Bethe states of the XXZ spin-$\frac{1}{2}$ chain with arbitrary boundary fields

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Abstract

Based on the inhomogeneous $T - Q$ relation constructed via the off-diagonal Bethe Ansatz, the Bethe-type eigenstates of the XXZ spin-$\frac{1}{2}$ chain with arbitrary boundary fields are constructed. It is found that by employing two sets of gauge transformations, proper generators and reference state for constructing Bethe vectors can be obtained respectively. Given an inhomogeneous $T - Q$ relation for an eigenvalue, it is proven that the resulting Bethe state is an eigenstate of the transfer matrix, provided that the parameters of the generators satisfy the associated Bethe Ansatz equations.

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1 Introduction

In this paper we focus on constructing the Bethe-type eigenstates (Bethe states) of the quantum XXZ spin-$\frac{1}{2}$ chain with arbitrary boundary fields, defined by the Hamiltonian

$$H = \sum_{j=1}^{N-1} \{ \sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y + \cosh \eta \sigma_j^z \sigma_{j+1}^z \} + \vec{h}_1 \cdot \vec{\sigma}_1 + \vec{h}_N \cdot \vec{\sigma}_N$$

where $\sigma_j^\alpha (\alpha = x, y, z)$ is the Pauli matrix on site $j$ and along $\alpha$ direction, and $\alpha_\pm, \beta_\pm, \theta_\pm$ are the boundary parameters associated with the boundary fields. The model has played a fundamental role in the study of quantum integrable system [1, 2, 3] with boundaries. Moreover, it has many applications in the non-perturbative analysis of quantum systems appearing in string and super-symmetric Yang-Mills (SYM) theories [4] (and references therein), low-dimensional condensed matter physics [5] and statistical physics [6, 7]. However, the Bethe Ansatz solution of the model for generic values of boundary fields has challenged for many years since Sklyanin’s elegant work [3], and many efforts had been made [8, 9, 10, 11, 12, 13, 14, 15, 16, 17] to approach this nontrivial problem.

The off-diagonal Bethe Ansatz (ODBA) provides an efficient method [18] for solving the eigenvalue problem of integrable models with generic integrable boundary conditions. Several long-standing models [18, 19, 20, 21, 22, 23, 24] including the XXZ spin-$\frac{1}{2}$ chain have since been solved via this method. The central point is to construct a proper $T-Q$ relation [25, 26], which immediately leads to the Bethe Ansatz solution for the eigenvalues, with an extra off-diagonal (or inhomogeneous) term based on their functional relations. An interesting issue in this framework is how to retrieve the Bethe states from the obtained spectrum. Indeed, significant progress has been achieved in this aspect recently. For example, based on the inhomogeneous $T-Q$ relation obtained in [19], the Bethe states of the open XXX spin chain was conjectured in [27] via the modified algebraic Bethe ansatz and then proven in [28]. Alternatively, a set of eigenstates of the inhomogeneous XXZ transfer matrix was
derived in [16, 29] via the separation of variables (SoV) method [30]. However, how to get the homogeneous limit (if there is any) of those SoV states is still an open problem. It is also interesting that the eigenstates in homogeneous limit can be classified by the representation of the q-Onsager algebra [31, 32].

For the open XXZ chain, when the boundary fields are all along $z$-direction (or the diagonal boundaries), the corresponding Bethe states were constructed by the algebraic Bethe Ansatz method [3, 33]. The unparallel boundary fields break the $U(1)$-symmetry (i.e., the total spin is not conserved any more). This makes the problem of constructing Bethe vectors rather unusual because of the absence of an obvious reference state. So far, the Bethe states could only be obtained for some constrained boundary parameters. When the boundary parameters obey a constraint [8, 9], which is already in $U(1)$-symmetry-broken case, the associated Bethe states were constructed [9] within the framework of the generalized algebraic Bethe Ansatz [25, 34]. Very recently, based on the inhomogeneous $T − Q$ relation and small sites analysis of the model with triangular boundaries, the corresponding Bethe states are conjectured [35] and proven in [36]. In this paper we study the Bethe states of the transfer matrix for the quantum XXZ spin-$1/2$ chain with arbitrary boundary fields based on the inhomogeneous $T − Q$ relation of the eigenvalues obtained by ODBA.

The paper is organized as follows. Section 2 serves as an introduction of our notations and the ODBA solutions of the model. In section 3, after introducing the gauge transformations and the associated left (right) state, we compute the associated commutation relations among the matrix elements of the two gauged double-row monodromy matrices, and their actions on the associated state. In section 4, two particular gauge transformations are chosen according to the boundary parameters of $K$-matrices respectively. Based on the chosen parameters of the resulting transformations, the Bethe-type eigenstates of the transfer matrix are constructed. In section 5, we summarize our results and give the concluding remarks. Some useful formulae and technical proofs are given in Appendices A-C respectively.
2 ODBA solution

Let $V$ be a two-dimensional vector space. For the XXZ spin chain with generic boundaries, the associated $R$-matrix and the reflection matrices $K^\mp(u)$ \([37, 38]\) read

$$ R(u) = \frac{1}{\sinh \eta} \begin{pmatrix} \sinh(u+\eta) & 0 & 0 & 0 \\ 0 & \sinh u & \sinh \eta & 0 \\ 0 & \sinh \eta & \sinh u & 0 \\ 0 & 0 & 0 & \sinh(u+\eta) \end{pmatrix}, \quad (2.1) $$

$$ K^- (u) = \begin{pmatrix} K^-_{11}(u) & K^-_{12}(u) \\ K^-_{21}(u) & K^-_{22}(u) \end{pmatrix}, $$

$$ K^-_{11}(u) = 2 (\sinh(\alpha_-) \cosh(\beta_-) \cosh(u) + \cosh(\alpha_-) \sinh(\beta_-) \sinh(u)), $$

$$ K^-_{22}(u) = 2 (\sinh(\alpha_-) \cosh(\beta_-) \cosh(u) - \cosh(\alpha_-) \sinh(\beta_-) \sinh(u)), $$

$$ K^-_{12}(u) = e^\theta^- \sinh(2u), \quad K^-_{21}(u) = e^{-\theta^-} \sinh(2u), \quad (2.2) $$

and

$$ K^+(u) = K^-(-u-\eta)|_{(\alpha_-, \beta_-, \theta_-) \rightarrow (-\alpha_+, -\beta_+, \theta_+)}; \quad (2.3) $$

where $\eta$ is the crossing parameter, and $\alpha_\mp, \beta_\mp, \theta_\mp$ are the boundary parameters associated with boundary fields (see (1.1)). The $R$-matrix is a solution of the quantum Yang-Baxter equation (QYBE)

$$ R_{12}(u_1 - u_2)R_{13}(u_1 - u_3)R_{23}(u_2 - u_3) = R_{23}(u_2 - u_3)R_{13}(u_1 - u_3)R_{12}(u_1 - u_2), \quad (2.4) $$

and $K^+(u)$ satisfy the following reflection equations (RE)

$$ R_{12}(u_1 - u_2)K^-_{11}(u_1)R_{21}(u_1 + u_2)K^-_{22}(u_2) $$

$$ = K^-_{22}(u_2)R_{12}(u_1 + u_2)K^-_{11}(u_1)R_{21}(u_1 - u_2), \quad (2.5) $$

and

$$ R_{12}(u_2 - u_1)K^+_{11}(u_1)R_{21}(-u_1 - u_2 - 2\eta)K^+_{22}(u_2) $$

$$ = K^+_{22}(u_2)R_{12}(-u_1 - u_2 - 2\eta)K^+_{11}(u_1)R_{21}(u_2 - u_1). \quad (2.6) $$

Here and below we adopt the standard notations: for any matrix $A \in \text{End}(V)$, $A_j$ is an embedding operator in the tensor space $V \otimes V \otimes \cdots$, which acts as $A$ on the $j$-th space and
as identity on the other factor spaces; \( R_{ij}(u) \) is an embedding operator of \( R \)-matrix in the tensor space, which acts as identity on the factor spaces except for the \( i \)-th and \( j \)-th ones.

We introduce the “row-to-row” (or one-row) monodromy matrices \( T_0(u) \) and \( \hat{T}_0(u) \), which are \( 2 \times 2 \) matrices with elements being operators acting on the tensor space \( V^\otimes N \),

\[
T_0(u) = R_{0N}(u - \theta_N)R_{0N-1}(u - \theta_{N-1}) \cdots R_{01}(u - \theta_1), \tag{2.7}
\]

\[
\hat{T}_0(u) = R_{10}(u + \theta_1)R_{20}(u + \theta_2) \cdots R_{N0}(u + \theta_N). \tag{2.8}
\]

Here \( \{ \theta_j | j = 1, \cdots, N \} \) are the inhomogeneous parameters. For open spin chains, one needs to consider the double-row monodromy matrix \( \mathcal{U}_0(u) \)

\[
\mathcal{U}_0(u) = T_0(u)K^\tau_0(u)\hat{T}_0(u). \tag{2.9}
\]

The double-row transfer matrix \( t(u) \) is thus given by

\[
t(u) = \text{tr}_0(K^+_0(u)\mathcal{U}_0(u)). \tag{2.10}
\]

The QYBE (2.4) and REs (2.5) and (2.6) lead to the fact that the transfer matrices with different spectral parameters commute with each other \([3]\): \([t(u), t(v)] = 0\). Then \( t(u) \) serves as the generating functional of the conserved quantities of the corresponding system, which ensures the integrability of the open spin chain.

The Hamiltonian (1.1) is expressed in terms of the transfer matrix (2.10) with the \( K \)-matrices (2.2) and (2.3) by

\[
H = \sinh \eta \frac{\partial \ln t(u)}{\partial u} |_{u=0,\theta_j=0} - N \cosh \eta - \tanh \eta \sinh \eta. \tag{2.11}
\]

It was proven in [20] that for generic \( \{ \theta_j \} \) the transfer matrix given by (2.10) for arbitrary boundary parameters satisfies the following operator identities

\[
t(\theta_j) t(\theta_j - \eta) = a(\theta_j) d(\theta_j - \eta) \times \text{id}, \tag{2.12}
\]

\[
t(-u - \eta) = t(u), \quad t(u + i\pi) = t(u), \tag{2.13}
\]

\[
t(0) = -2^3 \sinh \alpha_- \cosh \beta_- \sinh \alpha_+ \cosh \beta_+ \cosh \eta \times \prod_{l=1}^{N} \frac{\sinh(\eta - \theta_l) \sinh(\eta + \theta_l)}{\sinh^2 \eta} \times \text{id}, \tag{2.14}
\]

\[
t\left(\frac{i\pi}{2}\right) = -2^3 \cosh \alpha_- \sinh \beta_- \cosh \alpha_+ \sinh \beta_+ \cosh \eta
\]
\[
\lim_{u \to \pm \infty} t(u) = -\frac{\cosh(\theta_+ - \theta_+) e^{\pm [(2N+4)u+(N+2)\eta]}}{2^{2N+1} \sinh^{2N} \eta} \times \text{id} + \ldots,
\]

where the functions \( a(u) \) and \( d(u) \) are given by

\[
a(u) = -2^2 \frac{\sinh(2u+2\eta)}{\sinh(2u+\eta)} \sinh(u-\alpha_-) \cosh(u-\beta_-) \sinh(u-\alpha_+) \cosh(u-\beta_+) \bar{A}(u), \quad (2.17)
\]

\[
d(u) = a(-u-\eta), \quad \bar{A}(u) = \prod_{l=1}^{N} \frac{\sinh(u-\theta_l + \eta) \sinh(u+\theta_l + \eta)}{\sinh^{2} \eta}.
\]  

The above operator relations lead to that the corresponding eigenvalue of the transfer matrix, denoted by \( \Lambda(u) \), enjoys the following properties

\[
\Lambda(\theta_j) \Lambda(\theta_j - \eta) = a(\theta_j) d(\theta_j - \eta), \quad j = 1, \ldots, N, \quad (2.19)
\]

\[
\Lambda(-u - \eta) = \Lambda(u), \quad \Lambda(u + i\pi) = \Lambda(u), \quad (2.20)
\]

\[
\Lambda(0) = -2^3 \sinh \alpha_- \cosh \beta_- \sinh \alpha_+ \cosh \beta_+ \cosh \eta \prod_{l=1}^{N} \frac{\sinh(\eta - \theta_l) \sinh(\eta + \theta_l)}{\sinh^{2} \eta}, \quad (2.21)
\]

\[
\Lambda\left(\frac{i\pi}{2}\right) = -2^3 \cosh \alpha_- \sinh \beta_- \cosh \alpha_+ \sinh \beta_+ \cosh \eta
\]

\[
\times \prod_{l=1}^{N} \frac{\sinh\left(\frac{i\pi}{2} + \theta_l + \eta\right) \sinh\left(\frac{i\pi}{2} + \theta_l - \eta\right)}{\sinh^{2} \eta},
\]  

\[
\lim_{u \to \pm \infty} \Lambda(u) = -\frac{\cosh(\theta_- - \theta_+) e^{\pm [(2N+4)u+(N+2)\eta]}}{2^{2N+1} \sinh^{2N} \eta} + \ldots
\]

\[
\Lambda(u), \text{ as an entire function of } u, \text{ is a trigonometric polynomial of degree } 2N + 4. \quad (2.24)
\]

Each solution of (2.19)-(2.24) can be given in terms of the following inhomogeneous \( T - Q \) relation [19, 20, 39, 40] \(^3\)

\[
\Lambda(u) = a(u) \frac{Q(u-\eta)}{Q(u)} + d(u) \frac{Q(u+\eta)}{Q(u)}
\]

\[
+ \frac{2c \sinh(2u) \sinh(2u+2\eta)}{Q(u)} \bar{A}(u) \bar{A}(-u-\eta), \quad (2.25)
\]

\(^3\)The inhomogeneous \( T - Q \) relation (2.23) corresponds to the special case (i.e., \( M = 0 \)) of the general ones in [19], which was first proposed for the \( XXX \) case and its validity for the \( XXZ \) case was also pointed out in [20]. The relation was then confirmed by the SoV method [41] for the \( XXZ \) case and its generalization to higher spin case was given in [40, 42].
where $c$ is a constant depending on the boundary parameters

\[ c = \cosh(\alpha_+ + \beta + \alpha + \beta + (1 + N)\eta) - \cosh(\theta_+ - \theta_+), \quad (2.26) \]

and the $Q$-function is given by

\[ Q(u) = \prod_{j=1}^{N} \frac{\sinh(u - \lambda_j) \sinh(u + \lambda_j + \eta)}{\sinh \eta \sinh \eta}, \quad (2.27) \]

with the parameters $\{\lambda_j\}$ satisfying the associated Bethe ansatz equations (BAEs)

\[ a(\lambda_j)Q(\lambda_j - \eta) + d(\lambda_j)Q(\lambda_j + \eta) + 2c \sinh 2\lambda_j \sinh(2\lambda_j + 2\eta)\bar{A}(\lambda_j)\bar{A}(-\lambda_j - \eta) = 0, \quad j = 1, \ldots, N. \quad (2.28) \]

We shall show in Section 4 that for each solution of (2.19)-(2.24), one can construct a corresponding Bethe-type eigenstate (see (4.13) below) of the transfer matrix (2.10) with the eigenvalue given by (2.25). Therefore the relations (2.19)-(2.24) (or the inhomogeneous $T-Q$ relation (2.25)) indeed completely characterize the spectrum of the transfer matrix.

Some remarks are in order. There exist various possible ways \[19\] to parameterize the solution of (2.19)-(2.24), but they are all equivalent to each other because of the finite number of solutions. For generic boundary parameters, the minimal degree of the $Q$-polynomial is $N$, while the degree of the $Q$-polynomial may be reduced to a small value in case of the inhomogeneous term (or the third term in (2.25)) vanishing. In this case the $T-Q$ relation becomes a homogeneous one (the well-known Baxter’s $T-Q$ relation). This happens in case of $U(1)$ symmetry or in degenerate cases \[9\], for which the transfer matrix can be diagonalized in smaller blocks.

3 Gauge transformations and the associated operators

A particular set of gauge transformation (the six-vertex version of the vertex-face correspondence), which have played a key role to construct the associated Bethe states, was proposed in \[9\]. Recently, such gauge transformation was adopted in constructing the SoV eigenstates \[29\] and the Bethe states \[35\] for the open chains. In this paper, we use two sets of such gauge transformation and the inhomogeneous $T-Q$ relation (2.25) to construct the Bethe states for the quantum XXZ spin-$\frac{1}{2}$ chain with arbitrary boundary fields.
Following [9], let us introduce two column vectors as follows

\[
X_m(u|\alpha) = \left( e^{-[u+(\alpha+m)\eta]} \right), \quad Y_m(u|\alpha) = \left( e^{-[u+(\alpha-m)\eta]} \right),
\]

where \( \alpha \) and \( m \) are two arbitrary complex parameters. For generic \( \alpha \) and \( m \), the two vectors are linearly independent. Thus one can introduce the following gauge matrices

\[
\bar{M}_m(u|\alpha) = \left( X_m(u|\alpha), \ Y_m(u|\alpha) \right), \quad \bar{M}_m^{-1}(u) = \left( \frac{\bar{Y}_m(u|\alpha)}{\bar{X}_m(u|\alpha)} \right),
\]

where

\[
\bar{X}_m(u|\alpha) = \frac{e^{u+\alpha\eta}}{2\sinh m\eta} \left( 1, \ -e^{-[u+(\alpha+m)\eta]} \right), \quad \bar{Y}_m(u|\alpha) = \frac{e^{u+\alpha\eta}}{2\sinh m\eta} \left( -1, \ e^{-[u+(\alpha-m)\eta]} \right),
\]

\[
\tilde{X}_m(u|\alpha) = \frac{e^{\eta\sinh(m-1)\eta}}{\sinh(m-1)\eta} X_m(u|\alpha), \quad \tilde{Y}_m(u|\alpha) = \frac{e^{\eta\sinh(m+1)\eta}}{\sinh(m+1)\eta} Y_m(u|\alpha),
\]

\[
\hat{X}_m(u|\alpha) = \frac{e^{-\eta\sinh(m+2)\eta}}{\sinh(m+1)\eta} X_m(u|\alpha), \quad \hat{Y}_m(u|\alpha) = \frac{e^{-\eta\sinh(m-2)\eta}}{\sinh(m-1)\eta} Y_m(u|\alpha).
\]

We remark that the vectors \( X_m(u|\alpha) \) and \( \bar{X}_m(u|\alpha) \) only depend on \( \alpha + m \), while the vectors \( Y_m(u|\alpha) \) and \( \bar{Y}_m(u|\alpha) \) only depend on \( \alpha - m \), up to a scaling factor.

These column and row vectors satisfy some intertwining relations [9], which are listed in Appendix A (see (A.1)-(A.28) below). These relations allow us to introduce the following gauged operators and the associated \( K^+ \)-matrix

\[
\overline{\mathcal{W}}(m, \alpha|u) = \left( \begin{array}{cc}
\overline{\mathcal{A}}_m(u|\alpha) & \overline{\mathcal{B}}_m(u|\alpha) \\
\overline{C}_m(u|\alpha) & \overline{D}_m(u|\alpha)
\end{array} \right),
\]

\[
\overline{\mathcal{K}}^+(m, \alpha|u) = \left( \begin{array}{cc}
\overline{K}^+_{11}(m, \alpha|u) & \overline{K}^+_{12}(m, \alpha|u) \\
\overline{K}^+_{21}(m, \alpha|u) & \overline{K}^+_{22}(m, \alpha|u)
\end{array} \right),
\]

\[
\overline{\mathcal{K}}^+(m+2, \alpha|u) = \left( \begin{array}{cc}
\overline{Y}_m(-u|\alpha)K^+(u)X_m(u|\alpha) & \overline{Y}_{m+2}(-u|\alpha)K^+(u)Y_m(u|\alpha) \\
\overline{X}_{m-2}(-u|\alpha)K^+(u)X_m(u|\alpha) & \overline{X}_{m}(-u|\alpha)K^+(u)Y_m(u|\alpha)
\end{array} \right).
\]

8
With the help of the relations \((A.29)-(A.31)\), we can rewrite the transfer matrix \((2.10)\) in terms of the above gauged operators and \(K\)-matrix, namely,

\[
t(u) = \text{tr} \left\{ K^+(u) \mathcal{U}(u) \right\} = \mathcal{K}_{11}^+(m, \alpha|u) \mathcal{C}_m(u|\alpha) + \mathcal{K}_{21}^+(m, \alpha|u) \mathcal{D}_m(u|\alpha) + \mathcal{K}_{12}^+(m, \alpha|u) \mathcal{C}_m(u|\alpha) + \mathcal{K}_{22}^+(m, \alpha|u) \mathcal{D}_m(u|\alpha)
\]

\[
= \text{tr} \left\{ \mathcal{V}(m, \alpha|u) K^+(m, \alpha|u) \right\}.
\]

\[(3.11)\]

The QYBE \((2.4)\), the RE \((2.5)\) and the intertwining relations given in Appendix A allow us to derive the commutation relations among the matrix elements of \(\mathcal{U}(m, \alpha|u)\). Here we present some relevant relations for our purpose:

\[
\mathcal{C}_m(u_1|\alpha) \mathcal{C}_{m+2}(u_2|\alpha) = \mathcal{C}_m(u_2|\alpha) \mathcal{C}_{m+2}(u_1|\alpha),
\]

\[(3.12)\]

\[
\left[ \mathcal{D}_{m-2}(u_2|\alpha), \mathcal{D}_{m-2}(u_1|\alpha) \right] = \frac{\sinh(m\eta + u_1 + u_2) \sinh \eta \mathcal{C}_{m-2}(u_1|\alpha) \mathcal{D}_m(u_2|\alpha)}{\sinh(m\eta \sinh(u_1 + u_2 + \eta))} - \frac{\sinh(m\eta + u_1 + u_2) \sinh \eta \mathcal{C}_{m-2}(u_2|\alpha) \mathcal{D}_m(u_1|\alpha)}{\sinh(m\eta \sinh(u_1 + u_2 + \eta))}.
\]

\[(3.13)\]

\[
\mathcal{D}_{m-2}(u_2|\alpha) \mathcal{C}_{m-2}(u_1|\alpha) = \frac{\sinh(u_1 - u_2 + \eta) \sinh(u_1 + u_2) \mathcal{C}_{m-2}(u_1|\alpha) \mathcal{D}_m(u_2|\alpha)}{\sinh(u_1 + u_2 + \eta)} - \frac{\sinh(u_1 - u_2 + \eta) \sinh(u_1 + u_2) \mathcal{C}_{m-2}(u_2|\alpha) \mathcal{D}_m(u_1|\alpha)}{\sinh(u_1 + u_2 + \eta)}.
\]

\[(3.14)\]

\[
\left[ \mathcal{D}_m(u_2|\alpha), \mathcal{C}_m(u_1|\alpha) \right] = \frac{\sinh(m+1) \sinh \eta \sinh(m\eta - u_1 + u_2) \sinh(u_1 + u_2 + 2\eta)}{\sinh(m+2) \sinh(m+1) \eta \sinh(u_1 - u_2) \sinh(u_1 + u_2 + \eta)} \mathcal{C}_m(u_1|\alpha) \mathcal{D}_{m+2}(u_2|\alpha) - \mathcal{C}_m(u_2|\alpha) \mathcal{D}_{m+2}(u_1|\alpha).
\]

\[(3.15)\]

The proof of the above relations is relegated to Appendix B.

Let us introduce the following left local states of the \(n\)-th site in the lattice:

\[
\langle \omega; m, \alpha|_n = \mathcal{X}_{m+n-N-1}(\theta_n|\alpha), \quad n = 1, \ldots, N,
\]

\[(3.16)\]

where the row vector \(\mathcal{X}_m(u)\) is given by \((3.5)\). Further, we introduce the following global state from the above local states,

\[
\langle \alpha + m| = 2^N e^{-\sum_{l=1}^N \theta_l - \alpha N \eta} \prod_{l=1}^N \sinh(m - l) \eta \bigotimes_{n=1}^N (\omega; m, \alpha|_n).
\]

\[(3.17)\]
The explicit expression (3.5) of the row vector $\overline{X}_m(u)$ implies that the above state does only depend on $\alpha + m$. Following the method in [9, 43, 44], after some tedious calculation, we obtain the actions of the gauged operators $\overline{\mathcal{E}}_m(u|\alpha)$, $\overline{\mathcal{A}}_m(u|\alpha)$ and $\overline{\mathcal{B}}_m(u|\alpha)$ on the state (3.17) as follows:

$$
\langle \alpha + m | \overline{\mathcal{E}}_m(u|\alpha) \rangle = K_{21}^-(m - N, \alpha|u) \frac{\sinh(m + 2)\eta}{\sinh(m + 2 - N)\eta} \prod_{j=1}^{N} \frac{\sinh(u - \theta_j + \eta)\sinh(u + \theta_j)}{\sinh^2 \eta} \langle \alpha + m + 2 |, \tag{3.18}\rangle
$$

$$
\langle \alpha + m | \overline{\mathcal{A}}_m(u|\alpha) \rangle = K_{22}^-(m - N, \alpha|u) \prod_{j=1}^{N} \frac{\sinh(u - \theta_j + \eta)\sinh(u + \theta_j + \eta)}{\sinh^2 \eta} \langle \alpha + m | + K_{21}^-(m - N, \alpha|u) \prod_{j=1}^{N} \frac{\sinh(u - \theta_j + \eta)}{\sinh \eta} \langle \alpha + m + 1 | \overline{B}_{m+1}(u|\alpha), \tag{3.19}\rangle
$$

$$
\langle \alpha + m | \overline{\mathcal{B}}_m(u|\alpha) \rangle = \frac{\sinh(2u - (m - 1)\eta)}{\sinh(2u + \eta)\sinh(1 - m)\eta} \prod_{j=1}^{N} \frac{\sinh(u - \theta_j + \eta)\sinh(u + \theta_j)}{\sinh^2 \eta} \left\{ K_{22}^-(m - N, \alpha|u) \prod_{j=1}^{N} \frac{\sinh(u - \theta_j + \eta)\sinh(u + \theta_j + \eta)}{\sinh^2 \eta} \langle \alpha + m | + K_{21}^-(m - N, \alpha|u) \prod_{j=1}^{N} \frac{\sinh(u - \theta_j + \eta)}{\sinh \eta} \langle \alpha + m + 1 | \overline{B}_{m+1}(u|\alpha) \right\} + F(u). \tag{3.20}\rangle
$$

Here we have introduced the gauged $K^-$-matrix

$$
\overline{K}^-(l', \alpha|u) = \left( \begin{array}{c} K_{11}^-(l', \alpha|u) \\ K_{12}^-(l', \alpha|u) \\ K_{21}^-(l', \alpha|u) \\ K_{22}^-(l', \alpha|u) \end{array} \right) = \left( \begin{array}{c} \overline{Y}_v(u|\alpha)K^{-}(u)\hat{X}_{v-2}(-u|\alpha) \\ \overline{Y}_v(u|\alpha)K^{-}(u)\hat{X}_{v}(-u|\alpha) \\ \overline{Y}_v(u|\alpha)K^{-}(u)\hat{X}_{v+2}(-u|\alpha) \\ \overline{X}_v(u|\alpha)K^{-}(u)\hat{X}_{v-2}(-u|\alpha) \\ \overline{X}_v(u|\alpha)K^{-}(u)\hat{X}_{v}(-u|\alpha) \end{array} \right), \tag{3.21}\rangle
$$

with $l' = m - N$, and the gauged operator $\overline{B}_m(u|\alpha)$ is given by

$$
\overline{B}_m(u|\alpha) = \overline{Y}_{m-N+1}(-u|\alpha)\hat{T}(u)\hat{Y}_{m+1}(-u|\alpha). \tag{3.22}\rangle
$$

The extra term $F(u)$ in (3.20) actually vanishes at the points $\{-\theta_j | j = 1, \ldots, N\}$, namely,

$$
F(-\theta_j) = 0, \quad j = 1, \ldots, N. \tag{3.23}\rangle
$$

This fact gives rise to the following important relations

$$
\langle \alpha + m | \overline{\mathcal{A}}_m(-\theta_j|\alpha) \rangle = \frac{\sinh((m - 1)\eta + 2\theta_j)\sinh \eta}{\sinh(m - 1)\eta \sinh(2\theta_j - \eta)} \langle \alpha + m | \overline{\mathcal{A}}_m(-\theta_j|\alpha) \rangle. \tag{3.24}\rangle
$$
The associated right state (c.f. (3.17)), which only depends on $\alpha + m$, is given by

$$|\alpha + m\rangle = \bigotimes_{n=1}^{N} X_{m+N-n+1}(\theta_n|\alpha\rangle),$$

(3.25)

and the associated gauged operators are

$$\mathcal{U}(m, \alpha|u) = \left( \begin{array}{cc} \mathcal{A}_m(u|\alpha) & \mathcal{B}_m(u|\alpha) \\ \mathcal{C}_m(u|\alpha) & \mathcal{D}_m(u|\alpha) \end{array} \right),$$

$$= \left( \begin{array}{cc} \tilde{Y}_{m-2}(u|\alpha) \mathcal{U}(u) X_{m}(-u|\alpha) & \tilde{Y}_{m}(u|\alpha) \mathcal{U}(u) Y_{m}(-u|\alpha) \\ \tilde{X}_{m}(u|\alpha) \mathcal{U}(u) X_{m}(-u|\alpha) & \tilde{X}_{m+2}(u|\alpha) \mathcal{U}(u) Y_{m}(-u|\alpha) \end{array} \right).$$

(3.26)

The matrix elements of the above gauged monodromy matrix acting on the state (3.25) were given in [9]. Here we present some relevant ones

$$\mathcal{C}_m(u|\alpha)|\alpha + m\rangle = K_{21}(l, \alpha|u) \frac{\sinh(m + N - 1)\eta}{\sinh(m - 1)\eta} \times \prod_{j=1}^{N} \frac{\sinh(u - \theta_j)\sinh(u + \theta_j + \eta)}{\sinh^2 \eta}|\alpha + m - 2\rangle,$$

(3.27)

$$\mathcal{A}_m(u|\alpha)|\alpha + m\rangle = K_{11}(l, \alpha|u) \prod_{j=1}^{N} \frac{\sinh(u - \theta_j + \eta)\sinh(u + \theta_j + \eta)}{\sinh^2 \eta}|\alpha + m\rangle$$

$$+ K_{21}(l, \alpha|u) \prod_{j=1}^{N} \frac{\sinh(u + \theta_j + \eta)}{\sinh \eta} B_{m-1}(u|\alpha)|\alpha + m - 1\rangle,$$

(3.28)

with $l = m + N$. Here another gauged $K^-$-matrix is (c.f., (3.21))

$$K^-(l, \alpha|u) = \left( \begin{array}{cc} K_{11}^-(l, \alpha|u) & K_{12}^-(l, \alpha|u) \\ K_{21}^-(l, \alpha|u) & K_{22}^-(l, \alpha|u) \end{array} \right),$$

$$= \left( \begin{array}{cc} \tilde{Y}_{l-2}(u|\alpha) K^-(-u|\alpha) & \tilde{Y}_{l}(u|\alpha) K^-(-u|\alpha) \\ \tilde{X}_{l}(u|\alpha) K^-(u|\alpha) & \tilde{X}_{l+2}(u|\alpha) K^-(-u|\alpha) \end{array} \right).$$

(3.29)

and the gauged operator $B_{m}(u|\alpha)$ is given by

$$B_{m}(u|\alpha) = \tilde{Y}_{m-1}(u) T(u) Y_{m+N-1}(u).$$

(3.30)

4 Bethe states

Up to now, the parameters $\alpha$ and $m$ in the definitions of the gauged operator $\mathcal{U}(m, \alpha|u)$ in (3.9) and the associated $K$-matrix $\mathcal{K}^+(m, \alpha|u)$ in (3.10) (resp. $\mathcal{U}(m, \alpha|u)$ in (3.26) and the
associated $K$-matrix $K^-(m, \alpha|u)$ in (3.29) are arbitrary. The works in \cite{27,28,35} shed light on the two important facts to construct the Bethe-type eigenstates of the $U(1)$-symmetry-broken integrable models: (1) The inhomogeneous $T-Q$ relation plays a central role in constructing the Bethe states because it enables one in this case to tell the wanted term from the unwanted ones within the framework of the algebraic Bethe Ansatz method; (2) It also suggests that in order to construct the right Bethe states of the transfer matrix (2.10), one may choose the two parameters $\alpha$ and $m$ according to the boundary parameters $\alpha_+, \beta_+$ and $\theta_+$ to construct the generators (resp. according to the boundary parameters $\alpha_-, \beta_-$ and $\theta_-$ to seek the associated reference state).

For this purpose, let us choose the gauge parameters in (3.10) as follows

\[
\begin{aligned}
\alpha \eta & \overset{\text{def}}{=} \alpha^{(l)} \eta = \eta - \theta_+ + i \frac{\pi}{2} \mod (2i\pi), \\
\m \eta & \overset{\text{def}}{=} m^{(l)} \eta = \alpha_+ + \beta_+ - i \frac{\pi}{2} \mod (2i\pi).
\end{aligned}
\]

(4.1)

In this particular choice of the gauged parameters, the corresponding gauged $K$-matrix $\overline{K}^+(m, \alpha|u)$ given by (3.10) becomes diagonal

\[
\overline{K}^+(m^{(l)}, \alpha^{(l)}|u) = \text{Diag}(\overline{K}^+_{11}(m^{(l)}, \alpha^{(l)}|u), \overline{K}^+_{22}(m^{(l)}, \alpha^{(l)}|u)),
\]

(4.2)

where the non-vanishing matrix elements read

\[
\overline{K}^+_{11}(m^{(l)}, \alpha^{(l)}|u) = \frac{-2e^{-u}}{\cosh(\alpha_+ + \beta_+)} \sinh(u + \alpha_+ + \eta) \cosh(u + \beta_+ + \eta) \cosh(\alpha_+ + \beta_+ - \eta),
\]

(4.3)

\[
\overline{K}^+_{22}(m^{(l)}, \alpha^{(l)}|u) = \frac{2e^{-u}}{\cosh(\alpha_+ + \beta_+)} \sinh(u - \alpha_+ + \eta) \cosh(u - \beta_+ + \eta) \cosh(\alpha_+ + \beta_+ + \eta).
\]

(4.4)

In this case the the transfer matrix (2.10) (see also (3.11)) can be rewritten as

\[
t(u) = \text{tr} \{ K^+(u) \mathcal{Z}(u) \} = \text{tr} \left\{ \overline{\mathcal{W}}(m^{(l)}, \alpha^{(l)}|u) \overline{K}^+(m^{(l)}, \alpha^{(l)}|u) \right\}
= \overline{K}^+_{11}(m^{(l)}, \alpha^{(l)}|u) \overline{\mathcal{W}}_{m^{(l)}}(u|\alpha^{(l)}) + \overline{K}^+_{22}(m^{(l)}, \alpha^{(l)}|u) \overline{\mathcal{W}}_{m^{(l)}}(u|\alpha^{(l)}).
\]

(4.5)

Direct calculation shows that the following identity holds

\[
\overline{K}^+_{22}(m^{(l)}, \alpha^{(l)}|u) + \frac{\sinh \eta \sinh((m^{(l)} - 1)\eta - 2u)}{\sinh(2u + \eta) \sinh(m^{(l)} - 1)\eta} \overline{K}^+_{11}(m^{(l)}, \alpha^{(l)}|u)
= 2e^{-u} \frac{\sinh(2u + 2\eta)}{\sinh(2u + \eta)} \sinh(u - \alpha_+) \cosh(u - \beta_+).
\]

(4.6)

\(^{4}\text{Construction of left Bethe states is straightforward with a similar procedure.}\)
Then let us choose the gauge parameters in (3.29) such that the following relation is satisfied

\[(m^{(r)} + \alpha^{(r)})\eta = -\theta_+ + \alpha_+ + \beta_+ - N\eta + i\pi \mod (2i\pi). \tag{4.7}\]

In this case the corresponding gauged $K$-matrix $K^-(m^{(r)} + N, \alpha^{(r)}|u)$ given by (3.29) becomes up-triangular with the matrix element $K^-_{11}(m^{(r)} + N, \alpha^{(r)}|u)$ fixed, namely,

\[K^-_{21}(m^{(r)} + N, \alpha^{(r)}|u) = 0, \quad K^-_{11}(m^{(r)} + N, \alpha^{(r)}|u) = -2e^u \sinh(u - \alpha_-) \cosh(u - \beta_-. \tag{4.8}\]

Although neither the parameter $\alpha^{(r)}$ nor $m^{(r)}$ is fixed by the up-triangularity condition of $K^-(m^{(r)}, \alpha^{(r)}|u)$, the sum of the two parameters is unique as shown in (4.7). This allows us to define a unique reference state $|\Omega\rangle$,

\[|\Omega\rangle = |\alpha^{(r)} + m^{(r)}\rangle, \tag{4.9}\]

where the state $|\alpha^{(r)} + m^{(r)}\rangle$ is defined by (3.25) with the parameter $\alpha + m$ fixed by the boundary parameters (see (4.7)). It should be noted that the reference state $|\Omega\rangle$ is rather different from that used in algebraic Bethe ansatz (namely, the all spin-up or spin-down state \[3, 26\]).

Let $|\Psi\rangle$ be an eigenstate of the transfer matrix $t(u)$ with an eigenvalue $\Lambda(u)$, namely,

\[t(u) |\Psi\rangle = \Lambda(u) |\Psi\rangle. \tag{4.10}\]

Due to the fact that the left states $\{ |(\alpha^{(l)}, m^{(l)}; \theta_{p_1}, \cdots, \theta_{p_n}|n = 0, \cdots, N, 1 \leq p_1 < p_2 < \cdots < p_n \leq N\}$ given by (C.1) form a basis of the dual Hilbert space, the eigenstate $|\Psi\rangle$ is completely determined (up to an overall scalar factor) by the following scalar products \[13, 28\]

\[F_n(\theta_{p_1}, \cdots, \theta_{p_n}) = \langle \alpha^{(l)}, m^{(l)}; \theta_{p_1}, \cdots, \theta_{p_n}|\Psi\rangle, \quad n = 0, \cdots, N. \tag{4.11}\]

After a tedious calculation, we have that the above scalar products are given by

\[F_n(\theta_{p_1}, \cdots, \theta_{p_n}) = \prod_{j=1}^n \left\{ \frac{-\sinh(2\theta_{p_j} - \eta)\Lambda(-\theta_{p_j})e^{-\theta_{p_j}}}{2\sinh(2\theta_{p_j} - 2\eta)\sinh(\theta_{p_j} + \alpha_+ ) \cosh(\theta_{p_j} + \beta_+ )} \right\} F_0, \]

\[n = 0, 1, \cdots, N, \tag{4.12}\]

where $F_0 = \langle \alpha^{(l)}, m^{(l)}|\Psi\rangle$ is an overall scalar factor. The proof of the above relations is relegated to Appendix C.
Following the method developed in [28], we propose that the Bethe-type eigenstate of the transfer matrix (2.10) for the present model is given by

$$\left| \lambda_1, \cdots, \lambda_N \right> = \overline{g}_{m(l)}(\lambda_1 | \alpha^{(l)} ) \overline{g}_{m(l) + 2}(\lambda_2 | \alpha^{(l)} ) \cdots \overline{g}_{m(l) + 2(N-1)}(\lambda_N | \alpha^{(l)} ) | \Omega \rangle, \quad (4.13)$$

where the two parameters $\alpha^{(l)}$ and $m^{(l)}$ are given by (4.1) and the $N$ parameters $\{ \lambda_j | j = 1, \cdots, N \}$ satisfy the BAEs (2.28). We shall show that the chosen reference state $| \Omega \rangle$ given by (4.9) indeed makes the conditions (4.12) fulfilled. For an eigenvalue $\Lambda(u)$ given by the inhomogeneous $T-Q$ relation (2.25), its value at the point $-\theta_j$ takes a simple form:

$$\Lambda(-\theta_j) = a(-\theta_j) \frac{Q(-\theta_j - \eta)}{Q(-\theta_j)}, \quad j = 1, \cdots, N. \quad (4.14)$$

The above relations and the equations (C.2) imply that the conditions (4.12) are equivalent to the following requirements on the reference state:

$$\langle \alpha^{(l)}, m'; \theta_{p_1}, \cdots, \theta_{p_n} | \Omega \rangle = \prod_{j=1}^{n} \left\{ 2e^{-\theta_{p_j}} \sinh(\theta_{p_j} + \alpha_-) \cosh(\theta_{p_j} + \beta_-) \overline{A}(-\theta_{p_j}) \right\} \frac{F_0}{G_0},$$

$$n = 0, 1, \cdots, N, \quad (4.15)$$

where $m' = m^{(l)} + 2N$ and the overall coefficient $G_0$ independent upon $n$ is

$$G_0 = \prod_{j=1}^{N} g_0(\lambda_j | m^{(l)} + 2(j - 1), \alpha^{(l)} ), \quad (4.16)$$

with function $g_0(u|m, \alpha)$ given by (C.4). Actually, the above conditions uniquely determine the reference state $| \Omega \rangle$ up to a scalar factor. Direct calculation shows that the state $| \Omega \rangle$ given by (4.9) indeed satisfies the conditions (4.15). The proof is relegated to Appendix C. Finally, we conclude that the Bethe state $| \lambda_1, \cdots, \lambda_N \rangle$ becomes an eigenstate of the transfer matrix $t(u)$ with the eigenvalue $\Lambda(u)$ given by (2.25) provided that the reference state $| \Omega \rangle$ is given by (4.9) and the $N$ parameters $\{ \lambda_j | j = 1, \cdots, N \}$ satisfy the BAEs (2.28).

From the definitions (3.1)-(3.8) of the gauge matrices, it is clear that both the reference state $| \Omega \rangle$ and the generators $\overline{g}_{m(l) + 2j}(u | \alpha^{(l)} )$ have well-defined homogeneous limits: $\{ \theta_j \to 0 \}$. This implies that the homogeneous limit of the Bethe state (4.13) exactly gives rise to the corresponding Bethe state of the homogeneous XXZ spin-$\frac{1}{2}$ chain with arbitrary boundary fields, where the associated $T-Q$ relation and BAEs are given by (2.25) and (2.28) with $\{ \theta_j = 0 \}$. It would be interesting to study the relation between our Bethe states and the eigenstates proposed in [29] for which the homogeneous limit is still unclear.
5 Conclusions

It should be emphasized that constructing the Bethe state of $U(1)$-symmetry-broken models had challenged for many years because of the lacking of the inhomogeneous $T - Q$ relations such as (2.25). The idea of this paper to construct the Bethe state is to search for two gauge transformations such that one makes the resulting $K^+$-matrix to be diagonal and the other makes the resulting $K^-$-matrix up-triangular. Then we find that the two parameters $m^{(l)}$ and $\alpha^{(l)}$ of the first gauge transformation must obey the following equations

$$\begin{cases}
\sinh(\alpha_+ + \beta_+) = \sinh(\theta_+ + (\alpha^{(l)} - 1)\eta + m^{(l)}\eta), \\
\sinh(\alpha_+ + \beta_+) = \sinh(\theta_+ + (\alpha^{(l)} - 1)\eta - m^{(l)}\eta),
\end{cases}$$

(5.1)

while the parameters of the second gauge transformation have to satisfy the relation

$$\sinh(\alpha_- + \beta_-) + \sinh(\theta_- + (m^{(r)} + \alpha^{(r)})\eta + N\eta) = 0.$$  

(5.2)

The equation (5.1) is to determine the generators $\mathcal{E}_{m^{(l)}+2j}(u|\alpha^{(l)})$, while the equation (5.2) is to choose the associated reference state (such as (4.9)). It is found that besides the solution given by (4.1) and (4.7) there exist three other solutions of (5.1) and (5.2). Each of the three solutions gives rise to a set of Bethe states with eigenvalues parameterized by a $T - Q$ relation of the form (2.25) by replacing $\alpha_\pm, \beta_\pm$ with $\pm\alpha_\pm, \pm\beta_\pm$. Nevertheless, different types of inhomogeneous $T - Q$ relations [14, 19] only give different parameterizations of the eigenvalues of the transfer matrix but not new solutions. We note that for the degenerate case considered in [9], the present method may not work but the Bethe states can be obtained via generalized algebraic Bethe Ansatz.

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Appendix A: Intertwining relations

We list some intertwining relations (or face-vertex correspondence relations in [9]) which are useful to construct the reference state and the commutation relations among the gauged operators:

\[
R_{12}(u_1 - u_2)X^1_{m+2}(u_1)X^2_{m+1}(u_2) = \frac{\sinh(u_1 - u_2 + \eta)}{\sinh \eta} X^2_{m+2}(u_2)X^1_{m+1}(u_1), \quad (A.1)
\]

\[
R_{12}(u_1 - u_2)X^1_m(u_1)Y^2_{m-1}(u_2) = \frac{\sinh(u_1 - u_2)\sinh(m-1)\eta}{\sinh \eta \sinh m\eta} Y^2_m(u_2)X^1_{m+1}(u_1) \nonumber \\
+ \frac{\sinh(m\eta + u_1 - u_2)}{\sinh m\eta} X^2_m(u_2)Y^1_{m-1}(u_1), \quad (A.2)
\]

\[
R_{12}(u_1 - u_2)Y^1_m(u_1)X^2_{m+1}(u_2) = \frac{\sinh(u_1 - u_2)\sinh(m+1)\eta}{\sinh \eta \sinh m\eta} X^2_m(u_2)Y^1_{m-1}(u_1) \nonumber \\
+ \frac{\sinh(m\eta - u_1 + u_2)}{\sinh m\eta} Y^2_m(u_2)X^1_{m+1}(u_1), \quad (A.3)
\]

\[
R_{12}(u_1 - u_2)Y^1_{m-2}(u_1)Y^2_{m-1}(u_2) = \frac{\sinh(u_1 - u_2 + \eta)}{\sinh \eta} Y^2_{m-2}(u_2)Y^1_{m-1}(u_1), \quad (A.4)
\]

\[
R_{12}(u_1 - u_2)\hat{X}^2_{m-1}(u_2)\hat{X}^1_m(u_1) = \frac{\sinh(u_1 - u_2 + \eta)}{\sinh \eta} \hat{X}^2_{m-1}(u_2)\hat{X}^1_m(u_1), \quad (A.5)
\]

\[
R_{12}(u_1 - u_2)\hat{X}^2_{m-2}(u_2)\hat{X}^1_{m-1}(u_1) = \frac{\sinh(u_1 - u_2)\sinh(m+1)\eta}{\sinh \eta \sinh m\eta} \hat{X}^2_{m-2}(u_2)\hat{X}^1_{m-1}(u_1) \nonumber \\
+ \frac{\sinh(m\eta - u_1 + u_2)}{\sinh m\eta} \hat{X}^2_{m-2}(u_2)\hat{X}^1_{m-1}(u_1), \quad (A.6)
\]

\[
R_{12}(u_1 - u_2)\hat{Y}^2_{m+1}(u_2)\hat{X}^1_{m-2}(u_1) = \frac{\sinh(u_1 - u_2)\sinh(m-1)\eta}{\sinh \eta \sinh m\eta} \hat{Y}^2_{m+1}(u_2)\hat{X}^1_{m-2}(u_1) \nonumber \\
+ \frac{\sinh(m\eta + u_1 - u_2)}{\sinh m\eta} \hat{Y}^2_{m+1}(u_2)\hat{X}^1_{m-2}(u_1), \quad (A.7)
\]

\[
R_{12}(u_1 - u_2)\hat{Y}^2_{m+1}(u_2)\hat{Y}^1_m(u_1) = \frac{\sinh(u_1 - u_2 + \eta)}{\sinh \eta} \hat{Y}^2_{m+1}(u_2)\hat{Y}^1_m(u_1), \quad (A.8)
\]

\[
\hat{X}^1_{m-1}(u_1)\hat{X}^2_{m-2}(u_2)R_{12}(u_1 - u_2) = \frac{\sinh(u_1 - u_2 + \eta)}{\sinh \eta} \hat{X}^2_{m-1}(u_2)\hat{X}^1_{m-2}(u_1), \quad (A.9)
\]

\[
\hat{X}^1_{m-1}(u_1)\hat{Y}^2_m(u_2)R_{12}(u_1 - u_2) = \frac{\sinh(u_1 - u_2)\sinh(m+1)\eta}{\sinh \eta \sinh m\eta} \hat{Y}^2_{m+1}(u_2)\hat{X}^1_m(u_1) \nonumber \\
+ \frac{\sinh(m\eta + u_1 - u_2)}{\sinh m\eta} \hat{X}^2_{m-1}(u_2)\hat{Y}^1_m(u_1), \quad (A.10)
\]

\[\text{In fact these vectors depend also on } \alpha \text{ but as this parameter will not vary in the following relations, in this appendix we omit this argument for simplicity temporarily.}\]
\[
\tilde{\Upsilon}^1_{m+1}(u_1)\tilde{X}^2_m(u_2)R_{12}(u_1 - u_2) = \frac{\sinh(u_1 - u_2) \sinh(m - 1) \eta \tilde{X}^2_{m-1}(u_2) \tilde{\Upsilon}^1_m(u_1)}{\sinh \eta \sinh m \eta} + \frac{\sinh(m \eta - u_1 + u_2) \tilde{Y}^1_{m+1}(u_2) \tilde{X}^1_m(u_1)}{\sinh m \eta}, \tag{A.11}
\]

\[
\tilde{\Upsilon}^1_{m+1}(u_1)\tilde{Y}^1_{m+2}(u_2)R_{12}(u_1 - u_2) = \frac{\sinh(u_1 - u_2 + \eta) \tilde{Y}^1_{m+1}(u_2) \tilde{\Upsilon}^1_m(u_1)}{\sinh \eta}, \tag{A.12}
\]

\[
\tilde{X}^1_{m+1}(u_1)\tilde{X}^2_m(u_2)R_{12}(u_1 - u_2) = \frac{\sinh(u_1 - u_2 + \eta) \tilde{X}^2_m(u_2) \tilde{X}^1_{m+1}(u_1)}{\sinh \eta}, \tag{A.13}
\]

\[
\tilde{X}^1_{m+1}(u_1)\tilde{Y}^2_{m-2}(u_2)R_{12}(u_1 - u_2) = \frac{\sinh(u_1 - u_2) \sinh(m + 1) \eta \tilde{X}^2_{m+1}(u_2) \tilde{X}^1_{m-2}(u_1)}{\sinh \eta \sinh m \eta} + \frac{\sinh(m \eta + u_1 - u_2) \tilde{Y}^1_{m+1}(u_2) \tilde{Y}^1_{m-2}(u_1)}{\sinh m \eta}, \tag{A.14}
\]

\[
\tilde{Y}^1_{m-1}(u_1)\tilde{X}^2_{m-2}(u_2)R_{12}(u_1 - u_2) = \frac{\sinh(u_1 - u_2) \sinh(m - 1) \eta \tilde{X}^2_{m+1}(u_2) \tilde{Y}^1_{m-2}(u_1)}{\sinh \eta \sinh m \eta} + \frac{\sinh(m \eta - u_1 + u_2) \tilde{Y}^1_{m-1}(u_2) \tilde{Y}^1_{m+2}(u_1)}{\sinh m \eta}, \tag{A.15}
\]

\[
\tilde{Y}^1_{m-1}(u_1)\tilde{Y}^2_{m-2}(u_2)R_{12}(u_1 - u_2) = \frac{\sinh(u_1 - u_2 + \eta) \tilde{Y}^2_{m-1}(u_2) \tilde{Y}^1_{m}(u_1)}{\sinh \eta}, \tag{A.16}
\]

\[
\tilde{X}^2_m(u_2)R_{12}(u_1 - u_2)X^1_m(u_1) = \frac{\sinh(u_1 - u_2) \sinh(m - 1) \eta \tilde{X}^2_{m-1}(u_2) X^1_m(u_1)}{\sinh \eta \sinh m \eta}, \tag{A.17}
\]

\[
\tilde{X}^2_m(u_2)R_{12}(u_1 - u_2)Y^1_m(u_1) = \frac{\sinh(u_1 - u_2 + \eta) \tilde{X}^2_m(u_2) Y^1_{m+1}(u_1)}{\sinh \eta} + \frac{\sinh(m \eta - u_1 + u_2) \tilde{Y}^2_{m+1}(u_2) X^1_m(u_1)}{\sinh m \eta}, \tag{A.18}
\]

\[
\tilde{Y}^2_m(u_2)R_{12}(u_1 - u_2)X^1_m(u_1) = \frac{\sinh(u_1 - u_2 + \eta) \tilde{Y}^2_{m-1}(u_2) X^1_{m-1}(u_1)}{\sinh \eta} + \frac{\sinh(m \eta + u_1 - u_2) \tilde{X}^2_{m-1}(u_2) Y^1_{m-1}(u_1)}{\sinh m \eta}, \tag{A.19}
\]

\[
\tilde{Y}^2_m(u_2)R_{12}(u_1 - u_2)Y^1_m(u_1) = \frac{\sinh(u_1 - u_2) \sinh(m + 1) \eta \tilde{Y}^2_{m+1}(u_2) Y^1_{m-1}(u_1)}{\sinh \eta \sinh m \eta}, \tag{A.20}
\]

\[
\tilde{X}^1_{m+1}(u_1)R_{12}(u_1 - u_2)X^2_{m+1}(u_2) = \frac{\sinh(u_1 - u_2) \sinh(m + 1) \eta X^2_m(u_2) \tilde{X}^1_{m+2}(u_1)}{\sinh \eta \sinh m \eta}, \tag{A.21}
\]

\[
\tilde{X}^1_{m+1}(u_1)R_{12}(u_1 - u_2)Y^2_{m-1}(u_2) = \frac{\sinh(u_1 - u_2 + \eta) \tilde{Y}^2_{m-2}(u_2) \tilde{X}^1_{m+1}(u_1)}{\sinh \eta} + \frac{\sinh(m \eta + u_1 - u_2) \tilde{Y}^2_{m-2}(u_2) \tilde{X}^1_{m-2}(u_1)}{\sinh m \eta}, \tag{A.22}
\]

\[
\tilde{Y}^1_{m-1}(u_1)R_{12}(u_1 - u_2)X^2_{m+1}(u_2) = \frac{\sinh(u_1 - u_2 + \eta) \tilde{X}^2_{m+2}(u_2) \tilde{Y}^1_{m}(u_1)}{\sinh \eta}, \tag{A.23}
\]
Moreover, the vectors also enjoy the following orthonormal relations:

\[
\tilde{Y}_{m-1}(u_1) R_{12}(u_1 - u_2) Y_{m-1}(u_2) = \frac{\sinh(u_1 - u_2) \sinh(m - 1) \eta}{\sinh \eta \sinh m \eta} Y_m^2(u_2) \tilde{Y}_{m-2}(u_1), \\
\tilde{X}_{m-1}(u_1) R_{12}(u_1 - u_2) \tilde{X}_{m-1}(u_2) = \frac{\sinh(u_1 - u_2) \sinh(m + 1) \eta}{\sinh \eta \sinh m \eta} X_m^2(u_2) \tilde{X}_{m-2}(u_1), \\
\tilde{X}_{m-1}(u_1) R_{12}(u_1 - u_2) \tilde{Y}_{m+1}(u_2) = \frac{\sinh(u_1 - u_2 + \eta) \tilde{Y}_m^2(u_2) \tilde{X}_{m-1}(u_1)}{\sinh \eta} \\
+ \frac{\sinh(m \eta + u_1 - u_2)}{\sinh m \eta} \tilde{X}_{m-2}(u_2) \tilde{Y}_{m-1}(u_1), \\
\tilde{Y}_{m+1}(u_1) R_{12}(u_1 - u_2) \tilde{X}_{m-1}(u_2) = \frac{\sinh(u_1 - u_2 + \eta) \tilde{X}_m^2(u_2) \tilde{Y}_{m+1}(u_1)}{\sinh \eta} \\
+ \frac{\sinh(m \eta - u_1 + u_2)}{\sinh m \eta} \tilde{Y}_{m+2}(u_2) \tilde{X}_{m-1}(u_1), \\
\tilde{Y}_{m+1}(u_1) R_{12}(u_1 - u_2) \tilde{Y}_{m+1}(u_2) = \frac{\sinh(u_1 - u_2) \sinh(m - 1) \eta}{\sinh \eta \sinh m \eta} \tilde{Y}_{m+2}(u_2) \tilde{Y}_{m-1}(u_1),
\]

where \(X_m^1(u), X_m^2(u)\) are embedding vectors in the 1-st and 2-nd tensor space, respectively.

Moreover, the vectors also enjoy the following orthonormal relations:

\[
\begin{align*}
\nabla_m(u) X_m(u) &= 1, & \nabla_m(u) Y_m(u) &= 0, \\
\nabla_m(u) X_m(u) &= 0, & \nabla_m(u) Y_m(u) &= 1, \\
X_m(u) \tilde{Y}_m(u) + Y_m(u) \tilde{X}_m(u) &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \\
\tilde{Y}_{m-1}(u) X_{m+1}(u) &= 1, & \tilde{Y}_{m-1}(u) Y_{m-1}(u) &= 0, \\
\tilde{X}_{m+1}(u) X_{m+1}(u) &= 0, & \tilde{X}_{m+1}(u) Y_{m-1}(u) &= 1, \\
X_{m+1}(u) \tilde{Y}_{m-1}(u) + Y_{m-1}(u) \tilde{X}_{m+1}(u) &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \\
\nabla_{m+1}(u) \tilde{X}_{m-1}(u) &= 1, & \nabla_{m+1}(u) \tilde{Y}_{m+1}(u) &= 0, \\
\nabla_{m-1}(u) \tilde{X}_{m-1}(u) &= 0, & \nabla_{m-1}(u) \tilde{Y}_{m+1}(u) &= 1, \\
\tilde{X}_{m-1}(u) \tilde{Y}_{m+1}(u) + \tilde{Y}_{m+1}(u) \tilde{X}_{m-1}(u) &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
\end{align*}
\]

**Appendix B: Commutation relations**

Using QYBE (2.3) and the RE (2.5), one may derive that

\[
R_{12}(u_1 - u_2) \mathcal{U}_1(u_1) R_{21}(u_1 + u_2) \mathcal{U}_2(u_2) = \mathcal{U}_2(u_2) R_{21}(u_1 + u_2) \mathcal{U}_1(u_1) R_{12}(u_1 - u_2). \tag{B.1}
\]
Multiplying the above equation with \(X_{m+1}^1(u_1)X_m^2(u_2)\) from the left and \(\hat{X}_{m+1}^1(-u_1)\hat{X}_{m+2}^2(-u_2)\) from the right, and using the relations (A.5) and (A.9), we arrive at (3.12). Similarly, multiplying (B.1) with \(X_{m}^1(u_1)X_m^2(u_2)\) from the left and \(\hat{Y}_{m+1}^1(-u_1)\hat{Y}_{m+2}^2(-u_2)\) from the right and using the intertwining relations (A.1)-(A.28), one can obtain the relation (3.13) (or (3.14)). Using the similar method and the relation (3.13), one can further check (3.15).

**Appendix C: Proof the Bethe state**

There are several ways \([27, 28, 35, 36]\) to show that the state \(|\lambda_1, \ldots, \lambda_N\rangle\) constructed by (4.13) is an eigenstate of the transfer matrix (2.10). Here we adopt the method developed in \([28]\) to demonstrate it.

**C.1 The proof of (4.12)**

For arbitrary parameters \(\alpha, m\) let us introduce the following left states\(^6\) parameterized by the \(N\) inhomogeneous parameters \(\{\theta_j\}\):

\[
\langle \alpha, m; \theta_{p_1}, \ldots, \theta_{p_n} | = \langle \alpha + m | \overline{D}_m(-\theta_{p_1}|\alpha) \ldots \overline{D}_m(-\theta_{p_n}|\alpha),
\]

\[1 \leq q_1 < q_2 < \ldots < q_n \leq N, \quad n = 0, 1, \ldots, N. \quad (C.1)
\]

The commutation relations (3.13), (3.14) and (3.18) imply that

\[
\langle \alpha, m; \theta_{p_1}, \ldots, \theta_{p_n} | C_m(u|\alpha) = g(u, \{\theta_{p_1}, \ldots, \theta_{p_n}\}) \langle \alpha, m+2; \theta_{p_1}, \ldots, \theta_{p_n}, \rangle, \quad (C.2)
\]

where

\[
g(u, \{\theta_{p_1}, \ldots, \theta_{p_n}\}) = g_0(u|m, \alpha) \prod_{j=1}^{n} \frac{\sinh(u + \theta_{p_j} + \eta) \sinh(u - \theta_{p_j})}{\sinh(u - \theta_{p_j} + \eta) \sinh(u + \theta_{p_j})}, \quad (C.3)
\]

and

\[
g_0(u|m, \alpha) = K_{21}(m-N;\alpha|u) \frac{\sinh(m+2)\eta}{\sinh(m+2-N)\eta} \times \prod_{j=1}^{N} \frac{\sinh(u - \theta_j + \eta) \sinh(u + \theta_j)}{\sinh^2 \eta}. \quad (C.4)
\]

\(^6\)Such states were used as a basis to construct the SoV eigenstates of the XXZ open chain \([29]\). Here we use two different gauge transformations respectively for the left and right reference states to reach the Bethe states.
The above equations also lead to the following fact

\[ \langle \alpha, m; \theta_{p_1}, \ldots, \theta_{p_n} | \overline{C}_m(\theta_{p_j}) | \alpha \rangle = 0, \quad j \neq 1, \ldots, n. \]  

(C.5)

Keeping the particular choice of the parameters (4.11) and the simple decomposition (4.5) of the transfer matrix, one can derive the following recursive relations (see (C.6) below) by considering the quantity of \( \langle \alpha^{(l)}, m^{(l)}; \theta_{p_1}, \ldots, \theta_{p_n} | t(-\theta_{p_{n+1}}) | \Psi \rangle \),

\[
\Lambda(\theta_{p_{n+1}}) F_n(\theta_{p_1}, \ldots, \theta_{p_n}) \\
= K_{11}^+(m^{(l)}, \alpha^{(l)}) - \theta_{p_{n+1}}) \langle \alpha^{(l)}, m^{(l)}; \theta_{p_1}, \ldots, \theta_{p_n} | \mathcal{A}_m^{(l)}(-\theta_{p_{n+1}} | \alpha^{(l)}) | \Psi \rangle \\
+ K_{22}^+(m^{(l)}, \alpha^{(l)}) - \theta_{p_{n+1}}) F_{n+1}(\theta_{p_1}, \ldots, \theta_{p_n}, \theta_{p_{n+1}}).
\]

The definitions (3.9) and (3.26) of the two gauged double-row monodromy matrices and the relations (3.27) and (3.28) allow us to express the operators \( \overline{C}_m(u|\alpha^{(l)}) \) and \( \overline{D}_m(u|\alpha^{(l)}) \) in terms of some linear combinations of \( \mathcal{A}_m(u|\alpha^{(l)}) \), \( \mathcal{B}_m(u|\alpha^{(l)}) \), \( \mathcal{C}_m(u|\alpha^{(l)}) \), and \( \mathcal{D}_m(u|\alpha^{(l)}) \)

Iterating the above recursive relations, we arrive at the relations (4.12).

C.2 The proof of the reference state

Due to the fact that the particular choice (4.7) of the parameters \( m^{(r)}, \alpha^{(r)} \) makes the matrix element \( K_{21}^{(r)}(m^{(r)} + N, \alpha^{(r)} | u) \) vanishes (see (4.8)), we can derive the following relations from (3.27) and (3.28)

\[ \mathcal{C}_m^{(r)}(u|\alpha^{(r)})|\Omega\rangle = 0, \]  

(C.7)

\[ \mathcal{A}_m^{(r)}(u|\alpha^{(r)})|\Omega\rangle = K_{11}^{(r)}(m^{(r)} + N, \alpha^{(r)} | u) \overline{A}(u) | \Omega \rangle. \]  

(C.8)

The definitions (3.9) and (3.26) of the two gauged double-row monodromy matrices and the relations (A.29)-(A.31) allow us to express the operators \( \overline{C}_m(u|\alpha^{(l)}) \) and \( \overline{D}_m(u|\alpha^{(l)}) \) in terms of some linear combinations of \( \mathcal{A}_m(u|\alpha^{(l)}) \), \( \mathcal{B}_m(u|\alpha^{(l)}) \), \( \mathcal{C}_m(u|\alpha^{(l)}) \), and \( \mathcal{D}_m(u|\alpha^{(l)}) \)

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respectively, namely,
\[
\overline{\mathcal{C}}_{m'}(-u|\alpha^{(l)}) = \mathcal{X}_{m'}(-u|\alpha^{(l)})X_{m(r)}(-u|\alpha^{(r)})\mathcal{A}_{m(r)}(-u|\alpha^{(r)})\mathcal{Y}_{m(r)}(u|\alpha^{(r)})\tilde{X}_{m'}(u|\alpha^{(l)}) \\
+ \mathcal{X}_{m'}(-u|\alpha^{(l)})Y_{m(r)-2}(-u|\alpha^{(r)})\mathcal{C}_{m(r)}(-u|\alpha^{(r)})\mathcal{Y}_{m(r)}(u|\alpha^{(r)})\tilde{X}_{m'}(u|\alpha^{(l)}) \\
+ \mathcal{X}_{m'}(-u|\alpha^{(l)})X_{m(r)+2}(-u|\alpha^{(r)})\mathcal{B}_{m(r)}(-u|\alpha^{(r)})\mathcal{X}_{m(r)}(u|\alpha^{(r)})\tilde{X}_{m'}(u|\alpha^{(l)}) \\
+ \mathcal{X}_{m'}(-u|\alpha^{(l)})Y_{m(r)}(-u|\alpha^{(r)})\mathcal{D}_{m(r)}(-u|\alpha^{(r)})\mathcal{X}_{m(r)}(u|\alpha^{(r)})\tilde{X}_{m'}(u|\alpha^{(l)}), \quad (C.9)
\]

\[
\overline{\mathcal{D}}_{m'}(-u|\alpha^{(l)}) = \mathcal{X}_{m'}(-u|\alpha^{(l)})X_{m(r)}(-u|\alpha^{(r)})\mathcal{A}_{m(r)}(-u|\alpha^{(r)})\mathcal{Y}_{m(r)}(u|\alpha^{(r)})\tilde{Y}_{m'+2}(u|\alpha^{(l)}) \\
+ \mathcal{X}_{m'}(-u|\alpha^{(l)})Y_{m(r)-2}(-u|\alpha^{(r)})\mathcal{C}_{m(r)}(-u|\alpha^{(r)})\mathcal{Y}_{m(r)}(u|\alpha^{(r)})\tilde{Y}_{m'+2}(u|\alpha^{(l)}) \\
+ \mathcal{X}_{m'}(-u|\alpha^{(l)})X_{m(r)+2}(-u|\alpha^{(r)})\mathcal{B}_{m(r)}(-u|\alpha^{(r)})\mathcal{X}_{m(r)}(u|\alpha^{(r)})\tilde{Y}_{m'+2}(u|\alpha^{(l)}) \\
+ \mathcal{X}_{m'}(-u|\alpha^{(l)})Y_{m(r)}(-u|\alpha^{(r)})\mathcal{D}_{m(r)}(-u|\alpha^{(r)})\mathcal{X}_{m(r)}(u|\alpha^{(r)})\tilde{Y}_{m'+2}(u|\alpha^{(l)}). \quad (C.10)
\]

The vanishing condition \((C.5)\) implies that
\[
\langle \alpha^{(l)}, m'; \theta_{p_1}, \ldots, \theta_{p_{n+1}} | \alpha^{(l)} \rangle |\Omega\rangle = 0, \quad n = 0, 1, \ldots, N - 1. \quad (C.11)
\]

Keeping the relations \((C.7)\) and \((C.8)\) in mind and using the above equations and the explicit expressions \((3.1), (3.5)-(3.8)\), after a tedious calculation, we can derive the following recursive relations
\[
\langle \alpha^{(l)}, m'; \theta_{p_1}, \ldots, \theta_{p_{n+1}} | \Omega\rangle = \bar{K}_{11}(m^{(r)} + N, \alpha^{(r)} - \theta_{p_{n+1}})\bar{A}(-\theta_{p_{n+1}}) \\
\times \langle \alpha^{(l)}, m'; \theta_{p_1}, \ldots, \theta_{p_{n}} | \Omega\rangle \\
\times 2e^{-\theta_{p_{n+1}}} \sinh(\theta_{p_{n+1}} + \alpha_-) \cosh(\theta_{p_{n+1}} + \beta_-)\bar{A}(-\theta_{p_{n+1}}) \\
\times \langle \alpha^{(l)}, m'; \theta_{p_1}, \ldots, \theta_{p_{n}} | \Omega\rangle, \quad n = 0, 1, \ldots, N - 1. \quad (C.12)
\]

Iterating the above recursive relations, we have
\[
\langle \alpha^{(l)}, m'; \theta_{p_1}, \ldots, \theta_{p_{n}} | \Omega\rangle = \prod_{j=1}^{n} \left\{ 2e^{-\theta_{p_{j}}} \sinh(\theta_{p_{j}} + \alpha_-) \cosh(\theta_{p_{j}} + \beta_-)\bar{A}(-\theta_{p_{j}}) \right\} \langle \alpha^{(l)} + m' | \Omega\rangle, \\
\quad n = 0, 1, \ldots, N.
\]

Comparing the above relations with the conditions \((4.15)\), we conclude that the state \(|\Omega\rangle\) given by \((4.9)\) is indeed the reference state which we are looking for. Therefore, the Bethe state \(|\lambda_1, \ldots, \lambda_N\rangle\) given by \((1.13)\) with the reference state \(|\Omega\rangle\) given by \((4.9)\) becomes an eigenstate of the transfer matrix \(t(u)\) with the eigenvalue \(\Lambda(u)\) given by \((2.25)\) provided that the \(N\) parameters \(\{\lambda_j | j = 1, \ldots, N\}\) satisfy the BAEs \((2.28)\).
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