An efficient solution procedure for constraint-free elastic structure subject to self-equilibrium static loading

Xiaohu Zhang¹, Qin Sun¹ and Ke Liang¹
¹ School of Aeronautics, Northwestern Polytechnical University, Xi’an, Shaanxi, 710072, China
*zhangxiaohu0406@163.com

Abstract. The stiffness matrix would be singular in finite element static analysis of free structure, so the analysis would fail. To solve this problem, we can obtain the total displacement of free structure under some well-posed constraint, and then use the projector matrix constructed by rigid body modes (RBM) to separate the elastic displacement from the total displacement. Another point, the traditional RBM matrix is only linear, which isn’t satisfy the large rotational condition, so a new construction of RBM matrix based on rigid body motion is proposed. It can be found that the nonlinear projector matrix can be reduced to linear projector matrix in some cases. The numerical results show the feasibility of the proposed method and the differences in linear and nonlinear analysis.

1. Introduction
The analysis of actual engineering structures often involves the lack of the basic constraint or the completely unconstrained condition (called free structure). The free structures include all kinds of aircraft, road vehicles and so on. The displacement \( u \) of free structure is composed of two parts: the elastic displacement \( u_e \) and the rigid body displacement \( u_r \). The elastic displacement indicates the load carrying capacity of a structure, which is the main design parameter in a static analysis. A lot of research work[1-5] has been done on solution technologies for the elastic deformation of free structure.

It is known to all that the finite element analysis based on the displacement method must has a complete and suitable basic support as a precondition, otherwise we have to eliminate the singularity of the structural stiffness matrix using the projector matrix which is constructed by rigid body displacements[6]. For free structure, the solution technique used in the MSC.NASTRAN software is inertial releasing[7]. The ZAERO software uses the so-called modal approach in the static aeroelastic/trim analysis[8]. In the Finite Element Tearing and Interconnecting (FETI) method [9], the rigid body modes (RBM) matrix is used to solve the floating subdomain, but this method destroys the sparsity of floating subdomain stiffness matrix and can only be applied to linear static analysis.

In this paper, for the calculation of elastic displacement of free structure subjected to a self-equilibrium loading, a construction method of projector matrix, which can satisfy the large rotational condition, is proposed based on the principle of minimum stability equilibrium. This new projector matrix can guarantee the geometric shape of free structure during rigid body movement. The effectiveness of the method is verified by the numerical examples of free beam.

2. Elastic displacement projector matrix
The finite element stiffness matrix \( K \) of free structure is a real symmetric matrix with singularity. For \( n \)-order stiffness matrix \( K \), its \( N \) eigenvectors can be divided into two parts: the elastic modal space \( \phi_{ei} \)
(i=1,2,...,n-Nr) and the rigid body modal space $\phi_j (j=1,2,...,Nr)$, where Nr is the number of rigid body degrees of freedom of free structure. Thus, the elastic displacement and the rigid body displacement can be expressed as the linear superposition of the elastic modal and the rigid body modal respectively.

$$u_e = \sum_{j=1}^{Nr} \alpha_j \phi_{ej}, \quad u_r = \sum_{j=1}^{Nr} \beta_j \phi_{rj}$$

Since the eigenvectors of a real symmetric matrix are orthogonal to each other[10], the total, elastic and rigid body displacement of free structure have the following relationship,

$$\|u\|^2 = \|u_e\|^2 + \|u_r\|^2 + 2 \sum_{j=1}^{Nr} \sum_{i=1}^{N_r-n} (\alpha_j \beta_i \phi_{ej} \phi_{rj}) = \|u_e\|^2 + \|u_r\|^2 \geq \|u\|^2$$

It is known from the principle of virtual work that the real elastic displacement is the minimum energy solution of the stable and equilibrium structure. Therefore, the elastic displacement $u_e$ of free structure is unique, while the total displacement $u$ varies with the rigid body motions $u_r$. Now we start with the scalar function $S$ associated with a rigid body motion state,

$$S(\alpha) = (u-u_0)^2 - (u-R\alpha)^2$$

Where, $\alpha$ is the rigid body motion of the optional reference point. In three dimensions, $\alpha$ is $(u_c, v_c, w_c, \theta_x, \theta_y, \theta_z)^T$ which describe translations along and rotations about the three coordinate axis. $R$ is the RBM matrix of free structure and also a function of $\alpha$. $R\alpha$ indicates the rigid body motions of all structural nodes. As an unknown quantity, when $\alpha$ minimizes the value of $S$, $u-R\alpha$ is the elastic displacement of free structure. So, let $\partial S/\partial \alpha=0$ and mark $\partial (R\alpha)/\partial \alpha$ as $R_\alpha$, one can obtain $\alpha$,

$$\alpha = (R_\alpha^T R)^{-1} R_\alpha^T u$$

Then $u_r$ and $u_e$ can be written separately as,

$$u_r = R\alpha = R (R_\alpha^T R)^{-1} R_\alpha^T u$$

$$u_e = u - u_e = Pu$$

$$P = I - R (R_\alpha^T R)^{-1} R_\alpha^T$$

where $P$ has the property $P=P^2$ and acts as a filter to extract the elastic displacement when it operates on $u$.

3. Rigid body motion matrix

The rigid body motions of each node of free structure is linear superposition of the rigid body translation $(u_c, v_c, w_c)^T$ with the reference point and the rigid body rotation $\Psi(\theta_x, \theta_y, \theta_z) \times r$ around a certain axis passing through the reference point, in figure 1. $(u_c, v_c, w_c, \theta_x, \theta_y, \theta_z)^T$ is the rigid body motion of the reference point and $r$ is the position vector that from the reference point to the node of free structure.
The rigid body motions of free structure can be written as,

\[ \mathbf{u}_r = \mathbf{R} \boldsymbol{\alpha} = \begin{bmatrix} I_3 & \cdots & I_3 \\ I_3^T & \cdots & I_3^T \end{bmatrix}^T \begin{bmatrix} \mathbf{u}_c \\ \mathbf{w}_c \\ \boldsymbol{\theta}_x \\ \boldsymbol{\theta}_y \\ \boldsymbol{\theta}_z \end{bmatrix} \tag{7} \]

where \( I_3 \) is a 3×3 identity matrix and \( \mathbf{X}_r \) takes the form,

\[
\mathbf{X}_r = \begin{bmatrix}
0 & \frac{\left(x-x_0\right)\left(y-y_0\right)+\left(z-z_0\right)\sin\left(\theta_z\right)}{\theta_z} & \frac{\left(x-x_0\right)\left(z-z_0\right)+\left(y-y_0\right)\cos\left(\theta_z\right)-\left(y-y_0\right)\sin\left(\theta_z\right)}{\theta_z} \\
\frac{\left(y-y_0\right)\cos\left(\theta_z\right)-\left(y-y_0\right)\sin\left(\theta_z\right)}{\theta_y} & 0 & \frac{\left(x-x_0\right)\sin\left(\theta_z\right)+\left(y-y_0\right)\cos\left(\theta_z\right)-\left(y-y_0\right)\sin\left(\theta_z\right)}{\theta_z} \\
\frac{\left(z-z_0\right)\cos\left(\theta_z\right)-\left(z-z_0\right)\sin\left(\theta_z\right)}{\theta_z} & \frac{\left(z-z_0\right)\cos\left(\theta_z\right)-\left(z-z_0\right)\sin\left(\theta_z\right)}{\theta_z} & 0
\end{bmatrix}
\]

For the linear analysis of small rotations, \((\theta_x, \theta_y, \theta_z)^T\) tends to infinity, \( \mathbf{R} \) can degenerate into a constant matrix which is consistent with the RBM matrix[9] constructed by the principle of force balance, and \( \mathbf{R}_p \) is equal to \( \mathbf{R} \). For the nonlinear analysis of finite rotation, \( \mathbf{R} \) varies nonlinearly with the rotational parameter. The nonlinear least squares principle can be used to solve the rotational parameter iteratively by equation (8),

\[ a_{\text{x,y,l}} - a_{\text{z}} = \left(\mathbf{R}_p^{-1} \mathbf{R}_n^{-1} \right) \left( \mathbf{u} - \mathbf{R} \mathbf{a}_z \right) \tag{8} \]

In summary, for free structure subjected to self-equilibrium static loading, we can firstly get the total displacement by constraining a reference set of DOF that makes the free structure in the simplest statically determinate state. And secondly, the real elastic displacement can be separated from the total displacement by the projective operator \( \mathbf{P} \). However, it is often difficult to select the reference set of DOF for complex structures.

4. Numerical examples

The computational procedure of the elastic displacement \( \mathbf{u}_e \) of free structure is given above. Next, the feasibility of this method is verified by numerical examples in two aspects, one is about the linear strain solution \( \mathbf{u} \) of free structure and the other is about the nonlinear strain solution.

4.1. Linear Strain Solution of Free-Structure

Take two-dimensional free-space beam as an example, the geometrical length, height and thickness are \( L=100\text{mm}, h=10\text{mm} \) and \( t=1\text{mm} \) respectively, the material elasticity modulus is \( E=70\text{GPa} \), the
poisson's ratio is $\mu=0.33$ and the quadrilateral element is applied. The external self-equilibrium loads are shown in figure. 2.

Figure 2. Schematic diagram of self-equilibrium load for two-dimensional free-space beam.

Two different statically determinate constraints are applied to the finite element model: one is to fix the midpoint (node 6), the other is to constrain the $u$ and $v$ direction of node 1 and $u$ direction of node 12. Then the total displacement solutions $u$ under the two different boundary conditions are obtained respectively by finite element calculation. The numerical results of displacement separation are shown in figure 3 and figure 4 respectively.

Figure 3. Displacement separation of the linear displacement solution $u$ under the first constraint.

Figure 4. Displacement separation of the linear displacement solution $u$ under the second constrain.
The numerical results show that, in the linear strain analysis, the total displacement solutions $u$ under different statically determinate constraints are different, but the elastic deformation displacement $u_e$, extracted from different displacement solution $u$ by degenerate linear projector matrix $P$, are unique. The elastic displacement is completely consistent with the numerical results obtained by the generalized flexibility matrix technique in reference [4]. In addition, the geometrical area of each element is enlarged by 0.595% due to its rigid body rotation in the second constraint state.

4.2. Nonlinear Strain Solution of Free-Structure

Different from linear strain analysis, the rigid body displacement is a nonlinear function of the rotational parameter $\theta$ when the structure has large deformation, such as large deflection, large rotation, etc. Take the three-dimensional beam in free space as an example, its geometrical length $L=100\text{mm}$, the section area is $9\text{mm}^2$, the material elasticity modulus $E=70\text{GPa}$, the Poisson’s ratio $\mu=0.33$ and the beam element is adopted. The external self-equilibrium load is applied as shown in figure 5.

![Figure 5. Schematic diagram of self-equilibrium load of free beam in space.](image)

Also, two different statically determinate constraints are applied to the finite element model: the first one is to fix the midpoint (node 6), the second one is to constrain the end point (node 1). Then the total displacement solutions $u$ under different boundary conditions are obtained using the geometrical nonlinear finite element method with large rotational small deformation. The numerical results of displacement separation, using the nonlinear $R$, $R_p$ and $P$, are shown in figure 6 and figure 7.

![Figure 6. Displacement separation of the nonlinear displacement solution $u$ while the midpoint is fixed.](image)

![Figure 7. Displacement separation of the nonlinear displacement solution $u$ while the end point is fixed.](image)
The numerical results show that, in nonlinear strain analysis, the total displacement solutions \( u \) under different statically determinate constraint are different. However, compared with linear strain analysis, the elastic displacement \( u_e \), extracted by the nonlinear projector matrix \( P \), are no longer unique. Obviously, in nonlinear analysis with large rotation and small deformations, the structure configuration varies with loading and the structure is always balanced at new configuration. Due to the change of structure configuration, the original equilibrium force system in undeformed configuration will turn to be non-equilibrium. The supporting force will be produced at the constraint set while structure is in non-equilibrium force system and will be different in different well-posed constraints. On the other hand, the rigid body displacement extracted from the nonlinear strain solutions is guaranteed, that is, the geometric shape of free structure will not change during the rigid body movement.

It is worth mentioning that the rigid body displacement of free structure reflects its spatial motion, while the elastic displacement of free structure reflects the deformation relative to the free structure itself. These two kinds of displacements are independent of each other. In linear analysis, the elastic displacement calculated by \( u-u_r \) is a measurement in the global coordinate system. However, in nonlinear analysis, the elastic displacement calculated by \( u-u_r \) is the relative deformation measured in a local coordinate system moving with free structure.

5. Conclusion
The above analysis shows the feasibility of the proposed method. Now, the main work is summarized as follows,
1. This paper derives the projective operator matrix which can be used for linear and nonlinear analysis (including the degeneration of small rotation angle problems).
2. The solution of free structure in linear strain analysis is proposed. It’s more simple and practical compared with the method in reference [5]. The proposed method does not change the software algorithm, so long as to make the free structure in well-posed constraint and do some post treatment work on displacement solution \( u \).
3. For the large-rotational nonlinear strain analysis, this method is not universal, but for the relatively simple problems that the pattern of structural distortion can be determined accurately, the correct results can still be obtained by imposing constraint on some appropriate nodes and doing some post treatment work.

References
[1] Felippa, C.A., Park, K.C. (1997) A direct flexibility method. Computer Methods in Applied Mechanics & Engineering, 149:319 - 337.
[2] Hesse, H., Palacios, R. (2012) Consistent structural linearisation in flexible-body dynamics with large rigid-body motion. Computers & Structures, 110-111:1-14.
[3] Hesse, H., Murua, J., Palacios, R. (2012) Consistent Structural Linearization in Flexible Aircraft Dynamics with Large Rigid-Body Motion. Aiaa Structures, Structural Dynamics and Materials Conference, 52:528-538.
[4] Liu, W.Y., Chen, X., Li, Y.L., Yang, P.Y. (2012) Quasi-static Finite Elements Analysis for Unconstrained Structures. Fire Control & Command Control, 37:138-140,147.
[5] Silverberg, Larry, M., Park, Sungtae. (1990) Interactions between rigid-body and flexible-body motions in maneuvering spacecraft. Journal of Guidance Control & Dynamics, 13: 73-81.
[6] Felippa, C.A., Park, K.C., Filho, M.R.J. (1998) The construction of free-free flexibility matrices as generalized stiffness inverses. Computers and Structures, 68:411-418.
[7] Caffrey, J.P., Lee, J.M. (1994) MSC/NASTRAN Linear Static Analysis: User's Guide, Version 68.
[8] ZAERO Theoretical Manual, Software Package, Ver. 8.2. ZONA Technology Inc., 2008.
[9] Li, B., Yang, Z.C. (2007) A localized FETI method for structural parallel analysis. Structure & Environment Engineering. 34:1-7

[10] Zhang, X.D. (2013) Matrix Analysis And Applications. Tsinghua university press, Beijing.