Robustness of Greenberger–Horne–Zeilinger and W states against Dzyaloshinskii-Moriya interaction

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Abstract In this article, the robustness of tripartite Greenberger–Horne–Zeilinger (GHZ) and W states is investigated against Dzyaloshinskii-Moriya (i.e. DM) interaction. We consider a closed system of three qubits and an environmental qubit. The environmental qubit interacts with any one of the three qubits through DM interaction. The tripartite system is initially prepared in GHZ and W states, respectively. The composite four qubits system evolve with unitary dynamics. We detach the environmental qubit by tracing out from four qubits, and profound impact of DM interaction is studied on the initial entanglement of the system. As a result, we find that the bipartite partitions of W states suffer from entanglement sudden death (i.e. ESD), while tripartite entanglement does not. On the other hand, bipartite partitions and tripartite entanglement in GHZ states do not feel any influence of DM interaction. So, we find that GHZ states have robust character than W states. In this work, we consider generalised GHZ and W states, and three $\pi$ is used as an entanglement measure. This study can be useful in quantum information processing where unwanted DM interaction takes place.

Keywords Greenberger–Horne–Zeilinger · W state · Tripartite system · Robustness · Dzyaloshinskii-Moriya interaction · Three $\pi$ · Entanglement sudden death

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1 Introduction

Entanglement is a fascinating concept in quantum mechanics which has no classical description [1,2]. It is the phenomenon which gives thrust to new technological development in quantum information processing. The roots of entanglement come from the debate of Einstein and Schrödinger in the Era of 1935 [3,4]. This concept attracted the attention of quantum community when its first application as teleportation is proposed in a seminal paper by Bennett et al. [5], which has experimental verification also in many quantum systems [6]. Up to till date, entanglement has applications in almost all the streams such as in information security, game theory, image processing, quantum secrete sharing, superdense coding and quantum machine learning and others [7–13]. The problem starts with entanglement when entangled quantum systems interact with the environment, which may destroy it. If entanglement suddenly dies for a finite time, then this phenomenon is called entanglement sudden death (i.e. ESD), which is investigated by Yu and Eberly [14,15]. The quantum systems which suffer from ESD for a particular time of interval may not be useful for quantum applications. So, quantum community is always interested to search such systems which can persist long entanglement. By keeping in view of application part of quantum systems, it is important to investigate the situations for ESD in bipartite to multipartite cases. There are many literatures available on ESD in bipartite cases with Dzyaloshinskii-Moriya (i.e. DM) interaction [16–23]. Here, in the present paper, we investigate this phenomenon in bipartite and tripartite case with DM interaction. DM interaction has an important role in quantum information processing, which takes place between two qubits connected through a ligand in a crystal [24–26]. If all the three entities (i.e. two qubits and ligand) are collinear, then DM interaction that becomes zero. Here, we express the possibility of unwanted DM interaction may arise when a foreign qubit adjusts in the vacancy defect present in any crystal. Recently, it has been found that this interaction also play an important role to produce skyrmions [27–30], which are expected key candidates for future data storage devices. The impact of this interaction has been studied in varieties of quantum spin chains even with thermal conditions [31,32].

The motivation of this study come from our previous work done in bipartite cases [19–23]. In this present work, we consider a tripartite closed system of three qubits and an environmental qubit. The environmental qubit establishes DM interaction with any one of the qubit of the tripartite system and disturbs the entanglement. The tripartite system is prepared initially in generalised Greenberger–Horne–Zeilinger (GHZ) and W states, and environmental qubit is prepared in pure state [33–35]. Both the states have significant role in quantum information and always production of these in different physical systems, knowing their properties and dynamics are the subject of study in many situations [36–41]. In past few years, the tripartite ESD is studied in pure and mixed GHZ and W states in different paradigms [42,43]. To study the dynamics in tripartite system considered in the paper, we trace out the environmental qubit and investigate the disturbance of DM interaction and the state of environmental qubit on initial entanglement of the system. We find that the state of environmental qubit does not disturb the entanglement in the system irrespective of whether the system is prepared in GHZ or W states. It is only DM interaction which has periodic influence on entanglement. It periodically amplifies the entanglement and sometimes kills it in
many quantum states. DM interaction produces ESD only in bipartite partitions of W states, while tripartite entanglement does not suffer from ESD. On the contrary, neither the bipartite partitions nor the tripartite entanglement in GHZ states face ESD, because the present study reveals that these states are not fragile with respect to (w.r.t.) the environmental DM interaction. The outline of the paper is as follows.

In Sect. 2, we discuss the Hamiltonian of the system, time evolution, GHZ and W states. Section 3 presents the discussion on three $\pi$ measure for entanglement. Further, Sect. 4 is divided in two parts: In first part, we consider the case 1 with W states and in second part, case 2 is considered with GHZ states. Lastly, the conclusion is presented in Sect. 5.

## 2 Hamiltonian, time evolution, GHZ and W states

In this study, the goal is to investigate the disturbance of DM interaction by taking an environmental qubit on tripartite system. Let consider tripartite system formed by qubits A, B and C, which takes place in $2 \otimes 2 \otimes 2$ dimensional Hilbert space. The environmental qubit (named D) interacts with any one qubit (assumed qubit C) of the tripartite system through DM interaction. Here, we consider this interaction in z direction. The pictorial representation of the system is shown in Fig. 1. So, the Hamiltonian is framed as given below

$$H_z = D_z. (\sigma^X_C \otimes \sigma^Y_D - \sigma^Y_C \otimes \sigma^X_D),$$  \hspace{1cm} (1)

where $\sigma^X_C$ and $\sigma^Y_C$ are Pauli matrices for qubit (C) and $\sigma^X_D$, $\sigma^Y_D$ are Pauli matrices of qubit (D). Further, $D_z$ is the DM interaction strength in z direction. The above Hamiltonian is easily diagonalisable by using the method of eigendecomposition. Further, the unitary time evolution operator can be computed as

$$U(t) = e^{-iHt}. \hspace{1cm} (2)$$
The action of this operator will be used to obtain the time evolution of the tripartite system. By using this operator, the time evolution density matrix of the composite system formed by four qubits A, B, C and D can be obtained as below

$$\rho(t) = U(t)\rho(0)U(t)^\dagger,$$

where $\rho(0) = \rho_s(0) \otimes \rho_e(0)$. The factor $\rho_s(0)$ corresponds to the initial state of tripartite system and $\rho_e(0)$ is the initial density matrix of the environmental qubit. The environmental qubit is prepared in pure state given as

$$|\psi\rangle = c_0|0\rangle + c_1|1\rangle,$$

with

$$|c_0|^2 + |c_1|^2 = 1.$$

Here, we mention that the tripartite system initially prepared in generalised GHZ and W states, respectively. The generalised GHZ states for three qubits can be written as

$$|\psi_G\rangle = g_0|000\rangle + g_1|111\rangle,$$

with

$$|g_0|^2 + |g_1|^2 = 1.$$

These states mapped to bipartite disentangled states when any one qubit is lost, so does not carry pairwise entanglement and have only three-way entanglement. On the other hand, the W state have opposite property. The generalised W states are given below

$$|\psi_w\rangle = w_0|001\rangle + w_1|010\rangle + w_2|100\rangle,$$

with

$$|w_0|^2 + |w_1|^2 + |w_2|^2 = 1.$$

These states are mapped to bipartite entangled states when any one of the qubit is lost and carry pairwise entanglement. Here, first, we focus on tripartite entanglement in W states and later for entanglement in its bipartite partitions. To study the tripartite entanglement, one needs mathematical tool to measure it. Here, we use three $\pi$ measures for pure states [44,45], which is given in the next section which have connection with negativity [46–50]. Negativity is a good measure of entanglement in bipartite systems.

### 3 The entanglement measure (three $\pi$)

There are many measures for entanglement, but few of these do not fit to measure entanglement in some tripartite quantum states. For example, three tangle in terms of concurrence $C$ are able to measure the tripartite entanglement in GHZ state but fails in W states and in many other pure states. The concurrence $C$ in a density matrix $\rho$ is given by

$$C(\rho) = \max\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\},$$
where $\lambda_i, i = 1, 2, 3, 4$, are the square roots of the eigenvalues in decreasing order of $\rho \rho^*$, $\rho^*$ is the spin flip density matrix given as

$$\rho^* = (\sigma^y \otimes \sigma^y) \rho^* (\sigma^y \otimes \sigma^y).$$  \hfill (11)

Here, $\rho^*$ is the complex conjugate of the density matrix $\rho$ and $\sigma^y$ is the Pauli Y matrix. The matrix $\rho \rho^*$ is used to calculate the concurrence. This concurrence in tripartite systems is used to calculate the tangle, which is a tripartite entanglement measure. The three tangle inspired by CKW inequality \cite{51} in terms of concurrence is defined as

$$\tau_{ABC} = C_{A|BC}^2 - C_{AB}^2 - C_{AC}^2,$$  \hfill (12)

where $C_{A|BC}$ is the concurrence between qubit A and qubits B and C together. $C_{AB}$ is the concurrence between qubits A and B. $C_{AC}$ is the concurrence between qubits A and C. Here, the calculated expression of the three tangle for a generalised pure three qubit states $|\psi\rangle = \sum_{i,j,k} x_{i,j,k} |ijk\rangle$, is given as

$$\tau_{ABC}(|\psi\rangle) = 4|4a_3 - 2a_2 + a_1|.$$  \hfill (13)

where

$$a_3 = x_{000}x_{110}x_{101}x_{011} + x_{111}x_{001}x_{010}x_{100},$$  \hfill (14)

$$a_2 = x_{000}x_{111}x_{011}x_{100} + x_{000}x_{111}x_{101}x_{010}$$
$$+ x_{000}x_{111}x_{110}x_{001} + x_{011}x_{101}x_{010}x_{100},$$  \hfill (15)

$$a_1 = x_{010}x_{111}^2 + x_{011}x_{110}^2 + x_{010}x_{101}^2 + x_{100}x_{111}^2.$$  \hfill (16)

Calculating the quantity $\tau_{ABC}$ by using the (13) for the W states given in Eq. (8), we get $\tau_{ABC}(|\psi_w\rangle) = 0$. While for GHZ states given in Eq. (6), this quantity is obtained as $\tau_{ABC}(|\psi_G\rangle) = 4g_0g_1^2 > 0$. For the special case of GHZ state ($|\psi_G\rangle = \frac{1}{\sqrt{2}} |000\rangle + \frac{1}{\sqrt{2}} |111\rangle$), it is obtained as $\tau_{ABC}(|\psi_G\rangle) = 1 > 0$. So we recall here that three tangle becomes zero for W states and does not seem a good measure for tripartite entanglement in W states. So for our work, we use an entanglement measure called three $\pi$, which is greater than zero for W and GHZ states and capture the feature of measurement of entanglement in both the states. The three $\pi$ measure is defined as the average of three kinds of residual entanglement by concentrating on different nodal qubits in tripartite system. The idea of residual entanglement come from CKW like monogamy inequality for negativity \cite{51}. Let we concentrate on qubit A (i.e. nodal qubit A) in tripartite system as shown in Fig. 1, then the CKW monogamy inequality with negativity is given as

$$N_{A|BC}^2 \geq N_{AB}^2 + N_{AC}^2,$$  \hfill (17)

where the negativity is defined as “The absolute sum of all negative eigenvalues of the density matrix after the perse partial transpose is taken”.

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Further, evaluation of Eq. (17) with equality sign leads as

$$\pi_A = N_{A|BC}^2 - N_{AB}^2 - N_{AC}^2.$$ (18)

The term $\pi_A$ is called residual entanglement in tripartite system, when nodal qubit is considered as A. Similarly, when the nodal qubit is considered B and C, respectively, the terms of residual entanglement are given as

$$\pi_B = N_{B|AC}^2 - N_{BA}^2 - N_{BC}^2.$$ (19)

$$\pi_C = N_{C|AB}^2 - N_{CA}^2 - N_{CB}^2.$$ (20)

In the above equations, $N_{A|BC}$ is the negativity measure of entanglement of qubit A with qubits B and C together. In other words, the quantity expresses that how much qubit A is entangled with qubits B and C together. Similar meanings are carried out by the terms $N_{B|AC}$ and $N_{C|AB}$ when nodal qubits are considered B and C, respectively. Further, $N_{AB}$ is the bipartite negativity between partition AB of the tripartite system. And similarly, the meaning exists for $N_{BC}$ and $N_{CA}$ for bipartite partitions BC and CA. From the symmetry of entanglement, the following factors are equal in Eqs. (18), (19) and (20).

$$N_{AB} = N_{BA}.$$ (21)

$$N_{BC} = N_{CB}.$$ (22)

$$N_{AC} = N_{CA}.$$ (23)

In general, ($\pi_A \neq \pi_B \neq \pi_C$) implies that residual entanglement varies under the permutations of nodal qubits A, B and C in tripartite system. Finally, the three $\pi$ measure is given as the average of the terms $\pi_A$, $\pi_B$ and $\pi_C$. This average does not change under permutations of qubits A, B and C; it is given below.

$$\frac{1}{3}(\pi_A + \pi_B + \pi_C).$$ (24)

We have calculated three $\pi$ for generalised W states given in Eq. (8) which is obtained as

$$\frac{4}{3} \left[ w_2^2 \sqrt{w_2^4 + 4w_1^2w_0^2} + w_1^2 \sqrt{w_1^4 + 4w_2^2w_0^2} + w_0^2 \sqrt{w_0^4 + 4w_3^2w_1^2} - w_2^4 - w_1^4 - w_0^4 \right].$$ (25)

This factor achieves the values greater than zero for W states and has the ability to capture the measurement of entanglement in the states. For the special case of W states ($|\psi_w\rangle = \frac{1}{\sqrt{3}}|001\rangle + \frac{1}{\sqrt{3}}|010\rangle + \frac{1}{\sqrt{3}}|100\rangle$), this factor acquires the value $0.549364 > 0$. Here, we mention that three $\pi$ for GHZ states achieves the value 1. So it is concluded that the three $\pi$ is greater than zero for both the pure three qubit W and GHZ states. This measure also satisfies the properties which must be adopted by a good entanglement measure. These properties are given below.
(a) It must be zero for fully separable or bi-separable states and must acquire nonzero value for fully entangled states.

(b) It must be non-increasing under local operations and classical communication (LOCC).

(c) It must be invariant under local unitary operations (i.e. LU).

Here, we recall the quantity three $\pi$ given in Eq. (24) which is invariant under the permutations of nodal qubits A, B and C and also invariant under the three simultaneous LU transformations. This measure satisfies all the properties given in (a), (b) and (c). For lucid mathematical proof of the above properties, one may refer to Refs. [45,52].

4 Reducing the system and reckoning three $\pi$ measure

In this section, we find out the reduced density matrices for tripartite system. In Sect. 2, we framed the Hamiltonian of the system and time evolution density matrix for four qubits A, B, C and D as given in Eq. (3). The equation involves the initial density matrix of the system as $\rho(0)$. Here, we mention that first we prepare the system in generalised W states and later in GHZ states. So we divide the study into two cases.

4.1 Case 1: Initial preparation with W states

We first prepare the system in generalised W states as given in Eq. (8) and find out the corresponding density matrix $\rho_w(0)$. The initial density matrix is obtained as $(\rho(0) = \rho_w(0) \otimes \rho_e$, which is plugged in Eq. (3). Further, we detach the environmental qubit by taking the partial trace operation over the basis of environmental qubit D, and reduced density matrix $Rd_{ABC}$ is obtained as

$$Rd_{ABC} = [a_{ij}]_{8 \times 8}. \quad (26)$$

The matrix involves the terms $(a_{32}, a_{23}, a_{52}, a_{25}, a_{53}, a_{35}, a_{22}, a_{33}, a_{55})$, and rest of the terms are zero. The nonzero terms of the matrix are given below

$$a_{23} = a_{32} = (|c_0|^2 + |c_1|^2)\sqrt{1 - w_1^2 - w_2^2(w_2 \sin[2Dt] + w_1 \cos[2Dt])},$$

$$a_{25} = a_{52} = -(|c_0|^2 + |c_1|^2)\sqrt{1 - w_1^2 - w_2^2(w_1 \sin[2Dt] - w_2 \cos[2Dt])},$$

$$a_{35} = a_{53} = \frac{1}{2}(|c_0|^2 + |c_1|^2)(w_2^2 - w_1^2)(\sin[4Dt] + 2w_1w_2 \cos[4Dt]),$$

$$a_{22} = -(|c_0|^2 + |c_1|^2)(-1 + w_1^2 + w_2^2),$$

$$a_{33} = (|c_0|^2 + |c_1|^2)(w_1 \sin[2Dt] + w_2 \cos[2Dt])^2,$$

$$a_{55} = (|c_0|^2 + |c_1|^2)(w_1 \sin[2Dt] - w_2 \cos[2Dt])^2.$$

While calculating reduced density matrix, every term of the matrix involves the factor $|c_0|^2 + |c_1|^2$. So by applying the normalisation condition given in Eq. (5), the reduced density matrix is free from probability amplitude terms (i.e. $c_0, c_1$) of environmental

qubit D. Hence, the state of environmental qubit D does not have any impact on entanglement carried out by tripartite system. While on the other hand, the parameter of DM interaction strength \((D_z = D)\) is involved in the reduced density matrix. The probability amplitudes of W states \((w_0, w_1\) and \(w_2)\) also contribute in \(Rd_{ABC}\).

For simplifying the calculations, we have replaced the term \(w_0\) by \(\sqrt{1 - w_1^2 - w_2^2}\) in \(Rd_{ABC}\). Our goal is to reckon the entanglement measure (three \(\pi\)) given in (24), which demand to calculate the terms \(\pi_A, \pi_B\) and \(\phi_c\). First, we focus on \(\pi_A\) given in Eq. (18), which further needs the evaluation of the terms \(N_{A|BC}, N_{AB}\) and \(N_{AC}\). To obtain \(N_{A|BC}\), we take partial transpose of the matrix \(Rd_{ABC}\) w.r.t. qubit A. It is given as

\[
Rd_{ABC}^{TA} = \text{Tr}_A[Rd_{ABC}].
\]  

Similarly for \(N_{AB}\) and \(N_{AC}\), first, we obtain reduced density matrices \(R_{AB}\) and \(R_{AC}\), by tracing out the qubits C and B respectively. Further, the partial transposition is taken as

\[
R_{AB}^{TA} = \text{Tr}_A[Rd_{AB}], \quad (28)
\]

\[
R_{AC}^{TA} = \text{Tr}_A[Rd_{AC}]. \quad (29)
\]

The negativity is defined as the absolute sum of negative eigenvalues of the partially transposed matrix. To calculate the negativities \(N_{A|BC}, N_{AB}\) and \(N_{AC}\), we need the eigenvalues spectrum of partially transposed matrices, which helps to simulate the negativities. The eigenvalues spectrum of matrices \(Rd_{ABC}^{TA}, Rd_{AB}^{TA}\) and \(Rd_{AC}^{TA}\) are obtained as given below.

| Transposed matrix | Eigenvalue spectrum |
|-------------------|---------------------|
| \(Rd_{ABC}^{TA}\)  | \([0, 0, 0, x, (1-x), \sqrt{x(1-x)}, \sqrt{x(1-x)}]\) |
| \(Rd_{AB}^{TA}\)    | \([x, y, p, p]\)  |
| \(Rd_{AC}^{TA}\)    | \([x, 1 - w_1^2 - w_2^2, q, q]\) |

with

\[
x = (w_1 \sin[2Dt] - w_2 \cos[2Dt])^2, \quad y = (w_2 \sin[2Dt] + w_1 \cos[2Dt])^2
\]

\[
p = \frac{1}{4}[2 - 2w_1^2 - 2w_2^2 - \sqrt{2p_1}], \quad q = \left(\frac{1}{8} (4(e w_2 + f w_1)^2 - 2q_1)\right)
\]

where, \(p_1 = -4aw_2w_3^3 + 4aw_3^3w_1 - (b - 3)w_1^4 + w_1^2(6(b + 1)w_2^2 - 4) - (b - 3)w_2^4 - 4w_2^2 + 2\), \(q_1 = 4w_2w_3(a + 10c) + w_1^4(b + 20d - 13) - 2w_1^2(3b + 13)w_2^2 + 8(d - 1)\) + \(w_2^4(b - 20d - 13) - 8cw_2w_1((d - 5)w_2^2 + 4) + 32f^2w_2^2\) with

\[
a = \sin[8Dt], \quad b = \cos[8Dt], \quad c = \sin[4Dt]
\]

\[
d = \cos[4Dt], \quad e = \sin[2Dt], \quad f = \cos[2Dt].
\]

The eigenvalues spectrum depends on the parameters \(w_1, w_2\) and \(Dt\). By varying these parameters, the negativities \(N_{A|BC}, N_{AB}\) and \(N_{AC}\) are obtained, which are used
to calculate the term $\pi_A$ given in Eq. (18). By using the above similar procedure, we calculate the another negativities involved in the equations Eqs. (19) and (20), which are further used to evaluate the terms $\pi_B$ and $\pi_C$. Finally, the terms $\pi_A$, $\pi_B$ and $\pi_C$ lead to the calculation of three $\pi$ measure given in Eq. (24). We here recall that, three $\pi$ measure also involves the parameters $Dt$, $w_1$ and $w_2$. By varying these parameters, we have to obtain the results.

We plot the tripartite entanglement (i.e. three $\pi$) for generalised W states for different values of the parameter $Dt$ in Fig. 2. We mention that the parameter $Dt$ can be assumed as DM interaction strength with unit time of interval. In the absence of DM interaction with $Dt = 0$, we find that, the three $\pi$ achieves positive values for generalised W states. As the DM interaction strength increases, the amplitude of three $\pi$ oscillates, because three $\pi$ involves oscillatory terms $\sin[ Dt ]$ and $\cos[ Dt ]$, which can also be observed in eigenvalues spectrum of partially transposed matrices. To make the results more clear, we plot the two-dimensional plots as shown in Fig. 2 and Fig. 3. In subfigures of Fig. 3, we plot three $\pi$ with less range of amplitude to visualise sudden death results. In Fig. 3, DM interaction strength produces the periodic oscillations in three $\pi$. The entanglement falls to zero level and smoothly rises; it do not stay at zero level for finite time. So no tripartite ESD is found in W states.

In search of ESD produced by DM interaction in generalised W states, we also study the bipartite entanglement in three partitions of tripartite system ABC. These partitions are $AB$, $AC$ and $BC$. The negativities in these partitions $N_{AB}$, $N_{AC}$ and $N_{BC}$ are plotted with different values of the set of parameters ($Dt$, $w_2$) vs. $w_1$ in Figs. 4 and 5. The quantities $N_{AB}$, $N_{AC}$ and $N_{BC}$ are plotted by the dotted blue, dashed orange and red graphs respectively. Looking at sub figures of Figs. 4 and 5, we find that the entanglement in partition BC is strongly present with the range ($0 \leq Dt \leq 0.7$).

![Fig. 2 Plot of three $\pi$ (i.e. T-$\pi$) with different values of parameter $Dt$ (i.e. $Dt = 0.0, 0.2, 0.5, 0.7$)](image-url)
Fig. 3 Two-dimensional plots corresponding to Fig. 2, with different values of parameter $D_t$ and $w_2$.

Fig. 4 Plots of bipartite entanglement in generalised W states with different values of parameter $D_t$ and $w_2$. The quantities $N_{AB}$, $N_{AC}$ and $N_{BC}$ are represented with Dotted blue, Dashed orange and thick red graph, respectively.

While on the other hand, the entanglement in partition AC is very much fragile and suffers from ESD. The entanglement starts to die in partition AC at $w_1 = 0.4$ and $w_1 = 0.5$. However, the partition AB has initial entanglement zero, which sometimes overlap with the sudden death part of partition AC. This absence of entanglement in partitions AB and AC is filled by the entanglement present in partition BC, which maintains the total tripartite entanglement greater than zero in the system, because there is no tripartite sudden death of entanglement. Further in Fig. 5, we observed that both the partitions AB and AC simultaneously suffer from ESD at $w_1 = 0.45$ with $(D_t = 0.6, w_2 = 0.8)$, while this absence of entanglement is filled by the entanglement present in the partition BC. We have found the threshold value of the parameter $D_t$ after which the periodic behaviour of DM interaction starts. This value
is $Dt = 0.8$. At this threshold value, the entanglement in partition AC immediately rises and achieves the value 1. Again observing sub figures in Fig. 5, the ESD in partition BC takes place at $w_1 = 0.6$ with $(Dt = 0.8, w_2 = 0.1)$. It is because of the periodic behaviour of DM interaction. Further, the ESD zone has slight shift at $w_1 = 0.3$ with $(Dt = 0.9, w_2 = 0.1)$. In the absence of entanglement in partition $BC$, the entanglement in partition $AC$ achieves higher value, which maintain the value of tripartite entanglement greater than zero in the system. On the other hand, we also observe that the entanglement in partition $AB$ is very less fragile and accepts very less fluctuations. We also study the specific class of W states which is widely used in quantum information and have symmetric properties. This state can be obtained by putting $(w_0 = w_1 = w_2 = \frac{1}{\sqrt{3}})$ in Eq. (8). The entanglement in this state with increasing values of parameter $Dt$ is given below

| $Dt$ | 0.0  | 0.1  | 0.2  | 0.3  |
|------|------|------|------|------|
| Entanglement | 0.549364 | 0.471871 | 0.281458 | 0.0819103 |

| $Dt$ | 0.4  | 0.5  | 0.6  | 0.7  | 0.8  |
|------|------|------|------|------|------|
| Entanglement | 0.000557363 | 0.106773 | 0.312717 | 0.492105 | 0.547631 |

Observing the results given in the above table reveals that as DM interaction strength increases, the value of entanglement decreases up to 0.000557363, which starts to rise immediately after $Dt = 0.4$ and the state does not suffer from ESD. We plot further results in Fig. 6. In sub figures of Fig. 6, we obtained the points where all the graphs cut each other for different values of the set of parameters $(Dt, w_1, w_2)$. These values are shown in the table below,
Fig. 6  Plots of bipartite entanglement in generalised W states with different values of parameter $Dt$ and $w_2$. The quantities $N_{AB}$, $N_{AC}$ and $N_{BC}$ are represented with *Dotted blue*, *Dashed orange* and *thick red graph*, respectively.

| $Dt$ | $w_1$ | $w_2$ | Entanglement |
|------|------|------|-------------|
| 2.9  | 2.9  | 0.8  | 0.4         |
| 6.5  | 0.8  | 0.4  | 0.4         |
| 7.2  | 0.75 | 0.5  | 0.4         |
| 9.1  | 0.6  | 0.7  | 0.4         |

We observe that, for these values, the entanglement in all the partitions AB, AC and BC achieves same amplitude as 0.4 and no bipartite partition goes under ESD. So, for these values, the entanglement is equally distributed in the partitions AB, AC and BC.

### 4.2 Case 2: Initial preparation with GHZ states

In this section, we study the dynamics of tripartite system when it is initially prepared in generalised GHZ states. We apply the procedure to obtain the reduced density matrix as adopted in case 1; after extensive calculations, the reduced density matrix of the system ABC is obtained as given below

$$
\begin{bmatrix}
(|c_0|^2 + |c_1|^2) g_0^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
(|c_0|^2 + |c_1|^2) g_0^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
(|c_0|^2 + |c_1|^2) g_0^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
$$

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We find that the reduced density matrix does not involve the parameter $Dt$, which becomes zero in calculations. The matrix only involves the probability amplitudes of environmental qubit C. Further, the factor $|c_0|^2 + |c_1|^2$ is involved with every terms of the matrix, so by applying the normalisation condition from Eq. (5), the probability amplitudes vanish from the reduced density matrix and it maps to initial generalised GHZ states. So we conclude, these states neither affected by DM interaction nor by the state of environmental qubit D. Now, it is obvious that the tripartite and bipartite entanglement have not been affected in GHZ states. Hence, GHZ state exhibit the robust character against DM interaction.

5 Conclusion

In the present paper, the robustness of GHZ and W states is investigated against the Dzyaloshinskii-Moriya interaction. We prepare the tripartite system in generalised GHZ and W states, respectively. By taking an environmental qubit which interacts with any one of the qubit of tripartite system, we study the dynamics of entanglement in GHZ and W states against the DM interaction. In this study, three $\pi$ is used as an entanglement measure. We find the state of environmental qubit does not contribute in the dynamics irrespective the system is prepared in either GHZ or W states. It is the DM interaction which only disturb the entanglement. Further, we find the tripartite entanglement does not face ESD, when system is initially prepared in W states. While on the other hand, the bipartite entanglement based on negativity in W states goes under periodic ESD. This periodic behaviour takes place with threshold value of the parameter $Dt = 0.8$. Tripartite entanglement in W states is more robust against DM interaction, while bipartite partitions are fragile. On the other hand, GHZ sates do not get affected neither by DM interaction nor by state of environmental qubit. So, we find GHZ states are more robust than W states against DM interaction. This study may be useful in quantum information processing where the unwanted DM interaction takes place.

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