The theory of quasiregular mappings in metric spaces: progress and challenges

Chang-Yu Guo

Dedicated to Seppo Rickman and Jussi Väisälä on the occasion of their 80th birthdays

Abstract. We survey the recent developments in the theory of quasiregular mappings in metric spaces. In particular, we study the geometric porosity of the branch set of quasiregular mappings in general metric measure spaces, and then, introduce the various natural definitions of quasiregular mappings in general metric measure spaces, and give conditions under which they are quantitatively equivalent.

Mathematics Subject Classification (2010). Primary 30C65, 57M12; Secondary 58C06.

Keywords. branched coverings, quasiconformal mappings, quasiregular mappings, spaces of bounded geometry, doubling metric spaces.

1. Introduction

A continuous mapping \( f : X \to Y \) between topological spaces is said to be a branched covering if \( f \) is discrete, open and of locally bounded multiplicity. Recall that \( f \) is open if it maps each open set in \( X \) to an open set \( f(X) \) in \( Y \); \( f \) is discrete if for each \( y \in Y \) the preimage \( f^{-1}(y) \) is a discrete subset of \( X \); \( f \) has locally bounded multiplicity if for each \( x \in X \), there exists a neighborhood \( U_x \) of \( x \) and a positive constant \( M_x \) such that \( N(f, U_x) \leq M_x < \infty \), where \( N(f, U_x) := \sup_{y \in Y} |f^{-1}(y) \cap U_x| \) is the multiplicity of \( f \) in \( U_x \). The latter assumption can be dropped off if \( X \) and \( Y \) have certain manifold structure \( [18] \). Nonconstant holomorphic functions between connected Riemann surfaces are typical examples of branched coverings.

For a branched covering \( f : X \to Y \) between two metric spaces, \( x \in X \) and \( r > 0 \), set

\[
H_f(x, r) = \frac{L_f(x, r)}{l_f(x, r)},
\]

This work was completed with the support of the Swiss National Science Foundation (No. 153599).
where
\[ L_f(x, r) := \sup_{y \in Y} \{ d(f(x), f(y)) : d(x, y) = r \}, \]
and
\[ l_f(x, r) := \inf_{y \in Y} \{ d(f(x), f(y)) : d(x, y) = r \}. \]
Then the linear dilatation function of \( f \) at \( x \) is defined pointwise by
\[ H_f(x) = \limsup_{r \to 0} H_f(x, r). \]

**Definition 1.1.** A branched covering \( f : X \to Y \) between two metric measure spaces is termed **metrically \( H \)-quasiregular** if the linear dilatation function \( H_f \) is finite everywhere and essentially bounded from above by \( H \).

If \( f : X \to Y \), in Definition 1.1, is additionally assumed to be a homeomorphism, then \( f \) is called **metrically \( H \)-quasiconformal**. We will call \( f \) a metrically quasiregular or quasiconformal mapping if it is metrically \( H \)-quasiregular or \( H \)-quasiconformal for some \( H \in [1, \infty) \).

Quasiregular mappings were first introduced by Reshetnyak in 1966 [26], where he actually used the (equivalent) analytic formulation of quasiregularity. The analytic foundation of the theory of quasiregular mappings were laid after a sequence of his papers from 1966 to 1969. A deep fact he discovered is that analytic quasiregular mappings are branched coverings [27]. The whole theory of quasiregular mapping were significantly advanced in a sequence of papers from the Finnish school in the late 1960s [21, 22, 23]. See also the beautiful paper [20] for a nice survey on the development of the field.

For quasiconformal mappings, there is also a geometric definition, which makes uses of the modoulus of curve families. One of the most fundamental results in the theory of quasiconformal mappings, due to the deep works of Gehring, Väisälä and many others, is that all the three different definitions of quasiconformality are quantitatively equivalent. According to a later remarkable result of Heinonen and Koskela [13], one can even relax limsup to liminf in the metric definition of quasiconformality. The similar result holds in the theory of quasiregular mappings as well, but the proofs are much more involved [28].

Due the numerous successful applications of the theory quasiconformal mappings in geometric group theory (for instance [24]), the fundation of the theory of metrically quasiconformal mappings has been laid [14] in the general framework of metric measure spaces with controlled geometry.

All of the three definitions of quasiconformality can be generalized in a natural way to the setting of metric measure spaces. A remarkable fact, after a number of seminal works [15, 31, 32, 33, 34, 35], is that the quantitative equivalences of quasiconformality extends to a large class of metric spaces.

The recent advances in analysis on metric spaces [18, 19, 12] promotes a general theory of quasiregular mappings in the setting of metric spaces,

---

\(^1\)A **metric measure space** is defined to be a triple \((X, d, \mu)\), where \((X, d)\) is a separable metric space and \(\mu\) is a nontrivial locally finite Borel regular measure on \(X\).
whereas a complete understanding of the various definitions of quasiregularity plays an important role. In this survey, we examine the minimal assumptions on metric spaces $X$ and $Y$ and a branched covering $f : X \to Y$ guarantee $f$ being quasiregular according to different definitions.

2. Preliminaries on metric spaces

A main theme in analysis on metric spaces is that the infinitesimal structure of a metric space can be understood via the curves that it contains. The reason behind this is that we can integrate Borel functions along rectifiable curves and do certain non-smooth calculus akin to the Euclidean spaces.

Let $(X, d)$ be a metric space. A curve (or path) in $X$ is a continuous map $\gamma : I \to X$, where $I \subset \mathbb{R}$ is an interval. We call $\gamma$ compact, open, or half-open, depending on the type of the interval $I$.

Given a compact curve $\gamma : [a, b] \to X$, we define the variation function $v_\gamma : [a, b] \to [a, b] \to \mathbb{R}$ by

$$v_\gamma(s) = \sup_{a \leq a_1 \leq b_1 \leq \cdots \leq a_n \leq b_n \leq s} \sum_{i=1}^{n} d(\gamma(b_i), \gamma(a_i)).$$

The length $l(\gamma)$ of $\gamma$ is defined to be the variation $v_\gamma(b)$ at the end point $b$ of the parametrizing interval $[a, b]$. If $\gamma$ is not compact, its length $l(\gamma)$ is defined to be the supremum of the lengths of the compact subcurves of $\gamma$.

A curve is said to be rectifiable if its length $l(\gamma)$ is finite, and locally rectifiable if each of its compact subcurves is rectifiable. For any rectifiable curve $\gamma$ there are its associated length function $s_\gamma : I \to [0, l(\gamma)]$ and a unique 1-Lipschitz map $\gamma_s : [0, l(\gamma)] \to X$ such that $\gamma = \gamma_s \circ s_\gamma$. The curve $\gamma_s$ is the arc length parametrization of $\gamma$.

When $\gamma$ is rectifiable, and parametrized by arclength on the interval $[a, b]$, the integral of a Borel function $\rho : X \to [0, \infty]$ along $\gamma$ is

$$\int_\gamma \rho \, ds = \int_0^{l(\gamma)} \rho(\gamma_s(t)) \, dt.$$

Similarly, the line integral of a Borel function $\rho : X \to [0, \infty]$ over a locally rectifiable curve $\gamma$ is defined to be the supremum of the integral of $\rho$ over all compact subcurves of $\gamma$.

A curve $\gamma$ is absolutely continuous if $v_\gamma$ is absolutely continuous. Via the chain rule, we then have

$$\int_\gamma \rho ds = \int_a^b \rho(\gamma(t))v_\gamma'(t)dt. \quad (2.1)$$

Let $X = (X, d, \mu)$ be a metric measure space. Let $\Gamma$ a family of curves in $X$. A Borel function $\rho : X \to [0, \infty]$ is admissible for $\Gamma$ if for every locally rectifiable curves $\gamma \in \Gamma$, $\int_\gamma \rho \, ds \geq 1$. 

$$\int_\Gamma \rho \, ds \geq 1. \quad (2.2)$$
The $p$-modulus of $\Gamma$, $p \geq 1$, is defined as

$$\text{Mod}_p(\Gamma) = \inf_{\rho} \left\{ \int_X \rho^p \, d\mu : \rho \text{ is admissible for } \Gamma \right\}.$$ 

A family of curves is called $p$-exceptional if it has $p$-modulus zero. We say that a property of curves holds for $p$-almost every curve if the collection of curves for which the property fails to hold is $p$-exceptional.

Let $X = (X, d, \mu)$ be a metric measure space and $Z = (Z, d_Z)$ be a metric space.

A Borel function $g : X \to [0, \infty]$ is called an upper gradient for a map $f : X \to Z$ if for every rectifiable curve $\gamma : [a, b] \to X$, we have the inequality

$$\int_\gamma g \, ds \geq d_Z(f(\gamma(b)), f(\gamma(a))).$$  \hfill (2.3)

If inequality (2.3) holds for $p$-almost every compact curve, then $g$ is called a $p$-weak upper gradient for $f$. When the exponent $p$ is clear, we omit it.

A $p$-weak upper gradient $g$ of $f$ is minimal if for every $p$-weak upper gradient $\tilde{g}$ of $f$, $\tilde{g} \geq g$ $\mu$-almost everywhere. If $f$ has an upper gradient in $L^p_{\text{loc}}(X)$, then $f$ has a unique (up to sets of $\mu$-measure zero) minimal $p$-weak upper gradient. We denote the minimal upper gradient by $|\nabla f|$.

Real-valued Sobolev spaces based on upper gradients were used to great success \cite{4} and explored in-depth in \cite{29}. They have been extended to the metric-valued setting in \cite{15,16}. There are several equivalent ways to define the Sobolev spaces of mappings between metric measure space and a simple definition is as follows. Let $f : X \to Y$ be a continuous map. Then $f$ belongs to the Sobolev space $N_{1,p}^{1,p}(X, Y)$, $1 \leq p < \infty$, if for each relatively compact open subset $U \subset X$, the map $f$ has an upper gradient $g \in L^p(U)$ in $U$, and there is a point $x_0 \in U$ such that $u(x) := d_Y(u(x), x_0) \in L^p(U)$.

A Borel regular measure $\mu$ on a metric space $(X, d)$ is called a doubling measure if every ball in $X$ has positive and finite measure and there exists a constant $C_{\mu} \geq 1$ such that

$$\mu(B(x, 2r)) \leq C_{\mu} \mu(B(x, r)) \quad \text{(2.4)}$$

for each $x \in X$ and $r > 0$. We call the triple $(X, d, \mu)$ a doubling metric measure space if $\mu$ is a doubling measure on $X$. We call $(X, d, \mu)$ an Ahlfors $Q$-regular space, $1 \leq Q < \infty$, if there exists a constant $C \geq 1$ such that

$$C^{-1} r^{Q} \leq \mu(B(x, r)) \leq C r^{Q} \quad \text{(2.5)}$$

for all balls $B(x, r) \subset X$ of radius $r < \text{diam} \, X$.

We say that a metric measure space $(X, d, \mu)$ admits a $(1,p)$-Poincaré inequality if there exist constants $C \geq 1$ and $\tau \geq 1$ such that

$$\int_B |u - u_B| d\mu \leq C \text{diam}(B) \left( \int_B g^p d\mu \right)^{1/p} \quad \text{(2.6)}$$

for all open balls $B$ in $X$, for every function $u : X \to \mathbb{R}$ that is integrable on balls and for every upper gradient $g$ of $u$ in $X$. 
The \((1, p)\)-Poincaré inequality can be thought of as a requirement that a space contains “many” curves, in terms of the \(p\)-modulus of curves in the space. For more information on the Poincaré inequalities, see [10, 16].

A metric measure space \((X, d, \mu)\) is said to have \(Q\)-bounded geometry if it is Ahlfors \(Q\)-regular and supports a \((1, Q)\)-Poincaré inequality. See [14] for more information on metric spaces of bounded geometry.

Throughout the paper, we assume that \(X\) and \(Y\) are locally compact, complete, connected, locally connected metric spaces.

3. Size of the branch set

The main obstacle in establishing the theory of quasiregular mappings in general metric spaces lies in the branch set \(B_f\), i.e., the set of points in \(X\) where \(f: X \to Y\) fails to be a local homeomorphism. The difficulty is somehow hidden in the Euclidean planar case, as the celebrated Stöllow factorization theorem asserts that a quasiregular mapping \(f: \Omega \to \mathbb{R}^2\) admits a factorization \(f = \varphi \circ g\), where \(g: \Omega \to g(\Omega)\) is quasiconformal and \(\varphi: g(\Omega) \to \mathbb{R}^2\) is analytic. This factorization, together with relatively complete understanding of the structure of analytic functions in the plane, connects quasiregular and quasiconformal mappings strongly. In particular, the branch set \(B_f\) of a quasiregular mapping \(f: \Omega \to \mathbb{R}^2\) is discrete.

In higher dimensions or more general metric measure spaces, the branch set of a quasiregular mappings can be very wild, for instance, it might contain many wild Cantor sets, such as the Antoine's necklace [17], of classical geometric topology. This makes the homeomorphic theory and the non-homeomorphic theory substantially different. Indeed, the most delicate part of establishing the theory of quasiregular mappings in various settings as mentioned above is to show that the branch set and its image have null measure. For a survey on the topological property of the branch set as well as open problems in this direction, see [11].

Regarding the Hausdorff dimension of \(B_f\) and its image \(f(B_f)\) in the Euclidean setting, a well-known result of Gehring and Väisälä [?] says that for each \(n \geq 3\) and each pair of numbers \(\alpha, \beta \in [n-2, n)\), there exists a quasiregular mapping \(f: \mathbb{R}^n \to \mathbb{R}^n\) such that
\[ \dim_{\mathcal{H}} B_f = \alpha \quad \text{and} \quad \dim_{\mathcal{H}} f(B_f) = \beta. \]

On the other hand, by the result of Sarvas [30], for a non-constant \(H\)-quasiregular mapping \(f: \Omega \to \mathbb{R}^n, n \geq 2,\) between Euclidean domains,
\[ \dim_{\mathcal{H}} f(B_f) \leq n - \eta \quad (3.1) \]
for some constant \(\eta = \eta(n, H) > 0\). Yet another well-known result of Bonk and Heinonen [3] says that for a non-constant \(H\)-quasiregular mapping \(f: \Omega \to \mathbb{R}^n, n \geq 2,\) between Euclidean domains,
\[ \dim_{\mathcal{H}} B_f \leq n - \eta \quad (3.2) \]
for some constant \(\eta = \eta(n, H) > 0\).
The recent development in analysis on metric spaces [18, 19, 12] promotes a general theory of quasiregular mappings beyond the Riemannian spaces, whereas a deeper understanding of the branch set of a quasiregular mapping is rather necessary.

Assume that $X$ and $Y$ are doubling metric spaces, which are also topological $n$-manifolds, that $X$ is linearly locally contractible\(^2\), and that $Y$ has bounded turning\(^3\). A special case of [8, Theorem 1.1] reads as follows.

**Theorem 3.1.** If $H_f(x) \leq H$ for every $x \in X$, then $B_f$ and $f(B_f)$ are countably $\delta$-porous, quantitatively. Moreover, the porosity constant can be explicitly calculated.

Recall that a set $E \subset X$ is said to be $\alpha$-porous if for each $x \in E$,

$$\liminf_{r \to 0} r^{-1} \sup \{ \rho : B(z, \rho) \subset B(x, r) \setminus E \} \geq \alpha.$$ 

A subset $E$ of $X$ is called countably ($\sigma$-)porous if it is a countable union of ($\sigma$-)porous subsets of $X$.

Since porous sets have Hausdorff dimension strictly less than $Q$, quantitative, in an Ahlfors $Q$-regular space, we have the following immediate corollary.

**Corollary 3.2.** If $X$ and $Y$ are Ahlfors $Q$-regular, and $H_f(x) < \infty$ for all $x \in X$, then $\mathcal{H}^Q(B_f) = \mathcal{H}^Q(f(B_f)) = 0$. Moreover, if $H_f(x) \leq H$ for all $x \in X$, then

$$\max \{ \dim_{\mathcal{H}}(B_f), \dim_{\mathcal{H}}(f(B_f)) \} \leq Q - \eta < Q,$$

where $\eta$ depends only on $H$ and the data of $X$ and $Y$. Moreover, $\eta$ can be explicitly calculated.

The assumptions in Theorem 3.1 or Corollary 3.2 is rather sharp as the following example indicates.

**Example** ([9]). For each $n \geq 3$, there exist an Ahlfors $n$-regular metric space $X$ that is homeomorphic to $\mathbb{R}^n$ and supports a $(1, 1)$-Poincaré inequality, and a $1$-quasiregular mapping (indeed $1$-BLD) $f: X \to \mathbb{R}^n$, such that

$$\min \{ \mathcal{H}^n(B_f), \mathcal{H}^n(f(B_f)) \} > 0.$$ 

4. The pullback factorization

In this section, we briefly explore the pullback factorization introduced (due to M. Williams) in [9]. One feature of the Stoilow factorization is that we can factorize a quasiregular mapping $f$ into the composition of an “analytically nice” (high regularity) mapping $\pi$ with a “topologically nice” (homeomorphism) mapping $g$.

Let $f: X \to Y$ be a branched covering.

---

\(^2\)A metric space is *linearly locally contractible* if each ball of radius $r$ contracts in a ball with the same center and radius $\lambda r$ for some $\lambda \geq 1$.

\(^3\)A metric space $X$ has *c-bounded turning* if every pair of points $x_1, x_2 \in X$ can be joined by a continuum $E \subset X$ such that $\text{diam } E \leq cd(x_1, x_2)$. 
4.1. The pullback metric
The "pullback metric" \( f^*d_Y : X \times X \to [0, \infty) \) is defined as follows:

\[
f^*d_Y(x_1, x_2) = \inf_{x_1, x_2 \in \alpha} \text{diam} \left( f(\alpha) \right),
\]

(4.1)

where the infimum is taken over all continua \( \alpha \) joining \( x_1 \) and \( x_2 \) in \( X \).

It is immediate from the definition that \( f^*d_Y \) satisfies the triangle inequality. Moreover, the connectivity assumption on \( X \) guarantees that it is finite. Thus the discreteness of \( f \) implies that \( f^*d_Y \) is indeed a genuine metric. We denote by \( X^f \) or \( f^*Y \) the metric space \((X, f^*d_Y)\).

4.2. Canonical factorization
In what follows, let \( g : X \to X^f \) be the identity map, and let \( \pi : X^f \to Y \) satisfy \( \pi \circ g = f \), so that on the level of sets, \( f = \pi \). We refer this canonical factorization as the pullback factorization for \( f \).

\[
\begin{array}{ccc}
X & \xrightarrow{g} & X^f \\
\downarrow{f} & & \downarrow{\pi} \\
Y
\end{array}
\]

Figure 1. The canonical pullback factorization

Under appropriate conditions, the metric space \( X^f \) and the branched covering \( \pi \) are well behaved. In order to make precise these nice behaviors, we need to recall some of the nice mapping classes between metric spaces.

**Definition 4.1.** A branched covering \( f : X \to Y \) between two metric spaces is said to be an \( L\)-BLD, or a mapping of \( L\)-bounded length distortion, \( L \geq 1 \), if

\[
L^{-1}l(\alpha) \leq l(f \circ \alpha) \leq Ll(\alpha)
\]

for all non-constant curves \( \alpha \) in \( X \).

The definition of BLD mappings is clearly only interesting if the metric spaces \( X \) and \( Y \) have a reasonable supply of rectifiable curves and so the most natural setting in which we study such mappings is that of quasiconvex metric spaces.

When \( X \) and \( Y \) have bounded turning, the natural branched analog of a bi-Lipschitz homeomorphism is what we call a mapping of bounded diameter distortion, which is defined in analogy with BLD mappings:

**Definition 4.2.** A branched covering \( f : X \to Y \) between two metric spaces is said to be an \( L\)-BDD, or a mapping of \( L\)-bounded diameter distortion, \( L \geq 1 \), if

\[
L^{-1} \text{diam}(\alpha) \leq \text{diam}(f \circ \alpha) \leq L \text{diam}(\alpha)
\]

\[\]
for all non-constant curves $\alpha$ in $X$.

It follows directly from the definition of arc-length that an $L$-BDD mapping is $L$-BLD as well.

As we will see in a moment, the metric space $X^f$ retains the original topology of $X$ and is often rather well-behaved: having 1-bounded turning and inheriting many metric and geometric properties from $Y$. Moreover, the branched covering $\pi$ from the pullback factorization is easily seen to be 1-Lipschitz and 1-BDD (and a fortiori 1-BLD).

4.3. Fine properties of the pullback metric

In this section, we fix a branched covering $\pi: X \to Y$ between two metric spaces. However, we will typically consider the space $X^\pi$ (or $\pi^*Y$) by endowing the set $X$ with the pullback metric $\pi^*d_Y$.

For each metric $d$ on a topological space $X$, the length metric $l^d(z_1, z_2)$ is given by infimizing the lengths of all paths joining $z_1$ and $z_2$. For a metric space $X = (X, d_X)$, we denote by $X^l$ the length space $(X, l^d_X)$.

For simplicity, we will formulate many of our basic results for the case that $\pi$ is proper (i.e. the preimage of each compact set is compact) and surjective, and $N = N(\pi, X) < \infty$. Up to the end of this section, we additionally assume the metric space $Y$ is proper, i.e. closed bounded balls are compact.

Recall that a continuous mapping $f: X \to Y$ between two metric spaces is said to be $c$-co-Lipschitz if for all $x \in X$ and $r > 0$,\

$$B(f(x), r) \subset f(B(x, cr)).$$

The basic properties of the pullback metric are formulated in the next proposition. Recall that for a continuous mapping $\pi: X \to Y$, $U(x, \pi, r)$ denotes the $x$-component of $\pi^{-1}(B(\pi(x), r))$.

**Proposition 4.3 (Section 4.3, [9]).** 1. The metric space $X^\pi$ is a proper metric space, homeomorphic to $X$ via the identity mapping $g$. Open and closed balls in $X^\pi$ are connected and $X^\pi$ have 1-bounded turning. The projection mapping $\pi: X^\pi \to Y$ is 1-Lipschitz, 1-BDD and for each $z \in X^\pi$,\

$$B(z, r) \subset U(z, \pi, r) \subset B(z, 2r). \quad (4.2)$$

2. If $Y$ has $c$-bounded turning, then $\pi$ is $c$-co-Lipschitz and is locally $c$-bi-Lipschitz on each set $X_k := \{z \in X^\pi : N(\pi(z, X) = k\}$. If, additionally, $Y$ is Ahlfors $Q$-regular with constant $c_2$, then $X^\pi$ is Ahlfors $Q$-regular with constant $c^Qc_2N$. Moreover, for each $k = 1, \ldots, N$, and at each Lebesgue point $z$ of $X_k$, we may take the pointwise constant of $Q$-regularity to be $c^Qc_2$.

3. If $Y$ is $c$-LLC-2 and $X$ has no local cut points\(^5\), then $X^\pi$ is locally 2c-LLC.

\(^5\)A point $x$ in a metric space $X$ is called a local cut point if $U \setminus \{x\}$ is disconnected for some neighborhood $U$ of $x$.
4. If $l_{d_Y}$ and $d_Y$ induces the same topology on $Y$, then the metrics $\pi^*d_Y$, $l_{\pi^*d_Y}$, $\pi^*l_{d_Y}$, and $l_{\pi^*l_{d_Y}}$ induces the same topology on $X$. The length of a curve in $X^\pi$ is the same with respect to any of these four metrics, and moreover,

$$\pi^*d_Y \leq \pi^*l_{d_Y} \leq l_{\pi^*d_Y} \leq l_{\pi^*l_{d_Y}} \leq (2N-1)\pi^*l_{d_Y}.$$ 

In particular, the metric space $\pi^*(Y^l)$ is $(2N-1)$-quasiconvex and if $Y$ is $c$-quasiconvex, then $X^\pi$ is $(2N-1)c$-quasiconvex.

4.4. Fine properties of the pullback factorization

Fix any branched covering $f : X \to Y$ and the pullback factorization $f = \pi \circ g$, where $g : X \to X^f$ is the identity mapping and $\pi : X^f \to Y$ is the branched covering given by $\pi = f$.

By Proposition 4.3, $\pi$ is a 1-BDD mapping and thus we have factorized $f$ into a composition of a homeomorphism and a 1-BDD “projection”. Note that while on the level of sets, we are factoring out the identity, the mapping $g$ will typically not be an isometry. But we have seen already, the projection mapping $\pi$ can be thought of as being as close to an isometry as possible. Thus, philosophically, we have factored $f$ into a geometric equivalence composed with a topological one.

Furthermore, as a result of the fact that $\pi$ is 1-BDD, $f$ and $g$ share many geometric properties.

Proposition 4.4 (Section 4.4, [9]).

1) The mapping $f$ is $L$-BDD/BLD if and only if $g$ is $L$-BDD/BLD.

2) Suppose that $Y$ is $c$-quasiconvex, and that $f$ is $L$-Lipschitz and $L$-BLD with $N = N(f, X) < \infty$. Then $f$ is $cNL$-BDD.

Thus much of the theory of BLD and BDD mappings reduces to the study of the pullback metric. Similarly, it turns out that under various definitions and for many different levels of generality, $g$ is quasiconformal if and only if $f$ is quasiregular. Thus one obtains a canonical factorization of a quasiregular mapping into a composition of a quasiconformal mapping with a 1-BDD mapping. This is particularly useful in extending the theory of quasiregular mappings to the metric setting, as the quasiconformal theory has at present advanced much further than its branched counterpart in this generality.

5. Quasiregular mappings between metric spaces

Fix two metric measure spaces $X = (X, d_X, \mu)$ and $Y = (Y, d_Y, \nu)$. Assume that $\Omega \subset X$ is a domain, and that $f : \Omega \to \Omega \subset Y$ is an onto branched covering.

In what follows, we fix the pullback factorization $f = \pi \circ g$ as in Section 4, where $g : \Omega \to \Omega^f$ is the identity mapping and $\pi : \Omega^f \to \Omega$ is the 1-BDD projection. We equip $\Omega^f$ with the Borel regular measure $\lambda = \pi^*\nu = g_*f^*\nu$ and simply write $\Omega^f$ for the metric measure space $(\Omega^f, f^*d_Y, \lambda)$. 

5.1. Definitions of quasiregularity in general metric measure spaces

In this section, we introduce the different definitions of quasiregularity in general metric measure spaces.

Set \( h_f(x) = \liminf_{r \to 0} H_f(x, r) \).

**Definition 5.1 (Weak metrically quasiregular mappings).** A branched covering \( f: \tilde{\Omega} \to \Omega \) is said to be *weakly metrically \( H \)-quasiregular* if it satisfies

i). \( h_f(x) < \infty \) for all \( x \in \tilde{\Omega} \);

ii). \( h_f(x) \leq H \) for \( \mu \)-almost every \( x \in \tilde{\Omega} \).

For a branched covering \( f: X \to Y \), the volume Jacobian is defined by

\[
J_f := \frac{d(f^*\nu)}{d\mu},
\]

where the pullback measure \( f^*\nu \) on \( X \) is given by

\[
f^*\nu(A) = \int_Y N(y, f, A) d\nu(y).
\]

**Definition 5.2 (Analytically quasiregular mappings).** A branched covering \( f: \tilde{\Omega} \to \Omega \) is said to be *analytically \( K \)-quasiregular* if \( f \in N_{loc}^{1,Q}(\tilde{\Omega}, \Omega) \) and

\[
|\nabla f(x)|^Q \leq K J_f(x)
\]

for \( \mu \)-a.e. \( x \in \tilde{\Omega} \).

The geometric definition requires some modulus inequalities between curve families.

**Definition 5.3 (Geometrically quasiregular mappings).** A branch covering \( f: \tilde{\Omega} \to \Omega \) is said to be *geometrically \( K \)-quasiregular* if it satisfies the \( K_O \)-inequality, i.e., for each open set \( \tilde{\Omega}_0 \subset \tilde{\Omega} \) and each path family \( \Gamma \) in \( \tilde{\Omega}_0 \subset \tilde{\Omega} \), if \( \rho \) is a test function for \( f(\Gamma) \), then

\[
\text{Mod}_Q(\Gamma) \leq K \int_{\Omega} N(y, f, \tilde{\Omega}_0) \rho^Q(y) d\nu(y).
\]

We will refer to the metric definition \( (M) \), the weak metric definition \( (m) \), the analytic definition \( (A) \), and the geometric definition \( (G) \) as elements of the *forward definitions*.

Next, we introduce the elements from the *inverse definitions*: the inverse metric definition \( (M^*) \), the inverse weak metric definition \( (m^*) \), the inverse analytic definition \( (A^*) \), and the inverse geometric definition \( (G^*) \).
For each $x \in X$,
\[ H_f^*(x, s) = \frac{L_f^*(x, s)}{l_f^*(x, s)}, \]
where
\[ L_f^*(x, s) = \sup_{z \in \partial U(x, f, s)} d(x, z) \]
and
\[ l_f^*(x, s) = \inf_{z \in \partial U(x, f, s)} d(x, z). \]

The inverse linear dilatation function of $f$ at $x$ is defined pointwise by
\[ H_f^*(x) = \limsup_{s \to 0} H_f^*(x, s). \]
Similarly, the weak inverse linear dilatation function of $f$ at $x$ is
\[ h_f^*(x) = \liminf_{s \to 0} H_f^*(x, s). \]

**Definition 5.4 (Inverse metrically quasiregular mappings).** A branched covering $f: \tilde{\Omega} \to \Omega$ between two metric measure spaces is termed inverse metrically $H$-quasiregular if the inverse linear dilatation function $H_f^*$ is finite everywhere and essentially bounded from above by $H$.

**Definition 5.5 (Inverse weak metrically quasiregular mappings).** A branched covering $f: \tilde{\Omega} \to \Omega$ between two metric measure spaces is termed inverse weak metrically $H$-quasiregular if it satisfies

i). $h_f^*(x) < \infty$ for all $x \in \tilde{\Omega}$;

ii). $h_f^*(x) \leq H$ for $\mu$-almost every $x \in \tilde{\Omega}$.

Using the pullback factorization, we introduce the following new class of inverse analytically quasiregular mappings.

**Definition 5.6 (Inverse analytically quasiregular mappings).** A branched covering $f: \tilde{\Omega} \to \Omega$ is said to be inverse analytically $K$-quasiregular if $g^{-1} \in N_{1, \text{loc}}^1(\tilde{\Omega}^f, \Omega)$ and
\[ |\nabla g^{-1}|(z)^Q \leq K J_{g^{-1}}(z) \]
for $\lambda$-a.e. $z \in \tilde{\Omega}^f$.

The inverse geometric definition also relies on certain inequalities for the modulus of curve families.

**Definition 5.7 (Inverse geometric quasiregular mappings).** A branched covering $f: \tilde{\Omega} \to \Omega$ is said to be inverse geometrically $K$-quasiregular if it satisfies the $K_f$-inequality or the Poletsky’s inequality, i.e., for every curve family $\Gamma$ in $\tilde{\Omega}$, we have
\[ \text{Mod}_Q(f(\Gamma)) \leq K \text{Mod}_Q(\Gamma). \]

We also introduce the following strong inverse geometrically quasiregular mappings, which can be viewed as a generalized Väisälä’s inequality.
Definition 5.8 (Strong inverse geometrically quasiregular mappings). A branched covering \( f : \tilde{\Omega} \to \Omega \) is said to be a strong inverse geometric \( K \)-quasiregular mapping if it satisfies the following generalized Väisälä’s inequality: For each open subset \( \tilde{\Omega}_0 \subset \tilde{\Omega} \), each curve family \( \Gamma \) in \( \tilde{\Omega}_0 \), \( \Gamma' \) in \( \Omega \), and for each \( \gamma' \in \Gamma' \), there are curves \( \gamma_1, \ldots, \gamma_m \in \Gamma \) and subcurves \( \gamma'_1, \ldots, \gamma'_m \) of \( \gamma' \) such that for each \( i = 1, \ldots, m \), \( \gamma'_i = f(\gamma_i) \), and for almost every \( s \in [0, l(\gamma')] \), \( \gamma_i(s) = \gamma_j(s) \) if and only if \( i = j \), then
\[
\text{Mod}_Q(\Gamma') \leq \frac{K}{m} \text{Mod}_Q(\Gamma).
\]

5.2. Equivalences of definitions of quasiregularity

By [15, Theorem 9.8] and [2, Theorem 1.1], when \( f : \tilde{\Omega} \to \Omega \) is a homeomorphism, and \( \tilde{\Omega} \) and \( \Omega \) have \( Q \)-bounded geometry, the inverse (metric, weak metric, analytic, geometric) definitions for \( f \) are, quantitatively, the forward (metric, weak metric, analytic geometric) definitions for \( f^{-1} \). Moreover, in this case, each of these definitions is further equivalent to the local quasisymmetry, quantitatively.

Using the pullback factorization, we have

**Theorem 5.9 (Theorem A, [9]).** The following conclusions hold:

i). \( f \) is analytically \( K_O \)-quasiregular if and only if it is geometrically \( K_O \)-quasiregular. Similarly, \( f \) is inverse analytically \( K_I \)-quasiregular if and only if it is strong inverse geometrically \( K_I \)-quasiregular.

ii). If \( \tilde{\Omega} \) and \( \Omega \) are Ahlfors \( Q \)-regular, and \( Y \) has \( c \)-bounded turning, then either of the following two conditions

a). \( h_f(x) \leq h \) for all \( x \in \tilde{\Omega} \);

b). \( h^*_f(x) \leq h \) for all \( x \in \tilde{\Omega} \),

implies that \( f \) is analytically \( K_O \)-quasiregular and inverse analytically \( K_I \)-quasiregular, with both constants \( K_O \) and \( K_I \) depending only on the constant of Ahlfors \( Q \)-regularity, and on \( c \) and \( h \).

iii). If both \( \tilde{\Omega} \) and \( \Omega \) have \( Q \)-bounded geometry, then all of the metric, geometric and analytic definitions are quantitatively equivalent.

Theorem 5.9 generalizes the earlier results of [15] [2] [34] [35] about quasiconformal mappings in a natural form to that of quasiregular mappings. Moreover, if \( f \) satisfies the \( K_I \)-inequality with \( \nu(f(B_f)) = 0 \), then \( f \) satisfies the standard Väisälä’s inequality with the same constant [9].

5.3. Branched quasisymmetric mappings

We next define a proper subclass of metrically quasiregular mappings that carry similar global metric information, but less restrictive as those BDD mappings.

**Definition 5.10 (Branched quasisymmetric mappings).** Let \( f : X \to Y \) be a branched covering. We say that \( f \) is branched quasisymmetric (BQS) if there exists a homeomorphism \( \eta : [0, \infty) \to [0, \infty) \) such that
\[
\frac{\text{diam } f(E)}{\text{diam } f(F)} \leq \eta\left(\frac{\text{diam } E}{\text{diam } F}\right)
\]
for all intersected continua $E, F \subset X$.

**Remark 5.11.** If $f: X \to Y$ is a homeomorphism between two metric spaces that have bounded turning and if $f$ satisfies the inequality (5.1), then $f$ is quasisymmetric, quantitatively. See [9].

Similar as quasiconformal mappings are locally quasisymmetric in spaces of bounded geometry, we have the following branched version.

**Theorem 5.12.** Let $f: X \to Y$ be a weak metrically $H$-quasiregular mapping such that $N = N(f, X) < \infty$. Assume that both $X$ and $Y$ have locally $Q$-bounded geometry. Then $f$ is locally $\eta$-branched quasisymmetric, quantitatively, with $\eta$ depending only on $H, N$, and the data of $X$ and $Y$.

**Remark 5.13.** In Theorem 5.12, the homeomorphism $\eta$ depends, quantitatively on the multiplicity $N$. In general, one can not get rid of this dependence from the theorem, as the simple analytic function $z \mapsto z^k$ indicated.

6. Concluding remarks

All the preceding works on the theory of quasiregular mapping or quasiconformal mappings require the spaces in question to have quantitative bounded geometry (i.e., both Ahlfors regularity and the Poincaré inequality with quantitative data), in order to get a rich theory.

On the other hand, much of the theory has been extended to the general equiregular subRiemannian manifolds [7, 6] or even certain non-equiregular subRiemannian manifolds [1]. Equiregular subRiemannian manifolds do not necessarily have quantitative bounded geometry and non-equiregular subRiemannian manifolds are merely doubling (typically not Ahlfors regular).

The standard assumptions nowadays in analysis on metric spaces are doubling and Poincaré (i.e., the metric measure spaces are doubling and support abstract Poincaré inequalities). It is of great interest to know whether one can build the theory of quasiconformal/quasiregular mappings in such metric spaces. In particular, we would like to know whether the quantitative equivalence of definitions of quasiconformality/quasiregularity as in Theorem 5.9 remains valid, and is further quantitatively equivalent to the local quasisymmetry/branched quasisymmetry as in Theorem 5.12 in the more general setting. Note that the question is unknown even for mappings from $X$ to $\mathbb{R}^2$ [25, Question 17.3].

**Acknowledgment**

Many thanks to wonderful event “International Conference on Complex Analysis and Related Topics, The 14th Romanian-Finnish Seminar”, where part of this work has been done.
References

[1] C. Ackermann, *An approach to studying quasiconformal mappings on generalized Grushin planes*, Ann. Acad. Sci. Fenn. Math. 40 (2015), no. 1, 305-320.

[2] Z. Balogh, P. Koskela and S. Rogovin, *Absolute continuity of quasiconformal mappings on curves*, Geom. Funct. Anal. 17 (2007), no. 3, 645-664.

[3] M. Bonk and J. Heinonen, *Smooth quasiregular mappings with branching*, Publ. Math. Inst. HautesÉtudes Sci. No. 100 (2004), 153-170.

[4] J. Cheeger, *Differentiability of Lipschitz functions on metric measure spaces*, Geom. Funct. Anal. 9 (1999), no. 3, 428-517.

[5] F.W. Gehring, *Rings and quasiconformal mappings in space*, Trans. Amer. Math. Soc. 103 (1962), 353-393.

[6] C.Y. Guo and T. Liimatainen, *Equivalence of quasiregular mappings on subRiemannian manifolds via the Popp extension*, preprint 2016.

[7] C.Y. Guo, S. Nicolussi Golo and M. Williams, *Quasiregular mappings between subRiemannian manifolds*, preprint 2015.

[8] C.Y. Guo and M. Williams, *Porosity of the branch set of discrete open mappings with controlled linear dilatation*, preprint 2015.

[9] C.Y. Guo and M. Williams, *Geometric function theory: the art of pullback factorization*, preprint 2016.

[10] P. Hajłasz and P. Koskela, *Sobolev met Poincaré*, Mem. Amer. Math. Soc. 145 (2000), no. 688.

[11] J. Heinonen, *The branch set of a quasiregular mapping*, Proceedings of the International Congress of Mathematicians, Vol. II (Beijing, 2002), 691-700, Higher Ed. Press, Beijing, 2002.

[12] J. Heinonen and S. Keith, *Flat forms, bi-Lipschitz parameterizations, and smoothability of manifolds*, Publ. Math. Inst. Hautes Études Sci. No. 113 (2011), 1-37.

[13] J. Heinonen and P. Koskela, *Definitions of quasiconformality*, Invent. Math. 120 (1995), 61-79.

[14] J. Heinonen and P. Koskela, *Quasiconformal maps in metric spaces with controlled geometry*, Acta Math. 181 (1998), 1-61.

[15] J. Heinonen, P. Koskela, N. Shanmugalingam and J.T. Tyson, *Sobolev classes of Banach space-valued functions and quasiconformal mappings*, J. Anal. Math. 85 (2001), 87-139.

[16] J. Heinonen, P. Koskela, N. Shanmugalingam and J.T. Tyson, *Sobolev spaces on metric measure spaces: an approach based on upper gradients*, Cambridge Studies in Advanced Mathematics Series, 2015.

[17] J. Heinonen and S. Rickman, *Quasiregular maps $S^3 \to S^3$ with wild branch sets*, Topology 37 (1998), no. 1, 1-24.

[18] J. Heinonen and S. Rickman, *Geometric branched covers between generalized manifolds*, Duke Math. J. 113 (2002), no. 3, 465-529.

[19] J. Heinonen and D. Sullivan, *On the locally branched Euclidean metric gauge*, Duke Math. J. 114 (2002), no. 1, 15-41.

[20] G. Martin, *The theory of quasiconformal mappings in higher dimensions*, I. Handbook of Teichmüller theory. Vol. IV, 619-677, IRMA Lect. Math. Theor. Phys., 19, Eur. Math. Soc., Zürich, 2014.
[21] O. Martio, S. Rickman and J. Väisälä, *Definitions for quasiregular mappings*, Ann. Acad. Sci. Fenn. Ser. A I No. 448 (1969), 40 pp.

[22] O. Martio, S. Rickman and J. Väisälä, *Distortion and singularities of quasiregular mappings*, Ann. Acad. Sci. Fenn. Ser. A I No. 465 (1970), 13 pp.

[23] O. Martio, S. Rickman and J. Väisälä, *Topological and metric properties of quasiregular mappings*, Ann. Acad. Sci. Fenn. Ser. A I No. 488 (1971), 31 pp.

[24] G.D. Mostow, *Strong rigidity of locally symmetric spaces*, Annals of Mathematics Studies, No. 78. Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1973. v+195 pp.

[25] K. Rajala, *Uniformization of two-dimensional metric surfaces*, Invent. Math., to appear.

[26] Yu.G. Reshetnyak, *Estimates of the modulus of continuity for certain mappings*, Sibirsk. Mat. Z., 7 (1966), 1106-1114; English transl. in Siberian Math. J., 7 (1966), 879-886.

[27] Yu.G. Reshetnyak, *Space mappings with bounded distortion*, Translations of Mathematical Monographs, 73. American Mathematical Society, Providence, RI, 1989.

[28] S. Rickman, *Quasiregular Mappings*, Ergeb. Math. Grenzgeb. (3) 26, Springer, Berlin, 1993.

[29] N. Shanmugalingam, *Newtonian spaces: an extension of Sobolev spaces to metric measure spaces*, Rev. Mat. Iberoamericana 16 (2000), no. 2, 243-279.

[30] J. Sarvas, *The Hausdorff dimension of the branch set of a quasiregular mapping*, Ann. Acad. Sci. Fenn. Ser. A I Math. 1 (1975), no. 2, 297-307.

[31] J. Tyson, *Quasiconformality and quasisymmetry in metric measure spaces*, Ann. Acad. Sci. Fenn. Ser. A I Math., 23 (1998), 525-548.

[32] J. Tyson, *Analytic properties of locally quasisymmetric mappings from Euclidean domains*, Indiana Univ. Math. J., 49 (2000), no. 3, 995-1016.

[33] J. Tyson, *Metric and geometric quasiconformality in Ahlfors regular Loewner spaces*, Conform. Geom. Dyn. 5 (2001), 21-73 (electronic).

[34] M. Williams, *Geometric and analytic quasiconformality in metric measure spaces*, Proc. Amer. Math. Soc. 140 (2012), no. 4, 1251-1266.

[35] M. Williams, *Dilatation, pointwise Lipschitz constants, and condition N on curves*, Michigan Math. J. 63 (2014), no. 4, 687-700.

Chang-Yu Guo
Department of Mathematics
University of Fribourg
Chemin du Musee 23
CH-1700 Fribourg, Switzerland
e-mail: changyu.guo@unifr.ch