Torsion in Gauge Theory

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I. INTRODUCTION

In the formulation of a physical theory in curved spacetime, the normal procedure is to replace the ordinary derivative with the corresponding covariant derivative. For a gauge theory in the Riemannian spacetime, because the connection is symmetric, the normal procedure yields a field strength tensor $F_{\mu\nu}$ in the form of its flat-space expression, which is gauge symmetric. But in a Riemann-Cartan spacetime with torsion, the connection being non-symmetric, this same procedure gives rise to an additional torsion term in the gauge field strength tensor that violates gauge symmetry. Torsion naturally appears in the Einstein-Cartan-Kibble-Sciama theory of gravitation. The potential conflict of torsion with gauge symmetry was already noticed by Kibble in his original paper, and has since been discussed by many authors. Kibble suggested that, to preserve gauge symmetry, one should forgo the covariant derivative in favor of the ordinary derivative in the definition of the field strength $F_{\mu\nu}$ for massless gauge theories, while for massive vector fields covariant derivatives should be adopted. This view was further emphasized by Hehl and collaborators in their influential 1976 review paper. We address the question of whether this deviation from normal procedure by forgoing covariant derivatives in curved spacetime with torsion could give rise to inconsistencies in the theory, such as the quantum renormalizability of a realistic interacting theory. We demonstrate in this note the one-loop renormalizability of a realistic gauge theory of gauge bosons interacting with Dirac spinors, such as the SU(3) chromodynamics, for the case of a curved Riemann-Cartan spacetime with totally antisymmetric torsion. This affirmative confirmation is one step towards providing justification for the assertion that the flat-space definition of the gauge field strength should be adopted as the proper definition.

II. SCIAMA-KIBBLE SCHEME

The genesis of the Kibble-Sciama theory can be traced back to the formulation of the Dirac equation in curved spacetime by Weyl and Fock. The vierbein fields $e^a_{\mu}$ were introduced by Weyl and Fock to provide local coordinate basis for defining the Dirac spinor, and the spin connection field $\omega^{ab}_{\mu}$ as the gauge potential for the SO(3,1) group of local Lorentz transformations of the Dirac spinor. Utiyama demonstrated that Ein-
stein’s Riemannian theory of gravitation can be regarded as a gauge theory of the SO(3,1) Lorentz group when the corresponding gauge potential, the spin connection \( \omega^a_{\mu} \), is identified with the Ricci coefficients of rotation [19] in terms of the vierbein fields \( e^a_{\mu} \). Sciama and Kibble [1, 2] took the step of treating the spin connection field \( \omega^a_{\mu} \), in the spirit of a genuine Lorentz group gauge theory, as independent dynamic variable to be determined by the theory, instead of being identified with the Ricci coefficients. The coupling of the spin connection to the Dirac spinors, for example, gives rise to torsion.

The metric tensor \( g_{\mu\nu} \) in the Kibble-Sciama scheme is defined by

\[
g_{\mu\nu} = \eta_{ab} e^a_\mu e^b_\nu, \tag{1}
\]

where \( \eta_{ab} = (1,-1,-1,-1) \), and the covariant derivatives with respect to both local Lorentz transformations and general coordinate transformations, for generic \( \chi^\lambda, \chi^\nu, \) are defined according to

\[
\nabla_\mu \chi_\lambda = \chi_{\alpha, \mu} - \omega^b_{a\mu} \chi^b_\lambda + \Gamma^\lambda_{\nu\mu} \chi^\nu, \tag{2}
\]

and

\[
\nabla_\mu \chi_\nu = \chi_{\alpha, \mu} + \omega^b_{b\mu} \chi^b_\nu - \Gamma^\lambda_{\nu\mu} \chi^\lambda. \tag{3}
\]

Kibble [1] chose the affine connection

\[
\Gamma^\lambda_{\mu\nu} = e^a_\lambda (e^a_\mu, e^b_\nu + \omega^a_{b\nu} e^b_\mu), \tag{4}
\]

so that it is metric compatible, meaning

\[
\nabla_\lambda e^a_\mu = 0, \tag{5}
\]

\[
\nabla_\lambda e^a_\mu = 0, \tag{6}
\]

and, consequently,

\[
\nabla_\lambda g^{\mu\nu} = 0, \tag{7}
\]

\[
\nabla_\lambda g_{\mu\nu} = 0. \tag{8}
\]

In the presence of torsion, which is defined as

\[
C^\lambda_{\mu\nu} = \Gamma^\lambda_{\mu\nu} - \Gamma^\lambda_{\nu\mu}, \tag{9}
\]

the metric compatibility relations (7) and (8) imply that the connection is of the general form:

\[
\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\rho} (g_{\mu\nu, \rho} + g_{\nu\rho, \mu} - g_{\mu\nu, \rho}) + Y^\lambda_{\mu\nu}, \tag{10}
\]

where the contortion tensor \( Y^\lambda_{\mu\nu} \) is given by

\[
Y^\lambda_{\mu\nu} = \frac{1}{2} (C^\lambda_{\mu\nu} + C^\lambda_{\nu\mu} + C^\lambda_{\mu\nu}). \tag{11}
\]

III. SYSTEM OF GLUONS INTERACTING WITH QUARKS IN SCIAMA-KIBBLE SCHEME

For notational convenience of presentation, we shall consider the specific case of the SU(3) chromodynamics, in which the gauge gluons interacting with a triplet of massless spinor quarks. Let the gauge field be denoted by \( A_\mu^a \), where the index \( a \) runs from 1 to 8. It is convenient to adopt the group algebraic notation

\[
A_\mu = A_\mu^a T^a, \tag{12}
\]

where \( T^a \), for concreteness, are the familiar 3x3 \( \frac{1}{2} \lambda_2 \) Gell-Mann matrices satisfying the algebra, with the totally antisymmetric \( f^{abc} \) being the SU(3) group structure constants,

\[
[T^a, T^b] = i f^{abc} T^c. \tag{13}
\]

In flat space, the field strength is given by

\[
F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu]. \tag{14}
\]

In curved space, the natural definition for the field strength is to follow the normal procedure of replacing the partial derivative by the appropriate covariant derivative, like

\[
\partial_\mu A_\nu \rightarrow \partial_\mu A_\nu - \Gamma^\lambda_{\nu\mu} A_\lambda. \tag{15}
\]

The additional torsion term in (15) violates gauge invariance. To preserve gauge symmetry, an alternative is to forgo the torsion term in (15) and adopt the flat-space expression (14) as the definition of the field strength \( F_{\mu\nu} \).

We now consider the system of SU(3) gauge bosons interacting with spinor quarks in the background of the curved Riemann-Cartan space as described above in the Kibble-Sciama scheme. We first check the consistency of the system of field equations with the two alternative versions of the field strength \( F_{\mu\nu} \), (14) and (16). For convenience, we shall consider a massless spinor quark, which is denoted by \( \psi \). The action for the system is of the form

\[
W = \int d^4x h[-\frac{1}{4} F_{\mu\nu} F_{\mu\nu} + \frac{1}{4}(\bar{\psi}\gamma^a e^a_\mu D_\mu \psi - \bar{\psi} D_\mu i\gamma^a e^a_\mu \psi)], \tag{17}
\]

where \( h = \det e^a_\mu \), and

\[
D_\mu = \partial_\mu - i \frac{1}{4} g_{ab} \omega^{ab}_\mu + ig A_\mu, \tag{18}
\]
\[ D_\mu = \frac{\partial}{\partial x_\mu} - igA^a_\mu T^a + \frac{i}{4} \sigma_{ab} \omega^{ab}_{\mu}, \]

with \( \sigma_{ab} = \frac{i}{2}(\gamma_a, \gamma_b) \) [23]. We note that \( F^{\mu\nu} = g^{\mu\lambda} g^{\nu\rho} F_{\lambda\rho} \). For proper normalization of the \( F^{\mu\nu} F_{\mu\nu} \) term in the Lagrangian, there should be a factor of \( \frac{1}{\sqrt{2}} \), which is defined by

\[ tr(T^a T^b) = C_2(R) \delta^{ab}. \]

For convenience, we have omitted this normalization factor, but it will be taken into account when we consider renormalization counter terms. The Lagrangian in the action (17) is scale invariant. The Dirac equation (22) is thus expected to be scale invariant. We have, by its construction according to (8), the connection \( \Gamma^\lambda_{\mu\nu} \) has the following scale transformation property

\[ \Gamma^\lambda_{\mu\nu} \rightarrow \Gamma^\lambda_{\mu\nu} + \delta^\lambda_\mu \Lambda_{\nu}, \]

where \( \Lambda \) is given in (19). We know that the Lagrangian in the action (17) is scale invariant. The Dirac equation (22) is thus expected to be scale invariant. We have, by its construction according to (8), the connection \( \Gamma^\lambda_{\mu\nu} \) has the following scale transformation property

\[ \Gamma^\lambda_{\mu\nu} \rightarrow \Gamma^\lambda_{\mu\nu} + \delta^\lambda_\mu \Lambda_{\nu}, \]

which implies

\[ C^\lambda_{\nu\mu} \rightarrow C^\lambda_{\nu\mu} + 3 \Lambda_{\nu}. \]

We denote

\[ B_\mu = \frac{1}{3} C^\lambda_{\nu\mu}. \]

It transforms as an effective Weyl gauge field for local scale transformations [22][24]

\[ B_\mu \rightarrow B_\mu + \Lambda_{\mu}. \]

The Dirac equation (22) is then expressed as

\[ i\gamma^\mu e^\mu_a (D_\mu + \frac{3}{2} B_\mu) \psi = 0. \]

So, indeed, the massless Dirac equation written in this form shows explicit scale invariance, and with the proper scale weight \( \frac{3}{2} \) for the Dirac field \( \psi \). The Euler-Lagrange equation for the gauge field is obtained straightforwardly. It is of the form

\[ (\nabla_\nu + 3B_\nu)F^{\mu\nu} = gJ^\mu, \]

where the covariant derivative \( \nabla_\nu \) is defined as in

\[ \nabla_\nu F^{\mu\nu} = \partial_\nu F^{\mu\nu} + \Gamma^\mu_{\nu\lambda} F^{\lambda\nu} + \Gamma^\nu_{\mu\lambda} F^{\lambda\mu} - ig[A_\nu, F^{\mu\nu}], \]

and the current \( J_\mu \) given by

\[ J_\mu = \bar{\psi} \gamma^\mu e^\mu_a T^a \psi / T^a. \]

In the presence of torsion, the field equation (28) is not gauge invariant. We would like to check whether current conservation is valid and whether the system of field equations, namely (27) and (28), are mutually consistent. As a consequence of the Dirac equation (27) and its hermitian conjugate equation for \( \bar{\psi} \), it is straightforward to verify that the current \( J_\mu \) given by (30) is indeed conserved,

\[ (\nabla_\mu + 3B_\mu)J_\mu = 0. \]

Consistency of (28) with this current conservation equation (31), which follows directly from the Dirac equation (27), requires that

\[ (\nabla_\mu + 3B_\mu)(\nabla_\nu + 3B_\nu)F^{\mu\nu} = 0. \]

Making use of the anti-symmetry of \( F^{\mu\nu} \), it is straightforward, though tedious, to show that

\[ (\nabla_\mu + 3B_\mu)(\nabla_\nu + 3B_\nu)F^{\mu\nu} = -R^{\mu\nu}_{\rho\sigma} F^{\rho\sigma} + \frac{1}{2} C^{\rho\sigma}_{\nu} \nabla_\mu F^{\rho\sigma} + \frac{1}{2} C^{\mu\nu}_{\sigma} \nabla_\sigma F^{\rho\sigma} + \frac{1}{2} F^{\rho\sigma} (\nabla_\mu B_\nu - \nabla_\nu B_\mu). \]
For the right-hand side of (33) to vanish, it is necessary, due to its structure, that the second term vanishes. That is, we have to set $C^\mu_{\rho\nu} = 0$. This results in $B_\mu = 0$, and $R^\mu_{\rho\nu}$, being symmetric in $\rho$ and $\nu$ because the connection $F^\lambda_{\mu\nu}$ now reduces to the Christoffel connection. The three terms on the right-hand side of (29) then all vanish. Consistency of the two field equations of the system (24) and (25) is thus seen to require the vanishing of torsion. The upshot is that the system of field equations is inconsistent for the gauge non-symmetric version (16) of $F_{\mu\nu}$.

V. CASE (II) GAUGE SYMMETRIC $F_{\mu\nu}$

We next consider the case of gauge symmetric $F_{\mu\nu}$

\[ F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu], \]

which is the version with the torsion term removed. With this expression for $F_{\mu\nu}$ in the action (17), the field equation for the Dirac field $\psi$ remains the same as (27), resulting in the same current conservation equation (31), while the field equation for the Maxwell field (28) is replaced by

\[ (\nabla_\nu + 3B_\nu)F^{\mu\nu} - \frac{1}{2}C^\mu_{\rho\nu} F^{\rho\nu} = J^\mu. \] (34)

Consistency of (34) with (31) requires that $(\nabla_\mu + 3B_\mu)$ operating on the left-hand side of (34) vanishes. The result of operating on the first term on the left-hand side of (34) is already found and given by (33). Operating on the second term yields the contribution

\[ -\frac{1}{2}C^\mu_{\rho\nu} \nabla_\mu F^{\rho\nu} - \frac{1}{2}\nabla_\mu C^\rho_{\mu\nu} F^{\rho\nu} - \frac{3}{2}B_\mu C^\mu_{\rho\nu} F^{\rho\nu}. \] (35)

Summing the two contributions given in (33) and (35) yields

\[ -R^\mu_{\rho\nu\sigma} F^{\rho\sigma} + \frac{1}{2}[3(B_{\nu,\mu} - B_{\mu,\nu}) - \nabla_\mu C^\nu_{\rho\nu}] F^{\rho\nu}. \] (36)

In the presence of torsion, the antisymmetric part of $R^\mu_{\rho\nu\sigma}$ does not vanish, and explicit evaluation gives the result

\[ \frac{1}{2}(R^\mu_{\rho\nu\sigma} - R^\mu_{\sigma\nu\rho}) = \frac{1}{2}[3(B_{\nu,\rho} - B_{\rho,\nu}) - \nabla_\mu C^\lambda_{\rho\nu}] . \] (37)

The two contributions from operating $(\nabla_\mu + 3B_\mu)$ on the two right-hand side terms of (34) miraculously cancel each other out and the final result is zero. Consistency of the field equations is thus established. The un-pleasing $C^\mu_{\rho\nu} F^{\rho\nu}$ term in the field equation (34) looks formidable, but it actually helped save consistency. We have thus seen that the system of classical field equations of chromodynamics in the curved Riemann-Cartan space is self consistent when the gauge field strength is defined by the gauge symmetric expression (14), while it is not for the gauge non-symmetric version (16). The latter version is thus ruled out, even at the classical level. We next check whether the gauge symmetric version (13) of the interacting gauge theory, chromodynamics, is one-loop renormalizable.

VI. ONE-LOOP RENORMALIZATION BY BACKGROUND-FIELD METHOD

The background-field method \cite{13, 17} is ideally suited to the computation of effective interaction in curved spaces. It has been used to study the renormalization property of gauge theories in curved Riemannian space-time by various authors \cite{27, 28, 31}, establishing renormalizability at one-loop level and beyond. In the case of Riemann-Cartan spacetime, there does not seem to exist investigations in the literature of the renormalizability question of gauge theories. The question in focus is whether torsion could create complications, a question we would like to study. Based on the background-field method, there is the unified super-space computation \cite{32} of the one-loop renormalization counter terms, treating both gauge bosons and Dirac fermions within the framework of the Schwinger-DeWitt proper-time representation of the propagator functions \cite{13, 32}. Rather than using this elegant framework for evaluating the renormalization counter terms, we will instead combine the normal treatment based on heat-kernel technique with 't Hooft’s algorithm \cite{17, 29} for extracting one-loop divergences. The one-loop renormalization counter terms arise from four types of loops, the boson gluon loop, the ghost loop, the spinor quark loop, and the mixed gluon-quark loops (quark self energy loop and gluon-quark vertex loops). For the gluon, ghost and quark loops we follow Toms’ treatment, which is based on the heat-kernel method (a variant of the Schwinger-DeWitt proper-time method) and dimensional regularization, while for the mixed gluon-quark loops, we make use of the ‘t Hooft algorithms \cite{15, 29}. The divergent part of the one-loop effective action is given by an integral of the coefficient $[a_2]$ of the heat-kernel expansion \cite{13, 28}. Its explicit expression is given by DeWitt \cite{13} and Gilkey \cite{26} in the case of Riemannian space-time. In the case of Riemann-Cartan space-time, the presence of torsion makes the evaluation of the corresponding $[a_2]$ quite involved, and there does not seem to exit a definitive result for a general torsion. The special case of totally antisymmetric torsion has been carefully studied by Yajima \cite{33}. It is Yajima’s result that we will make use of, and we will thus restrict ourselves to the special case of totally antisymmetric torsion, for which the effective Weyl gauge field vanishes, namely, $B_\mu = 0$.

Let’s denote the classical background fields by $\eta$ and $\hat{A}_\mu$, which satisfy the field equations (22) and (28), respectively. We replace in the action (17) the field $A_\mu$ by $A_\mu + \hat{A}_\mu$, and $\psi$ by $\psi + \eta$. In the respective sums, $A_\mu$ and $\psi$ (and $\bar{\psi}$) are regarded as quantum fields, while $\hat{A}_\mu$
and $\xi$ as classical fields. We remind ourselves that in the action (17) the gauge field strength $F_{\mu\nu}$ is defined by the gauge symmetric expression (14), namely,

$$ F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu]. $$

We will also need to add to the action the gauge-fixing term and the corresponding Faddeev-Popov ghost term [34]. The gauge-fixing term is chosen in accordance to the Landau-DeWitt gauge condition and is given by [27]

$$ W_{(GF)} = \int d^4x [\frac{1}{2} (\bar{\psi}_\mu A^\mu)^2], $$

where

$$ \bar{\psi}_\mu A^\mu = \partial_\mu A^\mu + \Gamma^\mu_{\lambda\rho} A^\lambda - ig[\tilde{A}_\mu, A^\mu]. $$

The gauge-fixing action (38) brakes gauge symmetry if only the quantum field $A^\mu$ undergoes gauge transformation, but can be made gauge covariant under suitably combined gauge transformations of both $A^\mu$ and $\tilde{A}_\mu$. The corresponding Faddeev-Popov ghost term can be obtained by changing the integration "variable" in the path integral and is given by

$$ W_{(ghost)} = \int d^4x \bar{\zeta} \left( - \bar{\nabla}_\mu \bar{\psi}_\mu - \bar{\nabla}_\mu A^\mu - A^\mu \bar{\nabla}_\mu \zeta \right), $$

where the ghost fields $\bar{\zeta}$ and $\zeta$ are Grassmann scalars and carry the same color index as $A_\mu$. The gauge-fixing action $W_{(GF)}$ and Faddeev-Popov ghost action $W_{(ghost)}$ are to be added to the action $W$, given by (17), to form the total action. Expand the action in powers of the quantum fields $A_\mu$ and $\psi$. The coefficients of terms linear in quantum fields vanish, as a result of the classical gauge symmetric expression (14), namely,

$$ \Gamma_{fermion-loop} = -i \ln \text{Det}(i\gamma^\mu \hat{D}_\mu). $$

In order to make use of the heat-kernel technique, while the heat equation is of second order in the differential operator, we need to re-formulate $\Gamma_{fermion-loop}$ so that the differential operator in the determinant is of second order. This can be accomplished by replacing in (45) the linear differential operator $i\gamma^\mu \hat{D}_\mu$ by its square $(i\gamma^\mu \hat{D}_\mu)^2$ and multiplying an overall factor of $\frac{1}{2}$, namely,

$$ \Gamma_{fermion-loop} = -\frac{1}{2} \ln \text{Det}[(i\gamma^\mu \hat{D}_\mu)^2]. $$

The square in the determinant can be expressed as [22]

$$ -(g^{\mu\nu} \hat{D}_\mu \hat{D}_\nu - \frac{i}{2} \sigma^\mu\nu C^\lambda_{\mu\nu} \hat{D}_\lambda + Z), $$

where

$$ Z = \frac{1}{4} R + \frac{i}{8} \gamma_5 \tilde{R} + \frac{i}{2} \sigma^\mu\nu (R_{\mu\nu} + i \tilde{F}_{\mu\nu}), $$

with

$$ R = R^{\mu\nu}_{\mu\nu}, $$

$$ \tilde{R} = \varepsilon^{\mu\nu\rho\lambda} R_{\mu\nu\rho\lambda}, $$

$$ R_{\mu\nu} = R^{\lambda}_{\mu\lambda\nu}. $$

We note that when torsion vanishes, $\tilde{R} = 0$, $R_{\mu\nu} = R_{\nu\mu}$, and $Z$ in (48) reduces to $\frac{1}{4} R$, a well recognized result for Riemannian spacetime. The linear derivative term in (47), which is proportional to the torsion tensor, can be absorbed into the quadratic derivative term by redefining the covariant derivative

$$ \hat{D}_\mu = \partial_\mu - \frac{i}{4} \gamma_{a\mu} \gamma^{a\mu} + i g \hat{A}_\mu. $$

The fermion-loop effective action (46) thus becomes

$$ \Gamma_{fermion-loop} = -\frac{1}{2} \ln \text{Det}[g^{\mu\nu} \hat{D}_\mu \hat{D}_\nu + X], $$

with

$$ X = Z - D_\mu Q^\mu - Q_\mu Q^\mu, $$
where
\[ Q_\mu = -\frac{i}{4} \sigma^{\lambda\rho} C_{\lambda\mu}. \] (52)

The divergent part of the fermion-loop effective action (48) is of the form
\[ \text{Div} \Gamma_{\text{fermion-loop}} = \frac{1}{\epsilon} \int d^4 x htr[a_2](x), \] (53)
where \( \epsilon = (4\pi)^2(n - 4) \) and the corresponding kernel for \( a_2 \) is \( g^{\mu\nu} \dot{D}_\mu \dot{D}_\nu + X \) in (50). For Riemann-Cartan spacetime and in the case of totally antisymmetric torsion tensor, the \( a_2 \) corresponding to the differential operator in (47) has been obtained by Yamada [33]. It is given by, as adopted with our metric,
\[ [a_2] = \frac{1}{2} \tilde{W}^{\mu\nu} \tilde{W}_{\mu\nu} + \frac{1}{16}(R^{(o)\mu\nu\lambda\rho} R^{(o)}_{\mu\nu\lambda\rho} - R^{(o)\mu\nu} R^{(o)}_{\mu\nu} - \frac{1}{2} \tilde{D}_\mu \tilde{D}^\mu \frac{1}{3}(\frac{2}{3}R - X) + \frac{1}{2} \{ \frac{1}{3}R - X \}^2, \] (54)
where \( R^{(o)\mu\nu\lambda\rho} \), etc. are the Riemannian curvature tensors, and \( \tilde{W}_{\mu\nu} \) is defined [33] according to
\[ [\tilde{D}_\mu, \tilde{D}_\nu] \psi = (\tilde{W}_{\mu\nu} + C^\lambda_{\mu\nu} \tilde{D}_\lambda) \psi. \] (55)

We remark that the disentanglement with the definition of \( \tilde{W}_{\mu\nu} \) of the torsion term in (55) is crucial in assuring gauge symmetry in the final result. With the definition in (47), \( \tilde{W}_{\mu\nu} \) is computed to be
\[ \tilde{W}_{\mu\nu} = -\frac{1}{2} \sigma^{\alpha\beta}[R^{(o)}_{\alpha\beta\mu\nu} + \frac{3}{2}(\tilde{\nabla}_\mu C_{\alpha\beta\nu} - \tilde{\nabla}_\nu C_{\alpha\beta\mu} - C^\lambda_{\mu\nu} C_{\alpha\beta\lambda} + C^\lambda_{\alpha\beta} C_{\mu\lambda\nu}] + i\tilde{F}_{\mu\nu}, \] (56)
where \( \tilde{\nabla}_\mu \) is the Riemannian covariant derivative. We point out one important aspect of (53) is that the torsion term in (53) is not involved in the definition of \( \tilde{W}_{\mu\nu} \). This ensures the clean appearance of the gauge invariant \( \tilde{F}_{\mu\nu} \) term in \( \tilde{W}_{\mu\nu} \) without involvement of the torsion tensor. Trace over the spinor and quark indices of the quark field \( \psi \) of the product \( \tilde{W}^{\mu\nu} \tilde{W}_{\mu\nu} \) term appearing in (54) is given by
\[ \frac{1}{12} tr \tilde{W}^{\mu\nu} \tilde{W}_{\mu\nu} = -\frac{1}{8} \Sigma_{\alpha\beta\mu\nu} \Sigma_{\alpha\beta\mu\nu} - \frac{1}{3} g^2 tr \tilde{F}^{\mu\nu} \tilde{F}_{\mu\nu}, \] (57)
where
\[ \Sigma_{\alpha\beta\mu\nu} = R^{(o)}_{\alpha\beta\mu\nu} + \frac{3}{2}(\tilde{\nabla}_\mu C_{\alpha\beta\nu} - \tilde{\nabla}_\nu C_{\alpha\beta\mu} - C^\lambda_{\mu\nu} C_{\alpha\beta\lambda} + C^\lambda_{\alpha\beta} C_{\mu\lambda\nu}). \] (58)

As our main interest is in the renormalizability of gauge theory in the Riemann-Cartan spacetime, we will concentrate on the gauge-field terms in (53). Explicit evaluation shows that these terms come from the first and last terms on the right-hand side of (54). The contribution from the first term is contained in (57). The contribution from the last term is in
\[ tr \frac{1}{2} x^2 = g^2 tr \tilde{F}^{\mu\nu} \tilde{F}_{\mu\nu} + \text{gravitational terms}. \]

The renormalization counter term for the gluon field due to the fermion loop is the sum of the two contributions and is given by
\[ \text{Div} \Gamma_{\text{fermion-loop}} = \frac{1}{3} \int d^4 x h(\frac{2}{3} g^2 tr \tilde{F}^{\mu\nu} \tilde{F}_{\mu\nu} + \text{gravitational terms}). \] (59)

**VIII. GLUON AND GHOST LOOPS**

The relevant terms for the gluon loop in \( W^{(2)} \) are
\[ W_{\text{gluon-loop}} = \int d^4 x h(-\frac{1}{2} \tilde{F}^{\mu\nu} \tilde{F}_{\mu\nu} - 2i A_\mu [\tilde{F}^{\nu\mu}, A_\nu] - \frac{1}{2} (\nabla_\mu A^\mu)^2). \] (60)

while that for the ghost loop is
\[ W_{\text{ghost-loop}} = \int d^4 x h(\nabla_\mu \nabla_\nu A^\mu - i[A_\mu, A_\nu]). \] (61)

Up to a total derivative, \( W_{\text{gluon-loop}} \) can be expressed in the following form,
\[ \int d^4 x \frac{1}{2} h A^\mu (g_{\mu\nu} \nabla_\lambda \nabla^\lambda - \nabla_\nu \nabla_\mu + \nabla_\mu \nabla_\nu + 2 C_{\mu\lambda\nu} \nabla_\mu + C_{\mu\lambda\nu} C_{\mu\lambda\nu} + g \tilde{F}_{\mu\nu}) A^\nu. \] (62)

where \( \nabla_\mu A^\mu \) is defined as in
\[ \nabla_\mu A_\nu = \nabla_\mu A_\nu - i[A_\mu, A_\nu]. \] (63)

In (62), the \( AFA \) product term is understood to be the product \( f^{abc} A_\mu A_\lambda A_\rho \). On account of
\[ [\nabla_\mu, \nabla_\nu] A^\rho = R_{\rho\mu\nu} A^\nu + C^\rho_{\mu\nu} \nabla_\rho A^\nu - i g [\tilde{F}_{\mu\nu}, A^\rho], \] (64)
we can express (62) in the form
\[ \frac{1}{2} h A^\mu (g_{\mu\nu} \nabla_\lambda \nabla^\lambda + C_{\mu\lambda\nu} \nabla_\lambda + \nabla^\lambda C_{\mu\lambda\nu} - C_{\mu\lambda\nu} C_{\nu\lambda\rho} + R_{\nu\mu} + 2 g \tilde{F}_{\mu\nu}) A^\nu. \] (65)

The term linear in derivative in (64), which is brought about by torsion, is to be absorbed into the quadratic derivative term by defining a modified connection
\[ \tilde{\nabla}_\mu = \nabla_\mu + \frac{1}{2} C^{\lambda}_{\mu\nu}, \] (66)
with the corresponding covariant derivative expressed as \( \nabla_\mu \). The gluon-loop action (62) can be written as
\[ W_{\text{gluon-loop}} = \int d^4 x \frac{1}{2} h A^\mu (g_{\mu\nu} \nabla_\lambda \nabla^\lambda + X_\mu) A^\nu, \] (67)
where
\[ X_{\mu\nu} = R_{\nu\mu\delta\rho} + 2 g f^{abc} \tilde{F}_{\mu\nu}^{ab}. \] (68)
The gluon-loop effective action $\Gamma_{\text{gluon-loop}}$ is then given by

$$\Gamma_{\text{gluon-loop}} = \frac{1}{2} \text{ndet}(g_{\mu\nu} \tilde{\nabla}_\alpha \tilde{\nabla}^\alpha + X').$$  (69)

Its divergent part is

$$\text{Div}\Gamma_{\text{gluon-loop}} = -\frac{1}{\epsilon} \int d^4 x \text{tr}[(a_2')(x)],$$  (70)

where $[a_2']$ is the asymptotic expansion coefficient corresponding to the kernel appearing in (67), with a structure similar to that for the fermion case, namely, as in (54). Again, we will concentrate on the corresponding first and last terms in (54) that give rise to gauge-field terms. The corresponding $W'_\mu$ can be obtained from calculating $[\tilde{\nabla}_\mu, \tilde{\nabla}_\nu]A_\lambda$, which can be neatly expressed as

$$[\tilde{\nabla}_\mu, \tilde{\nabla}_\nu]A_\lambda = \tilde{\nabla}^\rho R^\rho_{\lambda\mu\nu} A_\lambda - i[F_{\mu\nu}, A_\lambda] + C^\rho_{\mu\nu} \tilde{\nabla}^\rho A_\lambda,$$  (71)

where $R^\rho_{\lambda\mu\nu}$ is the curvature tensor formed with $\Gamma^\lambda_{\mu\nu}$ as the connection. From (71) we obtain

$$(W'_\mu)^{\rho\sigma} = R^\rho_{\lambda\mu} \delta^{\sigma\lambda} + g f^{\rho\sigma\lambda} \tilde{\nabla}^\lambda.$$  (72)

We then obtain contribution from the first term,

$$\text{tr}[(1/2) W'_\mu W^{\mu\nu}] = -\frac{1}{4} g^2 f^{\rho\sigma\lambda} \tilde{\nabla}^\rho F^{\lambda}_{\mu\nu} F^\sigma_{\mu\nu} + \text{gravitational terms.}$$  (73)

Define $C_2(G)$ by

$$f^{abc}T_a T_b = \frac{i}{2} C_2(G) T_c.$$  (74)

We then have

$$\text{tr}[(1/2) W'_\mu W^{\mu\nu}] = -\frac{1}{4} g^2 \frac{C_2(G)}{C_2(R)} \text{tr} F^{\mu\nu} F^\mu_{\nu} + \text{gravitational terms},$$  (75)

where $C_2(R)$ is the normalization factor given by (20). The contribution from the last term can be similarly calculated and is given by

$$\text{tr} \frac{1}{2} X_{\mu\nu} X^{\mu\nu} = \frac{2}{3} \frac{C_2(G)}{C_2(R)} \text{tr} F^{\mu\nu} F^\mu_{\nu} + \text{gravitational terms.}$$  (76)

The divergent pole term of the gluon loop is due to the sum of the contributions in (75) and (76) and given by

$$\text{Div}\Gamma_{\text{gluon-loop}} = -\frac{1}{\epsilon} \int d^4 x \text{tr}[5/3 \frac{C_2(G)}{C_2(R)} \text{tr} F^{\mu\nu} F^\mu_{\nu} + \text{gravitational terms}].$$  (77)

The effective action due to the loop of the complex Grassmann ghost field is given by

$$\Gamma_{\text{ghost-loop}} = -2\text{ndet}(-\tilde{\nabla}_\mu \tilde{\nabla}^\mu).$$  (78)

Its divergent part can be similarly calculated, taking into account that the ghost field is a complex Grassmann scalar and there is no spinor index to sum over, is given by

$$\text{Div}\Gamma_{\text{ghost-loop}} = \frac{1}{\epsilon} \int d^4 x \text{tr}[-\frac{1}{6} g^2 \frac{C_2(G)}{C_2(R)} \text{tr} F^{\mu\nu} F^\mu_{\nu} + \text{gravitational terms}].$$  (79)

The final result of the divergent parts due to the quark-, gluon-, and ghost-loops is the sum of (59), (77) and (79)

$$\text{Div}\Gamma_{\text{loops}} = \frac{1}{\epsilon} \int d^4 x g^2 \frac{4 C_2(R) - 11 C_2(G)}{6 C_2(R)} \text{tr} F^{\mu\nu} F^\mu_{\nu} + \text{gravitational terms.}$$  (80)

We recall that we have for convenience omitted the normalization factor of $\frac{1}{2 C_2(R)}$ for the $F^{\mu\nu} F^\mu_{\nu}$ term in the original Lagrangian in (17). Thus, when we consider renormalization constants, this normalization factor should be similarly omitted, namely, by dropping $C_2(R)$ in the denominator in the above equation. The resulting result for the gauge-field term, we note, is compatible with earlier results for the Riemannian spacetime. In our specific case of quantum chromodynamics without additional flavor, $C_2(G) = 3$ and $C_2(F) = \frac{1}{2}$.

**IX. MIXED LOOPS**

These are the gluon-quark vertex and quark self-energy loops, which contain both internal quark and gluon lines. We will use ’t Hooft’s algorithm [13, 24] to find the renormalization counter terms. Following ’t Hooft’s procedure, we make the substitutions in the action $W^{(2)}$ in (41)

$$\bar{\psi} \rightarrow \bar{\psi},$$

$$\psi \rightarrow \gamma^\mu \hat{D}_\mu \xi,$$

where $\hat{D}_\mu = \partial_\mu - \frac{i}{2} \sigma_{\alpha\beta} \omega^{\alpha\beta}_{\mu}$, namely, the covariant derivative $D_\mu$ without the gauge-potential term. The fermion part of the Lagrangian in (41) becomes

$$h[-g^\mu\nu \bar{\psi} \hat{D}_\mu \hat{D}_\nu \bar{\xi} \xi + \frac{3}{2} \sigma^{\mu\nu} C_\lambda_{\mu\nu} \hat{D}_\lambda \bar{\xi} + \bar{\psi} i \gamma^\mu i g A_\mu i \gamma^\nu \hat{D}_\nu \xi]$$

Applying ’t Hooft’s algorithm to this fermion Lagrangian and the gluon part of the Lagrangian as given in (67), we have, in ’t Hooft’s notation,

$$(\alpha)_X^\lambda = i \gamma^\lambda i g T^2_\eta,$$
\[
\tilde{\beta}^\lambda = \bar{\eta} i \gamma^\lambda i i g T^\lambda
\]

\[
(N^\mu)^{abc}_X = g^{\mu\nu} (\Gamma^\rho_{\lambda\nu} \delta^{ac} + g f^{abc}_d A^\rho_d g^\rho)
\]

where \( \gamma^\mu = e^\mu_X \gamma^a \). According to the algorithm, the renormalization counter terms due to the mixed loops are the sum of the following four terms:

\[
\frac{2}{\epsilon} \frac{1}{2} \tilde{\beta}^\mu \partial_\mu \alpha
\]

\[
\frac{2}{\epsilon} \frac{1}{2} [\tilde{\beta}^\mu (\frac{i}{4} \sigma_{ab} \omega_{ab}^\mu - \frac{i}{4} \sigma^\lambda \omega_{\lambda\mu})] \alpha
\]

\[
\frac{2}{\epsilon} \frac{1}{2} N^\mu \frac{1}{2} \tilde{\beta}^\mu \alpha
\]

\[
\frac{2}{\epsilon} \frac{1}{2} \tilde{\beta}^\mu \frac{1}{2} i \gamma^\nu A^\nu^\mu i \alpha
\]

With the help of the relations

\[
\gamma_a \omega^{ab}_{\mu} \epsilon^b = \frac{i}{4} \omega_{bc\mu} [\sigma^{bc}, \gamma^\nu]
\]

\[
\partial_\mu \gamma^\nu = \frac{i}{4} \sigma_{ab} \omega^{ab \mu} - \Gamma^\nu \lambda \mu \gamma^\lambda
\]

\[
\partial_\mu \gamma^\nu = \frac{i}{4} \sigma_{ab} \omega^{ab \mu} + \Gamma^\nu \mu \gamma^\lambda
\]

the sum of the four terms is given by

\[
\frac{2}{\epsilon} [\bar{\eta} g^2 (T^\mu T^\mu) i \gamma^\mu (\partial_\mu - \frac{i}{4} \sigma_{ab} \omega_{ab}^\mu) \gamma^\mu] - \bar{\eta} g^3 (f^{abc}_d T^\mu T^\mu) A^\mu_{abc} i \gamma^\mu \bar{\eta} + \bar{\eta} g^3 (T^\mu T^\mu) i \gamma^\mu \bar{\eta}
\]

In addition to \( C_2(R) \) defined by (20) and \( C_2(G) \) defined by (74), we further define \( C_2(F) \) by [53]

\[
T^\mu T^\mu = C_2(F) I.
\]

We note that \( C_2(F) \) and \( C_2(R) \) are related. In our specific case here, \( C_2(F) = \frac{4}{3} C_2(R) = \frac{4}{3} \). It can be easily shown that

\[
f^{abc}_d T^\mu T^\mu = \frac{i}{2} C_2(G) T^\lambda
\]

\[
T^\mu T^\mu = \frac{1}{2} C_2(G) T^\lambda + C^2(F) T^\lambda
\]

The sum (89) becomes,

\[
\frac{2}{\epsilon} C_2(F) g^2 \bar{\eta} i \gamma^\mu (\partial_\mu - \frac{i}{4} \sigma_{ab} \omega_{ab}^\mu + ig \hat{A}_\mu) \eta
\]

This is the final result for the renormalization counter terms due to the gluon-quark vertex and quark self-energy loops. It is also compatible with the earlier result [29, 24] for the Riemannian spacetime.

X. CONCLUSIONS

We have in this note deliberated on the compatibility of torsion with gauge symmetry in a realistic interacting gauge theory, namely, quantum chromodynamics of gluons interacting with quarks in Riemann-Cartan spacetime. We have demonstrated that the system of classical field equations are consistent with the choice of the gauge invariant definition of \( F_{\mu\nu} \), which is the flat-space expression, while inconsistent with the choice of the gauge non-invariant version, which is the one with covariant derivatives replacing the ordinary derivatives in the flat-space expression. To further substantiate the choice of the gauge invariant version of \( F_{\mu\nu} \), we have investigated the quantum renormalizability at on-loop level, to make sure that torsion does not somehow get entangled with gauge symmetry at a level beyond the classical. The heat kernel technique being an essential method in our treatment, we restrict ourselves to the special case of totally antisymmetric torsion, as the general case is much more complicated and there is lack of reliable study on the corresponding heat kernel. We would like to note that the results of Yajima and collaborators [33] on the heat kernel in the presence of torsion are essential for our results.

With regard to the renormalization counter terms for the gluon field and the quark field, our one-loop results are contained in (80) and (91). It is seen that the counter terms are in the same gauge invariant forms as the original terms in the Lagrangian. Except for the gravitational counter terms, which we have omitted, the pattern of counter terms for the gluon and quark fields in the present case of Riemann-Cartan spacetime is exactly the same as in the previously studied case of Riemannian spacetime [23, 29]. We will hence not repeat here defining the renormalized constants. The conclusion, of course, is that the theory, with the choice of the gauge invariant version of \( F_{\mu\nu} \), is renormalizable and gauge symmetry preserved at the one-loop level.

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