ON THE WEAK SOLUTION $u \in C_{1-\alpha}(I,E)$ OF A FRACTIONAL–ORDER WEIGHTED CAUCHY TYPE PROBLEM IN REFLEXIVE BANACH SPACES

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Abstract. In this paper, we study the existence of a weak solution $u \in C_{1-\alpha}(I,E)$ of the nonlinear weighted Cauchy type problem of fractional-order.

1. Introduction

In this paper, we study the existence of solutions, in the Banach space $C_{1-\alpha}[I,E]$, for the nonlinear weighted Cauchy-type problem of the following type

$$
\left\{
\begin{array}{l}
D^\alpha u(t) = f(t,u(t)), \quad t > 0, \quad \alpha \in (0,1) \\
t^{1-\alpha} u(t)|_{t=0} = b, \quad b > 0.
\end{array}
\right.
$$

This problem has been studied by many authors for example in ([4]), the author supposed that the function $f(t,u)$ is continuous on $R^+ \times R$, $|f(t,u)| \leq t^\mu e^{-\sigma t} \psi(t)|u|^m$, $\mu \geq 0$, $m > 1$, $\sigma > 0$, $\psi(t)$ is a continuous function on $R^+$. Also; In ([2]–[3]) the author proved the existence of $L_1$ and $L_p$ solution of the same problem respectively.

2. Preliminaries

Let $L_1(I)$ be the space of Lebesgue integrable functions on the interval $I = [0,1]$. Unless otherwise stated, $E$ is a reflexive Banach space with norm $\|\cdot\|$ and dual $E^*$. We will denote by $E_w$ the space $E$ endowed with the weak topology $\sigma(E,E^*)$ and denote by $C(I,E)$ the space of continuous functions defined on $I = [0,1]$ with norm

$$
\|u\|_C = \sup_{t\in[0,1]} \|u(t)\|.
$$

Also; define the space $C_{1-\alpha}(I,E)$ by

$$
C_{1-\alpha}(I,E) = \{ u : t^{1-\alpha} u(t) \text{ is continuous on } I = [0,1] \}.
$$

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with norm
\[ ||u||_{C^{1-\alpha}} = ||t^{1-\alpha}u||_{C}. \]

We recall that the fractional integral operator of order \( \alpha > 0 \) with left-hand point \( a \) is defined by (see [9], [14], [15] and [20])
\[
I^\alpha_a u(t) = \int_a^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} u(s) \, ds.
\]

**DEFINITIONS.** Let \( E \) be a Banach space and let \( u : I \to E \). Then

1. \( u(,) \) is said to be weakly continuous (measurable) at \( t_0 \in I \) if for every \( \varphi \in E^* \) we have \( \varphi(u(,)) \) continuous (measurable) at \( t_0 \).

2. A function \( h : E \to E \) is said to be weakly sequentially continuous if \( h \) takes weakly convergent sequences in \( E \) to weakly convergent sequences in \( E \).

Note that:

1. If \( u \) is weakly continuous on \( I \), then \( u \) is strongly measurable (see [7]), hence weakly measurable.

2. In reflexive Banach spaces weakly measurable functions are Pettis integrable (see [1], [7] and [13] for the definition) if and only if \( \varphi(u(,)) \) is Lebesgue integrable on \( I \) for every \( \varphi \in E^* \).

Now, we present some auxiliary results that will be needed in this paper. Firstly, we state O’Regan fixed point theorem ([12]).

**THEOREM 2.1.** Let \( E \) be a Banach space with \( Q \) a nonempty, bounded, closed, convex, equicontinuous subset of \( C[I,E] \). Suppose \( T : Q \to Q \) is weakly sequentially continuous and assume \( TQ(t) \) is weakly relatively compact in \( E \) for each \( t \in I \), holds. Then the operator \( T \) has a fixed point in \( Q \).

The following theorems can be found in [5], [22] and [10] respectively:

**THEOREM 2.2.** (Dominated convergence theorem for Pettis integral) Let \( u : I \to E \). Suppose there is a sequence \( (u_n) \) of Pettis integrable functions from \( I \) into \( E \) such that \( \lim_{n \to \infty} \varphi(u_n) = \varphi(u) \) a.e. for \( \varphi \in E^* \). If there is a scalar function \( \psi \in L_1(I) \) with \( ||u_n(\cdot)|| < \psi(\cdot) \) a.e. for all \( n \), then \( u \) is Pettis integrable and
\[
\int_I u_n(s) \, ds \to \int_I u(s) \, ds \quad \text{weakly} \quad \forall \ t \in I.
\]

**THEOREM 2.3.** A subset of a reflexive Banach space is weakly compact if and only if it is closed in the weak topology and bounded in the norm topology.

**THEOREM 2.4.** Let \( Q \) be a weakly compact subset of \( C[I,E] \). Then \( Q(t) \) is weakly compact subset of \( E \) for each \( t \in I \).
Finally, we state some results which is an immediate consequence of the Hahn-Banach theorem.

**Theorem 2.5.** Let $E$ be a normed space with $u_0 \neq 0$. then there exists a $\varphi \in E^*$ with $\|\varphi\| = 1$ and $\varphi(u_0) = |u_0|$. 

**Theorem 2.6.** If $u_0 \in E$ is such that $\varphi(u_0) = 0$ for every $\varphi \in E^*$, then $u_0 = 0$.

Now consider the fractional-order integral equation

$$u(t) = b t^{\alpha - 1} + \int_{0}^{t} \frac{(t-s)^{\alpha - 1}}{\Gamma(\alpha)} f(s, u(s)) \, ds, \quad t \in [0, 1].$$

(2) For each continuous function $f$ and sequentially continuous for each $y \in E$, the fractional order Pettis integral of $\chi_{[0,1]}$ is a function of bounded variation. Thus, according to Lemma 3.2, for any $r > 0$, the weak closure of the range of $f(I \times B_r)$ is weakly compact in $E$ (or equivalently; there exists an $M_r$ such that $\|f(t, u)\| \leq M_r$ for all $(t, u) \in I \times B_r$).

**Example 2.1.** Let $T$ be the interval $[0, 1]$ and define $f : T \to L^\infty(T)$ by $f(t) = \chi_{[0,1]}$. This function is weakly measurable and for each $\varphi \in L^\infty_*$, we have $\varphi f \in L_1$ (each $\varphi f$ is a function of bounded variation). Thus, according to Lemma 3.2, $I^\alpha f$ exists. Also, the fractional order Pettis integral of $f$ exists see [6, 16, 18].

**Definition 2.1.** By a weak solution of (2) we mean a function $u \in C_{1-\alpha}[I, E]$ such that for all $\varphi \in E^*$

$$\varphi(u(t)) = b t^{\alpha - 1} + \int_{0}^{t} \frac{(t-s)^{\alpha - 1}}{\Gamma(\alpha)} \varphi(f(s, u(s))) \, ds, \quad t \in [0, 1].$$
3. Fractional-order integrals in reflexive Banach spaces

Here, we define the fractional-order integral operator in reflexive Banach spaces. Definition given below is an extension of such a notion for real-valued functions.

**Definition 3.1.** Let \( u : I \rightarrow E \) be a weakly measurable function, such that \( \varphi(u(.)) \in L_1(I) \), and let \( \alpha > 0 \). Then the fractional (arbitrary) order Pettis integral (shortly FPI) \( I^\alpha u(t) \) is defined by

\[
I^\alpha u(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} u(s) \, ds.
\]

In the above definition the sign "\( \int \)" denotes the Pettis integral.

**Lemma 3.1.** [16] Let \( u : I \rightarrow E \) be a weakly measurable function, such that \( \varphi(u(.)) \in L_1(I) \), and let \( \alpha > 0 \). The fractional (arbitrary) order Pettis integral

\[
I^\alpha u(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} u(s) \, ds
\]

exists for almost every \( t \in I \) as a function from \( I \) into \( E \) and \( \varphi(I^\alpha u(t)) = I^\alpha \varphi(u(t)) \).

**Lemma 3.2.** [17] Let \( u : I \rightarrow E \) be weakly continuous function on \([0, 1]\). Then, FPI of \( u \) exists for almost every \( t \in [0, 1] \) as a weakly continuous function from \([0, 1]\) to \( E \). Moreover,

\[
\varphi(I^\alpha u(t)) = I^\alpha \varphi(u(t)), \quad \text{for all } \varphi \in E^*.
\]

**Definition 3.2.** [13] Let \( u : I \rightarrow E \). We define the fractional-Pseudo derivative (shortly FPD) of \( u \) of order \( \alpha \in (n-1, n) \), \( n \in N \) by

\[
\frac{d^\alpha}{dt^\alpha} u(t) = D^n I^{n-\alpha} u(t).
\]

In the above definition the sign "\( D \)" denotes the Pseudo differential operator.

**Lemma 3.3.** [21] Let \( u : [0, 1] \rightarrow E \) be weakly continuous function on \([0, 1]\) such that the real-valued function \( I^{n-\alpha} \varphi u \) is \( n \)-times differentiable. Then, the FPD of \( u \) of order \( \alpha \in (n-1, n) \) exists.

**Definition 3.3.** A function \( u : I \rightarrow E \) is called Pseudo solution of (1) if \( u \in C_{1-\alpha}[I,E] \) has FPD of order \( \alpha \in (0, 1) \), \( I^{1-\alpha} u(t)|_{t=0} = b \), \( b > 0 \) and satisfies

\[
\frac{d}{dt} \varphi(I^{1-\alpha} u(t)) = \varphi(f(t,u(t))), \quad \text{a.e. on } [0, 1], \quad \text{for each } \varphi \in E^*.
\]

Now, for the properties of the integrals of fractional-orders in reflexive spaces we have the following lemma (see [16]):
LEMMA 3.4. Let \( u : I \to E \) be weakly measurable and \( \varphi(u(.)) \in L_1(I) \). If \( \alpha, \beta \in (0, 1) \), we have:

1. \( I^{\alpha}I^{\beta}u(t) = I^{\alpha+\beta}u(t) \) for a.e. \( t \in I \).
2. \( \lim_{\alpha \to 1} I^{\alpha}u(t) = I^1u(t) \) weakly uniformly on \( I \) if only these integrals exist on \( I \).
3. \( \lim_{\alpha \to 0} I^{\alpha}u(t) = u(t) \) weakly in \( E \) for a.e. \( t \in I \).
4. If, for a fixed \( t \in I \), \( \varphi(u(t)) \) is bounded for each \( \varphi \in E^* \), then \( \lim_{t \to 0} I^{\alpha}u(t) = 0 \).

4. Main result

In this section we present our main result by proving the existence of solution of equation (2) in \( C_{1-\alpha}[I, E] \).

Let \( E \) be a reflexive Banach space. And let

\[ E_r = \left\{ u \in C_{1-\alpha}[I, E] : ||u||_{C_{1-\alpha}} < b + \frac{M_r}{\Gamma(1+\alpha)} \right\} \]

We will consider the set

\[ B_r = \{ u(t) \in E : u \in E_r, \ t \in I \}. \]

Now, we are in a position to formulate and prove our main result.

THEOREM 4.1. Let the assumptions (1)–(3) are satisfied, then equation (2) has at least one weak solution \( u \in C_{1-\alpha}[I, E] \).

Proof. Let us define the operator \( T \) as

\[ Tu(t) = b t^{\alpha-1} + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,u(s)) \, ds, \ t \in [0,1]. \]

We will solve equation (2) by finding a fixed point of the operator \( T \).

We will prove that

\[ T : C_{1-\alpha}[I, E] \to C_{1-\alpha}[I, E]. \]

First note that from assumption (2), we get that for each \( u \in C_{1-\alpha}[I, E] \), \( f(., u(.)) \) is weakly measurable on \( I \). Since \( f \) has weakly compact range, then \( \varphi(f(., u(.))) \) is Lebesgue integrable on \( I \) for every \( \varphi \in E^* \) and thus the operator \( T \) is well defined.

Now, we show that if \( u \in C_{1-\alpha}[I, E] \), then \( Tu \in C_{1-\alpha}[I, E] \). Note that there exists \( r > 0 \) with \( ||u||_{C_{1-\alpha}} = \sup_{t \in I} ||t^{1-\alpha}u(t)|| < b + \frac{M_r}{\Gamma(1+\alpha)} \).

Now assumption (3) implies that

\[ ||f(t,u(t))|| \leq M_r \text{ for } t \in [0,1]. \]

Let \( t, \tau \in [0,1] \) with \( t > \tau \). Without loss of generality, assume \( t^{1-\alpha}Tu(t) - \tau^{1-\alpha}Tu(\tau) \neq 0 \). Then there exists (a consequence of Theorem 2.5) \( \varphi \in E^* \) with \( ||\varphi|| = 1 \) and

\[ ||t^{1-\alpha}Tu(t) - \tau^{1-\alpha}Tu(\tau)|| = \varphi(t^{1-\alpha}Tu(t) - \tau^{1-\alpha}Tu(\tau)). \]
Thus

\[
||t^{1-\alpha}Tu(t) - \tau^{1-\alpha}Tu(\tau)|| \leq \left| t^{1-\alpha} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \phi(f(s,u(s))) \, ds \right| - \tau^{1-\alpha} \int_0^\tau \frac{(\tau-s)^{\alpha-1}}{\Gamma(\alpha)} \phi(f(s,u(s))) \, ds \\
\leq \left| \int_0^\tau \frac{t^{1-\alpha}(t-s)^{\alpha-1} - \tau^{1-\alpha}(\tau-s)^{\alpha-1}}{\Gamma(\alpha)} \phi(f(s,u(s))) \, ds \right| \\
+ \left| \int_\tau^t \frac{t^{1-\alpha}(t-s)^{\alpha-1}}{\Gamma(\alpha)} \phi(f(s,u(s))) \, ds \right| \\
\leq \frac{Mr}{\Gamma(\alpha)} \left( \int_0^\tau |t^{1-\alpha}(t-s)^{\alpha-1} - \tau^{1-\alpha}(\tau-s)^{\alpha-1}| \, ds \right) \\
+ \int_\tau^t |t^{1-\alpha}(t-s)^{\alpha-1}| \, ds \\
\leq \frac{Mr}{\Gamma(1+\alpha)} \left( 2(t-\tau)^\alpha + |t-\tau| \right).
\]  

(3)

which proves that \( Tu \in C_{1-\alpha}[I,E] \).

Now, let

\[
Q = \left\{ u \in E_r : (\forall t, \tau \in I) ||t^{1-\alpha}u(t) - \tau^{1-\alpha}u(\tau)|| \leq \frac{Mr}{\Gamma(1+\alpha)} \left( 2(t-\tau)^\alpha + |t-\tau| \right) \right\},
\]

Note that \( Q \) is nonempty, closed, bounded, convex and equicontinuous subset of \( C_{1-\alpha}[I,E] \). Now, we claim that \( T : Q \to Q \) and is weakly sequentially continuous. If this is true then according to Theorem 2.3, \( TQ \) is bounded in \( C_{1-\alpha}[I,E] \) (hence, Theorem 2.4, implies \( TQ(t) \) is weakly relatively compact in \( E \) for each \( t \in I \)) and the result follows immediately from Theorem 2.1. It remains to prove our claim. First we show that \( T \) maps \( Q \) into \( Q \). To see this, note that the inequality (3) shows that \( TQ \) is norm continuous. Now, take \( u \in Q \); without loss of generality, we may assume that \( t^{1-\alpha}I^\alpha f(t,u(t)) \neq 0 \), then, by Theorem 2.5, there exists \( \varphi \in E^* \) with \( ||\varphi|| = 1 \) and \( ||t^{1-\alpha}I^\alpha f(t,u(t))|| = \varphi(t^{1-\alpha}I^\alpha f(t,u(t))) \). Thus

\[
||t^{1-\alpha}Tu(t)|| \leq b + \frac{Mr}{\Gamma(1+\alpha)},
\]

(4)

therefore

\[
||Tu||_{C_{1-\alpha}} < b + \frac{Mr}{\Gamma(1+\alpha)}.
\]

Thus \( T : Q \to Q \). Finally, we will show that \( T \) is weakly sequentially continuous. To see this, let \( \{u_n\}_{n=1}^\infty \) be a sequence in \( Q \) and let \( u_n(t) \to u(t) \) in \( E_w \) for each \( t \in [0,1] \). Recall [10] that a sequence \( \{u_n\}_{n=1}^\infty \) is weakly convergent in \( C[I,E] \) if and only if it is weakly pointwise convergent in \( E \). Fix \( t \in I \). From the weak sequential continuity of \( f(t,.) \), the Lebesgue dominated convergence theorem (see assumption (3)) for the
Pettis integral [5] implies for each $\varphi \in E^*$ that $\varphi(Tu_n(t)) \to \varphi(Tu(t))$ a.e. on $I$, $Tu_n(t) \to Tu(t)$ in $E_w$. So $T : Q \to Q$ is weakly sequentially continuous. The proof is complete. □

Now, we are looking for sufficient conditions to ensure the existence of Pseudo solution to the nonlinear weighted Cauchy-type problem (1).

Note that, the following theorem is a generalization of the results of §3.3 in [8]:

**Theorem 4.2.** If $f : I \times B_r \to E$ satisfies the assumptions of Theorem 4.1, then the nonlinear weighted Cauchy-type problem (1) has a fractional-Pseudo derivative (FPD) $u \in C_{1-\alpha}[I, E]$. 

**Proof.** Let us remark, that by assumptions (2), (3) the FPI of $f$ of order $\alpha > 0$ exists and 

$$\varphi(I^\alpha f(t,u(t))) = I^\alpha \varphi(f(t,u(t))), \text{ for all } \varphi \in E^*.$$ 

Let $u$ be a solution of equation (2), then

$$u(t) = bt^{\alpha-1} + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,u(s)) \, ds, \quad t \in [0,1].$$

It is clear that

$$t^{1-\alpha}u(t)|_{t=0} = b.$$ 

Furthermore, we have 

$$u(t) = bt^{\alpha-1} + I^\alpha f(t,u(t))$$

since $u \in C_{1-\alpha}[I, E]$, then $\varphi(I^{1-\alpha}u(t)) = I^{1-\alpha} \varphi(u(t))$, for all $\varphi \in E^*$ (see Lemma 3.2). From equation (5), we deduce that

$$\varphi(u(t)) = bt^{\alpha-1} + \varphi(I^\alpha f(t,u(t))).$$

Operating by $I^{1-\alpha}$ on both sides of the equation (6) and using the properties of fractional calculus in the space $L_1[0,1]$ (see [19] and [20]) result in

$$I^{1-\alpha} \varphi(u(t)) = b_1 + I \varphi(f(t,u(t))).$$

Therefore,

$$\varphi(I^{1-\alpha}u(t)) = b_1 + I \varphi(f(t,u(t))).$$

Thus

$$\frac{d}{dt} \varphi(I^{1-\alpha}u(t)) = \varphi(f(t,u(t))) \text{ a.e. on } [0,1].$$

That is $u$ has the FPD of order $\alpha \in (0,1)$ and $u$ is a solution of the differential equation (1). Conversely, let $u(t)$ be a solution of (1), integrate both sides, then

$$I^{1-\alpha} \varphi(u(t)) - I^{1-\alpha} \varphi(u(t))|_{t=0} = I \varphi(f(t,u(t))),$$
operating by $I^\alpha$ on both sides of the last equation, then

$$I\varphi(u(t)) - I^\alpha C = I^{1+\alpha} \varphi(f(t, u(t))),$$

differentiate both sides, then

$$\varphi(u(t)) - C_1 t^{\alpha-1} = I^\alpha \varphi(f(t, u(t))),$$

from the initial condition, we find that $C_1 = b$, then we obtain (2), i.e. Problem (1) and equation (2) are equivalent to each other. □

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