A NOTE ON SINGULAR EQUIVALENCES AND IDEMPOTENTS

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Abstract. Let Λ be an Artin algebra and let e be an idempotent in Λ. We study certain functors which preserve the singularity categories. Suppose \( \text{pd } \Lambda e\Lambda e < \infty \) and \( \text{id }_{\Lambda \tan \Lambda (e)} \Lambda (e) < \infty \), we show that there is a singular equivalence between \( e\Lambda e \) and \( \Lambda \).

1. Introduction

Let \( R \) be a left noetherian ring. Let \( \text{mod } R \) be the category of finitely generated left \( R \)-modules and let \( \text{proj } R \) be the full subcategory of projective left \( R \)-modules. A complex in \( \text{mod } R \) is called \textit{perfect} if it is quasi-isomorphic to a bounded complex in \( \text{proj } R \). The \textit{singularity category} of \( R \) is the Verdier quotient of the bounded derived category of \( \text{mod } R \) by the thick subcategory of perfect complexes [2, 10]. If there is a triangle equivalence between the singularity categories of two rings \( R \) and \( S \), then such an equivalence is called a \textit{singular equivalence} [4].

Let \( e \) be an idempotent in \( R \). There is a recollement

\[
\text{mod } R / \langle e \rangle \quad \xrightarrow{i} \quad \text{mod } R \quad \xleftarrow{i} \quad \text{mod } eRe.
\]

Here, \( i \) takes any \( M \) in \( \text{mod } R \) to \( eM \) and \( i_\lambda \) takes any \( N \) in \( \text{mod } eRe \) to \( Re \otimes_{eRe} N \). Suppose

(1) \( \text{pd } eRe eRe < \infty \), and

(2) every \( M \in \text{mod } R \) annihilated by \( e \),

then \( i \) induces a singular equivalence between \( R \) and \( eRe \); see [3].

In the present paper, we investigate the singular equivalence induced by the functor \( i_\lambda \). We have the following

**Theorem I.** Let \( R \) be a left noetherian ring and \( e \) be an idempotent in \( R \). Suppose

(1) \( \text{pd } Re eRe < \infty \), and

(2) every \( M \in \text{mod } R \) admits a projective resolution \( P \) such that \( P^{-i} \in \text{add } Re \) for every sufficiently large \( i \),

then \( i_\lambda \) induces singular equivalence between \( eRe \) and \( R \).

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Let \( \Lambda \) be an Artin algebra and \( e \) be an idempotent in \( \Lambda \). The following conditions are studied in [11].

\[
\begin{align*}
(\alpha) \quad & \text{id}_{\Lambda} \frac{\Lambda/(e)}{\text{rad } \Lambda/(e)} < \infty \quad (\beta) \quad \text{pd}_{e\Lambda e} \Lambda < \infty \\
(\gamma) \quad & \text{pd} \frac{\Lambda/(e)}{\text{rad } \Lambda/(e)} < \infty \quad (\delta) \quad \text{pd} \Lambda e_{e\Lambda e} < \infty
\end{align*}
\]

It turns out that

(1) \((\alpha)\) and \((\beta)\) hold if and only if \((\gamma)\) and \((\delta)\) hold;
(2) \(i\) induces a singular equivalence if and only if \((\beta)\) and \((\gamma)\) hold.

As a complement, we have the following

**Theorem II.** Let \( \Lambda \) be an Artin algebra and \( e \) be an idempotent in \( \Lambda \). Then \( i_\lambda \) induces a singular equivalence if and only if \((\alpha)\) and \((\delta)\) hold.

## 2. Singularity categories

Let \( R \) be a left noetherian ring. Let \( K^{-,b}(\text{proj } R) \) be the homotopy category of bounded above complexes in \( \text{proj } R \) with bounded cohomologies. It is a triangulated category, whose translation functor \( \Sigma \) is the shift of complexes [12].

A complex \( P \) in \( K^{-,b}(\text{proj } R) \) is perfect if there is an integer \( \ell \) such that the \((-i)\)-th coboundary of \( P \) is projective for every \( i \geq \ell \). Let \( K^b(\text{proj } R) \) be the full subcategory of perfect complexes.

The singularity category of \( R \) is the quotient triangulated category

\[
D_{\text{sg}}(R) := K^{-,b}(\text{proj } R)/K^b(\text{proj } R).
\]

Let \( \text{mod } R \) be the projectively stable category of \( \text{mod } R \). For any \( M \) in \( \text{mod } R \), take a projective precover \( \pi : P \to M \); the syzygy \( \Omega M \) of \( M \) is the kernel of \( \pi \). There is a functor

\[
\Omega : \text{mod } R \to \text{mod } R
\]

sending a module \( M \) to the syzygy \( \Omega M \). For any \( M \) in \( \text{mod } R \), take a projective resolution \( pM \) of \( M \). There is a functor

\[
p : \text{mod } R \to D_{\text{sg}}(R)
\]

sending a module \( M \) to the projective resolution \( pM \). For any \( \ell \geq 0 \), there is a natural isomorphism

\[
p\Omega^\ell(M) \cong \Sigma^{-\ell}pM.
\]

**Lemma 2.1** ([4, Lemma 2.1]). For any \( X \in D_{\text{sg}}(R) \), there is \( n \geq 0 \) and \( M \in \text{mod } R \) such that

\[
\Sigma^{-n}X \cong pM.
\]

**Lemma 2.2** ([9, Exemple 2.3]). For any \( M, N \in \text{mod } R \), there is an isomorphism

\[
\lim_{\ell \geq 0} \text{Hom}_R(\Omega^\ell M, \Omega^\ell N) \cong \text{Hom}_{D_{\text{sg}}(R)}(pM, pN).
\]

Let \( e \) be an idempotent in \( R \). There is a functor

\[
i_\lambda : \text{mod } eRe \to \text{mod } R
\]
such that $i_\lambda(M) = Re \otimes_{eRe} M$ for every $M \in \mod eRe$. It is a full faithful functor which takes projectives to projectives. The functor $i_\lambda$ restricts to a full faithful functor $i'_\lambda$ such that

$$i'_\lambda : \proj eRe \xrightarrow{\cong} \add Re \xrightarrow{\subseteq} \proj R.$$ 

Here, we denote by $\add Re$ the full subcategory of summands of finite direct sums of copies of $Re$.

We now restate and prove Theorem I.

**Theorem 2.3.** Suppose $\pd Re < \infty$, then $i_\lambda$ induces a full faithful triangle functor

$$D_{\sg}(i_\lambda) : D_{\sg}(eRe) \to D_{\sg}(R).$$

Moreover, $D_{\sg}(i_\lambda)$ is a triangle equivalence if and only if for every $M \in \mod R$, there is an integer $\ell$ and a projective resolution $P$ of $M$ such that $P^{-i} \in \add Re$ for every $i \geq \ell$.

**Proof.** Let $X$ be in $K^{-b}(\proj eRe)$ which is exact at degree $\leq \ell$. Then $i_\lambda(X)$ is exact at degree $\leq \ell - \pd Re < \infty$. Therefore $i_\lambda$ induces a triangle functor

$$K^{-b}(i_\lambda) : K^{-b}(\proj eRe) \to K^{-b}(\proj R).$$

Since $i_\lambda$ is full faithful, $K^{-b}(i_\lambda)$ is also full faithful. Since $i_\lambda$ preserves perfect complexes, it induces a triangle functor $D_{\sg}(i_\lambda)$ such that

$$D_{\sg}(i_\lambda) : D_{\sg}(eRe) \xrightarrow{\cong} K^{-b}(\add Re)/K^b(\add Re) \to D_{\sg}(R).$$

Let $f : X \to Y$ be a morphism in $K^{-b}(\proj R)$, where $X^i$ is zero for every $i \leq \ell$ and $Y$ belongs to $K^{-b}(\add Re)$. Let $\sigma_{\geq \ell} Y$ be the stupid truncation of $Y$ at degree $\geq \ell$, then it belongs to $K^b(\add Re)$. Since $f$ factors through $\sigma_{\geq \ell} Y$, by [8, Proposition 10.2.6] the functor $D_{\sg}(i_\lambda)$ is full faithful.

By Lemma 2.1 and 2.2, the singularity category of $R$ is triangle equivalent to the stabilization of the stable category $\mod R$. Then the denseness of $D_{\sg}(i_\lambda)$ follows from [5, Corollary 2.13].

□

3. **Triangular matrix rings**

Let $T$ and $S$ be two rings and $M$ be an $S$-$T$-bimodule. We consider the triangular matrix ring

$$R = \begin{pmatrix} T & 0 \\ M & S \end{pmatrix}.$$ 

Following [1, III.2] a left $R$-module is given by

$$(X, Y, \phi) = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \mid x \in X, y \in Y \right\},$$

where $X$ is a left $T$-module, $Y$ is a left $S$-module and $\phi : M \otimes_T X \to Y$ is a left $S$-module map. The action is given by

$$\begin{pmatrix} t \\ m \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} tx \\ \phi(m \otimes x) + sy \end{pmatrix}$$

for every $t \in T$, $s \in S$ and $m \in M$.

Let $e = \diag(0, 1)$ be in $R$. Then $Re = eRe$ and

$$\add Re = \{(0, Y, 0) \mid Y \in \proj S\}.$$
Lemma 3.1. Let $X$ be a left $T$-module and $Y$ be a left $S$-module.

1. $\text{pd}_S Y = \text{pd}_R (0, 0, 0)$;
2. $(X, M \otimes_T X, 1)$ is a projective left $R$-module if and only if $X$ is a projective left $T$-module.

We have the following: compare [3, Theorem 4.1].

Proposition 3.2. Let $T$ and $S$ be left noetherian rings, and let $M$ be an $S$-$T$-bimodule such that $SM$ is finitely generated. Assume that $T$ has finite left global dimension, then there is a triangle equivalence
\[
D_{sg} \left( \begin{pmatrix} T & 0 \\ M & S \end{pmatrix} \right) \simeq D_{sg}(S).
\]

Proof. One checks that the triangular matrix ring $R$ is left noetherian. Let $(X, Y, \phi)$ be in $\text{mod} R$. Since $\text{pd}_T X$ is finite, by Lemma 3.1 there is a projective resolution $P$ of $(X, Y, \phi)$ such that $P^{i-1} \in \text{add Re}$ for every $i > \text{pd}_T X$. Then by Theorem II there is a triangle equivalence between $D_{sg}(R)$ and $D_{sg}(S)$. □

Example 3.3 (see [4], Proposition 4.1). Let $k$ be a field, $\Lambda$ be a finite dimensional $k$-algebra and $M$ be a finite dimensional left $\Lambda$-module. Then $M$ is a $\Lambda$-$k$-bimodule. The one-point extension of $\Lambda$ by $M$ is the triangular matrix algebra
\[
\Lambda[M] = \begin{pmatrix} k & 0 \\ M & \Lambda \end{pmatrix}.
\]

By Theorem II there is a singular equivalence between $\Lambda$ and $\Lambda[M]$.

4. Artin algebras

Let $\Lambda$ be an Artin algebra. We need the following well known

Lemma 4.1. Let $M$ be in $\text{mod} \Lambda$ and $P_M$ be a minimal projective resolution of $M$. For any semi-simple $\Lambda$-module $S$ and any $i \geq 0$ there is an isomorphism
\[
\text{Ext}^i_\Lambda (M, S) \cong \text{Hom}_\Lambda (P_M^{-i}, S).
\]

Proof. For $i = 0$, it is obvious. For $i = 1$, let $K$ be the $(-1)$-th coboundary of $P_M$. There is an exact sequence
\[
K \to P_M^0 \xrightarrow{\pi} M.
\]
Applying $\text{Hom}_\Lambda (-, S)$ to it, we obtain an exact sequence
\[
\text{Hom}_\Lambda (M, S) \xrightarrow{\pi^*} \text{Hom}_\Lambda (P_M^0, S) \to \text{Hom}_\Lambda (K, S) \to \text{Ext}^1_\Lambda (M, S).
\]
Since $\pi$ is a projective cover and $S$ is semi-simple, $\pi^*$ is surjective. Then
\[
\text{Hom}_\Lambda (K, S) \cong \text{Ext}^1_\Lambda (M, S).
\]
Since $P_M^{-1}$ is a projective cover of $K$, there is an isomorphism
\[
\text{Ext}^1_\Lambda (M, S) \cong \text{Hom}_\Lambda (P_M^{-1}, S).
\]
By shifting one proves the isomorphism for $i \geq 2$. □
For any \( P \in \text{proj} \Lambda \), we have the following
\[
(4.1) \quad P \in \text{add} \Lambda e \iff \text{Hom}_\Lambda(P, \frac{\Lambda/e}{\text{rad} \Lambda/e}) = 0.
\]

We now restate and prove Theorem II.

**Theorem 4.2.** Let \( \Lambda \) be an Artin algebra and \( e \) be an idempotent in \( \Lambda \). Suppose \( \text{pd} \Lambda e_{\text{rad} \Lambda} < \infty \), then \( \text{D}_{\text{sg}}(i_\lambda) \) is a triangle equivalence if and only if \( \text{id} \frac{\Lambda/e}{\text{rad} \Lambda/e} < \infty \).

**Proof.** “\( \Rightarrow \)” Let \( P \) be a minimal projective resolution of \( \Lambda/\text{rad} \Lambda \). If \( \text{D}_{\text{sg}}(i_\lambda) \) is dense, by Theorem I there is \( \ell \geq 0 \) such that \( P_i \in \text{add} \Lambda e \) for every \( i > \ell \). By Lemma 4.1 and (4.1) we have
\[
\text{Ext}_\Lambda^i(\frac{\Lambda/e}{\text{rad} \Lambda/e}) = 0.
\]
Then \( \text{id} \frac{\Lambda/e}{\text{rad} \Lambda/e} \) is finite.

“\( \Leftarrow \)” Let \( M \) be in \( \text{mod} \Lambda \) and \( P_M \) be a minimal projective resolution of \( M \). If \( \text{id} \frac{\Lambda/e}{\text{rad} \Lambda/e} \) is finite, by Lemma 4.1 we have \( P_i \in \text{add} \Lambda e, \forall i > \text{id} \frac{\Lambda/e}{\text{rad} \Lambda/e} \).

From Theorem I we infer that \( \text{D}_{\text{sg}}(i_\lambda) \) is a triangle equivalence. \( \Box \)

We end this section by an example.

Let \( \Lambda \) be an Artin algebra. Let \( S \) be a semi-simple left \( \Lambda \)-module with \( \text{id} \Lambda S \leq 1 \). Denote by
\[
\perp S = \{ M \in \text{mod} \Lambda \mid \text{Hom}_\Lambda(M, S) = \text{Ext}_\Lambda^1(M, S) = 0 \}
\]
the perpendicular category of \( S \) in \( \text{mod} \Lambda \). By Lemma 4.1 a finitely generated left \( \Lambda \)-module \( M \) belongs to \( \perp S \) if and only if \( M \) admits a projective presentation
\[
P^{-1} \to P^0 \to M
\]
such that both \( P^{-1} \) and \( P^0 \) belong to \( \perp S \).

Let \( e \) be an idempotent in \( \Lambda \) such that \( \text{proj} \Lambda \cap \perp S \) coincides with \( \text{add} \Lambda e \). Recall the functor \( i_\lambda \) decomposes into
\[
i_\lambda \colon \text{mod} \Lambda e \cong \perp S \subseteq \text{mod} \Lambda.
\]

We have the following; compare [6 Proposition 2.13].

**Proposition 4.3.** Keep the notation as previous.
(1) \( \text{gl.dim } \Lambda e \leq \text{gl.dim } \Lambda \leq \text{gl.dim } \Lambda e + 2 \);
(2) there is a singular equivalence between \( \Lambda e \) and \( \Lambda \).

**Proof.** (1) Let \( \Omega^2 M \) be the minimal second syzygy for \( M \) in \( \text{mod} \Lambda \). Since \( \text{id} \Lambda S \leq 1 \), \( \text{Ext}_\Lambda^i(M, S) = 0 \) for every \( i \geq 2 \). By Lemma 4.1 \( \Omega^2 M \in \perp S \).

Since \( i_\lambda \) preserves projective resolutions, for any \( N \) in \( \text{mod} \Lambda e \) we have
\[
\text{pd}_{\Lambda e} N = \text{pd}_\Lambda i_\lambda(N).
\]
Then
\[
\text{pd}_\Lambda M \leq \text{pd}_\Lambda \Omega^2 M + 2 = \text{pd}_{\Lambda e} i_\lambda(\Omega^2 M) + 2.
\]

(2) From [7 Proposition 1.1] we infer that \( \perp S \) is an exact subcategory of \( \text{mod} \Lambda \). Then the right \( \Lambda e \)-module \( \Lambda e \) is projective. By Theorem I there is a singular equivalence between \( \Lambda e \) and \( \Lambda \). \( \Box \)
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