Towards a renormalization theory for quasi-periodically forced one dimensional maps II. Asymptotic behavior of reducibility loss bifurcations

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Abstract

In this paper we are concerned with quasi-periodic forced one dimensional maps. We consider a two parametric family of quasi-periodically forced maps such that the one dimensional map (before forcing) is unimodal and it has a full cascade of period doubling bifurcations. Between one period doubling and the next one it is known that there exist a parameter value where the $2^n$-periodic orbit is superattracting. In a previous work we proposed an extension of the one-dimensional (doubling) renormalization operator to the quasi-periodic case. We proved that, if the family satisfies suitable hypotheses, the two parameter family has two curves of reducibility loss bifurcation around these parameter values. In the present work we study the asymptotic behavior of these bifurcations when $n$ grows to infinity. We show that the asymptotic behavior depends on the Fourier expansion of the quasi-periodic coupling of the family. The theory developed here provides a theoretical explanation to the behavior that can be observed numerically.

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1 Introduction

This is the second of a series of papers (together with [15, 16]) where we propose an extension of the one dimensional renormalization theory for the case of quasi-periodic forced maps. Each of these papers is self contained, but highly interrelated with the others. An more detailed exposition can be found in [13]. In [15] we give the definition of the operator for the case of quasi-periodic maps and we use it to prove the existence of reducibility loss bifurcations when the coupling parameter goes to zero. In this paper we use the results obtained there to study the asymptotic behavior of these bifurcations when the period of the attracting set goes to infinity. Our quasi-periodic extension of the renormalization operator is not complete in the sense that several conjectures must be assumed. In [16] we include the numerical evidence which support our conjectures and we show that the theoretical results agree with the behavior observed numerically. In [16] we also include a numerical study of the asymptotic behavior of the reducibility loss bifurcations which will be summarized in the forthcoming section 1.1.

The classic one dimensional renormalization theory provides an explanation to the behavior observed in the cascades of period doubling bifurcations. Concretely, given a typical one parametric family for unimodal maps \( \{f_\alpha\}_{\alpha \in I} \) one observes numerically that there exists a sequence of parameter values \( \{d_n\}_{n \in \mathbb{N}} \subset I \) such that, the attracting periodic orbit of the map undergoes a period doubling bifurcation. Between one period doubling and the next one there exists also a parameter value \( s_n \), for which the critical point of \( f_{s_n} \) is a periodic orbit with period \( 2^n \). One can also observe that

\[
\lim_{n \to \infty} \frac{d_n - d_{n-1}}{d_{n+1} - d_n} = \lim_{n \to \infty} \frac{s_n - s_{n-1}}{s_{n+1} - s_n} = \delta = 4.66920\ldots
\]

This reveals two important phenomena. The first one is the self-renormalizable structure of the bifurcation diagram. Since the limit converges, it indicates that there exists a scale factor of \( \delta \) between one bifurcation and the next. The second one is the universality, in the sense that the limit \( \delta \) does not depend on the family considered.

Renormalization theory provides a theoretical explanation to this phenomenon. The literature on this topic is quite extensive, some remarkable works are [3, 4, 8, 17, 10, 2], we also refer the reader to the books [12, 11] and references therein.

In this paper we are interested in the analog of renormalization and universality problem for the case of quasi-periodic forced one dimensional maps. In [14] we have given numerical evidences of self-similarity of the bifurcation diagram and universality. These numerical evidences are described in section 1.1 below. In this paper we provide a theoretical explanation to the behavior observed numerically.
1.1 Numerical observations on renormalization and universality for quasi-periodically forced maps

Consider \( \{g_{\alpha, \varepsilon}\}_{(\alpha, \varepsilon) \in J \subset \mathbb{R}^2} \) a two parametric family of quasi-periodic maps in the cylinder \( T \times \mathbb{R} \), such that it has the form

\[
\begin{align*}
\bar{\theta} &= \theta + \omega, \\
\bar{x} &= f_\alpha(x) + \varepsilon h_{\alpha, \varepsilon}(\theta, x),
\end{align*}
\]

with \( \omega \) a Diophantine number, \( \alpha \) and \( \varepsilon \) parameters, \( h \) a periodic with respect \( \theta \) and \( \{f_\alpha\}_{\alpha \in J} \) a family of one dimensional maps having a complete cascade of period doubling bifurcations as the family described before. As before, let \( \{d_n\}_{n \in \mathbb{N}} \subset I \) be the parameter values where the attracting periodic orbit of the map undergoes a period doubling bifurcation and \( \{s_n\}_{n \in \mathbb{N}} \subset I \) the values for which the critical point of \( f_{s_n} \) is a periodic orbit with period \( 2^n \). The paradigmatic example for this type of maps is the Forced Logistic Map (FLM for short), where the uncoupled one dimensional family is the logistic map, \( f_\alpha(x) = \alpha x(1 - x) \) with \( \alpha \in [0, 4] \). Nevertheless the results that we obtain are applicable to a wider class of maps.

In [5] we computed some bifurcation diagrams in terms of the dynamics of the attracting set. We have taken into account different properties of the attracting set, as the Lyapunov exponent and, in the case of having a periodic invariant curve, its period and reducibility. The reducibility loss of an invariant curve is not a bifurcation in the classical sense, it is only a change in the spectral properties of the transfer operator associated to the curve (see [6]). Despite of this, it can be characterized as a bifurcation (see definition 2.3 in [5]) and it will be considered as such for the rest of this paper. The numerical computations in the cited work reveal that the parameter values for which the invariant curve doubles its period are contained in regions of the parameter space where the invariant curve is reducible, as sketched in figure 1. Taking into account the properties of universality and self renormalization of the Logistic Map, one might look for similar phenomena in the bifurcation diagram of the FLM.

Let \( s_n \) be the parameter value where the critical point of the uncoupled family \( \{f_\alpha\}_{\alpha \in I} \) is periodic with period \( 2^n \). Numerical computations (see [5]) revealed that from every parameter value
\((\alpha, \varepsilon) = (s_n, 0)\) two curves are born. These curves correspond to reducibility-loss bifurcations of the \(2^n\)-periodic invariant curve. In \[15\] we proved that these curves really exist under suitable hypotheses. Assume that these two curves can be locally expressed as \((s_n + \alpha'_n \varepsilon + O(\varepsilon^2), \varepsilon)\) and \((s_n + \beta'_n \varepsilon + O(\varepsilon^2), \varepsilon)\). Numerical experiments in \[5, 16\] show that the slopes depend on \(\omega\), i.e. \(\alpha'_n = \alpha'_n(\omega)\) and \(\beta'_n = \beta'_n(\omega)\), and also show that \(\beta'_n(\omega) = -\alpha'_n(\omega)\) for the examples studied numerically. In \[15\] we give explicit expressions of these slopes in terms of the quasi-periodic forced renormalization operator, for both \(\alpha'_n(\omega)\) and \(\beta'_n(\omega)\). In this paper we focus only on \(\alpha'_n(\omega)\), but the discussion for \(\beta'_n(\omega)\) is completely analogous.

The slopes \(\alpha'_n(\omega)\) can be used for the numerical detection of universality and self-renormalization phenomena. If the bifurcation diagram is self renormalizable one should have that \(\alpha'_n(\omega)/\alpha'_{n-1}(\omega)\) converges to a constant. In general, this is not true due to the fact that when the period is doubled, the rotation number of the system also is. Then one should look for renormalization properties between the bifurcation diagram of the family for rotation number \(\omega\) and the bifurcation diagram of the same family for rotation number \(2\omega\). This is sketched in figure \[1\]. In \[14\] we do a numerical study for the case of the Forced Logistic Map and some modifications of it. Concretely we consider the family of maps in the cylinder \(T \times \mathbb{R}\) defined by:

\[
\begin{align*}
\bar{\theta} &= \theta + \omega, \\
\bar{x} &= \alpha x(1-x) + \varepsilon g(\theta, x),
\end{align*}
\]  

with \(\omega\) a Diophantine number.

In \[14\] we did the following discoveries.

- **First numerical observation:** the sequence \(\alpha'_n(\omega)/\alpha'_{n-1}(\omega)\) is not convergent in \(n\). But, for \(\omega\) fix, one obtains the same sequence for any family of quasi-periodic forced maps, with a quasi-periodic forcing of the type \(g(\theta, x) = f_1(x) \cos(\theta) + f_2(x) \sin(\theta)\).

- **Second numerical observation:** the sequence \(\alpha'_n(\omega)/\alpha'_{n-1}(2\omega)\) is convergent in \(n\) when the quasi-periodic forcing of the type \(g(\theta, x) = f_1(x) \cos(\theta) + f_2(x) \sin(\theta)\). The limit depends on \(\omega\) and on the particular family considered.

- **Third numerical observation:** the two previous observations are not true when the quasi-periodic forcing is of the type \(g_\eta(\theta, x) = f_1(x) \cos(\theta) + \eta f_2(x) \cos(2\theta)\) when \(\eta \neq 0\). But the sequence \(\alpha'_n(\omega)/\alpha'_{n-1}(2\omega)\) associated to the map \(3\) with \(g = g_\eta\) is \(\eta\)-close to the same maps with \(g = g_0\).

In this paper we give a theoretical explanation in terms of the dynamics of the quasi-periodic renormalization operator. In section \[2\] we review the concepts and results from \[15\] that are necessary for this. In section \[3.2\] we review the study of the asymptotic behavior of the sequences \(\alpha'_n(\omega)/\alpha'_{n-1}(\omega)\) to the dynamics of the quasi-periodically forced renormalization operator. In sections \[3.3\], \[3.4\] and \[3.5\] we give a theoretical explanation to each of the three numerical observations described above.
2 Review on quasi-periodic renormalization

Here we summarize the ideas and results developed in [15] which are essential for the discussion. Consider a quasi-periodic forced map like

\[ F : \mathbb{T} \times I \rightarrow \mathbb{T} \times I \]

\[ (\theta, x) \mapsto (\theta + \omega, f(\theta, x)), \]

(4)

with \( f \in C^r(\mathbb{T} \times I, I) \). To define the renormalization operator it is only necessary that \( r \geq 1 \).

For simplicity the exposition done here is restricted to the analytic case. Along section 2.1 it is not necessary to require \( \omega \) to be Diophantine, but it will be necessary in section 2.2.

Note that the map \( F \) is completely determined by the couple \((\omega, f)\). From now on we consider \( \omega \) fixed and we focus only on the function \( f \). The definition of the operator is done in a perturbative way, in the sense that it is only applicable to maps \( f(\theta, x) = g(x) + h(\theta, x) \) with \( g \) renormalizable in the one dimensional case and \( h \) small.

2.1 Definition of the operator and basic properties

Preliminary notation

Let \( \mathcal{W} \) be an open set in the complex plane containing the interval \( I_\delta = [-1 - \delta, 1 + \delta] \) and let \( \mathbb{B}_\rho = \{z = x + iy \in \mathbb{C} \text{ such that } |y| < \rho\} \). Then consider \( \mathcal{B} = \mathcal{B}(\mathbb{B}_\rho, \mathcal{W}) \) the space of functions \( f : \mathbb{B}_\rho \times \mathcal{W} \rightarrow \mathbb{C} \) such that:

1. \( f \) is holomorphic in \( \mathbb{B}_\rho \times \mathcal{W} \) and continuous in the closure of \( \mathbb{B}_\rho \times \mathcal{W} \).
2. \( f \) is real analytic.
3. \( f \) is 1-periodic in the first variable, i. e. \( f(\theta + 1, z) = f(\theta, z) \) for any \((\theta, z) \in \mathbb{B}_\rho \times \mathcal{W}\).

This space, endowed with the supremum norm, is a Banach space.

Let \( \mathcal{R}\mathcal{H}(\mathcal{W}) \) denote the space of functions real analytic functions such that are holomorphic in \( \mathcal{W} \), continuous in the closure of \( \mathcal{W} \). This is also a Banach space with the supremum norm.

Consider the operator

\[ p_0 : \mathcal{B} \rightarrow \mathcal{R}\mathcal{H}(\mathcal{W}) \]

\[ f(\theta, x) \mapsto \int_0^1 f(\theta, x) d\theta. \]

(5)

Let \( \mathcal{B}_0 \) the natural inclusion of \( \mathcal{R}\mathcal{H}(\mathcal{W}) \) into \( \mathcal{B} \) then we have that \( p_0 \) as a map from \( \mathcal{B} \) to \( \mathcal{B}_0 \) is a projection \(((p_0)^2 = p_0)\).

Set up of the one dimensional renormalization operator.

First let us give a concrete definition of the one dimensional renormalization operator before extending it to the quasi-periodic case. Actually, we tune the definition of the operator given in [8] in order to be able to add a quasi-periodic perturbation.
Given a small value $\delta$, let $\mathcal{M}_\delta$ denote the subspace of $\mathcal{RH}(\mathbb{W})$ formed by the even functions $\psi$ which send the interval $I_\delta = [-1-\delta, 1+\delta]$ into itself, and such that $\psi(0) = 1$ and $x\psi'(x) < 0$ for $x \neq 0$.

Set $a = \psi(1)$, $a' = (1+\delta)a$ and $b' = \psi(a')$. We can define $\mathcal{D}(\mathcal{R}_\delta)$ as the set of $\psi \in \mathcal{M}_\delta$ such that $a < 0$, $1 > b' > -a'$, and $\psi(b') < -a'$.

We define the renormalization operator, $\mathcal{R}_\delta : \mathcal{D}(\mathcal{R}_\delta) \rightarrow \mathcal{M}_\delta$ as

$$\mathcal{R}_\delta(\psi)(x) = \frac{1}{a} \psi \circ \psi(ax).$$

where $a = \psi(1)$.

For maps $\psi \in \mathcal{D}(\mathcal{R}_\delta)$ such that $\psi(a\mathbb{W}) \subset \mathbb{W}$ we have that $\mathcal{R}_\delta(\psi)$ is well defined.

For convenience, we introduce the following working hypothesis.

**H0)** There exists an open set $\mathbb{W} \subset \mathbb{C}$ containing $I_\delta$ and a function $\Phi \in \mathcal{B} \cap \mathcal{X}_0$ such that $\phi = p_0(\Phi)$ is a fixed point of the renormalization operator $\mathcal{R}_\delta$ and such that the closure of both $a\mathbb{W}$ and $\phi(\Phi)(a\mathbb{W})$ is contained in $\mathbb{W}$ (with $a := \Phi(1)$).

In [9], it is claimed that the hypothesis $\textbf{H0}$ is satisfied by the set

$$\left\{ z \in \mathbb{C} \text{ such that } |z^2 - 1| < \frac{5}{2} \right\}.$$ 

This set used by Lanford is more convenient in his study since he works in the set of even holomorphic functions. In the numerical computations from [16] we use as $\mathbb{W}$ the disc centered at $\frac{1}{5}$ with radius $\frac{3}{2}$, and we check the hypothesis $\textbf{H0}$ numerically (without rigorous bounds).

**Definition of the renormalization operator for quasi-periodically forced maps**

Consider the space $\mathcal{X} \subset \mathcal{B}$ defined as:

$$\mathcal{X} = \{ f \in C^r(\mathbb{T} \times I_\delta, I_\delta) | p_0(f) \in \mathcal{M}_\delta \}.$$ 

Consider also the decomposition $\mathcal{X} = \mathcal{X}_0 \oplus \mathcal{X}_0^\perp$ given by the projection $p_0$. In other words, we have $\mathcal{X}_0 = \{ f \in \mathcal{X} | p_0(f) = f \}$ and $\mathcal{X}_0^\perp = \{ f \in \mathcal{X} | p_0(f) = 0 \}$. Note that from the definition of $\mathcal{X}$ follows that $\mathcal{X}_0$ is an isomorphic copy of $\mathcal{M}_\delta$.

Given a function $g \in \mathcal{X}$, we define the **quasi-periodic renormalization** of $g$ as

$$[\mathcal{T}_\omega(g)](\theta, x) := \frac{1}{\hat{a}} g(\theta + \omega, g(\theta, \hat{a}x)),$$

where $\hat{a} = \int_0^1 g(\theta, 1)d\theta$.

Then we have that there exist a set $\mathcal{D}(\mathcal{T})$, open in $(p_0 \circ \mathcal{T}_\omega)^{-1}(\mathcal{M}_\delta)$, where the operator is well defined, in the sense that $\hat{a} \neq 0$. Moreover this set contains $\mathcal{D}_0(\mathcal{T})$, the inclusion of $\mathcal{D}(\mathcal{R})$ in $\mathcal{B}$. By definition we have that $\mathcal{T}_\omega$ restricted to $\mathcal{D}_0(\mathcal{T})$ is isomorphically conjugate to $\mathcal{R}$, therefore the fixed points of $\mathcal{R}$ extend to fixed points of $\mathcal{T}_\omega$. Assume that $\textbf{H0}$ holds and let $\Phi$ be the fixed point given by this hypothesis. Then we have that there exists $U \subset \mathcal{D}(\mathcal{T}) \cap \mathcal{B}$, an open neighborhood of $\Phi$, such that $\mathcal{T}_\omega : U \rightarrow \mathcal{B}$ is well defined. Moreover we have that $\mathcal{T}_\omega$ is Fréchet differentiable for any $\Psi \in U$.
Fourier expansion of $D\mathcal{T}_\omega(\Psi)$.

Let $\Psi$ be a function in a neighborhood of $\Phi$ (given in hypothesis $H_0$) where $\mathcal{T}_\omega$ is differentiable. Additionally assume that $\Psi \in \mathcal{D}_0(\mathcal{T}_\omega)$.

Given a function $f \in \mathcal{B}$ we can consider its complex Fourier expansion in the periodic variable

$$f(\theta, z) = \sum_{k \in \mathbb{Z}} c_k(z) e^{2\pi k \theta i},$$  \hspace{1cm} (8)

with

$$c_k(z) = \int_0^1 f(\theta, z) e^{-2\pi k \theta i} d\theta.$$

Then we have that $D\mathcal{T}_\omega$ “diagonalizes” with respect to the complex Fourier expansion, in the sense that we have

$$[D\mathcal{T}_\omega(\Psi)f](\theta, z) = D\mathcal{R}_\delta[c_0](z) + \sum_{k \in \mathbb{Z}\setminus\{0\}} \left( [L_1(c_k)](z) + [L_2(c_k)](z) e^{2\pi k \omega i} \right) e^{2\pi k \theta i},$$  \hspace{1cm} (9)

where

$$L_1 : \mathcal{RH}(\mathbb{W}) \to \mathcal{RH}(\mathbb{W})$$

$$g(z) \mapsto \frac{1}{a} \psi' \circ \psi(a z) g(a z),$$

and

$$L_2 : \mathcal{RH}(\mathbb{W}) \to \mathcal{RH}(\mathbb{W})$$

$$g(z) \mapsto \frac{1}{a} g \circ \psi(a z),$$

with $\psi = p_0(\Psi)$ and $a = \psi(1)$.

An immediate consequence of this diagonalization is the following. Consider

$$B_k := \{ f \in \mathcal{B} \mid f(\theta, x) = u(x) \cos(2\pi k \theta) + v(x) \sin(2\pi k \theta), \text{ for some } u, v \in \mathcal{RH}(\mathbb{W}) \},$$  \hspace{1cm} (10)

then we have that the spaces $B_k$ are invariant by $D\mathcal{T}_\omega$ for any $k > 0$.

Moreover $D\mathcal{T}_\omega(\Psi)$ restricted to $B_k$ is conjugate to $L_{k\omega}$, where $L_{\omega}$ is the defined as

$$L_{\omega} : \mathcal{RH}(\mathbb{W}) \oplus \mathcal{RH}(\mathbb{W}) \to \mathcal{RH}(\mathbb{W}) \oplus \mathcal{RH}(\mathbb{W})$$

$$\begin{pmatrix} u \\ v \end{pmatrix} \mapsto \begin{pmatrix} L_1(u) \\ L_1(v) \end{pmatrix} + \begin{pmatrix} \cos(2\pi \omega) & -\sin(2\pi \omega) \\ \sin(2\pi \omega) & \cos(2\pi \omega) \end{pmatrix} \begin{pmatrix} L_2(u) \\ L_2(v) \end{pmatrix},$$  \hspace{1cm} (11)

Then we have that the understanding of the derivative of the renormalization operator in $\mathcal{B}$ is equivalent to the study of the operator $L_{\omega}$ for any $\omega \in \mathbb{T}$.

Properties of $L_{\omega}$

Given a value $\gamma \in \mathbb{T}$, consider the rotation $R_{\gamma}$ defined as

$$R_{\gamma} : \mathcal{RH}(\mathbb{W}) \oplus \mathcal{RH}(\mathbb{W}) \to \mathcal{RH}(\mathbb{W}) \oplus \mathcal{RH}(\mathbb{W})$$

$$\begin{pmatrix} u \\ v \end{pmatrix} \mapsto \begin{pmatrix} \cos(2\pi \gamma) & -\sin(2\pi \gamma) \\ \sin(2\pi \gamma) & \cos(2\pi \gamma) \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix},$$  \hspace{1cm} (12)
then we have that $L_\omega$ and $R_\gamma$ commute for any $\omega, \gamma \in \mathbb{T}$.

This has some consequences on the spectrum of $L_\omega$. Concretely we have that any eigenvalues of $L_\omega$ (different from zero) is either real with geometric multiplicity even, or a pair of complex conjugate eigenvalues. On the other hand $L_\omega$ depends analytically on $\omega$, which (using theorems III-6.17 and VII-1.7 of [7]), imply that (as long as the eigenvalues of $L_\omega$ are different) the eigenvalues and their associated eigenspaces depend analytically on the parameter $\omega$.

Finally, doing some minor changes on the domain of definition, we can prove the compactness of $L_\omega$. Recall that the compactness of an operator implies that its spectrum is either finite or countable with 0 on its closure (see for instance theorem III-6.26 of [7]).

2.2 Reducibility loss and quasi-periodic renormalization

Given a map $F$ like (4) with $f \in \mathcal{B}$ and $\omega \in \mathbb{T}$ we denote by $f^n: \mathbb{T} \times \mathbb{R} \rightarrow \mathbb{R}$ the $x$-projection of $F^n(x, \theta)$. Equivalently $f^n$ can be defined through the recurrence

$$f^n(\theta, x) = f(\theta + (n-1)\omega, f^{n-1}(\theta, x)).$$  \hspace{1cm} (13)

From this point on, whenever $\omega$ is used, it is assumed to be Diophantine. Denote by $\Omega = \Omega_{\gamma, \tau}$ the set of Diophantine numbers, that is the set of $\omega \in \mathbb{T}$ such that there exists $\gamma > 0$ and $\tau \geq 1$ such that

$$|q\omega - p| \geq \frac{\gamma}{|q|^\tau}, \quad \text{for all} \ (p, q) \in \mathbb{Z} \times (\mathbb{Z} \setminus \{0\}).$$

Additionally, we will need to assume that the following conjecture is true.

**Conjecture A.** The operator $T_\omega$ (for any $\omega \in \Omega$) is an injective function when restricted to the domain $\mathcal{B} \cap D(T)$. Moreover, there exist $U$ an open set of $D(T)$ containing $W^u(\Phi, R) \cup W^s(\Phi, R)^\perp$ where the operator $T_\omega$ is differentiable.

In [15] we discuss the difficulties for proving this conjecture, and in [16] we show that the results obtained assuming this conjecture are coherent with the numerical computations. Whenever the conjecture [A] is needed for a result it is explicitly stated in the hypotheses.

**Consequences for a two parametric family of maps**

Consider a two parametric family of maps $\{c(\alpha, \varepsilon)\}_{(\alpha, \varepsilon) \in A}$ contained in $\mathcal{B}$, with $A = [a, b] \times [0, d]$ and $a$, $b$ and $d$ are real numbers (with $a < b$ and $0 < d$). We assume that the dependency on the parameters is analytic.

Consider the following hypothesis on the family of maps.

**H1)** The family $\{c(\alpha, \varepsilon)\}_{(\alpha, \varepsilon) \in A}$ uncouples for $\varepsilon = 0$, in the sense that the family $\{c(\alpha, 0)\}_{\alpha \in [a, b]}$ does not depend on $\theta$ and it has a full cascade of period doubling bifurcations. We assume that the family $\{c(\alpha, 0)\}_{\alpha \in [a, b]}$ crosses transversely the stable manifold of $\Phi$, the fixed point of the renormalization operator, and each of the manifolds $\Sigma_n$ for any $n \geq 1$, where $\Sigma_n$ is

$^1$Here $W^u(\Phi, R)$ and $W^s(\Phi, R)$ are considered as the inclusion in $\mathcal{B}$ of the stable and the unstable manifolds of the fixed point $\Phi$ (given by $\text{H0}$) by the map $R$ in the topology of $\mathcal{B}_0$. 



8
the inclusion in $\mathcal{B}$ of the set of one dimensional unimodal maps with a super-attracting $2^n$ periodic orbit.

In other words, we assume that the family $c(\alpha, \varepsilon)$ can be written as,

$$c(\alpha, \varepsilon) = c_0(\alpha) + \varepsilon c_1(\alpha, \varepsilon),$$

with $\{c_0(\alpha)\}_{\alpha \in [a, b]} \subset \mathcal{B}_0$ having a full cascade of period doubling bifurcations.

Given a family $\{c(\alpha, \varepsilon)\}_{(\alpha, \varepsilon) \in A}$ satisfying the hypothesis $\textbf{H1}$, let $\alpha_n$ be the parameter value for which the uncoupled family $\{c(\alpha, 0)\}_{\alpha \in [a, b]}$ intersects the manifold $\Sigma_n$. Note that the critical point of the map $c(\alpha_n, 0)$ is a $2^n$-periodic orbit. Our main achievement in [15] is to prove that from every parameter value $(\alpha_n, 0)$ there are born two curves in the parameter space, each of them corresponding to a reducibility loss bifurcation. Now we introduce some technical definitions in order to give a more precise statement of this result.

Let $\mathcal{RH}(\mathbb{B}_\rho, \mathbb{W})$ denote the space of periodic real analytic maps from $\mathbb{B}_\rho$ to $\mathbb{W}$ and continuous in the closure of $\mathbb{B}_\rho$. Consider a map $f_0 \in \mathcal{B}$ and $\omega \in \Omega$, such that $f$ has a periodic invariant curve $x\omega$ of rotation number $\omega$ with a Lyapunov exponent bounded by certain $-K_0 < 0$. Using lemma 3.6 in [15] we have that there exist a neighborhood $V \subset \mathcal{B}$ of $f_0$ and a map $x \in \mathcal{RH}(\mathbb{B}_\rho, \mathbb{W})$ such that $x(f)$ is a periodic invariant curve of $f$ for any $f \in V$. Then we can define the map $G_1$ as

$$G_1: \quad \Omega \times V \rightarrow \mathcal{RH}(\mathbb{B}_\rho, \mathbb{W})$$

$$(\omega, g) \mapsto \mathcal{D}_x g (\theta, \omega, [x(\omega, g)](\theta)) \mathcal{D}_x g (\theta, [x(\omega, g)](\theta)).$$

(14)

On the other hand, we can consider the counterpart of the map $G_1$ in the uncoupled case. Given a map $f_0 \in \mathcal{B}_0$, consider $U \subset \mathcal{B}_0$ a neighborhood of $f_0$ in the $\mathcal{B}_0$ topology. Assume that $f_0$ has an attracting 2-periodic orbit $x_0 \in I$. Let $x = x(f) \in \mathbb{W}$ be the continuation of this periodic orbit for any $f \in U$. We have that $x$ depends analytically on the map, therefore it induces a map $x: U \rightarrow \mathbb{W}$. Then if we take $U$ small enough we have an analytic map $x: U \rightarrow \mathbb{W}$ such that $x[f]$ is a periodic orbit of period 2. Now we can consider the map

$$\widehat{G}_1: \quad U \subset \mathcal{B}_0 \rightarrow \mathbb{C}$$

$$f \mapsto \mathcal{D}_x f(x[f]) \mathcal{D}_x f(x[f]).$$

(15)

Note that $\widehat{G}_1$ corresponds to $G_1$ restricted to the space $\mathcal{B}_0$ (but then $\widehat{G}_1(f)$ has to be seen as an element of $\mathcal{RH}(\mathbb{B}_\rho, \mathbb{W})$).

Consider the sequences

$$\omega_k = 2\omega_{k-1}, \quad \text{for } k = 1, \ldots, n - 1.$$  

$$f^{(n)}_k = \mathcal{R} \left( f^{(n)}_{k-1} \right), \quad \text{for } k = 1, \ldots, n - 1.$$  

$$u^{(n)}_k = \mathcal{D} \mathcal{R} \left( f^{(n)}_{k-1} \right) u^{(n)}_{k-1}, \quad \text{for } k = 1, \ldots, n - 1.$$  

$$v^{(n)}_k = \mathcal{D} \mathcal{T} \omega_{k-1} \left( f^{(n)}_{k-1} \right) v^{(n)}_{k-1}, \quad \text{for } k = 1, \ldots, n - 1.$$  

(16)

with

$$f^{(n)}_0 = c(\alpha_n, 0), \quad u^{(n)}_0 = \partial_\alpha c(\alpha_n, 0), \quad v^{(n)}_0 = \partial_\varepsilon c(\alpha_n, 0).$$

(17)
Consider the following hypothesis.

**H2)** The family \( \{c(\alpha, \varepsilon)\}_{(\alpha, \varepsilon) \in A} \) is such that

\[
DG_1 \left( \omega_{n-1}, f_{n-1}^{(n)} \right) DT_{\omega_{n-2}} \left( f_{n-2}^{(n)} \right) \cdots DT_{\omega_0} \left( f_0^{(n)} \right) \partial_c c(\alpha_n, 0),
\]

has a unique non-degenerate minimum (respectively maximum) as a function from \( T \) to \( \mathbb{R} \), for any \( n \geq n_0 \).

Consider a family of maps \( \{c(\alpha, \varepsilon)\}_{(\alpha, \varepsilon) \in A} \) such that the hypotheses **H1** and **H2** are satisfied and \( \omega_0 \in \Omega \). If the conjecture \( A \) is true, then theorem 3.8 in [15] asserts that there exists \( n_0 \) such that, for any \( n \geq n_0 \), there exist two bifurcation curves around the parameter value \((\alpha_n, 0)\), such that they correspond to a reducibility-loss bifurcation of the \( 2^n \)-periodic invariant curve. Moreover, these curves are locally expressed as \((\alpha_n + \alpha_n'(\omega)\varepsilon + o(\varepsilon), \varepsilon)\) and \((\alpha_n + \beta_n'(\omega)\varepsilon + o(\varepsilon), \varepsilon)\) with

\[
\alpha_n'(\omega) = -\frac{m \left( DG_1 \left( \omega_{n-1}, f_{n-1}^{(n)} \right) v_{n-1}^{(n)} \right)}{DG_1 \left( f_{n-1}^{(n)} \right) u_{n-1}^{(n)}}, \quad (18)
\]

and

\[
\beta_n'(\omega) = -\frac{M \left( DG_1 \left( \omega_{n-1}, f_{n-1}^{(n)} \right) v_{n-1}^{(n)} \right)}{DG_1 \left( f_{n-1}^{(n)} \right) u_{n-1}^{(n)}}, \quad (19)
\]

where \( G_1 \) and \( \hat{G}_1 \) are given by equations (14) and (15), and \( m \) and \( M \) are the minimum and the maximum as operators, that is

\[
m : \mathcal{R}(B_{\rho}, \mathbb{C}) \to \mathbb{R} \\
g \mapsto \min_{\theta \in T} g(\theta).
\]

and

\[
M : \mathcal{R}(B_{\rho}, \mathbb{C}) \to \mathbb{R} \\
g \mapsto \max_{\theta \in T} g(\theta).
\]

Now we can go back to the hypothesis **H2**, which is not intuitive. Actually we can introduce a stronger condition which is much more easy to check. Moreover this condition is automatically satisfied by maps like the Forced Logistic Map. Consider a family of maps \( \{c(\alpha, \varepsilon)\}_{(\alpha, \varepsilon) \in A} \) as before, satisfying hypothesis **H1**.

**H2’** The family \( \{c(\alpha, \varepsilon)\}_{(\alpha, \varepsilon) \in A} \) is such that the quasi-periodic perturbation \( \partial_c c(\alpha, 0) \) belongs to the set \( B_1 \) (see equation (10)) for any value of \( \alpha \) (with \( (\alpha, 0) \in A \)).

Proposition 3.10 in [15] asserts that **H2’** implies **H2**.
3 Universality for q.p. forced maps

In [14] we have done a numerical study of the asymptotic behavior of the reducibility loss directions \( \alpha'_i(\omega) \) of the FLM. This study is summarized in section 1.1. Concretely we have done three different numerical observation on this asymptotic behavior, to which we refer as first, second and third numerical observations. On the other hand, formula (18) provides an explicit expression for the reducibility loss directions \( \alpha'_i(\omega) \) in terms of the quasi-periodic renormalization operator. In this section we propose three different conjectures on the dynamics of the quasi-periodic renormalization operator which provide a suitable explanation to the numerical observations.

Due to the periodicity of the maps considered, the quasi-periodic renormalization has an intrinsic rotational symmetry. In section 3.1 we reduce the symmetry by taken a suitable section, in a process analogous to a Poincaré section.

In section 3.2 we reduce the problem to the dynamics of the q.p. renormalization operator. To do this it is necessary to introduce conjecture B, in which we assume that the normal behavior of the operator for the iterates close to the stable and the unstable manifold is described by the linearization of the operator in the fixed point.

Consider \( B_1 \) the space given by (10) for \( k = 1 \). In section 3.3 we study the linearized dynamics of the renormalization operator but restricted to the space \( B_1 \). We use some symmetries of the map to perform some kind of “Poincaré section” of the operator. Then we introduce conjecture C in which we require the “Poincaré map” to be contractive. Finally we present theorem 3.6 which gives a theoretical explanation to the first numerical observation described in section 1.

In section 3.4 we prove that, under appropriate hypotheses, the behavior associated to the first numerical observation implies the behavior associated to the second observation. In this section we introduce conjecture D which is necessary to check that the appropriate hypotheses are satisfied in the case of the Forced Logistic Map.

In section 3.5 we analyze what happens when a map does not satisfy hypothesis H2', as it happened in sections 3.3 and 3.4. This analysis provides an explanation to the third numerical observation.

All proofs have been moved to the end of their respective subsections to make the presentation clearer.

3.1 Rotational symmetry reduction

Given a function \( g : T \times I_\delta \rightarrow I_\delta \) in \( B \) we can consider the function \( \tilde{g} \) defined as \( \tilde{g}(\theta, x) = g(\theta + \gamma, x) \) for some \( \gamma \in T \). Maps like (4) determined by \( f = g \) or by \( f = \tilde{g} \) exhibit essentially the same dynamics, although (from the functional point of view) they are not the same map. For example they have different Fourier expansion. Roughly speaking, this fact induce a rotational symmetry on the derivative of the quasi-periodic renormalization operator \( T'_\omega \). To follow with our study we need to remove this symmetry from the problem.
Given $\gamma \in T$, consider the following auxiliary function

$$t_\gamma : \mathcal{B} \rightarrow \mathcal{B}$$

$$v(\theta, z) \mapsto v(\theta + \gamma, z).$$

(22)

Let $\mathcal{B}_1$ be the subspace of $\mathcal{B}$ defined by (10) for $k = 1$. The space $\mathcal{B}_1$ is indeed the image of the projection $\pi_1 : \mathcal{B} \rightarrow \mathcal{B}$ defined as

$$[\pi_1(v)](\theta, x) = \left(\int_0^1 v(\theta, x) \cos(2\pi x) d\theta\right) \cos(2\pi \theta) + \left(\int_0^1 v(\theta, x) \sin(2\pi x) d\theta\right) \sin(2\pi \theta).$$

(23)

Given $x_0 \in \mathbb{W} \cap \mathbb{R}$ and $\theta_0 \in T$ we can also consider the sets

$$\mathcal{B}'_1 = \mathcal{B}'_1(\theta_0, x_0) = \{ f \in \mathcal{B}_1 | f(\theta_0, x_0) = 0, \partial_\theta f(\theta_0, x_0) > 0 \},$$

and

$$\mathcal{B}' = \mathcal{B}'(\theta_0, x_0) = \{ f \in \mathcal{B} | \pi_1(f) \in \mathcal{B}'_1 \}.$$

**Proposition 3.1.** For a fixed $x_0 \in \mathbb{W} \cap \mathbb{R}$ and $\theta_0 \in T$, we have that $\mathcal{B}'_1(\theta_0, x_0)$ is an open subset of a codimension one linear subspace of $\mathcal{B}_1$. Moreover for any $v \in \mathcal{B}_1 \setminus \{0\}$ there exists a unique $\gamma_0 \in T$ such that $t_{\gamma_0}(v) \in \mathcal{B}'_1(\theta_0, x_0)$. Therefore for any $v \in \mathcal{B}$ such that $\pi_1(v) \in \mathcal{B}_1 \setminus \{0\}$ there exists a unique $\gamma_0 \in T$ such that $t_{\gamma_0}(v) \in \mathcal{B}'(\theta_0, x_0)$.

Consider a two parametric family of maps $\{c(\alpha, \epsilon)\}_{(\alpha, \epsilon) \in \mathbb{A}}$ contained in $\mathcal{B}$ satisfying the hypotheses H1 and H2 as in section 2.2. Consider also the reducibility loss bifurcation curves associated to the $2^n$-periodic orbit with slopes given by (18) and (19). The goal of this section is to use proposition 3.1 to express formulas (18) and (19) in terms of vectors in $\mathcal{B}'_1(\theta_0, x_0)$. The case $\beta_n^\epsilon(\omega)$ is omitted from now on in the discussion since it is completely analogous to the case considered here, one only has to replace the appearances of a minimum by a maximum.

Consider the sequences $\{\omega_k\}$, $\{f^{(n)}_k\}$ and $\{u^{(n)}_k\}$ given by (16) and (17). Consider now the sequence

$$\tilde{v}^{(n)}_k = t_\gamma(\tilde{v}^{(n)}_{k-1}) \left( DT_{\omega_{k-1}} \left( f^{(n)}_{k-1} \right) \tilde{v}^{(n)}_{k-1} \right)$$

for $k = 1, ..., n - 1,$

(24)

and

$$v^{(n)}_0 = t_{\gamma_0}(\partial_\epsilon c(\alpha, 0)),$$

(25)

where $\gamma(\tilde{v}^{(n)}_{k-1})$ and $\gamma_0$ are chosen such that $\tilde{v}^{(n)}_k$ belongs to $\mathcal{B}'(\theta_0, x_0)$ for $k = 0, 1, ..., n$. If the projection of $DT_{\omega_{k-1}} \left( f^{(n)}_{k-1} \right) \tilde{v}^{(n)}_{k-1}$ in $\mathcal{B}_1$ is non zero, then $\gamma(\tilde{v}^{(n)}_{k-1})$ is uniquely determined and the vectors $\tilde{v}^{(n)}_k$ are well defined.

**Theorem 3.2.** Consider a family of maps $\{c(\alpha, \epsilon)\}_{(\alpha, \epsilon) \in \mathbb{A}}$ such that the hypotheses H1 and H2 are satisfied. Assume also that $\omega_0 \in \Omega$ and that the conjecture $\mathbb{A}$ is true. Let $\{\omega_k\}$, $\{f^{(n)}_k\}$ and $\{u^{(n)}_k\}$ be defined by (16) and (17) and $\tilde{v}^{(n)}_k$ be defined by (24) and (25). Assume also that the projection of $DT_{\omega_{k-1}} \left( f^{(n)}_{k-1} \right) \tilde{v}^{(n)}_{k-1}$ in $\mathcal{B}_1$ (given by (23)) is non zero.

Then the slopes $\alpha'_n$ of the reducibility loss bifurcations given by (18) can be also written as

$$\alpha'_n(\omega) = -m \left( DG_1 \left( \omega_{n-1}, f^{(n)}_{n-1} \right) \tilde{v}^{(n)}_{n-1} \right),$$

(26)

$$D\hat{G}_1 \left( f^{(n)}_{n-1} \right) u^{(n)}_{n-1},$$

where $G_1$, $\hat{G}_1$ and $m$ are given by equations (14), (15) and (20).
Proofs

Lemma 3.3. Consider the function $t_\gamma$ given by (22).

1. For any $f \in \mathcal{B}$ and $\gamma_1, \gamma_2 \in \mathbb{T}$,
   \[ t_{\gamma_1 + \gamma_2}(v) = t_{\gamma_1} \circ t_{\gamma_2}(v). \]

2. For any $f \in \mathcal{B}$ and $\gamma \in \mathbb{T}$ we have $\|t_\gamma(f)\| = \|f\|$ (recall that the norm of $\mathcal{B}$ considered is the supremum norm in $\mathbb{B}_\rho \times \mathbb{W}$).

3. Let $T_\omega : \mathcal{D}(\mathcal{T}) \to \mathcal{B}$ be the renormalization operator, and $\Phi$ a fixed point. Then we have that $t_\gamma$ and the differential of $T_\omega$ in the fixed point commute. In other words, we have
   \[ [t_\gamma \circ DT_\omega(\Phi)](v) = [DT_\omega(\Phi)](t_\gamma(v)), \]
   for any $v \in \mathcal{B}$ and $\gamma \in \mathbb{T}$.

Proof. The first point follows easily since
   \[ [t_{\gamma_1 + \gamma_2}(v)](\theta, z) = v(\theta + \gamma_1 + \gamma_2, z) = [t_{\gamma_2}(v)](\theta + \gamma_1, z) = [t_{\gamma_1}(t_{\gamma_2}(v))](\theta, z). \]

For the second point of the proposition, recall that the norm considered in $\mathcal{B}$ is the supremum norm in the set $\mathbb{B}_\rho \times \mathbb{W}$. Using the invariance of this set by a translation on the first variable we have have
   \[ \|v\| = \sup_{\mathbb{B}_\rho \times \mathbb{W}} |v(\theta, x)| = \sup_{\mathbb{B}_\rho \times \mathbb{W}} |v(\theta + \gamma, x)| = \|t_\gamma(v)\|. \]

Let us focus now in the third point of the proposition. Given $v \in \mathcal{B}$ consider its complex Fourier expansion on the $\theta$ variable.
   \[ v(\theta, z) = \sum_{k \in \mathbb{Z}} c_k(z) e^{2\pi k\theta i}. \]

Then we have that the complex Fourier expansion of the map $t_\gamma(v)$ is given by
   \[ [t_\gamma(v)](\theta, z) = \sum_{k \in \mathbb{Z}} \left( e^{2\pi k\gamma_1 i} c_k(z) \right) e^{2\pi k\theta i}. \]

Using this, the expansion of $DT_\omega(\Phi)v$ given by equation (9) and the linearity of the operators $L_1$ and $L_2$ we have
   \[ [DT_\omega(\Phi)t_\gamma(v)](\theta, z) = DR_\delta(c_0)(z) + \sum_{k \in \mathbb{Z} \setminus \{0\}} \left[ L_1 \left( e^{2\pi k\gamma_1 i} c_k(z) \right) + L_2 \left( e^{2\pi k\gamma_1 i} c_k(z) \right) e^{2\pi k\omega i} \right] e^{2\pi k\theta i} \]
   \[ = DR_\delta(c_0)(z) + \sum_{k \in \mathbb{Z} \setminus \{0\}} \left[ L_1(c_k(z)) + L_2(c_k(z)) e^{2\pi k\omega i} \right] e^{2\pi k(\theta + \gamma)i} \]
   \[ = [t_\gamma(DT_\omega(\Phi)(v))](\theta, z). \]

\[ \square \]
Proof of proposition 3.1. Consider the map $Ev : B_1 \rightarrow \mathbb{R}$ the evaluation map defined as $Ev(v) = v(\theta_0, x_0)$. Note that the evaluation of a map in a given point is differentiable as a function (see proposition 2.4.17 in [1]). Then we have that set $B_1' = \{(\theta_0, x_0) \mid Ev^{-1}(0, v) \}$ is an open subset of the set $Ev^{-1}(0)$, which is a codimension one Banach space. We also have that $Ev^{-1}(0)$ is a linear subspace because $Ev(\cdot)$ is a linear function.

Let us focus now on the second part of the proposition. Given $v \in B_1$ we have that $v(\theta, z) = A(z) \cos(2\pi \theta) + B(z) \sin(2\pi \theta)$.

with $A$ and $B$ in $\mathcal{R}(\mathbb{W})$. We have that
\[ [t_\gamma(v)(\theta, 0, x_0) = A(x_0) \cos(2\pi (\theta_0 + \gamma)) + B(x_0) \sin(2\pi (\theta_0 + \gamma))] = \hat{A} \cos(2\pi \gamma) + \hat{B} \sin(2\pi \gamma), \]
with $\hat{A} = A(x_0) \cos(2\pi \theta_0) + B(x_0) \sin(2\pi \theta_0)$ and $\hat{B} = B(x_0) \cos(2\pi \theta_0) - A(x_0) \sin(2\pi \theta_0)$. Then taking $\gamma_0 = \frac{1}{2\pi} \arctan \left(-\frac{A}{B}\right)$ and $\gamma_1 = \gamma_0 + \frac{1}{2}$ we have that $t_{\gamma_0}(v)$ and $t_{\gamma_1}(v)$ belong to $Ev^{-1}(0)$ but only one of them belongs to $B_1'(\theta_0, x_0)$. \hfill \Box

Proof of theorem 3.2. We will need the following lemma for the proof.

Lemma 3.4. Consider the function $t_\gamma$ defined in equation (22) and the set $\Sigma_1$ of one dimensional unimodal maps such that its critical point is a two periodic orbit. Then for any $\omega \in \Omega$, $f \in \Sigma_1$ and $v \in B$ we have that
\[ m(DG_1(\omega, f) v) = m(DG_1(\omega, f) t_\gamma(v)), \]
for any $\gamma \in \mathbb{T}$.

Proof. Let $\tilde{t}_\gamma : C(T, \mathbb{R}) \rightarrow C(T, \mathbb{R})$ be the operator defined as $\tilde{t}_\gamma(p)(\theta) = p(\theta + \gamma)$. Note that for any function $p : T \rightarrow \mathbb{R}$ and $\gamma \in \mathbb{T}$ we have that
\[ m(p) = \min_{\theta \in T} p(\theta) = \min_{\theta \in T} p(\theta + \gamma) = \min_{\theta \in T} \tilde{t}_\gamma(p)(\theta) = m(\tilde{t}_\gamma(p)). \]

Hence,
\[ m(DG_1(\omega, f) v) = \min_{\theta \in T} \tilde{t}_\gamma(DG_1(\omega, f) v)). \]

Since $f \in \Sigma_1$ we have that $DG_1(\omega, f)$ is explicitly given by proposition 3.12 in [15]. Using this it is easy to check that $\tilde{t}_\gamma \circ DG_1(\omega, f) v = DG_1(\omega, f) t_\gamma(v)$. Applying this to the equation above, the result follows. \hfill \Box

Using lemma 3.4 we have
\[ m(DG_1(\omega_k, f^*_k), v_k) = m(DG_1(\omega_k, f^*_k), t_\gamma(v_k)), \]
with $\gamma$ any value in $\mathbb{T}$. Since the value $\gamma$ is arbitrary, we can choose $\gamma = \gamma_k + \tilde{\gamma}$ with $\gamma_k$ and $\tilde{\gamma}$ any values in $\mathbb{T}$. Recall now that the values $\omega_n$ and $v_n$ are defined by the recurrence (27). Using these recurrences and the first and third properties of proposition 3.3 we have that
\[ t_{\gamma_k}(v_k) = t_{\gamma_k} \left( DT_{\omega_k}^{-1}(\Phi) v_k \right) = t_{\gamma_k} \left( DT_{\omega_k}^{-1}(\Phi) \tilde{e}_{k-1} \right). \]

This can be reproduced at every step of the recurrence in such a way that the sequence $v_k$ for $k = 0, \ldots, n$ can be replaced by the sequence $t_{\gamma_k}(v_k)$ without loss of generality.

By hypothesis we have that the projection of $T_{\omega_k}^{-1} (f_{k-1}^{(n)}) \tilde{e}_{k-1}^{(n)}$ in $B_1$ is non zero. We can apply proposition 3.1 and then the values of $\gamma_k$ can be chosen in such a way that $t_{\gamma_k}(T_{\omega_k}^{-1} (f_{k-1}^{(n)}) \tilde{e}_{k-1}^{(n)})$ belongs to $B_1'$ for any $k \geq 0$. \hfill \Box
3.2 Reduction to the dynamics of the renormalization operator

Consider a two parametric family of maps \( \{ c(\alpha, \varepsilon) \}_{(\alpha, \varepsilon) \in A} \) contained in \( B \) satisfying the hypotheses \( H1 \) and \( H2 \) as in section 2.2. Consider also the reducibility loss bifurcation curves associated to the \( 2^n \)-periodic orbit with slopes \( \alpha'_n(\omega) \) and \( \beta'_n(\omega) \) given by (18) and (19). As in section 3.1 we omit the case concerning \( \beta'_n(\omega) \) since it is completely analogous to the case concerning \( \alpha'_n(\omega) \). The goal of this section is to reduce the problem of describing the asymptotic behavior of \( \alpha'_n(\omega, c_1)/\alpha'_{n-1}(\omega, c_1) \) to the dynamics of the quasi-periodic renormalization operator.

**Definition 3.5.** Given two sequences \( \{ r_i \}_{i \in \mathbb{Z}^+} \) and \( \{ s_i \}_{i \in \mathbb{Z}^+} \) in a Banach space, we will say that they are **asymptotically equivalent** if there exists \( 0 < \rho < 1 \) and \( k_0 \) such that

\[
\| r_i - s_i \| \leq k_0 \rho^i \quad \forall i \in \mathbb{Z}^+.
\]

We will commit an abuse of notation and denote this equivalence relation by \( s_i \sim r_i \) instead of \( \{ r_i \}_{i \in \mathbb{Z}^+} \sim \{ s_i \}_{i \in \mathbb{Z}^+} \).

Let us remark that it should be more precise to speak about geometric asymptotic equivalence, but the word geometrically has been omitted for simplicity.

Given a family \( \{ c(\alpha, \varepsilon) \}_{(\alpha, \varepsilon) \in A} \) satisfying the hypotheses \( H1 \) and \( H2 \) as before and a fixed Diophantine rotation number \( \omega_0 \), consider \( \omega_k \), \( f_k^{(n)} \), \( u_k^{(n)} \) given by (16) and \( \tilde{v}_k^{(n)} \) given by (24), with \( f_0^{(n)} \) and \( u_0^{(n)} \) given by (17) and \( \tilde{v}_0^{(n)} \) given by (25). Note that \( f_k^{(n)} \), \( u_k^{(n)} \) and \( \tilde{v}_k^{(n)} \) depend on \( c \), the family of maps considered, and the vectors \( v_k^{(n)} \) depend also on the initial value of the rotation number \( \omega_0 \). In general this dependence will be omitted to keep the notation simple. If two different families or two different values of the rotation number should be considered then we will make the dependence explicit.

Let \( \alpha^* \) denote the parameter value such that the family \( \{ c(\alpha, 0) \}_{(\alpha, 0) \in A} \) intersects with \( W^u(\Phi, \mathcal{R}) \) and \( f_j^* \) denote the intersection of \( W^u(\Phi, \mathcal{R}) \) with the manifold \( \Sigma_j \). Consider then

\[
\omega_k = \begin{cases} 2\omega_{k-1}, & \text{for } k = 1, \ldots, n-1, \\ D\mathcal{R}(\Phi)u_{k-1}, & \text{for } k = 1, \ldots, [n/2] - 1, \\ D\mathcal{R}(f_{n-k}^*)u_{k-1}, & \text{for } k = [n/2], \ldots, n-1. \end{cases}
\]

(27)

\[
v_k = \begin{cases} t_{\gamma(\tilde{v}_{k-1})} (D\mathcal{T}_{\omega_{k-1}}(\Phi)u_{k-1}) , & \text{for } k = 1, \ldots, [n/2] - 1, \\ t_{\gamma(\tilde{v}_{k-1})} (D\mathcal{T}_{\omega_{k-1}}(f_{n-k}^*)u_{k-1}) , & \text{for } k = [n/2], \ldots, n-1. \end{cases}
\]

with

\[
u_0 = \partial_\alpha c(\alpha^*, 0), \quad v_0 = t_{\gamma_0} (\partial_\varepsilon c(\alpha^*, 0)),
\]

and \( \gamma(\tilde{v}_{s-1}) \) and \( \gamma_0 \) are chosen such that \( \tilde{v}_k^{(n)} \) belongs to \( B'_1(\theta_0, x_0) \) for any \( s = 1, \ldots, n \).

**Conjecture B.** For any family of maps \( \{ c(\alpha, \varepsilon) \}_{(\alpha, \varepsilon) \in A} \) satisfying \( H1 \) and \( H2 \), assume that

\[
\frac{\tilde{v}_{n-1}^{(n)}}{\| \tilde{v}_{n-1}^{(n)} \|} \sim \frac{v_{n-1}}{\| v_{n-1} \|},
\]

with \( \tilde{v}_{n-1}^{(n)} \) and \( v_{n-1} \) given by (24) and (27). Also assume that there exists a constant \( C > 0 \) such that

\[
\| v_{n-1} \| > C \text{ for any } n > 0.
\]
Finally assume that there exists a constant $C_0 > 0$ such that

$$|m(DG_1(\omega_{n-1}, f_1^*, \frac{v_{n-1}}{\|v_{n-1}\|}))| > C_0,$$

for any $n \geq 0$ and $\omega_0$ Diophantine, where $m$ is given by (20), $G_1$ by (14) and $\{f_1^*\} = W^u(\mathcal{R}, \Phi) \cap \Sigma_1$.

In other words we assume that the asymptotic behavior of the vectors $\tilde{v}_{n-1}^{(n)}$ is determined by the linearization of the renormalization operator in the fixed point. Moreover we assume that the modulus of the vector does not decrease to zero. In figure 2 we have a schematic representation of the orbit $f_n$ with respect to the fixed point $\Phi$ and its stable and unstable manifolds $W^s(\Phi, \mathcal{R})$ and $W^u(\Phi, \mathcal{R})$. We have that the orbit of $f_0^{(n)}$ corresponds to a passage near a saddle point. Note that the initial point $f_0^{(n)}$ is always in $\{c(\alpha, 0)\}_{(\alpha, 0) \in A}$, the final point $f_{n-1}^{(n)}$ is always in $\Sigma_1$ for any $n$, and the orbit of the points spends more and more iterates in a neighborhood of $\Phi$ when $n$ is increased.

To justify conjecture B, let us remark that the initial point $f_0^{(n)} = c(\alpha_n, 0)$ corresponds to the family $c(\alpha, 0)$ intersected with the manifold $\Sigma_n$. On the other hand, the point $f_{n-1}^{(n)}$ corresponds to the intersection of $\mathcal{R}^n(c(\alpha, 0))$ with $\Sigma_1$. When $n$ is increased we have that $f_0^{(n)}$ converges to the intersection of $c(\alpha, 0)$ with the stable manifold $W^s(\Phi, \mathcal{R})$ and $f_{n-1}^{(n)}$ converges to the intersection or $\Sigma_1$ with the unstable manifold $W^u(\Phi, \mathcal{R})$. Moreover when $n$ is increased the intermediate points $f_k^{(n)}$ spend more and more iterates in an arbitrarily small neighborhood of $\Phi$. Actually, this is the typical behavior of passages close to a saddle fixed point. Then we can expect that the asymptotic behavior of the vectors $\tilde{v}_{n-1}^{(n)}$ is determined by the dynamics of the fixed point. The last part of the conjecture can be understood as a kind of uniform transversality of the vectors $\frac{v_{n-1}}{\|v_{n-1}\|}$ with respect to the manifold defined by the zeros of this function. This conjecture is checked numerically for the case of the Forced Logistic Map in [16].

Finally we will need the following extension of the hypothesis $H2$. 

Figure 2: Representation of the dynamics of $\mathcal{R}$ around its fixed point $\Phi$, see the text for more details.
H3) Consider a two parametric family of maps \( \{ c(\alpha, \varepsilon) \} \) where \( A \subset \mathbb{R}^2 \) satisfying H1 and H2 and a fixed Diophantine rotation number \( \omega_0 \). Consider also \( \omega_n \) and \( v_n \) given by (27) and the point \( \{ f_1^{(n)} \} = W^u(\mathcal{R}, \Phi) \cap \Sigma_1 \). We assume that \( DG_1(\omega_{n-1}, f_1^{(n)}) \) has a unique non-degenerate minimum for any \( \omega_0 \in \Omega \) and \( n \geq 0 \). Assume also that the projection of \( DT_{\omega_{n-1}} \left( f_{k-1}^{(n)} \right) \) is non zero.

Using the notation and the hypotheses introduced so far, we have the following results on the asymptotic behavior of the quotients \( \alpha'_{n}(c, \omega_0) \).

**Theorem 3.6.** Let \( \{ c(\alpha, \varepsilon) \} \) be a two parametric family of q.p. forced maps satisfying H1, H2 and H3. Suppose that \( \omega_0 \) is Diophantine (\( \omega_0 \in \Omega \)). Consider the loss of reducibility directions \( \alpha'_{n}(c, \omega_0) \) and the sequences \( v_n \) and \( v_n \) given by (27). Additionally assume that conjectures A and B are true. Then we have that

\[
\frac{\alpha'_{n}(c, \omega_0)}{\alpha'_{n-1}(c, \omega_0)} \sim \delta^{-1} \cdot \frac{m \left( DG_1 \left( \omega_{n-1}, f_1^{*}, \frac{v_{n-1}}{\|v_{n-1}\|} \right) \right)}{m \left( DG_1 \left( \omega_{n-2}, f_1^{*}, \frac{v_{n-2}}{\|v_{n-2}\|} \right) \right)} \cdot \left\| DT_{\omega_{n-2}}(f_2^{(n)}) \frac{v_{n-2}}{\|v_{n-2}\|} \right\|, \tag{28}
\]

where \( m \) is given by (27), \( G_1 \) by (27), \( \{ f_1^{(n)} \} = W^u(\mathcal{R}, \Phi) \cap \Sigma_1 \) is the intersection of the unstable manifold of \( \mathcal{R} \) at the fixed point \( \Phi \) with the manifold \( \Sigma_1 \) and \( \delta \) is the universal Feigenbaum constant.

The interpretation of this result, which will become clearer in section 3.3, is the following. Let \( c_1 \) and \( c_2 \) be two families of q.p. forced maps satisfying H1, H2 and H3 and \( \omega_0 \) a Diophantine number. Consider the loss of reducibility directions \( \alpha'_{n}(c_1, \omega_0) \) associated to each family of maps, as well as the sequences \( v_n(c_1, \omega_0) \) given by the recurrence (27) with \( v_0(c_1, \omega_0) = \partial_{\varepsilon} c_1(\alpha^*, 0) \).

Then, to show that

\[
\frac{\alpha'_{n}(\omega_0, c_1)}{\alpha'_{n-1}(\omega_0, c_1)} \sim \frac{\alpha'_{n}(\omega_0, c_2)}{\alpha'_{n-1}(\omega_0, c_2)},
\]

it is enough prove that

\[
\frac{v_k(\omega_0, c_1)}{\|v_k(\omega_0, c_1)\|} \sim \frac{v_k(\omega_0, c_2)}{\|v_k(\omega_0, c_2)\|}.
\]

**Proofs**

To prove theorem 3.6, it is necessary to introduce the following technical lemmas on the equivalence relation \( \sim \).

**Lemma 3.7.** Given four different sequences \( \{ r_i^{(1)} \}, \{ r_i^{(2)} \}, \{ s_i^{(2)} \} \) and \( \{ s_i^{(2)} \} \) all of them in \( \ell^\infty(\mathbb{R}) \), assume that

\[
r_i^{(1)} \sim r_i^{(2)} \text{ and } s_i^{(1)} \sim s_i^{(2)}.
\]

Then we have that

\[
r_i^{(1)} s_i^{(1)} \sim r_i^{(2)} s_i^{(2)}.
\]

**Proof.** We have that

\[
| r_i^{(1)} s_i^{(1)} - r_i^{(2)} s_i^{(2)} | = | r_i^{(1)} (s_i^{(1)} - s_i^{(2)}) + s_i^{(2)} (r_i^{(1)} - r_i^{(2)}) | \leq | r_i^{(1)} | | s_i^{(1)} - s_i^{(2)} | + | s_i^{(2)} | | r_i^{(1)} - r_i^{(2)} |.
\]

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From \( \{r^{(1)}_i\} \in \ell^\infty(\mathbb{R}) \) and \( \{s^{(2)}_i\} \in \ell^\infty(\mathbb{R}) \) it follows that there exist a constant \( K_0 \) such that \( |r^{(1)}_i| < K_0 \) and \( |s^{(2)}_i| < K_0 \). Using (29) and the bound above the lemma follows easily. \( \square \)

**Lemma 3.8.** Let \( \{r_i\} \) and \( \{s_i\} \) be two different sequences of real numbers with \( r_i \sim s_i \) and \( s_i > C_0 \) for any \( i \geq n_0 \). Then we have \( r_i \sim s_i \).

**Proof.** If follows easily from \( r_i \sim s_i \) and the following bound

\[
\left| \frac{r_i}{s_i} - 1 \right| = \left| \frac{1}{s_i} \right| |r_i - s_i| \leq \frac{1}{C_0} |r_i - s_i|.
\]

\( \square \)

**Lemma 3.9.** Let \( B \) be a Banach space and \( N \) a normed space. Consider that we have \( \{f_n\}_{n \geq 0} \) a sequence on \( B \) such that \( f_n \sim f \), with \( f \in B \). Also consider \( \{u_n\}_{n \geq 0} \) a sequence of vectors on \( N \) and a function \( G : B \times N \rightarrow \mathbb{R} \). Assume that \( G \) is differentiable w.r.t the first variable in a neighborhood \( V \) of \( f \) and

\[
\left\| \frac{\partial}{\partial x_1} G(g, u_n) v \right\| \leq C \|v\|,
\]

for any \( n \geq n_0, g \in V \) and \( v \in B \).

Then we have that

\[ G(f_n, u_n) \sim G(f, u_n). \]

**Proof.** From \( f_n \sim f \) we have that \( f_n \) tends to \( f \) with a geometric rate. In other words, we have that

\[ f_n = f + \Delta_n, \text{ with } \Delta_n \in B \text{ and } \|\Delta_n\| < k_0 \rho^n, \]

with \( k_0 > 0 \) independent of \( n \) and \( \rho < 1 \).

In particular we have that \( f_n \) belongs to a neighborhood \( V \) of \( f \) for any \( n \geq n_0 \). We can consider the auxiliary functions \( H_n : [0, 1] \rightarrow \mathbb{R} \) given as \( H_n(t) = G(f + t\Delta_n, u_n) \). If we apply the mean value theorem to \( H_n \) we have that there exist a real value \( r_n \in (0, 1) \) such that

\[ G(f_n, u_n) = G(f, u_n) + \frac{\partial}{\partial x_1} G(f + r_n\Delta_n, u_n) \Delta_n. \]

Remark that for any \( n \geq n_0 \) we have that \( f + r_n\Delta_n \) belongs to the neighborhood \( V \) of \( f \). Therefore we can apply the bound (30) given by hypothesis, then we have that

\[
|G(f_n, u_n) - G(f, u_n)| = |G(f + \Delta_n, u_n) - G(f, u_n)|
\]

\[ = |\frac{\partial}{\partial x_1} G(f + r_n\Delta_n, u_n) \Delta_n| \leq C \|\Delta_n\| \leq K_0 \rho^n, \]

for any \( n \geq n_0 \). \( \square \)

**Proof of theorem 3.6.** To simplify the expression of \( \alpha'_n(\omega_0, c) \) in terms of the (q.p. forced) renormalization operator let us consider the following functions,

\[
L : \mathcal{D}(\mathcal{R}) \times B_0 \rightarrow \mathbb{R}
\]

\[
(f, u) \mapsto \hat{D} \hat{G}_1(f) u, \quad (31)
\]

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and

\[ K : \Omega \times \mathcal{D}(\mathcal{T}) \times \mathcal{B} \to \mathbb{R} \]
\[ (\omega, f, v) \mapsto m(DG_1(\omega, f) v), \]

where \( m, G_1 \) and \( \hat{G}_1 \) are the functions given by (20), (14) and (15).

Note that the map \( L \) is linear on the component \( u \) and non-linear but smooth with respect to the component \( f \). On the other hand, we have that \( kK(\omega, f, v) = K(\omega, f, kv) \). If \( DG_1(\omega, f, v) \) has a unique minimum as a function from \( \mathcal{T} \) to \( \mathbb{R} \) then we have that \( K \) is differentiable in a neighborhood \( V \subset \mathcal{D}(\mathcal{T}) \times \mathcal{B} \) of \((f, v)\) (see appendix A in [10]).

Note that theorem 3.2 is applicable to the family \( \{c(\alpha, \varepsilon)\}_{(\alpha, \varepsilon) \in A} \). We can replace the value of \( \alpha'_n(\omega_0) \) given by (26) and, rearranging the terms, we obtain

\[
\frac{\alpha'_n(\omega_0)}{\alpha'_n(\omega_0)} = \frac{K(\omega_n-1, f_{n-1}^\ast, \tilde{v}_{n-1}^\ast)}{K(\omega_n, f_n^\ast, \tilde{v}_n^\ast)} \cdot \frac{L(f_{n-1}, u_{n-1})}{L(f_n, u_n)}.
\]

Consider

\[
A_n(c) = \frac{L(f_n, u_n)}{L(f_{n-1}, u_{n-1})}, \quad B_n(c) = \frac{K(\omega_n-1, f_{n-1}^\ast, \tilde{v}_{n-1}^\ast)}{K(\omega_n, f_n^\ast, \tilde{v}_n^\ast)}.
\]

Using lemma 3.7 it is enough to prove that \( \{A_n(c)\}_{n \geq 0} \), \( \{B_n(c)\}_{n \geq 0} \in \ell^\infty(\mathbb{R}) \) and

\[
A_n(c) \sim \delta^{-1}, \quad B_n(c) \sim \frac{K(\omega_n-1, f_n^\ast, \frac{v_{n-1}}{\parallel v_{n-1} \parallel})}{K(\omega_n, f_n^\ast, \frac{v_n}{\parallel v_n \parallel})} \cdot \|DT_{\omega_n-1}(\Phi) \frac{v_{n-1}}{\parallel v_{n-1} \parallel}\|.
\]

Recall that \( f_{k-1} \) corresponds to \( \mathcal{R}^n(\{c(\alpha, 0)\}) \cap \Sigma_1 \). Using that the family \( \{c(\alpha, 0)\} \) crosses transversaly the stable manifold of the fixed point \( \Phi \) of \( \mathcal{R} \) and that the set \( \Sigma_1 \) crosses transversely the unstable one dimensional manifold of \( \Phi \), we have that \( f_{k-1} \) converges geometrically to \( \{f_1^\ast\} = W^u(\mathcal{R}, \Phi) \cap \Sigma_1 \), where \( W^u(\mathcal{R}, \Phi) \) is the unstable manifold of \( \mathcal{R} \) at the fixed point \( \Phi \). Concretely we have that \( f_{k-1} \sim f^\ast_1 \).

Recall that \( L(f, u) = DG_1(f)u \), therefore we have that \( DfL(f, u)v = D^2G_1(f)(u, v) \). We can apply now lemma 3.9 to \( L(f^\ast_{n-1}, u_{n-1}^\ast, u_{n-1}^\ast) \), then we have that

\[
L\left(f_{n-1}^\ast, \frac{u_{n-1}^\ast}{\parallel u_{n-1}^\ast \parallel}\right) \sim L\left(f^\ast_1, \frac{u_{n-1}^\ast}{\parallel u_{n-1}^\ast \parallel}\right).
\]

On the other hand, we have that \( u_k^\ast \) converges geometrically to \( u_0 = \partial_{\alpha} c(\alpha^\ast, 0) \) with \( \alpha^\ast \) the parameter value for which the family \( \{c(\alpha, 9)\}_{\alpha, 0} \in A \) intersects with \( W^s(\Phi, \mathcal{R}) \). Then, using the \( \lambda \)-lemma, we have that \( \frac{u_{k-1}^\ast}{u_{k-1}^\ast} \) converges to \( e^{u(f^\ast_1)} \), the unitary tangent vector to \( W^u(\Phi, \mathcal{R}) \) at the point \( f^\ast_1 \). With the use of the \( \lambda \)-lemma we also have that \( \|u_{k-1}^\ast\| \) behaves asymptotically as
\[ \delta^k \text{ when } k \text{ goes to infinity, with } \delta \text{ the unstable eigenvalue of } DR(\Phi). \]  
Concretely we have that \[ \|u_{n-1}\| > C_0 \text{ for any } n \geq n_0. \]  
Then we can multiply and divide \( A_n(c) \) by \( \|u_{n-2}\| \) and \( \|u_{n-1}\| \) and write
\[
A_n(c) = \frac{L \left( f^{(n-1)}_{n-2}, \frac{u^{(n-1)}_{n-2}}{\|u^{(n-1)}_{n-2}\|} \right)}{L \left( f^{(n)}_{n-1}, \frac{u^{(n)}_{n-1}}{\|u^{(n)}_{n-1}\|} \right)} \cdot \frac{\|u^{(n-1)}_{n-2}\|}{\|u^{(n)}_{n-1}\|}.
\]  
(34)

Also recall that \( L \) is linear in the second component and \( \frac{u^{(k)}_{k-1}}{\|u^{(k)}_{k-1}\|} \sim e^{u^*(f^*_1)} \). Then it follows that
\[
L \left( f^*_1, \frac{u^{(n)}_{n-1}}{\|u^{(n)}_{n-1}\|} \right) \sim L \left( f^*_1, e^{u^*(f^*_1)} \right).
\]  
(35)

Note that the term \( L \left( f^*_1, e^{u^*(f^*_1)} \right) \) is constant, and it is different from zero since \( \Sigma_1 \) crosses transversely \( W^u(\Phi, R) \). Then we have that
\[
L \left( f^{(n-1)}_{n-2}, \frac{u^{(n-1)}_{n-2}}{\|u^{(n-1)}_{n-2}\|} \right) \sim 1.
\]

We also have that \( \frac{u^{(k)}_{k-1}}{\|u^{(k)}_{k-1}\|} \sim \delta^k \), therefore \( \frac{\|u^{(n-1)}_{n-2}\|}{\|u^{(n-1)}_{n-1}\|} \sim \delta^{-1} \). Applying this to (34) we have
\[ A_n(c) \sim \delta^{-1}. \]  
Additionally, this implies that \( A_n(c) \in \ell^\infty(\mathbb{R}). \)

Now we focus on the asymptotics of \( B_n(c) \). We follow the same arguments used for the study of \( A_n(c) \). Using that \( CK(\omega, f, v) = K(\omega, f, Cv) \) for any constant \( C > 0 \) and the fact that (due to conjecture \( B^* \)) there exists \( C > 0 \) such that \( \|v^{(n)}_{n-1}\| > C \) for any \( n \), we can rearrange the terms on the expression of \( B_n(c) \) in such a way that we have
\[
B_n(c) = \frac{K \left( \omega^{(n)}_{n-1}, f^{(n)}_{n-1}, \frac{v^{(n)}_{n-1}}{\|v^{(n)}_{n-1}\|} \right)}{K \left( \omega^{(n-1)}_{n-2}, f^{(n-1)}_{n-2}, \frac{v^{(n-1)}_{n-2}}{\|v^{(n-1)}_{n-2}\|} \right)} \cdot \frac{\|v^{(n-1)}_{n-1}\|}{\|v^{(n-1)}_{n-2}\|},
\]  
(36)

Consider the term \( \frac{\|v^{(n-1)}_{n-1}\|}{\|v^{(n-1)}_{n-2}\|} \), note that we can use the same argument as we used for \( \frac{\|u^{(n-1)}_{n-2}\|}{\|u^{(n-1)}_{n-1}\|} \) to conclude that
\[
\frac{\|v^{(n)}_{n-1}\|}{\|v^{(n-1)}_{n-2}\|} \sim \frac{\|v^{(n-1)}_{n-1}\|}{\|v^{(n-2)}_{n-2}\|} = \left\| T_{1/2} \left( D T_{1/2} f^*_2 \frac{v_{n-2}}{\|v_{n-2}\|} \right) \right\| = \left\| D T_{1/2} f^*_2 \frac{v_{n-2}}{\|v_{n-2}\|} \right\|.
\]

On the other hand, using the hypothesis \( H3 \) and applying lemma 3.9 to \( K \left( \omega^{(n)}_{n-1}, f^{(n)}_{n-1}, \frac{v^{(n)}_{n-1}}{\|v^{(n)}_{n-1}\|} \right) \), we have
\[
K \left( \omega^{(n)}_{n-1}, f^{(n)}_{n-1}, \frac{v^{(n)}_{n-1}}{\|v^{(n)}_{n-1}\|} \right) \sim K \left( \omega^{(n)}_{n-1}, f^*_1, \frac{v^{(n)}_{n-1}}{\|v^{(n)}_{n-1}\|} \right).
\]
Using the hypothesis $H3$ again we have that $K \left( \omega_{n-1}, f^*_1, \frac{v_{n-1}}{\| v_{n-1} \|} \right)$ is differentiable with respect to the third component. Then using the mean value theorem and conjecture $B$ it can be shown (by means of an analog argument to the one used in the proof of lemma $3.9$) that

$$K \left( \omega_{n-1}, f^*_1, \frac{v_{n-1}}{\| v_{n-1} \|} \right) \sim K \left( \omega_{n-1}, f^*_1, \frac{v_{n-1}}{\| v_{n-1} \|} \right).$$

It is only left to check that $B_n(c) \in \ell^\infty(\mathbb{R})$. Using the last part of conjecture $B$ we have that the sequence given as $\left\{ 1/K \left( \omega_{n-2}, f^*_1, \frac{v_{n-2}}{\| v_{n-2} \|} \right) \right\}_{n \geq 0}$ is bounded. On the other hand, using the definition of the operator $K$ given by (32) and the proposition 3.21 in [15] it follows that $\left\{ K \left( \omega_{n-1}, f^*_1, \frac{v_{n-1}}{\| v_{n-1} \|} \right) \right\}_{n \geq 0}$ is also a bounded sequence. Note that $DT_{\omega}(\Phi)$ is a bounded operator for any $\omega \in \mathbb{T}$, therefore we have that $\left\{ \left\| DT_{\omega_{n-2}}(f^*_2) \frac{v_{n-2}}{\| v_{n-2} \|} \right\| \right\}_{n \geq 0} \in \ell^\infty(\mathbb{R})$. Using lemma 3.7 it follows that $B_n(c) \in \ell^\infty(\mathbb{R})$, which finishes the proof.

### 3.3 Theoretical explanation to the first numerical observation

Consider a two parametric family of maps $\{c(\alpha, \varepsilon)\}_{(\alpha, \varepsilon) \in A}$ contained in $\mathcal{B}$ satisfying the hypotheses $H1$, $H2$, and $H3$. Consider also $\omega_0$ a Diophantine rotation number for the family. As in the previous section we are concerned with the asymptotic behavior of the reducibility loss directions $\alpha'_n(\omega_0, c)$.

Due to theorem 3.6 we have that the values $\frac{\alpha'_{n}(\omega_0, c)}{\| \alpha'_{n-1}(\omega_0, c) \|}$ depend only on the sequences $\omega_n$ and $v_n$ given by equation (27), with $v_0 = t_{\gamma_0} \left( \partial_c c(\alpha^*, 0) \right)$, $\gamma_0$ such that $v_0 \in \mathcal{B}'$ and $\alpha^*$ the parameter value for which the family intersects $W^s(\mathcal{R}, \Phi)$. The behavior of vectors $v_n$ is described by the dynamics of the following operator,

$$L : \mathbb{T} \times \mathcal{B}' \to \mathbb{T} \times \mathcal{B}'$$

$$(\omega, v) \mapsto \left( 2\omega, \frac{t_{\gamma(v)} \left( DT_{\omega}(\Phi)v \right)}{\| t_{\gamma(v)} \left( DT_{\omega}(\Phi)v \right) \|} \right), \quad (37)$$

where $\gamma$ is chosen such that $t_{\gamma(v)} \left( DT_{\omega}(\Phi)v \right)$ belongs to $\mathcal{B}'$.

In this section we focus the case where $\{c(\alpha, \varepsilon)\}_{(\alpha, \varepsilon) \in A}$ satisfies also hypothesis $H2'$. In such a case, we have that $v_0 = \partial_c c(\alpha^*, 0)$ belongs to $\mathcal{B}_1$ the linear subspace of $\mathcal{B}$ given by (10) for $k = 1$. Due to proposition 2.16 in [15] the space $\mathcal{B}_1$ is invariant by the iterates of $DT_{\omega}(\Psi)$.

Consider $\mathcal{L}_\omega$ is the map defined by equation (11) (this is the restriction of $DT_{\omega}(\Psi)$ to $\mathcal{B}_1$). Let us define

$$\mathcal{L}'_\omega : \mathcal{B}'_1 \to \mathcal{B}'_1$$

$$v \mapsto t_{\gamma(v)} \circ \mathcal{L}_\omega(v), \quad (38)$$

where $\gamma(v)$ is chosen such that $t_{\gamma(v)} \circ \mathcal{L}_\omega(v) \in \mathcal{B}'_1$. Note that, due to proposition $3.1$ above, the value $\gamma$ is unique. Actually, we can use this map to induce the following one on $\mathbb{T} \times \mathcal{B}'_1$,

$$L_1 : \mathbb{T} \times \mathcal{B}'_1 \to \mathbb{T} \times \mathcal{B}'_1$$

$$(\omega, v) \mapsto \left( 2\omega, \frac{\mathcal{L}'_\omega(v)}{\| \mathcal{L}'_\omega(v) \|} \right). \quad (39)$$
This is the restriction of \( L_1 \) to \( \mathbb{T} \times B'_1 \).

In \cite{16} we present numerical evidences which suggest that the following conjecture is true.

**Conjecture C.** There exists an open set \( V \subset B'_1 \) (independent of \( \omega \)) such that the second component of the map \( L'_1 \) given by \( \left[ 39 \right] \) is contractive (with the supremum norm) in the unit sphere and it maps the set \( V \) into itself for any \( \omega \in \mathbb{T} \). Additionally we will assume that the contraction is uniform for any \( \omega \in \mathbb{T} \), in the sense that there exists a constant \( 0 < \rho < 1 \) such that the Lipschitz constant associated to the second component of the map \( L'_1 \) is upper bounded by \( \rho \) for any \( \omega \in \mathbb{T} \).

Consider

\[
\text{Rot}(V) = \{ v \in B_1 | t_\gamma(v) \in V \subset B'_1 \text{ for some } \gamma \in \mathbb{T} \}.
\]  

The following result gives a theoretical explanation to the first numerical observation described in the introduction (section \ref{1.1}).

**Theorem 3.10.** Consider \( \{c_1(\alpha, \varepsilon)\} \) and \( \{c_2(\beta, \varepsilon)\} \) two different families of two parametric maps satisfying the hypotheses \( \text{H1, H2'} \) and \( \text{H3} \). Assume that conjectures \( \text{A, B and C} \) are true. Let \( \alpha^* \) and \( \beta^* \) be the parameter values where each family \( c_1(\alpha, 0) \) and \( c_2(\beta, 0) \) intersects \( W^s(R, \Phi) \), the stable manifold of the fixed point of the renormalization operator. Assume that \( \partial_\varepsilon c_1(\alpha^*, 0) \) and \( \partial_\varepsilon c_2(\beta^*, 0) \) belong to \( \text{Rot}(V) \).

Then, for any \( \omega_0 \in \Omega \), we have that

\[
\frac{\alpha'_n(\omega_0, c_1)}{\alpha'_{n-1}(\omega_0, c_1)} \sim \frac{\alpha'_n(\omega_0, c_2)}{\alpha'_{n-1}(\omega_0, c_2)},
\]  

where \( \alpha'_i(\omega_0, c_i) \) are the reducibility loss directions associated to each family \( c_i \) for the rotation number of the system equal to \( \omega_0 \).

**Proofs**

**Lemma 3.11.** Let \( c_1 \) and \( c_2 \) be two families of q.p. forced maps satisfying the hypotheses of theorem \ref{3.14} and \( \omega_0 \) a Diophantine number. Consider the loss of reducibility directions \( \alpha'_n(c_i, \omega_0) \) associated to each family of map, as well as \( v_n(c_i, \omega_0) \) given by the recurrence \( \left[ 27 \right] \).

If

\[
\frac{v_k(\omega_0, c_1)}{\|v_k(\omega_0, c_1)\|} \sim \frac{v_k(\omega_0, c_2)}{\|v_k(\omega_0, c_2)\|},
\]  

then we have that

\[
\frac{\alpha'_n(\omega_0, c_1)}{\alpha'_{n-1}(\omega_0, c_1)} \sim \frac{\alpha'_n(\omega_0, c_2)}{\alpha'_{n-1}(\omega_0, c_2)}.
\]  

**Proof.** Using the same arguments of theorem \ref{3.6} it follows that the sequences

\[
\left\{ K(\omega_{n-1}, f_1^*, v'_{n-1}(\omega_0, c_i)) \right\}_{n>0},
\]

\[
\left\{ 1/K(\omega_{n-2}, f_1^*, v'_{n-2}(\omega_0, c_i)) \right\}_{n>1},
\]

and

\[
\left\{ \|DT_{\omega_{n-2}}(f_2^*)v'_{n-2}(\omega_0, c_i)\| \right\}_{n>1},
\]

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belong to \( \ell^\infty(\mathcal{R}) \) for \( i = 1, 2 \).

On the other hand, using the condition \([42]\) given by hypothesis and using the differentiability of \( K \) is not difficult to see that
\[
K \left( \omega_k, f_1^*, \frac{v_k(\omega_0, c_1)}{\|v_k(\omega_0, c_1)\|} \right) \sim K \left( \omega_k, f_1^*, \frac{v_k(\omega_0, c_2)}{\|v_k(\omega_0, c_2)\|} \right),
\]
for \( k = n - 1, n - 2 \). Then using that \( DT_{\omega_{n-2}}(f_2^*) \) is linear we have that
\[
\left\| DT_{\omega_{n-2}}(f_2^*) \frac{v_{n-2}(\omega_0, c_1)}{\|v_{n-2}(\omega_0, c_1)\|} \right\| \sim \left\| DT_{\omega_{n-2}}(f_2^*) \frac{v_{n-2}(\omega_0, c_2)}{\|v_{n-2}(\omega_0, c_2)\|} \right\|.
\]

Finally, we can apply \([28]\) and lemma \([3,7]\) to conclude that \([43]\) holds.

**Proof of theorem \([3,10]\)**. Using lemma \([3,11]\) it is enough to prove that \([42]\) holds.

Given \( \Psi \) a point where \( \mathcal{T}_\omega \) is differentiable, consider the map
\[
H_{\Psi, \omega} : \mathcal{B}_1' \rightarrow \mathcal{B}_1'
\]
\[
v \mapsto t_{\gamma(v)}(DT_{\omega}(\Psi)v),
\]
where \( \gamma(v) \) is chosen such that \( t_{\gamma(v)}(DT_{\omega}(\Psi)v) \in \mathcal{B}_1' \). Let \( \Phi \) be the fixed point of the quasi-periodic renormalization operator, then \( H_{\Phi, \omega} \equiv \mathcal{L}_\omega' \).

Using conjecture \([C]\) and the differentiability of \( \mathcal{T}_\omega \) we have that there exists a neighborhood of \( \Phi \) such that, for any \( v_1, v_2 \in V \subset \mathcal{B}_1' \) and \( \omega \in \mathbb{T} \), we have
\[
\left\| \frac{H_{\Psi, \omega}(v_1)}{\|H_{\Psi, \omega}(v_1)\|} - \frac{H_{\Psi, \omega}(v_2)}{\|H_{\Psi, \omega}(v_2)\|} \right\| < \tilde{\rho} \left\| \frac{v_1}{\|v_1\|} - \frac{v_2}{\|v_2\|} \right\|,
\]
with \( 0 < \tilde{\rho} < 1 \). Note that the invariance of \( V \) given by conjecture \([C]\) also extends to this neighborhood of \( \Phi \).

Consider \( f_j^* = \Sigma_j \cap W^u(\mathcal{R}, \Phi) \), since \( \Sigma_j \) accumulate to \( W^u(\mathcal{R}, \Phi) \), we have that there exists \( j_0 \) such that \( f_j^* \) belong to a neighborhood of \( \Phi \) arbitrarily small. Using this fact, the definition of \( v_k(\omega_0, c_i) \) given by \([27], (41)\) and the differentiability of \( \mathcal{T}_\omega \) it is not difficult to check that
\[
\left\| \frac{v_n(\omega_0, c_1)}{\|v_n(\omega_0, c_1)\|} - \frac{v_n(\omega_0, c_2)}{\|v_n(\omega_0, c_2)\|} \right\| \leq K_0 \rho^{n-j_0} \left\| \frac{v_0(\omega_0, c_1)}{\|v_0(\omega_0, c_1)\|} - \frac{v_0(\omega_0, c_2)}{\|v_0(\omega_0, c_2)\|} \right\|
\]
where \( K_0 \) a constant. This implies that \([42]\) holds, which finishes the proof.

**Remark 3.12**. To prove theorem \([3,10]\) conjecture \([C]\) can be slightly relaxed. On the one hand, the existence of the open set \( V \subset \mathcal{B}_1' \) such that \( L_1(\omega, V) \subset \{2\omega\} \times V \) can be replaced for an open set \( V(\omega) \) depending on \( \omega \) such that \( L_1(\omega, V) \subset \{2\omega\} \times V(2\omega) \) for each \( \omega \in \mathbb{T} \). On the other hand, the contracitivity on the second component can be replaced by the following condition. There exists constants \( 0 < \rho < 1, K > 0 \) and \( n_0 \in \mathbb{N} \) (independent of \( \omega \)), such that
\[
\| \pi_2 \left[ (L_1')^n(\omega, u) \right] - \pi_2 \left[ (L_1')^n(\omega, v) \right] \| < K \rho^n
\]
for any \( (\omega, u), (\omega, v) \in \{\omega\} \times V(\omega) \) (with \( u \) and \( v \) unitary vectors) and any \( n > n_0 \).
3.4 Theoretical explanation to the second numerical observation

In this section we give a theoretical explanation of the second numerical observation described in section 1.1.

Given a two parametric family of maps \( \{ c(\alpha, \varepsilon) \} \) satisfying hypotheses H1 and H2 let \( \alpha'_n(\omega, c) \) denote the slope of the reducibility loss bifurcation associated to the \( 2^n \) periodic invariant curve of the family. Let \( \{ \mathcal{T}_\omega(c(\alpha, \varepsilon)) \} \) denote the family defined as the renormalization \( \{ c(\alpha, \varepsilon) \} \). This is, let \( f = f_{\alpha, \varepsilon} : \mathbb{B}_p \times \mathbb{W} \to \mathbb{W} \) be the map which defines \( c(\alpha, \varepsilon) \), then \( \mathcal{T}_\omega(c(\alpha, \varepsilon)) \) is given by the map \( g : \mathbb{B}_p \times \mathbb{W} \to \mathbb{W} \) defined as \( g = \mathcal{T}_\omega(f) \). Let \( \alpha^* \) be the parameter value for which \( \{ c(\alpha, 0) \} \) intersects \( \mathcal{W}^*(\Phi, \mathcal{R}) \). For \( (\alpha, \varepsilon) \) close enough to \( (\alpha^*, 0) \) we have that \( \mathcal{T}_\omega(f_{\alpha, \varepsilon}) \) is well defined, then \( \mathcal{T}_\omega(c) \) is also well defined family for \( (\alpha, \varepsilon) \in \tilde{A} \subset A \) a neighborhood of \( (\alpha^*, 0) \in A \). If the family \( \{ c(\alpha, \varepsilon) \} \) has as associated rotation number \( \omega \), then the family \( \{ \mathcal{T}_\omega(c(\alpha, \varepsilon)) \} \) has as a rotation number \( 2\omega \).)

**Theorem 3.13.** Assume that there exists \( B_0 \) a set of two parametric families (satisfying the hypotheses H1 and H2) such that:

1. For any \( c_1 \) and \( c_2 \) in \( B_0 \), we have that
   \[
   \frac{\alpha'_n(\omega, c_1)}{\alpha'_{n-1}(\omega, c_1)} \sim \frac{\alpha'_n(\omega, c_2)}{\alpha'_{n-1}(\omega, c_2)}.
   \]

2. For any family \( \{ c(\alpha, \varepsilon) \} \in A \in B_0 \) we have that \( \{ \mathcal{T}_\omega(c(\alpha, \varepsilon)) \} \in \tilde{A} \in B_0 \).

3. For any value \( \omega \) we have that \( \alpha'_n(\omega, c)/\alpha'_n(2\omega, c) \) is a bounded sequence.

Then \( \alpha'_n(\omega, c)/\alpha'_{n-1}(2\omega, c) \) converges to a constant value.

We want to use this theorem and theorem 3.10 to give an explanation of the second of the numerical observations described in section 1.1. To do that we need to introduce a new conjecture. This conjecture will be also used in section 3.5 to explain the third numerical observation.

**Conjecture D.** Consider \( \mathcal{L}_\omega \) the map given by (11) and \( \omega_0 \in \Omega \). Given \( v_{0,1} \) and \( v_{0,2} \) two vectors in \( \mathcal{R}\mathcal{H}(\mathbb{W}_\rho) \oplus \mathcal{R}\mathcal{H}(\mathbb{W}_\rho) \setminus \{ 0 \} \), consider the sequences

\[
\begin{align*}
\omega_k &= 2\omega_{k-1} & \text{for } k = 1, \ldots, n - 1, \\
v_{k,1} &= \mathcal{L}_{\omega_{k-1}}(v_{k-1,1}) & \text{for } k = 1, \ldots, n - 1, \\
v_{k,2} &= \mathcal{L}_{2\omega_{k-1}}(v_{k-1,2}) & \text{for } k = 1, \ldots, n - 1.
\end{align*}
\tag{45}
\]

There exist constant \( C_1 \) and \( C_2 \) such that

\[
C_1 \frac{\| v_{0,2} \|}{\| v_{0,1} \|} \leq \frac{\| v_{n,2} \|}{\| v_{n,1} \|} \leq C_2 \frac{\| v_{0,2} \|}{\| v_{0,1} \|},
\]

for any \( n \geq 0 \).

In [16] we include numerical evidences which suggest that this conjecture is true. It can be interpreted as a uniform growth condition on \( \mathcal{L}_\omega \).
Corollary 3.14. Consider \( \{ c(\alpha, \varepsilon) \}_{(\alpha, \varepsilon) \in A} \) a two parametric maps satisfying the hypotheses H1, H2' and H3. Assume that conjectures [A, B, C] and [D] are true. Let \( \alpha^* \) be the parameter values for which the family \( \{ c(\alpha, 0) \}_{(\alpha, 0) \in A} \) intersects \( W^s(\mathcal{R}, \Phi) \). Consider \( \text{Rot}(V) \) the set given by \( (40) \) and assume that \( \partial_c c(\alpha^*, 0) \in \text{Rot}(V) \).

Then for any \( \omega_0 \in \Omega \) we have that

\[
\lim_{n \to \infty} \frac{\alpha'_n(\omega_0, c_1)}{\alpha'_{n-1}(2\omega_0, c_1)},
\]

exists.

Proofs

Proof of theorem 3.13. To prove the result we will need the following lemmas

Lemma 3.15. Given a sequence \( \{ s_i \}_{i \in \mathbb{Z}_+} \) such that \( s_i \sim s_{i-1} \) (in other words that its equivalent to the same sequence shifted by one position) then it converges to a limit.

Proof. We will see that \( \{ s_i \}_{i \in \mathbb{Z}_+} \) is a Cauchy sequence.

Consider a positive integer \( N_0 \) fixed, then for any (positive integer) \( N \) we have that

\[
s_{N_0} - s_{N_0+N} = \sum_{i=N_0}^{N_0+N-1} s_i - s_{i+1}.
\]

Using this and the triangular inequality we have that

\[
|s_{N_0} - s_{N_0+N}| \leq \sum_{i=N_0}^{N_0+N-1} |s_i - s_{i+1}|.
\]

Now we can use the hypothesis \( s_i \sim s_{i+1} \) and bound term by term the equation above to obtain

\[
|s_{N_0} - s_{N_0+N}| \leq \sum_{i=N_0}^{N_0+N-1} K_0 \rho^i \leq K_0 \rho^{N_0} \frac{1}{1 - \rho}.
\]

Therefore \( \{ s_i \}_{i \in \mathbb{Z}_+} \) is a Cauchy sequence.

Lemma 3.16. Consider a two parametric family of maps \( \{ c(\alpha, \varepsilon) \}_{(\alpha, \varepsilon) \in A} \) satisfying hypotheses H1 and H2. Let \( \alpha'_n(\omega, c) \) denote the slope of the reducibility loss bifurcation associated to the 2\( n \) periodic invariant curve of the family. Then we have \( \alpha'_n(\omega, c) = \alpha'_{n-1}(2\omega, \mathcal{T}_\omega(c)) \).

Proof. The proof relies on the concepts introduced in sections 3.1 and 3.2 of [15], concretely let us consider the set \( \mathcal{Y}^+_i(\omega) \) introduced there. We have that \( \alpha'_i(\omega, c) \) is the slope at \( \varepsilon = 0 \) of the curve in \( A \) defined by \( \{ c(\alpha, \varepsilon) \}_{(\alpha, \varepsilon) \in A} \cap \mathcal{Y}^+_i(\omega) \). On the other hand, \( \alpha'_{i-1}(2\omega, \mathcal{T}_\omega(c)) \) is the tangent direction of the curve in \( \mathcal{A} \subset A \) defined by \( \{ \mathcal{T}_\omega(c(\alpha, \varepsilon)) \}_{(\alpha, \varepsilon) \in \mathcal{A}} \cap \mathcal{Y}^+_{i-1}(2\omega) \).

Let \( \alpha_i \) be the parameter value for which \( \{ c(\alpha, 0) \}_{(\alpha, 0) \in A} \) intersects \( \Sigma_i = \mathcal{Y}^+_i(\omega) \cap \mathcal{B}_0 \). Using lemma 3.7 in [15] we have that \( \mathcal{T}_\omega(\mathcal{Y}^+_i(\omega) \cap D(T)) = \mathcal{Y}^+_{i-1}(2\omega) \cap \text{Im}(\mathcal{T}_\omega) \), then we have that \( \{ c(\alpha, \varepsilon) \}_{(\alpha, \varepsilon) \in A} \cap \mathcal{Y}^+_i(\omega) \) and \( \{ \mathcal{T}_\omega(c(\alpha, \varepsilon)) \}_{(\alpha, \varepsilon) \in \mathcal{A}} \cap \mathcal{Y}^+_{i-1}(2\omega) \) are exactly the same set around \( \alpha_i \). Then their slope at \( \varepsilon = 0 \) also coincide.

\( \square \)
Lemma 3.17. Consider \( \{r_i\}_{i \in \mathbb{Z}_+} \) and \( \{s_i\}_{i \in \mathbb{Z}_+} \) two sequences of real numbers, with \( r_i \sim s_i \). Consider also third sequence \( \{K_i\}_{i \in \mathbb{Z}_+} \) which is bounded. Then we have \( K_ir_i \sim K_is_i \).

Proof. The proof follows easily from the definition of \( \sim \).

Now we can focus on the proof of theorem 3.13. Using lemma 3.15 is enough to prove that \( \frac{\alpha_i'(\omega, c)}{\alpha_{i-1}'(2\omega, c)} \sim \frac{\alpha_{i-1}'(\omega, c)}{\alpha_{i-2}'(2\omega, c)} \). Due to lemma 3.16 we have that \( \alpha_i'(\omega, c) = \alpha_{i-1}'(2\omega, T_\omega(c)) \), therefore we have
\[
\frac{\alpha_i'(\omega, c)}{\alpha_{i-1}'(2\omega, c)} = \frac{\alpha_{i-1}'(2\omega, T_\omega(c))}{\alpha_{i-1}'(2\omega, c)}.
\]

If we multiply and divide the fraction by \( \alpha_{i-2}'(2\omega, T_\omega(c)) = \alpha_{i-1}'(\omega, c) \) we have
\[
\frac{\alpha_i'(\omega, c)}{\alpha_{i-1}'(2\omega, c)} = \frac{\alpha_{i-1}'(2\omega, T_\omega(c))}{\alpha_{i-2}'(2\omega, T_\omega(c))} \cdot \frac{\alpha_{i-1}'(\omega, c)}{\alpha_{i-1}'(2\omega, c)}.
\]  \hspace{1cm} (47)

Using the fist and the second hypotheses and theorem 3.10 we have
\[
\frac{\alpha_{i-1}'(2\omega, T_\omega(c))}{\alpha_{i-2}'(2\omega, T_\omega(c))} \sim \frac{\alpha_{i-1}'(2\omega, c)}{\alpha_{i-2}'(2\omega, c)}.
\]

Consider now two general sequences \( \{r_i\}_{i \in \mathbb{Z}_+} \) and \( \{s_i\}_{i \in \mathbb{Z}_+} \), with \( r_i \sim s_i \), and a third sequence \( \{K_i\}_{i \in \mathbb{Z}_+} \) which is bounded. Then is not hard to see that \( K_ir_i \sim K_is_i \).

By hypothesis we have that \( \left\{ \frac{\alpha_{i-1}'(\omega, c)}{\alpha_{i-2}'(2\omega, c)} \right\}_{i \in \mathbb{Z}_+} \) is bounded, then we can apply lemma 3.17 to (47) to obtain
\[
\frac{\alpha_i'(\omega, c)}{\alpha_{i-1}'(2\omega, c)} \sim \frac{\alpha_{i-1}'(2\omega, c)}{\alpha_{i-2}'(2\omega, c)} \cdot \frac{\alpha_{i-1}'(\omega, c)}{\alpha_{i-1}'(2\omega, c)}.
\]  \hspace{1cm} \Box

Proof of corollary 3.14. Set \( B_0 \) the set of two parametric families such that satisfy \( H1 \) and \( H2' \). The result follows applying theorem 3.13. Let us check that the hypotheses of the theorem are satisfied.

Condition 1 of theorem 3.13 is satisfied thanks to theorem 3.10.

If a family \( \{c(\alpha, \varepsilon)\}_{(\alpha, \varepsilon) \in A} \) satisfies \( H1 \), we have that \( \{c(\alpha, 0)\}_{(\alpha, 0) \in \tilde{A}} \) has a full cascade of period doubling bifurcations (in the sense described in \( H1 \)). Then \( \{T_\omega(c(\alpha, 0))\}_{(\alpha, 0) \in \tilde{A}} = \{R(c(\alpha, 0))\}_{(\alpha, 0) \in \tilde{A}} \) also has a full cascade of period doubling bifurcations. Then \( \{T_\omega(c(\alpha, \varepsilon))\}_{(\alpha, \varepsilon) \in \tilde{A}} \) also satisfies \( H1 \). If a family \( \{c(\alpha, \varepsilon)\}_{(\alpha, \varepsilon) \in A} \) satisfies \( H2' \) then \( \{T_\omega(c(\alpha, \varepsilon))\}_{(\alpha, \varepsilon) \in \tilde{A}} \) also does due to the invariance of \( B_1 \) by \( D T_\omega(\Psi) \). We have that condition 2 of theorem 3.13 is also satisfied.

If we apply theorem 3.8 to \( \alpha_i'(\omega, c) \) and \( \alpha_i'(2\omega, c) \) we obtain
\[
\frac{\alpha_i'(\omega, c)}{\alpha_i'(2\omega, c)} \sim \frac{\alpha_{i-1}'(\omega, c)}{\alpha_{i-1}'(2\omega, c)} \cdot \frac{\alpha_{i-1}'(\omega, c)}{\alpha_{i-1}'(2\omega, c)}.
\]  \hspace{1cm} (48)
Using the same arguments used in the proof of theorem 3.6 to (48) one obtains:

\[
\frac{\alpha'_i(\omega, c)}{\alpha'_i(2\omega, c)} \sim \frac{m \left( DG_1(\omega_{n-1}, f'_1) \right)}{\|v_{i-1}(\omega, c)\|} \frac{\|v_{i-1}(\omega, c)\|}{\|v_{i-1}(2\omega, c)\|}
\]  

(49)

Using conjecture \( \mathbf{B} \) we have that \( m \left( DG_1(\omega_{n-1}, f'_1) \right) \) is bounded away from zero. Then the boundedness of \( \frac{\alpha'_i(\omega, c)}{\alpha'_i(2\omega, c)} \) only depends on the boundedness \( \frac{\|v_{i-1}(\omega, c)\|}{\|v_{i-1}(2\omega, c)\|} \), which is given by conjecture \( \mathbf{D} \).

Then we have that condition \( \mathbf{I} \) of theorem 3.13 is also satisfied.

\[ \square \]

3.5 Theoretical explanation to the third numerical observation

In sections 3.3 and 3.4 we focussed the discussion on the asymptotic behavior for families satisfying hypothesis \( \mathbf{H2}' \). The aim of this section is to illustrate what happens with maps that satisfy hypotheses \( \mathbf{H1}, \mathbf{H2} \) and \( \mathbf{H3} \), but not \( \mathbf{H2}' \). This is the main difference between the family of maps considered in the first and second numerical observations of section 1.1 and the family considered in the third one.

Let \( \{c(\alpha, \epsilon)\}_{(\alpha, \epsilon) \in A} \) be a two parametric family of maps satisfying hypotheses \( \mathbf{H1}, \mathbf{H2} \) and \( \mathbf{H3} \). Let \( \alpha'_n(\omega, c) \) denote the slope of the reducibility loss bifurcation associated to the \( 2^n \) periodic invariant curve of the family. Finally consider \( \omega_0 \) a Diophantine rotation number for the family. Let \( \alpha^* \) be the parameter value for which \( \{c(\alpha, 0)\}_{(\alpha, 0) \in A} \) intersects \( W^s(\Phi, \mathcal{R}) \). Additionally assume that

\[
\partial_c c(\alpha^*, 0) = v_{0,1} + v_{0,2} \text{ with } v_{0,i} \in \mathcal{B}_i, \quad i = 1, 2,
\]

where the spaces \( \mathcal{B}_i \) are given by (10).

In the third numerical observation of section 1.1 we have considered the family \( c \) as above with

\[
v_{0,1} = f_1(x) \cos(\theta), \quad v_{0,2} = \eta f_2(x) \cos(2\theta).
\]

(50)

As the family depends on \( \eta \), we denote by \( c_\eta \) this concrete family. This parameter \( \eta \) is considered in addition to the parameters \( \alpha \) and \( \epsilon \) of the family. In other words, for each \( \eta \geq 0 \), \( c_\eta \) is a two parametric family of maps. Numerical computations in [14] suggest that the sequence \( \alpha'_n(\omega_0, \eta) / \alpha'_{n-1}(\omega_0, \eta) \) (for \( \eta > 0 \)) is not asymptotically equivalent to \( \alpha'_n(\omega_0, c_0) / \alpha'_{n-1}(\omega_0, c_0) \), but both sequences are \( \eta \)-close to each other. Here \( c_0 \) denotes the family \( c_\eta \) for \( \eta = 0 \). We first discuss why they are not asymptotically equivalent.

Due to theorem 3.6 we have that the values \( \frac{\alpha'_n(\omega, c)}{\alpha'_{n-1}(\omega_0, c_\eta)} \) depend only on the sequences \( \omega_n \) and \( v_n \) given by equation (27), with \( v_0 = t_{\gamma_0} (\partial_c c(\alpha^*, 0)) \), \( \gamma_0 \) such that \( v_0 \in \mathcal{B}' \) and \( \alpha^* \) the parameter value for which the family intersects \( W^s(\mathcal{R}, \Phi) \).

Due to theorem 3.6 we have that the values \( \frac{\alpha'_n(\omega, c_\eta)}{\alpha'_{n-1}(\omega_0, \eta)} \) depend only on the sequences \( \omega_n \) and \( v_n \) given by (27), with \( v_0 = t_{\gamma_0} (\partial_c c(\alpha^*, 0)) \), \( \gamma_0 \) such that \( v_0 \in \mathcal{B}' \) and \( \alpha^* \) the parameter value for which the family intersects \( W^s(\mathcal{R}, \Phi) \). Recall that the space \( \mathcal{B}_1 \) and \( \mathcal{B}_2 \) are invariant by \( DT_{c_\eta}(\Phi) \)
(see proposition 2.16 in [15]). We have that $v_n$ can be written as

$$v_n = v_{n,1} + v_{n,2},$$

with

$$\omega_k = 2\omega_{k-1} \quad \text{for} \quad k = 1, \ldots, n - 1,$$

$$v_{k,1} = t_{\gamma(v)}(L_{\omega_{k-1}}(v_{k-1,1})) \quad \text{for} \quad k = 1, \ldots, n - 1,$$

$$v_{k,2} = t_{\gamma(v)}(L_{2\omega_{k-1}}(v_{k-1,2})) \quad \text{for} \quad k = 1, \ldots, n - 1,$$

where $v_{0,1}$ and $v_{0,2}$ are given by (50) and the value $\gamma(v_{n-1})$ is chosen such that $t_{\gamma(v)}(DT_{\omega}(\Phi)v_{n-1})$ belongs to $B'$. Note that, since $c_1$ and the projection $\pi_1$ given by (23) commute, $\gamma(v_{n-1})$ only depends on $v_{n-1,1}$. Then we have that the vectors $\frac{v_{n-1,1}(c_1)}{\|v_{n-1,1}(c_1)\|}$ and $\frac{v_{n-1,1}(c_2)}{\|v_{n-1,1}(c_2)\|}$ are asymptotically equivalent for any $\nu_1 \neq \nu_2$. Despite of these, the vectors $\frac{v_{n-1,2}(c_1)}{\|v_{n-1,2}(c_1)\|}$ and $\frac{v_{n-1,2}(c_2)}{\|v_{n-1,2}(c_2)\|}$ will not be (in general) asymptotic equivalents. This explains why the universal behavior of the sequence $\alpha'_n(\omega_0, c_1)/\alpha'_{n-1}(\omega_0, c_1)$ ceases for $\eta > 0$.

**Remark 3.18.** If we have a family with $v_0 \in B_j \oplus B_k$ (with $j \neq k$) instead of $v_0 \in B_1 \oplus B_2$, then the same discussion can be adapted with minor modifications.

**Remark 3.19.** Consider $\tilde{c}$ an arbitrary two parametric family satisfying the hypotheses H1, H2 and H3. If the Fourier expansion w.r.t $\theta$ of $\partial_\theta c(\alpha^*, 0)$ has non-trivial Fourier nodes for different orders of the expansion, then one should not expect it to exhibit the universal behavior of the Forced Logistic Map, since the same argument used for the family $c_1$ would be applicable.

To explain why quotients $\alpha'_n(\omega_0, c_1)/\alpha'_{n-1}(\omega_0, c_1)$ and $\alpha'_n(\omega_0, c_0)/\alpha'_{n-1}(\omega_0, c_0)$ are $\eta$-close we have the following result.

**Theorem 3.20.** Consider $\{c_\eta(\alpha, \varepsilon)\}_{(\alpha, \varepsilon) \in A}$ a family of maps satisfying the hypotheses H1, H2 and H3 for any $\eta_0 \geq \eta \geq 0$ (with $\eta_0 \neq 0$ fixed). Assume that conjectures A, B and D are true.

Then there exist $\tilde{\eta}_0$ sufficiently small such that, for any $\eta \geq \eta_0$, we have that

$$\left| \alpha'_n(\omega_0, c_1) - \alpha'_n(\omega_0, c_0) \right| \leq O(\eta) + O(\rho^\eta),$$

where $\rho$ is the constant $0 < \rho < 1$ associated to the asymptotic equivalence relation $\sim$.

**Proof of theorem 3.20.** We need the following lemma.

**Lemma 3.21.** Assume the same hypotheses as in theorem 3.20. Consider $v_k$, $v_{k,1}$ and $v_{k,2}$ given by (51) and (52), with $v_{0,1}$ and $v_{0,2}$ given by (50). Then we have that

$$\left\| \frac{v_k}{\|v_k\|} - \frac{v_{k,1}}{\|v_{k,1}\|} \right\| < \frac{2C\eta}{1 - C\eta}.$$

**Proof.** If we use $v_n = v_{n,1} + v_{n,2}$, rearrange the sums, and we apply the triangular inequality, then we have

$$\left\| \frac{v_n}{\|v_n\|} - \frac{v_{n,1}}{\|v_{n,1}\|} \right\| = \left\| \frac{v_{n,1}}{\|v_{n,1}\|} - \frac{v_{n,1} + v_{n,2}}{\|v_{n,1} + v_{n,2}\|} \right\| v_{n,1} + \frac{v_{n,2}}{\|v_{n,1} + v_{n,2}\|} \right\| \leq \frac{\|v_{n,1} - v_{n,1} + v_{n,2}\|}{\|v_{n,1} + v_{n,2}\|} + \frac{\|v_{n,2}\|}{\|v_{n,1} + v_{n,2}\|}.$$

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Note that to deduce the last equation it is necessary to check that \(\|v_{n,1}\| > 0\). This is true due to conjecture [B]. Recall that if the assumption is true, we have that there exists a constant \(C'\) such that \(\|v_n\| > C'\). If \(\eta\) is small enough we have that \(\eta C < 1\), therefore \(\|v_n\| > \|v_{n,2}\|\). Then we have
\[
\|v_{n,1}\| = \|v_{n,1} + v_{n,2} - v_{n,2}\| \geq \|v_n\| - \|v_{n,2}\| \geq C' - \eta C\|v_{n,1}\|.
\]
Which yields
\[
\|v_{n,1}\| \geq \frac{C'}{1 + \eta C},
\]
therefore we have that \(\|v_{n,1}\| > 0\).

Using the reverse triangular inequality we have
\[
\|\|v_{n,1}\| - \|v_{n,1} + v_{n,2}\|\| \leq \|v_{n,2}\| \leq \eta C\|v_{n,1}\|.
\]
On the other hand, if \(\eta\) is small enough we have that \(\eta C < 1\), therefore \(\|v_{n,2}\| < \|v_{n,1}\|\) and consequently we have that
\[
\|v_{n,1} + v_{n,2}\| \geq \|v_{n,1}\| - \|v_{n,2}\| \geq (1 - C\eta)\|v_n\|.
\]
Applying the two last inequalities to equation (54) the result follows.

Using lemma [3.21] we have that
\[
\frac{v_n}{\|v_n\|} = \frac{v_{n,1}}{\|v_{n,1}\|} + O(\eta).
\] (55)

Using theorem [3.6] and the definition of the equivalence relation \(\sim\) follows
\[
\frac{\alpha_n'(c, \omega_0)}{\alpha_{n-1}'(c, \omega_0)} = \delta^{-1} \cdot \frac{m(DG_1(\omega_{n-1}, f_1^*, \frac{v_{n-1}}{\|v_{n-1}\|}))}{m(DG_1(\omega_{n-2}, f_1^*, \frac{v_{n-2}}{\|v_{n-2}\|}))} \cdot \|DT_{\omega_{n-2}}(f_2^*) \frac{v_{n-2}}{\|v_{n-2}\|}\| + O(\rho^n)\] (56)

with \(DG_1\), \(m\) and \(f_1^*\) given by the hypotheses of the theorem.

Since the hypotheses of theorem [3.6] are satisfied, we have that \(m(DG_1(\omega_k, f_1^*, \cdot))\) and \(\|DT_{\omega_k}(f_2^*)(\cdot)\|\) are differentiable functions. Using this and (55) we obtain
\[
m(DG_1(\omega_k, f_1^*, \frac{v_k}{\|v_k\|})) = m(DG_1(\omega_k, f_1^*, \frac{v_{k,1}}{\|v_{k,1}\|})) + O(\eta), \text{ for } k = n - 2, n - 1,\] (57)

and
\[
\|DT_{\omega_{n-2}}(f_2^*) \frac{v_{n-2}}{\|v_{n-2}\|}\| = \|DT_{\omega_{n-2}}(f_2^*) \frac{v_{n-2,1}}{\|v_{n-2,1}\|}\| + O(\eta)\]. (58)

Replacing (57) and (58) into (56) follows easily
\[
\frac{\alpha_n'(c, \omega_0)}{\alpha_{n-1}'(c, \omega_0)} = \delta^{-1} \cdot \frac{m(DG_1(\omega_{n-1}, f_1^*, \frac{v_{n-1,1}}{\|v_{n-1,1}\|}))}{m(DG_1(\omega_{n-2}, f_1^*, \frac{v_{n-2,1}}{\|v_{n-2,1}\|}))} \cdot \|DT_{\omega_{n-2}}(f_2^*) \frac{v_{n-2,1}}{\|v_{n-2,1}\|}\| + O(\eta) + O(\rho^n).
\]
Using this and theorem 3.6 on the family $c_0$ we have

$$
\frac{\alpha'_n(c_0, \omega_0)}{\alpha'_{n-1}(c_0, \omega_0)} = \delta^{-1} \cdot \frac{m(DG_1(\omega_{n-1}, f_1^*, \frac{v_{n-1,1}}{\|v_{n-1,1}\|}))}{m(DG_1(\omega_{n-2}, f_1^*, \frac{v_{n-2,1}}{\|v_{n-2,1}\|}))} \cdot \left\| D\tau_{\omega_{n-2}}(f_2^*) \frac{v_{n-2,1}}{\|v_{n-2,1}\|} \right\| + O(\rho^n).
$$

Using the two last equations the result follows. \qed

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