Multiparticle States and the Hadron Spectrum on the Lattice

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(Dated: November 1, 2018)

Abstract

The Clebsch-Gordan decomposition is calculated for direct products of the irreducible representations of the cubic space group. These results are used to identify multiparticle states which appear in the hadron spectrum on the lattice. Consideration of the cubic space group indicates how combinations of both zero momentum and non-zero momentum multiparticle states contribute to the spectrum.

PACS numbers: 11.15.Ha,12.38.Gc
I. INTRODUCTION

In Lattice QCD with sufficiently light quarks, the only stable particles are flavored pseudoscalar \((J^P = 0^-)\) mesons and \(J^P = \frac{1}{2}^+\) baryons. At unphysically heavy quark masses, the lowest single particle resonance states in other quantum number channels become stable once they fall below decay thresholds, \(e.g.\ \rho \rightarrow \pi \pi\) or \(\Delta \rightarrow p \pi\). Historically, Lattice QCD has been used to determine the masses of these stable ground state resonances which are then extrapolated to physical light quark regime to estimate the physical masses of the unstable resonances [1, 2, 3, 4, 5, 6, 7]. To extract the masses of excited state resonances, or ground state resonances above threshold, in Lattice QCD is a more involved process that essentially incorporates the study of multiparticle scattering states in a finite box [8, 9, 10, 11, 12, 13, 14, 15, 16]. See [17] for a recent review.

There have been many previous studies in the still nascent field of multiparticle states in Lattice QCD. These include \(\pi \pi\) scattering [18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29], \(NN\) scattering [21, 22, 30], heavy-light meson scattering [31], searching for pentaquark resonances in \(KN\) scattering [32, 33, 34, 35, 36, 37, 38, 39, 40, 41], and hadronic decays [42, 43, 44, 45].

There have also been several studies of excited state resonance masses in Lattice QCD: Roper resonance and other excited baryons [46, 47, 48, 49, 50, 51, 52, 53, 54, 55], charmonium and bottomonium [56, 57, 58, 59, 60, 61, 62], heavy-light mesons [63]. As mentioned above, the masses of all these resonances lie somewhere within the discrete energy levels of the multiparticle scattering states and must be disentangled. It is usually not sufficient to just compute two-point correlation functions between single particle operators unless it can be clearly demonstrated that the overlaps of the operators with the multiparticle states are small enough that only resonances contribute to the correlator.

In essentially all of the calculations referenced above, the operators used to compute correlation functions were constructed to transform irreducibly under the symmetry group of continuum QCD Hamiltonian. It is well known [64] that these operators need not transform irreducibly under the symmetry group of the lattice QCD Hamiltonian. When calculating ground state masses, ignoring this fact usually does not lead to confusion. One possible exception is the \(I(J^P) = \frac{1}{2}(3^+)\) and \(\frac{1}{2}(5^+)\) baryons, whose lowest-lying resonances should correspond to the experimentally observed \(N(1720)\) \(P_{13}\) and \(N(1680)\) \(F_{15}\), respectively [65].

When calculating properties of single particle resonances at non-zero momentum, estab-
lishing continuum quantum number assignments will be more difficult than for resonances in the rest frame \[66, 67\]. In this work, we demonstrate for each set of quantum numbers in the center-of-mass frame what two-particle decompositions are possible, including states with non-zero relative momentum.

II. CLEBSCH-GORDAN DECOMPOSITION

We wish to calculate the decomposition of the direct product of irreducible representations of $\mathcal{T}_{\text{lat}}^3 \rtimes O^D_h$ into a direct sum of irreducible representations. Using the character table for $\mathcal{T}_{\text{lat}}^3 \rtimes O^D_h$, the character of a group element $g \in \mathcal{T}_{\text{lat}}^3 \rtimes O^D_h$ in the direct product representation $\Gamma_i \otimes \Gamma_j$ is given as $\chi_{\Gamma_i,\Gamma_j}(g) = \chi_{\Gamma_i}(g)\chi_{\Gamma_j}(g)$. The irreducible representations $\Gamma_k$ and character table for $\mathcal{T}_{\text{lat}}^3 \rtimes O^D_h$ are described in \[66\]. Then, the multiplicity $m$ that an irreducible representation $\Gamma_k$ is contained in the direct product representation $\Gamma_i \otimes \Gamma_j$ is given by:

$$m = \frac{1}{|G|} \sum_g \chi_{\Gamma_k}(g)^*\chi_{\Gamma_i,\Gamma_j}(g)$$

(1)

where the sum is taken over all group elements $g$, and $|G|$ is the order of the group. This formula applies to finite lattices, and we take the limit of $m$ as the lattice size becomes arbitrarily large.

The irreducible representations of $\mathcal{T}_{\text{lat}}^3 \rtimes O^D_h$ are labeled by the magnitude of a lattice momentum $\mathbf{k}$ and by $\alpha$, which labels an irreducible representation of the little group of $\mathbf{k}$. The irreducible representations labeled by lattice vectors $\mathbf{k}$ and $\mathbf{k}'$ are equivalent if there is a group element $g \in O^D_h$ such that $\mathbf{k}' = g\mathbf{k}$, and the set of such $\mathbf{k}'$ is called the star of $\mathbf{k}$. Thus, the inequivalent irreducible representations of $\mathcal{T}_{\text{lat}}^3 \rtimes O^D_h$ are labeled by a star, denoted $|\mathbf{k}|$ in analogy with the continuum notation (we must be careful with vectors such as $\mathbf{k} = (3, 0, 0)$ and $\mathbf{k}' = (2, 2, 1)$ since even though $|\mathbf{k}| = |\mathbf{k}'|$, they are not in the same star in the discrete group). As we expect, we see that linear momentum is conserved, i.e. the product of two representations with momenta $|\mathbf{k}_1|$ and $|\mathbf{k}_2|$ gives only representations labeled by $|\mathbf{k}|$ which are the sum of some vector in the star of $\mathbf{k}_1$ and some vector in the star of $\mathbf{k}_2$. Thus, the direct product of two irreducible representations of $\mathcal{T}_{\text{lat}}^3 \rtimes O^D_h$ contain irreducible representations labeled by $|\mathbf{k}| = 0$ (the irreducible representations of $O^D_h$) if and only if $|\mathbf{k}_1| = |\mathbf{k}_2|$. In this paper we are mainly concerned with such direct products of irreducible representations which have overlap with zero momentum states.
TABLE I: Clebsch-Gordan decomposition for $O^D$. The table for $O^D_h$ adds a parity $g$ or $u$ to each irreducible representation

| ⊗ | $A_1$ | $A_2$ | $E$ | $T_1$ | $T_2$ | $G_1$ | $G_2$ | $H$ |
|---|---|---|---|---|---|---|---|---|
| $A_1$ | $A_1$ | $A_2$ | $E$ | $T_1$ | $T_2$ | $G_1$ | $G_2$ | $H$ |
| $A_2$ | $A_1$ | $E$ | $T_2$ | $T_1$ | $G_2$ | $G_1$ | $H$ |
| $E$ | $A_1$ $A_2$ | $T_1$ $T_2$ | $T_1$ $T_2$ | $H$ | $H$ | $G_1$ $G_2$ | $+$ |
| $E$ | $E$ | $T_1$ $T_2$ | $T_1$ $T_2$ | $G_1$ $H$ | $G_2$ $H$ | $G_1$ $G_2$ | $+$ |
| $T_1$ | $A_1$ $E$ | $A_2$ $E$ | $G_1$ $H$ | $G_2$ $H$ | $G_1$ $G_2$ | $+$ |
| $T_2$ | $A_1$ $E$ | $G_2$ $H$ | $G_1$ $H$ | $G_1$ $G_2$ | $+$ |
| $G_1$ | $A_1$ $T_1$ | $A_2$ $T_2$ | $E$ $T_1$ $T_2$ |
| $G_2$ | $A_1$ $T_1$ | $E$ $T_1$ $T_2$ |
| $H$ | $A_1$ $A_2$ | $E$ $2T_1$ | $2T_2$ |

For the case where $|k_1| = |k_2| = 0$, we have the Clebsch-Gordan decomposition for products of the representations of $O^D_h$ which is given in Tab. II. The single valued irreducible representations of $O^D$ are labeled $A_1, A_2, E, T_1, T_2$ and the double valued representations are labeled $G_1, G_2, H$ using the Mulliken convention. The correspondence of these lattice states to continuum spin states is well-known and given in Tab. II [64]. If we include parity to form the group $O^D_h$, then each representation carries either the label $g$ or $u$ which correspond to positive and negative parity respectively. These labels are omitted in the table since they follow the same multiplication rules that hold in the continuum: $g \cdot g = u \cdot u = g$ and $g \cdot u = u \cdot g = u$.

III. MULTIPARTICLE STATES

From Tabs. II and III we see that multiparticle states for the lowest energy states for each spin behave as in the continuum. In fact, we can generate much of Tab. II using...
TABLE II: The decomposition of continuum SU(2) spin states $J$ to the representations of $O^D$ and the continuum $O(2)^D$ states $m_j$ to the various little groups.

| $J/m_j$ | $O^D$ | $\text{Dic}_4$ | $\text{Dic}_3$ | $\text{Dic}_2$ | $C_4$ | $C_2$ |
|---------|-------|---------------|---------------|---------------|-------|-------|
| 0$^+$   | $A_{1g}$ | $A_1$         | $A_1$         | $A_1$         | $A_1$ | $A$   |
| 0$^-$   | $A_{1u}$ | $A_2$         | $A_2$         | $A_2$         | $A_2$ | $A$   |
| $\frac{1}{2}$ | $G_1$ | $E_1$         | $E_1$         | $E$           | $E$   | $2B$  |
| 1       | $T_1$  | $E_2$         | $E_2$         | $B_1 \oplus B_2$ | $A_1 \oplus A_2$ | $2A$  |
| $\frac{3}{2}$ | $H$    | $E_3$         | $B_1 \oplus B_2$ | $E$           | $E$   | $2B$  |
| 2       | $E \oplus T_2$ | $B_1 \oplus B_2$ | $E_2$         | $A_1 \oplus A_2$ | $A_1 \oplus A_2$ | $2A$  |
| $\frac{5}{2}$ | $G_2 \oplus H$ | $E_3$         | $E_1$         | $E$           | $E$   | $2B$  |
| 3       | $A_2 \oplus T_1 \oplus T_2$ | $E_2$         | $A_1 \oplus A_2$ | $B_1 \oplus B_2$ | $A_1 \oplus A_2$ | $2A$  |
| $\frac{7}{2}$ | $G_1 \oplus G_2 \oplus H$ | $E_1$         | $E_1$         | $E$           | $E$   | $2B$  |
| 4       | $A_1 \oplus E \oplus T_1 \oplus T_2$ | $A_1 \oplus A_2$ | $E_2$         | $A_1 \oplus A_2$ | $A_1 \oplus A_2$ | $2A$  |

Tab. II and the continuum rules for addition of angular momentum. For example, continuum spins $1 \otimes 2 = 1 \oplus 2 \oplus 3$ and the corresponding lattice representations $T_1 \otimes (E \oplus T_2) = (T_1 \oplus T_2) \oplus (A_2 \oplus E \oplus T_1 \oplus T_2) = T_1 \oplus (E \oplus T_2) \oplus (A_2 \oplus T_1 \oplus T_2)$. The continuum relations hold, of course, because taking the direct sum of the lattice irreducible representations in Tab. II gives equivalent representations to the SU(2) irreducible representations, so they must follow the same multiplication rules. For spins which lie in a single lattice irreducible representation, there is no ambiguity in the combinations. Thus, on the lattice, the combination of low spin single particle states is identical to the continuum. The $\pi$ with $J^P = 0^-$ lies in the irreducible representation $A_{1u}$, and the $\pi \pi$ multiparticle state lies in $A_{1u} \otimes A_{1u} = A_{1g}$, which as expected corresponds to $J^P = 0^+$. Similarly, for the vector meson $\rho(770)$ which lies in $T_{1u}$, then the state $\rho \pi$ lies in $T_{1g}$ which corresponds to $J^P = 1^+$. Higher spin states are less straightforward since multiple lattice irreducible representations appear for each spin. A spin 2 continuum state could lie in either the $E$ or $T_2$ representations on the lattice, which leads to different possibilities for multiparticle states.
For the continuum example $1 \otimes 2$, then if the spin 2 state lies in $E$, the combined state is $T_1 \oplus T_2$ whereas if it were in $T_2$, the combined state would be $A_2 \oplus E \oplus T_1 \oplus T_2$. In both cases, we get combinations of the representations $A_2, E, T_1,$ and $T_2$, all of which have their lowest spins corresponding to 1, 2, or 3, but the identification of a particular spin state to each is difficult without further information.

When combining states with non-zero momentum, the continuum relations are not as easily recovered. Tabs. III - VII show the zero-momentum representations in the decomposition of products of irreducible representations labeled by the possible lattice momenta. Here, the continuum representations are no longer labeled by $J$, but by the projection of $J$ along the momentum vector, $m_j$. However, we know that a particle with a given $m_j$ has $J \geq m_j$, so the lowest spin state for a given irreducible representation of $T^3_{\text{lat}} \rtimes O^D_h$ will be $J = m_j$. In addition, the reduced symmetry of the little groups leads to fewer distinct lattice irreducible representations than at zero momentum. Since the continuum irreducible representations are mapped to fewer lattice representations, it is more difficult to assign a particular spin to a given lattice irreducible representation.

Thus, for low spins on the lattice we expect to see the same multiparticle states as we would in the continuum, but as we go to higher spins or non-zero momentum deviations will occur from the continuum behavior. For example, from Tab. III a $J^P = 2^+$ state in the continuum can lie in either $E_g, T_{2g}$, or some combination of both on the lattice. In the continuum, an $f_2(1270)$ meson with $J^P = 2^+$ has the decay modes $\pi\pi, 4\pi,$ and $K\bar{K}$. Thus we expect to see multiparticle states in the lattice spectrum corresponding to these decay modes. From Tab. III we see that the multiparticle state $\{n, 0, 0\}; A_2 \otimes \{n, 0, 0\}; A_2$ corresponding to these decay modes occurs in the $E_g$ channel, but not in the $T_{2g}$ channel. We must calculate exactly how the particular spin 2 continuum state we are interested in subduces to the lattice to determine whether these multiparticle states will appear. In this case, it is possible that states we expect from the continuum rules for addition of angular momentum would be absent from the lattice spectrum.

Another example where the lattice multiparticle states cannot be predicted from the continuum behavior occurs for $J^P = \frac{5}{2}^-$. As for spin 2, a continuum spin $\frac{5}{2}$ state can lie in more than one lattice representation, either $G_{2u}$ or $H_u$. However, if we consider multiparticle states in the $H_u$ channel, then both states which go to spin $\frac{3}{2}$ in the continuum limit and states which go to spin $\frac{5}{2}$ in the continuum limit will appear, since at any finite lattice
spacing these states may have the same lattice quantum numbers (i.e. correspond to the
$H_u$ irreducible representation). Again, we must know how our $J^P = \frac{5}{2}$ state subduces to
the lattice in order to determine exactly which multiparticle states will occur in the lattice
spectrum. Here, multiparticle states are present in the lattice spectrum which we would not
predict from the continuum states. As we consider higher spins, these types of ambiguities
become common.

IV. CONCLUSION

We have calculated the Clebsch-Gordan decomposition for the cubic space group,
$T^{3}_{lat} \rtimes O^{P}_{h}$, which determines the allowed multiparticle states for each lattice irreducible
representation, including states with non-zero momentum. For states with low spin, we
find that the lattice states mirror the continuum behavior since these continuum irreducible
representations remain irreducible on the lattice.

For higher spins or non-zero momentum, the continuum relations are not as easily re-
covered. Since multiple continuum spins lie in each lattice irreducible representation, mul-
tiparticle states appear which we would not expect from the continuum behavior. In the
continuum limit, we should recover the correct multiparticle states, but at any finite lattice
spacing these effects will play a role. In general, continuum states must be subduced to the
lattice irreducible representations in order to correctly predict which multiparticle states
will appear on the lattice.

V. ACKNOWLEDGMENTS

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TABLE III: Clebsch-Gordan decomposition for products of the irreducible representations of $T_{\alpha\beta}^D \times O_h^D$ labeled by $|\langle n, 0, 0 \rangle|$. Only the zero momentum representations are given.

| $\otimes$ | $A_1$ | $A_2$ | $B_1$ | $B_2$ | $E_1$ | $E_2$ | $E_3$ |
|----------|-------|-------|-------|-------|-------|-------|-------|
| $A_1$    | $A_{1g} \oplus E_g \oplus T_{1g} \oplus A_{1u} \oplus A_{2g} \oplus E_g \oplus T_{2g} \oplus A_{2u} \oplus G_{1g} \oplus H_g \oplus T_{1g} \oplus T_{2g} \oplus G_{2g} \oplus H_g$ | $T_{1u}$ | $E_u$ | $T_{2u}$ | $E_u$ | $G_{1u} \oplus H_u$ | $T_{1u} \oplus T_{2u}$ | $G_{2u} \oplus H_u$ |
| $B_1$    | $A_{1g} \oplus E_g \oplus T_{1g} \oplus A_{1u} \oplus A_{2g} \oplus E_g \oplus T_{2g} \oplus A_{2u} \oplus G_{1g} \oplus H_g \oplus T_{1g} \oplus T_{2g} \oplus G_{1g} \oplus H_g$ | $T_{1u}$ | $E_u$ | $T_{2u}$ | $G_{1u} \oplus H_u$ | $T_{1u} \oplus T_{2u}$ | $G_{1u} \oplus H_u$ |
| $E_1$    | $A_{1g} \oplus E_g \oplus G_{1g} \oplus G_{2g} \oplus A_{2g} \oplus E_g \oplus 2T_{1g} \oplus 2T_{2g} \oplus 2H_g \oplus G_{1u} \oplus T_{1g} \oplus 2T_{2g} \oplus A_{1u} \oplus E_u \oplus G_{2u} \oplus 2H_u \oplus 2T_{1u} \oplus 2T_{2u} \oplus A_{2u} \oplus E_u$ | $T_{1u}$ | $T_{2u}$ |
| $E_2$    | $A_{1g} \oplus A_{2g} \oplus G_{1g} \oplus G_{2g} \oplus 2E_g \oplus T_{1g} \oplus 2H_g \oplus G_{1u} \oplus 2T_{2g} \oplus A_{1u} \oplus G_{2u} \oplus 2H_u \oplus A_{2u} \oplus 2E_u$ | $T_{1u}$ | $T_{2u}$ |
| $E_3$    | $A_{1g} \oplus E_g \oplus 2T_{1g} \oplus A_{1u} \oplus E_u$ | $T_{1u}$ | $T_{2u}$ |

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| ⊗ | A₁ | A₂ | B₁ | B₂ | E |
|---|---|---|---|---|---|
| A₁ | $A_{1g} \oplus E_g \oplus T_{2g}$ | $T_{1g}$ | $T_{2g}$ | $A_{2g} \oplus E_g \oplus T_{1g}$ | $T_{1g}$ | $T_{2g}$ | $G_{1g}$ | $G_{2g}$ | $G_{1u}$ | $G_{2u}$ | $2H_g$ | $G_{1u}$ | $G_{2g}$ | $G_{1g}$ | $G_{2u}$ | $2H_u$ |
|   | $T_{1u}$ | $E_u$ | $T_{2u}$ | $T_{1u}$ | $T_{2u}$ | $A_{2u}$ | $E_u$ | $T_{1u}$ | $T_{2u}$ | $A_{2u}$ | $E_u$ | $T_{1u}$ | $T_{2u}$ | $A_{2u}$ | $E_u$ | $T_{1u}$ | $T_{2u}$ |
| A₂ | $A_{1g} \oplus E_g \oplus T_{2g}$ | $T_{1g}$ | $T_{2g}$ | $A_{2g} \oplus E_g \oplus T_{1g}$ | $G_{1g}$ | $G_{2g}$ | $G_{1u}$ | $G_{2u}$ | $2H_g$ | $G_{1u}$ | $G_{2g}$ | $G_{1g}$ | $G_{2u}$ | $2H_u$ |
|   | $T_{1u}$ | $E_u$ | $T_{2u}$ | $T_{1u}$ | $T_{2u}$ | $A_{2u}$ | $E_u$ | $T_{1u}$ | $T_{2u}$ | $A_{2u}$ | $E_u$ | $T_{1u}$ | $T_{2u}$ | $A_{2u}$ | $E_u$ | $T_{1u}$ | $T_{2u}$ |
| B₁ | $A_{1g} \oplus E_g \oplus T_{2g}$ | $T_{1g}$ | $T_{2g}$ | $T_{1u}$ | $T_{2u}$ | $A_{1u}$ | $E_u$ | $T_{1u}$ | $T_{2u}$ | $A_{1u}$ | $E_u$ | $T_{1u}$ | $T_{2u}$ | $A_{1u}$ | $E_u$ | $T_{1u}$ | $T_{2u}$ |
| B₂ | $A_{1g} \oplus E_g \oplus T_{2g}$ | $T_{1g}$ | $T_{2g}$ | $T_{1u}$ | $T_{2u}$ | $A_{1g}$ | $E_g$ | $T_{1u}$ | $T_{2u}$ | $A_{1g}$ | $E_g$ | $T_{1u}$ | $T_{2u}$ | $A_{1g}$ | $E_g$ | $T_{1u}$ | $T_{2u}$ |
| E  | $A_{1g}$ | $A_{2g}$ | $2E_g$ | $3T_{1g}$ | $3T_{2g}$ | $A_{1u}$ | $A_{2u}$ | $2E_u$ | $3T_{1u}$ | $3T_{2u}$ |

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TABLE V: Zero-momentum irreducible representations for $|(n,n,n)|$. 

| ⊗  | $A_1$ | $A_2$ | $E_2$ | $B_1$ | $B_2$ | $E_1$ |
|----|-------|-------|-------|-------|-------|-------|
| $A_1$ | $A_{1g} \oplus T_{2g}$ | $A_{2g} \oplus T_{1g}$ | $E_g \oplus T_{1g}$ | $H_g \oplus H_u$ | $H_g \oplus H_u$ | $G_{1g} \oplus G_{2g}$ |
|      | $A_{2u} \oplus T_{1u}$ | $A_{1u} \oplus T_{2u}$ | $T_{2g} \oplus E_u$ | $H_g \oplus G_{1u}$ | $G_{2u} \oplus H_u$ |       |
| $A_2$ | $A_{1g} \oplus T_{2g}$ | $E_g \oplus T_{1g}$ | $H_g \oplus H_u$ | $H_g \oplus H_u$ | $H_g \oplus G_{1u}$ | $G_{2u} \oplus H_u$ |
|      | $A_{2u} \oplus T_{1u}$ | $T_{2g} \oplus E_u$ | $G_{1u} \oplus G_{2g}$ | $H_g \oplus G_{1u}$ | $G_{2u} \oplus H_u$ |       |
|      | $T_{1u} \oplus T_{2u}$ |       |       |       |       |       |
| $E_2$ | $A_{1g} \oplus A_{2g}$ | $G_{1g} \oplus G_{2g}$ | $G_{1g} \oplus G_{2g}$ | $G_{1g} \oplus G_{2g}$ | $G_{1u} \oplus G_{1u}$ | $3H_g \oplus G_{1u}$ |
|      | $E_g \oplus 2T_{1g}$ | $H_g \oplus G_{1u}$ | $H_g \oplus G_{1u}$ | $G_{1u} \oplus G_{1u}$ | $G_{2u} \oplus H_u$ | $G_{2u} \oplus H_u$ |
|      | $2T_{2g} \oplus A_{1u}$ | $G_{2u} \oplus H_u$ | $G_{2u} \oplus H_u$ | $G_{2u} \oplus H_u$ | $G_{2u} \oplus H_u$ |       |
|      | $A_{2u} \oplus E_u$ | $G_{2u} \oplus H_u$ |       |       |       |       |
|      | $2T_{1u} \oplus 2T_{2u}$ |       |       |       |       |       |
| $B_1$ | $A_{2g} \oplus T_{1g}$ | $A_{1g} \oplus T_{2g}$ | $E_g \oplus T_{1g}$ | $A_{1u} \oplus T_{2u}$ | $A_{2u} \oplus T_{1u}$ | $T_{2g} \oplus E_u$ |
|      | $A_{1u} \oplus T_{2u}$ |       |       | $T_{1u} \oplus T_{2u}$ | $T_{1u} \oplus T_{2u}$ |       |
| $B_2$ | $A_{2g} \oplus T_{1g}$ | $E_g \oplus T_{1g}$ | $A_{1u} \oplus T_{2u}$ | $T_{2g} \oplus E_u$ | $T_{1u} \oplus T_{2u}$ |       |
|      | $A_{1u} \oplus T_{2u}$ |       |       |       |       |       |
| $E_1$ | $A_{1g} \oplus A_{2g}$ | $E_g \oplus 2T_{1g}$ | $2T_{2g} \oplus A_{1u}$ | $A_{2u} \oplus E_u$ | $2T_{1u} \oplus 2T_{2u}$ |       |
|      | $E_g \oplus 2T_{1g}$ | $2T_{1u} \oplus 2T_{2u}$ | $A_{2u} \oplus E_u$ | $2T_{1u} \oplus 2T_{2u}$ |       |       |

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| $\otimes$ | $A_1$ | $A_2$ | $E$ |
|---|---|---|---|
| $A_1$ | $A_{1g} \oplus A_{2g} \oplus 2E_g \oplus T_{1g} \oplus T_{2g} \oplus 2T_{1u} \oplus 2T_{2u}$ | $2T_{1g} \oplus 2T_{2g} \oplus A_{1u} \oplus A_{2u} \oplus 2T_{1u} \oplus 2T_{2u}$ | $2G_{1g} \oplus 2G_{2g} \oplus 4H_g \oplus 2G_{1u} \oplus 2G_{2u} \oplus 4H_u$ |
| $A_2$ | $A_{1g} \oplus A_{2g} \oplus 2E_g \oplus T_{1g} \oplus T_{2g} \oplus 2T_{1u} \oplus 2T_{2u}$ | $2E_u \oplus T_{1u} \oplus T_{2u} \oplus 2E_u \oplus T_{1u} \oplus T_{2u}$ | $2G_{1g} \oplus 2G_{2g} \oplus 4H_g \oplus 2G_{1u} \oplus 2G_{2u} \oplus 4H_u$ |
| $E$ | $A_{1g} \oplus A_{2g} \oplus 2E_g \oplus T_{1g} \oplus T_{2g} \oplus 2T_{1u} \oplus 2T_{2u}$ | $T_{1g} \oplus 2T_{1u} \oplus T_{2u} \oplus A_{1u} \oplus A_{2u} \oplus E_u \oplus 2T_{1u} \oplus 2T_{2u}$ | $2A_{1g} \oplus 2A_{2g} \oplus 2A_{2u} \oplus 2E_u \oplus 2T_{1g} \oplus 2T_{2g} \oplus 4T_{1u} \oplus 4T_{2u}$ |
| $E$ | $A_{1g} \oplus A_{2g} \oplus 2E_g \oplus T_{1g} \oplus T_{2g} \oplus 2T_{1u} \oplus 2T_{2u}$ | $A_{1u} \oplus E_u \oplus T_{1u} \oplus 2T_{2u} \oplus A_{2u} \oplus E_u \oplus 2T_{1u} \oplus T_{2u}$ | $4E_u \oplus 6T_{1u} \oplus 6T_{2u}$ |
| $E$ | $A_{1g} \oplus A_{2g} \oplus 2E_g \oplus T_{1g} \oplus T_{2g} \oplus 2T_{1u} \oplus 2T_{2u}$ | $A_{1u} \oplus E_u \oplus T_{1u} \oplus 2T_{2u} \oplus A_{2u} \oplus E_u \oplus 2T_{1u} \oplus T_{2u}$ | $4E_u \oplus 6T_{1u} \oplus 6T_{2u}$ |

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