Constructive Tensor Field Theory: The $T^4_3$ Model

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Abstract

We build constructively the simplest tensor field theory which requires some renormalization, namely the rank three tensor theory with quartic interactions and propagator inverse of the Laplacian on $U(1)^3$. This superrenormalizable tensor field theory has a power counting almost similar to ordinary $\phi_4^2$. Our construction uses the multiscale loop vertex expansion (MLVE) recently introduced in the context of an analogous vector model. However to prove analyticity and Borel summability of this model requires new estimates on the intermediate field integration, which is now of matrix rather than of scalar type.

1 Introduction

Colored tensor models [1, 2] were proposed first as an improvement of group field theory [3, 4]. A key progress over earlier tensor models [5, 6, 7] is that they admit a $1/N$ expansion [8, 9, 10]. $U(N)^\otimes D$ invariant actions and observables for a pair of rank $D$ complex conjugate tensor fields of dimension $N$ are in one to one correspondence with $D$-regular edge-colored bipartite graphs [11]. Such interactions generalize the invariant matrix interactions used in matrix models [12] and in matrix based field theories such as the Kontsevich model [13] or the renormalizable asymptotically safe non-commutative Grosse-Wulkenhaar model over the four dimensional Moyal space [14]. Random tensor models with such invariant actions (called “uncolored” models [19]) are the effective theories coming from colored models when integrating out all tensor fields save one. They also admit a $1/N$ expansion which is universal in a certain precise mathematical sense [11].

The tensor track [20, 21, 22] is the proposal to use the infinite dimensional space of tensor invariant interactions as a new theory space [23] for the quantization of gravity in dimensions higher than 2. In particular it proposes to study renormalization group flows in this space [24] in the hope to discover interesting new random geometries. Indeed the Feynman graphs of rank $D$ tensor theory are $(D+1)$-regular edge-colored bipartite graphs dual to triangulations of (pseudo)-manifolds of dimension $D$. Conversely any $D$-dimensional pseudo-manifold is dual to infinitely many Feynman graphs of a rank $D$ tensor model. Hence the perturbation expansion of tensor field theories performs a sum over all $D$-dimensional (pseudo)-manifolds. Moreover this expansion sums also over discretized metrics. In particular tensor amplitudes without further data ponder equilateral triangulations exactly with a discretized form of the Einstein-Hilbert action [25]. It should also be noticed that adding group field theoretic projectors to tensor models leads to tensor amplitudes which are spin foams, achieving second quantization of loop quantum gravity [26].

To launch and study a renormalization group flow in the tensor theory space requires to introduce convenient cutoffs allowing for scale decomposition. The most convenient field-theoretic way to do this is to introduce as propagator an inverse Laplacian that softly breaks the $U(N)^\otimes D$ invariance and to use the heat-kernel regularization. This procedure can be justified also out of perturbative renormalization considerations [27].

Tensor field theories [28] have been therefore defined as random tensor models with tensor invariant interactions and such a Laplacian-based propagator. The tensorial $1/N$ expansion is the essential tool which

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allows for power counting and for renormalization of such theories, just like the matrix $1/N$ expansion does in the Grosse-Wulkenhaar theory [14].

Superrenormalizable and renormalizable tensor field theories come in essentially two versions. The basic version has no group field theoretic projectors [28, 29, 30] and can be considered the field theoretic version of random tensor models which sum over equilateral triangulations simply equipped with the graph-distance metric. The more sophisticated version equipped with additional group-field theoretic projectors [31, 32, 33] uses a different metric which incorporates the usual simplicity constraints of group field theory, hence should be properly called tensor group field theory. In both cases renormalizable models have the generic property of being asymptotically free [29, 35, 36]. This is up to now the physically most interesting result of the tensor track, since it allows to envision geometrogenesis [37] as a cosmological scenario [38] of tensor theories.

Constructive field theory [39] is a set of techniques to resume perturbative quantum field theory and obtain a rigorous definition of quantities such as the Schwinger functions of interacting renormalizable models. The loop vertex expansion (LVE) [40, 41, 42, 43] is a constructive tool well-adapted to the control of non-local theories in a single renormalization group slice. It is also particularly efficient for the non-perturbative construction of random tensor models [44, 45, 46]. A multiscale loop vertex expansion or MLVE has been recently defined and tested on a vector field theory in [47]. To include renormalization, this MLVE adds to the usual Bosonic layer of the LVE a Fermionic layer (Mayer-type expansion [50, 51, 52]). It has been used to revisit the standard construction of the $\phi^4_2$ theory [48].

It is therefore natural to extend the constructive program to tensor field theories. This is what we do in this paper for the simplest such theory which requires some infinite renormalization, namely the $U(1)$ rank-three model with inverse Laplacian propagator and quartic interactions, which we nickname $T^3$. It can also be considered as an ordinary field theory on the torus $T^3$, but with non-local quartic interactions which break rotation invariance. It turns out that the $T^3$ model requires to add to the MLVE of [47] several additional non-trivial arguments, since the tensor propagator links the indices of the tensor together and the intermediate fields are matrices rather than scalars.

The plan of the paper is the following. In section 2 we recall the model and its intermediate field representation and we introduce the standard multiscale analysis [49, 28] to perform renormalization.

In section 3 we perform the MLVE itself, which expresses the connected functions of the theory as a two-level tree expansion, with both Bosonic and Fermionic links. We also state our main theorem which is the convergence of this expansion, allowing to prove existence of the ultraviolet limit of the theory and its Borel summability in a certain cardioid-like domain of the coupling constant.

In section 4 we gather the proofs of the theorem. The Fermionic integrals are exactly similar to those of [47] and bounded in the same way. We decompose then the Bosonic blocks into perturbative and non perturbative parts which we evaluate separately thanks to a Cauchy-Schwarz inequality. The non-perturbative part requires to bound a determinant which is new compared to [47]; this is done through a combination of norms and trace bounds. The perturbative part requires a parametric representation of resolvents factors which allows strand factorization and resolvent bounds in the style of [45]. Concluded by a relatively standard perturbative bound on convergent graphs with scales constraints, this part delivers the key power counting factors which ultimately beat the combinatorics of the expansion in the same manner than in [47].

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1 The tensor theory space is different from the “Einsteinian” theory space studied in the asymptotic safety program [34], which is the space of diffeo-invariant functions of a metric $g_{\mu\nu}$ on a fixed $\mathbb{R}^4$ topology. Therefore there is absolutely no contradiction between existence of a non-Gaussian fixed point in Einsteinian space and asymptotic freedom in the tensorial space. The uv asymptotically free tensorial flow can lead in the infrared to one or presumably several phase transitions which could create a background random space with effective local properties similar to $\mathbb{R}^4$. The same flow rewritten in new effective variables could then look as if it emerges out of the vicinity of an asymptotically safe fixed point on this effective background space.
2 The Model

2.1 Laplacian, Bare and Renormalized Action

Consider a pair of conjugate rank-3 tensor fields $T_n, \bar{T}_n$ with $n = \{n_1, n_2, n_3\} \in \mathbb{Z}^3$, and $\bar{n} = \{\bar{n}_1, \bar{n}_2, \bar{n}_3\} \in \mathbb{Z}^3$. They belong respectively to the tensor product $H = H_1 \otimes H_2 \otimes H_3$ and to its dual, where each $H_c$ is an independent copy of $\ell_2(\mathbb{Z}) = L_2(U(1))$, and the color or strand index $c$ takes values $c = 1, 2, 3$. Indeed by Fourier transform these fields can be considered also as ordinary scalar fields $T(\theta_1, \theta_2, \theta_3)$ and $\bar{T}(\bar{\theta}_1, \bar{\theta}_2, \bar{\theta}_3)$ on the three torus $T_3 = U(1)^3$ [28].

We introduce the normalized Gaussian measure

$$d\mu_C(T, \bar{T}) = \left( \prod_{n,\bar{n} \in \mathbb{Z}^3} \frac{dT_n d\bar{T}_n}{2\pi} \right) \det(C)^{-1} e^{-\sum_n T_n C_{n,\bar{n}}^{-1} \bar{T}_n}$$

(2.1)

where the covariance $C$ is the inverse of the Laplacian on $T_3$ plus a unit mass term

$$C_{n,\bar{n}} = \frac{\delta_{n,\bar{n}}}{n_1^2 + n_2^2 + n_3^2 + 1}.$$  

(2.2)

The partition function is then

$$Z(g) = \int e^{-\frac{g}{2} \sum_c V_c(T, \bar{T})} d\mu_C(T, \bar{T}),$$  

(2.3)

where $g$ is the coupling constant and

$$V_c(T, \bar{T}) = \sum_{n,\bar{n},m,\bar{m}} \left( T_n \bar{T}_n \prod_{c' \neq c} \delta_{n_{c'},\bar{n}_{c'}} \right) \delta_{n,\bar{m}} \delta_{m,\bar{n}} \left( T_m \bar{T}_m \prod_{c' \neq c} \delta_{m_{c'},\bar{m}_{c'}} \right)$$

(2.4)

are the three quartic interaction terms of random tensors at rank three. This model is the simplest interacting tensor field theory. Indeed it has smallest rank (three), smallest interaction degree (quartic), is symmetric under independent unitary transforms in each of the three spaces of the tensor product $H \otimes H \otimes H$ and globally symmetric under color permutations. Remark that all three quartic interactions are melonic at rank 3. This is no longer true at higher rank [46].

The model has a power counting almost similar to the one of ordinary $\phi^4_2$ [63, 54]. It has a divergent self-loop for each color $c$, which corresponds to a mass renormalization and also both a linearly and a logarithmically divergent vacuum counterterm corresponding to the two vacuum connected graphs with a single vertex (see Figure 1).

![Figure 1](image)

Figure 1: From left to right, the divergent self-loop, the convergent self loop and the two vacuum connected graphs with a single vertex.

It can be renormalized easily, defining the renormalized quartic interaction of color $c$ as:

$$V_c^R(T, \bar{T}) = \sum_{n,\bar{n},m,\bar{m}} \left( T_n \bar{T}_n \prod_{c' \neq c} \delta_{n_{c'},\bar{n}_{c'}} - D^c_{n,\bar{n}} \right) \delta_{n,\bar{m}} \delta_{m,\bar{n}} \left( T_m \bar{T}_m \prod_{c' \neq c} \delta_{m_{c'},\bar{m}_{c'}} - D^c_{m,\bar{m}} \right) - E^c,$$  

(2.5)
where
\[ D_{n, \tilde{n}_c}^c = \delta_{n_c, \tilde{n}_c} \sum_{n_c, n_c' \neq c} \frac{1}{n_1^2 + n_2^2 + n_3^2 + 1} \] (2.6)
is the mass counterterm for color \( c \), and
\[ E^1 = \sum_{n_1, n_1', n_2, n_3} \frac{1}{n_1^2 + n_2^2 + n_3^2 + 1} \] (2.7)
is the vacuum logarithmically divergent counter term for color 1, the two others \( E^2 \) and \( E^3 \) being obtained by color permutation. Remark indeed that the linearly divergent vacuum counter term is correctly renormalized when taking into account the product of the two \( D \) factors in (2.5).

We decompose the three interactions \( V_c^R, c = 1, 2, 3 \), by introducing three intermediate Hermitian matrix fields \( \sigma^c \) acting on \( \mathcal{H}_c \). To simplify the formulas we put \( g = \lambda^2 \) and write
\[ e^{-\frac{\lambda^2}{2} V_c^R(T, \bar{T})} = \int e^{i\lambda (\sigma^1 T_1 + \sigma^2 T_2 + \sigma^3 T_3)} e^{\frac{\lambda^2}{2} \sum E^c c} d\nu(\sigma^c). \] (2.8)
where \( d\nu(\sigma^c) \) is the normalized Gaussian independently identically distributed measure of covariance 1 on the Hermitian matrix \( \sigma^c \).

\( Z(g) \) is now a Gaussian integral over \( (T, \bar{T}) \), hence can be evaluated as
\[ Z(g) = \int \prod_c d\nu(\sigma^c) \int \left( \prod_{n, \tilde{n} \in \mathbb{Z}^3} \frac{dT_n d\bar{T}_{\tilde{n}}}{2\pi} \right) \det(C)^{-1} e^{-i\lambda \sum_c Tr_c(\sigma^c D^c) + \frac{\lambda^2}{2} \sum E^c} \]
\[ \times \exp \left[ -\sum_{n\tilde{n}} T_n \bar{T}_{\tilde{n}} \left( C_{n\tilde{n}}^{-1} - i\lambda \sum_c \sigma^c_{n\tilde{n}} \prod_{c' \neq c} \delta_{n_{c'}, \tilde{n}_{c'}} \right) \right] \] (2.9)
\[ = \int \prod_c d\nu(\sigma^c) e^{-i\lambda \sum_c Tr_c(\sigma^c D^c) + \frac{\lambda^2}{2} \sum E^c} \]
\[ \det \left[ I - i\lambda C(\sigma^1 \otimes \mathbb{1}_2 \otimes \mathbb{1}_3 + \mathbb{1}_1 \otimes \sigma^2 \otimes \mathbb{1}_3 + \mathbb{1}_1 \otimes \mathbb{1}_2 \otimes \sigma^3) \right]^{-1}. \] (2.10)
where \( \text{Tr}_c \) means trace over \( \mathcal{H}_c \), \( \mathbb{1} \) is the identity on \( \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3 \) and \( \mathbb{1}_c \) is the identity over \( \mathcal{H}_c \). Now we use
\[ \sum_c \text{Tr}_c(\sigma^c D^c) = \text{Tr}[C \bar{\sigma}] \] (2.11)
where \( \bar{\sigma} = \sigma^1 \otimes \mathbb{1}_2 \otimes \mathbb{1}_3 + \mathbb{1}_1 \otimes \sigma^2 \otimes \mathbb{1}_3 + \mathbb{1}_1 \otimes \mathbb{1}_2 \otimes \sigma^3 \), and \( \text{Tr} \) means trace on the tensor product \( \mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3 \), to write in a more compact form
\[ Z(g) = \int d\nu(\bar{\sigma}) e^{\frac{\lambda^2}{2} \sum E^c - \text{Tr log}_2[I - i\lambda C \bar{\sigma}]} \]
\[ = \int d\nu(\bar{\sigma}) e^{-\text{Tr log}_2[I - i\lambda C \bar{\sigma}]} + \frac{\lambda^2}{2} \int d\nu(\bar{\sigma}) \text{Tr}(C \bar{\sigma})^2. \] (2.12)
where \( \text{log}_2(1 - x) \equiv x + \log(1 - x) = O(x^2), d\nu(\bar{\sigma}) = \prod_c d\nu(\sigma^c) \), and we remarked that \( \sum E^c \) is nothing but the expectation value \( \int d\nu(\bar{\sigma}) \text{Tr}(C \bar{\sigma})^2 \).

The above formulas do not make sense yet. We have to introduce an ultraviolet cutoff to replace these ill-defined expressions by well-defined functional integrals. Then we shall prove that the ultraviolet limit which removes this cutoff is well defined and Borel summable in \( g = \lambda^2 \) for functions such as \( \log Z \) and the cumulants of the theory. A natural “sharp momentum cutoff” would replace each \( \mathcal{H}_c \) by a finite dimensional space \( \mathcal{H}_c^N \) of dimension \( 2N + 1 \), hence restricts to frequencies \( n^c \in [-N, N] \). A propagator with such a sharp momentum cutoff has no dual direct space decrease. Hence we prefer to use below the parametric cutoff of

\[ ^2 \text{We do not need this spatial decrease in this paper but its absence could become a problem for the future construction of similar models at rank 4 or 5.} \]
multiscale renormalization group analysis \[49\] which is also the one used for renormalization analysis in \[28\]. It has dual direct space decay and is more familiar in the constructive field theory context \[49\]. However all results of the next section could be reproduced without much difficulty for sharp momentum cutoffs. This is left to the reader.

### 2.2 Slices and Intermediate Field Representation

We fix an integer \(M > 1\), and we slice the propagator as usual in the multislice analysis \[47\] \[49\], defining the ultraviolet cutoff as a maximal slice index \(j_{max}\),\(^3\) writing \(C = C_{\leq j_{max}} = \sum_{j=0}^{j_{max}} C_j\) with in momentum space

\[
C_0(n, \bar{n}) = \delta_{n, \bar{n}} \int_1^\infty e^{-\alpha(n_1^2 + n_2^2 + n_3^2 + 1)} \, d\alpha \leq \delta_{n, \bar{n}} e^{-|n_1| - |n_2| - |n_3|},
\]

\[
C_j(n, \bar{n}) = \delta_{n, \bar{n}} \int_{M^{-j-1}}^{M^{-j}} e^{-\alpha(n_1^2 + n_2^2 + n_3^2 + 1)} \, d\alpha \leq \delta_{n, \bar{n}} e^3 M^2 M^{-2j} e^{-\alpha M^{-j} |n_1| + |n_2| + |n_3|} \quad \text{for } j \geq 1.
\]

From now on we shall not try to take care of the exact value of numerical constants throughout this paper. We shall use the time-honored constructive practice to write \(O(1)\) for any inessential numerical constant such as the factor \(e^3 M^2\) in (2.14).

For technical reasons we also impose a second auxiliary sharp cutoff \(|n|, |\bar{n}| \leq N_{\text{max}}\) with \(N_{\text{max}} \gg M_{\text{max}}\) so that our tensors live in a finite dimensional vector space. This allows all traces used later in this paper to be well-defined and finite. The ultraviolet limit for quantities such as the free energy will be reached by letting first \(N_{\text{max}} \to \infty\), then \(j_{max} \to \infty\). We define \(C_{\leq j}(t_j) = \sum_{k<j} C_k + t_j C_j\) where \(t_j \in [0, 1]\) is an interpolation parameter for the \(j\)-th scale of the propagator, and the log-divergent counter term with cutoff \(j\) as

\[
E_{\leq j} = \frac{\lambda^2}{2} \int d\nu(\sigma) \Tr(C_{\leq j}(1)\sigma)^2,
\]

the interaction with cutoff \(j\) is

\[
V_{\leq j} = \Tr \log_2 [I - i\lambda C_{\leq j}(1)\sigma] - E_{\leq j}.
\]

Remark that there is a constant \(c\) such that

\[
0 \leq E_{\leq j} \leq cj, \quad 0 \leq E_{\leq j} - E_{\leq j-1} \leq c.
\]

We also introduce the resolvent with cutoff \(j\)

\[
R_{\leq j}(t_j) = \frac{1}{I - i\lambda C_{\leq j}(t_j)\sigma}.
\]

The specific part of the interaction which should be attributed to the scale \(j\) is the sum over all loop vertices with at least one propagator at scale \(j\) and all others at scales \(\leq j\). Hence it is given by

\[
V_j = V_{\leq j} - V_{\leq j-1} = \Tr \log_2 [I - i\lambda C_{\leq j}(1)\sigma] - \Tr \log_2 [I - i\lambda C_{\leq j}(0)\sigma] - (E_{\leq j} - E_{\leq j-1})
\]

\[
= \lambda^2 \int_0^1 dt_j \left[ \Tr(R_{\leq j}(t_j)C_{\leq j}(t_j)\sigma C_{\leq j}(\sigma)) - \int d\nu(\sigma) \Tr(C_{\leq j}(t_j)\sigma C_{\leq j}(\sigma)) \right].
\]

In this formula it is clear that the second term (the vacuum counterterm) exactly cancels the expectation value of the leading order term of the first term, in which the resolvent is replaced by \(I\).

As in \[47\] we encode this factorization of the interaction through Grassmann numbers as

\[
Z(g, j_{max}) = \int d\nu(\sigma) \prod_{j=0}^{j_{max}} e^{-V_j} = \int d\nu(\sigma) \left( \prod_{j=0}^{j_{max}} d\mu(\bar{\chi}_j, \chi_j) \right) e^{-\sum_{j=0}^{j_{max}} \bar{\chi}_j W_j(\sigma) \chi_j},
\]

where \(d\mu(\bar{\chi}, \chi) = d\bar{\chi} d\chi \, e^{-\bar{\chi} \chi}\) is the standard normalized Grassmann Gaussian measure with covariance 1 and \(W_j(\sigma) = e^{-V_j} - 1\).

\(^3\)Beware we choose the convention of lower indices for slices, as in \[47\], not upper indices as in \[49\].
3 The Multiscale Loop Vertex Expansion

We perform now the two-level jungle expansion of [47]. For completeness we summarize the main steps, referring to [47] for details.

Considering the set of scales \( S = [0, j_{\text{max}}] \), we denote \( \mathbb{I}_S \) the \( |S| \) by \( |S| \) identity matrix. Then we rewrite the partition function as:

\[
Z(g, j_{\text{max}}) = \sum_{n=0}^{\infty} \frac{1}{n!} \int d\nu_S e^{-W}, \quad d\nu_S = d\nu(\sigma) \, d\mu_g(\{\bar{\chi}_j, \chi_j\}), \quad W = \sum_{j=0}^{j_{\text{max}}} \bar{\chi}_j W_j(\bar{\sigma}) \chi_j. \tag{3.1}
\]

The first step expands to infinity the exponential of the interaction:

\[
Z(g, j_{\text{max}}) = \sum_{n=0}^{\infty} \frac{1}{n!} \int d\nu_S (-W)^n. \tag{3.2}
\]

The second step introduces Bosonic replicas for all the vertices in \( V = \{1, \ldots, n\} \):

\[
Z(g, j_{\text{max}}) = \sum_{n=0}^{\infty} \frac{1}{n!} \int d\nu_{S,V} \prod_{a=1}^{n} (-W_a), \tag{3.3}
\]

so that each vertex \( W_a \) has now its own set of three Bosonic matrix fields \( \bar{\sigma}^a = \{(\sigma^1)^a, (\sigma^2)^a, (\sigma^3)^a\} \). The replicated measure is completely degenerate between replicas (each of the three colors remaining independent of the others):

\[
d\nu_{S,V} = d\nu_V(\{\bar{\sigma}^a\}) \, d\mu_g(\{\bar{\chi}_j, \chi_j\}), \quad W_a = \sum_{j=0}^{j_{\text{max}}} \bar{\chi}_j W_j(\bar{\sigma}^a) \chi_j. \tag{3.4}
\]

The obstacle to factorize the functional integral \( Z \) over vertices and to compute \( \log Z \) lies in the Bosonic degenerate blocks \( 1_V \) and in the Fermionic fields. In order to remove this obstacle we need to apply two successive forest formulas [55, 56], one Bosonic, the other Fermionic. The main difference with [47] is that the Bosonic forest will be \textit{three-colored} since there are three colors for the intermediate matrix fields.

To analyze the block \( 1_V \) in the measure \( dv \) we introduce coupling parameters \( x_{ab} = x_{ba}, x_{aa} = 1 \) between the Bosonic vertex replicas. Since there are three colors, and since the interpolation parameters are color-blind, we obtain a sum over three-colored forests. Representing Gaussian integrals as derivative operators as in [47] we have

\[
Z(g, j_{\text{max}}) = \sum_{n=0}^{\infty} \frac{1}{n!} \left[ e^{\frac{1}{2} \sum_{k=1}^{n} x_{ab} \sum_{c=1}^{3} \frac{\partial}{\partial \sigma^c_{\ell_B}} \frac{\partial}{\partial \sigma^c_{\ell_B}} + \sum_{j=0}^{j_{\text{max}}} \frac{\partial}{\partial \chi_{\ell_B}} \frac{\partial}{\partial \chi_{\ell_B}} \right] n \left( \prod_{a=1}^{n} (-\sum_{j=0}^{j_{\text{max}}} \bar{\chi}_j W_j(\bar{\sigma}^a) \chi_j) \right) \left(\chi_{\ell_B}^x \sigma_{\ell_B}^x\right). \tag{3.5}
\]

The third step applies the standard Taylor forest formula of [55, 56] to the \( x \) parameters. We denote by \( F^3_{B} \) a three-colored Bosonic forest with \( n \) vertices labelled \( \{1, \ldots, n\} \). It means an acyclic set of edges over \( V \) in which each edge \( \ell_B \) has a specific color \( c(\ell) \in \{1, 2, 3\} \). For \( \ell_B \) a generic edge of the forest we denote by \( a(\ell_B), b(\ell_B) \) the end vertices of \( \ell_B \). The result of the Taylor forest formula is:

\[
Z(g, j_{\text{max}}) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{F^3_B} \int_0^1 \left( \prod_{\ell_B \in F^3_B} d\nu_{F^3_B} \right) \left[ e^{\frac{1}{2} \sum_{k=1}^{n} X_{ab}(w_{\ell_B}) \sum_{c=1}^{3} \frac{\partial}{\partial \sigma^c_{\ell_B}} \frac{\partial}{\partial \sigma^c_{\ell_B}} + \sum_{j=0}^{j_{\text{max}}} \frac{\partial}{\partial \chi_{\ell_B}} \frac{\partial}{\partial \chi_{\ell_B}} \right] \prod_{\ell_B \in F^3_B} \left( \frac{\partial}{\partial (\sigma^c(\ell_B))} \frac{\partial}{\partial (\sigma^c(\ell_B))} \right) n \left( \prod_{a=1}^{n} (-\sum_{j=0}^{j_{\text{max}}} \bar{\chi}_j W_j(\bar{\sigma}^a) \chi_j) \right) \left(\chi_{\ell_B}^x \sigma_{\ell_B}^x\right), \tag{3.6}
\]

where \( X_{ab}(w_{\ell_B}) \) is the infimum over the parameters \( w_{\ell_B} \) in the unique path in the forest \( F^3_B \) connecting \( a \) to \( b \). This infimum is set to 1 if \( a = b \) and to zero if \( a \) and \( b \) are not connected by the forest [55, 56].

The colored forest \( F^3_B \) partitions the set of vertices into blocks \( B \) corresponding to its connected components. In each such block the edges of \( F^3_B \) form a spanning tree. Remark that such blocks can be reduced
to single vertices. Any vertex \( a \) belongs to a unique Bosonic block \( B \). Contracting every Bosonic block to an “effective vertex” we obtain a reduced set which we denote \( \{ n \}/F_B \).

The fourth step introduces replica Fermionic fields \( \chi^B_{j a} \) for these blocks of \( F^3_B \) (i.e. for the effective vertices of \( \{ n \}/F^3_B \)) and replica coupling parameters \( y_{BB'} = y_{BB'} \). The fifth and last step applies (once again) the forest formula, this time for the \( y \)'s, leading to a set of Fermionic edges \( L_F \) forming an (uncolored) forest in \( \{ n \}/F^3_B \) (hence connecting Bosonic blocks). Denoting \( L_F \) a generic Fermionic edge connecting blocks and \( B(L_F), B'(L_F) \) the end blocks of the Fermionic edge \( L_F \) we follow exactly the same steps than in [47] and obtain a two level-jungle formula [56] in which the first level is three-colored and the second level is uncolored. The result writes

\[
Z(g,j_{max}) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{J_1 = 0}^{j_{max}} \cdots \sum_{J_n = 0}^{j_{max}} \int d\nu_J \int dw_J \partial_J \left[ \prod_{B \in B} \left( \prod_{a \in B} (W_{ja}(\vec{a}) \chi^B_{ja} \chi^B_{ja}) \right) \right], \tag{3.7}
\]

where

- the sum over \( J \) runs over all two-level jungles, the first level of which is three-colored, hence over all ordered pairs \( J = (F^3_B, F_F) \) of two (each possibly empty) disjoint forests on \( V \), such that \( F^3_B \) is a three colored-forest, \( F_F \) is an uncolored forest and \( J = F^3_B \cup F_F \) is still a forest on \( V \). The forests \( F^3_B \) and \( F_F \) are the Bosonic and Fermionic components of \( J \). Bosonic edges \( \ell_B \in F^3_B \) have a well-defined color \( c(\ell) \in \{1, 2, 3\} \) and Fermionic edges \( \ell_F \in F_F \) are uncolored.

- \( \int dw_J \) means integration from 0 to 1 over parameters \( w_\ell \), one for each edge \( \ell \in J \), namely \( \int dw_J = \prod_{\ell \in J} \int_0^1 dw_\ell \). There is no integration for the empty forest since by convention an empty product is 1. A generic integration point \( w_J \) is therefore made of \( |J| \) parameters \( w_\ell \in [0,1] \), one for each \( \ell \in J \).
\[ \partial J = \prod_{\ell_B \in \mathcal{F}_B} \left( \frac{\partial}{\partial (\sigma^c(\ell_B))} \right)^a \left( \frac{\partial}{\partial (\sigma^e(\ell_B))} \right)^b \prod_{\ell_F \in \mathcal{F}_F} \delta_{j_B}( \frac{\partial}{\partial \chi_{j_B}^{B(d)}} ) \prod_{\ell_F \in \mathcal{F}_F} \delta_{j_F}( \frac{\partial}{\partial \chi_{j_F}^{B(e)}} ) + \frac{\partial}{\partial \chi_{j_F}^{B(e)}} \frac{\partial}{\partial \chi_{j_F}^{B(d)}} ) = \frac{\partial}{\partial \chi_{j_F}^{B(e)}} \frac{\partial}{\partial \chi_{j_F}^{B(d)}} , \] (3.8)

where \( B(d) \) denotes the bosonic block to which the vertex \( d \) belongs.

- The measure \( d\nu J \) has covariance \( X(w_{\ell_B}) \otimes 1_S \) on bosonic variables and \( Y(w_{\ell_F}) \otimes 1_S \) on fermionic variables, hence

\[
\int d\nu J F = \left[ \frac{1}{e} \sum_{a,b} X_{ab}(w_{\ell_B}) \sum_{c,e} \frac{\partial}{\partial (\sigma^c)} + \sum_{B,B'} Y_{BB'}(w_{\ell_F}) \sum_{a \in B, b \in B'} \delta_{ja} \frac{\partial}{\partial \chi_{ja}} \frac{\partial}{\partial \chi_{ja}} \right]_{\sigma=\bar{\chi}=\chi=0} .
\] (3.9)

- \( X_{ab}(w_{\ell_B}) \) is the infimum of the \( w_{\ell_B} \) parameters for all the bosonic edges \( \ell_B \) in the unique path \( P_{a \rightarrow b} \) from \( a \) to \( b \) in \( \mathcal{F}_B \). The infimum is set to zero if such a path does not exist and to 1 if \( a = b \).

- \( Y_{BB'}(w_{\ell_F}) \) is the infimum of the \( w_{\ell_F} \) parameters for all the fermionic edges \( \ell_F \) in any of the paths \( P_{a \rightarrow b} \cup \mathcal{F}_F \) from some vertex \( a \in B \) to some vertex \( b \in B' \). The infimum is set to 0 if there are no such paths, and to 1 if such paths exist but do not contain any fermionic edges.

Remember that a main property of the forest formula is that the symmetric \( n \times n \) matrix \( X_{ab}(w_{\ell_B}) \) is positive for any value of \( w_{\ell_B} \), hence the Gaussian measure \( d\nu J \) is well-defined. The matrix \( Y_{BB'}(w_{\ell_F}) \) is also positive, with all elements between 0 and 1. Since the slice assignments, the fields, the measure and the integrand are now factorized over the connected components of \( \bar{J} \), the logarithm of \( Z \) is easily computed as exactly the same sum but restricted to two-level spanning trees (whose first level is three-colored):

\[
\log Z(g, j_{\text{max}}) = \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\text{tree}} \sum_{j_1=1}^{j_{\text{max}}} \cdots \sum_{j_n=1}^{j_{\text{max}}} \int d\nu J \int d\nu J \prod_B \prod_a \left( W_{ja}(\sigma^a) \chi_{ja}^B \chi_{ja}^B \right) ,
\] (3.10)

where the sum is the same but conditioned on \( \bar{J} = \mathcal{F}_B \cup \mathcal{F}_F \) being a spanning tree on \( V = [1, \cdots, n] \).

Our main result is

**Theorem 3.1** Fix \( \rho > 0 \) small enough. The series (3.10) is absolutely and uniformly in \( j_{\text{max}} \) convergent for \( g \) in the small open cardioid domain \( \text{Card}_\rho \) defined by \( |g| < \rho \cos^2(\text{Arg } g)/2 \) (see Figure 4). Its ultraviolet limit \( \log Z(g) = \lim_{j_{\text{max}} \to \infty} \log Z(g, j_{\text{max}}) \) is therefore well-defined and analytic in that cardioid domain; furthermore it is the Borel sum of its perturbative series in powers of \( g \).

![Figure 4: A Cardioid Domain](image)

The rest of the paper is devoted to the proof of this Theorem.
4 The Bounds

4.1 Grassmann Integrals

The Grassmann Gaussian part of the functional integral [3.10] is also treated exactly as in [47], resulting in the same computation:

\[
\int \prod_{a \in B} (d\chi_a^B d\bar{\chi}_a^B) e^{-\sum_a \chi_a^B \bar{\chi}_a^B} \prod_{i \in F \subset (a,b)} \delta_{i,j} \left( \frac{\chi_a^B}{\chi_b^B} + \frac{\bar{\chi}_a^B}{\bar{\chi}_b^B} \right) \]

where \( k = |F| \), the sum runs over the \( 2^k \) ways to exchange an \( a_i \) and a \( b_j \), and the \( Y \) factors are (up to a sign) the minors of \( Y \) with the lines \( b_1 \ldots b_k \) and the columns \( a_1 \ldots a_k \) deleted. The most important factor in (4.1) is \( (\prod_{a \in B} \prod_{a \neq b} (1 - \delta_{j,ab})) \) which means that the scales obey to a hard core constraint inside each block. Positivity of the \( Y \) covariance means as usual that the \( Y \) minors are all bounded by 1 [57, 47], namely for any \( a_1, \ldots, a_k \) and \( b_1, \ldots, b_k \),

\[
|Y_{a_1 \ldots a_k}^{b_1 \ldots b_k}| \leq 1. \tag{4.2}
\]

4.2 Bosonic Integrals

The main problem is now the evaluation of the Bosonic integral in (3.10). Since it factorizes over the Bosonic blocks, it is sufficient to bound separately this integral in each fixed block \( B \). In such a block the Bosonic forest \( F_B^c \) restricts to a three-colored Bosonic tree \( T_B^{3c} \), and the Bosonic Gaussian measure \( d\nu \) restricts to \( d\nu_B \) defined by

\[
\int d\nu_B F_B = \left[ e^{\frac{1}{2} \sum_{a,b \in B} X_{ab}(w_{ij})} \sum_{\bar{\sigma} \in \bar{\sigma}^B} \prod_{a \in B} \frac{\partial}{\partial \bar{\sigma}^a} F_B \right]_{\sigma = 0}. \tag{4.3}
\]

The Bosonic integrand \( F_B = \prod_{a \in B} (\sigma_{ij})^{1/2} (\sigma_{ij}^{(a)})^{1/2} \prod_{a \in B} \left( W_{ja} (\sigma_{ij}) \right) \) can be written in shorter notations as

\[
F_B = \prod_{a \in B} \left[ \prod_{s \in S_a^B} \partial \sigma_{ja} W_{ja} \right]. \tag{4.4}
\]

where \( S_a^B \) runs over the set of all edges in \( T_B^{3c} \) which end at vertex \( a \), hence \( |S_a^B| = d_a(T_B^{3c}) \), the degree or coordination of the tree \( T_B^{3c} \) at vertex \( a \). To each element \( s \) is therefore associated a well-defined color and well-defined matrix elements (which have to be summed later after identifications are made through the edges of \( T_B^{3c} \)).

When \( B \) has more than one vertex, since \( T_B^{3c} \) is a tree, each vertex \( a \in B \) is touched by at least one derivative and we can replace \( W_{ja} = e^{-V_{ja}} - 1 \) by \( e^{-V_{ja}} \) (the derivative of 1 giving 0) and write

\[
F_B = \prod_{a \in B} \left[ \prod_{s \in S_a^B} \partial \sigma_{ja} e^{-V_{ja}} \right]. \tag{4.5}
\]

We can evaluate the derivatives in (4.5) through the Faà di Bruno formula:

\[
\prod_{s \in S} \partial \sigma_{ja} f(g(\sigma)) = \sum_{\pi} f^{[\pi]}(g(\sigma)) \prod_{b \in \pi} \left( \prod_{s \in b} \partial \sigma_{ja} g(\sigma) \right), \tag{4.6}
\]

where \( \pi \) runs over the partitions of the set \( S \) and \( b \) runs through the blocks of the partition \( \pi \). In our case \( f \), the exponential function, is its own derivative, hence the formula simplifies to

\[
F_B = \prod_{a \in B} e^{-V_{ja}} \left[ \sum_{\pi} \prod_{b \in \pi} \left( \prod_{s \in b} \partial \sigma_{ja} (-V_{ja}) \right) \right]. \tag{4.7}
\]
where \( \pi^a \) runs over partitions of \( S_B^a \) into blocks \( b^a \).

We recall that
\[
V_j = \lambda^2 \int_0^1 dt_j \left[ \text{Tr}(R_{\leq j} C_{\leq j} \bar{\sigma} C_j \bar{\sigma}) - \int d\nu(\bar{\sigma}) \text{Tr}(C_{\leq j} \bar{\sigma} C_j \bar{\sigma}) \right],
\]
using the shortened notations \( C_{\leq j} \) for \( C_{\leq j}(t_j) \) and \( R_{\leq j} \) for \((1 - i\lambda[|C_{\leq j}(t_j)|\bar{\sigma}])^{-1} \). The second term is a constant which does not depend on \( \bar{\sigma} \). Hence remembering that \( \partial \sigma \) really stand for a derivative with well defined color and matrix elements \( \partial \sigma^c_{n^c, \bar{n}^c} \), we get
\[
\partial \sigma_1(-V'_j) = -\lambda^2 \int_0^1 dt_j \text{Tr} \left( R_{\leq j} C_{\leq j} \left[ i\lambda \Delta^1 R_{\leq j} C_{\leq j} \bar{\sigma} C_j \bar{\sigma} + \bar{\sigma} C_j \Delta^1 + \Delta^1 C_j \bar{\sigma} \right] \right),
\]
\[
\prod_{s=1}^k \partial \sigma_s(-V_j) = -\lambda^2 \sum_{\tau} \int_0^1 dt_j \text{Tr} \left( \prod_{s=1}^k i\lambda R_{\leq j} C_{\leq j} \Delta^{(s)} \right) R_{\leq j} C_{\leq j} \bar{\sigma} C_j \bar{\sigma}
\]
\[
+ \prod_{s=2}^k i\lambda R_{\leq j} C_{\leq j} \Delta^{(s)} \right) R_{\leq j} C_{\leq j} \left[ \Delta^{(1)} C_j \bar{\sigma} + \bar{\sigma} C_j \Delta^{(1)} \right]
\]
\[
+ \prod_{s=3}^k i\lambda R_{\leq j} C_{\leq j} \Delta^{(s)} \right) R_{\leq j} C_{\leq j} \Delta^{(2)} C_j \Delta^{(1)} \right) \text{ for } k \geq 2.
\]

In the second formula the sum over \( \tau \) runs over the permutations of \([1, k]\) and \( \Delta^s \), defined as \( (\Delta^s)^{m,n} = \frac{i\lambda}{\partial \sigma_{n^c, \bar{n}^c}} \), is the tensor product of the identity matrix on colors \( c' \neq c_s \) with the matrix on color \( c_s \), with zero entries everywhere except at position \( n^c_s, \bar{n}^c_s \) where it has entry one. These formulas express the derivatives of the trace as a sum over all cycles with exactly \( k \) derivatives, and at most two remaining numerator \( \sigma \) fields. Accordingly they have \( k \) to \( k + 2 \) numerator propagators, and one of them must be a \( C_j \), hence exactly of scale \( j \).

The Bosonic integral in a block can be written therefore in a simplified manner as:
\[
\int d\nu B = \sum_G \int d\nu a \prod_{a \in B} e^{-V_a(\nu)} A_G(\sigma).
\]

where we gather the result of the derivatives as a sum over graphs \( G \) of corresponding amplitudes \( A_G(\sigma) \).

These graphs \( G \) are still forests, with effective loop vertices, one for each \( b^a \in \pi^a, a \in B \), each of them expressed as a trace of a product of three-stranded operators by \( (4.9) \), with \( k = |b^a| \). Each such effective vertex of \( G \) bears at most two \( \sigma \) insertions plus exactly \( |b^a| \Delta \) insertions, which are contracted together via the colored edges of the tree \( T_{B^a} \). Each corner (here we define a corner as the pair made of two consecutive \( \Delta^s \) or of a \( \Delta \) and a neighboring \( \sigma \) insertion) of the vertices bears a \( R_{\leq j}, C_{\leq j} \) operator, except one distinguished corner which bears a \( C_{j,c} \) operator and no resolvent.

Note that to each initial \( W_{j, a} \) may correspond several effective loop vertices \( V_{bs} \), depending of the partitioning of \( S_B^a \) in \( (4.7) \). Therefore although at fixed \( |B| \) the number of (colored) edges \( E(G) \) for any \( G \) in the sum \( (4.10) \) is exactly \( |B| - 1 \), the number of connected components \( c(G) \) is not fixed but simply bounded by \( |B| - 1 \) (each edge can belong to a single connected component). Similarly the number \( V(G) = c(G) + E(G) \) of effective loop vertices of \( G \) is not fixed, and simply obeys the bounds
\[
|B| \leq V(G) \leq 2(|B| - 1).
\]

From now on we shall simply call “vertices” the effective loop vertices of \( G \), as we shall no longer meet the initial \( W_{j, a} \) vertices.

When the block \( B \) is reduced to a single vertex \( a \), we have a simpler contribution for which, an important cancellation occurs due to the presence of the logarithmically divergent counter term in \( V_j \). More precisely
**Lemma 4.1**

4.3 Non-Perturbative Bound

with respect to one \(\sigma\) scalar product between the \(\Delta\) and \(\Delta'\) where in the third line we used from the non-perturbative factor:

$$d\nu$$ with respect to the positive measure interaction, and can be treated therefore exactly as the ones with two or more vertices.

Again to perturbatively convergent graphs with at least two vertices multiplied by the exponential of the contribution of a single vertex corresponds therefore expected this formula shows that the vacuum expectation value of the graph made of a single vertex has been successfully canceled by the counter term. The contribution of a single vertex corresponds therefore again to perturbatively convergent graphs with at least two vertices multiplied by the exponential of the interaction, and can be treated therefore exactly as the ones with two or more vertices.

In all cases (including the single isolated blocks treated in (4.12)) we apply a Cauchy-Schwarz inequality with respect to the positive measure \(d\nu_B\) to separate the perturbative part “down from the exponential” from the non-perturbative factor:

$$|\int d\nu_B F_B| \leq \sum_G \left( \int d\nu_B \prod_a e^{2|V_{j_a}(\sigma_a)|} \right)^{1/2} \left( \int d\nu_B |A_G(\sigma)|^2 \right)^{1/2}.$$ \hspace{1cm} (4.13)

4.3 Non-Perturbative Bound

**Lemma 4.1** For \(g\) in the cardioid domain \(\text{Card}_p\) we have

$$|V_j(\sigma)| \leq \rho \left[ \text{Tr}(C_{\leq j} \bar{\sigma} C_j \bar{\sigma}) + c \right].$$ \hspace{1cm} (4.14)
Proof Using the bound \((2.17)\) for the counter term we have
\[
|V_j(\sigma)| \leq |g| e^{+g} \int_0^1 dt_j |\text{Tr}(C_j \delta R_{j\leq j} C_{\leq j} \delta)|
\]
\[
= |g| e^{+g} \int_0^1 dt_j |\text{Tr}(C_{j\leq j}^{1/2} \delta C_j \delta C_{\leq j}^{1/2} C_{\leq j}^{-1/2} R_{j\leq j} C_{\leq j}^{1/2})|)
\]
\[
\leq \rho e + \rho \int_0^1 dt_j |\text{Tr}(C_{\leq j}^{1/2} \delta C_j \delta C_{\leq j}^{1/2})| = \rho [\int_0^1 dt_j |\text{Tr}(C_{\leq j}^{1/2} \delta C_j \delta)| + c].
\]
(4.15)

Simply notice that for \( A \) positive\(^4\) Hermitian and \( B \) bounded we have \( |\text{Tr} AB| \leq \|B\| \text{Tr} A \). Indeed if \( B \) is diagonalizable with eigenvalues \( \mu_i \), computing the trace in a diagonalizing basis we have \( |\sum_i A_{ii} \mu_i| \leq \max_i |\mu_i| |\sum_i A_{ii}| \); if \( B \) is not diagonalizable we can use a limit argument. We can now remark that for any Hermitian operator \( L \) we have, if \( |\text{Arg} g| = |\phi| < \pi \), \( \|I - i \sqrt{g} L\| \leq \cos^{-2}(\phi/2) \). We can therefore apply these arguments to \( A = C_{j\leq j}^{1/2} \delta C_j \delta C_{\leq j}^{1/2} \) (which is Hermitian positive) and \( B = C_{\leq j}^{-1/2} R_{j\leq j} C_{\leq j}^{1/2} \). Indeed
\[
|B| = |L_{j\leq j}| = \|\sum_i (1 - \sum_i A_{ii} \mu_i) \delta C_{\leq j}^{1/2} \delta C_{\leq j}^{1/2} | \leq \cos^{-2}(\phi/2).
\]
(4.16)

We conclude since in the cardioid \( |g| \cos^{-2}(\phi/2) \leq \rho \).

We can now bound the first factor in the Cauchy-Schwarz inequality (4.13).

**Theorem 4.2 (Bosonic Integration)** For \( \rho \) small enough and for any value of the \( w \) interpolating parameters
\[
\left( \int dv_B \prod_{a \in B} e^{2|V_a(\sigma^a)|} \right)^{1/2} \leq e^{O(1)\rho |B|}.
\]
(4.17)

**Proof** The term \( \prod_{a \in B} e^{2|V_a(\sigma^a)|} \) is simply \( e^{\rho |B|} \). Applying Lemma 4.1 we get
\[
\int dv_B \prod_{a \in B} e^{2|V_a(\sigma^a)|} \leq e^{\rho \rho |B|} \int dv_B e \langle \sigma, Q \sigma \rangle
\]
(4.18)

where \( Q \) is a symmetric positive matrix in the big vector space \( V \) which includes color, components and vertices indices \( a \in B \). This big space has dimension \( N_V = 3|B|N_{\text{max}}^2 \). Hence the matrix \( Q \) is an \( N_V \) by \( N_V \) matrix. More precisely \( Q \) is defined by the equation:
\[
\langle \sigma, Q \sigma \rangle = \sum_{a \in B} <\sigma^a, Q^a \sigma^a> + <\sigma^a, Q^a \sigma^a> \equiv 2 \rho \int_0^1 dt_j \rho |\text{Tr}(C_{\leq j} \delta C_{j} \delta C_{\leq j}^\dagger)|
\]
(4.19)

Hence \( Q = \sum_{a \in B} Q^a \), where \( Q^a \) is the \( N_V \) by \( N_V \) matrix with all elements zero except the \( 3N_{\text{max}}^2 \) by \( 3N_{\text{max}}^2 \) which have both vertex indices equal to \( a \). These non zero elements form the \( 3N_{\text{max}}^2 \) by \( 3N_{\text{max}}^2 \) positive symmetric matrix \( Q^a \) with matrix elements
\[
Q^a_{\cdot m,n; c'c'n'} = Q^a_{\cdot m,n; c'c'n'} + Q^a_{\cdot m,n; c'c'n'}
\]
\[
Q^a_{\cdot m,n; c'c'n'} = 2 \rho \delta_{c,c'} \int_0^1 dt_j \rho \rho |\text{Tr}(C_{\leq j} \delta C_{j} \delta C_{\leq j}^\dagger)|
\]
(4.20)

where the \( D \) factors are defined respectively as the color-diagonal and color off-diagonal part of a bubble with two propagators of slices \( j \) and \( k \):
\[
D_{j,k,1}(m,n) = \sum_{m_2,m_3} C_k(m,m_2,m_3)C_j(m,n,m_2,m_3),
\]
(4.21)
\[
D_{j,k,2}(m,m') = \sum_{m_3} C_k(m,m_3,m')C_j(m,m',m_3).
\]
(4.22)

\(^4\)We usually simply say positive for “non-negative”, i. e. each eigenvalue is strictly positive or zero.
The big matrix $Q$ has elements $Q_{a,c,m,n; a',c',m',n'} = \delta_{a,a'}Q^a_{c,m,n; c',m',n'}$. Using the bounds (2.14) it is easy to check that

$$D_{j,k,1}(m, n) \leq O(1)M^{-2j}e^{-M^{-j}|n|}e^{-M^{-j}|m|},$$
$$D_{j,k,2}(m, m') \leq O(1)M^{-2j-k}e^{-M^{-k}(|m|+|m'|)}.$$  \hfill (4.23)

**Lemma 4.3** The following bounds hold uniformly in $j_{\text{max}}$ and $N_{\text{max}}$

$$\mathrm{Tr} \ Q^a \leq O(1)\rho,$$  \hfill (4.24)
$$\|Q^a\| \leq O(1)\rho j_{\text{a}}M^{-2j_{\text{a}}}.$$ \hfill (4.25)

**Proof** The first bound is easy. Since we compute a trace, only $Q^{a,1}$ contributes and the bound follows from (4.21) which implies that $\sum_{m,n} D_{j,1}(m, n) \leq O(1)$. Since $Q^{a,3}$ is diagonal both in component and color space, from (4.21) we deduce that $\sup_{m,n} D_{j,1}(m, n) \leq O(1)jM^{-2j}$, hence

$$\|Q^{a,1}\| \leq O(1)\rho j_{\text{a}}M^{-2j_{\text{a}}}.$$ \hfill (4.26)

Finally to bound $\|Q^{a,2}\|$ we use first a triangular inequality to sum over the 6 pairs of colors $c, c'$ and over $k$

$$\|Q^{a,2}\| \leq 12\rho \sum_{k=0}^j \|E_{j_{\text{a}},k,2}\|$$ \hfill (4.27)

where $E_{j_{\text{a}},k,2}$ is the (component space) matrix with matrix elements

$$E_{j_{\text{a}},k,2}(m, n; m', n') = \delta_{m,m'}\delta_{n,n'}D_{j_{\text{a}},k,2}(m, m').$$ \hfill (4.28)

The operator norm of $E_{j_{\text{a}},k,2}$ is bounded by its Hilbert Schmidt norm

$$\|E_{j_{\text{a}},k,2}\|_2 = \left(\sum_{m,m'} D_{j_{\text{a}},k,2}(m, m')^2\right)^{1/2} \leq O(1)M^{-2j_{\text{a}}} - k[M^{2k}]^{1/2} = O(1)M^{-2j_{\text{a}}}.$$ \hfill (4.29)

Hence

$$\|Q^{a,2}\| \leq O(1)\rho j_{\text{a}}M^{-2j_{\text{a}}}.$$ \hfill (4.30)

hence gathering (4.26) and (4.30) proves (4.25). \hfill $\square$

The covariance $X$ of the Gaussian measure $d\nu_{\Psi}$ is also a symmetric matrix on the big space $V$, but which is the tensor product of the identity in color and component space times the matrix $X_{ab}(w_{\ell_{\Psi}})$ in the vertex space. Defining $A = XQ$, we have

**Lemma 4.4** The following bounds hold uniformly in $j_{\text{max}}$ and $N_{\text{max}}$

$$\mathrm{Tr} \ A \leq O(1)\rho |B|,$$ \hfill (4.31)
$$\|A\| \leq O(1)\rho.$$ \hfill (4.32)

**Proof** Since $Q = \sum_{a \in B} Q^a$ we find that

$$\mathrm{Tr} \ A = \sum_{a \in B} \mathrm{Tr} XQ^a = \sum_{a \in B} X_{aa}(w_{\ell_{\Psi}}) \mathrm{Tr} Q^a = \sum_{a \in B} \mathrm{Tr} Q^a \leq O(1)\rho |B|.$$ \hfill (4.33)

where in the last inequality we used (4.24). Furthermore by the triangular inequality and (4.25)

$$\|A\| \leq \sum_{a \in B} \|XQ^a\| = \sum_{a \in B} X_{aa}(w_{\ell_{\Psi}})\|Q^a\| = \sum_{a \in B} \|Q^a\| \leq \sum_{j=0}^\infty O(1)\rho jM^{-2j} \leq O(1).$$ \hfill (4.34)

where we used the fundamental fact that all vertices $a \in B$ have different scales $j_a$. \hfill $\square$

We can now complete the proof of Theorem 4.2. By (4.32) for $\rho$ small enough the series $\sum_{n=1}^\infty (\mathrm{Tr} A^n)/n$ converges and we have

$$\int d\nu_{\Psi} \ e^{\langle a, Q\sigma \rangle} = [\det(1 - A)]^{-1/2} = e^{-1/2} \mathrm{Tr} \log(1 - A) = e(1/2) \sum_{n=1}^\infty (\mathrm{Tr} A^n)/n \leq e(1/2) \mathrm{Tr} A[\sum_{n=1}^\infty \|A\|^{n-1}] \leq eO(1)\rho |B|.$$ \hfill (4.35)

$\square$
4.4 Graph Bounds

We still have to bound the second factor in (4.13), namely \( \left( \int d\nu^{|B|} |A_G(\sigma)|^2 \right)^{1/2} \). We recall that at fixed \(|B|\), the graphs \( G \) are forests with \( EG = |B| - 1 \) (colored) edges joining \( V(G) = e(G) + E(G) \) (effective) vertices, each of which has a weight given by (4.9). The number of connected components \( e(G) \) is bounded by \(|B| - 1\), hence (4.11) holds.

This squared amplitude can be represented as the square root of an ordinary amplitude but for a graph \( G' = G \cup G' \) which is the (disjoint) union of the graph \( G \) and its mirror conjugate graph \( G' \) of identical structure but on which each operator has been replaced by its Hermitian conjugate. This overall graph \( G'' \) has thus twice as many vertices, edges, resolvents, \( \bar{\sigma}^a \) insertions and connected components than the initial graph \( G \).

To evaluate the amplitude \( A_{G''} = \int d\nu^{|B|} |A_G(\sigma)|^2 \), we first delete every \( \bar{\sigma}^a \) insertion using repeatedly integration by parts

\[
\int \sigma_{\mu}^a \cdot F(\bar{\sigma})d\nu(\sigma) = \int \frac{\partial}{\partial \sigma_{\mu}^a} F(\bar{\sigma})d\nu(\sigma). \tag{4.36}
\]

The derivatives \( \frac{\partial}{\partial \sigma_{\mu}^a} \) will act on any resolvent \( R_{\mu} \) or remaining \( \bar{\sigma}^a \) insertion of \( G'' \), creating a new contraction edge. When it acts on a resolvent, it creates a new corner bearing a \( R \) insertion, \( \bar{\sigma}^a \) insertion using repeatedly integration by parts, \( |B| - 1 \) factors as a sum over scale assignments \( \mu \), hence will be no problem using the huge decay factors of (4.14).

Theorem 4.5 (Graph bound) The amplitude of a graph \( \mathfrak{G} \) with scale attribution \( \mu \) is bounded by

\[
|A(\mathfrak{G}_\mu)| \leq |O(1)|^p \nu^{E(\mathfrak{G})} M^{-\frac{1}{2}} \sum_{\lambda \in G \lambda(\nu)}. \tag{4.37}
\]

Proof We work at fixed value of each \( \sigma \), and denote \( \mathfrak{C} \) the connected components of a graph \( \mathfrak{G} \), thus \( A(\mathfrak{G}_\mu) = \prod_\mathfrak{C} A(\mathfrak{C}_\mu) \). The amplitude \( A(\mathfrak{C}_\mu) \) of a connected component \( \mathfrak{C} \) can be bounded by iterated Cauchy-Schwarz inequalities [44, 46] using the formula

\[
|\langle \alpha | B \otimes B' \otimes 1^{\otimes p} | \beta \rangle| \leq ||B|| ||B'|| \sqrt{\langle \alpha | \alpha \rangle \sqrt{\langle \beta | \beta \rangle}}, \tag{4.38}
\]

where the scalar product \( \langle f | g \rangle \) means the scalar product in the natural three stranded Hilbert space \( \mathcal{H} \).

First, we rewrite the operators \( R_{\leq j} C_{j'} \) or \( C_{j'} R^*_{{\leq j}} \) (with \( j' \leq j \)) as

\[
R_{\leq j} C_{j'} = C_{j'}^{1/2} B_{j',j}^{1/2} C_{j}^{1/2}, \quad C_{j'} R^*_{{\leq j}} = C_{j'}^{1/2} B_{j',j}^* C_{j}^{1/2}, \tag{4.39}
\]

where \( B_{j',j} = C_{j'}^{-1/2} R_{\leq j} C_{j'}^{1/2} \).

Remember that by (4.10) the operators \( B \) defined by (4.39) are bounded in norm by \( \cos^{-1}(\phi/2) \) for \( g = |g|e^{i\phi} \) in the cardioid.

We then choose a spanning tree for each connected component \( \mathfrak{C} \), and order the operators \( B \) at the corners along the clockwise contour walk of the tree. For simplicity we consider first a connected component with an even number \( 2n \) of operators \( B \), which are then labeled from \( B_1 \) to \( B_{2n} \).
We choose $B_1$ and the antipodal operator $B_{n+1}$ as the marked operators $B$ and $B'$ to apply \eqref{eq:4.38}. Hence we split the tree in two parts, according to the unique path going from the corner 1 to the corner $n$. The vector $\alpha$ is made of everything on the left on the splitting line, and the vector $\beta$ is made on everything on the right. The identity $1^{\otimes p}$ comes from all loop edges which cross the splitting line, as in \cite{36}.

The graphs $\langle \alpha | \alpha \rangle$ and $\langle \beta | \beta \rangle$ have the same structure of plane trees decorated with loop edges and operators on the corners, but each one has only $(2n - 2)$ $B$-operators left. Each graph now has two “cleaned” corners, which bear a $C_j$, but no $B$ operator. We can repeat the same cleaning process on each new graph by ordering the $(2n - 2)$ $B$-operators along the clockwise contour walk of the new graph, then choosing a new pair of antipodal operators as $B$ and $B'$. Repeating the process $n$ times gives thus a geometric mean over $2^n$ final completely cleaned graphs $g$ bearing no $B$ at all, times a product of norms of $B$ operators, which are all bounded by $\cos^{-1}(\phi/2)$ for $g = |g|e^{i\phi}$ in the cardioid. Since these graphs no longer have any dependence on $\sigma$, the normalized measure $\int d\nu(\sigma)$ simply evaluates to 1, and we are left with a perturbative bound:

\[
|A(\mathcal{C}_\mu)| \leq \prod_{i=1}^{2n} ||B_i|| \left( \prod_g A(g) \right)^{1/2n} \leq \left( \frac{\cos \frac{\phi}{2}}{2} \right)^{-2n} \left( \prod_g A(g) \right)^{1/2n}.
\] (4.40)

The Cauchy-Schwarz process keeps track of a number of items. Indeed, at every iteration, each vertex and edge of the bounded graph gives respectively two vertices and two edges of the next-stage graphs. The $C_j$ operators follow the same rule, and each $C_j$ operator of the original graph will generate $2^n$ identical $C_j$ operators in the final graphs, which repartition is a priori unknown. Finally, each vertex of the original graph bearing at least one $B$ operator, each vertex of the final graph has been cut at least once and is thus mirror-symmetric.

In a final graph $g$, each corner bears a $C_j$ or $C_j'$ operator, and strands represent contractions of their indices. The integrand of those operators can be factorized as a tensor product of operators each acting on one color, for which we have the bound

\[
C_j(n, \bar{n}) = \delta_{n, \bar{n}} \int_{M^{-2(j-1)}}^{M^{-2j}} e^{-\alpha(n_i^2 + n_{\bar{i}}^2 + n_{\bar{i}}^2)} d\alpha
\]

\[
= \int_{M^{-2j}}^{M^{-2(j-1)}} d\alpha \prod_{i=1}^{\eta} e^{-\alpha n_i^2} \delta_{n_i, \bar{n}_i}
\]

\[
\leq O(1) M^{-2j} \prod_{i} e^{-M^{-j}|\eta_i|} \delta_{n_i, \bar{n}_i} \quad \text{for} \quad j \geq 1.
\] (4.41)

For a final graph $g$ with $2n$ corners that we index by $\eta$, and denoting $a(\eta)$ the vertex of the original graph that bore the operator,

\[
A(g) = |\lambda|^{2E(g)} \prod_{\eta}^{\{\bar{n}\}} C^{(n)}(n_\eta \bar{n}_\eta) \prod_{\text{strands } s} \delta_{n_\eta, \bar{n}_\eta}
\]

\[
\leq |\lambda|^{2E(g)} \prod_{\eta}^{\{\bar{n}\}} O(1) \left( \sum_{i}^{} e^{-M^{-j\eta_i}|\eta_i|^2} \delta_{n_\eta, \bar{n}_\eta} \right) \prod_{\text{strands } s} \delta_{n_\eta, \bar{n}_\eta}.
\] (4.42)

where $j_\eta \in \{0...j_\eta(\eta)\}$, and $f$ are the faces of color $i$. In the bound, the $C$ operators being removed, the faces are closed cycles of $\delta$ operators multiplied by exponential factors. Hence only one index $n_f$ remains for each colored face $f$. Thus the amplitude of a final graph $g$ is bounded by

\[
A(g) \leq |O(1)|\lambda|^{2E(g)} M^{-2\sum_{j_\eta} j_\eta} \prod_{f} \frac{1}{\sum_{n_f \in f} M^{-j_\eta}}
\]

\[
\leq |O(1)||g||^{E(g)} M^{\sum_{j_\eta n_f} j_\eta - 2\sum_{j_\eta} j_\eta}.
\] (4.43)

Corners of final graphs that were generated by distinguished corners (without resolvents) of the original graph will be denoted $\eta^* \in H^*$, as opposed to regular corners $\eta$. Those corners bear $C_{j_\eta(\eta^*)}$ operators that
For any tree, the relationship between the number we want to keep track of. Other corners bear $C$ operators of scale $j_\eta \leq j_{a(i)}$. The amplitude is thus bounded by

$$
|A(\mathcal{C}_\mu)| \leq \left[ \cos \frac{\phi}{2} \right]^{-2n} \left( \frac{\prod \left[ O(1) |g| \right]^{E(g)}}{\mathcal{B}_\mu} \right)^{\frac{1}{2}} \frac{1}{M} \sum_{\mathcal{g}} \left[ \sum f \eta_{\min}(f) - 2 \sum j_\eta + \frac{1}{2} \sum_{H^*} j_{\eta*} \right]
$$

where we use the conservation of the number of distinguished corners during the Cauchy-Schwarz process $\sum_{\mathcal{g}} \left[ \sum_{H^*} \eta_{a(i)} \right] = 2^n \sum_{\mathcal{g}} j_{a(i)}$, along with the fact that for any graph, $2n < 2E(\mathcal{C})$.

For a connected component with an odd number $2n + 1$ of $B$ operators, we first proceed to a slightly asymmetric Cauchy-Schwarz splitting of the graph, choosing $B^1$ and $B^{n+1}$ as $B$ and $B'$. Both scalar product graphs will then have an even number of $B$ operators and the previous results stand.

**Lemma 4.6** For any connected components $\mathcal{C}_\mu$ with final graphs $g$,

$$
\sum_{\mathcal{g}} \left[ \sum f \eta_{\min}(f) - 2 \sum j_\eta + \frac{1}{2} \sum_{H^*} j_{\eta*} \right] \leq 0.
$$

**Proof** A final graph consists of the gluing of two mirror symmetric graphs along a path whose ends are undistinguished corners $\eta \notin H^*$. Thus a final graph bears at least two undistinguished corners. Therefore,

$$
\sum_{f} 1 - 2 \sum_{\eta} 1 + \frac{1}{2} \sum_{H^*} 1 = F - 2C + \frac{1}{2} |H^*| \leq F - \frac{3}{2} C - 1.
$$

For any tree, the relationship between the number $C$ of corners $\eta$ and the number $F$ of faces $f$ is $F - C = 3$.

This can be proved starting with a single isolated vertex and adding extra vertices and edges one by one. Each new vertex and edge comes with two new faces and two new corners and the isolated vertex had three faces and no corner.

Any loop edge adds two corners and may increase or decrease the number of faces by one. Thus, for a tree $T$ decorated with $L$ loop edges $\ell \in \mathcal{L}$,

$$
(F - \frac{3}{2} C - 1)_{T+L} \leq (F - \frac{3}{2} C - 1)_T - 2L = 3 - V - 2L.
$$

For any graph with at least 3 vertices, or with at least one loop edge, this is lower than 0. A pathological final graph cannot be a single vertex without edges, because final graphs have at least two corners. A final graph composed of two vertices and no loops can only arise from a graph which had two consecutive corners bearing $B$ operators, separated by an edge of the chosen tree, before the last iteration of the Cauchy-Schwarz process.

If a mirror-symmetric graph has only two resolvents, then those resolvents are mirror symmetric and therefore on each side of the symmetry axis, which is a path between two “cleaned” corners (bearing no $B$ operators), therefore there is at least one corner between them. Therefore, a graph with only two remaining resolvents, which are on consecutive corners, cannot arise from the bounding process. Theorem 4.6.2 says that there are only two families of original graphs with less than four $B$ operators, two being separated only by an edge. We will call them $S_2$ and $S_3$ and deal with them with an adapted Cauchy-Schwarz bound that avoid pathological final graphs (Fig. 5).

Therefore, for any final graph, the $j$s brought by corners (2 for undistinguished ones, and 3/2 for distinguished ones) is large enough to cancel the number of $j_{\text{min}}$ brought by the faces. However, each $j_{\text{min}}$ must be canceled individually by a higher $j$.

First, we consider a distinguished corner of scale $j_{a(i)}$. Such a corner is generated by a corner without $B$ operator and thus cannot be used in a Cauchy-Schwarz bound. Therefore, a distinguished corner $\eta^*$ cannot be on a leaf of the final graph, and each vertex being mirror symmetric, they carry an even number of distinguished corners. If a vertex only bears distinguished corners, it is then made of $2k$ replicas of the same corner (and thus brings $3k j_{a(i)}$), and has degree $2k$. A corner of degree $2k$ can belong to at most $3 + k$ faces.
Figure 6: The graphs $S_3$ (left) and $S_2$ (right) with dashed lines representing the Cauchy-Schwarz splitting used to avoid pathological graphs. Dotted corners bear $B$ operators. When the dashed line crosses an un-dotted corner, the propagator $C$ must be rewritten $C^{1/2}1C^{1/2}$ and the identity matrix $1$ is used instead of a $B$ operator.

For $k > 1$, the $3k j_a$ are enough to cancel the $j_{min}$ of all faces the vertex belongs to. For $k=1$ (vertex of degree 2), if the two edges are of different colors, the vertex belongs to only three faces, that are canceled out by the $3j_a$. If the two edges are of the same color $c$, the vertex can belong to two distinct faces of color $c$. If any of those faces also goes through a vertex of degree two bearing two undistinguished corners (and thus bringing $4j_\eta$, enough to cancel all the faces running through it), or a vertex of degree $\geq 2$, its $j_{min}$ will be canceled out by this vertex. If both those faces run only through vertices of degree two bearing only distinguished corners, then the final graph must be a closed cycle of vertices of degree two bearing only distinguished corners, which is impossible. Therefore, $\frac{3}{2} \sum_{H^*} j_{\eta^*}$ is enough to cancel out every potential faces with $j_{min} = j_\eta(\eta^*)$.

For undistinguished corners, the situation is actually better. Each vertex of degree $> 1$ brings enough $j_\eta$ to cancel each faces it belongs to. Only the leaf has one more face than $j_\eta$s. However, one face running through a leaf will also run through its only neighboring vertex, which is of degree two or more (recall that the two-leaves-graph is excluded), and will be canceled out by this vertex. If the neighbor is of degree two, then it has only 3 faces running through it. If it is of degree $> 2$, then it has more than enough $j_\eta$.

Therefore on any final graph, all the $j_{min}$ can be canceled individually by a $j_\eta$, hence we have

$$\sum_{\theta} \left[ \sum_f j_{min}(f) - 2 \sum_\eta j_\eta + \frac{1}{2} \sum_{H^*} j_{\eta^*} \right] \leq 0,$$

and thus,

$$|A(\Phi_\mu)| = \prod_{\mathcal{C}_\mu} |A(\mathcal{C}_\mu)| \leq |O(1)\rho|^{E(\mathcal{C})} M^{-\frac{1}{2}} \sum_{v \in G} j_{a(v)}.$$

(4.49)

5 Conclusion

Once decay in the maximal scale at each vertex has been garnered by (4.49) the remaining sum over scale attributions $\mu$ is completely standard [49]. Similarly the auxiliary sums such as those over $\tau$ and the other terms in (4.9), over partitions $\pi$ in (4.7) (hence over the choice of $G$) and over $\sigma$ contractions (hence over the choice of $\mathcal{C}$) cannot endanger convergence, exactly as in [48, 49]. Indeed the key observation is that in a block $B$ since all slice indices are different and since we cleaned first a distinguished propagator $C_j$, whose decay cannot have disappeared in (4.49), any small power of the product $\prod_{j \in B} e^{-\frac{1}{2} \sum_{v \in G} j_{a(v)}}$ is still smaller than $e^{-O(1)|B|^2}$ for some small $O(1)$, hence amply sufficient to beat any fixed power of $|B|$, such as those generated by the previous sums.

Combinatorial estimates are also exactly similar to those of [47] except for the fact that counting the colored two-level trees requires an additional factor $3^{F_B^{|B|}}$ to choose the colors of Bosonic edges. Hence
Proposition 5.1 The number of two level trees with a three-colored first level over \( n \geq 1 \) vertices is bounded by \( 12^n n^{n-2} \).

Uniform Taylor remainder estimates at order \( p \) are required to complete the proof of Borel summability \[58\] in Theorem 3.1. They correspond to further Taylor expanding beyond trees up to graphs with excess (i.e., number of cycles) at most \( p \). The corresponding mixed expansion is described in detail in \[45\]. The main change is to force for an additional \( p! \) factor to bound the cycle edges combinatorics, as expected in the Taylor uniform remainders estimates of a Borel summable function.

The main theorem of this paper clearly also extends to cumulants of the theory, introducing ciliated trees and graphs as in \[45\]. This is left to the reader. Indeed in tensor theories the relation between such cumulants and ciliated trees in the intermediate field representation is complicated, involving in the general case graphical branching of the cilia and Weingarten functions \[45, 46\], and could detract the reader’s attention from what is new in the tensor field theory case.

The next tasks in constructive tensor field theories would be to treat the \( T_4^4 \) and \( T_5^5 \), which correspond in level of difficulty respectively to \( \phi_4^4 \) and \( \phi_4^4 \) in the ordinary quantum field theory context with local interactions. This would clearly require a much more precise phase space cell expansion. The reward is that ultimately, in contrast with \( \phi_4^4 \), a renormalizable tensor field theory such as \( T_5^5 \) should exist non-perturbatively without cutoffs, since it is asymptotically free \[20\].

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