MATROIDS FROM HYPERSIMPLEX SPLITS

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Abstract. A class of matroids is introduced which is very large as it strictly contains all paving matroids as special cases. As their key feature these split matroids can be studied via techniques from polyhedral geometry. It turns out that the structural properties of the split matroids can be exploited to obtain new results in tropical geometry, especially on the rays of the tropical Grassmannians.

1. Introduction

The purpose of this paper is to introduce, to characterize and to exploit a new class of matroids, which we call split matroids. We will argue that there are good reasons to study these matroids for the sake of matroid theory itself. Additionally, however, they also give rise to a large and interesting class of tropical linear spaces. In this way we can use split matroids to answer some questions which previously arose in the investigation of tropical Grassmannians [SS04] and Dressians [HJJS09, HJS12].

The split matroids are motivated via polyhedral geometry in the following way. For a given matroid $M$ the convex hull of the characteristic vectors of the bases of $M$ is the matroid polytope $P(M)$. The hypersimplices $\Delta(d,n)$ are the matroid polytopes corresponding to the uniform matroids $U_{d,n}$. If $M$ has rank $d$ and $n$ elements, the matroid polytope $P(M)$ is a subpolytope of $\Delta(d,n)$. Studying matroids in polyhedral terms goes back to Edmonds [Edm70].

A split of a polytope is a subdivision with precisely two maximal cells. These subdivisions are necessarily regular, and the cells are matroid polytopes. The hyperplane spanned by the intersection of the two maximal cells is the corresponding split hyperplane. Clearly this hyperplane determines the split, and it yields a facet of both maximal cells. As our first contribution we show the following converse. Each facet of a matroid polytope $P(M)$ corresponds to either a hypersimplex facet or a hypersimplex split (Proposition 7). We call the latter the split facets of $P(M)$. The hypersimplex facets correspond to matroid deletions and contractions, and the hypersimplex splits have been classified in [HJ08]. Now the matroid $M$ is a split matroid if the split facets of $P(M)$ satisfy a compatibility condition. We believe that these matroids are interesting since they form a large class but feature...
stronger combinatorial properties than general matroids. “Large” means that they comprise
the paving matroids and their duals as special cases (Theorem 19). It is conjectured that
asymptotically almost all matroids are paving matroids [MNWW11] and [Oxl11, 15.5.8]. In
particular, this would imply that almost all matroids are split. Section A in the appendix
provides statistical data based on a census of small matroids which has been obtained by
Matsumoto, Moriyama, Imai and Bremner [MMIB12].

We characterize the split matroids in terms of deletions and contractions, i.e., in pure
matroid language (Theorem 11 and Proposition 15). This way it becomes apparent that the
basic concepts of matroid splits and split matroids make several appearances in the matroid
literature. For instance, a known characterization of paving matroids implicitly makes use of
this technique; see [Oxl11, Prop. 2.1.24]. Splits also occur in a recent matroid realizability
result by Chatelain and Ramírez Alfonsín [CRA14]. Yet, to the best of our knowledge, so
far split matroids have not been recognized as an interesting class of matroids in their own
right.

One motivation to study matroid polytopes comes from tropical geometry; see Maclagan
and Sturmfels [MS15]. Tropical geometry is related to the study of an algebraic variety
defined over some field with a discrete valuation, and a tropical variety is the image of
such a variety under the valuation map. In particular, a tropical linear space corresponds
to a polytopal subdivision of the hypersimplices where each cell is a matroid polytope;
see De Loera, Rambau and Santos [DLRS10] for general background on subdivisions of
polytopes. The Dressian Dr(\(d,n\)) is the polyhedral fan of lifting functions for the (regular)
matroid subdivisions of \(\Delta(d,n)\). By definition this is a subfan of the secondary fan. In
general, Dr(\(d,n\)) has maximal cones of various dimensions, i.e., it is not pure. In work of
Dress and Wenzel [DW92] these lifting functions occur as “valuated matroids”. Using split
matroids we provide exact asymptotic bounds for \(\dim \text{Dr}(d,n)\) (Theorem 31).

A tropical linear space is realizable if it arises as the tropicalization of a classical linear space.
It is known from work of Speyer [Spe05, Spe09] that the realizability of tropical linear spaces
is related with the realizability of matroids. Here we give a first characterization of matroid
realizability in terms of certain tropical linear spaces (Theorem 35). The subset of Dr(\(d,n\))
which corresponds to the realizable tropical linear spaces is the tropical Grassmannian.
The latter is also equipped with a fan structure, which is inherited from the Gröbner fan
of the \((d,n)\)-Plücker ideal. Yet it is still quite unclear how these two fan structures are
related. Here we obtain a new structural result by showing that, via split matroids, one can
construct very many non-realizable tropical linear spaces which correspond to rays of the
Dressian (Theorem 41). It was previously unknown if any such ray exists. The Dressian
rays correspond to those tropical linear spaces which are most degenerate. Once they are
known it is “only” necessary to determine the common refinements among them to describe
the entire Dressians. In this way the rays yield a condensed form of encoding. It is worth
noting that the Dressians have far fewer rays than maximal cones. For instance, Dr(3,8)
has 4748 maximal cones but only twelve rays, up to symmetry [HJS12, Theorem 31].
2. Matroid polytopes and their facets

Throughout this paper let $M$ be a matroid of rank $d$ with ground set $[n] = \{1, 2, \ldots, n\}$. Frequently, we use the term $(d,n)$-matroid in this situation. We quickly browse through the basic definitions; further details about matroid theory can be found in the books of Oxley [Oxl11] and White [Whi86]. We use the notation of Oxley [Oxl11] for specific matroids and operations. The matroid $M$ is defined by its bases. They are $d$-element subsets of $[n]$ which satisfy an abstract version of the basis exchange condition from linear algebra. Subsets of bases are called independent, and a dependent set which is minimal with respect to inclusion is a circuit. An element $e \in [n]$ is a loop if it is not contained in any basis, and it is a coloop if it is contained in all the bases. Let $S$ be a subset of $[n]$. Its rank, denoted by $\text{rk}(S)$, is the maximal size of an independent set contained in $S$. The set $S$ is a flat if for all $e \in [n] - S$ we have $\text{rk}(S + e) = \text{rk}(S) + 1$. The entire ground set and, in the case of loop-freeness, also the empty set are flats; the other flats are called proper flats. The set of flats of $M$, partially ordered by inclusion, forms a geometric lattice, the lattice of flats. The matroid $M$ is connected if there is no separator set $S \subsetneq [n]$ with $\text{rk}(S) + \text{rk}([n] - S) = d$. A connected $(d,n)$-matroid decomposes in a direct sum of an $(r,m)$-matroid $M'$ and a rank $d-r$ matroid $M''$ on $\{m + 1, \ldots, n\}$, i.e, a basis is the union of a basis of $M$ and a basis of $N$. We write $M' \oplus M''$ for the direct sum.

For a flat $F$ of rank $r$ we define the restriction $M|F$ of $F$ with respect to $M$ as the matroid on the ground set $F$ whose bases are the sets in the collection

$$\{\sigma \cap F \mid \sigma \text{ basis of } M \text{ and } #(\sigma \cap F) = r\} \ .$$

Dually, the contraction $M/F$ of $F$ with respect to $M$ is the matroid on the ground set $[n] - F$ whose bases are given by

$$\{\sigma \setminus F \mid \sigma \text{ basis of } M \text{ and } #(\sigma \cap F) = r\} \ .$$

The restriction $M|F$ is a matroid of rank $r$, while the contraction $M/F$ is a matroid on the complement of rank $d-r$.

Via its characteristic function on the elements, a basis of $M$ can be read as a 0/1-vector of length $n$ with exactly $d$ ones. The joint convex hull of all such points in $\mathbb{R}^n$ is the matroid polytope $P(M)$ of $M$. A basic reference to polytope theory is Ziegler’s book [Zie00]. It is immediate that the matroid polytope of any $(d,n)$-matroid is contained in the $(n-1)$-dimensional simplex

$$\Delta = \left\{ x \in \mathbb{R}^n \left| x_1 \geq 0, x_2 \geq 0, \ldots, x_n \geq 0, \sum_{i=1}^n x_i = d \right. \right\} \ .$$

Combinatorial properties of $M$ directly translate into geometric properties of $P(M)$ and vice versa. For instance, Edmonds [Edm70, (8) and (9)] gave the exterior description

$$P(M) = \left\{ x \in \Delta \left| \sum_{i \in F} x_i \leq \text{rk}(F), \text{ where } F \text{ ranges over the set of flats} \right. \right\} \ .$$
of the matroid polytope $P(M)$ in terms of the flats. The set

$$P_M(F) := \left\{ x \in P(M) \ \middle| \sum_{i \in F} x_i = \text{rk}(F) \right\}$$

is the face of $P(M)$ defined by the flat $F$. Clearly, some flats lead to redundant inequalities. A flacet of $M$ is a flat which defines a facet of $P(M)$ and which is minimal with respect to inclusion among all flats that define the same facet. They have been characterized in purely combinatorial terms by Fujishige [Fuj84, Theorems 3.2 and 3.4] and, independently, by Feichtner and Sturmfels [FS05, Propositions 2.4 and 2.6] as follows.

**Proposition 1.**

(i) The dimension of $P(M)$ equals $n$ minus the number of connected components of $M$.

(ii) A proper flat $F$ whose restriction $M|F$ and contraction $M/F$ both are connected is a flacet of $M$.

(iii) For each proper flat $F$ we have $P_M(F) = P(M|F) \times P(M/F) = P(M|F \oplus M/F)$.

**Remark 2.** Proposition 1(ii) characterizes the facets of a connected matroid. For a disconnected matroid the notion of a flacet is somewhat subtle. First, in the disconnected case there are proper hyperplanes which contain the entire matroid polytope. Such a hyperplane is not facet defining and the corresponding flat is not a flacet. Second, for any given facet the defining inequality is never unique. In our definition we choose a specific representative by picking the inclusion minimal flat. If a flat is a direct sum $F \oplus G$ then $P_M(F \oplus G)$ is the intersection of the two faces $P_M(F)$ and $P_M(G)$. In particular, the restriction to a flacet is always connected, while the contraction is not.

The hypersimplex $\Delta(d,n)$ is the matroid polytope of the uniform matroid $U_{d,n}$ of rank $d$ on $n$ elements. Its vertices are all the 0/1-vectors of length $n$ with exactly $d$ ones. As $\Delta(d,n)$ is the intersection of the unit cube $[0,1]^n$ with the hyperplane $\sum x_i = d$, the $2n$ facets of $[0,1]^n$ give rise to a facet description for $\Delta(d,n)$. In this case the flats are the $n$ flats with one element. The matroid polytope of any $(d,n)$-matroid is a subpolytope of $\Delta(d,n)$. The following converse, obtained by Gel’fand, Goresky, MacPherson and Serganova, is a fundamental characterization. The vertex-edge graph of the $(d,n)$-hypersimplex is called the Johnson graph $J(d,n)$. This is a $[d \cdot (n - d)]$-regular undirected graph with $\binom{n}{d}$ nodes; each of its edges corresponds to the exchange of two bits.

**Proposition 3** ([GGMS87, Theorem 4.1]). A subpolytope $P$ of $\Delta(d,n)$ is a matroid polytope if and only if the vertex-edge graph of $P$ is a subgraph of the Johnson graph $J(d,n)$.

In the subsequent sections we will be interested in polytopal subdivisions of hypersimplices and, more generally, arbitrary matroid polytopes. The following concept is at the heart of our deliberations. A split of a polytope $P$ is a polytopal subdivision $\Sigma$ of $P$ with exactly two maximal cells. The two maximal cells share a common codimension-1-cell, and its affine span is the split hyperplane of $\Sigma$. 


Proposition 4 ([HJ08, Lemma 5.1]). For any proper non-empty subset \( S \subseteq [n] \) and any positive integer \( \mu < d \) with \( d - \#S < \mu < n - \#S \) the \((S, \mu)\)-hyperplane equation
\[
\mu \sum_{i \in S} x_i = (d - \mu) \sum_{j \not\in S} x_j
\]
defines a split of \( \Delta(d, n) \). Conversely, each split of \( \Delta(d, n) \) arises in this way.

The split equation above is given in its homogeneous form. Since the hypersimplices are not full-dimensional this can be rewritten in many ways. For instance, taking \( \sum_i x_i = d \) into account yields the inhomogeneous equation
\[
\sum_{i \in S} x_i = d - \mu ,
\]
which is equivalent to (2). Note that the equation (3) has a similar shape as the inequalities in the exterior description (1) of the matroid polytopes. A direct computation shows that the intersection of \( \Delta(d, n) \) with the \((S, \mu)\)-hyperplane is the product of hypersimplices
\[
\Delta(d - \mu, S) \times \Delta(\mu, [n] - S) ,
\]
where we use a complementary pair of subsets of \([n]\) (instead of cardinalities) in the second arguments of the hypersimplex notation to fix the embedding into \( \Delta(d, n) \) as a subpolytope.

Remark 5. By [HJ08, Observation 3.1] a hyperplane \( H \) which separates an arbitrary polytope \( P \) defines a split of \( P \) if and only if \( H \) does not intersect any edge of \( P \) in its relative interior: Clearly, if \( H \) separates any edge of \( P \) it does not define a subdivision of \( P \) without new vertices. Conversely, if no edge of \( P \) gets separated then \( H \) induces a split with the two maximal cells \( P \cap H^+ \) and \( P \cap H^- \), where \( H^+ \) and \( H^- \) are the two affine halfspaces defined by \( H \). In view of Proposition 3 we conclude that the (maximal) cells of any split of a hypersimplex form matroid polytopes. See also [HJJS09, Proposition 3.4].

We want to express Proposition 4 in terms of matroids and their flats.

Lemma 6. Let \( F \) be a proper flat such that \( 0 < \text{rk}(F) < \#F \). If there is an element \( e \) in \([n] - F\) which is not a coloop then the \((F, d - \text{rk}(F))\)-hyperplane defines a split of \( \Delta(d, n) \). In this case the intersection of \( \Delta(d, n) \) with that split hyperplane equals
\[
\Delta(\text{rk}(F), F) \times \Delta(d - \text{rk}(F), [n] - F) ,
\]
and, in particular, the face \( P_M(F) = P(M|F) \times P(M/F) \) is the intersection of \( P(M) \) with the split hyperplane.

Proof. Pick an element \( e \in [n] \) in the complement of \( F \) which is not a coloop. This yields \( \text{rk}([n] - e) = d \), whence the submodularity of the rank function implies
\[
\#F - \text{rk}(F) \leq \#F - \text{rk}(F) + \#([n] - (F + e)) - \text{rk}([n] - (F + e))
\]
\[
\leq \#([n] - e) - \text{rk}([n] - e)
\]
\[
= n - 1 - d.
\]
With our assumption $0 < \text{rk}(F) < \#F$ we obtain
\[ d - \#F < d - \text{rk}(F) \leq n - \#F - 1, \]
which is precisely the condition in Proposition 4 for $S = F$ and $\mu = d - \text{rk}(F)$. This means that the $(F, d - \text{rk}(F))$-hyperplane defines a split of $\Delta(d, n)$. The intersection with $\Delta(d, n)$ can be read off from (4). □

The value $d - \text{rk}(F)$ is determined by the flat $F$, whence we will shorten the notation of $(F, d - \text{rk}(F))$-hyperplane to $F$-hyperplane. Throughout the rest of this paper we will assume that $n \geq 2$, i.e., $M$ has at least two elements. If $M$ is additionally connected, this forces that $M$ does not have any loops or coloops. The relevance of the previous lemma for the investigation of matroid polytopes stems from the following observation.

**Proposition 7.** Suppose that $M$ is connected. Each facet of $P(M)$ is defined by the $F$-hyperplane for some flat $F$ with $0 < \text{rk}(F) < \#F$, or it is induced by one of the hypersimplex facets. In particular, the facets of $P(M)$ are either induced by hypersimplex splits or hypersimplex facets.

*Proof.* Consider an arbitrary facet $\Phi$ of the polytope $P(M)$. From (1) we know that $\Phi$ is either induced by an inequality of the form $\sum_{i \in F} x_i \leq \text{rk}(F)$ for some flat $F$ of $M$, or $\Phi$ corresponds to one of the non-negativity constraints. The latter yield hypersimplex facets, and the same also holds for the singleton flats. We are left with the case where $F$ has at least two elements.

The connectivity implies that $M$ has no coloops, as we assumed that $M$ has at least two elements. Suppose that $\text{rk}(F) = \#F$. Then the restriction $M|F$ to the flat consists of coloops and thus is disconnected. Since $M$ is connected, this implies that the hyperplane $\sum_{i \in F} x_i = \text{rk}(F)$ cuts out a face of codimension higher than one. A similar argument works if $\text{rk}(F) = 0$ as in this case the contraction $M/F$ is disconnected. We conclude that $0 < \text{rk}(F) < \#F$. Now the claim follows from Lemma 6. □

We call a facet $F$ a split facet if the $F$-hyperplane is a split of $\Delta(d, n)$. Notice that Lemma 6 explains this notion in matroid terms.

**Example 8.** Let $S$ be the matroid on $n = 6$ elements and rank $d = 2$, with the three non-bases 12, 34 and 56; i.e., $S$ has exactly twelve bases. We call this matroid the snowflake matroid for its relationship with the snowflake tree discussed in Example 29 below. The pairs 12, 34 and 56 form flats of rank one. The matroid polytope $P(S)$ has nine facets: the six non-negativity constraints $x_i \geq 0$, together with $x_1 + x_2 \leq 1$, $x_3 + x_4 \leq 1$ and $x_5 + x_6 \leq 1$. These are split facets, written as in (3).

Two splits of a polytope $P$ are compatible if their split hyperplanes do not meet in a relatively interior point of $P$.  

**Definition 9.** The $(d, n)$-matroid $M$ is a split matroid if its split facets form a compatible system of splits of the affine hull of $P(M)$ intersected with the unit cube $[0, 1]^n$.  

The matroid polytopes of the \((d,n)\)-split matroids are exactly those whose faces of codimension at least two are contained in the boundary of the \((d,n)\)-hypersimplex. The notion of a split matroid is a bit subtle in the disconnected case, which we will look into next. See also Proposition 15 (which characterizes the connected components of a split matroid) and Example 17 below.

**Lemma 10.** Let \(M\) be a split matroid which is disconnected. Then each connected component of \(M\) is a split matroid, too.

**Proof.** Let \(C\) be some connected component of the \((d,n)\)-matroid \(M\). Assume that \(M|C\) has \(n' = \#C\) elements and rank \(d'\). Let \(F\) and \(G\) be two distinct split facets of the connected matroid \(M|C\). Notice that this can only happen if \(M|C\) is not uniform. Now \(F\) is a flat of \(M\), and Lemma 6 gives us the \(F\)-hyperplane \(H_F\) which yields a split of \(\Delta(d,n)\) and a valid inequality of \(P(M)\). Notice that we may assume that \([n] \setminus C\) contains an element which is not a coloop. We have

\[
H_F \cap \Delta(d,n) = \Delta(rk(F), F) \times \Delta(d - rk(F), [n] - F) = \Delta(rk(F), F) \times \Delta(d' - rk(F), C - F) \times \Delta(d - d', [n] - C).
\]

That intersection contains interior points of \(\Delta(d,n)\), which is why this defines a facet of \(P(M)\). By construction this defines a split facet of \(M\). The same applies to \(G\), yielding another split hyperplane \(H_G\), which also yields a split facet of \(M\). Since \(M\) is a split matroid these two split facets of \(M\) are compatible. The explicit description in (5) shows that the split facets \(F\) and \(G\) of \(M|C\) are compatible, too. We conclude that \(M|C\) is a split matroid. \(\square\)

We conclude that it suffices to analyze those split matroids which are connected. The following characterization of split matroids does not require any reference to polyhedral geometry.

**Theorem 11.** Let \(M\) be a connected matroid. The matroid \(M\) is a split matroid if and only if for each split facet \(F\) the restriction \(M|F\) and the contraction \(M/F\) both are uniform.

**Proof.** Assume that \(M\) is a split matroid and \(F\) is a split facet. Let \(r\) be the rank of \(F\). As \(F\) does not correspond to a hypersimplex facet we know that \(r < d\). Hence \(F\) is not the entire ground set \([n]\). In particular, all conditions for Lemma 6 are satisfied. Moreover, the intersection of any two facets of the matroid polytope \(P(M)\) is contained in the boundary of the hypersimplex \(\Delta(d,n)\). This implies that the intersection of the split hyperplane of \(F\) with \(P(M)\) coincides with the intersection of that hyperplane with \(\Delta(d,n)\). By Lemma 6 we have that \(M|F\) is the uniform matroid of rank \(r\) on the set \(F\), and \(M/F\) is the uniform matroid of rank \(d - r\) on the set \([n] - F\).

To prove the converse, let \(F\) and \(G\) be two distinct split facets of \(M\) with uniform restrictions and contractions. We need to show that the hypersimplex splits corresponding to \(F\) and \(G\) are compatible. By Proposition 1(iii) and Lemma 6 we have

\[
P_M(F) = P(M|F) \times P(M/F) = \Delta(rk(F), F) \times \Delta(d - rk(F), [n] - F).
\]
This implies that $P_M(F)$ is exactly the intersection of the $F$-hyperplane with $\Delta(d,n)$. In particular, since the $G$-hyperplane is a valid inequality for $P_M(F)$, the $F$- and $G$-hyperplanes do not share any points in the relative interior of $\Delta(d,n)$. This means that the corresponding hypersimplex splits are compatible. \[\square\]

**Remark 12.** Equation (6) says that the face $P_M(F)$ corresponding to a flacet $F$ of split a matroid is the matroid polytope of a partition matroid, i.e., a direct sum of uniform matroids.

A flat is called *cyclic* if it is a union of circuits. This notion gives rise to yet another cryptomorphic way of defining matroids; see [BdM08, Theorem 3.2]. A matroid whose cyclic flats form a chain with respect to inclusion is called *nested*. Such matroids will play a role in Section 4 below.

**Proposition 13.** Each flacet $F$ of $M$ with at least two elements is a cyclic flat. This property holds even if $M$ is not connected.

*Proof.* Let $F$ be a flacet of $M$. The restriction $M|F$ is connected, even if $M$ itself is not connected, see also Remark 2. Thus for each $e \in F$ there exists a circuit $e \in C \subseteq F$ in $M|F$ that connects $e$ with another element of $F$. This circuit of $M|F$ is a minimal dependent set in $M$. Hence $F$ a cyclic flat. \[\square\]

The compatibility relation among the hypersimplex splits was completely described in [HJ08, Proposition 5.4]. The following is a direct consequence. Notice that this characterization of split compatibility is a tightening of the submodularity property of the rank function.

**Proposition 14.** Assume that $M$ is connected. Let $F$ and $G$ be two distinct split flacets. The splits obtained from the $F$- and the $G$-hyperplane are compatible if and only if
\[
\#(F \cap G) + d \leq \text{rk}(F) + \text{rk}(G).
\]
For instance, this condition is satisfied if $F \cap G$ is an independent set and $F + G$ contains a basis.

*Proof.* The $F$- and the $G$-hyperplane both define splits. [HJ08, Proposition 5.4] states that two splits are compatible if and only if exactly one of the following four inequalities hold.
\[
\begin{align*}
\#(F \cap G) &\leq \text{rk}(F) + \text{rk}(G) - d \\
\#(F - G) &\leq \text{rk}(F) - \text{rk}(G) \\
\#(G - F) &\leq \text{rk}(G) - \text{rk}(F) \\
\#([n] - F - G) &\leq d - \text{rk}(F) - \text{rk}(G).
\end{align*}
\]
We will show that the last three conditions never hold for a connected matroid.

We denote by $H \subseteq F \cap G$ the inclusion maximal cyclic flat that is contained in $F \cap G$. Then $c := \#(F \cap G) - H$ is the number of coloops in $M|(F \cap G)$. By Proposition 13 the flacet $F$ is a cyclic flat, too. Now [BdM08, Theorem 3.2] implies that
\[
\#(F - G) = \#(F - H) - c > \text{rk}(F) - \text{rk}(H) - c = \text{rk}(F) - \text{rk}(G \cap F) \geq \text{rk}(F) - \text{rk}(G).
\]
Similarly we get \( \#(G - F) > \text{rk}(G) - \text{rk}(F) \). The submodularity of the rank function yields
\[
\#([n] - (F + G)) + \text{rk}(F) + \text{rk}(G) - d \geq \text{rk}([n] - (F + G)) + \text{rk}(F + G) + \text{rk}(F \cap G) - d
\geq \text{rk}([n]) - d + \text{rk}(F \cap G)
\geq 0.
\]
In the above equality holds if and only if the matroid is the direct sum \( F \oplus G \oplus ([n] - (F + G)) \) and the set \([n] - (F + G)\) consists of coloops.

If \( F \cap G \) is independent and \( F + G \) has full rank \( d \) we have
\[
(7) \quad \#(F \cap G) = \text{rk}(F \cap G) + \text{rk}(F + G) - d \leq \text{rk}(F) + \text{rk}(G) - d.
\]

**Proposition 15.** A matroid \( M \) is a split matroid if and only if at most one connected component is a non-uniform split matroid and all other connected components are uniform.

**Proof.** We only need to discuss the case that \( M \) is disconnected. First assume that \( M \) is a direct sum of uniform matroids and at most one non-uniform split matroid \( M|C \). Let \( F \) and \( G \) be a split facets of \( M \). By assumption the \( F \)-hyperplane does not separate the matroid polytope of any of the uniform matroids. Hence \( F \) is a facet of \( M|C \). Similarly is \( G \) a facet of the split matroid \( M|C \). In particular, the intersection of the \( F \)-hyperplane with the \( G \)-hyperplane restricted to \( P(M|C) \) contains no interior point of \( P(M|C) \). This implies that the intersection of the \( F \)-hyperplane with the \( G \)-hyperplane contains no interior point of \( P(M) = P(M|C) \times P(M/C) \).

Now assume that \( M \) is a disconnected split \( (d, n) \)-matroid. From Lemma 10 we know that each connected component is a split matroid. Let \( C_1, C_2 \) be two connected components of \( M \), and let \( F, G \) be a split facets of \( C_1 \) and \( C_2 \), respectively. These split facets exist if and only if neither \( M|C_1 \) nor \( M|C_2 \) is uniform. Let \( x_F \in P(M|C_1) \) be a point on the relative interior of the facet defined by \( \sum_{i \in F} x_i = \text{rk}(F) \). Similarly, let \( x_G \in P(M|C_2) \) be a point on the relative interior of facet defined by \( G \). Finally, let \( x_H \) be a point in the relative interior of \( P(M/(C_1 + C_2)) \).

We have seen in Lemma 10 that the \( F \)-hyperplane is a facet of \( P(M) \). Hence \( F \) a facet of \( M \), and \( G \) is similar. By construction the point \( (x_F, x_G, x_H) \in P(M|C_1) \times P(M|C_2) \times P(M/(C_1 + C_2)) \) lies in the interior of \( P(M) \) as well as on the \( F \)- and \( G \)-hyperplanes. We conclude that the facets \( F \) and \( G \) are incompatible. Since this cannot happen in a split matroid, we may conclude that either \( M|C_1 \) or \( M|C_2 \) are uniform. \( \square \)

**Example 16.** For instance, the direct sum of the \((2, 4)\)-matroid with five bases, which is a split matroid, with an isomorphic copy is not a split matroid.

**Example 17.** The 12-, 34- and the 56-hyperplanes, corresponding to the split facets of the snowflake matroid \( \mathcal{S} \) from Example 8 are pairwise compatible. For instance, we have \( \#(\{1, 2\} \cap \{3, 4\}) = 0 \leq 1 + 1 - 2 \). This shows that the snowflake matroid is a split matroid; see also Figure 1a below. Note that the direct sum of the snowflake matroid with a coloop \( U_{1,1} \) is a split matroid, too. In particular, the 12- and 34-hyperplanes do not intersect in
the interior of $\Delta(2, 6) \times \Delta(1, 1)$. However, they do intersect in the interior of $\Delta(3, 7)$, as $\#((\{1,2\} \cap \{3,4\}) = 0 > 1 + 1 - 3$ shows.

**Example 18.** For a different kind of example consider the $(3,6)$-matroid with the eight non-bases $134, 234, 345, 346, 156, 256, 356$ and $456$. This matroid has exactly the two facets 34 and 56. The 34- and the 56-hyperplanes are not compatible. Hence this is not a split matroid.

A rank-$d$ matroid whose circuits have either $d$ or $d + 1$ elements is a *paving* matroid. It is conjectured that asymptotically almost all matroids are paving; see [Oxl11, Conjecture 15.5.10] and [MNWW11, Conjecture 1.6]. A paving matroid whose dual is also paving is called *sparse paving*. It is known that a matroid is paving if and only if there is no minor isomorphic to the direct sum of the uniform matroid $U_{2,2}$ and $U_{0,1}$; see [Oxl11, page 126].

The following is a geometric characterization of the paving matroids.

**Theorem 19.** Suppose that the $(d,n)$-matroid $M$ is connected. Then $M$ is paving if and only if it is a split matroid such that each split facet has rank $d - 1$.

**Proof.** Let $M$ be paving, and let $F$ be a split facet. Then $F$ is a corank-1 flat of $M$, i.e., $F$ is a proper flat of maximal rank $d - 1$. Since there are no circuits with fewer than $d$ elements, the restriction $M|F$ is a uniform matroid of rank $d - 1$. The contraction $M/F$ is a loop-free matroid of rank 1, and thus uniform. By Theorem 11 we find that $M$ is a split matroid, and each split facet of $M$ has rank $d - 1$.

Conversely, let $M$ be a matroid such that the split facets correspond to a compatible system of splits of $\Delta(d, n)$ such that, moreover, each split facet is of rank $d - 1$. Let $F$ be such a split facet. Then, by Lemma 6 we have $P_M(F) = \Delta(d - 1, F) \times \Delta(1, [n] - F)$. It follows that the restriction $M|F$ does not have a circuit with fewer than $d$ elements.

Now consider a set $C$ of size $d - 1$ or less which is contained in no split facet, and let $D \subseteq [n] - C$ be some set of size $d - \#C$ in the complement of $C$. Let $\bar{x} = e_{C+D}$. Then, for any facet $F$, we have

$$\sum_{i \in F} \bar{x}_i = \sum_{i \in F \cap C} \bar{x}_i + \sum_{i \in F \cap D} \bar{x}_i < \#C + d - \#C = d.$$  

as $C$ is not contained in $F$. This shows that $\bar{x}$ satisfies the facet inequality $\sum_{i \in F} x_i \leq d - 1$. Further, the inequalities imposed by the hypersimplex facets also hold, and so $\bar{x}$ is contained in $P(M)$. Since $\bar{x} = e_{C+D}$ is a vertex of $\Delta(d, n)$ it follows that it must also be a vertex of the subpolytope $P(M)$. Therefore, $C + D$ is a basis of $M$, whence $C$ is an independent set. We conclude that $M$ does not have any circuit with fewer than $d$ elements. Any circuit of a rank-$d$ matroid with more than $d$ elements has exactly $d + 1$ elements. This is why $M$ is a paving matroid. \hfill $\Box$

**Remark 20.** Each split facet of a paving matroid $M$ corresponds to a partition matroid, and the split facets are precisely the corank-1 flats of $M$ that contain a circuit; see also Remark 12. In this way, the split facets of a paving matroid implicitly occur in the matroid literature, e.g., in the proof of [Oxl11, Prop. 2.1.24].
We want to look into a construction which yields very many split matroids. Let \( \sigma \) be some \( d \)-element subset of \([n]\). That is, \( \sigma \) is a basis of the uniform matroid \( U_{d,n} \), and \( e_\sigma = \sum_{i \in \sigma} e_i \) is a vertex of \( \Delta(d,n) \). It neighbors in the Johnson graph \( J(d,n) \) lie on the \((\sigma, d-1)\)-hyperplane in \( \Delta(d,n) \). More precisely, from (4) we can see that the convex hull of the neighbors of \( e_\sigma \) equals \( \Delta(d-1, \sigma) \times \Delta(1, [n] - \sigma) \), which is the product of a \((d-1)\)-simplex and an \((n-d-1)\)-simplex. The resulting split is called the vertex split with respect to \( \sigma \) or \( e_\sigma \). Two vertex splits are compatible if and only if the two vertices do not span an edge. In this way the compatible systems of vertex splits of \( \Delta(d,n) \) bijectively correspond to the stable sets in the Johnson graph \( J(d,n) \). The following observation is similar to [BPvdP15, Lemma 8].

**Corollary 21.** Again let \( M \) be a \((d,n)\)-matroid which is connected. Then \( M \) is sparse paving if and only if the conclusion of Theorem 19 holds and additionally the splits are vertex splits.

**Proof.** For each rank \( d-1 \) split facet \( F \) of the split matroid \( M \) we have \( M/F = U_{1,[n]-F} \) and \( M|F = U_{d-1,F} \). The dual of \( M \) is a matroid of rank \( n-d \) on \( n \) elements. The matroid polytope \( P(M^*) \) is the image of \( P(M) \) under coordinate-wise transformation \( x_i \mapsto 1 - x_i \).

It follows that the split facets of \( M^* \) are the complements of the split facets of \( M \). Thus, for the split facet \([n] - F \) in \( M^* \), we obtain

\[
M^*|([n] - F) = (M/F)^* = U_{1,[n]-F}^* = U_{n-\#F-1,[n]-F} \\
M^*/([n] - F) = (M|F)^* = U_{d-1,F}^* = U_{\#F-d+1,F}.
\]

This implies that \( M^* \) is paving if and only if each split facet \( F \) has cardinality \( d \). \( \square \)

The following two examples illustrate the differences between paving and split matroids. The class of split matroids is strictly larger. In contrast to the class of paving matroids the class of split matroids is closed under dualization.

**Example 22.** The \((\{1, 2, 3, 4\}, 2)\)-hyperplane yields a split of the hypersimplex \( \Delta(4,8) \). The two maximal cells correspond to split matroids which are not paving nor are their duals.

Yet there are still plenty of matroids which are not split.

**Example 23.** Up to symmetry there are 15 connected matroids of rank three on six elements. Among these there are exactly four which are non split. One such example is the nested matroid given by the columns of the matrix

\[
\begin{pmatrix}
1 & 1 & 0 & 1 & \lambda \\
0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix},
\]

where \( \lambda \neq 0, 1 \). This matroid is realizable over any field with more than two elements.

Knuth gave the following construction for stable sets in Johnson graphs [Knu74]. Due to Corollary 21 this is the same as a compatible set of vertex splits, which arise from the split facets of a sparse paving matroid.
Example 24. The function
\[(x_1, \ldots, x_n) \mapsto \sum_{i=1}^{n} i \cdot x_i \mod n\]
defines a proper coloring of the nodes of \(J(d, n)\) with \(n\) colors. Each color class forms a stable set, and there must be at least one stable set of size at least \(\frac{1}{n} \binom{n}{d}\).

For special choices of \(d\) and \(n\) larger stable sets in \(J(d, n)\) are known.

Example 25. Identifying a natural number between 0 and \(2^k - 1\) with its binary representation yields a 0/1-vector of length \(k\). All quadruples of such vectors that sum up to 0 modulo 2 form a stable set \(S\) in \(J(4, 2^k)\) of size \(n(n-1)(n-2)/24\), where \(n = 2^k\). Fixing one vector and restricting to those quadruples in \(S\) which contain that vector gives a stable set of size \((n-1)(n-2)/6\) in \(J(3, 2^k - 1)\). The latter construction also occurs in [Duk04, Theorem 3.1] and [HJJS09, Theorem 3.6].

In a way, the sparse paving matroids are those split matroids which are the easiest to get at. We sum up our discussion in the following characterization.

Theorem 26. The following sets are in bijection with one another:

(i) The split facets of sparse paving connected matroids of rank \(d\) on \(n\) elements,
(ii) the cyclic flats of sparse paving connected matroids of rank \(d\) on \(n\) elements,
(iii) the sets of compatible vertex splits of \(\Delta(d, n)\),
(iv) the stable sets of the graph \(J(d, n)\),
(v) the sets of binary vectors of length \(n\) with constant weight \(d\) and Hamming distance at least 4.

Proof. Each split facet of a sparse paving matroid \(M\) is a cyclic flat by Proposition 13. The proof of Theorem 19 shows that cyclic flats of rank \(d - 1\) are split facets of \(M\). Further, the cyclic flats of a connected paving matroid are those of rank \(d - 1\), the empty set and the entire ground set \([n]\). This establishes that (i) and (ii) are equivalent.

Corollary 21 is exactly the equivalence of (i) and (iii).

By Proposition 14 two vertex splits of \(\Delta(d, n)\) are compatible if and only if the two vertices do not span an edge. The compatible systems of vertex splits of \(\Delta(d, n)\) bijectively correspond to the stable sets in the vertex-edge graph of \(\Delta(d, n)\), which is the Johnson graph \(J(d, n)\). This means that (iii) is equivalent to (iv).

The vertices of the hypersimplex \(\Delta(d, n)\) are all binary vectors of length \(n\) with constant weight \(d\). The Hamming distances of two such vectors \(v\) and \(w\) is the number of coordinates where \(v_i \neq w_i\). This number is twice the distance of the vertices in the Johnson graph \(J(d, n)\). Note that odd numbers do not occur as Hamming distances. Hamming distance at least 4 means that the vertices are not adjacent in \(J(d, n)\). This yields the equivalence of (iv) and (v).

A table with lower bounds on the maximal size of such a set for \(n \leq 28\) is given in [BSSS90, Table I-A]. Notice that this data also gives lower bounds on the total number of \((d, n)\)-matroids; see, e.g., [BPvdP15].
In this section we want to exploit the structural information that we gathered about split matroids to derive new results about tropical linear spaces, the tropical Grassmannians and the related Dressians [SS04, HJJS09]. We begin with some basics on general polyhedral subdivisions; see [DLRS10] for further details.

Let \( P \) be some polytope. A polytopal subdivision of \( P \) is regular if it is induced by a lifting function on the vertices of \( P \). Examples are given by the Delaunay subdivisions where the lifting function is the Euclidean norm squared. The lifting functions on \( P \) which induce the same polytopal subdivision, \( \Sigma \), form a relatively open polyhedral cone, the secondary cone of \( \Sigma \). The secondary fan of \( P \) comprises all secondary cones. The inclusion relation on the closures of the secondary cones of \( P \) imposes a partial ordering, and this is dual to the set of regular polytopal subdivisions of \( P \) partially ordered by refinement. The secondary fan has a non-trivial lineality space which accounts for the various choices of affine bases. Usually we will ignore these linealities. In particular, whenever we talk about dimensions we refer to the dimension of a secondary fan modulo its linealities.

A tropical Plücker vector \( \pi \in \mathbb{R}^{\binom{n}{d}} \) is a lifting function on the vertices of the hypersimplex \( \Delta(d,n) \) such that the regular subdivision induced by \( \pi \) is a matroid subdivision, i.e., each of its cells is a matroid polytope. The cells of the dual of a matroid subdivision that correspond to loop-free matroid polytopes form a subcomplex. This subcomplex of that matroid subdivision is the tropical linear space defined by \( \pi \). The Dressian \( \text{Dr}(d,n) \) is the subfan of the secondary fan of the hypersimplex \( \Delta(d,n) \) comprising the tropical Plücker vectors. According to Remark 5 each split of a hypersimplex is a regular matroid subdivision and hence it defines a ray of the corresponding Dressian.

Let \( M \) be a \((d,n)\)-matroid. The matroid polytope \( P(M) \) is a subpolytope of \( \Delta(d,n) \). Restricting the tropical Plücker vectors to vertices of \( P(M) \) and looking at regular subdivisions of \( P(M) \) into matroid polytopes gives rise to the Dressian \( \text{Dr}(M) \) of the matroid \( M \); see [HJJS09, Section 6]. The rank of any subset \( S \) of \([n]\) coincides with the rank of the flat spanned by \( S \). Restricting the rank function of \( M \) to all subsets of \([n]\) of a fixed cardinality \( k \) yields the \( k \)-rank vector of \( M \). The dual-rank function of \( M \) is the rank function of \( M^* \), the dual matroid of \( M \), and the corank function is the difference between \( d \) and the rank function. The \( k \)-corank vector of \( M \) is the map

\[
\rho_k(M) : \binom{[n]}{k} \to \mathbb{N}, \ S \mapsto d - \text{rk}_M(S).
\]

The regular subdivision of \( \Delta(k,n) \) with lifting function \( \rho_k(M) \) is the \( k \)-corank subdivision induced by the matroid \( M \). Usually we will omit the size \( k \) in those definitions if \( k \) equals \( d \). The following known result says that the \( k \)-corank subdivision is a matroid subdivision.

**Lemma 27.** The \( k \)-corank vector \( \rho_k(M) \) of the \((d,n)\)-matroid \( M \) is a \((k,n)\)-tropical Plücker vector. Moreover, the matroid polytope \( P(M) \) occurs as a cell in the \( k \)-corank subdivision induced by \( M \). That cell is maximal if and only if \( M \) is connected.
Proof. Speyer showed that $\rho_k(M)$ is a tropical Plücker vector such that the matroid polytope $P(M)$ occurs as a cell [Spe05, Proposition 4.5.5]. The dimension of that cell can be read off from Proposition 1. □

Example 28. With $d = 2$ and $n = 4$ let $M$ be the matroid with the five bases 12, 13, 14, 23 and 24. We pick $k = d = 2$. The rank of the unique non-basis 34 equals 1, whence $\rho_2(M) = (0, 0, 0, 0, 1)$. The matroid subdivision induced by $\rho_2(M)$ splits the hypersimplex $\Delta(2, 4)$ into two Egyptian pyramids. Every subset of $\{1, 2, 3, 4\}$ with cardinality $k = 3$ contains a basis, and thus $\rho_3(M) = (0, 0, 0, 0)$. There are no loops in $M$, whence for $k = 1$ the corank vector $\rho_1(M)$ equals $(1, 1, 1, 1)$. Here and below the ordering of the $k$-subsets of $[n]$ in the corank vectors is lexicographic.

Example 29. The corank subdivision of the matroid $S$ in Example 8 is a matroid subdivision of $\Delta(2, 6)$ whose tropical linear space is the snowflake tree. Hence the name snowflake matroid for $S$. See Figure 1a for a visualization.

By Proposition 7 the facets of any matroid polytope are either hypersimplex facets or induced by hypersimplex splits. In the following we will be interested in the set of hypersimplex splits arising from the split facets of a given matroid. The next result explains what happens if that matroid is a split matroid.

**Proposition 30.** Let $M$ be a split $(d,n)$-matroid which is connected. Then the corank vector $\rho(M)$ is contained in the relative interior of a simplicial cone of $\text{Dr}(d,n)$, and the dimension of that cone is given by the number of split facets of $M$. In particular, $\rho(M)$ is a ray if and only if it induces a split of $\Delta(d,n)$. This is the case if and only if $M$ is a nested matroid with exactly three cyclic flats.

**Proof.** Let $H$ be the set of hypersimplex splits corresponding to the split facets of $M$. By definition the splits in $H$ are compatible. Since each subset of a compatible set of splits is again compatible it follows that the secondary cone spanned by $H$ is a simplicial cone.
Recall that \( M \) is nested if the cyclic flats form a chain. The empty set and \([n]\) are two cyclic flats in any connected matroid. Assume that the matroid \( M \) is nested with precisely three cyclic flats. Then the third cyclic flat \( F \) induces the only split, since the restriction \( M|F \) and the contraction \( M\setminus F \) are uniform matroids.

Conversely, if the matroid \( M \) is split with a unique split facet \( F \), then obviously \( \emptyset \subsetneq F \subsetneq [n] \). Each circuit \( C \) of \( M \) with fewer than \( d+1 \) elements leads to valid inequality of the polytope \( P(M) \). This inequality separates \( P(M) \) from those vertices of the hypersimplex with \( x_i = 1 \) for \( i \in C \). Hence, the only split facet \( F \) contains the circuit \( C \). The restriction \( M|F \) is a uniform matroid and thus \( \operatorname{rk}(C) = \operatorname{rk}(F) \). We get that \( F \) is the closure of \( C \). Hence we may conclude that \( M \) is nested. \( \square \)

Our next result generalizes [HJJS09, Thm. 3.6], which settled the case \( d = 3 \).

**Theorem 31.** For the dimension of the Dressian we have
\[
\frac{1}{n} \binom{n}{d} - 1 \leq \dim \operatorname{Dr}(d,n) \leq \binom{n-2}{d-1} - 1 .
\]

**Proof.** Speyer showed that the spread of any matroid subdivision of the hypersimplex \( \Delta(d,n) \), i.e., its number of maximal cells, does not exceed \( \binom{n-2}{d-1} \) [Spe05, Thm. 3.1]. The dimension of a secondary cone of a subdivision \( \Sigma \) is the size of a maximal linearly independent family of coarsest subdivisions which are refined by \( \Sigma \). As each (coarsest) subdivision has at least two maximal cells, the dimension of the secondary cone is at most the spread minus one. This follows from the fact that at least \( k \) (linearly independent) rays are necessary in order to generate a cone of dimension \( k \). It follows that \( \dim \operatorname{Dr}(d,n) \leq \binom{n-2}{d-1} - 1 \). The lower bound is given by Knuth’s construction of stable sets in \( J(d,n) \); see Example 24. \( \square \)

This gives the following asymptotic estimates.

**Corollary 32.** For fixed \( d \) the dimension of the Dressian \( \dim \operatorname{Dr}(d,n) \) is of order \( \Theta(n^{d-1}) \). Further, the asymptotic dimension of the Dressian \( \dim \operatorname{Dr}(d,2d) \) is bounded from below by \( \Omega(4^d d^{-3/2}) \) and bounded from above by \( O(4^d d^{-1/2}) \).

**Proof.** For fixed \( d \) the lower and the upper bound in Theorem 31 both grow as fast as \( n^{d-1} \) asymptotically. Stirling’s formula yields that the binomial coefficient \( \binom{2d}{d} \) grows like \( 2^{2d}/\sqrt{\pi d} \). Specializing the bounds in Theorem 31 to \( n = 2d \) thus yields
\[
\Omega\left(\frac{2^{2d-1}}{d \sqrt{\pi d}}\right) \leq \dim \operatorname{Dr}(d,2d) \leq O\left(\frac{2^{2d-2}}{\sqrt{\pi(d-1)}}\right) .
\]

Now the lower and the upper bound differ by a multiplicative factor of
\[
\frac{d \sqrt{d}}{2 \sqrt{d-1}} ,
\]
which tends to \( d/2 \) when \( d \) goes to infinity. \( \square \)

The following example shows that not all matroid subdivisions are induced by a corank function.
Example 33. The matroid subdivision $\Sigma$ of the hypersimplex $\Delta(2,6)$ induced by the lifting vector $(3,2,1,0,0,2,1,0,0,2,1,1,2,2,3)$ is not a corank subdivision. We give a hint how this claim can be verified. This subdivision $\Sigma$ has exactly $4$ maximal cells, which come as two pairs of isomorphic cells. One can check that $\Sigma$ does not agree with the corank subdivision induced by any of these maximal cells. The lifting-vector is obtained from a metric caterpillar tree with six leaves and unit edge lengths, see Figure 1b. Notice that the subdivision $\Sigma$ is realizable by a tropical point configuration, while the corank subdivision induced by the snowflake matroid $S$ is not; see [HJS12].

Tropical geometry studies the images under the valuation map of algebraic varieties over fields with a discrete valuation; see, e.g., [MS15, Chapter 3]. Let $K\{t\}$ be the field of formal Puiseux series over an algebraically closed field $K$. The valuation map $\mathrm{val}: K\{t\} \to \mathbb{R} \cup \{\infty\}$ sends a Puiseux series to the exponent of the term of lowest order. Each Puiseux series over an algebraically closed field $K$ sends a Puiseux series to the exponent of the term of lowest order. Each point of $K\{t\}$ induces a valuation ring which forms a valuation domain.

Example 4.5.4. We indicate a short proof for the sake of completeness.

Proposition 34. Let $\pi$ be a $(d,n)$-tropical Plücker vector which can be lifted to an ordinary Plücker vector over $K\{t\}$. Then the cells in the subdivision of $\Delta(d,n)$ induced by $\pi$ necessarily correspond to matroids which are realizable over $K$.

Proof. By our assumption there exists an ordinary Plücker vector $p$ which valuates to $\pi$. We can pick a matrix $A \in K\{t\}^{d \times n}$ such that for each $d$-set $I$ of columns we have $\det A_I = p_I$. It follows that $\mathrm{val}(\det A_I) = \pi_I$. Note that the matrix $A$ is not unique.

Let $M$ be the matroid corresponding to a cell. Up to a linear transformation we may assume that $\pi$ is non-negative, and we have $\pi_I \geq 0$ if and only if $I$ is a basis of $M$. We will show that $A$ can be chosen such that the valuation of each entry is non negative.

We apply Gaussian elimination to the $n \geq d$ columns of $A$. This way the classical Plücker vector associated with $A$ is multiplied with a non-zero scalar. Thus the tropical Plücker vector $\pi$ is modified by adding a multiple of the all-ones vector. In each step, among the possible pivots pick one whose valuation is minimal. Let $\gamma$ be the product of all pivot elements, and let $c t^\gamma$ for $c \neq 0$ be the term of lowest order. By construction $g = \mathrm{val}(c t^\gamma) = \mathrm{val}(\gamma)$ is a
lower bound for the valuations of the minors of $A$, which is actually attained. Since $\pi$ is non-negative and since $\pi_I = 0$ if $I$ is a basis we conclude that $g = 0$.

Including possibly trivial pivots with 1 we obtain exactly $d$ pivots, one for each row of $A$. Multiplying each row with the inverse of the lowest order term of the corresponding pivot does not change $\pi$. The resulting matrix $A'$ is a realization with entries whose valuations are non-negative. Hence we can evaluate the matrix $A' \in K\{t\}^{d \times n}$ at $t = 0$. This gives us the matrix $B \in K^{d \times n}$ with the constant terms of $A'$. The matrix $B$ realizes $M$ since $\det B_I = 0$ if and only if the lowest order term of $\det A'_I$ is constant in $t$. \qed

Our next goal is to prove a characterization of matroid realizability in terms of tropical Plücker vectors. In the proof we will use a standard construction from matroid theory which will also reappear further below. The free extension of the $(d,n)$-matroid $M$ by an element $f \notin [n]$ is the $(d,n+1)$-matroid which arises from $M$ by adding $f$ to the ground set such that it is independent from each $(d-1)$-element subset of $[n]$.

**Theorem 35.** Let $M$ be a $(d,n)$-matroid. The corank vector $\rho(M)$ can be lifted to an ordinary Plücker vector over $K\{t\}$ if and only if $M$ is realizable over $K$.

**Proof.** Let $\rho(M)$ be realizable. Since $P(M)$ occurs as a cell in the matroid subdivision induced by $\rho(M)$ the matroid $M$ is realizable due to Proposition 34.

Conversely, let us assume that the matroid $M$ is realizable and the matrix $B \in K^{d \times n}$ is a full rank realization. The matrix $B$ has only finitely many entries, and these generate some extension field $L$ of the prime field of $K$. The field $L$ may or may not be transcendental, but it is certainly not algebraically closed. Hence there exists an element $\alpha \in K - L$ which is algebraic over $L$ of degree at least $n$. The vector $B \cdot (1, \alpha, \ldots, \alpha^{n-1})^\top$ is $L$-linearly independent of any $d-1$ columns of $B$. We infer that even the free extension of $M$ is realizable over $K$. After altogether $n$ free extensions we obtain a matrix $C \in K^{d \times n}$ such that the block column matrix $[B|C]$ is a realization of the $n$-fold free extension of $M$. We define $A := B + t \cdot C$, which is a $d \times n$-matrix with coefficients in $K\{t\}$.

For any $d$-subset $I$ of $[n]$ and for any subset $S \subseteq I$ we denote by $D(S) \in K\{t\}^{d \times n}$ the matrix whose $k$-th column is the $k$-th column of $B$ if $k \in S$ and $t$ times the $k$-th column of $C$ otherwise. Then

$$\det A_I = \det(B_I + t \cdot C_I) = \sum_{S \subseteq I} \det D_I(S).$$

Further, by choice of $C$, we have $\det D_I(S) = 0$ if and only if $S$ is a dependent set in $M$, and $\val(\det D_I(S)) = d - \# S$ if $S$ is independent. For a fixed set $S \subseteq I$ the Puiseux series $\det D_I(S)$ has a term $c(S)t^{g(S)}$ of lowest order, and we have $g(S) = \val(\det D_I(S)) = d - \# S$.

The field $K$ is an $L$-vector space, and the set

$$\{c(S) \mid S \text{ independent subset of } I\}$$

of leading coefficients is linearly independent over $L$. This is why we obtain $\val \det A_I = d - \rk(I)$, i.e., cancellation does not occur. That is, the ordinary Plücker vector of the matrix $A$ tropicalizes to $\rho(M)$. \qed
4. Rays of the Dressian

The purpose of this section is to describe a large class of tropical linear spaces, which are tropically rigid, i.e., they correspond to rays of the corresponding Dressian. Before we can define a special construction for matroids we first browse through a few standard concepts.

Let $M$ be a connected matroid of rank $d$ with $[n]$ as its set of elements. The parallel extension of $M$ at an element $e \in [n]$ by $s \not\in [n]$ is the $(d,n+1)$-matroid whose flats are either flats of $M$ which do not contain $e$ or sets of the form $F + s$, where $F$ is a flat containing $e$. Among all connected extensions the parallel extension is the one in which the shortest length of a circuit that contains the added element is minimal. In fact, that length equals two. Similarly, the free extension is characterized by the following property: Any circuit that contains the added element has length $d+1$, and this is the maximal length of such a circuit.

In general a coextension of $M$ is the dual of an extension applied to the dual matroid $M^*$. That is, a coextension of a $(d,n)$-matroid is a $(d+1,n+1)$-matroid. Finally, a series-extension is a parallel coextension.

**Definition 36.** The series-free lift of $M$, denoted as $\Lambda M$, is the matroid of rank $d+1$ with $n+2$ elements obtained as the series-extension of $M'$ at $f$ by $s$, where $M'$ is the free extension of $M$ by $f$.

Note that $\Lambda M$ is connected as $M$ is connected. In the sequel we want to show that the corank subdivision of $\Lambda M$ yields a ray of the Dressian $Dr(d+1,n+2)$, whenever $M$ is a $(d,n)$-split matroid. Let us first determine the rank function and the bases of $\Lambda M$. We write $fs$ as shorthand for the two-element set $f + s = \{f,s\}$.

**Lemma 37.** The set $B$ of size $d+1$ is a basis in $\Lambda M$ if and only if one of the following conditions hold:

(i) $fs \subseteq B$ and $rk_M(B - fs) = d - 1$, or
(ii) $f \in B$ and $s \not\in B$ and $rk_M(B - f) = d$, or
(iii) $f \not\in B$ and $s \in B$ and $rk_M(B - s) = d$.

Further, the rank of $S \subseteq [n] + fs$ is given by

\[
rk_{\Lambda M}(S) = \min\{rk_M(S - fs) + \#(fs \cap S), d + 1\}.
\]

The split flacets of $\Lambda M$ are those of $M$ and additionally $[n]$, the ground set of $M$.

**Proof.** Clearly each basis in $\Lambda M$ contains at least $f$ or $s$. Conversely, any basis $B$ of $M$ extends to a basis of $\Lambda M$ with either $f$ or $s$. A circuit of the free extension $M'$ of $M$ by $f$ that contains $f$ has size $d+1$. Hence each circuit of $\Lambda M$ that contains $f$ and $s$ has length $d+2$. In particular, this implies that each independent set $B$ in $M$ of size $d-1$ together with $fs$ forms a basis of $\Lambda M$. Any set which is dependent over $M$ is also dependent over $\Lambda M$.

The formula for the rank function is a direct consequence of the description of the bases. We see that there is no circuit of length at most $d$, that contains $f$, $s$ or both. Proposition 13 says that there is no flacet that contains $f$ or $s$. Contracting the set $[n]$ in $\Lambda M$ yields the
uniform matroid of rank 1 on the two-element set $fs$, and this is connected. For $S$ a subset of $[n] + fs$ and any set $F \neq [n]$ that does not contain $fs$ we have
\[
\begin{align*}
\text{rk}_{\Lambda(M)/F}(S) &= \text{rk}_{\Lambda M}(S + F) - \text{rk}_{\Lambda M}(F) \\
&= \min \{ \text{rk}_M(S + F - fs) + \#(fs \cap S), d + 1 \} - \text{rk}_M(F) \\
&= \min \{ \text{rk}_{M/F}(S - fs) + \#(fs \cap S), d - \text{rk}_M(F) + 1 \} \\
&= \text{rk}_{\Lambda(M/F)}(S)
\end{align*}
\]
The matroid $\Lambda(M/F) = (\Lambda M)/F$ is connected if and only if $M/F$ is connected. The restriction $\Lambda(M[F])$ coincides with $M[F]$. Both the restriction and contraction on $F$ are connected in $M$ if and only if they are connected in $\Lambda M$. We conclude that the split facets of $\Lambda M$ are precisely the ones in our claim. \hfill \Box

Our next goal is to describe the maximal cells of the corank subdivision induced by $\Lambda M$. To this end we first define the matroid $\Lambda^* M$ as the free coextension of $M$ by $f$, followed by the parallel extension at $f$ by $s$. We call $\Lambda^* M$ the parallel-cofree lift of $M$. This new construction is related to the series-free lift by the equality
\[
\Lambda^* M = (\Lambda(M^*))^*.
\]
A direct computation shows that the rank function is given by
\[
\begin{align*}
\text{rk}_{\Lambda^* M}(S) &= \min \{ \text{rk}_{\Lambda M}(S) + \#(fs - S) - 1, \# S \} \\
&= \min \{ \text{rk}_M(S - fs) + 1, \# S \}.
\end{align*}
\]
\[(9)\]

One maximal cell of the corank subdivision induced by $\Lambda M$ is obvious, namely the matroid polytope $P(\Lambda M)$. This is the case as $M$, and thus also $\Lambda M$, is connected. Here is another one.

**Lemma 38.** The corank subdivision of $\Lambda^* M$ coincides with the corank subdivision of $\Lambda M$. Hence the matroid polytope $P(\Lambda^* M)$ is a maximal cell of the corank subdivision of $\Delta(d + 1, n + 2)$ induced by $\Lambda M$. Further, the cells $P(\Lambda M)$ and $P(\Lambda^* M)$ intersect in a common cell of codimension one.

**Proof.** Let $S$ be a subset of $[n] + fs$ of size $d + 1$. We have $\text{rk}_M(S - fs) \leq d$. From (9) we deduce that $\text{rk}_{\Lambda^* M}(S) = \text{rk}_M(S - fs) + 1 \leq d + 1 = \# S$, while Lemma 37 gives $\text{rk}_{\Lambda M}(S) = \text{rk}_M(S - fs) + \#(fs \cap S) \leq \#(S - fs) + \#(fs \cap S) = d + 1$. Combining these two arrive at the equation $\text{rk}_{\Lambda^* M}(S) = \text{rk}_{\Lambda M}(S) - \#(fs \cap S) + 1$. This implies
\[
\rho(\Lambda^* M) + 1 = \rho(\Lambda M) + x_f + x_s.
\]
As a consequence the corank subdivision of $\Lambda^* M$ coincides with the corank subdivision of $\Lambda M$. The common bases of the matroids $\Lambda M$ and $\Lambda^* M$ are the bases of the direct sum $M \oplus U_{1,fg}$. The corresponding matroid polytope yields the desired cell of codimension one. \hfill \Box

For each split facet $F$ of $M$ we let $N_F$ be the connected $(d + 1, n + 2)$-matroid with elements $[n] + fs$ which has the following list of cyclic flats: $\emptyset$, $[n] - F$ of rank $d - \text{rk}(F)$, $[n] - F + fs$ of rank $d + 1 - \text{rk}(F)$ and $[n] + fs$ of rank $d + 1$. 

Note that these sets form a chain. This chain has a rank 0 element, the ranks are strictly increasing, and for each set the rank is less than the size. Hence these sets form the cyclic flats of a matroid. Its rank function is given by \( \text{rk}(S) = \min \{ \text{rk}(G) + \#(S - G) \mid G \text{ is a cyclic flat} \} \); see [BdM08]. Hence, the rank function of \( N_F \) satisfies
\[
(10) \quad \text{rk}_{N_F}(S) = \min \{ d+1, \#(S), \#(S \cap F) + d+1 - \text{rk}_M(F), \#(S \cap (F + fs)) + d - \text{rk}_M(F) \}.
\]
This is a nested matroid with exactly two split facets, namely \([n] - F\) and \([n] - F + fs\). The corresponding hypersimplex splits are not compatible, i.e., \( N_F \) is not a split matroid. The following result compares the corank in \( \Lambda M \) with the corank in \( N_F \).

**Proposition 39.** For each split facet \( F \) of \( M \) and any set \( S \subseteq [n] + fs \) with \( \#(S) = d + 1 \) we have
\[
(11) \quad d + 1 - \text{rk}_{\Lambda M}(S) + \text{rk}_M(F) - \#(S \cap F) \geq d + 1 - \text{rk}_{N_F}(S).
\]

**Proof.** Since the size of \( S \) equals \( d + 1 \) the equation \((10)\) simplifies to
\[
d + 1 - \text{rk}_{N_F}(S) = \max \{ 0, \text{rk}_M(F) - \#(S \cap F), \text{rk}_M(F) + 1 - \#(S \cap F) - \#(S \cap fs) \}
\]
if we subtract both sides from \( d + 1 \). That expression is the corank of \( S \) in the nested matroid \( N_F \). This corank function gives the \((d + 1, n + 2)\)-tropical Plücker vector \( \rho(N_F) \). In the sequel we will make frequent use of the inequality
\[
(12) \quad \text{rk}_M(S - fs) \leq \text{rk}(F) + \#(S - F - fs) = \#(S - fs) - \#(S \cap F) + \text{rk}_M(F),
\]
which is a consequence of the fact that \( F \) is a cyclic flat of \( M \).

To prove \((11)\) we distinguish three cases. First, if neither \( f \) nor \( s \) are in \( S \) the inequality \((11)\) is equivalent to
\[
(13) \quad d + 1 - \text{rk}_M(S) + \text{rk}_M(F) \geq \max \{ \#(S \cap F), \text{rk}_M(F) + 1 \},
\]
as \( \text{rk}_{\Lambda M}(S) = \text{rk}_M(S) < d + 1 \) by \((8)\). The inequality \((13)\) follows from \( \text{rk}_M(S) \leq d \) and \((12)\) with \( \#(S - fs) = d + 1 \). Second, if \( \#(fs \cap S) = 1 \), again by applying \((8)\) the inequality \((11)\) is equivalent to
\[
d - \text{rk}_M(S - fs) + \text{rk}_M(F) \geq \max \{ \#(S \cap F), \text{rk}_M(F) \},
\]
which holds due to the same arguments as in the first case with \( \#(S - fs) = d \). Third, in the remaining case we have \( s, f \in S \), which yields \( \text{rk}_M(S - fs) \leq \#(S - fs) = d - 1 \). This implies that the inequality \((11)\) is equivalent to
\[
(14) \quad d - 1 - \text{rk}_M(S - fs) + \text{rk}_M(F) - \#(S \cap F) \geq \max \{ 0, \text{rk}_M(F) - \#(S \cap F) \}.
\]
If the maximum on the right hand side is attained at \( \text{rk}_M(F) - \#(S \cap F) \) that inequality holds trivially. We are left with the situation where the maximum on the right is attained solely by zero. This means that \( \text{rk}_M(F) < \#(S \cap F) \), which yields
\[
(15) \quad d - \#(S \cap F) + \text{rk}_M(F) \geq d - 1 \geq \text{rk}_M(S - fs).
\]
If \( \text{rk}_M(S - fs) < d - 1 \) then \((14)\) is immediate. So we may assume that \( \text{rk}_M(S - fs) = d - 1 \). From Lemma 37 we deduce that \( S \) is a basis of \( \Lambda M \). Since \( F \) is also a facet of \( \Lambda M \) we get \( \text{rk}_M(F) \geq \#(S \cap F) \). However, this contradicts \( \text{rk}_M(F) < \#(S \cap F) \), and we conclude that
the case where the maximum to the right of (14) cannot be attained at zero only. This final
contradiction completes our proof. □

Lemma 40. Let $M$ be a $(d,n)$-split matroid. Then for each split facet $F$ of $M$ the matroid
polytope $P(N_F)$ is a maximal cell of the corank subdivision of $\Delta(d+1,n+2)$ induced by $\Lambda M$.
Further, the cell $P(N_F)$ shares a split facet with $P(\Lambda M)$ and another one with $P(\Lambda^* M)$.

Proof. We want to show that equality holds in (11) if $S$ is a basis of $N_F$. In other words
the corank lifting of $N_F$ agrees with the corank lift of $\Lambda M$ on $P(N_F)$, up to an affine
transformation. Moreover, the bases of $N_F$ are lifted to height zero, while the lifting function
is strictly positive on all other bases; see inequality (11). This implies that $P(N_F)$ is a
maximal cell in the corank subdivision of $\Lambda M$.

The matroid $M$ is split, hence the contraction $M/F$ on the facet $F$ is a uniform matroid
of rank $d - \text{rk}(F)$. Therefore, the rank function satisfies

$$\text{rk}_M(S + F - fs) - \text{rk}_M(F) = \min \{ \#(S - F - fs), d - \text{rk}_M(F) \}.$$  

With Lemma 37 we get

$$\text{rk}_{\Lambda M}(S) \leq \text{rk}_{\Lambda M}(S + F)$$

(16)

$$= \min \{ \text{rk}_M(S + F - fs) + \#(S \cap fs), d + 1 \}$$

$$= \min \{ \#(S - F) + \text{rk}_M(F), d + \#(S \cap fs), d + 1 \}.$$  

The set $[n] - F$ is a facet of rank $d - \text{rk}_M(F)$ in $N_F$. For any basis $S$ of $N_F$ we get

$$\text{rk}_M(F) + 1 + \#(S - F) \leq d + 1 = \#(S - F) + \#(S \cap F).$$  

This implies that $\#(S \cap F) \geq \text{rk}_M(F) + 1$. Together with the inequality (16) we get

$$d + 1 - \text{rk}_{\Lambda M}(S) - \#(S \cap F) + \text{rk}(F) \leq 0.$$  

This means that equality holds in (11) whenever $S$ is a basis of $N_F$.

As a consequence $P(N_F)$ is a maximal cell of the corank subdivision of $\Lambda M$. Clearly
$P(N_F)$ intersects $P(\Lambda M)$ in a codimension-1-cell that is contained in

$$P_{\Lambda M}(F) = P_{N_F}([n] - F + fs).$$  

By Lemma 38 the same kind of argument holds for $\Lambda^* M$. That is, $P(N_F)$ intersects $P(\Lambda M)$
in a codimension-1-cell that is contained in $P_{\Lambda^* M}(F + fs) = P_{N_F}([n] - F).$ □

From the above we know that, for a split matroid $M$, the matroid polytopes of $\Lambda M$,
$\Lambda^* M$ and the nested matroid $N_F$ for each facet of $M$ form maximal cells of the corank
subdivision induced by $\Lambda M$. The following result describes the corresponding tropical linear
space completely.
Theorem 41. Let $M$ be a connected $(d,n)$-split matroid. Then the corank vector $\rho(\Lambda M)$ is a ray in the Dressian $Dr(d+1, n+2)$. Moreover, it can be lifted to an ordinary Plücker vector over $K\ll(t\rr)$ if and only if $M$ is realizable over $K$.

Proof. Let $\Sigma$ be the matroid subdivision of $\Delta(d+1, n+2)$ induced by $\rho(\Lambda M)$. By Lemma 27, Lemma 38 and Lemma 40 the matroid polytopes $P(\Lambda M)$, $P(\Lambda^* M)$ and $P(N_F)$, for each facet of $M$, form maximal cells of $\Sigma$. Further, those results show that for each facet of these three kinds of matroids there are precisely two maximal cells in that list which contain that facet. Since the dual graph of $\Sigma$ is connected this shows that these are all the maximal cells of $\Sigma$.

Moreover, for each facet $F$ of $M$, the three maximal cells $P(\Lambda M)$, $P(\Lambda^* M)$ and $P(\Lambda N_F)$ form a triangle in the tropical linear space. It follows from [HJS12, Proposition 28] that $\Sigma$ does not admit a non-trivial coarsening, i.e., $\rho(\Lambda M)$ is a ray of the secondary fan and thus of the Dressian.

Finally, by Theorem 35, the tropical Plücker vector $\rho(\Lambda M)$ can be lifted to an ordinary Plücker vector over $K\ll(t\rr)$ if and only if $\Lambda M$ is realizable over $K$. As $K$ is algebraically closed a matroid is realizable over $K$ if and only if any free extension or any series extension is realizable. □

Another general construction for producing tropical Plücker vectors and thus tropical linear spaces arises from point configurations in tropical projective tori. This has been investigated in [HJS12], [Rin13] and [FR15]. In the latter reference the resulting tropical linear spaces are called Stiefel tropical linear spaces. These two constructions are not mutually exclusive; there are Stiefel type rays which also arise via Theorem 41. Complete descriptions of the Dressians $Dr(3,n)$ are known for $n \leq 8$. All their rays are of Stiefel type or they arise from connected matroids of rank two via Theorem 41.

Via our method non-realizable matroids of rank three lead to interesting phenomena in rank four. In particular, the following consequence of the above answers [HJS12, Question 36].

Corollary 42. The Dressian $Dr(d,n)$ contains rays which do not admit a realization in any characteristic for $d = 4$ and $n \geq 11$ as well as for $d \geq 5$ and $n \geq 10$. There are rays of the Dressian $Dr(4,9)$ that are not realizable in characteristic 2 and others that are not realizable in any other characteristic.

Proof. The non-Pappus $(3,9)$-matroid and the Vamos $(4,8)$-matroid are not realizable in any characteristic. Both are connected and paving and hence split. The construction in Theorem 41 leads to non-realizable rays in $Dr(4,11)$ and $Dr(5,10)$. Each free extension or coextension of such a matroid is again connected and split. Thus we obtain non-realizable rays in all higher Dressians.

Applying Theorem 41 to the Fano and the non-Fano $(3,7)$-matroids we obtain two rays in $Dr(4,9)$. The first one is realizable solely in characteristic 2, whereas the other one is realizable in all other characteristics. □
Figure 2. Projection of the corank subdivision of $\Delta(3, 8)$ induced by $\Lambda S$ or, equivalently, induced by $\Lambda^* S$. There are five maximal cells, one of which is almost entirely hidden in the picture.

Example 43. Once again consider the snowflake matroid $S$ from Examples 8 and 29. The corank vector of the series-free lift $\Lambda S$ is a ray in $\text{Dr}(3, 8)$. Since $S$ has three split flacets the corank subdivision has $3 + 2 = 5$ maximal cells. This is the, up to symmetry, unique ray of $\text{Dr}(3, 8)$ which does not arise from point configuration in the tropical projective 2-torus; see [HJS12, Fig. 7]. A projection of this subdivision to three dimensions is shown in Figure 2.

5. Concluding remarks and open questions

It would be interesting to characterize the split matroids in terms of their minors. To this end we have the following contribution.

Proposition 44. The class of split matroids is closed under duality as well as under taking minors.

Proof. The matroid polytope $P(M^*)$ of the dual $M^*$ of a $(d, n)$-matroid $M$ is the image of $P(M) \subset \mathbb{R}^n$ under the the coordinate-wise transformation $x_i \mapsto 1 - x_i$. In particular, $P(M^*)$ is affinely isomorphic with $P(M)$. In view of Proposition 15 we may assume that $M$ is connected. In this case any flacet $F$ of $M$ is mapped to the flacet $[n] - F$ of $M^*$. The compatibility relation among the splits is preserved under affine transformations. It follows that $M^*$ is split if and only if $M$ is.

Assume that $M$ is a split matroid. Next we will show that the deletion $M|([n] - e)$ of an element $e \in [n]$ is again split. Since we already know that the class of split matroids is closed under duality it will follow that the class of split matroids is minor closed.

Let $F$ be a split flacet of $M|([n] - e)$. The $F$-hyperplane separates at least one vertex of $\Delta(d, [n] - e)$ from $P(M|([n] - e))$. This implies that the closure of $F$ in $M$ is a split flacet of $M$. For that closure there are two possibilities. So either $F$ or $F + e$ is a split flacet of $M$.

Let us suppose that $F$ and $G$ are two split flacets of $M|([n] - e)$ which are incompatible. That is, there is some point $x$ in the relative interior of $\Delta(d, [n] - e)$ which lies on the $F$-
and $G$-hyperplanes. We aim at finding at a contradiction by distinguishing four cases which arise from the two possibilities for the closures of the two facets $F$ and $G$.

First, suppose that $F$ and $G + e$ are split facets of $M$. Then there exists some element $h \in G - F$, for otherwise $e$ would be in the closure of $F$ in $M$. For each $\varepsilon > 0$ we define the vector $\hat{x} \in \mathbb{R}^n$ with

\begin{equation}
\hat{x}_e = \varepsilon, \quad \hat{x}_h = x_h - \varepsilon \quad \text{and} \quad \hat{x}_i = x_i \text{ for all other elements } i.
\end{equation}

If $\varepsilon > 0$ is sufficiently small then the vector $\hat{x}$ is contained in the relative interior of $\Delta(d, n)$. By construction $\hat{x}$ lies on the $F$- and $(G + e)$-hyperplanes, so that the corresponding splits are not compatible. This contradicts that $M$ is a split matroid.

The second case where $F + e$ and $G$ are split facets of $M$ is symmetric to the previous. Thirdly suppose that $F$ and $G$ are split facets of $M$. Assume that $M|([n] - e)$ is connected. Then we have $\#(F \cap G) + d > \text{rk}(F) + \text{rk}(G)$ from Proposition 14, and the same result implies that $F$ and $G$ are incompatible split facets of $M$. Again this is a contradiction to $M$ being split. So we assume that $M|([n] - e)$ is disconnected. Then there exists an element $h \in [n] - F - G - e$, and we may construct a relatively interior point $\hat{x} \in \Delta(d, n)$ as in (17). As before this leads to a contradiction to the assumption that $M$ is a split matroid.

In the fourth and final case $F + e$ and $G + e$ are split facets of $M$. As in the third case the desired contradiction arises from Proposition 14, provided that $M|([n] - e)$ is connected. It remains to consider the situation where $M|([n] - e)$ is disconnected. Then we can find elements $f \in F - G$, $g \in G - F$ and $h \in [n] - F - G - e$. As a minor variation to (17) we let

\begin{equation}
\hat{x}_e = \varepsilon, \quad \hat{x}_f = x_f - \varepsilon, \quad x_g = x_g - \varepsilon \quad \text{and} \quad \hat{x}_i = x_i \text{ for all other elements } i.
\end{equation}

The vector $\hat{x}$ lies on the $(F + e)$- and $(G + e)$-hyperplanes, as well as in the relative interior of $\Delta(d, n)$. This entails that the facets $F + e$ and $G + e$ are incompatible, and this concludes the proof. \hfill \Box

So it is natural to ask for the following.

**Question A.** What are the forbidden minors for the split matroids?

We want to list what we know about this question. The only disconnected minimal excluded minor is the $(4,8)$-matroid in Example 16. One can show that the rank of a connected excluded minor must be at least 3. The class of split matroids is also closed under dualization. Hence the number of elements is at least 6. There are precisely four excluded minors of rank 3 on 6 elements, up to symmetry. One of them is the matroid in Example 18, and a second one is its dual. The third example is the nested matroid $\Lambda(\Lambda U_{1,2})$; see Example 23. Finally, the fourth case has an extra split and is represented by the vectors: $(1,0,0), (1,0,0), (0,1,0), (1,1,0), (0,0,1), (1,0,1)$.

Here is another class of matroids of recent interest; see, e.g., Fife and Oxley [FO17]. A **laminar** family $\mathcal{L}$ of subsets of $[n]$ satisfies for all sets $A, B \in \mathcal{L}$ either $A \cap B = \emptyset$, $A \subseteq B$ or $B \subseteq A$. Furthermore, let $c$ be any real valued function on $\mathcal{L}$, and this is called a **capacity function.** A set $I$ is an independent set of the laminar matroid $L = L([n], \mathcal{L}, c)$ if $\#(I \cap A) \leq c(A)$ for all $A \in \mathcal{L}$. Here the triplet $([n], \mathcal{L}, c)$ is called a **presentation** of $L$. By [FO17, Theorem 2.7] each loop-free laminar matroid has a unique canonical presentation.
where the laminar family is the set of closures of the circuits, and the capacity function assigns to each set in the laminar family its rank. The class of split matroids and the class of laminar matroids are not contained in one another: The Fano matroid is a split matroid, but it is not laminar as it has closed circuits of size three which share exactly one element. On the other hand the nested matroid from Example 23 is not split. However, each nested matroid is laminar [FO17, Proposition 4.4].

It may be of general interest to look at tropical linear spaces where the matroidal cells correspond to matroids from a restricted class. For instance, Speyer [Spe09] looks at series-parallel matroids, and he conjectures that the tropical linear spaces arising from them maximize the $f$-vector. Tropical linear spaces all of whose maximal cells come from split matroids are necessarily one-dimensional, i.e., they are trees. For instance, this is always the case for $d = 2$.

Conceptually, it would be desirable to be able to write down all rays of all the Dressians and the tropical Grassmannians. Due to the intricate nature of matroid combinatorics, however, it seems somehow unlikely that this can ever be done in an explicit way. The next best thing is to come up with as many ray classes as possible. In [HJS12] tropical point configurations are used as data, whereas here we look at split matroids and their corank subdivisions. A third class of rays comes from the nested matroids. However, their analysis is beyond the scope of the present paper. It can be shown that the corank subdivision of a connected matroid $M$ is a “$k$-split” in the sense of Herrmann [Her11] if and only if $M$ is a nested matroid with $k + 1$ cyclic flats. The proof for this claim will be given elsewhere.

All known rays of the Dressians arise from corank vectors of various matroids. So the following is another obvious challenge.

**Question B.** Is there a ray in any Dressian that does not induce a corank subdivision?

A *polymatroid* is a polytope associated with a submodular function. This generalizes matroids given by their rank functions. Since splits are defined for arbitrary polytopes there is an obvious notion of a “split polymatroid”. It seems promising to investigate them.

Both polymatroids and tropical Plücker vectors are closely related to “integral discrete functions” which occur in discrete convex analysis; see, e.g., Murota [Mur03]. In that language a tropical Plücker vector is the same as an “$M$-concave function” on the vertices of the underlying matroid polytope. It would be interesting to investigate the notation of splits and realizability in terms of $M$-convexity. Hirai took a first step in this direction in [Hir06], where he studies splits of “polyhedral convex functions”.

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Appendix A. Some Matroid Statistics

Matsumoto, Moriyama, Imai and Brenner classified matroids of small rank with few elements [MMIB12]. A summary is given in Table 1 below. Based on the census of [MMIB12] we determined the percentages of paving and split matroids. The results are given in Table 2. That computation employed polymake [GJ00], and the results are accessible via the new database at db.polymake.org. In all tables we marked entries with — that have not been computed due to time and memory constraints.

Filtering all 190214 matroids of rank 4 on 9 elements for paving, sparse paving and splits matroids took about 2000 sec with polymake version 3.1 (AMD Phenom II X6 1090T with 3.6 GHz single-threaded, running openSUSE 42.1). We expect that the computation for all (4, 10)-matroids, which is the next open case, would take much more than 600 CPU days.

Example 45. All matroids of rank $d$ on $d + 2$ elements are split matroids. Table 2a shows that most of these are not paving.

Table 1. The number of isomorphism classes of all matroids of rank $d$ on $n$ elements, see [MMIB12, Table 1]

| $d \setminus n$ | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|-----------------|---|---|---|---|---|---|----|----|----|
| 2               | 7 | 13| 23| 37| 58| 87| 128| 183| 259|
| 3               | 4 | 13| 108|325|1275|10037|298491|31899134|
| 4               | 1 | 5 | 23 |108|940|190214|4886380924|— |— |— |
| 5               | 1 | 6 | 37 |325|190214|— |— |— |— |— |
| 6               | 1 | 7 | 58 |1275|4886380924|— |— |— |— |— |
| 7               | 1 | 8 | 87 |10037|— |— |— |— |— |— |
| 8               | 1 | 9 | 128|298491|— |— |— |— |— |— |
| 9               | 1 | 10| 183|31899134|— |— |— |— |— |— |
| 10              | 1| 11| 259|— |— |— |— |— |— |— |
| 11              | 1| 12|— |— |— |— |— |— |— |— |

References

[BdM08] Joseph E. Bonin and Anna de Mier. The lattice of cyclic flats of a matroid. Ann. Comb., 12(2):155–170, 2008.

[BPvdP15] Nikhil Bansal, Rudi A. Pendavingh, and Jorn G. van der Pol. On the number of matroids. Combinatorica, 35(3):253–277, 2015.

[BSSS90] Andries Brouwer, James B. Shearer, Neil Sloane, and Warren D. Smith. A new table of constant weight codes. IEEE Trans. Inform. Theory, 36(6):1334–1380, 1990.

[CRA14] Vanessa Chatelain and Jorge Luis Ramírez Alfonsín. Matroid base polytope decomposition II: Sequences of hyperplane splits. Adv. in Appl. Math., 54:121–136, 2014.

[DLRS10] Jesús A. De Loera, Jörg Rambau, and Francisco Santos. Triangulations, volume 25 of Algorithms and Computation in Mathematics. Springer-Verlag, Berlin, 2010. Structures for algorithms and applications.

[Duk04] Mark Dukes. On the number of matroids on a finite set. Sém. Lothar. Combin., 51:Art. B51g, 12, 2004.
Table 2. The percentage of paving and split matroids among the isomorphism classes of all matroids of rank $d$ on $n$ elements

| $d \backslash n$ | 4     | 5     | 6     | 7     | 8     | 9     | 10    | 11    | 12    |
|----------------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| 2              | 57    | 46    | 43    | 38    | 36    | 33    | 32    | 30    | 29    |
| 3              | 50    | 31    | 24    | 21    | 21    | 30    | 52    | 78    | 91    |
| 4              | 100   | 40    | 22    | 17    | 34    | 77    | −     | −     | −     |
| 5              | 100   | 33    | 14    | 12    | 63    | −     | −     | −     | −     |
| 6              | 100   | 29    | 10    | 14    | −     | −     | −     | −     | −     |
| 7              | 100   | 25    | 7     | 17    | −     | −     | −     | −     | −     |
| 8              | 100   | 22    | 5     | 19    | −     | −     | −     | −     | −     |
| 9              | 100   | 20    | 4     | 16    | −     | −     | −     | −     | −     |
| 10             | 100   | 18    | 3     | −     | −     | −     | −     | −     | −     |
| 11             | 100   | 17    |       |       |       |       |       |       |       |

| $d \backslash n$ | 4     | 5     | 6     | 7     | 8     | 9     | 10    | 11    | 12    |
|----------------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| 2              | 100   | 100   | 100   | 100   | 100   | 100   | 100   | 100   | 100   |
| 3              | 100   | 100   | 89    | 75    | 60    | 52    | 61    | 80    | 91    |
| 4              | 100   | 100   | 100   | 75    | 60    | 82    | −     | −     | −     |
| 5              | 100   | 100   | 100   | 100   | 60    | 82    | −     | −     | −     |
| 6              | 100   | 100   | 100   | 100   | 52    | −     | −     | −     | −     |
| 7              | 100   | 100   | 100   | 100   | 61    | −     | −     | −     | −     |
| 8              | 100   | 100   | 100   | 100   | 80    | −     | −     | −     | −     |
| 9              | 100   | 100   | 100   | 100   | 91    | −     | −     | −     | −     |
| 10             | 100   | 100   | 100   | 100   | 100   | −     | −     | −     | −     |
| 11             | 100   | 100   | 100   | 100   | 100   | −     | −     | −     | −     |

[DW92] Andreas W. M. Dress and Walter Wenzel. Valuated matroids. Adv. Math., 93(2):214–250, 1992.
[Edm70] Jack Edmonds. Submodular functions, matroids, and certain polyhedra. In Combinatorial Structures and their Applications (Proc. Calgary Internat. Conf., Calgary, Alta., 1969), pages 69–87. Gordon and Breach, New York, 1970.
[FO17] Tara Fife and James Oxley. Laminar matroids. European J. Combin., 62:206–216, 2017.
[FR15] Alex Fink and Felipe Rincón. Stiefel tropical linear spaces. J. Combin. Theory Ser. A, 135:291–331, 2015.
[FS05] Eva Maria Feichtner and Bernd Sturmfels. Matroid polytopes, nested sets and Bergman fans. Port. Math. (N.S.), 62(4):437–468, 2005.
[Fuj84] Satoru Fujishige. A characterization of faces of the base polyhedron associated with a submodular system. J. Oper. Res. Soc. Japan, 27(2):112–129, 1984.
[GGMS87] Izrail’ M. Gel’fand, Mark Goresky, Robert D. MacPherson, and Vera V. Serganova. Combinatorial geometries, convex polyhedra, and Schubert cells. Adv. in Math., 63(3):301–316, 1987.
[GJ00] Ewgenij Gawrilow and Michael Joswig. polymake: a framework for analyzing convex polytopes. In Polytopes—combinatorics and computation (Oberwolfach, 1997), volume 29 of DMV Sem., pages 43–73. Birkhäuser, Basel, 2000.
[Her11] Sven Herrmann. On the facets of the secondary polytope. J. Combin. Theory Ser. A, 118(2):425–447, 2011.
[Hir06] Hiroshi Hirai. A geometric study of the split decomposition. Discrete Comput. Geom., 36(2):331–361, 2006.
[HJ08] Sven Herrmann and Michael Joswig. Splitting polytopes. Münster J. Math., 1:109–141, 2008.
[HJJS09] Sven Herrmann, Anders Jensen, Michael Joswig, and Bernd Sturmfels. How to draw tropical planes. Electron. J. Combin., 16(2, Special volume in honor of Anders Björner): Research Paper 6, 26, 2009.
[HJS12] Sven Herrmann, Michael Joswig, and David Speyer. Dressians, tropical Grassmannians and their rays. Forum Mathematicum, pages 389–411, 2012.
[Knu74] Donald E. Knuth. The asymptotic number of geometries. J. Combin. Theory Ser. A, 16:398–400, 1974.
Yoshitake Matsumoto, Sonoko Moriyama, Hiroshi Imai, and David Bremner. Matroid enumeration for incidence geometry. *Discrete Comput. Geom.*, 47(1):17–43, 2012.

Dillon Mayhew, Mike Newman, Dominic Welsh, and Geoff Whittle. On the asymptotic proportion of connected matroids. *European J. Combin.*, 32(6):882–890, 2011.

Diane Maclagan and Bernd Sturmfels. *Introduction to Tropical Geometry*, volume 161 of Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2015.

Kazuo Murota. *Discrete Convex Analysis: Monographs on Discrete Mathematics and Applications 10*. Society for Industrial and Applied Mathematics, Philadelphia, PA, USA, 2003.

James Oxley. *Matroid theory*, volume 21 of *Oxford Graduate Texts in Mathematics*. Oxford University Press, Oxford, second edition, 2011.

Felipe Rincón. Local tropical linear spaces. *Discrete Comput. Geom.*, 50(3):700–713, 2013.

David E. Speyer. *Tropical geometry*. ProQuest LLC, Ann Arbor, MI, 2005. Thesis (Ph.D.)—University of California, Berkeley.

David E. Speyer. A matroid invariant via the $K$-theory of the Grassmannian. *Adv. Math.*, 221(3):882–913, 2009.

David Speyer and Bernd Sturmfels. The tropical Grassmannian. *Adv. Geom.*, 4(3):389–411, 2004.

Neil White, editor. *Theory of matroids*, volume 26 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1986.

Günter M. Ziegler. Lectures on 0/1-polytopes. In *Polytopes—combinatorics and computation (Oberwolfach, 1997)*, volume 29 of *DMV Sem.*, pages 1–41. Birkhäuser, Basel, 2000.

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