ON CONFORMAL INVARIANCE OF ISOTROPIC GEODESICS

M.A. Akivis and V.V. Goldberg

Abstract

We consider real isotropic geodesics on manifolds endowed with a pseudoconformal structure and their applications to the theory of lightlike hypersurfaces on such manifolds, to the geometry of four-dimensional conformal structures of Lorentzian type, and to a classification of the Einstein spaces.

0. Introduction. It is well-known that geodesics on a Riemannian or pseudo-Riemannian manifold are not invariant with respect to conformal transformations of a Riemannian metric. However, in many problems of geometry and physics connected with the theory of pseudo-Riemannian manifolds, the isotropic geodesics (i.e., geodesics that are tangent to isotropic cones at each of their points) arise.

In the present paper we prove that the isotropic geodesics are invariant with respect to conformal transformations of a pseudo-Riemannian metric. This is a reason that it is appropriate to consider the isotropic geodesics on a manifold endowed with a pseudoconformal structure.

The isotropic geodesics arise naturally when one studies lightlike hypersurfaces that are also conformally invariant. We prove that such hypersurfaces possess a foliation formed by isotropic geodesics. In another terminology this fact is given in [DB 96]. We also consider the isotropic geodesics on manifolds endowed with a pseudoconformal structure of signature $(1, 3)$ and apply the obtained results to a classification of the Einstein spaces. Note that the geometry of the conformal structures of signature $(1, 3)$ was also considered in the recent paper [AZ 95].

1. Preliminaries. Let $V^n_q = (M, g)$ be a pseudo-Riemannian manifold of signature $(p, q)$, where $p + q = n = \dim M$ and $0 < q < n$.

We associate with any point $x \in M$ its tangent space $T_x(M)$, and define the frame bundle whose base is the manifold $M$ and the fibers are the families of vectorial frames $\{e_1, \ldots, e_n\}$ in $T_x(M)$ defined up to a transformation of the general linear group $GL(n)$. Let us denote by $\{\omega^1, \ldots, \omega^n\}$ the co-frame dual to the frame $\{e_1, \ldots, e_n\}$:

$$\omega^i(e_j) = \delta^i_j, \quad i, j = 1, \ldots, n.$$ 

Then an arbitrary vector $\xi \in T_x(M)$ can be written as

$$\xi = \omega^i(\xi)e_i.$$ 

The forms $\omega^i$ can be considered as differential forms on the manifold $M$ if we assume that $\xi = dx$ is the differential of the point $x \in M$. Thus the form $g$ can be written as

$$g = g_{ij}\omega^i\omega^j. \quad (1)$$

The equation

$$g_{ij}\omega^i\omega^j = 0 \quad (2)$$

defines the foliation of isotropic cones $C_x$ of the manifold $M$.

It is well-known that on the manifold $M$ an invariant torsion-free affine connection $\gamma$ is defined by means of the basis forms $\omega^i$ and connection forms $\omega^i_j$ satisfying the following structure equations:

$$d\omega^i = \omega^k \wedge \omega^i_k, \quad d\omega^i_j = \omega^k_j \wedge \omega^i_k + R^i_{jkl} \omega^k \wedge \omega^l \quad (3)$$

where $R^i_{jkl}$ is the curvature tensor of the connection $\gamma$ (see, for example, [C 28]). This invariant torsion-free affine connection $\gamma$ is called the Levi-Civita connection if it satisfies the equation

$$\nabla g_{ij} := dg_{ij} - g_{kj} \omega^k_i - g_{ik} \omega^k_j = 0. \quad (4)$$

The operator $\nabla$ is called the operator of covariant differentiation with respect to this connection. The Levi-Civita connection is uniquely determined on $(M, g)$.

Consider a conformal transformation of a pseudo-Riemannian metric, that is, consider another pseudo-Riemannian metric

$$\overline{g} = \sigma g = \sigma g_{ij}\omega^i\omega^j, \quad (5)$$

where $\sigma = \sigma(x)$ is a function of a point $x \in M$ and $\sigma(x) > 0$. The metric tensor of this new metric $\overline{g}$ has the form

$$\overline{g}_{ij} = \sigma g_{ij}. \quad (6)$$

We assume that the basis forms $\omega^i$ are not changed.

Denote by $\overline{\gamma}$ the Levi-Civita connection defined by the metric $\overline{g}$ and by $\overline{\nabla}$ the operator of covariant differentiation with respect to the connection $\overline{\gamma}$. Then

$$\overline{\nabla}\overline{g}_{ij} = 0. \quad (7)$$

It is easy to prove that the connection forms $\overline{\omega}^i_j$ of the new connection $\overline{\gamma}$ are expressed in terms of the connection forms $\omega^i_j$ of the connection $\gamma$ as follows:

$$\overline{\omega}^i_j = \omega^i_j + \frac{1}{2}(\delta^i_j \sigma_k \omega^k + \sigma_j \omega^i - \sigma_i \omega^j), \quad (8)$$

where the quantities $\sigma_k$ are defined by the equation

$$d \log \sigma = \sigma_k \omega^k. \quad (9)$$
and
\[ \omega_j = g_{jk} \omega^k, \quad \sigma^j = g^{jk} \sigma_k. \]  

2. Existence of isotropic geodesics. Consider geodesics on a pseudo-Riemannian manifold \( M \). By means of the basis forms \( \omega^i \) and connection forms \( \omega^i_j \) the equations of geodesics on \( M \) can be written in the form
\[ d\omega^i + \omega^j \omega^i_j = \alpha \omega^i, \]  
where \( \alpha \) is a 1-form, and \( d \) is the symbol of ordinary (not exterior) differentiation.

An isotropic geodesic on the manifold \( M \) is a geodesic that is tangent to the isotropic cone \( C_x \) at each of its points \( x \). In addition to equations (10), such geodesics also satisfy equation (2) of the isotropic cone \( C_x \).

We will now prove the following result which confirms the existence of isotropic geodesics:

**Theorem 1** If a geodesic of the manifold \( M \) is tangent to the isotropic cone at one of its points \( x_0 \), then this curve is tangent to the isotropic cones at any other of its points \( x \); that is, this curve is an isotropic geodesic.

**Proof.** The geodesic \( x = x(t) \) in question is uniquely defined by the system of differential equations (11) and initial conditions \( x(t_0) = x_0 \) and \( \frac{dx}{dt}|_{t=t_0} = a_0 \). Since by hypothesis the geodesic is tangent to the isotropic cone at the point \( x_0 \), we have
\[ g^0_{ij} a^i_0 a^j_0 = 0, \]
where by \( g^0_{ij} \) we denote the values of the components of the metric tensor \( g_{ij} \) at the point \( x_0 \). The last condition can be rewritten in the form
\[ (g_{ij} \omega^i \omega^j)|_{x=x_0} = 0, \]  
(12)
since we have \( \omega^i = a^i dt \) along the curve \( x = x(t) \).

Differentiating the left-hand side of equation (2) and taking into account equations (4) and (11), we find that
\[ d(g_{ij} \omega^i \omega^j) = 2\alpha g_{ij} \omega^i \omega^j, \]
where the differentiation is carried out along the curve \( x = x(t) \). Along this curve, the 1-form \( \alpha \) is a total differential and can be written in the form \( \alpha = d\varphi \), where \( \varphi = \varphi(t) \). Thus the last equation can be written in the form
\[ d(g_{ij} \omega^i \omega^j) = 2d\varphi \cdot g_{ij} \omega^i \omega^j. \]

Integrating this equation, we find that
\[ g_{ij} \omega^i \omega^j = C e^{2\varphi}. \]
But since for \( t = t_0 \) condition (12) holds, we find that \( C = 0 \) and that
\[
g_{ij} \omega^i \omega^j = 0
\]
everywhere along the curve \( x = x(t) \), so this curve is an isotropic geodesic. □

It follows from Theorem 1 that in the pseudo-Riemannian manifold \( V^n_q \), where \( q > 0 \), through any point \( x \) and along any isotropic direction emanating from this point, there passes one and only one isotropic geodesic.

Note that the usual model of space-time in general relativity is a four-dimensional pseudo-Riemannian manifold with signature \((1,3)\) (Lorentzian signature) (e.g., see [Ch 83], Ch. 2, §11). Isotropic geodesics of this manifold are curves of propagation of light impulses. Hence they are important in this theory.

3. Conformal invariance of isotropic geodesics. On the manifold \( V^n_q = (M, \mathcal{F}) \) the equations of geodesic lines has the form
\[
d\omega^i + \omega^j \xi^j_i = \omega^i.
\]
(13)
Substituting the values (8) of the forms \( \xi^j_i \) into equations (13), we obtain the equations of geodesics in the connection \( \mathcal{F} \) in the form
\[
d\omega^i + \omega^j \omega^i_j - \frac{1}{2} \sigma^i g_{jk} \omega^j \omega^k = (\alpha - d \log \sigma) \omega^i.
\]
(14)
Comparing equations (11) and (14), we see that under conformal transformation of a pseudo-Riemannian metric, geodesics do not remain invariant. The reason for this is the third term on the left-hand side of equation (14) containing \( \sigma^i \). However, there are two cases where equation (14) defines the same curves as equation (11). First of all, this happens if \( \sigma^i = 0 \), that is, if \( \sigma = \text{const} \). In this case equation (14) coincides with equation (11) with \( \sigma = \alpha \), and all geodesics are transformed into geodesics. But this case is not so interesting, since the conformal transformation has a very special form if \( \sigma = \text{const} \). Second, equation (14) defines the same curves as equation (11) if \( g_{jk} \omega^j \omega^k = 0 \), that is, if the geodesic is isotropic. In this case equations (14) and (11) coincide if \( \sigma = \alpha + d \log \sigma \). Thus we have proved the following result:

**Theorem 2** Under the general conformal transformation of a pseudo-Riemannian metric on a manifold \( M \), isotropic geodesics and only such geodesics remain invariant. □

4. The conformal structure \( CO(n-1, 1) \) The conformal structure \( CO(n-1, 1) \) on a manifold \( M \) of dimension \( n \) is a set of conformally equivalent pseudo-Riemannian metrics with the same signature \((n-1,1)\). Such a structure is called conformally Lorentzian.

A metric \( g \) can be given on \( M \) by means of a nondegenerate quadratic form
\[
g = g_{ij} du^i du^j,
\]
where $u^i, i = 1, \ldots, n$, are curvilinear coordinates on $M$, and $g_{ij}$ are the components of the metric tensor $g$.

A pseudoconformal structure on a manifold $M$ is the collection of all pseudo-Riemannian metrics obtained from a fixed pseudo-Riemannian metric by conformal transformations. In other words, we can say that a conformal structure on a manifold $M$ is defined by means of a relatively invariant quadratic form

$$g = g_{ij} du^i du^j.$$

The equation $g = 0$ defines in the tangent space $T_x(M)$ a cone $C_x$ of second order called the isotropic cone. Thus the conformal structure $CO(n-1,1)$ can be given on the manifold $M$ by a field of cones of second order.

The cone $C_x \subset T_x(M)$ remains invariant under transformations of the group $G = SO(n-1,1) \times H$, where $SO(n-1,1)$ is the special $n$-dimensional pseudoorthogonal group of signature $(n-1,1)$ (the connected component of the unity of the pseudoorthogonal group $O(n-1,1)$), and $H$ is the group of homotheties. Thus the conformal structure $CO(n-1,1)$ is a $G$-structure defined on the manifold $M$ by the group $G$ indicated above. For the conformal structure $CO(n-1,1)$ the isotropic cone is real.

As in Section 1, we associate with any point $x \in M$ its tangent space $T_x(M)$, and define the frame bundle whose base is the manifold $M$ and the fibers are the families of vectorial frames $\{e_1, \ldots, e_n\}$ in $T_x(M)$. If $\{\omega^1, \ldots, \omega^n\}$ is the co-frame dual to the frame $\{e_1, \ldots, e_n\}$, then the form $g$ can be written as

$$g = g_{ij} \omega^i \omega^j$$

(see Section 1).

The structure equations of the $CO(n-1,1)$-structure can be reduced to the following form (see [AG 96], Section 4.1):

$$d\omega^i = \theta \wedge \omega^i + \omega^j \wedge \theta^i_j,$$  \hspace{1cm} (16)

$$d\theta = \omega^i \wedge \theta_i,$$  \hspace{1cm} (17)

$$d\theta^i_j = \theta_j \wedge \omega^i + \theta^k_j \wedge \theta^i_k + g_{jk} \omega^k \wedge g^{il} \theta_l + C^i_{jkl} \omega^k \wedge \omega^l,$$  \hspace{1cm} (18)

$$d\theta_i = \theta_i \wedge \theta + \theta^j_i \wedge \theta_j + C^i_{jkl} \omega^j \wedge \omega^k,$$  \hspace{1cm} (19)

and the metric tensor $g_{ij}$ satisfies the equations

$$dg_{ij} - g_{ik} \theta^k_j - g_{kj} \theta^k_i = 0.$$  \hspace{1cm} (20)

Note that in equations (16)–(20) the forms $\omega^i$ are defined in a first-order frame bundle, the 1-forms $\theta^i_j$ and a scalar 1-form $\theta$ in a second-order frame bundle, and a covector form $\theta_i$ in the third-order frame bundle.
For $C_{ijkl} = C_{ijk} = 0$, equations (16)–(20) coincide with the structure equations of the pseudoconformal space $C^n$. For this reason the object \{\[C_{ij}, C_{ij}^k\] is called the curvature object of the conformal structure $CO(n - 1, 1)$.

The quantities $C_{ijkl}$ form a $(1, 3)$-tensor which is called the Weyl tensor or the tensor of conformal curvature of the structure $CO(n - 1, 1)$.

The quantities $C_{ijkl}$ and $C_{ijk}$, occurring in equations (18) and (19), satisfy the conditions:

$$C_{ijkl} = -C_{ikjl}, C_{ijk} = -C_{ikj}, C_{ijkl} = 0.$$  \hfill (21)

These quantities and the tensor $g_{ij}$ also satisfy some algebraic and differential equations (see [AG 96], Section 4.1).

5. Isotropic geodesics on lightlike hypersurfaces of a manifold $M$ endowed with $CO(n - 1, 1)$-structure. A lightlike hypersurface $V^{n-1}$ on a manifold $M$ of dimension $n$ endowed with a $CO(n - 1, 1)$-structure of Lorentzian signature $(n - 1, 1)$ is a hypersurface which is tangent to the isotropic cone $C_x$ at each point $x \in V^{n-1}$.

Let $T_x(V^{n-1})$ be the tangent subspace to $V^{n-1}$. In $T_x(M)$ we choose a vectorial frame $\{e_1, \ldots, e_n\}$ in such a way that its vector $e_1$ has the direction of the generator of the isotropic cone $C_x$ along which the subspace $T_x(V^{n-1})$ is tangent to $C_x$; the vector $e_n$ also has a direction of a generator of the cone $C_x$ that does not belong to the subspace $T_x(V^{n-1})$, and we locate the vectors $e_a$ in the $(n - 2)$-dimensional subspace of intersection of $T_x(V^{n-1})$ and the subspace tangent to $C_x$ along $e_n$. Then

$$(e_1, e_1) = (e_n, e_n) = 0, \quad (e_a, e_1) = (e_a, e_n) = 0, \quad (e_a, e_b) = g_{ab}, (e_1, e_n) = -1,$$  \hfill (22)

where the parentheses denote the scalar product of vectors in $T_x(M)$ defined by the quadratic form (15), and $a, b = 2, \ldots, n - 1$. The last relation in (22) is a result of an appropriate normalization of the vectors $e_1$ and $e_n$.

With respect to the chosen moving frame, the fundamental form $g$ of $M$ can be reduced to the expression

$$g = g_{ab} \omega^a \omega^b - 2 \omega^1 \omega^n, \quad a, b = 2, \ldots, n - 1,$$  \hfill (23)

and the quadratic form $\bar{g}$ defining the conformal structure on $V^{n-1}$ has the form

$$\bar{g} = g_{ab} \omega^a \omega^b$$  \hfill (24)

and is of signature $(n - 2, 0)$, that is, the form $\bar{g}$ is positive definite. The isotropic cone $C_x$ is determined by the equation $g = 0$. Thus, the components $g_{ij}$ of the tensor $g$ are the entries of the following matrix:

$$(g_{ij}) = \begin{pmatrix} 0 & 0 & -1 \\ 0 & g_{ab} & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$  \hfill (25)
Equations (20) and (25) imply that

\[\begin{align*}
\theta^a_1 &= \theta^n_1 = 0, \\
\theta^n_a &= g_{ab}\theta^b_1, \\
dg_{ab} - g_{ac}\theta^c_b - g_{cb}\theta^c_a &= 0.
\end{align*}\]  (26)

Since the vectors \(e_1\) and \(e_a\) of the frame \(\{e_1, e_a, e_n\}\) of \(T_x(M)\) belong to \(T_x(V^{n-1})\), and the vector \(e_n\) does not belong to \(T_x(V^{n-1})\), for \(x \in V^{n-1}\) we have

\[dx = \omega_1 e_1 + \omega^a e_a.\]

This means that our hypersurface \(V^{n-1}\) is defined by the following Pfaffian equation:

\[\omega^n_0 = 0,\]  (27)

and the forms \(\omega^1, \omega^a, a = 2, \ldots, n-1,\) are basis forms of the hypersurface \(V^{n-1}\).

Taking exterior derivative of equation (27) by means of (16), we obtain the exterior quadratic equation

\[\omega^a \wedge \theta^n_a = 0.\]

Applying Cartan’s lemma to this equation, we find that

\[\theta^a_a = \lambda_{ab}\omega^b, \quad \lambda_{ab} = \lambda_{ba}.\]  (28)

It follows from equations (26) and (28) that

\[\theta^a_1 = g^{ab}\lambda_{bc}\omega^b,\]  (29)

where \(g^{ab}\) is the inverse tensor of the tensor \(g_{ab}\).

An isotropic geodesic on the manifold \(M\) is a geodesic which is tangent to the isotropic cone \(C_x\) at each of its points \(x\). We will prove now the following theorem:

**Theorem 3** A lightlike hypersurface \(V^{n-1} \subset CO(n-1, 1)\) carries a foliation formed by isotropic geodesics.

**Proof.** Since the vectors \(e_1\) form an isotropic vector field on a hypersurface \(V^{n-1}\), the equations of the isotropic foliation on \(V^{n-1}\) have the form

\[\omega^a = 0.\]  (30)

Thus equation (2) is satisfied.

Let us prove that the curves belonging to this foliation are isotropic geodesics. The isotropic geodesics on the conformal structure \(CO(n-1, 1)\) are determined
by equations (2) and (11). However, in Section 4 we changed the notations \( \omega_j^i \) for \( \theta_j^i \). Thus equations (11) can be written as
\[
d\omega^i + \omega^j \theta^i_j = \alpha \omega^i, \quad i, j = 1, \ldots, n,
\]
where as before \( \alpha \) is a 1-form, and \( d \) is the symbol of ordinary (not exterior) differentiation. In our moving frame equations (31) take the form
\[
\begin{align*}
d\omega_1^1 + \omega_1^a \theta_1^a + \omega_2^a \theta_2^a + \omega_3^a \theta_3^a + \omega_n^a \theta_n^a &= \alpha \omega_1^1, \\
d\omega_2^1 + \omega_1^b \theta_1^b + \omega_2^b \theta_2^b + \omega_3^b \theta_3^b + \omega_n^b \theta_n^b &= \alpha \omega_2^1, \\
d\omega_n^1 + \omega_1^n \theta_1^n + \omega_2^n \theta_2^n + \omega_3^n \theta_3^n + \omega_n^n \theta_n^n &= \alpha \omega_n^1.
\end{align*}
\]
(32)
But by means of equations (27) and (28), which are valid on \( V^{n-1} \), and equation (30) defining the isotropic foliation on \( V^{n-1} \), equations (32) are identically satisfied.

Note that a similar result in another terminology is given in [DB 96], p. 86.

6. Isotropic geodesics on a manifold \( M \) endowed with \( CO(1, 3) \)-structure. We now consider the pseudoconformal \( CO(1, 3) \)-structure. In a pseudoorthonormal frame its fundamental form \( g \) becomes
\[
g = - (\omega^1)^2 - (\omega^2)^2 - (\omega^3)^2 + (\omega^4)^2.
\]
Transformations of the tangent subspace \( T_x(M) \) preserving the form \( g \) make up the pseudoorthogonal group \( SO(1, 3) \) which is called the Lorentz group. The isotropic cone \( C_x \subset T_x(M) \) which is determined by the equation \( g = 0 \) remains invariant under transformations of the group \( G = SO(1, 3) \times H \) where \( H \) is the group of homotheties.

By means of the real transformation
\[
\begin{align*}
\frac{\omega^1 + \omega^4}{\sqrt{2}} &\rightarrow \omega^1, \\
\frac{\omega^1 - i \omega^3}{\sqrt{2}} &\rightarrow \omega^4, \\
\omega^2 &\rightarrow \omega^2, \\
\omega^3 &\rightarrow \omega^3
\end{align*}
\]
the form \( g \) can be reduced to the form
\[
g = 2 \omega^1 \omega^4 - (\omega^2)^2 - (\omega^3)^2.
\]
It is easy to see now that the cone \( g = 0 \) carries real one-dimensional generators (the straight lines \( \omega^1 = \omega^2 = \omega^3 = 0 \) and \( \omega^4 = \omega^2 = \omega^3 = 0 \) are examples of such generators) but does not carry two-dimensional generators.

We complexify the tangent space \( T_x(M) \) by setting \( CT_x(M) = T_x(M) \otimes \mathbb{C} \). Moreover, in \( CT_x(M) \), we will consider only such transformations that preserve its real subspace \( T_x(M) \) and the symmetry with respect to \( T_x(M) \).

Next, by means of the complex transformation
\[
\begin{align*}
\omega^1 &\rightarrow \omega^1, \\
\frac{\omega^2 + i \omega^3}{\sqrt{2}} &\rightarrow \omega^2, \\
\frac{\omega^2 - i \omega^3}{\sqrt{2}} &\rightarrow \omega^3, \\
\omega^4 &\rightarrow \omega^4,
\end{align*}
\]
we reduce the quadratic form $g$ to the form

$$g = 2(\omega^1\omega^4 - \omega^2\omega^3),$$  \hfill (33)

where the 1-forms $\omega^1$ and $\omega^4$ are real, and the 1-forms $\omega^2$ and $\omega^3$ are complex conjugate forms:

$$\overline{\omega}^1 = \omega^1, \quad \overline{\omega}^4 = \omega^4, \quad \overline{\omega}^3 = \omega^2.$$  \hfill (34)

It follows that the isotropic cone $g = 0$ carries two-dimensional complex conjugate plane generators.

A vectorial frame in the space $CT_3(M)$, in which the form $g$ on the $CO(1,3)$-structure reduces to form (33), satisfies the conditions

$$\overline{e}_1 = e_1, \quad \overline{e}_4 = e_4, \quad \overline{e}_2 = e_3.$$  \hfill (35)

Such a frame is called a Newman-Penrose tetrad (see [NP 62] and [Ch 83], Ch. 1, §8). In such a frame the vectors $e_1$ and $e_4$ are real, and the vectors $e_2$ and $e_3$ are complex conjugate.

Equations

$$\omega^1 + \lambda \omega^3 = 0, \quad \omega^2 + \lambda \omega^4 = 0$$  \hfill (36)

and

$$\omega^1 + \mu \omega^2 = 0, \quad \omega^3 + \mu \omega^4 = 0$$  \hfill (37)

determine two families of two-dimensional complex plane generators of the complexified cone $C_x$. On the $CO(1,3)$-structure, the parameters $\lambda$ and $\mu$ in equations (36) and (37) are complex coordinates on the projective lines $CP_\alpha$ and $CP_\beta$. These plane generators are, respectively, the $\alpha$-planes and the $\beta$-planes of the $CO(1,3)$-structure. If in equations (36) we replace all quantities by their conjugates, we obtain equations (37), where $\mu = \overline{\lambda}$. Thus there is a one-to-one correspondence between $\alpha$-planes and $\beta$-planes of these two families of plane generators of the cone $C_x$, and this correspondence is determined by the condition $\mu = \overline{\lambda}$.

Since to each point $x \in M$ of a real manifold $M$ carrying a $CO(1,3)$-structure there correspond two families of 2-planes, the family of $\alpha$-planes and the family of $\beta$-planes, determined by complex parameters $\lambda$ and $\mu$, two bundles, $E_\alpha = (M, CP_\alpha)$ and $E_\beta = (M, CP_\beta)$, arise on $M$, and these two bundles have the manifold $M$ as their common base and the families of complex plane generators of the cone $C_x$ as their fibers. These bundles are called the isotropic bundles of the $CO(1,3)$-structure.

Since $\mu = \overline{\lambda}$, the isotropic bundles $E_\alpha = (M, CP_\alpha)$ and $E_\beta = (M, CP_\beta)$ are complex conjugates: $\overline{E}_\beta = E_\alpha$.

Two complex conjugate generators of the cone $C_x$ determined by the parameters $\lambda$ and $\mu = \overline{\lambda}$ intersect one another along its real rectilinear generator. The equation of this generator can be found from equations (36) and (37) provided that $\mu = \overline{\lambda}$. Solving these equations, we find that

$$\omega^1 = \lambda \overline{\lambda} \omega^4, \quad \omega^2 = -\lambda \omega^4, \quad \omega^3 = -\overline{\lambda} \omega^4.$$
Hence the directional vector of the rectilinear generator can be written in the form
\[ \xi = \lambda \overline{e}_1 - \lambda e_2 - \overline{e}_3 + e_4. \] (38)

Since the basis vectors of the complexified space $CT_x$ satisfy relations (35), the vector $\xi$ is real. It depends on one complex parameter or two real parameters. Equation (38) can be considered as the equation of the director two-dimensional surface of the three-dimensional cone $C_x$ in the real space $T_x(M)$.

7. Structure equations of the CO (1, 3)-structure. In the adapted frame only the following components of the tensor $g_{ij}$ will be nonzero: $g_{14} = g_{41} = 1$, $g_{23} = g_{32} = -1$. In view of this, equations (20) imply that the forms $\theta^i_j$ satisfy the conditions
\[
\begin{align*}
\theta_4^1 = \theta_2^2 = \theta_3^3 = \theta_1^1 & = 0, \\
\theta_2^2 = \theta_4^1, \quad \theta_3^3 = \theta_1^2, \quad \theta_1^3 = \theta_2^1, \\
\theta_1^1 + \theta_4^4 & = 0, \quad \theta_2^2 + \theta_3^3 = 0.
\end{align*}
\] (39)

Now equations (16) take the form
\[
\begin{align*}
\rho \omega^1 &= (\theta - \theta^1_1) \wedge \omega^1 + \omega^2 \wedge \theta^1_2 + \omega^3 \wedge \theta^1_3, \\
\rho \omega^2 &= (\theta - \theta^2_2) \wedge \omega^2 + \omega^1 \wedge \theta^2_1 + \omega^3 \wedge \theta^2_3, \\
\rho \omega^3 &= (\theta + \theta^3_3) \wedge \omega^3 + \omega^1 \wedge \theta^3_1 + \omega^2 \wedge \theta^3_2, \\
\rho \omega^4 &= (\theta + \theta^4_4) \wedge \omega^4 + \omega^2 \wedge \theta^4_1 + \omega^3 \wedge \theta^4_2.
\end{align*}
\] (40)

By virtue of equations (39), among the forms $\theta^i_j$ only the forms $\theta^1_1, \theta^2_2, \theta^3_3, \theta^4_4, \theta^1_1, \theta^2_2$ are independent. If $\omega^i = 0$, these forms together with the 1-form $\theta$ are the invariant forms of a seven-parameter group $G \subset \text{GL}(4)$ that preserves the cone $C_x$ determined by equations $g = 0$ where $g$ is determined by equation (33).

To write structure equations (18) for the CO(1, 3)-structure, we consider its tensor of conformal curvature $C_{ijkl}$. This tensor has 21 essential nonvanishing components that satisfy 11 independent conditions (see [AG 96], Section 5.1):
\[
\begin{align*}
C_{1234} - C_{1324} + C_{1423} & = 0, \\
C_{1224} = C_{1334} = C_{1213} = C_{2434} & = 0, \\
C_{1314} - C_{1323} = C_{1424} - C_{2324} & = 0, \\
C_{1214} + C_{1223} = C_{1434} + C_{2334} & = 0, \\
C_{1414} = C_{2323} = C_{1234} + C_{1324} & = 0.
\end{align*}
\] (41)

Hence the tensor $C_{ijkl}$ has 10 independent components in all. We denote them as follows:
\[
\begin{align*}
C_{1212} = a_0, \quad C_{1214} = a_1, \quad C_{1234} = a_2, \quad C_{1434} = a_3, \quad C_{1434} = a_4, \\
C_{1313} = b_0, \quad C_{1314} = b_1, \quad C_{1324} = b_2, \quad C_{1424} = b_3, \quad C_{2424} = b_4.
\end{align*}
\] (42)
The remaining components of the tensor of conformal curvature $C_{ijkl}$ are expressible in terms of the above components (42).

Now we can write equations (17) and (18) for the $CO(1,3)$-structure in more detail. The former can be written as

$$d\theta = \omega^1 \land \theta_1 + \omega^2 \land \theta_2 + \omega^3 \land \theta_3 + \omega^4 \land \theta_4,$$

and by (39) and (42), the latter has the form

$$d\theta^1_1 = \theta_1 \land \omega^1 - \theta_2 \land \omega^2 + \theta^1_2 \land \theta^1_1 + \theta^1_3 \land \theta^1_3$$

$$-2[a_1 \omega^1 \land \omega^2 + a_2 (\omega^1 \land \omega^3 - \omega^2 \land \omega^3) + a_3 \omega^3 \land \omega^4]$$

$$+ b_1 \omega^1 \land \omega^2 + b_2 (\omega^1 \land \omega^4 + \omega^2 \land \omega^3) + b_3 \omega^2 \land \omega^4],$$

$$d\theta^2_2 = \theta_2 \land \omega^2 - \theta_3 \land \omega^3 - \theta^2_3 \land \theta^2_2 + \theta^2_4 \land \theta^2_4$$

$$-2[a_1 \omega^1 \land \omega^2 + a_2 (\omega^1 \land \omega^4 + \omega^2 \land \omega^3) + a_3 \omega^3 \land \omega^4]$$

$$- b_1 \omega^1 \land \omega^2 - b_2 (\omega^1 \land \omega^4 + \omega^2 \land \omega^3) - b_3 \omega^2 \land \omega^4],$$

$$d\theta^3_3 = \theta_3 \land \omega^3 + \theta_4 \land \omega^4 + (\theta^3_1 - \theta^3_2) \land \theta^3_4$$

$$+ 2[a_1 \omega^1 \land \omega^2 + b_1 (\omega^1 \land \omega^4 + \omega^2 \land \omega^3) + b_2 \omega^2 \land \omega^4],$$

$$d\theta^4_4 = \theta_1 \land \omega^2 + \theta_2 \land \omega^3 + \theta^4_1 \land \theta^4_1 + (\theta^4_3 + \theta^4_4) \land \theta^4_4$$

$$+ 2[a_1 \omega^1 \land \omega^2 + a_1 (\omega^1 \land \omega^4 - \omega^2 \land \omega^3) + a_2 \omega^3 \land \omega^4],$$

and

$$d\theta^3_1 = \theta_3 \land \omega^1 + \theta_4 \land \omega^2 + \theta^3_1 \land \theta^3_1 + (\theta^3_3 + \theta^3_4)$$

$$-2[a_2 \omega^1 \land \omega^2 + a_3 (\omega^1 \land \omega^4 - \omega^2 \land \omega^3) + a_4 \omega^3 \land \omega^4].$$

It follows from equations (44) and (45) that

$$d(\theta^1_1 + \theta^2_1) = 2\theta^1_1 \land \theta^1_1 + \theta_1 \land \omega^1 + \theta_2 \land \omega^2 - \theta_3 \land \omega^3 - \theta_4 \land \omega^4$$

$$-4[a_1 \omega^1 \land \omega^2 + a_2 (\omega^1 \land \omega^4 - \omega^2 \land \omega^3) + a_3 \omega^3 \land \omega^4]$$

and

$$d(\theta^1_1 - \theta^2_1) = 2\theta^1_1 \land \theta^1_1 + \theta_1 \land \omega^1 - \theta_2 \land \omega^2 + \theta_3 \land \omega^3 - \theta_4 \land \omega^4$$

$$-4[b_1 \omega^1 \land \omega^2 + b_2 (\omega^1 \land \omega^4 + \omega^2 \land \omega^3) + b_3 \omega^2 \land \omega^4].$$
Using notations (42), we will write now 10 differential equations that the independent components of the tensor of conformal curvature $C_{ijkl}$ satisfy:

$$
\begin{align*}
&\begin{cases}
    da_0 + 2a_0(\theta - \theta_1^1 - \theta_2^2) - 4a_1\theta_1^1 = a_0\omega^i, \\
    da_1 + a_1(2\theta - \theta_1^1 - \theta_2^2) - a_0\theta_1^3 - 3a_2\theta_1^3 = a_1\omega^i, \\
    da_2 + 2a_2\theta - 2a_1\theta_1^3 - 2a_3\theta_1^3 = a_2\omega^i, \\
    da_3 + a_3(2\theta + \theta_1^1 + \theta_2^2) - 3a_2\theta_1^3 - a_4\theta_1^3 = a_3\omega^i, \\
    da_4 + 2a_4(\theta + \theta_1^1 + \theta_2^2) - 4a_3\theta_1^3 = a_4\omega^i,
\end{cases}
\end{align*}
$$

We can see from (52) and (53) that when $\omega^i = 0$, the differentials of the components $a_u$, $u = 0, 1, 2, 3, 4$, of the tensor of conformal curvature are expressible only in terms of these components, and by the same token the same is true for the components $b_u$. In view of this, the tensor of conformal curvature of the structure $CO(1, 3)$ is decomposed into two subtensors $C_\alpha$ and $C_\beta$ with the components $a_u$ and $b_u$, respectively.

Equations (46)–(51) allow us to establish a geometric meaning of the subtensors $C_\alpha$ and $C_\beta$ of the tensor of conformal curvature of the $CO(1, 3)$-structure. The quantities $a_u$ are the components of the curvature tensor of the fiber bundle $E_\alpha$ formed by the first family of plane generators of the cones $C_x$, while the quantities $b_u$ are the components of the curvature tensor of the fiber bundle $E_\beta$ formed by the second family of plane generators of the cones $C_x$.

For the $CO(1,3)$-structure not all quantities occurring in equations (39), (40), and (41)–(49) are real. In particular, as we noted earlier, the basis forms $\omega^i$ satisfy the equations (34).

The forms $\theta_i^j$ occurring in equations (40) are invariant forms of a complex representation of the real six-parameter Lorentz group $SO(1, 3)$ that leaves invariant the cone $C_x$ determined by the equation $g = 0$ in the tangent space $T_x(M)$. The form $\theta$ is real, $\mathbf{\theta} = \theta$, since this form is an invariant form of the one-parameter group $H$ of real homotheties which also leaves invariant the cone $C_x$.

The following theorem is valid (see [AZ 95], Theorem 1):

**Theorem 4** On the $CO(1,3)$-structure, the complex forms $\theta_i^j$ occurring in equations (40) satisfy the following relations:

$$
\bar{\theta}_1^1 = \theta_1^1, \quad \bar{\theta}_2^2 = -\theta_2^2, \quad \bar{\theta}_1^3 = \theta_1^3, \quad \bar{\theta}_3^1 = \theta_3^1, \quad (54)
$$

12
the forms \( \theta_i \) satisfy the relations

\[
\bar{\theta}_1 = \theta_1, \quad \bar{\theta}_2 = \theta_3, \quad \bar{\theta}_3 = \theta_2, \quad \bar{\theta}_4 = \theta_4;
\]

and the components \( a_u \) and \( b_u \), \( u = 0, 1, 2, 3, 4 \), of the curvature tensors \( C_\alpha \) and \( C_\beta \) of the isotropic fiber bundles \( E_\alpha \) and \( E_\beta \) satisfy the relations

\[
\bar{a}_u = a_u.
\]

Let us state some consequences of relations (54)–(56) occurring in Theorem 4.

Equations (54) show that the complex forms \( \theta^i \) occurring in them are expressed in terms of precisely six linearly independent real forms. This number is equal to the number of parameters on which the Lorentz group depends. These six forms are real invariant forms of the group \( \text{SO}(1, 3) \).

Equations (55) show that among the forms \( \theta_i \) there are two real forms and two complex conjugate forms, and all four forms \( \theta_i \) are expressed in terms of four linearly independent real forms.

Finally, equations (56) show that the curvature tensors \( C_\alpha \) and \( C_\beta \) of the \( \text{CO}(1, 3) \)-structure are complex conjugates: \( \bar{C}_\beta = C_\alpha \). This matches the fact that the isotropic fiber bundles \( E_\alpha \) and \( E_\beta \) of the \( \text{CO}(1, 3) \)-structure are complex conjugates themselves: \( \bar{E}_\beta = E_\alpha \).

It follows that if one of the tensors \( C_\alpha \) or \( C_\beta \) of the \( \text{CO}(1, 3) \)-structure vanishes, the other one vanishes too. This implies that the \( \text{CO}(1, 3) \)-structure cannot be conformally semiflat without being conformally flat.

8. Geometric meaning of the curvature tensor. We will now establish a geometric meaning of the subtensors \( C_\alpha \) and \( C_\beta \) of the curvature tensor of the \( \text{CO}(1, 3) \)-structure. Let \( \xi = \xi^i e_i \), and let \( \eta = \eta^i e_i \) be two vectors in the tangent space \( T_x(M) \), and \( \xi \wedge \eta \) be the bivector defined by these two vectors. Consider two bilinear forms associated with this bivector:

\[
C(\xi \wedge \eta) = C_{ijkl} \xi^i \eta^j \xi^k \eta^l = C_{ijkl} \xi^{[i} \eta^{j]} \xi^{[k} \eta^{l]}
\]

and

\[
g(\xi \wedge \eta) = (g_{ik} g_{jl} - g_{il} g_{jk}) \xi^i \eta^j \xi^k \eta^l = (g_{ik} g_{jl} - g_{il} g_{jk}) \xi^{[i} \eta^{j]} \xi^{[k} \eta^{l]}.
\]

Their ratio

\[
K(\xi \wedge \eta) = \frac{C(\xi \wedge \eta)}{g(\xi \wedge \eta)}
\]

is the conformal curvature of the bivector which is called the conformal sectional curvature.

Since \( \alpha \)-planes and \( \beta \)-planes are isotropic bivectors, for them we have \( g(\xi \wedge \eta) = 0 \), and thus the expression \( K(\xi \wedge \eta) \) does not make sense for them.
Therefore we will consider for them only the numerator $C(\xi \wedge \eta)$ of this expression and will call it the relative conformal curvature of two-dimensional isotropic direction.

Let us denote the bivector $\xi \wedge \eta$ by $p$: $p = \xi \wedge \eta$, and compute $C(p)$ taking into account equations (41) and (42):

$$\frac{1}{4} C(p) = a_0(p^{12})^2 + 2a_1 p^{12}(p^{14} - p^{23}) + a_2[2p^{12}p^{34} + (p^{14} - p^{23})^2]$$

$$+ 2a_3 p^{34}(p^{14} - p^{23}) + a_4 (p^{34})^2$$

$$+ b_0 (p^{13})^2 + 2b_1 p^{13}(p^{14} + p^{23}) + b_2[-2p^{13}p^{42} + (p^{14} + p^{23})^2]$$

$$- 2b_3 p^{42}(p^{14} + p^{23}) + b_4 (p^{42})^2. \quad (57)$$

By (36), the $\alpha$-plane $\alpha(\lambda)$ is determined by the vectors

$$\xi_\lambda = e_3 - \lambda e_1 \quad \text{and} \quad \eta_\lambda = e_4 - \lambda e_2.$$  

Hence the coordinates of the bivector $p_\lambda = \xi_\lambda \wedge \eta_\lambda$ are the minors of the matrix

\[
\begin{pmatrix}
-\lambda & 0 & 1 & 0 \\
0 & -\lambda & 0 & 1
\end{pmatrix};
\]

they are

$$p^{12} = \lambda^2, \quad p^{13} = 0, \quad p^{14} = -\lambda, \quad p^{23} = \lambda, \quad p^{34} = 1, \quad p^{42} = 0.$$  

Substituting these expressions into equations (57), we find that

$$\frac{1}{4} C(p_\lambda) = a_0 \lambda^4 - 4a_1 \lambda^3 + 6a_2 \lambda^2 - 4a_3 \lambda + a_4 := C_\alpha(\lambda). \quad (58)$$

In exactly the same way, by virtue of (37), the $\beta$-plane $\beta(\mu)$ is determined by the vectors

$$\xi_\mu = e_2 - \mu e_1 \quad \text{and} \quad \eta_\mu = e_4 - \mu e_3.$$  

This implies that the coordinates of the bivector $p_\mu = \xi_\mu \wedge \eta_\mu$ are

$$p^{12} = 0, \quad p^{13} = \mu^2, \quad p^{14} = -\mu, \quad p^{23} = -\mu, \quad p^{34} = 0, \quad p^{42} = -1,$$

and the following formula holds:

$$\frac{1}{4} C(p_\mu) = b_0 \mu^4 - 4b_1 \mu^3 + 6b_2 \mu^2 - 4b_3 \mu + b_4 := C_\beta(\mu). \quad (59)$$

Thus the components of the subtensors $C_\alpha$ and $C_\beta$ of the tensor of conformal curvature of a $CO(1,3)$-structure are the coefficients of the polynomials $C_\alpha(\lambda)$ and $C_\beta(\mu)$, by means of which we can evaluate the relative curvature of the $\alpha$-planes $\alpha(\lambda)$ and $\beta$-planes $\beta(\lambda)$, respectively.
Those isotropic 2-planes of the structure $CO(1, 3)$ for which $C_\alpha(\lambda) = 0$ or $C_\beta(\mu) = 0$ are called the principal $\alpha$-planes or principal $\beta$-planes of the isotropic bundles $E_\alpha$ and $E_\beta$, respectively. Since polynomials (58) and (59) are of the fourth degree, it follows that, in general, the isotropic cone $C_x$ carries four principal $\alpha$-planes and the same quantity of principal $\beta$-planes if we count each of these planes as many times as its multiplicity.

By (58) and (59), the equations defining the parameters $\lambda$ and $\mu$ of the principal 2-planes of the $CO(1, 3)$-structure have the form

\[
\begin{align*}
  a_0\lambda^4 - 4a_1\lambda^3 + 6a_2\lambda^2 - 4a_3\lambda + a_4 &= 0, \\
  b_0\mu^4 - 4b_1\mu^3 + 6b_2\mu^2 - 4b_3\mu + b_4 &= 0.
\end{align*}
\]  

(60)

Since by (56) the coefficients of these equations are complex conjugate, their solutions $\lambda_p$ and $\mu_p$, $p = 1, 2, 3, 4$, are also complex conjugate, $\mu_p = \overline{\lambda}_p$. But as we have proved in Section 6, the intersection of two complex conjugate 2-planes of the $CO(1, 3)$-structure is a real generator of the cone $C_x$. The latter generator is determined by the vector

\[
\xi_p = \lambda_p\overline{\lambda}_p e_1 - \lambda_p e_2 - \overline{\lambda}_p e_3 + e_4.
\]  

(61)

Thus there arise real fields of principal directions on the manifold $M$.

Now we will prove the following result:

**Theorem 5** The integral curves of each of four fields of principal isotropic directions on a manifold $M$ endowed with a $CO(1, 3)$-structure are isotropic geodesics of the manifold $M$.

**Proof.** As we noted in Sections 1 and 2, the isotropic geodesics on a manifold $M$ endowed with a $CO(1, 3)$-structure are determined by equations (2) and (11). But by (38), in a specialized frame associated with a $CO(1, 3)$-structure the coordinates of isotropic vectors have the form

\[
\begin{align*}
  \xi^1 &= \lambda\overline{\lambda}, & \xi^2 &= -\lambda, & \xi^3 &= -\overline{\lambda}, & \xi^4 &= 1.
\end{align*}
\]  

(62)

As a result, equations (11) take the form

\[
\begin{align*}
  d(\lambda\overline{\lambda}) - \lambda\theta^1_2 - \overline{\lambda}\theta^1_3 &= \lambda\overline{\lambda}(\alpha - \theta^1_4), \\
  -d\lambda + \lambda\overline{\lambda}\theta^2_1 + \theta^3_1 &= -\lambda(\alpha - \theta^2_2), \\
  -d\overline{\lambda} + \lambda\overline{\lambda}\theta^3_1 + \theta^2_3 &= -\overline{\lambda}(\alpha + \theta^2_2), \\
  -\lambda\theta^3_3 - \overline{\lambda}\theta^2_1 &= \alpha + \theta^1_1.
\end{align*}
\]  

(63)

By relations (54), which the forms $\theta^i_j$ of the $CO(1, 3)$-structure satisfy, only two of equations (63), for example, the second and the fourth, are independent. Excluding the 1-form $\alpha$ from these equations, we find that

\[
-d\alpha + \lambda(\theta^1_1 + \theta^2_2) - \theta^1_3 + \lambda^2\theta^3_1 = 0.
\]  

(64)
Taking exterior derivative of this equation by means of (64) and (48)–(50), we will arrive again to equations (60). Moreover, the parameters $\lambda_p$, determining the vectors $\xi_p$ in equation (73), satisfy equation (64) since for $\lambda = \lambda_p$ equations (60) become identities. This means that the integral curves of the vector fields $\xi_p$ are isotropic geodesics of the manifold $M$. ■

Note also that the integral curves of the principal isotropic directions of the $CO(1,3)$-structure form isotropic geodesic congruences on the manifold $M$. In general, the manifold $M$ carries four such congruences.

9. **Classification of the Einstein spaces.** For the $CO(1,3)$-structure, equations (60), which by (34) are complex conjugates of one another, are connected with A. Z. Petrov’s classification of Einstein spaces.

We remind that an *Einstein space* is a four-dimensional pseudo-Riemannian manifold of signature $(1,3)$ whose curvature tensor $R_{ijkl}$ satisfies the condition

$$R_{jk} - \frac{1}{2} g_{jk} R = -\frac{8\pi G}{c^4} T_{jk}, \quad (65)$$

where $R_{jk} = R_{jk}^i$ is the Ricci tensor, $R = g^{jk} R_{jk}$ is the scalar curvature of the Riemannian manifold, $T_{jk}$ is the energy-momentum tensor, $G$ is the gravitational constant, and $c$ is the speed of light. Equation (65) is called the *Einstein equation*.

In empty space, that is, in a region of space-time in which $T_{ij} = 0$, the Einstein equation can be reduced to the form

$$R_{ij} = 0.$$ 

This implies that the curvature tensor of this space coincides with its Weyl tensor: $R_{ijkl}^i = C_{ijkl}^i$. This follows from the expression of the tensor $C_{ijkl}^i$ in terms of $R_{ijkl}^i, R_{jk},$ and $R$ (see [AG 96], Section 4.2).

The classification of Einstein spaces is connected with the structure of its tensor of conformal curvature. Hence this classification is of a conformal nature. This classification was first constructed by Petrov in [Pe 54] (see also [Pi 57]).

To give a geometric characterization of Einstein spaces of different types, we will also apply the principal isotropic congruences on the manifolds endowed with a $CO(1,3)$-structure.

Since for the $CO(1,3)$-structure, equations (60) determining the principal isotropic vector fields $\xi_p$ are complex conjugates, for classification of Einstein spaces it is sufficient to consider only one of these equations, for example, the first one. By means of this equation, this classification can be conducted as follows:

1. **Type I** of Petrov (we use the Penrose notation for types; see [Ch 83], Ch. 1, §9, or [PR 86], Ch. 8) is characterized by the fact that all roots of equation (60) are distinct. As a result, every isotropic cone $C_x$ carries four distinct principal directions, and the manifold $M$ carries four principal isotropic congruences.
2. Type II of Petrov is characterized by the fact that equation (60) has one double root and two simple roots. As a result, every isotropic cone $C_x$ carries three principal directions, one of which is double, and the manifold $M$ carries three principal isotropic congruences, one of which is double.

3. Type D of Petrov is characterized by the fact that equation (60) has two distinct double roots. Hence every isotropic cone $C_x$ carries two double principal directions, and the manifold $M$ carries two double principal isotropic congruences.

4. Type III of Petrov is characterized by the fact that equation (60) has one triple root and one simple root. As a result, every isotropic cone $C_x$ carries two principal directions, one of which is triple, and the manifold $M$ carries two principal isotropic congruences, one of which is triple.

5. Type N of Petrov is characterized by the fact that all four roots of equation (60) coincide. Hence every isotropic cone $C_x$ carries a quadruple principal direction, and the manifold $M$ carries a quadruple principal isotropic congruence.

References

[AG 96] Akivis, M.A., and V. V. Goldberg, Conformal differential geometry and its generalizations, John Wiley & Sons, New York, 1996, xiii+383 pp.

[AZ 95] Akivis, M. A., and B. V. Zayatuev, Geometry of isotropic bundles on a four-dimensional pseudoconformal structure $CO(1,3)$, Webs and Quasi-groups, Tver Gos. Univ., Tver, 1995, 44–61.

[Ch 83] Chandrasekhar, S., The mathematical theory of black holes, Clarendon Press, Oxford & Oxford University Press, New York, 1983, xxi+646 pp.

[DB 96] Duggal, K. L., and A. Bejancu, Lightlike submanifolds of semi-Riemannian manifolds and applications, Kluwer Academic Publishers, Amsterdam, 1996, 308 pp.

[NP 62] Newman, E. T., and R. Penrose, An approach to gravitational radiation by a method of spin coefficients, J. Math. Phys. 3 (1962), 566–578.

[PR 86] Penrose, R., and W. Rindler, Spinors and space-time, vol. 2: Spinor and twistor methods in space-time geometry, Cambridge Univ. Press, Cambridge, 1986, x+501 pp.

[Pe 54] Petrov, A. Z., Classification of spaces defined by gravitational fields, Kazan. Gos. Univ. Uchen. Zap. 114 (1954), no. 8, 55–69 (Russian);
English transl. in Trans. No. 29, Jet Propulsion Lab, California Inst. Tech., Pasadena, 1963.

[Pi 57] Pirani, F. A. E., \textit{Invariant formulation of gravitational radiation theory}, Phys. Rev. (2) \textbf{105} (1957), 1089–1099.

Authors’ addresses:

M.A. Akivis
Department of Mathematics
Ben-Gurion University of the Negev
P.O. Box 653
Beer Sheva 84105, Israel

V.V. Goldberg
Department of Mathematics
New Jersey Institute of Technology
University Heights
Newark, NJ 07102, U.S.A.