Motivated by developing a field-theoretic algebraic approach to the universal part of multi-stress tensor conformal blocks of a scalar four-point function in a class of higher-dimensional CFTs, we construct a mode operator, $L_m$, near the lightcone in $d = 4$ CFTs and show that it leads to a Virasoro-like commutator, including a regularized central-term. As an example, we describe how to reproduce the $d = 4$ single-stress tensor exchange contribution in the lightcone limit by a mode summation. We comment on possible generalizations.
1. Introduction

The existence of the Virasoro symmetry lies at the heart of two-dimensional conformal field theories. Such an infinite-dimensional stress-tensor algebra dictates universal behaviors of $d = 2$ systems, allowing computable multi-point correlation functions and critical exponents [1]. With the developments of the AdS/CFT correspondence [2–4] and the revitalized conformal bootstrap program (for a review and references see, e.g., [5]), there has been significant recent progress in using $d = 2$ Virasoro conformal blocks to understand quantum entanglement [6–8], chaotic dynamics [9–11], the eigenstate thermalization hypothesis [12–14], and to describe gravitational effects in AdS$_3$ with a black hole [15–22].

In $d > 2$, the conformal group is finite-dimensional and stress tensors generally do not form an algebra. One should not expect to find a model-independent way that universally captures the full stress-tensor contribution. Is it therefore completely hopeless if one desires to generalize a similar story to $d > 2$ in a certain way?

The motivation of the present work comes from recent growing evidence [23–30] indicating that a certain universality of multi-stress tensors in a large class of $d > 2$ CFTs (including holographic CFTs) appears in the limit where operators in a correlator approach each others lightcone or, equivalently, in the lowest-twist limit. We will mainly focus on CFTs with a large central charge and a scalar four-point function with two heavy and two light scalars. The universality, more precisely, means that the operator product expansion (OPE) coefficients of the lowest-twist multi-stress tensors are “protected”, in the sense that they are fixed by dimensions of scalars and the central charge $C_T$. The higher-twist OPE coefficients, on the other hand, can be contaminated by other model-dependent parameters. From the gravity side’s viewpoint, the universality implies that these coefficients are insensitive to higher-curvature terms in the bulk gravitational action, i.e. they can be determined by Einstein gravity. In $d = 2$, such universality can be explained by the Virasoro symmetry. The recent $d > 2$ results share intriguing similarities with $d = 2$ CFTs and we are motivated to search for a possible Virasoro-like derivation in the lightcone limit in $d > 2$ CFTs; we focus on $d = 4$ in this note.

A recent effort towards this direction was made in [24]. Based on the most general stress-tensor commutators consistent with the Poincaré algebra in local QFT [31], it was shown that, under an assumption on the Schwinger term, a Virasoro-like stress-tensor commutator emerges near the lightcone in $d = 4$ CFTs. Here, we would like to start to build a bridge between the stress-tensor commutator and the conformal block decomposition of a scalar four-point function. We shall also comment on a potential relationship between the Schwinger-term assumption and the validity of the lightcone universality in $d = 4$ CFTs.
To build a bridge between the \( d = 4 \) stress-tensor commutator and the scalar four-point correlator, it is desirable to construct an effective mode operator, similar to the generator \( L_m \) in \( d = 2 \). An immediate obstacle, however, is that the \( d = 4 \) stress-tensor commutator has a UV cut-off, \( \Lambda \), dependent central-term. (\( \Lambda \) has mass dimension one.) We will propose an \( \mathcal{L}_m \), defined near the lightcone, and show that, using the \( d = 4 \) stress-tensor commutator, it results in a Virasoro-like \([\mathcal{L}_m, \mathcal{L}_n]\). The basic picture of this construction is that we treat the additional two-dimensional transverse space as a thin layer with a thickness defined by a short-distance cut-off \( \epsilon \). The product \( \epsilon^2 \Lambda^2 \) gives a dimensionless constant whose value will be fixed by the conventional OPE coefficient of the single-stress tensor exchange. Introducing a thin region, instead of infinite transverse space, may be interpreted as a lightcone limit, where we arrange scalars to live on a \( d = 2 \) plane and the stress tensors contribute only near the plane.

We will describe how to use this \( d = 4 \) lightcone commutator \([\mathcal{L}_m, \mathcal{L}_n]\) to compute the single-stress tensor exchange in the lightcone limit by a direct mode summation. This computation generalizes the Virasoro-algebra derivation of the one-graviton contribution to the identity block in \( d = 2 \) CFTs described in [15].

The more general case, beyond single-stress tensor, becomes more involved partially because the stress-tensor-scalar, \( T \mathcal{O} \), OPE in \( d = 4 \) has a delicate structure. While the general story is left to future work, we will make some preliminary remarks on a possible multi-stress tensor generalization.

2. A mode operator in \( d = 4 \)

2.1. Stress-tensor commutator near the lightcone

We start with the stress-tensor commutation relation in \( d = 4 \) CFTs in Minkowski space-time \( ds^2 = -dx^+dx^- + dy^2 + dz^2 \) where \( x^\pm = t \pm x \). Using the tracelessness condition, one can write the relevant component of the stress tensor in the lightcone limit as \( T^{++} = -2(T_0^0 - T_1^0) - T_a^a \); \( a = 2, 3 \) denote transverse directions. An important point is that the purely-spatial components of the stress tensor generally do not admit a model-independent commutator [31]. However, in the case where stress tensors are inserted in a scalar correlator, the transverse components are suppressed in the lightcone limit. (By lightcone limit, we mean that we consider 4 scalars to lie on an \( x^+ - x^- \) plane with configuration \( \langle \mathcal{O}(\infty)\mathcal{O}(1)\mathcal{O}(x^+, x^-)\mathcal{O}(0) \rangle \), and then take \( x^- \to 0 \). We also send stress tensor’s
The dominating contribution in the lightcone limit is \([24]\)
\[
-i[\tilde{T}^{++}(x^+,x^a),\tilde{T}^{++}(x'^+,x'^a)] = -4\left(\tilde{T}^{++}(x^+,x^a) + \tilde{T}^{++}(x'^+,x'^a)\right)\partial_+ \delta^3
\]
\[
+ \frac{C_\gamma \pi^2}{60} \left(\Lambda^2 + \Delta\right) \partial_+ \Delta \delta^3 ,
\]
where \(\tilde{T}^{++} = -2(T_0^0 - T_1^0)\), \(\delta^3 = \delta(x^+ - x'^+)\delta^2(x^a - x'^a)\), and \(\Delta\) is a Laplacian. Note that the central-term contains a UV cut-off \(\Lambda\)-dependent piece. We have set \(x^- \to 0\) in the above commutator. More formally, one can write \(x^- = \epsilon\) and then focus on the leading small-\(\epsilon\) contribution.

The result (1) is valid only when the Schwinger-term in the stress-tensor commutator is a \(c\)-number. That is, the central-term in (1) is assumed to be the same as the expectation value of the stress-tensor commutator. A priori, however, there might be an additional operator Schwinger-term. It remains an interesting question to ask in what class of \(d = 4\) CFTs the Schwinger-term is effectively a \(c\)-number (in the lightcone limit) as it may be related to the validity of the universality in \(d = 4\) CFTs.

In what follows, we shall simply assume that we focus on the class of \(d = 4\) CFTs where the Schwinger-term is effectively a \(c\)-number and adopt (1).

### 2.2. A mode operator and \([\mathcal{L}_m, \mathcal{L}_n]\) in \(d = 4\)

To develop a Virasoro-like effective representation theory for the class of \(d = 4\) CFTs whose lowest-twist subsector has a universal meaning, one would like to explore possible constructions of an effective mode generator, denoted as \(\mathcal{L}_m\), which is defined via integrating the coordinates of a stress tensor out. Our goal here is to find an \(\mathcal{L}_m\) such that, when combined with the stress-tensor commutator near the lightcone in \(d = 4\) CFTs, it can lead to a commutator \([\mathcal{L}_m, \mathcal{L}_n]\) which (i) satisfies the Jacobi identity and (ii) has a regularized central-term.

Since the difference between \(\tilde{T}^{++}\) and \(T^{++}\) is suppressed in the lightcone limit, as mentioned, we simply adopt \(T^{++}\) in the following to have simpler expressions.

Let us Wick rotate to a Euclidean plane, \(ds^2(E) = dx^+_{(E)}dx^-_{(E)} + dy^2 + dz^2\), with complex coordinates \(x^\pm_{(E)} \equiv x^1 \pm ix^2\). (The subscript will be dropped.) We keep the extra two-dimensional transverse directions uncompactified.\(^1\) Consider the following ansatz:

\[
\mathcal{L}^j_k = \lim_{x^- \to \epsilon} \int \int dy \ dz \ f(y,z;j,k) \int \frac{dx^+}{2\pi i} (x^+)^{m+1} T^{++}(x^+,x^-,y,z) .
\]

\(^1\)In general, one can consider other geometries such as a torus.
Changing the power of $x^+$ corresponds to shifting $m$; we adopt $m + 1$ for later convenience. The smear function $f(y, z; j, k)$ generally can depend on new mode numbers, $j, k$, associated with transverse coordinates. The integrals along the transverse directions are necessary as the stress-tensor commutator contains Dirac delta-functions; just sending $y, z$ to zero in the stress tensor does not give a sensible commutator.

The Jacobi identity severely constrains the form of $f(y, z; j, k)$. We propose

$$L_m = -\frac{\pi}{2} \lim_{L, x^+ \to \epsilon} \int_0^L \int_0^L dy \, dz \int_0^{L^+} \frac{dx^+}{2\pi i} (x^+)^{m+1} T^{++} (x^+, x^-, y, z),$$

where a short-distance scale $\epsilon$ is introduced for the transverse directions. The stress-tensor contribution therefore comes only from a very thin region near a $d = 2$ plane.

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The stress-tensor commutator (1) and (3) give, in the lightcone limit,

$$[L_m, L_n] = (m - n) L_{m+n}$$

$$+ \frac{C_T \pi^3}{480} \Lambda^2 \epsilon^2 m(m^2 - 1) \delta_{m+n,0} - \frac{C_T \pi^3}{480} \epsilon^2 m(m^2 - 1)(m - 2)(m - 3) \delta_{m+n,2} (4)$$

where the Cauchy integral theorem was used. In this notation, $\delta_{m+n,0}$ has mass dimension $-m - n$, and $\delta_{m+n,2}$ has dimension $-m - n + 2$, both with magnitude unit. Keeping an explicit $\epsilon$ for the limit of $x^-$ is irrelevant in deriving (4), but it will be useful when we later consider a scalar correlator with a stress tensor inserted.

Here we consider that the large UV cut-off term suppresses the last piece of (4), which causes tension with the Jacobi identity. The product of the UV cut-off $\Lambda$ and the short-distance regulator $\epsilon$ provides a dimensionless parameter. This constant will be fixed (see below) by the conventional single-stress tensor OPE coefficient that contains $C_T$ defined via the stress-tensor two-point function, $\langle T_{\mu\nu}(x) T_{\lambda\rho}(0) \rangle = C_T \frac{I_{\mu\nu,\lambda\rho}}{x^8}$. (See [32] for related notations.) We arrive at an effective lightcone commutator:

$$[L_m, L_n] = (m - n) L_{m+n} + \alpha C_T m(m^2 - 1) \delta_{m+n,0}$$

with $\alpha = \frac{56}{405 \pi^2}$ in our convention.\(^4\)

\(^2\)One may adopt asymmetric limits, $x^- \to \epsilon, L \to b \epsilon$, but $b$ can be absorbed into the overall undetermined coefficient of the central-term discussed below. We use the conventional single-stress tensor OPE coefficient to fix the undetermined constant.

\(^3\)The radial ordering is implicit. In the lightcone limit, $\partial_+ T^{++}$ is also suppressed. In $d = 2$, the central-term of the stress-tensor commutator is finite. In terms of the generator $L_m$, the $d = 2$ stress-tensor commutator leads to the well-known Virasoro algebra of the form $[L_m, L_n]$.

\(^4\)If one first redefines $\Lambda \to \tilde{\Lambda}$ in (1) to absorb $C_T$, one needs to reintroduce $C_T$ via $\tilde{\Lambda}^2 \epsilon^2 \sim C_T$. This process looks ad hoc and we do not adopt here. However, it is interesting to note that a similar identification appears in the soft-theorem related literature: see (147) in [33]. There, it is argued that the central charge can be related to internal soft exchanges. I thank L. Fitzpatrick for a discussion.
It should be emphasized that unlike in $d = 2$ CFTs where the Virasoro algebra represents an exact symmetry, (5) is an effective description. In (5), we have ignored contributions suppressed in the lightcone limit with a large UV cut-off, and we have assumed a class of $d = 4$ CFTs with a $c$-number Schwinger-term.

3. A Virasoro-like derivation of $d = 4$ single-stress tensor exchange

The effective lightcone algebra (5) looks formally the same as the $d = 2$ Virasoro algebra. We may assume that, in a universal class of $d = 4$ CFTs, there exists a lowest-twist subsector where the associated intermediate states, $|\alpha\rangle_s$ ($s$ denotes a subspace), can be effectively organized into a Virasoro-like representation theory. In some sense, the lightcone limit acts like picking the holomorphic sector out and we do not need to introduce “$\bar{\mathcal{L}}_m$”.

Focusing on such a subspace, we may try to follow the terminology of the highest-weight representation in $d = 2$ CFTs: $L_0|\lambda\rangle = \lambda|\lambda\rangle$ and $L_m|\lambda\rangle = 0$ for $m \geq 1$. The modes $L_m$ with $m < 0$ generate descendants. The vacuum $|0\rangle$, preserving the maximal numbers of symmetries, is the associated state of the identity operator that has $\lambda = 0$. One important difference, however, is that $T\bar{\mathcal{O}}$ OPE structure in $d = 4$ is more delicate. Using the $T\bar{\mathcal{O}}$ OPE to express $L_m$ as a general differential operator will not be included in the present note. Here, we focus on a Virasoro-like derivation of the single-stress tensor exchange in the lightcone limit in $d = 4$. This derivation does not require knowing $L_m$ as a general differential operator because we can use the three-point function $\langle T\bar{\mathcal{O}}\rangle$ together with (5) and (3). The three-point function of the stress tensor with two scalar primaries in $d = 4$ is [32]

$$\langle T^{\mu\nu}(x_1)\mathcal{O}_\Delta(x_2)\mathcal{O}_\Delta(x_3)\rangle = \frac{c_{T\bar{\mathcal{O}}}}{x_{12}^4 x_{13}^4 x_{23}^{2\Delta-4}} \left(\frac{X^\mu X^\nu}{X^2} - \frac{\delta^{\mu\nu}}{4}\right), \quad X^\mu = \frac{x_{12}^\mu}{x_{12}^2} - \frac{x_{13}^\mu}{x_{13}^2}$$

with $c_{T\bar{\mathcal{O}}} = -\frac{2\Delta}{3\pi^2}$.

We will focus on the identity block in $d = 4$ CFTs at large $C_T$ with the heavy-light limit: $\Delta_H \sim C_T$, $\Delta_L \sim \mathcal{O}(1)$. The single-stress tensor exchange contribution, discussed below, may be computed without explicitly imposing these limits, but we will still formally adopt $\Delta_H$ and $\Delta_L$ in what follows, having in mind a potential generalization involving multi-stress tensors.

In the lightcone limit, we assume that the corresponding intermediate states can be

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5For explicit expressions at the first few orders in the OPE, see, for instance, Appendix B of [34].
effectively generated by the operator $L_m$ acting on the vacuum. Introduce a basis
\[ |a_0\rangle_T = \frac{L_m^+ |0\rangle}{\sqrt{N_m}}, \quad N_m = \langle L_m L_m^+ \rangle, \tag{7} \]
which formally represents a normalized one-graviton state. Assume $m > 1$ here. We may relate the single-stress tensor contribution to the conformal block in the lightcone limit to the following object:
\[ V_T = \lim_{\bar{z} \to 0} \sum_{m=2}^{\infty} \frac{\langle O_H(\infty) O_H(1) L_m^+ \rangle \langle L_m O_L(z, \bar{z}) O_L(0) \rangle}{N_m \langle O_H(\infty) O_H(1) \rangle \langle O_L(z, \bar{z}) O_L(0) \rangle}. \tag{8} \]
We have switched to the conventional variables $z, \bar{z}$ defined by $u = z \bar{z}, v = (1 - z)(1 - \bar{z})$ where $u, v$ are conformal cross-ratios. In the configuration (8), $x^+ = z, x^- = \bar{z}$ for the scalar and the scalar’s lightcone limit is $\bar{z} \to 0$. A non-trivial role is the normalization factor, $N_m$, as it will be determined by the central-term of the $L_m$ commutator. On the other hand, recall $L_m^+ = L_{-m}$ in $d = 2$; here we shall look for a similar relation, $\tilde{L}_m^+ \sim L_{-m}$.

The Hermitian conjugate of the $d = 4$ stress tensor in the radial quantization is
\[ T^\mu(x)^\dagger = T^{\mu}_{\lambda}(x) T^{\nu}_{\rho}(x) x^{-8} T^{\lambda \rho}(\frac{x}{x^2}) \quad \text{and} \quad T^\mu(x) = \delta^\mu_{\lambda} - 2 \frac{x^\mu x_{\lambda}}{x^2}. \tag{9} \]
Let us first compute the numerator of (8) using (9), (6), and (3). The computation is short but can be thorny as it involves a certain order of limits.

Denote $y_T, z_T$ as the transverse coordinates for $T^{++}$ and $r^2 = y_T^2 + z_T^2$. We have
\[ \frac{\langle L_m O_L(z, \bar{z}) O_L(0) \rangle}{\langle O_L(z, \bar{z}) O_L(0) \rangle} = -\frac{\pi}{2} \lim_{x \to \epsilon} \int_0^L \int_0^L dy_T dz_T \frac{\langle \delta_{\frac{dx^+}{2\pi i}} (x^+)^{m+1} T^{++}(x^+, x^-, y_T, z_T) O_L(z, \bar{z}) O_L(0) \rangle}{\langle O_L(z, \bar{z}) O_L(0) \rangle} = \Delta_L \lim_{x \to \epsilon} \int_0^L \int_0^L dy_T dz_T \left[ r^4 (z - \frac{y_T^2}{x^-})^{m-1} \right. \]
\[ \times \frac{(m - 2)(m - 3)(x^-)^2 z^2 + 6(m - 3)r^2 x^- z + 12r^4}{6\pi (x^-)^8 z^2} \bigg] + O(\bar{z}^2), \tag{10} \]
in a small $\bar{z}$ expansion. In performing the contour integral, we have picked the pole due to $O_L(z, \bar{z})$. We should consider the stress tensor’s lightcone limit as $x^- \to \epsilon$, instead of directly setting $x^- = 0$ from the start in this correlator computation. After performing the remaining integrals,
\[ \frac{\langle L_m O_L(z, \bar{z}) O_L(0) \rangle}{\langle O_L(z, \bar{z}) O_L(0) \rangle} = \frac{14\Delta_L (m - 2)(m - 3)}{135\pi} z^{m-1} \bar{z} + \text{subleading}. \tag{11} \]
A similar procedure, using (9), gives the following leading non-zero contribution:
\[ \frac{\langle O_H(\infty) O_H(1) L_m^+ \rangle}{\langle O_H(\infty) O_H(1) \rangle} = \frac{\Delta_H (m - 2)(m - 3)}{6\pi} \epsilon + \text{subleading}. \tag{12} \]
From the above explicit computations (11) and (12), we see that the leading contributions vanish at $m = 2$ and $m = 3$. We interpret that the vacuum is annihilated effectively by the operators $L_2^\dagger$ and $L_3^\dagger$. By effective, we mean $L_m$ is in a correlator with the lightcone limit imposed. Note we are interested in developing an effective representation theory in $d = 4$ CFTs where stress tensors are in a scalar correlator; conditions $L_2^\dagger|0\rangle = L_3^\dagger|0\rangle = 0$ without referring to the lightcone limit can be too strong. In $d = 2$, the corresponding (11) and (12) both give a factor of $(m - 1)$, but the lightcone limit is not necessary.

To define a normalizable basis with the normalization factor $N_m$, we shall use the commutator (5). We may first assume, in the lightcone limit, a formal relation $L_m^\dagger = f(m) L_{-m}$ with $m > 1$ where $L_{-m}$ does not annihilate the vacuum. One can compute

$$\frac{\langle \mathcal{O}_H(∞) \mathcal{O}_H(1) L_{-m} \rangle}{\langle \mathcal{O}_H(∞) \mathcal{O}_H(1) \rangle} = \frac{\Delta_H}{3\pi} \epsilon^2 + \text{subleading}. \quad (13)$$

The vanishing results obtained using $L_m^\dagger$ at $m = 2, 3$ then require $f(m) \sim (m - 2)(m - 3)$. However, as this ratio formally appears in both the numerator and the denominator of (8) it is irrelevant in computing the $d = 4$ scalar correlator. Thus, effectively, we write

$$\mathcal{V}_T = \lim_{\bar{z} \to 0} \sum_{m=4}^{\infty} \frac{\langle \mathcal{O}_H(∞) \mathcal{O}_H(1) L_{-m} \rangle \langle L_m \mathcal{O}_L(z, \bar{z}) \mathcal{O}_L(0) \rangle}{\langle L_m L_{-m} \rangle \langle \mathcal{O}_H(∞) \mathcal{O}_H(1) \rangle \langle \mathcal{O}_L(z, \bar{z}) \mathcal{O}_L(0) \rangle}. \quad (14)$$

where

$$\langle L_m L_{-m} \rangle = \frac{56}{405\pi^2} C_T m(m^2 - 1). \quad (15)$$

To have a non-vanishing final result in the limit $\epsilon \to 0$, we perform an overall rescaling

$$\tilde{\mathcal{V}}_T = \frac{1}{\epsilon^2} \mathcal{V}_T. \quad (16)$$

The $\epsilon^2$ factor can be related to the volume of the transverse space. We obtain, in the lightcone limit, $(\kappa_T = \frac{1}{4} \frac{\Delta_H \Delta_L}{C_T})$

$$\tilde{\mathcal{V}}_T = \kappa_T \sum_{m=4}^{\infty} \frac{(m - 2)(m - 3)}{m(m^2 - 1)} z^{m-1} \bar{z}$$

$$= \kappa_T \frac{3(z - 2)z - (6 + (z - 6)z) \ln(1 - z)}{z^2} \bar{z} = \frac{\Delta_H \Delta_L}{120 C_T} z^3 \, _1F_2(3, 3, 6, z) \bar{z}, \quad (17)$$

which is the single-stress tensor block in the lightcone limit in $d = 4$ CFTs [35].
4. Concluding remarks and outlook

We have described an alternative derivation of the single-stress tensor block in $d = 4$ CFTs using a Virasoro-like approach. The effective mode generator $L_m$ is defined by integrating the stress tensor near a $d = 2$ plane where scalars live. This picture suggests that one may deal with the UV divergence in the stress-tensor commutator via the finite product $\Lambda^2 k^2$. It would be interesting to see if there is another way of renormalizing the central-term of the stress-tensor commutator.

In a recent work [28], an ansatz has been proposed for the multi-stress tensor sector of the heavy-light scalar correlator in the lightcone limit. Assuming such an ansatz, the resulting OPE coefficients agree with the earlier holographic computation [23]. The proposed near-lightcone ansatz can be expressed as a sum of products of hypergeometric functions, which is quite similar to the $d = 2$ Virasoro vacuum block. It would be interesting to derive such a general pattern involving multi-stress tensors in $d = 4$ CFTs based on a Virasoro-like approach. Being optimistic, the fact that we are able to reproduce the $d = 4$ single-stress tensor block structure based on the operator $L_m$ perhaps hints that such an algebraic derivation for the more general case exists. It will be of interest to further develop an effective representation theory near the lightcone for this universal class of $d = 4$ CFTs.

A potentially important step, which we have not considered in the present note, is to link $L_m$ to a general differential operator acting on $\mathcal{O}$. In [34], using the $T\mathcal{O}$ OPE, the authors show how to recast the $d = 4$ averaged null energy (ANEC) operator as a differential operator, given as a series expansion and then resum. (See also [36–38] for recent related discussions.) Note that the ANEC operator can be related to $L_{-1}$, after integrating the transverse coordinates out. Considering a more general computation to obtain a differential form of $L_m$ in $d = 4$ CFTs with a large central charge can be useful.

A differential form of the operator $L_m$ should in principle allow one to compute the following more general object:

$$V_{Tk} = \lim_{\bar{z} \to 0} \frac{\langle O_H(\infty)O_H(1)\mathcal{P}^{(k)}O_L(z,\bar{z})O_L(0)\rangle}{\langle O_H(\infty)O_H(1)\rangle\langle O_L(z,\bar{z})O_L(0)\rangle},$$

with the $k$-stress tensors lightcone effective projector $\mathcal{P}$ given by

$$\mathcal{P}^{(k)} = \sum_{\{m_i,k_i\}} \frac{L_{m_1}^{k_1} \ldots L_{m_n}^{k_n} \langle 0 \mid L_{m_n}^{k_n} \ldots L_{m_1}^{k_1} \rangle}{N_{\{m_i,k_i\}}}.$$

A direct $k > 1$ computation is more complicated, but we expect that, similar to the $d = 2$ case, the computation can be simplified in the geodesic limit, $\Delta_L \to \infty$, leading to a
possible exponentiation in the lightcone limit. Moreover, it might be possible to derive certain near-lightcone null-state equations for this universal class of $d = 4$ CFTs via an algebraic approach. We hope to discuss these possibilities somewhere else.

It would be also interesting to explore other possible constructions of the lightcone operator $L_m$ that leads to an effective algebra and compare to the form proposed here.

Let us end by mentioning another general question that has not been addressed: what precisely is the validity of the lightcone universality in $d = 4$ CFTs?

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