DISCONTINUITY OF A DEGENERATING ESCAPE RATE

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Abstract. We look at degenerating meromorphic families of rational maps on $\mathbb{P}^1$—holomorphically parameterized by a punctured disk—and we provide examples where the bifurcation current fails to have a bounded potential in a neighborhood of the puncture. This is in contrast to the recent result of Favre-Gauthier that we always have continuity across the puncture for families of polynomials; and it provides a counterexample to a conjecture posed by Favre in 2016. We explain why our construction fails for polynomial families and for families of rational maps defined over finite extensions of the rationals $\mathbb{Q}$.

1. Introduction

Let $f_t$ be a holomorphic family of rational maps on $\mathbb{P}^1$ of degree $d > 1$, parameterized by the punctured unit (open) disk $D^* = \{t \in \mathbb{C} : 0 < |t| < 1\}$, and assume that the coefficients of $f_t$ extend to meromorphic functions on the unit disk $D = \{t \in \mathbb{C} : |t| < 1\}$. Let $a : D \to \mathbb{P}^1$ be a holomorphic map. In this article, we examine the potential function $g_{f_t,a}$ on $D^*$ (having the order $o(\log |t|)$ as $t \to 0$) for the bifurcation measure associated to the pair $(f, a)$. Our main result is that this potential function does not necessarily extend continuously across the puncture at $t = 0$.

The question of continuous extendability of $g_{f_t,a}$ across $t = 0$ arose naturally in the study of degenerating families of rational maps, and specifically in the context of equidistribution questions and height functions associated to the family $f_t$; see, e.g., [BD, Fa2]. Continuity of the potential at $t = 0$ was required to apply certain equidistribution theorems on arithmetic varieties (as in the proofs of the main results of [BD, DM, FG1, GY], and others). Moreover, when $a(t)$ parameterizes a critical point of $f_t$, the bifurcation measure and its potential are related to the structural stability of the family $f_t$ [De1, DF]. It is well known that continuity holds when $f_t$ has a uniform limit on the whole $\mathbb{P}^1$ as $t \to 0$, for any choice of $a$. It is also true when $f_t$ is any family of polynomials with coefficients meromorphic in $t$, again for any choice of $a$ [FG2].

To formulate the problem and our construction more precisely, we will work with $f_t$ in homogeneous coordinates: assume that we are given a family of homogeneous polynomial maps $\tilde{f}_t : \mathbb{C}^2 \to \mathbb{C}^2$ of degree $d$, where the coefficients are holomorphic functions on the entire disk $D$, such that for every $t \in D^*$, $\tilde{f}_t^{-1}(0,0) = \{(0,0)\}$ and $\tilde{f}_t$ projects to $f_t$ on $\mathbb{P}^1$.

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There exists a continuous plurisubharmonic escape rate
\[ G_f : \mathbb{D}^* \times (\mathbb{C}^2 \setminus \{(0,0)\}) \to \mathbb{R} \]
such that for each fixed \( t \in \mathbb{D}^* \), the current \( df \cdot G_f(t, \cdot) \) on \( \mathbb{C}^2 \setminus \{(0,0)\} \) projects to the measure of maximal entropy of \( f_t \) on \( \mathbb{P}^1 \). Given a holomorphic lift of \( a \) to \( \tilde{a} : \mathbb{D} \to \mathbb{C}^2 \setminus \{(0,0)\} \), we may write
\[ G_{\tilde{f}}(t, \tilde{a}(t)) = \eta \log \| t \| + g_{f,a}(t), \]
where \( \eta \in \mathbb{R} \) represents a “local height” for the pair \( (f,a) \), and the function \( g_{f,a} \) on \( \mathbb{D}^* \) satisfies
\[ g_{f,a}(t) = o(\log \| t \|) \]
as \( t \to 0 \) [De4; see §2.1]. The value of \( \eta \) and the subharmonic function \( g_{f,a} \) depend on the choices of \( \tilde{f} \) and \( \tilde{a} \), but \( g_{f,a} \) is uniquely determined up to the addition of a harmonic function on \( \mathbb{D}^* \) which is bounded near \( t = 0 \). The Laplacian \( \mu_{f,a} = \frac{1}{2\pi} \Delta g_{f,a} \) on \( \mathbb{D}^* \) is the bifurcation measure associated to the pair \((f,a)\) [DF, §3].

It turns out that the function \( g_{f,a} \) is always bounded from above near \( t = 0 \) (Lemma 2.1). In this article, we construct examples of pairs \((f,a)\) that satisfy
\[ \limsup_{t \to 0} g_{f,a}(t) = -\infty \]
to show that it need not be bounded from below, so in particular does not extend continuously across \( t = 0 \). In our examples, the maps \( f_t \) will converge to a rational map \( \varphi \) on \( \mathbb{P}^1 \) of degree \( < d \) as \( t \to 0 \) locally uniformly on \( \mathbb{P}^1 \setminus H \), where \( H \) is a non-empty finite set. The idea of the construction is to choose \( \varphi \) and \( a \) so that some sequence of iterates \( \varphi^{n_j}(a(0)) \) accumulates fast on \( H \) as \( n_j \to \infty \).

Furthermore, choosing \( a(t) \) to parameterize a critical point of the family \( f_t \), we obtain a counterexample to the continuity statement in [Fa2, Conjecture 1], in proving:

**Theorem 1.1.** For every integer \( d > 1 \), there exists a holomorphic family \( f_t \) of rational maps on \( \mathbb{P}^1 \) of degree \( d \), parameterized by \( t \in \mathbb{D}^* \), whose coefficients extend to meromorphic functions on \( \mathbb{D} \) but for which the bifurcation current associated to the family \( f_t \) fails to have a bounded potential in any punctured neighborhood of \( t = 0 \).

**Remark.** It will be clear from the proof that the family \( f_t \) can be chosen to be algebraic, in the sense that it extends to define a holomorphic family parameterized by \( t \) in a quasiprojective curve \( X \), with coefficients that are meromorphic on a compactification of \( X \).

The bifurcation current associated to the family \( f_t \) is equal to the Laplacian of the continuous and subharmonic function \( t \mapsto L(f_t) \) on \( \mathbb{D}^* \), where for each \( t \), \( L(f_t) \) is the Lyapunov exponent of \( f_t \) with respect to its unique measure of maximal entropy. For more details on \( L(f_t) \) and its relationship to \( g_{f,a} \), see Section 3.

The construction of examples of pairs \((f,a)\) for which \( g_{f,a} \) fails to extend continuously across \( t = 0 \) is laid out in Section 2. Our use of the Baire category theorem in the
construction is similar to that of [Fa1, Example 4], [Bu], or [DG] in the context of higher-dimensional (bi)rational maps. In Section 3 we give the proof of Theorem 1.1. In Section 4 we comment on why the strategy for producing these examples fails for families of polynomials and for rational maps on \(P^1\) defined over \(\mathbb{Q}\). We expect that a continuous extension of \(g_{f,a}\) to \(D\) always exists when the pair \((f,a)\) is algebraic and defined over \(\mathbb{Q}\), as is known for algebraic families of elliptic curves [Si2, Theorem II.0.1] and therefore also for Lattès maps on \(P^1\) [DM, Proposition 3.4]; see also [JR, Theorem A] in the context of (bi)rational maps in dimension 2. The bifurcation current associated to a family \(f\) was introduced in [De1]; its properties at infinity in the moduli space of quadratic rational maps (related to our Theorem 1.1) were studied in [BG].

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2. A recipe for discontinuity

In this section, we construct the examples for which \(g_{f,a}\) fails to extend continuously to the disk \(D\).

2.1. The potential is bounded from above. Suppose we are given a family of homogeneous polynomial maps \(\tilde{f}_t : \mathbb{C}^2 \to \mathbb{C}^2\) of degree \(d > 1\), where the coefficients are holomorphic functions on the entire disk \(D\), and such that for every \(t \in D^*\), we have \(\tilde{f}_t^{-1}(0,0) = \{(0,0)\}\) so \(\tilde{f}_t\) projects to a rational map \(f_t\) on \(P^1\) of degree \(d\). We let \(\tilde{a} : D \to \mathbb{C}^2 \setminus \{(0,0)\}\) be any holomorphic map, and let \(a : D \to P^1\) be its projection. For each \(n \in \mathbb{N}\), there is a unique non-negative integer \(o_n\) so that

\[ F_n(t) := t^{-o_n} \tilde{f}_t^n(\tilde{a}(t)) \]

is a holomorphic map from \(D\) to \(\mathbb{C}^2 \setminus \{(0,0)\}\). Choose any norm \(\| \cdot \|\) on \(\mathbb{C}^2\). The function \(g_{f,a}\) on \(D^*\) defined by (1.1) is the locally uniform limit on \(D^*\) of the sequence of continuous and subharmonic functions

\[ g_n(t) := \frac{1}{d^n} \log \|F_n(t)\| \quad \text{on } D, \]

as \(n \to \infty\), and the value \(\eta\) of (1.1) is given by

\[ \eta = \lim_{n \to \infty} \frac{o_n}{d^n}, \]

as explained in [De4 §3]. Note, in particular, that the function \(g_{f,a}\) is continuous and subharmonic on \(D^*\).

The following observation is not required in this section, but it will be useful in Section 3.

Lemma 2.1. The function \(g_{f,a}\) is bounded from above on \(\{0 < |t| \leq r\}\) for every \(r \in (0,1)\).
2.2. The ingredients for discontinuity. Let \( \varphi \in \mathbb{C}(z) \) be a rational map on \( \mathbb{P}^1 \) of degree \( e \geq 1 \), and suppose that there is a point \( a_0 \in \mathbb{C} \) such that \( \# \{ \varphi^n(a_0) : n \in \mathbb{N} \} = \infty \) and that \( \omega_{\varphi}(a_0) \cap \{ \varphi^n(a_0) : n \in \mathbb{N} \} \neq \emptyset \), where

\[
\omega_{\varphi}(a_0) := \bigcap_{N \in \mathbb{N}} \{ \varphi^n(a_0) : n > N \}
\]

is the \( \omega \)-limit set of \( a_0 \) under \( \varphi \). Then there exists \( N_0 \in \mathbb{N} \) so that \( \{ \varphi^n(a_0) : n \geq N \} \) is dense in \( \omega_{\varphi}(a_0) \) for all \( N \geq N_0 \).

Let \( \{ r_n \} \) be any sequence in \( \mathbb{R}_{>0} \) decreasing to 0 as \( n \to \infty \), which will be chosen appropriately later. It follows that the set

\[
U_N(a_0, \{ r_n \}) := \left( \bigcup_{n \geq N} \{ z \in \omega_{\varphi}(a_0) : [z, \varphi^n(a)] < r_n \} \right) \setminus \{ a_0, \varphi(a_0), \ldots, \varphi^{N-1}(a_0) \}
\]

is open and dense in \( \omega_{\varphi}(a_0) \) for all \( N \geq N_0 \). Here \( [\cdot, \cdot] \) denotes the chordal distance on \( \mathbb{P}^1 \). Therefore, by the Baire category theorem,

\[
B_{\varphi}(a_0, \{ r_n \}) := \bigcap_{N \geq N_0} U_N(a_0, \{ r_n \})
\]

is dense in \( \omega_{\varphi}(a_0) \).

Fix any \( h \in B_{\varphi}(a_0, \{ r_n \}) \cap \mathbb{C} \). Then \( \varphi^n(a_0) \neq h \) for all \( n \in \mathbb{N} \cup \{ 0 \} \), and there is a sequence \( n_j \to \infty \) such that

\[
0 < [\varphi^{n_j}(a_0), h] < r_{n_j}
\]

for all \( j \in \mathbb{N} \).

We consider the family

\[
f_t(z) := \varphi(z) \cdot \frac{z - h - \varepsilon t}{z - h + \varepsilon t}
\]

parameterized by \( t \in \mathbb{D}^* \), where \( \varepsilon > 0 \) is chosen so that \( \varphi \) has neither zeros nor poles in the set \( \{ z : 0 < |z - h| < \varepsilon \} \). Thus, \( f_t \) defines a holomorphic family of rational maps of degree \( d := e + 1 > 1 \). As \( t \to 0 \), the maps \( f_t \) converge locally uniformly to \( \varphi \) on \( \mathbb{P}^1 \setminus \{ h \} \).
2.3. An unbounded escape rate. Set now

\[ r_n = \exp(-n \, d^{n+1}) \]

for each \( n \in \mathbb{N} \). Working on \( \mathbb{C}^2 \), we define

\[ \tilde{f}_t(z, w) := (P(z, w)(z - (h + \varepsilon t)w), Q(z, w)(z - (h - \varepsilon t)w)) \]

for all \( t \in \mathbb{D} \), where \( P \) and \( Q \) are homogeneous polynomials of degree \( e = \deg \varphi \) such that \( \varphi(z) = P(z, 1)/Q(z, 1) \). Let \( \tilde{a} : \mathbb{D} \to \mathbb{C}^2 \setminus \{(0, 0)\} \) be any holomorphic map such that \( \tilde{a}(0) = (a_0, 1) \) and let \( a : \mathbb{D} \to \mathbb{P}^1 \) be its projection to \( \mathbb{P}^1 \).

Choose any norm \( \| \cdot \| \) on \( \mathbb{C}^2 \). As \( \varphi^n(a_0) \neq h \) for all \( n \geq 0 \), we see that \( \tilde{f}_0^n(\tilde{a}(0)) \neq (0, 0) \) for all \( n \geq 0 \). Therefore, as described in [De4, Proposition 3.1], we have \( \eta = 0 \) and the function \( g_{f,a} \) is given by the formula

\[ g_{f,a}(t) = \lim_{n \to \infty} \frac{1}{d^n} \log \| \tilde{f}_t^n(\tilde{a}(t)) \| \]

for \( t \in \mathbb{D}^* \) [De4, Proposition 3.1].

Set

\[ \Phi := (P, Q) \quad \text{and} \quad H(z, w) := z - hw \]

so that \( \tilde{f}_0 = (HP, HQ) \). For all \( n \geq 0 \), as \( \deg \Phi = e > 0 \), the iteration formula of [De3, Lemma 2.2] states that

\[ \tilde{f}_0^n = \left( P_n \cdot \prod_{k=0}^{n-1} ((\Phi^k)^* H)^{d^{n-k-1}}, Q_n \cdot \prod_{k=0}^{n-1} ((\Phi^k)^* H)^{d^{n-k-1}} \right), \]

where we set \( \Phi^n = (P_n, Q_n) \), so that

\[ \log \frac{\| \tilde{f}_0^n \|}{d^n} = \sum_{k=0}^{n-1} \log \frac{\| (\Phi^k)^* H \|}{d^{k+1}} + \frac{\log \| \Phi^n \|}{d^n} \]

on \( \mathbb{C}^2 \setminus \{(0, 0)\} \), and consequently,

\[ \log \frac{\| \tilde{f}_0 \circ \tilde{f}_0^n \|}{\| \tilde{f}_0^n \|} = \log \frac{\| (\Phi^n)^* H \|}{\| \Phi^n \|} + \log \frac{\| \Phi \circ \Phi^n \|}{\| \Phi^n \|} \]

on \( \mathbb{C}^2 \setminus \tilde{f}_0^{-n}(0, 0) \).

Note that \( \log \| \Phi \| \) is bounded on the unit sphere in \( \mathbb{C}^2 \), so the last term on the right-hand side of (2.4) is bounded on \( \mathbb{C}^2 \setminus \{(0, 0)\} \) uniformly in \( n \geq 0 \). The first term on the right-hand side of (2.4) is the log of [\( \varphi^n(\cdot), h \)], up to scaling of the metric [\( \cdot, \cdot \) ]; therefore, combined with (2.1), we see that there is a constant \( C \) so that

\[ \log \frac{\| \tilde{f}_0(\tilde{f}_0^n(\tilde{a}(0))) \|}{\| \tilde{f}_0^n(\tilde{a}(0)) \|} < C + \log(r_{n_j}) = C - n_j d^{n_j+1} \]

for all \( j \). For all \( j \), by continuity of \( \tilde{f}_t^n(\tilde{a}(t)) \) as a map from \( \mathbb{D} \) to \( \mathbb{C}^2 \setminus \{(0, 0)\} \), there is a radius \( \delta_j \in (0, 1/2) \) such that

\[ \sup_{|t| \leq \delta_j} \log \frac{\| \tilde{f}_t(\tilde{f}_t^n(\tilde{a}(t))) \|}{\| \tilde{f}_t^n(\tilde{a}(t)) \|} \leq C - n_j d^{n_j+1}. \]
On the other hand, we also have from (2.3) that
\[
\begin{align*}
g_{f,a}(t) &= \log \|\tilde{a}(t)\| + \sum_{k=0}^{\infty} \frac{1}{d^{k+1}} \log \frac{\|\tilde{f}_t^k(\tilde{a}(t))\|}{\|\tilde{f}_t^k(\tilde{a}(t))\|^d} \\
&= \log \|\tilde{a}(t)\| + \frac{1}{d^{n_j+1}} \log \frac{\|\tilde{f}_t^{n_j}(\tilde{a}(t))\|}{\|\tilde{f}_t^{n_j}(\tilde{a}(t))\|^d} + \sum_{k \neq n_j} \frac{1}{d^{k+1}} \log \frac{\|\tilde{f}_t^k(\tilde{a}(t))\|}{\|\tilde{f}_t^k(\tilde{a}(t))\|^d}
\end{align*}
\]
for each \(j\).

The following is elementary but useful:

**Lemma 2.2.** Let \(F_t = (P_t, Q_t)\) be any family of homogeneous polynomial maps of degree \(d \geq 2\), with coefficients that are bounded holomorphic functions of \(t\) in \(\mathbb{D}\). Then there is a constant \(C\) so that
\[
\|F_t(z, w)\| \leq C \|(z, w)\|^d
\]
for all \((z, w) \in \mathbb{C}^2 \setminus \{(0, 0)\}\) and all \(t \in \mathbb{D}\).

**Proof.** As \(F_t\) is homogeneous, it suffices to bound its values on the unit sphere in \(\mathbb{C}^2\). The result follows because the coefficients are bounded uniformly on \(\mathbb{D}\). \(\square\)

As a consequence of Lemma 2.2, we can bound all the terms in the final sum of (2.6) from above, uniformly on the disk \(\{|t| \leq 1/2\}\), and therefore there is a constant \(C'\) so that
\[
(2.7) \quad \sup_{|t| \leq 1/2} g_{f,a}(t) \leq C' + \frac{1}{d^{n_j+1}} \log \frac{\|\tilde{f}_t^{n_j}(\tilde{a}(t))\|}{\|\tilde{f}_t^{n_j}(\tilde{a}(t))\|^d}
\]
for every \(j\). Combined with (2.5), we conclude that there is another constant \(C\) so that
\[
\sup_{|t| \leq \delta_j} g_{f,a}(t) \leq C - n_j
\]
for every \(j\). Letting \(j \to \infty\) shows that
\[
\limsup_{t \to 0} g_{f,a}(t) = -\infty.
\]

### 2.4. Examples with degree \(d = 2\)

Fix any \(\theta \in \mathbb{R} \setminus \mathbb{Q}\), and let
\[
\varphi(z) = e^{2\pi i \theta} z.
\]
Set \(a_0 = 1\); the \(\omega\)-limit set \(\omega_\varphi(a_0)\) is the unit circle in \(\mathbb{C}\). Set \(r_n = \exp(-n 2^{n+1})\) for each \(n \in \mathbb{N}\), and define \(B_\varphi(1, \{r_n\})\) as above. Taking any \(h \in B_\varphi(1, \{r_n\})\) and setting \(\varepsilon = 1\), we define the family \(f_t\) as in (2.2). Then the potential function \(g_{f,a}\) fails to be bounded around \(t = 0\) for any holomorphic map \(a : \mathbb{D} \to \mathbb{P}^1\) with \(a(0) = 1\).

Note that only the Möbius transformations \(\varphi\) which are Möbius (i.e., \(\text{PSL}(2, \mathbb{C})\))-conjugate to an irrational rotation have recurrent orbits, as needed for the construction described above.
2.5. Examples in degree $> 2$, with a marked critical point. Fix an integer $d > 2$. For every $\theta \in \mathbb{R}$, the polynomial

\[ (2.8) \quad \phi(z) = e^{2\pi i \theta} \left( z - \frac{e^{2\pi i \theta} - (d - 1)}{(d - 1)^{(d-1)/(d-2)}} \right)^{d-1} \]

of degree $d - 1$ has a fixed point with multiplier $e^{2\pi i \theta}$ and its unique finite critical value at $z = 0$. Now fix $\theta$ to be irrational; the critical point $a_0 := e^{2\pi i \theta} - \frac{(d - 1)}{(d - 1)^{(d-1)/(d-2)}}$ of $\phi$ satisfies $\#\{\phi^n(a_0) : n \in \mathbb{N}\} = \infty$ and $a_0 \in \omega_{\phi}(a_0)$ [Ma]. Let

\[ r_n = \exp(-n d^{n+1}) \]

for each $n \in \mathbb{N}$ and fix any point $h \in B_\phi(a_0, \{r_n\})$. Choose any $\varepsilon \in (0, |a_0 - h|]$, and set

\[ f_t(z) = \phi(z) \cdot \frac{z - h - \varepsilon t}{z - h + \varepsilon t}, \]

which is a rational map on $\mathbb{P}^1$ of degree $d$ for all $t \in \mathbb{D}^*$. We let

\[ a(t) = a_0 \]

for all $t \in \mathbb{D}$, which satisfies $f'_t(a(t)) = 0$ for all $t \in \mathbb{D}^*$. (This is the reason for requiring the unique finite critical value $\phi(a_0)$ of $\phi$ to be 0.) It follows that $g_{f,a}$ fails to be bounded around $t = 0$.

2.6. Example in degree 2, with a marked critical point. We can produce examples of $(f, a)$ also for quadratic rational maps $f_t$ where $a(t)$ parameterizes a critical point of $f_t$, though we do not have as much flexibility as in higher degrees. For example, we have:

Lemma 2.3. Suppose $f_t(z) = \varphi(z)(z - h - t^n)/(z - h + t^n)$ is a family of quadratic rational maps, for some $h \in \mathbb{C}$, $n \in \mathbb{N}$, and a rational map $\varphi$ on $\mathbb{P}^1$ of degree 1, and suppose that $c_1, c_2 : \mathbb{D} \to \mathbb{P}^1$ are holomorphic maps parameterizing the two critical points of $f_t$. Then

\[ \lim_{t \to 0} c_1(t) = \lim_{t \to 0} c_2(t) = h. \]

Proof. As $\varphi$ has degree 1, it has no critical points of its own. On the other hand, $\lim_{t \to 0} f_t = \varphi$ locally uniformly on $\mathbb{P}^1 \setminus \{h\}$, so it must be that $c_1(t), c_2(t) \to h$ as $t \to 0$. \hfill \Box

In particular, if we wish to let $a(t)$ parameterize a critical point of $f_t$, then necessarily we will have $a(0) = h$, which was not allowed by the construction above.

However, let us fix our decreasing sequence as

\[ r_n = \exp(-(n - 1) 2^n), \]
for each \( n \in \mathbb{N} \), and apply the Baire Category Theorem now to the space of rotations \( z \mapsto e^{2\pi i \theta} z \) to find

\[
\theta_0 \in \left( \bigcap_{N \in \mathbb{N}} \bigcup_{n \geq N} \{ \theta \in \mathbb{R} : [1, e^{2\pi i n \theta}] < r_n \} \right) \setminus \mathbb{Q}.
\]

Then \( e^{2\pi i n \theta_0} \neq 1 \) for all \( n \in \mathbb{N} \), and there is a sequence \( n_j \to \infty \) as \( j \to \infty \) such that

\[
0 < [1, e^{2\pi i n_j \theta_0}] < r_{n_j}
\]

for all \( j \).

Now set \( \varphi_0(z) = e^{2\pi i \theta_0} z \), and

\[
f_t(z) = \varphi_0(z) \cdot \frac{z - 1 - t^2}{z - 1 + t^2}
\]

for \( t \in \mathbb{D}^* \). Note that we have used \( t^2 \) here rather than \( t \) in (2.2); this is so that we can holomorphically parameterize the critical points of \( f_t \). Indeed, the critical points of \( f_t \) are

\[
c_{\pm}(t) = 1 - t^2 \pm \sqrt{2t^4 - 2t^2} = 1 - t^2 \pm i\sqrt{2t\sqrt{1 - t^2}},
\]

which extend holomorphically on \( \mathbb{D} \) by setting \( c_{\pm}(0) = 1 \). Define the function \( a : \mathbb{D} \to \mathbb{C} \) by either \( c_+ \) or \( c_- \) so that \( f_t \) has the critical value

\[
v(t) := f_t(a(t)) = e^{2\pi i \theta_0} + O(t) \quad \text{as } t \to 0,
\]

which also extends holomorphically to \( \mathbb{D} \) by setting \( v(0) := v_0 := e^{2\pi i \theta_0} = \varphi_0(1) \). We also set

\[
\tilde{f}_t(z, w) := (e^{2\pi i \theta_0} z(z - (1 + t^2)w), (z - (1 - t^2)w)w)
\]

for all \( t \in \mathbb{D} \).

We will work with the pair \( (f, v) \). Then, since \( \varphi_0^n(v_0) \neq 1 \) for all \( n \in \mathbb{N} \cup \{0\} \) and since we also have

\[
0 < [1, \varphi_0^{n_j-1}(v_0)] < r_{n_j} = \exp(-(n_j - 1) 2^{(n_j-1)+1})
\]

for all \( j \), the arguments above go through exactly as before – applied to the sequence \( \{n_j - 1\}_j \) – to show that \( g_{f, v} \) fails to be bounded around \( t = 0 \).

Finally, if we set \( \tilde{v}(t) = (v(t), 1) \) and \( \tilde{a}(t) = (a(t), 1) \), then

\[
\tilde{f}_t(\tilde{a}(t)) = (a(t) - 1 + t^2) \tilde{v}(t).
\]

Note that \( a(t) - 1 + t^2 = \pm i\sqrt{2t\sqrt{1 - t^2}} \) on \( \mathbb{D} \), so that the function

\[
h(t) = \log |a(t) - 1 + t^2| - \log |t|
\]

on \( \mathbb{D}^* \) extends to a harmonic function on the disk \( \mathbb{D} \). We have

\[
G_{\tilde{f}}(t, \tilde{v}(t)) = g_{f, v}(t)
\]
on $\mathbb{D}^*$ from the definitions given in (1.1) and because the pair satisfies the hypotheses for (2.3). Consequently,

$$G_f(t, \tilde{a}(t)) = \frac{1}{2} G_f(t, \tilde{f}_t(\tilde{a}(t)))$$

$$= \frac{1}{2} \left( G_f(t, \tilde{v}(t)) + \log |a(t) - 1 + t^2| \right)$$

$$= \frac{1}{2} g_{f,v}(t) + \frac{1}{2} h(t) + \frac{1}{2} \log |t|$$

so that, by the definition of $g_{f,a}$ in (1.1), we have $\eta = 1/2$ and

$$g_{f,a}(t) = \frac{1}{2} g_{f,v}(t) + \frac{1}{2} h(t)$$

on $\mathbb{D}^*$. We conclude that the function $g_{f,a}$ also fails to be bounded near $t = 0$.

3. LYAPUNOV EXPONENTS AND THE BIFURCATION CURRENT

The Lyapunov exponent of an individual rational map $f$ on $\mathbb{P}^1$ of degree $> 1$, with respect to its unique measure $\mu_f$ of maximal entropy on $\mathbb{P}^1$, is the positive and finite quantity

$$L(f) = \int_{\mathbb{P}^1} \log |f'| d\mu_f,$$

where $| \cdot |$ is any choice of metric on the tangent bundle of $\mathbb{P}^1$.

Let $f_t$ be a holomorphic family of rational maps on $\mathbb{P}^1$ of degree $d > 1$ parameterized by $\mathbb{D}^*$ whose coefficients extend to meromorphic functions on $\mathbb{D}$. If all the critical points of $f_t$ are parameterized by holomorphic maps $c_1, \ldots, c_{2d-2} : \mathbb{D} \to \mathbb{P}^1$, then

$$(3.1) \quad L(f_t) = h(t) + \sum_{j=1}^{2d-2} g_{f,c_j}(t)$$

on $\mathbb{D}^*$, for a harmonic function $h$ on $\mathbb{D}^*$ satisfying $h(t) = O(\log |t|)$ as $t \to 0$ [De2, Theorem 1.4], [Fa2, Theorem C]. By the symmetry in the critical points in (3.1), this formula holds even if the critical points cannot be holomorphically parameterized on $\mathbb{D}^*$. The bifurcation current associated to the family $f_t$ can be given by

$$T_{\text{bif}} := \frac{1}{2\pi} \Delta L(f_t)$$

on $\mathbb{D}^*$, in the sense of distributions; the original definition of $T_{\text{bif}}$ in [De1] was based on the right hand side of (3.1). From [De2, Theorem 1.1], the support of $T_{\text{bif}}$ is equal to the bifurcation locus of the family $f_t$ in the sense of [MSS, Ly].

In particular, because the sum $\sum_j g_{f,c_j}$ in (3.1) is $o(\log |t|)$ near $t = 0$, we see that the bifurcation current $T_{\text{bif}}$ has a bounded potential if and only if the sum $\sum_j g_{f,c_j}$ is bounded near $t = 0$. 
Proof of Theorem 1.1. We give examples in an arbitrary degree $> 1$. First, let $f_t$ be the holomorphic family of quadratic rational maps on $\mathbb{P}^1$ parameterized by $D^*$ described in §2.6. As we have seen, neither of the functions $g_{f,c_{\pm}}$ extend continuously to $D$; indeed, both tend to $-\infty$ as $t \to 0$. Hence by (3.1), the bifurcation current for the family $f_t$ fails to have a potential bounded around $t = 0$.

Next, let $f_t$ be the holomorphic family of rational maps on $\mathbb{P}^1$ of degree $d > 2$ parameterized by $D^*$, described in §2.5. As we have seen, the constant map $a(t) \equiv a_0$ on $D$ satisfies $f_t'(a(t)) = 0$ for every $t \in D^*$, and the function $g_{f,a}$ tends to $-\infty$ as $t \to 0$. Taking an at most finitely ramified holomorphic covering $\pi : D^* \to D^*$ if necessary, all the critical points of $f_{\pi(s)}$ are parameterized by holomorphic maps $c_1, \ldots, c_{2d-2} : D \to \mathbb{P}^1$. We may assume the points are labeled so that $c_1 = a \circ \pi$ on $D$. By the formula (2.3) for $g_{f,c_1}$, we have $g_{f,a}(\pi(s))$ on $D^*$. On the other hand, for every $j \in \{2, \ldots, 2d-2\}$, the function $g_{f_{\pi(s)},c_j}$ is bounded from above on $\{0 < |s| \leq r\}$ for every $r \in (0,1)$, by Lemma 2.1. Hence the sum $\sum_j g_{f_{\pi(s)},c_j}(s)$ tends to $-\infty$ as $s \to 0$. Therefore, the bifurcation current associated to the family $f_t$ fails to have a potential bounded around $t = 0$. □

4. Limitations of the construction

To find the examples of Section 2, we used a rational map $\varphi \in \mathbb{C}(z)$ of degree $\geq 1$ and points $a_0, h \in \mathbb{P}^1(\mathbb{C})$ such that

$$0 < [\varphi^n(a_0), h] < r_{n_j}$$

in the chordal metric $[\cdot, \cdot]$, along a sequence $n_j \to \infty$, with $(r_n)$ chosen so that

$$\lim_{n \to \infty} \frac{\log r_n}{d^n} = -\infty.$$ 

Combining (2.7) with (2.5) guaranteed that $\lim_{t \to 0} g_{f,a}(t) = -\infty$. Looking carefully at the estimates, we see that the orbit $\{\varphi^n(a_0)\}$ needs only to satisfy a weaker divergence condition

$$(4.1) \quad \sum_{n=0}^{\infty} \frac{\log [\varphi^n(a_0), h]}{d^n} = -\infty,$$

with $\varphi^n(a_0) \neq h$ for all $n \in \mathbb{N} \cup \{0\}$, to achieve our conclusion with this method.

As observed in the Introduction, the function $g_{f,a}$ will always extend continuously to $\mathbb{D}$ when $f_t$ is a family of polynomials, by [FG2, Main Theorem]. Here we explain explicitly why our construction breaks down for polynomials.

Proposition 4.1. The construction of Section 2 cannot produce any pair $(f, a)$ such that for every $t \in \mathbb{D}^*$, $f_t$ is Möbius conjugate to a polynomial.
Proof. Suppose that \( f_t \) is a holomorphic family of rational maps of degree \( d > 1 \) parameterized by \( \mathbb{D}^* \), that for every \( t \in \mathbb{D}^* \), there exists \( A_t \in \text{PSL}(2, \mathbb{C}) \) such that \( A_t \circ f_t \circ A_t^{-1} \) is a polynomial, and that \( \lim_{t \to 0} f_t = \varphi \) locally uniformly on \( \mathbb{P}^1 \setminus \{ h \} \) for some \( h \in \mathbb{P}^1 \) and some \( \varphi \in \mathbb{C}(z) \) of degree \( d - 1 \) \((> 0)\). For every \( t \in \mathbb{D}^* \), the point \( p_t := A_t^{-1}(\infty) \) is a superattracting fixed point of \( f_t \) for which \( \deg p_t f_t = d \). We first claim that \( \lim_{t \to 0} p_t = h \); otherwise, there is a sequence \((t_j)\) in \( \mathbb{D}^* \) tending to 0 as \( j \to \infty \) such that there is the limit \( p := \lim_{j \to \infty} p_{t_j} \in \mathbb{P}^1 \setminus \{ h \} \). By the locally uniform convergence \( \lim_{t \to 0} f_t = \varphi \) on \( \mathbb{P}^1 \setminus \{ h \} \), \( \deg \varphi > 0 \), and the Argument Principle, this \( p \) must be a superattracting fixed point of \( \varphi \) for which \( \deg p \varphi = d \), contradicting \( \deg \varphi = d - 1 \). We next claim that \( \varphi^{-1}(h) = \{ h \} \); for, if there is a point \( q \in \mathbb{P}^1 \setminus \{ h \} \) for which \( \varphi(q) = h \), then by the first claim, the locally uniform convergence \( \lim_{t \to 0} f_t = \varphi \) on \( \mathbb{P}^1 \setminus \{ h \} \), \( \deg \varphi > 0 \), and the Argument Principle, for any \( t \in \mathbb{D}^* \) close enough to 0, there must exist a point \( q_t \in \mathbb{P}^1 \setminus \{ p_t \} \) (near \( q \)) for which \( f_t(q_t) = p_t \), contradicting \( \deg f_t = d \).

Suppose \( a : \mathbb{D} \to \mathbb{P}^1 \) is any holomorphic map with \( a(0) =: a_0 \neq h \). If \( d > 2 \) so that \( \deg \varphi = d - 1 > 1 \), then by the second claim, we have a constant \( C < 0 \) so that

\[
\log [\varphi^n(a_0), h] \geq C \cdot (d - 1)^n
\]

for all \( n \in \mathbb{N} \). If \( d = 2 \) so that \( \deg \varphi = d - 1 = 1 \), then by the second claim above, the orbit \( \{ \varphi^n(a_0) \} \) can accumulate to \( h \) and satisfy \( \varphi^n(a_0) \neq h \) for any \( n \geq 0 \) only if \( h \) is an attracting or parabolic fixed point of \( \varphi \). Hence we still have a constant \( C < 0 \) so that

\[
\log [\varphi^n(a_0), h] \geq C \cdot n
\]

for all \( n \in \mathbb{N} \). Therefore in both cases, the points \( a_0, h \in \mathbb{P}^1 \) cannot satisfy (4.1).

Working over the field \( \mathbb{C} \) of complex numbers allowed us to exploit the Baire Category Theorem in our construction. In fact, the construction is impossible over a field such as \( \overline{\mathbb{Q}} \).

Proposition 4.2. The construction of Section 2 cannot produce any pair \((f, a)\) such that the map \( \varphi \) and points \( a_0 \) and \( h \) are simultaneously defined over \( \overline{\mathbb{Q}} \).

Proof. Suppose \( f_t \) is any holomorphic family of rational maps of degree \( d > 1 \) parameterized by \( \mathbb{D}^* \) such that \( \lim_{t \to 0} f_t = \varphi \) locally uniformly on \( \mathbb{P}^1 \setminus \{ h \} \), for some \( \varphi \in \overline{\mathbb{C}}(z) \) of degree \( d - 1 \) and some \( h \in \mathbb{P}^1(\overline{\mathbb{Q}}) \). Fix any point \( a_0 \in \mathbb{P}^1(\overline{\mathbb{Q}}) \) such that \( \varphi^n(a_0) \neq h \) for all \( n \geq 0 \).

Suppose first that \( d > 2 \), so that \( \deg \varphi = d - 1 > 1 \). If there is \( A \in \text{PSL}(2, \overline{\mathbb{Q}}) \) such that either \( A \circ \varphi \circ A^{-1} \) or \( A \circ \varphi^2 \circ A^{-1} \) is a polynomial and that \( A(h) = \infty \), then we have a constant \( C < 0 \) so that

\[
\log [\varphi^n(a_0), h] \geq C(d - 1)^n
\]

for all \( n \in \mathbb{N} \). Otherwise, by [11, Theorem E], which uses the Roth theorem, we have the stronger result that

\[
\log [\varphi^n(a_0), h] = o((d - 1)^n)
\]

as \( n \to \infty \). Therefore, in both cases, \( \varphi, a_0, \) and \( h \) cannot satisfy (4.1)
Now suppose that \( d = 2 \) so that \( \deg \varphi = d - 1 = 1 \). Note that the orbit \( \{ \varphi^n(a_0) \} \) can accumulate to \( h \) and satisfy \( \varphi^n(a_0) \neq h \) for any \( n \geq 0 \) only if either \( h \) is an attracting or parabolic fixed point of \( \varphi \) (in \( \mathbb{P}^1(\mathbb{Q}) \)) or there exists \( A \in \text{PSL}(2, \mathbb{Q}) \) such that \( A \circ \varphi \circ A^{-1} \) is an irrational rotation \( z \mapsto \lambda z \), where \( \lambda \) is not a root of unity, \( \lambda \in \mathbb{Q} \), and \( |\lambda| = 1 \), with \( |A(h)| = |A(a_0)| = 1 \). In the former case, we have a constant \( C < 0 \) so that \( \log |\varphi^n(a_0), h| \geq C \cdot n \) for all \( n \in \mathbb{N} \). So \( \varphi, a_0, \) and \( h \) cannot satisfy (1.1).

In the latter case, we claim that we still have a constant \( C < 0 \) such that

\[
\log |\varphi^n(a_0), h| \geq C \cdot n
\]

for all \( n \in \mathbb{N} \); since \( A \in \text{PSL}(2, \mathbb{Q}) \) is biLipschitz with respect to \([\cdot, \cdot] \), we can assume that \( \varphi \) is an irrational rotation \( z \mapsto \lambda z \), where \( \lambda \) is not a root of unity, \( \lambda \in \mathbb{Q} \), and \( |\lambda| = 1 \), with \( h, a_0 \in \mathbb{Q} \) and \( |h| = |a_0| = 1 \). Fix a number field \( K \) so that \( \lambda, a_0, h \in K \), and denote by \( M_K \) the set of all places (i.e., equivalence classes of non-trivial either archimedean or non-archimedean absolute values) of \( K \). Recall that there are a family \( (N_v)_{v \in M_K} \) in \( \mathbb{N} \) and a family \( (|\cdot|_v)_{v \in M_K} \) of representatives \( |\cdot|_v \) of places \( v \) such that for every \( x \in K^* \), \( |x|_v = 1 \) for all but finitely many \( v \in M_K \) and \( \prod_{v \in M_K} |x|^N_v = 1 \). Then by the (strong) triangle inequality, we can choose a family of real numbers \( C_v \geq 1 \), \( v \in M_K \), such that \( |\lambda^n - h/a_0|_v \leq C^n_v \) for any \( v \in M_K \) and any \( n \in \mathbb{N} \) and that \( C_v = 1 \) for all but finitely many \( v \in M_K \). We also note that \( \lambda^n a_0 = \varphi^n(a_0) \neq h \) for all \( n \in \mathbb{N} \). Hence for every \( v_0 \in M_K \) and every \( n \in \mathbb{N} \), we have \( |\lambda^n - h/a_0|_{v_0} \geq (\prod_{v \in M_K} C_v^{-N_v})^n \). In particular, recalling that \( [z, w] = |z - w|[z, \infty][w, \infty] \) on \( \mathbb{C} \times \mathbb{C} \), there is a constant \( C < 0 \) such that

\[
\log |\varphi^n(a_0), h| = \log |\lambda^n - h/a_0| - \log 2 \geq C \cdot n
\]

for all \( n \in \mathbb{N} \). So the claim holds, and \( \varphi, a_0, \) and \( h \) cannot satisfy (1.1). \( \square \)

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