On Vague Computers

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Abstract

Vagueness is something everyone is familiar with. In fact, most people think that vagueness is closely related to language and exists only there. However, vagueness is a property of the physical world. Quantum computers harness superposition and entanglement to perform their computational tasks. Both superposition and entanglement are vague processes. Thus quantum computers, which process exact data without “exploiting” vagueness, are actually vague computers.

1 Introduction

Vagueness is something we all are familiar with. A very rough definition of vagueness is this: the property of objects or entities that lack definite shape, form, or character. For many years vagueness was considered just a linguistic phenomenon. This simply means that vagueness is part of our everyday expression and not something real, which turned out not to be true. In the linguistic realm, a property of some object is vague when it is not clear to which group or, more generally, category the object belongs to. Thus when Garnet is 1.68 m tall, it is not obvious if she is tall or not tall. Similarly, if Jim is 1.90 m tall, he might be classified as tall, in general, but as short when his height is compared to the height of the average NBA player. Now, if John’s height is 2.00 m and we are sure he is tall, then any other person whose height is slightly different (e.g., Mike, whose height is 1.98 m) is also considered tall, thus similar to John with respect to his height. But what exactly is slightly different? Obviously, this similarity degree is an indirect way to define vagueness. In particular, similarity between physical objects can be used to show that vagueness exists in the natural world. Although, elementary particles of the same kind (i.e., protons) are considered indistinguishable by most physicists, still there are some who argue that elementary particles are in fact distinguishable but also similar (e.g., see [5] for an overview). Provided this approach is valid, and I will say more on this later on, one can talk about vagueness in the physical reality.

Sometimes people confuse vagueness with ambiguity. For example, the sentence “Garnet ate the cookies on the couch” is ambiguous because one can understand it in more than one way. In particular, did Garnet eat the cookies that were on the couch or did she bring cookies that she later ate on the couch? Now contrast the previous sentence with “the room was gray,” which is vague because there are many shades of gray and it is not clear to which one is the color of the room. People also confuse imprecision with vagueness. For example, the sentence “bring me the cup” is not precise when there are many cups.

A modern computer are fed with exact data, processes them as such and delivers exact answers. Of course, this scheme is quite reasonable as we usually have specific problems and we want concrete answers to them, but in most, if not all cases, we do not care if the internal working involves vague data and operations as long as this does not affect the final result. Of course, it is widely assumed that the computational process does not involve any form of vagueness, yet there are error correction protocols because errors happen. But can we attribute these errors to vagueness? Furthermore, can we use vagueness constructively in the computational process? Or, in different words, is there room for vagueness in computation? As far as the second question is concerned, the answer is affirmative since there are realistic and not so realistic models of computation that employ vagueness [e.g., the fuzzy Turing machine is not a so realistic model, while fuzzy P systems and fuzzy chemical machines are
realistic models of computation (see [15] for details). In addition, it seems that there is a connection between quantum mechanics and vague computing.

The pillars of modern physics are quantum mechanics and general relativity (special relativity explains only the special case where motion is uniform). One could say that general relativity is the physics of the macrocosm while quantum mechanics is the physics of the microcosm. In different words, one could say that quantum mechanics helps us understand the behavior of molecules, atoms, and elementary particles while general relativity is the theory we use to explain phenomena near very massive objects, such as planets, stars, and galaxies (gravity weakens as we go away from massive objects). It is really weird that the two theories do not “mix”. So far all efforts to quantize gravity have failed! Quantum mechanics started when Max Karl Ernst Ludwig Planck explained the problem of the radiation of a black body in 1900. Roughly, he proposed that energy can have only certain discrete values something that helped him to solve this problem (see [2] among others for a short description of the genesis of quantum mechanics).

Quantum computing is making use of the laws of quantum mechanics, and quantum mechanics is explained by, among others, statistical probabilities (i.e., a combination of statistics and probability theory). Quantum mechanics was formalized in 1926 while probability theory was formalized in 1930 [3]. Until that time, probability theory was considered a prediction tool, something that most people still believe. For example, today many people think they can use probabilities to make educated bets at a blackjack table and other games of chance. Naturally, they use statistical probabilities. Of course, probability theory is not about chance and games. In mathematics, (pure) probabilities are ratios of the measure of subsets of a given set. Here the word “measure” means “counting” in case one deals with finite sets. However, when one has to deal with sets that contain an infinite number of elements (e.g., the set of integer numbers), then one must employ a suitable measuring process to “count” elements. Thus when one knows how to count the elements of a set, then one can calculate probabilities. Obviously, this has nothing to do with chance or randomness.

In what follows I will explore the connection between (mathematical models of) vagueness and quantum computing. In particular, after a concise introduction to fuzzy set theory, I will introduce possibility theory. Then I will discuss vagueness at the quantum level and I will explain how possibilities can replace probabilities in quantum mechanics thus giving rise to real vague computers.

2 Fuzzy Set Theory: A Mathematical Model of Vagueness

Fuzzy set theory is a mathematical model of vagueness that was introduced by Lotfi Askar Zadeh [16]. Fuzzy sets are a natural extension of ordinary sets. Zadeh defined fuzzy sets by generalizing the membership relationship. In particular, given a universe $X$, he defined a fuzzy subset of $X$ to be an object that is characterized by a function $A : X \rightarrow [0, 1]$. The value $A(x)$ specifies the degree to which an element $x$ belongs to $A$. Thus if $A$ denotes height and $x$ is Garnet, then $A(x)$ is the degree to which Garnet is tall.

A fuzzy set $A$ for which there is an $x \in X$ such that $A(x) = 1$ is called normalized.

Most newcomers tend to take fuzzy set theory for an alternative formulation of probability theory, nevertheless, this is not the case. For instance, there are probability theorists that still believe that fuzziness is unnecessary since they argue that probability theory can be used to solve all problems that can be tackled by fuzzy set theory. Zadeh [17] has argued that the two theories are complementary, that is, they are different facets of vagueness. Bart Kosko [10] and other researchers, including this author [15], have argued that fuzzy set theory is more fundamental than probability theory. According to Kosko fuzziness “measures the degree to which an event occurs, not whether it occurs. Randomness describes the uncertainty of event occurrence.” However, I do not plan to say anything more on this matter (a very detailed discussion is included in [15]). Instead, let me now present the basic operations between fuzzy subsets.

Assume that $A, B : X \rightarrow [0, 1]$ are two fuzzy subsets of $X$. Then, their union and their intersection are defined as follows:

$$ (A \cup B)(x) = \max\{A(x), B(x)\} $$

1 Speculations about higher dimensions, parallel universes, etc., will remain speculations until there is solid proof about their existence.
and

$$(A \cap B)(x) = \min\{A(x), B(x)\}. \quad (2)$$

Also, if $\bar{A}$ is the complement of the fuzzy subset $A$, then $\bar{A}(x) = 1 - A(x)$. More generally, it is quite possible to use functions other than min and max to define the intersection and the union of fuzzy subsets. These functions are known in the literature as $t$-norms and $t$-conorms, respectively. For more information on $t$-norms and $t$-conorms see [9] or any other textbook on fuzzy set theory.

In the years that followed the publication of Zadeh’s paper, various researchers proposed and defined various fuzzy structures (e.g., fuzzy algebraic structures, fuzzy topologies, etc.). For instance, the concept of fuzzy languages was introduced by E.T. Lee and Zadeh [11]:

**Definition 2.1** A fuzzy language $\lambda$ over an alphabet $S$ (i.e., an ordinary set of symbols) is a fuzzy subset of $S^*$. If $s \in S^*$, then $\lambda(s)$ is the grade of membership that $s$ is a member of the language.

**Example 2.1** Consider the following set that includes all the sequences of zeros followed by ones:

$$L = \{0^i1^j \mid i \neq j \text{ and } i, j > 0\}.$$

Then, the following function

$$\lambda(0^i1^j) = \begin{cases} j/i, & \text{if } i > j \\ i/j, & \text{otherwise} \end{cases}$$

defines a fuzzy language.

Ordinary set theory is built out of two predicates: membership and equality. This means that in a fuzzy theory of sets both the membership and the equality should be fuzzy. Unfortunately, and for unknown reasons, Zadeh fuzzified only the membership predicate whereas he left crisp the equality predicate, thus, making the resulting theory somehow incoherent. It is not difficult to fuzzify the equality predicate and Michael Barr [1] has provided a solution to this problem. In addition, he showed how to construct categories of “fuzzy” sets that form a topos. Interestingly, a topos is a non-fuzzy mathematical universe, thus, he showed how to actually embed “fuzzy” sets in such a universe. Although a topos is an intuitionistic universe, that is, a universe that is strongly connected to recursion theory, still it is one that has no respect for vagueness! This implies that it is necessary to define fuzzy universes, whatever this may mean.

### 3 From Probabilities to Possibilities

Most textbooks on quantum mechanics introduce the reader to the statistical interpretation of the theory in the first pages of the book (e.g., Griffiths’s [7] excellent textbook follows this convention). Of course, the reason is that the statistical interpretation plays a central role in quantum mechanics. Now, this interpretation is based on the pre-Kolmogorov probability theory and uses it for the estimation of the likelihood of various events. For example, if $\Psi(x, t)$ is the wave function of a particle that moves on a straight line and $a$ and $b$ are two points of this line, then

$$\int_a^b |\Psi(x, t)|^2 dx = \begin{cases} \text{the probability of finding the particle between } a \text{ and } b, \text{ at time } t. \end{cases} \quad (3)$$

Obviously, here we are talking about events that we cannot control and so they can be classified as random. This does not surprise anyone since nonspecialists perceive probabilities as a mathematical “measure” of how likely it is to see some event to happen. Statements like the following ones express exactly this view:

- it is quite probable that Bayern Munich will win the Champions League this season, or
- there is a 20% probability that it will rain tomorrow, or
• the probability of throwing two dice and obtaining two sixes is 1/36.

A rigorous and mathematically sound definition of probabilities have been given by Andrey Nikolaevich Kolmogorov in his *Analytical Methods of Probability Theory*, which was published in 1931. Kolmogorov’s formulation appeared almost 6 years after the formalization of quantum mechanics (see [3]).

Kolmogorov employed measure theory in order to rigorously define probability theory. In particular, a probability measure is a function taking sets as arguments that assigns the number 0 to the empty set and a nonnegative number to any other set. Also, it has to be countably additive. Thus given a nonempty set \( X \) and a nonempty class \( C \) of subsets of \( X \), and a function \( \mu : C \to [0, 1] \) such that

- \( \mu(\emptyset) = 0; \)
- \( \mu \left( \bigcup_{i=1}^{\infty} E_i \right) = \sum_{i=1}^{\infty} \mu(E_i) \) for any disjoint sequence \( \{E_i\} \) of sets in \( C \) whose union is also in \( C \);
- \( \mu(X) = 1; \)

then \( \mu \) is a probabilistic measure on \( C \).

As was outlined above, quantum mechanics is using probability theory to explain and predict physical phenomena. But one could use possibility theory to give the same explanations and predictions in a more natural way. In particular, Bart Kosko [10], a prominent fuzzy set theorist, argued in favor of the superiority of fuzzy set theory when compared to probability theory by saying that fuzziness “measures the degree to which an event occurs, not whether it occurs. Randomness describes the uncertainty of event occurrence.” Thus if the particle lies between \( a \) and \( b \), we need to know how likely it is for the particle to be at \( a \leq c \leq b \) and not whether it is between \( a \) and \( b \). Possibility theory is based on possibility measures, which are are based on fuzzy sets [18].

A possibility measure \( \pi \) is different from a probability measure in that

\[
\pi \left( \bigcup_{i=1}^{\infty} E_i \right) = \sup_{i=1}^{\infty} \pi(E_i). \tag{4}
\]

In simple words, the difference between the two approaches is that in probability theory one demands that the sum of probabilities for given event should be 1 whereas in possibility theory there should be at least one plausible event (i.e., one whose possibility is 1). And this is clearly closer to what actually happens. A particle that lies between \( a \) and \( b \) is definitely somewhere between them.

Starting from some measure one can define a corresponding integral. For example, when using a probabilistic measure one may define the Lebesgue integral. Similarly, using a possibility measure one can define the Sugeno integral. Assume that \( (X, F) \) is measurable space, where \( X \) is some set and \( F \) is a \( \sigma \)-algebra,\(^2\) \( \mu : F \to [0, +\infty] \) is continuous monotone measure,\(^3\) and \( G \) is the class of all finite nonnegative measurable functions.\(^4\) For any \( f \in G \), \( F_\alpha = \{x \mid f(x) \geq \alpha\} \) and \( F_\alpha^+ = \{x \mid f(x) > \alpha\} \), where \( \alpha \in [0, +\infty] \). Suppose that \( A \in F \) and \( f \in G \). Then the Sugeno integral of \( f \) on \( A \) with respect to \( \mu \) is defined by

\[
\int_A f \, d\mu = \sup_{\alpha \in [0, +\infty]} \left( \alpha \wedge \mu(A \cap F_\alpha) \right), \tag{5}
\]

When \( A = X \), the Sugeno integral is also denoted by \( \int f \, d\mu \). This form of integration could be used instead of the Lebesgue integral in equation (3) to compute the possibility of finding the particle between \( a \) and \( b \), at time \( t \).

### 4 Vagueness in the Physical Reality

If vagueness is not just part of our everyday expression, then there should be vague objects. But are there such objects?\(^5\) I will not give a “yes” or “no” answer but instead I would like to ponder about

\(^2\)\( F \) has to be a subclass of the power set \( 2^X \). Also, it must satisfy the following conditions: (a) \( X \in F \); (b) for all \( E, F \in F \), \( E - F \in F \); and (c) for all \( E_i \in F \), \( i = 1, 2, \ldots, \bigcup_{i=1}^{\infty} E_i \in F \).

\(^3\)\( \mu \) is monotone if and only if \( E, F \in F \) and \( E \subseteq F \) imply \( \mu(E) \leq \mu(F) \).

\(^4\)A function \( f : X \to (-\infty, +\infty) \) on \( X \) is measurable if and only if \( f^{-1}(B) = \{x \mid f(x) \in B\} \in F \) for any Borel set \( B \in B \). Now, assume that \( X \) is the real line. Then, the class of all bounded, left closed, and right open intervals, denoted by \( B \), is the class of Borel sets.

\(^5\)A mathematical response to this question has been recently given by Giangiacomo Gerla [6].
the length of the UK coastline. The British Cartographic Society does not give an exact answer on their web page. Instead, they give this answer: The true answer is: it depends! It depends on the scale at which you measure it. Benoit Mandelbrot [14] gave exactly this answer in 1967. So in a sense it is not exactly known what is inside the UK and what is outside. And of course it is quite possible that some objects may lie somewhere in the middle. Thus one could say that the UK is actually a vague object since its boundaries are rigid. Similarly, clouds are vague objects for exactly the same reasons. On the other hand, there are objects that appear to be genuine vague objects (e.g., think of heaps of grain or men with few hair), still most of them are classified as such because the terms that describe them are vague. However, there is a third approach to the problem of finding vague objects in Nature. In quantum mechanics, the “standard” view is that elementary particles are indistinguishable, nevertheless, not everybody shares this view. More specifically, Edward Jonathan Lowe [12], has argued against this view thus showing that vagueness exists in the subatomic level:

Suppose (to keep matters simple) that in an ionization chamber a free electron \(a\) is captured by a certain atom to form a negative ion which, a short time later, reverts to a neutral state by releasing an electron \(b\). As I understand it, according to currently accepted quantum-mechanical principles there may simply be no objective fact of the matter as to whether or not \(a\) is identical with \(b\). It should be emphasized that what is being proposed here is not merely that we may well have no way of telling whether or not \(a\) and \(b\) are identical, which would imply only an epistemic indeterminacy. It is well known that the sort of indeterminacy presupposed by orthodox interpretations of quantum theory is more than merely epistemic—it is ontic. The key feature of the example is that in such an interaction electron \(a\) and other electrons in the outer shell of the relevant atom enter an ‘entangled’ or ‘superposed’ state in which the number of electrons present is determinate but the identity of any one of them with \(a\) is not, thus rendering likewise indeterminate the identity of \(a\) with the released electron \(b\).

The idea behind this example is that “identity statements represented by ‘\(a = b\)’ are ‘ontically’ indeterminate in the quantum mechanical context” [4]. In different words, in the quantum mechanical context \(a\) is equal to \(b\) to some degree, which is one of the fundamental ideas behind fuzzy set theory. For a thorough discussion of the problem of identity in physics see [5].

5 Superposition and Entanglement Revisited

The well-known Schrödinger’s cat paradox (see [7]) is about a cat that is placed inside a box along with a Geiger counter. The box contains a tiny amount of a radioactive substance whose atoms may or may not decay within an hour. If there is a decay, it triggers the Geiger counter which, in turn, triggers a hammer that breaks a glass that contains a poison capable to kill the cat. The obvious question is: What would happen to the cat after exactly one hour? At the end of the hour the wave function of the cat would be

\[
\psi = \frac{1}{\sqrt{2}} \psi_{\text{alive}} + \frac{1}{\sqrt{2}} \psi_{\text{dead}}.
\]  

(6)

This implies that the cat is neither dead nor alive! Schrödinger regarded this as patent nonsense, however, I tend to disagree. The reason of course is that there are many things that are not either black or white. After all, this is exactly the essence of vagueness. Thus, a patient who is in coma is not exactly alive and not exactly dead. Regardless of our objections, superposition, that is, the ability of particles to be in more than one state at the same time, is what makes quantum computing really interesting.

In “classical” computing a bit is either the digit 0 or the digit 1. In quantum computing a qubit is a quantum system (typically a polarized photon, a nuclear spin, etc.) in which the two digits are represented by two quantum states: \(|0\rangle\) and \(|1\rangle\). These states are represented by the following matrices:

\[
|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.
\]  

(7)

Also, these two states are “basic” states (i.e., they form a basis of a Hilbert space) and any other state of the qubit can be written as a superposition \(\alpha|0\rangle + \beta|1\rangle\), where \(\alpha\) and \(\beta\) are complex numbers that are called normalization factors and they must obey the normalization condition \(|\alpha|^2 + |\beta|^2 = 1\). For
example, consider a photon that can be polarized in the $x$ direction or in the $y$ direction and assume that these states are represented by the vectors $|\uparrow\rangle$ and $|\rightarrow\rangle$, respectively, then one can use $|\uparrow\rangle$ for $|0\rangle$ and $|\rightarrow\rangle$ for $|1\rangle$.

The standard interpretation of $\alpha|0\rangle + \beta|1\rangle$ is that a particle is in states $|0\rangle$ or $|1\rangle$ with probability that depends on $\alpha$ and $\beta$. Of course, according to a layman’s interpretation of probability theory, these two numbers express the change that a particle is in one of these states. A fuzzy theoretic interpretation of this state is that the particle is in both states but with some degree. In fact, one can define a fuzzy set as follows:

$$\Psi(|0\rangle) = |\alpha|^2$$
$$\Psi(|1\rangle) = |\beta|^2$$

However, here there is no reason to demand that $|\alpha|^2 + |\beta|^2 = 1$. In fact, there is no reason to impose any restriction other than $|\alpha|^2 \leq 1$ and $|\beta|^2 \leq 1$. One may argue that these two restrictions are not that different, however, the fuzzy theoretic approach assumes that the particle is in fact in a state that is partly $|0\rangle$ and partly $|1\rangle$. In different words, $\alpha|0\rangle + \beta|1\rangle$ is like a shade of gray, where, for instance, $|0\rangle$ is like black and $|1\rangle$ is like white.

Assume $\Psi$ describes the state of quantum particle in superposition. Then, the superposition collapses upon a measurement, but the question is why this happens. Perhaps, the measurement forces a defuzzification of $\Psi$, that is, a process by which one gets bivalent data from multivalued data (in this case a vague state is transformed into a crisp one). But if defuzzification is possible, then one might expect that fuzzification is also possible. Indeed, the Hadamard gate is a mechanism that creates “vague” states as follows:

$$H|0\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$$
$$H|1\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$$

Thus superposition corresponds to the fuzzification of a quantum system by means of the $H$ operator, while measurement is a “natural” defuzzification process.

Entanglement is another important quantum mechanical phenomenon. Consider a physical system with two degrees of freedom, $A$ and $B$. The states of such a system belong to $\mathcal{E} = \mathcal{E}_A \otimes \mathcal{E}_B$. Some states can be expressed as

$$|\Psi\rangle = |\alpha\rangle \otimes |\beta\rangle.$$ (12)

However, there are states that cannot be factorized (i.e., they cannot be written as “products”). Such states are called entangled states. For example, the following is such a state:

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\left(|\alpha_1\rangle \otimes |\beta_1\rangle + |\alpha_2\rangle \otimes |\beta_2\rangle\right).$$ (13)

First of all, there are two “special” forms of entanglement, namely entanglement of cost, $E_C$, and entanglement of distillation, $E_D$, that vague in a particular case (see [8] for details). More generally, Lowe [13] proposed a thought experiment that showed that entanglement is vague. Assume that there are two determinately distinct electrons. One of them (call it $a$) is determinately absorbed by an atom and then becomes entangled with a single electron (call it $a^*$) determinately already in the atom. Because these electrons exist in an entangled state inside the atom they are not determinately distinct but of course we know that there are two of them. At some moment one electron is emitted and so one electron is still inside the atom and one is outside the atom. Since these two electrons were in an entangled state, it is impossible to tell which electron left the atom. In a nutshell, this is the root of vagueness in entanglement.

Quantum computing is so attractive because it is harnessing both superposition and entanglement to achieve its exponential computational power. Since both superposition and entanglement are vague in their nature, this means that quantum computers operate on vague data using vague operations.
Conclusions

I have briefly explained why vagueness is not only a linguistic phenomenon but also a property of the physical world. Also, it is a fact that quantum computers harness quantum mechanical properties of matter to perform their computations. These properties of matter have been shown to be vague, thus quantum computers internally employ vagueness, which makes them automatically vague computers. Of course, these vague computers process non-vague data in a non-vague way, however, it would be really interesting to see if processing vague data vaguely would broaden our understanding of computation. This is certainly an open problem and I think a very interesting one.

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