Quantum mean-field treatment of the dynamics of a two-level atom in a simple cubic lattice

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Abstract

The mean field approximation is used to investigate the general features of the dynamics of a two-level atom in a ferromagnetic lattice close to the Curie temperature. Various analytical and numerical results are obtained. We first linearize the lattice Hamiltonian, and we derive the self-consistency equation for the order parameter of the phase transition for arbitrary direction of the magnetic field. The reduced dynamics is deduced by tracing out the degrees of freedom of the lattice, which results in the reduction of the dynamics to that of an atom in an effective spin bath whose size is equal to the size of a unit cell of the lattice. It is found that the dephasing and the excited state occupation probability may be enhanced by applying the magnetic field along some specific directions. The dependence on the change of the temperature and the magnitude of spin is also investigated. It turns out that the increase of thermal fluctuations may reduce the occupation probability of the excited state. The entanglement of two such atoms that occupy non-adjacent cells is studied and its variation in time is found to be not much sensitive to the direction of the magnetic field. Entanglement sudden death and revival is shown to occur close to the critical temperature.
I. INTRODUCTION

The theoretical study of the physical properties of quantum many-body systems is a challenging task that attracted great amount of attention. These systems are known to exhibit rather complicated and rich features that lie at the center of modern applications such as those of solid-state physics [1], spintronics [2–4], plasma physics, quantum optics, and quantum information processing [5–10]. Among the most important phenomena that occur in systems with large degrees of freedom, the processes of dissipation, decoherence and dephasing are known to be of greater relevance in either theoretical and practical perspectives. As a matter of fact, the long-standing problem of the transition from the quantum world to the classical world is speculated to be the mere result of the decoherence of the system of interest [11]. Along with that, the dissipation of energy among the constituents of the composed system plays a crucial role in transport phenomena dealt with in the emerging field of quantum thermodynamics and quantum transport. These facts are in favor of dealing with real quantum systems as open quantum systems [12], that are part of or are interacting with their surrounding in, generally speaking, quite complicated manner.

The dynamics of a central spin system interacting with its surrounding is a fundamental paradigm that has been investigated by many authors in several contexts [13–31]. The most peculiar property of the evolution in spin systems resides in the non-Markovian nature of the mutual coupling between the central system and its environment; this is manifested in the form of a backflow of information from the environment to the system, due to memory effects, in contrast to the Markovian dynamics [12]. This is the reason for which a great deal of attention has been given to the problem of characterizing the non-Markovianity in open quantum systems [32–37]. On the other hand, the problem of the dynamics of an impurity in a spin lattice is another example of great relevance. In order to facilitate the study of the dynamics in such lattices, many techniques and approximations have been developed in the course of the last decades, whose applicability and accuracy vary depending on the physical problem at hand.

It should be stressed that the Green’s function technique, which has known a huge success in the theory of many-body systems, represents one of the most important tools that we dispose [38]. An other method widely used is the mean-field approximation which has been applied to spin lattices since the early years of the development of the quantum theory
of solids, in particular in dealing with phase transitions \[39\]. Recently, the dynamics of a two-level atom impurity that is coupled to a ferromagnetic lattice through Heisenberg XY interaction at zero temperature, has been investigated using spin wave theory and the Green’s function technique \[40\]. In this paper, we generalize the investigation to higher temperatures, typically close to the Curie temperature in three dimensions, where spin wave theory ceases to apply. For this purpose, we use the Mean-field approximation \[41\] in order to account for both the thermal fluctuations and their quantum counterpart.

The manuscript is organized as follows. In Section II we introduce the model Hamiltonian for the whole system, and we use the mean-field approximation to linearize the lattice Hamiltonian. There, we derive the self-consistency equation for the order parameter of the phase transition of the lattice, and we also derive the reduced density matrix of the two-level atom. Section III is devoted to the study of the dephasing dynamics, where we analyze the evolution in time of the coherences when the atom is prepared in a state that is a linear combination of its ground and excited states, and we investigate the effect of the applied magnetic field and the variation of the temperature on the coherences. Section IV deals with the populations of the excited and ground states when the atom is initially prepared in the ground state; the dependence of the dynamics on the magnetic field and the temperature is also analyzed. In Section V the evolution of the entanglement of two identical two-level atoms, that are located in non-adjacent cells, is studied along with its variation with respect to the model parameters. We end the paper with a short conclusion.

II. MODEL

A. Hamiltonian and mean-field approximation

We consider a three dimensional cubic lattice of \( N \) spin-\( S \) particles, where every particle interacts with its neighboring spins through Heisenberg ferromagnetic interactions. The Hamiltonian of the lattice is explicitly given by

\[
H_B = - \sum_{\langle i,j \rangle} J_{ij} (S_i^x S_j^x + S_i^y S_j^y + S_i^z S_j^z) - \sum_{i=1}^{N} (h_x S_i^x + h_y S_i^y + h_z S_i^z),
\]

where \( J_{ij} > 0 \) denote the coupling constants, and \( \langle i, j \rangle \) indicates that the interactions are restricted to the neighboring spins. In addition, the effect of a constant magnetic field \( \vec{h} \)
whose components are designated by \((h_x, h_y, h_z)\) is included.

In the center of a unit cell, we place a two-level spin-\(\frac{1}{2}\) atom (see Fig. 1), whose coupling to the lattice is described by the Hamiltonian

\[
H_I = S_0^x \sum_{i=1}^{\eta} \alpha_i S_i^x + S_0^y \sum_{i=1}^{\eta} \gamma_i S_i^y + S_0^z \sum_{i=1}^{\eta} \lambda_i S_i^z. \tag{2}
\]

In the above equation, \(S_0^k\) \((k \equiv x, y, z)\) are the components of the spin operator of the atom, and the constants \(\alpha_i\), \(\gamma_i\) and \(\lambda_i\) are its coupling constants to the \(i\)'th lattice spin. Notice that the summation is meant to be restricted to the \(\eta\) spins of the unit cell, namely, \(\eta = 8\) for the simple cubic lattice. The transition energy between the ground state and the excited state of the atom is denoted by \(\omega_0\), such that the free Hamiltonian of the latter reads:

\[
H_0 = \omega_0 S_0^z. \tag{3}
\]

FIG. 1. Schematic representation of the system studied: a two-level atom located in a unit cell of a ferromagnetic simple cubic lattice interacts with its neighbors. The arrow indicates the direction of the magnetic field.

We assume that the lattice is in thermal equilibrium at temperature \(T\). In the absence of the magnetic field, the ferromagnetic interactions tend to point the spins in the same direction up to the Curie temperature of the lattice, above which the latter undergoes a phase transition to the paramagnetic phase. When an external magnetic field is applied, the ferromagnetic phase is stabilized at even higher temperatures. In general, the phase transition is well described by its order parameter \(m\). For the present model, assuming translation invariance, the order parameter is given by

\[
m = \sqrt{m_x^2 + m_y^2 + m_z^2} \tag{4}
\]
where
\[ \vec{m} = \langle \vec{S}_i \rangle. \]  

(5)

Here, the expectation value is evaluated with respect to the Gibbs equilibrium state of the lattice, that is:
\[ \langle \vec{S}_i \rangle = \text{tr}(\vec{S}_i \, e^{-\beta H_B}/Z), \]

(6)

with \( \beta \) being the inverse temperature, and \( Z \) is the partition function of the lattice:
\[ Z = \text{tr}(e^{-\beta H_B}). \]

(7)

We now invoke the mean-field approximation which enables us to linearize the lattice Hamiltonian. The essence of latter approximation consists in neglecting higher order deviations from the mean values of the spin operators. As a result, the mean-field Hamiltonian of the lattice reads:
\[ H_{mf}^B = \sum_{\langle i,j \rangle} J_{ij} m^2 - 2 \sum_{\langle i,j \rangle} J_{ij} \left( m_x S_i^x + m_y S_i^y + m_z S_i^z \right) \]
\[ - \sum_{i=1}^{N} (h_x S_i^x + h_y S_i^y + h_z S_i^z). \]

(8)

Since we assumed translation invariance, we may further write:
\[ H_{mf}^B = N \sum_{\vec{\delta}} J_{\vec{\delta}} m^2 - 2 N \sum_{\vec{\delta}} J_{\vec{\delta}} \left( m_x S_i^x + m_y S_i^y + m_z S_i^z \right) \]
\[ - N \sum_{i=1}^{N} (h_x S_i^x + h_y S_i^y + h_z S_i^z), \]

(9)

where \( \sum_{\vec{\delta}} \) is the sum with respect to all vectors \( \vec{\delta} \) linking a given lattice spin to its neighbors.

The latter Hamiltonian can be put into the compact form
\[ H_{mf}^B = N \sum_{\vec{\delta}} J_{\vec{\delta}} m^2 - \left( 2 \vec{m} \sum_{\vec{\delta}} J_{\vec{\delta}} + \vec{h} \right) \sum_{i=1}^{N} \vec{S}_i. \]

(10)

The partition function corresponding to the above Hamiltonian reads:
\[ Z = e^{-\beta N \sum_{\vec{\delta}} J_{\vec{\delta}} m^2} \prod_{i=1}^{N} \text{tr} \left( e^{\beta \left( 2 \vec{m} \sum_{\vec{\delta}} J_{\vec{\delta}} + \vec{h} \right) \vec{S}_i} \right). \]

(11)

By a unitary transformation, we can show that the partition function is calculated as:
\[ Z = \exp\{-\beta N \sum_{\tilde{\delta}} J_{\tilde{\delta}} m^2\} \]

\[ \times \left[ 1 + \frac{2 \cosh \left( \frac{\beta (S+1)}{2} \| 2 \tilde{m} \sum_{\tilde{\delta}} J_{\tilde{\delta}} + \tilde{h} \| \right) \sinh \left( \frac{\beta S}{2} \| 2 \tilde{m} \sum_{\tilde{\delta}} J_{\tilde{\delta}} + \tilde{h} \| \right) }{\sinh \left( \frac{\beta S}{2} \| 2 \tilde{m} \sum_{\tilde{\delta}} J_{\tilde{\delta}} + \tilde{h} \| \right) } \right]^N. \] (12)

The free energy per spin of the lattice is given in terms of the partition function by

\[ F = -1/(N\beta) \ln Z. \] (13)

By minimizing the free energy with respect to \( m_x, m_y \) and \( m_z \), we obtain the following self-consistency equations:

\[ m_x = \frac{h_x + 2m_x \sum_{\tilde{\delta}} J_{\tilde{\delta}}}{\Lambda} \left\{ \cosh \left[ \frac{\beta}{2} (S+1) \Lambda \right] \left[ S \cosh(\beta \Lambda S) - \coth(\frac{\beta}{2} \Lambda) \sinh(\frac{\beta}{2} S \Lambda) \right] \right\} \frac{1}{\sinh(\frac{\beta}{2} (1 + 2S) \Lambda)}. \] (14)

\[ m_y = \frac{h_y + 2m_y \sum_{\tilde{\delta}} J_{\tilde{\delta}}}{\Lambda} \left\{ \cosh \left[ \frac{\beta}{2} (S+1) \Lambda \right] \left[ S \cosh(\beta \Lambda S) - \coth(\frac{\beta}{2} \Lambda) \sinh(\frac{\beta}{2} S \Lambda) \right] \right\} \frac{1}{\sinh(\frac{\beta}{2} (1 + 2S) \Lambda)}. \] (15)

\[ m_z = \frac{h_z + 2m_z \sum_{\tilde{\delta}} J_{\tilde{\delta}}}{\Lambda} \left\{ \cosh \left[ \frac{\beta}{2} (S+1) \Lambda \right] \left[ S \cosh(\beta \Lambda S) - \coth(\frac{\beta}{2} \Lambda) \sinh(\frac{\beta}{2} S \Lambda) \right] \right\} \frac{1}{\sinh(\frac{\beta}{2} (1 + 2S) \Lambda)}. \] (16)

where for ease of notation we have introduced the quantity

\[ \Lambda = \| 2 \tilde{m} \sum_{\tilde{\delta}} J_{\tilde{\delta}} + \tilde{h} \|. \]

The calculation of the components \( m_j \) is sufficient to fix the form of the mean-field Hamiltonian. From the above equations we infer that the order parameter \( m \) may be determined self-consistently through the equation:

\[ m = \left\{ \cosh \left[ \frac{\beta}{2} (S+1) \Lambda \right] \left[ S \cosh(\beta \Lambda S) - \coth(\frac{\beta}{2} \Lambda) \sinh(\frac{\beta}{2} S \Lambda) \right] \right\} \frac{1}{\sinh(\frac{\beta}{2} (1 + 2S) \Lambda)}. \] (17)
Equation (17) enables us to derive the Curie temperature of the lattice in zero external field, namely:

\[ T_c = \frac{2S(S + 1)}{3k_B} \sum_\delta J_\delta \]  

(18)

As was pointed out earlier, in zero magnetic field the lattice exhibits ferromagnetic order below the Curie temperature; however, when the magnetic field is applied, the ferromagnetic phase persists at temperatures larger than \( T_c \). The order of these temperatures depends on the strength of the magnetic field, and cannot be established directly from Eq. (17), since for nonzero \( \vec{h} \), the latter equation has always a non-zero solution for the order parameter \( m \).

That is the reason for which we shall restrict ourselves from here on to temperatures lower than the Curie temperature \( T_c \).

### B. Reduced dynamics

The evolution of the density matrix describing the state of the atom is given by

\[ \rho(t) = \frac{1}{Z} \text{tr}_B e^{-it(H_0 + H_B + H_I)} \rho(0) \otimes e^{-\beta(H_0 + H_B + H_I)} \]  

(19)

where \( \text{tr}_B \) designates the trace with respect to the lattice degrees of freedom, and \( \rho(0) \) is the initial state of the atom which is assumed initially uncorrelated with that of the lattice.

Taking into account the mean-field Hamiltonian, we may write:

\[ \rho(t) = \frac{1}{Z} \text{tr}_B e^{-it(\omega_0 S_z^0 + \sum_{i=1}^N \vec{\Gamma}_i \vec{S}_i)} \rho(0) \otimes \beta \left( \frac{2m}{\|\vec{h}\|} + 1 \right) \sum_{i=1}^N \vec{h} \vec{S}_i e^{it(\omega_0 S_z^0 + \sum_{i=1}^N \vec{\Gamma}_i \vec{S}_i)} \]  

(20)

where we use the notations:

\[ \vec{\Gamma}_i = -\left( \frac{2m}{\|\vec{h}\|} + 1 \right) \vec{h} + \vec{K}_i, \]  

(21)

\[ \vec{K}_i = (\alpha_i S_x^0, \gamma_i S_y^0, \lambda_i S_z^0). \]  

(22)

When the coupling of the atom is restricted to its nearest neighbors, we should have:

\[ K_i = \begin{cases} (\alpha_i S_x^0, \gamma_i S_y^0, \lambda_i S_z^0) & \text{for } i \leq \eta, \\ 0 & \text{for } i > \eta, \end{cases} \]  

(23)

where \( \alpha, \gamma \) and \( \lambda \) are the nearest-neighbor coupling constants. This implies that

\[ \vec{\Gamma}_i = \begin{cases} \vec{\Gamma} = -\left( \frac{2m}{\|\vec{h}\|} + 1 \right) \vec{h} + \vec{K} & \text{for } i \leq \eta, \\ \left( \frac{2m}{\|\vec{h}\|} + 1 \right) \vec{h} & \text{for } i > \eta. \end{cases} \]  

(24)
Afterward, by taking the trace in Eq. (20), we end up with:

$$\rho(t) = \frac{1}{Z} \text{tr}_B e^{-it\left(\omega_0 S^z_0 + \vec{r} \sum_{i=1}^{\eta} \vec{s}_i\right)} \rho(0) \otimes e^{\beta \left(2m \sum_{j} J_{\delta} + \|\vec{h}\| + 1\right)} \sum_{i=1}^{\eta} \vec{h} \vec{s}_i \ e^{i\left(\omega_0 S^z_0 + \vec{r} \sum_{i=1}^{\eta} \vec{s}_i\right)}$$

with

$$\tilde{Z} = \exp\{-\beta \eta \sum_{\delta} J_{\delta} m^2\}$$

$$\times \left[1 + \frac{2 \cosh\left[\frac{\beta(S+1)}{2} \left(2m \sum_{\delta} J_{\delta} + \|\vec{h}\|\right)\right] \sinh\left[\frac{\beta S}{2} \left(2m \sum_{\delta} J_{\delta} + \|\vec{h}\|\right)\right]}{\sinh\left[\frac{\beta}{2} \left(2m \sum_{\delta} J_{\delta} + \|\vec{h}\|\right)\right]} \right]^\eta,$$

whereas $\text{tr}_B$ denotes the trace with respect to the spins in the unit cell where the atom is located.

Consequently, the atom behaves as if it is interacting with an effective spin bath whose size is precisely the size of the aforementioned cell. The overall effect of the total lattice on the atom is encoded in the order parameter $m$, along with the coupling constants $J_\delta$, and the Curie temperature $T_c$. In the next section, we investigate the variation of the reduced density matrix and its dependence on the relevant parameters of the model, in particular the magnetic field and the temperature.

III. PURE DEPHASING

In the particular case when $\alpha = \gamma = 0$, which corresponds to the Ising interaction Hamiltonian

$$H_I = \lambda S^z_0 \sum_{i=1}^{\eta} S^z_i,$$

the central atom does not exchange energy with the lattice. The diagonal elements of its density matrix do not evolve in the course of the time, but preserve their initial values; this implies that for the present parameters of the model, both the ground and the excited states $|g\rangle$ and $|e\rangle$ of the atom are not affected by the lattice. The coherences in general do change with time, leading to pure dephasing dynamics of the atom. We shall mainly be interested in the dynamics in a simple cubic lattice, with nearest-neighbor coupling for which $\eta = 8$, and the Curie temperature reads

$$T_c = \frac{4S(S+1)J}{k_B}$$
FIG. 2. The evolution of the elements of the reduced density matrix corresponding to the initial state $\phi(0)$. For (a) and (b): $h_z=0.5\ J, h_x = h_y = 0$, $T = 2J$. For (c) and (d): $h_z=0.5\ J, h_x = h_y = 0$, $T = 2.5J$. The other parameters are $S = 1/2$, $\omega_0 = 2J$, $\alpha = \gamma = 0$, $\lambda = J$.

where $J$ refers to the nearest-neighbor coupling constant. Let us begin with a two-level atom that is embedded in a lattice of spin-$\frac{1}{2}$ atoms, that is $S = S_0 = \frac{1}{2}$. It can be shown that the off-diagonal element of the reduced density matrix evolves according to:

$$\rho_{12}(t) = \rho_{12}(0)e^{-i\omega_0 t}F(t)\eta$$  \hspace{1cm} (29)$$

with

$$F(t) = \frac{1}{2 \cosh\left(\frac{\beta}{2}(12Jm + ||\vec{h}||)\right)} \text{tr} \left[ \left( \cos\left(\frac{t\Lambda_-}{2}\right) + \frac{2i}{\Lambda_-}\sin\left(\frac{t\Lambda_-}{2}\right)H_- \right) \times \left( \cosh(\beta\Lambda/2) + \frac{2}{\Lambda}\sinh(\beta\Lambda/2)H_b \right) \left( \cos\left(\frac{t\Lambda_+}{2}\right) - \frac{2i}{\Lambda_+}\sin\left(\frac{t\Lambda_+}{2}\right)H_+ \right) \right]$$ \hspace{1cm} (30)$$
where the operators $H_b$ and $H_{\pm}$ along with the quantities $\Lambda_{\pm}$ are given by:

$$H_b = \left( \frac{12Jm}{\|\vec{h}\|} + 1 \right) \vec{h} \vec{S}, \quad (31)$$

$$H_{\pm} = \left( \frac{12Jm}{\|\vec{h}\|} + 1 \right) \vec{h} \vec{S} \pm \frac{\lambda}{2} S_z, \quad (32)$$

$$\Lambda_{\pm} = \sqrt{(12mJ + \|\vec{h}\|)^2 + \frac{\lambda^2}{4} \pm \lambda \left( \frac{12Jm}{\|\vec{h}\|} + 1 \right)}. \quad (33)$$

FIG. 3. The evolution of the elements of the reduced density matrix corresponding to the initial state $\phi(0)$. For (a) and (b): $h_x=0.5 \ J$, $h_y = 0$, $\omega_0 = 2J$. For (c) and (d): $h_x=0.5 \ J$, $h_z = h_y = 0$, $\omega_0 = J$. The other parameters are $S = 1/2$, $T = 2J$, $\alpha = \gamma = 0$, $\lambda = J$.

In Fig. 2 we display the real and the imaginary parts of $\rho_{12}$ when the atom is initially prepared in the state

$$\phi(0) = \frac{1}{\sqrt{2}} (|g\rangle + |e\rangle). \quad (34)$$

The plots correspond to a lattice with $S = \frac{1}{2}$ for uniform coupling of the atom at different temperatures. The magnetic field in this case is chosen in the $z$ direction. It can be seen that the coherence exhibits oscillatory decay and revival, which indicates the significance of the lattice memory effects on the dynamics of the atom that persist even at high temperature.
FIG. 4. The variation of the elements of the reduced density matrix corresponding to the initial state $\phi(0)$. For (a) and (b): $h_x = h_y = h_z = J/2\sqrt{3}$, $T = 2J$. For (c) and (d): $h_x = h_y = h_z = J/2\sqrt{3}$, $T = 2.5J$. The other parameters are $S = 1/2$, $\omega_0 = 2J$, $\alpha = \gamma = 0$, $\lambda = J$.

FIG. 5. The variation of the elements of the reduced density matrix corresponding to the initial state $\phi(0)$ for $h_x = h_y = J/2\sqrt{2}$, $T = 2J$, $S = 1/2$, $\omega_0 = 2J$, $h_z = 0$, $\alpha = \gamma = 0$, $\lambda = J$.

where the thermal fluctuations are much important. We recall that the state of a two-level atom may be described by the density matrix:

$$\rho(t) = \frac{1}{2} \begin{pmatrix} 1 + r_3(t) & r_1(t) - i r_2(t) \\ r_1(t) + i r_2(t) & 1 - r_3(t) \end{pmatrix},$$

where $(r_1, r_2, r_3)$ are the components of the Bloch vector $\vec{r}$. The latter satisfies the condition
$|\vec{r}| \leq 1$, with the equality being verified if the state is pure. Within this representation, the expectation values of the components of the spin operator of the atom at any moment $t$ are

$$\langle S_0^x(t) \rangle = \frac{r_1(t)}{2}, \quad \langle S_0^y(t) \rangle = \frac{r_2(t)}{2}, \quad \text{and} \quad \langle S_0^z(t) \rangle = \frac{r_3(t)}{2}.$$  

Hence, the initial state $\phi(0)$ evolves in the subsequent times to a state for which

$$\langle S_0^x(t) \rangle = \frac{1}{2} [\cos(\omega_0 t) \text{Re} F(t)^\eta + \sin(\omega_0 t) \text{Im} F(t)^\eta], \quad (36)$$

$$\langle S_0^y(t) \rangle = \frac{1}{2} [\sin(\omega_0 t) \text{Re} F(t)^\eta - \cos(\omega_0 t) \text{Im} F(t)^\eta], \quad (37)$$

$$\langle S_0^z(t) \rangle = 0. \quad (38)$$

We notice that the evolved state approaches in multiple instances the pure initial state. However, at high temperature the lapses of the time in which the atom evolves close to the fully mixed state are much larger. An other point worth observing is that the state of the atom evolves periodically close to the pure states $\psi = \frac{1}{\sqrt{2}} (|g\rangle \pm i|e\rangle)$. It turns out that for stronger values of the magnetic field, the periods of the oscillations decrease, and the decay and revival is much faster for both low and high temperatures.

A quite different situation occurs when the magnetic field points in the $x$ direction, see figure 3. Indeed, in this case the frequencies of oscillations have significantly decreased, and it turns out that the dynamics is much less sensitive to the variation of the temperature. Increasing the latter does not lead to a noticeable decrease of the off-diagonal elements. In order to check the significance of thermal fluctuations, we also plot the variation of the matrix elements for a smaller value of the proper energy $\omega_0$, leaving the other parameters unchanged. We find that the bar effect resides in a decrease in the frequency of the collapse and revival of the coherence; again, the dynamics is not affected much by the variation of the temperature. When the magnetic field points along the direction (1,1,1), we notice that the decoherence is again more sensitive to the temperature, but the revival of the coherence takes larger time to occur as displayed in figure 4. This leads to the conclusion that as long as the component of the magnetic field along the $z$-direction is nonzero, the dynamics displays a strong dependence on the temperature. To verify this assertion, we show in figure 5 the case when the magnetic field points within the $x−y$ plane. It is clear that this situation is quite similar to that corresponding to a magnetic field pointing in the $x$ direction.
For a two-level atom in a lattice whose atoms have spin $S = 1$, we find that

$$F(t) = \frac{1}{1 + 2 \cosh(\beta(12mJ + |\mathbf{h}|))} \text{tr} \left[ \left( 1 + \frac{1}{\Lambda_-} (\cos(t\Lambda_-) - 1)H_- + \frac{i}{\Lambda_-^2} \sin(t\Lambda_-)H_-^2 \right) \left( 1 + \frac{1}{\Lambda_+} (\cos(t\Lambda_+) - 1)H_+ + \frac{i}{\Lambda_+^2} \sin(t\Lambda_+)H_+^2 \right) \right]$$

(39)

Figures 6 and 7 display the evolution in time of the coherences when the lattice is composed of spin-1 particles for the directions of the magnetic fields corresponding to figures 2 and 4 respectively, for comparison with the case of spin-$\frac{1}{2}$. We clearly notice that as the temperature gets closer to the Curie temperature of the lattice, the loss of coherence is much important when the magnetic field is applied along the $z$-direction; actually for temperatures high enough the state of the atom becomes completely mixed during relatively long intervals of time. Despite this, there occurs periodic revival of the coherence, which
FIG. 7. The evolution of the elements of the reduced density matrix corresponding to the initial state \( \phi(0) \). For (a) and (b): \( h_x=0.5 \, J, \, h_z = h_y = 0, \, \omega_0 = 2J \). For (c) and (d): \( h_x=0.5 \, J, \, h_z = h_y = 0, \, \omega_0 = J \). The other parameters are \( S = 1, \, T = 7J, \, \alpha = \gamma = 0, \, \lambda = J \).

demonstrates, once again, the significance of the purely quantum memory effects of the lattice which are responsible of the backflow of information from the lattice to the atom even when the thermal fluctuations dominate. However, when the magnetic field points along the \( x \)-direction, the decay of the coherences is much very slow and they take on relatively large values periodically in the course of the time.

In order to get more insights into the nature of the dynamics, we introduce, in analogy with the decay rate \[12, 40\], the so-called dephasing rate
\[
\kappa(t) = -\frac{1}{|F(t)^\eta|} \frac{d}{dt} |F(t)^\eta|.
\]
(40)

From figure 8 we see that the increase of temperature leads to a significant increase of the decay rate when the magnetic field is along the \( z \)-axis for spin-\( 1/2 \) but the periods of oscillation remains unchanged. This is to be contrasted to the case where the magnetic field is pointing to the \( x \)-direction, where we see that the amplitude does not increase much but there occurs a slight decrease of the frequency of oscillation. For \( S = 1 \), when the magnetic field is directed along the \( z \)-axis, we see that the decay rate is not much affected by the change
FIG. 8. The dephasing rate as a function of time for different values of the temperature and the magnetic field. For (a): $S = \frac{1}{2}$, $h_z = 0.5J$ (slowly oscillating curve), $h_x = 0.5$ (quickly oscillating curve), and $T = 2J$. For (b): $S = \frac{1}{2}$, $h_z = 0.5J$ (slowly oscillating curve), $h_x = 0.5$ (quickly oscillating curve) and $T = 2.5J$. For (c): $S = 1$, $h_z = 0.5J$ (slowly oscillating curve), $h_x = 0.5$ (quickly oscillating curve) and $T = 7J$. For (d): $S = 1$, $h_z = 0.5J$ (slowly oscillating curve), $h_x = 0.5$ (quickly oscillating curve) and $T = 7.8J$. The other parameters are $h_y = 0$, $\alpha = \gamma = 0$ and $\lambda = J$.

of the temperature, whereas the frequency of oscillations decreases when the magnetic field points to the $x$-direction. Nevertheless, the non-Markovian character is clearly noticed from the oscillation of the decay rates between positive and negative values. This manifestation is more noticeable when the magnetic field has no component along the $z$ axis.

IV. ENERGY LEVEL TRANSITION

Consider now the case where the atom is initially prepared in its ground state

$$\phi(0) = |g\rangle,$$  \hspace{1cm} (41)
FIG. 9. The elements of the reduced density matrix corresponding to the initial ground state for:
\( h_z = 0.5J, h_x = h_y = 0, T = 2J \). The other parameters are \( S = 1/2, \omega_0 = 2J, \alpha = \gamma = \lambda = J \).

FIG. 10. The diagonal elements of the reduced density matrix corresponding to the initial ground
state for: \( h_z = 0.5J, h_x = h_y = 0, T = 2.5J \). The other parameters are \( S = 1/2, \omega_0 = 2J, \alpha = \gamma = \lambda = J \).

and is isotropically coupled to the atoms of the lattice, that is, we take \( \alpha = \delta = \gamma \). We shall
investigate the effect of the relevant parameters of the model on the exited state population,
i.e., we shall study the probability of transition of the atom to the excited state. The latter
is best described by the matrix element \( \rho_{11} \) or equivalently by the component \( r_3 \). The
transition occurs as a result of the interaction of the atom with the spins of the lattice,
leading to energy exchange between the two systems, which affects both the diagonal and
the off-diagonal elements of the reduced density matrix.

When the magnetic field points along the z-direction, the excited state population is given
by

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FIG. 11. The evolution in time of the element $\rho_{11}$ associated with the initial ground state. For (a): $h_z = 0.5J$, $h_x = h_y = 0$, $T = 5J$. For (b): $h_z = 0.5J$, $h_x = h_y = 0$, $T = 7J$. The other parameters are $S = 1$, $\omega_0 = 2J$, $\alpha = \gamma = \lambda = J$.

FIG. 12. The evolution in time of the element of the reduced density matrix associated with the initial ground state for: $h_x = 0.5J$, $h_z = h_y = 0$, $T = 2J$. The other parameters are $S = 1/2$, $\omega_0 = 2J$, $\alpha = \gamma = \lambda = J$.

$$
\rho_{11}(t) = \frac{1}{Z} \sum_{j=0,1/2} \eta^S \alpha^n \nu(j, \eta; S) \sum_{\ell=-j} \left( j(j+1) - \ell(\ell-1) \right) e^{\beta(12Jm+h_z)\ell} \\
\times \sin^2 \left[ \frac{1}{2} \sqrt{M_-}(j, \ell) \right] / \sqrt{M_-}(j, \ell) 
$$

(42)
where
\[ M_-(j, \ell) = \alpha^2(j(j + 1) - \ell(\ell - 1)) + \frac{1}{4}[2(h_z + 12Jm + \omega_0) + \alpha(2\ell - 1)]^2, \] (43)

and
\[ \tilde{Z} = \sum_{j=0}^{\eta S} \nu(j, \eta; S) \sum_{\ell=-j}^{j} e^{\beta(12Jm+h_z)\ell}. \] (44)

The off-diagonal element remains always zero, i.e. \( \rho_{12}(t) = 0 \). Notice that in the above equation \( \nu(j, \eta; S) \) denotes the degeneracy corresponding to the angular momentum \( j \) resulting from the addition of the spin operators of the unit cell. It is given by [41]:
\[
\nu(j, \eta; S) = \sum_{L_{-S+2}, L_{-S+3}, \ldots, L_{S}=0}^{\eta} \eta! \left( \frac{(2S-1)\eta + 2j + 1 - \sum_{\rho=2}^{2S} (2\rho - 1)L_{-s+\rho}}{S\eta + j + 1 - \sum_{\rho=2}^{2S} \rho L_{-s+\rho}} \right) \times \left[ (S\eta + j - \sum_{\rho=2}^{2S} \rho L_{-s+\rho})! (1 - S)\eta - j + \sum_{\rho=2}^{2S} (\rho - 1)L_{-s+\rho})! \right]^{-1} \] (45)

where the integers \( L_i \) assume values ranging from 0 to \( \eta \).

In figures 9 and 10, we display the variation in time of the elements of the reduced density matrix when the magnetic field is directed along the \( z \)-axis. We notice that there exists a
non-zero, yet small, probability that the atom occupies its excited state, which slightly increases with the temperature. It turns out that changing the strength of the magnetic field does not affect much the transition probability. On the other hand, the state of the atom remains mixed at all moments. The same occurs when the lattice is composed of spin-1 particles, but in this case we observe a slight decrease of the transition probability as depicted in Fig. 11 in contrast to one might expect at higher temperatures.

The probability of transition significantly increases when the magnetic field points in the $x$-direction, as illustrated in Figs. 12-14 for a lattice of spin-1/2. In this case the periods of oscillations are longer as compared with those in figures 9 and 10. Increasing the strength of the magnetic field induces slightly larger probability occupation of the excited state. This behavior is also produced when the magnetic field points in the $y$ direction. However, in contrast to the case where the magnetic field is parallel to the $z$-axis, we see that here, as the temperature increases, the occupation probability decreases. The real part of $\rho_{12}$ remains negative, while its imaginary part oscillates about zero. The opposite is true when the magnetic field points in the $y$-direction. Generally speaking, it is found that the more the magnetic field deviates from the $x$-$y$ plane the smaller the occupation probability is.
Let us consider now the instance in which the initial state of the atom is the state:

\[ \phi_s(0) = \frac{1}{\sqrt{2}}(|g⟩ + |e⟩). \] (46)

The initial occupation probability is the same for both the exited and the ground state. We find that the atom is most likely to be found in the excited state (see figure 15) unless the magnetic field is pointing in the \( y \) direction where the excited state occupation probability may decrease compared to its initial value as illustrated in Fig. 16. Hence the atom periodically undergoes transitions between these states, which persist at longer times due to the non-Markovian nature of the dynamics.

![Figure 15](image1.png)

FIG. 15. The excited state population as a function of time corresponding to the initial state \( \phi_s(0) \) with: (a) \( h_z = 0.5J \), \( h_x = h_y = 0 \), and (b) \( h_x = 0.5J \), \( h_x = h_y = 0 \). The other parameters are \( T = 2J \), \( S = 1/2 \), \( \omega_0 = 2J \), \( \alpha = \gamma = \lambda = J \).

![Figure 16](image2.png)

FIG. 16. The excited state population as a function of time corresponding to the initial state \( \phi_s(0) \) with: (a) \( h_y = 0.5J \), \( T = 2J \), \( h_x = h_z = 0 \), and (b) \( h_y = 0.5J \), \( h_x = h_z = 0 \), \( T = 2.5J \). The other parameters are \( S = 1/2 \), \( \omega_0 = 2J \), \( \alpha = \gamma = \lambda = J \).

To assess the effect of the spin magnitude when the magnetic field points in the \( x \)-direction, we suppose that the states \( |g⟩ \) and \( |e⟩ \) are degenerate, which amounts to choosing
\( \omega_0 = 0 \). When the central system is initially prepared in the state \( |g\rangle \), then the evolution in time of the reduced density matrix element \( \rho_{11} \) is given by

\[
\rho_{11}(t) = \frac{1}{2} \left[ 1 - \text{Re} \Psi(t) \right],
\]

with

\[
\Psi(t) = \frac{1}{Z} \sum_{j=0}^{\eta} \sum_{\ell=-j}^{j} \nu(j, \eta; S) e^{i(\eta + \beta \ell)(12Jm + h_x)} \times \left[ \cos \left( \frac{t}{2} \sqrt{M_+(j, \ell)} \right) - i \frac{(12Jm + h_x) + \alpha(\ell + \frac{1}{2})}{\sqrt{M_+(j, \ell)}} \sin \left( \frac{t}{2} \sqrt{M_+(j, \ell)} \right) \right] \times \left[ \cos \left( \frac{t}{2} \sqrt{M_-(j, \ell)} \right) - i \frac{(12Jm + h_x) + \alpha(\ell - \frac{1}{2})}{\sqrt{M_-(j, \ell)}} \sin \left( \frac{t}{2} \sqrt{M_-(j, \ell)} \right) \right]
\]

where \( M_+(j, \ell) \) is defined as above, and

\[
M_+(j, \ell) = \alpha^2(j(j+1) - \ell(\ell+1)) + \frac{1}{4} \left[ 2(h_x + 12Jm + \omega_0) + \alpha(2\ell + 1) \right]^2.
\]

It is clearly seen from Figure 17 that the system evolves at high temperatures to a state

![Figure 17](image)

**FIG. 17.** The excited state population as a function of time corresponding to the initial ground state \( \phi(0) \) with: (a) \( h_x = 0.5J, T = 5J, h_y = h_z = 0 \), and (b) \( h_x = 0.5J, h_y = h_z = 0, T = 7J \). The other parameters are \( S = 1, \omega_0 = 0, \alpha = \gamma = \lambda = J \).

where the ground and the excited state have equal statistical weights. This occurs in much faster rate as compared to the evolution in a lattice of spin-1/2 particles.

**V. ENTANGLEMENT OF TWO ATOMS LOCATED AT NON-ADJACENT CELLS**

Let us consider two two-level atoms occupying the centers of non-adjacent unit cells in the lattice, such that each cell does not share any spin with the other one. In such situation,
the effect of the lattice on either atom is reduced to the coupling to its own local unit cell. 
We further assume that the two-level systems do not interact with each other and that they are initially prepared in the Bell maximally entangled state:

\( |\Phi\rangle = \frac{1}{\sqrt{2}} (|gg\rangle + |ee\rangle). \) (51)

The initial density matrix corresponding to the latter state may be written as

\[ \rho(0) \equiv |\Phi\rangle \langle \Phi| = \frac{1}{4} \mathbb{I}_4 + \frac{1}{4} \sigma_z \otimes \sigma_z + \frac{1}{2} [\sigma_+ \otimes \sigma_+ + \sigma_- \otimes \sigma_-]. \] (52)

When the magnetic field points in the \( z \)-direction, we find after tracing out the lattice degrees of freedom that the density matrix evolves according to

\[ \rho(t) = \begin{pmatrix} 
\frac{1}{4} + \frac{\Xi_+(t)^2}{4} & 0 & 0 & \frac{\Psi(t)^2}{2} \\
0 & \frac{1}{4} + \frac{\Xi_+(t)\Xi_-(t)}{4} & 0 & 0 \\
0 & 0 & \frac{1}{4} + \frac{\Xi_+(t)\Xi_-(t)}{4} & 0 \\
\frac{\Psi^*(t)^2}{2} & 0 & 0 & \frac{1}{4} + \frac{\Xi_-(t)^2}{4} 
\end{pmatrix}, \] (53)

where

\[ \Xi_{\pm}(t) = \mp 1 \pm \frac{2}{Z} \sum_{j=0}^{n_S} \nu(j, \eta; S) \sum_{\ell=-j}^{j} e^{3\ell(12Jm+h_z)} \]

\[ \times \left\{ \cos \left[ \frac{\ell}{2} \sqrt{M_{\pm}(j, \ell)} \right]^2 - \frac{(12Jm + h_z) + \alpha(\ell \pm \frac{1}{2})}{M_{\pm}(j, \ell)} \sin \left[ \frac{\ell}{2} \sqrt{M_{\pm}(j, \ell)} \right]^2 \right\}. \] (54)

The amount of entanglement of a state \( \rho \) (whether mixed or not) of a bipartite two-level system may be quantified using the concurrence \( C(\rho) \) \cite{42, 43}, which is defined by

\[ C(\rho) = \max \{0, \sqrt{\nu_1} - \sqrt{\nu_2} - \sqrt{\nu_3} - \sqrt{\nu_4}\}, \] (55)

where \( \nu_1, \nu_2, \nu_3 \) and \( \nu_4 \) are the eigenvalues, in descending order, of the operator

\[ \tilde{\rho}(t) = \rho(t)(\sigma_y \otimes \sigma_y)\rho^*(t)(\sigma_y \otimes \sigma_y). \] (56)

The concurrence takes on values ranging from zero for separable states to one for maximally entangled states.

In figures 18-19 we display the variation in time of the concurrence for different values of the magnetic field and the temperature for a lattice of spin-1/2. It can be seen that close to the critical temperature there occurs a sudden death of the entanglement for both directions.
FIG. 18. The concurrence as a function of time corresponding to the initial state $|\Phi\rangle$ with: (a) $h_z = 0.5J$, $T = 2J$, $h_x = h_y = 0$, and (b) $h_z = 0.5J$, $h_x = h_y = 0$, $T = 2.5J$. The other parameters are $S = 1/2$, $\omega_0 = 2J$, $\alpha = \gamma = \lambda = J$.

FIG. 19. The concurrence as a function of time corresponding to the initial state $|\Phi\rangle$ with: (a) $h_x = 0.5J$, $T = 2J$, $h_y = h_z = 0$, and (b) $h_x = 0.5J$, $h_y = h_z = 0$, $T = 2.5J$. The other parameters are $S = 1/2$, $\omega_0 = 2J$, $\alpha = \gamma = \lambda = J$.

of the magnetic field. The decay and revival of the entanglement occur at comparable rates. Nevertheless, we can see that while the evolution of the entanglement is less dependent on the direction of the magnetic field it is very sensitive to the temperature, in contrast to the dephasing and the population evolution as discussed above. Again, the complete loss of entanglement and its revival is mainly due to the competition between thermal fluctuations and the memory effects of the lattice responsible for the back flow of information from the latter to the central atoms. For a lattice of spin-1 particles, we see from figure 20 that the decay of entanglement becomes much faster and its revival takes much longer times when the magnetic field points in the $z$-direction. In case where $\omega_0 = 0$, and the magnetic field
points in the $x$-direction, the reduced density matrix is calculated as:

$$
\rho(t) = \begin{pmatrix}
\frac{1+\text{Re}\Psi(t)^2}{4} & \frac{\Delta(t)-4\text{Im}\Psi(t)^2}{16} & \frac{\Xi_+(t)^2\Xi_-(t)^2-\text{Re}\Psi(t)^2}{16} & \frac{\Xi_+(t)^2\Xi_-(t)^2+\text{Re}\Psi(t)^2}{4} \\
\frac{\Delta(t)+4\text{Im}\Psi(t)^2}{16} & \frac{1-\text{Re}\Psi(t)^2}{4} & \frac{1-\text{Re}\Psi(t)^2}{4} & \frac{\Delta(t)+4\text{Im}\Psi(t)^2}{16} \\
\frac{\Delta(t)+4\text{Im}\Psi(t)^2}{16} & \frac{\Xi_+(t)^2\Xi_-(t)^2-\text{Re}\Psi(t)^2}{4} & \frac{\Delta(t)-4\text{Im}\Psi(t)^2}{16} & \frac{\Delta(t)-4\text{Im}\Psi(t)^2}{16} \\
\frac{\Xi_+(t)^2\Xi_-(t)^2+\text{Re}\Psi(t)^2}{4} & \frac{1-\text{Re}\Psi(t)^2}{4} & \frac{1-\text{Re}\Psi(t)^2}{4} & \frac{\Delta(t)+4\text{Im}\Psi(t)^2}{16}
\end{pmatrix},
$$

where $\Delta(t) = \Xi_+(t)^2 - \Xi_-(t)^2$. It turns out (see Fig. 21) that the entanglement displays here very similar behavior as that corresponding to a magnetic field that is directed along the $z$-direction, which confirms once again the results obtained above for spin-1/2.

FIG. 20. The concurrence as a function of time corresponding to the initial state $|\Phi\rangle$ with: (a) $h_z = 0.5J$, $T = 5J$, $h_x = h_z = 0$, and (b) $h_z = 0.5J$, $h_x = h_z = 0$, $T = 7J$. The other parameters are $S = 1$, $\omega_0 = 2J$, $\alpha = \gamma = \lambda = J$.

FIG. 21. The concurrence as a function of time corresponding to the initial state $|\Phi\rangle$ for $h_x = 0.5J$, $T = 7J$, $h_y = h_z = 0$, $S = 1$, $\omega_0 = 0$, $\alpha = \gamma = \lambda = J$. 

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VI. CONCLUSION

In conclusion we employed the meant-field approximation to study the dynamics of a two level atom that is coupled to the spins in a ferromagnetic lattice. This approximation along with the translation invariance of the lattice enabled us to reduce the $N+1$ body problem to that of $\eta + 1$ problem where $\eta$ is the number of spins in a unit cell. It is found that depending on the orientation of the applied magnetic field various features of the dynamics may occur. Specifically, for the dephasing dynamics, when the magnetic field points in the $z$ direction, the revival of the coherences is less important compared to when the field points in the $x$ or $y$ directions. In the former case the coherence are very sensitive to the variation of the temperature, whereas in the latter case, they are quite robust with respect to thermal fluctuations. When the lattice spins are equal to one, we found a clear increase of the rate of the loss of the coherence when the magnetic field is along the $z$-axis as compared to the spin one-half case. When the atom is prepared in its ground state, a magnetic field along the $z$ direction produces a small probability of transition to the excited state. This transition is rather enhanced by the thermal fluctuations. On the contrary, these fluctuations reduce the occupation probability of the excited state which turns out to be quite important when the magnetic field lies in the $x$-$y$ plane. The concurrence is used to quantify the entanglement of two atoms located at non-adjacent cells that are initially prepared in the Bell maximally entangled state. Its evolution is rather the same for all directions of the magnetic field. However it displays entanglement sudden death and revival close to the Currie temperature in either direction. All the obtained results show clearly the importance of memory effects for temperatures bellow the Curie temperature where the dynamics display non-Markovian features.

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