A short introduction to Asymptotic Safety

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Abstract: I discuss the notion of asymptotic safety and possible applications to quantum field theories of gravity and matter.

What is asymptotic safety?

We want to discuss the high energy behavior of a quantum field theory (QFT). Assume that a “theory space” has been defined by giving a set of fields, their symmetries and a class of action functionals depending on fields $\phi$ and couplings $g_i$. We will write $g_i = k^{d_i} \tilde{g}_i$, where $k$ is a momentum cutoff and $d_i$ is the mass dimension of $g_i$. The real numbers $\tilde{g}_i$ are taken as coordinates in theory space. Ideally the couplings $g_i$ should be defined in terms of physical observables such as cross sections and decay rates. In any case “redundant” couplings, i.e. couplings that can be eliminated by field redefinitions, should not be included. We also assume that a Renormalization Group (RG) flow has been defined on theory space; it describes the dependence of the action on an energy scale $k$ (or perhaps a “RG time” $t = \log k$). The action is assumed to have the form

$$\Gamma_k(\phi, g_i) = \sum_i g_i(k) O_i(\phi),$$

where $O_i$ are typically local operators constructed with the field $\phi$ and its derivatives, which are compatible with the symmetries of the theory. We identify theories with RG trajectories.

It can generically be expected that when $k$ goes to infinity some couplings $g_i(k)$ also go to infinity. What we want to avoid is that the dimensionless couplings $\tilde{g}_i$ diverge. In fact, there are famous examples such as QED and $\phi^4$ theory where this happens even at some finite scale $k_{\text{max}}$. Such
divergences signal a breakdown of the theory, and any theory where they occur can only hold for a finite energy range, and is said to be an “effective field theory”. In contrast, suppose that the RG flow admits a fixed point (FP), which is defined as a point $\tilde{\phi}^*_i$ where the beta functions of the dimensionless couplings vanish. An RG trajectory which ends (for $k \to \infty$) at the FP is free of such divergences; it is called a “renormalizable” or “asymptotically safe” (AS) trajectory and represents a UV complete theory [1]. The existence of such a trajectory is therefore a sufficient condition for the theory to be well behaved in the UV.

Now, let us try to count how many such trajectories there are in theory space. We define the “UV critical surface” associated to our FP to be the subset in theory space which is attracted towards it in the UV. Assuming that this surface is a smooth manifold, its dimension is equal to the dimension of its tangent space at the FP. The latter can be computed in the following way. Let $y_i = \phi_i - \phi^*_i$; then in the vicinity of the FP the flow can be linearized:

$$\frac{dy_i}{dt} = M_{ij} y_j ,$$

where

$$M_{ij} = \left. \frac{\partial \beta_i}{\partial \phi^*_j} \right|_{\phi^*_i} .$$

By a linear transformation $z_i = S_{ij} y_j$ we pass to coordinates in which $M$ is diagonal. Then the equation becomes

$$\frac{dz_i}{dt} = \lambda_i z_i ,$$

where $\lambda_i$ are the eigenvalues of $M$. The solutions of this equation are $z_i(t) = e^{\lambda_i t} z_i(0)$, so the coordinates $z_i$ for which $\lambda_i < 0$ are attracted towards the FP; they are called the “relevant” couplings. The coordinates for which $\lambda_i > 0$ are repelled and are called “irrelevant”. If an eigenvalue vanishes the corresponding coordinate is said to be “marginal” and its behavior cannot be determined by the linearized analysis. We will not consider such cases in the following, because they are not generic. The conclusion then is that the dimension of the UV critical surface is equal to the number of negative eigenvalues of $M$.

The condition of asymptotic safety requires that the theory has to lie in the UV critical surface of the FP. This leaves a number of free parameters that is equal to the dimension of this surface. Thus, the theory is more predictive when the critical surface has lower dimension. The ideal situation would be a theory with a one dimensional critical surface. In this case there would be a single renormalizable trajectory and once we have
determined the initial position at some scale $k$, the theory is completely determined. At the opposite extreme, if the UV critical surface was infinite dimensional, the theory would not be predictive. The intermediate case is a theory space with finite dimensional critical surface. Such a theory space would have the same good properties of a perturbatively renormalizable and asymptotically free theory, because it would be well behaved in the UV and it would have only a finite number of undetermined parameters.

It is useful to consider the example of the Gaussian FP, which corresponds to a free theory. The beta functions have the form

$$\frac{d\tilde{g}_i}{dt} = -d_i \tilde{g}_i + k^{-d_i} \beta_i .$$

(5)

The functions $\beta_i = d\tilde{g}_i/dt$ represent the loop corrections, which vanish at the Gaussian FP. In this case the eigenvalues of the matrix $M$ are given just by the canonical dimensions:

$$\lambda_i = -d_i .$$

(6)

The relevant couplings are the ones that are power counting renormalizable, and the critical surface consists of the power counting renormalizable actions. We see that the requirement of asymptotic safety is a generalization of the requirement of asymptotic freedom and renormalizability to the case when the FP does not correspond simply to a free theory. Of course the case of a non-Gaussian FP is harder to study. If it is not too far from the Gaussian FP, one may be able to study it using perturbation theory, but unlike asymptotically free theories, in this case perturbation theory does not get better and better as the energy increases.

**Gravity**

Gravity is the domain of fundamental physics where the problem of finding a UV completion is most acute, and so it is here that most work on asymptotic safety has concentrated, following the original suggestion of [2]. (For earlier reviews see [3].) I will now show that it is reasonable to expect that there exist asymptotically safe theories of gravity.

It is well known that general relativity can be treated as an effective quantum field theory [5, 6]. This means that it is possible to compute quantum effects due to graviton loops, as long as the momenta of the particles in

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1a complementary approach to the one discussed here consists in performing Monte Carlo simulations of discretized gravity. Significant advances have been made in recent years, also lending support to the general idea of nonperturbative renormalizability. See [4] and references therein.
the loops are cut off at some scale. For example, in this way it has been possible to unambiguously compute quantum corrections to the Newtonian potential [7]. The results are independent of the structure of any “ultraviolet completion”, and therefore constitute genuine low energy predictions of any quantum theory of gravity. When one tries to push this effective field theory to energy scales comparable to the Planck scale, or beyond, well-known difficulties appear. It is convenient to distinguish two orders of problems. The first is that the strength of the gravitational coupling grows without bound. For a particle with energy $p$ the effective strength of the gravitational coupling is measured by the dimensionless number $\sqrt{\tilde{G}}$, with $\tilde{G} = G p^2$. This is because the gravitational couplings involve derivatives of the metric. The consequence of this is that if we let $p \to \infty$, also $\tilde{G}$ grows without bound. The second problem is the need of introducing new counterterms at each order of perturbation theory. Since each counterterm has to be fixed by an experiment, the ability of the theory to predict the outcome of experiments is severely limited.

As we have seen in the previous section, the first problem could be fixed if $\tilde{G}$ had a FP. In order to see whether this is reasonable, imagine evaluating the beta function using perturbation theory at one loop. The coefficient\(^2\) of the Hilbert action is the square of Planck’s mass, $M^2_{\text{pl}} = 1/16\pi G$. In the quantum theory it is expected to diverge quadratically with the cutoff, leading to a beta function of the form

$$k \frac{d}{dk} M^2_{\text{pl}} = ck^2,$$

(7)

where $c$ is some constant. Then, the beta function of $G$ has the form

$$k \frac{dG}{dk} = -16\pi c G^2 k^2$$

and the beta function of $\tilde{G}$ is

$$k \frac{d\tilde{G}}{dk} = 2\tilde{G} - 16\pi c \tilde{G}^2.$$

(8)

This beta function has an IR attractive fixed point at $\tilde{G} = 0$ and also an UV attractive nontrivial fixed point at $\tilde{G}_* = 1/8\pi c$. In order to establish whether $c > 0$ one has to do a calculation. The dependence of $G$ on distance has been computed at one loop in the low energy effective field theory [8], leading to

$$16\pi c = \frac{167}{15\pi}.$$

\(^2\)We choose units such that $c = 1$ and $\hbar = 1$. Then everything has dimension of a power of mass.
This has the desired positive sign, but it is not a particularly memorable number: it depends on details of the way in which it is computed. Fortunately, one can show that for any reasonable cutoff it will always have the same sign, so if one loop perturbation theory is a good guide, \( \tilde{G} \) would indeed cease to grow at high energy and settle at some constant value of order one.

Of course such a value of \( \tilde{G} \) is quite large and it is not really clear that near this FP perturbation theory can be trusted. Furthermore, it is also known [9] that loop effects will induce terms with higher derivatives. So the next thing one could do is calculate the one loop beta functions in a theory containing four derivative terms, with an action of the general form

\[
\int d^4x \sqrt{g} \left[ 2Z\Lambda - ZR + \frac{1}{2\lambda} C^2 + \frac{1}{\xi} R^2 + \frac{1}{\rho} E \right],
\]

where \( C^2 \) is the square of the Weyl tensor, \( E \) the integrand of the Euler term,

\[
Z = \frac{1}{16\pi G}; \quad \frac{1}{\xi} = -\frac{\omega}{3\lambda}; \quad \frac{1}{\rho} = \frac{\theta}{\lambda}.
\]

Such calculations have a long history [10]. They were mostly based on dimensional regularization. More recently, we have repeated this calculation using a mass-dependent heat kernel regularization procedure [11]. The beta functions of the four-derivative terms are

\[
\beta_{\lambda} = -\frac{1}{(4\pi)^2} \frac{133}{10} \lambda^2; \quad \beta_{\xi} = -\frac{1}{(4\pi)^2} \left( 10\lambda^2 - 5\lambda \xi + \frac{5}{36} \right); \quad \beta_{\rho} = \frac{1}{(4\pi)^2} \frac{196}{45} \rho^2 \lambda.
\]

We see that the overall coupling \( \lambda \) is asymptotically free:

\[
\lambda(k) = \frac{\lambda_0}{1 + \lambda_0 \frac{1}{(4\pi)^2} \frac{133}{10} \log \left( \frac{k}{k_0} \right)},
\]

whereas the \( \omega \) and \( \theta \), which define the ratio of \( \xi \) and \( \rho \) to \( \lambda \) tend to the asymptotic limits \( \omega(k) \to \omega_* \approx -0.0228 \) and \( \theta(k) \to \theta_* \approx 0.327 \). On the other hand, the cosmological constant and Newton’s constant have the
beta functions

\[
\beta_{\tilde{\Lambda}} = -2\tilde{\Lambda} + \frac{1}{(4\pi)^2} \left[ \frac{1 + 20\omega^2}{256\pi\tilde{G}\omega^2} \lambda^2 + \frac{1 + 86\omega + 40\omega^2}{12\omega} \lambda\tilde{\Lambda} \right] \\
- \frac{1 + 10\omega^2}{64\pi^2\omega} \lambda + \frac{2\tilde{G}}{\pi} - q(\omega)\tilde{G}\tilde{\Lambda},
\]

(11)

\[
\beta_{\tilde{G}} = 2\tilde{G} - \frac{1}{(4\pi)^2} \left[ \frac{3 + 26\omega - 40\omega^2}{12\omega} \lambda\tilde{G} - q(\omega)\tilde{G}^2 \right],
\]

(12)

where \(q(\omega) = (83 + 70\omega + 8\omega^2)/18\pi\). The first few terms in these expressions agree with [10], but the last three terms of \(\beta_{\tilde{\Lambda}}\) and the last term of \(\beta_{\tilde{G}}\) are new. The flow in the invariant subspace \(\lambda = 0, \omega = \omega_*, \theta = \theta_*\) is

\[
\beta_{\tilde{\Lambda}} = -2\tilde{\Lambda} + \frac{2\tilde{G}}{\pi} - q_*\tilde{G}\tilde{\Lambda},
\]

(13)

\[
\beta_{\tilde{G}} = 2\tilde{G} - q_*\tilde{G}^2,
\]

(14)

where \(q_* = q(\omega_*) \approx 1.440\). This flow admits a FP with

\[
\tilde{\Lambda}_* = \frac{1}{\pi q_*} \approx 0.221, \quad \tilde{G}_* = \frac{2}{q_*} \approx 1.389.
\]

It is quite striking that in spite of the very different structure of the theory, the beta function of Newton’s constant is very similar to the one we found in Einstein’s theory. Again, the FP for \(\tilde{G}\) occurs at some value of order one. Nevertheless, it has been argued in [12] that since \(\lambda\), the true coupling constant in this theory, is asymptotically free, this result is reliable.

These calculations highlight the importance of using a mass dependent cutoff scheme: had we used dimensional regularization, we would not see the nontrivial FP. This is because dimensional regularization misses information about the power divergences. It is therefore not a convenient method to study the beta functions of dimensionful couplings.

In fact, even with dimensional regularization there is a somewhat roundabout way to see the effect of power divergences: they appear as logarithmic divergences in other dimensions. One can therefore recover this information by performing a dimensional continuation. In two dimensions \(G\) is dimensionless and its beta function can be extracted at one loop from the pole of a counterterm. It is \(-38G^2/3\) [13]. Then, one can perform the so-called \(\epsilon\) expansion, by studying the beta function as a function of the dimension \(d\). For \(d = 2 + \epsilon\), \(G\) has dimension \(\epsilon\), so \(\tilde{G} = Gk^\epsilon\). The first term in the \(\epsilon\) expansion gives

\[
\beta_{\tilde{G}} = \epsilon\tilde{G} - \frac{38}{3}\tilde{G}^2,
\]

(15)
so we recover the existence of a nontrivial FP in dimension $d > 2$. If we let $\epsilon = 2$ the FP occurs again at some positive value $\tilde{G}$. This was historically the first hint of asymptotic safety [2].

Both the one loop and the $\epsilon$ expansion give a FP which occurs in a regime where the approximation is not clearly reliable. It is for this reason that much of the recent work has been done using (some approximation to) an Exact RG Equation (ERGE), which has been first applied to gravity in [14, 15]. Without entering into details, suffice it to say that one can define a $k$-dependent effective action $\Gamma_k$ by introducing an IR cutoff $k$ in the functional integral, and that this functional obeys the equation

$$k \frac{d\Gamma_k}{dk} = \frac{1}{2} \text{Tr} \left[ \frac{\delta^2 \Gamma_k}{\delta \phi \delta \phi} + R_k \right]^{-1} k \frac{dR_k}{dk}.$$  

(16)

If $\Gamma_k$ has the form (1),

$$k \frac{d\Gamma_k}{dk} = \sum_i \beta_i O_i(\phi).$$  

(17)

Therefore, expanding the r.h.s. of (16) on the basis of operators $O_i$ one can read off the beta functions of the individual couplings $g_i$. This method has several advantages: (i) it works in any dimension, (ii) there is no need to introduce UV regulators, since the r.h.s. of (16) is finite, and (iii) it does not depend on the couplings being small. Of course, it is generally impossible to compute the beta functions of infinitely many couplings and so one has to truncate the sum to finitely many terms. For example, if we keep only the first two terms in (9) we find, for a cutoff of type “1b” [20]:

$$\beta_{\Lambda} = \frac{-2(1 - 2\tilde{\Lambda})^2 \tilde{\Lambda} + \frac{36 - 41\tilde{\Lambda} + 42\tilde{\Lambda}^2 - 600\tilde{\Lambda}^3}{72\pi} \tilde{G} + \frac{467 - 572\tilde{\Lambda}}{288\pi^2} \tilde{G}^2}{(1 - 2\tilde{\Lambda})^2 - \frac{29 - 9\tilde{\Lambda}}{72\pi} \tilde{G}},$$

$$\beta_{\tilde{G}} = \frac{2(1 - 2\tilde{\Lambda})^2 \tilde{\Lambda} - \frac{373 - 654\tilde{\Lambda} + 600\tilde{\Lambda}^2}{72\pi} \tilde{G}^2}{(1 - 2\tilde{\Lambda})^2 - \frac{29 - 9\tilde{\Lambda}}{72\pi} \tilde{G}}.$$  

One can still glean the one loop result, which is obtained by neglecting $\Lambda$ and setting the denominators to one. There has been a number of independent calculations, using different cutoffs and different gauges, and treating the ghosts in different ways, which give slightly different numbers but agree on the qualitative structure of the result [16, 17, 18, 19, 20, 21]. This method has been applied also to four-derivative gravity in [22], where a nontrivial FP with nonzero values for all the couplings is found.
In another direction, it has been possible to work out the beta functions for truncations of the form

\[ \Gamma_k = \sum_{i=0}^{n} g_i \int d^4x \sqrt{g} R^i. \]  

The case \( n = 2 \) was first examined in [23], while in [24, 25] the calculation was pushed up to \( n = 8 \). The results of these calculations can be summarized by the following tables, which give the position of the FP and the eigenvalues \( \lambda_i \) as functions of \( n \).

### Position of Fixed Point \((\times 10^{-3})\)

| \( n \) | \( \hat{g}_{0*} \) | \( \hat{g}_{1*} \) | \( \hat{g}_{2*} \) | \( \hat{g}_{3*} \) | \( \hat{g}_{4*} \) | \( \hat{g}_{5*} \) | \( \hat{g}_{6*} \) | \( \hat{g}_{7*} \) | \( \hat{g}_{8*} \) |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 5.23 | -20.1 |   |   |   |   |   |   |   |
| 2 | 3.29 | -12.7 | 1.51 |   |   |   |   |   |   |
| 3 | 5.18 | -19.6 | 0.70 | -9.7 |   |   |   |   |   |
| 4 | 5.06 | -20.6 | 0.27 | -11.0 | -8.65 |   |   |   |   |
| 5 | 5.07 | -20.5 | 0.27 | -9.7 | -8.03 | -3.35 |   |   |   |
| 6 | 5.05 | -20.8 | 0.14 | -10.2 | -9.57 | -3.59 | 2.46 |   |   |
| 7 | 5.04 | -20.8 | 0.03 | -9.78 | -10.5 | -6.05 | 3.42 | 5.91 |   |
| 8 | 5.07 | -20.7 | 0.09 | -8.58 | -8.93 | -6.81 | 1.17 | 6.20 | 4.70 |

### Eigenvalues of linearized flow

| \( n \) | \( \text{Re} \lambda_1 \) | \( \text{Im} \lambda_1 \) | \( \lambda_2 \) | \( \lambda_3 \) | \( \text{Re} \lambda_4 \) | \( \text{Im} \lambda_4 \) | \( \lambda_6 \) | \( \lambda_7 \) | \( \lambda_8 \) |
|---|---|---|---|---|---|---|---|---|---|
| 1 | -2.38 | -2.17 |   |   |   |   |   |   |   |
| 2 | -1.38 | -2.32 | -26.9 |   |   |   |   |   |   |
| 3 | -2.71 | -2.27 | -2.07 | 4.23 |   |   |   |   |   |
| 4 | -2.86 | -2.45 | -1.55 | 3.91 | 5.22 |   |   |   |   |
| 5 | -2.53 | -2.69 | -1.78 | 4.36 | 3.76 | 4.88 |   |   |   |
| 6 | -2.41 | -2.42 | -1.50 | 4.11 | 4.42 | 5.98 | 8.58 |   |   |
| 7 | -2.51 | -2.44 | -1.24 | 3.97 | 4.57 | 4.93 | 7.57 | 11.1 |   |
| 8 | -2.41 | -2.54 | -1.40 | 4.17 | 3.52 | 5.15 | 7.46 | 10.2 | 12.3 |

From these numbers one can draw several conclusions. First of all the FP exists for all truncations and secondly is relatively stable, in the sense that adding new terms to the truncations generally does not change very much the results of the lower truncation. Third, there are three negative eigenvalues, showing that the critical surface is three dimensional. In fact, knowing the eigenvectors of the matrix \( M \), one can write explicitly the linearized
equation of this surface. Using $g_0$, $g_1$ and $g_2$ as independent parameters,

\[
\begin{align*}
\tilde{g}_3 &= 0.00061243 + 0.06817374 \tilde{g}_0 + 0.46351960 \tilde{g}_1 + 0.89500872 \tilde{g}_2 \\
\tilde{g}_4 &= -0.00916502 - 0.83651466 \tilde{g}_0 - 0.20894019 \tilde{g}_1 + 1.62075130 \tilde{g}_2 \\
\tilde{g}_5 &= -0.01569175 - 1.23487788 \tilde{g}_0 - 0.72544946 \tilde{g}_1 + 1.01749695 \tilde{g}_2 \\
\tilde{g}_6 &= -0.01271954 - 0.62264827 \tilde{g}_0 - 0.82401181 \tilde{g}_1 - 0.64680416 \tilde{g}_2 \\
\tilde{g}_7 &= -0.00083040 + 0.81387198 \tilde{g}_0 - 0.14843134 \tilde{g}_1 - 2.01811163 \tilde{g}_2 \\
\tilde{g}_8 &= 0.00905830 + 1.25429854 \tilde{g}_0 + 0.50854002 \tilde{g}_1 - 1.90116584 \tilde{g}_2
\end{align*}
\]

This illustrates the predictivity of asymptotically safe theories: once the three parameters $g_0, g_1$ and $g_2$ have been measured at some scale by means of three experiments, everything else is determined and any further experiment is a test of the theory. Of course, the specific results of this calculation should not be taken too seriously: there are many important things that have been neglected here.

**Matter**

However hard it may be to prove the asymptotic safety of gravity, it would still not be enough: for applications to the real world one will have to show that a (possibly unified [26]) theory of all interactions is asymptotically safe. The strong interactions are already described by an asymptotically safe theory, and there are reasons to believe that this result is not ruined by the coupling to gravity [27]. The electroweak and Higgs sectors of the standard model are perturbatively renormalizable, but some of their beta functions are positive. This means that either new weakly coupled degrees of freedom manifest themselves at some scale, before the couplings blow up, or else the theory is consistent, but in a nonperturbative sense. The simplest realization of the latter behavior is AS. If the world is described by an AS theory, there are two main possibilities: one is that AS is an inherently gravitational phenomenon, in which case AS would manifest itself at the Planck scale $^3$; the other is that each interaction reaches the FP at its characteristic energy scale.

In the first case, one has to compute the effect of gravity on matter couplings and the effect of matter on the gravitational couplings. The effect of gravity on scalar couplings has been considered in [29, 30, 31], on gauge couplings in [32] and on Yukawa couplings in [33]. One possibility is that $^3$ this includes the possibility that due to the presence of large extra dimensions the effective Planck mass is much lower than $10^{19}$GeV. I refer to [28] for an analysis of this scenario.
the coupling to gravity makes all matter interactions asymptotically free, as conjectured long ago by Fradkin and Tseytlin [34]. There is some evidence that this can happen in some cases, with gravity preventing the Landau pole of scalar theory and QED [30, 32]. In this case the second part of the job, namely computing the effect of matter on gravity couplings, would be much simplified, because in order to establish the existence of a FP it would be enough to consider minimally coupled matter fields. This problem has been studied in [35], where it was found that the existence of a FP with desirable properties puts restrictions on the number of matter fields of each spin. In fact, for a large number of matter fields, the task is even simplified, and to leading order in a $1/N$ expansion one can prove the existence of a gravitational FP to all orders of the derivative expansion [36]. Things are more complicated if matter remains interacting also in the UV limit. One particularly striking possibility has been pointed out recently [38]: QED coupled to gravity seems to have two nontrivial FPs, in addition to the Gaussian one: at one gravity is interacting but QED is free, at the other they are both interacting. The latter has a lower dimensional critical surface and is therefore more predictive: on a renormalizable trajectory ending at this FP, the low energy value of the fine structure constant can in principle be calculated.

In the second case, matter and gravity would be separately AS. Then, one would have to prove that electroweak theory somehow heals itself of its UV problems. At the moment, there are two approaches to this idea: the first, motivated by the formal analogies between gravity and the nonlinear sigma models, is that a Higgsless version of the standard model could be AS. Some partial calculations support this view [39]. It has been shown recently that this possibility is compatible with electroweak precision data [40]. See also [41] for comments. The other possibility is that a suitably balanced theory of coupled scalars and fermions with potential and Yukawa couplings exhibits AS [42]. In both cases the Higgs VEV, which is the source of the masses of all pointlike particles, would run linearly above some scale, restoring scale invariance. This would affect the physics of the Higgs, which is being explored at LHC, making this by far the most exciting possibility from the point of view of possible experimental signatures.

**Cosmology and time**

It is generally expected that a quantum theory of gravity should be able to solve the puzzles that remain open in classical general relativity, for example the fate of spacetime near a singularity. Furthermore, a scale dependence of couplings (such as Newton’s constant or the cosmological
constant) may well have an effect on the cosmological evolution, even at relatively late stages. For these reasons, cosmology, and especially very early cosmology, is probably the most promising domain of application of asymptotically safe gravity.

The most popular way of applying the RG to cosmology consists in identifying the cutoff scale \( k \) with some characteristic cosmological parameter (usually the Hubble scale \( H(t) = \dot{a}(t)/a(t) \)) and then replacing the constant gravitational couplings \( (G, \Lambda...) \) by their scale-dependent counterparts, making the gravitational couplings effectively time-dependent [43]. This substitution can be done in a solution, in the equations of motion or directly in the action, with different results. Consider for example the effect of "RG-improving" Einstein’s equations [44]:

\[ G_{\mu\nu} = 8\pi G(k)T_{\mu\nu} - \Lambda(k)g_{\mu\nu} . \]  

(19)

For simplicity we assume a spatially flat Friedmann-Robertson-Walker metric with scale factor \( a(t) \) and an energy momentum tensor in the form of a perfect fluid \( T^{\mu\nu} = \text{diag}(-\rho, p, p, p) \) with equation of state \( p(\rho) = w\rho \). Both \( G_{\mu\nu} \) and \( R_{\mu\nu} \) can be expressed in terms of the Hubble rate:

\[ R_{tt} = -3(\dot{H} + H^2) , \quad R = R^\mu_\mu = 6(\dot{H} + 2H^2) , \quad G_{tt} = 3H^2 , \]

so that the \((tt)\)-component and the trace of Einstein’s equations become

\[ 3H^2 = 8\pi G\rho + \Lambda , \]

(20)

\[ 6(\dot{H} + 2H^2) = 8\pi G\rho (1 - 3w) + 4\Lambda . \]

(21)

Choose the cutoff \( k = \xi H \), for some real number \( \xi \) of order one. Then Newton’s constant and the cosmological constant become functions of time: \( G = G(\xi H) \), \( \Lambda = \Lambda(\xi H) \), whose form is fixed by the renormalization group equations. To simplify, let us assume that we are at sufficiently high \( k \) such that we may assume that the (dimensionless) couplings \( \tilde{G} \) and \( \tilde{\Lambda} \) are at their fixed point values. Then \( G = \tilde{G}_*/(\xi^2 H^2) \) and \( \Lambda = \tilde{\Lambda}_* \xi^2 H^2 \). One then looks for inflationary de Sitter solutions

\[ a(t) = a_0 e^{Ht} ; \quad H = \text{constant} , \]

(22)

or power law solutions

\[ a(t) = a_0 t^p ; \quad H = \frac{p}{t} . \]

(23)

The equations admit power law solutions with

\[ p = \frac{2}{(3 - \tilde{\Lambda}_* \xi^2)(1 + w)} . \]

(24)
Let us set \( w = 1/3 \), as appropriate for ultrarelativistic matter. We see that for \( 1/2 < \tilde{\Lambda}_* \xi^2/3 < 1 \) the solution has inflationary character \((p > 1)\), with the acceleration becoming stronger as \( \tilde{\Lambda}_* \xi^2/3 \) increases. For \( \tilde{\Lambda}_* \xi^2/3 = 1 \) (and any \( w > -1 \)) the exponent diverges. We observe that this condition is equivalent to the equation \( R = 4\Lambda \) written in the FP regime; the corresponding solution is a de Sitter universe. Similar conclusions have been shown to hold also for the fixed point of \( f(R) \) gravity [45], and first steps towards a calculation of the spectrum of fluctuations have been made in [46]. A general qualitative analysis of the cosmological dynamics in the presence of running couplings has been given recently in [47].

This approach raises several issues. One is that inflation is supposed to occur at energies considerably lower than the Planck scale, so that the approximation of being close to the fixed point may actually not be warranted [48]. Another issue is the exit from inflation. Presumably this would happen when the RG trajectory departs from the immediate neighborhood of the fixed point, but a detailed study has not been done so far. Perhaps more worrisome is the nonconservation of the matter energy-momentum tensor. From Friedmann’s equations one obtains a modified conservation equation

\[
\dot{\rho} + 3H(\rho + p) = -\frac{1}{8\pi G}(\dot{\Lambda} + 8\pi \rho \dot{G})
\]

We see that the time variation of the couplings, which follows from the time dependence of the cutoff, gives rise to nonconservation of the energy. One may try to interpret this in terms of the energy and momentum of the field modes that have been removed from the system by coarse graining. Bonanno and Reuter actually turn this into a positive feature [44]: they show that, under reasonable assumptions, the energy transferred to the matter system through the decay of the cosmological constant over the age of the universe is of the correct order of magnitude to explain the entropy of the cosmic background radiation.

In order to avoid these issues, Weinberg follows a different approach [49]. He writes the Friedmann equations following from the most general effective action that is local in curvatures and covariant derivatives of curvatures, and looks for de Sitter solutions. He argues that a choice of cutoff of the order of \( H \) may be a reasonable compromise between the conflicting requirements of avoiding large radiative corrections to the field equations, and the Einstein-Hilbert truncation being a resonable approximation. In this approach the exit from inflation should be signalled by an instability of the solution. Unfortunately explicit calculations based on known properties of the fixed point of pure gravity seem to show too much instability, leading to a number of \( e \)-foldings that is too small.
Aside from these attempts to apply asymptotic safety to inflationary cosmology, one may try to make connection also to other ideas. One important fact is that physics at a fixed point is scale invariant \(^4\). Even though the fixed point Lagrangian contains dimensionful couplings, these scale with energy according to their canonical dimension so that all observable quantities have power law dependences. Under these circumstances, defining a clock becomes impossible even in principle and the notion of time loses its operational meaning \([50]\). Although one may still be able to define separate points, time intervals and distances become meaningless. In this sense, one may argue that a fixed point leads to a notion of minimal distance \([51]\). This is also in line with the view that the metric geometry “melts down” near the big bang, but the conformal geometry remains well defined. In fact it is worth noting that if the infrared behavior of gravity was also governed by a fixed point, as conjectured in \([52]\), then one would have scale invariance at both ends of the cosmological evolution. This would lend support to Penrose’s Conformal Cyclic Cosmology \([53]\).

**Discussion, summary and prospects**

I have presented some evidence that a theory of gravity and perhaps of all interactions is AS. None of the calculations performed so far can be said to be a proof, but the qualitative agreement of the results in all the approximations makes this by now a rather plausible scenario. If this was true, we would have an UV complete theory remaining within the familiar domain of QFT. It is important to appreciate the differences between this and other popular approaches to quantum gravity.

AS is a “bottom up” approach to quantum gravity: the discussion starts within the theory space of an effective field theory, and goes on to note that if the world corresponds to a trajectory of a special type, then the effective description can be pushed to arbitrarily high energy. An AS theory is simply the continuation of an effective theory to higher energy scales. As a result, an AS theory has the great advantage that if it exists, it is almost automatically in agreement with our knowledge of the low energy world. This is in contrast to string theory and loop quantum gravity, which are “top down” approaches. For them, making a connection with known low energy phenomenology is proving a very hard issue.

There is obviously a price to pay for this. On one hand, in a nonperturbative context it is hard to obtain reliable results and hard proofs. Further-

\(^4\)due to the complex critical exponents, one may only have invariance under a discrete subgroup of scale transformations.
more, the action of the FP theory seems to contain infinitely many terms with nonzero couplings, making it unwieldy at best. It is in principle possible that the description of the fixed point could be simplified by a suitable change of variables (perhaps along the lines of [54]). Then, the AS QFT may turn out to be equivalent to one of the top down theories. In that case it would be enough to establish the equivalence in the vicinity of the Planck scale. From there downwards, one would just follow the RG as in any effective field theory.

This remark applies also to the scenario of “emergent gravity”. According to a popular point of view, gravity is not a fundamental interaction but rather the effective description of some underlying microscopic dynamics that may have little to do with the geometry [55, 56, 57]. It is often said that in this case attempts at formulating a quantum theory of gravity in terms of metric degrees of freedom are misplaced. As discussed in [58], even if gravity at very high energies was described by some as yet unknown theory with non-metric degrees of freedom, from some energy scale downwards it can be described by an effective theory of the metric, and in this effective theory couplings will run according to the RG as discussed above. At sufficiently low energy we would therefore be again in the theory space discussed in section 2. If there is a FP in this theory space, then the RG trajectory that describes emergent gravity will approach its UV critical surface at low energies, so that even in this case the notion of AS would prove to be a useful tool.

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