G₁-class elements in a Banach algebra

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Abstract
Let A be a complex unital Banach algebra with unit 1. An element a ∈ A is said to be of G₁-class if
\[ \| (z - a)^{-1} \| = \frac{1}{d(z, \sigma(a))} \quad \forall z \in \mathbb{C} \setminus \sigma(a). \]
Here d(z, σ(a)) denotes the distance between z and the spectrum σ(a) of a. Some examples of such elements are given and also some properties are proved. It is shown that a G₁-class element is a scalar multiple of the unit 1 if and only if its spectrum is a singleton set consisting of that scalar. It is proved that if T is a G₁ class operator on a Banach space X, then every isolated point of σ(T) is an eigenvalue of T. If, in addition, σ(T) is finite, then X is a direct sum of eigenspaces of T.

Keywords Banach algebra · Spectrum · G₁-class · Pseudospectrum · Spectral radius

Mathematics Subject Classification 46B99 · 47A05

1 Introduction
Let T be a normal operator on a complex Hilbert space H and λ a complex number not lying in the spectrum σ(T) of T. Then it is known that the distance between λ and σ(T) is given by \[ \frac{1}{\| (λ - T)^{-1} \|}. \] It is also known that there are many other operators that are not normal but still satisfy this property. Putnam called such operators as
operators satisfying $G_1$ condition and investigated properties of such operators in [8, 9]. In particular, he proved that if $T$ is a $G_1$ class operator, then every isolated point of $\sigma(T)$ is an eigenvalue of $T$ and every $G_1$ class operator on a finite dimensional Hilbert space is normal.

In this note we extend this concept of $G_1$ class operators to operators on a Banach space and more generally to elements of a complex Banach algebra and investigate the properties of such elements. The next section contains some preliminary definitions and results that are used throughout. In Sect. 3, we give definition of a $G_1$ class element in a complex unital Banach algebra, give some examples and prove a few elementary properties of such elements. In particular, it is proved that every element of a uniform algebra is of $G_1$ class and conversely if every element of a complex unital Banach algebra $A$ is of $G_1$ class, then $A$ is commutative, semisimple and hence isomorphic and homeomorphic to a uniform algebra. The last section deals with the spectral properties of $G_1$ class elements and contains the main results of this note. In particular, it is proved that if $T$ is a $G_1$ class operator on a Banach space $X$, then every isolated point of $\sigma(T)$ is an eigenvalue of $T$. Further, if, in addition, $\sigma(T)$ is finite, then $X$ is a direct sum of eigenspaces of $T$. In this sense $T$ is “diagonalizable” and hence this result can be considered to be an analogue of the Spectral Theorem for such operators.

An overall aim of such a study can be to obtain an analogue of the Spectral Theorem for $G_1$ class operators. Though at present we are far away from this goal, the present results can be considered a small step in that direction. Next natural step should be to try to prove a similar result for compact operators of $G_1$ class. Another way of looking at this study is an attempt to answer the following question: “To what extent does the spectrum of an element determine the element?” This question has a long and interesting history. It has appeared under different names at different times such as “Spectral characterizations”, “hearing the shape of a drum” [2], “$T = I$ problem” [13] etc. The results in this note say that the spectrum of a $G_1$ class element gives a fairly good information about that element.

We shall use the following notations throughout this article. Let $B(z_0, r) := \{z \in \mathbb{C} : |z - z_0| < r\}$, the open disc with the centre at $z_0$ and radius $r$, $D(z_0, r) := \{z \in \mathbb{C} : |z - z_0| \leq r\}$, the closed disc with the centre at $z_0$ and radius $r$, $\Omega + D(0, r) = \bigcup_{\lambda \in \Omega} D(\lambda, r)$ for $\Omega \subseteq \mathbb{C}$ and $d(z, K) = \inf \{|z - k| : k \in K\}$, the distance between a complex number $z$ and a closed set $K \subseteq \mathbb{C}$.

Let $\partial \Omega$ denote the boundary of a set $\Omega \subseteq \mathbb{C}$. $\mathbb{C}^{n \times n}$ denotes the space of square matrices of order $n$ and $B(X)$ denotes the set of bounded linear operators on a Banach space $X$.

### 2 Preliminaries

Since our main objects of study are certain elements in a Banach algebra, we shall review some definitions related to a Banach algebra. Many of these definitions can be found in the book [1]. Some material in this section is also available in the review article [6].
Definition 2.1  Spectrum: Let $A$ be a complex unital Banach algebra with unit $1$. For $\lambda \in \mathbb{C}$, $\lambda \cdot 1$ is identified with $\lambda$. Let $\text{Inv}(A) = \{ x \in A : x \text{ is invertible in } A \}$ and $\text{Sing}(A) = \{ x \in A : x \text{ is not invertible in } A \}$. The spectrum of an element $a \in A$ is defined as:

$$\sigma(a) := \{ \lambda \in \mathbb{C} : \lambda - a \in \text{Sing}(A) \}$$

The spectral radius of an element $a$ is defined as:

$$r(a) := \sup \{ |\lambda| : \lambda \in \sigma(a) \}$$

Its value is also given by the Spectral Radius Formula,

$$r(a) = \lim_{n \to \infty} \|a^n\|^\frac{1}{n} = \inf_{n} \|a^n\|^\frac{1}{n}$$

The complement of the spectrum of an element $a$ is called the resolvent set of $a$ and is denoted by $\rho(a)$.

Thus when $A = C(X)$, the algebra of all continuous complex valued functions on a compact Hausdorff space $X$ and $f \in A$, then the spectrum $\sigma(f)$ of $f$ coincides with the range of $f$.

Similarly when $A = \mathbb{C}^{n \times n}$, the algebra of all square matrices of order $n$ with complex entries and $M \in A$, the spectrum $\sigma(M)$ of $M$ is the set of all eigenvalues of $M$.

Definition 2.2  Numerical range Let $A$ be a Banach algebra and $a \in A$. The numerical range of $a$ is defined by

$$V(a) := \{ f(a) : f \in A', f(1) = 1 = \| f \| \},$$

where $A'$ denotes the dual space of $A$, the space of all continuous linear functionals on $A$.

The numerical radius $v(a)$ is defined as

$$v(a) := \sup \{ |\lambda| : \lambda \in V(a) \}$$

Let $A$ be a Banach algebra and $a \in A$. Then $a$ is said to be Hermitian if $V(a) \subseteq \mathbb{R}$.

If $A$ is a $C^*$ algebra(also known as $B^*$ algebra), then an element $a \in A$ is Hermitian if and only if it is self-adjoint [1].

Definition 2.3  Spatial numerical range

Let $X$ be a Banach space and $T \in B(X)$. Let $X'$ denote the dual space of $X$. The spatial numerical range of $T$ is defined by

$$W(T) = \{ f(Tx) : f \in X', \| f \| = f(x) = 1 = \| x \| \}.$$
$\overline{Co} W(T) = V(T)$

where $\overline{Co} E$ denotes the closure of the convex hull of $E \subseteq \mathbb{C}$.

The following theorem gives the relation between the spectrum and numerical range.

**Theorem 2.4** Let $A$ be a complex unital Banach algebra with unit $1$ and $a \in A$. Then the numerical range $V(a)$ is a closed convex set containing $\sigma(a)$.

Thus $\overline{Co}(\sigma(a)) \subseteq V(a)$. Hence $r(a) \leq v(a) \leq \|a\| \leq ev(a)$.

A proof of this can be found in [1].

**Corollary 2.5** Let $A$ be a complex unital Banach algebra with unit $1$ and $a \in A$. If $a$ is Hermitian, then $\sigma(a) \subseteq \mathbb{R}$.

We now discuss another important and popular set related to the spectrum, namely pseudospectrum. We begin with its definition.

**Definition 2.6** Pseudospectrum Let $A$ be a complex Banach algebra, $a \in A$ and $\epsilon > 0$. The $\epsilon$-pseudospectrum $\Lambda_\epsilon(a)$ of $a$ is defined by

$$
\Lambda_\epsilon(a) := \{ \lambda \in \mathbb{C} : \| (\lambda - a)^{-1} \| \geq \epsilon^{-1} \}
$$

with the convention that $\| (\lambda - a)^{-1} \| = \infty$ if $\lambda - a$ is not invertible.

This definition and many results in this section can be found in [5]. The book [11] is a standard reference on Pseudospectrum. It contains a good amount of information about the idea of pseudospectrum, (especially in the context of matrices and operators), historical remarks and applications to various fields. Another useful source is the website [12].

The following theorems establish the relationships between the spectrum, the $\epsilon$-pseudospectrum and the numerical range of an element of a Banach algebra.

**Theorem 2.7** Let $A$ be a Banach algebra, $a \in A$ and $\epsilon > 0$. Then

$$
d(\lambda, V(a)) \leq \frac{1}{\| (\lambda - a)^{-1} \|} \leq d(\lambda, \sigma(a)) \quad \forall \lambda \in \mathbb{C} \setminus \sigma(a).
$$

Thus

$$
\sigma(a) + D(0; \epsilon) \subseteq \Lambda_\epsilon(a) \subseteq V(a) + D(0; \epsilon).
$$

A proof of this Theorem can be found in [5].

The following theorem gives the basic information about the analytical functional calculus for elements of a Banach algebra.

**Theorem 2.8** Let $A$ be a Banach algebra and $a \in A$. Let $\Omega \subseteq \mathbb{C}$ be an open neighbourhood of $\sigma(a)$ and $\Gamma$ be a contour that surrounds $\sigma(a)$ in $\Omega$. Let $H(\Omega)$ denote the set of all analytic functions in $\Omega$ and let $P(\Omega)$ denote the set of all
polynomials in \( z \) with \( z \in \Omega \). We recall the definition of \( \tilde{f}(a) \) in the analytical functional calculus as

\[
\tilde{f}(a) = \frac{1}{2\pi i} \int (z - a)^{-1} f(z) \, dz
\]

(3)

Then the map \( f \rightarrow \tilde{f}(a) \) is a homomorphism from \( H(\Omega) \) into \( A \) that extends the natural homomorphism \( p \rightarrow p(a) \) of \( P(\Omega) \) into \( A \) and

\[
\sigma(\tilde{f}(a)) = \{ f(z) : z \in \sigma(a) \}
\]

A proof of this Theorem can be found in [1].

3 \( G_1 \)-class elements

In this section, we give definition, some examples and elementary properties of \( G_1 \)-class elements. It is possible to view this definition as motivated by considering the question of equality in some of the inclusions given in Theorem 2.7.

Definition 3.1 Let \( A \) be a Banach algebra and \( a \in A \). We define \( a \) to be of \( G_1 \)-class if

\[
\| (z - a)^{-1} \| = \frac{1}{\text{d}(z, \sigma(a))} \quad \forall z \in \mathbb{C} \setminus \sigma(a).
\]

(4)

Remark 3.2 The idea of \( G_1 \)-class was introduced by Putnam who defined it for operators on Hilbert spaces. (See [8, 9].) It is known that the \( G_1 \)-class properly contains the class of seminormal operators (that is, the operators satisfying \( TT^* \leq T^*T \) or \( T^*T \leq TT^* \)) and this class properly contains the class of hyponormal operators (that is, the operators satisfying \( T^*T \leq TT^* \)) which, in turn, properly contains the class of normal operators. (See [8, 9].) Using the Gelfand- Naimark theorem [1], we can make similar statements about elements in a \( C^* \) algebra.

\( G_1 \)-class operators on a finite dimensional Hilbert space are normal [8].

In particular, normal elements are hyponormal. In general, the equation (4) may hold, for every \( z \in \mathbb{C} \setminus \sigma(a) \), for an element \( a \) of a \( C^* \)-algebra even though \( a \) is not normal.

For example, we may consider the right shift operator \( R \) on \( \ell^2(\mathbb{N}) \). It is not normal but \( \Lambda(R) = \sigma(R) + D(0, 1) = D(0, 1 + \epsilon) \forall \epsilon > 0 \). The operator \( R \) is, however, a hyponormal operator.

We now deal with a natural question: What are \( G_1 \)-class elements in an arbitrary Banach algebra?

The following lemma is elementary and gives a characterization of a \( G_1 \) class element in terms of its pseudospectrum.

Lemma 3.3 Let \( A \) be a Banach algebra and \( a \in A \). Then
\[ \Lambda_{\epsilon}(a) = \sigma(a) + D(0, \epsilon) \quad \forall \epsilon > 0 \] (5)

iff \( a \) is of \( G_1 \)-class.

A proof of this Lemma can be found in [5].

As one may expect, most natural candidates to be \( G_1 \)-class elements are scalars, that is, scalar multiples of the identity 1.

**Theorem 3.4** Let \( A \) be a complex Banach algebra with unit 1 and \( a \in A \).

(i) If \( a = \mu \) for some complex number \( \mu \), then \( a \) is of \( G_1 \)-class and \( \sigma(a) = \{ \mu \} \).

(ii) If \( a \) is of \( G_1 \)-class, then \( za + \beta \) is also of \( G_1 \)-class for every complex numbers \( z, \beta \).

(iii) If \( a \) is of \( G_1 \)-class and \( \sigma(a) = \{ \mu \} \), then \( a = \mu \).

A proof of this is straightforward. It also follows easily from Lemma 3.3 and Corollary 3.17 of [5]. We include it here for the sake of completeness.

**Proof** (i) Let \( a = \mu \) for some complex number \( \mu \). Then clearly \( \sigma(a) = \{ \mu \} \). Hence for all \( z \in \mathbb{C} \setminus \sigma(a) \), we have \( z \neq \mu \). Thus \( \| (z - a)^{-1} \| = \frac{1}{|z - \mu|} = \frac{1}{d(z, \sigma(a))} \). This shows that \( a \) is of \( G_1 \)-class.

(ii) Next suppose that \( a \) is of \( G_1 \)-class and \( b = za + \beta \) for some complex numbers \( z, \beta \). We want to prove that \( b \) is of \( G_1 \)-class. If \( z = 0 \), then it follows from (i). So assume that \( z \neq 0 \). Let \( w \notin \sigma(b) = \{ wz + \beta : z \in \sigma(a) \} \). Then \( z := \frac{w - \beta}{z} \notin \sigma(a) \) and since \( a \) is of \( G_1 \)-class, \( \| (z - a)^{-1} \| = \frac{1}{d(z, \sigma(a))} \). Now \( \| (w - b)^{-1} \| = \| (wz + \beta - (za + \beta))^{-1} \| = \frac{1}{|z|} \| (z - a)^{-1} \| = \frac{1}{|z|d(z, \sigma(a))} = \frac{1}{d(z, \sigma(za))} = \frac{1}{d(w, \sigma(b))} \). This shows that \( b \) is of \( G_1 \)-class.

(iii) Suppose \( a \) is of \( G_1 \)-class and \( \sigma(a) = \{ \mu \} \). Let \( b = a - \mu \). Then by (ii), \( b \) is of \( G_1 \)-class and \( \sigma(b) = \{ 0 \} \). Let \( \epsilon > 0 \) and \( C \) denote the circle with the centre at 0 and radius \( \epsilon \) traced anticlockwise. Then for every \( z \in C \),

\[ \| (z - b)^{-1} \| = \frac{1}{d(z, \sigma(b))} = \frac{1}{|z - 0|} = \frac{1}{\epsilon}. \]

Also

\[ b = \frac{1}{2\pi i} \int_{C} z(z - b)^{-1} dz \]

Hence \( \| b \| \leq \frac{1}{2\pi} 2\pi \epsilon \frac{1}{\epsilon} = \epsilon \). Since this holds for every \( \epsilon > 0 \), we have \( b = 0 \), that is \( a = \mu \). \( \square \)

**Remark 3.5** The above Theorem has a relevance in the context of a very well known classical problem in operator theory known as the \( T = I? \) problem. This problem asks the following question: Let \( T \) be an operator on a Banach space. Suppose \( \sigma(T) = \{ 1 \} \). Under what additional conditions can we conclude \( T = I? \) A survey article [13] contains details of many classical results about this problem.

From the above Theorem it follows that if \( T \) is of \( G_1 \)-class and \( \sigma(T) = \{ 1 \} \), then we can conclude that \( T = I \). In other words \( \| T \| \) is of \( G_1 \)-class” works as an additional condition in the “\( T = I \) problem”.
Next we show that every Hermitian idempotent element is of $G_1$-class. A version of this result was included in the thesis [4].

**Theorem 3.6** Let $A$ be a complex unital Banach algebra with unit $1$ and $a \in A$. If $a$ is a Hermitian idempotent element, then $a$ is of $G_1$-class. Also, if $a$ is of $G_1$-class and $\sigma(a) \subseteq \{0, 1\}$, then $a$ is a Hermitian idempotent.

**Proof** Suppose $a$ is a Hermitian idempotent element. If $a = 0$ or $a = 1$, then $a$ is of $G_1$ class by (i) of Theorem 3.4. Next, let $a \neq 0, 1$. Then $\sigma(a) = \{0, 1\}$ and by Theorem 1.10.17 of [1], $\|a\| = r(a) = 1$. Now Corollary 3.18 of [5] implies that $\Lambda(a) = D(0, \epsilon) \cup D(1, \epsilon)$ for every $\epsilon > 0$. Hence $a$ is of $G_1$ class by Lemma 3.3.

Next suppose $a$ is of $G_1$-class and $\sigma(a) \subseteq \{0, 1\}$. If $\sigma(a) = \{0\}$, then $a = 0$ by (iii) of Theorem 3.4. Similarly, if $\sigma(a) = \{1\}$, then $a = 1$. So assume that $\sigma(a) = \{0, 1\}$. Then by Lemma 3.3, $\Lambda(a) = D(0, \epsilon) \cup D(1, \epsilon)$ for every $\epsilon > 0$. Hence by 3.18 of [5], $a$ is a Hermitian idempotent element. 

The abundance or scarcity of $G_1$-class elements in a given Banach algebra depends on the nature of that Banach algebra. There exist extreme cases, that is, there are Banach algebras in which every element is of $G_1$-class. On the other hand, there are also Banach algebras in which the scalars are the only elements of $G_1$-class. We shall see examples of both types below. Before that, we need to review a relation between the spectrum and numerical range of an element of $G_1$-class. Recall that the numerical range of an element of a Banach algebra is a compact convex subset of $\mathbb{C}$ containing its spectrum, and hence it also contains the closure of the convex hull of the spectrum. The next proposition shows that the equality holds in case of elements of $G_1$-class.

**Proposition 3.7** Let $A$ be a complex unital Banach algebra and $a \in A$. Suppose $a$ is of $G_1$-class. Then $V(a) = \overline{Co}(\sigma(a))$, the closure of the convex hull of the spectrum of $a$ and $\|a\| \leq er(a)$.

A proof of this can be found in [5].

**Corollary 3.8** Let $A$ be a complex unital Banach algebra. Suppose $a \in A$ is of $G_1$-class and $\sigma(a) \subseteq \mathbb{R}$. Then $a$ is Hermitian.

This is in fact a generalization of Theorem A of [10] (See also [7]) where this result is proved for operators on a Hilbert space.

It is shown in the next theorem that every element in a uniform algebra is of $G_1$-class. Also a partial converse of this statement is proved. We may recall that a uniform algebra is a unital Banach algebra satisfying $\|a\|^2 = \|a^2\|$ for every $a \in A$. Every complex uniform algebra is commutative by a theorem of Hirschfeld and Zelazko [1]. Then it follows by Gelfand theory [1] that such an algebra is isometrically isomorphic to a function algebra, that is, a uniformly closed subalgebra of $C(X)$ that contains the constant function $1$ and separates the points of $X$, where $X$ is the maximal ideal space of $A$.

**Theorem 3.9** (See also Theorem 3.15 of [5]) Let $A$ be a complex unital Banach algebra with unit $1$. 

$\square$ Springer
(i) If $A$ is a uniform algebra, then every element in $A$ is of $G_1$-class.
(ii) If every element of $A$ is of $G_1$-class, then $A$ is commutative, semisimple and hence isomorphic and homeomorphic to a uniform algebra.

**Proof** (i) The Spectral Radius Formula implies that $\|a\| = r(a)$ for every $a \in A$. Now let $a \in A$ and $\lambda \notin \sigma(a)$. Then

\[
\| (\lambda - a)^{-1} \| = r((\lambda - a)^{-1}) \\
= \sup \{ |z| : z \in \sigma((\lambda - a)^{-1}) \} \\
= \sup \left\{ \frac{1}{|\lambda - \mu|} : \mu \in \sigma(a) \right\} \\
= \frac{1}{\inf \{ |\lambda - \mu| : \mu \in \sigma(a) \}} \\
= \frac{1}{d(\lambda, \sigma(a))}
\]

This shows that $a$ is of $G_1$-class.

(ii) By Proposition 3.7, $\|a\| \leq er(a)$ for all $a \in A$. Hence $A$ is commutative by a theorem of Hirschfeld and Zelazko [1]. Also, the condition $\|a\| \leq er(a)$ for all $a \in A$ implies that $A$ is semisimple and hence the spectral radius $r(.)$ is a norm on $A$. Clearly, $r(a^2) = (r(a))^2$ for every $a \in A$. Hence $A$ is a uniform algebra under this norm. Also the inequality $r(a) \leq \|a\| \leq er(a)$ for all $a \in A$ implies that the identity map is a homeomorphism between these two algebras. □

Next we consider an example of a Banach algebra in which scalars are the only elements of $G_1$-class.

**Example 3.10** (See also Example 2.16 and Remark 2.20 of [3])

Let $A = \left\{ a \in \mathbb{C}^{2 \times 2} : a = \begin{bmatrix} \alpha & \beta \\ 0 & \alpha \end{bmatrix} \right\}$ with the norm given by $\|a\| = |\alpha| + |\beta|$.

Suppose $a = \begin{bmatrix} \alpha & \beta \\ 0 & \alpha \end{bmatrix} \in A$ is of $G_1$-class. Then since $\sigma(a) = \{ \alpha \}$, it follows by Theorem 3.4(iii) that $a = \alpha$. (This means $\beta = 0$.)

### 4 Spectral properties of $G_1$-class elements

In this section, we show that $G_1$-class elements have some properties that are very similar to the properties of normal operators on a complex Hilbert space. For example, if $H$ is a complex Hilbert space, $T$ is a normal operator on $H$ and $\lambda$ is an isolated point of $\sigma(T)$, then $\lambda$ is an eigenvalue of $T$. We show that a similar property holds for a bounded operator of $G_1$-class on a Banach space. For that we need the following theorem about isolated points of the spectrum of a $G_1$-class element in a Banach algebra.
Theorem 4.1 Let $A$ be a complex unital Banach algebra with unit $1$. Suppose $a$ is of $G_1$-class and $\lambda$ is an isolated point of $\sigma(a)$. Then there exists an idempotent element $e \in A$ such that $ae = \lambda e$ and $\|e\| = 1$.

Proof If $\sigma(a) = \{\lambda\}$, then by Theorem 3.4(iii), $a = \lambda$ and we can take $e = 1$.

Next assume that $\sigma(a) \setminus \{\lambda\}$ is nonempty. Let $D_1$ and $D_2$ be disjoint open neighbourhoods of $\lambda$ and $\sigma(a) \setminus \{\lambda\}$ respectively. Define

$$f(z) = \begin{cases} 1 & \text{if } z \in D_1 \\ 0 & \text{if } z \in D_2 \end{cases}$$

Then $f$ is analytic in $D_1 \cup D_2$. Let $e = \tilde{f}(a)$. Then since $f^2 = f$, we have $e^2 = e$, that is, $e$ is an idempotent element and $\|e\| \geq 1$. To prove other assertions, choose $\epsilon > 0$ in such a way that for every $z \in \Gamma_1 := \{w \in \mathbb{C} : |w - \lambda| = \epsilon\}$, $\lambda$ is the nearest point of $\sigma(a)$ and $\Gamma_1 \subseteq D_1$. Then for every such $z$, $d(z, \sigma(a)) = |z - \lambda| = \epsilon$, hence $\|(z - a)^{-1}\| = \frac{1}{\epsilon}$. Now let $\Gamma_2$ be any closed curve lying in $D_2$ and enclosing $\sigma(a) \setminus \{\lambda\}$ and let $\Gamma = \Gamma_1 \cup \Gamma_2$. Then

$$e = \tilde{f}(a) = \frac{1}{2\pi i} \int_{\Gamma} f(z)(z - a)^{-1} \, dz = \frac{1}{2\pi i} \int_{\Gamma_1} (z - a)^{-1} \, dz$$

Hence

$$\|e\| \leq \frac{1}{2\pi} \frac{1}{\epsilon} = \frac{1}{2\pi \epsilon} = 1$$

This shows that $\|e\| = 1$.

Now define $g(z) = (z - \lambda)f(z)$. Then $|g(z)| \leq \epsilon$ for all $z \in \Gamma_1$. Note that

$$ae - \lambda e = \tilde{g}(a) = \frac{1}{2\pi i} \int_{\Gamma} g(z)(z - a)^{-1} \, dz = \frac{1}{2\pi i} \int_{\Gamma_1} g(z)(z - a)^{-1} \, dz$$

Hence

$$\|ae - \lambda e\| \leq \frac{1}{2\pi} \epsilon \frac{1}{\epsilon} = \frac{1}{2\pi \epsilon} = \epsilon$$

Since this holds for every $\epsilon > 0$, we have $ae - \lambda e = 0$. \qed

Corollary 4.2 Let $X$ be a complex Banach space, $T \in B(X)$ be of $G_1$-class and $\lambda$ be an isolated point of $\sigma(T)$. Then $\lambda$ is an eigenvalue of $T$.

Proof By Theorem 4.1, there exists an idempotent element $P \in B(X)$ such that $\|P\| = 1$ and $TP = \lambda P$. Clearly $P$ is a nonzero projection operator on $X$. Let $x \neq 0$ be an element of the range $R(P)$ of $P$. Then $P(x) = x$. Hence $T(x) = TP(x) = \lambda P(x) = \lambda x$. Thus $\lambda$ is an eigenvalue of $T$. \qed

Some ideas in the proof of the next theorem can be compared with the proof of Theorem C in [10] that deals with similar results about hyponormal operators on a Hilbert space.
Theorem 4.3 Let A be a complex unital Banach algebra with unit 1. Suppose a is of $G_1$-class and $\sigma(a) = \{\lambda_1, \ldots, \lambda_m\}$ is finite. Then there exist idempotent elements $e_1, \ldots, e_m$ such that

1. $\|e_j\| = 1$, $ae_j = \lambda_j e_j$ for $j = 1, \ldots, m$, $e_j e_k = 0$ for $j \neq k$, $e_1 + \cdots + e_m = 1$

and

$$a = \lambda_1 e_1 + \cdots + \lambda_m e_m.$$  

2. If $p$ is any polynomial, then

$$p(a) = p(\lambda_1)e_1 + \cdots + p(\lambda_m)e_m.$$  

3. In particular,

$$(a - \lambda_1) \cdots (a - \lambda_m) = 0.$$  

4. If $\lambda$ is a complex number such that $\lambda \neq \lambda_j$ for $j = 1, \ldots, m$, then

$$(\lambda - a)^{-1} = \frac{1}{\lambda - \lambda_1} e_1 + \cdots + \frac{1}{\lambda - \lambda_m} e_m.$$

5. If a function $f$ is analytic in a neighbourhood of $\sigma(a)$, then

$$\tilde{f}(a) = f(\lambda_1)e_1 + \cdots + f(\lambda_m)e_m.$$  

Proof If $m = 1$, then by Theorem 3.4(iii), $a = \lambda_1$. Hence we can take $e_1 = 1$ and all the conclusions follow trivially. Next we assume $m > 1$. Let $D_1, \ldots, D_m$ be mutually disjoint neighbourhoods of $\lambda_1, \ldots, \lambda_m$ respectively and let $D = \bigcup_{j=1}^m D_j$. Now for each $j = 1, \ldots, m$, define a function $f_j$ on $D$ by

$$f_j(z) = \begin{cases} 
1 & \text{if } z \in D_j \\
0 & \text{if } z \not\in D_j
\end{cases}$$

Let $e_j = \tilde{f}_j(a)$. Then it follows as in Theorem 4.1 that each $e_j$ is an idempotent, $\|e_j\| = 1$ and $ae_j = \lambda_j e_j$. Since for $j \neq k$, $f_j f_k = 0$, we have $e_j e_k = 0$. Further $f_1 + \cdots + f_m = 1$ implies $e_1 + \cdots + e_m = 1$. Next

$$a = a1$$

$$= a(e_1 + \cdots + e_m)$$

$$= ae_1 + \cdots + ae_m$$

$$= \lambda_1 e_1 + \cdots + \lambda_m e_m.$$  

This proves (1).

Now since $e_j^2 = e_j$ for each $j$ and $e_j e_k = 0$ for $j \neq k$, we have
and in general for any power \( k \),
\[
    a^k = \lambda_1^k e_1 + \cdots + \lambda_m^k e_m.
\]

It follows easily from this that for any polynomial \( p \), we have
\[
    p(a) = p(\lambda_1)e_1 + \cdots + p(\lambda_m)e_m.
\]

Thus (2) is proved.

Now consider the polynomial \( p \) given by
\[
    p(z) = (z - \lambda_1) \cdots (z - \lambda_m). \quad \text{Then}
    p(\lambda_j) = 0 \quad \text{for each} \quad j.
\]
Hence \( p(a) = 0 \), that is, \( (a - \lambda_1) \cdots (a - \lambda_m) = 0 \). This completes the proof of (3).

Now suppose \( \lambda \) is a complex number such that \( \lambda \neq \lambda_j \) for \( j = 1, \ldots, m \). Let
\[
    b = \frac{1}{\lambda - \lambda_1} e_1 + \cdots + \frac{1}{\lambda - \lambda_m} e_m.
\]

Then in view of (1), we have
\[
    (\lambda - a)b = [(\lambda - \lambda_1)e_1 + \cdots + (\lambda - \lambda_m)e_m] \left[ \frac{1}{\lambda - \lambda_1} e_1 + \cdots + \frac{1}{\lambda - \lambda_m} e_m \right] = 1
\]

Similarly, we can prove \( b(\lambda - a) = 1 \) implying (4).

Next suppose a function \( f \) is analytic in a neighbourhood \( \Omega \) of \( \sigma(a) \) and \( \Gamma \) is a closed curve lying in \( \Omega \) and surrounding \( \sigma(a) \). Then
\[
    \tilde{f}(a) = \frac{1}{2\pi i} \int_{\Gamma} f(z)(z-a)^{-1} dz
    = \frac{1}{2\pi i} \int_{\Gamma} f(z) \left[ \frac{1}{z - \lambda_1} e_1 + \cdots + \frac{1}{z - \lambda_m} e_m \right] dz
    = \left( \frac{1}{2\pi i} \int_{\Gamma} f(z) \frac{dz}{z - \lambda_1} \right) e_1 + \cdots + \left( \frac{1}{2\pi i} \int_{\Gamma} f(z) \frac{dz}{z - \lambda_m} \right) e_m
    = f(\lambda_1)e_1 + \cdots + f(\lambda_m)e_m
\]

\[\blacksquare\]

**Remark 4.4** Note that the conclusions (2) and (4) of the above Theorem are special cases of (5).

Now we apply the above Theorem to a bounded operator on a Banach space.

**Theorem 4.5** Let \( X \) be a complex Banach space. Suppose \( T \in B(X) \) is of \( G_1 \)-class and \( \sigma(T) = \{\lambda_1, \ldots, \lambda_m\} \) is finite. Then

1. Each \( \lambda_j \) is an eigenvalue of \( T \). In fact, there exist projections \( P_j \) such that for each \( j \), the range of \( P_j \) is the eigenspace corresponding to the eigenvalue \( \lambda_j \) and
$X$ is the direct sum of these eigenspaces. In other words, $T$ is “diagonalizable”. Also $\|P_j\| = 1$ and $TP_j = \lambda_j P_j$ for each $j$, $P_j P_k = 0$ for $j \neq k$,

$$P_1 + \cdots + P_m = I$$

and

$$T = \lambda_1 P_1 + \cdots + \lambda_m P_m.$$  

(2)

$$(T - \lambda_1 I) \cdots (T - \lambda_m I) = 0.$$  

(3) If a function $f$ is analytic in a neighbourhood of $\sigma(T)$, then

$$\tilde{f}(T) = f(\lambda_1)P_1 + \cdots + f(\lambda_m)P_m.$$  

Proof It follows from Corollary 4.2 that each $\lambda_j$ is an eigenvalue of $T$. The existence and properties of projections $P_j$ follow from Theorem 4.3. Let $X_j = R(P_j)$, the range of $P_j$. The property $TP_j = \lambda_j P_j$ implies that $X_j$ is the eigenspace of $T$ corresponding to the eigenvalue $\lambda_j$ for each $j$. Also $P_j P_k = 0$ for $j \neq k$ implies that $X_j \cap X_k = \{0\}$ for $j \neq k$. It follows from

$$P_1 + \cdots + P_m = I$$

that $X$ is the sum of $X_j$. This shows that $X$ is the direct sum of these eigenspaces.

Remark 4.6 Let $X$ and $T$ be as in the above Theorem. Since the conclusion (1) says that $X$ has a basis consisting of eigenvectors of $T$ and $T$ is a linear combination of projections, it can be called Spectral Theorem for such operators. Similarly, the conclusion (2) is an analogue of the Caley-Hamilton Theorem. If, in particular, $X$ is a Hilbert space, then every projection of norm 1 is orthogonal and hence Hermitian (self-adjoint). Thus each $P_j$ is self-adjoint and hence $T$ is normal. This last result is also proved in [9] and [10]. On the other hand a compact operator of $G_1$-class defined on a Hilbert space need not be normal. See [10] for an example.

Suppose $X$ is finite dimensional. Then the above Theorem says that every $G_1$-class operator on $X$ is diagonalizable.

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