One-phase free-boundary problems with degeneracy

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Abstract
In this paper, we study local minimizers of a degenerate version of the Alt–Caffarelli functional. Specifically, we consider local minimizers of the functional $J_Q(u, \Omega) := \int_{\Omega} |\nabla u|^2 + Q(x)^2 \chi_{\{u > 0\}} \, dx$ where $Q(x) = \text{dist}(x, \Gamma)\gamma$ for $\gamma > 0$ and $\Gamma$ a $C^{1,\alpha}$ submanifold of dimension $0 \leq k \leq n - 1$. We show that the free boundary may be decomposed into a rectifiable set, on which we prove upper Minkowski content estimates, and a degenerate cusp set about which little can be said in general with the current techniques. Work in the theory of water waves and the Stokes wave serves as our inspiration, however the main thrust of this paper is to study the geometry of the free boundary for degenerate one-phase Bernoulli free-boundary problems in the context of local minimizers.

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1 Introduction

In the groundbreaking paper [1] Alt and Caffarelli studied the existence and regularity of minimizers of the functional,

\[ J_Q(u, \Omega) := \int_{\Omega} |\nabla u|^2 + Q^2(x)\chi_{|u|>0} \, dx \]  

(1.1)

for \( \Omega \subset \mathbb{R}^n \) an open, connected set with locally Lipschitz boundary \( \partial \Omega \) such that the boundary satisfies \( \mathcal{H}^{n-1}(\partial \Omega) > 0 \). Minimization happens in the class,

\[ K_{u_0, \Omega} := \{ u \in W^{1,2}(\Omega) | u - u_0 \in W^{1,2}_0(\Omega) \}, \]

for a \( u_0 \in W^{1,2}(\Omega) \) satisfying \( u_0 \geq 0 \), under the assumption that \( Q \) was bounded and measurable.

In this paper, we study local minimizers of (1.1) where \( Q(x) = \text{dist}(x, \Gamma)^\gamma \) for any \( \gamma > 0 \), where \( \Gamma \) is a \( k \)-dimensional \( C^{1,\alpha} \) submanifold in \( \mathbb{R}^n \), with \( 0 \leq k \leq n - 1 \). In their original paper [1], the authors prove the regularity of the free boundary \( \partial \{ u > 0 \} \cap \Omega \) under the assumption that the function \( Q \) satisfies the following conditions:

1. \( Q \in C^{0,\alpha}(\Omega) \)
2. \( 0 < Q_{\min} \leq Q(x) \leq Q_{\max} < \infty \).

This assumption plays a crucial role in proving non-degeneracy of the minimizing function \( u \) near the free boundary and the non-degeneracy of the free boundary itself. For example, if \( n = 2 \) then near a free boundary point \( x_0 \), we can write

\[ u(x) = Q(x_0)\langle x - x_0, \eta \rangle_+ + O(|x - x_0|) \]

for some unit vector \( \eta \). Thus, if \( 0 < Q(x_0) \) the blow-up of \( u \) at \( x_0 \) is a piece-wise linear function and hence the blow-up of \( \partial \{ u > 0 \} \) is flat. On the other hand, if \( Q(x_0) = 0 \) then we see that \( u \) cannot have piece-wise a linear blow-up at \( x_0 \), and therefore all blow-ups of \( \partial \{ u > 0 \} \) are not flat. We decompose \( \partial \{ u > 0 \} \) into the regular set, \( \text{reg}(\partial \{ u > 0 \}) \), where blow-ups of \( \partial \{ u > 0 \} \) are piece-wise linear and the singular set, \( \text{sing}(\partial \{ u > 0 \}) \), where blow-ups are not piece-wise linear. We shall refer to the case when \( 0 < Q_{\min} \leq Q \) as the non-degenerate case.

In higher dimensions, the non-degenerate case becomes more complicated, since there exist blow-ups which are not piece-wise linear. However, the following results are known.

**Theorem 1.1** Let \( 0 < Q_{\min} \leq Q \leq Q_{\max} \) be \( C^{0,\alpha} \) in \( B_2(0) \subset \mathbb{R}^n \). Suppose that \( u \) is a local minimizer of \( J_Q(\cdot, B_2(0)) \). Then \( \partial \{ u > 0 \} \cap B_2(0) \) may be decomposed into

\[ \partial \{ u > 0 \} = \text{reg}(\partial \{ u > 0 \}) \cup \text{sing}(\partial \{ u > 0 \}) \]

i. (11) The regular set, \( \text{reg}(\partial \{ u > 0 \}) \), is relatively open and can be locally written as the graph of a \( C^{1,\beta} \) function.

ii. (11) In \( n = 2 \), \( \text{sing}(\partial \{ u > 0 \}) = \emptyset \).

iii. (22) If \( k^* \) is the first dimension that there exists a non-linear one-homogeneous minimizer of \( J_1(\cdot, \mathbb{R}^n) \), then \( \dim_{\mathcal{H}}(\text{sing}(\partial \{ u > 0 \})) \leq n - k^* \).

iv. (4) \( k^* \geq 4 \).

\[ \text{Springer} \]
v. \((\{15\})\) \(k^* \geq 5\).

vi. \((\{10\})\) \(k^* \leq 7\)

Furthermore, if \(Q\) is \(C^{k,\alpha}\) (resp. smooth) for some \(k \geq 1\) and \(0 < \alpha < 1\), then \(\text{reg}(\partial\{u > 0\})\) is locally the graph of a \(C^{k+1,\beta}\) (resp. smooth) function \([1]\).

The results of [1] have inspired a countless papers and generalizations, including two-phase versions [2], fractional Laplacian versions [6] and [9], \(p\)-Laplacian versions [8], and elliptic versions [19], and versions for almost-minimizer [11] [7], just to name a few. Much interest has centered around the exact value of \(k^*\) in the non-degenerate case, and in [12], Edelen and Engelstein prove that \(\text{sing}(\partial\{u > 0\})\) is \((n - k^*)\)-rectifiable and satisfies certain upper Minkowski content estimates.

However, virtually all work on the geometry of the related free boundaries has proceeded under the assumption of non-degeneracy on \(Q\). Indeed, to the author’s knowledge [3] is the first instance in which the degenerate case, i.e. \(Q_{\min} = 0\), was considered. In [3], Arama and Leoni investigate absolute minimizers and assume \(n = 2\), \(Q(x_1, x_2) = \sqrt{(c - x_2)^2 + \Omega}\) is a rectangle, and symmetric boundary conditions on \(\partial\Omega\). Subsequent work for \(n \geq 3\), \(0 < \gamma\), and \(Q(x')_n = (c - x_n)^{\gamma}_+\) has been carried out in [13] [14], though also only for absolute minimizers under similar assumptions of symmetry. These investigations were carried out to investigate the theory of the water waves and the Stokes wave, which are usually merely critical points of \(J_Q(\cdot, \Omega)\), in the context of minimizers. No work has been done for other functions \(Q\).

Inspired by the work of [3] and [13], we study the fine-scale structure of \(\text{sing}(\partial\{u > 0\})\) for local minimizers in the degenerate case for a natural class of \(Q(x) = \text{dist}(x, \Gamma)^\gamma\). In particular, we make no assumptions of symmetry and allow \(\Gamma\) to be non-flat. Since our questions are essentially local and Alt and Caffarelli [1], Weiss [22], and Edelen and Engelstein [12] have proven detailed partial regularity results on \(\text{reg}(\partial\{u > 0\})\) and \(\text{sing}(\partial\{u > 0\})\) when \(Q_{\min} \geq c > 0\), we restrict our investigation to questions on the infinitesimal structure of \(\partial\{u > 0\} \cap \{Q = 0\}\).

1.1 Main results

Our most basic result allows us to decompose \(\partial\{u > 0\} \cap \Gamma\) into two sets which we must treat very differently.

**Lemma 1.2** Let \(0 < \gamma < \infty\) and \(0 \leq k \leq n - 1\) be an integer. Let \(\Gamma\) be a \(k\)-dimensional \(C^{1,\alpha}\) submanifold. If \(u\) is a local minimizer of (1.1) with \(Q(x) = \text{dist}(x, \Gamma)^\gamma\) then the following holds.

1. For all \(x \in \partial\{u > 0\} \cap \Gamma\), the \(Q\)-density

\[
\Phi(x, 0^+) := \lim_{r \to 0^+} \frac{1}{\omega_{n-1}r^{n-1}2^\gamma} \int_{B_r(x)} Q^2 \chi_{\{u > 0\}} dx \tag{1.2}
\]

exists.

2. There exist constants \(0 < c(n, \gamma) < C(n, \gamma) < \int_{B_1(0)} |x_n|^{2\gamma} dx\) such that we may decompose \(\partial\{u > 0\} \cap \Gamma\) as

\[\partial\{u > 0\} \cap \Gamma = S \cup \Sigma,\]

where \(\Sigma = (\partial\{u > 0\} \cap \Gamma) \cap \{x : \Phi(x, 0^+) = 0\}\) and

\[S = (\partial\{u > 0\} \cap \Gamma) \cap \{x : \Phi(x, 0^+) \in [c(n, \gamma), C(n, \gamma)]\} .\]
Moreover, $\partial \{ u > 0 \} \cap \Gamma \subset \text{sing}(\partial \{ u > 0 \})$, since all blow-ups at points in $\Sigma$ are $(1 + \gamma)$-homogeneous and all blow-ups at points in $\Sigma$ vanish identically.

The existence of the $Q$-density is proven in Lemma 6.1. The lower bound $0 < c(n, \gamma)$ is proven in Lemma 6.2. The upper bound is proved in Lemma 8.3.

For lack of a better term, the set $\Sigma$ shall be referred to as the set of degenerate singularities, and the set $S$ shall be called the set of non-degenerate singularities.

The main results of this paper concern the non-degenerate set $S$. Specifically, we prove upper Minkowski content estimates on the “effective” strata of $S$ using the powerful techniques of [18]. Roughly speaking, the strata $S^j_{\epsilon, r_0}$ is the set of points $x \in \Gamma \cap \partial \{ u > 0 \}$ such that for all $r_0 < r \leq \text{dist}(x, \partial \Sigma)$ the function $u$ is “$\epsilon$-far” in $B_r(x)$ from all homogeneous functions which are translation invariant along a $(k + 1)$-dimensional linear subspace. See Sect. 2.3 for details and rigorous definitions.

**Theorem 1.3** Let $n \geq 2$, $k$ be an integer such that $0 \leq k \leq n - 1$, and let $0 < \gamma$. Let $\Gamma \subset \mathbb{R}^n$ be a $k$-dimensional $(1, M)$-C$^1,\alpha$ submanifold such that $0 \in \Gamma$, and let $Q(x) = dist(x, \Gamma) \gamma$.

If $u$ is an $\epsilon_0$-local minimizer of $J_Q(\cdot, B_2(0))$ in the class $K_{u_0, B_2(0)}$ and $\| \nabla u_0 \|^2_{L^2} \leq \Lambda$ and $\sup_{B_2(0)} u_0 = A < \infty$. Then, for any $0 < \epsilon, \rho$ and any radius $r$ such that $\rho \leq r$,

$$\text{Vol}(B_r(S^j_{\epsilon, \rho} \cap B_1(0))) \leq C(n, j, \epsilon, \gamma, M, \epsilon_0, A) r^{n-j}.$$

Furthermore, $S^j_{\epsilon} = \bigcap_{0 < \rho} S^j_{\epsilon, \rho}$ is countably $j$-rectifiable. Consequently, for all $0 \leq j \leq k$ the strata $S^j := \bigcup_{0 < \epsilon} S^j_{\epsilon}$ are countably $j$-rectifiable.

Thanks to Lemma 1.2(2), we are able to prove the following containment result.

**Lemma 1.4** ($\epsilon$-containment) Under the hypotheses of Theorem 1.3, $S \cap B_1(0)$ is closed and satisfies the following containment relationship. If $k = n - 1$, then there exists an $0 < \epsilon(n, \gamma, \alpha, M, \Lambda, A, \epsilon_0)$ such that $S \subset S^{n-2}_{\epsilon}.$

Theorem 1.3 and Lemma 1.4 immediately imply the following corollary.

**Corollary 1.5** Under the hypotheses of Theorem 1.3, if $k = n - 1$, $\text{dim}_{\text{M}}(S) \leq n - 2$ and

$$\mathcal{H}^{n-2}(S \cap B_1(0)) \leq M^{n-2}(S \cap B_1(0)) \leq C(n, \gamma, \alpha, M, \Lambda, A, \epsilon_0).$$

We note that when $k < n - 1$ similar estimates are obtainable, but are meaningless, as there exists simple examples in which $S = \Gamma \cap B_1(0)$. When $k = n - 1$, these result say that the non-degenerate singular set $S$ cannot be too spread out, since it must “sit in space” like an $(n - 2)$-dimensional submanifold, but also cannot be too concentrated at any scale, either.

### 1.1.1 Cusps

A central concern in the degenerate case is the formation of degenerate singularities, i.e. cusps. Since the classical estimates in [1] which are essential to regularity become vacuous in regions where $Q$ vanishes, the standard analytic techniques of establishing weak geometric regularity (i.e., interior ball conditions) do not work near $\Gamma$. New—or at least different—ideas are needed.

The potential development of cusps leads to some notable differences between the degenerate case and the non-degenerate case. See, Lemma 5.6 and Remark 5.7.

In this paper, we apply analytic techniques to prove the following preliminary result on the degenerate singular set $\Sigma$.
Lemma 1.6 If \( u \) is a local minimizer of \( J_Q(\cdot, B_2(0)) \), then \( \partial \{ u > 0 \} \cap B_1(0) \) is a set of finite perimeter. In particular, the set \( \Sigma = \Gamma \cap \partial \{ u > 0 \} \setminus S \) has \( H^{n-1} \)-measure zero.

Remark 1.7 Subsequent to writing this paper, the author and Lisa Naples have proven some general conditions under which \( \Sigma = \emptyset \). In particular, when \( n \geq 2 \), \( 0 \leq k \leq n-1 \) is an integer, \( 0 < \gamma \), and \( \Gamma \subset \mathbb{R}^n \) a flat \( k \)-dimensional submanifold, then \( \Sigma = \emptyset \) for all \( \epsilon_0 \)-local minimizers of \( J_Q(\cdot, B_2(0)) \) for \( Q(x) = \text{dist}(x, \Gamma)^\gamma \). See [17].

Notwithstanding the *post facto* non-existence of cusps under these circumstances, the techniques in this paper may prove useful in other circumstances when cusps cannot be eliminated.

1.2 Strategy and organization

The overall strategy of the paper is to employ the tools and techniques of [18] and to prove the density “gap” in Lemma 1.2(2). This density “gap” allows us to prove the \( \epsilon \)-containment results in Corollary 1.4. The key ingredients leading to these results are: non-degeneracy of \( u \) and local Lipschitz estimates depending upon \( Q(x) = \text{dist}(x, \Gamma)^\gamma \). With these results, one is able to show that at non-degenerate singularities the the Weiss \((1 + \gamma)\)-density is almost monotone and points have \((1 + \gamma)\)-homogeneous blow-ups. The degree of homogeneity roughly follows from the fact that \( Q \) is \( \gamma \)-homogeneous and \( Q \sim |\nabla u| \).

In Sect. 2, we introduce the quantitative stratification from [5] which is necessary to the machinery of [18], as well as state some basic estimates on \( C^{1,\alpha} \) submanifolds that are necessary to establish the almost-monotonicity formula for the Weiss \((1 + \gamma)\)-density on \( \partial \{ u > 0 \} \cap \Gamma \). Section 3 is dedicated to recapping the results of Alt and Caffarelli [1] and Arama and Leoni [3]. In particular, these comparison techniques prove the non-degeneracy and local Lipschitz bounds adjusted to the function \( Q \).

Section 4 proves the almost-monotonicity of the Weiss \((\gamma + 1)\)-density for local minimizers and variational solutions. With the almost-monotonicity formula, we then prove strong compactness and the existence of \((1 + \gamma)\)-homogeneous blow-ups in Sect. 5. Note that the degeneracy of \( Q \) plays an important role in the compactness results (see Remark 5.7). In Sect. 6, we prove the lower bounds in the density “gap” in the definition of \( S \), see Lemma 6.2.

Section 7 is devoted to carefully following the argument of [12], which applied the techniques of [18] to the non-degenerate case to prove Theorem 1.3, noting all the changes necessary to adapt them to the degenerate case. In Sect. 8, we prove the containment results which prove Lemma 1.4. Finally, in Sect. 9, we prove that the positivity set \( \{ u > 0 \} \) is a set of finite perimeter, and hence \( \Sigma \) has \( H^{n-1} \)-measure zero.

2 Preliminaries

In this section, we discuss some basic definitions and elementary results.

2.1 \( C^{1,\alpha} \) geometry

In this subsection, we record some elementary observations on the submanifolds \( \Gamma \) which are necessary to prove the almost-monotonicity results in Lemma 4.3.
**Definition 2.1** Let \( n \in \mathbb{N} \) and \( 0 \leq k \leq n - 1 \). A set \( \Gamma \) is locally a \( k \)-dimensional \( C^{1,\alpha} \) submanifold if for every \( x \in \Gamma \), there is a radius \( 0 < r_x \) such that

\[
\Gamma \cap B_{r_x}(x) = \text{graph}_{Tx\Gamma}(f_x)
\]

for some function, \( f_x \in C^{1,\alpha}(\mathbb{R}^k; \mathbb{R}^{n-k}) \). We will use \([f]_{\alpha, B_{r_x}(x)}\) to denote the H"{o}lder seminorm of \( Df \) in \( B_{r_x}(x) \subseteq \mathbb{R}^k \). That is,

\[
[f]_{\alpha,B_{r_x}(x)} := \sup_{z,y \in B^k_{r_x}(x), z \neq y} \frac{|Df(y) - Df(z)|}{|y - z|^\alpha}.
\]

We shall call \( \Gamma \) a \((r, M)\)-\(C^{1,\alpha}\) submanifold of dimension \( 0 \leq k \leq n - 1 \), if for every \( x \in \Gamma \), we may take \( r_x = r \) and \([f]_{\alpha, B_{r}(x)} \leq M \). For \( \Gamma \subseteq \mathbb{R}^n \) a \((r, M)\)-\(C^{1,\alpha}\) submanifold of dimension \( 0 \leq k \leq n - 1 \), we denote

\[
[\Gamma]_{\alpha} := \sup\{[f_x]_{\alpha, B_{r}(x)} : x \in \Gamma\} \leq M.
\]

**Lemma 2.2** Let \( \Gamma \subseteq \mathbb{R}^n \) be a \((1, M)\)-\(C^{1,\alpha}\) submanifold of dimension \( 0 \leq k \leq n - 1 \). Then, for any \( x \in \Gamma \) with defining function \( f : B_1(x) \cap (x + T_x\Gamma) \to \mathbb{R}^{n-k} \), for all \( y \in B_1(x) \cap (x + T_x \Gamma) \),

\[
|f(y) - (y, 0)| = [\Gamma]_\alpha |y|^{1+\alpha}.
\]

**Proof** By translation and rotation, we may assume that \( x = 0 \) and \( \frac{\partial}{\partial x_j} f(0) = 0 \) for \( 1 \leq j \leq k \). That is, \( T_x\Gamma = \mathbb{R}^k \hookrightarrow \mathbb{R}^n \). We calculate,

\[
|f(y) - (y, 0)| = \left| \int_0^y Df(z) \cdot \frac{z}{|y|} dz \right| \\
\leq |y| \max_{z \in 0y} ||Df(z)|| = [\Gamma]_\alpha |y|^{1+\alpha}.
\]

\( \square \)

**Remark 2.3** The function, \( \text{dist}(\cdot, \Gamma) : \mathbb{R}^n \to \mathbb{R} \), is a Lipschitz function with Lipschitz constant 1. By Rademacher’s theorem, \( \nabla(\text{dist}(\cdot, \Gamma)) \) exists \( \mathcal{H}^n \)-a.e.. Furthermore, for \( x \in \mathbb{R}^n \) such that \( \nabla(\text{dist}(x, \Gamma)) \) exists, there exists a unique minimizing point \( y \in \Gamma \) such that \( \text{dist}(x, \Gamma) = |x - y| \) and

\[
\nabla(\text{dist}(x, \Gamma)) = \frac{x - y}{|x - y|}.
\]

**Definition 2.4** We define the function, \( \pi_{\Gamma} : \Omega \rightarrow \Gamma \), as follows. For \( x \) such that \( \text{dist}(\cdot, \Gamma) \) is differentiable at \( x \), we define \( \pi_{\Gamma}(x) := y \), where \( y \in \Gamma \) is the unique minimizing point such that \( \text{dist}(x, \Gamma) = |x - y| \). By Remark 2.3, this is sufficient to define the function \( \mathcal{H}^n \)-a.e..

**Remark 2.5** Let \( \Gamma \subseteq \mathbb{R}^n \) be a locally \( C^{1,\alpha} \) submanifold of dimension \( 0 \leq k \leq n - 1 \). Then, for every \( x \in \Omega \) such that \( \pi_{\Gamma}(x) \) is defined,

\[
x - \pi_{\Gamma}(x) \perp T_{\pi_{\Gamma}(x)}\Gamma.
\]

**Lemma 2.6** Let \( \Gamma \subseteq \mathbb{R}^n \) be an \((r_0, M)\)-\(C^{1,\alpha}\) submanifold of dimension \( 0 \leq k \leq n - 1 \). Let \( x_0 \in \Gamma \), then for every \( x \in B_{r_0}(x_0) \) such that \( \pi_{\Gamma}(x) \) is defined,

\[
(\pi_{\Gamma}(x) - x_0) \cdot \frac{x - \pi_{\Gamma}(x)}{|x - \pi_{\Gamma}(x)|} \leq 8[\Gamma]_\alpha r_0^{1+\alpha}.
\]
Proof} We begin by choosing coordinates so that $\Gamma$ is a graph of $f : \mathbb{R}^k \to \mathbb{R}^{n-k}$ over $x_0 + T_{x_0} (\Gamma)$ where $x_0 = 0$ and $\frac{\partial}{\partial x_i} f(0) = 0$ for all $i = 1, 2, \ldots, k$. We shall use the notation, $\vec{\eta} = \frac{x - \pi_{\Gamma}(x)}{|x - \pi_{\Gamma}(x)|}$. Let $y \in \mathbb{R}^k$ be such that $\pi_{\Gamma}(x) = (y, f(y)) \in \mathbb{R}^k \times \mathbb{R}^{n-k}$. We decompose

$$\pi_{\Gamma}(x) = \vec{a} + \vec{b}$$

where $\vec{a} = (y, 0)$ and $\vec{b} = (0, f(y))$. Thus $\vec{b} \cdot \vec{\eta} |\vec{b}| \leq |\Gamma| |y|^{1+\alpha}$. Next, we choose a vector $\vec{\eta}_0 \in N_0 \Gamma$, the normal bundle to $\Gamma$ at 0, such that $|\vec{\eta} - \vec{\eta}_0| \leq |\Gamma| y^{\alpha}$.

$$\vec{a} \cdot \vec{\eta} \leq \vec{a} \cdot (\vec{\eta} + \vec{\eta}_0 - \vec{\eta}_0)$$

$$\leq \vec{a} \cdot \vec{\eta}_0 + \vec{a} \cdot (\vec{\eta} - \vec{\eta}_0)$$

$$\leq |\Gamma| |y|^{1+\alpha}.$$ 

Since $|y| \leq 2r_0$ and $0 < \alpha \leq 1$, we have the desired inequality. \hfill \Box

**Lemma 2.7** (Manifold compactness) Let $\Gamma_i$ be a sequence of $k$-dimensional $(1, M \cdot C^{1,\alpha})$ submanifolds satisfying $0 \in \Gamma_i$ for all $i \in \mathbb{N}$. There is a $k$-dimensional $(1, M \cdot C^{1,\alpha})$ submanifold $\Gamma \subset \mathbb{R}^n$ with $0 \in \Gamma$ such that for every $0 < R < \diam (B)$,

$$\Gamma_j \cap B_R(0) \to \Gamma \cap B_R(0)$$

in the Hausdorff metric on compact subsets.

This lemma is is a consequence of the compactness of the Grassmanian and the compactness of $C^{1,\alpha}$ functions with bounded seminorm $\lfloor f \rceil_{a, B_1(0)}$. The rest is left to the reader.

**Lemma 2.8** Let $\Gamma$ be a $k$-dimensional $(1, \frac{1}{4})C^{1,\alpha}$ submanifold satisfying $0 \in \Gamma$. Then, there is a constant $C_1 < \infty$, such that

$$\mathcal{H}^a(B_{2^{-i}}(\Gamma) \cap B_1(0)) \leq C_1 2^{-i(n-k)}.$$ 

**Proof** This fact holds more generally for Ahlfors regular sets. We argue below for $\Gamma$, specifically. We begin with an initial estimate,

$$\mathcal{H}^a(B_1(\Gamma) \setminus B_{\frac{1}{2}}(\Gamma) \cap B_1(0)) \leq \omega_k \omega_{n-k}.$$

Note that since $\Gamma$ is assumed to be $(1, \frac{1}{4})C^{1,\alpha}$ submanifold, by Lemma 2.2, we have that

$$(B_1(0) \cap B_1(\Gamma) \setminus B_{\frac{1}{2}}(\Gamma)) \subset B_1(0) \setminus B_{\frac{1}{2}}(T_0 \Gamma),$$

and hence $\mathcal{H}^a(B_1(0) \cap B_1(\Gamma) \setminus B_{\frac{1}{2}}(\Gamma)) \leq \omega_k \omega_{n-k}.$

We now iterate this estimate at dyadic scales. Note that since $\mathcal{H}^a (B_2(0) \cap \Gamma) \leq 2^k \omega_k \sqrt{\Gamma + [\Gamma]_{a}} = 2^k \omega_k$, an $r$-net in $\Gamma$ consists of at most $C(n, k) \omega_k r^{k}$ points. Furthermore, if $x \in B_1(0) \cap \{x : \text{dist}(x, \Gamma) \in (2^{-i}, 2^{-i+1})\}$, then there must exist a $y \in \Gamma \cap B_{1+2^{-i+2}}(0)$ such that $x \in B_{2^{-i+1}}(y)$. Additionally, we note that if $x \in \Gamma$, then $[\Gamma^{x, r}]_{a} \leq [\Gamma]_{a} r^{\alpha}$. Therefore, if we take an $2^{-i}$-net in $\Gamma \cap B_2(0)$ and apply our previous result to $B_1(0)$ and $\Gamma^{x, 2^{-i+1}}$, we obtain

$$\mathcal{H}^a(B_1(0) \cap B_{2^{-i+1}}(\Gamma) \setminus B_{2^{-i}}(\Gamma)) \leq C(n, k) \omega_k 2^{ik}(\omega_k \omega_{n-k}) 2^{(-i+1)n}$$

$$\leq C(n, k) \omega_k^2 \omega_{n-k}^2 2^{n} 2^{-(n-k)}.$$ 

This proves the estimate with $C_1 = C(n, k) \omega_k^2 \omega_{n-k} 2^n$. \hfill \Box
2.2 Minimizers and local minimizers

**Definition 2.9** Let \( \Omega \) be an open set with Lipschitz boundary satisfying \( \mathcal{H}^{n-1}(\partial \Omega) > 0 \). Suppose that \( Q \in C^{0,\alpha}(\Omega) \) such that \( 0 \leq Q \). A function \( u \) is a **minimizer** of

\[
J_Q(u, \Omega) := \int_{\Omega} |\nabla u|^2 + Q^2(x) \chi_{\{u > 0\}} \, dx,
\]

in the class \( K_{u_0, \Omega} := \{ u \in W^{1,2}(\Omega) | u - u_0 \in W^{1,2}_0(\Omega) \} \) for a \( u_0 \in W^{1,2}(\Omega) \) satisfying \( u_0 \geq 0 \), if for every other function \( v \in K_{u_0, \Omega} \),

\[
J_Q(u, \Omega) \leq J_Q(v, \Omega).
\]

For \( 0 < \epsilon_0 \), a function \( u \) is called an **\( \epsilon_0 \)-local minimizer** of \( J_Q(\cdot, \Omega) \) if

\[
J_Q(u, \Omega) \leq J_Q(v, \Omega)
\]

for every \( v \in K_{u, \Omega} \) satisfying

\[
\|\nabla (u - v)\|^2_{L^2(\Omega)} + \| \chi_{\{u > 0\}} - \chi_{\{v > 0\}} \|_{L^1(\Omega)} < \epsilon_0.
\]

When we do not need to quantify such things, \( \epsilon_0 \)-local minimizers will simply be called local minimizers. Clearly, minimizers are local minimizers for all \( 0 < \epsilon_0 \). And, for all \( \Omega' \subset \Omega \), local minimizers in \( K_{u, \Omega} \) are local minimizers in \( K_{u, \Omega'} \). We shall often speak of \( u \) as a local minimizer without reference to the class \( K_{u, \Omega} \).

**Theorem 2.10** ([1] Theorem 1.3) Let \( \Omega \) be an open set with Lipschitz boundary satisfying \( \mathcal{H}^{n-1}(\partial \Omega) > 0 \). Suppose that \( Q \in C^{0,\alpha}(\Omega) \) such that \( 0 \leq Q \), and let \( u_0 \in W^{1,2}(\Omega) \) be non-negative satisfying \( J_Q(u_0, \Omega) < \infty \). Then, minimizers of the functional \( J_Q(\cdot, \Omega) \) in the class \( K_{u_0, \Omega} \) exist.

2.3 Quantitative stratification

Stratification is used in dimension-reduction arguments of the kind introduced by Federer and Almgren. It was applied to the non-degenerate case by Weiss in [22] to show Theorem 1.1(iii.), among other results. These dimension-reduction techniques have been augmented into powerful “effective” versions, first by Cheeger and Naber [5] to study manifolds with Ricci curvature bounded below, and then greatly strengthened by Naber and Valtorta [18] in the context of minimal surfaces. Following the deep analogy between minimal surfaces and local minimizers of (1.1) established by [1], this “effective” version was used by Edelen and Engelstein [12] to address local minimizers of the non-degenerate case, as well as two-phase and vector versions.

The key to the improvement introduced by Cheeger and Naber is the quantitative control introduced in the form of a quantitative stratification. See Definition 2.12, below.

**Definition 2.11** (Symmetric functions and rescalings) Given an integer \( 0 \leq j \leq n - 2 \), a function \( u \in C^0(\mathbb{R}^n) \) is called **\( j \)-symmetric** if \( u \) is homogeneous and there is a linear subspace \( V \) with \( \text{dim}(V) = j \) such that for all \( y \in V \),

\[
u(x) = u(x + y).
\]
For a function $u \in W^{1,2}_{loc}(B_2(0)) \cap C^0(B_2(0))$ and a non-trivial ball $\overline{B_r(x)} \subset B_2(0)$, we define the rescaling of $u$ at $x$ at scale $0 < r$ by,

$$T_{x,r}u(y) := \frac{u(x + ry)}{\left( \int_{\partial B_1(0)} u(x + ry)^2 d\sigma(y) \right)^{\frac{1}{2}}}.$$  

(2.3)

We shall say that $u$ is $(j, \epsilon)$-symmetric in $B_r(x)$ if there exists a non-trivial $j$-symmetric function $\phi$ such that

$$\|T_{x,r}u - T_{0,1}\phi\|_{L^2(B_1(0))} \leq \epsilon.$$  

(2.4)

**Definition 2.12** (Quantitative stratification) Let $n \geq 2$ and $0 \leq k \leq n - 1$ be integers. Let $0 < \gamma < \infty$, $\Gamma$ be a $k$-dimensional $C^{1,\alpha}$ manifold, and $Q(x) = \text{dist}(x, \Gamma)^\gamma$. If $u$ is a local minimizer of $(1.1)$, we make the following definitions.

For each $\epsilon > 0$, $r_0 > 0$, and integer $0 \leq j \leq n - 1$, we define the $(j, \epsilon, r_0)$-strata $S^{j}_{\epsilon,r_0}$ as follows,

$$S^{j}_{\epsilon,r_0} := \{ x \in \Gamma \cap \partial \{ u > 0 \} : u \text{ is not } (k + 1, \epsilon)\text{-symmetric in } B_r(x) \text{ for all } r_0 \leq r \leq \text{dist}(x, \partial \Omega) \}.$$  

That is, $x \in S^{j}_{\epsilon,r_0}$ if and only if $\|T_{x,r}u - T_{0,1}\phi\|_{L^2(B_1(0))} \geq \epsilon$ for all non-trivial $(j + 1)$-symmetric functions $\phi$. We shall use the notation $S^j_{\epsilon} := S^j_{\epsilon,0}$. Observe that $S^j_{\epsilon} := \bigcup_{0<\epsilon} S^j_{\epsilon}$ is the traditional, qualitative $j$-stratum. Where we compare the strata of different functions, we shall use the notation $S^j_{\epsilon,r_0}(u)$.

**Remark 2.13** The quantitative strata defined above are closed under $L^2_{loc}$ convergence of the underlying functions. In addition, they enjoy the following properties.

1. $S^0 \subset S^1 \subset \ldots \subset S^{n-2} \subset S^{n-1}$ and $S \subset S^{n-2}$.
2. For all $\delta < \epsilon$ and $r < R$, $S^{j}_{\epsilon,r} \subset S^{j}_{\delta,R}$. Furthermore, for integers $0 < j < k \leq n - 1$, $S^{j}_{\epsilon,r} \subset S^{k}_{\epsilon,r}$.

We note that by definition, $\partial \{ u > 0 \} \cap \text{reg}(\partial \{ u > 0 \}) \subset S^{n-1}$.

### 3 Minimizers and integral average growth estimates

In this section, we recall several important results for local minimizers of $J_Q(\cdot, \Omega)$. The main results in this section are the integral average growth estimates first established in [1] which are crucial to establishing local Lipschitz estimates and non-degeneracy of the functions. The results of this section are limited to local minimizers.

#### 3.1 Basic properties

**Lemma 3.1** (Outer Variation, [1] Lemma 2.2 and Lemma 2.3) Suppose that $Q \in C^{0,\alpha}(\Omega)$ such that $0 \leq Q$. Minimizers and local minimizers of $J_Q(\cdot, \Omega)$ in the class $K_{\mu,\Omega}$ are subharmonic ($\Delta u \geq 0$ in $\Omega$) in the sense of distributions. That is, for all $\phi \geq 0$, $\phi \in C^\infty_c(\Omega)$,

$$\int_\Omega \nabla u \cdot \nabla \phi dx \leq 0.$$  

$$\text{Lemma 3.1}$$

$\text{Lemma 3.1}$
Moreover, $u$ is harmonic in the sense of distributions in the interior of $\{u > 0\}$. And, we may choose a representative of $u \in W^{1,2}(\Omega)$ which is defined point-wise by subharmonicity

$$u(x) = \lim_{r \to 0} \int_{B_r(x_0)} u \, dx.$$  

In this case, $u \in L^\infty_{loc}(\Omega)$ satisfying

$$0 \leq u(x) \leq \text{ess sup}_{\Omega} u_0.$$

We now record the Noether Equations (inner variation) associated to being a local minimizer of $J_Q(\cdot, \Omega)$. See [20] Lemma 9.5 for the details of the calculation when $\gamma = 0$.

**Lemma 3.2** (Inner variation) Let $0 \leq k \leq n - 1$ be integers. Let $0 < M < \infty$, $0 < \gamma$, $\Gamma$ be a $k$-dimensional $(1,M)$-$C^{1,\alpha}$ submanifold, and $Q(x) = \text{dist}(x, \Gamma)^\gamma$. For $u$ a local minimizer of $J_Q(\cdot, \Omega)$, and any $\phi \in C_0^1(\Omega; \mathbb{R}^n)$,

$$0 = \int_{\Omega} \left( |\nabla u|^2 + Q^2 \chi_{\{u > 0\}} \right) \text{div}(\phi) \, dx - \int_{\Omega} 2\nabla u \cdot \nabla \phi + \chi_{\{u > 0\}} \nabla (Q^2(x)) \cdot \phi \, dx$$

$$= \int_{\Omega} \left( |\nabla u|^2 + Q^2 \chi_{\{u > 0\}} \right) \text{div}(\phi) \, dx$$

$$- \int_{\Omega} 2\nabla u \cdot \nabla \phi + 2\gamma \text{dist}(x, \Gamma)^{2\gamma-1} \chi_{\{u > 0\}} \nabla (\text{dist}(x, \Gamma)) \cdot \phi \, dx. \quad (3.1)$$

### 3.2 Integral average growth estimates

The techniques used in [1] to establish the non-degeneracy of a local minimizer $u$ rely upon comparing $u$ with two other functions. First, we will need to compare $u$ with the harmonic extension of $u$ in a ball $B_r(0)$. Second, we will need to compare $u$ with a function $w = \min\{u, v\}$ in $B_r(0)$ for

$$v(x) = \left( \sup_{y \in B_r(0)} \{u(y)\} \right) \max \left\{ 1 - \frac{|x|^{2-n} - r^{2-n}}{|xy|^{2-n} - r^{2-n}}, 0 \right\}.$$

**Remark 3.3** Since $\|\nabla v\|_{L^2}^2 \leq C(s, n) \sup_{y \in B_r(0)} \{u^2(y)\} \gamma n - 2$ and harmonic functions are energy minimizers, for every $\epsilon_0$-local minimizer $u$, there is a uniform scale

$$r_0 = r_0(n, \sup_{\Omega} u_0, \|\nabla u\|_{L^2}, \epsilon_0) \quad (3.2)$$

at which we can apply these arguments. We shall refer to this scale $r_0$ as the standard scale.

**Theorem 3.4** ([1] Lemma 3.2) Let $n \geq 2$, and let $u$ be a $\epsilon_0$-local minimizer of $J_Q(\cdot, \Omega)$. There is a constant, $C_{\text{max}}(n) > 0$ such that for every $0 < r < r_0$ and every ball $B_r(x) \subset \Omega$ if

$$\int_{\partial B_r(x)} u \, d\sigma > C_{\text{max}} r \cdot \max_{y \in B_r(x)} Q(y)$$

then $u > 0$ in $B_r(x)$. In particular, if $\mathcal{H}^n(\{u = 0\} \cap B_r(x)) > 0$ then

$$\int_{\partial B_r(x)} u \, d\sigma < C_{\text{max}} r \cdot \max_{y \in B_r(x)} Q(y).$$
We now argue that functions which satisfy the preceding properties are Lipschitz continuous.

**Lemma 3.5** (cf. [1] Corollary 3.3) Let $n \geq 2$, $\Omega \subset \mathbb{R}^n$ be an open bounded set, and $Q \in C^{0,\alpha}(\Omega)$ be a non-negative function. Let $u$ be an $\epsilon_0$-local minimizer $J_Q(\cdot,\Omega)$. Let $r_0$ be the standard scale. Let $\Omega_r = \{ x \in \Omega : \text{dist}(x, \partial \Omega) > r \}$. Then, $u$ enjoys the following properties.

1. $u$ is harmonic in the classical sense in any $B_r(x) \subset \Omega$ for which $\mathcal{H}^{n}(\{u = 0\} \cap B_r(x)) = 0$.
2. $\{u > 0\}$ is open.
3. $u$ is locally Lipschitz, satisfying

$$
\|\nabla u\|_{L^\infty(\Omega_{2r_0})} \leq \max\{C(n) \frac{\|u\|_{L^1(\Omega_{r_0})}}{r_0^{n+1}}, C(n)C_{\max} \max_{y \in B_{r_0}(\partial \Omega \setminus \{u > 0\})} \{Q(y)\}\}.
$$

**Proof** First, observe that since $u$ is a local minimizer, it must be that for all $0 < r \leq r_0$ $J_Q(u, B_r(x_0)) \leq J_Q(h^{x_0,r}_u, B_r(x_0))$, where $h^{x_0,r}_u$ is the harmonic extension of $u$ in $B_r(x_0)$. Thus, using the orthogonality of harmonic functions,

$$
\int_{B_r(x_0)} \left| \nabla u - \nabla h^{x_0,r}_u \right|^2 dx \leq \int_{B_r(x_0)} Q^2(x) \chi_{\{u = 0\}}(x) dx.
$$

Therefore, if $\mathcal{H}^{n}(\{u = 0\} \cap B_r(x)) = 0$, then $u = h^{x_0,r}_u$. This proves (1).

To see that $\{u > 0\}$ is open, let $x_0 \in \Omega \cap \partial \{u > 0\}$. By (1) and the Maximum Principle, there must be an $0 < \rho$ such that $\mathcal{H}^{n}(\{u = 0\} \cap B_r(x)) > 0$ for all $0 < r < \rho$. Therefore, we may let $r \to 0$, we may invoke Lemma 3.1 and the pointwise definition of $u(x_0)$ to obtain

$$
u(x_0) \leq \int_{\partial B_r(x_0)} ud\mathcal{H}^{n-1} \leq C_{\max} r \max_{y \in B_r(x_0)} \{Q(y)\} \leq C_{\max} r \max_{y \in \Omega} \{Q(y)\} \to 0,
$$

where the last line follows from the fact that $|Q|$ is bounded since $Q \in C^{0,\alpha}(\Omega)$ and $\Omega$ is bounded. Thus, $\{u > 0\} \cap \partial \{u > 0\} = \emptyset$ and $\{u > 0\}$ is open.

The local Lipschitz bound follows from standard harmonic estimates. We break into two cases. For any $x_0 \in \{u > 0\} \cap \Omega_{2r_0} \setminus B_{r_0/2}(\partial \{u > 0\})$, we estimate

$$
|\nabla u(x_0)| \leq C_n \frac{2^{n+1}}{r_0^{n+1}} \int_{B_{r_0/2}(x_0)} ud\mathcal{H}^{n-1} \leq C_n \frac{1}{r_0^{n+1}} \|u\|_{L^1(\Omega_{r_0})}.
$$
Now, consider \( x_0 \in \{ u > 0 \} \cap \Omega_{2r_0} \cap B_{r_0}/2(\partial \{ u > 0 \}) \). Let \( 0 < \delta \leq r_0/2 \) and \( x \in \partial \{ u > 0 \} \) be such that \( |x - x_0| = \text{dist}(x_0, \partial \{ u > 0 \}) = \delta \). Then, \( u \) is harmonic in \( B_\delta(x_0) \), and

\[
|\nabla u(x_0)| \leq C_n \frac{1}{\delta^{n+1}} \int_{B_\delta(x_0)} u \, dx
\]

\[
\leq C_n \frac{1}{\delta^{n+1}} \int_{B_{2\delta}(x)} u \, dx
\]

\[
\leq C(n) \frac{1}{\delta^{n+1}} \int_{0}^{2\delta} \int_{\partial B_{\gamma}(x)} u \, d\gamma n^{-1} \, dr
\]

\[
\leq C(n) \frac{1}{\delta^{n+1}} \max_{y \in B_{2\delta}(x)} \{ Q(y) \} \int_{0}^{2\delta} \omega_{n-1} \, dr
\]

\[
\leq C(n) C_{\text{max}} \max_{y \in B_{2\delta}(x)} \{ Q(y) \}
\]

\[
\square
\]

**Corollary 3.6** Let \( 0 \leq k \leq n - 2 \), and let \( \Gamma \) be a \( k \)-dimensional \((1, M)\)-\( C^{1,\alpha} \) submanifold such that \( \Gamma \cap \Omega \neq \emptyset \). Let \( 0 < \gamma \), \( Q(x) = \text{dist}(x, \Gamma)' \), and \( u \) a \( \epsilon_0 \)-local minimizer of \( J_Q(., \Omega) \) with standard scale \( 0 < r_0 \). If \( x_0 \in \partial \{ u > 0 \} \), then for all \( 0 < r < \frac{1}{2} r_0 \)

\[
||\nabla u||_{L_{\infty}(B_r(x_0))} \leq C(n) r \max_{w \in B_{2r}(x_0)} \{ Q(w) \}
\]

And, for all \( x \in \{ u > 0 \} \cap B_r(x_0) \)

\[
|\nabla u(x)| \leq C(n) 2^\gamma \max\{\text{dist}(x, \partial \{ u > 0 \}), \text{dist}(x, \Gamma)\}'.
\]

**Proof** This corollary follows immediately from our choice of \( Q \) and noting that in the penultimate line of Lemma 3.5, we have for \( \delta = \text{dist}(x_0, \partial \{ u > 0 \}) \)

\[
\max_{y \in B_{2\delta}(x)} \{ Q(y) \} \leq 2^\gamma \max\{\text{dist}(x, \partial \{ u > 0 \}), \text{dist}(x, \Gamma)\}'.
\]

\[
\square
\]

**Lemma 3.7** ([1] Lemma 3.4) Let \( u \) be an \( \epsilon_0 \)-local minimizer of \( J_Q(., \Omega) \). Let \( s \in (0, 1) \) be fixed. Then, for all \( 0 < r \leq \frac{1}{2} r_0 \) and all \( B_{2r}(x_0) \subset \Omega \), there is a constant \( C_{\text{min}} = C(n, s) \) such that if

\[
\int_{\partial B_r(x_0)} u \, d\sigma \leq r C_{\text{min}} \min_{y \in B_{\epsilon_0}(x_0)} \{ Q(y) \},
\]

then \( u = 0 \) on \( B_{sr}(x_0) \). In particular, if \( \{ u > 0 \} \cap B_{sr}(x_0) \neq \emptyset \), then

\[
\int_{\partial B_r(x_0)} u \, d\sigma > r C_{\text{min}} \min_{y \in B_{\epsilon_0}(x_0)} \{ Q(y) \}.
\]

**Corollary 3.8** (Non-degeneracy of functions) Let \( \Gamma \) be a \((1, \frac{1}{4})\)-\( C^{1,\alpha} \) submanifold of dimension \( 0 \leq k \leq n - 1 \) such that \( 0 \in \Gamma \). Let \( Q(x) = \text{dist}(x, \Gamma)' \). Let \( u \) be a \( \epsilon_0 \)-local minimizer of \( J_Q(., B_2(0)) \). Then for every \( 0 < r < r_0 \) and \( x \in \{ u > 0 \} \cap (B_{r_0}(\partial \{ u > 0 \}) \setminus B_r(\partial \{ u > 0 \})) \),

Then there is a constant \( 0 < C(n, \gamma) < 1 \) such that

\[
u(x) \geq C(n, \gamma) r \max\{Q(x), (r/2)'\}.
\]
Proof We break the proof into two cases. First, suppose that $B_{\frac{1}{2}r}(x) \cap \Gamma = \emptyset$. Then since $u > 0$ on $B_{\frac{1}{2}r}(x)$, then by Lemma 3.7 with $s = 1/2$, we have that

$$u(x) = \int_{\partial B_{\frac{1}{2}r}(x)} u d\sigma \geq C_{\min} \left( \frac{1}{2} r \min_{w \in B_{\frac{1}{2}r}(x)} \{Q(w)\} \right) \geq C_{\min} \left( \frac{1}{2} r \right)^{2\gamma + 1}.\]

If, on the other hand, $B_{\frac{1}{2}r}(0) \cap \Gamma \neq \emptyset$, then let $y \in \Gamma \cap B_{\frac{1}{2}r}(x)$. By Lemma 2.6, we have that $\Gamma \cap B_{\frac{1}{2}r}(x) \subset B_{\frac{1}{2}r+s} (L)$ for some affine $k$-plane $L$ intersecting $B_{\frac{1}{2}r}(x)$. Thus, we may find a point $y' \in \partial B_{\frac{1}{2}r}(x)$ such that $B_{\frac{1}{2}r}(y') \cap \Gamma = \emptyset$. Note that we may choose $y'$ such that $Q(y') = \max_{w \in \partial B_{\frac{1}{2}r}(x)} \{Q(w)\}$. Thus,

$$u(x) = \int_{\partial B_{\frac{1}{2}r}(x)} u d\sigma \geq (\omega_{n-1} (r/2)^{n-1})^{-1} \int_{\partial B_{\frac{1}{2}r}(x) \cap B_{\frac{1}{2}r}(y')} u d\sigma \geq (\omega_{n-1} (r/2)^{n-1})^{-1} \int_{\partial B_{\frac{1}{2}r}(x) \cap B_{\frac{1}{2}r}(y')} \left( \int_{\partial B_{\gamma/16}(z)} u(w) d\sigma (w) \right) d\sigma (z) \geq C_{\min} \frac{r}{16^{n-1}} \min_{y' \in \partial B_{\frac{1}{2}r}(y')} \{Q(w)\} \geq C_{\min} \frac{16^{1-n} r}{8}.\]

\[\square\]

Lemma 3.9 (Interior Balls) Let $\Gamma$ be a $(1, \frac{1}{2})$-$C^{1,\alpha}$ submanifold of dimension $0 \leq k \leq n-1$ such that $0 \in \Gamma$. Let $Q(x) = \text{dist}(x, \Gamma)$. Suppose that $u$ is an $\epsilon_0$-local minimizer of $J_Q (\cdot, B_2(0))$ and $x \in \partial \{u > 0\}$. Then for every $0 < r \leq r_0$, if $B_r(x) \subset B_1(0) \setminus \Gamma$, then there exists a point $y \in \partial \{u > 0\} \cap \partial B_{\frac{1}{2}r}(x)$ and a constant

$$0 < c(n, \min_{B_{1/2r}(x)} \{Q(w)\}, \max_{B_{1/2r}(x)} \{Q(w)\}) < \frac{1}{2}$$

such that for all $z \in B_{\frac{1}{2}r}(y)$,

$$u(z) \geq C_{\min}(n) r \min_{B_{\frac{1}{2}r}(x)} \{Q(w)\},$$

where $C_{\min}(n)$ is the constant from Lemma 3.7.

Proof By Lemma 3.7 with $s = 1/2$,

$$\int_{\partial B_{\frac{1}{2}r}(x)} u d\sigma \geq C_{\min}(n) \frac{1}{2} r \min_{w \in B_{\frac{1}{2}r}(x)} \{Q(w)\}.\]

Thus, there must exist a point $y \in \partial B_{\frac{1}{2}r}(x) \cap \{u > 0\}$ such that

$$u(y) \geq C_{\min} \frac{1}{2} r \min_{w \in B_{\frac{1}{2}r}(x)} \{Q(w)\}.\]

By Lemma 3.6, $\text{Lip}(u|_{B_{1/2r}(x)}) \leq C(n) r \max_{B_{1/2r}(x)} \{Q(w)\}$, it must be the case that for

$$c = \frac{C_{\min} \min_{w \in B_{\frac{1}{2}r}(x)} \{Q(w)\}}{4C(n) \max_{B_{1/2r}(x)} \{Q(w)\}},$$

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on $B_{cr}(z)$ we have that $u(z) \geq C_{min}(n) \frac{1}{d} r \min_{\mathcal{J}^n} [Q(w)]$, as claimed. \qed

4 Weiss density

The Weiss densities were first introduced in [22] Theorem 3.1 for critical points of more general, two-phase versions of non-degenerate $J_Q(\cdot, \Omega)$ functionals. Since then Weiss-type densities, also called boundary adjusted energy functionals, have become an important tool in Bernoulli-type free boundary problems. Their key property is that they give a monotonicity formula for critical points of $J_Q(\cdot, \Omega)$, thereby extending the analogy between minimal surfaces and Bernoulli free-boundary problems established by Alt and Caffarelli [1] for regularity to the analysis of singularity.

In [21] Theorem 3.5, the authors extend this monotonicity formula to so-called “weak solutions” to a free boundary problem associated to $J_Q(\cdot, \Omega)$ for the case $Q(x', x_n) = |x_n|$. [3] applied this monotonicity formula to local minimizers of $J_Q(\cdot, \Omega)$ for $n = 2, \gamma = 1/2$, and $\Gamma$ flat. The main result of this section, Lemma 4.3 below, extends this monotonicity formula to the cases $0 < \gamma < \infty$ and $Q(x) = \text{dist}(x, \Gamma)^\gamma$ where $\Gamma$ is a $k$-dimensional $(1, M)\cdot C^{1,\alpha}$ submanifold. The calculation largely follows [21] Theorem 3.5, with the additional error carried around by fact that we now allow $\Gamma$ to be non-flat in a controlled way.

At the end of this section, we give a brief discussion of Lemma 4.3 and Corollary 4.4 in comparison with the standard results for $\gamma = 0$. See Remark 4.5.

For the remainder of this section, we assume $n \geq 2$ and $\Gamma$ a $(1, M)\cdot C^{1,\alpha}$ submanifold of dimension $k \in \{0, 1, \ldots, n - 1\}$ such that $\Gamma \cap \Omega \neq \emptyset$. Furthermore, assume that $0 < \gamma < \infty$ and $Q(x) = \text{dist}(x, \Gamma)^\gamma$.

Definition 4.1 (Weiss Density) For any open set $\Omega \subset \mathbb{R}^n$ and any function $u \in W^{1,2}(\Omega) \cap C(\Omega)$, we define the Weiss $(1 + \gamma)$-density with respect to $J_\Omega(\cdot, \Omega)$ at a point $x_0 \in \Omega$ and scale $0 < r$ such that $B_r(x_0) \subset \Omega$ as follows.

$$W^{1,2}_{\gamma+1}(x_0, r, u, \Gamma) = \frac{1}{r^{n+2\gamma}} \int_{B_r(x_0)} |\nabla u|^2 + Q^2(\cdot) \chi_{u > 0} dx - \frac{\gamma + 1}{r^{n+2\gamma}} \int_{\partial B_r(x_0)} u^2 d\sigma.$$ 

Remark 2.4 The Weiss density is invariant in the following senses. For $u_{x,r}(\gamma) = \frac{1}{r^{\gamma}} u(ry + x)$, and $\Gamma^{x,r} = \frac{1}{r}(\Gamma - x_0)$,

$$W^{1,2}_{\gamma+1}(0, 1, u_{x,r}, \Gamma^{x,r}) = W^{1,2}_{\gamma+1}(x, r, u, \Gamma)$$

Lemma 4.3 Let $u$ is local minimizer of $J_Q(\cdot, \Omega)$ and $x_0 \in \Gamma \cap \partial\{u > 0\}$. For almost every $0 < r \leq \frac{1}{2} \text{dist}(x, \Omega^d)$ we have the following formula.

$$\frac{d}{dr} W^{1,2}_{\gamma+1}(x_0, r, u, \Gamma) = \frac{2}{r^{n+2\gamma}} \int_{\partial B_r(x_0)} (\nabla u \cdot \eta - \frac{\gamma + 1}{r} u)^2 + \frac{1}{r^{n+1+2\gamma}} 2\gamma \int_{B_r(x_0)} \text{dist}(x, \Gamma)^{2\gamma - 1} \frac{\chi_{u > 0}}{\chi_{u > 0}} \frac{x - \pi_\Gamma(x)}{|x - \pi_\Gamma(x)|} \cdot (\pi_\Gamma(x) - x_0) dx. \quad (4.1)$$

Moreover,

$$\frac{d}{dr} W^{1,2}_{\gamma+1}(x_0, r, u, \Gamma) \geq \frac{2}{r^{n+2\gamma}} \int_{\partial B_r(x_0)} (\nabla u \cdot \eta - \frac{\gamma + 1}{r} u)^2 d\sigma - 16\gamma |\Gamma|_d r^{\alpha - 1}. \quad (4.2)$$

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where $\eta$ is the unit outer normal.

Our first task is to use the inner variation formula to rewrite $U'(r)$. Let $\phi_\tau \in C^\infty_c(B_r(0))$ be a function which satisfies the following conditions:

$$\phi_\tau = 1 \text{ in } B_{r-\tau}(0), \quad \nabla \phi_\tau(x) = -\frac{x}{\tau |x|} + o(\tau) \text{ in } B_r(0) \setminus B_{r-\tau}(0).$$

For example, a suitable modification of $\phi(x) = \max\{0, \min\{1, \frac{r-|x|}{\tau}\}\}$ suffices. Define a vector field $\xi_\tau(x) = x\phi_\tau(x)$. Observe that

$$\text{div}(\xi_\tau(x)) = n\phi_\tau(x) + x \cdot \nabla \phi_\tau(x)$$

and

$$D\xi_\tau(x) = \phi_\tau(x) I d_n + x \otimes \nabla \phi_\tau(x).$$

Therefore, plugging $\xi_\tau$ into the Noether equations of Lemma 3.2 have

$$0 = \int_\Omega |\nabla u|^2 + Q^2 \chi_{[u>0]}(n \phi_\tau(x) + x \nabla \phi_\tau(x)) dx$$

$$- 2 \int_\Omega |\nabla u|^2 \phi_\tau(x) + (\nabla u \cdot x) (\nabla u \cdot \nabla \phi_\tau(x)) dx$$

$$+ 2\gamma \int_\Omega \text{dist}(x, \Gamma)^{2\gamma - 1} \chi_{[u>0]} \nabla (\text{dist}(x, \Gamma)) \cdot \xi_\tau(x) dx.$$

Letting $\tau \to 0$, we have

$$0 = \int_{B_r(0)} |\nabla u|^2 + Q^2 \chi_{[u>0]} dx - r \int_{\partial B_r(0)} |\nabla u|^2 + Q^2 \chi_{[u>0]} d\sigma$$

$$- 2 \int_{B_r(0)} |\nabla u|^2 dx + 2r \int_{\partial B_r(0)} (\nabla u \cdot \eta)^2 d\sigma$$

$$+ 2\gamma \int_{B_r(0)} \text{dist}(x, \Gamma)^{2\gamma - 1} \chi_{[u>0]} \frac{x - \pi\Gamma(x)}{|x - \pi\Gamma(x)|} \cdot x dx.$$

Splitting the last term, by writing $x = x - \pi\Gamma(x) + \pi\Gamma(x)$, we obtain

$$2\gamma \int_{B_r(0)} \text{dist}(x, \Gamma)^{2\gamma - 1} \chi_{[u>0]} \frac{x - \pi\Gamma(x)}{|x - \pi\Gamma(x)|} \cdot x dx$$

$$= 2\gamma \int_{B_r(0)} \text{dist}(x, \Gamma)^{2\gamma} \chi_{[u>0]} dx$$

$$+ 2\gamma \int_{B_r(0)} \text{dist}(x, \Gamma)^{2\gamma - 1} \chi_{[u>0]} \frac{x - \pi\Gamma(x)}{|x - \pi\Gamma(x)|} \cdot \pi\Gamma(x) dx.$$
Thus, we obtain the following equation

\[
0 = (n + 2\gamma) \int_{B_r(0)} |\nabla u|^2 + Q^2 \chi_{\{|u|>0\}} \, dx - r \int_{\partial B_r(0)} |\nabla u|^2 + Q^2 \chi_{\{|u|>0\}} \, d\sigma \\
- (2 + 2\gamma) \int_{B_r(0)} |\nabla u|^2 \, dx + 2r \int_{\partial B_r(0)} (\nabla u \cdot \eta)^2 \, d\sigma \\
+ 2\gamma \int_{B_r(0)} \text{dist}(x, \Gamma)^{2\gamma-1} \chi_{\{|u|>0\}} \frac{x - \pi_\Gamma(x)}{|x - \pi_\Gamma(x)|} \cdot \pi_\Gamma(x) \, dx.
\]

(4.5)

We now using (4.5) to re-write \(U'(r)\). Recalling (4.3), (4.5) gives

\[
U'(r) = \frac{1}{r^{n+1+2\gamma}} \left( 2r \int_{\partial B_r(0)} (\nabla u \cdot \eta)^2 \, d\sigma - (2 + 2\gamma) \int_{\partial B_r(0)} u \nabla u \cdot \eta \, d\sigma \\
+ 2\gamma \int_{B_r(0)} \text{dist}(x, \Gamma)^{2\gamma-1} \chi_{\{|u|>0\}} \frac{x - \pi_\Gamma(x)}{|x - \pi_\Gamma(x)|} \cdot \pi_\Gamma(x) \, dx \right).
\]

Now, using the Divergence theorem one obtains that \(\int_{B_r(0)} |\nabla u|^2 \, dx = \int_{\partial B_r(0)} u \nabla u \cdot \eta \, d\sigma\) for almost every \(0 < r\). Whence, for every such radius

\[
U'(r) = \frac{1}{r^{n+1+2\gamma}} \left( 2r \int_{\partial B_r(0)} (\nabla u \cdot \eta)^2 \, d\sigma - (2 + 2\gamma) \int_{\partial B_r(0)} u \nabla u \cdot \eta \, d\sigma \\
+ 2\gamma \int_{B_r(0)} \text{dist}(x, \Gamma)^{2\gamma-1} \chi_{\{|u|>0\}} \frac{x - \pi_\Gamma(x)}{|x - \pi_\Gamma(x)|} \cdot \pi_\Gamma(x) \, dx \right).
\]

Now, since, \(\frac{d}{dr} W_{\gamma+1}(0, r, u, \Gamma) = U'(r) - (\gamma + 1) V'(r)\), we calculate as follows.

\[
\frac{d}{dr} W_{\gamma+1}(0, r, u, \Gamma) \\
= \frac{1}{r^{n+1+2\gamma}} \left( 2r \int_{\partial B_r(0)} (\nabla u \cdot \eta)^2 \, d\sigma - (2 + 2\gamma) \int_{\partial B_r(0)} u \nabla u \cdot \eta \, d\sigma \\
+ 2\gamma \int_{B_r(0)} \text{dist}(x, \Gamma)^{2\gamma-1} \chi_{\{|u|>0\}} \frac{x - \pi_\Gamma(x)}{|x - \pi_\Gamma(x)|} \cdot \pi_\Gamma(x) \, dx \right)
\]

\[
- (\gamma + 1) \left( \frac{2}{r^{n+2+2\gamma}} \int_{\partial B_r(0)} u^2 \, d\sigma + \frac{1}{r^{n+1+2\gamma}} \int_{\partial B_r(0)} 2u \nabla u \cdot \eta \, d\sigma \right)
\]

\[
= \frac{2}{r^{n+1+2\gamma}} \left( 2r \int_{\partial B_r(0)} (\nabla u \cdot \eta)^2 \, d\sigma - (2 + 2\gamma) \int_{\partial B_r(0)} u \nabla u \cdot \eta \, d\sigma \\
+ 2\gamma \int_{B_r(0)} \text{dist}(x, \Gamma)^{2\gamma-1} \chi_{\{|u|>0\}} \frac{x - \pi_\Gamma(x)}{|x - \pi_\Gamma(x)|} \cdot \pi_\Gamma(x) \, dx \right)
\]

\[
+ \frac{1}{r^{n+1+2\gamma}} \left( 2\gamma \int_{B_r(0)} \text{dist}(x, \Gamma)^{2\gamma-1} \chi_{\{|u|>0\}} \frac{x - \pi_\Gamma(x)}{|x - \pi_\Gamma(x)|} \cdot \pi_\Gamma(x) \, dx \right).
\]

This proves (4.1).

To finish the lemma, we note that for any \(y \in \Gamma\), if \(x \in B_r(0)\) is such that \(y = \pi_\Gamma(x)\), then \(x - y \in N_r \Gamma\). We denote \(\frac{x - y}{|x - y|} = \eta(y)(\pi_\Gamma(x))\). Hence, we over-estimating as follows.

\[
\left| \frac{2\gamma}{r^{n+1+2\gamma}} \int_{B_r(0)} \text{dist}(x, \Gamma)^{2\gamma-1} \chi_{\{|u|>0\}} \frac{x - \pi_\Gamma(x)}{|x - \pi_\Gamma(x)|} \cdot \pi_\Gamma(x) \, dx \right| \\
\leq \frac{2\gamma}{r^{n+2}} \int_{B_r(0)} |\eta(y)(\pi_\Gamma(x))| \cdot \pi_\Gamma(x) \, dx \\
\leq \frac{2\gamma}{r^{n+2}} \int_{B_r(0)} 8[\Gamma]_\alpha r^{1+\alpha} \, dx \leq 16\gamma[\Gamma]_\alpha r^{1+\alpha-1},
\]

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where the penultimate inequality follows from Lemma 2.6. This gives (4.2).

**Corollary 4.4** Let $\Gamma$ be as above, and let $u$ be a local minimizer of $J_Q(\cdot, \Omega)$, and $x_0 \in \Gamma \cap \partial\{u > 0\}$. Then, for any $0 < r < R < r_0$,

$$W_{\gamma+1}(x_0, r, u, \Gamma) \leq W_{\gamma+1}(x_0, R, u, \Gamma) + C(\gamma, \alpha)[\Gamma]_\alpha R^\alpha.$$  \hspace{1cm} (4.6)

where $c(\gamma, \alpha) = 16^n \frac{\gamma}{a}$. Furthermore, $\lim_{r \to 0^+} W_{\gamma+1}(x_0, r, u, \Gamma) = W_{\gamma+1}(x_0, 0^+, u, \Gamma)$ exists and $W_{\gamma+1}(x_0, 0^+, u, \Gamma) \in [-c(n, \gamma), c(n)]$.

**Proof** The inequality (4.6) follows from integrating (4.2) and rearranging. To see that the limit exists, we first observe that by Corollary 3.6,

$$\frac{1}{r^{n-2+2(\gamma+1)}} \int_{B_r(x_0)} |\nabla u|^2 + Q^2(x) \chi_{\{u > 0\}} \, dx \leq r^{-n-2\gamma} \omega_n C(n) r^{n+2\gamma} \leq C(n).$$

Now, suppose for the sake of contradiction that $r_i \to 0$ and $a < b$ are accumulation points of $W_{\gamma+1}(x_0, r_i, u, \Gamma)$. Then for $r_i$ sufficiently small, $C(\gamma, \alpha) r_i^\alpha < |a - b|/2$ and $|W_{\gamma+1}(x_0, r_i, u, \Gamma) - a| < |a - b|/2$ and we obtain a contradiction.

**Remark 4.5** (Comparing $\gamma = 0$ and $\gamma \neq 0$) When $\gamma = 0$ and $u$ is a local minimizer of $J_1(\cdot, \Omega)$ and $W_1$ is the Weiss 1-density associated to $J_1(\cdot, \Omega)$, one can verify by direct computation that

$$\frac{d}{dr} W_1(x_0, r, u) = \frac{1}{r^n} \int_{\partial B_r(x_0)} \left( |\nabla u \cdot \eta - \frac{1}{r} u |^2 + n \int_{\partial B_r(x_0)} (W_1(x_0, r, z_u) - W_1(x_0, r, u)) \right) \, d\sigma.$$

where $z_u$ is the 1-homogeneous extension of $u|_{\partial B_r(x_0)}$. From here, one typically uses the Noether equations obtained from the inner variation to obtain an “equipartition of energy” result and rewrites

$$\frac{d}{dr} W_1(x_0, r, u) = \frac{2}{r^n} \int_{\partial B_r(x_0)} \left( \nabla u \cdot \eta - \frac{1}{r} u \right)^2 \, d\sigma.$$

See [20] Chapter 9 for details in the $\gamma = 0$ case, which hold mutatis mutandis when $0 < \gamma$.

When $0 < \gamma$, one may obtain a similar expression, using the $(1 + \gamma)$-homogeneous extension. Following the same direct computation and assuming that $\Gamma$ is flat, if $Q(x) = \text{dist}(x, \Gamma)^{\gamma}$, $u$ is a local minimizer of $J_Q(\cdot, \Omega)$, and $x_0 \in \Gamma \cap \partial\{u > 0\}$ one obtains

$$\frac{d}{dr} W_{1+\gamma}(x_0, r, u, \Gamma) = \frac{2}{r^{n+2\gamma}} \int_{\partial B_r(x_0)} \left( \nabla u \cdot \eta - \frac{1+\gamma}{r} u \right)^2 \, d\sigma + \frac{n + 2\gamma}{r^{n+1+2\gamma}} \int_{\partial B_r(x_0)} u^2 \, d\sigma,$$

where $z_u$ is now the $(1 + \gamma)$-homogeneous extension of $u|_{\partial B_r(x_0)}$. Indeed, combined with the weak geometry (Lemma 3.9) and the Lipschitz bound (Corollary 3.6), this expression is sufficient to obtain the results of Corollary 4.4.

However, this form is less convenient than (4.1) for several reasons. First, it obscures the role of the geometry of $\Gamma$ which we allow to be non-flat. While the errors incurred are controllable for non-flat $\Gamma$, for the purposes of this paper it is easier to follow [21] Theorem 3.5, apply the Noether Equations first, and avoid introducing $(1 + \gamma)$-homogeneous extensions.
5 Compactness and blow-ups

In this section, we show that $\epsilon_0$-local minimizers of $J_Q(\cdot, \Omega)$ satisfy nice compactness properties. In particular, we highlight Lemma 5.6, which describe the convergence of the positivity sets near $\Gamma = \{Q = 0\}$. Note that because of the degeneracy of $Q$, we do not obtain that the positivity sets of convergent subsequences of functions converge in the local Hausdorff metric. See Remark 5.7. This behavior is markedly different from known behavior when $\gamma = 0$. The rest of the results follow from standard techniques.

As a consequence of the compactness result (Theorem 5.4, below) the second collection of results in this section are results on the existence and characterization of blow-ups. In particular, in Lemma 5.12 it is shown that if $Q(x) = \text{dist}(x, \Gamma)^\gamma$ then blow-ups of local minimizers at points in $\Gamma \cap \partial\{u > 0\}$ are $(1 + \gamma)$-homogeneous.

First, we define the following rescalings.

**Definition 5.1** (Rescalings) Let $n \geq 2$. For any function $f : \mathbb{R}^n \to \mathbb{R}$, $x \in \mathbb{R}^n$, and $0 < r < \infty$ we define the rescalings

$$f_{x,r}(y) := \frac{f(ry + x)}{r^{\gamma+1}}.$$  

For any set $K \subset \mathbb{R}^n$ we define the rescalings,

$$K^{x,r} := \frac{1}{r}(K - x).$$

**Remark 5.2** We define these rescalings in addition to the rescalings of (2.3), because whereas the rescalings $T_{x,r}u$ are used to define the quantitative stratification, they do not obviously work with the structure of local minimizers. If $0 \leq k \leq n - 1$ is an integer, $\Gamma$ is a $k$-dimensional $C^{1,\alpha}$ submanifold, $0 < \gamma$, and $Q(x) = \text{dist}(x, \Gamma)^\gamma$ then by change of variables it is clear that if $u$ is an $\epsilon_0$-minimizer of $J_Q(\cdot, \Omega)$, then $u_{x,r}$ is an $\epsilon_0 \min\{r^{-2\gamma}, r^{-n}\}$-minimizer of $J_{Q_{x,r}}(\cdot, \Omega^{x,r})$. Furthermore, if $Q^{x,r}(y) = \text{dist}(y, \Gamma^{x,r})^\gamma$.

**Remark 5.3** We note that since $\Gamma$ is a $k$-dimensional $C^{1,\alpha}$ submanifold, for any sequence $r_j \to r \in [0, \infty)$ and any $x_0 \in \Gamma$, the sequences $\Gamma^{x_0,r_j} \to \Gamma^{x_0,r}$ locally in the Hausdorff metric on compact subsets. When $r = 0$, we will denote $\Gamma^{x_0,0}$ by $\Gamma^\infty$ and note that $\Gamma^\infty = T_{x_0}\Gamma$ a $k$-dimensional linear subspace of $\mathbb{R}^n$.

**Theorem 5.4** (Compactness) Let $n \geq 2$ and $0 \leq k \leq n - 1$ be integers. Let $0 < \epsilon_0$ and $0 < \gamma < \infty$. Let $\Gamma_i$ be a $(1, M)$-$C^{1,\alpha}$ submanifold satisfying $0 \in \Gamma_i$. Let $Q_i(x) = \text{dist}(x, \Gamma_i)^\gamma$, and let $\Omega_i$ be a Lipschitz domain with $B_\frac{1}{2}(0) \subset \Omega_i$.

Suppose that $\{u_i^\gamma\}_{\Gamma_i}$ is a sequence of $\epsilon_0$-local minimizer of $J_{Q_i}(\cdot, \Omega_i)$ with standard scale $r_{0,i}(\epsilon_0, u_i, \Omega_i) = 1$ which satisfies

$$0 \in \partial\{u_i > 0\} \cap \Gamma_i.$$  

Furthermore, let $\{r_i\}_{\Gamma_i}$ be a sequence of numbers $0 < r_i \leq 2$ such that and $r_i \to r \in [0, 2)$. Write $B = B_r(0)$ if $r > 0$ and $B = \mathbb{R}^n$ if $r = 0$.

Then, there is a subsequence, $r_j \to r$, a $k$-dimensional $(1, M)$-$C^{1,\alpha}$ submanifold $\Gamma$, and a function $u \in C^{0,1}_{\text{loc}}(B) \cap W^{1,2}_{\text{loc}}(B)$ such that,

1. $u_{0,r_j} \to u$ in $C^{0,1}_{\text{loc}}(B) \cap W^{1,2}_{\text{loc}}(B)$.
(2) For every $0 < R < \text{diam}(B)$,

$$
\Gamma_j^{0,r_j} \cap B_R(0) \to \Gamma \cap B_R(0)
$$

in the Hausdorff metric on compact subsets.

(3) For every $0 < R < \text{diam}(B)$ and any $\epsilon > 0$, there is an $N \in \mathbb{N}$ such that for all $j \geq N$,

$$
\partial\{u > 0\} \cap B_R(0) \subset B_\epsilon(\partial\{u_{0,r_j}^i > 0\}).
$$

(4) For any $0 < R < \text{diam}(B)$ and any $\epsilon > 0$, there is an $N \in \mathbb{N}$ such that for all $j \geq N$,

$$
\partial\{u_{0,r_j}^i > 0\} \cap B_R(0) \cap \{u > 0\} \subset B_\epsilon(\partial\{u > 0\}).
$$

Similarly, for any $0 < R < \text{diam}(B)$ and any $\epsilon > 0$, there is an $N \in \mathbb{N}$ such that for all $j \geq N$,

$$
\partial\{u_{0,r_j}^i > 0\} \cap B_R(0) \cap \{u = 0\} \subset B_\epsilon(\partial\{u > 0\} \cup \Gamma).
$$

(5) $\chi_{\{u_{0,r_j}^i > 0\}} \to \chi_{\{u > 0\}}$ in $L^1_{\text{loc}}(B)$.

(6) The function $u$ is harmonic in $\{u > 0\}$.

The proof of Theorem 5.4 is broken up into the following lemmata. We note that (2) is given as Lemma 2.7, and that (5) follows immediately from (3) and (4) and the fact that $\Gamma$ is a lower-dimensional submanifold.

**Lemma 5.5** Let $u^i$, $\Gamma_i$, $r_i \to r$, and $B$ as in Theorem 5.4. There is a subsequence $r_j \to r$ and a function $u \in C^{0,1}_{\text{loc}}(B) \cap W^{1,2}_{\text{loc}}(B)$ such that

1. $u_{0,r_j}^i \to u$ in $L^2_{\text{loc}}(B)$ and $C^{0,1}_{\text{loc}}(B)$.
2. $\nabla u_{0,r_j}^i \to \nabla u$ in $L^2_{\text{loc}}(B; \mathbb{R}^n)$.

**Proof** Fix a $0 < R < \text{diam}(B)$. Since $0 \in \Gamma_i \cap \partial\{u^i > 0\}$, the local Lipschitz bounds in Corollary 3.6 immediately implies that,

$$
||\nabla u_{0,r_i}^i||_{L^2(B_R(0))} = \left( \int_{B_R(0)} \left( \frac{\nabla u^i(r_j y)}{r_i} \right)^2 dy \right)^{\frac{1}{2}} \leq C(n) R^{\gamma} R^2.
$$

Thus, $\nabla u_{0,r_i}^i$ are uniformly bounded in $L^2(B_R(0); \mathbb{R}^n)$. It remains to show that $u_{0,r_i}^i$ are uniformly bounded in $L^2(B_R(0))$. By Corollary 3.6 and the assumption that $0 \in \Gamma_i \cap \partial\{u^i > 0\}$, for any $0 < r$ we bound

$$
\int_{B_r(0)} |u^i|^2 dx \leq C(n) r^{n+2(\gamma+1)}.
$$

Thus, $||u_{0,r_i}^i||_{L^2(B_R(0))} \leq C R^{\gamma+1} R^2$. By Rellich compactness, then, there exists a function $u$ and a subsequence $r_j \to 0$ such that,

$$
u_{0,r_j}^j \to u \text{ in } L^2_{\text{loc}}(B), \quad \nabla u_{0,r_j}^j \to \nabla u \text{ in } L^2_{\text{loc}}(B; \mathbb{R}^n).
$$
Note that by the uniform local Lipschitz bounds of $u_{0,r_j}^j$, we have that $u_{0,r_j}^j$ is locally Lipschitz continuous. Thus, up to a further subsequence $u_{0,r_j}^j \to u$ in $C^{0,1}(B_R(0))$ by Arzela-Ascoli. Diagonalizing for a sequence of $\mathcal{R} \not
earrow \text{diam}(B)$ proves the full claim. \hfill \Box

**Lemma 5.6** For $u^j$, $\Gamma_j$, $r_j \to r$, and $B$ as in Theorem 5.4, the following holds. Let $u_{0,r_j}^j$ be a subsequence as in Lemma 5.5 such that $\Gamma_j^{0,r_j} \to \Gamma$ as in Theorem 5.4(2).

For every $0 < R < \text{diam}(B)$ and any $\epsilon > 0$, there is an $N \in \mathbb{N}$ such that for all $j \geq N$, such that,

$$\partial \{u > 0\} \cap B_R(0) \subseteq B_\epsilon(\partial \{u_{0,r_j}^j > 0\})$$

$$\partial \{u_{0,r_j}^j > 0\} \cap B_R(0) \cap \{u > 0\} \subseteq B_\epsilon(\partial \{u > 0\})$$

$$\partial \{u_{0,r_j}^j > 0\} \cap B_R(0) \cap \{u = 0\} \subseteq B_\epsilon(\partial \{u > 0\} \cup \Gamma).$$

**Remark 5.7** We note that we do not have convergence of the free boundaries locally in the Hausdorff metric on compact subsets. It is a priori possible that the positivity sets, $\{u_{0,r_j}^j > 0\}$ have tendrils which reach out into $\{u = 0\}$ and cleave close to $\Gamma_j$ which thin as $j \to \infty$ and vanish in the limit. This problem is connected to the degeneracy of $Q$ and the problem of establishing interior balls in balls centered on $\Gamma \cap \partial \{u_0 > 0\}$.

**Proof** Let $0 < R < \text{diam}(B)$ and $1 \geq \epsilon > 0$ be given. We prove the first containment. Assume for the sake of contradiction that $x' \in \partial \{u > 0\}$, but that $x' \notin B_\epsilon(\partial \{u_{0,r_j}^j > 0\})$ for all $j \in \mathbb{N}$. Suppose there is a subsequence $j'$ such that $x' \in \{u_{0,j'}^j > 0\}$. By Corollary 3.8, we have that $u_{0,r_j}^{j'}(x') \geq C(n, \gamma) \epsilon^{1+\gamma}$.

Since $x' \in \partial \{u > 0\}$ and $u_{0,r_j}^j \to u$ in $C^{0,1}_{loc}(B)$, we have that

$$0 = u(x') = \lim_{j' \to \infty} u_{0,j'}^{j'}(x') \geq C(n, \gamma) \epsilon^{1+\gamma} > 0.$$ 

This is absurd. On the other hand, there exists a subsequence $j'$ such that $x' \in \{u_{0,j'}^{j'} = 0\}$, we obtain a similar contradiction. Indeed, since $x' \in \partial \{u > 0\}$ there must exist some $y \in B_\epsilon(x')$ such that $u(y) > 0$. Therefore, by $C^{0,1}$ convergence, there must exist a number $N$ sufficiently large such that $u_{0,j'}^{j'}(y) > 0$ for all $j' \geq N$. This forces $B_\epsilon(x') \cap \partial \{u_{0,r_j}^j > 0\} \neq \emptyset$ and proves the claim.

To see the second containment result, we assume for the sake of contradiction that there exists a sequence of $j \in \mathbb{N}$, such that there exists an $x_j \in B_R(0) \cap \partial \{u_{0,r_j}^j > 0\} \cap \{u > 0\}$ for which

$$x_j \notin B_\epsilon(\partial \{u > 0\}).$$

By passing to a further subsequence, we may assume that $x_j \to x' \in \overline{B_R(0)} \cap \{u > 0\}$. Since $u_{0,r_j}^j \to u$ in $C^{0,1}_{loc}(\mathbb{R}^n)$ which forces $0 < u(x') = \lim_{j \to \infty} u_{0,r_j}^j(x_j) = 0$. This is a contradiction. Hence, the claim follows.

The third containment follows from an analogous argument. We assume for the sake of a contradiction that there exists a sequence of $j \in \mathbb{N}$, for which we can find a point $x_j \in B_R(0) \cap \partial \{u_{0,r_j}^j > 0\} \cap \{u = 0\}$ for which

$$x_j \notin B_\epsilon(\partial \{u > 0\} \cup \Gamma_j^{0,r_j}).$$
However, by Lemma 3.9, and our choice of $Q$, we have that for all $x_j$, there is a ball $B_{c\epsilon}(y_j) \subset B_{c/2}(x_j) \cap \{u^0_{0,r_j} > 0\}$. Passing to a further subsequence $j'$, we may assume that $x_{j'} \to x'$ and $y_{j'} \to y'$. By $C^{0,1}$ convergence, then, $B_{c\epsilon}(y') \subset \{u > 0\}$. This contradicts the assumptions that $x_{j'} \in \{u = 0\}$ and $\text{dist}(x_{j'}, \partial\{u > 0\}) > \epsilon$ for all $j'$.

\[ \square \]

**Lemma 5.8** Let $u^i, r_j$, and $u$ be as in Theorem 5.4. Then $u$ is harmonic in $\{u > 0\}$.

**Proof** Let $x' \in \{u > 0\}$ with $\text{dist}(x', \partial\{u > 0\}) = 3\epsilon$. For sufficiently large $j$, we have that $B_{\epsilon}(x') \subset \{u_{0,r_j} > 0\}$. Since $u^i_{0,r_j}$ is harmonic in $B_{\epsilon}(x')$, and the $W^{1,2}(B_{2\epsilon}(x'))$ norm of $u^i_{0,r_j}$ are uniformly bounded, it is a classical result that $u^i_{0,r_j}$ converge in $C^\infty(B_{2\epsilon}(x'))$.

Thus, the harmonicity of $u$ at $x'$ is a consequence of $u$ satisfying the mean value property in $B_{\epsilon}(x')$. That is, for all $y' \in B_{\epsilon}(x')$,

\[ u(y') = \lim_{j \to \infty} u^i_{0,r_j}(y') = \lim_{j \to \infty} \int_{B_{\epsilon}(y')} u^i_{0,r_j} d\sigma = \int_{B_{\epsilon}(y')} u d\sigma. \]

\[ \square \]

**Lemma 5.9** Let $u^i, u$, and $r_m$ be as in Theorem 5.4. Then there exists a subsequence such that $u^i_{0,r_j} \to u$ strongly in $W^{1,2}_{\text{loc}}(\mathbb{R}^n)$.

**Proof** By Lemma 5.5, we may reduce to a subsequence such that $u^i_{x_0,r_j} \rightharpoonup u$ in $W^{1,2}_{\text{loc}}(\mathbb{R}^n)$. To show strong convergence, it suffices to show that for every $\phi \in C^\infty_c(\mathbb{R}^n)$

\[ \limsup_{j \to \infty} \int |\nabla u^i_{x_0,r_j}|^2 \phi dx \leq \int |\nabla u|^2 \phi dx. \]

Using the fact that $u$ and $u^i_{x_0,r_j}$ are harmonic in their positivity sets and the senses of convergence of $u^i_{x_0,r_j} \to u$ in Lemma 5.5, we obtain by integration by parts,

\[ \int |\nabla u^i_{x_0,r_j}|^2 \phi dx = - \int u^i_{x_0,r_j} \nabla u^i_{x_0,r_j} \cdot \nabla \phi dx \]

\[ \rightarrow - \int u \nabla u \cdot \nabla \phi dx = \int |\nabla u|^2 \phi dx. \]

It follows that $u^i_{x_0,r_j} \to u$ strongly in $W^{1,2}_{\text{loc}}(\mathbb{R}^n)$.

\[ \square \]

This concludes the proof of Theorem 5.4.

**Definition 5.10** (Blow-ups) Let $n \geq 2$ and $0 \leq k \leq n - 1$ be integers. Let $0 < \gamma < \infty$. Let $\Gamma$ be a $(1, M)$-C$^{1,\alpha}$ submanifold satisfying $0 \in \Gamma$. Let $Q(x) = \text{dist}(x, \Gamma)^{\gamma}$, and let $\Omega$ be a Lipschitz domain.

If $u$ is a local minimizer of $J_Q(\cdot, \Omega)$, $x_0 \in \Gamma \cap \partial\{u > 0\} \cap \Omega$, and $r_j \to 0$ any subsequential limit $u_{x_0,r_j} \to u_{x_0,0}$ in the senses of Theorem 5.4 is called a blow-up of $u$ at $x_0$.

**Corollary 5.11** Let $u$ be a local minimizer of $J_Q(\cdot, B_1(0))$ for $\Gamma$ a k-dimensional $(1, M)$-C$^{1,\alpha}$ submanifold with $x_0 \in \Gamma \cap \partial\{u > 0\} \cap B_1(0)$. Let $r_j \to 0$ and $u_{x_0,r_j} \to u_{x_0,0}$ in the senses of Theorem 5.4. Then for any fixed $0 < r < \infty$,

\[ \lim_{j \to \infty} W_{\gamma + 1}(0, r, u_{x_0,r_j}, \Gamma^{x_0,r_j}) = W_{\gamma + 1}(0, r, u_{x_0,0}, T_{x_0, \Gamma}). \]
The corollary follows immediately from Theorem 5.4(1) and (5).

**Lemma 5.12** (Homogeneity) Let \( x_0 \in S \) for a local minimizer \( u \), and let \( u_{x_0,0} \) be a blow-up as in Definition 5.10. The function \( u_{x_0,0} \) is \((\gamma + 1)\)-homogeneous and non-trivial.

**Proof** We recall that the Weiss density is invariant under the dilation \( W_{\gamma+1}(x_0, r, u, \Gamma) = W_{\gamma+1}(0, 1, u_{x_0,r}, \Gamma^{x_0,r}) \). Therefore, for all \( 0 < r_1 < r_2 < \infty \) and all sufficiently small \( 0 < r < \text{dist}(0, \partial \Omega) \),

\[
W_{\gamma+1}(0, r_1, u_{x_0,r_1}, \Gamma^{0,r_1}) - W_{\gamma+1}(0, r_2, u_{x_0,r_2}, \Gamma^{0,r_2})
\]

\[
= \int_{r_1}^{r_2} \frac{1}{s^{n+2+2\gamma}} \int_{\partial B_s} (\nabla u_{0,r_1}(x) \cdot x - (\gamma + 1)u_{0,r_1}(x))^2 d\sigma(x) ds
\]

\[
+ \int_{r_1}^{r_2} \frac{1}{s^{n+2+2\gamma}} \int_{B_s(0)} 2(\nabla u_{0,r_1}(x) \cdot x - (\gamma + 1)u_{0,r_1}(x))^2 \chi_{\{u_{0,r_1} > 0\}} (\frac{x - \pi_{1,0,r_1}(x)}{|x - \pi_{1,0,r_1}(x)|}) \sigma(x) ds.
\]

We now calculate the limit as \( j \to \infty \), in two ways. First, we recall that \( \Gamma^{0,r_j} \) converges to a \( k \)-dimensional linear subspace locally in the Hausdorff metric. Thus, the function

\[
\frac{(x - \pi_{1,0,r_j}(x)) \cdot \pi_{1,0,r_j}(x)}{|x - \pi_{1,0,r_j}(x)|} \to 0
\]

in \( L^1(B_1(0)) \). Recalling Corollary 5.11 and taking the limit as \( j \to \infty \), then, we obtain the following expression.

\[
W_{\gamma+1}(0, r_1, u_{x_0,0}, \Gamma^{x_0,0}) - W_{\gamma+1}(0, r_2, u_{x_0,0}, \Gamma^{x_0,0})
\]

\[
= \int_{r_1}^{r_2} \frac{1}{s^{n+2+2\gamma}} \int_{\partial B_s} (\nabla u(x) \cdot x - (\gamma + 1)u(x))^2 d\sigma(x) ds.
\]

We now show that this expression is zero. Because for any sequence \( r_j \to 0 \),

\[
\lim_{j \to \infty} W_{\gamma+1}(x_0, r_j, u, \Gamma) = W_{\gamma+1}(x_0, 0^+, u, \Gamma),
\]

it must be the case that for any \( c \in (0, 1) \),

\[
\lim_{j \to \infty} W_{\gamma+1}(x_0, cr_j, u, \Gamma) = \lim_{j \to \infty} W_{\gamma+1}(0, c, u_{x_0,r_j}, \Gamma^{x_0,cr_j})
\]

\[
= W_{\gamma+1}(0, c, u_{x_0,0}, T_{x_0} \Gamma)
\]

where the last equality is obtained by Corollary 5.11. Comparing the two limits proves that

\[
0 = W_{\gamma+1}(0, r_1, u_{x_0,0}, \Gamma^{x_0,0}) - W_{\gamma+1}(0, r_2, u_{x_0,0}, \Gamma^{x_0,0})
\]

\[
= \int_{r_1}^{r_2} \frac{1}{s^{n+2+2\gamma}} \int_{\partial B_s} (\nabla u_{x_0,0}(x) \cdot x - (\gamma + 1)u_{x_0,0}(x))^2 d\sigma(x) ds.
\]

Thus, \( u_{x_0,0} \) is \((\gamma + 1)\)-homogeneous in \( B_{r_2}(0) \setminus B_{r_1}(0) \). Repeating this for \( r_1 \to 0 \) and \( r_2 \to \infty \) shows that \( u_{x_0,0} \) is \((\gamma + 1)\)-homogeneous in \( \mathbb{R}^n \). It is non-trivial because \( x_0 \in S \) and so by assumption \( \{u_{x_0,0} > 0\} \text{ is non-trivial.} \)

**Corollary 5.13** Fix \( 0 < \gamma \) and \( 0 \leq k \leq n - 1 \). Let \( \Omega_i \) be a sequence of \( \Omega \)-local minimizers of \( J_{\Omega_i} \) such that \( Q_i(x) = \text{dist}(x, \Gamma_i)\) for \( \Gamma_i \) a \( k \)-dimensional \((1, \frac{4}{\gamma})\)-submanifold with \( 0 \in \Gamma_i \). Assume that the standard scales \( r_{0,i} = 1 \). For a subsequence \( u_i \to u \) and their corresponding \( \Gamma_i \to \Gamma \) in the senses of Theorem 5.4, the limiting function \( u \) satisfies the Noether equations (3.1) and the almost monotonicity equations of Lemma 4.3 and Corollary 4.4.
Proof By assumption, \( u_i \to u \) strongly in \( W^{1,2}(B_1(0)) \cap C^{0,1}(B_1(0)) \) and \( \Gamma_i \cap B_1(0) \to \Gamma \cap B_1(0) \) in the Hausdorff metric on compact subsets. The first point implies that 
\[
|\nabla u_i|^2 \to |\nabla u|^2
\]
in \( L^1(B_1(0)) \). The second point implies that \( \text{dist}(\cdot, \Gamma_i) \to \text{dist}(\cdot, \Gamma) \) in \( L^1(B_1(0)) \cap C^{0,1}(B_1(0)) \). Therefore, \( \text{dist}(\cdot, \Gamma_i)^{2\gamma} \to \text{dist}(\cdot, \Gamma)^{2\gamma} \) in \( L^1(B_1(0)) \). Thus, to show that \( u \) satisfies the Noether Equations (3.1) it is sufficient to show that 
\[
\text{dist}(\cdot, \Gamma_i)^{2\gamma - 1} \to \text{dist}(\cdot, \Gamma)^{2\gamma - 1}
\]
in \( L^1(B_1(0)) \) and \( \nabla(\text{dist}(\cdot, \Gamma_i)) \to \nabla(\text{dist}(\cdot, \Gamma)) \) in \( L^1(B_1(0)) \).

We first we note that for any \( k \)-dimensional \( (1, \frac{1}{4}) \)-\( C^{1, \alpha} \) submanifold \( \Gamma_i \) satisfying \( 0 \in \Gamma_i \), using Lemma 2.8, we may estimate

\[
\int_{B_1(0)} \text{dist}(x, \Gamma_i)^{2\gamma - 1} \, dx \leq C(k, n) \sum_{i=1}^{\infty} 2^{-i(n-k)/2}(-i+1)(2\gamma-1) < \infty
\]

\[
\int_{B_{2^{-N}}(\Gamma_i)} \text{dist}(x, \Gamma_i)^{2\gamma - 1} \, dx \leq C(k, n) \sum_{i=N}^{\infty} 2^{-i(n-k)/2}(-i+1)(2\gamma-1)
\]

Thus, for any \( k \)-dimensional \( (1, \frac{1}{4}) \)-\( C^{1, \alpha} \) submanifold \( \Gamma_i \) satisfying \( 0 \in \Gamma_i \), given any \( \epsilon > 0 \), there is \( \delta > 0 \) such that \( \int_{B_\delta(\Gamma_i)} \text{dist}(x, \Gamma_i)^{2\gamma - 1} \, dx < \epsilon \), and that this \( \delta = \delta(\epsilon) \). That is, this estimate is independent of \( i \in \mathbb{N} \).

Now, let \( \epsilon > 0 \) and let \( \delta = \delta(\epsilon) > 0 \). Since \( \Gamma_i \to \Gamma \) locally in the Hausdorff metric on compact subsets, for all sufficiently large \( i \in \mathbb{N} \), we have that \( B_{\frac{\delta}{2}}(\Gamma_i) \subset B_\delta(\Gamma) \). In particular, for all sufficiently large \( i \in \mathbb{N} \), both functions \( \text{dist}(\cdot, \Gamma_i)^{2\gamma - 1} \) and \( \text{dist}(\cdot, \Gamma)^{2\gamma - 1} \) are uniformly Lipschitz in \( B_1(0) \setminus B_\delta(\Gamma) \) and converge in \( L^1(B_1(0)) \). Hence,

\[
\lim_{i \to \infty} ||\text{dist}(\cdot, \Gamma_i)^{2\gamma - 1} - \text{dist}(\cdot, \Gamma)^{2\gamma - 1}||_{L^1(B_1(0))} = \lim_{i \to \infty} \int_{B_1(0) \cap B_\delta(\Gamma)} |\text{dist}(x, \Gamma_i)^{2\gamma - 1} - \text{dist}(x, \Gamma)^{2\gamma - 1}| \, dx
\]

\[
+ \lim_{i \to \infty} \int_{B_1(0) \setminus B_\delta(\Gamma)} |\text{dist}(x, \Gamma_i)^{2\gamma - 1} - \text{dist}(x, \Gamma)^{2\gamma - 1}| \, dx \leq 2\epsilon.
\]

Since \( \epsilon > 0 \) was arbitrary, \( \text{dist}(\cdot, \Gamma_i)^{2\gamma - 1} \to \text{dist}(\cdot, \Gamma)^{2\gamma - 1} \) in \( L^1(B_1(0)) \).

That \( \nabla(\text{dist}(\cdot, \Gamma_i)) \to \nabla(\text{dist}(\cdot, \Gamma)) \) in \( L^1(B_2(0); \mathbb{R}^n) \) follows immediately from Rellich-Kondrakov compactness, noting that \( \nabla(\text{dist}(\cdot, \Gamma_i)) \) are uniformly bounded in \( W^{1,2}(B_2(0)) \). This concludes the proof that \( u \) satisfies the Noether equations (3.1). Following the proof of Lemma 4.3 and Lemma 4.4 verifies that \( u \) satisfies the almost monotonicity equations (4.1), (4.2), and (4.6) with \( [\Gamma]_a \leq \limsup_{i \to \infty} [\Gamma_i]_a \leq \frac{1}{4} \).

\[\square\]

6 Density lower bounds

We now prove some important results about the Weiss density. One of the key results in an energy “gap” result in Lemma 6.2. This next lemma justifies the notion of \( W_{\gamma + 1} \) as a “density.”

Lemma 6.1 Fix \( 0 < \gamma \) and \( 0 \leq k \leq n - 1 \). Let \( Q(x) = \text{dist}(x, \Gamma)^\gamma \) for \( \Gamma \) a \( k \)-dimensional \( (1, \frac{1}{4}) \)-\( C^{1, \alpha} \) submanifold with \( 0 \in \Gamma \). Let \( u \) be a local minimizer of \( J_Q(\cdot, \Omega) \). Let \( x_0 \in \)
\[\Gamma \cap \partial\{u > 0\}.\] Then,
\[W_{y+1}(x_0, 0^+, u, \Gamma) = \lim_{r \to 0^+} \frac{1}{r^{n+2\gamma}} \int_{B_r(x_0)} Q^2(x) \chi_{\{u > 0\}} dx.\]

Therefore, \[W_{y+1}(x_0, 0^+, u, \Gamma) \in [0, c_n],\] where \[c_n = \int_{B_1(0)} \text{dist}(x, T_{x_0} \Gamma)^{2\gamma} dx.\] In particular, \[W_{y+1}(x_0, 0^+, u, \Gamma) = 0\] implies that every blow-up \[u_{x_0, 0} \equiv 0.\] Furthermore, the function \[x \mapsto W_{y+1}(x, 0, u, \Gamma)\] is upper semicontinuous when restricted to \[\Gamma \cap \partial\{u > 0\}.\]

**Proof** By translation and scaling, we may assume that \(x_0 = 0\) and that the standard scale of \(c_0\)-local minimizers in \(K_{u, \Omega} = 1\). Therefore, Theorem 5.4 implies there exists a function \(v\) and a sequence of radii \(r_j \to 0\), such that \(u_{x_0, r_j} \to v\) strongly in \(W^{1,2}(\mathbb{R}^n)\) and in \(C^{0,1}(\mathbb{R}^n)\). Thus, we calculate,
\[
\lim_{r \to 0^+} W_{y+1}(0, 1, u_{x_0, r}, \Gamma^{x_0, r}) = \lim_{j \to \infty} W_{y+1}(0, 1, u_{x_0, r_j}, \Gamma^{x_0, r_j})
= \frac{1}{\int_{B_1(0)} |\nabla v|^2 - (\gamma + 1) \int_{\partial B_1(0)} v^2 d\sigma + \lim_{j \to \infty} \int_{B_1(0)} \text{dist}(x, \Gamma^{x_0, r})^{2\gamma} \chi_{\{u_{x_0, r_j} > 0\}} dx}
= \frac{1}{\lim_{j \to \infty} r^{n+2\gamma}} \int_{B_r(x_0)} \text{dist}(x, \Gamma)^{2\gamma} \chi_{\{u > 0\}} dx.
\]

where the last inequality follows from the fact that \(v\) is a \((\gamma + 1)\)-homogeneous function. We note that while the limit function \(v\) may *a priori* depend upon the subsequence \(r_j\), Lemma 4.3 implies that the limit on the left-hand side exists. Therefore, the limit on the right-hand side is unique, even if \(v\) may not be.

Upper semicontinuity, in general, holds for limits of monotone increasing functions. We prove it here for the almost monotone Weiss density. Let \(x_0 \in \partial\{u > 0\} \cap B_1(0).\) Let \(\delta > 0\) and let \(0 < r(x, \delta)\) be such that \(W_{y+1}(x_0, r, u, \Gamma) \leq W_{y+1}(x_0, 0, u, \Gamma) + \delta.\) Then for \(x \in \Gamma \cap \partial\{u > 0\} \cap B_1(0)\) such that \(H^n(B_r(x) \Delta B_r(x_0)) \leq \delta r^n, |x - x_0| \leq \delta,\) and \(r \leq \delta^{\frac{1}{2}},\)
\[W_{y+1}(x, 0^+, u, \Gamma) \leq W_{y+1}(x, r, u, \Gamma) + C[\Gamma]_0 r^\alpha
\leq W_{y+1}(x_0, r, u, \Gamma) + |B_r(x) \Delta B_r(x_0)| \frac{C r^{2\gamma}}{r^{n+2(\gamma+1)}}
+ C r^{n-1} |x - x_0|^{2(\gamma+1)} + C[\Gamma]_0 r^\alpha
\leq W_{y+1}(x_0, r, u, \Gamma) + C \delta
\leq W_{y+1}(x_0, 0, u, \Gamma) + (C + 1) \delta.
\]

Thus, \(\limsup_{x \to x_0} W_{y+1}(x, 0, u, \Gamma) \leq W_{y+1}(x_0, 0, u, \Gamma).\)

**Lemma 6.2** (Density lower bound) *Let \(\Gamma\) be a \(k\)-dimensional \((1, \frac{1}{2})\)-\(C^{1,\alpha}\) submanifold. Let \(u\) be a local minimizer of \(J_Q(\cdot, B_2(0))\) for \(Q(x) = \text{dist}(x, \Gamma)^{\gamma}.\) Let \(x \in \partial\{u > 0\} \cap \Gamma.\) There is a constant, \(0 < c_0(n, \gamma),\) such that if \(W_{y+1}(x, 0^+, u, \Gamma) \neq 0,\) then \(W_{y+1}(x, 0^+, u, \Gamma) \geq c_0.\)

**Proof** The proof relies upon non-degeneracy of the functions \(u,\) and in particular the interior ball condition of Lemma 3.9. We note that by Lemma 5.12, every blow-up \(u_{x, 0^+}\) is homogeneous, and \(\{u_{x, 0^+} > 0\}\) is a cone. Therefore, we observe that if \(\{u_{x, 0^+} > 0\} \cap B_1(0) \not\subset B_r(T_x \Gamma)\) for some \(c = c(n, \gamma) > 0,\) then Lemma 3.9 and Lemma 6.1 gives the desired result. The strategy of the proof will be to show that such a constant \(c(n, \gamma) > 0\) must exist.
We argue by contradiction. Let $L$ be any $(n - 1)$-dimensional linear subspace which contains $T_x \Gamma$. Suppose that $\{u_{x,0^+} > 0\} \cap B_1(0)$ is compactly contained within the tubular neighborhood $B_c(T_x \Gamma) \subset B_c(L)$. Define the region $D_c$ as follows,

$$D_c = B_1(0) \cap \{(r, \theta) \in \mathbb{R}_+ \times \mathbb{S}^{n-1} : (1, \theta) \in \partial B_1(0) \cap B_c(L)\}.$$  

That is, $D_c$ is the intersection of the cone over $\partial B_1(0) \cap B_c(L)$ and $B_1(0)$. We note that $\{u_{x,0^+} > 0\} \cap B_1(0) \subset D_c$.

We define an auxiliary function $h$ as the solution to the following Dirichlet problem,

$$\begin{cases}
\Delta h = 0 & \text{in } D_c \\
h = 0 & \text{on } \partial D_c \cap B_1(0) \\
h = C & \text{on } \partial D_c \cap \partial B_1(0)
\end{cases}$$

By the local Lipschitz bounds of Corollary 3.6 and the Maximum Principle, for sufficiently large $C(n, \gamma)$ in the definition of $h$, $u_{x,0^+}(y) \leq h(y)$ for all $y \in \{u_{x,0^+} > 0\} \cap B_1(0)$.

We now do some quick geometric analysis on the auxiliary function $h$. By standard harmonic theory, since $D_c$ is a cone in $B_1(0)$ the Almgren frequency function,

$$N(0, r, h) = \frac{r \int_{B_r(0)} |\nabla h|^2 dx}{\int_{\partial B_r(0)} h^2 d\sigma}$$

is a monotone non-decreasing function of $0 < r < 1$. Furthermore the blow-up sequence $T_{0, r} h \rightarrow h_\infty$ strongly in $W^{1,2}(B_1(0))$ as $r \rightarrow 0^+$ for a function $h_\infty$ which is homogeneous of degree $N(0, 0^+, h) = \lim_{r \rightarrow 0} N(0, r, h)$ and $\Delta h_\infty = 0$ in $\{h_\infty > 0\}$, the cone over $\partial B_1(0) \cap B_c(T_x \Gamma)$. See [16] which proves these for star-shaped domains such as $D_c$.

We note that $u_{x,0^+}(y) \leq h(y)$ for all $y \in \{u_{x,0^+} > 0\} \cap B_1(0)$ it must be the case that $N(0, 0^+, h) \leq \gamma + 1$. We will show that for sufficiently small $0 < c = c(n, \gamma)$ in the definition $D_c$, $N(0, 0^+, h) \geq \gamma + 1$.

We begin with the symmetries of $\{h_\infty > 0\} \subset \mathbb{R}^n$. Choose a unit normal $\vec{v}$ to the hyperplane $L$. Since $\{h_\infty > 0\}$ is invariant with respect to the rotations which preserve the hyperplane $L$ and is homogeneous, the function $h_\infty$ may be written as a function of two variables,

$$h_\infty(\rho, \theta) = R(\rho)T(\theta)$$

where $\rho = |\vec{x}|$ and $\theta \in [-\frac{\beta}{\gamma}, \frac{\beta}{\gamma}]$, is the angle that the vector $\vec{x}$ raises above $L$, with positive $\theta$ chosen in the $\vec{v}$ direction. We may write out the Laplace equation in terms of these variables,

$$\Delta h = h_{\rho \rho} + \frac{n-1}{\rho} h_\rho + \frac{1}{\rho^2} \Delta_{S^{n-1}} h = h_{\rho \rho} + \frac{n-1}{\rho} h_\rho + \frac{1}{\rho^2} h_{\theta \theta} - \frac{(n-2) \tan(\theta)}{\rho^2} h_\theta = 0.$$

Since $h_\infty(\rho, \theta) = R(\rho)T(\theta)$, we obtain

$$\begin{align*}
0 &= R'' + (n-1) \frac{1}{\rho} R' - \lambda \frac{1}{\rho^2} R \\
0 &= T'' - (n-2) \tan(\theta) T' - \lambda T.
\end{align*}$$

for some $\lambda \in \mathbb{R}$. We note that $R(\rho) = \rho^c(\lambda)$ for $0 < c(\lambda) = -\frac{(n-2)+\sqrt{(n-2)^2+4\lambda}}{2} < \infty$. This forces $0 < \lambda < \infty$ and shows that $c(\lambda) \rightarrow \infty$ as $\lambda \rightarrow \infty$. We now consider the function
T. By Steiner symmetrization with respect to the hyperplane $L$ we see that we may assume that for $0 < \beta$ small enough, $T'(\theta) \leq 0$ for $0 \leq \theta \leq \frac{\beta}{2}$. Therefore, if we consider a scalar multiple of $h_\infty$ (which we relabel as $h_\infty$) such that $\max\{h_\infty(y) : y \in \partial B_1(0)\} = 1$ it must be that $\max\{T(\theta) : \theta \in [-\frac{\beta}{2}, \frac{\beta}{2}]\} = 1$. Hence, we are able to estimate,

$$|T''(\theta)| \leq \lambda$$

for all $0 \leq \theta \leq \frac{\beta}{2}$.

By a second order Taylor expansion without remainder, for any $\theta \in [0, \beta/2]$

$$T(\theta) = 1 + T'(0)(\theta) + \frac{1}{2} T''(\theta')\theta^2$$

for some $\theta' \in [0, \theta]$. Since $T(\frac{\beta}{2}) = 0$ and $T'(0) = 0$ there must be a point $0 \leq \theta' \leq \frac{\beta}{2}$ such that $\frac{8}{\beta^2} = |T''(\theta')| \leq \lambda$. Therefore, as $c \to 0$ in the definition of $D_c$, $\beta \to 0$ and $\lambda \to \infty$, which implies that the degree of homogeneity of $h_\infty c(\lambda) \to \infty$, as well. However, this contradicts assumption that $c(\lambda) \leq 1 + \gamma$. Thus, we see that

$$c(n, \gamma) \leq \sin\left(\frac{\sqrt{2}}{\sqrt{(1 + \gamma)(n - 2) + (1 + \gamma)^2}}\right) \text{ implies } \lambda \geq \frac{8}{(1 + \gamma)(n - 2) + (1 + \gamma)^2}$$

and $c(\lambda) \geq 1 + \gamma$. This proves the lemma.

\[\square\]

### 6.1 Corollaries of the density lower bound

**Lemma 6.3** Let $\Gamma$ be a $k$-dimensional $C^{1,\alpha}$-submanifold in $\mathbb{R}^n$, $0 \in \Gamma$, and let $Q(x) = \text{dist}(x, \Gamma)$. Let $u$ be a local minimizer of $J_Q(\cdot, B_2(0))$, then the set of non-degenerate singular points $\partial\{u > 0\} \cap \Gamma \setminus \Sigma$ is closed.

**Proof** We note that $\Gamma \cap \partial\{u > 0\}$ is closed. The claim follows from the fact that the function

$$x \mapsto W_{\gamma+1}(x, 0^+, u, \Gamma)$$

is upper-semicontinuous in $\Gamma \cap \partial\{u > 0\} \cap B_1(0)$. Thus, if $\{x_i\}_i \subset \Gamma \cap \partial\{u > 0\}$ then there is an $x \in \Gamma \cap \partial\{u > 0\}$ and a subsequence $x_j$ such that $x_j \to x$ and $W_{\gamma+1}(x_j, 0^+, u, \Gamma) \leq W_{\gamma+1}(x, 0^+, u)$. By Lemma 6.2, $W_{\gamma+1}(x_j, 0^+, u) \geq c_0 > 0$. Thus, $x \notin \Sigma$. \[\square\]

This next lemma allows us to find a scale at which the scalings $T_{x,r}u$ and $u_{x,r}$ are comparable.

**Lemma 6.4** Let $\Gamma$ be a $k$-dimensional $(1, M)-C^{1,\alpha}$ submanifold with $0 \in \Gamma$. Let $B_2(0) \subset \Omega$. Let $u$ be an $\epsilon_0$-local minimizer of $J_Q(\cdot, \Omega)$ for $Q(x) = \text{dist}(x, \Gamma)$. Suppose that $x \in \partial\{u > 0\} \cap \Gamma$. There exists an $0 < \eta_0(n, \gamma, \alpha)$ such that if $M \leq \eta_0$, then there exists $a < C_2(n, \gamma)$ such that if $x \in S$, then for every scale $0 < r < r_0$ and every $B_2(x) \subset \Omega$,

$$\int_{\partial B_1(0)} u_{x,r}^2(y) d\sigma(y) \geq C_2.$$

In particular, then, there exist a pair of constants $0 < c(n, \gamma) < C(n, \gamma) < \infty$ such that

$$c(n, \gamma)u_{x,r}(u) \leq T_{x,r}u(x) \leq C(n, \gamma)u_{x,r}. \quad (6.1)$$
Proof By Lemma 6.2 and Corollary 4.4, if \([\Gamma]_{\alpha} \leq M \leq c_0(n, \gamma) \frac{1}{2C(n, \gamma, \alpha)}\), then we have that \(W_{\gamma+1}(x, 0^+, u, \Gamma) \geq c_0\) and

\[
W_{\gamma+1}(x, R, u) \geq \frac{1}{2}c_0,
\]

for all \(0 \leq R \leq 1\). Therefore,

\[
\frac{1}{2}c_0 \leq W_{\gamma+1}(x, R, u, \Gamma) \leq \frac{1}{R^{n+2\gamma}} \int_{B_R(x)} |\nabla u|^2 + Q^2(y) \chi_{\{u > 0\}} dy \leq \frac{2C(n) \max_{y \in [u > 0] \cap B_R(x)} \{\text{dist}(y, \Gamma)\}^2 \gamma H^n([u > 0] \cap B_R(x))}{R^{n+2\gamma}}.
\]

Note, then, that if \(\max_{y \in [u > 0] \cap B_R(x)} \{\text{dist}(y, \Gamma)\} \leq CR\) for \(C\) sufficiently small with respect to \(c_0(n, \gamma)\), then we have a contradiction. Thus, for all \(0 < R < 1\), there is a point \(y \in [u > 0] \cap B_R(x)\) such that \(\text{dist}(y, \Gamma) \geq C(n, \gamma) R\). For example, \(C(n, \gamma)\) may be taken to be a scalar multiple of \(c(n, \gamma)\) in Lemma 6.2.

Now, if \(B_{\frac{1}{4}} C(n, \gamma) R(y) \cap \partial \{u > 0\} = \emptyset\), then Lemma 3.9 implies that \(u(y) \geq C(n, \gamma) R^{1+\gamma}\).

By the Maximum Principle, we can find a point \(y_R \in \partial B_{R} \cap \partial \{u > 0\}\) such that \(u(y_0) \geq C(n, \gamma) R^{1+\gamma}\). By the Lipschitz bound on \(u\), there exists a ball \(C(n, \gamma) R\) in which \(u \geq \frac{1}{2} C(n, \gamma) R^{1+\gamma}\). Integrating,

\[
\int_{\partial B_R(x)} u^2(y) d\sigma(y) \geq C(n, \gamma) R^{2(1+\gamma)}.
\]

On the other hand, if \(y_0 \in B_{\frac{1}{4}} CR \cap \partial \{u > 0\}\), then by Lemma 3.7,

\[
\int_{\partial B_{\frac{1}{4}} C(n, \gamma) R(y_0)} u(y) d\sigma(y) \geq C_{\min}(\frac{1}{4} C(n, \gamma) R)^{1+\gamma}.
\]

Thus, there must exist a point \(z \in \partial B_{\frac{1}{4}} C(n, \gamma) R(y_0) \cap \{u > 0\}\) such that

\[
u(z) \geq C_{\min}(\frac{1}{4} C(n, \gamma) R)^{1+\gamma}.
\]

Since by Lemma 3.6, \(\text{Lip}(u_{B_{\frac{1}{4}} C(n, \gamma) R(y_0)}) \leq C(n) R^{\gamma}\), it must be the case that there exists a ball \(B_{C(n, \gamma) R(z)}\) such that \(u \geq C(n, \gamma) R^{1+\gamma}\) on \(B_{C(n, \gamma) R(z)}\). Thus, the desired lower bound holds,

\[
\int_{\partial B_R(x)} u^2(y) d\sigma(y) \geq c(n, \gamma) R^{2(1+\gamma)}.
\]

The upper bound follows from Corollary 3.6. \(\square\)

**Corollary 6.5** There is an \(0 < r_2(n, \gamma, \alpha, [\Gamma]_{\alpha}, \epsilon_0, \Lambda, A)\) such that the statement of Theorem 5.4 holds with rescalings \(T_{x,r} u\) replacing rescalings \(u_{x,r}\) for all \(0 < r < r_2\).

**Proof** For the \(\eta_0(n, \gamma, \alpha) > 0\) the constant given by Lemma 6.4, we let \(0 < r_1\) be a constant such that \([\Gamma]_{\alpha} r_1^{\gamma} \leq \eta_0\). Under this condition, note that if \(x_0 \in \Gamma\), \([\Gamma^{x_0} r_1]_{\alpha} \leq \eta_0\) for all \(0 < r < r_1\). We define,

\[
r_2 = \min\{r_1, r_0\}
\]

where \(r_0\) is the standard scale for \(\epsilon_0\)-local minimizers in the class \(K_{\alpha_0, \Omega}\). \(\square\)
Remark 6.6 (Non-degeneracy) Let $\Gamma$ be a $k$-dimensional $C^{1,\alpha}$-submanifold in $\mathbb{R}^n$, $0 \in \Gamma$, and let $Q(x) = \text{dist}(x, \Gamma)^2$. If $\{u_i\}_1$ are a sequence of local minimizers of $J_{Q}(\cdot, B_2(0))$ with standard scale identically $r_0 = 1$, $x_i \in S(u_i)$, $0 < r_i \leq r_2$ is a sequence of scales, then for any subsequential limit function $u_0$ obtained through Lemma 5.4, $u_0$ is non-degenerate in the sense that (6.1) holds.

7 Proof of Theorem 1.3

Let $u$ be an $\epsilon_0$-local minimizer in the class $K_{u_0, B_2(0)}$ with $\|\nabla u\|_{L^2(B_2(0))} \leq \Lambda$ and $\sup_{B_2(0)} u_0 \leq A$. Let $0 < \epsilon, 0 < \rho \leq r$, and the integer $0 \leq j \leq k$ in $S^j_{\epsilon, \rho}$ be given. The strategy of the proof is to consider a scale $0 < r_3$ at which we can employ the techniques of [12]. If $r_3 \leq r$, we can easily cover $S^j_{\epsilon, \rho} \cap B_1(0)$ by $C(n, k)r_3^{-k}$ balls of radius $r$. If, on the other hand, $r \leq r_3$, then, in each ball, we follow the proof of [12] Theorem 1.11, using Theorem 5.4 and Lemma 5.9 applied to the rescalings $T_{\epsilon, r}u$ in place of [12] Theorem 1.3, Lemma 7.3 in place of [12] Lemma 3.1, and Lemma 7.6 in place of [12] Lemma 3.3. The rest of the argument of [12] follows verbatim, save that where Edelen and Engelstein need to make the Holder seminorm $[\Gamma]_a$ small, we need to make the Holder seminorm $[\Gamma]_a$ small. We define $0 < r_3$ in Definition 7.7.

For the benefit of the reader, we adumbrate the proof, below. Following [18], the proof falls broadly into two parts. The first part consists of two Reifenberg-type results, the so-called Discrete Reifenberg and Rectifiable Reifenberg Theorems, which provide the necessary packing estimates. These theorems are purely geometric measure theoretic results, which we restate, below. We refer the interested reader to [18] for their proofs.

Theorem 7.1 (Discrete Reifenberg [18]) Let $\{B_{r_q}(q)\}_q$ be a collection of disjoint balls with $q \in B_1(0)$ with radii $0 < r_q \leq 1$ and let $\mu$ be the packing measure $\mu = \sum_q r_q^k \delta_q$, for $\delta_q$ the Dirac measure at $q$. There exist constants $\delta_{RD}(n) > 0$ such that if

$$\int_0^r \int_{B_{r_q}(q)} \rho_{\mu, 2}(z, s)^2 d\mu \frac{ds}{s} \leq \delta_{RD} r^k$$

for all $0 < r \leq 1$ and all $x \in B_1(0)$, then $\mu(B_1(0)) = \sum_q r_q^k \leq C_{DR}(n)$.

Theorem 7.2 (Rectifiable Reifenberg [18]) Let $S \subset \mathbb{R}^n$. There exists a constant $\delta_{RR}(n) > 0$ such that if

$$\int_0^{2r} \int_{B_r(x)} \rho_{\mathcal{H}^k | S, 2}(z, s) d\mathcal{H}^k | S(z) \frac{ds}{s} \leq \delta_{RR} r^k$$

for all $x \in B_1(0)$ and $0 < r \leq 1$, then $S$ is countably $k$-rectifiable and $\mathcal{H}^k(S \cap B_r(x)) \leq C_{RR}(n) r^k$ for all $x \in B_1(0)$ and $0 < r \leq 1$.

The second part of the argument consists of constructing a finite number of collections of balls to which we may apply these theorems. Broadly speaking, the construction of these covers rely upon three observations and a complicated stopping-time argument. Below, we state the three necessary observations, with careful details of any changes that need to be made from the proof in [12]. However, we shall only sketch the stopping-time argument, as the details follow [12] verbatim.
7.1 Necessary ingredients

The observations are a kind of quantitative stability. These type of results follow a predictable “limit-compactness” argument by contradiction.

Briefly, Lemma 7.3 is a quantitative version of Lemma 5.12, wherein we showed that if \( W_{\gamma+1}(x, r, u, \Gamma) \) is constant for \( 0 < r \leq 1 \), then \( u \) is \((\gamma + 1)\)-homogeneous, and therefore is \((0, 0)\)-symmetric in \( B_1(x) \). In 7.3, below, we show that if \( W_{\gamma+1}(x, r, u) \) almost constant, then \( u \) is almost symmetric in \( B_1(x) \).

**Lemma 7.3** (Quantitative Rigidity) Fix \( 0 \leq k \leq n - 1 \). Let \( \delta > 0 \) and let \( u \) be an \( \epsilon_0 \)-local minimizer of \( J_{\Gamma}(\cdot, B_2(0)) \) with \( \Gamma \) a \( k \)-dimensional \( C^{1, \alpha} \) submanifold. Assume that \( 0 < r_0 = 1 \) and that \( 0 \in \Gamma \cap \partial \{ u > 0 \} \). Then there is a \( \gamma_1 = \gamma(n, \delta) > 0 \) such that if \( [\Gamma]_\alpha \leq \gamma_1 \), and

\[
W_{\gamma+1}(0, 1, u, \Gamma) - W_{\gamma+1}(0, \gamma_1, u, \Gamma) \leq \gamma_1,
\]

then \( u \) is \((0, \delta)\)-symmetric in \( B_1(0) \).

**Proof** We argue by contradiction. Suppose that there were a sequence of \( \gamma_i \to 0 \) and \( u_i \) solutions to \( J_{\Gamma}(\cdot, B_2(0)) \) with \( \Gamma_i \) a \( k \)-dimensional \( C^{1, \alpha} \) submanifold such that \( 0 \in \Gamma_i \cap \partial \{ u_i > 0 \} \), \( [\Gamma_i]_\alpha \leq \gamma_i \), and

\[
W_{\gamma+1}(0, 1, u_i, \Gamma_i) - W_{\gamma+1}(0, \gamma_i, u_i, \Gamma_i) \leq \gamma_i,
\]

but \( u_i \) is not \((0, \delta)\)-symmetric in \( B_1(0) \) for any \( i \in \mathbb{N} \). By Theorem 5.4, we may pass to a subsequence under which

1. \( \Gamma_j \to \Gamma \) a \( k \)-dimensional linear subspace,
2. \( u_j \to u \) strongly in \( W^{1, 2}(B_2(0)) \) and in \( C^{0, 1}(B_2(0)) \).
3. By the aforementioned convergence, \( W_{\gamma+1}(0, r, u_i, \Gamma_i) \to W_{\gamma+1}(0, r, u, \Gamma) \).

By Corollary 5.13, the function \( u \) satisfies the monotonicity formula, and by Corollary 5.11 we have that

\[
W_{\gamma+1}(0, 1, u, \Gamma) - W_{\gamma+1}(0, 0^+, u, \Gamma) = \int_0^1 \frac{1}{s^{a+1+2\gamma}} \int_{\partial B_r(0)} (\nabla u \cdot x - (\gamma + 1)u)^2 d\sigma(x) ds = 0.
\]

Hence, arguing as in Lemma 5.12, we have that \( u \) is \((\gamma + 1)\)-homogeneous and is \((0, 0)\)-symmetric by the non-degeneracy of Remark 6.6. Since \( u_i \to u \) in \( L^2(B_2(0)) \), this contradicts the assumption that each \( u_i \) was not \((0, \delta)\)-symmetric. \( \square \)

The next observation is quite similar in nature. It is a quantitative version of the following fact: if \( u \) is homogeneous with respect to the origin and translation invariant (i.e., symmetric) along a linear subspace \( L \), and \( u \) is homogeneous with respect to a point \( x \notin L \), then \( u \) is symmetric with respect to \( \text{span}\{x, L\} \). In brief, the lemma below says that if \( 0 \in S^k_\epsilon \) and \( u \) is far from being \((k + 1)\)-symmetric in \( B_1(0) \), but \( u \) is also almost symmetric in \( B_1(0) \), then \( S^k_\epsilon \) must be contained in the tubular neighborhood of a \( k \)-plane.

**Lemma 7.4** Let \( \epsilon > 0 \). Let \( u \) be an \( \epsilon_0 \)-local minimizer of \( J_{\Gamma}(\cdot, B_{10r}(x)) \) for \( 0 < 10r \leq r_0 \), \( Q(x) = \text{dist}(x, \Gamma) \), and \( \Gamma \) a \( k \)-dimensional \( (1, \delta) \)-\( C^{1, \alpha} \) submanifold. There is a \( \delta(n, k, \gamma, \epsilon) > 0 \) such that if

\[
\begin{align*}
&u \text{ is } (0, \delta) \text{-symmetric in } B_{8r}(x) \\
u \text{ is NOT } (k + 1, \epsilon) \text{-symmetric in } B_{8r}(x),
\end{align*}
\]

\( \square \)
then for any finite Borel measure $\mu$,

$$
\beta_{k,2}(x, r)^2 \leq \frac{C(n, \alpha, \gamma, \epsilon)}{r^k} \int_{B_r(x)} W_{r+1}(y, 8r, u, \Gamma) - W_{r+1}(y, r, u, \Gamma) + [\Gamma]_{0}(8r)^{\epsilon} d\mu(y).
$$

**Proof** Fix the ball $B_r(x)$ and let $\mu$ be a finite Borel measure. We shall use the notation $A_{r, R}(x)$ to denote the annulus $B_R(x) \setminus B_r(x)$.

This lemma follows from the analysis of the following non-negative quadratic form,

$$
B(v, w) := \int_{B_r(x)} (v \cdot (y - X))(w \cdot (y - X)) d\mu(y)
$$

where $X = \int_{B_r(x)} y d\mu(y)$ is the center of mass. We note that $B$ has an orthogonal eigenbasis $v_1, \ldots, v_n$ with associated eigenvalues $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \geq 0$. Note that

$$
\beta_{k,2}(x, r)^2 = \frac{\mu(B_r(x))}{r^{2k}}(\lambda_{k+1} + \ldots + \lambda_n).
$$

The first estimate we need is that for sufficiently small $\delta > 0$, and any orthonormal basis, $\{v_i\}_i^n$, there exists a constant $0 < C(n, \gamma, \alpha, \epsilon) < \infty$,

$$
\frac{1}{C(n, \gamma, \alpha, \epsilon)} \leq r^{-n-2} \int_{A_{3r, 4r}(x)} \sum_{i=1}^{k+1} (v_i \cdot \nabla u(z))^2 dz
$$

This follows from a straightforward “limit-compactness” argument, since for a $(0,0)$-symmetric function if $\sum_{i=1}^{k+1} (v_i \cdot \nabla u(z))^2 = 0$ then $u$ is $(k+1, 0)$-symmetric if it is non-trivial. Note that non-triviality follows from non-degeneracy. The details follow [12] Theorem 5.1 verbatim, substituting Theorem 5.4 and Theorem 5.9 in place of [12] Theorem 1.3.

The second estimate requires greater detail. We begin by observing that for any integers $0 \leq i \leq n$ and any $z$,

$$
\lambda_i (v_i \cdot \frac{1}{r^\gamma} \nabla u(z)) = B(v_i, \frac{1}{r^\gamma} \nabla u(z))
$$

$$
= \int_{B_r(x)} (v_i \cdot (y - X))(\frac{1}{r^\gamma} \nabla u(z) \cdot (y + z - z - X)) d\mu(y)
$$

$$
- \frac{(\gamma + 1)}{r^\gamma} u(z) \int_{B_r(x)} v_i \cdot (y - X) d\mu(y)
$$

$$
= \int_{B_r(x)} (v_i \cdot (y - X))(\frac{1}{r^\gamma} \nabla u(z) \cdot (y - z) - \frac{(\gamma + 1)}{r^\gamma} u(z)) d\mu(y)
$$

$$
\leq \lambda_i^{\frac{1}{2}} \left( \int_{B_r(x)} \frac{1}{r^\gamma} \nabla u(z) \cdot (y - z) - \frac{(\gamma + 1)}{r^\gamma} u(z) d\mu(y) \right)^{\frac{1}{2}}.
$$
Thus, $\lambda_i (v_i \cdot \nabla u(z))^2 \leq \left( \frac{1}{r^2} \int_{B_r(x)} |\nabla u(z) \cdot (\gamma - z) - (\gamma + 1) u(z)|^2 d\mu(y) \right)^{\frac{1}{2}}$. We continue calculating.

$$\lambda_i r^{-n-2} \int_{A_{3r,4r}(x)} (v_i \cdot \nabla u(z))^2 dz$$

$$= r^{-n-2} \int_{A_{3r,4r}(x)} \left( \frac{1}{r^{2\gamma}} \int_{B_r(x)} |\nabla u(z) \cdot (\gamma - z) - (\gamma + 1) u(z)|^2 d\mu(y) \right) dz$$

$$\leq \frac{n+2+2\gamma}{r} \int_{B_r(x)} \int_{A_{3r,4r}(x)} \frac{1}{|z - \gamma|^{n+2+2\gamma}} |\nabla u(z) \cdot (\gamma - z) - (\gamma + 1) u(z)|^2 dz d\mu(y)$$

$$= \frac{n+2+2\gamma}{r} \int_{B_r(x)} \int_{A_{1r,8r}(x)} \frac{1}{|z - \gamma|^{n+2+2\gamma}} |\nabla u(z) \cdot (\gamma - z) - (\gamma + 1) u(z)|^2 dz d\mu(y)$$

$$= \frac{n+2}{r} \int_{B_r(x)} \int_1^{8r} \left( \frac{d}{d\rho} W_{\gamma+1}(y, \rho, u, \Gamma) + 16\gamma [\Gamma]_a r^{\alpha-1} \right) d\rho d\mu(y)$$

$$\leq C(n, \gamma, \alpha) \int_{B_r(x)} W_{\gamma+1}(y, 8r, u, \Gamma) - W_{\gamma+1}(y, r, u, \Gamma) + [\Gamma]_a r^{\alpha} d\mu(y).$$

With these two estimates, we have,

$$\beta_{\mu,2}^k(x, r)^2 \leq \frac{\mu(B_r(x))}{r^k} n\lambda_{k+1}$$

$$\leq \frac{\mu(B_r(x))}{r^k} nC(n, \epsilon, \gamma, \alpha) \left( \sum_{i=1}^{k+1} \frac{\lambda_i}{r^{n+2}} \int_{A_{3r,4r}(x)} (v_i \cdot \nabla u(z))^2 dz \right)$$

$$\leq \frac{\mu(B_r(x))}{r^k} C(n, \gamma, \alpha, \epsilon, k) \left( \int_{B_r(x)} W_{\gamma+1}(y, 8r, u, \Gamma) - W_{\gamma+1}(y, r, u, \Gamma) + [\Gamma]_a r^{\alpha} d\mu(y) \right)$$

$$\leq \frac{1}{r^k} C(n, \gamma, \alpha, \epsilon, k) \int_{B_r(x)} W_{\gamma+1}(y, 8r, u, \Gamma) - W_{\gamma+1}(y, r, u, \Gamma) + [\Gamma]_a r^{\alpha} d\mu(y).$$

This proves the lemma. \[\square\]

The last crucial observation is a dichotomy which is a quantitative version of the geometry in the homogeneous case. Simply stated, if $u$ is $k$-symmetric, but not $(k+1)$-symmetric, then $\partial \{ u > 0 \}$ can be contained in either a $k$- or $(k-1)$-dimensional subspace.

**Lemma 7.5** (Key Dichotomy) Let $0 \leq k \leq n - 1$ be an integer, and let $\Gamma$ by a $k$-dimensional $(1, M)$-$C^{1,\alpha}$ submanifold. Let $u$ be an $e_0$-local minimizer of $J_{Q\Gamma}$ such that $r_0 = 2$, $0 \in \partial \{ u > 0 \} \cap \Gamma$. Let $0 < \rho < 1$, $0 < \gamma' < 1$, and $0 < \eta' < 1$ be small, fixed constants. Let $\sup_{x \in B_{1}(0)} W_{\gamma+1}(z, 2, u, \Gamma) \leq E \in (0, E_0]$ Then there exists a constant $\eta_1(n, \gamma, \alpha, \epsilon, \rho, \gamma_1, \eta, E_0) \leq \rho$ such that if $0 < \eta \leq \eta_1$ and $M \leq \eta_1$ then either

i. $W_{\gamma+1}(x, 2, u, \Gamma) \geq E - \eta'$ for all $x \in S^k \cap B_{1}(0)$, or,

ii. There exists a $(k - 1)$-dimensional affine subspace $L^{k-1}$ such that $\{ x \in B_{1}(0) : W_{\gamma+1}(x, 2\eta, u, \Gamma) \geq E - \eta' \} \subset B_{\rho}(L^{k-1})$.

Note that we may take $E_0 = C(n)$ per the proof of Corollary 4.4. The proof of this lemma is immediate from the following lemma.

**Lemma 7.6** Fix $0 \leq k \leq n - 1$. There are constants,

$$\eta_1(n, \epsilon, \rho, \gamma_1, \eta', \alpha, \gamma) \ll \rho$$

and $\beta(n, k, \eta', \rho, \gamma_1, \epsilon, \alpha) < 1$,
such that the following holds: Let $u$ be a minimizer to $J_Q(\cdot, B_4(0))$ with $\Gamma$ a $k$-dimensional $C^{1,\alpha}$ submanifold such that $0 \in \Gamma \cap \partial \{ u > 0 \}$ and $|\Gamma|_\alpha \leq \Lambda$. Assume that

$$\sup_{x \in B_1(0)} W_{\gamma+1}(x, 2, u, \Gamma) \leq E \in [0, c_n].$$

Suppose that $\eta \leq \eta_1$ and $|\Gamma|_\alpha \leq \eta$, and there are points $y_0, \ldots, y_j \in B_1(0) \cap \Gamma \cap \partial \{ u > 0 \}$ satisfying,

$$y_i \notin B_\rho((y_0, \ldots, y_{i-1}))$$

$$W_{\gamma+1}(y_i, 2\eta, u, \Gamma) \geq E - \eta, \quad \forall i = 0, 1, \ldots, j.$$  

Then, writing $L = (y_0, \ldots, y_j)$, for all $x \in B_\beta(L) \cap B_1(0)$,

$$W_{\gamma+1}(x, \gamma_1 \rho, u, \Gamma) \geq E - \eta'$$

and

$$S_{\epsilon, \eta}^j \cap B_1(0) \subset B_\beta(L).$$

**Proof** There are two claims. We argue both by contradiction. Suppose that the first fails. That is, suppose that there were a sequence of $u_i$ solutions to $J_Q(\cdot, B_2(0))$ with $\Gamma$, a $k$-dimensional $C^{1,\alpha}$ submanifold such that $0 \in \Gamma_i \cap \partial \{ u_i > 0 \}$, $|\Gamma_i|_\alpha \leq \eta_i \to 0$, and collections $E_j, y_{ij}, \eta_j$, and $\beta_j$ which satisfy the hypotheses. Suppose that $\beta_j \to 0$ but that for each $j \in \mathbb{N}$ there is an $x_j \in B_{\beta_j}(L_j) \cap B_1(0)$ such that

$$W_{\gamma+1}(x_j, \gamma_i, u_i, \Gamma_i) \leq E - \eta'.$$

By Theorem 5.4, we may pass the a subsequence and obtain a function $u$ such that,

1. $u_j \to u$ strongly in $W^{1,2}(B_4(0))$ and in $C^{0,1}(B_3(0))$.
2. $\Gamma_i \to \Gamma$ in the Hausdorff metric and $\Gamma$ is a $k$-dimensional linear subspace.
3. $E_j \to E$, $y_{ij} \to y_i$, $L_j \to L$, $x_j \to x \in B_1(0) \cap L$.

We note that since $\rho$ is fixed, the $y_i$ span $L$. We note that by Corollary 5.11 and Corollary 5.13, $u$ satisfies,

1. $\sup_{z \in B_1(0)} W_{\gamma+1}(z, 2, u, \Gamma) \leq E$.
2. $W_{\gamma+1}(x, \gamma_1 \rho, u, \Gamma) \leq E - \eta'$.
3. $W_{\gamma+1}(y_i, 0^+, u, \Gamma) \geq E$ for each $i = 0, 1, 2, \ldots, j$.

Since $\Gamma$ is flat, Lemma 4.3 gives that the Weiss density is monotone increasing. Thus, $u$ is $(\gamma + 1)$-homogeneous at $y_i$. Therefore, $u$ is translation invariant along $L$ in $B_{1+\delta} \subset \cup_i B_2(y_i)$ for some $\delta > 0$. In particular, then, $W_{\gamma+1}(x, 0^+, u, \Gamma) = E$, which is a contradiction.

We now argue the second claim. That is, fix $\beta$, and assume that we have sequences $u_j, E_j, y_{ij}, L_j$, which satisfy the hypotheses of the lemma, and $\eta_j \to 0$ such that for $j$ there exists some $x_j \in S^j_{\epsilon, \eta_j}(u_j) \cap B_1(0) \setminus B_\beta(L_j)$. By Theorem 5.4, we may pass to a subsequence such that $u_j, E_j, y_{ij}, L_j$ converge as above. Again, we have that for some $\delta > 0$, the limit function $u$ will be $k$-symmetric with respect to $L$ in $B_{1+\delta}(0)$. Since the limit point $x \in B_1(0) \setminus B_\beta(L)$, any blow-up of $u$ at $x$ will be $(k+1)$-symmetric. In particular, for some $r > 0$, $u_j$ will be $(k+1, \epsilon)$-symmetric in $B_r(x)$. This is a contradiction. \qed

**Definition 7.7** (The scale $0 < r_3$) We define the base scale $0 < r_3$ using the constants from the lemmata, above. Let the $0 < \delta$ in Lemma 7.3 be the $0 < \delta$ from Lemma 7.4. Let $0 < \eta'$ from Lemma 7.6 be the $0 < \gamma_1(n, \delta)$ from Lemma 7.3. Let $0 < r_2(n, \gamma, \alpha, |\Gamma|_\alpha, \epsilon_0, \Lambda, A)$
be as in Corollary 6.5. We define $0 < r_3$ to satisfy the following two conditions. First, we let $0 < r_3$ be a constant which satisfies

$$0 < r_3 \leq \frac{1}{20} r_2,$$

where $r_2$ is as in Corollary 6.5. Second, let $0 < r_3$ be sufficiently small so that in each ball $x \in \Gamma, \Gamma^{x \cdot r_3}_\alpha \leq \min\{\gamma_1, \delta, \eta_1\}$, where $\eta_1(n, \epsilon, \rho, \gamma_1, \eta', \alpha, \gamma)$ is as in Lemma 7.6. This makes $0 < r_3 = r_3(n, k, \gamma, \alpha, [\Gamma]_\alpha, \epsilon_0, \Lambda, A, \epsilon, j)$.

With these observations in hand, the argument becomes procedural, and the construction of the desired coverings follows the proof in Edelen-Engelstein verbatim, with Theorem 5.4 and Lemma 5.9 in place of [12] Theorem 1.3, Lemma 7.3 in place of [12] Lemma 3.1, and Lemma 7.6 in place of [12] Lemma 3.3.

In particular, these observations and Theorem 7.1 allow us to prove the following lemma.

**Lemma 7.8** Let $0 \leq k \leq n - 1$ and $0 < \gamma, \alpha$ be fixed. Let $\Gamma$ be a $k$-dimensional $(1, M)\cdot C^{1,\alpha}$ submanifold such that $0 \in \Gamma$, and let $Q(x) = \text{dist}(x, \Gamma)\gamma$. Let $u$ be an $\epsilon_0$-local minimizer for $J(u, B_3(0))$ with $0 < r_0 = 2$, and let $E = \sup_{y \in B_1(0)} W_{y + 1}(u, 2, y, \Gamma)$.

Let $0 < R$. There is an $\eta(n, \gamma, \alpha, \epsilon, M_0) > 0$ such that if $[\Gamma]_\alpha \leq \eta$ and $\{B_{\eta r_3}(p)\}_p$ is a collection of disjoint balls satisfying,

i. $W_{y + 1}(u, \eta r_3, \eta, \alpha) \geq E - \eta$.  
ii. $p \in S_{\epsilon, R}$.  
iii. $R \leq r_p \leq 1$.

then $\{B_{\eta r_3}(p)\}_p$ satisfies the packing estimate,

$$\sum_p r_p^\epsilon \leq C(n).$$

The proof follows that of [12] Lemma 6.1 *mutatis mutandis*. The rest of the proof of Theorem 1.3 relies upon constructing a finite sequence collections of balls whose union covers each of the quantitative strata $S_{\epsilon, R}^j$.

### 7.2 Cover construction

The idea is to use the dichotomy implied by Lemma 7.6 to set up a stopping-time argument. For ease of notation, we rescale. In any ball $B_{r_3}(x)$, we dilate everything to the ball $B_1(0)$, where $\Gamma^{x \cdot r_3}$ satisfies the conditions of the previous lemmata. The dichotomy implies that either $S_{\epsilon, x} \cap B_1(0) \subset B_\beta(L) \cap B_1(0)$ for some $j$-dimensional affine linear subspace and the Weiss density does not have a large drop on this set, or, the drop in the Weiss density must be large for all points outside of a neighborhood of a lower-dimensional affine subspace $B_{\beta}(L^{j-1})$. In the former case, we call $B_1(0)$ a “good” ball. In the latter case, we call $B_1(0)$ a “bad” ball.

In a “good” ball, the procedure is to cover $B_\beta(L) \cap B_1(0)$ by balls of a smaller scale and then use the dichotomy to inductively refine a cover, stopping in “bad” balls, and refining the cover in “good” balls. The full details of the construction are in Section 7.1 of [12]. By construction, the collection of “good” balls and “bad” balls so produced satisfy the hypotheses of the Lemma 7.8.

In a “bad” ball, the construction becomes one step more complicated. In a “bad” ball, we again cover the set of points with small drop in Weiss density in the neighborhood $B_{\beta}(L^{j-1}) \cap$
B_1(0) by balls of a smaller scale, and we cover the rest of the ball in B_1(0) \backslash B_\rho (L^{j-1})$, as well. We sort the balls covering B_\rho (L^{j-1}) \cap B_1(0) into “good” and “bad” balls and inductively refine in “bad” balls. We stop in “good” balls. This creates three collections of balls. First, the collection of "bad" balls stemming from the covers of the tubular neighborhoods of the (j − 1)-dimensional subspaces. Second, the “good” balls stemming from the covers of the tubular neighborhoods of the (j − 1)-dimensional subspaces. Third, the balls which cover the complement, B_\rho (q) \backslash B_{\rho q} (L_q^{j-1}) in the collection of “bad” balls. The full details of the construction are in Section 7.2 of [12].

For the collections of “good” and “bad” balls in the “bad” ball construction, the lower-dimensional containment in neighborhoods of affine (j−1)-dimensional subspaces in Section 7.2 of [12].

Note that it relies upon the non-degeneracy of solutions.

8 Containment and density upper bound

In this section we focus upon containment results. Lemma 8.1, below, proves Lemma 1.4. Note that it relies upon the non-degeneracy of solutions.

Lemma 8.1 (Containment for Local Minimizers) Let \Gamma be a k-dimensional (1, \frac{1}{4})-C^{1,\alpha} submanifold such that 0 \in \Sigma. If Q(x) = \text{dist}(x, \Gamma)’ and u is an \epsilon_0-local minimizer of J_Q(·, B_2(0)) in the class K_{u_0, B_2(0)} for \|\nabla u_0\|_{L^2(B_2(0))} \leq \Lambda and sup_{y \in B_2(0)} u_0(y) \leq A, then, if k = n − 1 there exists an 0 < \epsilon < (n, \gamma, \alpha, M, \Lambda, A, \epsilon_0) such that S \subset S^{n-2}_\epsilon.

Proof We argue by contradiction. Fix \alpha > 0, \gamma > 0, and a positive number M. Suppose that for every i \in \mathbb{N}, there exists a n−1-dimensional C^{1,\alpha} submanifold \Gamma_i such that 0 \in \Gamma_i and |\Gamma_i|\alpha \leq \frac{1}{4}, a function u^{(i)} an \epsilon_0-local minimizer of J_{Q_i}(·, B_2(0)), where Q_i = \text{dist}(x, \Gamma_i)’ and a point x_i \in B_1(0) \cap \Gamma_i and a radius 0 < r_i \leq \text{dist}(x_i, \partial B_2(0)) ≤ 2 such that x_i \in \partial \{u_i > 0\} and there exists a normalized non-trivial (n−1)-symmetric function \phi_i such that \|u_{x_i,r_i} - \phi_i\|_{L^2(B_1(0))} \leq 2^{-i}. Then Theorem 5.4 and Lemma 2.7 imply that there is a (n−1)-dimensional C^{1,\alpha} submanifold \Gamma_0 such that |\Gamma_0|\alpha \leq \frac{1}{4} and a function u \in W^{1,2}(B_1(0)) such that, by passing to a subsequence,

\Gamma_i \cap B_2(0) \to \Gamma_0 \cap B_2(0) in the Hausdorff metric

u^{(i)}_{x_i,\min[r_i, r_0]} \to u in W^{1,2}(B_1(0)) and in C^{0,1}(B_1(0)).

Furthermore, u is (n−1, 0)-symmetric in B_1(0) because u is non-degenerate in the sense of Remark 6.6. Therefore, u must be a non-trivial piece-wise linear function. In particular, it must be 1-homogeneous. However, since by Corollary 3.6 for all y \in B_1(0)

|\nabla u^{(i)}_{x_i,\min[r_i, r_0]}(y)| \leq C(n)2^\gamma |y|^{1+\gamma}.
Therefore, for a radius $0 < \rho$ depending only upon $n$ and $\gamma$ there is a ball $B_\rho(0)$ such that

\[ |\nabla u_n^{(i)}(y)| \leq \frac{1}{2} c_n \text{ for all } y \in B_\rho(0), \]

where $c_n = |\nabla u|$ in $\{u > 0\}$. Since this holds for all $i$, we reach a contradiction with strong convergence in $W^{1,2}(B_1(0))$.

The upper bound on the $Q$-density $W_{1+\gamma}(x, 0^+, u, \Gamma)$ is a result of some weak results on $\{u = 0\}$ near $\partial\{u > 0\}$. Stronger results than those proved below are known for local minimizers. See, for example [11] in which an Exterior Ball Condition is proved for almost-minimizers. However, nothing so powerful is needed here.

**Lemma 8.2** (cf. [1] Lemma 3.7) Let $u$ be as in the hypotheses of Lemma 1.2. And let $0 < r_0$ be the standard scale. For all $0 < r < r_0$ and all $x \in \partial\{u > 0\}$, if $\text{dist}(x, \Gamma) \geq 2r$ then there is a constant $0 < C(n, \gamma)$ such that

\[ C(n, \gamma)r \left( \min_{w \in B_r(x)} \{Q(w)\} \right)^2 \omega_n r^n \leq \int_{B_r(x)} Q^2 \chi_{\{u > 0\}} dx. \]

**Proof** We note that for $h_u^{x, r}$ the harmonic extension of $u$ in $B_R(x)$ we have from comparison and the Poincaré inequality

\[ r^{-1} \int_{B_r(x)} (u - h_u^{x, r})^2 dx \leq \int_{B_r(x)} Q^2 \chi_{\{u = 0\}} dx. \]

By Corollary 3.6, $|\nabla u|, |\nabla h_u^{x, r}| \leq C(n) \max_{w \in B_r(x)} \{Q(w)\}$ and by Corollary 3.8 and Corollary 3.9

\[ |h_u^{x, r}(y)| \geq C(n, \gamma) r \min_{w \in B_r(x)} \{Q(w)\} \]

for a set of $y \in \partial B_r(x)$ with measure $(cr/2)^{n-1}$ for

\[ c = \frac{C_{\min \{w \in B_{2r}(x) \} \{Q(w)\}}}{4C(n) \max_{w \in B_{2r}(x)} \{Q(w)\}}. \]

Thus, $|h_u^{x, r}(x)| \geq \frac{c}{2^{n-1}} C(n, \gamma) r \min_{w \in B_r(x)} \{Q(w)\}$. Therefore there is a ball of radius $\rho \geq 2^{-n-1} C(n, \gamma) r \min_{w \in B_r(x)} \{Q(w)\}$ in which $|u - h_u^{x, r}| \geq \frac{c}{2^{n-1}} C(n, \gamma) r \min_{w \in B_r(x)} \{Q(w)\}$. The claim follows immediately recalling $(\min_{w \in B_r(x)} \{Q(w)\})^2 \leq 2^{2\gamma}.$

**Lemma 8.3** (Proof of the $Q$-density upper bound) For $u$ a local minimizer satisfying the hypotheses of Lemma 1.2. Then there is a $C(n, \gamma) < \int_{B_1(0)} |x|^2 \gamma dx$ such that $\Phi(x, 0^+) \leq C(n, \gamma)$ for all $x \in \Gamma \cap \partial\{u > 0\}$.

**Proof** We argue by contradiction. Suppose that there were a sequence of $u_i, x_i$ as in the hypotheses for which $\Phi(x_i, 0^+) \to \int_{B_1(0)} |x|^2 \gamma dx$. Letting $\tilde{u}_i$ be a blow-up of $u_i$ we may use Theorem 5.4 to pass to a subsequence which converges to a non-trivial $(1 + \gamma)$-homogeneous function $u$ for which $\int_{B_1(0)} Q^2 \chi_{\{u > 0\}} dx = \int_{B_1(0)} |x|^2 \gamma dx$. That is, $\{u > 0\}$ has full measure. There are two cases. First, if $\partial\{u > 0\} \cap \partial B_1(0) = \emptyset$ then $u$ is a function which is harmonic in $\mathbb{R}^n \setminus \{0\}$ which is continuous at $\{0\}$. Therefore, $\{0\}$ is removable and $u$ is a non-negative entire harmonic function with minimum achieved at $u(0) = 0$. This is absurd. The second case is that $\partial\{u > 0\} \cap \partial B_1(0) \neq \emptyset$. In this case, since $x_i \in \partial\{\tilde{u}_i > 0\} \cap \Gamma$ and $\partial\{\tilde{u}_i > 0\}$ converges locally to $\partial\{u > 0\}$ away from $\Gamma$ the exterior mass condition in Lemma 8.2 must be violated for sufficiently large $i$. This contradiction proves the claim.
9 Finite perimeter

In this section, we focus upon the set of cusps $\Sigma = \partial \{ u > 0 \} \cap \Gamma \setminus S_{\epsilon_0}$. The main result of this section is the following Lemma.

Lemma 9.1 Let $\Gamma$ be a $k$-dimensional $(1, \frac{1}{4})$-$C^{1,\alpha}$ submanifold such that $0 \notin \Sigma$. If $Q(x) = \text{dist}(x, \Gamma)$ and $u$ is a local minimizer of $J_Q(\cdot, B_2(0))$, then $\partial \{ u > 0 \} \cap B_1(0)$ is a set of finite perimeter. In particular, if $k = n - 1$, then set of cusp points

$$\Sigma = \Gamma \cap \partial \{ u > 0 \} \setminus S$$

has $\mathcal{H}^{n-1}$-measure zero.

Before proving Lemma 9.1, we need the results of the following lemmata.

Lemma 9.2 (Inwards-Minimality implies Finite Perimeter Estimate) Let $n \geq 2$ and fix $0 < \gamma$, $0 < \alpha \leq 1$, and an integer $0 \leq k \leq n-1$. Let $\Gamma$ be a $k$-dimensional $(1, \frac{1}{4})$-$C^{1,\alpha}$ submanifold. Let $Q(x) = \text{dist}(x, \Gamma)$ and $u$ be an $\epsilon_0$-local minimizer of $J_Q(\cdot, B_2(0))$ in the class $K_{\epsilon_0, B_2(0)}$ for $\| \nabla u_0 \|_{L^2(B_2(0))} \leq \Lambda$ and $\sup_{y \in B_2(0)} u_0(y) \leq A$. Let $0 < r_0$ be the standard scale. Then, for all $0 < r \leq \frac{1}{4} r_0$ there is a constant $0 < C < \infty$ such that

$$\int_{\{0 < u \leq \epsilon \} \cap B_r(x_0)} |\nabla u|^2 + Q^2(x) \chi_{\{0 < u \leq \epsilon \}} dx \leq C \epsilon,$$

for all $\epsilon \in (0, 1)$. Moreover, we may take

$$C = C(n)(\| \nabla u \|_{L^2(B_{2r}(x_0))} r^{\frac{n-2}{2}} + \epsilon r^{n-2}).$$

Proof Let $\phi \in C^\infty(\mathbb{R}^n)$ such that

$$\phi(x) = \begin{cases} 0 & x \in B_r(x_0) \\ 1 & x \in \mathbb{R}^n \setminus B_{2r}(x_0) \end{cases}$$

Note that we may choose $\phi$ to be radial, $\phi = \phi(|x - x_0|)$, such that $\phi > 0$ in $B_{2r}(x_0) \setminus B_r(x_0)$, $\phi < 1$ in $B_{2r}(x_0)$, and such that $|\nabla \phi| \leq \frac{2}{r}$.

For a fixed $\epsilon$, let $u_\epsilon = (u - \epsilon)_+$ and $\tilde{u}_\epsilon = \phi u + (1 - \phi) u_\epsilon$. Note that $\tilde{u}_\epsilon > 0$ in $B_{2r}(x_0) \setminus B_r(x_0)$ if $u > 0$, and $\tilde{u}_\epsilon > 0$ in $B_r(x_0)$ if and only if $u > \epsilon$. Then, we let

$$C = C(n)(\| \nabla u \|_{L^2(B_{2r}(x_0))} \| \nabla \phi \|_{L^2(B_r(x_0))} + \epsilon \| \nabla \phi \|_{L^2(B_{2r}(x_0))}^2)
\leq C(n)(\| \nabla u \|_{L^2(B_{2r}(x_0))} r^{\frac{n-2}{2}} + \epsilon r^{n-2}).$$

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then because $2r \leq r_0$, by the local minimality of $u$ we may compare
\[
0 \geq \left( \int_{B_{2r}(x_0)} |\nabla u|^2 + Q^2 \chi_{\{u > 0\}} dx \right) - \left( \int_{B_{2r}(x_0)} |\nabla \tilde{u}_\epsilon|^2 + Q^2 \chi_{\{\tilde{u}_\epsilon > 0\}} dx \right)
\]
\[
\geq \left( \int_{B_{2r}(x_0)} |\nabla u|^2 - \int_{B_{2r}(x_0)} |\nabla \tilde{u}_\epsilon|^2 dx \right) + \int_{B_r(x_0)} Q^2 \chi_{\{0 < u \leq \epsilon\}} dx
\]
\[
\geq \int_{B_{2r}(x_0) \cap \{0 < u \leq \epsilon\}} (1 - \phi^2) |\nabla u|^2 dx + \int_{B_r(x_0)} Q^2 \chi_{\{0 < u \leq \epsilon\}} dx - C\epsilon
\]
\[
\geq \int_{B_r(x_0) \cap \{0 < u \leq \epsilon\}} |\nabla u|^2 dx + \int_{B_r(x_0)} Q^2 \chi_{\{0 < u \leq \epsilon\}} dx - C\epsilon.
\]
\[\square\]

Lemma 9.3 (Locally finite perimeter bounds) Suppose that $D \subset \mathbb{R}^n$ is an open, bounded set, $Q \in C^{0,\alpha}(D)$, $0 < Q_{D,\min} \leq Q$, and that $\phi : D \to [0, +\infty]$ is a function in $W^{1,2}(D)$ such that there exists an $\varepsilon > 0$ and a $C > 0$ such that for all $0 < \varepsilon \leq \varepsilon$
\[
\int_{\{0 < \phi \leq \varepsilon\} \cap D} |\nabla \phi|^2 + Q^2(x) \chi_{\{0 < \phi \leq \varepsilon\}} dx \leq C\varepsilon.
\]
Then, $\text{Per}(\{\phi > 0\} \cap D) \leq C \frac{1}{\sqrt{Q_{D,\min}}}$, where $Q_{D,\min} = \min_{y \in D} Q(y)$. Furthermore, if $D = B_r(x)$, then $C$ may be taken to be $C_n \|\nabla u\|_{L^2(B_{2r}(x))} r^{n-2}$.

Proof
\[
\int_0^\varepsilon \mathcal{H}^{n-1}(\{\phi = t\} \cap D) dt = \int_{\{0 < \phi \leq \varepsilon\} \cap D} |\nabla \phi| dx
\]
\[
\leq |\{0 < \phi \leq \varepsilon\} \cap D|^{\frac{1}{2}} \left( \int_{\{0 < \phi \leq \varepsilon\} \cap D} |\nabla \phi|^2 dx \right)^{\frac{1}{2}}.
\]
We now bound each term by our assumption.
\[
\int_0^\varepsilon \mathcal{H}^{n-1}(\{\phi = t\} \cap D) dt \leq |\{0 < \phi \leq \varepsilon\} \cap D|^{\frac{1}{2}} \left( \int_{\{0 < \phi \leq \varepsilon\} \cap D} |\nabla \phi|^2 dx \right)^{\frac{1}{2}}
\]
\[
\leq \frac{Q_{D,\min}}{Q_{D,\min}} |\{0 < \phi \leq \varepsilon\} \cap D|^{\frac{1}{2}} \left( \int_{\{0 < \phi \leq \varepsilon\} \cap D} |\nabla \phi|^2 dx \right)^{\frac{1}{2}}
\]
\[
\leq \frac{Q_{D,\min}}{Q_{D,\min}} (C\varepsilon)^{\frac{1}{2}}
\]
\[
\leq C\varepsilon^{\frac{1}{2}} C_{Q_{D,\min}}^{-\frac{1}{2}} C_0^{\frac{1}{2}}.
\]
Now, let $0 < \varepsilon$ and let $\delta \in (0, \varepsilon]$ such that
\[
\mathcal{H}^{n-1}(\partial^+(\{\phi > \delta\} \cap D)) \leq \frac{1}{\varepsilon} \int_0^\varepsilon \mathcal{H}^{n-1}(\{\phi = t\} \cap D) dt
\]
\[
\leq C \frac{1}{\sqrt{Q_{D,\min}}}.
\]
Then, letting $\varepsilon \to 0$, $\delta \to 0$ and we obtain, $\mathcal{H}^{n-1}(\partial^+(\{\phi > 0\} \cap D)) \leq C \frac{1}{\sqrt{Q_{D,\min}}}$. 
\[\square\]
9.1 Proof of Lemma 9.1

Following Lemma 2.8, we decompose $B_1(0)$ into dyadic annular neighborhoods,

$$A_j = \left( B_{2^{-j}}(\Gamma) \setminus B_{2^{-j-1}}(\Gamma) \right) \cap B_1(0).$$

Now, using the estimate in Lemma 2.8, we can cover each $A_j$ by $C(n)2^j k$ balls of radius $2^{-j-3}$. We denote this collection by $B_j = \{ B_j \}$. For a ball $B_j$, we shall use $2B_j$ to denote the concentric dilate of $B_j$ by inflation factor 2. By Lemma 9.2, Lemma 9.3, and the Lipschitz estimate from Corollary 3.6 we obtain that for sufficiently large $j$ depending upon the “standard scale” $0 < r_0$,

$$\mathcal{H}^{n-1}(\partial \{ u > 0 \} \cap B_j) \leq \frac{1}{\sqrt{Q_{\min, B_j}}} C(j),$$

$$\leq C(j)(2^{-j-3})^{-\frac{1}{2} \gamma},$$

where $C(j)$ is as in Lemma 9.2

$$C(j) \leq C(n)(\|u\|_{L^2(B_j)}(2^{-j-3})^{\frac{n-2}{2}}) \leq C(n)(C(n)2^j(2^{-j})^{\frac{n+2}{2}}) \leq C(n)(C(n)2^{j/2}(2^{-j-3})^{\frac{n-2}{2}}).$$

Note that since we took $\epsilon \to 0$ in Lemma 9.3, the $\epsilon$ term in $C$ from Lemma 9.2 vanished.

Summing over each of the $C(n)2^j k$ in the collection $B_j$, we obtain the estimate,

$$\mathcal{H}^{n-1}(\partial \{ u > 0 \} \cap A_j) \leq C(n)2^{-j(n-k-1+\gamma)-3\frac{n-2}{2}+\frac{5}{2} \gamma}.$$

Summing over $j$ and recalling that $k \leq n - 1$, we obtain that

$$\mathcal{H}^{n-1}(\partial \{ u > 0 \} \cap B_{2r_0}(0)) = \mathcal{H}^{n-1}(\Gamma) + \mathcal{H}^{n-1}\left( \partial \{ u > 0 \} \cap \bigcup_{j=j_0}^{\infty} A_j \right)$$

$$\leq C(n, [\Gamma]_\alpha)\omega_{n-1} + C(n)\sum_{j=1}^{\infty} 2^{-j(n-k-1+\gamma)-3\frac{n-2}{2}+\frac{5}{2} \gamma}$$

$$\leq C(n, [\Gamma]_\alpha)\omega_{n-1} + C(n, \gamma)\sum_{j=1}^{\infty} 2^{-j(\gamma)} \leq C(n, [\Gamma]_\alpha, \gamma) < \infty.$$  

Since $\partial \{ u > 0 \} \cap B_1(0) \setminus B_{r-j-3}(\Gamma)$ has finite $\mathcal{H}^{n-1}$-measure for every integer $j$, and the previous estimate shows $\partial \{ u > 0 \} \cap B_1(0) \setminus B_{r-j-3}(\Gamma)$ has finite $\mathcal{H}^{n-1}$-measure for $j_0$ sufficiently large, $\{ u > 0 \}$ is a set of locally finite perimeter, and hence $\text{sing}(\partial \{ u > 0 \})$ has $\mathcal{H}^{n-1}$-measure zero.

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