Smoothing closed gridded surfaces embedded in $\mathbb{R}^4$ *

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Abstract
We say that a topological $n$-manifold $N$ is a cubical $n$-manifold if it is contained in the $n$-skeleton of the canonical cubulation $C$ of $\mathbb{R}^{n+k}$ ($k \geq 1$). In this paper, we prove that any closed, oriented cubical 2-manifold has a transverse field of 2-planes in the sense of Whitehead and therefore it is smoothable by a small ambient isotopy.

1 Introduction

The canonical cubulation $C$ of $\mathbb{R}^m$ is the decomposition into hypercubes which are the images of the unit cube $I^m = \{(x_1, \ldots, x_m) | 0 \leq x_i \leq 1\}$ by translations by vectors with integer coefficients [2].

Definition 1.1. Let $N$ be a topological $n$-manifold embedded in $\mathbb{R}^{n+k}$. We say that $N$ is a cubical manifold of codimension $k$ if it is contained in the $n$-skeleton of the canonical cubulation of $\mathbb{R}^{n+k}$. When $n = 2$ and $k = 2$, $N$ is called a gridded surface in $\mathbb{R}^4$.

Observe that a cubical manifold can be subdivided into simplices to become a PL-manifold. M. Boege, G. Hinojosa and A. Verjovsky proved in [2] the following theorem.

Theorem 1.2. Let $N$ be a closed and smooth $n$-dimensional submanifold of $\mathbb{R}^{n+2}$ such that it has a trivial normal bundle. Then $N$ can be deformed by an ambient isotopy into a cubical manifold.

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The goal of this paper is to prove a sort of reciprocal theorem for cubical manifolds of dimension two in $\mathbb{R}^4$ (gridded surfaces).

**Theorem 1.** Any closed, oriented, gridded surface $N$ in $\mathbb{R}^4$ is smoothable. More precisely $N$ admits a transverse field of 2-planes and therefore by a theorem of J. H. C. Whitehead there is an arbitrarily small topological isotopy that moves $N$ onto a smooth surface in $\mathbb{R}^4$ (see [12], [14]).

The relevance of this result is that there exist PL-manifolds which are not smoothable. For instance, Kervaire ([8]) constructed an example of a PL triangulable closed manifold $M$ of dimension 10 that does not admit any differentiable structure. Therefore the Kervaire manifold cannot be embedded as a codimension two cubical submanifold of $\mathbb{R}^{12}$. Theorem 1 contrasts with the result of M. H. Freedman stating that every homology 3-sphere embeds topologically in $\mathbb{R}^4$ and if its Rokhlin invariant is one the embedding can never be smooth (or even PL) since this would violate Rokhlin signature theorem for spin 4-manifolds [13], [5].

**Remark 1.3.** Theorem 1 is false for PL-surfaces contained in the 2-skeleton of a PL triangulation of $S^4$. For instance if $K \subset S^3$ is the trefoil knot and we consider the pair $(S^3, K)$ as a pair of PL-manifolds with respect to some PL-triangulation of the 3-sphere and $(S^4, \Sigma(K))$ is the suspension of the pair with the canonical suspended PL-structure, then there exists a subdivision of a PL-triangulation of $S^4$ such that $\Sigma(K)$ is embedded in the two skeleton of the triangulation. In this way we obtain a 2-dimensional PL knot $\Sigma(K) \hookrightarrow S^4$ which has two points (the vertices of the suspension) where the knot cannot be made PL-locally flat because if it were PL-locally flat one could take the cyclic branched covering of order 5 of $S^4$ along $\Sigma(K)$ and one can see that this implies that the Poincaré sphere would be the boundary of a PL homology disk and this contradicts Rokhlin’s theorem. This means that the cubic structure plays an important role in the theorem.

## 2 Transverse fields for gridded surfaces

Let $M$ be a topological $m$-manifold embedded in $\mathbb{R}^{m+k}$. We say that an affine $k$-plane $T \subset \mathbb{R}^{m+k}$ is transverse to $M$ at $p \in M$ if $p \in T$ and locally $M$ is the graph of a Lipschitz function $h : T^\perp(r) \to N$, where $T^\perp(r)$ denotes the $k$-disk of radius $r$ at $p$ in the affine plane $T^\perp \subset \mathbb{R}^{m+k}$ perpendicular to $T$ at $p$ (see [12]). The map $H : x \mapsto x + h(x)$ is a graph chart for $M$ at $p$ that it sends $T^\perp(r)$ homeomorphically onto a neighborhood of $p$ in $M$. Since $H$
and \( H^{-1} \) are Lipschitz, \( H \) is a \textit{Lipeomorphism}.

Let \( G = G(k, m+k) \) denote the Grassmann of \( k \)-planes in \( \mathbb{R}^{m+k} \). A continuous map \( \psi : M \rightarrow G \) such that each affine plane \( p + \psi(p) \) is transverse to \( M \) at \( p \) is a \textit{transverse field} for \( M \).

In \cite{14} was proved that if \( M \) admits a transverse field then it has an ambient smoothing: it is Lipeomorphic to a nearby \( C^\infty \) submanifold of \( \mathbb{R}^{m+k} \). We will use this result to prove Theorem 1.

Let \( N \) be a closed, oriented, gridded surface in \( \mathbb{R}^4 \) and consider \( x \in N \). Notice that if \( x \) lies in the interior of some face \( F \subset N \), then we take the neighborhood \( U_x \) of \( x \) given by \( U_x = \text{Int}(F) \). Now consider the plane \( P \) parallel to the support plane of \( F \) such that \( 0 \in P \). Then the map \( v_x : U_x \rightarrow G(2, 4) \) defined as \( y \mapsto y + P^\perp \), \( y \in U_x \), is an example of a locally transverse 2-field to \( N \) at \( x \), where \( P^\perp \) is the orthogonal plane to \( P \).

By the above, the only difficulty lies on the case where \( x \) is a vertex of the canonical cubulation \( C \).

\textbf{Definition 2.1.} Let \( M \) be a topological \( n \) manifold. We say that a point \( x \in M \) is \textit{topologically locally flat} or \textit{topologically locally tame} if there exists

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1.png}
\caption{The locally transverse vector field \( v_x \) at \( x \).}
\end{figure}
an open neighborhood $U$ of $x$ such that there is a homeomorphism of pairs: $(U, U \cap M) \sim (\text{Int}(\mathbb{B}^{n+2}), \text{Int}(\mathbb{B}^n))$. We say that $M$ is \textit{topologically locally flat} if all its points are topologically locally flat. We say that $M$ is \textit{Whitehead locally flat} if it admits a transverse plane field in the sense of Whitehead. If $M$ is Whitehead locally flat it is also topologically locally flat. The embedded Poincaré sphere in $\mathbb{R}^4$, $\Sigma \hookrightarrow \mathbb{R}^4$ is topologically locally flat but it is not PL (or Whitehead) locally flat.

We will discuss the case where $x$ is a vertex. We can assume without loss of generality and for the sake of simplicity that the vertex $x$ is the origin $0 = (0, 0, 0, 0) \in \mathbb{R}^4$ and take all the 2-faces of $N$, $F_1, F_2, \ldots, F_j$, that contain it. Notice that each edge of $N$ must belong only to two faces and since there are only eight edges whose end-point is 0 it follows that $j \leq 8$.

Consider the unitary canonical vectors on $\mathbb{R}^4$: $e_{\pm 1} = (\pm 1, 0, 0, 0), e_{\pm 2} = (0, \pm 1, 0, 0), e_{\pm 3} = (0, 0, \pm 1, 0)$ and $e_{\pm 4} = (0, 0, 0, \pm 1)$. We will use throughout this paper, the following notation for this kind of 2-faces:

$$F_{u,v} = \{ae_u + be_v : 0 \leq a, b \leq 1\}$$

where $e_u$ and $e_v$ ($u, v \in \{\pm 1, \pm 2, \pm 3, \pm 4\}, |u| \neq |v|$), denote the corresponding unitary canonical vectors.

**Definition 2.2.** Let $C^0$ be the union of all 2-faces $F \in C$ such that $0 \in F$ and let $N$ be a gridded surface. The intersection $\mathcal{F}(N) = N \cap C^0$ is called the \textit{squared-star} of $N$ at 0.

**Definition 2.3.** Let $\mathcal{F}$ be the set of all \textit{squared-stars} of all possible gridded surfaces which have a vertex at 0.

**Remark 2.4.** If the face $F_{a,b}$ belongs to $\mathcal{F}(N)$, since $N$ is a closed 2-manifold, there must exist two 2-faces $F_{a,c}, F_{b,c} \in \mathcal{F}(N)$. This allows us to describe $\mathcal{F}(N)$ as a finite path $\square \rightarrow \square \cdots \rightarrow \square$ of consecutive squares i.e. squares sharing an edge (square denotes one of the squares $F_{i,j} \in \mathcal{F}(N)$). For instance $F_{1,2} \rightarrow F_{1,3} \rightarrow F_{2,3}$ (see Figure 8). Moreover

**Examples 2.5.** The number of square faces of a squared-star $\mathcal{F} \in \mathcal{F}$ is any number from 3 to 8 as it is shown in the following examples.

1. $F_{1,2} \rightarrow F_{1,3} \rightarrow F_{2,3}$ (see Figure 8).
Figure 2: squared-star consisting on 3 squares.

Figure 3: squared-star consisting on 4 squares.

2. $F_{1,2} \rightarrow F_{1,-2} \rightarrow F_{2,3} \rightarrow F_{-2,3}$ (see Figure 3).
3. $F_{1,2} \rightarrow F_{1,-2} \rightarrow F_{2,3} \rightarrow F_{-1,3} \rightarrow F_{-1,-2}$ (see Figure 4).
4. $F_{1,-2} \rightarrow F_{-1,2} \rightarrow F_{1,-3} \rightarrow F_{2,-3} \rightarrow F_{-2,3} \rightarrow F_{1,-3}$ (see Figure 5).
5. $F_{1,2} \rightarrow F_{1,-2} \rightarrow F_{2,3} \rightarrow F_{-1,3} \rightarrow F_{-2,-3} \rightarrow F_{-1,4} \rightarrow F_{-3,4}$.
6. $F_{1,2} \rightarrow F_{1,-2} \rightarrow F_{-1,2} \rightarrow F_{-1,-3} \rightarrow F_{-2,-4} \rightarrow F_{-3,4} \rightarrow F_{3,4} \rightarrow F_{3,-4}$.

Now, we will determine the different squared-stars, up to rotations and reflections, in the set $\mathbb{F}$.

**Theorem 2.6.** There are 20 different elements in $\mathbb{F}$, up to rotations and reflections.

The analysis will be made based on the number of faces of a squared-star that appear in the same plane. To be able to do so, we will think a squared-star as a cycle path of degree 2, as follows. Consider the complete graph $K_8$ formed with the eight vertices labelled 1, $-1$, 2, $-2$, 3, $-3$, 4, $-4$ and edges $F(i, j), j \in \{\pm 1, \pm 2, \pm 3, \pm 4\}$. We would like to count the number of closed
paths (cycles) of length $n$, with $3 \leq n \leq 8$, which use the edge $F_{1,2}$, but do not use the edges $F_{-1,1}, F_{-2,2}, F_{-3,3}$ or $F_{-4,4}$ and such that every vertex in the path has degree 2.

**Observations.**

- There are 6 different planes: generated by the 6 pairs $e_u$ and $e_v$ ($u, v \in \{1, 2, 3, 4\}$).

- Any squared-star $F$ could have faces in those 6 different planes, so the maximum number of planes where there are faces of $F$ is 6.

- If any path has faces $F_{a,b}$ and $F_{c,d}$, with at least three of the numbers $a, b, c, d$ having different absolute value, then there are at least two faces of the squared-star in different planes.

- There is only one case, where there are 4 faces of a squared-star in the same plane: $F_{1,2} \rightarrow F_{2,-1} \rightarrow F_{2,-1} \rightarrow F_{-1,-2}$.

- Except for the previous observation, a squared-star can have maximum 3 faces in the same plane.
• A squared-star has three faces in the same plane if in the path we can find three edges of the form: $F_{a,b}, F_{b,-a}, F_{-a,-b}$.

• There are two ways that a squared-star can have two faces in the same plane: $F_{a,b} \rightarrow F_{b,-a}$ or $F_{a,b} \rightarrow F_{-a,-b}$.

We define the signature of a squared-star $F$ as $(a_1, a_2, \ldots, a_k)$, the numbers of faces of $F$ in different planes, where $1 \leq k \leq 6$ and $a_1 \geq a_2 \geq \cdots \geq a_k$. For example, the path $F_{1,2} \rightarrow F_{2,-3} \rightarrow F_{-3,-2} \rightarrow F_{-2,1}$ has signature $(2, 2)$, since the path has two faces in the plane generated by $e_1$ and $e_2$ and two faces in the plane generated by $e_2$ and $e_3$. Except for the one case mentioned in the observations, all the numbers in the signature are less than or equal to 3.

The signatures $(1, 1, 1)$ and $(1, 1, 1, 1)$ are realizable by $F_{a,b} \rightarrow F_{b,x} \rightarrow F_{x,a}$ and $F_{a,b} \rightarrow F_{b,c} \rightarrow F_{c,d} \rightarrow F_{d,a}$, respectively, if all the symbols have distinct absolute value.

**Lemma 2.7.** Any squared-star cannot have 5 single faces in 5 different planes.

**Proof.** First let assume that there is a squared-star with exactly 5 faces, all of them in 5 different planes, that is, with signature $(1, 1, 1, 1, 1)$. The corresponding path will have the form $F_{a_1,a_2} \rightarrow F_{a_2,a_3} \rightarrow F_{a_3,a_4} \rightarrow F_{a_4,a_5} \rightarrow F_{a_5,a_1}$. Since there are 5 symbols, at least two of them have the same absolute value, let say $a_i = -a_j$ with $3 \geq j - i \geq 2$. If $j - i = 2$, the path has edges $F_{a_i,a_{i+1}}$ and $F_{a_{i+1},-a_i}$, which are in the same plane, a contradiction. If $j - i = 3$, the path will have edges $F_{a_{i-1},a_i}$ and $F_{-a_i,a_{i+1}}$, a contradiction again.

The three cases left are when a squared-star has, besides the 5 faces in different planes, another single face in other plane, two faces in other plane or three faces in other plane. But the proof is the same as before, since we will have again in the path edges with at least 5 different symbols, and we will arrive to the same contradiction. □

Now, let characterize the squared-stars which have three faces in the same plane. Remember that a squared-star has three faces in the same plane, if in the path there are three faces of the form $F_{a,b}, F_{b,-a}, F_{-a,-b}$.

**Lemma 2.8.** The possible signatures of squared-stars that contains a number 3 are $(3, 1, 1)$, $(3, 1, 1, 1)$, $(3, 2, 1, 1)$, $(3, 3, 1, 1)$.  

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Again let $F$ be a squared-star that has 3 appearing in its signature, that is, contains three faces of the form $F_{a,b}$, $F_{b,-a}$, $F_{-a,-b}$. We will proceed depending on the number of faces in $F$. It is clear that $F$ has $n$ faces with $n \geq 4$.

If $F$ has 4 faces, they are $F_{a,b}$, $F_{b,-a}$, $F_{-a,-b}$ and $F_{-b,x}$, then it follows that $x = a$ and the signature of $F$ is $(4)$, a contradiction.

Assume that $F$ has 5 faces, that is, $F_{a,b} \rightarrow F_{b,-a} \rightarrow F_{-a,-b} \rightarrow F_{-b,t} \rightarrow F_{t,a}$. Since $|t| \neq |a|, |b|$, then the last two faces are in different planes, which are not the plane determined by $a$ and $b$, then the signature of $F$ is $(3, 1, 1)$. More generally, if $F$ has $n$ faces, then $\exists a, b, \ldots$ such that $|\{x \in F \mid x \neq a \land x \neq b \land x \neq t \land x \neq |a|, |b| \land |t| \neq |a|, |b|, |t| \neq |s| \land |t| \neq |s|\} \geq 3$.

Finally, if $F$ has 8 faces, it is $F_{a,b} \rightarrow F_{b,-a} \rightarrow F_{-a,-b} \rightarrow F_{-b,t} \rightarrow F_{t,s} \rightarrow F_{s,u} \rightarrow F_{u,a}$. As in the previous case, $t, s, u, w \in \{\pm p, \pm q\}$, then the path is $F_{a,b} \rightarrow F_{b,-a} \rightarrow F_{-a,-b} \rightarrow F_{-b,t} \rightarrow F_{t,s} \rightarrow F_{s,t} \rightarrow F_{-t,-s} \rightarrow F_{-s,a}$, with signature $(3, 3, 1, 1)$.

As we mentioned before, there are two ways that a squared-star can have two faces in the same plane: $F_{a,b}$, $F_{b,-a}$ or $F_{a,b}$, $F_{-a,-b}$, to the first one we will assign the number 2 in the signature and to the second one the number 2. Now let us find a similar result as Lemma 2.8 but when 2 appears in the signature but 3 does not.

Again let $F$ be a squared-star with $n$ faces, in whose signature appears 2 but not 3. If $F$ has faces $F_{-a,b}$, $F_{b,-a}$, then it must have another face $F_{-a,t}$, with $t \neq a$, then $n \geq 4$. Also let $N = \{\pm 1, \pm 2, \pm 3, \pm 4\} \{\pm a, \pm b\} = \{\pm p, \pm q\}$.

If $n = 4$, then $F$ is $F_{a,b} \rightarrow F_{b,-a} \rightarrow F_{-a,t} \rightarrow F_{t,a}$, which has signature $(2, 2)$. 

Proof.
If $n = 5$, then $\mathcal{F}$ is $F_{a,b} \to F_{b,-a} \to F_{-a,t} \to F_{t,s} \to F_{s,a}$, with $t, s \neq -b$, otherwise the signature of $\mathcal{F}$ has a number 3. Then $|s|, |t| \neq |a|, |b|$, which implies that the signature is $(2, 1, 1, 1, 1)$.

The case $n = 6$ gives the path $F_{a,b} \to F_{b,-a} \to F_{-a,t} \to F_{t,s} \to F_{s,u} \to F_{u,a}$, where $t, u \in N$. There are three cases. First if $s = -b$, the path is $F_{a,b} \to F_{b,-a} \to F_{-a,t} \to F_{t,-b} \to F_{-b,u} \to F_{u,a}$, and then depending whether $|t| = |u|$ or not, we get two possible signatures $(2, 2, 2)$ or $(2, 1, 1, 1, 1, 1)$.

If $s \neq -b$, then $t, s, u \in N$, and it follows that the path has the form $F_{a,b} \to F_{b,-a} \to F_{-a,\pm p} \to F_{\pm p,\pm q} \to F_{\pm q,\mp p}, F_{\mp p,a}$, which have signature $(2, 2, 2)$.

If $\mathcal{F}$ has 7 faces, the path is $F_{a,b} \to F_{b,-a} \to F_{-a,\pm p} \to F_{\pm p,\pm q} \to F_{\pm q,\mp p} \to F_{\mp p,a}$; that is, with signature $(3, 2, 1, 1)$, a contradiction.

In a similar way, we obtain for $s = -b$ or $u = -b$, signatures $(2, 2, 1, 1, 1)$ and $(2, 2, 1, 1, 1)$.

Now, let $\mathcal{F}$ be a squared-star with $n$ faces, in whose signature appears 2 but not 3. If $\mathcal{F}$ has edges $F_{a,b}, F_{b,-a}$, then it must have at least another 4 edges $F_{b,x}, F_{y,a}, F_{s,-b}$ and $F_{-a,t}$. That is, the paths of this kind with 6 edges have two possible forms

$$F_{a,b} \to F_{b,x} \to F_{s,-b} \to F_{-b,-a} \to F_{-a,t} \to F_{y,a}$$

or

$$F_{a,b} \to F_{b,x} \to F_{s,-a} \to F_{-a,-b} \to F_{-b,t} \to F_{y,a},$$

where $x = s$ and $t = y$. The cases $t = y = -x$ and $t = y \neq -x$, in both cases, give the signatures $(2, 2, 2)$, $(2, 2, 2)$ and $(2, 1, 1, 1, 1)$.

For the analysis we made for the case $n = 7$, we know that there are not signatures of the form $(2, 2, 2, 1)$. Let us prove that the signatures for $n = 7$, with at least two numbers 2, are $(2, 2, 1, 1, 1)$ and $(2, 2, 1, 1, 1)$. For that, now it is enough to analyze the case with at least one 2. We have seen that these paths have the form $F_{a,b} \to F_{b,x} \to F_{s,-b} \to F_{-b,-a} \to F_{-a,t} \to F_{y,a}$ or $F_{a,b} \to F_{b,x} \to F_{s,-a} \to F_{-a,-b} \to F_{-b,t} \to F_{y,a}$. We will work with the first.
path, and similarly it will follow for the second one.

If \( t = y \), we can add an edge connecting \( F_{b,x} \) and \( F_{a,-b} \), obtaining the path

\[
F_{a,b} \rightarrow F_{b,x} \rightarrow F_{x,s} \rightarrow F_{s,-b} \rightarrow F_{-b,-a} \rightarrow F_{-a,t} \rightarrow F_{t,a},
\]

where \( x, s \in \{ \pm 1, \pm 2, \pm 3, \pm 4 \}\backslash\{ \pm a, \pm b, t \} \). It follows that the path has signature \((2, 2, 1, 1, 1)\). In the same way, if \( x = s \) and adding an edge between \( F_{-a,t} \) and \( F_{y,a} \), we obtain the same signature \((2, 2, 1, 1, 1)\). Therefore there are not signatures for \( n = 7 \), with three numbers 2 (or \( \bar{2} \)) and one number 1.

Similarly there are not signatures for \( n = 8 \) with three numbers 2 and two numbers 1. That is, the paths with signatures \((2, 2, 2, 1)\) and \((2, 2, 2, 1, 1)\), (here 2 can be \( \bar{2} \) too) are not realizable. Now, let us state the result, that can be proved with the previous analysis.

**Lemma 2.9.** The signatures for paths of length \( n \leq 7 \), with at least one number 2 and not numbers 3 are

\[
(2, 2), (2, 2, 1, 1), (2, 2, \bar{2}), (\bar{2}, \bar{2}), (2, 2, 1, 1, 1),
\]

\[
(2, 2, 1, 1, 1), (2, 1, 1, 1, 1), (\bar{2}, 1, 1, 1, 1), (\bar{2}, 1, 1, 1, 1).
\]

We are ready to determine the different squared-stars, up to rotations and reflections, in the set \( \mathbb{F} \). As mentioned before, the analysis will be made based in the number of faces of a squared-star that appear in the same plane.

Given a number \( 3 \leq m \leq 8 \), we will find the different ways to write \( m \) as sum of numbers less than or equal to 3, and then find if those numbers in the sums form possible signatures. For example, two ways to write 5 are \( 2 + 2 + 1 \) and \( 3 + 2 \), but \((2, 2, 1)\) and \((3, 2)\) are not realizable signatures by Lemmas 2.8 and 2.9.

- For \( n = 3 \), the ways to write 3 as sums are \( 1 + 1 + 1 \), \( 2 + 1 \), and \( 3 \). The only possible signature for \( n = 3 \) is \((1, 1, 1)\), which is realizable by \( F_{a,b}, F_{b,x}, F_{x,a} \), since \( x \neq a, b, -b \). All these squared-stars are images under rotations or reflectios of each other. Then there is only one path of length 3, up to rotation and reflection.

- For \( n = 4 \), we have \( 4 = 4 = 3 + 1 = 2 + 1 + 1 = 2 + 2 = 1 + 1 + 1 + 1 \), for which only the signatures \((4)\), \((2, 2)\) and \((1, 1, 1, 1)\) are realizable, by Lemmas 2.8 and 2.9.
• For $n = 5$, we obtain two possible signatures $(3, 1, 1)$ and $(2, 1, 1, 1)$.

• For $n = 6$, the possible signatures are $(3, 1, 1, 1)$, $(2, 1, 1, 1, 1)$, $(2, 2, 2)$, $(2, 2, 2)$.

• For $n = 7$, the signatures are $(3, 2, 1, 1)$, $(2, 2, 1, 1, 1)$, $(2, \bar{2}, 1, 1, 1)$, $(2, 2, 1, 1, 1, 1)$.

• For $n = 8$, the signatures are $(3, 3, 1, 1)$, $(2, 2, 1, 1, 1, 1)$, $(2, 2, 2, 2)$, $(2, 2, 2, 2)$.

For the case $n = 8$, by Lemmas 2.7, 2.8 and 2.9, those are the only possible signatures and it is easy to see that they are realizable; next we will give examples of all of them. □

**Examples 2.10.** We will exhibit a path representative for each signature given above.

• $n = 3$. $F_{1,2} \rightarrow F_{1,3} \rightarrow F_{2,3}$.

• $n = 4$. Signature

  $(4): F_{1,2} \rightarrow F_{1,-2} \rightarrow F_{-1,2} \rightarrow F_{-1,-2}$

  $(2, 2): F_{1,2} \rightarrow F_{1,-2} \rightarrow F_{2,3} \rightarrow F_{-2,3}$

  $(1, 1, 1, 1): F_{1,2} \rightarrow F_{2,3} \rightarrow F_{3,4} \rightarrow F_{1,4}$

• $n = 5$. Signature

  $(3, 1, 1): F_{1,2} \rightarrow F_{-1,2} \rightarrow F_{-1,3} \rightarrow F_{-2,3} \rightarrow F_{1,-2}$

  $(2, 1, 1, 1): F_{1,2} \rightarrow F_{-1,2} \rightarrow F_{1,3} \rightarrow F_{3,4} \rightarrow F_{-1,4}$

• $n = 6$. Signature

  $(3, 1, 1, 1): F_{1,2} \rightarrow F_{-1,2} \rightarrow F_{-1,3} \rightarrow F_{-2,4} \rightarrow F_{3,4} \rightarrow F_{1,-2}$

  $(2, 1, 1, 1, 1): F_{1,2} \rightarrow F_{-1,2} \rightarrow F_{1,3} \rightarrow F_{-2,3} \rightarrow F_{-2,4} \rightarrow F_{1,4}$

  $(\bar{2}, 1, 1, 1, 1): F_{1,2} \rightarrow F_{-1,-2} \rightarrow F_{1,3} \rightarrow F_{-2,3} \rightarrow F_{2,4} \rightarrow F_{1,4}$

  $(2, 2, \bar{2}): F_{1,2} \rightarrow F_{-1,2} \rightarrow F_{1,3} \rightarrow F_{-1,-3} \rightarrow F_{-2,3} \rightarrow F_{-2,3}$

  $(\bar{2}, 2, \bar{2}): F_{1,2} \rightarrow F_{-1,-2} \rightarrow F_{1,3} \rightarrow F_{-2,3} \rightarrow F_{2,-3} \rightarrow F_{-1,-3}$

• $n = 7$. Signature
squared-stars from the topological and differentiable point of view. Vertex points of gridded surfaces is very complicated. Next, we will study As we can see from the previous section the combinatorial description at $3$ Smoothing cubulated closed $2$-manifolds

(3, 2, 1, 1): $F_{1,2} \rightarrow F_{-1,2} \rightarrow F_{-1,4} \rightarrow F_{3,-4} \rightarrow F_{3,4} \rightarrow F_{-2,-4} \rightarrow F_{1,-2}$

(2, 2, 1, 1, 1): $F_{1,2} \rightarrow F_{-1,2} \rightarrow F_{-1,4} \rightarrow F_{3,4} \rightarrow F_{-2,3} \rightarrow F_{-2,-3} \rightarrow F_{1,-3}$

(2, $\bar{2}$, 1, 1, 1): $F_{1,2} \rightarrow F_{-1,2} \rightarrow F_{1,-3} \rightarrow F_{-1,3} \rightarrow F_{-2,3} \rightarrow F_{2,-4} \rightarrow F_{-3,4}$

(2, $\bar{2}$, 1, 1, 1): $F_{1,2} \rightarrow F_{-1,-2} \rightarrow F_{1,-3} \rightarrow F_{-1,3} \rightarrow F_{-2,-3} \rightarrow F_{2,4} \rightarrow F_{3,4}$

• $n = 8$. Signature

(3, 3, 1, 1): $F_{1,2} \rightarrow F_{-1,2} \rightarrow F_{-1,4} \rightarrow F_{3,-4} \rightarrow F_{3,4} \rightarrow F_{-3,-4} \rightarrow F_{-2,-3} \rightarrow F_{1,-2}$

(2, 2, 1, 1, 1, 1): $F_{1,2} \rightarrow F_{2,-4} \rightarrow F_{-1,4} \rightarrow F_{1,-4} \rightarrow F_{3,4} \rightarrow F_{-2,3} \rightarrow F_{-2,-3} \rightarrow F_{1,-3}$

(2, $\bar{2}$, 1, 1, 1, 1): $F_{1,-4} \rightarrow F_{2,-4} \rightarrow F_{-1,2} \rightarrow F_{-1,4} \rightarrow F_{3,4} \rightarrow F_{-2,3} \rightarrow F_{-2,-3} \rightarrow F_{1,-3}$

(2, 2, $\bar{2}$): $F_{1,2} \rightarrow F_{-1,2} \rightarrow F_{-1,4} \rightarrow F_{3,4} \rightarrow F_{-2,3} \rightarrow F_{-2,-3} \rightarrow F_{1,-4} \rightarrow F_{-3,-4}$

(2, $\bar{2}$, 2, $\bar{2}$): $F_{1,2} \rightarrow F_{-1,-2} \rightarrow F_{1,3} \rightarrow F_{-1,-3} \rightarrow F_{2,4} \rightarrow F_{-3,4} \rightarrow F_{-2,-4} \rightarrow F_{3,-4}$

Remark 2.11. For a higher dimensional cubical $n$-manifold $N \subset \mathbb{R}^{n+2}$, with $n + 2 > 4$, we can also define the notion of cubical-star of a vertex (i.e. it is the union of all the $n$-dimensional cubes in $N$ which contain the vertex). The set $\mathcal{F}$ of all cubical-stars of all possible cubical knots which have a vertex $0$ becomes extremely complicated. However we have the following conjecture:

Conjecture 2.12. Any closed, oriented, cubical $n$-manifold $N$ in $\mathbb{R}^{n+2}$, $n > 2$, is smoothable. More precisely $N$ admits a transverse field of 2-planes and therefore by a theorem of J. H. C. Whitehead there is an arbitrarily small topological isotopy that moves $N$ onto a smooth manifold in $\mathbb{R}^{n+2}$ (see [12], [14]).

3 Smoothing cubulated closed 2-manifolds

As we can see from the previous section the combinatorial description at vertex points of gridded surfaces is very complicated. Next, we will study squared-stars from the topological and differentiable point of view.
Let us remember that we are considering a *gridded surface* \( N \subset \mathbb{R}^4 \), that is, \( N \) is contained in the scaffolding \( S \) of the canonical cubulation \( C \) of \( \mathbb{R}^4 \) and 0 belongs to \( N \).

**Definition 3.1.** We define the *squared-link* of 0, as the boundary of its squared-star \( \mathcal{F}(N) \) and it is denoted by \( \text{slk}(N) \).

**Remarks 3.2.**

1. Let \( D \) be the union of all the hypercubes \( Q_i \in \mathcal{C} \) such that \( 0 \in Q_i \). Notice that \( i = 1, 2, \ldots, 16 \). Let \( N \) be a gridded-surface such that \( O \in N \). Then its squared-star \( \mathcal{F}(N) \) is contained in \( D \) and the squared-link \( \text{slk}(N) \) is contained in the boundary of \( D \).

2. Remember that \( \mathcal{F}(N) = \bigcup_{i=1}^{n} F_i \), where \( F_i \) is a square (2-face). Since \( N \) is a topological manifold, we have that only two edges \( e_{i_1} \) and \( e_{i_2} \) of \( F_i \) belong to \( \text{slk}(N) \) (see Remark 2.4). Hence \( \text{slk}(N) \) consists on the union of an even number of edges. More precisely, this number is smaller or equal to sixteen.

**Theorem 3.3.** Let \( N \) be a gridded surface such that \( 0 \in N \). Then \( \text{slk}(N) \) is an unknotted simple closed curve.

**Proof.** Let \( \mathcal{F}(N) = \bigcup_{i=1}^{n} F_i \) be the squared-star of \( N \). We will prove that \( \mathcal{F}(N) \) is isotopic to the square \([-1, 1]^2\) on some canonical ab-plane; that is \( \mathcal{F}(N) \) is equivalent to \( F_{a,b} \to F_{a,-b} \to F_{-a,b} \to F_{-a,-b} \). We will do it using induction over the number of faces \( n \) of \( \mathcal{F}(N) \).

If \( \mathcal{F}(N) \) consists of 3 faces \( F_{a,b} \to F_{a,c} \to F_{b,c} \) (see Figure 8), we will apply the following claim.

**Claim.** Let \( \mathcal{F}(N) = \bigcup_{i=1}^{n} F_i \) be a squared-star such that \( F_1 = F_{a,b} \) and \( F_2 = F_{a,c} \) (\( |b| \neq |c| \)). Then \( \mathcal{F}(N) \) is isotopic to the squared-star \( \mathcal{F}(N_1) = F_{b,c} \cup (\bigcup_{i=3}^{n} F_i) \).

**Proof of Claim.** Notice that the faces \( F_{a,b} \) and \( F_{a,c} \) share a common edge \( e_a \). Consider the cube \( Q \) generated by the canonical vectors \( e_a, e_b \) and \( e_c \). Then \( Q \) is a 3-face contained in the cubulation \( C \) and \( Q \) possesses the 2-faces \( F_{a,b}, F_{a,c}, F_{b,c}, F'_{a,b} = F_{a,b} + e_c, F'_{a,c} = F_{a,c} + e_b \) and \( F'_{b,c} = F_{b,c} + e_a \) (see Figure 6). Notice that the faces \( F'_{a,b}, F'_{a,c} \) and \( F'_{b,c} \) do not contain the vertex 0.

Clearly our gridded surface \( N \) is ambient isotopic to a gridded surface \( N_1 = (N \setminus (F_{a,b} \cup F_{a,c})) \cup (F_{b,c} \cup F'_{a,b} \cup F'_{a,c} \cup F'_{b,c}) \) (see Figure 7). Observe that \( \mathcal{F}(N_1) = (\mathcal{F}(N) \setminus (F_{a,b} \cup F_{a,c})) \cup F_{b,c} \). □
By the previous claim, our squared-star $F(N)$ described by $F_{a,b} \to F_{a,c} \to F_{b,c}$ is isotopic to $F_{a,b} \to F_{a,c} \to F_{-a,b} \to F_{-a,c}$; i.e.

$$F_{a,b} \to F_{a,c} \to F_{b,c} \sim F_{a,b} \to F_{a,c} \to F_{-a,b} \to F_{-a,c}$$

(1)

If $n = 4$, then $F$ is described as a path as follows

$$F_{a,b} \to F_{a,c} \to F_{b,d} \to F_{c,d}.$$  

(2)

Then applying the above claim, we have that (2) is isotopic to

$$F_{a,b} \to F_{a,c} \to F_{b,c},$$

which by (1), is isotopic to $F_{a,b} \to F_{a,-b} \to F_{-a,b} \to F_{-a,-b}$.

We assume that the Theorem holds for any squared-star consisting on $k$ faces, $k \leq n - 1$. Suppose that $F(N)$ consists of $n$ faces, i.e.

$$F_{a_1,b_1} \to F_{a_2,b_2} \to \ldots \to F_{a_{n-1},b_{n-1}} \to F_{a_n,b_n}.$$  

(3)

Figure 6: All faces of the cube $Q$.

Figure 7: $N$ is ambient isotopic to $N_1$. 
Since $N$ is a manifold, there exists $a_i$, $1 \leq i \leq n - 1$ such that $a_i = a_n$. For simplicity we will assume that $i = n - 1$, hence the squared-star (3) is equal to
\[ F_{a_1,b_1} \rightarrow F_{a_2,b_2} \rightarrow \ldots \rightarrow F_{a_{n-1},b_{n-1}} \rightarrow F_{a_n,b_n}. \] (4)
By the above claim, the squared-star (4) is isotopic to
\[ F_{a_1,b_1} \rightarrow F_{a_2,b_2} \rightarrow \ldots \rightarrow F_{b_{n-1},b_{n-1}}, \] (5)
and applying the induction hypothesis on (4) and (1), we have that $\mathcal{F}(N)$ is isotopic to $F_{a,b} \rightarrow F_{-a,b} \rightarrow F_{-a,-b}$. Therefore, $\text{slk}(N)$ is isotopic to the boundary of the square $[-1,1]^2$ on $ab$-plane, hence $\text{slk}(N)$ is unknotted simple closed curve. □

**Corollary 3.4.** $\mathcal{F}(N)$ is topologically locally flat. □

**Theorem 3.5.** Let $N$ be a grid surface such that $0 \in N$. Then $N$ is Whitehead locally flat at 0.

**Proof.** Consider the squared-link $\text{slk}(N)$ of 0. By the Theorem 3.3, we know that $\text{slk}(N)$ is an unknotted simple closed curve, then there exists a smooth closed curve $J$ isotopic to $\text{slk}(N)$ such that $J$ is $C^0$-arbitrarily close to $\text{slk}(N)$. This is because we can round the corners at the vertices of $\text{slk}(N)$ in an arbitrarily small neighborhoods of them (see [4]). Hence $J$ is the boundary of a smooth disk $U_0$, $C^0$-arbitrarily close to $\text{Int}(\mathcal{F}(N))$.

Let $\psi : U_0 \times [0,1] \rightarrow \text{Int}(\mathcal{F}(N))$ be this isotopy. Then by [14] there exists a locally transverse 2-field $v(x)$ to each point of $x \in U_0$, hence $v(x)$ is also transverse to $y = \psi(x,1) \in \text{Int}(\mathcal{F}(N))$. Therfore, the result follows. □

We are ready to prove Theorem 1.

**Theorem 1.** Any closed, oriented, grid surface $N$ in $\mathbb{R}^4$ is smoothable. More precisely $N$ admits a transverse field of 2-planes and therefore by a theorem of J. H. C. Whitehead there is an arbitrarily small topological isotopy that moves $N$ onto a smooth surface in $\mathbb{R}^4$ (see [12], [14]).

**Proof.** We will prove that $N$ admits a global transverse 2-field and by [14] it will follow that $N$ is smoothable. We will construct a local transverse 2-field at each point $x \in N$ in such a way that we can define a global transverse 2-field at $N$. For the sake of simplicity, we divide each square $F \subset N$ of $C$ into $m^2$ squares $S_i$, $i = 1, 2, \ldots, m^2$; i.e. each edge of $N$ is subdivided into $m$ equal segments. Let $x \in N$. 

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Case 1. The point $x$ lies on some square $S \subset \text{Int}(F)$. Consider the plane $P$ parallel to the support plane of $F$. Then the map $v_S : S \rightarrow G(2,4)$ defined as $y \mapsto y + P^\perp$, $y \in S$, where $P^\perp$ is the orthogonal plane to $P$. Thus $v_S(y)$ is a local transverse 2-field at $y \in \text{Int}(S)$ and for $y \in \partial S$, we have that it is a plane orthogonal to the corresponding square face.

Case 2. The point $x$ lies on some square $S_1 \subset F_1$ such that $S_1$ intersects some edge $e \subset N \subset C$ and none point of $S_1$ is a vertex of $C$. We have that $e$ is the intersection of two faces $F_1$ and $F_2$ of $N$. Let $S_2 \subset F_2$ be a square of $N$ such that $S_1 \cap S_2 \subset e$. Consider $l_1$ and $l_2$ edges of $S_1$ and $S_2$, respectively; such that $l_1 \cap l_2 = z \in e$; so $l_1 \times l_2$ is a square. Let $C$ be the circle of radius $\frac{1}{m}$ and centered at the opposited vertex $z'$ to $z$ in $l_1 \times l_2$, then $C_{1,2} \subset C \cap l_1 \times l_2$ is an arc such that, by construction, $l_1$ and $l_2$ are tangent to it. Let $r_x = x - z'$ be a vector, where $x \in l_1 \cup l_2$, and let $e'$, $l_1'$ and $l_2'$ be vectors parallel to $e$, $l_1$ and $l_2$ respectively, at the origin. Observe that these three vectors are canonical vectors up to scale. Take the remaining canonical vector $v_{1,2}$ and consider the plane $P_x = \langle r_x, v_{1,2} \rangle$. In general, for any $y \in S_1 \cup S_2$, there exists $x \in l_1 \cup l_2$ such that $y - x$ is parallel to $e$, hence $P_y = \langle r_x, v_{1,2} \rangle$. We define $v_{S_1 \cup S_2} : S_1 \cup S_2 \rightarrow G(2,4)$ given by $y \mapsto y + P_y$, $y \in S_1 \cup S_2$. Then by construction, it is a local transverse 2-field at $y \in \text{Int}(S_1 \cup S_2)$ and for $y \in \partial(S_1 \cup S_2) \setminus \{l_1, l_1 + \frac{1}{m}e, l_2, l_2 + \frac{1}{m}e\}$, we have that $v_{S_1 \cup S_2}(y)$ is a plane orthogonal to the corresponding square face.

Case 3. The point $x$ is a vertex on the canonical cubulation $C$, then $x = \cap_{j=1}^r F_{ij}$, where $F_{ij}$ is a 2-face of $N$. We will suppose for simplicity that $x$ is the origin $0 = (0,0,0,0) \in \mathbb{R}^4$. As before, we will consider that $x = \cap_{j=1}^r S_j$, where $S_j \subset F_{ij}$ is a square; so $F_{m}(N) = \cup_{j=1}^r S_j$. Let $e_j = S_j \cap S_{j+1}$ $(r + 1 = 1)$ be an edge whose end point is $x_j$. Then $slk_m(N) = \partial F_{m}(N)$ consists on the union of a finite number of edges; in fact, two edges for
each square $S_j$. More precisely, $slk_m(N) = (\bigcup_{j=1}^r l_{j1} \cup (\bigcup_{j=1}^r l_{j2})$, where $l_{j1}, l_{j2} \subset S_j$, $l_{j1}$ is parallel to $e_{j+1}$ and $l_{j2}$ is parallel to $e_j$ ($r + 1 = 1$); in particular $x_j = l_{j2} \cap l_{j+1}$. Notice that using the same argument of case 2, we can define $v_{l_{j2} \cup l_{j+1}} : l_{j2} \cup l_{j+1} \to G(2, 4)$ such that it is a local transverse 2-field at $y \in \text{Int}(l_{j2} \cup l_{j+1})$ and for $y \in \partial(l_{j2} \cup l_{j+1})$, we have that $v_{l_{j2} \cap l_{j+1}}(y)$ is a plane orthogonal to the corresponding square face.

By the above, we can define $v_{slk_m(N)} : slk_m(N) \to G(2, 4)$ such that if $x \in l_{j2} \cup l_{j+1}$ then $v_{slk_m(N)}(x) := v_{l_{j2} \cup l_{j+1}}(x)$. Observe that by construction, $v_{slk_m(N)}$ is well-defined continuous transverse 2-field.

By Theorem 3.5, we know that there exists a local transverse 2-vector field $v_{F_m(N)} : F_m(N) \to G(2, 4)$, such that it is a local transverse 2-field at $y \in \text{Int}(F_m(N))$ and for $y \in \partial(S_1 \cup S_2)$, we have that $v_{F_m(N)}(y) = v_{slk_m(N)}(y)$.

Next, we define $\mathfrak{V} : N \to G(2, 4)$ as $\mathfrak{V}(y) := v_U(y)$ if $y \in U$, where $U$ can be either a square $S$ (case 1), two squares $S_1 \cup S_2$ (case 2) or a squared-star $F_m(N)$ (case 3). Notice that by construction $\mathfrak{V}(x)$ is a well-defined global transverse 2-field at $N$. The continuity of $\mathfrak{V}(x)$ follows from the fact that by construction $\mathfrak{V}(x) \to \mathfrak{V}(y)$ as $x \to y$. Therefore $N$ is smoothable. \[\square\]

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