SOME PROPERTIES OF EXTENDED REMAINDER OF BINET’S FIRST FORMULA FOR LOGARITHM OF GAMMA FUNCTION

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ABSTRACT. In the paper, we extend Binet’s first formula for the logarithm of the gamma function and investigate some properties, including inequalities, star-shaped and sub-additive properties and the complete monotonicity, of the extended remainder of Binet’s first formula for the logarithm of the gamma function and related functions.

1. INTRODUCTION

1.1. For positive numbers $x$ and $y$ with $y > x$, let

$$g_{x,y}(t) = \int_x^y u^{t-1} \, du = \begin{cases} \frac{y^t - x^t}{t}, & t \neq 0, \\ \ln y - \ln x, & t = 0. \end{cases}$$

(1)

The reciprocal of $g_{x,y}(t)$ can be rewritten as

$$\frac{1}{g_{x,y}(t)} = F_{a,b}(t) = \begin{cases} \frac{t}{e^{at} - e^{bt}}, & t \neq 0, \\ \frac{1}{b - a}, & t = 0, \end{cases}$$

where $a$ and $b$ are real numbers with $b > a$.

It is well-known [11, p. 11] that Binet’s first formula of $\ln \Gamma(x)$ for $x > 0$ is given by

$$\ln \Gamma(x) = \left(x - \frac{1}{2}\right) \ln x - x + \ln \sqrt{2\pi} + \theta(x),$$

(2)

where

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} \, dt$$

stands for Euler’s gamma function and

$$\theta(x) = \int_0^\infty \left(\frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2}\right) \frac{e^{-xt}}{t} \, dt$$

(3)

for $x > 0$ is called the remainder of Binet’s first formula for the logarithm of the gamma function.

In [29, 30], some inequalities and completely monotonic properties of the function $g_{x,y}(t)$ were established and applied to construct Steffensen pairs in [4], to refine Gautschi-Kershaw’s inequalities in [24, 25], and to study monotonic properties, logarithmic convexities and Schur-convexities of extended mean values $E(r, s; x, y)$ in [13, 14, 15, 16, 26, 31]. See also [17, 18] for related contents.
In recent years, some inequalities and monotonic properties of the function
\[
\frac{1}{t^2} - \frac{e^{-t}}{(1 - e^{-t})^2}
\]
for \( t > 0 \) and related ones were researched in [5, 10, 32] and related references therein. These results were used in [2, 8, 27, 28] to consider completely monotonic properties of remainders of Binet’s first formula, the psi function and related ones.

Recently, logarithmic convexities of \( g_{x,y}(t) \) and \( F_{a,b}(t) \) were found in [7, 23]. By virtue of these conclusions, some simple and elegant proofs for the logarithmic convexities, Schur-convexities of extended mean values \( E(r, s; x, y) \) were simplified in [6, 19].

1.2. Now it is very natural to ask a question: Are there any relationship between the above studies? Direct computation yields
\[
[g_{e^t, e^t}(t)]' = (\ln F_{a,b}(t))' = \frac{be^{bt} - ae^{at}}{e^{bt} - e^{at}} - \frac{1}{t} = \frac{b - a}{e^{(b-a)t} - 1} - \frac{1}{t} + b \triangleq \delta_{a,b}(t)
\]
for \( t \neq 0 \). Therefore, if taking \( a = -\frac{1}{2} \) and \( b = \frac{1}{2} \), then \( \delta_{-1/2,1/2}(t) \) for \( t > 0 \) equals the integrand in the remainder of Binet’s first formula for the logarithm of the gamma function and the first order derivative of \( \delta_{-1/2,1/2}(t) \) for \( t > 0 \) also equals the function (4). These relationships connect closely the above three seemingly unrelated problems.

If replacing \( \delta_{-1/2,1/2}(t) \) by \( \delta_{a,b}(t) \) on \( t \in (0, \infty) \) for \( b > a \) in (3), then a more problem emerges: How to calculate the improper integral
\[
\int_0^\infty \left[ \frac{b - a}{e^{(b-a)t} - 1} - \frac{1}{t} + b \right] \frac{e^{-tx}}{t} \, dt
\]
for \( b > a \)? The following Theorem 1 answers this question affirmatively.

**Theorem 1.** Let \( b > a, \alpha > 0 \) and \( x \in \mathbb{R} \) be real numbers. Then the improper integral (6) converges if and only if \( a + b = 0 \) and
\[
\int_0^\infty \frac{\alpha}{e^{\alpha t} - 1} - \frac{1}{t} + \frac{\alpha}{2} \right] \frac{e^{-tx}}{t} \, dt = \alpha \ln \Gamma \left( \frac{x}{\alpha} \right) - \left( x - \frac{\alpha}{2} \right) \ln \frac{x}{\alpha} + x - \frac{\alpha}{2} \ln(2\pi).
\]

**Remark 1.** It is easy to see that the formula (2) is the special case \( \alpha = 1 \) of (7). So we call (7) the extended remainder of Binet’s first formula for the logarithm of the gamma function \( \Gamma \).

1.3. It is well-known [8, 20, 22] that a function \( f \) is said to be completely monotonic on an interval \( I \) if \( f \) has derivatives of all orders on \( I \) and
\[
(-1)^n f^{(n)}(x) \geq 0
\]
for \( x \in I \) and \( n \geq 0 \), that a function \( f(x) \) is said to be star-shaped on \( (0, \infty) \) if
\[
f(\alpha x) \leq \alpha f(x)
\]
for \( x \in (0, \infty) \) and \( 0 < \alpha < 1 \), that a function \( f \) is said to be super-additive on \( (0, \infty) \) if
\[
f(x + y) \geq f(x) + f(y)
\]
for all \( x, y > 0 \), and that a function \( f \) is said to be sub-additive if \( -f \) is super-additive.

The function \( \delta_{a,b}(t) \) defined by (5) has the following properties.

**Theorem 2.** Let \( a \) and \( b \) be real numbers with \( a \neq b \), and let \( 0 < \tau < 1 \).

(1) The function \( \delta_{a,b}(t) \) is increasingly concave on \( (0, \infty) \) and convex on \( (-\infty, 0) \);
Remark 1. The gamma function $\Gamma$ by
\[ \ln \Gamma(x) = \ln \Gamma(x) - \ln \Gamma(1) = \ln \Gamma(x) \]
is valid on $(0, \infty)$; if either $a + b \neq 0$ or $ab \neq 0$, then the inequality (8) is sharp;
(3) If $a + b \geq 0$, then
\[ \tau \delta_{a,b}(t) < \delta_{a,b}(t) \]
is valid on $(0, \infty)$, i.e., the function $-\delta_{a,b}(t)$ is star-shaped; if $\max\{a, b\} \leq 0$, then the inequality (9) is reversed, i.e., the function $\delta_{a,b}(t)$ is star-shaped; if $a + b = 0$, then the inequality (9) is sharp.

Remark 2. Some properties of special cases of the function $\delta_{a,b}(t)$ and related ones have been investigated and applied extensively in $[5, 6, 7, 8, 10, 23, 29, 30, 32]$ and related references therein.

1.4. If denoting the extended remainder of Binet’s first formula for the logarithm of the gamma function $\Gamma$ by
\[ \theta_{\alpha}(x) = \int_{0}^{\infty} \left( \frac{\alpha}{e^{\alpha t} - 1} - \frac{1}{t} + \frac{\alpha}{2} \right) e^{-tx} \frac{dt}{t} \] (10)
for $\alpha > 0$ and $x > 0$, then formula (7) in Theorem 1 can be simplified as
\[ \theta_{\alpha}(x) = \alpha \theta \left( \frac{x}{\alpha} \right) \quad \text{or} \quad \theta_{\alpha}(\alpha x) = \alpha \theta(x). \]
This motivates us to study properties of $\theta_{\alpha}(x)$ and the function
\[ f_{p,q;\alpha}(x) = \theta_{\alpha}(px) - q\theta_{\alpha}(x) \]
on $(0, \infty)$, where $p > 0$, $\alpha > 0$ and $q \in \mathbb{R}$, which may be concluded as the following theorem.

**Theorem 3.** The extended remainder $\theta_{\alpha}(x)$ of Binet’s first formula for the logarithm of the gamma function $\Gamma$ satisfies
\[ \frac{(-1)^k}{(1 + \lambda)^2} \theta^{(k)}_{\alpha} \left( \frac{x}{1 + \lambda} \right) > \frac{(-1)^k}{2 \lambda^k} \left[ \frac{1}{\lambda} \theta^{(k)}_{\alpha} \left( \frac{x}{\lambda} \right) + \theta^{(k)}_{\alpha}(x) \right] \] (11)
for $x > 0$, $\lambda > 0$ with $\lambda \neq 1$, $k \geq 0$ and $\alpha > 0$.

The function $f_{p,q;\alpha}(x)$ is completely monotonic on $(0, \infty)$ if either $0 < p \leq 1$ and $q \leq 1$ or $p > 1$ and $q \leq \frac{1}{p}$; the function $-f_{p,q;\alpha}(x)$ is completely monotonic on $(0, \infty)$ if $p \geq 1$ and $q \geq 1$.

The function $-\theta_{\alpha}(x)$ is star-shaped and $\theta_{\alpha}(x)$ is sub-additive.

Remark 3. If taking $a = -\frac{1}{2}$ and $b = \frac{1}{3}$, then the results obtained in $[2, 3, 8, 27, 28]$ can be deduced directly from Theorem 3.

2. Proofs of Theorems

**Proof of Theorem 1.** By transformations of integral variables, it easily follows that
\[ \int_{\varepsilon}^{\infty} \left[ \frac{b - a}{e^{(b-a)t} - 1} - \frac{1}{t} + b \right] e^{-tx} \frac{dt}{t} \]
\[ = \int_{(b-a)\varepsilon}^{\infty} \left( \frac{b - a}{e^{u} - 1} - \frac{b - a}{u} + b \right) e^{-ux/(b-a)} \frac{du}{u} \]
\[ = (b - a) \int_{(b-a)\varepsilon}^{\infty} \left( \frac{1}{e^{u} - 1} - \frac{1}{u} + \frac{1}{2} \right) e^{-ux/(b-a)} \frac{du}{u} + \frac{a + b}{2} \int_{\varepsilon}^{\infty} e^{-ux/(b-a)} \frac{du}{u} \]
Consequently, the function \( \delta \) convexity and concavity of Lazarević's inequality in [1, p. 131] and [9, p. 300] tells us that
\[
\begin{align*}
\tau &= (b - a) \int_{(b-a)\varepsilon}^{\infty} \left( \frac{1}{e^u - 1} - \frac{1}{u} + \frac{1}{2} \right) \frac{e^{-ux/(b-a)}}{u} \, du \\
&\quad + \frac{a + b}{2} \int_{(b-a)\varepsilon}^{\infty} \frac{e^{-ux}}{u} \, du,
\end{align*}
\] (12)
where \( \varepsilon > 0 \).

By virtue of Binet’s first formula for \( \ln \Gamma(z) \) in (3), the integral in the first term of (12) may be calculated as
\[
\lim_{\varepsilon \to 0^+} \int_{(b-a)\varepsilon}^{\infty} \left( \frac{1}{e^u - 1} - \frac{1}{u} + \frac{1}{2} \right) \frac{e^{-ux/(b-a)}}{u} \, du = \int_{0}^{\infty} \left( \frac{1}{e^u - 1} - \frac{1}{u} + \frac{1}{2} \right) \frac{e^{-ux/(b-a)}}{u} \, du = \ln \Gamma \left( \frac{x}{b-a} \right) - \left( \frac{x}{b-a} - \frac{1}{2} \right) \ln \frac{x}{b-a} + \frac{x}{b-a} - \frac{1}{2} \ln(2\pi).
\] (13)
Furthermore, the second integral in (12) satisfies
\[
\lim_{\varepsilon \to 0^+} \int_{(b-a)\varepsilon}^{\infty} \frac{e^{-ux}}{u} \, du = \lim_{\varepsilon \to 0^+} \int_{(b-a)\varepsilon}^{\infty} t^{-1}e^{-t} \, dt = \int_{0}^{\infty} t^{-1}e^{-t} \, dt
\]
which is divergent. As a result, the improper integral (6) is convergent if and only if \( a + b = 0 \).

Taking \( a = -b \) in (6) and (13) and simplifying yields formula (7). The proof of Theorem 1 is complete.

\textbf{Proof of Theorem 2.} Straightforward computation gives
\[
\begin{align*}
\delta_{a,b}'(t) &= \frac{1}{t^2} - \frac{(a-b)^2 e^{(a+b)t}}{(e^{at} - e^{bt})^2}, \\
\delta_{a,b}''(t) &= \frac{(a-b)^3 e^{(a+b)t}(e^{at} + e^{bt})}{(e^{at} - e^{bt})^3} - \frac{2}{t^3} \\
&= \frac{2e^{3(a+b)t/2} (at - bt)}{t^3} \left( e^{at} - e^{bt} \right)^3 \left\{ \frac{e^{(a-b)t/2} + e^{(b-a)t/2}}{2} - \frac{e^{(a-b)t/2} - e^{(b-a)t/2}}{2} \right\}^3 \\
&= \frac{2e^{3(a+b)t/2}}{t^3} \left( \frac{at - bt}{e^{at} - e^{bt}} \right)^3 Q \left( \frac{a - b}{2} t \right).
\end{align*}
\]
Lazarević’s inequality in [1, p. 131] and [9, p. 300] tells us that
\[
Q(t) = \frac{e^{-t} + e^t}{2} - \left( \frac{e^t - e^{-t}}{2t} \right)^3 = \cosh t - \left( \frac{\sinh t}{t} \right)^3 < 0
\]
for \( t \in \mathbb{R} \) with \( t \neq 0 \). Hence \( \delta_{a,b}''(t) < 0 \) on \((0, \infty)\) and \( \delta_{a,b}''(t) > 0 \) on \((-\infty, 0)\). The convexity and concavity of \( \delta_{a,b}(t) \) are proved.

Since \( \delta_{a,b}''(t) < 0 \) on \((0, \infty)\), the derivative \( \delta_{a,b}'(t) \) is decreasing on \((0, \infty)\) for all real numbers \( a \) and \( b \) with \( a \neq b \). Since
\[
\delta_{a,b}'(t) = \frac{1}{t^2} - \frac{(a-b)^2 e^{(b-a)t}}{[1 - e^{(a-b)t}]^2} = \frac{1}{t^2} - \frac{(a-b)^2 e^{(b-a)t}}{[1 - e^{(a-b)t}]^2},
\]
it is easy to obtain that
\[
\lim_{t \to \infty} \delta_{a,b}'(t) = 0.
\]
Consequently, the function \( \delta_{a,b}'(t) \) is positive, and so \( \delta_{a,b}(t) \) is increasing, on \((0, \infty)\). This means that inequality (8) holds for \( 0 < \tau < 1 \) and \( t > 0 \).
From
\[
\delta_{a,b}(t) = \frac{b e^{bt} - a e^{at}}{e^{bt} - e^{at}} - \frac{1}{t} = \frac{b e^{(b-a)t} - a}{e^{(b-a)t} - 1} - \frac{1}{t} = \frac{b - a e^{(a-b)t}}{1 - e^{(a-b)t}} - \frac{1}{t},
\]

it follows easily that
\[
\lim_{t \to \infty} \delta_{a,b}(t) = \max\{a, b\}.
\]

L'Hôpital's rule gives
\[
\lim_{t \to 0^+} \delta_{a,b}(t) = \lim_{t \to 0^+} \frac{t(b e^{bt} - a e^{at}) - e^{bt} + e^{at}}{t(e^{bt} - e^{at})} = \lim_{t \to 0^+} \frac{b^2 t e^{bt} - 2 a^2 e^{at}}{(b e^{bt} - a e^{at}) + (e^{bt} - e^{at})/t} = \frac{a + b}{2}.
\]

For \(0 < \tau < 1\), let
\[
h_{a,b}(t) = \delta_{a,b}(\tau t) - \tau \delta_{a,b}(t)
\]
for \(t > 0\). It is obvious that
\[
\lim_{t \to 0^+} h_{a,b}(t) = \frac{(1 - \tau)(a + b)}{2} \quad \text{and} \quad \lim_{t \to \infty} h_{a,b}(t) = (1 - \tau) \max\{a, b\}.
\]

Since \(\delta_{a,b}'(t)\) is decreasing, then
\[
h'_{a,b}(t) = \tau [\delta_{a,b}'(\tau t) - \delta_{a,b}'(t)] > 0,
\]
and so \(h_{a,b}(t)\) is strictly increasing. If \(a + b \geq 0\), then \(h_{a,b}(t) > 0\) on \((0, \infty)\) and inequality (9) is valid. If \(\max\{a, b\} = 0\), then inequality (9) is reversed.

It is apparent that
\[
\lim_{t \to \infty} \frac{\delta_{a,b}(\tau t)}{\delta_{a,b}(t)} = 1
\]
if \(ab \neq 0\), which implies that inequality (8) is sharp. If \(a + b \neq 0\), then it is clear that
\[
\lim_{t \to 0^+} \frac{\delta_{a,b}(\tau t)}{\delta_{a,b}(t)} = 1,
\]
which also implies that inequality (8) is sharp.

By L'Hôpital's rule, it is not difficult to obtain that
\[
\lim_{t \to 0^+} \delta_{a,b}'(t) = \frac{(a - b)^2}{12}.
\]

If \(a + b = 0\) and \(a \neq b\), then
\[
\lim_{t \to 0^+} \frac{\delta_{a,b}(\tau t)}{\delta_{a,b}(t)} = \lim_{t \to 0^+} \frac{\tau \delta_{a,b}'(\tau t)}{\delta_{a,b}'(t)} = \tau,
\]
which means that inequality (9) is sharp. The proof of Theorem 2 is complete. □

Proof of Theorem 3. From the concavity of \(\delta_{a,b}(t)\) on \((0, \infty)\), it follows that
\[
\delta_{-\alpha/2,\alpha/2}(\frac{(1 + \lambda)t}{2}) > \frac{\delta_{-\alpha/2,\alpha/2}(\lambda t) + \delta_{-\alpha/2,\alpha/2}(t)}{2}
\]
for \(t > 0\) and positive numbers \(\alpha\) and \(\lambda \neq 1\). Multiplying by the factor \(t^{k-1} e^{-tx}\)
for any nonnegative integer \(k \geq 0\) and integrating from 0 to \(\infty\) on both sides of the above inequality yields
\[
\int_0^\infty \left[\frac{\alpha}{e^{\alpha(t+\lambda)t/2} - 1} - \frac{2}{(1+\lambda)t} + \frac{\alpha}{2}\right] t^{k-1} e^{-tx} \, dt
\]
It is clear that those of the function arguments that δ follow. Substituting formula (10) and its derivatives into the above inequalities leads to

\[
\int_0^\infty \left( \frac{\alpha}{e^{\alpha t} - 1} - \frac{1}{t} + \frac{\alpha}{2} \right) t^{k-1} e^{-tx} \, dt
\]

which can be rewritten by transformations of integral variables as

\[
\int_0^\infty \left( \frac{\alpha}{e^{\alpha u} - 1} - \frac{1}{u} + \frac{\alpha}{2} \right) \frac{u^k}{(1 + \lambda)^k} \cdot \frac{e^{-u/(1+\lambda)}}{u} \, du
\]

Further taking \( x = \alpha t \) if \( p > q \) and \( p < q < 1 \) if \( p < q \) ≤ \( \frac{1}{2} \), or 0 < \( q < 1 \) and \( q < \frac{1}{2} \), and \( q < \frac{1}{2} \) if \( p = q \), or \( q > \frac{1}{2} \), or \( q < \frac{1}{2} \). By virtue of properties of \( \delta_{\alpha/2,2,\alpha/2}(t) \) obtained in Theorem 2, it follows by standard arguments that

1. \( h_{p,q,\alpha}(t) \geq 0 \) if either \( 0 < p \leq 1 \) and \( q \leq 1 \), or \( p > 1 \) and \( q \leq \frac{1}{p} \), or \( 0 < q < 1 \) and \( q \leq \frac{1}{p} \);
2. \( h_{p,q,\alpha}(t) \leq 0 \) if \( p \geq 1 \) and \( q > 1 \).

It is clear that

1. if \( h_{p,0,\alpha}(t) \geq 0 \) then \( f_{p,0,\alpha}(x) \) is completely monotonic on \((0, \infty)\);
2. if \( h_{p,0,\alpha}(t) \leq 0 \) then \( -f_{p,0,\alpha}(x) \) is completely monotonic on \((0, \infty)\).

As a result, the completely monotonic properties of \( f_{p,0,\alpha}(x) \) is proved.

It is easy to see that the star-shaped properties of the function \( \theta_\alpha(x) \) follow from those of the function \( \delta_{\alpha/2,2,\alpha/2}(t) \) and formula (10).

In [12, p. 453], it was presented that a star-shaped function must be super-additive, therefore the function \( \theta_\alpha(x) \) is also sub-additive. The proof of Theorem 3 is complete.

**Remark 4.** Dividing both sides of (7) by \( \alpha > 0 \) gives

\[
\int_0^\infty \left( \frac{1}{e^{\alpha t} - 1} - \frac{1}{\alpha t + \frac{1}{2} \frac{e^{-x/t}}{t}} \right) \, dt = \ln \Gamma \left( \frac{x}{\alpha} \right) - \left( \frac{x}{\alpha} - \frac{1}{2} \right) \ln \frac{x}{\alpha} + \frac{x}{\alpha} - \frac{1}{2} \ln(2\pi). \tag{14}
\]

Further taking \( x = \alpha y \) in (14) yields

\[
\int_0^\infty \left( \frac{1}{e^{\alpha t} - 1} - \frac{1}{\alpha t + \frac{1}{2} \frac{e^{-\alpha y/t}}{t}} \right) \, dt = \ln \Gamma(y) - \left( \frac{y - \frac{1}{2}}{2} \right) \ln y + y - \frac{1}{2} \ln(2\pi). \tag{15}
\]
Since
\[
\int_0^\infty \left( \frac{1}{e^{\alpha t} - 1} - \frac{1}{t} + \frac{1}{2} \right) e^{-\alpha t} d\alpha = \int_0^\infty \left( \frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2} \right) e^{-t} dt,
\]
so the identity (7) is essentially equivalent to (2).

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