1 Introduction

The aim of the present work is to derive rigorous estimates for turbulent MHD flow quantities such as the size and anisotropy of the dissipative scales, as well as the transition between 2D and 3D state. To this end, we calculate an upper bound for the attractor dimension of the motion equations, which indicates the number of modes present in the fully developed flow. This method has already been used successfully to derive such estimates for 2D and 3D hydrodynamic turbulence as a function of the $L_\infty$ norm of the dissipation, as in [5]. We tackle here the problem of a flow periodic in the 3 spatial directions (spatial period $2\pi L$), to which a permanent magnetic field is applied. In addition, the detailed study of the dissipation operator provides more indications about the structure of the flow.

In section 2, we review the tools of the dynamical system theory as well as the results they have led to in the case of 3D hydrodynamic turbulence. Section 3 is devoted to the study of the set of modes which minimises the trace of the operator associated to the total dissipation in MHD turbulence (viscous and Joule). Eventually, the estimates for the attractor dimension and dissipative scales in MHD turbulence under strong magnetic field are derived in section 4 and compared to results obtained from heuristic considerations.

2 The Navier-Stokes equation as a dynamical system

We shall first explain the interest of studying the dynamical system associated with the Navier-Stokes equations. The quantity we are mostly interested in is the set of functions which ”attracts” any initial flow, in the sense of the limit when the time $t$ tends to infinity. Indeed, the dimension of this so-called global attractor is known to be high for turbulent flows,
but finite under the assumption that the Navier-Stokes equations do not produce any finite time singularity \[5\]. Physically, this indicates that an established homogeneous turbulent flow includes a finite number of vortices, which therefore cannot be smaller than the ratio of the the volume of the physical domain by the number of modes, precisely given by the attractor dimension \(d_M\). Evaluating an upper bound for \(d_M\) is thus a way to derive a lower bound for the size of the dissipative scales. This will be our purpose from now on.

### 2.1 Dimension of the attractor associated to the Navier-Stokes equation

To calculate the attractor dimension of a dynamical system (defined by an evolution equation of the kind \(\frac{\partial}{\partial t} u = F(u)\), we consider a solution \(u\) located on the attractor and an arbitrary number \(n\) of small independent disturbances \(\delta u_i / i \in \{1..n\}\). Note that ”small” is relative to the norm defined in the phase space, which is a space of functions in the case of the Navier-Stokes system. The subset spanned by these \(n\) independent vectors evolves as to be located within the attractor at infinite time. Therefore, if \(N > d_M\), the \(n\)-dimensional volume of this subset, defined as

\[
V_n(t) = |\delta u_1 \times .. \times \delta u_n| \quad (1)
\]

tends to 0 when \(t\) tends to infinity. This latter property is expressed by Constantin and Foias theorem \[3\].

In the vicinity of the attractor, the evolution operator can be linearised as \(F(u) = Au + O(||\delta u||^2)\) so that \(V_n(t)\) varies exponentially in time:

\[
V_n(t) = V_n(0) \exp(t\langle \text{Tr}(A_n) \rangle) \quad (2)
\]

The subscript \(n\) stands for projection of operator into \(n\)-dimensional subsets of the phase space. If \(\text{Tr}(A_n)\) is positive for at least one choice of \(n\) disturbances, then \(n\) is an upper bound for the attractor’s dimension because at least one \(n\)-volume would expand (see (2)). We shall therefore look for the maximum trace of the evolution operator associated to the Navier-Stokes equations for any arbitrary integer \(n\).

If \(\sigma\) is the electrical conductivity, \(\rho\) is the density, \(\nu\) is the kinematic viscosity, the motion equations for velocity \(u\), pressure \(p\) electric current density \(j\) can be written:

\[
(\partial_t + u \cdot \nabla)u + \frac{1}{\rho} \nabla p = \nu \nabla^2 u + \frac{\sigma}{\rho \nu} j \times B + f \quad (3)
\]

\[
\nabla . u = 0 \quad (4)
\]
where $f$ represents some forcing independent of the velocity field. The set of Maxwell equations as well as electric current conservation and the Ohm’s law are normally required to close the system. However, we assume here that the magnetic field is not disturbed by the flow. In other words, the magnetic diffusion is supposed to take place instantaneously at the time scale of the flow (”low magnetic Reynolds number” approximation).

In the literature, the inertial terms are often written as a bilinear operator $B(u, \delta u)$, and the dissipation, as a linear operator that we call $D_{Ho}$. One can guess from this equation, that the evolution of small volume of the phase space generated by a set of $n$ disturbances (as defined in section previously) results from the competition between inertial terms which tend to expand the volume by vortex stretching and dissipative terms which tend to damp the disturbances, and hence reduce the volume.

2.2 The case of hydrodynamic turbulence ($B = 0$)

The case without magnetic field ($B = 0$) has been investigated in 2 and 3 dimensions. In 2D, [5] found an upper bound for the attractor dimension which matches well the results obtained by Kolmogorov-like arguments. To this day, no rigorous estimate for the attractor dimension of the 3D problem precisely matches Kolmogorov’s prediction for the number of degrees of freedom. One of the main reasons is that unlike in 2D, it has not yet been proved that the velocity gradients remain finite at finite time, which lets the door open to possible singularities. However, one can work under the assumption that the flow remains regular at finite time and define the maximum local energy dissipation rate as:

$$\epsilon_\infty = \nu \langle \sup_u \sup_r \| \nabla u(r, t) \|^2 \rangle_t$$

(5)

Here, $\sup_u$ stands for the upper bound over the set of solutions $u$ in the phase space, whereas $\sup_r$ stands for the upper bound over the physical domain. Under this strong assumption, and using a typical large scale $L$, which can be extracted from the eigenvalue of the laplacian of smallest module $-\lambda_1$, such that $L = \lambda_1^{-1/2}$ an upper bound for the trace of the operator $B(., u)$ on any $n$-dimensional subspace of the phase space is presented in [4]:

$$|\text{Tr}(B(., u))| < \frac{\nu}{L^2} \left( \frac{\epsilon_\infty L^4}{\nu^3} \right)^{1/2}$$

(6)

Also, studying the sequence of eigenvalues of the dissipation operator (which reduces to a Laplacian in the absence of magnetic field) on a finite physical domain with appropriate boundary conditions, gives access to the trace of the dissipation operator (see for instance [5]) and provides an upper bound for the trace of the total evolution operator, on any $n$-dimensional subspace.
of the phase space:

\[ \text{Tr}( (B(, u) + \nu \nabla^2) P_n ) \leq \nu \lambda_1 n \left( \frac{\epsilon_L L^4}{\nu^3} \right)^{1/2} - cn^2 \]  

(7)

where \( c \) is a real constant of the order of unity. One can be sure that when \( n \) is such that the r.h.s. of (7) is negative, all \( n \)-volumes shrink, hence \( n > d_{3D} \) where \( d_{3D} \) is the attractor’s dimension (this is Constantin and Foias theorem [3]). It then comes from (7) that:

\[ d_{3D} \leq c_3 \left( \frac{\epsilon_L L^4}{\nu^3} \right)^{3/4} \]  

(8)

The bound (8) apparently matches the Kolmogorov estimate of \( \left( \frac{\epsilon L^4}{\nu^3} \right) \). Unfortunately, the maximum local dissipation defined in (5) could be much higher than the average dissipation rate used in the Kolmogorov theory [1]. Note that a more recent attempt to find an upper bound for \( d_{3D} \) [6] using the average dissipation \( \epsilon \) has led to \( d_{3D} < \left( \frac{\epsilon L^4}{\nu^3} \right)^{24/5} \). We will however still use (8) throughout the rest of this work as this bound turns out to be relevant when a strong magnetic field is applied to the flow (see section 4). Note also that the discrepancy between analytical estimates and heuristic results is due to the difficulty in getting estimates for the norms of the velocity gradients, as well as to the fact that the bound given here does not rely on the existence of a power-law spectrum, which makes it also valid for flows with a low Reynolds number, unlike the K41 theory [1].

In order to derive an estimate for the attractor dimension in the MHD case, our main task now consists in finding the minimum of the trace of the dissipation operator on all \( n \)-dimensional subspace, for arbitrary values of \( n \).

3 Properties of the modes minimising the dissipation

We shall now look for the set of \( n \) modes that achieve the minimum dissipation for any value of \( n \) and exhibit a few important properties of these modes. The dissipation operator is compact and self-adjoint, so its trace expresses as the sum of its eigenvalues. The next step is now to solve the eigenvalue problem of the dissipation operator and to sort the eigenvalues in ascending order. The sum of the \( n \) first actually achieves the minimum of the trace over all \( n \)-dimensional subset of the phase space. Using non dimensional dissipation operator and wavenumbers (normalised respectively
by $\nu^L$ and $L^{-1}$, the three spatial component of the eigenvector appear to be of the form:

$$U(x, y, z) = \sin(k_x x) \sin(k_y y) \sin(k_z z)/(k_x, k_y, k_z) \in N^3(k_x, k_y, k_z) \neq (0, 0, 0)$$

(9)

The eigenvalue associated to the mode $(k_x, k_y, k_z)$ expresses its dissipation rate and writes:

$$\lambda(k_x, k_y, k_z) = -(k_x^2 + k_y^2 + k_z^2) - Ha^2 \frac{k_z^2}{k_x^2 + k_y^2 + k_z^2}$$

(10)

where the square of the Hartmann number $Ha^2 = L^2B^2\frac{\sigma}{\rho \nu}$ represents the ratio of Joule to viscous dissipation. The function $\lambda(k_x, k_y, k_z)$ is convex so that if $\lambda_m$ is the largest eigenvalue (corresponding to the $n^{th}$ mode), all modes associated to smaller eigenvalues are located inside the area delimited by the curve $\lambda(k_{\perp}, k_z) = \lambda_m$ in the $(k_{\perp}, k_z)$ plane, where $k_{\perp} = \sqrt{k_x^2 + k_y^2}$, as shown on figure 1. The knowledge of the iso-$\lambda_m$ curve also provides the maximum values of the modes in the direction of the magnetic field $k_{z,m}$ and in the orthogonal direction $k_{\perp,m}$, the ratio of which is an indication of the anisotropy of the small scales.

These features can be used to calculate the $n$ first modes and the associated trace of the dissipation as a function of $n$ and $Ha$. This is done in the general case, using an iterative algorithm implemented on a computer. The shape of the iso-$\lambda_m$ is determined by the ratio $\frac{n}{\pi \sigma^2}$ (see figure 1). Intuitively, it indicates the relative importance of forcing versus dissipation (as a higher inertia tends to generate more modes, and hence, increase the dimension of the attractor). We notice that the smaller this number, the more modes are concentrated outside of the a cone of axis $(Oz)$. This behaviour has been pointed out both experimentally [2] and theoretically [8] for real flows, for which a strong magnetic field is known to result in turbulent modes being confined outside the Joule cone. For dominating electro-magnetic effects, the Joule cone extends to the whole space except from the horizontal plane $(kx, ky)$ : the flow becomes two-dimensional. This also occurs in the eigenvalue problem where two-dimensional modes appear to be the less dissipative ones. This allows us to find out whether the set of $n$ eigenmodes is purely two-dimensional (i.e when all the modes satisfy $k_z = 0$).

In the case of a distribution of a high number of 3D modes ($n \gg 1$) located outside of the Joule cone ($Ha < -\lambda_m < Ha^{2}$). An analytical expression for the trace of the dissipation, as well as the Joule cone angle $\theta_m$ can be found, by replacing the sum over the $n$ eigenvalues by an integral [7]:

$$\text{Tr}(D_{Ha}P_n) = \frac{2\sqrt{2}}{3\pi} n^{3/2} Ha^{1/2}$$

(11)

$$\sin \theta_m = \sqrt{\frac{2}{\pi}} n^{1/4} Ha^{-3/4}$$

(12)
Figure 1: Iso-\(\lambda\) curves in the plane \((k_\perp, k_z)\). One can see the three major types of mode distribution: the 2d state corresponds to a set of modes located on the \(k_\perp\) axis, the strongly anisotropic 3d state exhibits the Joule cone-like shape (the bordure of the Joule cone has been plotted in the case where all the modes are inside the curve designated by the vertical arrow) and the quasi-isotropic state is reached when the modes are enclosed inside curves located the furthest away from the origin. Axis units are arbitrary.
The geometrical shape of the cardioid yields the maximum wavenumbers in the z-direction and in the orthogonal direction:

\[
k_{⊥m} = \sqrt{-\lambda_m} = \frac{9^{1/4}}{\pi^{1/2}} n^{1/4} H a^{1/4}
\]

and

\[
k_{zm} = \frac{\lambda_m}{2 H a} = \frac{1}{\pi \sqrt{2}} n^{1/2} H a^{-1/2}
\]

and the set of minimal modes is two dimensional if and only if \( n < 2 \pi^2 H a \).

The properties of the eigenmodes of the dissipation operator and those of the real flow exhibit some striking similarities. We shall exhibit more of them using the full result on the estimate for the attractor dimension.

4 Bounds on turbulent MHD flow quantities

4.1 Analytical estimates

We shall now derive an estimate for the attractor dimension of the Navier-Stokes equation on a periodical domain. To this end, we add (6) to the result of the numerical calculation of the trace in order to get an upper bound for the expansion rate of the Volume of any \( n \)-dimensional subset located in the vicinity of the attractor. We recall that the attractor dimension is the smallest value of the integer \( n \) for which this expansion rate is negative. The results are plotted on figure 2 in the general case. In the case \( n \gg 1 \) and \( H a < -\lambda_m < H a^2 \), abound for the trace of the evolution operator can be expressed analytically by summing (6) and (11) so that so that using equations (13-14) we get an analytical upper bound for the attractor dimension, as well as upper bounds for the maximum wavenumbers:

\[
d_M \leq \frac{9 \pi^2 Re^4}{32 H a}
\]

\[
k_{⊥m} \leq \frac{\sqrt{3}}{2} Re
\]

\[
k_{zm} \leq \frac{3 Re^2}{8 H a}
\]

4.2 Heuristics on MHD turbulence of Kolmogorov type under strong field

Now, it is worth underlining again that estimates (15) are exact results, and come exclusively from the mathematical properties of the Navier-Stokes equations, without the involvement of any physical approximation. There is therefore considerable interest in comparing them with orders of magnitude obtained from heuristic considerations. Let us recall how the smallest scales
Figure 2: Attractor dimension as a function of $Ha$. dotted: strong field approximation. solid: numerics.
can be obtained in a more intuitive manner: in a 3D periodic flow where Joule dissipation is stronger than viscosity except at small scales ($Ha \gg 1$), it is usual to consider that a vortex in the inertial range, of typical velocity $U_v$ and scales $k_{\perp}$ and $k_z$, results from a balance between inertial and Lorentz forces, which implies:

$$\frac{k_z}{k_{\perp}} \sim \left( \frac{\sigma B^2 L}{\rho k_{\perp} U_v} \right)^{-1/2}$$

(18)

Moreover, one usually assumes that anisotropy remains the same at all scales [2], over the inertial range. Under this assumption, (18) implies $U_v(k_{\perp}) = U_0 k_{\perp}^{-1}$, where $U_0$ stands for a typical large scale velocity. This is usually expressed in terms of the energy spectrum as:

$$E(k_{\perp}) \sim k_{\perp}^{-1} U_v^2(k_{\perp}) \sim U_0^2 k_{\perp}^{-3}$$

(19)

As mentioned in introduction the $k^{-3}$ spectrum is a strong feature of this type of MHD turbulence. Using the dissipation defined as $\epsilon \sim \nu k_{\perp}^2 U(k_{\perp})^2 \sim \nu U_0^2$, the large scale velocity expresses as $U_0 \sim \sqrt{3\epsilon}$, and (18) can then be written:

$$\frac{k_z}{k_{\perp}} \sim \left( \frac{\epsilon L^4}{\nu^3} \right)^{1/4} \frac{1}{Ha} = N^{-1/2}$$

(20)

where the ratio $N$ is the interaction parameter, which represents the ratio between Lorentz forces and inertia. Eventually, the small scales are heuristically defined as the smallest possible structures of the inertial range which are not destroyed by viscosity, which means that they result from a balance between inertia and viscosity. This yields:

$$\frac{k_{zm}}{k_{zm}^2} \sim Ha^{-1}.$$  

(21)

Now combining (20) and (21) yields:

$$k_{\perp,\text{max}} \sim \left( \frac{\epsilon L^4}{\nu^3} \right)^{1/4}$$

(22)

$$k_{z,\text{max}} \sim \left( \frac{\epsilon L^4}{\nu^3} \right)^{1/2} \frac{1}{Ha}$$

(23)

from which the number of degrees of freedom of the flow can be estimated by counting the number of vortices in the of size $L/k_{\perp} \times L/k_{\perp} \times L/k_z$ in a $L \times L \times L \times$ box:

$$N_f \sim k_{\perp}^2 k_z \sim \frac{1}{Ha} \left( \frac{\epsilon L^4}{\nu^3} \right) \sim d_M$$

(24)

This suggests that our estimate for $d_M$ is sharp and yields the right order of magnitude for the small scales. Note also that (12) and (15) yield $\sin \theta_m = \frac{\sqrt{3}}{\sqrt{2}} N^{-1/2}$ which matches the heuristic prediction of [8] for the Joule cone angle.
5 Conclusion

Though they are not solution of the motion equations, the eigenmodes of the dissipation operator exhibit some strong similarities with what is known from the real flow. As these properties are derived under the only assumption that the solutions of the Navier-Stokes equations are regular, this gives some strong support to the assumptions on which former heuristic results rely. However, the estimates obtained might be improved if the estimate for the inertial terms is improved. Indeed, the dissipation defined in (5) is generally higher than the average dissipation used to derive the heuristic value of the $d_M$ so the mathematical estimate found for $d_M$ is somewhat too high compared to the heuristics. These rigorous estimates are however very encouraging as they feature the same dependence on the Hartmann number as the heuristic results. This confirms that the set of minimal modes of the dissipation operator do render well the MHD properties of the actual flow. Lastly, it is worth mentioning that more physical behaviour such as boundary layer velocity profiles could be recovered by performing some similar study with classical wall boundary conditions on the planes orthogonal to the magnetic field.

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References

[1] Kolmogorov A, N. Local structure of turbulence in an incompressible fluid at very high reynolds numbers. *Dokl. Akad. Nauk. SSSR*, 30:299–303, 1941.

[2] A. Alemany, R. Moreau, P. Sulem, and U. Frish. Influence of an external magnetic field on homogeneous MHD turbulence. *Journal de Mécanique*, 18(2):277–313, 1979.

[3] P. Constantin, C. Foias, O.P. Mannley, and R. Temam. Attractors representing turbulent flows. *Mem. Am. Math. Soc.*, 53,314, 1985.

[4] P. Constantin, C. Foias, O.P. Mannley, and R. Temam. Determining modes and fractal dimension of turbulent flows. *J. Fluid. Mech.*, 150:427–440, 1985.

[5] C.R Doering and J. D. Gibbons. *Applied analysis of the Navier-Stokes equation*. Cambridge University Press, 1995.

[6] J.D. Gibbon and E.S. Titi. Attractor dimension and small scales estimates for the three dimensional navier-stokes equations. *non linearity*, 10:109–119, 1997.
[7] A. Pothérat and T. Alboussière. Small scales and anisotropy in low Rm MHD turbulence. *Phys. Fluids*, 15(10): 1370-1380, 2003.

[8] Joël Sommeria and René Moreau. Why, how and when, MHD turbulence becomes two-dimensionnal. *J. Fluid Mech.*, 118:507–518, 1982.