Results on the symmetries of integrable fermionic models on chains

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We investigate integrable fermionic models within the scheme of the graded Quantum Inverse Scattering Method, and prove that any symmetry imposed on the solution of the Yang-Baxter Equation reflects on the constants of motion of the model; generalizations with respect to known results are discussed. This theorem is shown to be very effective when combined with the Polynomial $\hat{R}$-matrix Technique (PRT): we apply both of them to the study of the extended Hubbard models, for which we find all the subcases enjoying several kinds of (super)symmetries. In particular, we derive a geometrical construction expressing any $\mathfrak{gl}(2,1)$-invariant model as a linear combination of EKS and U-supersymmetric models. Furtherly, we use the PRT to obtain 32 integrable $\mathfrak{so}(4)$-invariant models. By joint use of the Sutherland’s Species technique and $\eta$-pairs construction we propose a general method to derive their physical features, and we provide some explicit results.

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I. INTRODUCTION

The issue of finding exact solutions for quantum models is strictly connected with the notion of integrability of their Hamiltonian $H$. Integrability actually consists in the possibility of finding a complete set of commuting observables, so that the eigenstates of $H$ can be univocally characterized by the values of the quantum numbers. Within this subject, the Quantum Inverse Scattering Method is a very powerful tool providing sets of commuting operators; it is based on the Yang-Baxter Equation (YBE), a functional equation for a $\mathbb{C}$-number matrix (the $R$-matrix), which can be thought of as a factorizability condition for the scattering matrix. Such techniques were originally applied to study non-linear PDE as well as spin systems, for which some remarkable results have been obtained. As to fermionic problems, the traditional approach was to use the Jordan-Wigner transformation to map them into spin systems, and then to study the latter by means of the QISM. More recently, new methods have been worked out to directly apply the QISM to systems of fermions: in the literature two main different approaches can be found to this aim; the first one makes use of algebraic representation theory to find solutions of the YBE, introducing a grading that accounts for the fact that the operators of fermionic systems can either commute or anticommute. The second method (fermionic $R$-matrix) is quite direct: it consists in dealing with an operator-valued YBE (instead of a $\mathbb{C}$-number YBE), so that everything is calculated directly in terms of fermionic operators. Despite the remarkable results obtained within each approach, interesting questions are still open, which we shall investigate in this paper.

In the first instance, the passage ‘$\mathbb{C}$-number’ $\leftrightarrow$ ‘fermionic operators’ should be clarified in order to use in a complementary way the two approaches cited above: in fact the algebraic methods are very powerful in providing solutions of the $\mathbb{C}$-number YBE, but their results are not always ‘translated’ into fermionic operators, so that the Hamiltonian and the constants of motion are not easily readable; on the other hand, the fermionic $R$-matrix formulation, working with operators instead of $\mathbb{C}$-numbers, is more direct but quite cumbersome on a computational point of view. In the preliminary sections of this paper we therefore clarify the mutual relationship between these two approaches: sec.II and sec.III provide systematic methods to represent fermionic operators with matrices and to derive fermion models from the $\mathbb{C}$-number YBE. In sec. IV some particular remarks are made on the structure of the graded tensor product, which is crucial to this interplay.

Secondly, a quite important open problem is concerned with the symmetries. Indeed, even if the algebraic methods allow to classify the solutions of the YBE according to several kinds of (super)symmetries, no much attention has been paid to investigate the effect that these (local) symmetries have on the constants of motion that the QISM determines. Although some results have been obtained with the fermionic $R$-matrix approach in a particular case, just a few attempts have been made to find general results on this issue, or to develop them. We point out that knowing the symmetries of the constants of motion of exact results is important when developing numerical approaches to study perturbations on such exact models; in fact, by fine-tuning the perturbation, one can control what happens to the

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Moreover, since for any homogeneous operators on a local vacuum, we can derive just exploiting the symmetry properties. It is therefore quite important, once a given symmetry has been realized to provide interesting physical insights, to find all the fermionic models (within a certain general class of Hamiltonians) that fulfill it, in order to deduce the properties of further models. Also, it is important to know how many and what kind of free parameters the imposition of a certain symmetry allows on a given class of models; in fact, even if the exact solution is not possible in general for all the values of the above free parameters, one can select which is the most suitable parameter for the investigation of a given physical feature, and see what happens when varying it while keeping the others fixed, the symmetry being always preserved.

We therefore devote all the central sections of our paper to present results on the symmetries. In particular in sec.V, concerned with the symmetries of the YBE equation itself, we extend the invariance properties of the YBE to the graded similarity transformations, which are the actually useful similarity transformations when dealing with fermionic systems. In section VI we prove a theorem assuring that the local symmetries imposed on the \( \hat{R} \)-matrix directly reflect on the constants of motion; here we shall generalize some known results by considering generic symmetries and possible additional signs in the generators of neighboring sites. In sec.VII we investigate the complementary question, namely the conditions under which the symmetries of the Hamiltonian reflect onto the \( \hat{R} \)-matrix; the Polynomial \( \hat{R} \)-matrix Technique (PRT) \[14\] is shown to provide interesting general answers to this issue. Finally, the two results of sec.VI and VII are combined together to investigate the specific case of the Extended Hubbard Models (EHM).

More precisely, in sec.VIII we exploit the matrix representation of sec.II to find all the EHM that fulfill several kinds of (super)symmetries (in particular, we provide a geometrical construction allowing to express any \( gl(2,1) \)-invariant model as a linear combination of the EKS model \[12\] and the U-supersymmetric model \[15\]); further, in analyzing the integrable subcases of each of the above symmetry classes, we realize that for most of them the \( \hat{R} \)-matrix is a first or second order polynomial; hence, we are able to deduce the symmetry properties of the constants of motion for such models (e.g. for EKS and U-supersymmetric models).

Finally, in sec. IX we focus on the \( so(4) \) symmetries which are proved to be particularly interesting in condensed matter physics. By means of the PRT we find all the models that are derivable by first degree polynomial \( \hat{R} \)-matrices, and through the theorem of section VI we show that the constants of motion are \( so(4) \)-invariant as well. All the models we find turn out to act as Generalized Permutators; this allows us to exploit the Sutherland’s Species technique, and to propose a general scheme for finding the spectrum and the ground state phase diagram. The joint use of the \( \eta \)-pairs construction (based on the \( so(4) \)-symmetry), leads to determine the eigenstates and the correlation functions for these models, even when arbitrary Coulomb repulsion and filling are allowed.

II. MATRIX REPRESENTATION FOR FERMIONIC OPERATORS

In the second quantization formulation, the behaviour of a system of fermions on a lattice is described by creation and annihilation operators, which are governed by the algebra:

\[
\{c_{i,s}, c_{j,s'}\} = 0 \quad \{c_{i,s}, c_{j,s'}^\dagger\} = \delta_{i,j} \delta_{s,s'}
\]

(1)

where \( s \) is the spin-\( J \) label assuming \( 2J + 1 \) possible values (\( J \) being a half odd integer for fermions). The local \( d \)-dimensional vector space \( V_j \) associated with the \( j \)-th site of the lattice is made up of vectors that are built by acting with creation operators on a local vacuum \( |0\rangle_j \):

\[
V_j = \text{Span} \left( |\alpha\rangle_j = h_j^{(\alpha)} |0\rangle_j , \alpha = 1 \ldots d \right)
\]

(2)

Here the \( h_j^{(\alpha)} \)'s are products of \( m \) creation operators \( c_{j,s}^\dagger \) with different \( s \) (\( m \) can take values from 0 to at most \( 2J + 1 \)). We shall typically deal with spin-\( \frac{1}{2} \) fermions, for which a 4-dimensional space is usually involved, so that:

\[
h_j^{(1)} = c_{j,s}^\dagger; h_j^{(2)} = c_{j,s}^\dagger; h_j^{(3)} = 1; h_j^{(4)} = c_{j,s}^\dagger c_{j,s}^\dagger.
\]

Due to the anticommutation relations \[8\], the space \( V_j \) has an intrinsic graduation; in fact \( V_j = V_j^{(1)} \oplus V_j^{(0)} \), where the odd (even) subspace \( V_j^{(1)} \) (\( V_j^{(0)} \)) is spanned by those vectors that are built with an odd (even) number of creation operators \( c_{j,s}^\dagger \). Similarly, the space \( \text{End}(V_j) \) of local linear operators on \( V_j \) is also graded; odd (even) vectors and operators are also said to have a parity \( p = 1 \) (\( p = 0 \)).

Moreover, since for any homogeneous \( \mathcal{O}_j^{(a)} , \mathcal{O}_j^{(b)} \in \text{End}(V_j) \) the relation \( p(\mathcal{O}_j^{(a)} \mathcal{O}_j^{(b)}) = p(\mathcal{O}_j^{(a)}) + p(\mathcal{O}_j^{(b)}) \) holds,
End(V) is actually a graded local algebra, which we shall denote $A_j$. Each operator $O_j^{(a)} \in A_j$ can be easily given a local representation in terms of a $d \times d$ matrix $O^{(a)}$ by making it act on each basis vector of the space $V_j$

$$O_j^{(a)} |\beta\rangle_j = (O^{(a)})_\beta^\alpha |\alpha\rangle_j$$

Here $(O^{(a)})_\beta^\alpha$ are C-numbers, $\alpha$ representing the row and $\beta$ the column of the matrix $O^{(a)}$. It is easily checked that this representation is faithful.

Let us now consider the fermionic problem on the whole lattice; let $N$ be the number of sites. The global algebra $A$ is the enveloping algebra of the sum of the local $A_j$‘s. Any global operator can be expressed in terms of linear combination of products of local operators:

$$O_{glob} \in \text{Span} \left( O_j^{(a_1)} O_j^{(a_2)} \ldots O_j^{(a_m)} \right)$$

where each $O_j^{(a_i)}$ is a generic single-site operator belonging to the local sub-algebra $A_j$ of the $j$-th site, and $m$ can run from 1 to $N$. In general, the local operators in (3) need neither appear in the order of the lattice sites nor be homogeneous elements of the algebra.

It is therefore easily understood that the basic tool to set up a matrix representation for operators like (4) is the global representation of a single-site operator $O_j^{(a)}$: since the latter is embedded in the global algebra, the matrix that represents it is actually a multi-index $(d^N \times d^N)$ matrix, which can be constructed in principle from the $d \times d$ local representation determining its action on the local space $V_j$ (see (3)). However, some caution has to be used in doing that, because of the graded structure of both the operators and the vectors. In particular, a graded tensor product must be used instead of an ordinary one.

Therefore, we first of all recall here some notions about multi-index matrices on graded euclidean vector spaces. A $n$-multi-index matrix $A$ is a $d^n \times d^n$ matrix, whose entries are C-numbers denoted by $A_{\alpha_1,\ldots,\alpha_n}^{\beta_1,\ldots,\beta_n}$, each greek index running from 1 to $d$. The upper $n$-multi-index $\{\alpha\} = (\alpha_1, \ldots, \alpha_n)$ represents the rows, while the lower $n$-multi-index $\{\beta\} = (\beta_1, \ldots, \beta_n)$ stands for the columns. The couple $(\alpha_j, \beta_j)$ describes the action of $A$ on the $j$-th local $C^d$ space. In each of these euclidean spaces, the canonical basis vectors $e_{\alpha}$ are supposed to be assigned a parity $\pi(\alpha)$ [10], so that $C^d$ is actually a graded vector space. This function $\pi$ is in principle completely arbitrary; however we shall choose in the following $\pi = p$, where $p$ is the intrinsic parity of the fermionic vectors $|\alpha\rangle_j$ of $V_j$ (see (3)).

The parity of a $n$-multi-index is defined as $p(\{\alpha\}) = p(\alpha_1) + \ldots + p(\alpha_n)$. An $n$-multi-index C-number matrix $A$ is said to be homogeneous with parity $p(A)$ if for all its non-vanishing entries $A_{\alpha}^{\beta}$ one has $p(\{\alpha\}) + p(\{\beta\}) = \text{const} = p(A)$.

Given a $n$-multi-index matrix $A$ and a $m$-multi-index matrix $B$ (not necessarily homogeneous), we define their graded tensor product as

$$(A \otimes^s B)_{\{\alpha\}}^{\{\beta\}} (\{\gamma\})^{\{\delta\}} = A_{\{\alpha\}}^{\{\beta\}} B_{\{\gamma\}}^{\{\delta\}} (-1)^{p(\{\beta\}) (p(\{\gamma\}) + p(\{\delta\}))}$$

The graded tensor product is a $(n+m)$-multi-index matrix. If $A$ and $B$ are homogeneous, then $A \otimes^s B$ is homogeneous with parity $p(A) + p(B)$. Although the associativity property $A \otimes^s (B \otimes^s C) = A \otimes^s (B \otimes^s C)$ holds, not any properties of the ordinary tensor product can be transferred in general to the graded tensor product; for instance, only if $B$ and $C$ are homogeneous one can state that

$$(A \otimes^s B)(C \otimes^s D) = (-1)^{p(B)p(C)} (AC \otimes^s BD)$$

Let us now come again to the problem of matrix representation of fermionic operators. In order to do that, we remind that for fermions a convention has to be specified to define the basis vectors of the global space $V_{glob} = V \otimes \ldots \otimes V_N$; we shall adopt the following one

$$|\alpha_1, \alpha_2, \ldots, \alpha_N \rangle = \frac{df}{h_1^{(\alpha_1)} \ldots h_N^{(\alpha_N)}} |0\rangle$$

where $|0\rangle$ is the global vacuum (defined through $c_{js}|0\rangle = 0 \ \forall \ i, s$), and $h_j^{(\alpha)}$ is the $\alpha$-th of the $d$ operators defining the state vectors at the $j$-th site (see (3)). The action of any global operator $O_{glob}$ on the global vectors is perfectly determined by the algebra (3). Its global representation $O_{glob}$ is defined through:

$$O_{glob} |\beta_1, \ldots, \beta_N \rangle = (O_{glob})_{\beta_1,\ldots,\beta_N}^{\alpha_1,\ldots,\alpha_N} |\alpha_1, \ldots, \alpha_N \rangle$$
The key step is to provide a matrix representation for single-site operators; it is straightforwardly realized that, with the choice \( \mathbb{1} \) for the basis vectors, the C-number \( d^N \times d^N \) matrix \( O_j^{(a)} \) representing \( O_j^{(a)} \) as a global operator turns out to be:

\[
O_j^{(a)} = \mathbb{1} \otimes \cdots \otimes \mathbb{1} O_j^{(a)} \otimes \cdots \otimes \mathbb{1}
\]

(9)

where \( O_j^{(a)} \) is the \( d \times d \) matrix representing the local action of \( O_j^{(a)} \) on the local space (see \( \mathbb{3} \)). Notice that, thanks to the associativity property cited above, this definition is unambiguous.

The matrix representation of an operator \( O = O_j^{(a_1)} O_j^{(a_2)} \ldots O_j^{(a_m)} \) is given by the product of the global representations of the single-site operators \( O_j^{(a_i)} \). In particular, if \( j_i = i \) (i.e. the order in which they appear matches the order of the \( h_j^{(a_j)} \)'s in the definition (\( \mathbb{3} \)) of the global vectors), then the matrix is simply given by \( O = O_j^{(a_1)} \otimes \cdots \otimes O_j^{(a_m)} \).

We wish to point out that the matrix representation introduced here also holds for non-homogeneous operators: this is very useful because one is not required to decompose a given operator into its even and odd components before arriving at its matrix representation.

A crucial role will be played in the following by Hubbard projectors \( E_j^b = |a\rangle_j \langle b| \). In the case of a 4-dim. local space they are explicitly given by the entries (\( a-\)th row and the \( b-\)th column) of the following matrix

\[
E_j = \begin{pmatrix}
\frac{n_j(1-n_j)}{n_j(1-n_j)} & c_j^\dagger (1-n_j) & c_j n_j^\dagger \\
-\frac{n_j(1-n_j)}{n_j(1-n_j)} & c_j^\dagger (1-n_j) & c_j n_j^\dagger \\
\frac{n_j(1-n_j)}{n_j(1-n_j)} & c_j c_j^\dagger & c_j c_j^\dagger \\
-\frac{n_j(1-n_j)}{n_j(1-n_j)} & c_j^\dagger (1-n_j) & c_j n_j^\dagger \\
\end{pmatrix}
\]

(10)

Each of the above entries is an homogeneous operator with parity \( p(E_j^b) = p(a) + p(b) \). The Hubbard projectors enjoy very important properties:

\[
\left[ E_j^b, E_k^c \right] = 0 \quad \forall j \neq k
\]

\[
E_j^b E_k^d = \delta^b_c E_j^d
\]

(11)

(12)

where \([X, Y] = XY - \left( -1 \right)^{p(X)p(Y)} YX\). The matrix representation of the Hubbard projectors will be denoted \( E_j^b \); it is given by eqn. (\( \mathbb{3} \)) where \( O_j^{(a)} \rightarrow E_j^b \), the \( d \times d \) matrix \( E_j^b \) having vanishing entries except for a 1 at row \( a \) and column \( b \). The \( E_j^b \)'s share the same properties (\( \mathbb{1} \)) and (\( \mathbb{2} \)) as the \( E_j^b \). Using the Hubbard projectors, one can express any single-site operator \( O_j^{(a)} \) just using its local representing matrix (see (\( \mathbb{3} \))) as follows

\[
O_j^{(a)} = (O_j^{(a)})_{\omega}^\alpha E_{\omega}^\gamma
\]

(13)

### III. THE QUANTUM INVERSE SCATTERING METHOD FOR FERMIONIC SYSTEMS

The Quantum Inverse Scattering Method (QISM) is a powerful tool for studying quantum integrability because it provides a set of mutually commuting operators. Within the QISM a key role is played by the so called \( \mathcal{L} \)-operator, an operator-valued matrix acting on an \( n \)-dimensional space termed auxiliary; the \( \mathcal{L} \)-operator is local, so that in discrete 1-dimensional systems one has a matrix \( \mathcal{L}_j \) for each site \( j \) of a chain. The nature of its entries \( \mathcal{L}^{\alpha}_{j,\beta} \) determines the kind of physical model one is dealing with (i.e. the \( \mathcal{L}^{\alpha}_{j,\beta} \) are spin operators for spin systems, fermionic operators for fermionic models, etc.); they also depend on two complex parameters which are referred to as spectral parameters \( u \) and \( v \). When the \( \mathcal{L}^{\alpha}_{j,\beta} \) belong to an ordinary algebra \( \mathcal{G}_j \), the standard formulation of QISM can be applied as follows: if the commutation rules of the \( \mathcal{L} \)-operator can be expressed in terms of a \( n^2 \times n^2 \) C-number matrix \( \tilde{R}(u, v) \) through the relation \( \tilde{R}(\mathcal{L}_j \otimes \mathcal{L}_j) = (\mathcal{L}_j \otimes \mathcal{L}_j) \tilde{R} \) (local realization of the Yang-Baxter algebra), then a global realization \( \tilde{R}(\mathcal{T} \otimes \mathcal{T}) = (\mathcal{T} \otimes \mathcal{T}) \tilde{R} \) can be deduced, where \( \mathcal{T}_\omega = \mathcal{L}_N \ldots \mathcal{L}_1 \) is the monodromy matrix. From this property a set of mutually commuting operators can be obtained (\( \mathbb{1} \)). The crucial step to pass from the local to the global
In doing so, one can easily realize that the presence of the additional signs in the definition (20) of the
At the same time the property (12) allows to show that the Yang-Baxter equation (19) is actually equivalent to (16).

Indeed, this turns out to be very natural when investigating the consistency conditions of eqn.(16); in fact, one can

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for arbitrary $G^s$-valued matrices $A$ and $B$ (where $G^s$ is the enveloping algebra of the locals $G^s_j$'s). It is possible to check that $\otimes_s$ is associative and that

as it is expected to be for even objects. Thanks to (12), one can state [5] that

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Indeed, this turns out to be very natural when investigating the consistency conditions of eqn.(16); in fact, one can mimic the non-graded case [4] and try to derive the Jacobi identities for the structure constants $\hat{A}_{\alpha\beta}^{\gamma\delta}$ by reducing a multiple product $L_j^\alpha \otimes_s L_j^\beta \otimes_s L_j^\gamma$ into its ‘reversed’ form $L_j^\gamma \otimes_s L_j^\delta \otimes_s L_j^\beta$ through repeated action of eqn.(16). A compact form is obtained only if $\tilde{R}$ is even too; moreover it can be shown that, despite the fact that graded algebras are involved, the obtained relation is non-graded

This C-number functional equation is known as the ordinary (because no extra signs appear) Yang-Baxter Equation (YBE). It is worth stressing that $\tilde{R}$ may depend on the spectral parameters in a completely general way, not only through their difference.

In analogy to what happens for spin models, the above scheme can also be approached backward, i.e. the YBE itself can be used to explicitly realize the structure [10], then yielding a set of mutually commuting fermionic operators. A peculiar feature of this procedure is that the dimension $n$ of the auxiliary space is taken to be equal to the dimension $d$ of the physical local space. Interestingly, one can prove [8] that, starting from an even (see eqn.(18)) solution of the YBE (10), one can construct an even L-operator as follows

where the $E_j^{\delta\gamma}$ are the Hubbard projectors. Here $\alpha$ gives the row and $\beta$ the column of the entry. Due to eqn.(10), the entries of two operators $L_j$ and $L_k$ of different sites fulfill

At the same time the property (12) allows to show that the Yang-Baxter equation (19) is actually equivalent to (16). In doing so, one can easily realize that the presence of the additional signs in the definition (20) of the $L$-operator is
crucial; therefore in the graded case, it is also customary to rewrite eqn. (17) for the variables \( \tilde{R}_{\beta \delta}^{\alpha \gamma} = (-1)^{p(\alpha)p(\gamma)} \tilde{R}_{\beta \delta}^{\alpha \gamma} \), obtaining the so-called graded Yang-Baxter equation \[1,10\]

\[
\tilde{R}_{\alpha_1 \beta_1}^{\alpha \beta}(u, v) \tilde{R}_{\alpha_2 \beta_2}^{\alpha \beta}(v, w) \tilde{R}_{\alpha_3 \beta_3}^{\alpha \beta}(w, u) (-1)^{p(\alpha_1)p(\alpha_2) + p(\beta_1)p(\beta_2)} = \\
\tilde{R}_{\alpha_2 \beta_2}^{\alpha \beta}(v, w) \tilde{R}_{\alpha_3 \beta_3}^{\alpha \beta}(w, u) \tilde{R}_{\alpha_1 \beta_1}^{\alpha \beta}(u, v) (-1)^{p(\alpha_2)p(\alpha_3) + p(\beta_2)p(\beta_3)} .
\]

Starting from eqn. (17), one can derive a set of commuting fermionic operators. It is worth emphasizing that, since \( \tilde{R}(u, v) \) is an even matrix, two relations are obtained

\[
\text{tr} \mathcal{T}_N(u, w), \text{tr} \mathcal{T}_N(v, w) = 0 \\
\text{str} \mathcal{T}_N(u, w), \text{str} \mathcal{T}_N(v, w) = 0
\]

where \( \text{tr} \) is the ordinary trace and \( \text{str} \) is the supertrace \( \text{str} = (-1)^{p(\alpha)} \mathcal{T}_N^{\alpha} \). Therefore it might seem that one has two kinds of conservation laws; however it is not by developing the equations (23) powers of the spectral parameters that one generates appropriate constants of motion, because neither of them yields local operators; in order to have such a property one usually requires to have a couple of values \((u_0, v_0)\) of the spectral parameters for which the solution of eqn. (19) reduces

\[
\tilde{R}_{\beta \delta}^{\alpha \gamma}(u_0, v_0) = \delta_{\beta}^{\alpha} \delta_{\delta}^{\gamma}
\]

If this is the case, a straightforward calculation shows that:

\[
Z(u, v) := (\text{str} \mathcal{T}_N(u_0, v_0))^{-1} \text{str} \mathcal{T}_N(u, v) \equiv (\text{tr} \mathcal{T}_N(u_0, v_0))^{-1} \text{tr} \mathcal{T}_N(u, v)
\]

and that the constants of motion defined as

\[
\mathcal{J}_n = \frac{d^n}{du^n} \ln Z(u, v)
\]

are local, in that \( \mathcal{J}_n \) is the sum of operators involving clusters of no more than \( n + 1 \) sites \[1\]. The logarithm is taken not only to obtain an additive eigenvalue spectrum in performing the Algebraic Bethe Ansatz \[22,23\], but also to avoid trivially commuting constants. Eqn. (24) shows that, although the transfer matrix is usually defined through the supertrace

\[
\tau(u, v) = \text{str} \mathcal{T}_N(u, v)
\]

the ordinary trace plays an equivalent role. However, taking the definition (27) does simplify the expression of the shift-operator, because \( \text{str} \mathcal{T}_N(u_0, v_0) = \mathcal{P}_{12}^{\gamma} \mathcal{P}_{23}^{\delta} \ldots \mathcal{P}_{N-1 N}^{\alpha} \), where \( \mathcal{P}_{jk}^{\beta} \) is in this case the graded permutator \( \mathcal{P}_{jk}^{\beta} = (-1)^{j \beta} \mathcal{E}_{\beta \gamma}^{\alpha} \mathcal{E}_{\gamma \delta}^{\alpha} \). Such a simple and compact form is not possible for the \( \text{tr} \mathcal{T}_N(u, v) \). A deep argument confirming the crucial role of the super-trace will be provided by the investigation of the symmetries (see section VI).

Let us consider in detail the first conserved quantity, usually interpreted as the Hamiltonian, which is of the form

\[
H = \sum_{j=1}^{N} H_{j+1}
\]

with periodic boundary conditions \( H_{N+1} = H_1 \). Each \( H_{j+1} \) is a two-site Hamiltonian given by

\[
H_{j+1} = (-1)^{p(\alpha)p(\beta) + p(\delta)} \partial_{\beta} \tilde{R}_{j+1}^{\alpha \beta}(u, v_0) \mathcal{E}_{j+1}^{\gamma} \mathcal{E}_{j+1}^{\delta}
\]

Writing down the matrix representation for the Hubbard projectors, it easily verified that the matrix representation \( H_{j+1} \) for the two-site Hamiltonian \[28\] is

\[
H_{j+1} = 1 \otimes \ldots \otimes 1 \mathcal{H}_{\text{site}} \otimes \ldots \otimes 1
\]

\[
(1 \otimes \ldots \otimes 1) H_{j+1} (1 \otimes \ldots \otimes 1) = \partial_{\beta} \tilde{R}_{j+1}^{\alpha \beta}(u, v_0) \mathcal{E}_{j+1}^{\gamma} \mathcal{E}_{j+1}^{\delta}
\]

Finally, we also remark that the fermionic operators \( \mathcal{R} \) and \( \tilde{\mathcal{R}} \) can be constructed as

\[
\mathcal{R}_{j \beta} = (-1)^{p(\gamma)p(\beta) + p(\delta)} \mathcal{R}_{j \beta}^{\gamma \delta}(u, v) \mathcal{E}_{j \alpha}^{\gamma} \mathcal{E}_{k \beta}^{\delta}
\]

\[
\tilde{\mathcal{R}}_{j \beta} = (-1)^{p(\gamma)p(\beta) + p(\delta)} \tilde{\mathcal{R}}_{j \beta}^{\gamma \delta}(u, v) \mathcal{E}_{j \alpha}^{\gamma} \mathcal{E}_{k \beta}^{\delta} = \mathcal{P}_{jk}^{\beta} \mathcal{R}_{j \beta}^{\gamma \delta}
\]

Using \( \tilde{\mathcal{R}}_{j \beta} \) and \( \mathcal{R}_{j \beta} \), one can rewrite eqns. (19) and (22) in equivalent forms

6
\[ \mathcal{H}_{j+1} = \partial_u \mathcal{C}_p \mathcal{R}_{\alpha \beta}^\gamma (u, v_0) \bigg|_{u=u_0} \]  

(34)

IV. SOME REMARKS ON THE GRADED TENSOR PRODUCT

Throughout the previous sections we used two different kinds of graded tensor product (gtp): the former was introduced to give matrix representations of fermionic operator (REP-gtp), the latter to adapt the QISM to fermionic systems (QISM-gtp). We defined and denoted them differently \((\otimes^s)\) and \((\otimes_\alpha)\) to emphasize their different roles. Indeed, in order to explicitly convert fermionic operator into matrices or to correctly interpret matrix results in terms of fermionic operators, it is worth distinguishing them; since in the literature they are often exchanged and the conventions yielding them are not always precised, we wish to comment about that. The graded tensor product \((\otimes^s)\) (see (3)) fits to represent the fermionic operators when the convention on the definition of the basis vectors is \((\otimes^s)\); however, had we chosen \(\{a_1, a_2, \ldots a_N\} \overset{def}{=} \{h^{(a_1)}_N \ldots h^{(a_1)}_1 \mid 0\},\) we would have got a product like \((\otimes_\alpha)\) (see (14)). The REP-gtp is therefore strictly related to the convention adopted on the basis vectors. On the contrary, the QISM-gtp has different origins: it must be introduced in order to obtain eqn.(13), which could not be derived from an ordinary tensor product. The actual explicit form \((\otimes_\alpha)\) used for the QISM-gtp just stems from the additional signs in the definition (24) of the \(L\)-operator; there the indices \(\alpha\) and \(\gamma\) are the rows of \(R\). This choice is customary in the literature; however, one could equivalently introduce such extra signs in correspondence with the columns, obtaining that the YBE (13) is equivalent to an expression similar to (18) where \((\otimes^s)\) is replaced by \((\otimes_\alpha)\).

Thus, one could in principle use either \((\otimes^s)\) or \((\otimes_\alpha)\) to define both REP-gtp and QISM-gtp; however, the use of two different definitions for the two graded tensor products leads to useful simplifications in the interpretation of the matrix results. In fact, the expression (28) for the two-site Hamiltonian \(\mathcal{H}_{j+1}\) contains some additional signs, which stem from the definition (24) of the \(L\)-operator (so, they are related to the choice of \((\otimes_\alpha)\) for QISM-gtp); as noticed in sec.III, when writing down the matrix representation \(\mathcal{H}_{j+1}\), the above additional signs cancel out with the signs coming from the matrix representation of the Hubbard projectors, provided that \((\otimes_\alpha)\) is used as REP-gtp. As a consequence, eqn.(29) contains no more extra signs. On the contrary, if the same definition \((\otimes^s)\) was adopted for both REP-gtp and QISM-gtp, further additional signs would add to the previous ones, yielding:

\[ (H_{2site})^{\alpha \beta}_{\gamma \delta} = (-1)^{p(\alpha)+p(\gamma)}(-1)^{q(\beta)+q(\delta)} \partial_u \mathcal{C}_p \mathcal{R}_{\alpha \beta}^\gamma (u, v_0) \bigg|_{u=u_0} \]  

(35)

which would contain ‘undesirable’ extra signs.

V. THE INVARIANCE PROPERTIES OF THE YBE

The invariance transformations of a given equation are those transformations that map solutions of the equation into other solutions of the same equation. In the case of the YBE the invariance properties are important to understand if two models are independently integrable or not. A very important example is supplied by the similarity transformations: if a system \(\mathcal{H}\) is integrable and we transform the constants of motions through a similarity operator \(\mathcal{A}_{\text{glob}}\) as follows \(J_n \rightarrow J'_n = \mathcal{A}_{\text{glob}} J_n \mathcal{A}_{\text{glob}}^{-1}\), the new operators yield a further integrable system. Nevertheless, it is not that obvious in general that the new system can be derived from an R-matrix because local structures may not be conserved; however, this is expected to be the case when the transformation is the product of the same local operator, \(i.e.\mathcal{A}_{\text{glob}} = \mathcal{A}_1 \ldots \mathcal{A}_N\), because one is reconstructed to study the local change \(\mathcal{H}_{j,j+1} \rightarrow (\mathcal{A}_j \mathcal{A}_{j+1}) \mathcal{H}_{j,j+1} (\mathcal{A}_j^{-1} \mathcal{A}_{j+1})\). Actually, for the non-graded case, Kulish and Sklyanin proved that this result can still be cast in the language of QISM by noticing that the eq.(19) is invariant under similarity transformation of the form \(A \otimes A;\) explicitly, if a

\(^1\)for instance, the transformed Hamiltonian may not be of the form \(\mathcal{H}' = \sum_{i=1}^N \mathcal{H}'_{i,i+1}\)
In the literature, quite powerful algebraic methods have been developed to classify the solutions of the YBE (19) and $A$ is a $n \times n$ invertible matrix, then the matrix $\tilde{R}' = (A \otimes A) \tilde{R} (A \otimes A)^{-1}$ is also a solution of (19).

However, as to fermionic systems, such a kind of similarity transformations are not suitable; in fact the matrix $A$ still represents a local operator $A_j$, but the matrices representing $A_j A_{j+1}$ are built up with the graded tensor product, i.e. are of the form $A \otimes^g A$; we shall refer to them as graded similarity transformations. The question therefore arises whether the YBE equation is actually invariant under such kind of transformations. To this purpose some facts must be taken into account:

i) In section III we pointed out that the Yang-Baxter Equations (19) can be set into other forms: we observed that when using the variable $\tilde{R} = P^g \tilde{R}$ where $P^g$ is the graded permutator $(P^g)^{\alpha \gamma}_{\beta \delta} = (-1)^{p(\alpha)p(\gamma)} \delta^\alpha_\beta \delta^\gamma_\delta - \text{the equation (23)}$ is obtained; in non-graded systems a widely used form is achieved rewriting (19) in terms of $R = P \tilde{R}$, where $P$ is the ordinary permutator $(P)^{\alpha \gamma}_{\beta \delta} = \delta^\alpha_\delta \delta^\gamma_\beta$, arriving at

$$R_{12}(u, v) R_{13}(u, w) R_{23}(v, w) = R_{23}(v, w) R_{13}(u, w) R_{12}(u, v).$$

Although all the above cited forms are equivalent (as long as the matrices are even), it is not trivial in general that if the equation in a given form fulfills a certain invariance property, the equation obtained with a change of variable fulfills the same invariance property. Now, it turns out that in the non-graded case the invariance property under similarity transformation $A \otimes A$ is actually satisfied from both (19) and (36); in fact the passages $\tilde{R} \rightarrow \tilde{R}'$ for the $R$-matrix (which is crucial for constructing integrable fermionic systems from the YBE), may not be conserved under such a similarity transformation. Therefore, in order for the transformed matrix $\tilde{R}'$ to actually generate a fermionic system, we must take care that $p(\tilde{R}') = 0$. Interestingly, we shall show that such a (physical) requirement is actually the only one that is necessary to prove that the (mathematical) invariance property holds. A simple condition to ensure (physical) requirement is actually the only one that is necessary to prove that the (mathematical) invariance property holds. A simple condition to ensure $p(\tilde{R}') = 0$ is to take a homogeneous $A$. In particular, one can obtain that $(\tilde{R}') = (A \otimes^g A) (A \otimes^g A)^{-1}$ is also even, then $\tilde{R}'$ is the product of 3 even matrices. Nevertheless, other looser conditions are also applicable in principle, depending on both the structure of $R$ and the local $A$ to be considered.

We can formulate our result as follows: let $A$ be an invertible $n \times n$ matrix and $\tilde{R}$ an even solution of (19); if the matrix $\tilde{R}'(u, v) = (A \otimes^g A) \tilde{R}(u, v) (A \otimes^g A)^{-1}$ is also even, then $\tilde{R}'$ is still a solution of (19).

Proof: First of all, it is easily realized that, due to the fact that $\tilde{R}$ is even, the ordinary tensor product in (19) can be replaced by the graded tensor product $\otimes^g$. Then, the following identities are the key for the proof

$$\langle A \otimes^g A \otimes^g A \rangle (I \otimes^g \tilde{R}') = (I \otimes^g \tilde{R}') (A \otimes^g A \otimes^g A)$$

(37)

$$(A \otimes^g A \otimes^g A)(\tilde{R} \otimes^g I) = (\tilde{R}' \otimes^g I) (A \otimes^g A \otimes^g A)$$

(38)

In order to prove them, one can multiply both equations on the left by $(A \otimes^g A \otimes^g A)^{-1} (A \otimes^g A \otimes^g A)$, and realize that, although the graded tensor product $(A \otimes^g A \otimes^g A)^{-1} \neq (A^{-1} \otimes^g A^{-1} \otimes^g A^{-1})$, it is always possible to write

$$(A \otimes^g A \otimes^g A)^{-1} = (I \otimes^g (A \otimes^g A)^{-1})(A^{-1} \otimes I \otimes I) = (I \otimes I \otimes^g A^{-1})(A \otimes^g A)^{-1} \otimes^g I)$$

Moreover, from the property (3), one can obtain that $(A \otimes^g B)(C \otimes^g D) = (A \otimes^g B)(C \otimes^g D)$, provided that $B$ (or $C$) is even, also if the remaining 3 matrices are not homogeneous. Finally, multiplying eqn.(19) by $A \otimes^g A \otimes^g A$ on the left, one obtains that $\tilde{R}'$ also fulfills eqn.(19).

VI. THE SYMMETRIES OF THE CONSTANTS OF MOTION

In the literature, quite powerful algebraic methods have been developed to classify the solutions of the YBE (19) according to several kinds of symmetries and supersymmetries. An important question is concerned with effects that these local symmetries have on the constants of motion $J_n$ derived from the QISM (eq.20). In the literature this issue has been examined for some specific fermionic models, such as the $t-J$ (11) and the EKS (12). The standard approach for these models is to perform an asymptotic expansion in the (additive) spectral parameter $u - v \rightarrow \infty$ of the monodromy matrix $T$, and to use the $RTT = TTR$ eq. (17) to extract from it the generators of the related superalgebra ($sp(2,1)$ for the $t-J$ and $u(2|2)$ for the EKS). The $J_n$ of these particular cases can thus be proved to share the symmetries of $\tilde{R}$.

In the authors’ opinion, however, no sufficiently general approach has been devoted to this subject. To this purpose,
in this section, by only making use of the existence of the shift-point condition \([\mathcal{F}]\), we shall prove that the local constraints imposed on the \(\hat{R}\)-matrix do reflect onto the whole set of constants of motion. We wish to stress that, with respect to the known results concerning specific models cited above, our theorem is more general in that: i) it is independent of the dimension of the \(R\)-matrix (i.e., on the dimension of the local vector space \(V_j\)); ii) it does not make any assumption on the superalgebra to be considered; iii) it is independent of the specific functional form of the \(R\)-matrix with respect to the spectral parameters: in particular, since it does not exploit any expansion on \(u - v\), it allows for non-additive \(\hat{R}\)-matrices, which are recently object of particular interest (see \([1]\)); iv) it considers also the case of ‘staggered’ operators (the case \(\sigma = -1\) below). Our results also improve and generalize some arguments provided in \([3]\) and \([4]\), with which we shall compare in the following.

Let us consider a single-site homogeneous operator \(\mathcal{X}_j\); as discussed in section II it is possible to write \(\mathcal{X}_j = X^\alpha_\beta \mathcal{E}_{j,\alpha}^\beta\), where (see \([13]\)) the matrix \(X\) is actually its local representation matrix \([3]\); we obviously have that \(p(X) = p(\mathcal{X}_j)\). We shall show that

\[
\begin{align*}
\left[ \hat{R}(u, v), X \otimes^s \mathbb{I} + \sigma \mathbb{I} \otimes^s X \right] &= 0 \quad \sigma = \pm 1 \\
\downarrow \\
\left[ \mathcal{J}_N, \sum_{i=1}^N \sigma_i \mathcal{X}_j \right] &= 0
\end{align*}
\]  

(39)

To prove this we shall proceed in four steps:

**Step 1** The condition \([3]\) implies the following relation for the \(\mathcal{L}\)-operator entries:

\[
\mathcal{L}_{j,\beta}^\alpha \ X_{j,\beta}^{\gamma'} + \sigma \mathcal{L}_{j,\beta}^\alpha (-1)^{p(\alpha)p(\beta)} \mathcal{X}_j = \mathcal{X}_j \mathcal{L}_{j,\beta}^\alpha (-1)^{p(\alpha)p(\beta)} + \sigma \mathcal{L}_{j,\beta}^{\delta'} \ X_{j,\delta}^{\alpha'} \quad \forall j
\]

(41)

where the dependence on \(u\) and \(v\) has been dropped to simplify the notation.

**Proof:** It is sufficient to develop \([3]\) in its matrix entries \((e.g., the rows (\gamma, \alpha) and columns (\beta, \delta))\) by means of \([1]\); then, by using the fact that \(\hat{R}_{j,\beta}^{\gamma\alpha} = (-1)^{p(\alpha)p(\gamma)} \hat{R}_{j,\beta}^{\gamma\alpha}\) and that the local \(d \times d\) matrix \(X\) has a definite parity \(p(X)\), one gets that eqn.\([3]\) is equivalent to

\[
\hat{R}_{j,\beta}^{\gamma\alpha} \ X_{j,\beta}^{\gamma'} + \sigma \hat{R}_{j,\beta}^{\gamma\alpha} \ X_{j,\beta}^{\delta'} (-1)^{p(\beta)} = (-1)^{p(\beta)} \ X_{j,\beta}^{\gamma'} \hat{R}_{j,\beta}^{\gamma\alpha} \ X_{j,\beta}^{\delta'} + \sigma \hat{R}_{j,\beta}^{\gamma\alpha} \ X_{j,\beta}^{\delta'}
\]

(42)

Multiplying this equation by \(\mathcal{E}_{j,\gamma}\) and summing up over \(\gamma\) and \(\delta\), one easily arrives at \([11]\) with the help of the identities \(\mathcal{L}_{j,\beta}^\alpha \ X_j = \hat{R}_{j,\beta}^{\gamma\alpha} \ X_{j,\gamma} \mathcal{E}_{j,\gamma}\) and \(\mathcal{X}_j \mathcal{L}_{j,\beta}^\alpha = \hat{R}_{j,\beta}^{\gamma\alpha} \ X_{j,\gamma} \mathcal{E}_{j,\gamma}\).

**Step 2** Using \([11]\) one can show that a similar relation holds for the monodromy matrix, i.e.

\[
\mathcal{T}_N^\alpha \ X_{j,\beta}^{\gamma'} + \mathcal{T}_N^\alpha (-1)^{p(\alpha)p(\beta)} \sum_{j=1}^N \sigma_j^\alpha \mathcal{X}_j = \sum_{j=1}^N \sigma_j^\alpha \mathcal{X}_j \cdot \mathcal{T}_N^\alpha (-1)^{p(\alpha)p(\beta)} + \sigma^N \mathcal{T}_N^\alpha \ X_{j,\beta}^{\gamma'}
\]

(43)

**Proof:** This can be done by induction; supposing that \([13]\) holds for chain with \(N\) sites, it can be seen that is also holds for a chain with \(N+1\) sites (for \(N = 1\) eqn.\([13]\) is nothing but eqn.\([11]\) itself). Indeed, it is sufficient to multiply eqn.\([13]\) by \(\mathcal{L}_{N+1,\alpha}^{\alpha'}\) on the left, sum up over \(\alpha\) and use eqn.\([11]\) with \(j \rightarrow N + 1\); finally, since eqn.\([21]\) implies that

\[
\mathcal{L}_{N+1,\alpha}^{\alpha'} \sum_{j=1}^N \mathcal{X}_j = (-1)^{p(\alpha)+p(\alpha')} \sum_{j=1}^N \mathcal{X}_j \mathcal{L}_{N+1,\alpha}^{\alpha'}
\]

\[
\mathcal{X}_{N+1} \mathcal{T}_N^{\delta'} = (-1)^{p(\delta') + p(\beta)} \mathcal{T}_N^{\delta'} \mathcal{X}_{N+1}
\]

one easily obtains \([13]\) with \(N \rightarrow N + 1\).

**Step 3** Eqn.\([13]\) implies that

\[
\text{if } \sigma = +1 \Rightarrow \left[ \tau(u, v), \sum_{j=1}^N \mathcal{X}_j \right] = 0 \quad \forall u, v
\]

(44)

\[
\text{if } \sigma = -1 \Rightarrow \{ \tau(u, v), \sum_{j=1}^N (-1)^{j} \mathcal{X}_j \} = 0 \quad \forall u, v
\]

(45)
Proof: In fact, if \( p(X) = 0 \) one takes the supertrace of (\ref{eq:43}), whereas if \( p(X) = 1 \) one can take the ordinary trace of (\ref{eq:43}); in both cases\(^2\) one obtains equations involving the super-trace of the monodromy matrix, namely (\ref{eq:43}-(\ref{eq:45}).

Step 4) Both eqn. (44) and (45) imply that
\[
\tau^{-1}(u, w) \tau(v, w), \sum_{i=1}^{N} \sigma^i \mathcal{X}_j = 0
\]
from which eqn. (46) is easily deduced due to eqn. (20).

Proof: For the case \( \sigma = +1 \) the proof is trivial; for the case \( \sigma = -1 \) one can observe that eqn. (43) implies that \( \{ \tau^{-1}(u, w), \sum_{i=1}^{N} (-1)^i \mathcal{X}_j \} = 0 \), from which
\[
[\tau^{-1}(u, w) \tau(v, w), \sum_{i=1}^{N} (-1)^i \mathcal{X}_j] = 2 \tau^{-1}(u, w) \{ \tau(v, w), \sum_{i=1}^{N} (-1)^i \mathcal{X}_j \} = 0
\]
This concludes the proof. It is worth pointing out that the relations obtained in the cases \( \sigma = 1 \) and \( \sigma = -1 \) (see eqns. (44) and (45) respectively) both involve the supertrace of the monodromy matrix. It would not be possible to obtain in general similar relations for the ordinary trace, contrary to what eqn. (25) could suggest. This simply confirms the role of the supertrace in fermionic systems; nevertheless, a more detailed inspection of the third step shows that in the case of even operators \( \mathcal{X}_j \), besides eqns. (44)-(45), one also has
\[
\text{if } \sigma = +1 \Rightarrow \left[ tr(\mathcal{T}(u, v)), \sum_{i=1}^{N} \mathcal{X}_j \right] = 0 \quad \forall \, u, v
\]
\[
\text{if } \sigma = -1 \Rightarrow \left[ tr(\mathcal{T}(u, v)), \sum_{i=1}^{N} (-1)^i \mathcal{X}_j \right] = 0 \quad \forall \, u, v
\]
meaning that, as long as only even operators are dealt with, the ordinary trace plays a perfectly analogous role as the supertrace.

Finally, it is worth stressing that eqn. (\ref{eq:49}) can be rewritten in terms of the fermionic \( \bar{R} \)-matrix, obtaining the equation \( [\bar{R}_{j, j+1}(u, v), \mathcal{X}_j + \sigma \mathcal{X}_{j+1}] = 0 \). Such form is quite appealing because the symmetries determined by \( \bar{R} \) are immediately readable at least for the first constant of motion \( \mathcal{J}_1 \), i.e. the Hamiltonian, thanks to eqn. (\ref{eq:34}).

We now want to point out that, for the subcase \( \sigma = +1 \), the result (\ref{eq:40}) was obtained in \cite{8} starting not from (\ref{eq:39}) but from a different kind of local condition on the \( R \)-matrix, which we report here:
\[
\bar{R}_{\alpha \gamma}^{\sigma \delta}(u, v) X_{\beta}^{\gamma'} + \bar{R}_{\beta \delta}^{\sigma \gamma}(u, v) X_{\gamma}^{\delta'} =
\]
\[
= (-1)^{p(X)(p(\alpha) + p(\beta))} X_{\alpha}^{\gamma}, \bar{R}_{\beta \delta}^{\sigma \gamma}(u, v) + (-1)^{p(X)(p(\alpha) + p(\beta))} X_{\gamma}^{\delta}, \bar{R}_{\alpha \delta}^{\sigma \gamma}(u, v)
\]
where as usual \( \bar{R}_{\gamma \delta}^{\sigma \alpha} = (-1)^{\sigma \alpha} \bar{R}_{\delta \gamma}^{\alpha \sigma} \).

Such an equation was proposed \cite{8} just in this form (it cannot be set in a compact form for \( \bar{R} \)) to explicitly suggest that, since the \( \mathcal{L} \)-operator is constructed with \( \bar{R} \) (see eqn. (20)), the local symmetry constraints should be imposed on \( \bar{R} \) and not on \( R \). However, it is worth emphasizing that, due to the fact that the constants of motion are derived from \( \mathcal{L} \) (see eqn. (25)) and not directly from the transfer matrix \( \tau \), the Hamiltonian is directly related to \( \bar{R} \) and not to \( R \) (see eqn. (29)). Indeed the constraint appearing in equation (\ref{eq:39}) is the natural requirement suggested by \( [\mathcal{H}_{j, j+1}, \mathcal{X}_j + \mathcal{X}_{j+1}] = 0 \). Anyway, unlike eqn. (\ref{eq:49}), eqn. (\ref{eq:39}) with \( \sigma = +1 \) can be equivalently imposed on \( \bar{R} \) or on \( R \). Moreover, it must also be observed that not any operator \( \mathcal{X}_j \) can be used to implement the condition (\ref{eq:40}) on the \( R \)-matrix. In fact, if one evaluates eqn. (\ref{eq:49}) for \( (u, v) = (u_0, v_0) \) and takes into account the fact that for such values we have \( \bar{R}_{\gamma \delta}^{\sigma \alpha} = (-1)^{\sigma \alpha} \delta_{\gamma}^{\delta} \delta_{\beta}^{\alpha} \) (see eqn. (24)), the following relation is obtained
\[
X_{\beta}^{\gamma} \delta_{\delta}^{\alpha} \left( 1 - (-1)^{p(X)(p(\beta))} \right) + X_{\delta}^{\gamma} \delta_{\beta}^{\alpha} \left( 1 - (-1)^{p(X)(p(\delta))} \right) = 0
\]
Eqn. (\ref{eq:54}) is identically satisfied when \( p(X) = 0 \); on the contrary, when \( \mathcal{X}_j \) is an odd operator, we can observe that there must exist at least a couple of indices \( \gamma, \beta \) with \( p(\beta) \neq p(\gamma) \) for which \( X_{\beta}^{\gamma} \neq 0 \); taking \( \alpha = \delta \) in eqn. (\ref{eq:54}) we obtain that

\(^2\)In the case \( \sigma = -1 \) one has to assume that the number of sites is even; this is quite customary in systems that fulfill symmetries of the kind \( \sum_{i=1}^{N} (-1)^i \mathcal{X}_j \) (see for instance \cite{24}).
solutions of the YBE $X_j^2(1 - (-1)^j) = 0$. This means that eqn. (49) only allows for those odd matrices $X_j$ that have non-vanishing entries in odd rows and even columns, which is a somehow 'asymmetric' constraint. This would not allow, for instance, all the odd operators of the $u(2,2)$ superalgebra of the EKS-model (see section VII for more details).

On the contrary, if one applies the same argument to eqn. (33) – or equivalently to eqn. (12) –, no constraints are obtained on $X_j$, so that any kind of odd operator $X_j$ can be used, as observed above; moreover, one can easily see that, if $X$ fulfills the equation (33), then $X^\dagger$ does as well, as it is expected to be the case when exploiting the hermiticity of the Hamiltonian in $[H_{j,j+1}, X_j + X_{j+1}] = 0$. Therefore the condition (33) seems to be more general than eqn. (10).

Finally, we wish to comment on the physical implication of the above theorem. In doing that we shall anticipate some notions on the extended Hubbard models that will be widely treated in the following. The reader non familiar with these models can find details in section VII or in the references henceforth given.

We first of all want to stress that the above theorem (33) ⇒ (10) generalizes the result obtained by Umeno, Shiroishi and Wadati on the ordinary Hubbard model. Indeed it is known that at half filling the Hamiltonian enjoys the $so(4)$ symmetry given by two orthogonal $su(2)$ sectors; the former is given by the spin, and the latter by the operators $\eta^- = \sum_{i=1}^{N} c_i^\dagger c^-_i$, $\eta^+ = (\eta^-)^\dagger$, and $\eta^z = \sum_{i=1}^{N} (n_{i^\dagger} + n_{i^\dagger} - 1)/2$. Using the technique of the fermionic $R$-matrix, the above authors not only realized that the $R$-matrix fulfills the $so(4)$-symmetry, but also showed that all the constants of motion enjoy such a symmetry; for the generators $\eta^\pm$ the proof was just based on realizing that $\{str T_N(u,v), \eta^\pm \} = 0$. In pass, we also remark that, since $\eta^\pm$ are even operators, one could also obtain that $\{tr T_N(u,v), \eta^\pm \} = 0$, as we observed above (see eqns (47) and (53)). A slight modification of the proof supplied in [6] actually confirms that.

Secondly, it is also worth remarking that, if the $R$-matrix is imposed to commute with the number of particles (i.e. with $X_j = n_j$), all the $J_n$ preserve the total number of particles; then the eigenvectors of the Fock space determined by the $J_n$ belong to fixed-$N$ subspaces; this is quite important because it ensures that such eigenstates can also be given a first quantization expression in terms of wave functions, which was not obvious a priori since the problem is formulated in the non-fixed particle number language of second quantization.

VII. THE TECHNIQUE OF POLYNOMIAL $R$-MATRICES AND ITS ROLE WITH RESPECT TO SYMMETRIES

The Polynomial $R$-matrix Technique (PRT) is somehow complementary (with respect to symmetries) to the theorem proved in sec.VI. Indeed the latter exploits the symmetries of the $R$-matrix to deduce the symmetries of the the constants $J_n$ (and in particular of the Hamiltonian); the PRT is a constructive method allowing to look for solutions of the YBE starting with a given Hamiltonian of interest; in particular if $\mathcal{H}$ belongs to a certain class of symmetry, the PRT precises sufficient conditions under which the $R$-matrix fulfills the same symmetries as $\mathcal{H}$. Using the PRT one straightforwardly obtain the result that, if the $R$-matrix is a first or a second order polynomial, any symmetry imposed on the Hamiltonian immediately reflects onto the $R$-matrix. This issue is of great interest for fermionic models, in that all the known models (apart from the ordinary Hubbard model, which has very peculiar $R$-matrix, in that it depends on two spectral parameters) have $R$-matrices that are first or second degree polynomials. We shall come again to this topic in next section.

Here we just briefly recall the main aspects of this method, which we shall use in combination with the theorem of sec. VI, in order to get further information about the symmetries. We consider for simplicity the case of additive $R$-matrices, i.e. matrices that depend on the spectral parameters through their difference, and search for polynomial solutions of the YBE

$$R(u) = I + uR^{(1)} + \ldots + \frac{u^p}{p!} R^{(p)}$$

where $R^{(i)}$, $i = 1, \ldots, p$ are matrices that do not depend on the spectral parameters; in the case of spin-1/2 fermions and 4-dimensional local spaces they are $16 \times 16$ matrices.

Inserting the expansion (51) into the YBE (14) one obtains a hierarchy of equations for the $R^{(i)}$'s. The advantage is that such equations are algebraic and not functional. They exhibit some interesting features (see (14)): first of all the highest degree term ($R^{(p)}$ in (51)) must fulfill the symmetric group equations

$$R^{(p)}_{23} R^{(p)}_{12} R^{(p)}_{23} = R^{(p)}_{12} R^{(p)}_{23} R^{(p)}_{12} \quad \left(R^{(p)}\right)^2 \propto I$$

In particular this implies that, for first degree polynomial $R$-matrices, the Yang-Baxter Equation is equivalent to the Symmetric Group equations.

In addition, the second degree coefficient $R^{(2)}$ is always explicitly given in terms of the first degree one $R^{(1)}$ as follows
where $\gamma$ is a $\mathbb{C}$-number. We recall that the first derivative of $\tilde{R}(u)$ with respect to spectral parameter must coincide with the 2-site Hamiltonian and therefore $\tilde{R}(1)$ must be the representing matrix of the 2-site hamiltonian one is interested in.

From the above discussion, it is easily seen that if $\tilde{R}(u)$ is a first or a second degree polynomial, it always fulfills the same symmetries of the Hamiltonian; in fact, if $\tilde{R}(u)$ is of first degree, it is made of the Identity and the 2-site Hamiltonian, so that the statement is trivial; moreover, thanks to eqn.$(53)$, this property also holds for second degree polynomials.

Combining this observation with the theorem of section VI one can therefore state that every time that an hamiltonian $\mathcal{H}$ is reproduced by a first or second order polynomial $\tilde{R}$-matrix, any of its symmetries is shared by the whole set of constants of motion $\mathcal{J}_n$. In the following we shall apply this sinergic combination to the study of the extended Hubbard models; in sec.VIII we deduce the symmetries of the EKS $[13,12]$ and the U-supersymmetric model $[17]$, which are reproduced by a first and a second degree polynomial $\tilde{R}$ respectively. In sec.IX we use the FRT to find integrable models that are so(4)-invariant and the theorem to deduce the symmetries of their constants of motion. The case of the AAS-model will also be discussed. However, we wish to emphasize the generality of the above observations, which can be applied not only to the extended Hubbard models but also for integrable models in general.

### VIII. SYMMETRIES IN THE EXTENDED HUBBARD MODELS

The Hubbard Model is a model of interacting electrons which was introduced to take into account the effect of correlations in narrow band insulators; its generalizations – the extended Hubbard models – are envisaged to investigate a wide number of physical phenomena such as metal-insulator transitions $[24]$, high-$T_c$ superconductivity $[13]$, quantum wires $[27]$, as well as quantum computation $[28]$. We shall consider here the following class of extended Hubbard models

$$\mathcal{H}^{EHM} = -\sum_{(j,k),s} \left[ t - X(n_{j,-s} + n_{k,-s}) + \tilde{X} n_{j,-s} n_{k,-s} \right] c_{j,s}^\dagger c_{k,s} + U \sum_j n_{j,\uparrow} n_{j,\downarrow} + \frac{V}{2} \sum_{(j,k)} n_j n_k + \frac{W}{2} \sum_{(j,k),s,s'} c_{j,s}^\dagger c_{k,s'} c_{j,s'} c_{k,s} + Y \sum_{(j,k)} c_{j,\uparrow}^\dagger c_{j,\downarrow}^\dagger c_{k,\downarrow} c_{k,\uparrow} + P \sum_{(j,k)} n_{j,\uparrow} n_{j,\downarrow} n_k + Q \sum_{(j,k),s,s'} n_{j,\uparrow} n_{j,\downarrow} n_{k,\uparrow} n_{k,\downarrow} + \mu \sum_j n_{j,s}$$

(54)

where $c_{j,s}^\dagger$ and $c_{j,s}$ are the usual fermionic operators (see $[10]$). Since electrons are spin $1/2$ fermions, the variable $s$ can now take two possible values $s = \{\uparrow, \downarrow\}$; the subscript $j$ labels the sites of a lattice $\Lambda$, because the creation and annihilation operators are thought of as referring to the set of Wannier functions. Moreover $n_{j,s} = c_{j,s}^\dagger c_{j,s}$ and $n_j = n_{j,\uparrow} + n_{j,\downarrow}$. The symbol $\langle j, k \rangle$ stands for ordered couples of nearest neighbors in $\Lambda$. In $(54)$ the term $t$ represents the band energy of the electrons, while the subsequent terms describe their Coulomb interaction energy in a narrow band approximation: $U$ parametrizes the on-site diagonal interaction, $V$ the neighboring site charge interaction, $X$ the bond-charge interaction, $W$ the exchange term, and $Y$ the pair-hopping term. Moreover, additional many-body coupling terms have been included in agreement with $[24]$. $\tilde{X}$ correlates hopping with on-site occupation number, and $P$ and $Q$ describe three and four-electron interactions. Finally $\mu$ is the chemical potential. The class $[24]$ depends on 10 parameters and is quite wide; in the literature some even more general class of models have been considered (see for instance $[20,31]$) in which the coupling constants are spin-dependent; however, it can be proved that the Hamiltonian $[24]$ is the most general single-band Hamiltonian that preserves the spin and the total charge, and that is isotropic (i.e. $\mathcal{H}_{j,j+1} = \mathcal{H}_{j+1,j}$). It therefore constitutes a fairly general starting point. The $su(2)$-spin generators explicitly read

$$S^+ = \sum_j c_{j,\uparrow}^\dagger c_{j,\downarrow} \quad S^- = \sum_j c_{j,\downarrow}^\dagger c_{j,\uparrow} \quad S^z = \frac{1}{2} \sum_j (n_{j,\uparrow} - n_{j,\downarrow})$$

(55)

while the $u(1)$-charge generator is simply given by the operator
\[ \eta^2 = \frac{1}{2} \sum_j (n_j \uparrow + n_j \downarrow - 1) \]  

(56)

In this section we shall investigate for this Hamiltonian several symmetries of the kind envisaged in section VI, i.e. we shall impose that \( \mathcal{H}^{EHM} \) commutes with a global generator in the form \( X = \sum_j \sigma_j^X \), where the single-site operators \( X_j \) are the generators of a local algebra or superalgebra; the \( \sigma = \pm 1 \) accounts for possible relative signs between the local generators of neighboring sites.

In this way we find all the mutual relations that the parameters \( (t, X, \bar{X}, U, V, W, Y, P, Q, \mu) \) must satisfy in order for these constraints to be fulfilled. To this purpose the matrix representation of fermionic operators developed in sec.II turns out to be a very useful tool, since it allows to study the symmetries directly on the \( \mathbb{C} \)-number matrices representing the Hamiltonian and the generators, so that one is reconducted use techniques of ordinary algebra. In particular, due to the fact that the \( \mathcal{H}^{EHM} \) involves nearest-neighbor interaction terms, the global constraint \( [\mathcal{H}^{EHM}, \sum_j \sigma_j^X X_j] = 0 \) is equivalent to the 2-site form

\[ [H_{2 \text{ sites}}^{EHM}(t, X, \bar{X}, U, V, W, Y, P, Q, \mu), X \otimes^\sigma I + \sigma I \otimes X] = 0 \]

(57)

where \( X \) is the \( 4 \times 4 \) local representing matrix of the generator \( X_j \), and \( H_{2 \text{ sites}}^{EHM} \) is the \( 16 \times 16 \) matrix representing the 2-site terms of (54), explicitly given in (59), where

\[
\begin{align*}
 h_{11}^{11} &= h_{22}^{22} = \mu + V - W & h_{12}^{12} &= h_{21}^{21} = \mu + V & h_{13}^{13} &= h_{31}^{31} = h_{23}^{23} = h_{32}^{32} = \mu/2 \\
h_{12}^{12} &= W & h_{24}^{44} &= Y & h_{13}^{13} &= h_{33}^{33} = -t & h_{41}^{41} &= h_{42}^{42} = t - 2X + \bar{X} \\
h_{14}^{14} &= h_{34}^{34} = h_{24}^{24} = \frac{2}{3} \mu + P + \frac{U}{2} + 2V - W & h_{34}^{34} &= h_{43}^{43} = \mu + \frac{U}{2} \\
h_{44}^{44} &= 2\mu + 4P + Q + U + 4V - 2W & h_{34}^{34} &= h_{43}^{43} = -h_{34}^{34} = -h_{23}^{23} = t - X
\end{align*}
\]

(58)

\[
H_{2 \text{ sites}}^{EHM} = \begin{pmatrix}
 h_{11}^{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & h_{12}^{12} & 0 & 0 & h_{21}^{21} & 0 & 0 & 0 & 0 & 0 & 0 & h_{14}^{14} & 0 & 0 & 0 & 0 \\
 0 & 0 & h_{13}^{13} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & h_{14}^{14} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - \\
 0 & h_{22}^{22} & 0 & 0 & 0 & 0 & h_{21}^{21} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & h_{23}^{23} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & h_{24}^{24} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - \\
 0 & 0 & h_{31}^{31} & 0 & 0 & 0 & 0 & h_{31}^{31} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & h_{32}^{32} & 0 & 0 & h_{32}^{32} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & h^0 & 0 & 0 & 0 & h^0 & 0 & 0 & 0 & 0 & 0 \\
 0 & h_{34}^{34} & 0 & h_{43}^{43} & 0 & 0 & 0 & 0 & 0 & h_{34}^{34} & 0 & 0 & 0 & 0 & 0 & 0 \\
 - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - \\
 0 & 0 & 0 & h_{41}^{41} & 0 & 0 & 0 & 0 & 0 & 0 & h_{41}^{41} & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & h_{42}^{42} & 0 & 0 & 0 & 0 & h_{42}^{42} & 0 & 0 & 0 & 0 \\
 0 & h_{43}^{43} & 0 & h_{43}^{43} & 0 & 0 & 0 & 0 & h_{43}^{43} & 0 & 0 & h_{43}^{43} & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

(59)

As observed above, the hamiltonian (54) contains 10 free parameters (actually 9, up to an overall multiplicative \( t \)); when imposing a symmetry on \( \mathcal{H}^{EHM} \) the number of independent parameters obviously reduces; for each (super)symmetry that we consider we shall also point out how many of them remain free and comment about it.

1) ‘doubly-occupied sites’-symmetry: It is a \( su(2) \oplus u(1) \oplus u(1) \) algebra which preserves, aside the spin \( su(2) \) and the charge \( u(1) \), the following \( u(1) \) generator

\[ \mathcal{K} = \sum_j (n_j \uparrow - \frac{1}{2})(n_j \downarrow - \frac{1}{2}) \]  

(60)

In terms of the parameters of (54) this yields just one constraint \( X = t \), leaving therefore 8 free parameters (apart from an overall multiplicative factor). This symmetry has been exploited for instance in [32][33] to obtain the phase diagram of the AAS-model (the subcase \( \bar{X} = V = W = Y = P = Q = 0 \)).

2) \( so(4) \)-symmetries: It is formed by two mutually commuting \( su(2) \) algebras; six generators are therefore involved: three of them are the usual spin components (55), while the remaining ones form another \( su(2) \) sub-algebra and read
The case $\sigma = -1$ (which we shall denote by $so(4)_{(-1)}$) was first considered by Yang [24] for the ordinary Hubbard model to investigate the symmetry at half-filled band. The case $\sigma = +1$ (denoted henceforth by $so(4)_{(+1)}$) has been envisaged for many models [13,32,33,35]. The generators of both $so(4)_{(-1)}$ and $so(4)_{(+1)}$ are also employed [24,33,14] to build up states having the property of Off-Diagonal Long Range Order, which is proved to imply superconductivity [37]. It can be proved that, in order for (57) to hold, the parameters must fulfill the following mutual relations $so(4)_{(-1)}$ (5 free parameters)

$$\begin{align*}
\eta_{\sigma j}^+ &= \sum_j \sigma^j c_{j,\uparrow} c_{j,\downarrow} \\
\eta_{\sigma j}^- &= \sum_j \sigma^j c_{j,\downarrow} c_{j,\uparrow} \\
\eta_{\sigma} &= \frac{1}{2} \sum_j (n_{j,\uparrow} + n_{j,\downarrow} - 1) \\
\end{align*}$$

(61)

The case $\sigma = -1$ (which we shall denote by $so(4)_{(-1)}$) was first considered by Yang [24] for the ordinary Hubbard model to investigate the symmetry at half-filled band. The case $\sigma = +1$ (denoted henceforth by $so(4)_{(+1)}$) has been envisaged for many models [13,32,33,35]. The generators of both $so(4)_{(-1)}$ and $so(4)_{(+1)}$ are also employed [24,33,14] to build up states having the property of Off-Diagonal Long Range Order, which is proved to imply superconductivity [37]. It can be proved that, in order for (57) to hold, the parameters must fulfill the following mutual relations $so(4)_{(-1)}$ (5 free parameters)

$$\begin{align*}
\tilde{X} &= 2X \\
V &= (W - Y - P)/2 \\
\mu &= Y - \frac{U}{2} \\
Q &= -2P \\
\end{align*}$$

(62)

$so(4)_{(+1)}$ (4 free parameters)

$$\begin{align*}
X &= t \\
\tilde{X} &= 0 \\
V &= (W + Y - P)/2 \\
\mu &= -Y - \frac{U}{2} \\
Q &= -2P \\
\end{align*}$$

(63)

It must be emphasized that the two fermionic realizations $so(4)_{(-1)}$ and $so(4)_{(+1)}$ are physically deeply different: for instance $so(4)_{(+1)}$ preserves the number of doubly occupied sites (because $\tilde{X} = t$), while the $so(4)_{(-1)}$ does not need to. Also, we observe that the number of free parameters that the two realizations allow when (57) is different (respectively 5 and 4); this is because, though it can be found a transformation mapping $so(4)_{(+1)}$ into $so(4)_{(-1)}$, such transformation does not map the Hamiltonian (54) into itself.

3) gl(2,1)-supersymmetries: they are formed by 8 generators; the even sector is a $su(2) \oplus u(1)$ subalgebra, made of the 3 generators (55) of the spin and that of the charge (57). The odd sector consists of the following four generators

$$\begin{align*}
Q_{\uparrow,(\sigma)} &= \sum_j \sigma^j [\alpha (1-n_{j,\downarrow}) c_{j,\uparrow} + \beta n_{j,\uparrow} c_{j,\downarrow}] \\
Q_{\downarrow,(\sigma)} &= \sum_j \sigma^j [\alpha (1-n_{j,\uparrow}) c_{j,\downarrow} + \beta n_{j,\downarrow} c_{j,\uparrow}] \\
Q^\dagger_{\uparrow,(\sigma)} &= (Q_{\uparrow,(\sigma)})^\dagger \\
Q^\dagger_{\downarrow,(\sigma)} &= (Q_{\downarrow,(\sigma)})^\dagger \\
\end{align*}$$

(64)

In this case the coupling constants appearing in (54) become functions of the parameters $\alpha$ and $\beta$ that determine such a linear combination. We shall distinguish three cases:

3a) $\alpha \neq 0$ and $\beta = 0$: In this case we have 3 free parameters $\tilde{X}, U$ and $Q$, whereas the other ones must assume the values

$$\begin{align*}
X &= t \\
W &= V = -\sigma t \\
Y &= -\sigma (t - \tilde{X}) \\
\mu &= 2\sigma t \\
P &= 0 \\
\end{align*}$$

3b) $\alpha = 0$ and $\beta \neq 0$: Also in this case the free parameters are $\tilde{X}, U$ and $Q$, the remaining ones being

$$\begin{align*}
X &= t \\
W &= -\sigma (t - \tilde{X}) \\
V &= -\sigma (t - \tilde{X}) + Q \\
Y &= -\sigma t \\
\mu &= -U \\
P &= -Q \\
\end{align*}$$

These two cases are related to atypical representations of the superalgebra $gl(2,1)$ (see [19][21]). The most interesting situation is therefore the following

3c) $\alpha \neq 0$ and $\beta \neq 0$: By denoting $b = \beta/\alpha$ we obtain the relations

$$\begin{align*}
X &= t - b(t + U/2) \\
\tilde{X} &= (1 - b)^2(t + U/2) \\
Y &= U/2 \\
W &= V = -\sigma [t - b^2(t + U/2)] \\
\mu &= 2\sigma t \\
P &= Q = 0 \\
\end{align*}$$

(65)

where $b$ is a non-vanishing real number. These relations yield two free parameters as long as $1 + \sigma U/2 \neq 0$; in particular the subcase characterized by $W = V = 0$ and $1 + \sigma U/2 \neq 0$ is known in the literature as the $U$-supersymmetric model, and is a 1-parameter model. On the contrary, when $1 + \sigma U/2 = 0$ no free parameters are left and the only possibility allowed is the so-called EKS-model

$$\begin{align*}
X &= t \\
\tilde{X} &= 0 \\
U &= -2\sigma t \\
V &= -\sigma t \\
W &= -\sigma t \\
Y &= -\sigma t \\
P &= 0 \\
Q &= 0 \\
\mu &= 2\sigma t \\
\end{align*}$$

(66)
Notice that in this case the values of the parameters are independent of \( b \); this means that the EKS-models commute with \([b, \sigma]\) for any \( b \neq 0 \) (see the case of \( u(2, 2) \) discussed below).

We also want to remark the relationships between the EKS-models and the U-supersymmetric ones; we remind that the U-supersymmetric is the subcase \( W = V = 0 \), while the EKS-model has \( W = -\sigma t \). First of all it is worth rewriting eqns.\((65)\) only in terms of the parameters of the Hamiltonian (i.e. eliminating \( b \)). To this purpose the couple \( X-W \) turns out to be suitable. Since for \( W \neq \sigma t \) we have two free parameters, whereas for \( W = -\sigma t \) we just have the EKS-models, the allowed values in the \( X-W \) plane are those that belong to the two half-planes \( W/t \leq -\sigma \), with the addition of the single point \((W/t = -\sigma; X/t = 1)\) representing the EKS model. In the former case eqns.\((63)\) are rewritten as (we divide by \( t \) to eliminate trivial overall factors)

\[
U/t = 2\sigma - 1 + (X/t - 1)^2 \quad \tilde{X}/t = (X/t + \sigma W/t)^2 \quad P = Q = 0
\]

\[
Y/t = \sigma - 1 + (X/t - 1)^2 \quad \mu/t = 2\sigma \quad \text{for}: W/t \neq -\sigma
\]

In the \( X-W \) plane the \( U \)-supersymmetric model is represented by the axis \( W = 0 \) (see fig.\[4\]): the relations for the other parameters are the following

\[
U/t = 2\sigma ((X/t)^2 - 2X/t) \quad \tilde{X}/t = (X/t)^2 \quad W = V = P = Q = 0
\]

\[
Y/t = \sigma ((X/t)^2 - 2X/t) \quad \mu/t = 2\sigma
\]

This pictorial characterization of the U-supersymmetric and EKS model also provides an interesting geometrical construction; in fact it can be shown \([38]\) that any model belonging to the 2-parameter \( \mathfrak{gl}(2,1) \)-class \((63)\) can be written as a linear combination of the EKS model and an appropriate U-supersymmetric model \( \mathcal{H}_{U_{ss}} \). In the \( X-W \) variables this property can be expressed as follows:

\[
\mathcal{H}_{\mathfrak{gl}(2,1)}(X, W) = [t + \sigma W] \mathcal{H}_{U_{ss}} - \sigma W \mathcal{H}_{EKS}
\]

The interesting fact is that the appropriate model \( \mathcal{H}_{U_{ss}} \) can be determined in a geometrical way; given any \( \mathfrak{gl}(2,1) \)-invariant model (characterized by a certain point \( P \) of coordinates \((X, W)\)) it is sufficient to trace the line from \( P \) to the point \( E = (X/t = 1; W/t = -\sigma) \) (which is the EKS-model), and determine the point \( A \) at which the above line intersects the axis \( W = 0 \) (see fig.\[4\]). This point precisely represents the \( \mathcal{H}_{U_{ss}} \)-model we need in eqn.\((63)\). Its coupling constants can be determined from its ascissa \( X_A \) by means of the relations \((58)\).

Eqn.\((69)\) is quite important in that it explicitly shows that the study of \( \mathfrak{gl}(2,1) \)-invariant models with \( W \approx -\sigma t \) actually consists in treating the U-supersymmetric model \( \mathcal{H}_{U_{ss}} \) as a perturbation of the EKS-model; on the contrary, in the limit \( W \approx 0 \), it is the EKS-model that acts as a perturbation on the known model \( \mathcal{H}_{U_{ss}} \).

4) \textit{so}(5)-symmetries: it is an algebra made of 10 generators; we shall consider the realization in terms of fermionic operators introduced in \([35]\)

\[
\begin{align*}
\mathcal{A}_3^{23} &= 2S^+ \\
\mathcal{A}_3^{14} &= 2\eta^+_{(1)} \\
\mathcal{A}_3^{12} &= \sum_j \sigma^j \epsilon_{j\downarrow} \\
\mathcal{A}_3^{21} &= \sum_j \sigma^j \epsilon_{j\downarrow} \\
\mathcal{A}_3^{13} &= -\sum_j \sigma^j \epsilon_{j\uparrow} \\
\mathcal{A}_3^{31} &= -\sum_j \sigma^j \epsilon_{j\uparrow}
\end{align*}
\]

This is a subcase of the algebra \( \textit{so}(4)_{(4,1)} \) defined above; the obtained relations among the parameters only allow the EKS-models \((70)\). The latter are therefore the only models of the extended Hubbard class \((54)\) that enjoy the \( \textit{so}(5) \)-symmetry; actually, there are other 1-d single-band models commuting with \((70)\) (see \([32]\) ), but they do not

\(^{3}\text{The first relation of (58) shows that, in spite of the name, the U-supersymmetric model is more suitably parametrized by }X\text{ than by }U.\text{ Indeed }U\text{ is a continuous function of }X\text{ (a parabola), while }X\text{ is not even a single-valued function of }U:\text{ this drawback both forces to consider only a sub-part of the parabola, and may also lead to non appropriate choices of the parametrization (like in (18)) that wrongly let think of discontinuities at }U = 0\text{ which actually do not exist.}
belong to the class [54] since they are not isotropic. Even if such models exhibit a superconducting behaviour, it should be observed that a more realistic description of the superconducting transition is reached when assuming that the Hamiltonian is isotropic and the wave function breaks the lattice symmetries (otherwise there is no break-up in fact).

5) $u(2,2)$-supersymmetry. It is made of 16 generators; the even sector contains the 3 generators of the spin and the 3 ones of the $so(4)_{(-1)}$ algebra, as well as the identity and the doubly-occupied sites generator [73]. The odd sector contains the following eight generators

$$\hat{Q}_{\uparrow}(\sigma) = \sum_j \sigma^j (1 - n_j \downarrow) c_j \uparrow$$
$$\hat{Q}_{\downarrow}(\sigma) = \sum_j \sigma^j (1 - n_j \uparrow) c_j \downarrow$$
$$\hat{Q}_{\uparrow}(\sigma) = \sum_j \sigma^j n_j \downarrow c_j \uparrow$$
$$\hat{Q}_{\downarrow}(\sigma) = \sum_j \sigma^j n_j \uparrow c_j \downarrow$$

The commutations with its generators only allow the solution [86] of the EKS models. If one considered the algebra $so(4)_{(-1)}$ instead of $so(4)_{(+1)}$ in the even sector, no extended Hubbard model [24] would be found.

We wish to make some remarks about the EKS-models, defined by eqn.(66). The original EKS-model introduced in [13] corresponds to the case $\sigma = +1$ of eqn. [34]; however, we have unified $\sigma = \pm 1$ in the definition since they often share the same kind of symmetry. Here we precise the conditions for this to happen. First of all, it is easily shown that the 2-site term of the EKS-models are respectively a graded permutator [13] (the case $\sigma = +1$, denoted $\mathcal{P}^g$), and an oppositely-graded permutator (the case $\sigma = -1$, denoted $\mathcal{P}^g$): their $16 \times 16$ matrix representations are $(P^g)_{\alpha \beta} = (-1)^{p(\alpha)p(\beta)} \delta_{\alpha}^{\delta} \delta_{\beta}^{\gamma}$ and $(\bar{P}^g)_{\alpha \beta} = (-1)^{p(\alpha)+1}p(\beta)+1) \delta_{\alpha}^{\delta} \delta_{\beta}^{\gamma}$ respectively. It can also be proved that, for any homogeneous operator $\mathcal{X}_j$,

$$\mathcal{P}_{jk}^g \mathcal{X}_j \mathcal{P}_{jk}^g = \mathcal{X}_k$$

Using (72) it is easy to realize that, if $X$ is the local representing matrix of $\mathcal{X}_j$

$$[P^g , X \otimes^s 1 + 1 \otimes^s X] = 0 \quad \forall X \text{ even and odd}$$

whereas

$$[\bar{P}^g , X \otimes^s 1 + 1 \otimes^s X] = 0 \quad \forall X \text{ even}$$
$$[P^g , X \otimes^s 1 - 1 \otimes^s X] = 0 \quad \forall X \text{ odd}$$

These relations imply that, when investigating the commutation of the Hamiltonian with generators of the form $\sum_j \sigma^j \mathcal{X}_j$, both the EKS-models are found to fulfill the commutations as long as additional $\sigma^j$ appears in odd generators (see for instance the cases of $gl(2,1)$, $so(5)$ and $u(2,2)$ symmetries); in particular $\mathcal{P}^g$ is obtained if $\sigma = +1$, while $\bar{P}^g$ when $\sigma = -1$. Indeed odd generators with $\sigma = +1$ can be mapped into those with $\sigma = -1$ through the transformation $c_j \to (-1)^j c_j$ (which leaves unaltered any even generator). On the contrary, when the sign $\sigma^j$ appears in even generators, we always obtain $\mathcal{P}^g$ for the subcase $\sigma = +1$, but not $\bar{P}^g$ in the subcase $\sigma = -1$ (see for instance the cases of the $so(4)$ symmetries). Indeed the case of symmetries with respect to even generators with $\sigma = -1$ is quite peculiar; in particular the study of the generators $\eta^\pm_{\alpha}(61)$ with $\sigma = -1$ is physically intriguing for reasons of energetic stability of the ground state [39]; we shall investigate more deeply this aspect in section VIII.

Here we want to briefly recall which are, among the several classes of symmetry found above, the subcases that (up to now) have been proved to be integrable. As discussed in the previous sections, the integrability of a nearest-neighbor Hamiltonian $\mathcal{H}$ is proved whenever it is possible to find a matrix $\hat{R}(u,v)$ which solves the YBE [13] and such that its first derivative with respect to the spectral parameter equals the representation of the local hamiltonian $\mathcal{H}_{i+1}$ (see eqn. [24]).

Focussing on the symmetries and the supersymmetries considered above, we remind that as to the ‘doubly occupied sites’-symmetry, the $\hat{R}$-matrix has been found for 96 models (subcases of which are known models). Within the 2-parameter class of $gl(2,1)$ invariant models the $\hat{R}$-matrix has been found for the subcases of the EKS-models [12] and the 1-parameter U-supersymmetric models [13-20]. For the $so(5)$-symmetry and the $u(2,2)$-supersymmetry the two EKS models are the only extended Hubbard models [24] that fulfill them, as observed before. So, their $\hat{R}$-matrices
are the ones cited just above. As to the \(so(4)\)-symmetry, the \(\hat{R}\)-matrix has been found for the half filled Hubbard model \([7]\).

As anticipated in sec.VIII, there is an intriguing feature shared by the \(\hat{R}\)-matrices of all the integrable models cited just above (apart from the Hubbard model, which has a non-additive \(\hat{R}\)-matrix): they are all polynomials in the difference of the spectral parameters; more precisely, the EKS-models have first degree polynomial \(\hat{R}\)-matrices, while the U-supersymmetric have a second degree one. Even if they are not usually written in the literature as polynomials (because the form in which they are given depends on the method used to find them as solutions of the Yang-Baxter equation), it is easy to realize that such \(\hat{R}\)-matrices can be cast into polynomials through a mere multiplication by a scalar function \(\phi\) or redefinition of the spectral parameters (see \([14]\)). This fact means that all the known models (apart from ordinary Hubbard) can be found by means of the Polynomial \(\hat{R}\)-matrix Technique discussed in sec.VIII. In particular we therefore have that all the constants of motion of the \(U\)-supersymmetric model enjoy the \(gl(2,1)\)-supersymmetry. As to the EKS models \([23]\) the conclusion is still more general; for the case \(\sigma = +1\) we have the graded permutator, and thus, using eqns.\((23)\) we deduce that the constants of motions \(J_n\) of this models fulfill any global symmetry \(\sum_j X_j\); for the case \(\sigma = -1\) we have an oppositely graded permutator and therefore (due to eqns.\((74)-(75)\)) we have that the constants of motion \(J_n\) commute with \(\sum_j X_j\) for any \(X\) even, and with \(\sum_j (-1)^j X_j\) for any \(X\) odd. Further, first degree polynomial \(\hat{R}\)-matrices for 96 models have been found in \([25]\), all these Hamiltonians are shown to fulfill the ‘doubly occupied symmetry’ \(su(2) \oplus u(1) \oplus u(1)\); now we can deduce that all the constants of motion of these models fulfill such symmetry as well.

Among the symmetries envisaged in this section, the case left to be considered on the point of view of integrability is the one of \(so(4)\)-invariant models. This class is very large (contains a number of free parameter), and in spite of the fact that it is quite interesting on a physical point of view, it has been investigated only partially up to now. We shall therefore devote the whole section IX to it. The PRT will be applied together with the theorem of section VI to find \(so(4)\)-invariant models and to deduce the symmetries of their constants \(J_n\).

IX. INTEGRABLE MODELS WITH \(SO(4)\)-SYMMETRIES

The class of \(so(4)\)-invariant extended Hubbard models (see eqns.\((24)\) and \((23)\)) is relevant in physics because, by exploiting its properties of symmetry, some interesting features on the eigenstates of these models can be easily deduced, such as the form of the correlation functions, which exhibit a Long Range Order. It is then important to have as many exact results as possible for such class. We recall that it is a rather wide class, since \(so(4)\)\(_{(-1)}\) allows 5 free parameters and \(so(4)\)\(_{(+1)}\) 4 free parameters.

Among the known \(so(4)\)-invariant models, the most famous is perhaps the ordinary Hubbard model at half filling \([24]\) (which corresponds to the subcase \(X = W = Y = P = 0\) of \(so(4)\)\(_{(-1)}\), see eqs.\((24)\)). The \(\hat{R}\)-matrix for this model was found in \([7]\): also it has been proved that its constants of motion fulfill such symmetry as well. Later, the EKS-models and the AAS model were also found, for which the phase diagrams were derived; they enjoy the \(so(4)\)\(_{(+1)}\)-symmetry. Moreover, in the model recently studied by Alcaraz and Bariev within the Coordinate Bethe Ansatz, an \(so(4)\)\(_{(-1)}\)-invariant subcase \((\eta = 0 \text{ and } \epsilon = +1\) in \([31]\)) can be identified; it corresponds to the 1-parameter subclass \(X = (1 - \sin \theta) t ; Y = -W = t \cos \theta\) and \(P = 0\) of eqs.\((22)\).

Motivated by the physical interest in the \(so(4)\)-symmetries, in this section we shall find further models enjoying such symmetry. More precisely, we shall make use of the polynomial \(\hat{R}\)-matrix technique to find all the integrable \(so(4)\)-invariant extended Hubbard models that are derived from a first degree polynomial \(\hat{R}(u)\) in the spectral parameter \(u\). We consider here both \(so(4)\)\(_{(+1)}\) and \(so(4)\)\(_{(-1)}\). The number of such models turns out to be 32, 16 with symmetry \(so(4)\)\(_{(-1)}\) and 16 with \(so(4)\)\(_{(+1)}\); the EKS and AAS models are shown to be among them. In providing the \(\hat{R}\)-matrix for all of them, we shall guarantee their integrability; moreover, by exploiting the result of the theorem proved in section VI, we will be able to show that not only the Hamiltonian but the whole set of constants of motion of these models are \(so(4)\)-invariant. All these models have the natural properties of preserving the total magnetization and charge and of being isotropic, since they belong to the class \([3]\) by construction.

In order to show how to find them, we have to use as \(\hat{R}^{(1)}\) the matrix \((59)\) of the extended Hubbard models, with the additional prescriptions \((22)\) and \((23)\) of the \(so(4)\)-symmetries, and impose that \(\hat{R}^{(1)}\) fulfills eqns.\((24)\). We give in the following the values of their coupling constants, as well as the \(\hat{R}\)-matrices that yield them. In order to simplify the notation we divide the 32 models into four subgroups according to vanishing or non-vanishing pair hopping and exchange amplitudes; they are denoted \(\mathcal{H}^{(a)}\), \(\mathcal{H}^{(b)}\), \(\mathcal{H}^{(c)}\) and \(\mathcal{H}^{(d)}\) respectively. The result are presented in a compact form: the case \(\sigma = +1\) gives the models that are invariant under \(so(4)\)\(_{(+1)}\) whereas \(\sigma = -1\) corresponds to the \(so(4)\)\(_{(-1)}\)-symmetric cases.

1st subgroup: \(\mathcal{H}^{(a)} = 8\) models with \(Y \neq 0\) and \(W \neq 0\)
\begin{align}
X &= t \quad \tilde{X} = (1 - \sigma)t \quad U = 2s_1t \quad V = s_1t \quad W = -s_2 \\
Y &= \sigma s_1 t \quad P = -(s_1 + s_2)t \quad Q = 2(s_1 + s_2)t \quad \mu = -2s_1t \\
\end{align}

\textbf{2}\textsuperscript{nd} subgroup: \( \mathcal{H}^{(b)} = 8 \) models with \( Y \neq 0 \) and \( W = 0 \)
\begin{align}
X &= t \quad \tilde{X} = (1 - \sigma)t \quad U = 2s_1t \quad V = (s_1 + s_2)t \quad W = 0 \\
Y &= \sigma s_1 t \quad P = -(s_1 + 2s_2)t \quad Q = 2(s_1 + 2s_2)t \quad \mu = -2s_1t \\
\end{align}

\textbf{3}\textsuperscript{rd} subgroup: \( \mathcal{H}^{(c)} = 8 \) models with \( Y = 0 \) and \( W \neq 0 \)
\begin{align}
X &= t \quad \tilde{X} = (1 - \sigma)t \quad U = 4s_1t \quad V = s_1t \quad W = s_2t \\
Y &= 0 \quad P = (-2s_1 + s_2)t \quad Q = (4s_1 - 2s_2)t \quad \mu = -2s_1t \\
\end{align}

\textbf{4}\textsuperscript{th} subgroup: \( \mathcal{H}^{(d)} = 8 \) models with \( Y = 0 \) and \( W = 0 \)
\begin{align}
X &= t \quad \tilde{X} = (1 - \sigma)t \quad U = 4s_1t \quad V = (s_1 + s_2)t \quad W = 0 \\
Y &= 0 \quad P = -2(s_1 + s_2)t \quad Q = 4(s_1 + s_2)t \quad \mu = -2s_1t \\
\end{align}

For the Hamiltonians \( \mathcal{H}^{(a)} \) the subcases with \( s_2 = -s_1 \) and \( \sigma = +1 \) are the two EKS models, while the subcase \( s_2 = -s_1 = 1 \) and \( \sigma = -1 \) is the model proposed in [31] with \( \eta = 0 \) and \( \epsilon = +1 \). No \( U \)-supersymmetric model appears, since one always has \( \tilde{X} \neq X^2 \). The case \( \mathcal{H}^{(d)} \) with \( s_2 = -s_1 \) and \( \sigma = +1 \) is the AAS-model [32, 33]. The models (76)-(77)-(78)-(79) can be obtained from the following \( \tilde{R} \)-matrices
\begin{align}
\tilde{R}(u) = I + u (H_{2\text{-sites}} + s_1 I) \\
\end{align}
where \( H_{2\text{-sites}} \), with \( i = a, b, c, d \) is the 2-site matrix (59) of coupling constants (76), (77), (78) and (79) respectively. Since the above \( \tilde{R} \)-matrices are first degree polynomials in \( u \), we have by construction that \( [\tilde{R}(u), X_{\text{so}(4)} \otimes \eta \mathbb{1} + \sigma \mathbb{I} \otimes \theta X_{\text{so}(4)}] = 0 \) where \( X_{\text{so}(4)} \) is the local representing matrix of the single-site generator \( E_{J_4}^3 = c_{J_4}^\dagger c_{J_4}^{\dagger} \), i.e. the \( 4 \times 4 \) matrix \( E_3^3 \) (see sec. II). Thus, thanks to the theorem of section VI, we can deduce that
\begin{align}
[\mathcal{J}_n, \sum_j \sigma_j c_j^\dagger c_{J_4}^{\dagger}] = 0. \\
\end{align}

We wish now to discuss how to deduce some physical properties of the models found above. We shall provide here a general scheme of techniques that allow to obtain information about the spectrum, the phase diagram of the ground state and the behaviour of the correlation functions for the 32 models presented above. Details on the application of the method to some specific case are given in separate publications [40, 43]. At the end of this section we provide as an example some results obtained for two models of group 2.

In the first instance we shall describe the method to derive the spectrum of the above models; such method is based on the observation that the Hamiltonian of the above 32 models (76) \( \div \) (79) can be proved to be of the form:
\begin{align}
\mathcal{H}^{(i)} = - \sum_j \Pi_{j, j+1} \\
\end{align}

(up to the trivial additive term \( s_1 t I \)) where \( \Pi_{j, j+1} \) is a 2-site generalized permutator. A 2-site generalized permutator (GP) acts on two neighboring sites exchanging some couples of states but leaving unchanged some other couples; more explicitly:
\begin{align}
\Pi_{j, j+1} \left\{ \alpha_j \left| \beta_j \right\rangle \right. j+1 = \theta_0 |\beta_j \left| \alpha_j \right\rangle + \theta^d |\alpha_j \left| \beta_j \right\rangle j+1 \\
\end{align}

\textsuperscript{4}As is clear from eq. (24), the \( \tilde{R} \)-matrix (80) yields \( \mathcal{H}^{(a)}, \mathcal{H}^{(b)}, \mathcal{H}^{(c)} \) and \( \mathcal{H}^{(d)} \) up to an additional energy shift \( s_1 t I \), which obviously changes nothing to the integrability of such models.
where $\theta^e_{\alpha\beta}$ and $\theta^d_{\alpha\beta}$ are two discrete-valued (0, +1 or -1) and ‘complementary’ ($|\theta^e_{\alpha\beta}| = 1 - |\theta^d_{\alpha\beta}|$) functions that identify the exchange / non-exchange respectively; notice that in both cases an additional sign is possible (if $\theta^e, \theta^d = -1$); moreover, $\theta^e_{\alpha\beta} = \theta^o_{\beta\alpha}$, due to hermiticity. These functions $\theta^e_{\alpha\beta}$ and $\theta^o_{\alpha\beta}$ completely determine the specific kind of GP one deals with.

In pass, it can be proved \[23\] that within the class of the extended Hubbard models \[54\] there are 96 (and only 96) models that can be cast in the form \[83\], with a suitable GP; the 32 models \[76\] \div \[23\] that we have presented here constitute the subclass of all the so(4)-invariant GP, and it is also possible to see that they cannot be mapped into other GP through any unitary transformation.

Remarkably, recognizing that a Hamiltonian has the structure \[81\] allows to exploit the Sutherland’s Species Technique \[40, 41\], which actually consists in regarding a GP as an ordinary permutator between the so-called Sutherland’s species (SS), the latter being groups of the $d$ local physical states. The groups of states are not determined by the original Fock Space but uniquely by the structure of the Hamiltonian; the SS therefore relate to the dynamical processes \[81\] where $\theta_{\alpha\beta}$ constitutes the subclass of all the $d$-fermionic SS. Different models that are recognized to be of the same SS-types also share the same structure of Coordinate or Algebraic Bethe Ansatz equations; one is therefore free to choose the type of SS:

| $1^{st}$ subgroup | $F^4$ ; $B^2F^2$ ; $B^4$ |
| $2^{nd}$ subgroup | $F^3$ ; $B F^2$ ; $B^2F$ ; $B^3$ |
| $3^{rd}$ subgroup | $F^3$ ; $B F^2$ ; $B^2F$ ; $B^3$ |
| $4^{th}$ subgroup | $B^2$ ; $B F$ ; $F^2$ |

where $B^n F^m$ characterizes a model with $n$ bosonic and $m$ fermionic SS. Different models that are recognized to be of the same SS-types also share the same structure of Coordinate or Algebraic Bethe Ansatz equations; one is therefore free to choose the type of SS.

The whole spectrum can be found in all these cases. However, it must be realized that the actual degeneracy of each eigenvalue depends on the way the Sutherland’s species are realized in terms of physical species in each specific model. This in turn enters the calculation of the partition function, determining different physical features (for the discussion in case of BF models see \[13\]).

Within the spectrum, the ground state is particularly worth of interest, because these Hamiltonians are expected to describe materials that exhibit peculiar physics at low temperatures. For the $F^P$ case, it can be shown \[41\] that the minimum of the energy, whose eigenvalue reads

$$\epsilon(\{n_i\}) = -2 \sum_{l=0}^{L-1} \cos \left(\frac{2\pi l}{L}\right) n_l$$  \hspace{1cm} (84)

where $n_l = 0, 1$ are quantum numbers and $L$ the length of the periodic chain.

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$$\epsilon_0 = 1 - \frac{2}{P} \int_0^1 dx \frac{x^{\frac{1}{2}-1}}{1-x}$$  \hspace{1cm} (85)

is reached at equal densities of all fermionic species. Interestingly such result, when applied to the $F^3$ and $F^4$ models in \[83\], implies that the ground state in these cases always contains doubly occupied sites, which are expected to model short coherence length pairs.

Also, again as far as the ground state is concerned, it has been shown \[10\] that Sutherland’s theorem (originally formulated \[11\] for ordinary permutators) can be extended to the generalized permutators, and it is thus possible to assert that, in the thermodynamic limit, the ground state energy (per site) $\epsilon_0$ of a $B^n F^m$ problem is equal to that of a $BF^m$ problem. The latter types become therefore the only relevant ones as to the issue of determining $\epsilon_0$.

We also want to stress another important general feature of the 32 models \[76\] \div \[79\]; their Hamiltonians $H^{(i)}$, aside the $so(4)$-symmetry, also preserve the number of doubly occupied sites (see section VII), since they all have $X = t$. 

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Hence, by adding to each $\mathcal{H}^{(i)}$ ($i = a, b, c, d$) further terms of on-site Coulomb repulsion and chemical potential with arbitrary coupling constants,

$$\mathcal{H}'^{(i)} = \mathcal{H}^{(i)} + U \sum_{j} n_{j\uparrow} n_{j\downarrow} + \mu \sum_{j} (n_{j\uparrow} + n_{j\downarrow}) , \quad i = a, b, c, d ,$$

(86)

the obtained models $\mathcal{H}'^{(i)}$ still preserve the number of spin-up, spin-down electrons separately, as well as the number of doubly occupied sites. Such property helps to diagonalize the Hamiltonian $\mathcal{H}'^{(i)}$ within each subspace of given eigenvalues of $S^2$, $\eta^2$ and $K$; this is interesting from a physical point of view, because one can investigate how the features of the model change when tuning either the parameter $U$ or the filling $n$ (which can be expressed in terms of $\mu$); the former is somehow an intrinsic energy unit for these models, and is actually the most meaningful parameter for the systems where strong electronic correlations are involved, as pointed out by Hubbard himself in his original paper [20]; the latter is also relevant in that it can model the degree of hole doping in the material. The two parameters $U$ and $n$ are expected to drive the transitions between physically different phases (Metallic, Insulator, Superconducting). The so(4)-symmetry of the part $\mathcal{H}^{(i)}$ of eq. (86) implies that

$$[\mathcal{H}'^{(i)}, \eta^\pm_{(\sigma)}] = (U - \frac{\mu}{2}) \eta^\pm_{(\sigma)} .$$

(87)

Eq. (87) is crucial in order to implement the so-called $\eta$-pairing mechanism for the construction of eigenstates of $\mathcal{H}'^{(i)}$: indeed, once a reference eigenstate $|\text{ref}\rangle$ of $\mathcal{H}'^{(i)}$ is given, further eigenstates $|\Psi_{(\sigma)}\rangle$ can be found by applying $\eta^\pm_{(\sigma)}$ on $|\text{ref}\rangle$, namely $|\Psi_{(\sigma)}\rangle = (\eta^\pm_{(\sigma)})^m |\text{ref}\rangle$. This property allows to reduct to the calculation of the correlation functions for the eigenstates $|\Psi_{(\sigma)}\rangle$ to those of the reference states $|\text{ref}\rangle$, which are in general easier to be computed. Within the class of the above 32 models, meaningful reference states can be found: the vacuum $|0\rangle$ is always one (the eigenstates $|\Psi_{(\sigma)}\rangle$ constructed on it are called pure $\eta$-pair states); the eigenstates $|U = \infty\rangle$ of the $U = \infty$ Hubbard model [36] are also reference states whenever $W = V = 0$; similarly, for the cases in which $W = V = \pm t$ the eigenstates $|t - J\rangle$ of the $t - J$-model [33][34] are reference states. In these cases the eigenstates $|\Psi_{(\sigma)}\rangle$ constructed on $|U = \infty\rangle$ or on $|t - J\rangle$ are referred to as mixed $\eta$-pair states. The correlation function for these reference states have been investigated in the literature (see for instance [15][33][34]).

The techniques outlined above are quite general and apply to all the 32 so(4)-models ($\Sigma 32$). As already observed, all the models $\mathcal{H}^{(i)}$ of a given $B^n F^m$-type share the same spectrum equation structure. However, this does not necessarily imply that the related $\mathcal{H}'^{(i)}$ have all the same physical features; indeed the presence of the on-site Coulomb repulsion $U$ has two main effects: i) it can change the eigenvalues and their degeneracy; ii) it can lead, according to how the SS are actually realized in terms of physical species, to quite different shapes of the ground state phase diagram as a function of $U$, even for models of the same type (see for instance [40] for the BF-type).

We just wish now to provide here a concrete example of the use of the scheme proposed in this section, by presenting results on the ground state of the models [33][34] with $\mathcal{H}^{(i)} = \mathcal{H}^{(i)}$ (subgroup 2), and $s_1 = s_2 = -1$; the coupling constants of these two models explicitly read $X = t$; $\tilde{X} = (1 - \sigma) t$; $Y = \sigma t$; $W = V = 0$; $Q = -2P = 2t$; $U$ and $\mu$ can be taken as arbitrary, as observed above. The part $\mathcal{H}^{(i)}$ is a GP for 3 Sutherland’s species, namely $F = \{|\uparrow\rangle, |\downarrow\rangle\}$, $B_1 = |0\rangle$, and $B_2 = |\downarrow\rangle$. Using the extension of Sutherland’s theorem one can derive the structure of the ground state phase diagram as a function of $U$ and $n$; details of this derivation can be found in [40]. The result is presented in fig[3].

The structure is the same for both models $\sigma = \pm 1$. Four regions can be recognized: in region I the ground state is made of doubly occupied and empty sites; the eigenvalue of the energy (per site) reads: $\epsilon_0 = (U/2 - 1) n$; the eigenvectors are pure $\eta$-pair states; the behavior of the two-particle reduced density matrix

$$(\rho_2)_{i,j} = \frac{m \langle \Psi_{(\sigma)} | \epsilon_{i\downarrow} \epsilon_{i\uparrow} \epsilon_{j\downarrow} \epsilon_{j\uparrow} | \Psi_{(\sigma)} \rangle m}{m \langle \Psi_{(\sigma)} | \Psi_{(\sigma)} \rangle m}$$

(88)

in this region is the following [15][33][34].

5To simplify the notation, we omitted any subscript $\sigma$ in denoting $\mathcal{H}^{(i)}$ and $\mathcal{H}'^{(i)}$: we precise that, in eq.(88), one has to take $\eta^\pm_{(+1)}$ (resp. $\eta^\pm_{(-1)}$) when $\mathcal{H}^{(i)}$ in $\mathcal{H}'^{(i)}$ contains $\sigma = +1$ ($\sigma = -1$); see eqns. (74) [(79)].
$$(\rho_2)_{i,j} \overset{|i-j|\to \infty}{\longrightarrow} e^{i\pi(1-\sigma)(i-j)} n_{\uparrow\downarrow}(1-n_{\uparrow\downarrow})$$  \hspace{1cm} (89)$$

where $n_{\uparrow\downarrow}$ is density of doubly occupied sites. As eq. (88) shows, in region I one has a Off-Diagonal Long-Range-Order in this state, which implies that the model is superconducting in this region. It is also possible to see that, with respect to the AAS model, where the pure $\eta$-pair phase is degenerate in the momentum of pairs [33], here the pair hopping term selects a specific momentum. This momentum (which is in principle observable through neutronic spectroscopy) is strictly related to the kind of so(4)-symmetry allowed by the model; in particular, for the case of so(4)$_{\pm 1}$ (i.e. $\sigma = +1$ in eqns.(77)) the selected momentum is $q = 0$, while for the case of so(4)$_{-1}$ (i.e. $\sigma = -1$ in eqns.(77)) the pairs have momentum $q = \pi$. The EKS model has 0-momentum pairs (see [15]).

In region II all the possible types of sites (empty, singly and doubly occupied) in the ground state are present; the energy reads

$$\epsilon_0 = (1 - \frac{U}{2}) \left( \frac{1}{\pi} \arccos \left( \frac{1}{2} - \frac{U}{4} \right) - n \right) - \frac{2}{\pi} \sqrt{1 - \frac{1}{4}(1 - \frac{U}{2})^2} ,$$ \hspace{1cm} (90)$$

and the eigenstate is mixed $\eta$-pair, constructed on the $|\text{ref}\rangle = |U = \infty\rangle$ states. The correlation function behaves in this case [33] as:

$$\langle \rho_2 \rangle_{i,j} \overset{|i-j|\to \infty}{\longrightarrow} e^{i\pi(1-\sigma)(i-j)} n_{\uparrow\downarrow}\frac{(1-n_{\uparrow\downarrow}-n_s)}{(1-n_s)^2} \langle (1-n_i)(1-n_j) \rangle_{U=\infty}$$  \hspace{1cm} (91)$$

where $n_s$ is the density of singly occupied sites. This region is again superconducting and is particularly interesting in that it survives up to positive values of $U$ (which are expected to be more physically meaningful, because the electronic on-site interaction should be repulsive, even if partially screened by the phononic effective attraction). This feature is shared by the other known models like the AAS and the EKS ones; however it must be observed that, around half-filling, the region II of our models raises up to $U^{max} = 6t$, the highest value of 1-D exactly known models. Finally, region III-a is metallic; the eigenvalue is $\epsilon_0 = -2\pi^{-1} \sin(\pi n)$, while the eigenstates are those of the $|U = \infty\rangle$ model; region III-b is the particle-hole transformed of region III-a ($\epsilon_0 = +2\pi^{-1} \sin(\pi n)$). At half-filling it is possible to show that a charge gap $\Delta = U - 6t$ exists, which makes the model an Insulator.

This is the picture of the ground state; for excited states one cannot make use of Sutherland’s theorem and is re-conducted to the spectrum of a whole $B^2F$ case. On the physical point of view, when the temperature is turned on, thermal fluctuations are expected to break the Long-Range-Order of the superconducting regions, according to Mermin-Wagner theorem. Nevertheless a Quasi-Long-Range Order can survive, because the decay of $\langle \rho_2 \rangle_{i,j}$ can preserve a tail over macroscopically observable distances.

Finally, we want to briefly comment on the mutual relationship between the 32 so(4)-invariant models $\mathcal{H}^{(i)}$. Within each of the 4 subgroups, there are transformations mapping some models into others; in particular a model $\mathcal{H}^{(i)}$ characterized by $(s_1, s_2, \sigma)$ in its parameters – see eqns.(10) and (11) – is connected to the one with $(-s_1, -s_2, \sigma)$ through the transformation $c_{i,s} \rightarrow (-)^s c_{i,s}$; analogously, a model with $(s_1, s_2, \sigma)$ in its parameters can be mapped into that with $(s_1, s_2, -\sigma)$ through $c_{i,s} \rightarrow [1 - (1 - (-)^s)n_{i,-s}]c_{i,s}$. Also, the transformation $c_{i,\uparrow\downarrow} \rightarrow (-1)^s c_{i,\uparrow\downarrow}$ maps the subgroup $\mathcal{H}^{(b)}$ into the $\mathcal{H}^{(c)}$.

An exhaustive classification of all these transformations is out of the purposes of the present paper, and will be treated in a forthcoming paper. Here we just want to emphasize the following aspects. In the first instance the above transformations act differently on neighboring sites; this implies that they are not graded similarity transformations, because (in matrix representation) they are of the form $A \otimes^s B$, and not $A \otimes^s A$ (see section V). Indeed, by exploiting the matrix representation developed in section II it is possible to see that none of the above models is connected to any other through a graded similarity transformation; therefore, as far as the $\bar{R}$-matrix is concerned, they are all independently integrable. Secondly, it is also possible to show that the models with different values of the parameter $Q$ cannot be connected by any kind of transformation (not only similarity). Thirdly, it should be pointed out that, even if some transformation maps a given model into another, it is not that obvious in general that the ground state of the former is mapped into the ground state of the latter. This confirms that, also for what concerns their physical stability properties, all the above models deserve a deep interest.

**X. CONCLUSIONS**

In the preliminary part of this paper we have clarified the relationship between the two approaches to fermionic integrable systems, by providing a systematic method to pass from fermionic operators to matrix representation
and from the C-number YBE to fermion models. Further, we have proved general results about the symmetries in integrable systems: we have treated the symmetries of the YBE, showing that it is invariant under graded similarity transformations. Also, we have proved, under quite general hypothesis, that the symmetries imposed on the $\bar{R}$-matrix directly reflect onto the whole set of the constants of motion $J_n$ that the QISM determines. The complementary question whether the symmetries of the Hamiltonian reflect on the $\bar{R}$-matrix has been widely developed with the PRT. These two general results are shown to be very effective when combined together: in particular we applied both of them to the study of the extended Hubbard models (EHM). By making use of the matrix representation discussed above, we found all the EHM that fulfill different kinds of symmetries and supersymmetries (‘doubly occupied sites’, $so(4)$, $gl(2,1)$, $so(5)$, $u(2,2)$), also including in the generators possible relative signs between two neighboring sites. In particular, for the 2-parameter subclass with $gl(2,1)$ supersymmetry, we have provided a geometrical construction expressing any $gl(2,1)$ model as a linear combination of the EKS-model and a U-supersymmetric model. Further, by showing that most of the integrable known model (such as the EKS and the U-supersymmetric and others) are reproduced by first/second degree polynomial $\bar{R}$-matrices, we deduce the symmetries of their constants of motion. Finally, focusing on the case of $so(4)$-symmetries (whose interest in condensed matter has been discussed), we exploited the PRT to find all the EHM that are derivable from first degree polynomial $\bar{R}$-matrices. The symmetries of the constants of motion of these models have also been discussed. Further, by making use of the Sutherland’s species technique, we have proposed a general scheme to derive the spectrum and other physical properties of these 32 models; in particular it has been observed that, by means of Sutherland’s theorem, the ground state energy can be determined for all of them. Finally, thanks to the $so(4)$ symmetry, the addition of arbitrary Coulomb repulsion and chemical potential terms to these integrable models can be used to implement the $\gamma$-pairs construction. We proposed an example of concrete application of the described techniques to two of the 32 models, for the ground state of which we have explicitly given the eigenvalue, the eigenvector and the pair correlation functions in the different region of the phase diagram (fig.2).

We also point out that all the extended Hubbard models investigated above can also be used as ‘bulk systems’ to which one can add appropriate boundary interaction terms; this would lead to new results in the context of models with Kondo impurities (see for instance [14]).

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FIG. 1. The $X$-$W$ plane to describe the 2-parameter class of $gl(2,1)$-invariant models. The coupling $X$ stands for correlated hopping amplitude, while $W$ is the exchange term. The allowed values are the two half planes $W/t > -\sigma$ and $W/t < -\sigma$, with the addition of the single point $E = (1, -\sigma)$ that represents the EKS models ($\sigma = \pm 1$). The 1-parameter subclass of U-supersymmetric models are represented by the $X$-axis ($W = 0$).

Any $gl(2,1)$-invariant model (point $P$) can be written as a linear combination of the EKS model and a particular U-supersymmetric model, whose parameters can be geometrically determined by tracing the line from $P$ to $E$, and finding the ascissa $X_A$ of the point $A$ in which it intersects the $X$-axis.
FIG. 2. Ground state phase diagram of the two models $X = 1; \tilde{X} = (1 - \sigma); Y = -\sigma; P = -1; Q = 2$ (subgroup 2, eq. (77)) from ref. [41]. The model exhibits an insulator-superconductor transition at $n = 1$, for $U_c = 6$. The dashed line is the EKS model, and the dotted line corresponds to the AAS model.