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Soliton sustainable socio-economic distribution

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Abstract. In the work presented, from close positions, we consider: 1) the question of the stability of socio-economic distributions; 2) the question of the possible mechanism for the formation of fractional power-law dependences in the Cobb /Douglas production function; 3) the introduction of a fractional order derivative for a general analysis of a fractional power function; 4) bringing in a state of mutual matching of the interest rate and the production function of Cobb/Douglas.

1. Introduction

According to the common view of the middle class, it is the social category in which people can make savings.

Figure 1. Man at the Crossroads (1934) was a fresco by Diego Rivera in New York City's Rockefeller Center
At the empirical level, it is well known that sustainable social state of society can be characterized with a "single hump" distribution of the population’s welfare. And the hump corresponds precisely to the very middle class. In the language of the simplest functions hump can be described by the formula depending on the frequency of the oscillation amplitude in the conditions close to resonance (Lorentz peak). However, it requires huge imagination to apply resonance model to the social phenomena.

Another, seemingly more natural concept of an isolated social hump can be described with a Gaussian distribution that is follows from the Central Limit Theorem. However, CLT doesn’t take into account transitions from one social group to another. Such “jumps” provide a dynamic balance and long-term stationarity of single-humped distribution.

2. Stability mechanism socioeconomic distributions

A well-known dynamic generalization (that allows a transition from one level of living to another) of a Gaussian distribution is the Wiener process. Consider discrete equidistant segments of the welfare, there it is assumed that all people with the level of life, whom are in the ± half step unit discrete scale "sit" in this segment. Also consider discrete time with the step τ (and variable of the scale of living standards t).

Suppose that at some point time t in the i-th node (corresponding to a standard of living (i+½) τ < L < (i+½)τ ) there are Nᵢ people. After time interval τ: firstly, when the probability of transition ω from (i - 1)-th node to the i-th number Nᵢ increase by +Nᵢ × ω and in view of the probability of transition ω from (i + 1)-th node in the i-th number Nᵢ also will increase and +Nᵢ × ω; second, when the transition probability ω with i-th node to the (i - 1)-th number Nᵢ should decrease by -Nᵢ×ω respectively, and i-th node on the (i+1)-th also decrease by -Nᵢ×ω. As a result, after the time step τ, the number of people with the standard of living in the vicinity of the i-th node will be: Nᵢ+ωδNᵢ = Nᵢ+2Nᵢ × ω × Nᵢ.∞ ω = Nᵢ+(Nᵢ-1×2Nᵢ+1)ω. The growth δNᵢ = (Nᵢ-1×2Nᵢ + Nᵢ+1)ω is proportional to the numerator of the finite-difference representation of the second derivative of the standard of living

\[ \frac{\partial^2 N}{\partial \tau^2} \]

Similarly, the instantaneous rate of change of the number of people with any specific standard of living as a result of unconscious "Brownian motion" to "better life" in the finite-difference form is

\[ \frac{\partial N}{\partial \tau} \]

Finally, both finite-difference formula can be reduced to the overall equation

\[ \frac{\omega \ell^2}{2} \times \frac{Nᵢ-1×2Nᵢ+Nᵢ+1}{\ell^2} = \tau \times \frac{\partial N_i}{\partial \tau} \]  

which under setting grid steps close to zero turns to the old/good Fokker/Planck. Fokker/Planck equation is a statistical form of an even older (Fourier work of the Great French Revolution) the heat equation (and then also the diffusion, viscosity and so on up to the Schrodinger equation for the amplitude of the non-Kolmogorov probability of non-relativistic particle localization). The solution in non-normalized form is presented as a model solution

\[ N(L,t) \sim \frac{1}{\sqrt{t}} e^{-\frac{L^2}{2\sigma^2}} = \frac{1}{\sqrt{t}} e^{-\frac{L^2}{\sigma^2(t)}} \]

where the standard deviation \( \sigma = \sigma(t) \) (in this case the deviation from average level of the welfare) increases with time as the

\[ \sigma(t) = \sqrt{\frac{\omega \ell^2}{2}} \]

(here and below the standard of living of the middle class is regarded as the zero point of reference throughout the life of the scale levels, thus "poor" corresponds to the L<0, "average" - L=0, « the rich » - L>0). The increase \( \sigma(t) = \sqrt{\frac{\omega \ell^2}{2}} \) means that over time, the Wiener process a random walk through the various levels of life will lead to the expansion of distribution’s hump full scale and asymptotically form a "flat" social distribution from the poorest to the richest. In practice, this is not observed, and hence the simplest Wiener process is not useful as a mechanism of sustainable humped distribution of living standards.
Complicate this scheme by entering the term $-\frac{\omega}{\tau}N$ that is responsible for the rate of drawdown on the $L$-th standard of living $\frac{\partial N(L)}{\partial t} = \frac{\omega}{\tau} \frac{\partial^2 N}{\partial L^2}$. In fact, this term corresponds to the individual consumption of the excess of income over expenditure, which leads to a reduction in the number of $L$-th level. However, considering the scheme in such form, people consume all savings and roll down to the low standard of life. Thus, along with the consumption it should be added the factor responsible for welfare growth on $L$-th level, respectively, in the equation with the opposite sign (i.e., with "+"). The principle of the expansion of smooth functions in power series suggests that if the previous term $-\frac{\omega}{\tau}N$, the following must be of type $\ell^2$. However, as is well known, the algebraic sum of a linear and quadratic contribution can always be reduced to a full square by a change in the given case of the origin $\ell$. Thus result will not differ from the case without consumption. The element responsible for the growth of wealth in the $L$-th level, should be sought in the form of an entire degree $\ell$ higher than 2, such as $2\ell^3$, which corresponds to the non-linear partial differential equation of parabolic type $\frac{\partial N}{\partial t} = \omega \frac{\partial^2}{\partial L^2} - \frac{\omega}{\tau} N - \frac{\omega}{\tau} N$. The equation is close to the famous soliton equations such as "nonlinear Schrödinger".

Since we are interested in the case of a stationary dynamic equilibrium $\frac{\partial N}{\partial t} = 0 \Rightarrow \ell^3 \frac{\partial^2 N}{\partial L^2} + 2\ell^3 - N = 0$ is not normalized and the solution is a steady soliton $N(L) \sim \frac{1}{c\ell(L - L_0)/\ell}$ providing humped distribution of the standard of living $L$, measured from the level of the middle class $L_0$. Meaning $\ell$ is already slightly different than it was in the consideration of discrete components on the scale $L$. Now $\ell$ is not the step between the nodes, which close to zero in transition from the finite-difference equations to the differential now $\ell$ is rather made dimensionless scale of the range of possible levels of life values.

General view of the equation $\frac{\partial N}{\partial t} = \omega \frac{\partial^2}{\partial L^2} - \frac{\omega}{\tau} N - \frac{\omega}{\tau} N$ allows, apparently, also make inference on the structure and formation of income/consumption, ensuring the stability of the distribution as a whole. According to the expression in the bracket income arises in the interaction of the three representatives of the social group ($+2\ell^3$), while the spending made individually ($-\ell^4$).

3. Mikrosemantickesky approach to the Cobb/Douglas model

Use the same approach for the semantic study of the form of the production function of the Cobb-Douglas $Y=aC^\alpha L^\beta$ obtained by the authors in 1926 [1] empirically. In this formula, $Y$ - output, $C$ - capital $L$ - labor invested in the production of goods, $0<\alpha<1, 0<\beta<1$ (or even a stronger restriction $0<\alpha+\beta<1$). Thus, due to the smaller exponents unit with increasing $C$ and $L$ partial derivatives of $Y = Y(C, L)$ for $C$ and $L$ are reduced. In particular the increase in production should decrease productivity $\frac{\partial Y}{\partial L} \rightarrow 0$ ("fatigue effect"), which was useful to explain without resorting to empiricism.

Assume, that for production of $Y$ is required to involve the amount of manpower $L$ broken (by the professionalism, educational and social level, etc.) on $m$ groups of performers. The probability of full utilization of labour resources (respectively perform manufacturing tasks) is the product of the probabilities of development work, specified for each selected group $w_i = \omega_i L_i$. As a result,
$W(L) \sim \prod_{i=1}^{m} w_i = \prod_{i=1}^{m} \omega_i L_i \sim \prod_{i=1}^{m} \omega_i L_i \sim \prod_{i=1}^{m} \epsilon_i \approx \langle L_i \rangle^m \sim L^m$ that $W(L) \sim L^m$ where $\langle L_i \rangle$ - the labour force, specified for each of the selected group and the average for all groups of performers.

On the other hand for the production of $Y$ good it is required to master the volume of the labour force $L$ in the sequential chain of operations of the $n$ links. It is obvious that the probability of performing manufacturing jobs is the product of the probability of its implementation in each area of the chain $p_i = p_{t_i}$, which in turn is proportional to the time spent on this operation. As a result

$P(L) \sim \prod_{i=1}^{n} p_i = \prod_{i=1}^{n} \rho t_i = \sim \prod_{i=1}^{n} \rho \int t_i = \prod_{i=1}^{n} \sim \langle t \rangle^n$, where $\langle t \rangle$ - time average for all transactions.

At the same time $W(L) \sim P(L)$ therefore, $L^m \sim \langle t \rangle^n$ and $\langle t \rangle \sim L^{m/n}$. Thus, the average time spent time on the individual transaction in proportion to the amount of reclaimed labour force $L$ in degree of $m/n$ which in the case of low qualification (the number of professional groups $m$ is less than the number of links of the production chain) would be less one

$$\left(\frac{m}{n} < 1 \right) & \left( \beta < 1 \right) \rightarrow \frac{m}{n} = \beta, \ldots.$$  

The total time $T$ spent on the production of the product obviously will be proportional to the average time spent on the individual transaction $T \sim \langle t \rangle$.

According to Marx, the cost of production $Y$ is determined by the $T$, averaged over all commodity producers. Then $Y \sim T \sim L^{m/n}$ ie, $Y \sim L^{m/n}$, that in fact in the case of $m < n$ is an integral part of the Cobb-Douglas formula $Y = aC^\alpha L^\beta \ldots$

4. Analysis of fractional powers dependencies such as the Cobb-Douglas and the concept of non-integer order derivatives

The presence in the formula of fractional powers of the Cobb-Douglas demonstrates a simple functional dependence. However, there may be more complicated in principle functional relationships $Y \sim \lambda L^\beta \rightarrow Y \sim \left( \lambda_1 L_1^\beta + \lambda_2 L_2^\beta + \lambda_3 L_3^\beta + \ldots \right)$. By analogy with the "integer" power series $f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$, where the coefficients of the respective powers of the integer argument can be defined in terms of derivatives of integer order $a_n = \frac{1}{n!} \left( \frac{d}{dx} \right)^n f(x) \bigg|_{x \rightarrow x_0}$, should introduce the concept of non-integer order derivatives that allow to find the coefficients of the corresponding non-integer powers of argument. To develop the analogy to non-integer degree first consider an alternative procedure for calculating the derivatives of the whole order

$$a_n = \lim_{x \rightarrow x_0} \frac{\sum_{n=0}^{n-1} a_n (x - x_0)^n}{(x - x_0)^n}.$$  

It is seen that, for example, for the derivative the first order, this procedure is reduced to the normal "asymptotically difference" scheme $f'(x_0) = a_1 = \lim_{x \rightarrow x_0} \frac{f(x) - a_0}{(x - x_0)}$.

Further, similar to the passage to the limit of the Fourier series for a periodic function to the integral of the Fourier integral for non-periodic introduce replacement power series

$\phi(x) = \int_{0}^{\infty} \alpha(\nu) (x - x_0)^\nu d\nu$.  

Continuing this analogy will lead to the definition of $\nu$-th coefficient $\alpha$
in an integrated power series expansion \( \alpha(\nu) = \lim_{n \to \infty} \frac{\phi(x) - \int_0^x \alpha(\zeta)(x-x_0)^\nu d\zeta}{(x-x_0)^\nu} \), which is proportional to the derivative of the \( \nu \)-th order of the function \( \phi(x) \) ie \( \left[ \frac{d}{dx} \right]^{\nu} \phi(x) = \alpha(\nu)\Gamma(\nu) \).

5. The modified law of compound interest and "right relationship" depreciation

Next, we consider the simplest model of expanded reproduction (approximately such that described in discrete form Marx in his Capital) \( \frac{d}{dt} Y = \gamma Y \) with the aim of comparing the law of monetary growth (here - \( \gamma \) macroeconomic parameter corresponding to the percentage of contributions from the produced by the moment of time \( t \) value \( Y \) for the modernization, which stimulating labour productivity \( Y ; \)). The obvious solution \( Y = Y_0 e^{\gamma t} \) to the recorded above ordinary differential equation is comparable to the usual law of compound interest \( F_r(n) = P_r(1+r)^n \), where \( - n \) time measured by the total number of treatment cycles, and the logarithmic growth factor \( \gamma \) production corresponds to the logarithm \( \ln(1+r) \), expressed through the interest rate \( r \). It is necessary to calculate a geometric progression, corresponding to the law of compound interest at the time forced the Europeans to move from the Roman nonpositional numbering system to the decimal positional, "brought" from India, a young mathematician Fibonacci (to XI ÷ XII century the Catholic Church strongly condemns usury, and in fact this only positional system used in Europe for scientific astronomical calculations were cumbersome ancient sumerian six decimal of calculation, which marks "and to this day" felt in the measures of angles and times). Comparison of \( Y = Y_0 e^{\gamma t} \) with \( F_r(n) = P_r(1+r)^n \) shows, that the law of compound interest is not a manifestation of voluntarism, but is a reflection of the objective (in accordance with the law of value Marx) dynamic harmonization macroeconomic growth in the total value of manufactured goods and the money supply (the geometric progression in the law of compound interest is the expression of a discrete exponential functional \( F_r(n) = P_r(1+r)^n = P_r e^{(\ln(1+r)} \), \( \gamma = \ln(1+r) = \text{const} \). However, the exponential growth in the value of goods in the course of expanded reproduction of production does not take into account "the effect of fatigue" in Cobb/Douglas model. For taking into account for this effect we should formally agree on money supply growth is not an exponentially growing number of products and dependence on wasted man/hours (ie, time \( t \)) replace the discrete \( L \) on continuous \( t \) formula: \( Y = \alpha C^\beta L^\delta = \lambda L^\delta \rightarrow Y = \lambda t^\delta \). Further, in the law of compound interest if should be made \( r = \text{const} \rightarrow r = r(t) \) (at which this law would have been modified formula Cobb/ Douglas \( r = r(t) = (t^{\beta/c} - 1) \rightarrow F_r(t) = P_r(1+r)^t = P_r(1+(t^{\beta/c} - 1)) = P_r t^\beta \leftrightarrow Y = \lambda t^\delta \). This fixed formally negotiate a dynamic growth of money supply and the total value of manufactured goods, but not according to the laws of expanded reproduction, and now, in accordance with the model Cobb/Douglas. Thus, taking into account the effect of fatigue balanced and, apparently, more just becomes a modified law of compound interest with a constantly decreasing rate \( r = r(t) = (t^{\beta/c} - 1) \). Apparently, not useless to compare the differential equation "normal" expanded reproduction model Cobb / Douglas, using the concept of elasticity (ie actually logarithmic derivative):

\[
\begin{align*}
\frac{d}{dt} Y &= \gamma Y \\
\frac{d}{dL} Y &= \frac{d}{dL} (\alpha C^\beta L^\delta) = \frac{\beta Y}{L} \\
\end{align*}
\Rightarrow
\begin{align*}
\frac{dY}{Y} &= \gamma dt \\
\frac{dY}{\beta dL} &= \frac{\beta Y}{L} \\
\end{align*}
\Rightarrow \gamma \Leftrightarrow \frac{\beta}{L}
We see that in the model of Cobb/Douglas effective rate of deductions for the modernization of production is instead of a constant (expanded reproduction $\gamma = \text{const}$) decreases in inverse proportion to the time worked $\gamma = \beta / t$. Perhaps this decline corresponds to the amortization (over time, the same money invested in productivity growth "is getting worse and worse working"). At least if it is, then the amortization (depreciation of equipment, as aging) should be calculated not according to the law of simple interest with a negative rate (arithmetic progression or linearly decreasing function of time) as an inverse function of time.

References

[1] Cobb C and Douglas P 1928 A theory of production American Economic Review 18, 139-250