Stationary states in a free fermionic chain from the Quench Action Method

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We employ the Quench Action Method (QAM) for a recently considered geometrical quantum quench: two free fermionic chains initially at different temperatures are joined together in the middle and let evolve unitarily with a translation invariant Hamiltonian. We show that two different stationary regimes are reached at long times, depending on the interplay between the observation timescale $T$ and the total length $L$ of the system. We show the emergence of a non-equilibrium steady state (NESS) supporting an energy current for observation time $T$ much smaller than the system size $L$. We then identify a longer time-scale for which thermalization occurs in a Generalized Gibbs Ensemble (GGE).

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I. INTRODUCTION

Quantum quenches are nowadays the paradigm to study the long-time dynamics of isolated many-body quantum systems. Their special interest relies in the possibility of an analytical treatment, especially in low dimensions and in their potential realization as experimental protocols with ultracold atoms. The real-time simulation of many-body quantum systems is also a challenge for sophisticated numerical algorithms. The main physical question regards the possibility of an effective thermodynamic description of a quantum system initially prepared with a density matrix, after unitary evolution with an Hamiltonian $\hat{H}$. Formally the long-time density matrix $\hat{\rho}_{\text{stat}}$ is defined from $\lim_{t\to\infty}\text{Tr}[\hat{\rho}_0 \hat{O}(t)] \equiv \text{Tr}[\hat{\rho}_{\text{stat}} \hat{O}]$ and this limit is believed to exist for any local operator $\hat{O}$. However the characterization of $\hat{\rho}_{\text{stat}}$ in terms of few macroscopic parameters of the system and the role of integrability are still open problems.

In a recent work, an unified approach, dubbed the Quench Action Method (QAM), has been proposed to theoretically address the characterization of the steady density matrix after an integrable quench. This method exploits the special properties of the thermodynamic limit $N \to \infty$, $L \to \infty$ at fixed density $N/L$, with $N$ and $L$ the particle numbers and the volume of the system respectively.

Known examples of applications are restricted to 1 + 1 dimensions when $\hat{H}$ is Bethe Ansatz solvable and we will also stick to this situation in this paper. Roughly speaking, in the thermodynamic limit the many-body Hilbert space trace, defining the expectation value of local observables, can be replaced by a functional integral over a set of smooth macroscopic densities $\rho_n(\phi)$ of single-particle momenta, weighted by an exponential factor $\exp(-\beta S_Q[\rho_n])$. The functional $S_Q[\rho_n]$ is called the Quench Action and is stationary at large times when all the $\rho_n(\phi)$'s equal $\rho_{\text{stat}}(\phi)$. The function $\rho_{\text{stat}}(\phi)$ identifies a representative pure state that reproduces all the macroscopic features of the mixed state $\hat{\rho}_{\text{stat}}$ reached by the system for large times. The QAM requires the knowledge of the thermodynamic limit of the overlaps between the many-body initial state and the eigenstates of the Hamiltonian $\hat{H}$, a non-trivial task even in one-dimensional systems, which has been solved in few particular cases.

Here, we discuss an application of the QAM to a recently considered geometrical quench: two identical quantum spin chains initially held at different inverse temperatures $\beta_{1,2}$ are joined together restoring translational invariance and evolved unitarily. Extensions to higher dimensions are also of current interest.

In order to illustrate how the QAM can be successfully employed to extract the stationary behavior of this system we focus on the simplest possible example: a free fermionic chain. Formally the steady state displays different regimes according to the order in which the large times and the thermodynamic limit are taken. If the linear size of the system $L$ is sent to infinity before the observation time $T$, $\hat{\rho}_{\text{stat}}$ supports a persistent energy current. Specifically, there will be a range of times $T \propto L/\nu_{\text{max}}$, where $\nu_{\text{max}}$ represents the speed of the fastest mode of the system, for which this Non-Equilibrium Steady State (NESS) is observable. For much longer times, boundaries start to be relevant in the dynamics and a complete time-reversal symmetric state is restored, where all the expectation values of local operators coincide with the mean between their thermal average in the two disconnected chains. This regime can be observed for $T \ll T_{\text{rev}}$, where $T_{\text{rev}} \propto L^2$ is the typical revival time of a free fermionic chain. As expected, thermalization occurs in a Generalized Gibbs Ensemble (GGE). As we will show, both the two regimes are captured by the QAM.

The paper is organized as follow. In Sec. II we introduce our model: the spin-1/2 XX-chain equivalent to a free-fermionic model after the Jordan-Wigner transformation. The overlaps between the Hamiltonian eigenstates before and after the quench are shown to be determinants and need to be properly regularized when taking the infinite volume limit $L \to \infty$. In Sec. III we carry out
II. THE MODEL

We consider two disconnected spin-1/2 XX chains with Hamiltonian \( H_0 = H_l + H_r \)
\[
\hat{H}_r = \frac{1}{2} \sum_{n=1}^{L} (\hat{\sigma}^x_n \hat{\sigma}^x_{n+1} + \hat{\sigma}^y_n \hat{\sigma}^y_{n+1}),
\]
\[
\hat{H}_l = \frac{1}{2} \sum_{n=1}^{L-1} (\hat{\sigma}^x_{-n} \hat{\sigma}^x_{n+1} + \hat{\sigma}^y_{-n} \hat{\sigma}^y_{n+1});
\]
where \( \hat{\sigma}_n^\alpha \) is a Pauli matrix at position \( n \in \mathbb{Z} \) and \( \alpha = x, y, z \). The model is equivalent to a free fermionic chain exploiting the Jordan-Wigner transformation
\[
\hat{H}_0 = \sum_{\lambda=\pi}^{\lambda=0} \int_0^\pi d\theta \varepsilon(\theta) \hat{\psi}^\dagger(\theta) \hat{\psi}(\theta),
\]
the operators \( \hat{c}_n \) satisfy canonical anticommutation relations \( \{\hat{c}_n, \hat{c}^\dagger_{n'}\} = \delta_{n,n'} \). For \( L \rightarrow \infty \), the quadratic form \( \hat{H}_0 \) can be easily diagonalized introducing single-particle fermionic operators in momentum space through \( \hat{c}_n = \sqrt{\frac{2}{\pi}} \int_0^\pi d\theta \sin[\theta(n-1/2)] \hat{\psi}(\theta) \) for positive values of \( n \) and \( \hat{c}_n = \sqrt{\frac{2}{\pi}} \int_0^\pi d\theta \sin[\theta(n-1/2)] \hat{\psi}(\theta) \) for negative. One finds
\[
\hat{H}_0 = \sum_{\lambda=\pi}^{\lambda=0} \int_0^\pi \varepsilon(\theta) \hat{\psi}^\dagger(\theta) \hat{\psi}(\theta),
\]
with \( \varepsilon(\theta) = -2 \cos \theta \) and the fermionic fields obeying
\[
\{ \hat{\psi}_{\theta/l}(\theta), \hat{\psi}_{\theta/l}(\theta') \} = \delta(\theta - \theta').
\]

When the two chains are joined together and the infinite volume limit is considered, the resulting Hamiltonian \( \hat{H} = \hat{H}_0 + \hat{c}_0^\dagger \hat{c}_1 + \hat{c}_1^\dagger \hat{c}_0 \) is translation invariant and can be diagonalized by Fourier transform defining \( \hat{c}_n = \int_{-\pi}^{\pi} \frac{d\phi}{2\pi} e^{in\phi} \hat{\psi}(\phi) \). It follows
\[
\hat{H} = \int_{-\pi}^{\pi} d\phi \varepsilon(\phi) \hat{\psi}^\dagger(\phi) \hat{\psi}(\phi),
\]
with canonically normalized fields
\[
\{ \hat{\psi}(\phi), \hat{\psi}(\phi') \} = \delta(\phi - \phi').
\]

From the explicit expressions of the local fermions \( \hat{c}_n \), we can derive the matrix elements \( M_{r/l}(\phi, \theta) = \langle 0 \hat{\psi}(\phi) \hat{\psi}_{r/l}(\theta)|0 \rangle \) between the single-particle fermionic operators before and after the quench; \( |0 \rangle \) denotes the Fock vacuum. Their explicit form is as follows
\[
M_r(\phi, \theta) = \frac{1}{2\pi i} \left[ \frac{e^{-i\theta/2}}{1 - e^{i(\phi - \theta - i\delta)}} - \frac{e^{i\theta/2}}{1 - e^{i(\phi + \theta - i\delta)}} \right], \quad (8)
\]
\[
M_l(\phi, \theta) = \frac{1}{2\pi i} \left[ \frac{e^{i\theta/2}}{1 - e^{i(\phi + \theta + i\delta)}} - \frac{e^{-i\theta/2}}{1 - e^{i(\phi - \theta + i\delta)}} \right], \quad (9)
\]
where \( \delta > 0 \) is a small positive quantity needed to ensure convergence; physically it can be interpreted as an infrared cutoff and we will see that the behavior of \( \delta \) at large \( T \) corresponds to different order of limits in \( T \) and \( L \). Using Cauchy theorem it is possible to show that in the limit \( \delta \rightarrow 0^+ \), \( (8, 9) \) are consistent with the canonical normalization \( (7) \). A basis for the \( N \)-particle sector of the fermionic Fock space associated to the Hamiltonian \( \hat{H} \) is \( \{\theta, \lambda \} = \{\theta_1^N, \ldots, \theta_N^N\} = \hat{\psi}^\dagger_1(\theta_1) \ldots \hat{\psi}^\dagger_N(\theta_N)|0\rangle \), with \( \lambda_i = l, r \). Similarly a basis for the \( N \)-particle Fock space of the fermions after the quench is \( \{\Phi\}_{\lambda N} = \{\phi_1, \ldots, \phi_N\} = \hat{\psi}^\dagger(\phi_1) \ldots \hat{\psi}^\dagger(\phi_N)|0\rangle \). Then applying the Wick theorem to multipoint correlation functions of the local fermions \( \hat{c}_n \) one obtains
\[
N(\Phi|\{\theta, \lambda\})_M = \delta_{N,M} \det[M_{\mu,\nu}(\phi_{\mu}, \theta_{\nu})]^N_{\mu,\nu=1}. \quad (10)
\]
The overlap between the \( N \)-particle states before and after the quench is the determinant of an \( N \times N \) matrix, whose elements are given in \( (8, 9) \).

III. THE QUENCH ACTION

We now derive the Quench Action for our protocol. Initially the two halves are disconnected and independently thermalized, the state of the system is then described by the density matrix \( \hat{\rho}_0 = Z^{-1}e^{-\beta_0 \hat{H}_0} \otimes e^{-\beta_0 \hat{H}_r} \), with \( Z \) a normalization constant.

Let us consider the expectation value of a local operator \( \hat{O} \), evolved in time with the Hamiltonian \( \hat{H} \). Since the theory is free we can fix the particle number \( N \) in the Fock spaces before and after the quench and consider the limit \( N \rightarrow \infty \) only at the very end. Such a treatment of the thermodynamic limit, in which the ratio \( N/L \) is always zero, fails to correctly treat interactions at finite particle density but is effective for free theories\([23]\). Formally we have
\[
\text{Tr}[\hat{O}(t)\hat{\rho}_0] = \sum_{\Phi,\Phi'} e^{-i(E_{\Phi} - E_{\Phi'})t} N \langle \Phi|\hat{O}|\Phi'\rangle_N \mathcal{D}(\Phi, \Phi'),
\]
where the sum is over all the \( N \)-particle states \( |\Phi\rangle_N \) with energy \( E_{\Phi} = \sum_{i=1}^{N} \varepsilon(\phi_i) \) and we defined
\[
\mathcal{D}(\Phi, \Phi') = \sum_{\{\theta, \lambda\}} P(\{\theta, \lambda\}) \langle \Phi|\{\theta, \lambda\}\rangle_N \langle \{\theta, \lambda\}|\Phi'\rangle_N.
\]
The probability measure of the state $\{|{\theta, \lambda}\rangle\rangle_N$ is factorized

$$P\{\theta, \lambda\} = \prod_{\mu=1}^N \frac{1}{\pi} \delta_{\lambda, \mu} | f_1(\theta_\mu) + \delta_{\lambda, \mu} f_r(\theta_{\mu}) \rangle$$

with $f_{1/r}(\theta) = 1/(1 + e^{\beta_{1/r}(\theta)}), \text{ the usual Fermi-Dirac distribution at inverse temperature } \beta_{1/r}$. The prefactor ensures the correct normalization since $\int_0^\pi d\theta f_{1/r}(\theta) = \pi/2$.

In order to compute $\mathcal{D}(\Phi, \Phi')$, we denote by $M_\mu(\phi_\mu | x_{\mu})$ the matrix elements $M_{\lambda,\mu}(\phi_\mu | x_{\mu})$ in (10) and observe that $\mathcal{D}(\Phi, \Phi')$ can be rewritten as the expectation value with respect to the probability measure (13) of the product of two determinants

$$\mathcal{D}(\Phi, \Phi') = \mathbb{E}[\det[M_\mu(\phi_{\mu} | x_{\mu})]_{\mu=1}^N \det[M_*^\mu(\phi_{\mu} | x_{\mu})]_{\mu=1}^N].$$

(14)

The notation $\mathbb{E}[\Phi]$ is a shorthand for

$$\mathbb{E}[\Phi] = \sum_{\lambda_1, \ldots, \lambda_N} \int_0^\pi d\theta_1 \ldots \int_0^\pi d\theta_N P\{\theta, \lambda\} g(\{\theta, \lambda\}).$$

Expanding the two determinants in (14) over permutations $\sigma, \omega \in S_N$ we get

$$\mathcal{D}(\Phi, \Phi') = \sum_{\sigma, \omega} (-1)^{\sigma+\omega} \mathbb{E} \left[ \prod_{\mu=1}^N M_{\sigma(\mu)}(\theta_\mu | x_{\mu}) M^*_{\omega(\mu)}(\theta_\mu | x_{\mu}) \right].$$

(16)

where the matrix elements $I(\phi, \phi')$ are now obtained expectation values in the single-particle space

$$I(\phi, \phi') = \sum_{z=1, r} \int_0^\pi \frac{d\theta}{\pi} M_{\lambda}(\phi, \theta) M^*_{\lambda}(\phi', \theta) f_{\lambda}(\theta).$$

(17)

We note in passing that (16) is known in Random Matrix Theory as Andréief identity and requires independent matrix elements, e.g. free theories.

The integrals in (17) can be computed exactly noticing that $M_{r/l}(\phi, \theta) = -M_{r/l}(\phi, -\theta)$ and $f_{r/l}(\theta) = f_{r/l}(-\theta)$. Then the integration domain can be extended to $\theta \in [0, 2\pi]$ and becomes the unit circle $|z| = 1$, in the complex variable $z = e^{i\phi}$. Applying Cauchy theorem, taking care of the essential singularity of the functions $f_{r/l}(z)$ at $z = 0$, one finds

$$I(\phi, \phi') = \frac{1}{4\pi^2} \left[ f_1(\phi') - f_r(\phi) \right] e^{i\phi} - e^{-i\phi} + f_1(\phi') + f_r(\phi) e^{i(\phi' - \phi) + 2\pi} - 1 + \frac{-f_1(\phi') - f_r(\phi)}{e^{i(\phi' - \phi)} - e^{-i(\phi' - \phi)} - e^{i(\phi' - \phi) - 2\pi} - 1}. \tag{18}$$

The matrix elements $I(\phi, \phi')$ have poles at $\phi = \phi' \pm 2\pi$, we remark that contrary to the single-particle overlaps no singularity is present for $\phi = -\phi'$. The formal Fock-space trace is in a free-theory a multidimensional integral over the angular variables $\phi_{\mu}$ and $\phi'_{\mu}, \mu = 1, \ldots, N$, defining the momenta of the $N$-particle states $|\Phi\rangle$ and $|\Phi'\rangle$. Let us denote those integrations shortly by $dN\phi$ and $dN\phi'$ and introduce a normalized momentum density by

$$\rho(\phi) = \frac{1}{N} \sum_{\mu=1}^N \delta(\phi - \phi_{\mu}), \tag{19}$$

analogously we will also consider a function $\rho'(\phi')$. In the spirit of the QAM the double sum in (11) is now replaced by a functional integral over the densities $\rho(\phi)$ and $\rho'(\phi')$.

$$\text{Tr}[\hat{\rho}_0 \hat{O}(t)] = \frac{N!}{\pi} \int dN\phi \int dN\phi' \int \mathcal{D}\rho \int \mathcal{D}\rho' \int \mathcal{D}\rho' J[\rho] J[\rho'] e^{iNt f d\phi(\rho - \rho') e^{i\rho(\phi) - \sum_{\mu=1}^N \delta(\phi - \phi_{\mu})}, \tag{20}$$

where $J[\rho]$ and $J[\rho']$ enforce the constraint. Following, they can be replaced by a functional integral over auxiliary imaginary functions $g(\phi)$ and $g'(\phi')$

$$J[\rho] \equiv \delta(N\rho(\phi) - N \sum_{\mu=1}^N \delta(\phi - \phi_{\mu})) = \int \mathcal{D}g e^{\frac{1}{N} \int \mathbb{E}[\Phi] [\rho_0(\phi) - \sum_{\mu=1}^N \delta(\phi - \phi_{\mu})]}, \tag{21}$$

We then expand the determinant in (20) over permutations $\sigma \in S_N$ and observe that the permutation sign $(-1)^{\sigma}$ can be absorbed inside the definition of the state $|\Phi\rangle = (-1)^{\sigma}|\sigma\Phi\rangle$, with $|\sigma\Phi\rangle \equiv |\phi_{\sigma(1)} \ldots \phi_{\sigma(N)}\rangle$. The integration over $dN\phi$ is now factorized and one ends up with a term of the form $\prod_{\mu=1}^N \int d\phi I(\phi, \phi_{\mu}) e^{-g(\phi)}$, that can be exponentiated and rewritten in terms of the density $\rho(\phi')$. Finally, similar manipulations of the remaining integration over $dN\phi'$ lead to

$$\text{Tr}[\hat{\rho}_0 \hat{O}(t)] = \int \mathcal{D}g \int \mathcal{D}g' \int \mathcal{D}\rho \int \mathcal{D}\rho' h[\rho'] e^{N S_{\phi}[\rho; g' - g]}, \tag{22}$$

where we absorbed a $(N!)^2$ prefactor as an inessential constant term inside $S_{\Phi}$. Notice that since $\hat{O}$ is a local operator the matrix element $N \langle \Phi | \hat{O}^{\dagger}\Phi\rangle_N$ is a smooth function of the set of momenta $\{\phi'\}$ that can be replaced for our purposes by a functional $h[\rho']$ appearing in (22). The reason is that local observables can only affect microscopic details of the densities $\rho', \rho$, which result in sub-leading contributions in the thermodynamic limit. A similar reasoning is standard in the study of equilibrium properties for one-dimensional integrable spin chains. The functional $S_{\Phi}$ is the Quench Action

$$S_{\Phi}[\rho; g, g'] = \int d\phi \left( g \rho + g' \rho' \right) + \log \int d\phi e^{-g'} \int d\phi \epsilon(\rho - \rho') + \int d\phi' \int g(\phi, \phi') e^{-g(\phi')}, \tag{23}$$

which is explicit time-dependent.
The integral in (25) can be computed in the t → ∞ limit by its saddle points, which physically identify for C ℑ pictured in figure. The region where ℑ limit. Considering then the variation with respect to ρ', with the aid of (24), we arrive at

\[ \delta S_\phi \bigg|_{\rho' = \rho_S} = -\log \rho_S'(\phi) - it\varepsilon(\phi) \]

implying the normalization of the stationary distribution \( \rho_S' \) and reproducing the familiar entropic term \( -\rho_S' \log \rho_S' \) when substituted back in (24). The stationarity condition for \( \rho \) simply identifies \( g_S = -it\varepsilon \) and notice that consistently with (21) this is a purely imaginary function. Considering then the variation with respect to \( \rho' \), with the aid of (24), we arrive at

\[ \frac{\delta S_\phi}{\delta \rho'} \bigg|_{\rho' = \rho_S} = -\log \rho_S'(\phi) - it\varepsilon(\phi) + \log \int d\phi' I(\phi', \phi) e^{it\varepsilon(\phi')} = 0. \]  

The integral in (25) can be computed in the t → ∞ limit extending it to the complex plane of \( z = e^{i\phi'} \). For large times, line integrals in the region \( \Im[\varepsilon(z)] > 0 \) are exponentially vanishing and we can replace the original contour on the unit circle by the contour \( C_t \), considered in Fig. 1. The final integration path consists of two closed curves: one inside the unit circle (red-colored in Fig. 1) for \( \Im(z) > 0 \), the other outside the unit circle (in blue in Fig. 1) when \( \Im(z) < 0 \). Looking at the expression (19), one realizes that the pole inside the unit circle, with residue proportional to \( f_r(\phi) \), contributes to the integral only of \( \phi > 0 \); vice-versa the pole outside the unit circle with residue proportional to \( f_l(\phi) \) must be taken into account only when \( \phi < 0 \). Taking the limit \( \delta \to 0^+ \) we conclude from (25)

\[ \rho_S'(\phi) = \frac{1}{\pi} [\Theta(\phi) f_r(\phi) + \Theta(\phi) f_l(\phi)], \]

where \( \Theta(\phi) \) is the Heaviside theta function. Finally, the variation with respect to \( g \) identifies \( \rho_S'(\phi) \) with \( \rho_S(\phi) \).

Any pure state defined by the momenta distribution in (26) would be a representative state for the factorized NESS encountered in free theories and CFT. Notice that the derivation requires \( \delta \to 0^+ \) faster than \( t \to \infty \), otherwise the contour integral along \( C_t \) would be vanishing. Physically this is related to the existence of the NESS for observation times \( T \ll L/v_{\text{max}} \). It is not difficult to show that the density distribution (26) implies a finite and constant energy flow between the two halves of the chain (13).

IV. NON-EQUILIBRIUM STEADY STATE

We now show how the NESS already found in (13) can be nicely re-derived imposing stationarity of the Quench Action (23) and is therefore exact in the thermodynamic limit.

Functional derivative of (23) with respect to \( g' \) gives the condition

\[ \frac{\delta S_\phi}{\delta g'} \bigg|_{g' = g, \rho' = \rho_S} = \rho_S' - \frac{e^{-g_S'}}{\int d\phi e^{-g_S'}} = 0, \]

implying the normalization of the stationary distribution \( \rho_S' \) and reproducing the familiar entropic term \( -\rho_S' \log \rho_S' \) when substituted back in (24). The stationarity condition for \( \rho \) simply identifies \( g_S = -it\varepsilon \) and notice that consistently with (21) this is a purely imaginary function. Considering then the variation with respect to \( \rho' \), with the aid of (24), we arrive at

\[ \frac{\delta S_\phi}{\delta \rho'} \bigg|_{\rho' = \rho_S} = -\log \rho_S'(\phi) - it\varepsilon(\phi) \]

V. THERMALIZATION TIME SCALE AND GGE

A different reasoning is required to obtain the stationary state observed in (23). In our protocol, where the initial state presents two subsystems that have a macroscopic energy difference, thermalization requires a time \( T_{\text{th}} \ll T_{\text{rev}} \) that however diverges with the system size and in particular \( T_{\text{th}} \gg L/v_{\text{max}} \). In this time regime, the previous approach is not correct since it assumed that the system was strictly speaking in the thermodynamic limit. At finite \( L \), the Hamiltonian (1) can still be diagonalized in a similar manner. The fermion momenta on the two halves are quantized according to \( \hat{\theta}^{(k)} = \frac{\pi k}{L/2 + 1} \), with \( k = 1, \ldots, L/2 \) and have single-particle dispersion relation \( \varepsilon(\theta^{(k)}) = -2\cos \theta^{(k)} \); the Hamiltonians \( H_{j/\tau} \) are then diagonal in the fermionic operators defined as

\[ \hat{\psi}_{r,k} = \frac{2}{\sqrt{L+2}} \sum_{j=1}^{L/2} \sin(\theta^{(k)}j) \hat{c}_j, \]

\[ \hat{\psi}_{l,k} = \frac{2}{\sqrt{L+2}} \sum_{j=0}^{L/2-1} \sin[\theta^{(k)}(j + L/2)] \hat{c}_j. \]

They satisfy canonical anticommutation relations \( \{\hat{\psi}_{l/\tau,k}, \hat{\psi}_{l/\tau,k}^\dagger\} = \delta_{k,k'} \). For large \( L \), the specific details
of the boundary conditions are immaterial and for simplicity we choose periodic boundary conditions for the full chain obtained after joining the two halves in the middle. One gets the momenta $\phi^{(m)} = \frac{2\pi m}{L}$, $m = -L/2 + 1, \ldots, L/2$ with dispersion relation $\varepsilon(\phi^{(m)}) = -2\cos\phi^{(m)}$ and the Hamiltonian becomes diagonal in the operators

$$\hat{\psi}_m = \frac{1}{\sqrt{L}} \sum_{j=-\frac{L}{2}+1}^{\frac{L}{2}} e^{ij\phi_m} \hat{c}_j, \quad \{\hat{\psi}_m, \hat{\psi}_n^\dagger\} = \delta_{m,n}.$$  \hfill (29)

We now reconsider (11). Although at finite $L$, a similar reasoning can be used to write $D(\Phi, \Phi') = N!\det[I_L(\phi_{\mu}, \phi'_{\nu})]_{\mu,\nu=1}^{N}$. Now $\Phi, \Phi'$ represents many-body configuration of particles with quantized momenta, while $I_L(\phi_{\mu}, \phi'_{\nu})$ replaces the result in (17) with the finite $L$ thermal average

$$I_L(\phi, \phi') = \frac{2}{L} \sum_{\lambda=1,r} \sum_k f_\lambda(\theta^{(k)}(\phi)) M_\lambda(\theta^{(k)}, \phi) M_\lambda^*(\theta^{(k)}, \phi')$$  \hfill (30)

where the prefactor comes from the normalization $\sum_\mu f_{1/r}(\theta_\mu) \simeq L/4$ and the finite $L$ overlaps take the form

$$M_r(\theta^{(k)}, \phi) = \frac{2}{\sqrt{L(L+2)}} \sum_{j=1}^{L/2} \sin(j\theta^{(k)}) e^{ij\phi},$$

$$M_l(\theta^{(k)}, \phi) = \frac{2}{\sqrt{L(L+2)}} \sum_{j=L/2+1}^{L} \sin(j\theta^{(k)}) e^{ij\phi}. $$  \hfill (31) \hfill (32)

The derivation of the quench action can formally proceed along the same lines described in sec. III. One arrives to an equation analogous to (25). One obtains

$$\rho'_S(\phi) = \sum_m I_L(\phi^{(m)}, \phi)e^{i(t(\varepsilon(\phi^{(m)}) - \varepsilon(\phi))}.$$  \hfill (33)

In the limit of large times, the sum is dominated by the value where the phase vanishes, i.e. $\rho'_S(\phi) \simeq \frac{L}{2\pi} I_L(\phi, \phi)$. The same conclusion follows taking the time average of (33), and recalling that in the thermodynamic limit a Jacobian factor $\frac{1}{2\pi}$ is produced passing from the discontinuous function (19) to the smooth density $\rho'_S(\phi)$. In order to compute $I_L(\phi, \phi)$ for the large $L$, we notice that normalization ensures that

$$\sum_k |M_{r,l}(\theta_k, \phi_m)|^2 = 1/2$$  \hfill (34)

Moreover for large $L$ the support of $|M_{r,l}|^2$ concentrates in a window $|\theta^{(k)} - \phi^{(m)}| = O(L^{-1})$. This two conditions are sufficient to see that

$$\rho'_S(\phi) = \frac{L}{2\pi} \lim_{L \to \infty} I_L(\phi, \phi) = \frac{1}{2\pi}(f_1(\phi) + f_r(\phi)).$$  \hfill (35)

This stationary state is clearly symmetric under parity $\phi \to -\phi$ and therefore no current is present. Moreover, this result is perfectly consistent with the GGE prediction. Indeed, the local quench we studied does not affect the expectation value of extensive conserved quantities, that therefore remain the sum of the contributions of the two halves, i.e. $Q_{\text{tot}} = Q_l + Q_r$. For the free system we are considering, this requires that all the occupation number at the same energy from the two halves simply sum up, or equivalently (25).

VI. CONCLUSIONS

In this paper we studied the behavior of two identical quantum XX chains, initially thermalized at different inverse temperatures $\beta_1/\lambda$ and then suddenly put in contact in the middle. We formulated the problem of determining the stationary state reached by the system at large times after the quench in terms of the recent QAM and showed that two different regimes are possible. A genuine NESS, characterized by time reversal symmetry breaking and the emergence of a stationary energy current, describes the chain for observation times $T \ll L/\nu_{\text{max}}$. At larger times the system is locally equivalent to a GGE: a state that indicates thermalization constrained to conservation laws where no current is flowing. These two scenarios are captured by the QAM and our analytic results completely agree with the numerics in [25]. We believe that the derivation of the Quench Action for this problem is a promising and necessary step in the attempt to generalize the calculation to interacting models; notably the Lieb Liniger gas. In particular the latter is currently under investigation by the authors.

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