Magnetic influence on classical dispersion

T. N. C. Mendes*, Reinaldo de Melo e Souza†, C. Farina*
Instituto de Física - UFRJ - CP 68528
Rio de Janeiro, RJ, Brasil - 21945-970.

Abstract

We discuss the Lorentz model for dispersion and absorption of radiation in dilute, linear and isotropic materials. Initially, with the purpose of making the paper as self-contained as possible, we reproduce the usual calculations concerning the interaction between the charged material oscillators and the electric field of the incident radiation, obtaining the main behavior of the reactive and dissipative electromagnetic properties of the materials. Thereafter, we also include the magnetic contribution of the Lorentz force to the equation of motion of the oscillators up to first order in $v/c$, which leads to some interesting results, like the approximately linear dependence of the refraction index with the radiation intensity and the appearance of a second region of anomalous dispersion around half the natural frequencies of the material.

* e-mail: tarciro@if.ufrj.br
† e-mail: reinaldo@if.ufrj.br
* e-mail: farina@if.ufrj.br
1 Introduction

As described by A. Pais, [1], Kirchoff’s 1859 pioneering work about the search of the spectral function of thermal radiation raised the discussion of the interaction between matter and radiation, an issue that, since then, has shown to be of great importance in many branches of physics, like quantum field theory, condensed matter physics and optical physics, to mention just a few. In this work, we discuss the Lorentz model, one of its most prodigious child. Born in the last years of the 19-th century, this model treats matter as possessing stable electronic charge configurations, held together by harmonic potentials, and that interact with electromagnetic fields of a given radiation. This model achieves amazing results describing optical and dissipative properties of material media (for instance, their dielectric constants, magnetic permeabilities and absorption/dispersion factors), specially those with low refractive index. Usually, when discussing classical dispersion theory, only the electric contribution is considered since this contribution is \( c/v \) times larger than the magnetic one. These issues are often covered by undergaduation textbooks in classical electromagnetism [2, 3], but are discussed in more advanced textbooks as well [4].

Here, we will show that the inclusion of the usually ignored magnetic term leads to quite interesting results as, for instance, the appearence of a second region of anomalous dispersion. We also present some numerical analysis in order to see if experimental verifications of the new effects discussed in this paper are plausible, particulary for electromagnetic radiation of high enough intensity.

We finish this brief introduction with a comment to motivate the search for other possibilities of occurrence of anomalous dispersion. Besides their intrinsic importance, it has been known for a long time that group velocity in regions of anomalous dispersion close to an absorption line may exceed the speed of light in vacuum and, under some circumstances, may become infinite or even negative [5, 6]. For a recent and pedagogical discussion on negative group velocity see [7] and references therein.

2 Lorentz’s model

For future convenience, and to make this article as self-contained as possible, we start with a brief review of the Lorentz model for the electric permittivity \( \epsilon(\omega) \) of linear and isotropic media. The interaction between charges and fields in this model is given by the Lorentz force. For the \( j \)-th charged particle with negative electric charge \(-e_j\) whose position at instant \( t \) is denoted by \( \mathbf{r}_j \), this force is given, in gaussian units, by

\[
F_j(r_j, t) = -e_j \left[ E(r_j, t) + \frac{v_j}{c} \times B(r_j, t) \right]
\]  

(1)

where \( E \) and \( B \) are, respectively, the electric and magnetic fields of the electromagnetic wave, \( c \) is the velocity of light in vacuum and \( v_j = \dot{r}_j \) is the velocity of the \( j \)-th particle at instant \( t \). Denoting by \( \mathbf{x}_j \) the difference between \( \mathbf{r}_j \) and the equilibrium position of the particle, denoted by \( \mathbf{r}_{j0} \), we are lead to the following equation of motion for such a particle

\[
\ddot{x}_j + 2\gamma \dot{x}_j + \omega_j^2 x_j = -\frac{e_j}{m_j} \left[ E_j(t) + \frac{\dot{x}_j}{c} \times B_j(t) \right]
\]  

(2)
where, in order to simplify the notation, $E_j(t)$ and $B_j(t)$ stand for the electric and magnetic fields at instant $t$ and position $r_{j0}$, that is, $E_j(t) = E(r_{j0}, t)$ and $B_j(t) = B(r_{j0}, t)$, and the quantities $\gamma_j$ and $\omega_j$ are the damping constant and the natural oscillation frequency associated to the $j$-th charged particle. In [2], we have assumed that $|x_j|$ is small enough so that the spatial variations of the electromagnetic fields over a distance of the order of the atom diameter can be neglected ($|x_j| \ll \lambda$, with $\lambda$ being the wavelength of the incident electromagnetic wave). In other words, the electromagnetic fields can be considered as if they were uniform in space for each charged particle (this is the so called dipole approximation). In fact, since we will need to talk about the polarization $P$ and magnetization $M$ of the material, which are macroscopic quantities defined as volumetric densities computed over a volume $\delta V$ which is very small macroscopically but big enough to contain many thousands of atoms, we shall assume that the condition $|x_j|^3 \ll \delta V / \ll \lambda^3$ is valid.

For simplicity, we consider a linearly polarized and monochromatic plane wave of angular frequency $\omega$ and choose the cartesian axis in such a way that the electromagnetic fields are written as

$$E_j(t) = E_\omega \cos(\omega t - \alpha_j) \hat{y}; \quad B_j(t) = E_\omega \cos(\omega t - \alpha_j) \hat{z}; \quad k = k \hat{x},$$

where $\alpha_j$ is a constant, $k = \omega/c$, $\hat{x}$, $\hat{y}$ and $\hat{z}$ constitute a right-handed orthonormal set and $E_\omega$ is the amplitude of the electromagnetic wave at position $r_{j0}$. It depends on $r_{j0}$, but this fact was not indicated explicitly to maintain the notation as simple as possible. Since the time averages to be taken do not depend on $\alpha_j$, we shall take $\alpha_j = 0$ without loss of generality (alternatively, we can get rid off $\alpha_j$ with a simple time translation $t \rightarrow t + \alpha_j/\omega$). With this in mind, equation (2) takes the form (in components)

$$\ddot{x}_j + 2\gamma_j \dot{x}_j + \omega_j^2 x_j = -\frac{\dot{y}_j e_j E_\omega}{m_j} \cos \omega t \tag{4}$$

$$\ddot{y}_j + 2\gamma_j \dot{y}_j + \omega_j^2 y_j = \left(1 - \frac{\dot{x}_j}{c}\right) \frac{e_j E_\omega}{m_j} \cos \omega t \tag{5}$$

$$\ddot{z}_j + 2\gamma_j \dot{z}_j + \omega_j^2 z_j = 0 \tag{6}$$

where $x_j = x_j \hat{x} + y_j \hat{y} + z_j \hat{z}$. Looking for stationary solutions only, we see that the motion of the particle is restricted to the $OX'Y'$ plane, perpendicular to the magnetic field. Restricting to non-relativistic cases, that is, assuming $|\dot{x}_j| \ll c$ and $|\dot{y}_j| \ll c$, equations (4) and (5) can be approximated, respectively, by

$$\ddot{x}_j + 2\gamma_j \dot{x}_j + \omega_j^2 x_j = 0 \quad \text{and} \quad \ddot{y}_j + 2\gamma_j \dot{y}_j + \omega_j^2 y_j = -\frac{e_j E_\omega}{m_j} \cos \omega t. \tag{7}$$

Hence, at this order, the stationary solutions of equations (4), (5) and (6), are

$$x_j(t) = 0; \quad y_j(t) = A_\omega \cos(\omega t + \phi_\omega); \quad z_j(t) = 0,$$

in which $A_\omega = E_j/Z_\omega$ and

$$E_j = -\frac{e_j E_\omega}{m_j}, \quad Z_\omega = \left[(\omega_j^2 - \omega^2)^2 + 4\gamma_j^2 \omega^2 \right]^{1/2}, \quad \cos \phi_\omega = \frac{\omega_j^2 - \omega^2}{Z_\omega}, \quad \sin \phi_\omega = -\frac{2\gamma_j \omega}{Z_\omega}. \tag{9}$$
Observe that the particle oscillates in the same direction of the electric field with the same angular frequency $\omega$, but shifted by a phase $\phi$. 

Suppose that in a region with a volume $\delta V$ (where the field variation is negligible by assumption) there are $n_j = N_j \delta V$ oscillators with charge $-e_j$, mass $m_j$ and angular frequency $\omega_j$. Therefore, the total number of oscillators in $\delta V$ is then $N_{\delta V} = \sum_j n_j = \sum_j N_j \delta V$ and the polarization due to the present field is 

$$P = -\sum_j N_j e_j x_j.$$ 

(10)

The mean energy per volume in a period, reversibly stored in the medium, is 

$$U_P = -\langle P \cdot E \rangle = E_\omega \sum_j \frac{N_j e_j E_j}{Z_\omega} (\cos(\omega t + \phi) \cos \omega t) = -\frac{1}{2} E_\omega^2 \sum_j \frac{N_j e_j^2}{m_j} \left( \frac{\omega_j^2 - \omega^2}{Z_\omega^2} \right).$$ 

(11)

On the other hand, this same energy may be written as 

$$U_P = -\frac{1}{2} \bar{P}_\omega E_\omega = -\frac{1}{2} \chi'(\omega) E_\omega^2,$$ 

(12)

where we used that $\bar{P}_\omega = \chi'(\omega) E_\omega$ for the Fourier transform of the polarization, with $\chi'(\omega)$ being the electric susceptibility of the medium. Comparing equations (11) and (12), we identify 

$$\chi'(\omega) = \sum_j \frac{N_j \alpha_j \omega_j^2 \left( \omega_j^2 - \omega^2 \right)}{\left( \omega_j^2 - \omega^2 \right)^2 + 4\gamma_j^2 \omega^2},$$ 

(13)

where $\alpha_j = e_j^2/m_j \omega_j^2$ is the static polarizability of the $j$-th oscillator. Comparing last equation to the following relation between polariazation and electric field in a linear and isotropic material (see, for instance, Chapter 4 of Ref. [4]) 

$$P(\omega) = \frac{3}{4\pi} \left( \frac{\epsilon(\omega) - 1}{\epsilon(\omega) + 2} \right) E(\omega),$$ 

(14)

we obtain 

$$\epsilon(\omega) = \frac{3 + 8\pi \chi'(\omega)}{3 - 4\pi \chi'(\omega)} \quad \implies \quad \epsilon(\omega) \simeq 1 + 4\pi \chi'(\omega),$$ 

(15)

where $\epsilon(\omega)$ is the dielectric “constant” of the material (it is not a constant since it is a function of $\omega$). Last approximation, $\chi'(\omega) \ll 1$, is valid specially for low density materials, as gases, for instance. Considering non-magnetic materials and neglecting the magnetic contribution, we have $\mu(\omega) = 1$, which makes the refractive index $n(\omega)$ approximately equal to 

$$n(\omega) = \sqrt{\epsilon(\omega)} \simeq 1 + 2\pi \chi'(\omega).$$ 

(16)

The behavior of $n(\omega) = \sqrt{\epsilon(\omega)}$ is shown in Figure [1] for a single natural frequency $\omega_0$ dominant in the medium. Between $\omega \simeq \omega_0 - \gamma_0$ and $\omega \simeq \omega_0 + \gamma_0$ we see a region of anomalous dispersion $(dn/d\omega < 0)$. 

4
The mean power per volume absorbed by the medium is given by

$$P(\omega) = -\sum_j N_j e_j^2 \langle \dot{x}_j \cdot E \rangle = -\sum_j \frac{N_j e_j^2}{m_j \gamma_j \omega} \omega E^2_\omega \sin(\omega t + \phi_\omega) \cos \omega t = \sum_j \frac{N_j e_j^2}{m_j \gamma_j \omega} \frac{\gamma_j \omega^2}{Z^2_\omega} E^2_\omega.$$ 

(17)

In terms of the mean value of the radiation intensity $I_\omega$, the above equation takes the form

$$P(\omega) = \frac{8\pi}{c} \omega \chi''(\omega) I_\omega, \quad \chi''(\omega) = \sum_j \frac{N_j \alpha_j \omega^2 \gamma_j \omega}{\left(\omega^2_j - \omega^2\right)^2 + 4\gamma^2_j \omega^2}; \quad \alpha_j = \frac{e_j^2}{m_j \omega^2},$$

(18)

reflecting the intuitive fact that the power is proportional to the intensity.

In Figure 1 we show the behavior of (18) as a function of the angular frequency with a fixed intensity. We considered also that the medium has only one dominating natural frequency $\omega_0$. We see that $P(\omega)$ is very small at all frequencies except those in the anomalous-dispersion region. Specially, for $\omega = \omega_0$ the electric field is in resonance with natural oscillators of the medium and the power absorbed is maximum.

![Figure 1: Refractive index $n(\omega) = \sqrt{\varepsilon(\omega)}$ (solid line) and absorbed power $P(\omega)$ (dashed line), for fixed intensity $I_\omega$ as functions of the field frequency, for a medium with only one natural frequency $\omega_0$ (for convenience, we used the ratio $\omega/\omega_0$). In both graphs we used a natural width $\gamma_0 = 0.1 \omega_0$. The values on the vertical axis are written in arbitrary units.](image)

3 Magnetic contribution to dispersion

In this section, we shall repeat the previous calculations but without neglecting the influence of the magnetic field. However, we shall compute only the first order correction (in order $|\dot{x}_j|/c$) to the stationary solutions written in [8]. As it will become evident in a moment, to obtain the new stationary corrections to these equations we
need to obtain only the new stationary solution for the component \( x_j(t) \), namely, to consider the previously neglected \( \dot{y}_j/c \) term on the rhs of equation (1). Hence, it suffices to substitute \( y_j(t) = A_\omega \cos(\omega t + \phi_\omega) \) into equation (1), which leads to

\[
\ddot{x}_j + 2\gamma_j \dot{x}_j + \omega_j^2 x_j = -\left[-\omega A_\omega \sin(\omega t + \phi_\omega)\right] \frac{e_j E_\omega}{m_j c} \cos(\omega t)
\]

\[
= -\frac{E_j^2 \omega}{c Z_\omega} \sin(\omega t + \phi_\omega) \cos(\omega t)
\]

\[
= -\frac{E_j^2 \omega}{2c Z_\omega} \left[ \sin(2\omega t + \phi_\omega) + \sin \phi_\omega \right]
\]

\[
= \frac{\gamma_j E_j^2 \omega^2}{c Z_\omega^2} - \frac{E_j^2 \omega}{2c Z_\omega} \sin (2\omega t + \phi_\omega)
\]

(19)

where we used the definitions \( A_\omega = E_j/Z_\omega = -e_j E_\omega/(m_j Z_\omega) \), the trigonometric identity \( \sin a \cos b = (1/2)[\sin(a + b) + \sin(a - b)] \) and that \( \sin \phi_\omega = -2\gamma_j \omega/Z_\omega \).

A simple inspection on the rhs of the previous differential equation allows us to state that its stationary solution will oscillate with angular frequency \( 2\omega \), rather than \( \omega \). Following the usual procedure of obtaining stationary solutions of damped harmonic oscillators under an external force given by a constant term plus a harmonic one, we can obtain in a straightforward way the expression for the stationary solution for \( x_j(t) \). For convenience, we write below also the stationary solution for \( y_j(t) \) up to first order in \( |\dot{x}_j|/c \):

\[
x_j(t) = \frac{\gamma_j}{\omega_j^2} \omega k A_\omega^2 - \frac{1}{2} k A_\omega A_{2\omega} \sin (2\omega t + \phi_\omega + \phi_{2\omega})
\]

(20)

\[
y_j(t) = A_\omega \cos(\omega t + \phi_\omega),
\]

(21)

where \( k = \omega/c \) and the quantities \( A_{2\omega}, Z_{2\omega} \) and \( \phi_{2\omega} \) can be obtained from equation (9) by just substituting \( \omega \) by \( 2\omega \). As we shall see, equation (21) is, indeed, the correct expression up to this order. But before justifying this statement, a few comments are in order:

(i) the stationary solution for \( z_j(t) \) remains \( z_j(t) = 0 \), so that, as mentioned before, the stationary motion occurs on the \( OXY \) plane, which is perpendicular to the magnetic field;

(ii) while the motion of the charged particle along the \( OY \) direction is an oscillation around an equilibrium position at \( y_j = 0 \), its motion along the \( OX \) direction is an oscillation around the equilibrium position \( x_j = \frac{\gamma_j}{\omega_j^2} \omega k A_\omega^2 \), which depends on the frequency of the electromagnetic fields. This dependence is similar as that of the absorbed power by the medium, namely: it is zero for \( \omega = 0 \) or \( \omega \to \infty \) and has its maximum at \( \omega = \omega_j \);

(iii) comparing the amplitudes of oscillation of equations (20) and (21) we see that the former is \( k A_{2\omega} \) times the latter. Hence, the oscillations in the direction of the electric field (\( OY \) direction) are greater than those along the direction of the wave vector \( k \) (\( OX \) direction).

Let us now estimate the error committed by neglecting the term \( \dot{x}_j/c \) in (5). Substituting (20) into (5), and writing the solution of the resulting differential equation
as \( y_j(t) = y_j^0(t) + \Delta y_j(t) \), where \( y_j^0(t) \) is given by expression [8], it can be shown that \( \Delta y_j(t) \) is given by

\[
\Delta y_j(t) = -\frac{1}{4} k^2 A_{2\omega}^2 A_{2\omega} \cos (\omega t + 2\phi_\omega + \phi_{2\omega}) - \frac{1}{4} k^2 A_{2\omega} A_{3\omega} A_{3\omega} \cos (2\omega t + \phi_\omega + \phi_{2\omega} + \phi_{3\omega}),
\]

(22)

where the quantities \( A_{3\omega} \) and \( \phi_{3\omega} \) can be determined in the same way as \( A_{2\omega} \) and \( \phi_{2\omega} \), described before. Looking at the last equation, and remembering that \( k^2 = \omega^2/c^2 \), it is not difficult to conclude that \( \Delta y_j(t) \) is of order \( |\bar{x}_j|^2/c^2 \) and hence it can be neglected if we maintain terms only up to first order in \( |\bar{x}_j|/c \). That is why \( \Delta y_j(t) \) is not present in equation (21).

The magnetic dipole momentum of the \( j \)-th oscillator is given by

\[
\mathbf{m}_j = -\frac{e_j}{2c} (\mathbf{x}_j \times \dot{\mathbf{x}}_j)
\]

\[
= -\frac{e_j}{2c} (x_j \dot{y}_j - y_j \dot{x}_j) \hat{z}
\]

\[
= -\frac{1}{4} e_j \mathcal{E}_j k^2 A_{2\omega} A_{2\omega} \hat{z} \left[ \cos(\omega t + \phi_\omega) \cos(2\omega t + \phi_\omega + \phi_{2\omega}) + \cos(\omega t + \phi_{2\omega}) - \frac{2\gamma_j \omega}{\omega_j^2} \frac{Z_{2\omega}}{Z_\omega} \sin(\omega t + \phi_\omega) \right],
\]

(23)

so that the corresponding medium magnetization is

\[
\mathbf{M} = \sum_j N_j \mathbf{m}_j
\]

(24)

and the medium mean energy per volume stored in the medium reads

\[
U_m = -\langle \mathbf{M} \cdot \mathbf{B} \rangle = -\sum_j N_j e_j^4 E_j^4 \frac{\omega^2}{4m_j^3 c^2 Z_j Z_{2\omega}} \left[ \frac{3}{4} \cos \phi_{2\omega} - \frac{\gamma_j \omega}{\omega_j^2} \frac{Z_{2\omega}}{Z_\omega} \sin \phi \right] = -\sum_j N_j e_j^4 E_j^4 \frac{\omega^2}{4m_j^3 c^2 Z_j^2} \left[ \frac{3}{4} \left( \frac{\omega_j^2 - 4\omega^2}{Z_{2\omega}^2} \right) + \frac{1}{2} \left( \frac{2\gamma_j \omega}{\omega_j Z_\omega} \right)^2 \right].
\]

(25)

Following a procedure totally analogous to that employed in Section 2 for the electric polarization, it is possible to write the magnetic energy as

\[
U_m = -\frac{1}{2} \bar{M}_\omega E_\omega \quad \Rightarrow \quad \bar{M}_\omega = \sum_j 3N_j e_j^4 E_j^3 \omega^2 \frac{Z_j^2}{Z_{2\omega}} \left[ \frac{\omega_j^2 - 4\omega^2}{Z_{2\omega}^2} + \frac{2}{3} \left( \frac{2\gamma_j \omega}{\omega_j Z_\omega} \right)^2 \right],
\]

(26)

where \( \mathbf{M} = \bar{M}_\omega \hat{z} \) is the mean induced magnetization parallel to the magnetic field. Comparing last equation to the relation between magnetization and the magnetic induction \( \mathbf{B} \) in a linear medium, namely, (see, for instance, Chapter 5 of Ref. [4])

\[
\mathbf{M}(\omega) = \frac{3}{4\pi} \left( \frac{\mu(\omega) - 1}{\mu(\omega) + 2} \right) \mathbf{B},
\]

(27)
we obtain, after some algebraic rearrangements,
\[
\mu(\omega) = \frac{3 + 8\pi\chi_m(\omega)}{3 - 4\pi\chi_m(\omega)} \quad \Rightarrow \quad \mu(\omega) \simeq 1 + 4\pi\chi_m(\omega),
\]
(28)
In the last equation, we defined
\[
\beta_j = \frac{e_j^4/m_j^3c^3}{\omega_j^4 \omega^2} = \frac{\alpha_j^2/m_j c^3}{\omega_j^4 \omega^2} \quad \text{and one could, in principle, think of} \chi_m(\omega) \text{as the magnetic susceptibility of the medium to the magnetic field, something analogous to the electric susceptibility } \chi'(\omega) \text{ defined by equation (12). Note that } \chi_m(\omega) \text{ has a dependence on the frequency of the electromagnetic wave which is much more complicated than that exhibited by the electric susceptibility } \chi'(\omega), \text{ as it is illustrated in Figure 2.}
\]

**Figure 2:** Electric and magnetic susceptibilities, given by equations (12) and (29) as functions of the ratio $\omega/\omega_0$. The dashed line stands for $\chi'(\omega)$ while the solid line corresponds to $\chi_m(\omega)$. For simplicity, we considered only one natural frequency, $\omega_0$, (and the corresponding natural line width $\gamma_0$). We have made $\gamma_0/\omega_0 = 0.1$ in both graphs. The values on the vertical axis are written in arbitrary units.

Differently from $\chi'(\omega)$, which varies smoothly with the frequency except in the range of anomalous dispersion (with a width $2\gamma_0$ centered at $\omega_0$), $\chi_m(\omega)$ shows an anomalous behavior around $\omega = \omega_0/2$. There is also a pronounced peak around $\omega = \omega_0$, similar to that appearing in the absorbed power by the medium, as shown in Figure 1 but whose intensity varies with $1/\gamma_0^2$ instead of $1/\gamma_0$.

In Figure 3 we plot some graphs for $\chi_m(\omega)$ versus $\omega/\omega_0$ for different values of $\gamma_0$. In the range $\frac{1}{2}\omega_0 < \omega < \omega_0$ we observe an uncommon behavior with the frequency, while for high frequencies, $\omega \gg \omega_0$, $\chi_m(\omega)$ goes to zero faster than $\chi'(\omega)$. It is worth mentioning that, for low frequencies, $\omega \ll \omega_0$, only $\chi_m(\omega)$ goes to zero.

In contrast to the electric permittivity $\epsilon(\omega)$, given by equation (15), the magnetic permeability $\mu(\omega)$ depends on the intensity $I_\omega$ of the electromagnetic wave, as can
be seen from equations (28) and (29). For $\chi_m(\omega) \ll 1$, which is generally the case, $\mu(\omega) - 1$ is proportional to $I_\omega$. Including the magnetic contribution, the refractive index takes the form

$$n(\omega) = \sqrt{\epsilon(\omega)} \mu(\omega) \implies n(\omega) \simeq 1 + 2\pi \left[ \chi'(\omega) + \chi_m(\omega) \right].$$

(30)

We see that the role of the magnetic field is to sum $2\pi \chi_m(\omega)$ to the rhs of equation (16), which introduces a linear dependence with the intensity of the electromagnetic wave for fixed frequencies. This term, however, is usually much smaller than the electric contribution, but may become relevant for values of intensity that make $\chi'(\omega)/\chi_m(\omega) \sim 1$. In the next section we discuss if there is a possibility of checking experimentally this magnetic contribution. Figure 4 shows the effect of this new magnetic term on the expression for the refractive index.

For low values of the intensity $I_\omega$, but still high enough to make the contribution due to $\chi_m(\omega)$ a relevant one, the frequency dependence of the refractive index is very similar to that shown in Figure 4 except for the appearance of a range of a secondary anomalous dispersion (SAD), centered at $\omega = \frac{1}{2}\omega_0$ (besides the usual anomalous dispersion around $\omega = \omega_0$). This fact may be viewed as a first signature of the influence of the intensity on the refractive index.

As the intensity increases, the behavior of $n(\omega)$, in the interval $\frac{1}{2}\omega_0 < \omega < \omega_0$, becomes more and more similar to that of $\chi_m(\omega)$. We also notice that the intersection between the curve for $n(\omega)$ and the line $n(\omega) = 1$, which occurs at $\omega = \omega_0$ for vanishing intensity, is shifted as the intensity is increased. Finally, for $\omega \ll \omega_0$ and $\omega > \omega_0$, $n(\omega)$ does not change appreciably when one varies the intensity.

In Figure 5 we see the relation between $n(\omega)$ and frequency for different values
Figure 4: Refractive index, given by equation (30), for different intensities of the incident electromagnetic wave and $\gamma_0/\omega_0 = 0.1 \omega_0$. The curve with solid line corresponds to an intensity 5 times smaller than the intensity used to draw the curve with dashed line and 10 times smaller than that used in the curve with dotted line. The values on the vertical axis are written in arbitrary units.

Figure 5: Refractive index, given by (30), as a function of $\omega/\omega_0$ for different values of $\gamma_0$ and fixed intensity. The solid line stands for $\gamma_0/\omega_0 = 0.03$, the dashed line, to $\gamma_0/\omega_0 = 0.05$ and the dotted line, to $\gamma_0/\omega_0 = 0.1$. The values on the vertical axis are in arbitrary units.

of $\gamma_0$, keeping the incident intensity fixed. As expected, as $\gamma_0$ becomes smaller and smaller, the secondary anomalous dispersion becomes more and more evident in the interval $\omega = \frac{1}{2} \omega_0$. 
4 Numerical estimatives

So far we have discussed the problem of the interaction between microscopic structures and electromagnetic radiation until first order in $v/c$, without worrying about numerical values that could sustain the possibility of experimental observation of those predictions (magnetic corrections to the reactive properties of substances). In this section, we want to analyze the limits for the parameters associated to the radiation (frequency and intensity) and to the system (natural frequencies) that make such predictions observable within the validity domain of this model.

Henceforth, we consider our material oscillators as electrons with electric charge $e = -4,803 \cdot 10^{-10}$ (erg cm)$^{1/2}$ and mass $m = 9,109 \cdot 10^{-28}$ g. We shall call them actives since they are the only charges that can interact with the incident radiation. For the sake of simplicity, we shall assume that there is only one natural frequency present, denoted by $\omega_0$, so every electron oscillates with the same frequency. Therefore, the sum in (12), (18) and (29) keeps only one term. We also admit that these oscillators are perfect harmonic oscillators, an assumption that leads to

$$\gamma_0 \equiv \frac{e^2 \omega_0^2}{3mc^2}.$$  

(31)

Strictly speaking, this model is applicable only to substances formed by active electrons that behave as harmonic oscillators. However, it can be successfully extended to those that actually are not formed by oscillators, but in these cases the parameters $\alpha_j, \beta_j, \omega_j$ and $\gamma_j$ have to be considered independent from each other, being adjusted through experimental data.

Let us, then, analyse the secondary anomalous dispersion. In the range where it occurs, we can approximate the magnetic susceptibility $\chi_m(\omega)$ by

$$\chi_m(\omega) \simeq \frac{2\pi}{3} N_0 \beta_0 \omega_0 I_w \frac{\omega_0 - 2\omega}{(\omega_0 - 2\omega)^2 + \gamma_0^2},$$  

(32)

while, in this range, the electric susceptibility is almost constant, and can be taken as being

$$\chi'(\omega) \simeq \frac{4}{3} N_0 \alpha_0.$$

(33)

We have seen that magnetic effects will become relevant for $|\chi'(\omega)/\chi_m(\omega)| \sim 1$. The maximum value of $|\chi_m(\omega)|$ occurs around $\frac{1}{2}(\omega_0 \pm \gamma_0)$ and is given by

$$|\chi_m| \simeq \frac{\pi}{3} N_0 \beta_0 I_w \frac{\omega_0}{\gamma_0} = \frac{\pi N_0 e^2}{m^2 \omega_0^5},$$

(34)

so that the minimum intensity needed is

$$I_{\text{min}} \sim \frac{4}{3} N_0 \alpha_0 \frac{I_w}{\gamma_0} = \frac{32\pi^2 mc^3}{3 \lambda_0^3} \simeq 260 \left(\frac{\mu m}{\lambda_0}\right)^3 \text{ GW/cm}^2.$$

(35)

These intensities are relatively small in the microwave region while they are very high in the optical region: for $\lambda_0 = 0.5 \mu m$ the intensity is something like 2.0TW/cm$^2$, making it hard to observe the secondary anomalous dispersion in that range of frequencies. Nevertheless, in the range from infrared to microwaves, an experimental observation of these magnetic effects seems feasible: for $\lambda_0 = 100 \mu m$, for instance, $I_{\text{min}} \sim 260 \text{ kW/cm}^2$, a value that can be achieved nowadays by lasers.
5 Conclusions and final remarks

In this work we have discussed the Lorentz model for dispersion and absorption of electromagnetic radiation in diluted, linear and isotropic material media. We started reviewing the usual computation of the electric permittivity of the material that describes the response of the material oscillators only to the electric field of a linearly polarized and monocromatic electromagnetic wave. In this simplified model, the magnetic field is not taken into account, since its contribution is already of order $v/c$. From this model, one obtains the main results about the electromagnetic reactive and dissipative properties of matter, namely: the dependence of the refractive index with the radiation frequency (dispersion); the existence of a region of strong anomalous dispersion for the frequency range $|\omega_j - \omega| \sim \gamma_j$ (which we call, for convenience, a primary anomalous dispersion), where $dn/d\omega < 0$; the absorbed/dissipated power is proportional to the intensity of radiation and varies quickly in the range $|\omega_j - \omega| \sim \gamma_j$, achieving its maximum value at resonance.

Then we discussed what are the first corrections to the previous model when we include the magnetic term of the Lorentz force into the equations of motion of the material oscillators. Despite this term does not change substantially the dissipative properties of the medium, it leads, when solved up to first order approximation in $v/c$, to quite interesting (and presumably observable) effects concerning the reactive properties of matter, namely: (i) the refractive index acquires a dependence on the intensity of the electromagnetic wave (besides its dependence on the wave frequency); (ii) the appearance of a secondary zone of anomalous dispersion in the region $|\frac{1}{2}\omega_j - \omega| \sim \gamma_j$ and of an intensification of the primary dispersion peak. This result breaks the monotonic increasing behavior of the refractive index for sufficiently high intensities, in the range $\frac{1}{2}\omega_j < \omega < \omega_j$.

The observation of the new magnetic effects just described in the resonance region ($\omega \sim \omega_j$) is frustrated due to the strong energy absorption. However, the situation may not be the same in the secondary anomalous dispersion zone, where absorption is very small, so that the required high intensities of the incident radiation may be achieved. In order to investigate the plausibility of making real experiments with the present technology with the purpose of observing the secondary anomalous dispersion, we have presented in the previous section some numerical estimatives. The observation of this secondary anomalous dispersion would be a signature of the dependence of the refractive index on the intensity of the electromagnetic wave.

As a final comment, we would like to say that, so far, we have considered only a linearly polarized incident radiation. However, one can use circularly polarized radiation as well. In this case, though the magnetization acquires a non-vanishing component along the direction of propagation of the electromagnetic wave, the final results remain the same, since the direction of propagation is perpendicular to the magnetic field, so that the contribution of this extra term vanish in the expression of $\langle M \cdot B \rangle$. 
References

[1] Abraham Pais, *Subtle is the Lord...: The Science and the Life of Albert Einstein*, (Oxford University Press, 1982).

[2] David J. Griffiths, *Introduction to Electrodynamics* (Prentice Hall, New Jersey, 1999), 3rd edition.

[3] Mark A. Heald and Jerry B. Marion, *Classical Electromagnetic Radiation* (Saunders College Publishing, New York, 1995).

[4] J.D. Jackson, *Classical Electrodynamics* (Wiley, New York, 1975), 2nd ed., Chapter 9.

[5] L. Brillouin, *Wave Propagation and Group Velocity* (Academic, New York, 1960). An explicit mention to negative group velocity can be found on page 122.

[6] R.Y. Chiao, “Superluminal (but causal) propagation of wave packets in transparent media with inverted atomic populations”, Phys. Rev. A 48 R34-R37 (1993).

[7] Kirk T. McDonald, “Negative group velocity”, Am. J. Phys. 69 607-614 (2001).