Generalized symmetric nonextensive thermostatistics and $q$-modified structures

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Abstract

We formulate a convenient generalization of the $q$-expectation value, based on the analogy of the symmetric quantum groups and $q$-calculus, and show that the $q \leftrightarrow q^{-1}$ symmetric nonextensive entropy preserves all of the mathematical structure of thermodynamics just as in the case of non-symmetric Tsallis statistics. Basic properties and analogies with quantum groups are discussed.

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In the last few years there has been much interest in nonextensive classical and quantum physics. The nonextensive statistical mechanics proposed by Tsallis [1,2], has been the source of inspiration for many investigations in systems which represent multifractal properties, long-range interactions and/or long-range memory effects [3]. On the other hand, quantum groups and the derived $q$-deformed algebraic structure such as $q$-oscillators, based on the deformation of standard oscillator commutation-anticommutation relations, have created considerable interest in mathematical physics and in several applications [4].

The investigations described above are only two apparently unrelated areas in nonextensive physics. Although a complete understanding of the connection between nonextensive statistics and $q$-deformed structure is still lacking, many papers are devoted to the study of a deep connection between these two non-extensive formalisms [5-9].

Tsallis statistics is a $q$-nonsymmetric formalism i.e., not invariant under $q \leftrightarrow q^{-1}$. Recently Abe [3] has employed a connection between Tsallis entropy and the non-symmetric Jackson derivative. Because the requirement of invariance under $q \leftrightarrow q^{-1}$ is very important in quantum groups [4], the above connection allows him to extend the Tsallis entropy to the $q$-symmetric one by means of a symmetric Jackson derivative. However, Abe has not extended the definition of the expectation value of an observable to the symmetric case and thus unable to formulate the thermostatistics which will preserve the Legendre transformation of standard thermodynamics in contrast to the Tsallis statistics which does. We would like to point out that Ref. [7] introduces a two-parameter modification for the entropy and for the expectation value of an observable but does not also produce a consistent formulation of thermostatistics.

The purpose of this letter is to show how the $q \leftrightarrow q^{-1}$ symmetric generalization of the Tsallis entropy together with a natural generalization of the $q$-expectation value produces a
thermostatistics that preserves the mathematical structure of standard thermodynamics and show that this property is a direct consequence of the generalization from the non-symmetric $q$-calculus to the symmetric one.

Before investigating the symmetric nonextensive thermostatistics, let us briefly review the fundamental properties of the Tsallis thermostatistics, which is based upon the following two postulates \[1,2\].

- A nonextensive generalization of the Boltzmann-Gibbs entropy (Boltzmann constant is set equal to unity)

$$S_q = \frac{1}{q-1} \left( 1 - \sum_{i=1}^{W} p_i^q \right) , \quad \text{with} \quad \sum_{i=1}^{W} p_i = 1 , \quad (1)$$

where $p_i$ is the probability of a given microstate among $W$ different ones and $q$ is a fixed real parameter. The new entropy has the usual properties of positivity, equiprobability, and reduces to the conventional Boltzmann–Gibbs entropy $S = -\sum_i p_i \ln p_i$ in the limit $q \to 1$.

- A generalized definition of internal energy

$$U_q = \sum_i \epsilon_i p_i^q \quad (2)$$

and, accordingly, a generalization of the $q$-expectation value of an observable $A$ which can be expressed as $\langle A \rangle_q \equiv \sum_i A_i p_i^q$. In the limit $q \to 1$, $\langle A \rangle_1$ corresponds to the standard mean value. This postulate plays a central role in the derivation of the equilibrium distribution and leads to the correct thermodynamic relations.

The deformation parameter $q$ measures the degree of nonextensivity of the theory. In fact, if we have two independent systems $A$ e $B$, such that the probability of $A + B$ is factorized into $p_{A+B}(u_A, u_B) = p_A(u_A) p_B(u_B)$, the global entropy is not simply the sum of their entropies and it is easy to verify that

$$S_q(A + B) = S_q(A) + S_q(B) + (1 - q)S_q(A)S_q(B) . \quad (3)$$

Another important property is that $S_q$ is consistent with Laplace’s maximum ignorance principle, i.e., if $p_i = 1/W, \ \forall i$ and $W \geq 1$ (equiprobability) one has the following extremum value \[2\]

$$S_q = \ln_q W , \quad (4)$$

where we have defined the generalized logarithmic function $\ln_q x$ as

$$\ln_q x = \frac{x^{1-q} - 1}{1 - q} . \quad (5)$$

In the limit $q \to 1$, $\ln_q x \to \ln x$ and Eq.(4) reproduces Boltzmann’s celebrated formula $S = \ln W$. Let us stress the importance of this result for the purpose of the following
discussion, because it defines the generalized logarithmic function in nonextensive statistics and plays a crucial role in the determination of the $q$-expectation value of an observable, consistent with the thermodynamic relations as we shall see in Eq.(8) below.

Working with the canonical ensemble, the probability distribution can be obtained by extremizing the entropy $S_q$ under fixed internal energy $U_q$ constraint and norm constraint ($\sum_i p_i = 1$). The outcome of this optimization procedure gives the result

$$p_i = \frac{[1 - (1 - q)\beta \epsilon_i]^{\frac{1}{1 - q}}}{Z_q},$$

where $Z_q$ is the partition function given by

$$Z_q = \sum_i [1 - (1 - q)\beta \epsilon_i]^{\frac{1}{1 - q}}.$$  \(6\)

Using the generalized expression (5) of the logarithmic function and Eqs.(2) and (7), it has been shown that \[2\]

$$U_q = -\frac{\partial}{\partial \beta} \ln_q Z_q.$$  \(8\)

On the basis of the above relation, the entire mathematical structure of the connection between standard statistical mechanics and thermodynamics is preserved by the generalization of the Tsallis entropy, the definition of the internal energy and replacing $\ln Z$ by $\ln_q Z_q$.

Recently Abe \[6\] has observed the connection between Tsallis entropy and Jackson derivatives which can be expressed as

$$S_q = -\partial_x^{(q)} \sum_i p_i^x \bigg|_{x=1} = -\sum_i \frac{p_i^q - p_i}{q - 1},$$

where

$$\partial_x^{(q)} f(x) = \frac{f(qx) - f(x)}{x(q - 1)},$$

is the Jackson derivative \[11\], which in the limit $q \to 1$, becomes the ordinary differential. The above connection is not just a coincidence but in fact it has been shown that the Jackson derivative can be identified with the generators of fractal and multifractal sets with discrete dilatation symmetries \[12\] and thus it is strictly related to Tsallis statistics.

The Jackson derivative in Eq.(10) is intimately connected with $q$-deformed structures in $q$-oscillator theory, signified by the $q$-basic number

$$[x]_q = \frac{q^x - 1}{q - 1}.$$  \(11\)

It has been shown that the pseudo-additivity property of the Tsallis entropy, displayed in Eq.(3), is also valid for the above $q$-basic number \[3\].
We now develop the \( q \)-symmetric theory of the Tsallis thermostatistics. In \( q \)-deformed structures, when one constructs the theory which is invariant under \( q \leftrightarrow q^{-1} \), the Jackson derivative has to be generalized to the form

\[
\mathcal{D}^{(q)}_x f(x) = \frac{f(qx) - f(q^{-1}x)}{x(q - q^{-1})},
\]

and correspondingly, it is possible to introduce the symmetric Tsallis entropy given by

\[
S^S_q = -\mathcal{D}^{(q)}_x \sum_{i} p_i^x \bigg|_{x=1} = -\sum_{i} \frac{p_i^q - p_i^{q^{-1}}}{q - q^{-1}}.
\]

The above expression for the symmetric Tsallis entropy satisfies a generalized pseudo-additivity property formally similar to the symmetric \( q \)-basic number defined by

\[
[x]^S_q = \frac{q^x - q^{-x}}{q - q^{-1}}.
\]

It is easy to show that the \( q \)-symmetric Tsallis entropy can be written in terms of the nonsymmetric Tsallis entropy in the compact form

\[
S^S_q = c_1 S_q + c_2 S_{q^{-1}},
\]

where \( c_1 \) and \( c_2 \) are two coefficients (always positive) which depend only on \( q \) and \( q^{-1} \),

\[
c_1 = \frac{q - 1}{q - q^{-1}}, \quad c_2 = \frac{1 - q^{-1}}{q - q^{-1}}, \quad c_1 + c_2 = 1.
\]

We wish to stress that the above expression is a consequence of the connection between the \( q \)-symmetric and the nonsymmetric \( q \)-structures. In fact we observe that the Jackson symmetric derivative in Eq.\((12)\) can be expressed in terms of the nonsymmetric ones

\[
\mathcal{D}^{(q)}_x = c_1 \partial^{(q)}_x + c_2 \partial^{(q^{-1})}_x.
\]

An analogous relation also holds for the \( q \)-basic number in quantum groups

\[
[x]^S_q = c_1 [x]_q + c_2 [x]_{q^{-1}}.
\]

Accordingly the above relations provide us with a recipe to obtain the symmetric generalization from the nonsymmetric structures.

As in the case of Tsallis entropy, the equiprobability distribution \( (p_i = 1/W, \ \forall i) \) can be derived by employing a new symmetric definition of the logarithmic function

\[
S^S_q = \ln^S_q W = c_1 \ln_q W + c_2 \ln_{q^{-1}} W.
\]

The above result is very important for the correct construction of a generalized symmetric thermostatistics. In fact this allows us to obtain the symmetric logarithm of the partition function (thermodynamic potential) in terms of a linear combination of the Tsallis one.
\[
\ln^S_q Z_q = c_1 \ln_q Z_q + c_2 \ln_{q^{-1}} Z_{q^{-1}} .
\]  
(20)

Because Tsallis statistics satisfies the condition in Eq. (3), it is also verified immediately in symmetric nonextensive statistics that

\[
U^S_q = -\frac{\partial}{\partial \beta} \ln^S_q Z_q ,
\]
(21)

if we choose the symmetric internal energy to be in the form

\[
U^S_q = c_1 U_q + c_2 U_{q^{-1}} \equiv \sum_i \left( (q-1) p_i^q - (q^{-1} - 1) p_i^{q^{-1}} \right) .
\]
(22)

The definition in Eq. (22) of the internal energy implies a generalized symmetric \( q \)-expectation value of a physical observable \( A \)

\[
\langle A \rangle^S_q = c_1 \langle A \rangle_q + c_2 \langle A \rangle_{q^{-1}} \equiv \sum_i A_i \left( (q-1) p_i^q - (q^{-1} - 1) p_i^{q^{-1}} \right) .
\]
(23)

where we note that the second relation on the right hand side is true only if \( A \) does not depend on \( q \) (such as the case of energy or particle number).

The above results are very important because we have a new generalized definition of the internal energy, which together with the symmetric Tsallis entropy, preserves all the thermodynamic relations (Legendre transformations). This follows directly from the \( q \leftrightarrow q^{-1} \) invariance of the \( q \)-deformed algebra, thus offering a closer connection between nonextensive Tsallis statistics and \( q \)-deformed structures.

Following the standard procedure, the probability distribution can be obtained by extremizing the entropy \( S^S_q \) under fixed internal energy \( U^S_q \) constraint and the norm constraint \( \sum_i p_i = 1 \). The result can be written as a linear combination

\[
p^S_i = c_1 \left[ \frac{1 - (1-q) \beta \epsilon_i}{Z_q} \right]^{1-q} + c_2 \left[ \frac{1 - (1-q^{-1}) \beta \epsilon_i}{Z_{q^{-1}}} \right]^{1-q^{-1}} .
\]
(24)

In the limit \( q \to 1 \), \( p^S_i \) reduces to the standard Maxwell-Boltzmann distribution. We note that the extremization procedure only establishes that the solution for the distribution function is a linear combination of Tsallis distribution evaluated at \( q \) and \( q^{-1} \). Eq. (24) is a reasonable choice and follows the prescription of the \( q \)-calculus. In fact in the \( q \)-oscillator theory the statistical distribution function can be written as the same linear combination of the non-symmetric ones \([4,13] \)

\[
f^S_q = c_1 f_q + c_2 f_{q^{-1}} ,
\]
(25)

where \( f_q \) and \( f_{q^{-1}} \) are the distribution functions in non-symmetric \( q \)-boson oscillators

\[
f_q = \frac{1}{e^{\beta \omega} - q} .
\]
(26)
In Fig. 1, we show the plot of the normalized probability function (24) against \( \beta \epsilon \) for different values of \( q \). Let us note that the above distribution has no cut-off as in Tsallis’s distribution for \( q < 1 \) and the high energy tail of the distribution is always enhanced compared to the Maxwell-Boltzmann distribution since \( p_i^S \) has the following power law behavior at high energy, \( p_i^S = a E^{1/(1-q)} + b E^{1/(1-q^{-1})} \).

In light of the above discussion, making a Legendre transform of the function \( \ln^S Z_q \) it is easy to verify the validity of the relation

\[
S_q^S = \beta U_q^S + \ln^S Z_q ,
\]

which implies the standard thermodynamic relation

\[
\frac{\partial S_q^S}{\partial U_q^S} = \frac{1}{T}
\]

and the \( q \)-deformed symmetric free energy is given by

\[
F_q^S = -\frac{1}{\beta} \ln^S Z_q = U_q^S - TS_q^S .
\]

All the above equations reduce to the standard thermodynamic relations in the limit \( q \to 1 \).

Finally, we note that Tsallis \([14]\) recently introduced a normalization procedure for the \( q \)-expectation value of an observable in order to remove some anomalies, such as non-additivity of the generalized internal energy and non-invariance of the probability distribution under the choice of origin of the energy spectrum. In the framework of the new generalization, all the results of the present investigation remain unaltered if we implement the normalization according to

\[
\tilde{U}_q^S = c_1 \tilde{U}_q + c_2 \tilde{U}_q^{-1} ,
\]

where \( \tilde{U}_q \) is the normalized Tsallis internal energy

\[
\tilde{U}_q = \frac{\sum_i \tilde{p}_i^q \epsilon_i}{\sum_i \tilde{p}_i^q} ,
\]

and \( \tilde{p}_i \) is the modified Tsallis distribution in the normalized \( q \)-expectation value given by \([14]\)

\[
\tilde{p}_i = \left[ 1 - (1-q)\beta(\epsilon_i - \tilde{U}_q)/\sum_j \tilde{p}_j^q \right]^{\frac{1}{1-q}} ,
\]

with

\[
\tilde{Z}_q = \sum_i \left[ 1 - (1-q)\beta(\epsilon_i - \tilde{U}_q)/\sum_j \tilde{p}_j^q \right]^{\frac{1}{1-q}} .
\]

In summary, we have shown that it is possible to extend Tsallis thermostatistics to the \( q \leftrightarrow q^{-1} \) symmetric generalization which preserves the Legendre transformation of standard
thermodynamics. This is achieved by introducing a $q$-symmetric expectation value which follows directly from the extension of the nonsymmetric $q$-deformed theory to the symmetric one. We thus establish a closer connection between nonextensive Tsallis statistics and $q$-deformed structures.

In conclusion, the relevance of the $q \leftrightarrow q^{-1}$ symmetry is well-known in $q$-oscillators from the mathematical structure as well as in applications [15–18]. Similarly we expect the symmetric Tsallis thermostatistics to be useful in many future investigations.

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FIG. 1. Plot of the symmetric normalized probability distribution, Eq. (24), versus $\beta \epsilon$ for $q = 1.0$ (Maxwell-Boltzmann distribution), solid line; $q = 1.5$ (equivalently $q^{-1} = 0.67$), long-dashed line and $q = 2.0$ (equivalently $q^{-1} = 0.5$), short-dashed line.