Star-product quantization and symplectic tomography

Olga V Manko

P. N. Lebedev Physical Institute, Leninskii Prospect 53, Moscow 119991, Russia
E-mail: omanko@sci.lebedev.ru

Received 8 January 2009
Accepted for publication 12 January 2009
Published 31 July 2009
Online at stacks.iop.org/PhysScr/T135/014004

Abstract
A review of the symplectic tomographic approaches within the framework of star-product quantization is presented. The classical statistical mechanics within the framework of the tomographic representation is considered. The kernels of the star product of functions—symbols of operators in classical and quantum mechanics, are presented.

PACS numbers: 42.50.−p, 03.65.Bz

1. Introduction
In quantum mechanics, the state is described by the wave function or density matrix, and the observables are described by operators acting in the Hilbert space.

In classical mechanics, the state of a system with fluctuations is described by the probability distribution function and observables are described by functions.

So, we have different languages for describing the quantum and classical nature, but to understand the nature of the system consisting of the classical and quantum parts, it is necessary to have the same language for both domains. Due to this, an idea appeared: to create the probability representation of quantum mechanics along with the probability representation of the classical states.

The tomographic probability representation of quantum mechanics was introduced in [1–3] and the tomographic probability representation of the classical states was introduced in [4–6]. In the probability representation, the quantum and classical states are described by the same objects—tomograms. Tomograms are positive measurable probability distribution functions of random variables, which are determined in an ensemble of the reference frames in the system’s phase space.

In order to describe observables by functions instead of operators, in quantum mechanics the quantization based on the star product of functions is used. In [7, 8], it was shown that the symplectic tomography scheme (probability representation) is a new example of quantization based on the star product of functions—symbols of operators. The operators determining the star-product quantization scheme and the kernel of the star product of symbols for symplectic tomography were obtained in the explicit form. The tomographic symbols of classical observables were discussed in [6]. The tomographic star-product kernel in classical mechanics was obtained in [9]. A scheme of quantization dual to the symplectic tomography scheme and its connection with the different nature of the density operators and operators–observables were discussed in [10].

The aim of the present paper is to present a review of the symplectic tomographic approaches in both the quantum and classical domains within the framework of star-product quantization.

2. Classical states
Let us consider a particle with one degree of freedom with unit mass. The position and velocity of the particle are \(-\infty < q < \infty\) and \(-\infty < \dot{q} < \infty\), respectively. The particle momentum is \(p = \dot{q}\) due to \(m = 1\). The particle state is identified with a point in the phase space (plane) with coordinates \(q\) and \(p\). The evolution of the particle state is described by a trajectory in the phase space \(q(t)\) and \(p(t)\).

Let us suppose that the classical particle is located inside some environment and the position \(q\) and momentum \(p\) fluctuate. In view of these fluctuations, the particle state is described by a probability distribution function \(f(q, p)\), which is nonnegative \(f(q, p) \geq 0\) and normalized:

\[
\int f(q, p) \, dq \, dp = 1. \tag{1}
\]

We consider a point in the phase space \((p, q)\). If one rotates the reference frame in the phase space by an angle \(\phi\) and then...
This function was called the symplectic tomogram. The physical meaning of the parameters \( \mu \) and \( v \) is that they describe an ensemble of rotated and scaled reference frames in which the position \( X \) is measured.

For \( \mu = \cos \varphi \) and \( v = \sin \varphi \), the marginal distribution (7) is the distribution for the homodyne-output variable used in optical tomography [12, 13].

Formula (7) can be inverted and the Wigner function of the state can be expressed in terms of the symplectic tomogram [1]

\[
W(q, p) = \frac{1}{2\pi} \int w(X, \mu, v) \exp[-i(\mu q + vp - X)] \, d\mu \, dv \, dX.
\]

Since the Wigner function determines completely the quantum state of a system and, on the other hand, this function itself is completely determined by the symplectic tomogram, one can use symplectic tomograms (positive and normalized) for describing quantum states that are probability distribution functions analogous to the classical ones.

The quantum state is given if the position probability distribution \( w(X, \mu, v) \) in an ensemble of rotated and squeezed reference frames in the phase space is given. Information contained in the symplectic tomogram \( w(X, \mu, v) \) is overcomplete. To determine the quantum state, it is enough to know the values of the symplectic tomogram of the state for arguments satisfying the condition \((\mu^2 + v^2 = 1)\), where \( \mu = \cos \varphi \). This representation of quantum mechanics, called probability representation, was introduced in [1–3]. In [14], it was shown that the density operator can be reconstructed from the symplectic tomogram

\[
\hat{\rho} = \frac{1}{2\pi} \int w(X, \mu, v)e^{i(x-\mu q-vp)} \, dX \, d\mu \, dv.
\]

4. General star-product scheme

In quantum mechanics, the observables are described by operators acting in the Hilbert space of the states. In order to consider the observables as functions in a phase space, first we describe following [7, 8] a general construction of a map of the operators onto the functions without any concrete realization of the map.

We consider an operator \( \hat{A} \) acting in a given Hilbert space. Let us suppose that we have a set of operators \( \hat{U}(x) \) acting in the Hilbert space \( H \), where the \( n \)-dimensional vector \( x = (x_1, x_2, \ldots, x_n) \) labels the particular operator in the set. We construct the \( c \)-number function \( f^\hat{A}_{c}(x) \) using the definition

\[
f^\hat{A}_{c}(x) = \text{Tr} (\hat{A}\hat{U}(x)).
\]

The function \( f^\hat{A}_{c}(x) \) is called the symbol of operator \( \hat{A} \), and the operators \( \hat{U}(x) \) are called dequantizers [10]. We suppose that this relation has an inverse. There exists a set of operators \( \hat{D}(x) \) acting in the Hilbert space such that

\[
\hat{A} = \int f^\hat{A}(x)\hat{D}(x) \, dx.
\]
The operators $\hat{D}(x)$ are called quantizers [10]. The formulae are self-consistent if the following property of the quantizers and dequantizers takes place:

$$\text{Tr} \left[ \hat{U}(x) \hat{D}(x') \right] = \delta(x - x').$$

(11)

Relations (9) and (10) determine the invertible map of the operator $\hat{A}$ onto the function $f_\hat{A}(x)$.

The most important property is the existence of the nonlocal product of two functions. We introduce the product (star product) of two functions $f_\hat{A}(x)$ and $f_\hat{B}(x)$ corresponding to two operators $\hat{A}$ and $\hat{B}$, respectively, by the relations

$$f_\hat{A}\hat{B}(x) = f_\hat{A}(x) * f_\hat{B}(x) = \text{Tr}(\hat{A} \hat{B} \hat{U}(x)).$$

(12)

The product of the operators in the Hilbert space is the associative product, $\hat{A}(\hat{B}\hat{C}) = (\hat{A}\hat{B})\hat{C}$, then the star product of functions—symbols of operators has to be associative too

$$f_\hat{A}(x) * (f_\hat{B}(x) * f_\hat{C}(x)) = (f_\hat{A}(x) * f_\hat{B}(x)) * f_\hat{C}(x).$$

(13)

The map provides the nonlocal product of two functions (star product)

$$f_\hat{A}(x) * f_\hat{B}(x) = \int f_\hat{A}(x') f_\hat{B}(x') K(x', x', x) \, dx' \, dx'.'$$

The kernel of the star product is linear with respect to the dequantizer and nonlinear in the quantizer operator

$$K(x', x', x) = \text{Tr} \left[ \hat{D}(x') \hat{D}(x') \hat{U}(x) \right].$$

The associativity condition for operator symbols means that the kernel of the star product of symbols of operators $K(x', x', x)$ satisfies the nonlinear equation [10]

$$\int K(x_1, x_2, y) K(y, x_3, x_4) \, dy = \int K(x_1, y, x_4) K(x_2, x_1, y) \, dy.$$

(14)

Now consider, following [10], another scheme

$$f^{(d)}_\hat{A}(x) = \text{Tr} \left[ \hat{A} \hat{D}(x) \right],$$

$$\hat{A} = \int f^{(d)}_\hat{A}(x) \hat{U}(x) \, dx.$$

(15)

We replace the quantizer and dequantizer by each other because the compatibility condition is valid in both the cases. We consider the quantizer–dequantizer pair as dual to the initial one

$$\hat{U}(x) = \hat{D}(x),$$

$$\hat{D}(x) = \hat{U}(x).$$

The interchange corresponds to a specific symmetry of the equation for associative product kernel. The star product of dual symbols $f^{(d)}_\hat{A}(x)$ and $f^{(d)}_\hat{B}(x)$ of two operators $\hat{A}$ and $\hat{B}$ is described by a dual integral kernel

$$K^{(d)}(x', x', x) = \text{Tr} \left[ \hat{U}(x') \hat{U}(x') \hat{D}(x) \right].$$

The dual kernel is another solution of the nonlinear equation (14).

5. Symplectic tomography

We consider the symplectic tomography scheme [1] as an example of the star-product quantization, following [7, 8]. In the symplectic tomography scheme, the tomographic symbol $f_\hat{A}(x)$ of the operator $\hat{A}$ is obtained by means of the dequantizer

$$\hat{U}(x, \mu, v) = \delta(x \hat{1} - \mu \hat{q} - v \hat{p}),$$

where vector $x = (X, \mu, v)$ has the coordinates that are real numbers and $\hat{1}$ is identity operator. The quantizer in symplectic tomography reads

$$\hat{D}(X, \mu, v) = \frac{1}{2\pi} \exp \left( iX \hat{1} - iv \hat{p} - i\mu \hat{q} \right).$$

The kernel of the star product of two tomographic symbols of operators $\hat{A}$ and $\hat{B}$ has the form [7]

$$K(X_1, \mu_1, v_1, X_2, \mu_2, v_2, X\mu, v) = \delta((v_1 + v_2) - \nu(\mu_1 + \mu_2)) \exp \left( \frac{i}{2} \left( v_1 \mu_2 - v_2 \mu_1 \right) + 2X_1 \nu + 2X_2 \nu + 2(X_1 + X_2) \nu \right).$$

It is worth noting the important properties of the described tomographic map. If the operator under consideration is a density operator $\hat{\rho}$, its tomographic symbol $w(X, \mu, v)$ is the standard probability density of continuous real variable $X$, i.e. the function called symplectic tomogram of the state $w(X, \mu, v) = \text{Tr} \hat{\rho} \delta(X \hat{1} - \mu \hat{q} - v \hat{p})$. It is nonnegative and normalized $\int w(X, \mu, v) \, dX = 1$. The mean value of the quantum observable $\hat{A}$ reads

$$\langle \hat{A} \rangle = \text{Tr} \hat{\rho} \hat{A} = \text{Tr} \int w(X, \mu, v) \hat{D}(X, \mu, v) \hat{A} \, dX \, d\mu \, dv.$$

Bearing in mind that the dual symbol is determined by formulae (15), we obtain

$$\langle \hat{A} \rangle = \int w(X, \mu, v) f^{(d)}_\hat{A}(X, \mu, v) \, dX \, d\mu \, dv.$$

Thus, the mean value of an observable $\hat{A}$ is given by the integral of the product of the tomographic symbol of the density operator and the symbol of the observable in the dual scheme. In fact, it can be shown that this observation is true in general.

6. Classical tomographic symbols

The reversible relationship between the tomographic symbol $w_j(X, \mu, v)$ of the probability distribution $f(q, p)$ in classical mechanics is determined by formulae (3) and (5) (see, for example, [6]). Here, we assumed for $f(q, p)$ the normalization condition $\int \int f(q, p) \, dq \, dp = 1$ analogously to the normalization condition for the Wigner function $W(q, p)$. According to [6], in classical mechanics one can introduce the operators $\hat{A}_d$ for which their formal Weyl
symbol $W_{A_0}(q, p)$ coincides with a classical observable $A(q, p)$, i.e.

$$W_{A_0}(q, p) = 2 \text{Tr} \hat{A} \hat{D}(2\alpha) \hat{I} = A(q, p). \quad (16)$$

The classical tomographic symbol for the observable $A(q, p)$ is the same as its quantum tomographic symbol if the Weyl symbol $W_A(q, p)$ for the observable $\hat{A}$ and the phase-space function $A(q, p)$ coincide. Really, expressions (16) also represent the relationship between the tomographic symbol of the observable $\hat{A}$ in quantum mechanics and its Weyl symbol $W_A(q, p)$ [6]. Consequently, we can consider the phase-space function $A(q, p)$ as the classical Weyl symbol of the observable $A(q, p)$ in classical mechanics. For example, the quantum tomographic symbol for a unity operator $\hat{1}$ and the classical tomographic symbol for unity operator [9] are

$$w_1(X, \mu, \nu) = -\pi |X| \delta(\mu) \delta(\nu).$$

Since in quantum mechanics Weyl symbols for the position operator $\hat{q}$ and momentum operator $\hat{p}$ are $c$-numbers $q$ and $p$, we have for both classical and quantum tomographic symbols

$$w_q(X, \mu, \nu) = \frac{\pi}{2} X |\delta'(\mu) \delta(\nu)|,$$

$$w_p(X, \mu, \nu) = \frac{\pi}{2} X |\delta(\mu) \delta'(\nu)|.$$

The commutative star-product kernel $K(X, \mu, v, X_1, \mu_1, v_1, X_2, \mu_2, v_2)$ for two classical tomographic symbols $w_{f1}(X_1, \mu_1, v_1)$ and $w_{f2}(X_2, \mu_2, v_2)$ was defined in [6]. It is of the form

$$K(X, \mu, v, X_1, \mu_1, v_1, X_2, \mu_2, v_2) = \frac{1}{(2\pi)^2} e^{i(X+X_1-X(v+v_2))/\nu} \delta(v(\mu_1+\mu_2) - \mu(\nu_1+\nu_2)).$$

The relationship between tomographic star-product kernels in quantum and classical mechanics reads [9]

$$K_{\text{quant}}(X, \mu, v, X_1, \mu_1, v_1, X_2, \mu_2, v_2) = K_{\text{cl}}(X, \mu, v, X_1, \mu_1, v_1, X_2, \mu_2, v_2) e^{i(\mu_1 v_1 - \mu_2 v_2)/2}.$$

7. Conclusions

To conclude, we formulate the main results of our study. We reviewed the generic approach of constructing operator symbols and their star product. We have shown that the probability representation of quantum states can be constructed using specific versions of star-product schemes. In the star-product scheme, the physical interpretation of dual structures is shown by using the example of symplectic tomography.

Acknowledgments

I thank the organizers of the XV Central European Workshop on Quantum Optics (Belgrade, May–June 2008) for their kind hospitality and the Russian Foundation for Basic Research for travel grant number 08-02-08125.

References

[1] Mancini S, Man’ko V I and Tombesi P 1995 Quantum Semiclass. Opt. 7 615
[2] Mancini S, Man’ko V I and Tombesi P 1996 Phys. Lett. A 213 1
[3] Mancini S, Man’ko V I and Tombesi P 1997 Europhys. Lett. 37 79
[4] Man’ko O V and Man’ko V I 1997 J. Russ. Laser Res. 18 407
[5] Man’ko O V and Man’ko V I 2000 J. Russ. Laser Res. 21 411
[6] Man’ko O V and Man’ko V I 2004 J. Russ. Laser Res. 25 477
[7] Man’ko O V, Man’ko V I and Marmo G 2002 J. Phys. A: Math. Gen. 35 699
[8] Man’ko O V, Man’ko V I and Marmo G 2001 Proc. 2nd Int. Symp. on Quantum Theory and Symmetries (Krakow, Poland, 2001) ed E Kapuschik and A Horzela (Singapore: World Scientific) p 126
[9] Man’ko O V, Man’ko V I and Pilyavets O V 2005 J. Russ. Laser Res. 26 129
[10] Man’ko O V, Man’ko V I, Marmo G and Vitale P 2007 Phys. Lett. A 360 522
[11] Wigner E 1932 Phys. Rev. 40 749
[12] Bertrand J and Bertrand P 1987 Found. Phys. 17 397
[13] Vogel K and Risken H 1989 Phys. Rev. A 40 2847
[14] D’Ariano G M, Mancini S, Man’ko V I and Tombesi P 1996 Quantum Semiclass. Opt. 8 1017
[15] Man’ko V I, Sharapov V A and Shchukin E V 2003 J. Russ. Laser Res. 24 180