ON GONCHAROV’S REGULATOR AND HIGHER ARITHMETIC CHOW GROUPS

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Abstract. In this paper we show that the regulator defined by Goncharov in [Gon05] from higher algebraic Chow groups to Deligne-Beilinson cohomology agrees with Beilinson’s regulator. We give a direct comparison of Goncharov’s regulator to the construction given by Burgos and Feliu in [BF09]. As a consequence, we show that the higher arithmetic Chow groups defined by Goncharov agree, for all projective arithmetic varieties over an arithmetic field, with the ones defined by Burgos and Feliu.

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Introduction

Let $X$ be an arithmetic variety over a field, i.e. a regular scheme which is flat and quasi-projective over an arithmetic field. Assuming that $X$ is projective, Goncharov introduced in [Gon05] the higher arithmetic Chow groups of $X$, $\hat{CH}^p(X,n)$, which fit in a long exact sequence of the form

$$
\cdots \rightarrow \hat{CH}^p(X,n) \xrightarrow{\zeta} CH^p(X,n) \xrightarrow{\rho} H_{\mathcal{D}}^{2p-n}(X,\mathbb{R}(p)) \xrightarrow{\alpha} \hat{CH}^p(X,n-1) \rightarrow \cdots
$$

Here, $\hat{CH}^p(X)$ denote the arithmetic Chow groups defined by Gillet and Soulé in [GS90], $H_{\mathcal{D}}^{2p-n}(X,\mathbb{R}(p))$ are the real Deligne-Beilinson cohomology groups and $(\mathcal{D}^*(X,*),d_D)$ is the Deligne complex of differential forms.

Goncharov’s definition left open the question whether the composition of the isomorphism $K_n(X)_{\mathbb{Q}} \cong \bigoplus_{p \geq 0} CH^p(X,n)_{\mathbb{Q}}$ given in [Blo86] with the morphism induced by $P$ agrees with Beilinson’s regulator. In addition, the possibility of defining pull-back morphisms and a product structure on $\bigoplus_{p,n} \hat{CH}^p(X,n)$ was also left open.

Later, in [BF09], Burgos and Feliu introduced a new definition of the higher arithmetic Chow groups, suitable for quasi-projective arithmetic varieties over a field. The main difference with Goncharov’s construction was the use of the Deligne complex of differential forms with logarithmic singularities instead of the Deligne complex of currents. This enabled one to have well-defined pull-backs and a product structure on $\bigoplus_{p,n} \hat{CH}^p(X,n)$. 

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The new definition was as well based on the definition of a regulator, which was shown to induce Beilinson’s regulator in $K$-theory.

In loc. cit., it was left a comparison of the two definitions of higher arithmetic Chow groups. In this paper, we show that both definitions agree in the case of proper arithmetic varieties over an arithmetic field. This is shown by a direct comparison of Goncharov’s regulator and the one introduced by Burgos and Felìu in [BF09].

The paper is organized as follows. The first four sections contain the required preliminaries. The first section contains the notation that will be used in the sequel. Section 2 and 3 cover the necessary background on Deligne-Beilinson cohomology and Bloch higher algebraic Chow groups respectively. In section 4 we review the construction of the regulator given by Burgos and Felìu. From section 5 to section 7, we find the core of this work. In section 5 we introduce the differential forms $T_m$ and a few properties are shown. A comparison of Wang’s forms and the differential forms given by Goncharov in his construction of the regulator is performed. In section 6, the comparison of regulators is done and finally, in section 7 we prove that the higher arithmetic Chow groups given by Goncharov and the ones given by Burgos and Felìu agree.

1. Notation

1.1. Notation on (co)chain complexes. We use the standard conventions on (co)chain complexes. By a (co)chain complex we mean a (co)chain complex over the category of abelian groups. The cochain complex associated to a chain complex $A_\ast$ is simply denoted by $A^\ast$ and the chain complex associated to a cochain complex $A^\ast$ is denoted by $A_\ast$.

The translation of a cochain complex $(A^\ast, d_A)$ by an integer $m$ is denoted by $A[m]^\ast$. Note that $A[m]^n = A^{m+n}$ and the differential of $A[m]^\ast$ is $(-1)^m d_A$. If $(A_\ast, d_A)$ is a chain complex, then the translation of $A_\ast$ by an integer $m$ is denoted by $A[m]_\ast$. In this case the differential is also $(-1)^m d_A$ but $A[m]^n = A_{n-m}$.

The simple complex associated to an iterated chain complex $A_\ast$ is denoted by $s(A)_\ast$ and the analogous notation is used for the simple complex associated to an iterated cochain complex (see [BKK07], §2 for definitions). The simple of a cochain map $A^\ast \xrightarrow{f} B^\ast$ is the cochain complex $(s(f)^\ast, d_{s(f)})$ with $s(f)^n = A^n \oplus B^{n-1}$, and differential $d_{s(f)}(a, b) = (d_A a, f(a) - d_B b)$. Note that this complex is the cone of $-f$ shifted by 1. There is an associated long exact sequence

\[\cdots \to H^n(s(f)^\ast) \to H^n(A^\ast) \xrightarrow{f} H^n(B^\ast) \to H^{n+1}(s(f)^\ast) \to \cdots\]

Equivalent results can be stated for chain complexes.

Following Deligne [Del71], given a cochain complex $A^\ast$ and an integer $n$, we denote by $\tau_{\leq n}A^\ast$, $\tau_{\geq n}A^\ast$ the canonical truncations of $A^\ast$ at degree $n$.

1.2. Cubical abelian groups and chain complexes. Let $C = \{C_n\}_{n\geq 0}$ be a cubical abelian group. We denote the face maps by $\delta_i^j : C_n \to C_{n-1}$, for $i = 1, \ldots, n$ and $j = 0, 1$, and the degeneracy maps by $\sigma_i : C_n \to C_{n+1}$, for $i = 1, \ldots, n + 1$. Let $D_n \subset C_n$ be the subgroup of degenerate elements of $C_n$.

By $C_\ast$ we denote the associated chain complex, that is, the chain complex whose $n$-th graded piece is $C_n$ and whose differential is given by $\delta = \sum_{i=1}^n \sum_{j=0,1} (-1)^{i+j} \delta_i^j$. We fix the normalized chain complex associated to $C_\ast$, $NC_\ast$, to be the chain complex whose $n$-th graded group is $NC_n := \bigcap_{i=1}^n \ker \delta_i^1$, and whose differential is $\delta = \sum_{i=1}^n (-1)^{i} \delta_i^1$. It is well-known that there is a decomposition of chain complexes $C_\ast \cong NC_\ast \oplus D_\ast$ giving an isomorphism of chain complexes $NC_\ast \cong C_\ast / D_\ast$. 
2. Deligne-Beilinson cohomology

In this paper we use the definitions and conventions on Deligne-Beilinson cohomology given in [Bur97] and [BKK07], §5.

As defined in [BKK07], §5.2, for any Dolbeault complex $A = (A, d_A)$ there is an associated cochain complex called the Deligne complex and denoted by $(\mathcal{D}^\ast (A, d_A), d_D)$.

Let $X$ be a complex algebraic manifold. Let $E^\ast_{R, c}(X)$ be the space of real smooth differential forms (resp. differential forms with compact support) of degree $n$ on $X$, and let $\mathcal{E}^\ast_{R}(X)$ be the space of real currents on $X$ of degree $n$, that is, the topological dual of $E^\ast_{R, c}(X)$. One denotes $\mathbb{R}(p) = (2\pi i)^p \cdot \mathbb{R} \subset \mathbb{C}$. Accordingly, we write $E^\ast_{R}(X, p) = (2\pi i)^p \cdot E^\ast_{R}(X)$ and $\mathcal{E}^\ast_{R}(X, p) = (2\pi i)^p \cdot \mathcal{E}^\ast_{R}(X)$.

When $X$ is equidimensional of dimension $d$, we write

\begin{equation}
D^\ast_{R}(X) = \mathcal{E}^\ast_{R}(X)[-2d](-d).
\end{equation}

Hence $D^\ast_{R}(X, p)$ is the topological dual of $E^{2d-n}_{R}(X, d - p)$.

2.1. Deligne complex of differential forms. Let $(E^\ast_{\log, R}(X), d)$ be the complex of real differential forms with logarithmic singularities along infinity [Bur94] and let $E^\ast_{\log, R}(X)(p)$ denote the vector space of real differential forms with logarithmic singularities along infinity, twisted by $p$. Since the complex $(E^\ast_{\log, R}(X), d)$ is a Dolbeault complex (see [BKK07], §5.2) we can consider the Deligne complex of differential forms with logarithmic singularities

\[
(D^\ast_{\log}(X, p), d_D) := (\mathcal{D}^\ast (E^\ast_{\log, R}(X), p), d_D).
\]

This complex is functorial on $X$. It computes the real Deligne-Beilinson cohomology of $X$, that is, there is an isomorphism

\[
H^n(D^\ast_{\log}(X, p)) \cong H^{2d-n}_D(X, \mathbb{R}(p)).
\]

Moreover, the Deligne-Beilinson cohomology product structure can be described by a cochain morphism on the Deligne complex (see [Bur97])

\[
\mathcal{D}^n(X, p) \otimes \mathcal{D}^m(X, q) \rightarrow \mathcal{D}^{n+m}(X, p + q).
\]

This product is graded commutative and satisfies the Leibniz rule, but it is only associative up to homotopy.

If $X$ is compact, then we simply denote by $\mathcal{D}^\ast(X, p)$ the Deligne complex of differential forms on $X$.

2.2. Currents. Assume that $X$ is compact and equidimensional of dimension $d$. The complex $D^\ast_{R}(X)$ has also a structure of Dolbeault complex, and hence, there is an associated Deligne complex denoted by

\[
(D^\ast_D(X, p), d_D) := (\mathcal{D}^\ast (D^\ast_{R}(X), p), d_D).
\]

The current associated to every differential form gives a quasi-isomorphism of Deligne complexes

\begin{equation}
\mathcal{D}^\ast(X, p) \xrightarrow{\int} D^\ast_D(X, p), \quad \alpha \mapsto [\alpha],
\end{equation}

where $[\alpha]$ is the current

\[
[\alpha](\omega) = \frac{1}{(2\pi i)^d} \int_X \omega \wedge \alpha.
\]
Therefore, the cohomology groups of $D^*_D(X,p)$ are isomorphic to the Deligne-Beilinson cohomology groups of $X$:

$$H^n(D^*_D(X,p)) \cong H^p_D(X,\mathbb{R}(p)).$$

Recall that we are using the conventions of [BKK07] concerning the twisting and the real structures. In particular, if $Y$ is a subvariety of codimension $p$ then the current integration along $Y$, denoted $\delta_Y$ is given by

$$\delta_Y(\omega) = \frac{1}{(2\pi i)^{d-p}} \int_Y \iota^* \omega,$$

for $\iota : \tilde{Y} \to X$ a resolution of singularities of $Y$. Hence, again using the conventions of [BKK07] we have $\delta_Y \in D^p_D(X,p)$ and is a representative of the class $\text{cl}(Y) \in H^p_D(X,p)$.

3. Bloch Higher Chow groups

We recall here the definition of higher algebraic Chow groups given by Bloch in [Blo86]. Initially, they were defined using the chain complex associated to a simplicial abelian group. An alternative definition can be given using a cubical presentation, as developed by Levine in [Lev94]. Both constructions are analogous, however, the cubical setting is more suitable to define products. These two settings are recalled here.

Let $k$ be a field and $\mathbb{P}^n$ the projective space of dimension $n$ over $k$. Fix $X$ to be an equidimensional quasi-projective algebraic scheme of dimension $d$ over the field $k$.

3.1. The simplicial Bloch complex. We recall first the definition of the higher Chow groups using a simplicial setting as given by Bloch in [Blo86]. We fix homogeneous coordinates $z_0, \ldots, z_n$ of $\mathbb{P}^n$ and put $\Delta^n = \mathbb{P}^n \setminus H_n$, where $H_n$ is the hyperplane defined by $z_0 + \cdots + z_n = 0$. The collection $\{\Delta^n\}_{n \geq 0}$ has a cosimplicial scheme structure, with coface maps denoted by $\partial^i$. The faces of $\Delta^n$ are closed subschemes which arise as the image of compositions of coface maps. A face of $X \times \Delta^n$ is a closed subscheme of the form $X \times F$, for $F$ a face of $\Delta^n$.

We denote by $Z^p_s(X,n)$ the free abelian group generated by the codimension $p$ closed irreducible subvarieties of $X \times \Delta^n$, which intersect properly all the faces of $X \times \Delta^n$.

Then, intersection with the face $\partial^i(X \times \Delta^{n-1})$ gives a map $\partial_i : Z^p_s(X,n) \to Z^p_s(X,n-1)$. Setting

$$\partial = \sum_{i=0}^n (-1)^i \partial_i : Z^p_s(X,n) \to Z^p_s(X,n-1),$$

then $(Z^p_s(X,*), \partial)$ is a chain complex of abelian groups. The higher algebraic Chow groups of $X$ are the homology groups of this complex:

$$CH^p(X,n) = H_n(Z^p_s(X,*)).$$

To emphasize explicitly that this higher algebraic Chow groups are given using the simplicial setting, we will write

$$CH^p_s(X,n) = H_n(Z^p_s(X,*)).$$
3.2. The cubical Bloch complex. Consider \( \mathbb{P}^1 \) the projective line over \( k \) and let \( \square = \mathbb{P}^1 \setminus \{1\} \cong A^1 \). The collection \( \{\square^n = \square \times \ldots \times \square\}_{n \geq 0} \) has a cocubical scheme structure, with coface maps denoted by \( \delta_i^n \) and codegeneracy maps denoted by \( \sigma_i^n \). Specifically, for \( i = 1, \ldots, n \), the coface and codegeneracy maps are defined as
\[
\delta_i^n(x_1, \ldots, x_{n-1}) = (x_1, \ldots, x_{i-1}, 0, x_i, \ldots, x_{n-1}),
\]
\[
\delta_0^n(x_1, \ldots, x_{n-1}) = (x_1, \ldots, x_{i-1}, \infty, x_i, \ldots, x_{n-1}),
\]
\[
\sigma_i^n(x_1, \ldots, x_n) = (x_1, \ldots, x_i, x_{i+1}, \ldots, x_n).
\]

Note that the cofaces are closed immersions and the codegeneracies are flat maps.

An \( r \)-dimensional face of \( \square^n \) is any closed subscheme of the form \( \delta_1^n \cdots \delta_r^n(\square^{n-r}) \). Faces of \( X \times \square^n \) are defined accordingly.

Let \( Z^p_c(X, n) \) be the free abelian group generated by the codimension \( p \) closed irreducible subvarieties of \( X \times \square^n \), which intersect properly all the faces of \( X \times \square^n \). The pull-back by the coface and codegeneracy maps of \( \square \) endow \( Z^p_c(X, \cdot) \) with a cubical abelian group structure. Let \((Z^p_c(X, \ast), \delta)\) be the associated chain complex and consider the normalized chain complex associated to \( Z^p_c(X, \ast) \), denoted by \( Z^p_c(X, \ast)_0 \) with
\[
Z^p_c(X, n)_0 := NZ^p_c(X, n) = \bigcap_{i=1}^n \ker \delta_i^n.
\]

Denote the homology groups of this complex by
\[
CH^p_c(X, n) = H_n(Z^p_c(X, \ast)_0).
\]

The subscript \( c \) refers to the use of the cubical setting. The higher Chow groups defined by Bloch are isomorphic to \( CH^p_c(X, n) \). That is, there is a natural isomorphism \([Lev94], \text{Theorem 4.7}]\):
\[
CH^p_c(X, n) \cong CH^p(X, n), \quad \forall n, p \geq 0.
\]

4. Burgos-Feliu construction of the Beilinson regulator

We review here briefly the construction of the Beilinson regulator given by Burgos and Feliu in [BF09].

Let \( X \) be a complex algebraic variety and let \( \square \) be as in [3.2]. For simplicity we will assume that \( X \) is proper although in [BF09] the construction is also done for open varieties. Consider the smooth compactifications of \( X \times \square^n \) given by \( X \times (\mathbb{P}^1)^n \). We denote \( D^n = X \times (\mathbb{P}^1)^n \setminus X \times \square^n \), which is a normal crossing divisor. Let \( E^*_X \times (\mathbb{P}^1)^n(\log D^n) \) be the complex of differential forms with logarithmic singularities along \( D \) [Bur94a].

In this paper we will denote
\[
E^*_X \times (\mathbb{P}^1)^n(\log D^n) := E^*_X \times (\mathbb{P}^1)^n(\log D^n)
\]
and we will call it the complex of differential forms on \( X \times \square^n \) with logarithmic singularities along infinity. Then, the Deligne complex associated to \( E^*_X \times (\mathbb{P}^1)^n(\log D^n) \), denoted \( D^*_X(\log D^n, p) \), computes the Deligne-Beilinson cohomology of \( X \times \square^n \), which, by homotopy invariance, agrees with the Deligne-Beilinson cohomology of \( X \).

**Remark 4.1.** The Deligne complex of differential forms on \( X \times \square^n \) with logarithmic singularities along infinity is usually defined by taking the limit over all compactifications of \( X \times \square^n \). In this work, however, since \( X \) is proper, we have a natural compactification of \( X \times \square^n \). The two different cochain complexes obtained, using the limit or with a fixed compactification, are quasi-isomorphic.
The description of the regulator uses some intermediate complexes that we describe in the following.

**The complex** $D^*_\log(X,p)_0$. For every $n,p \geq 0$, let $\tau D^*_\log(X \times \square^n,p)$ be the Deligne complex of differential forms in $X \times \square^n$, with logarithmic singularities at infinity, truncated at degree $2p$:

$$\tau D^*_\log(X \times \square^n,p) := \tau_{\leq 2p} D^*_\log(X \times \square^n,p).$$

The structural maps of the cocubical scheme $\square$ induce a cubical structure on $\tau D^*_\log(X \times \square^n,p)$ for every $r$ and $p$. Consider the 2-iterated cochain complex given by

$$D^r_{\delta,n}(X,p) = \tau D^r_{\log}(X \times \square^n,p)$$

and with differential $(d_D, \delta = \sum_{i=1}^n (-1)^i (\delta_0^i - \delta_1^i))$. Let

$$D^r_{\log}(X,p) = s(D^\tau_{\log,n}(X,p))$$

be the simple complex associated to the 2-iterated complex $D^\tau_{\log,n}(X,p)$.

For every $r,n$, let $\tau D^r_{\log}(X \times \square^n,p)_0 = N \tau D^r_{\log}(X \times \square^n,p)$ be the normalized complex and let

$$D^r_{\log,n}(X,p)_0 = \tau D^r_{\log}(X \times \square^n,p)_0.$$  

Denote by $(D^r_{\log,n}(X,p)_0, d_s)$ the associated simple complex.

**The complex** $H^p(\mathcal{X},*)_0$. Let $\mathcal{Z}^p$ be the set of all codimension $p$ closed subvarieties of $X \times \square^n$ intersecting properly the faces of $X \times \square^n$. When there is no source of confusion, we simply write $\mathcal{Z}^p$ or even $\mathcal{Z}^p$. Write

$$D^r_{\log}(X \times \square^n \setminus \mathcal{Z}^p) = \lim_{Z \in \mathcal{Z}^p} \frac{D^r_{\log}(X \times \square^n \setminus Z, p),}{\text{where here logarithmic singularities at infinity are defined by taking the limit over all possible compactifications.}}$$

Let $(D^r_{\log,n}(X \times \square^n,p), d_D)$ be the Deligne complex with supports

$$D^r_{\log,n}(X \times \square^n,p) = s(D^r_{\log}(X \times \square^n,p) \rightarrow D^r_{\log}(X \times \square^n \setminus \mathcal{Z}^p)).$$

The cohomology groups of this complex are denoted by $H^p_{\mathcal{Z}^p}(X \times \square^n, \mathbb{R}(p))$.

Consider the cubical abelian group

$$(4.2) \quad H^p(\mathcal{X},\cdot) := H^p_{\mathcal{Z}^p}(X \times \square^n, \mathbb{R}(p)),$$

with faces and degeneracies induced by those of $\square$. Let $H^p(\mathcal{X},*)_0$ be the associated normalized complex.

**The complex** $D^r_{\log,Z^p}(X,p)_0$. Let $D^r_{\log,Z^p}(X,p)_0$ be the 2-iterated cochain complex, whose component of bidegree $(r,-n)$ is

$$D^r_{\log,Z^p}(X,p)_0 := \tau_{\leq 2p} D^r_{\log,Z^p}(X \times \square^n,p)_0 = N \tau_{\leq 2p} D^r_{\log,Z^p}(X \times \square^n,p)_0,$$

and whose differentials are $(d_D, \delta)$. As usual, we denote by $(D^r_{\log,Z^p}(X,p)_0, d_s)$ the associated simple complex. Let $D^{2p}_{\log,Z^p}(X,p)_0$ be the chain complex whose $n$-th graded piece is $D^{2p}_{\log,Z^p}(X,p)_0$.

The main properties of the above complexes are summarized in the following result.
Proposition 4.3 ([BF09].) (1) The natural morphism of complexes
\[ \tau D_{\log}^*(X, p) = D_{\log}^*(X, p)_0 \rightarrow D_{\log}^*(X, p)_0 \]
is a quasi-isomorphism.
(2) There is an isomorphism of chain complexes
\[ f_1 : Z^p_c(X, *, 0) \otimes \mathbb{R} \xrightarrow{\sim} H^p(X, *), \]
sending every algebraic cycle \( z \) to its class \( \text{cl}(z) \).
(3) For every \( p \geq 0 \), the morphism
\[ D_{2p-n}^{2p-n}(X, p)_0 \xrightarrow{g_1} H^p(X, n)_0 \]
defines a quasi-isomorphism of chain complexes.

Proof. See [BF09], Corollary 2.9, Lemma 2.11 and Proposition 2.13. \( \square \)

Definition of the regulator. Consider the map of iterated cochain complexes defined by the projection onto the first factor
\[ D_{2p-n}^{2p-n}(X, p)_0 \xrightarrow{g_1} H^p(X, n)_0 \]
\[ \left( (\omega_n, g_n), \ldots, (\omega_0, g_0) \right) \mapsto \left[ [\omega_n, g_n] \right] \]
It induces a chain morphism
\[ D_{2p-n}^{2p-n}(X, p)_0 \xrightarrow{\rho} D_{2p-n}^{2p-n}(X, p)_0. \]
The morphism induced by \( \rho \) in homology, together with the isomorphisms of Proposition 4.3 induce a morphism
\[ \rho : CH^p(X, n) \rightarrow CH^p(X, n)_\mathbb{R} \rightarrow H^p_D(X, \mathbb{R}(p)). \]
By abuse of notation, all these morphisms are denoted by \( \rho \).
Observe that in the derived category of chain complexes, the morphism \( \rho \) is given by the composition
\[ Z^p_c(X, *)_0 \xrightarrow{f_1} H^p(X, *)_0 \xleftarrow{\sim} D_{2p-n}^{2p-n}(X, p)_0 \xrightarrow{\rho} D_{2p-n}^{2p-n}(X, p)_0. \]
The following result follows directly from the definitions.

Lemma 4.6. Let \( z \in CH^p(X, n) \), then
\[ \rho(z) = \left[ (\omega_n, \ldots, \omega_0) \right], \]
for any cycle \( (\omega_n, g_n), \ldots, (\omega_0, g_0) \in D_{2p-n}^{2p-n}(X, p)_0 \) such that \( [\omega_n, g_n] = \text{cl}(z) \). \( \square \)

Theorem 4.7 ([BF09], Thm. 3.5). Let \( X \) be an equidimensional complex algebraic manifold. Let \( \rho' \) be the composition of \( \rho \) with the isomorphism given by the Chern character of [Blo86]
\[ \rho' : K_n(X)_\mathbb{Q} \xrightarrow{\cong} \bigoplus_{p \geq 0} CH^p(X, n)_\mathbb{Q} \xrightarrow{\rho} \bigoplus_{p \geq 0} H^p_D(X, \mathbb{R}(p)). \]
Then, the morphism \( \rho' \) agrees with the Beilinson regulator. \( \square \)
We consider now the diagram of chain complexes
\[
C_* = \begin{pmatrix}
\mathcal{H}^p(X, \ast)_0 & \mathcal{D}^{2p-*}(X, p)_0 \\
\sim & \rho \\
D^{2p-*}_{\mathcal{H}, \mathcal{A}}(X, p)_0 & \mathcal{D}^{2p-*}_{\mathcal{H}, \mathcal{A}}(X, p)_0
\end{pmatrix}.
\]

We denote
\[
D^{2p-*}_{\mathcal{H}, \mathcal{A}}(X, p)_0 = s(C)[-1],
\]
the simple complex associated to the above diagram as in [BF09] §1.2, shifted by minus one. That is, an element of \(D^{2p-*}_{\mathcal{H}, \mathcal{A}}(X, p)_0\) is a triple \((\alpha_1, \alpha_2, \alpha_3)\) with \(\alpha_1 \in D^{2p-n-1}_{\mathcal{H}, \mathcal{A}}(X, p)_0\), \(\alpha_2 \in \mathcal{H}^p(X, n)_0\) and \(\alpha_3 \in D^{2p-n}_{\mathcal{H}, \mathcal{A}}(X, p)_0\), and the differential is given by
\[
d(\alpha_1, \alpha_2, \alpha_3) = (-d\alpha_1, d\alpha_2 + g_1(\alpha_1), d\alpha_3 - \rho(\alpha_1)).
\]

There is a quasi-isomorphism
\[
\beta : D^{2p-*}_{\mathcal{H}, \mathcal{A}}(X, p)_0 \xrightarrow{\sim} D^{2p-*}_{\mathcal{H}, \mathcal{A}}(X, p)_0
\]
given by \(\beta(\alpha) = (0, 0, \alpha)\). We identify
\[
H_*(D^{2p-*}_{\mathcal{H}, \mathcal{A}}(X, p)_0) = H^{2p-*}_{D}(X, \mathbb{R}(p))
\]
by means of this quasi-isomorphism. Then Beilinson’s regulator for higher Chow groups is given by the morphism, also denoted \(\rho\),
\[
\rho : Z^p_{\mathcal{A}}(X, \ast)_0 \longrightarrow D^{2p-*}_{\mathcal{H}, \mathcal{A}}(X, p)_0
\]
given by \(\rho(z) = (f_1(z), 0, 0)\).

5. Goncharov’s forms and Wang’s forms

5.1. The differential form \(T_m\). Let \(A^*\) be a Dolbeault algebra and let \(\mathcal{D}^*(A, \ast)\) be the associated Deligne algebra as given in [BKK07]. Let \(u_1, \ldots, u_m \in D^1(A, 1)\). Following [Wan92], for \(i = 1, \ldots, m\), we write
\[
S^i_m(u_1, \ldots, u_m) = (-2)^m \sum_{\sigma \in \mathfrak{S}_m} (-1)^{|\sigma|} u_{\sigma(1)} \partial u_{\sigma(2)} \wedge \cdots \wedge \partial u_{\sigma(i)} \wedge \partial u_{\sigma(i+1)} \wedge \cdots \wedge \partial u_{\sigma(m)},
\]
and
\[
T_m(u_1, \ldots, u_m) = \frac{1}{2m!} \sum_{i=1}^{m} (-1)^i S^i_m(u_1, \ldots, u_m).
\]

We will also write \(T_0 = 1\). The forms \(T_m(u_1, \ldots, u_m)\) will be called Wang’s forms.

**Proposition 5.3.** (1) The form \(T_m(u_1, \ldots, u_m)\) belongs to \(D^m(X, m)\).
(2) It holds
\[
T_m(u_1, \ldots, u_m) = \frac{1}{m!} \sum_{\sigma \in \mathfrak{S}_m} (-1)^{|\sigma|} u_{\sigma(1)} \cdots \cdot (u_{\sigma(m-1)} \bullet u_{\sigma(m)}).
\]
(3) There is a recursive formula
\[
d_D T_m(u_1, \ldots, u_m) = \sum_{i=1}^{m} (-1)^{i-1} d_D u_i \bullet T_{m-1}(u_1, \ldots, \hat{u_i}, \ldots, u_m).
\]
Proof. It is clear that \[2\] implies \[1\] and, by the Leibniz rule, also \[3\]. Nevertheless we will prove first \[3\] and use it to prove \[2\]. For this we will follow [Tak05] §5.2, but note that our \(S_m^i\) is \((-2)^m\) times the form denoted \(S_m^i\) there. Given elements \(u_1, \ldots, u_n \in \mathcal{D}^1(A,1)(= A^n(0))\), we will denote by \((u_1, \ldots, u_n)^{(i)}\) the piece of bidegree \((i, n-i)\) of \(du_1 \wedge \cdots \wedge du_n \in A^n\). Then

\[
S_m^i(u_1, \ldots, u_m) = (-2)^m(i-1)!(m-i)! \sum_{j=1}^{m} (-1)^{j+1} u_j(u_1, \ldots, \hat{u}_j, \ldots, u_m)^{(i-1)}.
\]

It is proved in [Tak05], Lemma 5.3 that

\[
\partial S_m^i(u_1, \ldots, u_m) = (-2)^m i!(m-i)!(u_1, \ldots, u_m)^{(i)}
\]

and

\[
\bar{\partial} S_m^i(u_1, \ldots, u_m) = (-2)^m(i-1)!(m-i+1)!(u_1, \ldots, u_m)^{(i-1)} - (i-1) \sum_{j=1}^{m} (-1)^i (-2\bar{\partial}) u_j \wedge S_{m-1}^i(u_1, \ldots, \hat{u}_j, \ldots, u_m).
\]

Then, using the definition of the differential in the Deligne complex,

\[
d\mathcal{D} T_m(u_1, \ldots, u_m) = \frac{1}{2m!} \sum_{i=1}^{m-1} (-1)^{i+1} \partial S_m^i(u_1, \ldots, u_m) + \frac{1}{2m!} \sum_{i=2}^{m} (-1)^{i+1} \bar{\partial} S_m^i(u_1, \ldots, u_m)
\]

\[
= (-2)^m \sum_{i=1}^{m-1} i!(m-i)!(u_1, \ldots, u_m)^{(i)}
\]

\[
+ (-2)^m \sum_{i=2}^{m} (i-1)!(m-i+1)!(u_1, \ldots, u_m)^{(i-1)}
\]

\[
+ \frac{1}{2m!} \sum_{i=1}^{m-1} (-1)^{i+1} (m-i) \sum_{j=1}^{m} (-1)^j (-2\bar{\partial}) u_j \wedge S_{m-1}^i(u_1, \ldots, \hat{u}_j, \ldots, u_m)
\]

\[
- \frac{1}{2m!} \sum_{i=2}^{m} (i-1)! \sum_{j=1}^{m} (-1)^j (-2\bar{\partial}) u_j \wedge S_{m-1}^i(u_1, \ldots, \hat{u}_j, \ldots, u_m)
\]

\[
= \frac{1}{2m!} \sum_{j=1}^{m-1} (-1)^{j-1} d\mathcal{D} u_j \wedge \left( \sum_{i=1}^{m-1} (-1)^i (m-i+1) S_{m-1}^i(u_1, \ldots, \hat{u}_j, \ldots, u_m) \right)
\]

\[
= \sum_{j=1}^{m-1} (-1)^{j-1} d\mathcal{D} u_j \wedge T_{m-1}(u_1, \ldots, \hat{u}_j, \ldots, u_m).
\]

In order to prove \[2\] write temporarily

\[
C_m(u_1, \ldots, u_m) = \frac{1}{m!} \sum_{\sigma \in S_m} (-1)^{\sigma} u_{\sigma(1)} \cdot (\cdots \cdot (u_{\sigma(m-1)} \cdot u_{\sigma(m)})).
\]

Then, it is easy to show by induction that the forms \(C_m\) are a linear combination of monomials of the form \(u_{\sigma(1)} \partial u_{\sigma(2)} \wedge \cdots \wedge \partial u_{\sigma(i)} \wedge \partial u_{\sigma(i+1)} \wedge \cdots \wedge \partial u_{\sigma(m)}\), for suitable
integers $i$ and permutations $\sigma$. Since they are invariant under the action of the symmetric group they are a linear combination of the forms $S'_m$. Say 

$$C_m(u_1, \ldots, u_m) = \sum_{i=1}^{m} \alpha_{i,m} S'_m(u_1, \ldots, u_m).$$

By the Leibniz rule the forms $C_m$ satisfy the relation \((5.4)\). In particular, $d_P C_m$ does not contain any term of the form $(u_1, \ldots, u_m)^{(i)}$. By \((5.5)\) and \((5.6)\) this implies that $\alpha_{i,m} = -\alpha_{i-1,m}$. Thus, we have to show that $\alpha_{1,m} = -1/(2m!)$. Given a differential form $\omega$ of degree $n$, we will denote 

$$\overline{F}^p \omega = \sum_{q \geq p} \omega^{(n-q,q)}.$$ 

Since $F^{m-1} C_m = \alpha_{1,m} S'_m$, to determine $\alpha_{1,m}$ we can compare $\overline{\partial} S'_m$ with $\overline{\partial} F^{m-1} C_m$. On the one hand 

$$\overline{\partial} S'_m = (-2)^m m! (u_1, \ldots, u_m)^{(i-1)}.$$ 

On the other hand, since for $a \in D^p(A, p)$ and $b \in D^q(A, q)$, it holds 

$$\overline{\partial} F^{p+q-1} (a \bullet b) = -2 \overline{\partial} F^{p-1} a \wedge \overline{\partial} F^{q-1} b,$$

we obtain 

$$\overline{\partial} F^{m+1} C_m(u_1, \ldots, u_m) = (-2)^{m-1} (u_1, \ldots, u_m)^{(i-1)}.$$ 

Therefore $\alpha_{1,m} = -1/(2m!)$, which concludes the proof of the proposition. \[\square\]

Given the inclusion, for $m \geq 1$, $D^m(A, m) \subset A^m_{R^{-1}}(m-1)$, we can view $T_m(u_1, \ldots, u_m)$ as an element of $A^m_{R^{-1}}(m-1)$. By the same techniques as in the proof of the previous proposition one can prove

**Proposition 5.7.** For $m > 1$, the following equation holds 

$$dT_m(u_1, \ldots, u_m) = 2^{m-1} \left( (u_1, \ldots, u_m)^{(m)} + (-1)^{m-1} (u_1, \ldots, u_m)^{(0)} \right)$$

$$+ 2 \sum_{i=1}^{m} (-1)^{i-1} \overline{\partial} u_i \wedge T(u_1, \ldots, \hat{u}_i, \ldots, u_m).$$

For $m = 1$, the following equation holds 

$$dT_1(u_1) = du_1 = \partial u_1 + \overline{\partial} u_1 = (u_1)^{(1)} + (u_1)^{(0)}.$$ 

\[\square\]

Let now $X$ be a proper complex algebraic manifold, $Y$ a closed integral subvariety of $X$ of codimension $p$, $\iota : \tilde{Y} \rightarrow X$ a resolution of singularities of $Y$ and $\mathbb{C}(Y) = \mathbb{C}(\tilde{Y})$ the function field of $Y$. For $f \in \mathbb{C}(Y)^{\times} = \mathbb{C}(Y) - \{0\}$, we write 

$$g(f) = \frac{-1}{2} \log f \tilde{f} \in D^1_{log}(\tilde{Y} \setminus \text{div } f, 1).$$

This is a Green form on $\tilde{Y}$ for the cycle $\text{div } f$. More precisely 

$$d_D g(f) = d_P \left( \frac{-1}{2} \log f \tilde{f} \right) - \delta_{\text{div } f} = -\delta_{\text{div } f}.$$

\((5.8)\)
Then, for $f_1, \ldots, f_m \in \mathbb{C}(Y)^\times$ and $1 \leq i \leq m$, we denote

\begin{equation}
S^i_m(f_1, \ldots, f_m) = S^i_m(g(f_1), \ldots, g(f_m))
\end{equation}

\begin{equation}
= \sum_{\sigma \in S_m} (-1)^{|\sigma|} \log |f_{\sigma(1)}|^2 \frac{df_{\sigma(2)}}{f_{\sigma(2)}} \wedge \cdots \wedge \frac{df_{\sigma(i)}}{f_{\sigma(i)}} \wedge \frac{d\overline{T}_{\sigma(i+1)}}{\overline{T}_{\sigma(i+1)}} \wedge \cdots \wedge \frac{d\overline{T}_{\sigma(m)}}{\overline{T}_{\sigma(m)}},
\end{equation}

and

\begin{equation}
T_m(f_1, \ldots, f_m) = \frac{1}{2m!} \sum_{i=1}^{m} (-1)^i S^i_m(f_1, \ldots, f_m).
\end{equation}

This is a differential form on $\tilde{Y}$, and has logarithmic singularities along $\text{div}(f_1) \cup \cdots \cup \text{div}(f_m)$. It is always locally integrable because, when $\text{div}(f_1), \ldots, \text{div}(f_m)$ have common components, the graded-commutativity of the product $\bullet$ assures us that the possible non locally integrable terms cancel each other. Note that, although now the definitions of $S^i_m$ and $T_m$ are overloaded, there is no possible confusion. In definitions (5.1) and (5.2) the arguments are elements of a Deligne algebra. By contrast, in definitions (5.9) and (5.10) the arguments are rational functions.

We will denote the current on $X$ associated to $T_m$ by

\begin{equation}
[T_m](f_1, \ldots, f_m) = \iota_*[T_m(f_1, \ldots, f_m)].
\end{equation}

This current belongs to $D_D^{2p+m}(X, p+m)$. Be aware of the conventions of §2.2 concerning the current associated to a differential form.

5.2. Goncharov’s differential forms. For any $f_1, \ldots, f_m$ rational functions on $X$, Goncharov has defined in [Gon05] differential forms $r_{m-1}(f_1, \ldots, f_m)$ as follows:

\begin{equation}
r_{m-1}(f_1, \ldots, f_m) = (-1)^m \sum_{0 \leq 2j+1 \leq m} c_{j,m} \text{Alt}_m \left( \log |f_1| d \log |f_2| \wedge \ldots \wedge \log |f_{2j+1}| \wedge \wedge \text{di arg } f_{2j+2} \wedge \ldots \wedge \text{di arg } f_m \right).
\end{equation}

Here the symbol $\sum_{0 \leq 2j+1 \leq m}$ means the sum over integers $j$ such that $0 \leq 2j+1 \leq m$, $c_{j,m}$ are the rational numbers

\begin{equation}
c_{j,m} = \frac{1}{(2j+1)!(m-2j-1)!},
\end{equation}

and $\text{Alt}_m$ stands for the alternating sum over the symmetric group $S_m$, i.e.,

\begin{equation}
\text{Alt}_m(F(f_1, \ldots, f_m)) = \sum_{\sigma \in S_m} (-1)^{|\sigma|} F(f_{\sigma(1)}, \ldots, f_{\sigma(m)}).
\end{equation}

Remark 5.12. The sign $(-1)^m$ appears due to the difference in sign on the differential of the Deligne complex as considered here and as considered by Goncharov in [Gon05].

Theorem 5.13. Goncharov’s form $r_{m-1}(f_1, \ldots, f_m)$ agrees with Wang’s form $T_m(f_1, \ldots, f_m)$.

Proof. Since $d \log |f| = \frac{1}{2}(\frac{df}{f} + \frac{df}{f})$ and $\text{di arg } f = \frac{1}{2}(\frac{df}{f} - \frac{df}{f})$, the $(i-1, m-i)$-part of the form

\begin{equation}
\text{Alt}_m (\log |f_1| d \log |f_2| \wedge \ldots \wedge d \log |f_{2j+1}| \wedge \text{di arg } f_{2j+2} \wedge \ldots \wedge \text{di arg } f_m)
\end{equation}

is equal to

\begin{equation}
\frac{1}{2m} \sum_{k=0}^{m-i} \binom{2j}{k} \binom{m-2j-1}{m-i-k} (-1)^{m-i-k} S^i_m(f_1, \ldots, f_m).
\end{equation}
Hence
\[ r_{m-1}(f_1, \ldots, f_m) = \frac{1}{2^m} \sum_{0 \leq j + l \leq m} \sum_{i=1}^{m-i} c_{j,m} \left( \frac{2^j}{k} \right) \binom{m - 2j - 1}{m - i - k} (-1)^{i+k} S^i_m(f_1, \ldots, f_m) \]
\[ = \frac{1}{2^m} \sum_{i=1}^{m} \sum_{k=0}^{m-i} \frac{(-1)^{i+k}}{k!(m - i - k)!} \sum_{k \leq 2j \leq k+1} \frac{1}{(2j+1)(2j-k)!(i+k-2j-1)!} S^i_m(f_1, \ldots, f_m). \]

By Lemma 5.14 we have
\[ r_{m-1}(f_1, \ldots, f_m) = \frac{1}{2^m} \sum_{i=1}^{m} \sum_{k=0}^{m-i} \frac{(-1)^{i+k}}{k!(m - i - k)!} \left( \sum_{l=0}^{k} \frac{(-1)^l k! 2^{i-l}}{(k-l)!(i+l)!} \right) S^i_m(f_1, \ldots, f_m) \]
\[ = \frac{1}{2^m} \sum_{i=1}^{m} \sum_{l=0}^{m-i} \frac{(-1)^l 2^{i-l}}{(i+l)!} \left( \sum_{k=\ell}^{m-i} \frac{(-1)^k}{k!(m - i - k - 1)!} \right) S^i_m(f_1, \ldots, f_m). \]

Note that
\[ n! \sum_{k=0}^{n} \frac{(-1)^k}{k!(n-k)!} = \sum_{k=0}^{n} (-1)^k \binom{n}{k} = \begin{cases} 0, & n > 0, \\ 1, & n = 0. \end{cases} \]

This follows from the equation \( (1 - x)^n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} x^k \), taking \( x = 1 \). Therefore
\[ r_{m-1}(f_1, \ldots, f_m) = \frac{1}{2^m} \sum_{i=1}^{m} \frac{(-1)^i 2^{m-1}}{m!} S^i_m(f_1, \ldots, f_m) = T_m(f_1, \ldots, f_m). \]

\[ \square \]

**Lemma 5.14.** For every pair of integers \( 0 \leq q \leq p \), we have
\[ \sum_{q \leq 2j \leq p} \frac{1}{(2j+1)(2j-q)!(p-2j)!} = \sum_{l=0}^{q} \frac{(-1)^l q! 2^{p-q+l}}{(q-l)!(p-q+l+1)!}. \]

**Proof.** We denote the left hand side of the equation above by \( A(q,p) \). The statement will be shown by induction on \( q \). When \( q = 0 \),
\[ A(0,p) = \sum_{0 \leq 2j \leq p} \frac{1}{(2j+1)!} = \frac{1}{(p+1)!} \sum_{0 \leq 2j \leq p} \binom{p+1}{2j+1}. \]

Since
\[ \frac{1}{2}((1 + x)^{p+1} - (1 - x)^{p+1}) = \sum_{0 \leq 2j \leq p} \binom{p+1}{2j+1} x^{2j+1}, \]
we obtain that (taking \( x = 1 \))
\[ A(0,p) = \frac{2^p}{(p+1)!} \]
as desired. Let us assume that the statement is true for \( q - 1 \) and let us now differentiate \( q \) times equation (5.15).
\[ \frac{1}{2}((1 + x)^{p+1} - (1 - x)^{p+1})^{(q)} = \frac{(p+1)p \cdots (p-q+2)}{(p+1)!}((1 + x)^{p-q+1} - (1 - x)^{p-q+1}). \]
On the other side, writing \( B = \left( \sum_{0 \leq 2j \leq p} \binom{p+1}{2j+1} x^{2j+1} \right)^{(q)} \), we have

\[
B = \sum_{q-1 \leq 2j \leq p} (2j+1)2j \cdots (2j-q+2) \binom{p+1}{2j+1} x^{2j-q+1}
= q \sum_{q-1 \leq 2j \leq p} 2j \cdots (2j-q+2) \binom{p+1}{2j+1} x^{2j-q+1}
+ \sum_{q \leq 2j \leq p} 2j \cdots (2j-q+1) \binom{p+1}{2j+1} x^{2j-q+1}
= \sum_{q-1 \leq 2j \leq p} q(p+1)! (2j+1)(2j-q+1)!(p-2j)! x^{2j-q+1}
+ \sum_{q \leq 2j \leq p} (p+1)! (2j+1)(2j-q)!(p-2j)! x^{2j-q+1}.
\]

where we decomposed the sums using \( 2j+1 = q + 2j - q + 1 \). Putting \( x = 1 \) and joining the two sides of equation \((5.15)\), we obtain

\[
(p+1)p \cdots (p-q+2)2^{p-q} = q(p+1)!A(q-1,p) + (p+1)!A(q,p).
\]

Hence, using the induction hypothesis, we have

\[
A(q,p) = \frac{2^{p-q}}{(p-q+1)!} - qA(q-1,p) = \frac{2^{p-q}}{(p-q+1)!} - \sum_{l=0}^{q-1} \frac{(q-1)!}{(q-l)!} \frac{2^{p-q+l}}{(p-q+l+2)!}
= \frac{2^{p-q}}{(p-q+1)!} + \sum_{l=1}^{q-1} \frac{(-1)^l}{(q-l)!} \frac{2^{p-q+l}}{(p-q+l+1)!}
= \sum_{l=0}^{q} \frac{(-1)^l}{(q-l)!} \frac{2^{p-q+l}}{(p-q+l+1)!}.
\]

\[ \square \]

5.3. Relation of currents. The assignment defined by \( T_m \) can be seen as a morphism of abelian groups

\[
T_m : \wedge^m \mathbb{C}(Y)^\times \to D^m(\overline{Y} \setminus Z^1, m),
\]

whereas the assignment defined by \([T_m]\) gives a morphism of abelian groups

\[
[T_m] : \wedge^m \mathbb{C}(Y)^\times \to D_{D}^{2p+m}(X, p+m),
\]

We are now interested in differentiating the current \([T_m]\). Assume \( Y \) is normal and \( Z \subset Y \) is a closed integral subscheme of codimension one. Then we can define the residue map

\[
\text{Res}_Z : \wedge^m \mathbb{C}(Y)^\times \to \wedge^{m-1} \mathbb{C}(Z)^\times
\]

by means of the valuation of \( \mathbb{C}(Y) \) with respect to \( Z \). For an arbitrary closed integral subscheme \( Y \) of \( X \), we define the current \((\mu_{m-1} \circ \text{Res})(f_1, \ldots, f_m) \) on \( X \) by

\[
([\mu_{m-1} \circ \text{Res}](f_1, \ldots, f_m) = \sum_{Z \in \mathcal{Y}^{(1)}} (i_Z)_* [\mu_{m-1} (\text{Res}_Z (\pi^* f_1, \ldots, \pi^* f_m))],
\]

where \( \overline{Y} \) is the normalization of \( Y \), \( \mathcal{Y}^{(1)} \) is the set of irreducible closed subvarieties of codimension one on \( \overline{Y} \) and \( i_Z : \overline{Z} \to X \) is the composition of a resolution of singularities of \( Z \) with the natural map to \( X \).
Then the differential of $T_m(f_1, \ldots, f_m)$ is described as follows ([Gon05], Proposition 2.8):

**Proposition 5.16.** Let $f_1, \ldots, f_m \in \mathbb{C}(Y)$. Then, as differential forms on $\tilde{Y}$ with logarithmic singularities, we have

\begin{equation}

dT_m(f_1, \ldots, f_m) = \frac{(-1)^m}{2} \left( \frac{df_1}{f_1} \wedge \cdots \wedge \frac{df_m}{f_m} + (-1)^{m-1} \frac{df_1}{f_1} \wedge \cdots \wedge \frac{df_m}{f_m} \right),

\end{equation}

and

\begin{equation}

d\mathcal{D}T_m(f_1, \ldots, f_m) = 0.
\end{equation}

As currents on $X$ we have, for $m > 1$,

\begin{equation}

d[T_m](f_1, \ldots, f_m) = \frac{(-1)^m}{2} \iota_* \left[ \frac{df_1}{f_1} \wedge \cdots \wedge \frac{df_m}{f_m} + (-1)^{m-1} \frac{df_1}{f_1} \wedge \cdots \wedge \frac{df_m}{f_m} \right] + ([T_{m-1}] \circ \text{Res})(f_1, \ldots, f_m),
\end{equation}

for $m = 1$,

\begin{equation}

d[T_1](f_1) = -\frac{1}{2} \iota_* \left[ \frac{df_1}{f_1} + \frac{df_1}{f_1} \right],
\end{equation}

and

\begin{equation}

d\mathcal{D}[T_m](f_1, \ldots, f_m) = -([T_{m-1}] \circ \text{Res})(f_1, \ldots, f_m).
\end{equation}

**Proof.** This is proved in ([Gon05] §2). Equations (5.17) and (5.18) follow directly from Propositions 5.3 and 5.7, using that $\partial \bar{\partial} \log f \bar{f} = 0$ for any holomorphic function $f$. When $\text{div} f_1 \cup \cdots \cup \text{div} f_m$ is a normal crossing divisor, and these divisors do not have any common components, then equations (5.19) and (5.21) follow from the same propositions using (5.8). The general case can be reduced to this one by using resolution of singularities.

□

There are two main examples of the construction of this section that we are interested in. The first one is the original definition of Wang’s forms, that are tailored to the cubical setting.

Let $\mathbb{C}^* = \mathbb{P}^1 \setminus \{0, \infty\}$. If $(x_i : y_i)$ are projective coordinates on the $i$-th projective line in $(\mathbb{P}^1)^m$, let $f_i = y_i/x_i$ be the rational function on $(\mathbb{P}^1)^m$. Then, Wang’s forms defined in [Wan92] (see also [BW98]) are given by

\begin{equation}

W_m = T_m(y_1/x_1, \ldots, y_m/x_m) \in \mathcal{D}_{\log}^m((\mathbb{C}^*)^m, m).
\end{equation}

In particular $W_0 = 1 \in \mathcal{D}_{\log}^0(\text{Spec}(\mathbb{C}), 0)$. We denote by $[W_n] \in \mathcal{D}_{\mathcal{D}}^m((\mathbb{P}^1)^m, m)$ the associated current, which we call Wang’s current. In this case Proposition 5.16 leads

**Theorem 5.23.** For every $m \geq 1$, Wang’s currents satisfy the relation

\begin{equation}

d\mathcal{D}[W_m] = \sum_{i=1}^{m} \sum_{j=0,1} (-1)^{i+j}(\delta^i_j)_{*}[W_{m-1}],
\end{equation}

where the maps $\delta^i_j$ are the structural maps of the cubical structure of $(\mathbb{P}^1)^m$. 
Proof. For \( i = 1, \ldots, m \) and \( j = 0,1 \), let \( D^i_j \) be the divisor image of the structural map \( \delta^i_j \). Then the only non zero residues of \( \frac{y_1}{x_1} \wedge \cdots \wedge \frac{y_m}{x_m} \) are

\[
\Res_{D^i_j} \left( \frac{y_1}{x_1} \wedge \cdots \wedge \frac{y_m}{x_m} \right) = (-1)^{i+j+1} \left( \frac{y_1}{x_1} \wedge \cdots \wedge \frac{y_i}{x_i} \wedge \cdots \wedge \frac{y_m}{x_m} \right) \bigg|_{D^i_j}.
\]

Therefore, the result follows directly from Proposition 5.16. \( \square \)

Wang’s forms have another property that will be useful when establishing the convergence of certain integrals.

**Proposition 5.24.** Let \( D_m \) be the divisor \( D_m = \{ x_i = y_i \} \) introduced in \( \S 4 \). Let \( f : X \rightarrow (\mathbb{C}^*)^m \) be a holomorphic map that factors through \( D_m \cap (\mathbb{C}^*)^m \). Then

\[
f^* W_m = 0.
\]

Proof. This follows from the definition of \( W_m \), because \( D_m = \bigcup_{i=1}^m \{ x_i = y_i \} \) and \( \log(1) = 0 \). \( \square \)

The second example is the original definition of Goncharov, that is tailored to the simplicial setting. Recall that we have fixed homogeneous coordinates \( z_0, \ldots, z_n \) of \( \mathbb{P}^n \) and we denoted \( \Delta^n = \mathbb{P}^n \setminus H_n \), where \( H_n \) is the hyperplane defined by \( z_0 + \cdots + z_n = 0 \).

We denote

\[
G_m = T_m \left( \frac{z_1}{z_0}, \ldots, \frac{z_m}{z_0} \right)
\]

and let \([G_m]\) be the associated current (called Goncharov’s current).

In this case Proposition 5.16 leads

**Theorem 5.26.** Goncharov’s currents satisfy the relation

\[
d_P [G_m] = \sum_{i=0}^m (-1)^i (\partial^i)_* [G_{m-1}],
\]

where the maps \( \partial^i \) are the structural maps of the semi-simplicial structure of \( \mathbb{P}^m \).

Proof. Let \( D_i \subset \mathbb{P}^m \) denote the divisor of equation \( z_i = 0 \). The result follows directly from the relations

\[
\Res_{D_0} \left( \frac{z_1}{z_0} \wedge \cdots \wedge \frac{z_m}{z_0} \right) = \Res_{D_0} \left( \frac{z_1}{z_0} \wedge \frac{z_2}{z_1} \wedge \cdots \wedge \frac{z_m}{z_1} \right) = - \left( \frac{z_2}{z_1} \wedge \cdots \wedge \frac{z_m}{z_1} \right) \bigg|_{D_0}
\]

and

\[
\Res_{D_i} \left( \frac{z_1}{z_0} \wedge \cdots \wedge \frac{z_m}{z_0} \right) = (-1)^{i-1} \left( \frac{z_1}{z_0} \wedge \cdots \wedge \frac{z_i}{z_0} \wedge \cdots \wedge \frac{z_m}{z_0} \right) \bigg|_{D_i}.
\]

\( \square \)

6. Algebraic cycles and the Beilinson regulator

In this section we compare the regulator defined by Goncharov to its cubical version, and show that the cubical version agrees with the Beilinson regulator, by comparing it to the construction given by Burgos and Feliu in [BF09].

Throughout this section, \( X \) will be an equidimensional compact complex algebraic manifold.
6.1. Goncharov’s conjecture. Let $D^p_D(X, p)$ be the Deligne complex of currents of $\mathbb{P}^1$. We denote by $\tau D^p_D(X, p) := \tau_{\leq p}D^p_D(X, p)$ the truncated complex.

For each integer $m \geq 0$, let $\pi_X : X \times \Delta^m \to X$ be the projection onto the first factor. To simplify the notation we will use the same symbol for these morphisms. In each case it will be clear which one is used. Let $z_0, \ldots, z_m$ be projective coordinates of $\Delta^m = \mathbb{P}^m \setminus H_m$. For a closed integral subscheme $Z \subset X \times \Delta^m$ of codimension $p$ which intersects properly with each face of $X \times \Delta^m$, let $\mathcal{Z}$ be the Zariski closure of $Z$ on $X \times \mathbb{P}^m$ and let $\iota : \mathcal{Z} \to X \times \mathbb{P}^m$ be the composition of a resolution of singularities of $\mathcal{Z}$ with the inclusion of $\mathcal{Z}$ in $X \times \mathbb{P}^m$. We define the current $\mathcal{P}_s^p(m)(Z) \in \tau D^{2p-m}_D(X, p)$ by

$$\mathcal{P}_s^p(m)(Z) = (\pi_X)_*\iota_* \left[ T_m \left( \frac{z_1}{z_0}, \ldots, \frac{z_m}{z_0} \mid \mathcal{Z} \right) \right] = (\pi_X)_* (\delta_Z \wedge G_m),$$

where $\delta_Z$ is the current integration along $Z$. Let

$$\mathcal{P}_s : Z^p_s(X, m) \to \tau D^{2p-m}_D(X, p)$$

be defined by $\mathcal{P}_s(Z) = \mathcal{P}_s^p(m)(Z)$ if $Z$ is as above, and extended to cycles $z \in Z^p_s(X, *)$ by linearity. Observe that if $m = 0$, $\mathcal{P}_s(z) = \delta_z$ is a closed current and therefore it belongs to the truncated complex. Remember that we are including the twist in the definition of the current associated to a differential form and the definition of the current associated to an algebraic cycle.

Theorem 5.26 implies that $d_\tau (\mathcal{P}_s^p(m)(z)) = \mathcal{P}_s^p(m-1)(\partial z)$ (see also [Gon05], Theorem 2.12). Hence, we have the following result.

Lemma 6.1. The morphism $\mathcal{P}_s$ is a chain morphism.

Goncharov has presented the following conjecture:

Conjecture 6.2. Let $X$ be an equidimensional compact complex algebraic manifold. The composition

$$K_n(X)_Q \xrightarrow{\approx} \bigoplus_{p \geq 0} CH^p_s(X, n)_Q \xrightarrow{\mathcal{P}_s} H^{2p-n}_D(X, p)$$

agrees with Beilinson’s regulator.

6.2. Cubical construction. We introduce here the cubical version of Goncharov’s regulator using Wang’s forms.

Let $\pi_X : X \times \Box^m \to X$ be here the projection onto the first factor. Let $(x_i : y_i)$ be homogeneous coordinates on the $i$-th factor of $(\mathbb{P}^1)^m$. For a closed integral subscheme $Z \subset X \times \Box^m$ of codimension $p$ which intersects properly with each face of $X \times \Box^m$, let $\mathcal{Z}$ be the Zariski closure of $Z$ on $X \times (\mathbb{P}^1)^m$ and let $\iota : \mathcal{Z} \to X \times (\mathbb{P}^1)^m$ be the composition of a resolution of singularities of $\mathcal{Z}$ with the inclusion of $\mathcal{Z}$ in $X \times (\mathbb{P}^1)^m$. We define the current $W^p(m)(Z) \in \tau D^{2p-m}_D(X, p)$ by

$$W^p(m)(Z) = (\pi_X)_*\iota_* \left[ T_m \left( \frac{y_1}{x_1}, \ldots, \frac{y_m}{x_m} \mid \mathcal{Z} \right) \right] = (\pi_X)_* (\delta_Z \wedge W_m).$$

This gives a map

$$Z^p_s(X, m)_0 \xrightarrow{\mathcal{P}_s} \tau D^{2p-m}_D(X, p)$$

(6.3)
defined by $\mathcal{P}_c(Z) = W^p(m)(Z)$, if $Z$ is as above, and extended to cycles $z \in Z^p_c(X, \ast)_0$ by linearity.

**Lemma 6.4.** The morphism $\mathcal{P}_c$ is a chain morphism.

**Proof.** This follows from Theorem [5.23]. □

We denote also by

$$\mathcal{P}_c : CH^p_c(X, n) \longrightarrow H^{2p-n}_D(X, \mathbb{R}(p))$$

the induced morphism.

### 6.3. Comparison of $\mathcal{P}_c$ and $\rho$.

We will now compare the map $\mathcal{P}_c$ with Beilinson’s regulator.

**Theorem 6.5.**

1. Given any differential form $\alpha \in D^{r-m}_A(X, p)$, the form $\alpha \cdot W_m$ is locally integrable as a singular form on $X \times (\mathbb{P}^1)^m$. Hence it defines a current $[\alpha \cdot W_m] \in D^{r-m}_D(X \times (\mathbb{P}^1)^m, p + m)$. Moreover

$$d_D[\alpha \cdot W_m] = [d_D\alpha \cdot W_m] + (-1)^r \sum_{j=0,1} \sum_{i=1}^m (-1)^{i+j}(\delta_j^i)^*(\alpha \cdot W_{m-1})$$

2. The assignment $\alpha \mapsto (\pi_X)_*[\alpha \cdot W_m]$ defines a morphism of complexes

$$D^{*}_A(X, p) \xrightarrow{\varphi} \tau D^*_D(X, p).$$

Hence, by composition, a morphism of complexes

$$D^{*}_A(X, p)_0 \xrightarrow{\varphi} \tau D^*_D(X, p).$$

3. If we identify $\tau D^*_D(X, p)$ with a subcomplex of $\tau D^*_D(X, p)$ via the morphism $[2,3]$, then the image of $\varphi$ is contained in $\tau D^*_D(X, p)$. By abuse of notation we will also denote by $\varphi$ the induced morphism

$$D^{*}_A(X, p)_0 \xrightarrow{\varphi} \tau D^*_D(X, p).$$

4. The morphism $\varphi$ is a quasi-inverse of the quasi-isomorphism given in Proposition [4.3] □

**Proof.** Statements [1] and [3] follow easily from Proposition [5.24] and the definition of $D^{*}_A(X, p)_0$, by using the techniques of [Bur94b] §3. We next prove that $\varphi$ is a morphism of complexes. Let $\alpha \in D^{r-m}_A(X, p)$. Then

$$\varphi(d\alpha) = \varphi(d_D\alpha) + (-1)^r \sum_{j=0,1} \sum_{i=1}^m (-1)^{i+j} \varphi((\delta_j^i)^*\alpha)$$

$$= (\pi_X)_*[d_D\alpha \cdot W_m] + (-1)^r \sum_{j=0,1} \sum_{i=1}^m (-1)^{i+j}(\pi_X)_*([((\delta_j^i)^*\alpha \cdot W_{m-1})])$$

$$= (\pi_X)_*(d_D([\alpha \cdot W_m]))$$

$$= d_D\varphi(\alpha).$$

For statement [1] if we denote by $\psi$ the quasi-isomorphism of Proposition [4.3] □ then, by definition, the composition $\varphi \circ \psi$ is the identity. □
Let \((\omega, g) \in D^{2p-m}_{\mathcal{A}, \mathcal{Z}}(X, p)_0\). Since we have defined \(D^\ast_{\mathcal{A}, \mathcal{Z}}(X, p)_0\) using truncated complexes, the pair \((\omega, g)\) is closed. Moreover, by the purity property of Deligne cohomology, there exists a cycle \(z \in Z^p(X, m)_0 \otimes \mathbb{R}\) such that \(\langle \omega, g \rangle = c(z)\). Let \(\delta_z\) be the associated current. Since the set where \(g\) is singular has codimension \(p\), \[\text{Bur94b}\] Corollary 3.8 implies that \(g\) is locally integrable on \(X \times \Box^m\). Then, \(\omega\) and \(g\) determine currents on the Deligne complex

\[D^\ast(D_X \times (\mathbb{P}^1)^m/D^m(X \times (\mathbb{P}^1)^m), p),\]

where \(D_X \times (\mathbb{P}^1)^m/D^m\) is the sheaf of currents defined, for instance in \[\text{Bur94b}\] after Definition 5.43. Moreover, by adapting the proof of \[\text{Bur94b}\] Theorem 4.4, to the above complex, one can prove that they satisfy the equation of currents

\[(6.6) \quad d_D[g] + \delta_z = [\omega].\]

Using again the techniques of the proof of \[\text{Bur94b}\] Theorem 4.4 one obtains

**Proposition 6.7.** Let \((\omega, g) \in D^{2p-m}_{\mathcal{A}, \mathcal{Z}}(X, p)_0\). Then, the differential form \(g \cdot W_m\) is locally integrable as a form on \(X \times (\mathbb{P}^1)^m\). Moreover,

\[d_D[g \cdot W_m] = [\omega \cdot W_m] - \delta_z \cdot W_m - [\delta g \cdot W_{m-1}].\]

Let now \((\omega, g) \in D^r_{\mathcal{A}, \mathcal{Z}}(X, p)_0\), with \(r < 2p\). Again, since the set where \(g\) is singular has codimension \(p\), \[\text{Bur94b}\] Corollary 3.8 implies that \(g\) is locally integrable on \(X \times \Box^m\). Then, \(\omega\) and \(g\) determine currents on the Deligne complex

\[D^\ast(D_X \times (\mathbb{P}^1)^m/D^m(X \times (\mathbb{P}^1)^m), p),\]

Moreover they satisfy the equations of currents

\[(6.8) \quad d_D[g] = [d_Dg], \quad d_D[\omega] = [d_D\omega].\]

and we have

**Proposition 6.9.** Let \((\omega, g) \in D^r_{\mathcal{A}, \mathcal{Z}}(X, p)_0\), with \(r < 2p\). Then, the differential form \(g \cdot W_m\) is locally integrable as a form on \(X \times (\mathbb{P}^1)^m\). Moreover,

\[d_D[g \cdot W_m] = [d_Dg \cdot W_m] + (-1)^{r-1}[\delta g \cdot W_{m-1}].\]

Let \(D^{2p-*}_{\mathcal{A}, \mathcal{H}}(X, p)_0\) be the complex \[\text{H.3}\]. Then the central result of this subsection is the following.

**Theorem 6.10.** The map

\[\psi : D^{2p-*}_{\mathcal{A}, \mathcal{H}}(X, p)_0 \longrightarrow \tau D^{2p-*}_D(X, p)\]

given by

\[\psi(z, (\omega, g), \alpha) = P(z) - (\pi_X)_*[g \cdot W_m] + \varphi(\alpha),\]
Theorem 6.11. For all $\omega, g \in D^{r-m}_{\mathcal{H}, \mathcal{I}}(X, p)_0$, is a morphism of complexes. Moreover, there is a commutative diagram

$$\begin{array}{ccc}
Z_c^p(X, *)_0 & \xrightarrow{P_c} & \tau D^{2p-*}D(X, p) \\
\downarrow^\rho & & \downarrow^\sim \\
D^{2p-*}_{\mathcal{H}}(X, p)_0 & \xrightarrow{\psi} & \tau D^{2p-*}D(X, p) \\
\sim & & \sim \\
D^{2p-*}_{\mathcal{H}}(X, p)_0 & \xrightarrow{\varphi} & \tau D^{2p-*}D(X, p)
\end{array}$$

Proof. If $z \in \mathcal{H}^p(X, *)_0$, then Lemma [6.4] implies that $d\psi((z, 0, 0)) = \psi(d(z, 0, 0))$. If $\alpha \in D^{2p-*}D(X, p)_0$, then Theorem [6.5] implies that $d\psi((0, 0, \alpha)) = \psi(d(0, 0, \alpha))$.

Let $(\omega, g) \in D^{2p-*}_{\mathcal{H}}(X, p)_0$. Let $z \in Z^p(X, m)_0 \otimes \mathbb{R}$ such that $[(\omega, g)] = cl(z)$, and let $\delta_z$ be the associated current. Then, using Proposition [6.7], we have

$$d\psi((0, (\omega, g), 0)) = -d_D(\pi_X)_*[g \cdot W_m]$$

$$= (\pi_X)_*[\delta_z \cdot W_m] + (\pi_X)_*[\delta g \cdot W_{m-1}] - (\pi_X)_*[\omega \cdot W_m],$$

and

$$\psi(d(0, (\omega, g), 0)) = \psi(((\omega, g)), -(\delta \omega, \delta g), -\omega)$$

$$= (\pi_X)_*[\delta_z \cdot W_m] + (\pi_X)_*[\delta g \cdot W_{m-1}] - (\pi_X)_*[\omega \cdot W_m].$$

If $(\omega, g) \in D^{r-m}_{\mathcal{H}, \mathcal{I}}(X, p)_0$, with $r < 2p$, using Proposition [6.9], we have

$$d\psi((0, (\omega, g), 0)) = -d_D(\pi_X)_*[g \cdot W_m]$$

$$= -(\pi_X)_*[d_D g \cdot W_m] + (-1)^r(\pi_X)_*[\delta g \cdot W_{m-1}],$$

and

$$\psi(d(0, (\omega, g), 0)) = \psi((0, -d_D \omega, -\omega + d_D g) - (-1)^r(\delta \omega, \delta g), -\omega)$$

$$= -(\pi_X)_*[d_D g \cdot W_m] + (-1)^r(\pi_X)_*[\delta g \cdot W_{m-1}].$$

The fact that the diagram is commutative follows directly from the definition of the maps involved.

As an immediate consequence of Theorem [6.10] we have

**Theorem 6.11.** For all $n, p \geq 0$ the morphisms

$$P_c, P : CH^p_c(X, n) \rightarrow H^{2p-*}D(X, \mathbb{R}(p))$$

agree.

6.4. Proof of the conjecture. At this point, we have seen that the cubical version of Goncharov’s construction agrees with the regulator defined by Burgos-Feliu. In this section we prove the Goncharov’s conjecture [6.2] by showing that the cubical and the simplicial constructions agree. This will be done through an intermediate complex consisting of both simplicial and cubical affine schemes.

Let $X$ be an equidimensional quasi-projective algebraic scheme of dimension $d$ over the field $k$. By a face of $X \times \square^n \times \Delta^m$ we understand any subset of the form $X \times F \times G$, where $F$ is a face of $\square^n$ and $G$ is a face of $\Delta^m$. 

□
Let \( Z^{p}_{cs}(X, n, m) \) be the free abelian group generated by the codimension \( p \) closed irreducible subvarieties of \( X \times \Delta^{n} \times \Delta^{m} \), which intersect properly all the faces of \( X \times \Delta^{n} \times \Delta^{m} \). Then \( Z^{p}_{cs}(X, n, m) \) has a simplicial structure with faces \( \partial_{i} \) and a cubical structure with faces \( \partial_{i}^{1} \). Let \( Z^{p}_{cs}(X, n, m) \) be the subgroup of \( Z^{p}_{cs}(X, n, m) \) consisting of those elements that lie in the kernel of \( \partial_{i}^{1} \) for all \( i = 1, \ldots, n \).

Consider the 2-iterated chain complex \( Z^{p}_{cs}(X, *, *)_{0} \) whose piece of degree \( (n, m) \) is \( Z^{p}_{cs}(X, n, m)_{0} \) and whose differentials are \( (\delta, \partial) \). Let \( Z^{p}_{cs}(X, *)_{0} \) denote the simple complex associated to \( Z^{p}_{cs}(X, *, *)_{0} \).

**Proposition 6.12** ([Lev94], Theorem 4.7). The natural morphisms

\[
Z^{p}_{c}(X, *) \xrightarrow{i_{s}} Z^{p}_{cs}(X, *)_{0}, \quad Z^{p}_{c}(X, *)_{0} \xrightarrow{i_{c}} Z^{p}_{cs}(X, *)_{0},
\]

are both quasi-isomorphisms.

This result is the key to show that the higher Chow groups defined using the cubical or the simplicial version agree. Moreover, it follows that the higher algebraic Chow groups can also be computed in terms of the complex \( Z^{p}_{cs}(X, *, *)_{0} \), that is \( CH^{p}(X, n) \cong H_{n}(Z^{p}_{cs}(X, *, *))_{0} \). As usual, we denote

\[
CH^{p}_{cs}(X, n) = H_{n}(Z^{p}_{cs}(X, *, *))_{0}.
\]

Assume that \( X \) is an equidimensional projective complex algebraic manifold. The strategy pursued to prove the conjecture is the following. We will construct a regulator map

\[
CH^{p}_{cs}(X, n) \xrightarrow{\mathcal{P}_{cs}} H^{2p-n}_{D}(X, \mathbb{R}(p))
\]

and show that there is a big commutative square

\[
\begin{array}{ccc}
CH^{p}_{cs}(X, n) & \xrightarrow{\mathcal{P}_{c}} & H^{2p-n}_{D}(X, \mathbb{R}(p)) \\
\leftarrow & \cong & \leftarrow \\
CH^{p}_{c}(X, n) & \xrightarrow{\mathcal{P}_{s}} & H^{2p-n}_{D}(X, \mathbb{R}(p)) \\
\end{array}
\]

Then, since \( \mathcal{P}_{c} \) is Beilinson’s regulator, it will follow that so are \( \mathcal{P}_{s} \) and \( \mathcal{P}_{cs} \).

**The morphism \( \mathcal{P}_{cs} \).** Recall again the projective coordinates \((x_{i} : y_{i})\) on the \( i\)-th projective line in \( \Delta^{n} \subset \mathbb{P}^{1} \) and the homogeneous coordinates \((z_{0} : \cdots : z_{m})\) on \( \Delta^{m} \subset \mathbb{P}^{m} \). We denote

\[
M_{n,m} = T_{n+m} \left( \frac{y_{1}}{x_{1}}, \ldots, \frac{y_{n}}{x_{n}}, \frac{z_{1}}{z_{0}}, \ldots, \frac{z_{m}}{z_{0}} \right),
\]

which is a differential form on \( \Delta^{n} \times \Delta^{m} \) with logarithmic singularities along \( \partial \Delta^{n} \times \Delta^{m} \) and \( \Delta^{n} \times \partial \Delta^{m} \). In particular, \( M_{n,0} = W_{n} \) and \( M_{0,m} = G_{m} \). Consider the group morphism

\[
Z^{p}_{cs}(X, n, m)_{0} \xrightarrow{\mathcal{P}_{cs}} \tau D^{2p-m}_{D}(X, p)
\]

if \( Z \) is an irreducible codimension \( p \) subvariety of \( X \times \Delta^{n} \times \Delta^{m} \) intersecting properly the faces of \( X \times \Delta^{n} \times \Delta^{m} \), analogous to the definition of \( \mathcal{P}_{c}, \mathcal{P}_{s} \).
Lemma 6.16. The morphism
\[ Z_{cs}^p(X,*)_0 \xrightarrow{P_{cs}} \tau D_{D}^{2p-*}(X,p) \]
is a chain morphism.

Proof. Denote by \([M_{n,m}]\) the current on \((\mathbb{P}^1)^n \times \mathbb{P}^m\) associated to \(M_{n,m}\). By Proposition 5.16, one can prove the equation of currents
\[ d_{D}[M_{n,m}] = \sum_{i=1}^{n} \sum_{j=0,1} (-1)^{i+j}(\delta_j \times 1)_{*}[M_{n-1,m}] + (-1)^{n} \sum_{i=0}^{m} (-1)^{i}(1 \times \partial)_{*}[M_{n,m-1}]. \]
The result follows easily from this relation. □

The proof of next lemma is straightforward.

Lemma 6.17. The diagrams
\[ \begin{array}{ccc}
Z^p_{cs}(X,*_0) & \xrightarrow{i_*} & \tau D_{D}^{2p-*}(X,p) \\
& & \\
& & \\
& & \\
Z^p_{cs}(X,*_0) & \xrightarrow{i_*} & \tau D_{D}^{2p-*}(X,p)
\end{array} \]
are commutative.

Theorem 6.18. Let \(X\) be an equidimensional projective complex algebraic manifold. Let \(P'_s\) be the composition of \(P_s\) with the isomorphism given by the Chern character of [Blo86]
\[ P'_s : K_n(X)_{\mathbb{Q}} \xrightarrow{\cong} \bigoplus_{p \geq 0} CH^p(X,n)_{\mathbb{Q}} \xrightarrow{P_s} \bigoplus_{p \geq 0} H^{2p-*}_{D}(X,\mathbb{R}(p)). \]

Then, the morphism \(P'_s\) agrees with Beilinson’s regulator.

Proof. From Lemma 6.17 we see that there is a commutative diagram (6.13). Then, the statement follows from Theorem 4.7 together with Theorem 6.11 □

7. Higher arithmetic Chow groups

In this last section, we use the comparison of regulators performed in the previous sections to show that the higher arithmetic Chow groups given by Goncharov agree with the ones given by Burgos and Feliu, for all proper arithmetic varieties.

Following [GS90], by an arithmetic variety \(X\) over a ring \(A\) we mean a regular scheme \(X\) which is flat and quasi-projective over an arithmetic ring \(A\).

Assume that \(X\) is a smooth proper variety defined over an arithmetic field \(F\). Then we obtain the associated real variety \(X_{\mathbb{R}} = (X_{\mathbb{C}}, F_{\infty})\), that is, a projective complex algebraic manifold \(X_{\mathbb{C}}\) equipped with an anti-holomorphic involution \(F_{\infty}\). We write
\[ D^*(X,p) = D^*_D(X_{\mathbb{C}},p)^{F_{\infty}=\text{Id}}, \quad D^*_D(X,p) = D^*_D(X_{\mathbb{C}},p)^{F_{\infty}=\text{Id}}. \]

Then we have the regulator map for \(X\):
\[ P_s : Z^p_s(X,n) \rightarrow \tau D_{D}^{2p-*}(X,p). \]

Let \(ZD^{2p}(X,p)\) be the space of cycles of degree \(2p\) in the Deligne complex of smooth differential forms on \(X\), considered as a chain complex concentrated in degree zero.
Definition 7.2 (Goncharov). Let
\[ \hat{Z}^p_G(X,*) = s \left( \begin{array}{ccc}
\tau D^{2p-*}_D(X,p) & [1] \\
Z^p_r(X,*) & ZD^{2p}(X,p)
\end{array} \right) \]
be the simple of the diagram. The higher arithmetic Chow groups of $X$ are defined to be the homology groups of this complex:
\[ \hat{CH}^p_G(X,n) = H_n(\hat{Z}^p_G(X,*)). \]
We define
\[ D^*_\mathcal{H}(X,p)_0 := D^*_\mathcal{H}(X,p)_0 F^*_\mathcal{H} = \text{id}. \]
We denote by $\beta : ZD^{2p}(X,p) \rightarrow D^{2p-*}(X,p)_0$ the map given by $\beta(\alpha) = (0,0,\alpha)$.

Definition 7.3 (Burgos-Feliu). Let
\[ \hat{Z}^p_{BF}(X,*) = s \left( \begin{array}{ccc}
D^{2p-*}_{\mathcal{H}}(X,p)_0 \\
Z^p_r(X,*)_0 & ZD^{2p}(X,p)
\end{array} \right) \]
be the simple of the diagram. The higher arithmetic Chow groups of $X$ as defined in [BF09] are the homology groups of this complex:
\[ \hat{CH}^p_{BF}(X,n) = H_n(\hat{Z}^p_{BF}(X,*)). \]

Theorem 7.4. For all $n,p \geq 0$ there are natural isomorphisms
\[ \hat{CH}^p_G(X,n) \rightarrow \hat{CH}^p_{BF}(X,n). \]

Proof. Let $\hat{CH}^p_c(X,n)$ and $\hat{CH}^p_{cs}(X,n)$ be the analogues of $\hat{CH}^p_G(X,n)$ defined using the cubical and the cubical-simplicial setting given by the morphisms $P_c$ and $P_{cs}$. It follows from Lemma 6.17 and Proposition 6.12 that there are natural isomorphisms
\[ \hat{CH}^p_G(X,n) \cong \hat{CH}^p_{cs}(X,n) \cong \hat{CH}^p_c(X,n). \]
By the commutative diagram of Theorem 6.10 there are natural isomorphisms
\[ \hat{CH}^p_{BF}(X,n) \cong \hat{CH}^p_c(X,n), \]
and the theorem is proved. \qed

As a consequence, we can transfer properties from one definition of higher arithmetic Chow groups to the other. In particular we obtain the following result.

Corollary 7.5. Let $X$ be a projective arithmetic variety over an arithmetic field and let $\hat{CH}^p_G(X,n)$ denote the higher arithmetic Chow groups defined by Goncharov.

- (Pull-back): Let $f : X \rightarrow Y$ be a morphism between two projective arithmetic varieties over a field. Then, there is a pull-back morphism
\[ \hat{CH}^p_G(Y,n) \rightarrow \hat{CH}^p_G(X,n), \]
for every $p$ and $n$, compatible with the pull-back maps on the groups $CH^p(X,n)$ and $H^p_D(X,\mathbb{R}(p))$. 
• (Product): There exists a product on
\[ \widehat{CH}_G^p(X, *) := \bigoplus_{p \geq 0, n \geq 0} \widehat{CH}_G^p(X, n), \]
which is associative, graded commutative with respect to the degree \( n \).

Proof. It follows from Theorem 7.4 together with the results in [BF09], where these properties are shown for \( \widehat{CH}^{BF}_G(X, n) \).

\[ \square \]

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