ON THE CONTACT CLASS IN HEEGAARD FLOER HOMOLOGY

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ABSTRACT. We present an alternate description of the Ozsváth-Szabó contact class in Heegaard Floer homology. Using our contact class, we prove that if a contact structure \((M, \xi)\) has an adapted open book decomposition whose page \(S\) is a once-punctured torus, then the monodromy is right-veering if and only if the contact structure is tight.

1. INTRODUCTION

In the paper [OS5], Ozsváth and Szabó defined an invariant of a contact 3-manifold \((M, \xi)\) which lives in the Heegaard Floer homology \(\widehat{HF}(−M)\) of the manifold \(M\) with reversed orientation. It is defined via the work of Giroux [Gi2], who showed that there is a 1-1 correspondence between isomorphism classes of open book decompositions modulo positive stabilization and isomorphism classes of contact structures on closed 3-manifolds. Ozsváth and Szabó associated an element in Heegaard Floer homology to an open book decomposition and showed that its homology class is independent of the choice of the open book compatible with the given contact structure. They also showed that this invariant \(c(\xi)\) is zero if the contact structure is overtwisted, and that it is nonzero if the contact structure is symplectically fillable. The contact class \(c(\xi)\) has proven to be extremely powerful at (i) proving the tightness of various contact structures and (ii) distinguishing tight contact structures, especially in the hands of Lisca-Stipsicz [LS1, LS2] and Ghiggini [Gh].

The goal of this paper is to introduce an alternate, more hands-on, description of the contact class in Heegaard Floer homology and to use it in the context of our program of relating right-veering diffeomorphisms to tight contact structures.

In [HKM2] we introduced the study of right-veering diffeomorphisms of a compact oriented surface with nonempty boundary (sometimes called a “bordered surface”), and proved that if \((S, h)\) is an open book decomposition compatible with a tight contact structure, then \(h\) is right-veering. In [HKM3] we continued the study of the monoid \(\text{Veer}(S, \partial S)\) of right-veering diffeomorphisms and investigated its relationship with symplectic fillability in the pseudo-Anosov case. We proved the following:

**Theorem 1.1.** Let \(S\) be a bordered surface with connected boundary and \(h\) be pseudo-Anosov with fractional Dehn twist coefficient \(c\). If \(c \geq 1\), then the contact structure \(\xi_{(S,h)}\) supported by \((S, h)\) is isotopic to a perturbation of a taut foliation. Hence \((S, h)\) is (weakly) symplectically fillable and universally tight if \(c \geq 1\).
Hence, when a contact structure is supported by an open book with “sufficiently” right-veering monodromy, it is symplectically fillable and therefore tight as a consequence of a theorem of Eliashberg and Gromov [El]. Unfortunately, a right-veering diffeomorphism with a small amount of rotation does not always correspond to a tight contact structure. In fact, any open book can be stabilized to a right-veering one (see Goodman [Go], as well as [HKM2]). However, we might optimistically conjecture that a minimal (i.e., not destabilizable) right-veering open book defines a tight contact structure. If we specialize to the case of a once-punctured torus, then we can use our description of the contact class to prove this conjecture.

**Theorem 1.2.** Let \((M, \xi)\) be a contact 3-manifold which is supported by an open book decomposition \((S, h)\), where \(S\) is a once-punctured torus. Then \(\xi\) is tight if and only if \(h\) is right-veering.

Very recently John Baldwin [Ba] also obtained results similar to Theorem 1.2.

The paper is organized as follows. In Section 2, we review the standard definition of \(c(\xi)\). Then, in Section 3 we describe the class \(EH(\xi) \in \widehat{HF}(-M)\), which arose in discussions between John Etnyre and the first author. We also prove that the class \(EH(\xi)\) equals the Ozsváth-Szabó contact class \(c(\xi)\), and hence \(EH(\xi)\) is a contact invariant. In Section 4, the class \(EH(\xi)\) is applied to contact structures with compatible genus one open book decompositions to prove Theorem 1.2.

2. Open Books and Ozsváth-Szabó Contact Invariants

In [OS1, OS2], Ozsváth and Szabó defined invariants of closed oriented 3-manifolds \(M\) which they called Heegaard Floer homology. Among the several versions of Heegaard Floer homology defined by Ozsváth and Szabó, we concentrate on the simplest one, namely \(\widehat{HF}(M)\). It is defined as the homology associated to a chain complex determined by a Heegaard decomposition of \(M\). Consider a Heegaard decomposition \((\Sigma, \alpha = \{\alpha_1, \ldots, \alpha_g\}, \beta = \{\beta_1, \ldots, \beta_g\})\) of genus \(g\). Here \(\Sigma\) is the Heegaard surface, i.e., a closed oriented surface of genus \(g\) which splits \(M\) into two handlebodies \(H_1\) and \(H_2\), \(\Sigma = \partial H_1 = -\partial H_2\), \(\alpha_i\) are the boundaries of the compressing disks of \(H_1\), and \(\beta_i\) are the boundaries of the compressing disks of \(H_2\). Then consider two tori \(\mathbb{T}_\alpha = \alpha_1 \times \cdots \times \alpha_g\) and \(\mathbb{T}_\beta = \beta_1 \times \cdots \times \beta_g\) in \(\text{Sym}^g(\Sigma)\). Also pick a basepoint \(z \in \Sigma\). The complex \(\widehat{CF}(M)\) is defined to be the free \(\mathbb{Z}\)-module generated by the points \(x = (x_1, \ldots, x_g)\) of \(\mathbb{T}_\alpha \cap \mathbb{T}_\beta\). The boundary map is defined by counting points in certain 0-dimensional moduli spaces of holomorphic maps of the unit disk into \(\text{Sym}^g(\Sigma)\). It is, very roughly, defined as follows. Denote by \(\mathcal{M}_{x,y}\) the 0-dimensional (after quotienting by the natural \(\mathbb{R}\)-action) moduli space of holomorphic maps \(u\) from the unit disk \(D^2 \subset \mathbb{C}\) to \(\text{Sym}^g(\Sigma)\) that (i) send \(1 \mapsto x, -1 \mapsto y, S^1 \cap \{\text{Im } z \geq 0\}\) to \(\mathbb{T}_\alpha\) and \(S^1 \cap \{\text{Im } z \leq 0\}\) to \(\mathbb{T}_\beta\), and (ii) avoid \(\{z\} \times \text{Sym}^{g-1}(\Sigma) \subset \text{Sym}^g(\Sigma)\). Then define

\[
\partial x = \sum_{\mu(x,y) = 1} \#(\mathcal{M}_{x,y}) y,
\]

where \(\mu(x,y)\) is the relative Maslov index of the pair and \(\#(\mathcal{M}_{x,y})\) is a signed count of points in \(\mathcal{M}_{x,y}\). The homology of this complex \(\widehat{HF}(M)\) is shown to be independent of the various choices made in the definition. In particular, it is independent of the choice of a “weakly admissible” Heegaard decomposition.

Each intersection point \(x\) in \(\mathbb{T}_\alpha \cap \mathbb{T}_\beta\) defines a \(\text{Spin}^c\) structure \(s_x\) on \(M\). If there is a topological disk from \(x\) to \(y\) which satisfies (i) and (ii) in the previous paragraph, then the two \(\text{Spin}^c\) structures
agree. Hence, the complex (as well as the homology of the complex) splits according to Spin<sup>e</sup> structures. The Heegaard Floer homology decomposes as a direct sum

$$ \widehat{HF}(M) = \oplus_s \widehat{HF}(M, s). $$

Given a contact structure $\xi$ on $M$, we denote the associated Spin<sup>e</sup> structure by $s_\xi$. Let $(S, h, K)$ be an open book decomposition of a manifold that is compatible with the contact structure $\xi$. Then Ozsváth and Szabó define in [OS5] an element $c(\xi) \in \widehat{HF}(-M, s_\xi)/(-1)$ by using a Heegaard splitting associated to the open book decomposition as follows. (At the time of the writing of the paper, the ±1 ambiguity still exists. It is possible, however, that a careful study of orientations would remove this ambiguity. The ±1 issue does not arise in Seiberg-Witten Floer homology.) To avoid writing ±1 everywhere, we either work with $\mathbb{Z}/2\mathbb{Z}$-coefficients or tacitly assume that $c(\xi)$ is well-defined up to a sign when $\mathbb{Z}$-coefficients are used. Consider the open book decomposition $(S, h, K)$, where $S$ is a surface of genus $g$ (here genus means the genus of the surface capped off with disks) with one boundary component $\partial S$, $h$ is a diffeomorphism of $S$ which is the identity on $\partial S$, and the pair $(M, K)$ is homeomorphic to $((S \times [0, 1])/\sim, (\partial S \times [0, 1])/\sim)$. The equivalence relation $\sim$ is generated by $(x, 1) \sim (h(x), 0)$ for $x \in S$ and $(y, t) \sim (y, t')$ for $y \in \partial S$, $t, t' \in [0, 1]$. From the above description of $M$ we immediately see an associated Heegaard splitting of $M$ by setting $H_1 = (S \times [0, \frac{1}{2}])/\sim$ and $H_2 = (S \times [\frac{1}{2}, 1])/\sim$. This gives a Heegaard decomposition of genus $2g$ with the splitting surface $\Sigma = S_{1/2} \cup -S_0$. A set of $2g$ properly embedded disjoint arcs $a_1, \ldots, a_{2g}$ which cut $S$ into a disk defines a set of compressing disks $a_i \times [0, \frac{1}{2}], i = 1, \ldots, 2g$, in $H_1$ and a set of compressing disks $a_i \times [\frac{1}{2}, 1], i = 1, \ldots, 2g$, in $H_2$. We then set $\alpha_i = \partial(a_i \times [0, \frac{1}{2}])$ and $\beta_i = \partial(a_i \times [\frac{1}{2}, 1])$, for $i = 1, \ldots, 2g$. See Figure 1.

This is, however, not the Heegaard splitting that Ozsváth and Szabó consider when defining $c(\xi)$. Instead they use a Heegaard surface that can be viewed simultaneously as a Heegaard surface for $M$ and for $M_0(K)$, the zero surgery along the binding $K$. The contact element in $\widehat{HF}(-M)$ can be seen on this Heegaard surface as the image of a class in $\widehat{HF}(-M_0(K))$ (or, equivalently, as the image of a class in $\widehat{HFk}(-M, K, F, -g)$). To construct such a splitting, take a disk $D \subset \text{int}(S)$ which is contained in a small neighborhood of $\partial S$, dig $D \times [0, \frac{1}{2}]$ out of $H_1$, and then attach it to $H_2$. This produces two new handlebodies $H_1'$ and $H_2'$. On $H_2'$ we keep the same set of $\beta$-curves $\beta_1, \ldots, \beta_{2g}$ as $H_2$ and add $\beta_0 = \partial D \times \{\frac{1}{2}\}$. Next, let $d$ be a short arc connecting between the two boundary components of $S - D$, and let $\{b_1, \ldots, b_{2g}\}$ be a set of arcs with endpoints on $\partial D$ which are “dual” to $\{a_1, \ldots, a_{2g}\}$. (By this we mean $a_{2i+1} \cap b_j = \emptyset$ if $j \neq 2i$ and $a_{2i+1} \cap b_j = \{x_{2i+1}\}$; also $a_{2i} \cap b_j = \emptyset$ if $j \neq 2i + 1$ and $a_{2i} \cap b_{2i+1} = \{x_{2i}\}$.) Then on $H_1'$, we let $\alpha_0 = \partial(d \times [0, \frac{1}{2}])$ and $\alpha_i = \partial(b_i \times [0, \frac{1}{2}])$. Also let $\alpha_0 \cap \beta_0 = \{x_0\}$.

These above choices determine a special point $x = (x_0, x_1, \ldots, x_{2g})$ in $\mathbb{T}_\alpha \cap \mathbb{T}_\beta \subset \text{Sym}^{2g+1}(\Sigma)$. (Here, $x_i$ means $(x_i, \frac{1}{2})$, for $i > 0$.) This point (after modifying the Heegaard diagram by winding in a region that does not affect $x$ to adjust for admissibility) defines the special cycle in Heegaard Floer homology. The homology class of $x$ is defined as the contact class $c(\xi)$ by Ozsváth-Szabó. They show that $\widehat{HFk}(-M, K, F, -g)$, the knot Floer homology for $(-M, K)$ at the lowest possible filtration level $-g$, is isomorphic to $\mathbb{Z}$ and is generated by $x$. Then $c(\xi)$ is defined to be the image of this generator in $\widehat{HF}(-M)$. For details, including the figures describing this decomposition and the corresponding generator of $c(\xi)$, see [OS5].
3. AN ALTERNATE DESCRIPTION OF THE CONTACT ELEMENT

3.1. Definition and main theorem. Let $S$ be a bordered surface whose boundary is not necessarily connected. Let $\{a_1, \ldots, a_r\}$ be a collection of disjoint, properly embedded arcs of $S$ so that $S - \bigcup_{i=1}^{r} a_i$ is a single polygon. We will call such a collection a basis for $S$. Observe that every arc $a_i$ of a basis is a nonseparating arc of $S$. Next let $b_i$ be an arc which is isotopic to $a_i$ by a small isotopy so that the following hold:
The endpoints of $a_i$ are isotoped along $\partial S$, in the direction given by the boundary orientation of $S$.

(2) $a_i$ and $b_i$ intersect transversely in one point in the interior of $S$.

(3) If we orient $a_i$ and $b_i$ is given the induced orientation from the isotopy, then the sign of the intersection $a_i \cap b_i$ is $+1$.

See Figure 2.

Let $M = M(S,h)$ be the 3-manifold with open book decomposition $(S,h)$. Recall the Heegaard decomposition for $M$ described in the previous section, where $\Sigma = S_{1/2} \cup -S_0$. We choose the compressing disks to be $\alpha_i = \partial (a_i \times [0,1])$ and $\beta_i = \partial (b_i \times [\frac{1}{2}, 1])$. We will sometimes write $\alpha_i = (a_i, a_i)$ and $\beta_i = (b_i, h(b_i))$, where the first entry is the arc on $S_{1/2}$ and the second entry is the arc on $S_0$. Let $x_i$ be the intersection point $(a_i \cap b_i) \times \{\frac{1}{2}\}$ lying in $S_{1/2} \subset \Sigma$, and let $z$ be the basepoint which sits on $S_{1/2}$ and lies outside the thin strips of isotopy between the $a_i$’s and the $b_i$’s. Then $(\Sigma, \beta, \alpha, z)$ gives a weakly admissible Heegaard diagram, namely every periodic domain has positive and negative components. This is due to the fact that every periodic domain which involves $\alpha_i$ crosses $x_i$, at which point the sign of the connected component of $\Sigma - \bigcup_{i=1}^{r} \alpha_i - \bigcup_{i=1}^{r} \beta_i$ changes.

Throughout this paper we use a product complex structure $J = Sym^r(j)$ on $Sym^r(\Sigma)$, where $j$ is some complex structure on $\Sigma$, and perturb the $\alpha$- and $\beta$-curves to attain transversality. This is done using the technique of [Oh], as sketched in Section 3.5 of [OS1]. We remark that moving the $\alpha$- and $\beta$-curves represents a subclass of the Hamiltonian isotopies of $\mathbb{T}_\alpha$ and $\mathbb{T}_\beta$ (i.e., we have fewer perturbations), so Theorem I of [Oh] does not carry over verbatim, but the proof technique carries without difficulty. Observe that if there is no holomorphic disk in a given homotopy class, then the moduli space of such disks is automatically Fredholm regular.

A $J$-holomorphic disk $u : D \rightarrow Sym^r(\Sigma)$ corresponds to a holomorphic map $\hat{u} : \hat{D} \rightarrow \Sigma$, where $\hat{D}$ is a branched cover of $D$. In the definition of the boundary map in the $HF$ theory, we only count holomorphic disks $u : D \rightarrow Sym^r(\Sigma)$ that miss $\{z\} \times Sym^{r-1}(\Sigma)$. Hence it follows that we only count $\hat{u}$ for which the image of $\hat{u}$ does not intersect $z \in \Sigma$. The intersection of any such $\hat{u}$ with $S_{1/2}$ is thus constrained to lie in the thin strips of isotopy of the $a_i$ to $b_i$. 

![Figure 2](image-url)
We claim that \( x = (x_1, \ldots, x_r) \in \widehat{CF}(\Sigma, \beta, \alpha, z) \) is a cycle, thanks to the fortuitous placement of the basepoint \( z \). (We write \( \widehat{CF}(\Sigma, \beta, \alpha, z) \) instead of \( \widehat{CF}(\Sigma, \alpha, \beta, z) \) to indicate homology on \( -M \).) Suppose \( \hat{u} \) contributes to \( \partial x \); in particular it is nonconstant. Let \( \delta_i \) be a short oriented arc of \( \partial \hat{D} \) which passes through a corner \( p_i \in \hat{D} \) for which \( \hat{u}(p_i) = x_i \). Then \( \hat{u}(\delta_i) \) first travels along \( \alpha_i \) and switches to \( \beta_i \) at \( x_i \). More explicitly, there is some \( t_0, \delta_i(t_0) \in \alpha_i \), such that \( \frac{d}{dt}(\hat{u} \circ \delta_i)(t_0) \neq 0 \) and points towards \( x_i \). Since the interior of \( \hat{D} \) is to the left of \( \delta_i \), by the openness of the holomorphic map, \( z \) would be contained in the image of \( \hat{u} \), a contradiction.

We define \( EH(S, h, \{a_1, \ldots, a_r\}) \) to be the homology class of the generator \( x \). The following is the main theorem of this section:

**Theorem 3.1.** \( EH(S, h, \{a_1, \ldots, a_r\}) \) is an invariant of the contact structure and equals \( c(\xi(S, h)) \).

In particular, \( EH(S, h, \{a_1, \ldots, a_r\}) \) is independent of the choice of basis, and it will often be denoted by \( EH(S, h) \).

In Theorem 3.1, we are not assuming that \( \partial S \) is connected.

**Examples:** To give some intuition for the class \( EH(S, h) \), we give three examples when \( S \) is an annulus. Refer to Figure 3. The leftmost diagram gives \( a \) and \( b \) on \( S_{1/2} \). The subsequent diagrams give \( S_0 \) for (1), (2), and (3) below (from left to right).

1. If \( h \) is the identity, then \((M, \xi)\) is the standard tight contact structure on \( S^1 \times S^2 \). Since there are two holomorphic disks from \( y \) to \( x \), it follows that \( EH(S, h) \neq 0 \). One of the holomorphic disks from \( y \) to \( x \) has been shaded in Figure 3.
2. If \( h \) is a positive Dehn twist about the core curve, then \((M, \xi)\) is the standard tight contact structure on \( S^3 \). Since \( x \) is the unique intersection point on \( \Sigma = T^2 \), \( EH(S, h) \neq 0 \).
3. If \( h \) is a negative Dehn twist about the core curve, then \((M, \xi)\) is an overtwisted contact structure on \( S^3 \). We have \( \partial y_1 = \partial y_2 = x \); hence \( EH(S, h) = 0 \).

**Figure 3.** Examples when \( S \) is an annulus.

The following lemma echoes our result in [HKM2], which states that \( \xi(S, h) \) is overtwisted if \( h \) is not right-veering.

**Lemma 3.2.** If \( h \) is not right-veering, then \( EH(S, h) = 0 \).
Proof. If \( h \) is not right-veering, then there exists an arc \( a_1 \) on \( S \) so that \( h(a_1) \) is to the left of \( a_1 \). If \( a_1 \) is nonseparating, then it can be completed to a basis \( \{a_1, \ldots, a_r\} \). There exists an intersection point \( y_1 \in \alpha_1 \cap \beta_1 \) and a unique (up to translation) holomorphic disk \( D \subset \Sigma \) from \( y_1 \) to \( x_1 \), where \( 1 \mapsto y_1, -1 \mapsto x_1, \partial D \cap \{y \geq 0\} \) maps to \( \beta_1 \) and \( \partial D \cap \{y \leq 0\} \) maps to \( \alpha_1 \). Since \( z \) forces any holomorphic disk \( \hat{u} : \hat{D} \to \Sigma \) which contributes to \( \partial(y_1, x_2, \ldots, x_n) \) to be constant near \( x_i, i = 2, \ldots, r \), all the \( \alpha_i \) and \( \beta_i, i = 2, \ldots, r \), are “used up”, and the only holomorphic disk that remains is the unique one from \( y_1 \) to \( x_1 \). Hence \( \partial(x_1, x_2, \ldots, x_n) = (x_1, x_2, \ldots, x_n) \).

If the arc \( a_1 \) is separating, then let us call its initial point \( p \). The arcs \( h(a_1) \) and \( a_1 \) must intersect at some point \( q \) in the interior of \( a_1 \); otherwise \( h(a_1) \) will cut off a strictly smaller subsurface of \( S \) inside a subsurface of \( S \) cut off by \( a_1 \). Let \( c \) be the subarc of \( a_1 \) from \( p \) to \( q \) and \( c' \) be the subarc of \( h(a_1) \) from \( p \) to \( q \). Then either \( c(c')^{-1} \) is separating or it is not. If \( c(c')^{-1} \) separates a region \( S' \) to the left of \( a_1 \), then there is a nonseparating arc \( b \subset S' \) which begins and ends at \( p \). On the other hand, if \( c(c')^{-1} \) is nonseparating, then we let \( b = c(c')^{-1} \). In either case, since \( b \) is strictly to the left of \( a_1 \) and strictly to the right of \( h(a_1) \), it follows that \( h(b) \) is strictly to the left of \( b \). \( \square \)

In view of Theorem 3.1 and the fact that every overtwisted contact structure admits an open book that is not right-veering, Lemma 3.2 immediately implies that \( c(\xi) = 0 \) for an overtwisted contact structure.

Proof of Theorem 3.1. Let us denote a positive Dehn twist about a closed curve \( \gamma \) by \( \phi_\gamma \). Assume \( \partial S \) is connected. We first prove the theorem for a special case, namely when \( h = \phi_\gamma^{n} \) with \( n > 0 \), in Section 3.2. Next, in Section 3.3 we prove that \( EH(S, h, \{a_1, \ldots, a_r\}) \) only depends on the isotopy class of \( h \) (relative to the boundary), and in Section 3.4 we show that \( EH(S, h, \{a_1, \ldots, a_r\}) \) is independent of the choice of basis by using handleslides. Then in Section 3.5 we prove that \( EH(S, h) \) is mapped to \( EH(S, \phi_\gamma^{-1} \circ h) \) under the natural map \( \hat{H} \hat{F}(-M(S, h)) \to \hat{H} \hat{F}(-M(S, \phi_\gamma^{-1} \circ h)) \) which corresponds to a Legendrian (+1)-surgery. We then start with \( \phi_\gamma^{n} \) with \( n \gg 0 \) and apply a sequence of negative Dehn twists until we reach the desired monodromy map \( h \). In Section 3.6 we reduce the case of multiple boundary components to the case when \( \partial S \) is connected. \( \square \)

3.2. Primordial Example. Let \( S \) be a once-punctured torus and \( h = \phi_\partial S \), i.e., a positive Dehn twist about \( \partial S \). The same argument works if \( S \) is a genus \( g \) surface with one boundary component and \( h = \phi_\partial S, n > 0 \).

The subarcs of \( \alpha_i \) and \( \beta_i \) that live in \( S_0 \) are given in Figure 4. We change notation and the constituent points of \( x \) representing \( EH(S, h) \) will be denoted \( x_0 = x'_0 \) and \( y_0 = y'_0 \) as in Figure 4. Although \( x_0 = x'_0 \) and \( y_0 = y'_0 \), strictly speaking, live on \( S_{1/2} \), we view them as sitting on \( \partial S_0 \). (Also, the points \( x_0 \) and \( x'_0 \), as well as \( y_0 \) and \( y'_0 \), are drawn as distinct points on \( \partial S_0 \), but we hope this will not cause any confusion for the reader.)

We then place the basepoint \( w \) on \( S_0 \) as indicated in Figure 4. Observe that \( z \) and \( w \) together represent the binding \( K \). The binding \( K \) is isotopic to the dotted curve \( \gamma_0 \) which consists of two subarcs \( c_1 \) and \( c_2 \) between \( z \) and \( w \), where \( c_1 \) intersects only \( \alpha \)-curves and \( c_2 \) intersects only \( \beta \)-curves. Then \( (\Sigma, \beta, \alpha, z, w) \) is a doubly-pointed Heegaard diagram for the knot Floer homology of \( K \).

If we stabilize this Heegaard splitting by digging a handle in \( S \times [0, \frac{1}{2}] \) which is parallel to the arc \( c_2 \), then we obtain a Heegaard surface \( \Sigma' \) on which we can see both \( -M \) and \( -M_0(K) \). See Figure 5. Here \( -M \) is given by \( \{\beta_0\} \cup \beta \) and \( \{\alpha_0\} \cup \alpha \), whereas \( -M_0(K) \) is given by \( \gamma = \{\gamma_0\} \cup \beta \).
and \( \{\alpha_0\} \cup \alpha \). (Here \( \gamma_0 \) is viewed as a curve that passes through the handle once.) The stabilization sends \( x = (x_0, y_0) \) to \( x' = (z_0, x_0, y_0) \), where \( z_0 \) is the intersection of the two new compressing curves \( \alpha_0 \) and \( \beta_0 \). 

As a first step in exploiting the Ozsváth-Szabó characterization of \( c(\xi) \), we show that the lowest filtration level is generated by \( x' = (z_0, x_0, y_0) \) as well as the other intersection points \( y = (z_0, x, y) \), where \( x \) and \( y \) live inside the dotted lines of Figure 4. The filtration level is computed by first letting \( F \subset \Sigma' \) be the domain bounded by \( \alpha_0 \) and \( \gamma_0 \) which does not intersect \( S_{1/2} \) (and hence lives mostly on \( S_0 \)). We additionally assume that \( F \) is oriented so that the surface \( \hat{F} \), obtained from \( F \) by capping off \( \partial F \), is an oriented fiber of the fibration of \( M_0(K) \). In order to find generators \( y \) which are at the lowest filtration level, we minimize \( \langle c_1(s_{y'}), \hat{F} \rangle \). Here \( y' = (z_0', x, y) \) and \( z_0' \) is the point on \( \alpha_0 \cap \gamma_0 \) which is close to \( z_0 \) and obtained by tensoring \( z_0 \) with the unique intersection point \( \Theta \in \beta_0 \cap \gamma_0 \) as in Figure 5. (Keep in mind that since we are dealing with \( \hat{HF} \) of \( -M \) and \( -M_0(K) \), the Heegaard surface is \( -\Sigma' \); otherwise our calculations will be off by a negative sign.)
To this end, we recall the first Chern class formula (Section 7.1 of [12]; for some details, see Rasmussen [2]:

\[
\langle c_1(s_{y'}), [A]\rangle = \chi(\mathcal{P}) - 2\pi_z(\mathcal{P}) + 2 \sum_{p \in \mathcal{P}} \pi_p(\mathcal{P})
\]

Here \([A] \in H_2(M_0(K), \mathbb{Z})\), \(s_{y'}\) is a Spin\(^c\) structure corresponding to \(y'\), \(\mathcal{P}\) is the periodic domain for \([A]\) (where we do not require that \(\mathcal{P}\) avoid \(z\)) and \(\chi\) is the Euler measure. Let \(\mathcal{D}\) be a component of \((-\Sigma') - \bigcup_1 \cup_1 \alpha_i - \bigcup_1 \gamma_i\). Then \(\pi_p(\mathcal{D})\) equals (i) \(1\) if \(p\) is in the interior of \(\mathcal{D}\), (ii) \(0\) if \(p\) does not intersect \(\mathcal{D}\), (iii) \(\frac{1}{2}\) if \(p\) is on an edge of \(\mathcal{D}\) (but not a corner), and (iv) \(\frac{1}{4}\) if \(p\) is on a corner of \(\mathcal{D}\). We then extend \(\pi_p\) linearly to \(\mathcal{P}\).

In the case at hand, the possible \(x\)'s and \(y\)'s are either in the interior of \(F\) or not in \(F\), and therefore they either contribute \(1\) or \(0\). On the other hand, \(\pi_z(\mathcal{P}) = -2\), \(\chi(\mathcal{P}) = -2g(S)\), and \(\pi_{c_0}(\mathcal{P}) = -1\) are constant, and it follows that \(\langle c_1(s_{y'}), [\hat{F}]\rangle = 2 - 2g(S)\) is the minimal value and it is attained when both \(x\) and \(y\) are not in \(F\). (In fact, \(\{\beta_0\} \cup \beta, \{\alpha_0\} \cup \alpha, z, w\) is a “sutured Heegaard diagram” in the sense of [3].)

The graded complex for calculating \(\widehat{HF}(M, K, -2)\) is generated by:

\[(z_0, x_0, y_0), (z_0, x_0, y_2), (z_0, x'_1, y_1), (z_0, x, y_1), (z_0, x_2, y_0), (z_0, x_2, y_2), (z_0, x_3, y_1)\].

Our task is to identify \(x' = (z_0, x_0, y_0)\) as a generator of \(\widehat{HF}(M, K, -2) \simeq \mathbb{Z}\). We will show that all the generators besides \(x'\) correspond to Spin\(^c\) structures which are different from that of the contact structure \(\xi\). An easy computation shows that \(H_2(M; \mathbb{Z}) \simeq \mathbb{Z}^2\) and is generated by tori \(T_0\) of the form \((\delta \times [0, 1]) / \sim\), where \(\delta\) is any nonseparating curve on \(S\) and \((x, 1) \sim (h(x), 0)\) as before. Since \(\xi\) is close to the foliation \(S \times \{t\}\) on \((S \times [0, 1]) / \sim\), it follows that \(\langle c(\xi), [T_0]\rangle = 0\). Now, let \(\delta_1\) be a \((0, 1)\)-curve on \(S\) and \(\delta_2\) be a \((1, 0)\)-curve. Then \([T_{\delta_1}]\) is given by the periodic domain \(\mathcal{P}_{\delta_1}\), which consists of two rectangles \(y_0y_2y_4y_2\) and \(y'_0y'_2y'_4y_2\) with opposite signs, shown in Figure 6. Similarly, \([T_{\delta_2}]\) is represented by \(\mathcal{P}_{\delta_2}\), consisting of \(x_0x_2x'_4x'_2\) and \(x'_0x'_2x_4x_2\) with opposite signs.

\[\text{Figure 6. Cover of a neighborhood of } \partial S.\]
Now refer to Figure 6 which is a cover of an annular neighborhood of $\partial S \subset S$. The dotted curve is (a lift of) $c_1c_2$. Points below the dotted curve are not in $F$, so only they have the proper filtration level to represent generators.

We will show $s_{(z_0,x,y)} \neq s_{(z_0,x_0,y_0)}$ if $(x,y) \neq (x_0,y_0)$, by showing that $\langle c(s_{(z_0,x,y)}), [T_{\delta_1}] \rangle \neq 0$ for $i = 1$ and 2 if $(x,y) \neq (x_0,y_0)$.

First consider the intersection points on the vertical lines starting at $x_0$ and at $x'_0$. Suppose that $\langle c(s_{(z_0,x,y)}), [T_{\delta_2}] \rangle = 0$. The rectangle $x_0'x_1'x_2'x_1$ of the periodic domain $\mathcal{P}_{\delta_2}$ contributes $\frac{1}{2}$ if $x = x_0$ or $x = x'_0$. Since there is no value of $y$ below the dotted curve with a contribution of $-\frac{1}{2}$ from the rectangle $x_0x_2x_4x_2'$ to cancel the $\frac{1}{2}$, the possibilities $x = x_0, x'_0$ are eliminated. Since $x_0x_2x_4x_2'$ gives a contribution of $-\frac{1}{2}$ to $x_1$ and $x_0'x_2x_4x_2$ contributes to $y_0, \frac{1}{2}$ to $y_1$ and 1 to $y_1$, the only generator containing $x_1$ that is allowed is $(z_0, x_1, y_1)$. Any generator containing $y_2$ is also disallowed since $x_0'x_2x_4x_2$ contributes 1 to $y_2$, and there is no $x$ value that will offset it from the $x_0x_2x_4x_2'$ rectangle. The only generator allowed to contain $y_1$ is again $(z_0, x_1, y_1)$. The same rectangle gives $x'_0$ a contribution of $-\frac{1}{2}$ that cannot be offset.

It therefore remains to consider the generator $(z_0, x_1, y_1)$, as well as pairs with $x = x_0$ or $x_2$. Moreover, the only possible $y$-coordinates are $y_0$ and $y_1$, and $(z_0, x_1, y_1)$ is the only option allowed for $y = y_1$. Now use the periodic domain $\mathcal{P}_{\delta_1}$. The rectangle $y_0y_2y_4y_0'$ contributes $-1$ to $(z_0, x_1, y_1)$, thus eliminating it as a possibility. The only other option different from $(z_0, x_0, y_0)$ is $(z_0, x_2, y_0)$ (since $y_2$ was banned) which gets a nonzero contribution from $y_0y_2y_4y_2'$.

To show how this argument generalizes to higher genus surfaces, let us examine the genus two case. The generators will have the form $(z_0, x, y, u, v)$, and there will be 8 intersection points on each vertical segment in a picture analogous to Figure 6. Denote the points on the boundary $u_0, v_0, u_0', v_0', x_0, y_0, x_0', y_0'$ going from right to left. Start by considering the rectangles $u_0u_2u_4u_6$ and $u_0'u_4'u_6u_2$. We eliminate $u_3, \ldots, u_7$ and all the $u'$ values besides $u_0'$, by noticing that there is no allowable $v$ value to offset the $\frac{1}{2}$ contribution from $u_0'u_4'u_6u_2$. The contribution of 1 from the same rectangle eliminates all values of $v$ other than $v_0$ and $v_1$ (though no $v_i'$ are yet disallowed). If $v = v_1$, only generators of the form $(z_0, x, y, u_1, v_1)$ are allowed.

Now use the periodic domain represented by the rectangles $v_0v_2v_4v_6$ and $v_0'v_4'v_6v_2$. The generators of the form $(z_0, x, y, u_1, v_1)$ get a contribution of $-1$ from $v_0v_2v_4v_6$ and there is no positive contribution from the allowable $x, y$ coordinates that can be gained from $v_0'v_4'v_6v_2$; therefore all the $(z_0, x, y, u_1, v_1)$ are eliminated. Next, $u_2$ gets a contribution of $-1$ from $v_0v_2v_4v_6$ that cannot be canceled since there is no $v$ value that gets a contribution of 1 needed from $v_0'v_4'v_6v_2$. It follows that $u_0$ is the only allowable $u$-coordinate. Generators $(z_0, x, y, u_0, v_i')$, $i \neq 0$, are eliminated since $v_i'$ gets a contribution of $\frac{1}{2}$ from $v_0'v_4'v_6v_2$ that cannot be canceled. Therefore we are left with $(z_0, x, y, u_0, v_0)$. The argument is now reduced to eliminating the possible $x, y$, coordinates, and this follows just as in the genus one argument given above.

This shows how the proof works for arbitrary genus. The inductive step is done in the same way by eliminating all extra options in the two new coordinates, thus reducing to the case of lower genus.

Since the contact invariant is the image of the generator of $\widehat{HF}(M, K, -2g)$ in $\widehat{HF}(-M)$, it follows that $c(\xi(s,h)) = EH(S, h)$. It is not hard to see how the above argument generalizes to the $h = \phi_0^S$, $n > 0$ case.

3.3. Isotopy. In this subsection we prove the following:
Lemma 3.3. If $h_t : S \rightarrow S$, $t \in [0, 1]$, is a 1-parameter family of diffeomorphisms which restrict to the identity on $\partial S$, then $EH(S, h_0, \{a_1, \ldots, a_r\}) = EH(S, h_1, \{a_1, \ldots, a_r\})$.

Proof. Let $\alpha_i = (a_i, a_i)$ and $\beta_i = (b_i, h_0(b_i))$. In other words, we fix the $\alpha_i$ and isotop the $\beta_i$. Observe that the $\beta_i$ remain constant on $S \times \{1\}$. According to Theorem 7.3 of [OS1], we can reduce to the case where $h_t$ is a Hamiltonian isotopy. Let $\Psi_t : \Sigma \rightarrow \Sigma$ be the Hamiltonian isotopy which restricts to the identity on $S \times \{1\}$ and restricts to $h_t$ on $S \times \{0\}$. We use the same notation for the induced isotopy on $Sym^r(\Sigma)$. Then the chain map $\Phi : \overline{CF}^{0, \alpha}(\beta_1^1, \alpha) \rightarrow \overline{CF}^{1, \alpha}$ is obtained by counting holomorphic disks $u : [0, 1] \times \mathbb{R} \rightarrow Sym^r(\Sigma)$ which satisfy

\[
\lim_{t \rightarrow +\infty} u(s + it) = x, \quad \lim_{t \rightarrow -\infty} u(s + it) = x', \quad u(0 + it) \in \Psi_t(T_\beta), \quad u(1 + it) \in T_\alpha, \quad \text{and avoid} \{z\} \times Sym^{r-1}(\Sigma).
\]

Here $x \in \overline{CF}^{0, \alpha}$ and $x' \in \overline{CF}^{1, \alpha}$. Now, if $x$ is unique $r$-tuple of points on $S \times \{1\}$ representing the generator of $EH(S, h_0, \{a_1, \ldots, a_r\}$, then the only holomorphic disk of the above type are constant holomorphic disks, due to the placement of the basepoint $z$. This implies that $EH(S, h_0, \{a_1, \ldots, a_r\})$ is mapped to $EH(S, h_1, \{a_1, \ldots, a_r\})$ under the isomorphism $\Phi : \overline{HF}^{0, \alpha}(\beta_1) \rightarrow \overline{HF}^{1, \alpha}$. \hfill \Box

3.4. Change of basis. In this subsection we prove the following proposition:

Proposition 3.4. $EH(S, h, \{a_1, \ldots, a_r\})$ is independent of the choice of basis $\{a_1, \ldots, a_r\}$.

Let $\{a_1, a_2, \ldots, a_r\}$ be a basis for $S$. After possibly reordering the $a_i$’s, suppose $a_1$ and $a_2$ are adjacent arcs on $\partial S$, i.e., there is an arc $\tau \subset \partial S$ with endpoints on $a_1$ and $a_2$ such that $\tau$ does not intersect any $a_i$ in $\text{int}(\tau)$. Define $a_1 + a_2$ to be the isotopy class of $a_1 \cup \tau \cup a_2$, relative to the endpoints. Then the modification $\{a_1, a_2, \ldots, a_r\} \mapsto \{a_1 + a_2, a_2, \ldots, a_r\}$ is called an arc slide.

Proposition 3.4 is immediate from the following two lemmas.

Lemma 3.5. $EH(S, h)$ is invariant under an arc slide $\{a_1, a_2, \ldots, a_r\} \mapsto \{a_1 + a_2, a_2, \ldots, a_r\}$.

Proof. Without loss of generality, consider the case where $S$ is a once-punctured torus. We show that the chain map which corresponds to an arc slide takes the representative of $EH(S, h, \{a_1, a_2\})$ determined by $x = (x_1, x_2)$ to the representative of $EH(S, h, \{a_1 + a_2, a_2\})$ determined by the intersection point $w = (w_1, w_2)$. Observe that an arc slide corresponds to a sequence of two handleslides for the corresponding Heegaard splitting.

Let $(\Sigma, \beta, \alpha, z)$ be the pointed Heegaard diagram corresponding to $a_i, b_i$ as described above, with $z$ a point in $S_{1/2}$ lying outside the thin strips of isotopy between $a_i$’s and $b_i$’s. If we slide $a_2$ over $a_1$ along a path parallel to $\partial S$, then we obtain a new pair $\gamma = \{\gamma_1, \gamma_2\}$, where $\gamma_1 = (a_1 + a_2, a_2, a_2)$ and $\gamma_2$ is a suitable pushoff of $(a_2, a_2)$ as in the proof of the invariance of Heegaard Floer homology under handleslides in [OS1]. Figure 7 depicts the case where $a_2$ is to the right of $a_1$ with respect to $\tau$; the case where $a_2$ is to the right of $a_1$ is treated similarly.

We claim that $(\Sigma, \gamma, \beta, \alpha, z)$ is a weakly admissible Heegaard triple-diagram. Recall that a triple-diagram is weakly admissible if each nontrivial triply-periodic domain which can be written as a sum of doubly-periodic domains has both positive and negative coefficients. First let us restrict to a neighborhood $R$ of the labeled regions of $\Sigma - \cup_i \alpha_i - \cup_i \beta_i - \cup_i \gamma_i$ on the right-hand side of Figure 7. Due to the placement of $z$, the only potential doubly-periodic region involving $\beta, \alpha$ on $R$ is $D_2 + D_3 - D_5 - D_6$. (Here $D_i$ is the domain labeled $i$.) Similarly, for $\gamma, \beta$ we have $D_1 + D_2 - D_4 - D_5$ and for $\alpha, \gamma$ we have $D_1 + D_6 - D_3 - D_4$. Taking linear combinations, we
have

\[ a(D_2 + D_3 - D_5 - D_6) + b(D_1 + D_2 - D_4 - D_5) + c(D_1 + D_6 - D_3 - D_4) \]
\[ = (b + c)D_1 + (a + b)D_2 + (a - c)D_3 - (b + c)D_4 - (a + b)D_5 + (-a + c)D_6. \]

Since the coefficients come in pairs, e.g., \( a + b \) and \( -(a + b) \), if any of \( a + b, b + c, a - c \) does not vanish, then the triply-periodic domain has both positive and negative coefficients. Hence, if any of \( \alpha_1, \beta_1 \) and \( \gamma_1 \) is used, then we are done. Otherwise, we may assume that none of \( \alpha_1, \beta_1 \) and \( \gamma_1 \) is used in the periodic domain. This allows us to erase all three, and apply the above considerations to \( \alpha_2, \beta_2, \) and \( \gamma_2 \). The verifications of weak admissibility of all other triple-diagrams in this paper are identical, and are omitted.

Let \( \Theta = (\Theta_1, \Theta_2) \) be the top generator of \( \widehat{HF}((S^1 \times S^2)) = \widehat{HF}(\alpha, \gamma) \). Define the map

\[ \psi : \widehat{HF}(\beta, \alpha) \otimes \widehat{HF}(\alpha, \gamma) \to \widehat{HF}(\beta, \gamma), \]

where \( \psi(y \otimes y') \) counts holomorphic triangles, two of whose vertices are \( y \in \widehat{CF}(\beta, \alpha) \) and \( y' \in \widehat{CF}(\alpha, \gamma) \). Then the isomorphism \( g : \widehat{HF}(\beta, \alpha) \cong \widehat{HF}(\beta, \gamma) \) is given by \( g(y) = \psi(y \otimes \Theta) \).

\[ \begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure7.png}
\caption{The first handleslide.}
\end{figure} \]

We claim that the representative \( x = (x_1, x_2) \) of \( EH(S, h, \{a_1, a_2\}) \) gets mapped to \( y = (y_1, y_2) \in \widehat{CF}(\beta, \gamma) \) given in Figure 7. By the placement of \( z \), we see that the unique holomorphic map \( \hat{u} \) which has \( x_1 \) and some \( \Theta_i \) as corners (and avoids \( z \)) must be a triangle with vertices \( x_1, \Theta_1, y_1 \). Now that \( \alpha_1, \beta_1, \) and \( \gamma_1 \) are used up, it easily follows that the unique holomorphic map \( \hat{u} \) which involves \( x_2 \) and \( \Theta_2 \) (and avoids \( z \)) is a triangle with vertices \( x_2, \Theta_2, y_2 \). This proves the claim.

Let us now consider the effect of the second handleslide, depicted in Figure 8. Let \( \delta = \{\delta_1, \delta_2\} \), where \( \delta_1 \) and \( \delta_2 \) are suitable pushoffs of \( (a_1 + a_2, h(a_1 + a_2)) \) and \( (a_2, h(a_2)) \), respectively. A similar argument as above shows that, under the map

\[ \widehat{HF}(\delta, \beta) \otimes \widehat{HF}(\beta, \gamma) \to \widehat{HF}(\delta, \gamma), \]
$\Theta \otimes y$ gets mapped to $w$. This shows that $x$ and $w$ determine the same element in Heegaard Floer homology, and consequently $EH(S, h, \{a_1, a_2\}) = EH(S, h, \{a_1 + a_2, a_2\})$. □

Figure 8. The second handleslide.

Lemma 3.6. Let $\{a_1, \ldots, a_r\}$ and $\{b_1, \ldots, b_r\}$ be two bases for $S$. Then there is a sequence of arc slides that takes $\{a_1, \ldots, a_r\}$ to $\{b_1, \ldots, b_r\}$.

We do not need to assume that $\partial S$ is connected.

Proof. We argue that we can reduce the total number of intersections of $\bigcup_i a_i$ and $\bigcup_i b_i$ by replacing $\{a_1, \ldots, a_r\}$ with $\{a'_1, \ldots, a'_r\}$, which is obtained from $\{a_1, \ldots, a_r\}$ by a sequence of arc slides. By inducting on the number of intersection points, this shows that we can perform a sequence of arc slides until $\bigcup_i a_i$ and $\bigcup_i b_i$ become disjoint. We then show that two disjoint bases can be brought one into another by a sequence of arc slides.

Let $P = S - \bigcup_i a_i$. Then $P$ is a polygon whose boundary $\partial P$ consists of $4r$ arcs, $2r$ of which are $a_i$ or $a_i^{-1}$ and $2r$ of which are arcs $\tau_1, \ldots, \tau_r$ of $\partial S$.

Suppose $(\bigcup_i a_i) \cap (\bigcup_i b_i) \neq \emptyset$, where we are assuming efficient intersections. After possibly reordering the arcs, there is a subarc $b_i^0 \subset b_i$ which starts on $\tau_1 \subset \partial S$ and ends on $a_1$, and whose interior $\text{int}(b_i^0)$ does not intersect $\bigcup_i a_i$. (In other words, $b_i^0$ is a properly embedded arc of $P$.) We may assume that $a_1$ is not adjacent to $\tau_1$; otherwise, isotop the relevant endpoint of $b_i$ along $\tau_1$. The subarc $b_i^0$ separates the polygon $P$ into two regions $P_1$ and $P_2$, only one of which contains a boundary arc that is labeled $a_i^{-1}$ (say $P_2$). We can then slide $a_1$ over all the arcs of type $a_i$ or $a_i^{-1}$ in the other region $P_1$, and obtain the new curve $a'_1$ as in Figure 9 so that the new basis $\{a'_1, a_2, \ldots, a_r\}$ has fewer intersections with $\bigcup_i b_i$. (Note that trying to slide over $a_1^{-1}$ presents a problem, so we must go the other way around.) There is one situation when the above strategy needs a little more thought, namely when $\partial P_2$ only intersects $a_1$ and $a_1^{-1}$ (among all the $a_i$ and $a_i^{-1}$). In this case, $b_1$ exits the polygon $P$ along $a_1$ and reenters through $a_1^{-1}$. Eventually we find a subarc of $b_i$ which starts on some $\tau_2$ and ends on an adjacent $a_i^{-1}$, a contradiction. We now apply the same procedure to $\{a'_1, a_2, \ldots, a_r\}$ and $\{b_1, \ldots, b_r\}$ until they become disjoint.
Figure 9. Simplifying the intersections of $\bigcup a_i$ and $\bigcup b_i$.

Now suppose that the two bases $\{a_1, \ldots, a_r\}$ and $\{b_1, \ldots, b_r\}$ are disjoint. We consider the polygon $P = S - \bigcup a_i$. Some of the $b_i$ arcs may be parallel to $a_j$ or $a_j^{-1}$. An arc $b_1$ that is not parallel to any of the $a_i$ will cut $P$ into two components $P_1$ and $P_2$, each containing more than one of $a_i, a_i^{-1}, i = 1, \ldots, r$. Recall that $b_1$ is nonseparating. One can easily verify that $b_1$ being nonseparating is equivalent to the existence of some $a_i$ such that $a_i \in P_1$ and $a_i^{-1} \in P_2$ (or vice versa). (If there is some $a_i$, then take an arc $c$ in $P$ from $a_i \subset P_1$ to $a_i^{-1} \subset P_2$. The closed curve in $S$ obtained by gluing up $c$ is dual to $b_1$.) If each such $a_i$ is parallel to some $b_j$, then $S - \bigcup b_i$ would be disconnected, so we could additionally assume that there is some $a_i$ which is not parallel to any $b_j$. Now we slide $a_i$ across all the arcs of type $a_j, a_j^{-1}$ in $P_1$ until it becomes parallel to $b_1$. □

3.5. Legendrian surgery. Let $\delta$ be a nonseparating curve and $\phi^{-1}_\delta$ be a negative Dehn twist about $\delta$. We now transfer $EH$ from $M = M_{(S, h)}$ to $M' = M_{(S, \phi^{-1}_\delta \circ h)}$. Recall that there is a natural map

$$ f : \widehat{HF}(-M) \to \widehat{HF}(-M'), $$

which arises from tensoring with the top generator $\Theta$ of $\widehat{HF}(\#(S^1 \times S^2))$.

**Proposition 3.7.** $f(EH(S, h)) = EH(S, \phi^{-1}_\delta \circ h)$.

**Proof.** By Proposition 3.4 we may take a basis $\{a_1, \ldots, a_r\}$ for $S$ so that $\delta$ is disjoint from $h(b_2), \ldots, h(b_r)$, intersects $h(b_1)$ exactly once, and is parallel to $h(b_2)$. Then the result of performing $(+1)$-surgery along $\delta$ (or, equivalently, a negative Dehn twist along $\delta$) is given by Figure 10.

The $\alpha$-curves and $\beta$-curves are as before, and we define the $\gamma$-curves as follows: Let $\gamma_1 = (b_1, \phi^{-1}_\delta \circ h(b_1))$ and $\gamma_i = (b_2, h(b_i))$ for $i > 1$. Let $\Theta \in \widehat{HF}(\gamma, \beta)$ be the top generator of $\#(S^1 \times S^2)$, given in Figure 10. Define the map

$$ \phi : \widehat{HF}(\gamma, \beta) \otimes \widehat{HF}(\beta, \alpha) \to \widehat{HF}(\gamma, \alpha), $$

where $\phi(y \otimes y')$ counts holomorphic triangles, two of whose vertices are $y$ and $y'$. Then the map $f : \widehat{HF}(\beta, \alpha) \to \widehat{HF}(\gamma, \alpha)$ is given by $f(y) = \phi(\Theta \otimes y)$. By the convenient placement of $z$, it follows...
that we only have small triangles in the Heegaard diagram. Hence if \([x] = EH(S, h, \{a_1, a_2\})\), then \(\phi([\Theta \otimes x]) = EH(S, \phi^{-1}_0 \circ h, \{a_1, a_2\})\). □

3.6. Multiple boundary components. Consider \((S, h)\) where \(S\) has disconnected boundary. For simplicity, assume \(S\) has two boundary components. Pick a basis \(\{a_1, \ldots, a_r\}\) for \(S\). Next consider \((S', h \# id)\), where \(S'\) is obtained from \(S\) by attaching a 1-handle between the two boundary components and we are extending \(h\) by the identity. If \(a_0\) is the cocore of the 1-handle, then \(\{a_0, \ldots, a_r\}\) is a basis for \(S'\). Our argument is similar to that of Lemma 4.4 of [OS3]. The natural map

\[
F_U : \widehat{HF}((-M_{(S,h)}) \# (S^1 \times S^2)) \to \widehat{HF}(-M_{(S,h)}),
\]

which corresponds to the cobordism \(U\) attaching a 3-handle as in Section 4.3 of [OS4], sends

\[
EH(S', h \# id, \{a_0, a_1, \ldots, a_r\}) \mapsto EH(S, h, \{a_1, \ldots, a_r\}).
\]

Since \(S'\) has only one boundary component, we already know that

\[
c(S', h \# id) = EH(S', h \# id, \{a_0, a_1, \ldots, a_r\}).
\]

Moreover, if \(\delta\) is a closed curve on \(S'\) which is “dual” to \(a_0\), then there is a natural map

\[
F_W : \widehat{HF}(-M_{(S,h)}) \to \widehat{HF}((-M_{(S,h)}) \# (S^1 \times S^2))
\]

which maps \(c(S, h)\) to \(c(S', h \# id)\). Here \((S, h)\) and \((S', \phi_0 \circ (h \# id))\) represent the same 3-manifold, and \(W\) is the cobordism corresponding to the Legendrian \((+1)\)-surgery. Finally, \(U \circ W \simeq [0, 1] \times M_{(S,h)}\), so

\[
c(S, h) = F_U \circ F_W(c(S, h)) = F_U(c(S', h \# id)) = EH(S, h, \{a_1, \ldots, a_r\}).
\]

4. Right-veering and holomorphic disks

In this section we prove Theorem 1.2.
Proof of Theorem 4.2. Let $S$ be a once-punctured torus.
Suppose first that $h$ has pseudo-Anosov monodromy. If the fractional Dehn twist coefficient $c \geq 1$, then the contact structure is already symplectically fillable and universally tight. It also follows that $c(\xi(S,h)) \neq 0$. If $c = \frac{1}{2}$, then $c(\xi(S,h)) \neq 0$ follows from Theorem 4.1 below. If $c \leq 0$, then $\xi$ is overtwisted since $S$ is not right-veering. (See [HKM2].)

If $h$ is periodic, then $\xi$ is right-veering if and only if $h$ is a product of positive Dehn twists by [HKM3].

If $h$ is reducible, then $c(\xi(S,h)) \neq 0$ follows from Theorem 4.3 below.

Theorem 4.1. Let $(S,h)$ be an open book decomposition for $M$, where $S$ is a once-punctured torus and $h$ is pseudo-Anosov with fractional Dehn twist coefficient $c = \frac{1}{2}$. Then $c(\xi(S,h)) = EH(S,h) \neq 0$, and hence the contact structure $\xi(S,h)$ is tight.

Proof. We show that $EH(S,h) \neq 0$ by choosing a basis for $S$ for which there are no holomorphic disks in the corresponding Heegaard diagram that map to the generator $x = (x_0,y_0)$ defining $EH(S,h)$.

The following lemma furnishes us with a convenient basis:

Lemma 4.2. Let $A \in SL(2, \mathbb{Z})$ be a matrix with $tr(A) < -2$. Then $A$ is conjugate in $SL(2, \mathbb{Z})$ to a matrix \[
\begin{pmatrix}
    a & b \\
    c & d
\end{pmatrix}
\] where $(a,c)$ and $(b,d)$ are in the third quadrant.

Proof. Let $\Lambda^s$ and $\Lambda^u$ be the stable and unstable laminations for $A$. The slopes of $\Lambda^s$ and $\Lambda^u$ will be written $slop(\Lambda^s)$ and $slop(\Lambda^u)$. (Recall that these slopes are irrational.) Let us consider the Farey tessellation on the hyperbolic unit disk $D^2$. Pick a vertex $s_1$ on the clockwise edge along $\partial D^2$ from $slop(\Lambda^s)$ to $slop(\Lambda^u)$, and pick a vertex $s_2$ on the counterclockwise edge from $slop(\Lambda^s)$ to $slop(\Lambda^u)$, so that there is an edge of the Farey tessellation between $s_1$ and $s_2$. (The existence of such a pair $s_1, s_2$ is an exercise.) Then $A(s_1)$ (resp. $A(s_2)$) is closer to $slop(\Lambda^s)$ than $s_1$ (resp. $s_2$) is. An oriented basis corresponding to $(s_1, s_2)$ will have the desired property. \[\square\]

With the choice of basis as above, we can represent $M = M_{(S,h)}$ by the Heegaard diagram below. We have drawn a picture of the diagram corresponding to $A = \begin{pmatrix}
-1 & -1 \\
-1 & -2
\end{pmatrix}$, but the same argument works for any such $A$ as described in the previous lemma. We prove that there is no holomorphic disk from any $y$ to $x = (x_0,y_0)$. Suppose on the contrary that there is such a holomorphic disk $u$. Let $\tilde{u} : \tilde{D} \to \Sigma$ be the corresponding holomorphic map to $\Sigma$. Assuming $\partial \tilde{D}$ is connected, it is given by a subarc of $a_1$ from some $x_i \in a_1 \cap h(a_2)$ to $x_0$, followed by a subarc of $h(a_1)$ from $x_0$ to some $y_j \in a_2 \cap h(a_1)$, followed by a subarc of $a_2$ from $y_j$ to $y_0$ (you either turn left or turn right at $y_j$), and then by a subarc of $h(a_2)$ from $y_0$ to $x_i$. If we lift $\partial \tilde{D}$ to the universal cover of the capped off surface $T^2 = S \cup D^2$, then in all cases we see that $\partial \tilde{D}$ is not contractible. This implies that $\partial \tilde{D}$ cannot bound a surface in $S$. We argue similarly when $\partial \tilde{D}$ has two components. It follows that the class $EH(S,h)$ of $x = (x_0,y_0)$ is nonzero. \[\square\]

Theorem 4.3. $EH(S,h) \neq 0$ if $h$ is reducible and right-veering.

Proof. Suppose $h$ is reducible. Let $g$ be an element of $Aut(S, \partial S)$ which is the minimally right-veering representative for the matrix $A = -id$. (In terms of positive Dehn twists, $g = (A_1A_2A_1)^2$.)
where $A_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $A_2 = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$.) After changing bases if necessary, $h = g^n \phi_\gamma^m$, where $n$ is a positive integer, $m$ is an integer, and $\phi_\gamma$ is a positive Dehn twist about a $(0,1)$-curve $\gamma$. If $m$ is nonnegative, then $h$ is a product of positive Dehn twists, and $EH(S,h) \neq 0$.

Suppose $m < 0$. It suffices to prove the theorem for $n = 1$, since the contact structures corresponding to larger $n$ are obtained from the $n = 1$ case by Legendrian surgery. Take a basis corresponding to slopes 0, $\infty$ and matrix $A = \begin{pmatrix} -1 & 0 \\ -m & -1 \end{pmatrix}$. Then $EH(S,h)$ is nonzero by the same method as in Theorem 4.1. 

□

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