Canonical and Lie-algebraic twist deformations of $\kappa$-Poincare and contractions to $\kappa$-Galilei algebras

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Abstract

We propose canonical and Lie-algebraic twist deformations of $\kappa$-deformed Poincare Hopf algebra which leads to the generalized $\kappa$-Minkowski space-time relations. The corresponding deformed $\kappa$-Poincare quantum groups are also calculated. Finally, we perform the nonrelativistic contraction limit to the corresponding twisted Galilean algebras and dual Galilean quantum groups.
1 Introduction

Recently, it has been suggested that the classical Poincaré invariance should be treated as an approximate symmetry in ultra-high energy regime and the relativistic space-time symmetries on Planck scale is deformed \cite{1}-\cite{4}. Besides, there are also arguments based on quantum gravity \cite{5}, \cite{6} and string theory models \cite{7}, \cite{8} which suggest that space-time at Planck length is quantum, i.e. it should be noncommutative. The simplest choice of the noncommutative space-time is the following

\[ [\hat{x}_\mu, \hat{x}_\nu] = i\theta_{\mu\nu} + i\theta^\rho_{\mu\nu}\hat{x}_\rho; \quad \theta_{\mu\nu}, \theta^\rho_{\mu\nu} - \text{const}. \] (1)

The first, simplest kind of noncommutativity ($\theta_{\mu\nu} \neq 0, \theta^\rho_{\mu\nu} = 0$ in formula (1)) was investigated in the Hopf-algebraic framework in \cite{9}-\cite{14}. It corresponds to the well-known canonical (soft) deformation of Poincare Hopf algebra obtained by twist procedure \cite{15}. The second type of space-time deformation ($\theta_{\mu\nu} = 0, \theta^\rho_{\mu\nu} \neq 0$) is directly associated with another modification of classical relativistic symmetries - the $\kappa$-deformed Poincare Hopf algebra \cite{16}, \cite{17}, which is an example of the Lie-algebraic kind of space-time noncommutativity.

In almost all considerations both modifications of Minkowski space - Lie- and soft-type - are considered separately. Here we ask about such a deformation of relativistic space-time symmetry, when both noncommutativities will appear together, i.e. for which in the formula (1) both coefficients $\theta_{\mu\nu}$ and $\theta^\rho_{\mu\nu}$ are different from zero.

The results of Zakrzewski’s (\cite{18}, \cite{19}) indicate how to look for such a generalized Hopf Poincare structure. The classical r-matrix related to such a modification of space-time symmetries should be a sum of r-matrices for $\kappa$-Poincare group and the one describing canonical twist. Besides, this extended r-operator should solve the modified Yang-Baxter equation the same as in the case of $\kappa$-deformed Poincare symmetry. In this way, one can see that the explicit form of a proper twist factor allows us to derive a deformation of new quantum group - canonically twisted $\kappa$-Poincare Hopf algebra. Moreover, its dual partner can be calculated by a canonical quantization scheme of corresponding extended Poisson-Lie structure \cite{20}.

It should be mentioned, that the above algorithm can be generalized to two other twist deformations of $\kappa$-Poincare algebra - Lie-type \cite{21} (see also \cite{22}) and quadratic-one \cite{21}. First of them leads to a Lie-algebraic noncommutativity of Minkowski space, and it introduces in natural way a second (apart of $\kappa$) mass-like parameter $\hat{\kappa}$. In the case of quadratic extension of $\kappa$-Poincare algebra the deformation parameter is dimensionless.

In this article we consider both the canonical and Lie-algebraic twist deformations of $\kappa$-Poincare symmetry. In second Section we recall necessary facts concerning the $\kappa$-deformed Poincare algebra and its dual quantum group. The canonical and Lie-algebraic deformations of $\kappa$-Poincare algebras and $\kappa$-Minkowski space-times are presented in Section 3 and 4, respectively. In Section 5 we find canonically and Lie-algebraically deformed $\kappa$-Poincare dual groups. Finally, the nonrelativistic contraction limits (\cite{23}-\cite{25}) to the twisted Galilean algebras and dual quantum groups \cite{26}, \cite{23} are performed in Section 6. The results are briefly discussed and summarized in the last Section.
2 κ-Poincare deformation - short review

2.1 κ-deformed Poincare algebra

The κ-deformed Poincare algebra $U_\kappa(P)$ is the associative and coassociative Hopf structure with generators $M_{\mu\nu}$ and $P_\mu$ satisfying the following relations \[27\] \((\eta_{\mu\nu} = (-, +, +, +))\)

\[
[M^{\mu\nu}, M^{\lambda\sigma}] = i \left( \eta^{\mu\sigma} M_{\nu\lambda} - \eta^{\nu\sigma} M_{\mu\lambda} + \eta^{\mu\lambda} M_{\nu\sigma} - \eta^{\nu\lambda} M_{\mu\sigma} \right),
\]

\[
[M^{ij}, P_k] = i \left( \delta^i_k P_j - \delta^j_k P_i \right),
\]

\[
[M^{i0}, P_j] = i \delta^i_j \left[ \frac{\kappa}{2} \left( 1 - e^{-\frac{2P_0}{\kappa}} \right) + \frac{1}{2\kappa} \vec{P} \right] - \frac{i}{\kappa} P^j P_j,
\]

\[
[M^{ij}, P_0] = 0, \quad [M^{i0}, P_0] = i P_i, \quad [P_\mu, P_\nu] = 0,
\]

with the coproducts, antipodes and counits defined by

\[
\Delta_\kappa(M^{ij}) = M^{ij} \otimes 1 + 1 \otimes M^{ij},
\]

\[
\Delta_\kappa(M^{i0}) = M^{i0} \otimes e^{-\frac{P_0}{\kappa}} + 1 \otimes M^{i0} - \frac{1}{\kappa} M^{ij} \otimes P_j,
\]

\[
\Delta_\kappa(P_0) = P_0 \otimes 1 + 1 \otimes P_0, \quad \Delta_\kappa(P_i) = P_i \otimes e^{-\frac{P_0}{\kappa}} + 1 \otimes P_i,
\]

\[
S_\kappa(M^{ij}) = -M^{ij}, \quad S_\kappa(M^{i0}) = -\left( M^{i0} + \frac{1}{\kappa} M^{ij} P_j \right) e^{\frac{P_0}{\kappa}},
\]

\[
S_\kappa(P_i) = -P_i e^{\frac{P_0}{\kappa}}, \quad S_\kappa(P_0) = -P_0, \quad \epsilon(P_\mu) = \epsilon(M^{\mu\nu}) = 0.
\]

The κ-deformed mass Casimir looks as follows

\[
C_\kappa = \left( 2\kappa \sinh \left( \frac{P_0}{\kappa} \right) \right)^2 - \vec{P}^2 e^{\frac{P_0}{\kappa}}.
\]

We see that in $U_\kappa(P)$ one can distinguish the following two Hopf subalgebras: non-deformed $O(3)$-rotation algebra and Abelian fourmomentum algebra. For $\kappa \to \infty$ the deformation disappears and we get the classical Poincare Hopf algebra $U_0(P)$.

It is well-known that the classical r-matrix corresponding to the above Hopf structure has the form \[18\], \[28\], \[29\]

\[
r_\kappa = \frac{1}{\kappa} M_{0\mu} \wedge P^\nu = \gamma^{\mu\nu;\alpha} M_{\mu\nu} \wedge P_\alpha; \quad \gamma^{\mu\nu;\alpha} = \frac{1}{2\kappa} (\delta^\mu_0 \eta^{\nu\alpha} - \delta^\nu_0 \eta^{\mu\alpha}),
\]

with $a \wedge b = a \otimes b - b \otimes a$. One can check that the matrix \[12\] with itself satisfies a modified Yang-Baxter equation (MYBE)

\[
[[r_\kappa, r_\kappa]] := [r_{\kappa 12}, r_{\kappa 13} + r_{\kappa 23}] + [r_{\kappa 13}, r_{\kappa 23}] = \frac{1}{\kappa^2} M_{\mu\nu} \wedge P^\mu \wedge P^\nu,
\]
where used in the above formula symbol \([ [ \cdot , \cdot ] ]\) denotes Schouten bracket while \(r_{\kappa 12} = \frac{1}{\kappa} M_{i0} \wedge P_i \wedge 1, r_{\kappa 13} = \frac{1}{\kappa} M_{i0} \wedge 1 \wedge P_i\) and \(r_{\kappa 23} = \frac{1}{\kappa} 1 \wedge M_{i0} \wedge P_i\).

2.2 \(\kappa\)-deformed Poincare group

The classical \(r\)-matrix (12) defines Poisson-Lie structure [20]. Its standard quantization procedure leads to a dual form of the Hopf algebra (2)-(10) - the \(\kappa\)-deformed Poincare group \(P_\kappa [28], [29]\). It is defined by the following

a) algebraic relations
\[
[\Lambda^\alpha_\beta, a^\rho] = -\frac{i}{\kappa}((\Lambda^\alpha_0 - \delta^\alpha_0)\Lambda^\rho_\beta + \eta^{\alpha\rho}(\Lambda_0^\beta - \eta_{0\beta})), \quad (14)
\]
\[
[a^\rho, a^\sigma] = -\frac{i}{\kappa}(\delta^\sigma_0 a^\rho - \delta^\rho_0 a^\sigma), \quad [\Lambda^\alpha_\beta, \Lambda^\delta_\rho] = 0, \quad (15)
\]

b) coproducts
\[
\Delta(\Lambda^\mu_\nu) = \Lambda^\mu_\alpha \otimes \Lambda^\alpha_\nu, \quad \Delta(a^\mu) = \Lambda^\mu_\nu \otimes a^\nu + a^\mu \otimes 1, \quad (16)
\]

c) antipodes and counits
\[
S(\Lambda^\mu_\nu) = \Lambda^\mu_\nu, \quad S(a^\mu) = -\Lambda^\mu_\nu a^\nu, \quad \epsilon(\Lambda^\mu_\nu) = \delta^\mu_\nu, \quad \epsilon(a^\mu) = 0. \quad (17)
\]

The used above generators \(\Lambda^\mu_\nu\) are dual to \(M^{\mu\nu}\) - Lorentz rotation generators
\[
<\Lambda^\mu_\nu, M^{\alpha\beta}> = (\eta^{\alpha\mu}\delta^\beta_\nu - \eta^{\beta\mu}\delta^\alpha_\nu), \quad (18)
\]
while \(a^\mu\) are dual to \(P_\mu\) (translations)
\[
< a^\mu, P_\nu > = \delta^\mu_\nu. \quad (19)
\]

It should be noted that the relations (16)-(17) remain undeformed as for the classical Poincare group \(P\).

2.3 \(\kappa\)-deformed Minkowski space

It is well-known (see e.g. [30]) that the deformed Minkowski space can be introduced as the quantum representation space (a Hopf module) for quantum Poincare algebra, equipped with a proper defined \(\star\)-multiplication of two arbitrary function. Such a \(\star\)-product should be consistent with the action of deformed symmetry generators satisfying suitably deformed Leibnitz (coproduct) rules. In the case of \(\kappa\)-deformation the \(\star_\kappa\)-multiplication looks as follows (see [31], [32] and references therein)
\[
f(x) \star_\kappa g(x) = \omega (\mathcal{O}_\kappa(x_\mu, \partial^\mu)(f(x) \otimes g(x))), \quad (20)
\]
where \(\omega(f(x) \otimes g(x)) = f(x)g(x)\) and the \(\star_\kappa\)-differential operator is given by
\[
\mathcal{O}_\kappa(x_\mu, \partial^\mu) := \exp(ix_\mu \gamma^\mu(\partial^\nu)), \quad (21)
\]
with
\[ \gamma^\mu(\partial^\nu) := \epsilon^\mu_{\rho\tau} \partial^\rho \otimes \partial^\tau + \frac{1}{12} \epsilon^\mu_{\rho\nu} \epsilon^\nu_{\lambda\mu} (\partial^\lambda \partial^\nu \otimes \partial^\rho \otimes \partial^\tau) + \cdots ; \] (22)
\[ c^i_0i = -c^i_{i0} = -\frac{1}{2\kappa} \text{ other } c^\mu_{\rho\tau} = 0 . \] (23)

Using the formula (20) in the case \( f(x) = x_\mu, \ g(x) = x_\nu \) we see that the \( \kappa \)-deformed Minkowski space-time takes the form
\[ [x_i, x_0]_{\kappa} = x_i^{*\kappa}x_0 - x_0^{*\kappa}x_i = \frac{i}{\kappa}x_i , \quad [x_i, x_j]_{\kappa} = 0 , \] (24)
and in the \( \kappa \to \infty \) limit it becomes classical.

3 Canonical twist deformation of \( \kappa \)-Poincare algebra

3.1 Extended classical r-matrix

Let us consider the following extension of classical r-matrix (12)
\[ r = r_\kappa + r_{\hat{\kappa}} + r_\xi , \] (25)
with
\[ r_{\hat{\kappa}} = \frac{1}{2\kappa} M_{12} \wedge P_0 , \] (26)
and
\[ r_\xi = \frac{\xi}{2} P_3 \wedge P_0 , \] (27)

where the formulas (26) and (27) describe Lie-algebraic and canonical twist deformations of \( \kappa \)-Poincare algebra, respectively. Due to the commutation relations \([ P_\mu, P_\nu \] = \[ M_{12}, P_3 \] = 0 (see (3) and (5)) we can see that both matrices \( r_{\hat{\kappa}} \) and \( r_\xi \) satisfy the classical Yang-Baxter equation (CYBE)
\[ [[ r_{\hat{\kappa}}, r_{\hat{\kappa}} ]] = [[ r_\xi, r_\xi ]] = 0 ; \] (28)

the mixed Schouten brackets vanish as well
\[ [[ r_{\hat{\kappa}}, r_\xi ]] = [[ r_\xi, r_{\hat{\kappa}} ]] = 0 . \] (29)

By explicit calculation one can check that
\[ [[ r_{\kappa}, r_{\cdot} ]] = [[ r_{\cdot}, r_{\kappa} ]] = 0 ; \quad r_{\cdot} = r_{\hat{\kappa}}, \ r_\xi , \] (30)

which together with the formulas (28) and (29) means that the extended r-matrix (25) satisfies the modified Yang-Baxter equation (13)
\[ [[ r, r ]] = \frac{1}{\kappa^2} M_{\mu\nu} \wedge P^\mu \wedge P^\nu . \] (31)
3.2 Canonical deformation of $\kappa$-Poincare algebra

In accordance with (30) one can consider the canonical ($r = r_\kappa + r_\xi$) deformation of enveloping $\kappa$-Poincare algebra $U_\kappa(\mathcal{P})$. As already mentioned in Introduction we can get such a modification of space-time relativistic symmetry by a proper ($\kappa$-deformed) twisting procedure.

First of all, let us introduce an element $\mathcal{F}_\xi \in U_\kappa(\mathcal{P}) \otimes U_\kappa(\mathcal{P})$ with the following linear term in series expansion with respect to the deformation parameter $\xi$

$$\mathcal{F}_\xi = 1 + ir_\kappa^{(1)} \otimes r_\xi^{(2)} + \cdots ; \quad r_\xi = r_\kappa^{(1)} \otimes r_\xi^{(2)}.$$  \hspace{1cm} (32)

Next, we define Drinfeld twist factor as the function (32) satisfying so-called $\kappa$-deformed cocycle condition \cite{33}

$$\mathcal{F}_{\xi_1 \otimes 1} \cdot (\Delta_\kappa \otimes 1) \mathcal{F}_\xi = \mathcal{F}_{\xi_2 \otimes 1} \cdot (1 \otimes \Delta_\kappa) \mathcal{F}_\xi,$$  \hspace{1cm} (33)

and the normalization condition

$$(\varepsilon \otimes 1) \mathcal{F}_\xi = (1 \otimes \varepsilon) \mathcal{F}_\xi = 1,$$  \hspace{1cm} (34)

with $\mathcal{F}_{\xi_1 \otimes 1} = \mathcal{F}_\xi \otimes 1$ and $\mathcal{F}_{\xi_2 \otimes 1} = 1 \otimes \mathcal{F}_\xi$. The solution of above equations has been found in \cite{19} and it looks as follows

$$\mathcal{F}_{\xi,\kappa} = \exp \left( i\kappa \frac{\xi}{2} P_3 \otimes \left( e^{-\frac{P_0}{\kappa}} - 1 \right) \right).$$  \hspace{1cm} (35)

One can easily see that in the limit $\xi \to 0$ factor $\mathcal{F}_{\xi,\kappa}$ goes to the unit operator

$$\lim_{\xi \to 0} \mathcal{F}_{\xi,\kappa} = 1,$$  \hspace{1cm} (36)

while in the case $\kappa \to \infty$ we get a standard canonical-twist element for the classical Poincare Hopf algebra $U_0(\mathcal{P})$

$$\mathcal{F}_{\xi,\infty} = e^{-i\frac{\xi}{2} P_3 \otimes P_0}.$$  \hspace{1cm} (37)

It is well-known that twist $\mathcal{F}_{\xi,\kappa}$ does not modify the algebraic part of $\kappa$-Poincare algebra (2)-(5) and counits, but it changes the coproducts (6)-(8) and antipodes (9), (10) according to

$$\Delta_{\mathcal{F}_{\xi,\kappa}}(a) = \mathcal{F}_{\xi,\kappa} \Delta_\kappa(a) \mathcal{F}_{\xi,\kappa}^{-1},$$  \hspace{1cm} (38)

$$S_{\mathcal{F}_{\xi,\kappa}}(a) = u(\kappa, \xi) S_\kappa(a) u^{-1}(\kappa, \xi),$$  \hspace{1cm} (39)

where $u(\kappa, \xi) = \sum f(1) S_\kappa(f(2))$, and where we use Sweedler’s notation $\mathcal{F}_{\xi,\kappa} = \sum f(1) \otimes f(2)$. Hence, using the formula

$$u(\kappa, \xi) = \exp \left( i\kappa \xi P_3 \left( \exp\left(\frac{P_0}{\kappa}\right) - 1 \right) \right),$$  \hspace{1cm} (40)
we obtain
\[
\Delta_{\mathcal{F}_{\xi,\kappa}}(P_0) = P_0 \otimes 1 + 1 \otimes P_0, \quad \Delta_{\mathcal{F}_{\xi,\kappa}}(P_i) = P_i \otimes e^{-\frac{P_0}{\kappa}} + 1 \otimes P_i, \tag{41}
\]
\[
\Delta_{\mathcal{F}_{\xi,\kappa}}(M^{ij}) = \Delta_{\kappa}(M^{ij}) + \frac{\xi}{2} \left( \delta^i_3 P_i - \delta^i_3 P_j \right) \otimes \left( e^{-\frac{P_0}{\kappa}} - 1 \right), \tag{42}
\]
\[
\Delta_{\mathcal{F}_{\xi,\kappa}}(M^{i0}) = \Delta_{\kappa}(M^{i0}) - \frac{\xi}{2} P_3 \otimes P_i e^{-\frac{P_0}{\kappa}} + \\
+ \frac{\xi}{2} \left( \delta^i_3 P_i - \delta^i_3 P_j \right) \otimes P_j \left( e^{-\frac{P_0}{\kappa}} - 1 \right), \tag{43}
\]
and
\[
S_{\mathcal{F}_{\xi,\kappa}}(P_0) = S_{\kappa}(P_0) = -P_0, \quad S_{\mathcal{F}_{\xi,\kappa}}(P_i) = S_{\kappa}(P_i) = -P_i e^{\frac{P_0}{\kappa}}, \tag{47}
\]
\[
S_{\mathcal{F}_{\xi,\kappa}}(M^{ij}) = S_{\kappa}(M^{ij}) - \kappa \xi \left( \delta^i_3 P_i - \delta^i_3 P_j \right) \cdot \left( \exp(P_0/\kappa) - 1 \right), \tag{48}
\]
\[
S_{\mathcal{F}_{\xi,\kappa}}(M^{i0}) = S_{\kappa}(M^{i0}) - \xi \left( \delta^i_3 P_i - \delta^i_3 P_j \right) P_j \cdot e^{\frac{P_0}{\kappa}} \cdot \\
\cdot \left( \exp(P_0/\kappa) - 1 \right) - \xi P_3 P_i e^{\frac{2P_0}{\kappa}} + \\
- \kappa \xi \left( \delta^i_3 \left[ \frac{\kappa}{2} \left( 1 - e^{-\frac{2P_0}{\kappa}} \right) + \frac{1}{2\kappa} \bar{P}^2 \right] - \frac{1}{\kappa} P_i P_3 \right), \tag{49}
\]
\[
\cdot e^{\frac{P_0}{\kappa}} \cdot \left( \exp(P_0/\kappa) - 1 \right). \tag{50}
\]

The algebraic relations (2)-(5) together with coproducts (41)-(46), antipodes (47)-(52) and classical counits (10) define the canonical twist deformation of \(\kappa\)-Poincare algebra \(\mathcal{U}_{\xi,\kappa}(\mathcal{P})\). We see, that for \(\xi \to 0\) one gets the \(\kappa\)-Poincare algebra \(\mathcal{U}_{\kappa}(\mathcal{P})\), which is in accordance with the formula (36), i.e. there is no twist transformation in such a case. For parameter \(\kappa \to \infty\), the algebra \(\mathcal{U}_{\xi,\kappa}(\mathcal{P})\) passes into well-known \(\theta^{\mu\nu}\)-Poincare Hopf structure [10], and this time, it agrees with the form of twist factor (37).

3.3 Canonical extension of \(\kappa\)-Minkowski space

Let us now find a noncommutative Minkowski space corresponding to the canonical deformation of \(\kappa\)-Poincare. As it was mentioned in the first section, our space-time can be defined as a quantum representation space for the extended quantum Poincare algebra
\(U_{\kappa,\xi}(P)\), equipped with a proper deformed \(\star\)-multiplication. We define our \(\star\)-product for arbitrary two functions depending on space-time coordinates as follows
\[
f(x) \star_{\kappa,\xi} g(x) = \omega \left( O_{\kappa,\xi}(x, \partial^\mu)(f(x) \otimes g(x)) \right),
\]
where the \(\star\)-operator \(O_{*,\kappa}(x, \partial^\mu)\) is given by the superposition of two \(\star\)-operators: for the \(\kappa\)-deformed r-matrix \(r_\kappa\) (see (21)), and for the canonical deformed matrix \(r_\xi\) (see twist factor (35)) [30]
\[
O_\xi(x, \partial) := F^{-1}_{\xi,\kappa}(x, \partial^\mu) = \exp \left( -i\kappa \frac{\xi}{2} \partial^3 \otimes \left( e^{-\frac{\xi}{\kappa}} - 1 \right) \right).
\]
Consequently, our operator takes the form
\[
O_{\kappa,\xi}(x, \partial^\mu) := O_\xi(x, \partial^\mu) \circ O_\kappa(x, \partial^\mu),
\]
and we obtain the following commutation relations
\[
[x_i, x_0]_{\kappa,\xi} = \frac{i}{\kappa} x_i + \frac{i\xi}{2} \partial^3_i, \quad [x_i, x_j]_{\kappa,\xi} = 0.
\]
The relations (56) define the canonically extended \(\kappa\)-Minkowski space-time \(M_{\kappa,\xi}\). We see that the soft deformation of \(\kappa\)-Poincare algebra introduces two kinds of noncommutativity: Lie-type associated with parameter \(\kappa\), and canonical type - corresponding to parameter \(\xi\). Of course, for \(\xi \to 0\) one gets the \(\kappa\)-deformed Minkowski space-time \(M_\kappa\), while in the \(\kappa \to \infty\) limit we obtain well-known \(\theta^{\mu\nu}\)-deformed Minkowski space \(M_\theta\) (see e.g. [10]).

4 Lie-algebraic twist deformation of \(\kappa\)-Poincare algebra

4.1 Deformation of algebra

In the case of Lie-algebraic deformation \((r = r_\kappa + r_\kappa)\) the twist factor has been found in [19]. Here we consider its antisymmetric form
\[
F_{\kappa,\kappa} = \exp \left( \frac{i}{2\kappa} M_{12} \wedge P_0 \right).
\]
By tedious calculation we get the following coproduct of deformed \(\kappa\)-Poincare algebra \(U_{\kappa,\kappa}(P)\)
\[
\Delta_{F_{\kappa,\kappa}}(P_0) = P_0 \otimes 1 + 1 \otimes P_0, \quad \Delta_{F_{\kappa,\kappa}}(P_3) = P_3 \otimes e^{-\frac{P_0}{\kappa}} + 1 \otimes P_3,
\]
\[
\Delta_{F_{\kappa,\kappa}}(P_1) = \Delta_\kappa(P_1) - \sin \left( \frac{P_0}{2\kappa} \right) \otimes P_2 + P_2 \otimes \sin \left( \frac{P_0}{2\kappa} \right) e^{-\frac{P_0}{\kappa}} +
\]
8
For Lie-algebraic deformation we define the

\[
\kappa
\]\(\text{The algebraic sector as well as the antipodes remain -1 + 1 = } \Delta \kappa_i + \left[ \kappa_2 \hat{\kappa} M \right] \Delta F M \left[ \sin F \Delta 12 \cos ij \left( \hat{\kappa}, \kappa \right) \hat{\kappa} \right] M \left( \hat{\kappa}, \kappa \right) \right] = \Delta \kappa \left[ \cos \left( \frac{P_0}{2\kappa} \right) - 1 \right] \), \quad (63)

\[
\Delta_{F,\kappa} (P_2) = \Delta_\kappa (P_2) + \sin \left( \frac{P_0}{2\kappa} \right) \otimes P_1 - P_1 \otimes \sin \left( \frac{P_0}{2\kappa} \right) e^{-\frac{P_0}{\kappa}} + \frac{1}{2\kappa} M^{12} \otimes P_1 + \frac{1}{2\kappa} P_i \otimes M^{12} e^{-\frac{P_0}{\kappa}} - i \left[ M^{10}, M^{12} \right] \otimes \sin \left( \frac{P_0}{2\kappa} \right) e^{-\frac{P_0}{\kappa}} + i \sin \left( \frac{P_0}{2\kappa} \right) \otimes [ M^{10}, M^{12} ] - \left[ [ M^{10}, M^{12} ], M^{12} \right] \otimes \left[ \cos \left( \frac{P_0}{2\kappa} \right) - 1 \right] e^{-\frac{P_0}{\kappa}} - \left[ \cos \left( \frac{P_0}{2\kappa} \right) - 1 \right] \otimes \left[ [ M^{10}, M^{12} ], M^{12} \right] = \Delta_\kappa \left( M^{12} \right) + \frac{1}{2\kappa} \left( \delta^{1i} P_2 - \delta^{2i} P_1 \right)
\]

\[
\Delta_{F,\kappa} (M^{ij}) = \Delta_\kappa (M^{ij}) - i \left[ M^{ij}, M^{12} \right] \wedge \sin \left( \frac{P_0}{2\kappa} \right) + \left[ [ M^{ij}, M^{12} ], M^{12} \right] \perp \left[ \cos \left( \frac{P_0}{2\kappa} \right) - 1 \right], \quad (64)
\]

\[
\Delta_{F,\kappa} (M^{i0}) = \Delta_\kappa (M^{i0}) - \frac{1}{2\kappa} M^{12} \otimes P_1 + \frac{1}{2\kappa} P_i \otimes M^{12} e^{-\frac{P_0}{\kappa}} - i \left[ M^{i0}, M^{12} \right] \otimes \sin \left( \frac{P_0}{2\kappa} \right) e^{-\frac{P_0}{\kappa}} + i \sin \left( \frac{P_0}{2\kappa} \right) \otimes [ M^{i0}, M^{12} ] - \left[ [ M^{i0}, M^{12} ], M^{12} \right] \otimes \left[ \cos \left( \frac{P_0}{2\kappa} \right) - 1 \right] e^{-\frac{P_0}{\kappa}} + \frac{1}{2\kappa} \left( \delta^{1i} P_1 + \delta^{2i} P_2 \right) \otimes M^{12} \left[ \cos \left( \frac{P_0}{2\kappa} \right) - 1 \right] e^{-\frac{P_0}{\kappa}} \right)
\]

\[
\Delta_{F,\kappa} (M^{12}) = \Delta_\kappa (M^{12}) - \frac{1}{2\kappa} \left( \delta^{1i} P_2 - \delta^{2i} P_1 \right) \otimes M^{12} \sin \left( \frac{P_0}{2\kappa} \right) e^{-\frac{P_0}{\kappa}} + \frac{1}{2\kappa} \left( \delta^{1i} P_1 + \delta^{2i} P_2 \right) \otimes M^{12} \left[ \cos \left( \frac{P_0}{2\kappa} \right) - 1 \right] e^{-\frac{P_0}{\kappa}} \right)
\]

The algebraic sector as well as the antipodes remain \(\kappa\)-deformed, i.e. \(S_{F,\kappa} (a) = S_\kappa (a)\) (see [9], [10]).

### 4.2 Two-parameter extension of \(\kappa\)-Minkowski space

For Lie-algebraic deformation we define the \(\ast\)-operator as follows

\[
\mathcal{O}_{\kappa,\hat{\kappa}} (x_{\mu}, \partial^\mu) := \mathcal{O}_\hat{\kappa} (x_{\mu}, \partial^\mu) \circ \mathcal{O}_\kappa (x_{\mu}, \partial^\mu);
\]

\[
(66)
\]
\[O_\kappa(x_\mu, \partial^\mu) := \mathcal{F}_{\kappa,\hat{\kappa}}^{-1}(x_\mu, \partial^\mu) = \exp \left( -\frac{i}{2\hat{\kappa}}(x_1 \partial^2 - x_2 \partial^1) \wedge \partial^0 \right), \tag{67}\]

and our \((\kappa, \hat{\kappa})\)-deformed Minkowski space takes the form

\[ [x_i, x_0]_{\kappa,\hat{\kappa}} = \frac{i}{\kappa} x_i + \frac{i}{\hat{\kappa}}(\delta^1_i x_2 - \delta^2_i x_1) , \quad [x_i, x_j]_{\kappa,\hat{\kappa}} = 0. \tag{68}\]

The relations (68) define the Lie-algebraic extension of \(\kappa\)-Minkowski space-time \(\mathcal{M}_{\kappa,\hat{\kappa}}\). We see that above deformation of \(\kappa\)-Poincare algebra introduces Lie-algebraic type of space-time noncommutativity corresponding to both parameters \(\kappa\) and \(\hat{\kappa}\). It should be also noted that in the \(\hat{\kappa} \to \infty\) limit we get the \(\kappa\)-deformed Minkowski space-time \(\mathcal{M}_\kappa\), while for \(\kappa \to \infty\) we obtain the Minkowski space for Lie-twisted Poincare algebra \(\mathcal{M}_\hat{\kappa}\). [22]

5 Canonical and Lie-algebraic twist deformation of \(\kappa\)-Poincare group

In accordance with the equation (31) one can define the corresponding to the matrix (25) Poisson-Lie structure as follows [20]

\[ \{ f, g \} = 2r^{AB} (X^R_A f X^R_B g - X^L_A f X^L_B g) . \tag{69}\]

The symbols \(X^R_A, X^L_A\) denote the right- and left-invariant vector fields on classical Poincare group \(\mathcal{P}\) given by

\[ X^\alpha_\beta = \Lambda^\mu_\alpha \frac{\partial}{\partial \Lambda^\mu_\beta} - \Lambda^\mu_\beta \frac{\partial}{\partial \Lambda^\mu_\alpha} , \quad X^\alpha = \Lambda^\mu_\alpha \frac{\partial}{\partial a^\mu} , \tag{70}\]

\[ X^\alpha_\beta = \Lambda^\nu_\beta \frac{\partial}{\partial \Lambda^\nu_\alpha} - \Lambda^\nu_\alpha \frac{\partial}{\partial \Lambda^\nu_\beta} + a^\beta \frac{\partial}{\partial a_\alpha} - a^\alpha \frac{\partial}{\partial a_\beta} , \quad X^\alpha = \frac{\partial}{\partial a_\alpha} . \tag{71}\]

If we calculate the Poisson brackets (69) with use of the formulas (12), (26) and (27), in a first step, and if we perform its standard quantization by replacing \(\{ \cdot, \cdot \} \to \frac{1}{i}[\cdot, \cdot]\), as a second step, then we obtain the following set of commutation relations

\[ [\Lambda^\alpha_\beta, a^\rho_\sigma] = -\frac{i}{\kappa}((\Lambda^\alpha_0 - \delta^\alpha_0)\Lambda^\rho_\beta + \eta^{\alpha\rho}(\Lambda_0\beta - \eta_0\beta)) + \tag{72}\]

\[ + \frac{1}{\hat{\kappa}}((\Lambda^\rho_0(\eta_{2\beta}\Lambda^\alpha_1 - \eta_{1\beta}\Lambda^\alpha_2) + \delta^\rho_0(\delta^\alpha_2\Lambda_{1\beta} - \delta^\alpha_1\Lambda_{2\beta})) , \tag{73}\]

\[ [a^\rho_\sigma, a^{\rho'}_{\sigma'}] = -\frac{i}{\kappa}(\delta^\rho_0 a^{\rho'} - \delta^\rho_0 a^{\rho'}) + \frac{i}{\hat{\kappa}}(\delta^\rho_0(\delta^\rho_2 a^{1} - \delta^\rho_1 a^{2}) + \tag{74}\]

\[ + \frac{i}{\hat{\kappa}}(\delta^\rho_0(\delta^\rho_1 a^{2} - \delta^\rho_2 a^{1}) + i\frac{\xi}{2}(\delta^\rho_3\delta^\rho_0 - \delta^\rho_0\delta^\rho_3) + \tag{75}\]

\[ + \frac{i}{2}(\Lambda^\rho_0\Lambda^\sigma_3 - \Lambda^\rho_3\Lambda^\sigma_0) , \quad [\Lambda^\alpha_\beta, \Lambda^\rho_\sigma] = 0 . \tag{76}\]
Next, if we define the ∗-operation in such a way that $\Lambda^\mu_\nu$ and $a^\mu$ are selfadjoint elements, we see that the above relations together with coproducts (16), counits and antipodes (17) give a Hopf ∗-algebra - the $(\hat{\kappa}, \xi)$-deformed $\kappa$-Poincare group $\mathcal{P}_{\kappa, \hat{\kappa}, \xi}$. In such a way for $\hat{\kappa} = \infty$ we get dual group to the canonically deformed algebra $\mathcal{U}_{\kappa, \xi}(\mathcal{P})$, while for $\xi = 0$ we obtain dual partner for $\mathcal{U}_{\kappa, \hat{\kappa}}(\mathcal{P})$.

It should be also noted that for $\kappa \to \infty$, $\hat{\kappa} \to \infty$ and $\xi \to 0$ we obtain the classical (undeformed) Poincare group $\mathcal{P}$. For $\kappa \to \infty$ and $\xi \to 0$ we get the Lie-algebraically twisted classical Poincare group [21], while in the case $\kappa \to \infty$ and $\hat{\kappa} \to \infty$ we obtain the canonical deformation of classical Poincare Hopf algebra [9].

6 Constructions to twisted $\kappa$-Galilei algebras and $\kappa$-Galilei groups

In this section we calculate the nonrelativistic contractions of Hopf structures derived in previous sections, i.e. we find their nonrelativistic counterparts - the canonical and Lie-algebraic twist deformations of $\kappa$-Galilei algebra.

6.1 Canonical deformation of $\kappa$-Galilei algebra

Let us introduce the following standard redefinition of Poincaré generators [34] (see also [35])

$$P_0 = \frac{\Pi_0}{c}, \quad P_i = \Pi_i, \quad M_{ij} = K_{ij}, \quad M_{i0} = cV_i,$$

where parameter $c$ describes the light velocity. We start with canonical twisted algebra $\mathcal{U}_{\xi, \kappa}(\mathcal{P})$, i.e. we introduce two parameters $\kappa$ and $\xi$ such that $\kappa = \bar{\kappa}/c$ and $\xi = \bar{\xi}c$. Next, one performs the contraction limit of algebraic part (2)-(5) and co-sector (41)-(46) in two steps (see e.g. [23]). Firstly, we rewrite the formulas (2)-(5) and (41)-(46) in term of the operators (77) and parameters $\kappa$, $\xi$. Secondly, we take the $c \to \infty$ limit, and in such a way, we get the following algebraic

$$\left[ K^{ij}, K^{kl} \right] = i \left( \delta^{il} K^{jk} - \delta^{jl} K^{ik} + \delta^{jk} K^{il} - \delta^{ik} K^{jl} \right),$$

$$\left[ K^{ij}, V^k \right] = i \left( \delta^{jk} V^i - \delta^{ik} V^j \right), \quad \left[ K^{ij}, \Pi_k \right] = i \left( \delta^j_k \Pi_i - \delta^i_k \Pi_j \right),$$

$$\left[ V_i, V_j \right] = 0 \quad \left[ V^i, \Pi_0 \right] = i \Pi_i \quad \left[ \Pi_\rho, \Pi_\sigma \right] = 0,$$

$$\left[ V^i, \Pi_j \right] = \delta^i_j \frac{1}{2\kappa} \Pi^2 - \frac{1}{\kappa} \Pi_i \Pi_j \quad \mathcal{C}_\kappa = \Pi^2 e^{\frac{\kappa}{r}},$$

and coalgebraic

$$\Delta_{\xi, \kappa}(\Pi_0) = \Pi_0 \otimes 1 + 1 \otimes \Pi_0 \quad \Delta_{\xi, \kappa}(\Pi_i) = \Pi_i \otimes e^{\frac{\kappa}{r}} + 1 \otimes \Pi_i.$$
\[ \Delta_{\xi,\kappa}(K^{ij}) = \Delta_{\kappa}(K^{ij}) + \frac{\xi}{2} (\delta^i_j \Pi_i - \delta^i_j \Pi_j) \otimes \left( e^{-\frac{n_0}{\kappa}} - 1 \right) , \quad (83) \]

\[ \Delta_{\xi,\kappa}(V^i) = \Delta_{\kappa}(V^i) - \frac{\xi}{2} \Pi_3 \otimes \Pi_i e^{-\frac{n_0}{\kappa}} + \]

\[ + \frac{\xi}{2} \left( \delta^i_3 \Pi_i - \delta^i_3 \Pi_j \right) \otimes \left( e^{-\frac{n_0}{\kappa}} - 1 \right) e^{-\frac{n_0}{\kappa}} , \quad (84) \]

\[ + \frac{\xi}{2} \left( \tilde{\delta}^i_3 \Pi_i - \tilde{\delta}^i_3 \Pi_j \right) \otimes \Pi_j \left( e^{-\frac{n_0}{\kappa}} - 1 \right) , \quad (85) \]

sectors, where \( \Delta_{\kappa}(a) = \Delta_{\kappa}(a) \). The antipodes look as follows

\[ S_{\xi,\kappa}(\Pi_0) = S_{\kappa}(\Pi_0) = -\Pi_0 , \quad S_{\xi,\kappa}(\Pi_i) = S_{\kappa}(\Pi_i) = -\Pi_i e^{\frac{n_0}{\kappa}} , \quad (87) \]

\[ S_{\xi,\kappa}(K^{ij}) = S_{\kappa}(K^{ij}) - \frac{\xi}{2} \Pi_3 \left( \tilde{\delta}^i_3 \Pi_i - \tilde{\delta}^i_3 \Pi_j \right) \otimes \left( e^{-\frac{n_0}{\kappa}} - 1 \right) , \quad (88) \]

\[ S_{\xi,\kappa}(V^i) = S_{\kappa}(V^i) - \frac{\xi}{2} \left( \tilde{\delta}^i_3 \Pi_i - \tilde{\delta}^i_3 \Pi_j \right) \Pi_j \left( e^{-\frac{n_0}{\kappa}} - 1 \right) , \quad (89) \]

\[ \cdot \left( e^{-\frac{n_0}{\kappa}} - 1 \right) - \frac{\xi}{2} \Pi_3 \Pi_i e^{\frac{n_0}{\kappa}} + \]

\[ - \frac{\xi}{2} \left( \tilde{\delta}^i_3 \Pi_i - \tilde{\delta}^i_3 \Pi_j \right) \otimes \left( e^{-\frac{n_0}{\kappa}} - 1 \right) , \quad (90) \]

\[ \cdot \left( e^{-\frac{n_0}{\kappa}} - 1 \right) - \frac{\xi}{2} \Pi_3 \Pi_i e^{\frac{n_0}{\kappa}} , \quad (91) \]

with \( S_{\kappa}(a) = S_{\kappa}(a) \). The relations (78)-(91) define the canonically twisted \( \kappa \)-Galilei algebra \( U_{\xi,\kappa}(G) \). One can see that for \( \xi \to 0 \) we get the \( \kappa \)-deformed Galilei group \( U_{\kappa}(G) \) firstly studied in [26] (see also [23]). In \( \frac{\pi}{\kappa} \to \infty \) limit we obtain the canonically deformed algebra \( U_{\xi}(G) \) found in [25]. Obviously, for \( \frac{\pi}{\kappa} \to \infty \) and \( \xi \to 0 \) one gets the undeformed Galilei quantum group \( U_0(G) \).

### 6.2 Lie-algebraic deformation of \( \kappa \)-Galilei algebra

In the case of Lie-algebraic modification of \( \kappa \)-Poincare algebra, we perform contraction with respect the parameters \( \kappa = \frac{\pi}{c} \) and \( \tilde{\kappa} = \frac{\pi}{\kappa} \). Due to the relations (58)-(65) we obtain the coproducts \( \Delta_{\pi,\kappa}(\Pi_\rho) \), \( \Delta_{\pi,\kappa}(K^{ij}) \) and \( \Delta_{\pi,\kappa}(V^i) \) such that \( \Delta_{\pi,\kappa}(a) = \Delta_{\pi,\kappa}(a) \). In this way we get the Lie-twisted Galilei algebra \( U_{\pi,\kappa}(G) \), which for \( \pi \to \infty \) passes into \( \kappa \)-deformed Galilei group \( U_{\pi}(G) \).

### 6.3 Canonical and Lie-algebraic deformation of \( \kappa \)-Galilei group

Finally, let us find the contraction of \( (\tilde{\kappa},\xi) \)-deformed Poincare group \( P_{\tilde{\kappa},\xi,\xi} \) (see (72)-(76) and (16), (17)). In this purpose we introduce the following redefinition of \( A^\mu_\nu, A^\mu_\nu \)
\[ \Lambda_0 = \left(1 + \frac{\tau^2}{c^2}\right)^{\frac{1}{2}}, \quad \Lambda^i_0 = \frac{v^i}{c}, \quad \Lambda^0_i = \frac{v^k R^k_i}{c}, \] (92)

\[ \Lambda^k_i = \left(\delta^k_i + \left(1 + \frac{\tau^2}{c^2}\right)^{\frac{3}{2}} - 1\right)\frac{v^k v^j}{\tau^2} R^j_i, \] (93)

\[ a^i = b^i, \quad a^0 = c\tau, \] (94)

where \( \{ R^i_j, v^i, \tau, b^i \} \) denote the generators of Galilei group. With use of the formulas (92)-(94) in the contraction limit \( c \to \infty \) we get

\[ [R^k_i, b^j] = -\frac{i}{\kappa}(v^k R^i_j - \delta^k_i v^\rho R^\rho_j) + \frac{1}{\kappa}v^i(\delta^k_2 R^k_1 - \delta^k_1 R^k_2), \] (95)

\[ [R^k_i, \tau] = \frac{1}{\kappa}(\delta^k_2 R^k_1 - \delta^k_1 R^k_2 - (\delta^k_2 R^k_1 - \delta^k_1 R^k_2)), \] (96)

\[ [v^i, b^j] = -\frac{i}{\kappa}(v^i v^j - \frac{1}{2} \delta^{ij} \tau^2), \quad [v^i, \tau] = -\frac{i}{\kappa}v^i - \frac{1}{\kappa}(\delta^i_2 v^1 - \delta^i_1 v^2), \] (97)

\[ [\tau, b^j] = -\frac{i}{\kappa}b^j + \frac{i}{\kappa}(\delta^i_2 b^1 - \delta^i_1 b^2) + \frac{\tau}{2}(R^i_3 + \delta^i_3), \] (98)

\[ [b^i, b^j] = i\xi(2v^i R^j_3 - R^i_3 v^j), \] (99)

\[ [R^i_j, R^k_l] = [v^i, R^k_l] = [v^i, v^j] = 0. \] (100)

The coproducts remain undeformed

\[ \Delta(R^i_j) = R^i_k \otimes R^k_j, \quad \Delta(v^i) = R^i_j \otimes v^j + v^i \otimes 1, \] (101)

\[ \Delta(\tau) = \tau \otimes 1 + 1 \otimes \tau, \quad \Delta(b^i) = R^i_j \otimes b^j + v^i \otimes \tau + b^i \otimes 1. \] (102)

The relations (95)-(102) with classical antipodes and counits define the \( (\hat{\kappa}, \xi) \)-deformed Galilei group \( G_{\kappa, \hat{\kappa}, \xi} \). As in the case of relativistic symmetries, for \( \hat{\kappa} = \infty \) we get dual group to the Galilei algebra \( U_{\kappa, \tau}(\mathcal{G}) \), while for \( \xi = 0 \) we obtain dual partner for the algebra \( U_{\kappa, \tau}(\mathcal{G}) \).

Finally, one should also notice that in the \( \hat{\kappa} \to \infty \) and \( \xi \to 0 \) limits we get the well-known \( \kappa \)-deformed Galilei group \( G_{\kappa, \tau}(\mathcal{G}) \) (see [36]), while for \( \kappa \to \infty \) and \( \xi \to 0 \) or \( \hat{\kappa} \to \infty \), we obtain the quantum Galilei groups recovered in [37].

### 7 Final remarks

In this article we introduced two twist extensions of \( \kappa \)-Minkowski spaces corresponding to soft and Lie-algebraic type of noncommutativity (see [56] and [68]). For such modified
space-times we find their quantum Poincare algebras and corresponding dual quantum groups. The nonrelativistic contractions are performed as well.

As it was mentioned in Introduction the Lie-algebraic twist introduces in natural way a second mass-like parameter of deformation. Consequently, in such a way, one can obtain a "modification" of so-called Doubly Special Relativity [38–40] with one fundamental mass parameter, by introducing a second observer-independent mass-like scale.

It should be also noted that this paper is only a starting point for a further investigation. For example, it is interesting to ask about the noncommutative field theory given on such generalized quantum Minkowski space-times. In particular, its formulation requires the construction of a proper differential calculus, a proper star product of fields, and a suitable deformation of statistics for creation/annihilation operators (see e.g. [41–44]). The above problems are now under considerations and they are postponed for further investigation.

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