Hölder stability in determining elastic coefficients of Biot’s system in poroelastic media

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Abstract

In this paper, we investigate an inverse problem of determining the four spatially varying elastic coefficients of Biot’s system simultaneously, i.e., the two Lamé parameters, the dilatational coupling factor and the bulk modulus, by a single measurement of data on a neighbourhood of the boundary. Following the idea of the B-K method, we prove the Hölder stability estimate of this inverse problem based on Carleman estimates.

1. Introduction

It is an important topic to determine coefficients of wave equations from measurements of boundary observations in inverse problems. In this research topic, there is a considerable number of papers dealing with the uniqueness and stability, for example, see [1–6] and the references therein. The analysis is based on the Carleman estimate near the boundary. A Carleman estimate, originally proposed by Carleman [7], is an inequality for the solution to a partial differential equation with the weighted $L^2$-norm. Since [7], the theory of Carleman estimates has been studied extensively (e.g., [8–11]). Since the Carleman estimate depends essentially on the type of differential equation, many difficulties arise in particular for boundary value problems of wave equations.

For works on Carleman estimates for scalar wave equations, we refer to [12–17]. In [17], Imanuvilov and Yamamoto modified the Bukhgeim-Klibanov (B-K) method [8] to determine a coefficient in the acoustic equation. This method has been applied to prove the uniqueness and stability in determining coefficients in elastic wave equations by Isakov, Wang and Yamamoto [18]. The Carleman estimate for Lamé system has also been investigated well (e.g., [19–23]). In [20], the uniqueness in inverse problems for the isotropic Lamé system by three measurements is given. In [22], the authors proved the uniqueness and stability in determination three coefficients using two measurements. In [21], Imanuvilov and Yamamoto proved it with a single measurement. In [19], the authors proved a logarithmic stability estimate for determining spatially varying density and two Lamé coefficients by a single measurement of data on an arbitrarily given subboundary.

The poroelastic model is a more real model describing wave propagation in fluid saturated porous media than the acoustic model and the elastic model. And Biot’s linear theory has long been regarded as its basis [24, 25]. Let $\Omega$ be an open and bounded domain in $\mathbb{R}^3$ with $C^\infty$ boundary $\Gamma = \partial \Omega$. Then the wave equations in poroelastic media (i.e., Biot’s system) in the source-free case can be written as

$$\begin{cases} ho_1 \partial_t^2 u^t + \rho_{12} \partial_t^2 u^f - \Delta_{\rho,\lambda} u^t - \nabla (q \text{ div } u^t) = 0, \\
\rho_{12} \partial_t^2 u^t + \rho_{22} \partial_t^2 u^f - \nabla (q \text{ div } u^f) - \nabla (r \text{ div } u^f) = 0,
\end{cases}$$

for $(x, t) \in Q := \Omega \times (-T, T)$, $T > 0$, (1.1)

with initial condition

$$\begin{align*}
(u^t(x, 0), \partial_t u^t(x, 0)) &= (\Phi^t(x), \Psi^t(x)), \\
(u^f(x, 0), \partial_t u^f(x, 0)) &= (\Phi^f(x), \Psi^f(x)),
\end{align*}$$

for $x \in \Omega$, (1.2)
where \( \nabla = (\partial_1, \partial_2, \partial_3)^T \) and \( \Delta_{\mu, \lambda} \) is the elliptic linear differential operator given by
\[
\Delta_{\mu, \lambda} v(x) = \mu(x) \Delta v(x) + (\mu(x) + \lambda(x)) (\nabla \cdot (\nabla v(x))) + (\nabla v(x)) \nabla \left( \lambda(x) + (\nabla v(x)) \nabla \mu(x) \right), \quad x \in \Omega,
\]
for \( v = (v_1, v_2, v_3)^T \). Throughout this paper, \( u^i = (u_1^i, u_2^i, u_3^i)^T \) and \( u^f = (u_1^f, u_2^f, u_3^f)^T \) denote, respectively, the solid frame and fluid phase displacement vectors at the location \( x \) and the time \( t \), where \( T \) denotes the transpose of matrices. In (1.1), \( \mu, \lambda, q \) and \( r \) are the four elastic coefficients. Moreover, specifically, \( \mu(x) \) and \( \lambda(x) \) are the Lamé parameters, \( q(x) \) is the dilatational coupling factor and \( r(x) \) is the bulk modulus. And \( \rho_{11}, \rho_{12}, \rho_{22} \) are the density parameters which can be expressed in terms of solid and fluid densities, the porosity and the tortuosity. We will assume the Lamé parameters \( \mu \) and \( \lambda \) satisfy
\[
\mu(x) > 0, \quad \lambda(x) + \mu(x) > 0, \quad \forall x \in \Omega. \tag{1.4}
\]

We can prove (e.g., [26]) that the system (1.1)–(1.2) possesses a unique solution with \( \Phi = (\Phi^i, \Phi^f) \in (H^2(\Omega))^6 \), \( \Psi = (\Psi^i, \Psi^f) \in (H^2(\Omega))^6 \) and suitable boundary conditions. Here \( H^m(\Omega) \) \((m \in \mathbb{N})\) is the usual function space (e.g., [27]). For given initial data \( \Phi \) and \( \Psi \), the suitable boundary condition and the coefficients \( \rho_{11}, \rho_{12}, \rho_{22} \), we denote the solution to the system (1.1)–(1.2) by \( \Phi = u = (u^i, u^f) \).

In this paper, we study the stability of the inverse problem for determining the four elastic coefficients simultaneously. To our best knowledge, the work on the inverse problem for the system (1.1) is not enough. In [28], Bellassoued and Yamamoto established a Hölder stability estimate for the source inverse problem rather than the coefficients inverse problem based on a new Carleman estimate for system (1.1). In [29], Bellassoued and Riahi established a Carleman estimate for Biot’s consolidation model [30] and proved the Lipschitz stability and the uniqueness in determining the coefficients. In [31], the authors proved the logarithmic stability of an inverse problem for Biot’s consolidation model in poro-elasticity for determining all coefficients simultaneously provided that initial data satisfy some nondegenerate condition. Both the Biot’s consolidation model and Biot’s system (1.1) involve poro-elasticity. However, they are different. The obvious difference is that there are the scalar temperature function and the consolidation effects in Biot’s consolidation model while they are not appeared in (1.1). In addition, the fluid phase displacement vector \( u^f \) in (1.1) also makes a difference, since we can only control \( \nabla \cdot u^f \) in the existing Carleman estimate.

Let \( \omega \subset \Omega \) be an arbitrary given subdomain such that \( \partial \omega \supset \partial \Omega \) and \( \omega_T = \omega \times (-T, T) \). The purpose of this article is to determine \( \mu(x), \lambda(x), q(x), r(x) \) by one measurement \( u(\mu, \lambda, q, r) \) in \( \omega_T \). We will follow the idea used in scale isotropic non-stationary Lamé system in [32] to prove the stability of this inverse problem. The main achievement of this paper is to prove a Hölder stability estimate for simultaneously determining the four elastic coefficients with a single observation (i.e., theorem 2.3). Moreover, a novel sufficient condition for the initial data is proposed originally.

2. The main result

In order to state the main result of this paper, we first introduce some assumptions. For \( x_0 \in \mathbb{R}^3 \setminus \bar{\Omega} \), we define the following set of the scalar coefficients for given constants \( m > 0 \) and \( \theta \in (0, 1) \):
\[
\mathcal{C}(m, \theta) = \left\{ c \in C^2(\bar{\Omega}), c(x) > c^* > 0, x \in \bar{\Omega}, \|c\|_{C^2(\bar{\Omega})} \leq m, \frac{\nabla c \cdot \theta(x-x_0)}{2c} \leq 1 - \theta \right\}. \tag{2.1}
\]
Now we introduce the following four assumptions A.1–A.4.

**Assumption A.1.** We assume that the coefficients
\[
(\rho_{ij})_{1 \leq i < j \leq 2}, \mu(x), \lambda(x), q(x), r(x) \in C^2(\bar{\Omega})
\]
satisfy
\[
\rho(x) = \rho_{11}\rho_{22} - \rho_{12}^2 > 0, \quad \lambda(x) \cdot r(x) - q^2(x) > 0, \quad \forall x \in \bar{\Omega}. \tag{2.2}
\]

**Assumption A.2.** Let \( A(x) \) be the matrix given by
\[
A(x) = \frac{1}{\rho} \begin{pmatrix} \rho_{22} & -\rho_{12} \\ -\rho_{12} & \rho_{11} \end{pmatrix} \begin{pmatrix} 2\mu + \lambda & q \\ q & r \end{pmatrix}.
\tag{2.3}
\]
We assume that \( A(x) \) have two distinct positive eigenvalues: \( \mu_1(x), \mu_2(x) > 0, \mu_1 \neq \mu_2 \). Moreover, we set \( \mu_3 = \left( \rho^{-1}\rho_{22} \right) \mu \) and assume \( \mu_1, \mu_2, \mu_3 \in \mathcal{C}(m, \theta) \).
Assumption A.3. For fixed functions \(\mu_0, \lambda_0, q_0, r_0\) on \(\omega\) and a given constant \(M > 0\), we set
\[
\Lambda = \{(\mu, \lambda, q, r) = (\mu_0, \lambda_0, q_0, r_0)\} \quad \text{in} \quad \omega, \quad ||u(\mu, \lambda, q, r)||_{W^{k-\infty}(\Omega)} \leq M.
\]
We assume \((\mu, \lambda, q, r) \in \Lambda\). Here we need the extra information of coefficients under consideration in a neighbourhood of \(\partial \Omega\) to guarantee the condition in lemma 3.3.

In order to simultaneously determine the four elastic coefficients, we choose the initial data \(\Phi, \Psi\) satisfying the following assumption A.4.

Assumption A.4. We set \(\varepsilon(\nu) = \frac{1}{\nu}((\nabla \nu)^T + \nabla \nu)\), and assume that the eigenvalues as well as the corresponding normalized eigenvectors of \(\varepsilon(\Phi^s)\) and \(\varepsilon(\Psi^s)\) have bounded derivatives with respect to \(x\). That is to say, if the scalar functions \(a(x)\), \(b(x)\) and the vector functions \(y(x)\), \(z(x)\) satisfy
\[
\varepsilon(\Phi^s)y = ay, \quad ||y||^2 = 1, \\
\varepsilon(\Psi^s)z = bz, \quad ||z||^2 = 1,
\]
then the derivatives of \(a(x)\), \(b(x)\), \(y(x)\) and \(z(x)\) with respect to \(x\) exist and are bounded. Moreover, we assume that
\[
\det \begin{pmatrix} 1 - x_3^2 \div \Phi^s & 2(x - x_0)^T \varepsilon(\Phi^s)(x - x_0) \\ 1 - x_3^2 \div \Phi^s & 2(x - x_0)^T \varepsilon(\Phi^s)(x - x_0) \end{pmatrix} = 0,
\]
\[
\det \begin{pmatrix} \div \Phi^s & \div \Phi^s \\ \div \Psi^s & \div \Psi^s \end{pmatrix} = 0, \quad \forall x \in \Omega.
\]

Remark 2.1. We remark that the matrices in (2.5)–(2.6) are both \(2 \times 2\) matrices, which is more simple in comparison with the previous condition in [19] (PP.1331). If the similar condition in [19] for the initial data is applied here, we need to introduce a \(18 \times 13\) full rank matrix.

Remark 2.2. If a symmetric matrix \(P(\zeta)\) is a continuously differentiable function of single variable \(\zeta\), then its eigenvalues are also continuously differentiable with respect to \(\zeta\), for which one can refer to [33] and theorem 6.8 in [34]. However, for a matrix with multiple variables, there exist some counter examples. Fortunately, we still can choose the initial data satisfying the assumption A.4. If we let \(\Phi', \Psi'\) be the form of \((\phi_1'(x_1), \phi_2'(x_2), \phi_3'(x_3))^T\) and \((\psi_1'(x_1), \psi_2'(x_2), \psi_3'(x_3))^T\) respectively, we can easily verify that \(\varepsilon(\Phi')\) and \(\varepsilon(\Psi')\) are diagonal matrices with eigenvalues \(\partial_1 \phi_1'(x_1)\) and \(\partial_1 \psi_1'(x_1), k = 1, 2, 3\). Then we can choose \(\Phi'\) and \(\Psi'\) satisfying (2.5) and (2.6) together. For example, if we take
\[
\Phi'(x) = \begin{pmatrix} x_1 \\ x_2 \\ 0 \\ 0 \end{pmatrix}, \quad \Psi'(x) = \begin{pmatrix} x_1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \Phi'(x) = \Psi'(x),
\]
and choose \(x_0 = (x_0^1, x_0^2, x_0^3)\) such that \(3(x_0^3 - x_1)^2 > |x_0 - x|^2\) for all \(x = (x_0, x_0, x_0) \in \Omega\), then the assumption A.4 is satisfied.

The purpose of this paper is to analyze the stability in the inverse problem of determining the four spatially varying elastic coefficients \(\mu(x), \lambda(x), q(x), r(x)\) in (1.1) with a single measurement of data on a neighbour of the boundary. The main result is given by theorem 2.3, which shows that a single measurement on a neighbour of the boundary is enough to determine the four spatially varying elastic coefficients.

Theorem 2.3. Let \(\omega\) be an arbitrary fixed subdomain of \(\Omega\) such that \(\partial \omega \supset \partial \Omega\) and \(T > 0\) sufficiently large for \(\Omega\), \(\omega\). If the assumption A.4 holds, then for any \((\mu, \lambda, q, r)\) and \((\tilde{\mu}, \tilde{\lambda}, \tilde{q}, \tilde{r})\) satisfying the assumptions A.1–A.4, there exist constants \(C > 0\) and \(\nu \in (0, 1)\) such that the following estimate holds
\[
||\mu - \tilde{\mu}||_{H^\nu(\Omega)} + ||\lambda - \tilde{\lambda}||_{H^\nu(\Omega)} + ||q - \tilde{q}||_{H^\nu(\Omega)} + ||r - \tilde{r}||_{H^\nu(\Omega)} \leq ||u(\mu, \lambda, q, r) - \tilde{u}(\tilde{\mu}, \tilde{\lambda}, \tilde{q}, \tilde{r})||_{H^\nu_{L^2}(\Omega)}.
\]

The remainder of the paper is to prove theorem 2.3.

3. Proof of theorem 2.3

In this section, we will prove theorem 2.3. There are two subsections in this section. In section 3.1, we present some necessary lemmas. Then in section 3.2, we show detail proof of theorem 2.3.
3.1. Necessary lemmas

We introduce some notations first. We set

\[ \partial_j = \frac{\partial}{\partial x_j}, \quad \nabla_{x,t} = (\partial_t, \partial_2, \ldots, \partial_n), \quad |\nabla_{x,t} v(x, t)| = \left( \left( \frac{\partial v}{\partial x} \right)^2 + \left( \frac{\partial v}{\partial t} \right)^2 \right)^{\frac{1}{2}}. \]

Let \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \) be a multi-index with \( \alpha_j \in \mathbb{N} \cup \{0\} \), we set \( \partial^\alpha_{x,t} = \partial^\alpha_1 \partial^\alpha_2 \cdots \partial^\alpha_n \), \( \alpha = \alpha_1 + \alpha_2 + \cdots + \alpha_n \).

For arbitrary fixed \( x_0 \not\in \Omega \), let \( \psi : \Omega \to \mathbb{R} \) be given by

\[ \psi(x) = |x - x_0|^2, \quad x \in \Omega, \]

and set

\[ \psi(x, t) = \psi(x) - \kappa t^2, \quad \varphi(x, t) = e^{\delta(x,t)}, \]

where \( \kappa \) is a chosen constant \([28]\) and \( \beta > 0 \). Moreover, set

\[ D^2 = \max_{x \in \Omega} \psi(x), \quad d^2 = 2 \min_{x \in \Omega} \psi(x), \quad D_0^2 = D^2 - d^2, \quad T_0 = \frac{D_0}{\sqrt{\kappa}}, \]

then for \( T > T_0 \) we have

\[ \varphi_{0}(x) = \varphi(x, 0) \geq d_0, \quad \varphi(x, \pm T) < d_0, \]

with \( d_0 = e^{4d^2} \). Thus for given \( \eta > 0 \), we can choose small \( \delta \) such that

\[ \varphi(x, t) \leq d_0 - \eta \equiv d_1, \quad \forall (x, t) \in \{(x, t) \in \Omega; |t| > T - \delta\}. \]

We denote \( N_{\tau,\varphi}(v) \) by

\[ N_{\tau,\varphi}(v) = \int_{\Omega} \tau \left( |\nabla_{x,t} v|^2 + |\nabla_{x,t} (\text{div } v)|^2 + |\nabla_{x,t} (\text{rot } v)|^2 \right) e^{2\tau x} \text{d}x \text{d}t \]

\[ + \int_{\Omega} \tau \left( |v|^2 + |\text{div } v|^2 + |\text{rot } v|^2 \right) e^{2\tau x} \text{d}x \text{d}t, \]

where \( v = (v^1, v^2) \) and \( \tau > 0 \).

The left hand side of Carleman estimate in lemma 3.2 in \([28]\) is in fact \( N_{\tau,\varphi}(\tilde{v}) \), where \( \tilde{v} = v \) multiplied by a cut-off function \( \eta(t) \) satisfying \( \eta(t) = 1 \) for \(|t| < T - 2\epsilon\) and \( \eta(t) = 0 \) for \(|t| > T - \epsilon\).

However, we can select \( \epsilon \) small enough such that \( \varphi(x, t) < d_1 \) for \(|t| > T - \epsilon\), then the difference between \( N_{\tau,\varphi}(\tilde{v}) \) and \( N_{\tau,\varphi}(v) \) can be absorbed by \( e^{2\tau T} \|\tilde{v}\|^2_{L^2(-T, T; H^2(\Omega))} \) when \( \tau \) is large. Therefore, inequality (3.3) is still true.

Lemma 3.1. \([28]\) Suppose the assumptions A.1–A.4 hold, then there exist positive constants \( \tau_0, C > 0 \) and \( C_0 > 0 \) such that the following inequality holds

\[ C N_{\tau,\varphi}(v) \leq \int_{\Omega} |G|^2 + |\nabla G|^2 e^{2\tau x} \text{d}x \text{d}t + e^{-C_0 T} \|v\|^2_{L^2(\Omega)} + e^{2\tau T} \|\text{div } v\|^2_{L^2(\Omega)}, \]

for any \( \tau \geq \tau_0 \) and any \( v \in (H^2(\Omega))^d \) and \( G = (G_1, G_2) \in (H^1(\Omega))^d \) satisfying

\[ \begin{align*}
\rho_1 \partial_t^2 v^1 + \rho_2 \partial_t^2 v^2 - \Delta u_{\alpha} v^1 - \nabla (q \text{ div } v^2) &= G_1, \\
\rho_2 \partial_t^2 v^1 + \rho_1 \partial_t^2 v^2 - \nabla (q \text{ div } v^1) - \nabla (r \text{ div } v^2) &= G_2, \quad (x, t) \in Q.
\end{align*} \]

Remark 3.2. The left hand side of Carleman estimate in lemma 3.2 in \([28]\) is in fact \( N_{\tau,\varphi}(\tilde{v}) \), where \( \tilde{v} = v \) multiplied by a cut-off function \( \eta(t) \) satisfying \( \eta(t) = 1 \) for \(|t| < T - 2\epsilon\) and \( \eta(t) = 0 \) for \(|t| > T - \epsilon\). However, we can select \( \epsilon \) small enough such that \( \varphi(x, t) < d_1 \) for \(|t| > T - \epsilon\), then the difference between \( N_{\tau,\varphi}(\tilde{v}) \) and \( N_{\tau,\varphi}(v) \) can be absorbed by \( e^{2\tau T} \|\tilde{v}\|^2_{L^2(-T, T; H^2(\Omega))} \) when \( \tau \) is large. Therefore, inequality (3.3) is still true.

Lemma 3.3. Suppose \( S(x) = (a_{ij}(x))_{1 \leq i, j \leq 3}, x \in \Omega \) is a real symmetric matrix, and its eigenvalues and the corresponding normalized eigenvectors have bounded derivatives with respect to \( x \), then there exists a positive constant \( C = C(\Omega, \beta, \alpha_0) > 0 \) such that the following estimate holds

\[ \tau \int_{\Omega} (|\nabla \varphi|^2 u + (\nabla \varphi)^2 S(x) \nabla \varphi \cdot v(x))e^{2\tau x} \text{d}x \leq C \int_{\Omega} (|f|^2 + u^2 + v^2) e^{2\tau x} \text{d}x, \]

for any \( u, v \in H^1(\Omega), u |_{\Gamma} = v |_{\Gamma} = 0 \) and \( f \in (L^2(\Omega))^3 \) satisfying

\[ \nabla u(x) + S(x) \nabla v(x) = f(x). \]

Proof. Since \( S(x) \) is a symmetric matrix, there exists an orthogonal matrix \( U(x) \) such that \( U^T SU = \text{diag}(\lambda_1, \lambda_2, \lambda_3) \). That is to say, there exists an orthonormal basis \( d_1(x), d_2(x), d_3(x) \in \mathbb{R}^3 \) such that

\[ d_k(x) S(x) = \lambda_k(x) d_k(x), \quad d_j \cdot d_k^T = \delta_{jk}, \quad j, k = 1, 2, 3. \]
Taking inner product on both sides of (3.6) with $d_k$, we obtain
\[ d_k \cdot \nabla u + \lambda_k d_k \cdot \nabla v = d_k \cdot f, \]  
and by the assumption of the existence of the derivatives of $\lambda_k$, we can rewrite (3.7) as
\[ d_k \cdot (\nabla u + \lambda_k v) = d_k \cdot f + (d_k \cdot \nabla \lambda_k) v. \]
We set $\gamma_k = d_k \cdot \nabla \vartheta, w_k = u + \lambda_k v$, then we have $\nabla \vartheta = \sum_{k=1}^{3} \gamma_k d_k^t$, and
\[ \int_{\Omega} \gamma_k^2 (d_k \cdot \nabla w_k) w_k e^{2\tau \vartheta} \, dx = -\int_{\Omega} \left( \nabla (\gamma_k^2 d_k^t) w_k^2 + \gamma_k^2 (d_k \cdot \nabla w_k) w_k + 2\tau \beta \varphi_0 \gamma_k (d_k \cdot \nabla \vartheta) w_k^2 \right) e^{2\tau \vartheta} \, dx, \]
by the Green formula. So there exists $C = C(\mathcal{S}, \Omega, \beta, x_0) > 0$ such that
\[ \tau \int_{\Omega} \sum_{k=1}^{3} \gamma_k^2 w_k^2 e^{2\tau \vartheta} \, dx \leq C \int_{\Omega} (|f|^2 + u^2 + v^2) e^{2\tau \vartheta(\vartheta)} \, dx. \]  
(3.8)

On the other hand, we have
\[ \sum_{k=1}^{3} \gamma_k^2 w_k^2 \geq \frac{1}{3} \left( \sum_{k=1}^{3} \gamma_k^2 w_k^2 \right) = \frac{1}{3} \left( \sum_{k=1}^{3} \gamma_k^2 u^2 + \sum_{k=1}^{3} \gamma_k^2 \lambda_k v^2 \right) = \frac{1}{3} \left( |\nabla \vartheta|^2 u^2 + (\nabla \vartheta)^t \mathcal{S}(\nabla \vartheta) \cdot (v(x))^2 \right). \]  
(3.9)

Combining (3.8) and (3.9), we obtain (3.5).

\[ \sum_{|\alpha| \leq 2} \| e^{\tau \nabla^\alpha \vartheta} v \|_{L^2(\Omega)} \leq C(\mathcal{S}, \Omega, \beta) \left( C(\mathcal{S}, \Omega, \beta, x_0) \right) \left( \| \nabla \vartheta \|_{L^2(\Omega)} + \| \nabla \vartheta \|_{L^2(\Omega)} + \| \nabla \vartheta \|_{L^2(\Omega)} \right), \]  
(3.10)

whenever $\tau > \tau_0$ and $v \in (H^2) \Omega$.

**Proof.** Note that
\[ \int_{\Omega} |\nabla \vartheta|^2 e^{2\tau \vartheta} \, dx = \sum_{k=1}^{3} \int_{\Omega} |\nabla \vartheta|^2 e^{2\tau \vartheta} \, dx = \sum_{k=1}^{3} \int_{\Omega} \left( \Delta \vartheta \vartheta + 2\tau \beta \varphi_0 (\nabla \vartheta) \cdot (\nabla \vartheta) \right) e^{2\tau \vartheta} \, dx \]
\[ = -\sum_{k=1}^{3} \int_{\Omega} \left( \Delta \vartheta \vartheta + 2\tau \beta \varphi_0 (\nabla \vartheta) \cdot (\nabla \vartheta) \right) e^{2\tau \vartheta} \, dx \]
\[ \leq \sum_{k=1}^{3} \int_{\Omega} \left( \frac{1}{\tau^2} |\nabla \vartheta|^2 + \tau^2 \vartheta \vartheta + \tau \beta \varphi_0 \left( \frac{1}{\tau^2} |\nabla \vartheta|^2 + \tau^2 \vartheta \vartheta \right) \right) e^{2\tau \vartheta} \, dx. \]  
(3.11)

Hence there exist constants $\tau_0$ and $C > 0$ depending on $\beta, x_0, \Omega$ such that
\[ \int_{\Omega} |\nabla \vartheta|^2 e^{2\tau \vartheta} \, dx \leq C \int_{\Omega} \left( \frac{1}{\tau^2} |\nabla \vartheta|^2 + \tau^2 \vartheta \vartheta \right) e^{2\tau \vartheta} \, dx = C \int_{\Omega} \left( \frac{1}{\tau^2} |\nabla \vartheta|^2 + \tau^2 \vartheta \vartheta \right) e^{2\tau \vartheta} \, dx, \]  
(3.12)

when $\tau > \tau_0$. Similarly,
\[ \int_{\Omega} |\nabla \vartheta|^2 e^{2\tau \vartheta} \, dx = -\sum_{k=1}^{3} \int_{\Omega} \left( \nabla \vartheta \vartheta + \tau \vartheta \vartheta + \tau \beta \varphi_0 \left( \frac{1}{\tau^2} |\nabla \vartheta|^2 + \tau^2 \vartheta \vartheta \right) \right) e^{2\tau \vartheta} \, dx \]
\[ = \sum_{k=1}^{3} \int_{\Omega} \left( \Delta \vartheta \vartheta + 2\tau \beta \varphi_0 (\nabla \vartheta) \cdot (\nabla \vartheta) \right) e^{2\tau \vartheta} \, dx \]
\[ \leq \sum_{k=1}^{3} \int_{\Omega} \left( \frac{1}{\tau^2} |\nabla \vartheta|^2 + \tau \beta \varphi_0 \left( \frac{1}{\tau^2} |\nabla \vartheta|^2 + \tau^2 \vartheta \vartheta \right) \right) e^{2\tau \vartheta} \, dx. \]  
(3.13)
Combining (3.12) and (3.13), we obtain
\[ \sum_{|\alpha| \leq 2} \int_{\Omega} |\partial_\alpha^2 \varphi|^2 e^{2\tau \varphi} \, dx \leq C \int_{\Omega} (|\Delta \varphi|^2 + \tau^6 |\varphi|^2) e^{2\tau \varphi} \, dx. \]

The proof is completed since \( \Delta \varphi = \nabla \text{div} \varphi - \text{rot(rot} \varphi) \).

If we multiply a function \( u \in (H^2(\Omega))^3 \) by a cut-off function \( \chi \in C_0^\infty(\Omega) \) satisfying \( 0 \leq \chi \leq 1 \) and \( \chi = 1 \) in \( \Omega \setminus \omega \), we immediately obtain the following corollary.

**Corollary 3.5.** There exist constants \( \tau_0 > 0 \) and \( C = C(\tau_0, \beta, \Omega, \omega, \chi_0) > 0 \) such that
\[ \sum_{|\alpha| \leq 2} \| e^{\tau \varphi} \partial_\alpha^2 u \|^2_{L^2(\Omega)} \leq C \left( \tau^6 \| e^{\tau \varphi} (\nabla \text{div} u) \|^2_{L^2(\Omega)} + \tau^6 \sum_{|\alpha| \leq 2} \| e^{\tau \varphi} \partial_\alpha^2 u \|^2_{L^2(\omega)} \right), \]

wherever \( \tau > \tau_0 \) and \( u \in (H^2(\Omega))^3 \).

**Lemma 3.6.** There exists a positive constant \( C > 0 \) such that the following estimate holds
\[ \int_\Omega |z(x, 0)|^2 e^{2\tau \varphi} \, dx \leq C \int_Q (|z(x, t)|^2 + |\partial_t z(x, t)|^2) e^{2\tau \varphi} \, dx dt, \]
for any \( z \in L^2(Q) \) such that \( \partial_t z \in L^2(Q) \).

**Proof.** We refer to Lemma 3.4 in the paper by Bellassoued and Yamamoto [28], which says:
\[ \int_\Omega |z(x, 0)|^2 \, dx \leq C \int_Q (\tau |z(x, t)|^2 + \tau^{-1} |\partial_t z(x, t)|^2) \, dx dt, \]
and substituting \( z(x, t) = e^{\tau \varphi(x, t)} \) yields (3.15).

**3.2. Proof of theorem 2.3**

For simplicity, we denote
\[ u_* = u - \bar{u}, \lambda_* = \lambda - \bar{\lambda}, \mu_* = \mu - \bar{\mu}, q_* = q - \bar{q}, r_* = r - \bar{r}. \]

Then obviously we have
\[ \begin{cases} 
\rho_{11} \partial_t^2 y_*^{(x)} + \rho_{12} \partial_t^2 y_*^{(y)} - \Delta_{\mu_*} y_*^{(x)} - \nabla (q_* \text{div} u_*^{(x)}) = \Delta_{\mu_*} y_*^{(x)} + \nabla (q_* \text{div} u_*^{(x)}), \\
\rho_{12} \partial_t^2 y_*^{(y)} + \rho_{22} \partial_t^2 y_*^{(y)} - \nabla (q_* \text{div} u_*^{(y)}) - \nabla (\bar{r} \text{div} u_*^{(y)}) = \nabla (q_* \text{div} u_*^{(y)}) + \nabla (r_* \text{div} u_*^{(y)}),
\end{cases} \]
and
\[ \partial_t^1 y_*^{(y)}(x, 0) = \partial_t^1 y_*^{(y)}(x, 0) = 0, \quad x \in \Omega, \quad k = 0, 1. \]

Moreover, we set
\[ y_\chi(x, t) = \partial_t^1 u_*^{(x)}(x, t), \quad z_\chi(x) = y_\chi(x, 0). \]

We follow the idea by Bukhgeim and Klibanov [8]. First we control \( N_{\tau, r}(y_\chi) \) by \( \| u_* \|^2_{H^\infty(\omega')} \) and the weighted \( H^2 \)-norm of \( \mu_*, \lambda_*, q_*, r_* \) in Q. More specifically, since \( y_\chi \) satisfies:
\[ \begin{cases} 
\rho_{11} \partial_t^2 y_\chi^{(x)} + \rho_{12} \partial_t^2 y_\chi^{(y)} - \Delta_{\mu_*} y_\chi^{(x)} - \nabla (q_* \text{div} y_\chi^{(x)}) = \Delta_{\mu_*} y_\chi^{(x)} + \nabla (q_* \text{div} y_\chi^{(x)}), \\
\rho_{12} \partial_t^2 y_\chi^{(y)} + \rho_{22} \partial_t^2 y_\chi^{(y)} - \nabla (q_* \text{div} y_\chi^{(y)}) - \nabla (\bar{r} \text{div} y_\chi^{(y)}) = \nabla (q_* \text{div} y_\chi^{(y)}) + \nabla (r_* \text{div} y_\chi^{(y)}),
\end{cases} \]
we have by Lemma 3.1 that
\[ C N_{\tau, r}(y_\chi) \leq \int_Q \sum_{|\alpha| \leq 2} (|\partial_\alpha^2 y_\chi^{(x)}|^2 + |\partial_\alpha^2 y_\chi^{(y)}|^2 + |\partial_\alpha^2 q_*|^2 + |\partial_\alpha^2 r_*|^2) e^{2\tau \varphi} \, dx dt + e^{C_{\tau, r} \tau} \| y_\chi \|^2_{L^\infty(\omega')} + e^{2\tau \tau} \| y_\chi \|^2_{H^2(\omega')} + e^{2\tau \tau} \| y_\chi \|^2_{H^2(\omega')}, \]
for \( \tau \) sufficiently large and \( k = 0, 1, 2, 3, 4 \). Here we have used the assumption \( u \in W^{k, \infty}(Q) \). Here \( W^{k, \infty}(Q) \) is the Sobolev space equipped with the norm
\[ \| u \|_{W^{k, \infty}(Q)} = \sum_{|\alpha| \leq k} \| \partial_\alpha^2 u \|_{L^\infty(Q)}. \]
Next we turn to estimate the weighted $H^2$-norm of $\mu \Phi$, $\lambda \Phi$, $q \Phi$, $r_s$ in $\Omega$. Note that

$$z_0(x) = z_1(x) = 0, \quad u(x, 0) = \Phi(x), \quad \partial_t u(x, 0) = \Psi(x).$$

Setting $t = 0$ in (3.18) for $k = 0, 1$, we obtain

$$
\begin{align*}
\nabla (\lambda \Phi + q \Phi^2) + 2\varepsilon (\Phi') \nabla \mu = \rho_{11} z_1^2 + \rho_{12} z_2^2 - (\Delta \Phi' + \nabla \Phi') \mu, \\
\nabla (q \Phi^2 + r_s \Phi^2) = \rho_{11} z_1^2 + \rho_{12} z_2^2,
\end{align*}
$$

(3.20)

and

$$
\begin{align*}
\nabla (\lambda \Phi + q \Phi^2) + 2\varepsilon (\Phi') \nabla \mu = \rho_{11} z_1^2 + \rho_{12} z_2^2 - (\Delta \Phi' + \nabla \Phi') \mu, \\
\nabla (q \Phi^2 + r_s \Phi^2) = \rho_{11} z_1^2 + \rho_{12} z_2^2.
\end{align*}
$$

(3.21)

From (3.20) we obtain the following further result ($|\alpha| \leq 2$):

$$
\begin{align*}
\nabla (\partial_\alpha^\nu \lambda \Phi + \partial_\alpha^\nu q \Phi^2) + 2\varepsilon (\Phi') \nabla (\partial_\alpha^\nu \mu) = \partial_\alpha^\nu (\rho_{11} z_1^2) + \partial_\alpha^\nu (\rho_{12} z_2^2) + R_c(\lambda, \mu, q, r), \\
\nabla (\partial_\alpha^\nu q \Phi^2 + \partial_\alpha^\nu r_s \Phi^2) = \partial_\alpha^\nu (\rho_{11} z_1^2) + \partial_\alpha^\nu (\rho_{12} z_2^2) + R_c(q, r),
\end{align*}
$$

(3.22)

where $R_c$, $R_2$ are the linear differential operators of order $|\alpha|$ with bounded coefficients. In view of the assumption A.3 and that $\lambda \Phi$, $\mu \Phi$, $q \Phi$, $r_s$ and their derivatives are all vanish in $\omega$, we apply lemma 3.3 to (3.22) and obtain:

$$
\begin{align*}
\tau \int_\Omega \sum_{|\alpha| \leq 2} (|\nabla \partial_\alpha^\nu \Phi|^2 + |\partial_\alpha^\nu r_s \Phi^2 + |\partial_\alpha^\nu q \Phi^2|)^2 e^{2\tau \gamma_0(x)} dx \\
\leq C \int_\Omega \sum_{|\alpha| \leq 2} (|\partial_\alpha^\nu z_1^2| + |\partial_\alpha^\nu z_2^2| + |\partial_\alpha^\nu q|)^2 e^{2\tau \gamma_0(x)} dx,
\end{align*}
$$

(3.23)

and

$$
\begin{align*}
\tau \int_\Omega \sum_{|\alpha| \leq 2} (|\nabla \partial_\alpha^\nu \Phi|^2 + |\partial_\alpha^\nu r_s \Phi^2 + |\partial_\alpha^\nu q \Phi^2|)^2 e^{2\tau \gamma_0(x)} dx \\
\leq C \int_\Omega \sum_{|\alpha| \leq 2} (|\partial_\alpha^\nu z_1^2| + |\partial_\alpha^\nu z_2^2| + |\partial_\alpha^\nu q|)^2 e^{2\tau \gamma_0(x)} dx.
\end{align*}
$$

(3.24)

We denote $K_1 = \sup_{x \in \Omega} \nabla \partial_\alpha^\nu | \nabla \partial_\alpha^\nu \Phi|$, then

$$
\begin{align*}
\tau \int_\Omega \sum_{|\alpha| \leq 2} \left( \frac{1}{2} (|\nabla \partial_\alpha^\nu \Phi|^2 + |\partial_\alpha^\nu r_s \Phi^2 + |\partial_\alpha^\nu q \Phi^2|) + K_1 \partial_\alpha^\nu q \Phi^2 \right) e^{2\tau \gamma_0(x)} dx \\
\leq C \int_\Omega \sum_{|\alpha| \leq 2} (|\partial_\alpha^\nu z_1^2| + |\partial_\alpha^\nu z_2^2| + |\partial_\alpha^\nu q|)^2 e^{2\tau \gamma_0(x)} dx,
\end{align*}
$$

(3.25)

By the same argument for (3.21) we finally obtain

$$
\begin{align*}
\tau \int_\Omega \sum_{|\alpha| \leq 2} (|\nabla \partial_\alpha^\nu \Phi|^2 + |\partial_\alpha^\nu r_s \Phi^2 + |\partial_\alpha^\nu q \Phi^2|)^2 e^{2\tau \gamma_0(x)} dx \\
\leq C \int_\Omega \sum_{|\alpha| \leq 2} (|\partial_\alpha^\nu z_1^2| + |\partial_\alpha^\nu z_2^2| + |\partial_\alpha^\nu q|)^2 e^{2\tau \gamma_0(x)} dx,
\end{align*}
$$

(3.26)

and

$$
\begin{align*}
\tau \int_\Omega \sum_{|\alpha| \leq 2} \left( \frac{1}{2} (|\nabla \partial_\alpha^\nu \Phi|^2 + |\partial_\alpha^\nu r_s \Phi^2 + |\partial_\alpha^\nu q \Phi^2|) + K_1 \partial_\alpha^\nu q \Phi^2 \right) e^{2\tau \gamma_0(x)} dx \\
\leq C \int_\Omega \sum_{|\alpha| \leq 2} (|\partial_\alpha^\nu z_1^2| + |\partial_\alpha^\nu z_2^2| + |\partial_\alpha^\nu q|)^2 e^{2\tau \gamma_0(x)} dx,
\end{align*}
$$

(3.27)

where $K_2 = \sup_{x \in \Omega} \nabla \partial_\alpha^\nu | \nabla \partial_\alpha^\nu \Phi|$. 


Now the selection of the initial condition \( \Phi, \Psi \) helps us move on. From the assumption A.4 we know that

\[
\begin{bmatrix}
|\nabla \Phi^2| \text{ div } \Phi^r & 2(\nabla \Phi)^T \varepsilon (\Phi^r) \nabla \Phi \\
|\nabla \Psi^2| \text{ div } \Psi^r & 2(\nabla \Psi)^T \varepsilon (\Psi^r) \nabla \Psi
\end{bmatrix}
\]

and

\[
\begin{bmatrix}
\text{div } \Phi^r & \text{ div } \Phi^r \\
\text{div } \Psi^r & \text{ div } \Psi^r
\end{bmatrix}
\]

are non-degenerate matrices for all \( x \in \mathbb{T} \). So there exists a constant \( C_1 \) such that

\[
\left\| \begin{bmatrix}
\partial_x^2 \lambda_x \\
\partial_x^2 \mu_x
\end{bmatrix}
\right\| \leq C_1 \left\| \begin{bmatrix}
|\nabla \Phi^2| \text{ div } \Phi^r + 2(\nabla \Phi)^T \varepsilon (\Phi^r) \nabla \Phi \\
|\nabla \Psi^2| \text{ div } \Psi^r + 2(\nabla \Psi)^T \varepsilon (\Psi^r) \nabla \Psi
\end{bmatrix} \right\| ,
\]

(3.28)

and

\[
\left\| \begin{bmatrix}
\partial_x^2 q_x \\
\partial_x^2 q_x
\end{bmatrix}
\right\| \leq C_1 \left\| \begin{bmatrix}
\text{div } \Phi^r \partial_x^3 q_x + \text{div } \Phi^r \partial_x^3 q_x \\
\text{div } \Psi^r \partial_x^3 q_x + \text{div } \Psi^r \partial_x^3 q_x
\end{bmatrix} \right\| .
\]

(3.29)

Combining (3.24)–(3.29) together, we have

\[
\tau \int_{\Omega} \sum_{|\alpha| \leq 2} (|\partial_x^2 \lambda_x|^2 + |\partial_x^2 \mu_x|^2) e^{2 r_{\gamma}(x)} dx \\
\leq C \int_{\Omega} \sum_{|\alpha| \leq 2} \sum_{k=1}^{3} (|\partial_x^2 z_x|^2 + |\partial_x^2 z_x|^2 + |\partial_x^2 q_x|^2 + |\partial_x^2 r_x|^2) e^{2 r_{\gamma}(x)} dx,
\]

(3.30)

and

\[
\tau \int_{\Omega} \sum_{|\alpha| \leq 2} (|\partial_x^2 \lambda_x|^2 + |\partial_x^2 \mu_x|^2 + 2C_1(K_1 + K_2) |\partial_x^2 q_x|^2) e^{2 r_{\gamma}(x)} dx \\
\leq C \int_{\Omega} \sum_{|\alpha| \leq 2} \sum_{k=1}^{3} (|\partial_x^2 z_x|^2 + |\partial_x^2 z_x|^2 + |\partial_x^2 q_x|^2 + |\partial_x^2 r_x|^2) e^{2 r_{\gamma}(x)} dx.
\]

(3.31)

Making (3.30) \((2C_1(K_1 + K_2) + 1) + (3.31)\) and letting \( \tau \) large enough, we have

\[
\tau \int_{\Omega} \sum_{|\alpha| \leq 2} (|\partial_x^2 \lambda_x|^2 + |\partial_x^2 \mu_x|^2 + |\partial_x^2 q_x|^2 + |\partial_x^2 r_x|^2) e^{2 r_{\gamma}(x)} dx \\
\leq C \int_{\Omega} \sum_{|\alpha| \leq 2} \sum_{k=1}^{3} (|\partial_x^2 z_x|^2 + |\partial_x^2 z_x|^2 + |\partial_x^2 q_x|^2 + |\partial_x^2 r_x|^2) e^{2 r_{\gamma}(x)} dx + C r^6 \int_{\Omega} \sum_{|\alpha| \leq 2} \sum_{k=1}^{3} (|\partial_x^2 z_x|^2 + |\partial_x^2 z_x|^2) e^{2 r_{\gamma}(x)} dx,
\]

(3.32)

where corollary 3.5 is applied in the last step. The remaining question is that we are lack of the information of \( z_k \) and \( \nabla \text{rot } z_k \) in \( \mathcal{N}_{r_{\gamma}}(y) \). Fortunately, we can fully use (3.18), (3.20) and (3.21) to make up for this. Applying the curl to the second equation of (3.20) and (3.21) respectively, we obtain

\[
\text{rot } (\rho_{11} z_k) + \text{rot } (\rho_{22} z_k) = 0, \quad k = 2, 3.
\]

The assumption A.1 implies \( |\rho_{12}| > 0 \) for all \( x \in \overline{T} \). So we have

\[
\text{rot } (z_{k}') = -\frac{\rho_{12}}{\rho_{22}} \text{rot } (z_k) + R_3 (z_k', z_k), \quad k = 2, 3,
\]

where \( R_3 \) is a zeroth-order linear differential operator with bounded coefficients. Now we carefully compute the integration with the integrand \( |\nabla \text{rot } z_k|^2 \) for \( k = 2, 3 \). Since \( \rho_{ij} \in C^2(\overline{\Omega}) \), we get

\[
\int_{\Omega} |\nabla \text{rot } z_k|^2 e^{2 r_{\gamma}(x)} dx = \int_{\Omega} \left| \nabla \left( -\frac{\rho_{12}}{\rho_{22}} \text{rot } (z_k) + R_3 (z_k', z_k) \right) \right|^2 e^{2 r_{\gamma}(x)} dx \\
\leq C \int_{\Omega} (|\nabla z_k|^2 + |\nabla z_k|^2) e^{2 r_{\gamma}(x)} dx.
\]
By the similar argument like \((3.11)\) and \((3.12)\) we obtain
\[
\int_{\Omega} |\nabla z_{k,t}^j|^2 e^{2\tau\gamma(x,k)} \, dx \leq \int_{\Omega} |\nabla (\chi z_{k,t}^j)|^2 e^{2\tau\gamma(x,k)} \, dx + \int_{\Omega} |\nabla((1 - \chi) z_{k,t}^j)|^2 e^{2\tau\gamma(x,k)} \, dx
\leq C \int_{\Omega} \left( \tau\|{\chi z_{k,t}^j}\|_1 + \frac{1}{\tau}\|{\Delta (\chi z_{k,t}^j)}\|_1 \right) e^{2\tau\gamma(x,k)} \, dx + C \int_{\Omega} \sum_{|\alpha| \leq 1} |\partial_x^\alpha z_{k,t}^j|^2 e^{2\tau\gamma(x,k)} \, dx
\leq C \int_{\Omega} \left( \tau\|{\chi z_{k,t}^j}\|_1 + \frac{1}{\tau}\|{\nabla \text{div}(z_{k,t}^j)}\|_1 + \frac{1}{\tau^2}\|{\nabla \text{rot}(z_{k,t}^j)}\|_1 \right) e^{2\tau\gamma(x,k)} \, dx
+ C \int_{\Omega} \sum_{|\alpha| \leq 1} |\partial_x^\alpha z_{k,t}^j|^2 e^{2\tau\gamma(x,k)} \, dx.
\]
So for large \(\tau > 0\) we deduce the following result
\[
\int_{\Omega} |\nabla \text{rot} z_{k,t}^j|^2 e^{2\tau\gamma(x,k)} \, dx \leq C \int_{\Omega} \left( |\nabla \text{rot} z_{k,t}^j|^2 + |\nabla z_{k,t}^j|^2 + \tau\|{\nabla \text{div}(z_{k,t}^j)}\|_1 \right) e^{2\tau\gamma(x,k)} \, dx
+ C \int_{\Omega} \sum_{|\alpha| \leq 1} |\partial_x^\alpha z_{k,t}^j|^2 e^{2\tau\gamma(x,k)} \, dx.
\]
(3.33)
Combining \((3.32)\) and \((3.33)\), we have
\[
\tau \int_{\Omega} \sum_{|\alpha| \leq 2} (|\partial_x^\alpha u_k| + |\partial_x^\alpha v_k| + |\partial_x^\alpha q_k| + |\partial_x^\alpha r_k|) e^{2\tau\gamma(x,k)} \, dx
\leq C \tau \int_{\Omega} \sum_{|\alpha| \leq 2} \left( \tau \|{\chi y_{k,t}^j}\|_1 + \tau\|{\nabla \text{div} y_{k,t}^j}\|_1 + \tau \|{\nabla \text{rot} y_{k,t}^j}\|_1 \right) e^{2\tau\gamma(x,k)} \, dx
\]
Furthermore, by lemma \(3.6\) we have
\[
\tau \int_{\Omega} \sum_{|\alpha| \leq 2} (|\partial_x^\alpha u_k| + |\partial_x^\alpha v_k| + |\partial_x^\alpha q_k| + |\partial_x^\alpha r_k|) e^{2\tau\gamma(x,k)} \, dx
\leq C \tau \int_{\Omega} \sum_{|\alpha| \leq 2} \left( \tau \|{\chi y_{k,t}^j}\|_1 + \tau\|{\nabla \text{div} y_{k,t}^j}\|_1 + \tau \|{\nabla \text{rot} y_{k,t}^j}\|_1 \right) e^{2\tau\gamma(x,k)} \, dx
+ \tau \int_{\Omega} \sum_{|\alpha| \leq 2} \left( |\partial_x^\alpha y_{k,t}^j| + |\partial_x^\alpha y_{k,t}^j| \right) e^{2\tau\gamma(x,k)} \, dx
\]
Again since \(|\rho_{2k}| \equiv 0\) for all \(x \in \Omega_{\tau}\), by the second equation of \((3.18)\) we have
\[
|y_{k+2,t}^j|^2 = |\rho_{2k}^{-1}(\nabla (\phi y_{k,t}^j) + \nabla (\phi y_{k,t}^j) + \nabla (q_{k,t}^j \Delta \partial_t u_k) + \nabla (r_{k,t}^j \Delta \partial_t u_k))|^2
\leq C \left( \|{\nabla \text{div} y_{k,t}^j}\|_1 + \|{\nabla \text{div} y_{k,t}^j}\|_1 + \|{\nabla \text{div} y_{k,t}^j}\|_1 + \|{y_{k,t}^j}\|_1^2 + \sum_{|\alpha| \leq 1} (|\partial_x^\alpha q_{k,t}^j| + |\partial_x^\alpha r_{k,t}^j|) \right),
\]
for \(k \geq 0\). Hence,
\[
\tau \int_{\Omega} \sum_{|\alpha| \leq 2} (|\partial_x^\alpha u_k| + |\partial_x^\alpha v_k| + |\partial_x^\alpha q_k| + |\partial_x^\alpha r_k|) e^{2\tau\gamma(x,k)} \, dx
\leq C \tau \int_{\Omega} \sum_{|\alpha| \leq 2} \left( |\partial_x^\alpha y_{k,t}^j| + |\nabla \text{div} y_{k,t}^j|_1 + |\nabla \text{div} y_{k,t}^j|_1 + |\nabla \text{rot} y_{k,t}^j|_1 \right)
+ \|{\nabla \text{rot} y_{k,t}^j}\|_1 e^{2\tau\gamma(x,k)} \, dx + C \tau \int_{\Omega} \sum_{|\alpha| \leq 1} (|\partial_x^\alpha q_{k,t}^j| + |\partial_x^\alpha r_{k,t}^j|) e^{2\tau\gamma(x,k)} \, dx
\leq C \int_{\Omega} \sum_{|\alpha| \leq 2} (|\partial_x^\alpha u_k| + |\partial_x^\alpha v_k| + |\partial_x^\alpha q_k| + |\partial_x^\alpha r_k|) e^{2\tau\gamma(x,k)} \, dx
\]
Combining (3.19) and (3.34), we have
\[
\tau \int_{Q} \sum_{|\alpha| \leq 2} (|\partial_{\alpha}^{x} \lambda_{A}|^{2} + |\partial_{\alpha}^{y} \mu_{A}|^{2} + |\partial_{\alpha}^{y} q_{A}|^{2} + |\partial_{\alpha}^{y} r_{A}|^{2}) e^{2\gamma \tau(x)} \, dx \\
\leq C \tau^{7} \int_{Q} \sum_{|\alpha| \leq 2} (|\partial_{\alpha}^{x} \mu_{A}|^{2} + |\partial_{\alpha}^{y} \lambda_{A}|^{2} + |\partial_{\alpha}^{y} q_{A}|^{2} + |\partial_{\alpha}^{y} r_{A}|^{2}) e^{2\gamma \tau(x)} \, dx d\tau \\
+ C \tau^{7} e^{C \tau^{2}} \|u_{A}\|_{H^{2}(\omega)}^{2} + C \sum_{k=0}^{4} \tau^{6-k} \|\gamma_{k}\|_{L^{2}(-T,T;H^{1}(\Omega))}^{2}.
\] (3.35)

Since \(\varphi_{a}(x) > \varphi(x, t)\) for \(t \neq 0\), by the Lebesgue theorem, we have
\[
\tau^{7} \int_{Q} \sum_{|\alpha| \leq 2} (|\partial_{\alpha}^{x} \lambda_{A}|^{2} + |\partial_{\alpha}^{y} \mu_{A}|^{2} + |\partial_{\alpha}^{y} q_{A}|^{2} + |\partial_{\alpha}^{y} r_{A}|^{2}) e^{2\gamma \tau(x)} \, dx d\tau \\
= \int_{Q} \sum_{|\alpha| \leq 2} (|\partial_{\alpha}^{x} \mu_{A}|^{2} + |\partial_{\alpha}^{y} \lambda_{A}|^{2} + |\partial_{\alpha}^{y} q_{A}|^{2} + |\partial_{\alpha}^{y} r_{A}|^{2}) e^{2\gamma \tau(x)} \, dx \\
= o(1) \int_{Q} \sum_{|\alpha| \leq 2} (|\partial_{\alpha}^{x} \lambda_{A}|^{2} + |\partial_{\alpha}^{y} \mu_{A}|^{2} + |\partial_{\alpha}^{y} q_{A}|^{2} + |\partial_{\alpha}^{y} r_{A}|^{2}) e^{2\gamma \tau(x)} \, dx
\]
as \(\tau \to \infty\). Therefore, we have
\[
\tau \int_{Q} \sum_{|\alpha| \leq 2} (|\partial_{\alpha}^{x} \lambda_{A}|^{2} + |\partial_{\alpha}^{y} \mu_{A}|^{2} + |\partial_{\alpha}^{y} q_{A}|^{2} + |\partial_{\alpha}^{y} r_{A}|^{2}) e^{2\gamma \tau(x)} \, dx \leq C \tau^{7} e^{C \tau^{2}} \|u_{A}\|_{H^{2}(\omega)}^{2} + C M \tau^{6 \gamma \tau}.
\] (3.36)

which implies that there exist positive constants \(a_{0}, \alpha_{2}\) and \(\tau_{0}\) such that
\[
\int_{Q} \sum_{|\alpha| \leq 2} (|\partial_{\alpha}^{x} \lambda_{A}|^{2} + |\partial_{\alpha}^{y} \mu_{A}|^{2} + |\partial_{\alpha}^{y} q_{A}|^{2} + |\partial_{\alpha}^{y} r_{A}|^{2}) \, dx \leq C (e^{\alpha_{2} \tau} \|u_{A}\|_{H^{2}(\omega)}^{2} + e^{-\tau_{0}}),
\] (3.37)
for all large \(\tau > \tau_{0}\). We replace \(C\) with \(Ce^{\alpha_{2} \tau}\) then (3.37) holds for all \(\tau > 0\). If \(\|u_{A}\|_{H^{2}(\omega)}^{2} = 0\), we can deduce that
\[
\int_{Q} \sum_{|\alpha| \leq 2} (|\partial_{\alpha}^{x} \lambda_{A}|^{2} + |\partial_{\alpha}^{y} \mu_{A}|^{2} + |\partial_{\alpha}^{y} q_{A}|^{2} + |\partial_{\alpha}^{y} r_{A}|^{2}) \, dx = 0.
\]
Else if \(\|u_{A}\|_{H^{2}(\omega)}^{2} = 0\) (we may assume \(\|u_{A}\|_{H^{2}(\omega)}^{2} < 1\), otherwise let \(\tau \to 0\) then the right hand side of (3.37) is not greater than \(2C \|u_{A}\|_{H^{2}(\omega)}^{2} = e^{\alpha_{2} \tau}\), we can choose \(\tau > 0\) such that \(e^{\alpha_{2} \tau} \|u_{A}\|_{H^{2}(\omega)}^{2} = e^{-\tau_{0}}\), then
\[
\int_{Q} \sum_{|\alpha| \leq 2} (|\partial_{\alpha}^{x} \lambda_{A}|^{2} + |\partial_{\alpha}^{y} \mu_{A}|^{2} + |\partial_{\alpha}^{y} q_{A}|^{2} + |\partial_{\alpha}^{y} r_{A}|^{2}) \, dx \leq 2C e^{\alpha_{2} \tau} \|u_{A}\|_{H^{2}(\omega)}^{2} = 2C \|u_{A}\|_{H^{2}(\omega)}^{2} e^{2\alpha_{2} \tau}.
\] (3.38)

Noting that \(\|u_{A}\|_{W^{2,\infty}(Q)} \lesssim 2M\) and the interpolation inequality, we have
\[
\|u_{A}\|_{H^{2}(\omega)}^{2} \lesssim C \|u_{A}\|_{H^{0}(\omega)}^{2} \|u_{A}\|_{H^{2}(\omega)}^{2}.
\] (3.39)

The proof of theorem 2.3 is completed. \(\square\)

4. Conclusions

Biot’s system is widely used to describe wave propagation in fluid saturated porous media. It is a more accurate model than the pure acoustic or elastic model. There are four important elastic coefficients in Biot’s system, i.e., two Lamé parameters, the dilatational coupling factor and the bulk modulus. We investigate the stability of the inverse problem for determining the four elastic coefficients simultaneously by a single measurement of data on a neighbourhood of the boundary. For the suitable initial data, we have proved the Holder stability of this inverse problem. We have developed new Carleman estimates since the existing Carleman estimates for Biot’s system are lack of the information of \(v^{T}\) and \(rot v^{T}\). We also have simplified the assumed condition in \(2 \times 2\) matrix form for the initial data. However, the physical meaning of the assumption is still a worth studying problem.

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