Bosonization at Finite Temperature and Anyon Condensation

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Abstract

An operator formalism for bosonization at finite temperature and density is developed. We treat the general case of anyon statistics. The exact $n$-point correlation functions, satisfying the Kubo-Martin-Schwinger condition, are explicitly constructed. The invariance under both vector and chiral transformations allows to introduce two chemical potentials. Investigating the exact momentum distribution, we discover anyon condensation in certain range of the statistical parameter. Another interesting feature is the occurrence of a non-vanishing persistent current. As an application of the general formalism, we solve the massless Thirring model at finite temperature, deriving the charge density and the persistent current.

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1 Introduction

Bosonization has become over the years an important tool for studying various two-dimensional field theory phenomena. It has been successfully applied to the Tomonaga-Luttinger liquid model [1], to quantum impurity problems [2] as well as in the theory of quantum Hall edge states [3], [4]. The possibility of describing in this framework also anyonic type excitations, attracts recently much attention. Apart from a few considerations [5], [6], generalized statistics were not very popular in the past. The discovery of the quantum Hall effect however, drastically changed the status of the subject, promoting it into a fundamental concept of modern condensed matter theory.

In the physical applications of bosonization, one has to deal essentially with systems at finite temperature and density. This problem has been usually studied by means of functional integration in the real or imaginary time formalism of thermal field theory (see for instance [7]). In spite of the progress reached along this way however, some relevant structural questions (e.g. the treatment of generalized statistics) are still open. There exist also some controversies, presumably related with genuine subtleties which appear in handling time-ordered paths in the complex time-plane. For all these reasons, we find important at this stage to test at work an alternative approach to bosonization at finite temperature and density - the real time operator formalism. This is precisely the goal of the present paper. We will work directly in infinite volume, thus avoiding some delicate problems related with the thermodynamic limit, especially in the presence of generalized statistics. The strategy will be to construct explicitly the thermal correlation functions of anyons, vector and chiral currents and to extract information about some physical quantities like the momentum distribution, the density and the persistent current. Our framework sheds new light on the role of the vector and chiral symmetries, which allow to introduce two independent chemical potentials. We expect the validity of our results to be quite general. Indeed, there are several indications ([8], [9], [10]) that the correlation functions we derive, capture the main universal features of one-dimensional gap-less quantum liquids, which support anyonic statistics and generalize the Tomonaga-Luttinger one [11], [12].

The paper is organized as follows. In Section 2 we collect some general algebraic properties of the free boson field and its dual, describing also the corresponding thermal representation. We also construct a set of anyon fields and establish their general properties. Section 3 is devoted to the derivation of the anyonic correlation functions at finite temperature and density. The symmetry content of these correlators, which satisfy the Kubo-Martin-Schwinger condition, is investigated. The momentum distribution of the left- and right-moving anyonic modes is determined and a remarkable condensation-like phenomenon is detected. In Section 4 we consider the bosonization of fermion fields. A
general family of solutions of the massless Thirring model, comprising also anyonic solutions, is considered in Section 5. We determine both the density and the persistent current of the model at finite temperature. Our derivation is free from any ambiguity and resolves the existing discrepancies in the literature about the coupling constant dependence of the density. The last section summarizes the main results and contains our conclusions.

2 From bosons to anyons

The fundamental building blocks for bosonization in 1+1 dimensions are the free massless scalar field \( \varphi \) and its dual \( \tilde{\varphi} \). These fields are required to satisfy the equations of motion

\[
g^{\mu \nu} \partial_\mu \partial_\nu \varphi (x) = g^{\mu \nu} \partial_\mu \partial_\nu \tilde{\varphi} (x) = 0 , \quad g = \text{diag} (1, -1) ,
\]

the relations

\[
\partial_\mu \tilde{\varphi} (x) = \varepsilon_{\mu \nu} \partial_\nu \varphi (x) , \quad \varepsilon_{01} = 1 ,
\]

and the equal-time canonical commutators

\[
[\varphi (x) , \varphi (y)]|_{x^0 = y^0} = [\tilde{\varphi} (x) , \tilde{\varphi} (y)]|_{x^0 = y^0} = 0 ,
\]

\[
[\partial_0 \varphi (x) , \varphi (y)]|_{x^0 = y^0} = [\partial_0 \tilde{\varphi} (x) , \tilde{\varphi} (y)]|_{x^0 = y^0} = -i \delta (x^1 - y^1) .
\]

Before constructing the finite temperature representation of \( \varphi \) and \( \tilde{\varphi} \), it is useful to recall some of their basic features which are representation independent (see also \([13], [14] \) and \([15] \)).

2.1 General algebraic aspects

One easily verifies that

\[
\varphi (x) = \frac{1}{\sqrt{2}} \int \frac{dk}{2\pi} \left[ a^* (k) e^{ik|x^0 - ikx^1|} + a (k) e^{-ik|x^0 + ikx^1|} \right] ; \quad (5)
\]

\[
\tilde{\varphi} (x) = \frac{1}{\sqrt{2}} \int \frac{dk}{2\pi} \varepsilon (k) \left[ a^* (k) e^{ik|x^0 - ikx^1|} + a (k) e^{-ik|x^0 + ikx^1|} \right] , \quad (6)
\]

obey Eqs. (1-2). Here \( \{a^* (k), a (k)\} \) are the generators of an associative algebra \( \mathcal{A}_- \) with involution \( * \) and identity element \( 1 \). The fields (5-6) satisfy the commutation relations (7-8) provided that

\[
[a (k) , a (p)] = [a^* (k) , a^* (p)] = 0 , \quad (7)
\]

\[
[a (k) , a^* (p)] = \Delta (k) 2\pi \delta (k - p) 1 \quad , \quad (8)
\]
\( \Delta(k) \) being a solution of
\[
|k| \Delta(k) = 1 .
\] (9)

There exists \[\{k^{-1} : \lambda > 0\}\] a one-parameter family of distributions \(|k|^{-1} \lambda \rangle\), which obey Eq. (4). A convenient representation is
\[
\int dk|k|^{-1} f(k) \equiv \lim_{\eta \downarrow 0} \frac{d}{d\eta} \eta \left( \frac{e^{\gamma E}}{\lambda} \right)^{\eta} \int dk|k|^\eta f(k) ,
\] (10)

where \( f \) is a test function of rapid decrease and \( \gamma_E \) is Euler’s constant. The infrared origin of the mass parameter \( \lambda \) is clear and well understood \[17\].

A direct consequence of Eqs. (5-9) are the commutation relations
\[
[\varphi(x_1), \varphi(x_2)] = [\bar{\varphi}(x_1), \bar{\varphi}(x_2)] = iD(x_1 - x_2) \mathbf{1} ,
\] (11)
\[
[\varphi(x_1), \bar{\varphi}(x_2)] = i\bar{D}(x_1 - x_2) \mathbf{1} ,
\] (12)

where
\[
D(x) = -\frac{1}{2} \varepsilon(x^0) \theta(x^2) ,
\] (13)
\[
\bar{D}(x) = \frac{1}{2} \varepsilon(x^1) \theta(-x^2) ,
\] (14)

are \( \lambda \)-independent. Eqs. (11,13) imply that \( \varphi \) and \( \bar{\varphi} \) are local fields. According to eqs. (12,14), \( \varphi \) and \( \bar{\varphi} \) are however not relatively local. It is worth stressing that this fact, known to be crucial for bosonization, generalized (anyon) statistics, duality and chiral super-selected sectors, is universal and does not depend on the representation of \( \mathcal{A}_- \). This observation is essential for finite temperature field theory, where instead of the standard Fock representation \( \mathcal{F} \) of \( \mathcal{A}_- \), one adopts a thermal representation \( \mathcal{T}_- \) (see sect. 2.2), induced by a Kubo-Martin-Schwinger (KMS) state \[18\] over \( \mathcal{A}_- \).

In what follows we will frequently use the basis of chiral right and left fields. In this basis locality is less transparent, but one gains nice factorization properties. Introducing the light-cone coordinates
\[ x^\pm = x^0 \pm x^1 , \] (15)

the chiral fields are defined by
\[
\varphi_R(x^-) \equiv \sqrt{2} \int_0^\infty \frac{dk}{2\pi} \left[ a^*(k)e^{ikx^-} + a(k)e^{-ikx^-} \right] = \varphi(x) + \bar{\varphi}(x) ,
\] (16)
\[
\varphi_L(x^+) \equiv \sqrt{2} \int_0^\infty \frac{dk}{2\pi} \left[ a^*(k)e^{-ikx^+} + a(k)e^{ikx^+} \right] = \varphi(x) - \bar{\varphi}(x) .
\] (17)
Combining eqs. (11-14,16,17), one finds
\[ [\varphi_{Z_1}(\zeta_1), \varphi_{Z_2}(\zeta_2)] = -i\varepsilon(\zeta_{12}) \delta_{Z_1Z_2} 1, \quad \zeta_{12} \equiv \zeta_1 - \zeta_2, \quad (18) \]
where \( Z = L, R \) and
\[ \delta_{Z_1Z_2} = \begin{cases} 1 & \text{if } Z_1 = Z_2 \\ 0 & \text{if } Z_1 \neq Z_2 \end{cases} \quad (19) \]

Let us denote by \( A_Z \) the associative algebra generated by \( \varphi_Z \). Given any smooth real valued function \( f_Z \), it is easily checked that the shift transformation
\[ \alpha_{f_Z} : \varphi_Z(\zeta) \mapsto \varphi_Z(\zeta) + f_Z(\zeta) 1, \quad (20) \]
leaves invariant both eqs. (1,2) and the commutation relations (18). Therefore, \( \alpha_{f_Z} \) is an automorphism of \( A_Z \). It can be given the form
\[ \alpha_{f_Z}[\varphi_Z(\zeta)] = e^{iQ_Z(f_Z)}\varphi_Z(\zeta) e^{-iQ_Z(f_Z)}, \quad (21) \]
where the charges \( Q_Z(f_Z) \) satisfy
\[ [Q_{Z_1}(f_Z_1), \varphi_{Z_1}(\zeta)] = -if_{Z_1}(\zeta) \delta_{Z_1Z_2} 1, \quad (22) \]
\[ [Q_{Z_1}(f_Z_1), Q_{Z_2}(f_Z_2)] = 0. \quad (23) \]
The automorphism (20) turns out to be essential for bosonization with non-vanishing chemical potential.

We now turn to generalized statistics, which can be discussed at purely algebraic level. In fact, let us concentrate on the set of fields parametrized by \( \xi = (\sigma, \tau) \in \mathbb{R}^2 \) and defined by
\[ A(x; \xi) \equiv \zeta_1(\zeta; \xi) \exp \left[ \frac{i\pi}{2} (\tau q_R - \sigma q_L) \right] : \exp \left\{ i\sqrt{\pi} \left[ \sigma \varphi_R(x^-) + \tau \varphi_L(x^+) \right] \right\} : , \quad (24) \]
where
\[ q_Z \equiv \frac{1}{\sqrt{\pi}} Q_Z(1), \quad (25) \]
and the normal ordering : : is taken with respect to the creators \( a^*(k) \) and annihilators \( a(k) \). The normalization \( \zeta_1(\zeta; \xi) \) will be fixed in what follows. Let us observe that due
to eqs. (22,23), the relative position of the two exponential factors in (24) is irrelevant. Notice also that under $^\ast$-conjugation,

$$A^\ast(x;\xi) = \frac{\overline{z}(\lambda;\xi)}{\overline{z}(\lambda;\xi)} A(x;\xi)$$  \hspace{1cm} (26)$$

Although one may consider expressions more general then (24), the field $A(x;\xi)$ will be enough for our purposes.

Using eqs. (18,22,23), we derive the exchange relation

$$A(x_1;\xi_1)A(x_2;\xi_2) = R(x_{12};\xi_1,\xi_2) A(x_2;\xi_2) A(x_1;\xi_1).$$  \hspace{1cm} (27)

The exchange factor $R$ reads

$$R(x_{12};\xi_1,\xi_2) = \exp\{ -i\pi [2(\sigma_1\sigma_2 + \tau_1\tau_2)D(x_{12}) + 2(\sigma_1\sigma_2 - \tau_1\tau_2)\tilde{D}(x_{12}) + (\sigma_1\tau_2 - \sigma_2\tau_1)] \}.$$  \hspace{1cm} (28)

The statistics of the field (24) is governed by the behavior of (28) for $\xi_1 = \xi_2 = \xi$ and space-like separated points $x_{12}^2 < 0$. By means of eqs. (13,14), one gets in this domain

$$A(x_1;\xi)A(x_2;\xi) = \exp[-i\pi(\sigma^2 - \tau^2)\varepsilon(x_1^1 - x_2^1)] A(x_2;\xi)A(x_1;\xi) .$$  \hspace{1cm} (29)

Therefore, $A(x;\xi)$ is an anyon field whose statistics parameter is

$$\vartheta(\xi) = \sigma^2 - \tau^2$$  \hspace{1cm} (30)

Bose or Fermi statistics are recovered when $\vartheta$ is an even or odd integer respectively. The remaining values of $\vartheta$ lead to Abelian braid statistics. The basic fact behind this implementation of generalized statistics, is clearly the relative non-locality of $\varphi$ and $\overline{\varphi}$.

In our discussion below we frequently use the chiral currents

$$j_z(\zeta) \equiv \frac{1}{\sqrt{\pi}} \partial \varphi_z(\zeta) .$$  \hspace{1cm} (31)

A straightforward computation, based on eq. (18), gives

$$[j_z(\zeta), A(x;\xi)] = 2\xi^z \delta(\zeta - x^z)A(x;\xi) ,$$  \hspace{1cm} (32)

where

$$\xi^Z = \begin{cases} \sigma & \text{if } Z = R \\ \tau & \text{if } Z = L \end{cases} \quad x^Z = \begin{cases} x^- & \text{if } Z = R \\ x^+ & \text{if } Z = L \end{cases}$$  \hspace{1cm} (33)
Moreover, differentiating (24) with respect to $x^Z$, one finds
\[ \frac{\partial}{\partial x^Z} A(x; \xi) = i\pi x^Z : j_x(x^Z) A(x; \xi) :. \] (34)

The energy-momentum tensor of the fields $\varphi$ and $\bar{\varphi}$ has two independent components, which read in the chiral basis
\[ \Theta_x(\zeta) = \frac{1}{4} : \partial \varphi_x(\zeta) \partial \varphi_x(\zeta) := \frac{\pi}{4} : j_x(\zeta) j_x(\zeta) :. \] (35)

Indeed, one can directly verify that
\[ [\Theta_x(\zeta_1), \varphi_x(\zeta_2)] = -i \partial \varphi_x(\zeta_2) \delta(\zeta_{12}). \] (36)

Finally, we would like to point out that the short distance expansion of the product $A^*(t, x_1; \xi) A(t, x_2; \xi)$ involves the chiral currents. In fact, normal ordering this product, one finds
\[ A^*(x_1; \xi) A(x_2; \xi) = |z(\lambda; \xi)|^2 \exp \left[ \pi \sigma^2 u(\lambda x_{12}) + \pi \tau^2 u(\lambda x_{12}^-) \right] \cdot \exp \left\{ i\sqrt{\pi} \left[ \sigma \varphi_R(x_2^-) - \sigma \varphi_R(x_1^-) + \tau \varphi_L(x_2^+) - \tau \varphi_L(x_1^+) \right] \right\} :., \] (37)

where
\[ u(\zeta) = -\frac{1}{\pi} \ln(i\zeta + \epsilon) = -\frac{1}{\pi} \ln |\zeta| - \frac{i}{2} \varepsilon(\zeta), \] (38)

the presence of the parameter $\epsilon > 0$ implies as usual the weak limit $\epsilon \to 0$. From (37) one derives the expansion
\[ C(x, \eta; \xi) \equiv \frac{1}{2} \left[ A^*(x + \eta; \xi) A(x; \xi) - A(x; \xi) A^*(x - \eta; \xi) \right] = \]
\[ i\pi |z(\lambda; \xi)|^2 e^{\pi(\sigma^2 u(\lambda \eta^-) + \tau^2 u(\lambda \eta^+))} \left[ \sigma \eta^- j_R(x^-) + \tau \eta^+ j_L(x^+) + O(\eta^+) + O(\eta^-) \right]. \] (39)

Eqs. (32,34,36) are useful for analyzing the symmetry content and the dynamics of $A(x; \xi)$, whereas (39) is convenient in the study of the short distance behavior.

Summarizing, we have introduced the basic algebraic tools for bosonization – the oscillator algebra $A_-$ and the related field algebras $A_R$ and $A_L$. We have established also some identities needed in the construction of fermions and, more generally, anyons. Since $A_-$ and $A_\varphi$ are infinite algebras, some of our manipulations above were formal. In what follows however, we shall deal with operator representations of $A_-$ and $A_\varphi$, in which the just mentioned identities hold at least in mean value on a suitable domain of the state space.
2.2 Thermal representations

We start this section by constructing a thermal representation $T_{\beta}$ of $A_{-}$, whose cyclic vector is the Gibbs (grand canonical) equilibrium state associated with

$$K_{\beta} \equiv H_{\beta} - \mu_{B} N_{B}, \quad \mu_{B} < 0.$$  \hspace{1cm} (40)

Here

$$H_{\beta} = \int \frac{dk}{2\pi} k^{2} a^{\dagger}(k)a(k), \quad N_{B} = \int \frac{dk}{2\pi} |k| a^{\dagger}(k)a(k),$$  \hspace{1cm} (41)

are the Hamiltonian and the number operator respectively, and $\mu_{B}$ is the chemical potential. The non-vanishing two-point correlation functions in this state are given by

$$\langle a^{\dagger}(p)a(q) \rangle^{\beta}_{\mu_{B}} = \frac{e^{-\beta(|p|-\mu_{B})}}{1 - e^{-\beta(|p|-\mu_{B})}} |p|^{-1} 2\pi \delta(p-q),$$  \hspace{1cm} (42)

$$\langle a(q)a^{\dagger}(p) \rangle^{\beta}_{\mu_{B}} = \frac{1}{1 - e^{-\beta(|p|-\mu_{B})}} |p|^{-1} 2\pi \delta(p-q),$$  \hspace{1cm} (43)

$\beta$ being the inverse temperature. Notice that the Bose-Einstein distribution appears as a factor in the right hand side of (42,43). The parameter $\mu_{B} < 0$ allows to work with well defined bosonic correlators. Dealing with the physical anyon fields, introduced in the next section, we will take the limit $\mu_{B} \to 0$. Let us also recall for comparison that

$$\langle a^{\dagger}(p)a(q) \rangle_{F} = 0, \quad \langle a(q)a^{\dagger}(p) \rangle_{F} = |p|^{-1} 2\pi \delta(p-q).$$  \hspace{1cm} (44)

in the Fock representation $F$ of $A_{-}$.

Referring to [18] for the rigorous argument, it is useful to sketch here a heuristic derivation of the expectation values (42,43). Since they are constrained by the commutator (8), it is enough to concentrate for instance on eq. (42). Setting

$$\langle a^{\dagger}(p)a(q) \rangle^{\beta}_{\mu_{B}} = \frac{\text{Tr} \left[ e^{-\beta K_{B}} a^{\dagger}(p)a(q) \right]}{\text{Tr} e^{-\beta K_{B}}},$$  \hspace{1cm} (45)

one finds

$$\langle a^{\dagger}(p)a(q) \rangle^{\beta}_{\mu_{B}} = \frac{\text{Tr} \left[ e^{-\beta K_{B}} a^{\dagger}(p)a(q) \right]}{\text{Tr} e^{-\beta K_{B}}} = e^{-\beta(|p|-\mu_{B})} \frac{\text{Tr} \left[ a^{\dagger}(p) e^{-\beta K_{B}} a(q) \right]}{\text{Tr} e^{-\beta K_{B}}} = e^{-\beta(|p|-\mu_{B})} \frac{\text{Tr} \left[ a^{\dagger}(p) e^{-\beta K_{B}} a(q) \right]}{\text{Tr} e^{-\beta K_{B}}} = e^{-\beta(|p|-\mu_{B})} \frac{\text{Tr} \left[ a^{\dagger}(p) e^{-\beta K_{B}} a(q) \right]}{\text{Tr} e^{-\beta K_{B}}},$$  \hspace{1cm} (46)
where eq. (8) and the cyclicity of the trace have been used. Eq. (46) can be solved for \( \langle a^*(p) a(q) \rangle_{\beta}^{\mu_B} \) and gives precisely (42).

Using the commutation relation (8), a generic correlation function can be reduced to

\[
\langle \prod_{i=1}^{m} a^*(p_i) \prod_{j=1}^{n} a(q_j) \rangle_{\beta}^{\mu_B},
\]

which can be evaluated by iteration from (see e.g. [18])

\[
\langle \prod_{i=1}^{m} a^*(p_i) \prod_{j=1}^{n} a(q_j) \rangle_{\beta}^{\mu_B} = \delta_{mn} \sum_{k=1}^{m} \langle a^*(p_1) a(q_k) \rangle_{\beta}^{\mu_B} \langle \prod_{i=2}^{m} a^*(p_i) \prod_{j=1}^{n} a(q_j) \rangle_{\mu_B},
\]

and the normalization condition

\[
\langle 1 \rangle_{\beta}^{\mu_B} = 1.
\]

As it should be expected, the correlators derived in this way satisfy the KMS condition corresponding to the automorphism of \( \mathcal{A}_- \)

\[
\alpha_s a^*(k) = a^*(k) e^{is|k|-\mu_B}, \quad \alpha_s a(k) = a(k) e^{-is|k|-\mu_B},
\]

generated by \( K_B \). One easily checks for example that

\[
\langle [\alpha_s a(q)] a^*(p) \rangle_{\beta}^{\mu_B} = \langle a^*(p) [\alpha_{s+i\beta} a(q)] \rangle_{\mu_B}^{\beta}.
\]

Summarizing, eqs. (42, 43, 48, 49) completely determine the thermal representation \( \mathcal{T}_- \) of \( \mathcal{A}_- \). Since the distribution \( |k|^{-1}_\lambda \) is not positive definite, \( \mathcal{T}_- \) has indefinite metric. This property is peculiar of the two-dimensional world and it is also present in the Fock representation [19] defined by eq. (44). We will address some of its aspects later on, when discussing the physical representation of the anyon field (24).

It is straightforward at this point to compute the correlation functions of \( \varphi_R \) and \( \varphi_L \). The mixed correlators vanish and we are left with \( \langle \varphi_Z (\zeta_1) \cdots \varphi_Z (\zeta_m) \rangle_{\beta}^{\mu_B} \), which define a thermal representation \( \mathcal{T}_z \) of the algebra \( \mathcal{A}_z \). One finds,

\[
\langle \varphi_Z (\zeta_1) \cdots \varphi_Z (\zeta_{2n+1}) \rangle_{\beta}^{\mu_B} = 0,
\]

\[
\langle \varphi_Z (\zeta_1) \cdots \varphi_Z (\zeta_{2n}) \rangle_{\beta}^{\mu_B} = \sum \langle \varphi_Z (\zeta_{k_1}) \varphi_Z (\zeta_{k_2}) \rangle_{\beta}^{\mu_B} \cdots \langle \varphi_Z (\zeta_{k_{2n-1}}) \varphi_Z (\zeta_{k_{2n}}) \rangle_{\mu_B}^{\beta},
\]

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where the summation runs over all permutations of the integers \(\{1, \ldots, 2n\}\), such that \(k_1 < k_3 < \ldots < k_{2n-1}\) and \(k_{2j-1} < k_{2j}\) for \(j = 1, \ldots, n\). The two-point function, which is the fundamental building block, reads

\[
\langle \varphi_z(\zeta_1) \varphi_z(\zeta_2) \rangle^\beta_{\mu_B} = w(\zeta_{12}; \beta, \mu_B) = 2 \int_0^\infty \frac{dk}{2\pi} |k|^{-1} \left[ \frac{e^{-\beta(k-\mu_B)}}{1 - e^{-\beta(k-\mu_B)}} e^{ik(\zeta_{12})} + \frac{1}{1 - e^{-\beta(k-\mu_B)}} e^{-ik(\zeta_{12})} \right].
\]

(54)

Since \(\lambda > 0\) and \(|\mu_B| > 0\), eq. (54) provides a well defined integral representation of \(w(\zeta; \beta, \mu_B)\). The full \(\lambda\)-dependence and the singularity in \(\mu_B = 0\) are captured by

\[
w(\zeta; \beta, \mu_B) = \frac{1}{\pi} \left\{ \frac{2}{|\mu_B|} \ln \frac{|\mu_B|e^{\gamma_E}}{\lambda} - \ln \left[ \frac{2i \sinh \left( \frac{\pi}{\beta} (\zeta - i\epsilon) \right) }{\lambda} \right] \right\} + O(\mu_B),
\]

(55)

where \(O(\mu_B)\) stands for \(\lambda\)-independent terms vanishing in the limit \(\mu_B \to 0\).

As already mentioned, the parameter \(\mu_B\) has been introduced exclusively for technical reasons. It has nothing to do with the chemical potentials for fermions and anyons. In order to recover the latter, we need the automorphism

\[
\alpha_{\mu_z} : \varphi_z(\zeta) \mapsto \varphi_z(\zeta) - \frac{\mu_z}{\sqrt{\pi}} \zeta, \quad \mu_z \in \mathbb{R},
\]

(56)

of \(A_z\), obtained by setting

\[
f_z(\zeta) = -\frac{\mu_z}{\sqrt{\pi}} \zeta,
\]

(57)

in eq. (20). By inspection, \(\alpha_{\mu_z}\) can not be implemented unitarily in \(T_z\). In fact,

\[
\langle \alpha_{\mu_z} [\varphi_z(\zeta_1) \cdots \varphi_z(\zeta_n)] \rangle^\beta_{\mu_B} = \langle \varphi_z(\zeta_1) \cdots \varphi_z(\zeta_n) \rangle^\beta_{\mu_B} + \frac{\mu_z^2}{\pi} \zeta_1 \zeta_n.
\]

(58)

Therefore, by applying \(\alpha_{\mu_z}\) one generates from \(T_z\) a new representation \(T_z(\mu_z)\) of \(A_z\), associated with the correlation functions \(\langle \alpha_{\mu_z} [\varphi_z(\zeta_1) \cdots \varphi_z(\zeta_n)] \rangle^\beta_{\mu_B}\), which can be obtained directly from eqs. (52-54) by the shift (56).

Concerning the correlators \(\langle \varphi_z(\zeta_1) \cdots \varphi_z(\zeta_n) \rangle^\beta_{\mu_B}\) of the chiral currents (31) in \(T_z(\mu_z)\), it follows from eqs. (52-55) that they are both \(\lambda\)-independent and regular in the limit \(\mu_B \to 0\). Defining

\[
\langle j_z(\zeta_1) \cdots j_z(\zeta_m) \rangle^\beta_{\mu_z} \equiv \lim_{\mu_B \to 0} \langle \alpha_{\mu_z} [j_z(\zeta_1) \cdots j_z(\zeta_m)] \rangle^\beta_{\mu_B},
\]

(59)
one has for instance

\[
\langle j_z(\zeta) \rangle^\beta_{\mu_z} = -\frac{\mu_z}{\pi}, \tag{60}
\]

\[
\langle j_z(\zeta_1) j_z(\zeta_2) \rangle^\beta_{\mu_z} = \frac{\mu_z^2}{\pi^2} - \frac{1}{\beta^2} \sinh^{-2}\left(\frac{\pi}{\beta} \zeta_{12} - \imath \epsilon\right) \tag{61}
\]

At this point, one can derive the correlation functions of \(\Theta_z(\zeta)\) from those of the currents, using the following weak limit representation

\[
\Theta_z(\zeta) = \frac{1}{4} \lim_{\zeta' \to \zeta} : \partial \varphi_z(\zeta') \partial \varphi_z(\zeta) : = \frac{1}{4} \lim_{\zeta' \to \zeta} \left[ \pi j_z(\zeta') j_z(\zeta) + \frac{1}{\pi(\zeta' - \zeta - \imath \epsilon)^2} \right], \tag{62}
\]

valid in \(\mathcal{T}_Z(\mu_z)\). Combining (61) and (62), one finds

\[
\langle \Theta_z(\zeta) \rangle^\beta_{\mu_z} = \frac{\mu_z^2}{4\pi} + \frac{\pi}{12\beta^2}. \tag{63}
\]

In conclusion, starting with the thermal representation \(\mathcal{T}_-\) of \(A_-\), we have constructed the representation \(\mathcal{T}_Z(\mu_z)\) of \(A_z\). We will show in what follows that \(\mathcal{T}_R(\mu_R)\) and \(\mathcal{T}_L(\mu_L)\) provide the basis for bosonization at finite temperature.

3 Anyons at finite temperature and density

In this section we adopt the tensor product \(\mathcal{T}_L(\mu_L) \otimes \mathcal{T}_R(\mu_R)\) for the construction of anyonic fields.

3.1 The physical representation - basic properties and correlation functions

By analogy with the zero temperature case, we concentrate on the \(q_z\)-symmetric anyon correlation functions. Because of eq. (62), they are characterized by two selection rules, namely

\[
\sum_{i=1}^n \sigma_i = 0, \quad \sum_{i=1}^n \tau_i = 0. \tag{64}
\]
Using (64), after some algebra one finds
\[
\begin{align*}
\langle \alpha_{\mu_Z} [A(x_1; \xi_1) \cdots A(x_n; \xi_n)] \rangle_{\mu_B}^\beta &= \\
&= \left[ \prod_{i=1}^{n} z(\lambda; \xi_i) \exp(-i\mu_R \sigma_i x_i^- - i\mu_L \tau_i x_i^+) \right] \exp \left[ -\pi v(\beta, \mu_B) \sum_{i=1}^{n} (\sigma_i^2 + \tau_i^2) \right] \cdot \\
&\exp \left\{ -\pi \sum_{\substack{i,j=1 \atop i<j}}^{n} \left[ \sigma_i \sigma_j w(x_{ij}; \beta, \mu_B) + \tau_i \tau_j w(x_{ij}; \beta, \mu_B) - \frac{i}{2} (\tau_i \sigma_j - \tau_j \sigma_i) \right] \right\},
\end{align*}
\]
where
\[
v(\beta, \mu_B) \equiv 2 \int_0^\infty \frac{dk}{\pi} |k|^{-1} \frac{e^{-\beta (|k| - \mu_B)}}{1 - e^{-\beta (|k| - \mu_B)}} = \frac{1}{\pi} \left( \frac{1}{\beta |\mu_B|} \ln |\mu_B| e^{\pi E_{\lambda}} + \frac{1}{2} \ln \frac{\beta \lambda}{2\pi} \right) + O(\mu_B).
\]
As in eq. (65), \(O(\mu_B)\) denotes \(\lambda\)-independent terms which vanish for \(\mu_B \to 0\). Conditions (64) immediately imply translation invariance of (65). A closer inspection reveals also that the right hand side of eq. (65) is:

- (i) \(\lambda\)-independent;
- (ii) well defined in the limit \(\mu_B \to 0\),

provided that
\[
z(\lambda; \xi) = \left( \frac{\lambda}{2\pi} \right)^{\frac{1}{2} (\sigma^2 + \tau^2)}.
\]
These features are shared also by the correlation functions
\[
\langle \alpha_{\mu_Z} [j_{z_1}(\xi_1) \cdots j_{z_m}(\xi_m) A(x_1; \xi_1) \cdots A(x_n; \xi_n)] \rangle_{\mu_B}^\beta,
\]
involving the chiral current (31). Therefore, from now on, we impose both (64) and (67) and define
\[
\begin{align*}
\langle j_{z_1}(\xi_1) \cdots j_{z_m}(\xi_m) A(x_1; \xi_1) \cdots A(x_n; \xi_n) \rangle_{\mu_B}^\beta = \\
\lim_{\mu_B \to 0} \langle \alpha_{\mu_Z} [j_{z_1}(\xi_1) \cdots j_{z_m}(\xi_m) A(x_1; \xi_1) \cdots A(x_n; \xi_n)] \rangle_{\mu_B}^\beta.
\end{align*}
\]
Eq. (53) implies

\[ \langle A(x_1; \xi_1) \cdots A(x_n; \xi_n) \rangle_{\mu_L, \mu_R}^\beta = \]

\[ \exp \left[ \frac{i\pi}{2} \sum_{i<j}^n (\sigma_i \tau_j - \tau_j \sigma_i) - i\mu_R \sum_{i=1}^n \sigma_i x_i^- - i\mu_L \sum_{i=1}^n \tau_i x_i^+ \right] \left( \frac{1}{2\pi} \right)^{\frac{1}{2} \sum_{i=1}^n (\sigma^2_i + \tau^2_i)} \]

\[ \prod_{i,j=1}^n \left[ \frac{i\beta}{\pi} \sinh \left( \frac{\pi}{\beta} x_{ij}^- - i\epsilon \right) \right]^{\sigma_i \sigma_j} \left[ \frac{i\beta}{\pi} \sinh \left( \frac{\pi}{\beta} x_{ij}^+ - i\epsilon \right) \right]^{\tau_i \tau_j} \]  

(70)

Then, a generic correlator of the type (59) can be obtained by iteration via

\[ \langle \hat{j}_{z_1} (\zeta_1) j_{z_2} (\zeta_2) \cdots j_{z_m} (\zeta_m) A(x_1; \xi_1) \cdots A(x_n; \xi_n) \rangle_{\mu_L, \mu_R}^\beta = \]

\[ - \left\{ \frac{\mu_{z_1}}{\pi} - \frac{i}{\beta} \sum_{j=1}^n \xi_j \coth \left[ \frac{\pi}{\beta} (\zeta_1 - x_j^0) - i\epsilon \right] \right\} \]

\[ \cdot \langle j_{z_2} (\zeta_2) \cdots j_{z_m} (\zeta_m) A(x_1; \xi_1) \cdots A(x_n; \xi_n) \rangle_{\mu_L, \mu_R}^\beta - \frac{1}{\beta^2} \sum_{j=2}^m \delta_{z_1 z_2} \sinh^{-2} \left( \frac{\pi}{\beta} \zeta_{ij} - i\epsilon \right) \]

\[ \cdot \langle j_{z_2} (\zeta_2) \cdots \hat{j}_{z_j} (\zeta_j) \cdots j_{z_m} (\zeta_m) A(x_1; \xi_1) \cdots A(x_n; \xi_n) \rangle_{\mu_L, \mu_R}^\beta , \]

(71)

the hat in the right hand side of (71) denotes that the corresponding current is omitted. One has for instance,

\[ \langle j_R (\zeta_1) j_R (\zeta_2) A(x_1; \xi_1) \cdots A(x_n; \xi_n) \rangle_{\mu_L, \mu_R}^\beta = \]

\[ \langle A(x_1; \xi_1) \cdots A(x_n; \xi_n) \rangle_{\mu_L, \mu_R}^\beta . \]

\[ \left\{ \frac{\mu_R^2}{\pi^2} - \frac{1}{\beta^2} \sinh^{-2} \left( \frac{\pi}{\beta} \zeta_{12} - i\epsilon \right) + i \frac{\mu_R}{\pi \beta} \sum_{j=1}^n \sigma_j \sum_{k=1}^2 \coth \left[ \frac{\pi}{\beta} (\zeta_k - x_j^-) - i\epsilon \right] \right\} . \]

(72)

Coming back to the points (i-ii) above, we observe that the property (i) is essential for our construction. Like at zero temperature, the \( \lambda \)-independence of (70) signals positivity. In fact, repeating with some obvious modifications the argument of reference [20], one can prove that the correlation functions (70) are positive definite, which is crucial for the physical interpretation. Point (ii) in turn, allows to eliminate \( \mu_R \) from the \( q_z \)-symmetric
correlators, in agreement with the fact that the relevant chemical potentials at anyonic level are $\mu_z$. This issue is discussed few lines below.

We are now in a position to perform an important step - the definition of the physical quantum fields $\{A^{\text{ph}}(x; \xi), j^{\text{ph}}_z(\zeta)\}$. Let us introduce for this purpose the set $\Xi = \{\xi_1, -\xi_1, ..., \xi_n, -\xi_n\}$ and let us consider the family $\{A(x; \xi) : \xi \in \Xi\}$. We define

$$
\langle j^{\text{ph}}_z(\zeta_1) \cdots j^{\text{ph}}_z(\zeta_m) A^{\text{ph}}(x_1; \xi_1) \cdots A^{\text{ph}}(x_n; \xi_n) \rangle_{\mu_L, \mu_R}^{\text{ph}, \beta} \equiv \\
\begin{cases} 
\langle j^{\text{ph}}_z(\zeta_1) \cdots j^{\text{ph}}_z(\zeta_m) A(x_1; \xi_1) \cdots A(x_n; \xi_n) \rangle_{\mu_L, \mu_R}^{\text{ph}, \beta} & \text{if (64) holds;} \\
0 & \text{otherwise.}
\end{cases}
$$

(73)

Being positive definite by construction, the functions (73) determine, via the reconstruction theorem [14], the fields $\{A^{\text{ph}}(x; \xi), j^{\text{ph}}_z(\zeta) : \xi \in \Xi\}$. The explicit form of the physical correlators (73) is quite remarkable and deserves a comment. Without current insertions, the equal-time $n$-point function is a finite temperature and density generalization of the Jastrow-Laughlin wave function [3]. We recall that the latter describes the ground state in several one-dimensional models, showing a Tomonaga-Luttinger liquid structure. In that context, the current insertions in (73) are associated with the charged excitations of the liquid. Let us mention also that (73) admits a well defined continuation to imaginary time, which still represents finite temperature and density version of the Jastrow-Laughlin wave function, though in two spatial dimensions.

The expectation values (73) are invariant under the transformations:

$$
A^{\text{ph}}(x; \xi) \mapsto A^{\text{ph}}((x^0 + t, x^1); \xi), \quad j^{\text{ph}}_z(\zeta) \mapsto j^{\text{ph}}_z(\zeta + t), \quad t \in \mathbb{R}; \quad (74)
$$

$$
A^{\text{ph}}(x; \xi) \mapsto e^{-is_R} A^{\text{ph}}(x; \xi), \quad j^{\text{ph}}_z(\zeta) \mapsto j^{\text{ph}}_z(\zeta), \quad s_R \in \mathbb{R}; \quad (75)
$$

$$
A^{\text{ph}}(x; \xi) \mapsto e^{-is_L} A^{\text{ph}}(x; \xi), \quad j^{\text{ph}}_z(\zeta) \mapsto j^{\text{ph}}_z(\zeta), \quad s_L \in \mathbb{R}. \quad (76)
$$

We denote the corresponding conserved charges by $H$ and $Q_Z$ and we consider the automorphism $\alpha_s$ generated by

$$
K \equiv H - \mu_L Q_L - \mu_R Q_R. \quad (77)
$$

Using that

$$
\alpha_s \left[ j^{\text{ph}}_z(\zeta) \right] = j^{\text{ph}}_z(\zeta + s), \quad \alpha_s \left[ A^{\text{ph}}(x; \xi) \right] = e^{is(\mu_R \sigma + \mu_L \tau)} A^{\text{ph}}((x^0 + s, x^1); \xi), \quad (78)
$$

one can verify that the physical correlation functions (73) obey the KMS condition relative to $\alpha_s$ with (inverse) temperature $\beta$. In particular,

$$
\langle A^{\text{ph}}(x_1; \xi_1) \cdots A^{\text{ph}}(x_m; \xi_m) \alpha_{s+\beta} \left[ A^{\text{ph}}(x_{m+1}; \xi_{m+1}) \cdots A^{\text{ph}}(x_n; \xi_n) \right] \rangle_{\mu_L, \mu_R}^{\text{ph}, \beta} = \\
\langle \alpha_s \left[ A^{\text{ph}}(x_{m+1}; \xi_{m+1}) \cdots A^{\text{ph}}(x_n; \xi_n) \right] A^{\text{ph}}(x_1; \xi_1) \cdots A^{\text{ph}}(x_m; \xi_m) \rangle_{\mu_L, \mu_R}^{\text{ph}, \beta}, \quad (79)
$$
for any $1 \leq m \leq n$. This result is fundamental for the physical interpretation of our construction.

At zero temperature (i.e. in the limit $\beta \to \infty$), the anyon correlation functions are in addition invariant with respect to dilatations

$$A_{\text{ph}}(x; \xi) \mapsto \varrho^{1/2(\sigma^2 + \tau^2)} A_{\text{ph}}(x; \xi), \quad \varrho > 0,$$

and Lorentz transformations

$$A_{\text{ph}}(x; \xi) \mapsto e^{\frac{1}{2}(\sigma^2 - \tau^2) \chi} A_{\text{ph}}(\Lambda(\chi)x; \xi), \quad \chi \in \mathbb{R},$$

where $\Lambda(\chi)$ is the 1+1 dimensional Lorentz boost with parameter $\chi$. From (80,81) we deduce the dimension

$$d(\xi) \equiv \frac{1}{2}(\sigma^2 + \tau^2), \quad (82)$$

and the Lorentz spin

$$l(\xi) \equiv \frac{1}{2}(\sigma^2 - \tau^2) = \frac{1}{2} \vartheta(\xi) \quad (83)$$

of $A_{\text{ph}}(x; \xi)$. In spite of the fact that transformations (80,81) do not define exact symmetries at finite temperature, we would like to mention that there exist an alternative conformal field theory approach to bosonization (see e.g. [21] and references therein).

Let us collect now some of the basic properties of $\{A_{\text{ph}}(x; \xi), j_{\text{ph}}(\xi) : \xi \in \Xi\}$, which follow from their correlation functions. Combining (26) and (67), one gets

$$A_{\text{ph}}^*(x; \xi) = A_{\text{ph}}(x; -\xi). \quad (84)$$

The exchange matrix and the statistical parameter of the physical anyon fields are given by (28) and (30) respectively, whereas (71) implies

$$[j_{\text{ph}}^z(\zeta), A_{\text{ph}}(x; \xi)] = 2\xi^Z \delta(\zeta - x^Z) A_{\text{ph}}(x; \xi), \quad (85)$$

where the notation (33) has been used. The translation of eq. (34) in terms of physical fields reads

$$i\pi^Z j_{\text{ph}}^z(x^Z) A_{\text{ph}}(x; \xi) = \frac{\partial}{\partial x^Z} A_{\text{ph}}(x; \xi). \quad (86)$$
where the normal product $\cdot : \cdots :$ is defined by
\[
\hat{\cdot} j^{ph}_z(x^Z) A^{ph}(x; \xi) \equiv \lim_{y \to x} \left\{ j^{ph}_z(y^Z) A^{ph}(x; \xi) + \frac{i \xi Z}{\beta} \coth \left[ \frac{\pi}{\beta} (y - x)^Z - i \epsilon \right] A^{ph}(x; \xi) \right\}.
\] (87)

The counterpart of eq. (62) is
\[
\Theta^{ph}_z(\zeta) = \frac{1}{4} \lim_{\zeta' \to \zeta} \left[ \pi j^{ph}_z(\zeta') j^{ph}_z(\zeta) + \frac{1}{\pi (\zeta' - \zeta - i \epsilon)^2} \right],
\] (88)
and using the physical correlation functions one finds:
\[
[\Theta^{ph}_z(\zeta), A^{ph}(x; \xi)] = -i \delta(\zeta - x^Z) \frac{\partial}{\partial x^Z} A^{ph}(x; \xi),
\] (89)
\[
[\Theta^{ph}_z(\zeta_1), \Theta^{ph}_z(\zeta_2)] = \delta'(\zeta_1) \delta(\zeta_2) [\Theta^{ph}_z(\zeta_1) + \Theta^{ph}_z(\zeta_2)] - i \frac{1}{24 \pi} \delta''(\zeta_1). \] (90)

Eqs. (88, 90) provide a consistency check on the physical correlation functions (73) and (88). Moreover, the $\delta''$-contribution in the right hand side of eq. (90) fixes the value of the central charge. As it should be expected on general grounds, the latter is $\beta$- and $\mu_z$-independent.

Finally, from eq. (82) one gets
\[
C^{ph}(x, \eta; \xi) \equiv \frac{1}{2} \left[ A^{ph*}(x + \eta; \xi) A^{ph}(x; \xi) - A^{ph}(x; \xi) A^{ph*}(x - \eta; \xi) \right] = i \pi (2\pi)^{-d(\xi)} e^{\pi[\sigma^2 u(\eta^-) + \tau^2 u(\eta^+)]} \left[ \sigma \eta^- j^{ph}_R(x^-) + \tau \eta^+ j^{ph}_L(x^+) + O(\eta^+) + O(\eta^-) \right].
\] (91)

This identity suggests that $A^{ph}$ and $j^{ph}_z$ are not independent and that one can recover $j^{ph}_z$ from $A^{ph}$ by point-splitting. For this purpose we assume that $\eta$ is a space-like vector ($\eta^- = \eta^+ \eta^- < 0$) and observe that the behavior of $C^{ph}(x, \eta; \xi)$ when $\eta \to 0$ is direction dependent. Setting
\[
Z_z(\eta; \xi) = \frac{(2\pi)^d(\xi)}{i\pi \eta^Z} e^{-\pi[\sigma^2 u(\eta^-) + \tau^2 u(\eta^+)]},
\] (92)
one gets from (91)
\[
\sigma j_R(x^-) = \lim_{\eta^- \to 0} \left[ \lim_{\eta^+ \to -\eta^-} Z_z(\eta; \xi) C^{ph}(x, \eta; \xi) \right], \quad (93)
\]
\[
\tau j_L(x^+) = \lim_{\eta^+ \to 0} \left[ \lim_{\eta^- \to -\eta^+} Z_z(\eta; \xi) C^{ph}(x, \eta; \xi) \right]. \quad (94)
\]
We stress that the two limits in the right hand side of (93) and (94) do not commute and must be taken in the prescribed order. One constructs in this way $j$ from $A$ provided that $\sigma \neq 0 (\tau \neq 0)$. We would like to mention that the inverse operation is also possible. $A(x; \xi)$ can be expressed in terms of $j_Z$, which allows to develop a framework for bosonization, totally based on the algebra of chiral currents. We refer for the zero temperature case to [13] and [22].

Summarizing, with any set $\Xi$ a thermal anyon representation $T(\Xi; \mu_L, \mu_R)$ is associated, which is defined by the expectation values (73). Let us consider the set $\Xi = \{\xi, -\xi, \xi', -\xi'\}$ with

$$\xi = (\sigma, \tau), \quad \xi' = (\tau, \sigma), \quad \vartheta(\xi) = \sigma^2 - \tau^2 \neq 0. \quad (95)$$

The corresponding representation will be denoted by $T(\sigma, \tau; \mu_L, \mu_R)$ and is essential in the bosonization of fermions.

In what follows, we will use exclusively the physical anyon representations constructed above, omitting for sake of notation simplicity the apex “ph”. One should always keep in mind however that $\{A(x; \xi), j_Z(\xi) : \xi \in \Xi\}$ and $\{A^{ph}(x; \xi), j^{ph}_Z(\xi) : \xi \in \Xi\}$ are different quantum fields.

### 3.2 Momentum distribution and anyon condensation

In order to get a deeper insight into the physical properties of the field $A(x; \xi)$, it is instructive to derive the relative momentum distribution. For this purpose we consider the Fourier transform

$$\tilde{W}^\beta(\omega, k; \xi) = \int_{-\infty}^{\infty} d^2x e^{i\omega x^0 - ikx^1} W^\beta(x; \xi) \quad (96)$$

of the two-point function

$$W^\beta(x_{12}; \xi) \equiv \langle A^*(x_1; \xi)A(x_2; \xi) \rangle^\beta_{\mu_L, \mu_R}. \quad (97)$$

Employing eqs. (70,73) one easily derives

$$\tilde{W}^\beta(\omega, k; \xi) = \frac{1}{2} \varrho \left( \frac{\omega}{2} + \frac{k}{2} + \mu_R \sigma; \sigma^2, \beta \right) \varrho \left( \frac{\omega}{2} - \frac{k}{2} + \mu_L \tau; \tau^2, \beta \right), \quad (98)$$

where

$$\varrho(k; \alpha, \beta) = \int_{-\infty}^{\infty} d\zeta e^{ik\zeta} \left[ 2i\beta \sinh \left( \frac{\pi}{\beta} \zeta - i\epsilon \right) \right]^{-\alpha}, \quad \alpha \geq 0. \quad (99)$$
Figure 1: The momentum distribution \( \varrho(k; \alpha, \beta = 1) \)

The evaluation of the integral (99) gives

\[
\varrho(k; \alpha, \beta) = \begin{cases} 
(\beta)^{1-\alpha} e^{\frac{i \pi}{2} k} \left| \frac{\Gamma \left( \frac{1}{2} \alpha + \frac{i \pi}{2} \beta k \right) \Gamma(\alpha)}{2\pi \delta(k)} \right|^{-1} & \text{for } \alpha > 0, \\
2\pi \delta(k) & \text{for } \alpha = 0.
\end{cases}
\] (100)

Since the contributions of the left- and right-moving modes factorize, it is convenient to consider first the particular cases

\[
\hat{W}^\beta(\omega, k; (\sigma, 0)) = 2\pi \delta(\omega - k) \varrho \left( k + \mu_R \sigma; \sigma^2, \beta \right),
\] (101)

and

\[
\hat{W}^\beta(\omega, k; (0, \tau)) = 2\pi \delta(\omega + k) \varrho \left( -k + \mu_L \tau; \tau^2, \beta \right).
\] (102)

Eqs. (101),(102) have a simple physical interpretation: the \( \delta \)-factors fix the dispersion relations, whereas the \( \varrho \)-factors give the momentum distributions. Eq. (101) for \( \sigma = \pm 1 \) and eq. (102) for \( \tau = \pm 1 \) provide useful checks. From the exchange relation (29) we
already know that the fields $A(x; (\pm 1, 0))$ and $A(x; (0, \pm 1))$ have Fermi statistics. In the next section we will show that they are actually canonical right and left chiral fermions. In agreement with this fact, setting $\alpha = 1$ in eq. (100), we get from (101,102)

\[ \hat{W}^\beta(\omega, k; (\pm 1, 0)) = 2\pi \delta(\omega - k) \frac{1}{1 + e^{-\beta(k \pm \mu_R)}} , \]  

(103)

\[ \hat{W}^\beta(\omega, k; (0, \pm 1)) = 2\pi \delta(\omega + k) \frac{1}{1 + e^{\beta(k \pm \mu_L)}} . \]  

(104)

As expected, the familiar Fermi distribution at finite temperature and chemical potential is recovered. The corresponding Fermi momenta are

\[ k_F = \mp \mu_R , \quad k_F = \pm \mu_L . \]  

(105)

Figure 2: Condensation-like behavior of $\varrho(k; \alpha = 10^{-1}, \beta = 1)$
Turning back to the general expression (100), we see that for $\alpha > 0$ the distribution $\varrho$ is a smooth positive function, whose asymptotic behavior is encoded in

$$\varrho(k; \alpha, \beta) \sim \frac{1}{\Gamma(\alpha)} \left( \frac{k}{2\pi} \right)^{\alpha-1}, \quad k \to \infty,$$

(106)

$$\varrho(k; \alpha, \beta) \sim e^{\beta k} \frac{1}{\Gamma(\alpha)} \left( -\frac{k}{2\pi} \right)^{\alpha-1}, \quad k \to -\infty.$$  

(107)

One can get a general idea about $\varrho$ from 3D plot in figure 1. In the range $\alpha \geq 1$, $\varrho$ is monotonically increasing on the whole line $k \in \mathbb{R}$. When $0 < \alpha < 1$, $\varrho$ increases monotonically for $k \leq 0$ and, according to eqs. (106,107), admits at least one local maximum for $k > 0$. Let us denote the position of the first one (when $k$ moves from 0 to $\infty$) by $k_c(\alpha, \beta)$. We have numerical evidence that this maximum is unique, but we have not an analytic proof of this statement. The plots of $\varrho$ (see figs. 1-2) indicate an interesting condensation-like behavior around $k_c$. The phenomenon is clearly marked for small values of the temperature and/or of the parameter $\alpha$ in the domain $0 < \alpha < 1$. We find it quite remarkable that for any fixed temperature, one can achieve an arbitrary sharp anyon condensation, taking a sufficiently small $\alpha$. See in this respect figure 3, where the momentum distribution is plotted for $\beta = 10^{-1}$.

Concerning the behavior of $\hat{W}^\beta(\omega, k; \xi)$ (see eq. (98)) when both $\sigma \neq 0$ and $\tau \neq 0$, 

\begin{figure}
\centering
\includegraphics[width=0.5\textwidth]{figure3.png}
\caption{The momentum distribution $\varrho(k; \alpha = 10^{-2}, \beta = 10^{-1})$}
\end{figure}
the above analysis implies condensation at

\[ \omega = k_c(\sigma^2, \beta) + k_c(\tau^2, \beta) - \mu_R \sigma - \mu_L \tau, \]  

\[ k = k_c(\sigma^2, \beta) - k_c(\tau^2, \beta) - \mu_R \sigma + \mu_L \tau, \]

provided that \( 0 < \sigma^2 < 1 \) and \( 0 < \tau^2 < 1 \).

It is worth mentioning that the condensation effect we discovered, does not contradict the Hohenberg-Mermin-Wagner (HMW) theorem about the absence of condensation in one space dimension. The point is that under very general conditions, any anyonic statistics can be equivalently described by a suitable exchange interaction with two- and three-body potentials, determined \[23\] by the exchange factor \( \text{(28)} \). When these potentials are confining, some assumptions of the HMW theorem are violated and condensation may occur \[24\], \[25\], even in one dimension.

In conclusion, we have shown that the right (left) moving modes of the anyon field \( A(x; \xi) \) condensate in the range \( 0 < \sigma^2 < 1 \) \((0 < \tau^2 < 1)\). Some applications of this phenomenon are currently under investigation \[26\].

4 A particular case - fermions

An useful testing ground for the bosonization scheme proposed above is provided by the free fermion field. The possibility to express the latter in terms of bosonic fields, was discovered long ago by Jordan and Wigner \[27\]. Using the general results of the previous section, we will discuss below some aspects of this phenomenon at finite temperature and density. Let us start by the 1+1 dimensional massless free Dirac equation

\[ \gamma^\nu \partial_\nu \psi(x) = 0, \]  

where

\[ \psi(x) = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \end{pmatrix}, \quad \gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \]

One easily verifies that

\[ \psi_1(x) = \int_0^{\infty} \frac{dk}{2\pi} \left[ d^*(k) e^{ikx^-} + b(k) e^{-ikx^-} \right] \equiv \psi_R(x^-), \]  

21
\[
\psi_2(x) = \int_{-\infty}^{0} \frac{dk}{2\pi} \left[ b(k) e^{i k x^+} - d^*(k) e^{-i k x^+} \right] \equiv \psi_L(x^+) ,
\]

(113)
obes eq. (110). Here \( \{b^*(k), b(k), d^*(k), d(k)\} \) are the generators of an associative algebra \( \mathcal{A}_+ \) (with involution * and identity element 1), which are assumed to satisfy the anti-commutation relations

\[
\begin{align*}
\{b(k), b(p)\} = \{b^*(k), b^*(p)\} = \{d(k), d(p)\} = \{d^*(k), d^*(p)\} &= 0 , \\
\{b(k), b^*(p)\} = \{d(k), d^*(p)\} &= 2\pi \delta(k - p) 1 .
\end{align*}
\]

(114)
Combining eqs. (112-113) and (114), one finds

\[
\{\psi(x_1), \psi(x_2)\} = 0 , \quad \{\psi(x_1), \psi^*(x_2)\} = \begin{pmatrix} \delta(x_{12}^+) & 0 \\ 0 & \delta(x_{12}^-) \end{pmatrix} ,
\]

(115)
which in particular imply the equal time canonical anti-commutation relations.

As a consequence of (110), both the vector and the axial currents

\[
\begin{align*}
    j_\mu(x) &= : \overline{\psi} \gamma_\mu \psi : (x) , & j_5^\mu(x) &= : \overline{\psi} \gamma_\mu \gamma^5 \psi : (x) , & \gamma^5 &\equiv \gamma^0 \gamma^1 ,
\end{align*}
\]

(116)
are conserved. Here \( \overline{\psi} \equiv \psi^\gamma \gamma^0 \) and the normal product is relative to the creators \( \{b(k), d^*(k)\} \) and the annihilators \( \{b(k), d(k)\} \). The duality relation

\[
\begin{align*}
    j_\mu(x) &= \varepsilon_{\mu\nu} j_\nu^\gamma(x) , & j_5^\mu(x) &= \varepsilon_{\mu\nu} j_5^\nu(x) ,
\end{align*}
\]

(117)
holds because of the identity \( \gamma_\mu \gamma^5 = \varepsilon_{\mu\nu} \gamma^\nu \). One has

\[
\begin{align*}
    [j_0(x), \psi(y)]|_{x^0=y^0} &= -\delta(x^1 - y^1) \psi(y) , \\
    [j_5^\gamma(x), \psi(y)]|_{x^0=y^0} &= -\delta(x^1 - y^1) \gamma^5 \psi(y) .
\end{align*}
\]

(118)
(119)
The conserved charges relative to \( j_\mu \) and \( j_5^\mu \) read

\[
\begin{align*}
Q &= \int_{-\infty}^{\infty} \frac{dk}{2\pi} [b^*(k)b(k) - d^*(k)d(k)] , & Q_5 &= \int_{-\infty}^{\infty} \frac{dk}{2\pi} \bar{\varepsilon}(k) [b^*(k)b(k) - d^*(k)d(k)] .
\end{align*}
\]

(120)
Let us consider now the thermal representation \( \mathcal{T}_+ \) of \( \mathcal{A}_+ \), whose cyclic vector is the Gibbs equilibrium state associated with

\[
K_F = H_F - \mu Q - \mu_5 Q_5 , \quad \mu, \mu_5 \in \mathbb{R}
\]

(121)
where

\[ H_F = \int_{-\infty}^{\infty} \frac{dk}{2\pi} |k| [b^*(k)b(k) + d^*(k)d(k)] , \tag{122} \]

is the Hamiltonian of the free Dirac field. We stress the presence of the chemical potential \( \mu_5 \), associated with the chiral symmetry. The non-vanishing two-point expectation values in this equilibrium state are:

\[
\langle b^*(p)b(q) \rangle_{\mu, \mu_5}^\beta = \frac{e^{-\beta |p - \mu - \epsilon(p)\mu_5|}}{1 + e^{-\beta |p - \mu - \epsilon(p)\mu_5|}} \frac{2\pi \delta(p - q)}{2}, \tag{123} \\
\langle b(p)b^*(q) \rangle_{\mu, \mu_5}^\beta = \frac{1}{1 + e^{-\beta |p - \mu - \epsilon(p)\mu_5|}} \frac{2\pi \delta(p - q)}{2}, \tag{124} \\
\langle d^*(p)d(q) \rangle_{\mu, \mu_5}^\beta = \frac{e^{-\beta |p + \mu + \epsilon(p)\mu_5|}}{1 + e^{-\beta |p + \mu + \epsilon(p)\mu_5|}} \frac{2\pi \delta(p - q)}{2}, \tag{125} \\
\langle d(p)d^*(q) \rangle_{\mu, \mu_5}^\beta = \frac{1}{1 + e^{-\beta |p + \mu + \epsilon(p)\mu_5|}} \frac{2\pi \delta(p - q)}{2}. \tag{126} 
\]

Like in the bosonic case, by using \( \langle 1 \rangle_{\mu, \mu_5}^\beta = 1 \) a generic \( n \)-point function can be expressed in terms of (123-126). The correlation functions of \( \psi \) are therefore fully determined. The ones involving \( \psi_a^* \) and \( \psi_a \) with the same value of the index \( a \), can be reduced by means of eqs. (113) to

\[
\langle \psi_1^*(x_1) \cdots \psi_n^*(x_n) \psi_1(y_1) \cdots \psi_1(y_1) \rangle_{\mu, \mu_5}^\beta = \delta_{mn} \det \langle \psi_1^*(x_1) \psi_1(y_1) \rangle_{\mu, \mu_5}^\beta = \]

\[
\delta_{mn} \exp \left[ i(\mu + \mu_5) \sum_{i,j=1}^n (x_i - y_j)^- \right] \det \frac{1}{2i\beta \sinh \left[ \frac{\pi}{\beta} (x_i - y_j)^- - i\epsilon \right]} , \tag{127} \\
\langle \psi_2^*(x_1) \cdots \psi_2^*(x_n) \psi_2(y_1) \cdots \psi_2(y_1) \rangle_{\mu, \mu_5}^\beta = \]

\[
\delta_{mn} \exp \left[ i(\mu - \mu_5) \sum_{i,j=1}^n (x_i - y_j)^+ \right] \det \frac{1}{2i\beta \sinh \left[ \frac{\pi}{\beta} (x_i - y_j)^+ + i\epsilon \right]} . \tag{128} 
\]

The expectation values involving both \( \psi_1 \) and \( \psi_2 \) factorize in the product of two correlators depending only on \( \psi_1 \) and \( \psi_2 \) respectively. For the currents (110), one gets

\[
\langle j_0(x) \rangle_{\mu, \mu_5}^\beta = \frac{\mu}{\pi}, \quad \langle j_0^5(x) \rangle_{\mu, \mu_5}^\beta = \frac{\mu_5}{\pi} . \tag{129} 
\]
It follows from eq. (117)

$$\langle j_1(x)\rangle_{\mu, \mu_5} = -\frac{\mu_5}{\pi},$$

(130)

which clarifies the physical meaning of the chemical potential $\mu_5$. It gives rise to a persistent current at finite temperature. We will discuss later on this interesting phenomenon in the context of the Thirring model.

Let us show now that the free fermion field considered above, has an equivalent bosonized description. For this purpose we take the representation $T(\sigma, 0; \mu_L, \mu_R)$ and set

$$\Psi(x; \sigma) = \left( \begin{array}{c} \Psi_1(x; \sigma) \\ \Psi_2(x; \sigma) \end{array} \right) \equiv \left( \begin{array}{c} A(x; (\sigma, 0)) \\ A(x; (0, \sigma)) \end{array} \right).$$

(131)

Since $\Psi_1$ and $\Psi_2$ depend on $x^-$ and $x^+$ respectively, the field $\Psi$ obviously satisfies eq. (110). According to eq. (30), the corresponding statistical parameter is

$$\vartheta = \sigma^2, \quad \sigma \neq 0,$$

(132)

showing that, in general, $\Psi$ obeys anyon statistics. The existence of anyonic solutions of the free Dirac equation (110) is not surprising. Let us recall in this respect that there is no genuine notion of spin in 1+1 dimensions, because the rotation group is trivial. Being associated with the one-parameter group of boost transformations, which is non-compact, the Lorentz spin $l(\xi)$ (see eq. (83)) may take any real value.

At this point we need the correlation functions of $\Psi$. From eqs. (70, 73), one gets

$$\langle \Psi_1^\dagger(x_1; \sigma) \cdots \Psi_1^\dagger(x_m; \sigma) \Psi_1(y_n; \sigma) \cdots \Psi_1(y_1; \sigma) \rangle_{\mu_L, \mu_R}^{\beta} = \delta_{mn} \exp \left[ i\mu_R \sigma \sum_{i,j=1}^n (x_i - y_j)^- \right]$$

$$\prod_{i,j=1}^n 2i\beta \sinh \left( \frac{\pi}{\beta} (x_{ij}^- - i\epsilon) \right) \sigma^2 \prod_{i,j=1}^n 2i\beta \sinh \left( \frac{\pi}{\beta} (y_{ij}^- - i\epsilon) \right) \sigma^2,$$

(133)

$$\langle \Psi_2^\dagger(x_1; \sigma) \cdots \Psi_2^\dagger(x_m; \sigma) \Psi_2(y_n; \sigma) \cdots \Psi_2(y_1; \sigma) \rangle_{\mu_L, \mu_R}^{\beta} = \delta_{mn} \exp \left[ i\mu_L \sigma \sum_{i,j=1}^n (x_i - y_j)^+ \right]$$

$$\prod_{i,j=1}^n 2i\beta \sinh \left( \frac{\pi}{\beta} (x_{ij}^+ - i\epsilon) \right) \sigma^2 \prod_{i,j=1}^n 2i\beta \sinh \left( \frac{\pi}{\beta} (y_{ij}^+ - i\epsilon) \right) \sigma^2,$$

(134)

$$\prod_{i,j=1}^n \left\{ \sinh \left[ \frac{\pi}{\beta} (x_i - y_j)^- + i\epsilon \right] \right\} \sigma^2$$

24
Since the mixed $\Psi$-correlators factorize like in the $\psi$-case, it is enough to compare eqs. (133,134) with eqs. (127,128). For this purpose we observe that the $i\epsilon$-prescription in the numerator of (133) and (134) is actually superfluous and the identity

$$\det \frac{1}{2i\beta \sinh \left[ \frac{\pi}{\beta} (x_i - y_j) - i\epsilon \right]} = \frac{\prod_{i < j}^{n} 2i\beta \sinh \left( \frac{\pi}{\beta} x_{ij} \right) \prod_{i > j}^{n} 2i\beta \sinh \left( \frac{\pi}{\beta} y_{ij} \right)}{\prod_{i,j=1}^{n} 2i\beta \sinh \left[ \frac{\pi}{\beta} (x_i - y_j) - i\epsilon \right]}.$$  \hspace{1cm} (135)

holds. The proof of Eq. (135) is given in the appendix. Eq. (135) generalizes

$$\det \frac{1}{(x_i - y_j - i\epsilon)} = \frac{\prod_{i < j}^{n} (x_{ij}) \prod_{i > j}^{n} (y_{ij})}{\prod_{i,j=1}^{n} (x_i - y_j - i\epsilon)}, \hspace{1cm} (136)$$

which makes possible the bosonization at zero temperature.

In view of (133), we conclude that the correlation functions of $\psi(x)$ and $\Psi(x; \sigma)$ coincide, provided that

$$\sigma^2 = 1,$$  \hspace{1cm} (137)

and

$$\mu_R = \frac{1}{\sigma} (\mu + \mu_\sigma), \hspace{0.5cm} \mu_L = \frac{1}{\sigma} (\mu - \mu_\sigma).$$  \hspace{1cm} (138)

In other words, the representations $\mathcal{T}(1,0; \mu + \mu_\sigma, \mu - \mu_\sigma)$ and $\mathcal{T}(-1,0; -\mu - \mu_\sigma, -\mu + \mu_\sigma)$ are equivalent and yield the bosonized version of the field $\psi$. For

$$\sigma^2 = 2n + 1, \hspace{0.5cm} n \geq 1$$  \hspace{1cm} (139)

$\Psi(x; \sigma)$ still possess Fermi statistics, but is not canonical. According to the results of the previous section, the momentum distributions of both $\Psi_1(x; \sigma)$ and $\Psi_2(x; \sigma)$ signal condensation in the range $0 < \sigma^2 < 1$.

For completing the Fermi-Bose correspondence, we have to find in $\mathcal{T}(\sigma,0; \mu_L, \mu_R)$ the counterparts $J_\mu$ and $J_\mu^5$ of the fermionic currents (116). Let us define

$$J_0(x) = -J_1^5(x) = -\frac{1}{2\sigma} [j_R(x^-) + j_L(x^+)],$$  \hspace{1cm} (140)

$$J_1(x) = -J_0^5(x) = \frac{1}{2\sigma} [j_R(x^-) - j_L(x^+)].$$  \hspace{1cm} (141)

25
These currents are conserved by construction. Moreover, eq. (71) implies
\[ [J_0(x), \Psi(y)]|_{x^a=y^a} = -\delta(x^1-y^1)\Psi(y), \tag{142} \]
\[ [J_5^0(x), \Psi(y)]|_{x^a=y^a} = -\delta(x^1-y^1)\gamma^5 \Psi(y), \tag{143} \]
which precisely reproduce eqs. (118,119). Notice that the normalization in (140,141) is uniquely fixed by requiring (142,143). Using (60), one finds
\[ \langle J_0(x) \rangle_{\mu_L, \mu_R}^\beta = \frac{1}{2\pi\sigma} (\mu_R + \mu_L), \quad \langle J_5^0(x) \rangle_{\mu_L, \mu_R}^\beta = \frac{1}{2\pi\sigma} (\mu_R - \mu_L). \tag{144} \]
As it should be expected, (129) and (144) are equivalent, if (137,138) are satisfied.

5 The Thirring model

The Thirring model \cite{28} has been extensively studied in the past. The classical equation of motion is
\[ i\gamma^\nu \partial_\nu \Psi(x) = g\pi \overline{\Psi}(x)\gamma_\nu \Psi(x)\gamma^\nu \Psi(x), \tag{145} \]
where \( g \in \mathbb{R} \) is the coupling constant and the factor \( \pi \) has been introduced for further convenience. Both vector and chiral currents,
\[ J_\nu(x) = \overline{\Psi}(x)\gamma_\nu \Psi(x), \quad J_5^\nu(x) = \overline{\Psi}(x)\gamma_\nu \gamma^5 \Psi(x), \tag{146} \]
are conserved, satisfy the duality relation
\[ J_5^\nu(x) = \varepsilon_{\nu\mu} J_\mu(x), \tag{147} \]
and have the following equal time Poisson brackets
\[ \{ J_0(0, x^1), \Psi(0, y^1) \}_{\text{P.B.}} = -\delta(x^1-y^1)\Psi(0, y^1), \tag{148} \]
\[ \{ J_5^0(0, x^1), \Psi(0, y^1) \}_{\text{P.B.}} = -\delta(x^1-y^1)\gamma^5 \Psi(0, y^1). \tag{149} \]
Introducing
\[ J_+(x) = J_0(x) + J_1(x), \quad J_-(x) = J_0(x) - J_1(x), \tag{150} \]
eq. (143) can be rewritten in the form
\[ 2i\partial_+ \Psi_1(x) = g\pi J_+(x)\Psi_1(x), \quad 2i\partial_- \Psi_2(x) = g\pi J_-(x)\Psi_2(x). \tag{151} \]
Our problem is now to quantize this system at finite temperature and chemical potentials \( \mu \) and \( \mu_5 \), associated with the charges \( Q \) and \( Q^5 \) generated by the currents (146). More precisely, we look for operators \{\Psi, J_+, J_-\} acting on a Hilbert space and satisfying the requirements:
• (a) there exist suitable normal products $N[J_+ \Psi_1](x)$ and $N[J_- \Psi_2](x)$, such that the quantum versions

$$2i\partial_+ \Psi_1(x) = g\pi N[J_+ \Psi_1](x), \quad 2i\partial_- \Psi_2(x) = g\pi N[J_- \Psi_2](x),$$

of $(151)$ hold;

• (b) the operators

$$J_0(x) = \frac{1}{2}[J_+(x) + J_-(x)], \quad J_1(x) = \frac{1}{2}[J_+(x) - J_-(x)],$$

are the components of a conserved current $J_\nu(x)$ and

$$[J_0(x), \Psi(y)]|_{x^0 = y^0} = -\delta(x^1 - y^1)\Psi(y);$$

• (c) the axial current $J_5(x)$ is related to $J_\nu(x)$ via the duality relation $(147)$ and it is also conserved;

• (d) the correlation functions of $\{\Psi, J_+, J_-\}$ obey the KMS condition relative to the automorphism

$$\alpha_s \Psi_1(x) = e^{is(\mu_+ + \mu_5)}\Psi_1(x^0 + s, x^1), \quad \alpha_s \Psi_2(x) = e^{is(\mu_- - \mu_5)}\Psi_1(x^0 + s, x^1),$$

$$\alpha_s J_\pm(x) = J_\pm(x^0 + s, x^1), \quad s \in \mathbb{R},$$

with (inverse) temperature $\beta$.

In the previous section we have actually solved the problem defined by (a-d) in the case $g = 0$. This solution suggests considering the representation $T(\sigma, \tau; \mu_L, \mu_R)$ (see eq. $(131)$) and setting

$$\Psi_1(x) \equiv A(x; \xi), \quad \Psi_2(x) \equiv A(x; \xi'),$$

which generalizes eq. $(131)$. The next step is to identify the normal product $N$ we are looking for, with $\cdots$. Then, defining

$$J_+(x) \equiv -\frac{1}{\sigma + \tau} j_L(x^+), \quad J_-(x) \equiv -\frac{1}{\sigma + \tau} j_R(x^-),$$

we see that $(150)$ implies $(152)$, provided that

$$2\tau(\sigma + \tau) = g.$$
In addition, inserting (157) in (153), we find that the requirements of point (b) above are satisfied. Let us focus on (c). The duality relation (147) defines a conserved chiral current \( J^5_\nu(x) \) in terms of \( J_\nu(x) \). A simple computation gives

\[
[J^5_0(x), \Psi(y)]_{x^0=y^0} = -\frac{\sigma - \tau}{\sigma + \tau} \delta(x^1 - y^1) \gamma^5 \Psi(y). \tag{159}
\]

Because of eq. (158), for \( g \neq 0 \) one has

\[
\sigma - \tau \neq \sigma + \tau, \tag{160}
\]

which shows the presence of a quantum correction in the right hand side of (159) with respect to (149). This is a known unavoidable feature [29], following from (a-b).

Finally, let us consider condition (d). From the results of Sect. 3.1 we deduce that it is satisfied, if the parameters \( \mu_z \) of \( T(\sigma, \tau; \mu_L, \mu_R) \) are fixed according to

\[
\mu_R = \frac{1}{\sigma + \tau} \mu + \frac{1}{\sigma - \tau} \mu_5, \quad \mu_L = \frac{1}{\sigma + \tau} \mu - \frac{1}{\sigma - \tau} \mu_5. \tag{161}
\]

Using eqs. (158) and (159) one can express \((\sigma, \tau)\) in terms of \((g, \psi)\). Two solutions are found:

\[
\sigma = \frac{g + 2\psi}{2\sqrt{g + \psi}}, \quad \tau = \frac{g}{2\sqrt{g + \psi}}, \tag{162}
\]

\[
\sigma' = -\frac{g + 2\psi}{2\sqrt{g + \psi}}, \quad \tau' = -\frac{g}{2\sqrt{g + \psi}}. \tag{163}
\]

Since \((\sigma, \tau) \in \mathbb{R}^2\), the coupling constant \(g\) and the statistical parameter \(\psi\) are constrained by

\[
\psi > -g. \tag{164}
\]

Anyonic solutions of the Thirring model have been proposed and investigated recently also in [30].

The correlation functions of the Thirring field \(\Psi\) follow directly from eq. (70) and coincide for both solutions (162) and (163). From eqs. (60,157,161-163) one can derive the expectation value of the Thirring current \(J_\nu(x)\) in the Gibbs state:

\[
\langle J_0(x) \rangle^{\beta}_{\mu, \mu_5} = \frac{\mu}{\pi(g + \psi)}, \quad \langle J_1(x) \rangle^{\beta}_{\mu, \mu_5} = -\frac{\mu_5}{\pi \psi}. \tag{165}
\]

Comparing eq. (165) to the free case (129,130), one observes nontrivial corrections induced by the interaction and the anyon statistic. Previous investigations ([31], [32], [33]) of the
charge density $\langle J_0(x) \rangle_{\mu, \nu_5}^\beta$ in the case $\vartheta = 1$, have given contradictory results. We confirm the result of [32] and [33] and extend it to the general class of anyon solutions with $\vartheta \neq 1$. In the functional integral framework of [32] and [33], the $g$-dependence of the charge density stems from topological contributions. It is worth stressing that our operator approach reproduces the same $g$-dependence in a flat 1+1 dimensional Minkowski spacetime.

To our knowledge, the appearance of a persistent current $\langle J_1(x) \rangle_{\mu, \nu_5}^\beta$ in the finite temperature Thirring model represents a novel feature. We would like to recall in this respect that persistent currents of quantum origin have been experimentally observed ([34], [35]) in mesoscopic rings placed in an external magnetic field. Such fields are absent in the two-dimensional world, but chiral symmetry, combined with duality still allow for a non-vanishing $\langle J_1(x) \rangle_{\mu, \nu_5}^\beta$. Notice that the persistent current (165) grows for small values of $\vartheta$, diverging in the limit $\vartheta \to 0$.

Concerning the energy-momentum tensor $T_{\mu\nu}(x)$ of the model, by means of eq. (89) we find that

$$T_{00}(x) = T_{11}(x) \equiv \Theta_L(x^+) + \Theta_R(x^-),$$
$$T_{01}(x) = T_{10}(x) \equiv \Theta_L(x^+) - \Theta_R(x^-),$$

(166)

generate on the Thirring field $\Psi$ the right transformations

$$[T_{0\nu}(x_1), \Psi(x_2)]|_{x_1^0 = x_2^0} = -i\partial_\nu \Psi(x_2) \delta(x_{12}^1).$$

(167)

Employing eqs. (72, 88, 101, 165) one derives the following energy and momentum densities:

$$\langle T_{00}(x) \rangle_{\mu, \nu_5}^\beta = \frac{\pi}{6\beta^2} + \frac{\mu_L^2 + \mu_R^2}{4\pi} = \frac{\pi}{6\beta^2} + \frac{\pi}{2}(g + \vartheta) \left( \langle J_0(x) \rangle_{\mu, \nu_5}^\beta \right)^2 + \left( \langle J_1(x) \rangle_{\mu, \nu_5}^\beta \right)^2,$$

(168)

$$\langle T_{01}(x) \rangle_{\mu, \nu_5}^\beta = \frac{\mu_L^2 - \mu_R^2}{4\pi} = \pi(g + \vartheta)\langle J_0(x) \rangle_{\mu, \nu_5}^\beta \langle J_1(x) \rangle_{\mu, \nu_5}^\beta.$$

(169)

When $\langle J_1(x) \rangle_{\mu, \nu_5}^\beta = 0$, the energy density coincides with the pressure and (168) gives rise to an equation of state, relating temperature, pressure and density.

6 Conclusions

In the present paper we have developed an operator formalism for bosonization at finite temperature and density, which applies not only to fermions, but covers the case of generalized statistics as well. The approach is systematic, works directly in infinite volume.
and produces an explicit solution. Starting from the conventional chiral field algebras \( \{ \mathcal{A}_Z : Z = R, L \} \), naturally associated with the free massless scalar field \( \varphi \) and its dual \( \bar{\varphi} \), we have constructed a set of fields \( A(x, \xi) \), parametrized by \( \xi = (\sigma, \tau) \in \mathbb{R}^2 \). For generic values of \( \xi \), \( A(x, \xi) \) exhibits Abelian braid statistics. The basic tool in our framework is the thermal representation \( \mathcal{T}_L(\mu_L) \otimes \mathcal{T}_R(\mu_R) \) of \( \mathcal{A}_L \otimes \mathcal{A}_R \), characterized by inverse temperature \( \beta \) and chemical potentials \( \mu_L \) and \( \mu_R \). We derived explicitly all correlation functions of the field \( A(x, \xi) \) in the representation \( \mathcal{T}_L(\mu_L) \otimes \mathcal{T}_R(\mu_R) \), verifying the KMS condition. The \( n \)-point function represents a generalization of the Jastrow-Laughlin wave function. The exact momentum distributions of the left- and right-moving modes, following from the 2-point function, reveal in certain range of the variables \( (\sigma, \tau) \) a clear condensation-like behavior. The parameters \( \mu_L \) and \( \mu_R \) are related to special shift authomorphisms of \( \mathcal{A}_L \) and \( \mathcal{A}_R \). Suitable combinations of \( \mu_L \) and \( \mu_R \) define the physical chemical potentials \( \mu \) and \( \mu_5 \), which are proportional to the vector- and chiral-charge densities. By duality, \( \mu_5 \neq 0 \) implies a non-vanishing persistent current.

The characteristic features of our finite temperature and density bosonization procedure have been illustrated on the example of the massless Thirring model. We established a general class of anyonic solutions, providing an unambiguous derivation of the charge density and demonstrating the presence of a persistent current.

The general framework proposed in the paper is mathematically self-consistent and is interesting not only from a purely theoretical point of view. It is widely accepted by now, that the excitations described by the anyon field \( A(x, \xi) \) are relevant for a class of low dimensional condensed matter systems in which the one-dimensional Luttinger liquid plays an important role. It is claimed for instance that the edge currents in the fractional quantum Hall effect can be described by a Luttinger liquid \( [1] \). Anderson’s \( [36] \) proposal for explaining high temperature superconductivity is based on the two-dimensional Hubbard model, whose ground state and low-energy excitations are also interpreted in terms of a Luttinger liquid. In this context, the phenomenon of anyon condensation and the appearance of a persistent current, discovered in this paper, deserve further attention. For application to quantum impurity problems, it will be interesting to extend to finite temperature and density the bosonization procedure on the half line, developed in \( [37] \). We hope to report on this subject in the near future.
Appendix

In this appendix we prove the determinant formulae (135,136). It is obvious that
\[
\det \frac{1}{(x_i - y_j)} = \frac{P(x, y)}{\prod_{i,j=1}^{n}(x_i - y_j)},
\]  
(170)
where \(P\) is a polynomial of degree \(n^2 - n\), \(n\) being the order of the determinant. Because of translation invariance, \(P\) depends only on the differences \(x_{ij}, y_{ij}\) and \(x_i - y_j\). Moreover, if \(x_i = x_j\) the determinant has two identical rows and vanishes, so \(P\) must be divisible by \(x_{ij}\); in the same way if \(y_i = y_j\) the determinant has two identical columns, so \(P\) must be divisible by \(y_{ij}\). We can write then
\[
P(x, y) = C \prod_{i,j=1}^{n} (x_{ij}) \prod_{i,j=1}^{n} (y_{ij})
\]  
(171)
where \(C\) is a constant, potentially depending on \(n\). Considering the particular case \(x_i = y_i + \epsilon_i\) and expanding in \(\epsilon_i\) one immediately checks that \(C = 1\). Formula (136) is a direct consequence of eqs. (170,171).

Concerning the determinant formula (135), put \(z_i = e^{x_i}, \quad w_i = e^{w_i}\). We have
\[
\det \frac{1}{\sinh(x_i - y_j)} = \det \frac{2z_i w_j}{z_i^2 - w_j^2} = \frac{\prod_{i=1}^{n} (2z_i w_i) \prod_{i<j}^{n} (z_i^2 - z_j^2) \prod_{i>j}^{n} (w_i^2 - w_j^2)}{\prod_{i,j=1}^{n} (z_i^2 - w_j^2)},
\]  
(172)
where the previous determinant formula has been used. Now
\[
z_i^2 - z_j^2 = 2z_i z_j \sinh(x_{ij}), \quad w_i^2 - w_j^2 = 2w_i w_j \sinh(y_{ij}),
\]  
(173)
and we get
\[
\frac{\prod_{i=1}^{n} (2z_i w_i)}{\prod_{i,j=1}^{n} (2z_i w_j)} \frac{\prod_{i<j}^{n} (z_i^2 - z_j^2)}{\prod_{i>j}^{n} (w_i^2 - w_j^2)} \frac{\prod_{i,j=1}^{n} \sinh(x_{ij})}{\prod_{i,j=1}^{n} \sinh(x_i - y_j)} \frac{\prod_{i,j=1}^{n} \sinh(y_{ij})}{\prod_{i,j=1}^{n} \sinh(y_i - y_j)}.
\]  
(174)
A trivial calculation shows that the first fraction equals 1 identically, so we finally obtain
\[
\det \frac{1}{\sinh(x_i - y_j)} = \frac{\prod_{i<j}^{n} \sinh(x_{ij}) \prod_{i>j}^{n} \sinh(y_{ij})}{\prod_{i,j=1}^{n} \sinh(x_i - y_j)}.
\]  
(175)
Eq. (135) is a straightforward consequence of this identity.
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