Thermodynamics of 2 + 1 dimensional Coulomb-Like Black Holes from Non Linear Electrodynamics with a traceless energy momentum tensor

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Abstract

In this work we study thermodynamics of 2+1-dimensional static black holes with a nonlinear electric field. Besides employing the standard thermodynamic approach, we investigate the black hole thermodynamics by studying its thermodynamic geometry. We compute the Weinhold and Ruppeiner metrics and compare the thermodynamic geometry with the standard description on the black hole thermodynamics. We further consider the cosmological constant as an additional extensive thermodynamic variable. In the thermodynamic equilibrium three dimensional space, we compute the efficiency of the heat engine and show that it is possible to be built with this black hole.

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I. INTRODUCTION

The three-dimensional models of gravity have been of great interest, because they are much simpler than the four-dimensional and higher-dimensional models of gravity and one can investigate more efficiently some of their properties which are shared by their higher dimensional analogs. The well known Bañados–Teitelboim–Zanelli (BTZ) black hole [1] is a solution to the Einstein equations in three dimensions with a negative cosmological constant, and it shares several features of the Kerr black hole [2]. Another model known as topologically massive gravity (TMG) was constructed by adding a Chern-Simons gravitational term to the action of three-dimensional GR [3], this model contains a propagating degree of freedom which corresponds to a massive graviton [3, 4], and also admits the BTZ (and other) black holes as exact solutions [5–7].

On the other hand, Bergshoeff, Hohm and Townsend considered an action with the standard Einstein-Hilbert term, and a specific combination of scalar curvature square term and Ricci tensor square one, which is known as BHT massive gravity [8–13]. BHT massive gravity is admitting interesting solutions [14–18], for further aspects see [19–23]. Nowadays, one could think that Lorentz invariance may not be fundamental or exact. So, by introducing a preferred foliation and terms that contain higher-order spatial derivatives can lead to significantly improved UV behavior, such gravity theory is known as Hořava gravity [24], and it admits a Lorentz-violating version of the BTZ black hole, along with black holes with positive and vanishing cosmological constant [25, 26]. Also, in three-dimensional theory of gravity it is possible to consider GR coupled to electromagnetic fields, such for Maxwell electrodynamics and also for consideration of nonlinear theory in which the Born-Infeld electrodynamics [27] is an example. Born-Infeld gravity has been a growing field of research and it is widely applied to black hole object. There are many exact solutions of charged black hole in different framework considering general relativity or modified gravity, see [28–35] and reference therein. One remarkable solution corresponds to regular charged black hole found by Ayon-Beato and Garcia in Ref. [36]. Currently, it is well known that the black holes are thermodynamic objects and the study of their properties have shown a growing interest in the literature from the seminal work about the laws of black hole mechanics [37]. On a modern point of view, the thermodynamic properties of AdS black hole are very important in order to gain some insights of AdS/CFT conjecture or more recently on de Sitter/conformal
field theory correspondence (dS/CFT) [38]. In this work, we consider a three-dimensional static black hole solution that arises from a nonlinear electrodynamics, and that satisfy the weak energy conditions, their electric field $E(r)$ is given by $E(r) = q/r^2$, thus it has the Coulomb form of a point charge in the Minkowski spacetime. The solutions describe charged (anti)–de Sitter spacetimes [39]. We will study the general formalism of such black holes and disclose corresponding thermodynamical properties by employing standard thermodynamic approach.

Besides standard thermodynamic descriptions, geometrical ideas was introduce to understand thermodynamics. Weinhold first introduced the geometrical concept into thermodynamics, he suggested a sort of Riemannian metric defined as the second derivatives of internal energy with respect to entropy and other extensive quantities of a thermodynamic system [40]. Ruppeiner further introduced a metric, defined as the second derivatives of entropy with respect to the internal energy and other extensive quantities of a thermodynamic system [41], which is conformally related to the Weinhold metric with the inverse temperature as the conformal factor. The Ruppeiner geometry has its physically meanings in the fluctuation theory of equilibrium thermodynamics [42]. Recently the Ruppeiner geometry was considered as a possible tool to disclose the possible microscopic structures of a black hole system [43]. Thus it is interesting to compare the thermodynamic geometry and the standard thermodynamic approach to understand better on the black hole thermodynamics. In this paper we will explore the thermodynamic geometry of the 2+1 dimensional AdS black hole with nonlinear electric field and understand better it thermodynamical properties.

Recently the study of black hole thermodynamics has been generalized to the extended phase space, where the cosmological constant is identified with thermodynamic pressure and its variations are included in the first law of black hole thermodynamics (for a review, please refer to [44]). In the extended phase space with cosmological constant and volume as thermodynamic variables, it was interestingly found that the system admits a more direct and precise coincidence between the first order small-large black hole phase transition and the liquid-gas change of phase occurring in fluids [45]. Considering the extended phase space, and hence treating the cosmological constant as a dynamical quantity, is a very interesting theoretical idea in disclosing possible phase transitions in AdS black holes [46]. More discussions in this direction can be found in existing references. The thermodynamics of D-dimensional Born-Infeld AdS black holes in the extended phase space was examined
in [35]. Here we will generalize the extended phase space thermodynamics discussion to 2+1 dimensional AdS black hole with nonlinear electric field. We are going to examine whether critical behavior in the extended phase space thermodynamics can show some special properties in such special spacetime.

The paper is organized as follows. In Sec. II we give a brief review of the background that we will study. In Sec. III we study the thermodynamics of the spacetime, by using the standard approach and the geometrothermodynamics point of view. In Sec. IV, we will generalize our discussion to the extended space thermodynamics. Finally, we conclude and discuss the results obtained in Sec. V.

II. GENERAL FORMALISM FOR 2+1 DIMENSIONAL GRAVITY PLUS NONLINEAR ELECTRODYNAMICS

We are considering the black hole solution arises from non-linear electrodynamics

\[ S = \int d^3x \sqrt{-g} \left( \frac{1}{16\pi} (R - 2\Lambda) + L(F) \right), \]  

where \( L(F) \) represents the electromagnetic lagrangian for non linear electrodynamics, and the electromagnetic tensor is written at the usual form from the vector potential \( A_\mu \) and the electromagnetic tensor as \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \). The variation respect to metric \( \delta g_{\mu\nu} \) and vector potential \( \delta A_\mu \) gives the equation of motions

\[ G_{\mu\nu} = -\Lambda g_{\mu\nu} + 8\pi T_{\mu\nu}, \]  

where the energy momentum tensor \( T_{\mu\nu} \) for the electromagnetic field is given by

\[ T_{\mu\nu} = g_{\mu\nu} L(F) - F_{\nu\rho} F_{\mu}^{\rho} L_{,F}. \]  

The electromagnetic equation is given by

\[ \nabla_\mu (F^{\mu\nu} L_{,F}) = 0, \]  

and the solution of these equations were found in [39] for the case where the energy momentum tensor has vanishing trace. Considering the following metric ansatz

\[ ds^2 = -f(r)dt^2 + f^{-1}(r)dr^2 + r^2 d\phi^2, \]
and the lapse function is given by

\[ f(r) = -M - \Lambda r^2 + \frac{4q^2}{3r}, \]  

(6)

the case \( \Lambda > 0 \) corresponds to an asymptotically de-Sitter spacetime, and the solution in this case show a cosmological horizon. For vanishing cosmological constant, it has an asymptotically flat solution coupled with a Coulomb-like field, that also show a cosmological horizon. The case of negative cosmological constant (\( \Lambda < 0 \)), represents a genuine black hole solution where its horizons corresponds to the solution of \( f(r) = -M - \Lambda r_h^2 + \frac{4q^2}{3r_h} = 0 \), its solutions are given by

\[ r_{h_1} = \frac{h}{3\Lambda} - \frac{M}{h}, \]

(7)

\[ r_{h_2} = -\frac{h}{6\Lambda} + \frac{M}{2h} + \frac{\sqrt{3}}{2} \left( \frac{h}{3\Lambda} + \frac{M}{h} \right), \]

(8)

\[ r_{h_3} = -\frac{h}{6\Lambda} + \frac{M}{2h} - \frac{\sqrt{3}}{2} \left( \frac{h}{3\Lambda} + \frac{M}{h} \right), \]

(9)

where \( h \) is given by

\[ h = \left( \frac{18q^2 + 3\sqrt{3}\left( \frac{M^3}{\Lambda} + 12q^4 \right)}{\Lambda^2} \right)^\frac{1}{3}. \]

(10)

In Fig. 1, we show the behaviour of the lapse function for different values of the cosmological constant, we can see the different behavior depending of the sign of \( \Lambda \). Thus, if \( \Lambda > 0 \) or \( \Lambda = 0 \), there is a cosmological horizon, and for \( \Lambda < 0 \) there are different ranges where there are two horizon black hole, single or naked singularities. In the following, we will consider

![Fig. 1: The behavior of the metric function \( f(r) \), with \( M = 1, q = 1 \), for different values of cosmological constant. Note that when \( \Lambda = -1 \), there is a naked singularity.](image)
the AdS case, and in order to explore the nature of the black holes solutions described by (6) we will show how the sign of the radical is crucial to see different solutions

\[ \alpha := \frac{M^3}{\Lambda} + 12q^4, \]  

(11)

here due to \( M > 0 \), the solution nature is essentially related to the comparison among the cosmological constant, black hole mass and electric charge. So:

- Case \( \alpha < 0 \) or \( 0 > \Lambda > -\frac{M^3}{12q^4} \), we get real solutions for the event horizon, and the solution represents a black hole with inner and outer horizon. This behavior is showed in Fig. 2.

![Graph showing the behavior of the metric function \( f(r) \), with \( M = 1, q = 1 \), for different values of the cosmological constant \( 0 > \Lambda > -\frac{M^3}{12q^4} \). Here we can see a two horizon black hole and at the particular value \( \Lambda = -\frac{1}{12} \) we obtain the extremal configuration.](image)

FIG. 2: The behavior of the metric function \( f(r) \), with \( M = 1, q = 1 \), for different values of the cosmological constant \( 0 > \Lambda > -\frac{M^3}{12q^4} \). Here we can see a two horizon black hole and at the particular value \( \Lambda = -\frac{1}{12} \) we obtain the extremal configuration.
Case $\alpha > 0$ or $\Lambda < -\frac{M^3}{12q^2}$. The solutions are one real and two complex and in general there are naked singularities, their behaviour is showed in Fig. 3.

![Graph](image)

**FIG. 3**: The behavior of the metric function $f(r)$, with $M = 1$, $q = 1$, for different values of the cosmological constant when $\Lambda < -\frac{M^3}{12q^2}$. Here we can see one real negative solution for horizon black hole, the other two are complex and the solution in this region represents naked singularities. Also we show in the plot the extremal case with $\Lambda = -\frac{1}{12}$.

- Case $\alpha = 0$ or $\Lambda = -\frac{M^3}{12q^2}$, we get one real root. The solution represents an extremal black hole. In Fig. 2 and Fig. 3 we show the extremal solution in both cases.

Finishing this section we would like to do a comparison with the charged BTZ black hole solution and discuss a little bit about the mayor differences between both solution in $2 + 1$ dimensions with linear and no linear electrodynamics interaction. The BTZ black hole is a solution to the Einstein- Maxwell theory in AdS space

$$f(r) = -2m + \frac{r^2}{l^2} - \frac{q^2}{2} \ln \left( \frac{r}{l} \right),$$

where $q$ and $m$ are the black hole charge and mass respectively, and $\Lambda = -\frac{1}{l^2}$ is the cosmological constant, with $l$ the AdS radius [1]. This case is very challenging to address, for example the asymptotic structure renders computation of the mass more problematic. It is necessary to introduce a renormalization procedure for computing the mass and therefore this solution exhibit one renormalization scale [47], and the solution reads as follows ,

$$f(r) = -2m_0 + \frac{r^2}{l^2} - \frac{q^2}{2} \ln \left( \frac{r}{r_0} \right),$$

where $m_0 = m + \frac{q^2}{2} \ln \left( \frac{r_0}{r_0} \right)$. In the case under study we can compute quasilocal energy and the quasilocal mass [39]. In fact, from (1), considering $\mathcal{L}(F) = C|F|^n$, $F = \frac{1}{4} F_{\mu \nu} F^{\mu \nu}$ and
The Euclidean continuation of the metric is given by
\[ ds^2 = N(r)^2 f(r) d\tau^2 + \frac{dr^2}{f(r)} + r^2 d\phi^2, \quad A_t(r) = -q/r, \quad (14) \]
where \( \tau = i t \) is the Euclidean time. The constant \( C \) is set to the value
\[ C = \frac{8|q|^{1/2}}{21^{1/4}3^{1/2}} \quad (15) \]
with this choice the metric function takes the value
\[ f(r) = -\Lambda r^2 - M + \frac{4q^2}{3r}, \quad (16) \]
and the conjugate momentum is given by
\[ \pi^r = \text{sgn}(F) \sqrt{g} C n |F|^{n-1} F^r. \quad (17) \]
Therefore, the Euclidean action is given by
\[ I_E = 2\pi \beta \int_{r_h}^{\infty} dr \left[ \frac{1}{2\pi} (f'(r) + 2\Lambda r) - \frac{(1 - 2n)([\pi^r]^2)^{2n/(2n-1)}}{2n \left( \frac{Cnr}{2n-1} \right)^{1/(2n-1)}} - A_t \partial_r \pi^r \right] + B. \quad (18) \]

Now, varying with respect to \( N, f, \pi^r \) and \( A_t \) we obtain the field equations
\[ \frac{1}{2\pi} (f'(r) + 2\Lambda r) - \frac{(1 - 2n)([\pi^r]^2)^{2n/(2n-1)}}{2n \left( \frac{Cnr}{2n-1} \right)^{1/(2n-1)}} = 0, \quad (19) \]
\[ N'(r) = 0, \quad (20) \]
\[ A'_t + \text{sgn}(\pi^r) N \left( \frac{2^{n-1} \pi^r}{Cnr} \right)^{1/(2n-1)} = 0, \quad (21) \]
\[ \partial_r \pi^r = 0, \quad (22) \]
without loss of generality we can set \( N(r) = 1 \) by a coordinate transformation, also we obtain \( \pi^r \) is a constant. These equations are consistent with the field equations obtained from the action (1) leading to the solution (14). The boundary term of the Euclidean action is given by
\[ \delta B = -2\pi \left[ \frac{1}{2\pi} \delta f + A_t \delta \pi^r \right]_{r_h}^{\infty}. \quad (23) \]
On the other hand, the variations of the field solutions at infinity are given by
\[ \delta f|_{\infty} = -\delta M, \quad A_t \delta \pi^r|_{\infty} = 0, \quad (24) \]
and at the horizon by
\[ \delta f|_{r_h} = -f'|_{r_h} \delta r_h = -\frac{4\pi}{\beta} \delta r_h, \quad A_t \delta \pi|_{r_h} = -\Phi \delta \pi|_{r_h}, \]
(25)
where \( \Phi = A_t(\infty) - A_t(r_h) = \frac{q}{r_h} \) is the electric potential. Then, we obtain
\[ I_E = \beta M - 4\pi r_h + 2\beta A_t(r_h) \pi^r, \]
(26)
and the mass, entropy and electric charge are respectively given by
\[ M = \frac{\partial I_E}{\partial \beta}|_\phi - \frac{\phi}{\beta} \frac{\partial I_E}{\partial \phi}|_\beta = M, \]
(27)
\[ S = \beta \frac{\partial I_E}{\partial \beta}|_\phi - I_E = 4\pi r_h, \]
(28)
\[ Q = -\frac{1}{\beta} \frac{\partial I_E}{\partial \phi}|_\beta = -2\pi \pi^r, \]
(29)
where
\[ \pi^r = -\text{sgn}(q) 2^{1/4} C n |q|^{1/2} = -\frac{2q}{\pi}. \]
(30)
Finally, we obtain the electric charge \( Q = 4q \). Thus, we can concluded that the analogous ADM mass is defined to be \( M(\infty) := M \), and therefore we do not need an extra renormalization procedure for the solution under study and also we do not need a renormalization mass scale, essentially the inclusion of non-linear term for the Maxwell Lagrangian interacting with gravity in 2 + 1 dimensions it is acting as a regulator and the problems arising from the logarithmic term disappears. On the other hand, the BTZ charged solution also exhibit a similar behavior, i.e it is possible to observe, a black hole with two horizons, naked singularities and the presence of extremal solutions depending of the values of charge and mass. In the next, we would like to study thermodynamics properties of like Coulomb black hole, and see the main differences with the thermodynamics description of BTZ black hole, in particular we will study possible phase transition, the reverse isoperimetric inequality, the thermodynamics curvature, extended thermodynamics and the like Coulomb black holes as a heat engine.

III. A SIGHT OF THE THERMODYNAMICS APPROACHES

A. Thermodynamics properties at the standard approach

The starting point in order to see the thermodynamics description corresponds to determine the entropy of the geometry, in this case we are supposing that our geometry satisfy Hawking-Bekenstein entropy-area, i.e \( S = \frac{A}{4} = 4\pi r_h \), where we can obtain the mass parameter as a
function of the entropy and charge. In Ref. [48] suggests that the mass $M$ of an AdS black hole can be interpreted as the enthalpy from classical thermodynamics, rather than as the total energy of the spacetime $M = H(S, q)$, this point will be very important when we will use the cosmological constant as a variable. Meanwhile we start with the mass of black hole, we obtained from (6) the following equation

$$M(S, q) = -\frac{\Lambda S^2}{16\pi^2} + \frac{16\pi q^2}{3S},$$

and using the conservation energy law or the first law of black holes mechanics

$$dM = TdS + \Phi dq,$$

and then we can obtain the thermodynamic variables as the temperature

$$T = \left(\frac{\partial M}{\partial S}\right)_\Phi = -\frac{\Lambda S}{8\pi^2} - \frac{16\pi q^2}{3S^2},$$

and the electric potential

$$\Phi = \left(\frac{\partial M}{\partial q}\right)_T = \frac{32\pi q S}{3}. $$

From Eq. (33), the temperature is positive when

$$3S^3 + \frac{128\pi^3 q^2}{3\Lambda} > 0, $$

that is a standard requirement of the mechanics laws of black holes, and at the case $q = 0$ we obtain the well known result $S_{BTZ} > 0$. Note that the temperature is vanished for $r_h = r_{extrem} = \left(-\frac{2q^2}{3\Lambda}\right)^{\frac{1}{3}}$, and for $r_h < r_{extrem}$ we obtain a negative temperature and therefore it corresponds to a region with nonphysical meaning where the thermodynamics description breakdown, see Fig. 4. In order to understand this result and study the thermodynamics stability of this solution, from (31) we compute the heat capacity, and then we can move forward at the understanding of the possible critical behavior of this 2+1 AdS dimensional black hole

$$C_q = T\left(\frac{\partial S}{\partial T}\right)_q = -S \left(\frac{128\pi^3 q^2 - 3\Lambda S^3}{256\pi^3 q^2 + 3\Lambda S^3}\right).$$

Now, from Eqs. (33) and (36) we can see that the temperature and the heat capacity are always positive when the condition (35) is satisfied. Then, this results implies that 2 + 1 like Coulomb black hole with a positive definite temperature must be a thermodynamically stable configuration. Fig. 5 shows the heat capacity for different values of electric charge as
FIG. 4: This plot shows behavior of the lapse function as a function of \( r \) (top panel) and the temperature as function of the event horizon \( r_h \) (bottom panel), here we are considering \( \Lambda = -0.5 \) and different values of electric charged \( q \). When \( q = 0 \) we obtain the temperature profile of BTZ black hole, and when \( q = (1/6)^{1/4} \) the extremal case.

a function of event horizon, similar solutions and description can read for BTZ rotating and charged black hole in Ref. [49] and [50–52]. Now, according to [53], one change at the sign of the heat capacity means an evidence of one instability or a phase transition among the black holes configuration, in the following we will study in more detail this point. Therefore, the main conclusion of Davies’s approach it is established the correlation between drastic change for the stability properties of a thermodynamic black hole system and the change of the sign of the heat capacity. Summarizing a negative heat capacity represents a region of instability whereas the stable domain is characterized by a positive heat capacity. At the

FIG. 5: This plot shows behavior of heat capacity as function of the event horizon \( r_h \), here we are considering \( \Lambda = -0.5 \) and different values of electric charged \( q \). When \( q = 0 \) we recover the heat capacity profile of static BTZ black hole.

reference of canonical ensemble, it is very well known that the black holes are one locally thermodynamical stable system if its heat capacity is positive or non vanishing. Therefore,
at the points where the heat capacity is vanishing or divergent there are a type one phase transition or one type two phase transition, respectively. Therefore we need to study the positivity of heat capacity $C_q > 0$ or also the positivity of $\partial S/\partial T$ or $(\partial^2 M/\partial S^2)$ with $T > 0$ as sufficient conditions to ensure the local stability of the black hole. From Figs. 4, 5, and

\[ \text{FIG. 6: This plot shows } \frac{\partial^2 M(S,q)}{\partial S^2} \text{ as function of the event horizon } r_h, \text{ here we are considering } \Lambda = -0.5 \text{ and different values of electric charged } q. \text{ When } q = 0 \text{ we obtain the } \frac{\partial^2 M(S,q)}{\partial S^2} \text{ profile of BTZ black hole.} \]

6 we can see that this black hole solution is locally stable from a thermal point of view, because the heat capacity $C_q$ is positive a free of divergent terms. Therefore we can see the heat capacity is a regular function for all real positive values where $r > r_h$ and also due to the positivity of $\frac{\partial^2 M(S,q)}{\partial S^2}$, we will back in more details for this point in the next section.

### B. Comparison with charged BTZ black hole

In this section we would like to establish the differences or similarities in the thermodynamics description between the charged BTZ black hole and the like Coulomb black hole with non linear electrodynamics interaction. So, by considering the charge BTZ metric (12), where $m$ is a integration constant related to the black hole mass $M$ through $M = \frac{m}{4}$, in the following we present the main results obtained in Refs. [51, 52]. Thus, the starting point is the enthalpy $H(S,q) = M(S,q)$ of the charge BTZ black hole, which is given by

\[ M(S,q) = \frac{1}{16} \left( \frac{8S^2|\Lambda|}{\pi^2} - q^2 \log \left( \frac{2S\sqrt{|\Lambda|}}{\pi} \right) \right), \quad (37) \]

and then the temperature yields

\[ T = \left( \frac{\partial M}{\partial S} \right)_\Phi = \frac{\Lambda S}{\pi^2} - \frac{q^2}{16S}, \quad (38) \]
and it is positive when
\[ 16S^2 \Lambda - \pi^2 q^2 > 0, \] (39)
which is a standard requirement of the mechanics laws of black holes, and at the case \( q = 0 \) we obtain the well known result \( S_{\text{BTZ}} > 0 \). Using the definition of the entropy \( S = \frac{\pi}{2} r_h \) we see the temperature is vanished for \( r_h = r_{\text{extrem}} \), and for \( r_h < r_{\text{extrem}} \) we obtain negative temperature and therefore a region with nonphysical meaning where the thermodynamics description breakdown. As in the non lineal like Coulomb case, in order to understand this

result and study the thermodynamics stability of this solution, from (37) we compute the heat capacity, and we can move forward at the understanding of the possible critical behavior and comparison of our results with charged BTZ black hole

\[ C_q = T \left( \frac{\partial S}{\partial T} \right)_q = -S \left( \frac{\pi^2 q^2 - 16\Lambda S^2}{\pi^2 q^2 + 16\Lambda S^2} \right). \] (40)

Now, from Eqs. (38) and (40) we can see that temperature and heat capacity are always positive when condition (39) is satisfied. Then, this results implies that like 2 + 1 BTZ charged black hole with a positive definite temperature must be a thermodynamically stable configuration. Fig. 8 shows the heat capacity for different values of electric charge as a function of event horizon, similar solutions and description can read for BTZ rotating and charged black hole in Ref. [49] and [50–52]. From this last result, we can see there are not main differences in the standard thermodynamics description between the BTZ charged black hole and the like Coulomb 2 + 1 dimensional black hole. Both black hole are thermodynamics stables. Finally, as the main section conclusion, we can say the black hole from non-linear
FIG. 8: This plot shows behavior of heat capacity as function of the event horizon $r_h$, here we are considering $\Lambda = -0.5$ and different values of electric charged $q$. When $q = 0$ we recover the heat capacity profile of static BTZ black hole.

electrodynamics in $2 + 1$ dimensions with a Coulomb like potential is a thermodynamically stable configuration at least using description of canonical ensemble, for a discussion of the ensemble dependency of BTZ charged black hole see [54].

C. Geometrothermodynamics point of view

In this section, we are looking the essential aspects of the called geometry of the thermodynamic phase space, this geometrical approach has been studied by Ruppeiner [42] and Weinhold [40] and after that there many references applying those ideas to black holes thermodynamics [49, 55–70]. In this approach it is possible to build a space analogy using the thermodynamics parameters and then it is allowed to define an appropriate metric tensors for this space, where the line element measures the distance between two neighbouring fluctuation states in the state space. Also, we can evaluate another geometric quantities as for example one key quantity it is the curvature that allow to study the thermodynamics properties and in particular critical behavior of the thermodynamics black hole systems, in comparison with Davies’s approach and also with the standard thermodynamics systems. Thermodynamics curvature can be shown important properties, and give us information about the nature of the interaction among the fundamental properties constituent of the system. It is well known for example for an ideal gas that the thermodynamics parameter space is flat and the thermodynamics curvature is vanished this reflect the fact that an ideal gas is a collection of non interacting particles. For a Van der Waals fluids we will find the thermodynamics curvature space is curve and the interpretation of this results implies
that we are in the presence of an interacting system, that could be attractive or repulsive depending of the sign of the thermodynamic curvature negative or positive, respectively. Another key point inside the thermodynamics curvature is given by its singularity structure. If there are singularities implies that the system has phases transition, because the phases transition are exactly at the point where the thermodynamics curvature is singular. Let us start wit the definition of Weinhold metric as the second derivatives of the internal energy with respect to extensive parameters \((S, q)\)

\[
g_{bc}^W = \frac{\partial^2 M(X^a)}{\partial X^b \partial X^c},
\]

where \(X^a = (S, x^{\hat{a}})\), \(S\) represents the entropy and \(x^{\hat{a}}\) all the other extensive variables. On the other side, the Ruppeiner metric can be defined as the second derivative of the entropy with respect to the internal energy and respect to the other extensive parameters

\[
g_{bc}^R = \frac{\partial^2 S(Y^a)}{\partial Y^b \partial Y^c},
\]

where \(Y^a = (M, y^{\hat{a}})\), \(M\) represent the mass and \(y^{\hat{a}}\) all the other extensive variables. Both metrics are related by a conformal transformation, such that \(ds_R^2 = \frac{1}{T} ds_W^2\). Then, we can compute both metrics in their natural coordinates. So, the Weinhold line element is given by

\[
ds_W^2 = \left( \frac{256\pi^3 q^2 - 3\Lambda S^3}{24S^3\pi^2} \right) dS^2 - \frac{64\pi q}{3S^2} dSdq + \frac{32\pi}{3S} dq^2.
\]

At this point, we would like to comment about the Weinhold metrics and its connection with the previous results found, about the comparison with the study of positivity of \(\frac{\partial^2 M(S, q)}{\partial S^2}\) the ensure the thermodynamics stability of the black hole under consideration. The Weinhold metric is essentially the Hessian matrix construct with the mass formula (31)

\[
H_{S, q}^M = \begin{pmatrix}
\frac{256\pi^3 q^2 - 3\Lambda S^3}{24S^3\pi^2} & -\frac{64\pi q}{3S^2} \\
-\frac{64\pi q}{3S^2} & \frac{32\pi}{3S}
\end{pmatrix},
\]

with this Hessian matrix we can performed the study of the thermodynamics stability at the grand canonical ensemble, and the standard extensive parameters are the entropy and the charge. Now, in order to study the stability in this case we compute the determinant of the Hessian matrix and we obtain \(\text{det}(H_{S, q}^M) = -\frac{4(256\pi^3 q^2 + \Lambda S^3)}{3\pi S^4}\). Therefore, we can conclude that in the grand canonical ensemble this black holes is stable and do not show any signal of
instability. At this point, we are seeing an evident ensemble dependency because from Eq. (36) we observer there is at least a region of instability starting where the \( C_q = 0 \) and after that \( C_q < 0 \), then according to Davies’s approach there is a type one phase transition. Also, the scalar Weinhold thermodynamics curvature is given by,

\[
R_W = -\frac{4\pi^2}{\Lambda S^2},
\]

which is regular and positive. In order to fix this controversy we are exploring different ways, first one it is to compute the Ruppeiner curvature and the second one is the extended thermodynamics space and consider the cosmological constant as another thermodynamics variable. In the next we are following the first one, and in this case the Ruppeiner element is given by

\[
ds^2_R = -\frac{1}{S} \left( \frac{256\pi^3 q^2 - 3S^3\Lambda}{128\pi^3 q^2 + 3S^3\Lambda} \right) dS^2 + \left( \frac{512\pi^3 q}{128\pi^3 q^2 + 3S^3\Lambda} \right) dS dq - \left( \frac{256\pi^3 S}{128\pi^3 q^2 + 3S^3\Lambda} \right) dq^2.
\]  

If we are looking for the thermodynamics stability we must to ensure that thermodynamics metric fluctuation must be positive. This required is written as \( g^R_{SS} > 0 \), \( g^R_{qq} > 0 \) and \( det(g_R) > 0 \), first two condition are satisfied in all point of \((S,q)\) space except for the point where the heat capacity becomes vanishing and the last condition \( det(g_R) = -\frac{768(256\pi^6 q^2 + \pi^3\Lambda S^3)}{(128\pi^3 q^2 - 3\Lambda S^3)^2} \). As we previously mentioned this could be the existence of a phase transition in the system. So, in order to see this point, we consider the scalar Ruppeiner thermodynamics curvature

\[
R_R = \frac{64\pi^3 q^2}{3\Lambda S^4} + \frac{384\pi^3 q^2}{128\pi^3 q^2 S - 3\Lambda S^4} - \frac{1}{S},
\]

that present a genuine divergence at

\[
128\pi^3 q^2 - 3\Lambda S^3 = 0,
\]

i.e when \( r_h = r_{\text{extrem}} \) is satisfied, see Fig (9). It is worth to mentioning that this point is exactly the point where the heat capacity is becoming vanishing, and now we can confirm that this black hole solution has a phase transition and also it represents a behaviour as an interacting systems from the microscopic point of view. This result shows that the Ruppeiner thermodynamic curvature in this case correctly describes the transition from a region with positive and well-defined temperature to a region with nonphysical negative temperature.
FIG. 9: This plot shows $R_R$ as a function of the event horizon $r_h$, here we are considering $\Lambda = -0.5$ and different values of electric charged $q$. When $q = 0$ we obtain the $R_R$ profile of BTZ static black hole.

In sum, we can see that, the Coulomb like black solution considering non linear Born Infeld electrodynamics in $2+1$ dimensions is showing a phase transition from the BH solution with two horizons to the extremal solution at the point $r_{\text{extrem}} = \left( -\frac{2q^2}{3\Lambda} \right)^{\frac{1}{3}}$ where from one side the heat capacity is vanishing and negative for lower distance than the $r_{\text{extrem}}$ and also we are seeing the temperature in this case are becoming negative and therefore with nonphysical sense. From the thermodynamics geometry point of view we see that, the Weinhold metric are no able to show the truly divergence of the system and therefore the phase transition. On the other hand Ruppeiner metric shows, there is a truly curvature divergence or singularity at the value $r_{\text{extrem}} = \left( -\frac{2q^2}{3\Lambda} \right)^{\frac{1}{3}}$, similar situation was reported for Kerr black holes in Ref. [71]. This value corresponds to the limit of vanishing temperature or the extremal limit for the geometry. In this sense, the unstable region ($C_q < 0$) corresponds to a nonphysical region with negative temperature. According to Davies the change of the sign of the heat capacity can be associated with a strong change of thermodynamics system stability a negative heat capacity represents a region of instability, also with negative temperature. Because the third law of thermodynamics is imposing the restriction of positive temperature for thermodynamics systems, we can see the thermodynamic description breaks down at the extremal limit. Similar results can bee seeing for the BTZ black hole [49], [50].

**D. Comparison with charged BTZ black hole**

In this case the Ruppeiner element is given by
\[ ds_{R}^{2}_{BTZ} = -\frac{1}{S} \left( \frac{\pi^2 q^2 + 16 S^2 \Lambda}{\pi^2 q^2 - 16 S^2 \Lambda} \right) \, dS^2 + \left( \frac{2\pi^2 q}{\pi^2 q^2 - 16 S^2 \Lambda} \right) \, dS \, dq - \left( \frac{2\pi^2 S \log \left( \frac{2S \sqrt{|\Lambda|}}{\pi} \right)}{\pi^2 q^2 - 16 S^2 \Lambda} \right) \, dq^2, \]

(49)

If we are looking for the thermodynamics stability we must ensure that thermodynamics metric fluctuation must be positive. This required is written as \( g_{SS}^{R} > 0, \ g_{qq}^{R} > 0 \) and \( \text{det}(g_{R}) > 0 \), first two condition are satisfied in all point of \((S, q)\) space except for the point where the heat capacity becomes vanishing and the last condition

\[ \text{det}(g_{R}^{BTZ}) = -\frac{2\pi^2 \left( \frac{\pi^2 q^2 \log \left( \frac{2\sqrt{|\Lambda|}}{\pi} \right)}{\pi^2 q^2 - 16 S^2 \Lambda} \right)}{(\pi^2 q^2 - 16 S^2 \Lambda)^2}, \]

it is showing a similar behaviors. As we mentioned previously this could be the existence of a phase transition in the system, and the scalar Ruppeiner thermodynamics curvature \([72]\) is given by,

\[ R_{R}^{BTZ} = \frac{A + B \log \left( \frac{2\sqrt{|\Lambda|}}{\pi} \right) + C \log^2 \left( \frac{2\sqrt{|\Lambda|}}{\pi} \right)}{\pi S \left( \pi^2 q^2 - 16 S^2 \Lambda \right) \left( 2\pi^2 q^2 \log \left( \frac{2\sqrt{|\Lambda|}}{\pi} \right) + q^2 + 32 S^2 \log \left( \frac{2\sqrt{|\Lambda|}}{\pi} \right) \right)^2}, \]

(50)

where constants \( A, B, C \) are

\[
A = 256\pi^3 \Lambda^2 q^2 S^4 + 1024\pi^2 \Lambda^2 q^2 S^4 - 1024\pi \Lambda^2 q^2 S^4 + 16\pi^5 \Lambda q^4 S^2 - 64\pi^4 \Lambda q^4 S^2 - 256\pi^3 \Lambda q^4 S^2 + 512\pi^2 \Lambda q^4 S^2 - \pi^7 q^6 + 4\pi^5 q^6 - 4096\pi \Lambda^3 S^6, \\
B = -256\pi^3 \Lambda^2 q^2 S^4 + 3072\pi^2 \Lambda^2 q^2 S^4 - 144\pi^5 \Lambda q^4 S^2 + 224\pi^4 \Lambda q^4 S^2 + 256\pi^3 \Lambda q^4 S^2 + \pi^7 q^6 - 28672\pi \Lambda^3 S^6 + 24576\Lambda^3 S^6, \\
C = 2048\pi^3 \Lambda^2 q^2 S^4 + 96\pi^5 \Lambda q^4 S^2 + 8192\pi \Lambda^3 S^6.
\]

Note that the Ruppeiner scalar shows a truly curvature divergence or singularity at the value \( r_h = r_{\text{extrem}} \). In the next section, we will consider the cosmological constant as an additional extensive thermodynamic variable that is called the extended thermodynamics description, in which the cosmological constant is treated as a thermodynamic variable and acts like a pressure term \([46, 48, 73, 74]\).
IV. EXTENDED THERMODYNAMICS DESCRIPTION

As we mention in previous section there is a dependence of the ensemble due to the expression for the heat capacity Eq.(36) and the determinant of Hessian matrix Eq. (44) are not compatible with the change of phase transition proposed by Davies approach. It does mean that we need to extended the thermodynamics space and consider the cosmological constant as another thermodynamics variable, and therefore the standard extensive parameters will be the entropy, the charge, and the cosmological constant. In order to see this extended thermodynamics we are considering the cosmological constant as source of a dynamical pressure using the relation \( P = -\frac{\Lambda}{8\pi} \) [73, 74]. Now, the mass of black hole will be understanding as the enthalpy and all thermodynamics description is given by the enthalpy functions \( H = H(S, P, q) \) and it reads as follow

\[
H = M(S, P, q) = \frac{PS^2}{2\pi} + \frac{16\pi q^2}{3S}, \tag{51}
\]

here conjugate quantity related to pressure could be interpreted as the thermodynamic volume,

\[
V = (\frac{\partial H}{\partial P})_{S,q} = 8\pi r_h^2. \tag{52}
\]

The first law of thermodynamics black holes now read as follow

\[
dH = TdS + PdV + \Phi dq, \tag{53}
\]

in the extended phase space. Before to analyze the non linear black hole under consideration, we think that it is worthwhile to study the case of the \((2 + 1)\)-dimensional BTZ black hole in order to establish important concepts and features. The extended thermodynamics of the BTZ black hole was studied in Ref. [73], the main results about it, are the black hole mass

\[
M = H(S, P) = \frac{4PS^2}{\pi}, \tag{54}
\]

where the entropy is defined as \( S = \frac{A}{4} \) and for the black hole area the author considered \( A = 2\pi r_h \), then the following equation of state was obtained

\[
P\sqrt{V} = \frac{\sqrt{\pi}T}{4}, \tag{55}
\]

that corresponds to the equation of state of an ideal gas, and the conclusion was the static BTZ black hole is associated with non interacting micro structure. Rotating BTZ black hole
were presented in Refs. [75], [51] [46], in this case the equation of states is written as follows,

\[ P = \frac{T}{v} + \frac{8J^2}{\pi v^4}, \quad (56) \]

where \( v = 4r_h \) is the specific volume. This equation can be interpreted as the Van der Waals fluid where the equation of states is given by [45]

\[ (P + \frac{a}{v^2})(v - b) = kT, \quad (57) \]

where \( v \) is the specific volume and \( k \) is the Boltzmann constant. Eq. (57) describes an interacting fluid, and also it is able to exhibit the critical behavior of the fluid. It was shown that the rotating BTZ black hole has a repulsive interaction and it does not have any critical thermodynamics behavior. In Ref. [51] authors did a deep study of lower dimensional black hole chemistry and studied in detail charged and rotating cases of BTZ black hole, one interesting point for our study, it is the inclusion of the reverse isoperimetric inequality [76], that conjectured the isoperimetric ratio

\[ \mathcal{R} = \left( \frac{(D-1)V}{\omega_{D-2}} \right)^{\frac{D-1}{D-2}} \left( \frac{\omega_{D-2}}{A} \right)^{\frac{1}{D-2}}, \quad (58) \]

always satisfied \( \mathcal{R} \geq 1 \) for the conjugate thermodynamics volume \( V \), and the horizon area \( A \), where \( \omega_d = \frac{2\pi^{\frac{d-1}{2}}}{\Gamma(\frac{d+1}{2})} \) that correspond to the area of a d-dimensional unit sphere. In Ref. [51] the authors found that for the rotating case \( \mathcal{R} = 1 \) and then the reverse isoperimetric inequality is saturated with the physical meaning that rotating BTZ black hole has maximal entropy. For the BTZ charged black hole, from Refs. [51] [46]

\[ M = H(S, P) = \frac{4PS^2}{\pi} - \frac{q^2}{32} \log \left( \frac{32PS^2}{\pi} \right), \quad (59) \]

where the entropy is defined as \( S = \frac{\pi}{r} \), then the following equation of state was obtained [46]

\[ P = \frac{T}{v} + \frac{q^2}{2\pi v^4}, \quad (60) \]

where \( v = 4r_h \) is the specific volume, therefore the charged BTZ black hole does not show a critical behaviour. For this charged case, authors in Ref. [51] showed a violation of the reverse isoperimetric inequality \( \mathcal{R} < 1 \) and therefore charged BTZ black hole is always superentropic, there are several references considering this point for different AdS black holes [35, 45, 77–82].
Now, for the black hole under consideration in our work we can computing the temperature and the we obtain the respective equation of state for the $2 + 1$ non lineal Coulomb black hole, as follow

$$P = \frac{\sqrt{\pi}T}{\sqrt{2V}} + \frac{4\sqrt{2}\pi q^2}{3V^{3/2}},$$

(61)

this equation can be interpreted as the Van der Waals fluid, and therefore $2 + 1$ like Coulomb black hole is associated with interacting repulsive micro structures and this results is consistent with the interpretation of non vanishing Ruppeiner curvature. From this equation also we can concluded there are not critical thermodynamics behavior and then the black hole does not have phase transitions associated, and at the limit $q = 0$, we obtain a similar equation of state than BTZ statics black hole, $P\sqrt{V} \propto T$. Also we can compute in our case the ratio of reverse isoperimetric inequality obtaining $\Re = \sqrt{2\pi} > 1$, and as we mention previously we do not need the introduction of any regulator due to the inclusion of non linear electrodynamics interaction.

• Holographic heat engine

Now we would like to study the efficiency of the $2 + 1$ non linear black hole as holographic heat engines. Following the analysis of PV criticality [83–88]. In Ref. [84] holographic heat engine were defined for extracting mechanical work from heat energy via the $PdV$ term present in the First Law of extended black hole thermodynamics $dH = TdS + PdV + \Phi dq$, where, the working substance is a black hole solution of the gravity system. Then using the volume, pressure, temperature and entropy we can compute heat energy and mechanical useful work. We start with equation of state (function of $P(V, T)$) and define an engine as a close path in P-V plane which receives a net input heat flow $Q_H$ and gives a net output heat flow $Q_C$, such that using the energy conservation law, $Q_H = W + Q_C$. It is a well known result that the efficiency of this kind of heat engines can be computed by $\eta = \frac{W}{Q_H} = 1 - \frac{Q_H}{Q_C}$, from classical thermodynamics, some of the classic cycles involve a pair of isotherms at temperature $T_H$ and $T_C$ ($T_H > T_C$) where there are isothermal expansion and compression while some net heat flow along each isobars by

$$Q = \int_{T_i}^{T_f} C_p dT,$$

(62)

and then we can compute mechanical work by $W = \int PdV$. In classical thermodynamics, there are several cycles that are used in heat engines and among them Carnot cycle has the
highest efficiency. This cycle consider two pairs of isothermal and adiabatic processes, and its efficiency is easy computed as $\eta = 1 - \frac{T_C}{T_H}$. Following Ref. [85], in order to constructing a heat engines in the background of a black hole, we chose simple heat cycle in this engine, with a pair of isotherms at high $T_H = T_1$ and low $T_C = T_2$ temperatures. Through the isochoric paths, we can connect these two temperatures, such as the Carnot cycle. In the process of isothermal expansion and compression, the heat absorbed is $Q_H$ and the heat discharged is $Q_C$, Fig. 10, and the efficiency of this cycle is given by the simple expression

$$\eta = 1 - \frac{M_3 - M_4}{M_2 - M_1},$$

then in cycle along the isochoric paths $V_1 = V_4$ and $V_2 = V_3$ and along the isobaric paths $P_1 = P_2$ and $P_3 = P_4$, then the expression for the engine efficiency is given by

$$\eta = \frac{3}{3P_1(S_2 + S_1)}S_1S_2(S_2 + S_1)$$

similar result was found in Ref. [89] in the context of non lineal electrodynamics black hole. We see, that our expression for the limit $q = 0$, we obtain the efficient of BTZ static black hole at the leading order as

$$\eta = 1 - \frac{T_C}{T_H} \sqrt{\frac{V_2}{V_4}},$$

consistent with the result was reported in Ref. [90].

![FIG. 10: Isothermal curves for the charged Non Linear Coulomb like black.](image)

V. CONCLUDING COMMENTS

In this paper we have studied the thermodynamics description of $2 + 1$ dimensional like Coulomb black holes from non linear electrodynamics and with the restriction to have
a traceless energy momentum in $2 + 1$ dimensions. This solution is interesting because the solution were obtained for a circularly symmetric metric describing a black holes with a Coulomb-like field, these solutions are asymptotically anti-de Sitter spacetimes. It is worthwhile to point out that the derived charged black holes possess finite mass contrary to the charged BTZ solutions, which because of the presence of the logarithmic term in the metric yields a divergent quasilocal mass, and therefore for our thermodynamics analysis we do not need to add a renormalization procedure or a renormalization scale for the black hole mass. From the thermodynamics approach we found, that the solution under consideration shows a stable behavior and where the temperature and the heat capacity are positive define in all the regions where $r_h > r_{\text{extrem}}$. In particular in this region the heat capacity $C_q$ is always positive and free of singular points, this point is usually interpreted as an indication that the black hole is a thermodynamically stable configuration where no phase transitions can occur. There is a region $r_h < r_{\text{extrem}}$ where the $C_q < 0$ and according to Davies’s approach this could indicated the presence of a type one phase transition, also it is important to remark that the temperature in this region is also negative and therefore it is considered as a non physical region, where the thermodynamics description breaks down. In order to contrast this results we used tools of the geometry of the thermodynamics phase space and we compute the Weinhold and Ruppeiner curvature. From Weinhold metrics we see that in the grand canonical ensemble the black hole is stable and free of divergence in a similar way to the the canonical ensemble description through the analysis of heat capacity. From Ruppeiner point of view, we found a non vanishing Rupeinner’s curvature which indicates that we are in front an interacting system. At this point, we also found a divergent point, this point is exactly where the heat capacity becomes vanishing, and now we can confirm that this black hole solution has a type one phase transition, but the region where $C_q < 0$ is nonphysical. In this sense, the unstable region corresponds to a nonphysical region with negative temperature. According to Davies the change of the sign of the heat capacity can be associated with a strong change of thermodynamic system stability and a negative heat capacity represents a region of instability, also with negative temperature. Because the third law of thermodynamics restricts a positive temperature for thermodynamic systems, we can see the the thermodynamic description breaks down in the extremal limit. Finally, in order to see the implication of this thermodynamic stability we considered the cosmological constant as source of a dynamical pressure using the relation $P = -\frac{\Lambda}{8\pi}$ and we perform the
thermodynamic study using the enthalpy function $H = H(S, P, q)$. Employing the first law of black hole mechanics we compute the equation of state

$$P = \frac{\sqrt{\pi} T}{\sqrt{2V}} + \frac{4\sqrt{2\pi} q^2}{3V^{3/2}}, \quad (66)$$

which can be interpreted as the Van der Waals fluid, and therefore we can concluded that the $2 + 1$ dimensional like Coulomb black hole is associated with an interacting repulsive micro structure and this result is consistent with the interpretation of non vanishing Ruppeiner curvature. From the Eq. (66) and its graphic in Fig. 9, we can conclude that there are not critical thermodynamics behavior and then the black hole does not have phase transitions associated, also we see that at the limit $q = 0$ we obtain a similar equation of state to the BTZ static black hole, $P\sqrt{V} \propto T$. Also, we computed the ratio of reverse isoperimetric inequality and we obtained $\Re = \sqrt{2\pi} > 1$, and as we mention previously we do not need the introduction of any regulator due to the inclusion of non lineal electrodynamics interaction. Finally, we built a heat engine in the background of this black hole, we chose simple heat cycle in this engine, with a pair of isotherms at high $T_H = T_1$ and low $T_C = T_2$ temperatures, and through the isochoric paths, we can connect these two temperatures, such as the Carnot cycle. In the process of isothermal expansion and compression, the heat absorbed is $Q_H$ and the heat discharged is $Q_C$ and we obtain an expression for the efficiency of this heat engine. This formula gives the correct one at the limit $q = 0$, corresponding to the static BTZ Black hole.

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[1] M. Banados, C. Teitelboim and J. Zanelli, “The Black hole in three-dimensional space-time,” Phys. Rev. Lett. 69 (1992) 1849 [hep-th/9204099].
[2] S. Carlip, “The (2+1)-Dimensional black hole,” Class. Quant. Grav. 12, 2853 (1995) [gr-qc/9506079].

[3] S. Deser, R. Jackiw and S. Templeton, “Topologically Massive Gauge Theories,” Annals Phys. 140, 372 (1982) [Annals Phys. 281, 409 (2000)] Erratum: [Annals Phys. 185, 406 (1988)].

[4] S. Deser, R. Jackiw and S. Templeton, “Three-Dimensional Massive Gauge Theories,” Phys. Rev. Lett. 48 (1982) 975.

[5] K. A. Moussa, G. Clement and C. Leygnac, “The Black holes of topologically massive gravity,” Class. Quant. Grav. 20, L277 (2003) [gr-qc/0303042].

[6] A. Garbarz, G. Giribet and Y. Vasquez, “Asymptotically AdS3 Solutions to Topologically Massive Gravity at Special Values of the Coupling Constants,” Phys. Rev. D 79, 044036 (2009) [arXiv:0811.4464 [hep-th]].

[7] Y. Vasquez, “Exact solutions in 3D gravity with torsion,” JHEP 1108, 089 (2011) [arXiv:0907.4165 [gr-qc]].

[8] E. A. Bergshoeff, O. Hohm and P. K. Townsend, “Massive Gravity in Three Dimensions,” Phys. Rev. Lett. 102 (2009) 201301 [arXiv:0901.1766 [hep-th]].

[9] E. A. Bergshoeff, O. Hohm and P. K. Townsend, “More on Massive 3D Gravity,” Phys. Rev. D 79 (2009) 124042 [arXiv:0905.1259 [hep-th]].

[10] E. A. Bergshoeff, O. Hohm and P. K. Townsend, “On Higher Derivatives in 3D Gravity and Higher Spin Gauge Theories,” Annals Phys. 325 (2010) 1118 [arXiv:0911.3061 [hep-th]].

[11] E. Bergshoeff, O. Hohm and P. Townsend, “On massive gravitons in 2+1 dimensions,” J. Phys. Conf. Ser. 229 (2010) 012005 [arXiv:0912.2944 [hep-th]].

[12] R. Andringa, E. A. Bergshoeff, M. de Roo, O. Hohm, E. Sezgin and P. K. Townsend, “Massive 3D Supergravity,” Class. Quant. Grav. 27 (2010) 025010 [arXiv:0907.4658 [hep-th]].

[13] E. A. Bergshoeff, O. Hohm, J. Rosseel, E. Sezgin and P. K. Townsend, “More on Massive 3D Supergravity,” Class. Quant. Grav. 28 (2011) 015002 [arXiv:1005.3952 [hep-th]].

[14] G. Clement, “Warped AdS(3) black holes in new massive gravity,” Class. Quant. Grav. 26 (2009) 105015 [arXiv:0902.4634 [hep-th]].

[15] E. Ayon-Beato, G. Giribet and M. Hassaine, “Bending AdS Waves with New Massive Gravity,” JHEP 0905 (2009) 029 [arXiv:0904.0668 [hep-th]].

[16] G. Clement, “Black holes with a null Killing vector in new massive gravity in three dimensions,” Class. Quant. Grav. 26 (2009) 165002 [arXiv:0905.0553 [hep-th]].
[17] E. Ayon-Beato, A. Garbarz, G. Giribet and M. Hassaine, “Lifshitz Black Hole in Three Dimensions,” Phys. Rev. D 80 (2009) 104029 [arXiv:0909.1347 [hep-th]].

[18] J. Oliva, D. Tempo and R. Troncoso, “Three-dimensional black holes, gravitational solitons, kinks and wormholes for BHT massive gravity,” JHEP 0907, 011 (2009) [arXiv:0905.1545 [hep-th]].

[19] W. Kim and E. J. Son, “Central Charges in 2d Reduced Cosmological Massive Gravity,” Phys. Lett. B 678 (2009) 107 [arXiv:0904.4538 [hep-th]].

[20] I. Oda, “Renormalizability of Massive Gravity in Three Dimensions,” JHEP 0905 (2009) 064 [arXiv:0904.2833 [hep-th]].

[21] Y. Liu and Y. -W. Sun, “On the Generalized Massive Gravity in AdS(3),” Phys. Rev. D 79 (2009) 126001 [arXiv:0904.0403 [hep-th]].

[22] M. Nakasone and I. Oda, “Massive Gravity with Mass Term in Three Dimensions,” Phys. Rev. D 79 (2009) 104012 [arXiv:0903.1459 [hep-th]].

[23] S. Deser, “Ghost-free, finite, fourth order D=3 ( alas) gravity,” Phys. Rev. Lett. 103 (2009) 101302 [arXiv:0904.4473 [hep-th]].

[24] P. Horava, “Quantum Gravity at a Lifshitz Point,” Phys. Rev. D 79, 084008 (2009) [arXiv:0901.3775 [hep-th]].

[25] T. P. Sotiriou, M. Visser and S. Weinfurtner, “Lower-dimensional Horava-Lifshitz gravity,” Phys. Rev. D 83, 124021 (2011) [arXiv:1103.3013 [hep-th]].

[26] T. P. Sotiriou, I. Vega and D. Vernieri, “Rotating black holes in three-dimensional Horava gravity,” Phys. Rev. D 90, no. 4, 044046 (2014) [arXiv:1405.3415 [gr-qc]].

[27] M. Born, L. Infeld, Proc. Roy. Soc. (London) A 144 (1934) 425.

[28] T. K. Dey, “Born-Infeld black holes in the presence of a cosmological [arXiv:hep-th/0406169 [hep-th]].

[29] R. G. Cai, D. W. Pang and A. Wang, “Born-Infeld black holes in (A)dS spaces,” Phys. Rev. D 70, 124034 (2004) [arXiv:hep-th/0410158 [hep-th]].

[30] G. Boillat, “Nonlinear electrodynamics - Lagrangians and equations of motion,” J. Math. Phys. 11, no.3, 941-951 (1970)

[31] S. Fernando and D. Krug, “Charged black hole solutions in Einstein-Born-Infeld gravity with a cosmological constant,” Gen. Rel. Grav. 35, 129-137 (2003) [arXiv:hep-th/0306120 [hep-th]].

[32] J. Jing and S. Chen, “Holographic superconductors in the Born-Infeld electrodynamics,” Phys.
Lett. B 686, 68-71 (2010) [arXiv:1001.4227 [gr-qc]].

[33] H. P. de Oliveira, “Nonlinear charged black holes,” Class. Quant. Grav. 11, 1469-1482 (1994).

[34] I. Gullu, T. C. Sisman and B. Tekin, “Born-Infeld extension of new massive gravity,” Class. Quant. Grav. 27, 162001 (2010) [arXiv:1003.3935 [hep-th]].

[35] D. C. Zou, S. J. Zhang and B. Wang, “Critical behavior of Born-Infeld AdS black holes in the extended phase space thermodynamics,” Phys. Rev. D 89, no.4, 044002 (2014) [arXiv:1311.7299 [hep-th]].

[36] E. Ayon-Beato and A. Garcia, Phys. Rev. Lett. 80, 5056-5059 (1998) [arXiv:gr-qc/9911046 [gr-qc]].

[37] J.M. Bardeen, B. Carter, S.W. Hawking, , Commun. Math. Phys. 31, 161-170 (1973).

[38] A. Strominger, “The dS/CFT correspondence,” Journal of High Energy Physics, vol. 2001, pp. 034–034, Oct. 2001.

[39] M. Cataldo, N. Cruz, S. del Campo and A. Garcia, “(2+1)-dimensional black hole with Coulomb-like field,” Phys. Lett. B 484, 154 (2000) [hep-th/0008138].

[40] F. Weinhold, J. Chem. Phys. 63, 2479 (1975); 63, 2484 (1975); 63, 2488 (1975); 63, 2496 (1975); 65, 559 (1976).

[41] G. Ruppeiner, Phys. Rev. A 20, 1608 (1979).

[42] G. Ruppeiner, “Riemannian geometry in thermodynamic fluctuation theory,” Rev. Mod. Phys. 67, 605 (1995) Erratum: [Rev. Mod. Phys. 68, 313 (1996)].

[43] S. W. Wei and Y. X. Liu, “Insight into the Microscopic Structure of an AdS Black Hole from a Thermodynamical Phase Transition,” Phys. Rev. Lett. 115, no.11, 111302 (2015) [erratum: Phys. Rev. Lett. 116, no.16, 169903 (2016)] [arXiv:1502.00386 [gr-qc]].

[44] D. Kubiznak, R. B. Mann and M. Teo, “Black hole chemistry: thermodynamics with Lambda,” Class. Quant. Grav. 34, no.6, 063001 (2017) [arXiv:1608.06147 [hep-th]].

[45] D. Kubiznak and R. B. Mann, “P-V criticality of charged AdS black holes,” JHEP 07, 033 (2012) [arXiv:1205.0559 [hep-th]].

[46] S. Gunasekaran, R. B. Mann and D. Kubiznak, “Extended phase space thermodynamics for charged and rotating black holes and Born-Infeld vacuum polarization,” JHEP 11 (2012), 110 [arXiv:1208.6251 [hep-th]].

[47] M. Cadoni, M. Meli and M. R. Setare, Class. Quant. Grav. 25, 195022 (2008) [arXiv:0710.3009 [hep-th]].
[48] D. Kastor, S. Ray and J. Traschen, “Enthalpy and the Mechanics of AdS Black Holes,” Class. Quant. Grav. 26 (2009), 195011 [arXiv:0904.2765 [hep-th]].

[49] H. Quevedo and A. Sanchez, “Geometrothermodynamics of asymptotically de Sitter black holes,” JHEP 09 (2008), 034

[50] M. Akbar, H. Quevedo, K. Saifullah, A. Sanchez and S. Taj, “Thermodynamic Geometry Of Charged Rotating BTZ Black Holes,” Phys. Rev. D 83, 084031 (2011) [arXiv:1101.2722 [gr-qc]].

[51] A. M. Frassino, R. B. Mann and J. R. Mureika, “Lower-Dimensional Black Hole Chemistry,” Phys. Rev. D 92, no.12, 124069 (2015) [arXiv:1509.05481 [gr-qc]].

[52] M. Dehghani, “Thermodynamics of (2 + 1)-dimensional charged black holes with power-law Maxwell field,” Phys. Rev. D 94, no.10, 104071 (2016)

[53] P. C. Davies, Reports on Progress in Physics 41, 1313 (1978).

[54] S. H. Hendi, S. Panahiyan and R. Mamasani, “Thermodynamic stability of charged BTZ black holes: Ensemble dependency problem and its solution,” Gen. Rel. Grav. 47, no.8, 91 (2015) [arXiv:1507.08496 [gr-qc]].

[55] T. Sarkar, G. Sengupta and B. Nath Tiwari, “On the thermodynamic geometry of BTZ black holes,” JHEP 11 (2006), 015 [arXiv:hep-th/0606084 [hep-th]].

[56] S. Ferrara, G. W. Gibbons and R. Kallosh, “Black holes and critical points in moduli space,” Nucl. Phys. B 500 (1997), 75-93 [arXiv:hep-th/9702103 [hep-th]].

[57] J. E. Aman and N. Pidokrajt, “Geometry of higher-dimensional black hole thermodynamics,” Phys. Rev. D 73 (2006), 024017 [arXiv:hep-th/0510139 [hep-th]].

[58] J. y. Shen, R. G. Cai, B. Wang and R. K. Su, “Thermodynamic geometry and critical behavior of black holes,” Int. J. Mod. Phys. A 22 (2007), 11-27 [arXiv:gr-qc/0512035 [gr-qc]].

[59] R. Banerjee, S. K. Modak and S. Samanta, “Second Order Phase Transition and Thermodynamic Geometry in Kerr-AdS Black Hole,” Phys. Rev. D 84 (2011), 064024 [arXiv:1005.4832 [hep-th]].

[60] D. Astefanesei, M. J. Rodriguez and S. Theisen, “Thermodynamic instability of doubly spinning black objects,” JHEP 08 (2010), 046 [arXiv:1003.2421 [hep-th]].

[61] S. W. Wei, Y. X. Liu, Y. Q. Wang and H. Guo, “Thermodynamic Geometry of Black Hole in the Deformed Horava-Lifshitz Gravity,” EPL 99 (2012) no.2, 20004 [arXiv:1002.1550 [hep-th]].

[62] H. Liu, H. Lu, M. Luo and K. N. Shao, “Thermodynamical Metrics and Black Hole Phase Transitions,” JHEP 12 (2010), 054 [arXiv:1008.4482 [hep-th]].

[63] C. Niu, Y. Tian and X. N. Wu, “Critical Phenomena and Thermodynamic Geometry of RN-AdS
Black Holes,” Phys. Rev. D 85 (2012), 024017 [arXiv:1104.3066 [hep-th]].

[64] S. W. Wei and Y. X. Liu, “Critical phenomena and thermodynamic geometry of charged Gauss-Bonnet AdS black holes,” Phys. Rev. D 87 (2013) no.4, 044014 [arXiv:1209.1707 [gr-qc]].

[65] R. Banerjee, B. R. Majhi and S. Samanta, “Thermogeometric phase transition in a unified framework,” Phys. Lett. B 767 (2017), 25-28 [arXiv:1611.06701 [gr-qc]].

[66] T. Vetsov, “Information Geometry on the Space of Equilibrium States of Black Holes in Higher Derivative Theories,” Eur. Phys. J. C 79 (2019) no.1, 71 [arXiv:1806.05011 [gr-qc]].

[67] H. Dimov, R. C. Rashkov and T. Vetsov, “Thermodynamic information geometry and complexity growth of a warped AdS black hole and the warped AdS₃/CFT₂ correspondence,” Phys. Rev. D 99 (2019) no.12, 126007 [arXiv:1902.02433 [hep-th]].

[68] K. Bhattacharya, S. Dey, B. R. Majhi and S. Samanta, “General framework to study the extremal phase transition of black holes,” Phys. Rev. D 99 (2019) no.12, 124047 [arXiv:1903.03434 [gr-qc]].

[69] Y. W. Han, Z. Q. Bao and Y. Hong, “Thermodynamic curvature and phase transitions from black hole with a Coulomb-like field,” Commun. Theor. Phys. 55, 599-601 (2011)

[70] Y. W. Han and G. Chen, “Thermodynamics, geometrothermodynamics and critical behavior of (2+1)-dimensional black holes,” Phys. Lett. B 714, 127-130 (2012) [arXiv:1207.5626 [gr-qc]].

[71] H. Quevedo and A. Vazquez, “The Geometry of thermodynamics,” AIP Conf. Proc. 977, no.1, 165-172 (2008) [arXiv:0712.0868 [math-ph]].

[72] Z. M. Xu, B. Wu and W. L. Yang, “Diagnosis inspired by the thermodynamic geometry for different thermodynamic schemes of the charged BTZ black hole,” [arXiv:2002.00117 [gr-qc]].

[73] B. P. Dolan, “Pressure and volume in the first law of black hole thermodynamics,” Class. Quant. Grav. 28 (2011), 235017 [arXiv:1106.6260 [gr-qc]].

[74] B. P. Dolan, “The cosmological constant and the black hole equation of state,” Class. Quant. Grav. 28, 125020 (2011) [arXiv:1008.5023 [gr-qc]].

[75] A. Ghosh and C. Bhamidipati, “Thermodynamic geometry and interacting microstructures of BTZ black holes,” Phys. Rev. D 101, no.10, 106007 (2020) [arXiv:2001.10510 [hep-th]].

[76] M. Cvetic, G. W. Gibbons, D. Kubiznak, and C. N. Pope, Phys.Rev. D84 024037 (2011).

[77] N. Altamirano, D. Kubiznak, R. B. Mann and Z. Sherkatghanad, “Thermodynamics of rotating black holes and black rings: phase transitions and thermodynamic volume,” Galaxies 2, 89-159 (2014) [arXiv:1401.2586 [hep-th]].

[78] W. Cong and R. B. Mann, “Thermodynamic Instabilities of Generalized Exotic BTZ Black
Holes,” JHEP 11, 004 (2019) [arXiv:1908.01254 [gr-qc]].

[79] R. Zhou, Y. X. Liu and S. W. Wei, “Phase transition and microstructures of five-dimensional charged Gauss-Bonnet-AdS black holes in the grand canonical ensemble,” [arXiv:2008.08301 [gr-qc]].

[80] P. Chaturvedi, S. Mondal and G. Sengupta, “Thermodynamic Geometry of Black Holes in the Canonical Ensemble,” Phys. Rev. D 98, no.8, 086016 (2018) [arXiv:1705.05002 [hep-th]].

[81] M. Mir, R. A. Hennigar, J. Ahmed and R. B. Mann, “Black hole chemistry and holography in generalized quasi-topological gravity,” JHEP 08, 068 (2019) [arXiv:1902.02005 [hep-th]].

[82] A. Belhaj, M. Chabab, H. El Moumni, K. Masmar, M. B. Sedra and A. Segui, “On Heat Properties of AdS Black Holes in Higher Dimensions,” JHEP 05, 149 (2015) [arXiv:1503.07308 [hep-th]].

[83] R. A. Hennigar, F. McCarthy, A. Ballon and R. B. Mann, “Holographic heat engines: general considerations and rotating black holes,” Class. Quant. Grav. 34, no.17, 175005 (2017) [arXiv:1704.02314 [hep-th]].

[84] C. V. Johnson, “Holographic Heat Engines,” Class. Quant. Grav. 31, 205002 (2014) [arXiv:1404.5982 [hep-th]].

[85] C. V. Johnson, “An Exact Efficiency Formula for Holographic Heat Engines,” Entropy 18, 120 (2016) [arXiv:1602.02838 [hep-th]].

[86] C. V. Johnson, “Born–Infeld AdS black holes as heat engines,” Class. Quant. Grav. 33, no.13, 135001 (2016) [arXiv:1512.01746 [hep-th]].

[87] S. Guo, Q. Q. Jiang and J. Pu, “Heat engine efficiency of the Hayward-AdS black hole,” [arXiv:1908.01712 [gr-qc]].

[88] P. K. Yerra and B. Chandrasekhar, “Heat engines at criticality for nonlinearly charged black holes,” Mod. Phys. Lett. A 34, no.27, 1950216 (2019) [arXiv:1806.08226 [hep-th]].

[89] L. Balart and S. Fernando, “Non-linear black holes in 2+1 dimensions as heat engines,” Phys. Lett. B 795, 638-643 (2019) [arXiv:1907.03051 [gr-qc]].

[90] J. X. Mo, F. Liang and G. Q. Li, “Heat engine in the three-dimensional spacetime,” JHEP 03, 010 (2017) [arXiv:1701.00883 [gr-qc]].