Exponential variational integrators for the dynamics of multibody systems with holonomic constraints

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Abstract. We here explore a geometric integrator scheme that is determined by a discretization of a variational principle using a higher-order Lagrangian that uses exponential type of interpolation functions. The resulting exponential variational integrators are here extended to conservative mechanical systems with constraints. To do so we first present continuous Euler-Lagrangian equations with holonomic constraints and then mimic the process for the discrete case. The resulting schemes are then tested to a typical dynamical multibody system with constraints, i.e the double pendulum showing the good long-time behavior when compared to other traditional methods.

1. Introduction

Geometric integration methods have been widely used for the simulation and/or optimization of mechanical systems. Since those consider the underlying geometric structure of the system, they general preserve the geometry of problem and thus simulations are more accurate [1, 2]. Variational integrators are a special kind of those methods which are symplectic and momentum preserving [1]. In order to solve conservative problems those methods are derived by mimicking the continuous Hamilton’s principle in a so called discrete version of it. Similarly for dissipative and/or controlled mechanical systems a discrete version of the Lagrange-d’Alembert principle can be used [3].

To improve the accuracy of the standard variational integrators, high order schemes have been proposed [4]. Their derivation relies on the approximation of the action on a finite-dimensional function space and a numerical quadrature formula. To that end in our previus work we have proposed exponential variational integrators [4, 5, 6, 7, 8]. Those have been derived using a discrete Lagrangian defined as a weighted sum of the evaluation of the continuous Lagrangian at a number of intermediate points. Their accurate results, especially for long term integration, has been compared with several variational integrators of similar type while it has been proven that they are unconditionally stable [4, 6].

To that end, in the present we extended the exponential variational integrators to conservative mechanical systems with holonomic constraints. We first present the continuous and the discrete version of the Euler-Lagrangian equations with holonomic constraints in Section 2. Those equations are then used for the proposed high order schemes that we describe in Section 3 and then tested for the solution of the double pendulum system.
2. Multibody system dynamics with holonomic constraints

In order to consider multibody system dynamics with holonomic constraints we will use Hamilton’s principle, i.e. variation of the action $S$ as

$$\delta S = \delta \int_{t_0}^{t_1} \left( L(q, \dot{q}, t) - \lambda^T \varphi(q, t) \right) dt = 0,$$

(1)

where $q, \dot{q}$ are the generalized displacements and velocities respectively. Function $L$ denotes the Lagrangian of the system and $\varphi$ is the constraint expression, i.e.

$$\varphi(q, t) = 0,$$

(2)

which is used in (1) via the Lagrange multiplier $\lambda$. The above variational method leads to the Euler-Lagrange equations

$$\frac{d}{dt} \left( \frac{\partial L(q, \dot{q}, t)}{\partial \dot{q}} \right) - \frac{\partial L(q, \dot{q}, t)}{\partial q} + \varphi^T(q, t) \lambda = 0$$

$$\varphi(q, t) = 0,$$

(3)

which can be then solved using several numerical methods [3]. Here we propose the use of a high order symplectic scheme of high order, that uses the advantages of variational integrators [4].

Following the steps of the continuous formalism we first divide the time integral $[t_0, t]$ into $N$ subintervals by the use of the time step $h$, i.e. $t_i = ih$ where $i = 0, 1, 2, \ldots, N$ and $N \in \mathbb{N}$. Doing so we can then define $q_i = q(t_i)$ and $\dot{q}_i = \dot{q}(t_i)$ as the discrete expressions of the configurations and velocities respectively. Discrete versions of the Lagrange multipliers can be also defined by using $\lambda_i = \lambda(t_i)$. For a smooth and finite dimensional configuration manifold $Q$ one can then define the discrete Lagrangian $L_d : Q \times Q \rightarrow \mathbb{R}$ as an approximation of a continuous action obtained as

$$L_d(q_k, q_{k+1}, h) \approx \int_{t_k}^{t_{k+1}} L(q, \dot{q}) dt.$$  

(4)

From the discrete variational principle, the solutions of the discrete system are determined from $L_d$ by extremizing the discrete action sum, keeping the endpoints $q_0$ and $q_N$ fixed. So the discrete Euler-Lagrange equations can be derived as

$$D_1 L_d(q_k, q_{k+1}) + D_2 L_d(q_{k-1}, q_k) - h \varphi^T(q_k) \lambda_k = 0$$

$$\varphi(q_{k+1}) = 0,$$

(5)

where $k = 1, \ldots, N - 1$. Finally the term $D_i L_d$ stands for the derivative with respect to the $i$-argument of $L_d$, see [3, 4, 5].

3. Exponential variational integrators

To extend the variational integrators to high order an approximation of the action integral along the curve segment between $q_k$ and $q_{k+1}$ must be considered. To that, as in (4), the discrete Lagrangian must depend only on the end points. For those intermediate points we can then define expressions for the configurations $q^j_k$ and velocities $\dot{q}_k^j$, $j = 0, \ldots, S - 1$, $S \in \mathbb{N}$, at time $t^j_k \in [t_k, t_{k+1}]$, by expressing the $t^j_k$ as $t^j_k = t_k + C^j_k h$ for $C^j_k \in [0, 1]$ such that $C^0_k = 0$, $C^{S-1}_k = 1$. Exponential variational integrators of high order can be then defined using [4, 5]

$$q^j_k = g_1(t^j_k)q_k + g_2(t^j_k)q_{k+1},$$

$$\dot{q}_k^j = \dot{g}_1(t^j_k)q_k + \dot{g}_2(t^j_k)q_{k+1},$$

(6)
where $h \in \mathbb{R}$ is the time step, while the functions $g_1$ and $g_2$ are \([4, 5, 7]\)
\[
g_1(t_k^j) = \sin \left( u - \frac{t_k^j - t_k}{h_k} u \right) (\sin u)^{-1}, \quad g_2(t_k^j) = \sin \left( \frac{t_k^j - t_k}{h_k} u \right) (\sin u)^{-1}. \tag{7}
\]

For the sake of continuity, the conditions $g_1(t_{k+1}) = g_2(t_k) = 0$ and $g_1(t_k) = g_2(t_{k+1}) = 1$ are required to be fulfilled \([9, 10]\).

For any different number of intermediate points, we can then define the discrete Lagrangian $L_d$ by a weighted sum of the form \([4, 5, 8]\)
\[
L_d(q_k, q_{k+1}, h_k) = \sum_{j=0}^{S-1} h_k w^j L \left( q(t_k^j), \dot{q}(t_k^j) \right), \tag{8}
\]
where, as can be readily proved, it holds $\sum_{j=0}^{S-1} w^j(C_k^j)^m = \frac{1}{m+1}$ with $m = 0, 1, \ldots, S - 1$ and $k = 0, 1, \ldots, N - 1$ \([4, 5]\).

By applying the above interpolation technique in combination with the expressions of (7) and following the analysis of \([4, 5]\), the parameter $u$ entering equations (7) must be chosen to be $u = \omega h$. For problems involving a definite frequency $\omega$ (such as the harmonic oscillator), the parameter $u$ can be easily computed. However, for the solution of orbital problems of the general $N$-body problem, where no unique frequency of the motion can be in general determined, a new parameter $u$ must be defined by estimating the frequency of the motion for any moving point mass \([4, 5]\).

4. Double pendulum
We now consider the planar double system of \([2]\). By considering $m_1, m_2, l_1, l_2$ the pendulum masses and lengths respectively, the kinetic energy of the system, using the configuration positions $q = [x_1 \ y_1 \ x_2 \ y_2]^T$, can be written as
\[
T(q, \dot{q}) = \frac{1}{2} \dot{q}^T M \dot{q}, \tag{9}
\]
where $M$ is the mass matrix which is given by $M = \text{diag}[m_1 \ m_2 \ m_2]$. The potential is
\[
V(q) = m_1 g y_1 + m_2 g y_2, \tag{10}
\]
for the gravitational constant $g$. For that problem the constraint equation of (3)
\[
\varphi(q) = \left[ (x_2 - x_1)^2 + (y_2 - y_1)^2 - l_1^2 \right] = 0. \tag{11}
\]
If we further for simplicity consider all masses and lengths to be ones, we can define the discrete Lagrangian of (8) for the double pendulum problem. That can be then used to the discrete Euler-Lagrange equations of (5) to obtain the solution of the physical problem.

In order to test the proposed method we consider the case where $S = 1$, i.e. one intermediate point at each time segments $[t_k, t_{k+1}]$ of the discrete Lagrangian (8). For that we first test the maximum error of the total energy $H$ of the system, $\max|\epsilon(H)|$, after 100 steps. In Figure 1 those errors are plotted and compared with the ones obtained using a standard Runge-Kutta method, see \([2]\). Although the methods do are not of the same order, for time steps $h = 10^{-2}$ and $h = 10^{-3}$ the energy errors obtained from the proposed are much more smaller. Same results have been obtained when comparing the maximum error of the constrained equation, noted by $\max|\epsilon(\varphi)|$, see Figure 2.
5. Conclusion and future work
In the present we extend the exponential variational integrators derived in our previous work in order to solve multibody system dynamics with constraints. Up to now those integrators have been completely defined and tested for conservative systems, showing their good behavior when compared to standard ones. To that end here we use the discrete Euler-Lagrangian equations with holonomic constraints. To those we define an appropriate discrete Lagrangian that describes the physical problem, while a discrete version of the constraints are also added. Preliminary simulation results obtained for the numerical solution of the double pendulum system show that proposed schemes can be a useful tool, while their good behavior when compared to other traditional methods has been seen even for low order methods.

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