Interactions and Asymptotics of Dispersive Shock Waves — Korteweg–de Vries Equation

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Abstract

The long-time asymptotic solution of the Korteweg–de Vries equation for general, step-like initial data is analyzed. Each sub-step in well-separated, multi-step data forms its own single dispersive shock wave (DSW); at intermediate times these DSWs interact and develop multiphase dynamics. Using the inverse scattering transform and matched-asymptotic analysis it is shown that the DSWs merge to form a single-phase DSW, which is the ‘largest’ one possible for the boundary data. This is similar to interacting viscous shock waves (VSW) that are modeled with Burgers’ equation, where only the single, largest-possible VSW remains after a long time.

Keywords: Shock wave interactions, Nonlinear phenomena, Solitons, Shock waves, KdV equation, Asymptotic methods

1. Introduction

Dispersive shock waves (DSWs) appear when dispersion dominates dissipation for step-like data; they have been seen in plasmas [1], fluids (e.g., undular bores) [2, 3], superfluids [4, 5, 6, 7], and optics [8, 9, 10, 11]. The Korteweg–de Vries (KdV) equation is the leading-order asymptotic equation for weakly dispersive and weakly nonlinear systems [12]. Each step in well-separated, multi-step data forms its own DSW; these DSWs interact and develop multiphase dynamics [13]. Here we show that these DSWs merge in the long-time limit to form a single-phase DSW, which is the ‘largest’ one possible for the boundary data. This is similar to interacting viscous shock waves (VSW) that are modeled with Burgers’ equation, where only the single, largest-possible VSW remains after a long time.

Figure 1: Numerical solutions of the KdV equation for the step and the vanishing initial data shown in gray. (a) For step data, there are three basic regions: exponential decay in region A, the DSW in region B with width $O(t)$ and height $O(1)$, and an oscillating tail in region C. (b) For vanishing data, there are four basic regions (see [18]). The collisionless shock in region III, which is analogous to the DSW in region B, has width $O(t^{2/3} / ( \log t)^{2/3} )$ and height $O((\log t)^{2/3} / t^{2/3})$.

Keywords:

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where $c$ and $\varepsilon$ are real, positive constants. Here, $\varepsilon$ corresponds to the size of the regularizing dispersive effects. (Since the KdV equation is Galilean invariant, we can transform any boundary conditions where $\lim_{x \to +\infty} u > \lim_{x \to -\infty} u$ to these boundary conditions. We use the IST method (see [14, 15, 16, 17]) and matched-asymptotic expansions (see [18, 19]) to find a large-time asymptotic solution.

Single-step data, such as a Heaviside function, evolve to form a single DSW. This DSW has three basic regions (Fig. 1a): a rapidly decaying region to the right of the DSW (region A); the central DSW (region B), which is a slowly varying cnoidal wave with a soliton train on its right and an oscillatory tail on its left; and a decaying, oscillatory region to the left of the DSW (region C). Similarly, each step in well-separated, multi-step data forms its own DSW (Fig. 1b); these DSWs eventually interact strongly and at intermediate times exhibit multiphase dy-

\[ u(x, t) \]
nematics (Figs. 2 and 3). In this letter we show, in the long-
time limit, that these DSWs eventually merge to form a single-
phase DSW (Fig. 2).

We use IST and matched-asymptotic methods to show that
general, step-like data go to a single-phase DSW in the large-
time limit. Individual, single-phase DSWs have been exten-
sively studied using wave averaging techniques (see [20,21,22
[23]), often referred to as Whitham theory [24,25]. The evolu-
tion of two-phase DSWs to a single-phase DSW was investi-
gated by Grava and Tian [26] in the zero-dispersion (ε → 0)
finitetime limit using Whitham theory and by Ablowitz et al.
[13] in the fixed-dispersion long-time limit using numerical and
asymptotic methods. Both zero-dispersion and large-time are
important, but different, limits; here we study the large-time
limit with fixed dispersion. By using the IST method we find
the asymptotic solution directly; in Whitham theory, we must
evolve the solution through intermediate times. Therefore, we
can investigate general, step-like initial data and the interaction
of DSWs without having to find the solution at intermediate
times.

The IST theory for step-like initial conditions was studied in
[27] and [28,29]. Hruslov [27], based on [30], gives the Gel’fand–
Levitan–Marchenko (GLM) integral equations and
investigates the soliton train associated with the DSW. Co-
en [28] and Cohen and Kappeler [29], using the methods in
[15,30], rigorously studied the properties of the scattering data,
rederived the GLM integral equations, and analyzed existence of
solutions corresponding to certain initial conditions. We state
the IST results that we need to find our asymptotic solution in
[3].

From these IST results, we use and suitably modify the
methods in [18,19] to find our long-time asymptotic solution.
Ablowitz and Segur [18,19] developed the IST and matched-
asymptotic methods for vanishing data (where c = 0); we mod-
ify them for step-like data (where c ≠ 0). The results of the
long-time asymptotic analysis in this letter are new. There are
elegant and powerful asymptotic methods based on Riemann–
Hilbert problems that depend on a parameter (here time, t); they
have been used to find the long-time asymptotic solution for
vanishing data (see [31,32] — see also [33] for a nonlinear
Schrödinger equation shock example. For our purposes here,
the matched-asymptotic method is sufficient.

In this letter: We give the required IST results in [2]. Then we
find the rapidly decaying solution in the region to the right of
the DSW (§3.1). This matches into the central DSW (§3.2),
which is a slowly varying cnoidal wave with a soliton train on
its right and an oscillatory tail on its left. Finally, we match the
decaying, oscillatory region to the left (§3.3) with the DSW. We
then compare this solution with: the solution of Burgers’ equa-
tion (§4.1), which is the leading-order asymptotic equation for
VSWs; the solution of the linear KdV equation with step-like
data (§4.2), and the solution of the KdV equation with vanish-
ing data (§4.3). Then we draw some conclusions (§4.4).

2. IST solution

The IST method transforms the initial data into scattering
data, evolves the scattering data in time, and then recovers the
solution from the evolved scattering data. First, we associate a
linear (Lax) pair with the nonlinear PDE, in this case (1). Then
we use the scattering equation of the linear pair to transform
the initial data into scattering data. The scattering data are then
evolved in time using the associated linear equation. Finally,
the solution is recovered using a linear integral equation, the
GLM integral equation, at any time.

The Lax pair associated with (1) is

\[ v_{xx} + \left( \frac{u}{6} + \lambda^2 \right) v v_x = 0, \]

\[ v_t = (u_x/6 + \gamma) v + (4\lambda^2 - u/3) v_x, \]

where \(\gamma\) is a constant. This linear pair is compatible (that is,
\(v_{xt} = v_{tx}\)) when \(u\) satisfies (1) and \(\lambda\) is isospectral (that is,
\(\partial \lambda / \partial t = 0\)). The eigenfunctions that satisfy (3a) are defined
using (2):

\[ \phi(x, \lambda) \sim e^{-i\lambda x/\epsilon}, \quad \bar{\phi}(x, \lambda) \sim e^{i\lambda x/\epsilon}, \]

as \(x \to -\infty\) and

\[ \psi(x, \lambda) \sim e^{i\lambda x/\epsilon}, \quad \bar{\psi}(x, \lambda) \sim e^{-i\lambda x/\epsilon}, \]

as \(x \to +\infty\), where \(\lambda \equiv \sqrt{\epsilon^2 - c^2}\). The branch cut of \(\lambda\) is taken
to be \(\lambda \in [-c, c]\); the branch cut of \(\lambda\) is taken to be \(\lambda \in [-ic, ic]\);
so \(\text{Im}(\lambda) \geq 0\) when \(\text{Im}(\lambda) \leq 0\). This branch cut is one of the
main differences between vanishing and step-like data.

The Wronskians \(W(\phi, \psi) = 2i\lambda/\epsilon\) and \(W(\phi, \bar{\psi}) = -2i\lambda/\epsilon\) are
constant. The scattering eigenfunctions and scattering data
\(a\) and \(b\) associated with (3a) satisfy

\[ \phi(x, \lambda) = a(\lambda, \lambda) \psi(x, \lambda) + b(\lambda, \lambda) \bar{\psi}(x, \lambda), \]

for \(\lambda \neq \lambda_j, \lambda_j \in \mathbb{R}\) (or, equivalently, \(|\lambda| > c, \lambda \in \mathbb{R}\)).

The scattering data can be written as \(2i\lambda/a = eW(\phi, \psi)\) and \(2i\lambda/b = eW(\bar{\psi}, \phi)\). We can use this to extend \(a\) to \(\lambda \in (-c, c), \) where \(\lambda,\)
Far to the DSW’s right, the contribution to \( \Omega \) from the reflection coefficient dominates and \([u(x,t) + 6c^2] \) is exponentially small.

Near the DSW’s right, the contribution to \( \Omega \) from the transmission coefficient dominates: the contribution from \( \lambda = 0 \) gives

\[
\Omega \sim \frac{-e^{-t(1+4c^2)/\varepsilon \pi} \sqrt{c}}{16 \sqrt{\pi} (6c - \xi/(2ct))^{3/2}} [H_2(0)t^{-3/2} + O(t^{-5/2})],
\]

where \( H_j(\lambda) \equiv \{ \partial^j/\partial \lambda^j \}e^{-\lambda} \). The terms in the Neumann series become disregarded when \([x + 2c^2 t + 3e/(4c) \log(6c^2 t - x)] = O(1) \), which is at the DSW’s right edge (cf. asymptotic principles discussed in \[35\]). The Neumann series can be summed, and we find that

\[
\Omega(x,t) \sim -6c^2 + 12c^2 \operatorname{sech}^2\left(\frac{\zeta - \zeta_0}{c}\right),
\]

where \( \zeta_0 = e/(2c) \log(32\pi^{1/2} /[H_2(0)c^{1/2}e^{1/2}]) \) and

\[
\zeta = -x - 2c^2 t - \frac{3e}{4c} \log(6c^2 t - x) + A_1(x/t)^{-1} + \cdots.
\]

(We omit \( A_1 \) due to length.) This provides the boundary condition on the DSW’s right.

This procedure gives the DSW’s phase, \( \zeta_0 \). This phase only depends on \( H_2(0) \) (since \( H_0(0) = H_1(0) = 0 \); the equivalent phase term \([18] \) Eq. (2.25c)\) for vanishing data is determined by \( r^{(\varepsilon)/(r^{(0)})} / r(0) \), where \( r \) is the corresponding reflection coefficient. The equivalent phase term in the shock solution associated with Burgers’ equation also depends on the initial data in a similar way (see \([41,44\) and \(36\).

### 3.2 DSW

We find the DSW using matched asymptotics analogous to \([13\]. First we make the variable change \( u(x,t) = -6c^2 + g(\xi,t) \) in \(1\), based on \(8\). Then we introduce the slow-variables \( Z \equiv \delta \xi \) and \( T \equiv \delta t \) (where \( \delta = O(\varepsilon^{-1}) \)) to get

\[
\varepsilon^2 \frac{\partial g}{\partial \xi} + gg_{\xi} - 4c^2 g_{\xi} = g_t
\]

\[
= \delta \left(3\varepsilon^2 (3e \xi g_{\xi} + gg_{\xi} - 12c^2 g_{\xi})/4c(8c^2 T + Z)\right) + \cdots.
\]

To leading order, \(11\) has the special solution (which can be found using the methods in \(37\))

\[
g(\xi,t) \sim 4c^2 - V + 4c^2 k^2 (1 - 2k^2)
\]

\[
+ 12k^2 V^2 \kappa^2 \ct^2 \left[k(\xi - \zeta_0 - V t), k\right],
\]

where \( \kappa(\zeta, k) \) is the Jacobian elliptic ‘cosine’ (see \([38\). Here, \( \kappa, k, \) and \( V \) are arbitrary constants when the right-hand-side of \(10\) is neglected but vary slowly in general. In the special case \( k = 1, \kappa = c/\varepsilon, \) and \( V = 0, g(\xi,t) = 12c^2 \sech^2 c(\xi - \zeta_0)/\varepsilon \) and exactly matches \(8\).

As in \([34\), we use the multiple-scales method to determine \( \kappa, k, \) and \( V \), which vary with the slow-variables \( Z \) and \( T \). This leads to three conservation laws, which determine \( \kappa, k, \) and \( V \).
We then use $O(\epsilon)$, which gives the diagonal system
\[
g(\theta, T) = g_0(\theta, Z, T) + \delta g_1(\theta, Z, T) + \delta^2 g_2(\theta, Z, T) + \cdots \quad \text{and group terms in like powers of } \delta.
\]
The solution of the $O(1)$ equation is $g_0(\theta, Z, T) = a(Z, T) + b(Z, T) \cos^2(2K(\theta - \theta_0), k(Z, T))$, where $K \equiv K(k(Z, T))$ is the complete elliptic integral of the first kind, $k^2 = b/[48c^2k^2K^2]$, and $a = 4c^2 - V - 2b/3 + b/(3k^2)$. Then we enforce the periodicity of $g_0(\theta, Z, T)$ in $\theta$ to eliminate secular terms (that is, terms that grow arbitrarily large); this gives the other two conservation laws, which we omit due to length. These three conservation laws determine $b$, $k$, and $V$.

If we make the variable change $b/k = 2 = (r_1 - r_2)/(r_2 - r_3)/(r_3 - r_4)$, and $V = 4e^2 - (r_1 + r_2 + r_3)/3$, then simplifying gives the diagonal system
\[
\frac{\partial r_i}{\partial t} + v_i(r_1, r_2, r_3) \frac{\partial r_i}{\partial x} = 0, \quad i = 1, 2, 3, \quad (12)
\]
where $v_1 = V + bK/[3(E - K)]$, $v_2 = V + b(1 - k^2)K/[3(1 - k^2)K]$, $v_3 = V - b(1 - k^2)K/(3k^2E)$, and $g_0(\theta, Z, T) = r_1 - r_2 + r_3 + 2(r_2 - r_1) \cos^2(2K(\theta - \theta_0), k)$. Whitham first found this diagonal system in [24] (see also [20, 13]).

For large-time, the solution tends to a self-similar solution: that is, $r_i = r_i(\chi)$ with $\chi \equiv Z/T \equiv \zeta/t$. The boundary conditions are satisfied when $r_1 = 0$ and $r_3 = 6c^2$. So (12) reduces to
\[
(v_2 - \chi)r_2'(\chi) = 0,
\]
which $v_2 = \chi$ satisfies. The numerical solution of this implicit equation for $r_2$ is plotted in Fig. 3. We can directly compute the right and left speed of the DSW: At the DSW’s right, we take the limit $r_2 \to r_3$ and get that $v_2 \to 0$ or $x \sim -2c^2t$ — the speed of the soliton train. At the DSW’s left, we take the limit $r_2 \to r_1$ and get that $v_2 \to 10c^2$ or $x \sim -12c^2t$.

\[\text{Figure 3: The value of } r_2(\chi) \text{ found numerically for } 0 < \chi < 10c^2, \text{ where } \chi \equiv \zeta/t. \text{ For comparison, we include } -r_2(\chi) \text{ as a dashed line and a numerical simulation of } u(x, t) \text{ in gray (inside the envelope of } r_2 \text{ and } -r_2) \text{ for a single-step at (a) } t/\epsilon = 10 \text{ and (b) } t/\epsilon = 200. \text{ Note that } \chi = 0 \text{ corresponds to } x \sim -2c^2t \text{ and } \chi = 10c^2 \text{ to } x \sim -12c^2t.\]\n
3.3. Trailing edge

On the DSW’s left, there is a decaying, slowly varying, oscillatory similarity-solution (as there is with vanishing data).

\[\text{Figure 4. Comparison with Burgers’ equation}\]

Burgers’ equation $u_t + uu_x - \nu w_{xx} = 0$, $\nu > 0$ is the leading-order asymptotic equation for VSWs. If we take initial data that go rapidly to the boundary conditions $\lim_{x \to -\infty} w(x, t) = 0$ and $\lim_{x \to +\infty} w(x, t) = -h^2$, then the long-time asymptotic solution is $w(x, t) = -(h^2/2)(1 + \tanh[h^2(x - x_0 + h^2t^2)/(4\nu)])$, where $x_0$ is a real constant that depends on the initial data [36]. So well-separated step data go to a single shock wave in the large-time limit for both Burgers’ and the KdV equation. For both, the boundary data determine its form and the initial data determine its location. Unlike with Burgers’ equation, the solution of the KdV equation with step-like data can also have a finite number of solitons, which move to the DSW’s right in the long-time limit.
4.2. Comparison with the linear KdV equation

The large-time asymptotic solution of (1) differs significantly from the linear problem ($\tilde{u}_0 + e^x \tilde{u}_{xxx} = 0$). While the nonlinear problem has a central DS region with strong nonlinearity over $|x| = O(t)$, the linear problem’s middle region is only over $|x| = O(t^{1/3})$. Indeed, the solution to the linear problem in the middle region is $\tilde{u}(x,t) \sim \tilde{u}_0(0) \int_{-\infty}^{x} A_i(\eta') d\eta'$, where $A_i(x)$ is the Airy function.

4.3. Comparison with vanishing boundary conditions

The large-time asymptotic solution of (4) for step-like data ($c \neq 0$) is also significantly different from that for vanishing data ($c = 0$). A collisionless shock (region III in Fig. 1b), which is analogous to the DSW, has width $O(\log t^{2/3})$; the DS region (region II in Fig. 1b) has width $O(t)$.

4.4. Conclusion

In this letter, we show that general, step-like initial data tend to a single-phase DSW for large-time. Therefore, well-separated, multi-step initial data eventually form a single-phase DS despite having multiphase dynamics at intermediate times (as indicated in Fig. 2). The asymptotic solution of the KdV equation for general, step-like data is new. The details of our calculations will be given in a separate paper; we anticipate that they can be applied to other integrable nonlinear PDEs with general, step-like data.

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