A REGULARIZATION-FREE APPROACH TO THE CAHN-HILLIARD EQUATION WITH LOGARITHMIC POTENTIALS

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Abstract. We introduce a regularization-free approach for the wellposedness of the classic Cahn-Hilliard equation with logarithmic potentials.

1. Introduction. Consider the 2D Cahn-Hilliard equation

\[
\begin{cases}
\partial_t u = \Delta \mu = \Delta (\nu \Delta u + F'(u)), & (t, x) \in (0, \infty) \times \Omega; \\
\left. u \right|_{t=0} = u_0, 
\end{cases}
\]

where \(\mu\) denotes the chemical potential, and \(u\) is the order parameter which corresponds to the rescaled local concentration in a binary mixture. For simplicity we shall take the domain \(\Omega = [-\frac{1}{2}, \frac{1}{2}]^2\) as a periodic torus in dimension two and note that other boundary conditions can also be covered with suitable modifications. We set the coefficient \(\nu > 0\) as a constant, although in general, it depends on the order parameter. The thermodynamic potential \(F : (-1, 1) \to \mathbb{R}\) is given by (see [4])

\[
F(u) = \frac{\theta}{2} \left( (1 + u) \ln(1 + u) + (1 - u) \ln(1 - u) \right) - \frac{\theta_c}{2} u^2, \quad 0 < \theta < \theta_c;
\]

\[
f(u) = F'(u) = -\theta_c u + \frac{\theta}{2} \ln \frac{1 + u}{1 - u}, \quad F''(u) = \frac{\theta}{1 - u^2} - \theta_c.
\]

Denote by \(u_+ > 0\) the positive root of the equation \(\frac{1}{a} \ln \frac{1 + u}{1 - u} = \frac{2\theta_c}{\theta}\). The potential \(F\) takes the form of a double-well with two equal minima at \(u_+\) and \(-u_+\) which are usually called binodal points. For \(u_s = (1 - \theta \theta_c)^{\frac{1}{2}}\), the region \((-u_s, u_s)\) where \(F''(u) < 0\) is called the spinodal interval. If \(\theta\) is close to \(\theta_c\), i.e., the quenching is shallow, one can expand near \(u = 0\) and obtain the usual quartic polynomial approximation of the free energy as

\[
F(u) \approx F_{\text{quartic}}(u) = \frac{\theta}{2} \cdot \frac{u^4}{6} + \left( \frac{\theta}{2} - \frac{\theta_c}{2} \right) u^2.
\]

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Alternatively, one can use \(1/(1-u^2) \approx 1+u^2\) to derive \(F''(u) \approx \theta(1+u^2) - \theta_c\). The standard double-well potential const \((u^2-1)^2\) corresponds to the choice \(\theta/\theta_c = 3/4\).

The system (1) is a gradient flow of a Ginzburg-Landau (GL) type energy functional \(\psi(u)\) in \(H^{-1}\), i.e.,

\[
\partial_t u = -\frac{\delta \psi}{\delta u} \bigg|_{H^{-1}} = \Delta \left( \frac{\delta \psi}{\delta u} \right)_{L^2},
\]

where \(\frac{\delta \psi}{\delta u} \bigg|_{H^{-1}}\) and \(\frac{\delta \psi}{\delta u} \bigg|_{L^2}\) denote the standard variational derivatives in \(H^{-1}\) and \(L^2\) respectively, and

\[
\psi(u) = \int_{\Omega} \left( \frac{1}{2} |\nabla u|^2 + F(u) \right) dx.
\]

Here the gradient term in the GL energy accounts for short range interactions in the material. It is derived by an approximation of a nonlocal term representing long range interactions [4]. A rigorous derivation of the nonlocal Cahn-Hilliard equation dates back to the work of Giacomin and Lebowitz [10, 11], which considered a lattice gas model with long range Kac potentials. Further results such as regularity and traveling waves on these and similar models can be found in [3, 9, 2] and the references therein.

Concerning the logarithmic Cahn-Hilliard equation with constant mobility, Elliott and Luckhaus in [8] considered the case of a multi-component mixture and proved (for the Neumann boundary condition) that if \(u_0 \in H^1\) satisfies \(\|u_0\|_\infty \leq 1\) with space average in \((-1,1)\), then there exists a unique global solution \(u \in C^0_t H^{-1} \cap L^\infty_t H^1_x\), \(\partial_t u \in L^2_{t,loc} H^{-1}_x\), \(\sqrt\nu \partial_t u \in L^2_{t,loc} H^1_x\) and \(\|u\|_\infty \leq 1\). Furthermore, the set \(\{|u| = 1\}\) has measure zero. The key idea in [8] is to work with a regularized problem where the logarithmic term is replaced by

\[
\phi_\epsilon(r) = \begin{cases} \ln r, & r \geq \epsilon; \\ \ln \epsilon - 1 + \frac{r}{\epsilon}, & r < \epsilon. \end{cases}
\]

In [7], Debussche and Dettori adopted a different regularization of \(F(u)\)

\[
F_N(u) = -\frac{\theta_c}{2} u^2 + \theta \sum_{k=0}^{N} \frac{u^{2k+2}}{(2k+1)(2k+2)}.
\]

For \(L^2\) or \(H^1\) initial data \(u_0\) with \(\|u_0\|_\infty \leq 1\) and \(m(u_0) \in (-1,1)\) (with Neumann or periodic boundary conditions), they proved the existence and uniqueness of solutions as well as continuity of the semigroup. In [16] Miranville and Zelik introduced another approximation by using viscous Cahn-Hilliard equations, namely

\[
\begin{cases}
\epsilon \partial_t u + (-\Delta)_N^{-1} \partial_t u = \Delta u - f(u) + \langle f(u) \rangle, & \epsilon > 0; \\
\partial_n u \bigg|_{\partial\Omega} = 0,
\end{cases}
\]

where \(\langle v \rangle := |\Omega|^{-1} \int_\Omega v(x) dx\) and \((-\Delta)_N^{-1}\) denotes the inverse Laplacian with Neumann boundary conditions acting on \(L^2_0(\Omega) = \{v \in L^2(\Omega) : \langle v \rangle = 0\}\). In [1] Abels and Wilke used a different approach based on the powerful theory of monotone operators. It is worthwhile pointing out that to show the subgradient \(\partial F(\epsilon)\) is single-valued (see Theorem 4.3 on P3183 of [1] and the proof) one still needs some suitable approximation of the potential by smooth ones (since the derivative goes to
Theorem 1.1. Let mild solutions (see Proposition 1). Roughly speaking, the main result of this note terms. The local wellposedness and uniqueness is then a breeze thanks to the use of 

One should note that as long as \( \|g\|_\infty < \infty \) we can guarantee \( \|u\|_\infty < 1 \) which corresponds to strict phase separation. The governing equation for \( g \) takes the form \( \partial_t g = -\Delta^2 g + O(\partial(e^{C_1 g} F_1(tanh g, g, \partial g) \partial^2 g)) + O(\partial^2 (e^{C_2 g} F_2(tanh g, g, \partial g))) + \cdots \), where \( C_1, C_2 \) are constants, \( F_1 \) and \( F_2 \) are polynomials, and \( \cdots \) represent similar terms. The local wellposedness and uniqueness is then a breeze thanks to the use of mild solutions (see Proposition 1). Roughly speaking, the main result of this note is the following.

**Theorem 1.1.** Let \( g_0 \in H^2(\Omega) \) and recall \( \Omega = [-1, 1]^2 \) is the periodic torus in 2D. Then there exists a unique global solution \( g \in C^0_t H^2 \). Moreover, for any \( t > 0 \), \( g(t, \cdot) \in H^k \) for all \( k \geq 2 \). In particular, there is strict phase separation for all \( t > 0 \).

The proof of Theorem 1.1 is subsumed in Proposition 1 and Theorem 2.1. Here to keep the argument light, we choose to work with subcritical data having \( H^2 \) regularity. To continue the local solution for all time we make use of the conservation law in conjunction with a bootstrap argument. This part of the argument is technical, and details are presented in the proof of Theorem 2.1.

**Notation.** For any real number \( a \in \mathbb{R} \), we denote by \( a^+ \) the quantity \( a + \epsilon \) for sufficiently small \( \epsilon > 0 \). The numerical value of \( \epsilon \) is unimportant, and the needed smallness of \( \epsilon \) is usually clear from the context. The notation \( a^- \) is similarly defined. This notation is particularly handy for interpolation inequalities.

For any two quantities \( X \) and \( Y \), we denote \( X \lesssim Y \) or \( X = O(Y) \) if \( X \leq CY \) for some constant \( C > 0 \). For any quantities \( X_1, X_2, \ldots, X_N \), we denote by \( C(X_1, \ldots, X_N) \) a positive constant depending on \( (X_1, \ldots, X_N) \).

For convenience we collect the identities for hyperbolic functions:

\[
\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}, \quad \cosh x = \frac{e^x + e^{-x}}{2}, \quad \text{sech}(x) = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}};
\]

\[
\frac{d}{dx} \tanh x = 1 - \tanh^2 x = \text{sech}^2(x), \quad \frac{1}{1 - \tanh^2 x} = \cosh^2 x.
\]
For any $f \in L^1(\Omega)$, we denote the mean value of $f$ as
$$\bar{f} = |\Omega|^{-1} \int_{\Omega} f(x)dx.$$

2. Analysis of the $g$ equation. We first derive the $g$ equation. Denote
$$\begin{cases}
u_t = \Delta K, \\
K = -\nu \Delta u - \theta \nu + \frac{\theta}{2} \log(\frac{1 + u}{1 - u}), \quad u \in (-1, 1).
\end{cases}$$

Define $g = \frac{1}{2} \log(\frac{1 + u}{1 - u})$. Then clearly
$$K = -\nu \Delta u - \theta \nu + \theta g.$$

Recall that $u = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \tanh(g)$. Then
$$\partial_t u = \text{sech}^2(g) \partial_t g = (1 - u^2) \partial_t g, \quad (\text{sech}^2 x = 1 - \tanh^2(x),$$
$$\Delta u = (1 - u^2) \Delta g + (2u^3 - 2u) |\nabla g|^2,$$
$$\Delta \partial_t u = (1 - u^2) \partial_t \Delta g + (2u^3 - 2u) \partial_t (|\nabla g|^2) - 2u \partial_t u \Delta g + (6u^2 - 2) \partial_t u |\nabla g|^2$$
$$= (1 - u^2) \partial_t \Delta g + (2u^3 - 2u) \partial_t (|\nabla g|^2) + (2u^3 - 2u) \partial_t g \Delta g$$
$$+ (-6u^4 + 8u^2 - 2) \partial_t g |\nabla g|^2,$$
$$\Delta^2 u = (1 - u^2) \Delta^2 g + (2u^3 - 2u) \Delta(|\nabla g|^2) + (2u^3 - 2u) \nabla \cdot (\nabla g \Delta g)$$
$$+ (-6u^4 + 8u^2 - 2) \nabla \cdot (\nabla g |\nabla g|^2) + (2u^3 - 2u) \partial_t g \Delta g$$
$$+ (-6u^4 + 8u^2 - 2) \nabla g \cdot (|\nabla g|^2) + (-6u^4 + 8u^2 - 2) \nabla g \cdot (\nabla g \Delta g)$$
$$+ (-24u^3 + 16u)(1 - u^2)|\nabla g|^4.$$}

Then the equation for $g$ takes the form
$$g_t = -\frac{\nu}{1 - u^2} \Delta^2 g - \frac{\theta}{1 - u^2} \Delta u + \frac{\theta}{1 - u^2} \Delta g$$
$$= -\nu \Delta^2 g + 2\nu \left( \Delta(|\nabla g|^2) + \nabla \cdot (\nabla g \Delta g) + \nabla g \cdot \nabla \Delta g \right)$$
$$- \nu (6u^2 - 2) \left( \nabla \cdot (\nabla g |\nabla g|^2) + \nabla g \cdot (|\nabla g|^2) + |\nabla g|^2 \Delta g \right)$$
$$+ \nu (24u^3 - 16u)|\nabla g|^4 - \theta \Delta g + 2\theta \nu |\nabla g|^2 + \frac{\theta}{1 - u^2} \Delta g.$$  \hfill (2)

One need not worry about the term $1/(1 - u^2)$ since
$$\frac{1}{1 - u^2} = \frac{1}{1 - \tanh^2(g)} = \cosh^2(g).$$

**Proposition 1** (Local wellposedness for the $g$-equation: subcritical data). Let the initial data $g_0 \in H^2(\Omega)$. There exists $T = T(\|g_0\|_{H^2}, \nu, \theta, \theta) > 0$ and a unique solution $g \in C([0, T], H^3) \cap L^2_t H^4$ to the equation (2). Furthermore due to smoothing the solution has higher regularity, i.e. $g \in C([0, T], \mathcal{H}^k)$ for any $k \geq 2$.

**Proof.** This is utterly standard, and we only sketch the details. To ease the notation we take $\nu = 1$. Roughly speaking, the $g$-equation can be rearranged to take the form
$$\partial_t g = -\Delta^2 g + O(\partial(e^{C_1g}F_1(\tanh g, g, g) \partial g)) + O(\partial^2(e^{C_2g}F_2(\tanh g, g, g))) + \cdots,$$
where $C_1, C_2$ are constants (we allow $C_1, C_2$ to be zero), $F_1$ and $F_2$ are polynomials, and $\cdots$ represent similar (and simpler) terms.
For the ease of reading, we explain how this is done for the first term. Other terms are similarly treated.

\[ u\partial^2(|\nabla g|^2) = \partial(u\partial(|\nabla g|^2)) - \partial u\partial(|\nabla g|^2) \]
\[ = O(\partial^2(u|\nabla g|^2)) + O(\partial u|\nabla g|^2) + O(\partial^2|\nabla g|^2) \]
\[ = O(\partial^2(u|\nabla g|^2)) + O(\partial((1 - u^2)(\partial g)^4)) + O((1 - u^2)\partial^2 g(\partial g)^2) + O(u(1 - u^2)(\partial g)^4). \]

Note that all these terms can be re-written as

\[ F_l(u)F_{l+1}(\partial g)\partial^2 g + \partial^l (F_3(u)F_4(\partial g)), \quad 0 \leq l \leq 2. \]

where \( F_l \) are polynomials.

In mild formulation (and dropping “similar terms” which are easier to handle), one can write

\[ g(t) = e^{-t\Delta^2}g_0 + \int_0^t \partial e^{-(t-s)\Delta^2} (e^{C_1g}F_1(tanh g, g, \partial g)\partial^2 g) ds \]
\[ + \int_0^t \partial^2 e^{-(t-s)\Delta^2} (e^{C_2g}F_2(tanh g, g, \partial g)) ds. \]

One can then derive

\[ ||g(t)||_{H^2} \lesssim ||g_0||_{H^2} + \int_0^t ((t-s)^{-\frac{1}{2}} + (t-s)^{-\frac{3}{2}})||e^{C_1g}F_1(tanh g, g, \partial g)\partial^2 g||_{L^2} ds \]
\[ + \int_0^t (t-s)^{-\frac{3}{2}} ||e^{C_2g}F_2(tanh g, g, \partial g)||_{L^2} ds \]
\[ + \int_0^t (t-s)^{-\frac{3}{2}} ||\partial(e^{C_2g}F_2(g, \partial g))||_{L^2} ds. \]

By Sobolev embedding and the fact that \( H^2 \) is an algebra (in 2D), we then get (below \( \theta_1 > 0, \theta_2 > 0, C > 0 \) are constants):

\[ \max_{0 \leq t \leq T} ||g(t)||_{H^2} \lesssim ||g_0||_{H^2} + (T^{\theta_1} + T^{\theta_2}) e^{C\max_{0 \leq t \leq T} ||g||_{H^2}}. \]

By taking \( T \) sufficiently small, one can then get contraction in the ball

\[ \{ g \in C([0,T],H^2) : \max_{0 \leq t \leq T} ||g(t)||_{H^2} \leq \text{const} \cdot ||g_0||_{H^2} \}. \]

The local solution is then easily constructed.

To get smoothing estimates, one can first estimate \(|t^{\frac{2}{4}}|||\nabla|^\eta g||_{L^\infty_T H^2}^\leq \) for some sufficiently small \( \eta > 0 \). The smallness of \( \eta \) is needed when we deal with the nonlinear term and absorb the fractional derivative into the kernel whilst keeping the integrability in time. The factor \( t^{\frac{2}{4}} \) is needed for the initial data. Bootstrapping then yields higher order smoothing estimates.

\[ \square \]

**Theorem 2.1** (Global wellposedness). Let the initial data \( g_0 \in H^2(\Omega) \). Then the corresponding local solution \( g \) constructed in Proposition 1 exists globally in time.

**Proof.** By using the smoothing effect, we may assume WLOG that the initial data \( g_0 \in H^k(\Omega) \) for all \( k \geq 2 \). For notational simplicity we shall set \( \nu = 1 \).

We divide the proof into several steps.

1) From energy conservation we have

\[ ||\nabla^{-1}u_t||_{L^2_x} \lesssim 1. \]
This implies
\[ \|\nabla K\|_{L^2_t} \lesssim 1. \]

2) Easy to check that \( K \) satisfies the equation:
\[ K_t = -\Delta^2 K - \theta_c \Delta K + \frac{\theta}{1 - u^2} \Delta K. \]

Multiplying both sides by \( -\Delta K \) and integrating by parts, we get
\[ \frac{1}{2} \partial_t (\|\nabla K\|_2^2) \leq -\|\Delta \nabla K\|_2^2 + \theta_c \|\Delta K\|_2^2 \]
\[ \leq -\|\Delta \nabla K\|_2^2 + \theta_c \|\Delta \nabla K\|_2 \|\nabla K\|_2 \]
\[ \lesssim \|\nabla K\|_2^2. \]

By using \( \|\nabla K\|_{L^2_t} \lesssim 1 \), one can then easily get the uniform bound
\[ \|\nabla K\|_{L^\infty_t L^2_x} \lesssim 1. \]

3) Control of \( \| g - \bar{g} \|_{L^\infty_t L^2_x} \). Recall
\[ K = -\Delta u - \theta_c u + \theta g. \]

Since \( \nabla u = (1 - u^2) \nabla g \), we have
\[ \int (K - \bar{K})(g - \bar{g}) dx = \int K \cdot (g - \bar{g}) dx \]
\[ = \int \nabla u \cdot \nabla g dx - \theta_c \int u(g - \bar{g}) dx + \theta \int (g - \bar{g})^2 dx \]
\[ \geq -\theta_c \int u(g - \bar{g}) dx + \theta \int (g - \bar{g})^2 dx. \]

A simple Cauchy-Schwartz using the fact \( \| K - \bar{K} \|_2 \lesssim \|\nabla K\|_2 \) then easily yields
\[ \| g - \bar{g} \|_{L^\infty_t L^2_x} \lesssim 1. \]

4) Control of the mean values \( \bar{g} \) and \( \bar{K} \). WLOG consider the case \( \bar{g} = M \geq 10 \).

Since \( \bar{u} \leq \frac{1}{M^2} \), we get
\[ \text{Leb}\{ x \in \Omega : g(x) \leq M/2 \} \lesssim M^{-2}. \]

Now
\[ \bar{u} \text{ Leb}(\Omega) = \int_{g(x) \geq \frac{M}{2}} dx + \int_{g(x) \geq \frac{M}{2}} (u(x) - 1) dx + \int_{g(x) < \frac{M}{2}} u(x) dx \]
\[ = \text{Leb}(\Omega) + \int_{g(x) \geq \frac{M}{2}} (u(x) - 1) dx + \int_{g(x) < \frac{M}{2}} (u(x) - 1) dx \]
\[ = \text{Leb}(\Omega) + O(e^{-\frac{M}{4}}) + O(\frac{1}{M^2}). \]

Since \( \bar{u} \) is preserved in time and \( |\bar{u}| < 1 \), the above easily implies that \( M \lesssim 1 \).

Thus we have proved \( |\bar{g}| \lesssim 1 \).

For the control of \( \bar{K} \), recall that
\[ K = -\Delta u - \theta_c u + \theta g. \]

Clearly then \( |\bar{K}| = | -\theta_c \bar{u} + \theta \bar{g}| \lesssim 1. \)
5) Control of $\|e^{C|g|}\|_{H^\infty}$, $\|\frac{1}{1-u^2}\|_{H^\infty}$, $\|\partial(\frac{1}{1-u^2})\|_{H^\infty}$ and $\|\partial^2(\frac{1}{1-u^2})\|_{H^\infty}$.

First since $K \in L^p_t H^1$, and we have the control of $\|g\|_{H^t}$, it is easy to check that

$$\|K\|_p \lesssim \sqrt{p}, \quad \forall 2 \leq p < \infty.$$  

By using (3) (multiply both sides by $|g|^{p-2}g$ and integrate by parts), we then get

$$\|g\|_p \lesssim \sqrt{p}.$$ 

This implies for any $C > 0$,

$$\|e^{C|g|}\|_{H^\infty} \lesssim 1.$$ 

Since $\frac{1}{1-u^2} \lesssim e^{2|g|}$, we also get $\|\frac{1}{1-u^2}\|_{H^\infty} \lesssim 1$.

By using (3), we also get $\|\Delta u\|_{H^\infty} \lesssim 1$. This easily implies

$$\|\partial^j(\frac{1}{1-u^2})\|_{H^\infty} \lesssim 1.$$ 

6) Control of $\|K\|_{L^\infty_t L^\infty_x}$, $\|g\|_{L^\infty_t L^\infty_x}$, and $\|g\|_{H^2}$.

Let $t_0 \geq 0$ be arbitrary. We then write

$$K(t) = e^{-(t-t_0)\Delta^2} K(t_0) - \theta e \int_{t_0}^t \Delta e^{-(t-s)\Delta^2} Kds + \theta \int_{t_0}^t e^{-(t-s)\Delta^2}(\frac{1}{1-u^2}\Delta K)ds.$$ 

Note that we can rewrite

$$\frac{1}{1-u^2}\Delta K = \Delta(\frac{1}{1-u^2}K) + O(\partial(\frac{1}{1-u^2})K)).$$ 

Taking $t = t_0 + 1$ and using the bounds on $\|K\|_{L^\infty_t L^\infty_x}$, $\|\partial^j(\frac{1}{1-u^2})\|_{L^\infty_t L^\infty_x}$, $0 \leq j \leq 2$, we then get

$$\|K(t_0 + 1)\|_{L^\infty_t L^\infty_x} \lesssim 1.$$ 

This implies $\|K\|_{L^\infty_t L^\infty_x} \lesssim 1$. By using (3) and a maximum principle argument, we also get $\|g\|_{L^\infty_t L^\infty_x} \lesssim 1$ and thus $\|\frac{1}{1-u^2}\|_{L^\infty_x} \lesssim 1$. Since $g = \frac{1}{2} \log(\frac{1}{1-u^2})$, $\|\frac{1}{1-u^2}\|_{L^\infty_x} \lesssim 1$ and $\|\Delta u\|_{H^2} \lesssim 1$, we get $\|g\|_{H^2} \lesssim 1$.

Since we have uniform control of $\|g\|_{H^2}$, by using the local theory and a bootstrap argument, we can then extend $g$ globally in time. \qed

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