Abstract

We consider a finite state set $S$ and a continuous time Markov Chain $X_t$, $t \geq 0$, taking values on $S$. We denote by $\Omega$ the set of paths $w$ taking values on $S$ (the elements $w$ are locally constant with left and right limits and are also right continuous on $t$). $P$ will denote the associated probability on $(\Omega, \mathcal{B})$ which we assume that is stationary. All functions $f$ we consider below are in the set $L^\infty(P)$.

From $P$ we are able to define a Ruelle operator $\mathcal{L}_t', t \geq 0$, acting on functions $f : \Omega \to \mathbb{R}$ of $L^\infty(P)$. Given $V : \Omega \to \mathbb{R}$, such that is constant in sets of the form $\{X_0 = c\}$, we define a modified Ruelle operator $\mathcal{L}_{V,t}', t \geq 0$, and we are able to show the existence of an eigenfunction and an eigen-probability $\rho_V$ on $\Omega$ associated to $\mathcal{L}_{V,t}', t \geq 0$.

We also show the follow property for the probability $\rho_V$: for any integrable $f \in L^\infty(P)$ and any real and positive $t$

$$\int e^{-\int_0^t (V \circ \Theta_s)(\cdot) ds} \left[ (\mathcal{L}_t(e^{\int_0^t (V \circ \Theta_s)(\cdot) ds} f)) \circ \theta_t \right] d\rho_V = \int f d\rho_V$$

This equation generalize for continuous time a similar one for discrete time systems (and which is quite important for understanding the KMS states of certain $C^*$-algebras).
1 Introduction

We would like to consider a continuous time stochastic process that maps the positive real line $\mathbb{R}^+ = \{ t \in \mathbb{R} : t \geq 0 \}$ on a finite set $S$ with $n$ elements, that we can simply write as $S = \{1, 2, \ldots, n\}$. Now take a $n$ by $n$ real matrix $L$ such that:

1) $0 < -L_{ii}$, for all $i \in S$,
2) $L_{ij} \geq 0$, for all $i \neq j$, $i \in S$,
3) $\sum_{i=1}^{n} L_{ij} = 0$ for all fixed $j \in S$.

We point out that, by convention, we are considering column stochastic matrices and not line stochastic matrices (see [N] section 2 and 3 for general references).

We denote by $P_t = e^{tL}$ the semigroup generated by $L$. The left action of the semigroup can be identified with an action over functions from $S$ to $\mathbb{R}$ (vectors in $\mathbb{R}^n$) and the right action can be identified with action on measures on $S$ (also vectors in $\mathbb{R}^n$).

The matrix $e^{tL}$ is column stochastic, since from the assumptions on $L$ follows that
\[
(1, \ldots, 1)e^{tL} = (1, \ldots, 1)(I + tL + \frac{1}{2}t^2L^2 + \cdots) = (1, \ldots, 1)
\]

It is well known that there exist a vector of probability $p_0 = (p_0^1, p_0^2, \ldots, p_0^n) \in \mathbb{R}^n$ such that $e^{tL}(p_0) = P_t p_0 = p^0$ for all $t > 0$. The vector $p_0$ is a right eigenvector of $e^{tL}$. All entries $p_0^i$ are strictly positive, as a consequence of hypothesis 1.

Now let us consider the space $\tilde{\Omega} = \{1, 2, \ldots, n\}^{\mathbb{R}^+}$ of all functions from $\mathbb{R}^+$ to $S$. In principle it could be enough for our purposes, but technical details in the construction of probability measures on such a space force us to use a restriction of it: We consider the space $\Omega \subset \tilde{\Omega}$ as the set of right-continuous functions from $\mathbb{R}^+$ to $S$. In this set we take the sigma algebra $\mathcal{B}$ generated by the cylinders of the form
\[
\{ w_0 = a_0, w_{t_1} = a_1, w_{t_2} = a_2, \ldots, w_{t_r} = a_r \},
\]
where $t_i \in \mathbb{R}^+$, $r \in \mathbb{Z}^+$, $a_i \in S$ and $0 < t_1 < t_2 < \ldots < t_r$. It is possible to endow $\Omega$ with a metric, the Skorohod-Stone metric $d$, which makes $\Omega$ complete and separable ([EK] section 3.5) but the space is not compact.

Now we can introduce a continuous time version of the shift map as follows: we define for each fixed $s \in \mathbb{R}^+$ the $\mathcal{B}$-measurable transformation $\Theta_s : \Omega \rightarrow \Omega$ given by $\Theta_s(w_t) = w_{t+s}$ (we remark that $\Theta_s$ is also a continuous transformation with respect to the Skorohod-Stone metric $d$).

For $L$ and $p_0$ fixed as above we denote by $P$ the probability on the sigma-algebra $\mathcal{B}$ defined for cylinders by
\[
P(\{ w_0 = a_0, w_{t_1} = a_1, \ldots, w_{t_r} = a_r \}) = P_{a_r a_{r-1}}^{t_r-t_{r-1}} \cdots P_{a_2 a_1}^{t_2-t_1} P_{a_1 a_0}^{t_1} p_0^{a_0}.
\]

For details of the construction of this measure the reader is refered to [B].
The probability $P$ on $(\Omega, \mathcal{B})$ is stationary in the sense that for any integrable function $f$ and any $s \geq 0$
\[ \int f(w)dP(w) = \int (f \circ \Theta_s)dP(w). \]

From now on the Stationary Process defined by $P$ is denoted by $X_t$ and all functions $f$ we consider are in the set $L^\infty(P)$.

There exist a version of $P$ such that for a set of full measure all elements $w$ are locally constant on $t$ on the right side with left and right limits and $w$ is right continuous on $t$. We consider from now on such $P$.

From $P$ we are able to define a continuous time Ruelle operator $L^t$, $t > 0$, acting on functions $f : \Omega \to \mathbb{R}$ of $L^\infty(P)$. It is also possible to introduce the endomorphism $\alpha_t : L^\infty(P) \to L^\infty(P)$ defined as
\[ \alpha_t(\varphi) = \varphi \circ \Theta_t, \quad \forall \varphi \in L^\infty(P) \]

Given $V : \Omega \to \mathbb{R}$, such that it is constant in sets of the form $\{X_0 = c\}$ (i.e., $V$ depends only on the value of $x(0)$), we are able to show the existence of a probability $\rho_V$ on $\Omega$ which is absolutely continuous with respect to $P$ and satisfies:

**Theorem A.** For any integrable $f \in L^\infty(P)$ and any positive $t$
\[ \int e^{-\int_0^t (V \circ \Theta_s)(\cdot)ds} \left[ (L^t (e^{\int_0^t (V \circ \Theta_s)(\cdot)ds}f)) \circ \theta_t \right] d\rho_V = \int f d\rho_V \]

The above functional equation is a natural generalization (for continuous time) of the similar one presented in [EL1] and [EL2]. We believe it will be important in the analysis of certain $C^*$ algebras, generated by the operators $\alpha$ and $L$, specially concerning the characterization of KMS states. We point out however that we are able to show this property of $\rho_V$ just for a quite simple function $V$ as above.

With the operators $\alpha$ and $L$ we can rewrite the theorem above as
\[ \rho_V(G_T^{-1}E_T(G_T \varphi)) = \rho_V(\varphi) \]
for all $\varphi \in L^\infty$ and all $T > 0$, where, as usual, $\rho_V(\varphi) = \int \varphi d\rho_V$, $E_T = \alpha_T L^T$ is in fact a projection on a subalgebra of $\mathcal{B}$ and $G_T : \Omega \to \mathbb{R}$ is given by
\[ G_T(x) = \exp(\int_0^T V(x(s))ds) \]

For the map $V : \Omega \to \mathbb{R}$, which is constant in cylinders of the form $\{w_0 = i\}$, $i \in \{1, 2, ..., n\}$, we denote by $V_i$ the corresponding value. We denote also by $V$ the diagonal matrix with the $i$-diagonal element equal to $V_i$.

We denote by $P^t_V = e^{t(L+V)}$. The Perron-Frobenius Theorem for such semigroup will be one of the main ingredients of the proof.

A related and more general result will appear in [LNT].
2 A continuous time Ruelle Operator

The infinitesimal generator $L$ defines a stochastic process taking values in $S = \{1, 2, ..., n\}$. Taking the stationary vector of probability we obtain a probability on the Skorohod space $\Omega$ which is denoted by $P$.

**Definition 2.1.** For $t$ fixed we define the operator $L^t : \mathcal{L}^\infty(\Omega, P) \to \mathcal{L}^\infty(\Omega, P)$ as follows:

$$L^t(\varphi)(x) = \int_{y \in \Theta_t^{-1}(x)} \varphi(y)d\mu^x_t(y)$$

**Remark 2.2.** The definition above can be rewritten as

$$L^t(\varphi)(x) = \int_{y \in D[0,t]} \varphi(yx)d\mu^x_t(yx)$$

where the symbol $yx$ means the concatenation of the path $y$ with the translation of $x$:

$$xy(s) = \begin{cases} 
  y(s) & \text{if } s \in [0,t) \\
  x(s-t) & \text{if } s \geq t 
\end{cases}$$

and $D[0,t)$ is the set of right-continuous functions from $[0,t)$ to $S$. This follows simply from the fact that, in this notation, $\Theta_t^{-1}(x) = \{yx : y \in D[0,t]\}$.

It is possible to shed some light on the meaning of this operator applying it to some simple functions. For example, we can see the effect of $L^t$ on a simple cylinder: Consider the sequence $0 = t_0 < t_1 < ... < t_{j-1} < t \leq t_j < ... < t_r$ and then take $f = I_{\{X_0 = a_0, X_1 = a_1, ..., X_r = a_r\}}$. Then, for a path $z \in \Omega$ such that $z_{t_{j-t}} = a_j, ..., z_{t_r-t} = a_r$ (the future condition) we have

$$L^t(f)(z) = \frac{1}{p_0^a} p^{t_{t_{j-1}} - t_{a_0}} p^{t_{a_1} - t_{a_0}} p_0^{a_0},$$

otherwise (i.e., if the path $z$ does not satisfy the condition above) we get $L^t(f)(z) = 0$.

Note that if $t_r < t$, then $L^t(f)(z)$ depends only on $z_0$. For example, if $f = I_{\{X_0 = i_0\}}$ then

$$L^t(f)(z) = \int_{y \in D[0,t]} I_{\{X_0 = i_0\}}(yx)d\mu^x_t(yx) = \mu^x_t([X_0 = i_0]) = \frac{1}{p_0^a} p^{t_{2i_0} - t_{i_0}} p_0^{i_0}$$

In the case $f = I_{\{X_0 = i_0, X_t = j_0\}}$, then $L^t(f)(z) = P^{t_{2i_0} - t_{i_0}} p_0^{j_0}$, if $z_0 = j_0$, and $L^t(f)(z) = 0$ otherwise.

Now we can show some properties of $L^t$. 

As usual we denote by $\mathcal{F}_s$ the sigma-algebra generated by $X_s$. We also denote by $\mathcal{F}^+_s$ the sigma-algebra generated $\sigma(\{X_u, s \leq u\})$. Note that a $\mathcal{F}^+_s$-measurable function $f(w)$ on $\Omega$ does depend of the value $w_s$.

We also denote by $I_A$ the indicator function of a measurable set $A$ in $\Omega$. 


Proposition 2.3. \( \mathcal{L}^t(1) = 1 \), where 1 is the function that maps every point in \( \Omega \) to 1.

**Proof:** Indeed

\[
\mathcal{L}^t(1)(x) = \int_{y \in \mathcal{D}[0,t]} 1(yx) d\mu^x_t(yx) = \int d\mu^x_t(yx) = \mu^x_t([X_t = x(0)]) =
\]

\[
\sum_{a=1}^{n} \mu^x_t([X_0 = a, X_t = x(0)]) = \frac{1}{p_x^{(0)}} \sum_{a=1}^{n} P^t_{x(0) a} p_a^0 = 1
\]

We can also define the dual of \( \mathcal{L}^t \), denoted by \((\mathcal{L}^t)^*\), acting on the measures. Then we get:

**Proposition 2.4.** For any positive \( t \) we have that \((\mathcal{L}^t)^*(P) = (P)\)

**Proof:** For a fixed \( t \) we have that \((\mathcal{L}^t)^*(P) = (P)\) because for any \( f \) of the form \( f = I_{\{X_0 = a_0, X_1 = a_1, \ldots, X_r = a_r\} \}, 0 = t_0 < t_1 < \ldots < t_j - t < t_j < \ldots < t_r \). we have

\[
\int \mathcal{L}^t(f) dP = \sum_{b=1}^{n} \int \mathcal{L}^t(f) dP(z) =
\]

\[
\sum_{b=1}^{n} \int I_{\{X_0 = b, X_{t_j-t} = a_j, \ldots, X_{t_r-t} = a_r\}}(z) dP(z) \cdot \frac{1}{p_0^b} P_{t_{j-1}}^{t_{j-1}} \ldots P_{a_2 a_1}^{t_{j-2}} P_{a_1 a_0}^{t_{j-1}} =
\]

\[
\sum_{b=1}^{n} P(\{X_0 = b, X_{t_j-t} = a_j, \ldots, X_{t_r-t} = a_r\}) \cdot \frac{1}{p_0^b} P_{t_{j-1}}^{t_{j-1}} \ldots P_{a_2 a_1}^{t_{j-2}} P_{a_1 a_0}^{t_{j-1}} =
\]

\[
\int f(w) dP(w).
\]

**Proposition 2.5.** Given \( t \in \mathbb{R}_+ \) and the functions \( \varphi, \psi \in \mathcal{L}^\infty(P) \) then we have

\[
\mathcal{L}^t(\varphi \times (\psi \circ \Theta_t))(z) = \psi(z) \times \mathcal{L}^t(\varphi)(z).
\]

**Proof:**

\[
\mathcal{L}^t(\varphi(\psi \circ \Theta_t))(x) = \int_{i \in \mathcal{D}[0,t]} \varphi(i x)(\psi \circ \Theta_t)(i x) d\mu^x_t(i) =
\]

\[
\psi(x) \int \varphi(i x) d\mu^x_t(i) = (\psi \mathcal{L}^t(\varphi))(x) = \psi(x) \mathcal{L}^t(\varphi)(x)
\]

since \( \psi \circ \Theta_t(i x) = \psi(x) \), independently of \( i \).  

\[\]
We just recall that the last proposition can be restated as
\[ L^t(\varphi \alpha_t(\psi)) = \psi L^t(\varphi) \]

Then we get:

**Proposition 2.6.** $\alpha_t$ is the dual of $L^t$ on $L^2(P)$.

**Proof:** From last two propositions
\[
\int L^t(f)g \, dP = \int L^t(f \times (g \circ \Theta_t)) \, dP = \int f \times (g \circ \Theta_t) \, dP = \int f \alpha_t(g) \, dP
\]
as claimed. \hfill \Box

Now we would like to obtain conditional expectations. For a given $f$ recall that the function $Z(w) = E(f|F^+_t)$ is the $Z$ (almost everywhere defined) $F^+_t$-measurable function such that for any $F^+_t$-measurable set $B$ we have $\int_B Z(w) \, dP(w) = \int_B f(w) \, dP(w)$.

**Proposition 2.7.** The conditional expectation is given by
\[ E(f|F^+_t)(x) = \int f \, d\mu^x_t \]

**Proof:** For $t$ fixed, consider a $F^+_t$-measurable set $B$. Then we have
\[
\int_B E(f|F^+_t) \, dP = \int_B \int f \, d\mu^w_t \, dP(w) = \int (I_B(w) \int f \, d\mu^w_t) \, dP(w) = \\
\int \int (f I_B) \, d\mu^w_t \, dP(w) = \int f(w) I_B(w) \, dP(w) = \int f \, dP,
\]
and the proposition is concluded. \hfill \Box

Now we can relate the conditional expectation with respect to the $\sigma$-algebras $F^+_t$ with the operators $cL^t$ and $\alpha_t$ as follows:

**Proposition 2.8.** $[L^t(f)](\Theta_t) = E(f|F^+_t)$ (i.e. $E = \alpha_t L^t$).

**Proof:** This follows from the fact that for any $B = \{X_{s_1} = b_1, X_{s_2} = b_2, \ldots, X_{s_u} = b_u\}$, with $t < s_1 < \ldots < s_u$, we have $I_B = I_A \circ \Theta_t$ for some measurable $A$ and
\[
\int_B L^t(f)(\Theta_t(w)) \, dP(w) = \int I_B(w) L^t(f)(\Theta_t(w)) \, dP(w) = \\
\int (I_A \circ \Theta_t)(w) L^t(f)(\Theta_t(w)) \, dP(w) = \int I_A(w) L^t(f)(w) \, dP(w) = \\
\int L^t(f(I_A \circ \Theta_t))(w) \, dP(w) = \int f(w) I_A(\Theta_t(w)) \, dP(w) = \int_B f(w) \, dP(w)
\]
\hfill \Box
3 The modified operator

We are interested in the perturbation by $V$ (defined above) of the $\mathcal{L}^t$ operator.

**Definition 3.1.** We define $G_t: \Omega \to \mathbb{R}$ as

$$G_t(x) = \exp \left( \int_0^t V(x(s)) ds \right)$$

**Definition 3.2.** We define the $G$-weighted transfer operator $\mathcal{L}_V^t: \mathcal{L}^\infty(\Omega, P) \to \mathcal{L}^\infty(\Omega, P)$ acting on measurable functions $f$ (of the above form) by

$$\mathcal{L}_V^t(f)(w) := \mathcal{L}_t(G_t f) = \mathcal{L}_t(e^{\int_0^t (V \circ \Theta_s)(.) ds} f) = \sum_{b=1}^n \mathcal{L}_t(e^{\int_0^t (V \circ \Theta_s)(.) ds} I_{\{X_t = b\}} f)(w)$$

Note that $e^{\int_0^t (V \circ \Theta_s)(.) ds} I_{\{X_t = b\}}$ does not depend on information larger than $t$. In the case $f$ is such that $t_r \leq t$ (in the above notation), then $\mathcal{L}_V^t(f)(w)$ depends only on $w(0)$.

The integration on $s$ above is over the open interval $(0, t)$.

We will consider soon an eigenfunction and an eigen-measure for such operator $\mathcal{L}_V^t$. But, first we need the following:

**Theorem 1.** ([S] page 111) We assume $S$ is finite. One can prove that for $L, p_0$ and $V$ fixed as above there exists

a) a unique positive function $u_V: \Omega \to \mathbb{R}$, constant equal to the value $u_V^i$ in each cylinder $X_0 = i$, $i \in \{1, 2, ..., n\}$, (we can see $u_V$ as $u_V: S \to \mathbb{R}$, or, as a vector in $\mathbb{R}^n$),

b) a unique probability vector $\mu_V$ in $\mathbb{R}^n$ (a probability over over the set $\{1, 2, ..., n\}$ such that $\mu_V(\{i\}) > 0$, $\forall i$), such that

$$\sum_{i=1}^n u_V^i(\mu_V)_i = 1,$$

c) a real positive value $\lambda(V)$,

such that

d) for any positive $s$

$$e^{-s\lambda(V)} u_V e^{s(L+V)} = u_V.$$ 

Moreover, for any $v = (v_1, ..., v_n) \in \mathbb{R}^n$

$$\lim_{t \to \infty} e^{-t\lambda(V)} v e^{t(L+V)} = \left( \sum_{i=1}^n v_i(\mu_V)_i \right) u_V,$$
e) for any positive \( t \)

\[
(P^t_V) \mu_V = e^{\lambda(V) t} \mu_V.
\]

From property e) it follows that

\[
(L + V)^* \mu_V = \lambda(V) \mu_V.
\]

From d) it follows that

\[
u_V(L + V) = \lambda(V) u_V.
\]

Note that when \( V = 0 \), then \( \lambda(V) = 0 \), \( \mu_V = p^0 \) and \( u_V \) is constant equal to 1.

In order to show the existence of \( u_V \), such that, \( u_V(L + V) = \lambda(V) u_V \) one add a constant to \( V \) in such way that all the entries of \( L + V \) are positive. This will imply d). For the case the space \( S \) is not finite see [LNT].

Now we return to our setting: for each \( i_0 \) and \( t \) fixed one can consider the probability \( \mu_{i_0}^t \) defined over the sigma-algebra \( \mathcal{F}_t^- = \sigma(\{X_s \leq t\}) \) with support on \( \{X_0 = i_0\} \) such that for cylinder sets with \( 0 < t_1 < \ldots < t_r \leq t \)

\[
\mu_{i_0}^t(\{X_0 = i_0, X_{t_1} = a_1, \ldots, X_{t_{r-1}} = a_{r-1}, X_t = j_0\}) = P_{t_0a_1}^t \cdots P_{a_{r-1}a_r}^t P_{a_rj_0}^t.
\]

The probability \( \mu_{i_0}^t \) is not stationary.

We denote by \( Q(j, i)_t \) the \( i, j \) entry of the matrix \( e^{t(L+V)} \), that is \((e^{t(L+V)})_{i,j}\).

It is known ([K] page 52 or [S] Lemma 5.15) that

\[
Q(j_0, i_0)_t = E_{X_0 = i_0} \{ e^{\int_0^t (V \circ \Theta_s)(w)ds} ; X(t) = j_0 \} = \\
\int I_{\{X_t = j_0\}} e^{\int_0^t (V \circ \Theta_s)(w)ds} d\mu_{i_0}^t(w).
\]

For example,

\[
\int I_{\{X_t = j_0\}} e^{\int_0^t (V \circ \Theta_s)(w)ds} dP = \sum_{i=1}^{n} Q(j_0, i)_t p^0_i
\]

In the particular case where \( V \) is constant equal 0, then \( p^0 = \mu_V \) and \( \lambda(V) = 0 \).

**Proposition 3.3.** \( f(w) = \frac{\mu_V(w)}{p^0(w)} = \frac{(\mu_V)_w(0)}{(p^0)_w(0)} \) is an eigenfunction for \( L^t_V \) with eigenvalue \( e^{\lambda(V)} \).

**Proof:** Note that \( \frac{\mu_V}{p^0} = \sum_{c=1}^{n} \frac{\mu_V(c)}{p^0(c)} I\{X_0 = c\}. \)

For a given \( w \), denote \( w(0) \) by \( j_0 \), then conditioning

\[
L^t_V(\frac{\mu_V}{p^0})(w) = \sum_{c=1}^{n} \sum_{b=1}^{n} L^t_V (\frac{\mu_V(c)}{p^0(c)} I\{X_0 = c\} I\{X_t = b\})(w).
\]
Consider \( c \) fixed, then for \( b = j_0 \) we have
\[
\mathcal{L}_V^t \left( I_{\{X_0 = c\}} I_{\{X_t = b\}} \right)(w) = \frac{Q(j_0, c) t}{p_0^b c_0} p_c^0,
\]
and for \( b \neq j_0 \), we have \( \mathcal{L}_V^t \left( I_{\{X_0 = c\}} I_{\{X_t = b\}} \right)(w) = 0. \)

Finally,
\[
\mathcal{L}_V^t \left( \frac{\mu_V}{p_0^b} \right)(w) = \sum_{c=1}^n \frac{\mu_V(c)}{p_0^b(c)} Q(j_0, c) t \frac{p_c^0}{p_0^b} = e^{t\lambda(V) \left( \frac{\mu_V}{p_0^b} \right)} = e^{t\lambda(V) \left( \frac{\mu_V}{p_0^b} \right)}(w),
\]
because \( e^{t(L+V)} (\mu_V) = e^{t\lambda(V)} (\mu_V) \).

Therefore for any \( t > 0 \) the function \( \frac{\mu_V}{p_0^b} \) (that depends only on \( w(0) \)) is an eigenfunction for the operator \( \mathcal{L}_V^t \) associated to the eigenvector \( e^{t\lambda(V)} \).

\[ \square \]

**Definition 3.4.** Consider now for each \( t \) the operator acting on \( g \) by
\[
\hat{\mathcal{L}}_V^t(g)(w) = \left[ \frac{p_0^b}{\mu_V} \mathcal{L}_V^t(e_j^0(V - \lambda(V)) \circ \Theta_s)(\cdot) ds \left( \frac{\mu_V}{p_0^b} \right)(g) \right](w)
\]
From the above \( \hat{\mathcal{L}}_V^t(1) = 1 \) for all positive \( t \).

Note that by conditioning, if \( g = I_{\{X_0 = a_0, X_{t_1} = a_1, X_{t_2} = a_2, X_{t_3} = a_3\}} \), with \( 0 < t_1 < t_2 < t \), then
\[
\hat{\mathcal{L}}_V^t(g)(w) = \frac{\mu_V(a_0)}{\mu_V(a_3)} e^{(t-t_2)(L+V-\lambda I)} e^{(t_2-t_1)(L+V-\lambda I)} e^{t_1(L+V-\lambda I)}
\]
for \( w \) such that \( w_0 = a_3 \), and \( \hat{\mathcal{L}}_V^t(g)(w) = 0 \) otherwise.

Moreover, for \( g = I_{\{X_0 = a_0, X_{t_1} = a_1, X_{t_2} = a_2, X_{t_3} = a_3\}} \), with \( 0 < t_1 < t < t_3 \), then
\[
\hat{\mathcal{L}}_V^t(g)(w) = \frac{\mu_V(a_0)}{\mu_V(a_2)} e^{(t-t_1)(L+V-\lambda I)} e^{t_1(L+V-\lambda I)}
\]
for \( w \) such that \( w_0 = a_2, w_{t_3-t} = a_3 \), and \( \hat{\mathcal{L}}_V^t(g)(w) = 0 \) otherwise.

Consider now the dual operator \( (\hat{\mathcal{L}}_V^t)^* \).

For \( t \) fixed consider the transformation in the set of probabilities \( \mu \) on \( \Omega \) given by
\( (\hat{\mathcal{L}}_V^t)^* (\mu) = \nu \).

**Theorem 2.** There exists a fixed probability measure \( \nu_V \) on \((\Omega, \mathcal{B})\) for such transformation \( (\hat{\mathcal{L}}_V^t)^* \). The stationary probability \( \nu_V \) does not depend on \( t \).

**Proof:** Denote by \( \nu = \nu_V \) the probability obtained in the following way, for
\[
g = I_{\{X_0 = a_0, X_{t_1} = a_1, X_{t_2} = a_2, ..., X_{t_r-1} = a_{r-1}, X_r = a_r\}},
\]
with $0 < t_1 < t_2 < ... < t_{s-1} < t \leq t_s < .. < t_r$, we define

$$
\int g(w) \, d\nu(w) = e^{(t_r-t_{r-1})(L+V-LI)} \cdots e^{(t_2-t_1)(L+V-LI)} e^{t_1 (L+V-LI)} \mu_V(a_0).
$$

This probability satisfies the Kolmogorov compatibility conditions because is defined via a semigroup (see chapter IV. 2 [BW])

In order to show that $\nu$ is a probability we have to use the fact that $\sum_{c \in S} \mu_V(c) = 1$.

On the other hand,

$$
z(w) = \hat{\mathcal{L}}_V^t(g)(w) = \frac{\mu_V(a_0)}{\mu_V(w_0)} e^{(t-t_{s-1})(L+V-LI)} \cdots e^{(t_2-t_1)(L+V-LI)} e^{t_1 (L+V-LI)},
$$

for $w$ such that $w_{t_{t-s}} = a_s, w_{t_{s-1}+s-t} = a_{s+1}, \ldots, w_{t_{t-r}} = a_r$, and $\hat{\mathcal{L}}_V^t(g)(w) = 0$ otherwise.

Note that $z(w) = \hat{\mathcal{L}}_V^t(g)(w)$ depends only on $w_0, w_{t_{s-1}+s-t}, \ldots, w_{t_{t-r}}$.

We have to show that for any $g$ we have $\int g \, d\nu = \int \hat{\mathcal{L}}_V^t(g)\,d\nu$.

Now,

$$
\int z(w) \, d\nu(w) = \int \sum_{c \in S} I_{\{X_0 = c, X_{t_s-t} = a_s, X_{t_{s+1}+s-t} = a_{s+1}, \ldots, X_{t_r-t} = a_r\}} z(w) \, d\nu(w) = \sum_{c \in S} \nu(\{X_0 = c, X_{t_s-t} = a_s, X_{t_{s+1}+s-t} = a_{s+1}, \ldots, X_{t_r-t} = a_r\})
$$

\begin{align*}
&\mu_V(a_0) e^{(t-t_{s-1})(L+V-LI)} \cdots e^{(t_2-t_1)(L+V-LI)} e^{t_1 (L+V-LI)} = \\
&\sum_{c \in S} \mu_V(c) e^{(t-t_{s-1})(L+V-LI)} \cdots e^{(t_2-t_1)(L+V-LI)} e^{t_1 (L+V-LI)}
\end{align*}

$$
\mu_V(a_0) e^{(t-t_{s-1})(L+V-LI)} \cdots e^{(t_2-t_1)(L+V-LI)} e^{t_1 (L+V-LI)} = \\
\mu_V(a_0) e^{(t_r-t_{r-1})(L+V-LI)} \cdots e^{(t_{s+1}-t_s)(L+V-LI)} e^{t_1 (L+V-LI)}
$$

$$
(\sum_{c \in S} e^{(t_{s-t})(L+V-LI)} e^{(t-t_{s-1})(L+V-LI)} \cdots e^{(t_2-t_1)(L+V-LI)} e^{t_1 (L+V-LI)}) = \\
\mu_V(a_0) e^{(t_r-t_{r-1})(L+V-LI)} \cdots e^{(t_{s+1}-t_s)(L+V-LI)} e^{t_1 (L+V-LI)}
$$

The claim for the general $g$ follows from the above result. □
Definition 3.5. Consider the probability $\rho_V = (g_V)^{-1}\nu_V$, where $g_V$ is chosen colinear to $\mu_V p_V$, in such way $\rho_V$ is a probability (not necessarily invariant) on $\Omega$.

It easily follows that $(L^t_V)^*(\rho_V) = e^{t\lambda_V}\rho_V$. The probability $\nu_V$ is invariant for $\theta_s$ with $s \geq 0$.

From last theorem follows easily:

Proposition 3.6. For any integrable $f, g \in L^\infty(P)$ and any positive $t$

$$\int g L^t_V(f)d\rho_V = \int L^t_V(f(g \circ \theta_t))d\rho_V = e^{t\lambda_V} \int f(g \circ \theta_t)d\rho_V.$$ 

Now we are in position to prove our main result. From $(L^t_V)^*(\rho_V) = e^{t\lambda_V}\rho_V$ it follows that the measure $\rho_V$ satisfies the important equation:

Theorem A. For any integrable $f \in L^\infty(P)$ and any positive $t$

$$\int e^{-\int_0^t(V \circ \Theta_s)(\cdot) ds} \left[ (L^t( e^{\int_0^t(V \circ \Theta_s)(\cdot) ds} f)) \circ \theta_t \right] d\rho_V = \int f d\rho_V$$

Proof:

$$\int e^{-\int_0^t(V \circ \Theta_s)(\cdot) ds} \left[ (L^t( e^{\int_0^t(V \circ \Theta_s)(\cdot) ds} f)) \circ \theta_t \right] d\rho_V =$$
$$e^{-t\lambda_V} \int [L^t(e^{-\int_0^t(V \circ \Theta_s)(\cdot) ds} f)] [L^t(e^{\int_0^t(V \circ \Theta_s)(\cdot) ds} f)] d\rho_V =$$
$$e^{-t\lambda_V} \int L^t(e^{\int_0^t(V \circ \Theta_s)(\cdot) ds} f) d\rho_V =$$
$$\int f d\rho_V$$

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