THE FUNDAMENTAL THEOREM OF CURVES AND CLASSIFICATIONS IN THE HEISENBERG GROUPS

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Abstract. We study the horizontally regular curves in the Heisenberg groups $H_n$. We show the fundamental theorem of curves in $H_n$ ($n \geq 2$) and define the orders of horizontally regular curves. We also show that the curve $\gamma$ is of order $k$ if and only if, up to a Heisenberg rigid motion, $\gamma$ lies in $H_k$ but not in $H_{k-1}$; moreover, two curves with the same order differ from a rigid motion if and only if they have the same invariants: $p$-curvatures and contact normality. Thus, combining with our previous work [1] we have completed the classification of horizontally regular curves in $H_n$ for $n \geq 1$.

1. Introduction

The Heisenberg group $H_n$, $n \geq 1$, is the space $\mathbb{R}^{2n+1}$ associated with the group multiplication

$$(x_1, \ldots, x_n, y_1, \ldots, y_n, z) \circ (\tilde{x}_1, \ldots, \tilde{x}_n, \tilde{y}_1, \ldots, \tilde{y}_n, \tilde{z}) = (x_1 + \tilde{x}_1, \ldots, x_n + \tilde{x}_n, y_1 + \tilde{y}_1, \ldots, y_n + \tilde{y}_n, z + \tilde{z} + \sum_{j=1}^n (y_j \tilde{x}_j - x_j \tilde{y}_j))$$

It is a $(2n+1)$-dimensional Lie group, and the space of all left invariant vector fields is spanned by the basic vector fields:

$$\hat{e}_j = \frac{\partial}{\partial x_j} + y_j \frac{\partial}{\partial z},$$
$$\hat{e}_{n+j} = \frac{\partial}{\partial y_j} - x_j \frac{\partial}{\partial z},$$
$$T = \frac{\partial}{\partial z},$$

for $1 \leq j \leq n$.

The Heisenberg group $H_n$ can be regarded as the $n$-dimensional CR manifold with zero Webster-curvature. For more details, the reader can refer the Appendix in [3] or [1][2][4][6]. We give the brief description of geometric structures on $H_n$: the standard contact bundle in $H_n$ is the subbundle $\xi$ of the tangent bundle $TH_n$, which is spanned by $\hat{e}_j$ and $\hat{e}_{n+j}$ for $1 \leq j \leq n$. The contact bundle can also be defined as the kernel of the contact form

$$\theta = dz + \sum_{j=1}^n (x_j dy_j - y_j dx_j).$$

The standard CR structure on $H_n$ is the almost complex structure defined on $\xi$ by

$$J(\hat{e}_j) = \hat{e}_{n+j}, \quad J\hat{e}_{n+j} = -\hat{e}_j.$$
Throughout the article, we regard the Heisenberg group $H_n$ with the standard pseudo-hermitian structure $(J, \theta)$ as a pseudohermitian manifold $(H_n, J, \theta)$. Denote the group of pseudohermitian transformations on $H_n$ by $PSH(n)$ which forms the group of rigid motions. An element in $PSH(n)$ is called a pseudohermitian transformation or a symmetry on $H_n$, which is a deffeomorphism $\Phi : H_n \to H_n$ preserving both the CR structure $J$ and the contact form $\theta$. More precisely, it satisfies

$$\Phi_* J = J\Phi_*, \quad \Phi^* \theta = \theta.$$  

Here are our settings for curves: suppose $\gamma : I \subset \mathbb{R} \to H_n$ is a parametrized curve defined by

$$\gamma(t) = (x_1(t), \cdots, x_n(t), y_1(t), \cdots, y_n(t), z(t)).$$

For $k = 1, \cdots, n$, the $k$th derivative $\gamma^{(k)}$ of the curve $\gamma$ has the natural decomposition

$$\gamma^{(k)}(t) = \gamma^{(k)}_\xi(t) + \gamma^{(k)}_T(t),$$

where $\gamma^{(k)}_\xi$ (resp. $\gamma^{(k)}_T$) is the orthogonal projection of $\gamma^{(k)}$ on the contact plane $\xi$ along $T$-direction (resp. on $T$ along $\xi$) with respect to the Levi metric. Recall that a curve is called horizontally regular if it has the non-vanishing first derivative in the horizontal part, $\gamma^{(1)}(t) \neq 0$ for all $t$. In [1] we show that any horizontally regular curve can be reparametrized by horizontal arc length $s$ with respect to the Levi metric, namely, $|\gamma^{(1)}(s)| = 1$ for all $s$. In the article, we use different parameters $t$ and $s$ to distinguish from being parametrized by arc length or not. Moreover, by identifying $H_n \cong \mathbb{C}^n \times \mathbb{R}$ and the natural projection,

$$\begin{array}{rcl}
H_n & \cong & \mathbb{C}^n \times \mathbb{R} \\
\gamma & \mapsto & (x_1(t), \cdots, x_n(t), y_1(t), \cdots, y_n(t), z(t)) \\
\pi & \mapsto & \mathbb{C}^n \cong \mathbb{R}^{2n}
\end{array}$$

we may rewrite the curve $\gamma$ in the real sense

$$\gamma(t) = (x_1(t), \cdots, x_n(t), y_1(t), \cdots, y_n(t), z(t)) \in \mathbb{R}^{2n} \times \mathbb{R},$$

with its projection on $\mathbb{R}^{2n}$

$$\alpha(t) = (x_1(t), y_1(t), \cdots, x_n(t), y_n(t)),$$

or equivalently in the complex sense

$$\beta(t) = (z_1(t), \cdots, z_n(t)) \in \mathbb{C}^n,$$

where $z_j(s) = x_j(s) + \sqrt{-1}y_j(s)$ for $1 \leq j \leq n$.

A key observation is that in case $H_1 \ni \gamma = (\beta, z)$, we have

$$\gamma' = (\beta', z') \in \mathbb{C} \times \mathbb{R}$$

$$= (x', y', z') \in \mathbb{R}^2 \times \mathbb{R}$$

$$= x'\dot{\epsilon}_1 + y'\dot{\epsilon}_2 + (z' - yx' + x'y)T.$$  

Thus, $\beta' \neq 0$ if and only if the curve $\gamma$ is horizontally regular in $H_1$. Given a curve in $H_n$, in general, we ask if one can establish the concept such that the curve is horizontally regular in any lower dimensional subspaces of $H_n$. 

Recall [4] a real linear subspace $P$ is totally real if and only if any vector $X \in P$ implies $JX \notin P$. Inspired by Griffiths [5], we generalize the definition of non-degenerate curves in $H_n$.

**Definition 1.1.** A **non-degenerate horizontally regular curve** in $H_n$ is a horizontally regular curve $\gamma(t) = (\beta(t), z(t)) \in \mathbb{C}^n \times \mathbb{R}$ satisfying

\[ W^{[n]}_\beta(t) := \gamma'_\xi(t) \wedge \cdots \wedge \gamma^{(n)}_\xi(t) \neq 0 \text{ for all } t, \]

and the set

\[ \{ \gamma'_\xi(t), \cdots, \gamma^{(n)}_\xi(t) \} \text{ is totally real for all } t. \]

The condition (1.0) ensures that we can always choose an oriented frame along the non-degenerate horizontally regular curve $\gamma$

\[ (\gamma(s); e_1(s), \cdots, e_n(s), e_{n+1}(s), \cdots, e_{2n}(s), T) \]

inductively satisfying the condition, for $1 \leq j \leq n$,

\[ e_1(s) \wedge \cdots \wedge e_j(s) = \pm \frac{\gamma'_\xi(s) \wedge \cdots \wedge \gamma^{(j)}_\xi(s)}{|\gamma'_\xi(s) \wedge \cdots \wedge \gamma^{(j)}_\xi(s)|} \]

and

\[ e_{n+j} = J e_j. \]

Along the curve $\gamma$, as in [1], we define the **p-curvatures** $\kappa_j(s)$, $1 \leq j \leq n$ and the **contact normality** $\tau(s)$ as

\[ \kappa_j(s) = \left( \frac{d e_j(s)}{ds}, e_{n+j}(s) \right), \text{ for } 1 \leq j \leq n - 1, \]

\[ \kappa_n(s) = \left( \frac{d e_n(s)}{ds}, e_{2n}(s) \right), \]

\[ \tau(s) = \left( \frac{d \gamma(s)}{ds}, T \right). \]

We point out that all quantities above are invariant under the group actions of $P\Sigma H(n)$. Our main theorem shows that those invariants completely determine the non-degenerate horizontally regular curve up to a Heisenberg rigid motion, which is analogous to the fundamental theorem of curves in $\mathbb{R}^n$.

**Theorem 1.2.** Given $(n + 1)$ smooth functions $\kappa_i(s)$ for $1 \leq i \leq n$ and $\tau(s)$, there exists a non-degenerate horizontally regular curve $\gamma(s) \in H_n$ parametrized by the horizontal arc-length $s$ such that the functions $\kappa_i$’s and $\tau$ are the p-curvatures and the contact normality of $\gamma$, respectively. In addition, two non-degenerate horizontally regular curves satisfying the same conditions above differ from a rigid motion in $P\Sigma H(n)$.

It is obvious that $\gamma(s)$ is a horizontal if $\gamma'(s) = \gamma'_\xi(s)$ for all $t \in I$, and therefore $\gamma(s)$ is horizontal if and only if $\tau(s) = 0$. We immediately have the corollary:

**Corollary 1.3.** Given smooth functions $\kappa_i(s), 1 \leq i \leq n$, there exists a horizontal curve $\gamma(s) \in H_n$ parametrized by the horizontal arc-length $s$ having $\kappa_i(s), 1 \leq i \leq n$ as its p-curvatures. In addition, two horizontal curves having the same p-curvatures differ from a rigid motion in $P\Sigma H(n)$.
Next we define the order of the curves. It is similar to the concept that spacial curves in $\mathbb{R}^3$ cannot be "squeezed" into any linear 2-dimensional subspaces, but planar curves can be. The order of a horizontal curve gives the minimal dimension of the subspaces in which the curve lives.

**Definition 1.4.** A horizontally regular curve $\gamma(t) = (\beta(t), z(t)) \in H_n$ is of order $k$, denoted by $\text{order}(\gamma) = k$, if there exists a positive integer $k \in [1, n]$ such that

$$
\begin{cases}
\beta'(t) \wedge \cdots \wedge \beta^{(k+1)}(t) = 0, \\
\beta'(t) \wedge \cdots \wedge \beta^{(k)}(t) \neq 0,
\end{cases}
$$

for all $t$. A horizontally regular curve is called degenerate in $H_n$ if $\text{order}(\gamma) < n$.

**Remark 1.5.** By Definition 1.4, any non-degenerate horizontally regular curve $\gamma$ is of the top order, $\text{order}(\gamma) = n$, and vice versa. In contrast to Theorem 1.2, two curves with different orders never lie in the same subspace of $H_n$, and hence they cannot be congruent to each other by any Heisenberg rigid motion.

We also characterize the degenerate horizontally regular curves of top order $(n-1)$. Similar to the fact that a planar curve in $\mathbb{R}^3$ can be "moved" to $xy$-plane, any degenerate regular curve $\gamma \in H_n$ can be acted by a symmetry of $PSH(n)$ to $H_{n-1}$.

**Proposition 1.6.** Let $\gamma(t) = (\beta(t), z(t)) \in H_n \cong \mathbb{C}^n \times \mathbb{R}$ be a degenerate horizontally regular curve. Then there exists a symmetry $\Phi \in PSH(n)$ such that $\Phi(\gamma) = \tilde{\gamma}$, where $\tilde{\gamma} = (\tilde{\beta}(t), z(t))$ is a horizontally regular curve with the projection $\tilde{\beta} = (\tilde{\beta}_1, \cdots, \tilde{\beta}_{n-1}, 0)$ of $\beta$ onto $\mathbb{C}^{n-1}$. Thus, we conclude that $\tilde{\gamma} \in H_{n-1} \subseteq H_n$.

**Remark 1.7.** In summary, we have the dichotomy to classify any horizontally regular curve $\gamma$ in $H_n$: let $\gamma(t) = (\beta(t), z(t)) \in H_n$. If $\gamma$ is non-degenerate, then by Theorem 1.2 it must be uniquely determined by the $p$-curvatures $\kappa_i(s)$, $1 \leq i \leq n$, and contact normality $\tau(t)$ up to a symmetry in $PSH(n)$; otherwise, the Wronskian $W^{[n]}(t) = 0$ somewhere and we keep checking if $W^{[n-1]}(t)$ is nonzero for all $t$. The nonzero condition implies that $\gamma$ is degenerate of order $(n-1)$. By using Proposition 1.6 and applying Theorem 1.2 to $H_{n-1}$, we obtain that $\gamma$ lies in the subspace $H_{n-1}$ but not in $H_{n-2}$, and is uniquely determined by the invariants $\kappa_1, \cdots, \kappa_{n-1}, \tau$. However, if $W^{[n-1]}(t) = 0$ somewhere, we have to check if $W^{[n-2]}(t)$ is nonzero again for all $t$. Repeating above processes, and finally we conclude that two curves with the same order $k \leq n-1$ differ from a symmetry in $PSH(k)$ if and only if both have same $k_i$, $i = 1, \cdots, k-1$ and $\tau$. Thus, we complete the classification of horizontally regular curves.

An interesting example is that the order of horizontal geodesics in $H_n$, $n \geq 1$, is always 1. By the processes described in Remark 1.7, the horizontal geodesics can always be embedded into $H_1$.

**Proposition 1.8.** Every horizontal geodesic $\gamma \in H_n$, $n \geq 1$, is of order 1 with constant $p$-curvatures and zero contact normality. Therefore, $\gamma$ is non-degenerate in $H_1$ and degenerate in $H_n$ for $n \geq 2$.

The structure of the paper is as follows: in Section 2 we recall the well-known theorems for existence and uniqueness. In Section 3 we derive the Darboux derivatives. In Section 4 we prove the Theorem 1.2. Finally, in Section 5, we characterize
the degenerate curves (Proposition 1.6), and, as an example, the order of horizontal geodesics will be calculated.

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2. Calculus on Lie groups

In the section, we shall give two well-known and essentially local results concerning smooth maps from manifold $M$ into the Lie group $G$. Given a connected smooth manifold $M$. Let $G \subset GL(n, R)$ be a matrix Lie group with Lie algebra $\mathfrak{g}$ and the (left-invariant) Maurer-Cartan form $\omega$ on $G$. The first result is the existence theorem:

**Theorem 2.1** ([5]). Suppose that $\phi$ is a $\mathfrak{g}$-valued one form on a simply connected manifold $M$. Then there exists a $C^\infty$-map $f : M \to G$ with $f^* \omega = \phi$ if and only if $d\phi + \phi \wedge \phi = 0$.

The second result states that the pull-back of the Maurer-Cartan form uniquely determines the map up to a group action:

**Theorem 2.2** ([5]). Given two maps $f, \tilde{f} : M \to G$, then $\tilde{f}^* \omega = f^* \omega$ if and only if $\tilde{f} = g \circ f$ for some $g \in G$.

We call the Lie algebra valued one-form $f^* \omega$ the Darboux derivative of the map $f : M \to G$.

3. Differential Invariants of Horizontally Regular Curves in $H_n$

Recall (equations (4.9)(4.10) in [2]) that any point $p \in H_n$ and element $\Phi_{p,R} \in PSH(n)$ have the corresponding representations respectively

$$p \in H_n \leftrightarrow X := \begin{pmatrix} 1 \\ p \end{pmatrix} \in \mathbb{R}^{2n+2},$$

$$\Phi_{p,R} \in PSH(n) \leftrightarrow M \in PSH(n),$$

satisfying the matrix multiplication

$$MX = \begin{pmatrix} 1 \\ \Phi_{p,R}(p) \end{pmatrix}.$$

Denote the indices

$$1 \leq a, b \leq 2n,$$

$$1 \leq j, k \leq n.$$

We also have the Maurer-Cartan form $\omega$ of $PSH(n)$ ([2], page 1104)

$$
\begin{pmatrix}
0 & 0 & 0 & 0 \\
\omega_k & \omega_k & \omega_n+k & 0 \\
\omega_n+k & \omega_n+k & \omega_n+j & 0 \\
\omega_{2n+1} & \omega_{n+j} & -\omega_j & 0
\end{pmatrix}.$$
where $\omega^k$, $\omega^j$, $\omega^{2n+1}$ are 1-forms on $PSH(n)$ satisfying $\omega^b_n = -\omega^a_n$, $\omega^{n+k}_n = -\omega^{n+j}_n$.

Let $(p; e_j, e_{n+j}, T)$ be an oriented frame at point $p \in H_n$. By identifying $PSH(n)$ with the space of all oriented frames on $H_n$,

$$M \in PSH(n) \leftrightarrow (p, e_j, e_{n+j}, T),$$

we have

$$(p, e_j, e_{n+j}, T) = (0, \dot{e}_j, \dot{e}_{n+j}, T) \cdot M,$$

where $\cdot$ denotes the matrix multiplication. Thus, one can derive the moving frame formulas (page 1105, [2])

\begin{align}
&dp = e_j \omega^j + e_{n+j} \omega^{n+j} + T \omega^{2n+1}, \\
&de_j = e_k \omega^j + e_{n+k} \omega^{n+k} + T \omega^{n+j}, \\
&de_{n+j} = e_k \omega^{n+j} + e_{n+k} \omega^{n+k} - T \omega^j, \\
&dT = 0.
\end{align}

(3.1)

Let $\gamma(s)$ be a horizontally regular curve with horizontal arc-length parameter $s$. Each point of $\gamma$ uniquely defines the oriented frame as $[\gamma]$ and we still denote the corresponding lifting $\tilde{\gamma} \in PSH(n)$ of $\gamma$ by

$$\tilde{\gamma}(s) = (\gamma(s), e_1(s), \cdots, e_n(s), e_{n+1}(s), \cdots, e_{2n}(s), T),$$

which is unique up to a $SO(2n)$ group action. Let $\omega$ be the Maurer-Cartan form of $PSH(n)$. We shall derive the Darboux derivative $\tilde{\gamma}^* \omega$ of the $\tilde{\gamma}$.

By the moving frame formulas (3.1),

$$d\gamma(s) = e_j(s)\tilde{\gamma}^* \omega^j + e_{n+j}(s)\tilde{\gamma}^* \omega^{n+j} + T \tilde{\gamma}^* \omega^{2n+1},$$

on the other hand, by the choice of the oriented frame

$$d\gamma(s) = \gamma'_{\xi}(s)ds + \gamma'_{\tau}(s)ds = e_1(s)ds + \gamma'_{\tau}(s)ds.$$ 

Comparing the components in above equations, we have

\begin{align}
\tilde{\gamma}^* \omega^1 &= ds, & \tilde{\gamma}^* \omega^\ell &= 0 & \text{for } 2 \leq \ell \leq 2n, \\
\tilde{\gamma}^* \omega^{2n+1} &= (\frac{d\gamma(s)}{ds}, T)ds = \tau(s)ds.
\end{align}

Again from (3.1), we have

\begin{align}
de_j(s) &= e_k(s)\tilde{\gamma}^* \omega^k + e_{n+k}(s)\tilde{\gamma}^* \omega_{n+k} + T \tilde{\gamma}^* \omega^{n+j} \\
&= e_k(s)\tilde{\gamma}^* \omega^k + e_{n+k}(s)\tilde{\gamma}^* \omega_{n+k}.
\end{align}

(3.3)

For $1 \leq j \leq n-1$, since $e_j(s)$ (resp. $de_j(s)$) is the linear combination of $\gamma_{\xi}^{(1)}(s), \cdots, \gamma_{\xi}^{(j)}(s)$, (resp. $\gamma_{\xi}^{(1)}(s), \cdots, \gamma_{\xi}^{(j+1)}(s)$), one has

$$\tilde{\gamma}^* \omega^j = 0, \text{ for } i > j + 1.$$ 

(3.4)

By (3.3) and the definition of p-curvatures, we also have

\begin{align}
\tilde{\gamma}^* \omega_{j+1}^{j+1} &= (\frac{d e_j(s)}{ds}, e_{j+1}(s))ds = \kappa_j(s)ds.
\end{align}

(3.5)

Similarly, for $j = n$,

\begin{align}
de_n(s) &= e_k(s)\tilde{\gamma}^* \omega^k + e_{n+k}(s)\tilde{\gamma}^* \omega_{n+k} + T \tilde{\gamma}^* \omega^{2n} \\
&= e_k(s)\tilde{\gamma}^* \omega^k + e_{n+k}(s)\tilde{\gamma}^* \omega_{n+k}.
\end{align}
In addition, since \( \omega_n^{n+k} = -\omega_k^n = \omega_n^{2n} \), \( \tilde{\gamma}^*\omega_n^{n+i} = \tilde{\gamma}^*\omega_n^{2n} = 0 \), for \( 1 \leq j \leq n - 1 \). One has

\[
(3.6) \quad \tilde{\gamma}^*\omega_n^{2n} = \left( \frac{d\omega_n(s)}{ds}, e_{2n}(s) \right) ds = \kappa_n(s) ds.
\]

By \( \{12\} \{14\} \{16\} \{18\} \{20\} \{22\} \) and use the anti-symmetric property, \( \omega_j^i = -\omega_i^j \), finally we reach the moving frame formulae for the curve \( \gamma(s) \)

\[
(3.7) \quad \begin{align*}
    d\gamma(s) &= e_1(s) ds + T\tau(s) ds, \\
    de_j(s) &= -e_{j-1}(s)\kappa_{j-1}(s) ds + e_{j+1}(s)\kappa_j(s) ds, \\
    de_n(s) &= -e_{n-1}(s)\kappa_{n-1}(s) ds + e_{2n}(s)\kappa_n(s) ds.
\end{align*}
\]

In conclusion, we obtain the Darboux derivative \( \tilde{\gamma}^*\omega \) of \( \tilde{\gamma} \)

\[
(3.8) \quad \begin{pmatrix}
0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
1 & 0 & -\kappa_1(s) & 0 & \cdots & 0 & 0 \\
0 & \kappa_1(s) & \ddots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \vdots & \vdots & -\kappa_{n-1}(s) & \ddots & \ddots & \ddots \\
0 & 0 & \cdots & 0 & \kappa_{n-1}(s) & 0 & \cdots & 0 & -\kappa_n(s) & 0 \\
0 & 0 & \cdots & 0 & 0 & -\kappa_1(s) & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \kappa_1(s) & 0 & \ddots & \ddots & \ddots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \vdots & \vdots & 0 & 0 & 0 & \cdots & \cdots & 0 & -\kappa_{n-1}(s) & \cdots \\
0 & 0 & \cdots & 0 & \kappa_n(s) & 0 & \cdots & \kappa_{n-1}(s) & 0 & 0 \\
\tau(s) & 0 & \cdots & \cdots & \cdots & \cdots & 0 & -1 & 0
\end{pmatrix} = \text{ds}.
\]

4. Proof of the main theorem

We show the proof of Theorem 1.2 in this section.

Proof. First we show the existence. Given \((n + 1)\) functions \( \kappa_i(s), 1 \leq i \leq n \), and \( \tau(s) \) defined on an open interval \( I \). Define a \( PSH(n) \)-valued one-form \( \varphi \) on \( I \) with entries \( \kappa_i \)'s, \( \tau \) as the one in \( (3.8) \). It is easy to show that \( d\varphi + \varphi \wedge \varphi = 0 \), and by Theorem 2.3 there exists a curve

\[
\tilde{\gamma}(s) = (\gamma(s); e_1(s), \cdots, e_n(s), e_{n+1}(s), \cdots, e_{2n}(s), T) \in PSH(n)
\]

such that \( \tilde{\gamma}^*\omega = \varphi \). Therefore, by the moving frame formula \( (3.7) \), we have

\[
\begin{align*}
    d\gamma(s) &= e_1(s) ds + T\tau(s) ds, \\
    de_j(s) &= -e_{j-1}(s)\kappa_{j-1}(s) ds + e_{j+1}(s)\kappa_j(s) ds, \\
    de_{n+1}(s) &= -e_{n+1}(s)\kappa_n(s) ds + e_{2n}(s)\kappa_n(s) ds, \\
    de_{n+2}(s) &= -e_{n+1}(s)\kappa_{n-1}(s) ds + e_{n+1}(s)\kappa_{n-1}(s) ds, \\
    de_{2n}(s) &= -e_n(s)\kappa_n(s) ds - e_{2n-1}(s)\kappa_n(s) ds,
\end{align*}
\]
for $1 \leq j \leq n - 1$, which implies that

$$e_1(s) = \gamma'_\xi(s),$$

$$\kappa_j(s) = \frac{de_j(s)}{ds}, e_{j+1}(s)), 1 \leq j \leq n - 1,$$

$$r_n(s) = \frac{de_n(s)}{ds}, e_{2n}(s)),$$

$$\tau(s) = \frac{d\gamma(s)}{ds}, T).$$

We have reached the proof of existence.

Next, for the uniqueness, suppose that $\gamma_1$ and $\gamma_2$ have the same $p$-curvatures $\kappa_j(s), 1 \leq j \leq n$ and the contact normality $\tau(s)$. By the moving frame formulas \[3.1\] we get

$$\tilde{\gamma}_1^* \omega = \tilde{\gamma}_2^* \omega.$$

Therefore, Theorem \[2.2\] implies that there exists an element $g \in PSH(n)$ such that $\tilde{\gamma}_2(s) = g \circ \tilde{\gamma}_1(s)$, and hence $\gamma_2(s) = g \circ \gamma_1(s)$ for all $s$. This completes the proof of uniqueness. \hfill $\square$

## 5. The degenerate case

We give the proof of Proposition 1.6.

**Proof.** Without lose of generality, we may assume order($\gamma$) = $n - 1$ and $\gamma(0) = (\beta(0), z(0)) = 0$. We observe that the second condition in \[1.9\] holds if and only if

$$\gamma'_\xi(s) \wedge \cdots \gamma^{(n-1)}_\xi(s) \wedge J\gamma_\xi(s) \wedge \cdots \wedge J\gamma^{(n-1)}_\xi(s) \neq 0.$$

At $s = 0$, we may take the orthonormal frame $e_1(0), \ldots, e_{n-1}(0)$ satisfying $e_1(0) \wedge \cdots \wedge e_k(0) = \gamma'_\xi \wedge \cdots \gamma^{(k-1)}_\xi \wedge e_k(0)$ for all $1 \leq k \leq n - 1$ such that $\text{span}_\mathbb{R}\{e_1(0), \ldots, e_{n-1}(0), Je_1(0), \ldots, Je_{n-1}(0)\} = \mathbb{C}^{n-1} \subset \mathbb{C}^n$. We also have the natural orthogonal decomposition

$$\mathbb{C}^n \ni \beta(s) = \beta'(s) + a(s)N$$

for some function $a(s)$, where $N$ is normal to $\mathbb{C}^{n-1}$ and $\beta' \in \mathbb{C}^{n-1}$. Since $\beta(0) \in \mathbb{C}^{n-1}$, we have the initial condition

$$a(0) = 0. \quad (5.1)$$

We shall claim that $a(s) = 0$ for all $s$, which implies that $\gamma(s) \in \mathbb{C}^{n-1}$. Since $\beta^{(j)}(0) \in \mathbb{C}^{n-1},$

$$a^{(j)}(0) = 0 \text{ for } 1 \leq j \leq n - 1. \quad (5.2)$$

On the other hand, by the assumption

$$0 = \beta' \wedge \cdots \wedge \beta^{(n)}$$

$$= (\beta' + a'N) \wedge \cdots \wedge (\beta^{(n)} + a^{(n)}N)$$

$$= (\beta' \wedge \cdots \wedge \beta^{(n)}) + \sum_{h=1}^{n} \beta' \wedge \cdots \wedge \beta^{(h-1)} \wedge (a^{(h)}N) \wedge \beta^{(h+1)} \wedge \cdots \wedge \beta^{(n)}$$

$$= (a'b_1 + a''b_2 + \cdots + a^{(n-1)}b_{n-1} + a^{(n)})(\beta' \wedge \cdots \wedge \beta^{(n-1)} \wedge N),$$
where we denote \( \tilde{\beta}(n) = \sum_{j=1}^{n-1} b_j \tilde{\beta}(j) \) for some smooth functions \( b_j(s) \) independent of \( a(j)'s \) and we have used the fact that the complex \( n \)-form \( \Lambda_{n=1}^{n-1} \tilde{\beta}(k) \) vanishes in \( \mathbb{C}^n \). Since the volume form \( (\Lambda_{n=1}^{n-1} \tilde{\beta}(k)) \wedge N \) is nonzero, together with (5.1), (5.2), we obtain the \( n \)-th order O.D.E. system with the initial conditions

\[
\begin{cases}
  a^{(n)}(s) + \sum_{j=1}^{n-1} b_j(s) a^{(j)}(s) = 0, \\
  a^{(j)}(0) = 0 \text{ for } j = 0, \ldots, n - 1.
\end{cases}
\]

Hence by the existence and uniqueness theorem of O.D.E. one has \( a(s) \equiv 0 \) for all \( s \), which implies \( \gamma(s) \in H_{n-1} \subset H_n \), and complete the proof. \( \square \)

Finally we calculate order \( \gamma = 1 \) if \( \gamma \) is a horizontal geodesic in \( H_n \) for any \( n \geq 1 \).

**Proof of Proposition 11.8.** In [7], the horizontal geodesic \( \gamma : I \to H_n \) satisfies the equation

\[
D_{\gamma'} \gamma'' + 2\lambda J(\gamma') = 0,
\]

for some constant \( \lambda \in R \).

Let \( \gamma(s) = (x_1(s), \ldots, x_n(s), y_1(s), \ldots, y_n(s), z(s)) \) be a horizontal geodesic with horizontal arc-length \( s \). Then, for \( 1 \leq j \leq n \), we have the following expression

\[
\begin{align*}
(5.3) \quad & x_j(s) = (x_0)_j + A_j \left( \frac{\sin(2\lambda s)}{2\lambda} \right) + B_j \left( \frac{1 - \cos(2\lambda s)}{2\lambda} \right), \\
(5.4) \quad & y_j(s) = (y_0)_j + A_j \left( \frac{1 - \cos(2\lambda s)}{2\lambda} \right) + B_j \left( \frac{\sin(2\lambda s)}{2\lambda} \right), \\
(5.5) \quad & z(s) = t_0 + \frac{1}{2\lambda} \left( s - \frac{\sin(2\lambda s)}{2\lambda} \right) + \sum_{j=1}^{n} \left\{ (A_j(x_0)_j + B_j(x_0)_j) \left( \frac{1 - \cos(2\lambda s)}{2\lambda} \right) \right. \\
& \quad \left. - (B_j(x_0)_j - A_j(x_0)_j) \left( \frac{\sin(2\lambda s)}{2\lambda} \right) \right\},
\end{align*}
\]

with the initial conditions \( x_j(0) = (x_0)_j \), \( y_j(0) = (y_0)_j \), and \( x'_j(0) = A_j \), \( y'_j(0) = B_j \) satisfying \( \sum_{j=1}^{n} (A_j^2 + B_j^2) = 1 \). By the decomposition (1.2)

\[
\gamma'(s) = (x'_1(s), \ldots, x'_n(s), y'_1(s), \ldots, y'_n(s), z'(s))
\]

\[
= \sum_{j=1}^{n} \left( x'_j(s) \frac{\partial}{\partial x_j} + y'_j(s) \frac{\partial}{\partial y_j} \right) + z'(s) \frac{\partial}{\partial z}
\]

\[
= \sum_{j=1}^{n} \left( x'_j(s) \tilde{e}_j + y'_j(s) \tilde{e}_{n+j} \right) + \sum_{j=1}^{n} \left( z'(s) + x_j y'_j - y_j x'_j \right) \frac{\partial}{\partial z},
\]

we have

\[
(5.6) \quad \gamma' = \sum_{j=1}^{n} \left( x'_j(s) \tilde{e}_j + y'_j(s) \tilde{e}_{n+j} \right),
\]

\[
\gamma' = \sum_{j=1}^{n} \left( z'(s) + x_j y'_j - y_j x'_j \right) T,
\]
where \( \frac{\partial}{\partial z} = T \). Since the geodesic is horizontal, the contact normality \( \tau(s) = 0 \).

Moreover, by (5.3), (5.4), (5.6)

\[
\gamma'_\xi(s) = \sum_{j=1}^{n} \left( (A_j \cos(2\lambda s) + B_j \sin(2\lambda s))\hat{e}_j + (-A_j \sin(2\lambda s) + B_j \cos(2\lambda s))\hat{e}_{n+j} \right),
\]

Note that \( |\gamma'_\xi(s)| = 1 \), we may take \( e_1 = \gamma'_\xi \). Taking the derivatives, we observe that

\[
\gamma''_\xi(s) = \sum_{j=1}^{n} \left( (-2\lambda A_j \sin(2\lambda s) + 2\lambda B_j \cos(2\lambda s))\hat{e}_j + (-2\lambda A_j \cos(2\lambda s) - 2\lambda B_j \sin(2\lambda s))\hat{e}_{n+j} \right),
\]

\[
\langle \gamma''_\xi(s), \gamma'_\xi(s) \rangle = 0,
\]

By Definition 1.4, we conclude that the order of geodesics is one. There is only one invariant, p-curvature, for \( \gamma \), namely,

\[
\kappa_1 = \langle \frac{de_1(s)}{ds}, e_2(s) \rangle \]

\[
= -2\lambda \sum_{j=1}^{n} \left\{ (A_j \sin(2\lambda s) - B_j \cos(2\lambda s))^2 + (A_j \cos(2\lambda s) + B_j \sin(2\lambda s))^2 \right\} \]

\[
= -2\lambda,
\]

where \( e_2 = Je_1 \).

\[ \square \]

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