Light-Like Wilson Line in QCD without Path Ordering

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Abstract—Unlike the Wilson line in QED the Wilson line in QCD contains path ordering. In this paper we get rid of the path ordering in the light-like Wilson line in QCD by simplifying all the infinite number of non-commuting terms in the SU(3) pure gauge. We prove that the light-like Wilson line in QCD naturally emerges when path integral formulation of QCD is used to prove factorization of soft and collinear divergences at all order in coupling constant in QCD processes at high energy colliders.

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1. INTRODUCTION

In Feynman diagrams the infrared divergences appear whenever the energy-momentum $k^\mu$ involved with the massless particle becomes very small. Similarly the collinear divergences occur when the momenta $\vec{k}, \vec{p}$ of two massless particles become parallel in the region $0 < k \ll p$. Typically the soft and collinear divergences occur in the Feynman diagrams due to momentum integration in the quantum loop diagrams involving massless propagators and due to momentum integration in the Feynman diagrams involving emission/absorption of massless particles. In quantum electrodynamics (QED) the massless particle is photon and in quantum chromodynamics (QCD) the massless particle is gluon. The soft and collinear divergences are severe in QCD than that in QED because massless gluons interact with each other whereas massless photons do not interact with each other. Since massless particle is always light-like one finds that soft and collinear divergences can be described by light-like Wilson line.

However, the physical quantities measured are all soft and collinear divergences free. Hence it is important to prove that all the non-canceling soft and collinear divergences in the perturbative Feynman diagrams are factorized in the definition of the (physical) gauge invariant non-perturbative quantities in QCD such as in the definition of the parton distribution function and fragmentation function at high energy colliders because the soft and collinear limit corresponds to long distance regime. This is done by supplying Wilson line in the definition of the parton distribution function and fragmentation function [1]. The factorization refers to separation of short-distance effects from the long-distance effects in quantum field theory.

The proof of factorization theorem in QCD is very non-trivial by using the diagrammatic method of QCD [2, 3] but it is enormously simplified by using the path integral method of QCD [4, 5]. The main idea behind the path integral method of QCD to prove factorization is to study the soft and collinear behavior of non-perturbative correlation function such as $\langle 0|\bar{\psi}(x)\psi(x')\bar{\psi}(x'')\psi(x''')...|0\rangle$ in QCD due to the presence of light-like Wilson line in QCD. Note that a light-like quark with light-like four-velocity $l^\mu$ produces SU(3) pure gauge potential at all the time-space points $x^\mu$ except at the spatial position $x$ transverse to the motion of the quark at the time of closest approach [2, 6, 7]. The soft and collinear divergences in Feynman diagrams in QCD can be studied by using Eikonal approximation for the propagators and vertices [1, 2, 8–15]. Hence due to the Eikonal approximation for soft and collinear divergences arising from the soft and collinear gluons interaction with the light-like quark, the light-like quark finds the gluon field $A^{\mu a}(x)$ as SU(3) pure gauge [4, 5]. The U(1) pure gauge

$$A^{\mu}(x) = \partial^\mu \alpha(x),$$

(1)
gives the light-like Wilson line in QED

$$e^{i \int_{\alpha} \frac{m^2}{2} d^4 A(x)}$$

(2)
which is used to study factorization of soft and collinear divergences in QED [8, 13]. In QCD the SU(3) pure gauge

$$T^a A^{\mu}(x) = \frac{1}{ig} \left[ \partial^\mu U(x) \right] U^{-1}(x),$$

(3)

$U(x) = e^{igT^a \omega(x)}$, where

$$T^a A^{\mu}(x) = \frac{1}{ig} \left[ \partial^\mu U(x) \right] U^{-1}(x),$$

(3)

$U(x) = e^{igT^a \omega(x)}$. 

\[417\]
gives the light-like Wilson line in QCD

$$\mathcal{P} e^{igT^a \int_{x_i}^{x_f} dx^\mu A^a_\mu(x)} = \exp \left[ i g T^a \int \frac{1}{2l_D [A(x_f)]} \frac{dA}{dg} \right] \tag{4}$$

which is used to study factorization of soft and collinear divergences in QCD [4, 5]. Note that, unlike the Wilson line in QED in Eq. (2) which does not contain path ordering $\mathcal{P}$, the Wilson line in QCD in Eq. (4) contains path ordering $\mathcal{P}$. In this paper we get rid of the path ordering $\mathcal{P}$ in the light-like Wilson line in QCD by simplifying all the infinite number of non-commuting terms in the SU(3) pure gauge in Eq. (3). We find that the light-like Wilson line in QCD without path ordering is given by

$$\mathcal{P} \exp \left[ ig \int x \ dx^\mu A^a_\mu(x) T^a \right] = \exp \left[ ig T^a \left[ \frac{1}{2l_D [A(x)]} \frac{dA}{dg} \right] \right] \times \exp \left[ -ig T^b \left[ \frac{1}{2l_D [A(x)]} \frac{dA}{dg} \right] \right] \tag{5}$$

where the right hand side of the above equation does not contain path ordering $\mathcal{P}$. In Eq. (5) the $D^a_{\mu}[A]$ is the covariant derivative, $l^a$ is the light-like four velocity and $A^\mu_a(x)$ is the SU(3) pure gauge in QCD, which unlike U(1) pure gauge $A^\mu(x)$ in QED, contains infinite powers of $g$ [6].

Since the light-like Wilson line in QCD does not depend on the path but depends only on the end points [4, 5] we find from Eq. (5) that the non-abelian phase or the gauge link in QCD without path ordering is given by

$$\mathcal{P} e^{igT^a \int dx^\mu A^a_\mu(x)} = \exp \left[ ig T^a \int \frac{1}{2l_D [A(x)]} \frac{dA}{dg} \right] \tag{6}$$

which is used to study factorization of soft and collinear divergences in QCD where the right hand side of the above equation does not contain the path ordering $\mathcal{P}$.

In this paper we will provide a derivation of Eq. (5). In [4] we have shown that the light-like Wilson line in QCD naturally emerges when path integral formulation of QCD is used to prove NRQCD factorization at all order in coupling constant. Hence we find that the light-like Wilson line in QCD naturally emerges when path integral formulation of QCD is used to prove factorization of soft and collinear divergences at all order in coupling constant in QCD processes at high energy colliders.

The paper is organized as follows. In section II we derive the light-like Wilson line in QCD without path ordering as given by Eq. (5). In section III we study the gauge transformation of the light-like Wilson line in QCD without path ordering. In section IV we prove that the light-like Wilson line in QCD naturally emerges when path integral formulation of QCD is used to prove factorization of soft and collinear divergences at all order in coupling constant in QCD processes at high energy colliders. Section V contains conclusions.

### 2. LIGHT-LIKE WILSON LINE IN QCD WITHOUT PATH ORDERING

The SU(3) pure gauge in QCD is given by Eq. (3) which contains infinite number of non-commuting terms. Simplifying all the infinite number of non-commuting terms in Eq. (3) we find that the SU(3) pure gauge $A^{\mu a}(x)$ is given by [6]

$$A^{\mu a}(x) = \partial^\mu \omega^a(x) \left[ \frac{g^{M(x)}}{g M(x)} \right] \tag{7}$$

where

$$M_{ab}(x) = f^{abc} \omega^c(x). \tag{8}$$

Expanding the exponential in Eq. (7) we find

$$A^{\mu a}(x) = \left[ \partial^\mu \omega^b(x) \right] \times [1 + \frac{g}{2} M(x) + \frac{g^2}{3!} M^2(x) + \frac{g^3}{4!} M^3(x) + ...]_{ab} \tag{9}$$

In QED the U(1) pure gauge potential produced by a point charge $e$ is linearly proportional to the electric charge $e$ [2, 6, 7], i.e.,

$$\partial^\mu \omega(x) = e. \tag{10}$$

Since $\omega(x)$ is linearly proportional to $e$ we find that $\omega^a(x)$ is linearly proportional to $g$ [6, 7]. Since $\omega^a(x)$ is linearly proportional to $g$ we write

$$\omega^a(x) = g \beta^a(x), \tag{11}$$
where $\beta^a(x)$ is independent of $g$. Using Eq. (11) in (9) we find

$$\frac{1}{g^2} A^{\mu a}(x) = \left[ \delta^{\mu \alpha} \beta^a(x) \right]$$

$$\times \left[ 1 + \frac{g^2}{2} N(x) + \frac{(g^2)^2}{3!} N^2(x) + \frac{(g^2)^3}{4!} N^3(x) + \ldots \right]_{ab},$$

where

$$N_{ab}(x) = f^{abc} \beta^c(x).$$

Multiplying $g^2 N_{ab}(x)$ in Eq. (12) we obtain

$$\left[ g N(x) A^{\mu a}(x) \right]_{ab}^a = \left[ \delta^{\mu \alpha} \beta^a(x) \right]$$

$$\times \left[ 1 + \frac{g^2}{2} N(x) + \frac{(g^2)^2}{3!} N^2(x) + \frac{(g^2)^3}{4!} N^3(x) + \ldots \right]_{ab}.$$ (14)

Adding $\delta^{\mu \alpha} \beta^a(x)$ in Eq. (14) we find

$$D^{\mu a}[A(x)] \beta^a(x) = \left[ \delta^{\mu \alpha} \beta^a(x) \right]$$

$$\times \left[ 1 + \frac{g^2}{2} N(x) + \frac{(g^2)^2}{3!} N^2(x) + \frac{(g^2)^3}{4!} N^3(x) + \ldots \right]_{ab},$$ (15)

where

$$D^{\mu a}_a[A(x)] = \delta^{\mu \alpha} \beta^a(x) + \bar{g}^{\mu \alpha \beta \gamma} A^\alpha_{\mu}(x).$$ (16)

Multiplying $g^2$ in Eq. (12) and then taking derivative with respect to $g^2$ we obtain

$$\frac{1}{2g^2} \frac{d}{dg} \left[ g A^{\mu a}(x) \right] = \left[ \delta^{\mu \alpha} \beta^a(x) \right]$$

$$\times \left[ 1 + \frac{g^2}{2} N(x) + \frac{(g^2)^2}{3!} N^2(x) + \frac{(g^2)^3}{4!} N^3(x) + \ldots \right]_{ab}.$$ (17)

Since right hand sides of Eqs. (15) and (17) are equal we find

$$D^{\mu a}[A(x)] \beta^a(x) = \frac{1}{2g^2} \frac{d}{dg} \left[ g A^{\mu a}(x) \right].$$ (18)

Converting $\beta^a(x)$ to $\omega^a(x)$ by using Eq. (11) we find from Eq. (18)

$$D^{\mu a}[A(x)] \omega^a(x) = \frac{1}{2g^2} \frac{d}{dg} \left[ g A^{\mu a}(x) \right].$$ (19)

Multiplying the same $x^\mu$ independent four vector $l^\mu$ in Eq. (19) we find

$$l^\mu \cdot \frac{d [g A^{\mu a}(x)]}{dg} = 2l \cdot D[A(x)] \omega^a(x).$$ (20)

Dividing $l \cdot D[A(x)]$ from left in Eq. (20) we obtain

$$\omega^a(x) = \left[ \frac{1}{2l \cdot D[A(x)]} \right]$$

$$\cdot \left[ \frac{1}{l \cdot D[A(x)]} \frac{d [g A^{\mu a}(x)]}{dg} \right]^a,$$ (21)

which gives the non-abelian phase

$$\Phi(x) = e^{ig T \omega^a(x)} = \exp \left[ ig T^{\mu a}(x) \right] \cdot \frac{1}{l \cdot D[A(x)]} \frac{d [g A^{\mu a}(x)]}{dg}.$$ (22)

From [4, 5] we find that the light-like Wilson line in QCD for soft and collinear divergences is given by

$$\mathcal{P} e^{ig \int_{x_f}^{x_i} dx^\mu A^a_{\mu}(x) T^a} = e^{ig T^{\mu a}(x_f)} e^{-ig T^{\mu a}(x_i)}$$

$$= \mathcal{P} e^{ig \int_{x_f}^{x_i} dx^\mu A^a_{\mu}(x_f) + \delta \int 0}.$$ (23)

Using Eq. (22) in Eq. (23) we find that the light-like Wilson line in QCD without path ordering is given by

$$\mathcal{P} \exp \left[ ig \int_{x_i}^{x_f} dx^\mu A^a_{\mu}(x) T^a \right]$$

$$= \exp \left[ ig T^{\mu a}(x_f) \right] \cdot \frac{1}{2l \cdot D[A(x)]} \frac{d [g A^{\mu a}(x_f)]}{dg}$$

$$\times \exp \left[ -ig T^{\mu a}(x_i) \right] \cdot \frac{1}{2l \cdot D[A(x_i)]} \frac{d [g A^{\mu a}(x_i)]}{dg},$$ (24)

which reproduces Eq. (5) where the right hand side does not contain the path ordering $\mathcal{P}$.

Since the light-like Wilson line in QCD does not depend on the path but depends only on the end points [4, 5] we find from Eqs. (23) and (22) that the non-abelian phase or the gauge link in QCD without path ordering is given by

$$\mathcal{P} e^{-ig \int_{x_f}^{x_i} dx^\mu A^a_{\mu}(x) T^a}$$

$$= \exp \left[ ig T^{\mu a}(x_f) \right] \cdot \frac{1}{2l \cdot D[A(x_f)]} \frac{d [g A^{\mu a}(x_f)]}{dg}$$

$$\times \exp \left[ -ig T^{\mu a}(x_i) \right] \cdot \frac{1}{2l \cdot D[A(x_i)]} \frac{d [g A^{\mu a}(x_i)]}{dg},$$ (25)

which reproduces Eq. (6) which is used to study factorization of soft and collinear divergences in QCD where the right hand side of the above equation does not contain the path ordering $\mathcal{P}$.

### 3. Non-Abelian Gauge Transformation of Light-Like Wilson Line in QCD without Path Ordering

In order to study the gauge transformation of the light-like Wilson line in QCD without path ordering
we proceed as follows. The non-abelian gauge transformation is given by

\[ T^a A^\mu_{a}(x) = U(x) T^a A^\mu_{a}(x) U^{-1}(x) \]

\[ + \frac{1}{ig} [\partial \mu U(x)] U^{-1}(x), \]  

(26)

where

\[ U(x) = e^{igT^a \omega^a(x)}. \]  

(27)

Since the matrices \( T^a \) are non-commuting we find from Eq. (27)

\[ T^a U^{-1}(x) = T^a e^{-igT^a \omega^a(x)} \]

\[ = T^a [1 + (-ig) T^b \omega^b(x) T^a + \frac{(-ig)^2}{2!} T^b T^c \omega^b(x) \omega^c(x) T^a + \frac{(-ig)^3}{3!} T^b T^c T^d \omega^b(x) \omega^c(x) \omega^d(x) T^a + \frac{(-ig)^4}{4!} T^b T^c T^d T^e \omega^b(x) \omega^c(x) \omega^d(x) \omega^e(x) T^a + \ldots] \]

(28)

By repeated use of the commutation relation

\[ [T^a, T^b] = if^{abc} T^c \]  

(29)

we find from Eq. (28)

\[ T^a U^{-1}(x) = \left[ T^a + (-ig) T^b \omega^b(x) T^a + \frac{(-ig)^2}{2!} T^b T^c \omega^b(x) \omega^c(x) T^a + \frac{(-ig)^3}{3!} T^b T^c T^d \omega^b(x) \omega^c(x) \omega^d(x) T^a + \frac{(-ig)^4}{4!} T^b T^c T^d T^e \omega^b(x) \omega^c(x) \omega^d(x) \omega^e(x) T^a + \ldots \right] \]

(30)

which gives after simplification

\[ T^a U^{-1}(x) = \left[ T^a + (-ig) T^b \omega^b(x) T^a + \frac{(-ig)^2}{2!} T^b T^c \omega^b(x) \omega^c(x) T^a + \frac{(-ig)^3}{3!} T^b T^c T^d \omega^b(x) \omega^c(x) \omega^d(x) T^a + \frac{(-ig)^4}{4!} T^b T^c T^d T^e \omega^b(x) \omega^c(x) \omega^d(x) \omega^e(x) T^a + \ldots \right] \]

\[ + \left[ 1 + (-ig) T^b \omega^b(x) + \frac{(-ig)^2}{2!} T^b T^c \omega^b(x) \omega^c(x) + \frac{(-ig)^3}{3!} T^b T^c T^d \omega^b(x) \omega^c(x) \omega^d(x) + \frac{(-ig)^4}{4!} T^b T^c T^d T^e \omega^b(x) \omega^c(x) \omega^d(x) \omega^e(x) + \ldots \right] \]

(31)

From Eq. (31) we find

\[ T^a U^{-1}(x) = U^{-1}(x) \left[ T^a + (-ig) if^{ade} \omega^d(x) T^e + \frac{(-ig)^2}{2!} if^{ade} \omega^d(x) if^{be} \omega^e(x) T^b + \frac{(-ig)^3}{3!} if^{ade} \omega^d(x) if^{be} \omega^e(x) if^{cf} \omega^f(x) T^c + \frac{(-ig)^4}{4!} if^{ade} \omega^d(x) if^{be} \omega^e(x) if^{cf} \omega^f(x) if^{dg} \omega^g(x) T^d + \ldots \right] \]

(32)

which gives

\[ U(x) T^a U^{-1}(x) = [e^{-\frac{g}{2} M(x)}]_{ab} T^b, \]  

(33)

where \( M_{ab}(x) \) is given by Eq. (8). From Eq. (33) we find

\[ U(x) T^a A^\mu_{a}(x) U^{-1}(x) = [e^{\frac{g}{2} M(x)}]_{ab} T^a A^\mu_{b}(x). \]  

(34)

Similarly by simplifying infinite number of non-commuting terms in \([\partial \mu U(x)] U^{-1}(x)\) we find [6]

\[ \frac{1}{ig} \sum_{ab} [\partial \mu U(x)] U^{-1}(x) = \left[ e^{\frac{g}{2} M(x)} - \frac{1}{g M(x)} \right]_{ab} [\partial \mu \omega^b(x)] T^a, \]  

(35)

where \( M_{ab}(x) \) is given by Eq. (8).
Hence by using Eqs. (34) and (35) in Eq. (26) we find

\[ A_{\mu}^{a}(x) = \left[ e^{gM(x)} \right]_{ab} A_{\mu}^{b}(x) + \left[ e^{gM(x)} - 1 \right] \left[ \partial_{\mu} \omega^{b}(x) \right] \] (36)

which is the finite gauge transformation in QCD where \( M_{ab}(x) \) is given by Eq. (8). Under infinitesimal gauge transformation we find from Eq. (36)

\[ A_{\mu}^{a}(x) = A_{\mu}^{a}(x) + g f^{abc} \omega^{c}(x) A_{\mu}^{b}(x) + \partial_{\mu} \omega^{a}(x) \] (37)

which is the infinitesimal gauge transformation in QCD which is familiar in the literature [16].

When \( A_{\mu}^{a}(x) \) is the SU(3) pure gauge we find by using Eq. (7) in (36) that

\[ A_{\mu}^{a}(x) = \left[ e^{2gM(x)} - 1 \right] \left[ \partial_{\mu} \omega^{a}(x) \right] \] (38)

By using Eq. (11) in (38) we find

\[ A_{\mu}^{a}(x) = \left[ e^{2gN(x)} - 1 \right] \left[ \partial_{\mu} \beta^{a}(x) \right] \] (39)

where \( N_{ab}(x) \) is given by Eq. (13) which is independent of \( g \) because \( \beta^{a}(x) \) is independent of \( g \). Multiplying the matrix \( gN(x) \) in Eq. (39) we obtain

\[ D_{\mu}^{a}[A^{a}(x)] \beta^{a}(x) = \left[ e^{2gN(x)} \right]_{ab} \left[ \partial_{\mu} \beta^{b}(x) \right] \] (40)

where

\[ D_{\mu}^{a}[A^{a}(x)] = \delta^{ab} \partial_{\mu} + g f^{abc} A_{\mu}^{c}(x). \] (41)

By multiplying \( g \) in Eq. (39) and then taking the derivative with respect to \( g \) we find

\[ \frac{d[g A_{\mu}^{a}(x)]}{dg} = 4g e^{2gN(x)} \left[ \partial_{\mu} \beta^{a}(x) \right] \] (42)

Using Eq. (40) in (42) we obtain

\[ \frac{d[g A_{\mu}^{a}(x)]}{dg} = 4g D_{\mu}^{a}[A^{a}(x)] \beta^{a}(x). \] (43)

By using Eq. (11) in (43) we find

\[ \frac{d[g A_{\mu}^{a}(x)]}{dg} = 4D_{\mu}^{a}[A^{a}(x)] \omega^{a}(x). \] (44)

By multiplying the same \( x^{\mu} \) independent four vector \( l^{\mu} \) in Eq. (44) we obtain

\[ l \cdot \frac{d[g A_{\mu}^{a}(x)]}{dg} = 4l \cdot D[A^{a}(x)] \omega^{a}(x). \] (45)

By dividing \( l \cdot D[A^{a}(x)] \) from left in Eq. (45) we find

\[ \left[ \frac{1}{2l \cdot D[A^{a}(x)]} l \cdot \frac{d[g A^{a}(x)]}{dg} \right]^{\mu} = 2\omega^{a}(x). \] (46)

Under the non-abelian gauge transformation as given by Eq. (26) we find from Eq. (22)

\[ \Phi'(x) = \exp \left[ ig T^{a} \left( - \frac{1}{2l \cdot D[A^{a}(x)]} l \cdot \frac{d[g A^{a}(x)]}{dg} \right)^{a} \right]. \] (47)

Hence from Eqs. (47), (46), (22) and (27) we find

\[ \Phi'(x) = U(x)\Phi(x), \quad \Phi'(x) = \Phi^{\dagger}(x)U^{-1}(x) \] (48)

which is the gauge transformation of the non-abelian phase in QCD under the non-abelian gauge transformation as given by Eq. (26).

From Eqs. (22), (23) and (48) we find that, under the non-abelian gauge transformation as given by Eq. (26), the light-like Wilson line in QCD transforms as

\[ \mathcal{P} e^{ig T^{a} \omega(x)} \]

which is the gauge transformation of the non-abelian gauge link in QCD under the non-abelian gauge transformation as given by Eq. (26).

From Eqs. (23) and (49) we find that, under the non-abelian gauge transformation as given by Eq. (26), the light-like Wilson line in QCD transforms as

\[ \mathcal{P} e^{ig T^{a} \omega(x)} \]

which is the gauge transformation of the non-abelian gauge link in QCD under the non-abelian gauge transformation as given by Eq. (26).

4. EMERGENCE OF LIGHT-LIKE WILSON LINE IN QCD IN THE PROOF OF FACTORIZATION THEOREM AT HIGH ENERGY COLLIDERS

Note that in [4] we have shown that the light-like Wilson line in QCD naturally emerges when path integral formulation is used to prove NRQCD factorization at all order in coupling constant in heavy quarkonium production. Similarly, in [5] we have shown that the light-like Wilson line in QCD naturally emerges when path integral formulation is used to prove factorization of soft and collinear divergences of the gluon distribution function at high energy colliders at all order in coupling constant. In this paper we will prove that the light-like Wilson line in QCD naturally emerges when path integral formulation is used to prove factorization of soft and collinear divergences of the quark distribution function at high energy colliders at all order in coupling constant.

The generating functional in the path integral method of QCD is given by [16, 17]

\[ Z[U, \eta, \bar{\eta}] = \int [dQ][d\psi][d\bar{\psi}] \det \left( \delta_{\mu} \frac{\partial^{a} Q^{\alpha \mu}}{\delta \omega^{b}} \right) \]

\[ \times \exp \left[ \int d^{4}x \left( \frac{\delta}{\mu^{2}} \right) \left( \psi^{\dagger} \gamma^{\mu} \partial_{\mu} \psi + \psi^{\dagger} \gamma^{\mu} m + 2g_{G} T^{a} \gamma^{\mu} \partial_{\mu} \psi \right) \right], \]
where \( J^{\mu a}(x) \) is the external source for the quantum gluon field \( Q^{\mu a}(x) \) and \( \bar{\eta}(x) \) is the external source for the Dirac field \( \psi(x) \) of the quark and

\[
F_{\mu \nu}^a[Q] = \partial_\mu Q_\nu^a(x) - \partial_\nu Q_\mu^a(x) + g f^{abc} Q_\mu^b(x) Q_\nu^c(x) - g f^{abc} A_\mu^b A_\nu^c, \tag{52}
\]

The light-like quark traveling with light-like four-velocity \( \Lambda^\mu \) produces SU(3) pure gauge potential \( A^{\mu a}(x) \) at all the time-space position \( \chi^\mu \) except at the position \( \vec{x} \) perpendicular to the direction of motion of the quark \( (\vec{v} \cdot \vec{x} = 0) \) at the time of closest approach \( 2, 6, 7 \). Hence the soft and collinear behavior of the non-perturbative correlation function in QCD due to the presence of light-like Wilson line in QCD can be studied by using path integral formulation of the background field method of QCD in the presence of SU(3) pure gauge background field \( 4, 5 \).

Background field method of QCD was originally formulated by 't Hooft [18] and later extended by Klueberg-Stern and Zuber [19, 20] and by Abbott [17]. This is an elegant formalism which can be useful to construct gauge invariant non-perturbative green's functions in QCD. This formalism is also useful to study quark and gluon production from classical chromo field [21] via Schwinger mechanism [22], to compute \( \beta \) function in QCD [23], to perform calculations in lattice gauge theories [24] and to study evolution of QCD coupling constant in the presence of chromo field [25].

It can be mentioned here that in soft collinear effective theory (SCET) [26] it is also necessary to use the idea of background fields [17] to give well defined meaning to several distinct gluon fields [9].

Note that a massive color source traveling at speed much less than speed of light can not produce SU(3) pure gauge field \( 2, 6, 7 \). Hence when one replaces light-like Wilson line with massive Wilson line one expects the factorization of soft/infrared divergences to break down. This in confirmation with the finding in [27] which used the diagrammatic method of QCD. In case of massive Wilson line in QCD the color transfer occurs and the factorization breaks down. Note that in case of massive Wilson line there is no collinear divergences.

The generating functional in the path integral formulation of the background field method of QCD is given by [17–19]

\[
Z[A, J, \eta, \bar{\eta}] = \int [dQ][d\bar{\eta}][d\psi]\det \left( \frac{\delta F^a_\mu(Q)}{\delta \omega^a} \right) e^{\int d^4x \left[ -\frac{1}{2} F_{\mu \nu}^a(Q) + \frac{1}{2} \delta^a_\mu \delta^b_\nu + g T^a i \bar{\psi} \gamma^\mu \gamma^\nu (A + Q) \psi + J \cdot \psi + \bar{\eta} \gamma_5 \psi \right]}, \tag{53}
\]

where the gauge fixing term is given by

\[
G^a(Q) = \partial_\mu Q^{\mu a} + g f^{abc} A_\mu^b Q^{\mu c} = D_\mu[A]Q^{\mu a} \tag{54}
\]

which depends on the background field \( A^{\mu a}(x) \) and

\[
F_{\mu \nu}^a[A + Q] = \partial_\mu [A_\nu^a + Q_\nu^a] - \partial_\nu [A_\mu^a + Q_\mu^a] + g f^{abc} [A_\mu^b + Q_\mu^b] [A_\nu^c + Q_\nu^c]. \tag{55}
\]

We have followed the notations of [17–19] and accordingly we have denoted the quantum gluon field by \( Q^{\mu a} \) and the background field by \( A^{\mu a} \).

Note that the gauge fixing term \( \frac{1}{2\alpha}(G^a(Q))^2 \) in Eq. (53) where \( G^a(Q) \) is given by Eq. (54) is invariant for gauge transformation of \( A^{\mu a}_\mu \):

\[
\delta A^{\mu a}_\mu = g f^{abc} A_\mu^b \omega^c + \partial_\mu \omega^a, \tag{56}
\]

(type I transformation)

provided one also performs a homogeneous transformation of \( Q^{\mu a}_\mu \) [17, 19]:

\[
\delta Q^{\mu a}_\mu = g f^{abc} Q^{\mu b}_\mu \omega^c. \tag{57}
\]

The gauge transformation of background field \( A^{\mu a}_\mu \) as given by Eq. (56) along with the homogeneous transformation of \( Q^{\mu a}_\mu \) in Eq. (57) gives

\[
\delta(A^{\mu a}_\mu + Q^{\mu a}_\mu) = g f^{abc} (A^{\mu b}_\mu + Q^{\mu b}_\mu) \omega^c + \partial_\mu \omega^a \tag{58}
\]

which leaves \( -\frac{1}{4} F_{\mu \nu}^a[A + Q] \) invariant in Eq. (53).

For fixed \( A^{\mu a}_\mu \), i.e., for

\[
\delta A^{\mu a}_\mu = 0, \tag{type II transformation} \tag{59}
\]

the gauge transformation of \( Q^{\mu a}_\mu \) [17, 19]:

\[
\delta Q^{\mu a}_\mu = g f^{abc} (A^{\mu b}_\mu + Q^{\mu b}_\mu) \omega^c + \partial_\mu \omega^a \tag{60}
\]

gives Eq. (58) which leaves \( -\frac{1}{4} F_{\mu \nu}^a[A + Q] \) invariant in Eq. (53).

It is useful to remember that, unlike QED [8], finding an exact relation between the generating functional \( Z[J, \eta, \bar{\eta}] \) in QCD in Eq. (51) and the generating functional \( Z[A, J, \eta, \bar{\eta}] \) in the background field method of
QCD in Eq. (53) in the presence of SU(3) pure gauge background field is not easy. The main difficulty is due to the gauge fixing terms which are different in both the cases. While the Lorentz (covariant) gauge fixing term $-\frac{1}{2\alpha}(\partial_{\mu}Q^{a\mu})^2$ in Eq. (51) in QCD is independent of the background field $A^{\mu}(x)$, the background field gauge fixing term $-\frac{1}{2\alpha}(G^a(Q))^2$ in Eq. (53) in the background field method of QCD depends on the background field $Q^a(x)$ which is given by Eq. (54) \cite{17–19}. Hence in order to study non-perturbative correlation function in the background field method of QCD in the presence of SU(3) pure gauge background field we proceed as follows.

By changing $Q \rightarrow Q - A$ in Eq. (53) we find

$$Z[A, J, \eta, \bar{\eta}] = e^{-\int d^4x J^\alpha A_{\alpha}} \left[ |dQ| |d\Psi| |d\bar{\Psi}| \right]$$

$$\det \left( \frac{\delta G^a_i(Q)}{\delta \omega^b} \right) e^{\int d^4x \left( -\frac{i}{2} \epsilon_{abc} Q^b \partial_{\mu} G^a(Q)^2 + J \cdot \bar{\Psi} \gamma^\mu \partial_{\mu} \Psi + g^{\mu\nu} \partial_{\mu} Q^a \Psi^a \right) \left[ \delta \omega^b \right]$$

(61)

where the gauge fixing term from Eq. (54) becomes

$$G^a_i(Q) = \partial_{\mu} Q^{a\mu} + g^{abc} A_{\mu} Q^{b\mu} - \partial_{\mu} A^{a\mu}$$

(62)

and Eq. (57) \cite{19, 17}] becomes

$$\delta Q^a_i = g^{abc} Q^b \omega^c + \partial_{\mu} \omega^a.$$ 

(63)

The Eqs. (62) and (63) can also be derived by using type II transformation which can be seen as follows. By changing $Q \rightarrow Q - A$ in Eq. (53) we find Eq. (61) where the gauge fixing term from Eq. (54) becomes Eq. (62) and Eq. (60) \cite{by using Eq. (59)] becomes Eq. (63). Hence we obtain Eqs. (61), (62) and (63) whether we use the type I transformation or type II transformation. Hence we find that we will obtain the same Eq. (84) whether we use the type I transformation or type II transformation.

\[ \int \frac{d^4x}{(2\pi)^4} \frac{1}{Z[0, J, \eta, \bar{\eta}]} \left[ |dQ'| |d\Psi'| |d\bar{\Psi}'| \right] \]

(67)

This is because a change of variables from unprimed to primed variables does not change the value of the integration. Under the finite transformation, using Eq. (66), we find

$$|dQ'| = |dQ| \det \left[ \frac{\partial Q'^a}{\partial \omega^b} \right] = |dQ| \det \left[ e^{\epsilon M(x)} \right]$$

(68)

$$= |dQ| \exp \left[ \text{Tr} \left( \ln e^{\epsilon M(x)} \right) \right] = |dQ|,$$

where we have used (for any matrix $H$)

$$\det H = \exp \left[ \text{Tr} \left( \ln H \right) \right].$$

The fermion field transforms as

$$\Psi'(x) = e^{\epsilon T^{\alpha}(x)} \Psi(x).$$

(70)

Using Eqs. (66) and (70) we find

$$|d\Psi'| |d\bar{\Psi}'| = |d\Psi| |d\bar{\Psi}|.$$ 

(71)

$$\Psi' \left[ i T^{\alpha} \partial_{\mu} - m + g T^{a\mu} Q^a_{\mu} \right] \Psi' = \Psi \left[ i T^{\alpha} \partial_{\mu} - m + g T^{a\mu} Q^a_{\mu} \right] \Psi,$$

$$F_{\mu\nu}^{a\alpha} = F_{\mu\nu}^{a\alpha}.$$
Using Eqs. (68) and (71) in Eq. (67) we find

\[
Z[A, J, \eta, \bar{\eta}] = e^{-\frac{i}{\hbar} \int^A \{ d\bar{\omega}[d\bar{\varphi}] + \varphi \}} 
\]

(72)

From Eq. (62) we find

\[
G^a_j(Q') = \partial^a_{\mu} Q^{\mu a} + \delta^{ab} A^b \partial^a_{\mu} Q^{\mu c} - \partial^a_{\mu} A^{\mu a}. 
\]

(73)

By using Eqs. (66) and (7) in Eq. (73) we find

\[
G^a_j(Q') = \partial^a_{\mu} [e^{\epsilon M(x)}]_{ab} Q^b_{\mu} (x) + \left[ \frac{e^{\epsilon M(x)} - 1}{g M(x)} \right] \partial^a_{\mu} \omega^b (x) 
\]

(74)

which gives

\[
G^a_j(Q') = \partial^a_{\mu} [e^{\epsilon M(x)}]_{ab} Q^b_{\mu} (x) 
\]

(75)

From Eq. (75) we find

\[
G^a_j(Q') = \partial^a_{\mu} [e^{\epsilon M(x)}]_{ab} Q^b_{\mu} (x) 
\]

(76)

which gives

\[
G^a_j(Q') = [e^{\epsilon M(x)}]_{ab} \partial^a_{\mu} Q^b_{\mu} (x) + \partial^a_{\mu} \partial^b_{\mu} [e^{\epsilon M(x)}]_{ab} 
\]

(77)

From [6] we find

\[
\frac{\partial^a_{\mu}}{\partial \omega^b} \left[ \frac{e^{\epsilon M(x)} - 1}{g M(x)} \right] T^a_{ab} \left[ e^{\epsilon M(x)} \right]_{cd}, 
\]

(78)

which in the adjoint representation of SU(3) gives (by using \( T^a_{bc} = -if^{abc} \))

\[
\frac{\partial^a_{\mu}}{\partial \omega^b} \left[ \frac{e^{\epsilon M(x)} - 1}{g M(x)} \right] g^{bac} \left[ e^{\epsilon M(x)} \right]_{cd}. 
\]

(79)

Using Eq. (79) in (77) we find

\[
G^a_j(Q') = [e^{\epsilon M(x)}]_{ab} \partial^a_{\mu} Q^b_{\mu} (x) 
\]

(80)

which gives

\[
(G^a_j(Q'))^2 = (\partial^a_{\mu} Q^{\mu a} (x))^2. 
\]

(81)

Since for \( n \times n \) matrices \( A \) and \( B \) we have

\[
\text{det}(AB) = (\text{det}A)(\text{det}B) 
\]

(82)

we find by using Eq. (80) that

\[
\text{det} \left[ \frac{\partial G^a_j(Q')}{\partial \omega^b} \right] = \text{det} \left[ \frac{\partial \left[ \frac{e^{\epsilon M(x)} - 1}{g M(x)} \right] T^a_{ab} \left[ e^{\epsilon M(x)} \right]_{cd}}{\partial \omega^b} \right] 
\]

(83)

\[
= \text{det} \left[ \frac{e^{\epsilon M(x)}}{g M(x)} \right]_{ac} \text{det} \left[ \frac{\partial G^a_j(Q')}{\partial \omega^b} \right] 
\]

(84)

Using Eqs. (81) and (83) in Eq. (72) we find

\[
Z[A, J, \eta, \bar{\eta}] = e^{-\frac{i}{\hbar} \int^A \{ d\bar{\omega}[d\bar{\varphi}] + \varphi \}} 
\]

(84)

From Eqs. (7) and (66) we find

\[
Q^a_{\mu} (x) = A^a_{\mu} (x) = [e^{\epsilon M(x)}]_{ab} Q^b_{\mu} (x), \quad M_{ab}(x) = f^{abc} \omega^c (x). 
\]

(85)
Note that Eqs. (84) and (85) are valid whether we use type I transformation [Eqs. (56) and (57)] or type II transformation [Eqs. (59) and (60)].

Since we have used Eq. (26) to study the gauge transformation of the Wilson line in QCD we will use type I transformation, see Eqs. (56) and (57), in the rest of the paper which gives for finite transformation \[ J_{\mu}^{\prime}(x) = e^{aM(x)}_{\mu} J_{\mu}^0(x), \quad M_{\mu\nu}(x) = f^{abc} \omega^c(x). \] (86)

From Eqs. (84), (85) and (86) we find

\[
\eta(x) = e^{igT^a(x)}\eta(x). (88)
\]

From Eqs. (70) and (88) we find

\[
\bar{\eta}^\prime \psi' = \bar{\eta} \psi, \quad \psi' = \psi \eta (89)
\]

When background field \(A^{ia}(x)\) is the SU(3) pure gauge as given by Eq. (3) we find from Eqs. (91), (93), (94), (86) and (88) that

\[
\langle 0 | \bar{\psi}(x) \psi(x') | 0 \rangle = \langle 0 | \bar{\psi}(x) \Phi(x) \Phi' \psi(x') | 0 \rangle \quad (95)
\]

which proves factorization of soft and collinear divergences at all order in coupling constant in QCD where [see Eq. (23) and \([4, 5]\)]

\[
\Phi(x) = \mathcal{P} \exp \left[ -igT^a \int_0^\infty d\lambda \int A^{ia}(x + \lambda \hat{n}) \right] = e^{igT^a a(x)} (96)
\]

is the non-abelian phase or the gauge-link in QCD.

From Eq. (95) we find that the correct definition of the quark distribution function at high energy colliders which is consistent with the number operator interpretation of the quark and is gauge invariant and is consistent with the factorization theorem in QCD is given by

\[
f_{q/P}(x) = \frac{1}{4\pi^2} \int dx' e^{-iP' x'} \times \left\langle \mathcal{P} \exp \left[ igT^a \int_0^\infty dz' A^{ia}(0, z', 0) \right] \psi(0) \right| P \right\rangle, (97)
\]

which is valid in covariant gauge, in light-cone gauge, in general axial gauges, in general non-covariant gauges and in general Coulomb gauge etc. respectively \([5]\). In Eq. (97) the \(\Phi(x)\) is the Dirac field of the quark and \(A^{ia}(x)\) is the SU(3) pure gauge background field as given by Eq. (3).

Hence we find from Eq. (97) and from \([4, 5]\) that the light-like Wilson line in QCD naturally emerges.
when path integral formulation of QCD is used to prove factorization of soft and collinear divergences at all order in coupling constant in QCD processes at high energy colliders.

5. CONCLUSIONS

Unlike the Wilson line in QED the Wilson line in QCD contains path ordering. In this paper we get rid of the path ordering in the light-like Wilson line in QCD by simplifying all the infinite number of non-commuting terms in the SU(3) pure gauge. We have proved that the light-like Wilson line in QCD naturally emerges when path integral formulation of QCD is used to prove factorization of soft and collinear divergences at all order in coupling constant in QCD processes at high energy colliders.

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