Maximizing robustness of point-set registration by leveraging non-convexity

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Abstract

Point-set registration is a classical image processing problem that looks for the optimal transformation between two sets of points. In this work, we analyze the impact of outliers when finding the optimal rotation between two point clouds. The presence of outliers motivates the use of least unsquared deviation, which is a non-smooth minimization problem over non-convex domain. We compare approaches based on non-convex optimization over special orthogonal group and convex relaxations. We show that if the fraction of outliers is larger than a certain threshold, any naive convex relaxation fails to recover the ground truth rotation regardless of the sample size and dimension. In contrast, minimizing the least unsquared deviation directly over the special orthogonal group exactly recovers the ground truth rotation for any level of corruption as long as the sample size is large enough. These theoretical findings are supported by numerical simulations.

Keywords: Least unsquared deviation, Robust point-set registration, Wahba’s problem, Special orthogonal group, Non-convex optimization

1 Introduction

In this paper, we study the problem of aligning two point clouds where one of the point clouds is subjected to gross corruption. More precisely, given a ground truth rotation $R_0 \in SO(d)$ and a set of indices $C \subseteq \{1, \ldots, N\}$, we model the two point clouds $\{x_i\}_{i=1}^N, \{y_i\}_{i=1}^N \subset \mathbb{R}^d$, where

$$x_i \sim \text{Unif}(S^{d-1}) \text{ for } 1 \leq i \leq N$$

independently, and

$$\begin{cases} y_i = R_0 x_i & \text{for } i \in C^c \\ y_i \sim \text{Unif}(S^{d-1}) & \text{for } i \in C \text{ independently.} \end{cases}$$

Here $\text{Unif}(S^{d-1})$ denotes uniform distribution over $S^{d-1}$, the unit sphere in dimension $d$, $C^c$ denotes the complement of $C$, $C^c := \{1, \ldots, N\} \setminus C$, and $SO(d)$ denotes the special orthogonal group

$$SO(d) = \{ R \in \mathbb{R}^{d \times d} \mid R^T R = I_d, \det R = 1 \}.$$
The model (1.2) implies the points $x_i$ for $i \in \mathcal{C}'$ can be aligned with $y_i$ from the same index set via applying some ground truth rotation $R_0$, while for $i \in \mathcal{C}$, $y_i$ and $x_i$ are generated independent of each other (hence cannot be aligned). Therefore $\mathcal{C}$ is the the index set of corrupted points, and we denote the corruption level as

$$p := \frac{|\mathcal{C}|}{N}.$$  \hfill (1.4)

When $p = 1$, the points in $\{y_i\}_{i=1}^N$ are all corrupted, while there is no corruption when $p = 0$. The goal of point-set registration is to recover the ground truth rotation $R_0$ given the point clouds $\{x_i\}_{i=1}^N, \{y_i\}_{i=1}^N$ when the index set of corrupted points, $\mathcal{C}$, is unknown, and the corruption level $p$ can take any value in $[0, 1)$.

In order to limit the influence of outliers when determining the rotation, we minimize the least unsquared deviation (LUD) [28] defined to be

$$L(A; \{x_i\}_{i=1}^N, \{y_i\}_{i=1}^N) := \frac{1}{N} \sum_{i=1}^N \|Ax_i - y_i\|_2, \quad A \in \mathbb{R}^{n \times n}. \hfill (1.5)$$

Therefore it is natural to determine the ground truth rotation $R_0$ via minimizing the LUD.

**Problem 1.1.** minimize $L(R; \{x_i\}_{i=1}^N, \{y_i\}_{i=1}^N)$ such that $R \in \text{SO}(d)$.

This problem is however, non-convex, due to the domain $\text{SO}(d)$.

### 1.1 Previous approaches

In $\mathbb{R}^3$, the point-registration problem without outliers can be formulated as the Wahba’s problem [27],[9], where the rotation $R_0$ is recovered via solving the least squares (LS)

$$\min_{R \in \text{SO}(d)} LS(R); \quad LS(R; \{x_i\}_{i=1}^N, \{y_i\}_{i=1}^N) := \frac{1}{N} \sum_{i=1}^N \|Rx_i - y_i\|_2^2. \hfill (1.6)$$

The Wahba’s problem is equivalent to the orthogonal Procrustes problem [22], with the additional constraint that the determinant of the solution equals to +1. The solution of (1.6) can be computed using the singular value decomposition (SVD) [8],[18]. Given that the solution of (1.6) is sensitive to outliers, in computer vision (1.6) is reformulated as a Maximum Consensus problem. The most common algorithm to solve the maximum consensus problem is random sample consensus (RANSAC) [11]. A survey of approximate and exact algorithms to solve the maximum consensus problem can be found in [6].

Another approach to deal with outliers is to change the loss function in (1.6) by Huber loss [25], LUD [28] or truncated-least-squares deviation [29]. The LUD has been proven to be more robust to outliers than (1.6) specifically in the context of robust registration [28], camera location recovery [17] and robust subspace recovery [19]. Observing the LUD is a convex function, one can obtain a convex problem from Problem 1.1 via applying a convex relaxation to the domain $\text{SO}(d)$. Several different kinds of relaxation can be applied. As a baseline, one can consider solving the unconstrained problem:

**Problem 1.2.** minimize $L(A; \{x_i\}_{i=1}^N, \{y_i\}_{i=1}^N)$ such that $A \in \mathbb{R}^{d \times d}$.

Compared with Problem 1.1 here the rotation group constraint is removed. Further improvement can be obtained if we solve:
Problem 1.3. minimize $L(A; \{x_i\}_{i=1}^N, \{y_i\}_{i=1}^N)$ such that $A \in \text{conv} \ SO(d)$.

Here the domain is relaxed to the convex hull of $SO(d)$, $\text{conv} \ SO(d)$, which admits characterization via positive semidefinite matrices of with size $2^{d-1} \times 2^{d-1}$ (exponential in $d$) [21]. This approach is used in [16] when optimizing over the convex hull of the special Euclidean group for two and three dimensions. However, this approach assumes that the proportion of outliers is rather small. Similarly, in [29], a tight semidefinite program relaxation using quaternions is proposed to minimize the truncated least squares deviation in $d = 3$. The dimension of the domain of the semidefinite relaxation depends quadratically in the sample size $N$.

On the other hand, recent years have seen many instances of non-convex optimization problems that have a rather benign optimization landscape. In these examples, either all critical points are saddle points and global optima, or the basin of convergence is large [24]. Therefore, a direct minimization via first order method, sometimes with the help of a cheap initialization, can solve the non-convex problem with optimality guarantees. Such benign behavior has been seen in a related problem of robust principal component analysis [17, 19]. As for our problem, in [25], an iterative reweighted least squares is proposed to solve Problem 1.1 with rather encouraging results. Thus we ask the natural question of whether Problem 1.1 admits a benign optimization landscape that allows the use of cheap first-order method instead of a convex relaxation with exponential complexity in $d$.

1.2 Our contributions

The contributions of this paper are two-fold:

- Although convex relaxation yields a surrogate optimization problem where the global optima can always be achieved, when $p$ is large it is possible that the solutions of Problem 1.2 and Problem 1.3 do not coincide with $R_0$. In this paper, we prove when $p$ is sufficiently large ($p \approx 0.6$), optimizing LUD over any convex set that contains $SO(d)$ does not recover the ground truth rotation $R_0$.

- Motivated by such observation, we solve Problem 1.1 using non-convex optimization to explicitly constrain the solution to be in $SO(d)$. We prove that by minimizing LUD in $SO(d)$ starting at almost any initialization point, one can always recover $R_0$ for any $p < 1$, as long as the sample size $N$ is sufficiently large. This is yet another example where the use of non-convex optimization is superior than convex relaxation approaches.

A summary of our contributions can be found in Fig. 1, where we compare the error for minimizing LS in (1.6) and solving Problem 1.1 and Problem 1.3 for dimension $d = 6$ and sample size $N = 1024$. Details of numerical implementation are provided in Section 6. Although it can be shown that minimizing LS asymptotically ensures $A^* \to R_0$ as $N \to \infty$, for finite sample size LS is sensitive to outliers. In Fig. 1, the empirical error between the ground truth rotation and the solution of LS is larger than $10^{-2}$, even when the corruption level is as low as $p = 0.1$. On the contrary, minimizing LUD over either $SO(6)$ or $\text{conv} \ SO(6)$ exactly recovers the ground truth when the corruption level is small. The maximum admissible corruption level to recover the ground truth is 0.6 if we optimize LUD over $\text{conv} \ SO(6)$, and 0.95 if we optimize over $SO(6)$. More interestingly, these thresholds seem to be independent of the dimension.

1.3 Summary of results

All universal constants are denoted by the notation $c$, unless in situations where we explicitly distinguish between different constants. Moreover we say an event happens with high probability,
Figure 1: Error for different methods. $A^*$ denotes the output of certain minimization procedure. Red: Solving Problem 1.1 Blue: Solving Problem 1.3 Brown: Solving the least-squares problem in (1.6).

when the probability of the event happening is larger than $1 - c/N$ for some universal constant $c$.

We start by characterizing the conditions under which the solution of convex relaxations of Problem 1.1 recovers the ground truth rotation. In particular we focus on convex relaxations of the optimization domain $\text{SO}(d)$, $\mathbb{R}^{d \times d}$, and $\text{conv SO}(d)$, i.e. Problem 1.2 and Problem 1.3 respectively.

We define the admissible corruption threshold

$$\tilde{p}(d) := \left(1 + \frac{B(d - 1, \frac{1}{2})}{B(d - 1, \frac{1}{2})} \right)^{-1}, \quad (1.7)$$

where $B(m, n)$ is the beta function $B(m, n) := 2 \int_0^{\pi/2} \sin^{2m-1} \theta \cos^{2n-1} \theta d\theta$. $\tilde{p}(d)$ is a decreasing function of $d$, such that $\tilde{p}(3) = 0.6$ and $\tilde{p}(d) \rightarrow (1 + 1/\sqrt{2})^{-1} \approx 0.5858$ as $d \rightarrow \infty$. We first establish when it is possible to recover the ground truth rotation via the baseline approach Problem 1.2 in Theorem 1.1.

**Theorem 1.1 (Exact recovery using unconstrained optimization, Problem 1.2).** Given data $\{x_i\}_{i=1}^N$, $\{y_i\}_{i=1}^N$ sampled from the distribution (1.1) and (1.2), if the corruption level

$$p < \tilde{p}(d) - O\left(d \sqrt{\frac{\log N}{N}}\right), \quad (1.8)$$

then, the ground truth rotation $R_0$ is the unique minimizer of $L(A; \{x_i\}_{i=1}^N, \{y_i\}_{i=1}^N)$ over $\mathbb{R}^{d \times d}$, with high probability.

One may wonder whether we can expect recovery for a larger corruption level $p$ than the bound in (1.8) using a tighter relaxation than Problem 1.2 i.e. Problem 1.3. We show in Theorem 1.2 that indeed the condition (1.8) is necessary, for the recovery of ground truth rotation via any convex relaxation.

**Theorem 1.2 (Recovery failure optimizing over conv $\text{SO}(d)$, Problem 1.3).** Given data $\{x_i\}_{i=1}^N$, $\{y_i\}_{i=1}^N$ sampled from the distribution (1.1) and (1.2), if the corruption level

$$p > \tilde{p}(d) + O\left(\sqrt{\frac{d \log N}{N}}\right), \quad (1.9)$$

then, the ground truth rotation $R_0$ is not the unique minimizer of $L(A; \{x_i\}_{i=1}^N, \{y_i\}_{i=1}^N)$ over $\mathbb{R}^{d \times d}$, with high probability.
then high probability the ground truth rotation \( R_0 \) is not a minimizer of \( L(A; \{x_i\}_{i=1}^N, \{y_i\}_{i=1}^N) \) over \( \text{conv} \ SO(d) \). Moreover, for any convex set \( \mathcal{P} \supseteq SO(d) \), let \( A_p^* \) be a minimizer of \( L(A; \{x_i\}_{i=1}^N, \{y_i\}_{i=1}^N) \) over \( \mathcal{P} \), then

\[
\| A_p^* - R_0 \|_F > O(\sqrt{p - \tilde{p}(d)}). \tag{1.10}
\]

Finally, we characterize the efficacy of solving Problem 1.1 for recovering the ground truth \( R_0 \). To this end, we identify the conditions under which a gradient descent on the manifold \( SO(d) \) converges to \( R_0 \). More precisely, we show that a dynamical system on \( SO(d) \) following a minimizing gradient flow of the LUD cost converges in finite time to the ground truth rotation given some mild assumptions on the initial point.

**Theorem 1.3** (Exact recovery in finite time optimizing over \( SO(d) \)) Let \( \{x_i\}_{i=1}^N, \{y_i\}_{i=1}^N \) be sampled from the distribution in (1.11) and (1.2), \( \partial_R L(R) \) be the Riemannian generalized gradient of \( L(R; \{x_i\}_{i=1}^N, \{y_i\}_{i=1}^N) \) at \( R \in SO(d) \). With high probability, all solutions of the dynamical system

\[
\frac{dR}{dt}(t) \in -\partial_R L(R), \tag{1.11}
\]

with initial condition \( R(0) \) such that

\[
\| \log(R_0^T R(0)) \|_2 < \pi - O \left( \left( \frac{d^3}{(1 - p)^2} \frac{\log(N)}{N} \right)^{1/4} \right) \tag{1.12}
\]

converge to \( R_0 \) in finite time, i.e. \( R(t) = R_0 \) for all \( t \geq T(\| \log(R_0^T R(0)) \|_2) \). Here

\[
T(s) := \frac{d}{1 - p} \left( \cosh^{-1} \left( \sec \left( \frac{s}{2} \right) \right) + c \left( \frac{d^3}{(1 - p)^2} \frac{\log(N)}{N} \right)^{1/4} \right), \quad s \in [0, \pi). \tag{1.13}
\]

In Theorem 1.3, the dynamical system (1.11) is a generalization of the differential equation when the gradient is nonsmooth. Additionally, condition (1.12) is equivalent to

\[
p < 1 - O \left( \frac{d}{(\pi - \| \log(R_0^T R(0)) \|_2)} \sqrt{\frac{d \log(N)}{N}} \right). \tag{1.14}
\]

Therefore, the maximum corruption ratio to ensure exact recovery solving Problem 1.1 in finite time goes to one as the sample size goes to infinity.

In the rest of the paper where there is no ambiguity, we simply write \( L(M; \{x_i\}_{i=1}^N, \{y_i\}_{i=1}^N) \) as \( L(M) \), though it should be understood that \( L(M) \) depends on the random variables \( \{x_i\}_{i=1}^N, \{y_i\}_{i=1}^N \). Because of the rotation invariance of the data uniformly distributed on \( S^{d-1} \), without loss of generality, we also fix the ground truth rotation \( R_0 = I \).

### 1.4 Organization

In Section 2, we present basic notations, definitions and theorems required for the rest of the paper. In Section 3, we study the properties of Problem 1.2 and Problem 1.3 and prove Theorem 1.1 and Theorem 1.2. In Section 4, we tailor some results in dynamical system theory in preparation for proving Theorem 1.3. In Section 5, we show that with a proper initialization, there is always exact recovery for any \( p < 1 \) when considering the noise model (1.2) therefore proving Theorem 1.3. In Section 6, we support our theoretical findings via numerical simulations.
2 Preliminaries

In this section, we introduce some background in concentration inequalities, dynamical systems, Riemannian manifolds, and ε-nets. Additional background in random variables uniformly distributed on $\mathbb{S}^{d-1}$ is given in Appendix B.

2.1 Concentration inequalities

Before introducing the concentration inequalities used in this paper, we first introduce the sub-gaussian norm of a random variable $X$:

$$\|X\|_{\psi_2} := \inf_t \left\{ t \mid \mathbb{E} \exp \left( \frac{X^2}{t^2} \right) \leq 2 \right\}. \quad (2.1)$$

Intuitively, this norm measures the spread of the distribution of a random variable $X$. If the distribution of $X$ has a large tail, then $\|X\|_{\psi_2}$ has to be large to ensure $\mathbb{E} \exp (X^2/\|X\|_{\psi_2}^2) \leq 2$. Indeed, for a Gaussian variable $X$ with variance $\sigma^2$, $\|X\|_{\psi_2} = c\sigma^2$ for some constant $c > 0$.

If $\|X\|_{\psi_2}$ is bounded, we say that $X$ is a sub-gaussian random variable. In addition if $\mathbb{E}X = 0$, $X$ being sub-gaussian is equivalent to

$$\mathbb{E} \exp (\lambda X) \leq \exp(c\lambda^2\|X\|_{\psi_2}^2) \quad (2.2)$$

for some universal constant $c > 0$.

We next introduce Talagrand’s inequality. First we need a few definitions. We use the radius and gaussian width to measure the size of a set $T \subset \mathbb{R}^d$:

$$\text{rad}(T) = \sup_{t \in T} \|t\|_2,$$
$$w(T) = \mathbb{E} \sup_{t \in T} \langle g, t \rangle,$$

(2.4)

where $g \in \mathbb{R}^n$ is a random vector with $\mathcal{N}(0,1)$ independently distributed entries. With these definitions, we introduce the following theorem.

**Theorem 2.1** (Hoeffding’s inequality, [26]). Let $X_1, \ldots, X_M$ be independent sub-gaussian random variables. Then for every $t \geq 0$

$$P \left\{ \left| \sum_{i=1}^M X_i - \mathbb{E}X_i \right| \geq t \right\} \leq \exp \left( \frac{-ct^2}{\sum_{i=1}^M \|X_i\|_{\psi_2}^2} \right) \quad (2.3)$$

for some universal constant $c > 0$.

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where $g \in \mathbb{R}^n$ is a random vector with $\mathcal{N}(0,1)$ independently distributed entries. With these definitions, we introduce the following theorem.

**Theorem 2.2** (Talagrand’s comparison inequality in tail bound form, [26]). Let $T \subset \mathbb{R}^n$ be a set and $\{X_t\}_{t \in T}$ be a random process indexed by elements in $T$, such that $X_0 = 0$. If for all $t, s \in T \cup \{0\}$

$$\|X_t - X_s\|_{\psi_2} \leq K\|t - s\|_2,$$

(2.5)

then for some universal constant $c > 0$,

$$P \left\{ \sup_{t \in T} |X_t| \leq cK(w(T) + \text{rad}(T) u) \right\} \geq 1 - 2\exp(-u^2). \quad (2.6)$$
2.2 Discontinuous dynamical systems

The differential equation (1.11) in Theorem 1.3 is called a differential inclusion, a term that generalizes dynamical systems when the forcing term is a set of functions. An introductory summary to the area can be found in [7], as well as classical books [3] and [10]. In Section 4, we derive finite time convergence for a particular set of differential inclusions in $SO(d)$. This results are based in generalizations of Lyapunov functions for differential inclusions in [4]. Numerical methods to solve differential inclusions can be found in [2].

Definition 2.1 (Differential inclusion, [2]). A differential inclusion is defined by

$$\frac{dz}{dt} \in \mathcal{F}(z(t)), \quad t \in [0,T], \quad z(0) = z_0,$$

where $z : \mathbb{R} \to \mathcal{D} \subseteq \mathbb{R}^n$ is a function of time, $\mathcal{B}(\mathbb{R}^n)$ is the collection of subsets of $\mathbb{R}^d$, $\mathcal{F} : \mathcal{D} \to \mathcal{B}(\mathbb{R}^n)$ is a set-valued map which associates to any point $z \in \mathcal{D}$ with a set $\mathcal{F}(z) \subset \mathbb{R}^n$ and $T > 0$.

Definition 2.2 (Solution of differential inclusion). A solution $z(t)$ of the differential inclusion (2.7) is absolutely continuous, i.e. it satisfies

$$z(t) = z(0) + \int_0^t \frac{dz(s)}{ds} ds$$

such that $\frac{dz}{dt}(t) \in \mathcal{F}(z(t))$ almost everywhere.

Definition 2.3 (Generalized gradient, [4]). Let $f : \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz function, then the generalized gradient of $f$ at $z$ is the set

$$\bar{\partial}f(z) := \text{conv}\left\{ \lim_{j \to +\infty} \nabla f(z_j) \mid z_j \to z, z_j \notin G, z_j \notin G_f \right\}$$

where $G$ is any set of zero measure in $\mathbb{R}^n$ and $G_f$ is the set of measure zero where the Euclidean gradient $\nabla f$ does not exists.

2.3 $SO(d)$ as a Riemannian manifold

$SO(d)$ can be seen as an embedded submanifold of $\mathbb{R}^{d \times d}$, or as a matrix Lie group. We follow the manifold optimization approach in [1] to define Riemannian gradient over $SO(d)$. Then we look at $SO(d)$ as a matrix Lie group to expose the close connection with skew-symmetric matrices of dimension $d$, following [14].

Definition 2.4 (Special orthogonal group). The special orthogonal group of dimension $d$, $SO(d)$ is defined as

$$SO(d) := \{ X \in \mathbb{R}^{d \times d} \mid X^\top X = I, \det(X) = 1 \}.$$  

(2.10)

Notice than the dimension of $SO(d)$ is $d(d-1)/2$.

Definition 2.5 (Tangent space of $SO(d)$). For $R \in SO(d)$, the tangent space of $SO(d)$ at a rotation $R$ is defined as

$$T_RSO(d) := R \cdot S_{\text{skew}}(d)$$

(2.11)

where the set of all skew-symmetric $d \times d$ matrices is defined as

$$S_{\text{skew}}(d) := \{ X \in \mathbb{R}^{d \times d} \mid -X = X^\top \}. $$

(2.12)
As an embedded submanifold of $\mathbb{R}^{d \times d}$, the Riemannian gradient of a function $f$ over $SO(d)$ is defined as the projection of the Euclidean gradient over $T_RSO(d)$.

**Definition 2.6** (Riemannian generalized gradient of $f$ over $SO(d)$). Given $f : SO(d) \to \mathbb{R}$ with locally Lipschitz extension $\bar{f} : \mathbb{R}^{d \times d} \to \mathbb{R}$ such that $f = \bar{f}|_{SO(d)}$, then the Riemannian generalized gradient of $f$ is defined as

$$\partial_R f(R) := R \cdot \text{skew}(R^\top (\bar{\partial f}(R)))$$

(2.13)

where $\text{skew}(A) = (A - A^\top)/2$ and $\bar{\partial f}$ is the Euclidean generalized gradient of $\bar{f}$ as defined in **Definition 2.3**.

We introduce the matrix exponential and the principal matrix logarithm.

**Definition 2.7** (Matrix exponential). For any $A \in \mathbb{C}^{d \times d}$, $\exp(A) = \sum_{k=0}^{\infty} \frac{A^k}{k!}$

**Definition 2.8** (Principal logarithm of a matrix). For any $A \in \mathbb{C}^{d \times d}$ with no negative eigenvalues, there exists a unique matrix $C \in \mathbb{C}^{d \times d}$ such that $\exp(C) = A$ and the imaginary part of the eigenvalues of $C$ are in the interval $(-\pi, \pi)$. The principal logarithm is defined as $\log(A) := C$

**Definition 2.9** (Geodesics on $SO(d)$). For any $R \in SO(d)$ and $A \in S_{\text{skew}}(d)$, the geodesic $\gamma : \mathbb{R} \to SO(d)$ such that $\gamma(0) = R$ and $\dot{\gamma}(0) = A \in T_RSO(d)$ is given by

$$\gamma(t) = R \cdot \exp(tA).$$

(2.14)

The matrix exponential map on $SO(d)$ is surjective [13, Chapter 11]. Therefore, given the set

$$B_{\text{skew}}(d) := \{S \in S_{\text{skew}}(d) \mid \|S\|_2 \leq 1\}.$$ 

(2.15)

then $SO(d) = \exp(\pi B_{\text{skew}}(d))$. This implies that it is possible to define a principal logarithm for all rotations with the image contained in $\pi B_{\text{skew}}(d)$.

Additionally, for any pair $R, Q \in SO(d)$, the geodesic $\gamma(t) = R \cdot \exp(t \log(R^\top Q))$ satisfies $\gamma(0) = R$ and $\gamma(1) = Q$. This remark provides a practical definition of the Riemannian distance in $SO(d)$.

**Definition 2.10** (Riemannian distance in $SO(d)$). Let $R, Q \in SO(d)$ then the Riemannian distance in $SO(d)$ is given by

$$D_{SO}(R, Q) := \| \log(Q^\top R) \|_F.$$ 

(2.16)

### 2.4 $\varepsilon$-nets

A discrete set $N_B^\varepsilon$ is called an $\varepsilon$-net for a set $P$ if for any $a \in P$, there exists $b \in N_B^\varepsilon$ such that, for some metric $D$, $D(a, b) \leq \varepsilon$.

**Theorem 2.3** ([26]). Let $N_B^\varepsilon$ be the smallest Euclidean $\varepsilon$-net of $B := \{x \in \mathbb{R}^q \mid \|x\|_2 \leq 1\}$ then

$$(\varepsilon^{-1})^q \leq |N_B^\varepsilon| \leq (3\varepsilon^{-1})^q.$$ 

**Theorem 2.4** ([26]). Let $K \subset L$. Let $N_K^\varepsilon$ be the $\varepsilon$-net of $K$ and $N_L^\varepsilon/2$ be the $\varepsilon/2$-net of $K$. Then

$$|N_K^\varepsilon| \leq |N_L^\varepsilon/2|.$$
3 Point-set registration via convex relaxation

We discuss the admissible corruption level to recover the ground truth rotation using a convex relaxation of $SO(d)$, i.e. solving Problem 1.2 and Problem 1.3. Let the admissible corruption threshold be defined as

$$\tilde{p}(d) := \left(1 + \frac{B(d-1, \frac{1}{2})}{B\left(\frac{d-1}{2}, \frac{1}{2}\right)}\right)^{-1}, \quad (3.1)$$

where $B(m, n)$ is the beta function. In Section 3.1, we prove Theorem 1.1 showing that if $p < \tilde{p}(d) - o(1)$ the ground truth rotation can be recovered for $N$ large enough. Similarly in Section 3.2, we prove Theorem 1.2 showing that the ground truth is not recovered when $p > \tilde{p}(d) + o(1)$.

The proof of Theorem 1.1 and Theorem 1.2 is based on constructing lower and upper bounds for $L(A) - L(I)$, for any matrix $A$. We use the following inequality.

Lemma 3.1. Let $u, v \in \mathbb{R}^n \setminus \{0\}$ then

$$u^T \frac{v}{\|v\|_2} \leq \|u + v\|_2 - \|v\|_2 \leq u^T \frac{v}{\|v\|_2} + \frac{1}{2} \sqrt{\|u\|_2^2 \|v\|_2^2 - (u^T v)^2} \quad (3.2)$$

Proof. See Appendix C \hfill \Box

3.1 Proof of Theorem 1.1: Regime where exact recovery is possible

We start this section by creating a lower bound of the expectation of $L(A) - L(I)$ in terms of the difference between $\tilde{p}(d)$ and $p$.

Lemma 3.2 (Success of Problem 1.2 in expectation). Let $A \in \mathbb{R}^{d \times d} \setminus I$ and $L(\cdot)$ the LUD as defined in (1.5), then

$$\mathbb{E}[L(A) - L(I)] \geq \frac{\|A - I\|_* (\tilde{p}(d) - p)}{d} \quad (3.3)$$

where $\| \cdot \|_*$ corresponds to the nuclear norm.

Proof. Using the definition of $L(\cdot)$ and Lemma 3.1, we have

$$L(A) - L(I) = \frac{1}{N} \sum_{i \in C^c} \| (A - I)x_i \|_2 + \frac{1}{N} \sum_{i \in C} \left( \| Ax_i - y_i \|_2 - \| x_i - y_i \|_2 \right)$$

$$\geq \frac{1}{N} \sum_{i \in C^c} \| (A - I)x_i \|_2 + \frac{1}{N} \sum_{i \in C} \left( (A - I)x_i, \frac{x_i - y_i}{\| x_i - y_i \|_2} \right), \quad (3.4)$$

Given the distribution of $\{x_i\}_{i=1}^N, \{y_i\}_{i=1}^N$ in (1.1) and (1.2), then

$$\mathbb{E}[L(A) - L(I)] \geq (1 - p)\mathbb{E}\| (A - I)x \|_2 + p \left( A - I, \mathbb{E}\left[ \frac{(x - y)(x - y)^T}{\| x - y \|_2} \right] \right) \quad (3.5)$$

for $x, y \in \text{Unif}(S^{d-1})$ independent. By Lemma B.2 we have

$$\mathbb{E}\left[ \frac{(x - y)(x - y)^T}{\| x - y \|_2} \right] = \frac{\mathbb{E}\| x - y \|_2^2}{d} I_d = \frac{2 B\left(\frac{d-1}{2}, \frac{1}{2}\right)}{d B\left(\frac{d-1}{2}, \frac{1}{2}\right)} I_d. \quad (3.6)$$
Similarly, Lemma B.1 provides the lower bound $\mathbb{E}\| (A - I)x \|_2 \geq \| A - I \|_s / d$. Then,

$$
\mathbb{E}[L(A) - L(I)] \geq \frac{\| A - I \|_s}{d} \left[ 1 - p \left( 1 + \frac{B (d - 1, \frac{1}{2})}{B (d - \frac{1}{2}, \frac{1}{2})} \right) \right]. \quad (3.7)
$$

\[\square\]

Lemma 3.2 implies that as $N \to \infty$, if $p < \tilde{p}(d)$,

$$
L(A) > L(I) \text{ for } A \neq I. \quad (3.8)
$$

Therefore the unique minimum of $L(\cdot)$ is $I$. Hence one can exactly recover the ground truth rotation via solving the convex problem if $p < \tilde{p}(d)$.

Now, we deduce a similar behavior when the sample size $N$ is finite. To prove Theorem 1.1, we show that the RHS of (3.4) is bounded away from zero with high probability for most configurations of $\{ x_i \}_{i=1}^N, \{ y_i \}_{i=1}^N$. We use the following result derived from Talagrand’s inequality (Theorem 2.2).

Lemma 3.3. Let $\{ x_i \}_{i=1}^n$ be a set of independent random variables in $\mathbb{R}^d$ and $\{ f_i \}_{i=1}^n$ a set of functions such that $f_i : \mathbb{R}^{d \times d} \times \mathbb{R}^d \to \mathbb{R}$ such that

$$
\| f_i(A, x_i) - f_i(B, x_i) \|_{\psi_2} \leq K \| A - B \|_F \quad (3.9)
$$

for all $\| A \|_F \leq 1, \| B \|_F \leq 1$, $i = 1, \ldots, n$. Then, there exist a universal constant $c$ such that with high probability

$$
\sup_{\| A \|_F \leq 1} \left| \sum_{i=1}^n f_i(A, x_i) - \mathbb{E} f_i(A, x_i) \right| \leq c \sqrt{n (d^2 K^2 + \log n)}. \quad (3.10)
$$

Proof. See Appendix D \[\square\]

We can now give the proof of Theorem 1.1.

Proof of Theorem 1.1. Lemma 3.2 and (3.4) imply that

$$
L(A) - L(I) \geq \frac{\| A - I \|_s (\tilde{p}(d) - p)}{d} + \frac{1}{N} \sum_{i \in \mathcal{C}} (\| (A - I)x_i \|_2 - \mathbb{E} \| (A - I)x_i \|_2) + \frac{1}{N} \sum_{i \in \mathcal{C}} \left( \langle (A - I)x_i, \frac{x_i - y_i}{\| x_i - y_i \|_2} \rangle - \mathbb{E} \left\langle (A - I)x_i, \frac{x_i - y_i}{\| x_i - y_i \|_2} \right\rangle \right). \quad (3.11)
$$

To bound the last two terms in the RHS of (3.11), we first show that each term satisfies condition (3.9) of Lemma 3.3. By triangle inequality, for $E, F \in \mathbb{R}^{d \times d}$

$$
\max_{x \in \mathbb{S}^{d-1}} \| Ex \|_2 - \| Fx \|_2 \leq \max_{x \in \mathbb{S}^{d-1}} \| (E - F)x \|_2 \leq \| E - F \|_F \quad (3.12)
$$

and

$$
\max_{x, u \in \mathbb{S}^{d-1}} | \langle Ex, u \rangle - \langle Fx, u \rangle | \leq \max_{x \in \mathbb{S}^{d-1}} \| (E - F)x \|_2 \leq \| E - F \|_F. \quad (3.13)
$$

Then $\| Ex \|_2 - \| Fx \|_2$ and $\langle Ex, u \rangle - \langle Fx, u \rangle$ are bounded random variables. Hence (A.1) implies for $i \in \mathcal{C}^c$,

$$
\| \| E x_i \|_2 - \| F x_i \|_2 \|_{\psi_2} \leq (\log 2)^{-1/2} \| E - F \|_F, \quad (3.14)
$$
and for $i \in C$,
\[
\left\| \left\langle E_{x_i}, \frac{x_i - y_i}{\|x_i - y_i\|_2} \right\rangle - \left\langle F_{x_i}, \frac{x_i - y_i}{\|x_i - y_i\|_2} \right\rangle \right\|_\psi \leq (\log 2)^{-1/2} \| E - F \|_F. \tag{3.15}
\]

Now, for any $E \in \mathbb{R}^{d \times d}$, we define
\[
X[E] := \sum_{i \in C} \left( \| E_{x_i} \|_2 - E \| E_{x_i} \|_2 \right) + \sum_{i \in C} \left( \left\langle E, \frac{x_i - y_i}{\|x_i - y_i\|_2} x_i^\top - E \left[ \frac{x_i - y_i}{\|x_i - y_i\|_2} x_i^\top \right] \right\rangle \right). \tag{3.16}
\]

Since $\| A - I \|_F > \| A - I \|_F$, then by inserting (3.16) in the RHS of (3.11) we get
\[
L(A) - L(I) \geq \frac{\| A - I \|_F}{d} \left( \tilde{p}(d) - p - O \left( \sqrt{\frac{(d^2 + \log N)}{N}} \right) \right). \tag{3.17}
\]

Using the result of Lemma 3.3, we have that
\[
\sup_{\|E\|_F = 1} |X[E]| \leq c \sqrt{N(d^2 + \log N)} \tag{3.18}
\]
with high probability. Therefore with high probability, for all $A \in \mathbb{R}^{d \times d}$
\[
L(A) - L(I) \geq \frac{\| A - I \|_F}{d} \left( \tilde{p}(d) - p - O \left( \sqrt{\frac{(d^2 + \log N)}{N}} \right) \right). \tag{3.19}
\]

Then, with high probability, for all $A \neq I$,
\[
L(A) > L(I) \quad \text{if} \quad p + O \left( d \sqrt{\frac{(d^2 + \log N)}{N}} \right) < \tilde{p}(d). \tag{3.20}
\]

Hence the only minimizer of Problem 1.2 in this regime is $I$. \hfill \Box

### 3.2 Proof of Theorem 1.2: Regime where exact recovery is impossible

To prove Theorem 1.2 we find a matrix $C \in \text{conv} \ SO(d)$ and $s > 0$ such that when $p > \tilde{p}(d) + o(1)$
\[
L(C) < L(A), \quad \text{for all } A \text{ s.t. } \| A - I \|_F < s. \tag{3.21}
\]
with high probability. This implies that, if the corruption level is large enough, exact recovery of the ground truth rotation is impossible when minimizing $L(\cdot)$ over $\text{conv} \ SO(d)$. First, we find such matrix $C$ that satisfies (3.21) in expectation.

**Lemma 3.4** (Failure of Problem 1.3 in expectation, Part 1.). Let $L(\cdot)$ the LUD as defined in (1.5), $p > \tilde{p}(d)$ and
\[
\lambda^* := \left( \frac{p - \tilde{p}(d)}{\tilde{p}(d) \cdot p} \right) \frac{B \left( \frac{d-1}{2}, \frac{1}{2} \right)}{B \left( \frac{d-2}{2}, \frac{1}{2} \right)} \leq 1 \tag{3.22}
\]
then
\[
\mathbb{E} [L((1 - \lambda^*)I) - L(I)] \leq -\frac{\lambda^*}{2p} (p - \tilde{p}(d)) < 0. \tag{3.23}
\]
Proof. We want to show \((1 - \lambda)I\) attains better cost than \(I\) for some \(\lambda > 0\). To this end, we find an upper bound for \(\mathbb{E}[L((1 - \lambda)I) - L(I)]\) for any \(\lambda > 0\) and then we get \(\lambda^*\) that minimizes this upper bound. By the definition of \(L(\cdot)\), we get

\[
L((1 - \lambda)I) - L(I) = (1 - p)\lambda + \frac{1}{N} \sum_{i \in C} \| - \lambda x_i + x_i - y_i \|_2 - \| x_i - y_i \|_2. \tag{3.24}
\]

Lemma 3.1 provides the upper bound for \(i \in C\)

\[
\| - \lambda x_i + x_i - y_i \|_2 - \| x_i - y_i \|_2 \leq -\lambda \frac{1}{2} \| x_i - y_i \|_2 + \lambda^2 \frac{1}{\| x_i - y_i \|_2}. \tag{3.25}
\]

Although \(\| x_i - y_i \|_2 \) or \(\| x_i + y_i \|_2 \) may be zero, if \(d \geq 3\), Lemma B.2 states

\[
\mathbb{E}\| x_i - y_i \|_2 = 2 \frac{B(d - 1, \frac{1}{2})}{B(d, \frac{1}{2})}, \quad \mathbb{E}\left[ \frac{1}{\| x_i - y_i \|_2} \right] = \frac{B(d - 1, \frac{1}{2})}{2B(d, \frac{1}{2})}. \tag{3.26}
\]

Therefore for all \(\lambda > 0\),

\[
\mathbb{E}[L((1 - \lambda)I) - L(I)] \leq -\lambda \frac{(p - \bar{p}(d))}{\bar{p}(d)} + \frac{p\lambda^2}{2} \frac{B(d - 1, \frac{1}{2})}{B(d, \frac{1}{2})}. \tag{3.27}
\]

In particular, taking \(\lambda = \lambda^*\) as defined in (3.22) the RHS of (3.27) is minimized. Moreover, since the Beta function satisfies

\[
B(q, 1/2) \leq B(r, 1/2) \text{ if } q \geq r \geq 1/2, \tag{3.28}
\]

then \(\lambda^* < 1\). Given that \(0, I \in \text{conv} \mathcal{SO}(d)\), then \((1 - \lambda^*)I \in \text{conv} \mathcal{SO}(d)^{\perp}\). \(\Box\)

Corollary 3.1 (Failure of Problem 1.3 in expectation, Part 2). Let \(L(\cdot)\) the LUD as defined in (1.5), \(\lambda^*\) as defined in (3.22) and \(p > \bar{p}(d)\). For all \(A \in \mathbb{R}^{d \times d}\) such that \(\| A - I \|_F \leq \sqrt{\lambda^*/4}\) then

\[
\mathbb{E}[L(A) - L((1 - \lambda^*)I)] \geq \frac{p - \bar{p}(d)}{\bar{p}(d)} \frac{\lambda^*}{4} > 0. \tag{3.29}
\]

Proof. Follows directly from Lemma 3.2 and Lemma 3.4 \(\Box\)

Corollary 3.1 implies that when \(N \to \infty\), any \(A^*\) minimizer of Problem 1.3 satisfies \(\| A^* - I \|_F > \lambda^* \sqrt{d/4} = O(p - \bar{p}(d))\).

We now prove Theorem 1.2

Proof of Theorem 1.2 Lemma 3.4 and (3.24) imply that

\[
L(I) - L((1 - \lambda^*)I) \geq \frac{\lambda^*}{2\bar{p}} (p - \bar{p}) - \frac{1}{N} \sum_{i \in C} (g(x_i, y_i, \lambda^*) - \mathbb{E}g(x_i, y_i, \lambda^*)) \tag{3.30}
\]

where \(g(x_i, y_i, \lambda^*) := \|(1 - \lambda^*)x_i - y_i\|_2 - \| x_i - y_i \|_2\). By triangle inequality,

\[
\max_{x, y \in S^{d-1}} \| (1 - \lambda^*)x - y \|_2 - \| x - y \|_2 \leq \max_{x \in S^{d-1}} \lambda^* \| x \|_2 = \lambda^*. \tag{3.31}
\]
Therefore (A.1) implies that \( \|g(x_i, y_i, \lambda^*)\|_{\psi_2} \leq (\log 2)^{-1/2} \lambda^* \) for all \( i \in \mathcal{C} \). Using Hoeffding’s inequality (Theorem 2.1) we get

\[
\left|\sum_{i \in \mathcal{C}} [g(x_i, y_i, \lambda^*) - \mathbb{E}g(x_i, y_i, \lambda^*)]\right| \leq c\lambda^* \sqrt{N \log N} \tag{3.32}
\]

with high probability. (3.32) implies that the LHS of (3.30) is bounded by

\[
L(I) - L((1 - \lambda^*)I) \geq \frac{\lambda^*}{2\bar{p}(d)} \left( p - \bar{p}(d) - O\left(\sqrt{\frac{\log N}{N}}\right)\right). \tag{3.33}
\]

From (3.19) in the proof of Theorem 1.1 we have with high probability, for all \( A \in \mathbb{R}^{d \times d} \)

\[
L(A) - L(I) \geq -\frac{\|A - I\|_F}{\sqrt{d} \bar{p}(d)} \left( p - \bar{p}(d) + O\left(\sqrt{\frac{d(\log N)}{N}}\right)\right), \tag{3.34}
\]

since \( \|A - I\|_* \leq \sqrt{d}\|A - I\|_F \). Hence, for all \( A \in \mathbb{R}^{d \times d} \) such that \( \|A - I\|_F \leq \sqrt{d}\lambda^* \),

\[
L(A) - L((1 - \lambda^*)I) \geq \frac{\lambda^*}{4} \left( p - \bar{p}(d) + O\left(\sqrt{\frac{d(\log N)}{N}}\right)\right), \tag{3.35}
\]

with high probability. Therefore, if \( p > \bar{p}(d) + O\left(\sqrt{d(\log N)N^{-1}}\right) \), then

\[
L(A) > L((1 - \lambda^*)I). \tag{3.36}
\]

with high probability. Hence if \( A^* \) is a minimizer of Problem 1.3 then \( \|A^* - I\|_F > \lambda^* \sqrt{d}/4 = O(p - \bar{p}(d)) \).

\[\square\]

4 Finite time dynamical systems in \( SO(d) \)

To prove Theorem 1.3 we want to show the dynamical system (1.11) converges in finite time, i.e. there exists \( T < \infty \) such that \( R(t) = I \) for all \( t > T \). In this section, we characterize the sufficient conditions for finite time convergence of any dynamical system in \( SO(d) \) of the form

\[
\frac{dR}{dt} \in R \cdot S(R), \quad S(R) \subseteq S_{\text{skew}}(d) \tag{4.1}
\]

in terms of the dynamics of \( \| \log R(t) \|_2 \). We start with an useful characterization of \( R \in SO(d) \) in terms of the principal angles of rotation. We denote the skew-symmetric matrix and the identity matrix in \( \mathbb{R}^{2 \times 2} \)

\[
A_2 := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad I_2 := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \tag{4.2}
\]

respectively.

**Lemma 4.1** (Planar decomposition in \( SO(d) \)). Any \( R \in SO(d) \) can be written as

\[
R - I = \sum_{i=1}^{\lfloor d/2 \rfloor} U_i (R_{\sigma_i} - I_2) U_i^T, \quad R_\theta := \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \tag{4.3}
\]

with \( \pi \geq \sigma_1 \geq \cdots \geq \sigma_{\lfloor d/2 \rfloor} \geq 0 \) and \( U = [ U_1 \ U_2 \ \ldots \ U_{\lfloor d/2 \rfloor} ] \) such that \( U^T U = I \) and \( U_i \in \mathbb{R}^{d \times 2} \).
Proof. For any rotation \( R \), we can define the principal logarithm such that \( \log R \in \pi B_{\text{skew}}(d) \), as defined in (2.15). Therefore, the SVD decomposition of \( \log R \) satisfies

\[
\log R = \sum_{i=1}^{\lfloor d/2 \rfloor} \sigma_i U_i A_2 U_i^T
\]

for \( \pi \geq \sigma_1 \geq \cdots \geq \sigma_{\lfloor d/2 \rfloor} \geq 0 \) and

\[
U = \begin{bmatrix} U_1 & U_2 & \ldots & U_{\lfloor d/2 \rfloor} \end{bmatrix}
\]

such that \( U^T U = I \). Notice that for any \( \theta \in [0, \pi] \),

\[
\exp(\theta \cdot A_2) = \cos(\theta) I_2 + \sin(\theta) A_2 = R_\theta.
\]

Then, using \( R = \exp(\log R) \), we get decomposition (4.3) for \( R \).

Lemma 4.1 provides a decomposition of any rotation of dimension \( d \) as a sum of planar rotations in orthogonal planes. Notice that if \( d \) is odd, (4.3) implies that \( R \) has at least one invariant direction. In addition, \( \sigma_1 = \| \log R \|_2 \) characterizes the largest angle of rotation of \( R \). Furthermore, \( \| \log R \|_2 \) provides the following bounds:

\[
\| \log R \|_2 \leq D_{SO}(R, I) \leq \sqrt{d} \| \log R \|_2.
\]

where \( D_{SO}(R, I) \) denotes the Riemannian distance between \( R \) and \( I \) as defined in (2.16).

Now we characterize the rate of change of \( \| \log R(t) \|_2 \) assuming that \( R(t) \) follows the dynamics (4.1).

Lemma 4.2 (Dynamics of \( \| \log R \|_2 \)). Let \( R(t) \in SO(d) \) be a solution of the differential inclusion (4.1), then \( \| \log R(t) \|_2 \) is absolutely continuous and satisfies

\[
\frac{d \| \log R(t) \|_2}{dt} \in \{ a \mid \exists S \in \mathcal{S}(R(t)), \langle S, U A_2 U^T \rangle = 2a \ \forall U \in \mathcal{U}(R(t)) \}\quad (4.7)
\]

almost everywhere, where

\[
\mathcal{U}(R) := \left\{ U \in \mathbb{R}^{d \times 2} \mid U^T U = I_2, \quad U^T (\log R) U = \| \log R \|_2 A_2 \right\}. \quad (4.8)
\]

Proof. See Appendix E

Ultimately, we want to show \( \| \log R(t) \|_2 \) converges to zero as \( t \) increases. A sufficient condition for this is summarized in the following lemma.

Lemma 4.3 (Finite-time convergence in \( SO(d) \)). Let \( R(t) \in SO(d) \) be a solution of the differential inclusion (4.1) with initial condition \( R(0) \). Suppose there exist \( g \in C^1[0, \pi] \) such that

\[
\max_{S \in \mathcal{S}(R(t)), U \in \mathcal{U}(R(t))} \langle S, U A_2 U^T \rangle \leq -2g(\| \log R(t) \|_2) \quad (4.9)
\]

holds for \( R(t) \neq I \) almost everywhere, \( g(s) > 0 \) for \( 0 \leq s \leq \| \log R(0) \|_2 \) and \( \mathcal{U}(R(t)) \) as defined in (4.8). Then for all \( t \geq T(\| \log R(0) \|_2) \), \( R(t) = I \), where

\[
T(s) := \int_0^s \frac{1}{g(u)} du < \infty. \quad (4.10)
\]
Proof. Lemma 4.2 implies that \( \| \log R(t) \|_2 \) is absolutely continuous, i.e.
\[
\| \log R(t) \|_2 = \| \log R(0) \|_2 + \int_0^t \frac{d}{d\tau} \| \log R(\tau) \|_2 d\tau \tag{4.11}
\]
and if \( \frac{d\| \log R(t) \|_2}{dt} \) exists when \( R(t) = I \) then
\[
\frac{d\| \log R(t) \|_2}{dt} \in \{ a \mid \exists S \in S(t), \langle S, U A_2 U^T \rangle = 2a \forall U \in U(I) \} \subseteq \{0\}. \tag{4.12}
\]
Therefore, condition (4.9) and \( g(s) > 0 \) for \( 0 \leq s \leq \| \log R(0) \|_2 \) imply
\[
\| \log R(t_2) \|_2 \leq \| \log R(t_1) \|_2 \quad \text{for} \quad t_1 \leq t_2, \tag{4.13}
\]
and we have strict inequality when \( R(t_1) \neq I \). In particular, if there exists \( T \) such that \( R(T) = I \), then \( R(t) = I \) for all \( t \geq T \).

When the initial condition is not \( I \), then (4.13) implies that for \( \| \log R(0) \|_2 > 0 \), we can upper bound the minimum time \( \tau \) such that \( \| \log R(\tau) \|_2 = 0 \) by
\[
\tau = \int_0^\tau ds \leq \int_0^\| \log R(0) \|_2 \frac{1}{g(u)} du =: T(\| \log R(0) \|_2). \tag{4.14}
\]
Then, \( R(t) = I \) for \( t \geq T(\| \log R(0) \|_2) \) as defined in (4.10).

In the rest of the paper, we will show finite time convergence of \( R(t) \) to \( I \) by checking the condition in Lemma 4.3.

5 Proof of Theorem 1.3: Exact recovery via non-convex optimization

We examine exact recovery via non-convex optimization for solving Problem 1.1, i.e. by explicitly restricting the transformation between the two point clouds to lie on \( SO(d) \). As mentioned previously, we take a dynamical system view-point of generalized gradient descent on the manifold \( SO(d) \) by considering the Riemannian gradient flow
\[
\frac{dR}{dt}(t) \in -R(t) \cdot \text{skew}(R(t)^T \partial L(R(t))) \subset T_{R(t)} SO(d) \tag{5.1}
\]
where \( \partial L(R) \) is the Euclidean generalized gradient of \( L(\cdot) \) given by
\[
\partial L(R) := \frac{1}{N} \sum_{i \in C^c} \alpha_i(R)x_i^T + \frac{1}{N} \sum_{i \in C} \beta_i(R)x_i^T, \tag{5.2}
\]
and \( \alpha_i's \) and \( \beta_i's \) are defined as
\[
\text{for } i \in C^c, \quad \alpha_i(R) := \begin{cases} \left\{ \frac{R x_i - x_i}{\| R x_i - x_i \|_2} \right\}, & \text{if } R x_i - x_i \neq 0 \\ \{ \theta_i \in \mathbb{R}^d \mid \| \theta_i \|_2 \leq 1 \} , & \text{otherwise}; \end{cases} \tag{5.3}
\]
\[
\text{for } i \in C, \quad \beta_i(R) := \begin{cases} \left\{ \frac{R x_i - y_i}{\| R x_i - y_i \|_2} \right\}, & \text{if } R x_i - y_i \neq 0 \\ \{ \theta_i \in \mathbb{R}^d \mid \| \theta_i \|_2 \leq 1 \} , & \text{otherwise}. \end{cases} \tag{5.4}
\]
Although $\alpha_i(R), \beta_i(R)$ are set-valued functions, we can also consider $\alpha_i(R), \beta_i(R)$ as single value functions, by associating each $\alpha_i, \beta_i$ with an arbitrary element from $\{\theta_i \in \mathbb{R}^d \mid \|\theta_i\|_2 \leq 1\}$ whenever its value is not uniquely defined.

Lemma 4.3 states that the differential inclusion (5.1) with initial condition $R(0)$ converges to $I$ in finite time if there exist $g : [0, \pi] \to \mathbb{R}$ continuously differentiable such that

$$\min_{U \in \mathcal{U}(R)} \left< \tilde{\partial} L(R), RU A_2 U^\top \right> \geq 2g(\| \log R \|_2) \tag{5.5}$$

holds for all $R \in SO(d) \setminus I$ and $g(s) > 0$ for $s \in (0, \| \log R(0) \|_2)$, where

$$\mathcal{U}(R) := \left\{U \in \mathbb{R}^{d \times 2} \mid U^\top U = I, \quad U^\top (\log R) U = \| \log R \|_2 A_2 \right\}. \tag{5.6}$$

In this section we show (5.5) holds for a particular $g$ in Lemma 5.1.

Lemma 5.1. With high probability, for all $R \in SO(d) \setminus I$ and $U \in \mathcal{U}(R)$

$$\left< \tilde{\partial} L(R), R U A_2 U^\top \right> \geq \frac{2(1-p)}{d} \cos \left( \frac{\| \log R \|_2}{2} \right) - O \left( \frac{d(\log N + d)}{N} \right) \tag{5.7}$$

Before proving Lemma 5.1 we first show that Theorem 1.3 directly follows from Lemma 4.3 and Lemma 5.1.

Proof of Theorem 1.3. Lemma 5.1 implies that there exists $\gamma > 0$ such that

$$\gamma = O \left( \frac{d}{1 - p} \sqrt{\frac{d(\log N + d)}{N}} \right) \tag{5.8}$$

and for all $R \in SO(d) \setminus I$ and $U \in \mathcal{U}(R)$

$$\cos \left( \frac{\| \log R \|_2}{2} \right) - \frac{d}{2(1-p)} \left< \tilde{\partial} L(R), R U A_2 U^\top \right> \leq \gamma \tag{5.9}$$

with high probability. Then, $g(s) = ((1-p)/d)(\cos(s/2) - \gamma)$ satisfies (5.5) with high probability. If $\| \log R(0) \|_2 < \pi(1 - \sqrt{\gamma})$, then $g(\| \log R(0) \|_2) > 0$ and Lemma 4.3 implies that $R(t) = I$ for all $t \geq T(\| \log R(0) \|_2)$, where

$$T(s) = \frac{d}{(1-p)} \int_0^s \frac{1}{\cos \left( \frac{\tau}{2} \right) - \gamma} d\tau$$

$$= \frac{2d}{(1-p)} \left( \cosh^{-1} \left( \sec \left( \frac{s}{2} \right) \right) + O\left( \sqrt{\gamma} \right) \right) \tag{5.10}$$

for $s \in [0, \| \log R(0) \|_2]$.

In the remaining of this section, we prove Lemma 5.1. First, in Section 5.1 we prove Lemma 5.2 that shows that inequality (5.7) holds in expectation. This leads to the proof of Lemma 5.1 in Section 5.2 where the proof follows from Lemma 5.4 (Section 5.2.1) and Lemma 5.5 (Section 5.2.2) where we estimate the deviation of the LHS of (5.7) from its expectation.
5.1 Expectation version of Lemma 5.1

In this section, we show that inequality (5.7) holds in expectation

Lemma 5.2 (Expectation version of Lemma 5.1). For all $R \in SO(d) \setminus I$ and $U \in U(R)$

\[
\frac{1}{N} \mathbb{E} \left( \sum_{i \in C} \alpha_i(R)x_i^T + \sum_{i \in C} \beta_i(R)x_i^T, RU A_2 U^T \right) \geq \frac{2(1-p)}{d} \cos \left( \frac{\| \log R \|_2}{2} \right). \tag{5.11}
\]

To prove Lemma 5.2, we first introduce the following lemma.

Lemma 5.3. Let $R \in SO(d) \setminus I$, then for all $U \in U(R)$ and $i \in C$

\[
\left\langle \alpha_i(R), R \cdot U A_2 U^T x_i \right\rangle \geq \| U A_2 U^T x_i \|_2^2 \cos \left( \frac{\| \log R \|_2}{2} \right) \tag{5.12}
\]

Proof. First, we consider the case when $Rx_i = x_i$. Then $x_i$ is an invariant direction of $R$. By Lemma 4.1, $(\log R)x_i = 0$. Since, $R \neq I$, then ker($UA_2 U^T$) $\supseteq$ ker($\log R$) and $UA_2 U^T x_i = 0$. Therefore (5.12) trivially holds.

Now, we consider the case when $Rx_i \neq x_i$. Then $\alpha_i(R) = (Rx_i - x_i)/\|Rx_i - x_i\|_2$. Since $UA_2 U^T \in S_{skew}$, then $\left\langle x_i, U A_2 U^T x_i \right\rangle = 0$. Therefore,

\[
\left\langle \frac{Rx_i - x_i}{\|Rx_i - x_i\|_2}, R \cdot U A_2 U^T x_i \right\rangle = \frac{-x_i^T R \cdot U A_2 U^T x_i}{\sqrt{2 - 2x_i^T Rx_i}}. \tag{5.13}
\]

By definition of $U(R)$, $R \cdot U = U \exp(\| \log R \|_2 A_2)$. Let $\hat{x}_i = U^T x_i$. Then,

\[
-x_i^T R \cdot U A_2 U^T x_i = -\hat{x}_i^T \exp(\| \log R \|_2 A_2) A_2 \hat{x}_i = -\sin(\| \log R \|_2) \hat{x}_i^T \hat{x}_i, \tag{5.14}
\]

and $\hat{x}_i^T \hat{x}_i = \| U A_2 U^T x_i \|_2^2$. Similarly, using the planar decomposition of $R$ in Lemma 4.1

\[
1 - x_i^T Rx_i = \sum_{j=1}^{[d/2]} x_i^T U_j (I_2 - \exp(\sigma_j A_2)) U_j^T x_i \leq 1 - \cos(\| \log R \|_2), \tag{5.15}
\]

since $\sigma_j \leq \| \log R \|_2 \leq \pi$ for all $j = 1, \ldots, [d/2]$. Inserting (5.14) and (5.15) in RHS of (5.13), we get

\[
\left\langle \frac{Rx_i - x_i}{\|Rx_i - x_i\|_2}, R \cdot U A_2 U^T \right\rangle \geq \frac{\sin(\| \log R \|_2) \| U A_2 U^T x_i \|_2^2}{2 \sin(\| \log R \|_2/2)}. \tag{5.16}
\]

Using Lemma 5.3, we can now prove Lemma 5.2.

Proof of Lemma 5.2. Let

\[
M = \frac{1}{N} \sum_{i \in C} \alpha_i(R)x_i^T + \frac{1}{N} \sum_{i \in C} \beta_i(R)x_i^T \tag{5.17}
\]
then Lemma 5.3 implies
\[ \langle M, R \cdot U A_2 U^\top \rangle \geq \cos \left( \frac{\| \log R \|_2}{2} \right) \frac{1}{N} \sum_{i \in C^c} \| U A_2 U^\top x_i \|_2^2 \]
\[ + \frac{1}{N} \sum_{i \in C^c} \langle \beta_i(R), R \cdot U A_2 U^\top x_i \rangle. \]  \hfill (5.18)

Given a fixed rotation \( R \), \( \beta_i(R) = (R x_i - y_i) / \| R x_i - y_i \|_2 \) almost everywhere. Additionally, for \( i \in C \), rotation invariance implies that \( R x_i \) and \( y_i \) are i.i.d random variables. Therefore, for \( i \in C \),
\[ E \left[ \beta_i(R) (R x_i)^\top \right] = \frac{1}{2} E \left[ \frac{(R x_i - y_i)(R x_i - y_i)^\top}{\| R x_i - y_i \|_2} \right] = \frac{E[\| x_i - y_i \|_2^2]}{2d} I \]  \hfill (5.19)
and \( E \langle \beta_i(R), R \cdot U A_2 U^\top x_i \rangle = 0. \)

On the other hand, for all \( i \in C^c \), \( E \| U A_2 U^\top x_i \|_2^2 = \| U A_2 U^\top \|_F^2 / d = 2/d. \) Hence
\[ E \langle M, R \cdot U A_2 U^\top \rangle \geq \frac{2(1 - p)}{d} \cos \left( \frac{\| \log R \|_2}{2} \right). \]  \hfill (5.20)

Lemma 5.2 and Lemma 4.3 imply that as \( N \to \infty \), for all \( p < 1 \) and \( \| \log R \|_2 < \pi \), the dynamical system (5.1) converges to \( I \) in finite time.

5.2 Proof of Lemma 5.1 via concentration inequalities

Lemma 5.3 provides the behavior in expectation of LHS of (5.7). To prove Lemma 5.1, we bound the deviation of (5.7) away from the population limit. We deal separately with the terms concerning corrupted and uncorrupted points in Lemma 5.4 and Lemma 5.5 respectively.

Lemma 5.4 (Deviation from expectation for uncorrupted points). Let \( C^c \) be the index set of uncorrupted points, then
\[ \sup_{\|A\|_F = 1} \left| \frac{1}{N} \sum_{i \in C^c} \| Ax_i \|_2^2 - E \| Ax_i \|_2^2 \right| \leq c \sqrt{\frac{d + \log N}{N}} \]  \hfill (5.21)
with high probability and \( c \) is a universal constant.

Lemma 5.5 (Deviation from expectation for corrupted points). Let \( C \) be the index set of corrupted points, then with high probability, for all choices of \( \beta_i(R) \) as defined in (5.4),
\[ \sup_{R \in SO(d), S \in B_{skew}} \left| \frac{1}{N} \sum_{i \in C} \langle RSx_i, \beta_i(R) \rangle \right| \leq c \sqrt{\frac{d \log(N)}{N}} \]  \hfill (5.22)
with universal constant \( c. \)

Lemma 5.1 is a direct consequence of Lemma 5.4 and Lemma 5.5.
Proof of Lemma 5.4. By (5.18) and Lemma 5.2, for $M$ defined as in (5.17) and any value choice of $\alpha_i(R), \beta_i(R)$,

$$
\langle M, R \cdot U A_2 U^\top \rangle \geq \cos \left( \frac{\| \log R \|_2}{2} \right) \left( \frac{2(1 - p)}{d} \right) + \cos \left( \frac{\| \log R \|_2}{2} \right) \left( \frac{1}{N} \sum_{i \in C} \| U A_2 U^\top x_i \|_2 - \mathbb{E}\| U A_2 U^\top x_i \|_2^2 \right) + \frac{1}{N} \sum_{i \in C} \left( R \cdot U A_2 U^\top x_i, \beta_i (R) \right).
$$

(5.23)

Since $U A_2 U^\top \in \mathcal{S}_{\text{skew}}$ such that $\| U A_2 U^\top \|_2 = 1$ and $\| U A_2 U^\top \|_F = 2$, Lemma 5.4 and Lemma 5.5 imply that with high probability, for all $U \in \mathcal{U}(R)$, and all value choices of $\alpha_i(R), \beta_i(R)$,

$$
\langle M, R \cdot U A_2 U^\top \rangle \geq \frac{2(1 - p)}{d} \cos \left( \frac{\| \log R \|_2}{2} \right) - O \left( \sqrt{\frac{d (\log N + d)}{N}} \right).
$$

(5.24)

5.2.1 Proof of Lemma 5.4: Deviation from expectation concerning uncorrupted points

In this section, the proof of Lemma 5.4 is detailed.

Proof of Lemma 5.4. We first show that $\| A x_i \|_2^2 - \mathbb{E}\| A x_i \|_2^2$ satisfies condition (3.9) of Lemma 3.3.

Let $S, T \in \mathbb{R}^{n \times n}$ such that $\| S \|_F = \| T \|_F = 1$, and $x, y \in \mathbb{S}^{d-1}$, then

$$
\| S x \|_2^2 - \| S y \|_2^2 - \| T x \|_2^2 + \| T y \|_2^2 = \left| \langle S^T S - T^T T, x x^\top - y y^\top \rangle \right|.
$$

(5.25)

By rewriting

$$
2 \langle S^T S - T^T T \rangle = (S - T)^\top (S + T) + (S + T)^\top (S - T),
$$

(5.26)

$$
2 \langle x x^\top - y y^\top \rangle = (x - y)(x + y)^\top + (x + y)(x - y)^\top,
$$

(5.27)

and using Cauchy-Schwartz inequality, we get

$$
\| S x \|_2^2 - \| S y \|_2^2 - \| T x \|_2^2 + \| T y \|_2^2 \leq \| S - T \|_F \| S + T \|_F \| x - y \|_2 \| x + y \|_2
\leq 4 \| S - T \|_F \| x - y \|_2.
$$

(5.28)

This implies that $f(x) = \| S x \|_2^2 - \| T x \|_2^2$ is a Lipschitz function on $\mathbb{S}^{d-1}$ with $\| f \|_{\text{Lip}} \leq 4 \| S - T \|_F$.

Then, by condition (A.3), $\| S x \|_2^2$ satisfies condition (3.9) with

$$
\| S x \|_2^2 - \| T x \|_2^2 - \mathbb{E} (\| S x \|_2^2 - \| T x \|_2^2) \|_{\psi_2} \leq c \frac{\| S - T \|_F}{\sqrt{d}}.
$$

(5.29)

Therefore, Lemma 3.3 implies

$$
\frac{1}{N} \sup_{\| S \|_F = 1} \left| \frac{(1 - p)^N}{\sum_{j=0}^N} \| S x_i \|_2^2 - \mathbb{E}\| S x_i \|_2^2 \right| \leq c \sqrt{\frac{d + \log N}{N}}
$$

(5.30)

with probability $1 - 2N^{-d}$.

\[
\Box
\]
5.2.2 Proof of Lemma 5.5: Deviation from expectation concerning corrupted points

We now outline the strategy of the proof. We first build an $\varepsilon$-net (Lemma 5.6) for the set $B_{\text{skew}}(d) := \{S \mid S = -S^\top, \|S\|_2 \leq 1\}$ and also $\mathcal{SO}(d)$. Then, we show that the term

$$\sum_{i \in \mathcal{C}} \langle RSx_i, \beta_i(R) \rangle, \quad R \in \mathcal{SO}(d), \quad S \in B_{\text{skew}}$$

(5.31)

behaves similarly within each neighborhood defined by the $\varepsilon$-nets. Next, for each pair of $(S, R)$ in $\varepsilon$-nets we bound the difference between (5.31) and its expectation, which also implies a similar behavior within a neighborhood of the pair $(S, R)$. Finally, an application of the union bound over all points in $\varepsilon$-net gives a global deviation bound of (5.31) for all $S, R$ simultaneously. This idea is summarized in Lemma 5.7, where Lemma 5.8 gives estimate to specific term in Lemma 5.7.

We construct an Euclidean $\varepsilon$-net for $\mathcal{SO}(d)$ and $B_{\text{skew}}(d)$ using the following lemma.

Lemma 5.6. Let $N_{\varepsilon}^A$ be an Euclidean $\varepsilon$-net of

$$A := \left\{ S \in \mathbb{R}^{d \times d} \mid \|\text{skew}(S)\|_2 \leq \frac{1}{\sqrt{2}}, \ S(i,j) = 0 \text{ if } i \geq j \right\}$$

(5.32)

then

$$N_{\varepsilon}^B := \left\{ \text{skew}(\sqrt{2} S) \mid S \in N_{\varepsilon}^A \right\}, \quad N_{\varepsilon}^{\pi \varepsilon} := \left\{ \exp \left( \pi \text{skew}(\sqrt{2} S) \right) \mid S \in N_{\varepsilon}^A \right\}$$

(5.33)

are Euclidean $\varepsilon$-net of $B_{\text{skew}}(d)$ and Euclidean $(\pi \varepsilon)$-net of $\mathcal{SO}(d)$ respectively, of size

$$|N_{\varepsilon}^{\pi \varepsilon}| \leq |N_{\varepsilon}^B| \leq |N_{\varepsilon}^A| \leq \left( 6 \sqrt{d} \varepsilon^{-1} \right)^{\frac{d(d-1)}{2}}.$$

(5.34)

Proof. See Appendix F.

We construct a cover of $\mathcal{SO}(d)$ by defining for all $Q_l \in N_{\varepsilon}^{\pi \varepsilon}$

$$\phi_{\pi \varepsilon}(Q_l) := \left\{ \exp(\pi \text{skew}(\sqrt{2}A)) \mid A \in A \text{ s.t. } \|A - A_l\|_F < \varepsilon, \ A_l \in N_{\varepsilon}^A, \ Q_l = \exp \left( \pi \text{skew}(\sqrt{2} A_l) \right) \right\}.$$

(5.35)

$\phi_{\pi \varepsilon}()$ maps the Euclidean $\varepsilon$-ball around $A_l$ in $A$ to a subset in $\mathcal{SO}(d)$ such that

$$\phi_{\pi \varepsilon}(Q_l) \subseteq \{ R \mid \|R - Q_l\|_F < \varepsilon \}.$$

(5.36)

To construct a cover of $B_{\text{skew}}(d)$, we consider the Euclidean ball

$$\{ S \in B_{\text{skew}}(d) \mid \|S - T_k\|_F < \pi \varepsilon \} \quad \text{for all} \quad T_k \in N_{\varepsilon}^B.$$

(5.37)

In the following lemma, we provide an upper bound of (5.31) for all $S \in B_{\text{skew}}(d)$ and $R \in \mathcal{SO}(d)$. The upper bound shows two sources of contribution to the deviation of (5.31):

1. The first two terms are the deviations of (5.31) from its expectation for points in $\varepsilon$-net.
2. The last term comes from the variation of (5.31) within each part of the partition defined by the $\varepsilon$-nets.
Here, for a rotation on the $\varepsilon$-net, instead of having set-valued $\beta_i$, for the sake of convenience we make it single-valued:

$$
\hat{\beta}_i(R) := \begin{cases} 
\frac{R x_i - y_i}{\|R x_i - y_i\|_2} & \text{if } R x_i - y_i \neq 0 \\
0 & \text{otherwise.}
\end{cases}
$$

(5.38)

Lemma 5.7 (Upper bound for (5.31)). Let $N_{A}^{\varepsilon/\pi}$ be an Euclidean $\varepsilon/\pi$-net of $A$, then for all $S \in B_{\text{skew}}(d)$ and $R \in SO(d)$, and all choices of $\hat{\beta}_i(R)$,

$$
\sum_{i \in C} \langle RS x_i, \beta_i(R) \rangle \leq 2 \varepsilon \ pN + \sup_{T_k \in N_{B}^{\varepsilon/\pi}} \sup_{Q_l \in N_{SO}^{\varepsilon/\pi}} \left| \sum_{i \in C} \langle Q_l T_k x_i, \hat{\beta}_i(Q_l) \rangle \right| + \sup_{Q_l \in N_{SO}^{\varepsilon/\pi}} \sup_{R \in \phi_\varepsilon(Q_l)} \sum_{i \in C} \|\beta_i(R) - \hat{\beta}_i(Q_l)\|_2.
$$

(5.39)

where $\hat{\beta}_i$’s are defined in (5.38).

Proof: Lemma 5.6 shows that $N_{B}^{\varepsilon/\pi}$ is an $\varepsilon/\pi$-net of $B_{\text{skew}}(d)$ and $N_{SO}^{\varepsilon/\pi}$ is an $\varepsilon$-net of $SO(d)$. Then, for any $S \in B_{\text{skew}}(d)$ and $R \in SO(d)$, there exists $T_k \in N_{B}^{\varepsilon/\pi}$ and $Q_l \in N_{SO}^{\varepsilon/\pi}$ such that $\|S - T_k\|_F < \varepsilon/\pi$ and $R \in \phi_\varepsilon(Q_l)$.

We rewrite

$$
\frac{1}{2} \left| \left( \langle Q_l T_k x_i, \hat{\beta}_i(Q_l) \rangle - \langle RS x_i, \beta_i(R) \rangle \right) - \left( \langle (Q_l T_k + RS) x_i, \hat{\beta}_i(Q_l) + \beta_i(R) \rangle \right) - \left( \langle (Q_l T_k + RS) x_i, \beta_i(Q_l) - \beta_i(R) \rangle \right) \right|,
$$

(5.40)

and using Cauchy-Schwartz inequality, we upper bound (5.40) by

$$
\|\langle Q_l T_k - RS x_i, \hat{\beta}_i(Q_l) \rangle\|_2 + \|\beta_i(R)\|_2 + \|Q_l T_k x_i\|_2 + \|RS x_i\|_2.
$$

(5.41)

We have that $\|\langle Q_l T_k - RS x_i, \hat{\beta}_i(Q_l) \rangle\|_2 \leq \|Q_l T_k - RS\|_F$. Also, by definition of $\beta_i(R)$ and $\hat{\beta}_i(R)$, $\|\beta_i(R)\|_2 \leq 1$ and $\|\hat{\beta}_i(R)\|_2 \leq 1$ for any $R \in SO(d)$. Similarly, by definition of $B_{\text{skew}}$, $\|T x_i\|_2 \leq 1$.

Therefore,

$$
\left| \langle Q_l T_k x_i, \hat{\beta}_i(Q_l) \rangle - \langle RS x_i, \beta_i(R) \rangle \right| \leq 2 \varepsilon + \|\hat{\beta}_i(Q_l) - \beta_i(R)\|_2.
$$

(5.42)

Then for any $R \in \phi_\varepsilon(Q_l)$ and $S \in B_{\text{skew}}(d)$, we get

$$
\langle RS x_i, \beta_i(R) \rangle \leq \langle Q_l T_k x_i, \hat{\beta}_i(Q_l) \rangle + 2 \varepsilon + \|\hat{\beta}_i(Q_l) - \beta_i(R)\|_2.
$$

(5.43)

Adding (5.43) over all the corrupted points $i \in C$, and taking supremum of the RHS of (5.43) over $N_{B}^{\varepsilon/\pi}$ and $N_{SO}^{\varepsilon/\pi}$, we get the result.

Next, we provide an upper bound for the last term in the RHS of (5.39).

Lemma 5.8. Let $\varepsilon = 0.5 N^{-\frac{d+2}{2(d-2)}}$, $N_{SO}^{\varepsilon}$ the $\varepsilon$-net of $SO(d)$ defined in (5.33), $\phi_\varepsilon(\cdot)$ defined in (5.35), $\hat{\beta}_i(\cdot)$ defined in (5.38), then

$$
\sup_{Q_l \in N_{SO}^{\varepsilon}} \sup_{R \in \phi_\varepsilon(Q_l)} \frac{1}{N} \sum_{i \in C} \|\beta_i(R) - \hat{\beta}_i(Q_l)\|_2 \leq c_1 d \sqrt{\frac{1}{N}}
$$

(5.44)

with probability $1 - N^{-c_2 d^2}$, for $c_1$ and $c_2$ universal constants.

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Proof. For each $Q_i \in \mathcal{N}_{SO}$, we construct the index set
\[
\mathcal{D}_\delta(Q_i) := \{ i \in \mathcal{C} : \|Q_i x_i - y_i\|_2 \leq \delta \}. \tag{5.45}
\]
If $i \in \mathcal{C}$ does not belong to $\mathcal{D}_\delta(Q_i)$, we can bound $\|\beta_i(R) - \tilde{\beta}_i(Q_i)\|_2$ using smoothness of $\beta_i$, as we will see later. Therefore, for all $R \in \phi_{\epsilon}(Q_i)$, we split the term in the LHS of (5.44) as
\[
\sum_{i \in \mathcal{C}} \|\beta_i(R) - \tilde{\beta}_i(Q_i)\|_2 \leq 2|\mathcal{D}_\delta(Q_i)| + \sum_{i \in \mathcal{C} \setminus \mathcal{D}_\delta(Q_i)} \|\beta_i(R) - \tilde{\beta}_i(Q_i)\|_2 \tag{5.46}
\]
using the fact that $\|\beta_i(R) - \tilde{\beta}_i(Q_i)\|_2 \leq 2$ for any value of $i$.

Consider $1_{\mathcal{D}_\delta(Q_i)}(i)$ the indicator function of $i \in \mathcal{D}_\delta(Q_i)$. Let $x, y \sim \text{Unif}(\mathbb{S}^{d-1})$. Then, rotation invariance and Lemma B.2 imply that for all $i \in \mathcal{C}$, $1_{\mathcal{D}_\delta(Q_i)}(i)$ is a Bernoulli random variable with probability
\[
p_{\delta} := P(||x - y||_2 < \delta) \leq \frac{\delta^d}{5\sqrt{d}}. \tag{5.47}
\]
Therefore, $1_{\mathcal{D}_\delta(Q_i)}(i)$ is a subgaussian random variable (A.2) with norm
\[
||1_{\mathcal{D}_\delta(Q_i)}(i)||_2^2 \leq (2 \log(5\sqrt{d}\delta^{-d}/2))^{-1}. \tag{5.48}
\]
Since $|\mathcal{D}_\delta(Q_i)| = \sum_{i \in \mathcal{C}} 1_{\mathcal{D}_\delta(Q_i)}(i)$, then by Hoeffding’s inequality (Theorem 2.1), for any $t > 0$
\[
\frac{|\mathcal{D}_\delta(Q_i)|}{N} \leq \frac{\delta^d}{5\sqrt{d}} + t \quad \text{with probability} \quad 1 - \exp \left( -\frac{cNt^2}{p \log \left( \frac{5\sqrt{d}}{2\delta^d} \right)} \right) \tag{5.49}
\]
for some universal constant $c > 0$. Notice that this bound only depends on $Q_i$ and not on other $R \in \phi_{\epsilon}(Q_i)$.

Now, we want to estimate the second term of the RHS of (5.46). First, for all $i \in \mathcal{C} \setminus \mathcal{D}_\delta(Q_i)$, $\|Q_i x_i - y_i\|_2 > \delta$. Then, for $\delta > \epsilon$, any $R \in \phi_{\epsilon}(Q_i)$ satisfies
\[
\|Rx_i - y_i\|_2 \geq \|Q_i x_i - y_i\|_2 - \|(R - Q)x_i\|_2 \geq \delta - \epsilon > 0. \tag{5.50}
\]
Therefore
\[
\beta_i(R) = \frac{Rx_i - y_i}{\|Rx_i - y_i\|_2} \quad \text{and} \quad \tilde{\beta}_i(Q_i) = \frac{Q_i x_i - y_i}{\|Q_i x_i - y_i\|_2}. \tag{5.51}
\]
Moreover,
\[
2\|\beta_i(R) - \tilde{\beta}_i(Q_i)\|_2 \leq \|Rx_i - y_i\|_2^{-1} + \|Q_i x_i - y_i\|_2^{-1} \|\|(R - Q)x_i\|_2
\]
\[
+ \|Rx_i - y_i\|_2^{-1} - \|Q_i x_i - y_i\|_2^{-1} \|\|(R + Q)x_i - 2y_i\|_2. \tag{5.52}
\]
Since $\|Rx_i - y_i\|_2^{-1}$ is differentiable in $\phi_{\epsilon}(Q_i)$, then
\[
\|\|Rx_i - y_i\|_2^{-1} - \|Q_i x_i - y_i\|_2^{-1}\| \leq \epsilon \max_{R \in \phi_{\epsilon}(Q_i)} \frac{\|\hat{R}x_i - y_i\|_2^3}{\|\|Rx_i - y_i\|_2^3}. \tag{5.53}
\]
Hence, for all $i \notin \mathcal{D}_\delta(Q_i)$ and $R \in \phi_{\epsilon}(Q_i)$, we get the bound
\[
\|\beta_i(R) - \beta_i(Q_i)\|_2 \leq \frac{\epsilon(\delta - \epsilon + 2)}{(\delta - \epsilon)^2}. \tag{5.54}
\]
Lemma 5.6 implies \( |N_{SO}^{c} | \leq (6 \sqrt{d} \pi^{-1})^{d(d-1)/2} \). In particular, taking \( \varepsilon = N^{-(d+2)/(2(d-2))} \), \( \delta = (2 \varepsilon)^{1/(d+2)} \) and \( t = d \sqrt{1.5 N^{-1} e^{-1}} \), by union bound over all \( Q_l \in N_{SO}^{c} \), we have that

\[
\sup_{Q_l \in N_{SO}^{c}} \sup_{R \in \phi_i(Q_l)} \frac{1}{N} \sum_{i \in C} \left\| \beta_i(R) - \bar{\beta}_i(Q_l) \right\|_2 \leq c_1 d \sqrt{\frac{1}{N}}
\]  

(5.55)

with probability \( 1 - N^{-c_2 d^2} \), for \( c_1 \) and \( c_2 \) universal constants.

We finally conclude this section with a proof of Lemma 5.5

Proof of Lemma 5.5. To control the second term in the RHS of (5.39), we use the fact that for a given \( Q_l \in N_{SO}^{c} \) and \( T_k \in N_{B}^{c}/\pi \),

\[
\left\langle Q_l T_k x_i, \tilde{\beta}_i(Q_l) \right\rangle = \left\langle Q_l T_k x_i, \frac{Q_l x_i - y_i}{\|Q_l x_i - y_i\|_2} \right\rangle \quad \text{almost everywhere for } i \in \mathcal{C}.
\]  

(5.56)

Lemma B.3 shows that for any \( S \in B_{skew}(d) \) and \( x, y \sim \text{Unif}(S^{d-1}) \) independent,

\[
\mathbb{E} \left[ \exp \left( \lambda \left\langle S x, \frac{x - y}{\|x - y\|_2} \right\rangle \right) \right] \leq \exp \left( \frac{\lambda^2}{4(d-1)} \right).
\]  

(5.57)

Then, rotation invariance and (2.2) imply

\[
\left\| \left\langle Q_l T_k x_i, \tilde{\beta}_i(Q_l) \right\rangle \right\|^2_{\psi_2} \leq \frac{c}{4(d-1)^{-1}}, \quad \text{for all } i \in \mathcal{C}.
\]  

(5.58)

Using Hoeffding’s inequality (Theorem 2.1), we get

\[
\left| \sum_{i \in C} \left\langle Q_l T_k x_i, \tilde{\beta}_i(Q_l) \right\rangle \right| \leq t \quad \text{with probability } \frac{1}{2} - \exp \left( - \frac{c_2 N(d-1)^2 t^2}{p} \right).
\]  

(5.59)

Lemma 5.6 implies \( |N_{SO}^{c} | \leq |N_{B}^{c}/\pi \| \leq (6 \sqrt{d} \pi^{-1})^{d(d-1)/2} \). Let \( \varepsilon = N^{-(d+2)/(2(d-2))} \) and \( t = 2 \sqrt{d \log(N) c_2^{-1}} N \). By union bound over all \( T_k \in N_{SO}^{c} \) and \( Q_l \in N_{SO}^{c} \), we get

\[
\sup_{T_k \in N_l, Q_l \in N_2} \left| \sum_{i \in C} \left\langle Q_l T_k x_i, \tilde{\beta}_i(Q_l) \right\rangle \right| \leq c \sqrt{\frac{d \log(N)}{N}}
\]  

(5.60)

with probability \( 1 - N^{-c_2 d^2} \).

From Lemma 5.7, we have that for any choice of \( \beta_i(R) \),

\[
\sup_{R \in \mathcal{SO}(d), S \in B_{skew}} \left| \sum_{i \in C} \left\langle RS x_i, \beta_i(R) \right\rangle \right| \leq 2 \varepsilon \ p N + \sup_{T_k \in N_{B}^{c}/\pi, Q_l \in N_{SO}^{c}} \left| \sum_{i \in C} \left\langle Q_l T_k x_i, \tilde{\beta}_i(Q_l) \right\rangle \right| \]

\[
+ \sup_{Q_l \in N_{SO}^{c}} \sup_{R \in \phi_i(Q_l)} \left| \sum_{i \in C} \left\| \beta_i(R) - \tilde{\beta}_i(Q_l) \right\|_2 \right|
\]  

(5.61)

Inserting (5.60) and the upper bound (5.44) in Lemma 5.8, we get that with high probability, for all choices of \( \tilde{\beta}_i(R) \),

\[
\sup_{R \in \mathcal{SO}(d), S \in B_{skew}} \frac{1}{N} \sum_{i \in C} \left[ \left\langle RS x_i, \beta_i(R) \right\rangle - \mathbb{E} \left[ \left\langle RS x_i, \beta_i(R) \right\rangle \right] \right] \leq c \sqrt{\frac{d \log(N)}{N}}.
\]  

(5.62)
6 Numerical Simulations

We compare the results of Theorem 1.1 and Theorem 1.2 with numerical solution of Problem 1.3 and the results of Theorem 1.3 with the solution of Problem 1.1. To solve Problem 1.3, we use CVX, a package for specifying and solving convex programs [13, 12]. We enforce the convex SO(d) constraint using the result [21, Theorem 1.3]

\[
\text{conv SO}(d) = \left\{ X \in \mathbb{R}^n : \begin{bmatrix} 0 & X \\ X^T & 0 \end{bmatrix} \preceq I_{2n}, \sum_{i,j=1}^{n} A^{(i,j)} [D X]_{ij} \preceq (n-2)I_{2n-1} \right\}, \tag{6.1}
\]

where \( D := \text{diag}(1, \ldots, 1, -1), A^{(i,j)} := -P_{\text{even}} \lambda_i \rho_j - P_{\text{even}}, \lambda_i := D_2^{\otimes i-1} \otimes A_2 \otimes D_2^{\otimes n-i}, \rho_i := I_2^{\otimes i-1} \otimes A_2 \otimes D_2^{\otimes n-i} \) and

\[
P_{\text{even}} := \frac{1}{2} \begin{bmatrix} 1 & \ 1 \\ \ 1 & -1 \end{bmatrix} \otimes I_2^{\otimes n-1} + \frac{1}{2} \begin{bmatrix} 1 & \ -1 \\ \ -1 & 1 \end{bmatrix} \otimes D_2^{\otimes n-1} \quad \text{for} \quad D_2 := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{6.2}
\]

To solve Problem 1.1, we implement a subgradient descent method, which can be viewed as a discretization of the dynamical system (5.1), using line search as specified in [20, Chapter 3.5]. Algorithm 1 shows a pseudocode of the implementation. The length of each step is controlled by \( \alpha_{max} \). To optimize over the manifold SO(d), the line search is done over the geodesic defined by the Riemannian gradient of \( L \) at \( R, \partial_R L \).

**Algorithm 1** Line-Search method over SO(d).

1: procedure OptimalRotation(\( \{x\}_{i=1}^{N}, \{y\}_{i=1}^{N}, R^0 \in \text{SO}(d), \text{tol} > 0, \alpha_{max} \))
2: for \( k \leftarrow 1, \ldots, \text{niter} \) do
3: \( \partial L^k \leftarrow 0 \)
4: for \( i \leftarrow 1, \ldots, N \) do
5: \( u^k_i \leftarrow R^{k-1} x_i - y_i \)
6: \( u_{\text{eps}} \leftarrow 0 \)
7: if \( \|u^k_i\|_2 < \text{tol} \) then
8: \( u_{\text{eps}} = \text{tol} \)
9: end if
10: \( u^k_i \leftarrow u^k_i / (u_{\text{eps}} + \|u^k_i\|_2) \)
11: \( \partial L^k \leftarrow dL^k + u^k_i x_i^T \)
12: end for
13: \( \partial_R L^k \leftarrow -(R^{k-1})^T \partial L^k - (\partial L^k)^T R^{k-1} / (2N) \)
14: \( \alpha_k \leftarrow \text{LINE-SEARCH}(f(\alpha) = L(R^{k-1} \exp(-\alpha \cdot \partial_R L^k); \{x\}_{i=1}^{N}, \{y\}_{i=1}^{N}, \alpha_{max}) \)
15: \( R^k = R^{k-1} \exp(-\alpha_k \cdot \partial_R L^k) \)
16: if \( \|R^k - R^{k-1}\|_F < \text{tol} \) then
17: break
18: end if
19: end for
20: end procedure

6.1 Uniformly distributed in \( \mathbb{S}^{d-1} \)

In this scenario, given a sample size \( N \) and corruption level, we generate \( \{x_i\}_{i=1}^{N}, \{y_i\}_{i=1}^{N} \) following the distribution specified in (1.1) and (1.2). We assume the ground truth rotation \( R_0 = I \).
6.1.1 Trade-off between corruption level and sample size

We first study the impact of the corruption level and the sample size in recovering the ground truth. We consider sample sizes $N \in [4, 1024]$ and corruption levels $p \in [0.1, 0.99]$. We generate 10 independent random samples of $\{x_i\}_{i=1}^N, \{y_i\}_{i=1}^N$ for each choice of $N$ and $p$. For each random sample we minimize $L(A)$ over $\text{conv } \mathcal{SO}(d)$ and $\mathcal{SO}(d)$, taking as initial point the solution of the least squares problem. We say we recover the ground truth if $\|I - A^*\| \leq 10^{-2}$, where $A^*$ is the minimizer. We then compute the empirical probability of exact recovery for each combination of $p, N$, as shown in Fig. 2.

![Figure 2: Empirical probability of exact recovery minimizing $L(A)$ over different domains. Data is uniformly distributed over $S^{d-1}$. The dark line is the theoretical upper bound of admissible corruption level in (1.8) and (1.14).](image)

We start by analyzing the probability of exact recovery solving Problem 1.3, i.e. minimizing $L(A)$ over $\text{conv } \mathcal{SO}(d)$. Fig. 2a and Fig. 2d show that the transition between exact recovery in all of the experiments and no recovery in any of them follows the bound (1.8) of Theorem 1.1, equals to $\tilde{p}(d) - c\sqrt{\log N/N}$.

Since the minimizer of $L(A)$ over $\text{conv } \mathcal{SO}(d)$, $A^*_{\text{conv } \mathcal{SO}(d)}$, is not necessarily a rotation, one can project $A^*_{\text{conv } \mathcal{SO}(d)}$ on $\mathcal{SO}(d)$ to find the closest rotation. Although the theorems in this paper do not provide any insight in this case, empirically Fig. 2b and Fig. 2e show that projecting the solution of Problem 1.3 over $\mathcal{SO}(d)$ does not significantly improve the probability of exact recovery when the corruption level is above $\tilde{p}(d)$.

Then, we inspect the probability of exact recovery solving Problem 1.1, i.e. minimizing $L(A)$ over $\mathcal{SO}(d)$. Fig. 2c and Fig. 2f show the transition between exact recovery in all of the experiments
and no recovery in any of them follows the bound (1.14), derived from Theorem 1.3 and equals to 
\[ p < 1 - c \sqrt{\frac{\log N}{N}}. \]
This means that we can always recover the ground truth regardless of the 
corruption level if we have enough samples.

6.1.2 Impact of initialization in dynamical system (5.1)

Exact recovery of the ground truth rotation minimizing \( L(A) \) over \( SO(d) \) depends on the selection 
of the initial point in the dynamical system (5.1). Our second experiment explores the influence 
of this selection. We consider four sample sizes, \( N \in \{24, 32, 64, 128\} \). For each \( N \), we generate 
a random sample \( \{x_i\}_{i=1}^{N}, \{y_i\}_{i=1}^{N} \) with \( d = 4 \) and \( p = 0.75 \). Then, for each sample, we solve 
the dynamical system (5.1) a hundred times, each time starting from a rotation \( R(0) \) chosen at 
random such that \( \| \log(R(0)) \|_2 \) is uniformly distributed in \([0, \pi)\). Given a starting point \( R(0) \), 
if the solution of the dynamical system \( R(t) \) converges to \( I \), we record the time \( T_{cvg} \) such that 
\( \| \log(R(T_{cvg})) \|_2 < 10^{-2} \), i.e. \( D_{SO}(R(T_{cvg}), I) \) is small. Otherwise, we include \( R(0) \) in the set of 
starting points without exact recovery. We also consider all the points in the trajectory \( R(t) \), and, 
for each of them, we compute \( T_{cvg} \) as if the dynamical system started from there.

![Figure 3: Empirical envelop of convergence time \( T_{cvg} \) minimizing \( L(A) \) over \( SO(4) \) with initial 
point \( R(0) \). The data is distributed uniformly in \( S^3 \) with corruption level \( p = 0.75 \). The red region 
corresponds to values of \( \| \log(R(0)) \|_2 \) where exact recovery is not guaranteed. The dark line is the 
thoretical bound of convergence time (1.13) in Theorem 1.3.
](image)

**Fig. 3** shows in blue an envelop of the convergence time to \( I, T_{cvg} \), in terms of \( \| \log(R(0)) \|_2 \), where \( R(0) \) is the starting point of the dynamical system (5.1). If empirically we observe a dynamical 
system with starting point \( R(0) \) do not converge to \( I \), then all the values greater or equal than 
\( \| \log(R(0)) \|_2 \) belong to the region of no exact recovery, denoted in red.

Comparing **Fig. 3a** through **Fig. 3d** we notice the region of exact recovery increases as the sample 
size \( N \) gets larger, as described in condition (1.12) of Theorem 1.3. Moreover, **Fig. 3d** shows that the 
convergence time \( T_{cvg} \) follows the curve 
\[
T(\| \log(R(0)) \|_2) = \frac{d}{1-p} \cosh^{-1}\left( \sec\left( \frac{\| \log(R(0)) \|_2}{2} \right) \right) + o(1),
\]
equals to the convergence time (1.13) in Theorem 1.3.

6.2 Stanford Bunny

Although the theorems proven in this paper assume that the data is uniformly distributed on \( S^{d-1} \), 
we are interested in the generalization of the bounds for other data distributions. For example, 
a common source of corruption is mislabeling of data pairs. We create a data set using the CT 
scan of the Stanford terra-cota bunny [23], [30]. We select the points with highest intensity, and 
we construct the set of coordinates \( \{b_j\}_{j=1}^{N_{Bn}} \subset \mathbb{R}^3 \) where \( N_{Bn} \approx 230K \) entries. We normalize the 
points to be centered at zero and to have a maximum length of one. This point cloud can be
modeled as the discrete distribution \( \rho_{Bn}(x) := \frac{1}{N_{Bn}} \sum_{j=1}^{N_{Bn}} \delta_{b_j}(x) \). For a given sample size \( N \), we generate samples \( \{x_i\}_{i=1}^{N}, \{y_i\}_{i=1}^{N} \) as specified in (1.1) and (1.2), except now we change from uniform distribution over a sphere to \( \rho_{Bn}(x) \).

\[
\sum_{j=1}^{N_{Bn}} \delta_{b_j}(x)
\]

Figure 4: Data generation using the Stanford Bunny. Uncorrupted points are denoted by blue circles, whereas corrupted ones are red triangles.

6.2.1 Trade-off between corruption level and sample size

In Fig. 5, we reproduce the experiment in Section 6.1.1, except now samples are drawn from \( \rho_{Bn}(x) \). Overall, there are not noticeable changes between Fig. 2 and Fig. 5.

![Figure 5](image)

Figure 5: Empirical probability of exact recovery minimizing \( L(A) \) over different domains. Data is uniformly distributed over Stanford Bunny, \( \rho_{Bn}(x) \). The dark line is the theoretical upper bound of admissible corruption level in (1.8) and (1.14).

We start by considering the solution of Problem 1.3, i.e. optimizing \( L(A) \) over \( \text{conv SO}(3) \). Fig. 5a shows there is no exact recovery in any of the experiments when \( p > \tilde{p}(3) \). Additionally the level set of exact recovery in all of the experiments follows the curve \( \tilde{p}(3) - c\sqrt{\log N/N} \), equals to the bound (1.8) of Theorem 1.1. Similar to Fig. 2b, Fig. 5b shows there is not substantial improvement in the probability of exact recovery when the solution of Problem 1.3 is projected on \( \text{SO}(3) \) to get back a rotation matrix.

Regarding the minimization of \( L(A) \) over \( \text{SO}(3) \), Fig. 5c shows the transition transition between exact recovery in all experiments and no recovery in any experiment follows the curve \( p < 1 - c\sqrt{\log N/N} \), equal to the bound (1.14), derived from Theorem 1.3. Similar as the data uniformly distributed over \( S^2 \), we can always recover the ground truth regardless of the corruption level if we have enough samples.
6.2.2 Impact of initialization in dynamical system (5.1)

In Fig. 6, we reproduce the experiment in Section 6.1.2, except now sampling from $\rho_{Bn}(x)$. Overall, Fig. 6 follows the same behavior as Fig. 3. First, the region where exact recovery is not guaranteed shrinks as the sample size grows, as shown by Fig. 6a through Fig. 6d. Second, in Fig. 6c and Fig. 6d we can see that the convergence time $T_{cvg}$ follows the curve 

$$T_{cvg}(\| \log(R(0)) \|_2) = \frac{d}{1-p} \cosh^{-1}\left(\sec\left(\| \log(R(0)) \|_2\right)\right) + o(1)$$

equals to the convergence time (1.13) in Theorem 1.3.

![Figure 6: Empirical envelop of convergence time $T_{cvg}$ minimizing $L(A)$ over $SO(3)$ with initial point $R(0)$. The data is distributed uniformly over Stanford Bunny, $\rho_{Bn}(x)$, with corruption level $p = 0.75$. The red region corresponds to values of $\| \log(R(0)) \|_2\|$ where exact recovery is not guaranteed. The dark line is the theoretical bound of convergence time (1.13) in Theorem 1.3](image)

7 Conclusion

We prove the point-set registration problem with outliers can be exactly solved by minimizing the least-unsquared-deviatiton (LUD) over $\mathbb{R}^d$ or conv $SO(d)$ only when the corruption level $p$ is less than $\tilde{p}(d) - o(1)$. On the other hand, we proved that we can exactly recover the ground truth rotation $R_0$ by minimizing the LUD over $SO(d)$ using the gradient flow (5.1) for any corruption level $p < 1$ and initial point $R(0)$ with $\| \log(R_0^T R(0)) \|_2 < \pi$ when the sample size $N$ is large enough. We showed these results are consistent with numerical simulations for data uniformly distributed on $S^{d-1}$ and on discrete points of Stanford bunny. In future work, we shall explore the extension of this theoretical bounds to arbitrary distributions.

Acknowledgments

The work of C. O. is partially supported by the Stanford Graduate Fellowship in Science & Engineering. The work of L.Y. is partially supported by the National Science Foundation under award DMS-1818449 and by the U.S. Department of Energy, Office of Science, Office of Advanced Scientific Computing Research, Scientific Discovery through Advanced Computing (SciDAC) program.

A Bounds on sub-gaussian norms for particular random variables

In this section, we provide some common examples of sub-gaussian random variables.
1. **Bounded random variables** [26] Chapter 2] any bounded random variable $X$ is sub-gaussian with

$$\|X\|_{\psi_2} \leq \frac{\|X\|_{\infty}}{\sqrt{\log 2}}$$  \hfill (A.1)

2. **Bernoulli random variables** [5] any $X$ Bernoulli random variable with $p \in (0, 1/2)$ is sub-gaussian with

$$\|X - \mathbb{E}X\|_{\psi_2}^2 = \frac{1 - 2p}{2(\log(1 - p) - \log(p))} \leq \frac{1}{2|\log(2p)|}$$  \hfill (A.2)

3. **Lipschitz function on the sphere** [26] Chapter 5] Let $f : S^{d-1} \to \mathbb{R}$ be a Lipschitz function with Lipschitz constant $\|f\|_{\text{Lip}}$. If $X \sim \text{Unif}(S^{d-1})$, then $f(X) - \mathbb{E}f(X)$ is sub-gaussian with

$$\|f(X) - \mathbb{E}f(X)\|_{\psi_2} \leq \frac{c\|f\|_{\text{Lip}}}{\sqrt{d}}.$$  \hfill (A.3)

for some universal constant $c > 0$.

## B Properties of $\text{Unif}(S^{d-1})$

In this section we list useful properties related with data uniformly distributed on the sphere on dimension $d$, $S^{d-1}$. To start, we use the fact that for $x \in \text{Unif}(S^{d-1})$

$$\mathbb{E}[xx^\top] = \frac{I_d}{d}.$$  \hfill (B.1)

as well as rotation invariance of the distribution to derive the following bounds.

**Lemma B.1.** Let $x \sim \text{Unif}(S^{d-1})$ then for any matrix $A \in \mathbb{R}^{d \times d}$

$$\frac{\|A\|_*}{d} \leq \mathbb{E}\|Ax\|_2 \leq \frac{\|A\|_F}{\sqrt{d}}.$$  \hfill (B.2)

**Proof.** For the lower bound, we consider the full SVD decomposition of $A = U_A \Sigma_A V_A^\top$, then by Cauchy-Schwartz inequality we get

$$\mathbb{E}\|Ax\|_2 = \mathbb{E}\|\Sigma_A V_Ax\|_2 \geq \mathbb{E}\left[x^\top V_A^\top \Sigma_A V_A x\right] = \frac{\text{Tr}(\Sigma_A)}{d} = \frac{\|A\|_*}{d}. \hfill (B.3)$$

For the upper bound, we use the concavity of the square root to get

$$\mathbb{E}\|Ax_1\|_2 \leq \sqrt{\text{Tr}(A^\top A \mathbb{E}[x_1x_1^\top])} = \frac{\|A\|_F}{\sqrt{d}}.$$  \hfill (B.4)

\[\square\]

For quantities depending on $x, y \sim \text{Unif}(S^{d-1})$ independently, it is handy to express $y$ in spherical coordinates with respect to $x$. Consider $U_{xy} := [u_{xy,1}, u_{xy,2}]$ such that $U_{xy}^\top U_{xy} = I$ and

$$x = U_{xy} \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad y = U_{xy} \begin{bmatrix} \cos \theta_1 \\ \sin \theta_1 \end{bmatrix}, \quad 0 \leq \theta_1 < \pi.$$  \hfill (B.5)
Some examples of this change of coordinates are
\[ ||x - y||_2 = 2 \sin \frac{\theta_1}{2}, \quad ||x + y||_2 = 2 \cos \frac{\theta_1}{2}, \quad \frac{x - y}{||x - y||_2} = \sin \frac{\theta_1}{2} x + \cos \frac{\theta_1}{2} u_{xy,2} \quad (B.6) \]
where \( u_{xy,2} \) is uniformly distributed on the sphere \( S^{d-2} \) orthogonal to \( x \). Now, to compute expectations, we use the area element of a sphere,
\[ d_{S^{d-1}} V = \sin^{d-2}(\theta_1) \sin^{d-3}(\theta_2) \cdots \sin(\theta_{d-2})d\theta_1 \cdots d\theta_{d-1} \quad (B.7) \]
and the definition of the Beta function,
\[ B \left( \frac{m + 1}{2}, \frac{n + 1}{2} \right) = 2 \int_0^{\pi/2} \sin^n \theta \cos^m \theta d\theta = \frac{\Gamma \left( \frac{m + 1}{2} \right) \Gamma \left( \frac{n + 1}{2} \right)}{\Gamma \left( \frac{m + n}{2} + 1 \right)} \quad (B.8) \]
where the Gamma function satisfies \( \Gamma(n) = (n - 1)! \) for any \( n \in \mathbb{N}^+ \). We list some useful results of expectations relating two independent random variables in \( S^{d-1} \) in the following lemma.

**Lemma B.2.** Let \( x, y \sim \text{Unif}(S^{d-1}) \) i.i.d. then
\[ \mathbb{E} ||x - y||_2 = 2 \frac{\beta \left( \frac{d - 1}{2}, \frac{1}{2} \right)}{\beta \left( \frac{d - 1}{2}, \frac{1}{2} \right)} \], \quad (B.9a)
\[ P(||x - y||_2 < \delta) \leq \frac{\delta d}{5 \sqrt{d}} \text{ for } \delta < 2, \quad (B.9b) \]
\[ \mathbb{E} \left[ \frac{1}{||x - y||_2 ||x + y||_2} \right] = \frac{\beta \left( \frac{d - 2}{2}, \frac{1}{2} \right)}{d \beta \left( \frac{d - 1}{2}, \frac{1}{2} \right)} I_d. \quad (B.9c) \]

**Proof.** Using the change of variables given by (B.6) and the area element (B.7), for (B.9a), we get
\[ \mathbb{E} ||x - y||_2 = 2 \frac{\int_0^{\pi} \sin \frac{\theta_1}{2} \sin^{d-2} \theta_1 d\theta_1}{\int_0^{\pi} \sin^{d-2} \theta_1 d\theta_1} = 2d^{-1} \frac{B \left( \frac{d}{2}, \frac{d - 1}{2} \right)}{B \left( \frac{d - 1}{2}, \frac{1}{2} \right)} \]
\[ = 2 \frac{\beta \left( \frac{d - 1}{2}, \frac{1}{2} \right)}{\beta \left( \frac{d - 1}{2}, \frac{1}{2} \right)}, \quad (B.10) \]
using the product identity of the Gamma function \( \Gamma(n)\Gamma(n + 1/2) = 2^{1-2n}\Gamma(1/2)\Gamma(2n) \).

Using the same change of variables in (B.6) for (B.9b), we get
\[ P(||x - y|| \leq \delta) = P \left( \sin \frac{\theta_1}{2} \leq \delta \right) = 2d^{-1} \frac{\int_{\delta/2}^{\pi/2} \sin^{d-1} \theta \cos^{d-2} \theta d\theta}{\int_0^{\pi/2} \sin^{d-2} \theta d\theta}. \quad (B.11) \]
Then, letting \( u = \sin^2 \theta \), and using \((1 - \delta^2/4) < 1\) we get,
\[ P(||x - y|| \leq \delta) \leq 2d^{-2} \frac{\int_0^{\delta^2/4} u^{d-1} du}{B \left( \frac{d - 1}{2}, \frac{1}{2} \right)} = \frac{\delta^d}{2d \beta \left( \frac{d - 1}{2}, \frac{1}{2} \right)} \leq \frac{\delta^d}{5 \sqrt{d}}. \quad (B.12) \]

For (B.9c), we use the representation on (B.6) to get
\[ \mathbb{E} \left[ \frac{1}{||x - y||_2 ||x + y||_2} \right] = \frac{\int_0^{\pi} \sin^{-1} \theta_1 \sin^{d-2} \theta_1 d\theta_1}{2 \int_0^{\pi} \sin^{d-2} \theta_1 d\theta_1} = \frac{B \left( \frac{d - 2}{2}, \frac{1}{2} \right)}{2 \beta \left( \frac{d - 1}{2}, \frac{1}{2} \right)}. \quad (B.13) \]
For \([B.9d]\), we notice that \(E[u_{xy,2}|x] = 0\) given that \(u_{xy,2}\) is uniformly distributed in the sphere orthogonal to \(x\). Therefore using \((B.9a)\), we get

\[
E \left[ \frac{x - y}{\|x - y\|_2} x^\top \right] = \frac{1}{d} B \left( d-1, 1/2 \right) I_d.
\]  

(B.14)

\[\square\]

At last, we are interested in inner products of the form

\[
\langle Sx, \frac{x - y}{\|x - y\|_2} \rangle \quad \text{where} \quad S \in S_{\text{skew}}(d).
\]  

(B.15)

Since \(S\) is skew-symmetric, then \(x^\top S x = 0\). Therefore if \(Sx \neq 0\), we can use \(Sx/\|Sx\|_2\) as a second orthonormal vector to express \(y\) in spherical coordinates. Assuming \(d \geq 3\), we consider

\[
U_{xSy} := [u_{xSy,1}, u_{xSy,2}, u_{xSy,3}], \quad U_{xSy}^\top U_{xSy} = I
\]  

(B.16)

such that for some \(0 \leq \theta_1, \theta_2 < \pi\)

\[
x = U_{xSy,1} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad Sx = \|Sx\|_2 U_{xSy,2} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad y = U_{xSy,3} \begin{bmatrix} \cos \theta_1 \\ \sin \theta_1 \cos \theta_2 \\ \sin \theta_1 \sin \theta_2 \end{bmatrix}.
\]  

(B.17)

Therefore, under this change of coordinates

\[
\langle Sx, \frac{x - y}{\|x - y\|_2} \rangle = \|Sx\|_2 \cos \frac{\theta_1}{2} \cos \theta_2.
\]  

(B.18)

**Lemma B.3.** Let \(x, y \in \text{Unif}(S^{d-1})\) i.i.d and \(S \in S_{\text{skew}}(d)\). If \(\|S\|_2 \leq 1\), then

\[
E \left[ \exp \left( \lambda \langle Sx, \frac{x - y}{\|x - y\|_2} \rangle \right) \right] \leq \exp \left( \frac{\lambda^2}{4(d-1)} \right), \quad \lambda > 0
\]  

(B.19)

**Proof.** To show \((B.19)\), we compute a bound for each of the moments using the change of variables in \((B.18)\) and the area element \((B.7)\),

\[
E \left[ \|Sx\|_2^r \cos^r \left( \frac{\theta_1}{2} \right) \cos(\theta_2)^r \right] \leq E \left[ \cos^r \frac{\theta_1}{2} \right] E [\sin^r \theta_2]
\]

\[
= \frac{\int_0^\pi \cos^r \left( \frac{\theta_2}{2} \right) \sin^{d-2} \theta_1 \ d\theta_1 \int_0^\pi \cos^r \theta_2 \sin^{d-3} \theta_2 \ d\theta_2}{\int_0^\pi \sin^{d-2} \theta_1 \ d\theta_1 \int_0^\pi \sin^{d-3} \theta_2 \ d\theta_2}.
\]  

(B.20)

Since \(\cos \theta_2\) is an odd function around \(\pi/2\), then \((B.20)\) is 0 for odd values of \(r\). For even values of \(r\),

\[
E \left[ \|Sx\|_2^r \cos^r \left( \frac{\theta_1}{2} \right) \cos(\theta_2)^r \right] \leq \frac{\Gamma(d-1) \Gamma \left( \frac{r+1}{2} \right)}{\Gamma \left( \frac{r}{2} + d - 1 \right)}
\]

(B.21)
Adding all the moments together and using the product identity of the Gamma function, we get

\[
\mathbb{E} \left[ \exp \left( \lambda \left( S_x, \frac{x - y}{\|x - y\|_2} \right) \right) \right] \leq \sum_{r=0}^{\infty} \frac{\lambda^{2r}}{\Gamma(2r + 1)} \frac{\Gamma \left( \frac{d}{2} + r + 1 \right)}{\Gamma \left( \frac{d}{2} + 1 \right)} \Gamma \left( r + d - 1 \right).
\]

(B.22)

Then, using the bound \((\Gamma (d - 1)) (\Gamma (r + d - 1))^{-1} < (d - 1)^{-r}\), we get (B.19).

\[\square\]

C Proof of Lemma 3.1: Bounds for \(\|u + v\|_2 - \|v\|_2\)

Using Taylor series around \(\|u\|_2 = 0\), we construct lower and upper bound for \(\|u + v\|_2 - \|v\|_2\), that are useful in dealing with the contribution of corrupted points to \(L(A) - L(I)\).

**Proof of Lemma 3.1.** Let \(\alpha \geq 0\) and \(\theta \in (0, \pi)\) such that \(\alpha = \|u\|_2/\|v\|_2\) and \(\cos \theta = (u^\top v)/\|u\|_2\|v\|_2\). Then, we can rewrite \(\|u + v\|_2 - \|v\|_2\) as a function of \(\alpha\)

\[f(\alpha) = \|v\|_2 \left( \sqrt{1 + 2\alpha \cos \theta + \alpha^2} - 1 \right).\]

(C.1)

Our goal is to use Taylor expansion of \(f(\alpha)\) around \(\alpha = 0\) and bound the high degree contributions. First note that

\[
\frac{df}{d\alpha}(\alpha) = \frac{\|v\|_2(\alpha + \cos \theta)}{\left(\sin^2 \theta + (\cos \theta + \alpha)^2\right)^{3/2}}, \quad \frac{d^2f}{d\alpha^2}(\alpha) = \frac{\|v\|_2 \sin^2 \theta}{\left(\sin^2 \theta + (\cos \theta + \alpha)^2\right)^{3/2}} \leq \frac{\|v\|_2}{\sin \theta}.
\]

(C.2)

Therefore,

\[\alpha\|v\|_2 \cos \theta \leq f(\alpha) \leq \alpha\|v\|_2 \cos \theta + \frac{\alpha^2}{2} \|v\|_2 \sin \theta.\]

(C.3)

Replacing the values of \(\alpha\) and \(\theta\), we get the bound. \[\square\]

D Proof of Lemma 3.3

**Proof of Lemma 3.3.** Let \(X_A\) be

\[X_A(\{x_i\}_{i=1}^n) := \sum_{i=1}^{n} f_i(A, x_i) - \mathbb{E} f_i(A, x_i).\]

(D.1)

Then, using Hoeffding’s inequality (Theorem 2.1), we get

\[\| (X_A - \mathbb{E} X_A) - (X_B - \mathbb{E} X_B) \|_{\psi_2}^2 \leq c^2 \sum_{i=1}^{n} \| (f_i(A, x_1) - \mathbb{E} f_i(A, x_1)) - (f_i(B, x_1) - \mathbb{E} f_i(B, x_1)) \|_{\psi_2}^2.\]

(D.2)
For any subgaussian random variable $X$, $\|X - \mathbb{E}X\|_{\psi_2} \leq \bar{c}\|X\|_{\psi_2}$ with absolute constant $\bar{c}$. Then, to satisfy condition (3.9) gives

$$\|(X_A - \mathbb{E}X_A) - (X_B - \mathbb{E}X_B)\|_{\psi_2}^2 \leq c^2 nK\|A - B\|_F^2. \quad (D.3)$$

To measure the size of $\mathcal{B}_1(d) := \{A \in \mathbb{R}^{d \times d} \mid \|A\|_F \leq 1\}$, let $G \in \mathbb{R}^{d \times d}$ be a random matrix where each entry distributes independent $\mathcal{N}(0, 1)$, then from the definitions in (2.4), the Gaussian width of $\mathcal{B}_1(d)$ is

$$w(\mathcal{B}_1(d)) = \mathbb{E}\sup_{\|A\|_F \leq 1} \|A, G\| = \mathbb{E}\|G\|_F = (\mathbb{E}\|G\|_F^2)^{1/2} = d, \quad (D.4)$$

and the radius of $\mathcal{B}_1(d)$ is $\text{rad}(\mathcal{B}_1(d)) = 1$. Since $X_A - \mathbb{E}X_A$ satisfies (2.5), Theorem 2.2 implies that

$$\mathbb{P} \left\{ \sup_{\|A\|_F \leq 1} |X_A| \leq c'K\sqrt{n}(d + u) \right\} \geq 1 - 2\exp(-u^2). \quad (D.5)$$

Taking $u = \sqrt{\log n/K}$ gives the desired bound.

\section*{E \ Proof of Lemma 4.2: Dynamics of $\|\log R(t)\|_2$}

We first introduce the following lemma.

\textbf{Lemma E.1 (Derivative of matrix logarithm).} For $A \in \mathbb{C}^{d \times d}$ with no eigenvalues in $\mathbb{R}^-$

$$\frac{d}{dt} \log(A) = \int_0^1 [s(A - I) + I]^{-1} \frac{dA}{dt} [s(A - I) + I]^{-1} ds. \quad (E.1)$$

\textbf{Proof.} For any $A \in \mathbb{C}^{d \times d}$ with no eigenvalues in $\mathbb{R}^-$, $\log A$ can be defined as \cite{15}

$$\log(A) = \int_0^1 (A - I) [s(A - I) + I]^{-1} ds. \quad (E.2)$$

Therefore, let $A$ be a function of $t$, then the derivative of $\log A$ with respect to $t$ is

$$\frac{d\log A}{dt} = \int_0^1 \frac{dA}{dt} [s(A - I) + I]^{-1} - s(A - I) [s(A - I) + I]^{-1} \frac{dA}{dt} [s(A - I) + I]^{-1} ds$$

$$= \int_0^1 [s(A - I) + I]^{-1} \frac{dA}{dt} [s(A - I) + I]^{-1} ds. \quad (E.3)$$

\textbf{Proof of Lemma 4.2.} This lemma is tailored from \cite{41} Lemma 1. Notice $\|\log(R(t))\|_2$ is absolutely continuous in any interval $I \subset \mathbb{R}$ because it is a composition of a locally Lipschitz function and an absolutely continuous function. Therefore $\frac{d\|\log R(t)\|_2}{dt}$ and $\frac{dR(t)}{dt}$ exists almost everywhere and

$$\frac{dR(t)}{dt} \in R(t) \cdot \mathcal{S}(R(t)) \quad \text{almost everywhere.} \quad (E.4)$$

To define $\|\log R(t)\|_2$, we consider the SVD of $\log R(t)$

$$\log R(t) = \sum_{j=1}^{[d/2]} \sigma_j(t)U_j(t)A_2U_j(t)^{\top} \quad (E.5)$$
with
\[ U(t) = \begin{bmatrix} U_1(t) & \ldots & U_{[d/2]}(t) \end{bmatrix} \]
such that \( \{U_j(t)\}_{j=1}^{[d/2]} \subset \mathbb{R}^{d \times 2} \), \( U(t)^	op U(t) = I \), and \( \pi \geq \sigma_1(t) \geq \cdots \geq \sigma_{[d/2]}(t) \geq 0 \). Then
\[
\| \log R(t) \|_2 := \max \{ \sigma_1(t), \ldots, \sigma_{[d/2]}(t) \}. \quad (E.6)
\]
To study \( \frac{d\| \log R(t) \|_2}{dt} \), we first consider the dynamics of all the singular values of \( \log R(t) \). Let \( \text{blockdiag}(\{V_j\}_{i=1}^k) \) be the block diagonal matrix with diagonal blocks \( V_i \). We denote
\[
\Sigma(t) := \text{blockdiag}\left( \sigma_1(t)I_2, \ldots, \sigma_{[d/2]}(t)I_2 \right) \quad (E.7)
\]
and
\[
V(t) := \begin{bmatrix} U_1(t)A_2^\top & \ldots & U_{[d/2]}(t)A_2^\top \end{bmatrix},
\]
then \( \log R(t) = U(t)\Sigma(t)V(t) ^\top \).
Recall that for any \( A \in \mathbb{R}^{d \times d} \) with SVD decomposition \( A = U_A\Sigma_AV_A ^\top \),
\[
\frac{d\Sigma_A}{dt} = I \odot \left( U_A ^\top \frac{dA}{dt} V_A \right) \quad (E.8)
\]
where \( \odot \) denotes the entry-wise product. Then, in particular, for \( \log R(t) \)
\[
\frac{d\Sigma}{dt} = I \odot \text{blockdiag} \left( \left\{ U_j ^\top \left( \frac{d\log R}{dt}(t) \right) U_j A_2 ^\top \right\}_{j=1}^{[d/2]} \right). \quad (E.10)
\]
To compute \( \frac{d\log R(t)}{dt} \), we use the fact that \( \frac{dR}{dt}(t) = R(t) S(t) \), \( S(t) \in \mathcal{S}(t) \), then \textbf{Lemma E.1} implies that
\[
\frac{d\log R}{dt} = \int_0^1 [s(R-I) + I]^{-1} RS [s(R-I) + I]^{-1} ds. \quad (E.11)
\]
To evaluate \( (E.11) \), we rewrite \( R(t) \) using the planar decomposition in \textbf{Lemma 4.1}
\[
R(t) - I_d = \sum_{i=1}^{[d/2]} U_i(t)(R_{\sigma_i}(t) - I_2)U_i(t) ^\top, \quad R_{\sigma_i} = \exp(\sigma_i A_2) \in \mathcal{SO}(2). \quad (E.12)
\]
Using decomposition \( (E.12) \), for all \( i = 1, \ldots, [d/2] \), we simplify the product
\[
U_i ^\top \left( \frac{d\log R}{dt} \right) U_i = \int_0^1 [s(R_{\sigma_i} - I) + I]^{-1} R_{\sigma_i}U_i ^\top S U_i [s(R_{\sigma_i} - I) + I]^{-1} ds \quad (E.13)
\]
Notice that \( U_i ^\top SU_i \in \mathcal{S}_{\text{skew}}(2) \) then \( U_i ^\top SU_i = \frac{1}{2} \langle S, U_i A_2 U_i ^\top \rangle A_2 \). Therefore \( (E.13) \) becomes
\[
U_i ^\top \left( \frac{d\log R}{dt} \right) U_i = \frac{1}{2} \langle S, U_i A_2 U_i ^\top \rangle \int_0^1 [s(R_{\sigma_i} - I) + I]^{-1} R_{\sigma_i}A_2 [s(R_{\sigma_i} - I) + I]^{-1} ds
\]
\[
= \frac{1}{2} \langle S, U_i A_2 U_i ^\top \rangle A_2, \quad (E.14)
\]
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given that \( A_2 \) commutes with any rotation in \( \mathcal{SO}(2) \).

Inserting (E.14) in RHS of (E.10) we get
\[
\frac{d\Sigma}{dt} = \text{blockdiag} \left\{ \frac{1}{2} \left[ S(t) U_j(t) A_2 U_j(t)^\top \right] I_2 \right\}_{j=1}^{[d/2]}.
\] (E.15)

Using the definition of \( \Sigma(t) \), we get
\[
\frac{d\sigma_i(t)}{dt} = \frac{1}{2} \left\langle S(t), U_i(t) A_2 U_i(t)^\top \right\rangle \quad i = 1, \ldots, \lfloor d/2 \rfloor.
\] (E.16)

Therefore, the generalize gradient of \( \| \log R \|_2 \) with respect to \( t \) is given by
\[
\frac{d}{dt} \| \log R(t) \|_2 = \text{conv} \left\{ \frac{1}{2} \left\langle S(t), U_i(t) A_2 U_i(t)^\top \right\rangle \right\}
\] (E.17)

as defined in (4.8). Then, for all \( t \) such that \( \frac{d}{dt} \| \log R(t) \|_2 \) exists, then RHS of (E.17) is a singleton. Therefore
\[
\frac{d}{dt} \| \log R(t) \|_2 \in \{ a \mid \exists S \in \mathcal{S}(t), \left\langle S, U A_2 U^\top \right\rangle = 2a \ \forall U \in \mathcal{U}(R(t)) \}
\] (E.19)

almost everywhere.

**F  Proof of Lemma 5.6: Constructing \( \varepsilon \)-net for \( \mathcal{SO}(d) \)**

We first introduce the following theorem.

**Lemma F.1** (Lipschitz constant of the matrix exponential in \( \mathcal{S}_{\text{skew}}(d) \)). *Let \( X, Y \) skew-symmetric matrices, then*
\[
\| \exp(X) - \exp(Y) \|_F \leq \| X - Y \|_F.
\] (F.1)

*Proof.* Let the directional derivative of matrix exponential of \( Z \) in direction \( X - Y \) be \[15\]
\[
d\exp_Z(X - Y) = \int_0^1 \exp((1-t)Z)(X - Y) \exp(tZ) \ dt.
\] (F.2)

By continuity of the matrix exponential, we have that for \( X, Y \in \mathcal{S}_{\text{skew}}(d) \), there exist \( Z \in \mathcal{S}_{\text{skew}}(d) \) such that
\[
\| \exp(X) - \exp(Y) \|_F \leq \left\| \int_0^1 \exp((1-t)Z)(X - Y) \exp(tZ) \ dt \right\|_F,
\] (F.3)
given that \( \mathcal{S}_{\text{skew}}(d) \) is a vector space. By triangle inequality,
\[
\| \exp(X) - \exp(Y) \|_F \leq \int_0^1 \| \exp((1-t)Z)(X - Y) \exp(tZ) \|_F \ dt = \| X - Y \|_F,
\] (F.4)
where the last equality follows from the fact that the exponential of a skew-symmetric matrix is a rotation, and the Frobenius norm is invariant under rotations. \( \square \)
Proof of [Lemma 5.6]. Recall $A := \{ S \in \mathbb{R}^{d \times d} \mid \| \text{skew}(S) \|_2 \leq 1/\sqrt{2}, \, S(i,j) = 0 \text{ if } i \geq j \}$. Let

$$\text{triu}(\cdot) : \mathbb{R}^{d \times d} \rightarrow \mathbb{R}^{d \times d}$$

be defined as

$$\text{triu}(A)(i,j) := \begin{cases} A(i,j) & \text{if } i \leq j \\ 0 & \text{otherwise} \end{cases} \quad (F.5)$$

Notice that for any $S \in \mathcal{B}_{\text{skew}}(d)$, $\text{triu}(2^{-1/2}S) \in A$. Therefore, there exists $\tilde{S} \in \mathcal{N}_A^\varepsilon$ such that $\left\| \text{triu} \left( 2^{-1/2}S \right) - \tilde{S} \right\|_F \leq \varepsilon$. Since $\| \text{skew}(\sqrt{2}\tilde{S}) \|_2 \leq 1$ then

$$\left\| S - \text{skew}(\sqrt{2}\tilde{S}) \right\|_F = \left\| \text{triu} \left( 2^{-1/2}S \right) - \tilde{S} \right\|_F \leq \varepsilon. \quad (F.6)$$

Similarly, for $R \in SO(d)$, let $S_R = \log(R)/\pi$. Then $R = \exp(\pi S_R)$ and $S_R \in \mathcal{B}_{\text{skew}}(d)$. Therefore, there exists $\tilde{S}_R \in \mathcal{N}_A^\varepsilon$ such that $\left\| S_R - \text{skew}(\sqrt{2}\tilde{S}_R) \right\|_F \leq \varepsilon$. Then, by [Lemma F.1]

$$\| R - \exp(\pi \text{skew}(\sqrt{2}\tilde{S}_R)) \|_F \leq \| \pi S_R - \pi \text{skew}(\sqrt{2}\tilde{S}_R) \|_F \leq \pi \varepsilon. \quad (F.7)$$

We can also provide an upper bound to the size of $\mathcal{N}_B^\varepsilon$ and $\mathcal{N}_SO^\varepsilon$ given the size of $\mathcal{N}_A^\varepsilon$. To estimate the size of $\mathcal{N}_A^\varepsilon$, we consider

$$A_F := \{ S \in \mathbb{R}^{d \times d} \mid \| S \|_F \leq \sqrt{d}, \, S(i,j) = 0 \text{ if } i \leq j \}. \quad (F.8)$$

Let $\mathcal{N}_{A_F}^{\varepsilon/2}$ be the smallest $\varepsilon/2$-net of $A_F$. Since $A \subset A_F$, [Theorem 2.4] implies that, for $\mathcal{N}_A^\varepsilon$ the smallest $\varepsilon$-net of $A$, $|\mathcal{N}_A^\varepsilon| \leq |\mathcal{N}_{A_F}^{\varepsilon/2}|$. Now, to know the size of $|\mathcal{N}_{A_F}^{\varepsilon/2}|$, notice that $A_F$ is an Euclidean ball of radius $\sqrt{d}$ in a vector space of dimension $d(d-1)/2$. Then, [Theorem 2.3] implies that

$$|\mathcal{N}_{SO}^\varepsilon| \leq |\mathcal{N}_B^\varepsilon| \leq |\mathcal{N}_A^\varepsilon| \leq |\mathcal{N}_{A_F}^{\varepsilon/2}| \leq \left( 6\sqrt{d} \varepsilon^{-1} \right)^{\frac{d(d-1)}{4}}. \quad (F.9)$$

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