White Noise Representation of Gaussian Random Fields

Zachary Gelbaum

Department of Mathematics
Oregon State University
Corvallis, Oregon 97331-4605, USA

Abstract

We obtain a representation theorem for Banach space valued Gaussian random variables as integrals against a white noise. As a corollary we obtain necessary and sufficient conditions for the existence of a white noise representation for a Gaussian random field indexed by a compact measure space. As an application we show how existing theory for integration with respect to Gaussian processes indexed by [0, 1] can be extended to Gaussian fields indexed by compact measure spaces.

Keywords: white noise representation, Gaussian random field, stochastic integral

1. Introduction

Much of literature regarding the representation of Gaussian processes as integrals against white noise has focused on processes indexed by \( \mathbb{R} \), in particular canonical representations (most recently see [8] and references therein) and Volterra processes (e.g. [1, 3]). An example of the use of such integral representations is the construction of a stochastic calculus for Gaussian processes admitting a white noise representation with a Volterra kernel (e.g. [1, 11]).

In this paper we study white noise representations for Gaussian random variables in Banach spaces, focusing in particular on Gaussian random fields indexed by a compact measure space. We show that the existence of a representation as an integral against a white noise on a Hilbert space \( H \) is equivalent to the existence of a version of the field whose sample paths lie almost surely in \( H \). For example as a consequence of our results a centered Gaussian process \( Y_t \) indexed by [0, 1] admits a representation

\[
Y_t = \int_0^1 h(t,z)dW(z)
\]

for some \( h \in L^2([0, 1] \times [0, 1], d\nu \times d\nu), \) \( \nu \) a measure on [0, 1] and \( W \) the white noise on \( L^2([0, 1], d\nu) \) if and only if there is a version of \( Y_t \) whose sample paths belong almost surely to \( L^2([0, 1], d\nu) \).

The stochastic integral for Volterra processes developed in [11] depends on the existence of a white noise integral representation for the integrator. If there exists an integral representation for a given Gaussian field then the method in [11] can be extended to define a stochastic integral with respect to this field. We describe this extension for Gaussian random fields indexed by a compact measure space whose sample paths are almost surely square integrable.

Email address: gelbaumz@math.oregonstate.edu (Zachary Gelbaum)
Section 2 contains preliminaries we will need from Malliavin Calculus and the theory of Gaussian measures over Banach spaces. In section 3, Theorem 1 gives our abstract representation theorem and Corollary 2 specializes to Gaussian random fields indexed by a compact measure space. Section 4 contains the extension of results in [11].

2. Preliminaries

2.1. Malliavin Calculus

We collect here only those parts of the theory that we will explicitly use, see [15].

**Definition 1.** Suppose we have a Hilbert space $H$. Then there exists a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a map $W: H \to L^2(\Omega, \mathbb{P})$ satisfying the following:

1. $W(h)$ is a centered Gaussian random variable with $E[W(h)^2] = \|h\|_H$
2. $E[W(h_1)W(h_2)] = \langle h_1, h_2 \rangle_H$

This process is unique up to distribution and is called the *Isornormal* or *White Noise Process* on $H$.

The classical example is $H = L^2[0, 1]$ and $W(h)$ is the Wiener-Ito integral of $h \in L^2$.

Let $\mathcal{S}$ denote the set of random variables of the form

$$F = f(W(h_1), \ldots, W(h_n))$$

for some $f \in C^\infty(\mathbb{R}^n)$ such that $f$ and all its derivatives have at most polynomial growth at infinity. For $F \in \mathcal{S}$ we define the derivative as

$$DF = \sum_{j=1}^n \partial_j f(W(h_1), \ldots, W(h_n))h_j.$$ 

We denote by $\mathcal{D}$ the closure of $\mathcal{S}$ with respect to the norm induced by the inner product

$$\langle F, G \rangle_D = E[FG] + E[\langle DF, DG \rangle_H].$$

($\mathcal{D}$ is usually denoted $\mathcal{D}^{1, 2}$.)

We also define a directional derivative for $h \in H$ as

$$D_h F = \langle DF, h \rangle_H.$$ 

$D$ is then a closed operator from $L^2(\Omega)$ to $L^2(\Omega, H)$ and $\text{dom}(D) = \mathcal{D}$. Further, $\mathcal{D}$ is dense in $L^2(\Omega)$. Thus we can speak of the adjoint of $D$ as an operator from $L^2(\Omega, H)$ to $L^2(\Omega)$. This operator is called the divergence operator and denoted by $\delta$.

$\text{dom}(\delta)$ is the set of all $u \in L^2(\Omega, H)$ such that there exists a constant $c$ (depending on $u$) with

$$|E[\langle DF, u \rangle_H]| \leq c\|F\|$$

for all $F \in \mathcal{D}$. For $u \in \text{dom}(\delta)$ $\delta(u)$ is characterized by

$$E[F\delta(u)] = E[\langle DF, u \rangle_H]$$

for all $F \in \mathcal{D}$.

For examples and descriptions of the domain of $\delta$ see [15], section 1.3.1.

When we want to specify the isonormal process defining the divergence we write $\delta^W$. We will also use the following notations interchangeably

$$\delta^W(u), \int udW.$$
2.2. Gaussian Measures on Banach Spaces

Here we collect the necessary facts regarding Gaussian measures on Banach spaces and related notions that we will use in what follows. For proofs and further details see e.g. [4, 5, 6, 13, 16]. All Banach spaces are assumed real and separable throughout.

**Definition 2.** Let $B$ be a Banach space. A probability measure $\mu$ on the borel sigma field $B$ of $B$ is called Gaussian if for every $l \in B^*$ the random variable $l(x) : (B, B, \mu) \to \mathbb{R}$ is Gaussian. The mean of $\mu$ is defined as

$$m(\mu) = \int_B x d\mu(x).$$

$\mu$ is called centered if $m(\mu) = 0$. The (topological) support of $\mu$ in $B$, denoted $B_0$, is defined as the smallest closed subspace of $B$ with $\mu$-measure equal to 1.

The mean of a Guassian measure is always an element of $B$, and thus it suffices to consider only centered Gaussian measures as we can then acquire any Gaussian measure via a simple translation of a centered one. For the remainder of the paper all measures considered are centered.

**Definition 3.** The covariance of a Gaussian measure is the bilinear form $C_\mu : B^* \times B^* \to \mathbb{R}$ given by

$$C_\mu(k, l) = \mathbb{E}[k(X)l(X)] = \int_B k(x)l(x)d\mu(x).$$

Any gaussian measure is completely determined by its covariance: if for two Gaussian measures $\mu, \nu$ on $B$ we have $C_\mu = C_\nu$ on $B^* \times B^*$ then $\mu = \nu$.

If $H$ is a Hilbert space then $C_\mu(f, g) = \mathbb{E}[\langle X, f \rangle \langle X, g \rangle] = \int_B \langle x, f \rangle \langle x, g \rangle d\mu(x)$ defines a continuous, positive, symmetric bilinear form on $H \times H$ and thus determines a positive symmetric operator $K_\mu$ on $H$. $K_\mu$ is of trace class and is injective if and only if $\mu(H) = 1$. Conversely, any positive trace class operator on $H$ uniquely determines a Guassian measure on $H$ [6]. Whenever we consider a Gaussian measure $\mu$ over a Hilbert space $H$ we can after restriction to a closed subspace assume $\mu(H) = 1$ and do so throughout.

We will denote by $H_\mu$ the Reproducing Kernel Hilbert Space (RKHS) associated to a Gaussian measure $\mu$ on $B$. There are various equivalent constructions of the RKHS. We follow [16] and refer the interested reader there for complete details.

For any fixed $l \in B^*$, $C_\mu(l, \cdot) \in B$ (this is a non trivial result in the theory). Consider the linear span of these functions,

$$S = \text{span}\{C_\mu(l, \cdot) : l \in B^*\}.$$

Define an inner product on $S$ as follows: if $\phi(\cdot) = \sum_i a_i C_\mu(l_i, \cdot)$ and $\psi(\cdot) = \sum_j b_j C_\mu(k_j, \cdot)$ then

$$< \phi, \psi >_{H_\mu} = \sum_{i=1}^n \sum_{j=1}^m a_i b_j C_\mu(l_i, k_j).$$

$H_\mu$ is defined to be the closure of $S$ under the associated norm $\| \cdot \|_{H_\mu}$. This norm is stronger than $\| \cdot \|_B$. $H_\mu$ is a dense subset of $B_0$ and $H_\mu$ has the reproducing property with reproducing kernel $C_\mu(l, k)$:

$$\langle \phi(\cdot), C_\mu(l, \cdot) \rangle_{H_\mu} = \phi(l) \quad \forall l \in B^*, \phi \in H_\mu.$$
Remark 1. Often one begins with a collection of random variables indexed by some set, \(\{Y_t\}_{t \in T}\). For example suppose \((T, \nu)\) is a finite measure space. Then setting \(K(s, t) = \mathbb{E}[Y_t Y_s]\) and supposing that application of Fubini-Tonelli is justified we have for \(f, g \in L^2(T)\)

\[
\mathbb{E}[(Y, f)|(Y, g)] = \int_T \int_T \mathbb{E}[Y_s, Y_t] f(s)g(t)dv d\nu = \langle K(s, t)(f), g \rangle
\]

where we denote \(\int_T K(s, t)f(s)dv(s)\) by \(K(s, t)(f)\). If one verifies that this last operator is positive symmetric and trace class then the above collection \(\{Y_t\}_{t \in T}\) determines a measure \(\mu\) on \(L^2(T)\) and the above construction goes through with \(C_{\mu}(f, g) = \langle K(s, t)(f), g \rangle\) and the end result is the same with \(H_\mu\) a space of functions over \(T\).

Define \(H_X\) to be the closed linear span of \(\{X(l)\}_{l \in B^*}\) in \(L^2(\Omega, \mathbb{P})\) with inner product 

\[
\langle X(l), X(l') \rangle_{H_X} = C_{\mu}(l, l')
\]

(again for simplicity assume \(X\) is nondegenerate). From the reproducing property we can define a mapping \(R_X\) from \(H_\mu\) to \(H_X\) given initially on \(S\) by

\[
R_X(\sum_{k=1}^n c_k C_{\mu}(k, \cdot)) = \sum_{k=1}^n c_k X(l)\]

and extending to an isometry. This isometry defines the isonormal process on \(H_\mu\).

In the case that \(H\) is a Hilbert space and \(\mu\) a Gaussian measure on \(H\) with covariance operator \(K\) it is known that \(H_\mu = \sqrt{K}(H)\) with inner product \(\langle \sqrt{K}(x), \sqrt{K}(y) \rangle_{H_\mu} = \langle x, y \rangle_H\).

It was shown in [12] that given a Banach space \(B\) there exists a Hilbert space \(H\) such that \(B\) is continuously embedded as a dense subset of \(H\). Any Gaussian measure \(\mu\) on \(B\) uniquely extends to a Gaussian measure \(\mu_H\) on \(H\). The converse question of whether a given Gaussian measure on \(H\) restricts to a Gaussian measure on \(B\) is far more delicate. There are some known conditions e.g. [7]. The particular case when \(X\) is a metric space and \(B = C(X)\) has been the subject of extensive research [14]. Let us note here however that either \(\mu(B) = 0\) or \(\mu(B) = 1\) (an extension of the classical zero-one law, see [4]).

From now on we will not distinguish between a measure \(\mu\) on \(B\) and its unique extension to \(H\) when it is clear which space we are considering.

3. White Noise Representation

3.1. The General Case

The setting is the following: \(B\) is a Banach space densely embedded in some Hilbert space \(H\) (possibly with \(B = H\)), where \(H\) is identified with its dual, \(H = H^*\). (A Hilbert space equal to its dual in this way is called a Pivot Space, see [2]).

The classical definition of canonical representation has no immediate analogue for fields not indexed by \(\mathbb{R}\), but the notion of strong representation does. Let \(L : H_\mu \to H\) be unitary. Then \(W_X(h) = R_X(L^*(h))\) defines an isonormal process on \(H\) and \(\sigma(\{W_X(h)\}_{h \in H}) = \sigma(H_X) = \sigma(\{X(l)\}_{l \in B^*})\) where the last inequality follows from the density of \(H\) in \(B^*\).

We now state our representation theorem.

Theorem 1. Let \(B\) be a Banach space, \(\mu\) a Gaussian measure on \(B\), and \(C_{\mu}\) the covariance of \(\mu\) on \(B^* \times B^*\). Then \(\mu\) is the distribution of a random variable in \(B\) given as a white noise integral of the form

\[
X(l) = \int h(l) dW.
\]
for some \( h : B^* \to H \) and a Hilbert space \( H \), where \( h|_H \) is a Hilbert-Schmidt operator on \( H \). Moreover, the representation is strong in the following sense: \( \sigma(\{W_X(h)\}_{h \in H}) = \sigma(\{X(l)\}_{l \in B^*}) \).

**Proof.** \( B \subset H = H^* \) as above. Let \( W_X \) be the isonormal process constructed above and \( C_\mu(l, k) \) the covariance of \( \mu \). Let \( L \) be a unitary map from \( H_\mu \) to \( H \) and define the function \( k_L(l) : B^* \to H \) by

\[
k_L(l) \equiv L(C_\mu(l, \cdot)).
\]

Consider the Gaussian random variable determined by

\[
Y(l) \equiv \int k_L(l)dW_X.
\]

We have

\[
\text{Cov}(Y(l_1), Y(l_2)) = \langle k_L(l_1), k_L(l_2) \rangle_H = \langle C_\mu(l_1, \cdot), C_\mu(l_2, \cdot) \rangle_{H_\mu} = C_\mu(l_1, l_2)
\]

so that \( \mu \) is the distribution of \( Y(l) \) and

\[
X(l) \equiv \int k_L(l)dW_X.
\]

It is clear that \( k_L \) is linear and if \( C_\mu(h_1, h_2) = \langle K(h_1), h_2 \rangle_H, \ h_1, h_2 \in H \), then from above

\[
k_L^*k_L = K.
\]

Because \( K \) is trace class this implies that \( k_L \) is Hilbert-Schmidt on \( H \).

From the preceding discussion we have \( \sigma(\{W_X(h)\}_{h \in H}) = \sigma(\{X(l)\}_{l \in B^*}) \). \( \square \)

**Remark 2.** While the statement of the above theorem is more general than is needed for most applications, this generality serves to emphasize that having a “factorable” covariance and thus an integral representation are basic properties of all Banach space valued Gaussian random variables.

**Remark 3.** The kernel \( h(l) \) is unique up to unitary equivalence on \( H \), that is if \( L' = U L \) for some unitary \( U \) on \( H \) \( L \) as above, then

\[
\int h_{L'}(l)dW \overset{d}{=} \int U(h_L(l))dW \overset{d}{=} \int h_L(l)dW.
\]

**Remark 4.** In the proof above,

\[
\langle k_L(l_1), k_L(l_2) \rangle_H = C_\mu(l_1, l_2)
\]

is essentially the “canonical factorization” of the covariance operator given in [17], although in a slightly different form.

**Remark 5.** In the language of stochastic partial differential equations, what we have shown is that every Gaussian random variable in a Hilbert space \( H \) is the solution to the operator equation

\[
L(X) = W
\]

for some closed unbounded operator \( L \) on \( H \) with inverse given by a Hilbert-Schmidt operator on \( H \).
3.2. Gaussian Random Fields

The proof of Theorem 1 has the following corollary for Gaussian random fields:

**Corollary 2.** Let \( X \) be a compact Hausdorff space, \( \nu \) a positive Radon measure and \( H = L^2(X, d\nu) \). If \( \{B_x\} \) is a collection of centered Gaussian random variables indexed by \( X \), then \( \{B_x\} \) has a version with sample paths belonging almost surely \( H \) if and only if

\[
B_x \overset{d}{=} \int h(x, \cdot) dW
\]

for some \( h : X \to H \) such that the operator \( K(f) = \int_X h(x, z) f(z) d\nu(z) \) is Hilbert-Schmidt. In this case (3.2) takes the form

\[
E[B_x B_y] = \int_X h(x, z) h(y, z) d\nu(z).
\]

In other words, the field \( B_x \) determines a Gaussian measure on \( L^2(X, d\nu) \) if and only if \( B_x \) admits an integral representation (3.3).

3.3. Some Consequences and Examples

In principle, all properties of a field are determined by its integral kernel. Without making an exhaustive justification of this statement we give some examples:

In Corollary 2 above, being the kernel of a Hilbert-Schmidt operator, \( h \in L^2(X \times X, d\nu \times d\nu) \). This means that we can approximate \( h \) by smooth kernels (supposing these are available). If we assume \( h(x, \cdot) \) is continuous as a map from \( X \) to \( H \) i.e.

\[
\lim_{x \to y} \|h(x, \cdot) - h(y, \cdot)\|_H = 0
\]

for each \( y \in X \) and let \( h_n \in C^\infty(X) \), \( h_n \overset{L^2}{\to} h \) it follows that \( \|h_n(x, \cdot) - h(x, \cdot)\|_H \to 0 \) pointwise so that if

\[
B^n_x = \int h_n(x, \cdot) dW
\]

we have

\[
E[B^n_x B^n_y] \to E[B_x B_y]
\]

point-wise. This last condition is equivalent to

\[
B^n \overset{d}{\to} B
\]

and we can approximate in distribution any field over \( X \) with a continuous (as above) kernel by fields with smooth kernels.

The kernel of a field over \( \mathbb{R}^d \) describes its local structure \( \square \): The limit in distribution of

\[
\lim_{r_n \to 0} \frac{X(t + c_n x) - X(t)}{r_n}
\]

is

\[
\lim_{r_n \to 0} \int \frac{h(t + c_n x) - h(t)}{r_n} dW
\]
where \( h \) is the integral kernel of \( X \), and this last limit is determined by the limit in \( H \) of

\[
\lim_{r_n \to 0} \lim_{c_n \to 0} \frac{h(t + c_n x) - h(t)}{r_n}.
\]

The representation theorem yields a simple proof of the known series expansion using the RKHS. The setting is the same as in Theorem 1.

**Proposition 3.** Let \( Y(l) \) be a centered Gaussian random variable in a Hilbert space \( H \) with integral kernel \( h(l) \). Let \( \{e_k\}_1^\infty \) be a basis for \( H \). Then there exist i.i.d. standard normal random variables \( \{\xi_k\} \) such that

\[
Y(l) = \sum_{k=1}^{\infty} \xi_k \Phi_k(l)
\]

where \( \Phi_k(l) = \langle h(l), e_k \rangle_H \) and the series converges in \( L^2(\Omega) \) and a.s.

**Proof.** For each \( l \)

\[
h(l) = \sum_{k=1}^{\infty} \Phi_k(l)e_k.
\]

We have

\[
Y(l) = \int \sum_{k=1}^{\infty} \Phi_k(l)e_k dW = \sum_{k=1}^{\infty} \Phi_k(l)\xi_k
\]

where \( \{\xi_k\} = \left\{ \int e_k dW \right\} \) are i.i.d. standard normal as \( \int dW \) is unitary from \( H \) to \( L^2(\Omega) \). As \( \{\Phi_k(l)\} \in l^2(\mathbb{N}) \) the series converges a.s. by the martingale convergence theorem. \( \square \)

4. **Stochastic Integration**

Combined with Theorem 1 above, [11] furnishes a theory of stochastic integration for Gaussian processes and fields, which we now describe for the case of a random field with square integrable sample paths as in Corollary 2.

Denote by \( \mu \) the distribution of \( \{B_x\} \) in \( H = L^2(X, d\nu) \) and as above the RKHS of \( B_x \) by \( H_\mu \subset H \). Let

\[
B_x = \int h(x, \cdot) dW
\]

and \( L^*(f) = \int h(x, y)f(y) d\nu(y) \). Then \( L^* : H \to H_\mu \) is an isometry and the map \( v \to R_B(L^*(v)) \equiv W(v) : H \to H_B \) (\( H_B \) is the closed linear span of \( \{B_x\} \) as defined in sec. 2) defines an isonormal process on \( H \). Denote this particular process by \( W \) in what follows.

First note that as \( H_\mu = L^*(H) \) and \( L \) is unitary, it follows immediately that \( \mathbb{D}_{H_\mu}^{1,2} = L^*(\mathbb{D}_H^{1,2}) \) where we use the notation in [13, 11] and the subscript indicates the underlying Hilbert space.
The following proof from [11] carries over directly: For a smooth variable $F(h) = f(B(L^*(h_1), ..., B(L^*(h_n)))$ we have

$$E(D^B(F), u)_{H_\mu} = \mathbb{E}\left(\sum_{k=1}^{n} f'(B(L^*(h_1), ..., B(L^*(h_n)))L^*(h_k), u)_{H_\mu}\right)$$

$$= \mathbb{E}\left(\sum_{k=1}^{n} f'(B(L^*(h_1), ..., B(L^*(h_n)))h_k, L(u))_{H}\right)$$

$$= \mathbb{E}\left(\sum_{k=1}^{n} f'(W(h_1), ..., W(h_n)))h_k, L(u))_{H}\right)$$

$$= \mathbb{E}(D^W(F), L(u))_{H}$$

which establishes

$$\text{dom}(\delta^B) = L^*(\text{dom}(\delta^W))$$

and

$$\int udB = \int L(u)dW.$$  

The series approximation in [11] also extends directly to this setting.

**Theorem 4.** If \{\Phi_k\} is a basis of $H_\mu$ then there exists i.i.d. standard normal \{\xi_k\} such that:

1. If $f \in H_\mu$ and

$$\int f dB = \sum_{k=1}^{\infty} \langle f, \Phi_k \rangle_{H_\mu} \xi_k \text{ a.s.}$$

2. If $u \in D_{H_\mu}$ then

$$\int udB = \sum_{k=1}^{\infty} ((u, \Phi_k)_{H_\mu} - (D^B_\Phi u, \Phi_k)_{H_\mu}) \text{ a.s.}$$

**Proof.** The proof of (1) and (2) follows that in [11]. \qed

**Remark 6.** For our purposes the method of approximation via series expansions above seems most appropriate. However in [1] a Riemann sum approximation is given under certain regularity hypotheses on the integral kernel of the process, and this could be extended in various situations as well.

**Remark 7.** The availability of the kernel above suggests the method in [1] whereby conditions are imposed on the kernel in order to prove an Ito Formula as promising for extension to more general settings.

5. References

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