Finite Mass Corrections to Leptonic Decay Constants in the Heavy Quark Effective Theory

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Abstract

We calculate finite mass corrections to leptonic decay constants in HQET using QCD sum rules. The results are given to two–loop accuracy including renormalization group analysis. The contribution of unknown two–loop anomalous dimension matrix–elements is estimated. We find that the $1/m_Q$ corrections to $f_P$, the decay constant of a pseudoscalar meson, amount $−(23 \pm 4 \pm 2)\%$ for the B meson. The first error comes from the sum rule uncertainty, the second one from unknown or not accurately known matrix elements in HQET. We obtain a ratio of vector to pseudoscalar decay constant of $1.00 \pm 0.07 + O(1/m_Q^2)$ for B mesons.
I. INTRODUCTION

QCD in the limit of infinite heavy quarks has received much interest over the past years (cf. [1] and references therein). The theory simplifies considerably in that limit and though it still does not allow a complete calculation of hadronic matrix elements, it poses some conditions on and gives relations between them that can be used to reduce the model–dependence of the analysis of experimental data. Its phenomenological relevance is however limited by the fact that quark masses are finite. Generally, matrix elements determined in the heavy quark effective theory (HQET) receive corrections due to the finite quark mass. Since we are dealing with a field theory that has to be renormalized, the corrections are not pure powers, but are modified by large logarithms of type \( \ln \frac{m_Q}{\mu} \) where \( m_Q \) is the heavy quark mass and \( \mu \) some hadronic scale. Physically, the logarithms are due to the radiation of high energetic gluons off the heavy quark. Although in certain cases some terms in the expansion can be shown to be small or absent (as is the case with Luke’s theorem [2]), in general they will introduce new unknown quantities order by order in the heavy quark expansion. If HQET is supposed to yield results of phenomenological relevance, it is thus of paramount importance to keep those corrections under control.

In the present paper we address the question of \( 1/m_Q \) corrections to one specific matrix element, the leptonic decay constant \( f_M \) of a pseudoscalar meson \( M \) with momentum \( p \), built of the heavy quark \( Q \) and the light anti–quark \( \bar{q} \):

\[
\langle 0 | \bar{q} \gamma_\mu \gamma_5 Q | M(p) \rangle = i f_M p_\mu. \tag{1.1}
\]

From the analysis of potential models [3], it was known long since that \( F \equiv f_M \sqrt{m_Q} \) approaches a constant as \( m_Q \) becomes infinitely heavy. The next step in unravelling the mysteries of \( f_M \) was the discovery that the simple square–root like behaviour gets modified by logarithmic terms [4]. After the development of the formal framework of heavy quark QCD as an effective theory [5], \( F \) was soon subject to numerical studies on the lattice [6] and by QCD sum rules [7–9]. After some fluctuations, the results now agree within the errors:

\[
F(\mu = m_B) \approx (0.4 – 0.6) \text{ GeV}^{3/2}. \tag{1.2}
\]

The value of \( f_B \), the decay constant of the B meson, that can be derived from the above value is however considerably bigger than values obtained in “full QCD” without reference to HQET [10]. Since \( f_B \) is an important quantity in many applications like \( B_0 – \bar{B}_0 \) mixing, e.g. (cf. [11]), its knowledge is of great importance and thus requires a careful analysis of finite mass corrections to \( F \). According to estimates of various authors, \( 1/m_Q \) corrections to the heavy quark limit amount to about 20% [3,11,12].

In a previous paper [7] we have treated higher order corrections based on a “higher twist” expansion of a QCD sum rule for \( f_B \). An explicit expansion in inverse powers of \( m_Q \) did not prove feasible since the expansion of all parameters inherent in the sum rule analysis in inverse powers of \( m_Q \) is by no means unique. Instead we defined the “correction term” to be the difference between QCD and HQET result. In the range \( m_Q > 3 \text{ GeV} \) the corrections proved to be dominated by the linear term in the heavy quark expansion. On the other hand, in [3] an attempt was made to compute \( 1/m_Q \) directly in HQET, but still relied heavily on
the expansion of sum rule parameters in $1/m_Q$. The resulting $1/m_Q$ terms are about a factor of three larger than the correction obtained in [7]. Thus we feel the necessity to complement our old calculation by a corresponding one where the $1/m_Q$ terms are investigated entirely in HQET. Moreover, such a calculation offers the possibility to include next-to-leading order renormalization group effects studied recently in [13].

Our paper is organized as follows: in Sec. II we recall the formal framework of HQET and give necessary definitions, Sec. III deals with the heavy quark expansion of correlation functions, in Sec. IV we attempt an approximate non-perturbative calculation of the relevant matrix elements in HQET by QCD sum rules. The last section, Sec. V, is devoted to the presentation of the results and their discussion.

II. PRELIMINARIES AND DEFINITIONS

Before starting with “heavy” calculations, let us recall the main features of HQET. The heavy quark field $h_v$ is defined by

$$P_+Q(x) = e^{-im_Q(v \cdot x)}h_v(x),$$

where

$$P_+ = \frac{1}{2}(1 + \gamma)$$

is the projector on the upper components of the heavy quark field $Q(x)$, $m_Q$ is the mass of the heavy quark, and $v$ its four-velocity. The contribution of the lower components to the QCD Lagrangian can be integrated out and the new effective Lagrangian be expanded in inverse powers of the heavy quark mass [14,15]:

$$\mathcal{L}_{HQET} = \bar{h}_v i(v \cdot D)h_v + \frac{1}{2m_Q} \mathcal{K} + \frac{1}{2m_Q} C_{mag}(\mu)S + O(1/m_Q^2).$$

At order $1/m_Q$, there are two operators: the kinetic energy operator

$$\mathcal{K} = \bar{h}_v(iD)^2h_v$$

whose matrix element over hadron states with equal velocities is related to the kinetic energy of the heavy quark within these states. Due to reparametrization invariance $\mathcal{K}$ does not get renormalized [16], whereas the chromomagnetic interaction operator,

$$S = \frac{1}{2} \bar{h}_v \sigma_{\mu\nu} g F^{\mu\nu} h_v$$

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1 By the heavy quark mass $m_Q$ we understand the pole mass whose precise definition will be given in (3.6).

2 A third one, $\bar{h}_v(iv \cdot D)^2 h_v$, can be shown to be of higher order in $1/m_Q$ by virtue of the equation of motion of the heavy quark field.
acquires an anomalous dimension $\gamma_{\text{mag}} = 6\alpha_s/(4\pi) + O(\alpha_s^2)$ [14] that gives rise to the Wilson coefficient $C_{\text{mag}}(\mu)$.

In HQET, the heavy quark does not take part in the dynamics of a process that is essentially determined by the hadron’s light degrees of freedom. As a consequence, the heavy quark spin becomes a “good” quantum number and the theory exhibits the famous spin–symmetry [3] which leads to a factorization of matrix elements into terms describing the spin–structure of the involved hadrons (and their interaction) and terms describing the dynamics of the process that is independent of the heavy quarks. The formal tool to handle that factorization is the so–called trace–formalism [17]. In this formalism and using the notation of [9], we define, making explicit the dependence on the renormalization scale $\mu$ whenever necessary,

$$\langle 0 | \bar{q} \Gamma h_v | M \rangle = \frac{1}{2} F(\mu) \text{Tr}\{\Gamma M\}. \quad (2.6)$$

$\mathcal{M}$ is the spin wave function of the HQET meson state $| M \rangle$:

$$\mathcal{M} = \sqrt{m_Q} P_+ \left\{ -i\gamma_5 \right. \text{ for } J^P = 0^-, \left. \not{\epsilon} \right. \text{ for } J^P = 1^- \right\}. \quad (2.7)$$

Here $\epsilon_\mu$ is the polarization vector of the vector meson.

At order $1/m_Q$ and neglecting radiative corrections, the leptonic decay constant $f_M$ is determined by (cf. [4])

$$f_M \sqrt{m_M} = F \left( 1 - d_M \frac{\bar{\Lambda}}{6m_Q} + \frac{G_K}{2m_Q} + d_M \frac{G_\Sigma}{m_Q} \right). \quad (2.8)$$

$\bar{\Lambda}$ is defined as the mass–difference of meson mass $m_M$ and quark mass $m_Q$ in the heavy quark limit, $m_M = m_Q + \bar{\Lambda} + O(1/m_Q)$. $G_K$ and $G_\Sigma$ are defined by

$$\langle 0 | i \int d^4 x K(x) \bar{q}(0) \Gamma h_v(0) | M \rangle = \frac{1}{2} F(\mu) G_K(\mu) \text{Tr}\{\Gamma M\},$$

$$\langle 0 | i \int d^4 x S(x) \bar{q}(0) \Gamma h_v(0) | M \rangle = \frac{1}{2} F(\mu) G_\Sigma(\mu) 2d_M \text{Tr}\{\Gamma M\}. \quad (2.9)$$

As already mentioned, at order $1/m_Q$ the spin–symmetry breaks down and matrix elements become dependent on the spin of the heavy particle as characterized by a scalar $d_M$ defined by

$$P_+ \bar{\sigma}_\alpha \beta \mathcal{M} \sigma^{\alpha \beta} = 2d_M \mathcal{M}, \quad (2.10)$$

yielding

\[\text{Of course the symmetry breaks down at order } 1/m_Q, \text{ but in a so to speak controlled manner that is induced by } S \text{ and only affects the traces over spin wave functions.}\]
\[ d_M = \begin{cases} 3 & \text{for } J^P = 0^-, \\ -1 & \text{for } J^P = 1^- \end{cases} \]  

(2.11)

Eq. (2.8) contains all information about \( f_M \) to order \( 1/m_Q \) and to lowest order in the strong coupling, but involves the three new matrix elements \( \Lambda, G_K, \text{and } G_\Sigma \). For \( \Lambda \) there exists a rigorous lower bound of 237 MeV \([18]\) that is however questioned in \([19]\). Neglecting \( 1/m_Q \) effects, the values of the \( b \)-quark mass quoted in literature, \( m_b = (4.6 - 4.8) \text{ GeV} \) \([20]\), yield \( \Lambda = (0.5 - 0.7) \text{ GeV} \). More refined analyses in HQET \([7,9]\) tend to the somewhat smaller value \( \Lambda = (0.5 - 0.6) \text{ GeV} \) which we will use in our paper.

The available values of \( G_K \) and \( G_\Sigma \) rely exclusively on the analysis given in \([9]\) that is similar in spirit to the one we are going to undertake but quite different in detail. We will come back to that point later. The results of \([9]\) read (at a renormalization scale of 5 GeV):

\[ G_K \simeq -4.4 \text{ GeV}, \quad G_\Sigma \simeq -0.05 \text{ GeV}. \]  

(2.12)

III. THE HEAVY QUARK EXPANSION OF CORRELATION FUNCTIONS

Our calculation of \( G_K \) and \( G_\Sigma \) relies on the analysis of two–point functions of the heavy–light current \( J^\Gamma = \bar{q} \Gamma h_v \) with insertions of \( \mathcal{K} \) and \( \mathcal{S} \), respectively. Since we will include radiative corrections to \( O(\alpha_s) \), an investigation of the renormalization–group behaviour proves necessary. In addition, at the required level of accuracy we need to match the effective theory to QCD, i.e. we have to require the equality of matrix elements in HQET to the corresponding ones in QCD at the matching scale \( \mu = m_Q \). As a by–product of this procedure, we will be able to check the heavy quark expansion explicitly at the two–loop level at the example of the two–point function of pseudoscalar currents. This will be done in the next section.

Let us first introduce a generic notation for correlation functions of two composite operators \( \mathcal{O}_1 \) and \( \mathcal{O}_2 \), either containing a heavy quark field,

\[ \langle \mathcal{O}_1 \mathcal{O}_2 \rangle \equiv i \int d^4x e^{i\omega v \cdot x} \langle 0 \mid T \mathcal{O}_1(x) \mathcal{O}_2(0) \mid 0 \rangle, \quad (3.1) \]

with four–momentum \( \omega v, v \) being the velocity of the heavy quark. Analogously, we define for the zero–momentum insertion of the operator \( \mathcal{X} \):

\[ \langle \mathcal{O}_1 \mathcal{X} \mathcal{O}_2 \rangle \equiv i^2 \int d^4x \, d^4y e^{i\omega v \cdot (x-y)} \langle 0 \mid T \mathcal{O}_1(x) \mathcal{X}(0) \mathcal{O}_2^\dagger(y) \mid 0 \rangle. \quad (3.2) \]

Employing the trace–formalism for heavy–light currents \( J^\Gamma_i = \bar{q} \Gamma_i h_v \) we can write:

\[ \langle J_{\Gamma_1} J_{\Gamma_2} \rangle = \frac{1}{2} \Pi(\omega, \mu) \text{ Tr}\{\Gamma_1 P_+ \Gamma_2 \hat{\sigma} \}. \quad (3.3) \]

\(^4\)Our definition deviates formally from the conventional one \( \langle J_{\Gamma_1} J_{\Gamma_2} \rangle = -\frac{i}{2} \Pi(\omega) \text{ Tr}(\Gamma_1 P_+ \Gamma_2) \), but coincides for “spin eigenstates” like \( \Gamma_i = i\gamma_5 \) or \( \Gamma_i = \gamma^\mu - v^\mu \) with \( \Gamma_i \hat{\sigma} = -\hat{\sigma} \Gamma_i \) and can be directly inferred from the structure of the relevant loop–integrals.
Recall that $\Pi(\omega, \mu)$ is independent of both $\Gamma_1$ and $\Gamma_2$, but depends on the renormalization scale $\mu$ due to the non–vanishing anomalous dimension of $J_\Gamma$. With $\mathcal{X} = \mathcal{K}$ or $\mathcal{S}$, (3.2) can be expressed as

$$
\langle J_{\Gamma_1}\mathcal{K} J_{\Gamma_2} \rangle \equiv T_K(\omega, \mu) \text{Tr}\{\Gamma_1 P_+ \Gamma_2 \psi\},
$$

$$
\langle J_{\Gamma_1}\mathcal{S} J_{\Gamma_2} \rangle \equiv T_\Sigma(\omega, \mu) d_M \text{Tr}\{\Gamma_1 P_+ \Gamma_2 \psi\}. (3.4)
$$

In QCD, the renormalization–group invariant two–point function of pseudoscalar currents with momentum $q$ is defined as

$$
\Pi_5 = i \int d^4 x e^{iqx} \langle 0 | \mathcal{T} m_{\overline{\text{MS}}} \bar{q}(x) i\gamma_5 Q(x) m_{\overline{\text{MS}}} \bar{Q}(0) i\gamma_5 q(0) | 0 \rangle. (3.5)
$$

Here the running $\overline{\text{MS}}$ mass $m_{\overline{\text{MS}}}$ enters whose relation to the pole mass is given by

$$
m_Q = m_{\overline{\text{MS}}}(\mu) \left[ 1 + \frac{\alpha_s(\mu)}{\pi} \left( \frac{4}{3} - 2 \ln \frac{m_Q}{\mu} \right) + O(\alpha_s^2) \right]. (3.6)
$$

$\Pi_5$ was calculated to two–loop accuracy in the deep Euclidean region $q^2 \ll 0$ in [21] for arbitrary quark masses. Once the heavy quark expansion of the pseudoscalar current is known, the expression for $\Pi_5$ can serve as check for both the cancellation of $\mu$ dependent terms in the effective theory and the values of the matching coefficients, i.e. the values of the short–distance expansion coefficients at $\mu = m_Q$.

As for the pseudoscalar current, its expansion reads (following the notation of [13]):

$$
m_{\overline{\text{MS}}} \bar{q} i\gamma_5 Q \cong m_Q \left( C(\mu) J_{i\gamma_5} + \frac{1}{2m_Q} \sum_i B_i(\mu) \mathcal{O}_i + \frac{1}{2m_Q} \sum_k A_k(\mu) T_k \right). (3.7)
$$

The above formula needs some explanation. First, “$\cong$” means equality only after taking matrix elements or insertion in Greens functions. Next, we have kept the multiplicative mass term in order to indicate that it affects the coefficient functions on the right hand side by virtue of Eq. (3.6). The leading order operator in HQET is the current $J_{i\gamma_5}$; at order $1/m_Q$, there are two local operators,

$$
\mathcal{O}_1 = \bar{q}i\gamma_5 i\not{\!D} h_v, \quad \mathcal{O}_2 = \bar{q}(-i\not{\!v} \cdot \not{\!D}) i\gamma_5 h_v, (3.8)
$$

and two non–local ones due to the insertion of the Lagrangian into the leading term:

$$
T_1 = i \int d^4 x T J_{i\gamma_5}(x) \mathcal{K}(0), \quad T_2 = i \int d^4 x T J_{i\gamma_5}(x) \mathcal{S}(0). (3.9)
$$

The operators have non–vanishing anomalous dimensions and mix under renormalization. The mixing is described by an $4 \times 4$ matrix $\hat{\gamma}^{\text{PS}}$ and the coefficient functions $\{B_1, B_2, A_1, A_2\} \equiv \vec{C}$ obey the renormalization–group equation

$$
\left( \frac{d}{d\mu} - \hat{\gamma}^{\text{PS}} \right) \vec{C} = 0. (3.10)
$$
Since $K$ does not get renormalized, it follows $A_1(\mu) = C(\mu)$. For $A_2$, we have $A_2(\mu) = C(\mu)C_{\text{mag}}(\mu)$ due to the non-zero anomalous dimension of $S$ (cf. (2.3)). $C_{\text{mag}}$ was calculated in [14,22] to be

$$C_{\text{mag}}(\mu) = 1 + \frac{\alpha_s}{\pi} \left( \frac{13}{6} + \frac{3}{2} \ln \frac{m_Q}{\mu} \right). \quad (3.11)$$

In order to determine the remaining coefficients, we proceed like Ref. [13] and calculate matrix-elements between on-shell quark states. The corresponding diagram is shown in Fig. 1. The on-shell spinors obey the Dirac equations $\slashed{p} u_Q(p) = m_Q u_Q(p)$ and $\slashed{p}' u_Q(p') = 0$. The heavy quark momentum $p$ can be written as $p = m_Q v + k$ where the off-shell momentum $k$ obeys $2m_Q v \cdot k + k^2 = 0$ in order to keep $u_Q$ on-shell. For the on-shell spinor in HQET the Dirac equation reads $\slashed{v} u_h = 0$, and it follows the relation $u_Q = \{ 1 + k/(2m_Q) \} u_h$. Taking into account only terms constant or linear in $p'$, loop-integrals in HQET vanish in dimensional regularization since there is no massive scale they could depend on. The diagram in Fig. 1, wave function renormalization and terms coming from the replacement of the running MS mass by the pole mass yield

$$C(\mu) = B_1(\mu) = 1 + \frac{\alpha_s}{\pi} \left( \ln \frac{m_Q}{\mu} - \frac{2}{3} \right), \quad B_2(\mu) = \frac{8}{3} \frac{\alpha_s}{\pi}. \quad (3.12)$$

The expression for $C(\mu)$ is well known and can be found in a somewhat different form in [23] apart from a slight error that was corrected in [7,8]. The first equality in (3.12) is consistent with the requirement of reparametrization invariance, the expression for $B_2$ is new.

The expansion of (3.5) to order $1/m_Q^2$ is now straightforward to derive. Using

$$\langle J_{\Gamma_1} i D_{\alpha} J_{\Gamma_2} \rangle = \frac{1}{6} \omega \Pi(\omega) \text{Tr} \{ (\gamma_\alpha - v_\alpha \slashed{p}) \Gamma_1 P_+ \Gamma_2 \}, \quad (3.13)$$

the insertion of (3.7) into (3.3) yields

$$\Pi_5 = m_Q^2 \left( C^2(\mu) \Pi(\omega, \mu) + \frac{1}{m_Q} \left[ \{ C^2(\mu) - B_2(\mu) C(\mu) \} \omega \Pi(\omega, \mu) + C^2(\mu) T_K(\omega, \mu) \right. \right.$$  

$$+ 3C^2(\mu) C_{\text{mag}}(\mu) T_{\Sigma}(\omega, \mu) \right] \}. \quad (3.14)$$

This expression is valid up to constants in $\omega$ that are related to contact terms and cannot be expanded in $1/m_Q$. The $\mu$-dependent terms on the right hand side must cancel in order to keep the result renormalization-group invariant. This cancellation will be checked in the next section.

One could now proceed by solving the renormalization-group equation (3.10) as was done in [9]. For our purposes, however, it proves more appropriate to investigate the behaviour of the two-point functions $\vec{\Pi} = \{ \omega \Pi(\omega), T_K(\omega), T_{\Sigma}(\omega) \}$ under renormalization-group. $\vec{\Pi}$ obeys the renormalization group equation

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5 We use anticommuting $\gamma_5$. 


\[
\left( \mu \frac{d}{d\mu} + \hat{\gamma} \right) \tilde{V}(\mu) = 0. \tag{3.15}
\]

The $3 \times 3$ anomalous dimension matrix $\hat{\gamma}$ can be obtained from the renormalization constant $\hat{Z}$ that relates the bare $\tilde{V}_{\text{bare}}$ to the renormalized one, $\tilde{V}_{\text{bare}} = \hat{Z} \tilde{V}$. Restricting ourselves to leading logarithmic accuracy (superscript LL), $\hat{\gamma}^{\text{LL}}$ can be obtained as $\hat{\gamma}^{\text{LL}} = \lim_{\epsilon \to 0} 2\epsilon \hat{Z}$ in the MS scheme using dimensional regularization in $D = 4 + 2\epsilon$ dimensions. For $\omega \ll 0$, $\tilde{V}$ can be treated within perturbation theory and we find

\[
\hat{\gamma}^{\text{LL}} = \frac{\alpha_s}{4\pi} \begin{pmatrix} -8 & 0 & 0 \\ \frac{32}{3} & -8 & 0 \\ -\frac{32}{3} & 0 & -2 \end{pmatrix} \equiv \frac{\alpha_s}{4\pi} \gamma. \tag{3.16}
\]

To that accuracy, the solution of Eq. (3.15) is

\[
\tilde{V}(m_Q) = \exp \left( -\frac{\gamma}{2\beta_0} \ln x \right) \tilde{V}(\mu) \tag{3.17}
\]

with $x = \frac{\alpha_s(\mu)}{\alpha_s(m_Q)}$. $\beta = -g_s [\beta_0 \alpha_s/(4\pi) + \beta_1 (\alpha_s/(4\pi))^2 + O(\alpha_s^3)]$ is the $\beta$ function of QCD. The matrix $\gamma$ cannot be diagonalized, but transformed to its Jordan form $\gamma_J$ via

\[
\gamma_J = T^{-1} \gamma T \tag{3.18}
\]

with

\[
T = \begin{pmatrix} 0 & -6 & 0 \\ -64 & 1 & 0 \\ 0 & -\frac{32}{3} & 1 \end{pmatrix}, \quad \gamma_J = \begin{pmatrix} -8 & 1 & 0 \\ 0 & -8 & 0 \\ 0 & 0 & -2 \end{pmatrix}. \tag{3.19}
\]

Using

\[
\exp \left( \begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix} \right) = \exp(a) \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \tag{3.20}
\]

we find

\[
\exp \left( -\frac{\gamma}{2\beta_0} \ln x \right) = x^{4/\beta_0} \begin{pmatrix} 1 & 0 & 0 \\ -\frac{16}{3\beta_0} \ln x & 1 & 0 \\ -\frac{16}{27} (x^{-3/\beta_0} - 1) & 0 & x^{-3/\beta_0} \end{pmatrix}. \tag{3.21}
\]

As expected, $\mathcal{V}_1$ renormalizes like $\Pi(\omega)$. For $\mathcal{V}_2$ and $\mathcal{V}_3$ we can infer the renormalization–group invariant quantities

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\( ^6 \)But note that the result is renormalization–scheme independent to that accuracy.
\[ \hat{V}_{2}^\text{LL} \equiv [\alpha_s(\mu)]^{4/\beta_0} \left( -\frac{16}{3\beta_0} \ln \alpha_s(\mu) \mathcal{V}_1(\mu) + \mathcal{V}_2(\mu) \right), \]

\[ \hat{V}_{3}^\text{LL} \equiv [\alpha_s(\mu)]^{1/\beta_0} \left( -\frac{16}{27} \mathcal{V}_1(\mu) + \mathcal{V}_3(\mu) \right). \]  

(3.22)

In order to extend the analysis to next–to–leading order (superscript NLO), one needs to include the anomalous dimension matrix \( \hat{\gamma} \) to two–loop accuracy. For \( \mathcal{V}_1 \) one finds \( [7,8,24] \)

\[ \hat{\mathcal{V}}_1^{\text{NLO}} \equiv [\alpha_s(\mu)]^{4/\beta_0} \left( 1 - S_{\text{HL}} \frac{\alpha_s(\mu)}{2\pi} \right) \mathcal{V}_1(\mu), \]  

(3.23)

where

\[ S_{\text{HL}} = \frac{\gamma_{00}^{\text{HL}}}{2\beta_0} \left( \frac{\gamma_1^{\text{HL}}}{\beta_0} - \frac{\beta_1}{\beta_0} \right). \]  

(3.24)

\( \gamma^{\text{HL}} = \gamma_0^{\text{HL}} \alpha_s/(4\pi) + \gamma_1^{\text{HL}} \{ \alpha_s/(4\pi) \}^2 + O(\alpha_s^3) \) is the anomalous dimension of the heavy–light current \( J_\Gamma \) calculated in \( [23,25] \). As for \( \hat{\mathcal{V}}_2 \) and \( \hat{\mathcal{V}}_3 \), some of the involved elements of \( \hat{\gamma} \) are still unknown to date. According to \( [13] \) one obtains

\[ \hat{\mathcal{V}}_2 = [\alpha_s(\mu)]^{4/\beta_0} \left( 1 - S_{\text{HL}} \frac{\alpha_s(\mu)}{2\pi} \right) \left\{ V_2(\mu) - V_1(\mu) \left( \frac{16}{3\beta_0} \ln \alpha_s(\mu) + \frac{8}{3} T_{34} \frac{\alpha_s(\mu)}{\pi} \right) \right\}, \]

\[ \hat{\mathcal{V}}_3 = [\alpha_s(\mu)]^{1/\beta_0} \left( 1 - 2S_{\text{HL}} \frac{\alpha_s(\mu)}{2\pi} + S_{\text{mag}} \frac{\alpha_s(\mu)}{4\pi} \right) \left\{ V_3(\mu) - \frac{16}{27} V_1(\mu) \left( 1 + \frac{3}{2} T_{32} \frac{\alpha_s(\mu)}{\pi} \right) \right\}. \]  

(3.25)

\( S_{\text{mag}} \) is the analogue of \( S_{\text{HL}} \) were \( \gamma^{\text{HL}} \) has to be replaced by the anomalous dimension \( \gamma^{\text{mag}} \) of the operator \( S \). Unknown are the matrix elements \( T_{32} \) and \( T_{34} \) as well as \( \gamma_{1}^{\text{mag}} \).

We now want to relate the two–point functions we have been dealing with in this section to the \( 1/m_Q \) corrections to \( F \) we are interested in. To that end, we saturate the correlation function \( \Pi_5 \) by hadronic states. Making explicit only the insertion of the ground–state, the meson \( M \), it follows with \( [1,11] \)

\[ \Pi_5 = \frac{m_{M}^4 f_{M}^2}{m_Q^2 (m_M^2 - q^2)} + \ldots \]  

(3.26)

where the dots denote insertions of radial excitations and multi–particle states. Analogously, we find from \( [2,9] \), choosing \( \mu = m_Q \):

\[ \Pi(\omega, m_Q) = \frac{F^2(m_Q)}{2(\Lambda - \omega)} + \ldots, \]

\[ T_K(\omega, m_Q) = \frac{F^2(m_Q) G_K(m_Q)}{2(\Lambda - \omega)} + \ldots, \]

\[ T_{\Sigma}(\omega, m_Q) = \frac{F^2(m_Q) G_{\Sigma}(m_Q)}{(\Lambda - \omega)} + \ldots \]  

(3.27)
For $T_K$ and $T_\Sigma$ the dots contain also additional double poles at $\omega = \bar{\Lambda}$ that are related to $1/m_Q$ corrections to $\bar{\Lambda}$ and are treated in detail in [26].

Identifying the leading terms in $1/m_Q$, we recover from (3.14) the well known relation for the pseudoscalar decay constant $f_P$

$$f^2_P m_P = F^2(m_Q) \left( 1 - \frac{4}{3} \frac{\alpha_s(m_Q)}{\pi} \right). \tag{3.28}$$

At order $1/m_Q$, we find

$$f_P \sqrt{m_P} = F(m_Q) \left( 1 - \frac{\alpha_s(m_Q)}{3} \right) \left[ 1 + \frac{1}{m_Q} \left\{ -\frac{1}{2} \left( 1 + \frac{8}{3} \frac{\alpha_s(m_Q)}{\pi} \right) \bar{\Lambda} + \frac{1}{2} G_K(m_Q) \right. \right.

$$

$$+ 3 \left( 1 + \frac{13}{6} \frac{\alpha_s(m_Q)}{\pi} \right) G_\Sigma(m_Q) \right\} \right]. \tag{3.29}$$

From an analogous consideration of the vector current, it follows for the vector decay constant $f_V$

$$f_V \sqrt{m_V} = F(m_Q) \left( 1 - \frac{\alpha_s(m_Q)}{3} \right) \left[ 1 + \frac{1}{m_Q} \left\{ \frac{1}{6} \left( 1 + \frac{4}{3} \frac{\alpha_s(m_Q)}{\pi} \right) \bar{\Lambda} + \frac{1}{2} G_K(m_Q) \right. \right.

$$

$$- \left( 1 + \frac{13}{6} \frac{\alpha_s(m_Q)}{\pi} \right) G_\Sigma(m_Q) \right\} \right]. \tag{3.30}$$

These expressions are the generalization of (2.8) to order $\alpha_s$. Eq. (3.29) coincides with the corresponding expression derived in [13].

The ratio of vector to pseudoscalar decay constant is then given by

$$\frac{f_V}{f_P} = 1 - \frac{2}{3} \frac{\alpha_s(m_Q)}{\pi} + \frac{1}{m_Q} \left\{ \frac{2}{3} \bar{\Lambda} \left( 1 + \frac{5}{3} \frac{\alpha_s(m_Q)}{\pi} \right) - 4G_\Sigma(m_Q) \left( 1 + \frac{3}{2} \frac{\alpha_s(m_Q)}{\pi} \right) \right\}. \tag{3.31}$$

From (3.29) we now can construct the invariant quantities

$$\hat{G}_K = G_K(\mu) - \bar{\Lambda} \left( \frac{16}{3\beta_0} \ln \alpha_s(\mu) + \frac{8}{3} T_{34} \frac{\alpha_s(\mu)}{\pi} \right),$$

$$\hat{G}_\Sigma = [\alpha_s(\mu)]^{-3/\beta_0} \left( 1 - S_{mag} \frac{\alpha_s(\mu)}{4\pi} \right) \left\{ G_\Sigma(\mu) - \frac{8}{27} \bar{\Lambda} \left( 1 + \frac{3}{2} T_{32} \frac{\alpha_s(\mu)}{\pi} \right) \right\}. \tag{3.32}$$

**IV. SUM RULES FOR $G_K$ AND $G_\Sigma$**

We now turn to the main part of this paper: the derivation of QCD sum rules for $G_K$ and $G_\Sigma$. As mentioned in Sec. I, a calculation of $G_K$ and $G_\Sigma$ from first principles is not feasible to date, nor is it for hadronic matrix elements in general. Instead one has to rely either on models invented *ad hoc* whose relation to the underlying field theory remains unclear or one is
restricted to numerical approximations to QCD that allow the inclusion of non-perturbative effects in a systematic way. One of these approximations is lattice QCD whose results can in principle be refined to arbitrary accuracy, but in practice are limited by available computer power and time. An other, less ambitious method is provided by QCD sum rules (cf. [27] for reviews). Originally invented for the calculation of vacuum–to–meson amplitudes in QCD [28], it found application likewise to meson–to–meson transition amplitudes [29] and could also be adapted to the study of HQET [7,8,24]. QCD sum rules rely on the field–theoretical aspects and features of QCD and were designed to make maximum use of known manifestations of non–perturbative QCD. Their advantage over lattice calculations is mainly their simplicity, whereas the main disadvantage may be seen in the fact that a systematic refinement to a given accuracy is not possible. The best accuracy one can hope to achieve is at the (10−20)% level.

Without going too much into details, we want to present here only the gross features of the method. QCD sum rules are based on the operator product expansion (OPE) of some correlation function, say \( \Pi(\omega) \), in terms of the so–called condensates, vacuum expectation values of certain operators that vanish when taken over the perturbative vacuum, but acquire non–zero values in the physical QCD vacuum and thus characterize its non–perturbative nature. The expansion is done in the not so deep Euclidean region of \( \omega \) where non–perturbative effects are already noticeable, but do not dominate yet:

\[
\sum_n \Pi^{(n)}(\omega) \langle O_n \rangle \equiv \sum_n \Pi^{(n)}(\omega) \langle 0|O_n|0 \rangle = \Pi(\omega).
\]

(4.1)

Examples for the vacuum condensates \( \langle O_n \rangle \) entering the OPE are \( \langle O_1 \rangle = 1 \), the unity operator, \( \langle O_3 \rangle = \langle \bar{q}q \rangle \), the (light) quark condensate, \( \langle O_4 \rangle = \langle \alpha_s G^2 / \pi \rangle \), the gluon condensate, and \( \langle O_5 \rangle = \langle \bar{q}g_s \sigma_{\mu\nu} G^{\mu\nu} q \rangle \), the mixed condensate. The coefficient functions \( \Pi^{(n)}(\omega) \), also called Wilson coefficients, can be calculated within perturbation theory and are determined by the short–distance behaviour of \( \Pi(\omega) \).

On the other hand, \( \Pi \) can be expressed via Cauchy’s theorem in terms of its singularities, which are situated on the positive real axis in \( \omega \) and are due to bound states causing poles and multi–particle states causing cuts. If we use as an input the information, that there is a pole due to the lowest–lying one–particle state, and its position, we can fit its residue, the square of the vacuum–ground–state transition amplitude, from the knowledge of \( \Pi \) in the not so deep Euclidean region. The short–comings, however, of such an approach are obvious: first we have to take it for granted that the OPE exists and makes sense. Besides, our knowledge of \( \Pi \) in the not so deep Euclidean region is not complete since one has to truncate the OPE at some point, which leads to errors in the determination of the residue. Moreover, there is not only the one–particle ground state, but also radial excitations and multi–particle states contribute to the singularities of \( \Pi \). Although it seems natural to assume that contributions of multi–particle states be suppressed by creation amplitudes, one still has to deal with both the unwanted contributions from radial excitations and unknown terms in the OPE stemming from higher–dimensional condensates.

To this end, it proved extremely useful in many applications to subject \( \Pi \) to a Borel transformation, a tool, that likewise was introduced in [28]. The Borel transform \( \hat{B}F \) of a function \( F \) depending on the Euclidean momentum \( \omega_E = -\omega \) is defined as
\begin{align}
\hat{B} F &= \lim_{\omega_E \to \infty, N \to \infty} \frac{1}{N!} (-\omega_E)^{N+1} \frac{d^{N+1}}{(d\omega_E)^{N+1}} F. 
\end{align}

(4.2)

The dimensionful quantity replacing $\omega$ is $t$, the Borel parameter. Applied to a dispersion relation over the physical spectral function $\rho^{\text{phys}}$, e.g., the Borel transformation yields

\begin{align}
\hat{B} \int ds \frac{\rho^{\text{phys}}(s)}{s - \omega} &= \frac{1}{t} \int ds \rho^{\text{phys}}(s) e^{-s/t}.
\end{align}

Thus contributions of $\rho^{\text{phys}}(s)$ become exponentially suppressed for large $s$, the suppression being controlled by the value of $t$. Contributions of higher dimensional condensates to the OPE \((4.1)\) get suppression factors in powers of $1/t$ and factorials. Roughly speaking, the Borel parameter serves as weight between perturbative ($\propto \langle O_1 \rangle$) and non–perturbative contributions.

Yet, one has to model $\rho^{\text{phys}}(s)$ for large values of $s$. At that point the quark hadron duality hypothesis enters. It states that integrals over $\rho^{\text{phys}}$ can be approximated to good accuracy by integrals over the perturbative spectral function $\rho^{\text{pert}}$, multiplying $\langle O_1 \rangle$, to wit

\begin{align}
\int ds \rho^{\text{phys}}(s) e^{-s/t} \approx \int ds \rho^{\text{pert}}(s) e^{-s/t}. 
\end{align}

(4.4)

Thus a suitable model for $\rho^{\text{phys}}$ for large values of $s$ is $\rho^{\text{phys}}(s) = \rho^{\text{pert}}(s)\Theta(s - \omega_0)$ where integration is understood afterwards. $\omega_0$ is the so–called continuum threshold and is unfortunately not related to fundamental quantities of QCD (or HQET), but should be less or equal to the position of the lowest–lying excitation, i.e. $\omega_0 \approx 1\text{ GeV}$. In the numerical analysis of sum rules one thus has to look for a value of $\omega_0$ and a sufficiently large range of $t$ where the sum rules becomes stable in these parameters. The accuracy of about \((10–20\%)\) mentioned above can, however, be worsened by a possibly strong dependence of the result on the precise values of the Borel parameter and the continuum threshold. Stated differently, too strong a dependence on that parameters is an indication for the unreliability of a sum rule. As we will see in the next section, that is fortunately not the case with the quantities we are interested in.

Keeping all these caveats in mind, we proceed with the derivation of QCD sum rules for $G_K$ and $G_\Sigma$. We write

\begin{align}
T_K &= \int ds \frac{\rho_K(s)}{s - \omega}, \quad T_\Sigma = \int ds \frac{\rho_\Sigma(s)}{s - \omega}, \quad (4.5)
\end{align}

do the OPE at the level of spectral densities and thus define

\begin{align}
\rho_K = \rho^{\text{pert}}_K + \rho^{(3)}_K \langle \bar{q}q \rangle + \rho^{(4)}_K \left( \frac{\alpha_s}{\pi} G^2 \right) + \rho^{(5)}_K \langle \bar{q}g_s \sigma_{\mu\nu} G^{\mu\nu} q \rangle + \ldots
\end{align}

(4.6)

and correspondingly for $\rho_\Sigma$. We use Feynman gauge for all diagrams involving only perturbative gluons; for the calculation of non–perturbative diagrams we choose the coordinate gauge $[30]$. The sign of the strong coupling is defined by $D_\mu = \partial_\mu - ig_s A_\mu^A \lambda^A / 2$. We apply dimensional regularization in $D = 4 + 2\epsilon$ dimensions and use the \underline{MS} renormalization scheme. For the perturbative contribution, we take into account the diagrams shown in Fig. \[3\] and find (at the renormalization scale $\mu$):
The loop–integrals involved in the calculation can be found in [26], e.g. The diagrams for the contribution of the quark condensate are shown in Fig. 3 and yield

$$\rho_{K}^{(3)} = 0, \quad \rho_{\Sigma}^{(3)} = \frac{2}{3} \frac{\alpha_s(\mu)}{\pi} \Theta(s).$$  \hfill (4.8)

In addition we have taken into account the contributions of the gluon and the mixed condensate shown in Fig. 4 that read

$$\rho_{K}^{(4)} = -\frac{1}{48} \delta(s), \quad \rho_{K}^{(5)} = \frac{3}{32} \delta'(s),$$

$$\rho_{\Sigma}^{(4)} = \frac{1}{48} \delta(s), \quad \rho_{\Sigma}^{(5)} = \frac{1}{48} \delta'(s).$$  \hfill (4.9)

From the result for $\Pi(\omega)$ given in [7,8,24] we obtain

$$\rho_{\omega \Pi} = \frac{3}{2\pi^2} s^2 \left\{ 1 + \frac{\alpha_s(\mu)}{\pi} \left( \frac{17}{3} + \frac{4}{9} \pi^2 - 2 \ln \frac{2s}{\mu} \right) \right\} \Theta(s) - \frac{1}{16} \delta(s) \langle \bar{q}g_\sigma \sigma_{\mu} G^\mu_{\nu} q \rangle.$$  \hfill (4.10)

We can now check all the formulæ obtained so far by means of Eq. (3.14). The expression for $\Pi_5$ including non–perturbative corrections can be found in [10], e.g. Here we concentrate on the perturbative part calculated in [21]. For its expansion to $O(1/m_Q^2)$, we recall that (3.14) is valid up to constants, so we make frequently use of partial integration to obtain the spectral density. Expanding the four–momentum as $q^2 = m_Q^2 + 2m_Q\omega + \omega^2$, we find

$$\rho_{\Pi_5}^{\text{pert}} = m_Q^2 \left\{\frac{3}{2\pi^2} s^2 \left\{ 1 + \frac{\alpha_s(\mu)}{\pi} \left( \frac{13}{3} - 2 \ln \frac{2s}{m_Q} + \frac{4}{9} \pi^2 \right) \right\} \right\} \Theta(s)$$

$$+ \frac{1}{m_Q} \frac{s^3}{\pi^2} \left\{ -\frac{3}{2} + \frac{\alpha_s(\mu)}{\pi} \left( -14 + 3 \ln \frac{2\omega}{m_Q} - \frac{2}{3} \pi^2 \right) \right\} \right\} \Theta(s).$$  \hfill (4.11)

Inserting $C(\mu), B_1(\mu)$ and $B_2(\mu)$ from (3.12) and $\rho_{\omega \Pi}^{\text{pert}}, \rho_{K}^{\text{pert}}, \rho_{\Sigma}^{\text{pert}}$ from above into the right hand side of (3.14), we indeed recover (4.11).

For the expansion of the full $\Pi_5$, note that the radiative correction to the Wilson coefficient multiplying the quark condensate was first calculated in [7,8] and reads before application of the Borel transformation:

$$\Pi_5^{(3)}(\bar{q}q) = \left( m_{\overline{\text{MS}}}^2(\bar{q}q) \right) \frac{m_Q^2}{q^2 - m_Q^2} \left\{ 1 + 2 \frac{\alpha_s}{\pi} \left( 1 + \frac{q^2 - m_Q^2}{q^2} \ln \frac{m_Q^2}{m_Q^2 - q^2} \right) \right\}. \hfill (4.12)$$
There remains only one slight complication to cope with, that is the structure of \( T_K \) and \( T_\Sigma \) expressed in terms of hadronic matrix elements. As mentioned at the end of the last section, both quantities develop apart from the single poles at \( \omega = \bar{\Lambda} \) we are interested in also double poles that stem from the contribution of three–point functions\(^8\) and can be obtained by inserting hadronic states between each of the operators in \( \langle J_{\Gamma_1} O J_{\Gamma_2} \rangle \). With \( K = \langle M | K | M \rangle / (2m_M) \) the invariant quantity \( \hat{T}_K \) defined in (3.25) can be expressed as

\[
\hat{T}_K = \hat{F}^2 \hat{G}_K \frac{\hat{K}}{2(\Lambda - \omega)} + \frac{\hat{F}^2 \hat{K}}{4(\Lambda - \omega)^2} + \text{higher states.} \quad (4.12)
\]

We can however isolate the single pole contribution by first applying a Borel transformation,

\[
\hat{B} \hat{T}_K = \frac{1}{4t^2} \hat{F}^2 \hat{K} e^{-\bar{\Lambda}/t} + \frac{1}{2t} \hat{F}^2 \hat{G}_K e^{-\bar{\Lambda}/t} + \ldots
\]

\[
= \frac{1}{2t} \hat{K} \hat{B} \hat{\Pi} + \hat{G}_K \hat{B} \hat{\Pi} + \ldots, \quad (4.13)
\]

where in the step from the first to the second equation we have made use of the Borel transform of Eq. (3.27). Taking then the derivative with respect to \( t \) yields

\[
\hat{G}_K = \frac{d}{dt} \frac{t \hat{B} \hat{T}_K}{B \hat{\Pi}}. \quad (4.14)
\]

The corresponding equation for \( \hat{G}_\Sigma \) is

\[
\hat{G}_\Sigma = \frac{1}{2} \frac{d}{dt} \frac{t \hat{B} \hat{T}_\Sigma}{B \hat{\Pi}}. \quad (4.15)
\]

The residues of the double pole contributions were calculated in [26]. Eqs. (4.14) and (4.15) are the final sum rules we are going to evaluate in the next section.

Let us close with a few remarks about the calculation performed in [9]. In our approach, the contributions of \( O(1) \) are completely separated from the ones of \( O(1/m_Q) \), because the matrix elements considered are different. So from our point of view it is not allowed to introduce a shift of the Borel parameter of order \( 1/m_Q \) in the sum rule for \( f_M \) as it was done in [9] since this would mix different orders in the inverse heavy quark mass. Since we have derived sum rules for \( G_K \) and \( G_\Sigma \) separately, there is no need to shift the value of the continuum threshold, either. Furthermore, the double pole contributions escaped the attention of Ref. [9] and thus the results obtained there are not values for \( G_K \) and \( G_\Sigma \), but mixtures of \( G_K \) and \( K \) and of \( G_\Sigma \) and \( \langle M | S | M \rangle \).

**V. RESULTS AND DISCUSSION**

In the numerical evaluation of the sum rules (4.14) and (4.15) we use the following values of the condensates at the renormalization point \( \mu = 1 \text{ GeV} \) [27]:

\(^8\)Such a mixing of two– and three–point functions was first considered in [31].
\[ \langle \bar{q}q \rangle = (-0.24 \text{ GeV})^3, \quad \left\langle \frac{\alpha_s}{\pi} G^2 \right\rangle = 0.012 \text{ GeV}^4, \]
\[ \langle \bar{q}g_s \sigma_{\mu\nu} G^{\mu\nu} q \rangle = 0.8 \text{ GeV}^2 \langle \bar{q}q \rangle. \quad (5.1) \]
For scales different from 1 GeV, we scale the condensates accordingly. For \( \alpha_s \), we use
\[ \alpha_s(\mu) = \frac{4\pi}{2\beta_0 \ln(\mu/\bar{\Lambda}_{\text{MS}}^{(n_f)}))} \left[ 1 - \frac{\beta_1}{\beta_0^2} \frac{\ln(2 \ln(\mu/\bar{\Lambda}_{\text{MS}}^{(n_f)})))}{2 \ln(\mu/\bar{\Lambda}_{\text{MS}}^{(n_f)}))} \right] \quad (5.2) \]
with \( \bar{\Lambda}_{\text{MS}}^{(4)} = 260 \text{ MeV} \), corresponding to \( \alpha_s(5 \text{ GeV}) = 0.20. \)

In Fig. 6 we show \( \hat{G}_K \) as function of \( t \) for \( \omega_0 = 1 \text{ GeV} \). Although \( \hat{T}_K \) is renormalization group invariant to next–to–leading order, we still have to choose one particular scale in building this quantity (cf. (3.25)). As argued in [7], the most natural choice is \( \mu = 2t \). In Fig. 6 we also show curves for \( \mu = 1 \text{ GeV} \) and \( \mu = 2 \text{ GeV} \). The scale–dependence still visible indicates the size of higher order effects in \( \alpha_s \). \( \hat{T}_K \) also depends on the unknown anomalous dimension matrix element \( T_{34} \) whose value is put zero in the figure. Varying the continuum threshold within \( \omega_0 = (1.0 - 1.2) \text{ GeV} \) and \( t \) within \( 0.35 \text{ GeV} < t < 0.6 \text{ GeV} \), as suggested by the analysis for \( F \) done in [7], and \( T_{34} \) in the range from \( -1 \) to 1, we obtain
\[ \hat{G}_K = - (1.1 \pm 0.2 \pm 0.2) \text{ GeV}. \quad (5.3) \]
The first error indicates the sum rule uncertainty in the “sum rule window” \( 0.35 \text{ GeV} < t < 0.6 \text{ GeV} \), the second one the uncertainty stemming from the variation of \( T_{34} \) which is rather large. From (3.32) we then obtain
\[ G_K(m_Q) = - (1.6 \pm 0.2 \pm 0.3 \pm 0.1) \text{ GeV}, \quad (5.4) \]
where the first two errors have the same meaning as before, the third one stems from varying \( \bar{\Lambda} \) in the range \( (0.5 - 0.6) \text{ GeV} \).

In Fig. 7, \( \hat{G}_\Sigma \) is displayed as function of \( t \) for \( \omega_0 = 1 \text{ GeV} \). As for \( \hat{G}_K \), we have varied \( \mu \) from 1 to 2 GeV. Although here two unknown matrix elements enter, \( S_{\text{mag}} \) and \( T_{32} \) whose values we have put to zero in the figure, their influence on the result is much smaller than that of \( T_{34} \) on \( G_K \) and we find
\[ \hat{G}_\Sigma = - (0.19 \pm 0.06 \pm 0.02) \text{ GeV}. \quad (5.5) \]
which corresponds to
\[ G_\Sigma(m_Q) = (0.042 \pm 0.034 \pm 0.023 \pm 0.030) \text{ GeV}. \quad (5.6) \]
The large errors in this result come from a cancellation of the central values of the two terms in (3.32).

In the last figure, Fig. 8, we show the resulting \( 1/m_Q \) corrections for \( f_P \sqrt{m_P} \) and \( f_V \sqrt{m_V} \), denoted as \( \delta F/F \), that are obtained from Eqs. (3.29) and (3.30). They yield\[ ]
\[ \text{Note that the errors of } G_\Sigma(m_Q) \text{ and } G_K(m_Q) \text{ are correlated.} \]
\[ f_P \sqrt{m_P} = F(m_Q) \left( 1 - \frac{2}{3} \frac{\alpha_s(m_Q)}{\pi} \right) \left( 1 - \frac{1}{m_Q} (1.07 \pm 0.19 \pm 0.07 \pm 0.02) \text{GeV} \right), \]

\[ f_V \sqrt{m_V} = F(m_Q) \left( 1 - \frac{4}{3} \frac{\alpha_s(m_Q)}{\pi} \right) \left( 1 - \frac{1}{m_Q} (0.78 \pm 0.08 \pm 0.09 \pm 0.04) \text{GeV} \right) \] (5.7)

with the same sequence of errors as before. Note that the error stemming from the unknown matrix elements is reduced as compared with (5.4) and (5.6) and that the sum rule uncertainty for \( f_V \) is smaller than that for \( f_P \) due to the better stability visible in Fig. 8.

Finally, we obtain from (3.31) (at the scale \( m_b \))

\[ \frac{f_V}{f_P} = 1.00 \pm 0.03 \pm 0.02 \pm 0.02 + O(1/m_Q^2), \] (5.8)

that is, a nearly cancellation of the \( 1/m_Q \) term, indicating that \( 1/m_Q^2 \) correction terms become important.

The result (5.7) agrees well with our old one obtained from the QCD sum rule for \( f_P \) [7]:

\[ f_P \sqrt{m_P} = F(m_Q) \left( 1 - \frac{2}{3} \frac{\alpha_s(m_Q)}{\pi} \right) \left( 1 - \frac{(0.8 - 1.1) \text{GeV}}{m_Q} \right). \] (5.9)

The remaining slight discrepancy is within the uncertainty of the method and/or could be due to \( 1/m_Q^2 \) correction terms. Our result also agrees well with values obtained from lattice calculations [6]. As far as [9] is concerned, our value for \( G_K(m_Q) \) is by nearly a factor three smaller, for \( G_\Sigma(m_Q) \) we obtain a different sign. The difference in sign affects mainly the ratio \( f_V/f_P \), where we find some disagreement with the lattice calculation [32] where \( f_V/f_P \) is found to be 1.11 \pm 0.05 at the scale of the B meson. This value, however, relies on a linear fit through four points in the mass range \((1.3 - 2.6) \text{ GeV} \) that has to be extrapolated to a mass of 5.28 GeV. Fitting a parable reduces the ratio already to 1.06 instead of 1.11. The remaining discrepancy could be due to \( 1/m_Q^2 \) terms that mimic a \( 1/m_Q \) behaviour in the restricted mass range considered in [32].

Concluding, we have found the sum rules for \( 1/m_Q \) corrections in HQET in complete agreement with conventional sum rules. There is however, a discrepancy in \( f_V/f_P \) compared to lattice calculations which might be due to \( 1/m_Q^2 \) correction terms.

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REFERENCES

[1] M.B. Wise, in *Proceedings of the 6th Lake Louise Winter Institute*, Lake Louise, 1991, edited by B. Campbell et al. (World Scientific, Singapore, 1991), p. 222; H. Georgi, in *Proceedings of the TASI 1991*, Boulder, 1991, edited by R.K. Ellis, C.T. Hill, and J.O. Sykken (World Scientific, Singapore, 1992), p. 589; M. Neubert, Preprint SLAC–PUB–6258 (1993).

[2] M.E. Luke, Phys. Lett. B 252, 447 (1990).

[3] Ya. I. Azimov, L.L. Frankfurt, and V.A. Khoze, JETP Lett. 24, 338 (1976); E.V. Shuryak, Nucl. Phys. B198, 83 (1982).

[4] M.A. Shifman and M.B. Voloshin, Sov. J. Nucl. Phys. 45, 292 (1987); 47, 511 (1988).

[5] H. Georgi, Phys. Lett. B 240, 447 (1990).

[6] A. Duncan et al., in Lattice 92, Nucl. Phys. B (Proc. Suppl.) 30, 433 (1993); C. Alexandrou et al., Preprint PSI–PR–92–27 (1992) [hep–lat/9211042]; C.W. Bernard, J.N. Labrenz, and A. Soni, Washington Preprint UW–PT–93–06 (1993) [hep–lat/9306009]; APE Collaboration (C.R. Allton et al.), Preprint LPTENS–93–12 (1993); UKQCD Collaboration (R.M. Baxter et al.), Edinburgh Preprint 93/526 (1993) [hep–lat/9308020].

[7] E. Bagan et al., Phys. Lett. B 278, 457 (1992).

[8] M. Neubert, Phys. Rev. D 45, 2451 (1992).

[9] M. Neubert, Phys. Rev. D 46, 1076 (1992).

[10] For a recent overview over QCD sum rule calculations, see C.A. Dominguez, Talk given at the Third Workshop on the Tau-Charm Factory, Marbella, Spain, 1-6 June 1993, Preprint SISSA Ref. 139/93/EP (1993) [hep–ph/9309260].

[11] B–Physics Working Group Report, edited by R. Aleksan and A. Ali, DESY Report 93–053 (1993).

[12] V. Eletsky and E. Shuryak, Phys. Lett. B 276, 191 (1992).

[13] M. Neubert, Preprint SLAC–PUB–6344 (1993) [hep–ph/9308303].

[14] A.F. Falk, B. Grinstein, and M.E. Luke, Nucl. Phys. B357, 185 (1991).

[15] T. Mannel, W. Roberts, and Z. Ryzak, Nucl. Phys. B368, 204 (1992).

[16] M. Luke and A.V. Manohar, Phys. Lett. B 286, 348 (1992).

[17] J.D. Bjorken, in *Proceedings of Les Rencontres de Physique de la Vallée d’Aoste*, La Thuile, 1990, edited by M. Greco (Editions Frontieres, Gif-sur-Yvette, 1990), p. 583; A. Falk et al., Nucl. Phys. B343, 1 (1990).

[18] Z. Guralnik and A.V. Manohar, Phys. Lett. B 302, 103 (1993).

[19] I.I. Bigi and N.G. Uraltsev, Preprint CERN–TH.7091/93 (1993) [hep–ph/9311337].

[20] M.B. Voloshin and Yu.M. Zaitsev, Sov. Phys. Usp. 30, 553 (1987); L.J. Reinders, Phys. Rev. D 38, 947 (1988); C.A. Dominguez and N. Paver, Phys. Lett. B 293, 197 (1992); P.B. Mackenzie, Talk presented at the Lepton–Photon Symposium, Cornell University, Aug. 10-15, 1993, Preprint FERMILAB–CONF–93/343–T (1993) [hep–ph/9311243]; S. Titard and F.J. Yndurain, Madrid Preprint UM–TH–93–25 (1993) [hep–ph/9310236].

[21] D.J. Broadhurst, Phys. Lett. B 101, 423 (1981).

[22] E. Eichten and B. Hill, Phys. Lett. B 243, 427 (1990).

[23] X. Ji and M.J. Musolf, Phys. Lett. B 257, 409 (1991).

[24] D.J. Broadhurst and A.G. Grozin, Phys. Lett. B 274, 421 (1992).

[25] D.J. Broadhurst and A.G. Grozin, Phys. Lett. B 257, 105 (1991).
[26] P. Ball and V.M. Braun, München Preprint TUM–T31–42/93 (1993) (hep-ph/9307291), to appear in Phys. Rev. D.

[27] L.J. Reinders, H. Rubinstein, and S. Yazaki, Phys. Rep. 127, 1 (1985); S. Narison, QCD Spectral Sum Rules (World Scientific, Singapore, 1989); Vacuum Structure and QCD Sum Rules, Current Physics Sources and Comments, Vol. 10, edited by M.A. Shifman (North–Holland, Amsterdam, 1992).

[28] M.A. Shifman, A.I. Vainshtein, and V.I. Zakharov, Nucl. Phys. B147, 385, 448, 519 (1979).

[29] B.L. Ioffe, A.V. Smilga, Phys. Lett. B 114, 353 (1982); Nucl. Phys. B 216, 373 (1983).

[30] V. Fateev, A. Schwartz, and Y. Tyupkin, Moscow Preprint FIAN–155 (1976); C. Cronström, Phys. Lett. B 90, 267 (1980).

[31] V.M. Braun, A.V. Kolesnichenko, Nucl. Phys. B283, 723 (1987).

[32] A. Abada et al., Nucl. Phys. B376, 172 (1992).
FIG. 1. Momentum configuration for the matching calculation of the pseudoscalar current. $p$ is the momentum of the heavy quark $Q$, $p'$ that of the massless quark $q$. Both quarks are on–shell.

FIG. 2. Perturbative contribution to $T_K$. Double lines denote heavy quarks, single lines light quarks and curly lines gluons. The crosses with circles denote the heavy–light vertex $\Gamma$, the black square the insertion of the operator $K$. All diagrams not drawn vanish.

FIG. 3. Diagrams contributing to the Wilson coefficient of the quark condensate in the OPE of $T_K$. Lines with crosses denote vacuum expectation values.
FIG. 4. Diagrams contributing to the Wilson coefficient of the gluon and the mixed condensate in the OPE of $T_K$. We use the coordinate gauge with the coordinates specified in the second diagram. All diagrams not drawn vanish.

FIG. 5. Diagrams contributing to $T_\Sigma$. The black square now denotes an insertion of the operator $S$. 

FIG. 6. \( \hat{G}_K \) as function of \( t \) for \( \omega_0 = 1 \text{GeV} \) and different scales \( \mu \). Solid line: \( \mu = 2t \), long–dashed line: \( \mu = 1 \text{GeV} \), short–dashed line: \( \mu = 2 \text{GeV} \).

FIG. 7. \( \hat{G}_\Sigma \) as function of \( t \) for \( \omega_0 = 1 \text{GeV} \) and different scales \( \mu \). Solid line: \( \mu = 2t \), long–dashed line: \( \mu = 1 \text{GeV} \), short–dashed line: \( \mu = 2 \text{GeV} \).

FIG. 8. The \( 1/m_Q \) corrections \( \delta F/F \) to \( F \) according to Eqs. (3.29) and (3.30) as function of \( t \) for \( \omega_0 = 1 \text{GeV} \). Solid line: pseudoscalar decay constant, dashed line: vector decay constant. The normalization scale is \( \mu = 2t \).