Permutation Orbifolds in the large $N$ Limit

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Abstract

The space of permutation orbifolds is a simple landscape of two dimensional CFTs, generalizing the well-known symmetric orbifolds. We consider constraints which a permutation orbifold with large central charge must obey in order to be holographically dual to a weakly coupled (but possibly stringy) theory of gravity in AdS. We then construct explicit examples of permutation orbifolds which obey these constraints. In our constructions the spectrum remains finite at large $N$, but differs qualitatively from that of symmetric orbifolds. We also discuss under what conditions the correlation functions factorize at large $N$ and thus reduce to those of a generalized free field in AdS. We show that this happens not just for symmetric orbifolds, but also for permutation groups which act “democratically” in a sense which we define.
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1 Introduction

1.1 AdS/CFT and the Space of CFT$_2$’s

The AdS/CFT correspondence provides, at least in principle, a completely non-perturbative definition of quantum gravity in asymptotically Anti-de Sitter Space. Weakly coupled theories of AdS gravity are typically dual to strongly coupled CFTs, making it difficult to use this correspondence to make precise statements about semi-classical gravity. The AdS$_3$/CFT$_2$ correspondence, however, provides the hope of something more. The constraints of conformal invariance in two dimensions are much stronger than in $d > 2$, allowing us to understand the duality even in the semi-classical limit. For example, the infinite Virasoro symmetry of two dimensional CFTs can be understood as the asymptotic symmetry algebra of three dimensional gravity [1]. Similarly, modular invariance – i.e. invariance under large conformal transformations – can be understood as coming from a sum over semi-classical saddle points of the gravitational path integral [2, 3].

In the AdS$_3$/CFT$_2$ dictionary, a CFT with central charge $c$ is dual to a theory of gravity with

$$c = \frac{3\ell}{2G},$$

where $\ell$ is the AdS radius and $G$ is Newton’s constant. The space of two dimensional CFTs can be regarded as a landscape of theories of quantum gravity in AdS$_3$, with many different types of matter content and values of the coupling constant $c$. A weakly coupled theory of gravity – that is, one where $G$ is small – is dual to a CFT with large central charge $c$. In this paper we will characterize a simple class of conformal field theories at large central charge and understand features of the corresponding gravity duals.

The set of two dimensional CFTs – while much simpler than in higher dimensions – is still quite complicated. The best understood theories are rational CFTs, which have small values of $c$. The Virasoro minimal models, with $c < 1$, are the most well known examples. Exactly solvable CFTs with $c > 1$ can also be constructed, but they are always rational with respect to some larger chiral algebra which includes not just the Virasoro generators but also some non-trivial $\mathcal{W}$-symmetries. In other words, rational CFTs may have $c > 1$, but the central charge will still be “small” compared to the size of the symmetry algebra. Most attempts to describe pure theories of quantum gravity in AdS (i.e. theories with only metric degrees of freedom) involve rational CFTs of some type.

In this paper we will take the opposite approach: rather than trying to study a simple, exactly solvable CFT which is dual to a particularly simple theory of gravity, we will attempt to characterize more generally the space of conformal field theories with large central charge.

The characterization of the space of large $c$ CFTs is an interesting problem in its own right. Conformal bootstrap techniques have proven useful in constraining the properties of these CFTs. For example, see [4, 5, 6, 7] for candidate duals to minimal model CFTs. We note that the extremal CFTs of Witten [7] are also rational CFTs in a rather trivial sense; they are chiral CFTs, so by definition are rational with respect to a sufficiently large chiral algebra.
general structure of 2d holographic CFTs [8, 9, 10, 11, 12, 13, 14, 15], as well as other aspects such as locality and thermalization [16, 17, 18]. Unfortunately a complete classification still seems out of reach. We will therefore consider only a particularly tractable corner of the landscape of 2d CFTs: the space of permutation orbifolds. The virtue of this approach is that, at least in principle, one can construct explicitly all theories in this subspace. The simplest examples are symmetric orbifolds, which appear as the dual CFTs in various string theory constrictions of AdS₃, including the D1-D5 system. Indeed, this family appears to include all known explicit examples of holographic CFTs with large central charge.

1.2 Holographic CFTs

In this paper we are interested in theories of gravity with a semi-classical limit. This means we will consider not a single CFT, but rather a family of CFTs labeled by a parameter \( N \) which is proportional to the central charge. The semiclassical limit is \( N \to \infty \). Although any individual CFT can be interpreted as a theory of gravity in AdS – at least in the sense that the CFT correlation functions can be regarded as scattering amplitudes for fields in asymptotically Anti-de Sitter space – the \( N \to \infty \) limit may not describe a well behaved weakly coupled theory of gravity. We emphasize that, in the present context, by a “weakly-coupled” theory of gravity we do not necessarily mean that the dual gravity theory is perturbative Einstein gravity coupled to matter. We only mean that – since the Planck length is small in AdS units – gravitational backreaction is negligible. For example, we will not require out theory to be local on the AdS scale; this would require additional constraints, such as the existence of a large gap in the CFT spectrum [19]. We would, for example, be happy to consider theories of gravity which have as their \( N \to \infty \) limit a classical \((g_s \to 0)\) string theory with string length of order the AdS scale. Such theories are expected to be dual to weakly coupled gauge theories in the large \( N \) limit.

We now ask what conditions we must impose on these theories in order for a well defined semi-classical limit to exist; see [20, 21, 22] for similar considerations. Our first constraint is that the number of states should remain finite in the large \( N \) limit. More precisely, if we let \( \rho_N(\Delta) \) be the number of states with energy (i.e. scaling dimension) \( \Delta \) in theory \( N \), then we demand that the limit

\[
\rho_\infty(\Delta) = \lim_{N\to\infty} \rho_N(\Delta)
\]

exists and is finite for any \( \Delta \). This is the statement that, once gravitational backreaction is turned off, the theory has only a finite number of degrees of freedom below a given energy. This property is satisfied by any of the familiar examples of AdS/CFT, including large \( N \) gauge theories. Indeed, once we take \( N \to \infty \), the resulting function \( \rho_\infty(\Delta) \) can be used to characterize the semi-classical gravity theory. In particular, the states counted by \( \rho_\infty(\Delta) \) are interpreted as perturbative excitations in AdS in the limit where gravitational back-reaction is neglected. In a typical string theory, for example,
one would expect a Hagedorn spectrum

$$\rho_\infty(\Delta) \approx \exp\{\beta_H \Delta\} \quad \text{as } \Delta \to \infty ,$$  \hspace{1cm} (3)

where the Hagedorn temperature $\beta_H$ is related to the string tension.\footnote{In many cases, the Hagedorn divergence can be naturally interpreted as the Hawking-Page transition between the thermal and black hole phases \cite{23,24}.} In a theory with fewer degrees of freedom $\rho_\infty(\Delta)$ would increase more slowly with $\Delta$. In particular, for a local quantum field theory in $d$ dimensions compactified down to AdS$_3$ we expect

$$\rho_\infty(\Delta) \approx \exp\{\beta \Delta^{(d-1)/d}\} \quad \text{as } \Delta \to \infty .$$  \hspace{1cm} (4)

It is worth emphasizing that the semi-classical density of states $\rho_\infty(\Delta)$ does not obey Cardy’s formula. In particular, the states which exhibit Cardy growth have $\Delta \gtrsim O(N)$, so are removed from the spectrum if we keep $\Delta$ fixed as $N \to \infty$. Indeed, these states are interpreted as BTZ black holes, which have very high energy in the limit $G \to 0$ where gravitational interactions are turned off.

Of course, we expect that holographic CFTs dual to semi-classical gravity should exhibit other features in addition to a finite spectrum. For example, we must also demand that the correlation functions remain finite in the large $N$ limit. More precisely, we require that for any $\Delta$, the spectrum of operators with energy $< \Delta$ must stabilize for sufficiently large $N$. Moreover, we require that the correlation functions for (correctly normalized) operators at finite separation will approach well defined, finite limits as $N \to \infty$.

Given these assumptions, we may then ask whether a given family of CFTs has other features which resemble semi-classical gravity in AdS. For example, we can ask whether the correlation functions factorize into products of two point functions at large $N$, signifying that the bulk theory reduces to a linearized theory of generalized free fields. In this paper we will restrict our attention to a family of theories where these questions can be addressed precisely: Permutation Orbifolds.

1.3 Permutation Orbifolds

The simplest way to construct a large central charge CFT is to take $N$ non-interacting copies of a given “seed” CFT $\mathcal{C}$ with Hilbert space $\mathcal{H}$ and central charge $c$. The resulting tensor product theory $\mathcal{C}^\otimes N$ has central charge $Nc$. Such theories are not, however, good candidate holographic duals of semi-classical gravity as they will typically have an infinite number of states at low energies. To obtain a finite number of states, we will consider orbifold theories. In particular, the product theory $\mathcal{C}^\otimes N$ has $S_N$ global symmetry that interchanges the various copies. One can quotient the theory by any subgroup $G_N \subseteq S_N$. A theory obtained in this manner is called a permutation orbifold and denoted

$$\mathcal{C}_{G_N} = \frac{\mathcal{C}^\otimes N}{G_N} .$$  \hspace{1cm} (5)
The orbifold theory will typically have two types of states: twisted sector and untwisted sector states. The untwisted sector is obtained by simply taking the subset of \( \mathcal{H}^\otimes N \) that is invariant under the action of \( G_N \). For the theory to have a well-defined \( N \to \infty \) limit, the total density of states has to be finite for any state of finite energy. This places strong constraints on the subgroups we can consider. For example, one can easily show that the cyclic group \( \mathbb{Z}_N \) does not have this property. As we will argue, subgroups with this property must have a finite number of orbits on \( K \)-tuples as \( N \to \infty \). They are called oligomorphic families of groups and we describe their properties in the following subsection.

The landscape of permutation orbifolds is, of course, just a tiny corner of the space of CFTs. It is, however, a rich enough space that it describes a variety of low energy spectra and correlation functions in the semi-classical limit. In the first part of this paper we will investigate the spectra of permutation orbifolds at large \( N \). We will establish several results for the spectrum which hold universally, and rule out the QFT growth of the type \([4]\). This extends results found previously in \([20, 21]\). We will also describe a new class of examples whose spectra exhibit novel and interesting features that differ from the more familiar symmetric products. We pay particular attention to the wreath product \( S_{\sqrt{N}} \wr S_{\sqrt{N}} \) and tensor product \( S_{\sqrt{N}} \times S_{\sqrt{N}} \) theories, and present several explicit results for their spectra. In the second part of the paper we will investigate correlation functions. We will show that symmetric product orbifold correlation functions factorize in the way expected of free field correlators, and describe the circumstances under which this feature generalizes to other permutation orbifolds at large \( N \).

Ultimately, our goal is to understand the statistics of the space of permutation orbifolds, and to describe features of “generic” conformal field theories in the large \( N \) limit, along the lines of \([23, 26]\). This would allow us to understand, for example, how likely it is that a randomly chosen family of CFTs happens be dual to a theory of weakly coupled gravity in AdS.

## 2 Permutation Orbifolds and their spectrum

### 2.1 Oligomorphic families \( G_N \)

We will first discuss the spectrum of permutation orbifolds. Starting out with an arbitrary seed theory \( \mathcal{C} \), we take its \( N \)-fold tensor product \( \otimes^N \mathcal{C} \). Regardless of any symmetries of \( \mathcal{C} \), \( \otimes^N \mathcal{C} \) is symmetric under permutations \( g \in S_N \). Using the standard orbifold construction \([27, 28]\), we can orbifold the theory by any permutation group \( G_N \subset S_N \). This means we project onto states \( \Phi \in \mathcal{H}^\otimes N \) which are invariant under \( G_N \). Since the resulting theory is no longer modular invariant, we need to add in so-called twisted sectors to restore modular invariance. For each conjugacy class \([g]\) in \( G_N \) we need to add one corresponding twisted sector. These twisted states of course also need to be invariant under \( G_N \), or more precisely under the centralizer \( C_g \) of \( g \). The net effect of an orbifold is thus not to so much to eliminate states, but rather to rearrange
them. Still, for our purposes this will often be enough. The important point is that for permutation orbifolds most (but not all) twisted states have weight \( \Delta \sim cN/12 \), and are therefore harmless in the \( N \to \infty \) limit.

We are interested in the spectrum \( \rho_{G_N}(\Delta) \) in the \( N \to \infty \) limit. Even though we will not do so, it should nonetheless be possible to make the notion of a limit of families of CFTs precise. The rough idea is to ensure that for any \( \Delta_1 \), both the spectrum and all correlation functions of states with \( \Delta < \Delta_1 \) converge. Note that this means that we are only interested in states with finite \( \Delta \) in the large \( N \) limit. Also note that for each \( N \), the theory will have Cardy behavior for \( \Delta \gg cN \), the onset of this behavior diverges with \( N \). The actual behavior we see will therefore be quite different from Cardy behavior, even for states whose weight \( \Delta \) is much higher than the central charge \( c \) of the seed theory \( \mathcal{C} \).

Not surprisingly, analyzing the untwisted sector is much easier than the twisted sectors. We will thus begin with the untwisted states.

### 2.1.1 Untwisted states

Let us now consider a family of permutation groups \( G_N \subseteq S_N \). We first need to describe the states of such a theory in some detail. To construct a generic untwisted state, we start with a state \( \phi \) in the underlying tensor theory \( \mathcal{C} \otimes N \). Such a state will consist of \( K \) factors which are in some non-vacuum states \( \varphi_i \) of the seed theory, while the rest of the factors will be in the vacuum. In particular, states of the \( N \to \infty \) theory will have finite weight \( \Delta \) only if almost all factors are in the vacuum. We can thus label such a state by an ordered \( K \)-tuple \( \vec{K} \) of distinct integers, and a \( K \)-vector \( \vec{\varphi} \) of states in the seed theory,

\[
\phi = \phi_{(\vec{K}, \vec{\varphi})}.
\]

The notation here is that the state \( \varphi_i \) is in factor \( K_i \), and all factors not specified by \( \vec{K} \) are in the vacuum. An advantage of this notation is that it does not depend explicitly on \( N \). Taking the \( N \to \infty \) limit on the level of such states is thus straightforward.

In the orbifolded theory, a generic state \( \phi \) that lives in the product Hilbert space \( \mathcal{H} \otimes N \) will obviously not survive the projection onto \( G_N \)-invariant states. In this context we will thus call \( \phi \) a prestate and use it to build actual states of the \( G_N \) orbifolded theory. Using the notation introduced above, \( G_N \) simply acts on \( \vec{K} \) in the natural way, and does not affect \( \vec{\varphi} \). We can project on an invariant state by summing over the images of \( G_N \), so that an actual state \( \Phi \) in the orbifold theory is given by the orbit of \( \vec{K} \),

\[
\Phi = \sum_{g \in G_N} \phi_{(g.\vec{K}, \vec{\varphi})}.
\]

To count the number of states, we are thus lead to counting the number of orbits of ordered \( K \)-tuples. Let us denote the number of orbits under \( G_N \) of ordered \( K \)-tuples of distinct elements by \( F_K(G_N) \). This is indeed the number of untwisted states coming from the states \( \vec{\varphi} \) if all states in \( \vec{\varphi} \) are distinct. If some of them are the same, then \( F_K(G_N) \) will overcount them. For instance, if all of them are equal, then the number
of states is given \( f_K(G_N) \), the number of orbits of (unordered) subsets of \( K \) distinct numbers. For a general \( \vec{\varphi} \), the number of states will lie somewhere in between. In general we have the relation
\[
f_K \leq F_K \leq K! f_K.
\] (8)

The detailed relation between \( f_K \) and \( F_K \) is in general very complicated and depends greatly on \( G_N \).

To get the total number of states of weight \( \leq \Delta \), we also need to count the number of possible \( \vec{\varphi} \). The important point here is that for any \( \Delta \) we will get a finite number of such configurations, which is independent of \( G_N \). In particular it does not depend on \( N \). The whole large \( N \) behavior of the spectrum is thus determined by the \( \vec{K} \)-orbits of \( G_N \).

Let us now consider the large \( N \) limit of a family \( G_N \). We need this family to converge in an appropriate sense, as mentioned above. We will require that the spectrum converges, i.e. that for any fixed \( \Delta_1 \) the limit \( N \to \infty \) leads to a finite spectrum of states with \( \Delta < \Delta_1 \). For that to happen, we need \( F_K(G_N) \) to converge to a finite number as \( N \to \infty \), which means that \( F_K(G_N) \) becomes independent of \( N \),
\[
F_K(G_N) = F_K \quad \text{for } N \text{ large enough.} \tag{9}
\]

A family of groups \( G_N \) which has this property is called \textit{oligomorphic}. From the remarks above, it follows that a family of permutation orbifolds has a finite number of untwisted states if it comes from an oligomorphic family of groups. From (8) it is clear that it does not matter whether we consider \( F_K \) or \( f_K \) here.

### 2.1.2 Twisted states

We now turn our attention to twisted states. The situation here is slightly more complicated, but it turns out that the end result is the same as in the untwisted sector: oligomorphic permutation orbifolds also lead to a finite number of twisted states.

Let us show this in more detail. The twisted sectors of the theory are roughly given by elements \( g \in G_N \). More precisely, they are given by conjugacy classes \([g]\). Being a permutation, \( g \) can be always written as a product of cycles. In the special case of \( S_N \), a conjugacy class \([g]\) is given by the number of cycles of different lengths,
\[
[g] = (1)^{N_1}(2)^{N_2} \cdots (s)^{N_s}
\] (10)

where \( \sum_n n N_n = N \). The centralizer is then \([g]\)
\[
C_{[g]} = S_{N_1} \times (\mathbb{Z}_2^{N_2} \rtimes S_{N_2}) \times \cdots \times (\mathbb{Z}_s^{N_s} \rtimes S_{N_s}) . \tag{11}
\]

Here the \( S_{N_n} \) permute the \( N_n \) cycles of length \( n \), and the \( \mathbb{Z}_n \) act as cyclic shifts within a cycle of length \( n \). For a general permutation group \( G \), \( g \) can still be written as a product of cycles, but its conjugacy classes are no longer in one-to-one correspondence with cycle lengths. The centralizer of \( g \) in \( G \) is a subgroup of (11). Note that the only part of this centralizer that grows with \( N \) are the permutations of the single cycles,
which will form a subgroup of $S_{N_1}$. In what follows we can thus afford to be imprecise with the other part of the centralizer.

Within a given cycle of length $n$, the ground state has weight

$$\Delta_n = \frac{c}{24} \left( n - \frac{1}{n} \right).$$

It follows immediately that to have finite weight in the large $N$ limit, almost all factors have to be in trivial cycles with $n = 1$. The situation is thus the same as in the untwisted sector, and we define the length $K$ of a state as the total number of factors which are in a non-trivial cycle or are not in the vacuum.

Let us generalize the notation introduced above to the twisted sector. Denote a pre-state $\phi_g$ of length $K$ in the twisted sector $g$ as a triple

$$\phi_g = \phi_{(P, \vec{K}, \vec{\phi})},$$

where $\vec{K}$ is again a $K$-tuple of distinct elements, $P$ is an integer partition of $K$ which we will represent by a vector $(\lambda_1, \ldots, \lambda_n)$, and $\vec{\phi}$ is again a $K$-tuple of states $\varphi_i$ of the seed theory. In particular $\vec{K}$ again describes the positions of the non-trivial factors. The new datum $P$ describes the cycles of the permutation element $g$, i.e. it determines which twisted sector the state is in. More precisely, the permutation $g$ is given by the cycle decomposition

$$g = \prod_{i=1}^{n} (K_{\mu_i+1}, K_{\mu_i+2}, \ldots, K_{\mu_i+\lambda_i}) \prod_{k \notin \vec{K}} (k)$$

where we defined $\mu_i := \sum_{j=1}^{i-1} \lambda_j$. The $\lambda_i$ thus encode the length of the cycles in $g$, and we fill up $g$ with single cycles.

An actual state $\Phi$ is again given by an orbit of $\phi_g$ under conjugation by $G_N$,

$$\Phi = A_\Phi^{-1/2} \sum_{h \in G_N} \phi_{(P, h \vec{K}, \vec{\phi})}.$$
this, we need to sum $\phi_{hgh^{-1}}$ over the centralizer $C_{hgh^{-1}} = hC_gh^{-1}$. Note that this does of course not effect $g$ as an element of $G_N$, since $(P, h.\vec{K})$ corresponds to the same permutation as $(P, \vec{K})$ for $h \in C_g$. It does however affect the state $\phi_{(P, \vec{K}, \vec{\varphi})}$, since it changes how the states $\varphi_i$ are assigned to the different factors. This is exactly how summing over the centralizer makes $\phi_g$ invariant. In total we thus get

$$\Phi \sim \sum_{h \in G_N/C_g} \sum_{hC_gh^{-1}} \phi_{hgh^{-1}}, \quad (17)$$

which is indeed the same as (15).

This establishes the desired result for the orbits of $G_N$. We have to be more careful when counting the actual states, since not every $\vec{\varphi}$ will give a state. In fact, $\vec{\varphi}$ is no longer a $K$-vector, but rather an $n$-vector instead. We assign a seed theory state to each cycle rather than each individual factor, since the Hilbert space $H_{(n)}$ of states in cycles of length $n$ is a subspace of the seed theory Hilbert space $H$ [30],

$$H_{(n)} \subset H. \quad (18)$$

If the centralizer is non-trivial, then $H_{(n)}$ is a proper subspace. This means that even with our new definition of $\vec{\varphi}$ we overcount the number of states, since not every $\phi_i \in H$ leads to a $C_g$ invariant state of the cycle. For our present purposes this does not matter, since the overcounting is as always independent of $N$.

This shows that oligomorphic permutation orbifolds also have a finite number of twisted states.

### 2.2 Oligomorphic groups

Let us point out that oligomorphic groups have been studied by mathematicians [31]. They are related to what we have defined above as oligomorphic families. We start out with a permutation group $G$ of an infinite countable set $\Omega$, say the natural numbers. $G$ is then said to be oligomorphic if for all $K$, it has only a finite number of orbits on $\Omega^K$, the set of $K$-tuples of elements of $\Omega$.

At least morally speaking the limit of an oligomorphic family $G_N$ should always give an oligomorphic group $G = G_\infty$. The converse is much less obvious. Given such an oligomorphic $G$, it is not clear in general how to construct an oligomorphic family $G_N$ whose orbifolds converge to $G$. One construction is the following: Let $G\{N\}$ be the setwise stabilizer of the set of the first $N$ elements, i.e. the subgroup that leaves that set invariant. Let $G(N)$ be the pointwise stabilizer of the first $N$ elements, i.e. the subgroup that leaves each of the first $N$ elements invariant. We can then define the quotient

$$G_N := G\{N\}/G(N), \quad (19)$$

which gives a well-defined permutation group on the first $N$ elements. For the symmetric group $S_\infty$ this gives the desired answer: $G_N$ is indeed exactly $S_N$. For a general oligomorphic group however, this construction does not even guarantee an oligomorphic
family, as the following example illustrates.

Take the group \(A = \text{Aut}(\mathbb{Q}, <)\) of order-preserving permutations of \(\mathbb{Q}\). This group for instance contains continuous, piecewise linear maps of strictly positive rational slope which are non-smooth only at rational points. One can show that this group is oligomorphic, and has indeed \(f_K(A) = 1\) and \(F_K(A) = K!\). The above construction however gives \(A_N = 1\), since the permutation that preserves the order of a finite number of elements is the identity \([32]\). The family \(A_N\) is then clearly not oligomorphic in our sense. We do not know if there is a way to construct an oligomorphic \(A_N\) whose orbifolds converge to \(A\) in an appropriate sense.

There is of course the question whether oligomorphic groups per say have a physical interpretation, for instance as the holographic dual to gravitational theories in \(AdS\) with strictly infinite radius, or at least as a tool to compute the leading order terms in a \(1/N\) expansion. In any case they should tell us interesting facts about what limits oligomorphic families can attain.

There are in fact several interesting theorems about the growth of \(f_K\) for oligomorphic groups \([33]\). It turns out that \(f_K\) grows either polynomially in \(K\), or faster than

\[
f_K > \exp(K^{1/2-t}) .
\]

It is not clear what this gap signifies physically. There are also examples of oligomorphic groups that have super-Hagedorn growth. The automorphism group of the random graph for instance has

\[
f_K \sim \exp(cK^2) .
\]

Again, it is not clear if this can be written as the limit of a oligomorphic family.

### 2.3 Examples of oligomorphic families

Let us now turn back to oligomorphic families. One way to construct such families is the wreath product \(A \wr B\) between two permutation groups \(A\) and \(B\). From a physicist’s point of view, orbifolding by \(A \wr B\) simply means we take the permutation orbifold \(B\) as the new seed theory, and then perform the permutation group \(A\). We can for instance define \(G_N = S_{\sqrt{N}} \wr S_{\sqrt{N}}\), which corresponds to an iterated symmetric orbifold of \(\sqrt{N}\). An alternate and more standard way of describing the action of \(S_{\sqrt{N}} \wr S_{\sqrt{N}}\) is the following: arrange the \(N\) factors into an \(\sqrt{N} \times \sqrt{N}\) matrix \(T_{ij}\). The \(i\)th symmetric group \(S_{\sqrt{N}}\) acts on the elements of the \(i\)-th row as \(T_{ij} \rightarrow T_{i\sigma(j)}\), and the overall \(S_{\sqrt{N}}\) permutes the rows. The same construction gives the group theoretic definition of a general wreath product \(A \wr B\).

The group \(S_{\sqrt{N}} \wr S_{\sqrt{N}}\) is oligomorphic, as can be seen from the following argument. Take a \(K\)-tuple, that is pick \(T\) with \(K\) non-vanishing entries. We can now use the various symmetric groups to move all non-vanishing entries to the first \(K\) columns, and then also to the first \(K\) rows. This shows that there can be at most \(K^2\) orbits. The same construction gives the group theoretic definition of a general wreath product \(A \wr B\).

Note that this group is much smaller than \(S_N\), since

\[
|S_{\sqrt{N}} \wr S_{\sqrt{N}}| = |S_{\sqrt{N}}|^{\sqrt{N}+1} \sim N^{(N+\sqrt{N})/2} .
\]
This suggests another construction. Arrange again the factors into a $\sqrt{N} \times \sqrt{N}$ matrix $T_{ij}$. However now act with just a single $S_{\sqrt{N}}$ on all the columns, and with another $S_{\sqrt{N}}$ on the rows, permuting the rows and columns as $T_{\sigma^1(i) \sigma^2(j)}$, so that both symmetric groups commute, giving a direct product $S_{\sqrt{N}} \times S_{\sqrt{N}}$. Note that even though we can write it as a direct product of two symmetric groups, the action on $N$ elements is very different from the standard action of those two groups. The same argument as before shows again that there are at most $\binom{K^2}{K}$ orbits of $K$-tuples, *i.e.* the group is again oligomorphic. This group is even smaller, having
\begin{equation}
|S_{\sqrt{N}} \times S_{\sqrt{N}}| \sim N^{\sqrt{N}}. \tag{23}
\end{equation}
We can generalize this even further by arranging the factors in a rank $d$ tensor $T_{i_1 i_2 \cdots i_d}$, and acting with the direct product $S_{N^{1/d}} \times \cdots \times S_{N^{1/d}}$ as $T_{\sigma^1(i_1) \sigma^2(i_2) \cdots \sigma^d(i_d)}$. The same argument as above shows that this group is again oligomorphic. The size of this group is
\begin{equation}
|S_{N^{1/d}} \times \cdots \times S_{N^{1/d}}| \sim N^{N^{1/d}}. \tag{24}
\end{equation}
This suggests that the ‘faster than polynomial growth’ criterion given in [20] is close to optimal.

### 2.4 Growth in the untwisted sector

#### 2.4.1 General considerations

Having established that $\rho_\infty(\Delta)$ exists and is finite, we now want to investigate its growth. In the cases we discuss, to leading order the result turns out to be universal, *i.e.* almost independent of the choice of the seed theory $\mathcal{C}$, with only its central charge $c$ entering sometimes. This may seem a bit surprising, so let us stress that this statement really only holds to leading order. More precisely, for $N$ large we investigate the regime
\begin{equation}
\rho_N(\Delta) \quad \text{for} \quad c \ll \Delta \ll cN. \tag{25}
\end{equation}
In the cases we consider, the main contribution to $\rho_N(\Delta)$ in this regime comes from states in the seed theory with $\Delta \gg c$, which are well in the Cardy regime. To leading order, their multiplicities are thus universally fixed by $c$.

In principle there are closed expressions for the partition function of any permutation orbifold [34]. Unfortunately in practice it is technically hard to extract the spectrum in the large $N$ limit. It is particularly difficult to get a handle on the twisted sectors. We will briefly return to this in section 2.5. For the moment let us concentrate on the untwisted sector, which at least gives a lower bound on the total number of states. To count these states, we will use the notation introduced in the previous section. Unfortunately this time we need to keep track of $N$-independent combinatorial factors, and in particular understand how $F_K$ and $f_K$ grow with $K$.

Let us first start with the simplest case. Let $\varphi_1$ be the lowest state of the seed theory of weight $\Delta_1$. The configuration $\vec{\varphi} = (\varphi_1, \ldots, \varphi_1)$ then of course gives states
\( \Phi \) of weight \( K \Delta_1 \). From these states alone the theory then has at least \( f_K \) states of weight \( K \Delta_1 \). In particular if \( f_K \) grows faster than exponential, we find that the theory has super-Hagedorn growth already from the untwisted sector.

Next say we get to choose the \( K \) states in \( \vec{\phi} \) out of a total of \( M \) states in the seed theory. Provided \( M \geq K \), there are \( M!/(M-K)! \) configurations \( \vec{\varphi} \) with distinct \( \varphi_i \), each of which contributes \( F_K \) orbits. To obtain the total number of states, we are however overcounting by a factor of \( K! \) since different permutations of the entries in \( \vec{\varphi} \) give the same states. The total number of states with different individual factors is thus

\[
F_K \binom{M}{K}.
\]  

If we want to keep track of states where some \( \varphi_i \) are the same, the combinatorics become more difficult. If \( M \gg K \) however, this almost never happens, so that we can neglect this effect. In that case (26) becomes

\[
F_K \frac{M^K}{K!}.
\]  

We stress again that this only applies if \( M \gg K \). This means that we cannot simply choose \( K \) as big as we want while keeping \( M \) fixed. If we want to pick a large \( K \), we need to include states with large \( \Delta \) to get a big enough \( M \).

### 2.4.2 \( S_N \) redux

As a warm up let us apply this to the symmetric orbifold. In the process we will rederive the growth behavior of the symmetric orbifold in the untwisted sector obtained in [20]. For the symmetric groups we have

\[
f_K = F_K = 1,
\]  

so each \( K \)-tuple has exactly one orbit. Let us first work out the contribution of the \( K \)-tuple states to states of weight \( \Delta \). For convenience set \( c = 3/(4\pi^2) \). We will also assume that we are always in the Cardy regime, an assumption whose consistency we will check in the end. The contribution from 1-tuples is then simply \( e^{\sqrt{\Delta}} \). The contribution from 2-tuples is

\[
\frac{1}{2!} \int d\delta e^{\sqrt{\Delta-\delta}} e^{\sqrt{\Delta-\delta}} \sim e^{\sqrt{2\Delta}}
\]  

where we have done a saddle point approximation, and the combinatorial prefactor takes care of the overcounting as described in [20]. Note that here we have assumed that almost all states are distinct, since otherwise the combinatorial factor would change. This is true, since the only case where states are the same are \( e^{\sqrt{\Delta/2}} \) states of weight \( \Delta/2 \), where the number of states is indeed much bigger than 2 if \( \Delta \) is large enough.
Similarly, a $K$-tuple contributes with
\[ \frac{1}{K!} e^{\sqrt{K \Delta}} \sim e^{\sqrt{K \Delta} - K \log K + K}. \] (30)

Again most states are distinct as long as there are many more states of weight $\Delta/K$ than $K$, i.e. as long as
\[ K \ll e^{\sqrt{\Delta/K}}. \] (31)

For a fixed $\Delta$, we can thus maximize (30) over $K$ to find where the maximal contribution to $\rho_\infty(\Delta)$ comes from. We find that it comes from tuples of length
\[ K \sim \frac{\Delta/4}{(\log \Delta/4)^2} \] (32)
and gives indeed the result obtained in [20]
\[ \exp \left( \frac{\Delta/4}{\log \Delta/4} \right). \] (33)

Note that from (32), $\Delta/K \to \infty$ for large $\Delta$, so that both (31) and using the Cardy formula are consistent.

2.4.3 $S_{\sqrt{N}} \wr S_{\sqrt{N}}$

Let us now turn to the wreath product $S_{\sqrt{N}} \wr S_{\sqrt{N}}$. Here we have $f_K = p_K$, the number of integer partitions of $K$. To see this, note that we can always use the permutations within the rows to move all non-trivial entries of the matrix all the way to the left, and then use the row permutation to order them in decreasing number, giving a Young diagram.

Of more interest is $F_K$. Note that this time the non-trivial entries are numbered. Using the same steps as above, $F_K$ is thus given by the number of different ways we can split $K$ distinct elements into different sets. Those are given by the Bell numbers $B_K$. Their asymptotic behavior is given by [35]
\[ B_K \sim \exp(K \log K - K \log \log K - K), \] (34)
i.e. they grow slightly slower than factorially. Plugging this into (26), and doing the saddle point approximation we obtain for the contribution of $K$-tuples
\[ e^{\sqrt{K \Delta} - K \log \log K}. \] (35)

The maximum contribution thus comes from states of length $K \sim \frac{\Delta/4}{(\log \log \Delta/4)^2}$ and gives a growth of the form
\[ \exp \left( \frac{\Delta/4}{\log \log \Delta/4} \right). \] (36)
The untwisted sector thus again has sub-Hagedorn growth. However, not surprisingly the growth is parametrically faster than for the symmetric orbifold.

### 2.4.4 $S_{\sqrt{N}} \times S_{\sqrt{N}}$

For the direct product $S_{\sqrt{N}} \times S_{\sqrt{N}}$, we have obtained the first few $f_K$ numerically:

$$f_K = 1, 3, 6, 16, 34, 90, 211 \ldots$$

(37)

We have plotted them in figure 1. They grow much faster than for the wreath product, and seems to fit an exponential quite well. Since we did not push our numerical computations very far, it is course possible that there are logarithmic corrections that make the growth slightly sub-exponential. If we assume that this is not the case, we can fit

$$f_K \sim e^{\alpha K} \quad \alpha = 0.88 \ldots$$

(38)

If the lightest non-vacuum state of the seed theory has weight $\Delta_1$, then the number of untwisted states of weight $\Delta$ grows at least as fast as

$$\rho_u(\Delta) > e^{\alpha \Delta/\Delta_1},$$

(39)

i.e. there is a Hagedorn transition already in the untwisted sector. The Hagedorn temperature here seems to depend on the seed theory, namely on the weight $\Delta_1$ of its lightest field. Note that this is only a lower bound for the growth of total number of states. It is possible (indeed probable, in our view) that the twisted states will show a super-Hagedorn behavior.

### 2.5 Growth in the twisted sector

Let us finally discuss twisted states. Estimating their growth is much more involved than for the untwisted states. Nonetheless, they are crucial for understanding the
growth behavior of all states, since in the examples we know, they rather than the untwisted states tend to give the dominant contribution. In particular for symmetric orbifolds they give a Hagedorn growth \[24\]
\[
\rho(\Delta) \sim e^{2\pi\Delta}.
\] (40)
Using the convention that
\[
Z(\beta) = \sum_{\Delta} \rho(\Delta)e^{-\beta(\Delta-c/24)},
\] (41)
the starting point for general permutation orbifolds is Bantay’s formula \[34\] \[35\]
\[
Z_{G_N}(\tau) = \frac{1}{|G_N|} \sum_{hg=gh} \prod_{\xi \in O(g,h)} Z(\tau_\xi).
\] (42)
The sum here is over all \(g, h \in G_N\) which commute. Such a commuting pair \(g, h\) generates an Abelian subgroup of \(S_N\), which or course has the natural permutation action on the integers \(1, 2, \ldots N\). In equation (42) we have denoted by \(O(g,h)\) the set of orbits of this action. For each orbit \(\xi \in O(g,h)\) we define the modified modulus \(\tau_\xi\) as follows. First, let \(\lambda_\xi\) be the size of the \(g\) orbit in \(\xi\), and \(\mu_\xi\) the number of \(g\) orbits in \(\xi\), so that \(\lambda_\xi \mu_\xi = |\xi|\). Let \(\kappa_\xi\) to be the smallest non-negative integer such that \(h^{\mu_\xi}g^{-\kappa_\xi}\) is in the stabilizer of \(\xi\). Then
\[
\tau_\xi = \frac{\mu_\xi \tau + \kappa_\xi}{\lambda_\xi}.
\] (43)
To write (42) in a maybe more familiar way, consider a fixed \(g\). This fixes a twisted sector, and the sum over \(h\) is then a sum over the centralizer \(C_g\) which projects onto the \(G_N\) invariant states in that twisted sector. Using \(|G_N| = |C_g||\{g\}|\), we can rewrite (42) as as sum over conjugacy classes
\[
Z_{G_N}(\tau) = \sum_{[g]} \frac{1}{|C_g|} \sum_{h \in C_g} \prod_{\xi \in O(g,h)} Z(\tau_\xi).
\] (44)
Let us consider a state given by a fixed \(g\) of finite length \(K\). The centralizer \(C_g\) of \(g\) in \(G_N\) is of course a subgroup of the centralizer of \(g\) in \(S_N\). We can therefore write
\[
C_g = C_g^{(1)} \times C_g^{(2)} \quad \text{with} \quad C_g^{(1)} \subset S_{N-K}, \quad C_g^{(2)} \subset (\mathbb{Z}_2^{N_2} \rtimes S_{N_2}) \times \cdots \times (\mathbb{Z}_2^{N_s} \rtimes S_{N_s}).
\] (45)
\footnote{To simplify notation, we write the partition function \(Z(\tau, \bar{\tau})\) simply as \(Z(\tau)\); despite this notation, the partition function is not assumed to be a holomorphic function of \(\tau\).}
Note that $C_g^{(2)}$ is independent of $N$. We thus have

$$
\frac{1}{|C_g|} \sum_{h \in C_g} \prod_{\xi \in O(g,h)} Z(\tau \xi) = \left( \frac{1}{|C_g^{(1)}|} \sum_{h_1 \in C_g^{(1)}} \prod_{\xi \in O(h_1)} Z(\xi|\tau) \right) \left( \frac{1}{|C_g^{(2)}|} \sum_{h_2 \in C_g^{(2)}} \prod_{\xi \in O(g,h_2)} Z(\tau \xi) \right).
$$

The first factor is simply the Polya enumeration formula, i.e., it computes the untwisted sector of a $C_g^{(1)}$ permutation orbifold on $N - K$ factors. If $G_N$ is oligomorphic, then $C_g^{(1)}$ is also oligomorphic. To see this, consider orbits under $G_N$ of $K + H$ tuples $(\tilde{H}, \tilde{K})$, where $\tilde{K}$ is the $K$-vector of all the factors in $g$. Since $G_N$ is oligomorphic, we know that there are at most $F_{K+H}$ such tuples which cannot be related by an element $a \in G_N$. Note that since $a$ leaves $\tilde{K}$ invariant, it is automatically of the form $C_g^{(1)} \times 1$.

But this shows that $G_g^{(1)}$ as a group acting on $N - K$ factors has at most $F_{H+K}$ orbits of $H$-tuples, which is independent of $N$, so that $C_g^{(1)}$ is indeed oligomorphic. From the arguments in the untwisted sector and the fact that the second factor in (46) is independent of $N$, it follows that in a given twisted sector the number of states remains finite. Since for a given weight $\Delta$, for an oligomorphic group $G_N$, there are only a finite number of twisted sectors that contribute, we have reestablished the original result that oligomorphic permutation orbifolds have indeed a finite number of states, even when including twisted states.

Expression (46) has a rather suggestive form. We will try to argue that the second factor grows at most as $e^{2\pi \Delta}$, that is in the same way as for the symmetric orbifold.

We could then write the total number of states schematically as

$$
\rho(\Delta) \sim F(\Delta)e^{2\pi \Delta},
$$

where $F(\Delta)$ is roughly the number of twisted sectors that can contribute states of weight $\Delta$. In particular (47) would imply that the growth behavior is mainly fixed by the number or conjugacy classes of $G_N$: if they grow more slowly than exponentially, then the phase diagram is the same as for the symmetric orbifold. If they grow more quickly, then that would change the phase diagram.

As a first step towards establishing (47), let us investigate the second factor in (42) more carefully. Fix a configuration $g$ with cycle lengths $L_i$. Let us concentrate for the moment on the term with the trivial centralizer element $h_2 = 1$. The contribution is then

$$
Z(\tau) = \prod_i Z(\frac{\tau}{L_i}).
$$

The number of states of weight $\Delta$ coming from states of weight $\Delta_i$ from the $i$th cycle, $\Delta = \sum_i \Delta_i$, is

$$
\rho_{G_N}(\Delta) = \prod_i \rho \left( L_i(\Delta_i - L_i \frac{c}{12}) + \frac{c}{12} \right),
$$

(49)
which of course vanishes unless for all $\Delta_i$

$$\Delta_i \geq c_{12}(L_i - \frac{1}{L_i}) \ .$$  \hspace{1cm} (50)

Assuming for the moment that we are in the Cardy regime for all the factors so that

$$\rho \sim \exp \left(2\pi \sqrt{c L_i (\Delta_i - L_i \frac{c}{12})/3}\right) \ ,$$  \hspace{1cm} (51)

we can evaluate the total contribution coming from all partitions $\Delta_i$ by saddle point approximation. The saddle point is given by

$$\Delta_i = L_i \left(\frac{c}{12} + \lambda\right) \ ,$$  \hspace{1cm} (52)

where the Lagrange multiplier is fixed by $\lambda = \Delta/L - c/12$ where $L = \sum_i L_i$, so that in total the contribution of this configuration is

$$\exp(2\pi \sqrt{c N/3(\Delta - c N/12)}) \ .$$  \hspace{1cm} (53)

This suggests that the maximum does not even depend on the specifics of the cycle decomposition, but only on its total length. It is maximized for $L = \frac{6\Delta}{c}$ giving indeed

$$e^{2\pi \Delta} \ .$$  \hspace{1cm} (54)

The issue is that we need to be more careful about the validity of applying the Cardy formula (51). Using (51) is valid only if

$$c \ll L_i \left(\Delta_i - L_i \frac{c}{12}\right) = L_i^2 \lambda = L_i^2 \frac{c}{12} \ ,$$  \hspace{1cm} (55)

\textit{i.e.} $L_i \gg 1$ for all $L_i$. This strengthens the result in [20]: To ensure (at least) Hagedorn growth at $\Delta$, it is sufficient to have an element in $G_N$ which has several cycles $L_i \gg 1$ such that $\sum_i L_i = L = 6\Delta/c$. The cycles $L_i$ themselves can be much shorter than $L$.

Note however that there are two major caveats here. First of all, the behavior for a $g$ which consists of many short cycles $L_i$ can be quite different, since in that case (55) may be violated, so that the Cardy formula may not apply. The main worry here is that this may lead to growth faster than (54). For example we can take $g$ to be given by $n$ cycles of length $L = 2$. The contribution of states of weight $\Delta_i = c/6$ to the state of total weight $\Delta = cn/6$ is then

$$\rho_{G_N}(\Delta) = \rho(c/12)^{\frac{6\Delta}{c}} \ .$$  \hspace{1cm} (56)

If we choose a seed theory with a large $\rho(c/12)$, then this seems to imply that we get indeed a faster growth than (54).

The reason that this probably does not happen is related to the second caveat.
We have so far only considered the term \( h = 1 \), that is, we have not projected to \( G_N \) invariant states. We expect that this projection will eliminate most of the states in (56), since for a \( g \) with so many short cycles, the centralizer group is very large. On the other hand for \( g \) consisting of only a few long cycles, \( C_g \) will be relatively small, and we expect (54) to hold to good approximation even after projecting to invariant states.

This makes it plausible that something like (47) could indeed be true. To prove it however clearly more work is needed.

### 2.6 Ramond and Neveu-Schwarz sectors

From our arguments it is clear that it is crucial that there is a vacuum in the theory, and that it is non-degenerate. For purely bosonic theories this is never an issue, since this is guaranteed by cluster decomposition. In theories with fermions we have to be somewhat more careful. For such theories this is still the case in the NS sector. In the Ramond sector however the situation is more complicated, since the ground state is no longer the vacuum, and therefore is no longer necessarily non-degenerate. In principle one can repeat the same analysis also in the Ramond sector, or possibly in some mixed NS-R sector. For \( N = 2 \) theories it is usually assumed that the results should be equivalent due to the spectral flow symmetry of the theory. This is however only the case if one keeps track of the \( U(1) \) charges. Once one specializes to different fugacities, there is no longer a guarantee that the results will agree. This is especially severe in the case at hand because spectral flow morally speaking shuffles states around by a distance \( c \), so that in the large \( c \) limit the process becomes even more drastic.

As an example for this phenomenon take for instance the symmetric orbifold of \( K3 \). In the NS-NS sector, the number of states remains of course perfectly finite, as \( S_N \) is oligomorphic. The free energy moreover is the contribution of the vacuum with at most finite corrections [24]. For the NS-R sector, the situation is different. The lowest lying state has degeneracy \( N \) coming from the \( N \) right-moving Ramond ground states. This then leads to a logarithmic correction to the free energy. For other permutation orbifolds, the difference may be even bigger. In [25] the number of ground states for the \( S_N \) orbifold of the \( K3 \) theory in the NS-R sector was found to be \( \sim e^{\sqrt{N}} \), which implies a correction of order \( O(\sqrt{N}) \) to the free energy, which is more than logarithmic corrections expected from supergravity. Note that all these states come from the untwisted sector. We on the other hand have found in (36) that the growth of untwisted states in the NS-NS sector is perfectly sub-Hagedorn, so that there is at most a finite \( O(1) \) correction to the vacuum contribution to the free energy. Although it is theoretically conceivable that the twisted states could change the behavior, the criterion given in [25] for the number of ground states is most likely not a necessary condition for having a Hawking-Page transition in the NS-NS sector.
3 Factorization for the Symmetric Orbifold

3.1 General setup

Let us now turn to correlation functions of permutation orbifolds. The computation of correlation functions is in general much harder than the counting of states done in the previous section. One technique is to go to the cover of the underlying correlations [36]. For symmetric orbifolds some correlation functions were indeed evaluated in [37, 38]. In the large $N$ limit, [39, 40, 41] argued that the cover method leads to a diagrammatic $1/N$ expansion. Luckily for us, we will not need such sophisticated methods. In fact we will argue that the leading contribution is always very simple for the permutation groups in question, and does not depend on the dynamics of the underlying seed theory.

Let us now discuss factorization of the correlation function in the large $N$ limit. Factorization means that any correlation function can be written as the sum of products of two point functions. To put it another way, any correlation function can be evaluated using Wick contractions, so that the theory is a generalized free field. From the gravity side we do indeed expect holographic CFTs to satisfy this property in the large $N$ limit — see e.g. [22]. Ultimately we want to understand what conditions this imposes on the permutation groups $G_N$. In this section, as a warm up we will prove that symmetric orbifolds indeed factorize in the large $N$ limit. This is of course expected, as famously they are dual to the $D1-D5$ system. In fact factorization for single cycle twist fields was already argued in [37].

As we have argued above, a general state $\Phi$ of length $K$ is given by fixing a $K$-tuple and summing over all its images under the action of $G_N$. To compute the correlation function of $n$ properly normalized fields, we thus evaluate a total of $|G_N|^n$ terms. A 2-point function has $|G_N|^2$ terms, and we will use it to fix the norm of $\Phi$ to 1. A 3-point function then has $|G_N|^3$ terms, so naively it seems like it should diverge as $N \to \infty$. It turns out however that a great many of those terms vanish, so that (at least in the cases we discuss below) the result remains finite.

In what follows it will be convenient to work with (unordered) $K$-sets $\mathcal{K}$ of distinct elements rather than with ordered $K$-tuples $\vec{K}$. As usual the two give the same result up to $N$-independent factors. It is useful to present such a set $\mathcal{K}$ pictorially as a row of black and white dots, e.g. represent

$$\mathcal{K} = \{1, 2, 5, 7, 8, 9\} \quad \text{as} \quad \underbrace{\bullet \circ \circ \circ \circ \bullet \circ \circ \circ \cdots \circ}_{N}. \tag{57}$$

In terms of the tensor product state, a white dot thus corresponds to a trivial factor, i.e. the corresponding factor is untwisted and in the vacuum. A black dots denotes a non-trivial factor, which means it is either untwisted, but not in the vacuum, or it is part of a twist cycle.

When computing a 3-point function, each term corresponds to three rows of the form (57) lined up below each other. Each of the $N$ columns then contains $i = 0, 1, 2$ or 3 non-trivial factors. Let us denote the number of columns with $i$ non-trivial factors by $n_i$. Note that if $n_1 > 0$, then this term directly vanishes: if the non-trivial factor is
untwisted, then it leads to a non-trivial 1-point function in the underlying seed theory, which vanishes. If it is part of a twisted cycle, then we know that the correlators vanish unless the twist sectors $g_i$ of the states involved satisfy

$$g_1 g_2 g_3 = 1 . \quad (58)$$

This is clearly impossible if only one state has a twist cycle in a given factor.

Let us now state the result for the symmetric orbifold: For properly normalized states $\Phi_i$, the total contribution of terms with total triple overlap $n_3$ to the sum in the 3-point function goes like

$$\sim O(N^{-n_3/2}) \quad (59)$$

for large $N$. This shows in particular that the only contributions that survive the large $N$ limit have $n_3 = 0$, so that all factors come as double overlaps, i.e. as 2-point functions. The theory thus indeed becomes free in the sense that all three point functions can be obtained by Wick contractions. In gauge theory language we can identify non-trivial cycles and non-trivial factors with single trace operators.

In the rest of this section we will show (59) for the symmetric orbifolds, and in section 4 we will discuss under what condition it also holds for other permutation orbifolds.

### 3.2 Untwisted correlation functions

As usual we will start with untwisted states. In our notation, a prestate $\phi$ is given by $\vec{K}$ and

$$\vec{\phi} = (\varphi_{k_1}, \ldots, \varphi_{k_I}, \ldots, \varphi_{k_I}) \quad (60)$$

with $\sum_{i=1}^I k_i = K$. The actual state is then obtained as a sum over $S_N$,

$$\Phi = (A_{\Phi})^{-1/2} \sum_{g \in S_N} \phi_{(g, \vec{K}, \vec{\phi})} . \quad (61)$$

We first need to fix the normalization $A_{\Phi}$ by computing the 2-point function

$$\langle \Phi | \Phi \rangle = (A_{\Phi})^{-1} \sum_{g_1, g_2 \in S_N} \langle \phi_{(g_1, \vec{K}, \vec{\phi})} | \phi_{(g_2, \vec{K}, \vec{\phi})} \rangle . \quad (62)$$

Clearly we can simply pull out the action of one of the $S_N$. We use it to fix the non-trivial factors of the first state to lie in the first $K$ factors, and obtain an overall factor of $|S_N| = N!$. Evaluating the terms coming from the permutations of the second $S_N$,
it is clear that they vanish unless they are of the form

\[ \phi_1 : \quad \begin{array}{cccccc}
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\end{array} \]

\[ \phi_2 : \quad \begin{array}{cccccc}
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\end{array} \]

If we take the \( \varphi_i \) to be orthonormal, we can evaluate the sum over \( S_N \) to fix the normalization constant as

\[ A_\Phi = N!(N-K)! \prod_i k_i! \sim N!(N-K)! . \]

(63)

In the last expression, \( N! \) comes from the order of the group \( S_N \), and \( (N-K)! \) is the order of the pointwise stabilizer of the first \( K \) elements. In what follows, we will often drop \( N \)-independent contributions if convenient, and use \( \sim \) to denote equality up to \( N \)-independent factors in the equations, just like we did in (63).

Now we turn to three point functions. Take three states \( \Phi_i \) of length \( K_1, K_2, K_3 \). Consider all terms in the sum with fixed \( n_3 \). Schematically, they look like

\[ \phi_1 : \quad \begin{array}{cccccc}
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\end{array} \]

\[ \phi_2 : \quad \begin{array}{cccccc}
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\end{array} \]

\[ \phi_3 : \quad \begin{array}{cccccc}
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\end{array} \]

For notational convenience we have used the overall \( S_N \) to make the non-trivial factors occupy only the first \( K_1 + K_2 - J \) columns; for the argument that follows this does not matter. Also note that for the same reasons as above \( n_1 = 0 \), so that this is the only non-vanishing type of contribution. \( J \) is fixed by

\[ K_3 = K_1 + K_2 - 2J + n_3 . \]

(64)

Let us now count the number of such terms in the sum over the three symmetric groups \( S_N^{(1)}, S_N^{(2)}, S_N^{(3)} \). As usual we fix the overall \( S_N \) and pull out a factor \( N! \) from \( S_N^{(1)} \). Next consider \( S_N^{(2)} \). Here we first distribute the \( K_2 - J \) factors of \( \phi_2 \) over \( N - K_1 \) slots for which there are no \( \phi_1 \) states, giving \((N-K_1)!/(N-K_1-K_2+J)\)! possibilities, and then distribute the trivial factors over the remaining slots, for which there are \((N-K_2)!\) possibilities. Finally, for \( \Phi_3 \), the positions of the non-trivial factors are fixed by the condition that they have to pair up with the remaining non-trivial factors of \( \Phi_{1,2} \), which leads to an \( N \)-independent combinatorial factor only. The trivial states
however can again be distributed in \((N - K_3)!\) different ways. In total there are thus
\[
\frac{N!(N - K_1)!(N - K_2)!(N - K_3)!}{(N - \frac{1}{2}(K_1 + K_2 + K_3 - n_3))!}
\]
terms. Combining this with the normalization factor \((A_1A_2A_3)^{-1/2}\), we get
\[
\sim \left(\frac{(N - K_1)!(N - K_2)!(N - K_3)!}{N!(\frac{1}{2}(K_1 + K_2 + K_3 - n_3))!}\right)^{1/2} \sim O(N^{-n_3/2}),
\]
which indeed establishes (65).

As a special case, take \(K_3 = 1\) with the only non-trivial state given by the energy-stress tensor \(T\). Naively our results imply that the three point function should vanish as \(N^{-1/2}\), which seems like a contradiction. Note however that in the above derivation we have chosen the states to be orthonormal, whereas the correct normalization for \(T\) is \(\langle T|T \rangle \sim c \sim N\). Using this normalization we need to multiply by \(N^{1/2}\) and so do indeed get that the 3pt function is finite,
\[
\langle OOT \rangle \sim O(1),
\]
which is consistent with the usual OPE of the energy-stress tensor.

### 3.3 Twisted sector

Let us now discuss twisted states. A twisted sector is given by a conjugacy class \([g]\) of \(S_N\). When computing correlation functions of states, we first need to average over all elements of the conjugacy class by picking a specific instance \(g \in [g]\) and then sum \(hgh^{-1}\) over \(h \in S_N/C_g\). We also need to project to an orbifold invariant state. This means we need to sum over the centralizer \(C_g\), i.e. all elements \(h\) which commute with \(g\). As we argued before, the combined action of these two on \(\vec{K}\) is just the standard action of the full group \(S_N\) on \(\vec{K}\). Since the argument for factorization only depended on counting the number of configurations, it is clear that the essentially the same argument will go through also for twisted states.

Let us start with the case where all non-trivial factors are twisted. For distinction we will denote these factors by \(\times\). The \(N\)-dependent part of the action of the centralizer \(C_g\) then simply factorizes through and cancels with the norm, so that we can simply take \(S_N\) to act in the standard way. The normalization of \(\Phi\) is then again given by
\[
A_\Phi \sim N!(N - K)!
\]
The three point function works out exactly like in the untwisted case: Consider all
terms with \( n_3 \) triple overlaps of twisted factors,

\[
\begin{align*}
\phi_1 & : \quad \underbrace{\times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \time
states. The important thing to note is that for two states of length $K_{1,2}$, all 3pt functions with $K_3 > K_1 + K_2$ vanish. This means that only states with a fixed, $N$ independent length run in the intermediate channels, even though of course for a given length there are still an infinite number of states which a priori give a non-vanishing contribution. The contributions for a given length, i.e. the infinite sum over all states of such length, can however be computed in a finite symmetric orbifold. This means that they are finite, and do not depend on $N$, which shows that all the $N$ dependence comes from the 3pt functions. It follows that the factorization arguments carry over to general $n$-point functions.

4 Factorization for general permutation groups

4.1 Factorization for oligomorphic groups

Let us now discuss factorization for general oligomorphic group. First we will rewrite much of the above in more group theoretic language. Let $G^K_N$ be the subgroup of $G_N$ stabilizing the set $\mathcal{K}$. Through the orbit-stabilizer theorem we can always relate this to the length of the orbit of $\mathcal{K}$, $O_N(\mathcal{K})$:

$$|G^K_N| = |G_N|/O_N(\mathcal{K}).$$  \hfill (73)

In particular note that $|G^K_N|$ is independent of which element of the orbit we choose. Using this new notation the normalization factor of a state $\Phi$ comes out to

$$A_\Phi \sim |G_N||G^K_N| = \frac{|G_N|^2}{O_N(\mathcal{K})}. \hfill (74)$$

The formula for the 3pt function can be obtained in a similar fashion following the procedure of $S_N$. Consider the again configuration

\[ \phi_1 : \begin{array}{c} K_1 \quad \vdots \quad K_2 - J \end{array} \quad \phi_2 : \begin{array}{c} J \quad \vdots \quad K_2 - J \end{array} \quad \phi_3 : \begin{array}{c} n_3 \quad \vdots \quad K_1 + K_2 - 2J \end{array} \]

where

$$J = \frac{1}{2}(K_1 + K_2 - K_3 + n_3). \hfill (75)$$

Note that this is only a very schematic picture of the situation: For general $G_N$ there is certainly no guarantee that we can move all the non-trivial factors all the way to the left. The position of the columns should therefore be understood up to permutation. We need to estimate the number of such terms. The sum over $G^1_N$ gives again $|G_N|$. 

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The sum over $G_N^2$ is more subtle. We will get $|G_N^K|$ for the vacuum states, but we must also sum over the different ways the non-trivial factors of $\phi_2$ can distribute themselves over the vacuum states of $\phi_1$. This number is given by the stabilizer of $K_1$ modulo the stabilizer of $K_1 \cup K_2 = K_1 \cup K_2 \cup K_3$,

$$|G_N^K|/|G_N^{K_1 \cup K_2 \cup K_3}|.$$  

The trivial factors of $\phi_3$ again give $|G_N^K|$. Including the normalization factors, the total contribution is thus

$$\left(\frac{O_N(K_1)O_N(K_2)O_N(K_3)}{|G_N|^{3/2}}\right)^{1/2} \frac{|G_N^K||G_N^{K_1}| |G_N^{K_2}| |G_N^{K_3}|}{|G_N^{K_1 \cup K_2 \cup K_3}|} = \frac{O_N(K_1 \cup K_2 \cup K_3)}{O_N(K_1)O_N(K_2)O_N(K_3))^{1/2}}.$$  

Let us now make an additional assumption. We will call $G_N$ democratic if for fixed $K$, all orbits have the same length up to $N$ independent factors. Up to factors the orbit length is thus only a function of $K$, so that we can write (77) as

$$\sim \frac{O_N(\frac{1}{2}(K_1 + K_2 + K_3 - n_3))}{(O_N(K_1)O_N(K_2)O_N(K_3))^{1/2}}.$$  

If $G_N$ is oligomorphic, we can estimate the average orbit length as

$$\langle O_N(K) \rangle = \binom{N}{K} f_K^{-1} \sim \binom{N}{K},$$  

where have used that $f_K$ is independent of $N$ for $N$ large enough. Democracy then implies that all orbits have average length,

$$O_N(K) \sim \langle O_N(K) \rangle \sim \binom{N}{K} \quad \forall |K| = K.$$  

Plugging this into (78) we recover the same result as for symmetric orbifolds,

$$\sim N^{\frac{1}{2}(K_1 + K_2 + K_3 - n_3) - \frac{1}{2}(K_2 + K_3 + K_1)} \sim N^{-n_3/2}.$$  

This shows that democratic oligomorphic groups factorize in the large $N$ limit.

4.2 Relaxing the assumptions

In the previous section we showed that democratic and oligomorphic are sufficient for factorization. We believe that those two conditions are too strong, and can be relaxed significantly. Let us discuss a few examples to support this belief.

First consider the cyclic orbifold. Untwisted states are given by $N$-tuples of states up to cyclic shifts. In this case we define the length $K$ as the number of non-trivial factors, which again has to remain finite for the state to have finite weight. Since $\mathbb{Z}_N$ is Abelian, the twisted sectors are simply given by elements of $\mathbb{Z}_N$, which is given by $N_n$.
cycles of length $n$ such that $nN_n = N$. The weight of the ground state of this sector is given by
\[ \Delta = \frac{c}{24} \sum_i (n - \frac{1}{n}) = \frac{c}{24} (N - N/n^2) \]  
(82)
which shows that only the untwisted sector with $n = 1$ has finite large $N$ limit. In what follows we can thus concentrate on the untwisted sector only.

In the untwisted sector, we have $O_N(K) = N$ for all $K > 0$, similarly for $O_N(K)$. The group is thus clearly democratic. Applying this to (78), we find that the correlator of three non-trivial fields goes like
\[ \sim N^{-1/2} . \]  
(83)
The only exception is if one of the states is the vacuum, in which case we are back at a two point function which of course goes as $\sim 1$. Cyclic theories thus also become free in the large $N$ limit, albeit in a somewhat trivial way. They do have an infinite number of states at fixed energy. This shows that there are non-oligomorphic orbifolds which factorize.

What happens if we relax the democratic assumption? It is then possible that different of orbits of $K$-tuples have vastly different lengths. An example of this is
\[ G_N = 1 \times S_{N-1} . \]  
(84)
The 1-tuples have two orbits, one of length 1, the other of length $N - 1$. Since this choice of $G_N$ gives a symmetric orbifold with an additional tensor factor, it is clear that the 3pt functions do not factorize, since the first factor does not.

Still, it seems very likely that also many non-democratic groups factorize. One guess would be that permutation groups factorize if they are transitive, i.e. if their natural action is transitive. This has the added advantage that they will have a unique energy-momentum tensor.

### 4.3 $S_{\sqrt{N}} \wr S_{\sqrt{N}}$

Let us now consider $S_{\sqrt{N}} \wr S_{\sqrt{N}}$. As we argued above, instances of orbits are given by partitions of $K$. For a given partition $P_K$, $\{k^l\}, l = 1 \ldots L$, $\sum_l k^l = K$, the stabilizer is given by
\[ |G_N^K| \sim |S_{\sqrt{N} - L}|S_{\sqrt{N}}|S_{\sqrt{N} - k^l}| \sim N^{\frac{1}{2}(\sqrt{N} - L + N - \sqrt{N}L + \sqrt{N}L - K)} \]  
(85)
so that the orbit length is given by
\[ O(K) \sim N^{\frac{1}{2}(K + L)} . \]  
(86)
This shows that the wreath product is not democratic, since the length of the orbit not only depends on $K$, but also on the number of columns $L$ of the partition. A quick
and dirty argument then shows that wreath product factorizes: For the configuration $K_t = K_1 \cup K_2 \cup K_3$ we have $K_t = \frac{1}{2}(K_1 + K_2 + K_3 - n_3)$ and $L_t = \frac{1}{2}(L_1 + L_2 + L_3 - n_3^2)$ where $n_3$ is the number of triple overlaps of non-trivial rows. From (57) we thus get in total

$$\sim O(N^{-\frac{1}{2}(n_3+n_3^2)}).$$  \hspace{1cm} (87)

Let us give a more careful argument for this using a detailed counting of the terms. Fix $K_1$ to be arranged as a partition of $K_1$ in the upper left corner, pulling out an overall factor of $|S_{\sqrt{N}} \wr S_{\sqrt{N}}|$. Next we take an image of $K_2$ which has $k_2^l$ non-trivial factors in the $l$-th row with $l = 1, \ldots, \sqrt{N}$. For $N$ large of course most rows will have $k_2^l = 0$. Consider configurations where there are $J$ non-trivial rows of $K_2$ in the first $L_1$ lines. There are $\sim (\sqrt{N-L_1})$ such configurations, each of which comes with an additional factor $|S_{\sqrt{N}}^{-L_1}|$ from distributing the trivial rows. Together this gives

$$\sim N^\frac{1}{2}(\sqrt{N}-J)$$  \hspace{1cm} (88)

ways of how the non-trivial rows of $K_2$ can be distributed. Next let us count for each row in how many ways the factors can be distributed. Take a configuration for which of the $k_2^l$ non-trivial factors in row $l$, $J'$ overlap with non-trivial factors of $K_1$. For the $l$-th row we then have $\sim (\sqrt{N-n_1^l})$ possible ways to distribute them, with an additional factor of $|S_{\sqrt{N}}^{-n_1^l}|$ from the trivial factor. Taking the product over all rows we get

$$\sim \prod_{l=1}^{\sqrt{N}} N^\frac{1}{2}(\sqrt{N}-J_l) = N^\frac{1}{2}(N-J_{tot}) \hspace{1cm} (89)$$

where $J_{tot}$ is the total number of $K_1$ and $K_2$ factors that overlap. Since 1pt functions vanish, the distribution of non-trivial $K_3$ factors is then again completely fixed up to $N$-independent factors, so that that we get an overall contribution of the stabilizer $|G_N^{K_3}|$. Putting this together with the normalization we get

$$\frac{(O_N(K_1)O_N(K_2)O_N(K_3))^{1/2}}{|G_N|^3}|G_N^{K_3}|N^\frac{1}{2}(N+\sqrt{N}-J_{tot})$$

$$= \frac{(O_N(K_1)O_N(K_2))^{1/2}}{O_N(K_3)^{1/2}|G_N|}N^\frac{1}{2}(N+\sqrt{N}-J_{tot})$$

$$= N^\frac{1}{2}(K_1+K_2+K_3)+L_1+L_2-L_3-2J = N^{-\frac{1}{2}(n_3+n_3^2)},$$

where the $n_3^2$ is the number of non-trivial row triple overlaps. This shows that the wreath product indeed factorizes.

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