Preheating is the stage of profuse production of out of equilibrium particles either by parametric amplification of quantum fluctuations or by the growth of spinodal instabilities prior to the reheating stage in inflationary cosmologies. Such a mechanism was recognized to be more efficient than single particle decay 1,2 in terms of reheating the post inflationary universe.

The origin of such a mechanism was traced to parametric amplification of quantum fluctuations by the evolution of the inflaton zero mode. In a previous article 3 we studied such processes, offering a detailed numerical analysis in several approximation schemes. We found some very interesting new phenomena, especially in the case of broken symmetry, where, for “slow-roll” initial conditions, the zero mode is driven back to the origin and most of the energy has been transferred to the non-equilibrium fluctuations. The quantitative features, such as time scales for the preheating stage, the rate of particle production and the total number of particles produced are strongly model dependent.

The possibility of symmetry restoration via this non-equilibrium fluctuations was put forth in reference 4, wherein such a mechanism was offered as an explanation for the unusual behavior found in 3. In a recent article 6 we provided a more thorough analysis of these new phenomena, focusing on the description in Minkowski space-time, within the large N approximation. This approximation provides a consistent, renormalizable, energy conserving scheme that permits an accurate numerical analysis of the evolution equations.

In this article we recognized that approximate schemes using the Mathieu equation to describe the stage of parametric amplification, though qualitatively correct in the sense that a solid intuition on the processes can be obtained from such an approximation, are quantitatively in error. We also recognized that the possibility of symmetry restoration is not borne out in the broken symmetry phase, ruling out the potential explanations proposed in ref. 7 for our previous results. In our recent article we have provided an analytic and numerical study for this phenomenon and explained its physical reasons.

In a recent comment to our article 8, L. Kofman, A. Linde and A. A. Starobinsky (KLS) present a criticism of our results. Which we analyze within the context of our article below.

I. THE SETTING:

As mentioned above, the phenomenon of preheating and reheating is strongly model dependent, as well as highly non-perturbative in nature. Our study focuses on the description of this phenomenon in Minkowski space to provide a detailed and deeper understanding within a simpler setting. The non-perturbative nature of this phenomenon requires that a consistent approximation scheme be invoked, and we used the large N expansion which is a non-perturbative scheme that can be consistently improved, unlike the Hartree approximation.

In this approximation, the dynamics, including back-reaction effects, can be summarized in the evolution equations for the expectation value of the zero mode and the fluctuations. We analyzed two different cases: unbroken and broken symmetry. The comment by KLS introduces expansion the of the universe, which certainly does not apply to our study.

A. Unbroken symmetry: Lamé vs. Mathieu

In the unbroken symmetry case we examined the case of very large energy density initially stored in the zero mode. In the large N limit the field expectation value \( \phi(t) = \langle \Phi(x,t) \rangle \) (with \( \langle \cdots \rangle \) being the expectation value in the translational invariant but non-equilibrium quantum state) obeys the evolution equation

\[
\ddot{\phi}(t) + M^2[\Phi] \phi(t) = 0,
\]

while the mode functions driving the quantum fluctuations obey:

\[
\ddot{\chi}_k + (k^2 + M^2[\Phi]) \chi_k(t) = 0
\]
the mode themselves through the quantum fluctuations. Therefore this is a highly non-linear integro-differential problem. We emphasized that the full backreaction problem is energy conserving and provided a detailed account of the renormalization aspects. Note that none of these issues were taken up by KLS in their analysis.

In the initial stage, all the energy is assumed to be in the zero mode of the field. This corresponds to choosing as initial state the “vacuum” for the $k \neq 0$ modes. For very weak coupling the back-reaction effect will be negligible up to a time we call the preheating time: we will estimate this time below. Thus for the early stages of the dynamics and for very weak coupling one can approximate

$$M^2[\Phi] \approx m^2 + \frac{\lambda}{2} \phi(t)^2 \quad (3)$$

Now, using the classical oscillating behaviour of $\phi(t)$ one is led by eq.(3) to an effective mass that oscillates in time. In this approximation (which is indeed very good for weak coupling and during the preheating stage, see (4)), eq.(3) exhibits parametric resonance as noticed first in ref. [1]. Namely, there are allowed and forbidden bands in $k^2$. The modes within the forbidden bands grow exponentially, whereas those in the allowed bands remain bounded in absolute value. The growth of the modes in the forbidden bands is interpreted as the production of particles when the back-reaction becomes important; this happens the approximation (3) breaks down, and the energy transferred from the zero mode to the produced particles is a large fraction of the initial zero mode energy.

In approximations that do not include the backreaction effects there is infinite particle production, since the particles is a large fraction of the initial zero mode energy.

This exponential, non-perturbative growth of quantum fluctuations will lead to strong backreaction effects once the contribution from the quantum fluctuations to $M^2[\Phi]$ becomes of the same order as the tree-level terms. When this happens the approximation (3) breaks down, and the energy transferred from the zero mode to the produced particles is a large fraction of the initial zero mode energy.

In approximations that do not include the backreaction effects there is infinite particle production, since the effect of draining energy from the zero mode is not taken into account. The full backreaction problem and the approximation used in our work maintains energy conservation, and thus displays the shutting down of the particle production when the back-reaction becomes important; this is the end of the preheating stage.

Now, in order to compute quantitatively the number of particles produced, the behavior of the zero mode is required. In ref. [1] $\phi(t)$ is approximated by a cosine in the calculations. The mode equations then become the Mathieu equation. As shown in ref. [3], the exact classical solution is actually a cn Jacobi function. The difference between this and a trigonometric function is profound.

Let us now compare the results from the exact mode solutions obtained in ref. [3] with the Mathieu equation approximation to it. In units where $m^2 = 1$ and setting $\eta(t) = \eta_0 \ cn \left( t, \sqrt{1 + \eta_0^2}, \bar{k} \right)$, one finds

$$\eta(t) = \eta_0 \ cn \left( t, \sqrt{1 + \eta_0^2}, \bar{k} \right), \quad (4)$$

where cn stands for the Jacobi cosine and we choose for initial conditions $\eta(0) = \eta_0, \ q(0) = 0$.

Inserting this form for $\eta(t)$ in eqs.(3) and (2) yields

$$\left[ \frac{d^2}{dt^2} + k^2 + 1 + \eta_0^2 \ cn^2 \left( t, \sqrt{1 + \eta_0^2}, \bar{k} \right) \right] \chi_k(t) = 0 . \quad (5)$$

This is the Lamé equation for a particular value of the coefficients that make it solvable in terms of Jacobi functions (see [3] and references therein). As shown in ref. [3], this equation has only one forbidden band for positive $k^2$, which runs from $k^2 = 0$ to $k^2 = \eta_0^2$. One can choose Floquet solutions of eq.(3) fulfilling the relation

$$U_k(t + 2\omega) = e^{iF(k)} U_k(t), \quad (6)$$

where the Floquet indices $F(k)$ are independent of $t$. In the forbidden band the $F(k)$ posses an imaginary part. Their exact form results [3]

$$F(k) = -2iK(\bar{k}) Z(2K(\bar{k}) v) + \pi$$

where $Z(\nu)$ is the Jacobi zeta function (see [3] and references therein) and $v$ is a function of $k$ in the forbidden band defined by

$$k = \frac{\eta_0}{\sqrt{2}} \ cn(2K(\bar{k}) v, k), \quad 0 \leq v \leq \frac{1}{2} \quad (7)$$

All these elliptic functions posses fastly convergent expansions in powers of the elliptic nome

$$q \equiv e^{-\pi K'((\bar{k})/K(\bar{k}))} .$$

Since $0 \leq \bar{k} \leq 1/\sqrt{2}$ [see eq.(3)], we have

$$0 \leq q \leq e^{-\pi} = 0.0432139 \ldots . \quad (8)$$

Then,

$$F(k) = 4i \pi q \ \sin(2\pi v) \left[ 1 + 2q \ \cos 2\pi v + O(q^2) \right] + \pi . \quad (9)$$

The imaginary part of this function has a maximum at

$$k = k_1 = \frac{1}{2} \eta_0 (1 - q) + O(q^2) \quad \text{where} \quad [3]$$

$$F \equiv \text{Im} F(k_1) = 4 \pi q + O(q^3) . \quad (10)$$

This simple formula gives the maximum of the imaginary part of the Floquet index in the forbidden band with a precision better than $8 \times 10^{-5}$. $q$ can be expressed in terms of $\eta_0$ as follows [3]

$$q = \frac{1}{2} \left( 1 + \eta_0^2 \right)^{1/4} - \frac{1}{2} \left( 1 + \eta_0^2 / 2 \right)^{1/4} .$$
with an error smaller than $\sim 10^{-7}$.

Let us now proceed to the Mathieu equation analysis of this problem. The cn Jacobi function can be expanded as (see [3] and references therein)

$$\text{cn}(z, k) = (1 - q) \cos(1 - 4q)z + q \cos 3z + O(q^2).$$

To zeroth order in $q$ we have

$$\eta(t)^2 = \frac{\eta_0^2}{2} \left[ 1 + \cos(2t \sqrt{1 + \eta_0^2}) \right] + O(q).$$

and $2\omega = \pi/\sqrt{1 + \eta_0^2} + O(q)$. Under such approximations eq. (3) becomes the Mathieu equation [8]

$$\frac{d^2y}{dz^2} + (a - 2\bar{q} \cos 2z) y(z) = 0,$$

where

$$a = 1 + \frac{k^2 - \frac{\eta_0^2}{\eta_0^2 + 1}}{\eta_0^2 + 1}, \quad \bar{q} = \frac{\eta_0^2}{4(\eta_0^2 + 1)}$$

and $z = \sqrt{\eta_0^2 + 1} t$. Notice that $0 \leq \bar{q} \leq 1/4$ in the present case. Eq. (11) possesses an infinite number of forbidden bands for $k^2 > 0$. The lower and upper band edges for the first band are given by [8]

$$k_{\text{in,}f}^2 = \frac{\eta_0^2}{4} \left[ 1 - \frac{\eta_0^2}{2^5(\eta_0^2 + 1)} + \frac{\eta_0^4}{2^{10}(\eta_0^2 + 1)} + \ldots \right],$$

$$k_{\text{sup}}^2 = \frac{\eta_0^2}{4} \left[ 3 - \frac{\eta_0^2}{2^5(\eta_0^2 + 1)} - \frac{\eta_0^4}{2^{10}(\eta_0^2 + 1)} + \ldots \right].$$

These values must be compared with the exact result for the Lamé equation given by [8]: $k_{\text{in,}f}^2 = 0$, $k_{\text{sup}}^2 = \frac{\eta_0^2}{2}$. Although the width of the band is well approximated by the Mathieu equation, its absolute position is not. The numerical values of the maximum of the imaginary part of the Floquet index are given in Table 1 and compared with the exact values from eq. (12).

| $\eta_0$ | $F_{\text{Lame}}$ | $F_{\text{Mathieu}}$ | %error |
|---|---|---|---|
| 1 | 0.2258$\ldots$ | 0.20$\ldots$ | 13% |
| 4 | 0.4985$\ldots$ | 0.37$\ldots$ | 35% |
| $\eta_0 \to \infty$ | $4\pi e^{-\eta} = 0.5430 \ldots$ | 0.39$\ldots$ | 39% |

TABLE I. The maximum of the imaginary part of the Floquet index $F$ for the Lamé equation and for its Mathieu approximation.

We see that the Mathieu approximation underestimates the exact result by a fraction ranging from 13% to 39%. The second forbidden band in the Mathieu equation yields $F_{\text{Mathieu}} = 0.086 \ldots$ for $\eta_0 \to \infty$. This must be compared with $F_{\text{Lame}} = 0$ corresponding to the fact that there is only one forbidden band in the Lamé equation.

In ref. [6], a large discrepancy between Lame and Mathieu Floquet indices has been reported within a different approximation scheme.

It is worth mentioning that differences in the Floquet indices such as those displayed, enter in the exponent. In the large N approximation we see that the discrepancy between Mathieu and Lame is very large and therefore cause very large errors in quantitative estimates of particle production and preheating time.

Although in some cases as found by KLS the Mathieu equation gives a reasonable estimate, in some other cases it clearly does not. This was the point raised in our article, each case must be treated in its own right.

For example, the number of particles produced during reheating is of the order of the exponential of $2F$ times the reheating time in units of $\pi/\sqrt{1 + \eta_0^2}$. An error of 25% in $F$ means an error of 25% in the exponent, so that one would find $10^9$ instead of $10^{12}$.

The Mathieu equation approximation would be exact in absence of the $\lambda \Phi^4$ inflaton self-coupling. That is, for the classical potential [8]

$$V = \frac{1}{2} m^2 \Phi^2 + g \sigma^2 \Phi^2,$$

one can consider as classical solution $\Phi(t) = \Phi_0 \cos(mt)$, $\sigma = 0$. However, the potential (12) is unstable under renormalization (a $\Phi^4$ counterterm is needed from the one-loop level). Hence, the $\lambda = 0$ choice is a fine-tuning not protected by any symmetry.

The mode equations (5) apply to the selfcoupled $\lambda \Phi^4$ scalar field. Models for reheating usually contain at least two fields: the inflaton and a lighter field $\sigma(x)$ in which the inflaton decays. For a $g \sigma^2 \Phi^2$ coupling, the mode equations for the $\sigma$ field take the form (8) (adapted to Minkowski space which is the focus of our article)

$$\ddot{V}_k + \left( k^2 + m_\sigma^2 + \frac{g}{\lambda} F[\eta(.)] \right) V_k(t) = 0$$

A new dimensionless parameter $\frac{4}{\lambda}$ appears here. Neglecting the $\sigma$ and $\Phi$ backreaction, we have

$$F[\eta(.)] \simeq \eta^2(t).$$

In ref. [6], we show that abundant particle production (appropriate for reheating) occurs even for $g = \lambda$.

In hep-ph/9608343, it is stated that “the main subject of investigation in [32] was the theory $m^2 \phi^2/2 + \lambda \phi^3/4$. There are two main regimes there: $\phi \ll m/\sqrt{\lambda}$ and $\phi \gg m/\sqrt{\lambda}$. The authors of [32] do not make any distinction between these two regimes because they neglect expansion of the universe. However, from [2] it follows that
in expanding universe there is no parametric resonance at all for \( \phi \ll m/\sqrt{\lambda} \). Therefore all results of \([3]\) related to parametric resonance in this regime do not give a correct description of reheating in the theory \( m^2 \phi^2/2 + \phi^4/4 \)."

Firstly, parametric resonance and universe expansion are independent processes that deserve separate investigation. Secondly, in our investigations in Minkowski spacetime \([\text{Ref.}]\) we provide analytic expressions which hold both for small and large amplitudes. One explicitly sees there that the parametric resonance dies in the limit of small amplitude.

Moreover, in ref. \([3]\) we studied the small amplitude case \( \eta << 1 \), in Minkowski space and compared explicitly with the one-loop and linear relaxation (single particle decay) results showing in detail how parametric amplification for large amplitudes merges with linear relaxation at small amplitudes.

Therefore, the above statement in \([\text{hep-ph}/9608341]\) cannot be applied to ref. \([3]\) neither to our previous works.

Eqs. (13)-(14) become Lamé equations when \( \eta(t) \) is approximated by the classical solution in Minkowski spacetime given by \([3]\). This Lamé equation is solvable in closed form when the couplings \( g \) and \( \lambda \) are related as follows

\[
\frac{2g}{\lambda} = n(n + 1), \quad n = 1, 2, 3, \ldots
\]

In those cases there are \( n \) forbidden bands for \( k^2 \geq 0 \). The Lamé equation exhibits an infinite number of forbidden bands for generic values of \( \frac{g}{\lambda} \). The Mathieu and WKB approximations have also been applied in the non-exactly solvable cases \([3]\). However, as the above analysis shows, (see Table I) these results cannot be trusted quantitatively. The only available precise method consists on accurate numerical calculations as those of ref. \([3]\) (where the precision was at least \( 10^{-6} \)).

As soon as the quantum fluctuations grow and cease to be negligible compared with the the classical contribution \([3]\), all the approximations discussed so far (Lamé, Mathieu, etc.) break down. This is the so-called preheating time \( t_{\text{pre}} \). One can estimate \( t_{\text{rea}} \) by equating the zero mode energy \([3]\) with the estimation of the quantum fluctuations derived from the unstable Floquet modes \([3]\). Such estimation yields accurate estimates when the Lamé Floquet indices are used \([3]\). However, as emphasized before, because of the non-perturbative nature of this time scale \( t_{\text{rea}} \approx \ln[1/\lambda] \) \([3]\), differences in the Floquet indices stemming from a Mathieu equation approximation to a Lamé equation lead to severe errors in the quantitative estimate of such a time scale.

Although a fairly accurate estimate of \( t_{\text{reh}} \) can be obtained via the Lamé equation, with the exact classical solution for the zero mode, in order to compute physical magnitudes beyond \( t_{\text{reh}} \), one must solve self-consistently the field equations including the back reaction. Clearly this requires a numerical treatment. In ref. \([3]\) this is done for the \( N \to \infty \) limit and in ref. \([3]\) to one-loop order. Such a study led to a very clear physical description of the non-equilibrium gas of created particles, and its equation of state \([3]\).

Thus, at the expense of re-iterating our conclusions and comments in our article, a Mathieu equation approximation provides a qualitative, intuitive description of parametric amplification, and the non-perturbative phenomena associated with particle production out of equilibrium. However, for a quantitative and trustworthy estimate of the physical time scales and production rates one must study the proper problem in its full complexity. Differences in Floquet indices, propagate exponentially making the qualitative estimates based on Mathieu unreliable whenever the Mathieu equation is not the proper description. Ultimately one must resort to a full numerical scheme to study the dynamics at long times, making sure that the approximations involved maintain energy conservation (or covariant conservation). Given that this phenomenon is not universal and strongly model dependent, one equation does not fit all the different situations and a careful and consistent analysis of each particular scenario is needed.

B. Broken Symmetry:

Our study of the dynamics in the broken symmetry case in connection with the possibility of symmetry restoration as advocated by KLS was sparked by the following statement in their article “Nonthermal Phase Transitions...” by KLS, Phys. Rev. Lett. 76, 1011, (1996) \([3]\) and again by Kofman \([\text{Ref.}]\) in his review article. In particular, on page 1012, second column, third paragraph of this article: “The mechanism of symmetry restoration described above is very general; in particular, it explains a surprising behavior of oscillations of the scalar field found numerically in the \( O(N) \)-symmetric model of Ref.\([6]\)”. This reference is to our previous article \([3]\), and there we studied the same situation that we re-analyzed in our latest article: Minkowski space-time, large \( N \) and broken symmetry phase, with the zero mode of the field beginning very close to the origin, i.e. with “slow-roll” initial conditions. It is precisely this situation that we studied in deeper detail in our latest article and concluded after a thorough analytic and numerical study that there is no symmetry restoration by the quantum fluctuations. Energy is conserved and the sum rule

\[
- 1 + \eta(\infty)^2 + g\Sigma(\infty) = 0
\]

which is nothing but a statement of massless pions and a consequence of Goldstone’s theorem, is satisfied. The final value \( \eta(\infty) \) is obtained dynamically, and depends on the initial condition that determines the total energy of the system which is conserved by the numerics to one part in \( 10^{10} \) which is our numerical accuracy.

We have provided exhaustive evidence for this behavior both analytically and numerically and showed unequivocally that there is no possibility of symmetry restora-
tion as envisaged by KLS and as quoted by in the situation that we studied.

We have learned privately that Cooper, Habib, Mottola and Kluger have obtained similar conclusions in their thorough and independent study of the broken symmetry phase, within the same approximation scheme. Their results are complementary to ours, Cooper et. al. study the strong coupling regime, but the behavior is consistent with the results obtained for the weak coupling case in our article.

In their comment, KLS seem to agree with our conclusion that in this particular situation studied in our articles, when the expectation value is released near the potential hill, symmetry restoration does not occur. We have pointed out in and more recently in that the notion of the effective potential is irrelevant for the dynamics and these effects should be understood as a dynamical change of the effective action. The minima of the effective action attained dynamically are very different from those of the effective potential.

In the situation of ‘chaotic initial conditions’ but with a broken symmetry tree level potential, the issue of symmetry breaking is more subtle. In this case the zero mode is initially displaced with a large amplitude and very high in the potential hill. The total energy density is non-perturbatively large. Classically the zero mode will undergo oscillatory behavior between the two classical turning points, of very large amplitude and the dynamics will probe both broken symmetry states. Even at the classical level the symmetry is respected by the dynamics in the sense that the time evolution of the zero mode samples equally both vacua.

This situation is reminiscent of finite temperature in which case the energy density is finite and above a critical temperature the ensemble averages sample both tree level vacua with equal probability thus restoring the symmetry. In the dynamical case, the “symmetry restoration” is just a consequence of the fact that there is a very large energy density in the initial state, much larger than the top of the tree level potential, thus under the dynamical evolution the system samples both vacua equally.

Thus the criterion for symmetry restoration when the tree level potential allows for broken symmetry states is that the energy density in the initial state be larger than the top of the tree level potential. That is when the amplitude of the zero mode is such that \( V(\eta_0) > V(0) \). In this case the dynamics will be very similar to the unbroken symmetry case, the amplitude of the zero mode will damp out, transferring energy to the quantum fluctuations via parametric amplification, but asymptotically oscillating around zero with a fairly large amplitude.

II. CONCLUSIONS

We here answered the points raised by KLS in hep-ph/9608341. We refer to our paper where in over 50 pages of text plus twenty figures, we provide a detailed and exhaustive analysis of the non-equilibrium issues in preheating both in the unbroken and broken symmetry phases, in Minkowski space-time, within a well defined non-perturbative, consistent, renormalizable and energy conserving scheme. We have given all the necessary technical details including the error estimates in the numerical study and included two lengthy appendices with the necessary details for the reader to follow all and every step of our analysis.

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