ON LENAGAN’S THEOREM FOR FINITE LENGTH BIMODULES

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Abstract. We offer a self-contained proof of Lenagan’s Theorem which does not rely on Goldie’s Theorem.

1. Introduction

Lenagan’s Theorem states that for a bimodule \( rM_A \), if \( rM \) has finite length and \( M_A \) is Noetherian, then \( M_A \) also has finite length. This theorem first appeared in [7], and as Lam writes

Indeed, although the argument above is quite short, it seemed to have used the full force of Goldie’s First Theorem, and it is not clear at all how one could have proved [it] otherwise. [6, p. 333]

One important consequence is that the left and right Artin radicals of a Noetherian ring agree. We can also regard Lenagan’s Theorem as a generalisation of the classical result that a left Artinian ring is right Artinian if and only if it is right Noetherian. This latter is a direct consequence of the Hopkins-Levitzki Theorem.

In fact, Lenagan’s Theorem was generalised by Crawley-Boevey [5]: if \( rM \) is Artinian and \( M_A \) is Noetherian, then \( rM \) and \( M_A \) both have finite length. We give a new proof of this result without recourse to Goldie’s Theorem. Instead it follows from a strengthening of the Hopkins–Levitzki Theorem. In a similar way we also obtain a result of Björk on subrings of semiprimary rings. The starting point for our proof is a result of Camps and Dicks on semilocal rings.

Let \( \Lambda \) be a ring, with Jacobson radical \( \text{rad}(\Lambda) \). We call \( \Lambda \) semilocal if \( \bar{\Lambda} := \Lambda/\text{rad}(\Lambda) \) is semisimple, and semiprimary if moreover \( \text{rad}(\Lambda) \) is nilpotent. It is well-known that the endomorphism ring of a finite length object in an abelian category is always semiprimary.

2. Finite length modules

We begin with a beautiful proof, due to Camps and Dicks [2]. We follow the proof in the second edition [3], which incorporates ideas from Camillo and Nielsen [4].

**Theorem 1.** Let \( M \) be an Artinian object in some abelian category, and set \( E := \text{End}(M) \). If \( \Lambda \subset E \) is a subring such that \( \Lambda^\times = \Lambda \cap E^\times \), then \( \Lambda \) is semilocal.

**Proof.** Observe first that for all \( x, y \in E \) we have

\[
\ker(x - xy) = \ker(x) \oplus \ker(1 - xy),
\]

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Proposition 2. Let \( \Lambda \) subring of semiprimary ring \( E \), and \( M \) a right \( E \)-module. Then \( M_\Lambda \) is Noetherian if and only if it is Artinian.

Proof. Since \( E \) is semiprimary, \( M_E \) has finite Loewy length, and hence we may assume that \( M_E \) is semisimple. Then \( M_\Lambda \) Artinian or Noetherian implies the same for \( M_E \), and hence we may assume that \( M_E \) is simple. Note that this already proves the result when \( \Lambda = E \); that is, \( M_E \) is Noetherian if and only if it is Artinian.

After taking the quotient by the annihilator of \( M \), we may assume that \( E = \mathbb{M}_n(\Delta) \) for a division ring \( \Delta \), and hence regard \( M \) as a \( \Delta-E \)-bimodule. We claim that, in this situation, \( \Lambda \) is semisimple, so \( M_\Lambda \) has finite length as above.

Since \( E_E \cong M_E^2 \), we know that \( E_\Lambda \), and hence also \( \Lambda_\Lambda \), is Artinian or Noetherian. If \( \Lambda \) is right Artinian, then it is semiprimary, and we are done. Assume therefore that \( E_\Lambda \) is right Noetherian.

Suppose \( x \in \Lambda \) has inverse \( y \in E \). For some \( d \) we have \( y^d \in \sum_{i<\infty} y^i \Lambda \), and thus \( y \in \sum_{i<\infty} x^{-d-i} \Lambda \subset \Lambda \). Since \( \Delta M \) has finite length, it follows from the theorem that \( \Lambda \) is semilocal. Also, \( E \cdot \mathfrak{rad}^n(\Lambda) \) form a descending chain of left \( \Delta \)-modules, so stabilises. Nakayama’s Lemma then gives \( E \cdot \mathfrak{rad}^n(\Lambda) = 0 \) for some \( n \). Thus \( \mathfrak{rad}^n(\Lambda) = 0 \), and \( \Lambda \) is semiprimary as claimed. \( \square \)

The next corollary appears as Theorem 3.11 in [1].

Corollary 3. Let \( E \) be semiprimary and \( \Lambda \subset E \) a subring. If \( E_\Lambda \) is Noetherian, then \( \Lambda \) is right Artinian.

Proof. Apply the proposition to \( M = E \). \( \square \)

Corollary 4 (Lenagan,Crawley-Boevey). Let \( \Gamma \)-module \( \Lambda \) be a bimodule such that \( \Gamma M_\Lambda \) is Artinian and \( M_\Lambda \) is Noetherian. Then \( \Gamma M \) and \( M_\Lambda \) both have finite length.

Proof. It is enough to prove that \( E := \text{End}_\Gamma(M) \) is semiprimary, since then \( M_\Lambda \) has finite length by the proposition, so \( \text{End}_\Lambda(M) \) is also semiprimary, and hence \( E \) has finite length by the proposition once more.

Now, the theorem tells us that that \( E \) is semilocal. Also, the \( \Gamma \)-submodules \( M \cdot \mathfrak{rad}^n(E) \) form a descending chain, so must stabilise. As \( M_\Lambda \) is Noetherian, so too is \( M_E \). Thus Nakayama’s Lemma gives \( M \cdot \mathfrak{rad}^n(E) = 0 \) for some \( n \), and hence that \( \mathfrak{rad}^n(E) = 0 \). \( \square \)
References

[1] Björk, J.-E.: Conditions which imply that subrings of semiprimary rings are semiprimary. J. Algebra 19, 384–395 (1971)
[2] Camps, R., Dicks, W.: On semilocal rings. Israel J. Math. 81, 203–211 (1993)
[3] Camps, R., Dicks, W.: On semilocal rings. 2nd edition. http://mat.uab.cat/~dicks/SemilocalNew.pdf (2010) Accessed 8 March 2019
[4] Camillo, V.P., Nielsen, P.P.: On a theorem of Camps and Dicks. In: Van Huynh, D., López-Permouth, S.R. (eds) Advances in ring theory. Trends in Mathematics, pp. 83–84. Birkhäuser, Basel (2010)
[5] Crawley-Boevey, W.W.: Modules of finite length over their endomorphism rings. In: Tachikawa, H., Brenner, S. (eds) Representations of algebras and related topics (Kyoto, 1990). London Math. Soc. Lecture Note Ser. 168, pp. 127–184. Cambridge Univ. Press, Cambridge (1992)
[6] Lam, T.Y.: Lectures on modules and rings. Graduate Texts in Math. 189. Springer-Verlag, New York (1999)
[7] Lenagan, T.H.: Artinian ideals in Noetherian rings. Proc. Amer. Math. Soc. 51, 499–500 (1975)

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