On General-$n$ Coefficients in Series Expansions for Row Spin-Spin Correlation Functions in the Two-Dimensional Ising Model

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We consider spin-spin correlation functions for spins along a row, $R_n = \langle \sigma_{0,0}\sigma_{n,0} \rangle$, in the two-dimensional Ising model. We discuss a method for calculating general-$n$ expressions for coefficients in high-temperature and low-temperature series expansions of $R_n$ and apply it to obtain such expressions for several higher-order coefficients. In addition to their intrinsic interest, these results could be useful in the continuing quest for a nonlinear ordinary differential equation whose solution would determine $R_n$, analogous to the known nonlinear ordinary differential equation whose solution determines the diagonal correlation function $\langle \sigma_{0,0}\sigma_{n,n} \rangle$ in this model.

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I. INTRODUCTION

Spin-spin correlation functions contain information about the degree of magnetic ordering in a spin model. The two-dimensional Ising model provides a valuable context in which one can obtain exact closed-form analytic expressions for these correlation functions. In thermal equilibrium at temperature $T$ on the square lattice, the partition function of the (zero-field, isotropic, spin-1/2, nearest-neighbor) Ising model is given by

$$Z = \sum_{\{\sigma\}} e^{-\beta H},$$  \hspace{1cm} (1.1)

where the Hamiltonian is

$$H = -J \sum_{nn} \sigma_r \sigma_{r'}.$$  \hspace{1cm} (1.2)

In Eq. (1.2), $\beta = 1/(k_B T)$; the sum is over nearest-neighbor $(nn)$ sites on the lattice; and $\sigma_r = \pm 1$ is a classical spin variable defined on each lattice site. Given the well-known mapping on a bipartite lattice between the ferromagnetic ($J > 0$) and antiferromagnetic ($J < 0$) spin-spin couplings, one can, with no loss of generality, take $J > 0$, and we will do this. In the following, we assume the thermodynamic limit. This model has a global $Z_2$ symmetry, which is spontaneously broken with the onset of a nonzero spontaneous magnetization $M$ as the temperature decreases below the critical temperature, $T_c$. The system undergoes a continuous, second-order phase transition at this critical temperature.

The two-dimensional Ising model has the appeal that many of its properties are known exactly. The free energy was calculated by Onsager [1], and an expression for the magnetization was first published by Yang [2]. A method for calculating spin-spin correlation functions in terms of Toeplitz determinants was developed by Kaufman and Onsager [3] and later extended by Montroll, Potts, and Ward [4]. Some reviews of the Ising model include [5, 6]. The critical behavior is known exactly; the thermal and magnetic exponents are $\nu_t = 1$ and $y_h = 15/8$, and hence the critical exponents $\nu, \alpha, \beta, \gamma$, etc., are known for models in the $d = 2, Z_2$ universality class of second-order phase transitions, of which the nearest-neighbor spin-1/2 Ising model is arguably the simplest example. Generalizations to anisotropic couplings, spin $s \geq 1$, and non-nearest-neighbor two-spin and multi-spin interactions preserving the $Z_2$ symmetry that do not cause frustration are in the same universality class. The critical behavior was elucidated in the context of continuous spatial dimensionality (above the lower critical dimensionality, $d = 1$) via the momentum-space renormalization group and $\epsilon$ expansion, where $\epsilon = 4 - d$ [7–9]. Further insight into the critical behavior of the $d = 2, Z_2$ universality class was obtained by means of conformal algebra methods [10], which showed that the critical behavior is described by a rational conformal field theory with central charge...
Although the 2D Ising model is classical, it can be related to a 1D quantum spin chain \([12]\). In addition to these works, some other studies relevant to spin-spin correlation functions in the two-dimensional Ising model include \([13]-[59]\).

While much is thus known about the two-dimensional Ising model, there are still interesting aspects to study. Among these are various properties of the spin-spin correlation functions. We denote the spin-spin correlation function as \(\langle \sigma_{\vec{r}} \sigma_{\vec{r}'} \rangle\), where \(\vec{r}\) and \(\vec{r}'\) are sites on the square lattice and \(\langle O \rangle\) denotes the thermal average of an operator \(O\). Given the homogeneity of the square lattice, one can, with no loss of generality, take one spin to be located at the origin and thus consider

\[
\langle \sigma_0 \sigma_{\vec{r}} \rangle \equiv C(\vec{r}) \ .
\]

We write \(\vec{r} = (m, n)\) so that \(C(\vec{r}) \equiv C(m, n)\). The spin-spin correlation function for two spins along a row is

\[
R_n \equiv C(n, 0) \equiv \langle \sigma_{0,0} \sigma_{n,0} \rangle \ .
\]

From the isotropy of the spin-spin couplings in \([1.2]\), it follows that the correlation functions for equidistantly separated spins along a row and column are equal: \(C(n, 0) = C(0, n)\). We denote the correlation function for spins along a diagonal of the lattice as

\[
D_n \equiv C(n, n) \equiv \langle \sigma_{0,0} \sigma_{n,n} \rangle \ .
\]

Note the symmetry relation \(C(n, n) = C(n, -n)\). In studying spin-spin correlation functions of the two-dimensional Ising model, one acknowledges that these are not universal in the sense of the renormalization group; that is, modifications of the model \((1.1)-(1.2)\) such as the generalization to spin \(s \geq 1\) and/or addition of (nonfrustrating) non-nearest-neighbor spin-spin or multispin interactions preserving the \(\mathbb{Z}_2\) symmetry would change \(\langle \sigma_{0,0} \sigma_{m,n} \rangle\) without changing the universality class of the phase transition. Nevertheless, these correlation functions contain useful information about the behavior of the model. We define the following notation:

\[
K = \beta J , \quad v = \tanh K , \quad x = v^2 , \quad z = e^{-2K} , \quad u = z^2 = e^{-4K} .
\]

Correlation functions are commonly expressed as functions of the variables

\[
k_> = \sinh^2(2K) , \quad k_< = \frac{1}{k_>} = \frac{1}{\sinh^2(2K)} .
\]

Recall that (as follows from duality) the critical point occurs at \(v_c = z_c = \sqrt{2} - 1\), i.e., \(K_c = J/(k_B T_c) = (1/2) \ln(\sqrt{2} + 1)\), at which point \(k_> = k_< = 1\). The high-temperature
series expansions of spin-spin correlation functions are commonly expressed as series in powers of $v$, while the low-temperature (LT) expansions on a bipartite lattice such as the square lattice considered here, are series in powers of $u$.

In Ref. [29], Jimbo and Miwa showed that $D_n$ can be calculated in terms of solutions to a (nonlinear, second-order) ordinary differential equation (ODE) of Painlevé VI type (see Appendix A). Subsequently, there has been a quest to find an analogous nonlinear ordinary differential equation whose solutions would yield the general spin-spin correlation function $C(m, n)$ in this model. However, as emphasized recently in [59], this is still an open problem. Even for $R_n$, to our knowledge, such a generalization of the Jimbo-Miwa ODE has not been found. Indeed, in the absence of an existence proof, it is not clear if such a (nonlinear, second-order) ODE whose solutions would yield the $R_n$, analogous to the Jimbo-Miwa Painlevé VI ODE for $D_n$ (see Appendix A), exists. Investigations into this can make use of exact calculations of correlation functions. The $D_n$ and $R_n$ can be expressed as Toeplitz determinants, and this method was used in [38, 39] to calculate these correlation functions for $n$ up to 6 and to present exact expressions for $n$ up to 5. Exact calculations of some other $C(m, n)$ were given in [41]. It was shown in [38] that $D_n$ is a homogeneous polynomial of degree $n$ in the complete elliptic integrals $K(k)$ and $E(k)$, where $k = k_>$ for $T \geq T_c$ and $k = k_<$ for $T \leq T_c$. The general structure of $R_n$ for the model of Eqs. (1.1), (1.2) was determined in [39] and is substantially more complicated, as will be reviewed below.

As shown in [46], the $C(m, n)$ for this model can be efficiently calculated recursively using certain quadratic relations [31] together with some initial inputs. Both of these methods yield specific correlation functions, e.g., $R_6$, $R_7$, etc. for higher $n$. In searching for a nonlinear ODE for $R_n$ analogous to the Jimbo-Miwa ODE for $D_n$, it would be convenient to use inputs that are general functions of $n$, rather than having to recursively compute $R_n$ for successive fixed values of $n$. For this purpose, high-temperature and low-temperature series expansions can be useful, if one knows general-$n$ expressions for the coefficients. However, standard procedures for calculating these series expansions are based on enumeration of graphs for a given correlation function $C(m, n)$ and, except for the first or second leading terms, do not normally yield expressions that are general functions of $(m, n)$. Here we focus on $R_n$. The leading term in the high-temperature series expansion of $R_n$ is $v^n$, and an elementary graphical enumeration yields the first higher-order term as $n(n+1)v^{n+2}$, but we are not aware of general-$n$ expressions for still higher-order terms in the literature. Similar comments apply for the low-temperature series expansion of this correlation function.

In this paper we shall discuss an approach that can yield general-$n$ coefficients of higher-order terms in high-temperature and low-temperature expansions of the row correlation functions $R_n$ for the two-dimensional Ising model defined by Eqs. (1.1)-(1.2) on the square
lattice. Our procedure makes use of exact calculations of individual \( R_n \). We illustrate the approach by computing general-\( n \) coefficients of several higher-order terms in high-temperature expansions of \( R_n \) and low-temperature expansions of \((R_n)_{\text{conn}}\). In addition to their intrinsic interest, this method and these results should be useful in the continuing endeavor to find a nonlinear ordinary differential equation for \( R_n \) analogous to the one derived for \( D_n \) by Jimbo and Miwa in \([29]\). Our work here is complementary to studies of form factor expansions for Ising correlation functions (e.g., \([28, 50–52, 55, 56]\)). It is also complementary to studies of properties of the Ising model susceptibility \( \chi \) (e.g., \([26, 44, 45, 47, 54, 56]\)), since the latter involves a sum over all connected spin-spin correlation functions, not just \( R_n \), via the relation

\[
\beta^{-1} \chi = \sum \vec{r} C(\vec{r})_{\text{conn}}.
\]

This paper is organized as follows. In Section II we review the general structural form for \( R_n \) obtained in \([39]\). In Sections III and IV we use the exactly calculated \( R_n \) from \([39]\) to infer general-\( n \) expressions for several coefficients of higher-order terms in high-temperature series for \( R_n \) and low-temperatures series for \((R_n)_{\text{conn}}\). Our conclusions are given in Section V. Some related results are included in appendices.

## II. STRUCTURE OF ROW CORRELATION FUNCTIONS

From our analysis in \([39]\), we inferred the following general structural form for the row correlation functions \( R_n \). These have different analytic forms \( R_{n,+} \) and \( R_{n,-} \) for \( T > T_c \) and \( T < T_c \), respectively (which are equal at \( T_c \)):

\[
R_{n,\pm} = B_n \sum_{\ell=0}^{n/2} \pi^{-2\ell} \sum_{s=0}^{2\ell} \mathcal{R}_{\ell-s,s}^{(n,\pm)}(k) \frac{E(k)^{2\ell}-s}{E(k)^{2\ell-s}K(k)^s},
\]

where \( k = k_> \) for \( T \geq T_c \) and \( k = k_< \) for \( T \leq T_c \) and

\[
R_{n,\pm} = B_n \sum_{\ell=0}^{n/2} \pi^{-\ell} \sum_{s=0}^{\ell} \mathcal{R}_{\ell-s,s}^{(n)}(k_>(k_> - 1)^{1/2}(-1)^{\ell-s} \delta_{s,0} E(k_)^{\ell-s}K(k_),
\]

where

\[
R_{n,+} = B_n \sum_{\ell=0}^{n/2} \pi^{-\ell} \sum_{s=0}^{\ell} \mathcal{R}_{\ell-s,s}^{(n)}(k_<)(k_< - 1)^{1/2}(-1)^{\ell-s} \delta_{s,0} E(k_<)^{\ell-s}K(k_<)^s,
\]

\[
R_{n,-} = B_n \sum_{\ell=0}^{n/2} \pi^{-\ell} \sum_{s=0}^{\ell} \mathcal{R}_{\ell-s,s}^{(n)}(k_<)(k_< - 1)^{1/2}(-1)^{\ell-s} \delta_{s,0} E(k_<)^{\ell-s}K(k_<)^s.
\]
In Eqs. (2.1)-(2.3), $B_n$ is a numerical prefactor; $p_n$ is an integer power\(^1\) and we define the compact notation
\[ \bar{K}(k) \equiv (k - 1)K(k) . \] (2.4)
The first five $B_n$ were given in \[39\], viz., $B_1 = B_2 = B_3 = 1$, $B_4 = 1/(3^2)$, and $B_5 = 1/(3^4)$ \[39\]. The sixth is $B_6 = 1/(3^6 \cdot 5^2)$. The values of the power $p_n$ in Eqs. (2.1)-(2.3) were listed (denoted as $q_n$) for $n$ up to 5 in \[39\]. Here we observe that for the known $p_n$ with $1 \leq n \leq 6$, the values are consistent with the general formula
\[ p_n = \left\lfloor \frac{n^2}{4} \right\rfloor_{\text{floor}} , \] (2.5)
where for $\nu \in \mathbb{R}$, $[\nu]_{\text{floor}}$ is the greatest integer $\leq \nu$. These powers $p_n$ are the same as the powers that occur in the general structural form for the diagonal correlation function $D_n$ that we found in \[38\]. In the remainder of the paper we will sometimes suppress the subscripts $\pm$ in the notation, with it being understood implicitly that $R_n \equiv R_{n,+}$ for $T \geq T_c$ and $R_n \equiv R_{n,-}$ for $T \leq T_c$. Note that although $K(k)$ is logarithmically divergent as $k \searrow 1$, this divergence is removed by the prefactor $(k - 1)$ in $\bar{K}(k)$. Indeed,
\[ \lim_{k \to 1} \bar{K}(k) = 0 , \] (2.6)
although the derivative $(d/dk)\bar{K}(k)$ is logarithmically divergent as $k \to 1$.

For a given $n$, the terms in $R_n$ can be divided into sets such that all of the terms in each set are homogeneous polynomials in $E(k)$ and $\bar{K}(k)$ of a given degree. We label this degree as the “level” of the set. For $R_{n,\pm}$ with odd $n$, as is evident from Eqs. (2.2) and (2.3), these terms are explicitly of the form $E(k)^{\ell-s}K(k)^s$ with $\ell$ in the range $0 \leq \ell \leq n$ and, for a given $\ell$, with $s$ in the range $0 \leq s \leq \ell$, where $k = k_\geq$ for $T \geq T_c$ and $k = k_\leq$ for $T \leq T_c$. The corresponding coefficients $\mathcal{R}_{\ell-s,s}^{(n)}$ in $R_{n,\pm}$ are polynomials in the respective $k$ elliptic modulus variables. For even $n$, only even-degree levels occur, running over $2\ell = 0, 2, ..., n$, as is evident in Eq. (2.1). Another difference between the $R_n$ with even and odd $n$ is that for odd $n$, the same coefficient polynomial $\mathcal{R}_{\ell-s,s}^{(n)}(k)$ occurs for $T \geq T_c$ and $T \leq T_c$ with the respective assignments $k = k_\geq$ and $k = k_\leq$, whereas for even $n$, the $\mathcal{R}_{2\ell-s,s}^{(n,+)}(k_\geq)$ and $\mathcal{R}_{2\ell-s,s}^{(n,-)}(k_\leq)$ are different functions of their respective arguments, $k_\geq$ and $k_\leq$. A third

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\(^1\) The quantities $B_n$ and $p_n$ and the dummy index $s$ were denoted $D_n$, $q_n$, and $r$ in \[39\]; here we relabel these to avoid confusion with our notation $D_n$ for $C(n,n)$ and $r = |\vec{r}|$. 
difference between the \( R_n \) for even and odd \( n \) is that the \( R_n \) for odd \( n \) contain a square-root prefactor, \((1 + k_\approx^{-1})^{1/2} = (1 + k_\approx)\), whereas the \( R_n \) for even \( n \) do not contain such square-root prefactors.

The general structural form that we inferred for \( R_n \) is considerably more complicated than the form that we had found in [38] for \( D_n \). These have different expressions \( D_{n,+} \) and \( D_{n,-} \) for \( T > T_c \) and \( T < T_c \) (which are equal at \( T = T_c \)):

\[
D_{n,\pm} = A_n \pi^{-n} k^{-2p_a - [1 - (1)^n]} \Theta(T - T_c)/2 \sum_{s=0}^{n} \mathcal{P}^{(n,\pm)}_{n-s,s}(k) (k^2 - 1)^{n-s} E(k)^n \left[(k^2 - 1) K(k)\right]^s, \tag{2.7}
\]

where again \( k = k_\approx \) if \( T \geq T_c \) and \( k = k_\approx \) if \( T \leq T_c \); and \( \Theta(x) \) is the Heaviside step function, defined as \( \Theta(x) = 1 \) if \( x > 0 \) and \( \Theta(x) = 0 \) if \( x \leq 0 \). One of the most striking differences is that \( D_n \) is a homogeneous polynomial of degree \( n \) in \( E(k) \) and \( K(k) \), while \( R_n \) has the multi-“level” structure of Eqs. (2.1), (2.2), and (2.3). Furthermore, calculations for \( D_n \) and the structural form presented in [38] apply for the general anisotropic case \( J_1 \neq J_2 \), with \( k_\approx = k_\approx^{-1} = \sinh(2K_1) \sinh(2K_2) \) and \( K_i = \beta J_i \), whereas in the anisotropic case, other correlation functions such as \( R_n \) would involve not just complete elliptic integrals of the first and second kinds, but also those of the third kind, as was already evident for \( R_1 \) [6, 58].

Our methods could be applied to this case in future work, although the series expansions would depend on two variables, e.g., \( v_i = \tanh K_i \) where \( K_i = \beta J_i \), \( i = 1, 2 \), or equivalently, \( s_i = \sinh(2K_i) \) with \( i = 1, 2 \) for the high-temperature expansions, and similarly on \( 1/s_i \) with \( i = 1, 2 \) for the low-temperature expansions.

III. GENERAL-\( n \) COEFFICIENTS IN THE HIGH-TEMPERATURE SERIES EXPANSION OF \( R_n \)

Here we report our new results on general-\( n \) coefficients of higher-order terms in the high-temperature Taylor series expansion of \( R_n \). This is analogous to the calculation of general-\( n \) coefficients in the HT expansion of \( D_n \) in [37] (see Appendix A), with the crucial difference that for \( D_n \) we were able to make use of the fact that the \( D_n \) can be determined in terms of solutions of the Painlevé VI ODE [29], whereas here no analogous (nonlinear) ODE for \( R_n \) is known. Hence, we make use of the \( R_n \) calculated in [39].

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2 We note some misprints in [4], [38], and [39]. In Eq. (A19) of [4], the expression \( -\frac{1}{2} \gamma_2 [F_{m_2,m_2} + F_{m_1,m_2-1}] \) should read \( -\frac{1}{2} \gamma_2 [F_{m_1,m_2+1} + F_{m_1,m_2-1}] \). In [38] there was a misprint in the overall sign of \( \mathcal{P}^{(5,-)}_{2,3} \), which should be reversed. In [39], the coefficient \( R_{3,1}^{(-)} \) should be multiplied by \((k_\approx - 1)\).
The standard procedure for calculating the high-temperature Taylor series expansion for
\( R_n = \langle \sigma_{0,0} \sigma_{n,0} \rangle \) enumerates the contributions from paths on the bonds of the lattice of minimal length and progressively greater lengths joining the points \((0,0)\) and \((n,0)\) (e.g., \[3\]). The number of bonds in the path is then the power of \( v \) in a given term in the series expansion, and the coefficient of each term is a positive integer. If one factors out an overall factor of \( v^n \) in the small-\( v \) expansion of \( R_n \), the rest of the series is a series in powers of \( v^2 \). This is an elementary consequence of the fact that if one reverses the sign of the spin-spin coupling \( J \), and hence the sign of \( K \) and of \( v = \tanh K \), then \( R_n \rightarrow (-1)^n R_n \). The lowest-order term, \( v^n \), in the small-\( v \) series expansion of \( R_n \), arises from the unique graph consisting of a straight path from \((0,0)\) to \((n,0)\), of length \( n \). Thus, the high-temperature expansion for \( R_n \) has the general form
\[
R_n = v^n \left[ 1 + \sum_{j=1}^{\infty} r_{n,2j} v^{2j} \right],
\] where the \( r_{n,2j} \) are positive integers. Aside from the leading \( v^n \) term, the paths contributing to all higher-order terms include bonds above the direct, horizontal path, and corresponding paths that are related to these by reflection about the horizontal axis. That is, for each path including bonds above the direct route along the horizontal axis joining the sites \((0,0)\) to \((n,0)\), there is a path that is obtained by this reflection process. Therefore, the \( r_{n,2j} \) are even integers.

To begin, we discuss the graphical derivation of the first subleading term in Eq. (3.1), namely, \( r_{n,2} v^{n+2} \). This term arises from paths of length \( n+2 \) bonds connecting the sites \((0,0)\) and \((n,0)\). An elementary enumeration counts these paths. The sites \((0,\ell)\) with \( 0 \leq \ell \leq n \) comprise \( n+1 \) vertices on the square lattice. One set of paths of length \( n+2 \) connecting \((0,0)\) and \((n,0)\) involves a \( 90^\circ \) turn upwards at one of these \( n+1 \) sites followed by a continuation along horizontal bonds, and then a \( 90^\circ \) turn downward and final continuation to the point \((n,0)\). For each such path, there is also a corresponding path obtained by reflecting about the horizontal axis, so that the first right-hand turn is downward instead of upward. There are \( \binom{n+1}{2} \) paths in the first set, and hence \( 2 \binom{n+1}{2} = n(n+1) \) paths of length \( n+2 \) joining the points \((0,0)\) and \((n,0)\). This simple combinatoric argument yields the coefficient, \( r_{n,2} \), of the \( v^{n+2} \) term in the high-temperature expansion of \( R_n \), namely
\[
r_{n,2} = n(n+1).
\] This is manifestly even, since either \( n \) or \( n+1 \) is even.

Now from high-temperature series expansions of our calculations of \( R_n \) for \( n \), we determine the following general-\( n \) expression for the next-to-next-leading-order coefficient, \( r_{n,4} \). For reference, we list these expansions for \( n \) up to 6 in Appendix \[B\]. Our method is motivated
by the structural form as polynomials in \( n \) that we obtained for coefficients of higher-order terms in the HT series for \( D_n \) in [37] (reviewed in Appendix A). We thus fit the respective \( O(v^{n+4}) \) terms in the HT series expansions of the exact expressions for \( R_n \) to a polynomial. This is an overconstrained fit, and we obtain the result

\[
r_{n,4} = \frac{n}{4}(n^3 + 2n^2 + 3n + 10).
\] (3.3)

Although there is an extensive literature on series expansions of quantities in the two-dimensional Ising model, we are not aware of this expression for \( r_{n,4} \) having appeared in this literature. An alternate approach to determining \( r_{n,4} \) would make use of an enumeration of all graphs that contribute to the \( O(v^{n+4}) \) term in the high-temperature expansion of \( R_n \) for arbitrarily great \( n \). This result is given to illustrate the method; clearly, one could proceed to calculate coefficients of more higher-order terms, \( r_{n,6} \), etc. Since the \( r_{n,2j} \) with \( 2j \geq 6 \) are higher-degree polynomials in \( n \), the procedure for calculating these polynomials via overconstrained fits requires the input of a larger number of row correlation functions. However, as emphasized, the value of this method is that the resultant coefficient applies for general \( n \) and hence is directly applicable to the search for a nonlinear differential equation whose solution would yield \( R_n \).

Despite the prefactor of 1/4, it is easy to show that the expression for \( r_{n,4} \) in Eq. (3.3) is an integer, and, furthermore, is even. This is proved by induction, starting from any of the known \( r_{n,4} \) values for \( 1 \leq n \leq 6 \), each of which is even. Given that there exists an \( n \) such that \( r_{n,4} \) is even, to carry out the inductive proof, one must prove that \( r_{n+1,4} \) is also even. This can be done by showing that the difference, \( r_{n+1,4} - r_{n,4} \), is even, i.e., \( r_{n+1,4} - r_{n,4} = 2p \) for some (positive) integer \( p \). We calculate

\[
r_{n+1,4} - r_{n,4} = (n + 2)(n^2 + n + 2).
\] (3.4)

Since the factor \( (n + 2) \) can be even or odd, we thus need to show that \( n^2 + n + 2 \) is always even. This follows directly by observing that \( n^2 + n + 2 = n(n + 1) + 2 \). Now \( n(n + 1) \) is manifestly even, since either \( n \) or \( n + 1 \) is even, and hence \( n(n + 1) + 2 \) is even. This completes the proof that the expression for \( r_{n,4} \) in Eq. (3.3) is an even (positive) integer.

One can also express these results equivalently as series expansions in powers of the variable \( \sqrt{k_>} \), using the relation (B2). It is convenient to introduce the variable \( \hat{k}_> = (1/4)k_> \) as in Eq. (B4). Then Eq. (3.1) can be written as

\[
R_n = \hat{k}_>^{n/2} \left[ 1 + \sum_{\ell=1}^{\infty} \tilde{r}_{n,\ell} \hat{k}_>^\ell \right],
\] (3.5)

where

\[
\tilde{r}_{n,1} = n^2
\] (3.6)
\[ \tilde{r}_{n,2} = \frac{1}{4} n(n - 1)(n^2 - n - 8). \quad (3.7) \]

While the coefficients \( r_{n,2}, r_{n,4}, \) and \( \tilde{r}_{n,1} \) are positive and monotonically increasing as functions of \( n \) in the interval \( n \geq 1 \), the behavior of \( \tilde{r}_{n,2} \) is more complicated. As a function of \( n \), with \( n \) generalized from integral values to real values in this interval \( n \geq 1 \), \( \tilde{r}_{n,2} \) decreases from zero at \( n = 1 \) through negative values, reaching a minimum of \(-4\) at \( n = (1/2)(1 + \sqrt{17}) = 2.56155 \) and then increases monotonically for larger \( n \), passing through zero again at \( n = (1/2)(1 + \sqrt{33}) = 3.27228 \). Thus, \( \tilde{r}_{n,2} \) is negative for \( n = 2 \) and \( n = 3 \), taking the values \( \tilde{r}_{2,2} = \tilde{r}_{3,2} = -3 \).

IV. **GENERAL-\( n \) COEFFICIENTS OF HIGHER-ORDER TERMS IN THE LOW-TEMPERATURE SERIES EXPANSION OF \((R_n)_{\text{conn}}\).**

In addition to its intrinsic interest, the spin-spin correlation function \( C(\vec{r}) \) is important because its limit as \( r \to \infty \) determines the (square of the) spontaneous magnetization:

\[ \lim_{r \to \infty} C(\vec{r}) = M^2, \quad (4.1) \]

where \( r \equiv |\vec{r}| \). The connected correlation function is then

\[ C(\vec{r})_{\text{conn.}} = C(\vec{r}) - M^2. \quad (4.2) \]

For \( T < T_c \) where the spontaneous magnetization is nonzero, an interesting question concerns the approach to the limit (4.1). For a given \( \vec{r} \), a quantitative measure of this approach is provided by the ratio

\[ A_{C(\vec{r})} = \frac{C(\vec{r})}{M^2} = 1 + \frac{C(\vec{r})_{\text{conn.}}}{M^2}, \quad (4.3) \]

In this section, we present our results on general-\( n \) coefficients of higher-order terms in the low-temperature Taylor series expansions of \((R_n)_{\text{conn.}} = R_n - M^2\) and \(A_{R_n} = R_n/M^2\). In calculating \((R_n)_{\text{conn.}}\), we make use of the result first published by Yang [2] for the spontaneous magnetization in the two-dimensional Ising model on the square lattice,

\[ M = (1 - k_-^2)^{1/8} = \frac{(1 + u)^{1/4}(1 - 6u + u^2)^{1/8}}{(1 - u)^{1/2}}. \quad (4.4) \]

The quantity \( M^2 \) has the resultant low-temperature Taylor series expansion

\[ M^2 = 1 - 4u^2 - 16u^3 - 64u^4 - 272u^5 - 1228u^6 - 5792u^7 - 28192u^8 - 140448u^9 \]
The property (4.1), together with the property that $M$ and $R_n$ are continuous functions of $u$, implies that as $n$ increases, the low-temperature (i.e., small-$u$) Taylor series expansion of $R_n$ must coincide with the small-$u$ expansion of $R_n - M^2$ to an increasingly high order, and the order of the first term in the small-$u$ expansion of $R_n - M^2$ must go to infinity as $n \to \infty$. From general arguments, for an Ising ferromagnet on (the thermodynamic limit of) a given lattice, $R_n$ is a monotonically decreasing function of $n$ for fixed temperature $T$, i.e., $R_n \geq R_{n+1}$, and hence $R_n \geq M^2$. (The two points at which this inequality is realized as an equality are (i) $T = 0$ for any $n$, where $R_n = M^2 = 1$, and (ii) $T = \infty$, where for $n \geq 1$, $R_n = M^2 = 0$.)

The low-temperature Taylor series expansions of the $R_n$ in powers of $u$ or $k_<$ match the corresponding expansions of $M^2$ to $O(u^{n+1}) = O(k_<^{n+1})$ inclusive. Thus, the LT series expansion of $R_n$ has the general form

\[(R_n)_{\text{conn.}} = 4u^{n+2}\left[1 + \sum_{j=1}^{\infty} \rho_{n,j} u^j\right] = 4\hat{k}_<^{n+2}\left[1 + \sum_{j=1}^{\infty} \tilde{\rho}_{n,j} \hat{k}_<^j\right]. (4.6)\]

Here it is convenient to use the rescaled variable $\hat{k}_< = (1/4)k_<$ (as defined in Eq. (B4)), since this yields integral coefficients $\tilde{\rho}_{n,j}$. Using LT expansions of the $R_n$ that we have calculated exactly, we apply the same polynomial fitting procedure that we used for the HT expansions. For reference, we list these LT expansions in Appendix C. We obtain the following general-$n$ expressions for the $u^{n+3}$ and $u^{n+4}$ terms in Eq. (4.6):

\[\rho_{n,1} = n^2 + 2n + 4, \quad (4.7)\]

and

\[\rho_{n,2} = \frac{1}{2}(n^4 + 4n^3 + 13n^2 + 26n + 32). \quad (4.8)\]

Equivalently, for the expansion of $(R_n)_{\text{conn.}}$ in terms of $\hat{k}_<$ in Eq. (4.6), we have

\[\tilde{\rho}_{n,1} = n^2 \quad (4.9)\]

and

\[\tilde{\rho}_{n,2} = \frac{1}{2}(n + 2)(n^3 - 2n^2 + n + 6). \quad (4.10)\]

By combining the LT expansion for $M^2$ with these results, one can thus obtain the corresponding general-$n$ LT expansion for $R_n$ up to $O(u^{n+4}) = O(\hat{k}_<^{n+4})$. We are not aware
of the expressions (4.7)-(4.10) having appeared before in the literature. These results are
given to illustrate the method and could be extended to higher order using additional \( R_n \)
correlation functions as input.

Note that, despite the prefactor of 1/2, the expression for \( \rho_{n,2} \) is an integer. We give an
inductive proof of this. First, this integral property holds for the LT series for \( R_1 \). Hence, it
is necessary and sufficient to show that with \( \rho_{n,2} \) being integral, so is \( \rho_{n+1,2} \). To do this, we
show that the difference \( \rho_{n+1,2} - \rho_{n,2} \) is integral. This difference is

\[
\rho_{n+1,2} - \rho_{n,2} = (n + 1)(2n^2 + 5n + 11),
\]

which is obviously integral. The same inductive method shows that \( \bar{\rho}_{n,2} \) is an integer.

Combining these results with the definition \( A_{R_n} = R_n / M^2 \) yields

\[
A_{R_n} = 1 + 4u^{n+2} \left[ 1 + \rho_{n,1}u + (\rho_{n,2} + 4)u^2 + O(u^3) \right] \\
= 1 + 4u^{n+2} \left[ 1 + (n^2 + 2n + 4)u + \frac{1}{2}(n^4 + 4n^3 + 13n^2 + 26n + 40)u^2 + O(u^3) \right].
\]

(4.12)

Equivalently, in terms of the \( \hat{k}_< \) variable,

\[
A_{R_n} = 1 + 4\hat{k}_<^{n+2} \left[ 1 + \bar{\rho}_{n,1}\hat{k}_< + (\bar{\rho}_{n,2} + 4)\hat{k}_<^2 + O(\hat{k}_<^3) \right] \\
= 1 + 4\hat{k}_<^{n+2} \left[ 1 + n^2\hat{k}_< + \frac{1}{2}(n^4 - 3n^2 + 8n + 20)\hat{k}_<^2 + O(\hat{k}_<^3) \right].
\]

(4.13)

V. CONCLUSIONS

In this paper we have discussed a method for obtaining general-\( n \) expressions for coefficients of higher-order terms in the high-temperature and low-temperature series expansions of the spin-spin correlation function \( R_n \) in the two-dimensional Ising model on the square lattice and have applied it to obtain general-\( n \) coefficients of several higher-order terms in these series. This method is complementary to the standard method for calculating these coefficients, which is via enumeration of graphs that contribute in a given order of expansion. It is also complementary to another method that was used in [37] for the high-temperature expansions of diagonal correlation functions \( D_n \), which was based on the property that the
\( D_n \) can be computed in terms of solutions to the Painlevé VI ordinary differential equation of \([29]\). In addition to the intrinsic interest in the general-\( n \) coefficients discussed here, they provide further inputs to the continuing quest to find a nonlinear ordinary differential equation whose solution would determine the \( R_n \).

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**Appendix A: General-\( n \) Coefficients in the High-Temperature Series Expansion of \( D_n \)**

The high-temperature series expansion of \( D_n \) in the Ising model on the square lattice has the form

\[
D_n = \sum_{j=n}^{\infty} c^{(D)}_{n,j} x^j,
\]

where \( x \equiv v^2 \) (c.f. Eq. (1.6)) and the subscript + in \( D_{n,+} \) is understood implicitly. An elementary combinatoric argument determines the coefficient of the leading-order term as

\[
c^{(D)}_{n,n} = \frac{(2n)!}{(n!)^2}.
\]

The series (A1) can equivalently be written as

\[
D_n = c^{(D)}_{n,n} x^n \left[ 1 + \sum_{\ell=1}^{\infty} r^{(D)}_{n,\ell} x^\ell \right],
\]

where the ratio \( r^{(D)}_{n,\ell} \) is given by

\[
r^{(D)}_{n,\ell} = \frac{c^{(D)}_{n,n+\ell}}{c^{(D)}_{n,n}}.
\]

Let us define a variable \( t \) as \( t = k^{-2} \) for \( T \geq T_c \) and \( t = k^{-2} \) for \( T \leq T_c \), and auxiliary functions \( \sigma_{n,\pm} \) as follows:

\[
\sigma_{n,+} = t(t-1) \frac{d \ln D_{n,+}}{dt} - \frac{t}{4}
\]
and
\[ \sigma_{n,-} = t(t-1) \frac{d \ln D_{n,-}}{dt} - \frac{1}{4}. \]  

(A6)

In [29], Jimbo and Miwa showed that the \( \sigma_n \) functions are solutions to the following ordinary differential equation of Painlevé VI type (where subscripts \( \pm \) on \( \sigma_n \) are understood implicitly for \( T \geq T_c \) and \( T \leq T_c \) and \( \sigma'_n \equiv d\sigma_n/dt \)):

\[ [t(t-1)\sigma''_n] - \sigma'_n - \sigma_n = 0. \]  

(A7)

The diagonal correlation functions \( D_{n,\pm} \) are then determined in terms of the \( \sigma_{n,\pm} \). In [37] with Ghosh, using the result from [29], we derived the general form of the nine terms beyond the leading term in the high-temperature expansion of \( D_n = D_{n,+} \). (In [37], the coefficients \( c^{(D)}_{n,j} \) and the ratios \( r^{(D)}_{n,l} \) were denoted as \( c_{n,j} \) and \( r_{n,n+j} \), respectively.) Our results in [37] included the following for the ratios \( r^{(D)}_{n,ell} \), in our present notation:

\[ r^{(D)}_{n,1} = 2n \]  

(A8)

\[ r^{(D)}_{n,2} = \frac{n(2n^2 + 3n + 5)}{n+1} \]  

(A9)

\[ r^{(D)}_{n,3} = \frac{2n(2n^3 + 5n^2 + 16n + 25)}{3(n+1)} \]  

(A10)

\[ r^{(D)}_{n,4} = \frac{4n^6 + 24n^5 + 103n^4 + 372n^3 + 943n^2 + 726n - 48}{6(n+1)(n+2)} \]  

(A11)

\[ r^{(D)}_{n,5} = \frac{4n^7 + 32n^6 + 183n^5 + 930n^4 + 4031n^3 + 10228n^2 + 6972n - 960}{15(n+1)(n+2)}, \]  

(A12)

and so forth for higher-order terms up to \( r^{(D)}_{n,9} \). It is interesting to note that, in an analogous manner, it was possible to use the fact that correlation functions for the transverse Ising quantum spin chain at critical field and \( T = 0 \) satisfy a Painlevé V equation to derive a number of properties of the correlation functions for these functions [33]-[36],[42, 43].

**Appendix B: High-Temperature Series for \( R_n \)**

For reference, in this appendix we list the high-temperature series that we calculate from our exact results for \( R_n \) with \( n \) up to 6. Note that \( R_1 \) was calculated in [3]. These series have the general form of Eq. (3.1). We first record some relations between the elliptic moduli.
$k_\uparrow = 1/k_\downarrow$ and the respective high- and low-temperature expansion variables $v$ and $z$. The latter two variables are dual to each other and satisfy

$$v = \frac{1 - z}{1 + z}, \quad \text{equivalently,} \quad z = \frac{1 - v}{1 + v}. \quad (B1)$$

Then

$$k_\uparrow = \left[ \frac{2v}{1 - v^2} \right]^2 \quad (B2)$$

and

$$k_\downarrow = \left[ \frac{2z}{1 - z^2} \right]^2 = \frac{4u}{(1 - u)^2}. \quad (B3)$$

It is convenient to introduce the rescaled quantities

$$\hat{k}_\uparrow = \frac{k_\uparrow}{4}, \quad \hat{k}_\downarrow = \frac{k_\downarrow}{4}. \quad (B4)$$

The high-temperature series expansions are

$$R_1 = v + 2v^3 + 4v^5 + 12v^7 + 42v^9 + 164v^{11} + 686v^{13} + 3012v^{15} + O(v^{17}) \quad (B5)$$

$$R_2 = v^2 + 6v^4 + 16v^6 + 46v^8 + 158v^{10} + 618v^{12} + 2618v^{14} + 11654v^{16} + O(v^{18}) \quad (B6)$$

$$R_3 = v^3 + 12v^5 + 48v^7 + 152v^9 + 506v^{11} + 1900v^{13} + 7902v^{15} + 35114v^{17} + O(v^{19}) \quad (B7)$$

$$R_4 = v^4 + 20v^6 + 118v^8 + 452v^{10} + 1564v^{12} + 5684v^{14} + 22726v^{16} + 98708v^{18} + O(v^{20}) \quad (B8)$$

$$R_5 = v^5 + 30v^7 + 250v^9 + 1200v^{11} + 4606v^{13} + 16920v^{15} + 65452v^{17} + 274422v^{19} + O(v^{21}) \quad (B9)$$

$$R_6 = v^6 + 42v^8 + 474v^{10} + 2862v^{12} + 12662v^{14} + 49282v^{16} + 189702v^{18} + 770190v^{20} + O(v^{22}). \quad (B10)$$

These series can be extended to higher $n$, but these are sufficient to illustrate our method.

These high-temperature series for $R_n$ can equivalently be expressed as series expansions in powers of the variable $\sqrt{k_\uparrow}$, using the relation $k_\uparrow = 1/k_\downarrow$, as defined Eq. $(B2)$. However, in contrast to the series in $v$, the series in powers of $\sqrt{k_\uparrow}$ have coefficients that vary in sign, and do not increase monotonically in magnitude; indeed, some terms have zero coefficients. We list these equivalent expansions here. It is convenient to use the rescaled quantity $\hat{k}_\uparrow = (1/4)k_\uparrow$ as defined Eq. $(B3)$, since this avoids fractional coefficients. We have

$$R_1 = \hat{k}_\uparrow^{1/2} \left[ 1 + \hat{k}_\uparrow + 5\hat{k}_\uparrow^3 - 4\hat{k}_\uparrow^4 + 44\hat{k}_\uparrow^5 - 60\hat{k}_\uparrow^6 + 469\hat{k}_\uparrow^7 - \ldots \right] \quad (B11)$$

$$R_2 = \hat{k}_\uparrow \left[ 1 + 4\hat{k}_\uparrow - 3\hat{k}_\uparrow^2 + 20\hat{k}_\uparrow^3 - 24\hat{k}_\uparrow^4 + 160\hat{k}_\uparrow^5 - 235\hat{k}_\uparrow^6 + 1556\hat{k}_\uparrow^7 - 2568\hat{k}_\uparrow^8 + \ldots \right] \quad (B12)$$

$$R_3 = \hat{k}_\uparrow^{3/2} \left[ 1 + 9\hat{k}_\uparrow - 3\hat{k}_\uparrow^2 + 28\hat{k}_\uparrow^3 + 8\hat{k}_\uparrow^4 + 153\hat{k}_\uparrow^5 + 233\hat{k}_\uparrow^6 + 1008\hat{k}_\uparrow^7 + 3588\hat{k}_\uparrow^8 + \ldots \right] \quad (B13)$$
\[ R_4 = \hat{k}_>^2 \left[ 1 + 16\hat{k}_> + 12\hat{k}_>^2 + 201\hat{k}_>^3 - 240\hat{k}_>^4 + 2332\hat{k}_>^5 - 3584\hat{k}_>^7 + 27280\hat{k}_>^8 + O(\hat{k}_>^9) \right] \] (B14)

\[ R_5 = \hat{k}_>^{5/2} \left[ 1 + 25\hat{k}_> + 60\hat{k}_>^2 - 75\hat{k}_>^3 + 561\hat{k}_>^4 - 699\hat{k}_>^5 + 4876\hat{k}_>^6 - 5420\hat{k}_>^7 + 45516\hat{k}_>^8 + O(\hat{k}_>^9) \right] \] (B15)

\[ R_6 = \hat{k}_>^3 \left[ 1 + 36\hat{k}_> + 165\hat{k}_>^2 - 140\hat{k}_>^3 + 821\hat{k}_>^4 + 276\hat{k}_>^5 + 3092\hat{k}_>^6 + 15440\hat{k}_>^7 - 2484\hat{k}_>^8 + O(\hat{k}_>^9) \right] \] (B16)

Note that in the square bracket for \( R_1 \) in Eq. (B11) there is no \( \hat{k}_>^2 \) term and in the square bracket for \( R_4 \) in Eq. (B14) there is no \( \hat{k}_>^3 \) term.

For reference, we list numerical values of the \( R_n \) for \( T \geq T_c \) in Table I. For comparison with the numerical values of \( R_n \) as \( T \to T_c \), the analytic values of \( (R_n)_{cr} \) with \( n \) up to 6 from [39] are as follows:

\[ (R_1)_{cr} = 2^{-1/2} = 0.707107 \] (B17)

\[ (R_2)_{cr} = 1 - \frac{2^2}{\pi^2} = \left( 1 - \frac{2}{\pi} \right) \left( 1 + \frac{2}{\pi} \right) = 0.594715 \] (B18)

\[ (R_3)_{cr} = 2^{3/2} \left( 1 - \frac{2^3}{\pi^2} \right) = 0.53579045 \] (B19)

\[ (R_4)_{cr} = 2^4 \left( 1 - \frac{2^4}{3^2 \pi^2} + \frac{2^8}{3^2 \pi^4} \right) = 0.497989 \] (B20)

\[ (R_5)_{cr} = 2^{15/2} \left( 1 - \frac{2^3 \cdot 19}{3^2 \pi^4} + \frac{2^9 \cdot 11}{3^4 \pi^4} \right) = 0.470724 \] (B21)

\[ (R_6)_{cr} = 2^{12} \left( 1 - \frac{2^2 \cdot 13 \cdot 31}{3 \cdot 5^2 \pi^2} + \frac{2^{10} \cdot 7 \cdot 13}{3^3 \cdot 5^2 \cdot \pi^4} - \frac{2^{22}}{3^6 \cdot 5^2 \pi^6} \right) \]

\[ = 0.449637 \] (B22)

Factorizations of these \( (R_n)_{cr} \) for even \( n \) were given in [39]; we have only shown the first of these factorizations, for \( R_2 \), here. See also [40].
Appendix C: Low-Temperature Series for \((R_n)_{\text{conn}}\).

For reference, we list here the low-temperature series expansions of the connected correlation functions \((R_n)_{\text{conn}}\) for \(n\) up to 6 here. These have the general form (4.6) and are as follows:

\[(R_1)_{\text{conn}} = 4u^3 + 28u^4 + 152u^5 + 780u^6 + 3972u^7 + 20348u^8 + 105192u^9 + 548792u^{10} + O(u^{11})\]  
\[\text{(C1)}\]
\[(R_2)_{\text{conn}} = 4u^4 + 48u^5 + 368u^6 + 2320u^7 + 13428u^8 + 74848u^9 + 410576u^{10} + 2238496u^{11} + O(u^{12})\]  
\[\text{(C2)}\]
\[(R_3)_{\text{conn}} = 4u^5 + 76u^6 + 832u^7 + 6648u^8 + 44852u^9 + 276456u^{10} + 1623704u^{11} + 9293292u^{12} + O(u^{13})\]  
\[\text{(C3)}\]
\[(R_4)_{\text{conn}} = 4u^6 + 112u^7 + 1712u^8 + 17584u^9 + 141756u^{10} + 988192u^{11} + 6317392u^{12} + 38365984u^{13} + O(u^{14})\]  
\[\text{(C4)}\]
\[(R_5)_{\text{conn}} = 4u^7 + 156u^8 + 3224u^9 + 42412u^{10} + 414228u^{11} + 3331068u^{12} + 23619120u^{13} + 154485248u^{14} + O(u^{15})\]  
\[\text{(C5)}\]
\[(R_6)_{\text{conn}} = 4u^8 + 208u^9 + 5632u^{10} + 93680u^{11} + 1111492u^{12} + 10437824u^{13} + 83409104u^{14} + 596805184u^{15} + O(u^{16}) \]  
\[\text{(C6)}\]

These series can equivalently be expressed in terms of the variable \(k_<\), using the relation (B3). As before, it is convenient to use the rescaled variable \(\hat{k}_< = (1/4)k_<\) as defined in Eq. (B4), since this avoids fractional coefficients. We have

\[(R_1)_{\text{conn}} = 4\hat{k}_<^3 + 4\hat{k}_<^4 + 36\hat{k}_<^5 + 52\hat{k}_<^6 + 384\hat{k}_<^7 + 668\hat{k}_<^8 + 4500\hat{k}_<^9 + 8820\hat{k}_<^{10} + O(\hat{k}_<^{11}) \]  
\[\text{(C7)}\]
\[(R_2)_{\text{conn}} = 4\hat{k}_<^4 + 16\hat{k}_<^5 + 64\hat{k}_<^6 + 192\hat{k}_<^7 + 908\hat{k}_<^8 + 2256\hat{k}_<^9 + 12704\hat{k}_<^{10} + O(\hat{k}_<^{12}) \]  
\[\text{(C8)}\]
\[(R_3)_{\text{conn}} = 4\hat{k}_<^5 + 36\hat{k}_<^6 + 180\hat{k}_<^7 + 440\hat{k}_<^8 + 2948\hat{k}_<^9 + 5604\hat{k}_<^{10} + 42808\hat{k}_<^{11} + 74980\hat{k}_<^{12} + O(\hat{k}_<^{13}) \]  
\[\text{16}\]
(R_4)_{conn.} = 4\hat{k}_5^6 + 64\hat{k}_5^7 + 504\hat{k}_5^8 + 1344\hat{k}_5^9 + 7720\hat{k}_5^{10} + 22912\hat{k}_5^{11} + 108608\hat{k}_5^{12} + 352256\hat{k}_5^{13} \\
\quad + O(\hat{k}_5^{14}) \quad \quad \quad \quad \quad \quad \quad \quad \quad \text{(C10)}

(R_5)_{conn.} = 4\hat{k}_6^7 + 100\hat{k}_6^8 + 1204\hat{k}_6^9 + 4900\hat{k}_6^{10} + 19224\hat{k}_6^{11} + 84708\hat{k}_6^{12} + 311588\hat{k}_6^{13} \\
\quad + 1230068\hat{k}_6^{14} + O(\hat{k}_6^{15}) \quad \quad \quad \quad \quad \quad \quad \quad \quad \text{(C11)}

(R_6)_{conn.} = 4\hat{k}_7^8 + 144\hat{k}_7^9 + 2496\hat{k}_7^{10} + 15872\hat{k}_7^{11} + 58484\hat{k}_7^{12} + 250896\hat{k}_7^{13} + 1104448\hat{k}_7^{14} \\
\quad + 3668416\hat{k}_7^{15} + O(\hat{k}_7^{16}) \quad \quad \quad \quad \quad \quad \quad \quad \quad \text{(C12)}

Combining Eqs. (C11)-(C6) with the LT series expansion for M^2, one obtains the LT series expansions for the full R_n correlation functions:

\[ R_1 = 1 - 4u^2 - 12u^3 - 36u^4 - 120u^5 - 448u^6 - 1820u^7 - 7844u^8 - 35256u^9 - 163484u^{10} - O(u^{11}) \quad \text{(C13)} \]

\[ R_2 = 1 - 4u^2 - 16u^3 - 60u^4 - 224u^5 - 860u^6 - 3472u^7 - 14764u^8 - 65600u^9 - 301700u^{10} - O(u^{11}) \quad \text{(C14)} \]

\[ R_3 = 1 - 4u^2 - 16u^3 - 64u^4 - 268u^5 - 1152u^6 - 4960u^7 - 21544u^8 - 95596u^9 - 435820u^{10} - O(u^{11}) \quad \text{(C15)} \]

\[ R_4 = 1 - 4u^2 - 16u^3 - 64u^4 - 272u^5 - 1224u^6 - 5680u^7 - 26480u^8 - 122864u^9 - 570520u^{10} - O(u^{11}) \quad \text{(C16)} \]

\[ R_5 = 1 - 4u^2 - 16u^3 - 64u^4 - 272u^5 - 1228u^6 - 5788u^7 - 28036u^8 - 137224u^9 - 669864u^{10} - O(u^{11}) \quad \text{(C17)} \]

\[ R_6 = 1 - 4u^2 - 16u^3 - 64u^4 - 272u^5 - 1228u^6 - 5792u^7 - 28188u^8 - 140240u^9 - 706644u^{10} - O(u^{11}) \quad \text{(C18)} \]

The equivalent series expansions, expressed in terms of the variable \( \hat{k}_< \), are

\[ R_1 = 1 - 4\hat{k}_<^2 + 4\hat{k}_<^3 - 20\hat{k}_<^4 + 36\hat{k}_<^5 - 172\hat{k}_<^6 + 384\hat{k}_<^7 - 1796\hat{k}_<^8 + 4500\hat{k}_<^9 - 20748\hat{k}_<^{10} + O(\hat{k}_<^{11}) \quad \text{(C19)} \]

\[ R_2 = 1 - 4\hat{k}_<^2 - 20\hat{k}_<^4 + 16\hat{k}_<^5 - 160\hat{k}_<^6 + 192\hat{k}_<^7 - 1556\hat{k}_<^8 + 2256\hat{k}_<^9 - 16864\hat{k}_<^{10} - 27392\hat{k}_<^{11} + O(\hat{k}_<^{11}) \quad \text{(C20)} \]

\[ R_3 = 1 - 4\hat{k}_<^2 - 24\hat{k}_<^4 + 4\hat{k}_<^5 - 188\hat{k}_<^6 + 180\hat{k}_<^7 - 2024\hat{k}_<^8 + 2948\hat{k}_<^9 - 23964\hat{k}_<^{10} + O(\hat{k}_<^{11}) \quad \text{(C21)} \]
\[ R_4 = 1 - 4k^2 - 24k^4 - 220k^6 + 64k^7 - 1960k^8 + 1344k^9 - 21848k^{10} + O(k^{11}) \]  
(C22)

\[ R_5 = 1 - 4k^2 - 24k^4 - 224k^6 + 4k^7 - 2364k^8 + 1204k^9 - 24668k^{10} + O(k^{11}) \]  
(C23)

\[ R_6 = 1 - 4k^2 - 24k^4 - 224k^6 - 2460k^8 + 144k^9 - 27072k^{10} + O(k^{11}) \]  
(C24)

We list numerical values of the \( R_n \) for \( T \leq T_c \) in Table II.

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TABLE I: Numerical values of the $R_n$ for $1 \leq n \leq 6$ and $T \geq T_c$ as functions of $k_\sigma$. For reference, the values of $v$ and $T/T_c$ corresponding to each value of $k_\sigma$ are also shown. In this and the other tables, the notation $a \times 10^{-n}$ means $a \times 10^{-n}$.

| $k_\sigma$ | $v$  | $T/T_c$ | $R_1$  | $R_2$  | $R_3$  | $R_4$  | $R_5$  | $R_6$  |
|------------|------|---------|--------|--------|--------|--------|--------|--------|
| 0          | 0    | $\infty$| 0      | 0      | 0      | 0      | 0      | 0      |
| 0.1        | 0.1543 | 8.828   | 0.1621 | 0.02746| 0.4837e-2| 0.8797e-3 | 1.642e-4| 0.3127e-4 |
| 0.2        | 0.2134 | 4.435   | 0.2349 | 0.05974| 0.01617| 0.4578e-2 | 1.338e-3| 0.4000e-3 |
| 0.3        | 0.2559 | 2.981   | 0.2950 | 0.09684| 0.03431| 0.01279 | 0.4926e-2| 1.939e-3 |
| 0.4        | 0.2897 | 2.260   | 0.3494 | 0.1389 | 0.06011| 0.02740 | 0.01290 | 0.6200e-2 |
| 0.5        | 0.3178 | 1.832   | 0.4013 | 0.1864 | 0.09463| 0.05059 | 0.02790 | 0.01569 |
| 0.6        | 0.3420 | 1.549   | 0.4525 | 0.2400 | 0.13935| 0.08507 | 0.05348 | 0.03426 |
| 0.7        | 0.3632 | 1.350   | 0.5045 | 0.3011 | 0.1965 | 0.1345 | 0.09462 | 0.06774 |
| 0.8        | 0.3820 | 1.203   | 0.5595 | 0.3723 | 0.2700 | 0.2046 | 0.1590 | 0.1256 |
| 0.9        | 0.3989 | 1.090   | 0.6210 | 0.4596 | 0.3684 | 0.3070 | 0.2617 | 0.2262 |
| 1          | 0.4142 | 1       | 0.7071 | 0.5947 | 0.5358 | 0.4980 | 0.4707 | 0.4496 |
TABLE II: Numerical values of the $R_n$ for $1 \leq n \leq 6$ and $T \leq T_c$ as functions of $k_<$ For reference, the values of $z$ and $T/T_c$ corresponding to each value of $k_<$ are also shown.

| $k_<$ | $z$  | $T/T_c$ | $R_1$  | $R_2$  | $R_3$  | $R_4$  | $R_5$  | $R_6$  |
|-------|------|---------|--------|--------|--------|--------|--------|--------|
| 0     | 0    | 0       | 0      | 0      | 0      | 0      | 0      | 0      |
| 0.1   | 0.1543 | 0.2940  | 0.9976 | 0.9975 | 0.9975 | 0.9975 | 0.9975 | 0.9975 |
| 0.2   | 0.2134 | 0.3811  | 0.9904 | 0.9800 | 0.9899 | 0.9898 | 0.9898 | 0.9898 |
| 0.3   | 0.2559 | 0.4593  | 0.9786 | 0.9769 | 0.9767 | 0.9767 | 0.9767 | 0.9767 |
| 0.4   | 0.2897 | 0.5351  | 0.9622 | 0.9580 | 0.95745| 0.9574 | 0.95735| 0.95735|
| 0.5   | 0.3178 | 0.6105  | 0.9410 | 0.9325 | 0.9310 | 0.9307 | 0.9306 | 0.9306 |
| 0.6   | 0.3420 | 0.6865  | 0.9144 | 0.8992 | 0.8958 | 0.8949 | 0.8946 | 0.8945 |
| 0.7   | 0.3632 | 0.7634  | 0.8817 | 0.8562 | 0.8491 | 0.8467 | 0.8458 | 0.8454 |
| 0.8   | 0.3820 | 0.8413  | 0.8412 | 0.8003 | 0.7864 | 0.78065| 0.7779 | 0.7765 |
| 0.9   | 0.3989 | 0.9202  | 0.7893 | 0.7245 | 0.6978 | 0.6842 | 0.6764 | 0.6716 |
| 1     | 0.4142 | 1       | 0.7071 | 0.5947 | 0.5358 | 0.4980 | 0.4707 | 0.4496 |