^i h\|_{L^\infty(C)} \leq A(A_0, C, \tau, d) \quad \text{for } i = 1, \ldots, d, \]
and
\[ \|T^j_{(\varepsilon)} h\|_{L^\infty(C)} \leq A(A_0, C, \tau, d) \quad \text{for } i = 1, \ldots, d \text{ and all } \varepsilon > 0. \]

**Proof** The arguments are very standard and we just sketch them. Since \( T^j \equiv T_{K^j,\mu} \) is bounded in \( L^2(\mu) \), then \( T_{K^j} \) (and its transpose) is bounded from \( M(C) \) into \( L^{1,\infty}(\mu) \) (see for example [21, Chapter 2]). Then we apply Lemma (4.2) to each smoothly truncated operator \( T^j_{(\varepsilon)} \) and we deduce the existence of a function \( h_\varepsilon : \text{Supp}\mu \to [0, 1] \) such that
\[ \int h_\varepsilon \, d\mu \geq \frac{1}{1 + \tau} \|\mu\| \]
and
\[ \|T^j_{(\varepsilon)} h\|_{L^\infty(C)} \leq A(A_0, C, \tau) \quad \text{for } j = 1, \ldots, d. \]

By a compactness argument in weak \( L^\infty(C) \) we deduce the existence of a single function \( h \) fulfilling the properties of the lemma.

We are ready to prove that \( \gamma \lesssim \gamma_{12} \) now. Let \( E \subset \mathbb{C} \) be compact. By (4.3) there exists a measure \( \mu \in \Sigma(E) \) such that \( \|C_\mu\|_{L^2(\mu)\to L^2(\mu)} \leq 1 \) and and \( \gamma(E) \asymp \mu(E) \). By Theorem 4.1, \( T_{K_1,\mu} \) and \( T_{K_2,\mu} \) are bounded in \( L^2(\mu) \). By (non-homogeneous) Calderón-Zygmund theory, then the operators \( T_{K^j} \) are bounded from \( M(C) \) to \( L^{1,\infty}(\mu) \) for \( s = 1, 2 \) (see [21, Chapter 2], for example). Then by Lemma 4.3 there exists some function \( h : E \to [0, 1] \) such that
\[ \int h \, d\mu \geq \frac{1}{2} \|\mu\| \]
and
\[ \|T_{K^{s,\mu}} h\|_{L^\infty(C)} \leq A \quad \text{for } s = 1, 2. \]

As a consequence, from the definition of \( \gamma_{12} \) we deduce that
\[ \gamma_{12}(E) \gtrsim \int h \, d\mu \gtrsim \mu(E) \gtrsim \gamma(E), \]
which completes the proof of \( \gamma_{12} \gtrsim \gamma \). \( \square \)
4.4 Proof of $\alpha \lesssim \alpha_{12}$

Let $E \subset \mathbb{C}$ be compact. By (4.4) there exists a measure $\mu \in \Sigma_0(E)$ such that $\|C_\mu\|_{L^2(\mu)} \leq 1$ and and $\alpha(E) \asymp \mu(E)$. Again by Theorem 4.1, we know that $T_{K_1, \mu}$ and $T_{K_2, \mu}$ are bounded in $L^2(\mu)$. Our next objective is to find some function $h : E \to [0, 1]$, supported on $E$, such that $\int h \, d\mu \geq c \mu(E)$ (with $c = c(\mathcal{L}) > 0$),

$$\|T_{K_s, \mu} h\|_{L^1(\mathbb{C})} \leq 1 \quad \text{for } s = 1, 2 \text{ and all } \varepsilon > 0, \quad (4.7)$$

$$\|T_{K_s} h\|_{L^1(\mathbb{C})} \leq 1 \quad \text{for } s = 1, 2, \quad (4.8)$$

and such that moreover both $T_{K_1, \mu} h$ and $T_{K_2, \mu} h$ can be extended continuously to the whole $\mathbb{C}$. Note that once we prove the existence of $h$ we are done, because from the definition of $\alpha_{12}$ we deduce that

$$\alpha_{12}(E) \gtrsim \int h \, d\mu \gtrsim \mu(E) \gtrsim \alpha(E),$$

as wished.

We will need a couple of additional auxiliary lemmas. The following result is proven (in more generality) in [22, Lemma 3]. The same result had been proved previously for the Cauchy transform in [8].

**Lemma 4.4** Let $\mu$ be a measure in $\mathbb{C}$ with compact support and linear growth and suppose that $\lim_{r \to 0} \frac{\mu(B(x, r))}{r} = 0$ for all $x \in \text{Supp}\mu$. Let $K$ be an odd kernel (i.e., $K(-z) = -K(z)$ for all $z \neq 0$) satisfying the conditions in (4.1). Suppose that $T_{K, \mu}$ is bounded in $L^2(\mu)$. Then, given $\delta > 0$ we can find $F \subset \text{Supp}\mu$ with $\mu(\mathbb{C}\setminus F) < \delta$ such that

(a) $\lim_{r \to 0} \frac{\mu(B(x, r) \cap F)}{r} = 0$ uniformly on $x \in \mathbb{C}$,

(b) $\lim_{r \to 0} \|T_{K, \mu} \|_{L^2(\mu|B(x, r) \cap F) \to L^2(\mu|B(x, r) \cap F)} = 0$, uniformly on $x \in \mathbb{C}$.

The next lemma is proven in [22, Lemma 8], in a more general context too.

**Lemma 4.5** Let $\mu$ be a measure in $\mathbb{C}$ with compact support and linear growth. Let $F = \text{Supp}\mu$ and, for $j = 1, \ldots, d$, let $K^j$ be a kernel satisfying the conditions in (4.1), and denote $T^j = T_{K^j, \mu}$, $T^j_{(\varepsilon)} = T_{K^j, \mu, (\varepsilon)}$ and $T^j_{(\ast)} = T_{K^j, \mu, (\ast)}$. Suppose that

(a) $\lim_{r \to 0} \frac{\mu(B(x, r))}{r} = 0$ uniformly on $x \in F$,

(b) $\lim_{r \to 0} \|T^j \|_{L^2(\mu|B(x, r)) \to L^2(\mu|B(x, r))} = 0$ uniformly on $x \in F$ for $j = 1, \ldots, d$.

Let $f$ be a bounded function supported on $F$ such that $\|T^j_{(\ast)} f\|_{L^\infty(\mathbb{C})} < \infty$ for all $j$, $\varepsilon > 0$. Then, given $0 < \tau \leq 1$, there exists $\delta > 0$ and a function $g$ supported on $F$ satisfying, for each $j = 1, \ldots, d$,

(i) $\int g \, d\mu = \int f \, d\mu$ and $0 \leq g \leq \|f\|_{L^\infty(\mu)} + \tau$,

(ii) $\|T^j_{(\ast)} g\|_{L^\infty(\mathbb{C})} \leq \|T^j_{(\ast)} f\|_{L^\infty(\mathbb{C})} + \tau$. 


On $C^1$-approximability of functions by solutions of second…

(iii) \[ |T^j_{(\varepsilon)} g(x) - T^j_{(\varepsilon)} g(y)| \leq \tau, \quad if \ |x - y| \leq \delta \text{ and } \varepsilon > 0, \]

(iv) and

\[ |T^j_{(\varepsilon)} g(x) - T^j_{(\varepsilon)} g(y)| \leq \sup_{\varepsilon' > 0} |T^j_{(\varepsilon')} f(x) - T^j_{(\varepsilon')} f(y)| + \tau, \quad \forall x, y \in \mathbb{C}, \varepsilon > 0. \]

Let us remark that, in fact, in [22, Lemma 8] the lemma above is stated for the “sharply” truncated operators $T^j_{(\varepsilon)}$, instead of smoothly truncated ones $T^j_{(\varepsilon)}$. However, the same proof also works for the operators $T^j_{(\varepsilon)}$. An analog of the lemma above for the Cauchy transform (with $d = 1$) with smooth truncations is proven in [8, Lemma 3.3].

Construction of $h$ We follow quite closely the arguments in [8, Lemma 3.4]. Let $\mu$ be as above, so that $\lim_{r \to 0} \mu(B(\varepsilon, r)) = 0$ for all $z \in \text{Supp} \mu$ and $T_1 = T_{K_1, \mu}$ and $T_2 = T_{K_2, \mu}$ are bounded in $L^2(\mu)$. By Lemma 4.4 there are subsets $F_j \subset E$ such that

\[ \lim_{r \to 0} \|T_{K_1, \mu} \|_{L^2(\mu; B(x, r) \cap F_j)} = 0 \quad \text{uniformly on } x \in \mathbb{C} \quad \text{and} \quad \|T_{K_1, \mu} \|_{L^2(\mu; B(x, r) \cap F_j)} = 0. \]

(4.9)

uniformly on $x \in \mathbb{C}$ for $j = 1, 2$, such that the set $E := \cap F_j$ satisfies $\mu(F) \geq \mu(E)/2$.

By Lemma 4.3 there exists function $h_1$ supported on $F$, with $0 \leq h_1 \leq 1$, $\|T_{j,(\varepsilon)} h_1\|_{L^\infty(\mathbb{C})} \leq 1$ for $j = 1, 2$, and $\int h_1 \, d\mu \geq A^{-1} \mu(F)$. We set $\delta_1 = 1$.

For $n \geq 1$, we set $\tau_n = 2^{-n}$, and given a positive bounded function $h_n$ supported on $F$ and $\delta_n > 0$, by means of Lemma 4.5 we construct a function $h_{n+1}$ also supported on $F$, so that $\int h_{n+1} \, d\mu = \int h_n \, d\mu$, $0 \leq h_{n+1} \leq \|h_n\|_{L^\infty(\mu)} + \tau_n$, such that for $j = 1, 2$,

\[ \|T_{j,(\varepsilon)} h_{n+1}\|_{L^\infty(\mathbb{C})} \leq \|T_{j,(\varepsilon)} h_n\|_{L^\infty(\mathbb{C})} + \tau_n, \quad \text{and moreover} \]

\[ |T_{j,(\varepsilon)} h_{n+1}(x) - T_{j,(\varepsilon)} h_{n+1}(y)| \leq \tau_n \quad \text{for } |x - y| \leq \delta_{n+1} \text{ and all } \varepsilon > 0 \quad (4.10) \]

(4.10)

(where $\delta_{n+1} \leq \delta_n$ is some constant small enough), and

\[ |T_{j,(\varepsilon)} h_{n+1}(x) - T_{j,(\varepsilon)} h_{n+1}(y)| \leq \sup_{\varepsilon' > 0} |T_{j,(\varepsilon')} h_n(x) - T_{j,(\varepsilon')} h_n(y)| + \tau_n \quad (4.11) \]

(4.11)

for all $x, y \in \mathbb{C}, \varepsilon > 0$.

Let $h$ be a weak * limit in $L^\infty(\mu)$ of a subsequence $\{h_{n_k}\}$. Clearly, $h$ is a positive bounded function such that

\[ \int h \, d\mu = \int h_1 \, d\mu \gtrsim \mu(F) \gtrsim \alpha(E) \quad (4.12) \]

(4.12)

Also, for each $\varepsilon > 0$ and $x \in \mathbb{C},$

\[ T_{j,(\varepsilon)} h_{n_k}(x) \to T_{j,(\varepsilon)} h(x) \quad \text{as } k \to \infty. \]
Since
\[ \|T_{j,(\varepsilon)} h_n\|_{L^\infty(\mathbb{C})} \leq \|T_{j,(\ast)} h_1\|_{L^\infty(\mathbb{C})} + \sum_{i=1}^{n-1} 2^{-i} \leq 2 \]

for all \( n \), we deduce \( \|T_{j,(\varepsilon)} h\|_{L^\infty(\mathbb{C})} \leq 2 \), which implies that
\[ \|T_{j,\varepsilon} h\|_{L^\infty(\mathbb{C})} \lesssim 1 \quad \text{for all } \varepsilon > 0, \quad (4.13) \]
by (4.2).

On the other hand, by (4.10), if \( |x - y| \leq \delta_n \), then
\[ |T_{j,(\varepsilon)} h_n(x) - T_{j,(\varepsilon)} h_n(y)| \leq 2^{-n} \]
for all \( \varepsilon > 0 \). From (4.11), for \( k \geq n \) we get
\[ |T_{j,(\varepsilon)} h_k(x) - T_{j,(\varepsilon)} h_k(y)| \leq \sup_{\varepsilon' > 0} |T_{j,(\varepsilon')} h_n(x) - T_{j,(\varepsilon')} h_n(y)| + \sum_{i=n}^{k-1} 2^{-i} \leq 2^{-n+2}, \]
assuming \( |x - y| \leq \delta_n \). Thus,
\[ |T_{j,(\varepsilon)} h(x) - T_{j,(\varepsilon)} h(y)| \leq 2^{-n+2} \quad \text{if } |x - y| \leq \delta_n. \quad (4.14) \]

Consider now the family of functions \( \{T_{j,(\varepsilon)} h\}_{\varepsilon > 0} \) on \( \bar{B}(0, R) \), where \( R \) is big enough so that \( E \subset B(0, \bar{R} - 1) \). This is a family of functions which is uniformly bounded and equicontinuous on \( \bar{B}(0, R) \), by (4.14). By the Ascoli-Arzelà theorem, there exists a sequence \( \{\varepsilon_n\}_n \), with \( \varepsilon_n \to 0 \), such that \( T_{j,(\varepsilon_n)} h \) converges uniformly on \( \bar{B}(0, R) \) to some continuous function \( g_j \). It is easily seen that \( g_j \) coincides with \( T_j h \) \( L^2 \)-a.e. in \( B(0, R) \) and \( \|g_j\|_{L^\infty(\bar{B}(0,R))} \leq 2 \). By continuity, it is clear then that \( g_j \) and \( T_j \mu \) coincide on \( \bar{B}(0, R) \) \( \setminus \) \( E \). Since \( T_j h \) is also continuous in \( \mathbb{C} \setminus \bar{B}(0, R - \frac{1}{2}) \), we deduce that the function which equals \( g_j \) on \( E \) and \( T_j h \) in the complement of \( E \) is continuous in the whole complex plane, as wished. Together with (4.12) and (4.13), this shows that \( c_0 h \) satisfies the required properties stated at the beginning of this section, for some constant \( c_0 > 0 \) depending at most on \( K_1 \) and \( K_2 \). \( \square \)

5 The \( C^1 \)-approximation criteria for classes of functions and the \( C^1 \)-approximation by \( L \)-polynomials

It is worth mentioning (see, for instance, [1, Theorem 1.12] and its proof) that the zero sets for capacity \( \alpha_1 \) (or \( \alpha \)) are precisely the sets of \( C^1 \)-removable singularities for solutions of the equation \( \mathcal{L} u = 0 \).

Standard arguments (see, for instance, [13, Proof of Theorem 6.1]) allow to deduce from Theorem 1.2 or Theorem 3.1 the following \( C^1 \)-approximation criterion for “classes of functions”.
Theorem 5.1 For a compact set $X$ in $\mathbb{C}$ the following conditions are equivalent:

(a) $A^1_L(X) = C^1_L(X)$;
(b) $\alpha_1(D \setminus X^0) = \alpha_1(D \setminus X)$ for any bounded open set $D$;
(c) there exist $A > 0$ and $k \geq 1$ such that

$$\alpha_1(B(a, \delta) \setminus X^0) \leq A\alpha_1(B(a, k\delta) \setminus X)$$

for each disc $B(a, \delta)$.

From the last theorem, Theorem 1.1 and Vitushkin’s criteria for uniform rational approximations (see [6, Ch. V, §2–3] or [14, §1], including the definitions of $R(X)$ and $A(X)$) the next corollary follows directly.

Corollary 1 For a compact set $X$ in $\mathbb{C}$ the following conditions are equivalent:

(a) $A^1_L(X) = C^1_L(X)$;
(b) $R(X) = A(X)$.

Applying [8, Theorem 1.3] and the previous result, we obtain also the following corollary.

Corollary 2 Let $X$ be a compact set in $\mathbb{C}$ with inner boundary $\partial_i X$. If $\alpha(\partial_i X) = 0$ then $A^1_L(X) = C^1_L(X)$.

For a compact set $X$ in $\mathbb{C}$ and a function $f$ of class $C^1$ in some neighbourhood of $X$, define the $C^1$-Whitney norm of $f$ on $X$ [23]:

$$||f||_{1X} = \inf\{||F||_1 : F \in BC^1(\mathbb{C}), F|_X = f|_X, \nabla F|_X = \nabla f|_X\}.$$

Denote by $\mathcal{P}_L$ the space of all polynomials $p$ of real variables, such that $\mathcal{L}p \equiv 0$ (see [12, Proposition 2.1]). The following analog of the well known Mergelyan theorem [24] is a direct corollary of Theorems 1.1 and 5.1, the fact that $\alpha_1(D) \asymp \text{diam}(D)$ for any domain $D$ (because the same holds for the capacity $\alpha$), and Runge-type theorems (see [25, Theorem 3 and Proposition 2]).

Theorem 5.2 For a compact set $X$ in $\mathbb{C}$ the following conditions are equivalent:

(a) for each $f \in C^1_L(X)$ and $\varepsilon > 0$ there is $p \in \mathcal{P}_L$ with $||f - p||_{1X} < \varepsilon$;
(b) $\mathbb{C} \setminus X$ is connected.

This result strengthens [12, Theorem 1.1 (4)], since the norm $|| \cdot ||_{1X}$ is stronger than the norm

$$||f||_{1wX} = \max\{||f||_X, ||\nabla f||_X\},$$

considered in [12].

As a plan for our subsequent work in this themes we formulate the following conjecture.
Conjecture 1  Theorems 1.1 and 1.2 have their direct analogs for all dimensions (in $\mathbb{R}^N$ for all $N \in \{3, 4, \ldots\}$). In analog of Theorem 1.1 we just have to take, instead of the capacities $\alpha$ and $\gamma$, the $C^1$- and $\text{Lip}_1$-harmonic capacities respectively (see [22]). In analog of Theorem 1.2 we define $\mathcal{O}_B^L(f)$ as in [3].

Compliance with ethical standards

Conflict of interest  The authors declare that they have no conflict of interest.

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