Anomalous Josephson effect in planar noncentrosymmetric superconducting devices

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In two-dimensional electron systems with broken inversion and time-reversal symmetries, a Josephson junction reveals an anomalous response: the supercurrent is nonzero even at zero phase difference between two superconductors. We consider details of this peculiar phenomenon in the planar double-barrier configurations of hybrid circuits, where the noncentrosymmetric normal region is described in terms of the paradigmatic Rashba model of spin-orbit coupling. We analyze this anomalous Josephson effect by means of both the Ginzburg-Landau formalism and the microscopic Green’s functions approach in the clean limit. The magnitude of the critical current is calculated for an arbitrary in-plane magnetic field orientation, and anomalous phase shifts in the Josephson current-phase relation are determined in terms of the parameters of the model in several limiting cases.

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I. INTRODUCTION

In Josephson junctions (JJ) of conventional s-wave superconductors, the supercurrent-phase relation \( j(\phi) \) is expected to obey rather general properties that depend neither on the junction’s geometry nor on the scattering processes taking place in the junction region, in other words, they apply to tunnel, ballistic, and diffusive junctions [1]. (i) The first basic property follows from the \( 2\pi \) periodicity of the superconducting order parameter, which implies that \( j(\phi) = j(\phi + 2\pi) \). (ii) The second property reflects the fact that changing the direction of the phase gradient applied across the junction reverses the direction of the superflow, \( j(\phi) = -j(-\phi) \), and therefore the supercurrent-phase relation is an odd function. (iii) The current must vanish at all integer phases modulo \( 2\pi \), namely, \( j(2\pi n) = 0 \) for \( n \in \mathbb{Z} \). This condition states an obvious thermodynamic requirement that a finite supercurrent is induced only by a nonzero phase gradient, so it vanishes for \( \phi = 0 \), and then by virtue of periodicity must vanish at other phases multiple of \( 2\pi \). (iv) The combination of the first two properties dictates that \( j(n\pi) = 0 \) for \( n \in \mathbb{Z} \); therefore it is sufficient to consider \( j(\phi) \) only in the interval \( 0 < \phi < \pi \). Additionally, it should be noted that in general, symmetries of the full Hamiltonian describing a Josephson junction, or their absence, can be related to the particular features in the pattern of the supercurrent-phase relation.

The anomalous Josephson effect (AJE), where the above-formulated properties of the supercurrent-phase relation are altered, can be realized in superconductors with broken time-reversal symmetry, leading to spontaneous currents. There are two kinds of systems where these effects have been discussed: (i) JJs between magnetic superconductors with unconventional pairing symmetry [2–5]; (ii) superconductor-ferromagnet-superconductor (SFS) junctions, and their more complex hybrids with additional noncollinear ferromagnetic layers and insulating barriers [6–11]. In particular, in the original work of Geshkenbein and Larkin [2] devoted to JJs based on heavy-fermion superconductors, the following current-phase relation was predicted:

\[
j(\phi) = j_1 \sin \phi + j_2 \cos \phi = j_c \sin(\phi + \phi_0), \tag{1}\]

where \( j_c = \sqrt{j_1^2 + j_2^2} \) is the critical current, and \( \phi_0 = \arctan(j_2/j_1) \) is the anomalous phase shift, whose microscopic form depends on the system under consideration and specific model assumptions. In general, the current-phase relation is not simply sinusoidal. Indeed, the contribution of higher-order harmonics may be nonnegligible, which is often the case at temperatures much lower than the critical. Therefore the generalized form of Eq. (1) can be presented as the Fourier series,

\[
j(\phi) = \sum_{n \geq 1} [j_{1n} \sin(n\phi) + j_{2n} \cos(n\phi)], \tag{2}\]

and contributions with \( j_{2n} \) are typically present as long as time-reversal symmetry is broken.

A different kind of AJE was proposed later in Refs. [12, 13]; see also important preceding works [14, 15]. The key insight of those works is that current-phase relation of the type Eq. (2) can be realized even in junctions of conventional superconductors when the normal layer between them is a noncentrosymmetric metal, i.e., with broken inversion symmetry. As a guiding example, calculations were presented for a weak link with Rashba-type spin-orbit coupling [16], and Eq. (1) was derived microscopically in the quasi-one-dimensional geometry. To

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the situation where the time reversal symmetry (TRS) in conventional JJs, the term “\(\phi_0\)-junction” was introduced [12].

In recent years we received compelling experimental verification of these anomalous Josephson phenomena in various heterojunctions [17–25]. These devices represent a diverse class of systems that differ from each other by material components, dimensionality, quality of contacts, and purity of interlayers between superconducting banks, thus reflecting the prevalence and robustness of the aforementioned effects. Theoretical studies address a broad spectrum of questions related but not limited to (i) types of spin-orbit interaction, including spin-active interfaces; (ii) effects of impurities; and (iii) electronic band structure, in particular topological properties. There are a number of notable theoretical contributions to this topic, and we can highlight studies of the AJE in quantum dots [26, 27], semiconducting nanowires [28–34], and topological [35, 42] and nontopological systems [43, 44] that involve a combination of unconventional superconductors, topological surface or edge states, and ferromagnets.

The continuous improvement in the quality of materials, where the electron mean free path is comparable to or even may exceed the dimensions of the junction, call for the investigation of AJE in the clean limit, which thus far has received very limited theoretical attention. This task is accomplished in the present work, and the rest of the paper is organized as follows. In Sec. II we apply Ginzburg-Landau (GL) phenomenology to address the anomalous Josephson effect in the two-dimensional electron gas (2DEG) with Rashba-type spin-orbit coupling. Even though the GL formalism has its limitations, it gives us an opportunity to fully analytically investigate the field dependence of the critical current and the phase shift in the two-dimensional geometry. The principal results of this work are presented in Sec. III, where we develop a microscopic theory of the AJE based on the Gor’kov equations for two complementary junction geometries. This analysis expands previous considerations of the AJE that exploited semiclassical approximations for ballistic (Eilenberger limit) and diffusive (Usadel limit) systems. In Sec. IV we provide summary of our findings in comparison to earlier related works.

II. GINZBURG-LANDAU FORMALISM

To elucidate the unusual properties of the AJE, we start with the simple Ginzburg-Landau (GL) model before delving into the microscopic calculation. The geometry that we consider is depicted schematically in Fig. 1 in which a normal region of two-dimensional electron gas (2DEG) of length \(L\) and width \(d\) is flanked by two conventional s-wave superconducting banks. Let us consider the situation where the time reversal symmetry (TRS) in the system is broken by an in-plane magnetic field \(\mathbf{h}\) and the space inversion symmetry in the normal region is broken by the presence of a Rashba-type spin-orbit coupling (SOC) term [10] given by \(\alpha[\sigma \times \mathbf{p}] \cdot \mathbf{n}\). Here \(\sigma\) is the Pauli spin matrix-vector, \(\mathbf{p}\) is the particle momentum, \(\mathbf{n}\) is the unit vector along the direction of the asymmetric potential gradient, and the parameter \(\alpha\) denotes the strength of the spin-orbit interaction, which has units of velocity.

In the presence of SOC, the GL free energy \(\Omega\) was derived by Edelstein [58] (see also related works Refs. [59, 62]):

\[
\Omega(\psi, \psi^*) = \int d\mathbf{r} \left[ a|\psi|^2 + b\frac{1}{2}|\psi|^4 + \frac{1}{4m}|\partial \psi|^2 - \epsilon (\mathbf{n} \times \mathbf{h}) \cdot (\psi^*(\partial \psi)^* + \psi^*(\partial \psi)) + \frac{\hbar^2}{8\pi} \right],
\]

where \(\psi(\mathbf{r})\) is the spatially inhomogeneous superconducting order parameter, \(\mathbf{h} = \nabla \times \mathbf{A}\) where \(\mathbf{A}\) is the vector potential, and \(\partial = -i\nabla - 2eA\) is the gauge invariant derivative. Here and in what follows, we work in the units \(\hbar = eB = c = 1\). This form of the GL functional applies to both clean and disordered superconductors. The difference is in the dependence of the expansion coefficients \(a, b, \epsilon\), and also the gradient term, on the strength of spin-orbit \(\alpha\), critical temperature \(T_c\) of a superconductor, and elastic scattering time \(\tau\) induced by disorder potential. The conventional part of the GL functional, namely, the first three terms in Eq. (3), weakly depends on the SOC. In contrast, the coefficient \(\epsilon = \alpha^2{v_F \over 2p_F} \int d\tau (2\pi^2 T_c^2 \tau)\), with \(v_F\) and \(p_F\) being the Fermi velocity and the Fermi momentum, respectively, depends sensitively on \(\alpha\). The asymptotic form of the function \(f_{\alpha}\) is established for two- and three-dimensional superconductors in various limiting cases, see Refs. [68, 69, 72] for details. Both parameters \(\alpha p_F^2 T_c\) and \(T_c \tau\) can be of the order of unity in materials.

The free energy functional in Eq. (3) must be minimized with respect to the order parameter \(\psi^*\) and the
vector potential $A$ to get the equilibrium GL equations. Therefore, varying $\psi^*$ with respect to $\psi^*$ and $A$ and setting that variation equal to zero for arbitrary variations $\delta \psi^*$ and $\delta A$, the two GL equations are obtained in the form
\[
\frac{1}{4m} \partial^2 \psi(r) - \frac{1}{2m} Q \cdot \partial \psi(r) + a \psi(r) + b |\psi(r)|^2 \psi(r) = 0,
\]
with the boundary condition,
\[
[(\partial - Q)\psi] \cdot n_b = 0.
\]

The unit vector $n_b$ is the normal vector at the system boundary. The wave vector $Q = 4m e [n \times h]$ represents an emergent scale for this GL theory with Rashba SOC and Zeeman field. Its presence induces a spatially modulated helical superconducting phase given by $\psi \propto e^{iQr}$.

We recall that the boundary conditions [Eq. (6)] on these equations are obtained from the condition that the surface integrals in the variation $\delta \Omega$ are zero. As a result of this condition, the normal component of the supercurrent density $j$ at the boundary of the superconductor with vacuum is $j \cdot n_b = 0$.

This framework was used in the original work $[12]$ to derive the anomalous Josephson current in a quasi-one-dimensional geometry with rigid boundary conditions. Below we extend these results to a full two-dimensional geometry with an arbitrarily oriented field in the plane of the 2DEG and also with more general boundary conditions:
\[
[(\partial - Q)\psi] \cdot n_b = \frac{1}{l_t} \partial \psi.
\]

Here $l_t$ is the extrapolation length to the point outside the boundary at which the order parameter $\psi$ would vanish if it maintained the slope it had at the surface $[13]$. The value of $l_t$ depends on the nature of the material to which the interface is made, approaching zero for a magnetic material or in the case of a high density of defects in the interface (Dirichlet boundary condition), and infinity for an insulator or vacuum (Neumann boundary condition), with normal metals lying in between.

To calculate the Josephson current in the geometry of Fig. 1, we can neglect the orbital effect of the field. We also neglect the nonlinear term proportional to $b$ in Eq. (4). The solution for $\psi(x, y)$ can be written as the series
\[
\psi = e^{iQx} \sum_n \left( A_n e^{x/\xi_n} + B_n e^{-x/\xi_n} \right) \cos[q_n(y + d/2)]
\]
where
\[
q_n = \pi n / d, \quad \xi_n^{-1} = \sqrt{4ma - Q^2 + q_n^2}.
\]

with the boundary condition $\psi \big|_{y = \pm \frac{d}{2}} = 0$. This can be simplified to get the form of the anomalous Josephson current in a full two-dimensional geometry as a function of the Zeeman field, plotted as a function of the Zeeman field, $Q_y$, for several different values of the sample width $d$, which is measured in the units of the superconducting coherence length $\xi$ and for a fixed ratio $L/\xi = 5$.

\[
\frac{j_0(Q_y)}{j_0(0)} = \frac{\sqrt{Q_x^2 + Q_y^2}}{Q_y} = 4m e \sqrt{h_x^2 + h_y^2} = 4m e h_y. \quad \text{The expansion coefficients $A_n$ and $B_n$ are easily calculated by using appropriate boundary conditions from Eq. (7) for the two SN interfaces,}
\]
\[
\left( \frac{\partial \psi}{\partial x} - iQ_x \psi \right) \big|_{y = \pm \frac{d}{2}} = \pm \frac{1}{l_t} |\psi_0| e^{\mp i \frac{Q_x}{2}},
\]

where, in the superconducting banks to the left and right of the normal region, $\psi(x < -d/2, y) = |\psi_0| e^{+i \frac{Q_x}{2}}$ and $\psi(x > d/2, y) = |\psi_0| e^{-i \frac{Q_x}{2}}$. In these solution we require $4ma > Q^2$, because the smallest value of $|n| = 0$. This is equivalent to the condition $Q \xi < 1$, which means that the length scale characterized by the inverse of the wave function $Q \xi$ is greater than the superconducting coherence length $\xi$. Equation (13) can now be used to find the current density:
\[
j_x(y) = \frac{e}{2mi} \left( \psi^* \frac{\partial \psi}{\partial x} - \psi \frac{\partial \psi^*}{\partial x} \right) - \frac{eQ_x}{m} |\psi|^2.
\]

Since the current density is inhomogeneous, we are interested in a current density averaged over the sample width $d$:
\[
j(\phi) = \frac{1}{d} \int_{-d/2}^{+d/2} j_x(y) dy = \frac{e}{2mi} \sum_{n \in Z} \left( \frac{2}{\xi_n} \left( B_n^* A_n - A_n^* B_n \right) \right).
\]

This can be simplified to get the form of the anomalous Josephson effect in the $\phi_0$ junction
\[
j(\phi) = j_c(Q_y) \sin(\phi + \phi_0).
\]

In this model, the critical current density $j_c$ and the
anomalous phase shift $\phi_0$ are given by

$$\frac{j_c(Q_y)}{j_c(0)} = \frac{\sin^2(Q_yd/2)}{(Q_yd)^2} + \sum_{n \geq 1} \frac{4(Q_yd)^2 [1 - (-1)^n \cos(Q_yd)]}{(Q_yd^2 - n^2\pi^2)^2} \xi_n \sinh(L/\xi_0) \xi_0 \sinh(L/\xi_n)$$

(14)

and

$$\phi_0 = Q_xL = 4\pi Eh_yL,$$

(15)

where $\xi_n$ is given by Eq. (9). From Eqs. (14) and (15) we note that the Zeeman field $h$ has dual effect: parallel to the SN boundary component, $Q_x \propto h_y$, defines the anomalous phase shift $\phi_0$, while the perpendicular to the SN interface component, $Q_y \propto h_x$, governs the current density $j_c$ modulation, see Fig. 2 for the illustration. These results can be further generalized to superconducting leads with Rashba coupling and Zeeman field. In that case, the vector $Q$ in the final expressions should be replaced by the difference between the corresponding vectors $Q$ in the superconducting and normal parts.

The anomalous phase shift matches with the earlier result of Ref. [12]. The critical current can be also recovered if we single out the $n = 0$ contribution from the sum over $n \in Z$ in Eq. (14), take the $d \rightarrow 0$ limit and consider the rigid boundary conditions, i.e.,

$$\psi(x = -\frac{L}{2}, y) = \psi_0 e^{+i\frac{\pi}{2}}$$

and

$$\psi(x = \frac{L}{2}, y) = \psi_0 e^{-i\frac{\pi}{2}}.$$

III. MICROSCOPIC APPROACH

In the context of various possible Josephson microconstrictions, the example considered in the previous section corresponds to the case of SINIS junctions, where “T” denotes an insulating tunnel barrier. The microscopic calculations of supercurrent-phase relations in such devices were originally carried out in several important works. In Ref. [64] a tunneling Hamiltonian was used and the critical current was calculated for the diffusive limit of transport. In Ref. [65] boundary conditions were derived in the semiclassical limit, and applications to the Josephson effect were given based on the solution of Usadel equations. In Ref. [66], the general solution for ballistic electronic transport through double-barrier junctions was elaborated in the phase-coherent limit (see also review Ref. [1] and additional references therein that expand on the topic). Following these works, we consider below planar SINIS junctions in a particular model of extended tunnel barriers with the focus on the anomalous phase shifts.

A. Model Hamiltonian

The total Hamiltonian for the SINIS system under consideration consists of three main parts:

$$\hat{H} = \hat{H}_S + \hat{H}_N + \hat{V}_T.$$

(16)

Here $\hat{H}_S = \hat{H}_L + \hat{H}_R$ represents left-(L) and right-(R) conventional s-wave superconducting electrodes described within the BCS model with equal pairing gaps $\Delta_L = |\Delta| e^{i\phi_L}$ and $\Delta_R = |\Delta| e^{i\phi_R}$, and overall phase difference $\phi = \phi_L - \phi_R$.

The Hamiltonian of the normal layer, $\hat{H}_N$, describes a two-dimensional metal in the presence of a Zeeman field and with a Rashba term:

$$\hat{H}_N = \sum_{\sigma \sigma'} \int dr \Psi_{N\sigma}^\dagger (r) \hat{h}_{\sigma\sigma'} \Psi_{N\sigma'}^\dagger (r),$$

(17)

with

$$\hat{h}_{\sigma\sigma'} = \frac{\hbar^2}{2m} \delta_{\sigma\sigma'} + \alpha \hat{p} \times \hat{n} \cdot \sigma + h \cdot \sigma.$$  

(18)

This is exactly the same model as used in Sec. II that led to an effective GL-free energy in Eq. (3). In accordance with the usual convention, $\Psi_{N\sigma}$ and $\Psi_{N\sigma'}$ in Eq. (17) describe electron creation and annihilation field operators, and $\sigma$ is the spin index. For the geometry of Fig. 1 with the $z$ axis normal to the plane of the metal, the unit vector $\hat{n}$ has only a $z$ component. Therefore if we consider a Zeeman field in the $xy$ plane, $h = (h \cos \varphi, h \sin \varphi, 0)$, the single-particle operator $\hat{h}$ in Eq. (18) can be written as the following matrix:

$$\hat{h} = \left( \begin{array}{cc} \frac{\hbar^2}{2m} + \frac{\hbar^2}{2m} & \alpha \hat{p}_y + i \alpha \hat{p}_x + h e^{-i\varphi} \\ \alpha \hat{p}_y - i \alpha \hat{p}_x + h e^{+i\varphi} & \frac{\hbar^2}{2m} + \frac{\hbar^2}{2m} \end{array} \right).$$

(19)

The coupling between the electrodes and the normal region is described by the tunneling part of the Hamiltonian. For the model of an extended tunnel junction that is translationally invariant along $y$ we have

$$\hat{V}_T = \hat{X}_{LN} + \hat{X}_{NR} + \hat{X}_{LN}^\dagger + \hat{X}_{NR}^\dagger$$

(20)

with [67]

$$\hat{X}_{LN} = t \sum_{\sigma} \int dr_{b1} \frac{\partial}{\partial r_n} \Psi_{L\sigma}^\dagger (r_{b1}) \frac{\partial}{\partial r_n} \Psi_{N\sigma} (r_{b1}),$$

(21a)

$$\hat{X}_{NR} = t \sum_{\sigma} \int dr_{b2} \frac{\partial}{\partial r_n} \Psi_{N\sigma}^\dagger (r_{b2}) \frac{\partial}{\partial r_n} \Psi_{R\sigma} (r_{b2}).$$

(21b)

The integrals are taken along the junction interfaces, and $r_n$ is normal to the boundary. The normal derivatives have to be understood as being taken from the side.
where the corresponding function is defined. In momentum space, tunnel matrix elements have the form $t_{kk'} \propto t_{k2k_2'} \delta(k_y - k'_y)$. In the normal state, the tunnel conductance is given by $g = \frac{32}{3\pi} e^2 t^2 \mu B S_d$, where $S_d$ is the total area of the interface. A similar model without the normal derivatives would imply longitudinal momentum independent tunneling matrix elements. The rationale for the extended model is discussed extensively in Ref. 67.

In the calculations below, we work in perturbation theory with respect to the Zeeman field and strength of spin-orbit interaction as compared to the Fermi energy, namely, $\{h, \alpha p_F\} \ll \varepsilon_F$. In particular, this enables us to neglect the effect of the field on the suppression of the order parameter in the leads. For the hierarchy of relevant length scales we explore various relations between the thermal length $l_T = v_F/T$, the superconducting coherence length $\xi$, the spin-orbit length $l_{so} = 1/\alpha m a$, and the distance $L$ between the tunnel contacts.

B. Josephson current

The operator of the current flowing from $L$ to $R$ is given by a commutator through the equation of motion,

$$I_{L \rightarrow R} = e \tilde{N}_L = ie \{ \hat{V}_T, \hat{N}_L \},$$

(22)

where $\tilde{N}_L$ is the operator of the electron number in the left lead. Therefore, $\tilde{I}_{L \rightarrow R} = -ie \left( \hat{X}_{LN} - \hat{X}_{LN}^\dagger \right)$, and consequently, the thermal average for the expectation value of the tunneling current can be written as follows:

$$\langle \tilde{I}_{L \rightarrow R} \rangle = I_{L \rightarrow R} = 2e \text{Im} \langle \hat{X}_{LN} \rangle.$$

(23)

To calculate this average, we use the interaction picture representation

$$I_{L \rightarrow R} = 2e \text{Im} \left[ \frac{\text{Tr} \left( \hat{X}_{LN} e^{-\beta \left( \hat{H}_0 - \mu \hat{N} \right)} \hat{U}_t(\beta) \right)}{\text{Tr} \left( e^{-\beta \left( \hat{H}_0 - \mu \hat{N} \right)} \hat{U}_t(\beta) \right)} \right],$$

(24)

where $\hat{H}_0$ denotes the system Hamiltonian without the tunneling part, and $\hat{U}_t(\beta) = \text{e}^{\frac{1}{\beta} \int_0^\beta \hat{V}_T(\tau) d\tau}$, where $T_\tau$ denotes time-ordering in imaginary time.

In the expression of Eq. (24), the first nonvanishing contribution to the current appears in fourth order in the tunneling matrix element $t$. We thus expand the exponential in powers of $\hat{V}_T$ and determine that the following combinations of $\hat{X}$'s and $\hat{X}^\dagger$'s contribute to the current:

$$I_{L \rightarrow R} \equiv I = -e \text{Im} \left[ \int_0^\beta d\tau_1 d\tau_2 d\tau_3 \right.$$

$$\left. \langle T_\tau \hat{X}_{NR}(\tau_1) \hat{X}_{NR}^\dagger(\tau_2) \hat{X}_{LN}(\tau_3) \hat{X}_{LN}^\dagger(0) \rangle \right].$$

(25)

At this point we apply Wick's theorem to construct field operators and express them in terms of the normal and anomalous Gor'kov Green's functions $[65, 66]$: $G_{\sigma \sigma'}(r, r', \tau - \tau') = -\left( T_\tau \Psi_\sigma(r, \tau) \Psi_\sigma^\dagger(r', \tau') \right)$, $F_{\sigma \sigma'}(r, r', \tau - \tau') = \left( T_\tau \Psi_\sigma(r, \tau) \Psi_\sigma(r', \tau') \right)$, and $F_{\sigma \sigma'}^\dagger(\tau, \tau') = \left( T_\tau \Psi_\sigma^\dagger(\tau, r) \Psi_\sigma(r, \tau') \right)$. Further, passing to the Fourier transform in terms of Matsubara frequencies we obtain

$$I = -2e t^4 \text{Im} \left[ T \sum \int \int dr_{b1} dr_{b2} dr_{b3} dr_{b4} dr_{b5} dr_{b6} \right.$$

$$\left. \frac{\partial^2 F_{-\sigma_1, \sigma_2}(r_{b1}, r_{b2}, i\omega_n)}{\partial x_{b1}^2 \partial x_{b2}^2} \right.$$

$$\left. \frac{\partial^2 F_{-\sigma_1, -\sigma_2}(r_{b3}, r_{b4}, -i\omega_n)}{\partial x_{b3}^2 \partial x_{b4}^2} \right] \text{Im} \left[ \int d(r_{b5} - r_{b6}^\dagger) e^{-ik_b(r_{b5} - r_{b6}^\dagger)} G(r, r', i\omega_n) \right].$$

(26)

This expression can be further simplified in our geometry. First, we introduce a more condensed notation for the Green’s functions, i.e., $F(r, r', \tau) = -F_{-\uparrow, \uparrow}(r, r', \tau) = F_{-\downarrow, \downarrow}(r, r', \tau)$, and similarly $F_{-\uparrow, \downarrow}(r, r', \tau) = -F_{-\downarrow, \uparrow}(r, r', \tau) = F_{-\downarrow, \downarrow}(r, r', \tau)$. Second, we take a partial momentum Fourier transform along the direction parallel to the surface of the tunnel boundary. For example, with $r_b = (x, r^\Vert)$ and $r_b^\dagger = (x', r'^\Vert)$,

$$G(x, x', k^\Vert, i\omega_n) = \int d(r_{b5} - r_{b6}^\dagger) e^{-i\mathbf{k}_b \cdot (r_{b5} - r_{b6}^\dagger)} G(r, r', i\omega_n).$$

(27)

As a result, in the mixed position-momentum representation, Eq. (26) for the current density $j(\phi) = I/S_d$ through the interface with the area $S_d$ reduces to

$$j(\phi) = -2e t^4 \text{Im} \left[ T \sum \sum_{\mathbf{r}_b} \int \mathbf{k}_1 P \left( -\frac{L}{2}, \frac{L}{2}, \mathbf{k}_1, \mathbf{i} \omega_n \right) \right.$$

$$\left. \times \frac{\partial^2 F_{-\uparrow, \uparrow}(\mathbf{r}_b, \mathbf{r}_b^\dagger, \mathbf{i} \omega_n)}{\partial x_{\mathbf{r}_b} \partial x_{\mathbf{r}_b}^\dagger} \right] \text{Im} \left[ \int d(r_{b5} - r_{b6}^\dagger) e^{-i\mathbf{k}_b \cdot (r_{b5} - r_{b6}^\dagger)} G(r, r', \mathbf{i} \omega_n) \right].$$

(28)

where the kernel function under the integral is defined by

$$P = \sum_{\sigma} \left[ \frac{\partial^2 G_{+ \sigma, -\sigma}^\dagger}{\partial x_{\mathbf{r}_b} \partial x_{\mathbf{r}_b}^\dagger} - \frac{\partial^2 G_{+ \sigma, -\sigma}^\dagger}{\partial x_{\mathbf{r}_b} \partial x_{\mathbf{r}_b}^\dagger} \frac{\partial^2 G_{- \sigma, -\sigma}^\dagger}{\partial x_{\mathbf{r}_b} \partial x_{\mathbf{r}_b}^\dagger} \right].$$

(29)

Here we used a short-hand notation $G_{\sigma \sigma'}^\pm = G_{\sigma \sigma'}(-\frac{L}{2}, \frac{L}{2}, \pm \mathbf{k}_1, \pm \mathbf{i} \omega_n)$. We note that a similar formula for the supercurrent appeared in Ref. 56. The differences are (i) the model for tunnel barriers, and (ii) effects of SOC were then studied numerically. Equation (28) has a transparent diagrammatic representation, which is depicted in Fig. 3. Therefore the problem of finding the current-phase relation and an anomalous phase shift is reduced to deriving $P \left( -\frac{L}{2}, \frac{L}{2}, \mathbf{k}_1, \mathbf{i} \omega_n \right)$. This defines our next task, which is the determination of the Green’s function in the normal region.
For the finite-size system with \( x \), each term satisfies Eq. (30) at the matrix differential equation,

\[
\begin{pmatrix}
  i\omega_n - \varepsilon_y + \frac{1}{2m} \frac{\partial^2}{\partial x^2} & -\alpha k_y - \frac{\partial}{\partial x} - he^{-i\phi} \\
-\alpha k_y + \frac{\partial}{\partial x} - he^{i\phi} & \omega_n - \varepsilon_y + \frac{1}{2m} \frac{\partial^2}{\partial x^2}
\end{pmatrix} \hat{G} = \mathbf{I} \delta(x - x'),
\]

(30)

where \( \varepsilon_y = k_y^2/2m - \varepsilon_F \), \( \mathbf{I} \) is the \( 2 \times 2 \) identity matrix, and the \( 2 \times 2 \) Green’s function matrix \( \hat{G} \) in the spin space is given by

\[
\hat{G} = \begin{pmatrix} G_{\uparrow\uparrow}(x, x', k_y, \omega_n) & G_{\uparrow\downarrow}(x, x', k_y, \omega_n) \\ G_{\downarrow\uparrow}(x, x', k_y, \omega_n) & G_{\downarrow\downarrow}(x, x', k_y, \omega_n) \end{pmatrix}.
\]

(31)

The rigid boundary conditions are of the form

\[
G_{\sigma\sigma'} \left( \pm \frac{L}{2}, x', k_y, \omega_n \right) = G_{\sigma\sigma'} \left( \pm \frac{L}{2}, k_y, \omega_n \right) = 0.
\]

(32)

For the finite-size system with \( \frac{L}{2} < x < \frac{L}{2} \), the general solution can be written down as a series expansion in eigenfunctions

\[
G(x, x', k_y, \omega_n) = \sum_{\lambda, \lambda'} C_{\lambda, \lambda'} e^{\lambda(x+L/2) - \lambda'(x'-L/2)} \quad \text{when} \quad x < x'
\]

\[
\sum_{\lambda, \lambda'} C_{\lambda, \lambda'} e^{\lambda(x-L/2) - \lambda'(x'+L/2)} \quad \text{when} \quad x > x'
\]

(33)

Here each term satisfies Eq. (30) at \( x \neq x' \), when its right-hand side is zero. Therefore, \( \lambda \) and \( \lambda' \) are the solutions of the characteristic equation,

\[
\det \hat{\Xi} = 0
\]

(34)

C. Green’s functions

For the planar 2D geometry, with the identification of \( k_x \rightarrow k_y \), the Green’s function in the normal layer satisfies the matrix differential equation,

\[
\begin{pmatrix}
  (i\omega_n - \varepsilon_y + \frac{1}{2m} \frac{\partial^2}{\partial x^2}) - \alpha k_y - \frac{\partial}{\partial x} - he^{-i\phi} \\
-\alpha k_y + \frac{\partial}{\partial x} - he^{i\phi} - (\omega_n - \varepsilon_y + \frac{1}{2m} \frac{\partial^2}{\partial x^2})
\end{pmatrix} \hat{G} = \mathbf{I} \delta(x - x'),
\]

(30)

The rigid boundary conditions are of the form

\[
\begin{pmatrix} G_{\uparrow\uparrow}(x, x', k_y, \omega_n) & G_{\uparrow\downarrow}(x, x', k_y, \omega_n) \\ G_{\downarrow\uparrow}(x, x', k_y, \omega_n) & G_{\downarrow\downarrow}(x, x', k_y, \omega_n) \end{pmatrix} \mathbf{I} = 0.
\]

(31)

The general solution can be written down as a series expansion in eigenfunctions

\[
G(x, x', k_y, \omega_n) = \sum_{\lambda, \lambda'} C_{\lambda, \lambda'} e^{\lambda(x+L/2) - \lambda'(x'-L/2)} \quad \text{when} \quad x < x'
\]

\[
\sum_{\lambda, \lambda'} C_{\lambda, \lambda'} e^{\lambda(x-L/2) - \lambda'(x'+L/2)} \quad \text{when} \quad x > x'
\]

(33)

Here each term satisfies Eq. (30) at \( x \neq x' \), when its right-hand side is zero. Therefore, \( \lambda \) and \( \lambda' \) are the solutions of the characteristic equation,

\[
\det \hat{\Xi} = 0
\]

(34)
to the situation of a weak SOC and a high temperature, when $l_{so} \gg L \gg l_T$. An integration of the analytical expressions in Eq. (28) leads to the supercurrent density of the form

$$ j(\phi) = j_0 \left[ \sin(\phi) \cos \left( \frac{2\hbar_y L}{v_F} \right) + \frac{\alpha}{v_F} \cos(\phi) \sin \left( \frac{2\hbar_y L}{v_F} \right) \right], $$

(40)

which can be rewritten as Eq. 1 with the phase shift

$$ \phi_0 = \arctan \left[ \frac{\alpha}{v_F} \tan \left( \frac{2\hbar_y L}{v_F} \right) \right] \approx \frac{2\alpha \hbar_y L}{v_F^2}, $$

(41)

provided that $\alpha \ll v_F$ and $\hbar_y \ll v_F/L$. The amplitude of the current scales as $j_0 \propto \sqrt{\frac{v_F}{L}} e^{-L/\Delta}$, and such an exponential falloff is characteristic for the high-temperature regime. At lower temperatures, $T \ll v_F/L \ll \Delta$, we find parametrically the same phase shift, which differs from the expression above only by a numerical factor. However, the temperature dependence of the critical current changes significantly and scales logarithmically with the system size $j_0 \propto \ln(v_F/LT)$. This result, however, does not hold in the limit of $T \to 0$. The reason is that our oversimplified treatment of the tunneling at SIN boundaries misses the proximity-induced spectral gap $E_g(\phi)$ in the normal region. When temperature becomes smaller than the so-called minigap, $T < E_g$, the logarithmic dependence is cut off. Furthermore, as the gap depends on the phase across the junction, $E_g \propto \cos(\phi/2)$, this leads to a nonsinusoidal skewed current-phase relation near the gap closure $\phi \sim \pi$, namely, $j_0 \propto \sin(\phi) \ln \left[ \frac{v_F/L}{E_g(0)} \cos(\phi/2) \right]$, see Refs. 69-71. These considerations suggest that unlike the critical current, phase shift has weak temperature dependence, although due to the limitations of the model we can’t access the temperature regime below the minigap to make a definitive statement in that limit.

Earlier works 54, 55 demonstrated that a linear scaling of $\phi_0$ with the strength of the spin-orbit coupling and system size at weak fields is not robust. It was shown that $\phi_0$ is extremely sensitive to the boundary conditions in comparison to the sensitivity to the type of scattering in the bulk of the normal layer. Indeed, for transparent interfaces one finds $\phi_0 \propto L^2$. This lead us to consider an alternative version of the model that in-part mimics an extended (rather than rigid) system. For that purpose we took an SINIS junction in which the size of the normal layer in $x$ direction is much larger than the distance $L$ between the two tunnel barriers, i.e., the superconducting leads and the normal parts are in different planes, and the tunneling occurs between those planes, see Fig. 4 for the illustration. In this setting, the Green’s function is translationally invariant, $G_{\sigma,\sigma'}(x - x', \mathbf{k}_\parallel, i\omega_n)$, which simplifies analytical calculations. The expression for the current can be derived for this geometry, and it takes a form similar to Eq. (28), albeit with the different kernel function under the integral, namely,

$$ j(\phi) \propto \Im \left[ T \sum_{i\omega_n} \int_{k_y} \mathcal{P} \left( -\frac{L}{2}, \frac{L}{2}, k_y, i\omega_n \right) \right. $$

$$ \times \mathcal{F} \left( L \frac{L}{2}, k_y, i\omega_n \right) \mathcal{F}^\dagger \left( -\frac{L}{2}, -\frac{L}{2}, k_y, i\omega_n \right), $$

(42)

where $P = \text{Tr} \left[ \sigma_y \mathcal{G} + \sigma_y \mathcal{G}' \right]$, with the notation $\mathcal{G}_\pm = \mathcal{G}(-L/2, L/2, \pm k_y, \pm i\omega_n)$ and with the superscript $T$ denoting matrix transposition. For simplicity, we took the tunnel matrix elements to be independent of $k_y$. In this spatially extended model, we derived the following result for the anomalous phase shift in the current-phase relation,

$$ \phi_0 = \frac{\hbar_y}{v_F} \left[ \frac{L}{l_{so}} + \frac{1}{k_y^2 l_T l_{so}} \right. $$

$$ - \frac{l_{so}}{2L} \left[ 1 - \sqrt{\frac{1}{1 + 4(k_F l_T / l_{so})^2} \cos \left( \frac{2L}{l_{so}} + \psi \right) } \right], $$

(43)

which is applicable in the long junction limit, $L \ll l_T$, and $\psi \approx l_T/l_{so}$. This expression enables us to consider additional limiting cases. If $l_{so} \ll l_T \ll L$ or $\alpha k_F \gg T$, the first contribution in Eq. (13) is clearly dominant and we again recover the conventional results $\phi_0 \sim \alpha \hbar_y L/L^2$. If instead $l_T \ll l_{so} \ll L$, the quartic-root term is of the order $\sim 1$ but the prefactor is still suppressed, and again we find the standard relation $\phi_0 \sim L$. Lastly, in the limit $l_T \ll L \ll l_{so}$, the root term is still of order unity but the cosine term should be treated carefully and expanded up to the fourth order to recover the leading behavior, where we find

$$ \phi_0 \approx \frac{2}{3} (k_F L)^2 \left( \frac{\alpha}{v_F} \right)^2 \frac{\alpha \hbar_y L}{v_F^2}. $$

(44)

Interestingly, this reproduces the behavior known from the context of ballistic junctions with transparent interfaces 55. To generalize these results beyond the perturbation theory in the Zeeman field, and in a broader range of parameters, one has to rely on the numerical solution. In Fig. 6 we sketch the characteristic dependence of the anomalous phase shift at higher fields and at different spin-orbit to Fermi velocity ratios.
This gives a broader perspective on the problem and helps us to place our study in the context of existing earlier works. We limit such comparative analysis only to the case of the Rashba model of SINIS devices as other systems may introduce additional features not discussed in our work. From Ref. [54] we deduce that in the disordered limit and at temperatures $T > \Delta(T)$, the anomalous phase shift takes the form

$$
\phi_0 \approx \frac{\kappa_h^2}{2\kappa_T^2} \tanh(\kappa_\alpha L) \left[ \frac{\kappa_T L}{\tanh(\kappa_T L)} \pm 1 \right].
$$

Here $\kappa_h = \sqrt{2h/D}$, $\kappa_T = \sqrt{2\pi T/D}$, $\kappa_\alpha = \tau \alpha^2 m^2/4$, with $\tau$ being the elastic scattering time on the quenched short-range disorder potential and $D = v_F^2/\tau/2$ being the corresponding diffusion constant. The plus/minus sign in Eq. (45) describes different boundary conditions: plus stands for tunnel barriers while minus for transparent interfaces. Equation (45) shows that for long junctions, $\kappa_T L \gg 1$, the result for $\phi_0$ is the same in both cases. This limit is simultaneously compatible with the condition $\kappa_\alpha L << 1$, and therefore the phase shift has an unusual length dependence, $\phi_0 \sim (\alpha h L/v_F^2)\sqrt{l/l_\alpha} \sim L^2$, where $l = v_F T$ is the elastic mean free path. In contrast, for short junctions, $\kappa_T L \ll 1$, the situation is different. In the case of tunnel barriers, one finds $\phi_0 \sim (\alpha h L/v_F^2)(l/l_\alpha)^3 \propto L$. Instead, for transparent interfaces, one recovers the same result as in the clean limit Eq. (44), as parameters of the disorder surprisingly cancel out. The physical picture behind this coincidental result is not immediately obvious. The result only suggests that in short junctions boundary conditions play a decisive role, and a finite barrier resistance of the SN contacts enhances the value of $\phi_0$.

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[1] A. A. Golubov, M. Yu. Kupriyanov, and E. Il'ichev, “The current-phase relation in Josephson junctions,” Rev. Mod. Phys. 76, 411–469 (2004).
[2] V. B. Geshkenbein and A. I. Larkin, “The josephson effect in superconductors with heavy fermions,” JETP Lett. 43, 395 (1986).
[3] Sungkit Yip, “Josephson current-phase relationships with unconventional superconductors,” Phys. Rev. B 52, 3087–3090 (1995).
[4] Manfred Sigrist, “Time-Reversal Symmetry Breaking States in High-Temperature Superconductors,” Progress of Theoretical Physics 99, 899–929 (1998) [https://academic.oup.com/ptp/article-pdf/99/6/899/534514/99-6-899.pdf]
[5] Satoshi Kashiyawa and Yukio Tanaka, “Tunnelling effects on surface bound states in unconventional supercon-
Critical-current oscillations as a function of the exchange field and thickness of the ferromagnetic metal (f) in an s-f-s josephson junction,” JETP Lett. 35, 178 (1982).

V. V. Ryazanov, V. A. Oboznov, A. Yu. Rusanov, A. V. Vereteninnikov, A. A. Golubov, and J. Aarts, “Coupling of two superconductors through a ferromagnet: Evidence for a π junction,” Phys. Rev. Lett. 86, 2427–2430 (2001)

A. I. Buzdin, “Proximity effects in superconductor–ferromagnet heterostructures,” Rev. Mod. Phys. 77, 935–976 (2005)

V. Braude and Yu. V. Nazarov, “Fully developed triplet proximity effect,” Phys. Rev. Lett. 98, 077003 (2007)

M. Houzet and A. I. Buzdin, “Long range triplet josephson effect through a ferromagnetic trilayer,” Phys. Rev. B 76, 060504 (2007)

E. C. Gingrich, Bethany M. Niedzielski, Joseph A. Glick, Alexandre Assouline, Cheryl Feuillet-Palma, Nico-I. V. Krive, L. Y. Gorelik, R. I. Shekhter, and A. A. Reynoso, Gonzalo Usaj, C. A. Balseiro, D. Feinberg, and M. Avignon, “Anomalous josephson effect injunctions with spin polarizing quantum point contacts,” Phys. Rev. Lett. 101, 107001 (2008)

A. A. Reynoso, Gonzalo Usaj, C. A. Balseiro, D. Feinberg, and M. Avignon, “Anomalous josephson current in junctions with spin polarizing quantum point contacts,” Phys. Rev. Lett. 101, 107001 (2008)

I. V. Krive, L. Y. Gorelik, R. I. Shekhter, and M. Jonson, “Chiral symmetry breaking and the josephson current in a ballistic superconductor–quantum wire–superconductor junction,” Low Temperature Physics 30, 398–404 (2004), https://doi.org/10.1063/1.1739160.

I. V. Krive, A. M. Kadigrobov, R. I. Shekhter, and M. Jonson, “Influence of the rashba effect on the josephson current through a superconductor/luttinger liquid/superconductor tunnel junction,” Phys. Rev. B 71, 214516 (2005)

Yu. A Bychkov and E. I. Rashba, “Properties of a 2d electron gas with lifted spectral degeneracy,” Sov. Phys. JETP Lett. 39, 78 (1984).

D. B. Szombati, S. Nadj-Perge, D. Car, S. R. Plissard, E. P. A. M. Bakkers, and L. P. Kouwenhoven, “Josephson φ0-junction in nanowire quantum dots,” Nature Physics 12, 568–572 (2016)

Alexandre Assouline, Cheryl Feuillet-Palma, Nicolas Bergal, Tianzhen Zhang, Alireza Mottaghizadeh, Paola Atkinson, Marco Aprili, and Hervé Aubin, “Spin-orbit induced phase-shift in Bi2Se3 josephson junctions,” Nature Communications 7, 1212 (2020)

C. Guarcello, R. Citro, O. Durante, F. S. Bergeret, A. Iorio, C. Sanz-Fernández, E. Strambini, F. Giazotto, and A. Braggio, “rf-SQUID measurements of anomalous josephson effect,” Phys. Rev. Research 2, 023165 (2020)

Elia Strambini, Andrea Iorio, Ofelia Durante, Roberta Citro, Cristina Sanz-Fernández, Claudio Guarcello, Ilya V. Tokatly, Alessandro Braggio, Mirko Rocci, Nadia Ligato, Valentina Zannier, Lucia Sorba, F. Sebastian Bergeret, and Francesco Giazotto, “A josephson phase battery,” Nature Nanotechnology 15, 656–660 (2020)

Tom Dvir, Ayelet Zalic, Einik Holm Fyhn, Morten Amundsen, Takashi Taniguchi, Kenji Watanabe, Jacob Linder, and Hadar Steinberg, “Planar graphene-nbse josephson junctions in a parallel magnetic field,” Phys. Rev. B 103, 115401 (2021)

H. Idzuchi, F. Pientka, K. F. Huang, K. Harada, Ö. Gül, Y. J. Shin, L. T. Nguyen, N. H. Jo, D. Shindo, R. J. Cava, P. C. Canfield, and P. Kim, “Unconventional supercurrent phase in ising superconductor josephson junction with atomically thin magnetic insulator,” Nature Communications 12, 5332 (2021).

D. Margineda, J. S. Claydon, F. Qejvanaj, and C. Checkley, “Anomalous josephson effect in andreev interferometers,” (2021).

C Baumgartner, L Fuchs, A Costa, Jordi Picó-Cortés, S Reinhardt, S Gronin, G C Gardner, T Lindemann, M J Manfra, P E Faria Junior, D Kochan, J Fabian, N Paradiso, and C Strunk, “Effect of rashba and dres-selhaus spin–orbit coupling on supercurrent rectification and magnetochiral anisotropy of ballistic josephson junctions,” Journal of Physics: Condensed Matter 34, 154005 (2022)

A. Zazunov, R. Egger, T. Jonckheere, and T. Martin, “Anomalous josephson current through a spin-orbit coupled quantum dot,” Phys. Rev. Lett. 103, 117004 (2009)

Aldo Brunetti, Alex Zazunov, Arijit Kundu, and Reinhold Egger, “Anomalous josephson current, incipient time-reversal symmetry breaking, and majorana bound states in interacting multilevel dots,” Phys. Rev. B 88, 144515 (2013)

Tomohiro Yokoyama, Mikio Eto, and Yuli V. Nazarov, “Josephson current through semiconductor nanowire with spin–orbit interaction in magnetic field,” Journal of the Physical Society of Japan 82, 054703 (2013), https://doi.org/10.7566/JPSJ.82.054703.

Teemu Ojanne, “Topological π josephson junction in superconducting rashba wires,” Phys. Rev. B 87, 100506 (2013)

Tomohiro Yokoyama, Mikio Eto, and Yuli V. Nazarov, “Anomalous josephson effect induced by spin-orbit interaction and zeeman effect in semiconductor nanowires,” Phys. Rev. B 89, 195407 (2014)

G Campagnano, P Lucignano, D Giuliano, and A Tagliacozzo, “Spin–orbit coupling and anomalous josephson effect in nanowires,” Journal of Physics: Condensed Matter 27, 205301 (2015)

S. V. Mironov, A. S. Mel’nikov, and A. I. Buzdin, “Double path interference and magnetic oscillations in cooper pair transport through a single nanowire,” Phys. Rev. Lett. 114, 227001 (2015).

Bin-He Wu, Xu-Yu Feng, Chao Wang, Xiao-Feng Xu, and Chun-Rui Wang, “Anomalous direct-current josephson effect in semiconductor josephson junctions,” Chin. Phys. Lett. 33, 017401 (2016).

Konstantin N. Nesterov, Manuel Houzet, and Julia S. Meyer, “Anomalous josephson effect in semiconducting nanowires as a signature of the topologically nontrivial...
Yukio Tanaka, Takehito Yokoyama, and Naoto Nagaosa, “Manipulation of the majorana fermion, andreev reflection, and josephson current on topological insulators,” Phys. Rev. Lett. 103, 107002 (2009).

Jacob Linder, Yukio Tanaka, Takehito Yokoyama, Asle Sudbø, and Naoto Nagaosa, “Interplay between superconductivity and ferromagnetism on a topological insulator,” Phys. Rev. B 81, 184525 (2010).

Annica M. Black-Schaffer and Jacob Linder, “Magnetization dynamics and majorana fermions in ferromagnetic josephson junctions along the quantum spin hall edge,” Phys. Rev. B 83, 220511 (2011).

Jennifer Nussbaum, Thomas L. Schmidt, Christoph Bruder, and Rakesh P. Tiwari, “Josephson effect in normal and ferromagnetic topological-insulator junctions: Planar, step, and edge geometries,” Phys. Rev. B 90, 045413 (2014).

Bo Lu, Keiji Yada, A. A. Golubov, and Yukio Tanaka, “Anomalous josephson effect in d-wave superconductor junctions on a topological insulator surface,” Phys. Rev. B 92, 100503 (2015).

Fabrizio Dolcini, Manuel Houzet, and Julia S. Meyer, “Topological josephson φ0 junctions,” Phys. Rev. B 92, 035428 (2015).

Pasquale Marra, Roberta Citro, and Alessandro Braggio, “Signatures of topological phase transitions in josephson current-phase discontinuities,” Phys. Rev. B 93, 220507 (2016).

Alexander Yuzin, Mohammad Alidoust, and Daniel Loss, “Josephson junction through a disordered topological insulator with helical magnetization,” Phys. Rev. B 93, 214502 (2016).

E. V. Bezuglyi, A. S. Rozhavsky, I. D. Vagner, and P. Wyder, “Combined effect of zeeman splitting and spin-orbit interaction on the josephson current in a superconductor-two-dimensional electron gas-superconductor structure,” Phys. Rev. B 66, 052508 (2002).

O. Diintrova and M. V. Feigel’man, “Two-dimensional s-n-s junction with the rashba spin-orbit interaction,” JETP 129, 742 (2006).

Artem V. Galaktionov, Mikhail S. Kalenkov, and Andrei D. Zaikin, “Josephson current and andreev states in superconductor–half metal–superconductor heterostructures,” Phys. Rev. B 77, 094520 (2008).

P. M. R. Brydon, Boris Kastening, Dirk K. Morr, and Dirk Manske, “Interplay of magnetism and triplet superconductivity in a josephson junction,” Phys. Rev. B 77, 104504 (2008).

R. Greim, M. Eschrig, G. Metallidis, and Gerd Schön, “Spin-dependent cooper pair phase and pure spin supercurrents in strongly polarized ferromagnets,” Phys. Rev. Lett. 102, 227005 (2009).

F. Konschelle and A. Buzdin, “Magnetic moment manipulation by a josephson current,” Phys. Rev. Lett. 102, 017001 (2009).

Jun-Feng Liu and K. S. Chan, “Relation between symmetry breaking and the anomalous josephson effect,” Phys. Rev. B 82, 125305 (2010).

Jun-Feng Liu and K. S. Chan, “Anomalous josephson current through a ferromagnetic trilayer junction,” Phys. Rev. B 82, 184533 (2010).

I Margaris, V Paltoglou, and N Flytzanis, “Zero phase difference supercurrent in ferromagnetic josephson junctions,” Journal of Physics: Condensed Matter 22, 445701 (2010).

Jun-Feng Liu, Kwok Sum Chan, and Jun Wang, “Anomalous josephson current through a ferromagnet–semiconductor hybrid structure,” Journal of the Physical Society of Japan 80, 124708 (2011).

https://doi.org/10.1143/JPSJ.80.124708

Andres A. Reynoso, Gonzalo Usaj, C. A. Balseiro, D. Feinberg, and M. Avignon, “Spin-orbit-induced chirality of andreev states in josephson junctions,” Phys. Rev. B 86, 214519 (2012).

F. S. Bergeret and I. V. Tokatly, “Theory of diffusive φ0 josephson junctions in the presence of spin-orbit coupling,” EPL (Europhysics Letters) 110, 57005 (2015).

François Konschelle, Ilya V. Tokatly, and F. Sebastian Bergeret, “Theory of the spin-galvanic effect and the anomalous phase shift φ0 in superconductors and josephson junctions with intrinsic spin-orbit coupling,” Phys. Rev. B 92, 125443 (2015).

Ashbjørn Rasmussen, Jeroen Danon, Henri Suominen, Fabrizio Nichel, Morten Kjaergaard, and Karsten Flensberg, “Effects of spin-orbit coupling and spatial symmetries on the josephson current in sns-junctions,” Phys. Rev. B 93, 155406 (2016).

M. A. Silaev, I. V. Tokatly, and F. S. Bergeret, “Anomalous current in diffusive ferromagnetic josephson junctions,” Phys. Rev. B 95, 184508 (2017).

Victor M Edelstein, “The ginzburg - landau equation for superconductors of polar symmetry,” Journal of Physics: Condensed Matter 8, 339–349 (1996).

K. V. Samokhin, “Magnetic properties of superconductors with strong spin-orbit coupling,” Phys. Rev. B 70, 104521 (2004).

Ol’ga Dimitrova and M. V. Feigel’man, “Theory of a two-dimensional superconductor with broken inversion symmetry,” Phys. Rev. B 76, 014522 (2007).

Manuel Houzet and Julia S. Meyer, “Quasiclassical theory of disordered rashba superconductors,” Phys. Rev. B 92, 014509 (2015).

Victor M. Edelstein, “Ginzburg-landau theory for impure superconductors of polar symmetry,” Phys. Rev. B 103, 094507 (2021).

Michael Tinkham, Introduction to Superconductivity, 2nd ed. (Dover Publications, New York, 2004).

L. G. Aslamazov, A. I. Larkin, and Yu. N Ovchinnikov, “Josephson effect in superconductors separated by a normal metal,” Sov. Phys. JETP 28, 171 (1969).

M. Yu. Kuprianov and V. F. Lukichev, “Influence of boundary transparency on the critical current of dirty ss’s structures,” Sov. Phys. JETP 67, 1163 (1988).

A. Brinkman and A. A. Golubov, “Coherence effects in double-barrier josephson junctions,” Phys. Rev. B 61, 11297–11300 (2000).

E. Prada and F. Sols, “Entangled electron current through finite size normal-superconductor tunneling structures,” The European Physical Journal B - Condensed Matter and Complex Systems 40, 379–396 (2004).

Gerald D. Mahan, Many-Particle Physics, 3rd ed. (Kluwer academis/Plenum Publishers, New York, 2000).

P.W. Brouwer and C.W.J. Beenakker, “Anomalous temperature dependence of the supercurrent through a chaotic josephson junction,” Chaos, Solitons and Frac-
[70] Alex Levchenko, Alex Kamenev, and Leonid Glazman, “Singular length dependence of critical current in superconductor/normal-metal/superconductor bridges,” Phys. Rev. B 74, 212509 (2006).

[71] Colin M. Whisler, Maxim G. Vavilov, and Alex Levchenko, “Josephson currents in chaotic quantum dots,” Phys. Rev. B 97, 224515 (2018).