The $b$-functions for prehomogeneous vector spaces of commutative parabolic type and generalized universal Verma modules

Atsushi KAMITA

Department of Mathematics
Osaka City University
Osaka, 558-8585 Japan
E-mail: kamita@sci.osaka-cu.ac.jp

Abstract. We shall give a uniform expression and a uniform calculation for the $b$-functions of prehomogeneous vector spaces of commutative parabolic type, which were previously calculated by case-by-case analysis. Our method is a generalization of Kashiwara’s approach using the universal Verma modules. We shall also give a new proof for the criterion of the irreducibility of the generalized Verma module in terms of $b$-functions due to Suga [9], Gyoja [1], Wachi [11].

0 Introduction

In this paper we deal with the $b$-functions of the invariants on the flag manifolds $G/P$. In the case where $P$ is a Borel subgroup, Kashiwara [4] determined the $b$-functions by using the universal Verma modules. Our method is a generalization of Kashiwara’s approach.

Let $g$ be a simple Lie algebra over the complex number field $C$, and let $G$ be a connected simply-connected simple algebraic group with Lie algebra $g$. Fix a parabolic subalgebra $p$ of $g$. We denote the reductive part of $p$ and the nilpotent part of $p$ by $l$ and $n$ respectively. Let $L$ be the subgroup of $G$ corresponding to $l$. Let $R$ be the symmetric algebra of the commutative Lie algebra $p/[p,p]$. For a Lie algebra $a$ we set $U_R(a) = R \otimes_C U(a)$ where $U(a)$ denotes the enveloping algebra of $a$. The canonical map $p \to R$ induces a one-dimensional $U_R(p)$-module $R_c$. Let $C_\mu$ be the one-dimensional $p$-module with weight $\mu$. Set $R_{c+\mu} = R_c \otimes_C C_\mu$. Then $R_{c+\mu}$ is a one-dimensional $U_R(p)$-module.

For a character $\mu$ of $p$ we regard $\mu$ as a weight of $g$, and let $V(\mu)$ be the irreducible $g$-module with highest weight $\mu$. We assume that the weight $\mu$ of $g$ is dominant integral. We define a $U_R(g)$-module homomorphism

$$\iota: U_R(g) \otimes_{U_R(p)} R_{c+\mu} \to U_R(g) \otimes_{U_R(p)} (R_c \otimes_C V(\mu))$$

by $\iota(1 \otimes 1) = 1 \otimes 1 \otimes v_\mu$, where $v_\mu$ is the highest weight vector of $V(\mu)$. For a $U_R(g)$-module homomorphism $\psi$ from $U_R(g) \otimes_{U_R(p)} (R_c \otimes_C V(\mu))$ to $U_R(g) \otimes_{U_R(p)} R_{c+\mu}$ the composite $\psi \iota$
is the multiplication on $U_R(g) \otimes_{U_R(p)} R_{c+\mu}$ by an element $\xi$ of $R$:

$$U_R(g) \otimes_{U_R(p)} R_{c+\mu} \xrightarrow{\iota} U_R(g) \otimes_{U_R(p)} R_{c+\mu} \xrightarrow{\xi \text{id}} U_R(g) \otimes_{U_R(p)} R_{c+\mu}. \tag{0.1}$$

The set $\Xi_\mu$ consisting of all $\xi \in R$ induced by homomorphisms from $U_R(g) \otimes_{U_R(p)} (R_c \otimes_C V(\mu))$ to $U_R(g) \otimes_{U_R(p)} R_{c+\mu}$ as above is an ideal of $R$. We can construct a particular element $\psi \in \text{Hom}_{U_R(g)}(U_R(g) \otimes_{U_R(p)} (R_c \otimes_C V(\mu)), U_R(g) \otimes_{U_R(p)} R_{c+\mu})$ by considering the irreducible decomposition of $V(\mu)$ as a $p$-module (see Section 3 below), however, the corresponding $\xi_\mu \in \Xi_\mu$ is not a generator of $\Xi_\mu$ in general. Note that Kashiwara [4] gave the generator of $\Xi_\mu$ when $P$ is a Borel subgroup.

Let $\psi \in \text{Hom}_{U_R(g)}(U_R(g) \otimes_{U_R(p)} (R_c \otimes_C V(\mu)), U_R(g) \otimes_{U_R(p)} R_{c+\mu})$ and let $\xi \in \Xi_\mu$ be the corresponding element. Then as in Kashiwara [4] we can define a differential operator $P(\psi)$ on $G$ satisfying

$$P(\psi)f^{\lambda+\mu} = \xi(\lambda)f^\lambda$$

for any character $\lambda$ of $p$ which can be regarded as a dominant integral weight of $g$. Here, $f^\lambda$ denotes the invariant on $G$ corresponding to $\lambda$ (see Section 3 below) and $\xi$ is regarded as a function on $\text{Hom}(p, C)$.

In the rest of Introduction we assume that the nilpotent radical $n$ of $p$ is commutative. Then the pair $(L, n)$ is a prehomogeneous vector space via the adjoint action of $L$. In this case there exists exactly one simple root $\alpha_0$ such that the root space $g_{\alpha_0}$ is in $n$. We denote the fundamental weight corresponding to $\alpha_0$ by $\varpi_0$.

We define an element $\xi_0 \in R$ by

$$\xi_0(\lambda) = \prod_{\eta \in \text{Wt}(\varpi_0) \setminus \{\varpi_0\}} \left( (\lambda + \rho + \varpi_0, \lambda + \rho + \varpi_0) - (\lambda + \rho + \eta, \lambda + \rho + \eta) \right) \quad (\lambda \in C\varpi_0),$$

where $\text{Wt}(\varpi_0)$ is the set of the highest weights of irreducible $l$-submodules of $V(\varpi_0)$, and $\rho$ is the half sum of positive roots of $g$.

**Theorem 0.1.** We have $\xi_0 = \xi_{\varpi_0}$, and the ideal $\Xi_{\varpi_0}$ of $R$ is generated by $\xi_0$.

We denote by $\psi_0$ the homomorphism satisfying $\psi_0 \iota = \xi_0 \text{id}$.

Let $n^-$ be the nilpotent part of the parabolic subalgebra of $g$ opposite to $p$. We can define a constant coefficient differential operator $P'(\psi_0)$ on $n^- \simeq \exp(n^-)$ by

$$(P'(\psi_0)f)|_{\exp(n^-)} = P'(\psi_0)(f|_{\exp(n^-)}).$$

**Theorem 0.2.** If the prehomogeneous vector space $(L, n)$ is regular, then $P'(\psi_0)$ corresponds to the unique irreducible relative invariant of $(L, n)$, and $b(s) = \xi_0(s\varpi_0)$ is its $b$-function.
Moreover, using the commutative diagram (1.1) for $\xi_0$ and $\psi_0$ we give a new proof of
the following criterion of the irreducibility of the generalized Verma module due to Suga [9],
Gyoja [1], Wachi [11]:

$$U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} \mathbb{C}_{s_0 \varpi_0} \text{ is irreducible } \iff \xi_0((s_0 - m) \varpi_0) \neq 0 \text{ for any positive integer } m.$$  

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1 Prehomogeneous Vector Spaces

In this section we recall some basic facts on prehomogeneous vector spaces (see Sato and
Kimura [7]).

**Definition 1.1.**  (i) For a connected algebraic group $G$ over the complex number field $\mathbb{C}$
and a finite dimensional $G$-module $V$, the pair $(G, V)$ is called a prehomogeneous vector
space if there exists a Zariski open orbit in $V$.

(ii) We denote the ring of polynomial functions on $V$ by $\mathbb{C}[V]$. A nonzero element $f \in \mathbb{C}[V]
$ is called a relative invariant of a prehomogeneous vector space $(G, V)$ if there exists a
character $\chi$ of $G$ such that $f(gv) = \chi(g)f(v)$ for any $g \in G$ and $v \in V$.

(iii) A prehomogeneous vector space is called regular if there exists a relative invariant $f
$ such that the Hessian $H_f = \text{det}(\partial^2 f/\partial x_i \partial x_j)$ is not identically zero, where $\{x_i\}$ is a
coordinate system of $V$.

We call algebraically independent relative invariants $f_1, f_2, \ldots, f_l$ basic relative invariants
if for any relative invariant $f$ there exist $c \in \mathbb{C}$ and $m_i \in \mathbb{Z}$ such that

$$f = cf_1^{m_1} \cdots f_l^{m_l}.$$  

Assume that $(G, V)$ is a prehomogeneous vector space such that $G$ is reductive. Then
the dual space $V^*$ of $V$ is also a prehomogeneous vector space by $\langle gv^*, v \rangle = \langle v^*, g^{-1}v \rangle$, where
$\langle , \rangle$ is the natural pairing of $V^*$ and $V$. If $f \in \mathbb{C}[V]$ is a relative invariant of $(G, V)$
with character $\chi$, then there exists a relative invariant $f^*$ of $(G, V^*)$ with character $\chi^{-1}$. For
$h \in \mathbb{C}[V^*]$ we define a constant coefficient differential operator $h(\partial)$ by

$$h(\partial) \exp(v^*, v) = h(v^*) \exp(v^*v),$$  

where $v \in V$ and $v^* \in V^*$. Then there exists a polynomial $b(s) \in \mathbb{C}[s]$ such that

$$f^*(\partial)f^{s+1} = b(s)f^s.$$  

This polynomial is called the $b$-function of $f$. It is known that $\deg b = \deg f$ (see [3]).

2 Generalized Universal Verma Modules

Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$ with Cartan subalgebra $\mathfrak{h}$. Let $\Delta \subset \mathfrak{h}^*$ be the root
system and $W \subset \text{GL}(\mathfrak{h})$ the Weyl group. For $\alpha \in \Delta$ we denote the corresponding root space
by $\mathfrak{g}_\alpha$. We denote the set of positive roots by $\Delta^+$ and the set of simple roots by $\{\alpha_i\}_{i \in I_0}$, where $I_0$ is an index set. Let $\rho$ be the half sum of positive roots of $\mathfrak{g}$. We set

$$n^\pm = \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_\pm \alpha, \quad b^\pm = \mathfrak{h} \oplus n^\pm.$$  

For $i \in I_0$ let $h_i \in \mathfrak{h}$ be the simple coroot and $\varpi_i \in \mathfrak{h}^*$ the fundamental weight corresponding to $i$. We denote the longest element of $W$ by $w_0$. Let $(\ , \ )$ be the $W$-invariant nondegenerate symmetric bilinear form on $\mathfrak{h}^*$. We denote the irreducible $\mathfrak{g}$-module with highest weight $\mu \in \sum_{i \in I_0} \mathbb{Z}_{\geq 0} \varpi_i$ by $V(\mu)$ and its highest weight vector by $v_\mu$. For a Lie algebra $\mathfrak{a}$ we denote the enveloping algebra of $\mathfrak{a}$ by $U(\mathfrak{a})$.

For a subset $I \subset I_0$ we set

$$\Delta_I = \Delta \cap \sum_{i \in I} \mathbb{Z} \alpha_i, \quad l_I = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta_I} \mathfrak{g}_\alpha,$$

$$n_I^\pm = \bigoplus_{\alpha \in \Delta^+ \setminus \Delta_I} \mathfrak{g}_\pm \alpha, \quad p_I^\pm = l_I \oplus n_I^\pm,$$

$$\mathfrak{h}_I = \mathfrak{h} / \sum_{i \in I} \mathbb{C} h_i, \quad \mathfrak{h}_I^* = \sum_{i \in I_0 \setminus I} \mathbb{C} \varpi_i.$$

Let $W_I$ be the subgroup of $W$ generated by the simple reflections corresponding to $i \in I$. We denote the longest element of $W_I$ by $w_I$. Let $\mathfrak{h}_{I, +}^*$ be the set of dominant integral weights in $\mathfrak{h}_I^*$. For $\mu \in \mathfrak{h}_I^*$ we define a one-dimensional $U(p_I^+)$-module $C_{I, \mu}$ by

$$C_{I, \mu} = U(p_I^+) / (U(p_I^+) n^+ + \sum_{h \in \mathfrak{h}} U(p_I^+) (h - \mu(h)) + U(p_I^+) (n^- \cap l_I)).$$

We denote the canonical generator of $C_{I, \mu}$ by $1_{\mu}$. Set $M_I(\mu) = U(\mathfrak{g}) \otimes_{U(p_I^+)} C_{I, \mu}$. We denote the irreducible $p_I^+$-module with highest weight $\mu \in \sum_{i \in I} \mathbb{Z}_{\geq 0} \varpi_i + \sum_{j \in I} \mathbb{Z} \varpi_j$ by $W(\mu)$.

Let $G$ be a connected simply-connected simple algebraic group with Lie algebra $\mathfrak{g}$. We denote the subgroups of $G$ corresponding to $\mathfrak{h}, \mathfrak{h}_I^+, l_I, n_I^+$ by $T, B_I^+, L_I, N_I^+$ respectively.

Let $R_l$ be the symmetric algebraic of $h_I$, and define a linear map $c : \mathfrak{h} \to R_l$ as the composite of the natural projection from $\mathfrak{h}$ to $h_I$ and the natural injection from $h_I$ to $R_l$. Set $U_{R_l}(\mathfrak{a}) = R_l \otimes_{\mathbb{C}} U(\mathfrak{a})$ for a Lie algebra $\mathfrak{a}$.

We set for $\mu \in \mathfrak{h}_I^*$

$$R_{l, c+\mu} = U_{R_l}(p_I^+) / (U_{R_l}(p_I^+) n^+ + \sum_{h \in \mathfrak{h}} U_{R_l}(p_I^+) (h - c(h) - \mu(h)) + U_{R_l}(p_I^+) (n^- \cap l_I)).$$

We denote the canonical generator of $R_{l, c+\mu}$ by $1_{c+\mu}$.

**Definition 2.1.** For $\mu \in \mathfrak{h}_I^*$ we call $M_{R_l}(c + \mu) = U_{R_l}(\mathfrak{g}) \otimes_{U_{R_l}(p_I^+)} R_{l, c+\mu}$ a generalized universal Verma module.
Note that $M_{R_I}(c)$ is the universal Verma module in Kashiwara [4]. For $\lambda \in \mathfrak{h}_I^*$ we regard $\mathbb{C}$ as an $R_I$-module by $c(h_i)1 = \lambda(h_i)$. Then we have

$$\mathbb{C} \otimes_{R_I} M_{R_I}(c + \mu) = M_I(\lambda + \mu).$$

The next lemma is obvious.

**Lemma 2.2.** End$_{U_{R_I}(\mathfrak{g})}(M_{R_I}(c + \mu)) = R_I$.

For $\mu \in \mathfrak{h}_I^*$ we define a $U_{R_I}(\mathfrak{g})$-module homomorphism

$$\iota_\mu : M_{R_I}(c + \mu) \longrightarrow U_{R_I}(\mathfrak{g}) \otimes_{U_{R_I}(\mathfrak{g})} (R_{I,c} \otimes \mathbb{C} V(\mu))$$

by $\iota_\mu(1 \otimes 1_{c+\mu}) = 1 \otimes 1_c \otimes v_\mu$. We denote by $\Xi_\mu$ the ideal of $R_I$ consisting of $\xi$ such that there exists a homomorphism $\psi \in \text{Hom}_{U_{R_I}(\mathfrak{g})}(U_{R_I}(\mathfrak{g}) \otimes_{U_{R_I}(\mathfrak{g})} (R_{I,c} \otimes \mathbb{C} V(\mu)), M_{R_I}(c + \mu))$ satisfying $\psi v_\mu = \xi$ id. Let us give a particular element $\xi_\mu$ of $\Xi_\mu$ for $\mu \in \mathfrak{h}_I^{*+}$.

**Lemma 2.3.** For $\mu_1, \mu_2 \in \sum_{i \in I} \mathbb{Z}_{\geq 0} \varpi_i + \sum_{j \notin I} \mathbb{Z} \varpi_j$ we define a function $p_{\mu_1, \mu_2}$ on $\mathfrak{h}_I^*$ by

$$p_{\mu_1, \mu_2}(\lambda) = (\lambda + \rho + \mu_1, \lambda + \rho + \mu_1) - (\lambda + \rho + \mu_2, \lambda + \rho + \mu_2),$$

which is regarded as an element of $R_I$. Then we have

$$\text{Ext}^1_{U_{R_I}(\mathfrak{g})}(U_{R_I}(\mathfrak{g}) \otimes_{U_{R_I}(\mathfrak{g})} (R_{I,c} \otimes \mathbb{C} W(\mu_1)), U_{R_I}(\mathfrak{g}) \otimes_{U_{R_I}(\mathfrak{g})} (R_{I,c} \otimes \mathbb{C} W(\mu_2))) = 0.$$

**Proof.** The action of the Casimir element of $U(\mathfrak{g})$ on $U_{R_I}(\mathfrak{g}) \otimes_{U_{R_I}(\mathfrak{g})} (R_{I,c} \otimes \mathbb{C} W(\mu))$ is given by the multiplication by $p_\mu \in R_I$, where $p_\mu(\lambda) = (\lambda + \rho + \mu, \lambda + \rho + \mu) - (\rho, \rho)$ for $\lambda \in \mathfrak{h}_I^*$. Using this action, we can easily check that $p_{\mu_1, \mu_2} = p_{\mu_1} - p_{\mu_2}$ is an annihilator.

**Lemma 2.4.** For any $\mu \in \mathfrak{h}_I^{*+}$ there exist $\mathfrak{p}_I^+$-submodules $F_1, F_2, \ldots, F_r$ of $V(\mu)$ and weights $\eta_1, \eta_2, \ldots, \eta_{r-1} \in \sum_{i \in I} \mathbb{Z}_{\geq 0} \varpi_i + \sum_{i \notin I_0 \setminus I} \mathbb{Z} \varpi_i$ satisfying the following conditions:

(i) $\mathcal{C} v_\mu = F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_r = V(\mu)$.

(ii) $F_{i+1}/F_i \cong W(\eta_i)^{\oplus N_i}$ for some positive integer $N_i$.

(iii) $\eta_i \neq \eta_j$ for $i \neq j$.

**Proof.** For a non-negative integer $m$ we set

$$P(m) = \{ \lambda \in \mathfrak{h}^* \mid \mu - \lambda = \sum_{i \in I_0} m_i \alpha_i \text{ and } \sum_{i \notin I} m_i = m \} \text{ and } V_m = \bigoplus_{\lambda \in P(m)} V(\mu)_\lambda,$$

where $V(\mu)_\lambda$ is the weight space of $V(\mu)$ with weight $\lambda$. Then $V_m$ is an $I$-module, and we have the irreducible decomposition

$$V_m = \tilde{W}^{\eta_{m,1}}_{m_1} \oplus \cdots \oplus \tilde{W}^{\eta_{m,t_m}}_{m, t_m} \oplus \mathbb{C}^{m_0} \oplus \cdots \oplus \mathbb{C}^{m_{t_0}}.$$
where $\bar{W}(\eta)$ is the irreducible $\mathfrak{g}_I$-module with highest weight $\eta$, and $\eta_{m,i} \neq \eta_{m,j}$ for $i \neq j$. For $1 \leq i \leq t_m$ we define a $p_{x_I}^+$-submodule $F_{m,i}$ of $V(\mu)$ by

$$F_{m,i} = V_0 \oplus \cdots \oplus V_{m-1} \oplus \bar{W}(\eta_{m,1}) \oplus \cdots \oplus \bar{W}(\eta_{m,i}) \oplus N_{m,i}.$$  

Then we have the sequence

$$\mathbb{C}v_\mu = F_{0,1} \subset F_{1,1} \subset F_{2,1} \subset F_{m,1} \subset F_{m,2} \subset \cdots \subset F_{m,t_m} \subset \cdots \subset F_{r,t_r} = V(\mu).$$

It is clear that the above sequence satisfies the conditions (ii) and (iii). \hfill $\square$

For $\mu \in \mathfrak{h}_{I,+}^*$, we fix the sequence $\{F_1, F_2, \ldots, F_r\}$ of $p_{x_I}^+$-submodules of $V(\mu)$ satisfying the conditions of Lemma 2.4, and set $\xi_\mu = \prod_{i=1}^{r-1} p_{\mu, \eta_i} \in R_I$.

**Theorem 2.5.** For $\mu \in \mathfrak{h}_{I,+}^*$ we have $\xi_\mu \in \Xi_\mu$.

**Proof.** It is clear that $U_{R_I}(\mathfrak{g}) \otimes_{U_{R_I}(p_{x_I}^+)} (R_{I,c} \otimes \mathbb{C} F_1) \simeq M_{R_I}(c + \mu)$. Let $\iota_j$ be the canonical injection from $U_{R_I}(\mathfrak{g}) \otimes_{U_{R_I}(p_{x_I}^+)} (R_{I,c} \otimes \mathbb{C} F_j)$ into $U_{R_I}(\mathfrak{g}) \otimes_{U_{R_I}(p_{x_I}^+)} (R_{I,c} \otimes \mathbb{C} F_{j+1})$. We show that there exists a commutative diagram

$$U_{R_I}(\mathfrak{g}) \otimes_{U_{R_I}(p_{x_I}^+)} (R_{I,c} \otimes \mathbb{C} F_j) \xrightarrow{\iota_j} M_{R_I}(c + \mu)$$

by the induction on $j$. Assume that there exists a commutative diagram (2.1) for $j \geq 1$. From the exact sequence

$$0 \longrightarrow U_{R_I}(\mathfrak{g}) \otimes_{U_{R_I}(p_{x_I}^+)} (R_{I,c} \otimes \mathbb{C} F_j) \xrightarrow{\iota_j} U_{R_I}(\mathfrak{g}) \otimes_{U_{R_I}(p_{x_I}^+)} (R_{I,c} \otimes \mathbb{C} F_{j+1}) \longrightarrow U_{R_I}(\mathfrak{g}) \otimes_{U_{R_I}(p_{x_I}^+)} (R_{I,c} \otimes \mathbb{C} F_{j+1}/F_j) \longrightarrow 0,$$

we have a long exact sequence

$$0 \longrightarrow \text{Hom}_{U_{R_I}(\mathfrak{g})}(U_{R_I}(\mathfrak{g}) \otimes_{U_{R_I}(p_{x_I}^+)} (R_{I,c} \otimes \mathbb{C} F_{j+1}/F_j), M_{R_I}(c + \mu)) \longrightarrow \text{Hom}_{U_{R_I}(\mathfrak{g})}(U_{R_I}(\mathfrak{g}) \otimes_{U_{R_I}(p_{x_I}^+)} (R_{I,c} \otimes \mathbb{C} F_{j+1}), M_{R_I}(c + \mu)) \longrightarrow \text{Hom}_{U_{R_I}(\mathfrak{g})}(U_{R_I}(\mathfrak{g}) \otimes_{U_{R_I}(p_{x_I}^+)} (R_{I,c} \otimes \mathbb{C} F_j), M_{R_I}(c + \mu)) \longrightarrow \text{Ext}^1_{U_{R_I}(\mathfrak{g})}(U_{R_I}(\mathfrak{g}) \otimes_{U_{R_I}(p_{x_I}^+)} (R_{I,c} \otimes \mathbb{C} F_{j+1}/F_j), M_{R_I}(c + \mu)) \longrightarrow \cdots.$$  

Since $F_{j+1}/F_j \simeq W(\eta_{j+1}) \oplus N_j$, we have $\delta(p_{\mu, \eta_j} \psi_j) = p_{\mu, \eta_j} \delta(\psi_j) = 0$ by Lemma 2.3. Hence there exists an element $\psi_{j+1} \in \text{Hom}_{U_{R_I}(\mathfrak{g})}(U_{R_I}(\mathfrak{g}) \otimes_{U_{R_I}(p_{x_I}^+)} (R_{I,c} \otimes \mathbb{C} F_{j+1}), M_{R_I}(c + \mu))$ such that
\[ \psi_{j+1} = p_{\mu,n_j} \psi_j. \] Hence we have the commutative diagram

\[
\begin{array}{ccc}
U_{R_I}(\mathfrak{g}) \otimes_{U_{R_I}(\mathfrak{p}_I^+)} (R_{I,c} \otimes \mathbb{C} F_1) & \longrightarrow & M_{R_I}(c + \mu) \\
\downarrow \psi_j & & \downarrow \Pi_{i=1}^{\ell} p_{\mu,n_i} \\
U_{R_I}(\mathfrak{g}) \otimes_{U_{R_I}(\mathfrak{p}_I^+)} (R_{I,c} \otimes \mathbb{C} F_j) & \longrightarrow & M_{R_I}(c + \mu) \\
\downarrow \psi_j & & \downarrow \psi_{j+1} \\
U_{R_I}(\mathfrak{g}) \otimes_{U_{R_I}(\mathfrak{p}_I^+)} (R_{I,c} \otimes \mathbb{C} F_{j+1}) & \longrightarrow & M_{R_I}(c + \mu).
\end{array}
\]

In particular \( \psi_{r \mu} = \xi_\mu \). Therefore \( \xi_\mu \in \Xi_\mu \). □

Let \( \psi_\mu \) be a homomorphism from \( U_{R_I}(\mathfrak{g}) \otimes_{U_{R_I}(\mathfrak{p}_I^+)} (R_{I,c} \otimes \mathbb{C} V(\mu)) \) to \( M_{R_I}(c + \mu) \) satisfying \( \psi_{r \mu} = \xi_\mu \). Note that \( \psi_\mu \) is non-zero since \( \xi_\mu \neq 0 \).

**Remark 2.6.** (i) In general \( \xi_\mu \) is not a generator of the ideal \( \Xi_\mu \). For example let \( \mathfrak{g} \) be a simple Lie algebra of type \( G_2 \). We take the simple roots \( \alpha_1 \) and \( \alpha_2 \) such that \( \alpha_1 \) is short.

If \( I = \{2\} \) and \( \mu = \omega_1 \), then we have \( \xi_\mu = (c(h_1) + 1)(c(h_1) + 2)(c(h_1) + 3)(2c(h_1) + 5) \) up to constant multiple. But \( (c(h_1) + 1)(2c(h_1) + 5) \in \Xi_\mu \).

(ii) For \( I = \emptyset \) it is shown in Kashiwara [4] that \( \Xi_\mu \) is generated by

\[
\prod_{\alpha \in \Delta^+} \prod_{j=0}^{\mu(h_\alpha) - 1} (c(h_\alpha) + \rho(h_\alpha) + j),
\]

where \( h_\alpha \) is the coroot corresponding to \( \alpha \).

## 3 Semi-invariants

Let \( \lambda \) be a dominant integral weight. We regard the dual space \( V(\lambda)^* \) as a left \( \mathfrak{g} \)-module via \( \langle xv^*, v \rangle = \langle v^*, -xv \rangle \) for \( x \in \mathfrak{g}, v^* \in V(\lambda)^* \) and \( v \in V(\lambda) \). We denote the lowest weight vector of \( V(\lambda)^* \) by \( v^*_\lambda \). We normalize \( v^*_\lambda \) by \( \langle v^*_\lambda, v_\lambda \rangle = 1 \).

**Definition 3.1.** We define a regular function \( f^\lambda \) on \( G \) by \( f^\lambda(g) = \langle v^*_\lambda, gv_\lambda \rangle \).

For \( b^\pm \in B^\pm \) and \( g \in G \) we have

\[
f^\lambda(b^- gb^+) = \lambda^- (b^-) \lambda^+(b^+) f^\lambda(g),
\]

where \( \lambda^\pm \) is the character of \( B^\pm \) corresponding to \( \lambda \). This function \( f^\lambda \) is called \( B^- \times B^+ \)-semi-invariant. Note that \( f^{\lambda_1 + \lambda_2} = f^{\lambda_1} f^{\lambda_2} \).

Let \( \mu \in \mathfrak{h}^*_+ \). We take a basis \( \{v_{\mu,j}\}_{0 \leq j \leq n} \) of \( V(\mu) \) consisting of weight vectors such that \( v_{\mu,0} = v_{\mu} \) is the highest weight vector and \( v_{\mu,n} \) is the lowest. We denote the dual basis of \( V(\mu)^* \) by \( \{v^*_\mu,j\} \). For a \( U_{R_I}(\mathfrak{g}) \)-module homomorphism

\[
\psi : U_{R_I}(\mathfrak{g}) \otimes_{U_{R_I}(\mathfrak{p}_I^+)} (R_{I,c} \otimes \mathbb{C} V(\mu)) \longrightarrow M_{R_I}(c + \mu)
\]

there exist a homomorphism \( \phi : V(\mu) \longrightarrow V(\lambda) \) and \( \lambda_1 \in \Lambda^+ \) such that

\[
\psi(g) = f^{\lambda_1}(g) \phi \bigg|_{v_{\mu} \mapsto v^*_\lambda}.
\]
we define elements $Y'_j \in U_{R_I}(n^-_I)$ for $0 \leq j \leq n$ by
\[
\psi(1 \otimes 1_c \otimes v_{\mu,j}) = Y'_j \otimes 1_{c+\mu},
\]
and define an element $\xi \in \Xi_\mu$ by $\xi = \psi_{\mu}$. Note that $Y'_0 = \xi$. Let $\pi : R_I \to U(\sum_{i \notin I} C h_i)$ be the algebra isomorphism defined by $\pi(c(h_i)) = h_i - \mu(h_i)$ for $i \notin I$. Set $\pi(\sum a_j \otimes y_j) = \sum_j y_j \pi(a_j)$ for $a_j \in R_I$ and $y_j \in U(n^-_I)$. Clearly we have $y \otimes 1_{c+\mu} = \pi(y) \otimes 1_{c+\mu} \in M_{R_I}(c+\mu)$ $(y \in U_{R_I}(n^-_I))$. We set $Y_j = \pi(Y'_j)$. We define differential operators $P_\mu(\psi)$ and $\tilde{P}_\mu(\psi)$ on $G$ by
\[
(P_\mu(\psi) \varphi)(g) = \sum_{j=0}^{n} \langle gv^*_{\mu,j}, v_{\mu,0} \rangle (R(Y_j)\varphi)(g),
\]
\[
(\tilde{P}_\mu(\psi) \varphi)(g) = \sum_{j=0}^{n} \langle gv^*_{\mu,j}, v_{\mu,n} \rangle (R(Y_j)\varphi)(g),
\]
where $R(y)$ $(y \in U(\mathfrak{g}))$ is the left invariant differential operator induced by the right action of $G$ on itself. Then we have the following theorem.

**Theorem 3.2.** Let $\mu \in \mathfrak{h}_{I,+}^*$ and $\psi \in \text{Hom}_{U_{R_I}(\mathfrak{g})}(U_{R_I}(\mathfrak{g}) \otimes U_{R_I}(\mathfrak{g}^*_{\mu,n}), M_{R_I}(c+\mu))$. Then we have
\[
P_\mu(\psi)f^{\lambda+\mu} = \xi(\lambda)f^\lambda
\]
for any $\lambda \in \mathfrak{h}_{I,+}^*$. Here $\xi$ is the element of $\Xi_\mu$ defined by $\xi = \psi_{\mu}$.

We omit the proof since it is almost identical to the one for [4, Theorem 2.1].

For a dominant integral weight $\lambda$ we define a function $\tilde{f}^\lambda$ on $G$ by
\[
\tilde{f}^\lambda(g) = \langle v^*_{w_0\lambda}, gv_{\lambda} \rangle,
\]
where $v^*_{w_0\lambda}$ is the highest weight vector which is normalized by $\langle v^*_{w_0\lambda}, \tilde{w}_0v_{\lambda} \rangle = 1$ and $\tilde{w}_0 \in N_G(T)$ is a representative element of $w_0 \in W = N_G(T)/T$. Since $\tilde{f}^\lambda(\tilde{w}_0g) = f^\lambda(g)$, we obtain the following lemma.

**Lemma 3.3.** Let $\lambda, \mu \in \mathfrak{h}_{I,+}^*$. For any $g \in G$ we have $(\tilde{P}_\mu(\psi)\tilde{f}^\lambda)(\tilde{w}_0g) = (P_\mu(\psi)f^\lambda)(g)$.

By Theorem 3.2 we have the following corollary.

**Corollary 3.4.** Let $\mu \in \mathfrak{h}_{I,+}^*$. We have
\[
\tilde{P}_\mu(\psi)\tilde{f}^{\lambda+\mu} = \xi(\lambda)\tilde{f}^\lambda
\]
for any $\lambda \in \mathfrak{h}_{I,+}^*$. Here $\xi$ is the element of $\Xi_\mu$ defined by $\xi = \psi_{\mu}$.
4 Commutative Parabolic Type

In the remainder of this paper we assume that

$$I = I_0 \setminus \{i_0\}$$

and that the highest root $\theta$ of $\mathfrak{g}$ is in $\alpha_{i_0} + \sum_{i \neq i_0} \mathbb{Z}_{\geq 0} \alpha_i$. Then it is known that $[n_I^+, n_I^+] = \{0\}$ and the pairs $(L_I, n_I^+)$ are prehomogeneous vector spaces via the adjoint action, which are called of commutative parabolic type. The all pairs $(\mathfrak{g}, i_0)$ of commutative parabolic type are given by the Dynkin diagrams of Fig. 1. Here the white vertex corresponds to $i_0$.

The pairs $(\mathfrak{g}, i_0)$ such that the corresponding prehomogeneous vector spaces are regular are as follows: $(A_{2n-1}, n), (B_n, 1), (C_n, n), (D_n, 1), (D_{2n}, 2n)$ and $(E_7, 7)$. Then it is seen that $w_0 \alpha_{i_0} = -\alpha_{i_0}$.

Since $n_I^-$ is identified with the dual space of $n_I^+$ via the Killing form, the symmetric algebra $S(n_I^-)$ is isomorphic to $\mathbb{C}[n_I^+]$. By the commutativity of $n_I^-$ we have $S(n_I^-) = U(n_I^-)$. Hence $\mathbb{C}[n_I^-]$ is identified with $U(n_I^-)$. 

Fig. 1: Commutative Parabolic Type
Set $\gamma_1 = \alpha_{i_0}$. For $i \geq 1$ we take the root $\gamma_{i+1}$ as a minimal element in
\[ \Gamma_i = \{ \alpha \in \Delta^+ \setminus \Delta_I \mid \alpha + \gamma_j \notin \Delta \text{ and } \alpha - \gamma_j \notin \Delta \cup \{0\} \text{ for all } j \leq i \}. \]

Let $r = r(g)$ be the index such that $\Gamma_{r(g)-1} \neq \emptyset$ and $\Gamma_{r(g)} = \emptyset$. Note that $(\gamma_i, \gamma_j) = 0$ for $i \neq j$. It is known that all $\gamma_i$ have the same length (see Moore [5]). Set $\mathfrak{h}^- = \sum_{i=1}^{r} \mathfrak{C} h_{\gamma_i}$, where $h_{\gamma_i}$ is the coroot corresponding to $\gamma_i$. For $1 \leq i \leq r$ we set $\lambda_i = -(\gamma_1 + \cdots + \gamma_i)$. Then we have the following lemmas.

**Lemma 4.1 (Moore [5]).** For $\beta \in \Delta^+ \cap \Delta_I$ the restriction $\beta|_{\mathfrak{h}^-}$ is as follows.

(i) $\beta|_{\mathfrak{h}^-} = 0$. Then $\beta \pm \gamma_i \notin \Delta$ for any $i$.

(ii) $\beta|_{\mathfrak{h}^-} = \frac{\gamma_i}{2}|_{\mathfrak{h}^-}$. Then $\beta \pm \gamma_i \notin \Delta$ for any $i \neq j$.

(iii) $\beta|_{\mathfrak{h}^-} = \frac{\gamma_i - \gamma_k}{2} |_{\mathfrak{h}^-}$ ($j > k$). Then $\beta \pm \gamma_i \notin \Delta$ for any $i \neq j, k$ and $\beta + \gamma_j \notin \Delta$.

Set $D = \{\alpha_i \mid i \in I\}$. For a subset $S$ of $\Delta$, $S(\mathfrak{h}^-)$ is defined by
\[ S(\mathfrak{h}^-) = \{ \beta \in \sum_{i=1}^{r} \mathfrak{Q}_{\gamma_i} \mid \beta|_{\mathfrak{h}^-} = \alpha_i|_{\mathfrak{h}^-} \text{ for } \alpha \in S \}. \]

**Lemma 4.2 (Moore [5], Wachi [11]).** (i) If $(L_I, n^+_I)$ is regular, then we have
\[
\begin{align*}
D(\mathfrak{h}^-) &= \left\{ \frac{1}{2}(\gamma_i+1 - \gamma_i) \mid 1 \leq i \leq r - 1 \right\} \cup \{0\} \\
\Delta_I \cap \Delta^+(\mathfrak{h}^-) &= \left\{ \frac{1}{2}(\gamma_j - \gamma_i) \mid 1 \leq i \leq j \leq r \right\} \\
\Delta^+ \setminus \Delta_I(\mathfrak{h}^-) &= \left\{ \frac{1}{2}(\gamma_j + \gamma_i) \mid 1 \leq i \leq j \leq r \right\}.
\end{align*}
\]

(ii) If $(L_I, n^+_I)$ is not regular, then we have
\[
\begin{align*}
D(\mathfrak{h}^-) &= \left\{ \frac{1}{2}(\gamma_i+1 - \gamma_i) \mid 1 \leq i \leq r - 1 \right\} \cup \left\{ -\frac{1}{2}\gamma_r \right\} \cup \{0\} \\
\Delta_I \cap \Delta^+(\mathfrak{h}^-) &= \left\{ \frac{1}{2}(\gamma_j - \gamma_i) \mid 1 \leq i \leq j \leq r \right\} \cup \left\{ -\frac{1}{2}\gamma_i \mid 1 \leq i \leq r \right\} \\
\Delta^+ \setminus \Delta_I(\mathfrak{h}^-) &= \left\{ \frac{1}{2}(\gamma_j + \gamma_i) \mid 1 \leq i \leq j \leq r \right\} \cup \left\{ \frac{1}{2}\gamma_i \mid 1 \leq i \leq r \right\}.
\end{align*}
\]

**Lemma 4.3.** If $(L_I, n^+_I)$ is not regular, then $j \in I$ such that $\alpha_j|_{\mathfrak{h}^-} = -\frac{\gamma_r}{2}|_{\mathfrak{h}^-}$ is unique.

**Proof.** Assume that for $j_1 \neq j_2$ we have $\alpha_{j_1} = \alpha_{j_2} = -\frac{\gamma_r}{2}$ on $\mathfrak{h}^-$. Let $h \in \mathfrak{h}^-$. Since $(\alpha_{j_1} + \alpha_{j_2})(h) = -\gamma_r(h)$, $\alpha_{j_1} + \alpha_{j_2} \notin \Delta$ by Lemma 4.2. So we have $(\alpha_{j_1}, \alpha_{j_2}) = 0$. Since $(\gamma_r, \alpha_{j_1}) > 0$ and $(\gamma_r + \alpha_{j_1}, \alpha_{j_2}) > 0$, $\beta = \gamma_r + \alpha_{j_1} + \alpha_{j_2} \in \Delta$. In particular $\beta \in \Delta^+ \setminus \Delta_I$. For any $1 \leq k \leq r$, $(\beta - \gamma_k)(h) = -\gamma_k(h)$, hence $\beta - \gamma_k \in \sum_{i \in I} \mathbb{Z}_{\geq 0}\alpha_i$ is not a root. Clearly $\beta + \gamma_k \notin \Delta$. Therefore $\beta \in \Gamma_{r+1}$. But by definition $\Gamma_{r+1} = \emptyset$. Thus $j_1 = j_2$. \qed
Lemma 4.4. $w_I \lambda_r = w_0 \omega_{i_0} - \omega_{i_0}$.

Proof. Assume that $(L_I, n^+_I)$ is regular. Then we have $w_I(w_0 \omega_{i_0} - \omega_{i_0}) = -2w_I \omega_{i_0} = -2 \omega_{i_0}$. By using Lemma 4.2 we can check $\lambda_r(h_i) = -2\delta_{i,i_0}$ easily (see Wachi [11]). Hence $\lambda_r = -2 \omega_{i_0}$, and the statement holds. Next we assume that $(L_I, n^+_I)$ is not regular. Then there exists an index $j_0 \in I$ such that $\alpha_{j_0} = -\frac{\gamma_r}{2}$ as functions on $\mathfrak{h}^-$. By Lemma 4.3 this index $j_0$ is unique. Similarly to the regular case we have $\lambda_r = \omega_{j_0} - 2 \omega_{i_0}$. Let $j'_0 \in I$ such that $w_I w_0 \alpha_{i_0} = \alpha_{j'_0}$. Then we have $w_I(w_0 \omega_{i_0} - \omega_{i_0}) = \omega_{j'_0} - 2 \omega_{i_0}$, and we can check $\gamma_r + \alpha_{j'_0} \in \Delta$ by the direct calculation. Hence $\alpha_{j'_0} = -\frac{\gamma_r}{2}$ on $\mathfrak{h}^-$ from Lemma 4.2. Therefore we have $j'_0 = j_0$.

The following fact is known (see [2], [8], [10]).

Lemma 4.5. As an $\text{ad}(I)$-module, $U(n^-_I)$ is multiplicity free, and

$$U(n^-_I) = \bigoplus_{\mu \in \sum_{i=1}^r Z \geq 0 \lambda_i} I(\mu),$$

where $I(\mu)$ is an irreducible $I$-submodule of $U(n^-_I)$ with highest weight $\mu$.

Let $f_i \in U(n^-_I)$ be the highest weight vector of $I(\lambda_i)$. Since $U(n^-_I)$ is naturally identified with the symmetric algebra $S(n^-_I)$, we can determine the degree of $f \in U(n^-_I)$. If $f \in U(n^-_I)$ is a weight vector with weight $\mu = -d \alpha_{i_0} + \sum_{i \in I} Z \leq 0 \alpha_i$, then $f$ is homogeneous and $\deg f = d$. In particular $\deg f_i = i$.

For $\mu \in \mathfrak{h}^*_{I,+}$ we take the lowest weight vector $v_{w_0 \mu}$ of $V(\mu)$. Then the $U_R^I(\mathfrak{g})$-module $U_R^I(\mathfrak{g}) \otimes_{U_{R^I}(\mathfrak{p}^+_I)} (R_{I,c} \otimes_{\mathbb{C}} V(\mu))$ is generated by $1 \otimes 1_c \otimes v_{w_0 \mu}$. There exists a non-zero element $u_{\mu} \in U_R^I(n^-_I)$ such that $\psi(1 \otimes 1_c \otimes v_{w_0 \mu}) = u_{\mu} \otimes 1_{c+\mu}$, where $\psi$ is a $U_R^I(\mathfrak{g})$-module homomorphism defined in Section 2. Since $y(1 \otimes 1_c \otimes v_{w_0 \mu}) = 0$ for any $y \in I_I \cap n^-$, $u_{\mu} \in U_R^I(n^-_I)$ is a lowest weight vector with weight $w_0 \mu - \mu$ as an $\text{ad}(I)$-module. By Lemma 4.3 such a lowest weight vector is unique up to constant multiple. Therefore $u_{\mu} = a_{\mu} u_{\mu}^0$ where $a_{\mu} \in R_I \setminus \{0\}$ and $u_{\mu}^0 \in U(n^-_I)$ is the unique lowest weight vector with weight $w_0 \mu - \mu$.

If $x(1 \otimes 1_c \otimes v_{w_0 \mu}) = 0$ for $x \in U_R^I(\mathfrak{g})$, then we have $x u_{\mu}^0 \otimes 1_{c+\mu} = 0$ since $a_{\mu} \neq 0$. Hence we can define a $U_R^I(\mathfrak{g})$-module homomorphism $\psi^0_{\mu}$ from $U_R^I(\mathfrak{g}) \otimes_{U_{R^I}(\mathfrak{p}^+_I)} (R_{I,c} \otimes_{\mathbb{C}} V(\mu))$ to $M_{R^I}(c + \mu)$ by

$$\psi^0_{\mu}(x(1 \otimes 1_c \otimes v_{w_0 \mu})) = x u_{\mu}^0 \otimes 1_{c+\mu}$$

for any $x \in U_R^I(\mathfrak{g})$. We set $\xi_{\mu}^0 = \psi^0_{\mu} x_{\mu} \in \Xi_{\mu}$.

Conversely, from the uniqueness of $u_{\mu}^0$ we have

$$\psi(1 \otimes 1_c \otimes v_{w_0 \mu}) = a u_{\mu}^0 \otimes 1_{c+\mu} = a \psi^0_{\mu}(1 \otimes 1_c \otimes v_{w_0 \mu}) \quad (a \in R_I)$$

for any $\psi \in \text{Hom}_{U_R^I(\mathfrak{g})} (U_R^I(\mathfrak{g}) \otimes_{U_{R^I}(\mathfrak{p}^+_I)} (R_{I,c} \otimes_{\mathbb{C}} V(\mu)), M_{R^I}(c + \mu))$. Therefore we have the following.

Proposition 4.6. Let $\mu \in \mathfrak{h}^*_{I,+}$. We have $\Xi_{\mu} = R_I \xi_{\mu}^0$. 

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We call the above homomorphism $\psi^0_\mu$ the minimal map in this paper.

Let $\tilde{f}_r$ be the lowest weight vector of the irreducible $I_l$-module generated by $f_r$.

**Proposition 4.7.** Let $\mu = m\varpi_{i_0} \in \mathfrak{h}^*_l$. Under the identification $\exp : \mathfrak{n}_l^- \simeq N_l^-$ we have

$$\langle \tilde{P}_\mu(\psi^0_\mu)\varphi \rangle|_{\mathfrak{n}_l^-} = \tilde{f}_r (\varphi|_{\mathfrak{n}_l^-}).$$

**Proof.** Let $\{v_i\}_{0 \leq i \leq n}$ be a basis of $V(\mu)$ consisting of weight vectors such that $v_n$ has the lowest weight $\omega_0 \mu$. We denote the dual basis by $\{v_i^*\}$. We define elements $Y_i \in U_{R_l}(\mathfrak{n}_l^-)$ by $\psi^0_\mu(1 \otimes 1_c \otimes v_i) = Y_i \otimes 1_{c+\mu}$. Set $Y_i = \pi(Y_i')$ Then we have

$$\langle \tilde{P}_\mu(\psi^0_\mu)\varphi \rangle(g) = \sum_{i=0}^{n} \langle gv^*_i, v_n \rangle (R(Y_i)\varphi)(g).$$

For $g \in N_l^-$ we have $\langle gv^*_i, v_n \rangle = \delta_{i,n}$. Therefore it is sufficient to show that

$$(4.1) \quad R(Y_n) = \tilde{f}_r^m(\varphi)$$

By the definition of $\psi^0_\mu$, $Y_n$ is the lowest weight vector of $\text{ad}(I_l)$-module $U(\mathfrak{n}_l^-)$ with weight $\omega_0 \mu - \mu = m(\omega_0 \varpi_{i_0} - \varpi_{i_0})$. By Lemma 4.3 the weight of $\tilde{f}_r$ is $\omega_0 \varpi_{i_0} - \varpi_{i_0}$. Hence we have $Y_n = \tilde{f}_r^m$ up to constant multiple. Since $\mathfrak{n}_l^-$ is commutative, we have $R(y) = y(\varphi)$ for any $y \in U(\mathfrak{n}_l^-)$. Hence the equation (1.1) holds. \qed

For $1 \leq p \leq r = r(\mathfrak{g})$ we set

$$\Delta^+_l(p) = \{ \beta \in \Delta^+ | \beta|_{h^-} = \gamma_j + \gamma_k \}_{h^-} \quad \text{for some } 1 \leq j \leq k \leq p \}.$$

By Lemma 4.3 we have $\Delta^+_l(p) \subset \Delta^+ \setminus \Delta_l$. We define subspaces $\mathfrak{n}^+_l(p)$ of $\mathfrak{g}$ by $\mathfrak{n}^+_l(p) = \sum_{\beta \in \Delta^+_l(p)} \mathfrak{g}_{\pm \beta}$. Set $I(p) = \{ \mathfrak{n}^+_l(p), \mathfrak{n}^-_l(p) \}$ and $I_l(p) = \{ i \in I | \mathfrak{g}_{\alpha_i} \subset I(p) \}$. Then we have the following.

**Lemma 4.8 (see Wallach [12] and Wachi [11]).** Set $\mathfrak{g}(p) = \mathfrak{n}^-_l(p) \oplus I(p) \oplus \mathfrak{n}^+_l(p)$. Then $\mathfrak{g}(p)$ is a simple subalgebra of $\mathfrak{g}$ with simple roots $\{ \alpha_{i_0} \} \cup \{ \alpha_i | i \in I(p) \}$, and the pair $(\mathfrak{g}(p), I_l(p))$ is of regular commutative parabolic type. For any $1 \leq j \leq p$ we have $f_j \in U(\mathfrak{n}^-_l(p))$, and $f_p$ is a basic relative invariant of $(L(p), \mathfrak{n}^+_l(p))$, where $L(p)$ is the subgroup of $L_l$ corresponding to $I_l(p)$.

This fact will be used in the subsequent sections.

## 5 Regular Type

In this section we assume that the prehomogeneous vector space $(L_l, \mathfrak{n}^+_l)$ is regular. We take $\gamma_i, \lambda_i$ and $f_i$ $(1 \leq i \leq r = r(\mathfrak{g}))$ as in Section 4. Then we have $w_0 \varpi_{i_0} = -\varpi_{i_0}$ and the highest weight vector $f_r \in U(\mathfrak{n}_l^-) \simeq \mathbb{C}[\mathfrak{n}^+_l]$ is the unique basic relative invariant of $(L_l, \mathfrak{n}^+_l)$ with character $2\varpi_{i_0}$. In particular $f_r \in U(\mathfrak{n}_l^-)$ is also the lowest weight vector as an $I_l$-module.
Lemma 5.3. From Corollary 3.4 we have therefore the statement holds by Proposition 4.7.

Proposition 5.1. Let $\delta(b)$ be the $b$-function of the basic relative invariant of $(L_I, n_I^\gamma)$. Then for $m \in \mathbb{Z}_{>0}$ we have

$$\xi_{m\omega_{i_0}}^0(s\omega_{i_0}) = b(s + m - 1)b(s + m - 2) \cdots b(s)$$

up to constant multiple.

Proof. For any $l \in L_I$ and $n \in n_I^\gamma$ we have

$$\tilde{f}_{\omega_{i_0}}(l \exp(n)l^{-1}) = (w_0\omega_{i_0} - \omega_{i_0})(l)\tilde{f}_{\omega_{i_0}}(\exp(n)) = -2\omega_{i_0}(l)\tilde{f}_{\omega_{i_0}}(\exp(n)).$$

Thus $\tilde{f}_{\omega_{i_0}}|_{n_I^\gamma}$ is the basic relative invariant of $(L_I, n_I^\gamma)$ under the identification $n_I^\gamma \simeq N_I^\gamma$. Hence we have

$$f_r(\delta)^m\tilde{f}(m+s)_{i_0}|_{n_I^\gamma} = f_r(\delta)^m(\tilde{f}_{\omega_{i_0}}|_{n_I^\gamma})^s+m = b(s + m - 1)b(s + m - 2) \cdots b(s)\tilde{f}^s_{\omega_{i_0}}|_{n_I^\gamma}.$$ From Corollary 3.4 we have

$$\tilde{P}_{m\omega_{i_0}}(\psi_{m\omega_{i_0}}^0)\tilde{f}(s+m)\omega_{i_0} = \xi_{\omega_{i_0}}^0(s\omega_{i_0})\tilde{f}^s_{\omega_{i_0}}.$$ Therefore the statement holds by Proposition 4.7.

In the rest of this section we shall show that $\xi_{\omega_{i_0}} = \xi_{\omega_{i_0}}^0$ up to constant multiple.

Lemma 5.2. For any $1 \leq j \leq r$ we have $w_I\gamma_j = \gamma_{r-j+1}$.

Proof. By Lemmas 4.1 and 4.2 we have $\gamma_r + \alpha_i \notin \Delta$ for any $i \in I$. Therefore $\gamma_r$ is the highest weight of the irreducible ad$(l_I)$-module $n_I^\gamma$. Since $\alpha_{i_0} = \gamma_1$ is the lowest weight of $n_I^\gamma$, $w_I\gamma_1 = \gamma_r$. Let $1 < i \leq \lceil \frac{r}{2} \rceil$. Assume that $w_I\gamma_j = \gamma_{r-j+1}$ for $1 \leq j \leq i - 1$. Since $j \neq i$, we have $\gamma_{r-i+1} \pm w_I\gamma_j = \gamma_{r-i+1} \pm \gamma_{r-j+1} \notin \Delta \cup \{0\}$. Hence $w_I\gamma_{r-i+1} \pm \gamma_j \notin \Delta \cup \{0\}$, and we have $w_I\gamma_{r-i+1} \in \Gamma_i$. By definition $\gamma_i \leq w_I\gamma_{r-i+1}$. So we have $w_I\gamma_i \geq \gamma_{r-i+1}$. Let us show that $w_I\gamma_i \leq \gamma_{r-i+1}$. By Lemma 4.2 there exist $\gamma_k$ and $\gamma_l$ such that $k \leq l$ and $w_I\gamma_i(h) = \frac{2k+\mu}{2}(h)$ for any $h \in \mathfrak{h}^-$. In particular $(w_I\gamma_i, \gamma_l) > 0$. Now we have $(w_I\gamma_i, \gamma_m) = (\gamma_l, w_I\gamma_m) = (\gamma_l, \gamma_{r-m+1}) = 0$ for $r-i+2 \leq m \leq r$. Hence $l \leq r-i+1$. Since $(w_I\gamma_i, \gamma_l) > 0$, $\gamma_l - w_I\gamma_i \in \Delta \cup \{0\}$. For $h \in \mathfrak{h}^-$ we have $(\gamma_l - w_I\gamma_i)(h) = \frac{2l-2k}{2}(h)$. By Lemma 4.2 $\gamma_l - w_I\gamma_i \in \Delta^+ \cup \{0\}$. Therefore we have $w_I\gamma_i \leq \gamma_l \leq \gamma_{r-i+1}$.

Hence the lowest weight $w_I\lambda_{r-1}$ of the irreducible component $I(\lambda_{r-1})$ of $U(n_I^\gamma)$ is $\lambda_r + \alpha_{i_0}$.

Lemma 5.3. For any $1 \leq p \leq r = r(g)$ we have

$$e_{i_0}f_p \otimes 1_{c+\mu} \in U_{R_l}(I_l \cap n^-)(f_p \otimes 1_{c+\mu}) \subset M_{R_l}(c + \mu),$$

where $e_{i_0} \in g_{\alpha_{i_0}} \setminus \{0\}$.
PROOF. By Lemma 4.3 it is sufficient to show that the statement holds for \( p = r \). We define 
\[ e_{i_0}(f_r \otimes 1_{c+\mu}) = y \otimes 1_{c+\mu}. \]
Since \( f_r \) is the lowest weight vector of the \( \text{ad}(I_f) \)-module \( U(n^-) \) and \([e_{i_0}, I_f \cap n^-] = \{0\}\), \( y \) is the lowest weight vector as an \( \text{ad}(I_f) \)-module. Moreover the weight of \( y \) is \( \lambda_r + \alpha_{i_0} = w_I \lambda_{r-1} \), which is the lowest weight of the irreducible component \( I(\lambda_{r-1}) = \text{ad}(U(I_f))f_{r-1} \). Therefore \( y \otimes 1_{c+\mu} \in U_{R_l}(I_f \cap n^-)(f_{r-1} \otimes 1_{c+\mu}) \). \[ \square \]

**Corollary 5.4.** Let \( u \in U(n^+) \) with weight \( k\alpha_{i_0} + \sum_{i \in I} m_i \alpha_i \). Then we have 
\[ uf_r \otimes 1_{c+\mu} \in U_{R_l}(I_f \cap n^-)(f_{r-k} \otimes 1_{c+\mu}) \]

**Proof.** We shall show the statement by the induction on \( k \). If \( k = 0 \), then the statement is clear. Assume that \( k > 0 \), and the statement holds for \( k - 1 \). We write \( u = \sum_j u_j e_{i_0} u'_j \), where \( u_j \in U(I_f \cap n^+) \) and \( u'_j \in U(n^+) \). Then the weight of \( u'_j \) is in \((k - 1)\alpha_{i_0} + \sum_{i \in I} Z_{\geq 0} \alpha_i \), and hence we have 
\[ uf_r \otimes 1_{c+\mu} \subseteq \sum_j u_j e_{i_0} U_{R_l}(I_f \cap n^-)(f_{r-k+1} \otimes 1_{c+\mu}) \subseteq U_{R_l}(I_f)(e_{i_0} f_{r-k+1} \otimes 1_{c+\mu}). \]
Here note that \([e_{i_0}, U_{R_l}(I_f \cap n^-)] = 0\). By Lemma 5.3 we have 
\[ e_{i_0} f_{r-k+1} \otimes 1_{c+\mu} \subseteq U_{R_l}(I_f \cap n^-)(f_{r-k} \otimes 1_{c+\mu}) \]
Therefore we obtain 
\[ uf_r \otimes 1_{c+\mu} \subseteq U_{R_l}(I_f)(f_{r-k} \otimes 1_{c+\mu}) = U_{R_l}(I_f \cap n^-)(f_{r-k} \otimes 1_{c+\mu}) \]
\[ \square \]

**Theorem 5.5.** We have \( \xi_{\varpi_i_0} = \prod_{j=1}^r p_{\varpi_i_0, \lambda_j + \varpi_i_0} \in C^\times \xi_{\varpi_i_0}^0, \) where \( C^\times = C \setminus \{0\} \).

**Proof.** Let \( v_{-\varpi_i_0} \) be the lowest weight vector of \( V(\varpi_i_0) \). Since \( f_r \) is the lowest weight vector of \( U(n^-) \) with weight \(-2\varpi_0 \), we have \( u_{\varpi_i_0}^0(1 \otimes 1_c \otimes v_{-\varpi_i_0}) = f_r \otimes 1_{c+\varpi_i_0} \). It is clear that \( \varpi_i_0 - w_0 \varpi_i_0 = 2 \varpi_i_0 = -\lambda_r \in r\alpha_{i_0} + \sum_{i \in I} Z_{\geq 0} \alpha_i \).

Set \( P(j) = \{ \lambda \mid \varpi_i_0 - \lambda \in j\alpha_{i_0} + \sum_{i \in I} Z_{\geq 0} \alpha_i \} \). We define an \( I_f \)-submodule \( V_j \) of \( V(\varpi_i_0) \) by 
\[ V_j = \bigoplus_{\lambda \in P(j)} V(\varpi_i_0) \lambda \]
(cf. Section 2). Note that \( V_j \neq 0 \) for \( 0 \leq j \leq r \). We take the irreducible decomposition of \( V_j \)
\[ V_j = \tilde{W}(\eta_{j,1}) \oplus \cdots \oplus \tilde{W}(\eta_{j,t_j}), \]
\[ 14 \]
where $\tilde{W}(\eta)$ is an irreducible $I_f$-module with highest weight $\eta$. Let $v_{j,k}$ be the highest weight vector of $\tilde{W}(\eta_{j,k})$. There exists an element $u_{j,k} \in U(n^+)$ such that $u_{j,k}v_{j,k} = v_{j,k}$. Then the weight of $u_{j,k}$ is in $(r-j)\alpha_{i_0} + \sum_{i \in I} Z \geq \alpha_i$. By Corollary 5.4 we have
\[\psi^0(1 \otimes 1_c \otimes v_{j,k}) = u_{j,k} \psi^0(1 \otimes 1_c \otimes v_{j,k}) = u_{j,k}f_j \otimes 1_{c+w_{i_0}} \in U_{R_l}(I_f \cap n^-)f_j \otimes 1_{c+w_{i_0}}.\]
Since $v_{j,k}$ is the highest weight vector, we have
\[\psi^0(1 \otimes 1_c \otimes v_{j,k}) \in R_I(f_j \otimes 1_{c+w_{i_0}}).\]
In particular $\eta_{j,k} = \lambda_j + w_{i_0}$, and the irreducible decomposition of $V(w_{i_0})$ as an $I_f$-module is given by
\[V(w_{i_0}) = \bigoplus_{j=0}^r \tilde{W}((\lambda_j + w_{i_0})^{\oplus N_j}),\]
where we set $\lambda_0 = 0$. Therefore we have $\xi_{w_{i_0}} = \prod_{j=1}^r p_{\lambda_j + w_{i_0} - \xi_{w_{i_0}}}$, which is regarded as a polynomial function on $Cw_{i_0}$. Since $\deg p_{\lambda_j + w_{i_0} - \xi_{w_{i_0}}} = 1$ for $j \geq 1$, we have $\deg \xi_{w_{i_0}} = r$. Now we have $\deg \xi^0_{w_{i_0}} = \deg \xi(w_{i_0}) = \deg f_r = r$. From Proposition 4.6 we have $\xi_{w_{i_0}} \in C \xi^0_{w_{i_0}}$, hence the statement holds.

For $1 \leq i < j \leq r$ we set $c_{i,j} = \sharp \{ \alpha \in \Delta_I \cap \Delta^+ \mid \alpha|_{l_h} = \frac{2i-\gamma_i}{2} \gamma_i|_{l_h} \}$. It is known that $c_{i,j} = \sharp \{ \alpha \in \Delta^+ \setminus \Delta_I \mid \alpha|_{l_h} = \frac{2i-\gamma_i}{2} \gamma_i|_{l_h} \}$ and this number is independent of $i$ or $j$ (see [13]). Set $c_0 = c_{i,j}$. Then we have $2p_{\rho, \gamma_j} = d_0(1 + c_0(j - 1))$, where $d_0 = (\alpha_{i_0}, \alpha_{i_0})$. In particular $2p_{\rho, \gamma_j} = jd_0(1 + \frac{j - 1}{2}c_0)$. Since $(\gamma_i, \gamma_j) = \delta_{i,j}d_0$, we have $(w_{i_0}, w_{i_0}) = (\lambda_j + w_{i_0}, \lambda_j + w_{i_0})$ for $1 \leq j \leq r$. Hence we have
\[p_{\lambda_j + w_{i_0}}(s w_{i_0} + \rho, \lambda_j) = jd_0(s + 1 + \frac{j - 1}{2}c_0).\]

6 Non-regular Type

Assume that the prehomogeneous vector space $(L_I, n^+_I)$ is not regular. We take $\gamma_i$, $\lambda_i$ and $f_i$ ($1 \leq i \leq r = r(\tilde{g})$) as in Section 2. For $\mu = m w_{i_0} \in h^+_I$ we denote by $\tilde{\nu}_\mu$ the highest weight vector of the irreducible $I_f$-submodule of $V(\mu)$ generated by the lowest weight vector of $V(\mu)$. The weight of $\tilde{\nu}_\mu$ is $w_I \mu$. We take $u \in U_{R_I}(n^-_I)$ as $\psi^0_{\mu}(1 \otimes 1_c \otimes \tilde{\nu}_\mu) = u \otimes 1_{c+\mu}$. By definition of $\psi^0_{\mu}$ we have $u \in U(n^-_I)$. Moreover $u$ is the highest weight vector of $U(n^-_I)$ with weight $w_I w_{0 \mu} - \mu = w_I(w_{0 \mu} - \mu)$. By Lemma 4.6 we have $w_I(w_{0 \mu} - \mu) = m \lambda_r$. Therefore we have $u = f^m_r$. Set $\xi^0_{\mu} = \psi^0_{\mu} \in \tilde{R}$. Therefore
\[p_{\lambda_j + w_{i_0}}(s w_{i_0} + \rho, \lambda_j) = jd_0(s + 1 + \frac{j - 1}{2}c_0).\]

We define subalgebras $g(r)$, $l(r)$ and $n^+_r$ of $g$ as in Lemma 4.8. We set $\tilde{g}^+ = l(r) \oplus n^+_r$. We denote by $\tilde{V}(\mu)$ the irreducible $g(r)$-module with highest weight $\mu$. Let $\tilde{I}_0$ be an index set of simple roots of $g(r)$, that is, $\tilde{I}_0 = I(r) \cup \{ i_0 \}$ (see Lemma 4.8). We set $\tilde{I} = I(r)$ and $\tilde{g} = g(r)$ for simplicity. Let $R$ be an enveloping algebra of $\sum_{i \in \tilde{I}_0} \mathbb{C}h_i/\sum_{i \in \tilde{I}} \mathbb{C}h_i$. Since we have the canonical identification $R_I \simeq \tilde{R}$, a $U_{R_I}(\tilde{g})$-submodule
\[\tilde{M}(c + \mu) = U_{R_I}(\tilde{g}) \otimes_{U_{R_I}(\tilde{g}^+)} R_{I_c + \mu}\]
of $M_{R_I}(c + \mu)$ is a generalized universal Verma module associated with $\tilde{g}$. We define an element $\xi_0^\mu$ of $\tilde{R} \simeq R_I$ by the multiplication map on $\tilde{M}(c + \mu)$ induced by the minimal map

$$\tilde{\psi}_\mu^0 : U_{R_I}(\tilde{g}) \otimes_{U_{R_I}(\tilde{p}^+)} (R_{I,c} \otimes_{\mathbb{C}} \tilde{V}(\mu)) \to \tilde{M}(c + \mu).$$

Then we have the following.

**Proposition 6.1.**
(i) Under the identification $\tilde{R} \simeq R_I$ we have $\xi_0^\mu = \tilde{\xi}_0^\mu$ for $\mu \in \mathfrak{h}_I^*$.  

(ii) $\xi_{\varpi_{i_0}} \in \mathbb{C} \xi_{\varpi_{i_0}}^0$.

**Proof.**
(i) We have $U(\tilde{g})v_\mu \simeq \tilde{V}(\mu)$, and $\tilde{v}_\mu$ is its lowest weight vector. Since we have $\psi_\mu^0(1 \otimes 1 \otimes \tilde{v}_\mu) = f^0_m \otimes 1_{c + \mu}$, the restriction $\psi_\mu^0$ on $U_{R_I}(\tilde{g}) \otimes_{U_{R_I}(\tilde{p}^+)} (R_{I,c} \otimes_{\mathbb{C}} U(\tilde{g})v_\mu)$ is $\tilde{\psi}_\mu^0$. Hence we have $\xi_0^\mu \otimes 1_{c + \mu} = \psi_\mu^0(1 \otimes 1 \otimes v_\mu) = \tilde{\psi}_\mu^0(1 \otimes 1 \otimes v_\mu) = \xi_0^\mu \otimes 1_{c + \mu}$.

(ii) Since the pair $(\tilde{g}, i_0)$ is of regular type, we have $\deg \xi_{\varpi_{i_0}}^0 = r$ (see the proof of Theorem 5.5). Similarly to the proof of Theorem 5.5 we can show that $\deg \xi_{\varpi_{i_0}} = r$. By (ii) we have $\deg \xi_{\varpi_{i_0}} = \deg \xi_{\varpi_{i_0}}^0$. Since $\varpi_{i_0} = R_I\xi_{\varpi_{i_0}}^0$ and $\xi_{\varpi_{i_0}} \in \mathbb{C} \xi_{\varpi_{i_0}}^0$, we have $\xi_{\varpi_{i_0}} \in \mathbb{C} \xi_{\varpi_{i_0}}^0$. 

As a result, we have the following.

**Theorem 6.2.** For any pair $(\mathfrak{g}, i_0)$ of commutative parabolic type, the ideal $\varpi_{i_0}$ is generated by $\xi_{\varpi_{i_0}}$.

### 7 Irreducibility of Verma Modules

Let $(L_I, n_I^-)$ be a prehomogeneous vector space of commutative parabolic type. Set $\{i_0\} = I_0 \setminus I$. In this section we give a new proof of the following well-known fact (Suga [9], Gyoja [3], Wachi [11]).

**Theorem 7.1.** Let $\lambda = s_0 \varpi_{i_0} \in \mathfrak{h}_I^*$. $M_I(\lambda)$ is irreducible if and only if $\xi_{\varpi_{i_0}}^0 (\lambda - m\varpi_{i_0}) \neq 0$ for any $m \in \mathbb{Z}_{>0}$.

We take $f_i \in U(n_I^-)$ ($1 \leq i \leq r = r(\mathfrak{g})$) as in Section 4.

**Lemma 7.2.** Let $e_i^-$ be a nonzero element of $\mathfrak{g}_{-\alpha_i}$. There exist $y_0, y_1, \ldots, y_t \in n_{(r)}^-$ and $i_1, \ldots, i_t \in I_{(r)}$ such that

$$f_{r-t} = \sum_{k=0}^{t} y_k \text{ad}(e_{i_k}^- \cdots e_{i_1}^-) f_{r-1},$$

and $\text{ad}(e_{i_k}^-) f_{r-1} = 0$ for $2 \leq k \leq r$.

This lemma is proved by direct calculations for each case. In [3] there are explicit decompositions of quantum counterparts $f_{q,r}$ of $f_r$ satisfying the properties of Lemma 7.2. We can get the decomposition (7.1) from the quantum counterpart via $q = 1$. For example, in the case of type $A$, $f_r$ is a determinant and the decomposition (7.1) corresponds to a cofactor decomposition.
**Proposition 7.3.** Let $a_i \in \mathbb{Z}_{\geq 0}$ and let $a_{i+1}, \ldots, a_r \in \mathbb{Z}_{\geq 0}$. There exists $u \in U(n^-)$ such that

$$uf_i^{a_i}f_{i+1}^{a_{i+1}} \cdots f_r^{a_r} \otimes 1_\lambda = f_i^{a_i-1}f_{i+1}^{a_{i+1}+1} \cdots f_r^{a_r} \otimes 1_\lambda.$$  

**Proof.** By Lemmas 1.8 and 7.2 we have

$$f_{i+1} = \sum_{k=0}^t y_k \text{ad}(u_k)f_i,$$

where $y_k \in n^-_{(i+1)}$ and $u_k = e_{j_k}^- \cdots e_{j_1}^-$ such that $j_1, \ldots, j_t \in I_{(i+1)}$ and $\text{ad}(e_{j_l}^-)f_i = 0$ for $l > 1$.

Note that for $p > i$ we have $y_k \in n^-_{(p)}$ and $j_k \in I_{(p)}$. Since $f_p$ is the lowest weight vector of an ad($I_{(p)}$)-module $U(n^-_{(p)})$, we have

$$\text{ad}(u_k)(f_i^{a_i}f_{i+1}^{a_{i+1}} \cdots f_r^{a_r}) = (\text{ad}(u_k)f_i^{a_i})f_{i+1}^{a_{i+1}} \cdots f_r^{a_r}.$$

If $k \geq 1$, then we have

$$\text{ad}(u_k)(f_i^{a_i}) = a_i \text{ ad}(e_{j_1}^- \cdots e_{j_t}^-)((\text{ad}(e_{j_l}^-)f_i)f_i^{a_i-1}) = a_i (\text{ad}(u_k)f_i)f_i^{a_i-1}.$$  

Hence for $u = y_0 + a_i^{-1}\sum_{k=1}^t y_k u_k$, we have

$$uf_i^{a_i}f_{i+1}^{a_{i+1}} \cdots f_r^{a_r} \otimes 1_\lambda = y_0 f_i^{a_i}f_{i+1}^{a_{i+1}} \cdots f_r^{a_r} \otimes 1_\lambda + \sum_{k=1}^t y_k (\text{ad}(u_k)f_i)f_i^{a_i-1}f_{i+1}^{a_{i+1}+1} \cdots f_r^{a_r} \otimes 1_\lambda = f_i^{a_i-1}f_{i+1}^{a_{i+1}+1} \cdots f_r^{a_r} \otimes 1_\lambda.$$

\[\square\]

**Corollary 7.4.** Let $K(\neq 0)$ be a submodule of $M_I(\lambda)$ for $\lambda \in \mathfrak{h}^*_I$. We have $f_r^nM_I(\lambda) \subset K$ for $n \gg 0$.

**Proof.** If $K = M_I(\lambda)$, then the statement is clear. Assume that $\{0\} \neq K \subsetneq M_I(\lambda)$. By Lemma 1.3 any highest weight vector of $M_I(\lambda)$ as an $I_I$-module is given by the following form:

$$f_1^{a_1} \cdots f_r^{a_r} \otimes 1_\lambda.$$  

Since $K$ has the highest weight vector as an $I_I$-module, there exists an element $f_1^{a_1} \cdots f_r^{a_r} \otimes 1_\lambda \in K$ such that $(a_1, \ldots, a_r) \neq 0$. By Proposition 7.3 for $n \gg 0$ there exists $u \in U(n^-)$ such that

$$f_r^n \otimes 1_\lambda = u(f_1^{a_1} \cdots f_r^{a_r} \otimes 1_\lambda) \in K.$$

Hence for any $y \in U(n^-_I)$ we have

$$f_r^n(y \otimes 1_\lambda) = yf_r^n \otimes 1_\lambda \in K,$$

and the statement holds.  

\[\square\]
Let us prove Theorem 7.1 by using the commutative diagram

\[
\begin{CD}
M_{R_1}(c + \mu) @>\iota_m>> M_{R_1}(c + \mu) \\
U_R(g) \otimes_{U_R(p_1^+)} (R_{I,c} \otimes_{C} V(\mu)) @>\psi_0^\mu>> M_{R_1}(c + \mu),
\end{CD}
\]

(7.2)

where \(\mu \in h^*_T\).

Set \(\lambda = s_0 \varpi_{i_0}\). We denote the highest weight vector of \(V(\mu)\) by \(v_\mu\). Let \(\tilde{v}_\mu\) be the highest weight vector of the irreducible \(I_T\)-module generated by the lowest weight vector of \(g\)-module \(V(\mu)\). For a positive integer \(m\), we set \(\mu^m = m \varpi_{i_0}\). Considering the functor \(C \otimes_{R_I} (\cdot)\), where \(C\) has the \(R_I\)-module structure via \(c(h_i)1 = (\lambda - \mu)(h_i)\), we obtain the following commutative diagram from (7.2):

\[
\begin{CD}
M_I(\lambda) @>\iota_m>> M_I(\lambda) \\
U(g) \otimes_{U(p_1^+)} (C_{I,\lambda - \mu} \otimes_{C} V(\mu)) @>\psi_0^\mu>> M_I(\lambda),
\end{CD}
\]

where \(\iota_m(1 \otimes 1_\lambda) = 1 \otimes 1_{\lambda - \mu} \otimes \tilde{v_\mu}\) and \(\psi_0^\mu(1 \otimes 1_{\lambda - \mu} \otimes \tilde{v_\mu}) = f_r^m \otimes 1_\lambda\).

Assume that \(M_I(\lambda)\) is irreducible. Since \(\psi_0^\mu \neq 0\), we have \(\text{Im}\psi_0^\mu = M_I(\lambda)\). The weight space of \(U(g) \otimes_{U(p_1^+)} (C_{I,\lambda - \mu} \otimes_{C} V(\mu))\) with weight \(\lambda\) is \(C(1 \otimes 1_{\lambda - \mu} \otimes \tilde{v_\mu})\), hence there exists \(a \in C \setminus \{0\}\) such that

\[1 \otimes 1_\lambda = \psi_0^\mu(a \otimes 1_{\lambda - \mu} \otimes \tilde{v_\mu}) = a\psi_0^\mu \iota_m(1 \otimes 1_\lambda) = a\xi_0^\mu (\lambda - \mu) \otimes 1_\lambda \neq 0.\]

By Propositions 5.3 and 6.3, we have

\[\xi_0^\mu (\lambda - \mu) = \xi_{\varpi_{i_0}}^0 (\lambda - \varpi_{i_0}) \xi_{\varpi_{i_0}}^0 (\lambda - 2 \varpi_{i_0}) \cdots \xi_{\varpi_{i_0}}^0 (\lambda - m \varpi_{i_0}).\]

Therefore we have \(\xi_{\varpi_{i_0}}^0 (\lambda - m \varpi_{i_0}) \neq 0\) for any \(m \in Z_{>0}\).

Conversely, we assume that \(\xi_{\varpi_{i_0}}^0 (\lambda - m \varpi_{i_0}) \neq 0\) for any \(m \in Z_{>0}\). We set

\[N(m) = U(g) \otimes_{U(p_1^+)} (C_{I,\lambda - \mu_m} \otimes_{C} V(\mu_m)).\]

Since \(\xi_{\mu_m}^0 (\lambda - \mu_m) = \xi_{\varpi_{i_0}}^0 (\lambda - \varpi_{i_0}) \xi_{\varpi_{i_0}}^0 (\lambda - 2 \varpi_{i_0}) \cdots \xi_{\varpi_{i_0}}^0 (\lambda - m \varpi_{i_0}) \neq 0\), we have

\[\psi_0^\mu (\xi_{\mu_m}^0 (\lambda - \mu_m)^{-1} \otimes 1_{\lambda - \mu_m} \otimes \tilde{v_{\mu_m}}) = \xi_{\mu_m}^0 (\lambda - \mu_m)^{-1} \psi_0^\mu \iota_m(1 \otimes 1_\lambda) = 1 \otimes 1_\lambda.\]

Hence \(\psi_0^\mu\) is surjective, and we have an isomorphism

\[N(m)/\text{Ker}\psi_0^\mu \simeq M_I(\lambda) : \overline{1 \otimes 1_{\lambda - \mu_m} \otimes \tilde{v_{\mu_m}}} \mapsto f_r^m \otimes 1_\lambda\]
for any $m$. Under this identification we have
\[
1 \otimes 1_{\lambda - \mu_{n+1}} \otimes v_{\mu_{n+1}} = f_r^n \otimes 1_{\lambda} = f_r^n (f_r \otimes 1_{\lambda}) = f_r^n (1 \otimes 1_{\lambda - \mu_{1}} \otimes v_{\mu_{1}}).
\]
Let $K \neq 0$ be a submodule of $M_I(\lambda)$. By Corollary 7.4 for $n \gg 0$ we have
\[
1 \otimes 1_{\lambda - \mu_{n+1}} \otimes v_{\mu_{n+1}} = f_r^n (1 \otimes 1_{\lambda - \mu_{1}} \otimes v_{\mu_{1}}) \in K.
\]

Hence we have
\[
M_I(\lambda) = N(n+1)/\text{Ker}v_{n+1}^0 = U(\mathfrak{g})_1 \otimes 1_{\lambda - \mu_{n+1}} \otimes v_{\mu_{n+1}} \subset K.
\]

Therefore $K = M_I(\lambda)$, and $M_I(\lambda)$ is irreducible. We complete the proof of Theorem 7.1.

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