1. Introduction

The Bach tensor is a basic object in four-dimensional conformal geometry. It is a conformally invariant trace-free symmetric 2-tensor involving 4 derivatives of the metric which is of particular interest because it vanishes for metrics which are conformal to Einstein metrics, and because it arises as the first variational derivative of the conformally invariant Lagrangian \( \int |W|^2 \), where \( W \) denotes the Weyl tensor. A generalization of the Bach tensor to higher even dimensional manifolds was indicated in [FG1]. This “ambient obstruction tensor”, which, suitably normalized, we denote by \( O_{ij} \), is also a trace-free symmetric 2-tensor which is conformally invariant and vanishes for conformally Einstein metrics. It involves \( n \) derivatives of the metric on a manifold of even dimension \( n \geq 4 \). In this paper we give the details of the derivation and basic properties of the obstruction tensor and provide a characterization generalizing the variational characterization of the Bach tensor in four dimensions. We also give an invariant-theoretic classification of conformally invariant tensors up to invariants which are quadratic and higher in curvature which illuminates the fundamental nature of the obstruction tensor.

Our higher dimensional substitute for \( \int |W|^2 \) is the integral of Branson’s \( Q \)-curvature ([B]). This \( Q \)-curvature is a scalar quantity defined on even-dimensional Riemannian (or pseudo-Riemannian) manifolds. It is not a pointwise conformal invariant like \( |W|^2 \), but it does have a simple transformation law under conformal change which implies that its integral over a compact manifold is a conformal invariant. In dimension 4, one has \( 6Q = -\Delta R + R^2 - 3|\text{Ric}|^2 \), where \( R \) denotes the scalar curvature and \( \Delta = \nabla^i \nabla_i \). Since the Pfaffian in 4 dimensions is a multiple of \( R^2 - 3|\text{Ric}|^2 + \frac{3}{2}|W|^2 \), it follows that \( \int Q \) is a linear combination of the Euler characteristic and \( \int |W|^2 \), so the variational derivatives of \( \int Q \) and \( \int |W|^2 \) are multiples of one another. It follows from a result announced by Alexakis [A1] that also in higher dimensions, \( \int Q \) is a linear combination of the Euler characteristic and the integral of a pointwise conformal invariant. However explicit formulae are not available.

Our variational characterization is:

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\end{itemize}
Theorem 1.1. If $g^t$ is a 1-parameter family of metrics on a compact manifold $M$ of even dimension $n \geq 4$, then

$$ \left( \int_M Q \, dv \right)^{\cdot} = (-1)^{n/2} \frac{n-2}{2} \int_M \mathcal{O}_{ij} \dot{g}^{ij} \, dv, $$

where $\cdot = \partial_t|_{t=0}$ and $\mathcal{O}_{ij}$ and $dv$ on the right hand side are with respect to $g^0$.

In [FG1], $\mathcal{O}_{ij}$ arose as the obstruction to the existence of a smooth formal power series solution for the ambient metric associated to the given conformal structure, a Ricci-flat metric in 2 higher dimensions homogeneous with respect to dilations. As described in [FG1], the ambient metric is equivalent to a Poincaré metric, a metric in 1 higher dimension with constant negative Ricci curvature having the given conformal structure as conformal infinity, and the obstruction tensor may alternately be viewed as obstructing smooth formal power series solutions for a Poincaré metric. It is the latter formulation that we use in this paper, for it is the Poincaré metric that provides the link between $Q$-curvature and the obstruction tensor. Specifically, we use the result of [GZ] that the integral of the $Q$-curvature is equal to a multiple of the log term in the volume expansion of a Poincaré metric. We then calculate the variation of the log term coefficient by a simplified version of the method of Anderson [An] for expressing the variation of volume as a boundary integral. A different calculation of the first variation of the log coefficient in the volume expansion is given in [HSS].

The existence of the obstruction tensor gives rise to the questions of whether there are other conformally invariant tensors lurking in the shadows, and whether there is some kind of odd-dimensional analogue. Of course, one may construct further invariants from known ones by taking tensor products and contracting. However, the following result shows that up to quadratic and higher terms in curvature, the Weyl tensor (or Cotton tensor in dimension 3) and the obstruction tensor are the only irreducible conformally invariant tensors.

Theorem 1.2. A conformally invariant irreducible natural tensor of $n$-dimensional oriented Riemannian manifolds is equivalent modulo a conformally invariant natural tensor of degree at least 2 in curvature with a multiple of one of the following:

- $n = 3$: the Cotton tensor $C_{ijk} = P_{ij,k} - P_{ik,j}$
- $n = 4$: the self-dual or anti-self dual Weyl tensor $W^{\pm}_{ijkl}$ or the Bach tensor $B_{ij} = \mathcal{O}_{ij}$
- $n \geq 5$ odd: the Weyl tensor $W_{ijkl}$
- $n \geq 6$ even: the Weyl tensor $W_{ijkl}$ or the obstruction tensor $\mathcal{O}_{ij}$

Here the trace-modified Ricci tensor $P_{ij}$ is defined by

$$ (n - 2)P_{ij} = R_{ij} - \frac{R}{2(n-1)} g_{ij}, $$

(1.1)
and the terminology used in the statement of the Theorem is explained in §4. Theorem 1.2 is an easy consequence of the classification of conformally invariant linear differential operators on the sphere due to Boe-Collingwood ([BC]).

In §2 we show how to derive the obstruction tensor in terms of a Poincaré metric and establish its basic properties. In §3 we prove Theorem 1.1 and in §4 we prove Theorem 1.2.

We are grateful to Mike Eastwood and Andi Čap for helpful discussions concerning the Boe-Collingwood classification of invariant operators.

2. The Obstruction Tensor

In this section we provide the details of the background about the obstruction tensor. We show how it arises as the obstruction to the existence of a smooth formal power series solution for a Poincaré metric associated to the given conformal structure and derive its properties from this characterization. We also show that this definition may be reformulated in terms of a formal solution to one higher order involving a log term.

Let $M$ be a manifold of dimension $n\geq 3$ with smooth conformal structure $[g]$ of signature $(p,q)$ and let $X$ be $n+1$-manifold with boundary $M$. All our considerations are local near a point of $M$. We are interested in conformally compact metrics $g_+$ of signature $(p+1,q)$ on $X$ with conformal infinity $[g]$. This means that if $x$ is a smooth defining function for $M$, then $x^2 g_+ |_{TM} \in [g]$. If $n$ is odd, then for any conformal class $[g]$ there are metrics $g_+$ with $x^2 g_+$ a formal smooth power series such that $\text{Ric} g_+ = -ng_+$ to infinite order. However if $n$ is even, the obstruction tensor obstructs the existence of formal smooth solutions at order $n-2$. (Throughout this paper, when we say that a tensor is $O(x^s)$, we mean that all components of the tensor are $O(x^s)$ in a smooth coordinate system on $X$.)\footnote{One could alternately consider metrics of signature $(p,q+1)$ for which $\text{Ric} g_+ = ng_+$. This formulation is equivalent to ours via the change $g_+ \rightarrow -g_+$.}

**Theorem 2.1.** If $n \geq 4$ is even, there exists a metric $g_+$ with $x^2 g_+$ smooth such that $g_+$ has $[g]$ as conformal infinity and $\text{Ric} g_+ + ng_+ = O(x^{n-2})$. $g_+$ is unique modulo $O(x^{n-2})$ up to a diffeomorphism of $X$ which restricts to the identity on $M$. The tensor $\text{tf}(x^{2-n}(\text{Ric} g_+ + ng_+)|_{TM})$ on $M$ is independent of the choice of such $g_+$, where $\text{tf}$ denotes the trace-free part with respect to $[g]$. We define the obstruction tensor

\begin{equation}
O = c_n \text{tf}(x^{2-n}(\text{Ric} g_+ + ng_+)|_{TM}), \quad c_n = \frac{2^{n-2}(n/2-1)!^2}{n-2}.
\end{equation}

Then $O_{ij}$ has the properties:

1. $O$ is a natural tensor invariant of the metric $g = x^2 g_+ |_{TM}$; i.e., in local coordinates the components of $O$ are given by universal polynomials in
the components of $g$, $g^{-1}$ and the curvature tensor of $g$ and its covariant derivatives. The expression for $O_{ij}$ takes the form

$$O_{ij} = \Delta^{n/2-2} \left( P_{ij,k}^k - P_{k,ij}^k \right) + \text{lots}$$

$$= (3-n)^{-1} \Delta^{n/2-2} W_{ijkl,k} + \text{lots},$$

where $\Delta = \nabla^i \nabla_i$, $W$ denotes the Weyl tensor of $g$, and lots denotes quadratic and higher terms in curvature involving fewer derivatives.

(2) One has

$$O_{ii} = 0 \quad O_{ij,j} = 0.$$  

(3) $O_{ij}$ is conformally invariant of weight $2-n$; i.e. if $0 < \Omega \in C^\infty(M)$ and $\hat{g}_{ij} = \Omega^2 g_{ij}$, then $\hat{O}_{ij} = \Omega^{2-n} O_{ij}$.

(4) If $g_{ij}$ is conformal to an Einstein metric, then $O_{ij} = 0$.

Proof. There are discussions of the asymptotics of Poincaré metrics in the literature, but we provide a self-contained treatment.

We shall work with metrics in a normal form. Lemma 5.2 and the subsequent paragraph in [GL] imply that if one is given a conformally compact metric $g_+$ which is asymptotically Einstein in the sense that $\text{Ric}_{g_+} + n g_+ = O(x^{-1})$ and a representative metric $g \in [g]$, there is an identification of a neighborhood of $M$ in $X$ with $M \times [0, \varepsilon)$ such that $g_+$ takes the form

$$g_+ = x^{-2}(dx^2 + g_x)$$

for a 1-parameter family $g_x$ of metrics on $M$ with $g_0 = g$.

It is straightforward to calculate $E = \text{Ric}_{g_+} + n g_+$ for $g_+$ of the form (2.3). We use Greek indices to label objects on $X$, Latin indices for $M$, and 0 for $\partial_x$ so that in an identification $X \cong M \times [0, \varepsilon)$ as above, a Greek index $\alpha$ corresponds to a pair $(i,0)$. One obtains:

(2.4) $2x E_{ij} = -x g''_{ij} + x g^{kl} g'_{ik} g'_{jl} - \frac{x}{2} g^{kl} g'_{kl} g'_{ij} + (n-1) g'_{ij} + g^{kl} g'_{kl} g'_{ij} + 2x \text{Ric}(g_x)_{ij}$

(2.5) $E_{i0} = \frac{1}{2} g^{kl} (\nabla_l g'_{ik} - \nabla_i g'_{kl})$

(2.6) $E_{00} = -\frac{1}{2} g^{kl} g''_{kl} + \frac{1}{4} g^{kl} g^{pq} g'_{kp} g'_{lq} + \frac{1}{2} x^{-1} g^{kl} g'_{kl},$

where $'$ denotes $\partial_x$, we have suppressed the subscript on $g_x$, and $\nabla$ and $\text{Ric}$ denote the Levi-Civita connection and Ricci curvature of $g_x$ for fixed $x$.

One can determine the derivatives of $g_x$ inductively to solve the equation $E_{ij} = O(x^{n-2})$, beginning with the prescription $g_0 = g$. Differentiating (2.4) $s-1$ times
and setting $x = 0$ gives
\begin{equation}
\partial_x^{s-1}(2x E_{ij})|_{x=0} = (n-s)\partial_x^s g_{ij} + g^{kl} \partial_x^s g_{kl} g_{ij} + \text{(terms involving } \partial_x^k g_{ij} \text{ with } k < s) \tag{2.7}
\end{equation}

For $s \neq n$, $2n$, the operator $\eta_{ij} \rightarrow (n-s)\eta_{ij} + g^{kl} \eta_{kl} g_{ij}$ is invertible on symmetric 2-tensors at each point of $\mathcal{M}$. It follows inductively that one uniquely obtains a metric $g_+$ mod $O(x^{n-2})$ of the form \((2.3)\) by the requirement $E_{ij} = O(x^{n-2})$. Moreover, the derivatives of $g_+$ at $x = 0$ of order less than $n$ are all natural tensor invariants of the initial representative metric $g$.

The vanishing of the remaining components of $E$ to the correct order is deduced via the Bianchi identity. The Bianchi identity for Ricci curvature of $g_+$ states
\begin{equation}
 g_+^{\alpha\beta} \nabla_+^\gamma E_{\alpha\beta} = 2g_+^{\alpha\beta} \nabla_\alpha E_{\beta\gamma},
\end{equation}
where $\nabla_+$ denotes the Levi-Civita connection of $g_+$. Taking separately $\gamma = 0$ and $\gamma = i$ and writing this in terms of the connection $\mathcal{E}$ of $g_x$ gives the following two equations:
\begin{equation}
g^{jk} E'_{jk} = 2 \nabla^j E_{0j0} + (\partial_x + g^{jk} g'_{jk} - 2(n-1)x^{-1}) E_{00} \tag{2.8}
\end{equation}

\begin{equation}
\partial_i E_{00} + \nabla_i E_{0}^j - 2 \nabla^j E_{ij} = 2(\partial_x + \frac{1}{2} g^{jk} g'_{jk} - (n-1)x^{-1}) E_{i0}. \tag{2.9}
\end{equation}

We claim that $E_{00} = O(x^{n-2})$ and $E_{i0} = O(x^{n-1})$. These follow by induction on the statement that $E_{00} = O(x^{s-1})$ and $E_{i0} = O(x^s)$ for $0 \leq s \leq n-1$. The case $s = 0$ is immediate from \((2.7)\) and \((2.3)\). Suppose the statement is true for some $s, s \leq n-2$. Write $E_{00} = \lambda x^{s-1}$ and recall $E_{ij} = O(x^{n-2})$. In \((2.8)\), we have $g^{jk} E'_{jk} = O(x^{n-3}) = O(x^{s-1})$, $\nabla^j E_{0j} = O(x^s)$, and $g^{jk} g'_{jk} = O(1)$. Calculating \((2.8)\) mod $O(x^{s-1})$ thus gives $(s-2n+1)\lambda x^{s-2} = O(x^{s-1})$, which implies $\lambda = O(x)$ so $E_{00} = O(x^s)$ as desired. Now write $E_{i0} = \mu_i x^s$ and calculate \((2.9)\) mod $O(x^s)$. One obtains similarly $(s+1-n)\mu_i x^{s-1} = O(x^s)$. Since $s \leq n-2$ it follows that $\mu_i = O(x)$ so $E_{i0} = O(x^{s+1})$, completing the induction.

This proves the first sentence of the statement of Theorem \(2.1\): existence of a formal solution to order $n-2$. The second sentence, uniqueness of $g_+$ up to diffeomorphism, follows from the fact that any metric $g_+$ can be put into the form \((2.3)\) by a diffeomorphism together with the uniqueness of the determination of $g_+$ in the form \((2.3)\) as above. The definition \((2.1)\) of $\mathcal{O}$ depends on a choice of defining function $x$ to first order, equivalently on the choice of a conformal representative, but is otherwise diffeomorphism invariant. So in order to establish the independence of $\mathcal{O}$ on the freedom in $g_+$ at order $n-2$ and the naturality of $\mathcal{O}$, it suffices to consider $g_+$ of the form \((2.3)\). These conclusions now follow from \((2.7)\): taking $s = n$ shows that the trace-free part of $x^{2-n} E_{ij}|_{x=0}$ is given purely in terms of the previously determined terms.
The tensor $\mathcal{O}_{ij}$ is trace-free by definition. The fact that $\mathcal{O}_{ij,j} = 0$ can be established by consideration of (2.8) and (2.9) as follows. Consider the approximately Einstein metric $g_+$ mod $O(x^{n-2})$ constructed inductively above which satisfies $E_{ij} = O(x^{n-2})$, $E_{i0} = O(x^{n-1})$, $E_{00} = O(x^{n-2})$. Although vanishing of the trace-free part of $E_{ij}$ at order $n - 2$ is obstructed by $\mathcal{O}_{ij}$, one sees from (2.7) that one can solve for the trace to ensure that $g^{ij}E_{ij} = O(x^{n-1})$. This is sufficient to allow one to conclude exactly as above from (2.8) that $E_{00} = O(x^{n-1})$. Now consider (2.9). One finds this time that the right hand side is already $O$ for $g_{ij}$, that if $Ric(g) = 0$ there is no obstruction to existence of a smooth formal solution at order $n - 2$, so it must be that $\mathcal{O}_{ij} = 0$, carrying out the calculations gives the following explicit formulae. Define $\hat{\mathcal{O}}_{ij} = \Omega^2 g_{ij} - \Omega^{2-n} \mathcal{O}_{ij}$. The fact that the same tensor arises when calculated in the normal forms determined by different conformal representatives is implicit in the invariance of the definition under diffeomorphisms.

The conformal invariance of $\mathcal{O}_{ij}$ follows immediately from its definition: the rescaling $\hat{g}_{ij} = \Omega^2 g_{ij}$ corresponds to $\hat{x} = \Omega x + O(x^2)$, which by (2.4) gives $\hat{\mathcal{O}}_{ij} = \Omega^{2-n} \mathcal{O}_{ij}$. In particular there is no obstruction to existence of a smooth formal solution at order $n - 2$, and the Bianchi identity (2.2) once gives $\hat{\mathcal{O}}_{ij}x^{n-2}$ mod $O(x^{n-1})$ and calculating mod $O(x^{n-1})$ gives $\mathcal{O}_{ij,j} = 0$ as desired.

To finish the proof of Theorem 2.1, it remains to derive the principal part of $\mathcal{O}_{ij}$. This can be done by keeping track of the leading term in the inductive derivation above. As described above, the derivatives $\partial_x^s(g_{ij})|_{x=0}$ for $1 \leq s \leq n - 1$ are determined inductively by setting $E_{ij} = 0$ and differentiating in (2.4), and the obstruction $\mathcal{O}_{ij}$ arises when trying to solve for $\partial_x^s(g_{ij})|_{x=0}$. Parity considerations show that these derivatives vanish for $s$ odd. Differentiating (2.4) once gives $\hat{\mathcal{O}}_{ij}|_{x=0} = -2P_{ij}$. Differentiating further and using the first variation of Ricci curvature

$$\hat{Ric}_{ij} = \frac{1}{2} (\hat{g}_{ik,j} k^k + \hat{g}_{jk,i} k^k - \hat{g}_{ij,k} k^k - \hat{g}_{k,k} k_{ij})$$

and the Bianchi identity $P_{ik} k^k = P_{ik} k^k$, one determines inductively that

$$\partial_x^{2m} g_{ij}|_{x=0} = 2 \frac{3 \cdot 5 \cdot 7 \cdots (2m - 1)}{(n - 4)(n - 6) \cdots (n - 2m)} (\Delta^{m-2} P_{k} k^k - \Delta^{m-1} P_{ij}) + \text{lots}$$

for $2 \leq m < n/2$. Using $E_{ij} = c_n^{-1} x^{n-2} \mathcal{O}_{ij}$ mod $O(x^{n-1})$ in (2.4) and differentiating $n - 1$ times then gives the first line of (2.2). The second follows from the fact that $W_{ki,jl} k^l = (3 - n) (P_{ij} k^k P_{k} k^j - P_{k} k^i P_{k} k^j k^k)$. □

The proof of Theorem 2.1 gives an algorithm for the calculation of $\mathcal{O}_{ij}$. For $n = 4, 6$, carrying out the calculations gives the following explicit formulae. Define
the Cotton and Bach tensors by:
\[ C_{ijk} = P_{ij,k} - P_{ik,j} \]
\[ B_{ij} = P_{ij,k} - P_{ik,j} - P^{kl}W_{kijl}. \]

Then when \( n = 4 \) one has \( O_{ij} = B_{ij} \) and when \( n = 6 \) one has
\[ O_{ij} = B_{ij,k} - 2W_{kijl}B^{kl} - 4P_{k}B_{ij} + 8P^{kl}C_{(ij)k,t} - 4C_{i}^{kl}C_{jkt}C_{ijkl} \]
\[ + 2C_{i}^{kl}C_{jkl} + 4P_{k}C_{ijkl} - 4W_{kijl}P_{m}P^{ml}. \]

If the obstruction tensor is nonzero, there are no formal smooth solutions to \( \text{Ric}(g_{+}) = -ng_{+} \) beyond order \( n-2 \). However, it is always possible to find solutions to all orders by including log terms in the expansion of \( g_{+} \). For our purposes it will suffice to consider solutions to one higher order. The obstruction tensor \( O_{ij} \) can then be characterized as the coefficient of the first log term.

**Theorem 2.2.** In the setting of Theorem 2.1 there is a solution \( g_{+} \) to \( \text{Ric}(g_{+}) + ng_{+} = O(x^{-1}\log x) \) of the form \( g_{+} = x^{-2}(dx^{2} + g_{x}) \), where \( g_{x} = h_{x} + r_{x}x^{n} \log x \) and \( h_{x} \) and \( r_{x} \) are smooth in \( x \). The coefficient \( r_{x} \) is uniquely determined at \( x = 0 \) and is given by \( n_{c}r_{0} = 2O \).

**Proof.** Fix a metric \( g_{0}^{x} = x^{-2}(dx^{2} + g_{0}^{x}) \) with \( g_{0}^{x} \) smooth which solves \( E_{0}^{x} = O(x^{-2}) \) as in Theorem 2.1. Set \( g_{x} = g_{0}^{x} + r_{x}x^{n} \log x + sx^{n} \). Substituting into (2.4) gives
\[ 2xE_{ij} = 2xE_{ij}^{0} + g_{ij}^{kl}r_{kl}g_{ij}(x^{n-1} \log x + x^{n-1}) \]
\[ - nr_{ij}x^{n-1} + ng_{ij}^{kl}r_{kl}g_{ij}x^{n-1} \equiv O(x^{n} \log x). \]

It is required that this expression vanish \( \mod O(x^{n} \log x) \). The requirement that there be no \( x^{n-1} \log x \) term forces \( g_{ij}^{kl}r_{kl}(x^{n}) = 0 \). Since \( c_{n} \text{tf}(x^{2-n}E_{ij}^{0})(x^{n}) = O_{ij} \), we must have \( n_{c}r_{ij}(x^{n}) = 2O_{ij} \). The trace of \( s_{ij} \) can be chosen to guarantee \( g_{ij}^{kl}(x^{2-n}E_{ij}^{0})(x^{n}) = 0 \); the trace-free part of \( s_{ij} \) remains arbitrary. With these choices, if we set \( h_{x} = g_{0}^{x} + sx^{n} \equiv O(x^{n+1}) \) and \( r_{x} = r \equiv O(1) \), we obtain \( g_{x} \) in the form required in the statement of the theorem satisfying \( E_{ij}^{x} = O(x^{n-1} \log x) \).

In the proof of Theorem 2.1 it was shown that \( g_{0}^{x} \) satisfies \( E_{0}^{x} = O(x^{-1}) \), \( E_{0}^{0} = O(x^{-2}) \). It is evident from this and (2.5) that \( g_{+} \) satisfies \( E_{0}^{x} = O(x^{n-1} \log x) \), \( E_{0}^{0} = O(x^{n-2} \log x) \). Arguing as in the proof of Theorem 2.1 one finds that (2.8) implies that in fact one has \( E_{0}^{0} = O(x^{n-1} \log x) \), completing the proof. \( \square \)

**3. Proof of Theorem 1.1**

For any metric \( g \) on \( M \), we consider a metric \( g_{+} = x^{-2}(dx^{2} + g_{x}) \) on \( M \times (0, \epsilon) \) given by Theorem 2.2 which satisfies \( \text{Ric}(g_{+}) + ng_{+} = O(x^{n-1} \log x) \). Recall that \( g_{x} \) was determined only up to addition of a trace-free tensor in the coefficient of \( x^{n} \) and up to addition of terms of order greater than \( n \). For definiteness, we specify...
where the $g^{(2m)}$ for $m < n/2$ are those coefficients derived in Theorem 2.1, and we take $g^{(n)}$ to be the multiple of $g$ determined in the proof of Theorem 2.2. Then the metric $g_\pm$ is completely determined by $g$, and of course satisfies $\text{Ric}(g_\pm) + ng_\pm = O(x^{n-1}\log x)$.

The proof of Theorem 1.1 depends on the volume expansion of an asymptotically Einstein metric; see [G]. The volume form of $g_\pm$ is

$$dv_{g_\pm} = x^{-n-1} \left( \frac{\det g_x}{\det g} \right)^{1/2} dv_g dx.$$  

From (3.1) and the fact that $g^{ij} \mathcal{O}_{ij} = 0$, it follows that

$$\left( \frac{\det g_x}{\det g} \right)^{1/2} = 1 + v^{(2)} x^2 + (\text{even powers}) + v^{(n)} x^n + \cdots,$$

where the $v^{(2j)}$ are locally determined invariant scalars given in terms of $g$ and its curvature, and $\cdots$ denotes terms vanishing to higher order. Integrating, it follows that for fixed $\epsilon_0$ we have the asymptotic expansion as $\epsilon \to 0$

$$\text{Vol}_{g_{\epsilon}}(\{\epsilon < x < \epsilon_0\}) = c_0 \epsilon^{-n} + c_2 \epsilon^{-n+2} + (\text{even powers}) + c_{n-2} \epsilon^{-2} + L \log \frac{1}{\epsilon} + O(1),$$

where $c_{2j} = (n-2j)^{-1} \int_M v^{(2j)} dv_g$ and $L = \int_M v^{(n)} dv_g$. The log term coefficient $L$ is invariant under conformal rescalings of $g$; see [G] for a proof.

The $Q$-curvature of a metric $g$ was originally defined by Branson [B] in terms of the zero-th order term of the conformally invariant $n$-th power of the Laplacian $P_n$ of [GJMS] by dimensional continuation. It is a scalar quantity with a particularly simple transformation law under conformal rescalings: if $\hat{g} = e^{2\Upsilon} g$, then $e^{n\Upsilon} \hat{Q} = Q + P_n \Upsilon$. Although $Q$ is not pointwise conformally invariant, it follows from the facts that $P_n$ is self-adjoint and annihilates constants that the integral $\int Q dv$ over a compact manifold is a conformal invariant. Characterizations of the $Q$-curvature in terms of Poincaré metrics were given in [GZ] and [FG2], and in terms of the ambient metric in [FH]. We refer to [B] and these references for background about $Q$-curvature. The main fact we will need here is the result of [GZ] (or see [FG2] for a simpler proof) that

$$(3.3) \int Q dv = k_n L, \quad k_n = (-1)^{n/2} n(n-2) c_n.$$

According to (3.3), in order to calculate the variation of $\int Q dv$ it suffices to compute $\dot{L}$. For this, we use a simplification of the method of Anderson [An] to rewrite the variation of volume as a boundary integral for variations through
Einstein metrics with fixed scalar curvature. In our case we need to estimate the errors resulting from the fact that our metrics are only asymptotically Einstein.

**Lemma 3.1.** Let $g^t$ be a 1-parameter family of metrics on a compact manifold $M$ and let $g^t_x = x^{-2}(dx^2 + g^t_x)$ be the corresponding asymptotically Einstein metrics on $M \times (0, \epsilon_0)$, where for each $t$, $g^t_x$ is constructed from $g^t$ as in (3.1). Set $X_\epsilon = \{ \epsilon < x < \epsilon_0 \}$. Then as $\epsilon \to 0$ we have

\[
(3.4) \quad \text{Vol}_{g^t_x}(X_\epsilon) = \frac{e^{1-n}}{2n} \int_{x=\epsilon} \left( -\frac{1}{2} g^t_{ij} g'^{kl} g^t_{ijkl} + x^{-1} g^t_{ij} \dot{g}^t_{ij} - (g^t_{ij} \dot{g}^t_{ij})' \right) dv_{g^t} + O(1).
\]

On the right hand side, $'$ denotes $\partial_x$ and $\cdot$ denotes $\partial_{t=0}$ as usual, and all $g = g^t_x$ are evaluated at $x = \epsilon$, $t = 0$ (after differentiation).

**Proof.** Since the metrics $g^t_x$ are asymptotically Einstein, we have uniformly in $t$ (and supressing the $t$-dependence of $g^t_x$):

\[
\text{Ric}_{g^t} = -n g^t + O(x^{n-1} \log x),
\]

\[
R_{g^t} = -n(n+1) + O(x^{n+1} \log x),
\]

\[
\dot{R}_{g^t} = O(x^{n+1} \log x).
\]

Therefore $\int_{X_\epsilon} \dot{R}_{g^t} dv_{g^t} = O(1)$ as $\epsilon \to 0$.

On the other hand, the usual formula for the first variation of scalar curvature gives

\[
\dot{R}_{g^t} = \dot{g}_{+\alpha\beta}^\alpha - g_{+\alpha}^\alpha_{+\beta} - \text{Ric}_{g^t}^\alpha_{+\alpha} + \dot{g}_{+\alpha}^\alpha - \dot{g}_{+\alpha}^\alpha_{+\beta} + n g_{+\alpha}^\alpha g_{+\alpha}^\beta + O(x^{n+1} \log x),
\]

where the covariant derivatives are with respect to the Levi-Civita connection of $g_+$ and indices are raised and lowered using $g_+$. Integrating gives

\[
\int_{X_\epsilon} (\dot{g}_{+\alpha\beta}^\alpha - g_{+\alpha}^\alpha_{+\beta}) dv_{g^t} + 2n \int_{X_\epsilon} \dot{d}v_{g^t} = O(1),
\]

so

\[
-2n \text{Vol}_{g^t}(X_\epsilon) = \int_{X_\epsilon} (\dot{g}_{+\alpha\beta}^\alpha - g_{+\alpha}^\alpha_{+\beta}) dv_{g^t} + O(1)
\]

\[
= \int_{\partial X_\epsilon} (\dot{g}_{+\alpha\beta}^\alpha - g_{+\alpha}^\alpha_{+\beta}) \nu^\beta_+ d\sigma_+ + O(1),
\]

where $\nu^\beta_+$ denotes the outward unit normal and $d\sigma_+$ the induced volume density. The integral over $x = \epsilon_0$ is $O(1)$, and on $x = \epsilon$ we have $\nu^\beta_+ = -\epsilon \delta^\beta_0$, $d\sigma_+ = \epsilon^n dv_{g_+}$. Also, $\dot{g}_{+\alpha\beta}$ vanishes if either $\alpha = 0$ or $\beta = 0$, and $\dot{g}_{+ij} = x^{-2} \ddot{g}_{ij}$. An easy computation relating the connections of $g_+$ and $g_x$ shows that

\[
\dot{g}_{+\alpha\beta}^\alpha - g_{+\alpha}^\alpha_{+\beta} = -\frac{1}{2} g^{ij} g^{kl} \dot{g}_{ijkl} + x^{-1} g^{ij} \ddot{g}_{ij} - (g^{ij} \dot{g}_{ij})',
\]
which gives (3.3).

Now \( \dot{L} \) occurs as the coefficient of \( \log \frac{1}{\epsilon} \) in the asymptotic expansion of the left hand side of (3.4). So we need to evaluate the \( \epsilon^{-1} \log \frac{1}{\epsilon} \) coefficient in the expansion of the integral on the right hand side. From \( dv_g = (\det g_x / \det g)^{1/2} dv_y \) and (3.2), it follows that the expansion of the volume form has no \( x^n \log x \) term, so does not contribute to this coefficient. Differentiating the volume form in \( t \), one concludes that also the expansion of \( g^{ij} \dot{g}_{ij} \) has no \( x^n \log x \) term. Therefore the second and third terms in the integrand also do not contribute to the \( \epsilon^{-1} \log \frac{1}{\epsilon} \) coefficient. The only contribution from the first term in the integrand comes from the log term in \( g^{ij}_t \). This gives

\[
2nc_n \dot{L} = \int_M \partial_{ij} g^{ij} dv_g,
\]

which combined with (3.3) gives Theorem 1.1.

### 4. Proof of Theorem 1.2

We consider natural tensor invariants of oriented \( n \)-dimensional Riemannian manifolds \((M, g)\) with values in a subbundle \( V \subset \otimes^k T^* M \) induced by a representation of \( SO(n) \) on an invariant subspace \( V \subset \otimes^k (\mathbb{R}^n)^* \); see [E] for a discussion of natural tensors. The components of a natural tensor are expressible as linear combinations of partial contractions of the metric, the volume form, the Riemannian curvature tensor, and its covariant derivatives. We say that a natural tensor is irreducible if it is nonzero and if \( V \) is irreducible as an \( SO(n) \)-module. We say that two natural tensors with values in subbundles \( V_1, V_2 \) are equivalent if they correspond under an isomorphism \( V_1 \cong V_2 \) induced by an \( SO(n) \)-module isomorphism \( V_1 \cong V_2 \) of the underlying subspaces.

The Ricci identity for commuting covariant derivatives does not preserve homogeneity degree, so the degree of a natural tensor as a polynomial in curvature and its derivatives is not well-defined. However, the space of natural tensors is filtered by degree and it does make sense to say that a natural tensor is of degree at least \( d \) in curvature for \( d \in \mathbb{N} \). A natural tensor \( T(g) \) is said to be conformally invariant of weight \( w \) if \( T(\Omega^2 g) = \Omega^w T(g) \) for \( 0 < \Omega \in C^\infty(M) \). The naturality of \( T \) implies that if \( \phi \) is any local diffeomorphism, then \( T(\phi^* g) = \phi^* T(g) \).

The idea of the proof of Theorem 1.2 is simple: linearizing a natural tensor at the usual metric on the sphere \( S^n \) gives a linear differential operator on infinitesimal metrics, and the conformal invariance implies that this differential operator satisfies an invariance property under conformal motions. A known theorem classifies such invariant differential operators and this classification in the linear case implies the classification up to quadratic and higher terms for natural tensors.

In more detail, let \( T(g) \) be an irreducible natural tensor which is conformally invariant of weight \( w \). Denote by \( g_0 \) the usual metric on \( S^n \). Define a linear differential operator \( T \) from the bundle \( S_0^2 T^* S^n \) of trace-free symmetric 2-tensors
on $\mathbb{S}^n$ to the bundle $\mathcal{V}$ on $\mathbb{S}^n$ by

$$T(h) = \frac{d}{dt} \mathcal{T}(g_0 + th)|_{t=0}$$

for $h$ a section of $S_0^2T^*\mathbb{S}^n$. We claim that if $\phi : \mathbb{S}^n \rightarrow \mathbb{S}^n$ is a conformal motion of $\mathbb{S}^n$ satisfying $\phi^*g_0 = \Omega^2g_0$ for $0 < \Omega \in C^\infty(\mathbb{S}^n)$, then

$$\Omega^wT(\phi^*h) = \phi^*(T(\Omega^2 \circ \phi^{-1} h)).$$

In fact,

$$\Omega^wT(g_0 + t\phi^*h) = \Omega^wT(\Omega^{-2}\phi^*g_0 + t\phi^*h) = \Omega^wT(\Omega^{-2}\phi^*(g_0 + t\Omega^2 \circ \phi^{-1} h))$$

$$= T(\phi^*(g_0 + t\Omega^2 \circ \phi^{-1} h)) = \phi^*T(g_0 + t\Omega^2 \circ \phi^{-1} h),$$

from which the claim follows by differentiation.

Let $G = O_e(n+1,1)$ denote the identity component of the conformal group and $P \subset G$ the isotropy group of a point on $\mathbb{S}^n$, so that $\mathbb{S}^n = G/P$. Then (4.1) states exactly that $T$ is a $G$-equivariant map between sections of the homogeneous bundles $S_0^2T^*\mathbb{S}^n(2)$ and $\mathcal{V}(w)$ on $G/P$, where the number in parentheses indicates the conformal weight of the homogeneous bundle. Such invariant differential operators between any irreducible homogeneous bundles on $\mathbb{S}^n$ have been completely classified; see (1.4) of [BC]. The classification in [BC] is formulated in terms of the homomorphisms of the generalized Verma modules dual to the homomorphisms of the modules of jets of sections of the homogeneous bundles induced by the differential operators. See [BE], [ES] and references cited there for elaboration and interpretation of this classification in the context of conformal geometry.

For our purposes it is sufficient to know all the invariant operators with domain $S_0^2T^*\mathbb{S}^n(2)$ and range any irreducible bundle. The bundle $S_0^2T^*\mathbb{S}^n(2)$ has regular integral infinitesimal character so fits into a generalized Bernstein-Gelfand-Gelfand complex of invariant operators (the so called deformation complex – see [GG] for a direct construction of this complex on a general conformally flat manifold). In odd dimensions, up to scale and equivalence, there is precisely one invariant operator acting on $S_0^2T^*\mathbb{S}^n(2)$: the linearized Cotton tensor in dimension 3 and the linearized Weyl tensor in higher dimensions. In even dimensions there are further operators. For $n \geq 6$ there is one more operator acting on $S_0^2T^*\mathbb{S}^n(2)$, which must be the linearized obstruction tensor since this is a nonzero operator acting between the appropriate bundles. The case $n = 4$ is exceptional because $S_0^2T^*\mathbb{S}^n(2)$ occurs at the edge of the middle diamond of the Hasse diagram, and there are three invariant operators acting on $S_0^2T^*\mathbb{S}^n(2)$, which can be identified with the linearized self-dual and anti-self-dual Weyl tensors and the linearized Bach tensor.

Theorem 1.2 is an immediate consequence. The linearization of a conformally invariant irreducible natural tensor is equivalent to a multiple of one of the invariant operators given by the Boe-Collingwood classification. By inspection, each such operator is the linearization of one of the conformally invariant natural tensors.
listed in the statement of the Theorem. But if two conformally invariant natural tensors have the same linearization on the sphere, their difference must be of degree at least 2 in curvature.

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