Coefficient Inequalities for Uniformly P-Valent Starlike and Convex Functions

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Introduction

Let \( U = \{ z \in \mathbb{C} : |z| < 1 \} \) denote the open unit disk and let \( A_p \) be the class of functions \( f(z) \) of the form \( f(z) = z^p + \sum_{n=p+1} a_n z^n, \) \( p \in \mathbb{N} = \{1, 2, \ldots \} \) which are analytic in the open unit disk \( U \). A function \( f \in A_p \) is said to be \( p \)-valent starlike of order \( \alpha \) \((0 \leq \alpha < 1)\), if
\[
\Re \left\{ \frac{zf'(z)}{f(z)} \right\} > \alpha, \quad z \in U.
\]
The class of all such functions is denoted by \( S^*_p(\alpha) \). A function \( f \in A_p \) is said to be \( p \)-valent convex of order \( \alpha \) \((0 \leq \alpha < p)\), if
\[
\Re \left\{ \frac{zf'(z)}{f(z)} \right\} > \alpha, \quad z \in U.
\]
Let \( K_p(\alpha) \) denote the class of all such functions. For \( p=1 \) we write \( A_1 = A \). Note that for \( p=1 \) the classes \( S^*_p(\alpha) \) and \( K_p(\alpha) \) are the usual classes of starlike and convex functions of order \( \alpha \) \((0 \leq \alpha < 1)\), respectively, and will be denoted by \( S^*(\alpha) \) and \( K(\alpha) \) respectively. For \( p=1 \) and \( \alpha=0 \), the classes \( S^*_p(\alpha) \) and \( K_p(\alpha) \) reduce to \( S^* \) and \( K \) respectively, which are the classes of starlike (with respect to the origin) and convex functions.

The Subclasses \( S_{DP}(\beta, \alpha) \) and \( K_{DP}(\beta, \alpha) \)

We begin this Section by remark that this article is motivated by the work of Owa et al. [1]. We now recall the definitions of the subclasses \( S_{DP}(\beta, \alpha) \) and \( K_{DP}(\beta, \alpha) \) of \( p \)-valent functions introduced and studied by Agnihotri and Singh [2].

A function \( f \in Ap \) is said to be in the class \( S_{DP}(\beta, \alpha) \) if
\[
\Re \left\{ \frac{zf'(z)}{f(z)} \right\} > \beta \alpha + p + \alpha, \quad z \in U,
\]
for some \( \beta \geq 0 \) and \( \alpha \) \((0 \leq \alpha < p)\).

A function \( f \in Ap \) is said to be in the class \( K_{DP}(\beta, \alpha) \) if
\[
\Re \left\{ \frac{zf'(z)}{f(z)} \right\} > \beta \alpha + p + \alpha, \quad z \in U,
\]
for some \( \beta \geq 0 \) and \( \alpha \) \((0 \leq \alpha < p)\). Note that if \( f(z) \in K_{DP}(\beta, \alpha) \) if and only if \( zf'(z) \in SD_{DP}(\beta, \alpha) \). Agnihotri and Singh [2] have shown some sufficient conditions for \( f \) to be in the classes \( SD_{DP}(\beta, \alpha) \) and \( KD_{DP}(\beta, \alpha) \).

The subclasses \( SD_{DP}(\beta, \alpha) \) and \( KD_{DP}(\beta, \alpha) \) which will also be denoted by \( SD(\beta, \alpha) \) and \( KD(\beta, \alpha) \) respectively were studied by Shams, Kulkarni and Jahangiri in [3]. They have obtained sufficient conditions for \( f \) to be in the classes \( SD(\beta, \alpha) \) and \( KD(\beta, \alpha) \).

Coefficient Inequalities

We now give coefficient inequalities for functions belonging to the subclasses \( SD_{DP}(\beta, \alpha) \) and \( KD_{DP}(\beta, \alpha) \). Our first result is contained in

**Theorem 3.1.** If \( f \in SD_{DP}(\beta, \alpha) \) with \( 0 \leq p\beta \leq \alpha < p \), then
\[
f \in S_{DP}(\alpha - p\beta, 1 - p\beta) \quad \text{and} \quad f \in S_{DP}(\alpha - p\beta, 1 - p\beta).
\]

**Proof:** We know that \( |f(z)| \leq |z| \) for any complex number \( z \). Therefore \( f \in SD_{DP}(\beta, \alpha) \) gives us
\[
\Re \left\{ \frac{zf'(z)}{f(z)} \right\} > \beta \Re \left\{ \frac{zf'(z)}{f(z)} \right\} - p + \alpha.
\]
From this we get
\[
\Re \left\{ \frac{zf'(z)}{f(z)} \right\} > \frac{\alpha - p\beta}{1 - p\beta}, \quad z \in U.
\]

Now, if \( 0 \leq p\beta \leq \alpha < p \), then it follows that
\[
0 \leq \frac{\alpha - p\beta}{1 - p\beta} < p,
\]
and if \( \beta > \frac{p + \alpha}{2p} \), then we have
\[
-p < \frac{p\beta - \alpha}{\beta - 1} \leq 0.
\]
Thus, \( 0 \leq \frac{a - pb}{b - 1} < p \).

For \( p = 1 \), we obtain the following corollary due to Owa, Polatoğlu, and Yuvaz [1].

**Corollary 3.1:** If \( f \in SD_{\vartheta}(\beta, \alpha) \) with \( 0 \leq pb \leq \alpha \) then \( f \in \mathcal{S} \left( \frac{a - \beta}{1 - \beta} \right) \).

Next, we state the corresponding result for functions belonging to the subclass \( KD_{\mathcal{P}}(\beta, \alpha) \).

**Theorem 3.2:** If \( f \in KD_{\mathcal{P}}(\beta, \alpha) \) with \( 0 \leq pb \leq \alpha < p \) then \( f \in K_{\mathcal{P}}(\frac{a - \beta}{1 - \beta}) \) and if \( \beta > \frac{p + a}{2p} \), then \( f \in K_{\mathcal{P}}(\frac{a - \beta}{1 - \beta}) \).

**Proof:** Proof is similar to the proof of Theorem 3.1.

The following corollary is due to Owa, Polatoğlu, and Yuvaz [1] for \( p = 1 \).

**Corollary 3.2:** If \( f \in KD_{\mathcal{P}}(\beta, \alpha) \) with \( 0 \leq \beta \leq \alpha \) then \( f \in K_{\mathcal{P}}(\frac{a - \beta}{1 - \beta}) \).

We now state the main theorem of this paper.

**Theorem 3.3:** If \( f \in SD_{\vartheta}(\beta, \alpha) \) then \( |a_{n+1}| \leq \frac{2(p - \alpha)}{n - \beta} \sum_{j=0}^{\infty} q_{2j}z^{2j} \) and \( |a_{p+1}| \leq \frac{2(p - \alpha)}{n - \beta} \sum_{j=0}^{\infty} q_{2j}z^{2j} \) \( (n \geq 2) \).

**Proof:** We know that if \( f \in SD_{\vartheta}(\beta, \alpha) \), then \( f(\frac{sz}{1 - s}) = \frac{a - \beta}{1 - \beta} (z \in U) \).

Define a function \( q(z) \) by
\[
q(z) = (1 - \beta) \frac{zf'(z)}{f(z)} - (\alpha - pb) (z \in U).
\]

Note that \( q \) is analytic in \( U \) with \( q(0) = 1 \) and \( R(q(z)) > 0 \) if \( q(z) = 1 + q_{1}z + q_{2}z^{2} + \ldots \).

then we can write
\[
\frac{zf'(z)}{f(z)} = \frac{a - \beta}{1 - \beta} + \frac{p - a}{1 - 1} \sum_{j=1}^{\infty} q_{2j}z^{2j}.
\]

or
\[
\frac{zf'(z)}{f(z)} = f(z) \left( p + \frac{p - a}{1 - \beta} \sum_{j=1}^{\infty} q_{2j}z^{2j} \right), \quad (q_{0} = 1).
\]

From this, we obtain
\[
a_{n+1} = \left( \frac{p - a}{1 - \beta} \right) \left( q_{a} + a_{j+1} + a_{j+2}q_{2j+1}z^{2j+1} + \cdots + a_{n} + n \cdot q_{n} \right). \quad (3.7)
\]

From the coefficient estimates for Carathéodory functions [4], we know that \( |q_{a}| \leq 2 \) for all \( n \geq 1 \).

Making use of it in (3.7), we see that
\[
|a_{n+1}| \leq \frac{2(p - \alpha)}{n - \beta} \left( 1 + |a_{p+1}| + |a_{p+2}| + \cdots + |a_{n+1}| \right). \quad (3.8)
\]

Therefore, for \( n = 1 \), we have
\[
|a_{p+1}| \leq \frac{2(p - \alpha)}{1 - \beta}, \quad (3.9)
\]

which proves (3.3). Now for \( n = 2 \), we obtain
\[
|a_{p+1}| \leq \frac{2(p - \alpha)}{1 - \beta} (1 + |a_{p+1}|).
\]

This shows that (3.4) holds for \( n = 2 \). For \( n = 3 \), we see that
\[
|a_{p+1}| \leq \frac{2(p - \alpha)}{1 - \beta} \left( 1 + |a_{p+1}| \right) \leq \frac{2(p - \alpha)}{1 - \beta} \left( 1 + \frac{2(p - \alpha)}{1 - \beta} \right)
\]

Thus, (3.4) holds for \( n = 3 \). Next, we assume that (3.4) is true for \( n - k \) and therefore
\[
|a_{p+1}| \leq \frac{2(p - \alpha)}{1 - \beta} \left( 1 + \frac{2(p - \alpha)}{1 - \beta} \right) \leq \frac{2(p - \alpha)}{1 - \beta} \left( 1 + \frac{2(p - \alpha)}{1 - \beta} \right)
\]

This shows that (3.4) is true for \( n = k + 1 \). Hence, by the principle of mathematical induction, (3.4) holds for all \( n \geq 2 \).

**Remark 3.1:** Taking \( p = 1 \) in Theorem 3.3, we obtain
\[
|a_{n+1}| \leq \frac{2(1 - a)}{n! - \beta} \sum_{j=0}^{\infty} (1 + 2(p - a)) \beta^{j} \quad (n \geq 2) \quad (3.10)
\]

which was given by Owa, Polatoğlu and Yuvaz [1].

**Remark 2.2:** Taking \( p = 1 \) and \( \beta = 0 \) in Theorem 3.3, we have
\[
|a_{n+1}| \leq \frac{1}{n!} \sum_{j=0}^{\infty} j! \beta^{j} \quad (n \geq 1), \quad \text{which was proven by Robertson [5].}
\]

We know that \( f \in KD_{\mathcal{P}}(\beta, \alpha) \) if and only if \( zf' \in SD_{\mathcal{P}}(\beta, \alpha) \) [2]. Thus, we have

**Theorem 4.1:** If \( f(\vartheta) \in KD_{\mathcal{P}}(\beta, \alpha) \) then
\[
|a_{n+1}| \leq \frac{2(p - a)}{(n + 1)! - \beta} \sum_{j=0}^{\infty} (1 + 2(p - a)) \beta^{j} \quad (n \geq 2). \quad (3.13)
\]

**Proof:** For \( f \in KD_{\mathcal{P}}(\beta, \alpha) \) we know \( zf'(z) = \sum_{n=1}^{\infty} a_{n} z^{n+1} \in SD_{\mathcal{P}}(\beta, \alpha) \).

Therefore
\[
|zf'(z)| \leq \frac{2(p - a)}{(n + 1)! - \beta} \sum_{j=0}^{\infty} (1 + 2(p - a)) \beta^{j} \quad (z \in U).
\]

Define a function \( r(z) \) by
\[
(r(z)) = (1 - \beta) \frac{zf'(z)}{f(z)} - (\alpha - pb) (z \in U). 
\]

Note that \( r \) is analytic in \( U \) with \( r(0) = 1 \) and \( R(r(z)) > 0 \) if \( r(z) = 1 + q_{1}z + q_{2}z^{2} + \ldots \).

Then we can write
\[
\frac{zf'(z)}{f(z)} = \frac{a - \beta}{1 - \beta} + \frac{p - a}{1 - 1} \sum_{j=1}^{\infty} q_{2j}z^{2j}.
\]

or
\[
\frac{zf'(z)}{f(z)} = f(z) \left( p + \frac{p - a}{1 - \beta} \sum_{j=1}^{\infty} q_{2j}z^{2j} \right), \quad (q_{0} = 1).
\]

From this, we obtain
\[
r_{n+1} \left( \frac{2(p - a)}{n - \beta} \sum_{j=0}^{\infty} q_{2j}z^{2j} \right). \quad (3.7)
\]

Making use of it in (3.7), we see that
\[
|a_{n+1}| \leq \frac{2(p - a)}{n - \beta} \left( 1 + |a_{p+1}| + |a_{p+2}| + \cdots + |a_{n+1}| \right). \quad (3.8)
\]

Therefore, for \( n = 1 \), we have
\[
|a_{p+1}| \leq \frac{2(p - a)}{1 - \beta}, \quad (3.9)
\]

which proves (3.3). Now for \( n = 2 \), we obtain
\[
|a_{p+1}| \leq \frac{2(p - a)}{1 - \beta} (1 + |a_{p+1}|) \leq \frac{2(p - a)}{1 - \beta} \left( 1 + \frac{2(p - a)}{1 - \beta} \right)
\]

This shows that (3.4) is true for \( n = 2 \). For \( n = 3 \), we see that
\[ |a_p| \leq \frac{2(p - a)}{(n + 1)} \prod_{j=0}^{n-2} (j - 2) \quad (n \geq 1), \]

which was proven by Owa et al. [1].

**Remark 3.4:** Taking \( p = 1 \) and \( \beta = 0 \) in Theorem 3.4, we get

\[ |a_p| \leq \frac{2(1 - a)}{(n + 1)} \prod_{j=0}^{n-2} (j - 2) \quad (n \geq 2) \]

which was proven by Robertson [5].

**Theorem 3.5:** If \( f \in SD_p(\beta, \alpha) \)

\[
\max \left\{ 0, \left| f'(z) \right| - \frac{2(p - a)}{|p + 1|} \prod_{j=0}^{n-2} (j + 1) \right\} \leq |f'(z)|
\]

and

\[
\left| f'(z) \right| \leq \frac{2(p + 1)(p - a)}{|p|} |f|
\]

**Proof:** Proof follows from the fact that

\[
f(z) = z^p + \sum_{k=1}^{\infty} a_k z^k, \quad p = 1, 2, \ldots \]

and using Theorem 3.3.

**Corollary 3.3:** If \( f \in KD_p(\beta, \alpha) \) then

\[
\max \left\{ 0, \left| f'(z) \right| - \sum_{k=0}^{n} \frac{2(p - a)}{(n + 1) |p + 1|} \prod_{j=0}^{n-2} (j + 1) \left| f^{(k)}(z) \right| \right\} \leq |f(z)|
\]

and

\[
\left| f'(z) \right| \leq \frac{2(p + 1)(p - a)}{|p|} |f|
\]

**Proof:** Proof follows from the fact that

\[
f(z) = z^p + \sum_{k=1}^{\infty} a_k z^k, \quad p = 1, 2, \ldots
\]

References

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