Multiple-degree-of-freedom rotor model with total unbalance and possible contact on stator

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Abstract. Mathematical model for the flexible rotor which has static and dynamic unbalance and enabled slip rolling on the compliant stator is developed. Also, rotation inertia, gyroscopic effect, bearings characteristics, internal hysteresis, and external viscous friction are included. The matrix approaches are used. The proposed model allows investigating the dynamic effect of total unbalance and stator displacements as an elastic and solid body on basis of the ordinary differential equations.

1. Introduction
The most of issues in engineering practice associated with the shaft or rotor vibration are due to the destructive effect of the unbalance and stator on the system. Therefore a large number of scientific papers are devoted to the mathematical modeling of rotor systems with the presence of unbalance or continuous rubbing on the stator. It is enough to refer, for example, to articles [1-8], as well as to the lists of references cited in them. However, the search for the simulation of dynamic unbalance (oblique disk seating on the shaft) or contact elasticity with the inertia of stator rotation very probably will result in failure.

The originality of this work is to build the unbalanced flexible rotor-elastic bearings-foundation system briefly called the rotor system and also the unbalanced flexible rotor-elastic bearings-rigid stator-elastic supports-foundation system which is simply called the rotor-stator system. If there is direct contact in the system between the flexible rotor and the compliant stator represented as an additional linear elastic coupling (spring between them) by the contact stiffness $k_c$ then it is called the contact system, otherwise – contactless system.

2. Total unbalance simulation
So, let the center of rotor mass $G$ not coincide with the geometric center of disk $O$ and locate in a disk plane at the distance $e$ from point $O$ (Figure 1). In addition, let the disk is mounted on the shaft with a tilt, i.e. with an existence of angle $e_1$ between disk axis and axis of the undeformed shaft $z$ or the same between vector $I_0\omega$ and tangent $z_1$ to axis of the bent shaft at point $O$. Finally, let a plane passing through the tangent $z_1$ and vector $OG=\epsilon$ be at an angle $e_2$ to vector $e_1\omega$ which is perpendicular to the plane passing through the tangent $z_1$ and the disk axis i.e. containing the angle $e_1$. The values $e$, $e_1$, $e_2$ characterize the total unbalance of the given rotor.

At the existence of linear and angular eccentricities, the motion equations of the light shaft with massive disk [9] should contain the accelerations and speeds of center of disk mass instead of the accelerations and speeds of geometric disk center:

\[ ... \]
\[ m\ddot{x}_G + k_{11}x + k_{12}\phi_y = 0 \]
\[ m\ddot{y}_G + k_{11}y + k_{12}\phi_y = 0 \]
\[ I\ddot{\phi}_x + I\omega\dot{\phi}_x + k_{21}x + k_{22}\phi_y = 0 \]
\[ I\ddot{\phi}_y - I\omega\dot{\phi}_y + k_{21}y + k_{22}\phi_y = 0 \]

where \( x_G, y_G, \phi_{\text{xy}}, \phi_{\text{yx}} \) are coordinates of point \( G \).

Internal friction is not taken into account because at rotation speed equal to whirl speed an elastic line of the bent rotor due to the unbalanced forces rotates around a support line so that the rotor does not rotate “inside” the elastic line, there are no changes in stress signs and, therefore, internal friction does not “work”.

![Figure 1](attachment:figure1.png)

**Figure 1.** The motion of unbalanced rigid disk on the flexible shaft and unbalanced forces.

In view of geometric reasons (figure 1):

\[ x_G = x + e\cos\omega t \]
\[ y_G = y + e\sin\omega t \]
\[ \dot{\phi}_G = \dot{\phi}_G + e\omega \cos(\pi/2 - \omega t - e_2) \]
\[ -\dot{\phi}_G = -\dot{\phi}_G + e\omega \sin(\pi/2 - \omega t - e_2) \]

From here:

\[ \ddot{x}_G = \dddot{x} - e\omega^2 \cos\omega t \]
\[ \ddot{y}_G = \dddot{y} - e\omega^2 \sin\omega t \]
\[ \ddot{\phi}_G = \dddot{\phi} + e\omega^2 \sin(\omega t + e_2) \]
\[ \ddot{\phi}_G = \dddot{\phi} + e\omega^2 \cos(\omega t + e_2) \]

Corresponding replacements yields the following differential equations of dynamics for unbalanced single-disk flexible rotor:

\[ m\ddot{x} + k_{11}x + k_{12}\phi_y = me\omega^2 \cos\omega t \]
\[ m\ddot{y} + k_{11}y + k_{12}\phi_y = me\omega^2 \sin\omega t \]
\[ I\ddot{\phi}_x + I\omega\dot{\phi}_x + k_{21}x + k_{22}\phi_y = (I_0 - I)e\omega^2 \cos(\omega t + e_2) \]
\[ I\ddot{\phi}_y - I\omega\dot{\phi}_y + k_{21}y + k_{22}\phi_y = (I_0 - I)e\omega^2 \sin(\omega t + e_2) \]

At “compressed” displacements:

\[ m\ddot{z} + k_{11}z + k_{12}\phi_0 = me\omega^2 e^{i\omega t} \]
\[ I\ddot{\phi}_0 - iI\omega\dot{\phi}_0 + k_{21}z + k_{22}\phi_0 = (I_0 - I)e_1 (\cos e_2 + i\sin e_2) \omega^2 e^{i\omega t} \]
where $z = x + iy$, $\phi_0 = \phi_x + i\phi_y$.

At the entry of matrices and vectors:

\[
\begin{pmatrix}
\frac{\partial^2}{\partial x^2} & 0 \\
0 & \frac{\partial^2}{\partial y^2}
\end{pmatrix} + \begin{pmatrix}
0 & 0 \\
0 & 0
\end{pmatrix} = \begin{pmatrix}
k_{11} & k_{12} \\
k_{21} & k_{22}
\end{pmatrix} \begin{pmatrix}
\frac{\partial^2}{\partial x^2} & 0 \\
0 & \frac{\partial^2}{\partial y^2}
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 \\
0
\end{pmatrix} = \begin{pmatrix}
k_{11} & k_{12} \\
k_{21} & k_{22}
\end{pmatrix} \begin{pmatrix}
\frac{\partial^2}{\partial x^2} & 0 \\
0 & \frac{\partial^2}{\partial y^2}
\end{pmatrix}
\]

\[
\frac{\partial^2}{\partial x^2} + \begin{pmatrix}
k_{11} & k_{12} \\
k_{21} & k_{22}
\end{pmatrix} \frac{\partial^2}{\partial y^2} = \begin{pmatrix}
0 \\
0
\end{pmatrix}
\]

\[
\begin{pmatrix}
\frac{\partial^2}{\partial x^2} & 0 \\
0 & \frac{\partial^2}{\partial y^2}
\end{pmatrix} = \begin{pmatrix}
k_{11} & k_{12} \\
k_{21} & k_{22}
\end{pmatrix} \begin{pmatrix}
\frac{\partial^2}{\partial x^2} & 0 \\
0 & \frac{\partial^2}{\partial y^2}
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 \\
0
\end{pmatrix} = \begin{pmatrix}
k_{11} & k_{12} \\
k_{21} & k_{22}
\end{pmatrix} \begin{pmatrix}
\frac{\partial^2}{\partial x^2} & 0 \\
0 & \frac{\partial^2}{\partial y^2}
\end{pmatrix}
\]

The worst case is $e_2 = 0$ i.e. actions of the linear and angular unbalances coincide in phase, and, as a result, deflection of the flexible rotor will be greatest. The most favorable case is $e_2 = 180^\circ$.

### 3. Flexible rotor-compliant stator system simulation

To research the dynamics of a rotor in contact with a stator, it is necessary to construct a vibrational model having two or more degrees of freedom i.e. containing at least several inertial, elastic and damping elements.

#### 3.1. The rotor system or the rigid stator case

Any axisymmetric rotor system is easy and advantageous (from the viewpoint of performing sufficiently accurate calculations) to represent as a combination of the following three finite elements (figure 2).

**Figure 2.** Finite elements of rotor: rigid disk (1), beam element (2), linearized short bearing (3) including generalized forces and displacements.

1. Rigid disk with concentrated mass $m_i$ and transverse $I_i$ and polar $I_{0i}$ moments of inertia. The equation of the two-dimensional motion of such disk rotating with the constant angular velocity $\omega_i$ in the fixed reference and matrix form:

\[
\begin{pmatrix}
m_y \\
n_y
\end{pmatrix} \begin{pmatrix}
\ddot{x}_i \\
\ddot{y}_i
\end{pmatrix} + \begin{pmatrix}
0 & 0 \\
0 & 0
\end{pmatrix} \begin{pmatrix}
\ddot{x}_i \\
\ddot{y}_i
\end{pmatrix} = \begin{pmatrix}
\{f_i\} \\
\{g_i\}
\end{pmatrix}
\]

where

\[
\begin{pmatrix}
\{x\} \\
\{y\}
\end{pmatrix} = \begin{pmatrix}
x_i \\
y_i
\end{pmatrix}, \quad \{f_i\} = \begin{pmatrix}
P_{xi} \\
P_{yi}
\end{pmatrix}, \quad \{g_i\} = \begin{pmatrix}
P_{x0} \\
P_{y0}
\end{pmatrix}, \quad \begin{pmatrix}
m_y \\
n_y
\end{pmatrix} = \begin{pmatrix}
m_i & 0 \\
0 & I_{0i}
\end{pmatrix}, \quad \begin{pmatrix}
0 & 0 \\
0 & I_{0i}
\end{pmatrix}.
\]

The transverse oscillations of rotors occur in a form of circular or elliptical precession,
therefore it is necessary to take into account the generalized forces and displacements of the rotor elements in two mutually perpendicular directions.

2. Rotating beam element with length \(l\) and radius \(r\) and distributed inertial-elastic parameters and internal damping of the hysteresis type. The motion equation of such a beam element in the adopted form [10]:

\[
\begin{bmatrix}
[m] & [0] \\
[0] & [m]
\end{bmatrix}
\begin{bmatrix}
\dot{x}_i \\
\dot{y}_i
\end{bmatrix} + \omega
\begin{bmatrix}
[0] & [g] \\
[-g] & [0]
\end{bmatrix}
\begin{bmatrix}
x_i \\
y_i
\end{bmatrix} + \frac{-\eta_h}{\sqrt{1 + \eta_h^2}}
\begin{bmatrix}
[0] & [k] \\
[-k] & [0]
\end{bmatrix}
\begin{bmatrix}
x_i \\
y_i
\end{bmatrix} = \begin{bmatrix}
f_{x_i} \\
f_{y_i}
\end{bmatrix},
\]

where

\[
\begin{bmatrix}
x_i \\
y_i
\end{bmatrix} = \begin{bmatrix}
x_i \\
\varphi_{y_{i+1}} \\
x_{i+1} \\
\varphi_{y_{i+1}} \\
y_i \\
\varphi_{y_{i+1}}
\end{bmatrix}^{T},
\]

\[
\begin{bmatrix}
f_{x_i} \\
f_{y_i}
\end{bmatrix} = \begin{bmatrix}
F_{x_i} \\
M_{x_{i+1}} \\
M_{y_{i+1}} \\
F_{y_{i+1}} \\
M_{x_{i+1}} \\
M_{y_{i+1}}
\end{bmatrix}^{T},
\]

\[
[m] = \frac{m_0 l}{420(1+s)^2}
\]

\[
[k] = \frac{E J}{l^4(1+s)}
\]

Here any element of the matrices is a generalized force at \(i\) node arising from the unit acceleration or velocity or displacement in this or \(i+1\) node.

In this general case the element is a Timoshenko beam i.e. besides the classical processes of elasticity (submatrices \([k]\)) and precession inertia including translational displacements (submatrices \([m]\)) and gyroscopic action (submatrices \([g]\)) are taken into account rotation inertia (submatrices \([mr]\)) and shear deformation (\(s\)-members). If a ratio of the length to diameter is greater than one and less than three, then it is permissible to neglect the influence of shear deformation i.e. to use the Rayleigh beam. If \(\frac{l}{r} > 3\), then consideration without inertia of rotation i.e. of the Bernoulli-Euler beam is possible.

It is assumed that the beam element has a dissipation that does not depend on the frequency. The stress is ahead of the corresponding deformation on an angle \(\phi\) determined by the material. The hysteresis loop in the stress-deformation plane repeats an ellipse which area is proportional \(\sin \phi\) and characterizes the value of the dissipated energy so that

\[
\frac{2 h}{h^2} \sin 1 \phi \eta_h \eta = +
\]

where \(\eta_h\) is the coefficient of hysteresis losses in the beam material.

3. Linearized short bearing characterized by the equation of the form:

\[
\begin{bmatrix}
\eta_h k_b & 0 \\
0 & \eta_h k_b
\end{bmatrix}
\begin{bmatrix}
\dot{x}_j \\
\dot{y}_j
\end{bmatrix} + \begin{bmatrix}
k_b & \eta_h k_b \\
-\eta_h k_b & k_b
\end{bmatrix}
\begin{bmatrix}
x_j \\
y_j
\end{bmatrix} = \begin{bmatrix}
Q_{x_j} \\
Q_{y_j}
\end{bmatrix},
\]

i.e. with isotropic transverse stiffness and zero angular stiffness and viscous damping expressed by coefficient \(\eta\) and hysteresis losses in the bearing materials expressed by coefficient \(\eta_h b\).

In general, the bearing also prevents the angular displacements of the rotor and is determined by 16
stiffness coefficients and 16 inelastic resistance coefficients proportional to speeds of the transverse-angular deformation. Angular coefficients are essential for a rigid rotor or for the higher vibration forms of a flexible rotor. However in such cases, the so-called “long bearings” are often replaced by a pair of closely spaced short bearings.

As a rule, only the radial coefficients play a significant role:

\[
\begin{bmatrix}
  c_{xx} & c_{xy} \\
  c_{yx} & c_{yy}
\end{bmatrix}
\begin{bmatrix}
  \dot{x}_j \\
  \dot{y}_j
\end{bmatrix}
+ \begin{bmatrix}
  k_{xx} & k_{xy} \\
  k_{yx} & k_{yy}
\end{bmatrix}
\begin{bmatrix}
  x_j \\
  y_j
\end{bmatrix}
= \begin{bmatrix}
  Q_{xj} \\
  Q_{yj}
\end{bmatrix},
\]

but even in this classical bearing model not all of them are used. The essential side coefficients of stiffness \( k_{xy}, k_{yx} \) and viscous damping \( c_{xy}, c_{yx} \) are characteristic for a fluid film bearing. At the same time, it should be borne in mind that usually such bearing is significantly non-conservative i.e. its rigidity matrix is significantly asymmetric. The difference of the side stiffness coefficients \( \Delta k_{ij} = k_{xy} - k_{yx} \) determines an action direction of “circular” resultant on the rotor which in the case of a lot of \( \Delta k_{ij} \) in comparison with the main stiffness coefficients \( k_{xx}, k_{yy} \) leads to a rotor circulation (whirl), at \( k_{xy} > k_{yx} \) to the forward, at \( k_{xy} < k_{yx} \) to the reverse. The latter is directly related to the appearance of the phenomenon of dynamic instability of the rotor on bearings of liquid friction (in the form of its significant direct asynchronous whirl).

It should be noted here that a bearing with equal main stiffness coefficients \( k_{xx} = k_{yy} \) and small side stiffness coefficients \( k_{xy} \approx k_{yx} \approx 0 \) in comparison with \( k_{xx}, k_{yy} \) is called isotropic. Bearing with equal main and also with essential and almost identical side stiffness coefficients \( k_{xy} \approx k_{yx} \neq 0 \) is called orthotropic. A bearing with unequal main \( (k_{xx} \neq k_{yy}) \) and close side \( (k_{xy} \approx k_{yx}) \) stiffness coefficients is called anisotropic, and if \( k_{xy} \neq k_{yx} \) then it is simply non-conservative.

It is also important to note that the contact place of the rotor with stator is formally analogous to the short support devoid of viscous friction, therefore:

\[
\begin{bmatrix}
  k_i & \eta_h k_i \\
  -\eta_h k_i & k_i
\end{bmatrix}
\begin{bmatrix}
  x_k \\
  y_k
\end{bmatrix}
= \begin{bmatrix}
  Q_{ik} \\
  Q_{ik}
\end{bmatrix},
\]

where \( \eta_h \) is hysteresis losses coefficient in the materials of contact surfaces of the rotor and the stator.

Since the coupling of rotor elements is made in the axial direction, a transition from the local coordinates of elements to the common coordinates of the rotor system is trivial. It does not contain geometric transformations and reduces to simple sequential numbering of generalized displacements/forces. A composing rule of the rotor system is that if two elements are connected, then the displacements of adjacent ends must be equal, and the corresponding generalized forces must be summed. This technique is called the elements assembly and discussed in many manuals on matrix structural analysis.

The finished result is a system of linear differential equations with constant coefficients of the form:

\[
[M]_n \{q\} + \{C\}_n \{\dot{q}\} + \{K\}_n \{q\} = \{F\} = 0,
\]

where \( n \) is the number of freedom degrees in the plane is \( 2p \), where \( p \) is the nodes number and \( (p-1) \) is the number of beam elements. Vectors of order \( n \times 1 \) are formed from the translational and angular displacements of nodes occurring in a plane or from components of unbalanced forces acting in the plane to certain nodes. Submatrices of size \( n \times n \) mean an assembly of submatrices or finite element constants.
Inertia \([M]\) and gyroscopic moments \([G]\) and stiffness \([K]\) submatrices for beam elements are the band matrices:

\[
[C]_{b\times n} = \begin{bmatrix}
    c_{11} & c_{12} & c_{13} & c_{14} & 0 & 0 \\
    c_{21} & c_{22} & c_{23} & c_{24} & 0 & 0 \\
    c_{11} & c_{12} & c_{33} & c_{34} & c_{13} & c_{14} \\
    c_{41} & c_{42} & c_{43} & c_{44} & c_{23} & c_{24} \\
    0 & 0 & c_{31} & c_{32} & c_{33} & c_{34} & c_{12} \\
    0 & 0 & c_{41} & c_{42} & c_{43} & c_{44} & c_{22} \\
    \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
    0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots
\end{bmatrix},
\]

where each two-digit index means the row number and column number of the matrices \([m]\), \([g]\) and \([k]\).

Inertial \([M_d]\) and gyroscopic \([G_d]\) disk submatrices, as well as the bearing stiffness submatrix \([K_b]\), are diagonal \([\cdot\cdot\cdot]\) so their union with the submatrices of the beam elements can be expressed as:

\[
[C]_{b\times n} + [\cdot\cdot\cdot]_{n\times n} = \begin{bmatrix}
    \cdots & + C_i & \cdots & + C_{i+1} \\
    \cdots & + C_{2i} & \cdots & + C_f \\
    \cdots & + C_{2j} & \cdots & + C_f \\
    \cdots & + C_{2k-1} & \cdots & + C_f \\
    \cdots & \cdots & \cdots & \cdots
\end{bmatrix},
\]

Here under each two-digit index is meant the row number and column number of the combined matrices which are determined by one or another node number. The constant \(C_i\) equals \(m_i\), the constant \(C_{i+1}\) equals to \(I_i\) or \(I_0i\), the constant \(C_j\) equals to \(k_{bj}\).

The constant \(C_k\) equaled to \(k_k\) is added to the stiffness matrix of the rotor system \([K]_{n\times n} = [K] + [K_b]\) when the \(k\)-th or other cross-section of the flexible rotor contacts with the elastic surface of the stationary stator (Figure 3). The resulting stiffness matrix of the contact rotor system is \([K]_{n\times n} = [K] + [K_b] + [K_{k}]\), where \([K_{k}]\) is the \(\kappa\)-filled diagonal matrix, \(\kappa\) is the number of contact points.

**Figure 3.** The arbitrary rotor system and its \(p\) nodes.

Since the system is axisymmetric a more compact form (1) is admissible by combining the matrix equations

\[
[M]_{n\times n}\{\dot{q}_i\} + \eta_i[K_b]\{\dot{q}_i\} + \omega[G]\{\dot{q}_i\} + [K]_{n\times n}\{q_i\} + \left(\eta_i[K] + \eta_{0b}[K_b]\right)\{q_i\} = \{F_i\}
\]

\[
i[M]_{n\times n}\{\dot{q}_i\} + i\eta_i[K_b]\{\dot{q}_i\} - \omega[G]\{\dot{q}_i\} + i[K]_{n\times n}\{q_i\} - i\left(\eta_i[K] + \eta_{0b}[K_b]\right)\{q_i\} = i\{F_i\},
\]

where \([M]_{n\times n} = [M] + [M_d],\ [G] = [G] + [G_d].\)
As a result, determining displacements in the \(xz\) plane are real and displacements in the \(yz\) plane are imaginary, i.e. considering the vectors of complex generalized coordinates \(\{q\}_{\text{rel}} = \{q_x\} + i\{q_y\}\) and forces \(\{F\}_{\text{rel}} = \{F_x\} + i\{F_y\}\), it will be:

\[
\begin{align*}
[M]_{\text{rel}}\{\dot{q}\}_{\text{rel}} + (\eta_i[K_i] - i\omega[G_i])\{\dot{q}\}_{\text{rel}} + \{K\}_{\text{rel}} - i(\eta_h[K_h] + \eta_b[K_b])\{q\}_{\text{rel}} = \{F\}_{\text{rel}}.
\end{align*}
\]

(2)

In the above equations, the elements displacements are assumed to be small (gyroscopic moments and positional forces are linearized). Although the theory is limited to small amplitudes (of elements vibrations), it gives surprisingly good results for large amplitudes (of rotor oscillations in general). In this case, the basic rule is simple: the number of used beam elements must be one higher than the number of the interesting highest frequency and mode of the natural rotor oscillations. The sufficiency of a small number of freedom degrees for accurate simulation is explained by the weak dependence of the dynamic rotor reaction on higher forms of rotor oscillations.

3.2. Rotor-stator system or the case of the elastically fixed stator

In modeling of practically used axisymmetric stator subsystem, i.e. including a relatively rigid frame and uniformly flexible elements of fastening to the foundation, it is appropriate to choose its discrete submission as the concentrated mass \(m_s\) and transverse moment of inertia \(I_s\) at the \(p+1\) node with linear constraints by total for the plane stiffness \(k_s = k_1 + k_2\) and viscous damping \(\eta_v k_s = \eta_v k_1 + \eta_v k_2\) (Figure 4). Usually, in rotor machines, the stator stiffness on deflections of its cross-sections from planes perpendicular to the rotation axis of undeformed rotor exceeds in two or more times the bending stiffness of the rotor.

![Figure 4. The arbitrary rotor-stator system and its \(p+1\) nodes.](image)

Let coordinates of the stator inertia center are \(x_s, y_s, \phi_{xs}, \phi_{ys}\), and its translational/angular oscillations as a rigid body on the compliant supports are small, then the equations of its motion in transverse directions and around the mass center (the \(p+1\) point) will be:

\[
\begin{align*}
& m_s\ddot{x}_s + \eta_v k_1 (\dot{x}_s - l_1\dot{\phi}_{ys}) + \eta_v k_2 (\dot{x}_s + l_2\dot{\phi}_{ys}) + k_1 (x_s - l_1\phi_{ys}) + k_2 (x_s + l_2\phi_{ys}) = 0 \\
& m_s\ddot{y}_s + \eta_v k_1 (\dot{y}_s - l_1\dot{\phi}_{ys}) + \eta_v k_2 (\dot{y}_s + l_2\dot{\phi}_{ys}) + k_1 (y_s - l_1\phi_{ys}) + k_2 (y_s + l_2\phi_{ys}) = 0 \\
& I_s\ddot{\phi}_{xs} - \eta_v k_1 (\dot{x}_s - l_1\dot{\phi}_{ys})l_1 + \eta_v k_2 (\dot{x}_s + l_2\dot{\phi}_{ys})l_2 - k_1 (x_s - l_1\phi_{ys})l_1 + k_2 (x_s + l_2\phi_{ys})l_2 = 0 \\
& I_s\ddot{\phi}_{ys} - \eta_v k_1 (\dot{y}_s - l_1\dot{\phi}_{ys})l_1 + \eta_v k_2 (\dot{y}_s + l_2\dot{\phi}_{ys})l_2 - k_1 (y_s - l_1\phi_{ys})l_1 + k_2 (y_s + l_2\phi_{ys})l_2 = 0
\end{align*}
\]

Obviously, this approach ensures the smallest number of necessary freedom degrees for a rotor-
stator system. System vectors and matrices are:

\[
\begin{align*}
\{q_x\}_{p=2}^\infty &= \begin{bmatrix} q_x \end{bmatrix}, \quad \{q_y\}_{p=2}^\infty = \begin{bmatrix} q_y \end{bmatrix}, \\
\{F_x\}_{p=1}^\infty &= \begin{bmatrix} F_x \end{bmatrix}, \quad \{F_y\}_{p=1}^\infty = \begin{bmatrix} F_y \end{bmatrix}, \\
[M]_{p=2}^{2n+2} &= \begin{bmatrix} m & I_x \end{bmatrix}
\end{align*}
\]

\[
\alpha \{G\}_{p=2}^{2n+2} = \omega \begin{bmatrix} G \\ 0 \end{bmatrix}, \quad \eta_k[K]_{p=2}^{2n+2} = \begin{bmatrix} \eta_h \end{bmatrix}, \quad \eta_{ho}[K_h]_{p=2}^{2n+2} = \begin{bmatrix} \eta_{ho} \end{bmatrix},
\]

\[
\begin{align*}
\eta_k[K]_{p=2}^{2n+2} &= \begin{bmatrix} \eta_k \\ \cdots \\ \eta_k \\ \cdots \end{bmatrix} = \begin{bmatrix} k_{2(n+1)-2(n+1)-1} & k_{2(n+1)-2(n+1)} & k_{2(n+1)-1} & k_{2(n+1)} \\ k_{2(n+1)} & k_{2(n+1)-1} & k_{2(n+1)-1} & k_{2(n+1)} \\ \vdots & \ddots & \ddots & \ddots \\ k_{2(n+1)-1} & k_{2(n+1)} & k_{2(n+1)-1} & k_{2(n+1)} \end{bmatrix}
\end{align*}
\]

\[
\begin{align*}
[K]_{p=2}^{2n+2} &= \begin{bmatrix} [K] \\ \cdots \\ [K] \\ \cdots \end{bmatrix} = \begin{bmatrix} k_{2(n+1)-2(n+1)-1} & k_{2(n+1)-2(n+1)} & k_{2(n+1)-1} & k_{2(n+1)} \\ k_{2(n+1)} & k_{2(n+1)-1} & k_{2(n+1)-1} & k_{2(n+1)} \\ \vdots & \ddots & \ddots & \ddots \\ k_{2(n+1)-1} & k_{2(n+1)} & k_{2(n+1)-1} & k_{2(n+1)} \end{bmatrix}
\end{align*}
\]

\[
\begin{align*}
\mathbf{0} = \mathbf{M} \{\dot{q}\} + (\eta_k[K] - i\alpha \{G\}) \{\dot{q}\} + \{[K] - i(\eta_h[K] + \eta_{ho}[K_h])\} \{q\} = \alpha^2 \omega^2 \begin{bmatrix} F_x \end{bmatrix} + \{F_y\},
\end{align*}
\]

where \( \{F_y\} = \begin{bmatrix} \cdots & m e_i & (I_{0i} - I_i) e_{ij} \cos e_{2j} & \cdots \end{bmatrix}^T, \quad \{F_x\} = \begin{bmatrix} \cdots & 0 & (I_{0i} - I_i) e_{ij} \sin e_{2j} & \cdots \end{bmatrix}^T.\]

**Conclusion**

Equations (1) or (2) can be directly used to determine the response of synchronously whirling rotor on unbalanced forces. So, in the case of (2):

\[
\mathbf{0} = \mathbf{M} \{\dot{q}\} + (\eta_k[K] - i\alpha \{G\}) \{\dot{q}\} + \{[K] - i(\eta_h[K] + \eta_{ho}[K_h])\} \{q\} = \alpha^2 \omega^2 \begin{bmatrix} F_x \end{bmatrix} + \{F_y\},
\]

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