The inversion formulae for automorphisms of polynomial algebras and differential operators in prime characteristic

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Abstract

Let $K$ be an arbitrary field of characteristic $p > 0$, let $A$ be one of the following algebras: $P_n := K[x_1, \ldots, x_n]$ is a polynomial algebra, $\mathcal{D}(P_n)$ is the ring of differential operators on $P_n$, $\mathcal{D}(P_n) \otimes P_m$, the $n$’th Weyl algebra $A_n$, the $n$’th Weyl algebra $A_n \otimes P_m$ with polynomial coefficients $P_m$, the power series algebra $K[[x_1, \ldots, x_n]]$, $T_{k_1, \ldots, k_n}$ is the subalgebra of $\mathcal{D}(P_n)$ generated by $P_n$ and the higher derivations $\partial_i^{[j]}$, $0 \leq j < p^{k_i}$, $i = 1, \ldots, n$ (where $k_1, \ldots, k_n \in \mathbb{N}$), $T_{k_1, \ldots, k_n} \otimes P_m$, an arbitrary central simple (countably generated) algebra over an arbitrary field. The inversion formula for automorphisms of the algebra $A$ is found explicitly.

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1 Introduction

Let $K$ be an arbitrary field, $P_n := K[x_1, \ldots, x_n]$ be a polynomial algebra over $K$, and $\tilde{P}_n := K[[x_1, \ldots, x_n]]$ be an algebra of formal power series over $K$.

In characteristic zero, there are several different inversion formulae for $\sigma \in \text{Aut}_K(P_n)$ ([2], [10], [1], [13], [3]), they are based on different ideas. Besides applications the interest in inversion formulae stems mainly from three different sources: in Algebra - to solve the Jacobian Conjecture; in Differential Equations - to study solutions of differential equations (and their dependence on parameters) where on various stages of finding solutions new coordinates (substitutions) are used; in Analysis - any set of ‘good’ functions $x'_1, \ldots, x'_n$ in variables $x_1, \ldots, x_n$ with nonzero Jacobian, $\det(\frac{\partial x'_i}{\partial x_j}) \neq 0$, defines (locally) coordinates. If, in addition, the functions $x'_1, \ldots, x'_n$ depend on a set of parameters $\lambda$ then properties and behaviour of the functions

$$x_1 = x_1(x'_1, \ldots, x'_n; \lambda), \ldots, x_n = x_n(x'_1, \ldots, x'_n; \lambda)$$

on $\lambda$ are of great importance (and difficult to study). A ‘nice’ inversion formula can help greatly with that sort of questions. One should also mention the Number Theory especially its connections with differential operators where the coefficients of differential operators carry valuable information about the number theoretic question they have connection with - the inversion formulae in [3] and in the present paper are given predominantly via differential operators (or/and powers of derivations in the noncommutative setting).

In characteristic zero, the ring of differential operators $\mathcal{D}(P_n)$ on the polynomial algebra $P_n$ is canonically isomorphic to the Weyl algebra $A_n$ (this is not the case in prime characteristic). The counter part to the Jacobian Conjecture for polynomial algebras is the Problem of Dixmier for the Weyl algebras $A_n$ which asks whether an algebra endomorphism of $A_n$ is an automorphism ([7], Problem 1). Recent results show that these two problems are essentially the same ($DP_n \Rightarrow JC_n$, [2]; $JC_{2n} \Rightarrow DP_n$, [12] and [6], see also [4]). One can even amalgamate $DP_n$ and $JC_m$ into a single question about algebra endomorphisms of the algebra $A_n \otimes P_m$, [3], though this question is equivalent to $DP_n + JC_m$, [3]. The inversion formula for $\sigma \in \text{Aut}_K(A_n \otimes P_m)$ was found in [4], it was used to prove the equivalence just mentioned. In [3], it was shown that the algebras $\{A_n \otimes P_m\}$ are the only algebras for which the type of questions like $JC$ or $DP$ makes sense (i.e. both $JC$ or $DP$ can be reformulated as questions about certain commuting locally nilpotent derivations, the algebras $\{A_n \otimes P_m\}$ are the only algebras that have them).

In characteristic $p > 0$, before the present paper I have known no inversion formula neither for polynomials (or series) or the Weyl algebras or the ring of differential operators $\mathcal{D}(P_n)$. An attempt was made by Tsuchimoto [11], Proposition 1, to find such a formula for the Weyl algebra $A_n$ using the reduced trace of its division algebra $\text{Frac}(A_n)$ (in more detail, the ‘inversion formula’ is found for $\sigma \in \text{Aut}_K(A_n)$ up to $\sigma^{-1}|Z(A_n)$ where $Z(A_n)$ is the centre of the Weyl algebra, it is a polynomial algebra in $2n$ variables).

In the present paper, the inversion formula is found (in which only algebraic operations are present like addition or multiplication) for the algebras mentioned in the Abstract. We
generalized the approach of the paper [3]. The prime characteristic case is more difficult
than the characteristic zero case and many more situations occur.

The paper is organized as follows: in the first four sections a necessary machinery is
developed which later is applied in finding the inversion formulae.

An idea of finding inversion formula. Briefly, it is to express the identity map
via ‘algebraic’ operations (i.e. operations which are well-behaved when applying an algebra
automorphism): let \( A \) be an algebra over a field \( K \) and \( \sigma \) be an algebra automorphism of
\( A \), let \( \{ x^\alpha \} \) be a \( K \)-basis of \( A \) and suppose that the identity map \( \text{id}_A : A \rightarrow A \) can be
written as

\[
\text{id}_A(\cdot) = \sum \lambda_{\alpha,y_\alpha}(\cdot) x^\alpha
\]

where \( \lambda_{\alpha,y_\alpha} : A \rightarrow K \) are ‘algebraic’ maps depending ‘algebraically’ on a certain set of
elements \( y_\alpha \). Applying \( \sigma \) one has also the presentation

\[
\text{id}_A(\cdot) = \sum \lambda_{\alpha,\sigma(y_\alpha)}(\cdot) \sigma(x^\alpha)
\]

where \( \sigma(\lambda_{\alpha,y_\alpha}) = \lambda_{\alpha,\sigma(y_\alpha)} \) since \( \lambda_{\alpha,\sigma(y_\alpha)} \) is an ‘algebraic’ map. Then applying \( \sigma^{-1} \) and using
the fact that \( \lambda_{\alpha,\sigma(y_\alpha)}(A) \subseteq K \) we have the ‘formula’ for the inverse map

\[
\sigma^{-1}(\cdot) = \sum \lambda_{\alpha,\sigma(y_\alpha)}(\cdot) x^\alpha.
\]

To realize this simple idea different algebras require different means. For central simple
algebras it is ‘easy to do it’ as the Density Theorem provides such a presentation as a limit
in the finite topology of certain maps given explicitly (on the other extreme, for commutative
algebras one has to use differential operators).

Example. Let \( M_n(K), n \geq 2 \), be the matrix algebra over a field \( K \), it is a central simple
finite dimensional algebra. Let \( \sigma \in \text{Aut}_K(M_n(K)) \). It it well-known that \( \sigma(x) = sxs^{-1} \) for
some non-singular matrix \( s \). Therefore, \( \sigma^{-1}(x) = s^{-1}xs \). For any \( x = (x_{ij}) \in M_n(K) \),

\[
x = \sum_{i,j=1}^{n} x_{ij} e_{ij} = \sum_{i,j=1}^{n} (x_{ij}E)e_{ij} = \sum_{i,j=1}^{n} (\sum_{k=1}^{n} e_{ki}x_{jk})e_{ij}
\]

where \( e_{ij} \) are the matrix units and \( E \) is the identity matrix. Equivalently, \( \text{id}_{M_n(K)}(\cdot) = \sum_{i,j=1}^{n} (\sum_{k=1}^{n} e_{ki}(\cdot)e_{jk})e_{ij} \). Hence, we have the inversion formula given explicitly

\[
\sigma^{-1}(\cdot) = \sum_{i,j=1}^{n} (\sum_{k=1}^{n} \sigma(e_{ki})(\cdot)\sigma(e_{jk}))e_{ij}.
\]

One can easily verify that this is the inversion formula: for \( \sigma(x) = sxs^{-1} \),

\[
\sum_{i,j=1}^{n} (\sum_{k=1}^{n} \sigma(e_{ki})x\sigma(e_{jk}))e_{ij} = \sum_{i,j=1}^{n} (\sum_{k=1}^{n} se_{ki}s^{-1}xse_{jk}s^{-1})e_{ij} = \sum_{i,j=1}^{n} s(\sum_{k=1}^{n} e_{ki}\sigma^{-1}(x)e_{jk})s^{-1}e_{ij}
\]

\[
= \sum_{i,j=1}^{n} s(\sigma^{-1}(x)_{ij}E)s^{-1}e_{ij} = \sum_{i,j=1}^{n} \sigma^{-1}(x)_{ij}e_{ij} = \sigma^{-1}(x).
\]
2 Locally nilpotent derivations and their nil algebras

This section is about the structure of algebras that admit certain locally nilpotent derivations. These derivations appear naturally in almost all the algebras we consider in the paper, and results of this section are the key ones in finding various inversion formulae.

Iterative $\delta$-descents. Let $A$ be an algebra over a field $K$ and let $\delta$ be a $K$-derivation of the algebra $A$. For any elements $a, b \in A$ and a natural number $n$, an easy induction argument gives

$$\delta^n(ab) = \sum_{i=0}^{n} \binom{n}{i} \delta^i(a)\delta^{n-i}(b).$$

It follows that the kernel $A^\delta := \ker \delta$ of $\delta$ is a subalgebra (of constants for $\delta$) of $A$ and the union of the vector spaces $N := N(\delta, A) = \cup_{i \geq 0} N_i$, $N_i := \ker \delta^{i+1}$, is a positively filtered algebra ($N_iN_j \subseteq N_{i+j}$ for all $i, j \geq 0$), so-called, the nil algebra of $\delta$. Clearly, $N_0 = A^\delta$ and $N := \{a \in A | \delta^n(a) = 0 \text{ for some natural } n = n(a)\}$.

A $K$-derivation $\delta$ of the algebra $A$ is a locally nilpotent derivation if for each element $a \in A$, $\delta^n(a) = 0$ for all $n > 1$. A $K$-derivation $\delta$ is locally nilpotent iff $A = N(\delta, A)$.

Definition. Let $\delta$ be a $K$-derivation of an algebra $A$ over an arbitrary field $K$. A finite or infinite sequence $y = \{y[i], 0 \leq i \leq l - 1\}$ of elements in $A$ where $y[0] := 1$ is called an iterative sequence of length $l$ if

$$y[i]y[j] = \binom{i+j}{i} y[i+j], \ 0 \leq i, j \leq l - 1, \ i + j \leq l - 1. \tag{1}$$

A sequence $y = \{y[i], 0 \leq i \leq l - 1\}$ is called a $\delta$-descent if $y[0] := 1$ and

$$\delta(y[i]) = y[i-1], \ 0 \leq i \leq l - 1, \ y[-1] := 0. \tag{2}$$

Definition. If both conditions (1) and (2) hold then the sequence $y$ is called an iterative $\delta$-descent of length $l$.

Lemma 2.1 Let $A$ be an algebra over an arbitrary field $K$, $\delta$ be a $K$-derivation of $A$.

1. If $\{x[i], i \geq 0\}$ is a $\delta$-descent then $N(\delta, A) = \oplus_{i \geq 0} A^\delta x[i] = \oplus_{i \geq 0} x[i]A^\delta$ and $N_n = \oplus_{i=0}^{n} A^\delta x[i] = \oplus_{i=0}^{n} x[i]A^\delta$ for all $n \geq 0$.

2. If $\delta^l = 0$ for some $2 \leq l < \infty$, and $\{x[i], 0 \leq i < l\}$ is a $\delta$-descent of length $l$. Then $\delta^{l-1} \neq 0$ and $N(\delta, A) = \oplus_{i=0}^{l-1} A^\delta x[i] = \oplus_{i=0}^{l-1} x[i]A^\delta$ and $N_j = \oplus_{i=0}^{j} A^\delta x[i] = \oplus_{i=0}^{j} x[i]A^\delta$ for all $0 \leq j < l$.

Proof. 1. Clearly, $N' := \sum_{i \geq 0} A^\delta x[i] \subseteq N := N(\delta, A) = \cup_{n \geq 0} N_n$ since all $x[i] \in N$ and $A^\delta \subseteq N$. Let us show that the sum in the definition of $N'$ is a direct one. Suppose this is not the case, then there is a nontrivial relation of degree $n > 0$,

$$c_0 + c_1 x[1] + \cdots + c_n x[n] = 0, \ c_i \in A^\delta, \ c_n \neq 0.$$
We may assume that the degree \( n \) of the relation above is the least one. Then applying \( \delta \) to the relation above we obtain the relation of smaller degree (a contradiction):

\[
c_1 + c_2x^{[1]} + \cdots + c_nx^{[n-1]} = 0.
\]

So, \( N' = \oplus_{i \geq 0} A^\delta x^i \). It remains to prove that \( N = N' \). It suffices to show that all subspaces \( N_i \) belong to \( N' \). We use induction on \( i \). The base of the induction is trivial since \( N_0 = A^\delta \). Suppose that \( i > 0 \), and \( N_{i-1} \subseteq N' \). Let \( u \) be an arbitrary element of \( N_i \). Then \( \delta(u) \in N_{i-1} \subseteq N' \), hence \( \delta(u) = \sum_{j=0}^{i-1} c_jx^{[j]} = \delta(\sum_{j=0}^{i-1} c_jx^{[j+1]}) \) for some \( c_i \in A^\delta \).

Hence, \( u - \sum_{j=0}^{i-1} c_jx^{[j+1]} \in A^\delta \subseteq N' \), and so \( u \in N' \), which proves that \( N_i \subseteq N' \), and so \( N = N' \). It is obvious that \( N_n = \oplus_{i=0}^n A^\delta x^i \), \( n \geq 0 \).

Repeating the argument above for the space \( N'' := \sum_{i \geq 0} x^i A^\delta \) we conclude that \( N = N'' = \oplus_{i \geq 0} x^i A^\delta \) and \( N_n = \oplus_{i=0}^n x^i A^\delta \), \( n \geq 0 \).

2. \( \delta^{i-1}(x^{[i-1]}) = 1 \neq 0 \) hence \( \delta^{i-1} \neq 0 \). For the rest, repeat literally the same arguments as in the proof of statement 1. \( \square \)

There are elements \( a_{k,j}^i, b_{k,j}^i \in A^\delta \) such that

\[
x^i x^j = \sum_{k=0}^{i+j} a_{k,j}^i x^k = \sum_{k=0}^{i+j} x^k b_{k,j}^i, \quad i, j \geq 0.
\]

The elements \( a_{k,j}^i, b_{k,j}^i \) are unique (Lemma 2.2). If the field \( K \) has characteristic zero and the element \( x := x^i \) is fixed then the elements \( x^i, i \geq 2 \) can be chosen as \( x^i = \frac{x^i}{i!} \) (in general, the elements \( x^i \) are highly non-unique but it is not difficult to describe all possibilities: if \( \{x^i\} \) is another choice of the elements \( \{x^i\} \) then there exist infinite number of scalars \( \lambda_1, \lambda_2, \ldots \) such that \( x^i = x^i + \sum_{j=1}^i \lambda_j x^{i-j} \), and vice versa). In the characteristic zero case, one can say more about the algebra \( N(\delta, A) \) (see Lemma 2.2).

The projection homomorphisms \( \phi \) and \( \psi \). Given a ring \( R \) and its derivation \( d \). The Ore extension \( R[x; d] \) of \( R \) is a ring freely generated over \( R \) by \( x \) subject to the defining relations: \( xr = rx + d(r) \) for all \( r \in R \). \( R[x; d] = \oplus_{i \geq 0} Rx^i = \oplus_{i \geq 0} x^i R \) is a left and right free \( R \)-module. Given \( r \in R \), a derivation \( (\text{ad } r)(s) := [r, s] = rs - sr \) of \( R \) is called an inner derivation of \( R \).

**Lemma 2.2** \( \square \) Let \( A \) be an algebra over a field \( K \) of characteristic zero and \( \delta \) be a \( K \)-derivation of \( A \) such that \( \delta(x) = 1 \) for some \( x \in A \). Then \( N(\delta, A) = A^\delta[x; d] \) is the Ore extension with coefficients from the algebra \( A^\delta \), and the derivation \( d \) of the algebra \( A^\delta \) is the restriction of the inner derivation \( \text{ad } x \) of the algebra \( A \) to its subalgebra \( A^\delta \). For each \( n \geq 0 \), \( N_n = \oplus_{i=0}^n A^\delta x^i = \oplus_{i=0}^n x^i A^\delta \).

If the algebra \( A \) is commutative or the element \( x \) is central the result above is old and well-known by specialists (in both cases, the algebra \( A \) is a polynomial algebra \( A^\delta[x] \)).

The element \( x \) from Lemma 2.2 yields the iterative \( \delta \)-descent \( \{x^i \} := \frac{x^i}{i!}, i \geq 0 \) of infinite length.
**Theorem 2.3** Let $A$ be an algebra over an arbitrary field $K$, $\delta$ be a locally nilpotent $K$-derivation of the algebra $A$, and \{x^i, i \geq 0\} be an iterative $\delta$-descent. Then the $K$-linear maps $\phi := \sum_{i \geq 0} (-1)^i x^i \delta^i, \psi := \sum_{i \geq 0} (-1)^i \delta^i (\cdot) x^i : A \to A$ satisfy the following properties.

1. The maps $\phi$ and $\psi$ are homomorphisms of right and left $A^\delta$-modules respectively.

2. The maps $\phi$ and $\psi$ are projections onto the algebra $A^\delta$ (see Lemma 2.1):

\[
\phi : A = A^\delta \oplus A_+ \to A^\delta \oplus A_+, \quad a + b \mapsto a, \text{ where } a \in A^\delta, b \in A_+ := \oplus_{i \geq 1} x^i A^\delta,
\]

\[
\psi : A = A^\delta \oplus A_+ \to A^\delta \oplus A_+, \quad a + b \mapsto a, \text{ where } a \in A^\delta, b \in A_+ := \oplus_{i \geq 1} A^\delta x^i.
\]

In particular, $\text{im}(\phi) = \text{im}(\psi) = A^\delta$ and $\phi(y) = \psi(y) = y$ for all $y \in A^\delta$.

3. $\phi(x^i) = \psi(x^i) = 0$ for all $i \geq 1$.

4. For each $a \in A$, $a = \sum_{i \geq 0} x^i \phi(\delta^i(a)) = \sum_{i \geq 0} \psi(\delta^i(a)) x^i$.

5. If, in addition, the elements $\{x^i\}$ are central then the maps $\phi$ and $\psi$ are $A^\delta$-algebra homomorphisms.

**Remark.** If $\text{char}(K) = 0$ this is Theorem 2.2, [3].

**Proof.** Let us prove the theorem, say, for $\phi$, for the map $\psi$ arguments are literally the same with obvious minor modifications. The map $\phi$ is well-defined since $\delta$ is a locally nilpotent derivation, it is a homomorphism of right $A^\delta$-modules (by the very definition of $\phi$), and $\phi(y) = y$ for all $y \in A^\delta$. For each $j \geq 1$,

\[
\phi(x^j) = \sum_{i=0}^{j} (-1)^i x^i \delta^{j-i} = \sum_{i=0}^{j} (-1)^i \binom{j}{i} x^j = (1 - 1)^j x^j = 0. \quad (3)
\]

So, the map $\phi$ is a projection onto $A^\delta$.

For each $a = \sum_{i \geq 0} x^i a_i \in A = \oplus_{i \geq 0} x^i A^\delta$ where $a_i \in A^\delta$, we have $\phi(\delta^i(a)) = a_i$, hence $a = \sum_{i \geq 0} x^i \phi(\delta^i(a))$.

If, in addition, the elements $\{x^i\}$ are central then the maps $\phi$ and $\psi$ are $A^\delta$-algebra homomorphisms since $A_+$ is a (two-sided) ideal of the algebra $A$. $\square$

The derivation $\delta$ from Theorem 2.3 is locally nilpotent but not nilpotent. The next corollary is a similar result but for a nilpotent derivation $\delta$.

**Corollary 2.4** Let $A$ be an algebra over a field $K$, $\delta$ be a nilpotent $K$-derivation of the algebra $A$ such that $\delta^l = 0$ for some $l \geq 2$, and $\{x^i, 0 \leq i < l\}$ be an iterative $\delta$-descent. Then the $K$-linear maps $\phi := \sum_{i=0}^{l-1} (-1)^i x^i \delta^i, \psi := \sum_{i=0}^{l-1} (-1)^i \delta^i (\cdot) x^i : A \to A$ satisfy the following properties.

1. The maps $\phi$ and $\psi$ are homomorphisms of right and left $A^\delta$-modules respectively.
2. The maps \( \phi \) and \( \psi \) are projections onto the algebra \( A^\delta \) (see Lemma 2.7):

\[
\phi : A = A^\delta \oplus A_+ \to A^\delta \oplus A_+, \quad a + b \mapsto a, \quad \text{where} \quad a \in A^\delta, \ b \in A_+ := \oplus_{i=1}^{l} x^i A^\delta,
\]

\[
\psi : A = A^\delta \oplus A_+ \to A^\delta \oplus A_+, \quad a + b \mapsto a, \quad \text{where} \quad a \in A^\delta, \ b \in A_+ := \oplus_{i=1}^{l} A^\delta x^i.
\]

In particular, \( \text{im}(\phi) = \text{im}(\psi) = A^\delta \) and \( \phi(y) = \psi(y) = y \) for all \( y \in A^\delta \).

3. \( \phi(x[i]) = \psi(x[i]) = 0 \) for all \( i \geq 1 \).

4. For each \( a \in A \), \( a = \sum_{i=0}^{l-1} x[i] \phi(\delta^i(a)) = \sum_{i=0}^{l-1} \psi(\delta^i(a)) x[i] \).

5. If, in addition, the elements \( \{x[i]\} \) are central and such that \( x[i] x[j] \in A_+ \) for all \( i \) and \( j \) with \( i + j \geq 1 \) then the maps \( \phi \) and \( \psi \) are \( A^\delta \)-algebra homomorphisms.

**Proof.** Repeat the proof of Theorem 2.3. \( \square \)

The ring of differential operators \( \mathcal{D}(P_n) \) on a polynomial algebra. If \( \text{char}(K) = p > 0 \) then the ring \( \mathcal{D}(P_n) \) of differential operators on a polynomial algebra \( P_n := K[x_1, \ldots, x_n] \) is a \( K \)-algebra generated by the elements \( x_1, \ldots, x_n \) and commuting higher derivations \( \partial_i^{[k]} := \frac{\partial^k}{\partial x_i^k}, \ i = 1, \ldots, n \) and \( k \geq 1 \), that satisfy the following defining relations:

\[
[x_i, x_j] = [\partial_i^{[k]}, \partial_j^{[l]}] = 0, \quad \partial_i^{[k]} \partial_j^{[l]} = \binom{k+l}{k} \partial_{ij}^{[k+l]}, \quad [\partial_i^{[k]}, x_j] = \delta_{ij} \partial_i^{[k-1]}, \quad (4)
\]

for all \( i, j = 1, \ldots, n \) and \( k, l \geq 1 \) where \( \delta_{ij} \) is the Kronecker delta and \( \partial_i^{[0]} := 1 \). \( \partial_i^{[1]} = \partial_i = \frac{\partial}{\partial x_i} \in \text{Der}_K(P_n), \ i = 1, \ldots, n \). It is convenient to set \( \partial_i^{[-1]} := 0 \). The algebra \( \mathcal{D}(P_n) \) is a simple algebra. Note that the algebra \( \mathcal{D}(P_n) \) is not finitely generated and not (left or right) Noetherian, it does not satisfy finitely many defining relations.

\[
\mathcal{D}(P_n) = \bigoplus_{\alpha, \beta \in \mathbb{N}^n} K x^\alpha \partial^{[\beta]} = \bigoplus_{\alpha, \beta \in \mathbb{N}^n} K \partial^{[\beta]} x^\alpha, \quad \text{where} \quad x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}, \ \partial^{[\beta]} := \partial_1^{[\beta_1]} \cdots \partial_n^{[\beta_n]}. \quad (5)
\]

The algebra \( \mathcal{D}(P_n) \) admits the canonical filtration \( F := \{F_i\}_{i \geq 0} \) where \( F_i := \bigoplus_{|\alpha| + |\beta| \leq i} K \partial^{[\beta]} x^\alpha, \ |\alpha| := \alpha_1 + \cdots + \alpha_n, \ \dim_K(F_i) = \binom{i+n}{2n} \) for all \( i \geq 0 \), \( \mathcal{D}(P_n) = \bigcup_{i \geq 0} F_i \), \( F_0 = K \) and \( F_i F_j \subseteq F_{i+j} \) for all \( i, j \geq 0 \).

For each \( i, j, \ldots, n \), the inner derivations \( -\text{ad} \ x_i \) of the algebra \( \mathcal{D}(P_n) \) and the higher derivations \( \{\partial_i^{[j]}\} \) satisfy the conditions of Theorem 2.4 (i.e. the sequence \( \{\partial_i^{[j]} \}, j \geq 0 \) is an iterative \( (-\text{ad} \ x_i) \)-descent), and the derivations \( \partial_i := \partial_i^{[1]} \) of the algebra \( P_n \) satisfy the conditions of Corollary 2.4 (\( \partial_i^2 = 0 \), \( \partial_i(x_i) = 1 \), and \( \frac{x_i^j}{j!}, 0 \leq j < p \) is an iterative \( \partial_i \)-descent of length \( p \)).

Let \( A \) be an algebra over a field \( K \) of characteristic \( p > 0 \). For each \( x \in A \),

\[
(\text{ad} \ x)^p = \text{ad} \ (x^p)
\]

since, for any \( a \in A \), \( \text{ad} (x^p)(a) = [x^p, a] = \sum_{i=1}^{p} \binom{p}{i} (\text{ad} \ x)^i (a) x^{p-i} = (\text{ad} \ x)^p(a) \).
Theorem 2.5 Let $A$ be an algebra over a field $K$ of characteristic $p > 0$, $\delta$ be a $K$-derivation of the algebra $A$ such that $\delta^p = 0$ and $\delta(x) = 1$ for some element $x \in A$. Then

1. $A = \oplus_{i=1}^{p-1} A^\delta x^i = \oplus_{i=0}^{p-1} x^i A^\delta$ and $A = N(\delta, A) = \cup_{n=0}^{p-1} N_n$ where $N_n = \oplus_{i=0}^n A^\delta x^i = \oplus_{i=0}^n x^i A^\delta$ for $n = 0, 1, \ldots, p - 1$.

2. $[x, A^\delta] \subseteq A^\delta$.

3. Let $A := A^\delta[t; d]$ be an Ore extension of the algebra $A^\delta$ where $d := \text{ad}(x)|_{A^\delta}$. Then $t^p - x^p$ is a central element of the algebra $A$ and the algebra $A$ is canonically isomorphic to the factor algebra $A/(t^p - x^p)$.

Remark. If, in addition, the algebra $A$ is commutative, the first statement of Theorem 2.5 is Theorem 27.3, [9].

Proof. Statement 1 is a particular case of Lemma 2.1(2) with the iterative $\delta$-descent $\{x^i := x^i/t^i, 0 \leq i < p\}$ of length $p$.

For each $c \in A^\delta$, $\delta([x, c]) = [\delta(x), c] + [x, \delta(c)] = [1, c] + [x, 0] = 0$, hence $[x, A^\delta] \subseteq A^\delta$ and $d$ is a derivation of the algebra $A^\delta$. This proves statement 2.

It remains to prove statement 3. Consider the Ore extension $A := A^\delta[t; d]$. There is a natural algebra epimorphism $A \to A$, $t \mapsto x$, $c \mapsto c$ for all $c \in A^\delta$. The epimorphism is obviously an $A^\delta$-module epimorphism. Since both one-sided ideals $A(t^p - x^p)$ and $(t^p - x^p)A$ of the algebra $A$ belong to the kernel of the epimorphism and, obviously, there are isomorphisms $A/A(t^p - x^p) \simeq A$ and $A/(t^p - x^p)A \simeq A$ of left and right $A^\delta$-modules respectively (induced by the epimorphism $A \to A$), we must have the equality $A(t^p - x^p) = (t^p - x^p)A$. Since, for each $c \in A^\delta$, the degree in $t$ of the commutator $[t^p - x^p, c] \in A$ is strictly less than $p$ we must have $[t^p - x^p, c] = 0$ as the commutator belongs to the kernel of the epimorphism above and, by statement 1, $A = \oplus_{i=0}^{p-1} A^\delta x^i$. One can prove this fact also directly using (5) repeatedly:

$$[t^p - x^p, c] = (\text{ad } t)^p(c) - [x^p, c] = (\text{ad } x)^p(c) - [x, x^p] = [x^p, c] - [x^p, c] = 0.$$ 

Now, $[t^p - x^p, t] = [t, x^p] = d(x^p) = [x, x^p] = 0$. This means that the element $t^p - x^p$ belongs to the centre of the algebra $A$ (since it commutes with generators of $A$) and $A \simeq A/(t^p - x^p)$. □

The next corollary describes the algebras from Lemma 2.5.

Corollary 2.6 Let $A$ be an algebra over a field $K$ of characteristic $p > 0$. Then the following statements are equivalent.

1. There exists a $K$-derivation $\delta$ of the algebra $A$ such that $\delta^p = 0$ and $\delta(x) = 1$ for some element $x \in A$.

2. The algebra $A$ is isomorphic to the factor algebra $A/A(x^p - \alpha)$ of an Ore extension $A = B[x; d]$ at the central element $x^p - \alpha$ for some $\alpha \in B^d$ where $d \in \text{Der}_K(B)$ such
that $d^p = \text{ad}(\alpha)$ (more precisely, for any choice of $\alpha \in B^d$ and a derivation $d$ of $B$ such that $d^p = \text{ad}(\alpha)$, the element $x^p - \alpha$ is a central element of $A$). In this case, the derivation $\delta$ (from statement 1) may be chosen as follows: $\delta(B) = 0$ and $\delta(x) = 1$ (then $\delta^p = 0$).

**Proof. (1 $\Rightarrow$ 2) Theorem 2.5**

$(2 \Rightarrow 1)$ First, let us prove that the element $x^p - \alpha$ is central. For any $b \in B$, $[x^p, b] = d^p(b) = [\alpha, b]$ since $d^p = \text{ad}(\alpha)$, and so $[x^p - \alpha, b] = 0$. Finally, $[x^p - \alpha, x] = d(\alpha) = 0$ since $\alpha \in \ker d$. So, the element $x^p - \alpha$ commutes with the generators $B$ and $x$ of the algebra $A$. Therefore, it is a central element of the algebra $A$.

Consider the derivation $\delta \in \text{Der}_B(A)$ given by the rule $\delta(x) = 1$. It is well-defined since, for each $b \in B$, $\delta([x, b]) = [1, b] + [x, \delta(b)] = 0 = \delta(d(b))$ and $\delta(x^p - \alpha) = px^{\alpha-1} - \delta(\alpha) = 0$. Since $\delta^p(x^i) = 0$ for all $i = 1, \ldots, p - 1$, we must have $\delta^p = 0$. □

### 3 Commuting locally nilpotent derivations with iterative descents of maximal length

**Definition.** Let $\delta$ be a nonzero locally nilpotent $K$-derivation of an algebra $A$ over an arbitrary field $K$ and $y := \{y^{[i]}\}$ be an iterative $\delta$-descent. We say that $y$ is of maximal length (or has maximal length) if $y$ has infinite length in the case when $\delta$ is not nilpotent derivation, and $y$ has length $l$ if $\delta^l = 0$ and $\delta^{l-1} \neq 0$.

Let $A$ be an algebra over a field $K$, $\delta := \{\delta_i, i \in I\}$ be a non-empty set of commuting, locally nilpotent $K$-derivations of the algebra $A$ such that

(i) for each $a \in A$, $\delta_i(a) = 0$ for almost all $i \in I$ (i.e. all but finitely many $i$), and

(ii) for each $i \in I$, there is an iterative $\delta_i$-descent $x_i := \{x_i^{[j]}\}$ of maximal length, say $l_i$, such that $\{x_i^{[j]}\} \subseteq \cap_{i \neq k \in I} A^{b_i}$.

**Remark.** If the set $I$ is finite then the condition (i) is vacuous.

The set $\delta$ is a disjoint union of two subsets $\delta = \mathbf{n} \cup I$ where the set $\mathbf{n}$ contains all the *nilpotent* derivations and the set $I$ contains all the *non-nilpotent* derivations. It is possible that one of these sets is an empty set.

Let $\mathbb{N} = \{0, 1, 2, \ldots\}$ be the monoid of natural numbers, $\mathbb{N}^{(I)}$ be the direct sum of $I$’th copies of the monoid $\mathbb{N}$,

$$E = E(\delta) := \{\alpha = (\alpha_i) \in \mathbb{N}^{(I)} | 0 \leq \alpha_i \leq l_i - 1 \text{ where } l_i \text{ is the length of } x_i\},$$

it is the set of all ‘possible exponents’ for $x^{[\alpha]}$ (see below). Without loss of generality we may assume that the set $I$ is a well-ordered set with respect to an ordering $<$. For each $\alpha = (\alpha_i) \in E$, consider the *ordered* product,

$$x^{[\alpha]} := \begin{cases} 1 & \text{if } \alpha = 0, \\ x_i^{[\alpha_i]} \ldots x_i^{[\alpha_n]} & \text{if } \alpha \neq 0 \text{ and } \end{cases}$$
\( i_1 < \cdots < i_n \), and \( \alpha_{i_1}, \ldots, \alpha_{i_n} \) are the only nonzero coordinates of \( \alpha \), the set \( \{ \alpha_{i_1}, \ldots, \alpha_{i_n} \} \) is called the support of \( \alpha \).

For each \( i \in I \), consider the maps \( \phi_i, \psi_i : A \to A \) from Theorem 2.3 and Corollary 2.4:

(a) if \( \{ x_i^j, j \geq 0 \} \) is an infinite \( \delta_i \)-descent then

\[
\phi_i := \sum_{j=0}^{\infty} (-1)^j x_i^j \delta_i^j \quad \text{and} \quad \psi_i := \sum_{j=0}^{\infty} (-1)^j \delta_i^j(x_i^j),
\]

(b) if \( \{ x_i^j, 0 \leq j \leq l_i - 1 \} \) is a finite \( \delta_i \)-descent then

\[
\phi_i := \sum_{j=0}^{l_i-1} (-1)^j x_i^j \delta_i^j \quad \text{and} \quad \psi_i := \sum_{j=0}^{l_i-1} (-1)^j \delta_i^j(x_i^j).
\]

**Theorem 3.1** Let \( A \) be an algebra over a field \( K \), \((I, <)\) be a non-empty well-ordered set, \( \delta := \{ \delta_i, i \in I \} \) be a set of commuting locally nilpotent \( K \)-derivation of the algebra \( A \) such that, for each \( a \in A \), \( \delta_i(a) = 0 \) for almost all \( i \in I \). Suppose that, for each \( i \in I \), there exists an iterative \( \delta_i \)-descent \( \{ x_i^j \} \) of maximal length such that \( \{ x_i^j \} \subseteq \cap_{i \neq k \in I} A_{\delta_i}^k \). Then

1. \( A = \oplus_{\alpha \in E} x^\alpha A^\delta = \oplus_{\alpha \in E} A^\delta x^\alpha \) where \( A^\delta := \cap_{i \in I} A_{\delta_i} \).

2. Consider the ordered products of maps

\[
\phi := \prod_{i \in I} \phi_i, \quad \psi := \prod_{i \in I} \psi_i : A \to A,
\]

given by the rules: \( \phi(1) := 1, \psi(1) := 1 \), and, for each \( 0 \neq \alpha \in E \), with \( \text{supp}(\alpha) = \{ \alpha_{i_1}, \ldots, \alpha_{i_n} \} \) where \( i_1 < \cdots < i_n \),

\[
\phi(x^\alpha) := \phi_{i_1} \cdots \phi_{i_n}(x_{i_1}^{\alpha_{i_1}} \cdots x_{i_n}^{\alpha_{i_n}}), \quad \psi(x^\alpha) := \psi_{i_1} \cdots \psi_{i_n}(x_{i_1}^{\alpha_{i_1}} \cdots x_{i_n}^{\alpha_{i_n}}).
\]

Then the maps \( \phi \) and \( \psi \) are homomorphisms of right and left \( A^\delta \)-modules respectively.

3. The maps \( \phi \) and \( \psi \) are projections onto the algebra \( A^\delta \):

\[
\phi : A = A^\delta + A_+ \to A^\delta + A_+, \quad a + b \mapsto a, \quad \text{where} \quad a \in A^\delta, \ b \in A_+ := \oplus_{0 \neq \alpha \in E} x^\alpha A^\delta,
\]

\[
\psi : A = A^\delta + A_+ \to A^\delta + A_+, \quad a + b \mapsto a, \quad \text{where} \quad a \in A^\delta, \ b \in A_+ := \oplus_{0 \neq \alpha \in E} A^\delta x^\alpha.
\]

In particular, \( \text{im}(\phi) = \text{im}(\psi) = A^\delta \) and \( \phi(y) = \psi(y) = y \) for all \( y \in A^\delta \).

4. \( \phi(x^\alpha) = \psi(x^\alpha) = 0 \) for all \( 0 \neq \alpha \in E \).

5. For each \( a \in A \), \( a = \sum_{\alpha \in E} x^\alpha \phi(\delta^\alpha(a)) = \sum_{\alpha \in E} \psi(\delta^\alpha(a))x^\alpha \) where \( \delta^\alpha := \prod_{i \in I} \delta_i^{\alpha_i} \), a finite product.

6. If, in addition, all the elements \( \{ x_i^j \} \) are central and for each finite length sequence \( \{ x_i^j, 0 \leq j < l_i \} \) : \( x_i^j x_i^k \in A_+ \) for all \( j \) and \( k \) with \( j + k \geq l_i \), then the maps \( \phi \) and \( \psi \) are \( A^\delta \)-algebra homomorphisms.
Proof. We have obvious symmetry between \( \phi \) and \( \psi \), and between statements about left \( A^\delta \)-modules and right \( A^\delta \)-modules. Let us, say, prove the statements about the map \( \phi \) and about right \( A^\delta \)-modules.

1. The sum \( A' := \sum_{\alpha \in E} x^{[\alpha]} A^\delta \) is a direct sum, \( A' := \oplus_{\alpha \in E} x^{[\alpha]} A^\delta \), as follows at once from the fact that \( \delta^\alpha(x^{[\alpha]}) = 1 \) for all \( \alpha \in E \). We have to prove that \( A' = A \). Let \( a \in A \). We have to show that \( a \in A' \). If \( a \in A^\delta \subseteq A' \) then there is nothing to prove. So, let \( a \not\in A^\delta \). By the assumption, there are only finitely many derivations, say \( \delta_1, \ldots, \delta_n \), such that \( \delta_i(a) \neq 0 \) (to save on notation we may assume that \( \{1 < \cdots < n\} \subseteq I \)). Applying step by step either Theorem 2.3 or Corollary 2.3 we have (where \( A^{\delta_1, \ldots, \delta_k} := \cap_{j=1}^k A^{\delta_j} \))

\[
A = \bigoplus_{i_1} x_1^{[i_1]} A^\delta_1 = \bigoplus_{i_1, i_2} x_1^{[i_1]} x_2^{[i_2]} A^\delta_1, \delta_2 = \cdots = \bigoplus_{i_1, \ldots, i_n} x_1^{[i_1]} \cdots x_n^{[i_n]} A^\delta_1, \ldots, A^\delta_n, \tag{7}
\]

we have used the facts that the derivations \( \delta_1, \ldots, \delta_n \) commute and that \( \delta_i(x_j^{[\lambda]}) = 0 \) if \( i \neq j \).

The element \( a \) is a unique finite sum \( a = \sum x_1^{[i_1]} \cdots x_n^{[i_n]} \lambda_{i_1, \ldots, i_n} \) for some \( \lambda_{i_1, \ldots, i_n} \in A^{\delta_1, \ldots, \delta_n} \). For each \( k \in I \setminus \{1, \ldots, n\} \),

\[
0 = \delta_k(a) = \sum x_1^{[i_1]} \cdots x_n^{[i_n]} \delta_k(\lambda_{i_1, \ldots, i_n}),
\]

therefore \( \delta_k(\lambda_{i_1, \ldots, i_n}) = 0 \) by (7), i.e. all \( \lambda_{i_1, \ldots, i_n} \in A^\delta \). This proves the equality \( A = A' \).

2 - 4. By the very definition, the map \( \phi \) is a homomorphism of right \( A^\delta \)-modules, \( \phi(1) := 1 \), and, for each \( 0 \neq \alpha \in E \) with \( \text{supp}(\alpha) = \{\alpha_1, \ldots, \alpha_n\} \) and \( i_1 < \cdots < i_n \),

\[
\phi(x^{[\alpha]}) = \phi_{i_1}(x_1^{[\alpha_{i_1}]}) \cdots \phi_{i_n}(x_n^{[\alpha_{i_n}]}) = \delta_{0, \alpha_{i_1}} \cdots \delta_{0, \alpha_{i_n}} = \delta_{0, \alpha} \quad \text{(the Kronecker delta)}.
\]

Now, statements 2 - 4 follow. 

5. For each \( a \in A \), we have a finite sum \( a = \sum x^{[\alpha]} \lambda_{\alpha} \) with \( \lambda_{\alpha} \in A^\delta \) (statement 1). Since \( \lambda_{\alpha} = \phi(\delta^\alpha(a)) \), we have \( a = \sum x^{[\alpha]} \phi(\delta^\alpha(a)) \), and so statement 5.

6. The assumptions of statement 6 guarantee that \( A_+ = I \) is an ideal of \( A \), hence \( \phi \) is an \( A^\delta \)-algebra homomorphism. \( \square \)

Infinitely iterated Ore extensions. Let \( R \) be a ring, \( d = \{d_i : i \in I\} \) be a non-empty (not necessarily finite) set of derivations of the ring \( R \), \( r = (r_{ij}) \) be a skew symmetric \( I \times I \) matrix (possibly of infinite size) with entries from \( R \) (\( r_{ij} = -r_{ji} \) and \( r_{ii} = 0 \)) such that

\[
[d_i, d_j] = \text{ad}(r_{ij}) \quad \text{and} \quad d_i(r_{jk}) = -d_k(r_{ij}) + d_j(r_{ik}) \quad \text{for all} \ i, j \in I.
\]

For each finite subset, say \( J = \{1, \ldots, n\} \), of \( I \) consider the iterated Ore extension

\[
R_J := R[t_1; d_1] \cdots [t_n; d_n] = R_G[t_n; d_n], \quad G := \{1, \ldots, n - 1\},
\]

where the derivation \( d_n \) of the ring \( R \) extended to a derivation (denoted in the same fashion) of the ring \( R_G \) by the rule \( d_n(t_i) = r_{ni} \). It is an easy exercise to verify that \( d_n \) is then well-defined: if \( n = 1 \) then there is nothing to prove. For \( n > 1 \), by induction on \( n \) one may assume that \( R_G \) is well-defined. Then the ring \( R_G \) has the following defining relations:

\[
[t_i, t_j] = r_{ij} \quad \text{and} \quad [t_i, r] = d_i(r), \quad 0 \leq i < j \leq n - 1, \ r \in R.
\]
Now, the extended $d_n$ respects these relations:
\[
d_n([t_i, t_j]) = [d_n(t_i), t_j] + [t_i, d_n(t_j)] = [r_{ni}, t_j] + [t_i, r_{nj}] = -d_j(r_{ni}) + d_i(r_{nj}) = d_n(r_{ij}),
\]
\[
d_n([t_i, r]) = [d_n(t_i), r] + [t_i, d_n(r)] = r_{ni}r + d_id_n(r) = (d_n, d_i + d_id_n)(r) = d_n d_i (r).
\]

Clearly, $J \subseteq L$ implies $R_J \subseteq R_L$ for finite subsets $J, L \subseteq I$. The infinitely iterated Ore extension $R[t; d, r]$ is the union (the direct limit) of the rings $R_I$. Without loss of generality one may assume that the set $I$ is a well-ordered set, $(I, <)$. Then

\[
R[t; d, r] = \bigoplus_{\alpha \in \mathbb{N}^{(|I|)}} R t^\alpha = \bigoplus_{\alpha \in \mathbb{N}^{(|I|)}} t^\alpha R,
\]
where $t^\alpha$ is the ordered product $t_{i_1}^{\alpha_1} \cdots t_{i_n}^{\alpha_n}$ where $\text{supp}(\alpha) = \{\alpha_{i_1}, \ldots, \alpha_{i_n}\}$ and $i_1 < \ldots < i_n$.

The partial derivatives $\partial_i := \frac{\partial}{\partial t_i} \in \text{Der}_R(R[t; d, r])$ are well-defined $R$-derivations of the ring $R[t; d, r]$ (since they respect the defining relations). So, $\partial := \{\partial_i, i \in I\}$ is the set of commuting locally nilpotent $R$-derivations of the ring $R[t; d, r]$ such that $\partial_i(t_j) = \delta_{ij}$. If $R$ is an algebra over a field of characteristic $p > 0$ then $\partial_i^p = 0$, $i \in I$.

**Lemma 3.2** Let the ring $A := R[t; d, r]$ be as above. Suppose, in addition, that $R$ is an algebra over a field of characteristic $p > 0$, $r_{ij} = [x_i, x_j]$ and $d_i = \text{ad}(x_i)$ for some elements $x_i \in R$. Then

1. $t_i^p - x_i^p$, $i \in I$, are central elements of $A$.
2. The ideal $a := (t_i^p - x_i^p, i \in I)$ of $A$ is $\partial_j$-invariant for all $j \in I$ ($\partial_j(a) \subseteq a$).
3. Let $A := A/a$ and $\delta_i \in \text{Der}_R(A)$: $a + a \rightarrow \partial_i(a) + a$. Then $\delta = \{\delta_i, i \in I\}$ is the set of commuting $R$-derivations of $A$ such that $\delta_i^p = 0$, $\delta_i(x_j) = \delta_{ij}$, and, for each $a \in A$, $\delta_i(a) = 0$ for almost all $i$.

Proof. 1. $\text{ad}(t_i^p) = \text{ad}(t_i)^p = \text{ad}(x_i)^p = \text{ad}(x_i^p)$, hence $\text{ad}(t_i^p - x_i^p) = 0$, i.e. $t_i^p - x_i^p$ is a central element of $A$. The rest is obvious. □

The next theorem is the converse to Lemma 3.2.

**Theorem 3.3** Let $A$ be an algebra over a field $K$ of characteristic $p > 0$, $\delta = \{\delta_i, i \in I\}$ be a non-empty set of commuting $K$-derivations of the algebra $A$ such that $\delta_i^p = 0$, $\delta_i(x_j) = \delta_{ij}$ (the Kronecker delta) for a set $x = \{x_i, i \in I\}$ of elements of $A$, and, for each $a \in A$, $\delta_i(a) = 0$ for almost all $i$. Then

1. $A = \bigoplus_{\alpha \in \mathcal{N}} A^\delta x^\alpha = \bigoplus_{\alpha \in \mathcal{N}} x^\alpha A^\delta$ where $\mathcal{N} := \{\alpha \in \mathbb{N}^{(|I|)} \mid \text{all } \alpha_i < p\}$ and $x^\alpha := x_{i_1}^{\alpha_1} \cdots x_{i_n}^{\alpha_n}$ where $\text{supp}(\alpha) = \{\alpha_{i_1}, \ldots, \alpha_{i_n}\}$ and $i_1 < \ldots < i_n$ (where $(I, <)$ is the well-ordered set).
2. $r_{ij} := [x_i, x_j] \in A^\delta$ and $[x_i, A^\delta] \subseteq A^\delta$ for all $i, j \in I$.
3. $A \cong A^\delta[t; d, r]/(t_i^p - x_i^p, i \in I)$ where $t = \{t_i, i \in I\}$, $d = \{d_i := \text{ad}(x_i), i \in I\}$, and $r = (r_{ij}), \{t_i^p - x_i^p, i \in I\}$ are central elements of the ring $A^\delta[t; d, r]$. 


4. Each derivation \( \delta_i \) is induced by the partial \( A^\delta \)-derivatives \( \partial_i := \frac{\partial}{\partial u_i} \) of the ring \( A^\delta[t; d, r] \).

**Proof.** 1. Theorem 3.1 since \( E = N \).

2. For any \( i, j \in I \), \( \delta_j([x_i, A^\delta]) \subseteq [\delta_{ij}, A^\delta] + [x_i, \delta_j(A^\delta)] = \delta_{ij}A^\delta \). For \( i, j, k \in I \), \( \delta_k([x_i, x_j]) = [\delta_{ki}, x_j] + [x_i, \delta_{kj}] = 0 \), i.e. \( r_{ij} \in A^\delta \).

3. The ring \( A := A^\delta[t; d, r] \) is well-defined since \( [d_i, d_j] = [\text{ad}(x_i), \text{ad}(x_j)] = \text{ad}([x_i, x_j]) \) and \( d_i(r_{jk}) = [x_i, [x_j, x_k]] = [[x_i, x_j], x_k] + [x_j, [x_i, x_k]] = -d_k(r_{ij}) + d_j(r_{ik}) \), the Jacobi identity.

Each element \( u := t_i^p - x_i^p \) belongs to the centre of the ring \( A \) since it commutes with the generators of \( A \): for each \( \lambda \in A^\delta \), \( [t_i^p, \lambda] = \text{ad}(t_i)^p(\lambda) = \text{ad}x_i)^p(\lambda) = [x_i^p, \lambda] \), and so \( [u, \lambda] = 0 \); for each \( j \in I \),

\[
[t_i^p, t_j] = \text{ad}(t_i)^p(t_j) = \text{ad}(t_i)^p-1([t_i, t_j]) = \text{ad}(t_i)^p-1([x_i, x_j]) = \text{ad}x_i)^p-1([x_i, x_j]) = (ad x_i)^p(x_j) = [x_i^p, x_j] = [x_i^p, t_j],
\]

and so \( [u, t_j] = 0 \). \( \Box \)

### 4 A formula for the inverse of an automorphism that preserves the ring of invariants

Let \( A, \delta := \{ \delta_i, i \in I \} \), and \( \{ x_i^{[j]} \} \) be as in Theorem 3.1 Suppose that a \( K \)-algebra automorphism \( \sigma \in \text{Aut}_K(A) \) preserves the ring of invariants \( A^\delta \), that is \( \sigma(A^\delta) = A^\delta \). Let \( \sigma_\delta := \sigma|_{A^\delta} \in \text{Aut}_K(A^\delta) \). Suppose that we know explicitly the inverse \( \sigma^{-1} \) and the twisted derivations \( \delta' := \{ \delta'_i := \sigma_\delta \sigma^{-1}, i \in I \} \), then we can write down explicitly a formula for the inverse automorphism \( \sigma^{-1} \) (Theorem 4.1). This is a very simplified version of the strategy we will follow in almost all the examples later. Since \( A = \oplus_{\alpha \in E} A^\delta x^{[\alpha]} = \oplus_{\alpha \in E} x^{[\alpha]} A^\delta \) (Theorem 3.1), the automorphism \( \sigma \) is uniquely determined by its restriction \( \sigma_\delta \) to the ring of invariants \( A^\delta \) and the images of the iterative descents \( x_i' := \{ x_{i'}^{[j]} := \sigma(x_i^{[j]}) \}, i \in I \). Note that the derivations \( \delta' = \{ \delta'_i \} \) and their iterative descents \( \{ x_i' \} \) do satisfy the conditions of Theorem 3.1 with \( A^{\delta'} = \sigma(A^\delta) = A^\delta \). For each \( i \in I \), let

\[
\phi_i' := \sum_{k=0}^{l_i-1} (-1)^k \frac{(x_i')^k}{k!} (\delta'_i)^k, \quad \psi_i' := \sum_{k=0}^{l_i-1} (-1)^k \frac{(\delta'_i)^k}{k!} (x_i')^k,
\]

\[
\phi_\sigma := \phi' := \prod_{i \in I} \phi_i', \quad \psi_\sigma := \psi' := \prod_{i \in I} \psi_i,'
\]

be the corresponding maps from Theorem 3.1. The maps \( \phi_\sigma, \psi_\sigma : A \to A \) are projections onto the subalgebra \( A^{\delta'} = A^\delta \) of \( A \), and, for any \( a \in A \),

\[
a = \sum_{\alpha \in E} x^{[\alpha]} \phi_\sigma(\delta'^\alpha(a)) = \sum_{\alpha \in E} \psi_\sigma(\delta'^\alpha(a)) x^{[\alpha]}.
\]
Then applying $\sigma^{-1}$ to these equalities we finish the proof of the next theorem.

**Theorem 4.1** Let $A$, $\delta = \{\delta_i\}$, $\delta' = \{\delta'_i\}$, $\{x_i\}$, $\{x'_i\}$, and $\sigma$ be as above. Then, for each $a \in A$,

$$\sigma^{-1}(a) = \sum_{\alpha \in E} x^{[\alpha]} \sigma^{-1}_\delta(\phi_\sigma(\delta'^\alpha(a))) = \sum_{\alpha \in E} \sigma^{-1}_\delta(\psi_\sigma(\delta'^\alpha(a)))x^\alpha.$$ 

### 5 An inversion formula for an automorphism of a central simple algebra

In this section, $K$ is an *arbitrary* field and $A$ is a *central simple $K$-algebra* (central means that the centre of $A$ is $K$). For simplicity, let us assume that the algebra $A$ is *countably generated*, for example $A$ is a finitely generated algebra, the general case is considered at the end of the section. Then $A = \cup_{i \geq 0} A_i$ is a union of an ascending chain of *finite dimensional* subspaces: $K \subseteq A_0 \subseteq A_1 \subseteq \cdots$.

The algebra $E := \text{End}_K(A)$ of all $K$-linear maps from $A$ to itself is a *topological* algebra with respect to the *finite topology* where neighbourhoods of the zero map is given by the ascending chain of subspaces

$$E_0 \supseteq E_1 \supseteq \cdots \supseteq E_i := \{f \in E \mid f(A_i) = 0\} \supseteq \cdots, \cap_{i \geq 0} E_i = 0.$$ 

So, an element $f$ of $E$ is 'small' if it annihilates 'big' $A_i$. Note that this description of the finite topology works only for algebras that has no more than countable basis over $K$. For the general case, the reader is referred to the book of Jacobson [8].

Let $A$ be the image of the $K$-algebra homomorphism

$$A \otimes A^{op} \to E, \ a \otimes b \mapsto (x \mapsto axb),$$

where $x \in A$ and $A^{op}$ is the *opposite* algebra to $A$. By the *Density Theorem*, $A$ is a *dense* subalgebra in $E$, i.e. for any map $f \in E$ and any $i \geq 0$, there are elements $a_j, b_j \in A$ such that $f(a) = \sum_j a_jab_j$ for all $a \in A_i$.

Let $e_1 := 1, e_2, \ldots$ be a $K$-basis for $A$ such that for each $i \geq 0$, $e_1, e_2, \ldots, e_{n_i}$ is a $K$-basis for $A_i$. Clearly, $n_0 \leq n_1 \leq \ldots$. For each $i \geq 0$ and $j$ such that $1 \leq j \leq n_i$, consider the $K$-linear map

$$p_{ij} : A_i = \bigoplus_{k=1}^{n_i} Ke_k \to A_i = \bigoplus_{k=1}^{n_i} Ke_k, \ \sum_{k=1}^{n_i} \lambda_k e_k \mapsto \lambda_j e_1 = \lambda_j.$$ 

By the Density Theorem, there are elements $\{a_{ij\nu}, b_{ij\nu}, \nu \in N_{ij}\}$ such that $p_{ij}(a) = \sum_{\nu \in N_{ij}} a_{ij\nu}ab_{ij\nu}$ for all $a \in A_i$. Then, for each $a \in A_i$, $a = \sum_{j=1}^{n_i} p_{ij}(a)e_j$, or, equivalently, for each $a \in A$,

$$a = \sum_{j=1}^{n_i} p_{ij}(a)e_j, \ i \gg 0.$$ 

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Applying $\sigma$ and denoting $\sigma(a)$ by $a$, we see that for each $a \in A$,

$$a = \sum_{j=1}^{n_i} p'_{ij}(a)e_j, \quad i \gg 0,$$

where $p'_{ij}(\cdot) = \sum_{\nu \in N_{ij}} \sigma(a_{ij\nu})(\cdot)\sigma(b_{ij\nu})$. Applying $\sigma^{-1}$, we prove the next theorem (use the fact that $p'_{ij}(A) \subseteq K$).

**Theorem 5.1** (The inversion formula) Let $A$ be the central simple countably generated algebra over the field $K$ and $\sigma \in \text{Aut}_K(A)$. Then, for each $a \in A$,

$$\sigma^{-1}(a) = \sum_{j=1}^{n_i} p'_{ij}(a)e_j, \quad i \gg 0,$$

where $p'_{ij}(\cdot) = \sum_{\nu \in N_{ij}} \sigma(a_{ij\nu})(\cdot)\sigma(b_{ij\nu})$.

**Corollary 5.2** (The inversion formula for a central simple finite dimensional algebra) If, in addition, the algebra $A$ is finite dimensional over $K$. Then $A = A_i$ for some $i$ and, for each $a \in A$,

$$\sigma^{-1}(a) = \sum_{j=1}^{n_i} p'_{ij}(a)e_j.$$

Let $A$ be an arbitrary central simple $K$-algebra. Fix a covering $A = \bigcup_{i \in I} A_i$ of $A$ by a set of finite dimensional subspaces $A_i$ such that $K \subseteq A_i$ for all $i \in I$. For each $i \in I$, fix a basis $\{e_j^i\}$ for the vector space $A_i$ with $e_1^i := 1$, and then define the maps $p_{ij} : A_i \to A_i$ in the same way as before, $p_{ij}(\sum \lambda_k e_k^i) = \lambda_j e_1^i = \lambda_j \in K$. By the Density Theorem, there are elements $\{a_{ij\nu}, b_{ij\nu}\}$ such that $p_{ij}(a) = \sum_{\nu} a_{ij\nu}b_{ij\nu}$ for all $a \in A_i$. Then each map $p'_{ij} := \sigma(p_{ij}) = \sum_{\nu} \sigma(a_{ij\nu})(\cdot)\sigma(b_{ij\nu}) : \sigma(A_i) \to \sigma(A_i)$ has the image $K$ since $\sigma(K) = K$, and, for each $a \in A$,

$$a = \sum_{j=1}^{n_i} p'_{ij}(a)e_j^i, \quad i \gg 0,$$

where ‘$i \gg 0$’ means that for all $i$ such that $a \in \sigma(A_i)$. Applying $\sigma^{-1}$, we have $\sigma^{-1}(a) = \sum p'_{ij}(a)e_j^i$ for $i \gg 0$. This means that Theorem 5.1 holds with the new meaning of ‘$i \gg 0$’.

6 The inversion formula for an automorphism of a polynomial algebra

In this section, $K$ is a field of characteristic $p > 0$ and $P_n := K[x_1, \ldots, x_n]$ is a polynomial algebra in $n$ variables over the field $K$.

Using the defining relations (1) for the algebra of differential operators $\mathcal{D}(P_n)$ on $P_n$ and its simplicity, one can verify that, for each natural number $k \geq 1$, there exists a $K$-algebra monomorphism of $\mathcal{D}(P_n)$ (which is obviously not an isomorphism) given by the rule

$$f_k : \mathcal{D}(P_n) \to \mathcal{D}(P_n), \quad x_i \mapsto x_i^k, \quad \partial_i^{[a]} \mapsto \partial_i^{p[a]}, \quad i = 1, \ldots, n. \quad (10)$$
Proved this conjecture for certain algebraic\( {\text{ and } K}\) each \( Z\) its restriction to the centre \( x\). Problem of Dixmier\( \ [{7}\), Problem 1\): Note that each monomorphism \( D\) monomorphism automatically a \( K\)-algebra homomorphism is a (well-defined) homomorphism of \( P\) polynomial subalgebra for a given polynomial \( u\) \( u\) \( \cdots + d_k p^k\) where \( 0 \leq d_j < p\) and \( d_k \neq 0\), one has \( \phi_i(u) = \phi_{i,k} \cdots \phi_{i,0}(u)\). For any fixed element \( u \in P_n\), almost all maps \( \phi_{i,k}\) in the product \( K\)-algebra homomorphism which is a projection onto the field \( K\). Let \( u \in P_n\) and \( \deg_{x_i}(u) = d_{i,0} + d_{i,1} p + \cdots + d_{i,k} p^k\) be the degree of the polynomial \( u\) in \( x_i\) where \( 0 \leq
Theorem 6.1  For any $\phi$, the maps $d_x$ of the canonical generators, $d_x u$ where, in the formula above, $\partial$ derivations $\partial$ for the higher derivations $\sigma$ act as the identity map on $u$.

Recall that, for $\alpha, \beta \in \mathbb{N}^n$,

$$\partial^{[\alpha]}(x^\beta) = \left( \frac{\beta}{\alpha} \right) x^{\beta - \alpha}, \quad \left( \frac{\beta}{\alpha} \right) := \prod_{i} \left( \frac{\beta_i}{\alpha_i} \right),$$

(13)

where, in the formula above, $x_i^t := 0$ for all negative integers $t$ and all $i$.

The next result is a kind of the Taylor formula in prime characteristic.

**Theorem 6.1**  For any $a \in P_n$,

$$a = \sum_{\alpha \in \mathbb{N}^n} \phi(\partial^{[\alpha]}(a))x^\alpha.$$  

Proof. If $a = \sum \lambda_\alpha x^\alpha, \lambda_\alpha \in K$, then, by (13) and (12), $\phi(\partial^{[\alpha]}(a)) = \lambda_\alpha$. $\square$

Equivalently, the identity map $\text{id}$ on $P_n$ can be written as

$$\text{id}(\cdot) = \sum_{\alpha \in \mathbb{N}^n} \phi(\partial^{[\alpha]}(\cdot))x^\alpha.$$  

(14)

Let $\sigma$ be a $K$-automorphism of the polynomial algebra $P_n$, it is uniquely determined by the images of the canonical generators, $x'_1 := \sigma(x_1), \ldots, x'_n := \sigma(x_n)$. The Jacobian of the automorphism $\sigma$, that is $\Delta := \det(\frac{\partial x'_i}{\partial x_j})$, must be a nonzero scalar, i.e. $\Delta \in K^* := K\setminus\{0\}$. The derivations $\partial'_1 := \frac{\partial}{\partial x'_1}, \ldots, \partial'_n := \frac{\partial}{\partial x'_n} \in \text{Der}_K(P_n)$ can be written as

$$\partial'_j := \Delta^{-1}\text{det} \begin{pmatrix} \frac{\partial\sigma(x_1)}{\partial x'_1} & \cdots & \frac{\partial\sigma(x_1)}{\partial x'_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial\sigma(x_n)}{\partial x'_1} & \cdots & \frac{\partial\sigma(x_n)}{\partial x'_n} \end{pmatrix}, \quad j = 1, \ldots, n,$$

(15)

where we have ‘dropped’ $\sigma(x_j)$ in the determinant $\det(\frac{\partial\sigma(x_k)}{\partial x'_l})$.

Let $\partial^{[\alpha]} := \frac{\partial^{[\alpha]}}{\partial^0} = \frac{\partial^{[\alpha]}}{\partial^1} \cdots \frac{\partial^{[\alpha]}}{\partial^{[0]}}$ (where $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$) be the corresponding higher derivations for the new choice of generators for the polynomial algebra $P_n$; that is $x'_1, \ldots, x'_n$. Now, we consider the projections (11) and (12) for the generators $x'_1, \ldots, x'_n$ of $P_n$. For each $i = 1, \ldots, n$,

$$\phi'_i := \cdots \phi'_{i,k} \cdots \phi'_{i,1} \phi'_{i,0} : P_n \to P_n, \quad \phi'_{i,k} := \sum_{j=0}^{p-1} (-1)^j \frac{x'_i^{p^k j}}{j!} \frac{\partial^{[p^k j]}}{\partial^j} \phi_{i,k}, \quad k \geq 0.$$  

(16)

The maps $\phi'_i$ commute. Let

$$\phi_\sigma := \phi' := \prod_{i=1}^n \phi'_i.$$  

(17)
Theorem 6.2 (The Inversion Formula) For each \( \sigma \in \text{Aut}_K(P_n) \) and \( a \in P_n \),
\[
\sigma^{-1}(a) = \sum_{\alpha \in \mathbb{N}^n} \phi_\sigma(\partial^{[\alpha]}(a))x^\alpha.
\]

Proof. By Theorem 6.1 \( a = \sum_{\alpha \in \mathbb{N}^n} \phi_\sigma(\partial^{[\alpha]}(a))x^\alpha \), then applying \( \sigma^{-1} \) we have the result
\[
\sigma^{-1}(a) = \sum_{\alpha \in \mathbb{N}^n} \phi_\sigma(\partial^{[\alpha]}(a))\sigma^{-1}(x^\alpha) = \sum_{\alpha \in \mathbb{N}^n} \phi_\sigma(\partial^{[\alpha]}(a))x^\alpha. \quad \square
\]

7 The inversion formula for \( \sigma \in \text{Aut}_K(D(K[x_1, \ldots, x_n])) \)

In this section, \( K \) is a field of characteristic \( p > 0 \) and \( D := D(P_n) \) is the ring of differential operators on the polynomial algebra \( P_n := K[x_1, \ldots, x_n] \) over the field \( K \). The algebra \( D \) is a central simple countably generated (but not finitely generated) algebra over \( K \). The results of Section 5 show that the inversion formula for \( \sigma \in \text{Aut}_K(D) \) can be written in the most economical way - using only addition and multiplication. Clearly, \( D = D(K[x_1]) \otimes \cdots \otimes D(K[x_n]) \), the tensor product of algebras. For each \( i = 1, \ldots, n \), the inner derivation \( \delta_i := -\text{ad } x_i \) of the algebra \( D \) is a locally nilpotent derivation, \( D^{\delta_i} = D(K[x_1]) \otimes \cdots \otimes K[x_i] \otimes \cdots \otimes D(K[x_n]) \), and \( \{\partial_i^{[j]}, j \geq 0\} \) is the iterative \( \delta_i \)-descent (of maximal length) since
\[
(-\text{ad } x_i)(\partial_i^{[j]}) = \partial_i^{[j-1]} \quad \text{and} \quad \partial_i^{[j]}\partial_i^{[k]} = \binom{j + k}{j}\partial_i^{[j+k]} \quad \text{for all } j, k \geq 0.
\]

By Theorem 2.3 there are two projections onto \( D^{\delta_i} \) (where \( a \in D^{\delta_i} \)):
\[
\phi_i = \sum_{k \geq 0} \partial_i^{[k]}(ad x_i)^k : D = D^{\delta_i} \oplus D_{+,i} \to D^{\delta_i} \oplus D_{+,i}, a + b \mapsto a, b \in D_{+,i} = \oplus_{k \geq 1} \partial_i^{[k]}D^{\delta_i},
\]
\[
\psi_i = \sum_{k \geq 0} (ad x_i)^k(\cdot)\partial_i^{[k]} : D = D^{\delta_i} \oplus D_{+,i} \to D^{\delta_i} \oplus D_{+,i}, a + b \mapsto a, b \in D_{+,i} = \oplus_{k \geq 1} D^{\delta_i}\partial_i^{[k]}.
\]

The maps \( \phi_i \) and \( \psi_i \) are homomorphisms of right and left \( D^{\delta_i} \)-modules respectively. The maps \( \phi_i \) (resp. \( \psi_i \)) commute and their products yield projections onto the polynomial subalgebra \( P_n \) of the ring of differential operators \( D(P_n) \),
\[
\phi_n \cdots \phi_1 : D = P_n \oplus D_+ \to P_n \oplus D_+, a + b \mapsto a, a \in P_n, b \in D_+ = \oplus_{\alpha \neq 0} \partial^{[\alpha]}P_n, \quad (18)
\]
\[
\psi_1 \cdots \psi_n : D = P_n \oplus D_+ \to P_n \oplus D_+, a + b \mapsto a, a \in P_n, b \in D_+ = \oplus_{\alpha \neq 0} P_n \partial^{[\alpha]}. \quad (19)
\]

For each \( i = 1, \ldots, n \), and each \( k \geq 0 \), the inner derivation \( \text{ad } \partial_i^{[k]} \) of \( D \) preserves the subalgebra \( P_{n,i,k} := K[x_1, \ldots, x_{i-1}, x_i^{p^k}, x_{i+1}, \ldots, x_n] \) (i.e. \([\partial_i^{[k]} : P_{n,i,k}] \subseteq P_{n,i,k} \) and \( P_{n,i,k}^{\text{ad } \partial_i^{[k]}} = P_{n,i,k} \)).
Theorem 7.1  For any $D$ operators projections $\phi$ The maps

$$\phi_{n+i} := \cdots \phi_{n+i,k} \cdots \phi_{n+i,1} \phi_{n+i,0} : P_n \to P_n, \quad \phi_{n+i,k} := \sum_{j=0}^{p-1} (-1)^j \frac{x^{p-j}}{j!} (\text{ad} \partial_i)^j, \quad k \geq 0,$$

is a projection onto the subalgebra $P_{n,i} := K[x_1, \ldots, x_i, \ldots, x_n]$ of the polynomial algebra $P_n$.

$$\phi_{n+i} : P_n = P_{n,i} \oplus P_n x_i \to P_n = P_{n,i} \oplus P_n x_i, \quad a + bx_i \mapsto a, \text{ where } a \in P_{n,i}, \quad b \in P_n.$$

The maps $\phi_{n+i}$ commute and their product

$$\phi_{n+1} \cdots \phi_{2n} : P_n = K \oplus P_{n,+} \to P_n = K \oplus P_{n,+}, \quad \sum_{\alpha \in \mathbb{N}^n} \lambda_\alpha x^\alpha \to \lambda_0, \quad (\lambda_\alpha \in K),$$

is a $K$-algebra homomorphism which is a projection onto the field $K$ where $P_{n,+} := \sum_{i=1}^n P_n x_i$. The maps $\phi_{n+i}$ are well-defined also as maps from the algebra $D$ to itself as it follows from the decomposition $D = \oplus_{\alpha,\beta \in \mathbb{N}^n} K x^\alpha \partial[\beta]$ and $[\Pi]$. Note that the maps $\phi_i, \phi_j (1 \leq i, j \leq 2n)$ commute unless $|i - j| = n$. The following maps (where $a := \sum \lambda_{\alpha,\beta} x^\alpha \partial[\beta] \in D$, $\lambda_{\alpha,\beta}, \lambda'_{\alpha,\beta} \in K$):

$$\phi := \phi_{2n} \cdots \phi_{n+1} \phi_n \cdots \phi_1 : D = K \oplus D_+ \to K \oplus D_+, \quad a \mapsto \lambda_{0,0}, \quad D_+ := \bigoplus_{\alpha+\beta \neq 0} K \partial[\beta] x^\alpha,$$

$$\psi := \phi_{2n} \cdots \phi_{n+1} \psi_1 \cdots \psi_1 : D = K \oplus D_+ \to K \oplus D_+, \quad a' \mapsto \lambda'_{0,0}, \quad D_+ := \bigoplus_{\alpha+\beta \neq 0} K x^\alpha \partial[\beta],$$

are projections onto $K$.

The next result is a kind of a noncommutative Taylor formula for the ring of differential operators $D(P_n)$ on $P_n$ in prime characteristic.

**Theorem 7.1** For any $a \in D(P_n)$,

$$a = \sum_{\alpha,\beta \in \mathbb{N}^n} (-1)^{|\alpha|} \phi(\delta^\beta(a) \partial[\alpha]) \partial[\beta] x^\alpha = \sum_{\alpha,\beta \in \mathbb{N}^n} \psi(\partial[\alpha] \delta^\beta(a)) x^\alpha \partial[\beta],$$

where $\delta^\beta := \prod_{i=1}^n (-\text{ad} x_i)^{\beta_i}$ and $|\alpha| := \alpha_1 + \cdots + \alpha_n$.

**Remark.** The element $\delta^\beta(a) \partial[\alpha] = \delta^\beta(a) \partial[\alpha]$ (resp. $\partial[\alpha] \delta^\beta(a) = \partial[\alpha] \delta^\beta(a)$) is the product in $D(P_n)$ of the elements $\delta^\beta(a)$ and $\partial[\alpha]$ (resp. $\partial[\alpha]$ and $\delta^\beta(a)$).

**Proof.** If $a = \sum \lambda_{\beta,\alpha} \partial[\beta] x^\alpha = \sum \lambda'_{\alpha,\beta} x^\alpha \partial[\beta]$ where $\lambda_{\beta,\alpha}, \lambda'_{\alpha,\beta} \in K$, then we must prove that $(-1)^{|\alpha|} \phi(\delta^\beta(a) \partial[\alpha]) = \lambda_{\beta,\alpha}$ and $\psi(\partial[\alpha] \delta^\beta(a)) = \lambda'_{\alpha,\beta}$. Since $D(P_n) = D(K[x_1]) \otimes \cdots \otimes$
Let \( \partial_1^{[i]}(a) \partial_1^{[i]} \) be a \( \sigma \)-automorphism of the algebra \( \mathcal{D} \). Then the elements \( x_i' := \sigma(x_i), \partial_i^{[k]} := \sigma(\partial_i^{[k]}), i = 1, \ldots, n, k \geq 1 \), are another canonical generators for the algebra \( \mathcal{D} \) (that satisfy the defining relations (4)). Let \( \phi' \) and \( \psi' := \psi \) be the maps as in (21) and (22) but for the new canonical generators \( x_i', \partial_i^{[k]} \) (one has to put ('') everywhere in the formulae).

**Theorem 7.2** (The inversion formula for \( \sigma \in \text{Aut}_K(\mathcal{D}(P_n)) \)) For each \( \sigma \in \text{Aut}_K(\mathcal{D}(P_n)) \) and \( a \in \mathcal{D}(P_n) \),

\[
\sigma^{-1}(a) = \sum_{\alpha, \beta \in \mathbb{N}^n} (-1)^{|\alpha|} \phi(\sigma^{[\beta]}(a)\partial^{[\alpha]}\partial^{[\beta]}x^\alpha) = \sum_{\alpha, \beta \in \mathbb{N}^n} \psi(\sigma^{[\alpha]}\partial^{[\beta]}(a))x^\alpha\partial^{[\beta]},
\]

where \( \delta^{\beta} := \prod_{i=1}^n (-\text{ad } x_i')^{\beta_i} \).

**Proof.** By Theorem 7.1

\[
a = \sum_{\alpha, \beta \in \mathbb{N}^n} (-1)^{|\alpha|} \phi(\sigma^{[\beta]}(a)\partial^{[\alpha]}\partial^{[\beta]}x^\alpha) = \sum_{\alpha, \beta \in \mathbb{N}^n} \psi(\sigma^{[\alpha]}\partial^{[\beta]}(a))x^\alpha\partial^{[\beta]}.
\]

Now, applying \( \sigma^{-1} \) to these equalities we have the result. \( \square \)

## 8 The inversion formula for \( \sigma \in \text{Aut}_K(\mathcal{D}(P_n) \otimes P_m) \)

In this section, \( K \) is a field of characteristic \( p > 0 \) and \( \mathcal{D} := \mathcal{D}(P_n) \) is the ring of differential operators on the polynomial algebra \( P_n := K[x_1, \ldots, x_n] \) over the field \( K \),
The inversion formula (Theorem 8.2) for \( \sigma \in \text{Aut}_K(A) \) is a consequence of the results of the previous two sections, very briefly, the main reason for that is that the algebra \( \mathcal{D}(P_n) \) is central, i.e., the centre of the algebra \( \mathcal{D}(P_n) \) is \( K \). Therefore, the centre of the algebra \( A \) is \( P_m \), and so \( \sigma(P_m) = P_m \). Note that \( A = \bigoplus_{\alpha,\beta \in \mathbb{N}^m, \gamma \in \mathbb{N}^m} K \partial^{[\beta]}x^\alpha y^\gamma = \bigoplus_{\alpha,\beta,\gamma \in \mathbb{N}^m} K \partial^{[\beta]} x^\alpha y^\gamma \).

The maps \( \phi \) and \( \psi \) from (21) and (22) respectively make sense for the algebra \( A \) by simply extending ‘scalars’ in the natural way (\( \mathcal{D}_{Q_m}(P_m \otimes Q_m) \simeq \mathcal{D}_K(P_n) \otimes Q_m \) where \( Q_m := K(y_1, \ldots, y_m) \) is the field of rational functions). Let us denote them by \( \phi_D \) and \( \psi_D \) respectively. The maps

\[
\phi_D : A = P_m \oplus A_+ \to P_m \oplus A_+, \quad a \mapsto \lambda_0, \quad A_+ := \bigoplus_{|\alpha|+|\beta|\neq 0} P_m \partial^{[\beta]} x^\alpha,
\]

\[
\psi_D : A = P_m \oplus A_+ \to P_m \oplus A_+, \quad a' \mapsto \lambda_0', \quad A_+ := \bigoplus_{|\alpha|+|\beta|\neq 0} P_m x^\alpha \partial^{[\beta]},
\]

are projections onto the centre \( P_m \) of \( A \) where

\[
a = \sum \lambda_{\beta \alpha} \partial^{[\beta]} x^\alpha, \quad a' = \sum \lambda'_{\alpha \beta} x^\alpha \partial^{[\beta]},
\]

\( \lambda_{\beta \alpha}, \lambda'_{\alpha \beta} \in P_m \). Let \( \phi_{P_m} : P_m \to P_m \) be the map (12).

Now, the next theorem is a direct consequence of Theorem 6.1 and Theorem 7.1.

**Theorem 8.1** For any \( a \in A := \mathcal{D}(P_n) \otimes P_m \),

\[
a = \sum_{\alpha,\beta \in \mathbb{N}^m, \gamma \in \mathbb{N}^m} (-1)^{|\alpha|} \phi_{P_m}(\partial^{[\gamma]}(\phi_D(\delta^{[\beta]}(a)\partial^{[\alpha]})))\partial^{[\beta]} x^\alpha y^\gamma
\]

\[
= \sum_{\alpha,\beta,\gamma \in \mathbb{N}^m} \phi_{P_m}(\partial^{[\gamma]}(\psi_D(\partial^{[\alpha]}\delta^{[\beta]}(a))))x^\alpha \partial^{[\beta]} y^\gamma,
\]

where \( \delta^{[\beta]} \) is as in Theorem 7.1 and \( \partial^{[\gamma]} \) is as in Theorem 6.1.

Let \( \sigma \in \text{Aut}_K(A) \). Then its restriction to the centre \( Z(A) = P_m \) of the algebra \( A \) is an automorphism, say \( \tau \in \text{Aut}_K(P_m) \). Let \( \phi_{P_m,\sigma} \) be the map (17) for the restriction \( \tau \). Similarly, let \( \phi_D, \psi_D \) be the maps given by the same formula as in Theorem 7.2 but for the algebra \( A = \mathcal{D} \otimes P_m \) rather than \( \mathcal{D} \), let us stress it again that this means that we extend the ‘scalars’ from \( K \) to \( P_m \) (of course, one can extend the field from \( K \) to the field of rational functions \( Q_m := K(x_1, \ldots, x_m) \) and repeat all the arguments from Section 7 over this bigger field). One should stress that the formulae for \( \phi_D, \psi_D \) are the same, the only new thing is that ‘scalars’ of all the elements and of the inner derivations involved are in \( P_m \), rather than \( K \).

**Theorem 8.2** (The inversion formula for \( \sigma \in \text{Aut}_K(\mathcal{D}(P_n) \otimes P_m) \)) For each \( \sigma \in \text{Aut}_K(\mathcal{D}(P_n) \otimes P_m) \) and \( a \in \mathcal{D}(P_n) \otimes P_m \),

\[
\sigma^{-1}(a) = \sum_{\alpha,\beta \in \mathbb{N}^m, \gamma \in \mathbb{N}^m} (-1)^{|\alpha|} \phi_{P_m,\sigma}(\partial^{[\gamma]}(\phi_D(\delta^{[\beta]}(a)\partial^{[\alpha]})))\partial^{[\beta]} x^\alpha y^\gamma
\]

\[
= \sum_{\alpha,\beta \in \mathbb{N}^m, \gamma \in \mathbb{N}^m} \phi_{P_m,\sigma}(\partial^{[\gamma]}(\psi_D(\partial^{[\alpha]}\delta^{[\beta]}(a))))x^\alpha \partial^{[\beta]} y^\gamma.
\]
Applying \( \sigma \) a formula for an automorphism of the algebra \( D \) operators nominial and the ring of differential operator's case - simply because the ring of differential \( \sigma \) the inversion formula for an automorphism of the manner as in this case.

Let \( \phi_{\gamma}(\phi_{\alpha}(a) \partial^{[\alpha]}(a))) \partial^{[\beta]}x^{\alpha}y^{\gamma} \)

Applying \( \sigma^{-1} \) yields the result. \( \square \)

Remark. It makes sense to stress that basically the process of finding the inversion formula for an automorphism of the algebra \( A := D \otimes P_m \) collapses to two cases - the polynomial and the ring of differential operator's case - simply because the ring of differential operators \( D(P_n) \) is a central algebra. We will see later that for the Weyl algebra \( A_n \) this is not the case - the centre of \( A_n \) is big, it is a polynomial algebra in \( 2n \) variables, and the problem of finding the inversion formula for \( \sigma \in Aut_K(A_n \otimes P_m) \) can not be reduced to the cases : \( \sigma \in Aut_K(A_n) \) and \( \sigma \in Aut_K(P_m) \) (see Section 9) in such a straightforward manner as in this case.

9 The inversion formula for an automorphism of the \( n \)'th Weyl algebra \( A_n \) (and of \( A_n \otimes K[y_1, \ldots, y_m] \))

Let \( K \) be a field of characteristic \( p > 0 \). The \( n \)'th Weyl algebra \( A_n = A_n(K) \) is a \( K \)-algebra generated by \( 2n \) generators \( q_1, \ldots, q_n, p_1, \ldots, p_n \) which are subject to the defining relations:

\[
[p_i, q_j] = \delta_{ij}, \quad [p_i, p_j] = [q_i, q_j] = 0 \quad \text{for all} \quad i, j = 1, \ldots, n,
\]

where \( \delta_{ij} \) is the Kronecker delta, \( [a, b] := ab - ba \). The Weyl algebra \( A_n = \oplus_{\alpha, \beta \in \mathbb{N}^n} Kp^\beta q^\alpha = \oplus_{\alpha, \beta \in \mathbb{N}^n} Kq^\alpha p^\beta \) (where \( p^\beta := p_1^{\beta_1} \cdots p_n^{\beta_n} \) and \( q^\alpha := q_1^{\alpha_1} \cdots q_n^{\alpha_n} \)) is a Noetherian algebra which is a free finitely generated module over its centre \( Z(A_n) = K[x_1, \ldots, x_{2n}] \), the polynomial algebra in \( 2n \) variables

\[
x_1 := q_1^p, \ldots, x_n := q_n^p, x_{n+1} := p_1^p, \ldots, x_{2n} := p_n^p.
\]

Clearly, \( A_n = \oplus_{\alpha, \beta \in \mathbb{N}^n} Z(A_n)p^\beta q^\alpha = \oplus_{\alpha, \beta \in \mathbb{N}^n} Z(A_n)q^\alpha p^\beta \) where \( \mathbb{N} := \{\alpha \in \mathbb{N}^n \mid 0 \leq \alpha_1 < p, \ldots, 0 \leq \alpha_n < p\} \), and \( A_n = K\langle p_1, q_1 \rangle \otimes \cdots \otimes K\langle p_n, q_n \rangle \simeq A_1^{2n} \), the tensor product of \( n \) copies of the first Weyl algebra. Let \( P_m := K[x_{2n+1}, \ldots, x_{2n+m}] \) be a polynomial algebra in \( m \) variables and \( P_0 := K \). Let \( A := A_n \otimes P_m \) be the tensor product of algebras. In particular, \( A = A_n \) if \( m = 0 \). The generators \( q_1, \ldots, q_n, p_1, \ldots, p_n, x_{2n+1}, \ldots, x_{2n+m} \) will be called the canonical generators of the algebra \( A \). Let \( r \) be one of the canonical generators of the Weyl algebra \( A_n \) (i.e. \( r \neq x_{2n+i} \) for all \( i \)) and let \( A_r \) be the subalgebra of \( A \) generated by \( r^p \) and all its canonical generators except \( r \). Clearly, \( (ad r)^p = ad(r^p) = 0 \) since the element \( r^p \) belongs to the centre \( Z \) of the algebra \( A \) (where \( ad r \) is the inner derivation of
the algebra \(A\). By Corollary 2.4 for each \(i = 1, \ldots, n\), the maps
\[
\Phi_i := \sum_{j=0}^{p-1}(-1)^j q_j^i (\text{ad} \, p_i)^j : A = \bigoplus_{j=0}^{p-1} A_{q_i^j} \to A,
\]
\[
\Psi_i := \sum_{j=0}^{p-1}(-1)^j (\text{ad} \, p_i)^j(q_j^i) : A = \bigoplus_{j=0}^{p-1} A_{q_i^j} q_i^j \to A,
\]
are projections onto the subalgebra \(A_{q_i^j}\) of \(A\), and the maps
\[
\Phi_{n+i} := \sum_{j=0}^{p-1} q_j^i (\text{ad} \, q_i)^j : A = \bigoplus_{j=0}^{p-1} A_{q_i^j} q_i^j \to A,
\]
\[
\Psi_{n+i} := \sum_{j=0}^{p-1} (\text{ad} \, q_i)^j(q_j^i) : A = \bigoplus_{j=0}^{p-1} A_{q_i^j} p_i^j \to A,
\]
are projections onto the subalgebra \(A_{q_i^j}\) of \(A\). Then their compositions
\[
\Phi := \Phi_{2n} \Phi_{2n-1} \cdots \Phi_1 : A = \bigoplus_{\alpha, \beta \in N} Z q^\alpha p^\beta \to A,
\]
(23)
\[
\Psi := \Psi_{2n} \Psi_{2n-1} \cdots \Psi_1 : A = \bigoplus_{\alpha, \beta \in N} Z p^\beta q^\alpha \to A,
\]
(24)
are projections onto the centre \(Z := Z(A_n) \otimes P_m\) of the algebra \(A\). The centre \(Z\) is a polynomial algebra \(P_s = K[x_1, \ldots, x_{2n}, x_{2n+1}, \ldots, x_s]\) in \(s := 2n + m\) variables, let \(\phi_Z\) be the map as in (12) in the case of the polynomial algebra \(Z = P_s\). Then the maps
\[
\phi := \phi_Z \Phi : A \to A, \quad a = \sum_{\alpha, \beta \in N, \gamma \in N^s} \lambda_{\alpha \beta \gamma} q^\alpha p^\beta x^\gamma \mapsto \lambda_0, \quad (\lambda_{\alpha \beta \gamma} \in K),
\]
(25)
\[
\psi := \phi_Z \Psi : A \to A, \quad a = \sum_{\alpha, \beta \in N, \gamma \in N^s} \lambda_{\beta \alpha \gamma} p^\beta q^\alpha x^\gamma \mapsto \lambda_0, \quad (\lambda_{\beta \alpha \gamma} \in K),
\]
(26)
are projections onto the field \(K\). The inner derivations \(\text{ad} \, q_1, \ldots, \text{ad} \, q_n, \text{ad} \, p_1, \ldots, \text{ad} \, p_n\) of the algebra \(A\) commute.

**Theorem 9.1** For any \(a \in A\),
\[
a = \sum_{\alpha, \beta \in N, \gamma \in N^s} \phi_Z(\partial[\gamma](\Phi(\delta^{[\alpha \beta]}(a))))q^\alpha p^\beta x^\gamma = \sum_{\alpha, \beta \in N, \gamma \in N^s} \phi_Z(\partial[\gamma](\Psi(\delta^{[\alpha \beta]}(a))))p^\beta q^\alpha x^\gamma,
\]
where \(\partial[\gamma] := \prod_{i=1}^{\gamma_1} \delta^{[\gamma_i]}_i, \delta^{[\alpha \beta]} := (\alpha! \beta!)^{-1} \prod_{i=1}^{\alpha} (\text{ad} \, p_i)^{\alpha_i} \prod_{j=1}^{\beta} (-\text{ad} \, q_j)^{\beta_j}.

**Proof.** If \(a = \sum_{\alpha, \beta \in N, \gamma \in N^s} \lambda_{\alpha \beta \gamma} q^\alpha p^\beta x^\gamma = \sum_{\alpha, \beta \in N, \gamma \in N^s} \lambda'_{\beta \alpha \gamma} p^\beta q^\alpha x^\gamma \in A\) where \(\lambda_{\alpha \beta \gamma}, \lambda'_{\beta \alpha \gamma} \in K\), then \(\phi_Z(\partial[\gamma](\Phi(\delta^{[\alpha \beta]}(a)))) = \lambda_{\alpha \beta \gamma}\) and \(\phi_Z(\partial[\gamma](\Psi(\delta^{[\alpha \beta]}(a)))) = \lambda'_{\beta \alpha \gamma}\). \(\square\)

Let \(\sigma \in \text{Aut}_K(A)\) and let \(q'_i := \sigma(q_i), \quad p'_i := \sigma(p_i), \quad i = 1, \ldots, n, \) and \(x'_j := \sigma(x_j), \quad j = 1, \ldots, s\). Then the elements \(q_1, \ldots, q'_n, p'_1, \ldots, p'_n, x'_{2n+1}, \ldots, x'_s\) are another choice of the canonical generators for the algebra \(A\) and \(x'_1, \ldots, x'_s\) are generators for the polynomial algebra \(Z = P_s\). Let \(\phi_{Z, \sigma}, \Phi_{\sigma}\) and \(\Psi_{\sigma}\) be the maps (12), (23) and (24) for the new choice of the canonical generators: for, we have to put (') everywhere.
Theorem 9.2 (The inversion formula for $\sigma \in \text{Aut}_K(A)$) For any $\sigma \in \text{Aut}_K(A)$ and $a \in A$, 
\[
\sigma^{-1}(a) = \sum_{\alpha,\beta \in N, \gamma \in N^s} \phi_{Z,\sigma}(\vartheta^{[\gamma]}(\Phi_{\sigma}(\delta^{[\alpha,\beta]}(a))))q^{\alpha}p^{\beta}x^{\gamma}
\]
\[
= \sum_{\alpha,\beta \in N, \gamma \in N^s} \phi_{Z,\sigma}(\vartheta^{[\gamma]}(\Psi_{\sigma}(\delta^{[\alpha,\beta]}(a))))p^{\beta}q^{\alpha}x^{\gamma},
\]
where $\vartheta^{[\gamma]} := \prod_{i=1}^{2n} \vartheta^{[\gamma]}_i$, $\delta^{[\alpha,\beta]} := (\alpha!\beta!)^{-1}\prod_{i=1}^{n}(\operatorname{ad} p'_i)^{\alpha_i}\prod_{j=1}^{n}(-\operatorname{ad} q'_j)^{\beta_j}$.

Proof. By Theorem 9.1
\[
a = \sum_{\alpha,\beta \in N, \gamma \in N^s} \phi_{Z,\sigma}(\vartheta^{[\gamma]}(\Phi_{\sigma}(\delta^{[\alpha,\beta]}(a))))q^{\alpha}p^{\beta}x^{\gamma}
\]
\[
= \sum_{\alpha,\beta \in N, \gamma \in N^s} \phi_{Z,\sigma}(\vartheta^{[\gamma]}(\Psi_{\sigma}(\delta^{[\alpha,\beta]}(a))))p^{\beta}q^{\alpha}x^{\gamma},
\]
then applying $\sigma^{-1}$ proves the result. $\square$

10 The inversion formula for $\sigma \in \text{Aut}_{K,c}(K[[x_1, \ldots, x_n]]$

The notations of Section 6 remain fixed. Let $\widehat{P}_n$ be a series algebra in $n$ variables $x_1, \ldots, x_n$ over a field $K$ of characteristic $p > 0$. The algebra $\widehat{P}_n$ is a completion of the polynomial algebra $P_n := K[x_1, \ldots, x_n]$ with respect to the $m$-adic topology given by the powers of the maximal ideal $m := (x_1, \ldots, x_n)$ of the algebra $P_n$. So, the algebra $\widehat{P}_n$ is a complete local algebra with maximal ideal $\widehat{m} := \widehat{P}_n m$.

The higher derivations $\vartheta^{[j]}_i \in \mathcal{D}(P_n)$ of the polynomial algebra $P_n$ are continuous maps in the $m$-adic topology, $\vartheta^{[j]}_i(m^k) \subseteq m^{k-j}$ for all $k \geq 0$ where $m^{-l} := P_n$ for $l \geq 0$. Hence, so are the maps $\phi_{i,k}$ from (11). It follows from the definition of the maps $\phi_i$ (see (11)) that they are well-defined and continuous in the $m$-adic topology, hence so is the map $\phi$ from (12) as their finite product. We denote by the same symbols $\phi_{i,k}, \phi_i, \phi, \vartheta^{[j]}_i, \vartheta^{[\alpha]}_i$, etc, the unique extensions of these continuous maps to the (necessarily continuous) maps from $\widehat{P}_n$ to itself (note that $\phi_{i,k} \cdots \phi_{i,0}(\sum_{j=0}^{p^{k+1}-1} a_j x_j^i) = a_0$ where $a_j \in K[[x_1, \ldots, x_{i-1}, x_i^{p^{k+1}}, x_{i+1}, \ldots, x_n]])$.

The maps $\phi_i$ commute, and the map
\[
\phi : \widehat{P}_n = K \oplus \widehat{m} \to K \oplus \widehat{m}, \quad \sum_{\alpha \in N^n} \lambda_\alpha x^{\alpha} \mapsto \lambda_0, \quad (\lambda_\alpha \in K) \tag{27}
\]
is projection onto $K$.

Theorem 10.1 For any $a \in \widehat{P}_n$, 
\[
a = \sum_{\alpha \in N^n} \phi(\vartheta^{[\alpha]}(a))x^{\alpha}.
\]
Proof. If \( \alpha = \sum \lambda_\alpha x^\alpha, \lambda_\alpha \in K \), then \( \phi(\partial^{[\alpha]}(a)) = \lambda_\alpha. \) □

Let \( \sigma : \hat{P}_n \to \hat{P}_n \) be a continuous \( K \)-algebra endomorphism such that \( \sigma(\hat{m}) \subseteq \hat{m} \) (this is, in fact, a part of the definition of continuous endomorphism) and such that its Jacobian \( \Delta := \det(\frac{\partial x_i}{\partial x_j}) \) is a unit of the algebra \( \hat{P}_n \) where \( x'_1 := \sigma(x_1), \ldots, x'_n := \sigma(x_n). \) Then obviously \( \sigma \in \text{Aut}_{K,c}(\hat{P}_n) \) where \( \text{Aut}_{K,c}(\hat{P}_n) \) is the group of continuous automorphisms of the algebra \( \hat{P}_n \) (if \( \tau \in \text{Aut}_{K,c}(\hat{P}_n) \) then \( \tau(\hat{m}) \subseteq \hat{m} \), by definition).

Example. Let \( \sigma \) be a \( K \)-algebra endomorphism of the polynomial algebra \( P_n \) such that \( \sigma(\hat{m}) \subseteq \hat{m} \) and its Jacobian \( \Delta \in K^\times \). Then the \( \sigma \) can be extended uniquely to a continuous \( K \)-automorphism of the algebra \( \hat{P}_n \).

Given a continuous automorphism of the algebra \( \hat{P}_n \) then its Jacobian is automatically a unit of the algebra \( \hat{P}_n \), as follows immediately from the chain rule.

So, let \( \sigma \in \text{Aut}_{K,c}(\hat{P}_n) \). Let us define continuous maps \( \partial'_j, \partial'^{[\alpha]}_j, \phi'_i, \phi_{\sigma} : \hat{P}_n \to \hat{P}_n \) in the same way as in [13], [16], and [17] respectively.

**Theorem 10.2** (The inversion formula for \( \sigma \in \text{Aut}_{K,c}(\hat{P}_n) \)) For any \( \sigma \in \text{Aut}_{K,c}(\hat{P}_n) \) and \( a \in \hat{P}_n, \)

\[ \sigma^{-1}(a) = \sum_{\alpha \in \mathbb{N}^n} \phi_{\sigma}(\partial'^{[\alpha]}(a))x^\alpha. \]

Proof. By Theorem [10.1] \( a = \sum_{\alpha \in \mathbb{N}^n} \phi_{\sigma}(\partial'^{[\alpha]}(a))x^\alpha \), then applying \( \sigma^{-1} \) we have the result. □

11 The inversion formula for \( \sigma \in \text{Aut}_K(T_{k_1,\ldots,k_n} \otimes P_m) \)

Let \( K \) be a field of characteristic \( p > 0 \), \( \mathcal{D} := \mathcal{D}(P_n) \) be the ring of differential operators on the polynomial algebra \( P_n := K[x_1,\ldots,x_n] \). For each \( \mathbf{k} = (k_1,\ldots,k_n) \in \mathbb{N}^n \), consider the \( K \)-subalgebra \( T_\mathbf{k} := T_{k_1,\ldots,k_n} \) of \( \mathcal{D} \) generated by the polynomial algebra \( P_n \) and the elements \( \partial^{[p_j]}_i, i = 1,\ldots,n, j_1 < k_1,\ldots,j_n < k_n \). The algebras \( T_\mathbf{k} \) play an important role in studying the ring \( \mathcal{D} \) and its modules. Then

\[ T_\mathbf{k} = \bigoplus_{\alpha \in \mathcal{N}} P_n \partial^{[\alpha]} = \bigoplus_{\alpha \in \mathcal{N}} \partial^{[\alpha]} P_n, \]

where \( \mathcal{N} := \{ \alpha = (\alpha_i) \in \mathbb{N}^n \mid \alpha_1 < p^{j_1},\ldots,\alpha_n < p^{j_n} \} \). One can easily verify that the algebra \( T_{1,\ldots,1} \) is canonically isomorphic to the factor algebra \( A_n/(p^{j_1},\ldots,p^{j_n}) \) of the Weyl algebra \( A_n \). Note that if \( k_i = 0 \) then there is no element \( \partial^{[p_j]}_i \) with \( j < 0 \). In order to accommodate this border case, we will assume that \( k_1 \geq 1,\ldots,k_n \geq 1 \) but instead of the algebra \( T_\mathbf{k} \) we consider the algebra

\[ T := T_\mathbf{k} \otimes P_m, \quad P_m := K[x_{n+1},\ldots,x_{n+m}], \quad s := n + m, \]
(it is obvious that for a general \( k \), the algebra \( T_k \) is of this type). Let \( P_s := K[x_1, \ldots, x_s] \).

Then the algebra

\[
T = \bigoplus_{\alpha \in \mathbb{N}} P_\alpha \partial^{[\alpha]} = \bigoplus_{\alpha \in \mathbb{N}} \partial^{[\alpha]} P_s
\]

is a left and right free finitely generated \( P_s \)-module of rank \( p^{k_1 + \cdots + k_n} \) with the centre \( Z := Z(T) = K[x_1^{p_{k_1}}, \ldots, x_n^{p_{k_n}}, x_{n+1}, \ldots, x_s] \), a polynomial algebra in \( s \) indeterminates. The algebra

\[
T = \bigoplus_{\alpha, \beta \in \mathbb{N}} Z x^\alpha \partial^{[\beta]} = \bigoplus_{\alpha, \beta \in \mathbb{N}} Z \partial^{[\beta]} x^\alpha
\]

is a free finitely generated \( Z \)-module of rank \( p^{2(k_1 + \cdots + k_n)} \). Let us consider the set \( \delta := \{ \delta_i := -\operatorname{ad} x_i, i = 1, \ldots, n \} \) of commuting nilpotent \( K \)-derivations of the algebra \( T \), \( \delta^{[p^{k_i}]} = (-1)^{p_{k_1}} \operatorname{ad} x_i^{p_{k_i}} = 0 \) since \( x_i^{p_{k_i}} \in Z \). The set \( \partial^{[j]} := \{ \partial^{[j]}_i, 0 \leq j < p^{k_i} \} \) is the iterative \( \delta_i \)-descent of maximal length. Clearly, \( T^{\delta} = P_s \). The set \( \delta \) satisfies the conditions of Theorem 3.1 let

\[
\prod_{i=1}^n \phi_i : T \to T, \quad \phi_i := \sum_{k=0}^{p_i-1} \partial^{[k]}_i (\operatorname{ad} x_i)^k,
\]

\[
\prod_{i=1}^n \psi_i : T \to T, \quad \psi_i := \sum_{k=0}^{p_i-1} (\operatorname{ad} x_i)^k \partial^{[k]}_i,
\]

be the corresponding maps from (3), these are the projections onto \( P_s \) as in Theorem 3.1 (3).

For each \( i = 1, \ldots, n \) and each \( k = 0, \ldots, k_i - 1 \), the inner derivation \( \operatorname{ad} \partial^{[p^{k_i}]}_i \) of the algebra \( T \) is a nilpotent derivations: \( (\operatorname{ad} \partial^{[p^{k_i}]}_i)^p = (\operatorname{ad} \partial^{[p^{k_i}]}_i)^p = \operatorname{ad} 0 = 0 \), and the subalgebra of \( P_s \),

\[
P_{s,i,k} := K[x_1, \ldots, x_{i-1}, x_i^{p_i}, x_{i+1}, \ldots, x_n] \otimes P_m
\]

is \( \operatorname{ad} \partial^{[p^{k_i}]}_i \)-invariant, the kernel of the derivation \( \operatorname{ad} \partial^{[p^{k_i}]}_i \) in \( P_{s,i,k} \) is equal to \( P_{s,i,k+1} \). Note that \( \{ x_i^{p_j}, 0 \leq j < p \} \subseteq P_{s,i,k} \) is the iterative \( \operatorname{ad} \partial^{[p^{k_i}]}_i \)-succession of maximal length \( p \). Applying repeatedly Corollary 2.4, we have the projection

\[
\phi_{n+i} := \phi_{n+i,k_i-1} \cdots \phi_{n+i,1} \phi_{n+i,0} : P_s \to P_s, \quad \phi_{n+i,k} := \sum_{j=0}^{p-1} (-1)^j \frac{x_i^{p_j}}{j!} (\operatorname{ad} \partial^{[p^{k_i}]}_i)^j,
\]

onto the subalgebra \( P_{s,i,k_i} \) of \( P_s = \bigoplus_{j=0}^{p-1} P_{s,i,k_i} x_i^j \). The maps \( \phi_{n+1}, \ldots, \phi_{2n} \) commute and their product

\[
\prod_{i=1}^n \phi_{n+i} : P_s \to P_s
\]

is the projection onto \( Z \) where \( P_s = \bigoplus_{\alpha \in \mathbb{N}} Z x^\alpha \).
In order to avoid a clash of notations, let
\[ X_1 := x_1^{p_1}, \ldots, X_n := x_n^{p_n}, X_{n+1} = x_{n+1}, \ldots, X_s = x_s. \]

Then \( Z = K[X_1, \ldots, X_s] \) is a polynomial algebra in \( s \) variables. For each variable \( X_\nu \), let \( \{\partial^{[j]}_\nu, j \geq 0\} \subseteq D(Z) \subseteq \text{End}_K(Z) \) be the corresponding higher derivations (with respect to the variable \( X_\nu \)).

Let \( \phi_Z \) be the map from (12) in the case of the polynomial algebra \( Z = K[X_1, \ldots, X_s] \). Then the maps
\[
\phi_Z \phi_s \cdots \phi_{n+1} \phi_n \cdots \phi_1 : T \to T, \quad a = \sum_{\alpha, \beta \in N, \gamma \in N^s} \lambda_{\beta \alpha \gamma} \partial^{[\beta]} \gamma x^\alpha X^\gamma \mapsto \lambda_0, \quad (\lambda_{\beta \alpha \gamma} \in K),
\]
\[
(29)
\]
\[
\phi_Z \phi_s \cdots \phi_{n+1} \psi_n \cdots \psi_1 : T \to T, \quad a = \sum_{\alpha, \beta \in N, \gamma \in N^s} \lambda_{\alpha \beta \gamma} x^\alpha \partial^{[\beta]} \gamma X^\gamma \mapsto \lambda_0, \quad (\lambda_{\alpha \beta \gamma} \in K),
\]
\[
(30)
\]
are projections onto the field \( K \),
\[
T = \bigoplus_{\alpha, \beta \in N, \gamma \in N^s} Kx^\alpha \partial^{[\beta]} \gamma X^\gamma = \bigoplus_{\alpha, \beta \in N, \gamma \in N^s} K\partial^{[\beta]} \gamma x^\alpha X^\gamma.
\]

The maps
\[
\phi := \phi_s \cdots \phi_{n+1} \phi_n \cdots \phi_1 : T \to T, \quad a = \sum_{\alpha, \beta \in N} a_{\beta \alpha} \partial^{[\beta]} \alpha x^\alpha \mapsto a_0, \quad (a_{\beta \alpha} \in Z),
\]
\[
(31)
\]
\[
\psi := \phi_s \cdots \phi_{n+1} \psi_n \cdots \psi_1 : T \to T, \quad a = \sum_{\alpha, \beta \in N} a_{\alpha \beta} x^\alpha \partial^{[\beta]} \mapsto a_0, \quad (a_{\alpha \beta} \in Z),
\]
\[
(32)
\]
are projections onto the centre \( Z \) (see (28)).

**Theorem 11.1** For any \( a \in T \),
\[
a = \sum_{\alpha, \beta \in N, \gamma \in N^s} (-1)^{|\alpha|} \phi_Z (\partial^{[\gamma]} (\phi (\delta^{[\beta]} (a) \partial^{[\alpha]} ))) \partial^{[\beta]} \gamma x^\alpha X^\gamma
\]
\[
= \sum_{\alpha, \beta \in N, \gamma \in N^s} \phi_Z (\partial^{[\gamma]} (\psi (\partial^{[\alpha]} \delta^{[\beta]} (a) ))) x^\alpha \partial^{[\beta]} \gamma X^\gamma,
\]
where \( \delta^{[\beta]} := \prod_{j=1}^n (-\text{ad} x_j)^{\beta_j}. \)

**Proof.** If \( a = \sum_{\alpha, \beta \in N} \lambda_{\beta \alpha} \partial^{[\beta]} \alpha x^\alpha = \sum_{\alpha, \beta \in N} \lambda'_{\alpha \beta} x^\alpha \partial^{[\beta]} \in T \) where \( \lambda_{\beta \alpha}, \lambda'_{\alpha \beta} \in Z \), then it suffices to prove that \( (-1)^{|\alpha|} \phi (\delta^{[\beta]} (a) \partial^{[\alpha]} ) = \lambda_{\beta \alpha} \) and \( \psi (\partial^{[\alpha]} \delta^{[\beta]} (a) ) = \lambda'_{\alpha \beta} \) since, for any \( z = \sum_{\gamma \in N^s} z_\gamma X^\gamma, \ z_\gamma \in K \), we have \( \sum_{\gamma \in N^s} \phi_Z (\partial^{[\gamma]} (z) ) = z_0 \). It suffices to prove the
statements for \( n = 1 \) (for an arbitrary \( n \), repeat the arguments below \( n \) times). So, let \( n = 1 \). Recall that \( \partial_1^{[0]} := 1 \) and \( \partial_1^{[s]} := 0 \) for all negative integers \( s \).

\[
(-1)^i \phi(\partial_1^{[i]}(a)\partial_1^{[j]}) = (-1)^i \phi((-\text{ad} \ x_1)^i(\sum_{\alpha, \beta \in N} \lambda_{\alpha} \partial_1^{[\beta-a]} x_1^{\alpha} \partial_1^{[j]})) = (-1)^i \phi(\sum_{\alpha \in N} \lambda_{\alpha} \partial_1^{[\beta-a+j]} x_1^{\alpha}) = (-1)^i(-1)^i \lambda_{ji} = \lambda_{ji},
\]

\[
\psi(\partial_1^{[i]}(a)) = \psi((-\text{ad} \ x_1)^i(a)) = \psi(\partial_1^{[i]}((-\text{ad} \ x_1)^i(\sum_{\alpha, \beta \in N} x_{\alpha, \beta} \partial_1^{[\beta-a+j]}))) = \psi(\sum_{\alpha \in N} x_{\alpha, \beta} \partial_1^{[\beta-a+j]} x_1^{\alpha}) = \psi(\sum_{\alpha \in N} x_{\alpha, \beta} \partial_1^{[\beta-a+j]} x_1^{\alpha}),
\]

Let \( \sigma \in \text{Aut}_K(T) \) and let \( x_1' := \sigma(x_1) \), \( \partial_1'^{[\mu]} := \sigma(\partial_1^{[\mu]}) \), \( i = 1, \ldots, n \), \( 0 \leq s_i < k_i \), and \( X_j' := \sigma(X_j) \), \( j = 1, \ldots, s \). Then \( Z = K'[X_1', \ldots, X_s'] \). Let \( \phi_{Z, \sigma}, \phi_{\sigma} \) and \( \psi_{\sigma} \) be the maps \([12], [31] \) and \([32] \) for the new choice of the canonical generators: for, we have to put '(' everywhere.

**Theorem 11.2** (The inversion formula for \( \sigma \in \text{Aut}_K(T) \)) For any \( \sigma \in \text{Aut}_K(T) \) and \( a \in T \),

\[
\sigma^{-1}(a) = \sum_{\alpha, \beta \in N, \gamma \in N} (-1)^{[\alpha]} \phi_{Z, \sigma}(\partial_1^{[\gamma]}(\phi_{\sigma}(\delta^{[\beta]}(a)\partial_1^{[\alpha]})))\partial_1^{[\beta]} x_1^{\alpha} X_1^{\gamma}
\]

\[
= \sum_{\alpha, \beta \in N, \gamma \in N} \phi_{Z, \sigma}(\partial_1^{[\gamma]}(\psi_{\sigma}(\partial_1^{[\alpha]}(\delta^{[\beta]}(a))))x_1^{\alpha} \partial_1^{[\beta]} X_1^{\gamma}.
\]

where \( \partial_1^{[\gamma]} := \prod_{i=1}^{s} \partial_1^{[\gamma_i]} \) and \( \delta^{[\beta]} := \prod_{j=1}^{n}(-\text{ad} \ x_j')^{\beta_j} \).

**Proof.** By Theorem [11.1]

\[
\sigma^{-1}(a) = \sum_{\alpha, \beta \in N, \gamma \in N} (-1)^{[\alpha]} \phi_{Z, \sigma}(\partial_1^{[\gamma]}(\phi_{\sigma}(\delta^{[\beta]}(a)\partial_1^{[\alpha]})))\partial_1^{[\beta]} x_1^{\alpha} X_1^{\gamma}
\]

\[
= \sum_{\alpha, \beta \in N, \gamma \in N} \phi_{Z, \sigma}(\partial_1^{[\gamma]}(\psi_{\sigma}(\partial_1^{[\alpha]}(\delta^{[\beta]}(a))))x_1^{\alpha} \partial_1^{[\beta]} X_1^{\gamma},
\]

then applying \( \sigma^{-1} \) proves the result. \( \square \)
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