Abstract. We describe a representation of the $q$–hypergeometric functions of one variable in terms of correlators of vertex operators made out of free scalar fields on the Riemann sphere.

1. Introduction

The $q$–hypergeometric functions [1] are of great interest in modern mathematical physics because of the clues that they are expected to give in connection with the development of the theory of difference equations and of quantum and non-commutative geometry. The main idea is to consider the $q$–hypergeometric functions as associated with the quantum analog of the Riemann sphere, which in turn is supposed to provide an interpolation between the Riemann sphere itself, its non-compact analogue — the upper half-plane — and the $p$–adic counterparts of the latter, represented for example, by discrete spaces like Bruhat Tits trees. Among other things, it is hoped that this approach will lead to a better understanding of the geometry of the simplest quantum group $Sl(2)_q$. It could also suggest interesting generalizations of $q$–hypergeometric functions which could reflect the properties of both the generic quantum groups and the quantum analogs of spaces of arbitrary topology. An immediate application would be a theory of integrable hierarchies, containing both differential and difference equations, i.e. both the KP and Toda-like systems. One could hope to get a deeper understanding of old powerful techniques like the various versions of the Bethe ansatz, the Yang–Baxter equations and the theory of lattice integrable systems like $XYZ$–model, as well as to establish their explicit relation to the theory of KP and Toda hierarchies. Two different aspects of this promising program have been recently discussed in some details in refs. [2] and [3].

The purpose of this letter is much more modest: it is to attract attention to the possible role of the free field formalism in the future development of these ideas.
free 2–dimensional massless holomorphic quantum fields have their pair correlators given by
\[ \langle \phi^\mu(t)\phi^\nu(t') \rangle = \delta^{\mu\nu} \log(t - t') + \text{regular terms} \]
and all other correlators estimated by the Wick rule.\(^1\) They are known to play a central role in all the theories listed above as subjects to be unified by the theory of \(q\)–special functions. Indeed, string models with the Riemann sphere, the upper half-plane or Bruhat-Tits trees as world sheets, are usually described in terms of free fields (see [4] for the least known case of \(p\)–adic strings). Quantum groups arise naturally in the study of rational conformal theories (see the already cited review [2] and references therein). These in turn, are describable in terms of free fields (i.e. in the Feigin–Fucks [5] or Dotsenko–Fateev [6] formalism), either through the minimal models [7] or the Wess–Zumino–Novikov–Witten model and its reductions [8]. We note that the last model is intimately related to the coadjoint-orbit approach in group theory [9]. As to the integrable hierarchies, they are identified with the theory of free fields through the concepts of infinite Grassmannian and \(\tau\)–function [10], which are further related to the theory of random matrices and orthogonal polynomials (see [11] for a review). Integrability appears also reflected in the topological properties of the moduli spaces [12] and this brings us back to the starting point: strings living on various Riemann surfaces. It should be mentioned that there has been recently a lot of interest [13–17] in free field realizations of quantum (affine) algebras as tools for solving the \(q\)–Knizhnik–Zamolodchikov equations that are obeyed for instance by the correlation functions of the \(XXZ\) spin chain [18,19].

While in the general context of string theory (and integrable hierarchies), the free fields with arbitrary boundary conditions are important, there are special cases when the “simplest” free fields — those on the Riemann sphere — are of interest. In particular, this could be a nice place from where to begin the study of quantum geometry. In this case formula (1) is exact: no “regular terms” appear on the r.h.s. It is in this situation that \(q\)–hypergeometric functions arise and we shall here restrict our considerations to this case.

The ordinary \((q = 1)\) hypergeometric functions can be represented as correlators of the “spherical” free fields, or, more exactly, of the “vertex operators” \(V_\alpha(z) = e^{i\alpha\phi(z)}\), (here \(\alpha\phi = \sum_\mu \alpha^\mu \phi^\mu\)) and the “screening charges” \(Q_{\gamma,C} = \oint_C V_\gamma(t)dt\).\(^2\)

We shall now present this construction in a form that will have a clear generalization to the \(q\)–hypergeometric case.

\(^1\)A short terminological comment may be in order. A field is called free whenever its correlators obey the Wick rule. We use the word “massless” for two-dimensional scalar fields on the Riemann sphere, whenever the pair correlators are (linear combinations of) logarithms. Note that \(2d\) Lorentz invariance is not required. This definition allows to consider not only “holomorphic” fields, as in eq. (1), but also somewhat more sophisticated free fields like those of (44).

\(^2\)This construction can be related to the “orbit construction”, where the same functions are represented as the matrix elements \(\langle M|e^{\beta_+ J_+}e^{\beta_- J_-}|N\rangle\), with \(J_\pm\) the raising and lowering operators associated with the positive and negative roots of certain Lie groups, and with \(\langle M\rangle\) and \(|N\rangle\) belonging to some representation space of the group. In this approach the screening charges are essentially included in the definition of \(\langle M\rangle\) and \(|N\rangle\) (see [20] for some details). For the \(q\)–generalization of this “orbit construction” see [21].
2. Integral — and free field representations of ordinary hypergeometric functions

The ordinary hypergeometric functions of one variable are defined by the following series

\[ _rF_s(a_1, \ldots, a_r; b_1, \ldots, b_s; z) \equiv \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_r)_n}{(b_1)_n \cdots (b_s)_n} \frac{z^n}{n!} \]

where

\[ (a)_n \equiv \frac{\Gamma(a + n)}{\Gamma(a)} = a(a + 1) \cdots (a + n - 1). \]

Among these functions, two are elementary

\[ _0F_0(z) = e^z \]

\[ _1F_0(a; z) = \sum_{n=0}^{\infty} \frac{\Gamma(a + n)}{\Gamma(a)n!} z^n = \sum_{n=0}^{\infty} \frac{\Gamma(1 - a)}{\Gamma(1 - a - n)n!} (-z)^n = (1 - z)^{-a} \]

while the others are in general transcendental: \(_0F_1(b; z)\) is for instance related to Bessel functions. Either one of the functions \(_0F_0(z)\) or \(_1F_0(a; z)\) can be used as starting point for a recursive construction of an integral representation of the functions \(_rF_s\). This construction involves three elementary steps:

\[ _rF_s \rightarrow _{r+1}F_{s+1} \]

\[ _{r+1}F_s \rightarrow _rF_s \]

\[ _{s+1}F_r \rightarrow _sF_r. \]

By combining these one clearly can transform any \(_rF_s\) into any other \(_r'F_{s'}\). The operations (6)–(8) are explicitly realized as follows.

**Step (a).**

\[ _{r+1}F_{s+1}(a_1, \ldots, a_{r+1}; b_1, \ldots, b_{s+1}; z) = \]

\[ \frac{1}{\hat{B}(a_{r+1}, b_{s+1})} \int_0^1 dt \frac{t^{a_{r+1}+1}}{(1 - t)^{b_{s+1} - a_{r+1} - 1}} _{r}F_{s}(a_1, \ldots, a_r; b_1, \ldots, b_s; tz). \]

This identity (which was discussed long ago [22]) is a simple corollary of the definition (2) and of the integral representation of the beta function:

\[ \hat{B}(a, b) \equiv \int_0^1 dt \frac{t^{a-1}}{(1 - t)^{b-a-1}} = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a + b)}. \]

We are using the unconventional notation \(\hat{B}(a, b) \equiv B(a, b - a)\) to simplify the formulas. Indeed, what is needed is the linear operation \(t^n \rightarrow (a_{r+1})_{n+1}/(b_{s+1})_n\) and this is provided by

\[ \int_0^1 dt \frac{t^{a_{r+1}+n+1}}{(1 - t)^{b_{s+1} - a_{r+1} - 1}} = \]

\[ \frac{\Gamma(a_{r+1} + n)(b_{s+1} - a_{r+1})}{\Gamma(b_{s+1})} \hat{B}(a_{r+1}, b_{s+1}) \]
The steps (7) and (8) are easily made explicit from observing that \( \lim_{N \to \infty} (N)_n / N^n = 1 \). From this we get

\[
{rF}_s(a_1, \ldots, a_r; b_1, \ldots, b_s; z) = \lim_{N \to \infty} {r+1F}_s(a_1, \ldots, a_r, a_{r+1} = N; b_1, \ldots, b_s; z/N) = \lim_{N \to \infty} {rF}_{s+1}(a_1, \ldots, a_r; b_1, \ldots, b_s, b_{s+1} = N; Nz).
\]

For our purposes, it will suffice to interpret all these formulas as relations between formal series.

Although (6–7) (or (9)–(12)) are enough to reproduce from \(_0F_0\) or \(_1F_0\) the entire set of hypergeometric functions, there exist different ways of realizing the steps (6–8). For example: \( rF_s \to r+1F_s \) (i.e. \( t^n \to (a_{r+1})_n \)) can be obtained from

\[
(r+1F_s(a_1, \ldots, a_{r+1}; b_1, \ldots, b_s; z)) = \frac{1}{\Gamma(a_{r+1})} \int_0^\infty dt e^{-t} t^{a_{r+1}} rF_s(a_1, \ldots, a_r; b_1, \ldots, b_s; tz)
\]

while \( rF_s \to rF_{s+1} \) (i.e. \( t^{-n} \to 1/(b_{s+1})_n \)) can be gotten from

\[
(rF_{s+1}(a_1, \ldots, a_r; b_1, \ldots, b_{s+1}; z)) = \frac{1}{\Gamma(1-b_{s+1})} \int_0^\infty dt e^{-t} t^{-b_{s+1}} rF_s(a_1, \ldots, a_r; b_1, \ldots, b_s; z/t).
\]

Let us also present two other versions of (9):

\[
(r+1F_{s+1}(a_1, \ldots, a_{r+1}; b_1, \ldots, b_{s+1}; z)) = \frac{1}{\Gamma(a_{r+1})} \int_1^\infty dt t^{-b_{s+1}} (t-1)^{b_{s+1}-a_{r+1}-1} rF_s(a_1, \ldots, a_r; b_1, \ldots b_s; z/t)
\]

\[
= \frac{z^{-a_{r+1}}}{\Gamma(a_{r+1})} \int_0^z dt t^{a_{r+1}-1} (1-t/z)^{b_{s+1}-a_{r+1}-1} rF_s(a_1, \ldots, a_r; b_1, \ldots b_s; t).
\]

We now return to free fields and note that

\[
\langle V_{\vec{\alpha}}(z)V_{\vec{\alpha}'}(z') \rangle = \langle e^{i\vec{\alpha} \cdot \vec{\phi}(z)} e^{i\vec{\alpha}' \cdot \vec{\phi}(z')} \rangle = (z - z')^{-\vec{\alpha} \cdot \vec{\alpha}'}.
\]

In particular

\[
\langle e^{i\vec{\alpha}_1 \phi(1)} e^{i\vec{\alpha}_x \phi(z)} \rangle = (1 - z)^{-\vec{\alpha}_1 \cdot \vec{\alpha}_x} = _1F_0(\vec{\alpha}_1 \cdot \vec{\alpha}_x; z).
\]

In conjunction with formulas (11) and (12), this means that all the functions \( s+1F_s \) (i.e. those with \( r - s = 1 \)) can be immediately represented as integrals of correlators of the vertex operators, while all the other \( rF_s \) (with \( r - s \neq 1 \)) can be obtained as limits of these \( s+1F_s \). One might note that precisely those \( (q)\)–hypergeometric functions with \( r - s = 1 \) seem to have the most interesting applications. Whether this is fortuitous or has something to do with their more natural relation with
the free field formalism is an interesting question. Explicitly, the representation of \( s+1F_s \) is as follows:

\[
(18) \quad s+1F_s(a_1, \ldots, a_{s+1}; b_1, \ldots, b_s; z) = z^{1-b_s} \frac{t_1^{1-b_s-1}}{B(a_s, b_s-1)} \int_0^t dt s^a t_s^{a_s-1}(z - t_s)^{b_s-a_s+1-1}
\]

\[
\times \frac{t_1^{1-b_1}}{B(a_2, b_1)} \int_0^{t_2} dt_1 t_1^{a_2-1}(t_2 - t_1)^{b_1-a_2+1} \ldots \frac{t_s^{1-b_s-1}}{B(a_s, b_s-1)} \int_0^{t_s} dt_{s-1} t_{s-1}^{a_{s-1}-1}(t_s - t_{s-1})^{b_{s-1}-a_{s-1}+1-1}
\]

\[
= z^{1-b_s} \prod_{j=1}^s \left( \int_0^{t_{j+1}} dt_j t_j^{a_{j+1}-b_j}(t_{j+1} - t_j)^{b_j-a_{j+1}+1-1} \right) t_1^{b_1-1}(1 - t_1)^{-a_1}
\]

\[
= \left( \prod_{j=1}^s \frac{1}{B(a_{j+1}, b_j)} \right) \langle V_{\hat{\alpha}_s}(z)V_{\hat{\alpha}_0}(0)V_{\hat{\alpha}_1}(1) \prod_{j=1}^s \int_0^{t_{j+1}} dt_j V_{\hat{\gamma}_j}(t_j) \rangle.
\]

We have put \( t_{s+1} = z \). It is convenient to set \( t_0 = 0 \), \( t_{s+2} = 1 \) and \( \hat{\alpha}_0 = \gamma_0 \), \( \hat{\alpha}_s = \gamma_{s+1} \), \( \hat{\alpha}_1 = \gamma_{s+2} \). The \( \gamma_s \) should then be chosen so that

\[
(19) \quad \gamma_i \cdot \gamma_j = (a_j - b_i + 1)\delta_{j,i+1} \quad \text{for} \quad 1 \leq i < j \leq s + 1
\]

\[
\gamma_j \cdot \gamma_{s+2} = a_1 \delta_{j,1} \quad \text{for} \quad 1 \leq j \leq s + 1
\]

\[
\gamma_1 \cdot \gamma_0 = 1 - a_2
\]

\[
\gamma_j \cdot \gamma_0 = b_{j-1} - a_{j+1} \quad \text{for} \quad 2 \leq j \leq s
\]

\[
\gamma_{s+1} \cdot \gamma_0 = b_s - 1.
\]

Of course, unless \( s = 1 \), these conditions can not be solved with \( \hat{\alpha}_s \) and \( \hat{\gamma}_s \) that have only one component. In general \( (s > 1) \), these variables should be multicomponent vectors, i.e. the number of free fields involved (or the number of possible values for the indices \( \mu \) and \( \nu \) in (1)) should be at least \( s \) or \( s+1 \). In fact, the vectors \( \gamma_j \) with \( 1 \leq j \leq s+1 \), can be chosen proportional to the simple roots of the Lie algebra \( \text{sl}(s+2) \). The points 0, 1 (and implicitly \( \infty \)) on the Riemann sphere are obviously distinguished in the integral representation of the hypergeometric functions. It can be assumed that they are fixed by a rational \( \text{SL}(2, \mathbb{C}) \) transformation — a symmetry of the Riemann sphere, otherwise, the formulas would give hypergeometric functions with arguments of the form \((z - z_0)(z_1 - z_{\infty})/(z - z_{\infty})(z_1 - z_0)\).

With the extension of the above results to \( q \)-series in mind, it is practical to rewrite eq. (18) in a slightly different form. Let \( \gamma_{ij} \equiv \gamma_i \cdot \gamma_j \) and replace \( V_{\gamma_j}(t) \) in (18) by \( e^{i\phi_j(t)} \) where the free fields \( \phi_j \) are the following linear combinations of the fields \( \phi^\mu \): \( \phi_j(t) = \gamma_j^\mu \phi^\mu(t) \). It follows that

\[
(20) \quad \langle \hat{\phi}(t)\hat{\phi}(t') \rangle = e_{\gamma_{ij}} \log(t - t')
\]
In order to obtain a representation of the hypergeometric functions which is valid for real values of the argument \( z \) between 0 and 1, it is enough to require that (20) be true only for real \( t \) and \( t' \) such that \( t > t' \). Then one has

\[
(21) \quad \sum_{s=0}^{\infty} s+1 F_s(a_1, \ldots, a_{s+1}; b_1, \ldots, b_s; z = t_{s+1}) \cdot \prod_{j=1}^{s} \hat{B}(a_j+1b_j) = \prod_{j=1}^{s} \left( \sum_{t_j+1}^{s+2} dt_j \prod_{j=1}^{s+2} e^{i\hat{\phi}_j(t_j)} \right).
\]

3. INTEGRAL REPRESENTATIONS OF THE \( q \)-HYPERGEOMETRIC FUNCTIONS

We now turn to the \( q \)-hypergeometric functions. We shall adopt the following definition\(^\dagger\) for the \( q \)-analog of the functions given in (2):

\[
\varphi_s(a_1, \ldots, a_r; b_1, \ldots, b_s; z) = \sum_{n=0}^{\infty} \frac{(a_1 | q)_n \ldots (a_r | q)_n}{(b_1 | q)_n \ldots (b_s | q)_n} \frac{z^n}{(1 | q)_n} \left[ (-1)^n q^n(n-1)/2 \right]^{1+s-r},
\]

where

\[
(a | q)_n \equiv \frac{\Gamma_q(a+n)}{\Gamma_q(a)} = \frac{(q^a, q)_n}{(1-q)_n} = \prod_{k=1}^{n} \frac{1-q^{a+k-1}}{1-q},
\]

and in particular,\(^\dagger\)

\[
(1 | q)_n = \frac{(q, q)_n}{(1-q)_n} = \prod_{k=1}^{n} \frac{1-q^k}{1-q}.
\]

We are using here the standard notation for the \( q \)-shifted factorials: \((z, q)_n \equiv \prod_{k=1}^{n} (1-zq^{k-1}) \). The \( q \)-gamma function \( \Gamma_q \) is defined so that \( \Gamma_q(z+1) = \frac{1-q^z}{1-q} \Gamma_q(z), \Gamma_q(1) = 1 \). In the limit \( q \to 1 \), \( (a | q)_n \to (a)_n \).

The two “elementary” \( q \)-functions are:

\[
o\varphi_0(z) = \sum_{n=0}^{\infty} \frac{q^{n(n-1)/2}}{(q, q)_n} (-1-qz)^n = ((1-qz, q)_\infty
\]

\[
\equiv E_q [-1-qz] = \frac{1}{e_q(1-qz)}
\]

\[
1\varphi_0(a, z) \equiv \frac{1}{(1, z|a)} \equiv \frac{(zq^a, q)_\infty}{(z, q)_\infty} = \prod_{k=1}^{\infty} \frac{1-zq^{k+a-1}}{1-zq^k}.
\]

\(^\dagger\)Our definition of the \( q \)-hypergeometric functions which is not conventional is such that in the limit \( q \to 1 \), \( r\varphi_s(a_1, \ldots, a_r; b_1, \ldots, b_s; z) \to rF_s(a_1, \ldots, a_r; b_1, \ldots, b_s; z) \). It is close to the definition given in the second book of ref. [1]. The presence of the factor \( [(-1)^n q^n(n-1)/2]^{1+s-r} \) is to ensure that a series of the form (22) is obtained when limits that change the difference \( r-s \) are performed. (See the third book of ref. [1]). Note that this factor is absent when \( r = s+1 \).

\(^\dagger\)Note an amusing matrix-integral representation for this quantity [23]:

\[
\begin{align*}
\frac{q^{n^2/2}}{(q, q)_n} &= \frac{q^{n^2/2}}{\prod_{k=1}^{n}(1-q^k)} \sim \int dH [dU] e^{-m^2trH^2 + trHUH^\dagger}.
\end{align*}
\]

Here \( H \) and \( U \) are the Hermitean and unitary \( n \times n \) matrices respectively, \([dU]\) denotes the Haar measure on \( U(n) \), while \( q = m^2 - \sqrt{m^2 - 1} \). Interesting implications of this representation are beyond the scope of the present letter.
We shall make extensive use of the \( q \)-integral defined by

\[
\int_{0}^{1} d_q t f(t) \equiv (1 - q) \sum_{n=0}^{\infty} f(q^n)q^n.
\]

Some simple formulas involving this integral are:

\[
\int_{A}^{B} d_q t f(t) = \int_{0}^{1} d_q t [B f(B t) - A f(At)];
\]

\[
\int_{q^k}^{1} d_q t f(t) = (1 - q) \sum_{n=0}^{\infty} [f(q^n) - q^k f(q^{n+k})] = (1 - q) \sum_{n=0}^{k-1} f(q^n)q^n,
\]

provided \( k \) is integer; in particular

\[
\int_{q}^{1} d_q t f(t) = (1 - q) f(1).
\]

The function \( \Gamma_q \) has the integral representation [24]:

\[
\Gamma_q(z) = \int_{0}^{1/(1-q)} d_q t t^{z-1} E_q[-q(1 - q)t^z].
\]

Especially useful will be the following representation for the \( q \)-beta function:

\[
\hat{B}_q(a, b) \equiv B_q(a, b - a) \equiv \frac{\Gamma_q(a) \Gamma_q(b - a)}{\Gamma_q(b)}
\]

\[
= \int_{0}^{1} d_q t t^{a-1}(1, t_q)^{b-a-1} = \int_{0}^{1} d_q t t^{a-1} \frac{(t_q, q)_\infty}{(t_q^{b-a}, q)_\infty}.
\]

The three operations that are the \( q \)-generalizations of (6)–(8) are realized as follows. The linear transformation \( t^n \rightarrow \frac{(a_n + 1 | q)_n}{(b_{n+1} | q)_n} \) which is required to effect \( r \varphi_s \rightarrow r+1 \varphi_{s+1} \) is easily obtained from (31):

\[
\int_{0}^{1} d_q t t^{a_r+1+n-1}(1, t_q)^{b_{s+1} - a_{r+1} n - 1}
\]

\[
= \frac{\Gamma_q(a_{r+1} + n) \Gamma_q(b_{s+1} - a_{r+1})}{\Gamma_q(b_{s+1} + n)} = \frac{(a_{r+1} | q)_n}{(b_{s+1} | q)_n} \hat{B}_q(a_{r+1}, b_{s+1}).
\]

It is then immediate to derive the following formula:

\[
r+1 \varphi_{s+1}(a_1, \ldots, a_{r+1}; b_1, \ldots, b_{s+1}; z) =
\]

\[
\frac{1}{\hat{B}_q(a_{r+1}, b_{s+1})} \int_{0}^{1} d_q t t^{a_{r+1}-1}(1, qt)^{b_{s+1} - a_{r+1} - 1} \varphi_s(a_1, \ldots, a_r; b_1, \ldots, b_s; tz) =
\]

\[
\frac{z^{-a_{r+1}}}{\hat{B}(a_{r+1})} \int_{0}^{1} d_q t t^{a_{r+1}-1}(1, qt/z)^{b_{s+1} - a_{r+1} - 1} \varphi_s(a_1, \ldots, a_r; b_1, \ldots, b_s; t).}
The realizations of the two remaining operations follow from the fact that

\[ \lim_{N \to \infty} \left( \frac{\log N}{\log q} \right)^n \frac{n}{q^n} = (-1)^n q^{n(n-1)/2}. \]

One then verifies that

\[ r \varphi_s(a_1, \ldots, a_r; b_1, \ldots, b_s; z) = \]

\[ \lim_{N \to \infty} r+1 \varphi_s \left( a_1, \ldots, a_r, a_{r+1} = \frac{\log N}{\log q}; b_1, \ldots, b_s, \frac{(1-q)z}{N} \right) = \]

\[ \lim_{N \to \infty} r_{s+1} \varphi_s \left( a_1, \ldots, a_r; b_1, \ldots, b_s, b_{s+1} = \frac{\log N}{\log q}; \frac{Nz}{1-q} \right). \]

4. Free-field representation of the \( q \)-hypergeometric functions

We now proceed to the free field interpretation of the above formulas. As in the case \( q = 1 \), it will prove most natural for the functions \( s+1 \varphi_s \) (i.e. for those with \( r-s = 1 \)). From the free field expression of \( s+1 \varphi_s \), the corresponding representations of all the other \( r \varphi_s \) will be obtained by repeated use of the limits defined in (35). The main ingredient is of course the free field representation of the basic function \( 1 \varphi_0(a; z) = 1/(1, z) \frac{(zq^a, q)\infty}{(z, q)\infty} \).

To begin, we note that taking the ordinary logarithm of \((z, q)\infty\) we get

\[ \log(z, q)\infty = \sum_{n=0}^{\infty} \log(1 - zq^n) = \frac{1}{1-q} \int_0^1 \frac{dq}{t} \log(1 - t z) \]

\[ = \frac{1}{1-q} \langle \phi(1) \int_0^1 \frac{dq}{t} \phi(tz) \rangle \]

where \( \phi(z) \) is the original free field that satisfies (1). Now with the help of (26) and (27) we arrive at

\[ -\log \frac{(zq^\alpha, q)\infty}{(z, q)\infty} = \frac{1}{1-q} \langle \phi(1) \int_{q^\alpha}^1 \frac{dq}{t} \phi(tz) \rangle \]

\[ = \frac{1}{(1-q)z} \int_q^1 \frac{dq}{t} \phi(t) \int_{q^\alpha}^1 \frac{dq}{t} \phi(tz) \). \]

This leads to the following definition of vertex operators:

\[ V_\alpha(z, q) \equiv V_\alpha \{ \phi(z), q \} = \frac{1}{1-q} : \exp \int_{q^\alpha}^1 \frac{dq}{t} \phi(tz) : = \frac{1}{1-q} : \exp i \Phi_\alpha(z) : \]

\[ V_{\bar{\alpha}}(z, q) \equiv \Pi_\mu V_{\alpha^\mu} \{ \phi^\mu(z), q \}. \]

Equation (37) can now be interpreted as the statement that

\[ 1 \varphi_0(a; z) = \frac{1}{[a]!} = \langle V_1(1, q)V_\alpha(z, q) \rangle. \]
Together with the integral representations of the previous section, this relation allows us to represent any \( q \)-hypergeometric function in the form of a correlator of free fields. (The screening charges \( Q_{\vec{\gamma},c} = \int_0^c d_t V_{\vec{\gamma}}(t, q) \) are now essentially double \( q \)-integrals.)

It is interesting to record the mode expansion of the free field \( \Phi_\alpha(z) \). The free fields \( \phi^\mu(z) \) on the Riemann sphere admit the expansion

\[
\phi^\mu(z) = -\sum_{n\neq 0} \frac{a^\mu_n}{n} z^n + a^\mu_0 \log z + a^\mu
\]

where \( a^\mu \) and \( a^\mu_n \ n \in \mathbb{Z} \) satisfy the commutation relations

\[
[a^\mu_n, a^\nu_m] = \delta^{\mu\nu} \delta_{n+m,0} \delta^n \delta^m

[a^\mu_0, a^\nu] = \delta^{\mu\nu}
\]

and generate the Heisenberg algebra. This algebra has a Fock space representation with the vacuum \( |0\rangle \) defined by \( a^\mu_0 |0\rangle = 0, n \geq 0 \). From (41) and (42) follows that

\[
\langle \phi^\mu(z) \phi^\nu(w) \rangle = \langle 0 | \phi^\mu(z) \phi^\nu(w) | 0 \rangle = \delta^{\mu\nu} \log(z - w)
\]

From (38), we find that

\[
\Phi_\alpha(z) = (1 - q) \sum_{n\neq 0} \frac{a^\mu_n}{n} z^{-n} \frac{(1 - q^{-\alpha n})}{(1 - q^{-n})} - a^\mu_0 \log q^\alpha.
\]

Now as was the case for \( q = 1 \), it is again convenient to introduce a new set of massless free fields \( \tilde{\Phi}_i(t) \) that satisfy here

\[
\langle \tilde{\Phi}_i(t) \tilde{\Phi}_j(t') \rangle \equiv \log \left( \frac{t^{\gamma_{ij}}}{(1 + \frac{q t'}{t})^{[\gamma_{ij}]}} \right)
\]

for real \( t > t' \)

with \( \gamma_{ij} = \vec{\gamma}_i \cdot \vec{\gamma}_j \) still the symmetric matrix defined from the algebraic conditions (19). In this case however, the requirement that \( t \) and \( t' \) are real and \( t > t' \) is less trivial. The point is that the expression on the r.h.s. of (44) is not symmetric under the exchange of the pairs \( i, t \) and \( j, t' \). (This is in contrast with the \( q = 1 \) situations, see (1).) One easily sees that the fields \( \tilde{\Phi}_i \) here have the following mode expansion

\[
\tilde{\Phi}_i(z) = -\sum_{n=0} \frac{\hat{a}^i_n}{n} z^{-n} + \hat{a}^i_0 + \hat{a}^i
\]

with the operators \( \hat{a}^i_n, n \in \mathbb{Z} \) and \( \hat{a}^i \) satisfying the commutation relations

\[
[\hat{a}^i_n, \hat{a}^j_m] = -\delta_{n+m,0} q^{[n]([\gamma_{ij}]+1)/2} \frac{[\gamma_{ij} n/2]}{[n/2]} [\hat{a}^i_n, \hat{a}^j] = \gamma_{ij}.
\]

The symbol \([x] \) stands for \([x] = (q^x - q^{-x})/(q - q^{-1}) \). In terms of these \( \tilde{\Phi}_i(t) \), the free field representation of the \( q \)-hypergeometric functions is completely analogous.
to that of the functions $sF_s$. Explicitly, for $0 < z < 1$, $z \in \mathbb{R}$,

$$
(47) \quad s+1 \varphi_s(a_1, \ldots, a_{s+1}; b_1, \ldots, b_s; z = t_{s+1}) \prod_{j=1}^{s} \hat{B}_q(a_{j+1}, b_j) = 
$$

$$
z^{-a_{s+1}} \prod_{j=1}^{s} \left( \int_{0}^{t_{j+1}} dq t_j^j a_{j+1} - a_j - 1 \left( 1, \frac{q t_j}{t_{j+1}} \right) b_j - a_{j+1} - 1 \right) t_1^{a_1}(1, t_1)^{-[a_1]}
= \prod_{j=1}^{s} \int_{0}^{t_{j+1}} dq t_j \left( \prod_{j=0}^{s+2} e^{i \hat{\Phi}_j(t_j)} \right)
$$

For representations of the $q$–hypergeometric functions of an arbitrary complex argument $z$, proper holomorphic analogs of the fields $\hat{\Phi}_i$ are required.

To sum up, we have described a surprisingly simple free field representation of the $q$–hypergeometric functions. It is related to the integral representation of the functions $s+1 \varphi_s$ with the integrand given as the product of $q$–powers of linear functions.

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