ON COMPLETE DEGENERATIONS OF SURFACES WITH ORDINARY SINGULARITIES IN $\mathbb{P}^3$

V.S. KULIKOV AND VIK.S. KULIKOV

Abstract. We investigate the problem of existence of degenerations of surfaces in $\mathbb{P}^3$ with ordinary singularities into plane arrangements in general position.

Introduction.

In the article we investigate degenerations of surfaces in $\mathbb{P}^3$ with ordinary singularities. To begin, consider the classical prototype of this situation, namely, degenerations of plane algebraic curves. As is known, any smooth projective curve can be projected to $\mathbb{P}^2$ onto a nodal curve $C$ — a curve with ordinary double points — nodes (singularities of type $A_1$). According to Severi Theorem ([1],[2]) any nodal plane curve of degree $m$ can be degenerated into an arrangement of $m$ lines in general position $\mathcal{L} = \bigcup_{i=1}^{m} L_i \subset \mathbb{P}^2$. Such degeneration defines a subset $S_0$ of the set $S$ of double points of the curve $\mathcal{L}$, which consists of the limit double points of the curve $C$. Conversely, for any subset $S_0 \subset S$ there exists a degeneration of nodal curves (may be, reducible) for which $S_0$ is the set of limit double points ([3]), in other words, there exists a smoothing of double points $S \setminus S_0$ of the curve $\mathcal{L}$ (preserving double points lying in $S_0$).

In dimension 2, the analog of nodal curves are surfaces in $\mathbb{P}^3$ with ordinary singularities. Only such singularities appear under generic projections of a smooth surface $X \subset \mathbb{P}^r$ to $\mathbb{P}^3$ (see, for example, [4] and [5]). Let $Y \subset \mathbb{P}^3$ be a surface of degree $m$ with ordinary singularities. This means that the singular set $\text{Sing}Y$ is the double curve $D$; the curve $D$ itself is smooth except the finite set of triple points $T$; the generic point $y \in D$ is nodal on $Y$ (locally defined by equation $xy = 0$); the points $y \in T$ are triple points of $Y$ (locally, $xyz = 0$); besides, $Y$ has a finite set of pinch points at which $Y$ is locally defined by equation $x^2 = y^2z$.

\footnote{This research was partially supported by grants of NSh-9969.2006.1, RFBR 08-01-00095 and RFBR 06-01-72017-MNTI-a. The research was started during the stay of the second author in Centro di Ricerca Matematica Ennio de Giorgi (Program: Groups in Algebraic Geometry).}
Maximally degenerate (reducible) surface of degree $m$ with ordinary singular-
ities is an arrangement $\mathcal{P} = P_1 \cup \cdots \cup P_m \subset \mathbb{P}^3$ of $m$ planes in general position. In this case the double curve $\text{Sing} \mathcal{P} = \mathcal{L}$ consists of $\binom{m}{2}$ lines $L_{i,j} = P_i \cap P_j$, 
\[ \mathcal{L} = \bigcup_{1 \leq i < j \leq m} L_{i,j}, \]
on which there lie $\binom{m}{3}$ triple points, 
\[ T = \bigcup T_{i,j,k}, \quad T_{i,j,k} = P_i \cap P_j \cap P_k, \quad 1 \leq i < j < k \leq m. \]
A degeneration of surfaces (or, in general, of varieties) of given type is a
one-dimensional family, or for brevity, its zero fibre, generic fibre of which is a
surface of given type (usually, smooth). We consider families, all fibres of which
are surface in $\mathbb{P}^3$ with ordinary singularities. A degeneration is a surface, which
has ”more” singularities than generic fibre. More precisely, a degeneration of a
surface $Y \subset \mathbb{P}^3$ with ordinary singularities is a flat family of embedded surfaces
$Y_u \subset \mathbb{P}^3$ with ordinary singularities, parametrized by points $u \in U$ of a smooth
curve $U$ (or even, of a disk $U \subset \mathbb{C}$), such that
(i) for the generic point $u \in U$ the fibres $Y_u$ have singularities of the same
type as $Y = Y_{u_1}$, and the fibre $Y_{u_0}$ over the point $u_0 = 0$ is called
degenerate;
(ii) there is a flat family $D_u \subset Y_u$, where $D_u$ is the double curve of the
surface $Y_u$ for $u \neq u_0$, and the curve $D = D_{u_0} \subset Y_{u_0}$ is called the limit
double curve;
(iii) there is a flat family $T_u, u \in U$, where $T_u \subset D_u$ is the set of triple points
of the surface $Y_u$ for $u \neq u_0$, and $T_{u_0} = T_3$ is the set of triple points of the
curve $D$.
A degeneration is called complete if the degenerate fibre $Y_{u_0} = \mathcal{P}$ is a plane
arrangement in general position.
A complete degeneration defines a limit double curve $D \subset \mathcal{L}$. Conversely,
if a line arrangement $D \subset \mathcal{L}$ is selected and there is a complete degeneration
of surfaces $Y_u \subset \mathbb{P}^3$ (not necessary irreducible) with ordinary singularities such
that $D_{u_0} = D$, then we say that the plane arrangement $\mathcal{P}$ is smoothed outside
of $D$.
As in the case of curves, there are two questions. Whether every surface
$Y \subset \mathbb{P}^3$ with ordinary singularities can be completely degenerated? Whether
every line arrangement $D \subset \mathcal{L}$ of a plane arrangement $\mathcal{P}$ can be smoothed
outside of $D$?
The problem of degeneracy of smooth surfaces $X \subset \mathbb{P}^N$ into a plane arrangements
was investigated previously, but in another setting, which is analogous to
the Zeuthen’s problem for curves (the degenerate surfaces are surfaces of Zappa,
or so called Zappatics; a survey of obtained results about Zappatic surfaces can be found in [6], see also [7]).

In spite of the fact that in many cases surfaces with ordinary singularities can be completely degenerated (see § 3 and § 6), we shall show in the article that in general the answers to the questions formulated above are in negative. The reason for impossibility to completely degenerate a surface with ordinary singularities can lie as in the fact that it is impossible to degenerate the double curve of the surface to a line arrangement \( D \) (see § 5), just as in the fact that the corresponding limit pair \( (P, D) \) does not exist, although the double curve of the surface can be degenerated to a line arrangement (see § 8.5). In addition we show that there exist pairs \( (P_m, D) \), which can not be smoothed outside of a fixed line arrangement \( D \). In some cases such pairs can not be smoothed for all \( m = \deg P_m \) (see § 7.2), and in the other cases such pairs can be smoothed only for sufficiently large \( m \) (see § 7.1).

1. Expression of numerical characteristics of a surface in terms of degeneration

There are two classical ways to study smooth surfaces \( X \subset \mathbb{P}^r \): to consider its generic projection to \( \mathbb{P}^3 \) or onto \( \mathbb{P}^2 \) respectively. We consider a few more general situation. We begin with a surface \( Y \subset \mathbb{P}^3 \) with ordinary singularities, and the normalization \( n : X \rightarrow Y \) of \( Y \) and its composition with projection of \( Y \) onto \( \mathbb{P}^2 \) define a generic covering \( X \rightarrow \mathbb{P}^2 \). The definition of a complete degeneration makes it possible to express numerical characteristics of the surface \( Y \) in terms of its degeneration. Invariants of a surface \( X \) can be expressed as by means of numerical characteristics of the surface \( Y \), just as by means of numerical characteristics of a generic covering \( X \rightarrow \mathbb{P}^2 \). This gives an expression of numerical characteristics of the covering in terms of degeneration.

1.1. Numerical characteristics of a surface with ordinary singularities.
Let \( Y \subset \mathbb{P}^3 \) be an irreducible surface with ordinary singularities, \( D = D_1 \cup \cdots \cup D_k \) the double curve \( g_i = g(D_i) \) its geometric genus, \( d_i = \deg D_i \) the degree of an irreducible component \( D_i \), \( T \) the set of triple points, \( \Omega \) the set of pinch points. The main numerical characteristics of the imbedding \( Y \subset \mathbb{P}^3 \) are:
- \( m = \deg Y \) – the degree of the surface,
- \( k \) – the number of irreducible components of the double curve,
- \( \bar{g} = \sum_{i=1}^{k} g_i \) – the geometric genus of the double curve,
- \( \bar{d} = \sum_{i=1}^{k} d_i \) – the degree of the double curve,
- \( t = \#(T) \) – the number of triple points,
- \( \omega = \#(\Omega) \) – the number of pinch points.

The collection of numerical data type \( (Y) = (m, \bar{d}, k, \bar{g}, t) \) we call the type of the surface \( Y \).
Let \( n : X \rightarrow Y \) be a normalization of the surface \( Y \). The surface \( X \) is smooth. Its invariants are expressed in terms of numerical characteristics of the surface \( Y \) by the following formulae (see [5]):

the intersection number of the canonical class is equal to
\[
K_X^2 = m(m - 4)^2 - (5m - 24)d + 4(\bar{g} - k) + 9t,
\]
(1)

the topological Euler number \( e(X) = c_2(X) \) is equal to
\[
e(X) = m^2(m - 4) + 6m - (7m - 24)d + 8(\bar{g} - k) + 15t,
\]
(2)

From these formulae and the Noether’s formula we obtain the Euler characteristic \( \chi(O_X) \):
\[
\chi(O_X) = \frac{1}{6}m(m^2 - 6m + 11) - (m - 4)d + (\bar{g} - k) + 2t.
\]
(3)

The number of pinches is equal to (see [5]):
\[
\omega = 2d(m - 4) - 6t - 4(\bar{g} - k).
\]
(4)

As is known (8), the arithmetic genus of a reduced curve \( C \) is equal to
\[
p_a(C) = g + \delta - r + 1,
\]
(5)

where \( g \) is its geometric genus, \( \delta \) is the sum of \( \delta \)-invariants of the singularities, \( r \) is the number of irreducible components. For the double curve \( D \), which has only triple points (\( \delta = 3 \)), formula (5) gives
\[
p_a(D) = \bar{g} + 3t - k + 1.
\]
(6)

1.2. Numerical data for description of a curve \( D \subset \mathcal{L} \). Let \( \mathcal{P} \subset \mathbb{P}^3 \) be an arrangement of \( m \) planes in general position, \( \mathcal{L} = \bigcup_{1 \leq i < j \leq m} L_{ij} \) its double curve. Let a curve \( \mathcal{D} \subset \mathcal{L} \) be the union of \( \bar{d} \) lines \( L_{ij} \), and \( \mathcal{R} \) the union of \( d \) remainder lines (the curves \( \mathcal{D} \) and \( \mathcal{R} \) also are called line arrangements),
\[
\mathcal{L} = \mathcal{D} \cup \mathcal{R}, \quad \deg \mathcal{D} = \bar{d}, \quad \deg \mathcal{R} = d, \quad d + \bar{d} = \binom{m}{2}.
\]
(7)

With respect to the partition \( \mathcal{L} = \mathcal{D} \cup \mathcal{R} \) the triple points of the curve \( \mathcal{L} \) are decomposed into 4 types:
\[
\mathcal{T} = \mathcal{T}_3 \sqcup \mathcal{T}_2 \sqcup \mathcal{T}_1 \sqcup \mathcal{T}_0,
\]
where \( \mathcal{T}_3 \) consists of those points, which are triple on the curve \( \mathcal{D} \); \( \mathcal{T}_2 \) consists of those points, which are double on the curve \( \mathcal{D} \); \( \mathcal{T}_1 \) consists of those points, which are non-singular on the curve \( \mathcal{D} \) (and are double on the curve \( \mathcal{R} \)); \( \mathcal{T}_0 \) consists of those points, which do not lie on \( \mathcal{D} \) (and are triple on the curve \( \mathcal{R} \)). Denote by \( \tau_3, \tau_2, \tau_1, \) and \( \tau_0 \) the number of points in corresponding sets. We have
\[
\tau = \tau_3 + \tau_2 + \tau_1 + \tau_0 = \binom{m}{3}.
\]
(8)
By the formula for arithmetic genus (5), we have
\[ p_a(D) = \tau_2 + 3\tau_3 - \bar{d} + 1. \] (9)
(the invariant \( \delta = 1 \) for double points, and \( \delta = 3 \) for triple points).

The numbers \( \tau_i \) are related as follows. On each line \( L_{i,j} = P_i \cap P_j \) there are \( m - 2 \) triple points — the intersection points with planes \( P_k, k \neq i, j \). Let us sum the numbers of triple points of \( \mathcal{L} \) lying on \( \bar{d} \) lines \( L_{i,j} \subset D \). On one hand, we get \( (m - 2)\bar{d} \). On the other hand, we get \( 3\tau_3 + 2\tau_2 + \tau_1 \), because the triple points of \( D \) are counted 3 times, the double points — 2, and non-singular — 1. We get
\[ \tau_1 + 2\tau_2 + 3\tau_3 = (m - 2)\bar{d}. \] (10)
the analogous calculation for the curve \( \mathcal{R} \) gives
\[ \tau_2 + 2\tau_1 + 3\tau_0 = (m - 2)d. \] (11)

The data collection type \((\mathcal{P}, \mathcal{D}) = (m, \bar{d}, k, \tau_2, \tau_3)\) is called the type of pair \((\mathcal{P}, \mathcal{D})\). Here \( k \) is the number of connected components of the curve \( D \setminus T_3 \), where \( T_3 \) is the set of triple points of \( D \).

1.3. The graph of a line arrangement. As is known, one can associate to an algebraic variety, and in particular, to a divisor with normal crossings, a polyhedron by the rule: the vertices correspond to the irreducible components; two vertices are connected by an edge if the components have a non empty intersection; three vertices span a triangle if the corresponding components have a non empty intersection and etc. In particular, to an arrangement \( \mathcal{P} \) of \( m \) planes in general position in \( \mathbb{P}^3 \), we associate a polyhedron, which is the two-dimensional skeleton of a \((m - 1)\)-dimensional simplex in \( \mathbb{R}^{m-1} \) (or a standard \((m - 1)\)-simplex in \( \mathbb{R}^m \)). Denote by \( \Gamma(\mathcal{L}) \) a graph, which is the one-dimensional skeleton of this polyhedron.

To any line arrangement \( \mathcal{D} \subset \mathcal{L} \) we associate a graph \( \Gamma(\mathcal{D}) \) — the graph of the curve \( \mathcal{D} \), which is a subgraph of \( \Gamma(\mathcal{L}) \), consisting of the union of edges corresponding to the lines \( L_{i,j} \subset D \). If \( \mathcal{D} \subset \mathcal{L} \) is a line arrangement, then we denote by \( \mathcal{R} \) the complementary line arrangement in \( \mathcal{L} \).

We say that a graph \( \Gamma \) is realizable, if it satisfies the following conditions: it doesn’t contain isolated vertices, it doesn’t contain simple loops (that is, edges with the same source and endpoint), and any two vertices are connected by at most one edge. It is obvious, a graph \( \Gamma \) is the graph of a set of double lines \( \mathcal{D} \) of a plane arrangement \( \mathcal{P} \) in general position if and only if \( \Gamma \) is realizable and the number of its vertices is not more than \( \deg \mathcal{P} \).

The graph \( \Gamma(\mathcal{D}) \) codes numerical characteristics of \( \mathcal{D} \). The number of vertices of \( \Gamma(\mathcal{D}) \) is equal to the number \( \bar{d} \) of lines composing the curve \( \mathcal{D} \). The number \( \tau_3 \) of triple points of \( \mathcal{D} \), obviously, is equal to the number of triangles of \( \Gamma(\mathcal{D}) \).
(by definition, a triangle in a graph is three vertices and three edges connecting these vertices).

Denote by \( v(P) \) the valence of a vertex \( P \) of \( \Gamma(D) \), that is, the number of edges outgoing of \( P \).

**Lemma 1.** The number of the double points of the curve \( D \) is equal to

\[
\tau_2 = \sum_{P \in \Gamma(D)} \frac{1}{2} v(P)(v(P) - 1) - 3\tau_3 \tag{12}
\]

**Proof.** A vertex \( P \) of valence \( v(P) \) corresponds to a plane, on which \( v(P) \) lines lie. These lines intersect at \( \frac{1}{2} v(P)(v(P) - 1) \) double points. The sum of numbers of these double points is the number of double points of \( D \), if \( D \) has no triple points. If \( D \) has triple points, then each triple point \( T_{i,j,k} = P_i \cap P_j \cap P_k \), being a double point on each of three planes \( P_i, P_j, P_k \), contribute 3 to the first summand of the formula (12). To calculate \( \tau_2 \), we have to remove this contribution, that is, to subtract \( 3\tau_3 \). □

A graph \( \overline{\Gamma}(R) \), consisting of all vertices of the graph \( \Gamma(L) \) and all edges of the graph \( \Gamma(R) \) is called an augmentation of \( \Gamma(R) \) (with respect to \( L \)). A graph \( \Gamma(R) \) is called augmented, if \( \Gamma(R) = \overline{\Gamma}(R) \).

Let \( D \subset L \) be the limit double curve of a complete degeneration of a surface \( Y \) with ordinary singularities in \( \mathbb{P}^3 \). It is not difficult to prove the following lemma.

**Lemma 2.** The number of irreducible components of a surface \( Y \) with ordinary singularities in \( \mathbb{P}^3 \) is equal to the number of connected components of the graph \( \overline{\Gamma}(R) \). In particular, a surface \( Y \) is irreducible, if the graph \( \overline{\Gamma}(R) \) is connected.

In connection with this lemma, a line arrangement \( D \), more precisely, a pair \( (L,D) \) is called irreducible, if the graph \( \overline{\Gamma}(R) \) is connected.

In the case of a connected graph we use a symbol

\[
\Gamma^{d,\tau_3,\nu_1,\nu_2,...}
\]

for the type of the graph \( \Gamma(D) \) (or for the line arrangement \( D \) itself). Here \( \nu_1, \nu_2,... \) are the numbers of vertices of valence 1,2 and etc., \( d \) is the degree of \( D \), that is, the number of edges of the graph, \( \tau_3 \) is the numbers of triple points of \( D \), that is, the number of triangles in \( \Gamma(D) \). If the graph \( \Gamma(D) \) is not connected, then we use the analogous notation for connected components and separate data for each component by parentheses. If several components have the same data, we use a multiplicative way of writing. For example, if \( D \) consists of three connected components, two of which are chains of two lines, and the third consists of four lines, three of which intersect in a triple point and the fourth line intersect two of the three lines, then the graph \( \Gamma(D) \) has type \( \Gamma^{(2,0)^2(4,1)}_{(2,1)^2(1,2,1)} \) (see Fig.1).
1.4. **Numerical characteristics of the limit double curve** $\mathcal{D} \subset \mathcal{L}$. The connection between numerical characteristics of double curves $\mathcal{D}$ and $\mathcal{D}$ of a complete degeneration $Y_u, u \in U$, is given by the following

**Proposition 1.** There are following equalities for the limit double curve $\mathcal{D} \subset \mathcal{L}$ of a complete degeneration $Y_u, u \in U$, of surfaces with ordinary singularities:

\[
\bar{d} = \deg \mathcal{D}; \quad (13)
\]
\[
\tau_3 = t; \quad (14)
\]
\[
\tau_2 = \bar{d} + \bar{g} - k; \quad (15)
\]
\[
\omega = 2\tau_1; \quad (16)
\]
\[
k = \# C(\mathcal{D} \setminus T_3), \quad (17)
\]

where $\# C(\mathcal{D} \setminus T_3)$ is the number of connected components of the curve $\mathcal{D} \setminus T_3$.

Thus, if a surface $Y = Y_{u_1}$ is irreducible, then the types: type($Y$) and type($\mathcal{P}, \mathcal{D}$), uniquely restore each other.

**Proof.** Formula (13) is valid because the degree $\deg D_u$ is constant for a flat family $D_u$. Analogously, formula (14) follows from the flatness of the family $T_u$. Formula (15) follows from the constancy of the arithmetic genus $p_a(D_u)$ in a flat family $D_u$ and from formulae (6) and (9). To prove (16) substitute the expression for $(\bar{g} - k)$ from (15) to formula (4). We have

\[
\omega = 2\bar{d}(m - 4) - 6\tau_3 + 4(\bar{g} - \tau_2) = 2\bar{d}(m - 2) - 4\tau_2 - 6\tau_3,
\]

and from (10) we obtain that $\omega = 2\tau_1$.

Formula (17) follows from flatness of the family of curves $D_u \setminus T_u$ and from constancy of the number of connected components in such a family. $\square$

The number $k$ of irreducible components of the double curve $\mathcal{D} \subset Y$ (formula (17)) can be expressed in terms of the graph $\Gamma(\mathcal{D})$ of the limit curve $\mathcal{D}$ as follows. By definition, a path in $\Gamma(\mathcal{D})$ is a sequence of edges, for which the initial vertex of the next edge is the the endpoint of the previous edge. A path is called prohibited, if two of its successive edges are sides of a triangle in $\Gamma(\mathcal{D})$. Call two edges of the graph $\Gamma(\mathcal{D})$ equivalent, if they are edges of some non prohibited path, and continue this relation to equivalence relation on the set edges of the graph $\Gamma(\mathcal{D})$. Immediately one obtains the following lemma.
Lemma 3. The number $k$ of connected components of the curve $D$ is equal to the number of equivalence classes of edges of the graph $\Gamma(D)$.

The formulae (14) and (15) permit to express the invariants of the surface $X$ in terms of numerical characteristics of the pair $(P, D)$:

\[
K^2_X = m(m - 4)^2 - (5m - 20)d + 4\tau_2 + 9\tau_3, \tag{18}
\]
\[
e(X) = m^2(m - 4) + 6m - (7m - 16)d + 8\tau_2 + 15\tau_3, \tag{19}
\]
or, taking into account (10),

\[
K^2_X = m(m - 4)^2 + 10\tilde{d} - 5\tau_1 - 6\tau_2 - 6\tau_3, \tag{20}
\]
\[
e(X) = m^2(m - 4) + 6m + 2\tilde{d} - 7\tau_1 - 6\tau_2 - 6\tau_3 \tag{21}
\]

It follows from the Noether’s formula and formula (8) that

\[
\chi(O_X) = \tilde{d} + \tau_0 - \frac{1}{2} m(m - 1). \tag{22}
\]

2. Generic projections onto plane

2.1. The discriminant of projection. Let $Y \subset \mathbb{P}^3$ be a surface (not necessary irreducible) of degree $m$ with ordinary singularities and let $\text{pr} : \mathbb{P}^3 \to \mathbb{P}^2$ be a linear projection with the center at a point $o \notin Y$. Choose coordinates in $\mathbb{P}^3$ such that $o = (0 : 0 : 0 : 1)$. Then the projection $\text{pr}$ is given by

\[
(x_1 : x_2 : x_3 : x_4) \mapsto (x_1 : x_2 : x_3) \in \mathbb{P}^2
\]

and $Y$ has an equation

\[
h(x_4) = x_4^m + \sum_{j=0}^{m-1} a_j(x_1, x_2, x_3)x_4^j = 0.
\]

The discriminant $\Delta(x_1, x_2, x_3)$ of the polynomial $h(x_4)$, as a polynomial in $x_4$, is a homogeneous polynomial in variables $x_1, x_2, x_3$ of degree $m(m - 1)$, and defines in $\mathbb{P}^2$ a discriminant divisor $\Delta$ of $p = \text{pr}|_Y$, given by equation $\Delta(x_1, x_2, x_3) = 0$.

Proposition 2. Let a projection $\text{pr} : \mathbb{P}^3 \to \mathbb{P}^2$ be such that

(i) the composition $f = p \circ n : X \to \mathbb{P}^2$ is unramified over generic points of components of the image $\bar{D} = \text{pr}(D)$ of the double curve $D \subset Y$, where $n : X \to Y$ is a normalization;

(ii) the ramification index of $f$ at generic points of its ramification curve $R \subset X$ is equal to two;

(iii) the restriction of $f$ to the curve $R \subset X$ is of degree one.
Then the polynomial \( \Delta(x_1, x_2, x_3) \) factors as:

\[
\Delta(x_1, x_2, x_3) = \beta(x_1, x_2, x_3) \cdot \rho(x_1, x_2, x_3)^2,
\]

where polynomials \( \beta(x_1, x_2, x_3) \) and \( \rho(x_1, x_2, x_3) \) have no multiple factors, the equation of the branch curve \( f(R) = B \subset \mathbb{P}^2 \) of \( f \) is \( \beta(x_1, x_2, x_3) = 0 \), and \( \rho(x_1, x_2, x_3) = 0 \) is the equation of the curve \( \bar{D} \).

**Proof.** The surface \( Y \) is covered by three charts \( U_i = \{ x_i \neq 0 \}, i = 1, 2, 3, U_i = \mathbb{C}^3 \). In each of these charts the projection \( pr \) is of the form \( (x, y, z) \mapsto (x, y) \). For example, in chart \( U_3 = \{ x_3 \neq 0 \} \) one takes \( \frac{x_1}{x_3} = x, \frac{x_2}{x_3} = y, \frac{x_4}{x_3} = z \). In this chart the surface \( Y \) has equation

\[
h(z) = z^m + \sum_{j=0}^{m-1} \bar{a}_j(x, y) z^j = 0,
\]

and the equation of the discriminant divisor is \( \Delta(x, y, 1) = 0 \).

It follows from conditions on the projection \( pr \) that for each point \( q \), lying in one of the three charts isomorphic to \( \mathbb{C}^2 \) (except for a finite set of points — the singular points of \( B \) and the images of pinches and triple points of \( Y \)), there are an analytic neighborhood \( V \subset \mathbb{C}^2 \) of this point and such analytic coordinates \( u, v \) that the polynomial \( h(z) \) is a product of \( m \) factors of the form \( z - \alpha_j(u, v) \) if \( q \notin B \), or of \( m - 2 \) factors of the same form and one of the form \( (z - g(u, v))^2 - u \) if \( q \in B \), where \( \alpha_j(u, v), g(u, v) \) are some analytic functions and \( u = 0 \) is a local equation of the curve \( B \). Furthermore, as is known, the discriminant of a polynomial \( h(z) = \prod_{j=1}^{m}(z - \alpha_j) \) is equal to \( \prod_{i<j}(\alpha_i - \alpha_j)^2 \). Consequently, the discriminant \( \Delta(x, y, 1) \) vanishes only at the points of \( B \) and \( \bar{D} = pr(D) \). Moreover, as \( Y \) has only ordinary singularities, it follows from the conditions on the projection \( pr \) that the equations of the irreducible components of the curve \( \bar{D} \) appear in \( \Delta(x, y, 1) \) with multiplicity two, and in the equations of the irreducible components of \( B \) appear with multiplicity one. \( \square \)

In particular, if \( Y = \mathcal{P} \) is a plane arrangement, then the discriminant divisor of restriction of the projection \( pr \) to \( \mathcal{P} \) equals \( \Delta = 2pr(L) = 2\mathcal{L} \). Moreover, if the projection \( pr \) satisfies the conditions of Proposition 2 for each surface \( Y_u, u \in U \), of a complete degeneration of surfaces with ordinary singularities, then we obtain three flat families of curves: \( \bar{D}_u = pr(D_u), B_u \) and \( \Delta_u = B_u + 2\bar{D}_u \), where \( \bar{D}_{u_0} = pr(\mathcal{D}) = \bar{D} \) and \( B_{u_0} = 2pr(\mathcal{R}) = 2\mathcal{R} \).
2.2. Generic coverings of the plane and generic projections of surfaces with ordinary singularities. Let $X$ be a smooth algebraic surface, not necessarily irreducible. Recall ([9]) that a finite covering $f : X \to \mathbb{P}^2$ is called generic, if it is like a generic projection of a projective surface onto the plane, that is, satisfies the following properties

(i) the branch curve $B \subset \mathbb{P}^2$ is cuspidal, that is, has ordinary cusps and nodes, as the only singularities;
(ii) $f^*(B) = 2R + C$, where the ramification curve $R$ is smooth, and the curve $C$ is reduced;
(iii) $f|_R : R \to B$ is the normalization of $B$.

**Proposition 3.** Let $X$ be a smooth irreducible projective surface. Then the branch curve $B \subset \mathbb{P}^2$ of a generic covering $f : X \to \mathbb{P}^2$ of degree $m \geq 2$ is irreducible.

**Proof.** The statement is obvious if deg $f = 2$. Let deg $f \geq 3$. A generic covering $f : X \to \mathbb{P}^2$ branched along a curve $B$ defines (and is defined by) an epimorphism $\mu : \pi_1(\mathbb{P}^2 \setminus B) \to S_m$ to the symmetric group $S_m$ such that the image $\mu(\gamma)$ of each geometric generator $\gamma \in \pi_1(\mathbb{P}^2 \setminus B)$ (that is, of each simple circuit around the curve $B$) is a transposition in $S_m$ (see, for example, [9]). If $B$ splits into the union of two curves $B_1$ and $B_2$, then $B_1$ and $B_2$ meet each other transversally at non-singular points, since $B$ has only nodes and ordinary cusps, as the only singularities. Therefore (see, for example, [10]) the elements of the group $\Gamma_1$, generated by the simple circuits around $B_1$, commute with the elements of the group $\Gamma_2$, generated by the simple circuits around $B_2$. On the other hand, it is easy to see that the elements of two non-trivial subgroups $\mu(\Gamma_1) \subset S_m$ and $\mu(\Gamma_2) \subset S_m$, generated by transpositions, can not generate $S_m$, if the elements of the group $\mu(\Gamma_1)$ commute with the elements of the group $\mu(\Gamma_2)$. \hfill \Box

As a corollary of Proposition 3 we obtain that the number of irreducible components of the branch curve $B$ of a generic covering $f : X \to \mathbb{P}^2$ is equal to the number of irreducible components $X_i$ of $X$ such that deg $f|_{X_i} \geq 2$.

The main numerical characteristics of a generic covering $f$ of projective plane by an irreducible surface are:

$m = \deg f$ – the degree of the covering,
$g = g(B)$ – the geometric genus of the branch curve,
$2d = \deg B$ – the degree of the branch curve,
$n$ – the number of nodes of $B$,
$c$ – the number of cusps.
The invariants of a surface \(X\) are expressed in terms of numerical characteristics of a generic covering by the following formulae (see \([9]\)):

\[
K_X^2 = 9m - 9d + g - 1,
\]

\[
e(X) = 3m + 2(g - 1) - c,
\]

(23)

(24)

Now let \(Y \subset \mathbb{P}^3\) be a surface with ordinary singularities and \(n : X \to Y\) be its normalization. A projection \(pr : \mathbb{P}^3 \to \mathbb{P}^2\) is called \textit{generic with respect to} \(Y\), if it is generic for the double curve \(D \subset Y\) (and, in particular, \(pr(D)\) has only \(t\) triple points and some nodes as singularities) and the composition \(f = p \circ n : X \to \mathbb{P}^2\) is a generic covering of the plane, where \(p = \text{pr}|_Y\).

**Proposition 4.** Let \(Y_u, u \in U, \dim U = 1\), be a complete degeneration of surfaces of degree \(m\) with ordinary singularities. Then for for almost all generic projections \(pr : \mathbb{P}^3 \to \mathbb{P}^2\) of the plane arrangement \(\mathcal{P} = Y_{u_0}\) the projections \(pr\) are generic with respect to \(Y_u\) for all \(u \in U\), may be, except a finite number of values \(u\).

**Proof.** According to definition of a degeneration the surfaces \(Y_u, u \in U\), are the fibres of a restriction to \(Y \subset \mathbb{P}^3 \times U\) of the projection onto the second factor.

By definition of a generic projection, applied to a plane arrangement, it follows that there exist a finite covering \(\bigcup U_i = \mathbb{P}^2\) by small open balls \(U_i\) with centers at points \(p_i\) and an open neighborhood \(V\) of \(\mathcal{P}\) such that the number of connected components of the intersection \(pr^{-1}(U_i) \cap V\) equals respectively to: \(m\), if \(p_i \notin \mathcal{L} = \text{pr}(\mathcal{L})\), \(m - 1\), if \(p_i\) is a non-singular point of \(\mathcal{L}\), and \(m - 2\), if \(p_i\) is a singular point of \(\mathcal{L}\). For \(u\) sufficiently close to \(u_0\) the surface \(Y_u \subset V\) and hence for such \(u\) the restriction of \(pr\) to \(Y_u\) satisfies the conditions of Proposition [2]. It follows from flatness of the family of branch curves \(B_u\) that for almost all \(u \in U\) the curves \(B_u\) are reduced. According to [11] the restriction of a generic projection to \(Y_{u_1}\) (for fixed \(u_1\)) is a generic covering of the plane and, in particular, the curve \(B_{u_1}\) is cuspidal. Therefore, if the projection is generic simultaneously for \(\mathcal{P}\) and for \(Y_{u_1}\), then it follows from flatness of the family \(B_u\) that for almost all \(u\) the curves \(B_u\) have singularities not worse than the singularities of \(B_{u_1}\), that is, for almost all \(u \in U\) the curves \(B_u\) are also cuspidal and the restriction of the projection \(pr\) to \(Y_u\) is a generic covering of the plane. \(\Box\)

2.3. **Numerical data for description of a projection of a curve \(D \subset \mathcal{L}\) onto the plane.** Let \(pr : \mathbb{P}^3 \to \mathbb{P}^2\) be a generic projection of an arrangement of \(m\) planes \(\mathcal{P} \subset \mathbb{P}^3\). Then the curve \(\mathcal{L} = \text{pr}(\mathcal{L}) \subset \mathbb{P}^2\) has \(\tau\) triple points and

\[
\nu = \frac{1}{2} \binom{m}{2} \binom{m - 2}{2} = \frac{m(m - 1)(m - 2)(m - 3)}{8}
\]

(25)

double points — points of intersection of lines \(pr(L_{i,j})\) and \(pr(L_{k,l})\), which are projections of skew lines \(L_{i,j}\) and \(L_{k,l}\) of the line arrangement \(\mathcal{L} \subset \mathbb{P}^3\).
Let $\mathcal{L} = \mathcal{D} \cup \mathcal{R}$ be a partition of $\binom{m}{2}$ lines of $\mathcal{L}$ into two parts, containing $\bar{d}$ and $d$ lines respectively. Denote $\mathcal{D} = \text{pr}(\mathcal{D})$ and $\mathcal{R} = \text{pr}(\mathcal{R})$. Then $\nu$ double points of the curve $\mathcal{L}$ are decomposed into 3 sets: $\nu_2$ points, which are double on $\bar{D}$ (they are intersection points of $\mathcal{D}$ and $\mathcal{R}$); $\nu_1$ points, which do not lie on $\bar{D}$ (they are double points of $\bar{R}$),

$$\nu = \nu_2 + \nu_1 + \nu_0.$$  \hfill (26)

The curve $\bar{D}$, which is an arrangement of $\bar{d}$ lines, has $\tau_3$ triple points and $\tau_2 + \nu_2$ double points, $\tau_2$ of which have being on $\bar{D}$, and $\nu_2$ appeared under projection of skew lines of the curve $\mathcal{D}$. Applying formula (5) to the curve $\bar{D}$, we get

$$\frac{1}{2}(\bar{d} - 1)(\bar{d} - 2) = (\tau_2 + \nu_2) + 3\tau_3 - \bar{d} + 1$$
or

$$\frac{1}{2}\bar{d}(\bar{d} - 1) = \nu_2 + \tau_2 + 3\tau_3,$$ from where

$$\nu_2 = \frac{1}{2}\bar{d}(\bar{d} - 1) - \tau_2 - 3\tau_3.$$ \hfill (27)

The analogous formula for the curve $\bar{R}$ of degree $d$ with $\tau_0$ triple points and $\tau_1 + \nu_0$ double points gives

$$\nu_0 = \frac{1}{2}d(d - 1) - \tau_1 - 3\tau_0.$$ \hfill (28)

### 2.4. Expression of numerical characteristics of a covering in terms of degeneracy.

We expressed invariants $K^2_X$ and $e(X)$ of an irreducible surface $X$ in terms of numerical characteristics of a degeneracy of the surface $Y$. On the other hand, $K^2_X$ and $e(X)$ can be expressed in terms of numerical characteristics of a generic covering (see (23) and (24)). This gives an expression of numerical characteristics of the covering in terms of numerical characteristics of degeneration.

**Proposition 5.** If $Y_u$, $u \in U$, is a complete degeneration of an irreducible surface $Y$, then numerical characteristics of a generic projection $p : Y \to \mathbb{P}^2$ are expressed in terms of numerical characteristics of the degeneration by formulae:

$$\deg B = 2d,$$ \hfill (29)

$$g - 1 = 6\tau_0 + \tau_1 - d,$$ \hfill (30)

$$c = 6\tau_0 + 3\tau_1,$$ \hfill (31)

$$n = 4\nu_0.$$ \hfill (32)
Proof. First we prove (29), that is, that $d$ in the notation of degree $\deg B$ in section 2.2 equals to $d = \deg R$ in the formula (7). Let $L$ be a generic line in $\mathbb{P}^2$, and $\bar{L} = p^{-1}(L)$ be the corresponding hyperplane section of $Y$. Then $\bar{L}$ is an irreducible plane curve of degree $m$ with $\tilde{d} = \deg D$ nodes. Hence the geometric genus $g(\bar{L}) = \frac{(m-1)(m-2)}{2} - \tilde{d}$. On the other hand, $p : \bar{L} \to L$ is a covering of degree $m$ ramified at $\deg B = (B, L)$ points. Therefore, by Hurwitz formula, we have: $2g(\bar{L}) - 2 = -2m + \deg B$. It follows that

$$\deg B = (m - 1)(m - 2) - 2\tilde{d} - 2 + 2m = m(m - 1) - 2\tilde{d} = 2d.$$  

Let us prove formula (30). From (23) we have $g - 1 = K_X^2 - 9m + 9d$ and by virtue of (18)

$$g - 1 = m(m - 4)^2 + 10\tilde{d} - 5\tau_1 - 6\tau_2 - 6\tau_3 - 9m + 9d.$$  

Substituting $\tilde{d} = \frac{(m-1)(m-2)}{2} - d$ from (7) and $\tau_2 + \tau_3 = \tau - \tau_1 - \tau_0$ from (8), we obtain formula (30): $g - 1 = m(m - 4)^2 + 5m(m - 1) - 10d - 5\tau_1 - m(m - 1)(m - 2) + 6\tau_1 + 6\tau_0 - 9m + 9d = 6\tau_0 + \tau_1 - d$.

To calculate $c$ we use formula (24): $c = -e(X) + 3m + 2(g - 1)$. Substituting the expression for $e(X)$ from (21) and $g - 1$ from (30), we obtain:

$$c = -m^2(m - 4) - 6m - 2\tilde{d} + 7\tau_1 + 6\tau_2 + 6\tau_3 + 3m + 12\tau_0 + 2\tau_1 - 2d =$$

$$-m^2(m - 4) - 3m - 2(\tilde{d} + d) + 6(\tau_0 + \tau_1 + \tau_2 + \tau_3) + 6\tau_0 + 3\tau_1.$$  

Applying formulae (7) and (8), we obtain (31).

To calculate $n$ we use the formula for genus of a plane curve

$$\deg B(\deg B - 3) \frac{2}{2} = g - 1 + c + n.$$  

From here and from formulae (29), (30), and (31) we obtain

$$d(2d - 3) = 6\tau_0 + \tau_1 - d + 6\tau_0 + 3\tau_1 + n = 12\tau_0 + 4\tau_1 + n - d,$$

that is, $n - (d - 1) - 12\tau_0 - 4\tau_1 = 4\nu_0$ by virtue of formula (28). 

2.5. Degenerations of cubic surfaces. The geometric meaning of formulae for $c$ and $n$. Let $pr : Y \to \mathbb{P}^2$ be a generic projection of a surface $Y \subset \mathbb{P}^3$, $\deg Y = 3$.

It is known [5] that the irreducible surfaces with ordinary singularities of degree $m = 3$ are either smooth cubics or cubics, the double curve of which is a line.

If $Y$ is a smooth cubic, then, as is known, the branch curve $B \subset \mathbb{P}^2$ is a curve of degree 6 with six cusps (which in addition lie on a conic).

If the double curve $D$ is a line ($\tilde{d} = 1$), then the surface $Y$ has two pinches, $\omega = 2$, and the branch curve $B$ is a rational quartic ($2d = 4$) with three cusps ($c = 3$).
If a surface \( Y \) is reducible, then either \( Y = P \cup Q \) is the union of a plane and a quadric, or \( Y \) is the union of three planes.

Now consider a complete degeneration of a cubic \( Y \) into an arrangement of three planes \( \mathcal{P} = P_1 \cup P_2 \cup P_3 \) with the double curve \( \mathcal{L} = L_{1,2} \cup L_{2,3} \cup L_{3,1} \). The arrangement \( \mathcal{P} \) has one triple point \( s = L_{1,2} \cap L_{2,3} \cap L_{3,1} \). Let \( \mathcal{D} \) and \( \mathcal{R} \) be the limit double curve and the limit ramification curve respectively. The surface \( Y \) is obtained from \( \mathcal{P} \) by smoothing \( \mathcal{R} \) (smoothing outside of \( \mathcal{D} \)). We have 4 possibilities:

1) if the generic fibre \( Y \) is a smooth cubic, then \( \mathcal{L} = \mathcal{R}, \mathcal{D} = \emptyset \), that is, the double curve \( \mathcal{L} \) is smoothed completely; in this case from the triple point \( \bar{s} = p(T_0) \in \bar{\mathcal{R}} \) there appears 6 cusps (and no nodes) on the branch curve \( B \);

2) if the generic fibre \( Y \) is an irreducible cubic, whose double curve \( \mathcal{D} \) is a line, then \( \mathcal{D} = L_{1,2} \) (for example), and \( \mathcal{R} = L_{2,3} \cup L_{3,1} \), i.e two lines in \( \mathcal{L} \) are smoothed and one line remains double; the point \( s \in T_1 \); in this case from the point \( s \) there appear two pinches on \( Y \), and the point \( \bar{s} \) gives three cusps on the curve \( B \);

3) if the generic fibre \( Y = P \cup Q \), then \( \mathcal{R} = L_{1,2}, \mathcal{D} = L_{2,3} \cup L_{3,1} \); the point \( s \in T_2 \); the curves \( 2\mathcal{R} \) and \( \mathcal{D} \) are smoothed into plane conics, one of which becomes a ramification curve \( \mathcal{R} \), and the other becomes a double curve \( \mathcal{D} \); the curve \( B \) is a conic in \( \mathbb{P}^2 \).

4) Finally, if \( Y \) is a union of three planes, then \( \mathcal{L} = \mathcal{D}, \mathcal{R} = \emptyset \), and \( s \in T_3 \); the double curve \( \mathcal{L} \) is not smoothed at all.

Now let \( \mathcal{P} \) be a complete degeneration of a surface \( Y \) of degree \( m \) with ordinary singularities, \( \mathcal{D} \) and \( \mathcal{R} \) be the limit double curve and the limit ramification curve. If \( s \in T \) a triple point on \( \mathcal{P} \), then (locally) in a neighborhood of the point \( s \) the smoothing outside of \( \mathcal{D} \), or the regeneration, looks like as in the case of the regeneration of a cubic surface, and this explains obtained previously formulae in the following way.

Formula (31) is explained by the fact that the regeneration in the case \( s \in T_0 \) gives 6 cusps on the curve \( B \), and in the case \( s \in T_1 \) there appears 3 cusps.

Formula (16) is explained by the fact that pinches (by two) appear only from points \( s \in T_1 \).

Finally, formula (32) is explained as follows. Let \( q \) be one of \( \nu_0 \) double points of the curve \( \bar{\mathcal{R}}, q = \bar{L}_{i,j} \cap \bar{L}_{k,l} \), where \( L_{i,j} \) and \( L_{k,l} \) is a pair of skew lines. In a neighborhood of a line \( L_{i,j} \) the surface \( \mathcal{P} \) is given (locally) by the equation \( z^2 - x^2 = 0 \) and smoothed surface \( Y_u \) is given by an equation \( z^2 - x^2 = \varepsilon \). Projecting to the plane \( \mathbb{P}^2 \), we obtain two branches of the branch curve \( B_u \) close ("parallel") to the curve \( \bar{L}_{i,j} \). Analogously, for the curve \( L_{k,l} \) we obtain two branches of the branch curve \( B_u \) close to the line \( \bar{L}_{k,l} \). These two pairs of branches meet in four points. Thus, each of \( \nu_0 \) pairs of skew lines of the curve
3. Complete degeneracy of quartic surfaces

Let us analyze complete degenerations of surfaces of degree \( m = 4 \). If a surface \( Y \) is reducible, then it is obvious that complete degenerations of components of \( Y \) give a complete degeneration of \( Y \). Thus, we can assume the surface \( Y \) to be irreducible since the case \( m = 2 \) is trivial, and the case \( m = 3 \) has been described in section 2.5.

3.1. Degenerations of irreducible quartics with ordinary singularities. The description of all irreducible quartics \( Y \subset \mathbb{P}^3 \) with ordinary singularities can be found in [5]. There are 6 types of such surfaces in accordance with the types of the double curve \( D \subset Y \):

1) \( Y \) is smooth, that is, \( D = \emptyset \);
2) \( D \) is a line;
3) \( D \) is a plane conic;
4) \( D \) is a pair of skewed lines;
5) \( D \) is the union of three lines meeting at a point;
6) \( D \) is a rational normal curve of degree 3.

We show that in each of these cases there is a complete degeneration of \( Y \).

Let \( F(x) = 0 \) be an equation of \( Y \), and \( H_i(x) = 0, i = 1, \ldots, 4 \), be equations of four planes \( P_1, \ldots, P_4 \) in general position in \( \mathbb{P}^3 \).

In cases 1), 2), 4) and 5) we can obtain the desired complete degenerations \( Y_u, u \in \mathbb{C} \), in a form

\[
u \cdot F(x) + (1 - u) H_1(x) H_2(x) H_3(x) H_4(x) = 0,
\]

where in case 1) linear functions \( H_1(x), \ldots, H_4(x) \) are arbitrary and such that \( P_1, \ldots, P_4 \) are in general position; in case 2) \( H_1(x) = H_2(x) = 0 \) are equations of the line \( D \); in case 4) \( H_1(x) = H_2(x) = 0 \) and \( H_3(x) = H_4(x) = 0 \) are equations of skew lines forming \( D \); in case 5) \( H_1(x) H_2(x) H_3(x) = 0 \) is an equation of the union of three planes, the double curve of which is the union of three lines forming \( D \). In all cases the double curve \( L \) of the arrangement \( \mathcal{P} \) of four planes \( P_1, \ldots, P_4 \) contains the double curve \( D_u \subset Y_u \) and the limit double curve \( D \) coincides with \( D \).

In the case 3) the double curve \( D \) is a complete intersection of a plane and a quadric. We consider complete degenerations with a complete intersection double curve in general in section 6.

Consider case 6). As is known, all smooth space curves of degree 3 are projectively equivalent. Such a curve \( D \) has parametrization \( x = t, y = t^2, z = t^3 \) in appropriate affine coordinates. It is not a complete intersection and is defined by three equations: \( y = x^2, xy = z, y^2 = xz \). In homogeneous coordinates
(x_1 : x_2 : x_3 : x_4) = (x : y : z : 1) in \mathbb{P}^3 the curve \( D \) is the intersection of three quadrics
\[
x_2x_4 = x_1^2, \quad x_1x_2 = x_3x_4, \quad x_2^2 = x_1x_3.
\]

Consider a family of curves \( D_u : x = t, \ y = ut^2, \ z = u^3t^3 \). The curves \( D_u \) are given by three equations \( y = ux^2, u^2xy = z, uy^2 = xz \), or in homogeneous coordinates
\[
x_2x_4 - ux_1^2 = 0, \ x_3x_4 - u^2x_1x_2 = 0, \ x_1x_3 - ux_2^2 = 0.
\]
The family \( D_u \) defines a degeneration of the curve \( D \) (for \( u = 1 \)) to the curve \( D \) (for \( u = 0 \)), which is given by equations: \( x_2x_4 = 0, x_3x_4 = 0, x_1x_3 = 0 \). If \( P_i \) is a plane given by equation \( x_i = 0 \), \( L_{i,j} = P_i \cap P_j \), then \( D \) is a chain of lines \( L_{1,4} \cup L_{4,3} \cup L_{3,2} \).

Consider a family of surfaces \( Y_u \) given by equation
\[
u(x_2x_4 - ux_1^2)(x_3x_4 - u^2x_1x_2) + u(x_3x_4 - u^2x_1x_2)(x_1x_3 - ux_2^2) + (x_2x_4 - ux_1^2)(x_1x_3 - ux_2^2) = 0.
\]

For \( u \neq 0 \) the double curves are smooth space cubic curves \( D_u \), and for \( u = 0 \) the surface \( Y \) degenerates to a plane arrangement \( Y_0 = \{ x_1x_2x_3x_4 = 0 \} \) and the limit double curve is \( D \).

3.2. Description of possible arrangements of limit double curves. In case \( m = 4 \) the graph \( \Gamma(\mathcal{L}) \) consists of the union of all edges of a tetrahedron. By Lemma 2 a curve \( D \subset \mathcal{L} \) can be a limit double curve of a degeneration of an irreducible quartic with ordinary singularities if the graph \( \Gamma(\mathcal{R}) \) is connected. The graph \( \Gamma(\mathcal{R}) \) is obtained from \( \Gamma(\mathcal{L}) \) by removing some edges corresponding to lines \( L_{i,j} \subset D \). It is obvious that the graph \( \Gamma(\mathcal{R}) \) remains connected only in the following cases: 1) we remove nothing from \( \Gamma(\mathcal{L}) \); 2) we remove one edge from \( \Gamma(\mathcal{L}) \); 3) we remove two adjacent edges; 4) we remove two skew edges; 5) we remove three edges going out from one vertex; 6) we remove a chain of three edges. Removing of four edges leads to a non isolated vertex, that is, to a non connected graph \( \Gamma(\mathcal{R}) \).

Thus, up to renumbering of planes of \( P_1, \ldots, P_4 \), the curve \( D \) is one of the following:
1) \( D = \emptyset \);
2) \( D = L_{1,2} \);
3) \( D = L_{1,2} \cup L_{1,3} \);
4) \( D = L_{1,2} \cup L_{3,4} \);
5) \( D = L_{1,2} \cup L_{1,3} \cup L_{2,3} \);
6) \( D = L_{1,2} \cup L_{2,3} \cup L_{3,4} \).

In section 3.1 we showed that all such \( D \) are realized as limit double curves. Thus, we obtain the following
Theorem 1. For any surface of degree 4 with ordinary singularities in $\mathbb{P}^3$ there exists a complete degeneration.

If $\mathcal{P}$ is an arrangement of four planes in general position in $\mathbb{P}^3$, $\mathcal{D} \subset \mathcal{L}$ is any line arrangement, then there exists a complete degeneration of surfaces of degree 4 with ordinary singularities, for which $\mathcal{D}$ is the limit double curve.

4. Existence of line arrangements which are not limit

In the previous section we showed that in the case of surfaces with ordinary singularities of degree $m \leq 4$ the matters go as well as in the case of plane curves. Responses to two both questions, put in the introduction, are affirmative. But if $m \geq 5$, the matters do not stand so good. In this section we show that there are at least seven arrangements of double lines $\mathcal{D}$ in plane arrangements $\mathcal{P}$, deg $\mathcal{P} = 5$, such that $\mathcal{P}$ cannot be smoothed outside of of $\mathcal{D}$.

![Diagrams of line arrangements](image)

Fig. 2

Theorem 2. The line arrangements $\mathcal{D}$ with graphs, depicted on Fig. 2, are not limit for complete degenerations of surfaces with ordinary singularities of degree 5.

Proof. Assume that line arrangements with graphs, depicted on Fig. 2, are limit double curves for complete degenerations of surfaces $Y$ of degree $m = 5$ with ordinary singularities.

In all cases the graph $\Gamma(\mathcal{R})$ is connected and by Lemma 2 the surface $Y$, for which $\mathcal{D}$ is the limit double curve, is irreducible.

We can obtain numerical characteristics of curves $\mathcal{D}$. In our case $m = 5$ and therefore, $d + \bar{d} = \binom{5}{2}$, $\tau = \binom{5}{3}$. The number $\tau_3$ is equal to the number of triangles in $\Gamma(\mathcal{D})$. To calculate $\tau_2$ we use the formula (12); $\tau_1$ and $\tau_0$ are obtained from formulae (10), (11), and (8):

$$
\tau_1 + 2\tau_2 + 3\tau_3 = 3\bar{d}, \quad \tau_2 + 2\tau_1 + 3\tau_0 = 3d, \quad \tau_0 + \tau_1 + \tau_2 + \tau_3 = \tau.
$$
By \((30)\), \((32)\), and \((31)\) we can find numerical characteristics of the double curve \(D \subset Y\) and of the branch curve \(B\): the numbers \(\bar{g}, t, k\) we find from Proposition 1 and the numbers \(g, c, n\) we find from Proposition 5 applying formulae \((30)\), \((31)\), \((32)\), and \((28)\).

Consider each of the line arrangements \(D\) with graphs, depicted in Fig. 2, and show that in all cases we obtain a contradiction with assumption made above for the arrangements to be limit.

For a line arrangement of type \(\Gamma^{3,0}_{3,0,1}\) we have: \(d = 3, \tau_3 = 0\) and \(\tau_2 = 3\). By Proposition 1 we have \(k = 2, \bar{g} = 0\) and the double curve \(D \subset Y\) is a disconnected union of a line \(L_1\) and a plane conic \(Q\). Let \(Q\) lies in a plane \(P\) and the point \(A = L_1 \cap P\). As in the previous case, we see that any line \(L \subset P\), passing through \(A\), lies in the surface \(Y\) and, consequently, \(P \subset Y\). Again we obtain a contradiction with the irreducibility of \(Y\).

For a line arrangement of type \(\Gamma^{1,0}_{2,1}\) we have: \(d = 3, \tau_3 = 0\) and \(\tau_2 = 1\). By Proposition 1 we have \(k = 2, \bar{g} = 0\) and the double curve \(D \subset Y\) is a disconnected union of a line \(L_1\) and a plane conic \(Q\). Let \(Q\) lies in a plane \(P\) and the point \(A = L_1 \cap P\). As in the previous case, we see that any line \(L \subset P\), passing through \(A\), lies in the surface \(Y\) and, consequently, \(P \subset Y\). Again we obtain a contradiction with the irreducibility of \(Y\).

We consider a line arrangement of type \(\Gamma^{4,0}_{3,1,1}\) in Theorem 5 where it is shown that the curve \(D\) can not be the limit double curve of a complete degeneration of a surface \(Y\) of any degree \(m\), and not only \(m = 5\).

For a line arrangement of type \(\Gamma^{5,1}_{3,1,1}\) we have: \(d = 5, \tau_3 = 1, \tau_2 = 3\), and \(\tau_1 = 6\). By \((15)\) and \((19)\), we obtain \(K_X^2 = 1\) and \(e(X) = -1\). It is known from the classification of algebraic surfaces that if \(K_X^2 > 0\), then \(e(X) > 0\). Thus such surfaces do not exist.

A line arrangement of type \(\Gamma^{6,4}_{0,0,4}\) is a special case (for \(m = 5\)) of line arrangements which are considered below in Proposition 9.

For a line arrangement of type \(\Gamma^{6,2}_{1,1,3}\) we have: \(d = 6, d = 4, \tau_3 = 2, \tau_2 = 4, \tau_1 = 4, \tau_0 = 0\). The branch curve \(B\) is cuspidal and have numerical characteristics: \(\deg B = 8, g = 1, c = 12, n = 8\). Let us show that the curve \(\hat{B}\) dual to \(B\) also is cuspidal. By Plücker formula, we obtain \(\deg \hat{B} = 4\). Since \(g(\hat{B}) = 1\) and a curve of degree 4 has arithmetic genus \(p_a(\hat{B}) = 3\), then \(\hat{B}\) has either two singular points with \(\delta = 1\), or one singular point with \(\delta = 2\). The Milnor number \(\mu\) and \(\delta\) are connected by formula \(\mu = 2\delta - r + 1\), where \(r\) is
the number of branches. Singularities with $\delta = 1$ are either singularities of type $A_1$ – nodes, or of type $A_2$ – cusps, therefore, in the first case the curve $\hat{B}$ is cuspidal. The second case is impossible, since the singularity dual to $A_3$ is $A_3$, and dual to $A_4$ is $\mathbb{E}_8 : x^3 + y^2 = 0$. But a curve dual to $\hat{B}$ is the curve $B$, which has only nodes and cusps. Thus, the curve $\hat{B}$ is cuspidal. In this case it follows from Plücker formula that $3 \deg B - c = 3 \deg \hat{B} - \hat{c}$, where $\hat{c}$ is the number of cusps of $\hat{B}$. Therefore, $\hat{c} = 0$ and, consequently, the curve $\hat{B}$ is nodal and the number of node of $\hat{B}$ is equal to $\hat{n} = 2$. For such a curve $\hat{B}$ there exists a generic covering of the plane of degree $4 = \deg \hat{B}$, branched along $B$ (see [9]).

On the other hand, a generic projection of a surface with ordinary singularities in $\mathbb{P}^3$ onto $\mathbb{P}^2$ is a generic covering, in our case of degree 5. As is shown in [13], generic coverings, which are generic projections, are uniquely defined by their branch curves always, except the case of surfaces of degree 4 with singularities consisting of three lines intersecting in a point. We obtain a contradiction with the uniqueness of the covering.

5. Existence of surfaces not possessing complete degenerations

In this section we show that there exist surfaces in $\mathbb{P}^3$ with ordinary singularities which can not be degenerated into plane arrangements in general position.

5.1. Zeuthen’s problem. The problem of existence of a complete degeneration for any surface with ordinary singularities is closely connected with the problem of degeneration of its double curve $D$ into a line arrangement. In the case of a smooth space curve $D$ it is a famous Zeuthen’s problem. In [12], a negative solution of Zeuthen’s problem is obtained, that is, it is shown that there exist smooth projective curves $D \subset \mathbb{P}^3$, which can not be degenerated into a line arrangement with double points. One of such curves has degree 30 and genus 113. We call it Hartshorne’s curve.

**Theorem 3.** There exist surfaces in $Y \subset \mathbb{P}^3$ with ordinary singularities, which can not be degenerated into a plane arrangement in general position.

**Proof.** Let $D$ be the Hartshorne’s curve. In the next subsection we prove (Proposition [1]) that there exists a surface $Y \subset \mathbb{P}^3$ with ordinary singularities, the double curve of which is $D$. By [12], the curve $D$ can not be degenerated into a line arrangement with double points and, consequently, $Y$ can not be degenerated into a plane arrangement in general position. □

1In [13] this statement was formulated only for surfaces obtained as a generic linear projection into $\mathbb{P}^3$ of smooth surfaces imbedded into some $\mathbb{P}^N$. But the proof of this statement used only the assumption that surfaces in $\mathbb{P}^3$ have only ordinary singularities.
5.2. Existence of a surface with ordinary singularities, the double curve of which is any smooth curve. Let $D$ be a smooth (nor necessary irreducible) curve in $\mathbb{P}^3$. Then for any point $x \in D$ there exist a Zariski open set $U_x$ in $\mathbb{P}^3$ and two rational functions $f_{x,1}$ and $f_{x,2}$, regular in $U_x$ and such that $f_{x,1}$ and $f_{x,2}$ are local parameters in $U_x$ and the curve $D \cap U_x$ is defined by equations $f_{x,1} = f_{x,2} = 0$. Let $f_{x,j} = \frac{F_{x,j}}{G_{x,j}}$, where $F_{x,j}$ and $G_{x,j}$ are relatively prime homogeneous polynomials of degree $M_{x,j}$. The curve $D$ being compact, we can choose a finite covering of $D$ by open sets $U_{x,1}, \ldots, U_{x,k}$ such that polynomials $F_{x,i,j}$, $1 \leq i \leq k$, $j = 1, 2$, generate the homogeneous ideal of $D$. Set

$$M(D) = 2 \max \left( \max_{1 \leq i \leq k} M_{x,i,j} \right) + 1. \quad (33)$$

**Proposition 6.** For any smooth projective curve $D \subset \mathbb{P}^3$ and for any natural number $m \geq M = M(D)$, where $M(D)$ is the number defined in (33), there exists a projective surface $Y$ with ordinary singularities of degree $m$ in $\mathbb{P}^3$ such that $D = \text{Sing} Y$.

**Proof.** Let $\sigma : \tilde{\mathbb{P}}^3 \to \mathbb{P}^3$ be a monoidal transformation with center at $D$, $E = \sigma^{-1}(D)$ its exceptional divisor and $\tilde{P} = \sigma^*(P)$ the total preimage of the plane $P \subset \tilde{\mathbb{P}}^3$.

The surfaces $Y_{i,j} = \{ F_{x,i,j} = 0 \} \subset \mathbb{P}^3$, deg $Y_{i,j} = \deg F_{x,i,j} = M_{x,i,j}$, $j = 1, 2$, meet transversally along $D$ in $U_{x,i}$. It is obvious that for any plane $P$ in $\mathbb{P}^3$ the divisors

$$Y_{i,1} + Y_{i,2} + (m - M_{x,i,1} - M_{x,i,2})\tilde{P}$$

are zeroes of sections of the sheaf $\mathcal{O}_{\tilde{\mathbb{P}}^3}(m\tilde{P} - 2E)$ for $m \geq M$. From this it is easy to see that the linear system $| m\sigma^*(P) - 2E |$ is not empty, does not have fixed components and base points, and for any points $x, y \in \tilde{\mathbb{P}}^3$ there exists a divisor of this system such that it does not go through the point $y$ and is a smooth reduced surface at $x$. According to Bertini theorem the generic member $X$ of the linear system $| m\sigma^*(P) - 2E |$ is a smooth surface, and it is easy to see that its image $Y = \sigma(X)$ is a surface in $\mathbb{P}^3$ of degree $m$ with ordinary singularities along $D$. \[\square\]

6. Complete degeneracy of surfaces with complete intersection double curves

As has been shown in the previous section, in general the answer to the question about existence of a complete degeneration is in negative even in the case of a smooth double curve. In this section we show that the answer to this question is affirmative in the case when the double curve is complete intersection.
6.1. Equations of surfaces with complete intersection double curves.

The following proposition gives a description of equations of surfaces, double curves of which are complete intersections.

**Proposition 7.** Let an irreducible surface $Y \subset \mathbb{P}^3$ with ordinary singularities, $\deg Y = m$, given by equation $F(x) = 0$, has a smooth double curve $D = Y_1 \cap Y_2$, which is a complete intersection of surfaces $Y_1$ and $Y_2$ of degrees $m_1$ and $m_2$. Then the polynomial $F$ can be written in a form

$$F = AF_1^2 + BF_1F_2 + CF_2^2,$$

where $F_1(x) = 0$ and $F_2(x) = 0$ are equations of surfaces $Y_1$ and $Y_2$, and $A$, $B$, and $C$ are homogeneous polynomial.

Conversely, if $F$ is written in the form (34) and $m \geq 2m_1 + 1$ (let $m_1 \geq m_2$), and polynomials $A$, $B$, and $C$ are sufficiently general, then the surface $Y$ has only ordinary singularities and its double curve is a complete intersection.

**Proof.** Since the curve $D$ is a complete intersection, the homogeneous ideal $I(D)$ is generated by two elements, $I(D) = (F_1, F_2)$. Write $F$ in a form $F = K_1F_1 + K_1F_1$. It follows from transversality of intersection of surfaces $F_1 = 0$ and $F_2 = 0$ along $D$ that differentials $dF_1$ and $dF_2$ are linear independent at each point of $D$. In addition, the differential $dF$ equals to zero at each point of $D$, since $D$ is the double curve of $Y$. At each point of $D$ we have $dF = K_1dF_1 + K_2dF_2$ and it follows from linear independence of differentials $dF_1$ and $dF_2$ at these points that $K_1$ and $K_2$ belong to the ideal $I(D)$, that is, $K_1 = S_1F_1 + S_2F_2$ and $K_2 = S_3F_1 + S_4F_2$, where $S_i$ are some homogeneous polynomials. Substituting these expressions for $K_1$ and $K_2$ to the expression for $F$, we obtain the desired form of $F$. The second part of lemma follows from Proposition 6 and the estimate (33). □

6.2. Construction of a degeneration. We prove the following theorem.

**Theorem 4.** Surfaces with ordinary singularities, double curves of which are complete intersections, have complete degenerations.

**Proof.** Let the double curve $D$ of a surface $Y$ be given by equations $F_1 = 0$, $F_2 = 0$, deg $F_1 = m_1$, deg $F_2 = m_2$. Then, by Proposition 7, the surface $Y$ is given by an equation of the form

$$AF_1^2 + BF_1F_2 + CF_2^2 = 0.$$

Consider a deformation of this equation killing the first and the third terms, and transforming the second term to a product of linear forms. Let a family of surfaces $Y_u$, $u \in \mathbb{C}$, be given by an equations

$$(uB + (1 - u)\overline{B})(uF_1 + (1 - u)\overline{F_1})(uF_2 + (1 - u)\overline{F_2}) +$$

$$uA(uF_1 + (1 - u)\overline{F_1})^2 + uC(F_2 + (1 - u)\overline{F_2})^2 = 0,$$

(35)
where $F_1 = H_1 \cdot \ldots \cdot H_{m_1}$, $F_2 = H_{m_1+1} \cdot \ldots \cdot H_{m_1+m_2}$, $B = H_{m_1+m_2+1} \cdot \ldots \cdot H_m$ are products of linear forms such that zeroes of the forms are planes $P_1, \ldots, P_m$ in general position.

It is easy to see that for $u = 1$ the surface $Y_u$ coincides with $Y$. It is obvious also that for values $u$, close to 1, curves $D_u$, given by equations

$$uF_1 + (1-u)F_1 = uF_2 + (1-u)F_2 = 0,$$

are smooth and they are double curves of surfaces $Y_u$ with ordinary singularities. For $u = 0$ the degenerated fibre is given by an equation $H_1 \cdot \ldots \cdot H_m = 0$ and the limit double curve $D_0$ is given by equations

$$H_1 \cdot \ldots \cdot H_{m_1} = H_{m_1+1} \cdot \ldots \cdot H_{m_1+m_2} = 0. \quad (36)$$

In particular, for $m_1 = m_2 = 2$ the graph of the limit double curve $D = D_0$ is depicted in Fig. 3.

![Fig. 3](image)

7. Potentially limit and absolutely not limit line arrangements

Until now we have considered complete degenerations of surfaces with ordinary singularities for fixed degree $m$. The line arrangements $D \subset L_m$ which can be obtained in the degenerate fibre are called now $m$-limit curves. (Here $L_m$ is the double curve of an arrangement of $m$ planes in general position). In section 4 it was shown that there exist line arrangements $D$, which are not 5-limit. We want to investigate the dependence on $m$ of a curve $D$ to be $m$-limit.

If for some $m_0$ a curve $D$ is not $m_0$-limit, but for sufficiently big $m$ and for an embedding $D \subset L_m$, the curve $D$ is $m$-limit, then $D$ is called potentially limit. If for any embedding $D \subset L_m$ a curve $D$ is not $m$-limit, then such a line arrangement $D$ is called absolutely not limit.

7.1. Examples of potentially limit line arrangements. Let us show that a line arrangement $D$ of type $\Gamma_{3,0,1}^3$ is potentially limit. It was shown in Theorem 2 that $\Gamma_{3,0,1}^3$ is not 5-limit. Let us show that $\Gamma_{3,0,1}^3$ is $m$-limit for $m \geq 7$. The line arrangement $D$ consists of three lines $L_1 \cup L_2 \cup L_3$, which lie in a plane $P$ and are cut out by three planes defined by equations $H_1 = 0$, $H_2 = 0$, and $H_3 = 0$. Such an arrangement $D$ is limit for surfaces $Y$, double curves $D$ of which are complete intersections of a smooth cubic and a plane. Indeed, let $F_1(x) = 0$ be an equation of a smooth cubic $Y_1$, and $F_2(x) = 0$ be an equation of a plane $Y_2 = P$. Consider a surface $F(x) = 0$, where $F$ is defined by formula (34) and
consider a family of surfaces $Y_u$ defined by equation (35). It follows from (36) that the limit double curve $D$ is defined by equations $H_1H_2H_3 = F_2 = 0$ and it is of type $\Gamma^{3,0,1}_{3,0,1}$.

Analogously one can show that a line arrangement of type $\Gamma^{(1,0)(2,0)}_{(2)(2,1)}$ also is potentially limit.

7.2. **An example of absolutely not limit line arrangement.** The following theorem gives an example of absolutely not limit line arrangement. Let us note that this line arrangement is a degeneration of a smooth space curve of degree 4 and genus 1.

**Theorem 5.** The line arrangement $\mathcal{D}$ of type $\Gamma^{4,0}_{3,1,1}$ (see Fig. 2) is absolutely not limit.

**Proof.** Let $\mathcal{D}$ be $m$-limit for some $m$, that is, $\mathcal{D}$ is the limit double curve of a complete degeneration of surfaces $Y_u$, $u \in U$, defined by equations $F_u(x) = 0$. Calculations analogous to calculations in the proof of Theorem 2 give: $\bar{d} = 4$, $\tau_3 = 0$, and $\tau_2 = 4$. By Proposition 1 the double curve $D \subset Y$ is a smooth irreducible curve in $\mathbb{P}^3$ of degree $\bar{d} = 4$ and genus $\bar{g} = 1$. As is known, such curves $D$ are complete intersections of two quadrics, and, consequently, a polynomial $F$ defining the surface $Y$ can be written in the form (34):

$$F = AQ_1^2 + BQ_1Q_2 + CQ_2^2,$$

where $Q_1(x) = 0$, $Q_2(x) = 0$ are equations of these quadrics, and $A$, $B$, and $C$ are polynomials of degree $m - 4$.

Let us show that the family of surfaces $F_u(x) = 0$, $u \in U$, can be written in a form

$$F_u = A_uQ_{1,u}^2 + B_uQ_{1,u}Q_{2,u} + C_uQ_{2,u}^2$$

(may be, after base change). For this let us consider the universal family of surfaces given by equations of the form (37). The base of this family $\mathcal{F}$ is an open subset in the space of coefficients of forms $Q_1$, $Q_2$, $A$, $B$, $C$. Denote by $\mathcal{H}_{m,4,1}$ the space parametrizing the surfaces of degree $m$, the double curve of which is a smooth curve of degree 4 and genus 1. Obviously, we have a rational dominant map $\mathcal{F} \to \mathcal{H}_{m,4,1}$. The family of surfaces $F_u(x) = 0$ defines a map $U \to \mathcal{H}_{m,4,1}$. We can assume that $U \subset \mathcal{H}_{m,4,1}$. If a curve $\tilde{U} \subset \mathcal{F}$ is mapped to the curve $U$ we get a family of surfaces (38) parametrized by points of $\tilde{U}$.

A line arrangement $\mathcal{D}$ of type $\Gamma^{4,0}_{3,1,1}$ consists of three lines in a plane and a fourth line not in this plane and intersecting one of these lines. Such curves are degenerations of space elliptic curves of degree 4 (see [14]). But, if a curve $\mathcal{D}$ is the limit double curve of the family (38), then the double curves $D_u$ are given by a family of ideals $J_u = (Q_{1,u}, Q_{2,u})$. By [14], the degenerate curve to coincide with $\mathcal{D}$ it is necessary for $u = u_0$ the quadratic forms $Q_{1,u_0}$ and $Q_{2,u_0}$ to split into a product of linear forms with one common form: $Q_{1,u_0} = HH_1$.
and $Q_{2,u_0} = HH_2$. But, then it follows from (38) that the degenerate surface $Y_{u_0}$ contains a multiple plane $H = 0$, and we obtain a contradiction with the definition of a complete degeneration of surfaces with ordinary singularities. □

8. Virtual degeneracy

If for a surface $Y \subset \mathbb{P}^3$ of degree $m$ with ordinary singularities there exists a complete degeneration in the sense of the definition given in introduction, then, for brevity, we call $Y$ a completely degenerative surface. In section 5 we gave examples of surfaces, which are not completely degenerative.

8.1. Necessary conditions for complete degeneracy. Let us weaken the notion of degeneracy of a surface. Recall that in sections 1.1 and 1.2 we defined the type of an irreducible surface $Y \subset \mathbb{P}^3$ of degree $m$ with ordinary singularities and the type of a pair $(P, D)$ as collections of numerical data:

$$\text{type}(Y) = (m, \bar{d}, k, \bar{g}, t), \quad \text{type}(P, D) = (m, \bar{d}, k, \tau_2, \tau_3).$$

By Proposition 1, if $Y$ is a completely degenerative surface, then the types $\text{type}(Y)$ and $\text{type}(P, D)$ define each other. We call collections of numbers $(m, \bar{d}, k, \bar{g}, t)$ and $(m, \bar{d}, k, \tau_2, \tau_3)$ corresponding to each other if $\tau_3 = t$, and numbers $\tau_2$ and $\bar{g}$ are connected by formula (15):

$$\tau_2 = \bar{d} + \bar{g} - k.$$  

We call an irreducible surface $Y$ virtually degenerative if there exists an irreducible pair $(P, D)$ the type of which corresponds to the type of the surface $Y$. Thus, a surface $Y$ can be not degenerative by a trivial reason: there does not exist a pair $(P, D)$ of corresponding type.

Analogously, we call a pair $(P, D)$ virtually smoothable outside of $D$ (or we call a line arrangement $D$ virtually $m$-limit) if there exists an irreducible surface $Y$ of degree $m$ which has a type corresponding to the type of the pair $(P, D)$. Thus, the term ”virtual” says about fulfilment of necessary numerical conditions for existence of a complete degeneration.

The proof of Proposition 5 does not use essentially the fact of complete degeneracy of $Y$, but uses only arising from it connection between the type of $Y$ and the type of the limit pair $(P, D)$, given by formulae (14): $t = \tau_3$, and (15): $\bar{g} = \tau_2 - \bar{d} + k$. Therefore, in the case of virtual degeneracy an analog of Proposition 5 holds.

Proposition 8. Let the type $(m, \bar{d}, k, \bar{g}, t)$ of a surface $Y$ corresponds to the type $(m, \bar{d}, k, \tau_2, \tau_3)$ of a pair $(P, D)$. Then the same formulae (20) – (32), as in Proposition 5 hold for numerical characteristics of a generic projection $p : Y \to \mathbb{P}^2$. 
8.2. Examples of virtually $m$-limit, but not $m$-limit arrangements. Consider an irreducible pair $(\mathcal{P}, \mathcal{D})$, where $\mathcal{D}$ is an arrangement of type $\Gamma_{3,1,1}^{4,0}$ from Theorem 5. This pair has type $(\mathcal{P}, \mathcal{D}) = (m, 4, 1, 4, 0)$. In Theorem 5 it was proven that $\mathcal{D}$ is not limit for all $m$. But the line arrangement $\mathcal{D}$ is virtually $m$-limit for $m \geq 5$. Indeed, the type of the pair $(\mathcal{P}, \mathcal{D})$ has a corresponding data set $(m, 4, 1, 1, 0)$. Surfaces $Y$ are of type $(Y) = (m, 4, 1, 1, 0)$, that is, surfaces of degree $m$, the double curve $D$ of which is an elliptic curve of degree 4, exist. Since $D$ is a complete intersection of two quadrics $Q_1$ and $Q_2$, we can take a surface $Y$, given by equation $F = 0$, where the polynomial $F$ is defined by formula (37). Such surfaces are completely degenerative, but the limit double curve $D$ has a graph not of type $\Gamma_{3,1,1}^{4,0}$, but a graph of type depicted in Fig. 3.

8.3. Examples of virtually not limit line arrangements. Consider an arrangement of $m$ planes $\mathcal{P} = \mathcal{P}_m$ in general position in $\mathbb{P}^3$. Let $\mathcal{P}_{m-1} \subset \mathcal{P}_m$ be an arrangement of some $m - 1$ of these planes, and $\mathcal{D} = \mathcal{L}_{m-1}$ be the double curve of the surface $\mathcal{P}_{m-1}$. Then $\bar{d} = (m-1)/2$, $\tau_0 = \tau_2 = 0$, $\tau_1 = (m-1)/2$, $\tau_3 = (m-1)/3$ and the pair $(\mathcal{P}, \mathcal{D})$ has type

$$\text{type}(\mathcal{P}, \mathcal{D}) = \left( m, \left( \frac{m-1}{2} \right), \left( \frac{m-1}{2} \right), 0, \left( \frac{m-1}{3} \right) \right)$$

(39)

Proposition 9. If $(\mathcal{P}, \mathcal{D})$ is an irreducible pair of type (39) and $m \geq 5$, then $\mathcal{D}$ is not virtually limit.

Proof. Assume that $\mathcal{D}$ is virtually limit. Then there exists a surface $Y$ of type

$$\text{type}(Y) = \left( m, \left( \frac{m-1}{2} \right), \left( \frac{m-1}{2} \right), 0, \left( \frac{m-1}{3} \right) \right)$$

corresponding to the type of the pair $(\mathcal{P}, \mathcal{D})$.

Consider a generic projection of the surface $Y$ to the plane. Then, by Proposition 5 the branch curve $B$ of the generic projection has the following invariants: $\deg B = 2d = 2(m-1)$, $g - 1 = \binom{m-1}{2} - m + 1$, $c = 3\binom{m-1}{2}$, $n = 0$. But then the degree the curve $\bar{B}$ dual to the branch curve $B$ equals

$$\deg \bar{B} = \deg B(\deg B - 1) - 3c = 2(m-1)(2m-3) - 9\binom{m-1}{2} = (3 - \frac{1}{2}m)(m-1) \leq 2$$

for $m \geq 5$, and it is impossible.

8.4. Examples of surfaces completely degenerative only virtually. Consider a degenerative surface $Y$ from section 5. The double curve $D$ of $Y$ is the Hartshorne’s curve. The curve $D$ is smooth, has degree $\bar{d} = 30$ and genus $\bar{g} = 113$. Then $\text{type}(Y) = (m, 30, 1, 113, 0)$ and, consequently, $\tau_2 = \bar{d} + \bar{g} - k = 142$ and the collection corresponding to the type of $Y$ is $(m, 30, 1, 142, 0)$. Let us show that the surface $Y$ is virtually degenerative.
Proposition 10. For \( m \geq 31 \) there exists an irreducible pairs of type \((\mathcal{P}, \mathcal{D}) = (m, 30, 1, 142, 0)\).

Proof. Consider a graph \( \Gamma(\mathcal{D}) \) depicted on Fig. 4.

It has 31 vertices, two of which \((v_{12} \text{ and } v_{17})\) are of valence 12, three vertices \((v_{13}, v_{14}, v_{15})\) are of valence 3, the rest vertices are of valence 1. Applying (12), we find \( \tau_2 = 142 \). Therefore, \( \Gamma(\mathcal{D}) \) really is of the mentioned type. It is easy to see that the pair \((\mathcal{P}, \mathcal{D})\) is irreducible, that is, the graph of the complementary curve \( \mathcal{R} = \mathcal{L} \setminus \mathcal{D} \) is connected. \( \square \)

8.5. Examples of virtually not degenerative surfaces. Below we prove the existence of a surface \( Y \subset \mathbb{P}^3 \) with ordinary singularities, whose double curve \( D \) has unique triple point and consists of three components: two lines \( L_1 \) and \( L_2 \) and a conic \( Q \). Such a curve \( D \) lies on the union of two planes \( P_1 \) and \( P_2 \) such that \( L_1 \cup L_2 \subset P_1, Q \subset P_2, s = L_1 \cap L_2 \cap Q \in P_1 \cap P_2 = L, L_i \neq L \) for \( i = 1, 2 \) and \( Q \) transversally intersect the line \( L \).

Let us show that any surface \( Y \) with ordinary singularities with double curve \( D \), described above, can not be virtually completely degenerated. Indeed, the type of such a surface \( Y \) is type(\( Y \)) = \((m, 4, 3, 0, 1)\). If \( Y \) is virtually completely degenerated, then an irreducible pair \((\mathcal{P}, \mathcal{D})\) with type(\( \mathcal{P}, \mathcal{D} \)) = \((m, 4, 3, 1, 1)\) must exist, that is, there must exist four double lines \( \mathcal{D} \) of a plane arrangement \( \mathcal{P} \) having the following invariants: \( \tau_2 = \tau_3 = 1 \) and, moreover, after removing the triple point the curve \( \mathcal{D} \) is decomposed into three \((k = 3)\) connected components.

Note that there is a degeneration of the curve \( D \) into the union of four lines having one triple point and one double point (to obtain such a degeneration we need to degenerate the conic into a pair of lines lying in \( P_2 \) so that one of the lines passes through the point \( s \) and the other one does not).

Nevertheless, it is easy to see that there does not exist such union of double curves of any plane arrangement \( \mathcal{P} = \cup P_i \) in general position. Indeed, three lines meeting at the triple point must be pairwise intersections of three planes, say \( P_1 \), \( P_2 \), and \( P_3 \). It follows from conditions \( k = 3 \) and \( \tau_2 = 1 \) that the fourth
line must intersect one (and only one) of these lines (without loss of generality we can assume that the fourth line intersects the line $L_{1,2}$). It means that the fourth line is the intersection of the fourth plane $P_4$ of the arrangement $\mathcal{P}$ with one of two planes $P_1$ or $P_2$ and then this line must intersect either $L_{1,3}$ or $L_{2,3}$. But it is impossible, since $t_2 = 1$.

Let us prove the existence of a surface $Y$ with ordinary singularities, $\deg Y \geq 8$, the double curve of which is $D$. For this let us blow up the point $s$ in $\mathbb{P}^3$ and after that let us consecutively blow up the proper transforms of the curves $L_1$, $L_2$, and $Q$. Let $\sigma : \widetilde{\mathbb{P}}^3 \to \mathbb{P}^3$ be a composition of these monoidal transformations, $E, E_1, E_2, E_3 \subset \widetilde{\mathbb{P}}^3$ the proper transforms of the exceptional divisors of each of these blow ups, and let $\tilde{P} = \sigma^*(P)$ be the total inverse image of a plane $P \subset \mathbb{P}^3$.

Let us show that a generic member $\tilde{X}$ of the linear system $|m\tilde{P} - 3E - 2E_1 - 2E_2 - 2E_3|$ is a smooth surface if $m \geq 8$. For this let us choose a system of homogeneous coordinates in $\mathbb{P}^3$ as follows (remind that the systems of homogeneous coordinates in $\mathbb{P}^3$ are uniquely determined by the choice of ordered sets of four planes in general position). We take $P_1$ and $P_2$ as the first two coordinate planes. As the third plane, we take any plane $P_3 \neq P_2$ passing through $s$ and touching the conic $Q$ at this point. We choose the fourth coordinate plane $P_4$ such that it is in general position with $P_1$, $P_2$, and $P_3$ and touches the conic $Q$ at some point $s_1 \in Q$, $s_1 \neq s$. In the coordinate system chosen in such a way, the lines $L_i$, $i = 1, 2$, are given by equations

$$z_1 = (c_1z_2 + z_3) = 0, \quad c_1 \neq c_2,$$

and the curve $Q$ is given by

$$z_2 = (z_3z_4 - c_3z_1^2) = 0, \quad c_3 \neq 0.$$

The triple point $s$ of $D$ has coordinates $(0 : 0 : 0 : 1)$.

Denote by $F_i$ the following homogeneous polynomials: $F_1 = z_1$, $F_2 = z_3z_4 + c_1z_2z_4 - c_3z_1^2$, $F_3 = z_3z_4 + c_2z_2z_4 - c_3z_1^2$, $F_4 = z_4$ and consider a linear system of surfaces $\mathfrak{S}_{a_1, a_2, a_3} \subset \mathbb{P}^3$ given by homogeneous equation

$$F_1F_2F_3F_4^3 + a_1F_1^2F_2^2F_3^2 + a_2F_1^2F_2F_3^2F_4 + a_3F_2^3F_3^2 = 0.$$

Denote also by $x_i = \frac{z_i}{z_4}$ nonhomogeneous coordinates in the chart $V_4 = \mathbb{P}^3 \setminus P_4 \simeq \mathbb{C}^3$ and put $S_{a_1, a_2, a_3} = \mathfrak{S}_{a_1, a_2, a_3} \cap V_4$.

**Claim 1.** (i) For almost all points $(a_1, a_2, a_3)$ the proper transforms $\sigma^{-1}(S_{a_1, a_2, a_3}) \subset \sigma^{-1}(V_4)$ are nonsingular surfaces.

(ii) For $j = 1, 2, 3$ and for almost all points $(a_1, a_2, a_3)$ the intersections $\sigma^{-1}(S_{a_1, a_2, a_3}) \cap E_j$ are nonsingular 2-sections of the ruled surfaces $E_j$, and the intersections $\sigma^{-1}(s) \cap \sigma^{-1}(S_{a_1, a_2, a_3}) \cap E_j$ consist of two points.

(iii) The base locus of the linear system $\sigma^{-1}(S_{a_1, a_2, a_3})$ consists of three rational curves lying in $E$, their images after blow down of the divisors $E_1$, $E_2$, and $E_3$. 


are lines in the exceptional divisor $E' \simeq \mathbb{P}^2$ of the blow up of $s$. In some analytic neighborhood of $E'$, after the blow down the image of a generic surface of this linear system is decomposed into three irreducible nonsingular components each of which intersects transversally $E'$ along one of three lines in $E' \simeq \mathbb{P}^2$ being the image of the base locus of the linear system.

Proof. In coordinates $x_1$, $x_2$, $x_3$, the linear system of the surfaces $S_{a_1,a_2,a_3}$ is given by

$$
\begin{aligned}
x_1(x_3 + c_1 x_2 - c_3 x_1^2)(x_3 + c_2 x_2 - c_3 x_1^2) + a_1 x_1^2 (x_3 + c_1 x_2 - c_3 x_1^2)^2 + \\
a_2 x_1^2 (x_3 + c_2 x_2 - c_3 x_1^2)^2 + a_3 (x_3 + c_1 x_2 - c_3 x_1^2)^2 (x_3 + c_2 x_2 - c_3 x_1^2)^2 & = 0.
\end{aligned}
$$

Let us take new coordinates in $V_4$:

$$
\begin{align*}
y_1 & = x_1 \\
y_2 & = x_3 + c_1 x_2 - c_3 x_1^2 \\
y_3 & = x_3 + c_2 x_2 - c_3 x_1^2.
\end{align*}
$$

In these coordinates the linear system of the surfaces $S_{a_1,a_2,a_3}$ is given by

$$
y_1 y_2 y_3 + a_1 y_1^2 y_2^2 + a_2 y_1^2 y_3^2 + a_3 y_2^2 y_3^2 = 0,
$$

that is, for almost all points $(a_1, a_2, a_3)$ the surfaces $S_{a_1,a_2,a_3} \cap V_4$ are affine parts of the images of Veronese surfaces under a generic projection into $\mathbb{P}^3$ for which the claim is well known.

To show that a generic member $\tilde{X}$ of the linear system $| m \tilde{P} - 3E - 2E_1 - 2E_2 - 2E_3 |$ is a nonsingular surface for $m \geq 8$, let us note that the surfaces $\tilde{S}_{a_1,a_2,a_3} = \sigma^{-1}(S_{a_1,a_2,a_3})$ belong to the linear system $| 8 \tilde{P} - 3E - 2E_1 - 2E_2 - 2E_3 |$. It follows from Claim 1 that for any point $p \subset \mathbb{P}^3$, $p \neq s$, after the change of the coordinate plane $P_4$ by another plane not passing through $p$, the base locus of the linear system $| m \tilde{P} - 3E - 2E_1 - 2E_2 - 2E_3 |$ does not meet the proper transform $\sigma^{-1}(p)$. Consequently, for any $m \geq 8$ the linear system $| m \tilde{P} - 3E - 2E_1 - 2E_2 - 2E_3 |$ has the same base locus as for $m = 8$, since the linear system $| (m-8) \tilde{P} |$ does not have the base points. Finally, it follows from Bertini Theorem that the generic member $\tilde{X}$ of $| m \tilde{P} - 3E - 2E_1 - 2E_2 - 2E_3 |$ satisfies the same properties (i) — (iii) of Claim 1 as the generic member $\sigma^{-1}(S_{a_1,a_2,a_3})$ has. From this it follows that the image $Y = \sigma(\tilde{X})$ of the generic member $\tilde{X}$ is a surface with ordinary singularities the double curve of which is $D$.

9. Concluding remarks

In this section, we formulate some open questions related to the existence problem of complete degenerations of surfaces with ordinary singular points.
9.1. **Absolute and relative complete non degeneracy.** Let \( X \) be a smooth projective surface and \( g : X \to \mathbb{P}^3 \) some "immersion" (that is, \( Y = g(X) \) is a surface with ordinary singularities and the morphism \( g : X \to Y \) is a normalization of \( Y \)). In subsections 5.1 and 8.5 we have given examples of surfaces \( Y \) which can not be completely degenerate. The reason of impossibility to be completely degenerate can originate from that the "immersion" \( g \) is "bad" and for some other "immersion" of \( X \) its image, nevertheless, can be completely degenerated. It is possible the second case: for all "immersions" of a surface \( X \) in \( \mathbb{P}^3 \) it is impossible to completely degenerate its image \( Y \), in other words, the reason of impossibility of complete degenerations is in topology of \( X \). In the first case we say that \( X \) is *relatively completely degenerative*, and in the second case \( X \) is *absolutely completely non degenerative*. We say also that \( X \) is *absolutely completely degenerative* if for any "immersion" \( g \) its image \( g(X) = Y \) is completely degenerative.

**Problem 1.** (i) Do there exist absolutely completely non degenerative surfaces \( X \)?
(ii) Do there exist absolutely completely degenerative surfaces \( X \)?

In the case of negative answer on any of these problems the normalizations \( X \) of completely non degenerative and completely degenerative surfaces \( Y \), described in the article, would give examples of relatively completely degenerative surfaces.

9.2. **Problem of adjacency.** Denote by \( \mathcal{H}_{\text{type}}(Y) \subset \mathbb{P}^{(m+3)-1} \) the quasiprojective variety parametrising the surfaces with ordinary singularities of the same type as the type of a surface \( Y \), \( \deg Y = m \). Let \( \Pi_m \subset \mathbb{P}^{(m+3)-1} \) be the variety parametrising arrangements of \( m \) planes in \( \mathbb{P}^3 \) in general position, \( \dim \Pi_m = 3m \). It follows from complete degeneracy of \( Y \) that \( \Pi_m \) and the closure of \( \mathcal{H}_{\text{type}}(Y) \) have nonempty intersection.

**Problem 2.** Let \( Y \) be completely degenerative surface with ordinary singularities. Is it true that \( \Pi_m \) lies in the closure of \( \mathcal{H}_{\text{type}}(Y) \)?

This is a part of the following more general problem: to describe the natural stratification (according to the types of double curves) of the variety \( \mathcal{H}_m \) of surfaces in \( \mathbb{P}^3 \) with ordinary singularities and the adjacencies of these strata.
9.3. **Uniqueness complete degeneracy problem.** We say that two pairs 
\((\mathcal{P}, \mathcal{D}_1)\) and \((\mathcal{P}, \mathcal{D}_2)\) are **deformation equivalent** if these pairs are fibres of flat families \(\mathcal{D}_u \subset \mathcal{P}_u \subset \mathbb{P}^3\), \(u \in U\), of plane arrangements in general position and the configurations of double lines contained in the plane arrangements. It is obvious that pairs \((\mathcal{P}_1, \mathcal{D}_1)\) and \((\mathcal{P}_2, \mathcal{D}_2)\) are deformation equivalent if and only if \(\deg \mathcal{P}_1 = \deg \mathcal{P}_2\) and the graphs \(\Gamma(\mathcal{D}_1)\) and \(\Gamma(\mathcal{D}_2)\) are isomorphic.

Let \(Y\) be a surface with ordinary singularities. We say that \(Y\) has a **unique complete degeneration** if it is completely degenerative and any two its complete degenerations are deformation equivalent.

**Claim 2.** Any surface \(Y\) with ordinary singularities whose double curve is of degree not more than four possesses not more than unique complete degeneration

![Diagram](image-url)
Proof. All possible realizable graphs with the number of edges not more than four are depicted in Fig. 5. The type \((m, \overline{d}, k, \overline{g}, t)\) of a surface \(Y\), in which the plane arrangement \(P_m\) in general position can be smoothed outside of the configuration of corresponding double curve \(D\), is written under each graph.

One can see from this list of graphs that a surface \(Y\) could have more than one complete degeneration only in two cases when \(\text{type}(Y) = (m, 4, 2, 0, 0)\) or \(\text{type}(Y) = (m, 4, 1, 1, 0)\). But in the first case graphs \(\Gamma(D)\) have different types \(\Gamma^{(2,0)}_{(2,1)}\) and \(\Gamma^{(1,0)}_{(2,0)}\), and if a plane arrangement \(P\) is smoothed outside of \(D\) with graph \(\Gamma^{(2,0)}_{(2,1)}\), than the double curve \(D\) of \(Y\) consists of two irreducible components of degree two; and if the smoothing takes place outside of \(D\) with graph \(\Gamma^{(1,0)}_{(2,0)}\), then the irreducible components of \(D\) have degrees one and three, that is, in this case the regenerated surfaces \(Y\) have different (extended) types. In the second case the graphs \(\Gamma(D)\) have the same type \(\Gamma^{(4,0)}_{(3,1,1)}\), but plane arrangements \(P\) can not be smoothed outside of one of configurations of double curves corresponding to this type (see Theorem 5).

**Problem 3.** Does any surface \(Y\) possess not more than unique complete degeneration?

9.4. **Smoothings in symplectic case.** Above we gave many examples of pairs \((P, D)\) which can not be smoothed outside of \(D\). In some cases the obstructions to smoothing were purely topological (for example, the negativity of degree of the dual curve of the branch curve \(B \subset \mathbb{P}^2\) of generic projection to the plane of smoothed surface \(Y\)), in the other cases the obstructions, possibly, have algebraic geometry nature (for example, the pairs \((P, D)\) with the curve \(D\) whose graph is depicted in Fig. 4).

To understand better the nature of these obstructions, it is useful to generalize the problem of complete degenerations of algebraic surfaces with ordinary singularities to the case of symplectic varieties. Namely, we say that a compact real four dimensional subvariety \(M\) of \(\mathbb{C}\mathbb{P}^3\) is a *symplectic variety with ordinary singularities* if for each point \(p \in M\) there is a neighborhood \(V\) of \(p\) such that either variety \(V \cap M\) is decomposed into \(n\) smooth components \((n \leq 3)\) being symplectic submanifolds of \(\mathbb{C}\mathbb{P}^3\) with respect to the Fubini-Studzi symplectic form and meeting transversally along smooth symplectic surfaces ("double curves" of \(M\); in the case \(n = 3\) the point \(p\) is a triple point of \(M\)) or \(V \cap M\) is a complex analytic variety given in some complex analytic coordinates in \(V\) by equation \(x^2 - yz^2 = 0\) (and in this case the point \(p\) is called a pinch of \(M\)). A definition of complete degeneration of varieties with ordinary singularities can be also generalized for symplectic varieties.
Problem 4. Do there exist pairs $(P,D)$ such that the plane arrangement $P$ can not be smoothed outside of $D$ in the context of algebraic geometry, but this arrangement can be smoothed outside of $D$ in symplectic context?

References

[1] Severi F., Vorlesungen uber algebraische Geometrie, Anhang F. Leipzig: Teubner 1921 (resp. Johnson 1968).
[2] Harris J., On the Severi problem, Invent. Math. 84 (1986), 445-461.
[3] Greuel G.-M., Lossen C., Shustin E. Introduction to singularities and deformations. Springer Verlag, 2007.
[4] Mioshezon B., Complex surfaces and connected sums of complex projective planes. LNM, 603. (1977), Springer-Verlag, Berlin, Heidelberg, New York.
[5] Griffiths P. and Harris J., Principles of algebraic geometry. A Wiley-Interscience Publication, John Wiley & Sons, New York Chichester Brisbane Toronto, 1978.
[6] Calabri A., Ciliberto C., Flamini F., Miranda R., On degenerations of surfaces. arXiv:math/0310009v2 [math.AG] 9 May 2008.
[7] Calabri A., Ciliberto C., Flamini F., Miranda R., On the geometric genus of reducible surfaces and degenerations of surfaces to unions of planes. The Fano Conference, 277–312, Univ. Torino, Turin, 2004.
[8] Buchweitz R.-O., Greuel G.-M., The Milnor number and deformations of complex curve singularities. Invent.math. 58, 241-281 (1980).
[9] Kulikov Vik.S., On Chisini’s Conjecture. Izv. Math., 63:6 (1999), 1139-1170.
[10] Kulikov Vik.S., On the structure of the fundamental group of complement to algebraic curves in $\mathbb{C}^2$, Izv. Math., 40:2 (1993), 443 - 454
[11] Ciliberto C., Flamini F., On the branch curve of a general projection of a surface to a plane. [arXiv:0811.0467v1 [math.AG]] 4 Nov 2008.
[12] Hartshorne R., Families of curves in $P^3$ and Zeuthen’s problem, Mem. Amer. Math. Soc. 130 (1997), no. 617, viii+96 pp.
[13] Kulikov Vik.S., On Chisini’s Conjecture. II, Izv. Math., 72:5 (2008), 901 – 913.
[14] Gotzmann G., The irreducible components of Hilb$^4(\mathbb{P}^3)$, [arXiv:0811.3160v1 [math.AG]] 19 Nov 2008.

Moscow State University of Printing
E-mail address: vskulikov@mail.ru

Steklov Mathematical Institute
E-mail address: kulikov@mi.ras.ru