Simplified Quantum Optical Stokes observables and Bell’s Theorem

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We introduce a simplified form of Stokes operators for quantum optical fields that involve the known concept of binning. Behind polarization analyzer photon numbers (more generally intensities) are measured. If the value obtained in one of the outputs, say H, is greater than in the other one, V, then the value of the simplified Stokes operator is, say, 1, otherwise it is -1. For equal photon numbers we put 0. Such observables do not have all properties of the Stokes operators, but surprisingly can be employed in Bell type measurements, involving polarization analyzers. They are especially handy for states of undefined number of photons, e.g. squeezed vacuum. We show that surprisingly they can lead to quite robust violations of associated Bell inequalities.

I. INTRODUCTION

The discussion about what is the essence of quantumness, started with first attempts of formulating of quantum mechanics. With the emblematic paper of Einstein, Podolsky and Rosen [1] the problem of completeness of quantum mechanics became a point of discussion among scientific community. This started with the response by Bohr [2]. Many years later, after the paper of Bell [3] the challenge of revealing non-classicality, in terms of violation of local realism, has entered to the core of contemporary research. All that in the meantime gained in importance with the emergence of quantum information and communication.

The ultimate test of non-classicality is violation of Bell inequalities. This is now also the essence of testing of device-independent quantum communication protocols. Formulations of Bell’s theorem for situations of fixed numbers of particles have already a vast literature, and well established methods, see e.g. reviews [4–7]. However, if one moves to situations with undefined numbers of particles, still the situation is quite open. This is of course e.g. the case of general quantum optical fields. A lot of approaches are tested.

Polarization entanglement experiments are classic examples of experimental tests of Bell’s inequalities. The two photon experiments are a realization of two qubit-entanglement [8, 9]. A deceptively obvious step in the direction towards optical fields of undefined photon numbers is to use quantum Stokes observables. The usual definition of these runs as follows. If one assumes that the intensity of light is proportional to the photon number, then (standard) quantum Stokes observables are given by

$$\hat{S}_H = \hat{a}^{\dagger} \hat{a} - \hat{a}_{\perp}^{\dagger} \hat{a}_{\perp},$$

where 0 is the vacuum state (of the optical beam in question). It has been shown that such operators allow construction of stronger entanglement criteria and they are a handy tool for formulation of Bell inequalities. One of their properties, which is crucial in this case, is the fact that, such operators have spectrum which consists on all rational numbers from -1 to 1. That is, they have the basic property of observables which allows to derive the CHSH-Bell inequalities. Thus a derivation of a version of CHSH inequality applicable for such Stokes operators essentially is a replacement procedure. With the recent development of measurement techniques allowing photon number resolving detection [15],[16] the discussion about normalized Stokes parameters stops to be only theoretical and its use in experiments is becoming feasible.

Note that what makes Pauli operators so straightforwardly applicable for Bell inequalities is their dichotomic nature. One of the attempts to construct field operators of a similar property was the formulation of pseudo-spin operators. E.g. the z component of pseudo spin is (−1)^n, where n is total photon number in the given optical mode [17],[18]. The spectrum of pseudo-spin operators is the same as spectrum of Pauli matrices, but their use introduces great difficulties from experimental point of view. Even a loss of one photon (due to e.g. detector inefficiency) or a single dark count reverses the result of a
Action of sign function on Stokes operators can be regarded as some form of the binning strategy used in the context of polarization measurements. Binning strategies are e.g. used in homodyne schemes for observing non-classicality [22–25].

We shall call the new operators sign Stokes operators. We denote by \( G_1 \) the sign operator the eigenstates of which refer to \( \{D, A\} \) polarisation basis, and by \( G_2 \) and \( G_3 \) for respectively \( \{R, L\} \) and \( \{H, V\} \) bases.

From all that was said above, one can easily conclude that sign operators share some properties of Stokes and normalized Stokes operators. Also, once one has a photon-number resolving detection setup, the data collected in each run allows to compute the obtained values of each of Stokes operators for the given basis \( i \): standard, normalized, and sign ones, as they depend solely on the measured \( n_i \) and \( n_{i\perp} \). Thus, as we see, the new approach is in fact just a new form of data analysis. Further, in order to measure different sign operators \( G_i \), i.e. in order to move from \( i \) to \( i' \), it is enough to change the polarization analysis basis (see Appendix A). However, not all properties of quantum Stokes and normalized Stokes operators are shared by quantum sign Stokes operators.

A. Stokes vector formed out of sign Stokes operators

For the standard Stokes operators one can construct a Stokes vector i.e. \( \langle \hat{\Theta} \rangle_\psi = \langle \hat{\Theta}_1 \rangle_\psi, \langle \hat{\Theta}_2 \rangle_\psi, \langle \hat{\Theta}_3 \rangle_\psi \rangle \) where \( \vert \psi \rangle \) is an arbitrary state of the optical field. The norm of this vector fulfills: \( \| \langle \hat{S} \rangle_\psi \| \leq \langle \hat{\Theta}_0 \rangle_\psi \). We can construct an analogue vector for normalized Stokes operators and \( \| \langle \hat{S} \rangle_\psi \| \leq \langle \hat{S}_0 \rangle_\psi \leq 1 \) [13]. These norms remain invariant under unitary transformation \( U \) between mutually unbiased polarisation bases i.e.: \( \| \langle \hat{S} \rangle_\psi \| = \| \langle \hat{S}' \rangle_\psi' \| \) where \( \vert \psi' \rangle = U \vert \psi \rangle \) and \( \| \langle \hat{S}' \rangle_\psi' \| = \| \langle \hat{S} \rangle_\psi \| \).

Thus norm of Stokes vectors, standard and normalized, is constant under unitary of the triads polarization analysis bases. These features of Stokes operators play a key role in construction of entanglement indicators involving Stokes operators.

Such properties are not shared by sign operators. Let us construct a vector like triad \( \langle \hat{G} \rangle_\psi = \langle G_1 \rangle_\psi, \langle G_2 \rangle_\psi, \langle G_3 \rangle_\psi \rangle \). It can be shown that \( \| \langle \hat{G} \rangle_\psi \| \neq \| \langle \hat{G} \rangle_\psi \| \). It is enough to find one counterexample. Consider state \( \vert \psi \rangle = \vert 3_H, 0_V \rangle \) i.e. Fock state with 3 photons polarised horizontally. It can be easily checked that for this state \( \| \langle \hat{G} \rangle_\psi \| = 1 \). After performing a \( SO(2) \) rotation of \( \vert \psi \rangle \) by \( \pi/8 \) we get \( \| \langle \hat{G} \rangle_\psi \| \approx 1.5 \) Thus, \( \| \langle \hat{G} \rangle_\psi \| \neq \| \langle \hat{G} \rangle_\psi \| \), i.e. the norm is not an invariant of the unitary transformations and additionally it is not bounded by \( \langle \hat{G}_0 \rangle \). This fact prohibits one to use methods

We address those problems by another normalization scheme, see [20], based on the so-called binning, which we call Sign approach normalization. To obtain new operators we use the sign function and apply it to Stokes operators:

\[
\hat{G}_i = \text{sign}(\hat{n}_i - \hat{n}_{i\perp}) = \text{sign}(\hat{U}_i(\hat{n}_H - \hat{n}_V)U_i^\dagger),
\]

where indices \( H \) and \( V \) refer to horizontal and vertical polarizations and operator \( \hat{U}_i \) is a unitary transformation which transforms polarization modes \( H, V \) into another orthogonal pair of in general elliptic polarization modes \( i \)-th and \( i\perp \) (for examples see: Appendix A). From (2) we see that the eigenstates of the \( i \)-th operator are photon number states of a given mode of a given polarization i.e. \( \vert j \rangle \) and \( \vert k \rangle \). These kets symbolically represent photon number states: \( \vert j \rangle \) and \( \vert k \rangle \). The spectral form of (2) is given by:

\[
\hat{G}(i) = \sum_{k>j} \left( \langle k \vert \langle j \rangle \langle j \vert \langle k \rangle \right) - \left( \langle j \vert \langle k \rangle \langle j \vert \langle k \rangle \right).
\]

Formula (3) clearly shows that new operators are well-defined Hermitian ones and that \( \hat{G}_i \) have three eigenvalues \( \pm 1 \) and 0.
of construction of entanglement indicators presented in [13], which works via a simple replacement of Pauli operators in entanglement conditions for qubits, by Stokes operators, standard or normalized. Still, as we shall see there is no obstacle to use this method in the case of construction of Bell inequalities.

Rotational covariance of polarization variables is not a necessary feature required to derive Bell inequalities (also, see [26] for consequence of demanding exactly that). This allows one to construct CHSH and CH inequalities for fields with sign Stokes observables. The main goal of proposing new operators, as we have said earlier, is to enable detecting nonclassicality for optical fields of higher mean intensity than in case of normalized Stokes operators. Also questions such as resistance to noise and losses of new operators are of our concern.

B. CHSH inequality

We start with defining local hidden values which pre-determine output of a measurement of sign Stokes operators (2). We denote the local hidden variables by $\lambda$. The functions $I^X(i, \lambda)$ and $I^X(i, \lambda)$ give the predetermined outcomes of intensity measurements of polarizations $i$, $i \perp$ in the local beam for observer $X$. The measurement is done with tuneable setting in presence of $\lambda$. We define local hidden values for sign operators as $G^X(i, \lambda) = \text{sign}(I^X(i, \lambda) - I^X(i, \lambda))$. These local hidden values are $\pm 1$ and 0, thus one can use standard methods to derive CHSH inequality. We assign the following set of settings $i = \theta, \theta'$ for first observer and $i = \phi, \phi'$ for second observer. Resulting CHSH inequality reads:

$$|\langle G^1(\theta, \lambda)G^2(\phi, \lambda) + G^1(\theta, \lambda)G^2(\phi', \lambda) + G^1(\theta', \lambda)G^2(\phi, \lambda) - G^1(\theta', \lambda)G^2(\phi', \lambda)\rangle_{LHV}| \leq 2,$$

(4)

However, this inequality cannot be violated by states with a significant vacuum component, e.g. the (polarization) four mode squeezed vacuum state, which will be our working example, see next Sections. This situation is analogous to the case of normalized Stokes operators, see [14]. Following ideas of [14] we modify sign Stokes operators as follows:

$$\tilde{G}^X(i) \rightarrow \tilde{G}^X(i) = \tilde{G}^X(i) - |\Omega^X\rangle \langle \Omega^X|,$$

(5)

what allows for reduction of the impact of vacuum term which often appears with the highest probability. Also local hidden values need to be modified:

- $G^X(i, \lambda) = \text{sign}(I^X(i, \lambda) - I^X(i, \lambda))$ if $I^X(i, \lambda) + I^X(i, \lambda) \neq 0$
- $G^X(i, \lambda) = -1$ if $I^X(i, \lambda) + I^X(i, \lambda) = 0$

As this modification does not change local hidden values $G^X(i, \lambda) \in \{0, \pm 1\}$ we use the following CHSH inequality:

$$|\langle G^1(\theta, \lambda)G^2(\phi, \lambda) + G^1(\theta, \lambda)G^2(\phi', \lambda) + G^1(\theta', \lambda)G^2(\phi, \lambda) - G^1(\theta', \lambda)G^2(\phi', \lambda)\rangle_{LHV}| \leq 2.$$

(6)

![LHS of CHSH inequality based on sign operators](image)

**FIG. 1.** LHS of CHSH inequality based on sign operators (6) - blue curve, and CHSH inequality based on normalized Stokes operators [14] - green dashed curve - in a function of amplification gain $\Gamma$ for BSV state. We perform numerical cutoff on terms with more that 150 photons. The threshold values of amplification gain ($\Gamma_r$), such that for all $\Gamma < \Gamma_r$, CHSH inequalities are violated, are $\Gamma_r \approx 0.88$ for normalized Stokes operators [14] and $\Gamma_r \approx 2.16$ for sign Stokes operators. Thus, with sign Stokes operators the range of violation with respect to amplification gain is much larger that in case of normalized Stokes operators.

1. Violation of Bell inequality for four mode squeezed vacuum - asymptotic behaviour

We are going to analyze how the use of sign Stokes operators in CHSH inequality helps to reveal the non-classicality of quantum states. Our working example is $2 \times 2$ mode squeezed vacuum state (BSV) which is the generalization of EPR singlet. It reads

$$|\psi_+\rangle = \frac{1}{\cosh^2(\Gamma)} \sum_{n=0}^{\infty} \sqrt{n + 1} \tanh^n(\Gamma) |\psi^n\rangle,$$

(7)

where $\Gamma$ is amplification gain, and

$$|\psi^n\rangle = \frac{1}{\sqrt{n + 1}} \sum_{m=0}^{n} (-1)^m |n - m\rangle_{H_q} |m\rangle_{V_q} |n - m\rangle_{H_q} |n - m\rangle_{V_q}.$$

(8)

Subscripts $q = 1, 2$ in $H_q$ and $V_q$ specify to which of the two beams particular mode corresponds. Amplification
gain determines the intensity of pumping field and thus \( \Gamma \) sets expectation value of intensity of the BSV state.

Let \( \theta, \theta', \phi \) and \( \phi' \) be the settings used in CHSH inequality. These settings describe the angles by which the measurement polarization basis is rotated in relation to \( \{H,V\} \) basis by SO(2) rotation. We chose: \( \theta = 0, \theta' = \pi/4, \phi = \pi/8 \) and \( \phi' = -\pi/8 \). It was shown that these settings are optimal in case of violation of CHSH inequality with normalized Stokes operators for BSV [14].

FIG. 1 shows quantum predictions of LHS of CHSH inequality for sign Stokes operators (6) LHS of CHSH for normalized Stokes parameters for BSV taken from from [14] in a function of amplification gain \( \Gamma \). From now on we denote LHS of (6) for BSV by \( \langle \psi^o \vert CHSH \vert \psi^- \rangle \).

Sign Stokes operators allow for violation of CHSH for wider range of amplification gain that is up to \( \Gamma = 2 \). For normalized Stokes operators this value is significantly lower i.e. \( \Gamma_{tr} \approx 2.16 \). Thus with sign Stokes operators it is possible to reveal nonclassicality of BSV with higher value of expectation value of intensity (brighter light).

From the other hand in FIG. 1 we can see that for \( \Gamma \approx 2.1 \) LHS for sign Stokes operators drops down suspiciously suddenly. We presume that such a behaviour might be a consequence of a cutoff performed on BSV state (the superposition of \( \vert \psi^n \rangle \)'s is considered up to \( n = 150 \) photons). Such a statement requires further investigation.

However, before we pursue further we need to make a remark about the BSV state and sign and Stokes operators. First, BSV is invariant under any unitary operation performed on both observers. Thus, the measured expectation value depends only on the difference \( \phi - \phi' \).

Also, the components of BSV, states \( \vert \psi^n \rangle \) are orthogonal: \( \langle \psi^n \vert \psi^k \rangle = 0 \) for \( n \neq k \).

Note that standard, normalized and sign Stokes operators are composed from functions of photon number operators, that do not change number of photons. Thus, LHS of CHSH inequality consists of two terms: vacuum term, that is CHSH inequality averaged over vacuum component of BSV and non-vacuum term. The vacuum term can be easily calculated:

\[
\sum_{j=0,\theta',k=\phi,\phi'} \langle \Omega \vert \hat G^{1-}(j) \hat G^{2-}(k) \vert \Omega \rangle = \frac{2}{\cosh^4 \Gamma}. \tag{9}
\]

The non-vacuum terms \( \langle LHS_{nv} \rangle \) results from the expectation values of \( \langle \psi^n \rangle \). Note that as \( \Gamma \) increases the role of non-vacuum terms in LHS of CHSH inequality increases too. For small \( \Gamma \) the contribution of vacuum term dominant.

In FIG. 2 non-vacuum contribution to LHS of (6) is presented. The calculation is performed for BSV state truncated up to \( n = 150 \) - blue curve and \( n = 100 \) - green curve. Both curves asymptotically go to 2 (classical bound) up to some point for which they both start to decrease. Note that the curve for \( n = 100 \) decreases faster than the curve for \( n = 150 \). It is highly probable that the decrease is conditioned by not including components with high enough number of photons and the non-vacuum term of LHS of 6 goes asymptotically to 2 from the left. The vacuum term goes asymptotically to 0 from the right, see (9). Thus, our hypothesis is that CHSH inequality with sign Stokes operators is violated for BSV for any \( \Gamma \).

Below we present a reasoning based on numerical calculation.

Let us analyze expectation values (6) for states \( \vert \psi^n \rangle \) i.e. \( \langle CHSH \vert \psi^n \rangle \) and compare them with LHS of the analogue expression for normalized Stokes operators from [14].

FIG. 3 shows results for \( n = 1, \ldots, 100 \) for sign Stokes operators and normalized Stokes operators. For normalized Stokes operators only for \( n = 1 \) we get \( LHS \geq 2 \). For sign operators values of \( \langle CHSH \vert \psi^n \rangle \) concentrate around 2 with growing \( n \). More detailed analysis, see Appendix B, reveals two patterns: an oscillating one for odd \( n \)'s and a pattern converging to 2 from bellow for even \( n \)'s. The period of odd \( n \)'s is equal to \( T = 8 \) in the sense that points \( n = 2k + 1 \) and \( n = 2k + 1 + T \) where \( k \in \mathbb{Z} \) correspond to e.g. two adjacent maximums in the pattern. The even pattern also has internal structure repeatable with \( T = 8 \).

Periodicity of the pattern provide us the natural grouping of \( \langle \psi^o \rangle \) for the regarded problem. Let us examine a weighted average of \( \langle LHS \rangle_N^{(\Gamma)} \) for a given \( \Gamma \) over \( N \)-th period for \( \Gamma > 2 \) with weights \( w_n \):

\[
\langle LHS \rangle_N^{(\Gamma)} = \frac{\sum_{n=1}^{T+N(\Gamma)-1} \langle \psi^n \vert \psi^- \rangle}{\sum_{n=1}^{T+N(\Gamma)-1} \vert \psi^n \rangle \langle \psi^- \vert \psi^n \rangle}, \tag{10}
\]

where \( \vert \langle \psi^n \vert \psi^- \rangle \vert^2 = (n+1)\frac{\tanh \Gamma\cdot \Gamma}{\cosh^4 \Gamma} \). Fig. 4 shows values of \( \langle LHS \rangle_N^{(\Gamma)} \) for \( \Gamma = 1, 2, 3 \) and \( \Gamma \rightarrow \infty \).

We observe that \( \langle LHS \rangle_N^{(\Gamma)} \) is a decreasing function of \( \Gamma \). All calculated values of \( \langle LHS \rangle_N^{(\Gamma)} \) exceed 2 and violate the inequality (6). Also \( \langle LHS \rangle_N^{(\Gamma)} \) for any given \( \Gamma \) converges to 2 with growing \( N \). Moreover the curve corresponding to \( \Gamma \rightarrow \infty \) is the most relevant for our analysis because it bounds the \( \langle LHS \rangle_N^{(\Gamma)} \) from bellow.

Let us observe that non-vacuum term \( \langle LHS_{nv} \rangle \) can be written as the weighted average of \( \langle LHS \rangle_N^{(\Gamma)} \):

\[
LHS_{nv} = \sum_{N=1}^{\infty} \langle LHS \rangle_N^{(\Gamma)} \sum_{n=1+T+N(\Gamma)-1} \vert \langle \psi^n \vert \psi^- \rangle \vert^2. \tag{11}
\]

Assuming that there is no change in pattern of \( \langle CHSH \rangle_{\psi^n} \) as \( n \) increases (see Appendix B for the argumentation) we can bound from bellow the value of \( LHS_{nv} \) by replacing \( \langle LHS \rangle_N^{(\Gamma)} \) with 2:

\[
LHS_{nv} \geq \sum_{n=1}^{\infty} 2(n+1)\frac{\tanh 2n \Gamma}{\cosh^4 \Gamma} = 2(\tanh^2 \Gamma + \sech^2 \Gamma \tanh^2 \Gamma). \tag{12}
\]
Equality due to our assumptions should be only reached in the limit of $\Gamma \to \infty$. This is because in the regime of high values of $\Gamma$ only terms with high $n$ are significant and $\langle LHS \rangle_N^{(\Gamma)}$ from the assumption reach 2 only when $N \to \infty$. Expression (12) as expected has an asymptotic value 2. If we add the vacuum term (9) to the RHS of (12) we obtain constant function 2. Thus, finally we obtain:

$$\langle \psi_- | CHSH | \psi_- \rangle \geq 2.$$  (13)

This strongly suggest that violation of (6) with $|\psi_-\rangle$ is possible for any finite $\Gamma$ and numerically obtained existence of the threshold value of $\Gamma$ is due to computational limitations.

C. CHSH inequality with losses

One of the critical aspects of experimental realization of Bell experiments are detectors with high efficiency $\eta$ of detection. Here we will analyze the critical value of efficiency $\eta_c$ such that for $\eta < \eta_c$ one can not observe violation of (6). We model inefficient detectors following [14]. Such detector can be described as perfect detector ($\eta = 1$) in front of which there is beamsplitter with transitivity $\sqrt{\eta}$. We denote number of photons which reach detectors as $k$. From those photons only $\kappa \leq k$ counts are registered due to losses on beamsplitter. For such case probability of measuring $\kappa$ photons is given by the binomial distribution:

$$p(\kappa | k) = \binom{k}{\kappa} \eta^\kappa (1 - \eta)^{k-\kappa}. \quad (14)$$

From FIG. 5 we can see that $\eta_c$ for small $\Gamma$ sign and normalized Stokes operators behaves almost identically. However critical efficiency for sign Stokes operators grows slower with $\Gamma$ than for normalized Stokes operators. Also rate of growth of $\eta_c$ from some point starts to decrease with $\Gamma$ in case of sign operator, where it is not the case of normalized Stokes operators. Such change of rate of growth for higher $\Gamma$ should be expected because in case of high number of photons loss of one photon matter less.
D. CHSH inequality with noise

Another deviation from idealized case in experimental setup can be uncorrelated noise in a state. Let us consider model of white noise with noisy state of the form:

\[ \rho' = q \rho_{BSV} + \frac{1 - q}{4} (|\phi^+\rangle \langle \phi^+| + |\phi^-\rangle \langle \phi^-| + |\psi^+\rangle \langle \psi^+| + |\psi^-\rangle \langle \psi^-|), \]

where \(|\psi^\pm\rangle, |\phi^\pm\rangle\) are bright squeezed vacuum states corresponding to states from Bell basis and \(\rho_{BSV}\) is a density matrix of the BSV state. Value \(1 - q\) determines probability of measuring noise. FIG. 6 shows critical value \(q_c\) of parameter \(q\) for which there is no violation of (6) if \(q < q_c\) in case of sign and normalized Stokes operators. From FIG. 6 one can observe that sign Stokes operators have similar advantage as in case of losses. For small \(\Gamma\) the resistance for noise in case of sign Stokes operators is close to this obtained with normalized Stokes operators. However, new operators reach higher maximal resistance and \(q_c\) goes asymptotically to 1.

E. CH inequality

Going along with idea of sign operators and rate approach to CH inequality [14] we can construct new CH-like inequality. Let us start with rate operators \(\hat{R}_i(\theta) = \Pi \hat{n}_i(\hat{n}_i + n_{i\perp})\Pi\) used in [14]. This operator is simply first term of normalized Stokes operator (1). One can observe that it has eigenvalues in \([1/2, 1]\) for states \(|n\rangle_i |m\rangle_i\perp\) with \(n > m\) and in \([0, 1/2]\) for \(n < m\). In case of sign Stokes operators we have assigned eigenvalues to those two subspaces of states based on extreme eigenvalues of normalized Stokes operator (corresponding to polarization \(i\)) for those subspaces. The pattern was to take highest eigenvalue in case \(n > m\) and the lowest in the case \(n < m\). We can do the same thing with rate operator with the modification in which we add to second subspace states for \(n = m\). As a result we obtain dichotomic observable with eigenvalues: 0 and 1. One can observe that such observable is simply projector onto subspace \(n > m\):

\[ \hat{P}(i) = \sum_{n > m} |n\rangle_i |m\rangle_i\perp \langle n| \langle m|_i\perp. \]

From that follows that expectation values of \(\hat{P}^X(i)\) is equal to probability of the observer \(X\) to see \(n > m\). We shall denote by \((\langle \hat{P}^X(i)\hat{P}^Y(j)\rangle)\) the quantum joint probability of obtaining the same result \(n > m\) by observers \(X\) and \(Y\) for their respective polarization basis \(i\) and \(j\). Had the probabilities been classical, e.g. Kolmogorovian, and if setting choice at one station cannot directly influence the result on the other one (Einstein’s locality) the following Clause-Horne like Bell inequality must hold

\[ -1 \leq \langle \hat{P}^1_+(\theta)\hat{P}^2_+\phi(\phi) + \hat{P}^1_+(\theta)\hat{P}^2_+(\phi') + \hat{P}^1_+(\theta')\hat{P}^2_+(\phi) \rangle - \hat{P}^1_+(\theta')\hat{P}^2_+\phi(\phi) - \hat{P}^1_+(\theta)\hat{P}^2_+\phi(\phi) \leq 0. \]

as classical probabilities of four events \(A, A', B\) and \(B'\) satisfy the Clasuer-Horne inequality

\[ -1 \leq P(A, B) + P(A', B') \]

\[ + P(A', B) - P(A', B') - P(A) - P(B) \leq 0. \]

FIG. 7 shows expectation value of expression (17) and it’s rate counterpart for the same settings as in case of CHSH inequality. The ‘sign’ approach gives for all \(\Gamma\) while the rate approach gives a violation only for \(\Gamma < 0.8866\) which is the same case as for CHSH. Note that this CH inequality is not equivalent to CHSH inequality (6) (see Appendix B)
such inequality is not violated by BGHZ state as in case of normalized Stokes operators. We have to again modify sign Stokes operators:

$$\hat{G}_i^X \rightarrow \hat{G}_i^{X-} = \hat{G}_i^X - |\Omega_X\rangle \langle \Omega_X|.$$  \hspace{1cm} (21)

One can easily write modified local hidden values for such operators as in II B and obtain inequality:

$$|\langle G_1^{-1}(\lambda)G_2^{-2}(\lambda)G_3^{-3}(\lambda) - G_1^{-1}(\lambda)G_2^{-2}(\lambda)G_3^{-3}(\lambda) \rangle_{\text{LHV}}| \leq 2. \quad (22)$$

FIG. 8 presents LHS of inequality (22) and LHS of analogous inequality for normalized Stokes operators as a function of amplification gain $\Gamma$. Range of $\Gamma$ for which inequality is violated by BGHZ state in case of sign Stokes operators exceeds the range of applicability of the method used in approximating probability amplitudes for BGHZ state. We also stress that this result is more robust than in case of normalized Stokes operators.

IV. CONCLUSIONS

We have proposed new Stokes-like polarization observables for quantum optical fields which have a clear operational meaning. In presented examples the sign Stokes operators...
observables allow observation of Bell non-classicality of squeezed-vacuum-type states for pumping powers, for which normalized Stokes observables fail do so.

One could be tempted to use sign Stokes operators to derive entanglement indicators not based on Bell inequalities, that is in a form of separability conditions. However those operators do not poses properties which are commonly used in derivations of bounds for separable states. Thus, this requires a novel approach. Similar questions arise when one thinks of a steering condition involving sign Stokes operators.

Another question would be if there is a type of state for which normalized Stokes operators allows for violating of some Bell inequality and for which this is impossible using sign Stokes operators.

Finally, we state that we came across the reference [20], when our manuscript was already written. The paper [20] introduces the ‘sign’ approach for observables based on particle number measurements in two outputs of measuring device, however in a different context (BEC condensates), and does not relate these with Stokes operators.

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### Appendix A EXPERIMENTALLY FRIENDLY PROPERTIES OF SIGN OPERATORS

In laboratory one has polarizers and phase shifters and optically active elements to disposal for constructing polarization measurements. Such optical elements perform unitary transformations from which one can obtain transformations $\hat{U}(H \rightarrow i)$. It can be seen during consideration of how $\hat{U}(H \rightarrow i)$ acts on creation operators $a_H^\dagger$, $a_V^\dagger$ for $\{D,A\}$, $\{R,L\}$:

$$\begin{pmatrix} a_H^\dagger \\ a_V^\dagger \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & e^{i\phi} \end{pmatrix} \begin{pmatrix} a_H^\dagger \\ a_V^\dagger \end{pmatrix} , \quad (1)$$

where $\theta = \pi/4$ ($\theta = -\pi/4$) and $\phi = 0$ ($\phi = 3\pi/2$) for $\{D,A\}$ ($\{R,L\}$). First matrix is a rotation matrix which transformation can be realized by active element that uses for example Faraday effect to rotate polarization by angle $\theta$. The second is matrix of a phase shift $\phi$ in second polarization mode.

In general if we rotate polarization analyzer by some angle or add some phase shifter in measurement setup we perform unitary transformation on observable changing polarization basis in which we measure. So this transformation is $\hat{U}(i \rightarrow i') = \hat{U}_V \hat{U}_H^\dagger$. Sign Stokes operators under $\hat{U}(i \rightarrow i')$ transforms in the following way:

$$\begin{align*}
\hat{U}(i \rightarrow i') \hat{G}(i) \hat{U}^\dagger(i \rightarrow i') \\
= \text{sign}(\hat{U}(i \rightarrow i')) \hat{U}(H \rightarrow i) \\
\left(\hat{n}_H - \hat{n}_V\right) \hat{U}^\dagger(H \rightarrow i) \hat{U}^\dagger(i \rightarrow i') \\
= \text{sign}(\hat{U}(H \rightarrow i') (\hat{n}_H - \hat{n}_V) \hat{U}^\dagger(H \rightarrow i')) \\
= \hat{G}(i').
\end{align*} \quad (2)$$

In second line we have used fact that in spectral decomposition of $\hat{G}(i)$ unitary transformation acts only on projectors (preserving their orthogonality) and preserves eigenvalues and thus sign function has no impact on transformation. From this results we see that operators transforms into each other under unitary transformations. Also sign Stokes operators can be realized experimentally analogously to standard Stokes operators.

### Appendix B VIOLATION OF BELL INEQUALITIES WITH $|\psi^n\rangle$

Let us make more in depth analysis of violation of CHSH inequality (6) by $|\psi^n\rangle$. Fig. 10 shows violation of (6) by $|\psi^n\rangle$. We can observe two patterns occurring. The first pattern for odd $n$’s oscillates around bound with decreasing amplitude and period $T = 8$. The second pattern for even $n$’s converge to 2 from bellow with growing $n$. This pattern also have internal structure which is repeatable with $T = 8$ (increase, decrease, increase, increase). Note that only $|\psi^n\rangle$ with odd $n$ violate CHSH inequality and that in odd pattern for every $n$ which does
1.996
1.998
2.000
2.002
2.004
2.006
LHS

FIG. 10. \( \langle CHSH \rangle_{\psi^n} \) versus \( n \). Blue darker dots depict odd \( n \) and green dots stand for even \( n \).

Let us make argument why conservation of the pattern for higher \( n \)'s is expected. Note that \( |\psi^n\rangle \) can be written in the following form:

\[
|\psi^n\rangle = \frac{1}{n!\sqrt{n+1}}(\hat{a}^\dagger \hat{b}_2^\dagger - \hat{a}_2 \hat{b}_1^\dagger)^n |\Omega\rangle. \tag{3}
\]

Given some point in the pattern \( \langle CHSH \rangle_{\psi^n} \) to obtain next corresponding point in the pattern we have only to apply operator \( (\hat{a}^\dagger \hat{b}_2^\dagger - \hat{a}_2 \hat{b}_1^\dagger)^T \) to the state \( |\psi^n\rangle \) and normalize it by the factor \( \frac{n!\sqrt{n+1}}{(n+T)!\sqrt{n+T+1}} \). To obtain next \( k \)-th corresponding point we have to apply this operator \( k \) times. Thus, applying such operator has to preserve some internal symmetries. There is no reason for existence of \( k \) such that it suddenly stops to preserve those symmetries. Therefore, pattern should be continued for any period \( N \).

Fig. 11 presents LHS of CH inequality (17) for \( |\psi^n\rangle \). In this case there are also two patterns. The oscillating odd pattern with the same properties and convergent even pattern. However, in this case the even pattern goes to bound from above, and clearly have higher impact on violation of (17) by the BSV state. This shows That CH inequality (17) is not equivalent to CHSH inequality (6).

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