An upper bound on Euclidean embeddings of rigid graphs with 8 vertices

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Abstract

A graph is called (generically) rigid in $\mathbb{R}^d$ if, for any choice of sufficiently generic edge lengths, it can be embedded in $\mathbb{R}^d$ in a finite number of distinct ways, modulo rigid transformations. Here, we deal with the problem of determining the maximum number of planar Euclidean embeddings of minimally rigid graphs with 8 vertices, because this is the smallest unknown case in the plane. Until now, the best known upper bound was 128. Our result is an upper bound of 116 (notice that the best known lower bound is 112), and we show it is achieved by exactly two such graphs. We conjecture that the bound of 116 is tight.

1 Definitions

Given a graph $G = (V, E)$ with $|V| = n$ and a collection of edge lengths $d_{ij} \in \mathbb{R}^+$, a Euclidean embedding of $G$ in $\mathbb{R}^d$ is a mapping of its vertices to a set of points $p_1, \ldots, p_n$, such that $d_{ij} = \|p_i - p_j\|$, for all $\{i, j\} \in E$. We call a graph (generically) rigid in $\mathbb{R}^d$ iff, for generic edge lengths, it can be embedded in $\mathbb{R}^d$ in a finite number of ways, modulo rigid transformations (translations and rotations). A graph is minimally rigid in $\mathbb{R}^d$ iff it is no longer rigid once any edge is removed. The study of rigid graphs is motivated by numerous applications, mostly in robotics, mechanism and linkage theory, structural bioinformatics, and architecture. The main goal is to determine the maximum number of distinct planar Euclidean embeddings of minimally rigid graphs, up to rigid transformations, as a function of the number of vertices. We shall focus on the planar case.
2 Existing work

A general upper bound on the number of embeddings, for the planar case, is \((2n-4) \approx \frac{4^{n-2}}{\sqrt{\pi(n-2)}}\). The bound was obtained by exploiting results from complex algebraic geometry that bound the degree of (symmetric) determinantal varieties defined by distance matrices \([1, 2, 8]\). In \([11]\), mixed volumes (cf. Section 1.2) also yield an upper bound of \(4^{n-2}\), for the planar case.

In the table below we can see the lower and upper bounds we know \([5, 6, 7]\) for the maximum number of embeddings of rigid graphs with up to \(n = 10\) vertices. The most recent tight bound is for \(n = 7\) \([5]\).

| Number of vertices | Lower Bounds | Upper Bounds |
|--------------------|--------------|--------------|
| 3                  | 2            | 2            |
| 4                  | 4            | 4            |
| 5                  | 8            | 8            |
| 6                  | 24           | 24           |
| 7                  | 56           | 56           |
| 8                  | 112          | 128          |
| 9                  | 288          | 512          |
| 10                 | 576          | 2048         |

Up to \(n = 7\) we have exact values. Our result is to prove an upper bound of 116 to the number of planar embeddings of minimally rigid graphs with 8 vertices in the plane. We conjecture that the bound of 116 is tight.

3 Laman graphs and Henneberg steps

We will use a combinatorial characterization of rigidity in \(\mathbb{R}^2\), which is based on Laman graphs \([9, 10]\). A graph \(G = (V, E)\) with \(|V| = n\) is called Laman iff \(|E| = 2n - 3\) and all of its induced subgraphs on \(k < n\) vertices have \(\leq 2k - 3\) edges. It is a fundamental theorem (Maxwell 1864 - Laman 1970) that a graph is minimally rigid in \(\mathbb{R}^2\) iff it is a Laman graph. Moreover, we know that Laman graphs admit inductive constructions that begin with a triangle, followed by a sequence of so-called Henneberg steps (Henneberg-2 constructions).
Laman if it has a Henneberg-2 construction. Each Henneberg step adds to the graph a new vertex and a total number of two edges. A Henneberg-1 (or H1) step connects the new vertex to two existing vertices.

A Henneberg-2 (or H2) step connects the new vertex to three existing vertices having at least one edge among them, and this edge is removed.

A Laman graph is called H1 if it can be constructed using only H1 steps and it is called H2 otherwise.

Using Henneberg steps we can construct all rigid graphs with 8 vertices and classify them up to isomorphism.

By doing this, we have the following results

| Number of vertices | Number of H1-graphs | Number of H2-graphs |
|--------------------|---------------------|---------------------|
| 3                  | 1                   | 0                   |
| 4                  | 1                   | 0                   |
| 5                  | 3                   | 0                   |
| 6                  | 11                  | 2                   |
| 7                  | 61                  | 9                   |
| 8                  | 499                 | 109                 |

For example, these are the 9 H2-graphs with 7 vertices.
The second one has 56 embeddings (the upper bound for \( n = 7 \)).

We see that, in total, there are 608 rigid graphs with 8 vertices and from them, the 499 are H1-graphs. Since two circles intersect generically in two points, a H1-step at most doubles the number of embeddings and this is tight. It follows that a H1-graph on \( n \) vertices has \( 2^{n-2} \) embeddings. Therefore, the 499 graphs have \( 2^6 = 64 \) embeddings, which is smaller than the lower bound we have (112).

From the remaining 109 graphs, 77 have a vertex of degree 2, which means that they have at most \( 2 \cdot 56 = 112 \) embeddings, because 56 is the maximum number of embeddings of a graph with 7 vertices.

Therefore, there are only 32 graphs to consider.
Lemma. The five marked graphs (in the above drawing with some bold edges) have at most 64 embeddings.

Proof. The marked graphs can be constructed by combining two smaller rigid graphs. More specifically, we can take the rigid graph with 4 vertices and combine it with a rigid graph with 5 vertices in a way that they have a common vertex, and we add one more edge so that the whole construction is rigid. The resulting graph is isomorphic with one of the above marked graphs and moreover, it has at most $4 \cdot 8 \cdot 2 = 64$ embeddings, because there are 4 embeddings for the rigid graph with 4 vertices, 8 embeddings for the graph with 5 vertices and the additional edge allows only 2 embeddings for each combination: We consider a cycle (blue cycle in the figure below) with center a vertex of the rigid subgraph with 4 vertices and we have to count its intersections with a specific vertex (red vertex) of the second rigid subgraph with 5 vertices. There are 2 such intersections because the latter vertex moves on a circle (orange circle) centered at the common vertex of the two rigid subgraphs (see the figure below).

Therefore, we focus on the remaining 27 graphs.

4 Calley-Menger matrices

We can associate every graph with a Cayley-Menger matrix [3], which is a matrix like the following:

\[
\begin{bmatrix}
0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 0 & c1 & x3 & x4 & c5 & c6 & x10 \\
1 & c2 & 0 & c3 & c4 & x12 & x13 & x26 \\
1 & x13 & c23 & 0 & x34 & c35 & c36 & x37 & x38 \\
1 & x34 & c24 & x34 & 0 & c45 & c46 & x46 & x48 \\
1 & c54 & x56 & c56 & c56 & 0 & x56 & x57 & x58 \\
1 & c65 & x66 & x66 & c67 & x68 & 0 & c67 & x68 \\
1 & x77 & x77 & x78 & c78 & x78 & 0 & c78 & x78 \\
1 & x8 & x8 & x8 & x8 & x8 & x8 & 0 & x8
\end{bmatrix}
\]
where $c_{ij}$ correspond to the fixed distances and $x_{ij}$ are the unspecified distances.

For every graph we construct the corresponding Cayley-Menger matrix. We know that the number of embeddings of a graph is equal to the number of completions of the above matrix that have rank 4. This means that all the $5 \times 5$ minors that correspond to Cayley-Menger sub-matrices vanish, therefore we have a polynomial system with \( \binom{8}{4} = 70 \) equations and \( \binom{8}{2} - (2 \cdot 8 - 3) = 15 \) unknowns.

For every system (one for each graph), we will find a square subsystem with finite number of roots and for which if the unknown lengths are fixed, they uniquely define the configuration of the overall graph; this is the approach in [7]. We now introduce a powerful algebraic tool, namely the mixed volume of a well-constrained polynomial system; see, e.g., [4] for definitions and an efficient algorithm for its computation. The mixed volume of the square subsystem of minors is an upper bound on the number of its complex roots, hence it bounds the number of embeddings of the graph.

More specifically, this bound does not take into account solutions with zero coordinates, since it counts toric complex roots. However, a graph has embeddings with some zero length only when the input bar lengths form a singular set, in the sense that they would satisfy a non-generic algebraic dependency. For example, by letting some input distance be exactly 0, some graph may theoretically have infinitely many configurations. However, generically, it is impossible to have such an embedding, as discussed in [6].

For the above matrix, the following $6 \times 6$ subsystem has a finite number of solutions, mixed volume equal to 116, and it uniquely defines the configuration of the overall graph. Let $D[X]$ denote the principal minor of the matrix indexed by the rows/columns in $X$ and the first row/column of ones.

\[
\begin{align*}
D(2, 3, 4, 6)(x_{13}, x_{25}) &= 0 \\
D(2, 3, 4, 7)(x_{13}, x_{26}) &= 0 \\
D(2, 4, 6, 7)(x_{13}, x_{56}) &= 0 \\
D(3, 5, 6, 9)(x_{25}, x_{48}) &= 0 \\
D(5, 6, 8, 9)(x_{48}, x_{57}) &= 0 \\
D(2, 3, 6, 7)(x_{25}, x_{26}, x_{56}) &= 0
\end{align*}
\]

Therefore, this graph has at most 116 embeddings (this is the first circled graph in the image below).

If we do this for every graph we have the following results (numbers up-left of each drawing):
We see that there are only two graphs which exceed the lower bound we know (112), and they have at most 116 embeddings. This proves that a rigid graph with 8 vertices can have at most 116 embeddings (we also expect that this bound is tight).

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