MOST GRAPH BRAID GROUPS ARE NOT CLASSICAL BRAID GROUPS

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Abstract. We prove that the first Betti number of most graph braid groups is strictly greater than 1 and thus not isomorphic to any classical braid group. Additionally, we explicitly construct an embedding of a right-angled Artin group into a classical pure braid group. We then use this to construct an embedding of a graph braid group into a classical pure braid group. Finally we describe all homomorphisms from the classical braid group on at least 4 strands into right-angled Artin groups, and that the image is isomorphic to $\mathbb{Z}$.

Introduction

Let $n$ be a natural number and $X$ be a (connected) topological space, then the labeled $n$-point configuration space is the space of all ordered subsets consisting of $n$ distinct points of $X$. We can express such a configuration as $(x_1, x_2, \ldots, x_n) \in X^n$, where $X^n$ is the $n$-fold Cartesian product of $X$ with itself. We denote the labeled configuration space by $C_n(X)$, and we can construct this space by first defining the diagonal $\Delta_n(X)$ which is the set consisting of all the configurations where distinct points coincide. Using set notation we have:

$$\Delta_n(X) = \{(x_1, \ldots, x_n) \in X^n \mid x_i = x_j \text{ for some } i \neq j\},$$

and so we have $C_n(X) := X^n \setminus \Delta_n(X)$.

Now let $D^2$ be the 2-dimensional topological disk, then the fundamental group of $C^n(D^2)$ is the classical pure braid group, or notationally $PB_n := \pi_1(C^n(D^2))$. If $G$ is a graph, then $\pi_1(C^n(G))$ is the graph pure braid group and denoted by $PB_n(G)$. Similarly the unlabeled $n$-point configuration space is the space of all unordered subsets of $n$ distinct points of $X$, denoted by $UC^n(X)$. Additionally $UC^n(X)$ can be constructed by quotienting $C^n(X)$ by the action of the symmetric group which acts by permuting the indices. The classical braid group, denoted here by $B_n$, can be expressed as $\pi_1(UC^n(D^2))$. Also a graph braid group is defined as $\pi_1(UC^n(G))$, and it is denoted by $B_n(G)$.

Classical braid groups have been extensively studied and are well understood, and graph braid groups are well studied objects, with both having a variety of different applications. We will defer the discussion on known results for these classes of groups to examine them in more depth. However, because of their similar constructions, there is a natural question about how these groups are related.

The primary goal of this paper is to classify exactly which graph braid groups are isomorphic to classical braid groups. To prove this, we will be combining results of the presentations of tree braid groups in [7] with the construction of cycles from [14] to show that classical braid groups and graph braid groups usually have different first Betti numbers. Therefore they must have different first Homology groups and abelianizations. In this paper, we use $\cong$ to denote an isomorphism, $\approx$ to denote a homeomorphism, and $I = [0, 1] \subseteq \mathbb{R}$, to be the unit interval. Thus we will prove the following theorem:
Theorem 3.1. Let $G$ be a sufficiently subdivided graph. Then $B_n(G) \cong B_n'$ if and only if one of the following conditions hold:

1. $n' = 1$ and $n \geq 1$ and $G \approx I$,
2. $n' = n = 1$ and $G$ is a tree,
3. $n' = 2$ and $n = 1$ and $G$ contains exactly one properly embedded cycle,
4. $n' = n = 2$ and $G \approx K_{3,1},$
5. $n' = 2$ and $n \geq 1$ and $G \approx S^1$,

where a sufficiently subdivided graph is a graph which satisfies the conditions of Theorem 1.2.

A secondary goal of this paper is to derive a method to embed any graph braid group into some classical braid group. We do this by explicitly constructing an embedding of a right-angled Artin group into the classical braid group, then combining this with the embedding from graph braid groups into right-angled Artin groups proved by Crisp and Wiest. Let $\hat{B}_n$ denote the subgroup generated by the squares of the generators of the standard Artin presentation of $B_n$. Hence we prove the following:

Theorem 4.4. For a finite, simple graph $\Gamma$, the right-angled Artin group $A_\Gamma$ embeds in $\hat{B}_n$ for $n \geq 2|\Gamma| + 3 \left(\left(\begin{array}{c} |\Gamma| \\ 2 \end{array}\right) - E\right)$, where $E$ is the number of edges of $\Gamma$.

Immediately from this we obtain:

Theorem 4.6. Let $G$ be a finite graph, $\Gamma$ be the opposite (dual) line graph of $G$, and $n \in \mathbb{N}$, then $B_n(G)$ embeds in $\hat{B}_n$ where $l \geq 2|\Gamma| + 3 \left(\left(\begin{array}{c} |\Gamma| \\ 2 \end{array}\right) - E\right)$ and $E$ is the number of edges of $\Gamma$.

The other secondary goal of this paper was originally to embed $B_n$ for $n > 2$ into graph braid groups. However it turns out that this is impossible since graph braid groups are bi-orderable (since they embed into a right-angled Artin group) whereas all non-right-angled Artin groups are not. Thus every non-right-angled Artin group does not embed into a graph braid group. From this, we can easily show the following as a corollary of the Crisp and Wiest embedding.

Corollary 4.7. Let $n > 2$, then there does not exist an embedding of $B_n$ into $B_n(G)$ for any $n'$ and a graph $G$.

Yet for $n > 4$, we will explicitly describing all possible homomorphisms from $B_n$ to any given right-angled Artin group. Thus we have the following statement:

Theorem 4.13. Let $\Gamma$ be an arbitrary simple graph, $n > 4$, and $\Lambda: B_n \rightarrow A_\Gamma$ be a homomorphism. Then $\Lambda(\sigma_i) = \Lambda(\sigma_j)$ for all $i$ and $j$, where $\sigma_i$ is a generator of the Artin presentation (see (1) below).

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1. Background

1.1. Classical Braid Groups and Artin Groups. The classical braid group (also called the standard braid group) is usually described by isotopy classes rel endpoints of $n$-tuples of pair-wise disjoint paths (i.e. braids) in $D^2 \times I$. This forms a group, denoted by $B_n$, where the group operation is defined to be concatenation. Similarly we can describe the classical braid group by taking cross-sections of $D^2 \times I$, the standard braid group.
where the points representing the braids can be thought about moving in time as we go from 0 to 1. An individual point/path will be referred to as a strand. The strands at a fixed time form a configuration of $D^2$, thus the fundamental group of the configuration space is an equivalent definition. Furthermore, the classical braid group can be described by the convex rotations of $n$ points of $D^2$ placed in a convex configuration; this is known as the Birman-Ko-Lee presentation. We refer the reader to \[2\] and \[16\] for more on this presentation.

The relations given by the isotopy classes gives the most common presentation for the classical braid group. This presentation is called the (standard) Artin presentation since it was the original method which Artin described and used.

\[
\begin{align*}
B_n &= \left\langle \sigma_1, \sigma_2, \ldots, \sigma_{n-1} \right| \\
& \quad \sigma_{i+1}\sigma_i = \sigma_i \sigma_{i+1} \quad \text{for } 1 \leq i < n-1 \\
& \quad \sigma_i \sigma_j = \sigma_j \sigma_i \quad \text{if } |i-j| \geq 2 \rightangle
\end{align*}
\]

The braid group is an special case of Artin groups. An Artin group $A$ which has a presentation of the form

\[
A = \left\langle a_1, a_2, \ldots, a_n \right| a_i a_j a_i \ldots a_j a_i \ldots = a_j a_i a_j \ldots \text{for all } i \neq j \rightangle
\]

where $m_{ij} = m_{ji}$ is an integer which is at least 2 or $m_{ij} = \infty$ which denotes no relation between $a_i$ and $a_j$.

A special class of Artin groups are right-angled Artin groups. A right-angled Artin group is a finitely generated group which has a presentation where all relations are commutators between generators. Right-angled Artin groups are also known as graph groups since they can be represented by a graph which has a vertex for each generator and an edge with respect to each relation. Thus for a given finite, simple graph $\Gamma$, we denote the corresponding right-angled Artin group by $A_\Gamma$, and Droms [6] demonstrated that this corresponding group is unique up to isomorphism.

We refer the reader to [3] for more detailed information on right-angled Artin groups and their applications.

The following result, proved by Crisp and Paris, describes an important right-angled Artin subgroup of an Artin group.

**Theorem 1.1.** [4] Let $A$ be an arbitrary Artin group, and let $e_s \geq 2$ be an integer for every generator $\sigma_s \in A$. Then the set of elements $T = \{T_s = \sigma_s^{e_s} \mid \sigma_s \in A\}$ generates a subgroup of $A$ with the presentation

\[
\langle T \mid T_s T_t = T_t T_s \text{ if } m_{st} = 2 \rangle
\]

where $m_{st}$ defined the relation between $\sigma_s$ and $\sigma_t$ in $A$.

Note that this proves the Tits conjecture which is the special case when $e_s = 2$ for all $s$. Thus the subgroup $\hat{A} := \langle a_1^2, \ldots, a_n^2 \rangle$ is a right-angled Artin group, and we will refer to $\hat{A}$ as the canonical right-angled Artin group (of $A$).

A group $G$ is said to be bi-orderable if there exists a linear (total) ordering on the group which is invariant under left and right multiplication. In other words, for any $x, y \in G$ such that $x < y$, then $a \circ x < a \circ y$ for all $a, b \in G$. An important property of bi-orderable groups that we will use is that all subgroups are also bi-orderable. Paris showed in [17] that an Artin group is right-angled if and only if it is bi-orderable.

### 1.2. Classical Pure Braid Group

A classical pure braid group is a normal subgroup of a standard braid group in which the braids return to their initial positions. Similarly it can be described as the kernel of the map from the classical braid group
into the symmetric group. Thus the quotient map from $C^n(D^2)$ to $UC^n(D^2)$ induces the short exact sequence

$$1 \to \mathcal{PB}_n \to \mathcal{B}_n \to \Sigma_n \to 1.$$ 

Now we note that $\sigma_j$ corresponds to exchanging the position of the $j$-th strand with the $(j+1)$-th strand, thus $\sigma_j^2$ corresponds to all of the strand’s final positions remaining fixed. Hence $\sigma_j^2 \in \mathcal{PB}_n$, and therefore we have $\mathcal{B}_n \subseteq \mathcal{PB}_n$. The classical pure braid group additionally fits into the following short exact sequence which splits:

$$1 \to F_{n-1} \to \mathcal{PB}_n \to \mathcal{PB}_{n-1} \to 1.$$ 

Thus by iterating this, the classical pure braid group can be realized a semi-direct product of free groups. Since the free group is bi-orderable, it follows that the classical pure braid group is bi-orderable as well.

1.3. Graph Braid Groups. A graph braid group is similar to a classical braid group except instead of using $D^2$, it is defined on a graph $G$ (viewed as a 1-dimensional CW-complex). That is to say, a braid here is an isotopy class of endpoints of $n$-tuples of pairwise disjoint paths living in $G \times I$. Similarly we can look at the fundamental group of the configuration space as an equivalent definition. Unlike classical braid groups, graph braid groups in general are not known to have the nice, compact presentations, nor do they have as simple an intuition-based construction of relations. Additionally the configuration space is usually a difficult space to do calculations with since it is not compact. As a result, a simplified configuration space is constructed called the discretized (configuration) space, denoted $\mathcal{D}^n(G)$. It can be formed by taking $G^n$ as a cell complex and removing all (open) cells whose closure intersects the diagonal $\Delta_n(G)$. Equivalently we can define the set $\tilde{\Delta}_n(G)$ which consists of all configurations in which at least two strands are less than a full edge apart, and then $\mathcal{D}^n(G) = G^n \setminus \tilde{\Delta}_n(G)$. Finally we can construct $\mathcal{D}^n(G)$ by first taking the 0-skeleton which are all configurations with the strands on the vertices. Then we inductively build up the $i$-skeleton by connecting configurations in which $i$ strands are moving between vertices and the closure of the edges they are moving along are disjoint. See Figure 1 for a demonstration of these equivalent representations.

Now it is not always the case that this discretization has its fundamental group isomorphic to the original graph braid group, however Prue and Scrimshaw were able to precisely characterize the conditions upon which the discretization preserves the graph braid group. This was an improvement on the bounds originally proved by Abrams in his Ph.D. thesis[1] and has applications in robotics and motion planning[10]. However we can use these bounds to determine the smallest (in terms of the number of cells) discretized space we require to guarantee that we have the correct fundamental group. An essential vertex is a vertex of a graph whose degree is not equal to 2.

**Theorem 1.2 (Stable Equivalence).** Let $n > 1$ be an integer. For a connected graph $G$ with $|G| > 1$, the space $C^n(G)$ strongly deformation retracts onto $\mathcal{D}^n(G)$ if and only if

(A) each path connecting distinct essential vertices in $G$ has length at least $n - 1$, and

(B) each non-contractible path in $G$ connecting a vertex to itself has length at least $n + 1$.

This naturally extends to the existence of a deformation retraction from $UC^n(G)$ to $UD^n(G)$.
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Figure 1. Demonstrating the different representations for discretized spaces. On the left is $U^D(P_4)$, where $P_4$ is a path with 4 vertices, and the bold/shaded cells are the surviving cells which removing $\tilde{\Delta}_n(P_4)$. On the right is $U^D(H)$ where $H$ is the smallest graph homeomorphic to the letter $H$; the bold edges of the graph are the ones which contain a strand. We omit the 1-cells to streamline the figure.

Now if a graph $G$ satisfies both conditions (A) and (B), then the graph is said to be sufficiently subdivided. So let $G$ be a sufficiently subdivided graph, then many properties about $D^n(G)$ are known. Abrams demonstrated that the discretized space has a cubical CW cell structure and is locally CAT(0)[1]. Thus $\pi_1(D^n(G))$ has solvable word and conjugacy problems. Also, Ghrist showed that $D^n(G)$ is an Eilenberg-Maclane space of type $K(\pi_1(D^n(G)), 1)$[10]. Furthermore, Crisp and Wiest proved that $B_n(G)$ embeds in a right-angled Artin group so it is linear, bi-orderable, and residually finite[5]. Conversely, Sabalka constructs an embedding of a right-angled Artin group into some $B_n(G)$[19].

An important subgraph is a maximal tree, a tree which contains each vertex of some graph $\Gamma$. The opposite graph (also known as the dual graph) is the graph $\Gamma^{op}$ which has the same vertex set as $\Gamma$ and an edge when the corresponding vertices are not adjacent in $\Gamma$. The line-graph of $\Gamma$, which will be denoted by $L(\Gamma)$, is a graph whose vertices $e_i$ correspond to edges in $\Gamma$ and has an edge $\{e_i, e_j\}$ if the edges $e_i$ and $e_j$ are adjacent in $\Gamma$. (Thus $L(\Gamma)^{op}$ has an edge when $e_i$ and $e_j$ are disjoint.) An $n$-partite graph is a graph which can be partitioned into $n$ sets of vertices $U_j$ such that every vertex in $U_j$ are not connected but connected to all other vertices in the graph. Recall that $K_n$ denotes the complete graph on $n$ vertices and $K_{m,n}$ is the complete bipartite graph on $m$ and $n$ vertices.

2. Framework

In this section we will give the framework necessary to prove our main theorems. We begin by giving a short exposition of Forman’s discrete Morse theory, allowing us to describe the tree braid groups. Next we show which graph braid groups are isomorphic to the trivial group and $\mathbb{Z}$, and then we prove some basic facts about the classical braid group. Then we describe a portion of a method to algorithmically construct a graph braid group which was developed by Kurlin which we will use to add in cycles to the tree braid groups. Afterwards we will give a short proof for
some conditions which imply an isomorphism between graph braid groups and the classical braid group.

2.1. Forman’s Discrete Morse Theory and the Tree Lemma. We will now proceed to give a very simple exposition of Forman’s discrete Morse theory [2] to state the theorems that we will use. The results we derive from here form the backbone of characterizing the first homology of tree braid groups. Note, while this covers the basics of Forman’s discrete Morse theory, the author does not recommend anyone to attempt it from this paper alone. There is still more framework than what is presented here to establish before one can effectively use discrete Morse theory to study graph braid groups. What is presented here is only an introduction into the subject and to define the terminology used in the theorems cited. For a more thorough treatment, see [2] [13] [8].

We begin by first making a few definitions used in discrete Morse theory. Let \( X \) denote a finite CW cell-complex and \( \sigma \) and \( \tau \) be two open cells in \( X \). A partial function is a function \( f : A \rightarrow B \) which is defined on a subset of \( A \). We will write \( \sigma \leq \tau \) if \( \sigma \subseteq \tau \) where \( \bar{\tau} \) denotes the closure of \( \tau \) and \( \subseteq \) denotes a subcomplex. Similarly \( \sigma < \tau \) if \( \sigma \leq \tau \) and \( \sigma \neq \tau \). Now suppose \( \sigma \) is a face of \( \tau \) and \( \dim(\sigma) = k \). Let \( B \) be a closed \( k \)-ball, and let \( h : B \rightarrow X \) be a characteristic map for \( \tau \). The cell \( \sigma \) is a regular face of \( \tau \) if \( h : \overline{h^{-1}(\sigma)} \rightarrow \sigma \) is a homeomorphism, and \( \overline{h^{-1}(\sigma)} \) is a closed \( k \)-ball.

Define a discrete vector field \( W \) on \( X \) which consists of a sequence of partial functions \( W_i : K_i \rightarrow K_{i+1} \) which satisfy the following conditions:

(i) Each \( W_i \) is injective.
(ii) If \( W_i(\sigma) = \tau \), then \( \sigma \) is a regular face of \( \tau \).
(iii) \( \text{Im}(W_i) \cap \text{Dom}(W_{i+1}) = \emptyset \)

Now let \( W \) be a discrete vector field on \( X \). A \( W \)-path of dimension \( p \) is a sequence of \( p \)-cells \( \sigma_0, \sigma_1, \ldots, \sigma_r \) such that if \( W(\sigma_i) \) is undefined, then \( \sigma_{i+1} = \sigma_i \); otherwise \( \sigma_{i+1} \neq \sigma_i \) and \( \sigma_{i+1} < W(\sigma_i) \). The \( W \)-path is closed if \( \sigma_0 = \sigma_r \) and non-stationary if \( \sigma_1 \neq \sigma_0 \). A discrete vector field \( W \) is a Morse matching (also known as a discrete gradient vector field) if there does not exist any closed non-stationary paths.

Next construct a maximal tree \( T \) of \( G \) and choose a vertex \( * \in T^0 \) (the 0-skeleton of \( T \)) such that \( \deg * = 1 \) in \( T \). Embed \( T \) in the plane then walk around \( T \) along the left side of the edge, consecutively numbering the vertices as they are first encountered starting at 0 for \(*\). By walking along the left side, imagine the edges of \( T \) as a wall, then place your right hand on the wall (thus you are located on the left side of the wall with respect to your direction of motion) and continue walking until you return to your starting point. Since \( T \) is a maximal tree of \( G \), every vertex in \( G \) will be numbered. Define an ordering on the vertices as a comparison between the labeling of the vertices, for example \(*\) is the smallest vertex since \( 0 < k \).

For a given edge \( e \) of \( G \), let \( i(e) \) and \( t(e) \) denote the endpoints of \( e \). Orient each edge to go from \( i(e) \) to \( t(e) \) such that \( i(e) > t(e) \). For a vertex \( v \), let \( e(v) \) denote the unique edge in \( T \) such that \( i(e(v)) = v \).

Let \( e = \{c_1, c_2, \ldots, c_{n-1}, v\} \) be a cell of \( UD^0(G) \) containing the vertex \( v \). If \( e(v) \cap c_i = \emptyset \) for \( i = 1, 2, \ldots, n-1 \), then define the cell \( \{c_1, \ldots, c_{n-1}, e(v)\} \in UD^\eta(G) \) to be the elementary reduction of \( e \) from \( v \). If \( v \) is the smallest vertex in terms of the ordering on vertices such that there exists an elementary reduction form \( e \) to \( v \), then the reduction is a principal reduction.

Now define a function \( W \) on \( UD^\eta(G) \) by defining \( W_0(e^0) \) is the principal reduction of a 0-cell \( e^0 \) if it exists. Then proceed inductively to define \( W_i(e^i) \) for \( i > 0 \) and some \( i \)-cell \( e^i \) to be the principal reduction of \( e^i \) if it exists and \( e^i \notin \text{Im} W_{i-1} \),...
Most graph braid groups are not classical braid groups.

Otherwise $W_i$ is undefined. Farley and Sabalka proved that $W$ is a Morse matching on $UD^n(G)$.

**Theorem 2.1.** Let $G$ be a graph with a maximal tree $T$. If $T \neq G$, then assume the degree of each vertex of every deleted edge have degree 1 in $T$ (and that $T$ be sufficiently subdivided). Fix the Morse matching $W$ as previously stated. Let $D$ be the set of deleted edges in $G$. Then the Morse presentation $P_W$ has

\[
|D| + \sum_{v \in T} \sum_{\deg(v) > 2} \left[ \left( \frac{n + \deg(v) - 2}{n - 1} \right) - \left( \frac{n + \deg(v) - i - 1}{n - 1} \right) \right]
\]

generators.

Now we can compute the number of generators in the Morse theory presentation. In this presentation, if $G$ is a tree, then Theorem 5.3 in [7] states that all relations are commutators (although not necessarily of generators). Since the statement of this theorem involves specific notation, we will not state the exact wording of the theorem. Instead we will state the relations as follows:

\[
[\alpha, w\beta w^{-1}]
\]
\[
[x^{-1}\beta x', \alpha]
\]

where $w$ and $x$ are words, $x'$ is a word “similar” to $x$ (for our purposes this isn’t significant), and $\alpha$ and $\beta$ are generators. Since all exponent sums are 0, thus by Tietze’s theorem on finite presentations[15], no generator can be deleted. Thus Equation (2) gives the precise number of generators of the group.

### 2.2. Basic Lemmas

Here we will begin by completely characterizing the graph braid groups which are isomorphic to $\mathbb{1}$ (the trivial group) and $\mathbb{Z}$ by using a combination of the combinatorics of the graphs and the Morse presentation. A distinct path is a path in $G$ which does not contain an essential vertex in its interior and satisfies either (A) or (B) of Theorem 1.2.

**Lemma 2.2.** Suppose $n \geq 2$, that $G$ is a sufficiently subdivided graph, and $B_n(G) \cong \mathbb{Z}$. Then either $G \cong K_{3,1}$ and $n = 2$, otherwise $G \cong S^1$ for all $n \geq 2$.

**Proof.** If $G \cong K_{3,1}$, we note that $B_2(G) \cong B_2(K_{3,1})$ by Theorem 1.2 since $K_{3,1}$ is sufficiently subdivided for $n = 2$. Now it has been shown that $B_2(K_{3,1}) \cong \mathbb{Z}$ by explicit computation in [1] and Figure 2. Next we can note that on $C_k$ for

![Figure 2. A dance of two strands on $K_{3,1}$ and note that this shows $B_2(K_{3,1}) \cong \mathbb{Z}$ since the complex on the right is $UD^2(K_{3,1}) \cong S^1$.](image-url)
Lemma 2.3. Let \( |G| \geq n \geq 2 \). Then \( B_n(G) \cong 1 \) if and only if \( G \) is a path.

Proof. This is an immediate consequence of Theorem 2 and that \( \binom{n}{k} \geq \binom{n-1}{k} \) for all \( n, k > 0 \).

Now if \( n = 1 \), then \( B_1(G) = 1 \) if and only if \( G \) is a tree. To show this, note that \( \mathcal{D}^1(G) \cong G \), and only trees are contractible graphs. Therefore, in conjunction with Lemma 2.3, these are all possible graph braid groups which are isomorphic to \( B_1 \).

Hence almost immediately we can prove the following lemma.

Lemma 2.4. If one of the following is true:
- \( n \geq 1 \) and \( G \approx S^1 \),
- \( n = 2 \) and \( G \approx K_{3,1} \),
- \( n = 1 \) and \( G \) contains exactly one properly embedded cycle,

then \( B_n(G) \cong B_2 \).

Proof. Clearly \( B_2 \cong \mathbb{Z} \). Next Lemma 2.2 states that \( B_n(G) \cong \mathbb{Z} \) when \( G \) is a cycle or when \( n = 2 \) and \( G \approx K_{3,1} \). Finally since \( \mathcal{D}^1(G) \cong G \), it is easy to see that each edge not in a maximal tree of \( G \) will correspond to a generator. Thus any \( G \) which contains one embedded cycle will result in \( B_1(G) \cong \mathbb{Z} \). Therefore under any of these conditions, we have \( B_n(G) \cong \mathbb{Z} \cong B_2 \).

Now we will quickly derive a key fact, the abelianization of any classical braid group. This is a well-known result and is intended as a warm-up for computing the first homology of the tree braid groups.

Lemma 2.5. Let \( D^2 \) be a topological disk and \( n \geq 2 \), then \( H_1(UC^n(D^2)) \cong \mathbb{Z} \).

Proof. From [11], we know that the first homology group is isomorphic to the abelianization of the fundamental group. Therefore we look at the abelianization of \( \pi_1(C^n(D^2)) = B_n \). Thus we take the relation \( \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \), and applying Tietze transformations using the commutator relations, we have

\[
\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \\
\sigma_i^2 \sigma_{i+1} = \sigma_i \sigma_{i+1}^2, \\
\sigma_i = \sigma_{i+1}.
\]

Hence all generators are equivalent to each other, and by additional Tietze transformations, the abelianization of \( B_n \) is a free group of rank 1 for all \( n \geq 2 \).
Finally we will derive the abelianization for the tree braid groups.

**Proposition 2.6.** Let $T$ be a tree and $T \not \cong I$. If $n \geq 3$, otherwise if $n = 2$ and there exists 4 distinct paths, then the first Betti number is strictly greater than 1.

**Proof.** Taking the discrete Morse presentation of $B_n(T)$, every relation can be expressed as a commutator. Therefore the abelianization of $B_n(T)$ is free abelian of order $m$ where $m$ is the number of generators of $T$. It is easy to see that 4 distinct paths implies there is either 2 branching vertices or a vertex of degree 4. So from Equation (4), if there are at least 4 distinct paths or $n \geq 3$, then $m > 1$. \hfill \qed

Thus an immediate consequence is $B_n(G) \not \cong B_{n'}$ if the conditions of Proposition 2.6 are satisfied since their abelianizations are clearly not isomorphic.

### 2.3. Computation Proposition

Next we restate a portion of the results of Kurlin’s algorithmic construction of graph braid groups in [14], specifically what happens to the graph braid group when adding an edge which creates a cycle. We will use this in conjunction with what we have previously discussed with trees to construct all possible graphs.

We begin by stating the Seifert – van Kampen Theorem which Kurlin uses as his primary computational tool.

**Theorem 2.7 (Seifert – van Kampen Theorem).** If presentations $\pi_1(X) = \langle \beta | \lambda \rangle$ and $\pi_1(Y) = \langle \gamma | \mu \rangle$ are given, and $\pi_1(X \cap Y)$ is generated by (a vector of) words $\alpha$, then the group $\pi_1(X \cup Y)$ has the presentation $\pi_1(X \cup Y) = \langle \beta, \gamma | \lambda, \mu, \alpha_X = \alpha_Y \rangle$, where $\alpha_X$ and $\alpha_Y$ are obtained from the words $\alpha$ by rewriting them in the alphabet of $\beta$ and $\gamma$ respectively.

Now take some edge $d \in G$ with endpoints $d_0$ and $d_1$ such that $G - d$ is connected, and let $G - N(d)$ split into $k$ connected components. Here $N(H)$ denotes the set $H$ and all open edges attached to the endpoints of $H \subseteq G$. Thus $UD^{n-1}(G - N(d))$ consists of $k$ connected components which the $j$-th connected component will be denoted by $CC_j$. Now fix basepoints $a \in UD^{n-1}(G - d)$ and $b_j \in CC_j$. Denote $(b_j \times d) \subseteq UD^n(G)$ as the path from $b_j \times d_0$ to $b_j \times d_1$ while all other strands remain fixed at $b_i \in CC_i$ for $i \neq j$.

Let $x_i$ denote the $i$-th strand (note we can impose a distinction between strands in order to observe their motions in a loop). Also let $\epsilon_j$ and $\tau_j$ be paths from $a$ to $b_j \times d_0$ and to $b_j \times d_1$ respectively in $UD^n(G - d)$. Therefore $\epsilon_j(b_j \times d)\tau_j$ is a loop in $UD^n(G - d)$ with basepoint $a$.

**Proposition 2.8.** [14] Given the presentation $\pi_1(UD^n(G - d)) = \langle \alpha | \rho \rangle$ and the presentation $\pi_1(CC_j) = \langle \beta_j | \lambda_j \rangle$ where $j = 1, 2, \ldots, k$. The group $\pi_1(UD^n(G))$ is generated by $\alpha$ and $\delta_j = \epsilon_j(\beta_j \times d)\tau_j$ subject to $\rho = 1$ and

\begin{equation}
\gamma_j = \delta_j \kappa_j \delta_j^{-1}
\end{equation}

where $\gamma_j = \epsilon_j(\beta_j \{x_n = d_0\})\epsilon_j^{-1}$ and $\kappa_j = \tau_j(\beta_j \{x_n = d_1\})\tau_j^{-1}$.

### 3. The Main Theorem

We proceed by combining the above results in order to formally prove that only the trivial classical braid groups are also graph braid groups.

**Theorem 3.1.** $B_n(G) \cong B_{n'}$ if and only if one of the follow conditions hold:

1. $n' = 1$ and $n \geq 1$ and $G \cong I$,
2. $n' = n = 1$ and $G$ is a tree,
3. $n' = 2$ and $n = 1$ and $G$ contains exactly one properly embedded cycle,
4. $n' = n = 2$ and $G \cong K_{3,1}$,
(5) \( n' = 2 \) and \( n \geq 1 \) and \( G \approx S^1 \).

**Proof.** (\( \Rightarrow \))

Lemma 2.3 and Lemma 2.4.

(\( \Rightarrow \))

We begin by stating that in [13], it was shown that all non-planar graphs have a first homology group which has torsion, and therefore all non-planar graphs are not classical braid groups. Furthermore, Proposition 2.6 gives precise conditions in which \( B_n(T) \not\cong B_{n'} \) where \( T \) is a tree.

Now, subdivide \( G \) such that there exists a sufficiently subdivided maximal tree \( M \) where the degree of every vertex in \( G \) whose degree is at least 3 is equal to the degree of the corresponding vertex in \( M \). Therefore \( B_n(M) = \langle \alpha \mid \rho \rangle \) where every relator \( r \in \rho \) is a commutator, and in the abelianization, these relations become trivial as in Proposition 2.6.

Next let \( D \) be the set of deleted edges, that is edges in \( G \) that are not in \( M \). Similar to [14], we take an edge \( d \in D \) with vertices \( d_0 \) and \( d_1 \) and add it back into \( M \) to obtain the graph \( M' \). Thus by Proposition 2.8 adding back \( d \) adds the fundamental group relations:

\[
\mu_j = \delta_j \kappa_j \delta^{-1}_j
\]

where \( \mu_j, \kappa_j, \) and \( \delta_j \) are the same as in Proposition 2.8. Note that \( \delta_j \) defines a new generator, and in the abelianization, \( \delta_j \) is not present in this relation since the exponent sum of \( \delta_j \) is zero. Hence all relations in the abelianization are equating old words, and the generators \( \delta_j \) are free of all relations. Therefore the abelianization preserves at least 2 generators each time we add back any \( d \in D \). Hence \( B_n(G) \not\cong B_{n'} \) unless there is at most 1 generator in the group. Thus with Lemma 2.2 it is clear that in only the cases states above do we have an isomorphism.

Note the classical pure braid group relations (see [16]), all clearly trivialize in the abelianization. While this excludes a large number of graphs, it does not permit the same homological arguments for proving there does not exist an isomorphism.

### 4. Embeddings of Classical and Graph Braid Groups

In this section we will construct embeddings of graph braid groups into classical braid groups. We begin by constructing an embedding of some easier right-angled Artin groups into classical braid groups before deriving a simple algorithmic approach. From this point, we can easily construct a map by embedding a graph braid group into a right-angled Artin group into a classical braid group by using the results of Crisp and Wiest.

On the other hand, it is impossible to embed \( B_n \) (or any non-right-angled Artin group for that matter) for \( n > 2 \) into graph braid groups since graph braid groups are bi-orderable whereas \( B_n \) is not. However what we will do here is describe all possible homomorphisms of \( B_n \) for \( n > 4 \) to an arbitrary right-angled Artin group, thus reproving no such embedding exists.

#### 4.1. Graph Braid Groups into Classical Braid Groups

We first state a result of Crisp and Wiest which we will not only generalize, but give a constructive proof which likely uses a smaller (in terms of the number of strands) classical pure braid group.

**Theorem 4.1.** [5] If the graph \( \Gamma^{\text{op}} \) has a finite-sheeted covering which is planar, then \( A_\Gamma \) embeds in \( PB_l \) for sufficiently large \( l \).
Thus consider $\mathcal{B}_n$ such that $n \geq 2$, then take 2 generators, $\sigma_1$ and $\sigma_2$, of the presentation given in (1), and define $\psi_i = \sigma_i^2$. Then observe that the subgroup
\[ B_n \supseteq \hat{\mathcal{B}}_n \supseteq \langle \psi_1, \psi_2 \rangle \cong \mathbb{F}_2 \]
by Theorem 1.1. Recall that $\mathbb{F}_2$ contains all free groups on $k$ generators (including when $k$ is countably infinite), thus every classical pure braid group contains $\mathbb{F}_k$ for all $k$. Just using this, we can construct a number of right-angled Artin groups by looking at sets of generators which are free with respect to each other, but commute with all other generators. That is to say we can construct all right-angled Artin groups of the form $F_{k_1} \times F_{k_2} \times \cdots \times F_{k_m}$ where $F_{k_i}$ is a free group on $k_i$ generators and $k_i \in \mathbb{N} \cup \infty$. In terms of graphs, this embeds all $m$-partite graphs. Explicitly this will embed into $\hat{\mathcal{B}}_n$ where $n \geq 3m$ since we take neighboring pairs $\psi_i \psi_{j+1}$ for $F_2$ (which contains $F_{k_i}$) and skip the index $j + 2$ and $B_n$ has $n - 1$ generators.

An example of a more complex group structure we need to consider is $A_{C_4}$, where $C_4$ denotes the cycle with 4 vertices. We note that $\hat{\mathcal{B}}_5 \cong A_{P_4}$ (recall that $P_4$ is the path on 4 vertices) which is clearly not isomorphic to $A_{C_4}$. In fact, according to Corollary 9 in [12], the group $A_{C_4}$ does not even exist as a subgroup. Therefore right-angled Artin groups, in general, do not naturally embed into classical braid groups, and we need a method for a general embedding.

We will begin by defining a new method to join two groups together which we will call the couple product.

**Definition 4.2 (Couple product).** Let $G$ and $G'$ be two groups with the following presentations
\[ G = \langle g_1, \ldots, g_n \mid \rho_G \rangle \quad \quad \quad G' = \langle g'_1, \ldots, g'_m \mid \rho_{G'} \rangle. \]

We form the couple product $G \times G'$ along $g_1 g'_1, \ldots, g_k g'_k$, where $k \leq n, m$, by taking the subgroup of $G \times G'$ which is generated by the following:
\[ G \times G' \supseteq G \times G' := \left\langle g_1 g'_1, g_2 g'_2, \ldots, g_k g'_k, g_{k+1}, g_{k+2}, \ldots, g_n, g'_{m+1}, g'_{m+2}, \ldots, g'_{m+k} \right\rangle. \]

Now from this definition, there are three natural (essentially disjoint) subgroups of $G \times G'$. The first one is the coupled subgroup which is the subgroup generated by the elements coupled together (i.e. the elements $g_i g'_i$) which is denoted by $G \times G'$. Note that because $g_i g'_i = g'_i g_i$, we can define the coupled subgroups by subsets of generators from $G$ and $G'$. The next subgroup is the complement subgroup of $G$ which is generated by the generators in $G$ not in the coupled subgroup, and the last is the complement subgroup of $G'$.

**Proposition 4.3.** The couple product has the following properties:

1. The couple product is symmetric. That is to say $G \times G'$ coupled along $g_1 g'_1, \ldots, g_k g'_k$ is isomorphic to $G' \times G$ coupled along $g'_1 g_1, \ldots, g'_k g_k$.
2. The couple product is associative. That is to say $(G \times G') \times G'' \cong G \times (G' \times G'')$ where the final coupled subgroups are equal.
3. There exists a surjection from $G \times G'$ to $G$ which is the identity on $G$ and similarly to $G'$.
4. We have $G \times F_k$ coupled along $g_1 f_1, \ldots, g_k f_k$ is isomorphic to $F_k$ where $f_i$ is a generator of $F_k$.
5. The couple product $G \times F_k$ coupled along $g_1 f_1, \ldots, g_k f_k$ has the following presentation:
\[ G \times F_k = \langle g_1 f_1, \ldots, g_k f_k, g_{k+1}, \ldots, g_n \mid \hat{\rho}_G \rangle \]
where \( \hat{\rho}_G = \{ w \in G \mid q(w) = 1 \} \) with \( q(g_i) = g_i f_i \) for \( 1 \leq i \leq k \) and \( q(g_j) = g_i \) for \( k < i \leq n \).

**Proof.**

1. This comes from the fact that the couple product is a subgroup of \( G \times G' \cong G' \times G \). We leave the details as an exercise.
2. This comes from the fact that the couple product is a subgroup of \( G \times G' \times G'' \). We leave the details as an exercise.
3. Consider the natural surjection \( \pi_G: G \times G' \to G \), and it is clear that \( \pi_G \) restricted to \( G \times G' \) is a surjection from \( G \times G' \) to \( G \). A similar construction holds for \( G'' \).
4. From the definition of the Cartesian product, the generators of \( F_k \) commute with the generators of \( G \), thus we can write any word in \( G \times F_k \) in the form \( w_G w_{F_k} \) where \( w_G \) is some word in \( G \) and \( w_{F_k} \) is a word in \( F_k \). Thus consider the natural surjective map \( \phi: G \times F_k \to F_k \) that is a restriction of \( \pi_{F_k} \). Explicitly this map sends every element in \( G \) to \( 1_G \) (the identity element in \( G \)) and is the identity on \( F_k \). So all we need to show is \( \phi \) is also injective. Thus taking \( \phi(w_G w_{F_k}) = 1 \), it is clear from the Cartesian product that \( w_G \) cannot alter \( w_{F_k} \), so we must have \( w_{F_k} = 1_{F_k} \). However it is clear this only occurs when we have the trivial word since \( F_k \) is free, thus \( \phi \) is an isomorphism.
5. From above, we know that \( G \times F_k \cong F_k \) which clearly has the following presentation:

\[
G \times F_k = \langle g_1 f_1, \ldots, g_k f_k \mid \rangle
\]

Now the complement subgroup of \( F_k \) is the trivial group, so it does not affect the group. Thus we look at when we add the complement subgroup of \( G \) (thus are considering the full couple product). From above we know that if \( w \) is not a product of relations, then \( q(w) \) cannot be trivial. Now let \( r_G \) be a product of relations in \( G \) such that \( q(r_G) \neq 1 \). Thus we can decompose \( q(r_G) = r_G r_{F_k} \) where \( r_{F_k} \in F_k \). Note that \( r_G = 1 \), so \( q(r_G) = r_{F_k} \neq 1 \), and hence is not a relation since \( F_k \) is a free group. Thus \( \hat{\rho}_G \) will consist of products of relations of \( G \) whose image under \( q \) is the trivial word.

\[ \square \]

Thus from the first two properties, the couple product is well-defined by just stating the base groups and the coupled subgroups. Now we can also extend the concept of a couple product to subgroups of a particular group \( G \) in the following fashion. Consider two subgroups \( H, H' \subseteq G \) such that \( H \times H' \) is also a subgroup of \( G \). That is to say every element in \( H \) commutes with every element in \( H' \). Thus the couple product \( H \times H' \) coupled along \( h_1 h'_1, \ldots, h_k h'_k \) is also a subgroup of \( G \). We note from part (5), that in a right-angled Artin group \( A \), all exponent sums of relations are zero, thus the restrictions are trivial. Hence all relations in \( A \times F_k \) will consists of any such relations \( [g_i, g_j] \) when \( j > k \) and all \( i < n \) in \( A \) such that \( [g_i, g_j] \) was a relation in \( A \).

**Theorem 4.4.** For a finite, simple graph \( \Gamma \), the right-angled Artin group \( A_\Gamma \) embeds in \( \hat{B}_n \) for \( n \geq 2|\Gamma| + 3 \left( \left( \frac{|\Gamma|}{2} \right) - E \right) \), where \( E \) is the number of edges of \( \Gamma \).

**Proof.** Denote generators of \( A_\Gamma \) by \( g_i \) and by a slight abuse of notation, we will identify the generators with the vertices of \( \Gamma \). Let \( \Psi: A_\Gamma \to \hat{B}_l \) for some \( l \geq 2|\Gamma| \) be a homomorphism such that \( \Psi(g_i) = \psi_{2i-1} \) where \( \psi_j = \sigma_j^2 \). Next consider any edge \( e = \{ u, v \} \) in \( \Gamma^{op} \) where \( u \) and \( v \) are vertices of \( \Gamma^{op} \). Note that \( e \) represents that the two generators are supposed to be free with respect to each other in \( A_\Gamma \), and so we will construct a new map which effectively modifies their image under \( \Psi \).
Let $k$ denote the highest index which has a pre-image under $\Psi$, therefore we define the modified map $\Psi'$ by having $\Psi'(u) = \Psi(u)\psi_{k+2}$ and $\Psi'(v) = \Psi(v)\psi_{k+3}$. Now in $PB_{k+3}$, the subgroups $\Psi(A_1)$ and $F_2 \cong \langle \psi_{k+2}, \psi_{k+3} \rangle$ admit a Cartesian product $\Psi(A_1) \times F_2$ which is also a subgroup of $PB_{k+3}$. Next observe that $\Psi'(A_1)$ is actually the couple product $\Psi'(A_1) \times F_2$ along $\Psi(u)\psi_{k+2}$ and $\Psi(v)\psi_{k+3}$. Thus by Proposition 4.3, $\Psi'(u)$ and $\Psi'(v)$ do not commute, but $\Psi'$ preserves all other relations. Therefore proceed inductively along all edges of $\Gamma$, and by Proposition 4.3, the final image of $\Psi$ is unique up to isomorphism and isomorphic to $A$. Relations. Therefore proceed inductively along all edges of $\Gamma$, and by Proposition 4.3, the final image of $\Psi$ is unique up to isomorphism and isomorphic to $A$.

Now to show that $n$ can be as low as $2|\Gamma| + 3E^{op}$ where $E^{op}$ is the number of edges in $\Gamma^{op}$, note that the initial map $\Phi$ (or equivalently consider the final map under $A_G \cong \mathbb{Z}^{|\Gamma|}$) maps to the odd numbered generators $1, 3, \ldots, 2|\Gamma| - 1$. Now recall that each time we modify $\Phi$, we require three more generators for each edge, thus requiring an additional $3E^{op}$ number of generators. Hence we require $2|\Gamma| + 3E^{op} - 1$ number of generators and $E^{op} = \left(\frac{|\Gamma|}{2}\right) - E$.

Recall that $\overline{B}_{n} \subseteq PB_{n} \subseteq B_{n}$, so Theorem 4.3 also gives an embedding of a right-angled Artin group into the classical braid group. It is clear that if we fix $n$, then all right-angled Artin groups which satisfy the inequality $n \geq 2|\Gamma| + 3E^{op}$ will embed into $\overline{B}_{n}$.

We note that this $n$ here is likely not the most optimal since any $K_a \subseteq \Gamma^{op}$ for some $a$ has a more compact embedding as previously discussed. A good problem would be to give an optimal bound on $n$ such that an embedding exists. One simple way we can lower the bound on $n$ is by noting that the first $2|\Gamma| - 1$ generators of $\overline{B}_{n}$ only provide an initial image for $A_{\Gamma}$ and can be removed for any generator of $A_{\Gamma}$ that is not in the center.

Now we will state the theorem proved by Crisp and Wiest which gives an embedding from a graph braid group into a right-angled Artin group. However we omit the local isometry portion (thus slightly weaken the statement) for brevity.

**Theorem 4.5.** Let $G$ be a finite graph and $n \in \mathbb{N}$, then there exists an embedding $\Phi$ from $B_n(G)$ into $A_{\mathcal{L}(G)^{op}}$.

Now we have the following tools to prove that there always exists an embedding of a graph braid group, if the graph is finite, into some classical braid group.

**Theorem 4.6.** Let $G$ be a finite graph, $\Gamma = \mathcal{L}(G)^{op}$, and $n \in \mathbb{N}$, then $B_n(G)$ embeds in $\overline{B}_{l}$ where $l \geq 2|\Gamma| + 3\left(\frac{|\Gamma|}{2}\right) - E$ where $E$ is the number of edges of $\Gamma$.

**Proof.** Let $\Phi$ be the embedding defined in Theorem 4.5 which maps $B_n(G)$ into $A_{\Gamma}$. Then by Theorem 4.4, we can construct an embedding $\Psi: A_{\Gamma} \to PB_l$ for $l \geq 2|\Gamma| + 3\left(\frac{|\Gamma|}{2}\right) - E$ where $E$ is the number of edges of $\Gamma$. Therefore the map $\Psi \circ \Phi$ defines an embedding from $B_n(G)$ into $PB_l$. \qed

Note that we also have the following as a corollary of the embedding proved by Crisp and Wiest.

**Corollary 4.7.** Let $n > 2$, then there does not exist an embedding of $B_n$ into $B_{n'}(G)$ for any $n' \in \mathbb{N}$ and graph $G$.

**Proof.** Assume there exists an embedding $\Psi: B_n \to B_{n'}(G)$, and by Theorem 4.5 there exists an embedding $\Phi: B_{n'}(G) \to A_{\Gamma}$ for some simple graph $\Gamma$. Thus $\Phi \circ \Psi$ is an embedding from $B_n$ into $A_{\Gamma}$. This implies that $B_n$ is biorderable, but this is a contradiction. Therefore there does not exist an embedding of $B_n$ into $B_{n'}(G)$. \qed
4.2. Classical Braid Groups into Graph Braid Groups. First we will define a homomorphism which encodes the exponent sum of words in right-angled Artin group and use this to show some of the structure on the image of the classical braid group under any homomorphism.

**Definition 4.8** (Exponent sum homomorphism). Let \( \epsilon_k : A_\Gamma \to \mathbb{Z} \) for some \( n \) and simple graph \( \Gamma \) which takes a word \( w \in A_\Gamma \) and does the following:

\[
w = \prod_i q_i^{p_i} \mapsto \sum_{k=j_i} p_i
\]

Note it is clear that \( \epsilon_k(ww') = \epsilon_k(w') + \epsilon_k(w) \), and that \( \epsilon_k(w) \) is invariant under any representation of \( w \). Therefore \( \epsilon_k \) is a well-defined homomorphism and \( \epsilon_k \) is the exponent sum of the generator \( g_k \).

To avoid excess parentheses and redefinition, for the remainder of this section we will have \( \Lambda : B_n \to A_\Gamma \) for some \( n \in \mathbb{N} \) and arbitrary simple graph \( \Gamma \). We also will define \( \lambda_i := \Lambda(\sigma_i) \).

**Lemma 4.9.** Let \( \Lambda \) be a homomorphism, then \( \epsilon_k(\lambda_i) = \epsilon_k(\lambda_j) \) for all \( 1 \leq i, j, k < n \).

**Proof.** By applying \( \epsilon_k \) to the braid relation \( \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \) under \( \Lambda \), we have

\[
2\epsilon_k(\lambda_i) + \epsilon_k(\lambda_{i+1}) = \epsilon_k(\lambda_i) + 2\epsilon_k(\lambda_{i+1}).
\]

This implies \( \epsilon_k(\lambda_i) = \epsilon_k(\lambda_{i+1}) \), and therefore we have \( \epsilon_k(\lambda_i) = \epsilon_k(\lambda_j) \) for all \( 1 \leq i, j, k < n \).

Recall that a reduced word is the shortest length representation of a word. The support of a reduced word \( w \), which we will write as \( \text{supp}(w) \), in \( A_\Gamma \) is the set of generators (vertices) present in \( w \). The link of \( w \), denoted by \( \text{link}(w) \), is the set of vertices in \( \Gamma \) which are not in \( \text{supp}(w) \), but are adjacent to every vertex in \( \text{supp}(w) \). A word is cyclically reduced if it is the shortest word in its conjugacy class; thus we can write \( w = pu'p^{-1} \) where \( w' \) is cyclically reduced. Let \( \Gamma_{w'} \) denote the subgraph of \( \Gamma \) induced by \( \text{supp}(w') \). Also, we can express \( w' = e_1^{f_1} e_2^{f_2} \cdots e_m^{f_m} \) where the \( \text{supp}(e_k) \) is equal to the \( k \)-th connected component of \( (\Gamma_{w'})^{\text{link}} \). The set of all \( f_k \)'s are the pure factors of \( w \).

Now we will state a result of Servatius which we will use to describe the structure of commuting elements. Recall that the centralizer of a word \( w \) is the set of all words which commute with \( w \).

**Theorem 4.10** (Centralizer theorem). [20] Let \( \Gamma \) be a simple graph and \( p, u \in A_\Gamma \). The centralizer of an element \( u = pu'p^{-1} \), where \( u' \) is cyclically reduced, is conjugate by \( p \) to the subgroup of \( A_\Gamma \) generated by \( \text{link}(u) \) and the pure factors of \( u \).

Note that Theorem 4.10 states that if \( u \) commutes with \( u = pu'p^{-1} \), then we can express \( v = pu'\hat{p}^{-1} \) where \( \text{supp}(\hat{p}) \subseteq \text{link}(u) \) and \( v' \) is a product of the pure factors of \( u \), for some \( p \in A_\Gamma \). Also it implies that if \( f' \) is a pure factor in \( v \) such that \( \text{supp}(f') \cap \text{supp}(f_k) \neq \emptyset \) for some \( k \), then \( f' = f_k \). This is due to the fact that if \( f' \neq f_k \), then \( f' \) would not be a pure factor of \( u \), nor is it in \( \text{link}(u) \) since \( \text{supp}(f') \cap \text{supp}(f_k) \subseteq \text{supp}(u) \). Hence \( v \) is not in the centralizer, which is all elements which commute with \( u \) and a contradiction.

Now assume that \( \lambda_1 \) is not cyclically reduced, thus we express \( \lambda_1 = p\lambda'_{1}p^{-1} \). Now let \( i \) be such that \( i \geq 3 \), then clearly \( \lambda_i \) commutes with \( \lambda_1 \). Thus by Theorem 4.10 we must have \( \lambda_i = p\lambda'_{1}p^{-1} \) where \( \lambda'_{1} \) commutes with \( \lambda_1 \). Now let \( i = 4 \), then also by Theorem 4.10 we have \( \lambda_2 = p\lambda'_{2}p^{-1} \) to commute with \( \lambda_1 \). Thus we have \( \lambda_i = p\lambda'_{i}p^{-1} \) for all \( i \), and hence we can consider \( \lambda_i \) cyclically reduced without loss.
of generality. Note that we will implicitly be using a similar procedure as above to state if a structure holds for all \( |i - j| \geq 2 \), then it holds for all \( i \) and \( j \).

Notice that we can decompose a word into its pure factors, but here we will group the pure factors together in a way to make our computations easier. First define \( S_i = \{ g_k \in \text{supp}(\lambda_i) \mid \epsilon_k(\lambda_i) \neq 0 \} \), and then we can write \( \lambda_i = h_i \hat{h}_i \) where \( h_i \) is a product of pure factors which have non-trivial intersection with \( S_i \) and \( \hat{h}_i \) is everything else. That is to say if we express \( h_i = f_1^m f_2^m \cdots f_m^m \), then \( \text{supp}(f_j) \cap S_i \neq \emptyset \) for all \( j \). Note that this immediately implies \( \epsilon_k(h_i) = 0 \) for all \( k \).

Now what we will do is explore the structure of \( h_i \) and \( \hat{h}_i \) in the following two lemmas.

**Lemma** 4.11. Let \( \Lambda \) be a homomorphism and \( n > 4 \), then \( h_i = h_j \) for all \( i \) and \( j \).

**Proof.** Let \( S_i = S_j \) for all \( i \) and \( j \). Now let \( a_i \) be the pure factor in \( h_i \) such that \( \epsilon_k(a_i) \neq 0 \) for some \( k \), and note that since \( S_i = S_j \), we must have \( \text{supp}(a_i) \cap \text{supp}(a_j) \neq \emptyset \). Now assume that \( |i - j| \geq 2 \), and since \( \lambda_i \) commutes with \( \lambda_j \), by Theorem 4.10 we must have \( a_i = a_j \) for all \( i \) and \( j \).

Next let \( \varepsilon_i \) equal the exponent of \( a_i \) in \( h_i \). Thus taking exponent sum of the image of the braid relation as in Lemma 4.10 we have

\[
2\varepsilon_i + \varepsilon_{i+1} = \varepsilon_i + 2\varepsilon_{i+1},
\]

\[
\varepsilon_i = \varepsilon_{i+1}.
\]

Repeating this for all pure factors, we have \( h_i = h_j \) for all \( i \) and \( j \). \( \square \)

**Lemma** 4.12. Let \( \Lambda \) be a homomorphism and \( n > 2 \). Let \( \hat{h}_i \) and \( \hat{h}_j \) be cyclically reduced words, then \( \text{supp}(\hat{h}_i) = \text{supp}(\hat{h}_j) \) for all \( i \) and \( j \).

**Proof.** Suppose there exists an \( x \in \text{supp}(\hat{h}_i) \) such that \( x \notin \text{supp}(\hat{h}_{i+1}) \). Let \( A_{x,y} \) be the subgroup generated by \( x \) and \( y \) and let \( q_{x,y} \) be the natural projection map from \( A_{\Gamma} \rightarrow A_{x,y} \). Now since \( \hat{h}_i \) is cyclically reduced, there exists a \( y \in \text{supp}(\hat{h}_i) \) such that \( s = q_{x,y}(\hat{h}_i) \neq 1 \). Let \( \overline{\Lambda} \) to be the homomorphism which makes the following diagram commute:

\[
\begin{array}{ccc}
B_n & \xrightarrow{\Lambda} & A_{\Gamma} \\
\downarrow{\pi} & & \downarrow{q_{x,y}} \\
A_{x,y} & & \\
\end{array}
\]

Now note that \( \overline{\Lambda}(\hat{h}_{i+1}) = 1 \) since the exponent sum of \( y \) in \( \hat{h}_{i+1} \) is zero and \( x \) is not present by assumption. Therefore we have \( \overline{\Lambda} \) applied to \( \sigma_i \sigma_{i+1} \) implying \( s^2 = s \). Hence \( s = 1 \) which is a contradiction, and thus \( \text{supp}(\hat{h}_i) = \text{supp}(\hat{h}_j) \) for all \( i \) and \( j \). \( \square \)

Now we have proved some of the more technical structure of \( \text{Im}(\Lambda) \), we can combine these results to completely characterize \( \text{Im}(\Lambda) \).

**Theorem** 4.13. Let \( \Lambda \) be a homomorphism and \( n > 4 \), then \( \lambda_i = \lambda_j \) for all \( i \) and \( j \).

**Proof.** Suppose \( \hat{h}_i \) was not cyclicly reduced, then we would have the following \( \lambda_i = h_i p_i^{\varepsilon_i} p^{-1} \), but \( \hat{h}_i \) must commute with \( \hat{h}_i \) for all \( i \geq 2 \). This is due to the fact that \( \lambda_i \) must commute with \( \lambda_j \) and \( h_i = h_j \) which commutes totally with \( \hat{h}_1 \) and \( \hat{h}_i \). Thus by Theorem 4.10 we have \( \lambda_i = h_i p_i^{\varepsilon_i} p^{-1} \) for all \( i \). Thus we can consider \( \hat{h}_i \) to be cyclicly reduced without loss of generality.
Hence by Lemma 4.12, we have $\text{supp}(\hat{h}_i) = \text{supp}(\hat{h}_j)$ for all $i$ and $j$. Now we can apply the same logic of Lemma 4.11 to $\hat{h}_i$, so $\hat{h}_i = \hat{h}_j$ for all $i$ and $j$. Therefore in conjunction with the original statement of Lemma 4.11 we have $\lambda_i = \lambda_j$. \hfill $\square$

It is easy to see that $\text{Im}(\Lambda) \cong \mathbb{Z}$.

5. Closing remarks

The author believes it is possible to generalize the above proof of Theorem 4.4 to constructively define an embedding of an arbitrary right-angled Artin group into any particular class of Artin groups with “enough” commutativity. Now note that Theorem 4.4 is related to Theorem 1.1 since Theorem 1.1 is describing some of the right-angled Artin groups which embed in an arbitrary Artin group. However this does not readily imply that every right-angled Artin group embeds in some classical braid group as we saw in Section 4.1.

Yet Theorem 4.4 is an embedding of an arbitrary right-angled Artin group into $\hat{B}_n$. Thus this has potential ramifications with a conjecture of Batty and Goda that was stated in [12] as to which graphs are concealable. A graph $\Gamma$ is concealable if there exists a graph $\Omega$ such that $A_{\Gamma}$ embeds in $A_{\Omega}$ but $\Gamma$ does not embed in $\Omega$. Theorem 4.4 might give rise to a graph which is concealable that is not included in Theorem 21 in [12], or perhaps Theorem 21 could be used to give better bounds on the size of a classical pure braid group to embed a given right-angled Artin group.

Also, the embedding given by Theorem 4.4 into the canonical right-angled Artin group is not the only method to construct the embedding. We can move it outside of the pure braid group by using $\psi_i = \sigma_i \sigma_{i+1}$ which is in left-greedy normal form, so $\psi_i$ and $\psi_{i+1}$ generate $F_2$. However this would require a larger classical braid group to construct the embedding since we would have to skip 2 indices (i.e. we must take $\psi_{k+3}$ and $\psi_{k+4}$ for the next copy of $F_2$), but the copies of $F_2$ still commute with each other. In fact, as long as we can find the commutative copies of $F_2$, we can construct the embedding necessary for Theorem 4.4.

Furthermore the proofs of Lemma 4.9 and Lemma 4.12 primarily relies upon the fact that $m_{ij}$ is odd, thus we can generalize Theorem 4.13 into general Artin groups. Let $a$ and $b$ be vertices (generators) of an Artin system $A$ such that they are connected by a path of odd weighted edges (the $m_{ij}$ values) and by a weight 2 edge (i.e. $a$ and $b$ commute). Let $\Lambda: A \to A_{\Gamma}$ for any simple graph $\Gamma$, then it is easy to see that $\Lambda(a) = \Lambda(b)$.

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