The impact of multiplicative noise in SPDEs close to bifurcation via amplitude equations

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Abstract
This article deals with the approximation of a stochastic partial differential equation (SPDE) via amplitude equations. We consider an SPDE with a cubic nonlinearity perturbed by a general multiplicative noise that preserves the constant trivial solution and we study the dynamics around it for the deterministic equation being close to a bifurcation. Based on the separation of time-scales close to a change of stability, we rigorously derive an amplitude equation describing the dynamics of the bifurcating pattern. This allows us to approximate the original infinite dimensional dynamics by a simpler effective dynamics associated with the solution of the amplitude equation. To illustrate the abstract result we apply it to a simple one-dimensional stochastic Ginzburg–Landau equation.

Keywords: amplitude equation, multiplicative noise, multi-scale analysis, bifurcation, slow–fast system, stochastic partial differential equations

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1. Introduction

In this paper we study a class of stochastic partial differential equations (SPDEs) of the following form

\[ du(t) = (Au(t) + \varepsilon^2 Lu(t) + F(u(t)))dt + \varepsilon G(u(t))dW(t), \]

where $A$ is a non-positive self-adjoint operator with finite-dimensional kernel, $\varepsilon^2 Lu(t)$ is a small deterministic perturbation with $\varepsilon > 0$ measuring the distance to the change of stability. The nonlinearity $F$ is a cubic mapping, and $G(u)$ is Hilbert–Schmidt operator with $G(0) = 0$.
so that the constant $u = 0$ is a solution to equation (1). The noise is given via a (possibly infinite dimensional cylindrical) Wiener process $W$ on some stochastic basis.

Our aim is to study in the limit $\varepsilon \to 0$ the asymptotic dynamics of the solution $u(t)$ to equation (1) on the natural slow time-scale of order $\varepsilon^{-2}$.

Near a change of stability of the linearized operator $\mathcal{A} + \varepsilon^2 \mathcal{L}$, a natural separation of time-scales allows the original system to be transferred into slow dynamics on a dominant pattern, which couples to dynamics on a fast time scale. A reduced equation eliminating the fast variable and characterizing the behaviour of dominant modes significantly simplifies the dynamics to a stochastic differential equation (SDE). This equation identifies the amplitudes of dominant pattern and is often called amplitude equation.

Amplitude approximation plays a prominent role in qualitative analysis on the dynamics of stochastic systems near a change stability. For additive noise amplitude approximation for SPDEs has been studied in many cases starting from [1] and later [2–4]. See also [5–8] for related work.

For the case of SPDEs on unbounded domains the effective equation is no longer an SDE, but the reduced model is still given as an infinite dimensional SPDE. For details see [9] in the case of a simple one-dimensional noise, [10] for large domains and [6, 11] for results with space-time white noise and on an unbounded domain. Here we will focus on the case of bounded domains only.

Amplitude equations can be used to qualitatively describe the dynamics close to a change of stability. In [2] amplitude equations were used to give an approximation of the infinite-dimensional invariant measure for a Swift–Hohenberg equation, while in [2, 12, 13] ideas were presented that would allow to approximate random attractors or random invariant manifolds via amplitude equations. See also [5, 14] for results for other models with additive noise.

While many results for the approximation via amplitude equations were established for additive noise, the case of multiplicative noise is not that well studied. Only for the very special case of $G(u) = u$ and $W$ being a scalar Brownian-motion first results for amplitude equations were obtained in [12]. With this special case of noise the approximation of random invariant manifolds was studied in [15]. In first approximation the dynamics on the dominant space is given by a variant of the amplitude equation, while for the qualitative description of a random invariant manifold, one also needs an effective equation on the infinite dimensional remainder. See also [16, 17] or [18] using parameterizing manifolds introduced by [19], see also [20].

In the present paper our main contribution is the analysis in the case of general infinite-dimensional multiplicative noise. We will only treat the case with $G(0) = 0$, so there is no contribution by an additive noise, which would lead to a different scaling in $\varepsilon$ of the noise.

Under some smoothness assumptions on the diffusion coefficient $G$ and regularity conditions on the noise, we derive the amplitude equations of responding equation (1) and show rigorously, that it captures the essential dynamics of the dominant modes. We use the Taylor expansion of $G$ in order to directly determine the errors bounds between the solution of (1) and that of the amplitude equation which is only on the dominant modes.

The organization of this paper is as follows: in section 2, we formulate the abstract framework and some basic assumptions. Section 3 contains the main results of the paper as presented in theorem 3.1. In section 4, we give the proof of the main results. Section 5 is devoted to an illustrative example.
2. Setting and assumptions

Throughout the paper, we shall work in a separable Hilbert space \( \mathcal{H} \), endowed with the usual scalar product \( (\cdot, \cdot) \) and with the corresponding norm \( \| \cdot \| \). Concerning the leading operator \( A \) we shall assume the following conditions.

**Assumption 1 (Linear operator \( A \)).** Suppose \( A \) is a self-adjoint and non-positive operator on \( \mathcal{H} \) with eigenvalues \( \{-\lambda_k\}_{k \in \mathbb{N}} \) such that \( 0 = \lambda_1 \leq \cdots \leq \lambda_k \cdots \), satisfying \( \lambda_k \geq C k^m \) for all sufficiently large \( k \), positive constants \( m \) and \( C \). The associated eigenvectors \( \{e_k\}_{k=1}^\infty \) form a complete orthonormal basis in \( \mathcal{H} \) such that \( Ae_k = -\lambda_k e_k \).

By \( N \) we denote the kernel space of \( A \), which, according to assumption 1, has finite dimension \( n \) with basis \( \{e_1, \ldots, e_n\} \). By \( P_c \) we denote the orthogonal projector from \( \mathcal{H} \) onto \( N \) with respect to the inner product \( (\cdot, \cdot) \), and by \( P_s \) the orthogonal projector from \( \mathcal{H} \) onto the orthogonal complement \( S = \mathcal{N}^\perp \).

One standard example is with \( m = 4/d \) is the Swift–Hohenberg operator \( A = -(1 + \Delta)^2 \) on \( \mathcal{H} = L^2([-\pi, \pi]^d) \) subject to periodic boundary conditions. Similar is the Laplacian \( \Delta \) with \( m = 2/d \). But we could also treat more general equations and also coupled systems of SPDEs here.

**Remark 2.1.** Let us remark that the setting of a Hilbert space and \( A \) being a self-adjoint operator is mainly for simplicity of presentation, as many crucial properties about the \( \mathcal{H}^\alpha \)-spaces defined below, the projections \( P_c \) and \( P_s \) and the semigroup \( e^{At} \) generated by \( A \) follow in this setting as trivial lemmas. Otherwise we would need to formulate them as an assumption and verify them in the given application.

We can now define fractional Sobolev-spaces \( \mathcal{H}^\alpha = D((1-A)^{\alpha/2}) \) by using the domain of fractional powers of the operator \( A \):

**Definition 2.1.** For \( \alpha \in \mathbb{R} \), we define the space \( \mathcal{H}^\alpha \) as

\[
\mathcal{H}^\alpha = \left\{ \sum_{k=1}^\infty \gamma_k e_k : \sum_{k=1}^\infty \gamma_k^2 (\lambda_k+1)^\alpha < \infty \right\},
\]

which is endowed with the norm

\[
\left\| \sum_{k=1}^\infty \gamma_k e_k \right\|_\alpha = \left( \sum_{k=1}^\infty \gamma_k^2 (\lambda_k+1)^\alpha \right)^{1/2}.
\]

The operator \( A \) generates an analytic semigroup \( \{e^{At}\}_{t \geq 0} \) on any space \( \mathcal{H}^\alpha \), defined by

\[
e^{At} \left( \sum_{k=1}^\infty \gamma_k e_k \right) = \sum_{k=1}^\infty e^{-\lambda_k t} \gamma_k e_k, \quad t \geq 0,
\]

and admits the following estimate, which is a classical property for an analytic semigroup. Its proof is straightforward and omitted here.

**Lemma 2.1.** Under assumption 1, for all \( \beta \leq \alpha \), \( \rho \in (\lambda_n, \lambda_{n+1}] \), there exists a constant \( M > 0 \), which is independent of \( u \in \mathcal{H} \), such that for any \( t > 0 \)

\[
\|e^{At} P_s u\|_\alpha \leq Mt^{\frac{\alpha}{2\rho}} e^{-\rho t} \|P_s u\|_\beta.
\]
In addition, we impose the following conditions:

**Assumption 2 (Operator \( \mathcal{L} \)).** Let \( \mathcal{L} : \mathcal{H}^\alpha \to \mathcal{H}^{\alpha-\beta} \) for some \( \alpha \in \mathbb{R}, \beta \in [0,m) \) be a linear continuous mapping that commutes with \( P_\varepsilon \) and \( P_s \).

Let us remark, that it is sufficient for our assumption that \( \mathcal{L} \) is a differential operator of lower order than the operator \( A \), which commutes with \( A \).

Moreover, this assumption is crucial for our approach. If we do not assume that \( P_\varepsilon \) and \( P_s \) commute with \( \mathcal{L} \), then we expect an additional linear coupling of \( \psi \) and \( \hat{u} \) in our formal calculation below, which changes the result completely.

**Assumption 3 (Nonlinearity \( \mathcal{F} \)).** Assume that \( \mathcal{F} : (\mathcal{H}^\alpha)^3 \to \mathcal{H}^{\alpha-\beta} \), with \( \alpha \) and \( \beta \) as in assumption 2, is a trilinear, symmetric mapping and satisfies the following conditions, for some \( C > 0 \),

\[
\| \mathcal{F}(u,v,w) \|_{\alpha-\beta} \leq C \| u \|_\alpha \| v \|_\alpha \| w \|_\alpha \quad \text{for all } u, v, w \in \mathcal{H}^\alpha.
\] (3)

Moreover, we have on the space \( \mathcal{N} \) the stronger assumptions

\[
\langle \mathcal{F}_c(u), u \rangle \leq 0 \quad \text{for all } u \in \mathcal{N},
\] (4)

\[
\langle \mathcal{F}_c(u, u, w), w \rangle \leq 0 \quad \text{for all } u, w \in \mathcal{N},
\] (5)

and for some positive constants \( C_0, C_1, \) and \( C_2 \) we have for all \( u, v, w \in \mathcal{N} \) that

\[
\langle \mathcal{F}_c(u, v, w) - \mathcal{F}_c(v, u), u \rangle \leq -C_0 \| u \|^4 + C_1 \| v \|^4 + C_2 \| w \|^2 \| v \|^2.
\] (6)

Here, to ease notation, we use \( \mathcal{F}_c := P_\varepsilon \mathcal{F} \) and we define \( \mathcal{F}_s, \mathcal{L}_c \) and \( \mathcal{L}_s \) in a similar way. Moreover, we used \( \mathcal{F}(u) := \mathcal{F}(u, u, u) \) for shorthand notation.

**Assumption 4 (Wiener process).** Let \( U \) be a separable Hilbert space with inner product \( \langle \cdot, \cdot \rangle_U \). Let \( \{ W_t \}_{t \geq 0} \) be the cylindrical Wiener process on a stochastic base \( (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}) \) with covariance operator \( I \), the identity.

Formally, \( W \) can be written (see Da Prato and Zabczyk [21]) as the infinite sum

\[
W_t = \sum_{k \in \mathbb{N}} B_k(t) f_k,
\]

where \( \{ B_k(t) \}_{k \in \mathbb{N}} \) are mutually independent real-valued Brownian motions on stochastic base \( (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}) \), and \( \{ f_k \}_{k \in \mathbb{N}} \) is any orthonormal basis on \( U \).

We proceed with some further notation. Let \( V \) be another separable Hilbert space with inner product \( \langle \cdot, \cdot \rangle_V \). Let \( \mathcal{L}_2(U, V) \) denote the Hilbert space consisting of all Hilbert–Schmidt operators from \( U \) to \( V \), where the inner product is denoted by \( \langle \cdot, \cdot \rangle_{\mathcal{L}_2(U,V)} \), and the norm by \( \| \cdot \|_{\mathcal{L}_2(U,V)} \).

**Assumption 5 (Operator \( G \)).** Assume that \( G : \mathcal{H}^\alpha \to \mathcal{L}_2(U, \mathcal{H}^\alpha) \) satisfying \( G(0) = 0 \), with \( \alpha \) as in assumption 2, is Fréchet differentiable up to order 2 and fulfills the following conditions: for one \( r > 0 \) there exists a constant \( L_r > 0 \) such that

\[
\| G(u) \|_{\mathcal{L}_2(U, \mathcal{H}^\alpha)} \leq L_r \| u \|_\alpha,
\] (7)
\[ \|G'(u) \cdot v\|_{L^2(U,H^\alpha)} \leq I_v \|v\|_\alpha \]  

(8)

and

\[ \|G''(u) \cdot (v, w)\|_{L^2(U,H^\alpha)} \leq I_v \|v\|_\alpha \|w\|_\alpha, \]  

(9)

for all \( u, v, w \in \mathcal{H}^\alpha \) with \( \|u\|_\alpha \leq r \), where we use notations \( G'(u) \) and \( G''(u) \) denote the first and second Fréchet derivatives at point \( u \), respectively.

Let us remark that the assumption on the second Fréchet-derivative is only posed for simplicity of proofs when we bound terms like \( G(u) - G'(0) \cdot u \).

To give a meaning to problem (1), we adapt the concept of local mild solution as in [22].

**Definition 2.2 (Local mild solution).** An \( \mathcal{H}^\alpha \)-valued process \( \{u(t)\}_{t \in [0,T]} \) is called a local mild solution of problem (1) if for some stopping time \( \tau_{\text{ex}} \) we have on a set of probability 1 that \( \tau_{\text{ex}} > 0 \), \( u \in C^0([0,\tau_{\text{ex}}),\mathcal{H}^\alpha) \) (i.e., a process with continuous paths) and

\[ u(t) = e^{tA}u(0) + \int_0^t e^{(t-s)A} \left[ \varepsilon^2 \mathcal{L}u(s) + \mathcal{F}(u(s)) \right] ds + \varepsilon \int_0^t e^{(t-s)A} G(u(s)) dW(s) \]  

for all \( t \in (0, \tau_{\text{ex}}) \).

The proof of the existence and the uniqueness of a local mild solution is fairly standard under our assumptions, and hence is omitted here. But the precise statement that covers exactly the case needed here does not seem to be present in the literature, although the key idea is standard. For locally Lipschitz nonlinearities like \( \mathcal{F} \) and \( \mathcal{G} \) we use a cut-off argument at an arbitrarily large radius in the space \( \mathcal{H}^\alpha \).

We can choose the cut-off in a way that the nonlinearities \( \mathcal{F} \) and \( \mathcal{G} \) are globally Lipschitz and bounded. Thus for global existence and uniqueness of solutions we can rely on Banach’s fixed-point theorem, which can be found in the standard results [21, theorem 7.2] or [23, theorem 6.5, chapter 6]. The existence time \( \tau_{\text{ex}} \) is then defined as the time when the solution of the cut-off system crosses the radius of the cut-off.

Let us state the final result as a theorem without proof:

**Theorem 2.1.** Under assumptions 1–5 there exists for any given \( u(0) \in \mathcal{H}^\alpha \) a unique local mild solution of problem (1) in the sense of definition 2.2 such that \( \tau_{\text{ex}} = \infty \) or \( \lim_{T \to \tau_{\text{ex}}} \|u(t)\|_\alpha = \infty \).

The uniqueness in the previous theorem has to be understood in the sense of choosing a version of the solution, i.e. by changing it on null-sets. As we do not rely on the uniqueness, we do not go into details here.

**3. Formal derivation and the main result**

We consider the local mild solution \( u \) on the slow time-scale \( T = \varepsilon^2 t \) and assume that it is small of order \( \mathcal{O}(\varepsilon) \). Let us split it into

\[ u(t) = \varepsilon a(\varepsilon^2 t) + \varepsilon \psi(\varepsilon^2 t), \]  

(10)

with \( a \in \mathcal{N} \) and \( \psi \in \mathcal{S} \). By projecting and rescaling (1) to the slow time scale, we obtain

\[ da(T) = [\mathcal{L}a(T) + \mathcal{F}(a(T) + \psi(T))] dT + \frac{1}{\varepsilon} G_c(\varepsilon a(T) + \varepsilon \psi(T)) dW(T) \]  

(11)
and
\[ d\psi(T) = \left[ \frac{1}{\varepsilon^2} A_\varepsilon \psi(T) + L_\varepsilon \psi(T) + F_\varepsilon(a(T) + \psi(T)) \right] dT + \frac{1}{\varepsilon} G_\varepsilon(\varepsilon a(T) + \varepsilon \psi(T)) d\tilde{W}(T), \]
(12)

where \( \tilde{W}(T) := \varepsilon W(\varepsilon^{-2} T) \) is a rescaled version of the Wiener process. These equations can be written in the integral form using the mild formulation:
\[ a(T) = a(0) + \int_0^T L_\varepsilon a(\tau) d\tau + \int_0^T F_\varepsilon(a(\tau) + \psi(\tau)) d\tau + \frac{1}{\varepsilon} \int_0^T G_\varepsilon(\varepsilon a(\tau) + \varepsilon \psi(\tau)) d\tilde{W}_\tau, \]
(13)
and
\[ \psi(T) = e^{A_\varepsilon T} \psi(0) + \int_0^T e^{A_\varepsilon (T-\tau) \varepsilon^{-2}} L_\varepsilon \psi(\tau) d\tau + \int_0^T e^{A_\varepsilon (T-\tau) \varepsilon^{-2}} F_\varepsilon(a(\tau) + \psi(\tau)) d\tau + \frac{1}{\varepsilon} \int_0^T e^{A_\varepsilon (T-\tau) \varepsilon^{-2}} G_\varepsilon(\varepsilon a(\tau) + \varepsilon \psi(\tau)) d\tilde{W}_\tau. \]
(14)

We shall see later that \( \psi \) is small as long as \( a \) is of order one (see lemmas 4.1 and 4.2 for the precise statement). Only the term with the initial condition is not \( O(\varepsilon) \), but that one is only of order one for very small times. Thus by neglecting all \( \psi \)-dependent terms in (11) or (13) and expanding the diffusion we obtain the amplitude equation
\[ db(\tau) = L_\varepsilon b(\tau) d\tau + F_\varepsilon(b(\tau)) d\tau + [G_\varepsilon(0) \cdot b(\tau)] d\tilde{W}_\tau, \quad b(0) = a(0). \]
(15)
This is equivalent to the integral equation
\[ b(T) = a(0) + \int_0^T L_\varepsilon b(\tau) d\tau + \int_0^T F_\varepsilon(b(\tau)) d\tau + \int_0^T [G_\varepsilon(0) \cdot b(\tau)] d\tilde{W}_\tau. \]
(16)

With our main assumptions we have the following main result on the approximation by amplitude equation, which is proved later at the end of section 4.

**Theorem 3.1.** Let the assumptions 1–5 be satisfied and let \( u \) be the local mild solution of (1) with initial condition
\[ u(0) = \varepsilon a(0) + \varepsilon \psi(0), \]
where \( a(0) \in \mathcal{N}, \psi(0) \in \mathcal{S} \) and \( b \) is the solution of the amplitude equation (15) with \( b(0) = a(0) \). Then for any \( p > 1, T_0 > 0 \) and all small \( \kappa \in \left( 0, \frac{1}{10} \right) \), there exists a constant \( C > 0 \) such that for \( \| u(0) \|_\alpha \leq \varepsilon^{1-\kappa/3} \) we have
\[ \mathbb{P} \left( \sup_{\kappa \in [0, \varepsilon^{-2} T_0]} \| u(t) - \varepsilon b(\varepsilon^2 t) - \varepsilon Q(\varepsilon^2 t) \|_\alpha > \varepsilon^{2-19\kappa} \right) \leq C \varepsilon^p, \]
where
\[ Q(T) = e^{A_\varepsilon T \varepsilon^{-2}} \psi(0). \]

Let us first remark that we also proved in the previous result that for the time \( \tau_{ex} \) of existence of our local mild solution we have \( \tau_{ex} > T_0 \varepsilon^{-2} \) with high probability.
Moreover, let us remark that the additional term $e^{A_T\epsilon^{-2}} \psi(0)$ in the approximation is exponentially small in $\epsilon$ after any short time of order $\epsilon$ by the stability of the semigroup on $S$. This is an attractivity result for the space $N$. Let us state this result by ignoring the small $\kappa > 0$ in the exponent. If we start with solutions of order $O(\epsilon)$, then after a very short time the part of the solution orthogonal to $N$ is of order $O(\epsilon^2)$.

Note finally that we did not optimize the factor in front of the $\kappa$. Both the $19\kappa$ in the final error estimate and the $-\kappa/3$ in the bound on the initial condition are not optimal. We use $\kappa$ mainly for technical reasons and think of it as being very small.

Let us finally give some remarks on straightforward extensions of the result presented here.

**Remark 3.1 (Other nonlinear terms).** We could add terms of higher power to the nonlinearity in the SPDE like quartic or quintic, for example. Formally, they lead to terms that are of higher order in $\epsilon$ when compared to the cubic term and we do not expect to change the result. They are fully absorbed by the error terms.

In contrast to that quadratic nonlinear terms $B(u, u)$ are quite different. Formally, we obtain in the amplitude equation the additional terms $\frac{1}{\epsilon} B_c(a, a)$ and $\frac{2}{\epsilon} B_c(a, \psi)$. So either we need to change the scaling of the equation, consider smaller noise, and obtain an amplitude equation with quadratic nonlinearity, or alternatively (as $B_c(a, a) = 0$ in many applications) we have to identify the mixed term $B(a, \psi)$. Even if $\psi$ is small of order $O(\epsilon)$, then $2\epsilon B(a, \psi)$ is of order $O(1)$ and we need to identify how $\psi$ depends on $a$.

See [22] for a discussion in the case of additive noise.

**Remark 3.2 (Additive noise or quadratic diffusion).** The assumption that $G(0) = 0$ is crucial for our result, as for additive noise one sees already in the formal calculation above, that we need a different scaling. We expect to need $\epsilon^2 G(u) dW$ in (1) which leads to an additive noise term $G_c(0)dW$ in the amplitude equation. The proofs and the final theorem should nevertheless be very similar.

If we assume that not only $G(0) = 0$ but also $G'(0) = 0$, then we expect to have $G(a)dW$ in (1) which leads to $[G''(0) \cdot (b, b)]dW$ in the amplitude equation. Again the proofs should be similar, but for the error estimate we might need additional assumptions on the third derivative of $G$.

### 4. Estimates and proof

Before proving the main results, we need to state some technical lemmas used later in the proof. Also, we need to introduce a stopping time in connection with process $(a, \psi)$. This stopping time is equivalent to a cut-off in (1) at radius $\epsilon^{1-\kappa}$. Also this stopping time is the reason, why we only need local solutions for the SPDE.

**Definition 4.1.** For the $N \times S$-valued stochastic process $(a, \psi)$ satisfying system (13) and (14) we define, for some time $T_0 > 0$ and small exponent $\kappa \in \left(0, \frac{1}{12}\right)$, the stopping time $\tau^*$ as

$$
\tau^* := T_0 \wedge \inf \left\{ T > 0 : \|a(T)\|_\alpha > \epsilon^{-\kappa} \text{ or } \|\psi(T)\|_\alpha > \epsilon^{-\kappa} \right\}.
$$

Next, we will denote by $Q(T)$, $I(T)$, $J(T)$ and $K(T)$ the corresponding four terms arising in the right-hand side of (14), respectively, that is

$$
\psi(T) = Q(T) + I(T) + J(T) + K(T). \tag{17}
$$
Lemma 4.1. Let the assumptions 1–5 be satisfied. For any \( p > 0 \) and \( \tau^* \) from the definition 4.1, there exists a constant \( C > 0 \) such that
\[
\mathbb{E} \sup_{0 \leq T \leq \tau^*} \|I(T)\|_p^p \leq C e^{2p-\kappa p}
\] (18)

and
\[
\mathbb{E} \sup_{0 \leq T \leq \tau^*} \|J(T)\|_p^p \leq C e^{2p-3\kappa p}.
\] (19)

Proof. By (2) and definition 4.1, we first have for \( I \),
\[
\mathbb{E} \sup_{0 \leq T \leq \tau^*} \|I(T)\|_p^p \leq \mathbb{E} \sup_{0 \leq T \leq \tau^*} \left[ \int_0^T \|e^{A(T-\tau)^{-2}} \mathcal{L}_s \psi(\tau)\|_\alpha d\tau \right]^p
\]
\[
\leq C e^{2p} \mathbb{E} \sup_{0 \leq T \leq \tau^*} \left[ \int_0^T e^{-\varepsilon^2 \mu(T-\tau)^{-2}} \|\mathcal{L}_s \psi(\tau)\|_{\alpha-\beta} d\tau \right]^p
\]
\[
\leq C e^{2p} \mathbb{E} \sup_{0 \leq T \leq \tau^*} \left[ \int_0^T e^{-\varepsilon^2 \mu(T-\tau)^{-2}} \|\psi(\tau)\|_\alpha d\tau \right]^p
\]
\[
\leq C e^{2p} \mathbb{E} \sup_{0 \leq T \leq \tau^*} \left[ \int_0^T e^{-\varepsilon^2 \mu(T-\tau)^{-2}} \|\mathcal{L}_s \psi(\tau)\|_{\alpha-\beta} d\tau \right]^p
\]
\[
\leq C e^{2p} \mathbb{E} \sup_{0 \leq T \leq \tau^*} \left[ \int_0^T e^{-\varepsilon^2 \mu(T-\tau)^{-2}} \|\psi(\tau)\|_\alpha d\tau \right]^p
\]
so that (18) follows. In view of assumption 3, definition 4.1 and (2) we obtain for \( J \),
\[
\mathbb{E} \sup_{0 \leq T \leq \tau^*} \|J(T)\|_p^p \leq \mathbb{E} \sup_{0 \leq T \leq \tau^*} \left[ \int_0^T \|e^{A(T-\tau)^{-2}} \mathcal{F}_s (a(\tau) + \psi(\tau))\|_\alpha d\tau \right]^p
\]
\[
\leq C e^{2p} \mathbb{E} \sup_{0 \leq T \leq \tau^*} \left[ \int_0^T e^{-\varepsilon^2 \mu(T-\tau)^{-2}} \|\mathcal{F}_s (a(\tau) + \psi(\tau))\|_{\alpha-\beta} d\tau \right]^p
\]
\[
\leq C e^{2p} \mathbb{E} \sup_{0 \leq T \leq \tau^*} \left[ \int_0^T e^{-\varepsilon^2 \mu(T-\tau)^{-2}} \|a(\tau) + \psi(\tau)\|_\alpha \|\psi(\tau)\|_\alpha d\tau \right]^p
\]
\[
\leq C e^{2p} \mathbb{E} \sup_{0 \leq T \leq \tau^*} \left[ \int_0^T e^{-\varepsilon^2 \mu(T-\tau)^{-2}} \|a(\tau) + \psi(\tau)\|_\alpha \|\psi(\tau)\|_\alpha d\tau \right]^p
\]
\[
\leq C e^{2p} \mathbb{E} \sup_{0 \leq T \leq \tau^*} \left[ \int_0^T e^{-\varepsilon^2 \mu(T-\tau)^{-2}} \|a(\tau) + \psi(\tau)\|_\alpha \|\psi(\tau)\|_\alpha d\tau \right]^p
\]
The proof of lemma 4.1 is thus completed. \( \square \)

While for \( I \) and \( J \) we immediately had uniform bounds in time, for \( K \) we first establish bounds in \( L^p([0, \tau^*], \mathcal{H}^\alpha) \).

Lemma 4.2. Assume the setting of lemma 4.1. Then it holds for every \( p > 0 \) that
\[
\mathbb{E} \sup_{0 \leq T \leq \tau^*} \int_0^T \|K(\tau)\|_\alpha^p d\tau \leq C e^{p-\kappa p}.
\] (20)
Proof. Throughout this proof let $\lambda_0$ be a positive constant less than $\lambda_{n+1}$ but close to it. For any $p > 0$, it holds

\[
\mathbb{E} \sup_{0 \leq T < \tau} T \int_0^T \|K(\tau)\|_p^p d\tau = \mathbb{E} \sup_{0 \leq T < \tau} T \int_0^T \frac{1}{\varepsilon} \int_0^T e^{A(\tau - \rho)^2} G_0(\varepsilon a(\tau) + \varepsilon \psi(\tau)) d\bar{W}_\tau^p d\tau.
\]

By an application of the maximal inequality for stochastic convolutions [24] based on the Riesz–Nagy theorem (as $A_n + \lambda_0$ generates a contraction semigroup on $S$), the condition (7) for $G$, and the definition of $\tau^*$, we obtain

\[
\mathbb{E} \sup_{0 \leq T < \tau^*} T \int_0^T \frac{1}{\varepsilon} \int_0^T e^{A(\tau - \rho)^2} G_n(\varepsilon a(\tau) + \varepsilon \psi(\tau)) d\bar{W}_\tau^p d\tau
\]

\[
\leq \mathbb{E} \int_0^T \frac{1}{\varepsilon} \int_0^T e^{A(\tau - \rho)^2} G_n(\varepsilon a(\tau) + \varepsilon \psi(\tau)) d\bar{W}_\tau^p d\tau
\]

\[
= \mathbb{E} \int_0^T \frac{1}{\varepsilon} \int_0^T e^{A(\tau - \rho)^2} G_n(\varepsilon a(\tau) + \varepsilon \psi(\tau)) d\bar{W}_\tau^p d\tau
\]

\[
= \frac{1}{\varepsilon} \int_0^T e^{-\lambda_0^2 \varepsilon^2} \mathbb{E} \int_0^T \int_0^T \int_0^T \int_0^T e^{A(\tau - \rho)^2} G_n(\varepsilon a(\tau) + \varepsilon \psi(\tau)) d\bar{W}_\tau^p d\tau.
\]

where the constant may change from line to line, but it mainly depends on $p$, $T_0$, the bound on $G$, and $\lambda_0$. \hfill \Box

So we have seen in the previous lemmas that $\psi$ equals the process $Q$ plus a small term of order $\mathcal{O}(\varepsilon)$. Next, let us rewrite the equation (11) for $a$ as the amplitude equation plus an error term (or residual):

\[
a(T) = a(0) + \int_0^T [L_c a(\tau) + \mathcal{F}_c(a(\tau))] d\tau + \int_0^T G_c'(0) \cdot a(\tau) d\bar{W}_\tau + R(T),
\]

where the error term is given by

\[
R(T) = \int_0^T [3 \mathcal{F}_c(a(\tau), \psi(\tau), \psi(\tau)) + 3 \mathcal{F}_c(a(\tau), a(\tau), \psi(\tau)) + \mathcal{F}_c(\psi(\tau))] d\tau
\]

\[
+ \int_0^T \left[ \frac{1}{\varepsilon} G(\varepsilon a(\tau) + \varepsilon \psi(\tau)) - G_c'(0) \cdot a(\tau) \right] d\bar{W}_\tau.
\]
Let us now start to show that $R$ is small. We will show in lemma 4.7 that all terms in the previous equation are almost of order $O(\varepsilon)$. Let us start with:

**Lemma 4.3.** Assume the setting of lemma 4.1. For any $p > 0$, there exists a constant $C_p > 0$ such that

$$\mathbb{E} \sup_{0 \leq T \leq t^*} \left\| \int_0^T \mathcal{F}_c(a(\tau), \psi(\tau), \psi(\tau))d\tau \right\|_p^p \leq C_p \left( \varepsilon^{2p-7p} + \|\psi(0)\|^{2p-2p-\varepsilon p} \right).$$

**Proof.** It is direct to see that by brute force expansion of the cubic using $\psi = Q + I + K + J$ from (17) that

$$\int_0^T \mathcal{F}_c(a(\tau), \psi(\tau), \psi(\tau))d\tau = \int_0^T \mathcal{F}_c(a(\tau), Q(\tau), Q(\tau))d\tau + \int_0^T \mathcal{F}_c(a(\tau), I(\tau), I(\tau))d\tau$$

$$+ \int_0^T \mathcal{F}_c(a(\tau), J(\tau), J(\tau))d\tau + \int_0^T \mathcal{F}_c(a(\tau), K(\tau), K(\tau))d\tau$$

$$+ 2 \int_0^T \mathcal{F}_c(a(\tau), Q(\tau), I(\tau))d\tau + 2 \int_0^T \mathcal{F}_c(a(\tau), Q(\tau), J(\tau))d\tau$$

$$+ 2 \int_0^T \mathcal{F}_c(a(\tau), K(\tau), I(\tau))d\tau + 2 \int_0^T \mathcal{F}_c(a(\tau), K(\tau), J(\tau))d\tau$$

$$+ 2 \int_0^T \mathcal{F}_c(a(\tau), I(\tau), J(\tau))d\tau + 2 \int_0^T \mathcal{F}_c(a(\tau), J(\tau), K(\tau))d\tau$$

$$+ 2 \int_0^T \mathcal{F}_c(a(\tau), K(\tau), I(\tau))d\tau + 2 \int_0^T \mathcal{F}_c(a(\tau), J(\tau), K(\tau))d\tau$$

$$:= \sum_{k=1}^{10} R_{1,k}^4(T). \quad (22)$$

We will estimate each term separately, which will all be very similar, as $I, J, K$ are small (i.e., almost of order $O(\varepsilon)$). Only for $Q$ we need an additional averaging argument. First since all $\mathcal{H}^\kappa$- norms are equivalent on $\mathcal{N}$, we get

$$\mathbb{E} \sup_{0 \leq T \leq t^*} \left\| R_{1,1}(T) \right\|_p^p \leq C \mathbb{E} \sup_{0 \leq T \leq t^*} \left\| R_{1,4}(T) \right\|_{{\alpha-\beta}}^p$$

$$\leq C \mathbb{E} \sup_{0 \leq T \leq t^*} \left[ \int_0^T \left\| \mathcal{F}_c(a(\tau), Q(\tau), Q(\tau)) \right\|_{{\alpha-\beta}}d\tau \right]^p$$

$$\leq C \mathbb{E} \sup_{0 \leq T \leq t^*} \left[ \int_0^T \left\| a(\tau) \right\|_\alpha \left\| Q(\tau) \right\|_2^2d\tau \right]^p$$

$$\leq C \varepsilon^{-2p} \left[ \int_0^T \left\| e^{A_{r,z}-\varepsilon^2} \psi(0) \right\|_2^2d\tau \right]^p \leq C \varepsilon^{2p-2p-\varepsilon p} \left\| \psi(0) \right\|_2^{2p}.$$
Due to definition 4.1 and (18), we get
\[
\mathbb{E} \sup_{0 \leq T < T^*} \| R^{1,3}(T) \|_{\alpha}^p \leq C e^{-\kappa \mathcal{P}} \mathbb{E} \sup_{0 \leq T < T^*} \int_0^T \| I(\tau) \|_{\alpha}^2 d\tau \leq C e^{4p - 3\kappa p}.
\]

By proceeding with analogous arguments, we can show the following results for all other terms:
\[
\begin{align*}
\mathbb{E} \sup_{0 \leq T < T^*} \| R^{1,4}(T) \|_{\alpha}^p & \leq C p e^{2p - 3\kappa p}, \\
\mathbb{E} \sup_{0 \leq T < T^*} \| R^{1,5}(T) \|_{\alpha}^p & \leq C p e^{-\kappa \mathcal{P}} (\varepsilon^2 p \| \psi \|_{\alpha}^2 + \varepsilon^4 p - 2\kappa p), \\
\mathbb{E} \sup_{0 \leq T < T^*} \| R^{1,6}(T) \|_{\alpha}^p & \leq C p e^{-\kappa \mathcal{P}} (\varepsilon^2 p \| \psi \|_{\alpha}^2 + \varepsilon^4 p - 6\kappa p), \\
\mathbb{E} \sup_{0 \leq T < T^*} \| R^{1,7}(T) \|_{\alpha}^p & \leq C p e^{-\kappa \mathcal{P}} (\varepsilon^2 p \| \psi \|_{\alpha}^2 + \varepsilon^2 p - 2\kappa p), \\
\mathbb{E} \sup_{0 \leq T < T^*} \| R^{1,8}(T) \|_{\alpha}^p & \leq C p e^{-\kappa \mathcal{P}} (\varepsilon^4 p - 2\kappa p + \varepsilon^4 p - 6\kappa p), \\
\mathbb{E} \sup_{0 \leq T < T^*} \| R^{1,9}(T) \|_{\alpha}^p & \leq C p e^{-\kappa \mathcal{P}} (\varepsilon^4 p - 2\kappa p + \varepsilon^2 p - 2\kappa p),
\end{align*}
\]

and
\[
\mathbb{E} \sup_{0 \leq T < T^*} \| R^{1,10}(T) \|_{\alpha}^p \leq C p e^{-\kappa \mathcal{P}} (\varepsilon^4 p - 6\kappa p + \varepsilon^2 p - 2\kappa p).
\]

Collecting all estimates for terms appearing in (1) we finish the proof. \qed

By the same arguments which we used to derive lemma 4.3, we are able to achieve following results:

**Lemma 4.4.** Assume the setting of lemma 4.1. For any \( p > 0 \), there exists a constant \( C > 0 \) such that
\[
\mathbb{E} \sup_{0 \leq T < T^*} \| \int_0^T \mathcal{F}_c(a(\tau), a(\tau), \psi(\tau)) d\tau \|_{\alpha}^p \leq C_p \left( \varepsilon^{p - 5\kappa p} + \| \psi(0) \|_{\alpha}^2 \varepsilon^2 p - 2\kappa p \right).
\]

**Lemma 4.5.** Assume the setting of lemma 4.1. For any \( p > 0 \), there exists a constant \( C > 0 \) such that
\[
\mathbb{E} \sup_{0 \leq T < T^*} \| \int_0^T \mathcal{F}_c(\psi(\tau)) d\tau \|_{\alpha}^p \leq C_p \left( \varepsilon^{3p - 9\kappa p} + \| \psi(0) \|_{\alpha}^3 \varepsilon^2 p \right). \tag{23}
\]

**Proof.** As we noticed before, all norms in finite dimensional space \( \mathcal{N} \) are equivalent. Thanks to (3), we get
\[
\| \mathcal{F}_c(\psi(\tau)) \|_{\alpha} \leq C \left( \| Q(\tau) \| + \| I(\tau) \| + \| J(\tau) \| + \| K(\tau) \| \right)^3 \leq C \left( \| Q(\tau) \|^3 + \| I(\tau) \|^3 + \| J(\tau) \|^3 + \| K(\tau) \|^3 \right).
\]

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Thus, according to the Hölder inequality, this implies

\[
\mathbb{E} \sup_{0 \leq T \leq T^*} \left\| \int_0^T F_z(\psi(\tau))d\tau \right\|_p^p \leq C_p \mathbb{E} \sup_{0 \leq T \leq T^*} \left[ \int_0^T \left\| Q(\tau) \right\|_\alpha^3 d\tau \right]^p
\]

\[
+ C_p \mathbb{E} \sup_{0 \leq T \leq T^*} \left\{ \int_0^T \left[ \left\| I(\tau) \right\|_\alpha^{3p} + \left\| J(\tau) \right\|_\alpha^{3p} + \left\| K(\tau) \right\|_\alpha^{3p} \right] d\tau \right\}
\]

It is easy to check that the first term appearing in the right side of above inequality is bounded by \( C_p \varepsilon^{2p} \| \psi(0) \|_\alpha^{3p} \) for a constant \( C_p > 0 \). Due to lemmas 4.1 and 4.2 we can conclude that the second term is bounded by \( C_p \varepsilon^{3p-5} \). Therefore, we finish the proof and obtain (23).

\[\square\]

**Lemma 4.6.** Assume the setting of lemma 4.1. For any \( p > 0 \), there exists a constant \( C_p > 0 \) such that

\[
\mathbb{E} \sup_{0 \leq T \leq T^*} \left\| \int_0^T \left[ \frac{1}{\varepsilon} G_\varepsilon(\varepsilon a(\tau) + \varepsilon \psi(\tau)) - G_\varepsilon(0) \cdot a(\tau) \right] dW_\tau \right\|_\alpha^p \leq C_p \varepsilon^{p-3}.
\]

**Proof.** Using Burkholder–Davis–Gundy inequality, we have

\[
\mathbb{E} \sup_{0 \leq T \leq T^*} \left\| \int_0^T \left[ \frac{1}{\varepsilon} G_\varepsilon(\varepsilon a(\tau) + \varepsilon \psi(\tau)) - G_\varepsilon(0) \cdot a(\tau) \right] dW_\tau \right\|_\alpha^p
\]

\[
\leq C \mathbb{E} \left[ \int_0^{T^*} \left\| \frac{1}{\varepsilon} G_\varepsilon(\varepsilon a(\tau) + \varepsilon \psi(\tau)) - G_\varepsilon(0) \cdot a(\tau) \right\|_{L^2(U; H^{\alpha})}^2 d\tau \right]^{\frac{p}{2}}
\]

\[
\leq C \mathbb{E} \left[ \int_0^{T^*} \left\| \frac{1}{\varepsilon} G_\varepsilon(\varepsilon a(\tau) + \varepsilon \psi(\tau)) - G_\varepsilon(0) \cdot a(\tau) \right\|_{L^2(U; H^{\alpha})}^2 d\tau \right]^{\frac{p}{2}}. \tag{24}
\]

By using the Taylor formula, we obtain

\[
\frac{1}{\varepsilon} G_\varepsilon(\varepsilon a(\tau) + \varepsilon \psi(\tau)) - G_\varepsilon(0) \cdot a(\tau)
\]

\[
= \frac{1}{\varepsilon} \left[ \frac{1}{2} G(0) + G(0) \cdot (\varepsilon a(\tau) + \varepsilon \psi(\tau)) + \frac{1}{2} G''(\varepsilon \zeta(\tau)) \cdot (\varepsilon a(\tau) + \varepsilon \psi(\tau), \varepsilon a(\tau) + \varepsilon \psi(\tau)) - G'(0) \cdot a(\tau) \right]
\]

where \( \zeta(\tau) \) is a vector on the line segment connecting 0 and \( \varepsilon a(\tau) + \varepsilon \psi(\tau) \). Now, as a consequence of the condition (9), we have

\[
\left\| \frac{1}{\varepsilon} G(\varepsilon a(\tau) + \varepsilon \psi(\tau)) - G'(0) \cdot a(\tau) \right\|_{L^2(U; H^{\alpha})}^2 \leq C \left\| \varepsilon \psi \right\|_\alpha^2 + C \varepsilon^2 \left\| a(\tau) \right\|_\alpha^4 + C \varepsilon^2 \left\| \psi(\tau) \right\|_\alpha^4.
\]

Therefore, if we plug the estimate above into (24), we get

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we show the following uniform bound on its solution.

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\[ E \sup_{0 \leq T \leq \tau^*} \| \int_0^T \left[ \frac{1}{\varepsilon} G_c(\varepsilon a(\tau) + \varepsilon \psi(\tau)) - G'_c(0) \cdot a(\tau) \right] dW_\tau \|_\alpha^p \]

| 1 \varepsilon |
| C \varepsilon |
| \int_0^T 1_{[0, \tau^*)}(\tau)\|\psi(\tau)\|^2_\alpha^2 + \varepsilon^2\|a(\tau)\|^4_\alpha + \varepsilon^2\|\psi(\tau)\|^4_\alpha d\tau |

| C_p e^{-2p\varepsilon} + C_p E \left[ \int_0^T 1_{[0, \tau^*)}(\tau)\|\psi(\tau)\|^2_\alpha d\tau \right] \frac{\varepsilon^2}{2} |

where the last estimate follows from the definition of \( \tau^* \). From the expression for \( \psi(\tau) \) and Hölder’s inequality, we get

\[ E \left[ \int_0^T 1_{[0, \tau^*)}(\tau)\|\psi(\tau)\|^2_\alpha d\tau \right] \frac{\varepsilon^2}{2} \leq C_p E \left[ \int_0^T 1_{[0, \tau^*)}(\tau) \left( \|Q(\tau)\|^2_\alpha^2 + \|I(\tau)\|^2_\alpha^2 + \|J(\tau)\|^2_\alpha^2 + \|K(\tau)\|^2_\alpha d\tau \right) \frac{\varepsilon^2}{2} \right] \leq C_p E \left[ e^p \|\psi(0)\|^p_\alpha + \sup_{0 \leq \tau \leq \tau^*} \|I(\tau)\|^p_\alpha + \sup_{0 \leq \tau \leq \tau^*} \|J(\tau)\|^p_\alpha + \int_0^{\tau^*} \|K(\tau)\|^p_\alpha d\tau \right]. \]

Recalling lemma 4.1 and 4.2, we thus have

\[ E \sup_{0 \leq T \leq \tau^*} \left\| \int_0^T \left[ \frac{1}{\varepsilon} G_c(\varepsilon a(\tau) + \varepsilon \psi(\tau)) - G'_c(0) \cdot a(\tau) \right] dW_\tau \|_\alpha^p \leq C_p e^{-3p\varepsilon}. \]

Due to lemmas 4.3–4.6, we readily obtain the following estimate for the remainder \( R \) defined in (21).

**Lemma 4.7.** Assume the setting of lemma 4.1 and suppose furthermore that \( \|\psi(0)\|^\alpha_\alpha \leq \varepsilon^{-\frac{1}{2}k}. \) Then for any \( p > 0 \), there exists a constant \( C_p > 0 \) such that

\[ E \sup_{0 \leq T \leq \tau^*} \| R(T) \|^p_\alpha \leq C_p e^{-p\varepsilon}. \tag{25} \]

In what follows, we shall consider the amplitude equation (15) associated with (11) and we show the following uniform bound on its solution \( b \). This is crucial in order to remove the stopping time from the error estimate. Moreover note, that our assumptions do not imply global solutions for the SPDE, we rely on the existence of global solutions for the amplitude equation, which is also ensured by the following lemma.

Although the proof is standard and similar bounds of this type are well known, we state a proof in order to quantify the dependence on \( a(0) \) on the bound.

**Lemma 4.8.** Let the assumptions 1–5 be satisfied. For any \( p > 1 \), there exists a constant \( C_p > 0 \) such that

\[ E \sup_{0 \leq T \leq T_0} \| b(T) \|^p_\alpha \leq C_p \| a(0) \|^p_\alpha. \tag{26} \]

**Proof.** This proof is straightforward using Itô-formula for powers of the norm. For large \( p > 2 \) define the twice continuously differentiable function

\[ f(\cdot) = \| \cdot \|^p : \mathcal{H} \to \mathbb{R}. \tag{27} \]
Directly, for any \( x, h \in \mathcal{H} \) we have
\[
f'(x)h = p\|x\|^{p-2}\langle x, h \rangle
\]
and
\[
f''(x)(h, h) = p(p-2)\|x\|^{p-4}\langle x, h \rangle\langle x, h \rangle + p\|x\|^{p-2}\langle h, h \rangle \leq p(p-1)\|x\|^{p-2}\|h\|^2,
\]
so that
\[
\text{trace} \left[ f''(b(\tau))G'_{x}(0)b(\tau) \right] (G'_{x}(0)b(\tau))^T \]
\[
\leq p(p-1)\|b(\tau)\|^{p-2}\text{trace}\left[(G'_{x}(0)b(\tau))(G'_{x}(0)b(\tau))^T\right] \leq Cp(p-1)\|b(\tau)\|^p.
\]
(29)

Applying Ito’s formula [25, theorem 2.9] and (29) we obtain that
\[
\|b(T)\|^p \leq \|a(0)\|^p + p \int_0^T \|b(\tau)\|^{p-2}\langle \mathcal{L}_x b(\tau), b(\tau) \rangle d\tau + p \int_0^T \|b(\tau)\|^{p-2}\langle F_x b(\tau), b(\tau) \rangle d\tau
\]
\[
+ p \int_0^T \|b(\tau)\|^{p-2}\langle b(\tau), G'_{x}(0) \cdot b(\tau) d\tilde{W}_+ \rangle + \frac{1}{2}Cp(p-1) \int_0^T \|b(\tau)\|^p d\tau.
\]
Therefore, using assumption 2 and the bound on \( F \) from (4), we get
\[
\|b(T)\|^p \leq \|a(0)\|^p + p \int_0^T \|b(\tau)\|^{p-2}\langle b(\tau), G'_{x}(0) \cdot b(\tau) d\tilde{W}_+ \rangle + C_p \int_0^T \|b(\tau)\|^p d\tau. \tag{30}
\]
For any stopping time \( \mathcal{T} \leq T_0 \), by Burkholder–Davis–Gundy inequality, we obtain
\[
p\mathbb{E} \sup_{0 \leq \tau \leq \mathcal{T}} \int_0^\tau \|b(\tau)\|^{p-2}\langle b(\tau), G'_{x}(0) \cdot b(\tau) d\tilde{W}_+ \rangle \]
\[
\leq 3p\mathbb{E} \left[ \int_0^\tau \sum_{k=1}^\infty \|b(\tau)\|^{2p-4}\langle b(\tau), G'_{x}(0) \cdot b(\tau) e_k \rangle^2 d\tau \right] \]
\[
\leq C\mathbb{E} \left[ \sup_{0 \leq \tau \leq \mathcal{T}} \|b(\tau)\|^p \int_0^\tau \|b(\tau)\|^p d\tau \right] \leq \frac{1}{2} \mathbb{E} \sup_{0 \leq \tau \leq \mathcal{T}} \|b(\tau)\|^p + C_p \mathbb{E} \int_0^\mathcal{T} \sup_{0 \leq \tau \leq \mathcal{T}} \|b(s)\|^p d\tau,
\]
where we applied Young’s inequality in the final step. Therefore, using (30), we obtain
\[
\mathbb{E} \sup_{0 \leq \tau \leq \mathcal{T}} \|b(\tau)\|^p \leq 2\|a(0)\|^p + C_p \mathbb{E} \int_0^\mathcal{T} \sup_{0 \leq \tau \leq \mathcal{T}} \|b(s)\|^p d\tau.
\]
Note that we need to use a stopping time \( \mathcal{T} \) here, as initially, we do not know that the moments of \( b \) are finite. Thus we consider only the \( \mathcal{T} \) which ensures this.
Replacing now the stopping time \( \mathcal{T} \) by the minimum \( \mathcal{T} \wedge T \) for arbitrary \( T > 0 \), we obtain
\[
\mathbb{E} \sup_{0 \leq \tau \leq \mathcal{T} \wedge T} \|b(\tau)\|^p \leq 2\|a(0)\|^p + C_p \int_0^T \mathbb{E} \sup_{0 \leq \tau \leq \mathcal{T} \wedge T} \|b(s)\|^p d\tau.
\]
We thus derive by using Gronwall’s lemma
\[
\mathbb{E} \sup_{0 \leq t \leq T \wedge \tau} \| b(t) \|^p \leq 2 \| a(0) \|^p e^{C_p T}.
\]
As the equation above holds for arbitrary stopping time such that the moments are finite,
\[
\mathbb{E} \sup_{0 \leq t \leq T_0} \| b(\tau) \|^p \leq C_p \| a(0) \|^p.
\]
This finishes the proof. \(\square\)

The next step now is to remove the error from the equation for \( a \) to obtain the amplitude equation. We show an error estimate between \( a \) and the solution \( b \) of the amplitude equation.

**Lemma 4.9.** Assume the setting of lemma 4.1 and suppose furthermore that \( \| \psi(0) \|_{a_\alpha} \leq \varepsilon^{-\frac{1}{4}} \). For any \( p > 0 \), there exists a constant \( C_p > 1 \) such that
\[
\mathbb{E} \sup_{0 \leq t \leq T \wedge \tau} \| a(T) - b(T) \|^p \leq C_p \varepsilon^{p-18\kappa p}.
\]

**Proof.** For the proof we derive an equation for the error \( a - b \) and proceed similarly that for the boundary on \( b \). But as \( R \) (defined in (21)) is not differentiable in the Itô-sense, we first substitute \( \varphi := a - R \). Clearly, we have
\[
\varphi(T) = a(0) + \int_0^T \mathcal{L}_\epsilon(\varphi(\tau) + R(\tau))d\tau + \int_0^T \mathcal{F}_\epsilon(\varphi(\tau) + R(\tau))d\tau + \int_0^T G_\epsilon(0) \cdot (\varphi(\tau) + R(\tau))d\tilde{W}_{\tau}.
\]
Defining the error \( h := b - \varphi = b - a + R \), we get
\[
h(T) = \int_0^T \mathcal{L}_\epsilon h(\tau)d\tau - \int_0^T \mathcal{L}_\epsilon R(\tau)d\tau + \int_0^T \mathcal{F}_\epsilon(b(\tau))d\tau
\]
\[
- \int_0^T \mathcal{F}_\epsilon(b(\tau) - h(\tau) + R(\tau))d\tau + \int_0^T G_\epsilon(0)(h(\tau) - R(\tau))d\tilde{W}_{\tau}.
\]
Let \( f \) be the \( p \)th power of the norm as in (27). By using again (29), we have
\[
\text{trace}[f''(h(\tau))(G_\epsilon(0)(b(\tau) - R(\tau))(G_\epsilon(0)(b(\tau) - R(\tau)))^*)]
\leq C_p (p - 1) \| b(\tau) \|^{p-2} \| h(\tau) - R(\tau) \|^2.
\]
Applying Itô’s formula and using the estimate above, we obtain
\[
\| h(T) \| \leq p \int_0^T \| h(\tau) \|^{p-2} \langle \mathcal{L}_\epsilon h(\tau), \mathcal{L}_\epsilon h(\tau) \rangle d\tau - p \int_0^T \| h(\tau) \|^{p-2} \langle \mathcal{L}_\epsilon R(\tau), \mathcal{L}_\epsilon h(\tau) \rangle d\tau
\]
\[
+ p \int_0^T \| h(\tau) \|^{p-2} \langle \mathcal{F}_\epsilon(b(\tau) - h(\tau) + R(\tau)), h(\tau) \rangle d\tau
\]
\[
+ p \int_0^T \| h(\tau) \|^{p-2} \langle h(\tau), G_\epsilon(0)(h(\tau) - R(\tau))d\tilde{W}_{\tau} \rangle
\]
\[
+ \frac{1}{2} C_p (p - 1) \int_0^T \| h(\tau) \|^{p-2} \| h(\tau) - R(\tau) \|^2 d\tau.
\]
By condition (6) and Cauchy–Schwarz inequality, we derive
\[
\|h(T)\|^p \leq C_p \int_0^T \|h(\tau)\|^p \, d\tau + C_p \int_0^T \|h(\tau)\|^{p-1} \|R(\tau)\| \, d\tau \\
+ C_p \int_0^T \|h(\tau)\|^{p-2} \|R(\tau)\|^2 \, d\tau + C_p \int_0^T \|h(\tau)\|^{p-2} \|R(\tau)\|^4 \, d\tau \\
+ C_p \int_0^T \|h(\tau)\|^{p-2} \|b(\tau)\|^2 \, d\tau \\
+ p \int_0^T \|h(\tau)\|^{p-2} \langle h(\tau), G'_\xi(0) \cdot (h(\tau) - R(\tau)) \rangle dW_\tau.
\]

Therefore, collecting together (25), (26), (32) and (33), for any \( \mathcal{T} \in [0, T_0] \) we obtain
\[
\mathbb{E} \sup_{0 \leq T \leq \tau^* \wedge T} \left| p \int_0^T \|h(\tau)\|^{p-2} \langle h(\tau), G'_\xi(0) \cdot (h(\tau) - R(\tau)) \rangle dW_\tau \right| \\
\leq 3p \mathbb{E} \left[ \int_0^{\tau^* \wedge T} \|h(\tau)\|^{2p-4} \|h(\tau)\|^2 \|h(\tau) - R(\tau)\|^2 \, d\tau \right]^{\frac{1}{2}} \\
\leq C_p \mathbb{E} \left[ \int_0^{\tau^* \wedge T} \left( \|h(\tau)\|^{2p} + \|h(\tau)\|^{2p-2} \|R(\tau)\|^2 \right) \, d\tau \right]^{\frac{1}{2}}.
\]

Using again Young’s inequality implies
\[
\mathbb{E} \sup_{0 \leq T \leq \tau^* \wedge T} \left| p \int_0^T \|h(\tau)\|^{p-2} \langle h(\tau), G'_\xi(0) \cdot (h(\tau) - R(\tau)) \rangle dW_\tau \right| \\
\leq C_p \mathbb{E} \left[ \int_0^{\tau^* \wedge T} \|h(\tau)\|^{2p} \, d\tau \right]^{\frac{1}{2}} + C_p \mathbb{E} \left[ \int_0^{\tau^* \wedge T} \|R(\tau)\|^{2p} \, d\tau \right]^{\frac{1}{2}} \\
\leq \frac{1}{2} \mathbb{E} \sup_{0 \leq T \leq \tau^* \wedge T} \|h(T)\|^p + C_p \mathbb{E} \left[ \int_0^T \|h(\tau)\|^{p} \, d\tau \right] + C_p \mathbb{E} \left[ \int_0^{\tau^* \wedge T} \|R(\tau)\|^{2p} \, d\tau \right]^{\frac{1}{2}}.
\]

Therefore, collecting together (25), (26), (32) and (33), for any \( \mathcal{T} \in [0, T_0] \) we obtain
\[
\mathbb{E} \sup_{0 \leq T \leq \tau^* \wedge \mathcal{T}} \|h(T)\|^p \leq C_p \mathbb{E} \int_0^T \sup_{0 \leq \tau \leq \tau^* \wedge \mathcal{T}} \|h(\tau)\|^p \, d\tau + C_p e^{p^{-1}k_p}. 
\]
Using Gronwall’s lemma we can show
\[ \mathbb{E} \sup_{0 \leq T \leq \tau^*} \| h(T) \|^p \leq C_p e^{-18\kappa p} + \varepsilon p^{9\kappa p} \| a(0) \|^p \leq C_p e^{-18\kappa p}, \]
so that, in view of (25),
\[ \mathbb{E} \sup_{0 \leq T \leq \tau^*} \| a(T) - b(T) \|^p \leq \mathbb{E} \sup_{0 \leq T \leq \tau^*} \| b(T) \|^p + \mathbb{E} \sup_{0 \leq T \leq \tau^*} \| R(T) \|^p \leq C_p e^{-18\kappa p}. \]

\[ \square \]

**Remark 4.1.** Notice that by lemma 4.9, for any \( p > 0 \) and \( \kappa \in (0, \frac{1}{18}) \), we obtain
\[ \mathbb{E} \sup_{0 \leq T \leq \tau^*} \| a(T) \|^p \leq \mathbb{E} \sup_{0 \leq T \leq \tau^*} \| a(T) - b(T) \|^p + \mathbb{E} \sup_{0 \leq T \leq \tau^*} \| b(T) \|^p \leq C_p (1 + \| a(0) \|^p). \]
(34)

We can use this to show that \( \| a \| < \varepsilon^{-\kappa} \) on \([0, T_0]\) with probability almost 1.

Let us define the overall error between \( \varepsilon (b + Q) \) and \( a \) by
\[ R(T) := u(\varepsilon^{-2} T) - \varepsilon b(T) - \varepsilon Q(T) \]
\[ = \varepsilon [a(T) - b(T) + \psi(T) - Q(T)] \]
\[ = \varepsilon [a(T) - b(T) + I(T) + J(T) + K(T)]. \]
(35)

We already know that \( I \) and \( J \) are uniformly small. It remains to bound \( K \). For this we use the factorization method and start with the following stochastic integral.

**Lemma 4.10.** Assume the setting of lemma 4.9. For any \( p > 1 \), there exists a constant \( C_p > 0 \) such that
\[ \mathbb{E} \sup_{0 \leq T \leq T_0} \left\| \int_{0}^{T} e^{A(T-\tau) \gamma^{-2}} G_{\varepsilon}(0) \cdot b(\tau) d\tilde{W}_{\tau} \right\|^p \leq C_p e^{-\frac{4}{3} \kappa}. \]
(36)

**Proof.** For large \( p > 1 \), fix \( \gamma \in (0, 1/2) \). By the celebrated factorization method, if we set
\[ Y_\gamma(s) = \int_{0}^{s} (s - \gamma) e^{A(k-\gamma) \gamma^{-2}} G_{\varepsilon}(0) \cdot b(\tau) d\tilde{W}_{\tau}, \]
we have
\[ \int_{0}^{T} e^{A(T-\tau) \gamma^{-2}} G_{\varepsilon}(0) \cdot b(\tau) d\tilde{W}_{\tau} = C_p \int_{0}^{T} (T - \tau)^{-1} e^{A(T-\tau) \gamma^{-2}} Y_\gamma(\tau) d\tau \]
for some constant \( C_p > 0 \). Thus we obtain by Hölder inequality and the bounds on the semigroup on the space \( S \) that
\[ \left\| \int_{0}^{T} e^{A(T-\tau) \gamma^{-2}} G_{\varepsilon}(0) \cdot b(\tau) d\tilde{W}_{\tau} \right\|^p \]
\[ \leq C \left( \int_{0}^{T} (T - \tau)^{-1} e^{-A(T-\tau) \gamma^{-2}} \| Y_\gamma(\tau) \|_{S} d\tau \right)^p \]
\[ \leq C \left( \int_{0}^{T} (T - \tau)^{-1} e^{-A(T-\tau) \gamma^{-2}} \| Y_\gamma(\tau) \|_{S} d\tau \right)^{p-1} \int_{0}^{T} \| Y_\gamma(\tau) \|_{S}^p d\tau \]

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\[ \leq C e^{2\gamma -2} \left( \int_0^{T/\epsilon} (t-\tau)^{-(1-\gamma)p(p-1)} e^{-p\epsilon(t-\tau)^{2(p-1)}} d\tau \right)^{p-1} \int_0^T \| Y_s(\tau) \|_2^p d\tau \]

\[ \leq C e^{2\gamma -2} \int_0^T \| Y_s(\tau) \|_2^p d\tau. \]

Note that we need to fix \( p \gg 1 \) large or \( 1 \gg \gamma > 0 \) small in order to have an integrable pole in the previous estimate.

Moreover, by Burkholder–Davis–Gundy inequality, but now without the supremum in time we obtain for \( t \in [0, T_0] \),

\[ \mathbb{E} \| Y_s(t) \|_2^p \leq C \mathbb{E} \left( \int_0^t (t-\tau)^{-2} \| e^{A(t-\tau)} - 1 \|_2^p \cdot b(\tau) \|_2 \sup_{\mathbb{E}} \| b(\tau) \|_p^{-2} \right)^{p/2} \]

\[ \leq C \mathbb{E} \left( \int_0^t (t-\tau)^{-2} e^{-p\epsilon(t-\tau)^{2}} \| b(\tau) \|_2^p \right)^{p/2} \]

\[ \leq C \mathbb{E} \sup_{0 \leq \tau \leq T_0} \| b(\tau) \|_p^{2(1-\gamma)p/2} \leq C \mathbb{E} \sup_{0 \leq \tau \leq T_0} \| b(\tau) \|_p^{p(1-\gamma)p}. \]

This finally implies

\[ \mathbb{E} \sup_{0 \leq \tau \leq T_0} \left\| \int_0^T e^{A(t-\tau)} - 1 \|_2^p \cdot b(\tau) d\tilde{W}_\tau \right\|_p^p \leq C T_0 \mathbb{E} \sup_{0 \leq \tau \leq T_0} \| b(\tau) \|_p^{p(1-\gamma)p}. \]

As we can choose \( p \) arbitrarily large, we obtain via Hölder inequality that for any small \( \bar{\kappa} > 0 \) there is a constant such that

\[ \mathbb{E} \sup_{0 \leq \tau \leq T_0} \left\| \int_0^T e^{A(t-\tau)} - 1 \|_2^p \cdot b(\tau) d\tilde{W}_\tau \right\|_p^p \leq C \mathbb{E} \sup_{0 \leq \tau \leq T_0} \| b(\tau) \|_p^{p-\bar{\kappa}}, \]

so that, thanks to (26), we have

\[ \mathbb{E} \sup_{0 \leq \tau \leq T_0} \left\| \int_0^T e^{A(t-\tau)} - 1 \|_2^p \cdot b(\tau) d\tilde{W}_\tau \right\|_p^p \leq C e^{p-\frac{1}{2} \bar{\kappa}}. \]

\[ \square \]

In order to bound \( K \) we set

\[ M(T) := \int_0^T e^{\hat{A}(t-\epsilon)} \left( \hat{G}_s(\epsilon a(\tau) + \epsilon \psi(\tau)) - G_s(0) \cdot b(\tau) \right) d\tilde{W}_\tau, \]

we have from (17),

\[ K(T) = M(T) + \int_0^T e^{\hat{A}(t-\tau)} G_s(0) \cdot b(\tau) d\tilde{W}_\tau, \]

where we just bounded the integral on the right in lemma 4.10. It remains to bound \( M \). Here we proceed similarly to the previous lemma using factorization.

**Lemma 4.11.** Assume the setting of lemma 4.9. For any \( p > 1 \) there exists a constant \( C_p > 0 \) such that

\[ \mathbb{E} \sup_{0 \leq T \leq T_0} \| M(T) \|_p^p \leq C_p e^{p-2\alpha p}. \] (37)
Proof. We can follow exactly the proof of the previous lemma 4.10 but have to pay attention to the fact that the integrand in $M$ is only defined for $t \leq \tau^*$. Moreover, the integrand is due to the presence of $\psi$ and thus $K$ not uniformly bounded in time.

Define the integrand as

$$\Phi(\tau) = \frac{1}{\varepsilon}G_{\varepsilon}(\varepsilon a(\tau) + \varepsilon \psi(\tau)) - G_{\varepsilon}'(0) \cdot b(\tau)$$

We notice that by Taylor’s formula

$$\Phi(\tau) = G'(0) \cdot \psi(\tau) + G'(0) \cdot [a(\tau) - b(\tau)] + \frac{\varepsilon}{2}G''(\tilde{z}(\tau)) \cdot (a(\tau) + \psi(\tau), a(\tau) + \psi(\tau)),$$

where $\tilde{z}(\tau)$ is a vector on the line segment connecting $0$ and $\varepsilon a(\tau) + \varepsilon \psi(\tau)$. Therefore,

$$\|\Phi(\tau)\|_{L^2(U; N^n)} \leq C\|\psi(\tau)\|_{\alpha} + C\|a(\tau) - b(\tau)\|_{\alpha} + C\|a(\tau)\|^2_{\alpha} + C\|\psi(\tau)\|^2_{\alpha},$$

Note that for $\tau \leq \tau^*$ the right-hand side above is bounded by $C\varepsilon^{-k}$ uniformly in time.

We obtain following the lines of lemma 4.10

$$\mathbb{E} \sup_{0 \leq T \leq \tau^*} \|M(T)\|^p_{\alpha} \leq \mathbb{E} \sup_{0 \leq T \leq T_0} \|\int_0^T e^{A(t-T)} \cdot 1_{[0,\tau^*]}(\tau)\Phi(\tau) d\hat{W}_\tau\|^p_{\alpha} \leq C\varepsilon^{p-1} \mathbb{E} \sup_{0 \leq T \leq T_0} \|1_{[0,\tau^*]}(\tau)\Phi(\tau)\|^p_{L^2(U; H^\alpha)} \leq C\varepsilon^{p-1} \mathbb{E} \sup_{0 \leq T \leq \tau^*} \|\Phi(\tau)\|^p_{L^2(U; H^\alpha)},$$

so that

$$\mathbb{E} \sup_{0 \leq T \leq \tau^*} \|M(T)\|^p_{\alpha} \leq C\varepsilon^{p(1-2\kappa)}.$$

As a consequence of lemmas 4.1, 4.9, and 4.11, we have the following bound on $R$:

Lemma 4.12. Assume the setting of lemma 4.9. For any $p > 1$, there exists a constant $C_p > 0$ such that

$$\mathbb{E} \sup_{0 \leq T \leq \tau^*} \|R(T)\|^p_{\alpha} \leq C_p\varepsilon^{2p-18\kappa p}.$$ (39)

Moreover, we obtain a bound on $\psi$ which is uniform in time:

Lemma 4.13. Assume the setting in lemma 4.9. For any $p > 1$, there exists a constant $C_p > 0$ such that

$$\mathbb{E} \sup_{0 \leq T \leq \tau^*} \|\psi(T)\|^p_{\alpha} \leq C_p(\varepsilon^{\kappa/3} + \varepsilon^{p-18\kappa p}).$$ (40)

Before we proceed with the final error estimate, we comment on improved bounds on $\psi$ and $M$. We know by definition that

$$\psi(T) = Q(T) + \int_0^T e^{A(T-t)} \cdot G_{\varepsilon}'(0) \cdot b(\tau) d\hat{W}_\tau + M(T) + I(T) + J(T).$$
As \( I(T) + J(T) = O(\varepsilon^{2-3\kappa}) \) by lemma 4.1 and both \( M \) and the stochastic integral is uniformly in time by \( O(\varepsilon^{1-2\kappa}) \), we can show that \( \psi - Q \) is small uniformly in time. This improved bound on \( \psi \) can be used in the proof of lemma 4.11, to show that \( M \) is smaller of order \( O(\varepsilon^{1-2\kappa}) \). Thus we obtain

\[
\mathbb{E} \sup_{T \in [0, \tau^*]} \left| \psi(T) - Q(T) - \int_0^T e^{\lambda_0(\tau - \tau') \kappa} G_x'(0) \cdot b(\tau') d\tilde{W}_\tau \right| \leq C \varepsilon^{\kappa(2-3\kappa)}. \tag{41}
\]

Before proving the main theorem, we need to construct a subset of \( \Omega \), which enjoys nearly full probability.

**Definition 4.2.** For \( \kappa > 0 \) from the definition of \( \tau^* \) define the set \( \Omega^* \subset \Omega \) of all \( \omega \in \Omega \) such that all these estimates

\[
\sup_{0 \leq T \leq \tau^*} \|a(T)\| < e^{-\frac{1}{2}\kappa}, \quad \sup_{0 \leq T \leq \tau^*} \|R(T)\| < e^{2-19\kappa}
\]

and

\[
\sup_{0 \leq T \leq \tau^*} \|\psi(T)\|_{\alpha} < e^{-\frac{1}{2}\kappa}
\]

hold.

**Lemma 4.14.** The set \( \Omega^* \) has approximately probability 1.

**Proof.** Indeed, let \( \Omega^* \) be as in the definition 4.2. It easily follows that

\[
\mathbb{P}(\Omega^*) \geq 1 - \mathbb{P}(\sup_{0 \leq T \leq \tau^*} \|a(T)\| \geq e^{-\frac{1}{2}\kappa}) - \mathbb{P}(\sup_{0 \leq T \leq \tau^*} \|\psi(T)\|_{\alpha} \geq e^{-\frac{1}{2}\kappa})
\]

\[
- \mathbb{P}(\sup_{0 \leq T \leq \tau^*} \|R(T)\| \geq e^{2-19\kappa}).
\]

We get by using Chebyshev’s inequality, (34), (36), (39) and (40)

\[
\mathbb{P}(\Omega^*) \geq 1 - \left( e^{-\frac{1}{2}\kappa} \right)^{-q} \mathbb{E} \sup_{[0, \tau^*]} \|a(T)\|_{\alpha}^q - \left( e^{-\frac{1}{2}\kappa} \right)^{-q} \mathbb{E} \sup_{[0, \tau^*]} \|\psi(T)\|_{\alpha}^q
\]

\[
- (e^{2-19\kappa})^{-q} \mathbb{E} \sup_{[0, \tau^*]} \|R(T)\|_{\alpha}^q
\]

\[
\geq 1 - 2C_2 e^{\varepsilon/2} e^{-q/3} - C e^{(19\kappa - 2\varepsilon)(2-18\kappa)q}
\]

\[
\geq 1 - C e^{p},
\]

where we take for a given \( p \) the exponent \( q \) sufficiently large. \( \square \)

**Proof of Theorem 2.1.** From the definition of \( \Omega^* \) and \( \tau^* \), we have

\[
\Omega^* \subseteq \left\{ \sup_{0 \leq T \leq \tau^*} \|a(T)\|_{\alpha} < e^{-\kappa}, \sup_{0 \leq T \leq \tau^*} \|\psi(T)\|_{\alpha} < e^{-\kappa} \right\} \subseteq \{ \tau^* = T_0 \} \subseteq \Omega.
\]

This allows us to get on \( \Omega^* \) that

\[
\sup_{0 \leq T \leq T_0} \|R(T)\|_{\alpha} = \sup_{0 \leq T \leq \tau^*} \|R(T)\|_{\alpha} \leq C \varepsilon^{2-19\kappa}
\]
such that
\[
\mathbb{P} \left( \sup_{0 \leq T \leq T_0} \| \mathcal{R}(T) \|_\alpha \geq \varepsilon^{2-19c} \right) \leq 1 - \mathbb{P}(\Omega^*) \leq C \varepsilon^p,
\]
which, by recalling representation (35), completes the proof.

5. Example—Ginzburg–Landau/Allen–Cahn equation

A very simple example to illustrate the main result is the stochastic Ginzburg–Landau equation (or Allen–Cahn equation) with linear multiplicative noise on the interval \( D = [0, \pi] \) of the form
\[
\partial_t u = (\partial_x^2 + 1)u + \nu \varepsilon^2 u - u^3 + \varepsilon u \cdot \partial_t W(t).
\]

In the following we consider the Itô-representation of the SPDE above with sufficiently smooth noise in the standard Sobolev-space \( H^1_0(D) \), which is the space of functions with square integrable derivatives that satisfy Dirichlet boundary conditions.

The existence and uniqueness of global (i.e., \( \tau_{\text{ex}} = \infty \)) solutions is standard for this equation [23, theorem 6.5, chapter 3].

The deterministic equation has a forward pitchfork bifurcation at \( \nu = 0 \), and we scale the linear term to be close to bifurcation by choosing \( \nu \varepsilon^2 \). Moreover, we choose a noise strength of order \( \varepsilon \) in order to have both terms in the limiting equation of our main result.

We set
\[
A := \partial_x^2 + 1, \quad \mathcal{L} := \nu \mathcal{I}, \quad \mathcal{F} := -u^3
\]
and discuss \( G \) further below, after we defined the noise given by the Wiener process \( W \) below.

Suppose that the equation is posed with Dirichlet boundary condition. Let \( \mathcal{H} = L^2([0, \pi]) \) be the space of all square integrable real-valued functions which are defined on the interval \([0, \pi]\). In this situation the eigenvalues of \(-A\) are explicitly known to be \( \lambda_k = k^2 - 1 \) with associated eigenvectors \( e_k(x) = \sqrt{2} \sin(kx) = \delta \sin(kx), k = 1, 2, \ldots, \) and \( N = \text{span}\{e_1\} \). So assumption 1 is true with \( m = 2 \).

Clearly, assumption 2 holds true for example for any \( \alpha > 1/2 \) and \( \beta = 0 \), as for the norm in \( \mathcal{H}_\alpha \) we then have \( \|w\|_\alpha \leq C \|u\|_0 \|v\|_\alpha \). We will fix \( \alpha = 1 \) for simplicity. Note that our spaces \( \mathcal{H}_\alpha \) are (up to an equivalent norm) the same space as the Sobolev spaces \( H^1_0(D) \).

Note that on the one-dimensional space \( N \) the \( \mathcal{H}_\alpha \)-norm is just a multiple of the \( \mathcal{H} \)-norm. So that for \( u, w \in N \) the conditions described in assumption 3 are satisfied as follows:
\[
\langle \mathcal{F}_\varepsilon(u), u \rangle = -\int_0^\pi u^4(x) \, dx \leq 0, \quad \langle \mathcal{F}_\varepsilon(u, u, w) \rangle = -\int_0^\pi u^2(x)w^2(x) \, dx \leq 0.
\]
In addition, condition (6) is true for some positive constants \( C_0, C_1, C_2 \), as \( \mathcal{F} \) is a standard cubic nonlinearity.

Define \( f_k(x) := \frac{1}{2} e_k(x), k = 1, 2, \ldots, \) such that \( \{f_k\}_{k \in \mathbb{N}} \) is an orthonormal basis of \( \mathcal{H}^1 \). We consider in our application that \( W \) is standard cylindrical \( \mathcal{H}^1 \)-valued Wiener process and define a covariance operator \( Q \) defined by \( Qf_k = \alpha_k f_k, k = 1, 2, \ldots, \) satisfying \( \text{trace}(Q) = \sum_{k=1}^\infty \alpha_k = C_0 < \infty \). For the operator \( G \) defined as
\[
G(u)v := u \cdot Q^{1/2}v,
\]
we have
\[ \|G(u)\|_{L^2(H^1, H^1)}^2 = \sum_{k \in \mathbb{N}} \|u \cdot (Q^{1/2} e_k)\|_{H^1}^2 = \sum_{k \in \mathbb{N}} \alpha_k \|u \cdot e_k\|_{H^1}^2 \leq C \sum_{k \in \mathbb{N}} \alpha_k \|u\|_{H^1}^2 \|e_k\|_{H^1}^2 = C \|u\|_{H^1}^2 \text{trace}(Q) \leq C \|u\|_{H^1}^2 < \infty. \]

Therefore, \( G(\cdot) : H^1 \to L^2(H^1, H^1) \) is a Hilbert–Schmidt operator satisfying
\[ G' \left( u \right) \cdot v = v \cdot Q^{1/2} \quad \text{and} \quad G''(u) = 0, \]
so that assumption 5 holds.

Therefore, our main theorem states that the dynamics of (43) can well approximate by the amplitude equation, which is for \( b \in \mathcal{N} \) a stochastic ordinary differential equation of the form:
\[ db = [\nu b - P_cF(b)] dt + P_c[G'(0) \cdot b] dW. \]

Let us finally rewrite the amplitude equation for the actual amplitude of \( b \):
\[ b = \gamma \sin(\cdot) \]
Clearly, as \( P_c f = \int_0^\pi \sin(\gamma) f(\gamma) d\gamma \) we have \( P_cF(b) = -\frac{3}{4} \gamma^3 \sin(\cdot) \). Moreover,
\[ P_c[G'(0) \cdot b] dW = \gamma \sum_{k=1}^\infty P_c[\sin(\cdot)Q^{1/2} f_k] d\tilde{B}_k \]
\[ = \gamma \sum_{k=1}^\infty \sqrt{\alpha_k} P_c[\sin(\cdot) f_k] d\tilde{B}_k = \gamma \sum_{k=1}^\infty \sqrt{\alpha_k} \sigma_k d\tilde{B}_k \sin(\cdot) \]
as \( P_c[\sin(x) \sin(kx)] = \frac{2}{\pi} \int_0^\pi \sin(\gamma)^2 \sin(\gamma) dy \sin(x) = k \sigma_k \sin(x) \) with
\[ \sigma_k := \begin{cases} \frac{4 \cos(k\pi)}{\pi} - \frac{1}{k^2}, & k \neq 2, \\ 0, & k = 2. \end{cases} \]
Thus we obtain for the amplitude
\[ d\gamma = \left[ \nu \gamma - \frac{3}{4} \gamma^3 \right] dT + \gamma \Sigma^{1/2} d\beta, \]
for a standard real valued Brownian motion and noise strength \( \Sigma = \sum_{k=1}^\infty \alpha_k \sigma_k^2 \).

Thus as a limiting equation for the dynamics we obtain an one-dimensional SDE, which in its complex form is sometimes called Landau-equation. It describes here a forward-pitchfork bifurcation perturbed by additive noise.

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