One-dimensional Model of a Gamma Klystron

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Abstract

A new scheme for amplification of coherent gamma rays is proposed. The key elements are crystalline undulators — single crystals with periodically bent crystallographic planes exposed to a high energy beam of charged particles undergoing channeling inside the crystals. The scheme consists of two such crystals separated by a vacuum gap. The beam passes the crystals successively. The particles perform undulator motion inside the crystals following the periodic shape of the crystallographic planes. Gamma rays passing the crystals parallel to the beam get amplified due to interaction with the particles inside the crystals. The term ‘gamma klystron’ is proposed for the scheme because its operational principles are similar to those of the optical klystron. A more simple one-crystal scheme is considered as well for the sake of comparison. It is shown that the gamma ray amplification in the klystron scheme can be reached at considerably lower particle densities than in the one-crystal scheme, provided that the gap between the crystals is sufficiently large.

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I. INTRODUCTION

Development of coherent radiation sources for a wavelength below 0.1 nm (which corresponds to the photon energy of tens keV or higher), i.e. in the gamma ray range, is a challenging goal of modern physics. Such radiation will have many applications in basic science, technology and medicine. In particular, they will have a revolutionary impact in nuclear and solid state physics as well as in life sciences.

The present state-of-the-art lasers are capable for emitting electromagnetic radiation from the infrared to ultraviolet range of spectrum. X-ray free-electron lasers are currently under construction \[1, 2\]. Moving further, i.e. into gamma ray band, is not possible without new approaches and technologies.

One of the most promising approaches utilizes the spontaneous emission of gamma rays by ultrarelativistic charged particles undergoing channeling in periodically bent crystals. An undulator-type radiation is emitted in such a system in addition to the well-known channeling radiation due to the periodic motion of the particle that follows the shape of crystallographic planes or axes. The feasibility of gamma ray generation by ultrarelativistic positrons in planar channeling regime in crystals with periodically bent crystallographic planes was proven a decade ago \[3, 4\] (see also the review \[5\] and references therein). Such a device is known as a 'crystalline undulator'.\(^1\) Recently, the feasibility of the crystalline undulator utilizing the planar channeling of electrons was demonstrated \[7, 8\].

The operation of the crystalline undulator is based on the phenomenon of charged particle channeling in crystals \[9\] (see also the latest review \[10\]). Channeling takes place if a charged particle enters a crystal nearly parallel to a crystallographic axis or plane. It can be confined by the electrostatic potential of crystal atoms so that it moves along the axis or plane. It is remarkable that the particle follows the shape of the axis or the plane if they are bent. Therefore the particle trajectory inside the crystal can be controlled by choosing a suitable shape of the axes or planes \[11\].

If crystallographic axes and planes are periodically bent, the trajectories of channeling particles are approximately sinusoidal. This results into emission of electromagnetic radiation predominantly in forward direction \[3, 4, 5\]. The process of emission is very similar to

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\(^1\) The term 'crystalline undulator' was introduced in \[6\] but was not elaborated there.
that in the periodic magnetic field of an undulator \[12, 13, 14\]. However, the electrostatic fields inside a crystal are so strong that they are able to steer the particles much more effectively than even the most advanced superconductive magnets.\(^2\) Due to this fact, the period of crystal bending can be made two to four orders of magnitude smaller than the period of a conventional undulator.\(^3\) Therefore the particle in a periodically bent crystal will emit electromagnetic radiation with much shorter wavelength, i.e. in the gamma ray frequency range.

The emission of the gamma rays will become even more powerful if the positions of the charged particles in a crystalline undulator are correlated along the direction of their motion in such a way that the waves emitted by all particles have approximately the same phase. The resulting radiation is expected to be not only more powerful but also more coherent than the spontaneous radiation from uncorrelated particles. The physical picture of the coherent emission by correlated beam is similar to what happens in free electron lasers \[22, 23, 24, 25\]. Therefore, we use for this process the term ‘lasing effect in a crystalline undulator’.

First, we consider a one-crystal scheme. The feasibility of obtaining a lasing effect in one crystalline undulator has been studied in \[4\] within the quantum formalism. In the current work, this problem is revisited within the classical approach. Indeed, the emission of comparatively low energy photons by a high energy particle (i.e., when \(\hbar \omega / \varepsilon \ll 1\), with \(\hbar \omega\) and \(\varepsilon\) being, respectively, the photon and the particle energy) can be treated classically. The classical approach \[26\] has been widely used for the description of conventional free electron lasers (see, for instance \[23, 25\] and references therein).

Numerical estimation of the parameters of the positron-based crystalline undulator confirm the results of \[4, 5\]: extremely high densities of channeling positrons (\(\sim 10^{21} \text{ cm}^{-3}\) or higher) would be needed to obtain a lasing effect in one periodically bent crystal. This suggest the necessity to search for alternative schemes which would be capable of reaching

\(^2\) Indeed, the interplanar electrostatic field in crystals is typically of the order of 10 GeV/cm which is equivalent to the magnetic field of approximately 3000 T. The present state-of-the-art superconductive magnets produce the magnetic flux density of the order of 10 T with 45 T being currently the highest value obtained by combining superconductive and resistive magnets \[15\].

\(^3\) Periodical bending can be obtained by the propagation of an acoustic wave along the crystal \[3, 4\] or by making regularly spaced grooves on the crystal surface either by a diamond blade \[16, 17\] or by means of laser-ablation \[18\]. Another method is a deposition of Si\(_3\)N\(_4\) layers onto the surface of a Si crystal \[17\]. One more possibility is growing of Si\(_{1-x}\)Ge\(_x\) crystals \[19\] with a periodically varying Ge content \(x\) \[20, 21\].
Reconsideration of the one-crystal scheme was necessary to develop the classical formalism applicable for the description of the lasing effect in crystalline undulator in presence of the dechanneling phenomenon (i.e. the process of leaving the channeling regime due to scattering by crystal electrons and nuclei). This formalism is more convenient than the quantum one for the developing of alternative schemes. Additionally, the one-crystal scheme, being the simplest one, provide a good reference point for the comparison with more sophisticated set-ups.

We propose a two-crystal scheme in which the periodically bent crystals separated by a vacuum gap. The beam of charged particles passes both crystals successively. In the first crystal, a correlation is achieved between the momentum of the particle and its position along the beam direction. This correlation is transformed to a density modulation of the beam. The gamma radiation is generated mostly in the second crystal by the density-modulated beam. This scheme is similar to that of the optical klystron [27, 28], but is expected to operate in the gamma ray range. Therefore we call it a ‘gamma klystron’.

The article is organized as follows. In section II a one dimensional model of the one-crystal gamma ray amplifier is considered, the formalism is introduced, and numerical estimation are presented. The results of this section are used as a basis for the comparison in section III, where the main subject of our work — the gamma klystron — is considered. The results are summarized and discussed in section V.

II. ONE-CRYSTAL GAMMA RAY AMPLIFIER

As it has been already mentioned above, the lasing effect in a crystalline undulator takes place if the positions of the channeling particles along the beam direction are correlated in such a way that the electromagnetic waves emitted by all particle have approximately the same phase. This is accomplished by a spatial modulation (termed usually as 'bunching' [23, 24] or 'microbunching' [1]) of the particle density along the direction of the beam motion with the period equal to the wavelength of the emitted radiation. To obtain such a modulation, initial (seed) radiation from an external source is needed. This radiation may be generated by spontaneous emission of charged particles in a crystalline undulator or in the field of a infrared laser wave. In both cases the initial radiation has to be well collimated.
FIG. 1: A scheme of the one-crystal amplifier. A charged particle beam (solid lines) and initial (seed) radiation (solid wavy lines) enter a crystal with periodically bent crystallographic planes. The particle follow the shape of the crystallographic planes and move along nearly sinusoidal trajectories (wavy dashed lines). The radiation is amplified due to its interaction with the beam in the crystal (see also explanations in the text).

to ensure sufficient monochromaticity and coherence.

Under certain conditions which are discussed below, this radiation modulates the density of the particles channeling in a periodically bent crystal. Then the bunched beam produces additional radiation of the same wavelength.

In other words, the undulator amplifies the initial radiation. The amplifier is the only possible type of lasers in the hard x-ray and gamma ray range.⁴

The scheme of the gamma ray amplifier based on one crystalline undulator (we call it ‘one-crystal gamma ray amplifier’) is shown in figure 1. A charged particles beam and initial gamma radiation fall parallel to each other onto a periodically bent crystal of the length \( L \). The beam is aligned with the tangent of the bent crystallographic planes at the entrance. Therefore, the particles are captured in the channeling mode and move along the planes following their shape. In this section we study the conditions under which the interaction between the radiation and the channeling particles leads to amplification of the radiation.

⁴ In the optical range, the output radiation feedback is commonly used in free electron lasers instead of an external source. The feedback is carried out by the mirrors of an optical resonator. Such class of devices are called ‘oscillators’ [25]. This approach cannot be applied in the hard x-ray or gamma-ray range because of the absence of the appropriate mirrors.
FIG. 2: A positively charged particle trajectory inside a periodically bent crystal. Due to the repulsive potential of the atomic nuclei (filled circles) the particle gets confined inside the interplanar channel of the width $d$ and moves along the bent crystallographic planes following their nearly sinusoidal shape with the period $\lambda_u$ and the amplitude $a$ and emitting electromagnetic waves. Note that the scheme does not reflect the relative scale of $d$, $a$ and $\lambda_u$ adequately. In reality, these quantities satisfy the following double inequality $d \ll a \ll \lambda_u$.

A. Particle dynamics

Let us consider a one-dimensional model of the gamma ray amplifier. We assume the crystal and the beam to be infinitely wide in the $x$ and $y$ directions, which are perpendicular to the direction of the beam propagation (the $z$ axis).

The crystallographic planes are parallel to the $x$ axes and are bent in $y$-direction, so that particles channel in $z$-direction and oscillate along the $y$-axis (see figure 2).

In what follows the shape $y(z)$ of the periodically bent midplane is assumed to have a harmonic form:

$$y = a \cos(k_u z),$$  \hspace{1cm} (1)

where $k_u = 2\pi/\lambda_u$ with $\lambda_u$ being the undulator period.
The magnitude of the undulator period exceeds greatly the bending amplitude, i.e. $\lambda_u \gg a$ (or, equivalently, $k_u a \ll 1$) \[5\]. This inequality, which implies that the crystal deformation is an elastic one, follows from the important condition for a stable channeling in a periodically bent crystal. Stable channeling occurs if the maximum centrifugal force $F_{cf}$ in the channel is less than the maximum interplanar force $U_{\text{max}}'$, i.e. $C \equiv F_{cf}/U_{\text{max}}' < 1$. For an ultra-relativistic particle, the centrifugal force depends on the particle energy as $F_{cf} \approx \varepsilon/R_{\text{min}}$, where $R_{\text{min}} = \lambda_u^2/4\pi^2a$ is the minimum curvature radius of the channel with the bending profile given by (1). Hence, the condition reads \[3, 4\]

$$C = \frac{4\pi^2\varepsilon a}{U_{\text{max}}' \lambda_u^2} < 1.$$  

(2)

A thorough analysis carried out for a number of crystals \[5\] allowed one to conclude that the crystalline undulator is feasible for the values of the bending parameter $C$ lying within the interval $0.01 \ldots 0.2$. For these $C$ values one obtains $a \sim 10^{-5} \ldots 10^{-3} \lambda_u$.

Another important feature of a crystalline undulator concerns the relative magnitude of the bending amplitude $a$ and the interplanar spacing $d$. It was demonstrated (see \[3, 4, 5\]) that the operation of a crystalline undulator should be considered in the large-amplitude regime, which implies $d \ll a$ (typically, $a \sim 10 \ldots 10^2d$). This condition, in particular, allows one to neglect the oscillations of the particle in the channel. Thus, one can assume that under the action of the interplanar field, the particle follows the trajectory (1). Formally this means that the length $s$, measured along the trajectory, can be used as a generalized coordinate which uniquely characterizes the position of the particle. The corresponding conjugate variable is the tangent projection of the particle momentum,

$$p_s = \frac{m\dot{s}}{\left(1 - \dot{s}^2/c^2\right)^{1/2}},$$

(3)

where $m$ is the mass of the projectile and $\dot{s} \equiv ds/dt$.

The evolution of $p_s$ is due to the presence of the amplified radiation whose intensity is $E$. The equation of motion reads

$$\frac{dp_s}{dt} = e(En),$$

(4)

where $e$ is the charge of the projectile, and $n$ is the unit tangent vector, $n = (0, y'_z, 1)/\sqrt{1 + y'_z^2}$ (where $y'_z = dy/dz \propto k_u a \ll 1$).

The amplified radiation is sought in the form of a plane wave linearly polarized along the $y$ direction and characterized by the wavelength $\lambda$ and the frequency $\omega = ck$ (here $k = 2\pi/\lambda$...
and $c$ is the speed of light). Neglecting the attenuation$^5$ of the radiation in the crystal, one describes the only non-zero component of the electric field of the amplified wave as follows

$$E_y = E_0 \cos(kz - \omega t + \phi).$$  \hfill (5)

Substituting (5) into (4) and taking into account that $ak_u \ll 1$ one derives

$$\frac{dp_s}{dt} = -eE_0a k_u \left[ \sin \left( \psi + \frac{a^2 k_u(k + k_u)}{8} \sin(2k_u \kappa s) \right) \ight. \left. - \sin \left( \psi + \frac{a^2 k_u(k - k_u)}{8} \sin(2k_u \kappa s) - 2k_u \kappa s \right) \right],$$  \hfill (6)

where the following notations are used

$$\psi = (k + k_u) \kappa s - \omega t + \phi, \quad \kappa = 1 - \left( \frac{ak_u}{4} \right)^2.$$  \hfill (7, 8)

The energy exchange between the particle and the electromagnetic field is most effective when the phase $\psi$ stays nearly constant, otherwise the first sine term on the right-hand side of (6) oscillates, thus averaging out the energy exchange$^6$. The phase $\psi$ is constant provided the following resonant condition is fulfilled:

$$(k + k_u) \kappa s - \omega = 0.$$  \hfill (9)

Using (8) and expressing the velocity $\dot{s}$ in terms of the Lorentz factor $\gamma$ of the projectile, $\dot{s} = c(1 - \gamma^{-2})^{1/2} \approx c(1 - \gamma^{-2}/2)$ (the limit $\gamma \gg 1$ is assumed), one notices that (9) reduces to the following relation between $\gamma$ and the wave number $k = \omega/c$:

$$\gamma = \sqrt{\frac{2k}{k_u (4 - a^2 k k_u)}} \equiv \gamma_r.$$  \hfill (10)

Resolving this equation for $k$ one finds$^7$

$$k = \frac{4\gamma_r^2 k_u}{2 + K^2},$$  \hfill (11)

$^5$ It can be shown that the attenuation length $L_a$ of gamma rays and hard X rays (see figure 6 in [5]) is by approximately an order of magnitude larger than the dechanneling length $L_d$ that determines the optimum length of the crystalline undulator.

$^6$ The phase of the second sine term cannot be made constant. Therefore, this term always oscillates. Hence the main contribution to the energy exchange is due to the first sine term.

$^7$ In accordance with the general theory of free electron lasers (e.g., [23]) the amplification of the electromagnetic wave in an undulator can occur only at the frequencies corresponding to the harmonics of the spontaneous undulator radiation, $\omega_k = k \omega_1$ ($k = 1, 2, \ldots$). The frequency $\omega_1$ of the fundamental harmonic coincides with $\omega = kc$ which one finds from (11). We will consider the emission stimulation in the first harmonic only.
where

\[ K = \gamma r a_k u = 2\pi \gamma r a/\lambda_u \]  

is the undulator parameter.

It follows from (11) that \( k \ll k_u \) in the vicinity of the resonance (9). Therefore, putting \( k \pm k_u \approx k \) in (6), expanding then the right-hand side of the equation in the Fourier series and, finally, omitting the oscillating terms one arrives at

\[
\frac{dp_s}{dt} = -\frac{eE_0 a_k u}{2}J(\eta) \sin \psi
\]

where

\[
\eta = \frac{K^2/2}{2 + K^2}, \quad J(\eta) = J_0(\eta) - J_1(\eta),
\]

with \( J_{0,1}(\eta) \) standing for the Bessel functions.

Let us derive the equation which describes the evolution of the phase \( \psi \) (see (7)). Differentiating (3) one gets \( dp_s/dt = m\gamma^3 \dot{s} \approx m\gamma^3 \ddot{s} \). On the other hand, double differentiation of (7) yields \( \ddot{s} = (k + k_u) \kappa \ddot{s} \approx k \ddot{s} \). Combining these equations with (13) and noticing that \( s \approx ct \) one derives

\[
\frac{d^2\psi}{ds^2} = -\Omega^2 \sin(\psi),
\]

which has the form of the pendulum equation. The quantity \( \Omega \), defined according to

\[
\Omega^2 = \frac{eE_0 kK}{2mc^2\gamma_r^3}J(\eta),
\]

has the meaning of the oscillation frequency of the corresponding simple pendulum.

The derivative \( d\psi/ds \) can be related to the deviation of the particle energy \( \varepsilon = \gamma mc^2 \) from its resonance value \( \varepsilon_r = \gamma_r mc^2 \):

\[
\frac{d\psi}{ds} \equiv \zeta = \frac{4k_u}{2 + K^2} \frac{\gamma - \gamma_r}{\gamma_r}
\]

In what follows we consider the limit of small gain and small signal. The limit of small gain means that the change in the amplitude \( E_0 \) is much smaller than its initial value at the entrance. In other words, the amplitude and, consequently, the frequency \( \Omega \) are approximately constant along the undulator. The small signal limit means that \( E_0 \) is small
enough to ensure the inequality $\Omega L \ll 1$ ($L$ is the length of the undulator). In this case the pendulum equation (15) can be solved iteratively yielding

$$\psi(s) \approx \psi_0 + \zeta_0 s + (\Omega s)^2 \left( \frac{\sin(\psi_0 + \zeta_0 s)}{(\zeta_0 s)^2} - \frac{\sin(\psi_0)}{(\zeta_0 s)^2} - \frac{\cos(\psi_0)}{\zeta_0 s} \right)$$

(18)

$$\zeta(s) \approx \frac{\Omega^2}{\zeta_0} \left( \cos(\psi_0 + \zeta_0 s) - \cos(\psi_0) \right) + \zeta_0$$

(19)

where $\psi_0$ and $\zeta_0$ denote the quantities at the undulator entrance.

B. The gain factor

The gain factor $g(L)$ characterizes the relative increase in the energy of the amplified electromagnetic wave over the undulator length $L$. This quantity is defined as follows

$$g(L) = \frac{\Delta E}{E(0)},$$

(20)

where $E(0) = E_0^2/8\pi$ is the radiation energy density at the entrance point, and $\Delta E = E(L) - E(0)$ is the increase in the energy density (with $E(L)$ denoting the density at the exit from the undulator).

It follows from the energy conservation law that the radiated energy equals to the decrease in the energy of the channeling particles due to the radiative losses. Therefore, to calculate the gain factor one can analyze the radiative energy loss of the particles.

The energy density of the beam particles, $E_b(s)$, at the distance $s$ can be written as

$$E_b(s) = \langle \varepsilon(s) \rangle n(s).$$

(21)

Here $n(s)$ stands for the volume density of the channeling particles and $\langle \varepsilon(s) \rangle$ denotes the average energy of a particle at the distance $s$. The averaging procedure takes into account that the initial beam is not spatially modulated, i.e. that for different particles the instants of entry into the crystal are not correlated. In other terms, the particles are randomly distributed with respect to the phase $\psi \bigg|_{s=0} \equiv \psi_0$ at the entrance point (see [7]). On the other hand, (6) suggests that a particle’s interaction with the radiation field depends on the value of $\psi_0$. Therefore, to obtain $\langle \varepsilon(s) \rangle$ one averages the energy $\varepsilon(s) = mc^2\gamma(s)$ with respect to $\psi_0$: $\langle \varepsilon(s) \rangle = (2\pi)^{-1} \int_0^{2\pi} \varepsilon(s) d\psi_0$.

In a crystalline undulator the particles move in a medium. Due to collisions with the crystal constituents the channeling particle increases its transverse energy $\varepsilon_\perp = c^2p_\parallel^2/2\varepsilon$. At
some point $\varepsilon_\perp$ exceeds the interplanar potential barrier and the particle leaves the channel. The average distance traveled by a particle in a crystal until it dechannels is called the dechanneling length $L_d$. In a straight crystal $L_d$ depends on the crystal type, the crystallographic plane, the energy and the type of a projectile. In addition to these, $L_d$ acquires the dependence on the parameter $C$ [4] in a bent crystal. The dechanneling effect results in a gradual decrease in the volume density of the channeled particles with the penetration distance. Roughly, this decrease satisfies the exponential decay law [29]:

$$n(s) = A n_0 \exp(-s/L_d).$$  \hspace{1cm} (22)

Here $n_0$ is the beam density at the entrance of the crystal, and $A$ stands for the channel acceptance (i.e., the fraction of the incident particles which is captured into the channeling regime).

It follows from [21] that the change $d\mathcal{E}_b$ of the beam energy density over the interval $ds$ contains two terms. One of these, proportional to the derivative of $n(s)$, is due to the dechanneling process, whereas another one, proportional to $d\langle \varepsilon(s) \rangle/ds$, describes the radiative losses. It is exactly the latter term which is responsible for the change $d\mathcal{E}$ of the electromagnetic field energy. Therefore, one can write

$$\frac{d\mathcal{E}}{ds} = -mc^2 n(s) \frac{d\langle \gamma \rangle}{ds}. \hspace{1cm} (23)$$

The derivative $d\langle \gamma \rangle/ds$ is calculated using [15] and [17]. Then, integrating (23), one obtains

$$\Delta \mathcal{E} = \frac{mc^2 \gamma^3}{k} \Omega^2 \int_0^L n(s) \langle \sin \psi(s) \rangle ds.$$  \hspace{1cm} (24)

Using (18) and introducing the expansion in powers of $\Omega s \leq \Omega L \ll 1$, one carries out the averaging over $\psi_0$:

$$\langle \sin(\psi) \rangle = \frac{\Omega^2}{2\zeta_0^2} \left( \sin(\zeta_0 s) - \zeta_0 s \cos(\zeta_0 s) \right).$$  \hspace{1cm} (25)

Substituting (24) with (25) into (20) one derives the following formula for the gain factor:

$$g(L, \zeta_0) = 8\pi r_0^2 k u \eta J^2(\eta) \frac{1}{\zeta_0^2} \int_0^L n(s) \left( \sin(\zeta_0 s) - \zeta_0 s \cos(\zeta_0 s) \right) ds.$$  \hspace{1cm} (26)

Here $r_0 = e^2/mc^2$ is the classical radius of the particle. The second argument of the gain factor in the left-hand side is introduced to indicate its dependence on the quantity $\zeta_0$.  

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In the limit of a short crystal, \( L \ll L_d \), the dechanneling can be neglected. Then, putting \( n(s) \approx n(0) \) in (26), one arrives at

\[
g(\zeta, \zeta_0) |_{L \ll L_d} = -2\pi r_0 k_u \frac{L^3}{\gamma^3} A n_0 \eta J^2(\eta) \frac{d}{du} \left( \frac{\sin u}{u} \right)^2
\]

where \( u = \frac{\zeta_0 L}{2} \).

Equation (27) presents a well-known formula for the gain factor of the conventional free electron laser obtained within the small signal and small gain approximation (see e.g. [23]).

In reality, the dechanneling cannot be neglected. Indeed, the dechanneling length of ultra-relativistic positrons (these projectiles are the best candidates for the use in crystalline undulators, see discussion in [5]) measured in cm can be roughly estimated as \( L_d \sim 0.1 \varepsilon \) with the energy in GeV. Hence, the dechanneling length for positrons with \( \varepsilon \) within the GeV range does not exceed several millimeters. Therefore, the limit of a long crystal, \( L \gg L_d \) is of a great interest for an amplifier based on a crystalline undulator. \(^8\)

Assuming in (26) \( L \gg L_d \), one extends the upper limit of the integration to infinity and carries out the integral. The result reads

\[
g(L, \zeta_0) |_{L \gg L_d} = 16\pi r_0 k_u \frac{L_d^3}{\gamma^3} A n_0 \eta J^2(\eta) \frac{v}{(1 + v^2)^2},
\]

where \( v = \frac{\zeta_0 L_d}{2} \). The factor \( v/(1 + v^2)^2 \) attains its maximum of \( 3\sqrt{3}/16 \) at \( v = 1/\sqrt{3} \). Therefore, in the limit \( L \gg L_d \) the maximum gain (with respect to \( \zeta_0 \)) is reached for \( \zeta_0 = 1/\sqrt{3} L_d \). Using this value in (28) one obtains the gain factor \( g \) which is optimized with respect to \( \zeta_0 \) and does not depend on the crystal length:

\[
g \equiv g(L, \zeta_0) \bigg|_{L \gg L_d, \zeta_0 = 1/\sqrt{3} L_d} = 3\sqrt{3} \pi r_0 k_u \frac{L_d^3}{\gamma^3} A n_0 \eta [J(\eta)]^2.
\]

This formula coincides up to a numeric factor with the expressions obtained in [4, 5] within the quantum approach. The difference in the numeric factor is due to an additional approximation made in the cited papers.

\(^8\) In the consideration we neglect the effect of the radiation attenuation in the crystal. Therefore our results are valid if \( L_a \gg L \), where \( L_a \) is the attenuation length. In fact, \( L_a \) for the photon energies in the range \( 10^2 \ldots 10^3 \) keV is on the level of several cm [30].
C. Optimization of the crystal and beam parameters

The gain factor \( G \) depends on a number of parameters. Firstly, it depends on the bending period and amplitude. Apart from the obvious dependence of the right-hand side on the period via \( k_u = 2\pi \lambda_u \), the parameters \( \lambda_u \) and \( a \) enter via \( \eta \) (see (12) and (14)). Additionally, they enter the acceptance \( A \) and dechanneling length \( L_d \) (as discussed below).

The latter are also dependent on the type of the crystal and the crystallographic plane as well as on the particle energy \( \varepsilon_r = mc^2 \gamma_r \).

In this section we present the scheme which allows one to define the optimum values of these parameters, i.e. those which insure the largest possible gain for a given crystal and initial beam density \( n_0 \). Our analysis is restricted to a positron beam.

To facilitate the analysis it is convenient to express the quantities on the right-hand side of (29) in terms of the parameter \( C \) and the undulator parameter \( K \) introduced in (2) and (12), respectively. We remind that \( C \) stands for the ratio of the maximum centrifugal force acting on the particle in the periodically bent channel to the maximum gradient of the interplanar potential that keeps the particle in the channel. This definition implies that \( C < 1 \), otherwise all the particles will be thrown out of the channel by the centrifugal force. The lower limit, \( C = 0 \) corresponds to a straight channel, \( a = 0 \) (or \( \lambda_u = \infty \)). In this limit there is no undulator motion and, consequently, there is no radiation amplification. Thus, the largest gain corresponds to some optimum value of \( C \) lying between 0 and 1.

The \( C \)-dependence of the dechanneling length of an ultra-relativistic positron can be modeled as follows [4, 29]:

\[
\begin{align*}
L_d &= (1 - C)^2 L_{d0} \\
L_{d0} &= \gamma_r \frac{256}{9\pi^2} \frac{a_{\text{TF}} d}{r_0} \Lambda,
\end{align*}
\]

where \( L_{d0} \) stands for the dechanneling length in the straight channel, \( a_{\text{TF}} \) is the Thomas-Fermi radius of the crystal atom, and

\[
\Lambda = \ln(\sqrt{2\gamma_r mc^2 / I}) - 23/24
\]

is the 'Coulomb logarithm' which characterizes the ionization losses of an ultra-relativistic particle in amorphous media [31] (\( I \) denotes an average ionization potential of the atom).

Similarly, the acceptance \( A \) of a bent channel can be related to the acceptance \( A_0 \) of the
corresponding straight channel \[29\]:

\[ A = (1 - C)A_0 \]  \hspace{1cm} (32)

with \( A_0 \approx 1 - 2a_{TF}/d \).

Expressing the undulator wave number \( k_u \) via \( C \) and \( K \)

\[ k_u = \frac{C}{K} \frac{U'_\text{max}}{mc^2} \]  \hspace{1cm} (33)

one re-writes \[29\] in the form

\[ g = 3\sqrt{3}\pi r_0 \frac{U'_\text{max}}{mc^2} \left( \frac{L_{d0}}{\gamma_r} \right)^3 n_0 C(1 - C)^7 \left[ K^{-1}\eta J^2(\eta) \right]_{\eta = K^2/(4 + 2K^2)}. \]  \hspace{1cm} (34)

The formulas \[30\] and \[31\] suggests that the ratio \( L_{d0}/\gamma_r \) weakly (logarithmically) depends on the beam energy: an order of magnitude change in \( \gamma_r \) results in a less than 10% change of the ratio. Therefore, we assume this ratio to be constant for a given crystal. Then, the gain factor, in addition to the linear proportionality with respect to \( n_0 \), is a function of two independent variables, \( C \) and \( K \).

The factor \( C(1 - C)^7 \) reaches the maximum value of \( 7^7/8^8 \approx 0.05 \) at \( C = 1/8 \). The factor \( K^{-1}\eta J^2(\eta) \), as a function of the undulator parameter \( K \), attains its maximum of \( \approx 0.15 \) at \( K \approx 1.2 \). These are the optimum values of \( C \) and \( K \) which ensure the maximum gain. The latter is given by

\[ g_{\text{max}} = r_0 \frac{U'_\text{max}}{mc^2} \left( \frac{L_{d0}}{\gamma_r} \right)^3 n_0. \]  \hspace{1cm} (35)

The undulator period, \( \lambda_u^{\text{opt}} \), which corresponds to the optimal values of \( C \) and \( K \), one derives from \[33\]:

\[ \lambda_u^{\text{opt}} \approx 60 \frac{mc^2}{U'_\text{max}}. \]  \hspace{1cm} (36)

The optimal relativistic factor of the beam particles \( \gamma_r \) can be found from \[12\] and \[33\]:

\[ \gamma_r = \frac{K}{ak_u} = \frac{K^2 mc^2}{C \frac{aU'_\text{max}}{U'_\text{max}}} \approx 11.5 \frac{mc^2}{aU'_\text{max}}. \]  \hspace{1cm} (37)

Then the energy \( \hbar \omega \) of the emitted photons can be calculated using \[11\]:

\[ \hbar \omega = hck = \frac{2K^3}{C (1 + K^2/2) \frac{aU'_\text{max}}{U'_\text{max}}} \frac{hmc^3}{a^2U'_\text{max}} = 16 \frac{hmc^3}{a^2U'_\text{max}}. \]  \hspace{1cm} (38)
TABLE I: Parameters of gamma ray lasers in the optimum regime for different crystals and planes at the temperature $T = 4$ K. The notation $\alpha$ stands for the ratio $d/a \ll 1$.

| Crystal Plane  | $d$  | $U_{\text{max}}^\gamma$ | $\lambda_u^\text{opt}$ | $\varepsilon$ | $\hbar\omega$ | $n_0(g_{\text{max}} = 1)$ |
|----------------|------|-------------------------|-------------------------|-------------|---------------|------------------|
| C(diamond) (111) | 1.54 | 5.16                    | 59.4 37.7$\alpha$ 132$\alpha^2$ | 1.4 $\cdot$ 10$^{23}$ |
| C(graphite) (100) | 3.35 | 8.77                    | 35.0 10.2$\alpha$ 165$\alpha^2$ | 5.3 $\cdot$ 10$^{21}$ |
| Si (110)       | 1.92 | 4.98                    | 61.6 31.4$\alpha$ 89$\alpha^2$ | 1.2 $\cdot$ 10$^{23}$ |
| Si (111)       | 2.35 | 6.28                    | 48.8 20.4$\alpha$ 47$\alpha^2$ | 4.5 $\cdot$ 10$^{22}$ |
| Ge (110)       | 2.00 | 10.94                   | 28.0 13.7$\alpha$ 37$\alpha^2$ | 7.3 $\cdot$ 10$^{22}$ |
| Ge (111)       | 2.45 | 13.55                   | 22.6 9.1$\alpha$ 20$\alpha^2$ | 2.9 $\cdot$ 10$^{22}$ |
| W (110)        | 2.24 | 40.74                   | 7.5 3.3$\alpha$ 8$\alpha^2$ | 2.0 $\cdot$ 10$^{22}$ |

The wavelength of the produced radiation and the optimal relativistic factor of the beam particles $\gamma_r$ are not fixed by the choice of the optimum values of the parameters $K$ and $C$. They depend on the bending amplitude $a$. Changing $a$ while keeping the parameters $K$ and $C$ constant does not destroy the optimum regime.

The optimum values of $\lambda_u$ and related parameters for different crystals and positron beam are shown in Table I. The optimum values of the beam energy $\varepsilon$ and the photon energy $\hbar\omega$ depend on the ratio $\alpha = d/a$, which is of the order of 0.1.

The beam density which is needed to reach $g_{\text{max}} = 1$ has been estimated. As is seen from the last column of Table I, extremely high positron densities in the beam are needed to obtain a lasing effect in a simple one-crystal amplifier even in optimized regime. This is consistent with previous results obtained within the quantum formalism [4].

The main reason why an appreciable gain cannot be reached at lower densities is the fact that both the beam evolution and the emission of output radiation take place in one crystal whose length is limited by a few dechanneling lengths. A deeper density modulation of the channeling beam could be obtained in a longer crystalline undulator, but it would not lead to the increase of the radiation gain. The reason is the exponential decay of the density of the radiating particles because of the dechanneling process. This dilemma is resolved in the next section where the scheme of a two-crystal gamma ray amplifier - the gamma klystron - is presented.
III. TWO-CRYSTAL GAMMA RAY AMPLIFIER — THE KLYSTRON

The scheme of the gamma klystron is shown in figure 3. Two periodically bent crystals of the lengths \( L_1 \) and \( L_3 \), respectively, are separated by a vacuum gap of the length \( L_2 \). The beam of particles passes the both crystals successively. A correlation between the particle momentum and its position along the beam direction is created due to the interaction of the channeling particle with the seed radiation in the first crystal. This correlation is further transformed into the density modulation of the beam in the vacuum gap. Production of the output radiation takes place in the second crystal.

The idea of the crystal undulator based gamma klystron is very similar to that of optical klystron [27, 28].

A. Particle dynamics in the Gamma Klystron

In the first undulator, the phase evolves according to (15) and the solution is given by (18) and (19) so that the following expressions are valid at exit from the first undulator

\[
\begin{align*}
\zeta_1 &\approx \zeta_0 + \frac{\Omega^2}{\zeta_0} \cos(\psi_0 + \zeta_0 L_1) - \frac{\Omega^2}{\zeta_0} \cos(\psi_0), \\
\psi_1 &\approx \psi_0 + \zeta_0 L_1 + (\Omega L_1)^2 \left[ \frac{\sin(\psi_0 + \zeta_0 L_1)}{(\zeta_0 L_1)^2} - \frac{\sin(\psi_0)}{(\zeta_0 L_1)^2} - \frac{\cos(\psi_0)}{\zeta_0 L_1} \right].
\end{align*}
\]

(39)

In vacuum, the particles move along straight trajectories. Therefore, the the equation of motion reduces to \( d^2 \psi / ds^2 = 0 \). Hence,

\[
\begin{align*}
\zeta_2 &= \zeta_1, \\
\psi_2 &= \psi_1 + \zeta_1 L_2
\end{align*}
\]

(40)

at entrance of the second undulator.

In the second undulator, the phase evolves again according to (15) leading to

\[
\begin{align*}
\zeta &\approx \zeta_2 + \frac{\Omega^2}{\zeta_2} \cos(\psi_2 + \zeta_2 s) - \frac{\Omega^2}{\zeta_2} \cos(\psi_2), \\
\psi &\approx \psi_2 + \zeta_2 s + (\zeta_2 s)^2 \left[ \frac{\sin(\psi_2 + \zeta_2 s)}{(\zeta_2 s)^2} - \frac{\sin(\psi_2)}{(\zeta_2 s)^2} - \frac{\cos(\psi_2)}{\zeta_2 s} \right],
\end{align*}
\]

(41)

where \( s \) is measured from the entrance of the second undulator, \( 0 \leq s \leq L_3 \).

We assume that

\[ L_2 \gg L_1, L_3. \]

(42)
FIG. 3: A scheme of the gamma klystron. A beam of charged particles (solid lines) and initial (seed) radiation (the solid wavy lines) enter the first crystal with periodically bent crystallographic planes of the length $L_1$. The particle follow the shape of the crystallographic planes and move along nearly sinusoidal trajectories (the wavy dashed lines) inside the crystal. Interaction between the seed radiation and the beam in the first crystal gives rise to a correlation between the particle momentum and its position along $z$ axis. This correlation is transformed into a modulation of the beam density while the beam travels in the vacuum gap of the length $L_2$. Then output radiation is produced by density-modulated beam in the second crystal of the length $L_3$ (see text for details).

Therefore, $\Omega L_2$ or $\Omega^2 L_1 L_2$ have not to be small even though $\Omega L_1 \ll 1$ and $\Omega s \lesssim \Omega L_3 \ll 1$. Hence, neglecting the terms proportional to $\Omega L_1$ and $\Omega s$, but keeping the $\Omega L_2$ terms one derives

$$\sin \psi = \sin \left\{ \psi_0 + \zeta_0 (L_1 + L_2 + s) + \Omega^2 L_1 L_2 \frac{\cos(\psi_0 + \zeta_0 L_1) - \cos(\psi_0)}{\zeta_0 L_1} \right\}. \quad (43)$$

Averaging with respect to $\psi_0$ leads to:

$$\langle \sin \psi \rangle = -J_1 \left[ 2 \frac{\Omega^2 L_2}{\zeta_0} \sin \left( \frac{\zeta_0 L_1}{2} \right) \sin \left( \frac{L_1}{2} + L_2 + s \right) \right].$$

**B. Radiation Gain in the Gamma Klystron**

The increment of the radiation energy density in the second\textsuperscript{9} undulator is found as (cf. (24))

$$\Delta E = \frac{mc^2 \gamma^3}{k} \Omega^2 \int_0^{L_3} n(s) \langle \sin \psi(s) \rangle ds, \quad (44)$$

\textsuperscript{9} Utilizing of the klystron scheme makes sense only if the radiation gain in second crystal is much larger than in the first one. Therefore the gain in the first crystal is neglected.
where \( n(s) \) stands for the volume density of channeling particles in the second undulator. To calculate \( n(s) \), one has to take into account the channel acceptance \( A \) twice: at the entrances of the first and the second undulator\(^{10} \). Additionally, the decrease in the number of channeling particles due to dechanneling in both undulators has to be taken into consideration. Therefore, the density of channeling particles in the second undulator reads

\[
n(s) = A^2 \exp \left( - \frac{L_1 + s}{L_d} \right) n_0.
\]

Here \( n_0 \) is density of the particles at the entrance of the first undulator.

We consider the case \( L_3 \gg L_d \) so that the upper limit of the integral in (44) can be replaced by the infinity. One obtains after the integration

\[
\Delta E = -\frac{mc^2\gamma_3^3}{k^2} \Omega^2 A^2 n_0 \exp \left( -\frac{L_1}{L_d} \right) J_1 \left( \Omega^2 L_1 L_2 \frac{\sin u_1}{u_1} \right) \frac{L_d \sin [u_1 + 2u_2 + \arctan(v)]}{\sqrt{1 + v^2}},
\]

where \( u_i = \zeta_0 L_i / 2, \ i = 1, 2 \) and \( v = \zeta_0 L_d \).

The above expression is derived in the small-gain approximation, but not in the small-signal approximation, i.e. it is valid even when the argument of the Bessel function is of the order of one. In the following, however, we restrict our consideration to the weak signal regime to make a comparison with the one-crystal amplifier.

In the small-signal case, the radiation gain does not depend on the amplitude of the initial wave and has the form

\[
g(L_1, L_2, \infty) = \frac{\Delta E}{\overline{E}} = 8\pi r_0 k \frac{L_1 L_2 L_d}{\gamma_i^3} A^2 n_0 \exp \left( -\frac{L_1}{L_d} \right) \eta [J(\eta)]^2 h(u_1, u_2, v)
\]

with

\[
h(u_1, u_2, v) = -\frac{\sin u_1 \sin [u_1 + 2u_2 + \arctan(v)]}{u_1 \sqrt{1 + v^2}}.
\]

The optimum value of \( L_1 \) can be chosen by maximizing the function \( L_1 \exp (-L_1 / L_d) \). The maximum is reached at \( L_1 = L_d \).

For fixed values of \( L_1, L_2 \) and \( L_3 \), the variables \( u_1, u_2 \) and \( v \) depend only on single variable, \( \zeta_0 \). The optimum value of \( \zeta_0 \) is found by maximizing the function \(-h(u_1, u_2, v)\). Taking into account that \( u_2 \gg u_1, v \) due to (42) one finds that the maximum is reached at \( u_2 \approx -\pi/4 \) and is approximately equal to 1.

\(^{10}\) Strictly speaking the acceptances of two undulators are not exactly equal because of different angular distributions of particles at the entrances of the undulators. But this difference is small and is neglected in our calculations.
Therefore the maximum radiation gain which can be reached in the gamma klystron is

\[ g \equiv g(L_d, L_2, L_3) \bigg|_{L_3 < L_d} \sim -\pi/(2L_3) = \frac{8\pi}{e} r_0 k_u \frac{L_2 L^2}{\gamma^3} A^2 n_0 \eta [\mathcal{J}(\eta)]^2. \]  

(49)

Here \( e = 2.7182818 \ldots \) is the base of the natural logarithm. \(^{11}\)

Comparing this formula with (29) one sees that, for the same parameters of the crystals and the beam, the radiation gain in the gamma klystron exceeds the gain achievable in the one-crystal amplifier by the following factor

\[ \frac{g(\text{klystron})}{g(\text{one-crystal})} = \frac{8}{3\sqrt{3}e} \frac{L_2}{L_d}. \]  

(50)

This means that a significant gain can be obtained in the gamma klystron at much lower beam densities than in the one-crystal amplifier, provided that \( L_2 \) is essentially larger than \( L_d \).

**IV. MULTICASCADE AMPLIFIER**

So far we considered a hard X ray or gamma ray amplifier in the small gain regime. Estimations in section II C were made for the case \( g = 1 \), i.e. when the amplifier doubles the intensity of the electromagnetic radiation. Of cause, such a small gain is insufficient. A useful device should be able to amplify the signal by several orders of magnitude. There are two methods of increasing the gain. The first one is straightforward. It is merely increasing the density of the positron beam. This will switch the amplifier into the large gain regime. The potential of this method is, however, limited. For one crystal amplifier, one needs very high positron densities, \( n_0 \sim 10^22 \text{ cm}^{-3} \), to reach even \( g = 1 \). Using the klystron scheme will decrease the minimum necessary density by several orders of magnitude. Nonetheless, this quantity will likely remain at the edge of the capabilities of the accelerator technique and the sustainability of the crystalline materials. Therefore, increasing the positron density much beyond the minimum necessary level will be difficult.

This difficulty can be overcome by using the second method of increasing the gain: combining several klystrons into a multicascade amplifier. This scheme is shown in figure 4. Scattering the beam particles in the crystal decreases the beam intensity and increases its

\(^{11}\) The constant \( e \) in (49) and (50) should not be confused with the particle charge \( e \).
FIG. 4: A scheme of a multicascade hard X ray or gamma ray amplifier. Each cascade consists of a klystron fed with a separate positron bunch. The radiation passes the klystron successively. If the gain factor of each cascade is $g$, the radiation intensity will be amplified by the factor of $G = (1 + g)^N$, where $N$ is the number of the cascades. "BD" stands for the positron beam dump.

emittance. Therefore, each cascade should be fed by a 'fresh' positron bunch from the accelerator. The radiation passes each cascade successively and is being gradually amplified. The distance between the bunches has to be chosen in such a way that each bunch enters the crystal at the same time as the wavepackage of the radiation reaches the corresponding cascade. The resulting amplification factor will be $G = (1 + g)^N$, where $g$ is the gain of each cascade and $N$ is the number of cascades.

V. CONCLUSION

We analyzed two different schemes of the gamma ray amplifier. Both schemes are based on crystalline undulators - single crystals with periodically bent crystallographic planes exposed to a high energy charged particle beam. Due to the channeling phenomenon, ultra-relativistic charged particles move along nearly sinusoidal trajectories inside the crystal. Initial gamma radiation traveling through the crystal parallel to the beam is amplified due to interaction with the channeling particles.

We have shown, that the simplest one-crystal scheme would require extremely high (more than $10^{21}$ particles per cubic centimeter) positron beam densities to obtain significant amplification. Which makes practical realization of this scheme extremely challenging.

On the other hand, the two-crystal klystron scheme seems to be more promising. A significant gain could be obtained at much lower densities, provided that the distance $L_2$
between two crystals in the klystron exceeds greatly the dechanneling length $L_d$. The question how large the distance $L_2$ can be in a realistic device cannot be answered within the present approach. The simple one-dimensional model considered in the present paper does not put any restrictions on the distance $L_2$. In reality, the restrictions on $L_2$ come from the energy spread (longitudinal temperature) of the beam. This is, however a technical restriction that depends on the quality of the beam determined by the parameters of the accelerator. Physical restrictions are more important. They appear, for instance, due to the longitudinal velocity spread induced by the channeling oscillations and incoherent scattering in the first crystal or the beam divergence in the vacuum gap due to the volume charge.

Hence further analysis is needed to give the final answer about the feasibility of practical realization of the gamma klystron.

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