Research Article

A Novel Method for Solving KdV Equation Based on Reproducing Kernel Hilbert Space Method

Mustafa Inc, 1 Ali Akgül, 2 and Adem Kılıçman 3

1 Department of Mathematics, Science Faculty, Firat University, 23119 Elazığ, Turkey
2 Department of Mathematics, Education Faculty, Dicle University, 21280 Diyarbakır, Turkey
3 Department of Mathematics and Institute for Mathematical Research, University Putra Malaysia (UPM), Selangor, 43400 Serdang, Malaysia

Correspondence should be addressed to Adem Kılıçman; akilicman@putra.upm.edu.my

Received 19 September 2012; Revised 17 December 2012; Accepted 23 December 2012

Abstract

We propose a reproducing kernel method for solving the KdV equation with initial condition based on the reproducing kernel theory. The exact solution is represented in the form of series in the reproducing kernel Hilbert space. Some numerical examples have also been studied to demonstrate the accuracy of the present method. Results of numerical examples show that the presented method is effective.

1. Introduction

In this paper, we consider the Korteweg-de Vries (KdV) equation of the form

\[ u_t (x, t) + \varepsilon u (x, t) u_x (x, t) + u_{xxx} (x, t) = 0, \]

\[-\infty < x < \infty, \; t > 0,\]  

with initial condition

\[ u (x, 0) = f (x). \]

The constant factor \( \varepsilon \) is just a scaling factor to make solutions easier to describe. Most of the authors chose \( \varepsilon \) to be one or six. Some mathematicians and physicians investigated the exact solution of the KdV equation without having either initial conditions or boundary conditions [1], while others studied its numerical solution [2, 3].

The numerical solution of KdV equation is of great importance because it is used in the study of nonlinear dispersive waves. This equation is used to describe many important physical phenomena. Some of these studies are the shallow water waves and the ion acoustic plasma waves [4]. It represents the long time evolution of wave phenomena, in which the effect of nonlinear terms \( uu_x \) is counterbalanced by the dispersion \( u_{xxx} \). Thus it has been found to model many wave phenomena such as waves in enharmonic crystals, bubble liquid mixtures, ion acoustic wave, and magnetohydrodynamic waves in a warm plasma as well as shallow water waves [5, 6].

The KdV equation exhibits solutions such as solitary waves, solitons and recurrence [7]. Goda [8] and Vlietinck-thart [9] used the finite difference method to obtain the numerical solution of KdV equation. Soliman [2] used the collocation solution with septic splines to obtain the solution of the KdV equation. Numerical solutions of KdV equation were obtained by the variational iteration method, finite difference method [3, 10], and by using the meshless based on the collocation with radial basis functions [11]. Wazwaz presented the Adomian decomposition method for KdV equation with different initial conditions [12]. Syam [13] worked the ADM for solving the nonlinear KdV equation with appropriate initial conditions.

In present work, we use the following equation:

\[ v (x, t) = u (x, t) - u (x, 0), \]
Abstract and Applied Analysis

by transformation for homogeneous initial condition of (1) and (2), we get the following:

\[ v_1(x, t) + A(x, t)v(x, t) + B(x, t)v_x(x, t) + v_{xxx}(x, t) = f(x, v(x, t), v_x(x, t)), \]

where

\[ A(x, t) = ef^t(x), \]

\[ B(x, t) = ef(x), \]

\[ f(x, t, v(x, t), v_x(x, t)) = -ev(x, t)v_x(x, t) - ef(x)f^t(x) - f^{'''}(x). \]

In this paper, we solve (1) and (2) by using reproducing kernel method. The nonlinear problem is solved easily and elegantly without linearizing the problem by using RKM. The technique has many advantages over the classical techniques; mainly, it avoids linearization to find analytic and approximate solutions of (1) and (2). It also avoids discretization and provides an efficient numerical solution with high accuracy, minimal calculation, and avoidance of physically unrealistic assumptions. In the next section, we will describe the procedure of this method.

The theory of reproducing kernels was used for the first time at the beginning of the 20th century by Zaremba in his work on boundary value problems for harmonic and biharmonic functions [14]. Reproducing kernel theory has important application in numerical analysis, differential equations, probability and statistics [14, 15]. Recently, using the RKM, some authors discussed fractional differential equation, nonlinear oscillator with discontinuity, singular nonlinear two-point periodic boundary value problems, integral equations, and nonlinear partial differential equations [14, 15].

The efficiency of the method was used by many authors to investigate several scientific applications. Geng and Cui [16] applied the RKHSM to handle the second-order boundary value problems. Yao and Cui [17] and Wang et al. [18] investigated a class of singular boundary value problems by this method and the obtained results were good. Zhou et al. [19] used the RKHSM effectively to solve second-order boundary value problems. In [20], the method was used to solve nonlinear infinite-delay-differential equations. Wang and Chao [21], Li and Cui [22], and Zhou and Cui [23] independently employed the RKHSM to variable-coefficient partial differential equations. Geng and Cui [24] and Du and Cui [25] investigated to the approximate solution of the forced Duffing equation with integral boundary conditions by combining the homotopy perturbation method and the RKHSM. Lv and Cui [26] presented a new algorithm to solve linear fifth-order boundary value problems. In [27, 28], authors developed a new existence proof of solutions for nonlinear boundary value problems. Cui and Du [29] obtained the representation of the exact solution for the nonlinear Volterra-Fredholm integral equations by using the reproducing kernel space. Wu and Li [30] applied iterative reproducing kernel method to obtain the analytical approximate solution of a nonlinear oscillator with discontinuities. Inc et al. [15] used this method for solving Telegraph equation.

The paper is organized as follows. Section 2 introduces several reproducing kernel spaces and a linear operator. The representation in \( W(\Omega) \) is presented in Section 3. Section 4 provides the main results. The exact and approximate solutions of (1) and (2) and an iterative method are developed for the kind of problems in the reproducing kernel space. We have proved that the approximate solution uniformly converges to the exact solution. Some numerical experiments are illustrated in Section 5. We give some conclusions in Section 6.

2. Preliminaries

2.1. Reproducing Kernel Spaces

Definition 1 (reproducing kernel). Let \( E \) be a nonempty abstract set. A function \( K : E \times E \to C \) is a reproducing kernel of the Hilbert space \( H \) if and only if

(a) for all \( t \in E, K(\cdot, t) \in H \),

(b) for all \( v \in H, \langle \varphi(\cdot), K(\cdot, t) \rangle = \varphi(t) \).

Then we need some notation that we use in the development of the paper. In the next we define several spaces with inner product over those spaces. Thus the space is defined as

\[ W_2^4[0,1] = \left\{ v(x) \mid v(x), v'(x), v''(x), v'''(x), v^{(4)}(x) \text{ are absolutely continuous in } [0,1], v^{(4)}(x) \in L^2[0,1], x \in [0,1] \right\}. \]

The inner product and the norm in \( W_2^4[0,1] \) are defined, respectively, by

\[ \langle v(x), g(x) \rangle_{W_2^4} = \sum_{i=0}^{3} v^{(i)}(0) g^{(i)}(0) + \int_0^1 v^{(4)}(x) g^{(4)}(x) dx, \]

\[ \| v \|_{W_2^4} = \sqrt{\langle v, v \rangle_{W_2^4}}, v \in W_2^4[0,1]. \]

The space \( W_2^4[0,1] \) is a reproducing kernel space, that is, for each fixed \( y \in [0,1] \) and any \( v(x) \in W_2^4[0,1] \), there exists a function \( R_y(x) \) such that

\[ v(y) = \langle v(x), R_y(x) \rangle_{W_2^4}. \]
Similarly, we define the space
\[ W_2^2[0, T] = \left\{ v(t) \mid v(t), v'(t) \right\}, \]
are absolutely continuous in \([0, T], v''(t) \in L^2[0, T], t \in [0, T], v(0) = 0 \right\}.

The inner product and the norm in \(W_2^2[0, T]\) are defined, respectively, by
\[
\langle v(t), g(t) \rangle_{W_2^2} = \sum_{i=1}^{\infty} c_i(\gamma) x_i^{-1}, x \leq \gamma, \quad \langle v(t), g(t) \rangle_{W_2^2} = \sum_{i=1}^{\infty} d_i(\gamma) x_i^{-1}, x > \gamma.
\]

Further we define the space \(W(\Omega)\) as
\[
W(\Omega) = \left\{ v(x, t) \mid \frac{\partial^4 v}{\partial x^2 \partial t^2}, \text{ is completely continuous,} \right\},
\]
in \(\Omega = [0, 1] \times [0, T], \frac{\partial^4 v}{\partial x^2 \partial t^2} \in L^2(\Omega), v(x, 0) = 0 \}

and the inner product and the norm in \(W(\Omega)\) are defined, respectively, by
\[
\langle v(x, t), g(x, t) \rangle_W = \int_0^T \int_0^1 \left[ \frac{\partial^4 v}{\partial x^2 \partial t^2} g(x, t) + \frac{\partial^4 g}{\partial x^2 \partial t^2} v(x, t) \right] dx dt, \quad \langle v(x, t), g(x, t) \rangle_W = \int_0^T \int_0^1 \left[ \frac{\partial^4 v}{\partial x^2 \partial t^2} g(x, t) + \frac{\partial^4 g}{\partial x^2 \partial t^2} v(x, t) \right] dx dt,
\]
\[
\| v \|_W = \sqrt{\langle v, v \rangle_W}, \quad v \in W(\Omega).
\]

Now we have the following theorem.

**Theorem 2.** The space \(W_2^2[0, 1]\) is a complete reproducing kernel space and its reproducing kernel function \(Q_y(x)\) can be denoted by
\[
Q_y(x) = \begin{cases} 1 + xy + \frac{y}{2} x^2 - \frac{1}{6} x^3, & x \leq y, \\ 1 + xy + \frac{x}{2} y^2 - \frac{1}{6} y^3, & x > y. \end{cases}
\]
where
\[\begin{align*}
c_1(y) &= 1, \\
c_2(y) &= y, \\
c_3(y) &= \frac{1}{4}y^2, \\
c_4(y) &= \frac{1}{36}y^3, \\
c_5(y) &= \frac{1}{144}y^3, \\
c_6(y) &= -\frac{1}{240}y^5, \\
c_7(y) &= -\frac{1}{5040}y^7, \\
c_8(y) &= -\frac{1}{14400}y^9,
\end{align*}\]

\[\begin{align*}
d_1(y) &= 1 - \frac{1}{5040}y^7, \\
d_2(y) &= y + \frac{1}{720}y^6, \\
d_3(y) &= \frac{1}{4}y^2 - \frac{1}{240}y^5, \\
d_4(y) &= \frac{1}{36}y^3 + \frac{1}{144}y^4, \\
d_5(y) &= 0, \\
d_6(y) &= 0, \\
d_7(y) &= 0, \\
d_8(y) &= 0.
\end{align*}\]

Proof. Since
\[\begin{align*}
\langle v(x), R_y(x) \rangle_{W_2^1}
&= \sum_{i=0}^{8} R_{y}^{(i)}(0) + \int_0^1 v^{(4)}(x) R_{y}^{(4)}(x) \, dx,
\end{align*}\]

by iterative integrations by parts for (22) we have
\[\begin{align*}
\langle v(x), R_y(x) \rangle_{W_2^1}
&= \sum_{i=0}^{8} v^{(i)}(0) [R_{y}^{(i)}(0) - (-1)^{(3-i)} R_{y}^{(7-i)}(0)] \\
&+ \sum_{i=1}^{8} (-1)^{(3-i)} v^{(i)}(1) R_{y}^{(7-i)}(1) \\
&+ \int_0^1 v(x) R_{y}^{(8)}(x) \, dx.
\end{align*}\]

Note that property of the reproducing kernel
\[\begin{align*}
\langle v(x), R_y(x) \rangle_{W_2^1} = v(y).
\end{align*}\]

If
\[\begin{align*}
R_y(0) + R_{y}^{(7)}(0) &= 0, \\
R_y'(0) + R_{y}^{(6)}(0) &= 0, \\
R_y''(0) + R_{y}^{(5)}(0) &= 0, \\
R_y'''(0) - R_{y}^{(4)}(0) &= 0, \\
R_y^{(4)}(1) &= 0, \\
R_y^{(5)}(1) &= 0, \\
R_y^{(6)}(1) &= 0, \\
R_y^{(7)}(1) &= 0,
\end{align*}\]

then by (23) we obtain the following equation:
\[\begin{align*}
R_{y}^{(0)}(x) &= \delta(x - y), \\
R_{y}^{(8)}(x) &= 0;
\end{align*}\]

when \( x \neq y \),
\[\begin{align*}
R_{y}^{(0)}(x) &= \delta(x - y), \\
R_{y}^{(8)}(x) &= 0;
\end{align*}\]

and
\[\begin{align*}
\partial_y^k R_y(y) &= \partial_y^k R_y(y), \quad k = 0, 1, 2, 3, 4, 5, 6, 7.
\end{align*}\]

From (25)–(31), the unknown coefficients \( c_i(y) \) \( (i = 1, 2, \ldots, 8) \) can be obtained. Thus \( R_y(x) \) is given by
\[\begin{align*}
R_{y}(x) &= \left\{ \begin{array}{ll}
1 + xy + \frac{1}{4}y^2x^2 + \frac{1}{36}y^3x^3 + \frac{1}{144}y^4x^4 \\
- \frac{1}{240}y^5x^5 + \frac{1}{720}y^6x^6 - \frac{1}{5040}y^7x^7, & x \leq y, \\
- \frac{1}{240}x^2y^5 + \frac{1}{720}x^3y^6 - \frac{1}{5040}y^7x^7, & x > y.
\end{array} \right.
\end{align*}\]

**Theorem 3.** The \( W(\Omega) \) is a reproducing kernel space, and its reproducing kernel function is
\[\begin{align*}
K_{(y_0)}(x, t) &= R_y(x) r_y(t), \\
such that for any \( v(x, t) \in W(\Omega) \),
\end{align*}\]

\[\begin{align*}
v(y, s) &= \langle v(x, t), K_{(y_0)}(x, t) \rangle_{W'}, \\
K_{(y_0)}(x, t) &= K_{(x_0)}(y, s),
\end{align*}\]

where \( R_y(x) \), \( r_y(t) \) are the reproducing kernel functions of \( W_2^1[0, 1] \) and \( W_2^2[0, T] \), respectively.

Similarly, the space \( \tilde{W}(\Omega) \) is defined as
\[\begin{align*}
\tilde{W}(\Omega) &= \left\{ v(x, t) \right\}_{\Omega = [0, 1] \times [0, T], \; \frac{\partial^3 v}{\partial x^3 \partial t} \in L^2(\Omega)}.
\end{align*}\]
The inner product and the norm in $\hat{W}(\Omega)$ are defined, respectively, by
\[
\langle v(x,t), g(x,t) \rangle_{\hat{W}} = \sum_{i=0}^{T} \int_{0}^{1} \left[ \frac{\partial}{\partial t} \frac{\partial v(x,t)}{\partial t} g(0,t) \right] dt + \int_{0}^{1} \int_{0}^{1} \left[ \frac{\partial^2 v(x,t)}{\partial x^2} \frac{\partial^2 g(x,t)}{\partial t^2} \right] dx dt,
\]
and
\[
\|v\|_{\hat{W}} = \sqrt{\langle v, v \rangle_{\hat{W}}}, \quad v \in \hat{W}(\Omega).
\]
Then the space $\hat{W}(\Omega)$ is a reproducing kernel space and its reproducing kernel function $G_{(y,z)}(x,t)$ is
\[
G_{(y,z)}(x,t) = Q_y(x)Q_z(t).
\]

3. Solution Representation in $W(\Omega)$

On defining the linear operator $L : W(\Omega) \rightarrow \hat{W}(\Omega)$ as
\[
Lv = \nu_i(x,t) - 24 \left( \text{sech}^2 x \right) \nu_s(x,t) \nu_i(x,t) + 12 \left( \text{sech}^2 x \right) \nu_s(x,t) + v_{xxx}(x,t),
\]
model problem (1) changes to the following problem:
\[
Lv(x,t) = f(x,t, \nu, v_x), \quad (x,t) \in [0,1] \times [0,T] \subset \mathbb{R}^2, \quad v(x,0) = 0.
\]

Lemma 4. The operator $L$ is a bounded linear operator.

Proof. We have
\[
\|Lv\|_{\hat{W}}^2 = \frac{1}{i=0} \int_{0}^{T} \int_{0}^{1} \left[ \frac{\partial}{\partial x^2} L
\]
\[
\end{array}
\]
\[
\begin{align*}
\|L^n\|_{\hat{W}}^2 &= \frac{1}{i=0} \int_{0}^{T} \int_{0}^{1} \left[ \frac{\partial}{\partial x^2} L^n(0,t) \right]^2 dt + \int_{0}^{T} \int_{0}^{1} \left[ \frac{\partial^2 L^n(x,t)}{\partial t^2} \right]^2 dx dt \\
&= \frac{1}{i=0} \int_{0}^{T} \int_{0}^{1} \left[ \frac{\partial}{\partial x^2} L^n(0,t) \right]^2 dt + \int_{0}^{T} \int_{0}^{1} \left[ \frac{\partial^2 L^n(x,t)}{\partial t^2} \right]^2 dx dt,
\end{align*}
\]
since
\[
v(x,t) = \nu(x,t, K_{(x,z)}(x,t)),
\]
\[
Lv(x,t) = \nu(x,t, LK_{(x,z)}(x,t)),
\]
on using the the continuity of $K_{(x,z)}(x,t)$, we have
\[
|Lv(x,t)| \leq \|L\|_{\hat{W}} \|LK_{(x,z)}(x,t)|_{\hat{W}} \leq a_i \|v\|_{\hat{W}}.
\]
Similarly for $i = 0, 1$,
\[
\frac{\partial}{\partial x^2} Lv(x,t) = \nu(x,t, \frac{\partial^2}{\partial t} LK_{(x,z)}(x,t)),
\]
and then
\[
\left| \frac{\partial}{\partial x^2} Lv(x,t) \right| \leq e_i \|v\|_{\hat{W}},
\]
\[
\left| \frac{\partial}{\partial t} \frac{\partial}{\partial x^2} Lv(x,t) \right| \leq f_i \|v\|_{\hat{W}}.
\]
Therefore
\[
\|Lv(x,t)\|_{\hat{W}}^2 \leq \sum_{i=0}^{j} \left( e_i^2 + f_i^2 \right) \|v\|_{\hat{W}}^2 \leq a^2 \|v\|_{\hat{W}}^2.
\]
\]
Now, choose a countable dense subset $\{(x_1, t_1), (x_2, t_2), \ldots \}$ in $\Omega = [0,1] \times [0,T]$ and define
\[
\Phi_i(x,t) = G_{(x_i,t)}(x,t), \quad \Psi_i(x,t) = L^* \Phi_i(x,t),
\]
where $L^*$ is the adjoint operator of $L$. The orthonormal system $\{\Psi_i(x,t)\}_{i=1}^\infty$ of $W(\Omega)$ can be derived from the process of Gram-Schmidt orthogonalization of $\{\Psi_i(x,t)\}_{i=1}^\infty$ as
\[
\hat{\Psi}_i(x,t) = \sum_{k=1}^{i} \beta_{ik} \psi_k(x,t).
\]

Theorem 5. Suppose that $\{(x_i, t_i)\}_{i=1}^\infty$ is dense in $\Omega$; then $\{\Psi_i(x,t)\}_{i=1}^\infty$ is complete system in $W(\Omega)$ and
\[
\Psi_i(x,t) = L_{(y,z)} K_{(y,z)}(x,t),
\]
\[
\text{Proof. We have}
\]
\[
\Psi_i(x,t) = \langle L^* \Phi_i(x,t) \rangle(\nu, s), \quad K_{(x,z)}(y,s),
\]
\[
= \left. \nu(x,t, L_{(y,z)}(x,t)) \right|_{(y,z) = (x_i,t_i)},
\]
\[
L_{(y,z)} K_{(x,z)}(y,s),
\]
\[
L_{(y,z)} K_{(x,z)}(x,t),
\]
\[
L_{(y,z)}(y,s)\right|_{(y,z) = (x_i,t_i)},
\]
\[
\text{Clearly } \Psi_i(x,t) \in W(\Omega). \text{ For each fixed } v(x,t) \in W(\Omega), \text{ if}
\]
\[
\langle v(x,t), \Psi_i(x,t) \rangle_W = 0, \quad i = 1, 2, \ldots
\]
then
\[ \langle v(x, t), (L^* \Phi_i)(x, t) \rangle_W = \langle Lv(x, t), \Phi_i(x, t) \rangle_{W^*} = (Lv)(x_i, t_i) = 0, \quad i = 1, 2, \ldots. \]  
(51)

Note that \( \{(x_i, t_i)\}_{i=1}^{\infty} \) is dense in \( W(\Omega) \), hence, \( (Lv)(x, t) = 0 \). It follows that \( v = 0 \) from the existence of \( L^{-1} \). So the proof is complete. \( \square \)

**Theorem 6.** If \( \{(x_i, t_i)\}_{i=1}^{\infty} \) is dense in \( \Omega \), then the solution of (39) is

\[ v(x, t) = \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} f(x_k, t_k, v(x_k, t_k), \partial_x v(x_k, t_k)) \Psi_i(x, t). \]  
(52)

**Proof.** Since \( \{\Psi_i(x, t)\}_{i=1}^{\infty} \) is complete in \( W(\Omega) \), we have

\[ v(x, t) = \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} (v(x_k, t_k, \partial_x v(x_k, t_k), v(x_k, t_k))) \Psi_i(x, t). \]

(53)

Now the approximate solution \( v_n(x, t) \) can be obtained from the \( n \)-term intercept of the exact solution \( v(x, t) \) and

\[ v_n(x, t) = \sum_{i=1}^{n} \sum_{k=1}^{i} \beta_{ik} f(x_k, t_k, v(x_k, t_k), \partial_x v(x_k, t_k)) \Psi_i(x, t). \]  
(54)

Obviously

\[ \|v_n(x, t) - v(x, t)\| \longrightarrow 0, \quad (n \longrightarrow \infty). \]  
(55)

**4. The Method Implementation**

If we write

\[ A_i = \sum_{k=1}^{i} \beta_{ik} f(x_k, t_k, v(x_k, t_k), \partial_x v(x_k, t_k)). \]  
(56)

then (52) can be written as

\[ v(x, t) = \sum_{i=1}^{\infty} A_i \Psi_i(x, t). \]  
(57)

Now let \( (x_1, t_1) = 0 \); then from the initial conditions of (39), \( v(x_1, t_1) \) is known. We put \( v_0(x_1, t_1) = v(x_1, t_1) \) and define the \( n \)-term approximation to \( v(x, t) \) by

\[ v_n(x, t) = \sum_{i=1}^{n} B_i \Psi_i(x, t), \]  
(58)

where

\[ B_i = \sum_{k=1}^{i} \beta_{ik} f(x_k, t_k, v_{k-1}(x_k, t_k), \partial_x v_{k-1}(x_k, t_k)). \]  
(59)

In the sequel, we verify that the approximate solution \( v_n(x, t) \) converges to the exact solution, uniformly. First the following lemma is given.

**Lemma 7.** If \( v_n \xrightarrow{W} \tilde{v}, (x_n, t_n) \rightarrow (y, s) \), and \( f(x, t, v(x, t)) \) is continuous, then

\[ f(x_n, t_n, v_{n-1}(x_n, t_n), \partial_x v_{n-1}(x_n, t_n)) \rightarrow f(y, s, \tilde{v}(y, s), \partial_x \tilde{v}(y, s)). \]  
(60)

**Proof.** Since

\[ |v_{n-1}(x_n, t_n) - \tilde{v}(y, s)| = |v_{n-1}(x_n, t_n) - v_{n-1}(y, s) + v_{n-1}(y, s) - \tilde{v}(y, s)| \leq |v_{n-1}(x_n, t_n) - v_{n-1}(y, s)| + |v_{n-1}(y, s) - \tilde{v}(y, s)|. \]  
(61)

From the definition of the reproducing kernel, we have

\[ v_{n-1}(x_n, t_n) = \langle v_{n-1}(x, t), K_{(x_n,t_n)}(x, t) \rangle_W, \]  
(62)

\[ v_{n-1}(y, s) = \langle v_{n-1}(x, t), K_{(y,s)}(x, t) \rangle_W. \]

It follows that

\[ |v_{n-1}(x_n, t_n) - v_{n-1}(y, s)| = \left| \left( v_{n-1}(x, t), K_{(x_n,t_n)}(x, t) - K_{(y,s)}(x, t) \right) \right|. \]  
(63)

From the convergence of \( v_{n-1}(x, t) \), there exists a constant \( M \), such that

\[ \|v_{n-1}(x, t)\|_W \leq N \|\tilde{v}(y, s)\|_W, \quad \text{as} \quad n \geq M. \]  
(64)
At the same time, we can prove
\[ \| K_{(x_n,t_n)}(x,t) - K_{(y,s)}(x,t) \|_W \rightarrow 0, \quad \text{as} \quad n \rightarrow \infty \] (65)
using Theorem 3. Hence
\[ v_{n-1}(x_n,t_n) \rightarrow \tilde{v}(y,s), \quad \text{as} \quad (x_n,t_n) \rightarrow (y,s). \] (66)
In a similar way it can be shown that
\[ \partial_x v_{n-1}(x_n,t_n) \rightarrow \partial_x \tilde{v}(y,s), \quad \text{as} \quad (x_n,t_n) \rightarrow (y,s). \] (67)
So
\[ f(x_n,t_n; v_{n-1}(x_n,t_n), \partial_x v_{n-1}(x_n,t_n)) \rightarrow f(y,s; \tilde{v}(y,s), \partial_x \tilde{v}(y,s)). \] (68)
This completes the proof.

**Theorem 8.** Suppose that \( \| v_n \| \) is a bounded in (58) and (39) has a unique solution. If \( \{ (x_i,t_i) \}_{i=1}^{\infty} \) is dense in \( \Omega \), then the \( n \)-term approximate solution \( v_n(x,t) \) derived from the above method converges to the analytical solution \( v(x,t) \) of (39) and
\[ v(x,t) = \sum_{i=1}^{\infty} B_i \tilde{\Psi}_i(x,t), \] (69)
where \( B_i \) is given by (59).

**Proof.** First, we prove the convergence of \( v_n(x,t) \). From (58), we infer that
\[ v_{n+1}(x,t) = v_n(x,t) + B_{n+1} \tilde{\Psi}_{n+1}(x,t). \] (70)
The orthonormality of \( \{ \tilde{\Psi}_i \}_{i=1}^{\infty} \) yields that
\[ \| v_{n+1} \|^2 = \| v_n \|^2 + B_{n+1}^2. \] (71)
In terms of (71), it holds that \( \| v_{n+1} \| > \| v_n \| \). Due to the condition that \( \| v_n \| \) is bounded, \( \| v_n \| \) is convergent and there exists a constant \( c \) such that
\[ \sum_{i=1}^{\infty} B_i^2 = c. \] (72)
This implies that
\[ \{ B_i \}_{i=1}^{\infty} \in \ell^2. \] (73)
If \( m > n \), then
\[ \| v_m - v_n \|^2 = \| v_m - v_{m-1} + v_{m-1} - v_{m-2} + \cdots + v_n - v_n \|^2 \]
\[ = \| v_m - v_{m-1} \|^2 + \| v_{m-1} - v_{m-2} \|^2 + \cdots + \| v_n - v_n \|^2. \] (74)

On account of
\[ \| v_m - v_{m-1} \|^2 = B_m^2, \] (75)
consequently
\[ \| v_m - v_n \|^2 = \sum_{l=n+1}^{m} B_l^2 \rightarrow 0, \quad \text{as} \quad n \rightarrow \infty. \] (76)
The completeness of \( W(\Omega) \) shows that \( v_n \rightarrow \tilde{v} \) as \( n \rightarrow \infty \). Now, let we prove that \( \tilde{v} \) is the solution of (39). Taking limits in (58) we get
\[ \tilde{v}(x,t) = \sum_{i=1}^{\infty} B_i \Psi_i(x,t). \] (77)

Note that
\[ (L\tilde{v})(x,t) = \sum_{i=1}^{\infty} B_i L\tilde{\Psi}_i(x,t), \]
\[ (L\tilde{v})(x_j,t_j) = \sum_{i=1}^{\infty} B_i L\tilde{\Psi}_i(x_j,t_j) \]
\[ = \sum_{i=1}^{\infty} B_i \left( \tilde{\Psi}_i(x,t), \Phi_i(x,t) \right)_W \]
\[ = \sum_{i=1}^{\infty} B_i \left( \tilde{\Psi}_i(x,t), \Psi_i(x,t) \right)_W \]
\[ = \sum_{i=1}^{\infty} B_i \left( \tilde{\Psi}_i(x,t), \Phi_i(x,t) \right)_W \]
Therefore
\[ \sum_{i=1}^{j} B_i \left( \tilde{\Psi}_i(x,t), \tilde{\Psi}_j(x,t) \right)_W = B_j. \] (79)
In view of (71), we have
\[ L\tilde{v}(x_j,t_j) = f(x_j,t_j, u_{t_j-1}(x_j,t_j), \partial_u u_{t_j-1}(x_j,t_j)). \] (80)
Since \( \{(x_i,t_i)\}_{i=1}^{\infty} \) is dense in \( \Omega \), for each \( (y,s) \in \Omega \), there exists a subsequence \( \{(x_{i_j},t_{i_j})\}_{j=1}^{\infty} \) such that
\[ (x_{i_j},t_{i_j}) \rightarrow (y,s), \quad j \rightarrow \infty. \] (81)
We know that
\[ L\tilde{v}(x_{i_j},t_{i_j}) = f(x_{i_j},t_{i_j}, u_{t_{i_j-1}}(x_{i_j},t_{i_j}), \partial_u u_{t_{i_j-1}}(x_{i_j},t_{i_j})). \] (82)
Let \( j \rightarrow \infty \); by Lemma 7 and the continuity of \( f \), we have
\[ (L\tilde{v})(y,s) = f(y,s, \tilde{v}(y,s), \partial \tilde{v}(y,s)), \] (83)
which indicates that \( \tilde{v}(x,t) \) satisfy (39). This completes the proof.
Remark 9. In a same manner, it can be proved that
\[
\left\| \frac{\partial v_n(x,t)}{\partial x} - \frac{\partial v(x,t)}{\partial x} \right\| \to 0, \quad \text{as } n \to \infty, \quad (84)
\]
where
\[
\frac{\partial v(x,t)}{\partial x} = \sum_{i=1}^{\infty} B_i \frac{\partial \tilde{\Psi}_i(x,t)}{\partial x},
\]
\[
\frac{\partial v_n(x,t)}{\partial x} = \sum_{i=1}^{n} B_i \frac{\partial \tilde{\Psi}_i(x,t)}{\partial x}, \quad (85)
\]
where \(B_i\) is given by (59).

5. Numerical Results

In this section, two numerical examples are provided to show the accuracy of the present method. All computations are performed by Maple 16. Results obtained by the method are compared with exact solution and the ADM [13] of each example are found to be in good agreement with each others. The RKM does not require discretization of the variables, that is, time and space, it is not affected by computation round off errors and one is not faced with necessity of large computer memory and time. The accuracy of the RKM for the KdV equation is controllable and absolute errors are very small with present choice of \(x\) and \(t\) (see Tables 1, 2, 3, and 4 and Figures 1, 2, and 3). The numerical results that we obtained justify the advantage of this methodology.

Example 10 (see [13]). Consider the following KdV equation with initial condition
\[
u_t(x,t) + \varepsilon u(x,t) u_x(x,t) + u_{xxx}(x,t) = 0, \quad -\infty < x < \infty, \quad t > 0, \quad (86)
\]
\[
u(x,0) = 2\text{sech}^2 x, \quad -\infty < x < \infty
\]
with \(\varepsilon = 6\). The exact solution is \(u(x,t) = 2\text{sech}^2(x - 4t)\). If we apply (3) to (86), then the following (87) is obtained
\[
v_i(x,t) - 24\text{sech}^3 x \sinh x v(x,t) + 12\text{sech}^2 x v_{xx}(x,t) + v_{xxx}(x,t) + 12\text{sech}^2 x v_{xx}(x,t) + 6 v(x,t) v_x(x,t) + 32 \frac{\sinh x}{\cosh^3 x} + 48 \frac{\sinh^3 x}{\cosh^3 x} + 48 \text{sech}^5 x \sinh x, \quad (87)
\]
\[
v(x,0) = 0.
\]

Example 11 (see [13]). We now consider the KdV equation with initial condition
\[
u_t(x,t) + \varepsilon u(x,t) u_x(x,t) + u_{xxx}(x,t) = 0, \quad -\infty < x < \infty, \quad t > 0, \quad (88)
\]
\[
u(x,0) = 6\text{sech}^2 x, \quad -\infty < x < \infty.
\]
The exact solution is \( u(x, t) = 12((3 + 4 \cosh(2x - 8t) + \cosh(4x - 64t))/[3 \cosh(x - 28t) + \cosh(3x - 36t)]^2) \). If we apply (3) to (88), then the following (89) is obtained:

\[
v_i(x, t) = \frac{\sinh x}{\cosh x} - 72 \text{sech}^3 x \sinh xv(x, t) + 36 \text{sech}^2 x v_{xx}(x, t) + v_{xxx}(x, t)
\]

Using our method we choose 36 points on \([0, 1]\). We replace \( v \) with \( u \) for simplicity. In Tables 3 and 4, we compute the absolute errors \( |u(x, t) - u_n(x, t)| \) and the relative errors \( |u(x, t) - u_n(x, t)|/|u(x, t)| \) at the points \( \{(x_i, t_i) : x_i = t_i = i, i = 0.1, \ldots, 0.6\} \).

**Remark 12.** The problem discussed in this paper has been solved with Adomian method [13] and Homotopy analysis method [31]. In these studies, even though the numerical results give good results for large values of \( x \), these methods give away values from the analytical solution for small values of \( x \) and \( t \). However, the method is used in our study for large and small values of \( x \) and \( t \), results are very close to the analytical solutions can be obtained. In doing so, it is possible to refine the result by increasing the intensive points.
6. Conclusion
In this paper, we introduce an algorithm for solving the KdV equation with initial condition. For illustration purposes, we chose two examples which were selected to show the computational accuracy. It may be concluded that the RKM is very powerful and efficient in finding exact solution for wide classes of problem. The approximate solution obtained by the present method is uniformly convergent.

Clearly, the series solution methodology can be applied to much more complicated nonlinear differential equations and boundary value problems. However, if the problem becomes nonlinear, then the RKM does not require discretization or perturbation and it does not make closure approximation. Results of numerical examples show that the present method is an accurate and reliable analytical method for the KdV equation with initial or boundary conditions.

Acknowledgment
A. Kilíçman gratefully acknowledge that this paper was partially supported by the University Putra Malaysia under the ERGS Grant Scheme having project no. 5527068 and Ministry of Science, Technology and Inovation (MOSTI), Malaysia under the Science Fund 06-01-04-SFI050.

References
[1] P. G. Drazin and R. S. Johnson, Solitons: An Introduction, Cambridge University Press, Cambridge, UK, 1989.
[2] A. A. Soliman, "Collucationsolution of the KdV equation using septic splines," International Journal of Computer Mathematics, vol. 81, pp. 325–331, 2004.
[3] A. A. Soliman, A. H. A. Ali, and K. R. Raslan, "Numerical solution for the KdV equation based on similarity reductions," Applied Mathematical Modelling, vol. 33, no. 2, pp. 1107–1115, 2009.
[4] N. J. Zabusky, "A synergetic approach to problem of nonlinear dispersive wave propagation and interaction," in Proceeding of Symposium Nonlinear PDEs, W. Ames, Ed., pp. 223—258, Academic Press, New York, NY, USA, 1967.
[5] D. I. Korteweg-de Vries and G. de Vries, "On the change in form of long waves advancing in rectangular canal and on a new type of long stationary waves," Philosophical Magazine, vol. 39, pp. 422–443, 1895.
[6] C. Gardner and G. K. Marikawa, "The effect of temperature of the width of a small amplitude solitary wave in a collision free plasma," Communications on Pure and Applied Mathematics, vol. 18, pp. 35–49, 1965.
[7] R. K. Dodd, J. C. Eilbeck, J. D. Gibbon, and H. C. Morris, Solitons and Nonlinear Wave Equations, Academic Press, New York, NY, USA, 1982.
[8] K. Goda, "On stability of some finite difference schemes for the KdV equation," Journal of the Physical Society of Japan, vol. 39, pp. 229–236, 1975.
[9] A. C. Vliengenthart, "On finite difference methods for KdV equation," Journal of Engineering Mathematics, vol. 5, pp. 137–155, 1971.
[10] M. Inc, "Numerical simulation of KdV and mKdV equations with initial conditions by the variational iteration method," Chaos, Solitons and Fractals, vol. 34, no. 4, pp. 1075–1081, 2007.
[11] I. Dağ and Y. Dereli, "Numerical solution of KdV equation using radial basis functions," Applied Mathematical Modelling, vol. 32, pp. 535–546, 2008.
[12] A. M. Wazwaz, Partial Differential Equations and Solitary Waves Theory, Higher Education Press, Springer, London, UK, 2009.
[13] M. I. Syam, "Adomian decomposition method for approximating the solution of the KdV equation," Applied Mathematics and Computation, vol. 162, pp. 1465–1473, 2005.
[14] N. Aronszajn, "Theory of reproducing kernels," Transactions of the American Mathematical Society, vol. 68, pp. 337–404, 1950.
[15] M. Inc, A. Akgül, and A. Kilíçman, "Explicit solution of telegraph equation based on reproducing Kernel method," Journal of Function Spaces and Applications, vol. 2012, Article ID 984682, 23 pages, 2012.
[16] F. Geng and M. Cui, "Solving a nonlinear system of second order boundary value problems," Journal of Mathematical Analysis and Applications, vol. 327, pp. 1167–1181, 2007.
[17] H. Yao and M. Cui, "A new algorithm for a class of singular boundary value problems," Applied Mathematics and Computation, vol. 186, pp. 1183–1191, 2007.
[18] W. Wang, M. Cui, and B. Han, "A new method for solving a class of singular two-point boundary value problems," Applied Mathematics and Computation, vol. 206, pp. 721—727, 2008.
[19] Y. Zhou, Y. Lin, and M. Cui, "An efficient computational method for second order boundary value problemsof nonlinear differential equations," Applied Mathematics and Computation, vol. 194, pp. 357–365, 2007.
[20] X. Lü and M. Cui, "Analytic solutions to a class of nonlinear infinite-delay-differential equations," Journal of Mathematical Analysis and Applications, vol. 343, pp. 724–732, 2008.
[21] Y. L. Wang and L. Chao, "Using reproducing kernel for solving a class of partial differential equation with variable-coefficients," Applied Mathematics and Mechanics, vol. 29, pp. 129–137, 2008.
[22] F. Li and M. Cui, "A best approximation for the solution of one-dimensional variable-coefficient Burger’s equation," Numerical Methods for Partial Differential Equations, vol. 25, pp. 1353–1365, 2009.
[23] S. Zhou and M. Cui, "Approximate solution for a variable-coefficient semilinear heat equation with nonlocal boundary conditions," International Journal of Computer Mathematics, vol. 86, pp. 2248–2258, 2009.
[24] F. Geng and M. Cui, "New method based on the HPM and RKHSM for solving forced Duffing equations with integral boundary conditions," Journal of Computational and Applied Mathematics, vol. 233, no. 2, pp. 165–172, 2009.
[25] J. Du and M. Cui, "Solving the forced Duffing equations with integral boundary conditions in the reproducing kernel space," International Journal of Computer Mathematics, vol. 87, pp. 2088–2100, 2010.
[26] X. Lv and M. Cui, "An efficient computational method for linear fifth-order two-point boundary value problems," Journal of Computational and Applied Mathematics, vol. 234, no. 5, pp. 1551–1558, 2010.
[27] W. Jiang and M. Cui, "Constructive proof of existence for nonlinear two-point boundary value problems," Computers and Mathematics with Applications, vol. 59, no. 2, pp. 903–911, 2010.
[30] B. Y. Wu and X. Y. Li, "Iterative reproducing kernel method for nonlinear oscillator with discontinuity," *Applied Mathematics Letters*, vol. 23, no. 10, pp. 1301–1304, 2010.

[31] H. Jafari and M. A. Firoozjaee, "Homotopy analysis method for KdV equation," *Surveys in Mathematics and Its Applications*, vol. 5, pp. 89–98, 2010.
