Abstract. Let $K$ be a compact convex body in $\mathbb{R}^n$. For any affine line $L$, denote $\hat{\chi}_K(L) = \int_L \chi_K(x) \, dl(x)$, where $dl$ is the arc length measure, the X-ray transform of the characteristic function $\chi_K$, i.e., the length of the chord $K \cap L$. We prove that if $K$ is bounded by a $C^\infty$ real algebraic hypersurface $\partial K$ and the X-ray transform $\hat{\chi}_K(L)$ behaves, under small parallel translations of the line $L$ to the distance $t$, as the $m$-th root of a polynomial of $t$, for some fixed $m \in \mathbb{N}$, then $\partial K$ is an ellipsoid.

1. Introduction

This article is devoted to characterization bodies in $\mathbb{R}^n$ in integral-geometric terms and is motivated by study of so called polynomially integrable bodies. Let us explain this relation.

Given a bounded domain $K \subset \mathbb{R}^n$, denote $A_K(\xi,t), \xi \in \mathbb{R}^n, |\xi| = 1, t \in \mathbb{R}$, the sectional volume function, which equals to the $n-1$-dimensional volume of the cross-section of $K$ by the affine hyperplane $\{\langle \xi, x \rangle = t\}$. Here $\langle , \rangle$ is the inner product in $\mathbb{R}^n$.

In other words, $A_K(\xi,t)$ is the Radon transform of the characteristic function $\chi_K$:

$$A_K(\xi,t) = \int_{\langle \xi, x \rangle = t} \chi_K(x) \, dv_{n-1}(x) = \text{vol}_{n-1}(K \cap \{\langle \xi, x \rangle = t\}).$$

The body $K$ is called polynomially integrable [1], if $A_K(\xi,t)$ is a polynomial with respect to $t$ so long as the above cross-section is non-empty.

It was proved in [9] (see also [1, 2]) that the only polynomially integrable domains with $C^\infty$ boundary are solid ellipsoids in odd-dimensional spaces. In particular, the sectional volume function $A_K(\xi,t)$ is never a polynomial in $t$ when $n$ is even. Nevertheless, for any ellipsoidal domain $E \subset \mathbb{R}^{2k}$, the squared sectional volume function $A_E^2(\xi,t)$ does polynomially depend on $t$. Thus, in any dimension, the sectional volume functions of ellipsoids are either polynomials or radicals of polynomials with respect to $t$. We conjecture that this property fully characterizes ellipsoids, disregarding the parity of the dimension of the space:

Conjecture 1.1. Let $K \subset \mathbb{R}^n$ be a compact body with $C^\infty$ boundary $\partial K$. Suppose that for some $m \in \mathbb{N}$ the $m$-th power $A_K^m(\xi,t)$ of the sectional volume function is a polynomial in $t$, whenever $A_K(\xi,t) \neq 0$. Then $\partial K$ is an ellipsoid.
If ∂K is an ellipsoid, then, in the case \( n \) is odd, the function \( A_K(\xi, t) \) is a polynomial with respect to \( t \) of degree \( \frac{n-1}{2} \), i.e., the condition is satisfied with \( m = 1 \), while if \( n \) is even, then \( A_K(\xi, t) \) is the square root of a polynomial of degree \( n-1 \) and therefore the condition is fulfilled with \( m = 2 \).

**Remark 1.2.** If Conjecture is true then the similar version using \( k \)-dimensional affine cross-sections, \( 1 \leq k \leq n-1 \) is fixed, is true. It immediately follows by applying Conjecture to intersections of \( K \) with \( k+1 \)-dimensional affine hyperplanes.

In Theorem 2.1 of this article, we confirm Conjecture for \( n = 2 \) and under a priori assumption of algebraicity of the boundary \( \partial K \). Applying this result to two-dimensional sections yields a characterization of \( n \)-dimensional ellipsoids in terms of the chord length function \( \hat{\chi}_K(L + t\xi) \), i.e., to the situation corresponding in Remark 1.2 to arbitrary \( n \) and \( k = 1 \).

Let us start with discussion of the basic, two-dimensional, case. When \( n = 2 \) then the hyperplanes are affine straight lines \( L_{\xi,t} = R \cdot \xi + t\xi = \{ x \in \mathbb{R}^2 : \langle x, \xi \rangle = t \} \) and the sectional volume function \( A_K(\xi, t) \) boils down to the chord length function

\[
A_K(\xi, t) = \hat{\chi}_K(L_{\xi,t}) = \text{length of the chord } K \cap L_{\xi,t}.
\]  

We want to characterize those domains \( K \) for which \( \hat{\chi}_K(\xi, t) \) is an algebraic function of a simple form, namely, is a radical of a polynomial in \( t \).

There is a relation of the question under discussion with a well known Newton’s Lemma about ovals (see [4]). It says that the area cut off a planar domain \( K \) with smooth boundary by a straight line is never algebraic function of the parameters of the secant line. The area \( V_{\pm}^K(\xi, t) = \text{area } (K \cap \{ x, \xi \geq t \}) \) of a portion of \( K \) on one side of the line (the solid volume function) is just the primitive function of the chord length function. Therefore, if \( A_K(\xi, t) = \sqrt[n]{P_\xi(t)} \) where \( P_\xi(t) \) is a polynomial in \( t \) then

\[
V_{\pm}^K(\xi, t) = \int \sqrt[n]{P_\xi(t)} dt
\]

is an Abelian integral.

Thus, Newton’s lemma says that the solid area function \( V_K \) of a domain \( K \) in the plane is always transcendental, Theorem 2.1 specifies that among those transcendental functions, Abelian integrals (1.2) characterize ellipses.

Multi-dimensional generalization of Newton’s Lemma are related to Arnold conjecture about algebraically integrable domains in \( \mathbb{R}^n \) (see [3], [4], [10]).

## 2. Main result

Let \( K \) be a compact (connected) domain in \( \mathbb{R}^n \). Given an affine line \( L \subset \mathbb{R}^n \) and a unit vector \( \xi \in \mathbb{R}^n \), we will call the function

\[
\mu_{L, \xi}(t) = \hat{\chi}_K(L + t\xi) = \text{length of } K \cap (L + t\xi), \quad t \in \mathbb{R},
\]

the chord length function.

We will be considering domains \( K \) which boundary \( \Gamma = \partial K \) is a semi-algebraic curve. This means that \( \Gamma \) is a connected component of the zero locus of a polynomial \( Q \) with real coefficients. The polynomial \( Q \) is assumed irreducible over the field \( \mathbb{C} \).

**Theorem 2.1.** Let \( K \subset \mathbb{R}^n, \ n \geq 2, \) be a compact convex domains with \( C^\infty \) semi-algebraic boundary \( \partial K \). Suppose for any fixed affine line \( L \), \( L \cap \text{int}K \neq \emptyset \), and unit
vector $\xi$, the chord length function has for small $t$ the form

$$\mu_{L,\xi}(t) = \sqrt[n]{P_{L,\xi}(t)},$$

where $m \in \mathbb{N}$ and $P_{L,\xi}(t)$ a polynomial with respect to $t$:

$$P_{L,\xi}(t) = \sum_{j=1}^{N} a_j(L,\xi)t^j.$$  

Then $\partial K$ is an ellipsoid (and therefore a posteriori $m$ can be taken 2).

Remark 2.2. The convexity of the domains in Theorem 2.1 can be derived from the main condition for the chord length function (see [1]) and hence Theorem 2.1 is valid without assumption of convexity. However, for the sake of simplicity of the exposition, we a priori assume the domain $K$ to be convex.

Theorem 2.1 is, in a certain sense, similar to Theorem 2 from [8] which states that if there exists a function $f$ with the X-ray transform identically one then the domain is a ball.

3. Outline of the proof of Theorem 2.1

The idea of the proof is as follows.

First of all, it suffices to prove Theorem 2.1 with $n = 2$, then the statement for arbitrary $n$ follows by considering two-dimensional affine cross-sections. In the case $n = 2$, the chord function $\mu_{L,\xi}(t)$ turns to $\mu_{L,\xi}(t) = A_K(\xi,t)$ if we take $L = \xi^\perp = \{x \in \mathbb{R}^2 : \langle \xi, x \rangle = 0 \}$. Also, we will use notation $P(\xi, t)$ instead of $P_{L}(\xi, t)$.

The key point is to determine the degree of the polynomial $t \to P(\xi, t)$. First of all, we want to obtain an upper bound for the degree. For this purpose it suffices to understand the order of growth of $P(\xi, t)$ as $t \to \infty$. Since the information about values of the polynomial $P(\xi, t)$ for large real $t$, when the line $\{\langle \xi, x \rangle = t \}$ becomes disjoint from $K$, is unavailable, we extend $P(\xi, t)$ for complex $t$.

At this point, we use algebraicity of the boundary curve $\partial K$. This curve has a natural complexification, which is a complex algebraic curve in $\mathbb{C}^2$. This allows us to construct, in Section 4, analytic extension of the chord length function $A_K(\xi, t)$ to complex values $t \in \mathbb{C}$. Determining the growth of the analytic extension along regular paths going to $\infty$ delivers the upper bound $\deg t P \leq m$ (Section 5).

The lower bound for $\deg t P(\xi, t)$ (Section 6) follows much easier, from vanishing the chord length function $A_K(\xi, t)$ on tangent lines to $\partial K$. We show that at Morse points the order of vanishing is $\frac{1}{2}$ and hence $P(\xi, t) = A_K(\xi, t)$ vanishes at tangent lines to the order $\frac{m}{2}$. Since there are two tangent lines with the same normal vector $\xi$, we conclude that $\deg t P \geq m$.

In Section 7, we finish the proof of Theorem 2.1. Together with the upper bound, this implies $\deg t P = m$ and hence all zeros of $P(\xi, t)$ are delivered by tangent lines. Knowing zeros allows us to reconstruct the polynomial $P(\xi, t)$ up to a factor depending on $\xi$, and express the chord length function $A_K(\xi, t)$ via the supporting function of $K$. Then the range conditions (the first three power moments) for X-ray transform applied to the function $A_K$ imply that the supporting function of $K$ coincides with the supporting function of an ellipse.
4. Analytic continuation of the chord length function

Let \( n = 2 \). Let \( K \) be a domain satisfying the conditions of Theorem 2.1. We assume that the boundary \( \Gamma = \partial K \) is a non-singular real semi-algebraic curve, which means that there is a real irreducible polynomial \( Q(x_1, x_2) \) such that

\[
Q(x) = 0, \quad x = (x_1, x_2) \in \Gamma
\]

and \( \nabla Q(x) \neq 0, x \in \Gamma \).

Extend polynomial \( Q \) to the complex space \( \mathbb{C}^2 \) and denote \( \Gamma^\mathbb{C} \) the complex algebraic curve

\[
\{z = (z_1, z_2) \in \mathbb{C}^2 : Q(z) = 0\}.
\]

The domain \( K \) is regarded as a set in the real subspace \( \mathbb{R}^2 = \{z \in \mathbb{C}^2 : \text{Im}z_1 = \text{Im}z_2 = 0\} \), so that \( \partial K \subset \Gamma^\mathbb{C} \cap \mathbb{R}^2 \).

Given a real unit vector \( \xi \in \mathbb{R}^2 \) and \( t \in \mathbb{C} \) denote the complex affine line in \( \mathbb{C}^2 \):

\[
X(\xi, t) = \{z \in \mathbb{C}^2 : \langle \xi, z \rangle = \xi_1 z_1 + \xi_2 z_2 = t\}.
\]

**Lemma 4.1.** Fix a unit vector \( \xi_0 \in S^1 \). There is a finite set \( Z_0 \subset \mathbb{R} \) such that is \( t_0 \in \mathbb{R} \setminus Z_0 \) and the real affine line \( \{\langle \xi_0, x \rangle = t_0\} \) intersects transversally the curve \( \Gamma = \partial K \) at two points \( a \) and \( b \) the following is true. There is a path \( T \subset \mathbb{C} \), joining \( t_0 \) and \( \infty \), an open connected neighborhood \( U \subset \mathbb{C} \) of \( T \) and two holomorphic mappings \( F^a, F^b : U \to \mathbb{C}^2 \) such that

(i) \( F^a(t), F^b(t) \in X(\xi_0, t) \cap \Gamma^\mathbb{C} \) for all \( t \in U \).
(ii) If \( t \in U \cap \mathbb{R} \) then \( F^a(t), F^b(t) \in \Gamma \).
(iii) \( F^a(t_0) = a, F^b(t_0) = b \).

**Proof** Applying a suitable rotation in \( \mathbb{R}^2 \), we can assume for simplicity that

\[
\xi_0 = (0, 1).
\]
In this case, $X(\xi, t)$ is the complex line \{\(z_2 = t\)\} in $\mathbb{C}^2$ and \(a = (a_1, t), \ b = (b_1, t)\). The condition \(z \in X(\xi_0, t) \cap \Gamma^C\) translates in this case as $Q(z_1, t) = 0$.

Then Lemma 4.1 asserts, in fact, that if we consider the projection $$\pi : \Gamma^C \to \mathbb{C}, \ \pi(z_1, t) = t$$ then there is a path $\mathcal{T} \subset \mathbb{C}$, joining $0, \infty$ and a neighborhood $U$ of $\mathcal{T}$ such that the holomorphic mapping $\pi$ possesses two holomorphic sections $F^a, F^b : U \to \Gamma^C$ of $\pi$ over the set $U$ with the initial conditions (i), (ii). The existence of such sections follows from the path lifting property (see, e.g., [5], Proposition 11.6) of covering maps and from the fact that the projection $\pi$ is a covering outside of the finite set of poles and ramification points.

Let us give more extended analytic arguments, for the sake of self-sufficiency. Represent the polynomial $Q$, defining the complex algebraic curve $\Gamma^C$, in the form

$$Q(z_1, z_2) = q_0(z_2) + q_1(z_2)z_1 + \cdots + q_M(z_2)z_1^M,$$

where $q_j$ are polynomials of one complex variable and $q_M \neq 0$.

Consider the discriminant $D(t) = \text{Disc}_{z_2} Q_t$ of the polynomial $Q_t(z_1) := Q(z_1, t)$:

$$D(t) = q_M^2(t) \prod_{i<j}(r_i - r_j)^2,$$

where $r_i$ are the roots of the polynomial $Q_t$.

The discriminant $D(t)$ is a polynomial in coefficients $q_0(t), \ldots, q_M(t)$ and therefore is itself a polynomial in $t$. It cannot vanish identically. Indeed, if $D(t) \equiv 0$ then for any $t \in \mathbb{C}$ either $q_M(t) = 0$ or the polynomial $Q_t$ has at least one multiple zero, which is a common zero of $Q_t$ and $\frac{\partial Q}{\partial z_2}$. Since the number of zeros of $q_M$ is finite, the set of common zeros of $Q$ and $\frac{\partial Q}{\partial z_2}$ must be then an infinite set in $\mathbb{C}^2$. By Bezout theorem this means that the two polynomials have a common polynomial factor, which is impossible as $Q$ is irreducible and $\frac{\partial Q}{\partial z_2}$ is of less degree than $G$. Therefore $D(t)$ is a nontrivial polynomial and hence the discriminant set

$$Z := \{t \in \mathbb{C} : D(t) = 0\}$$

is finite. We set $Z_0 = Z \cap \mathbb{R}$.

Let $t_0$ be real, $t_0 \notin Z_0$. Let the line \{\(\langle \xi_0, x \rangle = t_0\)\} intersects $\Gamma$ at the points $a$ and $b$.

Now choose a smooth simply-connected path $\mathcal{T} \subset \mathbb{C} \setminus Z$, joining $t_0$ and $\infty$ and avoiding the discriminant set $Z$. Consider the equation

$$Q_t(z_1) = Q(z_1, t) = 0, \ t \in \mathcal{T}.$$ 

Since $q_M(t) \neq 0$ for $t \in \mathcal{T}$, we can divide by $q_M(t)$ and reduce the polynomial in the equation to the form:

$$(4.1) \quad Q_t(z_1) = p_0(t) + p_1(t)z_1 + \cdots + p_{M-1}(t)z_{M-1} + z_1^M = 0,$$

where the coefficients

$$(4.2) \quad p_j(t) = \frac{q_j(t)}{q_M(t)}, \ j = 0, \cdots, M, \ p_M(t) = 1,$$

are continuous functions on $\mathcal{T}$.

Thus, we deal with an algebraic monic equation for $z_1$ with the coefficients, continuously depending on the parameter $t \in \mathcal{T}$. For any $t \in \mathcal{T}$ we have $t \notin Z$, hence $D(t) \neq 0$, i.e., all the roots of $Q_t$ are simple.
The monodromy theorem [7, Thm. 16.2] implies that the algebraic equation (4.1) so completely solvable. This means that there is no monodromy on \( t \in T \) and there exist \( M + 1 \) continuous functions \( f_0(t), \ldots, f_M(t) \) on \( T \), satisfying equation (4.1):

\[
Q_t(f_j(t)) = 0, \ j = 0, \ldots, M.
\]

Since all the roots are simple, we have \( f_i(t) \neq f_j(t) \) for all \( t \in T \) and \( i \neq j \).

Let us explain this point in more details. Consider the spaces

\[ B = \{(p_0, \ldots, p_{M-1}) \in \mathbb{C}^M : p_0 + p_1 z_1 + \ldots + p_{M-1} z_1^{M-1} + z_1^M \ has \ no \ multiple \ roots \}. \]

Define the mapping

\[ \pi : E \to B \]

as follows: \( \pi(\lambda), \lambda \in E \) is the vector \( p = (p_0, \ldots, p_{M-1}) \) of the coefficients of the monic polynomial with the roots \( \lambda_j \), i.e.,

\[
\sum_{j=0}^{M} p_j \lambda^j = \prod_{j=0}^{M} (\lambda - \lambda_j), \ p_M = 1.
\]

The roots \( \lambda_j \) of the polynomial in the right hand side are symmetric functions of the coefficients \( p_j \) and by Implicit Function Theorem, \( \pi \) is a \((M+1)\)!-covering map.

Let \( \lambda^{(0)} = (\lambda_0^{(0)}, \ldots, \lambda_M^{(0)}) \in E \) be the roots of the polynomial \( Q_{\theta_0}(z_1) = Q(z_1, t_0) \), i.e., \( \pi(\lambda^{(0)}) = (p_0(t_0), \ldots, p_{M-1}(t_0)) \).

The mapping

\[ g(t) := (p_0(t), \ldots, p_{M-1}(t)) \in B, \ t \in T, \]

where \( p_j(t) \) are coefficients (4.2) of the polynomial \( Q_t \) defines a path

\[ g : T \to B \]

in the base space \( B \). The path lifting property of covering mappings (see, e.g., [5], Proposition 11.6; [7], Theorem 16.2) says that there is a lifting path

\[ f = (f_0, \ldots, f_M) : T \to E \]

such that \( \pi \circ f = g \) and \( f(t_0) = \lambda^{(0)} \).

Then the functions

\[ f_j(t), j = 0, \ldots, M \]

define the above claimed continuous family of roots of the polynomials \( Q_t \).

The point \( a = (a_1, t_0) \), \( b = (b_1, t_0) \) satisfy \( Q(a_1, t) = Q(b_1, t) = 0 \). Therefore, there are two branches, say, \( f_j(t) \). \( f_j(t) \) which take at \( t_0 \) the values \( a_1 \), \( b_1 \), correspondingly. Denote \( f_i = f^a \), \( f_j = f^b \).

Then we have

1. \( Q(f^a(t), t) = Q_t(f^a(t)) = 0, \ t \in T, \)
2. \( Q(f^b(t), t) = Q_t(f^b(t)) = 0, \ t \in T, \)
3. \( f^a(t_0) = a_1, \ f^b(t_0) = b_1. \)

For any fixed \( t \in \mathbb{C} \) we have

\[
Q(f^a(t), t) = 0, \ \frac{\partial Q}{\partial z_1}(f^a(t), t) \neq 0,
\]
because $f^a(t)$ is a simple root. The same is true for $f^b(t)$. By Implicit Function Theorem, there is a complex neighborhood $U^a_t$ of $t$ and a complex neighborhood $V^a_t$ of $f_a(t)$ such that for any $s \in U^a_t$ there exists a unique $w := f^a(s) \in V^a_t$ such that $Q(w, s) = 0$, and the function $w = f^a(s)$ is holomorphic in $U^a_t$.

Thus, given $t \in T$ the function $f^a(t)$ extends to the neighborhood $U^a_t$ as a holomorphic function. The union $U^a = \cup_{t \in T} U^a_t$ constitutes an open connected set containing the path $T$. The function $f^a(t), t \in T$ extends to $U^a$ as a holomorphic function. The extensions satisfies the same polynomial equation $Q(f^a(t), t) = 0, t \in U^a$. Similarly, we construct an open set $U^b$ and the holomorphic function $f^b$ in $U^b$ with analogous properties.

Now set

$$U = U^a \cap U^b,$$

$$f^a(t) = (f^a(t), t), \quad f^b(t) = (f^b(t), t).$$

Check properties (i)-(iii). The mapping $F^a : U \to \mathbb{C}^2$ is holomorphic in $U$, and by construction $Q(F^a(t)) = Q(f_a(t), t) = 0$ for all $t \in U$. Also, since $\xi = (0, 1)$, we have $\langle \xi, F^a(t) \rangle = t$, hence $F^a(t) \in \Gamma^\xi \cap X(\xi, t)$.

Furthermore, $F^a(t_0) = (f^a(t_0), t_0) = (a_1, t_0) = a$. For $t \in U \cap \mathbb{R}$ near $t_0$, the straight line $x_2 = t$ intersects $\partial K$ at a point $a_t = (a_{1,t}, t)$ close to $a$. Then $Q(a_{1, t}, t) = 0$ for $t$ in a neighborhood of $t_0$, i.e. the polynomial $Q$ vanishes on an open subarc of the real-analytic curve $\partial K$. By the uniqueness theorem for holomorphic functions, the identity holds for all real $t \in U$. Since the root is unique, we conclude that $a_t = F^a(t)$ and thus $F^a(t) \in \partial K \cap U$. Hence the set $U$ and the mapping $F^a$ satisfy all properties (i)-(iii). Similarly, we proceed with the mapping $F^b$.

Lemma is proved.

**Lemma 4.2.** Assume, as in Theorem 2.1, $A^m_K(\xi, t) = P(\xi, t)$, where $P(\xi, t)$ is a polynomial in $t$. Fix a unit vector $\xi_0 \in S^1$. Let the straight line $\langle \xi_0, x \rangle = t_0$ intersects $\partial K$ at the points $a$ and $b$ and $t_0 \notin Z_0$, where $Z_0$ is the finite exceptional set from Lemma 4.1. Construct the open set $U \subset \mathbb{C}$ and the holomorphic mappings $F^a, F^b : U \to \mathbb{C}^2$ as in Lemma 4.1. Then

$$P(\xi_0, t) = (\langle \xi_0, F^a(t) \rangle - \langle \xi_0, F^b(t) \rangle)^m, \quad t \in U,$$

where $\xi_0 = \frac{a - b}{|a - b|}$ is the vector orthogonal to $\xi_0$.

**Proof** By Lemma 4.1 (i), (ii), when $t \in U \cap \mathbb{R}$ then the segment $[F^a(t), F^b(t)]$ is just the chord $X(\xi_0, t) \cap K$. The length of the chord is

$$A_K(\xi, t) = |F^a(t) - F^b(t)| = \langle F^a(t) - F^b(t), \frac{F^a(t) - F^b(t)}{|F^a(t) - F^b(t)|} \rangle.$$

Also, $\langle F^a(t) - F^b(t), \xi_0 \rangle = t - t = 0$ and hence the second factor in the inner product is a unit vector orthogonal to $\xi_0$ and does not depend on $t$. By taking $t = t_0$, we find, due to $F^a(t_0) = a$, $F^b(t_0) = b$, that this vector is $\xi_0 = \frac{a - b}{|a - b|}$.

Then $P(\xi_0, t) = A^m_K(\xi_0, t) = (\langle F^a(t) - F^b(t), \xi_0 \rangle)^m$ for $t \in U \cap \mathbb{R}$. Since $P(\xi, t)$ and the function in the right hand side of the equality are holomorphic in $t \in U$, and the set $U$ is open and connected, the equality is satisfied for all $t \in U$ by the uniqueness theorem. Lemma is proved.
5. Upper bound of the degree of the polynomial $P(\xi, t)$

As before, $n = 2$. We fix $\xi_0 \in S^1$ and $t_0 \in \mathbb{R}$ such that the line $\{\xi_0, x\}$ meets $\Gamma = \partial K$ at the points $a, b$ and $t$ does not belong to the finite exceptional set $Z_0$ in Lemma 4.1. Let the path $T \subset \mathbb{C}$ and the open set $U \subset \mathbb{C}$, $T \subset U$, be as in Lemma 4.1.

In order to obtain the upper bound for the degree of the polynomial $P(\xi_0, t)$, it suffices to estimate the order of its growth as $t \to \infty$. According to the representation (4.4), the problem is reduced to understanding the behaviour of the mappings $F^a(t), F^b(t)$ as $t \to \infty$.

Let

$$Q = Q_0 + \cdots + Q_N$$

be the decomposition of the polynomial $Q$, defining the complexified boundary $\Gamma^C$, into homogeneous polynomials $Q_j, \deg Q_j = j, j = 1, \ldots, N = \deg Q$.

The leading homogeneous polynomial $Q_N(z_1, z_2)$ of two complex variables is completely reducible over $\mathbb{C}$:

$$Q(z_1, z_2) = \text{const} \prod_{j=1}^J (A_j z_1 + B_j z_2)^{m_j}, \quad \sum_{j=1}^J m_j = N = \deg Q.$$

**Lemma 5.1.** Suppose that $a_0 B_j - \beta_0 A_j \neq 0$ for any $j = 1, \ldots, J$, where $\xi_0 = (\alpha_0, \beta_0)$. Then $\frac{F^a(t)}{t}, \frac{F^b(t)}{t}$ are bounded for $t \in U$.

**Proof** The zero set $Q_N^{-1}(0)$ of the homogeneous polynomial $Q_N$ in $\mathbb{C}^2$ consists of $J$ complex lines $A_j z_1 + B_j z_2 = 0, j = 1, \ldots, J$, counting multiplicities. The condition for $\xi_0$ means that the affine complex line $X(\xi_0, 0)$ is none of those. Again, it would be convenient to apply rotation in the real plane $\mathbb{R}^2$ and make $\xi_0 = (0, 1)$. Then we have $a_0 = 0, \beta_0 = 1$ and $A_j \neq 0$ for all $j = 1, \ldots, J$.

Let $z \in Q_N^{-1}(0) \cap X(\xi_0, t)$. It means that $z_2 = t$ and $Q(z) = Q(z_1, t) = 0$, or, the same,

$$Q_0 + tQ_1(\frac{z_1}{t}, 1) + \cdots + t^{N-1}Q_{N-1}(\frac{z_1}{t}, 1) + t^NQ_N(\frac{z_1}{t}, 1) = 0.$$

Dividing both sides by $t^N$ yields

$$\Psi(w, s) := s^N Q_0 + s^{N-1} Q_1(w, 1) + \cdots + s Q_{N-1}(w, 1) + Q_N(w, 1) = 0,$$

where $w, s$ and $z_1, t$ are related by

$$w = \frac{z_1}{t}, \quad s = \frac{1}{t}.$$

Denote

$$w_j = -\frac{B_j}{A_j}$$

the root of the polynomial $Q_N(w, 1)$, of multiplicity $m_j$.

Consider the logarithmic residue

$$r_j(s) = \frac{1}{2\pi i} \int_{|w-w_j| = \varepsilon} \frac{\Psi'(w, s)}{\Psi(w, s)} dw,$$

where $\varepsilon$ is so small that $w_j$ is the only root of $\Psi(w, 0) = Q_N(w, 1)$ in the disc $|w - w_j| \leq \varepsilon$. Then

$$r_j(0) = m_j.$$
Thus, the total sum is

\[ r_1(0) + \cdots + r_J(0) = m_1 + \cdots + m_J = N. \]

By continuity, for every \( j = 1, \ldots, J \) and sufficiently small \( \varepsilon > 0 \) there exists \( \delta_j > 0 \) such that for \( |s| < \delta_j, \Psi(w, s) \neq 0 \) when \( |w - w_j| = \varepsilon \). Then the function \( r_j(s) \) is continuous in \( |s| < \delta_j \) and integer-valued, therefore \( r_1(s) + \cdots + r_J(s) = N \) for \( |s| < \delta = \min\{\delta_1, \ldots, \delta_J\} \). This means that for \( |s| < \delta \), all \( N \) roots, counting multiplicities, of the polynomial \( w \to \Psi(w, s) \) are located in \( \varepsilon \)-neighborhood of the set of zeros of the polynomial \( \Psi(w, 0) = Q_N(w, 1) \).

Going back to the variable \( z_1 = tw \) and the function \( Q(z_1, t) \) we conclude that if \( |t| > \frac{1}{|a|} \), then \( |\frac{1}{t} - w_j| < \varepsilon \) for some root \( w_j \) of \( Q_N(w, 1) \).

Since we consider the situation \( \xi_0 = (0, 1) \), the condition (i) in Lemma 5.1 means that \( F^a(t), F^b(t) \) have the form

\[ F^a(t) = (f^a(t), t), \quad F^b(t) = (f^b(t), t), \quad t \in U. \]

Now, by the construction in Lemma 4.1 if \( t \in U \) then \( Q_t(z_1) = Q(z_1, t) \) has only simple roots \( \Lambda(t) = \{ \lambda_0(t), \ldots, \lambda_M(t) \} \). Therefore, when \( |t| > \frac{1}{3} \) then \( \frac{\lambda_i(t)}{t} \) are in an \( \varepsilon \)-neighborhood of a root of \( Q_N(w, 1) \). Among the collection \( \Lambda(t) \) of the roots of \( Q_t \), there are two branches, say, \( \lambda_i(t), \lambda_j(t) \), determined by the initial conditions \( \lambda_i(t_0) = a, \lambda_j(t_0) = b \). They must coincide, correspondingly, with \( f^a(t) \) and \( f^b(t) \) and hence

\[ \left| \frac{f^a(t)}{t} - w_i \right| < \varepsilon, \quad \left| \frac{f^b(t)}{t} - w_j \right| < \varepsilon, \quad t \in U, |t| > \frac{1}{\delta}. \]

Since the set of roots \( w_i \) is finite, we conclude that \( \frac{f^a(t)}{t}, \frac{f^b(t)}{t}, t \in U \), are bounded. Lemma is proved.

**Lemma 5.2.** For all \( \xi \in S^1 \) but finite set of those, \( \deg_t P(\xi, t) \) is at most \( m \).

**Proof** The set of vectors \( \xi_0 \) which do not satisfy the condition of Lemma 5.1 is finite. Let \( \xi \neq \xi_0 \) be not such a vector and let \( t_0, a, b \), be as in Lemma 4.1. By Lemma 4.2, \( A^m_K(\xi, t) = P(\xi, t) = \left( (\xi_+, F^a(t) - F^b(t)) \right)^m \). Then by Lemma 5.1, \( \frac{P(\xi, t)}{t_m} \) is bounded, as \( t \to \infty \), \( t \in U \). Therefore, \( \deg_t P(\xi, t) \leq m \).

6. The zeros and exact degree of \( P(\xi, t) \)

In this section \( n = 2 \). Given \( \xi \in S^1 \), denote

\[ \rho_+(\xi) = \max_{x \in K} (\xi, x), \quad \rho_-(\xi) = \min_{x \in K} (\xi, x). \]

Denote \( M_\pm(\xi) \in \partial K \) the points where the maximum and minimum are attained:

\[ \rho_+(\xi) = (\xi, M_+(\xi)), \quad \rho_-(\xi) = (\xi, M_-(\xi)). \]

The lines \( (\xi, x) = \rho_\pm(\xi) \) are supporting lines to \( \partial K \) at the points \( M_\pm(\xi) \) and the vectors \( \pm \xi \) are the unit outward normal vectors \( \pm \xi = \nu_{M_\pm(\xi)} \) to the curve \( \partial K \) at the points \( M_\pm(\xi) \), correspondingly.

**Lemma 6.1.** The polynomial \( P \) has the form

\[ P(\xi, t) = c(\xi) (\rho_+(\xi) - t) \frac{\phi}{\bar{\phi}} (t - \rho_-(\xi)) \frac{\overline{\phi}}{\phi}, \quad c(\xi) > 0. \]
Proof By continuity, it suffices to establish the representation for almost all \( \xi \). Since \( \partial K \) is a non-singular algebraic curve, it is real-analytic. Then all points, except finite number of those, are Morse points. Therefore, only for a finite vectors \( \xi \), the points \( M_k(\xi) \) are non-Morse. Choose \( \xi \) such that this is the case.

Using rotation and translation, we can assume that \( \xi = (0,1) \) and \( M(\xi) = (0,0) \). The outward normal vector at \( M_-(\xi) \) is \( -\xi = (0,-1) \) and \( \rho_-(\xi) = 0 \). In a neighborhood of the point \( M_-(\xi) = (0,0) \) the curve \( \Gamma \) can be represented as a graph \( x_2 = \varphi(x_1) \) of a real-analytic function, with \( \varphi'(0,0) = 0, \varphi''(0,0) \neq 0 \):

\[
x_2 = \frac{1}{2} \varphi''(0,0)x_1^2 + o(x_1), \quad x_1 \to 0.
\]

Then the length of the chord \( x_2 = t \) is \( A_K(\xi, t) = 2\sqrt{\frac{m}{\varphi''(0,0)}} + o(\sqrt{t}), t \to 0 \).

It shows that the length of the chord, obtained by the parallel translation of a tangent line to the distance \( t \), behaves at \( \sqrt{t} \). This yields that if \( M_\pm(\xi) \in \Gamma \) are Morse points then

\[
A_K(\xi, t) = \text{const}(t - \rho_-(\xi))^\frac{1}{2} + o((t - \rho_-(\xi))^\frac{1}{2}), \quad t \to \rho_-(\xi).
\]

Similarly,

\[
A_K(\xi, t) = \text{const}((\rho_+\xi) - t)^\frac{1}{2} + o((\rho_+\xi) - t)^\frac{1}{2}, \quad t \to \rho_+(\xi).
\]

Thus, the polynomial \( P(\xi, t) = A_K^m(\xi, t) \) vanishes, to the order \( \frac{m}{2} \), at \( t = \rho_+(\xi) \) and \( t = \rho_-(\xi) \). This means, firstly, that \( m \) is even and, secondarily, that \( \text{deg}_t P(\xi, t) \geq m \).

According to the remark at the beginning of the proof, it holds for all \( \xi \in S^1 \) except for a finite set. Together with Lemma 5.2 it proves that \( \text{deg}_t P(\xi, t) = m \) for all \( \xi \) except for a finite set, and for those \( \xi \) we have \( P(\xi, t) = c(\xi)(\rho_+(\xi) - t)^\frac{m}{2} (t - \rho_-(\xi))^\frac{m}{2} \). By continuity, \( P(\xi, t) \) has the claimed representation for all \( \xi \in S^1 \).

7. Proof of Theorem 2.1

First of all, as it has been mentioned before, it suffices to prove Theorem 2.1 for \( n = 2 \). Indeed, each transversal intersection of \( K \) with two-dimensional affine plane produces a domain in this plane satisfying all the conditions of Theorem 2.1. If we could conclude that all such two-dimensional cross-sections are bounded by ellipses then the entire body \( K \) is bounded by an ellipsoid.

Thus, we assume that \( n = 2 \). Then we follow the arguments from [1]. The representation of the polynomial \( P(\xi, t) \) given by Lemma 6.1 yields

\[
(7.1) \quad A_K(\xi, t) = \sqrt{P(\xi, t)} = d(\xi)(\rho_+(\xi) - t)^\frac{1}{2} (t - \rho_-(\xi))^\frac{1}{2},
\]

where \( d(\xi) = \sqrt{c(\xi)} \). Representation (7.1) holds whenever \( t \in [\rho_-(\xi), \rho_+(\xi)] \), otherwise \( A_K(\xi, t) = 0 \).

The next step is applying the range conditions for Radon transform [6]. Function \( A_K(\xi, t) \) is the Radon transform of the characteristic function \( \chi_K \) and hence satisfy the moment conditions. Namely, the moments

\[
(7.2) \quad M_k(\xi) = \int_{\rho_-(\xi)}^{\rho_+(\xi)} A_K(\xi, t)t^k dt
\]

must be restriction to the unit circle \( S^1 \) of a homogeneous polynomial of degree \( k \).

Notice, that \( M_0(\xi) = \text{area} K = \text{const} > 0 \).
Then substituting (7.2) into (7.1) yields:

$$M_k(\xi) = d(\xi) \int_{\rho_-}^{\rho_+} (\rho_+(\xi) - t)^{\frac{k}{2}} (t - \rho_-(\xi))^{\frac{k}{2}} \, dt.$$ 

Denote

$$B(\xi) = \frac{1}{2} (\rho_+(\xi) + \rho_-(\xi)),$$

$$C(\xi) = \frac{1}{2} (\rho_+(\xi) - \rho_-(\xi)),$$

and perform the change of variables in the integral

$$s = t - B(\xi).$$

Since

$$(\rho_+(\xi) - t)(t - \rho_-(\xi)) = (C(\xi)^2 - s^2),$$

substitution in the integral yields:

$$M_k(\xi) = d(\xi) \int_{-C(\xi)}^{C(\xi)} (C(\xi)^2 - s^2)^{\frac{k}{2}} (s + B(\xi))^{k} \, ds.$$ 

Finally, the change of variable

$$s = C(\xi)v$$

leads to

$$M_k(\xi) = G(\xi) \int_{-1}^{1} (1 - v^2)^{\frac{k}{2}} (C(\xi)v + B(\xi))^{k} \, dv$$

where

$$G(\xi) = d(\xi)C(\xi).$$

Let us write the first three moments:

$$M_0(\xi) = G(\xi)\alpha_0,$$

$$M_1(\xi) = G(\xi)(C(\xi)\alpha_1 + B(\xi)\alpha_0),$$

$$M_2(\xi) = G(\xi)(C^2(\xi)\alpha_2 + 2C(\xi)B(\xi)\alpha_1 + B^2(\xi)\alpha_0),$$

where $\alpha_k = \int_{-1}^{1} (1 - v^2)^{\frac{k}{2}} v^k \, dv$.

Since $\alpha_1 = 0$ we have

$$B(\xi) = \frac{M_1(\xi)}{M_0(\xi)}.$$ 

By the range conditions, $M_0(\xi) = const, G(\xi) = const$ on $S^1$ and $M_1(\xi)$ extends from $S^1$ as a homogeneous linear polynomial. Therefore, $B(\xi)$ is the restriction to the unit circle of a linear form:

$$B(\xi) = \langle \xi, b \rangle,$$

where $b$ is a fixed vector.

Since $M_2(\xi)$ extends from $S^1$ as a homogeneous quadratic polynomial, $G(\xi)$ is constant and $B(\xi)$ is a linear form, the third equality in (7.4) implies that $C^2(\xi)$ is
the restriction of a (strictly positive) quadratic form. Applying a suitable rotation we can reduce \( C^2(\xi) \) to the form

\[
C^2(\xi) = c_1\xi_1^2 + c_2\xi_2^2, \quad c_1, c_2 > 0.
\]

Now, by translating \( K \) by the vector \( b \), we can make \( B(\xi) = 0 \) for all \( \xi \). Indeed, the supporting functions \( \rho_{\pm} \) transform, under the translation by the vector \( b \), as follows:

\[
\rho_+(\xi) \rightarrow \rho_+(\xi) - \langle \xi, b \rangle, \quad \rho_-(\xi) \rightarrow \rho_-(\xi) - \langle \xi, b \rangle.
\]

Then (7.3) shows that \( B(\xi) \) transforms to \( B(\xi) - \langle \xi, b \rangle = 0 \).

Thus, we can apply the translation \( K \) by the vector \( b \) and assume that \( B(\xi) = 0 \). Then from (7.3) we have \( \rho_+(\xi) = -\rho_-(\xi) \) and \( C(\xi) = \rho_+(\xi) \). From (7.5), we obtain

\[
\rho_+(\xi) = \sqrt{c_1\xi_1^2 + c_2\xi_2^2}.
\]

Thus, the supporting function \( \rho_+(\xi) \) of the body \( K \) coincides with the supporting function of the ellipse

\[
E = \{ \frac{x_1^2}{c_1} + \frac{x_2^2}{c_2} = 1 \}.
\]

Thus, we conclude that \( \partial K = E \). Theorem 2.1 is proved.

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