Conformal decomposition in canonical general relativity

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Abstract

A new canonical transformation is found that enables the direct canonical treatment of the conformal factor in general relativity. The resulting formulation significantly simplifies the previously presented conformal geometrodynamics and provides a further theoretical basis for the conformal approach to loop quantum gravity.

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I. BACKGROUND

The well-known Arnowitt-Deser-Misner (ADM) canonical variables for general relativity (GR) consist of the spatial metric $g_{ab}$ with the conjugate momentum $p^{ab}$. For bookkeeping purposes we will referred to these variables as the “G-variables”. The dynamical evolution is generated by the action of the form

$$\int dt \int d^3x \left[ p^{ab} \dot{g}_{ab} - NcH + N^a H_a \right]$$

(1)

where the over dot denotes a $t$-derivative, $N$ is the lapse function and $N^a$ the shift vector. This action contains the the momentum (diffeomorphism) constraint:

$$H_a = -2 \nabla_b p^b_a$$

(2)

and the Hamiltonian constraint:

$$\mathcal{H} = \mu^{-1} g_{abcd} p^{ab} p^{cd} - \mu R$$

(3)

were $\mu := \sqrt{\det g_{ab}}$, $R$ is the Ricci scalar of $g_{ab}$ and $g_{abcd} := (g_{ac}g_{bd} + g_{ad}g_{bc} - g_{ab}g_{cd})/2$. The Poisson bracket (PB) with respect to the G-variables $(g_{ab}, p^{ab})$ is denoted by $\{ \cdot, \cdot \}^G$.

In a recent paper [1] the ADM phase space has been extended to that consisting of York’s mean curvature “time” variable $\tau := (4/3)K$, where $K$ is the mean extrinsic curvature, with $\mu$ as momentum and conformal metric $\gamma_{ab}$ with momentum $\pi^{ab}$. Based on York’s conformal decomposition of tensors, a canonical transformation has been found to relate the G-variables to the “Γ-variables” $(\gamma_{ab}, \pi^{ab}; \tau, \mu)$ via

$$g_{ab} = \phi^4 \gamma_{ab}$$

(4)

$$p^{ab} = \phi^{-4} \pi^{ab} - \frac{1}{2} \phi^2 \mu_\gamma \gamma^{ab} \tau$$

(5)

where $\mu_\gamma := \sqrt{\det \gamma_{ab}}$ and

$$\phi := \left( \frac{\mu}{\mu_\gamma} \right)^{1/6}$$

(6)

is the conformal factor. Note that here $\phi$ is regarded as a function of the canonical variables $\mu$ and $\gamma_{ab}$. It can be seen from the above relations that a local rescaling of $\gamma_{ab}$ and $\pi^{ab}$ while
holding $\tau$ and $\mu$ leaves $g_{ab}$, $p^{ab}$ invariant. This redundancy of the $\Gamma$-variables is offset by the “conformal constraint”:

$$C = \gamma_{ab} \pi^{ab}$$  \hfill (7)

that generates conformal transformations through its PB with respect to the $\Gamma$-variables denoted by $\{\cdot, \cdot\}^\Gamma$. The tensor $\gamma_{ab}$ will play the role of the conformal metric, whereas the original $g_{ab}$ will stay as the physical metric.

The ADM diffeomorphism and Hamiltonian constraints then take the following forms:

$$\mathcal{H}_a = -2 \nabla_\gamma \pi^b_a + \mu \tau_a + 4(\ln \phi)_a C$$ \hfill (8)

$$\mathcal{H} = \frac{1}{\mu} \pi_{ab} \pi^{ab} - \frac{3}{8} \tau^2 \mu - \phi^2 \mu \gamma R + 8 \mu \phi \Delta \gamma \phi + \frac{\tau}{2} C - \frac{1}{2\mu} C^2.$$ \hfill (9)

The indices of $\pi^{ab}$ and $\pi_{ab}$ are raised or lowered by the conformal metric $\gamma_{ab}$ and its inverse $\gamma^{ab}$ respectively. Here we have used the Levi-Civita connection $\nabla_\gamma$, scalar curvature $R_\gamma$ and Laplacian $\Delta_\gamma := \gamma^{ab} \nabla_\gamma a \nabla_\gamma b$, associated with the conformal metric $\gamma_{ab}$.

It is useful to introduce the diffeomorphism constraint $C_a$ for the $\Gamma$-variables by

$$C_a := \mathcal{H}_a - 4(\ln \phi)_a C = -2 \nabla_\gamma \pi^b_a + \mu \tau_a.$$ \hfill (10)

Using the preservation of the PB relations by the canonical transformation from the $G$- to $\Gamma$-variables, the constraints $\mathcal{H}, C_a$ and $C$ can be explicitly shown to be first class [1, 2].

By using the relation

$$p^{ab} g_{ab} = -\tau \dot{\mu} + \pi^{ab} \dot{\gamma}_{ab} + 4(\ln \phi) \cdot C$$ \hfill (11)

the canonical action for GR in the $\Gamma$-variables can be written as

$$\int dt \int d^3x \left[ \pi^{ab} \dot{\gamma}_{ab} + \mu \dot{\tau} - N \mathcal{H} - N^a C_a - \Lambda C \right]$$ \hfill (12)

where $\Lambda$ is a Lagrange multiplier used to effect weakly vanishing of $C$. For further background information on the conformal decomposition in GR, see [1, 2, 3] and references therein.

II. CONFORMAL FACTOR AS A CANONICAL VARIABLE FOR GRAVITY

The structure of $\mathcal{H}$ in (9) is quite complicated, with $\phi$ being a highly nonlinear expression given in (6). If a canonical transformation can be found that retains $\gamma_{ab}$ as canonical variables
while turning $\phi$ into a canonical variable, then the conformal decomposition for canonical GR will be greatly simplified. Below we show that this can indeed be done by explicit construction of new a set of variables satisfying these criteria. To this end, consider the quantities $\tilde{\pi}^{ab}$ and $\bar{\pi}$ given by:

$$\tilde{\pi}^{ab} := \pi^{ab} - \frac{1}{2} \phi^6 \mu_{\gamma} \tau \gamma^{ab} \quad (13)$$

$$\bar{\pi} := -6 \phi^5 \mu_{\gamma} \tau = -8 \phi^5 \mu K = 4 \phi^{-1} g_{ab} p^{ab}. \quad (14)$$

From (14) we see that

$$\tau = -\frac{1}{6} \phi^{-5} \mu_{\gamma}^{-1} \bar{\pi}. \quad (15)$$

Using (13) and (15) we have

$$\pi^{ab} = \tilde{\pi}^{ab} - \frac{1}{12} \phi \bar{\pi} \gamma^{ab}. \quad (16)$$

In terms of these quantities, Eqs. (4) and (5) become simply

$$g_{ab} = \phi^4 \gamma_{ab} \quad (17)$$

$$p^{ab} = \phi^{-4} \bar{\pi}^{ab}. \quad (18)$$

It follows that

$$-\tau \mu = \bar{\pi} \dot{\phi} - \phi^6 \tau \mu_{\gamma} \quad (19)$$

and

$$\pi^{ab} \gamma_{ab} = \tilde{\pi}^{ab} \gamma_{ab} + \phi^6 \tau \mu_{\gamma}. \quad (20)$$

Therefore the relation

$$\pi^{ab} \gamma_{ab} - \tau \mu = \tilde{\pi}^{ab} \gamma_{ab} + \bar{\pi} \dot{\phi} \quad (21)$$

holds and it shows that the variables ($\gamma_{ab}, \tilde{\pi}^{ab}, \phi, \bar{\pi}$) are canonical. We shall refer to them as the “$\Phi$-variables” and denote the PB with respect to them by $\{\cdot, \cdot\}^\Phi$. Using (17), (18) we see the preservation of the PBs $\{A, B\}^G = \{A, B\}^\Phi$ for any $A$ and $B$ depending on the $\Phi$-variables through the $G$-variables. In particular, the Dirac algebra for $\mathcal{H}$ and $\mathcal{H}_a$ is preserved.

From (16) we see that the conformal constraint $\mathcal{C}$ given in (7) becomes

$$\mathcal{C} = \gamma_{ab} \tilde{\pi}^{ab} - \frac{1}{4} \phi \bar{\pi}. \quad (22)$$
The constraint $C$ generates the conformal transformation in the $\Phi$-variables via the following PB relations:

$$\{\gamma_{ab}(x), C(x')\}^\Phi = \gamma_{ab}(x) \delta(x,x') \tag{23}$$

$$\{\tilde{\pi}^{ab}(x), C(x')\}^\Phi = -\tilde{\pi}^{ab}(x) \delta(x,x') \tag{24}$$

$$\{\phi(x), C(x')\}^\Phi = -\frac{1}{4} \phi(x) \delta(x,x') \tag{25}$$

$$\{\tilde{\pi}(x), C(x')\}^\Phi = \frac{1}{4} \tilde{\pi}(x) \delta(x,x'). \tag{26}$$

From (17) and (18) we see that the physical 3-metric and its momentum are independent of $\tilde{\pi}$ and are conformally invariant:

$$\{g_{ab}(x), C(x')\}^\Phi = 0, \quad \{g_{ab}(x), \phi(x')\}^\Phi = 0 \tag{27}$$

$$\{p^{ab}(x), C(x')\}^\Phi = 0, \quad \{p^{ab}(x), \phi(x')\}^\Phi = 0 \tag{28}$$

which, along with the preservation of PBs, imply

$$\{\mathcal{H}(x), C(x')\}^\Phi = 0, \quad \{\mathcal{H}(x), \phi(x')\}^\Phi = 0 \tag{29}$$

$$\{\mathcal{H}_a(x), C(x')\}^\Phi = 0, \quad \{\mathcal{H}_a(x), \phi(x')\}^\Phi = 0. \tag{30}$$

It follows that $\mathcal{H}, \mathcal{H}_a, C$ form a set of first class constraints. To construct the explicit expressions of $\mathcal{H}$ and $\mathcal{H}_a$ in the $\Phi$-variables, substitute (17), (18) and (22) into (2) to get

$$\mathcal{H}_a = -2 \nabla_{\gamma b} \tilde{\pi}^b_a + \tilde{\pi} \phi_a + 4(\ln \phi)_a C \tag{31}$$

and substitute (17), (18) into (3) to get

$$\mathcal{H} = \phi^{-6} \mu_\gamma^{-1} \gamma_{abcd} \tilde{\pi}^{ab} \tilde{\gamma}^{cd} - \phi^2 \mu_\gamma R_\gamma + 8 \mu_\gamma \phi \Delta_\gamma \phi \tag{32}$$

where $\gamma_{abcd} := (\gamma_{ac} \gamma_{bd} + \gamma_{ad} \gamma_{bc} - \gamma_{ab} \gamma_{cd})/2$. As with the $\Gamma$-variables, it is useful to introduce the diffeomorphism constraint in the $\Phi$-variables as

$$C_a = \mathcal{H}_a - 4(\ln \phi)_a C = -2 \nabla_{\gamma b} \tilde{\pi}^b_a + \tilde{\pi} \phi_a \tag{33}$$

and adopt

$$\int dt \int d^3x \left[ \tilde{\pi}^{ab} \gamma_{ab} + \tilde{\pi} \phi - N\mathcal{H} - N^a C_a - \Lambda C \right] \tag{34}$$

as the canonical action for GR in the $\Phi$-variables.
It follows from the Dirac algebra for the original ADM constraints that the complete set of independent constraints \( \{ C, C_a, H \} \) is indeed first class, satisfying the following algebra:

\[
\{ C(x), C(x') \}^\Phi = 0 \\
\{ C_a(x), C_b(x') \}^\Phi = C_b(x) \delta_a(x, x') - (ax \leftrightarrow bx') \\
\{ C_a(x), C(x') \}^\Phi = C(x) \delta_a(x, x') \\
\{ H(x), H(x') \}^\Phi = \phi^{-4} \gamma^{ab}(C_a + 4(\ln \phi)_a C)(x) \delta_b(x, x') - (x \leftrightarrow x') \\
\{ C(x), H(x') \}^\Phi = 0 \\
\{ C_a(x), H(x') \}^\Phi = H(x) \delta_a(x, x').
\]

The above new formulation can be applied to simplify the spin-gauge treatment in \cite{1, 2} and to develop the conformal approach to loop quantum gravity.

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