The Liouville–Arnold–Nekhoroshev theorem for non-compact invariant manifolds

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Abstract.
Under certain conditions, generalized action-angle coordinates can be introduced near non-compact invariant manifolds of completely and partially integrable Hamiltonian systems.

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1 Introduction

Let us recall that an autonomous Hamiltonian system on a 2n-dimensional symplectic manifold is said to be completely integrable if there exist n independent integrals of motion in involution. By virtue of the classical Liouville–Arnold theorem [1, 6], such a system admits action-angle coordinates around a connected regular compact invariant manifold. In a more general setting, one considers Hamiltonian systems having partial integrability, i.e., k ≤ n independent integrals of motion in involution. The Nekhoroshev theorem for these systems [3, 8] generalizes both the Poincaré–Lyapunov theorem (k = 1) and the above mentioned Liouville–Arnold theorem (k = n). The Nekhoroshev theorem in fact falls into two parts. The first part states the sufficient conditions for an open neighbourhood of an invariant torus T^n k to be a trivial fibre bundle (see [3] for a detailed exposition). The second one provides this bundle with partial action-angle coordinates similarly to the case of complete integrability.

The present work addresses completely and partially integrable Hamiltonian systems whose invariant manifolds need not be compact. This is the case of any autonomous Hamiltonian system because its Hamiltonian, by definition, is an integral of motion. In the preceding papers, we have shown that, if an open neighbourhood of a non-compact invariant manifold of a completely integrable Hamiltonian system is a trivial bundle, it can be equipped with the generalized action-angle coordinates which bring a symplectic form into...
the canonical form [2, 4]. Here, we prove that, under certain conditions, an open neigh-
bourhood of a regular non-compact invariant manifold of a completely integrable system is
a trivial bundle (see parts (A) – (C) in the proof of Theorem 1) and, consequently, it can
be equipped with the generalized action-angle coordinates (see part (D) of this theorem).
Then, this result is extended to partially integrable Hamiltonian systems (see Theorem 3).
The proof of Theorem 3 mainly follows that of Theorem 1. Note that part (D) in the proof
of Theorem 1 can be simplified by the choice of a Lagrangian section \( \sigma \), but this is not the
case of partially integrable systems. This proof also shows that, from the beginning, one can
separate integrals of motion whose trajectories live in tori.

It should be emphasized that the results of Theorem 3 are not limited by the scope of
autonomous mechanics. Any time-dependent Hamiltonian system of \( n \) degrees of freedom
can be extended to an autonomous Hamiltonian system of \( n + 1 \) degrees of freedom which
has at least one integral of motion, namely, its Hamiltonian [2]. Thus, any time-dependent
Hamiltonian system can be seen as a partially integrable autonomous Hamiltonian system
whose invariant manifolds are never compact because of the time axis. Just the time is a
generalized angle coordinate corresponding to a Hamiltonian of this autonomous system.

One also finds reasons in quantum theory in order to introduce generalized action-angle
variables. In particular, quantization with respect to these variables enables one to include
a Hamiltonian in the quantum algebra [2, 5].

2 Completely integrable systems

Let \((Z, \Omega)\) be a \(2n\)-dimensional symplectic manifold, and let it admit \(n\) real smooth functions
\( \{F_\lambda\} \), which are pairwise in involution and independent almost everywhere on \( Z \). The latter
implies that the set of non-regular points, where the morphism

\[
\pi = \bigwedge_\lambda F_\lambda : Z \to \mathbb{R}^n
\]

fails to be a submersion, is nowhere dense. Bearing in mind physical applications, we agree
to think of one of the functions \( F_\lambda \) as being a Hamiltonian and of the other as first integrals
of motion. Accordingly, their common level surfaces are called invariant surfaces.

Let \( M \) be a regular invariant surface, i.e., the morphism \( \pi \) (1) is a submersion at all
points of \( M \) or, equivalently, the \( n \)-form \( \bigwedge dF_\lambda \) vanishes nowhere on \( M \). Hence, \( M \) is a closed
imbedded submanifold of \( Z \). There exists its open neighbourhood \( U \) such that the morphism
\( \pi \) is a submersion on \( U \), i.e.,

\[
\pi : U \to N = \pi(U)
\]

is a fibred manifold over an open subset \( N \subset \mathbb{R}^n \). The vertical tangent bundle \( VU \) of \( U \to N \)
coincides with the \( n \)-dimensional distribution on \( U \) spanned by the Hamiltonian vector fields
\( \vartheta_\lambda \) of the functions \( F_\lambda \). Integral manifolds of this distribution are components of the fibres
of \( \pi \). They are Lagrangian submanifolds of \( Z \). Let \( U \) be connected. Then \( N \) is a domain.
Without loss of generality, one can suppose that there exists a section of \( U \to N \).
If $M$ is connected and compact, we come to the conditions of the Liouville–Arnold theorem. If $M$ need not be compact, one should require something more.

**Theorem 1.** Let $M$ be a connected regular invariant manifold of a completely integrable Hamiltonian system $\{F_\lambda\}$ and let $U$ be an open neighbourhood as above. Let us additionally assume that: (i) all fibres of the fibred manifold $U \to N$ (2) are mutually diffeomorphic, (ii) the Hamiltonian vector fields $\vartheta_\lambda$ on $U$ are complete. Then, there exists a domain $N$ so that $U \to N$ is a trivial bundle

$$U = N \times (\mathbb{R}^{n-m} \times T^m),$$

(3)

provided with the generalized action-angle coordinates $(I_\lambda; x^a; \phi^i)$ such that the integrals of motion $F_\lambda$ depend only on the action coordinates $I_\alpha$ and the symplectic form $\Omega$ on $U$ reads

$$\Omega = dI_a \wedge dx^a + dI_i \wedge d\phi^i.$$  

(4)

**Proof.** (A) Since Hamiltonian vector fields $\vartheta_\alpha$ on $U$ are complete and mutually commutative, their flows assemble into the additive Lie group $\mathbb{R}^n$. This group is naturally identified with its Lie algebra, and its group space is a vector space coordinated by parameters $(s^\lambda)$ of the flows with respect to the basis $\{e_\lambda\}$ for its Lie algebra. This group acts in $U$ so that its generators $e_\lambda$ are represented by the Hamiltonian vector fields $\vartheta_\lambda$ and its orbits are fibres of the fibred manifold $U \to N$. Given a point $r \in N$, the action of $\mathbb{R}^n$ in the fibre $M_r = \pi^{-1}(r)$ factorizes as

$$\mathbb{R}^n \times M_r \to G_r \times M_r \to M_r$$

(5)

through the free transitive action in $M_r$ of the factor group $G_r = \mathbb{R}^n/K_r$, where $K_r$ is the isotropy group of an arbitrary point of $M_r$. It is the same group for all points because $\mathbb{R}^n$ is an Abelian group. Since the fibres $M_r$ are mutually diffeomorphic, all isotropy groups $K_r$ are isomorphic to the group $\mathbb{Z}^m$ for some fixed $m$, $0 \leq m \leq n$, and the groups $G_r$ are isomorphic to the additive group $\mathbb{R}^{n-m} \times T^m$. Let us show that the fibred manifold $U \to N$ (2) is a principal bundle with the structure group $G_0$, where we denote $\{0\} = \pi(M)$. For this purpose, let us determine isomorphisms $\rho_r : G_0 \to G_r$ of the group $G_0$ to the groups $G_r$, $r \in N$. Then, a desired fibrewise action of $G_0$ in $U$ is given by the law

$$G_0 \times M_r \to \rho_r(G_0) \times M_r \to M_r.$$  

(6)

(B) Generators of each isotropy subgroup $K_r$ of $\mathbb{R}^n$ are given by $m$ linearly independent vectors of the group space $\mathbb{R}^n$. One can show that there are ordered collections of generators $(v_1(r), \ldots, v_m(r))$ of the groups $K_r$ such that $r \mapsto v_i(r)$ are smooth $\mathbb{R}^n$-valued fields on $N$. Indeed, given a vector $v_i(0)$ and a section $\sigma$ of the fibred manifold $U \to N$, each field $v_i(r) = (s^\alpha(r))$ is the unique smooth solution of the equation

$$g(s^\alpha)\sigma(r) = \sigma(r), \quad (s^\alpha(0)) = v_i(0),$$

(7)
on an open neighbourhood of \{0\}. Without loss of generality, one can assume that this neighbourhood is \(N\). Let us consider the decomposition
\[ v_i(0) = B_i^a(0)e_a + C_i^k(0)e_k, \quad a = 1, \ldots, n - m, \quad k = 1, \ldots, m, \]
where \(C_i^k(0)\) is a non-degenerate matrix. Since the fields \(v_i(r)\) are smooth, there exists an open neighbourhood of \{0\}, say \(N\) again, where the matrices \(C_i^k(r)\) remain non-degenerate. Then, there is a unique linear morphism
\[
A_r = \begin{pmatrix} \text{Id} & (B(r) - B(0))C^{-1}(0) \\ 0 & C(r)C^{-1}(0) \end{pmatrix}
\]
of the vector space \(\mathbb{R}^n\) which transforms its frame \(v_a(0) = \{e_a, v_i(0)\}\) into the frame \(v_a(r) = \{e_a, v_i(r)\}\). Since it is also an automorphism of the group \(\mathbb{R}^n\) sending \(K_0\) onto \(K_r\), we obtain a desired isomorphism \(\rho_r\) of the group \(G_0\) to the group \(G_r\). Let an element \(g\) of the group \(G_0\) be the coset of an element \(g(s)\) of the group \(\mathbb{R}^n\). Then, it acts in \(M_r\) by the rule (6) just as the element \(g((A_r^{-1})^s)\) of the group \(\mathbb{R}^n\) does. Since entries of the matrix \(A\) (8) are smooth functions on \(N\), this action of the group \(G_0\) in \(U\) is smooth. It is free, and \(U/G_0 = N\). Then, the fibred manifold \(U \to N\) is a principal bundle with the structure group \(G_0\) which is trivial because \(N\) is a domain.

(C) Given a section \(\sigma\) of the principal bundle \(U \to N\), its trivialization \(U = N \times G_0\) is defined by assigning the points \(\rho^{-1}(g_r)\) of the group space \(G_0\) to points \(g_r, \sigma(r), g_r \in G_r\), of a fibre \(M_r\). Let us endow \(G_0\) with the standard coordinate atlas \((y^\lambda) = (t^a; \varphi^i)\) of the group \(\mathbb{R}^n\times\mathbb{T}^m\). We provide \(U\) with a desired trivialization (3) with respect to the coordinates \((J^\lambda; t^a; \varphi^i)\), where \(J^\lambda(u) = F^\lambda(u), u \in U\), are coordinates on the base \(N\). The Hamiltonian vector fields \(\vartheta_\lambda\) on \(U\) relative to these coordinates read
\[
\vartheta_a = \partial_a, \quad \vartheta_i = -(BC^{-1})^i_a \partial_a + (C^{-1})^i_k \partial_k.
\]
In particular, the Hamilton equation takes the form
\[
\dot{J}^\lambda = 0, \quad \dot{y}^\lambda = f^\lambda(J^\lambda).
\]

(D) Since fibres of \(U \to N\) are Lagrangian manifolds, the symplectic form \(\Omega\) on \(U\) is given by the coordinate expression
\[
\Omega = \Omega_{\alpha\beta} dJ_\alpha \wedge dJ_\beta + \Omega_{\alpha}^\beta dJ_\alpha \wedge dy^\beta.
\]
Let us bring it into the canonical form (4). The Hamiltonian vector fields \(\vartheta_\lambda\) obey the relations \(\vartheta_\lambda|\Omega = -dJ_\lambda\), which take the coordinate form
\[
\Omega^\beta_{\alpha} \delta^\alpha_\lambda = \delta^\alpha_\lambda.
\]
It follows that \(\Omega^\alpha_{\beta}\) is a nondegenerate matrix whose entries are independent of coordinates \(y^\lambda\). By virtue of the well-known Künneth formula for the de Rham cohomology of manifold product, the closed form \(\Omega\) (10) on \(U\) (3) is exact, i.e., \(\Omega = d\Xi\) where \(\Xi\) reads
\[
\Xi = \Xi^\alpha(J_\lambda, y^\lambda) dJ_\alpha + \Xi^i(J_\lambda) d\varphi^i.
\]
Because entries of \(d\Xi = \Omega\) are independent of \(y^\lambda\), we obtain the following.

(i) \(\Omega^\lambda_i = \partial^\lambda \Xi_i - \partial_i \Xi^\lambda\). Consequently, \(\partial_i \Xi^\lambda\) are independent of \(\varphi^i\), i.e., \(\Xi^\lambda\) are at most affine in \(\varphi^i\) and, therefore, are independent of \(\varphi^i\) since these are cyclic coordinates. Hence, \(\Omega^\lambda_i = \partial^\lambda \Xi_i\) and \(\partial_i \Omega = -d\Xi_i\). A glance at the last equality shows that \(\partial_i\) are Hamiltonian vector fields.

Moreover, the coordinates \(t^a\) are exactly the flow parameters \(s^a\). Substituting the expressions (13) into the conditions (11), we obtain

\[
\Xi = (-s^a + E^a(I_\lambda))dJ_a + E^i(J_\lambda)dJ_i + \Xi_i(J_j)d\varphi^i.
\]

Finally, put

\[
x^a = s^a - E^a(I_\lambda), \quad \phi^i = \varphi^i - E^i(I_\lambda)
\]

in order to obtain the desired action-angle coordinates

\[
I_a = J_a, \quad I_i = \Xi_i(J_j), \quad x^a(J_\lambda, s^a), \quad \phi^i(J_\lambda, \varphi^k).
\]

The shifts (14) correspond to the choice of a Lagrangian section \(\sigma\).

Let us remark that the generalized action-angle coordinates in Theorem 1 are by no means unique. For instance, the canonical coordinate transformations

\[
I_a = f_a(I'_\lambda), \quad I_i = I'_i, \quad x^a = \frac{\partial f_b}{\partial I'_a}x^b, \quad \phi^i = \phi^i + \frac{\partial f_a}{\partial I'_i}x^a.
\]

give new generalized action-angle coordinates on \(U\).
3 Partially integrable systems

Let a 2n-dimensional symplectic manifold \((Z, \Omega)\) admit \(k < n\) smooth real functions \(\{F_\lambda\}\), which are pairwise in involution and independent almost everywhere on \(Z\). Let us consider the morphism

\[
\pi = \times_{\lambda} F_\lambda : Z \to \mathbb{R}^k, \tag{16}
\]

and its regular connected common level surface \(W\). There exists an open connected neighbourhood \(U_W\) of \(W\) such that

\[
\pi : U_W \to V_W = \pi(U_W) \tag{17}
\]

is a fibred manifold over a domain \(V_W\) in \(\mathbb{R}^k\). Restricted to \(U_W\), the Hamiltonian vector fields \(\vartheta_\lambda\) of functions \(F_\lambda\) define a \(k\)-dimensional distribution and the corresponding regular foliation \(\mathcal{F}\) of \(U_W\). Its leaves are isotropic. They are located in fibres of the fibred manifold \(U_W \to V_W\) and, moreover, make up regular foliations of these fibres.

Let us assume that the foliation \(\mathcal{F}\) has a total transversal manifold \(S\) and its holonomy pseudogroup on \(S\) is trivial. Then, \(\mathcal{F}\) is a fibred manifold

\[
\pi_1 : U_W \to S' \tag{18}
\]

and \(S = \sigma(S')\) is its section \([7]\). Thereby, the fibration \(\pi\) \((17)\) factorizes as

\[
\pi : U_W \xrightarrow{\pi_1} S' \xrightarrow{\pi_2} V_W
\]

through the fibration \(\pi_1\) \((18)\). The map \(\pi_2\) reads \(\pi_2 = \pi \circ \sigma\) and, consequently, it is also a fibred manifold.

**Proposition 2.** Let us assume that there exists a domain \(N \subset S'\) such that: (i) the fibres of the fibred manifold \(\pi_1\) \((18)\) over \(N\) are mutually diffeomorphic, (ii) the Hamiltonian vector fields \(\vartheta_\lambda\) on \(U = \pi_1^{-1}(N)\) are complete. Then, there exists a domain in \(S'\), say \(N\) again, such that \(U \to N\) is a trivial principal bundle with the structure group \(\mathbb{R}^{k-m} \times T^m\).

**Proof.** The proof is a straightforward repetition of parts (A) – (B) in the proof of Theorem 1.

Furthermore, one can always choose the domain \(N\) in Proposition 2 as the domain of a fibred chart of \(\pi_2\). Following part (C) in the proof of Theorem 1, we can provide \(U \to N\) with the trivialization

\[
U = N \times \mathbb{R}^{k-m} \times T^m, \tag{19}
\]

coordinated by \((J_\lambda; z^A; y^\lambda)\) where: (i) \(J_\lambda(u) = F_\lambda(u), u \in U\), are coordinates on the base \(V\), (ii) \((J_\lambda; z^A)\) are coordinates on \(N\), and (iii) \((y^\lambda) = (t^a; \varphi^i)\) are coordinates on \(\mathbb{R}^{k-m} \times T^m\).

The Hamiltonian vector fields \(\vartheta_a\) on \(U\) with respect to these coordinates read

\[
\vartheta_a = \partial_a, \quad \vartheta_i = \vartheta_i^a(J_\lambda, z^A)\partial_a + \vartheta_i^k(J_\lambda, z^A)\partial_k. \tag{20}
\]
Since fibres of \( U \to N \) are isotropic, the symplectic form \( \Omega \) on \( U \) relative to the coordinates \((J_\lambda; z^A; y^\lambda)\) is given by the expression
\[
\Omega = \Omega^{\alpha\beta} dJ_\alpha \wedge dJ_\beta + \Omega^\beta dJ_\alpha \wedge dy^\beta + \Omega_A dJ_\lambda \wedge dz^A + \Omega_\lambda dJ_\lambda \wedge dA + \Omega_{A\beta} dA \wedge dy^\beta. \tag{21}
\]
The Hamiltonian vector fields \( \vartheta_\lambda \) obey the relations \( \vartheta_\lambda \mid \Omega = -dJ_\lambda \), which give the conditions
\[
\Omega^{\alpha\beta} \vartheta_\beta = \delta^{\alpha}_\lambda, \quad \Omega_{A\beta} \vartheta_\beta = 0.
\]
The first of them shows that \( \Omega^{\alpha\beta} \) is a non-degenerate matrix independent of coordinates \( y^\lambda \). Then, the second one implies \( \Omega_{A\beta} = 0 \). The rest is a minor modification of part (D) in the proof of Theorem 1.

The symplectic form \( \Omega \) (21) on \( U \) is exact, and the Liouville form is
\[
\Xi = \Xi^a(J_\lambda, z^B, y^\lambda) dJ_a + \Xi^i(J_\lambda, z^B) d\varphi^i + \Xi_A(J_\lambda, z^B, y^\lambda) dz^A.
\]
Since \( \Xi_a = 0 \) and \( \Xi_i \) are independent of \( \varphi^i \), one easily obtains from the relations \( \Omega_{A\beta} = \partial_A \Xi_\beta - \partial_\beta \Xi_A = 0 \) that \( \Xi_i \) are independent of coordinates \( z^A \), while \( \Xi_A \) are independent of coordinates \( y^\lambda \). Hence, the Liouville form reads
\[
\Xi = \Xi^a(J_\lambda, z^B, y^\lambda) dJ_a + \Xi_i(J_\lambda) d\varphi^i + \Xi_A(J_\lambda, z^B) dz^A
\]
(cf. (12)). Running through item (i), we observe that, in the case of a partially integrable system, one can also separate integrals of motion \( F_i \) whose Hamiltonian vector fields are tangent to invariant tori. Then, the Hamiltonian vector fields (20) take the form
\[
\vartheta_a = \partial_a, \quad \vartheta_i = \vartheta_i^k(J_\lambda, z^A) \partial_k.
\]
Following items (i) – (ii) of part (D), we obtain
\[
\Xi = (-s^a + E^a(J_\lambda, z^B))dJ_a + E^i(J_\lambda, z^B)dJ_i + \Xi_i(J_i) d\varphi^i + \Xi_A(J_\lambda, z^B) dz^A.
\]
Finally, the coordinates
\[
x^a = -s^a + E^a(J_\lambda, z^B), \quad I_i = \Xi_i(J_i), \quad I_a = J_a, \quad \phi^i = \varphi^i - E^i(J_\lambda, z^B) \frac{\partial J_i}{\partial I_i}
\]
bring \( \Omega \) into the form
\[
\Omega = dI_a \wedge dx^a + dI_i \wedge d\phi^i + \Omega_{AB}(I_\lambda, z^B) dz^A \wedge dz^B + \Omega^A(I_\lambda, z^B) dI_\lambda \wedge dz^A. \tag{22}
\]
Therefore, one can think of these coordinates as being partial generalized action-angle coordinates. The Hamiltonian vector fields of integral of motions with respect to these coordinates read
\[
\vartheta_a = \frac{\partial}{\partial x^a}, \quad \vartheta_i = \frac{\partial J_i}{\partial I_j} \frac{\partial}{\partial \phi^i}.
\]
Thus, we have proved the following.

Theorem 3. Given a partially integrable Hamiltonian system \( \{ F_\lambda \} \) on a symplectic manifold \((Z, \Omega)\), let \( W \) be its regular connected level surface, and let \( M \subset W \) be a leaf of the characteristic foliation \( \mathcal{F} \) of the distribution generated by the Hamiltonian vector fields \( \vartheta_\lambda \) of \( F_\lambda \). Let \( M \) have an open saturated neighbourhood \( U \subset Z \) such that: (i) the foliation \( \mathcal{F} \) of \( U \) admits a transversal manifold \( S \) and its holonomy pseudogroup on \( S \) is trivial, (ii) the leaves of this foliation are mutually diffeomorphic, (iii) Hamiltonian vector fields \( \vartheta_\lambda \) on \( U \) are complete. Then, there exists an open saturated neighbourhood of \( M \), say \( U \) again, which is a trivial bundle (19), provided with the particular coordinates \((I_\alpha; z^A; x^a; \phi^i)\) such that the integrals of motion \( F_\lambda \) depend only on the coordinates \( I_\alpha \) and the symplectic form \( \Omega \) on \( U \) is brought into the form (22).

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