The effect of measurements, randomly distributed in time, on quantum systems: stochastic quantum Zeno effect

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Abstract
The manifestation of measurements, randomly distributed in time, on the evolution of quantum systems are analyzed in detail. The set of randomly distributed measurements (RDM) is modeled within the renewal theory, in which the distribution is characterized by the probability density function (PDF) \( W(t) \) of times \( t \) between successive events (measurements). The evolution of the quantum system affected by the RDM is shown to be described by the density matrix satisfying the stochastic Liouville equation. This equation is applied to the analysis of the RDM effect on the evolution of a two-level system for different types of RDM statistics, corresponding to different PDFs \( W(t) \). Obtained general results are illustrated as applied to the cases of the Poissonian (\( W(t) \sim e^{-\lambda t} \)) and anomalous (\( W(t) \sim 1/t^{1+\alpha}, \alpha \leq 1 \)) RDM statistics. In particular, specific features of the quantum and inverse Zeno effects, resulting from the RDM, are thoroughly discussed.

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1. Introduction
The effect of measurements on the evolution of quantum systems has recently been studied very actively both experimentally and theoretically (for a comprehensive review see, for example, [1–4]). The interest to this problem was inspired by the pioneering paper [5] concerning the analysis of the specific feature of the manifestation of measurements which was called the Zeno effect, showing itself in the strong decrease of the decay rate of the quantum state with the increase of a number of measurements [5]. Since then a lot of work, analyzing different aspects of this effect, have been published.
Traditionally, the quantum Zeno effect is considered assuming a set of measurements to be distributed equidistantly in time (with constant time between measurements). The equidistant distribution results evidently in simplification of the mathematical treatment of the problem and experimental observation of the measurement effect [1–4].

Naturally, the majority of manifestations of the measurement effect (including the Zeno effect), found for equidistant distribution of measurements, are expected to occur in the case of irregular distribution as well. However, some additional analysis is certainly needed.

In this work we will discuss the measurement effect in the interesting special case of the irregular distribution, the case of measurements randomly distributed in time, called hereafter randomly distributed measurements (RDM). The random process of measurements is modeled within the renewal approach (RA) which treats the sequence of measurements as a stochastic set of renewals [6, 7]. In the RA the distributions of time intervals $t$ between successive renewals are assumed to be stochastically independent and are described by the probability density function (PDF) $W(t)$ (often denoted as $ψ(t)$ [6]).

The evolution of the quantum system, affected by the RDM of this type, is shown to be described by the stochastic Liouville equation (SLE) for the density matrix of the system [8, 9]. The SLE allows one to analyze the effect of the RDM for different types of PDFs $W(τ)$ in a fairly simple analytical form.

In our work general expressions for the density matrix of the system, affected by the RDM, are derived. With the use of these expressions some important specific features of the RDM effect are analyzed for different types of $W(t)$ behavior. In particular, the quantum Zeno effect, i.e. the decrease of the decay rate of the state with decreasing the average time between measurements $\bar{t}$, is predicted only in the case of rapidly decreasing PDFs $W(t)$ for which the average time between measurements $\bar{t} = \int_0^{\infty} dτ \tau W(τ)$ is finite. As to heavy tailed PDFs, which cannot be described by a finite $\bar{t}$ ($\bar{t} \to \infty$), for these PDFs the quantum Zeno effect is shown to be absent. Such a drastic difference of the RDM effect in these two cases appears to be conveniently treated in terms of the Zeno effect dependence on the characteristic rate $w_r$ of the Laplace transform $\tilde{W}(ε)$ as a function of the Laplace variable $ε$. The fact is that, unlike $\bar{t}$, the rate $w_r$ can be introduced independently of the mathematical form of $W(t)$ decrease, as will be shown in our work, though for rapidly decreasing $W(t)$ these two parameters are closely related: $\bar{t} \sim 1/w_r$.

To illustrate the general results we will discuss the case of the Poissonian distribution with $W(t) = e^{-w_r t}$, as an example of the RDM with rapidly decreasing PDF, and the case of anomalous heavy tailed distribution $W(t) \sim 1/τ^{1+\alpha}$ with $\alpha \leq 1$.

In particular, within the Poissonian model some characteristic properties of the RDM effect are studied which are typical for the case of rapidly decreasing PDF $W(t)$. These properties manifest themselves in specific features of the time dependence of the probability $p(t)$ to survive in the measured state. It is shown, for instance, that in the limit of small characteristic time $t_r = w_r^{-1}$ between measurements $p(t)$ is the exponentially decreasing function with the rate non-monotonically depending on $t_r$: the decreasing dependence at very small $t_r$, corresponding to the quantum Zeno effect, is changed by the increasing one, associated with the inverse Zeno effect [2–4, 10], as $t_r$ is increased. The time $t_{rm}$ of the change is determined by the parameters of the system.

In both Poissonian and anomalous RDM models, the manifestation of relaxation and, for example, the appearance and specific features of the quantum and inverse Zeno effects in the presence of relaxation are also analyzed by means of the proposed approach.

In conclusion, possible realistic examples of processes, in which the predicted effects can be observed, are thoroughly discussed. In particular, some realizations of both Poissonian and anomalous RDM in the processes with the participation of Brownian particles are considered.
2. Formulation of the problem

In this work we consider the effect of the RDM on the evolution of dynamic quantum systems, i.e. the systems in which relaxation is absent. The statistical properties of the RDM are treated within the renewal theory [6, 7], well known in statistical physics.

In order to analyze the RDM effect we, first, should clarify some details of this effect for the finite number, say, $n$ of measurements.

2.1. Measurement affected quantum dynamics

The dynamic quantum system is characterized by the wavefunction $|\psi(t)\rangle$ (the vector in the Hilbert space) whose time evolution is governed by the Schrödinger equation ($\hbar = 1$)

$$|\dot{\psi}\rangle = -iH|\psi\rangle, \quad \text{with} \quad H = H_0 + H_i. \quad (2.1)$$

In this equation, $H$ is the Hamiltonian of the system represented as a sum of the free and interaction parts, $H_0$ and $H_i$, respectively. The initial condition for equation (2.1) $|\psi(t = 0)\rangle = |\psi_i\rangle$ depends on the process considered (see below).

The time evolution of the wavefunction is usually described by expansion in some (complete) basis of functions $|\psi_j\rangle$ in the Hilbert space. In our further analysis, for the sake of convenience, we will assume $|\psi_j\rangle$ to be the basis of eigenfunctions of the free Hamiltonian $H_0$.

In accordance with the conventional von Neumann rule [2–4, 11], measurements, which show that the system is in some state suggested to be the eigenstate of $H_0$ with the wavefunction $|\psi_0\rangle (H_0|\psi_0\rangle = \omega_0|\psi_0\rangle)$, enable one to obtain the probability $p(t)$ to find the system in the state $|\psi_0\rangle$:

$$p(t) = |\langle \psi_0 | \psi(t) \rangle|^2. \quad (2.2)$$

In what follows we will analyze the probability $p(t)$ of survival in the state $|\psi_0\rangle$ after a set of measurements. Following [1, 3] the subtle problem of the evaluation of the multiple measurement effect will be treated assuming that ‘if every time the measurement has a positive outcome and the system is found in the initial state, the wavefunction “collapses” and the evolution starts anew from $|\psi_0\rangle$’. In this case the problem is to calculate the probability $p(t)$ to find the system, initially created in $|\psi_0\rangle$ (i.e. with $|\psi_i\rangle = |\psi_0\rangle$), in the same state $|\psi_0\rangle$ after the measurements.

It is worth noting that, in fact, the effect of measurements reduces to population and phase relaxation in the system under study (see below). In such a case, in general, the evolution of the system should be described with the density matrix $\rho(t)$ satisfying the Schrödinger-like equation but in the Liouville space [11]. In this space, the density matrix is represented as $\rho(t) = |\psi(t)\rangle\langle\psi(t)|$. The above-mentioned Schrödinger equation (in the Liouville space) for this matrix is similar to equation (2.1) but with the Hamiltonian operator $H$ replaced by the superoperator $\hat{H}$:

$$\frac{\text{d}\rho}{\text{d}t} = -i\hat{H}\rho, \quad \text{where} \quad \hat{H}\rho = [H, \rho] = H\rho - \rho H. \quad (2.4)$$
According to definition (2.2) the matrix elements of the superoperator $\hat{H}$ are given by
$$
\langle kk' | \hat{H} | jj' \rangle = \langle \psi_k | H | \psi_j \rangle \delta_{k'j'} - \langle \psi_j | H | \psi_k \rangle \delta_{k'j}.
$$
It is important to note that for the Hermitian Hamiltonian $H$ the superoperator $\hat{H}$ is also Hermitian (but in the Liouville space).

The effect of a measurement on the evolution of the quantum system is conveniently determined in terms of the projection superoperator $P_m = |00\rangle \langle 00|$, where $|00\rangle = |\psi_0\rangle |\psi_0\rangle$. In particular, after $n$ successive measurements at times $\{t_j\} (j = 0, \ldots, n)$ ordered as $t \geq t_n \geq t_{n-1} \geq \cdots \geq t_1 \geq t_0 = 0$, the density matrix $\rho(t)$ can be obtained with the formula
$$
\rho(t) = \hat{U}_n(t) \rho_0 \quad (n = 0, 1, \ldots),
$$
in which $t_n = (t_0, \ldots, t_0)$ is the vector of measurement times, $\hat{U}_n(t) = \hat{U}_0(t - t_0)$ and
$$
\hat{U}_{k,\gamma}(t) = \hat{U}_0(t - t_k) \prod_{j=1}^n \left( \hat{P}_m \hat{U}_0(t_j - t_{j-1}) \right)
$$
with $\hat{U}_0(t) = e^{-iHt}$ being the evolution superoperator (operator in the Liouville space) and $\rho_0 = |\psi_0\rangle \langle \psi_0| \equiv |00\rangle$.

With the density matrix, the observable under study, i.e. the probability of survival in the state $|\psi_0\rangle$ after $n$ measurements, is expressed as
$$
p_n(t) = \langle \psi_0 | \rho(t) | \psi_0 \rangle = \text{Tr}\left[ \hat{P}_m \hat{U}_n(t) \right],
$$
where the trace is evaluated over the states $|jj'\rangle$ in the Liouville space.

The probability $p_n(t)$ is known to depend not only on the parameters of the Hamiltonian $H$ but also on the number $n$ of measurements. The dependence is thoroughly analyzed in a large number of publications assuming of fixed time interval $\tau_n = t/n$ between measurements (see, for example, reviews [2–4, 11]). Below we will extend the analysis assuming the multiple measurements to be a stochastic processes (in general non-Markovian), which will be modeled within the RA [6, 7].

### 2.2. Renewal approach

#### 2.2.1. General formulas.

In the RA the times $t_n$ of events (measurements) are considered to be randomly distributed in time with intervals $\tau_n = t_n - t_{n-1}$ between successive events described as independent random variables with the monotonically decreasing PDFs $W(\tau_n)$, the same for all intervals. Note that $W(t)$ is defined only for positive $t$, with $W(t < 0) = 0$. To completely characterize the statistics of events one also needs the probability $P(t)$ that the interval between the successive events is greater than $t$: $P(t) = \int_t^{\infty} \mathrm{d}t' W(t')$, which is, naturally, normalized by the relation $P(0) = 1$.

In what follows we will mainly operate with the Laplace transforms denoted as
$$
\tilde{Z}(\epsilon) = \int_0^{\infty} \mathrm{d}t \ Z(t) \ e^{-\epsilon t}
$$
for any function $Z(t)$. In particular, worth noting is the relation $\tilde{P}(\epsilon) = [1 - \tilde{W}(\epsilon)]/\epsilon$ and suitable representations
$$
\tilde{W}(\epsilon) = [1 + \Phi(\epsilon)]^{-1} \quad \text{and} \quad \tilde{P}(\epsilon) = [\epsilon + \epsilon/\Phi(\epsilon)]^{-1}
$$
in terms of the auxiliary function $\Phi(\epsilon)$.

The analytical form of $\Phi(\epsilon)$ is completely determined by that of $W(t)$. In what follows, to specify the characteristic scale of $\Phi(\epsilon)$-dependence we will introduce the characteristic rate $w_r$, whose meaning will come from some particular examples of functions $\Phi(\epsilon)$ considered...
below. It is important that this rate can be introduced for both rapidly and anomalously slowly decreasing PDFs $W(t)$ (see below).

The effects analyzed in this work are essentially controlled by the small $\epsilon$ behavior of $\Phi(\epsilon)$. In general, we can only say that $\Phi(\epsilon) \sim \epsilon^{0}$. Some additional information on the $\Phi(\epsilon)$ behavior at $\epsilon \rightarrow 0$ can be obtained by the analysis of the long time dependence of $W(t)$.

1. If at $t \rightarrow \infty$ the PDF $W(t)$ decreases so rapidly that average time (first moment) $\overline{t} = \int_{0}^{\infty} dt tW(t) \sim w_{r}^{-1}$ is finite, at small $\epsilon$ the function $\Phi(\epsilon)$ can be represented as [6]

$$\Phi(\epsilon) \sim \epsilon^0 = o(\epsilon/w_{r}).$$

(2.10)

2. If, however, $W(t)$ is a heavy tailed function: $W(t \rightarrow \infty) \rightarrow 1/t^{1+\alpha}$ with $\alpha < 1$, and, therefore, $\overline{t}$ is infinite (does not exist), then $\Phi(\epsilon) \sim (\epsilon/w_{r})^{\alpha}$ [6, 7].

Within the RA, due to independence of renewals, the PDF $W_{n}(t)$ of $n$ events ($n \geq 0$) at times $\{t_{j}\} (j = 0, 1, \ldots, n)$, satisfying the relation $t_{n} \geq t_{n-1} \geq \cdots \geq t_{1} \geq t_{0} = 0$ and combined into the vector $t_{n} = (t_{n}, \ldots, t_{1})$, is given by

$$W_{t_{n+1}} = \prod_{j=1}^{n} W(t_{j} - t_{j-1}).$$

(2.11)

These functions completely describe the stochastic renewal process. In particular, with the use of $W_{n}$, the probability $\pi_{n}(t)$ to observe $n$ events in the time interval $(0, t)$ is expressed as [6]

$$\pi_{n}(t) = \int_{0}^{t} dt' P(t - t')W_{n}(t')dt' \ (n \geq 0).$$

(2.12)

In this formula $W_{n}(t)$ is the PDF of $n$ events, which for $n \geq 2$ is equal to $W_{n}$ integrated over all $t_{j}\leq\alpha-1$ (from 0 to $t_{j+1}$, respectively) except for $t_{n} = t$, and for $n = 0, 1$ is defined by $W_{1}(t) = \overline{W}(t)$, and $W_{0}(t) = \delta(t)$. The functions $\pi_{n}(t)$ can, evidently, be represented as

$$W_{n}(t) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} de \bar{W}^{n}(\epsilon) e^{\epsilon t} de.$$

(2.13)

Note that according to equation (2.13) $W_{n}(t) = \delta(t)$ and therefore $\pi_{n}(t) = P(t)$. In addition, it is worth noting that the probabilities $\pi_{n}(t)$ satisfy the normalization condition $\sum_{n=0}^{\infty} \pi_{n} = 1$.

2.2.2. Examples of renewal processes. In our further analysis special attention will be paid to some particular renewal processes corresponding to different distributions of times of renewals, i.e. different functional forms of the PDF $W(t)$.

**Poissonian distribution.** The most simple is the Poissonian model, corresponding to [6, 7]

$$P(t) = e^{-w_{r}t}, \quad W(t) = w_{r}e^{-w_{r}t},$$

(2.14)

so that $\bar{W}(\epsilon) = [1 + (\epsilon/w_{r})]^{-1}$ and therefore

$$\Phi(\epsilon) = \epsilon/w_{r}.$$  

(2.15)

**Equidistant distribution.** The model of equidistant distribution, in which

$$P(t) = \theta(t_{r} - t), \quad W(t) = \delta(t - t_{r}),$$

(2.16)

with $t_{r} = 1/w_{r}$, describes the set of events with the constant interval $t_{r}$ between successive ones [12]. This is just the model considered in almost all publications concerning the Zeno effect. In this model

$$\Phi(\epsilon) = e^{\epsilon t_{r}} - 1.$$  

(2.17)
Anomalous distribution. The anomalous model implies the Lévy-type distribution [13] of times between events with the heavy tailed behavior of $W(t) \sim 1/t^{1+\alpha}$ ($\alpha \leq 1$). One can find a number of such type of models which predict the same results for long time features of processes. In our analysis we use the simple one, for which [6–8]

$$P(t) = E_{\alpha}(-w_{\tau}^\alpha), \quad W(t) = -\dot{P}(t) \quad (\alpha \leq 1), \quad (2.18)$$

where $E_{\alpha}(-x^\alpha) = (2\pi i)^{-1} \int_{-\infty}^{\infty} dz e^{\pi z(1-\alpha)^{-1}}$ is the Mittag–Leffler function [13]. This model corresponds to

$$\Phi(\epsilon) = (\epsilon/w_{\tau})^\alpha. \quad (2.19)$$

It is easily seen that expression (2.18) predicts the heavy tailed behavior the PDF $W(t)$.

3. Stochastic Liouville equation

The above consideration shows that the RDM effect on the evolution of the quantum system under study is expressed in terms of the superoperator $\hat{U}(t)$ (equation (2.6)) averaged over the stochastic process of measurements. In what follows this averaged superoperator will be denoted as $\check{U}(t)$.

The problem of finding $\check{U}(t)$, which is a functional of type of (2.6) averaged over the renewal process, has already been discussed in the literature [8, 9]. It is shown to reduce to solving some equation for this operator, which is called the SLE.

For some types of renewal processes the averaging is essentially simplified. The most well-known example is the Poissonian process, simplification for which results from the Markovian nature of this process. In the RDM analysis the Poissonian process of quantum transitions (jumps) is just a realization of the Markovian process of ‘migrations’ (or jumps) over states $|jj\rangle$ of the quantum system under study (in the Liouville space). The treatment of this Poissonian ‘migration’ process within the widely accepted continuum time random walk approach [14, 15] shows that the PDF $\rho_{M}$ for the ‘migrating’ system satisfies the Markovian evolution equation

$$\dot{\rho}_{M} = -w_{\tau}(1 - \check{P}_{m})\rho_{M}. \quad (2.11)$$

The effect of such ‘migrations’ on quantum evolution is described by the SLE for the evolution superoperator $\dot{\check{U}}(t) = -iH\check{U}(t) - w_{\tau}(1 - \check{P}_{m})\check{U}(t)$ [16] (see section 4.3).

Fortunately, the SLE can be obtained for any type of renewal process. In the most general form the SLE is rigorously derived with the Markovian representation of the RA [8, 9]. In this work, however, we will restrict ourselves to the simplest variant of the RA for which the SLE can be obtained fairly easily. Below we will outline some details of the derivation.

The derivation is based on the fact that in the absence of measurements ($n = 0$) the evolution operator $\check{U}_{n}(t) = \check{U}_{0}(t)$, while for any number $n \geq 1$ of measurements the operator $\check{U}_{n}(t)$ (equation (2.6)) and the PDF of measurements $W_{n}$ (2.11) are represented as products of terms depending on differences of times $t_{j} - t_{j-1}$. In such a case we obtain the following formulas for corresponding average evolution operators $\check{U}_{n}(t)$:

$$\check{U}_{0}(t) = P(t)\check{U}_{0}(t) \quad \text{for} \quad n = 0$$

and

$$\check{U}_{n}(t) = \int_{0}^{t} dt_{n} P(t - t_{n})W_{n}\check{U}_{n-1}(t) \quad \text{for} \quad n \geq 1. \quad (3.1)$$

where $dt_{n} = \prod_{j=1}^{n} dt_{j}$. The convolution-like form of formulas for $\check{U}_{n}(t)$ results in a simple representation for the Laplace transforms of these functions:

$$\hat{\check{U}}_{n}(\epsilon) = \hat{P}(\hat{\Omega}_{n})[\hat{\check{P}}_{m}\hat{W}(\hat{\Omega}_{n})]^\alpha \quad \text{with} \quad \hat{\Omega}_{n} = \epsilon + i\hat{H}. \quad (3.2)$$
Summing up the contributions for different numbers of measurements we thus get the evolution operator \( \hat{U}(t) \) averaged over the renewal process [8, 9]

\[
\hat{U}(\epsilon) = \sum_{n=0}^{\infty} \hat{U}_m(\epsilon) = \hat{P}(\hat{\Omega}_e) [1 - P_m \hat{W}(\hat{\Omega}_e)]^{-1}
\]

(3.3)

where \( \hat{Q}_m = 1 - \hat{P}_m = \sum_{jj'} |jj'\rangle \langle jj'| \) is the superoperator of projection onto the subspace \( \{|jj'\rangle\} \), \( (jj' \neq 00) \).

The expression (3.4) can be treated as a solution of the non-Markovian SLE for \( \hat{U}(t) \)

\[
\hat{U}(t) = -i \hat{H} \hat{U} - \hat{Q}_m \int_0^t d\tau \hat{M}(\tau) e^{-i\hat{H} t} \hat{U}(t - \tau)
\]

(3.5)

in which

\[
\hat{M}(t) = \frac{1}{2\pi i} \frac{d}{dt} \int_{-\infty}^{\infty} de \ e^{\epsilon t} \Phi^{-1}(\epsilon)
\]

(3.6)

is the memory function whose analytical properties are essentially determined by those of the PDF \( \hat{W}(t) \).

The SLE (3.5) is not quite convenient for applications. In our further analysis we will mainly use expression (3.4) for the Laplace transform of the evolution operator.

In accordance with the general formula (2.7) the (average) probability \( p(t) \) of survival in the initial state \( (|\psi_0\rangle) \) is completely determined by the average evolution operator \( \hat{U}(t) \). In terms of the Laplace transforms the corresponding expression is written as

\[
\tilde{p}(\epsilon) = \text{Tr}[\hat{P}_m \hat{U}(\epsilon)] = \text{Tr}[\hat{P}_m \hat{U}(\epsilon) \hat{P}_m],
\]

(3.7)

where, similar to formula (2.7), trace is evaluated over the states \( |jj'\rangle \) in the Liouville space.

In the conclusion of this general analysis we will show that with the use of above-obtained formulas \( \tilde{p}(\epsilon) \) can be expressed in terms of the probability \( p_0(t) = (00)\hat{U}_0(t)(00) = (00)e^{-i\hat{H}t}(00) \) of survival in the initial state \( |\psi_0\rangle \) in the absence of measurements. According to equation (3.7) the Laplace transform \( \tilde{p}(\epsilon) \) is determined by the trace of the supermatrix

\[
\hat{\tilde{U}}_p(\epsilon) \equiv \hat{P}_m \hat{U}(\epsilon) \hat{P}_m = \sum_{n=0}^{\infty} \hat{\tilde{U}}_p(\epsilon),
\]

(3.8)

where \( \hat{\tilde{U}}_p(\epsilon) = \hat{P}_m \hat{U}_m(\epsilon) \hat{P}_m \). Each term \( \hat{\tilde{U}}_p(\epsilon) \) in the sum can be represented by formula

\[
\hat{\tilde{U}}_p(\epsilon) = \hat{P}_p(\epsilon) \hat{W}_p(\epsilon),
\]

(3.9)

In this expression we have introduced the supermatrices

\[
\hat{\tilde{X}}_p(\epsilon) = \hat{P}_m \hat{X}(\hat{\Omega}_e) \hat{P}_m \quad \text{for} \quad X = P, W
\]

(3.10)

directly related to the probability \( p_0(t) \), as is clear from relations \( \hat{P}_m \hat{X}(\hat{\Omega}_e) \hat{P}_m = \int_0^\infty dt \ e^{-\epsilon t} \hat{P}_m e^{-i\hat{H}t} \hat{X}(t) \hat{P}_m e^{i\hat{H}t} \hat{P}_m = \hat{P}_m p_0(t) \). Finally, we obtain

\[
\hat{\tilde{U}}_p(\epsilon) = \hat{P}_m \hat{P}_p(\epsilon) \hat{W}_p(\epsilon),
\]

(3.11)

where

\[
\hat{\tilde{X}}_p(\epsilon) = \int_0^\infty dt \ e^{-\epsilon t} X(t) \hat{P}_m p_0(t) \quad \text{for} \quad X = P, W.
\]

(3.12)
Substitution of formula (3.11) into equations (3.8) and (3.7) yields
\[ \tilde{p}(\epsilon) = \tilde{P}_m(\epsilon)[1 - \tilde{W}_p(\epsilon)]^{-1}. \]  

(3.13)

In principle, both formulas (3.7) (with equation (3.4)) and (3.13) are quite suitable for the analysis of the RDM effect on \( p(t) \). In what follows, however, we will mainly apply the formulation based on the first formula (3.7), though equation (3.13) will also be used to clarify some particular properties of the effect.

4. Zeno effect on two-level systems

In this section we will analyze the effect of the RDM on some model quantum system to illustrate the specific features of manifestation and treatment of the Zeno effect in this kind of measurements.

The simplest (though quite realistic) system, which enables one to significantly simplify mathematical problems, is the two-level system. It is very convenient for a detailed description of all important features of the Zeno effect [1–4].

4.1. Hamiltonian of the two-level system

In our analysis we will use the Hamiltonian in the form (see equation (2.1))
\[ H_0 = \epsilon(|1\rangle\langle 1| - |2\rangle\langle 2|), \quad H_i = \nu(|1\rangle\langle 2| + |2\rangle\langle 1|) \]  

(4.1)
in which \( \epsilon \) and \( \nu \) are positive real parameters. The measured state is assumed to be the state \( |1\rangle \), i.e. \( |\psi_0\rangle = |1\rangle \).

In the Liouville space the Hamiltonian is represented as a \( 4 \times 4 \) matrix:
\[ \hat{H}_0 = 2\epsilon(|12\rangle\langle 12| - |21\rangle\langle 21|), \]  
\[ \hat{H}_i = \nu(|11\rangle - |22\rangle)(|21\rangle - |12\rangle) + (|21\rangle - |12\rangle)(|11\rangle - |22\rangle). \]  

(4.2)
(4.3)
in the basis \( |jj\rangle = |j\rangle\langle j| \) (defined in equation (2.2)). In this basis the superoperator \( \hat{P}_m \), describing the effect of a measurement, is written as
\[ \hat{P}_m = |11\rangle\langle 11| \quad \text{and} \quad \hat{Q}_m = \sum_{jj'\neq 11} |jj\rangle\langle jj'| . \]  

(4.4)
The Laplace transform of the function under study, the survival probability \( \tilde{p}(\epsilon) \), can conveniently be evaluated with expression (3.7), which, as applied to the two-level system considered, is written as
\[ \tilde{p}(\epsilon) = \text{Tr}[\hat{P}_m \hat{U}(\epsilon)] = (|1\rangle\langle 1| \hat{U}(\epsilon)|1\rangle). \]  

(4.5)

With the above formulas at hand one can analyze the specific features of the Zeno effect for any type of the RDM.

4.2. General results

Here we obtain some general results, valid for quantum systems with arbitrary (but finite) number of levels, to clarify the manifestation of the analytical properties of the decreasing function \( W(t) \) in the specific features of the Zeno effect.

The most important property of function \( W(t) \) is the rate of decrease at long times which is determined by the behavior of \( \Phi(\epsilon) \) in the limit \( \epsilon \to 0 \) (section 2.2.1), i.e. at \( \epsilon \ll w_r \), where \( w_r \) is the above-defined rate, characterizing the time \( t_r = 1/w_r \) of the onset of the asymptotic long time behavior of \( W(t) \). Just in this limit, or more accurately in the limit of large \( w_r \),
when $\xi = \|\hat{\Omega}\|/w_r \ll 1$, one can demonstrate some important properties of the RDM effect. In principle, essential conclusions can be made without any particular assumptions on $\Phi(\epsilon)$-behavior except for the relation $\Phi(\epsilon) \to 0$. However, in the analysis it is suitable to keep in mind the approximation $\Phi(\epsilon \to 0) \approx (\epsilon/w_r)^{\alpha}$, with $\alpha \leq 1$, which is of particular interest for our further discussion.

The above-mentioned important general conclusions concern the properties of $\tilde{p}(\epsilon)$ at $\xi \ll 1$. In this limit we get from equations (3.4) and (4.5)

$$\tilde{p}(\epsilon) \approx \tilde{p}_\infty(\epsilon) = \text{Tr}[\hat{P}_m \hat{\Omega}_m^{-1} \Phi(\hat{\Omega}_m)]/(\|\Phi(\hat{\Omega}_m)\|11)$$

$$= (\|\hat{\Omega}_m^{-1} \Phi(\hat{\Omega}_m)\|11)/(\|\Phi(\hat{\Omega}_m)\|11).$$

This general expression is valid for systems with any number of levels, though it is presented in terms of the considered two-level model (4.2), (4.3).

Formula (4.6) can be derived from equations (3.4) and (4.5) by taking into account some characteristic properties of the matrix $\hat{L} = \Phi(\hat{\Omega}_m) + \hat{Q}_m$. For brevity, in our further study we will use the notation $\Phi(\hat{\Omega}_m) = \hat{\Phi}$. To clarify the derivation we will, first, analyze the specific features of two parts of $\hat{L}$: $\hat{L}_P = \hat{P}_m \hat{\Omega}_m \hat{P}_m = \hat{P}_m \hat{\Phi} \hat{\Omega}_m + \hat{\Phi} \hat{\Omega}_m \hat{P}_m$ and $\hat{L}_Q = \hat{Q}_m \hat{\Omega}_m \hat{Q}_m = \hat{Q}_m \hat{\Phi} \hat{\Omega}_m + \hat{\Phi} \hat{\Omega}_m \hat{Q}_m$, operating in subspaces $\{\{11\}\}$ and $\{\{j\neq 1\}\}$, respectively. The fact is that in the considered limit $\xi = \|\hat{\Omega}_m\|/w_r \ll 1$, corresponding to $\|\hat{\Phi}\| \ll 1$, the eigenvalue $\lambda_P = (\|\hat{\Phi}\|11) \ll 1$ of the matrix $\hat{L}_P$ is much smaller than $\lambda_Q$-eigenvalues $\lambda_Q \sim 1$, whose magnitudes are mainly determined by $\hat{Q}_m$. These estimations yield for the characteristic splitting of eigenstates $\delta \lambda = \lambda_Q - \lambda_P$ the value $\delta \lambda \sim 1$.

The obtained splitting appears to be much larger than the $\hat{L}$-induced interaction $\hat{L}_{QP} = \hat{P}_m \hat{\Omega}_m \hat{P}_m + \hat{Q}_m \hat{\Phi} \hat{\Omega}_m + \hat{\Phi} \hat{\Omega}_m \hat{Q}_m$ between the states of $\{\{11\}\}$ and $\{\{j\neq 1\}\}$ subspaces: $\|\hat{L}_{QP}\| \sim \|\hat{\Phi}\| \ll \delta \lambda$. This means that in the leading order in $\|\hat{\Phi}\|/\delta \lambda \sim \|\hat{\Phi}\| \ll 1$ the eigenstates and eigenvalues of the matrix $\hat{L}$ coincide with those of $\hat{L}_P + \hat{L}_Q$. Of special importance is the coincidence of the lowest eigenstate $\{|l\rangle\}$ of $\hat{L}$ with $\{\{11\}\}$: $\langle l \rangle \sim (\langle 11 \rangle + \sum_{j\neq 1} \zeta_{jj} \langle j \rangle \rangle$ with $\zeta_{jj} \sim \|\hat{\Phi}\| \ll 1$, which results in formula $\langle l|\hat{L}|l\rangle = (\langle 11 \rangle\hat{L}_{P}|11\rangle + (\|\hat{\Phi}\|))$ with $\langle 11 \rangle\hat{L}_{P}|11\rangle = (\langle 11 \rangle|\hat{\Phi}|11\rangle$. This formula implies that with the accuracy $\|\hat{\Phi}\| \ll 1$ the Green’s function $\hat{L}^{-1}$ is mainly determined by the contribution of the eigenstate $|l\rangle \sim |11\rangle$:

$$\hat{L}^{-1} = \hat{P}_m (\|\hat{\Phi}\|11)^{-1}[1 + O(\|\hat{\Phi}\|)].$$

Other eigenstates (which with high accuracy coincide with those of $\hat{L}_Q$) make the contribution much smaller than that of $|l\rangle$, since the corresponding eigenvalues $\lambda_{Q\ell} \sim 1$ are much larger than $\lambda_P = (\|\hat{\Phi}\|11) \ll 1$.

Substitution of expression (4.7) into equations (3.4) and (4.5) yields formula (4.6). Note that in the same way it can also be derived by means of equation (3.13).

This formula allows one to analyze the limiting behavior of the RDM effect in the limit $w_r \to \infty$, i.e. for very small characteristic time between measurements.

In particular, we can discuss the case of rapidly decreasing $W(t)$ for which $\Phi(\epsilon) \to 0$ (see equation (2.10)). Taking into account relation (2.10) one arrives at the estimation $\tilde{p}(\epsilon) = \epsilon^{-1}[1 + O(|\hat{\Omega}|/w_r)]$ which means that

$$p(t) = \lim_{w_r \to \infty} \frac{w_r}{r} = 1.$$  

The limiting relation (4.8) demonstrates the localization of the system in the measured state in the limit $w_r \to \infty$, which is associated with the quantum Zeno effect [2-4] expressed in terms of the RA.

The dependence of $p(t)$ on the average time $\bar{t} \sim w_r^{-1}$ between measurements is, in general, not universal. It is also determined by other parameters of the system, for example,
the parameters of the Hamiltonian. In such a case, to characterize the quantum Zeno effect as a function of \( w_r \), one can apply the average time \( t_Z(w_r) = \int_0^\infty dt \ p(t) = \tilde{p}(\epsilon = 0) \):

\[
t_Z = (11|\hat{\Omega}_e^{-1}\Phi(\hat{\Omega}_x)[\Phi(\hat{\Omega}_m)]^{-1}|11|_t \to 0
= \tilde{P}_p(\epsilon)[1 - \tilde{W}_p(\epsilon)]^{-1}|_t \to 0,
\]

where \( \Omega_e = \epsilon + i \hat{H} \). The second part of equations (4.9) is written with the use of expression (3.13) for \( \tilde{p}(\epsilon = 0) \) in which, however, the functions \( \tilde{P}_p(\epsilon) \) and \( \tilde{W}_p(\epsilon) \) are replaced with \( \tilde{X}_p(\epsilon) = \int_0^\infty dt \ e^{-\epsilon t} X(t) p_1(t) \) for \( X = P, W \). This replacement is made due to the change of notation for the measured state (1) instead of (0), according to which, to avoid possible confusions, \( p_0(t) \) should be replaced by \( p_1(t) = (11|e^{-\epsilon t}|11) \).

It is seen from the above definition of \( \tilde{P}_p(\epsilon) \) and \( \tilde{W}_p(\epsilon) \) that the value of \( t_Z \) is finite in the case of finite \( \int_0^\infty dt \ tW(\epsilon) = \int_0^\infty dt \ p(t) \).

In general, \( t_Z \) can be calculated only numerically. However, some conclusions on the specific features of this parameter can be made by the analysis of the limiting behavior of \( t_Z(w_r) \) at \( w_r \to \infty \) and \( w_r \to 0 \).

1. For \( w_r \to \infty \) we get the relation \( t_Z(w_r \to \infty \to \infty \) which follows from the definition of this parameter and the relation \( \tilde{p}(\epsilon) = e^{-\epsilon} \) valid in this limit.

2. In the opposite limit \( w_r \to 0 \) the parameter \( t_Z(w_r) \) also grows to infinity: \( t_Z(w_r \to 0 \sim 1/w_r \to \infty \). This dependence can be obtained by analyzing \( w_r \)-dependence of \( \tilde{P}_p(\epsilon = 0) \) and \( \tilde{W}_p(\epsilon = 0) \) at small \( w_r \). The fact is that the probability \( p_1(t) = (11|e^{-\epsilon t}|11) \), which determines the values of these two functions, can, in general, be represented as \( p_1(t) = \tilde{p}_1 + \delta p_1(t) \), where \( \tilde{p}_1 \) is independent of time and \( \delta p_1(t) \) is the oscillating part represented as a sum of harmonically oscillating functions. The most important for our analysis is \( \tilde{p}_1 \), which can be found by expansion of the evolution operator \( e^{-\epsilon tH} \) in the basis of eigenfunctions \( |\phi_j\rangle \) of the Hamiltonian \( H \): \( \tilde{p}_1 = \sum_j |\langle \phi_j | \phi_j \rangle|^2 < 1 \). With this representation for \( p_1(t) \) one can obtain the estimations \( \tilde{P}_p(0) \approx \tilde{p}_1 \int_0^\infty dt \ p(t) = \tilde{p}_1 \tilde{t} = \tilde{p}_1/w_r \) and \( \tilde{W}_p(0) \approx \tilde{p}_1 \int_0^\infty dt \ W(t) = \tilde{p}_1 < 1 \), in which the contribution of the oscillating part \( \delta p_1(t) \), negligibly small in the limit \( w_r \to 0 \), is ignored. Substitution of these relations into equation (4.9) leads to the above-mentioned limiting dependence \( t_Z(w_r \to 0) \approx \tilde{p}_1(1 - \tilde{p}_1)^{-1} w_r^{-1} \).

The above analysis shows that the behavior of \( t_Z(w_r) \) is non-monotonic with the minimum of this function at some \( w_r \), whose value is determined by the parameters of the system. The validity of such a conclusion will be demonstrated below as applied to the case of the Poissonian RDM distribution, as an example.

### 4.3. Poissonian distribution of measurements

In the case of the Poissonian RDM statistics, when \( \Phi(\hat{\Omega}_x) = \hat{\Omega}_x/w_r \), equation (3.4) for the Laplace transform \( \tilde{U}(\epsilon) \) is essentially simplified reducing to the SLE of Schrödinger type [16]

\[
(\hat{\Omega}_x + w_r \hat{Q}_m)\tilde{U}(\epsilon) \equiv (\epsilon + i\hat{H} + w_r \hat{Q}_m)\tilde{U}(\epsilon) = 1
\]

which is briefly discussed in section 3.

For the two-level system this equation is the system of four linear equations which can be solved analytically. We are not going to present the cumbersome expressions for matrix elements of \( \tilde{U}(\epsilon) \) but restrict ourselves to obtaining only one element corresponding to the
observable under study, the survival probability \( p(t) \) (see equation (2.7)). For the initial condition \( |\psi_0\rangle = |1\rangle \) the Laplace transform \( \tilde{p}(\epsilon) \) is given by

\[
\tilde{p}(\epsilon) = (11|\tilde{U}(\epsilon)|11) \\
= \frac{\epsilon + w_r + \tilde{w}(\epsilon)}{\epsilon^2 + \epsilon[w_r + 2\tilde{w}(\epsilon)] + w_r \tilde{w}(\epsilon)},
\]  

(4.11)

where

\[
\tilde{w}(\epsilon) = 2(\epsilon + w_r)v^2 /[(\epsilon + w_r)^2 + 4\epsilon^2].
\]  

(4.12)

Of special interest is, naturally, the limiting variant of the general formula (4.11) corresponding to the large average rate \( w_r \) of repetition of measurements. It is seen from this expression that in the limit \( w_r \gg v \) (and for \( \epsilon < w_r \) ) \( \tilde{p}(\epsilon) \approx (\epsilon + \tilde{w}_0)^{-1} \), i.e. for \( t > w_r^{-1} \)

\[
p(t) \approx e^{-\tilde{w}_0 t} \quad \text{with} \quad \tilde{w}_0 = \tilde{w}(0) = \frac{2w_r v^2}{w_r^2 + 4\epsilon^2}.
\]  

(4.13)

Note that in the considered limit \( w_r \gg v \) the rate \( \tilde{w}_0 \sim w_r(v/w_r)^2 \ll w_r \).

Equation (4.12) shows that in the limit of large \( w_r \) the decay of the survival probability turns out to be exponential with the rate \( \tilde{w}_0 \), in principle, non-monotonically depending on \( w_r \). The non-monotonic behavior, however, can correctly be described by equation (4.13) only in the case \( \epsilon \gg v \), when within the region of validity of this formula \( (w_r \gg v) \) there exists the subregion of \( w_r \) values, \( \epsilon \gg w_r \gg v \), in which \( \tilde{w}_0(w_r) \) is the increasing function of \( w_r \): \( \tilde{w}_0(w_r) \sim w_r \). The non-monotonic dependence \( \tilde{w}_0(w_r) \) can be treated as the acceleration of \( p(t) \)-decay by measurements at relatively low measurement rates \( w_r \ll \epsilon \) (associated with the inverse Zeno effect) followed by the deceleration at large \( w_r \gg \epsilon \): \( \tilde{w}_0(w_r) \sim 1/w_r \) (corresponding to the quantum Zeno effect).

In general, the specific features of the RDM effect can be demonstrated with the parameter \( t_Z(w_r) = \tilde{p}(0) \). The expression (4.11) yields simple formula for this function:

\[
t_Z = w_r^{-1} + \tilde{w}_0^{-1} = v^{-1}[1 + \frac{1}{2}(\tilde{w}_r^2 + 4\epsilon^2)]/\tilde{w}_r
\]  

(4.14)

with \( \tilde{w}_r = \epsilon/v \) and \( \tilde{w}_0 = w_r/v \). In agreement with general qualitative conclusions (section 4.2), \( t_Z(w_r) \) non-monotonically depends on \( w_r \) with \( t_Z \overset{w_r \to 0}{\approx} 1/w_r \to \infty, t_Z \overset{w_r \to \infty}{\approx} w_r/(2v^2) \to \infty \) and the minimum at

\[
\tilde{w}_r = (2 + 4\epsilon^2)^{1/2}.
\]  

(4.15)

For \( \epsilon \gg v \) in the limit \( w_r \gg v \) the non-monotonic behavior of \( t_Z(w_r) \approx \tilde{w}_0^{-1} \) with \( \tilde{w}_r = 2\epsilon \) indicates the occurrence of two regimes of the Zeno effect, quantum and inverse, mentioned above. It is seen that in the case \( \epsilon \gg v \) the coordinate \( \tilde{w}_r = 2\epsilon \) predicted by equation (4.15) coincides with that of the maximum of the rate \( \tilde{w}_0(w_r) \). The results of this analysis demonstrate that the parameter \( t_Z(w_r) \) is certainly useful for studying qualitative specific features of the RDM effect.

Characteristic properties of the behavior of the probability \( p(t) \equiv p(\tau|\tau_r) \) (at fixed \( \tau = tv \)) as a function of the dimensionless average time \( \tau_r = \tilde{w}_r^{-1} = v/w_r \) between measurements are shown in figure 1.

At very small \( \tau_r \), the probability \( p(\tau|\tau_r) \) monotonically approaches 1 as \( \tau_r \to 0 \), in agreement with predictions of the quantum Zeno effect. The above analysis with the SLE (4.10) clearly reveals the mechanism of the slowing down of 1 \( \to 2 \) transitions by the RDM. According to this equation the RDM effect on the evolution of the state \( |1\rangle \) is actually equivalent to the effect of the decay of the state \( |2\rangle \) with the rate \( w_r \), which is accompanied by dephasing (decay of the density matrix elements \( \langle 1|\rho|2 \rangle \) and \( \langle 2|\rho|1 \rangle \)) with the same rate \( w_r \). Just fast
dephasing in the limit \( w_r \to \infty \) leads to the strong reduction of the \( 1 \to 2 \) transition rate \( \sim v(v/w_r) \), associated with the quantum Zeno effect.

At intermediate values of \( \tau_r \), the function \( p(\tau | \tau_r) \) non-monotonically depends on \( \tau_r \), with the minimum at some \( \tau_{\min} = v/w_{\tau_{\min}} \). The position of the minimum is reasonably accurately estimated by means of \( \tilde{\omega}_{\tau_{\min}} / (4\varepsilon^2) \) for all values of \( \tilde{\varepsilon} = \varepsilon/v \) used in the numerical calculation: \( \tau_{\min} = v\tau_{\min} \approx \tilde{\omega}_{\tau_{\min}}^{-1} \). The accuracy of this estimation is especially good for the largest considered value \( \tilde{\varepsilon} = 2.5 \), as expected from equation (4.13) and the above analysis.

The analysis of expression (4.13) makes it possible to understand the reason of the appearance of this minimum. It results from the interplay between quantum oscillations in the system and RDM-induced dephasing. Just this interplay manifests itself in the factor \( \omega_r^2 + 4\varepsilon^2 \) in the denominator in formula (4.13) for \( \tilde{u}_0 \), which is responsible for the non-monotonic behavior of \( p(\tau_r) \).

In the region of validity of equation (4.13), i.e. for \( \tau_r = v/w_r \ll 1 \), the kinetics of \( p(\tau) \) decay is exponential. This means that the non-monotonic behavior of the dependence \( p(\tau_r) \) at a fixed \( \tau = vt \) stems from that of the rate \( \tilde{u}_0 / (w_r) \). Such a dependence of the rate is traditionally treated as a transition from the quantum Zeno effect (in the region of decreasing behavior of \( \tilde{u}_0(w_r) \)) to the inverse Zeno effect (when \( w_0(w_r) \) is the increasing function) [3]. In general, however, for non-exponential \( p(\tau) \)-dependence, it is not quite correct to associate...
the non-monotonic behavior of \( p(\tau_r) \) with any specific phenomena because of a large number of possible peculiarities of this behavior \( p(\tau) \)-kinetics in general.

4.4. Anomalous distribution of measurements

Of special interest is the case of the anomalous RDM, in which the PDF \( W(t) \) anomalously slowly decreases at large times \( t \): \( W(t) \sim \frac{1}{t^{1+\alpha}} \) with \( \alpha < 1 \). It is easily seen that this PDF cannot be characterized by the average time \( \bar{t} \) between measurements in its usual definition though, of course, this function still has the characteristic decay time (see below). In our further consideration we will apply the simplest Mittag–Leffler model for the anomalous PDF \( W(t) \), defined in equations (2.18) and (2.19). The existence and qualitative definition of the characteristic time \( t_r = \frac{1}{\omega_r} \) in this model is clear from equation (2.19).

For brevity in our analysis of the anomalous case we will restrict ourselves to the most interesting limit of high characteristic inverse time between measurements \( w_r \), corresponding to \( \|\tilde{W}\|/w_r \ll 1 \). In this limit \( \tilde{p}(\epsilon) \approx \tilde{p}_\infty(\epsilon) \) can be evaluated with formula (4.6), which in the applied Mittag–Leffler model is represented as [17]

\[
\tilde{p}_\infty(\epsilon) = \frac{\Gamma(1)\Omega_\epsilon^{-1}\eta}{\Gamma(1)\Omega_\epsilon^{-1}\eta} \approx \frac{(2\epsilon^2 + 1)e^{\alpha - 1} + \tilde{\Omega}_\epsilon^{-1}}{(2\epsilon^2 + 1)e^{\alpha} + \tilde{\Omega}_\epsilon^{-1}},
\]

(4.16)

where \( \tilde{\epsilon} = \epsilon/\bar{v} \) and

\[
\tilde{\Omega}_\epsilon^{\beta} = \left\{ \left( \epsilon + 2i\sqrt{\tilde{\epsilon}^2 + 1} \right)^\beta + \left( \epsilon - 2i\sqrt{\tilde{\epsilon}^2 + 1} \right)^\beta \right\}.
\]

(4.17)

It is easily seen that for the Poissonian RDM \( (\alpha = 1) \) \( \tilde{p}_\infty(\epsilon) \approx 1/\epsilon \) and \( p(t) = 1 \) as predicted in the presence of the quantum Zeno effect.

As for the anomalous RDM, the existence of the non-trivial limit \( \tilde{p}_\infty(\epsilon) \) itself indicates the violation of the characteristic Zeno effect behavior of \( \tilde{p}_\infty(\epsilon) \) in this case. In the limit \( \|\tilde{W}\|/w_r \ll 1 \) the function \( \tilde{p}_\infty(\epsilon) \) (and thus \( p(t) = p_\infty(t) \)) appears to be independent of \( w_r \) and determined by the parameters of the Hamiltonian only.

At short times \( t \ll \epsilon^{-1}, v^{-1} \) formula (4.16) predicts no transitions, i.e. \( \tilde{p}_\infty(\epsilon) \approx 1/\epsilon \) and \( p(t) \approx 1 \). In the limit of long times \( t \gg \epsilon^{-1}, v^{-1} \), however, one gets

\[
p_\infty(t) \sim A_\omega(t)t^{-\alpha},
\]

(4.18)

where \( A_\omega(t) \) is the oscillating function of time: \( A_\omega(t) = a_\omega + c_\omega \cos(2\tilde{E}t) + s_\omega \sin(2\tilde{E}t) \), in which \( \tilde{E} = v\sqrt{\tilde{\epsilon}^2 + 1} \) and \( a_\omega, c_\omega \) and \( s_\omega \) are the constants depending on \( \alpha \) and \( \tilde{\epsilon} \). This asymptotic expression can be derived by taking into account that the most slowly decreasing (and additive) contributions to the integral of the inverse Laplace transformation \( p_\infty(t) \sim (2\pi i)^{-1}\int_{c-i\infty}^{c+i\infty} d\epsilon \tilde{p}_\infty(\epsilon) e^{\epsilon t} \) come from singularities of \( \tilde{p}_\infty(\epsilon) \) (4.16) located at the imaginary axis of the complex plane of \( \epsilon \). The singularities (brunching points) are determined by the terms \( e^{\alpha - 1} \) and \( \tilde{\Omega}_\epsilon^{-1} \) in the numerator in equation (4.16). In the long time limit \( t \gg 1/\sqrt{\tilde{\epsilon}^2 + v^2} \) these singularities contribute independently and the evaluation of contributions reduces to the calculation of integrals of type \( (2\pi i)^{-1}\int_{c-i\infty}^{c+i\infty} d\epsilon e^{\alpha - 1} e^{\epsilon t} \sim t^{-\alpha} \), which leads to equation (4.18).

It is important to note that the transition from the anomalous case \( \alpha < 1 \) to the Poissonian one \( \alpha = 1 \) is fairly non-trivial. In order to clarify the details of this transition we will consider the case of \( \alpha \) close to 1, when \( \delta_\alpha = 1 - \alpha \ll 1 \). In this limit the onset of the inverse power-type kinetics (4.18) displaces to very long times and the major part of the kinetics \( p(t) \) reduces to the exponential one, which can be obtained by expanding \( \tilde{p}_\infty(\epsilon) \) in powers of \( \delta_\alpha \). For example, taking into account that for \( \delta_\alpha \ll 1 \), \( x^{\alpha - 1} \approx x(1 - \delta_\alpha \ln x) \), one can represent the
Figure 2. The dependence of the survival probability \( p_\infty(\tau) \) on \( \tau = t v \), calculated by the inverse Laplace transformation of \( \tilde{p}_\infty(\epsilon) \) (4.16), for anomalous RDM with different parameters \( \bar{\epsilon} = \epsilon/v \) and \( \alpha \): (a) \( \bar{\epsilon} = 0.94, \alpha = 0.1 \) (full line), \( \bar{\epsilon} = 0.53, \alpha = 0.3 \) (dashed line) and (b) \( \bar{\epsilon} = 0.53, \alpha = 0.92 \) (1), \( \bar{\epsilon} = 0.94, \alpha = 0.97 \) (2); \( \bar{\epsilon} = 0.53, \alpha = 0.97 \) (3); \( \bar{\epsilon} = 0.94, \alpha = 0.97 \) (4).

(a) The straight lines corresponding to the dependence \( p_\infty(t) = 1/(2.3 t^\alpha) \) are presented for the sake of demonstration of the asymptotic behavior of the exact dependence \( p_\infty(t) \). (b) The straight (dashed) lines correspond to the approximate dependence (4.19).

It is worth noting that the characteristic behavior (4.19) is determined by that of \( \tilde{p}_\infty(\epsilon) \) at small \( |\epsilon| \sim \Delta_0 \). In this region the terms of higher order in \( \Delta_0 = 1 - \alpha \), whose summarized contribution is denoted as \( O(\delta_0) \), result in the correction of the kinetics \( \sim \delta_0^2 \ln(1/\delta_0) \) which can be neglected in the limit \( \delta_0 \ll 1 \). These terms, non-analytical in \( \epsilon \), are responsible for the inverse time behavior of \( p_\infty(t) \) at long times.

Figure 2 shows the dependence \( p_\infty(t) \) for some particular sets of parameters of the model.

5. Effect of relaxation

5.1. General remarks

So far we have considered the RDM effect on dynamical systems only, though it is known that the relaxation can strongly modify the manifestation of this effect [2–4]. In particular,
in addition to the quantum and inverse Zeno regimes of the effect, analyzed above, there appears another one, in which no strong influence of measurements on the evolution kinetics is observed in the limit of short average time $\bar{t} \sim w^{-1} \rightarrow 0$. This regime can also be considered as a manifestation of the inverse Zeno effect [3, 4] though this is a matter of definition.

The influence of relaxation on the Zeno effect has already been analyzed within the model of equidistantly distributed measurements [2–4]. Here we will discuss this influence in the case of the RDM.

In general, the analysis of the relaxation in quantum systems is a difficult problem. In this work we will restrict ourselves to the simple Markovian case, in which the kinetic equation for the density matrix can be written as

$$\dot{\rho} = -\hat{\mathcal{L}}\rho, \quad \text{where} \quad \hat{\mathcal{L}} = i\hat{H} + \hat{R}. \quad (5.1)$$

Here $\hat{R}$ is the relaxation superoperator which can be evaluated in the short correlation time (or the Bloch–Redfield) approximation [18]. Some example of the relaxation model, leading to the particular expression for the superoperator $\hat{R}$ will be considered below.

It is easily seen that in the presence of relaxation all general formulas are similar to those obtained above for dynamic systems. The only difference is in the definition of $\Omega_\epsilon$. In the presence of relaxation

$$\tilde{\Omega}_\epsilon = \epsilon + \hat{\mathcal{L}} = \epsilon + \hat{R} + i\hat{H}. \quad (5.2)$$

To apply the results obtained in section 3 with the redefined $\tilde{\Omega}_\epsilon$ we need to specify the relaxation superoperator, i.e. the relaxation mechanism.

### 5.2. Simple relaxation model

In order to illustrate the main features of the manifestation of relaxation in the RDM effect it is sufficient to analyze the simplest relaxation models. In our analysis we will discuss the variant of the model, widely accepted in the magnetic resonance theory [18], in which

$$\hat{R} = w_d((11)-(22))(11)-(22) + w_p(12)(12) + |21)(21). \quad (5.3)$$

This operator describes population relaxation with the rate $w_d$ and dephasing with the rate $w_p$. The rates satisfy the relation $w_p \geq w_d$, which ensures positivity of the density matrix $\rho(t)$ during the evolution, described by equation (5.1) [19, 20].

In principle, the relaxation operator (5.3) can be represented in the Lindblad form [20], but this representation does not prove to be of real use for the analysis of the problem under study.

The model (5.3) allows one to analyze fairly easily the important properties of the relaxation effect for any type of the PDF $W(t)$.

#### 5.2.1. Poissonian distribution of measurements

As we have already mentioned above, the case of the Poissonian RDM can be considered as a simple example of renewal processes with rapidly decreasing PDF $W(t)$. The substitution of $\tilde{\Omega}_\epsilon$ (5.2) into equation (4.10) leads to the expression (4.11) for $\tilde{p}(\epsilon)$, but with $\bar{w}(\epsilon)$ replaced by $\bar{w}_d(\epsilon) = \bar{w}(\epsilon) + w_d$ and with $w_r$ changed by $w_r + w_p$ in the function $\bar{w}(\epsilon)$ itself:

$$\tilde{p}(\epsilon) = \frac{\epsilon + w_r + \bar{w}_d(\epsilon)}{\epsilon^2 + \epsilon[w_r + 2\bar{w}_d(\epsilon)] + w_d\bar{w}_d(\epsilon)} = \frac{1}{\epsilon + \bar{w}_d(\epsilon)(\epsilon + w_r)[\epsilon + w_r + \bar{w}_d(\epsilon)]}, \quad (5.4)$$
where
\[ \tilde{u}_d(\epsilon) = w_d + 2(\epsilon + w_{rp})v^2/[(\epsilon + w_{rp})^2 + 4\epsilon^2] \quad (5.5) \]
and \( w_{rp} = w_r + w_p \).

This expression enables us to analyze all possible effects of relaxation in the case of Poissonian distribution. Formula (5.4) predicts different types of \( \bar{p}(\epsilon) \) dependence (and thus \( p(t) \)) on the average time between measurements \( t = w_r^{-1} \). The important parameter, which essentially determines the dependencies of \( \bar{p}(\epsilon) \) and \( p(t) \) on \( w_r \), is
\[ \bar{u}_d^0 = \bar{u}_d(0) = w_d + 2w_{rp}v^2/(w_{rp}^2 + 4\epsilon^2). \quad (5.6) \]

In particular, the parameter \( t_Z \) is directly related to \( \bar{u}_d^0 \):
\[ t_Z = \bar{p}(0) = 1/w_r + 1/\bar{u}_d^0. \quad (5.7) \]

Here we summarize some most interesting limiting \( p(t) \) dependences on \( w_r \).

**Slow population relaxation, \( \bar{u}_d^0 \ll w_r \).** In the case of slow population relaxation (or fast repetition of measurements as is considered in section 4.3), when \( \bar{u}_d^0 \ll w_r \) (implying that \( v, w_d \ll w_r \)), formula (5.4) strongly simplifies predicting exponentially decreasing \( p(t) \):
\[ \bar{p}(\epsilon) \approx (\epsilon + \bar{u}_d^0)^{-1} \quad \text{and} \quad p(t) \approx e^{-\bar{u}_d^0 t} \quad (5.8) \]
for which \( t_Z = 1/\bar{u}_d^0 \). Depending on the relation between parameters of the system, these expressions predict different \( w_r \)-dependences of the RDM effect.

1. For fast dephasing, when \( w_p \gg w_r \) and \( w_{rp} \approx w_p \), we find that the \( p(t) \) decay rate and \( t_Z \) are independent of \( w_r \), i.e. of the measurements, which can be considered as the onset of the inverse Zeno regime [2–4]: \( \bar{u}_d^0 \approx w_d + 2\epsilon^2 w_p/(w_{rp}^2 + 4\epsilon^2) \).
2. For slow dephasing, when \( w_p \ll w_r \), the decay rate \( \bar{u}_d^0 \approx w_d + 2\epsilon^2 w_r/(w_r^2 + 4\epsilon^2) \) and therefore both quantum and inverse Zeno regimes are possible. For \( w_r > \epsilon \) we get the decreasing function \( \bar{u}_d^0(w_r) \approx w_d + 2\epsilon^2/w_r \), corresponding the quantum Zeno case, while in the opposite limit \( w_r < \epsilon \) we obtain the dependence \( \bar{u}_d^0(w_r) \approx w_d + (\epsilon^2 w_r)/(2\epsilon^2) \), which is associated with the inverse Zeno effect. Note that the above analysis of \( \bar{u}_d^0(w_r) \) dependence (and RDM effect, in general) is closely related to that of the rate \( \bar{u}_0(w_r) \) of the (exponential) \( p(t) \) decay in the absence of relaxation and for fast repeated measurements \( (w_r \gg v) \), which is described by equation (4.13).

**Fast population and phase relaxation \( w_d, w_p \gg w_r \).** In the limit of large population and phase relaxation rates \( w_d, w_p \gg w_r \) (recall that \( w_p \gg w_d \)), the case \( \bar{u}_d^0 > w_r \) is of special interest, in which the evolution kinetics consists of two stages: the stage of fast equilibration (at \( t \sim 1/\bar{u}_d^0 \)) and the stage of slow quasi-equilibrium evolution affected by measurement (at \( t \gg 1/w_r > 1/\bar{u}_d^0 \)). During the first fast stage the survival probability \( p(t) \) decreases from 1 to 1/2 according to \( p(t) \approx \frac{1}{2}(1 + e^{-\bar{u}_d^0 t}) \). After that (during the second most interesting stage) \( p(t) \) decreases exponentially:
\[ \tilde{p}(\epsilon) \approx \frac{1}{2}(\epsilon + \bar{u}_r)/2 \quad \text{and} \quad p(t) \approx \frac{1}{2} e^{-w_r t/2}, \quad (5.9) \]
so that \( t_Z = 1/w_r \) in agreement with the prediction of equation (5.7). Formulas (5.9) can also be derived with the use of equation (3.13) in which \( \tilde{P}_{p_r}(\epsilon) \) and \( \tilde{W}_{p_r}(\epsilon) \) should be replaced by \( \tilde{P}_{p_r}(\epsilon) \) and \( \tilde{W}_{p_r}(\epsilon) \), respectively, as it was mentioned in the analysis of equation (4.9). Taking into consideration that in the limit of fast relaxation after short time \( \sim \bar{u}_d^0 \) the survival probability \( p_1(t) = (|\rho(t)|) \) in the absence of measurements is given by \( p_1(t) \approx 1/2 \), we get
\[ \tilde{P}_{p_1}(\epsilon) = \int_0^\infty dt e^{-\epsilon t} P(t) p_1(t) \approx (\epsilon + w_r)^{-1} \]
\[ W_p(\epsilon) = \frac{1}{\pi} \int_0^\infty dt \, e^{-\epsilon t} W(t) p_1(t) \approx \frac{1}{2} w_r (\epsilon + w_r)^{-1}. \]

Substitution of these expressions into equation (3.13) leads to formula (5.9).

These formulas demonstrate that for \( w_d, w_p \gg w_r \), the effect of measurement shows itself in the inverse Zeno effect, in which the effect increases as \( w_r \) is increased.

5.2.2. Anomalous distribution of measurements. As in the case of dynamic systems, in our discussion of the anomalous RDM effect in the presence of relaxation we will also restrict ourselves to the analysis of the limit of short characteristic time between measurements \( w_r^{-1} \rightarrow 0 \), corresponding to \( \| \hat{\Omega}_r \| / w_r \ll 1 \).

Most clearly the effect of relaxation can be demonstrated in the limit of fast dephasing, when \( w_p \gg v \). In this limit the general kinetic equation (5.1) reduces to the system of balance equations, i.e. equations for populations of states.

For the considered two-level system (4.1) with relaxation superoperator (5.3) the balance equations for the vector \( p(t) \) of populations of states \( |1\rangle \) and \( |2\rangle \) can be written as

\[ \dot{p} = -\tilde{R}_d p \quad \text{where} \quad p(t) = p_1(t)|1\rangle + p_2(t)|2\rangle, \]

(5.10)

and

\[ \tilde{R}_d = \tilde{w}_d^0 (|1\rangle - |2\rangle)(|1\rangle - \langle 2|) \]

(5.11)

with

\[ \tilde{w}_d^0 = w_d + 2v^2 w_p / (w_p^2 + 4\epsilon^2). \]

(5.12)

The function under study \( \tilde{p}(\epsilon) \equiv \tilde{p}_\infty(\epsilon) \), which is the Laplace transform of the probability \( p_\infty(t) \equiv p_1(t) \) of survival in the state \( |1\rangle \), can be calculated with the use of expression (4.16) but with the supermatrix \( \tilde{\Omega}_d \) replaced by \( \hat{\Omega}_d \) defined in the reduced space of diagonal elements of the density matrix

\[ \tilde{p}_\infty(\epsilon) \approx \langle 1|\hat{\Omega}_d^{\omega - 1} |1\rangle / \langle 1|\hat{\Omega}_d |1\rangle, \]

(5.13)

\[ = \frac{e^{\omega - 1} + (\epsilon + 2\tilde{w}_d^0)^{\omega - 1}}{e^{\omega} + (\epsilon + 2\tilde{w}_d^0)^{\omega}}, \]

(5.14)

where \( \hat{\Omega}_d = \epsilon + \tilde{R}_d \).

This formula predicts the long tailed behavior \( p_\infty(t) \sim 1/t^\omega \) similar to that found in section 4.4 for dynamic systems. In the limit \( \alpha \rightarrow 1 \), however, the amplitude of the tail becomes negligibly small and the kinetics reduces to the evident exponential: \( p_\infty(t) = e^{-\tilde{w}_d^0 t} \).

Similar to the dynamic systems, the most important feature of the anomalous kinetics \( p_\infty(t) \) in the presence of relaxation consists in the independence of this kinetics of \( w_r \), i.e. of the characteristic time \( w_r^{-1} \) between measurements. It is worth noting, though, that this independence results from the existence of the nontrivial special limit \( w_r \rightarrow \infty \), manifesting itself in equation (4.16), rather than from the effect of relaxation itself.

6. Discussion and conclusions

In this work we have discussed the RDM effect on the evolution of quantum systems. The sequence of measurements is described as a renewal stochastic process [6, 7], whose specific properties are controlled by the PDF \( W(t) \) of time intervals \( t \) between measurements. The specific features of the RDM effect, which can be called the stochastic Zeno effect, are essentially determined by the analytical behavior of \( W(t) \) and properties of the quantum system under study, for example, the presence of relaxation or decay [3, 4, 10] (see below).
The analysis carried out in our work shows that in the case of rapidly decreasing PDF $W(t)$, the RDM effect on dynamic systems (without relaxation) is similar to that of measurements equidistantly distributed in time [1–4]. This effect is characterized by the average time $\bar{t} = \int_0^\infty dt W(t)$ between measurements or the average rate $w_r \sim 1/\bar{t}$ of the repetition of measurements. To demonstrate the specific features of the effect of the RDM with rapidly decreasing $W(t)$ on dynamic systems, we have analyzed the effect on the two-level system within the Poissonian model for the RDM. The model is shown to allow one to describe both quantum and inverse Zeno regimes of the RDM effect by very simple analytical expressions.

In particular, it is possible to easily treat the main manifestation of the quantum Zeno effect consisting in the reduction of the decay rate of the measured state with the decrease of the time between measurements. It is also shown that in the limit of frequently repeated measurements $w_r \gg \nu$ the RDM result in the exponential decay of the survival probability: $p(t) \approx \exp(-\bar{w} t) \ (\text{equation (4.13)})$, with the rate $\bar{w}(w_r)$ which is either the decreasing or the increasing function of $w_r$ depending on the relation between $w_r$ and the splitting of levels $\epsilon$. These two types of behavior correspond to the above-mentioned quantum and inverse Zeno regimes, respectively [3, 4, 10].

Of special interest is the anomalous case of heavy tailed PDF $W(t) \sim 1/(w_r t)^{1+\alpha}$ ($\alpha < 1$), in which the RDM effect appears to strongly differ from that for rapidly decreasing PDF. Note that in this case the average time $\bar{t}$ does not exist, but the PDF is still characterized by the specific time $t_r = w_r^{-1}$. For anomalous PDFs the Zeno effect is not observed: in the limit $t_r \to 0$ the survival probability $p(t) \equiv p_\infty(t)$ turns out to be a function of time $t$. Moreover the probability $p_\infty(t)$ is a very slowly decreasing function: $p_\infty(t) \sim 1/t^\alpha$.

The relaxation in the quantum system can strongly show itself in the effect of the RDM on the evolution of the system. In particular, in the case of rapidly decreasing $W(t)$, for not very large rates $w_r$ smaller than the characteristic relaxation rate $\bar{w}_d^0$ (see section 5.2), the survival probability $p(t)$ is demonstrated to exponentially decrease $p(t) \sim \exp(-\bar{w}_d t) \ (\text{equation (5.8)})$ with the rate $\bar{w}_d^0(w_r)$ either decreasing or increasing as $w_r$ is increased. The form of $\bar{w}_d^0(w_r)$ behavior depends on the value of $(w_r + w_p)/\epsilon$, where $w_p$ is the dephasing rate. Similar to the pure dynamic case (in the absence of relaxation) discussed above these types of dependences can be considered as a manifestation of quantum and inverse Zeno effects [3, 4, 10]. The acceleration of $p(t)$ decay by measurements, i.e. the inverse Zeno effect, is also found in the limit of very fast relaxation $w_d, w_p \gg w_r \ (\text{equation (5.9)})$.

In the case of anomalous $W(t)$ the relaxation in the system does not lead to any new specific features of the evolution kinetics of the system under study. Similar to the dynamic systems, for systems with relaxation in the limit of a high characteristic rate $w_r \to \infty$, the survival probability $p_\infty(t)$ is still a function of time slowly decreasing at large times: $p_\infty(t) \sim 1/t^\alpha$.

At the end of this section we would like to discuss some recent papers [21–23] concerning the theoretical analysis of quantum measurements and, in particular, the quantum Zeno effect within the approaches which has something in common with those applied in our work. In above-mentioned papers the kinetics of measurement-induced jump-like relaxation transitions in quantum systems under study are described both numerically by direct Monte Carlo modeling of stochastic quantum relaxation transition processes [21] and analytically by solving some Markovian kinetic equations [22, 23] for corresponding PDFs. As far as analytical approaches are concerned the analysis of Poissonian-like theoretical models of stochastic measurement-induced jumps between states made in [22] is of particular interest. It is worth noting though that the authors of this work restricted themselves only to the most general (and at some respects too formal) analysis of specific features of Poissonian jump processes, including the equivalent description of the process in terms of the SLE or corresponding
Ito-type stochastic differential equations. Interesting extension of the Markovian kinetic description of the problem within a quantum model of the measuring device is considered in [23]. The perturbative treatment of the system-device interaction allowed the authors to derive relatively simple Markovian (Bloch-type) kinetic equations which describe the evolution of the measurement-affected system. Results obtained in this work are, to some extent, related to some results of our study, concerning the Poissonian RDM (since, as we have mentioned above, the Markovian processes are directed related to Poissonian ones), but unfortunately the direct comparison is hardly possible because of the difference in model parameters used in both treatments.

The important contribution of our work, as compared to those discussed above, consists in the proposed new approaches, which enable us to significantly generalize the analysis of measurement effect on the evolution of quantum systems, allowing for the consideration of the RDM effects for different types of stochastic RDM processes (described within the RA) from conventional Poissonian one to anomalous. In addition, even in the Poissonian approach, which is related to some Markovian models applied in early works, the analysis is essentially extended by considering more thoroughly the dephasing effect of measurements and by using a more general relaxation model for investigating the manifestation of relaxation in the system.

Concluding our short analysis of obtained results we would like to discuss the possibilities of experimental observation of the effects predicted in our work. Of course, the question of great importance is whether it is possible to realize the RDM experimentally. Answering this question it is worth emphasizing the following points.

1. The RDM can be realized experimentally with the use of equipment which can randomize the set of measurements. Among possible variants of random processes the Poissonian one is, probably, the most simple for realization. At first sight the corresponding efforts will be of not very much use in reality; however, it is worth noting that thus obtained results on the RDM effect can be analyzed somewhat easier than those of equidistantly distributed measurements. The simplicity results from the possibility of the analysis with simple Markovian kinetic equations for any number of experiments. In addition, as was mentioned in the above discussion, the Poissonian RDM can be realized in the measurement processes themselves due to the stochastic nature of the measurement procedure in some particular experiments [21–23].

2. Perhaps, the most important fact is that the RDM can be realized by the process under study itself. The point is that in many cases the measurement is made within fairly small volume which can be a small part of the larger volume where the process occurs. For example, suppose that the measured system is a small Brownian particle with quantum internal degrees of freedom, undergoing stochastic motion within large volume. In its motion the particles cross the small measurement volume. Any visit of this volume can be assumed to result in the measurement. In such a scenario of measurements their statistics is represented by that of visits of the small measurement site.

The statistics of visits depends on the mechanism of motion of the particle. For instance, in the case of the particle, confined in a relatively small volume (of type of a cage), the statistics is close to Poissonian. For freely diffusing motion in the infinite space the statistics is quite well described by the anomalous variant of the RA with the heavy tailed PDF $W(t) \sim 1/t^{1+\alpha}$, in which $\alpha \leq 1$ is determined by the dimensionality $d_s$ of the space [8, 9].

3. As a possible modification of the procedure, described above in point 2, one can consider the ‘measurement’ of the quantum state of a Brownian particle by another large quencher particle whose effect on the quantum subsystem of the migrating Brownian particle can be treated as a measuring device. The statistics of measurements in such systems is,
actually, determined by that of re-encounters of the Brownian particle with the measuring quencher. The properties of this re-encounter statistics are well described by the RA, as mentioned above, with the specific features depending on the mechanism of migration.

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References

[1] Nakazato H, Namiki M and Pascazio S 1996 Int. J. Mod. Phys. B 10 247
[2] Namiki M, Pascazio S and Nakazato H 1997 Decoherence and Quantum Measurements (Singapore: World Scientific)
[3] Facchi P and Pascazio S 2008 J. Phys. A: Math. Theor. 41 493001
[4] Koshino K and Shimizu A 2005 Phys. Rep. 412 191
[5] Misra B and Sudarshan E C G 1977 J. Math. Phys. 18 756
[6] Weiss G H 1994 Aspects and Applications of the Random Walk (Amsterdam: North-Holland)
[7] Bouchaud J-B and Georges A 1990 Phys. Rep. 195 127
[8] Shushin A I 2001 Phys. Rev. E 64 051108
[9] Shushin A I 2003 Phys. Rev. E 67 061107
[10] Shushin A I 2008 Phys. Rev. E 77 031130
[11] Facchi P, Nakazato H and Pascazio S 2001 Phys. Rev. Lett. 86 2699
[12] von Neumann J 1955 Mathematical Foundation of Quantum Mechanics (Princeton, NJ: Princeton University Press)
[13] Shushin A I 2000 J. Chem. Phys. 112 9509
[14] West B J, Bologna M and Grigolini P 2002 Physics of Fractal Operators (New York: Springer)
[15] Scher H and Montroll E W 1975 Phys. Rev. B 12 2455
[16] Kubo R 1963 J. Math. Phys. 4 174
[17] Shushin A I 2005 JETP 101 577
[18] Shushin A I 2006 J. Phys. Chem. A 110 2345
[19] Abragam A 1961 The Principles of Nuclear Magnetism (Oxford: Clarendon)
[20] Haberkorn R 1976 Mol. Phys. 32 1491
[21] Gorini V, Kossakowski A and Sudarshan E C G 1976 J. Math. Phys. 17 821
[22] Power W L and Knight P L 1996 Phys. Rev. A 53 1052
[23] Ruseckas J and Kaulakys B 2006 Phys. Rev. A 73 052101