We develop a new interpolation scheme, based on harmonic inpainting, for reconstructing the
cosmic microwave background CMB temperature data within the Galaxy mask from the data outside
the mask. We find that, for scale-invariant isotropic random Gaussian fluctuations, the developed
algorithm reduces the errors in the reconstructed map for the odd-parity modes significantly for
azimuthally symmetric masks with constant galactic latitudes. For a more realistic Galaxy mask,
we find a modest improvement in the even parity modes as well.

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I. INTRODUCTION

After the first release of the cosmic microwave background (CMB) anisotropy data from the Wilkinson Mi-
crowave Anisotropy Probe (WMAP) in 2003, various types of “anomalies” in the CMB temperature anisotropy
on large angular scales have been reported[1, 2, 3, 4, 5, 6, 7, 8, 9, 10]. The origin of these anomalies – whether
they are cosmological, statistical fluke, or something else – is unknown[11, 12, 13, 14, 15, 16, 17, 18]

Recent analyses based on either a Bayesian or frequentist approach show that the statistical significance level
of some of the large-angle anomalies is sensitive to the treatment of the Galactic sky cut[19, 20]. The lack of
robustness mainly comes from the fact that we cannot use the full sky information, but can only use a part of
the sky outside the Galaxy mask. There is a way to construct a full sky map that minimizes the foreground
contamination, e.g., the Internal Linear Combination (ILC) technique[21, 22]; however, we cannot rule out a
potential residual foreground contamination in the ILC map, especially in the region that is very close to the
Galactic plane, due to a limited number of sky maps at different frequencies.

An alternative approach is to reconstruct the temperature data within the Galaxy mask from the information
available outside the mask. So far, various types of methods such as the direct inversion, the Wiener filtering,
and the power equalization filtering have been proposed for reconstructing the CMB anisotropy on the cut
sky[23]. The Gibbs sampling[24] provides a way to compute an ensemble of the Wiener filtered maps with a
statistical sample of power spectra fit to a given input sky map, which can be used to estimate the best-fitting
Wiener filtered map and the corresponding uncertainties; see[25] and references therein. All these methods are
based upon a linear transformation of the expansion coefficients of spherical harmonics on the cut sky, \( a_{lm}^{\text{cut}} \),
into the full sky coefficients, \( a_{lm}^{\text{full}} \), which are suitable for nearly all-sky data with a small sky cut. However, it
has been shown that the methods do not work properly for maps with a large sky cut[26, 27].

In order to reconstruct the data within the Galaxy mask, one needs to assume a prior on the properties of
the data. For instance, one may require the temperature anisotropies and their absolute values of the gradient
to have a Gaussian distribution. Then, one can find an optimal solution for reconstructing the data within
the masked region from available information outside the mask. Such an operation is called an “inpainting”, which
has been used by skilled museum or art workers for restoring damaged photographs, films, and paintings. In
recent years, various automatic inpainting algorithms based on partial differential equations or the variational
principle have been proposed and used for automatic image restoration and removal of occlusions[28, 29, 30].

In this paper, we formulate an algorithm based on harmonic inpainting for reconstructing smooth CMB data
within the Galaxy mask. In Sec. II, we develop a numerical scheme for implementing harmonic inpainting on
a unit sphere based on the boundary element method. In Sec. III, we use Monte Carlo simulations to estimate
the errors of the inpainted signal for azimuthally symmetric sky cuts as well as for a realistic non-symmetric
cut. In Sec. IV, we summarize our results.
\[ \lambda > 0 \text{ for } x \notin D, \quad \lambda = 0 \text{ for } x \in D. \]
the elements of the matrices are

\[ H_{ij} = \begin{cases} \tilde{H}_{ij} & i \neq j \\ \tilde{H}_{ij} - \frac{1}{2} & i = j, \end{cases} \]

where

\[ \tilde{H}_{ij} = \int_{\Gamma_j} \frac{\partial G(x_i, y_j)}{\partial n(x_j)} \sqrt{g} \, dy, \quad G_{ij} = \int_{\Gamma_j} G(x_i, y_j) \sqrt{g} \, dy. \quad (2.8) \]

In terms of the normal derivatives \( \{ q \} \) that are derived from Eq. (2.6), the reconstructed function \( u \) within the masked region, \( D \), is given by

\[ u(x) = -\int_{\partial D} G(x, y)q(y)\sqrt{g} \, dy + \int_{\partial D} H(x, y)u(y)\sqrt{g} \, dy, \quad x \in D. \quad (2.9) \]

### III. ERROR ESTIMATION

#### A. Application to azimuthally symmetric masks

In order to check the robustness of the algorithm developed in section II, we carry out Monte-Carlo simulations of random Gaussian fluctuations on a unit sphere, \( u_g \). We generate Monte Carlo realizations from a scale invariant power spectrum \( (n = 1) \),

\[ C_l = \langle |a_{lm}|^2 \rangle \propto \frac{1}{l(l+1)}, \quad (3.1) \]

which describes the large-angle CMB temperature fluctuations in the Einstein-de Sitter (EdS) universe.

We first consider azimuthally symmetric masks with constant galactic latitudes, i.e., we mask region below given galactic latitudes, \( |b| < \theta_0 \). For isotropic random fluctuations, modes with \( l \gtrsim \pi/\theta_0 \) are not affected by the azimuthal cut very much (except for the overall normalization). Therefore, in the following we shall use only multipoles with \( l < l_c \), where \( l_c = \pi/\theta_0 \). From Eq. (2.4), one can see that this choice of cutoff in harmonic space is roughly equivalent to choosing the control parameter, \( \lambda \), such that \( \lambda = (\pi/\theta_0)^2 = l_c^2 \). The number of elements on the boundary necessary for numerically solving the integral equation (2.5) depends on the size of the mask. For instance, we discretize the boundary of the mask into 80 linear elements for \( \theta_0 = 20^\circ \).

To measure the difference between the reconstructed fluctuation, \( u(x) \), and the original one, \( u_g(x) \), we use the \( L^2 \)-norm defined as

\[ (a[u])^2 \equiv \int_{S^2} (u[\theta, \phi])^2 \, d\Omega = \sum_{lm} (a^H_{lm})^2, \quad (3.2) \]

where \( d\Omega \) is the infinitesimal area element for a surface of unit sphere, and \( a^H_{lm} \) is the spherical harmonic coefficients of the reconstructed map. To reduce the sample variance, we generate 1000 realizations, each of which is masked by the azimuthally symmetric mask and is reconstructed by the developed algorithm. We then calculate the mean relative errors defined as

\[ \left( \frac{\Delta a}{a} \right)^2 \equiv \left< \frac{\int_{S^2} (u_g[\theta, \phi] - u[\theta, \phi])^2 \, d\Omega}{\int_{S^2} (u_g[\theta, \phi])^2 \, d\Omega} \right>. \quad (3.3) \]

where \( u_g(x) \) and \( u(x) \) are the underlying Gaussian fluctuation and the inpainted (reconstructed) fluctuation, respectively. For comparison, we also compute the relative errors for the fluctuations with the "naive" ansatz in which \( u = 0 \) within the masked region and \( u = u_0 \) outside the mask.

As one can see in Fig. II some of the missing features within the masked region are reconstructed by the inpainting. Improvements are more conspicuous for odd multipoles.

In Table II we compare the mean relative errors, \( (\Delta a/a)^2 \), at \( l = 1, 2, 3, 4, \) and 5, for two azimuthally symmetric masks, \( |b| < 20^\circ \) and \( 30^\circ \). Compared with the naive cases, our developed algorithm based on harmonic inpainting significantly improves reconstruction of odd multipoles. However, for azimuthally symmetric masks, we find that the inpainted even multipoles are identical with those in the naive case.
FIG. 1: Contour maps of one Gaussian realization (top), those with an azimuthally symmetric mask $|b| < 20^\circ$ (middle), and the inpainted maps (bottom) in Mollweide projections. The left figures show co-added multipoles (from $l = 1$ to $l = 5$) whereas the right figures show co-added odd multipoles ($l = 1, 3, 5$). The boundaries of the mask are shown in white lines.

| $|b| < 20^\circ$ | $l$ | 1 | 2 | 3 | 4 | 5 |
|-----------------|-----|---|---|---|---|---|
| naive           | 0.23| 0.21| 0.55| 0.39| 0.55|
| inpainted       | 0.13| 0.21| 0.31| 0.39| 0.32|

| $|b| < 30^\circ$ | $l$ | 1 | 2 | 3 | 4 | 5 |
|-----------------|-----|---|---|---|---|---|
| naive           | 0.39| 0.37| 0.76| 0.58| 0.62|
| inpainted       | 0.27| 0.37| 0.50| 0.58| 0.48|

TABLE I: Mean relative errors, $(\Delta a/a)^2$, of multipoles for azimuthally symmetric masks with constant galactic latitudes

This can be explained as follows. The Green’s function on a unit sphere is given by $G(x, y) = -Q_0(x \cdot y)/2\pi$, where $2Q_0(x \cdot y) = \ln[(1 + x \cdot y)/(1 - x \cdot y)]$; thus, we have $G(x, y) = G(-x, -y)$. In other words, the Green’s function is invariant under the spatial inversion. Therefore, if the boundary of the masked region is invariant under the spatial inversion, the order of the $2N$-dimensional element matrix $[G]$ is reduced to $N$, which kills the degree of freedom for even-parity modes.

The sky cut biases the estimation of the angular power spectrum when it is computed naively as $C_l(\text{est}) = \sum_m (a_{lm}^{HI})^2 / (2l + 1)$, which is often called the “pseudo $C_l$” [32]. The effect of mask needs to be deconvolved using, e.g., the MASTER method [33]. Here, we do not use the MASTER method, but study how much the bias is reduced when we use the naive $C_l$ estimation and the harmonic inpainting. In Figure 2, we show the ratio of the estimated $C_l$ divided by the input $C_l$, averaged over 1000 Monte Carlo simulations, for three azimuthally symmetric masks, $|b| < 10^\circ$, $20^\circ$, and $30^\circ$. We find modest improvements at odd multipoles, whereas no improvements are found at even multipoles as they are identical to those in the naive case.
FIG. 2: The estimated (pseudo) power spectrum, $C_l(\text{est}) = \sum |a_{lm}|^2/(2l+1)$, for azimuthally symmetric masks with $|b| < \theta_0 = 10^\circ, 20^\circ, \text{ and } 30^\circ$, divided by the input $C_l \propto (l(l+1))^{-1}$. The estimated $C_l(\text{est})$ have been averaged over 1000 Monte Carlo realizations, and plotted for the inpainted (reconstructed) fluctuations (solid lines) and for the naive ones (dashed lines).

B. Application to a realistic Galaxy mask

As we have seen, the developed algorithm based on harmonic inpainting fails to reconstruct even multipoles when the Galaxy mask is azimuthally symmetric. However, the shape of the Galactic foreground emission region is not perfectly azimuthally symmetric, and thus the corresponding Galaxy mask does not respect parity conservation. Therefore, we expect an improvement for even multipoles as well relative to the naive estimations, once realistic Galaxy masks are used.

To confirm this expectation, we have carried out Monte-Carlo simulations with a realistic Galaxy mask. In this study we also use a realistic sky signal as well: we have generated 1000 realizations of temperature maps from a ΛCDM model with cosmological parameters given by $(n, \Omega_m, \Omega_\Lambda) = (1, 0.24, 0.76)$. As for the Galaxy mask, we use the WMAP’s $Kp0$ mask[21].

Since the shape of the $Kp0$ mask is fairly irregular at the scale of pixels, the definition of the border is not so trivial like the azimuthally symmetric mask in Fig. 1. For instance, some observed pixels close to the mask are surrounded by two or more pixels, some others are completely surrounded by masked pixels, making the choice of the border arbitrary and unstable. To avoid this we have built a smoothed version of the $Kp0$ border. Our procedure consists of going (for a fixed $\phi$ sampled with a constant interval) along the coordinate $\theta$ from Galactic north to south until we find a masked pixel belonging to the $Kp0$ mask. Once we have found this pixel, we would go back by a step of 10 arcminutes and choose that pixel living on the border. The final set of $(\theta_i, \phi_i)$ defines our border. We found that this procedure regularizes the $Kp0$ border.

We choose the cutoff angular scale to be $l_c = 15$, which corresponds to the mean angular size of the mask in the polar direction. The boundary of the $Kp0$ mask are discretized into 80 linear elements $(\theta_i, \phi_i)$ with a constant interval $2\pi/40$ (Fig.3).

As in the case of the azimuthally symmetric mask, some of the missing features within the $Kp0$ mask are reconstructed by the inpainting. The improvements are more conspicuous for odd multipoles than even multipoles (Fig.4).

We show the mean relative errors for the naive estimation and those for the inpainted map with the $Kp0$ mask in Table II. We find that our algorithm improves reconstruction of both the odd and even multipoles, but the improvement for the even multipoles is modest (3 to 10%).

In order to find the polar-angle dependence, we compute the mean relative errors for each $(l, m)$ mode, defined by

$$\left(\frac{\Delta a}{a}\right)^2_{(l,m)} = \left\langle \frac{(a_{lm}^{\text{HI}} - a_{lm})^2}{(a_{lm}^{\text{HI}})^2} \right\rangle,$$

(3.4)

where $a_{lm}^{\text{HI}}$ and $a_{lm}$ are real expansion coefficients for the inpainted map and the original map, respectively.

We find that significant improvements come mainly from odd-parity modes with $m = \pm 1$ (Fig. 5). Lower multipoles are improved more by inpainting than higher ones. We also find that the biases in the pseudo $C_l$ are
FIG. 3: Meshes on the boundary of the \( K\sigma \) Galaxy mask, smoothed on scale \( \sim 10 \) arcmin.

FIG. 4: Contour maps of one Gaussian realization (top), those with a \( K\sigma \) mask (middle) and the inpainted maps (bottom) in Mollweide projections with galactic coordinates. The left figures show co-added multipoles (from \( l = 1 \) to \( l = 6 \)) whereas the middle and right figures show co-added odd multipoles (\( l = 1, 3, 5 \)) and co-added even multipoles (\( l = 2, 4, 6 \)), respectively.

reduced for both the odd and even multipoles (Fig. 6).

IV. SUMMARY

We have shown that our new algorithm based on harmonic inpainting offers a way to reconstruct the missing CMB temperature anisotropy data within the Galaxy mask, provided that the primordial fluctuation is Gaussian with a scale-invariant (\( n = 1 \)) spectrum. The method reduces the errors for multipoles with an odd-parity (odd \( l \)) significantly. While the method fails to reconstruct multipoles with an even parity (even \( l \)) for azimuthally symmetric masks, it can reduce the errors for multipoles with an even parity for a more realistic, non-azimuthally symmetric mask, such as the WMAP’s \( K\sigma \) mask. As we have shown, our method is useful for studying the dependence of the CMB at low multipoles given by the azimuthal parameter \( m \) of the harmonic coefficients, which is relevant to confirm or confute some of the anomalies at low multipoles found in the WMAP data.

Our new algorithm does not require any specific form of the power spectrum as a prior for reconstructing the missing data as long as the fluctuations obey a Gaussian distribution. Even when fluctuations are not Gaussian, our algorithm is expected to work as long as the fluctuations are sufficiently smooth and the gradients approximately obey a Gaussian distribution. Therefore, our method may be used to reconstruct other
observational data that is sufficiently smooth such as radio or infrared maps on a cut sky (a similar technique that can incorporate texture as well as the smooth part is studied in [34]).

Our analysis implies that low multipoles with an even parity are more likely to suffer from the effect of sky cut compared with those with an odd parity. In other words, even parity modes on large angular scales cannot be precisely reconstructed without assuming specific forms of the angular power. Therefore, the statistical significance of the anomalous features involving the CMB quadrupole may be much lower than claimed, as recent studies suggest [19, 20].

The feature of the inpainted 5 year WMAP data will be explored in the forthcoming paper (Inoue & Cabella 2008).

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FIG. 6: The estimated (pseudo) power spectrum, $C_l^{(est)} = \sum |a_{lm}|^2 / (2l + 1)$, for the $Kp\theta$ Galaxy mask, averaged over 1000 Monte Carlo realizations. The dashed and dotted lines show $C_l^{(est)}$ from the inpainted map and the map with the $Kp\theta$ Galaxy mask, respectively. The solid and the dash-dotted lines show the input $C_l$ and the full-sky $C_l^{(est)}$ estimated from 1000 realizations, respectively.

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