REMARK ON THE ALEXANDER POLYNOMIALS OF PERIODIC KNOTS

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To Yuko’s 29th birthday

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1. INTRODUCTION

Let \( p \) be a prime number and \( K \) a non-trivial knot in \( S^3 \) which has period \( p \). Then the Alexander polynomial \( \Delta_K(t) \) of such a knot must have some distinguished properties as Murasugi \cite{3} have revealed.

In this note, we shall show that under a certain assumption on the Alexander polynomial \( \Delta_K(t) \), it is uniquely determined only by \( p \).

The proof will be done by applying number theory to Murasugi’s work \cite{3} on a necessary condition on \( \Delta_K(t) \) for periodic knots \( K \).

2. RESULT

For a knot \( K \) in \( S^3 \), we denote by \( \Delta_K(t) \in \mathbb{Z}[t] \) the Alexander polynomial of \( K \) normalized such that \( \Delta_K(0) \neq 0 \) and the leading coefficient of it is positive.

Our main result is the following:

**Theorem 1.** Let \( p \) be an odd prime number and \( K \subseteq S^3 \) a non-trivial periodic knot of period \( p \). If \( \Delta_K(t) \) is monic and has degree \( p - 1 \), then we have

\[
\Delta_K(t) = \sum_{n=0}^{p-1} (-1)^n t^n = t^{p-1} - t^{p-2} + \cdots - t + 1.
\]

For the case where degree \( p - 1 \) Alexander polynomials of \( p \)-periodic knots have general leading coefficients, we will show the following:

**Theorem 2.** Let \( p \) be an odd prime number and define \( \Pi(p) \) to be the set of all the \( p \)-periodic knots in \( S^3 \) whose Alexander polynomial has degree \( p - 1 \). Also, for a finite set \( S \) of prime numbers which does not
contain $p$, we define the set $\mathcal{D}(p, S) \subseteq \mathbb{Z}[t]$ to be the collection of all the $\Delta_K(t)$’s such that $K \in \Pi(p)$ and the leading coefficient of $\Delta_K(t)$ is prime to the prime numbers outside $S$. Then $\mathcal{D}(p, S)$ is finite and

$$\#\mathcal{D}(p, S) \leq \left(\frac{p+1}{2}\right)^{3 \cdot \frac{3(p-1)}{2} + \#S(Q(\zeta_p))},$$

where $S(Q(\zeta_p))$ denotes the set of all the primes of the $p$-th cyclotomic field $Q(\zeta_p)$ lying over the prime numbers in $S$.

**Remark 1.**

(1) If $K$ is fibred, then $\Delta_K(t)$ is monic.

(2) For a non-trivial knot $K$ of prime period $p$, if the leading coefficient of $\Delta_K(t)$ is prime to $p$, then $\deg \Delta_K(t) \geq p - 1$ (See Davis and Livingston\[2, Cor. 4.2\]).

3. Proof of Theorem 1.

Let $T$ be a transformation of $S^3$ of order $p$ such that $T(K) = K$ and it acts on $K$ fixed point freely, and we denote by $B \subseteq S^3$ the set of the $\Delta^1_K(t)$'s such that $K \in \Pi(p)$ and the leading coefficient of $\Delta_K(t)$ is prime to the prime numbers outside $S$. Then $\mathcal{D}(p, S)$ is finite and

$$\#\mathcal{D}(p, S) \leq \left(\frac{p+1}{2}\right)^{3 \cdot \frac{3(p-1)}{2} + \#S(Q(\zeta_p))},$$

where $S(Q(\zeta_p))$ denotes the set of all the primes of the $p$-th cyclotomic field $Q(\zeta_p)$ lying over the prime numbers in $S$.

Let $T$ be a transformation of $S^3$ of order $p$ such that $T(K) = K$ and it acts on $K$ fixed point freely, and we denote by $B \subseteq S^3$ the set of the fix points of $T$. Then $B$ is the unknot and the quotient space $S^3/T$ is homeomorphic to $S^3$, and we let $\overline{K}$ and $\overline{B}$ be the quotient knots of $K$ and $B$ in $S^3/T$, respectively. We write for $\Delta_{\overline{K}\cup\overline{B}}(t, u) \in \mathbb{Z}[t, u]$ the two-variable Alexander polynomial of the link $\overline{K}\cup\overline{B}$ with $\Delta_{\overline{K}\cup\overline{B}}(t, u) \not\equiv t\mathbb{Z}[t, u] \cup u\mathbb{Z}[t, u]$ (Note that $\Delta_{\overline{K}\cup\overline{B}}(t, u)$ is defined up to $\pm 1$). We put $\lambda$ the linking number of $K$ and $B$.

It follows from Murasugi [3] that

(1) $$\Delta_K(t) = \Delta_{\overline{K}}(t) \prod_{i=1}^{p-1} \Delta_{\overline{K}\cup\overline{B}}(t, \zeta_p^i)$$

with a primitive $p$-th root of unity $\zeta_p$. Also, by using Murasugi’s congruence [3],

$$\Delta_K(t) \equiv \pm t^j \left(\frac{t^\lambda - 1}{t - 1}\right)^{p-1} \Delta_{\overline{K}}(t)^p \pmod{p}$$

for some $j \in \mathbb{Z}$, we derive $\lambda = 2$ and $\Delta_{\overline{K}}(t) = 1$, because $\Delta_{\overline{K}}(t) \mid \Delta_K(t)$ in $\mathbb{Z}[t]$ by (11), $\deg \Delta_K(t) = p - 1$, and the leading coefficient of $\Delta_K(t)$ is prime to $p$. Hence we may assume that

(2) $$\Delta_{\overline{K}\cup\overline{B}}(t, \zeta_p^i) = g(\zeta_p^i)t - h(\zeta_p^i)$$

with some $g(u), h(u) \in \mathbb{Z}[u]$ for $1 \leq i \leq p - 1$ by (11).

Because $\Delta_K(t)$ is monic, we see that $\eta_1 := g(\zeta_p^i)$ is a unit of the ring $\mathbb{Z}[\zeta_p]$ by the relation $\prod_{i=1}^{p-1} g(\zeta_p^i) = 1$, which comes from (11). Also, since
the constant term of $\Delta_K(t)$ is equal to 1, we find that $\eta_2 := h(\zeta_p)$ is a unit of $\mathbb{Z}[\zeta_p]$. Hence if we put $\varepsilon := \eta_1^{-1}\eta_2 \in \mathbb{Z}[\zeta_p][\times]$, then we have

$$
\Delta_K(t) = \prod_{i=1}^{p-1} g(\zeta_p^i) \prod_{i=1}^{p-1} (t - g(\zeta_p^i)^{-1}h(\zeta_p^i)) = \prod_{\sigma \in \text{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q})} (t - \sigma(\varepsilon)).
$$

On the other hand, it follows from the second Torres condition (See [4]) that

$$
\Delta_K(t) = \frac{t^a \zeta_p^b \Delta_K(t, \zeta_p)}{J(t)} \quad \text{for some } a, b \in \mathbb{Z}.
$$

Then we find from (2) that

$$
\varepsilon = \eta_1^{-1} \eta_2 \in \mathbb{Z}[\zeta_p][\times],
$$

and

$$
h(\zeta_p^{-1}) = -\zeta_p^b g(\zeta_p), \quad g(\zeta_p^{-1}) = -\zeta_p^b h(\zeta_p).$$

Therefore, if we denote by $J \in \text{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q})$ the complex conjugation, we have

$$
J(\varepsilon) = J(h(\zeta_p))/J(g(\zeta_p)) = h(\zeta_p^{-1})/g(\zeta_p^{-1}) = g(\zeta_p)/h(\zeta_p) = \varepsilon^{-1}.
$$

We need the following fact from the theory of cyclotomic fields (see [5, Prop.1.5]):

**Lemma 1.** For any $\varepsilon \in \mathbb{Z}[\zeta_p][\times]$, there exist $r \in \mathbb{Z}$ and $\varepsilon_0 \in \mathbb{Z}[\zeta_p + \zeta_p^{-1}][\times]$ such that

$$
\varepsilon = \zeta_p^r \varepsilon_0.
$$

By Lemma 1 and (3), we obtain

$$
\varepsilon^{-1} = J(\varepsilon) = J(\zeta_p)^r J(\varepsilon_0) = \zeta_p^{-r} \varepsilon_0 = \zeta_p^{-2r} \varepsilon,
$$

from which we derive

$$
\varepsilon = \pm \zeta_p^r.
$$

Because $\Delta_K(1) = \pm 1$ and $\Delta_K(-1) \neq 0$, we conclude that $\varepsilon = -\zeta_p^r$ with some $r \in \mathbb{Z}$ prime to $p$, and

$$
\Delta_K(t) = t^{p-1} - t^{p-2} + \cdots - t + 1
$$

by (3). Thus we have proved Theorem 1.

4. PROOF OF THEOREM 2

We will give the following proposition, from which we can easily derive Theorem 2:

**Proposition 1.** For a finite extension field $F/\mathbb{Q}$, a finite set $S$ of prime numbers, and positive integer $m$, we define the set $\mathcal{P}(F, S, m) \subseteq \mathbb{Z}[t]$ to be the collection of $\Delta_K(t)$'s for the knots $K$ in $S^3$ such that the leading coefficient of $\Delta_K(t)$ is prime to the prime numbers outside $S$, the splitting field $\text{Spl}(\Delta_K(t))$ of $\Delta_K(t)$ over $\mathbb{Q}$ is contained in $F$, and
the multiplicity of each zero of \( \Delta_K(t) \) is at most \( m \). Then \( \mathcal{P}(F, S, m) \) is finite and we have
\[
\#\mathcal{P}(F, S, m) \leq (m + 1)^{3\cdot \pi[F : K] + \#(S(F) \cup \infty(F))},
\]
where \( S(F) \) and \( \infty(F) \) denote the set of the primes of \( F \) lying over the prime numbers in \( S \) and that of the archimedean primes of \( F \), respectively.

**Proof.** Let \( \Delta_K(t) \in \mathcal{P}(F, S, m) \). Then we have
\[
\Delta_K(t) = a \prod_{i=1}^{d} (X - \alpha_i)^{m_i} \in \mathbb{Z}[t]
\]
for some \( a \in \mathbb{Z} \) which is prime to the prime numbers outside \( S \), \( 0 \leq d \in \mathbb{Z} \), distinct \( \alpha_i \)'s in \( F \), and \( 1 \leq m_i \leq m \).

Let \( p \not\in S(F) \) be any non-archimedean prime of \( F \) and \( v_p \) a \( p \)-adic valuation of \( F \). Since \( a\alpha_i \) is integral over \( \mathbb{Z} \) and \( v_p(a) = 0 \), we see
\[
v_p(\alpha_i) = v_p(a\alpha_i) \geq 0.
\]
Because the constant term and the leading coefficient of \( \Delta_K(t) \) are coincide, we have
\[
a = \pm \prod_{i=1}^{d} \alpha_i^{m_i},
\]
from which we derive
\[
0 = v_p(a) = \sum_{i=1}^{d} m_i v_p(\alpha_i).
\]
Therefore we find that \( v_p(\alpha_i) = 0 \) for \( 1 \leq i \leq d \), which means that \( \alpha_i \)'s are \( S(F) \)-units of \( F \).

On the other hand, \( a(1 - \alpha_i) = a - a\alpha_i \) is also integral over \( \mathbb{Z} \), we obtain
\[
v_p(1 - \alpha_i) = v_p(a(1 - \alpha_i)) \geq 0.
\]
Also, since \( \Delta_K(1) = \pm 1 \), we have
\[
a \prod_{i=1}^{d} (1 - \alpha_i)^{m_i} = \pm 1,
\]
from which we derive
\[
0 = v_p(\pm 1) = v_p(a) + \sum_{i=1}^{d} m_i v_p(1 - \alpha_i) = \sum_{i=1}^{d} m_i v_p(1 - \alpha_i).
\]
Hence \( v_p(1 - \alpha_i) = 0 \) for \( 1 \leq i \leq d \), which means that \( 1 - \alpha_i \)'s are also \( S(F) \)-units of \( F \).
Now we apply the following result from analytic number theory given by Evertse [1]:

**Lemma 2.** Let $F$ be a finite extension of $\mathbb{Q}$ and $T$ a finite set of non-archimedean primes of $F$. Then the number of solutions $(X, Y)$ of the equation

$$X + Y = 1$$

in the $T$-unit group of $F$ is at most $3 \cdot 7^{[F: \mathbb{Q}]} + \#(T \cup \infty_F)$.

As we have seen in the above, $(\alpha_i, 1 - \alpha_i)$ is a solution in the $S(F)$-unit group of $F$ of the equation $X + Y = 1$. Hence, it follows from Lemma 2 that the number of such $\alpha_i$’s is at most $3 \cdot 7^{[F: \mathbb{Q}]} + \#(T \cup \infty_F)$. Therefore we obtain

$$\#\mathcal{P}(F, S, m) \leq (m + 1)^{3 \cdot 7^{[F: \mathbb{Q}]} + \#(T \cup \infty_F)}.$$

Now we will derive Theorem 2 from Proposition 1. Assume $K \in \mathcal{D}(p, S)$. Then, as the proof of Theorem 1, we find that

$$\Delta_K(t) = \prod_{i=1}^{p-1} (g(\zeta_p)t - h(\zeta_p))$$

for some $g(u), h(u) \in \mathbb{Z}[u]$, since the leading coefficient of $\Delta_K(t)$ is prime to $p$ by $p \not\in S$ and $\deg \Delta_K(t) = p - 1$. Hence $\text{Spl}(\Delta_K(t)) \subseteq \mathbb{Q}(\zeta_p)$ and $\Delta_K(t) \in \mathcal{P}(\mathbb{Q}(\zeta_p), S, p^{-1})$ because $\Delta_K(t)$ has at least two distinct zeros. Therefore, applying Proposition 1, we complete the proof of Theorem 2 by using the facts $[\mathbb{Q}(\zeta_p) : \mathbb{Q}] = p - 1$ and $\#\infty_{\mathbb{Q}(\zeta_p)} = \frac{p-1}{2}$. \hfill \Box

**References**

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