Analytic Bethe ansatz and functional equations associated with any simple root systems of the Lie superalgebra $sl(r + 1|s + 1)$

Zengo Tsuboi *
Institute of Physics, University of Tokyo
Komaba 3-8-1, Meguro-ku, Tokyo 153 Japan

Abstract

The Lie superalgebra $sl(r + 1|s + 1)$ admits several inequivalent choices of simple root systems. We have carried out analytic Bethe ansatz for any simple root systems of $sl(r + 1|s + 1)$. We present transfer matrix eigenvalue formulae in dressed vacuum form, which are expressed as the Young supertableaux with some semistandard-like conditions. These formulae have determinant expressions, which can be viewed as quantum analogue of Jacobi-Trudi and Giambelli formulae for $sl(r + 1|s + 1)$. We also propose a class of transfer matrix functional relations, which is specialization of Hirota bilinear difference equation. Using the particle-hole transformation, relations among the Bethe ansatz equations for various kinds of simple root systems are discussed.

PACS numbers: 02.20.Qs, 02.20.Sv, 03.20.+i, 05.50.+q
Keywords: analytic Bethe ansatz, Lie superalgebra, solvable lattice model, transfer matrix, T-system, Young superdiagram

Journal-ref: Physica A 252 (1998) 565-585
DOI: 10.1016/S0378-4371(97)00625-0

*This paper was published in 1998 first. present address (on December 2009): Okayama Institute for Quantum Physics, 1-9-1 Kyoyama, Okayama 700-0015, Japan
1 Introduction

In reference [1], analytic Bethe ansatz [2, 3] was carried out systematically for fundamental representations of the Yangians $\mathcal{Y}(\mathcal{G})$ [4] associated with classical simple Lie algebras $\mathcal{G} = B_r, C_r$ and $D_r$. That is, eigenvalue formulas in dressed vacuum form were presented for the commuting transfer matrices of solvable vertex models. These formulae are Yangian analogues of the Young tableaux for $\mathcal{G}$ and obey some semi-standard like conditions. It had been proven that they do not have poles under the Bethe ansatz equation. Furthermore, for $\mathcal{G} = B_r$ case, these formulae were generalized [5] to the case of finite dimensional modules labeled by skew-Young diagrams $\lambda \subset \mu$. The eigenvalue formulae of the transfer matrices in dressed vacuum form labeled by rectangular Young diagrams $\lambda = \phi, \mu = (m^n)$ obey a class of functional relations, the T-system [6] (see also, references [7, 8, 9, 10, 11, 12, 13, 14]). Making use of the T-system, we are able to calculate [14] various kinds of physical quantities such as the correlation lengths of the vertex models and central charges of RSOS models. The T-system is not only a class of transfer matrix functional relations but also a two-dimensional Toda field equation on discrete space time. Solving it recursively, we can express its solutions in terms of pfaffians or determinants [5, 15, 16, 17].

In contrast to above mentioned successful story in the T-system and the analytic Bethe ansatz for simple Lie algebras, systematic treatment of them for Lie superalgebras [18] had not been studied yet until quite recently. Studying supersymmetric integrable models is significant not only in mathematical physics but also in condensed matter physics (see for example, reference [19]). For instance, the supersymmetric $t - J$ model received much attention in connection with high $T_c$ superconductivity. As is well known, there are several choices of simple root systems for a superalgebra. We can construct all the simple root systems, from any one of them by applying repeatedly the reflections with respect to the elements of the Weyl supergroup $\text{SW}(\mathcal{G})$ [20]. The simplest system of simple roots is so called distinguished one [18]. Recently we had executed [21] analytic Bethe ansatz associated mainly with the distinguished simple root system of the Lie superalgebra $\text{sl}(r + 1|s + 1)$ and then established functional relations for commuting family of transfer matrices.

The purpose of this paper is to extend our previous results [21] to any simple root systems of $\mathcal{G} = \text{sl}(r + 1|s + 1)$. One can reproduce many of the earlier results [21] if one set the grading parameters (2.11) to $p_a = 1 : 1 \leq a \leq r + 1; p_a = -1 : r + 2 \leq a \leq r + s + 2$. Throughout this paper, we often use similar notation presented in references [1, 5, 16, 21].

We execute analytic Bethe ansatz based upon the Bethe ansatz equation (3.1) associated with any simple root systems of $\text{sl}(r + 1|s + 1)$. The observation that the Bethe ansatz equation can be expressed by the root system of a Lie algebra
is traced back to reference [22] (see also, reference [23] for $sl(r+1|s+1)$ case). Moreover, Kuniba et.al. [5] conjectured that the left hand side of the Bethe ansatz equation (3.1) can be written as a ratio of some ‘Drinfeld polynomials’ [4]. In addition, extra signs appear in the Bethe ansatz equation. This is because in the supersymmetric models, the R-matrix satisfies the graded Yang-Baxter equation [24] and then the transfer matrix is defined as a supertrace of the monodromy matrix. There are several sets of Bethe ansatz equations corresponding to the fact that there are several choices of simple root systems for a Lie superalgebra. However these sets of the Bethe ansatz equations are connected with each other under the particle-hole transformation. In fact, the equivalence of these sets of the Bethe ansatz equations was established for $sl(1|2)$ case in references [25, 26] and for $sl(2|2)$ case in reference [27]. Then we discuss relations among these sets of the Bethe ansatz equations for $sl(r+1|s+1)$ and we point out that the particle-hole transformation is related with the reflection with respect to the element of the Weyl supergroup for odd simple root $\alpha$ with $(\alpha|\alpha) = 0$.

We introduce the Young superdiagram [28]. To put it more precisely, this Young superdiagram is different from the classical one in that it carries spectral parameter $u$. In contrast to ordinary Young diagram, there is no limitation on the number of rows. We define semi-standard like tableau on it. Making use of this tableau, we introduce the function $T_{\lambda \subset \mu}(u)$ (3.15), which should be the fusion transfer matrix whose auxiliary space is finite dimensional module of super Yangian $Y(sl(r+1|s+1))$ [29] or quantum affine superalgebra $U_q(sl(r+1|s+1)^{(1)})$ [30, 31], labeled by skew-
Young superdiagram $\lambda \subset \mu$. We can trace the origin of the function $T^1(u)$ back to the eigenvalue formula of transfer matrix of the Perk-Schultz model [32, 33, 34], which is a multi-component generalization of the six-vertex model (see also reference [23]). Furthermore, the function $T^1(u)$ reduces to the eigenvalue formula of transfer matrix derived by algebraic Bethe ansatz (For instance, reference [35]: $r = 1, s = 0$ case; reference [26]: $r = 0, s = 1$ case; references [36, 27]: $r = s = 1$ case). We prove pole-freeness of $T^a(u) = T_{(1^a)}(u)$, essential property in the analytic Bethe ansatz. Owing to the same mechanism presented in reference [5], the function $T_{\lambda \subset \mu}(u)$ has determinant expressions whose matrix elements are only the functions associated with Young superdiagrams with shape $\lambda = \phi; \mu = (m)$ or $(1^n)$. They can be viewed as quantum analogue of Jacobi-Trudi and Giambelli formulae for $sl(r+1|s+1)$. Then we can easily show that the function $T_{\lambda \subset \mu}(u)$ is free of poles under the Bethe ansatz equation (3.1). We present a class of transfer matrix functional relations among the above-mentioned eigenvalue formulae of transfer matrix in dressed vacuum form associated with rectangular Young superdiagrams. It is specialization of Hirota bilinear difference equation [37], which can be proved by the Jacobi identity.

The outline of this paper is given as follows. In section 2, we briefly review the Lie superalgebra $\mathcal{G} = sl(r+1|s+1)$. In section 3, we execute the analytic Bethe ansatz based upon the Bethe ansatz equation (3.1) associated with any simple root systems.
We prove pole-freeness of the function $T^a(u) = T_{(1^a)}(u)$. In section 4, we propose functional relations, the T-system, associated with the transfer matrices in dressed vacuum form defined in the previous section. In section 5, using the particle-hole transformation, relations among the sets of the Bethe ansatz equations for various kinds of simple root systems are discussed. Section 6 is devoted to summary and discussion.

2 The Lie superalgebra $sl(r + 1|s + 1)$

In this section, we briefly review the Lie superalgebra $G = sl(r + 1|s + 1)$. A Lie superalgebra [18] is a $\mathbb{Z}_2$ graded algebra $G = G_0 \oplus G_1$ with a product $[\ ,\ ]$, whose homogeneous elements $a \in G_\alpha, b \in G_\beta (\alpha, \beta \in \mathbb{Z}_2 = \{0, 1\})$ and $c \in G$ obey the following relations.

\[
[a, b] \in G_{\alpha + \beta}, \quad [a, b] = -(-1)^{\alpha \beta}[b, a], \quad [a, [b, c]] = [[a, b], c] + (-1)^{\alpha \beta}[b, [a, c]].
\]

We can divide the set of non-zero roots into the set of non-zero even roots (bosonic roots) $\Delta_0$ and the set of odd roots (fermionic roots) $\Delta_1$. For $sl(r + 1|s + 1)$ case, they have the following form

\[
\Delta_0 = \{\epsilon_i - \epsilon_j\} \cup \{\delta_i - \delta_j\}, i \neq j; \quad \Delta_1 = \{\pm(\epsilon_i - \delta_j)\}
\]

(2.2)

where $\epsilon_1, \ldots, \epsilon_r; \delta_1, \ldots, \delta_s, \delta_{s+1}$ are basis of dual space of the Cartan subalgebra with the bilinear form $(\ | \ )$ such that

\[
(\epsilon_i|\epsilon_j) = \delta_{i,j}, \quad (\epsilon_i|\delta_j) = (\delta_i|\epsilon_j) = 0, \quad (\delta_i|\delta_j) = -\delta_{i,j}.
\]

(2.3)

The Weyl group $W(G)$ of a Lie superalgebra $G$ is generated by the Weyl reflections with respect to the even roots:

\[
\omega_\alpha(\beta) = \beta - \frac{2(\alpha|\beta)}{(\alpha|\alpha)}\alpha
\]

(2.4)

where $\alpha \in \Delta_0, \beta \in \Delta_0 \cup \Delta_1$. Moreover the Weyl group $W(G)$ can be extended to the Weyl supergroup $SW(G)$ [20] by adding the reflections with respect to the odd roots:

\[
\omega_\alpha(\beta) = \begin{cases} 
\beta - \frac{2(\alpha|\beta)}{(\alpha|\alpha)}\alpha & \text{for } (\alpha|\alpha) \neq 0 \\
\beta + \alpha & \text{for } (\alpha|\alpha) = 0 \ \text{and } (\alpha|\beta) \neq 0 \\
\beta & \text{for } (\alpha|\alpha) = 0 \ \text{and } (\alpha|\beta) = 0 \\
-\alpha & \text{for } \beta = \alpha
\end{cases}
\]

(2.5)
where $\alpha \in \Delta_1, \beta \in \Delta_0 \cup \Delta_1$. Note that $\Delta_0$ and $\Delta_1$ are invariant under $\omega_\alpha \in \mathcal{W}(G)$; are not invariant under $\omega_\alpha \in \mathcal{SW}(G)$ with $(\alpha|\alpha) = 0$. There are several choices of simple root systems depending on choices of Borel subalgebras. The simplest system of simple roots is so-called distinguished one [18]. For example, the distinguished simple root system $\{\alpha_1, \ldots, \alpha_{r+s+1}\}$ of $sl(r+1|s+1)$ has the form

\[
\begin{align*}
\alpha_i &= \epsilon_i - \epsilon_{i+1} & i &= 1, 2, \ldots, r, \\
\alpha_{r+1} &= \epsilon_{r+1} - \delta_1 \\
\alpha_{j+r+1} &= \delta_j - \delta_{j+1}, & j &= 1, 2, \ldots, s
\end{align*}
\] (2.6)

where $\{\alpha_i\}_{i \not= r+1}$ are even roots and $\alpha_{r+1}$ is an odd root with $(\alpha_{r+1}|\alpha_{r+1}) = 0$. One can construct all the simple root systems, unequivalent with respect to $\mathcal{W}(G)$, from any one of them by applying repeatedly the reflections with respect to $\omega_\alpha \in \mathcal{SW}(G)$ with $(\alpha|\alpha) = 0$ (see, figure 1). We define the sets

\[
J = \{1, 2, \ldots, r+s+2\}
\] (2.7)

with the total order

\[
1 \prec 2 \prec \cdots \prec r+s+2.
\] (2.8)

Divide the set $J$ into two disjoint sets

\[
\begin{align*}
J &= J_+ \cup J_- \\
J_+ &= \{i_1, i_2, \ldots, i_{r+1}\} \\
J_- &= \{j_1, j_2, \ldots, j_{s+1}\}
\end{align*}
\] (2.9)

with the ordering

\[
i_1 \prec i_2 \prec \cdots \prec i_{r+1}, \quad j_1 \prec j_2 \prec \cdots \prec j_{s+1}.
\] (2.10)

For any element of $J$, we introduce the grading

\[
p_a = \begin{cases} 
1 & \text{for } a \in J_+ \\
-1 & \text{for } a \in J_-
\end{cases}
\] (2.11)

Using this grading parameters $\{p_j\}$, one can express Cartan matrix as follows

\[
(\alpha_k|\alpha_l) = (p_k + p_{k+1})\delta_{k,l} - p_{k+1}\delta_{k+1,l} - p_k\delta_{k,l+1}.
\] (2.12)
Figure 1: Dynkin diagrams for the Lie superalgebra $sl(2|2)$: A white circle expresses even root; a grey (a cross) circle expresses odd root $\alpha$ with $(\alpha|\alpha) = 0$. The topmost Dynkin diagram is the one associated with the distinguished simple root system. The odd root $\alpha$ attached to the root of the arrow is transfered to the one attached to the arrowhead by the reflection with respect to $\omega_\alpha \in SW(G)$. 
3 Analytic Bethe ansatz

Consider the following type of the Bethe ansatz equation (cf. references [23, 22, 5, 34]).

\[-P_a(u_k^{(a)} + \zeta_a) = \frac{(-1)^{\deg(\alpha_a)}}{P_a(u_k^{(a)} - \zeta_a)} \prod_{b=1}^{r+s+1} \frac{Q_b(u_k^{(a)}) + (\alpha_a|\alpha_b))}{Q_b(u_k^{(a)}) - (\alpha_a|\alpha_b))}, \tag{3.1}\]

\[Q_a(u) = \prod_{j=1}^{N_a} [u - w_j^{(a)}], \tag{3.2}\]

\[P_a(u) = \prod_{j=1}^{N_a} P_a^{(j)}(u), \tag{3.3}\]

\[P_a^{(j)}(u) = [u - w_j^{\delta_{a,1}}] \tag{3.4}\]

where \([u] = (q^u - q^{-u})/(q - q^{-1}); N_a \in \mathbb{Z}_{\geq 0}; u, w_j \in \mathbb{C}; a, k \in \mathbb{Z} \ (1 \leq a \leq r + s + 1, 1 \leq k \leq N_a), \zeta_1 = p_1\) and

\[\deg(\alpha_a) = \begin{cases} 0 & \text{for even root} \\ 1 & \text{for odd root} \end{cases} \tag{3.5}\]

\[\frac{1 - p_a p_{a+1}}{2}.\]

Particularly for distinguished simple root of \(sl(r+1|s+1)\), we have \(\deg(\alpha_a) = \delta_{a,r+1}\).

In the present paper, we suppose that \(q\) is generic. The left hand side of the Bethe ansatz equation (3.1) is connected with the quantum space. We suppose that it is the ratio of some ‘Drinfeld polynomials’ labeled by skew-Young diagrams \(\tilde{\lambda} \subset \tilde{\mu}\) (cf. reference [5]). For simplicity, we deal only with the case \(\tilde{\lambda} = \phi, \tilde{\mu} = (1)\). The generalization to the case for any skew-Young diagram will be accomplished by the empirical procedures given in reference [5]. The factor \((-1)^{\deg(\alpha_a)}\) of the Bethe ansatz equation (3.1) exists so as to make the transfer matrix to be the supertrace of the monodromy matrix. Note that the Bethe ansatz equation (3.1) is invariant under the Weyl group \(W(G)\) since \((\omega_\beta(\alpha)|\omega_\beta(\gamma)) = (\alpha|\gamma)\) for \(\alpha, \gamma \in \Delta_0 \cup \Delta_1\) and \(\beta \in \Delta_0\).

For any \(a \in J\), set

\[z(a; u) = \psi_a(u) \frac{Q_{a-1}(u + \sum_{j=1}^{a-1} p_j + 2p_a)Q_a(u + \sum_{j=1}^{a} p_j - 2p_a)}{Q_{a-1}(u + \sum_{j=1}^{a-1} p_j)Q_a(u + \sum_{j=1}^{a} p_j)} \tag{3.6}\]

where \(Q_0(u) = 1, Q_{r+s+2}(u) = 1\) and

\[\psi_a(u) = \begin{cases} P_1(u + 2p_1) & \text{for } a = 1 \\ P_1(u) & \text{for } a \in J - \{1\} \end{cases}. \tag{3.7}\]
In the present paper, we frequently express the function \( z(a; u) \) as the box \( \Box_{a} \), whose spectral parameter \( u \) will often be abbreviated. Under the Bethe ansatz equation (3.1), the following relations are valid

\[
\text{Res}_{u=-\sum_{j=1}^{b} p_{j}+u_{k}} (p_{b} z(b; u) + p_{b+1} z(b+1; u)) = 0, \quad b \in J - \{ r + s + 2 \}. \tag{3.8}
\]

It was pointed out [1] that the dressed vacuum form in the analytic Bethe ansatz for fundamental representations of Yangians \( Y(G) \) associated with simple Lie algebras \( G = B_r, C_r, D_r \) have similar structure of the crystal graph [38, 39]. This is also the case with \( G = sl(r+1|s+1) \). Actually, we can express the relation (3.8) schematically as follows:

\[
\begin{array}{c}
\text{Res}_{u=-\sum_{j=1}^{b} p_{j}+u_{k}} (p_{b} z(b; u) + p_{b+1} z(b+1; u)) = 0, \quad b \in J - \{ r + s + 2 \}.
\end{array}
\]

where the number \( b \) on the arrow represents the 'color' (superscript of \( u_{k}^{(b)} \)) of the common pole \(-\sum_{j=1}^{b} p_{j} + u_{k}^{(b)}\) of the functions \( z(b; u) \) and \( z(b+1; u) \).

We will use the functions \( T^{a}(u) \) and \( T_{m}(u) \) \((a, m \in \mathbb{Z}; u \in \mathbb{C})\) determined by the non-commutative generating series of the form

\[
(1 + z(r + s + 2; u)X)^{pr+s+2} \cdots (1 + z(r + 2; u)X)^{pr+2} \\
\times (1 + z(r + 1; u)X)^{pr+1} \cdots (1 + z(1; u)X)^{p_{1}}
= \sum_{a=-\infty}^{\infty} T^{a}(u + a - 1)X^{a}, \tag{3.10}
\]

\[
(1 - z(1; u)X)^{-p_{1}} \cdots (1 - z(r + 1; u)X)^{-pr+1} \\
\times (1 - z(r + 2; u)X)^{-pr+2} \cdots (1 - z(r + s + 2; u)X)^{-pr+s+2}
= \sum_{m=-\infty}^{\infty} T_{m}(u + m - 1)X^{m}, \tag{3.11}
\]

where \( X \) is a shift operator \( X = e^{2\partial_{u}} \). In particular, we have \( T^{0}(u) = 1; T_{0}(u) = 1; T^{a}(u) = 0 \) for \( a < 0; T_{m}(u) = 0 \) for \( m < 0 \). We note that the origin of the functions \( T^{1}(u), T_{1}(u) \) and the Bethe ansatz equation (3.1) with (2.12) trace back to the eigenvalue formula of transfer matrix and the Bethe ansatz equation for the Perk-Schultz model [34] but the vacuum part, some gauge factors and extra signs after some redefinition. (See also, reference [23]). As for the relation between fundamental \( L \) operator and the function \( T^{1}(u) \), see, for example, Appendix A in reference [21].

Let \( \lambda \subset \mu \) be a skew-Young superdiagram labeled by the sequences of non-negative integers \( \lambda = (\lambda_{1}, \lambda_{2}, \ldots) \) and \( \mu = (\mu_{1}, \mu_{2}, \ldots) \) such that \( \mu_{i} \geq \lambda_{i} : i = 1, 2, \ldots; \lambda_{1} \geq \lambda_{2} \geq \ldots \geq 0; \mu_{1} \geq \mu_{2} \geq \ldots \geq 0 \) and \( \lambda' = (\lambda'_{1}, \lambda'_{2}, \ldots) \) be the conjugate of \( \lambda \). We assign a coordinates \((i, j) \in \mathbb{Z}^{2}\) on this skew-Young superdiagram \( \lambda \subset \mu \) such that the row index \( i \) increases as we go downwards and the column index \( j \)
increases as we go from left to right and that \((1, 1)\) is on the top left corner of \(\mu\).

We define an admissible tableau \(b\) on the skew-Young superdiagram \(\lambda \subset \mu\) as a set of element \(b(i, j) \in J\) labeled by the coordinates \((i, j)\) mentioned above, with the following rule (admissibility conditions).

1. For any elements of \(J\),
   \[
   b(i, j) \preceq b(i, j + 1), \quad b(i, j) \preceq b(i + 1, j).
   \] (3.12)

2. For any elements of \(J_+\),
   \[
   b(i, j) \prec b(i + 1, j).
   \] (3.13)

3. For any elements of \(J_-\),
   \[
   b(i, j) \prec b(i, j + 1).
   \] (3.14)

Let \(B(\lambda \subset \mu)\) be the set of admissible tableaux on \(\lambda \subset \mu\). For any skew-Young superdiagram \(\lambda \subset \mu\), define the function \(T_{\lambda \subset \mu}(u)\) as follows

\[
T_{\lambda \subset \mu}(u) = \sum_{b \in B(\lambda \subset \mu)} \prod_{(i, j) \in (\lambda \subset \mu)} p_{b(i,j)}z(b(i, j); u - \mu_1 + \mu'_1 - 2i + 2j) \quad (3.15)
\]

where the product is taken over the coordinates \((i, j)\) on \(\lambda \subset \mu\). The following relations should be valid by the same reason mentioned in [5], that is, they will be verified by induction on \(\mu_1\) or \(\mu'_1\).

\[
T_{\lambda \subset \mu}(u) = \det_{1 \leq i, j \leq \mu_1}(T^{\mu'_1 - \lambda'_j - i + j}(u - \mu_1 + \mu'_1 - \lambda'_j + i + j - 1)) \quad (3.16)
\]

\[
= \det_{1 \leq i, j \leq \mu_1}(T_{\mu_j - \lambda_i + i - j}(u - \mu_1 + \mu'_1 + \mu_j - \lambda_i - i - j + 1)). \quad (3.17)
\]

For instance, for \(sl(2|1)\): \(\lambda = \phi; \mu = (2^1, 1^1); J_+ = \{1, 3\}; J_- = \{2\}\) case, we obtain

\[
T_{(2^1, 1^1)}(u) = -\begin{array}{|c|c|}
1 & 1 \\
2 & 3 \\
\end{array} + \begin{array}{|c|c|}
1 & 1 \\
2 & 2 \\
\end{array} - \begin{array}{|c|c|}
1 & 1 \\
2 & 3 \\
\end{array}
\]

\[
= P_1(u - 2)P_1(u + 2)\left\{ -P_1(u + 4)\frac{Q_1(u - 3)Q_2(u)}{Q_1(u + 3)Q_2(u - 2)} + P_1(u + 4)\frac{Q_1(u - 1)Q_2(u)}{Q_1(u + 3)Q_2(u - 2)} + P_1(u + 2)\frac{Q_1(u - 3)Q_2(u)Q_2(u + 4)}{Q_1(u + 3)Q_2(u - 2)Q_2(u + 2)} - P_1(u + 2)\frac{Q_1(u - 1)Q_2(u)Q_2(u + 4)}{Q_1(u + 3)Q_2(u - 2)Q_2(u + 2)} \right\}
\]
\[- P_1(u + 2) \frac{Q_1(u - 3)Q_2(u)Q_2(u + 4)}{Q_1(u + 1)Q_2(u - 2)Q_2(u + 2)}
+ P_1(u + 2) \frac{Q_1(u - 1)Q_2(u)Q_2(u + 4)}{Q_1(u + 1)Q_2(u - 2)Q_2(u + 2)}
+ P_1(u) \frac{Q_1(u - 3)Q_2(u + 4)}{Q_1(u + 1)Q_2(u - 2)}
- P_1(u) \frac{Q_1(u - 1)Q_2(u + 4)}{Q_1(u + 1)Q_2(u - 2)} \right \}
\right \}
= \begin{bmatrix}
T^2(u - 1) & T^3(u) \\
1 & T^1(u + 2)
\end{bmatrix}
\right \}
(3.18)
\]

where
\[
T^1(u) = \begin{bmatrix}
1 & -2 + 3 \\
2 & 1 + 2 - 3
\end{bmatrix}
= P_1(u + 2) \frac{Q_1(u - 1)}{Q_1(u + 1)} - P_1(u) \frac{Q_1(u - 1)Q_2(u + 2)}{Q_1(u + 1)Q_2(u)}
+ P_1(u) \frac{Q_2(u + 2)}{Q_2(u)},
\]
\[
T^2(u) = \begin{bmatrix}
1 & 1 & -2 + 2 - 3 \\
2 & 1 & 2
\end{bmatrix}
= P_1(u - 1) \left \{-P_1(u + 3) \frac{Q_1(u - 2)Q_2(u + 1)}{Q_1(u + 2)Q_2(u - 1)}
+ P_1(u + 3) \frac{Q_1(u)Q_2(u + 1)}{Q_1(u + 2)Q_2(u - 1)}
+ P_1(u + 1) \frac{Q_1(u - 2)Q_2(u + 3)}{Q_1(u + 2)Q_2(u - 1)} - P_1(u + 1) \frac{Q_1(u)Q_2(u + 3)}{Q_1(u + 2)Q_2(u - 1)} \right \},
\]
\[
T^3(u) = \begin{bmatrix}
1 & -1 & -2 + 2 + 2 \\
2 & 2 & 3
\end{bmatrix}
= P_1(u - 2) P_1(u) \left \{P_1(u + 4) \frac{Q_1(u - 3)Q_2(u + 2)}{Q_1(u + 3)Q_2(u - 2)}
- P_1(u + 4) \frac{Q_1(u - 1)Q_2(u + 2)}{Q_1(u + 3)Q_2(u - 2)}
- P_1(u + 2) \frac{Q_1(u - 3)Q_2(u + 4)}{Q_1(u + 3)Q_2(u - 2)} + P_1(u + 2) \frac{Q_1(u - 1)Q_2(u + 4)}{Q_1(u + 3)Q_2(u - 2)} \right \}
\left \} + \frac{T_{(23)}(u)}{P_1(u + 2)}
\Rightarrow
\]

Remark: If we drop the $u$ dependence of (3.16) and (3.17) for $p_1 = p_2 = \cdots =$
\[ p_{r+1} = 1, p_{r+2} = p_{r+3} = \cdots = p_{r+s+2} = -1, \]

they reduce to classical Jacobi-Trudi and Giambelli formulae for \( sl(r+1|s+1) \) \([28, 40]\), which give us classical (super) characters. This fact confirms ‘character limit’ \([1]\) of the eigenvalue formula for the transfer matrix.

**Remark 2:** In the case \( \lambda = \phi \) and \( s = -1 \), (3.16) and (3.17) reduce to the quantum analogue of Jacobi-Trudi and Giambelli formulae for \( sl_{r+1} \) \([7]\), which give us classical (super) characters. This fact confirms ‘character limit’ \([1]\) of the eigenvalue formula for the transfer matrix.

**Remark 3:** (3.16) and (3.17) have the same form as the quantum Jacobi-Trudi and Giambelli formulae for \( U_q(B_n^{(1)}) \) \([5]\), but the function \( T^a(u) \) is quite different as we can be easily seen from (3.10) and (3.11).

The following Theorem is a generalization of the Theorem in reference \([21]\). We will present a detailed proof here partly because it is essential in the analytic Bethe ansatz and partly because for reader’s convenience.

**Theorem 3.1** For any integer \( a \), the function \( T^a(u) \) is free of poles under the condition that the Bethe ansatz equation (3.1) is valid.

Proof. For simplicity, we assume that the vacuum parts are formally trivial, that is, the left hand side of the Bethe ansatz equation (3.1) is constantly \(-1\). We prove that \( T^a(u) \) is free of color \( b \) pole, namely, \( \text{Res}_{u=u_1^{(b)}+} \ldots T^a(u) = 0 \) for any \( b \in J - \{ r + s + 2 \} \) under the condition that the Bethe ansatz equation (3.1) is valid. The function \( z(c; u) = [c]_u \) with \( c \in J \) has the color \( b \) pole only for \( c = b \) or \( b+1 \), so we shall trace only \( b \) or \( b+1 \). Denote \( S_k \) the partial sum of \( T^a(u) \), which contains \( k \) boxes among \( b \) or \( b+1 \). Apparently, \( S_0 \) does not have color \( b \) pole.

Now we examine \( S_1 \), which is the summation of the tableaux of the following form

\[
\begin{array}{ccc}
\xi & b & \xi \\
b & \xi & b+1 \\
\zeta & \zeta & \\
\end{array}
\]

(3.22)

where \( \xi \) and \( \zeta \) are columns whose total length are \( a - 1 \) and they do not involve \( b \) and \( b+1 \). Thanks to the relation (3.8), color \( b \) residues in these tableaux (3.22) cancel each other under the Bethe ansatz equation (3.1). Then we deal with \( S_k \) only for \( 2 \leq k \leq a \) from now on.

The case \( b, b+1 \in J_+ \) : In this case, only the case for \( k = 2 \) should be considered because \( b \) or \( b+1 \) appear at most twice in one column. \( S_2 \) is the summation of the tableaux of the following form

\[
\begin{array}{ccc}
\xi & b & v \\
b & v-2 & \\
b+1 & v & \\
\zeta & & \\
\end{array}
\]

(3.23)
where \( \xi \) and \( \zeta \) are columns whose total length are \( a - 2 \), which do not involve \( b \) and \( b + 1 \): \( v = u + h_1 \); \( h_1 \) is some shift parameter; the function \( X_1 \) does not contain the function \( Q_b \). Obviously, \( S_2 \) is free of color \( b \) pole.

The case \( b \in J_+, b + 1 \in J_- : S_k(k \geq 2) \) is the summation of the tableaux of the following form

\[
\begin{array}{c|ccccc}
\xi & \overset{v}{\cdots} & b & v-2 & b+1 & \overset{v-2k+2}{\zeta} \\
\hline
\end{array}
\]

\[
= \frac{Q_{b-1}(v + \sum_{j=1}^{b-1} p_j + 2)Q_b(v + \sum_{j=1}^{b-1} p_j - 2k + 2)}{Q_{b-1}(v + \sum_{j=1}^{b-1} p_j)Q_b(v + \sum_{j=1}^{b-1} p_j + 2)} X_2
\tag{3.24}
\]

and

\[
\begin{array}{c|ccccc}
\xi & \overset{v}{\cdots} & b+1 & v-2 & b+1 & \overset{v-2k+2}{\zeta} \\
\hline
\end{array}
\]

\[
= \frac{Q_b(v + \sum_{j=1}^{b-1} p_j - 2k + 1)Q_{b+1}(v + \sum_{j=1}^{b-1} p_j + 2)}{Q_b(v + \sum_{j=1}^{b-1} p_j + 1)Q_{b+1}(v + \sum_{j=1}^{b-1} p_j - 2k + 2)} X_2
\tag{3.25}
\]

where \( \xi \) and \( \zeta \) are columns with total length \( a - k \), which do not involve \( b \) and \( b + 1 \); \( v = u + h_2 \); \( h_2 \) is some shift parameter; the function \( X_2 \) does not contain the function \( Q_b \). Obviously, color \( b \) residues in (3.24) and (3.25) cancel each other under the Bethe ansatz equation (3.1).

The case \( b \in J_-, b + 1 \in J_+ : S_k(k \geq 2) \) is the summation of the tableaux of the following form

\[
\begin{array}{c|ccccc}
\xi & \overset{v}{\cdots} & b & v-2k+4 & b & v-2k+2 \\
\hline
\end{array}
\]

\[
= \frac{Q_{b-1}(v + \sum_{j=1}^{b-1} p_j - 2k + 2)Q_b(v + \sum_{j=1}^{b-1} p_j + 1)}{Q_{b-1}(v + \sum_{j=1}^{b-1} p_j)Q_b(v + \sum_{j=1}^{b-1} p_j - 2k + 1)} X_3
\tag{3.26}
\]

\[
\times \frac{Q_{b+1}(v + \sum_{j=1}^{b-1} p_j - 2k)}{Q_{b+1}(v + \sum_{j=1}^{b-1} p_j - 2k + 2)} X_3
\]

\[12\]
and

\[
\begin{align*}
\begin{array}{c}
\xi \\
\vdots \\
\beta \\
\vdots \\
\zeta \\
\end{array}
\quad v
\quad =
\quad \frac{Q_{b-1}(v + \sum_{j=1}^{b-1} p_j - 2k)Q_{b}(v + \sum_{j=1}^{b-1} p_j + 1)}{Q_{b-1}(v + \sum_{j=1}^{b-1} p_j)Q_{b}(v + \sum_{j=1}^{b-1} p_j - 2k + 1)}X_3 \\
\end{align*}
(3.27)
\]

where \( \xi \) and \( \zeta \) are columns with total length \( a - k \), which do not involve \( b \) and \( b + 1 \); \( v = u + h_3 \); \( h_3 \) is some shift parameter; the function \( X_3 \) does not contain the function \( Q_b \). Obviously, color \( b \) residues in (3.26) and (3.27) cancel each other under the Bethe ansatz equation (3.1).

The case \( b, b + 1 \in J_- : S_k(k \geq 2) \) is the summation of the tableaux of the following form

\[
\begin{array}{c}
\xi \\
\vdots \\
\beta \\
\vdots \\
\zeta \\
\end{array}
\quad v
\quad =
\quad \frac{Q_{b-1}(v + \sum_{j=1}^{b-1} p_j - 2n)Q_{b}(v + \sum_{j=1}^{b-1} p_j + 1)}{Q_{b-1}(v + \sum_{j=1}^{b-1} p_j)Q_{b}(v + \sum_{j=1}^{b-1} p_j - 2n + 1)}
\times
\frac{Q_{b}(v + \sum_{j=1}^{b-1} p_j - 2k - 1)Q_{b+1}(v + \sum_{j=1}^{b-1} p_j - 2n)}{Q_{b}(v + \sum_{j=1}^{b-1} p_j - 2n - 1)Q_{b+1}(v + \sum_{j=1}^{b-1} p_j - 2k)}X_4, \quad 0 \leq n \leq k
\]
(3.28)

where \( \xi \) and \( \zeta \) are columns with total length \( a - k \), which do not involve \( b \) and \( b + 1 \); \( v = u + h_4 \); \( h_4 \) is some shift parameter and is independent of \( n \); the function \( X_4 \) does not have color \( b \) pole and is independent of \( n \). \( f(k, n, \xi, \zeta, u) \) has color \( b \) poles at \( u = -h_4 - \sum_{j=1}^{b-1} p_j + 2n - 1 + u^{(b)}_l \) and \( u = -h_4 - \sum_{j=1}^{b-1} p_j + 2n + 1 + u^{(b)}_l \) for \( 1 \leq n \leq k - 1 \); at \( u = -h_4 - \sum_{j=1}^{b-1} p_j + 1 + u^{(b)}_l \) for \( n = 0 \); at \( u = -h_4 - \sum_{j=1}^{b-1} p_j + 2k - 1 + u^{(b)}_l \) for \( n = k \). Evidently, color \( b \) residue at \( u = -h_4 - \sum_{j=1}^{b-1} p_j + 2n + 1 + u^{(b)}_l \) in \( f(k, n, \xi, \zeta, u) \) and \( f(k, n + 1, \xi, \zeta, u) \) for \( 0 \leq n \leq k - 1 \) cancel each other under the Bethe ansatz equation (3.1). Thus, under the Bethe ansatz equation (3.1), \( \sum_{n=0}^{k} f(k, n, \xi, \zeta, u) \) is free of color \( b \) poles, so is \( S_k \).  

Applying Theorem 3.1 to (3.16), one can show that \( T_{\lambda \subset \mu}(u) \) is free of poles under the Bethe ansatz equation (3.1). Thus each term in \( T_{\lambda \subset \mu}(u) \) has a counterterm which
Figure 2: The ‘Bethe-strap’ structure of the function $\mathcal{T}_{(2^1,1^1)}(u)$ for the Lie superalgebra $sl(2|1)$ with the grading $p_1 = 1, p_2 = -1, p_3 = 1$: The pair $(a, b)$ denotes the common pole $u_k^{(a)} + b$ of the pair of the tableaux connected by the arrow. This common pole vanishes under the Bethe ansatz equation. The topmost tableau corresponds to the ‘highest weight vector’, which is called the ‘top term’.

cancel the common pole under the Bethe ansatz equation (3.1). Furthermore the set of all the terms in $\mathcal{T}_{\lambda \subset \mu}(u)$ forms ‘Bethe-strap’ structure, which bears a resemblance to a weight space diagram. See Figure 2 and the relation (3.18) for $\lambda = \phi$ and $\mu = (2^1,1^1)$ case; the diagram (3.9) for $\lambda = \phi$ and $\mu = (1^1)$ case. Consult the references [1, 13] for detailed accounts on the ‘Bethe-strap’.

4 Functional equations

Consider the following Jacobi identity:

$$D \left[ \begin{array}{c} b \\ b \end{array} \right] D \left[ \begin{array}{c} c \\ c \end{array} \right] - D \left[ \begin{array}{c} b \\ c \end{array} \right] D \left[ \begin{array}{c} c \\ b \end{array} \right] = D \left[ \begin{array}{c} b \\ c \\ c \end{array} \right] D, \quad b \neq c$$

(4.1)
where $D$ is the determinant of a matrix and $D\begin{bmatrix} a_1 & a_2 & \cdots \\ b_1 & b_2 & \cdots \end{bmatrix}$ is its minor removing $a_\alpha$’s rows and $b_\beta$’s columns. Set $\lambda = \phi$, $\mu = (m^a)$ in (3.16). From the relation (4.1), we have

$$T_m^a(u-1)T_m^a(u+1) = T_{m+1}^a(u)T_{m-1}^a(u) + T_m^{a-1}(u)T_m^{a+1}(u) \quad (4.2)$$

where $a, m \geq 1$; $T_m^a(u) = T_{(m^a)}(u)$: $a, m \geq 1$; $T_m^a(u) = 1$: $m \geq 0$; $T_0^a(u) = 1$: $a \geq 0$. The functional equation (4.2) is a special case of Hirota bilinear difference equation [37]. For $s = -1$, this functional equation (4.2) reduces to a discretized Toda field equation of $A_r$ type. Furthermore, there is a restriction on it, which we consider below.

**Theorem 4.1** $T_{\lambda \subset \mu}^a(u) = 0$ if $\lambda \subset \mu$ contains a rectangular subdiagram with $r + 2$ rows and $s + 2$ columns.

Proof. Consider a tableau $b$ on this Young superdiagram $\lambda \subset \mu$. Decompose the set $J_+$ and $J_-$ (2.9) as a union of the disjoint sets:

$$J_+ = \bigcup_{k=1}^{\alpha} J_+^{(k)} : \quad J_+^{(k)} = \{i_1^{(k)}, i_2^{(k)}, \ldots, i_{a_k}^{(k)}\}, \quad (4.3)$$

$$J_- = \bigcup_{k=1}^{\alpha} J_-^{(k)} : \quad J_-^{(k)} = \{j_1^{(k)}, j_2^{(k)}, \ldots, j_{b_k}^{(k)}\}, \quad (4.4)$$

where we assumed, for any $k \in \{1, 2, \ldots, \alpha\}$,

$$i_\gamma^{(k)} = \sum_{\delta=1}^{k-1} (a_\delta + b_\delta) + \gamma : \quad \gamma \in \{1, 2, \ldots, a_k\}, \quad (4.5)$$

$$j_\gamma^{(k)} = \sum_{\delta=1}^{k-1} (a_\delta + b_\delta) + a_k + \gamma : \quad \gamma \in \{1, 2, \ldots, b_k\}. \quad (4.6)$$

Note that $J_+^{(1)} = \phi$ ($a_1 = 0$), if the minimal element in the set $J$ is a member of the set $J_-$; $J_-^{(\alpha)} = \phi$ ($b_\alpha = 0$), if the maximal element in the set $J$ is a member of the set $J_+$. On this rectangular subdiagram, consider a strip, which is a union of $a_k \times 1$ rectangular subdiagrams, $1 \times b_k$ rectangular subdiagrams and $1 \times 1$ square subdiagram. Fill this strip by the elements $\{i_t^{(k)}, j_t^{(k)}\}$ of $J$ so as to meet the admissibility conditions (i), (ii) and (iii) (see, Figure 3). For any $k \in \{1, 2, \ldots, \alpha\}$, we find

$$h_t^{(k)} \geq i_t^{(k)} : \quad t \in \{1, 2, \ldots, a_k\}; \quad (4.7)$$

$$i_t^{(k)} \geq j_t^{(k)} : \quad t \in \{1, 2, \ldots, b_k\}. \quad (4.8)$$

If $J_-^{(\alpha)} \neq \phi$ ($b_\alpha \neq 0$), there is no admissible element $y \in J$ since $l_t^{(\alpha)} \geq j_t^{(\alpha)} = r + s + 2 \in J_-$. If $J_-^{(\alpha)} = \phi$ ($b_\alpha = 0$), there is no admissible element $y \in J$ since
Figure 3: a strip in \((r + 2) \times (s + 2)\) rectangular subdiagram

\[ h^{(\alpha)}_{a} \geq i^{(\alpha)}_{a} = r + s + 2 \in J_{+}. \] Then there is no admissible tableau on this Young superdiagram.

Remark: The spectrum of fusion model was discussed in references [41, 42] from the point of view of representation theory and the corresponding theorem was discussed. As a corollary, we have

\[ T_{a}^{r+1}(u) = 0 \quad \text{for} \quad a \geq r + 2 \quad \text{and} \quad m \geq s + 2. \quad (4.9) \]

Applying the relation (4.9) to (4.2), we obtain

\[ T_{m}^{r+1}(u-1)T_{m+1}^{r+1}(u+1) = T_{m+1}^{r+1}(u)T_{m-1}^{r+1}(u) \quad m \geq s + 2, \quad (4.10) \]

\[ T_{a}^{r+1}(u-1)T_{a+1}^{r+1}(u+1) = T_{a+1}^{r+1}(u)T_{a-1}^{r+1}(u) \quad a \geq r + 2. \quad (4.11) \]

5 On the equivalence of the Bethe ansatz equations

Bares et. al. [25] showed that Lai’s [43] representation of the Bethe ansatz equation on the supersymmetric \(t-J\) model is equivalent to Sutherland ‘s one [44] under the
particle-hole transformation. Moreover, following reference [25], Essler and Korepin [26] showed that Sutherland’s [44] representation of the Bethe ansatz equation on the supersymmetric $t - J$ model is equivalent to the one originate from the grading $(p_1, p_2, p_3) = (-1, 1, -1)$ for Lie superalgebra $sl(1|2)$. Then the equivalence of three different sets of Bethe ansatz equations on the supersymmetric $t - J$ model, which originate from $\frac{3}{12}! = 3$ different gradings for $sl(1|2)$ was established. Futhermore Essler et. al. [27] established the equivalence of six different sets of Bethe ansatz equations on the supersymmetric extended Hubbard model, which originate from $\frac{4}{2!2!} = 6$ different gradings for Lie superalgebra $sl(2|2)$ (see, Figure 1). Now, following reference [25], we discuss relations among the sets of Bethe ansatz equations (3.1) for different $\frac{(r+s+2)!}{(r+1)!(s+1)!}$ gradings $\{p_j\}$ (2.11) or different sets of simple root systems inequivalent under the Weyl group $W(G)$ of Lie superalgebra $sl(r + 1|s + 1)$.

In this section, we assume that $q = 1$. For some $b$ ($2 \leq b \leq r + s$), we assume $p_b p_{b+1} = -1$. Namely, $b$ th simple root $\alpha_b$ is an odd root with $(\alpha_b|\alpha_b) = 0$. In this case, $b$ th Bethe ansatz equation in (3.1) has the following form

$$1 = \frac{Q_{b-1}(u^{(b)}_k - p_b)Q_{b+1}(u^{(b)}_k - p_{b+1})}{Q_{b-1}(u^{(b)}_k + p_b)Q_{b+1}(u^{(b)}_k + p_{b+1})}, \quad k = 1, 2, \ldots, N_b. \quad (5.1)$$

Define the polynomial

$$f(z) = Q_{b-1}(z + p_b)Q_{b+1}(z + p_{b+1}) - Q_{b-1}(z - p_b)Q_{b+1}(z - p_{b+1}). \quad (5.2)$$

Among the roots of the equation $f(z) = 0$, $N_b$ of which are $\{u^{(b)}_k\}_{1 \leq k \leq N_b}$. So \{f(u^{(b)}_k) = 0\}_{1 \leq k \leq N_b} reproduces the Bethe ansatz equation (5.1). Let the rest of the roots be $\{\tilde{u}^{(b)}_k\}_{1 \leq k \leq \tilde{N}_b}$. Then \{f(\tilde{u}^{(b)}_k) = 0\}_{1 \leq k \leq \tilde{N}_b} reduces to the Bethe ansatz equation of the form

$$1 = \frac{Q_{b-1}(\tilde{u}^{(b)}_k + p_b)Q_{b+1}(\tilde{u}^{(b)}_k + p_{b+1})}{Q_{b-1}(\tilde{u}^{(b)}_k - p_b)Q_{b+1}(\tilde{u}^{(b)}_k - p_{b+1})}, \quad k = 1, 2, \ldots, \tilde{N}_b. \quad (5.3)$$

Thanks to the residue theorem, the following relation holds

$$\sum_{j=1}^{N_b} \frac{1}{2\pi i} C_j \left[ d \frac{\log z - u_j^{(b-1)} - p_b}{z - u_j^{(b-1)} + p_b} \right] \left[ d \frac{\log f(z)}{dz} \right] = \sum_{j=1}^{N_b} \frac{\log u_j^{(b)} - u_j^{(b-1)} - p_b}{u_j^{(b)} - u_j^{(b-1)} + p_b} \quad (5.4)$$

where $C_j$ denotes contour around $u_j^{(b)}$. We assume the branch cut of the logarithm in the left hand side of (5.4) extends from $u_i^{(b-1)} - p_b$ to $u_i^{(b-1)} + p_b$. The left hand
side of the relation (5.4) can be rewritten as follows

\[-\sum_{j=1}^{N_b} \frac{1}{i} \log \frac{\tilde{u}_j^{(b)} - u_i^{(b-1)} - p_b}{\tilde{u}_j^{(b)} - u_i^{(b-1)} + p_b} + \frac{1}{i} \log \frac{f(u_i^{(b-1)} + p_b)}{f(u_i^{(b-1)} - p_b)}. \] (5.5)

Then the following relation holds

\[-1 = \frac{Q_{b-1}(u_i^{(b-1)} - p_b) - p_b)Q(b_i^{(b-1)} - p_b)}{Q_{b-1}(u_i^{(b-1)} - p_b) - p_b)Q(b_i^{(b-1)} + p_b)}, \quad l = 1, 2, \ldots, N_{b-1}. \] (5.6)

where \( \tilde{Q}_b(u) = \prod_{j=1}^{N_b} (u - \tilde{u}_j^{(b)}). \) Noting that the relation

\[\frac{Q_{b-1}(u_i^{(b-1)} + p_b - p_b)}{Q_{b-1}(u_i^{(b-1)} - p_b - p_b)} = \frac{Q_{b-1}(u_i^{(b-1)} + 2p_b)}{Q_{b-1}(u_i^{(b-1)} - 2p_b)}, \] (5.7)

we find that the \( b - 1 \)th Bethe ansatz equation in (3.1) has the form:

\[-1 = (-1)^{\deg(\alpha_{b-1})} \frac{Q_{b-2}(u_i^{(b-1)} - p_b) - p_b)Q(b_i^{(b-1)} + 2p_b)Q(b_i^{(b-1)} - p_b)}{Q_{b-2}(u_i^{(b-1)} + p_b - p_b)Q(b_i^{(b-1)} - 2p_b)Q(b_i^{(b-1)} + p_b)}, \quad l = 1, 2, \ldots, N_{b-1}. \] (5.8)

Combining these two equations (5.8) and (5.6), we obtain

\[-1 = (-1)^{\deg(\alpha_{b-1})} \frac{Q_{b-2}(u_i^{(b-1)} - p_b) - p_b)Q(b_i^{(b-1)} - 2p_b)Q(b_i^{(b-1)} + p_b)}{Q_{b-2}(u_i^{(b-1)} + p_b - p_b)Q(b_i^{(b-1)} - 2p_b)Q(b_i^{(b-1)} + p_b)}, \quad l = 1, 2, \ldots, N_{b-1}. \] (5.9)

where \( \deg(\alpha_{b-1}) = \deg(\alpha_{b-1}) + 1 \mod 2. \) The following relation is valid

\[\sum_{j=1}^{N_b} \frac{1}{2\pi i} \int_{C_j} \frac{dz}{z} \log \frac{z - u_i^{(b+1)} - p_b + 1}{z - u_i^{(b+1)} + p_b + 1} \frac{d}{dz} \log f(z) = \sum_{j=1}^{N_b} \frac{1}{i} \log \frac{\tilde{u}_j^{(b)} - u_i^{(b+1)} - p_b + 1}{\tilde{u}_j^{(b)} - u_i^{(b+1)} + p_b + 1} \] (5.10)

\[= -\sum_{j=1}^{N_b} \frac{1}{i} \log \frac{\tilde{u}_j^{(b)} - u_i^{(b+1)} - p_b + 1}{\tilde{u}_j^{(b)} - u_i^{(b+1)} + p_b + 1} + \frac{1}{i} \log \frac{f(u_i^{(b+1)} + p_b + 1)}{f(u_i^{(b+1)} - p_b + 1)} \]

where \( C_j \) denotes contour around \( \tilde{u}_j^{(b)}. \) We assume the branch cut of the logarithm in left hand side of (5.10) extends from \( u_i^{(b+1)} - p_b + 1 \) to \( u_i^{(b+1)} + p_b + 1. \) This equation reduces to the following equation:

\[-1 = \frac{Q_{b+1}(u_i^{(b+1)} + 2p_b + 1)Q(b_i^{(b+1)} - 2p_b + 1)Q(b_i^{(b+1)} + 2p_b + 1)}{Q_{b+1}(u_i^{(b+1)} - 2p_b + 1)Q(b_i^{(b+1)} + 2p_b + 1)Q(b_i^{(b+1)} + 2p_b + 1)}, \quad l = 1, 2, \ldots, N_{b+1}. \] (5.11)
\( b + 1 \) th Bethe ansatz equation in (3.1) has the form:

\[
-1 = (-1)^{\text{deg}(\alpha_{b+1})} \frac{Q_b(u_i^{(b+1)} - p_{b+1})Q_{b+1}(u_i^{(b+1)} + 2p_{b+2})Q_{b+2}(u_i^{(b+1)} - p_{b+2})}{Q_b(u_i^{(b+1)} + p_{b+1})Q_{b+1}(u_i^{(b+1)} - 2p_{b+1})Q_{b+2}(u_i^{(b+1)} + p_{b+2})}, (5.12)
\]

\( l = 1, 2, \ldots, N_{b+1} \).

Combining these two equations (5.11) and (5.12), we obtain

\[
-1 = (-1)^{\text{deg}(\tilde{\alpha}_{b+1})} \frac{\tilde{Q}_b(u_i^{(b+1)} + p_{b+1})Q_{b+1}(u_i^{(b+1)} - 2p_{b+1})Q_{b+2}(u_i^{(b+1)} - p_{b+2})}{\tilde{Q}_b(u_i^{(b+1)} - p_{b+1})Q_{b+1}(u_i^{(b+1)} + 2p_{b+2})Q_{b+2}(u_i^{(b+1)} + p_{b+2})}, (5.13)
\]

where \( \text{deg}(\tilde{\alpha}_{b+1}) = \text{deg}(\alpha_{b+1}) + 1 \mod 2 \). Note that the set of the equations (5.8), (5.1) and (5.12) is transferred to the equivalent set of the equations (5.9), (5.3) and (5.13) under the following transformation:

\[
(p_b, p_{b+1}, N_b, \{ u_k^{(b)} \}) \rightarrow (-p_b, -p_{b+1}, \tilde{N}_b, \{ u_k^{(b)} \}). \quad (5.14)
\]

One can also develop similar argument for \( p_1p_2 = -1 \) case using the function

\[
f(z) = P_1(z + p_1)Q_2(z + p_2) - P_1(z - p_1)Q_2(z - p_2) \quad (5.15)
\]

instead of the function (5.2) and for \( p_{r+s+1}p_{r+s+2} = -1 \) case using the function

\[
f(z) = Q_{r+s}(z + p_{r+s+1}) - Q_{r+s}(z - p_{r+s+1}) \quad (5.16)
\]

instead of the function (5.2). Then the set of the Bethe ansatz equations (3.1) is transferred to the equivalent set of the Bethe ansatz equations under the transformation (5.14) for \( p_b p_{b+1} = -1 \). Therefore, taking notice of a change of the grading \( \{ p_j \} \) or odd simple root \( \alpha_b \) with \( \langle \alpha_b | \alpha_b \rangle = 0 \) and applying the transformation (5.14) repeatedly to the set of the Bethe ansatz equations (3.1) with any one of the grading \( \{ p_j \} \) for \( sl(r+1|s+1) \), one can get the set of the Bethe ansatz equations with any other grading \( \{ p_j \} \) for \( sl(r+1|s+1) \). Furthermore, we note that the transformation (5.14) corresponds to the reflection \( \omega_b \in SW(G) \) for odd simple root \( \alpha_b \) with \( \langle \alpha_b | \alpha_b \rangle = 0 \). This fact follows from the relations: \( \langle \omega_{\alpha_b}(\alpha_b) | \omega_{\alpha_b}(\alpha_{b+1}) \rangle = -(-p_{b+1}), \langle \omega_{\alpha_b}(\alpha_{b-1}) | \omega_{\alpha_b}(\alpha_{b-1}) \rangle = p_{b-1} + (-p_b) \), etc.

### 6 Summary and discussion

In the present paper, we have executed analytic Bethe ansatz based upon the Bethe ansatz equations (3.1) with any simple root systems of the Lie superalgebra \( sl(r+1|s+1) \). Pole-freeness of eigenvalue formula of transfer matrix in dressed
vacuum form was shown for a wide class of finite dimensional representations labeled by skew-Young superdiagrams. Functional relation has been given especially for the eigenvalue formulae of transfer matrices in dressed vacuum form labeled by rectangular Young superdiagrams, which is a special case of Hirota bilinear difference equation with a restrictive relation. There are earlier results [21] for the distinguished simple root system of $sl(r + 1|s + 1)$, many of which are special case of the results in the present paper. We discussed how the set of the Bethe ansatz equations for any simple root system of $sl(r + 1|s + 1)$ is related to the one for any other simple root system of $sl(r + 1|s + 1)$ under the particle-hole transformation. And then, we pointed out that the particle-hole transformation is connected with the reflection with respect to the element of the Weyl supergroup for odd simple root $\alpha$ with $\langle \alpha|\alpha \rangle = 0$.

It should be emphasized that our method explained in the present paper is still valid even if such factors like gauge factor, extra sign (different from $(-1)^{\text{deg}(\alpha_{\alpha})}$ in (3.1)), etc. appear in the Bethe ansatz equation (3.1) as long as such factors do not influence the analytical property of the right hand side of the Bethe ansatz equation (3.1).

In reference [12], functional relations for any fusion type transfer matrices associated with any (not always rectangular) Young diagrams of simple Lie algebra $A_r$ was given. Similar functional relations for suitable boundary conditions will be also valid for $sl(r + 1|s + 1)$ case.

In reference [45], coincidence between the free field realization of the generators of $U_q(G(1))$ and eigenvalue formulae [1] of transfer matrices in dressed vacuum form in the analytic Bethe ansatz was discussed associated with classical simple Lie algebras $G$. As for a Lie superalgebra $G$ case, nobody has studied such a relation so far. A deeper inspection will be desirable.

It will be interesting problems to extend a similar analysis discussed in this paper for other Lie superalgebras, such as $B(m|n), C(n)$ and $D(m|n)$.

Finally we note that functional relations among fusion transfer matrices at finite temperatures have been given in the preprint [46] quite recently using quantum transfer matrix approach. In addition, these functional relations are transformed into TBA equations without using string hypothesis.

Acknowledgments
The author would like to thank Professor A. Kuniba for encouragement. He also thanks Dr J. Suzuki for discussions.

References

[1] A. Kuniba and J. Suzuki, Commun. Math. Phys. 173, 225 (1995).

[2] N. Yu. Reshetikhin, Sov. Phys. JETP 57, 691 (1983).
[3] N. Yu. Reshetikhin, *Lett. Math. Phys.* **14**, 235 (1987).

[4] V. G. Drinfel’d, *Sov. Math. Dokl* **36**, 212 (1988).

[5] A. Kuniba, Y. Ohta and J. Suzuki, *J. Phys. A Math. Gen.* **28**, 6211 (1995).

[6] A. Kuniba, T. Nakanishi and J. Suzuki, *Int. J. Mod. Phys.* **A9**, 5215 (1994).

[7] V. V. Bazhanov and N. Reshetikhin, *J. Phys. A Math. Gen.* **23**, 1477 (1990).

[8] I. Krichever, O. Lipan, P. Wiegmann and A. Zabrodin, hep-th/9604080; *Commun. Math. Phys.* **188**, 267 (1997).

[9] A. Kuniba, *J. Phys. A Math. Gen.* **27**, L113 (1994).

[10] A. Klümper and P. Pearce, *Physica* **A183**, 304 (1992).

[11] A. Kuniba and J. Suzuki, *J. Phys. A Math. Gen.* **28**, 711 (1995).

[12] O. Lipan, P. Wiegmann and A. Zabrodin, solv-int/9704015; *Mod. Phys. Lett.* **A12**, 1369 (1997).

[13] J. Suzuki, *Phys. Lett.* **A195**, 190 (1994).

[14] A. Kuniba, T. Nakanishi and J. Suzuki, *Int. J. Mod. Phys.* **A9**, 5267 (1994).

[15] A. Kuniba, S. Nakamura and R. Hirota, *J. Phys. A Math. Gen.* **29**, 1759 (1996).

[16] Z. Tsuboi and A. Kuniba, *J. Phys. A Math. Gen.* **29**, 7785 (1996).

[17] Z. Tsuboi, solv-int/9610011; *J. Phys. Soc. Jpn.* **66**, 3391 (1997).

[18] V. Kac, *Adv. Math.* **26**, 8 (1977).

[19] V. E. Korepin and F. H. L. Essler eds., *Exactly solvable models of strongly correlated electrons* (World Scientific, Singapore 1994).

[20] ^1^ V.K. Dobrev and V.B. Petkova, *Fortschr. d. Phys.* **35**, 537 (1987); ICTP Trieste preprint IC/85/29 (March 1985); V.V. Serganova, Appendix to the paper: D.A. Leites, M.V. Saveliev and V.V. Serganova, in: Proc. of Group Theoretical Methods in Physics, Yurmala, 1985 (Nauka, Moscow, 1985, in Russian) p. 377; English translation in VNU Sci. Press, Utrecht, 1986, p. 255.

[21] Z. Tsuboi, *J. Phys. A: Math. Gen.* **30**, 7975 (1997).

---

^1^ [a comment added on 12 December 2009] The notion of ‘odd reflection’ was introduced independently by these papers at the same time. We thank V. Dobrev for information on this.
[22] N. Yu. Reshetikhin and P. B. Wiegmann, *Phys. Lett.* B189, 125 (1987).

[23] P. P. Kulish, *J. Sov. Math* 35, 2648 (1986).

[24] P. P. Kulish and E. K. Sklyanin, *J. Sov. Math* 19, 1596 (1982).

[25] P. A. Bares, I. M. P. Carmelo, J. Ferrer and P. Horsch, *Phys. Rev.* B 46, 14624 (1992).

[26] F. H. L. Essler and V. E. Korepin, *Phys. Rev.* B 46, 9147 (1992).

[27] F. H. L. Essler, V. E. Korepin and K. Schoutens, cond-mat/9211001; *Int. J. Mod. Phys.* B 8, 3205 (1994).

[28] A. B. Balantekin and I. Bars, *J. Math. Phys.* 22, 1149 (1981).

[29] M. L. Nazarov, *Lett. Math. Phys* 21, 123 (1991).

[30] H. Yamane, *Publ. RIMS, Kyoto Univ.* 30, 15 (1994).

[31] H. Yamane, preprint q-alg/9603015 (1996).

[32] J. H. H. Perk and C. L. Schultz, in *Nonlinear Integrable Systems-Classical Theory and Quantum Theory*, eds. Jimbo M and Miwa T (World Scientific, Singapore 1983).

[33] J. H. H. Perk and C. L. Schultz, *Phys. Lett.* 84A, 407 (1981).

[34] C. L. Schultz, *Physica* A122, 71 (1983).

[35] A. Foerster and M. Karowski, *Nucl. Phys.* B 396, 611 (1993).

[36] F. H. L. Essler, V. E. Korepin and K. Schoutens, *Phys. Rev. Lett.* 68, 2960 (1992).

[37] R. Hirota, *J. Phys. Soc. Jpn.* 50, 3787 (1981).

[38] M. Kashiwara and T. Nakashima, *J. Algebra* 165, 295 (1994).

[39] T. Nakashima, *Commun. Math. Phys.* 154, 215 (1993).

[40] P. Pragacz and A. Thorup, *Adv. Math.* 95, 8 (1992).

[41] T. Deguchi and P. P. Martin, *Int. J. Mod. Phys.* A 7 Suppl. 1A, 165 (1992).

[2][comment added on 12 December 2009] This preprint was published in *Publ. Res. Inst. Math. Sci.* 35, 321 (1999); errata: *Publ. Res. Inst. Math. Sci.* 37, 615 (2001).
[42] P. P. Martin and V. Rittenberg, *Int. J. Mod. Phys. A* **7 Suppl. 1B**, 707 (1992).

[43] C. K. Lai, *J. Math. Phys.* **15**, 1675 (1974).

[44] B. Sutherland, *Phys. Lett. B* **12**, 3795 (1975).

[45] E. Frenkel and N. Reshetikhin, *Commun. Math. Phys.* **178**, 237 (1996).

[46] G. Jüttner, A. Klümper and J. Suzuki, *hep-th/9707074; Nucl. Phys. B* **512** 581 (1998).