Canonical solutions of the local moment problem

Vadym Adamyan and Igor M. Tkachenko

Abstract. The present paper is devoted to the local moment problem, which consists in finding of non-decreasing functions on the real axis having given first \(2n + 1\), \(n \geq 0\), power moments on the whole axis and also \(2m + 1\) first power moments on a certain finite axis interval. Considering the local moment problem as a combination of the Hausdorff and Hamburger truncated moment problems we obtain the conditions of its solvability and describe the class of its solutions with minimal number of growth points if the problem is solvable.

Mathematics Subject Classification (2000). Primary 30E05, 30E05; Secondary 82C70, 82D10.

Keywords. non-decreasing function, power moments, moment problems, orthogonal polynomials, canonical solutions, Nevanlinna formula, selfadjoint extensions.

1. Introduction

In many-body physical problems correlation functions of observables admit so far exact calculation only for infinitesimal time intervals and corresponding spectral distribution functions can be reconstructed from experimental data only for rather narrow spectral intervals. Attempts to extract from available data some useful information on correlation functions on the whole axis have led us to the following version of moment problem.

Given two sets of numbers (Hermitian matrices) \(a_0, ..., a_{2n}, b_0, ..., b_{2m}\) and an interval \([0, \Lambda]\), \(0 < \Lambda < \infty\). To find a set of non-decreasing (matrix) measures \(d\sigma(t)\), which satisfy the conditions:

This work was partly supported by the Erasmus Mundus grant EURO 1200752. Vadym Adamyan is grateful to the Universidad Politécnica de Valencia for regular hospitality.
The formulated problem is a special combination of the well known truncated Hausdorff and Hamburger moment problems [1, 2]. It is motivated by the fact that in reality the spectral distribution (matrix) function $\sigma(t)$ is accessible to observation only on a finite spectral interval but at the same time the values of a finite number of the first power moments of $d\sigma(t)$ can be found independently from exact asymptotic relations and sum rules.

In this paper we study the local moment problem only for scalar $\sigma(t)$. Its first part contains a special class of solutions of the truncated Hausdorff moment problem. Using as in [3] [4] [5] [6] the approach based on the extension theory of Hermitian operators we obtain in the next section the solvability criterion of the truncated Hausdorff problem, which is treated as a version of the Stieltjes problem [1], where the sought $\sigma(t)$ should be constant out of $(0, \Lambda)$.

In Section 3 we make clear here which among the canonical solutions of the truncated Stieltjes problem, that is solutions of minimal number of point mass, are constant out of $(0, \Lambda)$.

In a short Section 4 we present the Nevanlinna formula for description of the all canonical solutions of the truncated Hausdorff problem.

In the last section a solution $\sigma(t)$ of the local moment moment problem ie represented as a sum $\sigma(\Lambda)(t) + \sigma(\perp \Lambda)(t)$, where $\sigma(\Lambda)(t)$ and $\sigma(\perp \Lambda)(t)$ grows only on $[0, \Lambda]$ and out of $[0, \Lambda]$, respectively. The summand $\sigma(\Lambda)(t)$ on $[0, \Lambda]$ is nothing else but a solution of the Hausdorff problem for the given moments $b_0,..., b_{2m}$, while $\sigma(\perp \Lambda)(t)$ is a solution of the truncated Hamburger moment for the altered moments $a_0 - b_0,..., a_{2n} - b_{2m}$, which has no growth points on $[0, \Lambda]$. We find here for the latter problem, which we call the Hamburger problem with gap, the solvability conditions and describe its canonical solutions.

2. The solvability criterium of truncated Hausdorff moment problem

The starting point for the solution of the above local moment problem is the truncated Hausdorff moment problem. It is formulated as follows:

\begin{equation}
\{b_0, b_1, b_2, \ldots, b_{2m}\}, \quad m = 0, 1, 2, \ldots
\end{equation}
To find all distributions $\sigma(t)$ such that

$$\int_0^\Lambda t^kd\sigma(t) = b_k, \ k = 0, 1, 2, \ldots, 2m. \quad (2.2)$$

The formulation of the corresponding Stieltjes problem is similar, the only difference is that $\Lambda$ in (2.2) is replaced by $\infty$,

$$\int_0^\infty t^kd\sigma(t) = b_k, \ k = 0, 1, 2, \ldots, 2m. \quad (2.3)$$

Evidently, any solution of the Hausdorff problem is a special solution of the Stieltjes problem, for which there are no growth points of the $\sigma(t)$ on the half-axis ($\Lambda, \infty$). Therefore the criterium of solvability of the Stieltjes problem is only a necessary condition for the solvability of the Hausdorff problem.

**Theorem 2.1.** A system of real numbers (2.1) admits the representation (2.2) with non-decreasing $\sigma(t)$ if and only if

a) the Hankel matrix $\Gamma_m := (b_{k+j})_{k,j=0}^m$ is non-negative;

b) for any set of complex numbers $\xi_0, \ldots, \xi_r$, $0 \leq r \leq m - 1$, the condition

$$\sum_{j,k=0}^r b_{j+k+2}\xi_k\xi_j = 0 \quad (2.4)$$

implies

$$\sum_{j,k=0}^r b_{j+k+2}\xi_k\xi_j = 0; \quad (2.5)$$

c) the Hankel matrix $\Gamma_{m-1} := (b_{k+j+1})_{k,j=0}^{m-1}$ is non-negative and for any set $\xi_0, \ldots, \xi_r \in \mathbb{C}$, $0 \leq r \leq m - 1$, the condition

$$\sum_{j,k=0}^r b_{j+k+1}\xi_k\xi_j = 0 \quad (2.6)$$

implies (2.5);

d) the matrix $\Lambda \Gamma_{m-1} - \Gamma_{m-1}^{(1)}$ is non-negative definite.

**Proof.** Due to [4,5] the conditions a) - c) of the theorem is a criterion of solvability of the truncated Stieltjes moment problem. Therefore we need only to prove that the condition d), in addition to a) - c), is equivalent to the existence, for given moments, of those solutions of the Stieltjes problem, for which $\sigma(t) = const$ for $t > \Lambda$.

Notice that due to the conditions a) and c) of the theorem, the moments $b_j$ are non-negative, $b_j \geq 0$, $j = 0, \ldots, 2m$. Excluding the trivial case, when the sought $\sigma(t)$ may have only one point of growth at $t = 0$, from now on we will assume that all these numbers are strictly positive, i.e. $b_j > 0$, $j = 0, \ldots, 2m$. 


Suppose that a) - d) hold. In this case for a given set of real numbers $b_0, ..., b_{2m}$ by virtue of the conditions a)-c) the corresponding truncated Stieltjes moment problem has at least one solution $\sigma_0(t)$ which amounts to $s \leq m$ point masses $\mu_1, ..., \mu_s, \ min \mu_j > 0$, located at some points $t_1 < ... < t_s < \infty$ of the half-axis $[0, \infty)$ (and also $[0, \infty)$). (We will return to this issue later). Note that the distribution $\sigma_0(t)$ is at the same time a solution of the Hausdorff problem if and only if $t_s \leq \Lambda$. 

For an arbitrary set of complex numbers $\xi_0, \xi_1, ..., \xi_{m-1}$ and the polynomial

$$P(t) = \xi_0 + \xi_1 t + \xi_2 t^2 + \ldots + \xi_m t^{m-1} \quad (2.7)$$

the equalities $(2.2)$ for $\sigma_0(t)$ and the special form of this distribution result in equalities

$$\int_0^\infty (\Lambda - t) |P(t)|^2 \ d\sigma(t) = \Lambda \cdot \sum_{j,k=0}^{m-1} b_{j+k} \xi_k \xi_j - \sum_{j,k=0}^{m-1} b_{j+k+1} \xi_k \xi_j = \sum_{j=1}^s (\Lambda - t_j)^2 \mu_j.$$ 

$(2.8)$

By $(2.8)$ if $t_s \leq \Lambda$ then, evidently the matrix $\Lambda \cdot \Gamma_{m-1} - \Gamma_{m-1}^{(1)}$ is non-negative definite. 

Alternatively, if $t_s > \Lambda$, then for

$$Q(t) = \eta_0 + \eta_1 t + ... + \eta_{m-1} t^{m-1},$$

$$Q(t_1) = Q(t_2) = ... = Q(t_{s-1}) = 0, \ Q(t_s) = 1,$$

we see that

$$\Lambda \cdot \sum_{j,k=0}^{m-1} b_{j+k} \xi_k \eta_j - \sum_{j,k=0}^{m-1} b_{j+k+1} \eta_k \eta_j = \sum_{j=1}^s (\Lambda - t_j)^2 |Q(t_j)|^2 \mu_j = (\Lambda - t_s) \mu_s < 0,$$

what is incompatible with the condition d) of the theorem. □

3. Which canonical solutions of the truncated Stieltjes problem are also solutions of the truncated Hausdorff problem?

Let us assume that for a system of real numbers $(2.1)$ the conditions a) - c) of Theorem $(2.1)$ hold and let $\sigma(t)$ be some solution of the corresponding truncated Stieltjes problem. Taking the set $\mathcal{L}$ of continuous complex valued functions $f(t), \ 0 < t < \infty$, for which

$$\int_0^\infty |f(t)|^2 \ d\sigma(t) < \infty,$$ 

$(3.1)$

we will consider $\mathcal{L}$ as pre-Hilbert space with the bilinear functional

$$\langle f, g \rangle = \int_0^\infty f(t) \overline{g(t)} \ d\sigma(t), \ f, g \in \mathcal{L}.$$ 

$(3.2)$
as the scalar product. Due to the conditions (2.3), any polynomial
\[ f(t) = \xi_0 + \xi_1 t + \ldots + \xi_r t^r, \quad \xi_0, \ldots, \xi_r \in \mathbb{C}, \quad 0 \leq r \leq m, \] (3.3)
may be considered as an element of \( L \). We will denote the linear subset of such polynomials by \( P_m \).

Let \( L_0 \) be the subspace of \( L \) consisting of all functions \( f \) such that \( \|f\| := \sqrt{\langle f, f \rangle} = 0 \) and \( \hat{L} \) be the factor - space \( L \setminus L_0 \). For any class of elements \( \tilde{g} = f + L_0 \) of this factor space we set \( \|\tilde{g}\|_{\hat{L}} = \|f\| \). Taking the closure of \( \hat{L} \) with respect to this norm, we obtain the Hilbert space \( L^2_{\sigma} \). We keep the same symbol \( \langle ., . \rangle \) for the scalar product in \( L^2_{\sigma} \).

Let \( L_m \) be the subspace of \( L^2_{\sigma} \) generated by the subset of polynomials \( P_m \).

By (2.2) and (3.2) for \( f, g \in P_m \),
\[ f(t) = \sum_{r=0}^{m} \xi_r t^r, \quad g(t) = \sum_{r=0}^{m} \eta_r t^r, \quad \xi_0, \ldots, \xi_r \in \mathbb{C}, \] (3.4)
we have
\[ \langle f, g \rangle = \sum_{j,k=0}^{m} b_{j+k} \xi_j \eta_k. \] (3.5)

Therefore for all distributions \( \sigma(t) \) satisfying (2.2), the restrictions onto \( L_m \) of the scalar products in the corresponding spaces \( L^2_{\sigma} \) must coincide. Among non-decreasing functions \( \sigma(t) \) satisfying (2.2), those for which \( L^2_{\sigma} = L_m \) are referred to as canonical. It was proven in [4] that the set of canonical solutions of the truncated Stieltjes moment problem is non-empty whenever the latter is solvable, i.e. whenever the conditions a) - c) of the theorem hold. By (3.5), a canonical \( \sigma(t) \) is a non-decreasing function having only a finite number \( \leq m \) of growth points.

Take some canonical solution \( \tilde{\sigma}(t) \) of the truncated Stieltjes moment problem for the given set of moments and consider the self-adjoint operator \( \tilde{A} \) of multiplication by the independent variable \( t \) in the related space \( L^2_{\sigma} = L_m \). Take the class \( \tilde{\sigma}_0 \subset L_m \) containing the polynomial \( \tilde{\sigma}_0(t) \equiv 1 \) and the classes containing the polynomials \( \tilde{\sigma}_k(t) \equiv t^k \), \( 0 \leq k \leq m \). According to the definition of \( \tilde{A} \) we have the representation
\[ \tilde{\sigma}_k = \tilde{A}^k \tilde{\sigma}_0, \quad 0 \leq k \leq m. \] (3.6)

For the unity decomposition \( \tilde{E}_t, -\infty < t < \infty \), of \( \tilde{A} \), let us introduce a non-decreasing function \( \overline{\sigma}(t) \), \( -\infty < t < \infty \) of bounded variation
\[ \overline{\sigma}(t) := \langle \tilde{E}_t \tilde{\sigma}_0, \tilde{\sigma}_0 \rangle_{L_m}. \] (3.7)

By (3.6), (3.7), and (2.2)
\[ b_{j+k} = \langle \tilde{\sigma}_k, \tilde{\sigma}_j \rangle_{L_m} = \langle \tilde{A}^k \tilde{\sigma}_0, \tilde{A}^j \tilde{\sigma}_0 \rangle_{L_m} = \] (3.8)
\[ = \int_0^{\infty} t^{j+k} d \langle \tilde{E}_t \tilde{\sigma}_0, \tilde{\sigma}_0 \rangle_{L_m} = \int_0^{\infty} t^{j+k} d \overline{\sigma}(t), \quad 0 \leq j, k \leq m. \] (3.9)
Let us denote by $L_{m-1}$ the subspace of $L_m$ generated by polynomials of a degree $\leq m-1$. By definition of $\tilde{A}$ its restriction $A_0$ to the subspace $L_{m-1}$ is a symmetric operator which actually does not depend on the choice of the canonical solution of the truncated Stieltjes moment problem. Therefore each canonical solution $\tilde{\sigma}(t)$ of this problem generates some self-adjoint extension $\tilde{A}$ of $A_0$ in $L_m$. On the other hand, each canonical self-adjoint extension $\tilde{A}$ of $A_0$ in $L_m$ generates a certain solution $\tilde{\sigma}(t)$ of the truncated Stieltjes moment problem. By the above formulas such a solution is at the same time a solution of the Hausdorff problem if and only if the corresponding spectral function $E_\lambda$ has no points of growth on the half-axis $(\Lambda, \infty)$, i.e. if and only if $\Lambda \cdot I_{m-1} - \tilde{A}$, where $I_r$ is the unity operator in $L_r$, is a non-negative extension of $\Lambda \cdot I_{m-1} - A_0$. Such an extension of $\Lambda \cdot I_{m-1} - A_0$ may exist only if the operator $\Lambda \cdot I_{m-1} - A_0$ is itself non-negative, i.e. the quadratic form of $\Lambda \cdot I_{m-1} - A_0$ is non-negative. But this is the case, since by our assumptions for a class $\hat{f} \in L_{m-1}$ containing a polynomial

$$f(t) = \sum_{r=0}^{m-1} \xi_r t^r,$$

we have by (2.3)

$$\left\langle \hat{f}, \hat{f} \right\rangle_{L_{m-1}} = \sum_{j,k=0}^{m-1} b_{j+k} \xi_j \overline{\xi_k}, \quad \left\langle A_0 \hat{f}, \hat{f} \right\rangle_{L_{m-1}} = \sum_{j,k=0}^{m-1} b_{j+k+1} \xi_j \overline{\xi_k},$$

$$\left\langle [\Lambda \cdot I_{m-1} - A_0] \hat{f}, \hat{f} \right\rangle_{L_{m-1}} = \Lambda \sum_{j,k=0}^{m-1} b_{j+k} \xi_k \overline{\xi_j} - \sum_{j,k=0}^{m-1} b_{j+k+1} \xi_j \overline{\xi_k} \geq 0. \quad (3.11)$$

If $L_m = L_{m-1}$, i.e. if $\det \Gamma_m = 0$, then $A_0$ is a self-adjoint operator and in this case the truncated Stieltjes problem has a unique solution $\sigma_0(t)$, which is, in line with (3.7), generated by the spectral function $E_\lambda^0$ of $A_0$. Since

$$\Lambda \cdot I_{m-1} - A_0 \geq 0,$$

then $\sigma_0(t)$ is also the unique solution of the truncated Hausdorff problem.

To describe the class of canonical solutions of Hausdorff problem if $L_m \neq L_{m-1}$, i.e. if $\det \Gamma_m > 0$, we remind first how it is done in the less restrictive case of Stieltjes problem.

Note that the condition $\det \Gamma_m > 0$ according to which $\Gamma_m > 0$ yields also $\Gamma_m^{(2)} > 0$. Indeed, if the quadratic form in of $\Gamma_m^{(2)} \geq 0$ vanishes for some set of complex numbers

$$\xi_0, \ldots, \xi_{m-1}, \max_{0 \leq k \leq m-1} |\xi_k| > 0,$$

then, by the condition c) of the theorem, the quadratic form of matrix $\Gamma_m^{(2)} = (s_{j+k+2})_{j,k=0}^{m-1}$ also vanishes for the same set and hence $\Gamma_m^{(2)}$ is non-invertible. But $\Gamma_m^{(2)}$ is a diagonal block of positive definite matrix $\Gamma_m$, a contradiction.

Let $\mathcal{N} = L_m \ominus L_{m-1}$, $\dim \mathcal{N} = 1$ and $P_\mathcal{N}$ be the orthogonal projector onto the one-dimensional subspace $\mathcal{N}$. With respect to the representation of $L_m$ as the
orthogonal sum $L_{m-1} \oplus N'$, we can represent a self-adjoint extension $\tilde{A}$ of $A_0$ as a $2 \times 2$ block operator matrix

$$
\tilde{A} = \begin{pmatrix}
A_{00} & G^* \\
G & \tilde{H}
\end{pmatrix},
$$

(3.12)

where $A_{00}$ is a symmetric operator in $L_{m-1}$, the quadratic form of which coincides with that of $A_0$, $G = P_N A_{0|L_{m-1}}$ and $\tilde{H}$ is a self-adjoint operator in $N'$, which just specifies a certain extension $\tilde{A}$. By (3.11) $A_{00}$ is a positive definite operator. Using the Schur-Frobenius factorization we can represent $\tilde{A}$ in the form

$$
\tilde{A} = \begin{pmatrix}
I & 0 \\
GA_{00}^{-1} & I
\end{pmatrix}
\begin{pmatrix}
0 & \tilde{H} - GA_{00}^{-1}G^* \\
A_{00} & 0
\end{pmatrix}
\begin{pmatrix}
I & A_{00}^{-1}G^* \\
0 & I
\end{pmatrix}.
$$

(3.13)

By this representation the extension $\tilde{A} \geq 0$ if and only if $\tilde{H} \geq GA_{00}^{-1}G^*$. We see that those and only those self-adjoint operators $\tilde{H}$ in $N'$ which have form

$$
\tilde{H} = GA_{00}^{-1}G^* + Q, Q \geq 0,
$$

(3.14)

with a non-negative operator $Q$ in $N'$, generate non-negative extensions $\tilde{A}$ in $L_m$ of $A_0$, and thereby generate canonical solutions of the Stieltjes problem. But only those of them are solutions of the Hausdorff problem, for which the corresponding non-negative extension $\tilde{A}$ satisfies the condition

$$
\tilde{A} - \Lambda \cdot I_m \leq 0.
$$

(3.15)

To express the condition (3.15) in terms of given moments (2.1) let us consider the Schur-Frobenius representation for $\tilde{A} - \lambda \cdot I_m$ assuming that $\lambda > \Lambda$. Due to the condition d), this guarantees the invertibility of $A_{00} - \lambda \cdot I_{m-1}$. We have

$$
\tilde{A} - \lambda \cdot I_m = \begin{pmatrix}
I & 0 \\
GA_{00}^{-1} & I
\end{pmatrix}
\begin{pmatrix}
0 & \tilde{H} - G[A_{00} - \lambda \cdot I_{m-1}]^{-1}G^* \\
A_{00} - \lambda \cdot I_{m-1} & 0
\end{pmatrix}
\begin{pmatrix}
I & [A_{00} - \lambda \cdot I_{m-1}]^{-1}G^* \\
0 & I
\end{pmatrix}.
$$

(3.16)

By virtue of (3.16), an extension $\tilde{A}$ satisfies the condition

$$
\tilde{A} - \lambda \cdot I_m \leq 0
$$

(3.17)

if and only if $A_{00} - \lambda \cdot I_{m-1} < 0$, what is provided by the condition d), and

$$
\tilde{H} - \lambda \cdot I_N - G[A_{00} - \lambda \cdot I_{m-1}]^{-1}G^* < 0.
$$

(3.18)

Let us denote by $A_{\mu}$ the minimal non-negative extension of $A_0$, for which $Q = 0$ in (3.14). This and only this canonical extension is non-invertible. For $A_{\mu}$ the block $H$ is simply $GA_{00}^{-1}G^*$. The inequality (3.18) holds for some non-negative extension $\tilde{A}$ if and only it is true for the minimal extension $\tilde{A}_{\mu}$ in (3.14), that is if

$$
GA_{00}^{-1}G^* - \lambda I_N - G[A_{00} - \lambda \cdot I_{m-1}]^{-1}G^* \leq 0, \lambda \leq \Lambda.
$$

(3.19)
Since the function of $\lambda$ in the left hand side of (3.19) is non-increasing, then the extension $\tilde{A}_0$ satisfies the inequality (3.17) if and only if

$$\Lambda \geq G[\Lambda \cdot I_{m-1} - A_{00}]^{-1}G^* + GA_{00}^{-1}G^*.$$  \hspace{1cm} (3.20)

In what follows, $\{e_k\}_0^m$ denote the natural basis $\mathcal{B}$ in $L_m$ of monomials $\{t_k\}_0^m$. To represent $GA_{00}^{-1}G^*$ in a more explicit form we introduce in $L_m$ operators $P_{\Omega}$ and $T$ in $L_m$, which for the basis $\mathcal{B}$ act as multiplication by $(m+1) \times (m+1)$ matrices

$$P_{\Omega} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}, \quad T = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}.$$

The symmetric operator $A_0$ in $L_m$ is the restriction of $T$ to the subspace $L_{m-1}$. Let $\tilde{\Gamma}_{m-1}^{(1)}$ be the $(m+1) \times (m+1)$ block operator matrix

$$\tilde{\Gamma}_{m-1}^{(1)} = \begin{pmatrix} \Gamma_{m-1}^{(1)} & 0_{m,1} \\ 0_{1,m} & 0_{1,1} \end{pmatrix},$$  \hspace{1cm} (3.21)

where $0_{n,m}$ are the $n \times m$ null-matrices. Note that for $\xi \in L_{m-1}$ and any $\eta \in L_m$ we have

$$< A_0 \xi, \eta >_{L_m} = < T \xi, \eta >_{L_m} = \left( \tilde{\Gamma}_{m-1}^{(1)} \xi, \eta \right)_{C_m} + (P_{\Omega} \Gamma_m T \xi, \eta)_{C_m}$$

$$= < \Gamma_{m-1}^{-1} \tilde{\Gamma}_{m-1}^{(1)} \xi, \eta > + < \Gamma_{m-1}^{-1} P_{\Omega} \Gamma_m T \xi, \eta > .$$

Hence

$$A_0|_{C_{m-1}} = \Gamma_{m-1}^{-1} \tilde{\Gamma}_{m-1}^{(1)}|_{C_{m-1}} + \Gamma_{m-1}^{-1} P_{\Omega} \Gamma_m T|_{C_{m-1}}.$$  \hspace{1cm} (3.22)

By (3.22) any self-adjoint extension $\tilde{A}$ of $A$ in $L_m$ has the form

$$\tilde{A} = \Gamma_{m-1}^{-1} \tilde{\Gamma}_{m-1}^{(1)} P_{\Omega}^\perp + \Gamma_{m-1}^{-1} P_{\Omega} \Gamma_m T P_{\Omega}^\perp + \Gamma_{m-1}^{-1} P_{\Omega} T^* \Gamma_m P_{\Omega} + \Gamma_{m-1}^{-1} \tilde{H},$$  \hspace{1cm} (3.23)

where $P_{\Omega}^\perp = I - P_{\Omega}$,

$$\tilde{H} = \begin{pmatrix} 0 & 0 \\ 0 & H \end{pmatrix}.$$
and $H$ is some real number, which defines the extension $\tilde{A}$. In a more detailed form,

$$
\tilde{A} = \Gamma_m^{-1} \begin{pmatrix}
\Gamma_m^{(1)} & b_{m+1} \\
\vdots & \vdots \\
b_{m+1} & b_{2m} & H
\end{pmatrix} = (3.24)
$$

$$
= T + \Gamma_m^{-1} \begin{pmatrix}
0_{m,m} & b_{m+1} \\
\vdots & \vdots \\
0 & b_{2m} & H
\end{pmatrix}.
$$

Observe, as before, that the invertibility of $\Gamma_m$ and the condition c) of Theorem 2.1 guarantee the invertibility of the matrix $\Gamma_m^{(1)}^{-1}$. Write $\Gamma_m^{(1)}^{-1} = (s_{jk})_{j,k=0}^{m-1}$ and put

$$
\begin{pmatrix}
\Gamma_m^{(1)}^{-1} \\
0_{m+1}
\end{pmatrix}_{\text{cond}} = \begin{pmatrix}
\Gamma_m^{(1)}^{-1} & 0_{m,1} \\
0_{1,m} & 0_{1,1}
\end{pmatrix}
$$

By the above argument the operator defined by the block matrix $\tilde{A}$ is non-negative if and only if

$$
\tilde{H} - P_\Omega \Gamma_m TP_\Omega \begin{pmatrix}
\Gamma_m^{(1)}^{-1} \\
0_{m+1}
\end{pmatrix}_{\text{cond}} P_\Omega T^* \Gamma_m P_\Omega \geq 0,
$$

or, equivalently, if and only if

$$
H - \sum_{j,k=0}^{m-1} b_{m+j+1}s_{jk}b_{m+k+1} \geq 0.
$$

(3.26)

Since

$$
Q := \sum_{j,k=0}^{m-1} b_{m+j+1}s_{jk}b_{m+k+1}
$$

is positive, all numbers $H$ generating non-negative extensions $\tilde{A}$ and hence the solutions of the Stieltjes problem, must be positive definite and, moreover, satisfy the inequality $H \geq Q$. Notice that the requirement $\tilde{A} > 0$ excludes the equality in (3.26).

To express the inequality (3.20) in terms of the given moments $b_0, b_1, b_2, \ldots, b_{2m}$ remind that the operator $A_0$ can be represented as the operator of multiplication by the independent variable in the space $L_{m-1}$ of polynomials of degree $\leq m$ defined on the subspace $L_{m-1}$ of polynomials of degree $\leq m-1$. Let us denote by $d_k(t)$, $k = 0, \ldots, m$, the set of orthogonal polynomials in $L_0^2$ with respect to any
measure $d\sigma(t)$ satisfying (2.2),

$$
d_0(t) = \frac{1}{\sqrt{b_0}} \quad d_k(t) = \frac{1}{\sqrt{\Delta_k \Delta_{k-1}}} \det \begin{vmatrix} b_0 & b_1 & \ldots & b_k \\ b_1 & b_2 & \ldots & b_{k+1} \\ \vdots & \vdots & \ddots & \vdots \\ b_{k-1} & b_k & \ldots & b_{2k-1} \\ 1 & t & \ldots & i^k \end{vmatrix}, \quad k = 1, 2, \ldots, m,
$$

(3.27)

where

$$
\Delta_k = \det \begin{vmatrix} b_0 & b_1 & \ldots & b_k \\ b_1 & b_2 & \ldots & b_{k+1} \\ \vdots & \vdots & \ddots & \vdots \\ b_{k-1} & b_k & \ldots & b_{2k-1} \end{vmatrix}.
$$

(3.28)

The operator $A_{00}$ in $L_{m-1}$ acts thereafter on arbitrary polynomials $q \in L_{m-1}$ as follows

$$(A_{00}q)(t) = tq(t) - \beta_{m-1} \langle q, d_{m-1} \rangle_{L_{m-1}} d_m(t), \quad \beta_{m-1} = \frac{\sqrt{\Delta_m \Delta_{m-2}}}{\Delta_{m-1}}. \quad (3.29)$$

Hence, for $q \in L_{m-1}$ we have

$$(Gq)(t) = ([A_0 - A_{00}]q)(t) = \beta_{m-1} \langle q, d_{m-1} \rangle_{L_{m-1}} d_m(t), \quad (3.30)$$

and

$$
\left( [A_{00} - z I_{m-1}]^{-1} q \right)(t) = \frac{1}{t - z} \left[ q(t) - \frac{q(z)}{d_m(z)} d_m(t) \right]. \quad (3.31)
$$

For the scalar Hausdorff problem $\dim \mathcal{N} = \dim [L_m \ominus L_{m-1}] = 1$ and $d_m(t)$ is a unit vector in $\mathcal{N}$. Therefore in the scalar case by (3.30) and (3.31), and the condition (3.19), any measure $d\sigma(t)$ satisfying (2.2) has the form

$$
\beta_{m-1}^2 \int_0^\infty \left[ d_{m-1}(t) - d_{m-1}(0) d_m(t) \right] d_{m-1}(t) d\sigma(t) - \lambda
$$

$$
-\beta_{m-1}^2 \int_0^\infty \frac{1}{t - \lambda} \left[ d_{m-1}(t) - d_{m-1}(\lambda) d_m(t) \right] d_{m-1}(t) d\sigma(t) \leq 0, \quad \lambda \geq 0. \quad (3.32)
$$

Notice further that for any $\lambda, \mu$ in accordance to the Christoffel-Darboux identity for orthogonal polynomials [1]

$$
\beta_{m-1}^2 \frac{d_{m-1}(\lambda) d_m(\mu) - d_{m-1}(\mu) d_m(\lambda)}{\mu - \lambda} = \sum_{k=0}^{m-1} \det \begin{vmatrix} 0 & 1 & \lambda & \ldots & \lambda^{m-1} \\ 1 & b_0 & b_1 & \ldots & b_{m-1} \\ \mu & b_1 & b_2 & \ldots & b_m \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \mu^m & b_{m-1} & b_m & \ldots & b_{2m-2} \end{vmatrix}.
$$

(3.33)
By (3.33) one can rewrite (3.32) in the form

$$\beta_{m-1} \frac{d_{m-1}(\lambda)d_m(0) - d_{m-1}(0)d_m(\lambda)}{d_m(\lambda)d_m(0)} - \lambda \leq 0, \lambda \geq \Lambda. \quad (3.34)$$

By our assumptions the consecutive numbers $b_1, \ldots, b_{2m-1}$ can be considered as the moments

$$b^1_0 = b_1, \ldots, b^1_{2m-2} = b_{2m-1}$$

of a non-negative measure $\tau \sigma(t)$, where $d \sigma(t)$ is any solution of the truncated Stieltjes moment for the given sequence $b_0, \ldots, b_{2m}$. Let us denote by $d_k^1, k = 0, \ldots, m-1$, the system of orthogonal polynomials for the set of moments $b^1_0, \ldots, b^1_{2m-2}$ and by $\Delta_k^1$ the determinants $\|b^1_{p+q}\|_{p,q=0}$. It follows from (3.34) and (3.33) that the condition (3.19) can be represented in the equivalent form

$$\sqrt{\Delta_m \Delta^1_{m-2}} \frac{d^1_{m-1}(\lambda)}{\Delta_{m-1} \Delta^1_{m-1}} \leq 1. \quad (3.35)$$

The last inequality permits to specify the condition d) in the solvability criterion of the truncated Hausdorff problem.

**Theorem 3.1.** For the given system of moments $b_0, \ldots, b_{2m}$ satisfying conditions a) - c) of Theorem 2.1 there is at least one solution of the truncated Stieltjes problem with non-negative measure concentrated on the interval $[0, \Lambda]$, i.e., there is a solution of the truncated Hausdorff problem for the interval $[0, \Lambda]$ if and only if the matrix $\Lambda \Gamma_{m-1} - \Gamma^1_{m-1}$ is non-negative and for any $\lambda \leq \Lambda$ the inequality (3.35) holds.

Under the above conditions the truncated Hausdorff problem has unique solution if and only if

$$\sqrt{\Delta_m \Delta^1_{m-2}} \frac{d^1_{m-1}(\Lambda)}{\Delta_{m-1} \Delta^1_{m-1}} = 1. \quad (3.36)$$

4. **Description of canonical solutions of the truncated Hausdorff problem**

Let us assume that the conditions of Theorem 3.1 hold. We denote by $e_r(t), r = 1, \ldots, m$, the system of conjugate polynomials:

$$e_r(t) = \int_{-\infty}^{\infty} \frac{d_r(t) - d_r(t')}{t-t'} d\sigma(t'),$$

where $d \sigma(t)$ is any solution of the truncated Hamburger problem for the set of moments $b_0, \ldots, b_{2m}$. All canonical solutions (that is those generated by the self-adjoint extensions of the symmetric operator $A_0$ in $L_m$) of the truncated Stieltjes
moment problem are, according to [4], described by the formula
\[ \int_0^\infty \frac{d\sigma_H(t)}{t - z} = \frac{\epsilon_m(z)(R_H + z) - \epsilon_{m-1}(z)}{d_m(z)(R_H + z) - d_{m-1}(z)}, \quad (4.1) \]
where
\[ R_H = (\Gamma_m^{-1})_{mm}^{-1} \Lambda_m - H (\Gamma_m^{-1})_{mm}, \quad \Im z > 0, \quad (4.2) \]
and is \( H \) the parameter such that
\[ \Lambda_m = (\Gamma_m^{-1})_{m-1,m} - b_{m+1} (\Gamma_m^{-1})_{mm}^{-2} \quad (4.3) \]
\[ H = \tau + \sum_{j,k=0}^{m-1} b_{m+j+1} s_j b_{m+k+1}, \quad (4.4) \]
where \( \tau \) is any non-negative number. The application of the above arguments to the Hausdorff problem yields

**Theorem 4.1.** Among all canonical solutions of the truncated Stieltjes problem for a given set of moments \( b_0, \ldots, b_{2m} \) which satisfy the conditions a) - d) of Theorem 2.1, those and only those are canonical solution of the truncated Hausdorff problem for which the parameter \( \tau \) in (4.4) satisfies the condition
\[ 0 \leq \tau \leq \Lambda \left( 1 - \sqrt{\frac{\Delta_m \Delta_{m-2}^{1/2} d_{m-1}(\Lambda)}{\Delta_{m-1} \Delta_{m-1}^{1/2} d_{m}(\Lambda)}} \right). \quad (4.5) \]

**5. Truncated Hamburger problem with gap**

Having disposed of the problems related to the truncated Hausdorff problem, one can turn now directly to the local moment problem for a given interval \([0, \Lambda]\), i.e., the truncated Hamburger moment problem in which along with the first \( 2n + 1 \) moments
\[ a_k = \int_{-\infty}^\infty t^k d\sigma(t), \quad 0 \leq k \leq 2n, \quad (5.1) \]
of the sought measure \( d\sigma(t) \), its \( 2m + 1, \ n \leq m \), local moments
\[ b_k = \int_0^\Lambda t^k d\sigma(t), \quad 0 \leq k \leq 2m, \quad (5.2) \]
are given also.

A possible approach to the solution of the local moment problem consists in the representation of the sought measure \( d\sigma(t) \) as the sum
\[ d\sigma(t) = d\sigma_{\Lambda}(t) + d\sigma_{\Lambda}^\perp(t), \]
where the measure \( d\sigma_{\Lambda}(t) \) is concentrated on the segment \([0, \Lambda]\), while the function \( \sigma_{\Lambda}^\perp(t) \) has no growth points on \([0, \Lambda]\).
The retrieval of $d\sigma_\Lambda(t)$ is reduced to the above Hausdorff problem on the interval $[0, \Lambda]$ for the given set of moments $b_0, ..., b_{2m}$. The quest of $d\sigma_\perp(t)$ consists in the search of some special solutions $d\tilde{\sigma}(t)$ of the truncated Hamburger moment problem, which satisfy the additional restriction

$$\tilde{\sigma}(\Lambda - 0) - \tilde{\sigma}(+0) = 0,$$

(5.3)

for the set of moments

$$c_k = \int_0^\Lambda t^k d\tilde{\sigma}(t) = \int_{-\infty}^{\Lambda-0} t^k d\sigma_\perp(t) + \int_{\Lambda+0}^\infty t^k d\sigma_\perp(t) = \int_{-\infty}^\infty t^k [\sigma(t) - \sigma_\Lambda(t)] = a_k - b_k, \quad k = 0, ..., 2n.$$

(5.4)

We call the latter moment problem the truncated Hamburger moment problem with the gap $[0, \Lambda]$. We see that the local moment problem formulated above is reduced to the truncated moment problem with the gap.

The proposed approach to the solution of this problem consists in the selection among the solutions $d\tilde{\sigma}(t)$ of the truncated Hamburger moment problem for the set of moments $c_0, ..., c_{2n}$ of those satisfying additional condition (5.3). In this way, we notice first that the necessary condition of solvability of the Hamburger problem for the given moments is positive definiteness of the Hankel matrix

$$\tilde{\Gamma}_n = (c_{j+k})_{j,k=0}^{2n}.$$

We will assume further that this condition holds.

Let $d\tilde{\sigma}(t)$ be a solution of the problem with the gap. Since $t(t - \Lambda) \geq 0$ on $\mathbb{R} \setminus [0, \Lambda]$ then for any polynomial

$$P_{n-1}(t) = \xi_0 + \xi_1 t + \xi_2 t^2 + \ldots + \xi_{n-2} t^{n-2}$$

we have

$$\int_{-\infty}^\Lambda t(t - \Lambda)|P_{n-1}(t)|^2 d\tilde{\sigma}(t)$$

$$= \int_{-\infty}^\Lambda t(t - \Lambda)|P_{n-1}(t)|^2 d\tilde{\sigma}(t) + \int_{\Lambda}^{\infty} t(t - \Lambda)|P_{n-1}(t)|^2 d\tilde{\sigma}(t)$$

$$= \sum_{j,k=0}^{n-2} [c_{j+k+2} - \Lambda c_{j+k+1}] \xi_k \xi_j \geq 0.$$

Therefore the positive definiteness of the matrix

$$\tilde{\Gamma}_n^{(2)} - \Lambda \tilde{\Gamma}_n^{(1)} = (c_{j+k+2} - \Lambda c_{j+k+1})_{j,k=0}^{n-2}$$

is an additional necessary condition for the solvability of the Hamburger moment problem with the gap for a given moments $c_0, ..., c_{2n}$.

To find sufficient conditions of solvability and find a description of canonical solutions of the gap problem one can as above look at this problem from the point of view of the extension theory. In other words, taking the set of moments $c_0, ..., c_{2n}$...
one can consider the Hilbert space $L_n$ of polynomials $P(t) = \xi_0 + \xi_1 t + \xi_2 t^2 + \ldots + \xi_n t^n$ with the norm

$$\|P\| = \sqrt{\sum_{j,k=0}^{n} c_{j+k+2\xi_j \xi_k}}$$

and the symmetric operator $\tilde{A}_0$ in $L_n$ defined as the multiplication by $t$ operator on the subspace $L_{n-1} \subset L_n$ of polynomials of degree not exceeding $n - 1$. Remind that all solutions of the corresponding Hamburger problem are generated by the self-adjoint extensions $\tilde{A}$ of $\tilde{A}_0$. Any self-adjoint extension $\tilde{A}$ of $\tilde{A}_0$ is a special extension of $\tilde{A}_0$ onto the defect subspace $N_0 = L_n \ominus L_{n-1}$. In our case $\dim N_0 = 1$ and $N_0$ consists of polynomials, which are collinear to the orthogonal polynomial $p_n(t)$, which is associated with the set of moments $c_0, \ldots, c_{2n}$. It is easy to verify that $\tilde{A}$ is uniquely defined by the formula

$$\alpha_k = \left(\tilde{A}_0 p_k, p_k\right)_L, \quad \beta_k = \sqrt{\Delta_k}, \quad \Delta_k = \det(c_{j+k})_{j,k=0}^{k,n}, \quad k = 0, \ldots, n - 1,$$

$$\left(\tilde{A} p_n\right)(t) = \alpha_\tilde{A} p_n(t) + \beta_{n-1} p_{n-1}(t), \quad \text{(5.6)}$$

where $\alpha_\tilde{A}$ is a real parameter defining the extension $\tilde{A}$.

With no limitations on values of real $\alpha_\tilde{A}$ the expressions $\{5.6\}$ define the all self-adjoint extensions of $\tilde{A}_0$ in $L_n$ and generate in this way the all canonical solutions $\tilde{\sigma}(t)$ of the truncated Hamburger moment problem. Remind that they are described by the Nevanlinna formula

$$\int_{-\infty}^{\infty} \frac{d\tilde{\sigma}_{\alpha_\tilde{A}}(t)}{t - z} = \frac{q_n(z)(\alpha_\tilde{A} + z) - q_{n-1}(z)}{p_n(z)(\alpha_\tilde{A} + z) - p_{n-1}(z)}, \quad \text{Im} z \geq 0, \quad \text{(5.7)}$$

where $q_k(z)$ are corresponding conjugate polynomials,

$$q_k(z) = \int_{-\infty}^{\infty} \frac{p_k(t) - p_k(z)}{t - z} d\tilde{\sigma}_{\alpha_\tilde{A}}(t).$$

It follows from the Nevanlinna formula $\{5.7\}$ that those and only those $\alpha_\tilde{A}$ give the sought canonical solutions with $\Lambda$-gap for which the polynomials

$$M_\tilde{A}(t) = (t - \alpha_\tilde{A}) p_n(t) + \beta_{n-1} p_{n-1}(t) \quad \text{(5.8)}$$

have no zeros in $(0, \Lambda)$.

Note that in some cases the last condition may not be satisfied for any real $\alpha_\tilde{A}$, that is the Hamburger moment problem with given gap may be not solvable while the corresponding problem without the gap demand may have infinitely many solutions. Indeed, remember that for any real $\alpha_\tilde{A} \neq 0$ the zeros of polynomial $M_\tilde{A}(t)$ are real and simple and between any two zeros of $p_{n-1}(t)$ there is at least one
zero of $M_{\tilde{A}}(t)$. Hence, if $p_{n-1}(t)$ has two or more zeros in $(0, \Lambda)$ the corresponding Hamburger problem with this gap has no solutions.

To get the solvability condition of the truncated Hamburger problem with gap observe that any self-adjoint extension $\tilde{A}$ of $A_0$ in $L_n$ generates a self-adjoint extension $Q_{\tilde{A}}$ of the symmetric operator $Q_0$, which is defined on the subspace $L_{n-2} \subset L_n$ and acts as the operator of multiplication by the polynomial $t(t - \Lambda)$, $Q_{\tilde{A}}$ is simply $\tilde{A}(\tilde{A} - \Lambda \cdot I)$, where $I$ is the unity operator in $L_n$. At the same time $\tilde{A}$ generates a gap extension if and only if $\tilde{A}$ has no eigenvalues on the segment $[0, \Lambda]$, that is if and only if $Q_{\tilde{A}}$ is a positive operator.

Let $N_1$ denote the subspace $L_n - L_{n-2}$. With respect to the representation of $L_n = L_{n-2} \oplus N_1$, write an extension $Q_{\tilde{A}}$ in the block form

$$Q_{\tilde{A}} = \begin{pmatrix} Q_{00} & K \\ K^* & W_{\tilde{A}} \end{pmatrix}.$$ (5.9)

Remind that the $(n-2) \times (n-2)$ block $Q_{00}$ of $Q_{\tilde{A}}$ does not depend on the choice of the extension $\tilde{A}$ and by (5.5) it is a positive operator (positive definite matrix). Using the Schur-Frobenius factorization

$$Q_{\tilde{A}} = \begin{pmatrix} I \\ K^*Q_{00}^{-1} \end{pmatrix} \begin{pmatrix} Q_{00} & 0 \\ 0 & W_{\tilde{A}} - K^*Q_{00}^{-1}K \end{pmatrix} \begin{pmatrix} I \\ 0 \end{pmatrix} \begin{pmatrix} Q_{00}^{-1}K \\ I \end{pmatrix},$$ (5.10)

we see that under our assumptions the operator $Q_{\tilde{A}}$ is positive if and only if the operator $(2 \times 2$ matrix) $W_{\tilde{A}} - K^*Q_{00}^{-1}K$ is positive (positive definite).

To represent the positivity condition for the $2 \times 2$ matrix $W_{\tilde{A}} - K^*Q_{00}^{-1}K$ in an explicit form let us assume that operator $Q_{\tilde{A}}$ is invertible and write $Q_{\tilde{A}}^{-1}$ with respect to the splitting $L_n = L_{n-2} \oplus N_1$ in the block form

$$Q_{\tilde{A}}^{-1} = \begin{pmatrix} Y_{00} & X \\ X^* & Z_{\tilde{A}} \end{pmatrix}.$$ (5.11)

Note that

$$Z_{\tilde{A}}^{-1} = W_{\tilde{A}} - K^*Q_{00}^{-1}K.$$ (5.12)

Indeed, for any invertible block-matrix

$$L = \begin{pmatrix} A & C \\ C^* & B \end{pmatrix}$$

with invertible diagonal block $B$ a direct calculation shows that

$$\begin{pmatrix} 0 & 0 \\ 0 & B^{-1} \end{pmatrix} = L^{-1} - L^{-1} \begin{pmatrix} G_A^{-1} & 0 \\ 0 & 0 \end{pmatrix} L^{-1},$$ (5.13)

where $G_A$ is the left upper block of $L^{-1}$. Replacing in (5.13) $L^{-1}$ by $L$ and applying the obtained relation to $Q_{\tilde{A}}$ yields (5.12). Hence $W_{\tilde{A}} - K^*Q_{00}^{-1}K$ is positive definite if and only if the corresponding inverse matrix $Z_{\tilde{A}}$ is positive definite.

To obtain a condition that guarantees that $Z_{\tilde{A}} > 0$ in terms of given moments $c_0, \ldots, c_{2n}$ let us write down the matrix $\tilde{Z}_{\tilde{A}}$ of $Z_{\tilde{A}}$ for the basis of $e_1, e_2$ in $N_1$ of the orthogonal polynomials $p_{n-1}(t), p_n(t)$, respectively. To this end remember that
for any \( z \) which is not an eigenvalue of \( \tilde{A} \) the resolvent \( (\tilde{A} - zI)^{-1} \) acts on any polynomial \( g \in \mathbb{L}_n \) by formula

\[
\left[ (\tilde{A} - zI)^{-1} g \right](t) = \frac{g(t)M_{\tilde{A}}(z) - g(z)M_{\tilde{A}}(t)}{M_{\tilde{A}}(z)} \frac{1}{t - z}
\]  
(5.14)

with introduced as in (5.8) polynomial \( M_{\tilde{A}}(t) \). Applying, in particular, (5.14) to \( p_{n-1}(t) \) and \( p_n(t) \) with account of the Christoffel-Darboux identity for the polynomials \( \{p_k(t)\}_{0}^{n} \) we obtain:

\[
\left[ (\tilde{A} - zI)^{-1} p_{n-1} \right](t) = \frac{1}{M_{\tilde{A}}(z)} \left[ \frac{z - \alpha_{\tilde{A}}}{\beta_{n-1}} \sum_{k=0}^{n-1} p_k(z)p_k(t) + p_{n-1}(z)p_n(t) \right],
\]  
(5.15)

\[
\left[ (\tilde{A} - zI)^{-1} p_n \right](t) = \frac{1}{M_{\tilde{A}}(z)} \sum_{k=0}^{n} p_k(z)p_k(t).
\]  
(5.16)

As follows, the right lower \( 2 \times 2 \) block \( R_{\tilde{A}}(z) \) of the resolvent \( (\tilde{A} - zI)^{-1} \) for the basis \( \{p_k\}_{0}^{n} \) has form

\[
R_{\tilde{A}}(z) = \frac{1}{M_{\tilde{A}}(z)} \begin{pmatrix} z - \alpha_{\tilde{A}} & p_{n-1}(z) \\ \beta_{n-1} & p_n(z) \end{pmatrix},
\]  
(5.17)

Since

\[
Q_{\tilde{A}} = \frac{1}{\Lambda} \left[ (\tilde{A} - \Lambda \cdot I)^{-1} - \tilde{A}^{-1} \right],
\]

then by (5.17)

\[
Z_{\tilde{A}} = \frac{1}{M_{\tilde{A}}(\Lambda)M_{\tilde{A}}(0)} \times \begin{pmatrix} \frac{\Lambda - \alpha_{\tilde{A}}}{\beta_{n-1}}p_{n-1}(\Lambda)M_{\tilde{A}}(0) + \frac{\alpha_{\tilde{A}}}{\beta_{n-1}}p_{n-1}(0)M_{\tilde{A}}(\Lambda) & p_{n-1}(\Lambda)M_{\tilde{A}}(0) - p_{n-1}(0)M_{\tilde{A}}(\Lambda) \\ p_{n-1}(\Lambda)M_{\tilde{A}}(0) - p_{n-1}(0)M_{\tilde{A}}(\Lambda) & p_n(\Lambda)M_{\tilde{A}}(0) - p_n(0)M_{\tilde{A}}(\Lambda) \end{pmatrix}.
\]  
(5.18)

Setting

\[
h_n(\lambda, \mu) = \sum_{k=0}^{\star} p_k(\lambda)p_k(\mu)
\]

and applying the Christoffel-Darboux identity we deduce from the representation (5.18) that \( Z_{\tilde{A}} \) is positive definite if and only if the following inequalities hold:

\[
\frac{h_n(\Lambda, 0)}{M_{\tilde{A}}(\Lambda)M_{\tilde{A}}(0)} > 0,
\]  
(5.19)

\[
W(\alpha_{\tilde{A}}) = \frac{h_n(\Lambda, 0)}{M_{\tilde{A}}(\Lambda)M_{\tilde{A}}(0)} \left\{ - [h_n(\Lambda, 0) + h_{n-1}(\Lambda, 0)] \alpha_{\tilde{A}}^2 \\
+ [\Lambda h_n(\Lambda, 0)h_{n-1}(\Lambda, 0) + 2\beta_{n-1}p_n(\Lambda)p_{n-1}(0)h_{n-1}(\Lambda, 0)] \alpha_{\tilde{A}} \\
+ \beta_{n-1}p_{n-1}(0) \left[ \beta_{n-1}p_{n-1}(\Lambda) - \Lambda p_n(\Lambda) \right] \right\} > 0.
\]  
(5.20)

Summarizing the assertions obtained above yields the following
Theorem 5.1. For the given set of moments \(c_0, ..., c_{2n}\) the Hamburger moment problem with the gap \([0, \Lambda]\) is solvable and has infinitely many solutions if and only if

- the Hankel matrices \(\tilde{\Gamma}_n = (c_{j+k})_{j,k=0}^n\) and
  \[
  \tilde{\Gamma}_n^{(2)} - \Lambda\tilde{\Gamma}_n^{(1)} = (c_{j+k+2} - \Lambda c_{j+k+1})_{j,k=0}^{n-2}
  \]
  are positive definite;
- the inequality (5.19) holds;
- the roots of quadratic trinomial \(W(\alpha_{\tilde{A}})\) are real and different.

Under the above conditions the set of canonical solutions \(d\tilde{\sigma}_\tau(t)\) of the Hamburger problem with the gap \([0, \Lambda]\) is described by the Nevanlinna formula (5.7), where \(\alpha_{\tilde{A}}\) runs the segment of real axis where \(W(\alpha) \geq 0\).

References

[1] N.I. Akhiezer. The classical moment problem and some related questions in analysis, Hafner Publishing Company, N.Y. (1965).
[2] Krein M.G., Nudel’man A.A., The Markov moment problem and extremal problems, "Nauka", Moscow, 1973 (in Russian) (English translation: Translation of Mathematical Monographs AMS, 50 (1977)).
[3] V. Adamyan and I. Tkachenko. Solution of the Truncated Matrix Hamburger Moment Problem According to M.G. Krein. Operator Theory: Advances and Applications, vol. 118 (Proceedings of the Mark Krein International Conference on Operator Theory and Applications, vol.II, Operator Theory and Related Topics), Birkhäuser Verlag Basel, (2000), 32 - 51.
[4] V. Adamyan, I. Tkachenko and M. Urrea. Solution of the Stieltjes truncated moment problem, J. Applied Analysis, vol. 9, N.1 (2003) 57-74.
[5] V. Adamyan and I. Tkachenko. Solution of the Stieltjes Truncated Matrix Moment Problem, Opuscula Mathematica, v. 25/1 (2005), 5-24.
[6] V. Adamyan and I. Tkachenko. General Solution of the Stieltjes Truncated Matrix Moment Problem, Operator Theory: Advances and Applications v. 163 (2005), 1- 22.

Vadym Adamyan
Department of Theoretical Physics
Odessa National I.I. Mechnikov University
Dvoryanska 2
65044 Odessa
Ukraine
e-mail: vadamyan@onu.edu.ua
Igor M. Tkachenko
Instituto de matemática Pura y Aplicada
Universidad Politécnica de Valencia
Camino de Vera s/n
46022 Valencia
Spain)
e-mail: imtk@mat.upv.es