R-matrix Approach to Quantum Superalgebras $su_q(m \mid n)$

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Abstract

Quantum superalgebras $su_q(m \mid n)$ are studied in the framework of $R$-matrix formalism. Explicit parametrization of $L^+$ and $L^-$ matrices in terms of $su_q(m \mid n)$ generators are presented. We also show that quantum deformation of nonsimple superalgebra $su(n \mid n)$ requires its extension to $u(n \mid n)$.

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1 Introduction

In the course of studying quantum algebras, a lot of attention has been paid to the case of quantum superalgebras (QSA) recently (see, for example, ref. [1], [2]). These algebras provide solutions to the Yang-Baxter equation and therefore may serve as a source of new exactly solvable models in statistical mechanics [3]. It is also very interesting to study their relation to supergroups in the WZW models. Such models were considered in ref. [4] and their connection with superconformal models was established.

An intriguing relation between QSA and knot theory was discovered in ref. [5]. It was shown there, that the QSA $su_q(n \mid n)$ is related to the Alexander-Conway polynomial in much the same way as the quantum algebra $su_q(n)$ is related to the Jones polynomial. It still remains to be seen how special properties of the Alexander-Conway polynomial are related to the nonsimplicity of the superalgebra $su(n \mid n)$.

QSA can also be shown to emerge from the “new” solutions to the Yang-Baxter equation discovered recently in a series of papers in ref. [6]. In those papers, these solutions were not recognized as QSA because of the choice of parametrization of the matrices $L^{(+)}$ and $L^{(-)}$ in the framework of $R$-matrix formalism. With proper redefinition of these parameters, QSA can be easily demonstrated to be associated with these “new” solutions in very much the same way the "old" solutions are related to the usual quantum algebras. Here we develop a convenient parametrization of these matrices for the QSA $su_q(m \mid n)$. In contrast to the previous papers on this subject [6,9], we use ordinary- (instead of “super-”) $R$-matrices, which is in line with the approach of ref. [1]. Our treatment includes the case of nonsimple QSA $su_q(n \mid n)$, the special features of which will be displayed.

In following three sections we discuss the QSA $su_q(2 \mid 0)$, $su_q(0 \mid 2)$ and $su_q(1 \mid 1)$, which are the building blocks of our general construction. In section V we assemble these blocks in the matrices $L^{(+)}$ and $L^{(-)}$ for the QSA $su_q(m \mid n)$. In Appendix A we discuss possible choices of the $R$-matrix for that algebra, and in Appendix B we give a brief description of the QSA $su_q(2 \mid 1)$ as a specific example of our general result.
Quantum Superalgebra $su_q(2 \mid 0)$

Superalgebra $su_q(2 \mid 0)$ is, of course, the same as the algebra $su_q(2)$, which has been described, e.g. in ref.\[7\]. We repeat their analysis to establish notations that will make it easier to use as a building block for $su_q(m \mid n)$.

According to the $R$-matrix method, one introduces the upper- and lower diagonal $2 \times 2$ matrices $L^{(+)}$ and $L^{(-)}$. Their off-diagonal elements are raising and lowering generators of $su_q(2 \mid 0)$ while their diagonal elements are exponents of Cartan subalgebra generators.

The basic commutation relations between the elements of $L^{(+)}$ and $L^{(-)}$ are expressed through the following relations:\[7\]:

$$L^{(\pm)}_2 L^{(\mp)}_1 = R_{21} L^{(\mp)}_1 L^{(\pm)}_2 R_{21}^{-1}, \quad L^{(\mp)}_2 L^{(\pm)}_1 = R_{21} L^{(\pm)}_1 L^{(\mp)}_2 R_{21}^{-1}$$

Here $L^{(\pm)}_1 = L^{(\pm)} \otimes 1$, $L^{(\pm)}_2 = 1 \otimes L^{(\pm)}$ and $R_{21} = PRP$, $P$ is a permutation operator, so that

$$R_{21} = PRP = \begin{pmatrix} q & 1 \ & q - q^{-1} \\ 0 \ & 1 \ & q \end{pmatrix}$$

Eq. (1) implies the following commutation relations between the matrix elements of matrices $L^{(\pm)}$:

$$L^{(\pm)}_{11} L^{(\mp)}_{12} = q^{\pm 1} L^{(\mp)}_{12} L^{(\pm)}_{11}, \quad L^{(\pm)}_{22} L^{(\mp)}_{12} = q^{\pm 1} L^{(\mp)}_{12} L^{(\pm)}_{22} \quad (3)$$

$$L^{(\pm)}_{11} L^{(\mp)}_{21} = q^{\pm 1} L^{(\mp)}_{21} L^{(\pm)}_{11}, \quad L^{(\pm)}_{22} L^{(\mp)}_{21} = q^{\pm 1} L^{(\mp)}_{21} L^{(\pm)}_{22} \quad (4)$$

$$[L^{(\pm)}_{12}, L^{(\mp)}_{21}] = (q - q^{-1}) \left( L^{(\pm)}_{11} L^{(\mp)}_{22} - L^{(\pm)}_{22} L^{(\mp)}_{11} \right) \quad (5)$$

Eqs. (3) and (4) show that if $L^{(\pm)}_{12}$ and $L^{(\mp)}_{21}$ are proportional to the raising and lowering operators $X^+$ and $X^-$, then both $L^{(\pm)}_{11}$ and $L^{(\pm)}_{22}$ should be proportional to $q^{-H}$, while $L^{(\pm)}_{22}$ and $L^{(\pm)}_{11}$ should be proportional to $q^H$, where $H$ is a Cartan subalgebra element. Operators $X^+$, $X^-$ and $H$ satisfy the standard commutation relations of $su_q(2)$:

$$[H, X^+] = X^+, \quad [H, X^-] = -X^- \quad (6)$$
\[ [X^+, X^-] = \frac{q^{2H} - q^{-2H}}{q - q^{-1}} \]  

(7)

Our normalization for diagonal elements of matrices \( L^{(+)} \) and \( L^{(-)} \) differs from that of ref. [7]:

\[
L_{11}^{(\pm)} = q^{\mp \frac{1}{2} q^\mp H}, \quad L_{22}^{(\pm)} = q^{\mp \frac{1}{2} q^\mp H}
\]

(8)

The advantage of such normalization is that these matrices have simple forms in the fundamental representation of \( su_q(2 \mid 0) \),

\[
L_{11}^{(\pm)} = q \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad L_{22}^{(\pm)} = q \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},
\]

(9)

and can be easily generalized to the case of \( su_q(m \mid n) \).

To reconcile eqs. (8) and (7), we have to introduce the factors \((q - q^{-1})\) for \( L_{12}^{(+)\cdot} \) and \( L_{21}^{(-)\cdot} \) as well as an extra negative sign which we ascribe to \( L_{12}^{(+)} \) for reasons that we will explain in Section 5.

Thus we arrive at the following parametrization of the matrices \( L^{(+)} \) and \( L^{(-)} \):

\[
L^{(+)} = \begin{bmatrix}
- \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & (q^{-1} - q)X^+ \\
q \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} & - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}
\end{bmatrix}
\]

(10)

\[
L^{(-)} = \begin{bmatrix}
\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & 0 \\
q \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} & (q - q^{-1})X^- \end{bmatrix}
\]

(11)

where the matrices of the diagonal blocks should be interpreted in the sense of eq. (8) for arbitrary representations.
quantum algebra $su_q(0 \mid 2)$

The algebra $su_q(0 \mid 2)$ is, of course, isomorphic to $su_q(2 \mid 0)$. However the $R$-matrix that we shall use for this QSA is different from that of eq. (2). Actually there are two choices for this matrix in the literature:

$$R_{21} = \begin{pmatrix} -q^{-1} & \pm 1 \quad q - q^{-1} \\ \pm 1 & 0 \quad \pm 1 \\ & \quad & -q^{-1} \end{pmatrix}$$

The upper signs are advocated in ref.[6], while the lower ones - in ref.[2]. We will discuss the relation between these two possibilities in Appendix A. Here we choose the lower signs, because, as it will be clear in Section 5, they simplify parametrization of matrices $L^{(+)}$ and $L^{(-)}$ in terms of QSA generators for the general case.

With the choice of lower signs in eq.(12), commutation relations between matrix elements of $L^{(+)}$ and $L^{(-)}$, which follow from eq.(4), are:

$$L^{(\pm)}_{11} L^{(+)}_{12} = q^{\pm 1} L^{(+)}_{12} L^{(\pm)}_{11}, \quad L^{(\pm)}_{22} L^{(+)}_{12} = q^{\pm 1} L^{(+)}_{12} L^{(\pm)}_{22}$$

(13)

$$L^{(\pm)}_{11} L^{(-)}_{21} = q^{\pm 1} L^{(-)}_{21} L^{(\pm)}_{11}, \quad L^{(\pm)}_{22} L^{(-)}_{21} = q^{\pm 1} L^{(-)}_{21} L^{(\pm)}_{22}$$

(14)

$$[L^{(+)}_{12}, L^{(-)}_{21}] = (q^{-1} - q) \left( L^{(+)}_{11} L^{(-)}_{22} - L^{(+)}_{22} L^{(-)}_{11} \right)$$

(15)

As expected this is nothing but eqs.(2, 4, 5) with $q$ and $q^{-1}$ interchanged. This time eq. (13) and (14) show that if $L^{(+)}_{12}$ and $L^{(-)}_{21}$ are proportional to the raising and lowering operators $Y^+$ and $Y^-$, then both $L^{(+)}_{11}$ and $L^{(-)}_{22}$ should be proportional to $q^J$, while $L^{(-)}_{11}$ and $L^{(+)}_{22}$ should be proportional to $q^{-J}$, where $J$ is a Cartan subalgebra element. As in the case of $su_q(2 \mid 0)$, operators $Y^+, Y^-$ and $J$ satisfy commutation relations of $su_q(2)$:

$$[J, Y^+] = Y^+, \quad [J, Y^-] = -Y^-$$

(16)

$$[Y^+, Y^-] = \frac{q^{2J} - q^{-2J}}{q - q^{-1}}$$

(17)
Convenient normalization for diagonal elements of $L^+$ and $L^-$ is

$$L_{11}^{(\pm)} = q^{\pm \frac{1}{2}} q^{\pm J} = q \pm \frac{1}{2} q \pm \frac{1}{2} J = q \pm J,$$

where the $2 \times 2$ matrices are in the fundamental representation.

We multiply all matrix elements of $L^+$ and $L^-$ by a factor $(-1)^F$ for future convenience. Here $F$ is a fermionic number operator with eigenvalues $\frac{1}{2}$ and $-\frac{1}{2}$ respectively in the fundamental representations of $su_q(2 \mid 0)$ and $su_q(0 \mid 2)$. Obviously, an extra factor of $(-1)^F$ does not affect the commutation relations (13-17), because it commutes with all operators involved. Thus we get the following parametrization of $L^+$ and $L^-$:

$$L^+ = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} (q - q^{-1})(-1)^F Y^+$$

$$L^- = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} (q^{-1} - q)(-1)^F Y^-$$

where the diagonal blocks should be similarly interpreted as functions of the Cartan subalgebra as in eqs. (10, 11). For reasons that will become clear in Section 5, we give an extra negative sign to $L_{21}^{(-)}$. 

5
4 Quantum Algebra $su_q(1 \mid 1)$

We begin, as usual, by presenting the matrix $R_{21}$ for this algebra:

$$R_{21} = \begin{pmatrix} q & 1 & q - q^{-1} \\ 1 & 0 & 1 \\ 0 & 1 & -q^{-1} \end{pmatrix}$$

The corresponding commutation relations for the matrix elements of $L^{(+)}$ and $L^{(-)}$ are

\begin{align*}
(L^{(+)}_{12})^2 &= (L^{(-)}_{21})^2 = 0 \\
L^{(\pm)}_{11} L^{(\pm)}_{12} &= q^{\mp 1} L^{(\pm)}_{12} L^{(\pm)}_{11}, \quad L^{(\pm)}_{22} L^{(\pm)}_{12} = -q^{\mp 1} L^{(\pm)}_{12} L^{(\pm)}_{22} \\
L^{(\pm)}_{11} L^{(-)}_{21} &= q^{\mp 1} L^{(-)}_{21} L^{(\pm)}_{11}, \quad L^{(\pm)}_{22} L^{(-)}_{21} = -q^{\pm 1} L^{(-)}_{21} L^{(\pm)}_{22} \\
[L^{(\pm)}_{12}, L^{(-)}_{21}] &= (q - q^{-1}) \left( L^{(+)}_{11} L^{(-)}_{22} - L^{(+)}_{22} L^{(-)}_{11} \right)
\end{align*}

Algebra $su_q(1 \mid 1)$ includes three generators $Z^{+}, Z^{-}$ and $E$ with (anti-)commutation relations

\begin{align*}
[E, Z^{\pm}] &= [E, Z^{-}] = 0 \\
\{Z^{+}, Z^{-}\} &= \frac{q^{2E} - q^{-2E}}{q - q^{-1}}
\end{align*}

In the fundamental representation $E = \frac{I}{2}$, where $I$ is identity operator.

Since the operator $E$ commutes with all generators of $su_q(1 \mid 1)$, it is clear that this set of operators is not enough to satisfy eqs. (23) and (24). This deficit is a reflection of the degeneracy of the Killing scalar product in the superalgebra $su(1 \mid 1)$:

$$Str E Z^{\pm} = Str E Z^{-} = Str E^2 = 0$$

In order to resolve these difficulties, we introduce another generator which we shall identify with the fermion number operator $F$. It has eigenvalues $\frac{1}{2}$ and $-\frac{1}{2}$ in the fundamental representation of $su_q(1 \mid 1)$. The operator $F$ pairs with $E$ in the Killing scalar product:

$$Str E F = \frac{1}{2}$$
and thus removes the degeneracy. It has the following commutation relations with other generators:

$$[F, Z^+] = Z^+, \quad [F, Z^-] = -Z^-, \quad [F, E] = 0$$  \hspace{1cm} (30)

If we now choose $L^{(+)\pm}_{12}$ and $L^{(-)\pm}_{21}$ to be proportional respectively to $Z^+$ and $Z^-$, then we should set

$$L^{(+)\pm}_{11} = q^{\mp(F+E)} = q \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad L^{(-)\pm}_{22} = (-1)^{F} q^{\mp(F-E)} = (-1)^{F} q \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$  \hspace{1cm} (31)

The factor $(-1)^F$ in the second expression is responsible for negative sign in the second formulas of eqs. (23) and (24). A factor of $(-1)^F$ added to the expression for $L^{(+)\pm}_{12}$ will turn the commutator (27) into the anticommutator (27). Thus a complete parametrization of $L^{(+)\pm}$ and $L^{(-)\pm}$ is

$$L^{(+)\pm} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} (q-q^{-1}) (-1)^{F} Z^+ \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$  \hspace{1cm} (32)

$$L^{(-)\pm} = \begin{pmatrix} (q^{-1} - q) Z^- \end{pmatrix} (-1)^{F} q \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$  \hspace{1cm} (33)

Inclusion of an additional generator $F$ means that we actually produce the quantum deformation of nonspecial superalgebra $u_q(1 \mid 1)$ rather than $su_q(1 \mid 1)$. The same happens to other nonsimple superalgebras $su_q(n \mid n)$ which we discuss in the next section.

In ref. [8], anticommutation relations appear as a result of a graded tensor product, whereas in refs. [3, 2], the anticommutation relations appear through the use of “super-” $R$ matrices. Here, the ordinary $R$ matrices and the ordinary tensor product is used, but the parametrization of the $L^\pm$ matrices is supplemented by extra factors of $(-1)^F$.

7
5 Quantum Superalgebra $su_q(m \mid n)$

The possible $R$-matrices for QSA $su_q(m \mid n)$ are discussed in Appendix A. Our choice of matrix $R_{21}$ is a combination of matrices (2), (12) and (21). This means that if we want to permute two vectors in the fundamental representation, then we simply use one of these matrices depending on whether both vectors are bosonic, fermionic, or one is bosonic and the other is fermionic. The corresponding $R$-matrix in Eq. (2) is

$$R = \sum_i (-1)^{p_i} q^{1 - 2p_i} e_{II} \otimes e_{II} + \sum_{i \neq j} (-1)^{p_ip_j} e_{II} \otimes e_{JJ} + (q - q^{-1}) \sum_{I > J} e_{IJ} \otimes e_{JI}$$

(34)

In our notations $e_{IJ}$ is an $(m+n) \times (m+n)$ matrix with only the $(I, J)$ matrix element being equal to 1, all other matrix elements are zero. We also set $p_I = 0$ for bosons and $p_I = 1$ for fermions. We will use indices $I, J, \ldots$ for all $(m+n)$ components of the fundamental representation, indices $i, j, \ldots$ only for the $m$ bosonic variables and indices $\alpha, \beta, \ldots$ only for the $n$ fermionic ones.

We compose matrices $L^{(+)}$ and $L^{(-)}$ out of the matrices (10), (11), (19), (20), (32) and (33) in the same way as we composed matrix $R_{21}$:

$$L^{(+)}_{ii} = q^{-e_{ii}}, \quad L^{(+)}_{\alpha\alpha} = (-1)^F q^{-e_{\alpha\alpha}},$$

$$L^{(+)}_{i\alpha} = (q - q^{-1})(-1)^F Z_{i\alpha}^+, \quad L^{(+)}_{ij} = (q^{-1} - q)X_{ij}^+, \quad i < j$$

$$L^{(+)}_{\alpha\beta} = (q - q^{-1})(-1)^F Y_{\alpha\beta}^+, \quad \alpha < \beta$$

$$L^{(+)}_{IJ} = 0, \quad I > J$$

(35)

$$L^{(-)}_{ii} = q^{e_{ii}}, \quad L^{(-)}_{\alpha\alpha} = (-1)^F q^{e_{\alpha\alpha}},$$

$$L^{(-)}_{i\alpha} = (q^{-1} - q)Z_{i\alpha}^-, \quad L^{(-)}_{ij} = (q^{-1} - q)X_{ij}^-, \quad i > j$$

$$L^{(-)}_{\alpha\beta} = (q^{-1} - q)(-1)^F Y_{\alpha\beta}^-, \quad \alpha > \beta$$

$$L^{(-)}_{IJ} = 0, \quad I > J$$

(36)

The signs in front of the raising and the lowering operators $X^\pm, Y^\pm$ and $Z^\pm$ are chosen in such a way that when $q \to 1$, these operators tend to their classical counterparts. This
means that in the fundamental representation

\[(X, Y, Z)_{IJ}^{\pm} \rightarrow (X_d, Y_d, Z_d)_{IJ}^{\pm} = e_{IJ}\]

(37)

The generators of the classical superalgebra \(su(m | n)\) satisfy the (super-)commutation relations

\[
[(X_d, Y_d, Z_d)_{IJ}^{\pm}, (X_d, Y_d, Z_d)_{JK}^{\pm}] = (X_d, Y_d, Z_d)_{IK}^{\pm}
\]

(38)

The corresponding relations for the matrix elements of \(L^{(+)}\) and \(L^{(-)}\) are

\[
L_{IJ}^{(+)} L_{JK}^{(+)} - (-1)^{P_j(P_l+P_k)} L_{JK}^{(-)} L_{IJ}^{(+)} = (q^{-1} - q)(-1)^{P_lP_j} L_{IK}^{(+)} L_{IJ}^{(+)}, (I < J < K)
\]

(39)

\[
L_{IJ}^{(-)} L_{JK}^{(-)} - (-1)^{P_j(P_l+P_k)} L_{JK}^{(+)} L_{IJ}^{(-)} = (q - q^{-1})(-1)^{P_kP_j} L_{IK}^{(+)} L_{IJ}^{(-)}, (I > J > K)
\]

(40)

\[
L_{KI}^{(-)} L_{IJ}^{(+)} - (-1)^{P_l(P_j+P_k)} L_{IJ}^{(-)} L_{KI}^{(+)} = (q^{-1} - q)(-1)^{P_lP_k} L_{II}^{(+)} L_{IJ}^{(-)}, (I < J < K)
\]

(41)

\[
L_{KI}^{(-)} L_{IJ}^{(-)} - (-1)^{P_l(P_j+P_k)} L_{IJ}^{(+)} L_{KI}^{(-)} = (q - q^{-1})(-1)^{P_lP_k} L_{II}^{(+)} L_{IJ}^{(-)}, (I < J < K)
\]

(42)

\[
L_{KJ}^{(-)} L_{IK}^{(+)} - (-1)^{P_k(P_l+P_j)} L_{IK}^{(+)} L_{KJ}^{(-)} = \\
\left\{
\begin{array}{l}
(q^{-1} - q)(-1)^{P_lP_k} L_{IJ}^{(+)} L_{KK}^{(-)} \quad (I < J < K) \\
(q - q^{-1})(-1)^{P_lP_k} L_{KK}^{(+)} L_{IJ}^{(-)} \quad (J < I < K)
\end{array}
\right.
\]

(43)

\[
L_{JI}^{(-)} L_{KJ}^{(+)} = (-1)^{P_kP_l} (-1)^{P_j} q^{-1} L_{KJ}^{(+)} L_{JI}^{(-)}, (I < J < K)
\]

(44)

\[
L_{KJ}^{(-)} L_{IJ}^{(+)} = (-1)^{P_lP_k} (-1)^{P_j} q^{-1} L_{IJ}^{(+)} L_{KJ}^{(-)}, (I < J < K)
\]

(45)

Matching the classical limit of the eqs. (39-45) with the classical commutator (38) dictates the choice between factors \((q - q^{-1})\) and \((q^{-1} - q)\) for the operators \(X^{\pm}\) and \(Y^{\pm}\). The signs of the operators \(Z^{\pm}\) are not prescribed by these requirements and can be chosen arbitrarily.

The exponents of \(q\) appearing in diagonal elements \(L_{II}^{(+)}\) and \(L_{JJ}^{(-)}\) are not supertraceless. Therefore, strictly speaking, they are not the elements of \(su_q(m | n)\) Cartan subalgebra. To overcome this difficulty we multiply these matrix elements by factors of \(q^{n-m}\) and \(q^{m-n}\)
respectively. Such factors will render the exponents supertraceless without affecting the
commutation relations (3-5), (13-15) and (23-25).

This ends the process of parametrization of matrix elements of $L^{(+)}$ and $L^{(-)}$ in terms of $su_q(m \mid n)$ generators if $m \neq n$. However supertracelessness cannot be achieved if $m = n$. In this case the condition of supertracelessness of the original classical algebra $su(n \mid n)$ should be dropped, so that we deal in fact with algebra $u(n \mid n)$. Its Cartan subalgebra includes one generator with nonvanishing supertrace which can be identified with the fermion number operator $F$. If $m \neq n$, $F$ can be considered to be just an element of $su_q(m \mid n)$ Cartan subalgebra.

6 Conclusion

We considered the construction of quantum superalgebra $su_q(m \mid n)$ in the framework of $R$-matrix formalism. In contrast to the papers [2] and [3], we used ordinary (not super-) commutation relations between matrix elements of $L^{(+)}$ and $L^{(-)}$ while parametrizing them in terms of the generators of QSA $su_q(m \mid n)$. Thus it can easily be shown that the “special” solutions of Yang-Baxter equation, discussed in ref.[6], are related to QSA in the same way as ordinary solutions are related to quantum algebras through the Reshetikhin construction [9]. Therefore we conjecture that QSA can be used to generate all possible solutions to the Yang-Baxter equation.

Our study of nonsimple superalgebras $su(n \mid n)$ also revealed that their quantum deformation requires extending them to superalgebras $u(n \mid n)$, whose Cartan subalgebras include the fermionic number operator.

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Appendix A – Choice of $R$-matrix for QSA $su_q(m \mid n)$

Here we discuss possible choices of $R$-matrix for QSA $su_q(m \mid n)$. To simplify discussion, we will consider $\tilde{R}$-matrix, which is the product of permutation operator $P$ and original $R$-matrix:

$$\tilde{R} = PR$$  \hspace{1cm} (A.1)

The $\tilde{R}$-matrix, presented in ref.[2], is

$$\tilde{R} = \sum_{I \neq J} (-1)^{p_I p_J} e_JI \otimes e_{IJ} + \sum_{I} (-1)^{p_I} q^{1-2p_I} e_{II} \otimes e_{II} + (q - q^{-1}) \sum_{I \prec J} e_{II} \otimes e_{JJ}$$  \hspace{1cm} (A.2)

However, the $\tilde{R}$-matrix, in ref.[6] for “nonstandard” solutions of the Yang-Baxter equation, has a slightly different form:

$$\tilde{R}' = \sum_{I \neq J} e_{JI} \otimes e_{IJ} + \sum_{I} (-1)^{p_I} q^{1-2p_I} e_{II} \otimes e_{II} + (q - q^{-1}) \sum_{I \prec J} e_{II} \otimes e_{JJ}$$  \hspace{1cm} (A.3)

The factor of $(-1)^{p_I p_J}$ in the first sum of formula (A.2) seems to be quite reasonable: an extra negative sign arises through permutation of two fermionic vectors. This is why the $R_{21}$-matrix in eq.(12) with lower signs, which stems from the $\tilde{R}$-matrix in eq.(A.2), provides simple parametrization of $L^+$ and $L^-$ (the upper signs in eq.(12) would correspond to the $\tilde{R}$-matrix (A.3)).

To show the equivalence of the $\tilde{R}$-matrices in eqs.(A.2) and (A.3), let us consider the action of $\tilde{R}$-matrix on the tensor product $\otimes_{k=1}^N V_k$ of $N$ fundamental representations $V_k$ (N is any integer). Let us denote by $e_{I_k}$ ($1 \leq I_k \leq m + n$) the basis vectors of $V_k$.

Consider now operator $D$ which calculates the parity of number of “fermionic disorders” in basis vectors of tensor product:

$$D \otimes_{k=1}^N e_{I_k} = \otimes_{k=1}^N (-1)^{d(I_k)} e_{I_k}$$  \hspace{1cm} (A.4)
Here \( d_{\{l_k\}} \) is the number of fermionic disorders, i.e. \( d_{\{l_k\}} \) counts the number of pairs of indices \( k, l \), such that

\[
k < l, \quad m, I_l, I_k
\]  

(A.5)

Obviously, \( D \) commutes with operators \( e_{IJ} \otimes e_{IJ} \) for all possible values of \( I \) and \( J \), because these operators do not permute different vectors. \( D \) commutes also with operators \( e_{IJ} \otimes e_{IJ} \), because they permute pairs of vectors at least one of which is bosonic. However

\[
De_{\alpha\beta} \otimes e_{\beta\alpha}D^{-1} = -e_{\alpha\beta} \otimes e_{\beta\alpha}, \quad \alpha \neq \beta
\]

(A.6)

because operator \( e_{\alpha\beta} \otimes e_{\beta\alpha} \) permutes two different fermionic vectors, thus changing the parity of fermionic disorder number. Therefore we see that

\[
D \sum_{I \neq J} (-1)^{pIpJ}e_{IJ} \otimes e_{IJ}D^{-1} = \sum_{I \neq J} e_{IJ} \otimes e_{IJ}
\]

(A.7)

and

\[
D\tilde{R}D^{-1} = \tilde{R}'
\]

(A.8)

Eq.(A.8) shows equivalence of two \( \tilde{R} \)-matrices \( (A.2) \) and \( (A.3) \).

Appendix B – Quantum Superalgebra \( su_q(2 \mid 1) \)

As an example we give a brief description of the \( su_q(2 \mid 1) \) quantum superalgebra. The matrices \( L^\pm \) are parametrized as in eqs. \( (B.1) \), \( (B.2) \),

\[
L^+ = q^{-1} \begin{pmatrix}
q^{h_2} & (q^{-1}-q)\alpha^+ & (q-q^{-1})(-1)^Fb^+ \\
0 & q^{h_1+h_2} & (q-q^{-1})(-1)^F\beta^+ \\
0 & 0 & (-1)^Fq^{h_1+2h_2}
\end{pmatrix}
\]

(B.1)

\[
L^- = q \begin{pmatrix}
q^{-h_2} & 0 & 0 \\
(q-q^{-1})\alpha^- & q^{-h_1-h_2} & 0 \\
(q^{-1}-q)b^- & (q^{-1}-q)\beta^- & (-1)^Fq^{-h_1-2h_2}
\end{pmatrix}
\]

(B.2)


Eqs. (1) and (34) imply

\[
\begin{align*}
    b^+\alpha^+ - q\alpha^+ b^+ &= 0 \\
    [\beta^-,\alpha^-] &= b^- q^{-(h_1 + h_2)} \\
    \beta^- b^- + q^{-1} b^- \beta^- &= 0 \\
    \alpha^+ \beta^- - q\beta^- \alpha^+ &= 0 \\
    b^+ \beta^- + \beta^- b^+ &= \alpha^+ q^{-h_1 - 2h_2} \\
    \beta^+ b^- + b^- \beta^+ &= \alpha^- q^{h_1 + 2h_2}.
\end{align*}
\]

These relations reduce to the classical relations of su(2 | 1) in the classical limit. A different parametrization can be found in Ref. [2]

**Addendum**

After this paper was submitted, H. Saleur brought to our attention the paper ref. [8] which addresses the problem of su_q(1 | 1). The authors of ref. [8] use a graded tensor product in order to produce anticommutators in some of the relations of (1). We use an ordinary tensor product, which is in line with the approach taken in ref. [6]. However, we supplement their parametrization of L± matrices with an extra factor of \((-1)^F\) to achieve the same effect.

**References**

[1] P. Kulish, N. Reshetikhin, *Lett.Math.Phys.* **18** (1989) 143.

[2] M. Chaichian, P. Kulish, *Phys.Lett.* **234B** (1990) 72.

[3] S. Saleur, *Nucl.Phys.* **B336** (1990) 363.

[4] M. Bershadsky, H. Ooguri, *Phys.Lett.* **229B** (1989) 374.

[5] L. Kauffman, H. Saleur, *Free Fermions and the Alexander-Conway Polynomial*, preprint EFI 90-42.
[6] N. Jing, M.-L. Ge, Y.-S. Wu, *Lett.Math.Phys.* **21** (1991) 193.
Y. Cheng, M.-L. Ge, K. Xue, *New Solutions of Yang-Baxter Equation*, preprint ITP-SB-90-38 (1990);

[7] L. Faddeev, N. Reshetikhin, L. Takhtajan, *Quantization of Lie Groups and Lie Algebras*, preprint LOMI E-14-87 (1987)

[8] Li-Liao, Xing-Chang Song, *Mod.Phys.Lett.* **A6** (1991) 959.

[9] N. Reshetikhin, *Quantized Universal Enveloping Algebras, the Yang-Baxter Equation and Invariants of Links*, parts I and II, preprint LOMI E-4-87, E-17-87.