The topology of Birkhoff varieties

Luke Gutzwiller and Stephen A. Mitchell

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1 Introduction

Let $\mathcal{F} = \tilde{G}/P_I$ be an affine flag variety. Here $G$ is a simply-connected complex algebraic group with simple Lie algebra, $\tilde{G} = G(\mathbb{C}[z, z^{-1}])$ is the corresponding affine group, and $P_I$ is the parabolic subgroup associated to a subset $I$ of the set of Coxeter generators $\tilde{S}$ of the affine Weyl group $\tilde{W}$. Then $\mathcal{F}$ has two dual stratifications: the Schubert or Bruhat cell decomposition

$$\mathcal{F} = \bigsqcup_{\lambda \in \tilde{W}/\tilde{W}_I} e_{\lambda},$$

and the Birkhoff stratification

$$\mathcal{F} = \bigsqcup_{\lambda \in \tilde{W}/\tilde{W}_I} S_{\lambda}.$$

The Schubert cells $e_{\lambda}$ are the orbits of the Iwahori subgroup $\tilde{B}$, while the Birkhoff strata $S_{\lambda}$ are the orbits of the opposite Iwahori subgroup $\tilde{B}^-$. The cells and the strata are dual in the sense that $S_{\lambda} \cap e_{\lambda} = \{\lambda\}$, and the intersection is transverse. The closure of $e_{\lambda}$ is the affine Schubert variety $X_{\lambda}$. It has dimension $\ell^I(\lambda)$, where $\ell^I$ is the minimal length occurring in the coset $\lambda \tilde{W}_I$, and its cells are indexed by the lower order ideal generated by $\lambda$ in the Bruhat order on $\tilde{W}/\tilde{W}_I$. Dually, the closure of $S_{\lambda}$ is the Birkhoff variety $Z_{\lambda}$. It is an infinite-dimensional irreducible ind-variety with codimension $\ell^I(\lambda)$. Its Birkhoff strata are indexed by the upper order ideal generated by $\lambda$.

Thus the Birkhoff varieties may be viewed as analogous to the dual Schubert varieties from the classical setting, in which the role of $\mathcal{F}$ is played by a finite-dimensional flag variety. More generally, let $\mathcal{I}$ denote an upper order ideal in the Bruhat poset $\tilde{W}/\tilde{W}_I$. Then $Z_{\mathcal{I}} = \bigcup_{\lambda \in \mathcal{I}} S_{\lambda}$ is a finite union of Birkhoff varieties. Our main theorem shows that in one respect, the classical and affine cases differ dramatically.

**Theorem 1.1** Let $Z_{\mathcal{I}}$ be a finite union of Birkhoff varieties in the affine flag variety $\mathcal{F}$. Then $Z_{\mathcal{I}}$ is a deformation retract of $\mathcal{F}$. In particular, the inclusion $Z_{\mathcal{I}} \subset \mathcal{F}$ induces isomorphisms on ordinary and equivariant cohomology, with any coefficients.
The proof has two main ingredients. The first is the existence of a sort of “algebraic tubular neighborhood” of $Z_I$. Let $E_I = \bigcup_{\lambda \in I} e_{\lambda}$. Then $E_I$ is a Zariski open neighborhood of $Z_I$. Similarly, let $J$ be a proper lower order ideal in $\tilde{W}/\tilde{W}_I$, let $X_J = \bigcup_{\lambda \in J} e_{\lambda}$, and let $S_J = \bigcup_{\lambda \in J} S_{\lambda}$. Then $X_J$ is a finite union of Schubert varieties, and $S_{\lambda}$ is a Zariski open neighborhood of $X_J$. Then the following theorem holds for both affine and classical flag varieties.

**Theorem 1.2**

a) $Z_I$ is a deformation retract of $E_I$.

b) $X_J$ is a deformation retract of $S_J$.

Versions of part (b) appear to be known (see for example the special case discussed in [2]), but we are not aware of a proof or even a full statement of this theorem in the literature.

The second ingredient depends on the infinite-dimensionality of the Birkhoff strata, and has no analog in the classical case.

**Lemma 1.3** The punctured Birkhoff stratum $S_{\lambda} - \{\lambda\}$ is contractible.

Given these two ingredients, the main theorem follows by a formal downward induction over the Birkhoff filtration, using Whitehead’s theorem at the inductive step.

**Organization of the paper:** In §2 we summarize some basic notation, and introduce a well-known $\mathbb{C}^\times$-action or complex flow on $\mathcal{F}$ that will be used to construct our deformations. In §3 we study complex flows on ind-spaces and ind-varieties. The main result is a general criterion for deforming an ind-space into an invariant ind-subspace using a flow (Theorem 3.4). In §4 we study the structure of the affine analog $\tilde{U}$ of a maximal unipotent subgroup, and its opposite $\tilde{U}^\circ$. The main application is to show that punctured Birkhoff strata are contractible (Lemma 4.6). In §5 we construct our algebraic tubular neighborhoods (Theorem 5.1).

In §6 we prove the main theorem. We also compute the homology of the pairs $(E_I, E_I - Z_I)$ and $(S_J, S_J - X_J)$. These pairs can be viewed as algebraic normal Thom spaces of the ind-subvarieties $Z_I$, $X_J$ in $\mathcal{F}$. Finally, we make some remarks on torus-equivariant cohomology $H^*_T Z_I$. In particular, we prove one half of a Goresky-Kottwitz-MacPherson theorem (Proposition 6.5).

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# 2 Preliminaries

We use the following conventions throughout this paper:

All (co)homology groups are singular (co)homology groups with integer coefficients, unless otherwise specified.

Varieties over $\mathbb{C}$ are given the classical Hausdorff topology inherited from $\mathbb{C}^n$ or $\mathbb{P}^n$, which we call the *complex* topology. When the Zariski topology is used, it will be indicated explicitly. Likewise, ind-varieties have both a complex and a Zariski direct limit topology.

The term *deformation retract* means what some authors call *strong deformation retract*; i.e., the deformation fixes the subspace in question pointwise.
2.1 Notation

The group $G$. Let $G$ be a simply-connected complex algebraic group with simple Lie algebra, with maximal torus $T_\mathbb{C}$, Weyl group $W$, $S \subset W$ the simple reflections, root system $\Phi$, and simple roots $\alpha_s, s \in S$. Let $Q^\vee$ denote the coroot lattice. Let $B$ denote a Borel subgroup containing $T_\mathbb{C}$, and $U \subset B$ the unipotent radical. Let $B^-, U^-$ denote the opposite Borel and unipotent subgroups. We write $\mathfrak{g}, \mathfrak{u}$, and so on for the Lie algebras.

Affine groups. Let $\tilde{G} = G(\mathbb{C}[z,z^{-1}])$; this is the group of regular maps $\mathbb{C}^\times \rightarrow G$. Similarly $P = G(\mathbb{C}[z])$ is the group of regular maps $\mathbb{C} \rightarrow G$. We have subgroups $P \supset \tilde{B} \supset \tilde{U} \supset P^{(1)}$ defined as follows: The Iwahori subgroup is $\tilde{B} = \{ f \in P : f(0) \in B^- \}$; similarly $\tilde{U} = \{ f \in P : f(0) \in U^- \}$. Set $P^{(1)} = \{ f \in P : f(0) = 1 \}$. Let $P^- = G(\mathbb{C}[z^{-1}])$ denote the group of regular maps $\mathbb{P}^1 - \{ 0 \} \rightarrow G$. Analogs of the subgroups of $P$ are defined in the evident way; e.g., $\tilde{B}^- = \{ f \in P : f(\infty) \in B \}$, etc. Associated Lie algebras are written $\tilde{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[z,z^{-1}]$, and so on.

The group $\tilde{G}$ is an affine ind-group. Explicitly, in the case $G = SL_n \mathbb{C}$ we let $F_n SL_n \mathbb{C}[z,z^{-1}]$ denote the subset of matrices $A$ such that $A_{ij} = \sum_{k=-m}^{m} a_{ijk} z^k$. This defines a filtration $F_m$ by affine varieties that yields the affine ind-group structure. In the general case we choose a faithful representation $G \subset SL_n \mathbb{C}$ and set $F_n \tilde{G} = \tilde{G} \cap F_n SL_n \mathbb{C}[z,z^{-1}]$. It is easy to see that the affine ind-group structure obtained is independent of the choice of representation. For a more general Kac-Moody approach, see [8], §7.3.

Affine Weyl group. Let $\tilde{W}$ denote the affine Weyl group, with Coxeter generators $\tilde{S} = S \cup \{ s_0 \}$. The affine root system is $\tilde{\Phi} = \mathbb{Z} \times \Phi$. As simple system of positive roots we take $\{(0, -\alpha_s) : s \in S \} \cup \{(1, \alpha_0) \}$, where $\alpha_0$ is the highest root. If $\theta = (n, \alpha)$, let $r_\theta = r_{n,\alpha}$ denote the affine reflection associated to $(n, \alpha)$.

The affine roots occur as weights of the extended torus $\tilde{T}_{\mathbb{C}} = \mathbb{C}^\times \times T_{\mathbb{C}}$ acting on $\tilde{\mathfrak{g}}$. Here the extra factor $\mathbb{C}^\times$ is acting by loop rotation. Thus $\tilde{\Phi}$ is actually the set of so-called “real” roots; we will also need the “imaginary” roots $(n, 0), n \in \mathbb{Z} - \{ 0 \}$, which are the weights of the $\tilde{T}_{\mathbb{C}}$ action on $t_{\mathbb{C}} \otimes \mathbb{C} \cdot z^n \subset \tilde{\mathfrak{g}}$.

$\tilde{W}/\tilde{W}_I$ and Bruhat order. Let $\tilde{W}^I$ denote the set of minimal length representatives for the cosets $\tilde{W}/\tilde{W}_I$. For any $\sigma \in \tilde{W}$, let $\ell_I(\sigma)$ denote the $I$-length of $\sigma$; that is, the length of the minimal coset representative in $\sigma \tilde{W}_I$. Let $\mathcal{I}_\lambda$ (resp. $\mathcal{J}_\lambda$) denote the upper order ideal (resp. lower order ideal) generated by $\lambda$ in the Bruhat order $\leq$ on $\tilde{W}/\tilde{W}_I$. We write $\lambda \downarrow \mu$ when $\mu < \lambda$ and the $I$-lengths differ by 1.

Parabolic subgroups. Let $P_I \subset \tilde{G}$ denote the parabolic subgroup generated by $\tilde{B}$ and $I$. Then $P_I$ is the semi-direct product of a normal subgroup $\tilde{U}_I$ and a finite-dimensional subgroup $L_I$. Here $\tilde{U}_I \subset \tilde{U}$ plays the role of unipotent radical, and $L_I$ is the Levi factor. Similarly, the opposite parabolic $P_I^-$ generated by $B^-$ and $I$ is the semi-direct product of $L_I$ and a normal subgroup $\tilde{U}_I^-$. 

Affine flag varieties. An affine flag variety is homogeneous space of the form $\mathcal{F} = \tilde{G}/P_I$. It has a canonical structure of projective ind-variety ([8], 13.2.13-18, [9]). Set $\mathcal{U}_0 = \tilde{U}^- P / P$ and $\mathcal{U}_\lambda = \lambda \mathcal{U}_0$ for $\lambda \in \tilde{W}/\tilde{W}_I$ (note this is well-defined). Then the natural map $\mathcal{U}_I^- \rightarrow \mathcal{U}_0$ is
an isomorphism of ind-varieties, and the $U_{\lambda}$’s form a Zariski open cover of $\mathcal{F}$. The Birkhoff strata $S_{\lambda}$ are the orbits of $B^{-}$ on $\mathcal{F}$.

**Schubert and Birkhoff varieties.** It is easy to see that any infinite subset of $\tilde{W}'$ is cofinal for the Bruhat order (cf. [1], Proposition 7.1). Hence any proper lower order ideal $\mathcal{J}$ is finite, and $X_{\mathcal{J}} = \cup_{\sigma \in \mathcal{J}} e_{\sigma}$ is a finite union of Schubert varieties. If $\mathcal{J} = \mathcal{J}_{\lambda}$, this is just the Schubert variety $X_{\lambda}$. If $\mathcal{I}$ is any non-empty upper order ideal, then $Z_{\mathcal{I}} = \cup_{\sigma \in \mathcal{I}} S_{\sigma}$ is a finite union of Birkhoff varieties. When $\mathcal{I} = \mathcal{I}_{\lambda}$, this is just the Birkhoff variety $Z_{\lambda}$.

Define $\tilde{U}_{\lambda} = \tilde{U} \cap \lambda \tilde{U}^{-} \lambda^{-1}$, $\tilde{U}_{\lambda}' = \tilde{U} \cap \lambda \tilde{U}_{\lambda}^{-} \lambda^{-1}$, $\tilde{U}_{\lambda}^{+} = \tilde{U}^{-} \cap \lambda \tilde{U}^{-} \lambda^{-1}$. Thus $\tilde{U}_{\lambda}'$ and $\tilde{U}_{\lambda}^{+}$ are the isotropy groups of the $U$ and $\tilde{U}$ actions on $\lambda P_{\lambda} / P_{\lambda}$, while the group action defines isomorphisms $\tilde{U}_{\lambda} \cong e_{\lambda}$ and $\tilde{U}_{\lambda}^{+} \cong S_{\lambda}$.

### 2.2 The extended torus action and the flow

Let $\tilde{T}_{\mathbb{C}}$ denote the extended torus $\mathbb{C}^{\times} \times T_{\mathbb{C}}$. Then $\tilde{T}_{\mathbb{C}}$ acts on $\tilde{G}$: The constant torus valued loops $T_{\mathbb{C}}$ act by conjugation, while the extra factor $\mathbb{C}^{\times}$ acts by loop rotation. The action preserves parabolic subgroups and induces an algebraic group action $\tilde{T}_{\mathbb{C}} \times \mathcal{F} \longrightarrow \mathcal{F}$, with fixed point set $\tilde{W} / \tilde{W}_I$. The action also preserves $B^{-}$, Schubert cells, Birkhoff strata, etc. The action of $\tilde{T}_{\mathbb{C}}$ on a Schubert cell $e_{\lambda}$ is isomorphic to a linear action, with weights precisely the set of roots $\Phi_{\lambda}$, each occurring with multiplicity one. In particular, the weights are positive.

Now consider the action of the torus $\tilde{T} = \mathbb{C}^{\times} \times T_{\mathbb{C}}$ on $\mathcal{F}$. One can always find a rank one subtorus $\phi : \mathbb{C}^{\times} \longrightarrow \tilde{T}$ such that the induced $\mathbb{C}^{\times}$ action has the following properties:

(i) The fixed-point set is still $\tilde{W} / W$;
(ii) If $x \in e_{\lambda}$, then $\lim_{t \to 0} t \cdot x = \lambda$;
(iii) If $x \in S_{\lambda}$, then $\lim_{t \to \infty} t \cdot x = \lambda$.

To see this, identify $\text{Hom}(\mathbb{C}^{\times}, \tilde{T})$ with $\mathbb{Z} \times Q^{\vee}$ and write $\phi = (k, \gamma)$. We then have:

**Proposition 2.1** Suppose that (a) For all $\alpha \in \Phi^{+}$, $\alpha(\gamma) < 0$, and (b) $k > \max_{\alpha \in \Phi^{+}} |\alpha(\gamma)|$. Then $\phi = (k, \gamma)$ has properties (i)-(iii) above.

In particular, (i)-(iii) hold when $\gamma = -\sum_{\alpha \in \Phi^{+}} \alpha^{\vee}$ and $k = 2h - 1$, where $h$ is the Coxeter number.

**Proof:** Assumptions (a) and (b) ensure that $\mathbb{C}^{\times}$ acts on each cell $e_{\lambda}$ with positive weights, yielding (i) and (ii). Now suppose $x \in \tilde{U}^{-} \lambda P_{\lambda} / P_{\lambda}$. Since $\tilde{U}^{-}$ is generated by the root subgroups $U_{n, \alpha}$ with $(n, \alpha) \in \Phi^{-}$ [7], and $\mathbb{C}^{\times}$ acts on these with negative weights, it follows that $\lim_{t \to \infty} t \cdot x = \lambda$, proving (iii). For the last assertion of the proposition, let $\rho^{\vee} = \omega_{1}^{\vee} + ... + \omega_{r}^{\vee}$, where the $\omega_{i}^{\vee}$’s are the fundamental coweights and $r$ is the rank of $G$. Let $\alpha_{0} = \sum_{i=1}^{r} m_{i} \alpha_{i}$, where the $\alpha_{i}$’s are the simple positive roots and $\alpha_{0}$ is the highest root as usual. Then $\gamma = -2\rho^{\vee}$, verifying (a), while the max occurring in (b) is $\alpha_{0}(2\rho^{\vee}) = 2 \sum m_{i} = 2h - 2$.

Fix $\gamma$, $k$ as in the Proposition. We refer to the resulting $\mathbb{C}^{\times}$ action as the complex flow.
3 Ind-spaces, ind-varieties and $\mathbb{C}^\times$-actions

3.1 Ind-spaces and ind-varieties

An *ind-space* is a set $X$ equipped with a filtration $X_1 \subset X_2 \subset ...$ such that $X = \bigcup X_n$, each $X_n$ is a topological space, and $X_n$ is closed in $X_{n+1}$. We give $X$ the direct limit topology: A subset of $X$ is closed if and only if its intersection with each $X_n$ is closed. A morphism of ind-spaces is a map $f : X \to Y$ such that for every $n$ there exists $m$ with $f(X_n) \subset Y_m$ and $f : X_n \to Y_m$ continuous. In particular, $f$ is continuous. Two ind-space structures on the same space $X$ are *commensurate* if an isomorphism between them. Any subspace $A$ of an ind-space $X$ is an ind-space with $A_n = A \cap X_n$. Given any space $X$, we can form the *constant ind-space* with $X_n = X$ for all $n$. This embeds the category of spaces as a full subcategory of the category of ind-spaces.

An *ind-variety* is defined similarly, with the requirement that each $X_n$ is a complex algebraic variety and a closed subvariety of $X_{n+1}$. See [8] for a brief introduction to ind-varieties. Every ind-variety is an ind-space in the Zariski and complex topologies. An ind-variety is *irreducible* if it is irreducible as a topological space in the Zariski topology. If each filtrant $X_n$ is irreducible, then so is $X$. Conversely, if $X$ is irreducible then it admits a commensurate filtration $Y_n$ with each $Y_n$ irreducible. In fact any filtration with $Y_n$ an irreducible component of $X_n$ is a commensurate filtration.

If $H$ is an algebraic group, an *ind-$H$-variety* is an ind-variety $X$ equipped with compatible algebraic $H$-actions on each $X_n$. If $H$ is connected and $V$ is any $H$-variety, each irreducible component of $V$ is invariant under the action. It follows that if $H$ is connected, then any ind-$H$-variety has a commensurate filtration by irreducible $H$-invariant varieties.

A *group ind-variety*, or simply “ind-group”, is a group object $\Gamma$ in the category of ind-varieties. Note that the filtrants $\Gamma_n$ are not assumed to be subgroups. A connected ind-group is irreducible ([8], Lemma 4.2.5).

3.2 Ind-CW-complexes

An *ind-CW-complex* is an ind-space such that each $X_n$ admits a CW-structure having $X_{n-1}$ as a subcomplex. We do not assume these structures are compatible as $n$ varies, and indeed $X$ itself need not admit any CW-structure (see the example below). An *ind-CW pair* is a pair of ind-spaces $(X, A)$ such that each $X_n$ admits a CW-structure such that $X_{n-1}$ and $A_n$ are subcomplexes. In the complex topology an ind-variety is also an ind-CW-complex, by Hironaka’s theorem [6]. However, an ind-variety need not admit any CW-structure.

Example: For $n \geq 1$ let $H_n$ denote the hyperplane $x = 1/n$ in $\mathbb{C}^2$. Let $X_n$ denote the union of the coordinate axes and $H_1, ..., H_n$. Let $X = \bigcup_n X_n$, with its evident ind-variety structure. Then $X$ does not admit a CW-structure. To see this, suppose given a CW-structure on $X$, and let $p_n = (1/n, 0)$. Then no $p_n$ lies in a 2-cell, since $X - \{p_n\}$ is disconnected. Furthermore, only finitely many $p_n$’s can be vertices, since $p_n \to (0,0)$ in the direct limit topology, and the vertex set of a CW-complex has no limit points. More generally, a subset of a CW-complex whose intersection with each cell is finite has no limit points. Thus all but finitely many $p_n$’s must lie in a single 1-cell $e^1$. Let $\phi : (0,1) \to e^1$ be a homeomorphism.
Then for some \( n \) we have \( a_{-1} < a_0 < a_1 \in (0,1) \) with \( \phi(a_i) = p_{n+i}. \) This forces \( e^1 \cap H_n = p_n, \) since if the path \( \phi \) ever enters \( H_n - \{p_n\} \) then it must also exit through \( p_n, \) contradicting the injectivity of \( \phi. \) But if \( e^1 \cap H_n = p_n, \) then no vertex of \( H_n - \{p_n\} \) can be connected by a 1-cell to \( p_n. \) Hence the 1-skeleton of \( X \) is disconnected, contradicting the connectedness of \( X. \)

As the following two results illustrate, however, for many purposes ind-CW-complexes are just as good as CW-complexes.

**Lemma 3.1** Let \((X, A)\) be an ind-CW-pair. Then \((X, A)\) has the homotopy extension property.

*Proof:* See [5], Chapter 0 for a discussion of the homotopy extension property. Any CW-pair has the homotopy extension property; the lemma follows immediately by an induction argument, using the CW-pair \((X_{n+1}, X_n \cup A_{n+1})\) at the inductive step.

A *CW-space* is a space with the homotopy-type of a CW-complex.

**Proposition 3.2** Let \( X \) be an ind-CW-complex. Then \( X \) is a CW-space.

*Proof:* For any space \( Y, \) there is a CW-approximation \( \eta_Y : W(Y) \longrightarrow Y; \) that is, a CW-complex \( W(Y) \) and a weak equivalence \( \eta_Y \) (see [5], Chapter 4). In fact one can make \( W \) a functor and \( \eta \) a natural transformation from \( W \) to \( Id, \) by taking \( W(Y) \) to be the geometric realization of the singular complex of \( Y. \) Hence there is a functorial CW-approximation \( \eta_X : W(X) \longrightarrow X \) that is filtered by CW-approximations \( W(X_n) \longrightarrow X_n, \) with \( W(X_n) \) a subcomplex of \( W(X_{n+1}). \) By Whitehead’s theorem ([5], Theorem 4.5), each \( W(X_n) \longrightarrow X_n \) is a homotopy equivalence. Since each of the pairs \((X_{n+1}, X_n)\) and \((W(X_{n+1}), W(X_n))\) has the homotopy extension property, it follows by a standard argument that the direct limit map \( W(X) \longrightarrow X \) is also a homotopy equivalence (see [5], Proposition 4.1 and the paragraph following its proof).

Thus Whitehead’s theorem applies to ind-CW-complexes. In particular, we have:

**Corollary 3.3** Let \((X, A)\) be an ind-CW-pair, and suppose the inclusion \( i : A \subset X \) is a weak equivalence. Then \( A \) is a deformation retract of \( X. \) In particular \( A \) is a deformation retract of \( X \) if \( A, X \) are simply-connected and \( H_*i \) is an isomorphism.

*Proof:* By Whitehead’s theorem, \( i \) is a homotopy equivalence. Since \((X, A)\) has the homotopy extension property, the first conclusion follows from [5], Corollary 0.20. If \( A, X \) are simply-connected and \( H_*i \) is an isomorphism, then \( i \) is automatically a weak equivalence ([5], Corollary 4.33).
3.3 $\mathbb{C}^\times$ actions

Let $X$ be an ind-space with $\mathbb{C}^\times$ action such that each filtration $X_n$ is invariant under the action. We also call this a complex flow. If $W \subset X$ is open and $C$ is any subset of $X$, we say that $C$ flows to $W$ at zero if for every $n$ there is an $s > 0$ such that for all $|t| \leq s$ we have $t \cdot C_n \subset W_n$. We say that $C$ flows to $W$ at infinity if for every $n$ there is an $s > 0$ such that for all $|t| \geq s$ we have $t \cdot C_n \subset W_n$.

A closed $\mathbb{C}^\times$-invariant ind-subspace $A$ is strongly attractive at zero (resp. strongly attractive at infinity) if for every neighborhood $W$ of $A$ and $x \in X$, there is a neighborhood $U$ of $x$ that flows to $W$ at zero (resp. at infinity). Since the conditions “attractive at zero” and “attractive at infinity” are interchanged under the automorphism $t \mapsto t^{-1}$ of $\mathbb{C}^\times$, for the remainder of this section we will consider only the former case and call such a subspace strongly attractive.

Remark: Call $A$ weakly attractive in $X$ if the above condition merely holds pointwise, i.e., for every neighborhood $W$ of $A$ and $x \in X$, there is an $s > 0$ such for all $|t| \leq s$ we have $t \cdot x \in W$. This is a very weak condition that does not imply strongly attractive, even if $X$ is a compact constant ind-space and one adds the requirement that $\lim_{t \to 0} t \cdot x$ exists for all $x$. For example, take $X = \mathbb{P}^1$ with the standard $\mathbb{C}^\times$ action coming from diagonal matrices in $SL_2 \mathbb{C}$, and take $A$ to consist of the two fixed points $p_0, p_\infty$. Here we have labelled the points so that for any $x \not \in A$, $t \cdot x \to p_0$ (resp. $p_\infty$) as $t \to 0$ (resp. $\infty$). Then $A$ is weakly attractive but evidently not strongly attractive (take $W$ to be the union of disjoint neighborhoods of $p_0, p_\infty$, and take $x = p_\infty$). One can easily exhibit similar examples with $A$ connected, for example with $X = \mathbb{P}^2$ and $A = \mathbb{P}^1 \cup \mathbb{P}^1$.

By a regular neighborhood of a subspace $B$ in a space $Y$, we mean a neighborhood $W$ such that $B$ is a deformation retract of $W$.

**Theorem 3.4** Let $X$ be a $T_1$ (points are closed) ind-space with $\mathbb{C}^\times$ action. Suppose $A \subset X$ is strongly attractive and each $A_n$ has a regular neighborhood in $X_n$. Then the inclusion $i : A \hookrightarrow X$ is a weak equivalence. If in addition $(X, A)$ is an ind-CW pair, then $A$ is a deformation retract of $X$.

**Proof:** Recall that a weak equivalence is a map inducing a bijection on path-components, and an isomorphism on homotopy groups for any choice of basepoint. We will show that for any compact space $K$, the inclusion induces a bijection on homotopy classes $i_* : [K, A] \xrightarrow{\cong} [K, X]$. It is well-known, and easy to prove, that this implies $i$ is a weak equivalence.

Suppose that $X$ is a constant ind-space. Let $f : K \to X$ be a map, and let $W$ be a regular neighborhood of $A$. For each $k \in K$, choose a neighborhood $U_k$ of $k$ and $s_k > 0$ such that for all $|t| \leq s_k$, $t \cdot U_k \subset W$. Since $K$ is compact, $f(K)$ is covered by finitely many such neighborhoods, say $U_{k_1}, \ldots, U_{k_n}$. Taking $s = \min \{ s_{k_1}, \ldots, s_{k_n} \}$, we have $s \cdot f(K) \subset W$. Since $s \cdot f$ is homotopic to $f$, composing with the deformation of $W$ into $A$ shows that $f$ is homotopic to a map $g : K \to A$. Hence $i_*$ is surjective. Next suppose that $f_0, f_1 : K \to A$ are maps that become homotopic in $X$. Applying the preceding argument to the homotopy shows that $s \cdot f_0$ is homotopic to $s \cdot f_1$ in $A$, and hence $f_0$ is homotopic to $f_1$. This shows that $i_*$ is injective, and hence bijective.
In the general case, we conclude that each inclusion $A_n \subset X_n$ is a weak equivalence. Now let $\text{Map}(-, -)$ denote the set of continuous maps. Then it is well-known and easy to prove that for any $T_1$-ind-space $X$ and compact space $K$, the natural map

$$\text{colim}_n \text{Map}(K, X_n) \to \text{Map}(K, X)$$

is bijective (the $T_1$ hypothesis ensures that every compact subset of $X$ lies in some $X_n$). Hence $i_*: [K, A] \to [K, X]$ is a colimit of bijections and so is bijective.

Finally, if $(X, A)$ is an ind-CW-pair then $A$ is a deformation retract of $X$ by Corollary 3.3.

**Remark:** Note that the theorem fails miserably if one only assumes $A$ is weakly attractive (see the example in the previous remark).

When $X$ is an ind-variety, we always assume that the $\mathbb{C}^\times$-action is algebraic. The following technical lemma will be need in the proof of Theorem 5.1:

**Lemma 3.5** Let $f: (X, A) \to (Y, B)$ be a map of ind-variety pairs with $\mathbb{C}^\times$ action. Suppose $f: X \to Y$ is surjective and satisfies the following condition:

(*) $X$ is a union of ind-subvarieties $Z_\alpha$ such that for each $\alpha$, the restriction $f|_{Z_\alpha}$ is an isomorphism of ind-varieties onto a Zariski open ind-subvariety of $Y$.

Then if $A$ is strongly attractive in $X$, $B$ is strongly attractive in $Y$.

**Proof:** Let $W \subset Y$ be a neighborhood of $B$, and $y \in Y$. Choose $x \in f^{-1}y$. Then there is a neighborhood $U$ of $x$ that flows to $f^{-1}W$. Moreover, $x \in Z$ for some $Z = Z_\alpha$ as in the theorem. By Chevalley’s theorem each $f(Z_m) \cap Y_n$ is a constructible subset of $Y_n$, so for fixed $n$ we have $f(Z) \cap Y_n = f(Z_m) \cap Y_n$ for sufficiently large $m$ (see [8], exercise 7.3.E(2)). Let $V = f(U \cap Z)$. Then $V$ is complex open, since any isomorphism of varieties is a homeomorphism in the complex topology. Furthermore, for fixed $n$ and $m >> 0$ we have

$$V_n = f(U_m \cap Z) \cap Y_n = f(U \cap Z_m) \cap Y_n = f(U \cap Z) \cap Y_n,$$

where the third equality uses the fact that $f|_Z$ is injective. Then there is an $s > 0$ such that $t \cdot U_m \subset f^{-1}W$ for all $|t| \leq s$, and hence $t \cdot V_n \subset W$. Thus $V$ is a complex open neighborhood of $y$ that flows to $W$, as required.

## 4 $\tilde{U}$ and $\tilde{U}^-$ as ind-varieties

In this section we study the structure of $\tilde{U}$ and $\tilde{U}^-$ as ind-varieties. In particular, we construct filtrations by weighted cones. Our main applications are Lemma 4.6, showing that punctured Birkhoff strata are contractible, and Corollary 4.4, which will be used in the proof of Theorem 5.1. In fact it will suffice to consider $\tilde{U}$, for the following reason: Define $\delta: \tilde{G} \to \tilde{G}$ by $(\delta f)(z) = w_0 f(z^{-1}) w_0$, where $w_0$ is the longest element of $W$. Then $\delta$ is an ind-group automorphism exchanging $\tilde{U}$ and $\tilde{U}^-$. We will leave it to the reader to make the translation from $\tilde{U}$ to $\tilde{U}^-; in particular one must replace positive weights by negative weights, and limits as $z \to 0$ by limits as $z \to \infty$. 
4.1 Weighted cones

Let $V$ be a finite-dimensional representation of $\mathbb{C}^\times$. By a weighted cone we mean a nonempty, closed $\mathbb{C}^\times$-invariant subvariety of $V$. We will only be concerned with positively or negatively weighted cones. Since the two cases are exchanged under the automorphism of $z \mapsto z^{-1}$ of $\mathbb{C}^\times$, there is no loss of generality in restricting to positively weighted cones.

4.1.1 Filtrations by weighted cones

In this section we show that $\tilde{U}$ is filtered by positively weighted cones. Consider the finite-dimensional filtrations $F_m \subset \tilde{U}$, defined in §2. Recall that $F_m \tilde{U}$ is an affine variety, but not a subgroup. Recall also that $\tilde{U}$ is not unipotent but embeds in an inverse limit of unipotent groups. More precisely, it embeds in an inverse limit of the form $\lim_k U[k]$, where $U[k]$ is a maximal unipotent subgroup of the finite-dimensional group $G(\mathbb{C}[z]/z^k)$. This inverse system is compatible with the $\mathbb{C}^\times$ action. Moreover the exponential map $u[k] \rightarrow U[k]$ is a $\mathbb{C}^\times$-equivariant isomorphism of varieties, so we may identify $U[k]$ with a finite-dimensional representation of $\mathbb{C}^\times$.

The next proposition is a special case of [8], Proposition 7.3.7.

**Proposition 4.1** Fix $m$. Then for all $k >> 0$, the natural map $F_m \tilde{U} \rightarrow U[k]$ is a closed $\hat{T}_\mathbb{C}$-equivariant embedding of varieties.

We remark that the proof in our special case is quite easy. Using an embedding $G_\mathbb{C} \subset SL_n \mathbb{C}$ for some $n$, one first reduces to the case $G_\mathbb{C} = SL_n \mathbb{C}$. Then it is clearly sufficient to take $k > m$.

**Corollary 4.2** $F_m \tilde{U}$ is $\mathbb{C}^\times$-equivariantly isomorphic to a positively weighted cone. Moreover the inclusions $F_m \subset F_{m+1}$ are induced by inclusions of $\mathbb{C}^\times$ representations.

**Corollary 4.3** Let $\tilde{U}_\Theta$ be the ind-subgroup of $\tilde{U}$ associated to a bracket closed subset $\Theta$ of the positive affine roots (cf. [8], §6.1.1). Then $\tilde{U}_\Theta$ has a commensurate filtration by irreducible positively weighted cones.

**Proof:** It is clear that $\tilde{U}_\Theta$ is a connected ind-group, hence an irreducible ind-variety by [8], Lemma 4.2.5. Since $\tilde{U}_\Theta$ is $\mathbb{C}^\times$-invariant, it inherits a filtration by weighted cones and hence a commensurate filtration by irreducible weighted cones.

**Corollary 4.4** The identity element 1 is strongly attractive in $\tilde{U}$.

**Proof:** Consider the case $\tilde{U}$, and choose an open $V \subset \tilde{U}$ so that each $V_m$ is a neighborhood of 1 in $F_m \tilde{U}$ with compact closure. Then for any neighborhood $W$ of 1, $V$ flows to $W$ at zero. Since every $x \in \tilde{U}$ lies in such a $V$, this proves the corollary. The case $\tilde{U}^-$ is the same.
4.1.2 Joins

Let $X$ be a positively weighted cone in $V$. Choose a Hermitian metric invariant under the $S^1$ action, and let $S(V)$, $D(V)$ denote respectively the unit sphere and unit disc. Let $S(X) = X \cap S(V)$, $D(X) = X \cap D(V)$. Then it is clear that the map $(S(X) \times [0, \infty))/(S(X) \times \{0\}) \to X$ given by $(v, t) \mapsto t \cdot v$ (if $t > 0$) and $(v, 0) \mapsto 0$ is a $\mathbb{C}^\times$-equivariant homeomorphism.

In particular, $S(X)$ is an equivariant deformation retract of $X - 0$, and $D(X)$ is just the cone $CS(X)$ on $S(X)$—in the topologist’s sense, where $CY = (Y \times [0, 1])/(Y \times 0)$.

Now recall that the join of spaces $Y, Z$ is defined by $Y \star Z = (CY \times Z) \cup_{Y \times Z} (Y \times CZ)$. An elementary argument shows that $Y \star Z$ is a deformation retract of $(CY \times CZ) - (p, q)$, where $p, q$ are the cone points. Here we conclude:

**Lemma 4.5** Suppose $X \subset V$, $Y \subset W$ are positively weighted cones. Then $(X \times Y) - (0, 0)$ contains $S(X) \star S(Y)$ as a deformation retract.

4.2 Punctured Birkhoff strata

The flow shows immediately that a Birkhoff stratum $S_\lambda$ itself is contractible. *A priori*, however, there are no restrictions whatever on the homotopy type of a contractible space minus a point; one has only to think of the cone on a space minus the cone point. However:

**Lemma 4.6** Every punctured Birkhoff stratum $S_\lambda - \{\lambda\}$ is contractible.

Since $S_\lambda - \{\lambda\}$ is isomorphic as an ind-variety to $\bar{U}_\lambda$, with $\lambda$ corresponding to the identity $e$, it will be enough to prove:

**Lemma 4.7** For all $\lambda \in \hat{W}^I$, $\bar{U}_\lambda^+ - e$ and $\bar{U}_\lambda^- - e$ are contractible.

*Proof:* We consider the case $\bar{U}_\lambda^+ - e$. Suppose for convenience that $I = \emptyset$, so that $\hat{W}^I = \hat{W}$. Let $\mu \in \hat{W}$ satisfy $\ell(\lambda \mu) = \ell(\lambda) + \ell(\mu)$. Then by general results from [8] (see especially Theorem 5.2.3c) group multiplication defines an isomorphism of ind-varieties

$$\phi : \lambda \bar{U}_\mu \lambda^{-1} \times \bar{U}_{\lambda \mu} \to \bar{U}_\lambda.$$  

Note that $\phi$ can also be interpreted in terms of loc. cit., Lemma 6.1.3, writing $\hat{\Phi}^+$ as the disjoint union of suitable bracket closed subsets.

Now $\bar{U}_\mu$ is a finite-dimensional unipotent group of dimension $\ell(\mu)$, where $\ell(\mu)$ can be taken arbitrarily large. Moreover, there is an analogous isomorphism for general $I$. We conclude that for every $n > 0$ there exists $d \geq n$ such that $\bar{U}_\lambda^+$ has a commensurate filtration by varieties of the form $\mathbb{C}^d \times A_n$, where $\mathbb{C}^d$ and $A_n$ are positively weighted cones. Hence by Lemma 4.5, the corresponding filtrations of $\bar{U}_\lambda^+ - e$ have the homotopy type of $S^{2d-1} \star S(A_n)$, which in turn is homotopy equivalent to $S^{2d} \wedge S(A_n)$ ([5], Exercise 0.24) and hence is $(2d - 1)$-connected. Passing to the direct limit, $\bar{U}_\lambda^+ - e$ is $(2d - 1)$-connected. But $d$ can be taken arbitrarily large, and therefore $\bar{U}_\lambda^+ - e$ is weakly contractible. Since $\bar{U}_\lambda^+ - e$ is an ind-variety and hence an ind-CW-complex, this completes the proof.
5 Schubert and Birkhoff neighborhoods

Let $\mathcal{I}$ be an upper order ideal and $\mathcal{J}$ a lower order ideal for the Bruhat order on $\check{W}^I$. Then the Schubert neighborhood $\mathcal{E}_\mathcal{I} = \cup_{\lambda \in \mathcal{I}} e_\lambda$ is a Zariski open neighborhood of $Z_\mathcal{I}$, and the Birkhoff neighborhood $\mathcal{S}_\mathcal{J} = \cup_{\lambda \in \mathcal{J}} S_\lambda$ is a Zariski open neighborhood of $X_\mathcal{J}$ (see the appendix). Recall from §2 that $Z_\mathcal{I}$ is a finite union of Birkhoff varieties, while $X_\mathcal{J}$ is a finite union of Schubert varieties. Although we are mainly interested in the case of principal order ideals—i.e., in Birkhoff and Schubert varieties—the general case will be useful for later induction arguments.

**Theorem 5.1** a) Let $Z_\mathcal{I}$ be a finite union of Birkhoff varieties. Then $Z_\mathcal{I}$ is a deformation retract of its Schubert neighborhood $\mathcal{E}_\mathcal{I}$.

b) Let $X_\mathcal{J}$ be a finite union of Schubert varieties. Then $X_\mathcal{J}$ is a deformation retract of its Birkhoff neighborhood $\mathcal{S}_\mathcal{J}$.

**Proof:** For ease of notation, we write $Z, \mathcal{E}, X, \mathcal{S}$ in place of $Z_\mathcal{I}, \mathcal{E}_\mathcal{I}, X_\mathcal{J}, \mathcal{S}_\mathcal{J}$.

a) By Theorem 3.4, it is enough to show that $Z$ is strongly attractive in $\mathcal{E}$, or that for every Schubert variety $X = X_\lambda$, $Z \cap X$ is strongly attractive in $\mathcal{E} \cap X$. Let $f : \check{U} \times (Z \cap X) \rightarrow \mathcal{E} \cap X$ denote the map induced by the action of $U$ on $\mathcal{E}$. Note that $f$ is a $C^\infty$-equivariant map of pairs $(\check{U} \times (Z \cap X), \{1\} \times (Z \cap X)) \rightarrow (\mathcal{E} \cap X, Z \cap X)$, where $C^\infty$ acts on $\check{U} \times Z$ by $t \cdot (u, z) = (t u^{-1}, t z)$. We will prove (a) by showing that $f$ satisfies the hypotheses of Lemma 3.5.

Since $Z \cap X$ is compact, tubes of the form $V \times (Z \cap X)$ are cofinal among neighborhoods of $Z \cap X$ in $\check{U} \times (Z \cap X)$. It then follows from Corollary 4.4 that $\{1\} \times (Z \cap X)$ is strongly attractive in $\check{U} \times (Z \cap X)$. Next we show that $f$ satisfies condition (*) of Lemma 3.5. For each $\sigma \in I \cap J_\lambda$, the natural map $\check{U}_\sigma \times S_\sigma \rightarrow \mathcal{U}_\sigma$ is an isomorphism of ind-varieties, and restricts to an isomorphism $\check{U}_\sigma \times (S_\sigma \cap X) \cong \mathcal{U}_\sigma \cap X$ (see the Appendix). More generally, for any $g \in \check{U}$ we have $g \check{U}_\sigma \times (S_\sigma \cap X) \cong g \mathcal{U}_\sigma \cap X$. Since the ind-varieties $g \check{U}_\sigma \times S_\sigma$ cover $\check{U} \times Z$, this verifies condition (*).

b) The proof here is analogous to the proof of (a), using the flow at infinity. In this case we use the natural map $f : \check{U}^\rightarrow \times X \rightarrow \mathcal{S}$. Since $\check{U}^\rightarrow$ is an ascending union of negatively weighted cones, and $X$ is compact, we conclude as before that $\{1\} \times X$ is strongly attractive at infinity in $\check{U}^\rightarrow \times X$. Condition (*) of Lemma 3.5 is also verified as in (a), using the isomorphisms $\check{U}_\sigma^\rightarrow \times e_\sigma \rightarrow \mathcal{U}_\sigma$.

Variants of Theorem 5.1 can be obtained by intersecting with $C^\infty$-invariant closed ind-subvarieties of $\mathcal{F}$. In particular, we will need the following for the proof of Theorem 1.1:

**Theorem 5.2** If $\mathcal{I} \subset \mathcal{I}'$, then $Z_\mathcal{I}$ is a deformation retract of $\mathcal{E}_\mathcal{I} \cap Z_{\mathcal{I}'}$.

**Proof:** Since $Z_{\mathcal{I}'}$ is invariant under the flow, the proof of Theorem 5.1 shows that $Z_\mathcal{I}$ is strongly attractive in $\mathcal{E}_\mathcal{I} \cap Z_{\mathcal{I}'}$.

6 The homotopy-type of a Birkhoff variety

6.1 The main theorem

In this section we show that every finite union of Birkhoff varieties $Z_\mathcal{I}$ is a deformation retract of $\mathcal{F}$ (Theorem 6.1). To motivate this result, we point out that there are much
simpler examples of the same phenomenon. For instance, the ind-variety $\mathbb{P}^\infty$ has a Birkhoff filtration $\mathbb{P}^\infty \supset Z_1 \supset Z_2 \supset \ldots$ dual to its Schubert filtration $\mathbb{P}^n$. Writing $\mathbb{C}^\infty = \cup \mathbb{C}^n$ as usual, $Z_n$ is just the subvariety of lines orthogonal to $\mathbb{C}^n$, and hence is isomorphic to $\mathbb{P}^\infty$. Moreover it is easy to show that $Z_{n+1}$ is a deformation retract of $Z_n$, as follows: The ind-variety structure on $Z_n$ is given by the Richardson varieties $Z_n \cap \mathbb{P}^m$. But $Z_n \cap \mathbb{P}^m$ is just $\mathbb{P}^m - n$, and hence $(Z_n \cap \mathbb{P}^m)/(Z_{n+1} \cap \mathbb{P}^m) = S^{2(m-n)}$. Letting $m \to \infty$, we have $Z_n/Z_{n+1} = S^\infty$, which is contractible. Since $(Z_n, Z_{n+1})$ is a CW-pair, it follows that $Z_{n+1}$ is a deformation retract of $Z_n$. In principle one could apply the same method in the present context, but the Richardson varieties $Z_\lambda \cap X_\sigma$ are much more complicated. Hence we will take a somewhat different approach.

**Theorem 6.1** Let $Z \subset Z'$ be finite unions of Birkhoff varieties. Then $Z$ is a deformation retract of $Z'$. In particular, any Birkhoff variety $Z_\lambda$ is a deformation retract of $\mathcal{F}$.

**Proof:** By downward induction over the strata, we reduce at once to the case when $Z' - Z$ is a single stratum $S_\lambda$, and $Z'$ is a deformation retract of $\mathcal{F}$. Since $\mathcal{F}$ is simply-connected (it is a connected CW-complex with only even-dimensional cells), in particular $Z'$ is simply-connected. Let $\mathcal{E}$ be the Schubert neighborhood of $Z$. Then we have a diagram of open sets and inclusions

$$
\begin{array}{ccc}
S_\lambda - \{\lambda\} & \xrightarrow{i} & S_\lambda \\
\downarrow & & \downarrow \\
\mathcal{E} \cap Z' & \xrightarrow{j} & Z'
\end{array}
$$

where $S_\lambda, \mathcal{E} \cap Z'$ cover $Z'$ and have intersection $S_\lambda - \{\lambda\}$. Then $H_*i$ is an isomorphism by Lemma 4.6, so $H_*j$ is an isomorphism by excision. Since $S_\lambda - \{\lambda\}$, $S_\lambda$ and $\mathcal{E} \cap Z'$ are path-connected (for the last case see Proposition 7.8), we conclude similarly from the Seifert-van Kampen theorem that $\pi_1j$ is an isomorphism and hence $\mathcal{E} \cap Z'$ is simply-connected. But $Z$ is a deformation retract of $\mathcal{E} \cap Z'$ by Theorem 5.1a. Hence $Z$ is a deformation retract of $Z'$ by Corollary 3.3.

It follows, of course, that the inclusions $Z \subset \mathcal{F}$ induce isomorphisms on any homology or cohomology theory, including $\hat{T}$-equivariant theories. For emphasis we record the following cases explicitly.

**Corollary 6.2** For any finite union of Birkhoff varieties $Z$, $H^*\mathcal{F} \xrightarrow{\cong} H^*Z$, and $H_T^*\mathcal{F} \xrightarrow{\cong} H_T^*Z$.

**Remark:** It follows from the corollary that $H_*Z$ has finite type and is concentrated in even dimensions. This does not seem obvious *a priori*; neither property need hold for an ind-subvariety $Y$ of $\mathcal{F}$. 

12
6.2 Cohomology of \((S_\mathcal{J}, S_\mathcal{J} - X_\mathcal{J})\) and \((E_\mathcal{I}, E_\mathcal{I} - Z_\mathcal{I})\)

We next consider the pairs \((S_\mathcal{J}, S_\mathcal{J} - X_\mathcal{J})\) and \((E_\mathcal{I}, E_\mathcal{I} - Z_\mathcal{I})\), which can be viewed as “normal Thom spaces” of the subvarieties \(X_\mathcal{J}, Z_\mathcal{I}\) in \(\mathcal{F}\). Let \(\mathcal{I}\) be a nonempty upper order ideal and let \(\mathcal{J}\) be the complementary lower order ideal. Then

\[ \mathcal{F} = E_\mathcal{I} \cup S_\mathcal{J}, \]

\[ E_\mathcal{I} - Z_\mathcal{I} = E_\mathcal{I} \cap S_\mathcal{J} = S_\mathcal{J} - X_\mathcal{J}. \]

For ease of notation, we henceforth write \(E, S, Z, X\) for the corresponding spaces above.

**Proposition 6.3** \(H_\ast(S, S - X) = 0\). Hence \(H_\ast(S - X) \cong H_\ast X\).

**Proof:** We have

\[ H_\ast(S, S - X) \cong H_\ast(\mathcal{F}, \mathcal{E}) \cong H_\ast(\mathcal{F}, Z) = 0, \]

where the first isomorphism is by excision, the second by Theorem 5.2 and the third by Theorem 6.1. Thus \(H_\ast(S - X) \cong H_\ast S \cong H_\ast X\) by Theorem 5.2.

**Remark:** This result reflects the intuition that \(S\) is a sort of infinite-dimensional “vector bundle” over \(X\), and so should have contractible Thom space, while its “sphere bundle” should have contractible fibers.

Similarly, we have:

**Proposition 6.4** \(H_\ast(E, E - Z) \cong H_\ast(\mathcal{F}, X)\), and \(H_\ast(E - Z) \cong H_\ast X\).

**Proof:** We have

\[ H_\ast(E, E - Z) \cong H_\ast(\mathcal{F}, S) \cong H_\ast(\mathcal{F}, X). \]

**Remark:** Note that \(H_\ast(\mathcal{F}, X)\) is a free abelian group on the upper order ideal \(\mathcal{I}\), graded by twice the length as usual. In fact \(\mathcal{F}/X\) is a CW-complex whose cells are the Schubert cells corresponding to \(\mathcal{I}\), plus a basepoint. When \(\mathcal{I} = \mathcal{I}_\lambda\), \(\mathcal{F}/X\) has \(2\ell^\ast(\lambda)\)-skeleton the sphere \(e^\lambda_+\). This reflects the intuition that the pair \((E_\mathcal{I}, E_\mathcal{I} - Z_\mathcal{I})\) is the “Thom space” of the complex \(\ell^\mathcal{I}(\lambda)\)-dimensional “normal bundle” of \(Z_\lambda\) in \(L_G\). In cannot actually be such a Thom space, however, since it does not even have the right Poincaré series.
6.3 Remarks on equivariant cohomology

Let $Y$ be a space with an action of a compact torus $T$. The Goresky-Kottwitz-MacPherson (GKM) theory [3] characterizes the equivariant cohomology $H^*_T(Y;\mathbb{Q})$ in terms of the zero- and one-dimensional orbits—provided that $Y$ is sufficiently well-behaved as a $T$-space. In particular, some finiteness restriction on $Y$ is usually required, such as compactness, finite cohomological dimension, and/or finite orbit type. Since the spaces we are considering are noncompact, of infinite cohomological dimension, and of infinite orbit type, any attempt to extend the results of [3] must proceed with caution.

The case of the $\hat{T}$-action on $\mathcal{F}$ itself has been studied by a number of authors; see [8] and the references cited there, and [4]. Here the beautiful properties of the Schubert cell decomposition more than compensate for the infinite-dimensionality of $\mathcal{F}$; one can proceed by induction over the Schubert filtration. The result is as follows: Identify $H^*_T(\mathcal{F}^\hat{T})$ with the ring of $H^*_I$-valued functions on $\hat{W}$. Let $\mathcal{R}(\mathcal{F})$ denote the subring consisting of those functions $f$ such that whenever $r_\theta \sigma = \lambda$ for some positive affine root $\theta$, we have $f(\sigma) = f(\lambda) \mod c_\theta$, where $c_\theta$ is the first Chern class of the line bundle $\xi_\theta \triangleright B\hat{T}$ associated to $\theta$. Then $\mathcal{F}$ satisfies the GKM theorem (compare [3], 1.2.2); that is, restriction to the fixed point set defines an isomorphism $H^*_T(\mathcal{F};\mathbb{Q}) \cong \mathcal{R}(\mathcal{F};\mathbb{Q})$.

Now consider a Birkhoff subspace $Z = Z_T$. Again we are faced with an infinite-dimensional space, with the further complication that there is no Schubert cell structure. The natural filtration by Richardson varieties is not so easy to analyze. Instead we will use Theorem 6.1 to obtain half of the GKM theorem for $Z$. Note that $Z^\hat{T} = \mathcal{I}$, and that if $\sigma > \lambda \in \mathcal{I}$, then the unique one-dimensional orbit with $\sigma, \lambda$ as its poles lies in $Z_\lambda \cap X_\sigma \subset Z$.

**Proposition 6.5** For any Birkhoff subspace $Z = Z_T$, the restriction $i^* : H^*_T Z \longrightarrow H^*_T Z^\hat{T}$ is an injection into $\mathcal{R}(Z)$.

**Proof:** That $i^*$ has image contained in $\mathcal{R}(Z)$ is straightforward; the argument is as in [3] or [8]. It remains to show that $i^*$ is injective. Since $H^*_T \mathcal{F}$ is torsion-free, it suffices to prove this rationally. Let $j : Z \longrightarrow \mathcal{F}$ denote the inclusion. Then there is a commutative diagram

$$
\begin{array}{ccc}
H^*_T(\mathcal{F};\mathbb{Q}) & \cong & H^*_T(Z;\mathbb{Q}) \\
\downarrow & & \downarrow \\
\mathcal{R}(\mathcal{F};\mathbb{Q}) & \xrightarrow{i^*} & \mathcal{R}(Z;\mathbb{Q})
\end{array}
$$

Thus $i^*$ is injective if and only if $\mathcal{R}(j)$ is injective. Now $\mathcal{R}(j)$ amounts to taking a function $f : \hat{W}/\hat{W}_I \longrightarrow H^*_T$ and restricting it to the upper order ideal $\mathcal{I}$. Suppose that $f$ restricts to zero on $\mathcal{I}$, and let $\sigma \hat{W}_I \in (\hat{W}/\hat{W}_I) - \mathcal{I}$. Let $\mathcal{A}$ denote the set of all affine reflections. Since $\mathcal{A}$ is infinite and all isotropy groups of the action of $\hat{W}$ on $\hat{W}/\hat{W}_I$ are finite, the set $(\mathcal{A} r \hat{W}_I) \cap \mathcal{I}$ is infinite. But then $f(\sigma \hat{W}_I)$ is divisible by an infinite set of pairwise relatively prime elements of $H^2(B\hat{T};\mathbb{Q})$, and hence must be zero. This proves the proposition.

From the commutative diagram we also have $\text{Im} i^* = \text{Im} \mathcal{R}(j)$. Hence the full GKM theorem holds if and only if every function $f \in \mathcal{R}(Z;\mathbb{Q})$ extends to $\tilde{f} \in \mathcal{R}(\mathcal{F};\mathbb{Q})$.  

14
Remark: The $\hat{T}$-space $E \cap S$ provides a typical example of what can go wrong with equivariant cohomology in an infinite-dimensional setting. It is equivariantly formal in the sense of [3], since $H^*_E(E \cap S)$ is a free module $H^*_E \otimes H^*X$, but it has no $\hat{T}$-fixed points. Hence localization at the fixed point set and the GKM theorem fail for $E \cap S$.

7 Appendix: Basic properties of Birkhoff varieties

We assume given the standard refined Tits system structure on $\tilde{G}$; in particular, the Bruhat and Birkhoff decompositions ([7], [8]). Recall that $U_\lambda = \lambda U_0$ (§2), where $U_0 = \tilde{U}_I P_I / P_I$.

Proposition 7.1 The $U_\lambda$’s form a Zariski open cover of $\mathcal{F}$.

Proof: That the $U_\lambda$’s cover $\mathcal{F}$ follows from the Bruhat decomposition. By reducing to the case $G = SL_n \mathbb{C}$, it is not hard to show that $U_0$ is Zariski open (or see [8]). Since multiplication by any fixed $f \in \tilde{G}$ gives a morphism of ind-varieties from $\mathcal{F}$ to itself, it follows that all the $U_\lambda$’s are Zariski open.

Proposition 7.2 The natural maps $\tilde{U}_\lambda \times S_\lambda \rightarrow U_\lambda$ and $\tilde{U}_\lambda^- \times e_\lambda \rightarrow U_\lambda$ are isomorphisms of ind-varieties.

Proof: The first isomorphism amounts to the isomorphism of ind-varieties

$$\phi : (\tilde{U} \cap (\lambda \tilde{U}^- \lambda^{-1})) \times (\tilde{U}^- \cap (\lambda \tilde{U}^- \lambda^{-1})) \cong \lambda \tilde{U}^- \lambda^{-1}.$$ 

Here $\phi$ is group multiplication. That $\phi$ is bijective follows from the axioms for a refined Tits system; compare [8], p. 169 (7), as well as p. 227 (1). The methods there also show that $\phi$ is an isomorphism of ind-varieties.

Corollary 7.3 $U_\lambda \times (S_\lambda \cap X_\mu) \rightarrow U_\lambda \cap X_\mu$ is an isomorphism of varieties.

If $\mathcal{J}$ is a lower order ideal in $\tilde{W}S/W$, let $S_\mathcal{J} = \cup_{\lambda \in \mathcal{J}} S_\lambda$. If $\mathcal{I}$ is an upper order ideal, let $E_\mathcal{I} = \cup_{\lambda \in \mathcal{I}} e_\lambda$. And if $K$ is any subset of $\tilde{W}S/W$, let $U_K = \cup_{\lambda \in K} U_\lambda$.

Proposition 7.4 a) Let $\mathcal{J}$ be a lower order ideal. Then $X_\mathcal{J} \subset S_\mathcal{J} = U_\mathcal{J}$.

b) Let $\mathcal{I}$ be a upper order ideal. Then $Z_\mathcal{I} \subset E_\mathcal{I} = U_\mathcal{I}$.

Proof: In (a) we have $X_\mathcal{J} \subset U_\mathcal{J}$ and $S_\mathcal{J} \subset U_\mathcal{J}$ by Proposition 7.2. Now suppose $\lambda \in \mathcal{J}$; we show that $U_\lambda \subset S_\mathcal{J}$. Since $S_\mathcal{J}$ is $\tilde{U}^-$-invariant, it is enough to show $e_\lambda \subset S_\mathcal{J}$. But if $x \in e_\lambda \cap S_\mu$, then $\mu = \lim_{t \rightarrow \infty} t \cdot x \in \overline{\mathcal{I}}_\lambda$, so $\mu \leq \lambda$.

The proof of (b) is similar.

Corollary 7.5 Let $\mathcal{I}$, $\mathcal{J}$ be respectively upper and lower order ideals. Then

a) $S_\mathcal{J}$ is Zariski open and $Z_\mathcal{I}$ is Zariski closed.

b) $E_\mathcal{I}$ is Zariski open and $X_\mathcal{J}$ is Zariski closed.
Both statements follow immediately, using the fact that the complement of an upper order ideal is a lower order ideal and vice-versa.

**Proposition 7.6** \( S_\lambda = Z_\lambda \) and \( e_\lambda = X_\lambda \). Here the closure can be taken in either the Zariski topology or the classical topology.

**Proof:** By the corollary, we have \( S_\lambda \subset Z_\lambda \) and \( e_\lambda \subset X_\lambda \). The reverse inclusions reduce to showing that if \( \sigma \downarrow \eta \), then \( \sigma \in S_\eta \) and \( \eta \in i_\sigma \). Let \( r_\sigma \sigma = \eta \) and let \( SL_2^\sigma \subset \tilde{G} \) denote the corresponding \( SL_2 \) subgroup. Then \( SL_2^\sigma \cdot \sigma \) is an embedded \( \mathbb{P}^1 \), denoted \( \mathbb{P}^1_{\eta \sigma} \), with \( \sigma, \eta \in \mathbb{P}^1 \) and \( \mathbb{P}^1 - \{ \sigma, \eta \} \subset S_\sigma \cap e_\eta \). This proves the proposition.

**Proposition 7.7** If \( \lambda \leq \mu \), then \( Z_\lambda \cap X_\mu \) is irreducible of codimension \( \ell \lambda \) in \( X_\mu \).

**Proof:** Note that \( U_\lambda \cap X_\mu \) is irreducible, since it is Zariski open in the irreducible variety \( X_\mu \). Hence \( S_\lambda \cap X_\mu \) is irreducible of codimension \( \ell \lambda \) by Corollary 7.3. Since \( S_\lambda \cap X_\mu \) is Zariski dense in \( Z_\lambda \cap X_\mu \), the result follows.

**Proposition 7.8** Suppose \( I, J \) are respectively upper and lower order ideals in \( \tilde{W}^I \). If \( I \cap J \) is connected as a subgraph of the Hasse diagram of \( \tilde{W}^I \), then \( Z_I \cap X_J \) is path-connected. In particular \( Z_I \cap X_\lambda, Z_\lambda \cap X_J, Z_I, \) and \( X_J \) are all path-connected.

**Proof:** Clearly the strata and cells are path-connected. If \( I \cap J \) is connected in the Hasse diagram, then any two of its points can be joined by a sequence of \( \mathbb{P}^1_{\lambda \mu} \)'s (see the proof of Proposition 7.6) lying in \( Z_I \cap X_J \).

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