Research Article

Precise Asymptotics for Complete Integral Convergence under Sublinear Expectations

Qunying Wu

College of Science, Guilin University of Technology, Guilin 541004, China

Correspondence should be addressed to Qunying Wu; wqy666@glut.edu.cn

Received 21 December 2019; Accepted 28 February 2020; Published 5 May 2020

1. Introduction

The sublinear expectation space has advantages of modelling the uncertainty of probability and distribution. Therefore, limit theorems for sublinear expectations have raised a large number of issues of interest recently. Limit theorems are important research topics in probability and statistics. They were widely used in finance and other fields. Classical limit theorems only hold in the case of model certainty. However, in practice, such model certainty assumption is not realistic in many areas of applications because the uncertainty phenomena cannot be modeled using model certainty. Motivated by modelling uncertainty in practice, Peng [1] introduced a new notion of sublinear expectation. As an alternative to the traditional probability/expectation, capacity/sublinear expectation has been studied in many fields such as statistics, finance, economics, and measures of risk (see Denis and Martini [2]; Gilboa [3]; Marinacci [4]; Peng [1, 5–7], etc.). The general framework of the sublinear expectation in a general function space was introduced by Peng [1, 7, 8], and sublinear expectation is a natural extension of the classical linear expectation.

Because the sublinear expectation provides a very flexible framework to model sublinear probability problems, the limit theorems of the sublinear expectation have received more and more attention and research recently. A series of useful results have been established. Peng [1, 7, 8] constructed the basic framework, basic properties, and central limit theorem under sublinear expectations, Zhang [9–11] established the exponential inequalities, Rosenthal’s inequalities, strong law of large numbers, and law of iterated logarithm, Hu [12], Chen [13], and Wu and Jiang [14] studied strong law of large numbers, Wu et al. [15] studied the asymptotic approximation of inverse moment, Xi et al. [16] and Lin and Feng [17] studied complete convergence, and so on. In general, extending the limit properties of conventional probability space to the cases of sublinear expectation is highly desirable and of considerably significance in the theory and application. Because sublinear expectation and capacity is not additive, many powerful tools and common methods for linear expectations and probabilities are no longer valid, so that the study of the limit theorems under sublinear expectation becomes much more complex and difficult.

Since the concept of complete convergence of a sequence of random variables was introduced by Hsu and Robbins [18], there have been extensions in several directions. One of them is to discuss the precise rate, which is more exact than complete convergence. Precise asymptotics for complete convergence and complete moment convergence is one of
the most important problems in probability theory. Many related results have been obtained in the probabilistic space. Their recent results can be found in the studies of Heyde [19]; Liu and Lin [20]; Zhou [21]; Li [22]; Zhou [23]; Gut and Steinebach [24]; He and Xie [25]; Wang et al. [26–29]; Spatariu [30]; and Kong and Dai [31]. However, in sublinear expectations, due to the uncertainty of expectation and capacity, the precise asymptotics is essentially different from the ordinary probability space. The study of precise asymptotics of complete convergence and complete integral convergence for sublinear expectation have not been reported. The purpose of this paper is to establish asymptotics theorems under sublinear expectation and complete integral convergence for sublinear expectation and capacity, the precise asymptotics et al. [26–29]; Spaataru [30]; and Kong and Dai [31].

H beagivenmeasurablespace, and let $\mathcal{E}$ a space of “random variables.” In this case, we denote $\mathcal{E}$ ∈ $\Omega$.

From the definition, it is easily shown that for all $X, Y \in \mathcal{E}$,

$$\mathcal{E}(X + c) = \mathcal{E}X + c,$$

$$|\mathcal{E}(X + Y)| \leq \mathcal{E}|X + Y|,$$

$$\mathcal{E}(X - Y) \geq \mathcal{E}X - \mathcal{E}Y.$$  

If $\mathcal{E}Y = \mathcal{E}X + aY$, then $\mathcal{E}(X + aY) = \mathcal{E}X + a\mathcal{E}Y$ for any $a \in \mathbb{R}$.

In the next section, we summarize some basic notations and concepts and related properties under the sublinear expectations.

2. Preliminaries

We use the framework and notations of Peng [8]. Let $(\Omega, \mathcal{F})$ be a given measurable space, and let $\mathcal{H}$ be a linear space of real functions defined on $(\Omega, \mathcal{F})$ such that if $X_1, \ldots, X_n \in \mathcal{H}$, then $\phi(X_1, \ldots, X_n) \in \mathcal{H}$ for each $\phi \in C_{\text{Lip}}(\mathbb{R})$, where $C_{\text{Lip}}(\mathbb{R})$ denotes the linear space of (local Lipschitz) functions $\phi$ satisfying

$$|\phi(x) - \phi(y)| \leq c(1 + |x|^m + |y|^m)|x - y|, \quad \forall x, y \in \mathbb{R},$$

for some $c > 0, m \in \mathbb{N}$, depending on $\phi$. $\mathcal{H}$ is considered as a space of “random variables.” In this case, we denote $X \in \mathcal{H}$.

**Definition 1.** A sublinear expectation $\mathcal{E}$ on $\mathcal{H}$ is a function $\mathcal{E}: \mathcal{H} \rightarrow [-\infty, \infty]$ satisfying the following properties: for all $X, Y \in \mathcal{H}$,

(a) Monotonicity: If $X \geq Y$, then $\mathcal{E}X \geq \mathcal{E}Y$

(b) Constant preserving: $\mathcal{E}c = c$

(c) Subadditivity: $\mathcal{E}(X + Y) \leq \mathcal{E}X + \mathcal{E}Y$ whenever $\mathcal{E}X + \mathcal{E}Y$ is not of the form $\infty - \infty$ or $-\infty + \infty$

(d) Positive homogeneity: $\mathcal{E}(\lambda X) = \lambda \mathcal{E}X, \lambda \geq 0$

The triple $(\Omega, \mathcal{H}, \mathcal{E})$ is called a sublinear expectation space.

Given a sublinear expectation $\mathcal{E}$, let us denote the conjugate expectation $\mathcal{F}$ of $\mathcal{E}$ by

$$\mathcal{F}X := -\mathcal{E}(-X), \quad \forall X \in \mathcal{H}.$$  

From the definition, it is easily shown that for all $X, Y \in \mathcal{H}$,

$$\mathcal{E}X \leq \mathcal{E}X,$$

$$\mathcal{F}X = \mathcal{E}(-X), \quad \forall X \in \mathcal{H}.$$  

2 Mathematical Problems in Engineering
whenever the subexpectations are finite. A sequence \( \{X_n; n \geq 1\} \) of random variables is said to be identically distributed if for each \( i \geq 1 \), \( X_i \) and \( X_1 \) are identically distributed.

(ii) Independence: in a sublinear expectation space \( (\Omega, \mathcal{F}, \tilde{E}) \), a random vector \( Y = (Y_1, \ldots, Y_n) \), \( Y_i \in \mathcal{F} \), is said to be independent of another random vector \( X = (X_1, \ldots, X_m) \), \( X_i \in \mathcal{F} \), under \( \tilde{E} \) if for each test function \( \varphi \in C_{\mathcal{L}ip}(\mathbb{R}^m \times \mathbb{R}^n) \), we have \( \tilde{E}(\varphi(X, Y)) = \tilde{E}[\tilde{E}(\varphi(x, Y))|_{x=X}] \), whenever \( \tilde{E}(x) = \tilde{E}((\varphi(x, Y))) < \infty \) for all \( x \) and \( \tilde{E}((\varphi(X))) < \infty \).

\[ \forall(S_n \geq x) \leq \left( \max_{1 \leq k \leq n} X_k > y \right) + \exp \left( \frac{-x^2}{2(y + B_n)} \left( 1 + \frac{1}{2} \ln \left( 1 + \frac{y}{B_n} \right) \right) \right) \]  

\[ \forall \left( \tilde{E}X_k = \tilde{E}(-X_k) = 0 \right) \text{, then} 
\[ \forall(\max_{1 \leq k \leq n} X_k > y) \leq \frac{B_n}{x^2} \]  

Lemma 2. For any \( X \in \mathcal{F} \), we have

\[ \int_{f(3)}^{\infty} \text{V}(X^2 \text{ln}|X| > x) dx \leq C_v X^2 \text{ln}|X| \leq f(3) + \int_{f(3)}^{\infty} \text{V}(X^2 \text{ln}|X| > x) dx, \]
\[ \int_{f(3)}^{\infty} \text{V}(X^2 \text{ln}|X| > x) dx = \int_{f(3)}^{\infty} \text{V}(X^2 \text{ln}|X| > f(3) \text{ln}|x|) dx, \]
\[ 2y \text{ln} y \leq 2 \text{ln} y + y \leq 3 \text{ln} y, \]

for \( y \geq 3 \).

Therefore, (16) holds.

Here, we give the notations of G-normal distribution which is introduced by Peng [7].

\[ \text{Definition 3 (G-normal random variable). For } 0 \leq \sigma^2 \leq \sigma^2 < \infty, \text{ a random variable } \xi \text{ in a sublinear expectation space } \tilde{E}(\mathcal{F}, \mathcal{G}) \text{ is called a normal } \mathcal{N}(0, [\sigma^2, \sigma^2]) \text{ distributed random variable (write } \xi \sim \mathcal{N}(0, [\sigma^2, \sigma^2]) \text{ under } \tilde{E} \text{); if for any } \varphi \in C_{\mathcal{L}ip}(\mathbb{R}), \text{ the function } u(x, t) = \tilde{E}(\varphi(x + \sqrt{t} \xi)) \text{, } (x \in \mathbb{R}, t \geq 0) \text{ is the unique viscosity solution of the following heat equation:} \]

\[ \partial_t u - G(\partial_{xx} u) = 0, \]

\[ u(0, x) = \varphi(x), \]

where \( G(\alpha) = (\sigma^2 \alpha^+ - \sigma^2 \alpha^-)/2. \)

\[ \text{Lemma 3 (Theorem 3.3 and Remark 3.4 in Peng [7] (CLT)). Suppose that } \{X_n; n \geq 1\} \text{ is a sequence of independent and identically distributed random variables with } \tilde{E}(X_1) = \tilde{E}(-X_1) = 0. \text{ Write } \sigma^2 = \tilde{E}(X_1^2) \text{ and } \sigma^2 = \tilde{E}(X_1^2). \text{ Then, for any continuous function } \varphi \text{ satisfying } |\varphi(x)| \leq c(1 + |x|), \]
\[
\lim_{n \to \infty} \mathbb{E}(\phi\left(\frac{S_n}{\sqrt{n}}\right)) = \mathbb{E}(\phi(\xi)),
\]
where \(\xi \sim \mathcal{N}(0, [\sigma^2, \sigma^2])\) under \(\mathbb{E}\).

In particular, if \(\sigma = \sigma\), then Lemma 3 becomes a classical central limit theorem.

**Remark 1.** For any \(x > 0\), by \(\mathbb{E}(x X_t^2) = x^2 \sigma^2\), \(\mathbb{E}(x X_t^2) = x^2 \sigma^2\), and \(x \xi \sim \mathcal{N}(0, [x^2 \sigma^2, x^2 \sigma^2])\) under \(\mathbb{E}\), (19) becomes
\[
\lim_{n \to \infty} \mathbb{E}(\phi\left(\frac{x S_n}{\sqrt{n}}\right)) = \mathbb{E}(\phi(x \xi)).
\]

**Lemma 4** (Lemma 3 in Chen and Hu [32]). Suppose that \(\xi \sim \mathcal{N}(0, [\sigma^2, \sigma^2])\) under \(\mathbb{E}\). Let \(P\) be a probability measure and \(\phi\) be a bounded continuous function on \(\mathbb{R}\). If \(\{B_t\}_{t \geq 0}\) is a Brownian motion under \(P\), then
\[
\mathbb{E}(\phi(\xi)) = \sup_{\theta \in \Theta} \mathbb{E}_p \left[ \int_0^1 \theta_t dB_t \right],
\]
where
\[
\Theta = \{\theta; \theta_t\text{ is adapted process such that }\sigma \leq \theta_t \leq \sigma\},
\]
\(\mathcal{F}_t = \sigma[B_s; 0 \leq s \leq t] \vee \mathcal{N}, \mathcal{N}\) is the collection of \(P\) - null subsets.

From Peng [8], if \(\xi \sim \mathcal{N}(0, [\sigma^2, \sigma^2])\) under \(\mathbb{E}\), then for each convex function \(\phi\),
\[
\mathbb{E}(\phi(\xi)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(\xi) e^{-\xi^2/2} d\xi,
\]
but if \(\phi\) is a concave function, the above \(\mathbb{E}\) must be replaced by \(\mathbb{E}\). If \(\sigma = \sigma\), then \(\mathcal{N}(0, [\sigma^2, \sigma^2]) = \mathcal{N}(0, \sigma^2)\) which is a classical normal distribution.

In particular, notice that \(\phi(x) = |x|^p, p \geq 1\), is a convex function; taking \(\phi(x) = |x|^p, p \geq 1\), in (23), we get
\[
\mathbb{E}(\phi(\xi)) = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} x^p e^{-x^2/2} dx < \infty.
\]
(24) implies that
\[
G_\nu(|\xi|^p) = \int_0^{\infty} \mathbb{E}(\phi(\xi)^p) dx \leq 1
\]
(25) implies that
\[
G_\nu(|\xi|^p) = \frac{1}{\mathbb{E}(\xi^p)} \int_0^{\infty} \mathbb{E}(\phi(\xi)^p) dx < \infty \quad \text{for any } p \geq \frac{1}{2}.
\]

**Definition 4.** A sublinear expectation \(\mathbb{E}\) is called to be continuous if it satisfies

Continuity from below: \(\mathbb{E}(X_n) \mid \mathbb{E}(X)\) if \(0 \leq X_n \downarrow X\), where \(X_n, X \in \mathcal{H}\)
Continuity from above: \(\mathbb{E}(X_n) \mid \mathbb{E}(X)\) if \(0 \leq X_n \uparrow X\), where \(X_n, X \in \mathcal{H}\)

**Lemma 5.** Suppose that the conditions of Lemma 3 hold and \(\mathbb{E}\) is continuous, set \(\Delta_n(x) = \mathbb{V}(\mathbb{E}(\xi)) - \mathbb{V}(\mathbb{E}(\xi))\), here and later, \(\xi \sim \mathcal{N}(0, [\sigma^2, \sigma^2])\) under \(\mathbb{E}\); then,
\[
\Delta_n = \sup_{x \geq 0} |\Delta_n(x)| \to 0, \quad \text{as } n \to \infty.
\]

**Remark 2.** Lemma 5 is a powerful tool for studying the uniform convergence of the central limit theorem under sublinear expectations, which plays a key role in proving the theorems in this paper.

**Proof of Lemma 5.** If \(\sigma = \sigma\), then Lemma 3 is a classical central limit theorem. In the classical probability, (26) follows from the central limit theorem and an important fact that \(P(|\xi| \geq x)\) is a continuous function of \(x\). Therefore, we only need to prove the situation \(\sigma < \sigma\).

Obviously, \(\Delta_n(0) = 1 - 1 = 0\); thus, \(\Delta_n = \sup_{x \geq 0} |\Delta_n(x)|\).

For \(0 < \mu < 1\), let \(\phi(x)\) be a Lipschitz even function and nondecreasing for \(x \geq 0\) such that \(0 \leq \phi(x) \leq 1\), for all \(x\) and \(\phi(x) = 0\) if \(|x| \leq \mu\) and \(\phi(x) = 1\) if \(|x| > 1\). Then,
\[
I(|x| \geq 1) \leq \phi(x) \leq I(|x| \geq \mu).
\]

This combines equation (7), for \(x > 0\),
\[
\Delta_n(x) \leq \mathbb{E}\left[ \phi(\frac{S_n}{\sqrt{n} x}) - \mathbb{E}\left[ \phi(\frac{\xi}{x}) \right] \right] = \mathbb{E}\left[ \phi(\frac{S_n}{\sqrt{n} x}) - \mathbb{E}\left[ \phi(\xi) \right] + \mathbb{E}\left[ \phi(\xi) \right] - \mathbb{E}\left[ \phi(\frac{\xi}{x}) \right] \right] = \Delta_{n_1}(x) + \Delta_2(x) \leq \sup_{x \geq 0} |\Delta_{n_1}(x)| + \sup_{x \geq 0} \Delta_2(x),
\]
where \(\Delta_{n_1}(x) = \mathbb{E}\left[ \phi(\frac{S_n}{\sqrt{n} x}) \right] - \mathbb{E}\left[ \phi(\xi) \right] \) and \(\Delta_2(x) = \mathbb{E}\left[ \phi(\xi|x) \right] - \mathbb{E}\left[ \phi(\frac{\xi}{x}) \right] \geq 0\).

On the other hand,
\[
\Delta_n(x) \geq \mathbb{E}\left[ \phi(\frac{\mu S_n}{\sqrt{n} x}) \right] - \mathbb{E}\left[ \phi(\frac{\xi}{x}) \right] = \mathbb{E}\left[ \phi(\frac{\mu S_n}{\sqrt{n} x}) \right] - \mathbb{E}\left[ \phi(\frac{\mu \xi}{x}) \right] + \mathbb{E}\left[ \phi(\frac{\mu \xi}{x}) \right] - \mathbb{E}\left[ \phi(\frac{\xi}{x}) \right] = \Delta_{n_1}(x) - \Delta_2(x) \geq - \sup_{x \geq 0} |\Delta_{n_1}(x)| - \sup_{x \geq 0} \Delta_2(x).
\]

Thus,
\[
\Delta_n = \sup_{x \geq 0} |\Delta_n(x)| \leq \sup_{x \geq 0} |\Delta_{n_1}(x)| + \sup_{x \geq 0} \Delta_2(x).
\]

Therefore, in order to prove that (26), it suffices to show that
\[
\lim_{n \to \infty} \sup_{x > 0} |\Delta_{n_1}(x)| = 0,
\]
\[
\lim_{n \to \infty} \sup_{x > 0} \Delta_2(x) = 0.
\]
Write \( F_n(x) = \hat{E}(\phi(S_n/\sqrt{n}x)) \) and \( F(x) = \begin{cases} \hat{E}(\phi(\xi/x)) & x > 0 \\ 1 & x = 0 \end{cases} \).

Obviously, \( 0 \leq F_n(x) \) and \( F(x) \leq 1 \); \( F_n(x) \) and \( F(x) \) are nonincreasing functions on \([0, +\infty)\). Thus, for any \( x_0 > 0 \), the limit \( \lim_{x \to x_0} F(x) \) exists. Actually, taking \( x_n \downarrow x_0 \) and \( x_n \downarrow x_0 \), by continuity of \( \hat{E} \), we have

\[
\lim_{x \to x_0^+} F(x) = \lim_{x_n \downarrow x_0} \hat{E}\left(\frac{\xi}{x_n}\right) = \hat{E}\left(\frac{\xi}{x_0}\right) = F(x_0) \text{ from } 0 \leq \phi\left(\frac{\xi}{x_0}\right) \leq \phi(\xi).
\]

Furthermore, by (19) and Remark 1, there exists a number \( n_0 \) such that for \( n > n_0 \) and we have

\[
\left|F_n(x_k) - F(x_k)\right| < \frac{\varepsilon}{2}, \quad k = 1, \ldots, m - 1.
\]

If \( x_k \leq x < x_{k+1} \) (\( k = 1, \ldots, m - 1 \)), then for \( n > n_0 \), we get

\[
F_n(x) - F(x) \leq F_n(x_k) - F(x_k) + F(x_k) - F(x_{k+1}) < \varepsilon,
\]

\[
F_n(x) - F(x) \geq F_n(x_{k+1}) - F(x_{k+1}) + F(x_{k+1}) - F(x_k) > -\varepsilon.
\]

If \( 0 < x < x_1 \), then for \( n > n_0 \),

\[
F_n(x) - F(x) \leq 1 - F(x_1) < \varepsilon,
\]

\[
F_n(x) - F(x) = F_n(x_1) - F(x_1) + F(x_1) - F(x_1) = -\varepsilon.
\]

If \( x \geq x_m \), then for \( n > n_0 \),

\[
F_n(x) - F(x) \leq F_n(x_m) = F_n(x_m) - F(x_m) + F(x_m) < \varepsilon,
\]

\[
F_n(x) - F(x) \geq 0 - F(x_m) > -\varepsilon.
\]

Thus, \( |F_n(x) - F(x)| < \varepsilon \) for all \( x \) and \( n > n_0 \). That is, (31) holds.

Next, we prove that (32).

Because \( F \) is continuous on \([0, \infty)\), \( F \) is uniformly continuous on \([0, 2]\). Therefore, for any \( \varepsilon > 0 \), there is \( \delta > 0 \) (can be assumed \( \delta < 2 \)), such that \( \forall x_1, x_2 \in [0, 2] \); if \( |x_1 - x_2| < \delta \), then

\[ |F(x_1) - F(x_2)| < \varepsilon. \]

Let \( \max(1/2, 1 - \delta/2, \sigma/\bar{\sigma}) < \mu < 1 \), for any \( x \in (0, 1) \); we have \( 0 < x/\mu \leq 2x \leq 2 \) and \( |x/\mu - x| < \delta \). Hence,

\[
\Delta_2(x) = \hat{E}\left(\frac{\xi}{x}\right) - \hat{E}\left(\frac{\mu \xi}{x}\right) = F(x) - F\left(\frac{x}{\mu}\right) < \varepsilon.
\]

Therefore,

\[
\sup_{0 < x < 1} \Delta_2(x) \leq \varepsilon.
\]

For \( x > 1 \), let \( \Theta = \{ \theta; \theta \text{ is } \mathcal{F}_E \text{ adapted process such that } g(x) \leq \delta \leq \bar{\sigma}/x \} \) and \( \Theta^\mu = \{ \theta; \theta \text{ is } \mathcal{F}_E \text{ adapted process such that } \theta \leq \mu/\bar{\sigma}/x \} \); combining any \( \alpha > 0, \xi/\mu \sim N(0, [a^2/\alpha^2, \bar{\sigma}/\bar{\sigma}) \) under \( \hat{E} \). By Lemma 4,

\[
\Delta_2(x) = \hat{E}\left(\frac{\xi}{x}\right) - \hat{E}\left(\frac{\mu \xi}{x}\right) \leq \sup_{\theta \in \Theta} E_p\left(\int_0^1 \theta_1 \sigma dB_1\right) - \sup_{\theta \in \Theta} E_p\left(\int_0^1 \theta_1 \sigma dB_1\right)
\]

\[
\leq \sup_{\theta \in \Theta} E_p\left(\int_0^1 \theta_1 \sigma dB_1\right) - \sup_{\theta \in \Theta} E_p\left(\int_0^1 \theta_1 \sigma dB_1\right)
\]

\[
\leq \sup_{\theta \in \Theta} E_p\left(\int_0^1 \theta_1 \sigma dB_1\right) - \sup_{\theta \in \Theta} E_p\left(\int_0^1 \theta_1 \sigma dB_1\right)
\]

\[
\leq \sup_{\theta \in \Theta} E_p\left(\int_0^1 \theta_1 \sigma dB_1\right) - \sup_{\theta \in \Theta} E_p\left(\int_0^1 \theta_1 \sigma dB_1\right)
\]

\[
\leq \varepsilon.
\]

where \( c_\varphi \) is the Lipschitz constant of \( \varphi \).

Therefore,
\[\sup_{\alpha>1} \Delta_\alpha(x) \leq c_\alpha \sigma (1-\mu) \longrightarrow 0, \quad \mu \longrightarrow 1^- . \quad (43)\]

The combination of (41) and (32) is established. This completes the proof of Lemma 5. \(\Box\)

### 3. Main Results and Proofs

Our results are stated as follows.

**Theorem 1.** Let \(\{X, X_n; n \geq 1\}\) be a sequence of independent and identically distributed random variables in \((\Omega, \mathcal{F}, \bar{E})\). We assume that \(\bar{E}\) is continuous and

\[\bar{E}(X) = \bar{E}(-X) = 0,\]
\[\lim_{c \to -\infty} \bar{E}(X_1^2 - c)^+ = 0,\]
\[\bar{E}(X^2) = \bar{\sigma}^2 < \infty, \quad (44)\]
\[\bar{\epsilon}(X^2) = \bar{\sigma}^2,\]
\[C_\nu(X^2) < \infty.\]

Then,

\[\lim_{\varepsilon \to 0} \varepsilon^2 \sum_{n=1}^{\infty} \mathbb{P}(|S_n| \geq \varepsilon n) = C_\nu(\bar{\xi}^2), \quad (45)\]

here and later, \(\xi \sim \mathcal{N}(0, [\bar{\sigma}^2, \bar{\sigma}^2])\) under \(\bar{E}\).

**Theorem 2.** Under the conditions of Theorem 1, for \(0 \leq p < 2\),

\[\varepsilon^2 \sum_{n=1}^{\infty} \mathbb{P}(|S_n| \geq \varepsilon n) = \varepsilon^2 \sum_{n=1}^{\infty} \mathbb{P}(|\xi| \geq \varepsilon \sqrt{n}) + \varepsilon^2 \sum_{n=1}^{\infty} \mathbb{P}(|S_n| \geq \varepsilon n) - \mathbb{P}(|\xi| \geq \varepsilon \sqrt{n}) = I_1(\varepsilon) + I_2(\varepsilon). \quad (49)\]

Hence, in order to establish (45), it suffices to prove that

\[\lim_{\varepsilon \to 0} I_1(\varepsilon) = C_\nu(\bar{\xi}^2), \quad (50)\]
\[\lim_{\varepsilon \to 0} I_2(\varepsilon) = 0. \quad (51)\]

\[\lim_{\varepsilon \to 0} I_1(\varepsilon) = \lim_{\varepsilon \to 0} \varepsilon^2 \int_{-\varepsilon \sqrt{n}}^{\varepsilon \sqrt{n}} \mathbb{P}(|\xi| \geq \varepsilon \sqrt{x}) dx = \lim_{\varepsilon \to 0} \varepsilon^2 \int_{\varepsilon^2}^{\infty} \mathbb{P}(\xi^2 \geq y) dy \]
\[= \int_{0}^{\infty} \mathbb{P}(\xi^2 \geq y) dy = C_\nu(\bar{\xi}^2). \quad (52)\]

Without loss of generality, here and later, we assume that \(\bar{E}X^2 = 1\). Let \(M > 12\),

\[\lim_{\varepsilon \to 0} \varepsilon^2 \sum_{n \leq [M \varepsilon^{-2}]} \mathbb{P}(|S_n| \geq \varepsilon n) \geq 0 \quad \text{and} \quad \varepsilon^2 \sum_{n \geq [M \varepsilon^{-2}]} \mathbb{P}(|S_n| \geq \varepsilon n) \geq \varepsilon^2 \sum_{n \geq [M \varepsilon^{-2}]} \mathbb{P}(|\xi| \geq \varepsilon \sqrt{n}) = I_{21}(\varepsilon) + I_{22}(\varepsilon) + I_{23}(\varepsilon). \quad (53)\]

\[\lim_{\varepsilon \to 0} \varepsilon^2 \sum_{n \geq [M \varepsilon^{-2}]} \mathbb{P}(|S_n| \geq \varepsilon n) = \frac{2}{2 - p} C_\nu(\bar{\xi}^2). \quad (46)\]

For \(p = 2\), we have the following theorem.

**Theorem 3.** Let \(\{X, X_n; n \geq 1\}\) be a sequence of independent and identically distributed random variables in \((\Omega, \mathcal{F}, \bar{E})\). We assume that \(\bar{E}\) is continuous and

\[\bar{E}(X) = \bar{E}(-X) = 0,\]
\[\lim_{c \to -\infty} \bar{E}(X_1^2 - c)^+ = 0,\]
\[\bar{E}(X_1^2) = \bar{\sigma}^2 < \infty, \quad (47)\]
\[\bar{\epsilon}(X_1^2) = \bar{\sigma}^2,\]
\[C_\nu(X^2|X| < \infty.\]

Then,

\[\lim_{\varepsilon \to 0} \frac{1}{\ln \varepsilon^{-1}} \sum_{n=1}^{\infty} \mathbb{P}(n^2 \mathbb{P}(|X_n| \geq \varepsilon n)) = 2C_\nu(\bar{\xi}^2). \quad (48)\]

Remark 3. Theorems 1–3 extend the corresponding results obtained by Liu and Lin [20] from the probability space to sublinear expectation space.

**Proof of Theorem 1.** Note that

\[\mathbb{E}(X^2) = 1. \quad \text{Let} \quad M > 12,\]
\[\mathbb{E}(X^2|X| \geq M) = \mathbb{E}(X^2|X| \geq M) \leq \frac{2}{2 - p} C_\nu(\bar{\xi}^2). \]
Let $\Delta_n = \sup_{x \geq 0} [\mathbb{P}(|S_n| \geq \sqrt{n} x) - \mathbb{P}(|\xi| \geq x)]$, from (26), $\Delta_n \to 0$ as $n \to \infty$. So, by Toeplitz’s lemma, if $x_n \to x$, $\omega_i \geq 0$, and $\sum_{i=1}^n \omega_i \to \infty$, then

$$
\lim_{\epsilon \to 0} I_{21}(\epsilon) \leq M,
$$

(54)

$$
\lim_{\epsilon \to 0} \frac{\sum_{n \in [M \epsilon^{-2}]} \Delta_n}{M \epsilon^{-2}} = 0.
$$

Taking $\varphi$ as the proof process of Lemma 5, by (7), (27) and identically distributed of $X, X_i$, for any $x > 0$,

$$
\mathbb{P}(|X| \geq x) = \mathbb{E}[\varphi\left(\frac{X}{x}\right)] = \mathbb{E}\left[\varphi\left(\frac{X}{x}\right)\right] \leq \mathbb{P}(|X| \geq \mu x).
$$

(55)

Hence, for $n > M \epsilon^{-2} > 12 \epsilon^{-2}$, taking $x = en$ and $y = en/12$ in Lemma 1 (i),

$$
\mathbb{P}(S_n \geq en) \leq \sum_{i=1}^{n} \mathbb{P}(X_i \geq \frac{en}{12}) + \exp\left(-\frac{\epsilon^2 n^2}{2(e^2 \epsilon^2/12 + n)}\right) \left\{1 + \frac{2}{3} \ln\left(1 + \frac{\epsilon^2 n^2}{12}\right)\right\} \leq n \mathbb{P}(|X| \geq \frac{\mu en}{12}) + \frac{c}{\epsilon^2 n^2}.
$$

(56)

More generally, for any $x > 0$ and $n > M \epsilon^{-2}$, we have

$$
\mathbb{P}(S_n \geq (\epsilon + x)n) \ll n \mathbb{P}(|X| \geq \frac{\mu (\epsilon + x)n}{12}) + \frac{1}{(\epsilon + x)^4 n^2}.
$$

(59)

This implies from Markov’s inequality and (24) that

$$
I_{22}(\epsilon) + I_{23}(\epsilon) \leq \epsilon^2 \sum_{n > M \epsilon^{-2}} \mathbb{P}(|X| \geq \frac{\mu en}{12}) + \frac{1}{\epsilon^2 n^2} + \mathbb{E}[|\xi|]^4 \epsilon^2 \int_{\epsilon^2}^{\infty} x \mathbb{P}(|X| \geq cx)dx + M^{-1} \text{ (let } c\epsilon x = y) \sim c \int_{\epsilon^2}^{\infty} y \mathbb{P}(|X| \geq y)dy + M^{-1}.
$$

(60)

Let $\epsilon \to 0$ first; then, let $M \to \infty$; we get

$$
\lim_{\epsilon \to 0} (I_{22}(\epsilon) + I_{23}(\epsilon)) = 0,
$$

(61)

from (44) and (15).

From this, combining with (53) and (54), (51) is established. This completes the proof of Theorem 1. \(\square\)

$$
\epsilon^{-2} \sum_{n=1}^{\infty} \frac{1}{n^p} C_V(|S_n|^p I(|S_n| \geq en)) = \epsilon^2 \sum_{i=1}^{\infty} \mathbb{P}(|S_n| \geq en) + \epsilon^{2-p} \sum_{n=1}^{\infty} \frac{1}{n^p} \int_{en}^{\infty} p x^{p-1} \mathbb{P}(|S_n| \geq x)dx.
$$

(62)

Hence, from Theorem 1, in order to establish (46), it suffices to prove that

$$
\lim_{\epsilon \to 0} \epsilon^{2-p} \sum_{n=1}^{\infty} \frac{1}{n^p} \int_{en}^{\infty} p x^{p-1} \mathbb{P}(|S_n| \geq x)dx = \frac{p}{2-p} C_V(\xi^2).
$$

(63)
Let \( M \geq 12 \). Note that

\[
\varepsilon^{-p} \sum_{n=1}^{\infty} \frac{1}{n^p} \int_{c_n}^{\infty} p x^{p-1} \varphi(|S_n| \geq x) dx
\]

\[
= \varepsilon^{-p} \sum_{n=1}^{\infty} \frac{1}{n^p} \int_{c_n}^{\infty} p x^{p-1} \varphi(|\xi| \geq \frac{x}{\sqrt{n}}) dx
\]

\[
+ \varepsilon^{-p} \sum_{n \geq \lfloor Me^{-1} \rfloor} \frac{1}{n^p} \int_{c_n}^{\infty} p x^{p-1} \left( \varphi(|S_n| \geq x) - \varphi(|\xi| \geq \frac{x}{\sqrt{n}}) \right) dx
\]

\[
+ \varepsilon^{-p} \sum_{n \geq \lfloor Me^{-1} \rfloor} \frac{1}{n^p} \int_{c_n}^{\infty} p x^{p-1} \left( \varphi(|S_n| \geq x) - \varphi(|\xi| \geq \frac{x}{\sqrt{n}}) \right) dx
\]

\[
:= J_1(\varepsilon) + J_2(\varepsilon) + J_3(\varepsilon).
\]

Hence, in order to establish (63), it suffices to prove that

\[
\lim_{\varepsilon \to 0} J_1(\varepsilon) = \frac{p}{2 - p} C(\varepsilon^2), \quad (65)
\]

\[
\lim_{\varepsilon \to 0} J_2(\varepsilon) = 0, \quad (66)
\]

\[
\lim_{M \to \infty} \limsup_{\varepsilon \to 0} |J_3(\varepsilon)| = 0. \quad (67)
\]

We first prove (65); let \( y = x/\sqrt{n} \), then

\[
\lim_{\varepsilon \to 0} J_1(\varepsilon) = \lim_{\varepsilon \to 0} \varepsilon^{-p} \sum_{n=1}^{\infty} \frac{1}{n^p} \int_{c_n}^{\infty} p y^{p-1} \varphi(|\xi| \geq y) dy
\]

\[
= \lim_{\varepsilon \to 0} \varepsilon^{-p} \int_{1}^{\infty} \frac{1}{t^{p+2}} \int_{c_n}^{\infty} p y^{p-1} \varphi(|\xi| \geq y) dy
\]

\[
= \lim_{\varepsilon \to 0} \varepsilon^{-p} \int_{0}^{\infty} p y^{p-1} \varphi(|\xi| \geq y) dy
\]

\[
+ \frac{2p}{2 - p} \int_{0}^{\infty} p y^{p-1} \varphi(|\xi| \geq y) dy
\]

\[
- \lim_{\varepsilon \to 0} \frac{2p}{2 - p} \int_{0}^{\infty} p y^{p-1} \varphi(|\xi| \geq y) dy
\]

\[
= \frac{p}{2 - p} C(\varepsilon^2), \quad (68)
\]

from (25).

Now, we prove (66). Let \( b_n = (\sqrt{n} \Delta_n^{1/2p})^{-1} \). Then,
By Lemma 1 (ii), Markov’s inequality, and (24), we get

\begin{equation}
J_{22}(\varepsilon) \leq cn^{p/2} \int_{n}^{\infty} (y + \varepsilon)^{p-1} \left( \frac{n^2}{y + \varepsilon} \right) dy \ll n^{p/2-1} \int_{n}^{\infty} \frac{1}{y + \varepsilon} dy = cn^{p/2-1} \left( \frac{1}{\sqrt{n} \Delta_n^{p/2}} + \varepsilon \right)^{p/2} \ll \Delta_n^{1/p-1/2} \to 0 \quad \text{as} \quad n \to \infty.
\end{equation}

From (69)–(72) and (66) is established. Finally, we prove (67):

That is, (66) is established.

Let \( \varepsilon \to 0 \) first, then let \( M \to \infty \), we get (67) from (44) and (15). This completes the proof of Theorem 2. \( \square \)
Hence, by Theorem 1, in order to establish (48), it suffices to prove that
\[
\lim_{\epsilon \to 0} \frac{1}{\ln \epsilon^{-1}} \sum_{n=1}^{\infty} \frac{1}{n^2} \int_{n}^{\infty} 2x\nu\left(\{S_n \geq x\}\right)dx = 2C\nu(\xi^2). \quad (77)
\]
However,
\[
\frac{1}{\ln \epsilon^{-1}} \sum_{n=1}^{\infty} \frac{1}{n^2} \int_{n}^{\infty} 2x\nu\left(\{S_n \geq x\}\right)dx
\]
\[= \frac{1}{\ln \epsilon^{-1}} \sum_{n=1}^{\infty} \frac{1}{n^2} \int_{n}^{\infty} 2x\left(\nu\left(\{|\xi| \geq \frac{x}{\sqrt{n}}\}\right) - \nu\left(\{|\xi| \geq \frac{x}{\sqrt{n}}\}\right)\right)dx
\]
\[+ \frac{1}{\ln \epsilon^{-1}} \sum_{n=1}^{\infty} \frac{1}{n^2} \int_{n}^{\infty} 2x\left(\nu\left(\{|\xi| \geq \frac{x}{\sqrt{n}}\}\right) - \nu\left(\{|\xi| \geq \frac{x}{\sqrt{n}}\}\right)\right)dx
\]
\[= K_1(\epsilon) + K_2(\epsilon) + K_3(\epsilon).
\]
(78)

Hence, in order to establish (77), it suffices to prove that
\[
\lim_{\epsilon \to 0} K_1(\epsilon) = 2C\nu(\xi^2), \quad (79)
\]
\[
\lim_{\epsilon \to 0} K_2(\epsilon) = 0, \quad (80)
\]
\[
\lim_{n \to \infty} \lim_{\epsilon \to 0} \sup\{|K_3(\epsilon)| = 0. \quad (81)
\]

We first prove (79). Because \(\lim_{y \to 0^+} y|\ln y| = 0\), there is \(c_1 > 0\), such that \(y|\ln y| \leq c_1\) for any \(y \in (0, 1]\). Therefore, combining with Markov’s inequality and (24),

\[
\left| \int_{\epsilon}^{\infty} y \ln y \nu\left(\{|\xi| \geq y\}\right)dy \right| \leq \int_{0}^{1} y|\ln y|dy + \int_{1}^{\infty} y \ln y \frac{E|\xi|^3}{y^2} dy \leq c_1 + E|\xi|^3 \int_{1}^{\infty} \frac{\ln y}{y^2} dy = c < \infty.
\]

(82)

Thus, (79) follows
Now, we prove (80). Let $d_n = (\sqrt{n} \Delta_n^{1/4})^{-1}$. Then,

\[
|K_2(\varepsilon)| \leq \frac{1}{\ln \varepsilon^{-1}} \sum_{n \in \mathbb{M}_{[-1]}^{[1]}} \frac{1}{n^2} \int_{0}^{d_n} 2x \left| \mathbb{V}( |S_n| \geq x) - \mathbb{V}( |\xi| \geq \frac{x}{\sqrt{n}}) \right| dx \text{ (let } (y + \varepsilon)n = x) \\
= \frac{1}{\ln \varepsilon^{-1}} \sum_{n \in \mathbb{M}_{[-1]}^{[1]}} \int_{0}^{d_n} 2(y + \varepsilon) \left( \mathbb{V}( |S_n| \geq (y + \varepsilon)n) - \mathbb{V}( |\xi| \geq (y + \varepsilon)\sqrt{n}) \right) dy \\
\leq \frac{1}{\ln \varepsilon^{-1}} \sum_{n \in \mathbb{M}_{[-1]}^{[1]}} \int_{0}^{d_n} 2(y + \varepsilon) \Delta_n d y \\
+ \frac{1}{\ln \varepsilon^{-1}} \sum_{n \in \mathbb{M}_{[-1]}^{[1]}} \int_{d_n}^{\infty} 2(y + \varepsilon) \left( \mathbb{V}( |S_n| \geq (y + \varepsilon)n) + \mathbb{V}( |\xi| \geq (y + \varepsilon)\sqrt{n}) \right) dy \\
=: \frac{1}{\ln \varepsilon^{-1}} \sum_{n \in \mathbb{M}_{[-1]}^{[1]}} \frac{1}{n} (K_{21}(\varepsilon) + K_{22}(\varepsilon)),
\]

where

\[
K_{21}(\varepsilon) = n \int_{0}^{d_n} 2(y + \varepsilon) \Delta_n d y, \\
K_{22}(\varepsilon) = n \int_{d_n}^{\infty} 2(y + \varepsilon) \left( \mathbb{V}( |S_n| \geq (y + \varepsilon)n) + \mathbb{V}( |\xi| \geq (y + \varepsilon)\sqrt{n}) \right) dy.
\]

Since $n \leq M\varepsilon^{-2}$ implies $\varepsilon \sqrt{n} \leq \sqrt{M}$, one can easily obtain that

\[
K_{21}(\varepsilon) \leq \Delta_n n \left( \frac{1}{\sqrt{n} \Delta_n^{1/4}} + \varepsilon \right)^2 \leq \left( \Delta_n^{1/4} + \sqrt{M} \Delta_n^{1/2} \right)^2 \ll \Delta_n^{1/2} \rightarrow 0 \quad \text{as } n \rightarrow \infty.
\]

By (59), Markov’s inequality, (24), and $cn(d_n + \varepsilon) \geq c\sqrt{n}$, we get

\[
K_{22}(\varepsilon) \leq cn \int_{d_n}^{\infty} 2(y + \varepsilon) \left( n\mathbb{V}( |X| \geq cn(y + \varepsilon)) + \frac{1}{n^2 (y + \varepsilon)^2} + \frac{\mathbb{E}\xi^4}{n^2 (y + \varepsilon)^2} \right) dy \\
= c \int_{cn(d_n + \varepsilon)}^{\infty} t\mathbb{V}( |X| \geq t) dt + c \int_{d_n}^{\infty} \frac{1}{n(y + \varepsilon)^2} dy \quad \text{ (let } cn(y + \varepsilon) = t) \\
\ll \int_{c\sqrt{n}}^{\infty} t\mathbb{V}( |X| \geq t) dt + \frac{1}{n} \left( \frac{1}{\sqrt{n} \Delta_n^{1/4}} + \varepsilon \right)^{-2} \\
\rightarrow 0 \quad \text{as } n \rightarrow \infty.
\]
From (84)–(87) and 
\[ \sum_{n \in [M^{-2}]} 1/n = O(\ln \varepsilon^{-1}) \to \infty, \varepsilon \to 0, \] 
using Toeplitz’s lemma, \((80)\) follows

\[ |K_2(\varepsilon)| \leq \frac{1}{\ln \varepsilon^{-1}} \sum_{n \in [M^{-2}]} \frac{\Delta_n^{1/2} + K_{22}(\varepsilon)}{n} \leq \frac{\sum_{n \in [M^{-2}]} (\Delta_n^{1/2} + K_{22}(\varepsilon)/n)}{\sum_{n \in [M^{-2}]} (1/n)} \to 0, \quad \text{as} \quad \varepsilon \to 0. \] \hspace{1cm} (88)

Finally, we prove \((81)\) as follows:

\[ |K_3(\varepsilon)| \leq \frac{1}{\ln \varepsilon^{-1}} \sum_{n \in [M^{-2}]} \int_0^\infty 2x \left( \mathbb{V}(|S_n| \geq x) + \mathbb{V}(|\xi| \geq \frac{x}{\sqrt{n}}) \right) dx \left( \text{let} \ y = \frac{x}{n-\varepsilon} \right) \]

\[ = \frac{1}{\ln \varepsilon^{-1}} \sum_{n \in [M^{-2}]} \int_0^\infty 2(y + \varepsilon) \left( \mathbb{V}(|S_n| \geq (y + \varepsilon)n) + \mathbb{V}(|\xi| \geq (y + \varepsilon)\sqrt{n}) \right) dy. \] \hspace{1cm} (89)

By \((59)\), Markov’s inequality, and \((24)\),

\[ |K_3(\varepsilon)| \leq \frac{1}{\ln \varepsilon^{-1}} \sum_{n \in [M^{-2}]} \int_0^\infty (y + \varepsilon) \left( n\mathbb{V}(|X| \geq c(y + \varepsilon)n) + \frac{1}{(y + \varepsilon)^2n^2} + \frac{\mathbb{E}|\xi|^4}{(y + \varepsilon)^4n^2} \right) dy \]

\[ \leq \frac{1}{\ln \varepsilon^{-1}} \int_0^\infty \int_{\varepsilon c} \mathbb{V}(|X| \geq t) dt \int_0^{t/(cM-\varepsilon) - 1} \frac{1}{y + \varepsilon} dy + \frac{2cM^{-1}}{\ln \varepsilon^{-1}} \]

\[ \leq \frac{1}{\ln \varepsilon^{-1}} \int_0^\infty \mathbb{V}(|X| \geq t) dt \left( \ln \frac{\varepsilon^2}{cM} + \ln t + \ln \varepsilon^{-1} \right) dt + M^{-1} \]

\[ \leq \frac{1}{\ln \varepsilon^{-1}} \int_{\varepsilon c}^\infty t \ln \mathbb{V}(|X| \geq t) dt + \int_{\varepsilon c}^\infty t \mathbb{V}(|X| \geq t) dt + M^{-1} \].

Let \(\varepsilon \to 0\) first and then \(M \to \infty\); we get \((81)\) from \((47)\) and \((16)\). This completes the proof of Theorem 3. \(\square\)

### Data Availability

No data were used to support this study.

### Conflicts of Interest

The authors declare no conflicts of interest.

### Acknowledgments

This research was supported by the National Natural Science Foundation of China (11661029) and the Support Program of the Guangxi China Science Foundation (2018GXNSFAA281011).

### References

[1] S. Peng, “G-expectation, G-brownian motion and related stochastic calculus of itô type,” *Stochastic Analysis and Applications*, vol. 2, no. 4, pp. 541–567, 2007.

[2] L. Denis and C. Martini, “A theoretical framework for the pricing of contingent claims in the presence of model uncertainty,” *The Annals of Applied Probability*, vol. 16, no. 2, pp. 827–852, 2006.

[3] I. Gilboa, “Expected utility with purely subjective non-additive probabilities,” *Journal of Mathematical Economics*, vol. 16, no. 1, pp. 65–8810, 1987.
[4] M. Marinacci, "Limit laws for non-additive probabilities and their frequentist interpretation," Journal of Economic Theory, vol. 84, no. 2, pp. 145–195, 1999.

[5] S. G. Peng, "BSDE and related g-expectation," Pitman Research Notes in Mathematics Series, vol. 364, pp. 141–159, 1997.

[6] S. Peng, "Monotonic limit theorem of BSDE and nonlinear decomposition theorem of Doob-Meyers type," Probability Theory and Related Fields, vol. 113, no. 4, pp. 473–499, 1999.

[7] S. G. Peng, "Nonlinear expectations and stochastic calculus under uncertainty," 2010, https://arxiv.org/abs/1002.4546.

[8] S. Peng, "Survey on normal distributions, central limit theorem, Brownian motion and the related stochastic calculus under uncertainty," Science China Mathematics, vol. 59, no. 12, pp. 3050–3064, 2016.

[9] L. X. Zhang, "Strong limit theorems for extended independent and extended negatively dependent random variables under non-linear expectations," 2016, https://arxiv.org/abs/1608.00710.

[10] L. X. Zhang, "Exponential inequalities under the sub-linear expectations with applications to laws of the iterated log-rithm," Science China Mathematics, vol. 59, no. 12, pp. 2503–2526, 2016.

[11] L. Zhang, "Rosenthal’s inequalities for independent and negatively dependent random variables under sub-linear expectations with applications," Science China Mathematics, vol. 59, no. 4, pp. 751–768, 2016.

[12] C. Hu, "A strong law of large numbers for sub-linear expectation under a general moment condition," Statistics and Probability Letters, vol. 119, pp. 248–258, 2016.

[13] Z. Chen, "Strong laws of large numbers for sub-linear expectations," Science China Mathematics, vol. 59, no. 5, pp. 945–954, 2016.

[14] Q. Wu and Y. Jiang, "Strong law of large numbers and Chover’s law of the iterated logarithm under sub-linear expectations," Journal of Mathematical Analysis and Applications, vol. 460, no. 1, pp. 252–270, 2018.

[15] Y. Wu, X. Wang, and L. Zhang, "On the asymptotic approximation of inverse moment under sub-linear expectations," Journal of Mathematical Analysis and Applications, vol. 468, no. 1, pp. 182–196, 2018.

[16] M. Xi, Y. Wu, and X. Wang, "Complete convergence for arrays of rowwise END random variables and its statistical applications under sub-linear expectations," Journal of the Korean Statistical Society, vol. 48, no. 3, pp. 412–425, 2019.

[17] Y. Lin and X. Feng, "Complete convergence and strong law of large numbers for arrays of random variables under sublinear expectations," Communications in Statistics—Theory and Methods, pp. 1–17, 2019.

[18] P. L. Hsu and H. Robbins, "Complete convergence and the law of large numbers," Proceedings of the National Academy of Sciences, vol. 33, no. 2, pp. 25–31, 1947.

[19] C. C. Heyde, "A supplement to the strong law of large numbers," Journal of Applied Probability, vol. 12, no. 1, pp. 173–175, 1975.

[20] W. Liu and Z. Lin, "Precise asymptotics for a new kind of complete moment convergence," Statistics & Probability Letters, vol. 76, no. 16, pp. 1787–1799, 2006.

[21] Y. Zhao, "Precise rates in complete moment convergence for ρ-mixing sequences," Journal of Mathematical Analysis and Applications, vol. 339, no. 1, pp. 553–565, 2008.

[22] J. Li, "Precise asymptotics of moving average process under ϕ-mixing assumption," Journal of the Korean Mathematical Society, vol. 49, no. 2, pp. 235–249, 2012.