Anomaly-Free Supersymmetric Models in Six Dimensions

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Abstract

The conditions for the cancellation of all gauge, gravitational, and mixed anomalies of $N = 1$ supersymmetric models in six dimensions are reviewed and illustrated by a number of examples. Of particular interest are models that cannot be realized perturbatively in string theory. An example of this type, which we verify satisfies the anomaly cancellation conditions, is the K3 compactification of the $SO(32)$ theory with small instantons recently proposed by Witten. When the instantons coincide it has gauge group $SO(32) \times Sp(24)$. Two new classes of models, for which non-perturbative string constructions are not yet known, are also presented. They have gauge groups $SO(2n+8) \times Sp(n)$ and $SU(n) \times SU(n)$, where $n$ is an arbitrary positive integer.

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1 Introduction

Recent developments have made it clear that the various known “superstring theories” and their compactifications are actually recipes for constructing solutions to a unique underlying theory [1]–[9]. Even though this theory has not yet been properly formulated, the identification of various non-perturbative dualities has led to a much deeper understanding of the big picture. One important program is to characterize the moduli space of vacua as completely as possible. Superstring vacua are most easily understood when they have many unbroken supersymmetries, but a realistic vacuum should have no unbroken supersymmetries at all. A possible viewpoint is that we should work our way towards the study of realistic vacua in small steps starting from ones with a lot of supersymmetry. As the lessons at one stage are learned, we can build on that experience at the next stage with the number of supersymmetries cut in half. For a given number of unbroken supersymmetries, the number of non-compact space-time dimensions is also an issue, since the classification of vacua becomes richer and more subtle as this number is decreased.

The maximum possible number of supersymmetries is 32, corresponding to $N=1$ in eleven dimensions, $N=2$ in ten dimensions, $N=4$ in six dimensions, or $N=8$ in four dimensions. The next case of interest is 16 unbroken supersymmetries, corresponding to $N=1$ in ten dimensions, $N=2$ in six dimensions, or $N=4$ in four dimensions. In all of these cases we have a pretty good grasp of the complete moduli space of superstring vacua. This is not to imply that everything about these theories is understood, just that we can enumerate them and identify their massless spectra in four or more dimensions.

It now seems timely to make a concerted effort to classify possible vacua with 8 unbroken supersymmetries, corresponding to $N=1$ in six dimensions or $N=2$ in four dimensions. This is certainly a challenging problem and will take some time to sort out. For one thing, the pioneering work of Seiberg and Witten [10] has taught us that for $N=2$ models in four dimensions the quantum moduli space is different from the classical one. Another indication that the classification of such theories is a challenging problem is the fact that many, but not all, are given by compactification of a Type II theory on a Calabi–Yau manifold, and the classification of Calabi–Yau manifolds is still far from complete.

A somewhat more modest problem is to classify $N=1$ vacua in six dimensions. This problem should be more tractable for a number of reasons. First, their number should be far fewer. Second, the possibilities are significantly constrained by the requirements
of anomaly cancellation. To appreciate these two points, one should recall the situation when there are 16 supersymmetries. In that case the maximum dimension is ten, and the requirements of anomaly cancellation imply that in ten dimensions there are just two possibilities, corresponding to gauge groups $SO(32)$ or $E_8 \times E_8$. Recall that when this result was obtained [11], Type I superstrings were known, but it was not clear whether there was a theory that realized the $E_8 \times E_8$ group. This led to the discovery of the heterotic string theory shortly thereafter. Ten-dimensional N=1 models are completely specified at low energy by the choice of the gauge group. In the case of N=1 models in six dimensions, on the other hand, a complete characterization of the low energy dynamics also requires specifying the representation of the gauge group to which the massless hypermultiplets belong.

One obvious way to obtain N=1 models in six dimensions is to compactify either of the two N=1 ten-dimensional models on a K3 manifold. Aspects of this analysis have been discussed by a number of authors [12, 13, 14, 15]. The generic result is described in section 2 and the symmetry enhancement that is achieved at a special (Gepner) point in the K3 moduli space is described in section 3. All the models obtained in this way have a gauge group with rank less than or equal to 20. They can be understood within the framework of perturbative heterotic string theory using standard conformal field theory technology.

In recent work, Witten has shown that shown that in the case of the $SO(32)$ theory compactified on K3, small instantons can give rise to non-perturbative symmetry enhancement [16]. The largest gauge group that can be achieved in this way is $SO(32) \times Sp(24)$, which has rank 40. In section 4 we verify that this model satisfies the requirements of anomaly cancellation. This is a very non-trivial check. The anomaly analysis suggests that it is very difficult to break the $SO(32)$ part of the gauge group, and that nothing like this is going to work for $E_8 \times E_8$. After the fact, it is evident that this model could have been discovered by looking for new ways to satisfy the anomaly cancellation requirements. This lesson motivates exploring whether they have other non-trivial solutions. In section 5 we present two new classes of solutions for which the gauge group is $SO(2n+8) \times Sp(n)$ or $SU(n) \times SU(n)$. It is quite surprising that gauge groups of arbitrarily high rank can be consistent. Even though realizations of these solutions in string theory are not yet known, the previous experience with $E_8 \times E_8$ in ten dimensions suggests that it may be worthwhile looking for them.
2 Review of Anomaly Cancellation Conditions

$N = 1$ supersymmetry in six dimensions involves four types of massless multiplets. Classifying massless particles by representations of the little group $O(4) \approx SU(2) \times SU(2)$, and labelling $SU(2)$ representations by their multiplicities $(2J + 1)$, they are

- (i) gravity: $(3, 3) + 2(2, 3) + (1, 3)$
- (ii) tensor: $(3, 1) + 2(2, 1) + (1, 1)$
- (iii) vector: $(2, 2) + 2(1, 2)$
- (iv) hyper: $2(2, 1) + 4(1, 1)$

A general $N = 1$ model has massless content given by $(i) + n_T(ii) + n_V(iii) + n_H(iv)$. With the exception of the final paragraph, we will only consider the case $n_T = 1$, which is what one expects for heterotic string compactifications. The vector multiplets belong to the adjoint representation of the gauge group $G$, and so $n_V = \dim G$. The hypermultiplets belong to some representation $R$ of the group. CPT invariance requires that $R$ is a real representation. From this it follows that $n_H = \dim R$. (However, as explained in [10], if $R$ is pseudoreal, it can be realized by $\frac{1}{2} \dim R$ hypermultiplets.) The knowledge of $G$ and $R$ completely characterizes the low energy dynamics and goes a long way towards characterizing the associated string theory dynamics.

The requirement of cancellation of all gauge and gravitational anomalies is a stringent condition on the possible choices of $G$ and $R$. The anomalies are characterized by a formal 8-form (a characteristic class) made from the curvature and gauge field 2-forms. One requirement is the cancellation of the $\text{tr} R^4$ term, where $R$ is the curvature 2-form. This leads to the requirement

$$n_H = n_V + 244.$$  \hfill (1)

The general condition for cancellation of the remaining anomalies has been given previously [12, 15]. To keep things relatively simple, the result will be given for the special case that $G = \otimes G_\alpha$ is semi-simple, i.e., there are no $U(1)$ factors. Assuming that eq. (1) is satisfied, and normalizing the remaining anomaly 8-form so that the coefficient of $(\text{tr} R^2)^2$ is unity, gives

$$I = (\text{tr} R^2)^2 + \frac{1}{6} \text{tr} R^2 \sum_\alpha X_\alpha^{(2)} - \frac{2}{3} \sum_\alpha X_\alpha^{(4)} + 4 \sum_{\alpha < \beta} Y_{\alpha \beta},$$ \hfill (2)

Models that violate this condition were proposed in [13].
where

\[ X^{(n)}_\alpha = \text{Tr} F^n_\alpha - \sum n_i \text{tr} F^n_i \tag{3} \]

\[ Y_{\alpha\beta} = \sum_{ij} n_{ij} \text{tr} F^2_\alpha \text{tr} F^2_\beta. \tag{4} \]

The notation is as follows: The symbol \( \text{Tr} \) denotes a trace in the adjoint representation and \( \text{tr}_i \) denotes a trace in the representation \( R_i \) (of the simple group \( G_\alpha \)). \( n_i \) is the number of hypermultiplets in the representation \( R_i \) of \( G_\alpha \) and \( n_{ij} \) is the number in the representation \( (R_i, R_j) \) of \( G_\alpha \times G_\beta \). These numbers are usually positive integers, but can be half-integral when pseudo-real representations occur. The cancellation of the remaining gravitational, gauge, and mixed anomalies by the mechanism in [11] requires that the anomaly 8-form should factorize as

\[ I = (\text{tr} R^2 + \sum_\alpha u_\alpha \text{tr} F^2_\alpha)(\text{tr} R^2 + \sum_\alpha v_\alpha \text{tr} F^2_\alpha), \tag{5} \]

where \( u_\alpha \) and \( v_\alpha \) are numerical coefficients and \( \text{tr} F^2_\alpha \) is evaluated in a convenient (“fundamental”) representation of \( G_\alpha \).

Given a consistent model solving the anomaly cancellation conditions, one can trivially obtain other ones by enlarging the gauge group by an arbitrary gauge group \( G' \) and adding hypermultiplets in the adjoint representation of \( G' \). Models with this structure are reducible and will not be considered.

The best way to understand the meaning of eqs. (2–5) is by means of a few examples. The standard examples, which follow, correspond to compactification of the \( SO(32) \) and \( E_8 \times E_8 \) heterotic strings on a generic \( K3 \). In each case, one identifies the spin connection (which belongs to \( SU(2) \) in the case of \( K3 \)) with a suitable \( SU(2) \) subgroup of the gauge group. This means that one chooses a background \( SU(2) \) gauge bundle on the \( K3 \) with instanton number 24. Decomposing \( SO(32) \) as \( SO(28) \times SO(4) \) and identifying the spin connection with one of the two \( SU(2) \)'s in \( SO(4) = SU(2) \times SU(2) \), leaves an unbroken gauge group \( G = G_1 \times G_2 \) with \( G_1 = SO(28) \) and \( G_2 = SU(2) \). Then the multiplicities of various hypermultiplet representations can be read off from standard index-theorem formulas given in [12]. One finds

\[ R = 10(28, 2) + 65(1, 1), \tag{6} \]
where 20 of the singlets come from the gravitational sector and 45 from the matter sector of the ten-dimensional theory. The first term in $R$ is better thought of as 20 copies of $\frac{1}{2}(28, 2)$, since $(28, 2)$ is pseudoreal. Note that altogether $n_V = 378 + 3 = 381$ and $n_H = 560 + 65 = 625$, which satisfies eq. (1).

The construction ensures that the anomaly cancellation conditions are satisfied, but let’s verify them anyway. This requires the identities

$$\text{Tr} F^4 = (n - 8)\text{tr}F^4 + 3(\text{tr}F^2)^2$$  \hspace{1cm} (7)$$

$$\text{Tr} F^2 = (n - 2)\text{tr}F^2$$  \hspace{1cm} (8)$$

for $SO(n)$ and

$$\text{Tr} F^4 = 8(\text{tr}F^2)^2$$  \hspace{1cm} (9)$$

$$\text{tr} F^4 = \frac{1}{2}(\text{tr}F^2)^2$$  \hspace{1cm} (10)$$

$$\text{Tr} F^2 = 4\text{tr}F^2$$  \hspace{1cm} (11)$$

for $SU(2)$. Using these in eqs. (3) and (4), we obtain

$$X_1^{(2)} = 6\text{tr}F_1^2, \quad X_1^{(4)} = 3(\text{tr}F_1^2)^2$$  \hspace{1cm} (12)$$

$$X_2^{(2)} = -276\text{tr}F_2^2, \quad X_2^{(4)} = -132(\text{tr}F_2^2)^2$$  \hspace{1cm} (13)$$

$$Y_{12} = 10\text{tr}F_1^2\text{tr}F_2^2.$$  \hspace{1cm} (14)$$

The important point is that all $\text{tr}F^4$ terms cancel and the remaining expression factorizes as follows

$$I = (\text{tr}R^2 - \text{tr}F_1^2 - 2\text{tr}F_2^2)(\text{tr}R^2 + 2\text{tr}F_1^2 - 44\text{tr}F_2^2).$$  \hspace{1cm} (15)$$

The analysis of the $E_8 \times E_8$ model is quite similar. One can identify the spin connection with the $SU(2)$ factor in the decomposition $E_8 \supset E_7 \times SU(2)$ of one of the two $E_8$’s, leaving
an unbroken gauge group $G = G_1 \times G_2$, where $G_1 = E_8$ and $G_2 = E_7$. The massless hypermultiplets are

$$R = 10(1, 56) + 65(1, 1). \quad (16)$$

Only singlets of $E_8$ occur, which is therefore a “hidden sector” group. As before, $n_V = 381$ and $n_H = 625$. To analyze the anomaly formula, we need the identities

$$\text{Tr} F^4 = \frac{1}{100} (\text{Tr} F^2)^2 \quad (17)$$

for $E_8$ and

$$\text{Tr} F^4 = \frac{1}{6} (\text{tr} F^2)^2 \quad (18)$$

$$\text{Tr} F^2 = 3\text{tr} F^2 \quad (20)$$

for $E_7$. Here tr is evaluated in the $56$. Using these, one obtains a factorized anomaly

$$I = (\text{tr} R^2 - \frac{1}{30} \text{Tr} F_1^2 - \frac{1}{6} \text{tr} F_2^2)(\text{tr} R^2 + \frac{1}{5} \text{Tr} F_1^2 - \text{tr} F_2^2). \quad (21)$$

When one has anomaly-free models, such as the two given above, one can obtain many more by Higgsing. The unique way vector and hypermultiplets can become massive is for one of each to pair up to give a massive vector multiplet. Note that this preserves $n_H - n_V = 244$.

There is a triplet of $D$ terms, quadratic in the hypermultiplet scalar fields, for each generator of $G$. The $4n_H$ scalars can be given arbitrary vevs that maintain the vanishing of all $3n_V D$ terms. Generically, this breaks much of the gauge symmetry giving many more consistent solutions of the anomaly equations. In this way one probes different phases of a connected moduli space of vacua.

Let us consider examples of such Higgsing for the two models we have presented. Starting with $SO(28) \times SU(2)$, we can break it to $G = SO(N)$, with $N \leq 28$. The hypermultiplet representation then consists of $N - 8$ copies of the $N$ representation and the number of singlets required to maintain $n_H - n_V = 244$. For all these models the anomaly factorizes in the form

$$I = (\text{tr} R^2 - \text{tr} F^2)(\text{tr} R^2 + 2\text{tr} F^2). \quad (22)$$
The $E_8 \times E_7$ model can be Higgsed in a similar manner. Before describing that, let us consider the more general problem of $G = E_8 \times G'$, where $G'$ is an arbitrary semi-simple group and all hypermultiplets are $E_8$ singlets. In this case (assuming eq. (1)) one has

$$I = (\text{tr} R^2)^2 + \frac{1}{6} \text{tr} R^2 \text{Tr} F_1^2 - \frac{1}{150} (\text{Tr} F_1^2)^2 + A \text{tr} R^2 + B,$$

where $A$ and $B$ depend on $G'$. This can factorize if and only if

$$B = \frac{6}{49} A^2,$$

in which case we obtain

$$I = (\text{tr} R^2 - \frac{1}{30} \text{Tr} F_1^2 + \frac{1}{7} A)(\text{tr} R^2 + \frac{1}{5} \text{Tr} F_1^2 + \frac{6}{7} A).$$

Equation (21) is of this form.

Now consider Higgsing the $E_8 \times E_7$ model to obtain $E_8 \times E_6$. The $E_6$ representations that describe the hypermultiplets turn out to be $9 \cdot 27 + 9 \cdot 27 + 84 \cdot 1$. One finds that $A = -\frac{7}{3} \text{tr} F_2^2$ and $B = \frac{2}{3} (\text{tr} F_2^2)^2$, so that eq. (24) is satisfied. Further Higgsing to $E_8 \times SO(10)$ gives hypermultiplets $18 \cdot 10 + 8 \cdot 16 + 8 \cdot 16 + 101 \cdot 1$. This time $A = -7 \text{tr} F_2^2$ and $B = 6 (\text{tr} F_2^2)^2$, which also satisfies eq. (24). This sequence of models corresponds to the ones considered in [5].

### 3 Symmetry Enhancement

In the preceding example we gave “standard” K3 compactification models and showed how ones with less symmetry can be reached by Higgsing. One can also find models with more gauge symmetry. These correspond to special subclasses of K3’s (or orbifolds) that give symmetry enhancement – the so-called Gepner points. One could derive these from first principles or (more simply) guess the result by playing around with the anomaly formulas. I will choose the latter route. It will suffice to find the example with maximal symmetry enhancement, since any others, including those in the preceding section, can then be obtained by Higgsing.

I claim that the maximal symmetry that be obtained in this way is

$$E_8 \times E_7 \times [SU(2)]^5 \text{ or } SO(28) \times [SU(2)]^6.$$  

Both of these models have rank 20, which is the most that can be accommodated in a perturbative conformal field theory treatment of the heterotic string (for $N = 1$ and $D = 6$).
The possibility of a $[U(1)]^5$ symmetry enhancement appears in the literature, but (as far as I am aware) $[SU(2)]^5$ is new. The $E_8 \times E_7 \times [SU(2)]^5$ example, which has $n_V = 396$ and hence requires $n_H = 640$, works as follows. The hypermultiplet representation (suppressing the $E_8$ label, which is always a singlet) is

$$R = [(56,2,1,1,1) + (1;2,2,2,1)] + 4 \text{ perms.}$$

(27)

Note that this gives $n_H = 640$ without the need for any singlets. This is what one might expect for a maximally symmetric Gepner point. The quantities $A$ and $B$ in eq. (23) in this case turn out to be

$$A = \frac{1}{6} \left( -7\text{tr}F_1^2 - 84 \sum_{i=1}^{5} \text{tr}F_{2i}^2 \right)$$

$$B = -\frac{2}{3} \left( -\frac{1}{4}(\text{tr}F_1^2)^2 - 36 \sum_{i=1}^{5}(\text{tr}F_{2i}^2)^2 \right)$$

$$+ \ 4 \left( \text{tr}F_1^2 \sum_{i=1}^{5} \text{tr}F_{2i}^2 + 12 \sum_{i<j} \text{tr}F_{2i}^2 \text{tr}F_{2j}^2 \right).$$

(28)

These satisfy $B = \frac{6}{3}A^2$ as required.

It appears that all $N = 1$ $D = 6$ models that can be understood within the framework of perturbative heterotic string theory can be obtained by suitable Higgsing of these two models. So at this point we have two disconnected components for the $D = 6$ slice of the moduli space of vacua with eight unbroken supersymmetries (i.e., $N = 1$). But, as we will see, there is more.

4 Small Instantons

In a recent paper, Witten showed that in the case of the $SO(32)$ theory there are non-perturbative possibilities for symmetry enhancement associated with instantons of vanishing size [16]. They correspond to Dirichlet five-branes, which carry additional symplectic group symmetry. Since the total instanton number should be 24, the maximal possibility for symmetry enhancement is to have 24 coincident five-branes carrying an $Sp(24)$ gauge symmetry. The notation is that $Sp(k)$ refers to a compact group of rank $k$, which is sometimes called $USp(2k)$ by other authors (including myself on occasion). Its fundamental representation has dimension $2k$ and the adjoint has dimension $k(2k + 1)$. An antisymmetric tensor of dimension $k(2k - 1)$ is reducible into a singlet plus the rest. The most symmetric small
instanton model has \( G = G_1 \times G_2 \) with \( G_1 = SO(32) \) and \( G_2 = Sp(24) \). This has rank 40 and dimension \( n_V = 1672 \). Its existence cannot be understood in terms of conformal field theory, since the small instantons are inherently non-perturbative, as explained in [16]. Our purpose here is to examine anomaly cancellation for this model.

The number of hypermultiplets must be \( n_H = n_V + 244 = 1916 \). From the analysis in [16], we know that there is \( \frac{1}{2}(32, 48) \) corresponding to open strings with one end attached to the five-brane. Note that the factor of \( \frac{1}{2} \) is allowed here, because 48 is pseudoreal. In addition, there is an antisymmetric tensor representation corresponding to open strings with both ends attached to the five-brane. This gives \((1, 1127) + (1, 1)\). We now have 1896 hypermultiplets. Thus, there must also be 20 singlets, which are just the usual gravitational moduli for \( K3 \) compactification.

Now the anomaly analysis can be carried out using the \( Sp(k) \) identities

\[
\begin{align*}
\text{Tr} F^4 &= (2k + 8)\text{tr} F^4 + 3(\text{tr} F^2)^2 \\
\text{Tr} F^2 &= (2k + 2)\text{tr} F^2 \\
\text{tr} A F^4 &= (2k - 8)\text{tr} F^4 + 3(\text{tr} F^2)^2 \\
\text{tr} A F^2 &= (2k - 2)\text{tr} F^2,
\end{align*}
\] (29)

where \( A \) refers to the antisymmetric tensor representation. Using these formulas we have

\[
\begin{align*}
X_1^{(2)} &= 6\text{tr} F_1^2, \quad X_1^{(4)} = 3(\text{tr} F_1^2)^2 \\
X_2^{(2)} &= -12\text{tr} F_2^2, \quad X_2^{(4)} = 0 \\
Y_{12} &= \frac{1}{2} \text{tr} F_1^2 \text{tr} F_2^2.
\end{align*}
\] (30)

Substitution into the anomaly formula gives the factorized expression

\[
I = (\text{tr} R^2 - \text{tr} F_1^2)(\text{tr} R^2 + 2\text{tr} F_1^2 - 2\text{tr} F_2^2).
\] (31)

This is an impressive confirmation of Witten’s result. In particular, the cancellation of the \( \text{tr} F^4 \) terms for the \( Sp \) factor required the \( SO(m) \) factor to be \( SO(32) \) and the cancellation of the \( \text{tr} F^4 \) terms for the \( SO \) factor required the \( Sp(n) \) factor to be \( Sp(24) \) – no other \( m \) or \( n \) would work with these representations.

Let us now examine what other models can be reached by Higgsing this model. It is easy to see that the \( Sp(24) \) factor can be broken to a subgroup \( \otimes_{i=1}^n Sp(k_i) \) with \( \sum k_i = 24 \). In this case the \( \frac{1}{2}(32, 48) \) decomposes in the obvious way without any of these hypermultiplets
being eaten. All the hypermultiplets that are eaten come from the antisymmetric tensor, leaving a sum of antisymmetric tensors (including the singlets) for each of the $Sp(k_i)$ factors. Also, the 20 singlets of gravitational origin survive untouched. The anomaly analysis works as above with $tr F_2^2 \to \sum_{i=1}^{n} tr F_2^2$. This result is exactly as expected based on the analysis of [10].

The $SO(32) \times Sp(24)$ model can be generalized to the case of $n \leq 24$ coincident small instantons. This requires embedding $24 - n$ units of instanton number in the $SO(32)$. The resulting gauge group is $SO(8 + n) \times Sp(n)$. In this case the hypermultiplets consist of $\frac{1}{2}(8 + n, 2n) + \frac{24-n}{2}(1, 2n)$, as well as the antisymmetric tensor representation of $Sp(n)$. The number of singlets (besides the one associated with the antisymmetric tensor) is $20 + \frac{1}{2}(n - 24)(n - 21)$. The anomaly analysis works as before, and the result is again given by eqs. (30) and (31). The $Sp(n)$ can again be broken by Higgsing, as described in the preceding paragraph.

Another interesting question is whether small instantons can be accomodated in $E_8 \times E_8$ models. $SO(32)$ and $E_8 \times E_8$ models belong to a common moduli space after compactification of each of them on a circle. However, this is not the case for $K3$ compactification. This is the same as for type IIA and IIB theories, which join up after $S^1$ compactification, but remain distinct upon $K3$ compactification. This means that $E_8 \times E_8$ models on $K3$ do not have a dual type I description, and D-branes are meaningless for them. Wisely, ref. [16] made no claims for small instanton effects in $E_8 \times E_8$ models. From the point of view of anomaly equations, it is obvious that there are no solutions of the form $E_8 \times E_8 \times G'$, at least if one assumes that all hypermultiplets are singlets of both $E_8$’s. Non-singlet representations quickly lead to very large values of $n_H$, which appear unlikely to lead to any consistent new possibilities.

5 New Anomaly-Free Models

Maybe there are six-dimensional $N = 1$ string vacua that arise from other non-perturbative mechanisms. One way to identify candidates is to find new solutions of the anomaly cancellation conditions. A non-systematic search turned up two new classes of solutions of the anomaly conditions, which may be of some interest. For the first class, the gauge group is

$$G = SO(2n + 8) \times Sp(n),$$

\[3\]I am grateful to E. Witten for bringing this possibility to my attention.
which has rank $2n + 4$. Since $n$ is an arbitrary non-negative integer, the rank may be arbitrarily large. The hypermultiplet content of these models is given by

$$R = (2n + 8, 2n) + 272(1, 1).$$

(33)

Since $2n$ is pseudoreal, the first term is really two copies of $\frac{1}{2}(2n + 8, 2n)$. The number of singlets is determined from the requirement $n_H = n_V + 244$. Despite the superficial resemblance to the examples in the preceding section, there are two important differences. One is the factor of two mentioned above and the second is the absence of an antisymmetric tensor representation of the symplectic group. The anomaly analysis is easy using the formulas in the preceding sections. One finds the factorized result

$$I = (\text{tr} R^2 - 2 \text{tr} F_1^2 + \text{tr} F_2^2)(\text{tr} R^2 + 2 \text{tr} F_1^2 - 2 \text{tr} F_2^2),$$

(34)

for all values of $n$. This result depends on a number of “miracles,” so I expect it to have physical significance. The most straightforward Higgsing of these models simply decreases the value of $n$. This means that these models form a single connected structure. Roughly speaking, there ought to be an infinite-dimensional group that underlies all of them.

The second class of models has

$$G = SU(n) \times SU(n)$$

(35)

with hypermultiplets in the representation

$$R = (n, \bar{n}) + (\bar{n}, n) + 242(1, 1).$$

(36)

Using the $SU(n)$ formulas (and $\text{tr}_n = \text{tr}_{\bar{n}} = \text{tr}$)

$$\text{Tr} F^4 = 2n \text{tr} F^4 + 6(\text{tr} F^2)^2$$
$$\text{Tr} F^2 = 2n \text{tr} F^2,$$

(37)

one finds that anomaly factorizes as follows

$$I = (\text{tr} R^2 - 2 \text{tr} F_1^2 + 2 \text{tr} F_2^2)(\text{tr} R^2 + 2 \text{tr} F_1^2 - 2 \text{tr} F_2^2).$$

(38)

Thus, we have a second infinite family of models with unbounded rank. An intriguing feature of this class of models, not shared by the first one, is that the corresponding $N = 2$ four-dimensional gauge theory is superconformal (or finite).
The two classes of models do share another interesting feature. The gauge groups are the bosonic subgroups of simple Lie superalgebras $OSp(2n + 8|n)$ and $SU(n|n)$. Moreover, the non-singlet hypermultiplets are in correspondence with the odd elements of the superalgebras. These properties seem closely related to the observation in ref. [17] that the BPS states of certain $N = 2$ $D = 4$ models have a similar relationship to infinite superalgebras. It is also intriguing that the factorized anomalies only involve the ‘supertrace’ combinations $\text{tr} F_1^2 - \text{tr} F_2^2$. It is tempting to conjecture that the $OSp(2n + 8|n)$ class of models has an explanation in terms of unoriented open strings and the $SU(n|n)$ class of models has an explanation in terms of oriented open strings.

A. Dabholkar has pointed out that if one adds eight tensor multiplets to these models (for a total of nine), then eq. (1) is replaced by $n_H - n_V = 12$ and the $(\text{tr} R^2)^2$ term in the anomaly cancels. In this case the number of singlets becomes 40 in the $OSp(2n + 8|n)$ models and 10 in the $SU(n|n)$ models. Also, the absence of the $(\text{tr} R^2)^2$ term implies that the gauge invariant field strength $H$ (which describes the self-dual tensor of the gravity multiplet and one of anti-self-dual matter tensors) contains a Yang-Mills Chern–Simons term and no Lorentz Chern–Simons term.

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