Bimodal response in periodically driven diffusive systems

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We study the response of one dimensional diffusive systems, consisting of particles interacting via symmetric or asymmetric exclusion, to time-periodic driving from two reservoirs coupled to the ends. The dynamical response of the system can be characterized in terms of the structure factor. We find an interesting frequency dependent response – the current carrying majority excitons cyclically crosses over from a short wavelength mode to a long wavelength mode with an intermediate regime of coexistence. This effect being boundary driven, decays inversely with system size. Analytic calculations show that this behavior is common to diffusive systems, both in absence and presence of correlations.

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I. INTRODUCTION

Study of one dimensional driven diffusive systems (DDS) has drawn a lot of attention in recent years.[1] Exclusion processes[2] have been studied extensively as the paradigm models of DDS. Despite being simple, they exhibit rich variety of phases, exotic correlations, non-trivial variations of density-profiles and currents providing deep understanding of non-equilibrium transport. These non-equilibrium systems are usually driven by coupling them to reservoirs at the boundaries. The simplest example of DDS is a symmetric exclusion process (SEP) where hardcore particles on a one dimensional lattice hop to any of their nearest neighbors, if vacant, with equal rates. Open system of SEP driven by time-independent bias at the boundaries has been studied in details[3–5]; its density profiles[3], correlation functions[5], and current fluctuations[4] have been calculated. DDS with asymmetric bulk dynamics, the asymmetric simple exclusion process (ASEP) where particles preferably hop to one direction and the totally asymmetric simple exclusion process (TASEP) where particles can only hop forward have also been studied extensively with time independent bias at the boundaries.[6]

Systems with time dependent bias at the boundaries are also studied in different contexts. The most commonplace example of periodically driven diffusive system can be found in alternating current (AC) driven classical electrical circuits where electron transport is best described as Drude diffusion. Studies involving classical stochastic system of particles under the influence of time-periodic potentials[7] remained an active field that led to interesting phenomena like stochastic resonance[8]. In recent years, this field has received renewed interest in the context of Brownian ratchet models of motor-proteins[9, 10], ion-pumps associated with cell membrane[11], and classical particle-pump models[12, 13]. AC driven TASEP is recently studied in the context of transport of vehicular traffics on modern highways under the influence of periodically modulating red/green light signal phases[14].

In this paper, we focus on characterizing a system of one dimensional diffusing interacting particles driven by two reservoirs coupled to the system at its two ends. We characterize the response of this system in terms of structure factor. We find an interesting bi-modality of the majority excitons. At a given frequency, the majority excitons cyclically crosses over from a predominantly long wavelength mode to a short wavelength mode via an intermediate regime of bi-modality. Our analytic calculations show that this behavior, which stems from the diffusive nature of the system, persists even in presence of finite inter-particle correlations.

II. MODEL: PERIODIC DRIVE

The asymmetric simple exclusion process (ASEP) is defined on a one dimensional lattice where sites are labelled by $i = 0, 1, 2, \ldots N$. Each site can either be vacant or occupied by at most one particle; the corresponding site variables are $n_i = 0$ and $n_i = 1$ respectively. The particles can hop to the right (left) nearest neighbouring site, if vacant, with rate $p$ ($q$). A special case of the dynamics with $p = q$ denotes SEP. The local density $\rho_i = \langle n_i \rangle$ and other correlation functions of the system, thus evolve according to the following particle conserving dynamics (in the bulk),

$$1 \quad 0 \quad \frac{p}{q} \quad 0 \quad 1. \quad (1)$$

At the boundaries, the system is coupled to two reservoirs, which effectively fixes the density of the boundary sites $i = 0, N$. The time varying drive, in density is introduced as

$$i = 0 \text{ (left) : } \rho_0 = \rho_L + a(t)$$
$$i = N \text{ (right) : } \rho_N = \rho_R - a(t) \quad (2)$$

$\rho_L$ and $\rho_R$ are the densities of the left and right reservoirs respectively, and $a(t)$ is an external time dependent source term whose specific form is discussed later.
For simplicity we consider sinusoidal variation \( a(t) = \alpha \sin \omega t \) all through this study.

The time-evolution of density profile is then given by

\[
\frac{d\rho_i}{dt} = J_{i-1,i} - J_{i,i+1} = p(\rho_i - \rho_i + C_{i,i+1} - C_{i-1,i}) + q(\rho_i - \rho_i + C_{i-1,i} - C_{i,i+1})
\]

(3)

where \( J_{i,i+1} = p \rho_i q \rho_{i+1} - (p - q)C_{i,i+1} \) is the current through \( i \)-th bond and \( C_{1,i_1...i_k} = (n_{i_1} n_{i_2} ... n_{i_k}) \) are the \( k \)-point correlation functions. Note that Eq. (3) involves the 2-point correlations which themselves evolve as

\[
\frac{d}{dt} C_{i,i+1} = p C_{i-1,i+1} + q C_{i,i+2} - (p + q)C_{i,i+1} + (p - q)(C_{i,i+1,1} - C_{i-1,i,i+1})
\]

Again the above equation involves 3-point correlations. In general, time evolution of \( k \)-point correlation depends on \( (k + 1) \)-point correlations and thus it is difficult to solve these hierarchical set of equations. However, for SEP \( (p = q) \) time evolution of local density (Eq. (3)) gets decoupled from 2-point correlations. Thus, we begin our discussion by analysing this simplest case of \( p = q \) first, before going into the details of the generic case \( p \neq q \).

### III. SEP WITH AC DRIVE

SEP in the bulk \( (p = q = r) \) leads to a simple diffusive time evolution of the density profile

\[
\frac{d\rho_i}{dt} = r (\rho_{i+1} + \rho_{i-1} - 2\rho_i) \quad \text{for } i = 1, \ldots, N - 1.
\]

(4)

For DC bias \( a(t) = 0 \) and \( \rho_L \neq \rho_R \), the steady state density profile is linear \( \rho^* = \rho_L - (\rho_L - \rho_R) r \). To focus our attention entirely on AC response we set \( \rho_L = \rho_R = \rho \). The requirement that the occupation at each site is bounded between zero and one, leads to the following bound to the amplitude of the oscillatory density at the boundaries \( \alpha \leq \min[\rho, (1 - \rho)] \). The resulting density profile is

\[
\rho_i = \rho + \phi_i(t),
\]

(5)

where \( \phi_i(t) \) is a function arising entirely due to the AC driving. To get rid of the time dependent boundary conditions let us define a new variable \( u_i(t) = \phi_i(t) + f_0a(t) \), such that \( u_0 = 0 = u_N \). A linear function of \( i \), \( f_0 = (2i/N - 1) \) maintains these criteria. Thus the time evolution of \( u_i \),

\[
\frac{du_i}{dt} = r(u_{i-1} + u_{i+1} - 2u_i) + f_0 \frac{da(t)}{dt}, \quad 1 \leq i \leq N - 1.
\]

(6)

This set of coupled differential equations can be easily decoupled by the discrete Fourier sine transform (as \( u_0 = 0 = u_N \))

\[
u_i(t) = \sum_{j=0}^{N} \hat{u}_j(t) \psi_j(i)
\]

(7)

\[
\phi_i(\omega, \theta) = \sum_{j=1}^{N-1} \hat{\phi}_j(\omega, \theta) \psi_j(i)
\]

(12)

**FIG. 1:** (Color online) Comparison of analytical density profiles (5) and corresponding structure factors (13) (shown in the insets) with those obtained from the Monte-Carlo simulations (symbols). The parameters are \( N = 65, r = 1/2, \rho = 0.5, \alpha = 0.5 \) and \( \omega = 0.01 \).

where \( \psi_j(i) = \sin(k_ji) \) with \( j \) or \( k_j = \pi j/N \) denoting sine Fourier modes. In this basis Eq. (8) becomes (using the fact \( a(t) = \alpha \sin \omega t \))

\[
\frac{d}{dt} \tilde{u}_j(t) = \epsilon_j \tilde{u}_j(t) + \tilde{f}_j \alpha \omega \cos \omega t
\]

(8)

where \( \epsilon_j = 2r \cos(k_j - 1) \) and \( \tilde{f}_j \) is the discrete sine transform of \( f_i \), i.e.,

\[
\tilde{f}_j = \frac{2}{N} \sum_{i=1}^{N-1} f_i \psi_j(i) = -\left[ 1 + (-1)^j \right] \frac{1}{N} \cot \left( \frac{k_j}{2} \right).
\]

(9)

Eq. (8) has the general solution

\[
\tilde{u}_j(\omega, t) = \tilde{u}_j(0)e^{\epsilon_j t} + \frac{\alpha \omega \tilde{f}_j}{\epsilon_j^2 + \omega^2} \left( -\epsilon_j \cos \omega t + \omega \sin \omega t + \epsilon_j e^{\epsilon_j t} \right)
\]

(10)

Clearly the relaxation time of \( j \)-th mode is \( \tau_j = -1/\epsilon_j \), the longest relaxation time being \( \tau = \tau_{j=2} = (1/r)(N/2\pi)^2 \). \( \tau \) is interpreted as the relaxation time of the system. In the limit of \( t \gg \tau \) the dependence on initial condition \( \tilde{u}_j(0) \) drops out (as \( \epsilon_j \leq 0 \)) and we are left with a temporally periodic function of \( \theta = \omega t \),

\[
\tilde{u}_j(\omega, \theta) = \frac{\alpha \omega \tilde{f}_j}{\epsilon_j^2 + \omega^2} \left( -\epsilon_j \cos \theta + \omega \sin \theta \right). \quad \tau
\]

(11)

The AC response in density profile in the long time limit, can be obtained from the Fourier sum,
where the Fourier modes are

\[
\tilde{\phi}_j(\omega, \theta) = \tilde{u}_j - \tilde{f}_j \alpha(t)
\]

\[
= -\frac{2\alpha}{N} \frac{\sin k_j}{\epsilon_j^2 + \omega^2} (\omega \cos \theta + \epsilon_j \sin \theta)
\]

(13)

for \(j = \text{even integers}\), and \(\tilde{\phi}_j(\omega, \theta) = 0\) for \(j = \text{odd integers}\). The profile \(\phi_1(\omega, \theta)\) is independent of initial conditions and periodic in \(\theta = \omega t\), a feature typical of periodically driven systems. The density profile \(\rho_1\) for any given \(\omega\) and \(\theta\) can be calculated now using Eq. (12) and (13) in Eq. (1).

In Fig. 1 we compare this analytically obtained density profile \(\rho_1\) (main figure) and \(\tilde{\phi}_j^2\) (insets) for a system of size \(N = 65\) with those obtained from Monte-Carlo simulations for two different values of \(\theta = \pi/2, \pi\). The amplitude and frequency of the AC drive are taken as \(\alpha = 0.5\) and \(\omega = 0.01\) respectively, and the mean density is kept fixed at \(\rho = 0.5\).

The structure factor (square of the sine Fourier modes corresponding to the AC response) \(\tilde{\phi}_j^2\) is a function of \(\omega\) and \(\theta\), such that \(\tilde{\phi}_j^2(\omega, \theta + \pi) = \tilde{\phi}_j^2(\omega, \theta)\). So the relevant physical domain is \(0 < \omega < \infty, 0 \leq \theta \leq \pi\). In the following we analyse the behaviour of the dominating mode(s) of the structure factor in this domain.

It is clear from Eq. (13) that for any \(\omega > 0\), \(\tilde{\phi}_j = 0\) at \(k_j = 0\) and \(\pi\), indicating the presence of at least one maximum in \(\tilde{\phi}_j^2\) at some non-zero \(k_j < \pi\). However, \(\tilde{\phi}_j\) has another zero when \((\omega \cos \theta + \epsilon_j \sin \theta)\) vanishes resulting in an additional minimum in \(\tilde{\phi}_j^2\). This minimum occurs at a specific mode \(j\) for which \(\epsilon_j = -\frac{\omega}{\tan \theta}\). Note that the system can access this mode only when

\[
\tan^{-1} \left( \frac{\omega}{4R} \right) \leq \theta \leq \frac{\pi}{2}
\]

(14)

because \(\epsilon_j\) is bounded within the range \(-4r \leq \epsilon_j \leq 0\).

Thus, in this regime of \(\theta\) the structure factor will have at least two maxima.

Clearly there is a distinct region in the \(\omega-\theta\) plane, given by Eq. (14), where \(\tilde{\phi}_j^2\) has bimodal structure with two dominating modes (Fig. 2). Outside this region the structure factor will remain unimodal, as shown in Fig. 3. At any given frequency \(\omega\), the typical nature of the structure factor undergoes the following transformations: with increasing \(\theta\) the structure factor changes from unimodal to a bimodal behaviour with the second peak appearing at a higher value of \(k_j\) or shorter wavelength. Eventually at a larger \(\theta\) the second peak at higher \(k\) becomes stronger and the first peak at smaller \(k\) loses weight. At further higher \(\theta\) the small \(k\) peak disappears. It is to be noted that all the while both the peaks shift towards smaller \(k\)-values, such that the mode structure can eventually repeat itself at \(\theta \rightarrow \theta + \pi\). This behaviour is clearly visible in Fig. 3.

Note that the mode structure we obtain is due to boundary driving and thus must vanish in the thermodynamic limit of large \(N\). We indeed find \(\tilde{\phi}_j^2 \propto 1/N^2\) (Eq. (13)).

This mode behaviour in \(\omega-\theta\) plane is shown in Fig. 3. Here, the line of crossover is defined as

\[
\tilde{\phi}_{j_1}^2(\omega, \theta) = \tilde{\phi}_{j_2}^2(\omega, \theta)
\]

(15)

where \(j_1, j_2\) denote the positions of the two maxima of \(\tilde{\phi}_j^2(\omega, \theta)\). In general it is difficult to find this curve analytically. However, the limiting behaviour at large \(\omega\) is easy to obtain. Substituting \(-\epsilon_j \equiv s\ (0 < s < 4\alpha)\) we
find at large $\omega (\omega >> x)$

$$
\frac{N^2}{40\pi^2} S_2^2 (\omega, \theta) = \frac{s(4r - s)}{4r^2 \omega^2} (\omega \cos \theta - s \sin \theta)^2
$$

which has two maxima at

$$
s_{1,2} = \frac{1}{4} \left[ 6r + \omega \cot \theta \pm \sqrt{36r^2 - 4r\omega \cot \theta + \omega^2 \cot^2 \theta} \right].
$$

Thus we find that the above-mentioned crossover $[\phi_{\sigma_1}^2 (\omega, \theta) = \phi_{\sigma_2}^2 (\omega, \theta)]$ occurs at (see Fig. 3)

$$
\tan \theta = \frac{\omega}{2r}.
$$

(16)

### IV. ASEP WITH AC DRIVE

In this section we extend this study of periodic boundary drive to a system with asymmetric hop rate, i.e. $p \neq q$. Thus correlations are no longer negligible in time evolution of the density profile. We assume that $p = r + \nu$ and $q = r - \nu$, with $\nu$ breaking the symmetry of the hopping rates. For small $\nu \ll r$ we can expand the local density and correlations in a perturbation series

$$
\rho_i = \rho + \sum_{n=0}^{\infty} \nu^n \rho_i^{(n)} (t),
$$

$$
C_{i,j} = C + \sum_{n=0}^{\infty} \nu^n C_{i,j}^{(n)} (t),
$$

(17)

where $\rho$ denotes the mean density and $C = \rho(N + 1)/N$ is the two-point correlation, for $\nu = 0$ and in absence of AC driving (exact results for SEP). Note that $\rho_i^{(0)} (t)$ and $C_{i,j}^{(0)} (t)$ denote time evolution in the SEP-system in response to the AC drive, i.e., $\rho_i^{(0)} (t) = \phi_i (t)$ (as discussed in the previous section). The perturbation series lets us write down the time evolution at each order; up to the lowest order in $\nu$,

$$
\frac{d\rho_i^{(0)}}{dt} = r \Delta_i \rho_i^{(0)},
$$

$$
\frac{d\rho_i^{(1)}}{dt} = r \Delta_i \rho_i^{(1)} + \left( \rho_i^{(0)} - \rho_{i+1}^{(0)} \right) - 2 \left( C_{i-1,i}^{(0)} - C_{i,i+1}^{(0)} \right)
$$

and

$$
\frac{dC_{i,j}^{(0)}}{dt} = r (\Delta_i + \Delta_j) C_{i,j}^{(0)},
$$

$$
\frac{dC_{i,i+1}^{(0)}}{dt} = r \left( C_{i-1,i+1}^{(0)} + C_{i,i+2}^{(0)} - 2C_{i,i+1}^{(0)} \right).
$$

(18)

where we have used the notation $\Delta_i g_{i,j} = g_{i+1,j} + g_{i-1,j} - 2g_{i,j}$. The time evolution of correlations are non-local and thus in general difficult to solve. However, in this case, the following mean field solution turns out to exactly satisfy Eq. (18)

$$
C_{i,j}^{(0)} = \rho_i \rho_j |_{\nu=0} = \rho \left( \rho_i^{(0)} + \rho_j^{(0)} \right) + \rho_i^{(0)} \rho_j^{(0)}.
$$

(19)

This simplifies the time evolution of first order perturbation in density to

$$
\frac{d\rho_i^{(1)}}{dt} = r \Delta_i \rho_i^{(1)} + \left( 1 - 2\rho - 2\rho_i^{(0)} \right) \left( \rho_{i-1}^{(0)} - \rho_{i+1}^{(0)} \right),
$$

(20)

where $\rho_i^{(0)} (t) = \phi_i (t)$. Since $\rho_i^{(1)} (t)$ must satisfy the boundary conditions $\rho_0^{(1)} (t) = 0 = \rho_N^{(1)} (t)$, we use a discrete sine transformation:

$$
\rho_i^{(1)} (t) = \sum_{j=1}^{N-1} \tilde{\rho}_j^{(1)} (t) \psi_j (t).
$$

(21)

where, as before, $\psi_j (i) = \sin (kj/i)$ with $j$ or $k_j = \pi j/N$ denoting the sine Fourier modes. Now, the set of equations in Eq. (20) are decoupled, resulting in

$$
\frac{d\tilde{\rho}_j^{(1)}}{dt} = \begin{cases} 
\epsilon_j \tilde{\rho}_j^{(1)} (t) + (1 - 2\rho) \gamma_j & j = \text{odd} \\
\epsilon_j \tilde{\rho}_j^{(1)} (t) - 2\mu_j & j = \text{even}
\end{cases}
$$

(22)

where

$$
\gamma_j = -\frac{4}{N} \sum_l (-1)^j \phi_l \frac{\sin k_l \sin k_j}{\cos k_l - \cos k_j}
$$

(23)

$$
\mu_j = -\frac{1}{2} \sum_l (\tilde{\phi}_{j-l} + \tilde{\phi}_{j+l}) \sin k_l.
$$

(24)

The sums in the above equations extend only over even values of $l$ as $\phi_l = 0$ for odd $l$ (see Eq. (13)). Note that
unlike $\gamma_j$, which is proportional to $\alpha$, $\mu_j$ varies as $\alpha^2$ and can be neglected for $\alpha \ll 1$. Thus, in the small $\alpha$ limit $\hat{\rho}_j^{(1)}$ for even $j$ vanishes exponentially as $t \to \infty$ (from Eq. (22)), and for odd $j$

$$\hat{\rho}_j^{(1)}(\omega, \theta) = \frac{8\alpha(1-2\rho)}{N^2(\epsilon_j^2 + \omega^2)} \sum_{l} \frac{\sin^2 k_l}{\epsilon_l^2 + \omega^2} \frac{\sin k_j}{\cos k_l - \cos k_j} \left[ (\omega^2 - \epsilon_j \epsilon_l) \sin \theta - \omega(\epsilon_j + \epsilon_l) \cos \theta \right].$$

(25)

Finally, to the linear order in $\nu$, the density profile is given by

$$\rho_i = \rho + \sum_{j=\text{even}}^{N-1} \hat{\rho}_j \psi_j(i) + \nu \sum_{j=\text{odd}}^{N-1} \hat{\rho}_j^{(1)} \psi_j(i).$$

(26)

The asymmetric bulk dynamics in ASEP produces a drift in one preferable direction generating the odd Fourier modes which were absent in SEP, whereas the even modes for small $\alpha$ is identical to that of SEP. In the inset of Fig. 4 we compare the density profile calculated from Eq. (26) with that obtained from Monte-Carlo simulations for a system of size $N = 128$.

Next, we investigate, whether the structure factor of ASEP retains the bimodal feature in the $\omega - \theta$ plane. It is sufficient to check that for only the odd Fourier modes, i.e., $\hat{\rho}_j^{(1)}$, as even modes remain unaltered for small $\alpha$. Eq. (25) shows that the dominant contributions in the expression of $\hat{\rho}_j^{(1)}$ come from $l = j \pm 1$ (remember that $j$ is an odd integer and $l$ takes only even values in the sum). Thus in the large $N$ limit $\hat{\rho}_j^{(1)}$ vanishes at some intermediate $k_j$, apart from the two extreme values $k_j = 0, \pi$ which satisfies $\tan \theta = 2\omega \epsilon_j/(\omega^2 - \epsilon_j^2)$. This indicates that the odd component of the structure factor too has a bimodal structure.

In Fig. 5 we show structure factors calculated from the discrete sine transform of the numerically obtained density profile (shown in the inset). These structure factors, both for odd and the even modes, agree qualitatively with the perturbation calculation Eq. (13) and (26) (drawn with lines). The small discrepancy between the theory and simulation may be attributed to keeping the perturbation expansion up to first order, a mean field estimate of the correlation function (Eq. (13)), and the linearization leading to decoupling of modes (Eq. (24)). In fact the sinusoidal drive may force the system to dynamically cross the boundaries between different phases (high density, low density and maximal current phases) known to exist in ASEP driven by time-independent bias.

In summary, we observe that the density correlations in ASEP do not destroy the bimodal structure. The asymmetry in the bulk drive for ASEP generates independent odd modes in addition to the even modes present in the structure factor of SEP.

We close this section by commenting that the current across $i$-th bond, up to first order in $\nu$, can be expressed as

$$J_{i,i+1} = r(\phi_i - \phi_{i+1}) + \nu^2\rho(1 - \rho) + (1 - 2\rho)(\phi_i + \phi_{i+1}) - 2\phi_i\phi_{i+1} + r(\rho_i - \rho_{i+1}).$$

This is not equal on all bonds, both for SEP ($\nu = 0$) and ASEP ($\nu \neq 0$), showing that the system always remains far from steady state.

V. CONCLUSION

In this paper we have shown emergence of an interesting bi-modal structure in the structure factor of simple AC driven diffusive systems namely SEP and ASEP. Our main prediction is that in a diffusive system two modes of transport, one a long wavelength mode and another a short wavelength mode, remains operative and their relative weight gets exchanged periodically in time. This behaviour repeats modulo $\pi$ in $\theta = \omega t$. The region of coexistence of the two dominating modes shrinks in the $\theta - \omega$ plane with increase in driving frequency $\omega$.

This prediction can be directly tested in experiments on sterically stabilized colloids confined in narrow glass channels. It was shown experimentally, that a system of hard core colloidal particles of diameter $\sigma$ confined within a narrow channel of width $< 2\sigma$ is describable by SEP. Recent experiments have studied impact of directed external drive on colloids confined in narrow channel. In Ref. a DC drive on confined colloids were generated by tilting the table, so that the gravitational force could be utilized to drag the colloids. Similar setup may be used to test our prediction, e.g., by simply oscillating the table by a small amplitude one may generate an AC drive. Optical microscopy, or direct light scattering may be used to find structure factor of the system to test our predictions.

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