In this paper, we initiate the study of higher-dimensional Auslander–Reiten theory of self-injective algebras. We give a systematic construction of (weakly) $d$-representation-finite self-injective algebras as orbit algebras of the repetitive categories of algebras of finite global dimension satisfying a certain finiteness condition for the Serre functor. The condition holds, in particular, for all fractionally Calabi-Yau algebras of global dimension at most $d$. This generalizes Riedtmann’s classical construction of representation-finite self-injective algebras. Our method is based on an adaptation of Gabriel’s covering theory for $k$-linear categories to the setting of higher-dimensional Auslander–Reiten theory.

Applications include $n$-fold trivial extensions and (classical and higher) preprojective algebras, which are shown to be $d$-representation-finite in many cases. We also get a complete classification of all $d$-representation-finite self-injective Nakayama algebras for arbitrary $d$. 

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1. Introduction

In representation theory of finite-dimensional algebras, one of the fundamental problems is to classify representation-finite algebras, that is, algebras having only finitely many indecomposable modules up to isomorphism. A famous result by Gabriel [24] characterizes representation-finite
hereditary algebras over algebraically closed fields as path algebras of Dynkin quivers. Another well-known result is the classification of representation-finite self-injective algebras, due chiefly to Riedtmann [18, 60, 61] (see also [36, 66, 67]). Her result, in characteristic different from 2, can be summarized as follows (see, e.g., [62, Theorem 3.5], or [63, Theorem 8.1]):

**Theorem 1.1.**

(a) Let $\Lambda$ be a tilted algebra of Dynkin type and $\phi$ an admissible automorphism of the repetitive category $\widehat{\Lambda}$ of $\Lambda$. Then $\widehat{\Lambda}/\phi$ is a representation-finite self-injective algebra.

(b) Every representation-finite self-injective algebra over an algebraically closed field of characteristic different from 2 is obtained in this way.

An autoequivalence $\phi$ of a Krull–Schmidt category $C$ is said to be admissible if $\phi^i$ acts freely on isomorphism classes of indecomposable objects in $C$ for all $i > 0$. For $C = \widehat{\Lambda}$, this is equivalent to $\widehat{\Lambda}/\phi$ being a finite-dimensional algebra (see Lemma 3.4).

After Gabriel’s and Riedtmann’s seminal results, the theory of representation-finite algebras has had two culmination points. One is the paper [15] by Bautista–Gabriel–Roiter–Salmeron showing that it is a finite problem, concerning the combinatorics of ray categories, to determine whether or not a given algebra over an algebraically closed field is representation finite (see also [26, Chapters 13, 14]). The other one is the characterization of Auslander–Reiten quivers of representation-finite algebras by Igusa–Todorov [37] and Brenner [17] (see also [38]). In view of Auslander’s bijection [10] between representation-finite algebras and Auslander algebras (that is, algebras of global dimension at most two and dominant dimension at least two), the latter result can be seen as a structure theorem for Auslander algebras.

Seeking to generalize Auslander’s result, one is naturally led to consider $d$-Auslander–algebras, i.e., algebras of global dimension at most $d + 1$ and dominant dimension at least $d + 1$. Such algebras correspond bijectively to equivalence classes of so-called $d$-cluster-tilting modules [40] (see Definition 2.1 below). The question thus occurs if there are generalizations of the fundamental results about representation-finite algebras to algebras possessing a $d$-cluster-tilting module, that is, to $d$-representation-finite algebras. Significant progress in this and related directions has been made in recent years; see, for example, [5, 6, 22, 27, 28, 29, 31, 34, 35, 42, 43, 44, 46, 47, 48, 49, 55, 57, 58]. Amongst all $d$-Auslander–algebras, the ones corresponding to $d$-cluster-tilting modules of algebras that are self-injective are characterized by the property that the class of projective-injective modules is closed under the Nakayama functor. The present paper provides new tools to study this class of algebras, through the corresponding self-injective algebras and their $d$-cluster-tilting modules.

Our aim is to generalize Riedtmann’s construction from the point of view of higher-dimensional Auslander–Reiten theory. The fundamental idea is to construct $d$-representation-finite self-injective algebras as orbit algebras of the repetitive categories of certain algebras $\Lambda$ of finite global dimension, called $\nu_d$-finite algebras. This class of algebras includes $d$-representation-finite algebras of global dimension $d$ (which are a higher-dimensional analogue of representation-finite hereditary algebras) and, more generally, (twisted) fractionally Calabi–Yau algebras. The bounded derived category $\mathcal{D}^b(\Lambda)$ of a $\nu_d$-finite algebra $\Lambda$ has a $d$-cluster-tilting subcategory $\mathcal{U}$ of a certain nice form – which we call an orbital $d$-cluster-tilting subcategory.

Taking the orbit algebra of $\widehat{\Lambda}$ with respect to an automorphism $\phi$ that preserves $\mathcal{U}$ gives a $d$-representation-finite self-injective algebra $\widehat{\Lambda}/\phi$ (see Theorem 2.6). Large classes of self-injective algebras, including many instances of $n$-fold trivial extension algebras (Corollaries 2.9, 2.8, 2.11) and higher preprojective algebras (Corollary 2.12), can be shown to be $d$-representation-finite using this construction. We apply these results to give examples amongst canonical algebras of tubular type and tensor products of Dynkin quivers. Algebras of tame as well as wild representation type occur in these constructions. It is known by [23] that any module of a $d$-representation-finite self-injective algebra has complexity at most one, i.e., the dimensions of the terms in a minimal projective resolution of such a module are bounded from above by a constant. The results in this paper give an explicit construction of self-injective algebras with this rather particular property.

In the formulation of Riedtmann’s classification in terms of orbit algebras, Gabriel’s covering theory [25] plays an important role. Let $k$ be a field. A central idea in this theory is to extend the
notation of representation-finiteness from finite-dimensional algebras to \( k \)-linear categories, including repetitive categories, through the concept of local representation-finiteness. Crucially, Gabriel proved that local representation-finiteness is preserved under taking orbit categories. To develop an analogue of this theory, we introduce the notion of local \( d \)-representation-finiteness (see Definition 2.2), which is a categorical correspondent to \( d \)-representation-finiteness in algebras. One of our main results, Corollary 2.15, is that local \( d \)-representation-finiteness is preserved under taking orbit categories satisfying a natural invariance property. This, together with the observation that orbit categories of higher Nakayama algebras by Jasso and Külsheimer \[48\].

Throughout this article, \( d \) is a positive integer, \( k \) a field, and \( \Lambda \) a finite-dimensional \( k \)-algebra of finite global dimension. While most of the theory is valid for general fields, some results require additional assumptions. This will then be stated at each such instance. In order not to make the introductory parts of this paper overly technical, the definitions of some of the concepts used in the results quoted in the previous paragraph. Our strategy can be summarized as follows.

![Diagram](image)

The notion of cluster tilting gives a strong link between higher-dimensional Auslander–Reiten theory on one hand, and categorification of Fomin and Zelevinsky’s cluster algebras on the other. In our context, this connection comes into play in Section 4.3 through the use of quivers with potential, and in Section 6, where we use cluster categories to infer the existence of non-orbital 2-cluster-tilting subcategories in the derived categories of certain algebras of global dimension 2.

We remark that some of the results in the present paper have been applied in the construction of higher Nakayama algebras by Jasso and Külsheimer [48].

This paper is organized as follows. In Section 2.1–2.3, main results are stated. Section 2.1 contains our generalization of Riedtmann’s construction, which in Section 2.2 is applied to show \( d \)-representation-finiteness of some important classes of algebras. In Section 2.3 we give a general result saying \( d \)-cluster-tilting subcategories are, under certain conditions, preserved by Galois coverings. Section 2.4 contains explanations of concepts and notation used in the paper. Proofs of the results in Section 2 are given in Section 3. In Section 4, we further investigate some examples and applications of the general theory, while Section 5 features a complete characterization of all \( d \)-representation-finite self-injective Nakayama algebras. We conclude by posing some open questions, together with a few partial results, in Section 6.

2. Our results

Throughout this article, \( d \) is a positive integer, \( k \) a field, and \( \Lambda \) a finite-dimensional \( k \)-algebra of finite global dimension. While most of the theory is valid for general fields, some results require additional assumptions. This will then be stated at each such instance. In order not to make the introductory parts of this paper overly technical, the definitions of some of the concepts used in this section have been postponed to Section 2.4.

We start by defining the fundamental concepts of our investigation.

**Definition 2.1.** Let \( C \) be an abelian or a triangulated category, and \( A \) a finite-dimensional \( k \)-algebra.

(i) A full subcategory \( \mathcal{U} \) of \( C \) is \( d \)-cluster-tilting if it is functorially finite in \( C \) and

\[
\mathcal{U} = \{ X \in C \mid \Ext^i_C(\mathcal{U}, X) = 0 \text{ for all } 1 \leq i \leq d - 1 \}
\]

(ii) A finitely-generated module \( M \in \text{mod} \ A \) is called a \( d \)-cluster-tilting module if its additive closure \( \mathcal{M} \) forms a \( d \)-cluster-tilting subcategory of \( \text{mod} \ A \).

(iii) The algebra \( A \) is said to be \( d \)-representation-finite if it has a \( d \)-cluster-tilting module.
In the case $d = 1$, the classical situation is recovered: a 1-cluster-tilting $A$-module is nothing but an additive generator of $\text{mod} \ A$, and an algebra is 1-representation-finite if and only if it is representation finite in the classical sense. It is easy to see that if $C$ is a Frobenius category then the natural functor $C \rightarrow \hat{C}$ induces a bijection between $d$-cluster-tilting subcategories of $C$ and $\hat{C}$ respectively. Note that, in contrast with several earlier papers (e.g., [31, 32, 42, 43]) we do not assume $d$-representation-finite algebras to have global dimension at most $d$.

**Definition 2.2.** A locally bounded $k$-linear Krull-Schmidt category $C$ is locally $d$-representation-finite if $\text{mod} \ C$ has a locally bounded $d$-cluster-tilting subcategory.

We remark that a $d$-cluster-tilting subcategory $U$ of $\text{mod} \ C$ is locally bounded if, and only if, for any $x \in C$ there exist only finitely many isomorphism classes of indecomposable objects $U \in U$ such that $U(x) \neq 0$ (Lemma 3.1(b)). Thus, for $d = 1$ our definition is equivalent to the classical one [16, 2.2]. Clearly, a finite-dimensional $k$-algebra $A$ is $d$-representation-finite if and only if the category $\text{proj} \ A$ of finitely generated projective $A$-modules is locally $d$-representation-finite.

**2.1. Basic construction.** We denote by $\mathcal{D}^b(\Lambda)$ the bounded derived category of the $\text{mod} \Lambda$ of finitely generated right $\Lambda$-modules, and by $\hat{\Lambda}$ the repetitive category of $\Lambda$ (see Section 2.4). Then the category $\text{mod} \hat{\Lambda}$ of finitely presented $\Lambda$-modules is Frobenius and therefore, its stable category $\text{mod} \hat{\Lambda}$ is triangulated, with suspension functor given by the inverse of Heller’s syzygy functor $\Omega$. Moreover, there exists a triangle equivalence [30] (see also [69, Section 3.4])

$$\mathcal{D}^b(\Lambda) \cong \text{mod} \hat{\Lambda} \tag{2.1}$$

restricting to the identity functor on $\text{mod} \Lambda$, which is a full subcategory of both $\mathcal{D}^b(\Lambda)$ and $\text{mod} \hat{\Lambda}$. In what follows, we will often view this equivalence as an identification and, consequently, we generally make no distinction between subcategories of $\mathcal{D}^b(\Lambda)$ and $\text{mod} \hat{\Lambda}$.

Let $\phi$ be an automorphism of $\hat{\Lambda}$. We denote by $\phi_\ast : \text{mod} \hat{\Lambda} \rightarrow \text{mod} \Lambda$ the induced automorphism of the module category, defined by precomposition with $\phi^{-1}$, and by $F_\ast : \text{mod} \hat{\Lambda} \rightarrow \text{mod} (\hat{\Lambda}/\phi)$ the push-down functor, induced by the covering functor $F : \hat{\Lambda} \rightarrow \hat{\Lambda}/\phi$ (see Section 2.3). A subcategory $U$ of $\mathcal{D}^b(\Lambda) \cong \text{mod} \hat{\Lambda}$ is said to be $\phi$-equivariant if $U$ and $\phi_\ast (U)$ have the same isomorphism closure.

The following result is of fundamental importance for our investigation. It is obtained as an application of Corollary 2.15 in Section 2.3.

**Theorem 2.3.** Let $\Lambda$ be a basic finite-dimensional $k$-algebra of finite global dimension, and $\phi$ an admissible automorphism of the repetitive category $\hat{\Lambda}$.

(a) If $\mathcal{D}^b(\Lambda)$ contains a locally bounded $\phi$-equivariant $d$-cluster-tilting subcategory $U$, then the algebra $\hat{\Lambda}/\phi$ is $d$-representation-finite. The converse implication also holds in case $k$ is algebraically closed.

(b) If, in (a), $S$ is a cross-section of the $\phi_\ast$-orbits of $\text{ind} U$, then

$$V = (\hat{\Lambda}/\phi) \oplus \bigoplus_{X \in S} F_\ast (X)$$

is a basic $d$-cluster-tilting $(\hat{\Lambda}/\phi)$-module.

The following diagram illustrates the correspondence between $d$-cluster-tilting subcategories of $\mathcal{D}^b(\Lambda)$ and $\text{mod} (\hat{\Lambda}/\phi)$ underlying Theorem 2.3.

$$\begin{array}{c}
\mathcal{D}^b(\Lambda) \cong \text{mod} \hat{\Lambda} \xrightarrow{\text{nat}} \text{mod} \hat{\Lambda} \xrightarrow{F_\ast} \text{mod} (\hat{\Lambda}/\phi) \\
\bigcup \bigcup \\
U \leftarrow \text{add} \ V
\end{array}$$

**Remark 2.4.** In the setting of Theorem 2.3, a $\phi$-equivariant $d$-cluster-tilting subcategory $U$ of $\mathcal{D}^b(\Lambda)$ is locally bounded if and only if its preimage under the natural functor $\text{mod} \Lambda \rightarrow \text{mod} \hat{\Lambda}$ is locally bounded, if and only if the number of $\phi_\ast$-orbits of $\text{ind} U$ is finite. A proof of this is given at the end of Section 3.1.
Next, we give a systematic construction of pairs $\Lambda$ and $\phi$ satisfying the conditions in Theorem 2.3, and thus giving rise to self-injective algebras that are $d$-representation-finite. Let $\Lambda$ be a finite-dimensional $k$-algebra of finite global dimension, and let

\[
(2.2) \quad \nu = D \circ \mathbb{R} \text{Hom}_A (-, \Lambda) \simeq - \circ \Lambda^* \partial D \Lambda : \mathcal{D}^b(\Lambda) \to \mathcal{D}^b(\Lambda)
\]

be the Nakayama functor of $\mathcal{D}^b(\Lambda)$. Here $D = \text{Hom}_k (-, k)$ is the usual $k$-dual. The Nakayama functor satisfies the functorial isomorphism

\[
\text{Hom}_{\mathcal{D}^b(\Lambda)}(X, Y) \simeq D \text{Hom}_{\mathcal{D}^b(\Lambda)}(Y, \nu X),
\]

in other words, it is a Serre functor on $\mathcal{D}^b(\Lambda)$. We denote

\[
(2.3) \quad \nu_d = \nu \circ [-d] : \mathcal{D}^b(\Lambda) \to \mathcal{D}^b(\Lambda),
\]

and say that $\Lambda$ is $\nu_d$-finite if $\text{gl.dim}(\Lambda) < \infty$ and

\[
\nu_d^i (\mathcal{D}^{\geq 0}(\Lambda)) \subset \mathcal{D}^{\geq 1}(\Lambda)
\]

for some $i \geq 0$. Here $\mathcal{D}^{\geq 0}(\Lambda)$ denotes the full subcategory of $\mathcal{D}^b(\Lambda)$ formed by all objects $X \in \mathcal{D}^b(\Lambda)$ satisfying $H^j(X) = 0$ for all $j < \ell$. The property of being $\nu_d$-finite is preserved under derived equivalences [41, Lemma 5.6]. All algebras of global dimension at most $d - 1$, as well as all $d$-representation-finite algebras of global dimension $d$, are $\nu_d$-finite. In particular, path algebras of quivers of Dynkin type are $\nu_d$-finite for all $d \geq 1$. Note that $\nu_1 = \nu \circ [-1]$ is the Auslander–Reiten translation on $\mathcal{D}^b(\Lambda)$ [59, Proposition I.2.3], [30, Theorem I:4.6]. Hence, a hereditary algebra is $\nu_1$-finite if and only if it is representation finite.

From [41, Proposition 5.4, Lemma 5.6, Theorem 1.23] we get the following result, which plays an important role in the present paper. For any $T \in \mathcal{D}^b(\Lambda)$, set

\[
(2.4) \quad \mathcal{U}_d(T) = \text{add}\{\nu_d^i(T) \mid i \in \mathbb{Z}\} \subset \mathcal{D}^b(\Lambda).
\]

**Proposition 2.5.** If $\Lambda$ is $\nu_d$-finite and $T \in \mathcal{D}^b(\Lambda)$ is a tilting complex such that the algebra $\text{End}_{\mathcal{D}^b(\Lambda)}(T)$ has global dimension at most $d$, then $\mathcal{U}_d(T)$ is a locally bounded $d$-cluster-tilting subcategory of $\mathcal{D}^b(\Lambda)$.

We call a $d$-cluster-tilting subcategory $\mathcal{U}$ of $\mathcal{D}^b(\Lambda)$ orbital if it can be written as $\mathcal{U} = \mathcal{U}_d(T)$ for some $T$ as in Proposition 2.5. These will be our primary examples of $d$-cluster-tilting subcategories of $\mathcal{D}^b(\Lambda) \simeq \text{mod} \hat{\Lambda}$. The following result, which is an immediate consequence of Theorem 2.3, Remark 2.4 and Proposition 2.5, allows us to construct large classes of $d$-representation-finite self-injective algebras.

**Theorem 2.6.** Let $\Lambda$ be a finite-dimensional $\nu_d$-finite $k$-algebra, and $\phi$ an admissible automorphism of the repetitive category $\hat{\Lambda}$ of $\Lambda$.

(a) If $\mathcal{D}^b(\Lambda)$ contains a tilting complex $T$ satisfying $\text{gl.dim}(\text{End}_{\mathcal{D}^b(\Lambda)}(T)) \leq d$, such that $\mathcal{U}_d(T)$ is $\phi$-equivariant, then $\hat{\Lambda}/\phi$ is a $d$-representation-finite and self-injective.

(b) If, in (a), $S$ is a cross-section of the $\phi$-orbits of $\text{ind}(\mathcal{U}_d(T))$, then

\[
V = (\hat{\Lambda}/\phi) \oplus \bigoplus_{X \in S} F_i(X)
\]

is a basic $d$-cluster tilting $(\hat{\Lambda}/\phi)$-module.

Denote by $\tilde{\nu} = \nu^*_\Lambda$ the Nakayama automorphism of $\hat{\Lambda}$. If $\Lambda$ is a $\nu_d$-finite algebra of global dimension at most $d$, then $\mathcal{U}_d(\Lambda)$, viewed as a $d$-cluster-tilting subcategory of $\text{mod} \hat{\Lambda}$, has the form

\[
\mathcal{U}_d(\Lambda) = \text{add}\{\tilde{\nu}_i \Omega^{d+1} \} \mid i \in \mathbb{Z}\}.
\]

If this subcategory is $\phi$-equivariant then, by Theorem 2.6(a), it follows that $\hat{\Lambda}/\phi$ is a $d$-representation-finite self-injective algebra.

As an illustration, we give a simple example of an application of Theorem 2.6.
Example 2.7. Let $\Lambda$ be the finite-dimensional $k$-algebra given by the quiver $1 \overset{a}{\to} 2 \overset{b}{\to} 3$ with the relation $ab = 0$. Being derived equivalent to a hereditary algebra of Dynkin type $A_3$, the algebra $\Lambda$ is $\nu_2$-finite; moreover, $\text{gl.dim} \Lambda = 2$. Hence, $U_2(\Lambda)$ is a 2-cluster-tilting subcategory of $D^b(\Lambda)$, by Proposition 2.5. The Auslander–Reiten quiver of $D^b(\Lambda)$ is depicted below, with the objects of $U_2(\Lambda)$ written in circles. The object $X \in D^b(\Lambda)$ is the cone of a non-zero morphism $\frac{2}{3} \to \frac{1}{2}$.

The repetitive algebra $\tilde{\Lambda}$ of $\Lambda$ is given by the following quiver with mesh relations.

Then again, $\sigma_*$ acts as reflection in the central horizontal line, and $\tilde{\nu}_* \simeq \tau^{-3}$, where $\tau$ denotes the Auslander–Reiten translation in $\mod \tilde{\Lambda}$. Thus, for example, the automorphisms $\tilde{\nu}$ and $\tilde{\nu}^2$ of $\tilde{\Lambda}$ satisfy $(\tilde{\nu} \circ \sigma)_* = \mathcal{U}$ and $(\tilde{\nu}^2 \circ \sigma)_* = \mathcal{U}$ and consequently, the orbit algebras $\tilde{\Lambda} \!/ (\tilde{\nu})$ and $\tilde{\Lambda} \!/ (\tilde{\nu}^2)$ are 2-representation-finite self-injective. We remark that $\tilde{\Lambda} \!/ (\tilde{\nu}^2)$ is isomorphic to the preprojective algebra $\Pi_2(A_3)$ of the Dynkin diagram $A_3$, while $\tilde{\Lambda} \!/ (\tilde{\nu}^2)$ is the 2-fold trivial extension algebra of $\Lambda$ (see Section 2.2).

Another example illustrating Theorem 2.6 is given in Section 4.1.

2.2. Applications. Below we give some examples of how Theorem 2.6 can be used to show $d$-representation-finiteness of some well-known types of self-injective algebras. First, we show how the first part of Riedtmann’s result follows from Theorem 2.6.

Proof of Theorem 1.1(a). Let $\Lambda$ be a tilted algebra of Dynkin type and $\phi$ an admissible automorphism of $\Lambda$. Then the corresponding path algebra $H$ may be viewed as the endomorphism algebra of a tilting complex $T$ in $D^b(\Lambda)$, and $\text{gl.dim} \text{End}_{D^b(\Lambda)}(T) = \text{gl.dim} H \leq 1$. Moreover, since $U_1(H) = D^b(H) \simeq D^b(\Lambda)$, it follows that $U_1(T) = D^b(\Lambda)$, so $U_1(T)$ is clearly $\phi$-equivariant. Therefore $\Lambda \! / \! \phi$, is 1-representation-finite and self-injective by Theorem 2.6.
It is natural to view $d$-representation-finite algebras of global dimension $d$ as generalizations of representation-finite hereditary algebras. If $\Lambda$ is $d$-representation-finite and $\text{gl.dim} \, \Lambda = d$ then $\text{mod} \, \Lambda$ has a unique basic $d$-cluster-tilting module $M$, given as $M = \bigoplus_{i \geq 0} \tau_d^i(I)$, where $I$ is a basic injective cogenerator of $\Lambda$ and

$$\tau_d = D\text{Ext}^d_A(-, \Lambda) : \text{mod} \, \Lambda \rightarrow \text{mod} \, \Lambda$$

the $d$-Auslander–Reiten translation [41, Proposition 1.3].

Corollary 2.8 below extends a famous theorem of Tachikawa–Yamagata [64, 68] stating that the trivial extension algebras of representation-finite hereditary algebras are again representation finite.

The $n$-fold trivial extension algebra of $\Lambda$ is defined as $T_n(\Lambda) = \hat{\Lambda}/\hat{n}=\hat{\nu}_\Lambda$; where $\hat{\nu} = \nu_\Lambda^n$ is the Nakayama automorphism of $\Lambda$, and $n \geq 1$. In particular, $T_1(\Lambda)$ is the usual trivial extension algebra $T(\Lambda)$. For $n \geq 2$, $T_n(\Lambda)$ may be viewed as the matrix algebra

$$\begin{pmatrix}
\Lambda & D\Lambda \\
D\Lambda & \Lambda \\
\ddots & \ddots & \ddots & \ddots \\
\ddots & \ddots & \ddots & \ddots & \ddots \\
D\Lambda & \Lambda \\
\end{pmatrix},$$

where the multiplications $\Lambda \cdot D\Lambda$ and $D\Lambda \cdot D\Lambda$ are defined by the $\Lambda$–$\Lambda$-bimodule structure of $D\Lambda$.

We remark that $T_n(\Lambda)$ is a finite-dimensional self-injective $k$-algebra, and that there is a natural embedding $\text{mod} \, \Lambda \rightarrow \text{mod} \, T_n(\Lambda)$, given by the projection $T_n(\Lambda) \rightarrow \Lambda$ onto the first diagonal entry. The functor $(\nu_{T_n(\Lambda)})_* : \text{mod} \, T_n(\Lambda) \rightarrow \text{mod} \, T_n(\Lambda)$, induced by the Nakayama automorphism $\nu_{T_n(\Lambda)}$ of $T_n(\Lambda)$, satisfies $(\nu_{T_n(\Lambda)})_* \simeq D\text{Hom}_{T_n(\Lambda)}(-, T_n(\Lambda))$.

**Corollary 2.8.** Let $\Lambda$ be a basic $d$-representation-finite algebra of global dimension $d$, and $M$ the unique basic $d$-cluster-tilting $\Lambda$-module. Then, for any $\ell \geq 1$, the $d\ell$-fold trivial extension algebra $T_{d\ell}(\Lambda)$ is $d$-representation-finite, and the $T_{d\ell}(\Lambda)$-module

$$U = T_{d\ell}(\Lambda) \oplus \bigoplus_{i=0}^{\ell(d+1)-1} ((\nu_{T_{d\ell}(\Lambda)}), \Omega^i(M/\Lambda)) \oplus \bigoplus_{j=0}^{\ell-1} ((\nu_{T_{d\ell}(\Lambda)}), \Omega^{d+1})^j(\Lambda)$$

is a basic $d$-cluster-tilting module.

Setting $d = \ell = 1$ above recovers the classical result by Tachikawa–Yamagata. Another instance of Corollary 2.8 was given in Example 2.7, where we saw that $T_2(\Lambda) = \hat{\Lambda}/\hat{\nu}^2$ was 2-representation-finite.

We now consider a wider class of algebras. Let $a, b \in \mathbb{Z}_{>0}$. An algebra $\Lambda$ of finite global dimension is said to be fractionally $\frac{b}{a}$-Calabi–Yau if $\nu^a \simeq [b]$ holds in $D^b(\Lambda)$, and twisted fractionally $\frac{b}{a}$-Calabi–Yau if there exists an automorphism $\phi$ of $\Lambda$ such that $\nu^a \simeq [b] \circ \phi$, where $\phi_* : D^b(\Lambda) \rightarrow D^b(\Lambda)$ is the functor induced by the functor $\phi_* : \text{mod} \, \Lambda \rightarrow \text{mod} \, \Lambda$ (see Section 2.3). For example, any $d$-representation-finite algebra of global dimension $d$ is twisted fractionally Calabi–Yau [31, Theorem 1.1(a)].

**Corollary 2.9.** Let $\Lambda$ be a basic $k$-algebra that is twisted fractionally $\frac{b}{a}$-Calabi–Yau, and $d$ a positive integer satisfying $d \geq \text{gl.dim} \, \Lambda$. Set $g = \gcd(d+1, a+b$, and $n = \ell(ad-b)/g$, where $\ell \geq 1$. Then $T_n(\Lambda)$ is $d$-representation-finite, and

$$V = T_n(\Lambda) \oplus \bigoplus_{i=0}^{\ell(a+1)/g-1} ((\nu_{T_n(\Lambda)}), \Omega^{d+1})^i(\Lambda)$$

is a basic $d$-cluster-tilting $T_n(\Lambda)$-module.

**Example 2.10.** Let $\Lambda$ be a canonical algebra of tubular type, that is, an algebra of one of the four types listed in Figure 1.
Let \( \Lambda \) be of type \((3, \tau)\) then \(d \geq 2\). More generally, for a positive integer \(n\), \(\Lambda\) is said to be \(n\)-representation-finite, if \(\Lambda\) is of type \((2^r, \tau)\) and \(\text{gcd}(d+1)\) divides \(n\). In particular, for any \(\ell \geq 1\), \(\Lambda\) is of type \((2, 2, 2)\) then \(T_\ell(\Lambda)\) is 2-representation-finite, if \(\Lambda\) is of type \((3, 3, 3)\) then \(T_\ell(\Lambda)\) is 2-representation-finite, if \(\Lambda\) is of type \((2, 4, 4)\) then \(T_{2n}(\Lambda)\) is 3-representation-finite, and if \(\Lambda\) is of type \((2, 3, 6)\) then \(T_{2n}(\Lambda)\) is 2-representation-finite.

We give one more example of an application of Theorem 2.6 to a particular class of \(n\)-fold trivial extension algebras. Let \(r\) be a positive integer. A \(d\)-representation-finite algebra \(\Lambda\) of global dimension \(d\) is said to be \(r\)-homogeneous if \(n_{d}(\Lambda) \simeq \Lambda[-d]\) in \(\mathcal{D}^{b}(\Lambda)\) or, equivalently, \(\tau_{d}^{r-1}(DA) \simeq \Lambda\) in \(\text{mod } \Lambda\) [31].

**Corollary 2.11.** Let \(\Lambda\) be a basic \(r\)-homogeneous \(d\)-representation-finite algebra of global dimension \(d\). Then the \((r-1)\)-fold trivial extension algebra \(T_{r-1}(\Lambda)\) of \(\Lambda\) is \((r-1)(d+1)\)-representation-finite, and \(M = T_{r-1}(\Lambda) \oplus \Lambda\) is a basic \((r-1)(d+1)\)-cluster-tilting \(T_{r-1}(\Lambda)\)-module.

In particular, the trivial extension algebra of a 2-homogeneous \(d\)-representation-finite algebra is \((d+1)\)-representation-finite.

The next application is to higher preprojective algebras, which were introduced in [42, 43, 51]. For a finite-dimensional \(k\)-algebra \(\Lambda\) with \(\text{gl.dim } \Lambda \leq d\), the \((d+1)\)-preprojective algebra is defined as the tensor algebra

\[
\Pi = \Pi(\Lambda) = T_{\Lambda} \text{Ext}_{\Lambda}^{d}(DA, \Lambda)
\]

of the \(\Lambda\)-bimodule \(\text{Ext}_{\Lambda}^{d}(DA, \Lambda)\). Thus \(\Pi\) has a natural structure of a \(\mathbb{Z}\)-graded \(k\)-algebra: \(\Pi = \bigoplus_{j \geq 0} \Pi_{j}\). More generally, for a positive integer \(n\), we define the \(n\)-fold \((d+1)\)-preprojective algebra of \(\Lambda\) as the matrix algebra

\[
\Pi^{(n)} = \Pi^{(n)}(\Lambda) = (\Pi_{i-j+nZ})_{1 \leq i, j \leq n},
\]

These are tame algebras of polynomial growth [63], and moreover, they are \(\frac{2}{p}\)-Calabi–Yau, where \(p\) is given by the following list.

| type     | \((2, 2, 2)\) | \((3, 3, 3)\) | \((2, 4, 4)\) | \((2, 3, 6)\) |
|----------|---------------|---------------|---------------|---------------|
| \(p\)    | 2             | 3             | 4             | 6             |
| \(n\)    | \(\frac{2d-1}{\text{gcd}(4, d+1)}\) | \(\frac{3d-1}{\text{gcd}(6, d+1)}\) | \(\frac{4d-1}{\text{gcd}(8, d+1)}\) | \(\frac{6d-1}{\text{gcd}(12, d+1)}\) |
where $\Pi_{j+nZ} = \bigoplus_{d \in \mathbb{Z}} \Pi_{j+n}$. Note that $\Pi^{(1)} = \Pi$. We apply our construction to give the following result, generalizing parts of [27, Theorem 2.2],[28, Theorem 1] by Geiss–Leclerc–Schröer for the classical preprojective algebras (i.e., $d = 1$ and $f = 1$) and of [43, Corollaries 3.4 and 4.16] due to Iyama–Oppermann for the higher preprojective algebras ($f = 1$).

**Corollary 2.12.** Let $\Lambda$ be a $d$-representation-finite algebra of global dimension $d$ and $n$ a positive integer. Then the $n$-fold $(d + 1)$-preprojective algebra $\Pi^{(n)}(\Lambda)$ of $\Lambda$ is $(d + 1)$-representation-finite and self-injective.

The corollaries 2.8, 2.9, 2.11 and 2.12 will be proved in Section 3.3. An example illustrating Corollary 2.12 is given in Section 4.1.2.

### 2.3. Galois coverings and $d$-cluster-tilting

Our main result, Theorem 2.3, is largely a consequence of a more general result, Theorem 2.14 below. It is based on the classical theory of Galois coverings of $k$-linear categories, initiated by Gabriel [25, 26].

Let $\mathcal{C}$ be a skeletally small locally bounded $k$-linear Krull-Schmidt category and $G$ an admissible group of automorphisms of $\mathcal{C}$ (see Definition 2.16). Then the orbit category $\mathcal{C}/G$ (see Section 2.4) is again a skeletally small locally bounded $k$-linear Krull-Schmidt category. The natural functor

$$F : \mathcal{C} \to \mathcal{C}/G$$

induces an exact functor

$$F^* : \text{Mod}(\mathcal{C}/G) \to \text{Mod}\mathcal{C}$$

called the pull-up, given by $F^*(M) = M \circ F$. This functor has a left adjoint

$$F_* : \text{Mod}\mathcal{C} \to \text{Mod}(\mathcal{C}/G)$$

called the push-down [26], which is also exact. If $\text{Hom}_\mathcal{C}(-, y) \xrightarrow{(-, f)} \text{Hom}_\mathcal{C}(-, x) \to M \to 0$ is a projective presentation of $M \in \text{mod}\mathcal{C}$, then $F_* (M)$ is given as the cokernel of the induced map $\text{Hom}_{\mathcal{C}/G}(-, y) \xrightarrow{(-, F(f))} \text{Hom}_{\mathcal{C}/G}(-, x)$. In particular, $F_*$ induces a functor $F_* : \text{mod}\mathcal{C} \to \text{mod}(\mathcal{C}/G)$ between the categories of finitely presented modules of $\mathcal{C}$ and $\mathcal{C}/G$, respectively. For a more detailed presentation, see [14, Section 6].

**Remark 2.13.** In the above, the purpose assuming that $\mathcal{C}$ is skeletally small is to ensure that the collection of morphisms between two objects $X, Y \in \text{Mod}\mathcal{C}$ forms a set. As, in the present work, we only concern ourselves with finitely presented modules, this set-theoretic difficulty may be circumvented in the following way: For an infinite cardinal $\kappa$, let $\text{Mod}^\kappa \mathcal{C}$ be the category of $\mathcal{C}$-modules $Y$ admitting an epimorphism

$$\bigoplus_{x \in \mathcal{X}} \text{Hom}_\mathcal{C}(-, x) \to Y,$$

where $\mathcal{X}$ is a collection of objects in $\mathcal{C}$ with $|\mathcal{X}| \leq \kappa$. If $G$ is a group acting on $\mathcal{C}$, and $\kappa \geq |G|$, then $F^*(Y) \in \text{Mod}^\kappa \mathcal{C}$ for all $Y \in \text{Mod}^\kappa (\mathcal{C}/G)$, and $F_* (X) \in \text{Mod}^\kappa (\mathcal{C}/G)$ when $X \in \text{Mod}^\kappa \mathcal{C}$. Hence, the pull-up and push-down functors may be defined on the categories $\text{Mod}^\kappa (\mathcal{C}/G)$ and $\text{Mod}^\kappa \mathcal{C}$ instead of the full module categories $\text{Mod}\mathcal{C}$ and $\text{Mod}(\mathcal{C}/G)$:

$$F^* : \text{Mod}^\kappa (\mathcal{C}/G) \to \text{Mod}^\kappa \mathcal{C}, \quad F_* : \text{Mod}^\kappa \mathcal{C} \to \text{Mod}^\kappa (\mathcal{C}/G).$$

For this reason, in what follows, we will not assume the category $\mathcal{C}$ to be skeletally small.

For simplicity, if $U \subseteq \text{mod}\mathcal{C}$ is a full subcategory closed under isomorphisms, then $F_*(U) \subseteq \text{mod}(\mathcal{C}/G)$ denotes the isomorphism closure of the image of $F_*$, that is, the smallest full subcategory of $\text{mod}(\mathcal{C}/G)$ that is closed under isomorphism and contains all objects of the form $F_* (U)$ for $U \in U$. The preimage $F_*^{-1}(V)$ of a full subcategory $V \subset \text{mod}(\mathcal{C}/G)$ is the full subcategory of $\text{mod}\mathcal{C}$ consisting of all objects $U \in \text{mod}\mathcal{C}$ such that $F_* (U) \in V$.

**Theorem 2.14.** Let $\mathcal{C}$ be a locally bounded $k$-linear Krull-Schmidt category and $G$ a group acting admissibly on $\mathcal{C}$. 
(a) Assume that $G$ acts admissibly on $\mod C$, and let $U$ be a $G$-equivariant full subcategory of $\mod C$ such that $F_*(U)$ is functorially finite in $\mod (C/G)$. Then $U$ is a $d$-cluster-tilting subcategory of $\mod C$ if and only if $F_*(U)$ is a $d$-cluster-tilting subcategory of $\mod (C/G)$.

(b) Assume that the field $k$ is algebraically closed, and that the group $G$ is free abelian of finite rank. Then, for any $d$-cluster-tilting subcategory $V$ of $\mod (C/G)$, the full subcategory $F^{-1}_*(V)$ of $\mod C$ is a $G$-equivariant $d$-cluster-tilting subcategory.

From Theorem 2.14 one can deduce the following corollary, which is an analogue of classical results due to Gabriel [25, Lemma 3.3, Theorem 3.6].

**Corollary 2.15.** Let $C$ be a locally bounded $k$-linear Krull-Schmidt category and $G$ a free abelian group of finite rank, acting admissibly on $C$.

(a) The push-down functor $F_* : \mod C \to \mod (C/G)$ induces an injective map from the class of locally bounded $G$-equivariant $d$-cluster-tilting subcategories of $\mod C$ to the class of locally bounded $d$-cluster-tilting subcategories of $\mod (C/G)$.

(b) The above map is a bijection if $k$ is algebraically closed.

Corollary 2.15 implies that $C$ is locally $d$-representation-finite whenever $C/G$ is locally $d$-representation-finite. However, in contrast to the classical case, the converse is not true in general, because $d$-cluster-tilting subcategories of $\mod C$ are usually not $G$-equivariant. For example, let $C = \tilde{k}$ be the repetitive category of the field $k$. Then $\tilde{k} \simeq kA_\infty^{\infty}/I^2$, where $A_\infty^{\infty}$ is the linearly oriented quiver of type $A_\infty^{\infty}$, and $I \subset kA_\infty^{\infty}$ is the ideal generated by all arrows in $A_\infty^{\infty}$. The group $G = \mathbb{Z}$ acts on $\tilde{k}$ by translation in the quiver $A_\infty^{\infty}$, and $\tilde{k}/\mathbb{Z} \simeq T(k) \simeq k[X]/(X^2)$ is the algebra of dual numbers over $k$. But while $\tilde{k}$ is locally $d$-representation-finite for all $d \geq 1$ by Proposition 2.5, the algebra $\tilde{k}/\mathbb{Z}$ is (locally) $d$-representation-finite only for $d = 1$ (see Theorem 5.1).

2.4. Notation. In this section, we clarify our notational conventions, and recall some well-known concepts and results.

Let $C$ be a Hom-finite $k$-linear category. A $k$-linear autoequivalence $\nu = \nu_C : C \to C$ is a Nakayama functor if for all $x, y \in C$ there is a functorial isomorphism

$$\Hom_C(x, y) \simeq D\Hom_C(y, \nu x).$$

A typical example is the Nakayama functor (2.2) of $D^b(\Lambda)$. Any two Nakayama functors of $C$ are isomorphic. A category $C$ having a Nakayama functor is said to be self-injective. For example, when $C$ is the category $\proj A$ of finitely generated projective modules of a $k$-algebra $A$, then $C$ is self-injective if and only if $A$ is a self-injective algebra. In this case, the Nakayama functor of $C$ is the automorphism of $C$ induced by the classical Nakayama automorphism of the algebra $A$.

In case $C$ is a triangulated category, the Nakayama functor of $C$ is often called the Serre functor and denoted by $S = cS$. In this case, and given an integer $d$, we write

$$S_d = cS_d = cS \circ [-d].$$

A fundamental property of the Nakayama functor is that it commutes with any autoequivalence $\phi$ of $C$, up to isomorphism of functors:

$$(2.6) \quad \phi \nu \simeq \nu \phi.$$

This follows from the isomorphisms

$$\Hom_C(y, \nu 2x) \simeq D\Hom_C(\phi x, y) \simeq D\Hom_C(x, \phi^{-1} y) \simeq \Hom_C(\phi^{-1} y, \nu x) \simeq \Hom_C(y, \phi \nu x)$$

and Yoneda’s lemma.

Let $\Lambda$ be a finite-dimensional $k$-algebra. The trivial extension algebra $T(\Lambda)$ of $\Lambda$ is defined as $\Lambda \oplus DA$, with multiplication given by

$$(a, x) \cdot (b, y) = (ab, ay + xb)$$

for $a, b \in \Lambda$ and $x, y \in DA$. Setting $T(\Lambda)_0 = \Lambda$ and $T(\Lambda)_1 = DA$ defines a $\mathbb{Z}$-grading on $T(\Lambda)$. The repetitive category $\Lambda$ of $\Lambda$ is defined as the category $\proj^Z T(\Lambda)$ of finitely generated $\mathbb{Z}$-graded
projective $T(\Lambda)$-modules. Observe that $\hat{A}$ is self-injective, the Nakayama automorphism being given by degree shift by 1:

$$\nu_{\hat{A}} = (1).$$

From here on, we will write $\hat{\nu}$ to denote this functor.

We now recall some basic terminology for functor categories [9]. Let $C$ be a $k$-linear Krull-Schmidt category. A $C$-module is a contravariant additive functor from $C$ to the category $Ab$ of abelian groups. We denote by $\text{Mod}C$ the category of all $C$-modules, and by $\text{mod}C$ the full subcategory of $\text{Mod}C$ consisting of finitely presented modules. A module $M : C \to Ab$ is finitely presented if there exists an exact sequence

$$\text{Hom}_C(-, y) \to \text{Hom}_C(-, x) \to M \to 0$$

in $\text{Mod}C$. A full subcategory $\mathcal{U}$ of $\text{mod}C$ is called a generator-cogenerator if the $C$-modules $\text{Hom}_C(-, x)$ and $D \text{Hom}_C(x, -)$ belong to $\mathcal{U}$ for all $x \in C$. Moreover, $\text{proj}C$ and $\text{inj}C$ denote the full subcategories of $\text{mod}C$ consisting of projective and injective objects in $\text{mod}C$, respectively. We denote by $\text{ind} C$ a chosen set of representatives of the isomorphism classes of indecomposable objects in $C$, and by $\text{Supp} M$ the support of a $C$-module $M$, that is,

$$\text{Supp} M = \{x \in \text{ind} C \mid M(x) \neq 0\} \subset \text{ind} C.$$

Modules of a finite-dimensional $k$-algebra $A$ are identified with modules of the category $\text{proj}A$ of finitely generated projective $A$-modules.

The category $C$ is said to be locally bounded if for any $x \in \text{ind} C$, we have

$$\sum_{y \in \text{ind} C} (\dim_k \text{Hom}_C(x, y) + \dim_k \text{Hom}_C(y, x)) < \infty.$$

Note that when $C$ is locally bounded, the objects in $\text{mod}C$ are precisely the finite-length modules of $C$. Therefore we have a duality

$$(2.7) \quad D = \text{Hom}_k(-, k) : \text{mod}C \leftrightarrow \text{mod}(C^{\text{op}})$$

satisfying $D \circ D \simeq \text{Id}_{\text{mod}C}$.

An example of a locally bounded category with infinitely many indecomposable objects is the repetitive category $\hat{A}$ of a finite-dimensional $k$-algebra $A$.

Let $G$ be a group. A $G$-action on the category $C$ is an assignment $g \mapsto F_g$ of an automorphism $F_g : C \to C$ to each element $g \in G$, such that $F_g \circ F_h = F_{gh}$ for all $g, h \in G$. The action is said to be $k$-linear if each $F_g$ is $k$-linear.

**Definition 2.16.** Let $C$ be a $k$-linear Krull-Schmidt category, and $G$ a group. A $G$-action on $C$ is admissible if $gx \neq x$ holds for every $x \in \text{ind} C$ and $g \in G \setminus \{1\}$.

Note that any autoequivalence $\phi$ of $C$ gives rise to an automorphism of the skeleton $C'$ of $C$ and therefore a $\mathbb{Z}$-action on $C'$. In particular, an autoequivalence $\phi$ of $\hat{A}$ is admissible (as defined in the sentence following Theorem 1.1) if and only if the induced action of the additive group $\mathbb{Z}$ on a skeleton of $C$, given by $m \mapsto \phi^m$, is admissible.

Let $\phi$ be an autoequivalence of $C$, and $\phi^{-1}$ a chosen quasi-inverse of $\phi$. In case $\phi$ is an automorphism, we always assume $\phi^{-1}$ to be the inverse of $\phi$. We denote by

$$(2.8) \quad \phi_* : \text{mod}C \to \text{mod}C \quad \text{and} \quad \phi^* : \text{mod}C \to \text{mod}C$$

the induced automorphisms of the module category defined by $\phi_*(M) = M \circ \phi^{-1}$ and $\phi^*(M) = M \circ \phi$ respectively. Hence, any action of a group $G$ on $C$ induces a $G$-action and a $G^{\text{op}}$-action on $\text{mod}C$, defined by $g \mapsto g_*$ and $g \mapsto g^*$ respectively. Clearly, these actions induce actions on the stable module category and the derived category too. A subcategory $\mathcal{U}$ of $\text{mod}C$ or $\text{mod}C$ is said to be $G$-equivariant if $g_*\mathcal{U} = \mathcal{U}$ for all $g \in G$.

The following is an immediate consequence of Auslander–Reiten duality (see, e.g., [8, Theorem IV:2.13]) for $k$-linear categories.

**Proposition 2.17.** Let $C$ be a locally bounded self-injective $k$-linear category with Nakayama functor $\nu$. Then $\text{mod} C$ has a Serre functor $\nu_* \circ \Omega$. 
Since \( D^b(\Lambda) \simeq \text{mod} \tilde{\Lambda} \), uniqueness of the Serre functor gives us the following diagram, which is commutative up to isomorphism of functors:

\[
\begin{array}{ccc}
D^b(\Lambda) & \xrightarrow{\nu} & \text{mod} \tilde{\Lambda} \\
\downarrow x^{-1} & & \downarrow \rho \Omega
\end{array}
\]

Given an additive category \( \mathcal{C} \) and a \( G \)-action on \( \mathcal{C} \), the orbit category \( \mathcal{C}/G \) is defined as follows: the objects of \( \mathcal{C}/G \) are all object in \( \mathcal{C} \), and morphism sets are defined by

\[
\text{Hom}_{\mathcal{C}/G}(x, y) = \bigoplus_{g \in G} \text{Hom}_{\mathcal{C}}(x, gy)
\]

for any \( x, y \in \mathcal{C} \). The composition of \( a = (a_g)_{g \in G} \in \text{Hom}_{\mathcal{C}/G}(x, y) \) with \( b = (b_g)_{g \in G} \in \text{Hom}_{\mathcal{C}/G}(y, z) \), where \( a_g \in \text{Hom}_{\mathcal{C}}(x, gy) \) and \( b_g \in \text{Hom}_{\mathcal{C}}(y, gz) \), is given by

\[
ba = \left( \sum_{h \in G} h(b_{h^{-1}g})a_h \right)_{g \in G} \in \bigoplus_{g \in G} \text{Hom}_{\mathcal{C}}(x, gz) = \text{Hom}_{\mathcal{C}/G}(x, z).
\]

We remark that if \( \mathcal{C} \) is self-injective, then the orbit category \( \mathcal{C}/G \) is again self-injective. Notice also that \( \mathcal{C}/G \) is not necessarily idempotent complete, although the covering functor \( F : \mathcal{C} \to \mathcal{C}/G \) induces a bijection between \( \text{ind}(\mathcal{C})/G \) and \( \text{ind}(\mathcal{C}/G) \).

Let \( U \) be a full subcategory of an additive category \( \mathcal{C} \). The additive closure \( \text{add}U \) of \( U \) is the smallest full subcategory of \( \mathcal{C} \) which contains \( U \) and is closed under direct sums and direct summands.

The subcategory \( U \) is contravariantly finite in \( \mathcal{C} \) if every \( x \in \mathcal{C} \) has a right \( U \)-approximation, that is, a morphism \( f : y \to x \) with \( y \in U \) such that \( \text{Hom}_{\mathcal{C}}(u, f) : \text{Hom}_{\mathcal{C}}(u, y) \to \text{Hom}_{\mathcal{C}}(u, x) \) is surjective for all \( u \in U \). Dually, the concept of a covariantly finite subcategory is defined. A subcategory that is both contravariantly and covariantly finite in \( \mathcal{C} \) is said to be functorially finite in \( \mathcal{C} \).

3. Proofs of our results

The proof of Theorem 2.14 is the main objective of this section. Once in place, the remaining results in Section 2 follow relatively straightforwardly.

3.1. Lemmata. In this subsection, we collect some fundamental facts about orbit categories that will be used in the proofs of the theorems in Section 2. While some of these results have already appeared in the literature in some form (e.g., in \([7, 14, 25]\)), we include the results as well as their proofs here, for the convenience of the reader.

**Lemma 3.1.** Let \( \mathcal{C} \) be a \( k \)-linear Krull–Schmidt category.

(a) Let \( G \) be a group acting admissibly on \( \mathcal{C} \). Then the orbit category \( \mathcal{C}/G \) is locally bounded if and only if \( \mathcal{C} \) is locally bounded.

(b) Let \( \mathcal{C} \) be a locally bounded \( k \)-linear category, and \( U \) a full subcategory of \( \text{mod}\mathcal{C} \) which is a generator. Then the following conditions are equivalent.

(i) \( U \) is locally bounded.

(ii) \( U \) is locally bounded and functorially finite in \( \text{mod}\mathcal{C} \).

(iii) the set \( \{ U \in \text{ind}U \mid U(x) \neq 0 \} \) is finite for all \( x \in \text{ind}\mathcal{C} \).

(iv) The full subcategory \( U \) of \( \text{mod}\mathcal{C} \) is locally bounded and contravariantly finite.

***Proof.*** (a) Recall first that if \( x, y \in \text{ind}\mathcal{C} \) are isomorphic in \( \mathcal{C}/G \) if and only if \( y \simeq gx \) for some \( g \in G \). Since \( G \) acts freely on \( \text{ind}\mathcal{C} \), it follows that, for any \( x \in \text{ind}\mathcal{C} \),

\[
\sum_{y \in \text{ind}(\mathcal{C}/G)} \dim \text{Hom}_{\mathcal{C}/G}(x, y) = \sum_{y \in \text{ind}(\mathcal{C}/G)} \dim \left( \bigoplus_{g \in G} \text{Hom}_{\mathcal{C}}(x, gy) \right) = \sum_{y \in \text{ind}\mathcal{C}} \dim \text{Hom}_{\mathcal{C}}(x, y)
\]
and similarly that
\[ \sum_{y \in \text{ind}(C/G)} \dim \text{Hom}_{C/G}(y, x) = \sum_{y \in \text{ind} C} \dim \text{Hom}_C(y, x). \]

Hence \( C/G \) is locally bounded if and only if \( C \) is locally bounded.

(b) The implications (ii)\( \Rightarrow \) (i) and (ii)\( \Rightarrow \) (iv) are clear.

(i)\( \Rightarrow \) (ii): Take \( x \in \text{ind} C \) and let \( P = \text{Hom}_C(\cdot, x) \in \text{mod} C \). Then \( \{ U \in \text{ind} \mathcal{U} \mid U(x) \neq 0 \} = \{ U \in \text{ind} \mathcal{U} \mid \text{Hom}_C(P, U) \neq 0 \} \) and, since \( P \in \mathcal{U} \), locally boundedness of \( \mathcal{U} \) implies that this set is finite.

(iii)\( \Rightarrow \) (ii): Let \( X \in \text{mod} C \). The projective cover \( P \to X \) of \( X \) induces a monomorphism \( \text{Hom}_C(X, Y) \to \text{Hom}_C(P, Y) \) for any \( Y \in \text{mod} C \). Now, since the set \( \{ U \in \text{ind} \mathcal{U} \mid \text{Hom}_C(P, U) \neq 0 \} \) is finite, so is \( \{ U \in \text{ind} \mathcal{U} \mid \text{Hom}_C(X, U) \neq 0 \} \). Similarly, one proves that the set \( \{ U \in \text{ind} \mathcal{U} \mid \text{Hom}_C(U, Y) \neq 0 \} \) is finite.

(iv)\( \Rightarrow \) (iii): Consider the following auxiliary condition:

(iii)': The set \( \{ U \in \text{ind} \mathcal{U} \mid (\text{top} U)(x) \neq 0 \} \) is finite for all \( x \in \text{ind} C \).

First, we prove (iv)\( \Rightarrow \) (iii)'. Clearly, \( \mathcal{U} \) is contravariantly finite in \( \text{mod} C \). For \( x \in \text{ind} C \), let \( S_x \) be the simple \( C \)-module supported at \( x \), and \( a : U_x \to S_x \) its right \( \mathcal{U} \)-approximation. Since \( \mathcal{U} \) is locally bounded, it suffices to prove that

\[ \{ U \in \text{ind} \mathcal{U} \mid (\text{top} U)(x) \neq 0 \} \subset \{ U \in \text{ind} \mathcal{U} \mid \text{Hom}_{\mathcal{U}}(U, U_x) \neq 0 \} \cup \{ C(\cdot, x) \}. \]

Clearly \( C(\cdot, x) \) is the unique indecomposable projective \( C \)-module \( U \) satisfying \( (\text{top} U)(x) \neq 0 \). Let \( U \) be an indecomposable non-projective \( C \)-module satisfying \( (\text{top} U)(x) \neq 0 \). Then there exists a surjective morphism \( f : U \to S_x \), and a morphism \( g : U \to U_x \) satisfying \( f = ag \). Since \( U \) is non-projective, \( f \) is non-zero in \( \text{mod} C \), and so is \( g \). Thus \( \text{Hom}_{\mathcal{U}}(U, U_x) \neq 0 \), as desired.

Next, we prove (iii)' \( \Rightarrow \) (iii). Fix \( x \in \text{ind} C \). Since \( C \) is locally bounded, it suffices to prove

\[ \{ U \in \text{ind} \mathcal{U} \mid U(x) \neq 0 \} \subset \bigcup_{y \in \text{ind} C, \text{Hom}_C(x, y) \neq 0} \{ U \in \text{ind} \mathcal{U} \mid (\text{top} U)(y) \neq 0 \}. \]

If \( U \in \text{ind} \mathcal{U} \) satisfies \( U(x) \neq 0 \), then the projective cover \( f : P \to U \) satisfies \( P(x) \neq 0 \). Thus \( P \) has an indecomposable direct summand \( \text{Hom}_C(\cdot, y) \) satisfying \( \text{Hom}_C(x, y) \neq 0 \). Then \( (\text{top} P)(x) = (\text{top} P)(y) \neq 0 \).

We remark that the assumption that \( \mathcal{U} \) is a generator is only used in the proof of the implication (i)\( \Rightarrow \) (iii) above. Indeed, the conditions (ii)–(iv) in Lemma 3.1(b) are equivalent even without this assumption on \( \mathcal{U} \).

**Lemma 3.2.** Let \( \mathcal{U} \) be a \( k \)-linear Krull-Schmidt category, and \( \psi \) and \( \phi \) autoequivalences of \( \mathcal{U} \) such that \( \psi \phi(x) \simeq \phi \psi(x) \) for all \( x \in \text{ind} \mathcal{U} \). If \( \phi \) is admissible, and the number of \( \psi \)-orbits of \( \text{ind} \mathcal{U} \) is finite, then the number of \( \phi \)-orbits of \( \text{ind} \mathcal{U} \) is finite.

**Proof.** Since \( \phi \) is an autoequivalence, and \( \psi \phi(x) \simeq \phi \psi(x) \) for all \( x \in \text{ind} \mathcal{U} \), the action of \( \phi \) induces a permutation \( \sigma \) of the \( \psi \)-orbits of \( \text{ind} \mathcal{U} \): \( \sigma(\psi(x)) = \psi(\phi(x)) \), where \( \phi(x) \) is the \( \psi \)-orbit of \( x \). Since the number of \( \psi \)-orbits is finite, there exists a positive integer \( a \) such that \( \sigma^a = \text{id} \); that is, such that for every \( x \in \text{ind} \mathcal{U} \), there exists a number \( b_x \in \mathbb{Z} \) such that \( \phi^a(x) \simeq \psi^{b_x}(x) \). The number \( b_x \) must be non-zero for every \( x \), otherwise \( \phi^a(x) \simeq x \), contradicting admissibility. Let \( S \) be a set of representatives of the \( \psi \)-orbits of \( \text{ind} \mathcal{U} \), so that \( \text{ind} \mathcal{U} = \{ \psi^i(x) \mid x \in S, i \in \mathbb{Z} \} \). Since \( \phi^a(\psi^i(x)) \simeq \psi^{i + b_x}(x) \) holds for all \( x \in \text{ind} C \) and \( i \in \mathbb{Z} \), it follows that the finite set \( \{ \psi^i(x) \mid x \in S, 0 \leq i < |b_x| \} \) meets all \( \phi \)-orbits in \( \text{ind} \mathcal{U} \).

Under a few additional assumptions, one can show that there exist non-zero \( a, b \in \mathbb{Z} \) such that \( \phi^a(x) \simeq \psi^b(x) \) for all \( x \in \text{ind} \mathcal{U} \). We record it here though we do not use in this paper.

**Remark 3.3.** Suppose, in addition to the assumptions in Lemma 3.2, that \( \psi \) is admissible, that the orbit category \( \mathcal{U}/\psi \) is indecomposable and that for every \( x \in \text{ind} \mathcal{U} \) there exist only finitely many \( y \in \text{ind} \mathcal{U} \) such that \( \text{Hom}_U(x, y) \neq 0 \). Then there exist non-zero \( a, b \in \mathbb{Z} \) such that \( \phi^a(x) \simeq \psi^b(x) \) for all \( x \in \text{ind} \mathcal{U} \).
Proof. We use the same notation as in the proof of Lemma 3.2. It suffices to show that $b_x$ is the same for all $x \in \text{ind}U$. Assume that $x, y \in \text{ind}U$ and $\text{Hom}_U(x, y) \neq 0$. Then, for all $i \in \mathbb{Z}$,

$$0 \neq \text{Hom}_U(x, y) \simeq \text{Hom}_U(\phi^{ai}x, \phi^{ai}y) \simeq \text{Hom}_U(\psi^{b_i}x, \psi^{b_i}y) \simeq \text{Hom}_U(x, \psi^{(b_x-b_y)i}(y))$$

holds. Admissibility of $\psi$ together with the finiteness assumption implies $b_x = b_y$. Since $U/\psi$ is indecomposable, we have desired the assertion. \qed

Lemma 3.4. Let $\Lambda$ be a finite-dimensional $k$-algebra. For any automorphism $\phi$ of $\tilde{\Lambda}$, the following conditions are equivalent.

(a) the automorphism $\phi$ of $\tilde{\Lambda}$ is admissible;
(b) the category $\tilde{\Lambda}/\phi$ is Hom-finite;
(c) the set $\text{ind}(\tilde{\Lambda}/\phi)$ is finite.

Proof. (a)⇒(b): Immediate from Lemma 3.1(a).

(b)⇒(a): Let $x \in \text{ind}\tilde{\Lambda}$. If $\phi^i x \simeq x$ for some $i$, then $\text{End}_{\tilde{\Lambda}/\phi}(x) = \bigoplus_{i \in \mathbb{Z}} \text{Hom}_{\tilde{\Lambda}}(x, \phi^i x)$ is infinite dimensional, a contradiction.

(a)⇒(c): The $\tilde{\nu}$-orbits of $\text{ind}\tilde{\Lambda}$ are indexed by the indecomposable projective $T(\Lambda)$-modules, hence the number of orbits is finite. Moreover, we have $\tilde{\nu} \phi \simeq \phi \tilde{\nu}$ by (2.6) and since $\phi$ is admissible, Lemma 3.2 implies that the number of $\phi$-orbits of $\text{ind}\tilde{\Lambda}$ is finite, i.e., $\text{ind}(\tilde{\Lambda}/\phi)$ is finite.

(c)⇒(a): We may assume that $\Lambda$ is indecomposable as a ring. Then the category $\tilde{\Lambda}$ is indecomposable. Now if $f : x \to y$ is a non-zero morphism between two indecomposable objects in $\tilde{\Lambda}$, then $\phi^i(f) \in \text{Hom}_{\tilde{\Lambda}}(\phi^i x, \phi^i y)$ is non-zero for all $i \in \mathbb{Z}$. Since $\tilde{\Lambda}$ is locally bounded, it follows that the $\phi$-orbit of $y$ is finite if and only if so is the $\phi$-orbit of $x$. Since $\tilde{\Lambda}$ is indecomposable, either all $\phi$-orbits are finite, or all $\phi$-orbits are infinite. The assumption that $\text{ind}(\tilde{\Lambda}/\phi) = (\text{ind}\tilde{\Lambda})/\phi$ is finite implies that the orbits must be infinite, from which follows that $\phi$ is admissible. \qed

A full subcategory $U$ of $\text{mod}C$ is called $d$-rigid if $\text{Ext}^i_C(X, Y) = 0$ for all $i \in \{1, \ldots, d - 1\}$ and $X, Y \in U$. By definition, any $d$-cluster-tilting subcategory of $\text{mod}C$ is $d$-rigid. Denote by $(\text{mod}C)/G$ the orbit category of the $G$-action on $\text{mod}C$ given by $g \mapsto g_*$. A functor $F : C \to D$ between additive categories $C$ and $D$ is said to be an equivalence up to summands if it is full and faithful, and satisfies $\text{add}F(C) = D$. The natural functor $C \to \text{proj}C, x \mapsto \text{Hom}_C(\cdot, x)$ is an equivalence up to summands for any additive category $C$. It is an equivalence of categories if and only if $C$ is idempotent complete.

Lemma 3.5. Let $C$ be a locally bounded $k$-linear Krull–Schmidt category, and $G$ a group acting on $C$.

(a) The push-down functor $F_*$ induces functors $(\text{proj}C)/G \to \text{proj}(C/G)$ and $(\text{inj}C)/G \to \text{inj}(C/G)$, which are equivalences up to summands.
(b) For all $i \geq 0$ and $X, Y \in \text{mod}C$, there is a functorial isomorphism

$$\text{Ext}^i_{C/G}(F_*(X), F_*(Y)) \simeq \bigoplus_{g \in G} \text{Ext}^i_C(X, g_*Y).$$

(c) The functor $F_* : \text{mod}C \to \text{mod}(C/G)$ induces a fully faithful functor $F_* : (\text{mod}C)/G \to \text{mod}(C/G)$.
(d) Let $U \subset \text{mod}C$ be a $G$-equivariant full subcategory. Then $U$ is $d$-rigid if and only if $F_*(U) \subset \text{mod}(C/G)$ is $d$-rigid.

Proof. In view of the equivalence of categories $\text{proj}C \simeq C$, and the compatibility of this equivalence with the $G$-actions on $\text{proj}C$ and $C$ respectively, we have that

$$(\text{proj}C)/G \simeq C/G \to \text{proj}(C/G)$$

is an equivalence up to summands. The equivalence up to summands $(\text{inj}C)/G \to \text{inj}(C/G)$ follows similarly from the equivalence $\text{inj}C \simeq C$. This proves the statement (a).
For the remaining statements, it suffices to prove (b). It is well-known [14, Section 6] that there is an isomorphism $F^*F_* \simeq \bigoplus_{g \in G} g_*$ of functors $\text{Mod} \mathcal{C} \to \text{Mod} \mathcal{C}$. Thus for any $X, Y \in \text{mod} \mathcal{C}$, we have functorial isomorphisms
\[
\text{Hom}_{\mathcal{C} / G}(F_*(X), F_*(Y)) \simeq \text{Hom}_{\mathcal{C}}(X, F^*F_*(Y)) \simeq \bigoplus_{g \in G} \text{Hom}_{\mathcal{C}}(X, g_*Y),
\]
and so the assertion holds for $i = 0$. Now let $\cdots \xrightarrow{f_3} P_2 \xrightarrow{f_2} P_1 \xrightarrow{f_1} P_0 \to 0$ be a projective resolution of $X$ in $\text{mod} \mathcal{C}$. By (a) and the exactness of the functor $F_*$,
\[
\cdots \xrightarrow{F_*(f_3)} F_*(P_2) \xrightarrow{F_*(f_2)} F_*(P_1) \xrightarrow{F_*(f_1)} F_*(P_0) \to 0
\]
is a projective resolution of $F_*(X)$ in $\text{mod} (\mathcal{C} / G)$. Thus we have
\[
\text{Ext}^1_{\mathcal{C} / G}(F_*(X), F_*(Y)) = \frac{\text{Ker} \text{Hom}_{\mathcal{C} / G}(F_*(f_{i+1}), F_*(Y))}{\text{Im} \text{Hom}_{\mathcal{C} / G}(F_*(f_i), F_*(Y))} = \frac{\bigoplus_{g \in G} \text{Ker} \text{Hom}_{\mathcal{C}}(f_{i+1}, g_*Y)}{\text{Im} \bigoplus_{g \in G} \text{Hom}_{\mathcal{C}}(f_i, g_*Y)} = \bigoplus_{g \in G} \text{Ext}^1_{\mathcal{C}}(X, g_*Y).
\]
\[\square\]

We denote by $\text{rad}_\mathcal{C}$ the Jacobson radical of the category $\mathcal{C}$ (see, e.g., [8, A.3]). In the following lemma, we collect some basic properties of orbit categories; cf. [7, Section 2].

**Lemma 3.6.** Let $\mathcal{C}$ be a locally bounded $k$-linear Krull-Schmidt category, and $G$ a group acting admissibly on $\mathcal{C}$.

(a) We have $\text{rad}_{\mathcal{C} / G} = (\text{rad}_\mathcal{C}) / G$, that is, $\text{rad}_{\mathcal{C} / G}(x, y) = \bigoplus_{g \in G} \text{rad}_\mathcal{C}(x, gy)$ for all $x, y \in \mathcal{C}$. In particular, $\mathcal{C} / G$ is Krull-Schmidt, and $\text{End}_{\mathcal{C} / G}(x) / \text{rad}_{\mathcal{C} / G}(x, x) = \text{End}_{\mathcal{C}}(x) / \text{rad}_{\mathcal{C}}(x, x)$ for all $x \in \text{ind} \mathcal{C}$.

(b) Let $X \in \text{mod} \mathcal{C}$ be indecomposable. If $g_*X \neq X$ for all $g \in G \setminus \{1\}$, then $F_*(X) \in \text{mod} (\mathcal{C} / G)$ is indecomposable.

(c) For any $X \in \text{mod} \mathcal{C}$, the subgroup $G_X = \{g \in G \mid g_*X \simeq X\}$ of $G$ is finite. In particular, if $G$ is torsion free, then $G$ acts admissibly on $\text{mod} \mathcal{C}$.

(e) If $G$ acts admissibly on $\text{mod} \mathcal{C}$, then the push-down functor $F_* : \text{mod} \mathcal{C} \to \text{mod} (\mathcal{C} / G)$ preserves indecomposability.

**Proof.** (a) Set $I = (\text{rad}_\mathcal{C}) / G$. First, since $\mathcal{C}$ is locally bounded, any $x \in \mathcal{C}$ satisfies $\text{rad}_\mathcal{C}^n(x, -) = 0$ for some $n > 0$. Thus the ideal $I(x, x) = \bigoplus_{g \in G} \text{rad}_\mathcal{C}(x, gx)$ of $\text{End}_{\mathcal{C} / G}(x)$ is nilpotent, and hence $I$ is contained in $\text{rad}_{\mathcal{C} / G}$. It remains to show that the factor category $(\mathcal{C} / G) / I$ is semisimple. Setting $\overline{\mathcal{C}} = \mathcal{C} / \text{rad}_{\mathcal{C}}$, it is clear that $(\mathcal{C} / G) / I = \overline{\mathcal{C}} / G$. Since the natural functor $\mathcal{C} \to \overline{\mathcal{C}}$ gives a bijection $\text{ind} \mathcal{C} \simeq \text{ind} \overline{\mathcal{C}}$, it follows that $G$ acts on $\overline{\mathcal{C}}$ admissibly. From the semisimplicity of $\overline{\mathcal{C}}$ it follows that $\text{End}_{\overline{\mathcal{C}} / G}(x) \simeq \text{End}_{\mathcal{C}}(x)$ and $\text{Hom}_{\overline{\mathcal{C}} / G}(x, y) = 0$ hold for all non-isomorphic $x, y \in \text{ind} \overline{\mathcal{C}}$. Thus, $\overline{\mathcal{C}} / G$ is semisimple.

(b) Assume that $a_b \in \text{Hom}_\mathcal{C}(x, hy)$ is a retraction. Let $s : hy \to x$ be such that $a_b s = 1_{hy} \in \text{End}_\mathcal{C}(hy)$, and define $b = (b_g)_{g \in G} \in \text{End}_{\mathcal{C} / G}(y, x)$ by $b_{h^{-1}} = h^{-1}(s) \in \text{Hom}_\mathcal{C}(y, h^{-1}x)$ and $b_y = 0$ for $g \neq h^{-1}$. Hence $ab \in \text{End}_{\mathcal{C} / G}(y)$ satisfies $(ab)_y = 1_{hy}$, while for $g \neq 1$ we have $y \neq gy$, implying $(ab)_g \in \text{Hom}_\mathcal{C}(y, gy) = \text{rad}_\mathcal{C}(y, gy)$. Consequently, $ab - 1_{hy} \in \text{rad}_{\mathcal{C} / G}(y, x)$ by (a), so $ab \in \text{End}_{\mathcal{C} / G}(y)$ is invertible. Hence $a$ is a retraction.

(c) First, observe that from the assumptions follow that $G$ acts admissibly on $\mathcal{C}' = \{g_*X \mid g \in G\}$. Using Lemma 3.5(c) and applying the second identity in (a) to $\mathcal{C}'$, $x = X$, we get
\[
\text{End}_{\mathcal{C} / G}(F_*(X)) = \text{End}_{\text{mod} \mathcal{C} / G}(X) = \text{End}_{\mathcal{C}}(X),
\]
which is a division algebra, since $\text{End}_{\mathcal{C}}(X)$ is local. Hence, $\text{End}_{\mathcal{C} / G}(F_*(X))$ is local, so $F_*(X)$ is indecomposable.
(d) Since the group $G_X$ acts admissibly on the support $\text{Supp} X$ of $X$, which is a finite subset of $\text{ind} C$, it follows that $|G_X| \leq |\text{Supp} X|$.

(e) The first assertion follows from (c), and the second one follows from (d). \hfill \Box

**Lemma 3.7.** Let $C$ be a locally bounded $k$-linear Krull-Schmidt category, and $G$ a group acting on $C$. If $\mathcal{U} \subset \text{mod} C$ is a full subcategory such that $F_\ast(\mathcal{U})$ is functorially finite in $\text{mod}(C/G)$, then $\mathcal{U}$ is functorially finite in $\text{mod} C$.

**Proof.** We only prove that $\mathcal{U} \subset \text{mod} C$ is covariantly finite. Observe that, by Lemma 3.5(c), the image in $\text{mod}(C/G)$ of the push-down functor is equivalent to the orbit category $(\text{mod} C)/G$. Set $V = F_\ast(\mathcal{U}) \simeq \mathcal{U}/G$.

Fix $X \in \text{mod} C$. By our assumptions, there exists a left $V$-approximation $a : F_\ast(X) \rightarrow V$ in $\text{mod}(C/G)$, and $V = F_a(W)$ for some $W \in \mathcal{U}$. Viewing $a$ as a morphism in $(\text{mod} C)/G$, we write $a = (a_g)_{g \in G}$, where $a_g \in \text{Hom}_C(X, gW)$. Then the set $I = \{g \in G \mid a_g \neq 0\}$ is finite. Set $U = \bigoplus_{g \in I} gW \in \text{mod} C$ and $b = (a_g)_{g \in I} \in \text{Hom}_C(X, U)$. We show that $b$ is a left $\mathcal{U}$-approximation in $\text{mod} C$.

Take any $c \in \text{Hom}_C(X, Y)$ with $Y \in \mathcal{U}$. For $F_\ast(c) \in \text{Hom}_{C/G}(F_\ast(X), F_\ast(Y))$, there exists $d \in \text{Hom}_{C/G}(V, F_\ast(Y))$ such that $F_\ast(c) = da$, since $a$ is a left $V$-approximation in $\text{mod}(C/G)$.

Write $d = (d_g)_{g \in G}$, where $d_g \in \text{Hom}_C(W, gY)$. Then we have

$$c = \sum_{h \in G} h_\ast(d_{h^{-1}})ah = \sum_{h \in I} h_\ast(d_{h^{-1}})ah,$$

which shows that $c = (h_\ast(d_{h^{-1}}))_{h \in I} \in \text{Hom}_C(U, Y)$ satisfies $c = eb$. Thus the assertion follows. \hfill \Box

**Proof of Remark 2.4.** Let $\mathcal{U}' \subset \text{mod} \hat{\Lambda}$ be the preimage of $\mathcal{U}$ under the natural functor $\hat{\Lambda} \to \text{mod} \hat{\Lambda}$. From Lemma 3.5(c) follows that $\mathcal{U}'/\phi_\ast \simeq F_\ast(\mathcal{U}')$. As the $\phi$-action on $\text{mod} \hat{\Lambda}$ is admissible by Lemma 3.6(d), we have

$$\mathcal{U} \text{ is locally bounded} \iff \mathcal{U}' \text{ is locally bounded} \quad \text{by Lemma 3.1(b)},$$

$$\mathcal{U}'/\phi_\ast \simeq F_\ast(\mathcal{U}') \text{ is locally bounded} \quad \text{by Lemma 3.1(a)},$$

$$\text{ind}(\mathcal{U}/\phi_\ast) \text{ is finite} \quad \text{by Lemma 3.1(b)},$$

$$\text{ind}(\mathcal{U}/\phi_\ast) \text{ is finite} \quad \text{by Lemma 3.4, ind}(\hat{\Lambda}/\phi) \text{ is finite}. \quad \Box$$

3.2. **Proof of Theorem 2.14 and Corollary 2.15.** First we prove the ‘if’ part of Theorem 2.14(a).

**Proof of ‘if’ part of Theorem 2.14(a).** It follows from Lemma 3.5(d) and Lemma 3.7 that $\mathcal{U}$ is $d$-rigid and functorially finite in $\text{mod} C$.

Let $X \in \text{mod} C$. By Lemma 3.5(b) we have, for all $U \in \mathcal{U}$,

$$\text{Ext}^i_{C/G}(F_\ast(X), F_\ast(U)) \simeq \bigoplus_{g \in G} \text{Ext}^i_C(X, gU).$$

Since $g_\ast U \in \mathcal{U}$ for all $g \in G$, it follows that $\text{Ext}^i_C(X, U) = 0$ for all $U \in \mathcal{U}$ and $i \in \{1, \ldots, d-1\}$ if and only if $\text{Ext}^i_{C/G}(F_\ast(X), F_\ast(U)) = 0$ for all $U \in \mathcal{U}$ and $i \in \{1, \ldots, d-1\}$. As $F_\ast(\mathcal{U})$ is a $d$-cluster-tilting subcategory of $\text{mod}(C/G)$, this is true precisely when $F_\ast(X) \in F_\ast(\mathcal{U})$, that is, by Lemma 3.5(c), when $X \in \mathcal{U}$.

By a similar argument, one can show that $\text{Ext}^i_C(U, X) = 0$ for all $U \in \mathcal{U}$ and $i \in \{1, \ldots, d-1\}$ if and only if $X$ belongs to $\mathcal{U}$. \hfill \Box

For the proof of the rest of Theorem 2.14 we shall need a result, Proposition 3.9 below, which characterizes $d$-cluster-tilting subcategories of $\text{mod} C$ in terms of the existence of so-called $d$-almost split sequences. This is a generalization of [42, Proposition 2.4], which deals with the case of a finite-dimensional $k$-algebra.
Recall that a Hom-finite k-linear Krull-Schmidt category C is called a dualizing k-variet[y [11] if the functors $D : \text{Mod} C \to \text{Mod}(C^{\text{op}})$ and $D : \text{mod}(C^{\text{op}}) \to \text{mod} C$ induce dualities
\begin{align*}
D : \text{mod} C &\to \text{mod}(C^{\text{op}}) \quad \text{and} \quad D : \text{mod}(C^{\text{op}}) \to \text{mod} C
\end{align*}
respectively. In this case, mod $C$ is an abelian subcategory of Mod $C$ which is closed under kernels, cokernels and extensions, and has enough projective objects and injective objects. Moreover, all simple objects in Mod $C$ and Mod($C^{\text{op}}$) are finitely presented. This implies that for every indecomposable object $x \in C$ there exists a right almost split morphism $f : y \to x$, and a left almost split morphism $g : x \to z$. A morphism $f \in \text{Hom}_C(y, x)$ is said to be right almost split if it is not a retraction, and any non-retraction $g \in \text{Hom}_C(z, x)$ factors through $f$. Left almost split morphisms are defined dually.

For example, by (2.7) in Section 2.4, any locally bounded k-linear category is a dualizing k-variet[y. If C is a dualizing k-variet[y then so is mod $C$ [11, Proposition 2.6]. A k-linear triangulated category is a dualizing k-variet[y if and only if it has a Serre functor [45, Proposition 2.11]. Furthermore, any functorially finite subcategory of a dualizing k-variet[y is again a dualizing k-variet[y [40, Proposition 1.2].

**Proposition 3.8.** Let $C$ be a dualizing k-variet[y, and $\mathcal{U}$ a functorially finite subcategory of $\text{mod} C$.

(a) The category $\mathcal{U}$ is a dualizing k-variet[y; in particular, $\text{mod} \mathcal{U}$ is abelian.

(b) The following identities hold:
\[ \text{gl.dim}(\text{mod} \mathcal{U}) = \sup \{ \text{inj.dim} S \mid S \text{ is a simple } \mathcal{U}\text{-module} \} \]
\[ = \sup \{ \text{proj.dim} S \mid S \text{ is a simple } \mathcal{U}\text{-module} \} \]

**Proof.** (a) The assertion follows from the remarks in the paragraph preceding the proposition.

(b) Since any object $M \in \text{mod} \mathcal{U}$ has a minimal projective (respectively, injective) resolution in mod $\mathcal{U}$, we have that $\text{proj.dim} M \leq \ell$ (respectively, $\text{inj.dim} M \leq \ell$) if and only if $\text{Ext}^{\ell+1}_\mathcal{U} (M, S) = 0$ (respectively, $\text{Ext}^{\ell+1}_\mathcal{U} (M, S) = 0$) for any simple $\mathcal{U}$-module $S$. Thus $\text{gl.dim(\text{mod} \mathcal{U})}$ is the supremum of $\text{inj.dim} S$ (respectively, $\text{proj.dim} S$) for simple $\mathcal{U}$-modules $S$. □

The following result is a category version of [42, Proposition 2.4] (see also [40, Theorem 5.1(3)]).

**Proposition 3.9.** Let $C$ be a dualizing k-variet[y, and $\mathcal{U}$ an additively closed functorially finite d-rigid subcategory of $\text{mod} C$ that is a generator-cogenerator. The following conditions are equivalent.

(a) $\mathcal{U}$ is a d-cluster-tilting subcategory of $\text{mod} C$.

(b) For any $X \in \text{mod} C$, there exists an exact sequence
\[ 0 \to M_d \to \cdots \to M_2 \to M_1 \to X \to 0 \]
with $M_i \in \mathcal{U}$.

(c) For any indecomposable object $X \in \mathcal{U}$, there exists an exact sequence
\[ 0 \to M_d \to \cdots \to M_1 \to M_0 \xrightarrow{f} X \]
with $M_i \in \mathcal{U}$ and $f$ right almost split in $\mathcal{U}$.

(d) $\text{gl.dim(\text{mod} \mathcal{U})} \leq d + 1$.

**Proof.** (a)$\Rightarrow$(b) This is shown in [39, Proposition 2.2.2(1)$\Rightarrow$(2-0)].

(b)$\Rightarrow$(c) Let $X \in \mathcal{U}$ be an indecomposable object. Since $\mathcal{U}$ is a dualizing k-variet[y, there exists a right almost split morphism $f_0 : M_0 \to X$ in $\mathcal{U}$. Let $g : K \to M_0$ be kernel of $f_0$ in $\text{mod} C$. By our assumption (b), there exists an exact sequence $0 \to M_d \xrightarrow{f_d} \cdots \xrightarrow{f_2} M_2 \xrightarrow{f_1} M_1 \xrightarrow{b} K \to 0$ with $M_i \in \mathcal{U}$. Let $f_1 = gh : M_1 \to M_0$, then the sequence
\[ 0 \to M_d \xrightarrow{f_d} \cdots \xrightarrow{f_2} M_2 \xrightarrow{f_1} M_1 \xrightarrow{f_0} M_0 \xrightarrow{f_0} X \]
has the desired properties.

(c)$\Rightarrow$(d) Any simple $\mathcal{U}$-module $S$ can be written as $S = \text{Hom}_\mathcal{U}(-, X)/\text{rad}_\mathcal{U}(-, X)$ for some indecomposable object $X \in \mathcal{U}$. Applying the Hom-functor to the sequence given in the condition (c), and using that $\mathcal{U}$ is d-rigid, we get an exact sequence
\[ 0 \to \text{Hom}_\mathcal{U}(-, M_d) \to \cdots \to \text{Hom}_\mathcal{U}(-, M_0) \to \text{Hom}_\mathcal{U}(-, X) \to S \to 0. \]
So $S$ has projective dimension at most $d+1$, whence Proposition 3.8(b) shows that $\text{gl.dim}(\mod\mathcal{U}) \leq d + 1$. 

(d)$\Rightarrow$(b) Fix $X \in \mod\mathcal{C}$. Since $\mathcal{C}$ is a dualizing $k$-variety, we can take an injective copresentation $0 \to X \to I^0 \to I^1$ in $\mod\mathcal{C}$. This gives rise to an exact sequence 

$$0 \to \text{Hom}_C(-, X)|_{\mathcal{U}} \to \text{Hom}_\mathcal{U}(-, I^0) \to \text{Hom}_\mathcal{U}(-, I^1)$$

in $\mod\mathcal{U}$. Since $\mathcal{U}$ is a generator-cogenerator, $I^i$ belongs to $\mathcal{U}$ for $i = 0, 1$. Therefore the $\mathcal{U}$-modules $\text{Hom}_\mathcal{U}(-, I^i)$ are projective. Since $\text{gl.dim}(\mod\mathcal{U}) \leq d + 1$, the projective dimension of $\text{Hom}_\mathcal{C}(-, X)|_{\mathcal{U}}$ is at most $d - 1$. Taking a projective resolution

$$0 \to \text{Hom}_\mathcal{U}(-, M_d) \to \cdots \to \text{Hom}_\mathcal{U}(-, M_1) \to \text{Hom}_\mathcal{C}(-, X)|_{\mathcal{U}} \to 0,$$

of $\text{Hom}_\mathcal{C}(-, X)|_{\mathcal{U}}$, Yoneda’s Lemma gives us the desired sequence. \hfill $\Box$

Now we are ready to prove Theorem 2.14(a).

**Proof of the ‘only if’ part of Theorem 2.14(a).** Let $\mathcal{V} = F_*(\mathcal{U})$. It follows from Lemma 3.5(a) and (d) that $\mathcal{V}$ is a $d$-rigid generator-cogenerator and, by assumption, $\mathcal{V}$ is functorially finite in $\mod(C/G)$.

We shall show that $\mathcal{V}$ satisfies the condition (c) of Proposition 3.9. Since $G$ acts admissibly on $\mod\mathcal{C}$, Lemma 3.6(e) gives that the push-down functor $F_*$ preserves indecomposability, implying that $\mathcal{V} = \text{add}\mathcal{V}$. Moreover, any indecomposable object in $\mathcal{V}$ is isomorphic to $F_*(U)$ for some indecomposable object $U \in \mathcal{U}$. Let $f_0 : U_0 \to U$ be a right almost split map in $\mathcal{U}$, and $g : K \to U_0$ its kernel in $\mod\mathcal{C}$. Applying Proposition 3.9(b) to $\mathcal{U}$, we get an exact sequence

$$0 \to U_d \xrightarrow{f_d} \cdots \xrightarrow{f_2} U_1 \xrightarrow{b} K \to 0$$

in $\mod\mathcal{C}$ with $U_i \in \mathcal{U}$. Setting $f_1 = gh$, we obtain an exact sequence

$$0 \to U_d \xrightarrow{f_d} \cdots \xrightarrow{f_2} U_1 \xrightarrow{f_1} U_0 \xrightarrow{b} U.$$

Applying $F_*$ gives an exact sequence

$$0 \to F_*(U_d) \xrightarrow{F_*(f_d)} \cdots \xrightarrow{F_*(f_2)} F_*(U_1) \xrightarrow{F_*(f_1)} F_*(U_0) \xrightarrow{F_*(b)} F_*(U)$$

in $\mod(C/G)$, with $F_*(U_i) \in \mathcal{V}$ for all $i$.

It now suffices to show that the morphism $F_*(f_0) : F_*(U_0) \to F_*(U)$ is right almost split in $\mathcal{V}$. Let $a : F_*(X) \to F_*(U)$ be a morphism in $\mathcal{V}$, that is not a retraction. By Lemma 3.5(c), we may view $a$ as a morphism in the orbit category $(\mod\mathcal{C})/G$, whence $a = (a_g)_{g \in G}$, with $a_g \in \text{Hom}_C(X, g_*U)$. From Lemma 3.6(b) follows that $a_g$ is not a retraction for any $g \in G$. Since the map $g_* : U_0 \to g_*U$ is right almost split in $\mathcal{U}$, there exist $b_g : X \to g_*U_0$ such that $a_g = g_*b_g$ for all $g \in G$. Now $a = F_*(f_0)b$ holds, where $b = (b_g)_{g \in G} \in \text{Hom}_{C/G}(F_*(X), F_*(U_0))$. Hence $F_*(f_0)$ is a right almost split map in $\mathcal{V}$. This concludes the proof of Theorem 2.14(a). \hfill $\Box$

We turn now to the proof of Theorem 2.14(b). For this, the following result by Amiot and Oppermann will be needed.

**Lemma 3.10.** [3, Corollary 4.5] Let $A$ be a finitely generated $\mathbb{Z}$-graded algebra over an algebraically closed field $k$, and $M$ a finite-dimensional $A$-module such that $\text{Ext}_A^1(M, M) = 0$. Then there exists a $\mathbb{Z}$-graded $A$-module $N$ which is isomorphic to $M$ as an ungraded $A$-module.

From Lemma 3.10 we deduce the following proposition, which plays an important role in the proof of Theorem 2.14(b).

**Proposition 3.11.** Let $\mathcal{C}$ be a locally bounded $k$-linear Krull-Schmidt category, where $k$ is algebraically closed, and $G$ a finitely generated free abelian group, acting admissibly on $\mathcal{C}$. Then every module $M \in \mod(C/G)$ satisfying $\text{Ext}_{C/G}^1(M, M) = 0$ is in the essential image of the push-down functor $F_* : \mod\mathcal{C} \to \mod(C/G)$.
Proof. (i) First we prove the statement for the case $G = \langle \phi \rangle \simeq \mathbb{Z}$.

Let $M \in \text{mod}(\mathcal{C}/G)$ be a module satisfying $\text{Ext}_{\mathcal{C}/G}^1(M, M) = 0$. Since $\mathcal{C}$ is locally bounded, the support $\text{Supp} M$ of $M$ is finite. Replacing $\mathcal{C}$ by the full subcategory $\text{add}\{\phi^i x \mid i \in \mathbb{Z}, x \in \text{Supp } M\} \subseteq \mathcal{C}$, we may assume that $(\text{ind } \mathcal{C})/G$ is a finite set. Let $S$ be a complete set of representatives of the $G$-orbits of $\text{ind } \mathcal{C}$, and set $A = \text{End}_{\mathcal{C}/G}(y)$ for $y := \bigoplus_{x \in S} x$. Then $A$ is a finite-dimensional $\mathbb{Z}$-graded $k$-algebra, with grading defined by $A_i = \text{Hom}_\mathcal{C}(y, \phi^i y)$, $i \in \mathbb{Z}$. This gives us a commuting diagram of functors

$$
\begin{array}{ccc}
\text{mod } \mathcal{C} & \xrightarrow{\sim} & \text{mod}^G A \\
\downarrow F_* & & \downarrow \text{forget} \\
\text{mod}(\mathcal{C}/G) & \xrightarrow{\sim} & \text{mod } A \\
\end{array}
$$

in which the horizontal arrows are equivalences of categories. From Lemma 3.10 it now follows that $M \in \text{mod}(\mathcal{C}/G)$ is in the essential image of the push-down functor $\mathcal{C} \to \mathcal{C}/G$.

(ii) We now prove the statement for general case, by induction on the rank of the group $G$.

Take $\phi \in G$ such that $G' = G/\langle \phi \rangle$ is a free abelian group, and set $\mathcal{C}' = \mathcal{C}/\phi$. Clearly, it follows that $\mathcal{C}/G = \mathcal{C}'/G'$. Now $F : \mathcal{C} \to \mathcal{C}/G$ is the composition of the natural functors $F' : \mathcal{C} \to \mathcal{C}/\phi = \mathcal{C}'$ and $F'' : \mathcal{C}' \to \mathcal{C}'/G' = \mathcal{C}/G$. Since the action of $G$ on $\mathcal{C}$ is admissible, the category $\mathcal{C}'$ is a locally bounded $k$-linear Krull-Schmidt category, and the action of $G'$ on $\mathcal{C}'$ is again admissible. By the induction hypothesis, the module $M$ is in the essential image of $F'' : \text{mod } \mathcal{C}' \to \text{mod}(\mathcal{C}/G)$. Take $N \in \text{mod } \mathcal{C}'$ such that $M \simeq F'' N$. Then $\text{Ext}_{\mathcal{C}/G}^1(N, N) = 0$ by Lemma 3.5(b), and by (i), the module $N$ is contained in the essential image of $F'_* : \text{mod } \mathcal{C} \to \mathcal{C}'$. The assertion follows. \hfill \Box

We now have the tools needed to prove Theorem 2.14(b). The proof is similar to that of [4, Proposition 3.1].

Proof of Theorem 2.14(b). Let $\mathcal{V}$ be a $d$-cluster-tilting subcategory of $\text{mod}(\mathcal{C}/G)$. If $d = 1$ then $\mathcal{V} = \text{mod}(\mathcal{C}/G)$ and hence $F_*^{-1}(\mathcal{V}) = \text{mod } \mathcal{C}$, which is a $1$-cluster-tilting subcategory of itself. Assume that $d > 1$. Then $\mathcal{V}$ is $2$-rigid and thus, by Proposition 3.11, it is contained in the essential image of the push-down functor $F_*$. Thus $\mathcal{U} = F_*^{-1}(\mathcal{V})$ satisfies $F_*(\mathcal{U}) = \mathcal{V}$. Since $G$ acts admissibly on $\text{mod } \mathcal{C}$ by Lemma 3.6(d) and $\mathcal{U}$ is a $G$-equivariant full subcategory of $\text{mod } \mathcal{C}$ such that $F_*(\mathcal{U})$ is a $d$-cluster tilting subcategory of $\text{mod}(\mathcal{C}/G)$, it follows from Theorem 2.14(a) that $\mathcal{U}$ is a $d$-cluster tilting subcategory of $\text{mod } \mathcal{C}$.

We now show how Corollary 2.15 follows from Theorem 2.14.

Proof of Corollary 2.15. (a) By Lemma 3.5(c), any full subcategory $\mathcal{U} \subseteq \text{mod } \mathcal{C}$ satisfies $\mathcal{U}/G \simeq F_*(\mathcal{U}) \subseteq \text{mod}(\mathcal{C}/G)$. So if $\mathcal{U} \subseteq \text{mod } \mathcal{C}$ is a locally bounded $d$-cluster-tilting subcategory, then $F_*(\mathcal{U})$ is locally bounded by Lemma 3.1(a), and thus functorially finite in $\text{mod}(\mathcal{C}/G)$ by Lemma 3.1(b). Theorem 2.14(a) now implies that $F_*(\mathcal{U})$ is a $d$-cluster-tilting subcategory of $\text{mod}(\mathcal{C}/G)$.

(b) Assume that $k$ is algebraically closed. If $\mathcal{V}$ is a locally bounded $d$-cluster-tilting subcategory of $\text{mod}(\mathcal{C}/G)$, then $\mathcal{U} = F_*^{-1}(\mathcal{V})$ is a $G$-equivariant $d$-cluster-tilting subcategory of $\text{mod } \mathcal{C}$ by Theorem 2.14(b). Since $\mathcal{V} = F_*(\mathcal{U}) \simeq \mathcal{U}/G$, Lemma 3.1(a) implies that $\mathcal{U}$ is locally bounded. \hfill \Box

3.3. Proofs of other results. We start by proving Theorem 2.3.

Proof of Theorem 2.3. (a) First, observe that $\mathcal{A}/\phi$ is a finite-dimensional $k$-algebra by Lemma 3.4. Hence, for the first assertion, it suffices to show that $\mathcal{A}/\phi$ is locally $d$-representation-finite. If $\mathcal{U}$ is a locally bounded $\phi$-equivariant $d$-cluster-tilting subcategory of $\mathbb{D}^b(\mathcal{A})$ then, by Lemma 3.1(b)(iv) => (i), the preimage $\mathcal{U}'$ of $\mathcal{U}$ under the natural functor $\text{mod } \mathcal{A} \to \text{mod } \mathcal{A}/\phi \simeq \mathbb{D}^b(\mathcal{A})$ is a locally bounded $\phi$-equivariant $d$-cluster-tilting subcategory of $\text{mod } \mathcal{A}$. Now, Corollary 2.15(a) implies that $F_*(\mathcal{U}') \subseteq \text{mod } (\mathcal{A}/\phi)$ is a locally bounded $d$-cluster-tilting subcategory, so $\mathcal{A}/\phi$ is locally $d$-representation-finite.

The second assertion in Theorem 2.3(a) is immediate from Corollary 2.15 and the equivalence $\mathbb{D}^b(\mathcal{A}) \simeq \text{mod } \mathcal{A}$ (2.1).
(b) Any additive generator of \( F_\ast(U) \subset \mod(\hat{\Lambda}/\phi) \) is a \( d \)-cluster-tilting module of \( \hat{\Lambda}/\phi \), and elements of \( \ind F_\ast(U) \) correspond bijectively to orbits of the \( \phi \)-action on \( \ind U \). □

We now turn to the proofs of the corollaries 2.8, 2.9, 2.11 and 2.12. In relation to Corollary 2.8, observe that \( F \circ \hat{\nu} \simeq \nu_{T_n(\Lambda)} \circ F : \hat{\Lambda} \to \hat{\Lambda}/\hat{\nu}^n = T_n(\Lambda) \), where

\[
F : \hat{\Lambda} \to \hat{\Lambda}/\hat{\nu}^n
\]
is the covering functor. Hence \( F \circ \hat{\nu} \simeq (\nu_{T_n(\Lambda)})_\ast \circ F_* : \mod \hat{\Lambda} \to \mod T_n(\Lambda) \). Again, uniqueness of the Serre functor gives us the following diagram, which is commutative up to isomorphisms of functors:

\[
\begin{array}{ccc}
D^b(\Lambda) & \xrightarrow{\nu} & \mod \hat{\Lambda} \\
\downarrow \hat{\nu} & & \downarrow F_* \\
D^b(\Lambda) & \xrightarrow{\nu} & \mod T_n(\Lambda)
\end{array}
\]

As we shall see, \( d \)-representation-finiteness of \( T_{\ast d}(\Lambda) \) is a consequence of the functorial isomorphisms in (3.2). However, to show that the \( d \)-cluster-tilting module \( U \) in Corollary 2.8 is basic, we need the following technical observation.

**Lemma 3.12.** Let \( I \) be a set, and \( I_+ \) a subset of \( I \).

(a) Assume that \( f \) is a permutation of \( I \), satisfying \( f(I_+) \subset I_+ \), \( I = \bigcup_{i \in \mathbb{Z}} f^i(I_+) \) and \( \emptyset = \bigcap_{i \in \mathbb{Z}} f^i(I_+) \). Then for any \( i \in \mathbb{Z} \), the set \( I(f,i) := f^i(I_+) \setminus f^{i+1}(I_+) \) is a cross-section for the \( f \)-orbits of \( I \).

(b) Assume that \( f \) and \( g \) both satisfy the conditions in (a), and that \( fg = gf \). Then for any \( a, b > 0 \), the set \( \left( \bigcup_{i \leq a} I(f,i) \right) \cup \left( \bigcup_{n \geq i} I(g,n) \right) \) is a cross-section for the \( f^a g^b \)-orbits of \( I \).

**Proof.** (a) It is clear from the assumptions that for each \( x \in I \) there is a unique \( m \in \mathbb{Z} \) such that \( f^i(x) \in I_+ \) if and only if \( j \geq m \). This implies that \( I(f,0) \) is a cross-section for the \( f \)-orbits of \( I \), and therefore the same is true for \( I(f,i) = f^i(I(f,0)) \) for any \( i \in \mathbb{Z} \).

(b) As \( a, b > 0 \), we have \( f^a g^b(I_+) \subset f^a(I_+) \subset I_+ \). Therefore,

\[
\bigcup_{i \geq 0} (f^a g^b)^i(I_+) = \bigcup_{i \geq 0} (f^a g^b)^{-i} (I_+) = \bigcup_{i \geq 0} f^{-in} (g^{-bn}(I_+)) = \bigcup_{i \geq 0} f^{-in} (I_+) = I
\]

and hence \( I = \bigcup_{i \geq 0} (f^a g^b)^i(I_+) \). Similarly,

\[
\bigcap_{i \in \mathbb{Z}} (f^a g^b)^i(I_+) = \bigcap_{i \geq 0} (f^a g^b)^i(I_+) = \bigcap_{i \geq 0} f^{in} (g^{bn}(I_+)) \subset \bigcap_{i \geq 0} f^{in} (I_+) = \emptyset
\]

so \( \emptyset = \bigcap_{i \in \mathbb{Z}} (f^a g^b)^i(I_+) \).

The above shows that the permutation \( f^a g^b \) of \( I \) satisfies the conditions in (a). It follows that \( I(f^a g^b,0) = I_+ \setminus f^a g^b(I_+) \) is a cross-section for the \( f^a g^b \)-orbits of \( I \), and hence, so is

\[
g^{-b} \left( I(f^a g^b,0) \right) \setminus f^a g^b(I_+) = I_+ \setminus f^a g^b(I_+) \cup (g^{-b}(I_+) \setminus I_+) \] □

**Proof of Corollary 2.8.** From the assumptions follows that \( \Lambda \) is \( \nu_\ast \)-finite and \( \text{gl.dim} \Lambda \leq d \). Thus \( U = U_\delta(\Lambda) \) is a \( d \)-cluster-tilting subcategory of \( D^b(\Lambda) \simeq \mod \hat{\Lambda} \) by Proposition 2.5. Moreover, by [43, Theorem 3.1],

\[
\nu(U) = U \quad \text{and} \quad U(d) = U
\]
hold, because \( \Lambda \) is \( d \)-representation-finite and \( \text{gl.dim} \Lambda \leq d \). By (3.2),

\[
(\nu_\ast^d) \simeq ([\nu^{-1}]_\ast^d) = [\nu^{-d}][\nu^{-d}] = \nu^{-d} \nu^{-d} = \nu^{-d} \nu^{-d}\]

and hence, in particular, \( (\nu^d) \ast (U) = U \). Applying Theorem 2.6(a) to \( \phi = \nu^d \), it follows that \( \hat{\Lambda}/\nu^d = T_{\ast d}(\Lambda) \) is \( d \)-representation-finite. This proves the first statement.

For the second statement, we need to calculate a cross-section for the \( (\nu^d) \ast \)-orbits in \( \ind U \). Let \( I = \ind U \) and \( I_+ = \ind \left\{ \nu_\ast^{-i}(\Lambda) \mid i > 0 \right\} \). Then \( \nu^{-1} : I \to I \) and \( \nu : I \to I \) satisfy the assumptions
in Lemma 3.12(a), so $I(ν^{-1}_d, \nu) = \text{ind}(\text{add}\Lambda)$ and $I(ν, 0) = \text{ind}(\text{add}(M/Λ))$ are cross-sections for the $ν^{-1}_d$-action and the $ν$-action on $\text{ind} U$ respectively (cf. [43, Lemma 4.9]). As $\hat{ν}^d_*$, $ν^d_{\nu}$ by (3.3), Lemma 3.12(b) implies that
\[
\left( \bigsqcup_{0 \leq i < ℓ(d+1)} \text{ind} \left( \text{add} \left( ν^i(M/Λ) \right) \right) \right) \sqcup \left( \bigsqcup_{0 \leq j < ℓ} \text{ind} \left( \text{add} \left( ν^j_*(Λ) \right) \right) \right)
\]
is a cross-section for the orbits of the $\hat{ν}^d_*$-action on $\text{ind} U$. By (3.2), the image of this cross-section under the equivalence $D^b(Λ) \simeq \text{mod} \Lambda$ is isomorphic to
\[
S = \left( \bigsqcup_{0 \leq i < ℓ(d+1)} \text{ind} \left( (\hat{ν}_*Ω)^i(M/Λ) \right) \right) \sqcup \left( \bigsqcup_{0 \leq j < ℓ} \text{ind} \left( (\hat{ν}_*Ω^{d+1})^j(Λ) \right) \right)
\]
It follows that
\[
T_{dℓ}(Λ) \oplus \bigoplus_{V \in S} F_{ν}(V) \simeq T_{dℓ}(Λ) \oplus \left( \bigoplus_{i=0}^{ℓ(d+1)-1} \left( ν^{(d+1)}_*(Λ), ν^i(M/Λ) \right) \right) \oplus \left( \bigoplus_{j=0}^{ℓ-1} \left( ν^{(d+1)}_{d}(Λ), ν^{d+1}_j(Λ) \right) \right)
\]
is a basic $d$-cluster-tilting $T_{dℓ}(Λ)$-module. □

Proof of Corollary 2.9. The twisted $(b/a)$-Calabi–Yau property means that $ν^a \simeq [b]_φ : D^b(Λ) \to D^b(Λ)$ for some $φ ∈ \text{Aut}(Λ)$, and hence $ν^a_d \simeq [b − da]_φ ∈ φ_*$. By [33, Proposition 2.1.10(c)], $b/a < \text{gl.dim} Λ$ and since $\text{gl.dim} Λ ≤ d$, we get $b − da < 0$. Clearly, $φ_*(D^n(Λ)) = D^n(Λ)$, so it follows that $ν^a_d(D^n(Λ)) \subset D^n(Λ)$. This implies that the algebra $Λ$ is $ν_4$-finite.

Recall that $g = \text{gcd}(d + 1, a + b)$. Setting $p = (d + 1)/g$ and $q = (a + b)/g$, we get the following identities:
\[
(3.4) \quad ap − q = \frac{a(d + 1)}{g} − \frac{a + b}{g} = \frac{ad − b}{g},
\]
\[
(3.5) \quad p(a + b) = q(d + 1).
\]
By (2.9), there are functorial isomorphisms $\hat{ν}_*^a \simeq ν^a \circ [a] \simeq [b]_φ$, $φ_* \simeq [a + b]_φ$, and hence
\[
\hat{ν}_*^{ap} = ([a + b]_φ)_ψ \simeq [p(a + b)]_φ \circ φ_*(\hat{ν}_*^p) \simeq [q(d + 1)]_φ \circ φ_*(\hat{ν}_*^p)
\]
which, in turn, gives
\[
\hat{ν}_*^a = \hat{ν}_*^{ap} \circ φ_*(\hat{ν}_*^p) \simeq \hat{ν}_*^{ap} \circ \hat{ν}_*^{−ℓq} \circ [ℓq(d + 1)]_φ \circ φ_*(\hat{ν}_*^p) \simeq ν^{−ℓq} \circ φ_*(\hat{ν}_*^p).
\]
The subcategory $U_d(Λ) \subset D^b(Λ)$ is invariant under $ν_d$ and $φ_*$, and hence under $\hat{ν}_*$. By Theorem 2.6, this implies that $T_{dℓ}(Λ)$ is $d$-representation-finite. Since $φ_*(Λ) = Λ$, it also means that $\text{ind} \{ν^i_*(Λ) | 0 ≤ i ≤ ℓq\}$ is a cross-section for the $\hat{ν}_*^a$-orbits of $U_d(Λ)$. The identity (2.5) follows. □

Proof of Corollary 2.11. Since $Λ$ is $r$-homogeneous, $ν \simeq ν^{−r}_d φ_*$ for some $φ ∈ \text{Aut}(Λ)$ [31, Theorem 1.3]. Hence, $ν^{−r}_d \simeq ν^{−r}[(r − 1)d]_φ$, and thus
\[
(3.6) \quad \hat{ν}_*^{−1} \simeq ν^{−1}[r − 1] \simeq ν^{−1}[r − 1](r − 1)d \circ φ_∗ \simeq ν^{−1}_d[(r − 1)d]_φ
\]
on $\text{mod} \Lambda \simeq D^b(Λ)$. It follows that $\hat{ν}_*^{−1}(U_{r−1}((d+1))_φ) \simeq U_{(r−1)(d+1)}(Λ)$, whence $T_{r−1}(Λ)$ is $(r−1)(d+1)$-representation-finite by Theorem 2.6(a). Moreover, (3.6) implies that $\text{ind} Λ \subset \text{mod} \Lambda$ is a cross-section for the $\hat{ν}_*^{−1}$-orbits of $U_{r−1}((d+1))_φ$. Thus $T_{r−1}(Λ) ⊂ Λ$ is basic and $(r−1)(d+1)$-cluster-tilting. □

To prove Corollary 2.12, some additional results are needed.

Proposition 3.13. [43, Propositions 3.6, 4.2, Theorem 4.5] Let $T$ be a triangulated category with a Serre functor $T S$. Let $U$ be an $ℓ$-cluster-tilting subcategory of $T$ satisfying $U[ℓ] = U$. Then the following statements hold:
and thus we get triangle equivalences
\[ k \rightarrow \text{mod}_S(\Gamma) \rightarrow \text{mod}_S(\Gamma + 1) \rightarrow k. \]

**Lemma 3.14.** Let \( A \) be a finite-dimensional \( k \)-algebra of global dimension at most \( d \). Taking a skeleton \( U \) of \( \text{mod}_A(A) \) and regarding \( \nu_A \) as an automorphism of \( U \), the \( n \)-fold \((d+1)\)-preprojective algebra \( \Pi^{(n)} \) of \( A \) satisfies \( \text{proj} \Pi^{(n)} \simeq \text{mod}_U(\nu_{A})^n \).

**Proof.** This is an easy adaptation of the proof of [1, Proposition 4.7]. For the convenience of the reader, we give an outline of the argument here.

The Nakayama functor \( \nu : D^b(A) \rightarrow D^b(A) \) has a quasi-inverse \( \nu^{-1} = - \otimes_k \Lambda \), where \( \Theta = \text{Hom}_A(\Lambda, A) \).

Hence, the functor \( \nu_d = - \otimes_k \Lambda \), where \( \Theta = \text{Hom}_A(\Lambda, A)[d] \), is a quasi-inverse of \( \nu_d \). Since \( \text{gl.dim} A \leq d \), it follows that \( \text{Hom}(\nu_d^j(\Lambda)) = 0 \) whenever \( i < j \). For \( i \geq j \), we have

\[
\text{Hom}(\nu_d^j(\Lambda)) = \text{Hom}(\nu_d^i(\Lambda)) \simeq \text{Hom}(\nu_d^{i-j}(\Lambda)) \simeq H^0(\nu_d^{i-j}(\Lambda)) \simeq H^0(\nu_d^{-i-j}(\Lambda)) \simeq \text{proj} \Pi_{i-j} \simeq \Pi_{i-j}.
\]

where the fourth isomorphism comes from [1, Lemma 4.8].

It is now easy to verify that multiplication in the matrix algebra \( \Pi^{(n)} = (\Pi_{i-j+n})_{1 \leq i, j \leq n} \) corresponds to composition of morphisms in the orbit category \( U/\nu_{A}^n \). The equivalence \( \text{proj} \Pi^{(n)} \simeq U/\nu_{A}^n \) follows.

**Proof of Corollary 2.12.** Let \( \Gamma = \text{End}_A(\Pi) \). By [41, Theorem 1.14, Proposition 1.12(d)], the algebra \( \Gamma \) is \( \nu_{d+1} \)-finite and \( \text{gl.dim} \Gamma \leq d + 1 \) in the terminology of [41], \( \Lambda \) is \( d \)-complete and \( \Gamma \) is \((d+1)\)-complete). By [43, Theorem 4.7], there exists an equivalence of categories

\[ U\nu_{A}(\Lambda) \simeq \tilde{\Gamma}, \]

and thus we get triangle equivalences

\[ \text{mod}_U(\Lambda) \simeq \text{mod}_{\tilde{\Gamma}} \simeq D^b(\Gamma). \]

On the other hand, from Lemma 3.14 follows that

\[ \tilde{\Gamma}/\nu_{\Lambda,d} \simeq \text{mod}_U(\nu_{A,d}^n) \simeq \Pi^{(n)} \]

Applying Theorem 2.6(a) to \( \Gamma \), it is enough to show that the autoequivalence \( \nu_{A,d}^{n} = (\nu_{A,d})^{n} \) of \( \text{mod}_{\tilde{\Gamma}} \simeq D^b(\Gamma) \) satisfies \( (\nu_{A,d})^{n}(U_{d+1}(\Gamma)) = U_{d+1}(\Gamma) \).

**Proposition 3.13(c) gives a commutative diagram**

\[
\begin{array}{ccc}
D^b(\Gamma) & \longrightarrow & \text{mod}_U \\
\downarrow^{\nu_{A,d}^{n}} & & \downarrow^{(\nu_{A,d})^{n}} \\
D^b(\Gamma) & \longrightarrow & \text{mod}_U
\end{array}
\]

and thus we have \( U_{d+1}(\Gamma) = \nu_{A,d}^{n}(U_{d+1}(\Gamma)) = (\nu_{A,d})^{n}(U_{d+1}(\Gamma)) \), as desired.

See Section 4.1.2 for an example illustrating the proof of Corollary 2.12.

4. Examples and applications

4.1. A simple example. With the purpose of illuminating the general theory from a somewhat more concrete viewpoint, we show here how some of the main constructions work out in a particular, simple example.

For \( n \geq 3 \), let \( \Lambda_n = kA_{n}/I^{n-1} \), where \( A_{n} \) is linearly oriented of Dynkin type \( A_{n} \), and \( I \subset kA_{n} \) the ideal generated by all arrows of \( kA_{n} \).

\[ \Lambda_n : \begin{array}{cccccccc}
1 & \longrightarrow & 2 & \longrightarrow & \cdots & \longrightarrow & n-1 & \longrightarrow & n
\end{array} \]
The algebra $\Lambda_n$ is 2-representation-finite of global dimension 2 [43, Theorem 3.12], and the subcategory $\text{add}((\Lambda_n \oplus D\Lambda_n))$ of $\text{mod}\Lambda_n$ is the unique 2-cluster-tilting subcategory. Moreover, by Proposition 2.5, the subcategory

\[ \mathcal{U}_2(\Lambda_n) = \text{add}\{\nu^2_n(\Lambda_n) \mid i \in \mathbb{Z}\} = \text{add}\{(\Lambda_n \oplus D\Lambda_n)[2\ell] \mid \ell \in \mathbb{Z}\} \subset D^b(\Lambda_n) \]

is 2-cluster-tilting.

Let $D_n$ be a Dynkin quiver of type $D_n$. The Dynkin diagram $D_n$ has an automorphism of order 2 (unique for $n > 4$), which induces an automorphism $\sigma$ of $D^b(kD_n)$. Let $X \in D^b(kD_n)$ be an indecomposable object such that $\sigma(X) \not\cong X$. Then $T = \bigoplus_{i=0}^{n-1}(\nu_1\sigma)^i(X)$ is a tilting complex, and $\text{End}_{D^b(kD_n)}(T) \cong \Lambda_n$. Thus the algebra $\Lambda_n$ is derived equivalent to $kD_n$, via a triangle equivalence sending $P_j \in D^b(\Lambda_n)$ to $(\nu_1\sigma)^{n-j}(X) \in D^b(kD_n)$ (here, $P_j$ denotes the projective cover of the simple $\Lambda_n$-module supported at the vertex $j$).

4.1.1. Orbit algebra. The automorphism $\sigma$ of $D^b(\Lambda_n) \cong D^b(kD_n)$ satisfies (see [54, Theorem 4.1])

\[ [1] \cong \nu_1^{-3n}, \quad \text{and} \]

\[ P_j \cong \nu_1\sigma(P_{j-1}) \quad \text{for } 1 < j \leq n \]

Hence we have

\[ \nu_2 = \nu_1 \circ [-1] \cong \nu_1 \nu_1^{-3n} = (\nu_1\sigma)^n \]

which, together with (4.1), implies that

\[ \mathcal{U}_2(\Lambda_n) = \text{add}\{(\nu_1\sigma)^i(P_1) \mid i \in \mathbb{Z}\}. \]

The repetitive category $\hat{\Lambda}_n = \text{proj}^\mathbb{Z}T(\Lambda_n)$ of $\Lambda_n$ is given by the following infinite quiver with relations:

\[
\begin{array}{cccccccc}
& & 1 & & 2 & & \cdots & & n-1 & & n & & 2 & & \cdots & & 1(1) & & \cdots \\
\bowtie & \quad & (1) & \quad & (2) & \quad & \cdots & \quad & (n-1) & \quad & n-1 & \quad & \cdots & \quad & 2(1) & \quad & \cdots \\
\end{array}
\]

Here, $j$ denotes the projective $\Lambda_n$-module $P_j$ concentrated in degree 0, and $(i)$ degree shift by $i$.

The Nakayama automorphism $\tilde{\nu} \cong (1)$ of $\hat{\Lambda}_n$ induces an automorphism $\tilde{\nu}_n$ of $\text{mod}\hat{\Lambda}_n \cong D^b(\Lambda_n)$, which satisfies

\[ \tilde{\nu}_n \cong \nu \circ [1] = \nu_1 \circ [2] \cong \nu_1^{3-2n}. \]

Since $\sigma^2 = \mathbb{I}$ and $(\nu_1\sigma)(\mathcal{U}_2(\Lambda_n)) = \mathcal{U}_2(\Lambda_n)$, we get

\[ \tilde{\nu}_n^{2\ell}(\mathcal{U}_2(\Lambda_n)) = \nu_1^{6-4n}(\mathcal{U}_2(\Lambda_n)) = (\nu_1\sigma)^{6-4n}(\mathcal{U}_2(\Lambda_n)) = \mathcal{U}_2(\Lambda_n) \]

for all $\ell \geq 1$. By Theorem 2.6, this implies that the orbit algebra $\hat{\Lambda}_n/\tilde{\nu}^{2\ell}$ is 2-representation-finite (see Figure 2 for the case $\ell = 1$).

**Figure 2.** The algebra $\hat{\Lambda}_n/\tilde{\nu}^2$.

A 2-cluster-tilting module of $\hat{\Lambda}_n/\tilde{\nu}^{2\ell}$ can be constructed as follows: The preimage $\mathcal{U}$ of $\mathcal{U}_2(\Lambda_n)$ under the natural functor $\text{mod}\hat{\Lambda}_n \to \text{mod}\Lambda_n \cong D^b(\Lambda_n)$ is a 2-cluster-tilting subcategory of
mod $\hat{\Lambda}_n$, which is equivariant under the induced automorphism $\hat{\nu}_2^D : \text{mod} \hat{\Lambda}_n \to \text{mod} \hat{\Lambda}_n$. From Theorem 2.14 in Section 2.3, it follows that the subcategory $F_\nu(U) \subset \text{mod}(\hat{\Lambda}_n/\hat{\nu}_2^D)$ is 2-cluster-tilting. Now any additive generator of $F_\nu(U)$ is a 2-cluster-tilting $\hat{\Lambda}_n/\hat{\nu}_2^D$-module.

The quiver with relations of the algebra $T_2(\Lambda_n)$ is given in Figure 2, and the module category of $\hat{\Lambda}_4$ in Figure 3.

![Figure 3. The module category of $\hat{\Lambda}_4$, with the 2-cluster-tilting subcategory given by $\Lambda_4$ indicated in black (boxes represent projective-injective modules). The module category of $\hat{\Lambda}_4/\hat{\nu}_2^D$ is equivalent to the orbit category $(\text{mod} \hat{\Lambda}_4)/\tau^{10}$, where $\tau = \nu_1$ is the Auslander–Reiten translation on mod $\hat{\Lambda}_4$.](image)

4.1.2. Preprojective algebra. Again let $\Lambda = \Lambda_n = kA_n/I^{n-1}$. Here, we shall construct the 3-preprojective algebra $\Pi$ of $\Lambda$ and show that it is 3-representation-finite, along the lines of the proof of Corollary 2.12.

Setting $\Gamma = \text{End}_\Lambda(\Pi)$, we have

$$\Gamma = \text{End}_\Lambda(\Pi) \simeq \text{End}_\Lambda(D\Lambda) \simeq kA_{n-2}.$$  

From Equation (4.5), we know that $U_2(\Lambda) = \text{add}\{(\nu_1\sigma)^i(P_1) \mid i \in \mathbb{Z}\} \subset D^b(\Lambda)$, and it is now easy to verify that $U_2(\Lambda) \simeq k\Lambda_\infty/I^{n-1} \simeq \tilde{\Gamma}$, where $\Lambda_\infty$ is linearly oriented of type $A_\infty$:

$$\Lambda_\infty : \cdots \overset{a_{-\frac{n-1}{2}}}{\longrightarrow} \overset{a_{-\frac{n-3}{2}}}{\longrightarrow} \cdots \overset{a_0}{\longrightarrow} \overset{a_1}{\longrightarrow} \cdots \overset{a_2}{\longrightarrow} \cdots$$

Let $\psi$ be the automorphism of $\tilde{\Gamma}$ given by shift one step to the left in $\Lambda_\infty$: $\psi(i) = i - 1$ and $\psi(a_i) = a_{i-1}$ for all $i \in \mathbb{Z}$; then $\nu_1\sigma \simeq \psi$ as automorphisms of $U_2(\Lambda) \simeq \tilde{\Gamma}$. Since $\Pi \simeq U_2(\Lambda)/\nu_{\Lambda,2}$ and $\nu_{\Lambda,2} \simeq (\nu_1\sigma)^n \simeq \psi^n$ by (4.4), we now get that

$$\Pi \simeq \Lambda_\infty/\psi^n \simeq \hat{\Lambda}_n/I^{n-1},$$

where $\hat{\Lambda}_n$ is a cyclically oriented quiver of extended Dynkin type $A_{n-1}$. In other words, $\Pi$ is the self-injective Nakayama algebra with $n$ isomorphism classes of simple modules and Loewy length $n - 1$.

The induced automorphism $\psi_\ast$ of $\text{mod} \tilde{\Gamma} \simeq D^b(\Gamma)$ is isomorphic to $\nu_{\Gamma,1}$ – the Auslander–Reiten translation on $D^b(\Gamma)$. The category $D^b(\Gamma)$ is fractionally $\frac{n+1}{n-1}$-Calabi–Yau, meaning that $[n-3] \simeq \nu_{\Gamma}^{-n-1}$ as autoequivalences of $D^b(\Gamma)$, and hence $[-2] \simeq \nu_{\Gamma,1}^{-n-1}$. Consequently,

$$\nu_{\Gamma,3} \simeq \nu_{\Gamma,1} \circ [-2] \simeq \nu_{\Gamma,1} \circ \psi^n \simeq (\nu_{\Lambda,2}),$$

on $D^b(\Gamma) \simeq \text{mod} \tilde{\Gamma} \simeq \text{mod} U_2(\Lambda)$. Since $\text{gl.dim} \Gamma = 1 \leq n$, the subcategory $U_3(\Gamma)$ of $D^b(\Gamma)$ is 3-cluster-tilting, and $U_3(\Gamma) = \nu_{\Gamma,3}(U_2(\Gamma)) = U_3(\Gamma)$ by (4.7). It follows from Theorem 2.6 that $\Pi \simeq \tilde{\Gamma}/\psi^n$ is 3-representation-finite.

Let $U' \subset \text{mod} \tilde{\Gamma}$ be the preimage of $U_3(\Gamma)$ under the natural functor $\text{mod} \tilde{\Gamma} \to \text{mod} \tilde{\Gamma} \simeq D^b(\Gamma)$. Denoting by $F : \tilde{\Gamma} \to \Gamma/\nu_{\Gamma,3} \simeq \Pi$ the covering functor, the subcategory $F_\ast(U')$ of mod $\Pi$ is 3-cluster-tilting. See Figure 4 for a picture in the case $n = 5$. 
4.2. Trivial extensions of homogeneous $d$-representation-finite algebras. In the case of $\Lambda$ being an $r$-homogeneous $d$-representation-finite algebra of global dimension $d$, either Corollary 2.8 or 2.9 can be employed to find a basic $d$-cluster-tilting $T_d(\Lambda)$-module. Specializing Corollary 2.9 to this situation, we get the following simplified description.

**Proposition 4.1.** Let $\Lambda$ be an $r$-homogeneous $d$-representation-finite algebra of global dimension $d$, and $m$ a positive integer. Then

$$V = T_{dm}(\Lambda) \oplus \bigoplus_{i=0}^{m(dr-d+r)-1} \left((\nu_{T_{dm}(\Lambda)})^i, \text{rad } \Lambda \right)$$

is a basic $d$-cluster-tilting $T_{dm}(\Lambda)$-module.

**Proof.** By [31, Theorem 1.3], the algebra $\Lambda$ is twisted $d(\ell+1)$-Calabi–Yau. We apply Corollary 2.9 to the case $(a, b, \ell) = (r, d(r-1), mg)$, where $g = \gcd(d+1, a+b)$. Then $n = \ell(ad-b)/g$ is $dm$, and $\ell(a+b)/g-1$ is $m(dr-d+r)-1$. Thus the result follows.

It is easy to check that the $d$-cluster-tilting module $V$ in Proposition 4.1 coincides with the module $U$ given in Corollary 2.8. Indeed, $V \subset \text{add}(\text{rad } \Lambda_i) | i \in \mathbb{Z}$ holds as a subcategory of $\text{mod} T_{dm}(\Lambda)$ for $S_d = (\nu_{T_{dm}(\Lambda)})^{\text{rad } \Lambda}$. Since $\Lambda \in \text{add } U$ and any $d$-cluster-tilting subcategory is closed under $S_d$, it follows that $\text{add } V \subset \text{add } U$. But since $U$ and $V$ are basic $d$-cluster-tilting, we get $U \simeq V$.

The following result by Herschend and Iyama, presented here in a slightly generalized form, gives a rich source of homogeneous $d$-representation-finite algebras.

**Proposition 4.2.** For $i = 1, \ldots, n$, let $\Lambda_i$ be an $r$-homogeneous $d_i$-representation-finite algebra of global dimension $d_i$ such that $\Lambda_i/\text{rad } \Lambda_i$ is a separable $k$-algebra. Then $\bigotimes_{i=1}^n \Lambda_i$ is an $r$-homogeneous $d$-representation-finite algebra of global dimension $d$, where $d = \sum_{i=1}^n d_i$, and $\bigotimes_{j=0}^{r-1} \left(\tau_{d_1}^{j}, \Lambda_1 \otimes \cdots \otimes \tau_{d_n}^{j}, \Lambda_n \right)$ is a $d$-cluster-tilting module of $\bigotimes_{i=1}^n \Lambda_i$.

**Proof.** This result is proved in [31, Corollary 1.5] under the additional condition that $k$ is a perfect field, which is stronger than the algebras $\Lambda_i/\text{rad } \Lambda_i$ being separable. In the original proof, the assumption that $k$ is perfect is used to ensure that the algebra $\Lambda = \bigotimes_{i=1}^n \Lambda_i$ has finite global dimension or, equivalently, that the projective dimension of the $\Lambda$-module $\Lambda/\text{rad } \Lambda$ is finite. Our assumption that $\Lambda_i/\text{rad } \Lambda_i$ is a separable $k$-algebra is enough for this, since, by [20, Proposition 7.7], $\Lambda/\text{rad } \Lambda \simeq \bigotimes_{i=1}^n (\Lambda_i/\text{rad } \Lambda_i)$ holds, and the latter module clearly has finite projective dimension.

Let $Q_1, \ldots, Q_n$ be quivers of Dynkin type $A_m$ with symmetric orientation, in the sense that the orientation of the arrows of each quiver $Q_i$ is invariant under the canonical automorphism of order 2 of the Dynkin diagram $A_m$. In particular, this implies that $m$ is odd. The path algebras $\Lambda_i = kQ_i$ are $r$-homogeneous and 1-representation-finite for $r = (m+1)/2$. By Proposition 4.2, the algebra $\Lambda = \Lambda_1 \otimes \cdots \otimes \Lambda_n$ is $r$-homogeneous and $n$-representation-finite, and $\text{gl.dim } \Lambda = n$. By
Proposition 4.1, the module $V$ defined by Equation (4.8) is a basic $n$-cluster-tilting $T_{n\ell}(\Lambda)$-module. Moreover, noting that

$$((\nu_{n+1}(\Lambda)^{\otimes}2)^{i}(\Lambda) \simeq F_{*}(\nu_{1}^{i}(\Lambda)) \simeq F_{*}(\nu_{1}^{i}(\Lambda_{1}) \otimes \cdots \otimes \nu_{1}^{i}(\Lambda_{n}))$$

for the covering functor $F : \widehat{\Lambda} \to T_{n\ell}(\Lambda)$, we get an alternative presentation of the module $V$:

$$V \simeq T_{n\ell}(\Lambda) \oplus \bigoplus_{i=0}^{\ell(r_{n}+\ell)-1} F_{*}(\nu_{1}^{i}(\Lambda_{1}) \otimes \cdots \otimes \nu_{1}^{i}(\Lambda_{n})) .$$

(4.9)

**Example 4.3.** Consider the quiver $\mathbb{A}_{3}$ with arrows pointing towards the middle point. The path algebra $k\mathbb{A}_{3}$ is a 2-homogeneous 1-representation-finite algebra of global dimension 1, so the algebra $\Lambda = (k\mathbb{A}_{3})^{\otimes 2}$ is 2-homogeneous and 2-representation-finite of global dimension 2.

By Corollary 2.11, the trivial extension algebra $T(\Lambda)$ of $\Lambda$ is 3-representation-finite.

In addition, Proposition 4.1 implies that the 2-fold trivial extension algebra $T_{2}(\Lambda)$ of $\Lambda$ is 2-representation-finite, and

$$V = T_{2}(\Lambda) \oplus \bigoplus_{i=0}^{3} F_{*}(\nu_{1}^{i}(k\mathbb{A}_{3})^{\otimes 2})$$

(4.10)

is a basic 2-cluster-tilting $T_{2}(\Lambda)$-module.

**Figure 5.** The algebra $\Lambda = k\mathbb{A}_{3} \oplus k\mathbb{A}_{3}$, and the quivers of $T(\Lambda)$ and $T_{2}(\Lambda)$.

### 4.3. 3-preprojective algebras

Corollary 2.12 provides a rich source of $(d+1)$-representation-finite self-injective algebras, constructed from $d$-representation-finite algebras of global dimension $d$. Many instances of (1-fold) higher preprojective algebras have already been described in the literature, see, for example, [32, 42, 43, 46]. In the case $d = 2$, Keller [51] has given a description of the 3-preprojective algebras in terms of quivers with potential (see [21]), which we shall recall below. This context is well suited for computing explicit examples.

Let $k$ be an algebraically closed field, and $\Lambda$ a finite-dimensional $k$-algebra of global dimension at most 2. We denote by $\widehat{\Pi} = \widehat{\Pi}(\Lambda)$ the complete 3-preprojective algebra of $\Lambda$, that is, $\widehat{\Pi} = \prod_{i \geq 0} \Pi_{i}$, where $\Pi_{i}$ is the 3-preprojective algebra of $\Lambda$. Take a presentation $\Lambda = \widehat{kQ}/(r_{1}, \ldots, r_{\ell})$ of $\Lambda$ by a quiver $Q$ with a minimal set of relations $r_{1}, \ldots, r_{\ell}$, where $\widehat{kQ}$ is the complete path algebra of $Q$, and $(r_{1}, \ldots, r_{\ell})$ the closure of the ideal $(r_{1}, \ldots, r_{\ell})$ with respect to the (rad $\widehat{kQ}$)-adic topology. Denote by $s(r_{i})$ and $t(r_{i})$ the initial and terminal vertices of a relation $r_{i}$, respectively. A quiver with potential $(Q_{\Lambda}, W_{\Lambda})$ is defined as follows:

$$Q_{\Lambda} = Q \sqcup \{ a_{i} : t(r_{i}) \to s(r_{i}) \mid 1 \leq i \leq \ell \}, \quad W_{\Lambda} = \sum_{i=1}^{\ell} a_{i}r_{i} .$$

In this case, there is an isomorphism

(4.10) $\widehat{\mathcal{P}}(Q_{\Lambda}, W_{\Lambda}) \simeq \widehat{\Pi}$.
of $k$-algebras [51, Theorem 6.10], where $\hat{\mathcal{P}}(Q, W)$ is the complete Jacobi algebra [21]. Note that, by Lemma 3.14, the algebra $\Lambda$ is $\nu_2$-finite if and only if $\Pi$ (or, equivalently, $\hat{\Pi}$) is finite dimensional. In this case, we have $\Pi = \hat{\Pi} \simeq \hat{\mathcal{P}}(Q, W)$. Therefore $\Lambda$ is 2-representation-finite if and only if $\hat{\mathcal{P}}(Q, W)$ is a finite-dimensional self-injective algebra [32].

In [32], it was shown that a basic $k$-algebra $\Lambda$ is 2-representation-finite of global dimension 2 if and only if it is isomorphic to $\hat{\mathcal{P}}(Q, W)/\langle C \rangle$ for a self-injective quiver with potential $(Q, W)$ and a cut $C$. A large number of self-injective quivers with potential and corresponding 2-representation-finite algebras were also given. In this situation, Corollary 2.12 tells us that the $m$-fold 3-preprojective algebras $\Pi(m)$ of $\Lambda$ are self-injective and 3-representation-finite for all $m \geq 1$. Here, we shall only briefly mention one example.

**Example 4.4.** Let $\Lambda = (k\mathbb{A}_3)^{\otimes 2}$. The (1-fold) 3-preprojective algebra $\Pi$ of $\Lambda$ is isomorphic to the Jacobi algebra of the quiver with potential $(Q, W)$ described in Figure 6. The Nakayama automorphism of $\Pi$ has order two, and thus this algebra is not symmetric. In particular, it is not isomorphic to the trivial extension algebra $T(\Lambda)$ which, by Corollary 2.11, is also 3-representation-finite.

4.4. A family of examples of wild representation type. Let $m \geq 2$ and $A_m = k\mathbb{A}_m$, where $\mathbb{A}_m$ is a quiver of Dynkin type $A_m$ of arbitrary orientation. As the hereditary algebra $k\mathbb{A}_m$ is fractionally $\frac{m-1}{m+1}$-Calabi–Yau, it follows that $A_m$ is $\frac{2(m-1)}{m+1}$-Calabi–Yau. Applying Corollary 2.9 we get that the algebra $T_n(A_m)$, where $n = \ell \frac{(d+1)(m+1) - (3m-1)}{\gcd(d+1, 3m-1)}$ with $\ell \geq 1$, is $d$-representation-finite for any $d \geq 2$. In particular, setting $d = 2$ and $d = 3m - 2$, respectively, gives the following results.

**Proposition 4.5.** Let $m \geq 2$, $\ell \geq 1$ and $A_m = k\mathbb{A}_m$ as above.

(a) The algebra $T_d(A_m)$ is 2-representation-finite.

(b) The algebra $T_{m\ell}(A_m)$ is $(3m - 2)$-representation-finite.

Observe that the algebra $A_m$ is of wild representation type whenever $m \geq 4$ (see [53, Theorem 2.5]), and hence that the same holds for $T_n(A_m)$ for all $n \geq 1$. See Figure 7 for an illustration of the case of $\Lambda_3$ with linear orientation.

![Figure 6. Quiver with potential $(Q, W)$ associated to $\Lambda = (k\mathbb{A}_3)^{\otimes 2}$.](image)

![Figure 7. The quiver of the repetitive category $\hat{\Lambda}_3$ in the case of linear orientation.](image)
5. \(d\)-representation-finite self-injective Nakayama algebras

In this section, let \(\Gamma\) be a self-injective Nakayama algebra over an arbitrary field \(k\). Our aim is to prove the following theorem, which gives a necessary and sufficient condition for \(\Gamma\) to be \(d\)-representation-finite.

**Theorem 5.1.** Let \(\Gamma\) be a ring-indecomposable self-injective Nakayama \(k\)-algebra with \(n\) isomorphism classes of simple modules and Loewy length \(\ell \geq 2\). Then \(\Gamma\) is \(d\)-representation-finite if and only if at least one of the following two conditions is satisfied:

(a) \(\ell(d - 1) + 2 \mid 2n\);

(b) \(\ell(d - 1) + 2 \mid tn\), where \(t = \gcd(d + 1, 2(\ell - 1))\).

In relation to Theorem 5.1, we remark that the \(d\)-representation-finite Nakayama algebras of global dimension \(d\) have recently been classified by Vaso [65].

Our proof builds on the characterization of \(d\)-cluster-tilting objects in \(d\)-cluster categories of type \(A\) as \((d + 1)\)-angulations of regular polygons. Below, we briefly recapitulate the necessary background, mostly from [13].

Let \(H\) be a hereditary algebra. The \(d\)-cluster category \(C_d(H)\) of \(H\) (denoted in [13] by \(CQ^d\), with \(Q\) being the quiver \(Q\) of \(H\)) of \(H\) is defined as the orbit category \(C_d(H) = \mathcal{D}^b(H)/\nu_d\). The \(d\)-cluster category is triangulated [50]. It is easy to see that \(d\)-cluster-tilting subcategories of \(\mathcal{D}^b(H)\) correspond bijectively to basic \(d\)-cluster-tilting objects in \(C_d(H)\) (see [19, Proposition 2.2] for the case \(d = 2\)).

Let \(\ell\) and \(d\) be positive integers, and set

\[N = (d - 1)\ell + 2 = (d - 1)(\ell - 1) + (d + 1)\]

Let \(P_N\) be a regular \(N\)-gon with corners indexed by the numbers 0, \ldots, \(N - 1\) in the clockwise direction. We denote by \([x, y] = [y, x]\) the edge between corners \(x\) and \(y\) of \(P_N\). A \((d - 1)\)-diagonal of \(P_N\) is a diagonal that dissects \(P_N\) into a \(((d - 1)\ell' + 2)\)-gon and a \(((d - 1)(\ell - \ell') + 2)\)-gon, for some \(1 < \ell' < \ell\). An edge \([x, y]\) between corners \(x\) and \(y\) of \(P_N\) is a \((d - 1)\)-diagonal if and only if

\[|y - x| > 1 \quad \text{and} \quad |y - x| - 1 \in (d - 1)\mathbb{Z} \]

A partial \((d + 1)\)-angulation of \(P_N\) is a set of non-crossing \((d - 1)\)-diagonals of \(P_N\). The maximal (with respect to inclusion) elements of the set of partial \((d + 1)\)-angulations of \(P_N\) are the \((d + 1)\)-angulations of \(P_N\). A permutation \(\rho\) of the set of partial \((d + 1)\)-angulations of \(P_N\) is defined by rotation one step in the anti-clockwise direction in \(P_N\); i.e., \(\rho([x, y]) = [x - 1, y - 1]\) (where the numbers are interpreted modulo \(N\)).

Let \(K\) be an algebraically closed field, and \(H\) the path algebra \(KH_{\ell - 1}\) over \(K\) of the quiver \(A_{\ell - 1}\) of type \(A_{\ell - 1}\) with linear orientation.

**Proposition 5.2 ([13, Proposition 5.5], [56, Proposition 2.14]).** There is a bijection between the set \(\mathcal{X}\) of basic \(d\)-cluster-tilting objects of \(C_d(H)\), and the set \(\mathcal{Y}\) of \((d + 1)\)-angulations of \(P_N\). Under this bijection, the permutation of \(\mathcal{X}\) induced by the Auslander–Reiten translation \(\tau = \nu_1\) on \(C_d(H)\) corresponds to the permutation \(\rho^{d - 1}\) of \(\mathcal{Y}\).

We are now ready to embark upon the proof of Theorem 5.1. A first characterization of \(d\)-representation-finiteness of the algebra \(\Gamma\) follows readily from Proposition 5.2 together with our previous results.

**Proposition 5.3.** The algebra \(\Gamma\) is \(d\)-representation-finite if and only if there exists a \((d + 1)\)-angulation \(\mathcal{F}\) of \(P_N\) satisfying \(\rho^{n(d - 1)}(\mathcal{F}) = \mathcal{F}\).

**Proof.** Let \(K\) be an arbitrary field, and \(\Gamma' = K\tilde{A}_{n - 1}/I^\ell\), where \(\tilde{A}_{n - 1}\) is a cyclically oriented quiver of extended Dynkin type \(A_{n - 1}\), and \(I\) is the ideal of \(K\tilde{A}_{n - 1}\) generated by all arrows. Recall that the indecomposable modules of a Nakayama algebra are uniserial, and determined up to isomorphism by their tops and lengths, and that the almost split sequences are described by these data, independently of the ground field (see the proof of Theorem V:2:1 in [12]). In particular, the Auslander–Reiten quiver of \(\Gamma\) is isomorphic to that of \(\Gamma'\), and we have a bijection between the
isomorphism classes of indecomposable $\Gamma$-modules and those of $\Gamma'$. This bijection preserves the (non-)vanishing of Ext$^1$, and therefore $\Gamma$ is $d$-representation-finite if and only if $\Gamma'$ is $d$-representation-finite. Thus we can assume that $k$ is algebraically closed, and $\Gamma = k\hat{k}_{n-1}/I'$, without loss of generality.

As before, we write $H = k\hat{k}_{\ell-1}$ where $\hat{k}_{\ell-1}$ is linearly oriented. Then the repetitive category $\hat{H}$ of $H$ is equivalent to $k\hat{\Lambda}_\infty^\omega/I'$, where $\hat{\Lambda}_\infty^\omega$ is linearly oriented of type $A_\infty^\omega$ and $I$ is the ideal generated by all arrows in $\hat{\Lambda}_\infty^\omega$. Denoting by $\psi$ the automorphism of $\hat{H}$ given by a shift one step to the left in $\hat{\Lambda}_\infty^\omega$, we get $\Gamma \simeq \hat{H}/\psi^n$.

By Theorem 2.3, the algebra $\Gamma$ is $d$-representation-finite if and only if $D^b(H)$ has a $\psi^n$-equivariant $d$-cluster-tilting subcategory. In mod $\hat{H} \simeq D^b(H)$, we have $\psi_n \simeq \nu_1$ – the Auslander–Reiten translation on $D^b(H)$ – and this functor descends to $\nu_1$ on the $d$-cluster category $C_d(H)$. Since $d$-cluster-tilting subcategories in $D^b(H)$ correspond bijectively to basic $d$-cluster-tilting objects in $C_d(H)$, Proposition 5.2 now implies that $D^b(H)$ has a $\psi^n$-equivariant $d$-cluster-tilting subcategory if and only if there exists a $(d+1)$-angulation of $P_N$ that is invariant under $(\rho^{d-1})^n = \rho^n(d-1)$.

In view of Proposition 5.3, Theorem 5.1 follows directly from the following result.

**Proposition 5.4.** There exists a $(d+1)$-angulation $\mathcal{F}$ of $P_N$ such that $\rho^{n(d-1)}(\mathcal{F}) = \mathcal{F}$, if and only if at least one of the following conditions is satisfied:

(a) $N \mid 2n$;

(b) $N \mid tn$, where $t = \text{gcd}(d+1, 2(\ell - 1))$.

The remainder of this section is devoted to the proof of Proposition 5.4. To abbreviate notation, we write $a \land b$ for $\text{lcm}(a, b)$ and $a \lor b$ for $\text{lcm}(a, b)$, where $a, b \in \mathbb{Z}$.

**Lemma 5.5.** Let $q$ be a positive integer dividing $N$. The following statements are equivalent:

(a) There exists a $(d+1)$-angulation $\mathcal{F}$ of $P_N$, containing a $(d+1)$-angle $G$, such that $\rho^{N/q}(\mathcal{F}) = \mathcal{F}$ and $\rho^{N/q}(G) = G$;

(b) $q \mid (\ell - 1) \land (d+1)$.

**Proof.** In the proof, the number $s = (d+1)/q$ plays a role. Since $N = (d-1)(\ell - 1) + (d+1)$, we have $\frac{N}{q} - s = (d-1)\frac{\ell - 1}{q}$.

(a)⇒(b): Let $G_0 \subseteq \{0, 1, \ldots, N - 1\}$ be the set of corners of $G$. Without loss of generality, we may assume that $0 \in G_0$ and hence $iN/q \in G_0$ for any $i$. Setting $X = \{0, 1, \ldots, N/q - 1\}$, it follows that $G_0$ can be written as a disjoint union

$$G_0 = \bigsqcup_{i=0}^{q-1} \rho^{iN/q}(G_0 \cap X).$$

Since $|G_0| = d+1$, this implies that $q$ divides $d+1$, and $|G_0 \cap X| = (d+1)/q = s$.

Let $G_0 \cap X = \{x_1, \ldots, x_s\}$, where $0 = x_1 < x_2 < \cdots < x_s$, and set $x_{s+1} = N/q$. Then, for each $i = 1, \ldots, s$, the edge $[x_i, x_{i+1}]$ is either an outer edge or a $(d-1)$-diagonal, so $x_{i+1} - x_i \in (d-1)\mathbb{Z}$.

Consequently,

$$(d-1)\frac{\ell - 1}{q} = \frac{N}{q} - s = x_{s+1} - x_1 - s = \sum_{i=1}^{s} (x_{i+1} - x_i - 1) \in (d-1)\mathbb{Z}$$

and thus $q$ divides $\ell - 1$. We have proved that $q \mid (\ell - 1)$ and $q \mid (d+1)$, i.e., $q \mid (\ell - 1) \land (d+1)$.

(b)⇒(a): Let $G$ be the $(d+1)$-angle with corners

$$\bigsqcup_{i=0}^{q-1} \rho^{iN/q}\{0, 1, \ldots, s - 1\}.$$

Each edge of $G$ is either an outer edge or a $(d-1)$-diagonal, since

$$\frac{(i+1)N}{q} - \left(\frac{iN}{q} + s - 1\right) = 1 = \frac{N}{q} - s = (d-1)\frac{\ell - 1}{q} \in (d-1)\mathbb{Z}.$$
For each \( i = 0, \ldots, q - 1 \), the edge \([iN/q + s - 1, (i + 1)N/q]\) of \( G \) cuts out an \( N' \)-gon \( P_{N'}^{(i)} \) with corners \([iN/q + s - 1, iN/q + s, \ldots, (i + 1)N/q]\), where \( N' = (d - 1)\frac{t - 1}{q} + 2 \). Thus \( P_N \) is partitioned into \( G \) and \( P_{N'}^{(i)} \), \( i = 0, \ldots, q - 1 \), and \( P_{N'}^{(i)} = \rho^{N/q} \left( P_{N'}^{(i')} \right) \) for each \( i \). Any \((d + 1)\)-angulation \( \mathcal{T}' \) of \( P_{N'}^{(i)} \) gives a \((d + 1)\)-angulation

\[
\mathcal{T} = \bigcup_{i=0}^{q-1} \rho^{N/q}(\mathcal{T}' \cup \{[s-1, N/q]\})
\]

of \( P_N \), which is clearly \( \rho^{N/q} \)-invariant. This proves the statement (a).

Lemma 5.6. Assume that \( d \geq 2 \). The following statements are equivalent:

(a) there exists a \((d + 1)\)-angulation \( \mathcal{T} \) of \( P_N \) for which the centre point of \( P_N \) lies on a \((d - 1)\)-diagonal, and \( \rho^{N/2}(\mathcal{T}) = \mathcal{T} \);

(b) \( 2 \mid \ell \).

Proof. A \((d + 1)\)-angulation of the specified type exists if and only if there exists a \((d - 1)\)-diagonal passing through the centre of \( P_N \). This is equivalent to \( N \) being even and the edge \([0, N/2]\) a \((d - 1)\)-diagonal. Since \( \frac{N}{d} - 1 = \frac{(d - 1)\gamma}{2} \), this is equivalent to \( 2 \mid \ell \).

Lemma 5.7. Let \( r = (\ell - 1) \wedge (d + 1) \), \( t = (d + 1) \wedge 2(\ell - 1) \), and \( n \in \mathbb{Z} \). Then \( N \mid nr(d - 1) \) is equivalent to \( N \mid nt \).

Proof. In view of the computation

\[
N \wedge r(d - 1) = N \wedge [(\ell - 1) \wedge (d + 1)](d - 1) = [(d - 1)(\ell - 1) + (d + 1)] \wedge (\ell - 1)(d - 1) = (d + 1) \wedge (\ell - 1)\wedge (d - 1) = (d + 1) \wedge [(d + 1)(\ell - 1) - 2(\ell - 1)] = (d + 1) \wedge 2(\ell - 1) = t,
\]

the numbers \( N/t \) and \( r(d - 1)/t \) are integers and coprime to each other. Thus we have the following chain of equivalences:

\[
N \mid nr(d - 1) \iff \frac{N}{t} \mid n \frac{r(d - 1)}{t} \iff \frac{N}{t} \mid n \iff N \mid nt.
\]

Proof of Proposition 5.4. First, we prove that \( N \mid t n \) if and only if \( P_N \) has a \(\rho^{(d-1)}\)-invariant \((d + 1)\)-angulation that contains a \((d + 1)\)-angle \( G \) such that \( \rho^{(d-1)}(G) = G \). Define an integer \( q \) by \( N/q = N \wedge n(d - 1) \). Then a \((d + 1)\)-angulation (respectively, \((d - 1)\)-angle of \( P_N \) is \(\rho^{(d-1)}\)-invariant if and only if it is \(\rho^{N/q}\)-invariant. By Lemma 5.5, the existence of a \((d + 1)\)-angulation \( \mathcal{T} \) containing a \((d + 1)\)-angle \( G \) such that \( \rho^{N/q}(\mathcal{T}) = \mathcal{T} \) and \( \rho^{N/q}(G) = G \) is equivalent to \( q \mid r \), where \( r = (\ell - 1) \wedge (d + 1) \). Since \( q \) and \( \frac{n(d - 1)}{N/q} \) are coprime, we have

\[
q \mid r \iff q \mid r \left( \frac{n(d - 1)}{N/q} \right) \iff N \mid nr(d - 1) \iff N \mid tn,
\]

where the last equivalence comes from Lemma 5.7. Thus the claim holds.

Next, we show that if \( P_N \) has a \(\rho^{(d-1)}\)-invariant \((d + 1)\)-angulation \( \mathcal{T} \) such that \( \rho^{(d-1)}(G) \neq G \) for every \((d + 1)\)-angle \( G \), then \( N \nmid 2n \). Observe that such a \( \mathcal{T} \) must contain an edge that passes through the centre point of \( P_N \) and, consequently, this edge is fixed by \( \rho^{(d-1)} \). It follows that \( n(d - 1) \) is a multiple of \( N/2 \) but, since \( \rho^{(d-1)} \neq 1 \), not of \( N \). Hence \( \rho^{(d-1)} = \rho^{N/2} \) which implies, by Lemma 5.6, that the number \( \ell \) is even. Thus \( d - 1 \) and \( N/2 = (d - 1)\ell/2 + 1 \) are coprime, so \( \frac{N}{t} \mid n \) and consequently \( N \nmid 2n \) hold.

Finally, we prove that if \( N \nmid 2n \) then there exists a \(\rho^{(d-1)}\)-invariant triangulation of \( P_N \). If \( \ell \) is an even number, then, by Lemma 5.6, there exists a \(\rho^{N/2}\)-invariant \((d + 1)\)-angulation \( \mathcal{T} \) of \( P_N \). The assumption \( N \nmid 2n \) implies that \( \rho^n \) is a power of \( \rho^{N/2} \), and hence \( \rho^{(d-1)}(\mathcal{T}) = \mathcal{T} \).

Suppose that \( \ell \) is an odd number. If \( t \) is even, then \( N \) divides \( tn \) and we are done. Assume instead that \( t \) is odd. This implies that \( d + 1 \) is odd, whence \( N = (d - 1)\ell/2 + 1 \) is odd, too. Therefore,
$N \mid 2n$ implies $N \mid n$ and thus $N \mid n(d-1)$. Since every $(d+1)$-angulation of $P_N$ is invariant under $\rho^N = I$, the result follows. \hfill \Box

6. Open problems

Here, we point to some directions of further inquiry, and give some partial results.

In view of Corollaries 2.8, 2.9 and 2.11, it is natural to seek to understand for which numbers $n$ and $d$ the algebra $T_n(\Lambda)$ is $d$-representation-finite. To this end, we pose the following questions.

**Question 6.1.** For any finite-dimensional $k$-algebra $\Lambda$, set

$$RF(\Lambda) = \{(n, d) \in \mathbb{Z}_{>0} \times \mathbb{Z}_{>0} \mid T_n(\Lambda) \text{ is } d\text{-representation-finite}\}.$$  

1. Given an algebra $\Lambda$, describe the set $RF(\Lambda)$.

2. Is $\Lambda$ twisted fractionally Calabi–Yau whenever $RF(\Lambda)$ is non-empty?

By Corollary 2.9, we know that the converse of (2) is true. With respect to (1), we hypothesize that for given $\Lambda$ and $n \in \mathbb{Z}_{>0}$, there are at most finitely many positive integers $d$ such that $(n, d) \in RF(\Lambda)$. More generally, we make the following conjecture.

**Conjecture 6.2.** For a given finite-dimensional $k$-algebra $A$, there are only finitely many integers $d$ such that $A$ is $d$-representation-finite.

This expectation stems from the general phenomenon that, for a given $\Lambda$, the number of indecomposable summands in a $d$-cluster-tilting $A$-module tends to become smaller as $d$ increases.

If a self-injective algebra $\Gamma$ can be written as an orbit algebra $\tilde{\Lambda}/\phi$ for some algebra $\Lambda$ of finite global dimension, then Theorem 2.14, together with the equivalence $\mathcal{D}^b(\Lambda) \cong \text{mod} \tilde{\Lambda}$ (2.1) gives a strong connection between $d$-cluster-tilting subcategories of $\text{mod} \Gamma$ and $\mathcal{D}^b(\Lambda)$. A natural question is therefore to which extent this construction exhausts all $d$-representation-finite self-injective algebras.

**Question 6.3.** Assume that $k$ is an algebraically closed field of characteristic 0 (or sufficiently large). Is any $d$-representation-finite self-injective $k$-algebra $\Gamma$ isomorphic to $\tilde{\Lambda}/\phi$ for some $\nu_d$-finite algebra $\Lambda$ of finite global dimension and admissible automorphism $\phi$ of $\tilde{\Lambda}$?

Whenever $\Gamma \cong \tilde{\Lambda}/\phi$ as above, Corollary 2.15 gives a bijection between, on one hand, $d$-cluster-tilting $\Gamma$-modules and, on the other, $\phi$-equivariant locally bounded $d$-cluster-tilting subcategories $U$ of $\mathcal{D}^b(\Lambda) \cong \text{mod} \tilde{\Lambda}$.

Recall that for a $\nu_d$-finite algebra $\Lambda$, every tilting object $T \in \mathcal{D}^b(\Lambda)$ satisfying $\text{gl.dim(End}_{\mathcal{D}^b(\Lambda)}(T)) \leq d$ gives rise to an orbital $d$-cluster-tilting subcategory $U_d(T)$ of $\mathcal{D}^b(\Lambda)$. However, as we shall see below, this construction is not exhaustive in general. Therefore, understanding the structure of $d$-cluster-tilting subcategories of $\mathcal{D}^b(\Lambda)$ is an important step towards understanding $d$-representation-finiteness for orbit algebras $\tilde{\Lambda}/\phi$.

**Question 6.4.** Given an algebra $\Lambda$ of finite global dimension, which $d$-cluster-tilting subcategories of $\mathcal{D}^b(\Lambda)$ are orbital? In particular, for which pairs $(\Lambda, d)$ are all $d$-cluster-tilting subcategories of $\mathcal{D}^b(\Lambda)$ orbital?

The following observation gives a partial answer to Question 6.4.

**Proposition 6.5.** Let $\Lambda$ be an iterated tilted algebra. Then every 2-cluster-tilting subcategory of $\mathcal{D}^b(\Lambda)$ is orbital.

**Proof.** If $\Lambda$ is iterated tilted then it is derived equivalent to some hereditary algebra $H$. Let $\mathcal{C}(H) = \mathcal{D}^b(H)/\nu_2$ be the cluster category [19] of $H$, and $\pi : \mathcal{D}^b(H) \to \mathcal{C}(H)$ the natural functor. Suppose that $U \subset \mathcal{D}^b(\Lambda) \cong \mathcal{D}^b(H)$ is a 2-cluster-tilting subcategory. From [19, Proposition 2.2] follows that $U = \pi^{-1}(V)$ for some cluster-tilting object $V \in \mathcal{C}(H)$, and [19, Theorem 3.3(i)] gives that $V = \pi(T)$, where $T \in \mathcal{D}^b(H)$ is a tilting complex with $\text{gl.dim(End}_{\mathcal{D}^b(H)}(T)) \leq 2$. Hence, $U = \pi^{-1}(\pi(T)) = U_2(T)$. \hfill \Box
Let $k$ be an algebraically closed field of characteristic different from 2, and $\Gamma$ a finite-dimensional representation-finite self-injective $k$-algebra.

By Theorem 1.1, $\Gamma$ is isomorphic to $\Lambda/\phi$ for some representation-finite tilted algebra $\Lambda$ and autoequivalence $\phi$ of $\Lambda$. Since $\Lambda$ is a tilted algebra, there exists a triangle equivalence $\mathcal{D}^b(\Lambda) \simeq \mathcal{D}^b(\hat{H})$ for some hereditary representation-finite algebra $H$. The following is an immediate consequence of Proposition 6.5 and Theorem 2.3.

**Corollary 6.6.** In the above setting, the algebra $\Gamma$ is 2-representation-finite if and only if $\mathcal{D}^b(\Lambda)$ has a $\phi$-equivariant orbital 2-cluster-tilting subcategory.

The following two examples show that neither the assumption that $d = 2$, nor that $\Lambda$ is iterated tilted, can be removed from Proposition 6.5.

**Example 6.7.** Let $\Lambda$ be the path algebra a quiver of type $A_2$. Then $\mathcal{D}^b(\Lambda)$ has a 3-cluster-tilting subcategory (indicated with black dots in the figure below) that is not orbital.

\[ \cdots \xrightarrow{\alpha} \bigcirc \xrightarrow{\beta} \bigcirc \xrightarrow{\gamma} \bigcirc \xrightarrow{\beta} \bigcirc \xrightarrow{\alpha} \cdots \]

To any algebra $\Lambda$ of global dimension at most 2 is associated a cluster category $\mathcal{C}(\Lambda)$. The cluster category is triangle equivalent to the Verdier quotient per $\Gamma/\mathcal{D}^b(\Gamma)$ where $\Gamma = \Gamma(Q_{\Lambda}, W_{\Lambda})$ is the Ginzburg dg algebra of the quiver $(Q_{\Lambda}, W_{\Lambda})$ with the potential constructed in Section 4.3. There is a fully faithful functor $\mathcal{D}^b(\Lambda)/\nu_2 \to \mathcal{C}(\Lambda)$ [1, Section 4.3] giving rise to an isomorphism

\[ \text{End}_{\mathcal{C}(\Lambda)}(\Lambda) \simeq \bigoplus_{i \in \mathbb{Z}} \text{Hom}_{\mathcal{D}^b(\Lambda)}(\Lambda, \nu_2^{-i}(\Lambda)) = \Pi_2(\Lambda) \]

of $\mathbb{Z}$-graded $k$-algebras.

From the following result, we can infer the existence of algebras $\Lambda$ for which the bounded derived category $\mathcal{D}^b(\Lambda)$ contains a non-orbital 2-cluster-tilting subcategory.

**Proposition 6.8.** Let $\Lambda$ be an algebra of global dimension 2 that is cluster-equivalent to a hereditary algebra $H$. If every 2-cluster-tilting subcategory of $\mathcal{D}^b(\Lambda)$ is orbital, then $\Lambda$ is derived equivalent to $H$.

Note that $\nu_2$-finiteness is equivalent to the cluster category being Hom-finite, hence this property is preserved by cluster equivalence.

**Proof.** Let $F : \mathcal{C}(\Lambda) \to \mathcal{C}(H)$ be a triangle equivalence, and $\pi_\Lambda : \mathcal{D}^b(\Lambda) \to \mathcal{C}(\Lambda)$ and $\pi_H : \mathcal{D}^b(H) \to \mathcal{C}(H)$ the natural functors. The object $V = F^{-1}\pi_H(H) \in \mathcal{C}(\Lambda)$ is a 2-cluster-tilting object and hence $U = \text{add}(\pi_\Lambda^{-1}(V))$ is a 2-cluster-tilting subcategory of $\mathcal{D}^b(\Lambda)$ [4, Propositions 3.1, 3.2]. Take $T \in \mathcal{D}^b(\Lambda)$ to be a tilting complex such that $U = U_2(T)$, and set $\Lambda' = \text{End}_{\mathcal{D}^b(\Lambda)}(T)$. Then $\Lambda$ and $\Lambda'$ are derived equivalent and hence cluster equivalent. As $V = \pi_\Lambda(T)$, by (6.1) we get

\[ H = \text{End}_{\mathcal{C}(H)}(\pi_H(H)) \simeq \text{End}_{\mathcal{C}(\Lambda)}(V) \simeq \text{End}_{\mathcal{C}(\Lambda')}(\pi_{\Lambda'}(\Lambda')) \simeq \Pi_3(\Lambda') \]

Since $H$ is hereditary, it follows from (4.10) that so is the degree zero part $(\Pi_3(\Lambda'))_0 = \Lambda'$ of $\Pi_3(\Lambda')$. Thus $\Pi_3(\Lambda') = \Lambda'$ and hence $H = \Lambda' = \text{End}_{\mathcal{D}^b(\Lambda)}(T)$ holds, so $H$ and $\Lambda$ are derived equivalent.

Let $\Lambda$ be an algebra of global dimension 2 that is cluster equivalent to a hereditary algebra but not iterated tilted. Then Proposition 6.8 implies that the derived category $\mathcal{D}^b(\Lambda)$ contains a non-orbital 2-cluster-tilting subcategory. Examples of such algebras $\Lambda$ can be found in [2], in which a classification is given of all algebras of global dimension 2 that are cluster equivalent to a hereditary algebra of extended Dynkin type $\hat{A}$.

**Example 6.9.** [4, Example 8.8] Let $H = kQ$ and $\Lambda = kQ'/I$ be given as below:

\[ Q = \begin{tikzcd}
1 & 2 & 3 \\
\alpha & \beta & \gamma
\end{tikzcd} \quad \text{and} \quad Q' = \begin{tikzcd}
1 & 2 & 3 \\
\alpha & \beta & \gamma
\end{tikzcd}, \quad I = \langle ba \rangle.
The associated quivers with potential are \((\tilde{Q}, W) = (Q, 0)\) and \((\tilde{Q}', W')\), where

\[
\tilde{Q}' = \begin{tikzpicture}
  \node (1) at (0,0) {1};
  \node (2) at (1,1) {2};
  \node (3) at (2,0) {3};
  \node (4) at (2,-1) {d};
  \node (5) at (1,-1) {c};
  \node (6) at (0,-1) {a};
  \node (7) at (1,0) {b};
  \draw[->] (2) -- (1);
  \draw[->] (2) -- (3);
  \draw[->] (2) -- (6);
  \draw[->] (2) -- (7);
  \draw[->] (3) -- (4);
\end{tikzpicture}, \quad W' = dba.
\]

Now, \((\tilde{Q}', W')\) and \((\tilde{Q}, W)\) are related by a mutation at the vertex 2; hence,

\[
\mathcal{C}(\Lambda) = \mathcal{C}(\tilde{Q}, W') \simeq \mathcal{C}(\tilde{Q}, W) = \mathcal{C}(H)
\]

holds [1, 52]. Since the quiver \(Q'\) is not acyclic, the algebra \(\Lambda = kQ'/I\) is not iterated tilted. By Proposition 6.8, it follows that \(\mathcal{D}^b(\Lambda)\) has a non-orbital 2-cluster-tilting subcategory.

The 2-cluster-tilting subcategory \(\mathcal{U} = \pi^{-1}_A \pi_H(H)\) of \(\mathcal{D}^b(\Lambda)\), constructed in the proof of Proposition 6.8, is given in Figure 8.

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{figure8.png}
\caption{The 2-cluster-tilting subcategory \(\mathcal{U} = \pi^{-1}_A \pi_H(H)\) of \(\mathcal{D}^b(\Lambda)\).}
\end{figure}

**Acknowledgements**

The authors are indebted to Karin Erdmann, Andrzej Skowroński and Kunio Yamagata for helpful comments and suggestions about the paper. They also wish to thank Claire Amoit for drawing their attention to the example 6.9, and an anonymous referee for helpful comments.

**References**

[1] C. Amiot. Cluster categories for algebras of global dimension 2 and quivers with potential. Ann. Inst. Fourier (Grenoble) 59(6):2525–2590, 2009.
[2] C. Amiot and S. Oppermann. Algebras of acyclic cluster type: tree type and type \(\tilde{\AA}\). Nagoya Math. J., 211:1–50, 2013.
[3] C. Amiot, S. Oppermann. The image of the derived category in the cluster category. Int. Math. Res. Not. IMRN 4:733–760, 2013.
[4] C. Amiot and S. Oppermann. Cluster equivalence and graded derived equivalence. Doc. Math., 19:1155–1206, 2014.
[5] J. Arias, E. Backelin. Higher Auslander–Reiten sequences and t-structures. J. Algebra 459 (2016), 280–308.
[6] S. Asai. The Grothendieck groups and stable equivalences of mesh algebras. Algebr Represent Theor, 21:635–681, 2018.
[7] H. Asashiba. A covering technique for derived equivalence. J. Algebra 191 (1997), 382–415.
[8] I. Assem, D. Simson, and A. Skowroński. Elements of the representation theory of associative algebras. Vol. 1, volume 65 of London Mathematical Society Student Texts. Cambridge University Press, Cambridge, 2006. Techniques of representation theory.
[9] M. Auslander. Coherent functors. 1966 Proc. Conf. Categorical Algebra (La Jolla, Calif., 1965) pp. 189–231 Springer, New York.
[10] M. Auslander. Representation dimension of Artin algebras, in Selected works of Maurice Auslander. Part 1. Edited and with a foreword by Idun Reiten, Sverre O. Smalø, and Oyvind Solberg. American Mathematical Society, Providence, RI, 1999.
[11] M. Auslander and I. Reiten. Stable equivalence of dualizing \(R\)-varieties. Adv. Math. 12:306–366, 1974.
[12] M. Auslander, I. Reiten, and S. O. Smalø. Representation theory of Artin algebras, volume 36 of Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 1995.
[13] K. Baur and R. J. Marsh. A geometric description of \(m\)-cluster categories. Trans. Amer. Math. Soc., 360(11):5789–5803, 2008.
[14] R. Bautista and S. Liu. Covering theory for linear categories with application to derived categories. J. Algebra 406:173–225, 2014.
54. J. Miyachi and A. Yekutieli. Derived Picard groups of finite-dimensional hereditary algebras. Compositio Math., 129(3):341–368, 2001.
55. Y. Mizuno. A Gabriel-type theorem for cluster tilting. Proc. Lond. Math. Soc. (3) 108 (2014), no. 4, 836–868.
56. G. J. Murphy. Derived equivalence classification of m-cluster tilted algebras of type $A_n$. J. Algebra, 323(4):920–965, 2010.
57. S. Oppermann, H. Thomas. Higher-dimensional cluster combinatorics and representation theory. J. Eur. Math. Soc. (JEMS) 14 (2012), no. 6, 1679–1737.
58. A. Pasquali. Tensor products of higher almost split sequences. J. Pure Appl. Algebra 221 (2017), no. 3, 645–665.
59. I. Reiten, M. Van den Bergh. Noetherian hereditary abelian categories satisfying Serre duality. J. Amer. Math. Soc. 15 (2002), no. 2, 295–366.
60. C. Riedtmann. Representation-finite self-injective algebras of class $A_n$. In Representation theory, II (Proc. Second Internat. Conf., Carleton Univ., Ottawa, Ont., 1979), Lecture Notes in Math. 832, pages 449–520. Springer, Berlin, 1980.
61. C. Riedtmann. Representation-finite self-injective algebras of class $D_n$. Compositio Math. 49(2):231–282, 1983.
62. A. Skowroński. Selfinjective algebras: finite and tame type. In Trends in representation theory of algebras and related topics, volume 406 of Contemp. Math., pages 169–238. Amer. Math. Soc., Providence, RI, 2006.
63. A. Skowroński and K. Yamagata. Selfinjective algebras of quasitilted type. In Trends in representation theory of algebras and related topics, EMS Ser. Congr. Rep., pages 639–708. Eur. Math. Soc., Zürich, 2008.
64. H. Tachikawa. Representations of trivial extensions of hereditary algebras. In Representation theory, II (Proc. Second Internat. Conf., Carleton Univ., Ottawa, Ont., 1979), Lecture Notes in Math. 832, pages 579–599. Springer, Berlin, 1980.
65. L. Vaso. n-cluster tilting subcategories of representation-directed algebras. J. Pure Appl. Algebra, 223(5):2101–2122, 2019.
66. J. Waschbüsch. Symmetrische Algebren vom endlichen Modultyp. J. Reine Angew. Math. 321:78–98, 1981.
67. J. Waschbüsch. On self-injective algebras of finite representation type. In Monografías del Instituto de Matemáticas [Monographs of the Institute of Mathematics] 14, pages 1+59. Universidad Nacional Autónoma de México, México, 1983.
68. K. Yamagata. Extensions over hereditary Artinian rings with self-dualities. I. J. Algebra. 73(2):386–433, 1981.
69. K. Yamamura. Realizing stable categories as derived categories. Adv. Math. 248:784–819, 2013.

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