A new partial gauge fixing condition for the abelian projection is introduced. It is based on the lowest-lying eigenvector of a covariant Laplacian operator. This gauge is smooth and free of lattice Gribov copies. These properties are important for an unambiguous computation of the abelian projected gauge field configuration.

1. INTRODUCTION

The abelian projection [1] remains the most popular framework for lattice Monte Carlo studies of monopoles and confinement. There are two uncomfortable aspects to the partial gauge fixing involved, though. One is the apparent preference for one particular gauge, the Maximally Abelian Gauge (MAG) [2], the other is the presence of lattice Gribov copies.

The situation with respect to the first point can be loosely summarized as follows. The MAG “works well”, the temporal (“Polyakov”) gauge “works sometimes”, and the other gauges “do not work” (See the reviews [4, 5, 6] for results and interpretations.) This may have its origin in the different smoothness properties of these gauges. The MAG is a smooth gauge, the Polyakov gauge is smooth in the time direction only, while the other gauges may not be smooth enough. Smoothness means that the link matrices are relatively close to unity, which is important for interpreting them in terms of continuum gauge fields, and hence for extracting their abelian part.

Turning attention to the apparently preferred MAG then, one is confronted with the second issue: In practice, implementation of the MAG is complicated by the presence of lattice Gribov copies, corresponding to different local minima of the gauge fixing functional. Unambiguous computation of the gauge fixed configuration is impossible, and certain quantities such as the monopole density are fairly sensitive to this ambiguity [5, 8]. One would like to have a smooth gauge without lattice Gribov problem.

An analogous problem arises in conventional Landau gauge fixing. Its lattice implementation is smooth but there are lattice Gribov copies. To circumvent this problem, Vink and Wiese introduced “Laplacian gauge fixing” [9], which shares the smoothness properties of the Landau gauge but avoids lattice Gribov copies. In this proposal, the gauge transformation matrices are determined in terms of the lowest-lying eigenvectors of the covariant Laplacian in the background of the given gauge field configuration. Recently this method was studied in practice [10], and its perturbative formulation was given [11].

Here I take this idea over to the abelian projection, constructing a procedure for partial gauge fixing which is smooth and free from ambiguities. Section 2 introduces the gauge, discusses its expected merits and briefly describes its perturbative continuum formulation. Numerical results are presented in section 3.

2. LAPLACIAN ABELIAN GAUGE

To introduce the Laplacian Abelian Gauge (LAG) it is convenient to start from the MAG in its spin-model formulation [12]. I will limit myself to the SU(2) case here.

The Maximally Abelian Gauge [2] for a link configuration \( \{ U_{\mu,x} \} \) is defined as the configuration \( \{ U_{\mu,x} = \Omega_x U_{\mu,x} \Omega_x^{+} \} \) where \( \{ \Omega_x \} \) minimizes the functional

\[
\tilde{S}_{U}(\Omega) = \sum_{x,\mu} \left\{ 1 - \frac{1}{2} \text{Tr} \left[ \sigma_3 U_{\mu,x}^{\dagger} \sigma_3 U_{\mu,x} \right] \right\},
\]

which is the functional used in the abelian projection. The new gauge condition is based on the lowest-lying eigenvectors of a covariant Laplacian operator. This gauge is smooth and free of lattice Gribov copies. These properties are important for an unambiguous computation of the abelian projected gauge field configuration.
which can be written in the form

\[
S_U(\phi) = \sum_{x,\mu} \left\{ 1 - \frac{1}{2} \text{Tr} \left[ \Phi_x U_{\mu,x} \Phi_{x,\mu} U_{\mu,x}^+ \right] \right\}
\]

\[
= \sum_{x,\mu} \left\{ 1 - \sum_{a,b} \phi^a_{x,\mu} R^{ab}_{\mu,x} \phi^b_{x,\mu} \right\},
\]

with the definitions

\[
\Phi_x = \Omega^+_x \sigma_3 \Omega_x = \sum_{a=1}^3 \phi^a_{x} \sigma_a,
\]

\[
R^{ab}_{\mu,x} = \frac{1}{2} \text{Tr} \left[ \sigma_a U_{\mu,x} \sigma_b U_{\mu,x}^+ \right].
\]

\(R_{\mu,x}\) is the link matrix in the adjoint representation and \(\phi_x \in SU(2)/U(1) \simeq S^2\) is a three-vector of unit length, neutral under abelian gauge transformations, as expected. The gauge fixing functional \(S_U(\phi)\) is nothing but the latticized covariant kinetic action \(\int \frac{1}{2} (D_\mu \phi)^2\) of a spin field \(\phi\) in the background of the given gauge field. The spin configuration which minimizes this action determines the MAG. Finding this absolute minimum is difficult because of the length-one constraints on the individual spins. The usual iterative local algorithms often get stuck in a local minimum. This means that the result of the gauge fixing depends on the particular algorithm used and on the starting point on the gauge orbit: the gauge fixing procedure is ambiguous.

The idea of Laplacian gauge fixing is to:

1. Minimize \(S_U(\phi)\) without taking into account the constraints \(|\phi_x| = 1\). This amounts to finding the eigenvector \(\bar{\phi}_x^a\) belonging to the lowest eigenvalue \(\lambda\) of the covariant Laplacian \(-\Box_x (\phi_x)(R)\).

2. Write the solution \(\bar{\phi}_x^a\) as \(\bar{\phi}_x^a = \rho_x \bar{\phi}_x^a\) and take \(\bar{\phi}_x^a\) for the gauge transformation: \(\Phi_x = \sum \bar{\phi}_x^a \sigma_a = \Omega^+_x \sigma_3 \Omega_x\) (of course \(\Omega_x\) is determined up to the residual U(1) freedom only).

This procedure is unambiguous because the computation of eigenvectors can be done to the precision required, and because the procedure is gauge covariant by construction: under a gauge transformation \(V\) of the starting configuration \(U\), the Laplacian operator \(-\Box((VR(U))\)) and its eigenvectors transform accordingly, such that the gauge fixed configuration \(\bar{U}\) is unchanged (again, up to residual abelian gauge transformations).

The procedure is ill-defined only if the lowest two eigenvalues of the covariant Laplacian coincide, or if \(\bar{\phi}_x = 0\) for some \(x\). However, the set of such configurations has measure zero. In practice, we might get into problems when either the difference between the two lowest eigenvalues, or \(\phi_x\) for some \(x\), is zero within the numerical precision of the computer, but, as it turns out, this never occurs.

The possibility of a zero in \(\bar{\phi}\) (not necessarily at a lattice site) is actually quite interesting. It means that the gauge fixing is ill-defined at that point, which is precisely what identifies a magnetic monopole in the abelian projection. In fact, the 't Hooft-Polyakov monopole (dyon) in the radial gauge satisfies the continuum equivalent (see below) of the LAG, with the solution for \(\bar{\phi}\) equal to the Higgs field \((A_4)\) of this configuration (which has a zero at the origin)!

In the continuum limit, the stationarity conditions corresponding to the LAG are

\[
\sum_\mu \left\{ -\partial_\mu^2 + (\bar{A}_\mu^1)^2 + (\bar{A}_\mu^2)^2 \right\} \rho = \lambda \rho,
\]

\[
\sum_\mu \left\{ \partial_\mu \mp i \bar{A}_\mu^3 \right\} (\rho^2 \bar{A}_\mu^\pm) = 0.
\]

(Note that \(\rho(x)\) depends on the gauge orbit as a whole only.) This corresponds to minimization of the quantity

\[
\int_V \rho^2 \left[ (A_\mu^1)^2 + (A_\mu^2)^2 \right] \int_V \rho^2
\]

(its minimum value being the smallest eigenvalue \(\lambda\) of the covariant Laplacian) or, on the lattice,

\[
\sum_{x,\mu} \left\{ 1 - \rho_x \left( U_x^{(1)} \right) \rho_x^{(1)} \right\} \frac{1}{2} \text{Tr} \left[ \sigma_3 U_{\mu,x}^{(1)} \bar{\sigma}_3 U_{\mu,x}^{(1)} \right].
\]

Note that in all these formulas one recovers the corresponding expression for the MAG by setting \(\rho\) equal to 1. In fact, in the continuum limit \(\beta \to \infty\), the LAG-fixed configuration is characterized by \(\bar{U}_{\mu,x} \to 1\) and \(\rho_x \to 1\), and one sees that LAG and MAG converge to each other.

At finite \(\beta\), the function \(\rho\) acts like a kind of “measure of local smoothness”: eq. 13 suggests that \(\rho(x)\) will be small when \((A_\mu^1)^2(x) + (A_\mu^2)^2(x)\) wants to be large. For example, in the centre of the 't Hooft-Polyakov dyon the non-abelian field components \(A_{\mu}^{1,2}\) blow up while \(\rho = 0\). Such divergences thus bear a relatively low penalty in
the gauge fixing functional \( \mathcal{F} \); the requirement of smoothness appears to tell the LAG to treat monopoles more mildly than the MAG does.

3. NUMERICAL TESTS

The following table contains some results for a \( 6^4 \) lattice at various \( \beta \) values.

| \( \beta \) | 2.2 | 2.4 | 2.6 | 2.8 |
|---|---|---|---|---|
| \( \lambda_1 \) | 2.123(3) | 1.872(8) | 1.677(2) | 1.529(2) |
| \( \lambda_2 \) | 2.154(3) | 1.926(8) | 1.759(2) | 1.631(3) |
| \( \Delta \rho \) | 0.512(3) | 0.402(1) | 0.293(2) | 0.243(2) |
| \( \#_L \) | 492(6) | 149(13) | 28.2(8) | 6.6(4) |
| \( \#_M \) | 428(4) | 103(10) | 16.8(5) | 4.2(3) |

\( \lambda_{1,2} \) are the two lowest eigenvalues of the covariant Laplacian, and \( \Delta \rho \) is the average fluctuation of \( \rho \) over a configuration. For \( \beta \to \infty \), \( \lambda_1 \) tends to zero, \( \lambda_2 \) to one (becoming 24-fold degenerate), and \( \Delta \rho \) to zero. However, one should keep in mind that this limit is unphysical since it corresponds to zero physical volume.

\( \#_L \) and \( \#_M \) are the numbers of dual links transmitting monopole current, for LAG and MAG respectively. Note that the monopole numbers are somewhat higher in the LAG. This should not be interpreted as a sign that the LAG is “worse” than the MAG, as might be suggested by the empirical fact that “bad” gauges usually have very high monopole numbers associated to them. Rather, it may be regarded as an indication of the number of monopoles present in a situation of optimal smoothness.

Fig. 1 shows a comparison of the monopole locations as determined by the LAG and the MAG, for a randomly chosen configuration at \( \beta = 2.4 \). It is interesting to see the large overlap, which is a sign of the similarity between the two gauges.

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