GLOBAL WELL-POSEDNESS FOR FRACTIONAL HARTREE EQUATION ON MODULATION SPACES AND FOURIER ALGEBRA

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Abstract. We study the Cauchy problem for fractional Schrödinger equation with cubic convolution nonlinearity $i \partial_t u - (-\Delta)^{\alpha/2} u \pm (K \ast |u|^2) u = 0$ with Cauchy data in the modulation spaces $M^{p,q}(\mathbb{R}^d)$. For $K(x) = |x|^{-\gamma} (0 < \gamma < \min\{\alpha, d/2\})$, we establish global well-posedness results in $M^{p,q}(\mathbb{R}^d)(1 \leq p \leq 2, 1 \leq q < 2d/(d+\gamma))$ when $\alpha = 2, d \geq 1$, and with radial Cauchy data when $d \geq 2, \frac{2d}{2d-1} < \alpha < 2$. Similar results are proven in Fourier algebra $\mathcal{F}L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$.

1. Introduction

We study fractional nonlinear Schrödinger equation with cubic convolution nonlinearity

$$i \partial_t u - (-\Delta)^{\alpha/2} u = (K \ast |u|^2) u, u(x, 0) = u_0(x)$$

(1.1)

where $u: \mathbb{R}^d \times \mathbb{R}^d_x \to \mathbb{C}, u_0: \mathbb{R}^d \to \mathbb{C}$, $K$ denotes the Hartree kernel

$$K(x) = \frac{\lambda}{|x|^\gamma}, (\lambda \in \mathbb{R}, \gamma > 0, x \in \mathbb{R}^d),$$

(1.2)

and $\ast$ denotes the convolution in $\mathbb{R}^d$. The fractional Laplacian is defined as

$$\mathcal{F}[(-\Delta)^{\alpha/2} u](\xi) = |\xi|^{\alpha} \mathcal{F}u(\xi)$$

(1.3)

where $\mathcal{F}$ denotes the Fourier transform. The equation (1.1) is known as the Hartree type equation with fractional Laplacian, and we call it fractional Hartree equation (FHTE).

In recent years, there has been a great deal of interest [5, 15] in using fractional Laplacians to model physical phenomena. It was formulated by N. Laskin [15] as a result of extending the Feynman path integral from the Brownian-like to Lévy-like quantum mechanical paths. Specifically, when $\alpha = 1$, (1.1) can be used to describe the dynamics of pseudo-relativistic boson stars in the mean-field limit, and when $\alpha = 2$ the Lévy motion becomes Brownian motion. Thus, in the last couple of years many authors have studied the Cauchy problem for fractional Hartree equation with Cauchy data in Sobolev spaces, see for instance [8, 13] and the reference therein.

On the other hand, there has been a lot of ongoing interest to study Cauchy problem for nonlinear dispersive equations with the Cauchy data in modulation spaces $M^{p,q}(\mathbb{R}^d)$ (See Definition 2.1 below). Generally speaking, the Cauchy data in a modulation spaces are rougher than any given one in a fractional Bessel potential space and this low regularity is desirable

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in many situations. It is well known that \([9, 2, 20]\) Schrödinger propagator \(e^{-it(-\Delta)^{\alpha/2}}\) is bounded on \(M^{p,q}(\mathbb{R}^d)\) but most authors have studied local \([3]\) and global \([19]\) well-posedness (for small data) for a nonlinear dispersive equations with Cauchy data in \(M^{p,1}(\mathbb{R}^d)\). As one advantage to consider Cauchy data in \(M^{p,1}(\mathbb{R}^d)\) is the fact that it is an algebra under point wise multiplication. There has not been much progress to consider the Cauchy data in the modulation spaces \(M^{p,q}(1 < q \leq \infty)\). However, there has been an attempt to get global well-posedness results for a large data in modulation spaces but yet we do not know the complete answer to this, see for instance, open question raised by Baoxiang-Ruzhansky-Sugimoto \([17, p.280]\).

Taking these considerations into account, in this article, first we study global well-posedness result for fractional Hartree equation \((1.1)\) with Cauchy data in modulation spaces \(M^{p,q}(\mathbb{R}^d)\) (without any smallness assumption for the Cauchy data). We could include family of Banach spaces which may not be algebra under point wise multiplication, say \(M^{p,q}(\mathbb{R}^d)(q \neq 1)\). We denote the Banach space \(X\) of radial functions by \(X_{\text{rad}}\).

 Specifically, we obtain following results.

**Theorem 1.1.** Let \(K\) be given by \((1.2)\) with \(0 < \gamma < \min\{\alpha/d, 2d/(d+\gamma)\}\), and \(1 \leq p \leq 2, 1 \leq q < \frac{2d}{d+\gamma}\).

1. Assume that \(u_0 \in M^{p,q}_{\text{rad}}(\mathbb{R}^d), d \geq 2, \frac{2d}{2d-1} < \alpha < 2\). Then there exists a unique global solution of \((1.1)\) such that
   \[u \in C(\mathbb{R}, M^{p,q}_{\text{rad}}(\mathbb{R}^d)) \cap L^{4\alpha/\gamma}_{\text{loc}}(\mathbb{R}, L^{4d/(2d-\gamma)}(\mathbb{R}^d)).\]

2. Assume that \(u_0 \in M^{p,q}(\mathbb{R}^d), d \in \mathbb{N}, \alpha = 2\). Then there exists a unique global solution of \((1.1)\) such that
   \[u \in C(\mathbb{R}, M^{p,q}(\mathbb{R}^d)) \cap L^{8/\gamma}_{\text{loc}}(\mathbb{R}, L^{4d/(2d-\gamma)}(\mathbb{R}^d)).\]

We note that \(M^{p,q}(\mathbb{R}^d) \subset L^p(\mathbb{R}^d)(q \leq \min\{p, p'\})\) is sharp embedding and up to now we cannot get the global well-posedness of fractional Hartree equation in \(L^p(\mathbb{R}^d)(1 \leq p < 2)\) but in \(M^{p,q}(\mathbb{R}^d)\) (see Theorem 1.1). We also note that recently the author \([4, \text{Theorem 1.1}]\) have proved the global well-posedness result for the classical Hartree equation (that is \((1.1)\) with \(\alpha = 2\)) with Cauchy date in \(M^{1,1}(\mathbb{R}^d)\), and this result has been extended by Manna \([16, \text{Theorem 1.1}]\) with Cauchy data in \(M^{p,p}(\mathbb{R}^d)\) for \(1 \leq p < \frac{2d}{d+\gamma}\). Thus, our Theorem 1.1 extends the existing results in the literature so far. It is also worth noting following

**Remark 1.1.** The nonlinear Schrödinger equation (NLS)

\[i\partial_t u + \Delta u = |u|^2 u,\]

is locally \([3]\) well-posed in \(M^{p,1}(\mathbb{R}^d)\) but it is not yet clear whether it is globally well-posed or not (see for instance \([17, p.280]\)). Also, it is not clear whether NLS is locally well posed in \(M^{p,q}(\mathbb{R}^d)(q \neq 1)\). But in contrast fractional Hartree equation is globally well-posed in \(M^{p,q}(\mathbb{R}^d)\) (see Theorem 1.1).
Next we study fractional Hartree equation (1.1) with Cauchy data in Fourier algebra $\mathcal{F}L^1(\mathbb{R}^d)$ (See Section 2 below) and the space of square integrable functions. We note that for $s \geq d/2$, $H^s(\mathbb{R}^d) \hookrightarrow \mathcal{F}L^1(\mathbb{R}^d) \hookrightarrow M^{\infty,1}(\mathbb{R}^d)$. We recall recently Carles-Mouzaoui [6] have proved global well-posedness result for classical Hartree equation (that is (1.1) with $\alpha = 2$) with Cauchy data in $L^2(\mathbb{R}^d) \cap \mathcal{F}L^1(\mathbb{R}^d)$. We extend this result for the fractional Hartree equation. Specifically, we prove following

**Theorem 1.2.** Let $d \geq 2$, $\frac{2d}{2d-1} < \alpha < 2$, $0 < \gamma < \min\{\alpha, d/2\}$. Assume that $u_0 \in L^2_{\text{rad}}(\mathbb{R}^d) \cap \mathcal{F}L^1(\mathbb{R}^d)$, and let $K$ be given by (1.2) with $\lambda \in \mathbb{R}$. Then there exists a unique global solution of (1.1) such that

$$u \in C(\mathbb{R}, L^2_{\text{rad}}(\mathbb{R}^d)) \cap C(\mathbb{R}, \mathcal{F}L^1(\mathbb{R}^d)) \cap L^{4\alpha/\gamma}_\text{loc} (\mathbb{R}, L^{4d/(2d-\gamma)}(\mathbb{R}^d)).$$

This article is organized as follows. We will give basic notations and tools that will be used throughout this article, in Section 2. This includes all definitions of function spaces and some of their properties, and known lemmas. Section 3 is devoted to proving result concerning global well-posedness result for the Cauchy data in $L^2(\mathbb{R}^d)$. We shall see this will turn out to be one of the main tools to obtain global well-posedness results in the realm of modulation spaces and Fourier algebra. In Section 4 we shall prove our main Theorems 1.1 and 1.2.

## 2. Notations and Preliminaries

### 2.1. Notations

The notation $A \lesssim B$ means $A \leq cB$ for a some constant $c > 0$, whereas $A \simeq B$ means $c^{-1}A \leq B \leq cA$ for some $c \geq 1$. The symbol $A_1 \hookrightarrow A_2$ denotes the continuous embedding of the topological linear space $A_1$ into $A_2$. Let $I \subset \mathbb{R}$ be an interval. Then the norm of the space-time Lebesgue space $L^p(I, L^q(\mathbb{R}^d))$ is defined by

$$\|u\|_{L^p(I, L^q(\mathbb{R}^d))} = \|u\|_{L^p_t L^q_x} = \left( \int_I \|u(t)\|_{L^q_x}^p dt \right)^{1/p}.$$  

If there is no confusion, we simply write

$$\|u\|_{L^p(I, L^q)} = \|u\|_{L^p_t L^q_x} = \|u\|_{L^p_{t,x}}.$$  

The Schwartz space is denoted by $\mathcal{S}(\mathbb{R}^d)$ (with its usual topology), and the space of tempered distributions is denoted by $\mathcal{S}'(\mathbb{R}^d)$. For $x = (x_1, \cdots, x_d), y = (y_1, \cdots, y_d) \in \mathbb{R}^d$, we put $x \cdot y = \sum_{i=1}^d x_i y_i$. Let $\mathcal{F} : \mathcal{S}(\mathbb{R}^d) \rightarrow \mathcal{S}(\mathbb{R}^d)$ be the Fourier transform defined by

$$\mathcal{F}f(w) = \hat{f}(w) = \int_{\mathbb{R}^d} f(t)e^{-2\pi i t \cdot w} dt, \ w \in \mathbb{R}^d.  \ (2.1)$$

Then $\mathcal{F}$ is a bijection and the inverse Fourier transform is given by

$$\mathcal{F}^{-1}f(x) = f^\vee(x) = \int_{\mathbb{R}^d} f(w) e^{2\pi i x \cdot w} dw, \ x \in \mathbb{R}^d,  \ (2.2)$$

and this Fourier transform can be uniquely extended to $\mathcal{F} : \mathcal{S}'(\mathbb{R}^d) \rightarrow \mathcal{S}'(\mathbb{R}^d)$. We denote by $\mathcal{F}L^p(\mathbb{R}^d)$ the Fourier-Lebesgue space, specifically,

$$\mathcal{F}L^p(\mathbb{R}^d) = \{ f \in \mathcal{S}'(\mathbb{R}^d) : \hat{f} \in L^p(\mathbb{R}^d) \}.$$
The $\mathcal{F}L^p(\mathbb{R}^d)$-norm is denoted by
\[
\|f\|_{\mathcal{F}L^p} = \|\hat{f}\|_{L^p} \quad (f \in \mathcal{F}L^p(\mathbb{R}^d)).
\]
If $s \in \mathbb{R}$, we put $\langle \xi \rangle^s = (1 + |\xi|^2)^{s/2}$ ($\xi \in \mathbb{R}^d$), and define Sobolev space $H^s(\mathbb{R}^d)$ to be
\[
H^s(\mathbb{R}^d) = \{ f \in \mathcal{S}(\mathbb{R}^d) : \mathcal{F}^{-1}[\langle \cdot \rangle\mathcal{F}(f)] \in L^2(\mathbb{R}^d) \}.
\]

2.2. Modulation spaces. In 1983, Feichtinger [10] introduced a class of Banach spaces, the so called modulation spaces, which allow a measurement of space variable and Fourier transform variable of a function or distribution on $\mathbb{R}^d$ simultaneously using the short-time Fourier transform (STFT). The STFT of a function $f$ with respect to a window function $g \in \mathcal{S}(\mathbb{R}^d)$ is defined by
\[
(2.3) \quad V_g f(x, w) = \int_{\mathbb{R}^d} f(t)g(t-x)e^{-2\pi iw \cdot t} dt, \quad (x, w) \in \mathbb{R}^d
\]
whenever the integral exists. For $x, w \in \mathbb{R}^d$ the translation operator $T_x$ and the modulation operator $M_w$ are defined by $T_x f(t) = f(t-x)$ and $M_w f(t) = e^{2\pi iw \cdot t} f(t)$. In terms of these operators the STFT may be expressed as
\[
(2.4) \quad V_g f(x, y) = \langle f, M_y T_x g \rangle = e^{-2\pi i x \cdot y} (f \ast M_y g^*)(x)
\]
where $g^*(y) = g(-y)$ and $\langle f, g \rangle$ denotes the inner product for $L^2$ functions, or the action of the tempered distribution $f$ on the Schwartz class function $g$. Thus $V : (f, g) \rightarrow V_g(f)$ extends to a bilinear form on $\mathcal{S}(\mathbb{R}^d) \times \mathcal{S}(\mathbb{R}^d)$ and $V_g(f)$ defines a uniformly continuous function on $\mathbb{R}^d \times \mathbb{R}^d$ whenever $f \in \mathcal{S}(\mathbb{R}^d)$ and $g \in \mathcal{S}(\mathbb{R}^d)$.

**Definition 2.1** (modulation spaces). Let $1 \leq p, q \leq \infty$, $s \in \mathbb{R}$ and $0 \neq g \in \mathcal{S}(\mathbb{R}^d)$. The weighted modulation space $M_s^{p,q}(\mathbb{R}^d)$ is defined to be the space of all tempered distributions $f$ for which the following norm is finite:
\[
\|f\|_{M_s^{p,q}} = \left( \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} |V_g f(x, y)|^p dx \right)^{q/p} \langle y \rangle^{sq} dy \right)^{1/q},
\]
for $1 \leq p, q < \infty$. If $p$ or $q$ is infinite, $\|f\|_{M_s^{p,q}}$ is defined by replacing the corresponding integral by the essential supremum.

**Remark 2.1.** The definition of the modulation space given above, is independent of the choice of the particular window function. See [10] Proposition 11.3.2(c), p.233. If $s = 0$, we simply write $M_s^{p,q} = M_s^{p,q}$.

In the next lemma we collect basic properties of modulation spaces which we shall need later.

**Lemma 2.2.** Let $p, q, p_i, q_i \in [1, \infty]$ ($i = 1, 2$), $s_1, s_2 \in \mathbb{R}$. Then
1. $M_{s_1}^{p_1, q_1}(\mathbb{R}^d) \hookrightarrow M_{s_2}^{p_2, q_2}(\mathbb{R}^d)$ whenever $p_1 \leq p_2$ and $q_1 \leq q_2$, and $s_2 \leq s_1$.
2. $M_s^p(\mathbb{R}^d) \hookrightarrow L^p(\mathbb{R}^d) \hookrightarrow M_s^{p_2}(\mathbb{R}^d)$ holds for $q_1 \leq \min\{p, p'\}$ and $q_2 \geq \max\{p, p'\}$ with $\frac{1}{p} + \frac{1}{p'} = 1$. 

(3) $M^\min(p',2,p) \hookrightarrow \mathcal{F} L^p(\mathbb{R}^d) \hookrightarrow M^\max(p',2,p)(\mathbb{R}^d), \frac{1}{p} + \frac{1}{p'} = 1$.

(4) $\mathcal{S}(\mathbb{R}^d)$ is dense in $M^{p,q}(\mathbb{R}^d)$ if $p$ and $q < \infty$.

(5) The Fourier transform $\mathcal{F} : M^{p,p}(\mathbb{R}^d) \to M^{p,p}(\mathbb{R}^d)$ is an isomorphism.

(6) The space $M^{p,q}(\mathbb{R}^d)$ is a Banach space.

(7) The space $M^{p,q}(\mathbb{R}^d)$ is invariant under complex conjugation.

**Proof.** All these statements are well-known and the interested reader may find a proof in [11, 10, 21]. For the proof of statements (1), (2) and (3), see [11, Theorem 12.2.2], [18], and [7, Corollary 1.1] respectively. The proof for the statement (5) can be derived from the fundamental identity of time-frequency analysis:

$$V_g f(x,w) = e^{−2\pi i x \cdot w} V_{\hat{g}}(w,−x),$$

which is easy to obtain. The proof of the statement (7) is trivial, indeed, we have $\|f\|_{M^{p,q}} = \|\hat{f}\|_{M^{p,q}}$.

**Theorem 2.3.** Let $p, q, p_i, q_i ∈ [1,∞]$ $(i = 0, 1, 2)$. If $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_0}$ and $\frac{1}{q_1} + \frac{1}{q_2} = 1 + \frac{1}{q_0}$, then

$$M^{p_1,q_1}(\mathbb{R}^d) \cdot M^{p_2,q_2}(\mathbb{R}^d) \hookrightarrow M^{p_0,q_0}(\mathbb{R}^d);$$

with norm inequality $\|fg\|_{M^{p_0,q_0}} \lesssim \|f\|_{M^{p_1,q_1}}\|g\|_{M^{p_2,q_2}}$. In particular, the space $M^{p,q}(\mathbb{R}^d)$ is a pointwise $\mathcal{F} L^1(\mathbb{R}^d)$-module, that is, it satisfies

$$\|fg\|_{M^{p,q}} \lesssim \|f\|_{\mathcal{F} L^1}\|g\|_{M^{p,q}}.$$ 

**Proof.** The product relation (2.5) between modulation spaces is well known and we refer the interested reader to [3] and since $\mathcal{F} L^1(\mathbb{R}^d) \hookrightarrow M^{\infty,1}(\mathbb{R}^d)$, the desired inequality (2.6) follows.

For $f ∈ \mathcal{S}(\mathbb{R}^d)$, we define the fractional Schrödinger propagator $e^{it(-\Delta)^{\alpha/2}}$ for $t, \alpha ∈ \mathbb{R}$ as follows:

$$U(t)f(x) = e^{it(-\Delta)^{\alpha/2}}f(x) = \int_{\mathbb{R}^d} e^{i\xi |\xi|^{\alpha}}\hat{f}(\xi) e^{2\pi i \xi \cdot x} d\xi.$$  

The next proposition shows that the uniform boundedness of the Schrödinger propagator $e^{it(-\Delta)^{\alpha/2}}$ on modulation spaces. In fact, using [9, Theorems 1, 2] and Lemma [2.2][11], we have

**Proposition 2.4** ([9]). Let $\frac{1}{2} < \alpha \leq 2, 1 ≤ p, q ≤ \infty$. Then

$$\|U(t)f\|_{M^{p,q}} \leq (1 + t)^d \|f\|_{M^{p,q}}.$$  

Finally, we note that there is also an equivalent definition of modulation spaces using frequency-uniform decomposition techniques (which is quite similar in the spirit of Besov spaces), independently studied by Wang et al. in [20], which has turned out to be very fruitful in PDEs. For a brief survey of modulation spaces and nonlinear evolution equations, we refer the interested reader to [17] and for further reading from the PDEs viewpoint we refer to [21] and the references therein.
3. Global wellposedness in $L^2(\mathbb{R}^d)$

In this section we prove global well-posedness for fractional Hartree equation with Cauchy date in $L^2(\mathbb{R}^d)$. We shall see this result will turn out to be one of the key steps in establishing the global well-posedness results in modulation spaces and Fourier algebra.

**Definition 3.1.** A pair $(q, r)$ is $\alpha$–fractional admissible if $q \geq 2, r \geq 2$ and
\[
\alpha \frac{d}{q} = d \left(\frac{1}{2} - \frac{1}{r}\right).
\]

**Proposition 3.2.** ([12, Corollary 3.10]) Let $d \geq 2$ and $\frac{2d}{2d-1} < \alpha \leq 2$. Assume that $u_0$ and $F$ are radial. Then

1. For any $\alpha$–fractional admissible pair $(p, q)$, there exists $C_q$ such that
\[
\|U(t)\phi\|_{L_{t,x}^p} \leq C_q \|\phi\|_{L^2}, \forall \phi \in L^2(\mathbb{R}^d).
\]

2. Define
\[
DF(t, x) = \int_0^t U(t - \tau)F(\tau, x)d\tau.
\]

For all $\alpha$–fractional admissible pair $(p_i, q_i)$, $(i = 1, 2)$, there exists $C$ (constant) such that for all intervals $I \ni 0$,
\[
\|D(F)\|_{L_{t,x}^{p_i, q_i}} \leq C\|F\|_{L_{t,x}^{p'_i, q'_i}}, \forall F \in L^{p'_i}(I, L^{q'_i}).
\]

where $p'_i$ and $q'_i$ are Hölder conjugates of $p_i$ and $q_i$ respectively.

**Remark 3.1.** If $\alpha = 2$, we do not need radial assumption on initial data $u_0$ and on nonlinearity $F$ in the Strichartz estimate. Specifically, if $\alpha = 2$, Proposition 3.2 is true for any $d \geq 1, u_0$ and $F$. See [14].

We also need to work with the convolution with the Hartree potential $|x|^{-\gamma}$, so for the convenience of reader we recall:

**Proposition 3.3** (Hardy-Littlewood-Sobolev inequality). Assume that $0 < \gamma < d$ and $1 < p < q < \infty$ with
\[
\frac{1}{p} + \frac{\gamma}{d} - 1 = \frac{1}{q}.
\]
Then the map $f \mapsto |x|^{-\gamma} \ast f$ is bounded from $L^p(\mathbb{R}^d)$ to $L^q(\mathbb{R}^d)$:
\[
\|x|^{-\gamma} \ast f\|_{L^q} \leq C_{d, \gamma, p}\|f\|_{L^p}.
\]

Now in next proposition we shall prove global wellposedness in $L^2(\mathbb{R}^d)$ for the fractional Hartree equation (1.1).

**Proposition 3.4.** Let $d \geq 2, \frac{2d}{2d-1} < \alpha \leq 2$, and $K$ be given by (1.2) with $\lambda \in \mathbb{R}$ and $0 < \gamma < \min\{\alpha, d\}$. If $u_0 \in L^2_{rad}(\mathbb{R}^d)$, then (1.1) has a unique global solution
\[
u \in C(\mathbb{R}, L^2_{rad}(\mathbb{R}^d)) \cap L^{4\alpha/\gamma}_{loc}(\mathbb{R}, L^{4d/(2d-\gamma)}(\mathbb{R}^d)).
\]
In addition, its $L^2$–norm is conserved,
\[ \|u(t)\|_{L^2} = \|u_0\|_{L^2}, \quad \forall t \in \mathbb{R}, \]
and for all $\alpha$–fractional admissible pairs $(p, q), u \in L^p_{loc}(\mathbb{R}, L^q(\mathbb{R}^d))$.

**Proof.** By Duhamel’s formula, we write (1.1) as
\[ u(t) = U(t)u_0 - i \int_0^t U(t - \tau)(K * |u|^2)u(\tau)d\tau : = \Phi(u)(t). \]
Put $s = \frac{\alpha}{2}$. We introduce the space
\[ Y(T) = \{ \phi \in C([0, T], L^2_{rad}(\mathbb{R}^d)) : \|\phi\|_{L^\infty([0, T], L^2)} \leq 2\|u_0\|_{L^2}, \]
\[ \|\phi\|_{L^\infty([0, T], L^{\frac{4d}{4d-\gamma}})} \lesssim \|u_0\|_{L^2} \}
and the distance
\[ d(\phi_1, \phi_2) = \|\phi_1 - \phi_2\|_{L^\infty([0, T], L^{\frac{4d}{4d-\gamma}})}. \]
Then $(Y, d)$ is a complete metric space. Now we show that $\Phi$ takes $Y(T)$ to $Y(T)$ for some $T > 0$. We put
\[ q = \frac{8s}{\gamma}, \quad r = \frac{4d}{2d - \gamma}. \]
Note that $(q, r)$ is $\alpha$–fractional admissible and
\[ \frac{1}{q'} = \frac{4s - \gamma}{4s} + \frac{1}{q}, \quad \frac{1}{r'} = \frac{\gamma}{2d} + \frac{1}{r}. \]
Let $(\bar{q}, \bar{r}) \in \{(q, r), (\infty, 2)\}$. By Proposition 3.2 and Hölder inequality, we have
\[ \|\Phi(u)\|_{L^{\bar{q}, \bar{r}}_{t,x}} \lesssim \|u_0\|_{L^2} + \|(K * |u|^2)u\|_{L^{\frac{4s}{4s-\gamma}, \frac{2d}{\gamma}}_{L^q_{t,x}, L^r_{x}}} \]
\[ \lesssim \|u_0\|_{L^2} + \|(K * |u|^2)\|_{L^{\frac{4s}{4s-\gamma}, \frac{2d}{\gamma}}_{L^q_{t,x}, L^r_{x}}} \|u\|_{L^{q,r}_{t,x}}. \]
Since $0 < \gamma < \min\{\alpha, d\}$, by Proposition 3.3 we have
\[ \|K * |u|^2\|_{L^{\frac{4s}{4s-\gamma}, \frac{2d}{\gamma}}_{L^q_{t,x}, L^r_{x}}} = \|K * |u|^2\|_{L^{\frac{4s}{4s-\gamma}, \frac{2d}{\gamma}}_{L^q_{t,x}, L^r_{x}}} \]
\[ \lesssim \|u^2\|_{L^{\frac{2d}{\gamma}, \frac{2d}{\gamma}}_{L^q_{t,x}, L^r_{x}}} \lesssim \|u\|_{L^{q,c}_{t,x}}^{2} \lesssim T^{1 - \frac{\gamma}{2d}}\|u\|_{L^{q,c}_{t,x}}^{2}. \]
(In the last inequality we have used inclusion relation for the $L^p$ spaces on finite measure spaces: $\|\cdot\|_{L^p(X)} \leq \mu(X)^{\frac{1}{p} - \frac{1}{q}}\|\cdot\|_{L^q(X)}$ if measure of $X = [0, T]$ is finite, and $0 < p < q < \infty$.) Thus, we have
\[ \|\Phi(u)\|_{L^{\bar{q}, \bar{r}}_{t,x}} \lesssim \|u_0\|_{L^2} + T^{1 - \frac{\gamma}{2d}}\|u\|_{L^{q,r}_{t,x}}^{3}. \]
This shows that $\Phi$ maps $Y(T)$ to $Y(T)$. Next, we show $\Phi$ is a contraction. For this, as calculations performed before, first we note that

\[(3.1) \quad \| (K * |v|^2)(v - w) \|_{L^{2, r} \times L^{2, r}} \lesssim T^{1-\frac{2\gamma}{d}} \| v \|_{L^{q, r} \times L^{q, r}}^2 \| v - w \|_{L^{q, r} \times L^{q, r}}.
\]

Put $\delta = \frac{8s}{4s-\gamma}$. Notice that $\frac{1}{q'} = \frac{1}{2} + \frac{1}{q}$, and thus by Hölder inequality, we obtain

\[
\|(K * (|v|^2 - |w|^2))w\|_{L^{2, r} \times L^{2, r}} \lesssim \|K * (|v|^2 - |w|^2)\|_{L^{q, r} \times L^{q, r}}^2 \|w\|_{L^{q, r} \times L^{q, r}} \\
\lesssim \left( \|K * (v\bar{v} - \bar{w})\|_{L^{q, r} \times L^{q, r}} + \|K * \bar{w}(v - w)\|_{L^{q, r} \times L^{q, r}} \right) \|w\|_{L^{q, r} \times L^{q, r}} \\
\lesssim (\|v\|_{L^{q, r} \times L^{q, r}} + \|w\|_{L^{q, r} \times L^{q, r}}^2) \|v - w\|_{L^{q, r} \times L^{q, r}}.
\]

(3.2)

In view of the identity

\[(K * |v|^2)v - (K * |w|^2)w = (K * |v|^2)(v - w) + (K * (|v|^2 - |w|^2))w,
\]

(3.1), and (3.2) gives

\[
\|\Phi(v) - \Phi(w)\|_{L^{q, r} \times L^{q, r}} \lesssim \|(K * |v|^2)(v - w)\|_{L^{q, r} \times L^{q, r}} + \|(K * (|v|^2 - |w|^2))w\|_{L^{q, r} \times L^{q, r}} \\
\lesssim T^{1-\frac{2\gamma}{d}} \left( \|v\|_{L^{q, r} \times L^{q, r}}^2 \|w\|_{L^{q, r} \times L^{q, r}} + \|w\|_{L^{q, r} \times L^{q, r}}^2 \right) \|v - w\|_{L^{q, r} \times L^{q, r}}.
\]

Thus $\Phi$ is a contraction form $Y(T)$ to $Y(T)$ provided that $T$ is sufficiently small. Then there exists a unique $u \in Y(T)$ solving (1.1). The global existence of the solution (1.1) follows from the conservation of the $L^2$-norm of $u$. The last property of the proposition then follows from the Strichartz estimates applied with an arbitrary $\alpha$-fractional admissible pair on the left hand side and the same pairs as above on the right hand side.

In view of Remark 3.1 and exploiting the ideas from the previous proposition, we obtain

**Proposition 3.5** ([6]). Let $d \geq 1$, and $K$ be given by (1.2) with $\lambda \in \mathbb{R}$ and $0 < \gamma < \min\{2, d\}$. If $u_0 \in L^2(\mathbb{R}^d)$, then (1.1) has a unique global solution

\[u \in C(\mathbb{R}, L^2(\mathbb{R}^d)) \cap L^{8/\gamma}(\mathbb{R}, L^{4d/(2d-\gamma)}(\mathbb{R}^d)).\]

In addition, its $L^2$-norm is conserved,

\[\|u(t)\|_{L^2} = \|u_0\|_{L^2}, \forall t \in \mathbb{R},\]

and for all 2-fractional admissible pairs $(p, q), u \in L^p_{L^q(\mathbb{R}^d)}$.
4. Proofs of main results

4.1. Global well-posedness in $M^{p,q}(\mathbb{R}^d)$. In this subsection, we shall prove Theorem 4.1. For convenience of the reader, we recall

**Proposition 4.1 (II).** Let $d \geq 1$, $0 < \gamma < d$ and $\lambda \in \mathbb{R}$. There exists $C = C(d, \gamma)$ such that the Fourier transform of $K$ defined by (1.2) is

$$
\hat{K}(\xi) = \frac{\lambda C}{|\xi|^{d-\gamma}}.
$$

We start with decomposing Fourier transform of Hartree potential into Lebesgue spaces: indeed, in view of Proposition 4.1, we have

$$
\hat{K} = k_1 + k_2 \in L^p(\mathbb{R}^d) + L^q(\mathbb{R}^d),
$$

where $k_1 := \chi_{\{|\xi| \leq 1\}} \hat{K} \in L^p(\mathbb{R}^d)$ for all $p \in [1, \frac{d}{d-\gamma})$ and $k_2 := \chi_{\{|\xi| > 1\}} \hat{K} \in L^q(\mathbb{R}^d)$ for all $q \in (\frac{d}{d-\gamma}, \infty]$

**Lemma 4.2.** Let $K$ be given by (1.2) with $\lambda \in \mathbb{R}$, and $0 < \gamma < d$, and $1 \leq p \leq 2, 1 \leq q < \frac{2d}{d+\gamma}$. Then for any $f \in M^{p,q}(\mathbb{R}^d)$, we have

$$
\|(K \ast |f|^2)f\|_{M^{p,q}} \lesssim \|f\|_{M^{p,q}}^3.
$$

**Proof.** By (2.6) and (1.2), we have

$$
\|(K \ast |f|^2)f\|_{M^{p,q}} \lesssim \|K \ast |f|^2\|_{L^1,L^1} \|f\|_{M^{p,q}}
$$

(4.4)

$$
\lesssim \left(\|k_1\|_{L^1} \|f\|_{L^1} + \|k_2\|_{L^1} \|f\|_{L^1}\right) \|f\|_{M^{p,q}}.
$$

(4.5)

By (1.2) and Lemma 2.3(2), we have

$$
\|k_1|f|^2\|_{L^1} \lesssim \|k_1\|_{L^1} \||f|^2\|_{L^\infty}
$$

$$
\lesssim \||f|^2\|_{L^1} = \|f\|_{L^2}^2
$$

$$
\lesssim \|f\|_{M^{p,q}}^2.
$$

(4.6)

Let $1 < \frac{d}{d-\gamma} < r \leq 2, \frac{1}{r} + \frac{1}{r'} = 1$. Note that $\frac{1}{r_1} + 1 = \frac{1}{r_1} + \frac{1}{r_2}$, where $r_1 = r_2 := \frac{2r}{2r-1} \in [1, 2]$, and $r_1' \in [2, \infty]$ where $\frac{1}{r_1} + \frac{1}{r_1'} = 1$. Now using Young’s inequality for convolution, Lemma 2.2(3), Lemma 2.2(1), and Lemma 2.2(7), we obtain

$$
\|k_2|f|^2\|_{L^1} \lesssim \|k_2\|_{L^{r}} \|\hat{f}^2\|_{L^{r'}}
$$

$$
\lesssim \|\hat{f} \ast \hat{f}\|_{L^{r'}}
$$

$$
\lesssim \|\hat{f}\|_{L^{r_1}} \|\hat{f}\|_{L^{r_1}}
$$

$$
\lesssim \||f|^2\|_{M^{\min(r',2),r_1}} \lesssim \|f\|_{M^{2,r_1}}^2.
$$

Since $f : \left[\frac{d}{d-\gamma}, \infty\right] \to \mathbb{R}, f(r) = \frac{2r}{2r-1}$ is a decreasing function, by Lemma 2.2(1), we have

$$
\|k_2|f|^2\|_{L^1} \lesssim \|f\|^2_{M^{2,r_1}} \lesssim \|f\|^3_{M^{p,q}}.
$$

(4.7)

Combining (4.4), (4.5), and (4.6), we obtain (4.3). \qed
Lemma 4.3. Let $0 < \gamma < d$, and $1 \leq p \leq 2$, $1 \leq q < \frac{2d}{d-\gamma}$. For any $f, g \in M^{p,q}(\mathbb{R}^d)$, we have
\[
\|(K * |f|^2)f - (K * |g|^2)g\|_{M^{p,q}} \lesssim (\|f\|_{M^{p,q}}^2 + \|f\|_{M^{p,q}} \|g\|_{M^{p,q}} + \|g\|_{M^{p,q}}^2) \|f - g\|_{M^{p,q}}.
\]
Proof. By exploiting the ideas from the proof of Lemma 4.2, we obtain
\[
\|(K * |f|^2)(f - g)\|_{M^{p,q}} \lesssim \|K * |f|^2\|_{FL^1} \|f - g\|_{M^{p,q}}
\]
\[
\lesssim (\|k_1|f|^2\|_{L^1} + \|k_2|f|^2\|_{L^1}) \|f - g\|_{M^{p,p}}
\]
\[
\lesssim \|f\|_{M^{p,q}}^2 \|f - g\|_{M^{p,q}}.
\]
Let $1 \leq s < \frac{d}{d-\gamma} < t \leq 2, \frac{1}{s} + \frac{1}{t} = 1, \frac{1}{t} + \frac{1}{p} = 1$. We note that
\[
\|(K * (|f|^2 - |g|^2))g\|_{M^{p,q}} \lesssim \|K * (|f|^2 - |g|^2)\|_{FL^1} \|g\|_{M^{p,q}}
\]
\[
\lesssim (\|k_1|f|^2 - |g|^2\|_{L^s} + \|k_2|f|^2 - |g|^2\|_{L^t}) \|g\|_{M^{p,p}}
\]
\[
\lesssim (\|f|f|^2 - |g|^2\|_{L^t} + \|f|f|^2 - |g|^2\|_{L^t}) \|g\|_{M^{p,q}}.
\]
Let $1 \leq r \leq 2$, and \(\frac{1}{r} + \frac{1}{p} = 1\). Note that \(\frac{1}{r} + 1 = \frac{1}{r_1} + \frac{1}{r_2}\), where \(r_1 = r_2 := \frac{2p}{2p-1} \in [1, 2]\), and \(r' \in [2, \infty]\) where \(\frac{1}{r_1} + \frac{1}{r_2} = 1\). Now using Young’s inequality for convolution, and exploiting ideas performed as in the proof of Lemma 4.2, we obtain
\[
\|f|^2 - |g|^2\|_{L^{r'}} \lesssim \|(f - g)\hat{f}\|_{L^{r'}} + \|g(\hat{f} - \hat{g})\|_{L^{r'}}
\]
\[
= \|(f - g) * \hat{f}\|_{L^{r'}} + \|\hat{g} * \hat{f}\|_{L^{r'}}
\]
\[
\lesssim \|f - g\|_{FL^{r'}} \|\hat{f}\|_{L^{r'}} + \|\hat{g}\|_{L^{r'}} \|\hat{f}\|_{L^{r'}}
\]
\[
\lesssim \|f\|_{M^{2, r_1}} \|g\|_{M^{2, r_1}} \|f - g\|_{M^{2, r_1}}
\]
Using this, (4.8) gives
\[
\|(K * (|f|^2 - |g|^2))g\|_{M^{p,q}} \lesssim (\|f\|_{M^{p,q}} + \|g\|_{M^{p,q}}) \|g\|_{M^{p,q}}
\]
(4.9)
Now taking the identity
\[
(K * |f|^2)f - (K * |g|^2)g = (K * |f|^2)(f - g) + (K * (|f|^2 - |g|^2))g
\]
into our account, (4.7) and (4.9) gives the desired inequality. □

Proof of Theorem 1.1. By Duhamel’s formula, we note that (1.1) can be written in the equivalent form
\[
u(\cdot, t) = U(t - t_0)u_0 - iAF(u)
\]
where
\[
(\mathcal{A}v)(t, x) = \int_{t_0}^t U(t - \tau) v(t, x) d\tau.
\]
For simplicity, we assume that $t_0 = 0$ and prove the local existence on $[0, T]$.
Similar arguments also apply to interval of the form $[-T', 0]$ for proving local solutions.

We consider now the mapping

$$
\mathcal{J}(u) = U(t)u_0 - i \int_0^t U(t - \tau) [(K * |u|^2(\tau))u(\tau)] \, d\tau.
$$

By Proposition 2.3, we have

$$
\|U(t)u_0\|_{M^{p,q}} \leq C(1 + t)^d \|u_0\|_{M^{p,q}}
$$

for $t \in \mathbb{R}$, and where $C$ is a universal constant depending only on $d$.

By Minkowski’s inequality for integrals, Proposition 2.4, and Lemma 4.2, we obtain

$$
\left\| \int_0^t U(t - \tau)[(K * |u|^2(\tau))u(\tau)] \, d\tau \right\|_{M^{p,q}} \leq TC_T \| (K * |u|^2(t))u(t) \|_{M^{p,q}}
$$

(4.14)

where $C_T = C(1 + t)^d |\frac{1}{p} - \frac{1}{2}|$.

By (4.13) and (4.14), we have

$$
\|J \|_{C([0,T], M^{p,q})} \leq C_T \left( \|u_0\|_{M^{p,q}} + cT \|u\|_{M^{p,q}}^3 \right),
$$

(4.15)

for some universal constant $c$.

For $M > 0$, put $B_{T,M} = \{ u \in C([0,T], M^{p,q}(\mathbb{R}^d)) : \|u\|_{C([0,T], M^{p,q})} \leq M \}$, which is the closed ball of radius $M$, and centered at the origin in $C([0,T], M^{p,q}(\mathbb{R}^d))$. Next, we show that the mapping $\mathcal{J}$ takes $B_{T,M}$ into itself for suitable choice of $M$ and small $T > 0$. Indeed, if we let, $M = 2C_T \|u_0\|_{M^{p,q}}$ and $u \in B_{T,M}$, from (4.15) we obtain

$$
\|\mathcal{J}u\|_{C([0,T], M^{p,q})} \leq \frac{M}{2} + cC_TM^3.
$$

(4.16)

We choose a $T$ such that $cC_TM^2 \leq 1/2$, that is, $T \leq \bar{T}(\|u_0\|_{M^{p,q}}, d, \gamma)$ and as a consequence we have

$$
\|\mathcal{J}u\|_{C([0,T], M^{p,q})} \leq \frac{M}{2} + \frac{M}{2} = M,
$$

(4.17)

that is, $\mathcal{J}u \in B_{T,M}$. By Lemma 4.3 and the arguments as before, we obtain

$$
\|\mathcal{J}u - \mathcal{J}v\|_{C([0,T], M^{p,q})} \leq \frac{1}{2} \|u - v\|_{C([0,T], M^{p,q})}.
$$

(4.18)

Therefore, using the Banach’s contraction mapping principle, we conclude that $\mathcal{J}$ has a fixed point in $B_{T,M}$ which is a solution of (4.10).

Now we shall see that the solution constructed before is global in time. In fact, in view of Proposition 3.4 to prove Theorem 1.1(1), it suffices to prove that the modulation space norm of $u$, that is, $\|u\|_{M^{p,q}}$ cannot become unbounded in finite time. In view of (4.2) and to
use the Hausdorff-Young inequality we infer that, $q > \alpha$

raising the above estimate to the power provided that $0 \leq \gamma < d/2$, we let $1 < \frac{d}{d-\gamma} < q \leq 2$, and we obtain

$$\|u(t)\|_{L^{p,q}} \leq C_T \left( \|u_0\|_{L^{p,q}} + \int_0^t \| (K \ast |u(\tau)|^2) u(\tau) \|_{L^{p,q}} d\tau \right)$$

$$\leq C_T \left( \|u_0\|_{L^{p,q}} + \int_0^t \| K * |u(\tau)|^2 \|_{L^1} \| u(\tau) \|_{L^{p,q}} d\tau \right)$$

$$\leq C_T \|u_0\|_{L^{p,q}} + C_T \int_0^t \left( \|k_1\|_{L^1} \|u(\tau)\|_{L^2}^2 + \|k_2\|_{L^q} \|\hat{u}(\tau)\|_{L^{q'}}^2 \right) \|u(\tau)\|_{L^{p,q}} d\tau$$

$$\leq C_T \|u_0\|_{L^{p,q}} + C_T \int_0^t \|u(\tau)\|_{L^{p,q}} d\tau$$

$$+ C_T \int_0^t \|u(\tau)\|_{L^{2q}}^2 \|u(\tau)\|_{L^{p,q}} d\tau,$$

where we have used (2.6), Hölder’s inequality, and the conservation of the $L^2$-norm of $u$.

We note that the requirement on $q$ can be fulfilled if and only if $0 < \gamma < d/2$. To apply Proposition 3.1 we let $\beta > 1$ and $(2\beta, 2q)$ is $\alpha$-fractional admissible, that is, $\frac{\alpha}{2\beta} = d \left( \frac{1}{2} - \frac{1}{2q} \right)$ such that $\frac{1}{\beta} = \frac{\alpha}{d} \left( 1 - \frac{1}{q} \right) < 1$. This is possible provided $\frac{q-1}{q} < \frac{\alpha}{d}$; this condition is compatible with the requirement $q > \frac{d}{d-\gamma}$ if and only if $\gamma < \alpha$. Using the Hölder’s inequality for the last integral, we obtain

$$\|u(t)\|_{L^{p,q}} \leq C_T \|u_0\|_{L^{p,q}} + C_T \int_0^t \|u(\tau)\|_{L^{p,q}} d\tau$$

$$+ C_T \|u\|_{L^{2\beta([0,T],L^2)}}^2 \|u\|_{L^{\beta'([0,T],L^{p,q})}},$$

where $\beta'$ is the Hölder conjugate exponent of $\beta$. Put,

$$h(t) := \sup_{0 \leq \tau \leq t} \|u(\tau)\|_{L^{p,q}}.$$

For a given $T > 0$, $h$ satisfies an estimate of the form,

$$h(t) \leq C_T \|u_0\|_{L^{p,q}} + C_T \int_0^t h(\tau) d\tau + C_T C_0(T) \left( \int_0^t h(\tau) \beta' d\tau \right)^{\frac{1}{\beta'}},$$

provided that $0 \leq t \leq T$, and where we have used the fact that $\beta'$ is finite. Using the Hölder’s inequality we infer that,

$$h(t) \leq C_T \|u_0\|_{L^{p,q}} + C_1(T) \left( \int_0^t h(\tau) \beta' d\tau \right)^{\frac{1}{\beta'}}.$$

Raising the above estimate to the power $\alpha'$, we find that

$$h(t)^{\beta'} \leq C_2(T) \left( 1 + \int_0^t h(\tau) \beta' d\tau \right).$$
In view of Gronwall inequality, one may conclude that $h \in L^\infty([0,T])$. Since $T > 0$ is arbitrary, $h \in L^\infty_{\text{loc}}(\mathbb{R})$, and the proof of Theorem 1.2 follows.

**Proof of Theorem 1.2.** Taking Lemmas 4.2 and 4.3 and Proposition 3.5 into account, and exploiting ideas from Theorem 1.1(1), we can produce the proof.

### 4.2. Global well-posedness in $\mathcal{F}L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$

In this subsection, we shall prove Theorem 1.2. The following lemma is easy to observe:

**Lemma 4.4.** Let $f, g \in \mathcal{F}L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$.

1. The space $\mathcal{F}L^1(\mathbb{R}^d)$ is an algebra under point wise multiplication, with norm inequality
   \[ \|fg\|_{\mathcal{F}L^1} \leq \|f\|_{\mathcal{F}L^1} \|g\|_{\mathcal{F}L^1}. \]

2. For all $t \in \mathbb{R}$, the fractional Schrödinger propagator $e^{it(-\Delta)^{\alpha/2}}$ is unitary on $\mathcal{F}L^1(\mathbb{R}^d)$.

3. Let $K$ be given by (1.2) with $\lambda \in \mathbb{R}$, and $0 < q < d$. Then
   \[ \|(K \ast |f|^2)g - (K \ast |g|^2)f\|_{L^2 \cap \mathcal{F}L^1} \lesssim \|(f\|_{L^2 \cap \mathcal{F}L^1} + \|g\|_{L^2 \cap \mathcal{F}L^1})\|_t^t \|g - f\|_{L^2 \cap \mathcal{F}L^1}. \]

**Proof.** The proof of statement (1) follows from the Young’s inequality for the convolution. Next we note that $\|e^{it(-\Delta)}f\|_{\mathcal{F}L^1} = \|e^{it\xi}k\|_{\mathcal{F}L^1} = \|f\|_{\mathcal{F}L^1}$. This completes the proof of statement (2). For the proof of statement (3), see [3 Lemma 3.1].

Using Lemma 4.4 and standard fixed point arguments, following proposition is easy to prove, and the proof is left to the reader.

**Proposition 4.5.** Let $K$ be given by (1.2) with $\lambda \in \mathbb{R}$, and $0 < q < d$. If $u_0 \in L^2_{\text{rad}}(\mathbb{R}^d) \cap \mathcal{F}L^1(\mathbb{R}^d)$, then there exists $T > 0$ depending only on $\lambda, q, d$ and $\|u_0\|_{L^2 \cap \mathcal{F}L^1}$ and a unique solution of (1.1) such that $u \in C([0,T], L^2 \cap \mathcal{F}L^1)$.

**Proof of Theorem 1.2.** Taking Proposition 4.5 and Proposition 3.4 into account, to prove Theorem 1.2 it suffices to prove that the Fourrier algebra norm of $u$, that is, $\|u\|_{\mathcal{F}L^1}$ cannot become unbounded in finite time. In view of (1.2) and to use the Hausdorff-Young inequality we let $1 < \frac{d}{d-\gamma} < q \leq 2$, and we obtain

\[
\|u(t)\|_{\mathcal{F}L^1} \lesssim \|u_0\|_{\mathcal{F}L^1} + \int_0^t \|(K \ast |u(t)|^2)u(t)\|_{\mathcal{F}L^1} \, d\tau
\]

\[
\lesssim \|u_0\|_{\mathcal{F}L^1} + \int_0^t \|K \ast |u(\tau)|^2\|_{\mathcal{F}L^1} \|u(\tau)\|_{\mathcal{F}L^1} \, d\tau
\]

\[
\lesssim \|u_0\|_{\mathcal{F}L^1} + \int_0^t \left(\|k_1\|_{L^1} \|u(\tau)\|_{L^2}^2 + \|k_2\|_{L^q} \|u(\tau)\|_{L^q}^2\right) \|u(\tau)\|_{\mathcal{F}L^1} \, d\tau
\]

\[
\lesssim \|u_0\|_{\mathcal{F}L^1} + \int_0^t \|u(\tau)\|_{\mathcal{F}L^1} \, d\tau
\]

\[
\lesssim \|u_0\|_{\mathcal{F}L^1} + \int_0^t \|u(\tau)\|_{\mathcal{F}L^1} \, d\tau
\]

\[
\lesssim \|u_0\|_{\mathcal{F}L^1} + \int_0^t \|u(\tau)\|_{\mathcal{F}L^1} \, d\tau + \int_0^t \|u(\tau)\|_{L^2}^2 \|u(\tau)\|_{\mathcal{F}L^1} \, d\tau,
\]
where we have used Lemma 4.4 H"older’s inequality, and the conservation of the $L^2$–norm of $u$.

We note that the requirement on $q$ can be fulfilled if and only if $0 < \gamma < d/2$. To apply Proposition 3.1, we let $\beta > 1$ and $(2\beta, 2q)$ is $\alpha$–fractional admissible, that is, \( \frac{\alpha}{2\beta} = d \left( \frac{1}{2} - \frac{1}{2q} \right) \) such that \( \frac{1}{\beta} = \frac{d}{\alpha} \left( 1 - \frac{1}{q} \right) < 1 \). This is possible provided \( \frac{q-1}{q} < \frac{\alpha}{d} \): this condition is compatible with the requirement \( q > \frac{d}{d-\gamma} \) if and only if \( \gamma < \alpha \). Using the H"older’s inequality for the last integral, we obtain

\[
\|u(t)\|_{\mathcal{F}L^1} \lesssim \|u_0\|_{\mathcal{F}L^1} + \int_0^t \|u(\tau)\|_{\mathcal{F}L^1} d\tau 
+ C_T \|u\|_{L^{2\beta}(\mathbb{R}^d)}^2 \|u\|_{L^{\beta'}([0,T],\mathcal{F}L^1)}
\]

where $\beta'$ is the H"older conjugate exponent of $\beta$. Put,

\[
h(t) := \sup_{0 \leq \tau \leq t} \|u(\tau)\|_{\mathcal{F}L^1}.
\]

For a given $T > 0$, $h$ satisfies an estimate of the form,

\[
h(t) \lesssim \|u_0\|_{\mathcal{F}L^1} + \int_0^t h(\tau) d\tau + C_0(T) \left( \int_0^t h(\tau)^{\beta'} d\tau \right)^{\frac{1}{\beta'}},
\]

provided that $0 \leq t \leq T$, and where we have used the fact that $\beta'$ is finite. Using the H"older’s inequality we infer that,

\[
h(t) \lesssim \|u_0\|_{\mathcal{F}L^1} + C_1(T) \left( \int_0^t h(\tau)^{\beta'} d\tau \right)^{\frac{1}{\beta'}}.
\]

Raising the above estimate to the power $\beta'$, we find that

\[
h(t)^{\beta'} \lesssim C_2(T) \left( 1 + \int_0^t h(\tau)^{\beta'} d\tau \right).
\]

In view of Gronwall inequality, one may conclude that $h \in L^\infty([0,T])$. Since $T > 0$ is arbitrary, $h \in L^\infty_{loc}(\mathbb{R})$, and the proof of Theorem 1.2 follows. \hfill \Box

5. Concluding Remarks

(1) We have proved Theorem 1.1 with range $1 \leq p \leq 2$, $1 \leq q < \frac{2d}{d+\gamma}$, it would be interesting to know whether the range of $q$ can be improved in Theorem 1.1.

(2) In view of Proposition 2.4 and the fact that $\mathcal{F}L^1(\mathbb{R}^d) \hookrightarrow M^{\infty,1}(\mathbb{R}^d)$, it would be interesting to know whether the analogue of Theorem 1.2 is true or not for the Cauchy data in $M^{\infty,1}(\mathbb{R}^d) \cap L^2_{\text{rad}}(\mathbb{R}^d)$.

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