SYMMETRIES OF THE FREE SCHRODINGER EQUATION

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Abstract

An algorithm is proposed for research into the symmetrical properties of theoretical and mathematical physics equations. The application of this algorithm to the free Schrödinger equation permitted us to establish that in addition to the known Galilei symmetry the free Schrödinger equation possesses also the relativistic symmetry in some generalized sense. This property of the free Schrödinger equation permits the equation to be extended into the relativistic area of movements of a particle being studied.

1 Introduction

The symmetry properties of theoretical and mathematical physics equations contain the important information about objects of research. The relativization of space-time obtained by studying the space-time symmetries of Maxwell equations may be an example.

A number of approaches was proposed for studying symmetries: the method of replacing the variables [1], [2]; the classical Lie algorithm [3], [4]; the modified Lie algorithm [5], [6]; the algorithms of finding the generalized and non-Lie symmetries [3], [4], [7], [8]; the theoretical-algebraic approach [9], [10], [11]; the method for studying the conditional symmetries [12], [13], [14]; the renormgroup concept [15], [16]. The purpose of the present work is formulation of the algorithm [17], enabling to obtain additional information. The elements of this algorithm were elaborated by studying symmetries of D’Alembert and Schrödinger equations [18], [19], as well as by proving the Galilei symmetry of Maxwell equations [20], [21], [22] in a generalized sense. As an object of the research in this work, we choose the Schrödinger equation.

2 Formulation of the algorithm

Let us begin with a definition of symmetry which we shall call generalized one, and which we put in the base of the search for new symmetries.

Let the equation be given in the space $R^n(x)$

$$L\phi(x) = 0,$$

where $L$ is a linear operator.

Definition 1 By the symmetry of Eq. (1) we shall mean a set of operators $Q^{(p)}$, $p=1,2,\ldots,n,\ldots$ if any operator of the type $L^{(p-1)}Q^{(p)}$ transforms a solution $\phi(x)$ into another solution $\phi'(x) = L^{(p-1)}Q^{(p)}\phi(x)$. 


Definition leads to the operators $Q^{(p)}$ satisfying commutational relations of order $p$

$$[L [L \ldots [L, Q^{(p)}] \ldots ]]_{(p-\text{fold})} \phi(x) = 0$$ (2)

The generalized Definition:

- includes the understanding of symmetry also in the case that Eq. (2) is fulfilled on a set of arbitrary functions, i.e. that $[L [L \ldots [L, Q^{(p)}] \ldots ]]_{(p-\text{fold})} = 0$ [18];
- contains the standard understanding of symmetry $[L, Q^{(1)}] \phi(x) = 0$ with $p=1$ [7], [9];
- includes understanding of symmetry in quantum mechanics sense $[L, Q^{(1)}] = 0$ [23];
- differs from the standard one in accordance with which the framework of the latter by the operators of symmetry we must mean not the operators $Q^{(p)}$, but the operators $X^{(1)} = L^{(p-1)}Q^{(p)}$ [11].

The question is how practically to find them. In the present work it is made by analogy with the modified Lie algorithm [3], [4]. Below we shall consider the case with $p=2$.

Let us introduce a set of operators

$$Q^{(2)} = \xi^a_2(x) \partial_a + \eta^a_2(x),$$ (3)

which have the following commutation properties

$$[L [L, Q^{(2)}]] = \zeta_2(x)L$$ (4)

The given expression is operator’s version of the generalized Definition of symmetry with $p=2$. Here $\partial_a = \partial/\partial x^a$; $a = 0, 1, \ldots, n-1$; $\xi^a_2(x)$, $\eta^a_2(x)$, $\zeta_2(x)$ are unknown functions; the summation is carried out over a twice repeating index. The unknown functions may be found by equaling the coefficients at identical derivatives in the left and in the right sides of the ratios (4) and by integrating the resulting set of differential equations. This set we shall call the determining one by analogy with [6].

After integrating the general form of the operator $Q^{(2)}$ may be written as a linear combination of the basic elements $Q^{(2)}_\mu$ and $Q^{(2)}_\nu$, on which, by analogy with [3], we impose the condition of belonging to Lie algebras:

$$[Q^{(2)}_\mu, Q^{(2)}_\nu] = C_{\mu\nu\sigma}Q^{(2)}_\sigma$$ (5)

Here $C_{\mu\nu\sigma}$ are the structural constants; operators $Q^{(2)}_\sigma$ belong to the sets of operators $Q^{(2)}$. By integrating the Lie equations we shall transfer from the Lie algebras to the Lie groups

$$dx^a/\,d\theta = \xi^a(x'), \, x^a(\theta=0) = x^a,$$ (6)

where $a = 0, 1, \ldots, n-1$; $\theta$ is a group parameter [3], [4].

For the law of transforming the field $\phi(x)$ to be found, instead of integrating the corresponding Lie equations [3], [4]

$$d\phi'(x')/\,d\theta = \eta(x')\phi'(x'), \, \phi'(x')(\theta=0) = \phi(x)$$ (7)
we shall use the approach [20], [21] that we shall illustrate by example of one-component field. We shall introduce a weight function \( \Phi(x) \) into the field transformation law so that

\[ \phi'(x') = \Phi(x) \phi(x) \]  

We shall choose the function \( \Phi(x) \) so that Eq. (1) should transform into itself

\[ L' \phi'(x') = 0 \rightarrow x' = x'(x) \rightarrow L \phi(x) = 0, \]

\[ L' = L \]  

due to the additional condition, namely, compatibility of the set of the engaging equations

\[ A \Phi(x) \phi(x) = 0, \]
\[ L \phi(x) = 0 \]  

(10)
The former is obtained by replacing the variables in the initial equation \( L' \phi'(x') = 0 \). If \( A = L \), the symmetry will be called the classical or standard one; and the symmetry will be called the symmetry in generalized sense or the non-standard one if \( A \neq L \). By solving Set (10) the weight function \( \Phi(x) \) may be put in conformity to each field function \( \phi(x) \) for ensuring the transition (9).

Instead of solving Set (10) the weight function may be found on the base of the symmetry approach. As \( \phi'(x') = Q^{(1)}_o \phi(x') \) is a solution too, and \( \phi'(x') = \Phi(x) \phi(x) \) we have [20], [21]:

\[ \Phi(x) = \frac{\phi'(x' \rightarrow x)}{\phi(x)} \in \left\{ \frac{\phi(x' \rightarrow x)}{\phi(x)} ; \frac{Q^{(1)}_o \phi(x' \rightarrow x)}{\phi(x)} ; \frac{[L', Q^{(2)}_\mu] \phi(x' \rightarrow x)}{\phi(x)} ; \ldots \right\} \]  

(11)
Here the dots correspond to a consecutive action of the operators \( Q^{(1)}_o \) and \([L', Q^{(2)}_\mu]\) on a solution \( \phi(x') \). Thus, for the function \( \Phi(x) \) to be found it is necessary to turn to the unprimed variables in the primed solution \( \phi'(x') \) and to divide the result available by the unprimed solution \( \phi(x) \) [20].

After finding the weight functions \( \Phi(x) \) the task on the symmetry of Eq. (1) for one-component field may be thought as completed in the definite sense, namely: the set of the operators of symmetry and the appropriate Lie algebra are indicated for \( p=2 \); the group of symmetry is restored by the given algebra; the transformational properties of the field \( \phi(x) \) are determined with the help of the weight functions.

The symmetries found in such a way are distinguished by two symptoms from the ones found in accordance with the standard Lie algorithm in the classical or the modified versions: we use the commutational relations (11) of higher order than unit and the non-Lie condition of symmetry (10). The point is that the equation \( A \Phi(x) = 0 \) belonging to Set (10) is not Lie invariant because \( A \neq L \) in general case. However at the set of the solutions \{\( \Phi(x)\phi(x) \), \( \phi(x) \)\} equations (10) become compatible and it follows from \( L' \phi'(x') = 0 \) that \( L \phi(x) = 0 \). It is the non-Lie condition of symmetry (10) that is the reason for using the relation (8) instead of integrating the corresponding Lie equations (7).

Taking it into account, we note that stated algorithm allows generalization for the case of multicomponent field and symmetries of higher order than \( p=2 \) and consider the particular examples.
3 Application of the algorithm to the Schrödinger equation

Let $t$ designate the time; $x, y, z$ are the space variables; $m$ is the mass of a particle, $\phi(x)$ is the wave function. Then \[ L_s \phi(x) = (i\hbar \partial_t + \frac{\hbar^2}{2m} \Delta)\phi(x) = 0 \] (12)

We shall consider the given equation within the framework of the stated algorithm.

3.1 The symmetry of the type $p=1$. The Galilei symmetry of the Schrödinger equation

The symmetry properties of Eq. (12) with $p=1$ were investigated by Niderer \[5\], Fushchich and Nikitin \[6\], Hagen \[24\]. It has been established that the invariance algebra of Eq. (12) is the Lie algebra of the Schrödinger group $Sch_{13}$. The algebra generators commute on solutions with the operator $L_s$ from Eq. (12) and belong to the type of the operators $Q^{(1)}$ with $p=1$, for example

\[ [L_s, H_1]\phi(x) = 0 \] (13)

Here $H_1 = it\partial_x + (m/\hbar)x$ is the generator of the Galilei transformations

\[ x' = x - Vt, \quad y' = y, \quad z' = z, \quad t' = t, \] (14)

$V$ is the velocity of the inertial reference $K'$ relative to $K$. By means of Eq. (14) we may formulate the set of equations for finding the weight functions $\Phi(x)$:

\[ (L_s + i\hbar V \partial_x)\Phi(x)\phi(x) = 0; \]
\[ L_s \phi(x) = 0 \] (15)

Putting the free Schrödinger equation solution \[25\]

\[ \phi(x) = \exp\left[-\frac{i}{\hbar}(Et - \mathbf{x}.\mathbf{p})\right] \] (16)

into the first equation of the set (15) or into the formula (11) we have:

\[ \Phi(x) = \frac{\phi(x' \rightarrow x)}{\phi(x)} = \exp\left[-\frac{i}{\hbar}(-Et + xP)\right] \] (17)

Here $E = m\mathbf{v}^2/2$; $\mathbf{p} = m\mathbf{v}$; $E = mV^2/2$; $P = mV$; $\mathbf{v}$ is the speed of a particle. The function (17) coincides with the one obtained as a result of integrating the Lie equations \[3\], \[3\], \[24\].

Thus, as to the symmetry of the type $p=1$, the stated algorithm has resulted in the known conclusion: the Galilei transformations (14) are the symmetry transformations of the Schrödinger equation if the latter are accompanied by the transformation of the wave function (14) following the rule

\[ \phi'(x') = \Phi(x)\phi(x) = \exp\left[-\frac{i}{\hbar}(-Et + xP)\right]\phi(x) \] (18)

Similarly, the weight functions may be also found for the other space-time transformations which fall into the Schrödinger group. As a consequence of Set (13), Galilei symmetry of the Schrödinger equation is the non-standard one in sense of the terminology from Section 2.
3.2 The symmetry of the type p=2. Relativistic symmetry of the Schrödinger equation

The evidence for possible existence of the relativistic symmetry of the Schrödinger equation has already been obtained by Malkin and Man’ko [9], Fushchich and Segeda [26]. In particular, Ikin and n’ko showed that the Schrödinger equation with the Coulomb potential if considered in the momentum p-space could be reduced to the free D’Alembert equation in the special basis $q = f(p)$. Hence, it is conform-invariant in the q-space [9]. Fushchich and Segeda found the symmetry of the free Schrödinger equation with respect to the non-Lie, integro-differential operators which form the representation of the Lie algebra of the Lorentz group [26]. However below we shall not consider these works in details as the present paper is devoted to the investigation of the Lie invariance algebras in the 4-dimentional real space-time.

It is known that the generators of the Lorentz transformations $M_{0k} = x_0 \partial_k - x_k \partial_0$ ($x^0 = ct$, $x_k = -x^k$, $x^k = x, y, z$) and the operator $L_s$ of the Schrödinger equation satisfy the commutational relations of second order [19]:

$$[L_s, [L_s, M_{0k}]] = 0$$

(19)

Here $[L_s, M_{0k}] = \hbar (ic\partial_k + \hbar \partial_0 \partial_k/mc)$. Hence, the generators $M_{0k}$ are the symmetry operators of the Schrödinger equation in accordance with Definition 1 in Minkowski space if p=2 [19]. Therefore, besides the Galilei interpretation the Schrödinger equation admits also the relativistic one. Accordingly, we assume that the mass in Eq. (12) depends on the speed and follows the relativistic law $m = m_0 / \sqrt{1 - \beta^2}$, where $\beta = v/c$; $v$ is the velocity of a particle; $c$ is the speed of light; $m_0$ is the rest mass. Then we have:

$$L^*_r \phi(x) = (ih\partial_t + \frac{\hbar^2 \sqrt{1 - \beta^2}}{2m_0} \Delta)\phi(x) = (ih\partial_t + \frac{c^2 \hbar^2}{2W} \Delta)\phi(x) = 0$$

(20)

We shall call this equation the relativistic Schrödinger equation and give its solutions

$$\phi_1^r(x) = \exp[-i \frac{(\beta^2}{2Wt - P.x)] = \exp[-i \frac{mv^2}{2h}(t - \frac{nx}{v/2})]$$

(21)

$$\phi_2^r(x) = \exp[-i \frac{(Wt - \sqrt{2}}{\beta P.x)] = \exp[-i \frac{mc^2}{h}(t - \frac{nx}{c/\sqrt{2}})]$$

(22)

Here $W = mc^2$ and $P = mv$ are the relativistic energy and momentum of a particle; $n = v/v$ is the guiding vector of the velocity $v$. In the non-relativistic approach the solution $\phi_1^r(x)$ is reduced to the solution (16)

$$\phi_1^r(x, \beta \ll 1) \cong \phi_{1nr}^r(x) = \exp[-i \frac{m_0v^2}{2h}(t - \frac{nx}{v/2})]$$

(23)

The solution $\phi_2^r(x)$ takes the form

$$\phi_2^r(x, \beta \ll 1) \cong \phi_{2nr}^r(x) = \exp[-i \frac{m_0c^2}{h}(t - \frac{x.n}{c/\sqrt{2}})]$$

(24)
It is the new solution of the Schrödinger equation (14). It contains the information on the value of the particle rest mass and on the direction of its movement. Corresponding to this solution the wave (24) may be propagated with far higher speed \( c/\sqrt{2} \) than the velocity of a particle \( v \ll c \) and than the speed of the wave propagation \( v/2 \) from the solution (23).

Let us find the set of equations for the weight function \( \Phi(x) \) from Eq. (8). We shall introduce the Lorentz transformations

\[
x' = \frac{x - V t}{\sqrt{1 - V^2/c^2}}, \quad y' = y, \quad z' = z, \quad t' = \frac{t - x V/c^2}{\sqrt{1 - V^2/c^2}}
\]  
(25)

By replacing the variables in Eq. (21) and taking into account the transformational properties of the mass, the energy, the momentum, as well as the interrelations between the private derivatives we find

\[
[i \hbar (\partial_t + V \partial_x) + \frac{c^2 \hbar^2 (1 - V^2/c^2)}{2W(1 - \mathbf{V} \cdot \mathbf{v}/c^2)} [(\partial_x + V \partial_t/c^2)^2 + \partial_{yy} + \partial_{zz})] \Phi(x) \phi(x) = 0;
\]

\[L'_x \phi(x) = 0
\]

(26)

For the wave functions \( \phi_1(x) \) and \( \phi_2(x) \) from the formulas (21) and (22) the solutions of the Set (26) take the following forms:

\[
\Phi_1 = \exp\{-\frac{i}{2\hbar(1 - V^2/c^2)}\{(\beta'^2 - 2 \frac{V^2}{c^2} - \beta^2(1 - \frac{V^2}{c^2}))W - V(\beta'^2 - 2)P_x]\} t - (\beta'^2 - 2)(\frac{VW}{c^2} - \frac{V^2 P_x}{c^2})x\}];
\]

\[
\Phi_2 = \exp\{-\frac{i}{\hbar(1 - V^2/c^2)}\{(1 - \sqrt{2} \frac{\beta'^2}{\beta'^2})\frac{V^2 W}{c^2} - V P_x\} t + \frac{V}{c^2} (1 - \frac{\sqrt{2} \beta}{\beta'}) + \sqrt{2} (\frac{1 - \sqrt{2} \beta}{\beta'}) P_x x - (1 - \frac{\sqrt{2} \beta}{\beta'}) \frac{W V x}{c^2} + (1 - \frac{V^2}{c^2}) \sqrt{2} (\frac{1 - \sqrt{2} \beta}{\beta'}) (P_y y + P_z z)\}]
\]

(27)

Here \( \beta'^2 = [V^2(1 - \beta^2)/c^2 + \beta^2 - 2V \beta_x/c + V^2 \beta_x^2/c^2]/(1 - V \beta_x/c)^2 \); \( V \) is the velocity of frame \( K' \) relative to \( K \); \( c \) is the speed of light; \( \beta = v/c; \beta_x = v_x/c \).

In the non-relativistic approximation Set (26) is turned into Set (15), and the function \( \Phi_1(x, \beta \ll 1) \) from Eq. (27) coincides with the function (17). The non-relativistic limit of the function \( \Phi_2(x) \) is

\[
\Phi_2(\beta \ll 1) \cong \exp\{-i \sqrt{2} \frac{n_y c V (v_x - V)}{(V - V_x) V (v_x - V)} \{t - \frac{n_x (v - v') - V}{V (v_x - V)} x - \frac{n_y (v - v')}{V (v_x - V)} y - \frac{n_z (v - v')}{V (v_x - V)} z\}
\]

(29)

where \( v' = (v_x - 2V v_x + V^2)^{1/2} \).

In another case, when \( \beta = 1 \), the relativistic Schrödinger equation describes the propagation of the fields of wave functions corresponding to the movement of a massless particle with the velocity \( c \), the momentum \( \mathbf{P} = n \mathbf{P} \) and the energy \( W = c \mathbf{P} \):

\[
(i \hbar \partial_t + \frac{c^2 \hbar^2}{2W} \Delta) \phi(x) = (i \hbar \partial_t + \frac{c^2 \hbar^2}{2P^2} \Delta) \phi(x) = 0
\]

(30)
The solutions of this equation may be derived from the formulae (21) and (22) if \( \beta = 1 \).

\[
\phi_1^r(x, \beta = 1) = \exp\left[-i \frac{cP}{2 \hbar} (t - \frac{n \cdot x}{c/2})\right];
\]

\[
\phi_2^r(x, \beta = 1) = \exp\left[-i \frac{cP}{\hbar} (t - \frac{n \cdot x}{c/\sqrt{2}})\right]
\]

Here \( n = c/c \), the phase velocity of propagation for the first wave is equal to \( c/2 \) and \( c/\sqrt{2} \) for the second one. In both cases these velocities are less than the speed of light \( c \). As a result two wave functions with the different speeds of propagation and the different frequencies \( \omega_1 = cP/2\hbar \) and \( \omega_2 = cP/\hbar \) correspond to the same value of the momentum and the energy of a particle. The corresponding weight functions are:

\[
\Phi_1(x, \beta = 1) = \exp\left\{\frac{iVP}{2\hbar (1 - V^2/c^2)} [(n_x - \frac{V}{c})t + (1 - n_x \frac{V}{c})\frac{x}{c}]\right\};
\]

\[
\Phi_2(x, \beta = 1) = \exp\left\{\frac{i(\sqrt{2} - 1)VP}{\hbar (1 - V^2/c^2)} [(n_x - \frac{V}{c})t + (1 - n_x \frac{V}{c})\frac{x}{c}]\right\}
\]
Thus, the relativistic symmetry of the Schrödinger equation allows the possibility of existence of massless fields propagating with the speeds smaller than the speed of light. It permits the possibility of existence of some hypothetical massless particles with the set of continuous parameters (momentum $P$, energy $W$) and discrete ones ($c$, $c/2$, $c/\sqrt{2}$).

Within the framework of the accepted terminology the relativistic symmetry of the Schrödinger equation is the symmetry in generalized sense.

### 3.3 Maximal linear group symmetry of the Schrödinger equation with $p=2$

The maximal group of symmetry (the algebra of invariance) of an equation $L \phi(x) = 0$ means the symmetry group (the algebra of invariance) of the maximal dimension, permitted by the given equation in Lie sense. In the present work the concept of maximality is not unambiguous as it must be conformed with the value of the number $p$ in the commutational ratio $[L_s, Q]$. It is necessary to distinguish between the maximal dimensions of group or algebra with $p=1$, $p=2$ and etc.

For determining the dimensions of invariance algebras of the Schrödinger equation we shall start, by analogy with the modified Lie algorithm [5], [6] from the set of the determining differential equations corresponding to the conditions (4).

For example, first we will write the equations system in the case of $p=1$. By equaling the coefficients at identical derivatives in the left and the right sides of the ratios $[L_s, Q] = \zeta_1(x)L_s$, where $Q^{(1)} = \xi^a_1(x)\partial_a + \eta_1(x)$, we have:

\begin{align*}
\partial_1 \xi_1^1 &= \partial_2 \xi_1^2 = \partial_3 \xi_3^1, \\
\partial_1 \xi_1^2 + \partial_2 \xi_1^3 &= \partial_2 \xi_2^1 + \partial_3 \xi_2^3 = \partial_3 \xi_1^1 + \partial_1 \xi_3^3, \\
\partial_1 \xi_1^3 &= 0;
\end{align*}

\begin{align*}
L_s \xi_1^0 = i\hbar \zeta_1; \\
L_s \xi_1^k = -i(\hbar^2/m)\partial_k \eta_1, \quad k = 1, 2, 3;
\end{align*}

\begin{equation}
L_s \zeta_1 = 0
\end{equation}

Providing $\hbar = c = 1$, the system may be transformed into the system from [1]. The system solutions allow the general expression for symmetry operator $Q^{(1)}$ to be written as the linear combination of operators [3]:

\begin{equation}
Q^{(1)} = a^0 p_0 + a^k p_k + b^k J_k + d^k H_k + e M + g D + f K,
\end{equation}

where $a^0, a^k, b^k, d^k, e, g, f$ are the arbitrary numbers; $x^0 = t$, $x^k = -x_k = (x, y, z)$, $k=1,2,3$; $L_s = i\hbar \partial_0 + (\hbar^2/2m)\Delta$; $p_0 = i\partial_t$; $p_k = -i\partial_k$; $J_k = (xxp)_k$; $H_k = -tp_k + Mx_k$; $M = m/\hbar$; $D = 2tp_0 - x \partial_0 + 3i/2$; $K = t^2 - tD - Mx^2/2$ are the generators of Lie algebra of Schrödinger group $Sch_{13}$ [4], [3], [24]. From this it follows, that the maximal invariance algebra of Schrödinger equation with $p=1$ is the algebra Lie of the Schrödinger group $Sch_{13}$ [1], [4], [24].

In the case of the type $p=2$ symmetry, when $[L_s [L_s, Q^{(2)}]] = \zeta_2(x)L_s$ and $Q^{(2)}$ is the
operator (3), the analogue of Set (36) takes the form (27):

\[
\begin{align*}
\partial_1 \xi^1 & = 0; & \partial_2 \xi^1 & = -2\partial_1 \xi^2; & \partial_3 \xi^1 & = -2\partial_1 \xi^3; \\
\partial_1 \xi^2 & = -2\partial_2 \xi^1; & \partial_2 \xi^2 & = 0; & \partial_3 \xi^2 & = -2\partial_2 \xi^3; \\
\partial_1 \xi^3 & = -2\partial_3 \xi^1; & \partial_2 \xi^3 & = -2\partial_3 \xi^2; & \partial_3 \xi^3 & = 0; \\
\partial_1 \xi^2 + & \partial_2 \xi^2 + \partial_3 \xi^2 = 0; & \partial_3 \xi^2 & = 0; \\
L_s \partial_0 \xi^0 & = -(\hbar^2/2m)\Delta \partial_0 \xi^0 + (i\hbar^3/4m^2)\Delta^2 \xi^0 + \zeta_2, \\
L_s \partial_0 \xi^0 & = +(\hbar^2/2m)\Delta \partial_0 \xi^0, \\
L_s \partial_1 \xi^k & = -(\hbar^2/2m)(\Delta \partial_0 \xi^k + 4\partial_{k \alpha} \eta_2) + (i\hbar^3/4m^2)(\Delta^2 \xi^k + 4\Delta \partial_k \eta_2); \\
L_s \partial_1 \xi^k & = +\zeta_2/4; \\
L_s (\partial_j \xi^k + \partial_k \xi^j) & = -(\hbar^2/2m)\partial_{jk} \eta_2; \\
\partial_1 \eta_2 & = \partial_2 \eta_2 = \partial_3 \eta_2; \\
L_s \partial_0 \eta_2 & = +(i\hbar^3/4m^2)\Delta^2 \eta_2.
\end{align*}
\]  

(38)

Set (38) has the solution permitting the symmetry operator \(Q^{(2)}\) to be written as:

\[
Q^{(2)} = a^0 P_0 + a^k P_k + b^{0k} G_{0k} + b^{lm} G_{lm} + \ldots,
\]

(39)

where \(a^0, a^k, b^{0k}, b^{lm}\) are the arbitrary numbers; \(k, l, m = 1, 2, 3\); the summation is absent over a twice repeating index; \(P_0 = \partial_0, P_k = \partial_k, G_{0k} = x_0 \partial_k, G_{lm} = x_l \partial_m\) are the generators of the 20-dimensional Lie algebra

\[
[P_a, P_b] = 0; \quad [P_a, G_{bc}] = \delta_{ab} P_c; \quad [G_{ab}, G_{cd}] = \delta_{bc} G_{ad} - \delta_{ad} G_{cb}
\]  

(40)

Here \(\delta_{ab} = 0\) at \(a \neq b\), \(\delta_{aa} = 1, a, b = 0, 1, 2, 3\). The algebra generators satisfy the commutational ratios

\[
[L_s [L_s, P_a]] = 0; \quad [L_s [L_s, G_{ab}]] = 0
\]  

(41)

where \([L_s, P_a] = 0, [L_s, G_{ab}] = i\hbar \delta_{0a} \partial_b + (\hbar^2/2m)\delta_{ab} \partial_b\). Hence, the algebra (40) is the algebra of invariance of the Schrödinger equation when \(p=2\). It induces the 20-dimensional group IGL(4,R) of the non-uniform linear transformations of space-time variables in the space \(R^4(x)\):

\[
x^{a\ell} = A^a_b x^b + A^a
\]  

(42)

Here all the sixteen matrix elements \(A^a_b\) are different in general. As far as more total linear transformations do not exist in the space \(R^4(x)\), the group IGL(4,R) forms the maximal linear group of symmetry of the Schrödinger equation when \(p=2\). All the other space-time transformations, corresponding to the condition (1), are nonlinear ones. As an example, we shall show the transformations induced by the operators

\[
Q^{(2)}_1 = 2x^{02} \partial_0 + x^0 x^k \partial_k; \\
Q^{(2)}_{jk} = x^0 (x^j \partial_k - x^k \partial_j),
\]  

(43, 44)
where the functions $\xi_2^a$ and $\eta_2$ are the solutions of Set (8). The operators (13) and (14) satisfy the relations $[L_s[L_s, Q_1^{(2)}]] = 4i\hbar L_s, \ [L_s[L_s, Q_{jk}^{(2)}]] = 0$. We have by integrating the Lie equations (6) $(dx'/d\theta = 2x'^{02} \rightarrow 1/2x'^{0} = -\theta + 1/2x^{0}; \ dx^{kj}/d\theta = x'^{0}x'^{k'} \rightarrow dx^{k'}/x'^{k'} = x^{0}d\theta/(1 - 2\theta x^{0}); \ dx^{j'}/d\theta = -x'^{0}x'^{k'}, \ dx^{k'}/d\theta = x^{0}x^{j'} \rightarrow x^{j'}dx^{j'} = -x^{k'}dx^{k'} \rightarrow x^{j'^2 + x^{k'^2}} = x^{j^2 + x^{k^2}})$:

$$Q_1^{(2)} : x'^{0} = \frac{x^0}{1 - 2\Theta_1 x^0}; \ x' = \frac{x}{\sqrt{1 - 2\Theta_1 x^0}};$$

$$Q_{jk}^{(2)} : x'^{0} = x^0;$$

$$x^{j'} = x^j \cos(\Theta_{jk} x^0) - x^k \sin(\Theta_{jk} x^0);$$

$$x^{k'} = x^j \sin(\Theta_{jk} x^0) + x^k \cos(\Theta_{jk} x^0).$$

Here $\Theta_1$ and $\Theta_{jk}$ are the group parameters. As a result a theorem takes place.

**Theorem 1** The group $IGL (4, R)$ is the maximal linear group of symmetry of the free Schrödinger equation in the 4-dimensional real space-time $R^4(x)$.

Accordingly to Theorem the Galilei and relativistic symmetries are the natural properties of the Schrödinger equation, since the Galilei, Lorentz and Poincaré groups are the IGL(4,R) subgroups. The generators of these transformations may be constructed from the generators of the algebra (10) by their linear combinations. Since $[Q, [Q, P_a]] = 0, [Q, [Q, G_{ab}]] = 0$, where $[Q, P_a] = 0, [Q, G_{ab}] = 2(\delta_{a0} \delta_{b0} - \delta_{ab} \delta_{00})$, this group is the maximal group not only for the Schrödinger equation, but and for D’Alembert and Maxwell equations too. Thus, the IGL(4,R) group is the maximal linear symmetry group of the type p=2 both in the quantum mechanics, and in the classical electrodynamics.

### 3.4 The infinite algebra of invariance of the Schrödinger equation with $p \to \infty$

Let us turn to the formulae (8) and (11) from the algorithm of section 2. It can be seen, that any equation $L\phi(x) = 0$ for one-component field and, in particular, the Schrödinger equation has the property of invariance under arbitrary reversible transformations $x' = x'(x)$, $x = x(x')$ due to the following rations

$$L'\phi'(x') = 0 \rightarrow x' = x'(x), \ \phi'(x') = \Phi(x)\phi(x) \rightarrow A\Phi(x)\phi(x) = 0;$$

$$L\phi(x) = 0,$$

where the set of engaging equations $A\Phi(x)\phi(x) = 0, \ L\phi(x) = 0$ satisfies the condition of the compatibility, if the weight function is choosen in the form (11): $\Phi(x) = \phi(x')/\phi(x)$. The concept of maximality of a symmetry group then becomes uncertain in view of arbitrary of transformations $x' = x'(x)$.

In the case of the algebraic approach the given property of the Schrödinger equation results in that the symmetry operators of this equations may be constructed from elements of the infinite set of operators

$$\partial_a; \ x_b \partial_c; \ x_d x_g \partial_h; \ x_s x_p x_q \partial_t, \ldots$$

(48)
Here a, b, ..., q, t = 0, 1, 2, 3. These operators induce the infinite Lie algebra

$$[\partial_a, \partial_b] = 0;$$

$$[\partial_a, x_b \partial_c] = \delta_{ab} \partial_c;$$

$$[\partial_a, x_b x_c \partial_d] = \delta_{ab} x_c \partial_d + \delta_{ac} x_b \partial_d;$$

$$[\partial_a, x_b x_c x_d \partial_g] = \delta_{ab} x_c x_d \partial_g + \delta_{ad} x_b x_c \partial_g;$$

$$[x_a \partial_b, x_c \partial_d] = \delta_{bc} x_a \partial_d - \delta_{ad} x_c \partial_b;$$

$$[x_a \partial_b, x_c x_d \partial_g] = \delta_{bc} x_a x_d \partial_g + \delta_{bd} x_a x_c \partial_g - \delta_{ab} x_c x_d \partial_g;$$

$$[x_a \partial_b, x_c x_d x_g \partial_h] = \delta_{bc} x_a x_d x_g \partial_h + \delta_{bd} x_a x_c x_g \partial_h + \delta_{bg} x_a x_c x_d \partial_h - \delta_{ab} x_c x_d x_g \partial_h;$$

$$[x_a x_b \partial_c, x_d x_g \delta h] = \delta_{cd} x_a x_b x_g \delta h + \delta_{cg} x_a x_b x_d \delta g - \delta_{ah} x_b x_d x_g \delta c;$$

$$[x_a x_b \partial_c, x_d x_g x_h \partial f] = \delta_{cd} x_a x_b x_f x_h \partial f + \delta_{cg} x_a x_b x_f x_h \partial f +$$

$$\delta_{ch} x_a x_b x_d x_f x_g \partial h - \delta_{af} x_a x_d x_f x_c \delta c.$$

Let us assume a = 0 in the formula (18), and take into account the commutational ratios

$$[\triangle, \partial_a] = 0;$$

$$[\triangle, x_a \partial_b] = 2\delta_{ka} \partial_{kb};$$

$$[\triangle, x_a x_b \partial_c] = 2\delta_{ka} \delta_{kb} \partial_c + 2\delta_{kb} x_b \partial_{kc} + 2\delta_{ka} x_a \partial_{kc};$$

$$[\triangle, x_a x_b x_c \partial_d] = 2\delta_{ka} \delta_{kb} x_d \partial_{kd} + 2\delta_{ka} \delta_{kc} x_b \partial_{kd} +$$

$$2\delta_{kb} \delta_{kc} x_a \partial_d + 2\delta_{ka} x_b x_c \partial_{kd} + 2\delta_{kb} x_a x_c \partial_{kd} + 2\delta_{ka} x_a x_c \partial_{kd}.$$

It is possible to show then, that the generators (18) of the Lie algebra of (49) - (51) and the operator of the Schrödinger equation \( L_s \) satisfy the commutational ratios (2). We omit the details, whose example can be the formula

$$[L_s, [L_s, x_a x_b x_c \partial_d]]] = -6i h^3 \delta_{0a} \delta_{0b} \delta_{0c} \partial_d -$$

$$6(h^4/m)(\delta_{ka} \delta_{0b} \delta_{0c} + \delta_{kb} \delta_{0a} \delta_{0b} + \delta_{kc} \delta_{0a} \delta_{0b}) \partial_{kd} +$$

$$6i(h^5/m^2)(\delta_{ka} \delta_{j0} \delta_{0c} + \delta_{kb} \delta_{j0} \delta_{0a} + \delta_{kc} \delta_{j0} \delta_{0b}) \partial_{jkd} +$$

$$2(h^6/m^3)(\delta_{ka} \delta_{j0} \delta_{m0} + \delta_{kb} \delta_{j0} \delta_{m0} + \delta_{kc} \delta_{j0} \delta_{m0}) \partial_{jkm};$$

and demonstrate only the final result:

$$[L_s, \partial_a] = 0;$$

$$[L_s, [L_s, x_a \partial_b]] = 0;$$

$$[L_s, [L_s, [L_s, x_a x_b \partial_c]]] = 0;$$

$$[L_s, [L_s, [L_s, x_a x_b x_c \partial_d]]] = 0.$$
Theorem 2  The infinite Lie algebra of the operators (48) is the algebra of invariance of the free Schrödinger equation when $p \to \infty$.

In addition, it can be seen, that this algebra is the invariance algebra for the D’Alembert equation too.

4  The conclusion

The algorithm is proposed to study the symmetry properties of theoretical and mathematical physics equations of the type $L\phi(x) = 0$. The main distinctive features of this algorithm are:

- introduction of the operators of symmetry $Q^{(p)}$ satisfying the commutational ratios of higher order than $p=1$: $[L \ldots [L, Q^{(p)}] \ldots]_{(p - \text{fold})}\phi(x) = 0$;

- introduction of some weight function $\Phi(x)$ which is not a component of field into the law of field transformation $\phi'(x') = \Phi(x)\phi(x)$;

- interpretation of the compatibility of the set of equations $A\Phi(x)\phi(x) = 0$, $L\phi(x) = 0$, where $A\Phi(x)\phi(x) = 0$ is obtained by replacing the variables $x' = x'(x)$ in the initial equation $L'\phi'(x') = 0$, as the condition of transformation into itself of the initial equation $L'\phi'(x') = 0 \to L\phi(x) = 0$, $L' = L$.

The application of the algorithm to the Schrödinger equation (earlier to the D’Alembert and Maxwell equations) has allowed us to establish, that in addition to the known (standard, $p=1$) symmetry the Schrödinger equation has the relativistic symmetry (the D’Alembert and Maxwell equations have the Galilei symmetry) when $p=2$. This circumstance permits the Schrödinger equation to be extended to the area of relativistic movements with relativistic values of mass, energy and momentum of the particle under study. The Galilei and the relativistic symmetries are the particular realizations of the more general symmetry of equations with respect to the 20-dimensional group of non-uniform transformations $\text{IGL}(4,\mathbb{R})$ in the real space-time $R^4(x)$ with $p=2$. When $p \to \infty$, the Schrödinger and D’Alembert equations display the symmetry with respect to the infinite Lie algebra which contains the Lie algebras of the groups $\text{IGL}(4,\mathbb{R})$, Galilei, Lorentz and Poincaré as the subalgebras.

The relativistic symmetry of the Schrödinger equation permits the possibility of existence of some hypothetical massless particle, moving with the speed of light $c$ and characterized by the two wave fields propagating behind of the particle with the speeds $c/2$ and $c/\sqrt{2}$.

In summary, it is possible to state that the concept of symmetry is conventional. It depends on the definition of symmetry and the associated algorithm. Dividing the equations into the relativistic and the Galilei-invariant equations makes sense only in the case of the narrow understanding of symmetry when $p=1$. In more general case, when $p \geq 1$, equations have cumulative symmetrical properties complying with the principles of relativity in the relativistic, in the Galilei, as well as in the other versions. In accordance with F. Klein’s [28], we may state that the equation satisfies as many principles of relativity, as groups of symmetry exist for this equation.
5 Acknowledgments

The author is deeply grateful to Prof. V. I. Man’ko and Prof. V. I. Fushchich for discussing various fragments of the present work and valuable remarks.

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