Adversarial Robust Low Rank Matrix Estimation: Compressed Sensing and Matrix Completion

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Abstract: We consider robust low rank matrix estimation as a trace regression when outputs are contaminated by adversaries. The adversaries are allowed to add arbitrary values to arbitrary outputs. Such values can depend on any samples. We deal with matrix compressed sensing, including lasso as a partial problem, and matrix completion, and then we obtain sharp estimation error bounds. To obtain the error bounds for different models such as matrix compressed sensing and matrix completion, we propose a simple unified approach based on a combination of the Huber loss function and the nuclear norm penalization. Some error bounds obtained in the present paper are sharper than the past ones.

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1. Introduction

Sparse estimation is a well-studied topic in high-dimensional statistics. For sparse estimation of the linear regression coefficient, the $\ell_1$ penalization and its variants have been introduced by [66], [32], [72], [70], [71], [62], and [5]. In some studies, such as [42], [55], [60], [56], [6], [39], [41], and [34], sparse estimation of the linear regression coefficient was extended to low rank matrix estimation, mostly utilizing the nuclear norm penalization.

In this study, we consider sparse estimations of regression coefficient under the existence of malicious outlier. [57] considered the case that a part of outputs is adversarially contaminated. They dealt with the following model:

$$y_i = \langle x_i, \beta^* \rangle + \xi_i + \sqrt{n}\theta_i^*, \quad i = 1, \cdots, n, \quad (1.1)$$

where $\beta^* \in \mathbb{R}^d$ is the true coefficient vector, $\{x_i\}_{i=1}^n$ is a sequence of covariate vectors, $\langle \cdot, \cdot \rangle$ is the inner product and $\{\xi_i\}_{i=1}^n$ is a sequence of random noises. In addition, $\theta^* = (\theta_{11}^*, \cdots, \theta_{nn}^*)^\top$ is a vector of adversarial noises ($\sqrt{n}$ is used for normalization). An adversary is allowed to set any value to any position in $\theta^*$. Here we give a more explanation on the role of $\theta^*$. Let $I_I$ and $I_O$ be the index sets for uncontaminated and contaminated outputs, respectively, in other words, we have $\theta_{ii}^* = 0$ for $i \in I_I$ and $\theta_{ii}^* \neq 0$ for $i \in I_O$, respectively. Let $o$ be the number of elements of $I_O$. We allow the adversary can choose $I_O$ arbitrarily on the knowledge of $\{x_i\}_{i=1}^n$ and $\{\xi_i\}_{i=1}^n$ only with the constraint that $o/n$, which is a ratio of the contaminated samples by the adversary, is sufficiently small. We should note that inliers can lose their independence and outliers can be correlated to inliers because the values of $\theta_{ii}^*$ for $i \in I_O$ are not constrained and $I_O$ can be chosen freely.

For a vector $v$, let $\|v\|_2$ and $\|v\|_1$ be the $\ell_2$ and $\ell_1$ norms, respectively. [57] introduced the
following estimator:

\[
(\hat{\beta}, \hat{\theta}) \in \arg\min_{(\beta, \theta) \in \mathbb{R}^d, \mathbb{R}^n} \text{obj}(\beta, \theta),
\]

\[
\text{obj}(\beta, \theta) = \sum_{i=1}^{n} (y_i - \langle x_i, \beta \rangle - \sqrt{n} \theta_i)^2 + \lambda_\ast \| \beta \|_1 + \lambda_0 \| \theta \|_1.
\]

Then, \cite{57} got a high-probability error bound for \( \| \hat{\beta} - \beta^* \|_2 \), which is

\[
P \left\{ \| \hat{\beta} - \beta^* \|_2 \leq C_{x, \xi, \delta} \left( \sqrt{\frac{s \log d}{n}} + \frac{o}{\sqrt{n \log n}} \right) \right\} \geq 1 - \delta,
\]

where \( C_{x, \xi, \delta} \) is some constant depending on \( \delta \) and properties of \( \{x_i\}_{i=1}^{n} \) and \( \{\xi_i\}_{i=1}^{n} \) when \( \{x_i\}_{i=1}^{n} \) and \( \{\xi_i\}_{i=1}^{n} \) are drawn from Gaussian distribution. It is known that without adversarial noises, by using the \( \ell_1 \) penalization, we can get an error bound such as \( \sqrt{\frac{s \log d}{n}} \) up to constant factor \cite{59}.

We note that even when adversarial noises contaminate outputs, the error bound is only loosened by \( \sqrt{\frac{\log n}{n}} \) up to constant factor by using estimator (1.2). Model (1.1) and estimator (1.2) were also studied in \cite{20}. \cite{20} introduced new concentration inequalities and derived a sharper error bound than (1.3), which is

\[
P \left\{ \| \hat{\beta} - \beta^* \|_2 \leq C'_{x, \xi, \delta} \left( \sqrt{\frac{s \log d}{n}} + \frac{o}{\sqrt{n \log n}} \sqrt{\frac{\log n}{o}} \right) \right\} \geq 1 - \delta,
\]

where \( C'_{x, \xi, \delta} \) is some constant depending on \( \delta \) and properties of \( \{x_i\}_{i=1}^{n} \) and \( \{\xi_i\}_{i=1}^{n} \).

Recently, \cite{65} proposed a new estimator, which is a variant of (1.2). The estimator is a combination of (1.2) and SLOPE \cite{62, 4, 3}:

\[
(\hat{\beta}, \hat{\theta}) \in \arg\min_{(\beta, \theta) \in \mathbb{R}^d, \mathbb{R}^n} \text{obj}(\beta, \theta),
\]

\[
\text{obj}(\beta, \theta) = \sum_{i=1}^{n} (y_i - \langle x_i, \beta \rangle - \sqrt{n} \theta_i)^2 + \| \beta \|_b + \| \theta \|_z,
\]

where \( \| \cdot \|_b \) and \( \| \cdot \|_z \) are the SLOPE norms. The error bound of \cite{65} is

\[
P \left\{ \| \hat{\beta} - \beta^* \|_2 \leq C_{x, \xi} \left( \sqrt{\frac{s \log(d/s)}{n}} + \frac{o}{n \log n} + \frac{1 + \sqrt{\log(1/\delta)}}{\sqrt{n}} \right) \right\} \geq 1 - \delta,
\]

where \( C_{x, \xi} \) is some constant depending on properties of \( \{x_i\}_{i=1}^{n} \) and \( \{\xi_i\}_{i=1}^{n} \). \cite{65} also weaken the assumption to make it applicable to the case that \( \{x_i\}_{i=1}^{n} \) is drawn from an \( L \)-subGaussian distribution and \( \{\xi_i\}_{i=1}^{n} \) is drawn from a subGaussian distribution. We note that the error bound in (1.6) is shaper than in (1.4) and the constant of error bound in (1.6) does not depend on \( \delta \).

On the other hand, after optimizing about \( \theta \), from \cite{61}, (1.2) can be re-written as

\[
\hat{\beta} \in \arg\min_{\beta} \text{obj}_H(\beta),
\]

\[
\text{obj}_H(\beta) = \lambda^2_0 \sum_{i=1}^{n} H \left( \frac{y_i - \langle x_i, \beta \rangle}{\lambda_0 \sqrt{n}} \right) + \lambda_\ast \| \beta \|_1,
\]

where \( H(t) \) is the Huber loss function

\[
H(t) = \begin{cases} 
|t| - \frac{1}{2} & (|t| > 1) \\
\frac{t^2}{2} & (|t| \leq 1).
\end{cases}
\]
In the present paper, we analysis (1.7) rather than (1.2) and derive a sharper error bound

\[ P \left( \left\| \hat{\beta} - \beta^* \right\|_2 \leq C'_{x, \xi} \left( \frac{s \log(d/s)}{n} + \frac{o}{n} \sqrt{\log \frac{n}{o}} + \frac{1 + \sqrt{\log(1/\delta)}}{\sqrt{n}} \right) \right) \geq 1 - \delta, \tag{1.9} \]

where \( C'_{x, \xi} \) is some constant depending on properties of \( \{x_i\}_{i=1}^n \) and \( \{\xi_i\}_{i=1}^n \). We also weaken the assumptions on \( \{\xi_i\}_{i=1}^n \) from a subGaussian distribution to a heavy-tailed distribution with a finite absolute moment. We note that the error bound in (1.9) is sharper than that of (1.6), however, our estimator requires rough knowledge about the sparsity \( s \) and the number of contaminated outputs \( o \) in the tuning parameters, as mentioned in Theorem 2.2.

Sparse estimation can be considered not only on sparse vectors, but also on low rank matrices. In the present paper, we consider the following model

\[ y_i = \langle X_i, B^* \rangle + \xi_i + \sqrt{n}\theta_i^*, \quad i = 1, \ldots, n, \tag{1.10} \]

where \( B^* \in \mathbb{R}^{d_1 \times d_2} \) is the true (low rank) coefficient matrix, \( \{X_i\}_{i=1}^n \) is a sequence of covariate matrices. For a matrix \( M \), let \( \|M\|_* \) be the nuclear norm. To estimate \( B^* \), [65] considered the following more general estimator than (1.5):

\[ (\hat{B}, \hat{\theta}) \in \arg\min_{(B, \theta) \in (\mathbb{R}^d, \mathbb{R}^n)} \ obj(B, \theta), \tag{1.11} \]

\[ obj(B, \theta) = \sum_{i=1}^n \left( y_i - \langle X_i, B \rangle - \sqrt{n}\theta_i \right)^2 + \lambda_0 \|B\|_* + \|\theta\|_2. \]

[65] dealt with robust matrix compressed sensing, robust matrix completion, trace regression with matrix decomposition. On the other hand, to estimate the low rank matrix, we consider the following model:

\[ \hat{B} \in \arg\min_{B} obj_H(B), \tag{1.12} \]

\[ obj_H(B) = \lambda_0^2 \sum_{i=1}^n H \left( \frac{y_i - \langle X_i, B \rangle}{\lambda_0 \sqrt{n}} \right) + \lambda_0 \|B\|_* , \]

which is an extension of (1.7). In the present paper, we also derive a sharper error bound than [65] about robust matrix compressed sensing and robust matrix completion under a weaker condition.

The remainder of this paper is organized as follows. In Section 2, we exhibit our results. In Section 3, we explain related works. In Section 4, we digest our key propositions and lemmas. All of the proofs are postponed to the Appendix.

2. Results

2.1. Adversarial matrix compressed sensing

Before presenting our results, we introduce the matrix restricted eigenvalue (MRE) condition for the covariance matrix of random matrix, which is defined in Definition 2.1. The MRE condition is an extension of restricted eigenvalue condition for the covariance matrix of random vector introduced in [20]. This condition enables us to deal with the case where the covariance matrix of is singular.

Before defining the MRE condition, we prepare some notations. Let \( \text{Proj}_{V^\perp} \) be the orthogonal projection into a linear vector subspace \( V \) of Euclidean space and \( V^\perp \) be the orthogonal complement space of \( V \). For a matrix \( E \), let \( l_i(E) \) and \( r_i(E) \) be the left and right orthonormal singular vectors of \( E \), respectively. Let \( V_l(E) \) and \( V_r(E) \) be the linear spans of \( \{l_i(E)\} \) and \( \{r_i(E)\} \), respectively. For a matrix \( M \in \mathbb{R}^{d_1 \times d_2} \), we define

\[ P^*_E(M) = \text{Proj}_{V^\perp_l(E)}M\text{Proj}_{V^\perp_r(E)} \tag{2.1} \]
and

\[ P_E(M) = M - P_{E}^{\bot}(M). \]  

(2.2)

For a matrix \( M \), let \( \|M\|_F \) be the Frobenius norm and \( T_\Sigma \in \mathbb{R}^{d_1 \times d_2} \) is an operator such that \( T_\Sigma(M) = \text{vec}^{-1}(\Sigma^{\frac{1}{2}}\text{vec}(M)) \), where \( \text{vec}(\cdot) : \mathbb{R}^{d_1 \times d_2} \to \mathbb{R}^{d_1d_2} \) is the operator that unfolds a matrix to a vector.

**Definition 2.1** (MRE Condition). The matrix \( \Sigma \) is said to satisfy the matrix restricted eigenvalue condition \( \text{MRE}(r,c_0,\kappa) \) with a positive integer \( r \) and positive values \( c_0 \) and \( \kappa \), if

\[ \kappa\|P_E(M)\|_F \leq \|T_\Sigma(M)\|_F \]

for any matrix \( M \in \mathbb{R}^{d_1 \times d_2} \) and any matrix \( E \in \mathbb{R}^{d_1 \times d_2} \) with rank(\( E \)) \( \leq r \) such that

\[ \|P_{E}^{\bot}(M)\|_* \leq c_0\|P_E(M)\|_* . \]  

(2.4)

Next, we introduce an \( L \)-subGaussian random vector, which appeared in [52, 53, 65] and others.

**Definition 2.2** (\( L \)-subGaussian random vector). A random vector \( x \in \mathbb{R}^d \) with the mean \( \mathbb{E}x = \mu \) is said to be an \( L \)-subGaussian if for every \( v \in \mathbb{R}^d \) and every \( p \geq 2 \),

\[ \|\langle x - \mu, v \rangle\|_{\psi_2} \leq L (\mathbb{E}|\langle x - \mu, v \rangle|^2)^{\frac{1}{p}} , \]

(2.5)

where the norm \( \| \cdot \|_{\psi_2} \) is defined in Definition 2.3.

**Remark 2.1.** See Remark 1.3 of [53] for details on the difference between a subGaussian random variable and an \( L \)-subGaussian random variable. The \( L \)-subGaussian property enables us to use the Generic Chaining [64].

**Definition 2.3** (\( \psi_2 \)-norm). Let \( f \) be defined on the same probability space. Set

\[ \|f\|_{\psi_2} := \inf \left\{ \eta > 0 : \mathbb{E}\exp(f/\eta)^2 \leq 2 \right\} < \infty. \]

(2.6)

For adversarial matrix compressed sensing, we use the following assumption.

**Assumption 2.1.** Assume that

(i) \( \{\text{vec}(Z_i)\}_{i=1}^n \) is a sequence with independent random vectors drawn from \( L \)-subGaussian distributions, where \( \Sigma^{\frac{1}{2}}\text{vec}(Z_i) = \text{vec}(X_i) \) and \( \Sigma := \mathbb{E}\text{vec}(X_i)\text{vec}(X_i)^{\top} \) and \( \Sigma \) satisfies \( \text{MRE}(r,c_0,\kappa) \).

(ii) \( \{\xi_i\}_{i=1}^n \) is a sequence with independent random variables from a distribution whose absolute moments is bounded by \( \sigma \),

(iii) for \( i = 1, \cdots, n \), \( \mathbb{E}h \left( \frac{\xi_i}{\lambda_0\sqrt{n}} \right) \times X_i \) is the zero matrix.

Under Assumption 2.1, we have the following theorem.

**Theorem 2.1.** Suppose that Assumption 2.1 holds. Consider the optimization problem (1.12). Suppose that \( \lambda_0\sqrt{n} \geq 72L^4\sigma \) and \( \lambda_* = \text{mcs} \times \lambda_0\sqrt{n} \times L \times r_{\lambda_*} \), where

\[ r_{\lambda_*} = \rho \sqrt{\frac{d_1 + d_2}{n}} + \frac{1}{c_1\sqrt{r}} \frac{1 + \sqrt{\log(1/\delta)}}{\sqrt{n}} + \frac{1}{c_1\sqrt{r}} \frac{\sqrt{\log n}}{\delta} \]  

(2.7)

and where \( c_{\text{mcs}} \) is some numerical constant and \( c_1 = \frac{n+1}{\kappa} \) and \( \rho^2 \) is the maximum diagonal element of \( \Sigma \). Let

\[ r_{\text{mcs}} = c'_{\text{mcs}} \times \lambda_0\sqrt{n} \times L \times c_1\sqrt{r} \times r_{\lambda_*}, \]  

(2.8)

where \( c'_{\text{mcs}} \) is some numerical constant, and suppose \( r_{\text{mcs}} \leq \frac{1}{4\sqrt{3}L^2} \). Then, the optimal solution \( \hat{B} \) satisfies \( \|T_\Sigma(\hat{B} - B^*)\|_F < r_{\text{mcs}} \) with probability at least \( 1 - 3\delta \) for \( 0 < \delta < 1/7 \).
Remark 2.2. When $\alpha = 0$, in other words, there are no adversarial noises, the error bound (2.8) gets

$$O \left( c_\kappa \rho \sqrt{\frac{d_1 + d_2}{n}} + \frac{1 + \sqrt{\log(1/\delta)}}{\sqrt{n}} \right). \quad (2.9)$$

[55] considered no adversarial case with the random noises drawn from a Gaussian distribution and obtained the error bound

$$C_\delta \left( c_\kappa \rho \sqrt{\frac{d_1 + d_2}{n}} \right), \quad (2.10)$$

where $C_\delta$ is some constant which depends on $\delta$. In our error bound, the term involving $\delta$ and the term involving $r$ is separated and the assumption of random noise is weakened from a Gaussian distribution to a distribution with a finite absolute moment.

Remark 2.3. As we mentioned in Introduction, [65] also considered the estimation of $B^\ast$ from model (1.10). The main term of the error bound of [65] is

$$O \left( L \sigma' \rho \sqrt{\frac{d_1 + d_2}{n}} + L \sigma' \frac{1}{\sqrt{n}} + L \sigma' \frac{a \log n}{n} \right), \quad (2.11)$$

where $\sigma'^2$ is the second moment of $\{\xi_i\}_{i=1}^n$. Our error bound is sharper from the perspective of convergence rate and we weaken the assumption on $\{\xi_i\}_{i=1}^n$ because [65] requires that $\{\xi_i\}_{i=1}^n$ is drawn from a subGaussian distribution. On the other hand, our method requires the information $o$ for the tuning parameter $\lambda$, however, the estimator of [65] does not. This factor is important for the practical use. In addition, the dependence of constants of the result of [65] is better than ours. For example, when $\lambda_o$ is set to $72 L^4 \sigma$, our error bounds depend on $L^5$, however, the one of [65] depends on $L$. This point would also be important for the practical use.

2.2. Adversarial estimation of sparse linear regression coefficient (adversarial lasso)

Before presenting our results, we introduce the restricted eigenvalue condition for the covariance matrix of random vector (the RE condition), which was introduced by [20]. As we refer to in section 2.1 and the MRE condition is an extension of the RE condition, the MRE condition contains RE condition. For completeness, we introduce the RE condition here.

Definition 2.4 (RE Condition). The matrix $\Sigma$ is said to satisfy the restricted eigenvalue condition $\text{RE}(s,c_0,\kappa)$ with a positive integer $s$ and positive values $c_0$ and $\kappa$, if

$$\kappa \|v_J\|_2 \leq \|\Sigma^2 v\|_2 \quad (2.12)$$

for any vector $v \in \mathbb{R}^d$ and for any set $J \subset \{1, \cdots, d\}$ with $\text{Card}(J) \leq s$ such that

$$\|v_J\|_1 \leq c_0 \|v_J\|_1. \quad (2.13)$$

For adversarial lasso, we use the following assumption.

Assumption 2.2. Assume that

(i) $\{z_i\}_{i=1}^n$ is a sequence with independent random vectors drawn from an $L$-subGaussian distribution, where $\Sigma^2 z_i = x_i$ and $\Sigma = \mathbb{E} x_i x_i^\top$ and $\Sigma$ satisfies $\text{RE}(r,c_0,\kappa)$.

(ii) $\{\xi_i\}_{i=1}^n$ is a sequence with independent random variables from a distribution whose absolute moment is bounded by $\sigma$.

(iii) for $i = 1, \cdots, n$, $\mathbb{E} h \left( \frac{\xi_i}{\lambda_o \sqrt{n}} \right) x_i$ is the zero vector.
For a vector $v$, let $\|v\|_0$ be the number of non-zero entries of $v$. Under Assumption 2.2, we have the following theorem.

**Theorem 2.2.** Suppose that Assumption 2.2 holds. Consider the optimization problem (1.7). Assume that $\lambda_0 \sqrt{n} \geq 72L^4\sigma$ and $\lambda_* = c_{lasso} \times \lambda_0 \sqrt{n} \times L \times r_{\lambda_*}$, where

$$
r_{\lambda_*} = \rho \frac{\sqrt{s} \log(d/s)}{\sqrt{n}} + \frac{1}{c_{\kappa}} \frac{1 + \sqrt{\log(1/\delta)}}{\sqrt{n}} + \frac{1}{c_{\kappa} \sqrt{s}} \frac{\log(n)}{n},
$$

and $c_{lasso}$ is some numerical constant, $c_{\kappa} = \frac{c_{\kappa}^2 + 1}{\kappa}$, $s = \|\beta^*\|_0$ and $\rho^2$ is the maximum diagonal element of $\Sigma$. Let

$$
r_{lasso} = c_{lasso}' \times \lambda_0 \sqrt{n} \times L \times c_{\kappa} \sqrt{r} \times r_{\lambda_*},
$$

where $c_{lasso}'$ is some numerical constant, and suppose $r_{lasso} \leq \frac{1}{4\sqrt{3}L^2}$. Then, the optimal solution $\hat{\beta}$ satisfies $\|\Sigma^{-1}(\hat{\beta} - \beta^*)\|_2 < r_{lasso}$ with probability at least $1 - 3\delta$ for $0 < \delta < 1/7$.

**Remark 2.4.** When $o = 0$, in other words, there are no adversarial noises, the error bound (2.15) gets

$$
O \left( c_{\kappa} \rho \sqrt{s} \frac{\log(d/s)}{\sqrt{n}} + \frac{1 + \sqrt{\log(1/\delta)}}{\sqrt{n}} \right).
$$

[59] considered no adversarial case with the random noises drawn from a Gaussian distribution and obtained the error bound

$$
C_\delta \left( c_{\kappa} \rho \sqrt{s} \frac{\log(d/s)}{\sqrt{n}} \right),
$$

where $C_\delta$ is some constant which depends on $\delta$. In our error bound, the term involving $\delta$ is separated from the term involving $s$.

**Remark 2.5.** As we mentioned in Introduction, [65] also considered the estimation of $\beta^*$ from model (1.4). The main term of the error bound of [65] is

$$
O \left( L\sigma^2 \sqrt{s} \frac{\log(d/s)}{\sqrt{n}} + L\sigma \frac{1 + \sqrt{\log(1/\delta)}}{\sqrt{n}} + L\sigma^2 \frac{\log(n)}{n} \right),
$$

where $\sigma^2$ is the second moment of $\{\xi_i\}_{i=1}^n$. As in the case of matrix compressed sensing, our error bound is sharper from the perspective of the convergence rate and we weaken the assumption on $\{\xi_i\}_{i=1}^n$ because [65] requires that $\{\xi_i\}_{i=1}^n$ is drawn from a subGaussian distribution. On the other hand, our method requires the information $o$ for the tuning parameter $\lambda_*$, however, the estimator of [65] does not. This factor is important for practical use. In addition, the dependence of constants of the result of [65] is better than ours. For example, when $\lambda_0$ is set to $72L^4\sigma$, our error bounds depend on $L^5$, however, the one of [65] depends on $L$. This point would also be important for practical use.

### 2.3. Adversarial matrix completion

Before presenting the observation model, we introduce a sequence of ‘mask’ matrix, $\{E_i\}_{i=1}^n$. Assume that $\{E_i\}_{i=1}^n$ lies in

$$
\left\{ e_k(d_1)e_l(d_2)^T \mid 1 \leq k \leq d_1, 1 \leq l \leq d_2 \right\},
$$

where $d_1$ and $d_2$ are the dimensions of the matrices.
where $e_k(d_1)$ is the $d_1$-dimensional $k$-th unit vector and $e_l(d_2)$ is the $d_2$-dimensional $l$-th unit vector, i.e.,

$$e_k(d_1) = (0, \cdots, 0, 1, 0, \cdots, 0)^\top \in \mathbb{R}^{d_1},$$

$$e_l(d_2) = (0, \cdots, 0, 1, 0, \cdots, 0)^\top \in \mathbb{R}^{d_2}.$$  \hspace{1cm} (2.20)

Let $E_{ikl}$ be the $(k,l)$-component of $E_i$. Let $d_{mc} = \sqrt{d_1d_2}$. For matrix completion, we consider the following observation model:

$$y_i = \langle X_i, B^* \rangle + \xi_i + \sqrt{n\theta_i^*}, \quad i = 1, \cdots, n,$$

where $X_i = d_{mc}\varepsilon_iE_i$, where $\varepsilon_i \in \{-1, +1\}$ is a random sign and $\{E_i\}_{i=1}^n$ is a sequence with independent random variables such that

$$P[E_{ikl} = 1] = 1/d_{mc}^2.$$  \hspace{1cm} (2.21)

For a matrix $M$, let $\|M\|_\infty$ be the element-wise $\ell_\infty$-norm. We introduce the spikiness condition used in [56, 65, 34] and so on. For a nonzero matrix $M$, let

$$\alpha(M) := d_{mc} \frac{\|M\|_\infty}{\|M\|_F}. \hspace{1cm} (2.22)$$

The spikiness condition is given by the following definition.

**Definition 2.5.** The spikiness condition is given by $\|B^*\|_F \leq 1$, or $\|B^*\|_\infty \leq \alpha(B^*)/d_{mc}$.

**Remark 2.6.** Roughly speaking, the spikiness condition requires the true matrix $B^*$ does not have overly large elements.

Using the spikiness condition, we consider the optimal solution given by

$$\hat{B} \in \arg \min_{\|B\|_\infty \leq \alpha^*/d_{mc}} \text{obj}_H(B)$$

where we abbreviate $\alpha(B^*)$ to $\alpha^*$.

We derive two results about adversarial matrix completion. The assumptions of $\{\xi_i\}_{i=1}^n$ used for each result are different. The first assumption is the following.

**Assumption 2.3** (Assumption for adversarial matrix completion when random noises are drawn from a heavy-tailed distribution). We assume that

(i) Let $X_i = d_{mc}\varepsilon_iE_i$, where $\varepsilon_i \in \{-1, +1\}$ is a random sign and $\{E_i\}_{i=1}^n$ is a sequence with independent random variables such that

$$P[E_{ikl} = 1] = 1/d_{mc}^2.$$  \hspace{1cm} (2.26)

(ii) $\{\xi_i\}_{i=1}^n$ is a sequence with independent random variables from a distribution whose $\alpha$-th absolute moment is bounded by $\sigma_{\xi,\alpha}^2$; with $\alpha \geq 2$.

(iii) for $i = 1, \cdots, n$, $\mathbb{E}h \left( \frac{\xi_i}{\lambda_{\alpha\sqrt{n}}} \right) X_i$ is the zero matrix.

Under Assumption 2.3, we have the following Theorem.
Theorem 2.3. Suppose that Assumption 2.3 holds. Consider the optimization problem (2.25). Suppose
\[ \lambda_o \sqrt{n} \geq 2\sigma_{\xi, \alpha} \min \left\{ \left( \frac{n}{o} \right)^{\frac{1}{\alpha+1}}, \left( \frac{n}{rd_{mc} \log d_{mc}} \right)^{\frac{1}{\alpha}} \right\} \]  
and \( \lambda_* = c_{mc1} \times \frac{1}{\sqrt{n}} \times r_{\lambda_*} \), where
\[ r_{\lambda_*} = \sigma \xi \sqrt{\frac{rd_{mc}(\log d_{mc} + \log(1/\delta))}{n}} + \lambda_o \sqrt{\frac{rd_{mc}(\log d_{mc} + \log(1/\delta))}{n}} + \lambda_o \sqrt{\frac{n}{o}}. \]  
and \( c_{mc1} \) is some numerical constant. Let
\[ r_{mc1} = c_{mc1}' \times \alpha^* \times \left( r_{\lambda_*} + \alpha^* \sqrt{\frac{rd_{mc}(\log d_{mc} + \log(1/\delta))}{n}} + \sqrt{\frac{d_{mc} \log d_{mc}}{n}} + \alpha^* \left( \frac{\sigma_{\xi, \alpha}}{n} \right)^{\frac{1}{2(1+\alpha)}} \right), \]  
where \( c_{mc1}' \) is a numerical constant. Then, the optimal solution \( \hat{B} \) satisfies
\[ \| \hat{B} - B^* \|_F < r_{mc1} \]  
with probability at least \( 1 - 2\delta \).

Remark 2.7. When \( o = 0 \), in other words, there are no adversarial noises, the error bound (2.29) gets
\[ O \left( \sqrt{\frac{rd_{mc}(\log d_{mc} + \log(1/\delta))}{n}} \right), \]  
and this coincides with the bound in [56] from the perspective of convergence rate, which considered the case of no adversarial noise and Gaussian random noises. Our result is an extension of [56] because we consider adversarial noises and distribution with a finite \( \alpha \)-th absolute moment with \( \alpha \geq 2 \), which includes a Gaussian distribution.

Remark 2.8. We make a detailed consideration of the following term in (2.29)
\[ \lambda_o \sqrt{\frac{d_{mc}(\log d_{mc} + \log(1/\delta))}{n}}, \]  
when \( \lambda_o \sqrt{n} \) is equal to the lower bound \( 2\sigma_{\xi, \alpha} \min \left\{ \left( \frac{n}{o} \right)^{\frac{1}{\alpha+1}}, \left( \frac{n}{rd_{mc} \log d_{mc}} \right)^{\frac{1}{\alpha}} \right\} \).

For example, when \( \alpha = 2 \), which is the lower bound, we have
\[ 2\sigma_{\xi, 2} \sqrt{\frac{rd_{mc}(\log d_{mc} + \log(1/\delta))}{n}} \times \sqrt{d_{mc}(\log d_{mc} + \log(1/\delta))} \]  
when \( \min \left\{ \left( \frac{n}{o} \right)^{\frac{1}{1}}, \left( \frac{n}{rd_{mc} \log d_{mc}} \right)^{\frac{1}{2}} \right\} = \left( \frac{n}{o} \right)^{\frac{1}{2}} \). On the other hand, when \( \alpha \to \infty \), (2.31) limits to larger
\[ 2\sigma_{\xi, \alpha} \sqrt{\frac{rd_{mc}(\log d_{mc} + \log(1/\delta))}{n}} \]  
Consequently, when \( \alpha \) can be assumed to be large, the result would be acceptable in the view of the convergence rate.

Remark 2.9. We make a detailed consideration of the term \( \alpha^* \sqrt{\frac{\sigma_{\xi, \alpha}}{n}} \) in (2.29). Assume \( \alpha = 2 \) and \( \lambda_o \sqrt{n} = 2\sigma_{\xi, \alpha} \left( \frac{n}{o} \right)^{\frac{1}{1+\alpha}} \). We have
\[ \alpha^* \sqrt{2\sigma_{\xi, \alpha} \left( \frac{n}{o} \right)^{\frac{1}{2}}}. \]
On the other hand, when $\alpha \to \infty$ and we choose $\lambda_o \sqrt{n} = 2\sigma_{\xi,\alpha} \left( \frac{n}{\alpha} \right)^{\frac{1}{\alpha}}$, the term $\alpha^* \sqrt{\lambda_o \sqrt{n} \frac{o}{n}}$ approaches to

$$\alpha^* \sqrt{2\sigma_{\xi,\alpha} \left( \frac{o}{n} \right)^{\frac{1}{\alpha}}}.$$  \hspace{1cm} (2.35)

Before presenting the second assumption, we introduce a subWeibull distribution $[44, 69]$.

**Definition 2.6** (subWeibull random variable of order $\alpha$). Define $\sigma_{x,\psi,\alpha}$ for the random variable $x$ with $E_x = 0$ as

$$\sigma_{x,\psi,\alpha} := \inf \left\{ \eta > 0 : E \psi_{\alpha}(x) \leq 1 \right\} < \infty,$$

where $\psi_{\alpha}(x) = \exp \left( \frac{|x|^\alpha}{\eta^\alpha} \right) - 1$. The random variable $x$ with $E_x = 0$ is said to be a subWeibull random variable of order $\alpha$ if

$$\mathbb{P}(|x| \geq t) \leq 2 \exp \left( -\frac{t^\alpha}{\sigma_{x,\psi,\alpha}} \right).$$  \hspace{1cm} (2.36)

**Remark 2.10.** We see that a subWeibull random variable of order 2 is a subGaussian random variable and a subWeibull random variable of order 1 is a subexponential random variable.

The second assumption is the following.

**Assumption 2.4** (Assumption for adversarial matrix completion when random noise drawn from a subWeibull distribution). We assume (i) and (iii) in Assumption 2.3 and

(ii) $\{\xi_i\}_{i=1}^n$ is a sequence with independent random variables from a subWeibull random variables of order $\alpha$ with $\alpha \leq 2$. In addition, we denote the second moment of $\xi_i$ as $\sigma_i^2$ for $i = 1, \cdots, n$.

Under Assumption 2.4, we have the following Theorem

**Theorem 2.4.** Suppose that Assumption 2.4 holds. Consider the optimization problem (2.25).

Suppose

$$\lambda_o \sqrt{n} \geq 2\sigma_{\xi,\psi,\alpha} \min \left\{ \log \frac{n}{\alpha}, \log \frac{n}{r d_{mc} \log d_{mc}}, \right\}$$  \hspace{1cm} (2.37)

and $\lambda_* = c_{mc2} \times r_{\xi,\alpha}$, where

$$r_{\lambda,\alpha} = \sigma_{\xi} \sqrt{\frac{r d_{mc} \left( \log d_{mc} + \log(1/\delta) \right)}{n}} + \lambda_o \sqrt{\frac{d_{mc} \log d_{mc} + \log(1/\delta)}{\sqrt{n}}} + \sqrt{\lambda_o \sqrt{n} \frac{o}{n}},$$  \hspace{1cm} (2.38)

where $c_{mc2}$ is some numerical constant. Let

$$r_{mc2} = c'_{mc2} \times \alpha^* \times \left( r_{\lambda,\alpha} + \alpha^* \sqrt{\frac{r d_{mc} \left( \log d_{mc} + \log(1/\delta) \right)}{n}} + \sqrt{\frac{d_{mc} \log d_{mc}}{n}} + \alpha^* \sqrt{\frac{o}{n}} \right),$$  \hspace{1cm} (2.39)

where $c'_{mc2}$ is some numerical constant. Then, the optimal solution $\hat{B}$ satisfies $\| \hat{B} - B^* \|_F < r_0$ with probability at least $1 - 2\delta$.

**Remark 2.11.** We make detailed consideration of the term condition in (2.39).

$$\lambda_o \sqrt{r d_{mc} \left( \log d_{mc} + \log(1/\delta) \right)} \sqrt{n}$$  \hspace{1cm} (2.40)

when $\lambda_o \sqrt{n} = 2\sigma_{\xi,\psi,\alpha} \min \left\{ \log \frac{n}{\alpha}, \log \frac{n}{r d_{mc} \log d_{mc}} \right\}$.
For example, when the random noises drawn from subGaussian distribution ($\alpha = 2$), we have

$$\sqrt{\frac{rd_{mc}(\log d_{mc} + \log(1/\delta))}{n}} \times \sqrt{\frac{d_{mc}(\log d_{mc} + \log(1/\delta))}{n}} \sqrt{\log \frac{n}{\delta}}$$

or

$$\sqrt{\frac{rd_{mc}(\log d_{mc} + \log(1/\delta))}{n}} \times \sqrt{\frac{(\log d_{mc} + \log(1/\delta))}{n}} \sqrt{\log \frac{n}{rd_{mc} \log d_{mc}}}$$

and these term become small when $\sqrt{\frac{d_{mc}(\log d_{mc} + \log(1/\delta))}{n}} \sqrt{\log \frac{n}{\delta}} \leq 1$ and this condition would be not so strong.

**Remark 2.12.** We make detailed consideration of the following term $\alpha^* \sqrt{\lambda_o \sqrt{n} \frac{a}{n}}$ in (2.39). Assume $\alpha = 2$ and $\lambda_o \sqrt{n} = 2\sigma_{\xi, \psi_2} \sqrt{\log \frac{n}{o}}$. We have

$$2\alpha^* \sigma_{\xi, \psi_2} \sqrt{\frac{a}{n}} \sqrt{\log \frac{n}{o}}$$

**Remark 2.13.** [65] also considered the estimation of $B^*$ from model (1.10) under the condition of matrix completion. The main term of error bound of [65] is

$$\sqrt{\|\hat{B} - B^*\|^2_2 + \|\hat{\theta} - \theta^*\|^2_2} = O \left( (a^* + \sigma_{\xi, \psi_2}) \sqrt{\frac{d_{mc}(\log d_{mc} + \log(1/\delta))}{n}} + (a^* + \sigma_{\xi}) \sqrt{\phi \log \frac{n}{\delta}} \right).$$

We note that [65] considered non-uniform sampling (the mask matrix is slightly different), however, (2.44) is the result of assuming uniform sampling from (i) in Assumption 2.4. Our error bound is sharper from the perspective of the convergence rate when $\lambda_o$ is chosen to be equal $2\sigma_{\xi, \psi_2} \sqrt{\log \frac{n}{o}}$. We also weaken the assumption on $\{\xi_i\}_{i=1}^n$ because [65] requires that $\{\xi_i\}_{i=1}^n$ is drawn from a subGaussian distribution. As in the case of matrix compressed sensing, on the other hand, our method requires the information $o$ for the tuning parameters, however, the estimator of [65] does not. The dependence of constants of the result of [65] is different from ours. These points would be important for practical use.

3. Related works

3.1. Recent development of robust estimation

One of the recent lines of research on robust estimation in the presence of outliers was triggered by the epoch-making study of [11], which considered robust estimation of mean and various types of covariance matrices under Huber’s contamination. Since then, [11] has been followed by [24], [45] and many other papers. These papers dealt with robust mean estimation or covariance matrix estimation [25, 43, 26, 15, 17, 16, 29, 46, 36, 58, 22, 50, 21], robust PCA [2, 27], robust regression [28, 1, 47, 35] and so on. These studies are mainly interested in deriving sharp error bounds, deriving learning limits for each problem and reduction of computational complexity.

3.2. Adversarial matrix compressed sensing and adversarial lasso

In Section 3.2, we refer, topic by topic, the relevant papers that are not referred to in Sections 1 and 2.
3.2.1. Sparse coefficient estimation under existence of outliers

[13] and [47] considered linear regression under contamination on both \{y_i\}_{i=1}^{n} and \{x_i\}_{i=1}^{n} by adversarial noise. [13] provided pioneering research, however, their error bound is not sharp. [47] derived a relatively sharper error bound compared to [13]; however, their method requires the information about the \ell_2 norm of the length true coefficient vector \beta^*, which is not required for our method. [35] considered various regression problems with \{y_i\}_{i=1}^{n} and \{x_i\}_{i=1}^{n} sampled from normal distributions with Huber’s contamination. [35] derived the learning limit and error bound which coincides to the limit up to a constant factor. [35] applied ‘Tukey’s half-space depth’ and consumed exponential computational complexity.

3.2.2. Sparse estimation under heavy-tailed distribution

[63] dealt with linear regression problem when the random noises are drawn from a heavy-tailed distribution. [34] considered regression problems including matrix compressed sensing, lasso, matrix completion and reduced-rank regression with heavy-tailed covariates and random noises, proposing a new ‘shrinkage’ estimator. [19] considered linear regression problem with heavy-tailed random noises contaminated by some kinds of outliers.

3.2.3. SubGaussian estimator

As we refer to in Section 1, \delta, probability of success, appears in our error bounds like \sqrt{\log(1/\delta)/n}. Estimators who have error bounds like this are called ‘subGaussian estimators’. For example, [23], [54], [49], [19], [17], [46], [35], [50] constructed sub-Gaussian estimators which have some kinds of robustness for outliers or heavy-tailed distributions.

3.2.4. Dependency between \{x_i\}_{i=1}^{n} and \{\xi_i\}_{i=1}^{n}

Our methods and methods of [65] do not require the independence of \{x_i\}_{i=1}^{n} and \{\xi_i\}_{i=1}^{n}. Instead of the independence, our method requires \E h\left(\frac{\xi_i}{\sqrt{n}}\right)x_i = 0, and methods of [65] requires \E \xi_i x_i = 0. Previous works dealt with robust linear regression [28, 1, 47, 35] under the independence of covariates and random noises.

3.3. Adversarial matrix completion

Matrix completion was considered in [56] and [42]. Since then, and numerous works ([60], [39], [7] and so on) have been conducted. The adversarial matrix completion was considered by [65] with subGaussian noises. The error bound about adversarial noise term of the result of [65] is of the order \sqrt{\frac{\log(1/\delta)}{n}}, however, our result is slightly sharper as we stated in Remark 2.12. However, as in the case of adversarial lasso, the method of [65] does not require the information \o for the tuning parameter \lambda, although our method does.

Our method also works when the random noises are sampled from a heavy-tailed distribution, and we find out that our error bound about the error term varies according to the thickness of the error distribution. In the case of adversarial matrix compressed sensing or adversarial lasso, this phenomenon does not be confirmed. [30], [34] considered non-adversarial heavy-tailed noises. Their error bound is \(O(\sqrt{\frac{\log(1/\delta)}{n}})\) and requires no additional condition such that \(\lambda_0 \frac{\sqrt{\log(1/\delta)}}{n} \leq 1\), which is required in our method. Ensuring robustness against both heavy-tailed noise and adversarial noise with weaker additional conditions is left as a future study.

On the other hand, although we consider the matrix completion as a partial problem of the trace regression in the present paper, there is another formulation of matrix completion, which is
based on the Bernoulli model ([8], [10], [40], [14], [37], [18]) and so on. Matrix completion based on the Bernoulli model is out of scopes of the present paper, however, we give a brief introduction. In matrix completion based on the Bernoulli model, each entry of \( B^* + N \), where \( N \) is a matrix of random noise, is observed independently of the other entries with probability \( n/(d_1 d_2) \). In other words, assume \( S_{ij} \) are i.i.d. Bernoulli random variables of parameter \( p \) which is independent of \( N \), and we can observe \( Y \) with

\[
Y_{ij} = S_{ij}(B^*_{ij} + N_{ij}) \quad 1 \leq i \leq d_1, 1 \leq j \leq d_2.
\]

As pointed out by [9], the major difference between matrix completion as a trace regression and matrix completion based on the Bernoulli model is that matrix completion as a trace regression admit multiple sampling of each entry, however, in the case of matrix completion based on the Bernoulli model, each entry is sampled at most once.

### 3.4. Other topics

There are other topics that the present paper does not deal with; e.g. reduced-rank regression (multi-task regression) [38, 67, 61]. Both in theory and in applications, it is important to deepen the robustness aspect of these topics. It will be a future subject.

### 4. Main theorem, key propositions, lemmas and corollaries

We introduce the main theorem, key propositions, lemmas and corollaries. The adversarial lasso is a special case of adversarial matrix compressed sensing because, like the argument of Section 2.2 of [34], linear regression is a special case of trace regression, where \( B^* \) and \( X_i \), \( i = 1, \ldots, n \) are diagonal matrices. We note that for diagonal matrix \( M \in \mathbb{R}^{d \times d} \), we see that \( \|X_i\|_{\infty} = \|\text{diag}(X_i)\|_{\infty} \) and \( \|X_i\|_* = \|\text{diag}(X_i)\|_1 \). Therefore, in Section 4, we discuss the main theorem, adversarial matrix compressed sensing and adversarial matrix completion. The value of the numerical constant \( C \) shall be allowed to change from line to line.

#### 4.1. Main theorem

We introduce our main theorem in an informal style. The precise statements is seen in Theorem B.1.

**Theorem 4.1.** Consider the optimization problem (1.12). Suppose that \( \{\xi_i, X_i\}_{i=1}^n \) and \( \lambda_*, \lambda_0 \) satisfies the following inequalities:

\[
\lambda_0 \sum_{i=1}^n \frac{h}{\sqrt{n}} \left( \frac{\xi_i}{\lambda_0 \sqrt{n}} \right) \langle X_i, M \rangle \leq a_F r_{a,F} \|T(\Sigma(M))\|_F + a_r r_{a,s} \|M\|_s, \quad (4.1)
\]

\[
\sum_{i=1}^n \frac{\lambda_0}{\sqrt{n}} u_i \langle X_i, M \rangle \leq b_F r_{b,F} \|T(\Sigma(M))\|_F + b_r r_{b,s} \|M\|_s, \quad (4.2)
\]

\[
c_1 \|T(\Sigma(M))\|_F^2 - c_2 r_{c,F} \|T(\Sigma(M))\|_F - c_3 r_z \leq \lambda^2_0 \sum_{i=1}^n \left\{ -h \left( \frac{\xi_i - \langle X_i, M \rangle}{\lambda_0 \sqrt{n}} \right) + h \left( \frac{\xi_i}{\lambda_0 \sqrt{n}} \right) \right\} \frac{\langle X_i, M \rangle}{\lambda_0 \sqrt{n}} \quad (4.3)
\]

where \( c_1 > 0, r_{a,F}, r_{a,s}, r_{b,F}, r_{b,s}, a_F, a_r, b_F, b_r, c_1, r_{c,F}, r_z \geq 0 \) are some numbers and \( u = (u_1, \ldots, u_n)^\top \) is some \( \alpha \)-sparse vector such that \( \|u\|_{\infty} \leq c \) with some numerical constant \( c \) and \( M \in \mathbb{R}^{d_1 \times d_2} \) is some matrix such that \( \|M\|_F = r_0 \) for some number \( r_0 \) satisfying (4.6). Suppose that \( \lambda_* \) satisfy

\[
\lambda_* - C_* > 0, \quad (4.4)
\]
where
\[ C_s = \frac{\alpha F r_{a,F} + \sqrt{2} b_F r_{b,F}}{c_\kappa \sqrt{r}} + (a_s r_{a,*} + \sqrt{2} b_s r_{b,*}). \] (4.5)

Then, for the number \( r_0 \) such that
\[ c_2 r_{c,F} + C_{\lambda_*} + \frac{\sqrt{c_1 c_2 r_c}}{c_1} < r_0, \] (4.6)

where
\[ C_{\lambda_*} = (\alpha F r_{a,F} + \sqrt{2} b_F r_{b,F}) + (a_s r_{a,*} + \sqrt{2} b_s r_{b,*}) c_\kappa \sqrt{r} + \lambda_* c_\kappa \sqrt{r}, \] (4.7)

the optimal solution \( \hat{B} \) satisfies
\[ \| T_\Sigma (\hat{B} - B^*) \|_F \leq r_0. \] (4.8)

In the remaining of Section 4, we introduce some inequalities to prove (4.1)-(4.3) are satisfied with high probability and with appropriate value of \( c_1, r_{a,F}, r_{a,*}, r_{b,F}, r_{b,*}, a_F, a_s, b_F, b_s, c_2, r_{F,c}, r_c \) under the Assumptions 2.1, 2.2, 2.3 or 2.4.

### 4.2. Adversarial matrix compressed sensing

In Section 4.2, suppose that Assumption 2.1 holds.

**Lemma 4.1.** For \( 0 < \delta < 1/7 \) and for any matrix \( M \in \mathbb{R}^{d_1 \times d_2} \), with probability at least \( 1 - \delta \), we have
\[ \left| \frac{1}{n} \sum_{i=1}^{n} h \left( \frac{\xi_i}{\lambda_o \sqrt{n}} \right) \langle X_i, M \rangle \right| \leq CL \left\{ \rho \sqrt{\frac{d_1 + d_2}{n}} \| M \|_* + \sqrt{\frac{\log(1/\delta)}{n}} \| T_\Sigma (M) \|_F \right\}. \] (4.9)

**Remark 4.1.** [65] derived an upper bound of
\[ \left| \frac{1}{n} \sum_{i=1}^{n} \xi_i \langle X_i, M \rangle \right|. \] (4.10)

The L.H.S. of (4.9) is an extended one because \( \{\xi_i\}_{i=1}^{n} \) is `wrapped’ by the differential of the Huber loss function. Lemma 4.9 enables us to deal with a heavy-tailed \( \{\xi_i\}_{i=1}^{n} \).

**Corollary 4.1** (Corollary of Proposition 4 of [65]). We have for any \( o \)-sparse vector \( u \in \mathbb{R}^n \) and any matrix \( M \in \mathbb{R}^{d_1 \times d_2} \),
\[ \left| \sum_{i=1}^{n} u_i \frac{1}{\sqrt{n}} \langle X_i, M \rangle \right| \leq CL \left( 1 + \sqrt{\frac{\log(1/\delta)}{n}} \right) \| T_\Sigma (M) \|_F + \rho \sqrt{\frac{d_1 + d_2}{n}} \| M \|_* + \sqrt{\frac{\rho \log n}{o}} \| T_\Sigma (M) \|_F \| u \|_2 \] (4.11)

with probability at least \( 1 - \delta \).

**Remark 4.2.** Corollary 4.1 is derived by combining Proposition 4 of [65] and Remark 4 of [20]. Proposition 4 of [65] also plays an important role in both [65] and our analysis, however the roles in [65] and our analysis are different because we analyse (1.12) and [65] analyses (1.11).

**Proposition 4.1.** Let
\[ R_{mcs} = \{ \Theta \in \mathbb{R}^{d_1 \times d_2} \mid \| \Theta \|_* \leq c_\kappa \| T_\Sigma (\Theta) \|_F, \| T_\Sigma (\Theta) \|_F = r_{mcs} \}, \] (4.12)
where $r_{\text{mcs}}$ is a positive number such that $r_{\text{mcs}} \leq \frac{1}{4\sqrt{3L}}$. Assume that $\lambda_{0}\sqrt{n} \geq 72L^{4}\sigma$. Then, with probability at least $1 - \delta$, we have

$$
\inf_{\Theta \in \mathcal{R}_{\text{mcs}}} \left[ \sum_{i=1}^{n} \lambda_{0}^{2} \left\{ -h \left( \frac{\xi_{i} + (X_{i}, \Theta)}{\lambda_{0}\sqrt{n}} \right) + h \left( \frac{\xi_{i}}{\lambda_{0}\sqrt{n}} \right) \right\} \right] \geq \frac{1}{3} \|T_{\Sigma}(\Theta)\|_{F}^{2} - C \left( L\rho_{c} \sqrt{r} \frac{d_{1} + d_{2}}{n} + \sqrt{8\log(1/\delta)} \right) \|T_{\Sigma}(\Theta)\|_{F} - 5 \log(1/\delta) n. \tag{4.13}
$$

Remark 4.3. Proposition 4.1 implies restricted strong convexity of the Huber loss. Essentially, the same techniques found as in [12] are used to prove Proposition 4.1.

4.3. Adversarial matrix completion

Lemma 4.2. Suppose that Assumption 2.3 or Assumption 2.4 holds. For $\delta > 0$, with probability at least $1 - \delta$, we have

$$
\left\| \frac{\lambda_{0}}{\sqrt{n}} \sum_{i=1}^{n} h \left( \frac{\xi_{i}}{\lambda_{0}\sqrt{n}} \right) X_{i} \right\|_{\text{op}} \leq C \left( \sigma \sqrt{\frac{d_{mc}(\log d_{mc} + \log(1/\delta))}{n}} + \frac{\lambda_{0} d_{mc}(\log d_{mc} + \log(1/\delta))}{\sqrt{n}} \right). \tag{4.14}
$$

Remark 4.4. [65] derived an upper bound of

$$
\frac{1}{n} \left| \sum_{i=1}^{n} \xi_{i}(X_{i}, M) \right|. \tag{4.15}
$$

The L.H.S. of (4.9) is an extended one because $\{\xi_{i}\}_{i=1}^{n}$ is ‘wrapped’ by the differential of the Huber loss function. Lemma 4.9 enables us to deal with a heavy-tailed $\{\xi_{i}\}_{i=1}^{n}$.

Lemma 4.3. Suppose that Assumption 2.3 or Assumption 2.4 holds. Assume that for any $M \in \mathbb{R}^{d_{1} \times d_{2}}$,

$$
\|M\|_{\infty} \leq c_{m} \frac{1}{d_{mc}} \|M\|_{F}. \tag{4.16}
$$

for some number $c_{m}$. Then, for any $o$-sparse vector $u \in \mathbb{R}^n$ such that $\|u\|_{2} \leq 2\sqrt{\delta}$, we have

$$
\left| \sum_{i=1}^{n} \frac{\lambda_{0}}{\sqrt{n}} u_{i}(X_{i}, M) \right| \leq c_{m} \lambda_{0} \sqrt{n} \frac{o}{n} \|M\|_{F}. \tag{4.17}
$$

Remark 4.5. Lemma 4.3 corresponds to Corollary 4.1 in adversarial matrix compressed sensing. In the proof of Theorem 2.3 and Theorem 2.4, different strategies are used depending on the magnitude of the spikiness of $B - B^{*}$ like [56] and [34]. Assumption (4.17) is derived when the spikiness of $B - B^{*}$ is sufficiently small.

Corollary 4.2. Suppose that Assumption 2.3 holds. Let

$$
\mathcal{R}_{mc} = \{ \Theta \in \mathbb{R}^{d_{1} \times d_{2}} | \|\Theta\|_{\ast} \leq C \sqrt{r} \|\Theta\|_{F}, \|\Theta\|_{F} = r_{mc} \}, \tag{4.18}
$$

where $r_{mc}$ is some number. Suppose

$$
\|\Theta\|_{\infty} \leq \frac{1}{12r_{mc}} \frac{1}{d_{mc}} \|\Theta\|_{F}, \tag{4.19}
$$

$$
\|\Theta\|_{\infty} \leq 2 \frac{\alpha}{d_{mc}}, \tag{4.20}
$$

where $r_{mc}$ is some number.
and \( \lambda_o \sqrt{n} \geq 2\sigma_{\xi, o} \min \left\{ \left( \frac{n}{o} \right)^{\frac{1}{n+1}}, \left( \frac{n}{r_{dmc} \log d_{mc}} \right)^{\frac{1}{2}} \right\} \). Then, with probability at least \( 1 - \delta \), we have

\[
\inf_{\Theta \in \mathcal{R}_{mc}} \left\{ \lambda_o^2 \sum_{i=1}^{n} \left\{ -h \left( \frac{\xi_i - (X_i, \Theta)}{\lambda_o \sqrt{n}} \right) + h \left( \frac{\xi_i}{\lambda_o \sqrt{n}} \right) \right\} \langle X_i, \Theta \rangle \right\} 
\geq \frac{1}{3} \| \Theta \|_F^2 - C \alpha^* \left( \sqrt{r_{dmc} \log d_{mc} \log(1/\delta) n} + \sqrt{r_{dmc} \log d_{mc} n} + \left( \frac{\sigma}{n} \right)^{\frac{1}{2(1+\alpha)}} \right) \| \Theta \|_F - C \frac{\log(1/\delta)}{n}.
\]

(4.21)

**Corollary 4.3.** Suppose that Assumption 2.4 holds. Let

\[ \mathcal{R}_{mc} = \left\{ L \in \mathbb{R}^{d_1 \times d_2} \mid \|\Theta\|_* \leq C \sqrt{r} \|\Theta\|_F, \|\Theta\|_F = r_{mc} \right\}. \]

(4.22)

Suppose

\[
\|\Theta\|_\infty \leq \frac{1}{12r_{mc} d_{mc}} \|\Theta\|_F, \quad (4.23)
\]

\[
\|\Theta\|_\infty \leq 2 \alpha^* \frac{n}{d_{mc}}, \quad (4.24)
\]

and \( \lambda_o \sqrt{n} \geq 2\sigma_{\xi, \psi, min} \left\{ \log \frac{n}{\sigma}, \log \frac{n}{r_{dmc} \log d_{mc}} \right\} \). Then, with probability at least \( 1 - \delta \), we have

\[
\inf_{\Theta \in \mathcal{R}_{mc}} \left\{ \lambda_o^2 \sum_{i=1}^{n} \left\{ -h \left( \frac{\xi_i - (X_i, \Theta)}{\lambda_o \sqrt{n}} \right) + h \left( \frac{\xi_i}{\lambda_o \sqrt{n}} \right) \right\} \langle X_i, \Theta \rangle \right\} 
\geq \frac{2}{3} \| \Theta \|_F^2 - C \alpha^* \left( \sqrt{r_{dmc} \log d_{mc} \log(1/\delta) n} + \sqrt{r_{dmc} \log d_{mc} n} + \sqrt{\frac{\sigma}{n}} \right) \| \Theta \|_F - C \frac{\log(1/\delta)}{n}.
\]

(4.25)

**Remark 4.6.** In adversarial matrix completion, \( \{\text{vec}(X_i)\}_{i=1}^{n} \) is not a sequence of L-subGaussian random vector and we need to prove Corollaries 4.2 and 4.3 in a different way from Proposition 4.1. However, (4.19) and (4.20) or (4.23) and (4.24) fulfill the role of L-subGaussianness in the proof of Corollaries 4.2 and 4.3. Like Lemma 4.3, (4.19), (4.23) are derived when the spikiness of \( B - B^* \) is sufficiently small. Assumptions (4.20) and (4.24) are derived from the constraint of the optimization problem (2.25).

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Appendix
Appendix A: Structure of the appendix

In this appendix, we give the proofs of Theorems 2.1-2.4. In Section B, we give deterministic arguments without the randomness of covariates and random noises. In Sections C, D and E, we introduce and prove some properties about covariates and random noises. In these sections, we assume the randomness of covariates and random noises. In Section F, we prove Theorem 2.1 using the results of Sections B and C. In Section G, we prove Theorem 2.2 using the results of Sections B and D. In Sections H and I, we prove Theorems 2.3 and 2.4 using the results of Sections B and E. The value of the numerical constant \( C \) shall be allowed to change from line to line.

Appendix B: Deterministic arguments

In Section B, we give deterministic arguments without the randomness of covariates and random noises. In Section B.3, we state our main theorem in the deterministic forms. In Section B.1 and B.2 are prepared for Sections B.3. First, we introduce a basic lemma related to convexity.

**Lemma B.1.** For a differentiable function \( f(x) \), we denote its derivative \( f'(x) \). For any differentiable and convex function \( f(x) \), we have

\[
(f'(a) - f'(b))(a - b) \geq 0. \tag{B.1}
\]

**Proof.** From the definition of the convexity, we have

\[
f(a) - f(b) \geq f'(b)(a - b) \quad \text{and} \quad f(b) - f(a) \geq f'(a)(b - a). \tag{B.2}
\]

From the inequalities above, we have

\[
0 \geq f'(b)(a - b) + f'(a)(b - a) = (f'(b) - f'(a))(a - b) \Rightarrow 0 \leq (f'(a) - f'(b))(a - b). \tag{B.3}
\]

**B.1. Matrix restricted eigenvalue condition**

When the matrix \( \Sigma \) satisfies MRE\((r,c_0,\kappa)\), the following lemma is obtained.

**Lemma B.2.** Suppose that \( \Sigma \) satisfies MRE\((r,c_0,\kappa)\). Then, we have

\[
\|M\|_* \leq c_\kappa \sqrt{r} \|T_\Sigma(M)\|_F, \tag{B.4}
\]

where \( c_\kappa := \frac{c_0 + 1}{\kappa} \), for any \( M \in \mathbb{R}^{d_1 \times d_2} \) satisfying (2.4) for any \( E \in \mathbb{R}^{d_1 \times d_2} \) with rank\((E) \leq r\).

**Proof.** We have

\[
\|M\|_* = \|P_E(M)\|_* + \|P_{\perp E}(M)\|_*,
\]

\[
\leq (c_0 + 1)\|P_E(M)\|_* \tag{a}
\]

\[
\leq (c_0 + 1)\sqrt{r}\|P_E(M)\|_F \tag{b}
\]

\[
\leq \frac{c_0 + 1}{\kappa}\sqrt{r}\|T_\Sigma(M)\|_F \tag{c}
\]

where (a) follows from (2.4), (b) follows from the fact that \( \|A\|_* \leq \sqrt{r}\|A\|_F \) for a matrix such that rank\((A) \leq r \) and (c) follows from (2.3). \[\square\]
B.2. Relation between $\| \hat{B} - B^* \|_*$ and $\| T_\Sigma ( \hat{B} - B^* ) \|_F$ under MRE condition

In Section B.2, we show the main proposition to obtain the main theorem. This proposition enables us to treat the adversarial matrix compressed sensing and adversarial matrix completion in a unified approach. Let

$$ h(t) = \frac{d}{dt} H(t) = \begin{cases} t & |t| \leq 1 \\ \text{sgn}(t) & |t| > 1 \end{cases} $$

(B.6)

For a matrix $M$, let $\| M \|_{op}$ be the operator norm.

**Proposition B.1.** Let $\Theta_\eta = \eta ( \hat{B} - B^* )$. Consider the optimization problem (1.12). Assume that $\Sigma$ satisfies MRE$(r, c_0, \kappa)$ and assume that for any $\alpha$-sparse vector $u = (u_1, \cdots, u_n)^T$ such that $\| u \|_\infty \leq c$, where $c$ is some numerical constant,

$$ \frac{\lambda_0}{\sqrt{n}} \sum_{i=1}^n h \left( \frac{\xi_i}{\lambda_0 \sqrt{n}} \right) \langle X_i, \Theta_\eta \rangle \leq a_F r_{a,F} \| T_\Sigma ( \Theta_\eta ) \|_F + a_s r_{a,s} \| \Theta_\eta \|_s, $$

(B.7)

$$ \sum_{i=1}^n \frac{\lambda_0}{\sqrt{n}} u_i \langle X_i, \Theta_\eta \rangle \leq b_F r_{b,F} \| T_\Sigma ( \Theta_\eta ) \|_F + b_s r_{b,s} \| \Theta_\eta \|_s, $$

(B.8)

where $r_{a,F}, r_{a,s}, r_{b,F}, r_{b,s}, a_F, a_s, b_F, b_s \geq 0$ are some numbers. Suppose that $\lambda_*$ satisfy

$$ \lambda_* - C_s > 0, \quad \lambda_* - C_s > 0, $$

(B.9)

where

$$ C_s = \frac{a_F r_{a,F} + \sqrt{2} b_F r_{b,F}}{c_0 \sqrt{F}} + (a_s r_{a,s} + \sqrt{2} b_s r_{b,s}). $$

(B.10)

Then, we have

$$ \| P_{\hat{B}^*} ( \Theta_\eta ) \|_* \leq \frac{\lambda_* + C_s}{\lambda_* - C_s} \| P_{B^*} ( \Theta_\eta ) \|_* $$

(B.11)

**Proof.** Let $\hat{\Theta} = \hat{B} - B^*$ and

$$ Q'(\eta) = \frac{\lambda_0}{\sqrt{n}} \sum_{i=1}^n \{-h(r_{B*,i}) + h(r_{B*,i})\} \langle X_i, \hat{B} - B^* \rangle, $$

(B.12)

where

$$ r_{M,i} = \frac{y_i - \langle X_i, M \rangle}{\lambda_0 \sqrt{n}}. $$

(B.13)

From the proof of Lemma F.2. of [33], we have $\eta Q'(\eta) \leq \eta Q'(1)$ and this means

$$ \sum_{i=1}^n \frac{\lambda_0}{\sqrt{n}} \{-h(r_{B*,i} + \Theta_{\eta,i}) + h(r_{B*,i})\} \langle X_i, \Theta_\eta \rangle \leq \sum_{i=1}^n \frac{\lambda_0}{\sqrt{n}} \eta \{-h(r_{B*,i} + \Theta_{\eta,i}) + h(r_{B*,i})\} \langle X_i, \hat{\Theta} \rangle. $$

(B.14)

Let $\partial^* M$ be the sub-differential of $\| M \|_*$. Adding $\eta \lambda_* \left( \| \hat{B} \|_* - \| B^* \|_* \right)$ to both sides of (B.14), we have
\[
\sum_{i=1}^{n} \frac{\lambda_0}{\sqrt{n}} \left\{ -h(r_{B^*+\Theta_{n,i}}) + h(r_{B_{\tilde{B},i}}) \right\} \langle X_i, \Theta_\eta \rangle + \eta \lambda_0 \left( \| \tilde{B} \|_* - \| B^* \|_* \right)
\]
\[
\leq \sum_{i=1}^{n} \frac{\lambda_0}{\sqrt{n}} \eta \left\{ -h(r_{B^*+\Theta_{n,i}}) + h(r_{B_{\tilde{B},i}}) \right\} \langle X_i, \hat{\Theta} \rangle + \eta \lambda_0 \left( \| \tilde{B} \|_* - \| B^* \|_* \right)
\]
\[
\leq (a) \sum_{i=1}^{n} \frac{\lambda_0}{\sqrt{n}} \eta \left\{ -h(r_{B^*+\Theta_{n,i}}) + h(r_{B_{\tilde{B},i}}) \right\} \langle X_i, \hat{\Theta} \rangle + \eta \lambda_0 \langle \partial^* \tilde{B}, \hat{\Theta} \rangle
\]
\[
= \sum_{i=1}^{n} \frac{\lambda_0}{\sqrt{n}} \eta h(r_{B_{\tilde{B},i}}) \langle X_i, \hat{\Theta} \rangle + \eta \left\{ - \sum_{i=1}^{n} \frac{\lambda_0}{\sqrt{n}} h(r_{B^*+\Theta_{n,i}}) \langle X_i, \hat{\Theta} \rangle + \lambda_0 \langle \partial^* \tilde{B}, \hat{\Theta} \rangle \right\}
\]
\[
(b) \sum_{i=1}^{n} \frac{\lambda_0}{\sqrt{n}} \eta h(r_{B_{\tilde{B},i}}) \langle X_i, \hat{\Theta} \rangle = \sum_{i=1}^{n} \frac{\lambda_0}{\sqrt{n}} h(r_{B_{\tilde{B},i}}) \langle X_i, \Theta_\eta \rangle, \quad (B.15)
\]
where (a) follows from \( \| \tilde{B} \|_* - \| B^* \|_* \leq \langle \partial^* \tilde{B}, \hat{\Theta} \rangle \), which is the definition of the sub-differential, and (b) follows from the fact that \( \tilde{B} \) is an optimal solution of (1.12).

From the convexity of the Huber loss and Lemma B.1, the first term of the L.H.S. of (B.15) satisfies
\[
\sum_{i=1}^{n} \frac{\lambda_0}{\sqrt{n}} \left\{ -h(r_{B^*+\Theta_{n,i}}) + h(r_{B_{\tilde{B},i}}) \right\} \langle X_i, \Theta_\eta \rangle = \sum_{i=1}^{n} \left\{ h(r_{B^*+\Theta_{n,i}}) - h(r_{B_{\tilde{B},i}}) \right\} \{ h(r_{B^*+\Theta_{n,i}}) - h(r_{B_{\tilde{B},i}}) \} \geq 0.
\]
(B.16)

The R.H.S. of (B.15) can be decomposed as
\[
\sum_{i=1}^{n} \frac{\lambda_0}{\sqrt{n}} h(r_{B_{\tilde{B},i}}) \langle X_i, \Theta_\eta \rangle = \sum_{i \in I_I} \frac{\lambda_0}{\sqrt{n}} h(r_{B^*+\Theta_{n,i}}) \langle X_i, \Theta_\eta \rangle + \sum_{i \in I_O} \frac{\lambda_0}{\sqrt{n}} h(r_{B_{\tilde{B},i}}) \langle X_i, \Theta_\eta \rangle
\]
\[
= \sum_{i \in I_I} \frac{\lambda_0}{\sqrt{n}} \left( \frac{\xi_i}{\lambda_0 \sqrt{n}} \right) \langle X_i, \Theta_\eta \rangle + \sum_{i \in I_O} \frac{\lambda_0}{\sqrt{n}} h(r_{B_{\tilde{B},i}}) \langle X_i, \Theta_\eta \rangle
\]
\[
= \sum_{i=1}^{n} \frac{\lambda_0}{\sqrt{n}} \left( \frac{\xi_i}{\lambda_0 \sqrt{n}} \right) \langle X_i, \Theta_\eta \rangle + \sum_{i=1}^{n} \frac{\lambda_0}{\sqrt{n}} w_i \langle X_i, \Theta_\eta \rangle, \quad (B.17)
\]
where
\[
w_i = \begin{cases} 
0 & (i \in I_I) \\
\frac{\xi_i}{\lambda_0 \sqrt{n}} & (i \in I_O)
\end{cases}.
\]
(B.18)
and \( w = (w_1, \cdots, w_n) \). From (B.15), (B.16) and (B.17), we have
\[
0 \leq \sum_{i=1}^{n} \frac{\lambda_0}{\sqrt{n}} \left( \frac{\xi_i}{\lambda_0 \sqrt{n}} \right) \langle X_i, \Theta_\eta \rangle + \sum_{i=1}^{n} \frac{\lambda_0}{\sqrt{n}} w_i \langle X_i, \Theta_\eta \rangle + \eta \lambda_0 \left( \| B^* \|_* - \| \tilde{B} \|_* \right).
\]
(B.19)
Furthermore, we evaluate the right-hand side of (B.19). First, from (B.7),
\[
\sum_{i=1}^{n} \frac{\lambda_0}{\sqrt{n}} \left( \frac{\xi_i}{\lambda_0 \sqrt{n}} \right) \langle X_i, \Theta_\eta \rangle \leq a_{FR a,F} \| T_\Sigma(\Theta_\eta) \|_F + a_{*r a,*} \| \Theta_\eta \|_*.
\]
(B.20)
Second, from (B.8), we have
\[
\sum_{i=1}^{n} \frac{\lambda_0}{\sqrt{n}} w_i \langle X_i, \Theta_\eta \rangle \leq \sqrt{2} (b_{FR b,F} \| T_\Sigma(\Theta_\eta) \|_F + b_{*r b,*} \| \Theta_\eta \|_*).
\]
(B.21)
From (B.19), (B.20), (B.21) and the assumption $\|T_\Sigma(\Theta_\eta)\|_F \leq \frac{1}{c_\kappa \sqrt{n}} \|\Theta_\eta\|_\ast$, we have

$$0 \leq a_F r_{a,F} \|T_\Sigma(\Theta_\eta)\|_F + a_s r_{a,s} \|\Theta_\eta\|_s + \sqrt{2} (b_F r_{b,F} \|T_\Sigma(\Theta_\eta)\|_F + b_s r_{b,s} \|\Theta_\eta\|_s) + \eta \lambda_s \left(\|B^\ast\|_\ast - \|\hat{B}\|_\ast\right).$$

Furthermore, we see

$$0 \leq C_s \|\Theta_\eta\|_\ast + \eta \lambda_s \left(\|B^\ast\|_\ast - \|\hat{B}\|_\ast\right)$$

and the proof is complete. □

Combining Lemma B.2 with Proposition B.1, we can easily prove the following proposition, which shows a relation between $\|\Theta_\eta\|_\ast$ and $\|T_\Sigma(\Theta_\eta)\|_F$.

**Proposition B.2.** Assume the conditions used in Proposition B.1. Then, we obtain

$$\|\Theta_\eta\|_\ast \leq c_\kappa \sqrt{n} \|T_\Sigma(\Theta_\eta)\|_F.$$  \hspace{1cm} (B.24)

**Proof.** When $\|\Theta_\eta\|_\ast < c_\kappa \sqrt{n} \|T_\Sigma(\Theta_\eta)\|_F$, we obtain (B.24) immediately. When $\|\Theta_\eta\|_\ast \geq c_\kappa \sqrt{n} \|T_\Sigma(\Theta_\eta)\|_F$, from Proposition B.1, we see that $\Theta_\eta$ satisfies $\|P_{\hat{B}^\perp}(\Theta_\eta)\|_\ast \leq c_0 \|P_B(\Theta_\eta)\|_\ast$, that is, the condition (2.4). Hence, because $\Sigma$ satisfies MRE($r, c_0, \kappa$), we have the property (B.4) with $\Theta_\eta$, so that we see $\|\Theta_\eta\|_\ast \leq c_\kappa \sqrt{n} \|T_\Sigma(\Theta_\eta)\|_F$, and then the property (B.24) holds. □

**B.3. Main theorem**

**Theorem B.1.** Consider the optimization problem (1.12). Assume all the conditions used in Proposition B.1. Assume that

$$c_1 \|T_\Sigma(\Theta_\eta)\|_F^2 - c_2 r_{c,F} \|T_\Sigma(\Theta_\eta)\|_F - c_3 r_c \leq \lambda_c^2 \sum_{i=1}^n \left\{-h \left(\frac{\xi_i - (X_i, \Theta_\eta)}{\lambda_\varnothing \sqrt{n}}\right) + h \left(\frac{\xi_i}{\lambda_\varnothing \sqrt{n}}\right)\right\} \langle X_i, \Theta_\eta \rangle$$

where $c_1 > 0, c_2, c_3, r_{c,F}, r_c \geq 0$ are some numbers. Suppose that

$$c_2 r_{c,F} + C_{\lambda_s} + \sqrt{c_3 c_0 r_c} < r_0,$$  \hspace{1cm} (B.25)

where

$$C_{\lambda_s} = (a_F r_{a,F} + \sqrt{2} b_F r_{b,F}) + (a_s r_{a,s} + \sqrt{2} b_s r_{b,s}) c_\kappa \sqrt{n} + \lambda_s c_\kappa \sqrt{n}.$$  \hspace{1cm} (B.27)

Then, the optimal solution $\hat{B}$ satisfies

$$\|T_\Sigma(\hat{B} - B^\ast)\|_F \leq r_0.$$  \hspace{1cm} (B.28)
Proof. We prove Theorem B.1 in a manner similar to the proof of Lemma B.7 in [33] and the proof of Theorem 2.1 in [12].

For fixed \( r_0 > 0 \), we define

\[
\mathcal{B} := \{ B : \| T_\Sigma (B - B^*) \|_F \leq r_0 \}. \tag{B.29}
\]

We prove \( \hat{B} \notin \mathcal{B} \) by assuming \( \hat{B} \notin \mathcal{B} \) and deriving a contradiction. For \( \hat{B} \notin \mathcal{B} \), we can find some \( \eta \in [0, 1] \) such that \( \| T_\Sigma (\Theta_\eta) \|_F = r_0 \).

From (B.15), we have

\[
\sum_{i=1}^{n} \frac{\lambda_\eta}{\sqrt{n}} \left\{ -h(r_{B^*+\theta_\eta,i}) + h(r_{B^*,i}) \right\} \langle X_i, \Theta_\eta \rangle = \sum_{i=1}^{n} \frac{\lambda_\eta}{\sqrt{n}} h(r_{B^*,i}) \langle X_i, \Theta_\eta \rangle + \eta \lambda_* \left( \| B^* \|_* - \| \hat{B} \|_* \right). \tag{B.30}
\]

The L.H.S. of (B.30) can be decomposed as

\[
\sum_{i=1}^{n} \frac{\lambda_\eta}{\sqrt{n}} \left\{ -h(r_{B^*+\theta_\eta,i}) + h(r_{B^*,i}) \right\} \langle X_i, \Theta_\eta \rangle = \sum_{i \in I_0} \frac{\lambda_\eta}{\sqrt{n}} \left\{ -h(r_{B^*+\theta_\eta,i}) + h(r_{B^*,i}) \right\} \langle X_i, \Theta_\eta \rangle + \sum_{i \in I_1} \frac{\lambda_\eta}{\sqrt{n}} \left\{ -h(r_{B^*+\theta_\eta,i}) + h(r_{B^*,i}) \right\} \langle X_i, \Theta_\eta \rangle \tag{B.31}
\]

The first term of the R.H.S. of (B.30) can be decomposed as

\[
\sum_{i=1}^{n} \frac{\lambda_\eta}{\sqrt{n}} h(r_{B^*,i}) \langle X_i, \Theta_\eta \rangle = \sum_{i \in I_0} \frac{\lambda_\eta}{\sqrt{n}} h(r_{B^*,i}) \langle X_i, \Theta_\eta \rangle + \sum_{i \in I_1} \frac{\lambda_\eta}{\sqrt{n}} h(r_{B^*,i}) \langle X_i, \Theta_\eta \rangle \tag{B.32}
\]

From (B.30), (B.31) and (B.32), we have

\[
\sum_{i=1}^{n} \frac{\lambda_\eta}{\sqrt{n}} \left\{ -h \left( \frac{\xi_i - \langle X_i, \Theta_\eta \rangle}{\lambda_\eta \sqrt{n}} \right) + h \left( \frac{\xi_i}{\lambda_\eta \sqrt{n}} \right) \right\} \langle X_i, \Theta_\eta \rangle \leq \sum_{i=1}^{n} \frac{\lambda_\eta}{\sqrt{n}} h \left( \frac{\xi_i}{\lambda_\eta \sqrt{n}} \right) \langle X_i, \Theta_\eta \rangle + \sum_{i=1}^{n} \frac{\lambda_\eta}{\sqrt{n}} w'_i \langle X_i, \Theta_\eta \rangle + \eta \lambda_* \left( \| B^* \|_* - \| \hat{B} \|_* \right), \tag{B.33}
\]

where

\[
w'_i = \begin{cases} 0 & (i \in I_1) \\ h(r_{B^*+\theta_\eta,i}) - h \left( \frac{\xi_i + \langle X_i, \Theta_\eta \rangle}{\lambda_\eta \sqrt{n}} \right) & (i \in I_0) \end{cases} \tag{B.34}
\]
and \( w' = (w'_1, \ldots, w'_n) \). We evaluate each term of (B.33). From (B.25), the L.H.S. of (B.33) is evaluated as
\[
c_1 \|T_\Sigma(\Theta_\eta)\|_F^2 - c_2r_{c,F}\|T_\Sigma\Theta_\eta\|_F - c_3r_c \leq \sum_{i=1}^n \frac{\lambda_i}{\sqrt{n}} \left\{ -h \left( \frac{\xi_i}{\lambda_0 \sqrt{n}} \right) + h \left( \frac{\xi_i}{\lambda_0 \sqrt{n}} \right) \right\} \langle X_i, \Theta_\eta \rangle. 
\]
(B.35)

From (B.7) and (B.24) and Proposition B.2, the first term of the R.H.S. of (B.33) is evaluated as
\[
\sum_{i=1}^n \frac{\lambda_i}{\sqrt{n}} h \left( \frac{\xi_i}{\lambda_0 \sqrt{n}} \right) \langle X_i, \Theta_\eta \rangle \leq a_F r_{a,F} \|T_\Sigma(\Theta_\eta)\|_F + a_* r_{a,*} \|\Theta_\eta\|_* 
\leq (a_F r_{a,F} + a_* c_{\xi} \sqrt{r_{a,*}}) \|T_\Sigma(\Theta_\eta)\|_F. 
\]
(B.36)

From (B.8) and (B.24) and Proposition B.2, the second term of the R.H.S. of (B.33) is evaluated as
\[
\sum_{i=1}^n \frac{\lambda_i}{\sqrt{n}} w'_i \langle X_i, \Theta_\eta \rangle \leq \left( b_F r_{b,F} \sqrt{\sigma} \|T_\Sigma(\Theta_\eta)\|_F + b_* r_{b,*} \|\Theta_\eta\|_* \right) \|w'\|_2 
\leq \sqrt{2} \left( b_F r_{b,F} \|T_\Sigma(\Theta_\eta)\|_F + b_* c_{\xi} \sqrt{r_{b,*}} \|T_\Sigma(\Theta_\eta)\|_F \right). 
\]
(B.37)

From (B.24) and Proposition B.2, the third term of the R.H.S. of (B.33) is evaluated as
\[
\eta \lambda_* \left( \|B^*_\|_* - \|\hat{B}^*\|_* \right) \leq \eta \lambda_* \|\Theta_\eta\|_* \leq \lambda_* c_{\xi} \sqrt{r} \|T_\Sigma(\Theta_\eta)\|_F. 
\]
(B.38)

Combining the above four inequalities with (B.33), we have
\[
c_1 \|T_\Sigma(\Theta_\eta)\|_F^2 - c_2r_{c,F}\|T_\Sigma\Theta_\eta\|_F - c_3r_c \leq C_{\lambda_*} \|T_\Sigma(\Theta_\eta)\|_F. 
\]
(B.39)

From (B.39), \( \sqrt{A} + B \leq \sqrt{A} + \sqrt{B} \) for \( A, B > 0 \), we have
\[
\|T_\Sigma(\Theta_\eta)\|_F \leq \frac{c_2r_{c,F} + c_{\lambda_*} + \sqrt{c_3r_c}}{c_1} < r_0. 
\]
(B.40)

This contradicts \( \|T_\Sigma(\Theta_\eta)\|_F = r_0 \). Consequently, we have \( \hat{B} \in B \) and \( \|T_\Sigma(\hat{B} - B^*)\|_F < r_0 \)

\( \square \)

Appendix C: Tools for proving Theorem 2.1

In this section, suppose that Assumption 2.1 holds.

C.1. Derivation of (B.7) under the assumptions of Theorem 2.1

Lemma C.1. For \( 0 < \delta < 1/7 \), with probability at least \( 1 - \delta \), we have
\[
\left| \frac{1}{n} \sum_{i=1}^n h \left( \frac{\xi_i}{\lambda_0 \sqrt{n}} \right) \langle X_i, M \rangle \right| \leq CL \left\{ \rho \sqrt{\frac{d_1 + d_2}{n}} \|M\|_* + \sqrt{\frac{\log(1/\delta)}{n}} \|T_\Sigma(M)\|_F \right\}. 
\]
(C.1)

Proof. Let
\[
V_M = \{ M \in \mathbb{R}^{d_1 \times d_2} \mid \|T_\Sigma(M)\|_F = 1, \|M\|_* \leq r_* \}. 
\]
(C.2)

For any \( M, M' \in \mathbb{R}^{d_1 \times d_2} \), we have
\[
\left\| h \left( \frac{\xi_i}{\lambda_0 \sqrt{n}} \right) \langle X_i, M \rangle - h \left( \frac{\xi_i}{\lambda_0 \sqrt{n}} \right) \langle X_i, M' \rangle \right\|_{\psi_2} \leq \|\langle X_i, M \rangle - \langle X_i, M' \rangle\|_{\psi_2}. 
\]
(C.3)
where (a) follows from Hölder’s inequality and (b) follows from Lemma H.1 of [1].

Proof. From Proposition 4 of [1], we easily have the following corollary.

In Section C.2, we show Corollary C.1, which implies that (B.8) is satisfied with high probability under the assumptions of Theorem 2.1. Combining Proposition 4 of [65] and Remark 4 of [20], we easily have the following corollary.

Corollary C.1 (Corollary of Proposition 4 of [65]). We have for all $o$-sparse vector $\mathbf{u} \in \mathbb{R}^n$ and $M \in \mathbb{R}^{d_1 \times d_2}$,

$$\left| \sum_{i=1}^{n} \frac{1}{\sqrt{n}} X_i \mathbf{M} \right| \leq \left( 1 + \frac{\sqrt{\log(1/\delta)}}{\sqrt{n}} \right) \| T_\Sigma(M) \|_F + \rho \sqrt{\frac{d_1 d_2}{n}} \| \mathbf{M} \|_\ast + \sqrt{\frac{\log n}{\delta}} \| T_\Sigma(M) \|_F \| \mathbf{u} \|_2 \tag{C.7}$$

with probability at least $1 - \delta$.

Proof. From Proposition 4 of [65], for any $\mathbf{u} \in \mathbb{R}^n$ and $M \in \mathbb{R}^{d_1 \times d_2}$, we have

$$\left| \sum_{i=1}^{n} \frac{1}{\sqrt{n}} X_i \mathbf{M} \right| \leq \left( 1 + \frac{\sqrt{\log(1/\delta)}}{\sqrt{n}} \right) \| T_\Sigma(M) \|_F \| \mathbf{u} \|_2 + \| Z' \|_\infty \| \mathbf{M} \|_\ast + \| T_\Sigma(M) \|_F \sqrt{\frac{\log n}{\delta}} \sup_{\mathbf{g} \in \mathbb{R}^d \setminus \mathbb{B}_2} \frac{g_{\ast}}{\| \mathbf{u} \|_1} \tag{C.8}$$
where $g_d$ is the $d$-dimensional standard Gaussian vector $Z' \in \mathbb{R}^{d_1 \times d_2}$ is a random matrix whose entries are standard normal Gaussian. From Lemma H.1 of [55]

$$\|Z'\|_{\text{op}} \leq \rho \sqrt{d_1 + d_2} \tag{C.9}$$

When $u$ is $\alpha$-sparse, from Remark 4 of [20], we have

$$\sup_{b \in \mathbb{R}^2 \cap B_2^2 / \|u\|_1} g_d^\top b \leq C \sqrt{\|u\|_2}. \tag{C.10}$$

Combining above arguments, the proof is complete.

C.3. Derivation of (B.25) under the assumptions of Theorem 2.1

In Section C.3, we show Proposition C.1, which implies that (B.25) is satisfied with high probability under the assumptions of Theorem 2.1. This is partly proved in a similar manner to [48], [31], [12] and [63].

**Proposition C.1.** Let

$$R_{\text{mcs}} = \{ \Theta \in \mathbb{R}^{d_1 \times d_2} \mid \|\Theta\|_* \leq c_k \|T_{\Sigma}(\Theta)\|_F, \|T_{\Sigma}(\Theta)\|_F = r_{\text{mcs}} \}, \tag{C.11}$$

where $r_{\text{mcs}}$ is some number such that $0 \leq r_{\text{mcs}} \leq \frac{1}{4 \sqrt{3} L^2}$. Assume that $\lambda_0 \sqrt{n} \geq 72 L^4 \sigma$. Then, with probability at least $1 - \delta$, we have

$$\inf_{\Theta \in R_{\text{mcs}}} \left[ \sum_{i=1}^{n} \lambda_0^2 \left\{ -h \left( \frac{\xi_i + (X_i, \Theta)}{\lambda_0 \sqrt{n}} \right) - h \left( \frac{\xi_i}{\lambda_0 \sqrt{n}} \right) \right\} \frac{(X_i, \Theta)}{\lambda_0 \sqrt{n}} \right] \geq \frac{1}{3} \|T_{\Sigma}(\Theta)\|_F^2 - C \left( L \rho c_k \sqrt{r_{d_1 + d_2}} + \sqrt{\frac{\log(1/\delta)}{n}} \right) \|T_{\Sigma}(\Theta)\|_F - C \frac{\log(1/\delta)}{n}. \tag{C.13}$$

**Proof.** Let

$$u_i = \frac{\xi_i}{\lambda_0 \sqrt{n}}, \quad v_i = \frac{(X_i, \Theta)}{\lambda_0 \sqrt{n}}. \tag{C.14}$$

The left-hand side of (D.9) divided by $\lambda_0^2$ can be expressed as

$$\sum_{i=1}^{n} \left\{ -h \left( \frac{\xi_i - (X_i, \Theta)}{\lambda_0 \sqrt{n}} \right) + h \left( \frac{\xi_i}{\lambda_0 \sqrt{n}} \right) \right\} \frac{(X_i, \Theta)}{\lambda_0 \sqrt{n}} = \sum_{i=1}^{n} \{ -h (u_i - v_i) + h (u_i) \} v_i \tag{C.15}$$

From the convexity of the Huber loss and Lemma B.1, we have

$$\sum_{i=1}^{n} \{ -h (u_i - v_i) + h (u_i) \} v_i \geq \sum_{i=1}^{n} \{ -h (u_i - v_i) + h (u_i) \} v_i 1_{E_i}, \tag{C.16}$$

where $1_{E_i}$ is the indicator function of the event

$$1_{E_i} := \left( |u_i| \leq \frac{1}{2} \right) \cap \left( |v_i| \leq \frac{1}{2 \lambda_0 \sqrt{n}} \right). \tag{C.17}$$

Define the functions

$$\varphi(v) = \begin{cases} v^2 & \text{if } |v| \leq \frac{1}{2 \lambda_0 \sqrt{n}} \\ (v - 1/2)^2 & \text{if } \frac{1}{2 \lambda_0 \sqrt{n}} \leq v \leq 1/2 \\ (v + 1/2)^2 & \text{if } -1/2 \leq v \leq -\frac{1}{2 \lambda_0 \sqrt{n}} \text{ and } \psi(u) = I_{|u| \leq 1/2}. \end{cases} \tag{C.18}$$

Let

$$\varphi(v) = \begin{cases} v^2 & \text{if } |v| \leq \frac{1}{2 \lambda_0 \sqrt{n}} \\ (v - 1/2)^2 & \text{if } \frac{1}{2 \lambda_0 \sqrt{n}} \leq v \leq 1/2 \\ (v + 1/2)^2 & \text{if } -1/2 \leq v \leq -\frac{1}{2 \lambda_0 \sqrt{n}} \text{ and } \psi(u) = I_{|u| \leq 1/2}. \end{cases} \tag{C.18}$$
Let
\[ \sum_{i=1}^{n} \varphi(v_i) \psi(u_i) = \sum_{i=1}^{n} f_i(\Theta) \]  
(C.19)

with \( f_i(\Theta) = \varphi(v_i)\psi(u_i) \) and we have
\[ \sum_{i=1}^{n} \{-h(u_i - v_i) + h(u_i)\} v_i \geq \sum_{i=1}^{n} \{-h(u_i - v_i) + h(u_i)\} v_i I_{E_i} \]
\[ = \sum_{i=1}^{n} v_i^2 I_{E_i} \]
\[ \geq \sum_{i=1}^{n} \varphi(v_i)\psi(u_i) = \sum_{i=1}^{n} f_i(\Theta), \]  
(C.20)

where (a) follows from \( \varphi(v) \geq v^2 \) for \( |v| \leq 1/2 \). We note that
\[ f_i(\Theta) \leq \varphi(v_i) \leq \max \left( \frac{(X_i, \Theta)^2}{\lambda_0^2 n}, 1 \right). \]  
(C.21)

To bound \( \sum_{i=1}^{n} f_i(\Theta) \) from bellow, for any fixed \( \Theta \in \mathcal{R}(r_{mcs}) \), we have
\[ \sum_{i=1}^{n} f_i(\Theta) \geq \mathbb{E} f(\Theta) - \sup_{\Theta' \in \mathcal{R}_{mcs}} \left| \sum_{i=1}^{n} f_i(\Theta') - \mathbb{E} \sum_{i=1}^{n} f_i(\Theta') \right|. \]  
(C.22)

Define the supremum of a random process indexed by \( \mathcal{R}_{mcs} \):
\[ \Delta := \sup_{\Theta' \in \mathcal{R}_{mcs}} \left| \sum_{i=1}^{n} f_i(\Theta') - \mathbb{E} \sum_{i=1}^{n} f_i(\Theta') \right|. \]  
(C.23)

From (C.20) and (C.18), we have
\[ \mathbb{E} \sum_{i=1}^{n} f_i(L) \geq \sum_{i=1}^{n} \mathbb{E} \left| \frac{(X_i, \Theta)}{\lambda_0 \sqrt{n}} \right|^2 - \sum_{i=1}^{n} \mathbb{E} \left| \frac{(X_i, \Theta)}{\lambda_0 \sqrt{n}} \right|^2 I \left( \left| \frac{(X_i, \Theta)}{\lambda_0 \sqrt{n}} \right| \geq \frac{1}{2 \lambda_0 \sqrt{n}} \right) - \mathbb{E} \left| \frac{(X_i, \Theta)}{\lambda_0 \sqrt{n}} \right|^2 I \left( \left| \frac{\xi_i}{\lambda_0 \sqrt{n}} \right| \geq \frac{1}{2} \right). \]  
(C.24)

We evaluate the right-hand side of (C.24) at each term. First, we have
\[ \sum_{i=1}^{n} \mathbb{E} \left| \frac{(X_i, \Theta)}{\lambda_0 \sqrt{n}} \right|^2 f \left( \left| \frac{(X_i, \Theta)}{\lambda_0 \sqrt{n}} \right| \geq \frac{1}{2 \lambda_0 \sqrt{n}} \right) \leq \sum_{i=1}^{n} \sqrt{\mathbb{E} \left| \frac{(X_i, \Theta)}{\lambda_0 \sqrt{n}} \right|^4} \sqrt{\mathbb{E} I \left( \left| \frac{(X_i, \Theta)}{\lambda_0 \sqrt{n}} \right| \geq \frac{1}{2 \lambda_0 \sqrt{n}} \right)}, \]
\[ \leq \sum_{i=1}^{n} \sqrt{\mathbb{E} \left| \frac{(X_i, \Theta)}{\lambda_0 \sqrt{n}} \right|^4} \sqrt{\mathbb{P} \left( \left| (X_i, \Theta) \right| \geq \frac{1}{2} \right)}, \]
\[ \leq \sum_{i=1}^{n} 4 \sqrt{\mathbb{E} \left| \frac{(X_i, \Theta)}{\lambda_0 \sqrt{n}} \right|^4} \sqrt{\mathbb{E} (X_i, \Theta)^4}, \]
\[ \leq \frac{4L^2}{\lambda_0^2} r_{mcs}^2, \]
\[ \leq \frac{1}{3 \lambda_0^2} \| T_S(\Theta) \|^2_F. \]  
(C.25)
where (a) follows from Hölder’s inequality, (b) follows from the relation between indicator function and expectation, (c) follows from Markov’s inequality, (d) follows from Assumption 2.1 and (e) follows from $0 \leq r_{mcs} \leq \frac{1}{4\sqrt{3}L_r}$. Second, we have

$$\sum_{i=1}^{n} \mathbb{E} \left| \frac{\langle X_i, \Theta \rangle}{\lambda_o \sqrt{n}} \right|^2 f \left( \left| \frac{\xi_i}{\lambda_o \sqrt{n}} \right| \geq \frac{1}{2} \right) \leq \sum_{i=1}^{n} \mathbb{E} \left| \frac{\langle X_i, \Theta \rangle}{\lambda_o \sqrt{n}} \right|^4 \mathbb{E} f \left( \left| \frac{\xi_i}{\lambda_o \sqrt{n}} \right| \geq \frac{1}{2} \right)$$

$$\leq \sum_{i=1}^{n} \mathbb{E} \left| \frac{\langle X_i, \Theta \rangle}{\lambda_o \sqrt{n}} \right|^4 \mathbb{P} \left( \left| \frac{\xi_i}{\lambda_o \sqrt{n}} \right| \geq \frac{1}{2} \right)$$

$$\leq \sum_{i=1}^{n} \frac{2}{\lambda_o \sqrt{n}} \mathbb{E} \left| \frac{\langle X_i, \Theta \rangle}{\lambda_o \sqrt{n}} \right|^4 \mathbb{E} |\xi_i|$$

$$\leq \sum_{i=1}^{n} \frac{2\sigma}{\lambda_o \sqrt{n}} \mathbb{E} \left| \frac{\langle X_i, \Theta \rangle}{\lambda_o \sqrt{n}} \right|^4 \sqrt{\mathbb{E} |\xi_i|}$$

$$\leq \frac{1}{3\lambda_o^2} \|T_\Sigma(\Theta)\|_F^2,$$  \quad (C.26)

where (a) follows from Hölder’s inequality, (b) follows from relation between indicator function and expectation, (c) follows from Markov’s inequality, (d) follows from Assumption 2.1 and (e) follows from the definition of $\lambda_o$. Consequently, we have

$$\mathbb{E} \sum_{i=1}^{n} f_i(\Theta) \geq \frac{1}{3\lambda_o^2} \|T_\Sigma(\Theta)\|_F^2$$  \quad (C.27)

and

$$\sum_{i=1}^{n} \{-h(u_i - v_i) - h(u_i)\} v_i \geq \sum_{i=1}^{n} f_i(\Theta) \geq \frac{1}{3\lambda_o^2} \|T_\Sigma(\Theta)\|_F^2 - \Delta.$$  \quad (C.28)

Next we evaluate the stochastic term $\Delta$ defined in (C.23). From (C.21) and Theorem 3 of [51], with probability at least $1 - \delta$, we have

$$\Delta \leq 2\mathbb{E} \Delta + \sigma_f \sqrt{8 \log(1/\delta)} + \frac{18.5}{4} \log(1/\delta) \leq 2\mathbb{E} \Delta + \sigma_f \sqrt{8 \log(1/\delta)} + \frac{\log(1/\delta)}{\lambda_o^2 n},$$  \quad (C.29)

where $\sigma_f^2 = \sup_{\Theta \in \mathcal{K}_{mcs}} \sum_{i=1}^{n} \mathbb{E} \{f_i(\Theta) - \mathbb{E} f_i(\Theta)\}^2$. About $\sigma_f$, we have

$$\mathbb{E} \{f_i(\Theta) - \mathbb{E} f_i(\Theta)\}^2 \leq \mathbb{E} f_i^2(\Theta).$$  \quad (C.30)

From (C.21) and $0 < r \leq \frac{1}{4\sqrt{3}L_r}$, we have

$$\mathbb{E} f_i^2(\Theta) \leq \mathbb{E} \frac{(\langle X_i, \Theta \rangle)^4}{\lambda_o^2 n^2} = \frac{1}{\lambda_o^2 n^2} \|T_\Sigma(\Theta)\|_F^4.$$  \quad (C.31)

Combining this and (C.29), we have

$$\Delta \leq 2\mathbb{E} \Delta + \frac{1}{\lambda_o} \sqrt{6 \log(1/\delta)} \|T_\Sigma(\Theta)\|_F^2 + \frac{\log(1/\delta)}{\lambda_o^2 n}.$$  \quad (C.32)

From Symmetrization inequality (Lemma 11.4 of [5]), we have $\mathbb{E} \Delta \leq 2\mathbb{E} \sup_{\Theta \in \mathcal{K}_{mcs}} |G_\Theta|$, where

$$G_\Theta := \sum_{i=1}^{n} \theta_i \varphi \left( \frac{\langle X_i, \Theta \rangle}{\lambda_o \sqrt{n}} \right) \psi \left( \frac{\xi_i}{\lambda_o \sqrt{n}} \right),$$  \quad (C.33)
and \( \{ \varphi_i \}_{i=1}^n \) is a sequence of i.i.d. Rademacher random variables which is independent of \( \{ X_i, \xi_i \}_{i=1}^n \). We denote \( E^* \) as a conditional variance of \( \varphi_i \) given \( X_i, \xi_i \). From contraction principal (Theorem 11.5 of [5]), we have

\[
E^* \sup_{\Theta \in R_{mcs}} \left| \sum_{i=1}^n \varphi_i \psi \left( \frac{X_i}{\lambda_o \sqrt{n}} \right) \right| \leq E^* \sup_{\Theta \in R_{mcs}} \left| \sum_{i=1}^n \varphi_i \psi \left( \frac{X_i}{\lambda_o \sqrt{n}} \right) \right|.
\]

(C.34)

and from the basic property of the expectation, we have

\[
E \sup_{\Theta \in R_{mcs}} \left| \sum_{i=1}^n \varphi_i \psi \left( \frac{X_i}{\lambda_o \sqrt{n}} \right) \right| \leq E \sup_{\Theta \in R_{mcs}} \left| \sum_{i=1}^n \varphi_i \psi \left( \frac{X_i}{\lambda_o \sqrt{n}} \right) \right|.
\]

(C.35)

Since \( \varphi \) is \( \frac{1}{2 \lambda_o \sqrt{n}} \)-Lipschitz and \( \varphi(0) = 0 \), from contraction principal (Theorem 11.6 in [5]), we have

\[
E \sup_{\Theta \in R_{mcs}} \left| \sum_{i=1}^n \varphi_i \psi \left( \frac{X_i}{\lambda_o \sqrt{n}} \right) \right| \leq E \sup_{\Theta \in R_{mcs}} \left| \sum_{i=1}^n \varphi_i \psi \left( \frac{X_i}{\lambda_o \sqrt{n}} \right) \right|.
\]

(C.36)

From Lemma C.2 and the definition of \( R_{mcs} \), we have

\[
2 \lambda_o^2 E \Delta \leq CL \rho C.3 \leq CL \rho C.3 \leq \frac{d_1 + d_2}{n} \| \Theta \|_F \leq \frac{d_1 + d_2}{n} \| T_\Sigma(\Theta) \|_F.
\]

(C.37)

Combining (C.37) with (C.32) and (C.28), with probability at least \( 1 - \delta \), we have

\[
\sum_{i=1}^n \frac{\lambda_o^2}{2} \left\{ -h \left( \frac{X_i}{\lambda_o \sqrt{n}} \right) - h \left( \frac{X_i}{\lambda_o \sqrt{n}} \right) \right\} \frac{X_i}{\lambda_o \sqrt{n}} \geq \frac{1}{3} \| T_\Sigma(\Theta) \|_F^2 - C \left( L \rho C.3 \sqrt{\frac{d_1 + d_2}{n}} + \sqrt{\frac{8 \log(1/\delta)}{n}} \right) \| T_\Sigma(\Theta) \|_F - \frac{\delta \log(1/\delta)}{n}.
\]

(C.38)

To calculate the L.H.S. of (C.37), we introduce the following Lemma.

Lemma C.2. Assume that \( R_{mcs} \) is the same of the one in Proposition C.1. We have

\[
E \sup_{\Theta \in R_{mcs}} \left| \sum_{i=1}^n \varphi_i X_i \right| \leq CL \rho \sqrt{\frac{d_1 + d_2}{n}} \| \Theta \|_F.
\]

(C.39)

Proof. From Hölder’s inequality

\[
E \sup_{\Theta \in R_{mcs}} \left| \sum_{i=1}^n \varphi_i X_i \right| \leq E \left\| \sum_{i=1}^n \varphi_i \right\|_\infty \sup_{\Theta \in R_{mcs}} \left\| \Theta \right\|_F \leq E \left\| \sum_{i=1}^n \varphi_i \right\|_\infty \sup_{\Theta \in R_{mcs}} \left\| \Theta \right\|_F \leq E \left\| \sum_{i=1}^n \varphi_i \right\|_\infty \left\| \Theta \right\|_F.
\]

(C.40)

We calculate \( E \left\| \sum_{i=1}^n \varphi_i X_i \right\|_\infty \). For any \( \Theta, \Theta' \in \mathbb{R}^{d_1 \times d_2} \), we have

\[
\left\| \varphi_i \langle X_i, M \rangle - \varphi_i \langle X_i, M' \rangle \right\|_{\psi_2} \leq \| \langle X_i, M \rangle - \langle X_i, M' \rangle \|_{\psi_2}
\]

(C.41)

because \( \varphi_i = 1 \) and we see that \( \varphi_i \langle X_i, \Theta \rangle \) is a \( L \)-subGaussian distribution. From (C.3) and the fact that \( \{ \varphi_i \langle X_i, \Theta \rangle \}_{i=1}^n \) is a sequence of i.i.d. random variables, we have

\[
\left\| \frac{1}{\sqrt{n}} \sum_{i=1}^n \varphi_i \langle X_i, \Theta \rangle - \frac{1}{\sqrt{n}} \sum_{i=1}^n \varphi_i \langle X_i, \Theta' \rangle \right\|_{\psi_2} \leq L \left\| \frac{1}{\sqrt{n}} \sum_{i=1}^n \varphi_i \langle X_i, \Theta - \Theta' \rangle \right\|_{2}.
\]

(C.42)

From Exercise 8.6.4 of [68], Lemma H.1. of [55], we have

\[
E \left\| \sum_{i=1}^n \varphi_i X_i \right\|_\infty \leq \frac{CL}{\sqrt{n}} \| Z' \|_\infty \leq CL \rho \sqrt{\frac{d_1 + d_2}{n}}.
\]

(C.43)

where \( Z' \) is a random matrix whose entries are standard normal Gaussian. Combining the arguments above, the proof is complete. \( \square \)
Appendix D: Tools for proving Theorem 2.2

In Section D, suppose that Assumption 2.2 holds and we introduce some inequalities to prove Theorem 2.2. The inequalities are the special case of Lemma C.1, Corollary C.1, Proposition C.1 and Lemma C.2 because, like the argument of section 2.2 of [34], linear regression is a special case of trace regression, where $B^* \quad \text{and} \quad X_i, \ i = 1, \cdots, n$ are diagonal matrices because for some diagonal matrix $M \in \mathbb{R}^{d \times d}$, we see that $\|X_i\|_\infty = \|\text{diag}(X_i)\|_\infty$ and $\|X_i\|_* = \|\text{diag}(X_i)\|_1$.

D.1. Derivation of (B.7) under the assumptions of Theorem 2.2

In Section D.1, we show Lemma D.1, which implies that (B.7) is satisfied with high probability under the assumptions of Theorem 2.2.

Lemma D.1. For $0 < \delta < 1/7$, with probability at least $1 - \delta$, we have

$$\left| \frac{1}{n} \sum_{i=1}^{n} h \left( \frac{\xi_i}{\lambda_0 \sqrt{n}} \right) \langle x_i, v \rangle \right| \leq C L \rho \left\{ \sqrt{\frac{\log(d/s)}{n}} \|v\|_1 + \left\{ 1 + \frac{\sqrt{\log(1/\delta)}}{\sqrt{n}} + \frac{s \log(d/s)}{n} \right\} \|\Sigma_v^2 v\|_2 \right\} \quad (D.1)$$

Proof. Define a set $V_v$, where

$$V_v = \left\{ v \in \mathbb{R}^d \left| \|\Sigma_v^2 v\|_2 = 1, \|v\|_1 \leq r_1 \right. \right\} \quad (D.2)$$

For any $v, v' \in \mathbb{R}^d$, we have

$$\left| h \left( \frac{\xi_i}{\lambda_0 \sqrt{n}} \right) \langle x_i, v \rangle - h \left( \frac{\xi_i}{\lambda_0 \sqrt{n}} \right) \langle x_i, v' \rangle \right| \leq \|\langle x_i, v \rangle - \langle x_i, v' \rangle\|_{\psi_2} \quad (D.3)$$

because $\left| h \left( \frac{\xi_i}{\lambda_0 \sqrt{n}} \right) \right| \leq 1$ and we see that $h \left( \frac{\xi_i}{\lambda_0 \sqrt{n}} \right) \langle x_i, v \rangle$ is a $L$-subGaussian distribution. From (D.3) and the fact that $\left\{ h \left( \frac{\xi_i}{\lambda_0 \sqrt{n}} \right) \langle x_i, v \rangle \right\}_{i=1}^{n}$ is a sequence of i.i.d. random variables, for any $v, v' \in V_v$, we have

$$\left| \frac{1}{n} \sum_{i=1}^{n} h \left( \frac{\xi_i}{\lambda_0 \sqrt{n}} \right) \langle z_i, \Sigma_v^2 v \rangle - \frac{1}{n} \sum_{i=1}^{n} h \left( \frac{\xi_i}{\lambda_0 \sqrt{n}} \right) \langle z_i, \Sigma_v^2 v' \rangle \right| \leq L \frac{1}{\sqrt{n}} \left\| \Sigma_v^2 v - \Sigma_v^2 v' \right\|_2 \quad (D.4)$$

and from Exercise 8.6.5 of [68], with probability at least $1 - \delta$, we have

$$\sup_{v \in V_v} \left| \frac{1}{n} \sum_{i=1}^{n} h \left( \frac{\xi_i}{\lambda_0 \sqrt{n}} \right) \langle x_i, v \rangle \right| \leq C L \sqrt{n} \left\{ \mathbb{E} \sup_{v \in V_v} \langle z_i', \Sigma_v^2 v \rangle + \frac{\sqrt{\log(1/\delta)}}{\sqrt{n}} \sup_{v \in V_v} \sqrt{\langle z_i', \Sigma_v^2 v \rangle^2} \right\} , \quad (D.5)$$

where $z'_i$ is the $d$-dimensional standard normal Gaussian random vector. Combining the arguments above, we have with probability at least $1 - \delta$,

$$\sup_{v \in V_v} \left| \frac{1}{n} \sum_{i=1}^{n} h \left( \frac{\xi_i}{\lambda_0 \sqrt{n}} \right) \langle x_i, v \rangle \right| \leq C L \sqrt{n} \left\{ \mathbb{E} \sup_{v \in V_v} \langle z_i', \Sigma_v^2 v \rangle + \frac{\sqrt{\log(1/\delta)}}{\sqrt{n}} \sup_{v \in V_v} \sqrt{\langle z_i', \Sigma_v^2 v \rangle^2} \right\} \leq \left( a \right) \left\{ \rho \frac{\sqrt{\log(d/s)}}{\sqrt{n}} \|v\|_1 + \rho \frac{\sqrt{s \log(d/s)}}{\sqrt{n}} \log(1/\delta) \sup_{v \in V_v} \sqrt{\langle z_i', \Sigma_v^2 v \rangle^2} \right\} \leq \left( b \right) \left\{ \rho \frac{\sqrt{\log(d/s)}}{\sqrt{n}} \|v\|_1 + \rho \frac{\sqrt{s \log(d/s)}}{\sqrt{n}} + \log(1/\delta) \right\} , \quad (D.6)$$

where (a) follows from Proposition E.1 and E.2 of [3] and (b) follows from Lemma H.1 of [55]. Lastly, using peeling device (Lemma 5 of [20]), the proof is complete. □
D.2. Derivation of (B.8) under the assumptions of Theorem 2.2

In Section D.2, we show Corollary D.1, which implies that (B.8) is satisfied with high probability under the assumptions of Theorem 2.2. The following proposition is easily derived from Proposition 4 of [65], Remark 4 of [20], Proposition E.1 and E.2 of [3].

**Corollary D.1** (Corollary of Proposition 4 of [65]). We have for all o-sparse vector \( u \in \mathbb{R}^n \) and \( v \in \mathbb{R}^d \),

\[
\left| \sum_{i=1}^{n} u_i \frac{1}{\sqrt{n}} (x_i, v) \right| \leq CL \left\{ \frac{1 + \sqrt{\log(1/\delta)}}{\sqrt{n}} + \sqrt{\frac{s \log(d/s)}{n}} + \sqrt{\frac{\delta}{n}} \right\} \| v \|_2 \| u \|_2
\]

with probability at least \( 1 - \delta \).

**Proposition D.1.** Let

\[
\mathcal{R}_{\text{lasso}} = \left\{ \theta \in \mathbb{R}^d : \| \theta \|_1 \leq c_k \| \Sigma^{1/2} \|_2, \| \Sigma^{1/2} \theta \|_2 = r \right\},
\]

where \( r \) is a number such that \( 0 < r \leq \frac{1}{4 \sqrt{\lambda_o}} \). Assume that \( \lambda_o \sqrt{n} \geq 72L^3 \sigma \). Then, with probability at least \( 1 - \delta \), we have

\[
\inf_{\theta \in \mathcal{R}_{\text{lasso}}} \sum_{i=1}^{n} \lambda_i^2 \left\{ -h \left( \frac{\xi_i + (x_i, \theta)}{\lambda_o \sqrt{n}} \right) - h \left( \frac{\xi_i}{\lambda_o \sqrt{n}} \right) \right\} \langle x_i, \theta \rangle \geq \frac{1}{3} \| \Sigma^{1/2} \theta \|_2^2 - C \left( L \rho \sqrt{\frac{s \log(d/s)}{n}} + \sqrt{\frac{\log(1/\delta)}{n}} \right) \| \theta \|_2 - C \frac{\log(1/\delta)}{n}. \tag{D.9}
\]

Appendix E: Tools for proving Theorem 2.3 and Theorem 2.4

In Section E, we state some inequalities used to prove Theorem 2.3 and Theorem 2.4.

E.1. Derivation of (B.7) under the assumptions of Theorem 2.3 and Theorem 2.4

In Section E.1, we show Lemma E.1, which implies that (B.7) is satisfied with high probability under the assumptions of Theorem 2.3 or Theorem 2.4.

**Lemma E.1.** Suppose that Assumption 2.3 or Assumption 2.4 holds. For \( \delta > 0 \), with probability at least \( 1 - \delta \), we have

\[
\left\| \frac{\lambda_o}{\sqrt{n}} \sum_{i=1}^{n} h \left( \frac{\xi_i}{\lambda_o \sqrt{n}} \right) X_i \right\|_{\text{op}} \leq C \left( \sigma \sqrt{\frac{d_{mc} \log d_{mc} + \log(1/\delta)}}{n} + \lambda_o \frac{d_{mc}}{\sqrt{n}} (\log d_{mc} + \log(1/\delta)) \right).
\]

\[
\left( \frac{d_{mc}}{\sqrt{n}} \right) \| E_i \|_{\text{op}} \leq \lambda_o \left( \frac{d_{mc}}{\sqrt{n}} \right). \tag{E.3}
\]

Proof. Let \( U_i = \frac{\lambda_o}{\sqrt{n}} h \left( \frac{\xi_i}{\lambda_o \sqrt{n}} \right) X_i \). We confirm that \( U_i \) satisfies the conditions of Lemma 7 of [56] and then we apply Lemma 7 of [56] to \( \sum_{i=1}^{n} U_i \). From Assumption 2.3 or 2.4, we have \( E \sum_{i=1}^{n} U_i = 0 \). From the definition of \( X_i \), we have

\[
\left\| U_i \right\|_{\text{op}} = d_{mc} \left\| \left( \frac{\lambda_o}{\sqrt{n}} h \left( \frac{\xi_i}{\lambda_o \sqrt{n}} \right) \right) E_i \right\|_{\text{op}} \leq \lambda_o \left( \frac{d_{mc}}{\sqrt{n}} \right) \| E_i \|_{\text{op}}. \tag{E.2}
\]

and from the definition of \( h(\cdot) \) and \( E_i \), we have

\[
d_{mc} \left\| \left( \frac{\lambda_o}{\sqrt{n}} h \left( \frac{\xi_i}{\lambda_o \sqrt{n}} \right) \right) E_i \right\|_{\text{op}} \leq \lambda_o \left( \frac{d_{mc}}{\sqrt{n}} \right).
\]
On the other hand, from the definition of $h(\cdot)$, we have
\[
\max \left\{ \left\| \frac{\lambda_i^2}{n} \mathbb{E} h \left( \frac{\xi_i}{\lambda_i \sqrt{n}} \right)^2 X_i X_i^\top \right\|_{op}, \left\| \frac{\lambda_i^2}{n} \mathbb{E} h \left( \frac{\xi_i}{\lambda_i \sqrt{n}} \right)^2 X_i^\top X_i \right\|_{op} \right\}
\]
\[
\leq \frac{d_{mc}}{n^2} \max \left\{ \left\| \mathbb{E} \xi_i^2 E_i X_i^\top \right\|_{op}, \left\| \mathbb{E} \xi_i^2 E_i^\top E_i \right\|_{op} \right\}
\]
\[
= \frac{\sigma \xi^2 d_{mc}}{n^2}
\]
(E.4)

Applying Lemma 7 of [56], we have
\[
P \left[ \left\| \frac{\lambda_i}{\sqrt{n}} \sum_{i=1}^n h \left( \frac{\xi_i}{\lambda_i \sqrt{n}} \right) X_i \right\|_{op} \geq t \right] \leq d_1 d_2 \max \left\{ \exp \left( \frac{-t^2 n}{4 \sigma^2 d_{mc}^2} \right), \exp \left( \frac{-t \sqrt{n}}{2 \lambda o d_{mc}} \right) \right\}
\]
(E.5)

Setting $\delta = d_{mc}^2 \max \left\{ \exp \left( \frac{-t^2 n}{4 \sigma^2 d_{mc}^2} \right), \exp \left( \frac{-t \sqrt{n}}{2 \lambda o d_{mc}} \right) \right\}$, the proof is complete.

\section*{E.2. Derivation of (B.8) under the assumptions of Theorem 2.3 and Theorem 2.4}

In Section E.2, we show Lemma E.2, which implies that (B.8) is satisfied under the assumptions of Theorem 2.3 and Theorem 2.4.

\textbf{Lemma E.2.} Suppose that Assumption 2.3 or Assumption 2.4 holds. Assume that for any $M \in \mathbb{R}^{d_1 \times d_2}$,
\[
\|M\|_\infty \leq c_m \frac{1}{d_{mc}} \|M\|_F.
\]
(E.6)

for some number $c_m$. Then, for any $o$-sparse vector $u \in \mathbb{R}^n$ such that $\|u\|_2 \leq 2 \sqrt{d_1}$, we have
\[
\left\| \sum_{i=1}^n \frac{\lambda_i}{\sqrt{n}} u_i \langle X_i, M \rangle \right\| \leq c_m 2 \lambda o \sqrt{n} \|M\|_F.
\]

\textbf{Proof.} We re-write $|\sum_{i=1}^n u_i \langle X_i, M \rangle|$ as
\[
\left| u^\top \tilde{X} \text{vec}(M) \right|, \text{ where } \tilde{X} = \begin{pmatrix} \text{vec}(X_1)^\top \\ \vdots \\ \text{vec}(X_n)^\top \end{pmatrix}.
\]
(E.7)

From H"{o}lder’s inequality and the definition of $X_i$, we have
\[
\left| u^\top \tilde{X} \text{vec}(M) \right| \leq \|u\|_1 \|\tilde{X} \text{vec}(M)\|_\infty \leq \|u\|_1 \left\| \tilde{X} \right\|_\infty \|\text{vec}(M)\|_\infty = \|u\|_1 \left\| \tilde{X} \right\|_\infty \|M\|_\infty.
\]
(E.8)

From the definition of $X_i$ and (E.6), we have
\[
\|u\|_1 \left\| \tilde{X} \right\|_\infty \|M\|_\infty \leq \|u\|_1 \sqrt{d_1 d_2} \|M\|_\infty \leq c_m \|u\|_1 \|M\|_F \leq c_m \sqrt{d_1} \|u\|_2 \|M\|_F
\]
and the proof is complete.

\section*{E.3. Derivation of (B.25) under the assumptions of Theorems 2.3 and 2.4}

In Section E.3, we show Corollary E.1, which implies that (B.25) is satisfied with high probability under the assumptions of Theorem 2.3 and, we show Corollary E.2, which implies that (B.25) is satisfied with high probability under the assumptions of Theorem 2.4.
Corollary E.1. Suppose that Assumption 2.3 holds. Let
\[ \mathcal{R}_{mc} = \{ \Theta \in \mathbb{R}^{d_1 \times d_2} \mid \| \Theta \|_s \leq C \sqrt{r} \| \Theta \|_F, \| \Theta \|_F = r_{mc} \} , \]
where \( r_{mc} \) is some number. Suppose
\[
\| \Theta \|_\infty \leq \frac{1}{12 r_{mc} d_{mc}} \| \Theta \|_F \quad \text{(E.10)}
\]
\[
\| \Theta \|_\infty \leq \frac{\alpha^*}{d_{mc}} \quad \text{(E.11)}
\]
and
\[
\lambda_o \sqrt{n} \geq 2 \sigma_{\epsilon, \alpha} \min \left\{ \left( \frac{n}{\sigma} \right)^{\frac{1}{\alpha+1}}, \left( \frac{n}{r d_{mc} \log d_{mc}} \right)^{\frac{1}{\alpha}} \right\} . \quad \text{(E.12)}
\]
Then, with probability at least \( 1 - \delta \), we have
\[
\inf_{\Theta \in \mathcal{R}_{mc}} \left\{ \lambda_o^2 \sum_{i=1}^n \left\{ -h \left( \frac{\xi_i - \langle X_i, \Theta \rangle}{\lambda_o \sqrt{n}} \right) + h \left( \frac{\xi_i}{\lambda_o \sqrt{n}} \right) \right\} \langle X_i, \Theta \rangle \right\}
\geq \frac{2}{3} \| \Theta \|_F^2 - C \left( \alpha^* \sqrt{\frac{d_{mc}(\log d_{mc} + \log(1/\delta))}{n}} + \sqrt{\frac{d_{mc} \log d_{mc}}{n}} + \alpha^* \left( \frac{\alpha}{n} \right)^{\frac{1}{2(\alpha+1)}} \| \Theta \|_F - \frac{5 \log(1/\delta)}{n} \right) . \quad \text{(E.13)}
\]

Proof. The proof is almost identical to the one of Proposition C.1. However, because \( \{X_i\}_{i=1}^n \) for matrix completion is not \( L \)-subGaussian, we should calculate (C.25), (C.26) and (C.31) by another strategy and (C.37) also requires another strategy. We note that
\[
\mathbb{E}(X_i, \Theta)^2 = \| \Theta \|_F^2 = r_{mc}^2 . \quad \text{(E.14)}
\]
From the definition of \( X_i \) and (E.10), we have
\[
\mathbb{E}(X_i, \Theta)^4 \leq \mathbb{E}(X_i, \Theta)^2 \| X_i \|_s^2 \| \Theta \|_\infty^2 \leq \frac{1}{144} r_{mc}^2 . \quad \text{(E.15)}
\]
From the definition of \( X_i \) and (E.25), we have
\[
\mathbb{E}(X_i, \Theta)^4 \leq \mathbb{E}(X_i, \Theta)^2 \| X_i \|_s^2 \| \Theta \|_F^2 \leq 4 \alpha^* \| \Theta \|_F^2 . \quad \text{(E.16)}
\]
Instead of (C.25), we have
\[
\sum_{i=1}^n \mathbb{E} \left[ \left| \frac{\langle X_i, \Theta \rangle}{\lambda_o \sqrt{n}} \right| \frac{\mathbb{I}}{\mathbb{I} \left( \left| \frac{\langle X_i, \Theta \rangle}{\lambda_o \sqrt{n}} \right| \geq \frac{1}{2 \lambda_o \sqrt{n}} \right)} \right] \leq \sum_{i=1}^n \sqrt{n} \mathbb{E} \left( \left| \frac{\langle X_i, \Theta \rangle}{\lambda_o \sqrt{n}} \right| \right) \frac{\mathbb{P}}{\mathbb{P} \left( \left| \langle X_i, \Theta \rangle \right| \geq \frac{1}{2} \right)} \quad \text{\( \text{(a)} \)}
\leq \sum_{i=1}^n \sqrt{n} \mathbb{E} \left( \left| \frac{\langle X_i, \Theta \rangle}{\lambda_o \sqrt{n}} \right| \right) \mathbb{P} \left( \left| \langle X_i, \Theta \rangle \right| \geq \frac{1}{2} \right) \quad \text{\( \text{(b)} \)}
\leq \sum_{i=1}^n \sqrt{n} \mathbb{E} \left( \left| \frac{\langle X_i, \Theta \rangle}{\lambda_o \sqrt{n}} \right| \right) \mathbb{E}(X_i, \Theta)^4 \quad \text{\( \text{(c)} \)}
\leq \frac{1}{3 \lambda_o^2} \| \Theta \|_F^2 . \quad \text{\( \text{(d)} \)}
\]
where (a) follows from Hölder’s inequality, (b) follows from the relation between indicator function and expectation, (c) follows from Markov’s inequality and (d) follows from (E.15) and the definition
of $R_{mc}$. Second, we have

$$
\sum_{i=1}^{n} E \left[ \frac{(X_i, \Theta)}{\lambda_0 \sqrt{n}} \right]^2 I \left( \left| \frac{\xi_i}{\lambda_0 \sqrt{n}} \right| \geq \frac{1}{2} \right) \leq \sum_{i=1}^{n} \sqrt{E \left[ \frac{(X_i, \Theta)}{\lambda_0 \sqrt{n}} \right]^4} \sqrt{P \left( \left| \frac{\xi_i}{\lambda_0 \sqrt{n}} \right| \geq \frac{1}{2} \right)}
$$

(a) follows from Hölder’s inequality, (b) follows from relation between indicator function and expectation, (c) follows from Markov’s inequality, (d) follows from Assumption 2.3 and (e) follows from (E.16), and (f) follows from (E.12).

Instead of (C.31), from (E.11), we have

$$
E_{f_i^2}(\Theta) \leq E \left[ \frac{(X_i, \Theta)}{\lambda_0^2 n^2} \right]^4 \leq \frac{4 \alpha^2}{\lambda_0^2 n^2} \| \Theta \|_F^2
$$

(E.19)

and remembering $\sigma_2^2 = \sup_{\Theta \in R_{mc}} \sum_{i=1}^{n} E \{ f_i(\Theta) - E f_i(\Theta) \}^2$, we have

$$
\lambda_0^2 \times \sigma_2 \sqrt{8 \log(1/\delta)} \leq 4 \alpha^* \sqrt{\frac{\log(1/\delta)}{n}} \| \Theta \|_F.
$$

(E.20)

To obtain the result (E.13) by the strategy of the proof of Proposition C.1, the remaining term we must calculate is

$$
E \sup_{\Theta \in R_{mc}} \left| \sum_{i=1}^{n} \theta_i \frac{(X_i, \Theta)}{2 \lambda_0^2 n} \right|.
$$

(E.21)

From Lemma 6 of [56] and combining the definition of $R_{mc}$, we have

$$
E \sup_{\Theta \in R_{mc}} \left| \sum_{i=1}^{n} \theta_i \frac{(X_i, \Theta)}{2 \lambda_0^2 n} \right| \leq C \lambda_0 \left\{ \sqrt{d_{mc} \log d_{mc}} + \frac{d_{mc} \log d_{mc}}{n} \right\} \sup_{\Theta \in R_{mc}} \| \Theta \|_*
$$

$$
\leq C \lambda_0 \left\{ \sqrt{\frac{d_{mc} \log d_{mc}}{n}} + \frac{d_{mc} \log d_{mc}}{n} \right\} \| \Theta \|_F
$$

(E.22)

and combining $\alpha^* \geq 1$, the proof is complete.

**Corollary E.2.** Suppose that Assumption 2.4 holds. Let

$$
R_{mc} = \left\{ L \in \mathbb{R}^{d_1 \times d_2} | \| \Theta \|_* \leq C \sqrt{\| \Theta \|_F}, \| \Theta \|_F = r_{mc} \right\}.
$$

(E.23)

Suppose

$$
\| \Theta \|_\infty \leq \frac{1}{12 r_{mc} d_{mc}} \| \Theta \|_F,
$$

(E.24)

$$
\| \Theta \|_\infty \leq \frac{2 \alpha^*}{d_{mc}},
$$

(E.25)
and
\[ \lambda_0 \sqrt{n} \geq 2\sigma \varepsilon_\alpha \min \left\{ \log \frac{n}{\sigma}, \log \frac{n}{rd_{mc} \log d_{mc}} \right\}. \] (E.26)

Then, with probability at least 1 − \( \delta \), we have
\[
\inf_{\Theta \in \mathbb{R}^{mc}} \left[ \frac{\lambda_0^2}{n} \sum_{i=1}^{n} \left( -h\left( \frac{\xi_i - \langle X_i, \Theta \rangle}{\lambda_0 \sqrt{n}} \right) + h\left( \frac{\xi_i}{\lambda_0 \sqrt{n}} \right) \right) \right] \\
\geq \frac{2}{3} \| \Theta \|_F^2 - C \left( \alpha^* \sqrt{\frac{d_{mc} \log d_{mc} + \log(1/\delta)}{n}} + \sqrt{\frac{d_{mc} \log d_{mc}}{n}} + \alpha^* \sqrt{\frac{\sigma}{n}} \right) \| \Theta \|_F - 5 \log(1/\delta). \] (E.27)

Proof. The proof is almost identical to the one of Proposition C.1. Like Corollary E.1, because \( \{X_i\}_{i=1}^n \) for matrix completion is not L-subGaussian, we should calculate (C.25) and (C.31) in the same strategy of Corollary E.1. However, to calculate (C.26), we need another strategy than one of Corollary E.1 because of the difference of the assumption on random noise.

For the definition of \( X_i \), (E.25), we have
\[ \mathbb{E} \langle X_i, \Theta \rangle^4 \leq \mathbb{E} \langle X_i, \Theta \rangle^2 \| X_i \|_2^2 \| \Theta \|_\infty^2 \leq 2 \alpha^* \| \Theta \|_F^2. \] (E.28)

For (C.26), we have
\[
\sum_{i=1}^{n} \mathbb{E} \left| \frac{\langle X_i, \Theta \rangle}{\lambda_0 \sqrt{n}} \right|^2 \mathbb{I} \left( \left| \frac{\xi_i}{\lambda_0 \sqrt{n}} \right| \geq \frac{1}{2} \right) \leq \sum_{i=1}^{n} \mathbb{E} \left| \frac{\langle X_i, \Theta \rangle}{\lambda_0 \sqrt{n}} \right|^2 \mathbb{I} \left( \left| \frac{\xi_i}{\lambda_0 \sqrt{n}} \right| \geq \frac{1}{2} \right) \leq \sum_{i=1}^{n} \mathbb{E} \left| \frac{\langle X_i, \Theta \rangle}{\lambda_0 \sqrt{n}} \right|^2 \mathbb{I} \left( \left| \frac{\xi_i}{\lambda_0 \sqrt{n}} \right| \geq \frac{1}{2} \right) \leq \sum_{i=1}^{n} \mathbb{E} \left| \frac{\langle X_i, \Theta \rangle}{\lambda_0 \sqrt{n}} \right|^2 \mathbb{I} \left( \left| \frac{\xi_i}{\lambda_0 \sqrt{n}} \right| \geq \frac{1}{2} \right) \leq 2 \alpha^* \left( \sqrt{\frac{\sigma}{n}} + \sqrt{\frac{d_{mc} \log d_{mc}}{n}} \right) \| \Theta \|_F, \] (E.29)

where (a) follows from Hölder’s inequality, (b) follows from relation between indicator function and expectation, (c) follows from assumption, (d) follows from (E.26) and (e) follows from (E.28). \( \square \)

Appendix F: Proof of Theorem 2.1

Suppose that the assumptions of Theorem 2.1 hold. The proof is complete if we confirm the assumptions in Theorem B.1 with
\[ r_{mcs} = c_{mcs} \times \lambda_0 \sqrt{n} \times L \left( \frac{1 + \sqrt{\log(1/\delta)}}{\sqrt{n}} + c_\rho \sqrt{\frac{d_1 + d_2}{n}} + \frac{\sigma}{n} \sqrt{\log \frac{n}{\delta}} \right) \] (F.1)

are satisfied with probability at least 1 − 3\( \delta \). We divide the proof into four steps:

I. We confirm (B.7) and (B.8) are satisfied with probability at least 1 − 2\( \delta \).
II. We confirm (B.9) is satisfied.
III. We confirm (B.25) is satisfied with probability at least 1 − \( \delta \).
We confirm (B.26) is satisfied.

In this section, we set

\[ r_{a,F} = r_{b,F} = \frac{1 + \sqrt{\log(1/\delta)}}{\sqrt{n}} + \frac{\alpha}{n} \sqrt{\log n/\delta}, \quad r_{a,*} = r_{b,*} = \frac{d_1 + d_2}{n}, \]  

\[ r_{c,F} = \rho c \sqrt{\frac{d_1 + d_2}{n}} + \sqrt{\frac{\log(1/\delta)}{n}}, \quad r_c = \frac{\log(1/\delta)}{n} \]  

and \( C_{mcs} \) is some sufficiently large numerical constant.

**F.1. Proof of I**

Remember that (B.7) is

\[ \left| \frac{1}{\sqrt{n}} \sum_{i=1}^{n} h \left( \frac{\xi_i}{\lambda_0 \sqrt{n}} \right) \langle X_i, \Theta_\eta \rangle \right| \leq a_F r_{a,F} \left\| T_{\Sigma}(\Theta_\eta) \right\|_F + a_* r_{a,*} \left\| \Theta_\eta \right\|_* \]  

and (B.8) is

\[ \left| \sum_{i=1}^{n} \frac{\lambda_0}{\sqrt{n}} u_i(X_i, \Theta_\eta) \right| \leq b_F r_{b,F} \left\| T_{\Sigma}(\Theta_\eta) \right\|_F + b_* r_{b,*} \left\| \Theta_\eta \right\|_*. \]  

First, from Lemma C.1, we confirm (B.7) with

\[ a_F = C_{mcs} \lambda_0 \sqrt{nL}, \quad a_* = C_{mcs} \lambda_0 \sqrt{nL} \rho \]  

with probability at least \( 1 - \delta \). Second, from Corollary C.1 and \( \frac{a}{n} \leq 1 \) we confirm (B.8) with

\[ b_F = C_{mcs} \lambda_0 \sqrt{nL}, \quad b_* = C_{mcs} \lambda_0 \sqrt{nL} \rho, \]  

with probability at least \( 1 - \delta \). From union bound, we confirm (B.7) and (B.8) are satisfied with probability at least \((1 - \delta)^2\).

**F.2. Proof of II**

Remember we have

\[ \lambda_* = c_{mcs} \times \lambda_0 \sqrt{n} \times L \times \left( \frac{1}{c_\kappa \sqrt{F}} + \frac{1 + \sqrt{\log(1/\delta)}}{\sqrt{n}} + \rho \sqrt{\frac{d_1 + d_2}{n}} + \frac{\alpha}{c_\kappa \sqrt{F} n} \sqrt{\log n/\delta} \right) \]

\[ = c_{mcs} \times \lambda_0 \sqrt{n} \times L \times \left( \frac{r_{a,F}}{c_\kappa \sqrt{F}} + \rho r_{a,*} \right) \]  

for some sufficiently large numerical constant \( c_{mcs} \) and we have

\[ C_s = \frac{a_F + \sqrt{2b_F}}{c_\kappa \sqrt{F}} r_{a,F} + (a_* + \sqrt{2b_*}) r_{a,*} \]

\[ = (1 + \sqrt{2}) C_{mcs} \times \lambda_0 \sqrt{n} \times L \times \left( \frac{r_{a,F}}{c_\kappa \sqrt{F}} + \rho r_{a,*} \right) \]  

For a sufficiently large numerical constant \( c_{mcs} \), we see \( \lambda_* - C_s > 0 \).
F.3. Proof of III

Remember that (B.25) is
\[ c_1 \| T_\Sigma(\Theta_\eta) \|_F^2 - c_2 r_c, F \| T_\Sigma(\Theta_\eta) \|_F - c_3 r_c \leq \lambda_o \sum_{i=1}^n \left\{ - h \left( \frac{\xi_i - (X_i, \Theta_\eta)}{\lambda_o \sqrt{n}} \right) + h \left( \frac{\xi_i}{\lambda_o \sqrt{n}} \right) \right\} (X_i, \Theta_\eta) \]
(F.10)

and we can confirm (B.25) from Proposition C.1, \( L \geq 1 \) and the definition of \( \lambda_o \sqrt{n} \) with
\[ c_1 = \frac{1}{3}, \quad c_2 = C_{mcs} L, \quad c_3 = C_{mcs}^2 L^2. \]
(F.11)

F.4. Proof of IV

From the definition of \( \lambda_* \), \( \lambda_o \sqrt{n} \) and the values of \( a_F, a_*, b_F, b_*, c_1, c_2, c_3, r_{a,F}, r_{a,*}, r_{a,F}, r_{b,F}, r_{c,F}, r_c \), we have
\[ c_2 r_{c,F} + C_{\lambda_*} + \sqrt{c_1 c_3 r_c} \]
\[ \leq \frac{1}{c_1} \left( a_F + \sqrt{2b_F} r_{a,F} + (a_* + \sqrt{2b_*}) c_k \sqrt{T r_{a,*}} + \lambda_* c_n \sqrt{T} + c_2 r_{c,F} + \sqrt{c_1 c_3 r_c} \right) \]
\[ \leq C_{mcs}' \lambda_o \sqrt{n} \times L \left( r_{a,F} + \rho c_k \sqrt{T r_{a,*}} \right), \]
where \( C_{mcs}' \) is some sufficiently large constant. Remember that
\[ r_0 = c_{mcs}' \lambda_o \sqrt{n} \times L \left( \frac{1 + \sqrt{\log(1/\delta)}}{\sqrt{n}} c_k \rho \sqrt{r d_1 + d_2} \sqrt{n} + \frac{\sigma}{n} \right) \]
\[ = c_{mcs}' \lambda_o \sqrt{n} \times L \left( r_{a,F} + \rho c_k \sqrt{T r_{a,*}} \right). \]
(F.12)

For sufficiently large numerical constant \( c_{mcs}' \), we confirm (B.26).

Appendix G: Proof of Theorem 2.2

The proof of Theorem 2.2 is almost identical to that of Theorem 2.1. Therefore, we shall omit it.

Appendix H: Proof of Theorem 2.3

Suppose the assumptions of Theorem 2.3 hold. The proof is complete if we confirm the assumptions in Theorem B.1 with
\[ r_0 = c_{mcs}' \times \lambda_o \sqrt{n} \times L \left( \frac{\sigma \xi + \alpha^*}{n} \right) \]
\[ \left( \frac{\sqrt{d_{mc} \log d_{mc} + \log(1/\delta)}}{n} \lambda_o \sqrt{r d_{mc} \log d_{mc} + \log(1/\delta)} + \frac{d_{mc} \log d_{mc}}{n} \right) \]
\[ + \left( \frac{\lambda_o \sqrt{\alpha \sigma}}{n} + \alpha^* \frac{\sigma}{n} \right) \frac{\sqrt{d_{mc} \log d_{mc}}}{n}. \]
(H.1)

are satisfied with probability at least \( 1 - 2\delta \). We divide the proof into four steps:

I. We confirm (B.8) and (B.7) are satisfied with probability at least \( 1 - \delta \).
II. We confirm (B.9) is satisfied.
III. We confirm (B.25) is satisfied with probability at least \( 1 - \delta \).
IV. We confirm (B.26) is satisfied.
In Section H, we set

\[ r_a, F = 0, \quad r_{a,*} = \sigma \sqrt{\frac{d_{mc}(\log d_{mc} + \log(1/\delta))}{n}} + \lambda_0 \sqrt{\frac{d_{mc}(\log d_{mc} + \log(1/\delta))}{n}}, \quad r_b, F = \sqrt{\lambda_0 \sqrt{\frac{o}{n}} n}, \quad r_{b,*} = 0, \quad r_c, F = \alpha^* \sqrt{\frac{d_{mc}(\log d_{mc} + \log(1/\delta))}{n}} + \sqrt{\frac{d_{mc} \log d_{mc}}{n}} + \alpha^* \left( \frac{o}{n} \right)^{\frac{1}{2}(1+\alpha)}, \quad r_c = \frac{\log(1/\delta)}{n} \]

(H.2)

and \( C_{mc1} \) is some sufficiently large numerical constant.

Before the proceeding each steps above, we confirm that when

\[ \alpha(\Theta \eta) \geq \frac{1}{12r_0}, \]  

(H.3)

we have \( \|B^* - \hat{B}\|_F \leq 24\alpha^* r_0 \). From \( \alpha(\Theta \eta) \geq \frac{1}{12r_0} \), we have

\[ \frac{1}{r_0} \leq \alpha(\Theta \eta) = \frac{d_{mc}}{\|B^* - \hat{B}\|_\infty} \Rightarrow \|B^* - \hat{B}\|_F \leq 12 \sqrt{d_1 d_2} \times r_0 \|B^* - \hat{B}\|_\infty. \]  

(H.4)

From the spikiness condition and the constraint of the optimization problem and, we have \( \|B^*\|_\infty \leq \frac{d_{mc}}{2} \) and \( \|\hat{B}\|_\infty \leq \frac{d_{mc}}{2} \). Combining these inequalities and the triangular inequality, we have

\[ \|B^* - \hat{B}\|_F \leq 24\alpha^* r_0. \]  

(H.5)

Therefore, in the remaining part of Section H, we assume

\[ \alpha(\Theta \eta) \leq \frac{1}{12r_0}, \]  

(H.6)

**H.1. Proof of I**

Remember that (B.7) is

\[ \left| \frac{\lambda_0}{\sqrt{n}} \sum_{i=1}^n h \left( \frac{X_i, \Theta \eta}{\lambda_0 \sqrt{n}} \right) \right| \leq a_F r_{a,F} \|\Theta \eta\|_F + a_* r_{a,*} \|\Theta \eta\|_* \]  

(H.7)

and (B.8) is

\[ \left| \sum_{i=1}^n \frac{\lambda_0}{\sqrt{n}} u_i \langle X_i, \Theta \eta \rangle \right| \leq b_F r_{b,F} \|\Theta \eta\|_F + b_* r_{b,*} \|\Theta \eta\|_*. \]  

(H.8)

First, from Lemma E.1, we confirm (B.7) with

\[ a_F = 0, \quad a_* = C_{mc} \]  

(H.9)

with probability at least \( 1 - \delta \). Second, from (H.6), we can confirm (E.6):

\[ \|\Theta \eta\|_\infty \leq \frac{1}{12r_0} \|\Theta \eta\|_F \leq \frac{1}{12} \sqrt{\frac{\lambda_0 \sqrt{n}}{d_{mc}}} \|\Theta \eta\|_F. \]  

(H.10)

From Lemma E.2, we confirm (B.8) with

\[ b_F = \frac{1}{6}, \quad b_* = 0. \]  

(H.11)

Lastly, from union bound, we confirm (B.7) and (B.8) are satisfied with probability at least \( 1 - \delta \).
H.2. Proof of II

Remember we have

\[ \lambda_* = c_{mc1} \times \frac{1}{\sqrt{r}} \times \left( \sigma \xi \sqrt{r(d_1 + d_2) \{ \log(d_1 + d_2) \log(1/\delta) \}} \frac{\log d_{mc} + \log(1/\delta)}{\sqrt{n}} + \lambda_0 \frac{d_{mc} \log d_{mc} + \log(1/\delta)}{\sqrt{n}} + \sqrt{\lambda_0 \sqrt{n}} \right) \]

for a sufficiently large numerical constant \( c_{mc1} \) and we have

\[ C_s = \frac{a_F r_{a,F} + \sqrt{2} b_F r_{b,F}}{c \sqrt{r}} + (a_* r_{a,*} + \sqrt{2} b_* r_{b,*}) = \frac{\sqrt{2} r_{b,F}}{12 \sqrt{r}} + C_{mc1} r_{a,*} \leq \left( C_{mc1} + \frac{\sqrt{2}}{12} \right) \left( r_{a,*} + \frac{r_{b,F}}{\sqrt{r}} \right), \]

where we use the fact that we can set \( c_0 = 1, \kappa = 1 \) and \( c_\kappa = 2 \). Consequently, we confirm \( \lambda_* - C_s > 0 \).

H.3. Proof of III

Remember that (B.25) is

\[ c_1 \| \Theta_\eta \|_F^2 - c_2 r_{c,F} \| \Theta_\eta \|_F - c_3 r_c \leq \lambda_*^2 \sum_{i=1}^n \left\{ -h \left( \frac{\xi_i - \langle X_i, \Theta_\eta \rangle}{\lambda_0 \sqrt{n}} \right) + h \left( \frac{\xi_i}{\lambda_0 \sqrt{n}} \right) \right\} \] \[ \geq \left( \frac{2}{3} \| \Theta_\eta \|_F^2 - C \left( \alpha^* \sqrt{\frac{d_{mc} \log d_{mc} + \log(1/\delta)}{n}} + \sqrt{\frac{d_{mc} \log d_{mc}}{n}} + \alpha^* \left( \frac{\delta}{n} \right) \frac{\log(1/\delta)}{n} \right) \right) \] \[ \geq \frac{2}{3} \| \Theta_\eta \|_F^2 - C \left( \frac{\alpha^*}{\sqrt{n}} \frac{d_{mc} \log d_{mc} + \log(1/\delta)}{n} + \sqrt{\frac{d_{mc} \log d_{mc}}{n}} + \alpha^* \left( \frac{\delta}{n} \right) \frac{\log(1/\delta)}{n} \right) \]

for matrix compressed sensing. From (H.6) and the constraint of the optimization problem, we confirm that (E.10) and (E.11) and we have

\[ \inf_{\Theta \in \mathbb{R}^{mc}} \lambda_0^2 \sum_{i=1}^n \left\{ -h \left( \frac{\xi_i - \langle X_i, \Theta \rangle}{\lambda_0 \sqrt{n}} \right) + h \left( \frac{\xi_i}{\lambda_0 \sqrt{n}} \right) \right\} \] \[ \geq \frac{2}{3} \| \Theta \|_F^2 - C \left( \frac{\alpha^*}{\sqrt{n}} \frac{d_{mc} \log d_{mc} + \log(1/\delta)}{n} + \sqrt{\frac{d_{mc} \log d_{mc}}{n}} + \alpha^* \left( \frac{\delta}{n} \right) \frac{\log(1/\delta)}{n} \right) \] \[ \geq \frac{2}{3} \| \Theta_\eta \|_F^2 - C \left( \frac{\alpha^*}{\sqrt{n}} \frac{d_{mc} \log d_{mc} + \log(1/\delta)}{n} + \sqrt{\frac{d_{mc} \log d_{mc}}{n}} + \alpha^* \left( \frac{\delta}{n} \right) \frac{\log(1/\delta)}{n} \right) \]

with probability at least \( 1 - \delta \). We can confirm (B.25) by Corollary E.1 with

\[ c_1 = \frac{2}{3}, \quad c_2 = c_3 = C_{mc1}. \]

H.4. Proof of IV

From the definition of \( \lambda_* \), \( \lambda_0 \sqrt{n} \) and the values of \( a_F, a_*, b_F, b_*, c_1, c_2, c_3, r_{a,F}, r_{a,*}, r_{a,F}, r_{b,F}, r_{b,*}, r_{c,F}, r_c \), we have

\[ \frac{c_2 r_{c,F} + C_{mc1} + \sqrt{c_1 c_{mc1} r_c}}{c_1} \leq \frac{(a_F + \sqrt{2} b_F) r_{a,F} + (a_* + \sqrt{2} b_*) c_{mc1} \sqrt{r} r_{a,*} + \lambda_* \sqrt{r} + c_2 r_{c,F} + \sqrt{c_1 c_{mc1} r_c}}{c_1} \]

\[ \leq C_{mc1} \times (\sqrt{r} r_{a,*} + r_{b,F} + r_{c,F} + \sqrt{r_c}), \]

(H.17)
where $C'_m e_1$ is some sufficiently large constant and we use the fact that $\sqrt{r_c} \leq r_{x,F}$. Remember that $r_0$

$$r_0 = c'_m e_1 \times$$

$$\left( (\sigma_\xi + \alpha^*) \sqrt{\frac{rd_{mc}(\log d_{mc} + \log(1/\delta))}{n}} + \lambda_0 \sqrt{\frac{rd_{mc}(\log d_{mc} + \log(1/\delta))}{\sqrt{n}}} + \frac{\sqrt{r_{d_{mc}} d_{mc} \log d_{mc}}}{\sqrt{n}} + \left( \lambda_0 \sqrt{\frac{\sigma}{n}} + \alpha^* \left( \frac{o}{n} \right) \right) \right)$$

(H.18)

for sufficiently large numerical constant $c'_m e_1$, and we confirm (B.26).

**Appendix I: Proof of Theorem 2.4**

The proof of Theorem 2.4 is almost identical to that of Theorem 2.3. Therefore, we shall omit it.