Universality of the universal $R$-matrix and applications to quantum integrable systems

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Abstract
Results obtained by us are overviewed from a general set up. The universal $R$-matrix is exploited to obtain various important relations and structures involved in quantum group algebra, which are used subsequently for generating different classes of quantum integrable systems through a systematic scheme. This recovers known models as well as discovers a series of new ones.

Introduction:
Our basic aim is to construct a systematic scheme for generating different classes of quantum integrable systems (QIS) starting from some well known and universal structure related to quantised algebra (QA). We also attempt to answer the following intriguing questions related to QIS like,

- How a single QA may be related to diverse forms of nonlinear integrable models.
- What is the ‘commonness’ between seemingly widely different models like sine-Gordon (SG), derivative nonlinear Schrodinger (DNLS), massive Thirring model (MTM) etc. (all of them share the same quantum $R$-matrix!).

Our starting point in achieving our goal would be an universal object - the universal $R$-matrix $\hat{\mathcal{R}} \in U_q(g) \otimes U_q(g)$, which satisfies the ‘universal’ Yang-Baxter equation (UYBE): $R_{i\hat{i}} R_{i\hat{j}} R_{\hat{i}\hat{j}} = R_{\hat{j}i} R_{i\hat{j}} R_{i\hat{j}}$, where $\hat{i}$ indicates the infinite dimensional operator (quantum) space. We show
first that the universal $\mathcal{R}$-matrix is universal enough to yield the Faddeev-Reshitikhin-Takhtajan (FRT) algebra $[1]$ itself along with the explicit forms of $R^\pm, L^\pm$ involved in this algebra. Since our construction relates further these objects to quantum integrable systems through Yang-Baxterisation, forms of universal $\mathcal{R}$-matrix in a sense answer ultimately for the generated classes of integrable systems. Moreover it is demonstrated that the $\mathcal{R}$-matrix and the UYBE give also the explicit coproduct structures as well as the intertwining relation between coproducts. Frequently, all these important objects are defined separately and introduced as independent entries, though in many occasions their relation with the $\mathcal{R}$-matrix has been mentioned. We intend to derive here these relations in a consistent way and use them for our subsequent application to integrable systems.

Not to make things abstract let us keep before us the universal $\mathcal{R}$-matrix related to $U_q(sl(2))$, which has relatively simpler form:

$$ R = q^{2(H \otimes H)} \text{Exp}(q^H X^+ \otimes q^{-H} X^-) \quad (0.1) $$

Note now that all finite-dimensional representations of these matrices are upper-triangular in form. Therefore for concreteness we denote such universal $\mathcal{R}$-matrices as $\mathcal{R}^+$ and the related equation as UYBE $ (+ + + )$:

i) $R_{12}^+ R_{13}^+ R_{23}^+ = R_{23}^+ R_{13}^+ R_{12}^+$. It is not difficult to see that if we define $\mathcal{R}^- = \mathcal{P}(\mathcal{R}^+)^{-1} \mathcal{P}$, it would satisfy now the UYBE $ (- - - )$:

ii) $R_{12}^- R_{13}^- R_{23}^- = R_{23}^- R_{13}^- R_{12}^-$, and will be lower-triangular in form. More interestingly along with the above two universal equations another four UYBE of $ (+ - - ), (- + + ), (+ + - )$ and $(- + - )$ type would also equally hold:

iii) $R_{12}^- R_{13}^- R_{23}^- = R_{23}^- R_{13}^- R_{12}^+$,

iv) $R_{12}^- R_{13}^- R_{23}^+ = R_{23}^- R_{13}^+ R_{12}^+$,

v) $R_{12}^+ R_{13}^+ R_{23}^- = R_{23}^- R_{13}^+ R_{12}^+$,

vi) $R_{12}^- R_{13}^- R_{23}^+ = R_{23}^- R_{13}^- R_{12}^+$,

which can be proved easily using the permutation property of $\mathcal{P}$. At the same time one also verifies easily the nonvalidity of $ (+ - + )$ type UYBE (holds only for triangular Hopf algebras: $R_{12}^{-1} = R_{21}$).

**Reductions of UYBE and consequences:**

Now we show that different reductions of the above UYBE’s can yield different important relations relevant to quantum group. Here by without hat numbers we denote $2 \times 2$ dimensional vector spaces reduced from infinite dimensional operator (quantum) spaces. Notice that the reduction $\hat{1} \rightarrow$
1, 2 → 2, 3 → 3 yield interestingly the famous FRT algebra simply as a consequence of the above UYBE’s. Under this reduction the (+++) UYBE reduces to

\[ R_{12}^+ R_{13}^+ R_{23}^+ = R_{23}^+ R_{13}^+ R_{12}^+ , \]

(0.2)

where \( R_{12}^+ = R_{BGR}^+ \) is now a 4×4 matrix (braid group representation(BGR)) and \( R_{13}^+ = L_{1(3)}^+ \) and \( R_{23}^+ = L_{2(3)}^+ \) are 2×2 upper triangular matrices with operator valued elements. With these notations eqn. (2) clearly turns into one of the ((+ + +) type) FRT relations. The other FRT relations are similarly obtained from UYBE (+−−), (+ +−) etc., yielding thereby the corresponding quantised algebra as the quadratic relations among the elements of \( L^\pm \) acting in the quantum space \( \hat{3} \). Note that why an FRT relation of (+−+) type does not hold is also answered naturally by the nonvalidity of such UYBE ,as mentioned above.

On the other hand for the complementary reduction like \( \hat{1} \to \hat{1}, \hat{2} \to \hat{2}, \hat{3} \to \hat{3} \) we may derive the intertwining property of the universal \( R \)-matrix as well as the explicit coproduct structures again as a consequence of the above UYBE’s. For example UYBE (+++) yields \( R_{12}^+ (L_{1}^{(1)} \cdot L_{3}^{(2)}) = (L_{3}^{(2)} \cdot L_{3}^{(1)}) R_{12}^+ \) with the notation \( R_{13}^+ = L_{3}^{(1)} \) etc., where \( L_{3}^{(1)} \) is now a lower triangular matrix with operator elements \( \{ l_{ij}^{(1)} \} \) belonging to the quantum space \( \hat{1} \). It is not difficult to see that the above equation gives in a matrix form the intertwining property: \( R_{12}^+ \Delta(\{ l_{ij} \}) = \Delta(\{ l_{ij} \}) R_{12}^+ \) for all the elements \( \{ l_{ij} \} \). The explicit form of \( \Delta_{L} \) also yields naturally the explicit expressions for coproducts; relations for other elements being similarly obtained from equation of (+−−) type. We may observe here that there is another interesting way to get the coproduct structure by using an important property of (2) i.e., if \( L_{1(3)}^+ \) and \( L_{1(3')}^+ \) are two solutions of the FRT equations related to the quantum spaces \( \hat{3} \) and \( \hat{3}' \), respectively then \( L_{1(3)}^+ \cdot L_{1(3')}^+ = \Delta(L_{1(3)}^+) \) is also a solution, which thereby defines the coproduct structures for the elements of \( L^+ \). Similarly those related to \( L^- \) can be obtained. The importance of such coproduct is that it is neither related directly to the intertwining property nor to the universal \( R \)-matrix and reflects only the underlying Hopf algebraic structure. Therefore such coproducts may exist even when the universal \( R \) may not be found. We shall use this fact later for deriving coproduct structure of a new algebra (3-4), for which \( R \)-matrix is not known.
Scheme for generation of integrable models:

- Start with an universal $\mathcal{R}$-matrix and reduce it to obtain $R_{BGR}^\pm, L_\pm$ etc giving the objects of FRT algebra.
- Determine the underlying quantised algebra and the related coproducts, as described above.
- Yang-Baxterise the FRT algebra, i.e. include spectral parameters to construct Lax operators $L(\lambda)$ and quantum $R(\lambda, \mu)$-matrix (trigonometric type) of an integrable ancestor model. For our case such Yang-Baxterisation may be given in the form
  \[ R(\lambda - \mu) = \frac{\eta}{\xi} R^+ - \frac{\xi}{\eta} R^-, \quad L(\lambda) = \frac{1}{\xi} L^{(+)}, \eta = e^{i\mu}, \xi = e^{i\lambda}. \]
- Find different realisations of the quantised algebra which reduce the ancestor model to generate different quantum integrable systems. The realisations in bosonic variables $(u, p)$ or $(\psi, \psi^\dagger)$ with $[u, p] = 1$, $[\psi, \psi^\dagger] = i$ or $q$-bosonic variables: $(A, A^\dagger)$ with $AA^\dagger - q^{-1}A^\dagger A = q^N$ seems to yield physical models.
- Since new universal R-matrix can be obtained from the original one by ‘twisting’: \( \tilde{\mathcal{R}} = \mathcal{F}^{-1} \mathcal{R} \mathcal{F}^{-1} \), it yields in its turn new forms of $\tilde{R}_{BGR}^\pm, \tilde{L}_\pm$ giving new ancestor Lax operator with deformed trigonometric quantum $R$-matrix, which would yield finally new classes of integrable models.
- At $q \rightarrow 1$, when the underlying quantum algebra reduces to Lie algebra the above scheme also reduces consistently to generate corresponding classes of integrable models with rational quantum $R$-matrix. Thus in a systematic way we may construct directly the Lax operators along with the quantum $R$-matrices of different families of exactly integrable quantum systems at the lattice level.

However before moving further we should focus first on a shortcoming in our scheme by noticing that, if we start from the explicit form of universal $\mathcal{R}$-matrix (1) and follow the above scheme literally, we will end up by discovering only a particular class of integrable models like sine-Gordon, deformed sine-Gordon (and nonlinear Schrodinger equation at $q = 1$ limit), which are related to the standard quantum group algebra (QGA) $U_q(sl(2))$. However, we could overcome this difficulty at the cost of losing some generality in our scheme, which on the other hand would stimulate discovery of a new type of quantised algebra worthy of attention. Notice that at the reduced level UYBE (2), acquire some extra freedom, which allows a more general choice of $R_{13}^+ = L_{13}^+$ than that obtained directly from the universal $\mathcal{R}$ (1)
(which gives only the known FRT form related to standard QGA, as mentioned above). In fact one may choose \(L^+\) in a general 'upper-triangular' form as 
\[
L^+(+) = \begin{pmatrix}
\tau_1^+ & \tau_{21}^+ \\
\tau_{12}^- & \tau_2^-
\end{pmatrix},
\]
while \(L^-\)-matrix as lower-triangular one:
\[
L^-(−) = \begin{pmatrix}
\tau_1^- & \tau_{12}^- \\
\tau_{12}^+ & \tau_2^−
\end{pmatrix},
\]
with yet undefined operators \(\vec{\tau}\). Note however that the forms of \(R_{\pm}^{12} = R_{BGR}^{\pm}\) are kept unchanged, which keep track of the starting universal \(R\). Interestingly, such \(L^\pm\) operators yield now from FRT relations a new quadratic algebra
\[
[\tau_{12}, \tau_{21}] = -(q - q^{-1}) (\tau_1^+ \tau_2^- - \tau_2^+ \tau_1^-), \tag{0.3}
\]
\[
\tau_i^± \tau_{ij} = q^{±1} \tau_{ij} \tau_i^±, \quad \tau_i^± \tau_{ji} = q^{±1} \tau_{ji} \tau_i^±, \tag{0.4}
\]
for \(i, j = (1, 2); q = e^{i\alpha}\), with the Casimir operators
\[
D_1 = \tau_1^+ \tau_1^−, \quad D_2 = \tau_2^+ \tau_2^−, \quad D_3 = \tau_1^+ \tau_2^+ + \tau_1^- \tau_2^- \quad D_4 = 2 \cos \alpha (\tau_1^+ \tau_2^- + \tau_1^- \tau_2^+) - [\tau_{12}, \tau_{21}]_+, \tag{0.5}
\]
The related coproduct structure \((\Delta)\), antipode \((S)\) and the counit \((\epsilon)\) may be given respectively by
\[
\Delta(\tau_i^\pm) = \tau_i^\pm \otimes \tau_i^\pm, \quad \Delta(\tau_{21}) = \tau_1^+ \otimes \tau_{21} + \tau_{21} \otimes \tau_2^+, \quad \Delta(\tau_{12}) = \tau_2^- \otimes \tau_{12} + \tau_{12} \otimes \tau_1^+, \tag{0.6}
\]
\[
S(\tau_{21}) = -(\tau_1^+)^{-1} \tau_{21} (\tau_2^+)^{-1}, \quad S(\tau_{12}) = -(\tau_2^-)^{-1} \tau_{12} (\tau_1^-)^{-1}, \quad S(\tau_i^\pm) = (\tau_i^\pm)^{-1}, \tag{0.7}
\]
and \(\epsilon(\tau_i^\pm) = 1, \epsilon(\tau_{ij}) = 0\), which endow this algebra with Hopf algebraic properties, though quasi-triangularity and the existence of universal \(R\)-matrix seem not to hold for it. This quantised algebra may be considered as an extention of the trigonometric Sklyanin algebra (TSA) and for the particular realisation \(\tau_1^+ = \tau_2^- = q^{-H}, \quad \tau_1^- = \tau_2^+ = q^H, \quad \tau_{12} = (q - q^{-1})X^+, \tau_{21} = (q - q^{-1})X^−\), it reduces to the standard QGA. However the extended TSA allows in general a significantly richer spectrum of different other realisations, which as we see below can yield much wider classes of quantum integrable models, not reachable directly from the standard QGA.

Explicit construction of different classes of quantum integrable models:

Not going into details of the explicit calculations involved in the above scheme for generating known as well as new integrable quantum systems,
we mention only the main results obtained by us [2] and answer the questions raised above. Our results show that for the first class of models, which starts from (1), the generalised ancestor Lax operator is associated with the trigonometric $R$-matrix of the six-vertex model and at different realisations it yields field models like [2,3]

1. Sine-Gordon (SG) model  
2. Liouville model (LM)  
3. A derivative NLS (DNLS) model [2b])  
4. Massive Thirring model (MTM) (obtained as a fusion of DNLS [2e])

and exactly integrable quantum lattice models like

1. XXZ-spin chain  
2. Lattice SG (LSG) model  
3. Lattice Liouville model (LLM)  
4. Discrete DNLS and Discrete MTM in bosonic as well as $q$-bosonic realisations [2e]) etc.

At $q = 1$ limit the $R$-matrix reduces to its rational form and the corresponding ancestor model yields now

1. Nonlinear schrodinger equation (NLS) and its integrable lattice variant (LNLS)  
2. Toda chain (TC)  
3. XXX-spin chain etc.

Remarkably in addition to the above set of quantum integrable models another important class is generated if we start from a transformed universal $\tilde{R}$-matrix, which may be obtained from (1) by twisting with $\mathcal{F} = e^{i(Z \otimes H - H \otimes Z)}$, where $\theta$ is a parameter and $Z$ a central element of QGA. Two different situations may now occur depending on the representation of $Z$.

i) If under the reduction of the quantum spaces to finite dimensions, $Z \rightarrow I$, the resultant quantum $R$-matrix becomes ‘$\theta$-deformed’ trigonometric $R$-matrix and the corresponding ancestor model yields set of integrable systems like [2,3]

1. Ablowitz-Ladik model (ALM)  
2. 6V(1) spin chain  
3. Relativistic (quantum ) Toda chain (RTC) [2a])  
4. $\theta$-deformed LSG, LLM, discrete DNLS, MTM etc.

and also at the $q = 1$ limit : $\theta$-deformed LNLS, Toda chain etc.

ii) However if at finite dimensional reductions of $\hat{1}$ and $\hat{2}$ the central element $Z$ has eigenvalues $\lambda$ and $\mu$, respectively, it induces a ‘colour’ BGR as finite dimensional representation of the twisted universal $R$-matrix. This
necessitates the formulation of ‘colour’ FRT algebra etc. resulting finally integrable models with a new nonadditive spectral parameter dependent quantum $R$-matrix [2 c)].

Thus following our scheme we are able to construct integrable ancestor Lax operators and through consistent realisations generate various families of quantum integrable models representing known as well as new lattice and field models.

The models sharing the same quantum $R$-matrix are found to be the descendants of the same ancestor model, which thus explains nicely their ‘commonness’ and answers the question raised here.

The criterion for integrable nonlinearity seems to be dictated by different concrete realisations of the quantised algebra (3-4) through physical variables. Transition to continuum limit distorts such ‘exact’ nonlinearity.

Extension of the scheme presented here to higher dimensional ($N \times N$) Lax operators and also to other algebras like rotational, projective as well as supersymmetric algebras should be important directions of development.

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