Beurling-type invariant subspaces of the Poletsky–Stessin–Hardy spaces in the bidisc

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Abstract
The invariant subspaces of the Hardy space $H^2(\mathbb{D})$ of the unit disc are very well known; however, in several variables, the structure of the invariant subspaces of the classical Hardy spaces is not yet fully understood. In this study, we examine the structure of invariant subspaces of Poletsky–Stessin–Hardy spaces which are the generalization of the classical Hardy spaces to hyperconvex domains in $\mathbb{C}^n$. We showed that not all invariant subspaces of $H^2_{\theta}(\mathbb{D}^2)$ are of Beurling-type. To characterize the Beurling-type invariant subspaces of this space, we first generalized the Lax–Halmos Theorem to the vector-valued Poletsky–Stessin–Hardy spaces and then we gave a necessary and sufficient condition for the invariant subspaces of $H^2_{\theta}(\mathbb{D}^2)$ to be of Beurling-type.

Keywords Poletsky–Stessin–Hardy space · Beurling-type invariant subspace · Vector-valued Hardy spaces

Mathematics Subject Classification Primary 47A15 · Secondary 32C15

1 Introduction
In [3], Beurling described all invariant subspaces of the multiplication operator on the space $H^2(\mathbb{D})$ of the unit disk. In $H^2(\mathbb{D})$, all invariant subspaces are of Beurling-type, i.e. they are of the form $fH^2(\mathbb{D})$, where $f$ is an inner function of $H^2(\mathbb{D})$. However, for
the Hardy spaces of several variables the structure of the invariant subspaces cannot be characterized in such a simple form. Although it is quite clear that the Beurling-type subspaces are invariant, it is known that not all invariant subspaces are of this form. In [8], Jacewicz gave an example of an invariant subspace which can be generated by two functions but cannot be generated by a single function and Rudin [12] gave an example of an invariant subspace which cannot be generated by finite number of functions. There is a vast literature on the characterization of the Beurling-type invariant subspaces of $H^2(\mathbb{D}^2)$ and in this study, we are going to generalize one of these works given by Sadikov in [13]. The interested reader may check the brief survey of Yang [18] for the details of the invariant subspaces of $H^2(\mathbb{D}^2)$.

In 2008, Poletsky and Stessin introduced Poletsky–Stessin–Hardy spaces to extend the theory of Hardy spaces to hyperconvex domains in $\mathbb{C}^n$. The structure of these spaces is examined in detail in [1, 9, 14, 16]. Hence, it is natural to ask the invariant subspace problem in the case of Poletsky–Stessin–Hardy spaces. In the case of unit disk, Alan and Göğüş [1] showed that all invariant subspaces of the Poletsky–Stessin–Hardy space $H^2_u(\mathbb{D})$ are of Beurling-type. In this study, we are going to consider the multivariable case for the Poletsky–Stessin–Hardy space $H^2_u(\mathbb{D}^2)$ of the bidisc. First of all, using analogous methods to Jacewicz we will show that there exists an invariant subspace of $H^2_u(\mathbb{D}^2)$ which is not of Beurling-type. Then we are going to generalize the classical Lax–Halmos theorem to $H^2_u(\mathbb{D}^2)$ using the methods of vector-valued Hardy spaces. Lastly, we are going to characterize the Beurling-type invariant subspaces of $H^2_u(\mathbb{D}^2)$ by generalizing the ideas of Sadikov [13].

2 Preliminaries

In this section, we will give the structure of Poletsky–Stessin–Hardy spaces on the polydisc and the solution of the one dimensional version of the invariant subspace problem over these spaces. Before proceeding with Poletsky–Stessin–Hardy spaces, let us first recall the classical Hardy spaces of the polydisc given in [12]:

**Definition 2.1** Hardy spaces on the unit polydisc of $\mathbb{C}^n$ are defined for $1 \leq p \leq \infty$ as

$$H^p(\mathbb{T}^n) = \{f \in \mathcal{O}(\mathbb{T}^n) : \sup_{0 < r < 1} \left( \frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} |f(rz)|^p d\mu \right)^{1/p} < \infty \},$$

where $\mathbb{T}^n$ is torus and $\mu$ is the usual product measure on the torus. And

$$H^\infty(\mathbb{T}^n) = \{f \in \mathcal{O}(\mathbb{T}^n) : \sup_{z \in \mathbb{T}^n} |f(z)| < \infty \}.$$

Let $u : \Omega \to [\infty, 0)$ be a negative, continuous, plurisubharmonic exhaustion function for $\Omega$ and let

$$B(r) = \{z \in \Omega : u(z) < r\}, \quad r \in [\infty, 0),$$

and

\[\begin{align*}
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\end{align*}\]
S(r) = \{ z \in \Omega : u(z) = r \} , \ r \in [-\infty, 0),

with

\[ u_r(z) = \max\{ u(z), r \} , \ r \in (-\infty, 0). \]

In [5], Demailly introduced the Monge–Ampère measures supported over \( S(r) \) as

\[ \mu_{u,r} = (dd^c u_r)^n - \chi_{\Omega \setminus B(r)}(dd^c u)^n \ r \in (-\infty, 0), \]

where the differential operator \( dd^c \) is defined as \( dd^c = \frac{i}{2} \partial \bar{\partial} \).

Now let \( u : \Omega \to [-\infty, 0) \) be a continuous, plurisubharmonic exhaustion for \( \Omega \) and suppose that

\[ \limsup_{r \to 0} - \frac{1}{\varrho_\Omega} \mathcal{M} \mathcal{A}(u) \leq \infty. \]  

(2.1)

Then as \( r \) approaches to 0, \( \mu_{u,r} \) converges weakly to a positive measure \( \mu_u \) on \( \partial \Omega \). As a consequence of [14, Proposition 2.2.3], we know that the boundary Monge–Ampère measure \( d\mu_u \) is mutually absolutely continuous with respect to the Euclidean measure on the unit circle and we have,

\[ d\mu_u = \beta(\theta)d\theta \]  

(2.2)

for a positive \( L^1 \) function \( \beta \) which is defined as

\[ \beta(\theta) = \int_{\mathbb{D}} P(z, e^{i\theta}) dd^c u(z), \]

where \( P(z, e^{i\theta}) \) is the Poisson kernel of the unit disc \( \mathbb{D} \). Now we can introduce the Poletsky–Stessin–Hardy classes, which will be our main focus throughout this study. In [9], Poletsky and Stessin gave the definition of new Hardy type spaces using Monge–Ampère measures as

**Definition 2.2** \( H^p_u(\Omega) \) for \( p > 0 \), is the space of functions \( f \in O(\Omega) \) such that

\[ \limsup_{r \to 0^-} \int_{S_\varrho(r)} |f|^p d\mu_{u,r} < \infty. \]

The norm on these spaces is given by

\[ \|f\|_{H^p_u} = \left( \lim_{r \to 0^-} \int_{S_\varrho(r)} |f|^p d\mu_{u,r} \right)^{\frac{1}{p}}. \]

In Poletsky–Stessin–Hardy spaces of the unit disk, we have the inner-outer factorization similar to the classical Hardy spaces [15, Theorem 4.2], but throughout this study we will consider a special type of inner function which is defined in [1] as follows:
Definition 2.3 Let $u$ be a continuous, subharmonic exhaustion function for $\mathbb{D}$. A function $\phi \in H^2_u(\mathbb{D})$ is a $u$-inner function if $|\phi^*(\xi)|^2 \beta(\xi)$ equals to 1 for almost every $\xi \in \mathbb{T}$, where $\phi^*$ represents the boundary value function associated to the holomorphic function $\phi$ and $\beta$ is the function given in (2.2). (Throughout the rest of the text this notation will be used for the boundary value functions of the Poletsky–Stessin–Hardy classes and for the details of these boundary values, see [14].)

Remark 2.4 The set of $u$-inner functions is non-trivial. In fact, first of all we need to show that there is a holomorphic function $\tilde{u}$ such that $|\tilde{u}|^2 = \frac{1}{\sqrt{\beta}}$. Now since $\beta(\xi) = \int_{\mathbb{D}} P(z, \xi)dd^c u(z)$ it is a strictly positive function, $\beta(\xi) > c$ for some $c > 0$ so $\frac{1}{\sqrt{\beta}}$ is a bounded, positive function. Then by [12, Problem 3.5.1], we know that we have an analytic function $u \in H^\infty(\mathbb{D})$ and $|\phi^*| = \frac{1}{\sqrt{\beta}}$ a.e. on $\mathbb{T}$. Then $|\phi^*|^2 \beta = 1$ a.e. on $\mathbb{T}$ and $H^\infty(\mathbb{D}) \subset H^2_u(\mathbb{D})$ so $\phi \in H^2_u(\mathbb{D})$.

In the following sections our main focus will be on the Poletsky–Stessin–Hardy space, $H^2_u(\mathbb{D}^2)$ of the bidisc generated by the following special type of exhaustion function:

Let $u$ be an exhaustion function of the unit disc $\mathbb{D}$ with finite Monge–Ampère mass. Then the following plurisubharmonic function,

$$\tilde{u}(z, w) = \max\{u(z), u(w)\}$$

is an exhaustion for the unit bidisc $\mathbb{D}^2$. For this exhaustion function $\tilde{u}$, the corresponding boundary Monge–Ampère measure on the torus $\mathbb{T}^2$ is given as follows [14, Theorem 3.2.1]:

$$d\mu_{\tilde{u}}(\theta_1, \theta_2) = d\mu_u(\theta_1)d\mu_u(\theta_2) = \beta(\theta_1)\beta(\theta_2)d\theta_1d\theta_2.$$

Now for the sake of completeness let us define the Poletsky–Stessin–Hardy space $H^2_u(\mathbb{D}^2)$:

Definition 2.5 $H^2_u(\mathbb{D}^2)$ is the space of holomorphic functions $f \in \mathcal{O}(\mathbb{D}^2)$ such that

$$\limsup_{r \to 0^+} \int_{S_r(\mathbb{T})} |f|^2 d\mu_{\tilde{u},r} < \infty.$$ 

The norm on these Hardy–Hilbert spaces is given by

$$\|f\|_{H^2_u} = \left( \lim_{r \to 0^+} \int_{S_r(\mathbb{T})} |f|^2 d\mu_{\tilde{u},r} \right)^{\frac{1}{2}} = \left( \int_{\mathbb{T}} \int_{\mathbb{T}} |f^*(e^{i\theta_1}, e^{i\theta_2})|^2 d\mu_u(\theta_1)d\mu_u(\theta_2) \right)^{\frac{1}{2}}.$$
As before $f^*$ represents the boundary value of the holomorphic function $f$ and the discussion about the existence and various properties of this boundary value function can be found in [14].

One can also generalize the definition of a $u$-inner function to $\mathbb{D}^2$:

**Definition 2.6** Let $\bar{u}$ of $\mathbb{D}^2$ be the plurisubharmonic exhaustion function of $\mathbb{D}^2$. Then a holomorphic function $\phi \in H^2_{\bar{u}}(\mathbb{D}^2)$ is called $\bar{u}$-inner if

$$|\phi^*(\zeta, \eta)|^2 \beta(\zeta) \beta(\eta) = 1$$

a.e. on $\mathbb{T}^2$, where $d\mu_{\bar{u}}(\xi, \eta) = \beta(\xi)\beta(\eta)d\xi d\eta$.

**Definition 2.7** $M$ is called an invariant subspace of $H^2_{\bar{u}}(\mathbb{D}^2)$ if (a) $M$ is a closed linear subspace of $H^2_{\bar{u}}(\mathbb{D}^2)$ and (b) $f \in M$ implies $zf \in M$ and $wf \in M$, i.e., multiplication by polynomials maps $M$ into $M$. An invariant subspace $M$ is called Beurling type if it is of the form $M = \phi H^2(\mathbb{D}^2)$, where $\phi$ is $\bar{u}$-inner.

In one variable case, Alan and Göğüş [1, Theorem 3.2] extended the classical characterization of invariant subspaces to the Poletsky–Stessin–Hardy spaces as follows:

**Theorem 2.8** Let $M \neq \{0\}$ be an invariant subspace of $H^2_{\bar{u}}(\mathbb{D})$. Then there exists a $u$-inner function $\phi$ so that $M = \phi H^2(\mathbb{D})$.

### 3 Main results

In this section, using the method in [8], we first show that the Poletsky–Stessin–Hardy space on the bidisc has an invariant subspace which is not of the form $f H^2(\mathbb{D}^2)$ for any $f \in H^2_{\bar{u}}(\mathbb{D}^2)$ in contrast to one variable case. Before proceeding, we recall that $H^2(\mathbb{D}^2)$ can be seen as a closed subspace $H^2(\mathbb{T}^2)$ of the standard Lebesgue space $L^2(\mathbb{T}^2)$ which consists of the functions in $L^2(\mathbb{T}^2)$ with Fourier coefficients vanishing off a pair of nonnegative integers. To each function $f$ in $H^2(\mathbb{T}^2)$ with Fourier series $\sum_{m,n=0}^{\infty} a_{mn} e^{im\theta_1} e^{in\theta_2}$ we associate the function $\sum_{m,n=0}^{\infty} a_{mn} z^m w^n$ analytic on $\mathbb{D}^2$ which we also denote by $f$. For more details, see [12]. Note that since $H^2_{\bar{u}}(\mathbb{D}^2)$ is a subspace of $H^2(\mathbb{D}^2)$ by [14, p.54], every function in $H^2_{\bar{u}}(\mathbb{D}^2)$ also has the Fourier representation above.

**Theorem 3.1** There exists an invariant subspace $M$ of $H^2_{\bar{u}}(\mathbb{D}^2)$ which is of the form $M = f_1 H^2(\mathbb{D}^2) + f_2 H^2(\mathbb{D}^2)$ for some $f_1, f_2 \in H^2_{\bar{u}}(\mathbb{D}^2)$ but cannot be of the form $M = h H^2(\mathbb{D}^2)$ for any $h \in H^2_{\bar{u}}(\mathbb{D}^2)$.

**Proof** We choose $f_1(z, w) = \varphi(z) \varphi(w) q(z)$ and $f_2(z, w) = \varphi(z) \varphi(w) w$, where $\varphi$ is the non-vanishing $u$-inner function such that $H^2(\mathbb{D}) = \varphi H^2(\mathbb{D})$ and $q$ is a nonconstant,
singular inner function in $H^2(\mathbb{D})$, which means that $q$ never vanishes in $\mathbb{D}$ and has modulus one a.e. on $\mathbb{T}$. It is clear that $f_2 \in H^2_u(\mathbb{D}^2)$ and since

$$||f_1||^2_{H^2(\mathbb{D}^2)} = \int_T \int_T |\varphi^*(z)\varphi^*(w)q^*(z)|^2 d\mu_u(z)d\mu_u(w)$$

$$= \int_T \int_T |\varphi^*(z)|^2|\varphi^*(w)|^2|q^*(z)|^2 d\mu_u(z)d\mu_u(w)$$

$$= \int_T \int_T |q^*(z)|^2 dzdw$$

$$= \int_T dzdw < \infty,$$

$f_1 \in H^2_u(\mathbb{D}^2)$. Consider $M = f_1H^2(\mathbb{D}^2) + f_2H^2(\mathbb{D}^2)$. It is easily seen that $M$ is an invariant subspace of $H^2_u(\mathbb{D}^2)$.

Suppose that $M$ is of the form $M = hH^2(\mathbb{D}^2)$ for some $h \in H^2_u(\mathbb{D}^2)$. Let $H^2(S_1)$ denote the subspace of $L^2(\mathbb{T}^2)$ consisting of functions whose Fourier coefficients vanish off the half-plane $S_1 = \{(m,n) \in \mathbb{Z}^2 : m > 0\} \cup \{(0,n) \in \mathbb{Z}^2 : n \geq 0\}$. It is clear that $M_1 := hH^2(S_1)$ is the invariant subspace of $H^2_u(S_1)$. If $q$ has the form $q(z) = \sum_{m=0}^{\infty} a_m z^m$, then we see that

$$\varphi(z)\varphi(w)a_0 = \varphi(z)\varphi(w)\left(\sum_{m=0}^{\infty} a_m z^m - \sum_{m=1}^{\infty} a_m z^{m-1}w\right)$$

(3.1)

lies in $M_1$. Because $c_m = (m, -1) \in S_1$ for $m \geq 1$ and so $c_m f_2 = z^m w^{-1} f_2 \in M_1$. Now consider the subspace $M'_1 = M_1/(q(z)\varphi(w)) \subset H^2(S_1)$. Analogous to one-dimensional case, we have $H^2_u(S_1) = \varphi(z)\varphi(w)H^2(\mathbb{D}^2)$ [details about how one can obtain this representation can be found in the proof of Lemma (3.5) given further in this section]. Hence, we observe that $M'_1$ is an invariant subspace of $H^2(S_1)$ and by (3.1) we see that $M'_1$ contains all the constant functions. Therefore, $M'_1 = H^2(S_1)$ and this finally gives us that $hH^2(S_2) = M_1 = H^2_u(S_1)$. In particular, $hH^2(S_2) = H^2_u(S_2)$ for the half-plane $S_2 = \{(m,n) \in \mathbb{Z}^2 : n > 0\} \cup \{(m,0) \in \mathbb{Z}^2 : m \geq 0\}$.

Let $P$ be the orthogonal projection of $H^2_u(S_2)$ onto $H^2_u(\mathbb{D})$ (Notice that the Fourier coefficients of the element of $H^2_u(\mathbb{D})$ are zero for $m < 0$). The invariant subspaces of the form $f_1H^2(\mathbb{D}^2) + f_2H^2(\mathbb{D}^2)$ and $hH^2(\mathbb{D}^2)$ are the same. Since $S_2$ contains the set $\{(m,n) : m \geq 0, n \geq 0\}$, the invariant subspaces $f_1H^2(S_2) + f_2H^2(S_2)$ and $hH^2(S_2)$ are the same. These subspaces are denoted by $M_2(f_1, f_2)$ and $M_2(h)$, respectively. $P[M_2(f_1, f_2)]$ is the closed linear span of all $z^m \varphi(z)q(z)$, for $m \geq 0$, while $P[M_2(h)] = H^2_u(\mathbb{D})$. Thus, $\varphi(z)q(z)H^2(\mathbb{D}) = H^2_u(\mathbb{D})$. In view of the equality $H^2_u(\mathbb{D}) = \varphi(z)H^2(\mathbb{D})$, we have $qH^2(\mathbb{D}) = H^2(\mathbb{D})$, i.e., $q$ is outer in $H^2(\mathbb{D})$. This is contradiction and so $M$ cannot be of the form $M = hH^2(\mathbb{D}^2)$ for any $h \in H^2_u(\mathbb{D}^2)$. \hfill \square
As a consequence of this theorem, we have that not all invariant subspaces of $H^2_a(D^2)$ are Beurling-type. Then it is natural to ask the structure of Beurling type invariant subspaces of $H^2_a(D^2)$.

First of all, we need to recall the class of vector-valued analytic functions. Let $K$ be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|_K$. Then by $H^2(K)$ we mean the space of all $K$-valued holomorphic functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$ on $\mathbb{D}$ for which the quantity

$$\frac{1}{2\pi} \int_0^{2\pi} \| f(re^{i\theta}) \|_K^2 \, dr = \sum_{n=0}^{\infty} \| a_n \|_K^2 r^{2n}$$

remains bounded for $0 \leq r < 1$. Clearly, $H^2(K)$ is a Hilbert space under the inner product

$$\langle f, g \rangle_2 = \lim_{r \to 1} \frac{1}{2\pi} \int_0^{2\pi} \langle f(re^{i\theta}), g(re^{i\theta}) \rangle_K d\theta = \sum_{n=0}^{\infty} \langle a_n, b_n \rangle_K$$

for any $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ in the space. Now if $K$ is a reflexive Banach space then it has Fatou property, i.e. each $f \in H^1(K)$ has non-tangential limits on $\partial K$ ([2, p. 38]). Hence, we know that each $f \in H^2(K)$ has the radial limit $f^*$ as a Bochner measurable function and $f^* \in L^2_+(K)$, where $L^2_+(K)$ is the space of $L^2(K)$ functions whose negative Fourier coefficients are 0, and we also have $\| f \|_{H^2(K)} = \| f^* \|_{L^2_+(K)}$ (For details, see [17, p.183–186]).

On the other hand, if $B(K,K_1)$ denotes the algebra of all the bounded linear operators from $K$ to $K_1$, then by $H^\infty(B(K,K_1))$ we mean the algebra of bounded $B(K,K_1)$-valued holomorphic functions $\Theta$ on $\mathbb{D}$ in the norm $\| \Theta \|_\infty = \sup_{z \in \mathbb{D}} \| \Theta(z) \|_{B(K,K_1)} < \infty$. It is obvious that each $\Theta \in H^\infty(B(K,K_1))$ gives rise to a bounded linear operator from $H^2(K)$ into $H^2(K_1)$, namely to an element $\Theta$, we correspond an operator $\hat{\Theta} : H^2(K) \to H^2(K_1)$ that is defined by the formula

$$(\hat{\Theta}f)(z) = \Theta(z)f(z), \ z \in \mathbb{D}, \ f \in H^2(K). \quad (3.2)$$

An operator-valued $\Theta \in H^\infty(B(K,K_1))$ is called inner if $\Theta(e^{it})$ is an isometry from $K$ into $K_1$ for almost every $t$ or equivalently, the operator $\hat{\Theta}$ is an isometry.

The reader can find the details of vector-valued analytic functions in [10, 11, 17].

Analogously, we are going to define the vector-valued Poletsky–Stessin–Hardy spaces as follows:

**Definition 3.2** Let $K$ be a Hilbert space, $u$ be a continuous, subharmonic exhaustion function for $\mathbb{D}$. Then the vector-valued Poletsky–Stessin–Hardy space is defined as follows:

$$H^2_u(K) = \{ f : \mathbb{D} \to K, \text{holomorphic} : \sup_{r<0} \int_{S_r} \| f(z) \|_K^2 \, d\mu_{u,r}(z) < \infty \}.$$ 

Following step by step the same arguments from the scalar valued case one can easily see that $H^2_u(K) \subset H^2(K)$. Thus, we automatically inherit the radial boundary...
values from the classical Hardy space $H^2(K)$ and again just rewriting scalar value arguments we have the following boundary value characterization:

**Proposition 3.3** Let $f \in H^2_u(K)$ and $f^*$ be its radial boundary value function. Then

$\|f\|_{H^2_u(K)}^2 = \|f^*\|_{L^2_{rad}(K)}^2 = \int_T \|f^*(\xi)\|_{L^2_{rad}(K)}^2 d\mu_u(\xi)$.

**Proof** Directly follows from the scalar valued argument given in [14, Theorem 2.2.1].

Now, recall the Wold decomposition for isometries [17, p. 3, Theorem 1.1]: Let $V$ be an arbitrary isometry on a Hilbert space $H$. Then $H$ decomposes into an orthogonal sum $H = H_1 \oplus H_2$ such that $H_1$ and $H_2$ reduce $V$, the part of $V$ on $H_1$ is unitary and the part of $V$ on $H_2$ is a unilateral shift. This decomposition is uniquely determined, indeed we have

$$H_1 = \bigcap_{n=0}^{\infty} V^n H \quad \text{and} \quad H_2 = \bigoplus_{n=0}^{\infty} V^n E \quad \text{where} \quad E = H \ominus VH.$$  

The space $H_1$ or $H_2$ may be absent, i.e., equal to $\{0\}$.

If the Poletsky–Stessin–Hardy space over the bidisc is interpreted as the vector-valued analytic functions on the unit disc of complex plane, then invariant subspaces under the multiplication operator by the independent variable $z$ are described in terms of Lax-Halmos theorem.

**Theorem 3.4** A non-zero subspace $M$ of $H^2_u(H^2_u(\mathbb{D}))$ is invariant under the multiplication operator by the independent variable $z$ if and only if there exists a Hilbert space $E$ and an inner function $\Theta \in H^\infty(B(E, \varphi H^2_u(\mathbb{D})))$ such that $M = \hat{\Theta} H^2_u(E)$. This class of the functions $\Theta$ is denoted by $\{\Theta_M\}$.

First of all, by [1, Corollary 3.3] we know that there exists a $u$-inner function $\varphi \in H^2_u(\mathbb{D})$ so that $H^2_u(\mathbb{D}) = \varphi H^2(\mathbb{D})$. Now for the proof of the theorem we need the following results:

**Lemma 3.5** $H^2_u(H^2_u(\mathbb{D})) = \varphi H^2(H^2_u(\mathbb{D}))$ where $\varphi$ is the $u$-inner function which gives $H^2_u(\mathbb{D}) = \varphi H^2(\mathbb{D})$.

**Proof** Let $f \in \varphi H^2(H^2_u(\mathbb{D}))$. Then $f(z) = \varphi(z) h(z)$, where $h(z) \in H^2_u(\mathbb{D})$. Now

$$\int_T |f(z)|^2_{H^2_u(\mathbb{D})} d\mu_u(z) = \int_T |\varphi(z)|^2 |h(z)|^2_{H^2_u(\mathbb{D})} d\mu_u(z)$$

$$= \int_T |h(z)|^2_{H^2_u(\mathbb{D})} d\theta = |h|_{H^2_u(\mathbb{D})} < \infty.$$
Then $f \in H^2_u(H^2_u(\mathbb{D}))$ and $H^2(H^2(\mathbb{D})) \supseteq \varphi H^2(\mathbb{D})$. Conversely, let $f \in H^2_u(H^2_u(\mathbb{D}))$ and consider the function $h(z) = \frac{f(z)}{\varphi(z)}$. We want to show that $h(z)$ is in $H^2_u(H^2_u(\mathbb{D}))$.

First of all, since $f \in H^2_u(H^2_u(\mathbb{D}))$; for all $z \in \mathbb{D}$, $f(z) = f_z \in H^2_u(\mathbb{D})$ and $h(z) = \frac{f(z)}{\varphi(z)} = \frac{f_z}{\varphi}$.

Now since $f_z \in H^2_u(\mathbb{D})$, from the definition of $h(z)$, we have

$$
\int_T |h|^2 d\theta = \int_T |\varphi|^2 \int_D |f_z|^2 d\mu_u d\theta = \int_T \int_D |f_z|^2 d\mu_u d\mu_u = \int_T |f(z)|^2 d\mu_u \leq |f| H^2(\mathbb{D}) < \infty,
$$

where the second equality is obtained from the fact that $|\varphi|^2 = 1$ a.e. which gives $|\varphi|^2 d\mu_u = d\theta$. Hence we obtain $H^2_u(H^2_u(\mathbb{D})) \supseteq \varphi H^2(\mathbb{D})$.

**Remark 3.6** From this argument in fact one can deduce that for the $u$-inner function which satisfies $H^2_u(\mathbb{D}) = \varphi H^2(\mathbb{D})$, we also have $H^2(H^2(\mathbb{D})^2) = \varphi(z) \varphi(w) H^2(\mathbb{D})^2$.

**Lemma 3.7** [17, p. 195] Let $U_+$ and $U'_+$ be unilateral shifts on the (complex, separable) Hilbert spaces $R_+$ and $R'_+$, and let $U$ and $U'$ be the corresponding generating subspaces. Let $Q$ be a contraction of $R_+$ into $R'_+$ such that

$$
QU_+ = U'_+ Q.
$$

Then there exists a contractive analytic function $\{U, U', \Theta(\lambda)\}$ such that

$$
\Phi_+ U Q = \hat{\Theta} \Phi_+ U.
$$

That this function be inner is necessary and sufficient that $Q$ is an isometry from $R_+$ into $R'_+$. Here $\hat{\Theta}$ is the operator given in (3.2) and $\Phi_+ U$ is the Fourier representation of $R_+$ which is the unitary transformation from $R_+$ to $H^2(U)$ defined by

$$
\left[ \Phi_+ (\sum_{n=0}^\infty U^n a_n) \right] (\lambda) = \sum_{n=0}^\infty \lambda^n a_n \quad (a_n \in U, \ |\lambda| < 1).
$$

(For details, see [17, Sect. 3].)

**Proof of Theorem (3.4)** If $\Theta \in H^\infty(B(E, \varphi H^2_u(\mathbb{D})))$ is an inner function then the corresponding operator is isometric and hence $M = \hat{\Theta} H^2(E)$ is closed. Its invariance for the multiplication by $z$ is obvious.

Now let $M$ be an invariant subspace of $H^2_u(H^2_u(\mathbb{D}))$ under multiplication by $z$. Now first of all embedding $H^2_u(\mathbb{D})$ in $H^2(H^2_u(\mathbb{D}))$ as a subspace by identifying the element $\lambda \in H^2_u(\mathbb{D})$ with the constant function $\lambda(z) = \lambda$; $H^2_u(\mathbb{D})$ is then wandering for the multiplication operator by $z$ and
\[ H^2(H^2_u(\mathbb{D})) = \bigoplus_{n=0}^{\infty} \zeta^n H^2_u(\mathbb{D}) \]

and by Lemma (3.5), we have

\[ H^2_u(H^2_u(\mathbb{D})) = \phi H^2(H^2_u(\mathbb{D})) = \bigoplus_{n=0}^{\infty} \zeta^n (\phi H^2_u(\mathbb{D})). \]

Let \( V \) denote the restriction of the multiplication operator by \( z \) to the invariant subspace \( M \); this is an isometry on \( M \). We have

\[
\bigcap_{n=0}^{\infty} \mathcal{V}^n M \subset \bigcap_{n=0}^{\infty} \zeta^n H^2_u(H^2_u(\mathbb{D})) \subset \bigcap_{n=0}^{\infty} \zeta^n H^2_u(\mathbb{D}) = \{0\}
\]

and thus \( V \) has no unitary part so that the corresponding Wold decomposition is of the form \( M = \bigoplus_{n=0}^{\infty} \mathcal{V}^n E \), where \( E = M \ominus (VM) \). Let us now apply Lemma 2.7 to \( R_+ = M, \ U_+ = V, \ U = E, \ R'_+ = H^2_u(H^2_u(\mathbb{D})), \ U_+ = \) multiplication by \( z \), \( U' = \phi H^2_u(\mathbb{D}) \) and \( Q = \) the identical transformation of \( M \) into \( H^2_u(H^2_u(\mathbb{D})) \), then there exists an inner function \( \Theta \in H^\infty(\mathcal{B}(E, \phi H^2_u(\mathbb{D}))) \) such that

\[ \Phi^\phi H^2_u(\mathbb{D}) Q = \Theta \Phi^E \]

(3.3)

on \( M \). Since \( \phi H^2_u(\mathbb{D}) \) consists of the constant functions in \( H^2_u(H^2_u(\mathbb{D})) \), the Fourier representation of \( H^2_u(H^2_u(\mathbb{D})) \) with respect to multiplication by \( z \) is identity transformation. On the other hand, we have \( Qh = h \) for \( h \in M \). Thus, (3.3) reduces to \( h = \Theta \Phi^E h, \ h \in M \). We have \( \Phi^E M = H^2(E) \) since \( \Phi^E \) is the unitary operator given in Lemma 2.7 which takes \( R_+ \) to \( H^2(\mathcal{U}) \) and we have \( M = \Theta \Phi^E M = \Theta H^2(E) \) as claimed.

\[ \square \]

**Lemma 3.8** Vector-valued Poletsky–Stessin–Hardy space \( H^2_u(H^2_u(\mathbb{D})) \) is isometrically isomorphic to the Poletsky–Stessin–Hardy space \( H^2_u(\mathbb{D}^2) \) of bidisc.

**Proof** Let \( \tilde{u}(z, w) = \max\{u(z), u(w)\} \) be the exhaustion function for the bidisc \( \mathbb{D}^2 \) then we have the following isometric isomorphism between the Banach spaces \( H^2_u(H^2_u(\mathbb{D})) \) and \( H^2_u(\mathbb{D}^2) \):

Take \( g \in H^2_u(H^2_u(\mathbb{D})) \) then \( g(z) = g_z(w) \) for some \( g_z \in H^2_u(\mathbb{D}) \). Now consider the corresponding function \( \tilde{g} \) on \( \mathbb{D}^2 \) defined as \( \tilde{g}(z, w) = g_z(w) \) then using [14, Theorem 3.2.1] we have,

\[
\|\tilde{g}\|^2_{H^2_u(\mathbb{D}^2)} = \int_{\mathbb{T}^2} |\tilde{g}^*(\xi, \eta)|^2 d\mu_u(\xi, \eta) = \int_{\mathbb{T}} \int_{\mathbb{T}} |\tilde{g}^*(\xi, \eta)|^2 d\mu_u(\eta) d\mu_u(\xi) = \int_{\mathbb{T}} \|g_z\|^2_{H^2_u(\mathbb{D})} d\mu_u = \|g\|^2_{H^2_u(H^2_u(\mathbb{D}))}.
\]

\[ \square \]
Suppose that a subspace $M$ of $H^2_u(\mathbb{D}^2)$ which is invariant under the multiplication operators by independent variables $z$ and $w$ is of Beurling-type, i.e., $M$ is of the form $M = \phi H^2(\mathbb{D}^2)$ for some $u$-inner function $\phi$. Since $M$ is invariant under the multiplication by $z$, in view of Lemma (3.8) and Theorem (3.4), it can be described by the class of functions $\{\Theta_M\}$. However, the subspaces determined by this class of functions $\{\Theta_M\}$ are not generally of Beurling-type and the following theorem gives a condition for those subspaces which are defined by $\{\Theta_M\}$ to be of Beurling-type using the simple relation $H^2(H^2_u(\mathbb{D})) = H^2(N) \oplus H^2(N^\perp)$, where $N$ is a subspace of $H^2_u(\mathbb{D})$ and $N^\perp$ its orthogonal complement.

**Theorem 3.9** A subspace $M$ of $H^2_u(\mathbb{D}^2)$ is invariant under the multiplication operators by the independent variables $z$ and $w$ is Beurling-type if and only if there exists at least an operator valued holomorphic function $\Theta(z)$, $z \in \mathbb{D}$ in the class $\{\Theta_M\}$ such that for every $z_0 \in \mathbb{D}$ the operator $\Theta(z_0)$ on $H^2_u(\mathbb{D})$ commutes with the multiplication operator by $w$ in $H^2_u(\mathbb{D})$.

Before starting the proof of the main theorem, we need the following preliminary result [16, p. 34]:

Define

$$\tilde{a}(z) = \int_\mathbb{T} \frac{e^{i\theta} + z}{e^{i\theta} - z} \log(\beta(e^{i\theta}))d\theta$$

then the function

$$A(z) = e^{\tilde{a}(z)}$$

is a holomorphic function of the unit disc which extends smoothly to the unit circle with the property $|A^*(e^{i\theta})| = \beta(e^{i\theta})$.

**Theorem 3.10** [16] The space $H^p_u(\mathbb{D})$ is isometrically isomorphic to $H^p(\mathbb{D})$.

**Remark 3.11** The above-mentioned isomorphism is given as follows:

$$H^p_u(\mathbb{D}) \leftrightarrow H^p(\mathbb{D})$$

$$f \leftrightarrow A^1f.$$
and for $n \neq m$

$$\left\langle \frac{z^n}{\sqrt{A(z)}}, \frac{z^m}{\sqrt{A(z)}} \right\rangle = \int_{\mathbb{T}} \frac{e^{in\theta}e^{-im\theta}}{|A^*(e^{i\theta})|} d\mu_u = \int_{\mathbb{T}} e^{i(n-m)\theta} d\theta = 0$$

since $\{z^k\}_{k \geq 0}$ is an orthonormal basis for the classical Hardy space $H^2(\mathbb{D})$ which follows from the fact that $H^2(\mathbb{D})$ is the $L^2(\mathbb{T})$ closure of $\{e^{ik\theta}\}_{k \geq 0}$ by [6, Theorem 3.3]. As for completeness let $f \in H^2_u(\mathbb{D})$ such that

$$\int_{\mathbb{T}} f(e^{i\theta}) \frac{e^{-in\theta}}{\sqrt{A(z)}} d\mu_u(\theta) = 0$$

for all $n$. Then

$$\int_{\mathbb{T}} f(e^{i\theta}) \frac{e^{-in\theta}}{\sqrt{A(z)}} d\mu_u(\theta) = \int_{\mathbb{T}} f(e^{i\theta})A^*(e^{i\theta})e^{-in\theta} d\theta = 0.$$

By the previous theorem, we know that $f(z)A(z) \in H^2(\mathbb{D})$, so by [6, Theorem 3.4] we have $f(z)A(z) \equiv 0$ but by definition $A \neq 0$ hence we have $f \equiv 0$. Hence, the claim follows.

Now we will show the generalization of this result to bidisc case:

**Lemma 3.13** The set $\left\{ \frac{z^n w^m}{\sqrt{A(z)} \sqrt{A(w)}} \right\}_{n,m \geq 0}$ is an orthonormal basis of $H^2_u(\mathbb{D}^2)$.

**Proof** It is enough to show that $\left\{ \frac{z^n w^m}{\sqrt{A(z)} \sqrt{A(w)}} \right\}_{n,m \geq 0}$ is a complete orthonormal set in $H^2_u(\mathbb{D}^2)$ by [4, Theorem 4.13, p. 16]. First, we will show the orthonormality of this set:

$$\left\langle \frac{z^n w^m}{\sqrt{A(z)} \sqrt{A(w)}} \right\rangle_{H^2_u(\mathbb{D}^2)} = \left\langle \frac{z^n w^m}{\sqrt{A(z)} \sqrt{A(w)}} \right\rangle_{H^2_u(\mathbb{D}^2)} = \left\langle \frac{z^n w^m}{\sqrt{A(z)} \sqrt{A(w)}} \right\rangle_{H^2_u(\mathbb{D}^2)}$$

$$= \int_{\mathbb{T}^2} \frac{e^{i\theta_1}e^{i\theta_2}e^{-in\theta_1}e^{-im\theta_2}}{|A^*(e^{i\theta_1})| |A^*(e^{i\theta_2})|} d\mu_u$$

$$= \left( \int_{\mathbb{T}} \frac{d\mu_u(\theta_1)}{|A^*(e^{i\theta_1})|} \right) \left( \int_{\mathbb{T}} \frac{d\mu_u(\theta_2)}{|A^*(e^{i\theta_2})|} \right) = \int_{\mathbb{T}} \int_{\mathbb{T}} d\theta_1 d\theta_2 = 1$$

For any $(n_1, m_1), (n_2, m_2)$ such that $(n_1, m_1) \neq (n_2, m_2)$,
\[ \left( \frac{z_{1}^{\mu}w_{1}^{\mu}}{A(z)^2}, \frac{z_{2}^{\mu}w_{2}^{\mu}}{A(w)^2} \right) = \left( \int_{\mathbb{T}} e^{i(n_{1} - n_{2})\theta_{1}} d\mu_{u}(\theta_{1}) \right) \left( \int_{\mathbb{T}} e^{i(m_{1} - m_{2})\theta_{2}} d\mu_{u}(\theta_{2}) \right) = 0 \]

since \( \{z^{k}\}_{k \geq 0} \) is orthonormal in \( H^{2}(\mathbb{D}) \).

As for completeness let \( f(z, w) \in H^{2}_{u}(\mathbb{D}^{2}) \) be such that

\[ \int_{\mathbb{T}^{2}} f(e^{i\theta_{1}}, e^{i\theta_{2}}) e^{-i\theta_{1}} e^{-i\theta_{2}} d\mu_{u}(\theta_{1})d\mu_{u}(\theta_{2}) = 0 \]

for all \( n, m \). We claim that \( f \equiv 0 \). We have, by Fubini’s theorem,

\[ 0 = \int_{\mathbb{T}^{2}} f(e^{i\theta_{1}}, e^{i\theta_{2}}) e^{-i\theta_{1}} e^{-i\theta_{2}} d\mu_{u}(\theta_{1})d\mu_{u}(\theta_{2}) = \int_{\mathbb{T}} \left( \int_{\mathbb{T}} f(e^{i\theta_{1}}, e^{i\theta_{2}}) e^{-i\theta_{1}} d\mu_{u}(\theta_{1}) \right) e^{-i\theta_{2}} d\mu_{u}(\theta_{2}). \]

Now, using the fact that \( \{ \frac{w^{m}}{\sqrt{A(w)}} \} \) is an orthonormal basis for \( H^{2}_{u}(\mathbb{D}) \), we have that for all \( n \)

\[ \int_{\mathbb{T}} f(e^{i\theta_{1}}, e^{i\theta_{2}}) e^{-i\theta_{1}} \sqrt{\frac{\mu_{u}(\theta_{1})}{A^{*}(e^{i\theta_{1}})}} = 0 \quad \mu_{u} - a.e. \]

Let \( E_{n} \subset \mathbb{T} \) be the set of measure zero where the above equality does not hold and let \( E = \bigcup_{n} E_{n} \). Then for \( e^{i\theta_{2}} \not\in E \),

\[ \int_{\mathbb{T}} f(e^{i\theta_{1}}, e^{i\theta_{2}}) e^{-i\theta_{1}} \sqrt{\frac{\mu_{u}(\theta_{1})}{A^{*}(e^{i\theta_{1}})}} = 0 \]

for all \( n \), and thus again using the fact that \( \{ \frac{z^{n}}{\sqrt{A(z)}} \} \) is a complete orthonormal set in \( H^{2}_{u}(\mathbb{D}) \), we have that \( f(e^{i\theta_{1}}, e^{i\theta_{2}}) = 0 \) \( \mu_{u} \)-a.e. Therefore \( f = 0, \mu_{u} \)-a.e. which gives \( f \equiv 0 \). Hence, the claim follows. \( \square \)

**Lemma 3.14**  The set of all bounded linear operators on \( H^{2}_{u}(\mathbb{D}) \) commuting with the operators of multiplication by the independent variable \( z \) is the set of all multiplication operators by multipliers in \( H^{\infty}(\mathbb{D}) \).
The claim is clear since the commutant of the multiplication operator by independent variable on $H^2(\mathbb{D})$ is the set of all multiplication operators by multipliers in $H^\infty(\mathbb{D})$ by [7, Problem 116] and $H^2_\mu(\mathbb{D})$ is subspace of $H^2(\mathbb{D})$. □

**Proof of Theorem (3.9)** Suppose that there is a $\Theta$ in the class $\{\Theta_M\}$ such that for any fixed $z_0 \in \mathbb{D}$, $\Theta(z_0)$ commutes with the multiplication operator by $w$ in $H^2_\mu(\mathbb{D})$. Since, by Lemma (3.14), the commutant of the multiplication operator by $w$ in $H^2_\mu(\mathbb{D})$ is $H^\infty(\mathbb{D})$, it follows that $\Theta(z_0) \in H^\infty(\mathbb{D})$ for every $z_0 \in \mathbb{D}$. Let us note that the function $z \to \Theta(z)1$, where the function 1 in $H^2_\mu(\mathbb{D})$ is identically equal to 1, is an analytic function of $z$ taking values in $H^2_\mu(\mathbb{D})$. Hence, it follows that if $\phi = 1$, then $\Theta(z_0)1$ coincides with a function $\phi(z_0, w)$, and the family of functions $w \to \Theta(z_0)1$, $w \in \mathbb{D}$ is a family generated by an analytic function $\phi$. To obtain that $\phi$ is a $\tilde{u}$-inner function it is enough to show that the multiplication operator by $\phi$ in $H^2_\mu(\mathbb{D}^2)$ is an isometry. If $g \in H^2_\mu(\mathbb{D}^2)$, then for $g(z, w) = g_z(w)$, by [14], we have

$$||g||^2_{H^2_\mu(\mathbb{D}^2)} = \int_\mathbb{T} \int_\mathbb{T} |g^2(\xi, \eta)|^2 d\mu_\mu(\xi) d\mu_\mu(\eta) = \int_\mathbb{T} ||g_\xi||^2_{H^2_\mu(\mathbb{D})} d\mu_\mu(\eta).$$

Applying this to the function $\phi\varphi$, we obtain

$$||\phi\varphi||^2_{H^2_\mu(\mathbb{D}^2)} = \int_\mathbb{T} ||\phi_\eta\varphi_\eta||^2_{H^2_\mu(\mathbb{D})} d\mu_\mu(\eta)$$

and by assumption $\phi_\eta$ is an isometric operator for almost all $\eta$; therefore, $||\phi_\eta\varphi_\eta||_{H^2_\mu(\mathbb{D})} = ||\varphi_\eta||_{H^2_\mu(\mathbb{D})}$ for almost all $\eta$ and $||\phi\varphi||^2_{H^2_\mu(\mathbb{D}^2)} = ||\varphi||^2_{H^2_\mu(\mathbb{D})}$. Thus the operator $\Theta$ in $H^2(\mathbb{D}^2)$ and the multiplication operator by $\phi = \Theta1$ in $H^2_\mu(\mathbb{D}^2)$ are bounded operators which agree on vectors of the type $\xi^\mu w^l, k, l \geq 0$ under the canonical isomorphism between $H^2_\mu(\mathbb{D}^2)$ and $H^2_\mu(\mathbb{D})$. In Lemma (3.13) we have proved that the elements $\xi^\mu w^l, k, l \geq 0$ are dense in $H^2_\mu(\mathbb{D}^2)$. Hence $\phi = \Theta1$ and $\Theta$ correspond to each other.

For the converse direction now suppose that $M$ is a subspace generated by a $\tilde{u}$ -inner function $\phi$ then for almost any $\xi \in \mathbb{T}$, $\phi^2(\xi, \cdot)$ is a $\mu$-inner function in $H^\infty(\mathbb{D})$ and the radial boundary values of the operator valued function $\Theta(z)$, where $\Theta(z)$ is the operator of multiplication by the function $\phi$, is an isometry almost everywhere. Hence, the result follows. □

**References**

1. Alan, M.A., Göğüş, N.G.: Poletsky–Stessin Hardy spaces in the plane. Complex Anal. Oper. Theory 8(5), 975–990 (2014)
2. Aytuna, A.: Some results on HP-Spaces on strictly Pseudoconvex Domains. PhD Dissertation, University of Washington (1976)
3. Beurling, A.: On two problems concerning linear transformations in Hilbert space. Acta Math. 81, 17 (1948)
4. Conway, J.B.: A course in functional analysis, 2nd edn. Springer-Verlag, New York (1990)
5. Demailly, J.P.: Mesures de Monge–Ampère et Caractérisation Géométrique des Variétés Algébraiques Affines. Mémoire de la Société Mathématique de France 19, 1–124 (1985)
6. Duren, P.L.: Theory of HP spaces. Academic Press Inc., New York, London (1970)
7. Halmos, P.R.: A Hilbert Space Problem Book. Graduate texts in mathematics, 2nd edn. Springer-Verlag, New York, Berlin (1982)
8. Jacewicz, C.A.: A nonprincipal invariant subspace of the Hardy space on the torus. Proc. Am. Math. Soc. 31, 127–129 (1972)
9. Poletsky, E.A., Stessin, M.I.: Hardy and Bergman spaces on Hyperconvex domains and their composition operators. Indiana Univ. Math. J. 57, 2153–2201 (2008)
10. Radjavi, H., Rosenthal, P.: Invariant Subspaces. Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 77. Springer-Verlag, New York, Heidelberg (1973)
11. Rosenblum, M., Rovnyak, J.: Hardy classes and operator theory. Oxford Mathematical Monographs. Oxford Science PublicationsOxford Science PublicationsOxford Science PublicationsOxford Science Publications. The Clarendon Press, Oxford University Press, New York (1985)
12. Rudin, W.: Function Theory in Polydiscs, p. vii+188. W. A. Benjamin Inc, New York, Amsterdam (1969)
13. Sadikov, N.M.: Invariant subspaces in the Hardy space on a bidisk. Spectr. Theory Oper. Appl. 7, 186–200 (1986)
14. Şahin, S.: Monge–Ampère measures and Poletsky–Stessin Hardy spaces on bounded hyperconvex domains. PhD Dissertation, Sabancı University, (2014)
15. Şahin, S.: Poletsky–Stessin Hardy spaces on domains bounded by an analytic Jordan curve in C. Compl. Var. Elliptic Equ. 60(8), 1114–1132 (2015)
16. Shresta, K.: Poletsky–Stessin Hardy spaces on the unit disk. PhD Dissertation, Syracuse University, Dissertations-ALL. Paper 279 (2015)
17. Sz.-Nagy, B., Foias, C.: Harmonic Analysis of Operators on Hilbert Space. Akademiai Kiadó Budapest (1970)
18. Yang, R.: A Brief Survey of Operator Theory in $H^2(\mathbb{D}^2)$, Handbook of analytic operator theory, 223–258. Handb. Math. Ser, CRC Press/Chapman Hall, Boca Raton (2019)