Relativistic Brownian Motion

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Abstract:
We solve the problem of formulating Brownian motion in a relativistically covariant framework in $1+1$ and $3+1$ dimensions. We obtain covariant Fokker-Planck equations with (for the isotropic case) a differential operator of invariant d’Alembert form. Treating the spacelike and timelike fluctuations separately, we show that it is essential to take into account the analytic continuation of “unphysical” fluctuations in order to achieve these results.

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The program of stochastic quantization of Parisi and Wu\textsuperscript{6} assumes the existence of relativistic wave equations, and applies a statistical approach closely related to path integrals. The Wiener distribution of nonrelativistic Brownian motion, as is well known, is associated with (imaginary time) path integral formulations, and one would expect that covariant Brownian motion would similarly be associated with Parisi-Wu stochastic quantization.

Nelson\textsuperscript{1} has pointed out that the formulation of his stochastic mechanics in the context of general relativity is an important open question. The Riemannian metric spaces one can achieve, in principle, which arise due to nontrivial correlations between fluctuations in spacetime directions, could, in the framework of a covariant theory of Brownian motion, lead to spacetime pseudo-Riemannian metrics in the structure of diffusion and Schrödinger equations.

In this paper we shall study the structure a covariant theory of Brownian motion.

We first point out some of the obvious difficulties in reaching a covariant theory of Brownian motion, and indicate the directions we have chosen to solve these problems.

Brownian motion, thought of as a series of “jumps” of a particle along its path, necessarily involves an ordered sequence. In the nonrelativistic theory, this ordering is naturally provided by the Newtonian time parameter. In a relativistic framework, the Einstein time $t$ does not provide a suitable parameter. If we contemplate jumps in spacetime, to accommodate a covariant formulation, a possible spacelike interval between two jumps may appear in two orderings in different Lorentz frames. We therefore adopt the invariant parameter $\tau$ introduced by Stueckelberg\textsuperscript{7} in his construction of a relativistically covariant
classical and quantum dynamics. For the many body theory, Piron and Horwitz postulated that this parameter is universal, as the Newtonian time in the nonrelativistic theory, and drives the classical particle trajectories \( x_i^{\mu}(\tau) \) (worldlines labelled \( i = 1, 2, 3, ..., N \)) through equations of motion, and the evolution of the wave function in the Hilbert space \( L^2(R^{4N}) \), \( \psi_\tau(\{x_i^{\mu}\}) \) through the Stueckelberg-Schrödinger equation (the differential form of the action of a one-parameter unitary group with parameter \( \tau \)).

A second fundamental difficulty in formulating a covariant theory of Brownian motion lies in the form of the correlation function of the random variables of spacetime. The straightforward generalization of the usual Brownian correlation property to special relativity, i.e.,

\[
< dw_\mu(\tau) dw_\nu(\tau') > = \begin{cases} 
0 & \tau \neq \tau' \\
2\alpha\eta_{\mu\nu}d\tau & \tau = \tau',
\end{cases}
\]

contains the serious problem that \( < dw_0(\tau) dw_0(\tau) > < 0 \), which is impossible. Brownian motion in spacetime, however, should be a generalization of the nonrelativistic problem, constructed by observing the nonrelativistic process from a moving frame according to the transformation laws of special relativity. Hence, as a first step, the process taking place in space in the nonrelativistic theory should be replaced by a spacetime process in which the Brownian jumps are spacelike. The pure time (negative) self-correlation does therefore not occur. In order to meet this requirement, we shall use a coordinatization in terms of generalized polar coordinates which assure that all jumps are spacelike. In this case, one would expect a distribution function of the form \( e^{-\frac{\mu^2}{a^2}} \), where \( \mu \) is the invariant spacelike interval of the jump, and \( a \) is some constant. As we shall see, a Brownian motion based on purely spacelike jumps does not yield the correct form for an invariant diffusion process. We must therefore consider the possibility as well that, in the framework of relativistic dynamics, there are timelike jumps. In a frame in which the timelike jumps are pure time, the construction of the Gaussian distribution from the central limit theorem can again be applied. The distribution would be expected to be of the form \( e^{-\frac{\sigma^2}{b^2}} \), where \( \sigma \) is the timelike interval of these jumps, and \( b \) is some constant. By suitably weighting the occurrence of the spacelike process (which we take for our main discussion to be “physical”, since its nonrelativistic limit coincides with the usual Brownian motion) and an analytic continuation of the timelike process, we show that one indeed obtains a Lorentz invariant Fokker-Planck equation in which the d’Alembert operator appears in place of the Laplace operator of the 3D Fokker-Planck equation. One may, alternatively, consider the timelike process as “physical” and analytically continue the spacelike (“unphysical”) process to achieve a d’Alembert operator with opposite sign.

2. Brownian motion in 1+1 dimensions

We consider a Brownian path in 1+1 dimensions of the form

\[
dx^\mu(\tau) = K^\mu(x(\tau))d\tau + dw^\mu(\tau). \tag{2.1}
\]

We start by considering the second order term in the series expansion of a function of position of the particle on the world line, \( f(x^\mu(\tau) + \Delta x^\mu) \), involving the operator

\[
\mathcal{O} = \Delta x^\mu \Delta x^\nu \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial x^\nu}. \tag{2.2}
\]
We have remarked that one of the difficulties in describing Brownian motion in spacetime is the possible occurrence of a negative value for the second moment of some component of the Lorentz four vector random variable. If the Brownian jump is timelike, or spacelike, however, the components of the four vector are not independent, but must satisfy the timelike or spacelike constraint. Such constraints can be realized by using parametrizations for the jumps in which they are restricted geometrically to be timelike or spacelike. We now separate the jumps into spacelike jumps and timelike jumps accordingly, i.e., for the spacelike jumps,

$$\Delta x = \mu \cosh \alpha \quad \Delta t = \mu \sinh \alpha$$  \hspace{1cm} (2.3)

and for the timelike jumps,

$$\Delta x = \sigma \sinh \alpha \quad \Delta t = \sigma \cosh \alpha$$  \hspace{1cm} (2.3')

Here we assumed that the two sectors have the same distribution on the hyperbolic variable. We first look for the effects of a particle experiencing spacelike jumps only. In that case the operator $O$ takes the following form:

$$O = \mu^2 \left[ \cosh^2 \alpha \frac{\partial^2}{\partial x^2} + 2 \sinh \alpha \cosh \alpha \frac{\partial^2}{\partial x \partial t} + \sinh^2 \alpha \frac{\partial^2}{\partial t^2} \right]$$  \hspace{1cm} (2.4)

If the particle going under timelike jumps only we find the operator $O$ takes the following form:

$$O = \sigma^2 \left[ \sinh^2 \alpha \frac{\partial^2}{\partial x^2} + 2 \sinh \alpha \cosh \alpha \frac{\partial^2}{\partial x \partial t} + \cosh^2 \alpha \frac{\partial^2}{\partial t^2} \right]$$  \hspace{1cm} (2.5)

In order to obtain the relativistically invariant d’Alembert diffusion operator, the expression obtained in the timelike region must be subtracted from the expression for the spacelike region, and furthermore, the amplitudes must be identified. In the physical timelike region, the coefficient $\sigma^2$ is, of course, positive, and using the law of large numbers on the random distribution, one obtains a Gaussian distribution analogous to that of the spacelike case.

We see, however, that we can obtain the d’Alembert operator only by considering the analytic continuation of the timelike process to the spacelike domain. This procedure is analogous to the effect, well-known in relativistic quantum scattering theory, of a physical process in the crossed ($t$) channel on the observed process in the direct ($s$) channel. Although we are dealing with an apparently classical process, as Nelson has shown, the Brownian motion problem gives rise to a Schrödinger equation, and therefore contains properties of the quantum theory. We thus see the remarkable fact that one must take into account the physical effect of the analytic continuation of processes occurring in a non-physical, in this case timelike, domain, on the total observed behavior of the system.

In the non-stochastic case, Einstein’s relativity identifies $\Delta x/\Delta t$ with $p/E$, where $p$ and $E$ are the components of the energy-momentum four-vector of the particle. If we make an analogous identification, assigning these variables as properties of the fluctuations, then

$$\sigma^2 = (\Delta t)^2 - (\Delta x)^2 \propto (E^2 - p^2),$$  \hspace{1cm} (2.7)
defining a stochastic mass squared associated with the Brownian particle. If the relation between $E$ and $p$ becomes spacelike, the notion of stochastic mass can be retained under the transformation to an imaginary representation $E \rightarrow iE'$ and $p \rightarrow ip'$, for $E', p'$ real.

§ This transformation is similar to the analytic continuation $p \rightarrow ip'$ in nonrelativistic tunneling, for which the particle appears as an instanton.

With these assumptions, the cross-term in hyperbolic functions cancels in the sum, which now takes the form

$$\langle \mathcal{O} \rangle = \langle \mu^2 \rangle \left[ \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial t^2} \right]$$

Taking into account the drift term in (2.1), one then finds the relativistic Fokker-Planck equation

$$\frac{\partial D(x, \tau)}{\partial \tau} = \left\{ -\frac{\partial}{\partial x^\mu} K^\mu + \langle \mu^2 \rangle \frac{\partial^2}{\partial x^\mu \partial x^\mu} \right\} D(x, \tau),$$

where $\partial/\partial x^\mu$ operates on both $K^\mu$ and $D$.

We see that the procedure we have followed permits us to construct the Lorentz invariant d'Alembertian operator, as required for obtaining a relativistically covariant diffusion equation. Furthermore, since the expectation of $\sinh^2 \alpha, \cosh^2 \alpha$ could be infinite (e.g., for a uniform distribution on $\alpha$), the result we obtain in this way constitutes an effective regularization.

3. Brownian motion in $3+1$ dimensions

In the $3+1$ case, we again separate the jumps into timelike and spacelike types. The spacelike jumps may be parametrized, in a given frame, by

$$\Delta t = \mu \sinh \alpha$$
$$\Delta x = \mu \cosh \alpha \cos \phi \sin \vartheta$$
$$\Delta y = \mu \cosh \alpha \sin \phi \sin \vartheta$$
$$\Delta z = \mu \cosh \alpha \cos \vartheta$$

We assume the four variables $\mu, \alpha, \vartheta, \phi$ are independent random variables. In addition we demand in this frame that $\vartheta$ and $\phi$ are uniformly distributed in their ranges $(0, \pi)$ and $(0, 2\pi)$, respectively. In this case, we may average over the trigonometric angles, i.e., $\vartheta$ and $\phi$ and find that:

$$\langle \Delta x^2 \rangle_{\varphi, \vartheta} = \langle \Delta y^2 \rangle_{\varphi, \vartheta} = \langle \Delta z^2 \rangle_{\varphi, \vartheta} = \frac{\mu^2}{3} \cosh^2 \alpha$$
$$\langle \Delta t^2 \rangle_{\varphi, \vartheta} = \mu^2 \sinh^2 \alpha$$

$$\langle \Delta x^2 \rangle_{\varphi, \vartheta} = \langle \Delta y^2 \rangle_{\varphi, \vartheta} = \langle \Delta z^2 \rangle_{\varphi, \vartheta} = \frac{\mu^2}{3} \cosh^2 \alpha$$
$$\langle \Delta t^2 \rangle_{\varphi, \vartheta} = \mu^2 \sinh^2 \alpha$$
We may obtain the averages over the trigonometric angles of the timelike jumps by replacing everywhere in Eq. (3.2)

\[ \cosh^2 \alpha \leftrightarrow \sinh^2 \alpha \]
\[ \mu^2 \rightarrow \sigma^2 \]

to obtain

\[
\begin{align*}
<\Delta x^2>_{\phi,\vartheta} &= <\Delta y^2>_{\phi,\vartheta} = <\Delta z^2>_{\phi,\vartheta} = \frac{\sigma^2}{3} \sinh^2 \alpha \\
<\Delta t^2>_{\phi,\vartheta} &= \sigma^2 \cosh^2 \alpha,
\end{align*}
\]

where \( \sigma \) is a real random variable, the invariant timelike interval. Assuming, as in the 1+1 case, that the likelihood of the jumps being in either the spacelike or (virtual) timelike phases are equal, and making an analytic continuation for which \( \sigma^2 \rightarrow -\lambda^2 \), the total average of the operator \( \mathcal{O} \), including the contributions of the remaining degrees of freedom \( \mu, \lambda \) and \( \alpha \) is

\[
<\mathcal{O} >= \left( <\mu^2> <\sinh^2 \alpha > - <\lambda^2> <\cosh^2 \alpha > \right) \frac{\partial^2}{\partial \tau^2} + \frac{1}{3} \left( <\mu^2> <\cosh^2 \alpha > - <\lambda^2> <\sinh^2 \alpha > \right) \Delta
\]

If we now insist that the operator \( <\mathcal{O} > \) be invariant under Lorentz transformations (i.e. the d’Alembertian) we impose the condition

\[
<\mu^2> <\sinh^2 \alpha > - <\lambda^2> <\cosh^2 \alpha >= - \frac{1}{3} <\mu^2> <\cosh^2 \alpha > - <\lambda^2> <\sinh^2 \alpha >.
\]

Using the fact that \( <\cosh^2 \alpha > - <\sinh^2 \alpha > = 1 \), and defining \( \gamma \equiv <\sinh^2 \alpha > \), we find that

\[
<\lambda^2 > = \frac{1 + 4\gamma}{3 + 4\gamma} <\mu^2 >.
\]

The Fokker-Planck equation then takes on the same form as in the 1 + 1 case, i.e., the form (2.9). We remark that for the 1 + 1 case, one finds in the corresponding expression that the 3 in the denominator is replaced by unity, and the coefficients 4 are replaced by 2; in this case the requirement reduces to \( <\mu^2> = <\lambda^2> \) and there is no \( \gamma \) dependence.

We see that in the limit of a uniform distribution in \( \alpha \), for which \( \gamma \rightarrow \infty \), \( <\lambda^2> \rightarrow <\mu^2> \). In this case, the relativistic generalization of a nonrelativistic Gaussian distribution of the form \( e^{-\mu^2/\sigma^2} \) becomes of the form \( e^{-\mu^2/\sigma^2} \), which is Lorentz invariant. As in the 1 + 1 case, the result (3.6) corresponds to a regularization.

4. Conclusions and Discussion

We have constructed a relativistic generalization of Brownian motion, using an invariant world-time to order the Brownian fluctuations, and separated consideration of spacelike
and timelike jumps to avoid the problems of negative second moments which might otherwise follow from the Minkowski signature. Associating the Brownian fluctuations with an underlying dynamical process, one may think of $\gamma$ in the $3+1$ case as an order parameter, where the distribution function (over $\alpha$), associated with the velocities, is determined by the temperature of the underlying dynamical system (as we have remarked, the result for the $1+1$ case is independent of the distribution on the hyperbolic variable).

At equilibrium, where $\partial D/\partial \tau = 0$, the resulting diffusion equation turns into a classical wave equation which, in the absence of a drift term $K^\mu$, is the wave equation for a massless field. An exponentially decreasing distribution in $\tau$ of the form $\exp(-\kappa \tau)$ would correspond to a Klein-Gordon equation for a particle in a tachyonic state (mass squared $-\kappa$). We have considered the spacelike jumps as “physical” since they result in the usual Brownian motion in the nonrelativistic limit. If the timelike jumps were considered as “physical”, one would analytically continue the “unphysical” spacelike process. The resulting diffusion equation would have the opposite sign for the d’Alembert operator, and an exponentially decreasing distribution would then result in a Klein-Gordon equation in a timelike particle state.

Nelson$^1$ has shown that non-relativistic Brownian motion can be associated with a Schrödinger equation. Equipped with the procedures we presented here, constructing relativistic Brownian motion, Nelson’s methods can be generalized. One then can construct relativistic equations of Schrödinger (Schrödinger-Stueckelberg) type. The eigenvalue equations for these relativistic forms are also Klein-Gordon type equations. Moreover one can also generalize the case where the fluctuations are not correlated in different directions into the case where correlations exist, as discussed by Nelson$^1$ for three dimensional Riemannian spaces. In this case the resulting equation is a quantum equation in a curved Riemannian spacetime; as pointed out in ref.10, the eikonal approximation to the solutions of such an equation contains the geodesic motion of classical general relativity. The medium supporting the Brownian motion may be identified with an “ether”$^1$ (Nelson$^1$ has remarked that the self-interaction of charged particles might provide a mechanism for the Brownian motion) for which the problem of local Lorentz symmetry is solved. This generalization of Nelson’s method will be discussed elsewhere.

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