Minimally dominant elements of finite Coxeter groups

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Abstract

Recently, Lusztig constructed for each reductive group a partition by unions of sheets of conjugacy classes, which is indexed by a subset of the set of conjugacy classes in the associated Weyl group. Sevostyanov subsequently used certain elements in each of these Weyl group conjugacy classes to construct strictly transverse slices to the conjugacy classes in these strata, generalising the classical Steinberg slice, and similar cross sections were built out of different Weyl group elements by He-Lusztig.

In this paper we observe that He-Lusztig’s and Sevostyanov’s Weyl group elements share a certain geometric property, which we call minimally dominant; for example, we show that this property characterises involutions of maximal length. Generalising He-Nie’s work on twisted conjugacy classes in finite Coxeter groups, we explain for various geometrically defined subsets that their elements are conjugate by simple shifts, cyclic shifts and strong conjugations. We furthermore derive for a large class of elements that the principal Deligne-Garside factors of their powers in the braid monoid are maximal in some sense. This includes those that are used in He-Lusztig’s and Sevostyanov’s cross sections, and explains their appearance there; in particular, all minimally dominant elements in the aforementioned conjugacy classes yield strictly transverse slices. These elements are conjugate by cyclic shifts, their Artin-Tits braids are never pseudo-Anosov in the conjectural Nielsen-Thurston classification and their Bruhat cells should furnish an alternative construction of Lusztig’s inverse to the Kazhdan-Lusztig map and of his partition of reductive groups.

Contents

1 Introduction 2

2 Shifts and sequences of eigenspaces 12
   2.1 Quasiregular elements and the Coxeter plane 15
   2.2 Gradient flows 17
   2.3 Shifts 20

3 Minimally dominant elements 23
   3.1 Isotropy subgroups 23
   3.2 Bruhat cells 27
   3.3 Involutions 29

4 Powers of reduced braids 32
   4.1 Stability of the Deligne-Garside normal form 32
   4.2 Root inversion sets and sequences 35
   4.3 Mixed shifts and braid conjugation 40
   4.4 From braiding sequences to braid powers 44

References 48
1 Introduction

During the 1990s, the character tables of finite Iwahori-Hecke algebras were determined through case-by-case analysis and extensive computer calculations on the conjugacy classes of finite Coxeter groups [GP00]. More specifically, these algebras have a basis \( \{ T_w \} \) indexed by elements \( w \) of the corresponding finite Coxeter group \( W \), and Geck-Pfeiffer discovered properties of conjugacy classes of \( W \) which imply that the value of a character \( \chi \) is constant on these basis elements when they correspond to minimal length elements in the same conjugacy class, and moreover that those values can be used to compute \( \chi(T_w) \) for arbitrary elements \( w \) in this conjugacy class [GP93, GKP00]. Furthermore, Geck-Michel noticed that every conjugacy class of \( W \) contains minimal length elements whose powers in the braid monoid are particularly simple, which was subsequently used to determine the “rationality” of these characters [GM97, GKP00, He07]. These results on conjugacy classes were later reused in other domains involving Weyl group elements (e.g. Bruhat cells [EG04, CLT10, Lus11a], Deligne-Lusztig varieties [BR08, OR08], 0-Hecke algebras [He15] and partitions of the wonderful compactification [He07]), and in particular He-Lusztig applied them to construct cross sections in reductive groups out of elliptic Weyl group elements of minimal length [HL12].

Recently, He-Nie gave a simple, geometric proof of these statements by focusing on the eigenspace decomposition for \( w \) acting in the real reflection representation [HN12]. Meanwhile, Sevostyanov used similar data to construct transverse slices to conjugacy classes in reductive groups [Sev11], and verified that for any stratum in Lusztig’s partition [Lus15] a suitable choice yields slices which are strictly transverse to its conjugacy classes [Sev19]. His data can be recast in the same framework:

**Definition 1.1.** A twisted Coxeter group is a group generated by some automorphisms and some simple reflections of a Coxeter system. We may assume that each of these automorphisms \( \{ \delta_i \}_{i=1}^m \) preserves the set of these simple reflections \( \{ s_i \}_{i=1}^n \); more concretely, a twisted Coxeter group is then a semidirect product \( W = \Omega \rtimes W' \), where \( W = \langle s_i \rangle_{i=1}^n \) is a Coxeter group and \( \Omega = \langle \delta_i \rangle_{i=1}^m \) is a group acting on \( W \) by automorphisms, each of which preserves its simple reflections. We call an element \( \delta \) of \( \Omega \) a twist of \( W \) and denote elements of \( W \) by \( \delta w := (\delta, w) \). Given a twisted Coxeter group \( W \) (with such data) we always denote its untwisted part by \( \tilde{W} \).

When \( W \) is generated by all automorphisms and simple reflections, one obtains the group of automorphisms of the corresponding root (or reflection) system \( \mathfrak{R} \). Twists were originally motivated by the theory of Iwahori-Hecke algebras and non-split reductive groups but coincidentally, they also played a crucial rôle in He-Nie’s proof of the aforementioned statements for ordinary finite Coxeter groups.

**Example 1.2.** When the Coxeter system is irreducible and of finite type, their classification by Coxeter-Dynkin diagrams implies that every twist \( \delta \) satisfies \( \text{ord}(\delta) \leq 3 \).

**Definition 1.3.** We define the standard parabolic subgroups of a twisted Coxeter group \( W \) to be the subgroups of the form \( \Omega' \rtimes W' \), where \( W' \) is a standard parabolic subgroup of \( W \) and \( \Omega' \) is a subgroup of \( \Omega \) whose elements preserve (the set of simple reflections in) \( W' \). Their conjugates are called parabolic subgroups.

Let \( W \) be a twisted finite Coxeter group and let \( W' \) be a standard parabolic subgroup of \( \tilde{W} \). By a \( W' \)-orbit we then mean an orbit of \( W' \) on \( W \) under the restriction of the conjugation action to \( W' \); every conjugacy class in \( W \) is then partitioned into \( W' \)-orbits.

**Example 1.4.** In type \( A_n \), the only nontrivial twist coincides with conjugation by the longest element, so \( \tilde{W} \)-orbits in \( W \) agree with conjugacy classes.

**Example 1.5.** As twists permute Coxeter elements of minimal length\(^1\) and they are already conjugate under \( \tilde{W} \), it again follows that their \( \tilde{W} \)-orbit is the entire conjugacy class.

\(^1\)By a Coxeter element we mean a conjugate of the product (in any order) of all of the simple reflections in the group.
Example 1.6. Let $\delta$ be the twist in $D_4$ interchanging the third and fourth simple reflections and consider the twisted Coxeter group $W := \langle \delta \rangle \ltimes \tilde{W}$. Then $s_3s_1$ and $\delta(s_3s_1) = s_4s_1$ are not $\tilde{W}$-conjugate, so the conjugacy class of $s_3s_1$ splits into the $\tilde{W}$-orbits of $s_3s_1$ and $s_4s_1$.

Notation 1.7. Throughout this paper, we have for the sake of brevity employed symbols like $\pm, \geq, \max, \min$. This always means that there are two statements being made simultaneously: one where the top and left symbols are used at each such position, and another one for the bottom and right symbols. This duality is explained in Remark 1.16.

Definition 1.8. Let $W$ be a twisted Coxeter group. When the untwisted part $\tilde{W} \subseteq W$ is finite, we will call $W$ an untwisted finite Coxeter group and denote the longest element of $\tilde{W}$ by $w_0$. We denote its set of positive (resp. negative) roots by $\mathcal{R}_+$ (resp. $\mathcal{R}_-$), and for any element $w$ in $W$ we denote its (right) (root) inversion set by

$$\mathcal{R}_w := w^{-1}(\mathcal{R}_-) \cap \mathcal{R}_+ = \{ \beta \in \mathcal{R}_+ : w(\beta) \in \mathcal{R}_- \}.$$ Given an element $w$ of $W$, we say that a root is stable if its $w$-orbit consists solely of positive or solely of negative roots. We then denote by $\mathcal{R}^w_st$ the set of stable roots, i.e.

$$\mathcal{R}^w_st = \{ \beta \in \mathcal{R} : w^i(\beta) \notin \mathcal{R}_w \text{ for } 1 \leq i \leq \text{ord}(w) \}.$$ In particular, there is an inclusion $\mathcal{R}^w \subseteq \mathcal{R}^w_st$, where $\mathcal{R}^w$ denotes the set of roots that are fixed by $w$.

We say that $w$ is convex when $\mathcal{R}^w$ forms a standard parabolic subsystem, and then denote by $\text{pb}(w)$ its (braid) power bound: this is the unique element of $\tilde{W}$ which makes negative all positive roots except for those in $\mathcal{R}^w$. Thus $\text{pb}(w) = w_0w_{st}$, where $w_{st}$ now denotes the longest element of the standard parabolic subgroup corresponding to $\mathcal{R}^w_st$. If furthermore we have an equality $\mathcal{R}^w = \mathcal{R}^w_st$, then we say that the element $w$ is firmly convex.

The name braid power bound was derived from part (i) of Proposition D. In Proposition 4.21 we show that an element $w$ is convex if and only if the set of non-stable positive roots $\mathcal{R}_+ \setminus \mathcal{R}^w_st$ is convex (see Definition 4.16); this property plays an important role in the sequel to this paper [Malb].

Notation 1.9. The action of a Coxeter group $\tilde{W}$ on its real reflection representation $V$ (which is due to Tits [Bou68, §5.4.4]) extends to twisted Coxeter groups $W = \Omega \ltimes \tilde{W}$ by letting elements of $\Omega$ act on the simple roots as prescribed, and extending linearly. We will then denote the $W$-invariant inner product on $V$ by $(\cdot, \cdot)$. Given a standard parabolic subgroup $W'$, we will typically interpret its reflection representation $V_{W'}$ as sitting inside of $V$ through the span of the corresponding simple roots.

The rank of $W$ is the dimension of $V$ (which equals the number of simple reflections in $W$) and is sometimes denoted by rank($W$) or simply rk. For any other element $w'$ in $W$ we write $w' \geq w$ if and only if $\mathcal{R}_w \subseteq \mathcal{R}_{w'}$, extending the weak left Bruhat-Chevalley (partial) order on $W$ to a preorder on $\tilde{W}$; antisymmetry holds up to left-multiplication by twists. Let $V_w = \text{im}(id_V - w) \subseteq V$ denote the orthogonal complement to the subspace $V^w = \text{ker}(id_V - w) \subseteq V$ of $w$-fixed points in $V$.

Moreover, we denote the length of $w$ by $\ell(w) := |\mathcal{R}_w|$ and the number of roots it fixes by $\ell_f(w) := |\mathcal{R}^w|$. For any subset $O$ of $W$ we denote by $O^{cx}$ the subset of convex elements in $O$ and write $O_{\text{max/min}}$ for the subset of maximal/minimal length elements inside of $O$. If the length function is constant on $O$ then we will simply denote this value by $\ell(O)$, and similarly for $\ell_f(\cdot)$ and the order (or period) function $\text{ord}(\cdot)$.

Definition 1.10 ([HN12, §1.3, §2.1]). Let $W$ be a twisted finite Coxeter group and let $W' \subseteq V$ be a subset of its reflection representation. Then we let $W_{V'} \subseteq W$ denote the subgroup of $W$ consisting of elements fixing $V'$, and we denote the set of root hyperplanes containing $V'$ by $\mathcal{H}_{V'}$.

A vector $v$ in $V'$ is called a regular point of $V'$ if $\tilde{W}_v = \tilde{W}_{\{v\}}$, which by Steinberg’s fixed-point theorem [Ste64, Theorem 1.5] is equivalent to $\mathcal{H}_{V'} = \mathcal{H}_{\{v\}}$. If $V'$ is convex then such points form a dense open subset of $V'$, by finiteness of the number of root hyperplanes. Given an element $w$ of $W$, we denote by $\mathcal{H}^w := \mathcal{H}_{V_w}$ the subset of root hyperplanes corresponding to roots of $\mathcal{R}_w$.

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2 Throughout this paper we follow Bourbaki’s labelling for simple roots $\alpha_i$ in irreducible root systems [Bou68, §VI.4], so for $D_n$ the degree 3 vertex has label $n - 2$. Furthermore, we write $\alpha_1 \cdots \alpha_m := \sum_{j=1}^{m} \alpha_j$ if this sum of simple roots is a root.
Proposition 3.2 shows that an isotropy group $W_{V'}$ is a parabolic subgroup, and if the closure of the dominant Weyl chamber contains a regular point (or open subset) of $V'$ then $W_{V'}$ is moreover a standard parabolic subgroup.

**Definition 1.11.** Let $W = \Omega \times \hat{W}$ be a twisted finite Coxeter group and pick an element $w$ in $W$. As $w$ is a linear isometry it naturally decomposes the real reflection representation

$$V = \bigoplus_{\lambda(\lambda) \geq 0} V^w_{\lambda} = \bigoplus_{\lambda(\lambda) \geq 0} \bigoplus_{k} V^w_{\lambda,k}$$

into an orthogonal sum of subspaces: given a complex eigenvalue $\lambda = e^{2\pi i \theta}$ whose imaginary part $\lambda(\lambda)$ is nonnegative, the element $w$ acts as rotation with angle $2\pi \theta$ on the “real eigenspace”

$$V^w_{\lambda} := \{ v \in V : w(v) + w^{-1}(v) = 2\cos(2\pi \theta)v \},$$

whose complexification is the sum of the usual complex eigenspaces for $\lambda$ and $\lambda^{-1}$ in $V \otimes_{\mathbb{R}} \mathbb{C}$. We may further (non-uniquely) decompose $V^w_{\lambda}$ into smaller real eigenspaces $V^w_{\lambda,k}$ ($k \geq 1$) of dimension $\geq 1$ for $\lambda \in \{1,-1\}$ and of even dimension $\geq 2$ for all other $\lambda$. We will then consider sequences of distinct eigenspaces

$$\Theta = (V^w_{\lambda_0,k_0}, \ldots, V^w_{\lambda_1,k_1})$$

and leave out the underscore to denote the underlying set of eigenspaces by $\Theta$. Given an arbitrary set $\Theta = \{V_m, \ldots, V_1\}$ of eigenspaces of some element $w$, we denote its span by

$$V_\Theta := \bigoplus_{i=1}^m V_i.$$

Given a sequence of eigenspaces $\Theta = (V_m, \ldots, V_1)$, we define a filtration on this span $V_\Theta \subseteq V$ via $F_i := \sum_{j=1}^i V_j$ for $0 \leq i \leq m$. Then setting $W_i := W_{F_i}$ for $0 \leq i \leq m$, we obtain a filtration of parabolic subgroups [HN12, §5.2]

$$W_0 := W_{V_0} = W_m \subseteq \cdots \subseteq W_1 \subseteq W_0 = \hat{W}.$$ 

Equivalently, one obtains a sequence of root hyperplanes $H_{F_i}$; when $H_{F_i} = H_{F_{i-1}}$ (or $\hat{W}_i = \hat{W}_{i-1}$) one says that $V_i$ is redundant. We will say that a Weyl chamber $C$ is in good position with respect to $\Theta$ if the closure of this Weyl chamber contains a regular point of $F_i$ for each $i$. In Proposition 2.2 we prove that this is equivalent to the definition given in [HN12, §5.2].

Moreover, there is always at least one Weyl chamber in good position [HN12, Lemma 5.1] and typically we will require that the dominant Weyl chamber is one of them; as we’ve noted, this implies that the parabolic subgroups $W_i$ are standard parabolic subgroups. If indeed the dominant Weyl chamber is in good position and the set of hyperplanes $H_\Theta := H_{V_\Theta}$ containing $V_\Theta$ is a subset of $H_\Theta$ and then we say that $\Theta$ is braiding; if moreover $H_\Theta$ is the empty set then we say that it is complete.

We will say that an element $w$ is quasiregular if $H_{V_\Theta} \subseteq H_\Theta$ for some eigenvalue $\lambda$ of $w$. In Proposition 2.9 we explain that regular elements are quasiregular elements satisfying $H_{V_\Theta} = \emptyset$ for some $\lambda$ and in Proposition 2.17 we explain that there are only two Coxeter elements of minimal length in $W$ that admit a braiding sequence of eigenspaces, namely the bipartite ones.

Given a sequence of eigenspaces $\Theta = (V_m, \ldots, V_1)$ there is an underlying sequence of eigenvalues $\{\lambda_m, \ldots, \lambda_1\}$ whose principal arguments we will normalise to lie in $[0,1/2]$ and then denote by

$$(\theta_m, \ldots, \theta_1),$$

so $\lambda_j = e^{2\pi i \theta_j}$ with $0 \leq \theta_j \leq 1/2$ for each $1 \leq j \leq m$. If none of the eigenvalues of $\Theta$ equals 1, then we will call the sequence anisotropic.

Given a set of eigenspaces $\Theta$, we write $\text{eig}(\Theta) := \{\lambda_m, \ldots, \lambda_1\}$ for the corresponding set of eigenvalues. We denote by $\Theta_k$ the sequence obtained from the set $\Theta$ by merging eigenspaces if their eigenvalues agree, and then ordering them so that $\theta_m < \cdots < \theta_1$ (resp. $\theta_m > \cdots > \theta_1$); we will call such a sequence increasing/decreasing. We furthermore write $\Theta = \{\lambda_m, \ldots, \lambda_1\}$ when $\Theta = \{V^w_{\lambda_0}, \ldots, V^w_{\lambda_1}\}$. 

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Definition 1.12. Let $W$ be a twisted finite Coxeter group. An element $w$ of $W$ is called elliptic (or cuspidal or anisotropic) if it has no nonzero fixed points in the reflection representation of $W$, and a subset of $W$ is labelled thusly all of its elements have this property.

Sevostyanov’s data for his transverse slices is equivalent to a Weyl group element with an anisotropic braiding sequence of eigenspaces (with each eigenspace of dimension 1 if the eigenvalue is $-1$ and of dimension 2 otherwise; he puts eigenspaces with eigenvalue 1 at the end of the sequence, but in that position they can be safely ignored) [Sev11, §2]. On the other hand, He-Nie uses decreasing complete sequences [HN12, §5.2]. Thus for elliptic elements Sevostyanov’s transverse slice conditions are slightly more general, whereas as the sets of non-elliptic elements these two conditions describe are entirely disjoint. In case-by-case work Sevostyanov later furnished elements with an anisotropic braiding sequence which is decreasing, i.e. $\theta_m > \cdots > \theta_1 > 0$, and verified that the corresponding slice is strictly transverse [Sev19, §1].

Example 1.13. Let $w$ be a non-simple reflection in type $B_2$. Then the subspace $V^w$ of fixed points is nontrivial and contains roots but its intersection with the closure of the dominant Weyl chamber is $\{0\}$, so $w$ has no decreasing complete sequence of eigenspaces, whilst its only nontrivial eigenspace $V^w_0$ yields an anisotropic braiding one. If instead $w$ is a simple reflection then it has no anisotropic braiding sequence, whilst $\Theta := (V^w_1, V^w_1)$ is decreasing and complete.

Definition 1.14. Let $W$ be a twisted finite Coxeter group with dominant Weyl chamber $C$, pick a standard parabolic subgroup $W' \subseteq \tilde{W}$ and let $\mathcal{O}$ be a $W'$-orbit. Let $\Theta = (V_m, \ldots, V_1)$ be a sequence of eigenspaces of an element $w$ of $\mathcal{O}$. Given an element $\tau$ in $W'$, we then obtain a sequence of eigenspaces
\[ \tau(\Theta) := (\tau(V_m), \ldots, \tau(V_1)) \]
for the element $\tau w \tau^{-1}$ in $\mathcal{O}$, and we similarly define $\tau(\Theta)$. We may then define
\[ O^\Theta := \{ \tau w \tau^{-1} \in \mathcal{O} : \tau \in W' \text{ and } C \text{ contains a regular point of } V_{\tau(\Theta)} \} \]
and
\[ O_\Theta := \{ \tau w \tau^{-1} \in \mathcal{O} : \tau \in W' \text{ and } C \text{ is in good position with respect to } \tau(\Theta) \}, \]
and finally set $O^\Theta := O_{\Theta^{\leq}}$.

Rephrasing the usual definition of cyclic shifts sheds some light on the dualities that follow:

Definition 1.15. Let $W$ be a twisted Coxeter group, let $\mathcal{O}$ be a subset of $W$ and pick two elements $w$ and $w'$ in $\mathcal{O}$. When $\mathcal{O}$ is not mentioned we set $\mathcal{O} := W$ for the following notions:

- If there exists a sequence of elements $w' = w_{n+1}, \ldots, w_0 = w$ in $\mathcal{O}$ and simple reflections $s_{i_0}, \ldots, s_{i_n}$ such that
  \[ s_{i_j} s_{i_j} s_{i_j} = w_{j+1} \quad \text{and} \quad \ell(w_j) \leq \ell(w_{j+1}) \]
  holds for each $0 \leq j \leq n$, then we write $w \overset{\uparrow}{\rightarrow} w'$ in $\mathcal{O}$ and $w' \overset{\rightarrow}{\rightarrow} w$ in $\mathcal{O}$. If furthermore $\ell(w) = \ell(w')$ then we write $w \leftrightarrow w'$ in $\mathcal{O}$, and call the set of such $w'$ the simple shift class of $w$ in $\mathcal{O}$ [GP93, §1]. If on the other hand there is a strict inequality $\ell(w_j) < \ell(w_{j+1})$ for each $j$, we write
  \[ w \overset{\uparrow}{\rightarrow} w' \text{ in } \mathcal{O} \quad \text{and} \quad w' \overset{\rightarrow}{\rightarrow} w \text{ in } \mathcal{O}. \]
- If there exists a sequence of elements $w' = w_{n+1}, \ldots, w_0 = w$ in $\mathcal{O}$ and $\tau_n, \ldots, \tau_0$ in $\tilde{W}$ such that
  \[ \tau_j w_j \tau_j^{-1} = w_{j+1} \quad \text{and} \quad \ell(w_j) \leq \ell(w_{j+1}) \]
  holds for each $0 \leq j \leq n$, and furthermore for each such $j$ at least one of
  \[ \ell(\tau_j w_j) = \ell(w_j) + \ell(\tau_j) \quad \text{or} \quad \ell(w_j \tau_j^{-1}) = \ell(w_j) + \ell(\tau_j^{-1}) \]
  holds, then we write $w \overset{\uparrow}{\leftrightarrow} w'$ in $\mathcal{O}$. If furthermore $\ell(w_j) = \ell(w_{j+1})$ holds for each $j$, we write $w \overset{\uparrow}{\leftrightarrow} w'$ and call the set of such $w'$ the strong conjugacy class of $w$ in $\mathcal{O}$ [GP93, §1].
Given two elements this section naturally extend to twisted Coxeter groups, e.g. for the inclusion we will write 

\[ \ell(w_j) = \ell(w_{j+1}) \] and for each \( 0 \leq j \leq n \) at least one of

\[ \ell(\tau_j w_j) = \ell(w_j) - \ell(\tau_j) \quad \text{or} \quad \ell(w_j \tau_j^{-1}) = \ell(w_j) - \ell(\tau_j^{-1}) \]

holds, then we write \( w \sim w' \) in \( \mathcal{O} \). If furthermore \( \ell(w_j) = \ell(w_{j+1}) \) holds for each \( j \), we write \( w \sim w' \) and call the set of such \( w' \) the cyclic shift class of \( w \) in \( \mathcal{O} \) [BM97, Définition 3.16].

- If such a sequence \( \tau_n, \ldots, \tau_0 \) yields a combination of cyclic shifts and strong conjugations in \( \mathcal{O} \), we write \( w \sim w' \) in \( \mathcal{O} \) and call the set of such \( w' \) the mixed shift class of \( w \) in \( \mathcal{O} \).

- When \( W' \subseteq \tilde{W} \) is a standard parabolic subgroup then we write by \( W' \) to express that the conjugators all lie in \( W' \), e.g. for \( w \sim w' \) by \( W' \) we mean that the \( \tau_j \) all lie in \( W' \).

**Remark 1.16.** This duality between strong conjugations and cyclic shifts can be explicitly observed by multiplying with \( w_0 \) (e.g. [GKP00, §2.9]). If we furthermore pick a twist \( \delta \) such that \( \delta w_0 \) acts as \(-1\) then it also exhibits a duality between eigenvalues, in particular between the eigenvalues \( 1 \) and \(-1\).

**Definition 1.17.** Let \( W \) be a twisted finite Coxeter group and pick a standard parabolic subgroup \( W' \subseteq \tilde{W} \). Given two elements \( w \) and \( w' \) in \( W \) and in a sequence of eigenvalues \( \Theta \) of \( w \), we let \( \text{Tran}^{\Theta}_{W'}(w, w') \) (resp. \( \text{Tran}^{\Theta,*}_{W'}(w, w') \)) for \( * \in \{ \leftrightarrow, +, -, \times \} \) denote the set of sequences of elements \( \tau_n, \ldots, \tau_0 \) in \( W' \) such that the sequence of elements

\[ w, \quad \tau_0 w \tau_0^{-1}, \quad \ldots, \quad \tau_n \cdots \tau_0 w \tau_0^{-1} \cdots \tau_n^{-1} = w' \]

lies in \( \mathcal{O}_\Theta \) (resp. \( \mathcal{O}^\Theta \)) and consists of simple shifts when \( * = \leftrightarrow \), of strong conjugations when \( * = + \), etc. Multiplication then yields a natural projection map of *transporters*

\[ \text{Tran}^{\Theta}_{W'}(w, w') \subseteq \text{Tran}^{\Theta,*}_{W'}(w, w') \longrightarrow \text{Tran}^{\Theta,*}_{W'}(w, w') := \{ \tau \in W' : \tau w \tau^{-1} = w' \}. \]

If \( w = w' \) then we simplify this notation to \( \text{Tran}^{\Theta,*}_{W'}(w) \), \( \text{Tran}^{\Theta,*}_{W'}(w) \) and \( \text{Tran}^{\Theta,*}_{W'}(w) \) respectively. Furthermore when \( V_\Theta = V \) or \( W' = \tilde{W} \) we might drop \( \Theta \) or \( W' \) from notation.

**Notation 1.18.** Every Coxeter group \( \tilde{W} \) is the codomain of a natural projection map \( b \mapsto w_b \) coming from the corresponding Artin-Tits braid group \( Br_{\tilde{W}} \). Matsumoto’s theorem [Mat64] furnishes a natural section \( \tilde{w} \mapsto b_{\tilde{w}} \), embedding \( \tilde{W} \) (as a set) into the the associated braid monoid \( Br_{\tilde{W}} \subseteq Br_{\tilde{W}} \). This projection and this section naturally extend to twisted Coxeter groups, e.g. for the inclusion we will write

\[ W := \Omega \ltimes \tilde{W} \longrightarrow \Omega \ltimes Br_{\tilde{W}} :=: Br_{\tilde{W}} :=: Br_{\tilde{W}} \quad \text{and} \quad w = \delta \tilde{w} \longmapsto \delta b_{\tilde{w}} =: b_w, \]

and such braids \( b_w \) are often called reduced (or simple or minimal) braids. We will also write \( b_{i_1 \cdots i_s} := b_{s_i \cdots s_i} \) when \( s_i \cdots s_i \) is a reduced decomposition.

**Proposition A.** Let \( W \) be a twisted Coxeter group and pick two elements \( w \) and \( w' \) in \( W \).

(i) If \( w \sim w' \) via a sequence of elements \( \tau_n, \ldots, \tau_0 \) of \( W \), then \( (\tau_n \cdots \tau_0)(\mathfrak{g}_{st}^w) = \mathfrak{g}_{st}^{w'} \).

In particular, if \( w \) and \( w' \) are convex and one of them is firmly convex, then so is the other one.

(ii) Pick a standard parabolic subgroup \( W' \subseteq \tilde{W} \). If \( \tilde{W} \) is finite, then the projection map

\[ \text{Tran}^{\times}_{W'}(w, w') \longrightarrow \text{Tran}^{\times}_{W'}(b_w, b_{w'}) := \{ \tau \in Br_{\tilde{W}} : \tau b_w = b_{w'} \tau \} \]

sending a sequence of conjugators \( \tau_n, \ldots, \tau_0 \) to the braid \( b_{\tau_n \cdots \tau_0} \), surjects.

In particular, \( w \sim w' \) by \( W' \) if and only if \( b_w \) and \( b_{w'} \) are conjugate under the braid subgroup \( Br_{\tilde{W}} \).
Part (ii) is proven in the more general language of Garside categories, whereas part (i) will be used in §3.2 and in proving [Malb, Proposition (iii)].

**Example 1.19.** Consider the conjugates \( w = s_1 s_2 s_3 s_4 s_1 \) and \( w' = s_2 s_3 s_4 s_2 s_3 \) of the element \( s_2 s_3 s_4 \) in type \( A_4 \). Then \( \mathcal{R}_{w}^w = \emptyset \) but as \( \mathcal{R}_{w}^{w'} \) contains (the orbit of) \( \alpha_1 \), it now follows that the braids \( b_w \) and \( b_{w'} \) are not conjugate in the corresponding Artin-Tits braid group.

Elements of \( \text{Tran}^{-}(w) \) yield universal homeomorphisms of Deligne-Lusztig varieties ([DL76, Case 1 of Theorem 1.6] and [Lus11b, §0.6]). In his work on the relationship between character sheaves and the character theory of finite reductive groups, Lusztig conjectured that for an elliptic element of minimal length \( w \), the natural map from \( \text{Tran}^{++}(w) \subset \text{Tran}^{-}(w) \) onto the centraliser \( \text{Tran}(w) \) is a surjection [Lus11b, §1.2]. This statement also plays a rôle in work of Broué, Digne and Michel on consequences of the Broué abelian defect conjecture for finite reductive groups [BM97, DM14]. He-Nie subsequently proved this conjecture by showing more generally that for any other such element \( w' \), the set \( \text{Tran}^{++}(w, w') \) projects onto \( \text{Tran}(w, w') \) [HN12, Theorem 4.2].

Unwinding and advancing the work of [HN12, §1-§4], in Lemma 2.7 we will study elements in \( O^\Theta \) for sequences of eigenspaces \( \Theta \) satisfying \( \Theta_0 \subseteq \Theta^w \); this will be required for the second half of the following

**Proposition B.** Let \( W \) be a twisted finite Coxeter group. Let \( O \) be a \( \tilde{W} \)-orbit and let \( \Theta \) be a set of eigenspaces of some element \( w \) in \( O \).

(i) There exists an element \( w' \) in \( O^\Theta \) such that \( w \overset{\pm}{\rightarrow} w' \) in \( W \), \( w \overset{\mp}{\sim} w' \) in \( O^\Theta \) and \( w' \overset{\mp}{\sim} w \) in \( O^\Theta \).

Now assume that \( \Theta_0 \subseteq \Theta^w \).

(ii) Then \( O^\Theta \) is contained in \( O^\Theta_{\text{max/min}} \) and the length of these elements is given by

\[
\ell(O^\Theta_{\text{max/min}}) = 2 \sum_{i=1}^{\text{m}} \theta_i |\mathcal{F}_{F_{i-1}} \setminus \mathcal{F}_{F_i}|, \tag{1.2}
\]

where the filtration \( F_i \) and ordering on the arguments \( \theta_i \) are derived from the sequence \( \Theta_\pm \).

(iii) Suppose \( w, w' \) lie in \( O^\Theta_{\text{max/min}} \). Then the projection

\[
\text{Tran}^\Theta(w, w') \longrightarrow \text{Tran}(w, w') \quad \text{(resp.} \quad \text{Tran}^+(w, w') \longrightarrow \text{Tran}(w, w')\text{)}
\]

surjects, with

\[
* = \begin{cases} 
\pm & \text{if } \mp \neq 1 \notin \text{eig}(\Theta), \\
\mp & \text{if } V_\Theta = V, \\
\times & \text{otherwise.}
\end{cases} \quad \text{(resp.} \quad * = \begin{cases} 
\leftrightarrow & \text{if } \mp \neq 1 \notin \text{eig}(\Theta) \text{ and } V_\Theta = V, \\
\leftrightarrow & \text{if } \text{eig}(\Theta) \cap \{1, -1\} = \emptyset.
\end{cases} \tag{1.3}
\]

Formula (1.2) was proven for elliptic elements of minimal length in [Yun20, Theorem 2.3].

**Example 1.20.** From (iii) one obtains the usual statement that minimal length elements lie in the same strong conjugacy class, and furthermore lie in the same simple shift class when they are elliptic.

Dually, they also imply that maximal length elements lie in the same cyclic shift class, and furthermore lie in the same simple shift class if \(-1\) is not an eigenvalue.

**Example 1.21.** Let \( O \) be the conjugacy class of \( s_1 s_3 \) in type \( B_3 \). If we set \( \Theta = \{ -1 \} \) or \( \Theta = \{ e^{2\pi i / 3}, -1 \} \) then \( O^\Theta \) is

\[
\{ w_\circ s_1, w_\circ s_2 \}.
\]
so the involutions $w_{0}s_{1}$ and $w_{0}s_{2}$ lie in the same cyclic shift class, and indeed there are reduced decompositions $w_{0}s_{1} = xy$ and $w_{0}s_{2} = yx$ for e.g. $x := s_{3}s_{2}s_{1}s_{3}s_{4}$ and $y := s_{1}s_{2}$.

**Example 1.22.** Consider $w = s_{1}s_{2}s_{1}$ and $v = s_{2}s_{3}s_{2}$ in type $A_{3}$. Then $v$ lies in $\text{Tran}(w)$ but $\text{Tran}^{\times}(w)$ is generated by $w$; the centraliser of $b_{w}$ in the braid group is generated by $b_{121}$ and $b_{321}b_{123}$.

**Definition 1.23.** Let $W$ be a twisted finite Coxeter group. Let $O$ be a $\tilde{W}$-orbit and let $\Theta$ be all of its eigenvalues except 1, so $V_{\Theta} = V_{w}$ when we pick an element $w$ in $O$. We will sometimes write dom instead of $\Theta$, e.g.

$$O^{\text{dom}} := O^{\Theta},$$

so that $O_{\text{max/min}}^{\text{dom}} := O_{\text{max/min}}^{\Theta}$, and refer to their elements as dominant and maximally/minimally dominant respectively.

In other words, an element $w$ is dominant if and only if the closure of the dominant Weyl chamber contains an open subset of $V_{w}$. From Corollary 3.3 it follows that for elements of $W$ there are implications elliptic or has an anisotropic braiding sequence of eigenspaces $\rightarrow$ dominant $\rightarrow$ firmly convex $\rightarrow$ convex.

Some of the properties of maximally and minimally dominant elements follow from simple geometric statements, others from specialising Proposition B:

**Lemma.** Let $W$ be an irreducible twisted finite Coxeter group and let $O$ be a $\tilde{W}$-orbit.

(i) Then there is an equality of lengths $\ell(O_{\text{max}}^{\text{dom}}) = \ell(O_{\text{max}})$ and an inequality of lengths

$$\ell(O_{\text{min}}^{\text{dom}}) \geq \ell(O_{\text{min}}),$$

with equality if and only if $O$ is either elliptic or trivial (meaning, the orbit of the identity element).

(ii) There is an inclusion $O_{\text{dom}}^{\text{max/min}} \subseteq O_{\text{max/min}}^{\text{dom}}$, and for any pair of elements $w$ and $w'$ in $O_{\text{max/min}}^{\text{dom}}$ the natural projection map

$$\text{Tran}_{\text{dom},-}: (w, w') \mapsto \text{Tran}(w, w')$$

is surjective.

(iii) If $O$ is nontrivial and lies in $\tilde{W}$ then there are inequalities

$$\ell(O_{\text{min}}^{\text{dom}}) \geq \frac{[\mathcal{R}] - \ell(O)}{\text{ord}(O)} \quad \text{and} \quad \ell(O_{\text{min}}^{\text{dom}}) \geq \text{rank}(W),$$

where the first one is equality if and only if $O$ is quasiregular, the second one if and only if $O$ is the Coxeter class.

**Example 1.24.** The conjugates $w = s_{1}s_{2}s_{3}s_{4}s_{2}s_{3}s_{1}$ and $w^{-1}$ of $s_{2}s_{3}s_{4}$ in type $B_{4}$ are dominant, but are neither maximally nor minimally dominant; they form two distinct cyclic shift classes.

Richardson showed that the minimal length involutions of a finite Coxeter group are those involutions $w$ such that $\mathcal{R}_{w}$ agrees with a standard parabolic subsystem [Ric82, Theorem A(a)], and he used that to algorithmically classify conjugacy classes of involutions [Ric82, §3] in terms of the corresponding Coxeter-Dynkin diagram. A similar description for maximal length involutions was given by Hart-Rowley [PR04, Theorem 1.1(ii)], who also gave, for each conjugacy class of involutions in every type, explicit expressions for (and deduced from that the number of) maximal and minimal length elements [PR02]. We show that involutions have maximal length precisely when they are dominant, and generalise other statements regarding maximal and minimal length involutions to orbits of standard parabolic subgroups:

---

3In some books and papers, it is claimed that $w \sim w'$ if and only if $w \leftrightarrow w'$. However, conjugating these two elements in $B_{3}$ by simple reflections either leaves them unchanged or results in an element of lower length, thus providing a counterexample. This example also shows that the strong conjugations in [CLT10, Proposition 2.8(2) and 2.13(2)] must be replaced by cyclic shifts. The subsequent proofs [CLT10, Proposition 2.9 and 2.14] about Bruhat cells don’t seem to work for either type of shift, but they can be replaced with Proposition 3.12.
Notation 1.25. Let $W$ be a twisted finite Coxeter group and $W' \subseteq \hat{W}$ be a standard parabolic subgroup. Then we denote by $w^\bullet \mathcal{R}_+$ the subset of positive roots of the corresponding standard parabolic subsystem. Given an element $w$ in $W$, we let $\mathcal{R}_-^w := \mathcal{R}_- \cap V^w_-$ denote the set of roots on which it acts as $-1$.

Proposition C. Let $W$ be a twisted Coxeter group, pick a standard parabolic subgroup $W' \subseteq \hat{W}$, let $O$ be a $W'$-orbit consisting of involutions and pick an element $w$ in $O$.

(i) There exists an element $w'$ in $O_{\max/\min}$ such that $w \rightarrow w'$ by $W'$.
(ii) The following are equivalent:
   (a) The element $w$ lies in $O_{\max/\min}$.
   (b) The set of roots $\mathcal{R}_w \cap w^\bullet \mathcal{R}_+$ is given by
       \[ w^\bullet \mathcal{R}_+ \setminus \mathcal{R}_w \quad (\text{resp. } w^\bullet \mathcal{R}_+ \cap \mathcal{R}_-^w). \]  
   (c) $w$ satisfies the explicit description of (2.4), which is amenable to an algorithmic classification.
   (d) The closure of the dominant Weyl chamber contains an open subset of $V^w_\pm$.

In particular, there is an equality
\[ |\mathcal{R}| = \ell_f(O) + 2\ell(\mathcal{R}^{\text{dom}}). \]

Example 1.26. Let $\mathcal{R}$ be a root system and let $W$ be its finite Coxeter group. If $w$ is a reflection in some root, then $V_w$ equals the line through it. Thus the only dominant element in the conjugacy class $O$ of reflecting in a long (resp. short) root is reflecting in a root that lies in the closure of the dominant Weyl chamber; when $\mathcal{R}$ is crystallographic, it is well-known that this is the highest (resp. highest short) root.\(^4\) Thus the dual Coxeter number\(^5\) of the dual root system $\hat{W}$, let $\mathcal{O}$ be a twisted Coxeter group, pick a standard parabolic subgroup $\mathcal{O}' \subseteq \hat{W}$, and Thurston proved that this normal form is computable in quadratic time complexity and linear space complexity [ECH+92, Cha92]. The normal form is sometimes used to prove faithfulness of representations (e.g. [Kra02, Jen17]), and in type $A$ it can be used [BNG95, GMW11] to determine how elements lie in the Nielsen-Thurston classification [Thu88]; generically they are pseudo-Anosov (e.g. [Mah11]). The conjugacy problem and normal form have been generalised beyond Artin-Tits groups of finite type [DDG+15], and have recently been employed for studying braid group-based post-quantum cryptography (e.g. [MP19]).

\[^4\]Outside of type $A$ these are also the only convex reflections; in type $A$, the only other ones are the two simple reflections corresponding to the endpoints of the Coxeter-Dynkin diagram.

\[^5\]It follows from [Car70] that for Weyl groups the dual Coxeter number $h^\vee$ agrees with $(\ell(r_\beta)+3)/2$, where $r_\beta$ is the reflection in a long root $\beta$ lying in the closure of the dominant Weyl chamber; here we extend its definition to non-crystallographic root systems by defining it at such. For $H_3$, $H_4$ and $I_2(m)$ we then find that $h^\vee$ is given by 8, 24 and $|m/2| + 1$ respectively.
Notation 1.27. For any integer $i \geq 1$ and element $b$ in $\text{Br}^+$ we denote the product of the first $i$ Deligne-Garside factors by $DG_{i>0}(b) := DG_i(b) \cdots DG_2(b)DG_1(b)$, often implicitly identify a reduced braid like $DG_i(b)$ with the corresponding element of $\tilde{W}$.

In order to better understand the normal form of an arbitrary braid $b$ of $\text{Br}^+$ (and of $b_d^w$ in particular), we associate to it in §4.2 a subset of roots $\mathfrak{R}_w$ extending inversion sets of twisted Coxeter group elements, and show in Lemma 4.19 that this set has bounding properties.

Proposition D. For any element $w$ in a twisted finite Coxeter group and integer $d \geq 1$ we have an inclusion of sets of roots $\mathfrak{R}_{d^w} \subseteq \mathfrak{R}_{w}$. This then implies that for any $i \geq 1$

(i) there is an inclusion

$$\mathfrak{R}_{DG_i(b_d^w)} \subseteq \mathfrak{R}_+ \setminus (DG_{i-1}(b_d^w) \cdots DG_1(b_d^w)(\mathfrak{R}_{w})), \quad (1.4)$$

so in particular

$$\mathfrak{R}_{w} \subseteq \mathfrak{R}_{DG_i(b_d^w)} \subseteq \mathfrak{R}_+ \setminus \mathfrak{R}_{st} \quad \text{and} \quad \ell(DG_i(b_w^d)) \leq |\mathfrak{R}_+ \setminus \mathfrak{R}_{st}|.$$ (ii) Moreover, set

$$d := i(\lfloor |\mathfrak{R}_+ \setminus \mathfrak{R}_{st}| - \ell(w) \rfloor + 1),$$

then $DG_{i\geq}(b_w^d)$ has “stabilised”, i.e. for any $d' \geq d$ we have

$$DG_{i\geq}(b_w^{d'}) = DG_{i\geq}(b_w^d).$$

Example 1.28. Consider $w = s_3s_1s_2s_1$ in type $B_3$. Then $\mathfrak{R}_{w} = \mathfrak{R}_{w}^w = \{ \pm \alpha_{23} \}$ and by induction on $d \geq 1$ one finds

$$DG_3(b_w^{2d}) = b_3s_2b^{-1}_{w,s_3}s_1b_{121321},$$

$$DG_3(b_w^{2d+1}) = b_{321321}b^{-1}_{w,s_3}s_1b_{121321}.$$ (1.4)

Example 1.29. Consider $w = s_2s_4s_3s_2s_1$ in type $D_4$. Then $\mathfrak{R} = \mathfrak{R}^w$ but as $w^2(\alpha_3) = w(\alpha_4) = \alpha_3$ it now immediately follows that

$$w_{s_1s_2s_1} \geq DG(b_w^d)$$

for all $d \geq 0$. One may compute that this is an equality for $d > 1$, conversely implying that $\mathfrak{R}_{w}^w = \{ \alpha_3, \alpha_4 \}$.

It is explained in the sequel to this paper [Malb] that convex elements $w$ of twisted finite Weyl groups whose braid lifts attain the upper bound $\mathfrak{R}_{DG(b_w^d)} = \mathfrak{R}_+ \setminus \mathfrak{R}_{st}$ for at least one integer $d \geq 0$ yield transverse slices in the corresponding twisted reductive groups, generalising the slices of He-Lusztig and Sevostyanov. Proposition 4.21 explains that this upper bound is never attained when $w$ is not convex, so equivalently $w$ should be a convex element satisfying the braid equation

$$DG(b_w^d) = pb(w) \quad (\ast)$$

for some $d$; the previous proposition implies that satisfiability can be checked by picking any $d > |\mathfrak{R}_+ \setminus \mathfrak{R}_{st}| - \ell(w)$.

The original aim of this paper was to show that equation $(\ast)$ is easily computable and is satisfied by a large number of twisted finite Coxeter group elements, including those that appear in the cross section isomorphisms of He-Lusztig and Sevostyanov:

Theorem. Let $W$ be a twisted finite Coxeter group and pick an element $w$ in $W$.

(i) If $w$ has an anisotropic braiding sequence of eigenspaces, then it satisfies $(\ast)$ when $d \geq \text{ord}(w)$.

6The bounds for $i = 1$ will reprop in the sequel, using different techniques [Malb, Corollary 2.31].

10
(ii) If \( w \) is convex and \( w \sim w' \) in \( W^{\text{cx}} \) for an element \( w' \) satisfying (1) for some \( d \), then \( w \) satisfies (1) when

\[
d > |R_+ \setminus R_{\text{wt}}^w| - \ell(w).
\]

In particular, (1) holds for all maximally and minimally dominant elements.

(iii) If \( w \) is quasiregular, minimally dominant and lies in \( \tilde{W} \) then

\[
b_{w}^{\text{ord}(w)} = b_{\text{pb}(w)}^{-1} b_{\text{pb}(w)},
\]

and conversely elements in \( \tilde{W} \) satisfying this equation are quasiregular and satisfy \( \ell(w) = \ell(\text{O}_{\text{dom min}}) \).

(iv) If \( W \) is irreducible, then a nontrivial element lying inside of a parabolic subgroup of lower rank is never convex. In particular, the following are then equivalent if \( w \) is of minimal length in its conjugacy class:

(a) \( w \) is trivial or elliptic,

(b) \( w \) is convex,

(c) \( w \) is firmly convex and satisfies the braid equation (1) for some (or any)

\[
d > |R_+| - \ell(w). \tag{1.5}
\]

(v) If \( w \) has maximal or minimal length or is minimally dominant, then the Artin-Tits braid \( b_w \) is not pseudo-Anosov.

Remark 1.30. After replacing \( R_+ \) by \( R_{n-1} \) and the operators in (1.4) by their inverses, the exact same statements are true for the left Deligne-Garside normal form.

Proposition 4.44 explicitly computes the (right) Deligne-Garside normal form of \( b_{w}^{\text{ord}(w)} \) for any twisted finite Coxeter group element \( w \) that has a braiding sequence of eigenspaces. Some additional properties of this normal form (that require different techniques) are proven in the sequel [Malb].

This paper and its sequel indicate that conjugacy classes in reductive groups appear to be more closely related to minimally dominant elements than to minimal length elements [Malb, Corollary B]. This suggests that it might be more natural to construct Lusztig’s inverse to the Kazhdan-Lusztig map [Lus11a] and its refinement in [Lus15] from minimally dominant elements instead of minimal length elements; the end result would remain unchanged, according to the following

**Conjecture.** Suppose \( W \) is the Weyl group of a maximal torus contained in a Borel subgroup \( B \) of a twisted reductive group over an algebraically closed field. Let \( w \) be a minimal length element in \( W \) and let \( w' \) a minimally dominant element in the conjugacy class of \( w \). Then for any conjugacy class \( C \) of this reductive group,

\[
C \cap BwB \neq \emptyset \quad \text{if and only if} \quad C \cap Bw'B \neq \emptyset.
\]

My approach to this problem is outlined in §3.2.

Remark 1.31. Here are some other open problems:

(i) Does Proposition A(ii) extend to all twisted Coxeter groups?

(ii) Generalise Proposition B and the last part of Proposition C to standard parabolic subgroups of arbitrary rank (as in [Nie13]).

(iii) Fix a classical type (A, B, C or D) and pick a natural number \( n \). Is the number of convex/firmly convex/dominant/minimally-dominant/*-satisfying elements of reflection length \( n \) independent of rank(\( W \)) when rank(\( W \)) is sufficiently large?
(iv) Suppose $w$ is firmly convex and has maximal length, does there exist a dominant element $w'$ such that $w \leftrightarrow w'$ in $W^{\text{cx}}$?

(v) Given two firmly convex elements $w, w'$ in a (parabolic) $\tilde{W}$-orbit $\mathcal{O}$ such that $w \sim w'$ in $\mathcal{O}$, is it true that $w \sim w'$ in $\mathcal{O}^{\text{cx}}$?

(vi) Suppose that $w$ is firmly convex and satisfies the braid equation ($\ast$). Does there exist a dominant element $w'$ such that $w \sim w'$ in $W^{\text{cx}}$? (If so, this would imply that $\ell(w) \geq \ell(\mathcal{O}^{\text{dom}})$. For the conjugacy classes appearing in Sevostyanov’s case-by-case work [Sev19], that inequality follows from combining his calculations, this work and the main theorem of the sequel [Malb].)

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2 Shifts and sequences of eigenspaces

The main goal of this section is to prove Proposition B. Before we commence, let us prove now that our definition of “good position” (in Definition 1.11) is equivalent to the one provided in [HN12, §5.2]:

Notation 2.1. Given a set of hyperplanes $\mathcal{H}'$ in a reflection representation $V$, we write

$$V \setminus \mathcal{H}' := V \setminus \bigcup_{\mathfrak{h} \in \mathcal{H}'} \mathfrak{h}.$$

Proposition 2.2. Let $w$ be an element of a twisted finite Coxeter group with a sequence $\Theta = (V_m, \ldots, V_1)$ of eigenspaces in the reflection representation $V$. Let $F_i := \sum_{j=1}^{i} V_j$ denote corresponding filtration of $V$, as in Definition 1.11.

Then this sequence $\Theta$ is in good position w.r.t. the dominant Weyl chamber $C$, if and only if for each integer $i \in \{1, \ldots, m\}$ the closure of the connected component of $C$ in $V \setminus \mathcal{H}_{F_{i-1}}$ contains a regular point of the subspace $V_i$.

Proof. We induct on the length of the sequence: we assume that it is true $\ell < m$ and apply the induction hypothesis to the shortened sequence $(V_{m-1}, \ldots, V_1)$, which is in good position w.r.t. $C$. It then suffices to show that if the closure of the dominant Weyl chamber $C$ contains a regular point of $F_{m-1}$, then it contains a regular point of $F_m = F_{m-1} \oplus V_m$ if and only if the closure of the connected component of $C$ in $V \setminus \mathcal{H}_{F_{m-1}}$ contains a regular point of the subspace $V_m$. Let $v_{m-1}$ be a regular point of $F_{m-1}$ in $C$, so $v_{m-1} = \sum_{i=1}^{k_i} c_i \omega_i$ with each $c_i \geq 0$. Let $J$ denote the set of indices such that $c_j = 0$, or equivalently, the indices of simple roots whose hyperplanes lie in $\mathcal{H}_{F_{m-1}}$.

$\Leftarrow$: By assumption, $V_m$ has a regular point $v_m = \sum_{i=1}^{k} c_i' \omega_i$ with $c_i' \geq 0$ when $i$ lies in $J$. If a hyperplane does not contain both $v_{m-1}$ and $v_m$ (and hence $F_m$), then it only contains one linear combination of them (up to scalar); hence by finiteness of the number of hyperplanes, there exists a point $v := \varepsilon v_{m-1} + (1 - \varepsilon) v_m$ for $\varepsilon$ positive but arbitrarily close to zero, which is a regular point of $F_m$. But then

$$\alpha_i(v) \begin{cases} \approx \alpha_i(v_m) > 0 & \text{if } i \notin J, \\ = \varepsilon \alpha_i(v_{m-1}) \geq 0 & \text{if } i \in J. \end{cases}$$

The corresponding code is available upon request.
so \( v \) still lies in the closure of the dominant Weyl chamber.

\[ \Rightarrow : \] By assumption \( C \) contains an open subset of \( F_m = F_{m-1} \oplus V_m \) consisting of regular points; we may assume it is of the form \( U_m^{-1} \odot U_m \) for bounded open subsets \( U_m^{-1} \) of \( F_{m-1} \) and \( U_m \) of \( V_m \). Using that \( C \) also contains an open subset of \( F_{m-1} \), the next lemma (with \( V' := F_{m-1} \) and \( C' := F_{m-1} \cap \overline{C} \)) implies that we may actually assume that \( U_{m-1} \) lies inside of \( F_{m-1} \cap \overline{C} \); it translates by a point of \( F_{m-1} \cap \overline{C} \subseteq \overline{C} \), which keeps \( U_{m-1} \odot U_m \) in \( C \). It now follows that there is a regular point \( v_{m-1} \) of \( F_{m-1} \) lying in \( \overline{C} \) such that for some regular \( v_m \) in \( V_m \) their sum \( v = v_{m-1} + v_m \) is a regular point of \( F_m \) lying in \( \overline{C} \). If \( v_m = \sum_{i=1}^{\ell(v)} c_i \omega_i \) does not satisfy the requirements, so \( c'_i < 0 \) for some \( i \) in \( J \), then \( \alpha_i(v) = \alpha_i(v_m) < 0 \) which contradicts that \( v \) lies in \( \overline{C} \). \( \Box \)

**Lemma 2.3.** Let \( V' \) be a real vector space. If a convex cone \( C' \subseteq V' \) contains an open subset of \( V' \), then for any bounded open subset \( U \) of \( V' \) there exists a vector \( v \) in \( C' \) such that \( v + U \subseteq C' \).

**Proof.** Since the convex cone contains an open subset, it contains a ball. By multiplying with scalars, we can make the ball arbitrarily large, so \( C' \) also contains a ball the same size as a ball around the origin containing \( U \); now let \( v \) be the translate of the origin to the centre of this ball. \( \Box \)

The following lemma will be used several times throughout this paper:

**Definition 2.4.** If \( w, x, y \) are elements of a twisted Coxeter group such that \( b_w = b_xb_y \) (or equivalently, \( w = xy \) and \( \ell(w) = \ell(x) + \ell(y) \)), then we say that \( w = xy \) is a reduced decomposition.

**Notation 2.5.** When the conjugation action of an element \( x \) permutes the simple reflections of a standard parabolic subgroup \( W' \), we write \( \delta_x \) for the corresponding twist of \( W' \).

Given two Weyl chambers \( C, C' \) and a set of root hyperplanes \( \mathcal{H}' \), we denote by \( \mathcal{H}'(C, C') \) the subset of hyperplanes in \( \mathcal{H}' \) separating \( C \) from \( C' \).

**Lemma 2.6.** Let \( W \) be a twisted finite Coxeter group and let \( w \) be an element of \( W \).

(i) For any standard parabolic subgroup \( W' \subseteq \tilde{W} \) preserved by \( w \) under conjugation, the element \( w \) admits a unique reduced decomposition

\[ w = xy \]

into an element \( x \) in \( W \) and an element \( y \) in \( W' \), such that \( x \) is a minimal length double coset representative with respect to \( W' \) and permutes its simple roots; thus the action of \( x \) on \( V_{W'} \) agrees with \( \delta_x \).

(ii) If the simple roots of \( W' \) are all fixed by \( w \), then \( y \) is the identity element.

Now let \( \Theta = (V_m, \ldots, V_1) \) be a sequence of eigenspaces for \( w \) such that the dominant Weyl chamber \( C \) is in good position.

(iii) Factorise the element \( w = xy \) as in (i) for \( W' := \tilde{V}_1 \). Then

\[ \Theta' := (V_m \cap V_{W'}, \ldots, V_1 \cap V_{W'}) \]

is a sequence of eigenspaces (throwing away trivial ones) for the element \( \delta_x \) of the twisted finite Coxeter group \( \langle \delta_x \rangle \times W' \), such that the dominant Weyl chamber for \( \langle \delta_x \rangle \times W' \) is in good position.

(iii) Now factorise \( w = xy \) as in (i) for \( W' = \tilde{W}_\Theta \). Then these elements have length

\[ \ell(x) = 2 \sum_{i=1}^{m} \theta_i |\delta_{F_{i-1}} \setminus \delta_{F_i}|, \quad \ell(y) = \delta_{\Theta}(C, w(C)) \]
Proof. (i): We follow the proof of [HN12, Proposition 2.2]: let \( w = y'xy'' \) where \( y' \) and \( y'' \) lie in \( W' \) and \( x \) is a minimal length double coset representative for \( W' \). From

\[
(y'xy'')(y'xy'')^{-1} = wW'w^{-1} = W'
\]

it follows that \( x^{-1}W'x = W' \). The element \( x^{-1}y'x \) and hence also \( y := (x^{-1}y'x)y'' \) then lie in \( W' \). If \( s_i \) is a simple reflection of \( W' \) then \( xs_i = (xs_ix^{-1})x \) has length \( \ell(x) + 1 \), so as the element \( xs_i x^{-1} \) lies in \( W' \) it must have length 1.

For the final statement it suffices to note that a minimal coset representative does not make any simple root in \( V_W \) negative.

(ii): By the last remark in (i), \( \delta_x y \) acts as \( w \) on \( V_{W''} \subseteq V_W \), so it acts as the identity. Using that the dominant chamber in \( V_W \), is a fundamental domain for \( W' \) which is also preserved by \( \delta_x \), it follows that \( y \) acts as the identity.

(iii): Since \( w \) preserves \( V_1 \), its conjugation action preserves the standard parabolic subgroup \( W' \) so we may factorise as in (i). As it consequently permutes the simple reflections of \( W' \) it preserves \( V_{W'} \) so \( V_i \cap V_{W'} \) is indeed an eigenspace for \( w \) and \( \delta_x y \). If \( C \) contains an open subset of \( F_1 \) then as projection maps are open, the orthogonal projection of \( C \) to \( V_{W'} \) contains an open subset of \( F_1 \cap V_{W'} \) (which by orthogonality of eigenspaces is the projection of \( F_1 \)); but the orthogonal projection of \( C \) to \( V_{W'} \) is precisely the dominant Weyl chamber for \( W' \).

(iv): Using (iii), this now follows by induction on the length of \( \Theta \) from the case where \( \Theta \) consists of just one eigenspace, which is [HN12, Proposition 2.2].

From (ii) and (iv) we now deduce the first part of the following crucial

**Lemma 2.7.** Let \( W \) be a twisted finite Coxeter group, let \( O \) be a \( \tilde{W} \)-orbit and let \( \Theta = (V_m, \ldots, V_1) \) be a sequence of eigenspaces of some element \( w \) in \( O \), satisfying \( S_{\Theta} \subseteq \tilde{S}^w \).

(i) Let the corresponding sequence of normalised principal arguments be \( (\theta_m, \ldots, \theta_1) \), then

\[
\ell(O, \Theta) = 2 \sum_{i=1}^{m} \theta_i |\tilde{S}_{F_{i-1}} \setminus \tilde{S}_{F_i}|.
\]  

(ii) Suppose \( w, w' \) lie in \( O, \Theta \). Then the projection

\[
\text{Tran}^{\Theta, *}(w, w') \rightarrow \text{Tran}(w, w') \quad \text{(resp. \quad Tran}^*(w, w') \rightarrow \text{Tran}(w, w'))
\]

surjects, with

\[
* = \begin{cases} \pm & \text{if } \mp 1 \notin \text{eig}(\Theta), \\ \times & \text{otherwise}. \end{cases} \quad \text{(resp. \quad }* = \mp & \text{if } \text{eig}(\Theta) \cap \{1, -1\} = \emptyset). \]

When the element \( y \in \tilde{W}_{F_{m-1}} \) in the splitting \( w = xy \) for \( F_{m-1} \) in Lemma 2.6 equals \( \{\pm \text{id}\} \) or has no eigenvalues in \( \{\pm 1\} \) (e.g. if \( V_{F_{m-1}} \subseteq V_m \), which happens when \( V_0 = V \)), we only need the eigenvalues of \( (V_{m-1}, \ldots, V_1) \) here.

Proof. (i): For any \( w \) in \( O, \Theta \), factorise it into \( w = xy \) as in part (iv) of the previous lemma. From \( S_{\Theta} \subseteq \tilde{S}^w \) it follows that \( V_{W'} \subseteq V^w \), so part (ii) of that lemma implies that \( y \) is the identity.

Part (ii) of this lemma will be proven in §2.3.
2.1 Quasiregular elements and the Coxeter plane

In this subsection we briefly recall classical statements involving the eigenspace decomposition of regular elements and Coxeter elements, and recast them in our framework. It yields a helpful perspective on the relationship between Sevostyanov’s and He-Lusztig’s elements, illustrates how Lemma 2.7(i) can be applied and might serve as a suitable warm-up to some of the ideas that follow.

**Definition 2.8.** Recall that an element of a twisted finite Coxeter group is called regular if it has a regular eigenvector, i.e. a complex eigenvector in the complexified reflection representation which is not contained in any complexified root hyperplane.

We say that a complex eigenvector of an element $w$ is quasiregular if it is not contained in any complexified root hyperplane, unless that hyperplane comes from $H^w$.

**Proposition 2.9.** Let $w$ be an element of a twisted finite Coxeter group with complex eigenvalue $\lambda$, and let $H'$ be a subset of root hyperplanes in the real reflection representation. Then $H^w_{\lambda} \subseteq H'$ if and only if $w$ has a complex eigenvector for the eigenvalue $\lambda$ which is only contained in complexified root hyperplanes corresponding to those in $H'$.

In particular, $w$ satisfies $H^w_{\lambda} = \emptyset$ (resp. $H^w_{\lambda} \subseteq H^w$) for some $\lambda$ if and only if it is (quasi)regular.

**Proof.** Recall that the complexification of $V^w_{\lambda}$ is the sum of the complex eigenspaces corresponding to $\lambda$ and $\lambda^{-1}$, so in other words

$$V^w_{\lambda} \otimes_{\mathbb{R}} \mathbb{C} = (V \otimes_{\mathbb{R}} \mathbb{C})^w_{\lambda} \oplus (V \otimes_{\mathbb{R}} \mathbb{C})^w_{\lambda^{-1}}. \quad (2.2)$$

$\Rightarrow$: Pick a regular element in the complex eigenspace of $\lambda$ (for $V \otimes_{\mathbb{R}} \mathbb{C}$ and the complexified root hyperplanes). If it is contained in a complexified root hyperplane then by regularity that complex hyperplane contains the entire complex eigenspace. But since it came from a real subspace the complexified root hyperplane must also contain its complex conjugate, which by (2.2) the sum of these complex eigenspaces contains $V^w_{\lambda}$, which implies that the hyperplane came from $H'$.

$\Leftarrow$: The complexification of a hyperplane containing $V^w_{\lambda}$ contains the complexification $V^w_{\lambda} \otimes_{\mathbb{R}} \mathbb{C}$, which by (2.2) contains the complex eigenspace for the eigenvalue $\lambda$. \qed

**Example 2.10.** Reflections in type $B_2$ are quasiregular but not regular.

The following slightly generalises [Spr74, Theorem 6.4(i)]:

**Proposition 2.11.** Let $w = \delta \tilde{w}$ be an element of a twisted finite Coxeter group and suppose that it is quasiregular, so there exists an eigenvalue $\lambda = e^{2\pi i \theta}$ with $\theta \in (0, 1/2] \cup \{1\}$ such that $H^w_{\lambda} \subseteq H^w$. If $\delta^1/\theta = \text{id}$ then $\text{ord}(w) = 1/\theta$ and if furthermore $V^w_{\lambda}$ is in good position then

$$\ell(w) = \frac{|\mathcal{R}| - \ell_f(w)}{\text{ord}(w)}.$$

**Proof.** Since $\lambda$ is an eigenvalue of $w$, we have $\text{ord}(w) \geq 1/\theta$. As $w^{1/\theta}$ fixes $V^w_{\lambda}$ its twisted component is trivial, Steinberg’s theorem implies that it lies in $\tilde{W}_{V^w}$, but as before in the corresponding reflection representation its eigenspaces are spanned by roots of $\mathcal{R}^w$ and thus this element is the identity. The final formula follows from e.g. Lemma 2.7(i). \qed

The following statement also plays an important role in the sequel [Malb]:

**Proposition 2.12.** Let $V'$ be a nontrivial subspace of the reflection representation of an irreducible finite Coxeter group. Then the roots projecting nontrivially to $V'$ generate the root lattice.
From (2.3) we hence obtain a linear combination of those two roots.

Since \( \alpha \) does not lie in \( I_j \), it is orthogonal to the simple roots in \( I_j \), and furthermore there exists a simple root \( \alpha' \) in \( I_j \), such that \( (\alpha, \alpha') < 0 \). The induction hypothesis for \( \alpha' \) furnishes a positive root

\[
\beta' = c' \alpha' + \sum_{\alpha'' \in I_{j-1}} c'' \alpha'', \quad c' \in \mathbb{R}_{>0}, \quad c'' \in \mathbb{R}_{\geq 0},
\]

so from

\[
(\alpha, \beta') = (\alpha, c' \alpha') = c'(\alpha, \alpha') < 0
\]

we deduce that there exists a scalar \( c \in \mathbb{R}_{>0} \) such that \( \beta := \beta' + c \alpha \) is a positive root. As \( \alpha \) projects trivially to \( V' \), \( \beta \) does not, neither does \( \beta' \).

We now conclude that each simple root lies in the span of roots projecting nontrivially to \( V' \): given \( \alpha \) in \( I_{j+1} \), let \( \beta \) and \( \beta' \) be roots as we just constructed. Then \( \beta' \) and \( \beta \) project nontrivially to \( V' \), and \( \alpha \) is a linear combination of those two roots. \( \square \)

**Corollary 2.13.** Let \( V_0 \) and \( V_1 \) be two nontrivial subspaces of \( V \). Then

\[
\{ \text{roots projecting nontrivially to } V_0 \} \cap \{ \text{roots projecting nontrivially to } V_1 \} \neq \emptyset.
\]

**Proof.** Let \( \beta_1 \) be a root in the second set, then the proposition yields a root \( \beta_0 \) in the first set such that \( (\beta_0, \beta_1) \neq 0 \). This implies that \( \beta := \beta_0 + c \beta_1 \) is a root for some \( c \in \mathbb{R}^\times \). If \( \beta_0 \) projects trivially to \( V_1 \) and \( \beta_1 \) projects trivially to \( V_0 \), then \( \beta \) projects nontrivially to both of them. \( \square \)

**Proposition 2.14.** Let \( w \) be an element with a braiding sequence of eigenspaces \( \Theta' = (V_{m}, \ldots, V_1) \). If the positive principal argument of the eigenvalue of \( V_1 \) is not minimal amongst the positive principal arguments of the eigenvales of \( w \), then \( w \) does not have minimal length.

**Proof.** For each root hyperplane not in \( \mathcal{H}_\Theta \), there is an \( i \) such that it lies in \( \mathcal{H}_{F_{i-1}} \setminus \mathcal{H}_{F_i} \). Denote the corresponding argument by \( \theta_h := \theta_i \), and for \( h \in \mathcal{H}_\Theta \) set \( \theta_h = 0 \). Then the length formula (2.1) rewrites as

\[
\ell(w) = 2 \sum_{i=1}^{m} \theta_i | \mathcal{H}_{F_{i-1}} \setminus \mathcal{H}_{F_i} | = 2 \sum_{h \in \mathcal{H}_\Theta} \theta_h.
\]

(2.3)

Now let \( w' \) be a conjugate of \( w \) such that \( \Theta' = (V_{n}, \ldots, V_1) \) is in good position, where \( \Theta' \) is the set of all eigenspaces (or eigenvalues) of \( w' \), and let \( \theta'_h \) denote the corresponding arguments for root hyperplanes \( h \). Then there is an inequality \( \theta_h \geq \theta'_h \), for all root hyperplanes: for the roots projecting nontrivially to \( V_{m} \) but trivially to \( F_{i-1} := \sum_{j=1}^{i-1} V_j \), the argument is \( \theta'_i \) which is minimal amongst all possible arguments.

According to the previous corollary, there exists a root hyperplane \( h' \) in \( \mathcal{H}_F \setminus \mathcal{H}_{F_1} = \mathcal{H}_{F_0} \setminus \mathcal{H}_{F_1} \) which also lies in \( \mathcal{H}_{F_0} \setminus \mathcal{H}_{F_{i-1}} \). As by assumption the argument of \( V_1 \) is not minimal, we have a strict inequality \( \theta_h > \theta'_h \).

From (2.3) we hence obtain

\[
\ell(w) = 2 \sum_{h \in \mathcal{H}_\Theta} \theta_h > 2 \sum_{h \in \mathcal{H}_\Theta} \theta'_h = \ell(w').
\]

Much of the following statement was observed by Coxeter [Cox49] and was proven uniformly by Steinberg [Ste59]. At the time, it was mainly used to relate the number of roots to the Coxeter number and to the height of the highest root.
Theorem 2.15. Fix an irreducible finite Coxeter group.

(i) Its Coxeter elements are elliptic.

(ii) We may uniquely decompose the set of vertices of its Coxeter-Dynkin diagram (which is a tree) into two disconnected subsets. For each of these subsets, the corresponding simple reflections then commute, so their products yield two involutions $i_1$ and $i_2$, and then their product $w := i_1 i_2$ is a Coxeter element of minimal length.

(iii) There exists a plane in the reflection representation on which $i_1$ and $i_2$ act as reflections, so in particular this plane contains two lines on which they act as $-1$. Both of these lines intersect nontrivially with the closure of the dominant Weyl chamber.

(iv) Every root projects nontrivially to this plane, so this plane yields a complete sequence of eigenspaces of length one for their product $w$.

(v) Hence the order of $w$ on the reflection representation agrees with the order of $w$ on this plane, and the corresponding eigenvalue thus has minimal argument amongst the eigenvalues of $w$. Moreover, this eigenvalue occurs with multiplicity one.

Definition 2.16. For any Coxeter element, the plane corresponding to this eigenvalue is called its Coxeter plane. Its order is called the Coxeter number, and is typically denoted by $h$. The two Coxeter elements just constructed are sometimes called bipartite.

Proposition 2.17. The only Coxeter elements of minimal length that have a braiding sequences of eigenspaces, are the two whose Coxeter planes go through the dominant Weyl chamber.

Proof. There are $h$ lines in the Coxeter plane on which the reflections in the dihedral subgroup generated by $i_1$ and $i_2$ act as $-1$, yielding a decomposition of this plane into $2h$ fundamental domains. Since the lines on which $i_1$ and $i_2$ act as $-1$ both lie in the closure of the dominant Weyl chamber, and the segment between them lies in its interior, there are there are precisely $2h$ Weyl chambers containing an open subset of the Coxeter plane. The corresponding $2h$ conjugates of $w$ in the Coxeter group are the $2h$ conjugates of $w = i_1 i_2$ inside this dihedral subgroup, but these are all either $w$ or $w^{-1} = i_2 i_1$. The claim now follows from Proposition 2.14.

This implies that Sevostyanov’s set of elements only includes two Coxeter elements of minimal length; hence it does not include all of the elements that are used in the statement of He-Lusztig. However, He-Lusztig’s proof uses a trick (which we’ve replaced with a simpler braid monoid argument in part (ii) of the main Theorem) to reduce the cross section statement for elliptic elements of minimal length to Geck-Michel’s “good” elements [HL12, §3.1-§3.5]. As already mentioned, the case-free construction of such elements in [HN12, §5.2] is through decreasing, complete sequences of eigenspaces, which does yield a subset of Sevostyanov’s elliptic elements.

2.2 Gradient flows

The main novelty of this subsection is Lemma 2.25, which is pivotal in proving the new claims on cyclic shifts and strong conjugations. In the end we deduce Proposition B(i).

Proposition 2.18. Let $w$ be an element of a twisted finite Coxeter group, let $v$ be an element in the wall of the dominant Weyl chamber $C$ corresponding to the hyperplane $h_i$ of a simple reflection $s_i$, and let $n$ be a normal vector to $h_i$ at one of its elements, pointing towards $C$ (e.g., $n = \alpha_i$).

Then there are implications

\[ \pm(n, w(v)) > 0 \implies \ell(ws_i) \geq \ell(w), \]
\[ \pm(w(n), v) > 0 \implies \ell(s_iw) \geq \ell(w). \]
Proof. The claims are equivalent to the implications
\[ \ell(ws_i) \geq \ell(w) \quad \Rightarrow \quad \pm (n, w(v)) \geq 0, \]
\[ \ell(s_iw) \geq \ell(w) \quad \Rightarrow \quad \pm (w(n), v) \geq 0. \]

We shall prove the first one; the second one then follows by taking inverses. The length of \( ws_i \) is larger/smaller than the length of \( w \) if and only if the hyperplane \( h_i \) lies between \( w(C) \) and \( \pm C \), if and only if \( w(C) \) and \( \pm C \) lie on the same side w.r.t. \( h_i \), if and only if \( \pm (n, w(v)) \geq 0 \).

**Corollary 2.19.** Hence if the inequality
\[ (n, w(v)) + (w(n), v) \geq 0 \]
holds, then at least one of
\[ \ell(ws_i) \geq \ell(w) \quad \text{or} \quad \ell(s_iw) \geq \ell(w) \]
holds, and if both of those hold then the converse is true.

**Corollary 2.20** ([HN12, Lemma 1.1]). Moreover, consider the function
\[ f_w : V \to \mathbb{R}, \quad v \mapsto ||(\text{id}_V - w)(v)||^2 \]
and set \( w' = s_iws_i \).

(i) The gradient \( \nabla f_w \) at \( v \) satisfies
\[ (\nabla f_w(v), n) = 2(n - w(n), v - w(v)) = -2(n, w(v)) - 2(w(n), v), \]
(ii) and yields an implication
\[ (\nabla f_w(v), n) \leq 0 \quad \Rightarrow \quad \pm \ell(w') \geq \pm \ell(w). \]

**Proof.** (i): The first equality follows from expanding
\[ (\nabla f_w(v), n) = \lim_{t \to 0} \frac{f_w(v + tn) - f_w(v)}{t}, \]
and the second equality from using \( (w(n), w(v)) = (n, v) = 0 \).
(ii): This is equivalent to the implication
\[ \ell(w') = \ell(w) + 2 \quad \Rightarrow \quad (\nabla f_w(v), n) \leq 0, \]
which follows from (i) and the previous corollary.

One of the earliest observations that led to this paper was noticing that the corresponding gradient flow can be effectively restricted to sums of eigenspaces:

**Notation 2.21.** Let \( \Theta \) be a set of eigenspaces. We denote by \( V^\pm_\Theta \) the eigenspace in \( \Theta \) corresponding to the eigenvalue with maximal/minimal positive principal argument, as in Definition 1.11. Given an element \( v \) in \( V_\Theta \), we denote its projection to \( V^\pm_\Theta \) by \( v^\pm_\Theta \).

**Proposition 2.22** ([HN12, §1.5-§1.6]). Let the eigenvalues of \( w \) be \( \{e^{\pm 2\pi i \theta_0}, \ldots, e^{\pm 2\pi i \theta_m}\} \) with each \( 0 \leq \theta_j \leq 1/2 \), and denote the projection of an element \( v \) of \( V \) to the real eigenspace corresponding to \( e^{2\pi i \theta_j} \) by \( v_j \). Then this function \( f_w \) has a global gradient flow
\[ \Phi_w : V \times \mathbb{R} \to V, \quad (v, t) \mapsto \exp(2t(2 - w - w^{-1}))v = \sum_{j=1}^{m} \exp(4t(1 - \cos 2\pi \theta_j))v_j, \]
Proof. This follows from dimension considerations, as in the proof of [HN12, Proposition 1.2].

Proposition 2.23. For a generic point in $V_\Theta$, the corresponding flow generically consists of regular points of $V_\Theta$, and each of the nonregular points are regular in a hyperplane of $V_\Theta$.

Proof. This follows from dimension considerations, as in the proof of [HN12, Proposition 1.2].

Notation 2.24. Let $w$ be an element of a twisted finite Coxeter group $W$ and let $C$ denote the dominant Weyl chamber. Given another Weyl chamber $C'$ there is a unique element $\tau$ such that $\tau(C) = C'$; we write $w_{C'} := \tau w^{-1}$.

Lemma 2.25. Let $w$ be an element of a twisted finite Coxeter group $W$ and $\Theta$ be a subset of its eigenspaces such that the closure of the dominant Weyl chamber $C$ contains regular points of $V_\Theta$. Pick a generic regular point as in the previous statement, flow it in negative/positive direction until it meets a root hyperplane inside $V_\Theta$. Call the corresponding nonregular point $v$, and flow slightly further to a regular point $v'$.

Then there exists a Weyl chamber $C'$ whose closure contains $v$ and $v'$, satisfying

$$w \sim w_{C'}.$$

Proof. The assumption on nonregular points can be rephrased as saying that the hyperplanes containing $v$ but not $V_\Theta$ all intersect $V_\Theta$ in the same hyperplane of $V_\Theta$. The claims on the flow follow from dimension considerations; let $v$ be such a regular point satisfying the requirements. We now slightly perturb $v$ to a point $\tilde{v}$ in the interior of $C$, so that its flow goes to a point $\tilde{v}'$ nearby $v'$; generically, it will only meet the hyperplanes containing $\tilde{v}$ in regular points, and those intersection points can be made to lie arbitrarily close to $\tilde{v}$. Thus we obtain a path of Weyl chambers

$$C, s_{i_1}(C), \ldots, s_{i_k} \cdots s_{i_1}(C),$$

where the final Weyl chamber contains $\tilde{v}'$ and its closure thus contains $v'$. This is a “minimal path” of Weyl chambers: if we sufficiently “zoom in” on a gradient flow then it is approximately linear, so this path is derived from an approximately straight line between these two chambers.

Corollary 2.20(i) yields that

$$(w(n), \tilde{v}) + (n, w(\tilde{v})) \geq 0$$

for each normal vector $n$ of the hyperplanes pointing towards $C$. Thus at least one of

$$(w(n), \tilde{v} - w(\tilde{v})) = (w(n), \hat{v}) \geq 0 \quad \text{or} \quad (n, w(\tilde{v}) - v) = (n, w(\tilde{v})) \geq 0$$

holds. In fact, one of these inequalities holds for each $n$: since $\tilde{v}$ lies in $V_\Theta$ and this subspace is preserved by $id_V - w$, the element $\hat{v} - w(\tilde{v})$ still lies in $V_\Theta$. By the assumption on hyperplanes containing nonregular points, $\hat{v} - w(\tilde{v})$ lies on one side of such a hyperplane if and only if it lies on the same side of all of them. And then the same inequality is true with $\tilde{v}$ replaced by one of the new intersection points $\tilde{v}'$ since they lie arbitrarily close.

As $\tilde{v}'$ lies arbitrarily close to $\tilde{v}$ and $\hat{v}$ lies in these hyperplanes, we have

$$s_{i_1} \cdots s_{i_k} (\hat{v}) \approx s_{i_1} \cdots s_{i_k} (\tilde{v}) = \hat{v}.$$

The claim now follows inductively by applying Proposition 2.18 consecutively: if say the first equation holds, then for each $w_j := s_{i_1} \cdots s_{i_k} w$ we have

$$(w_j(n), \hat{v}) = (w(n), \hat{v}) \equiv (w(n), \tilde{v}) \geq 0,$$

and hence

$$\ell(s_{i_1} \cdots s_{i_k} w_j) = \ell(w_j) \pm 1 = \ell(w) \pm (j + 1).$$
Proof of Proposition B(i). We slightly generalise the reasoning of [HN12, Proposition 1.2], inducting on the length of $\Theta_+$ (or the rank of $W$). The closure $\overline{C}$ of $C$ contains an open subset of $V_\Theta$, and for a generic point $v'$ in this subset its projection $v'_w$ to $V_\Theta^\perp$ is nonzero and regular. The limit of a generic point $v$ inside of the chamber $C$ under the positive/negative flow might not lie in $V_\Theta^\perp$ (when the corresponding eigenvalue is not maximal among the eigenvalues of $w$), but by picking $v$ sufficiently close to $v'$ its half-line will pass by the half-line through $v_\pm$ with arbitrarily little distance between them, and thus the flow will pass through a chamber $C'$ whose closure contains an open neighbourhood of $v'_\pm$ in $V_\Theta^\perp$. Furthermore we can ensure that $v$ satisfies the statements in Proposition 2.23; when it meets the first hyperplane there is then a strict inequality

$$ \langle \nabla f_w(v), n \rangle \leq 0 $$

where $n$ is the normal vector to $v$, pointing towards the chamber the flow came from. Since the flow converges there is, for each chamber it goes through, a final time it is there before getting within the required distance of $v_{\pm}$, so if we only use those wall crossings then it is evident that the corresponding sequence of Weyl chambers from $C$ to $C'$ is finite. Corollary 2.20(ii) now implies that $w \overset{\pm}{\leadsto} w_{C'}$ in $W$, whilst Lemma 2.25 furnishes that $w \overset{\pm}{\sim} w_{C'}$, furnishing that $w \overset{\pm}{\sim} w_{C'}$ in $\mathcal{O}_\Theta$.

Thus by replacing $C$ with $C'$, we may now assume that the closure of $C$ contains a regular point $v'_\pm$ of $V_\Theta^\perp$, which is the first eigenspace of $\Theta_+$. By Proposition 2.6 there is a factorisation $w = x y$ where $y$ lies in the proper standard parabolic subgroup $\tilde{W}_{v'_\pm}$ of $W$ and $x$ permutes the indices of $\tilde{W}_{v'_\pm}$. Applying the induction hypothesis to the element $\delta_x y$ in $\langle \delta_x \rangle \times \tilde{W}_{C'}$ then implies that we can further shift the chamber so that it is in good position with respect to $\Theta_+$, yielding the first two claims. By starting at this chamber and following the flow in reverse direction, we get the final claim.

2.3 Shells

The main aim of this subsection is to generalise the proof of [HN12, §3.3] to yield Lemma 2.7(ii), and subsequently deduce the remainder of Proposition B. For the new claims on cyclic shifts and strong conjugations, we will need the following lemma:

Definition 2.26 ([HN12, §2.4]). Let $V'$ be a subspace of the reflection representation $V$. We will say that two Weyl chambers $C, C'$ are $V'$-adjacent if $\overline{C} \cap \overline{C'} \cap V'$ spans a codimension 1 subset of $V'$.

Lemma 2.27. Let $w$ be an element of a twisted finite Coxeter group. Let $V'$ be a real eigenspace in the reflection representation corresponding to an eigenvalue of $w$. Let $C$ and $C'$ be two Weyl chambers whose closures contain open subsets of $V'$, which lie in the same connected component of $V' \backslash \delta_{V'}$, and which are $V'$-adjacent.

If the eigenvalue is not 1 or $-1$ then either

$$ w \sim w_{C'} \quad \text{and} \quad w_{C'} \overset{\pm}{\sim} w $$

or

$$ w \overset{\pm}{\sim} w_{C'} \quad \text{and} \quad w_{C'} \sim w $$

holds. If the eigenvalue is $\pm 1$ then

$$ w \overset{\pm}{\sim} w_{C'} \quad \text{and} \quad w_{C'} \overset{\pm}{\sim} w $$

holds.

Proof. Let $\tau \in \tilde{W}$ be the unique element such that $\tau(C) = C'$. Pick a point in the interior of $\overline{C} \cap V'$. Assume that the eigenvalue is not $\pm 1$; since this point lies in $V'$, its action is described by a two-dimensional plane where $\overline{C} \cap \overline{C'} \cap V'$ is a hyperplane and where $\overline{C} \cap V'$ and $\overline{C'} \cap V'$ are chambers touching this hyperplane. Either $w$ rotates in the direction of the hyperplane (with angle strictly between 0 and $\pi$) or it rotates away from it.

Let’s assume the former. Since we may choose the point arbitrarily close to the hyperplane and the eigenvalue is not 1, it follows that $w(C)$ crosses this hyperplane, and that $w^{-1}(C)$ does not as the eigenvalue is not $-1$. The chamber $w(C)$ might not be equal to $C'$, but its projection to $C'$ is the same or lies further along
this rotation, so \( \mathcal{F}(C, C') \setminus \mathcal{F}_V \subseteq \mathcal{F}(C, w(C)) \setminus \mathcal{F}_V \). But by assumption on \( C' \) we have \( \mathcal{F}(C, C') \cap \mathcal{F}_V = \emptyset \), so \( \mathcal{F}(C, C') \subseteq \mathcal{F}(C, w(C)) \). Hence

\[
\mathcal{F}(C, \tau(C)) \cup \mathcal{F}(C', w\tau^{-1}(C')) = \mathcal{F}(C, C') \cup \mathcal{F}(C', w(C)) = \mathcal{F}(C, w(C'))
\]

which is equivalent to \( \ell(\tau) + \ell(w\tau^{-1}) = \ell(w) \), and hence \( \ell(w\tau^{-1}) = \ell(w) - \ell(\tau) \). Similarly,

\[
\mathcal{F}(C, \tau(C)) \cup \mathcal{F}(C', wC, \tau(C')) = \mathcal{F}(C, C') \cup \mathcal{F}(C', wC, \tau(C')) = \mathcal{F}(C, wC, \tau(C))
\]

yields \( \ell(\tau) + \ell(wC') = \ell(wC) \). This shows \( w \sim wC \) and \( wC \sim w \), the other case is proven analogously, and so is the case where the eigenvalue is \( \pm 1 \).

\[\square\]

**Proposition 2.28.** Let \( \Theta = (\ldots, V_1) \) be a sequence of eigenspaces of an element of a twisted finite Coxeter group, such that the dominant Weyl chamber \( C \) is in good position. If \( C' \) is a Weyl chamber lying in the same connected component as \( C \) in \( V \setminus \mathcal{F}_V \), then \( C' \) is in good position w.r.t. \( \Theta \) if and only if its closure contains a regular point of \( V_1 \).

In particular, for such \( C' \) there exists a sequence of Weyl chambers

\[ C = C_0, \ldots, C_m = C' \]

such that each Weyl chamber in this sequence is in good position with respect to \( \Theta \), and each pair \( C_i, C_{i+1} \) is \( V_1 \)-adjacent.

**Proof.** Since \( C \) and \( C' \) lie in the same connected component of \( V \setminus \mathcal{F}_V \), they definitely lie in the same connected component of \( V \setminus \mathcal{F}_F \) for \( i \geq 1 \). The first claim then follows from Proposition 2.2.

The second claim follows from the first, using [HN12, Lemma 2.4]. \[\square\]

**Proposition 2.29.** Let \( C, C' \) be two Weyl chambers. The unique (untwisted) Coxeter group element \( \tau \) sending \( C \) to \( C' \) fixes all points in \( \overline{C \cap C'} \), i.e.

\[ \overline{C \cap \tau(C)} = \overline{C \cap C'}. \]

**Proof.** We may assume that \( C \) is the dominant Weyl chamber, and that \( C' \) is not. The intersection \( \overline{C \cap C'} \) is then precisely the intersection of the hyperplanes separating \( C \) from \( C' \), and at least one of them \( h_i \) corresponds to a simple reflection \( s_i \). Thus we can decrease the distance from \( C \) to \( C' \) by crossing one such plane to \( C'' := s_i(C) \); as the intersection \( \overline{C \cap C'} \) lies in the wall \( h_i \), it is fixed by the reflection \( s_i \), and thus the new intersection \( \overline{C'' \cap C} \) contains \( \overline{C \cap C'} \). By induction we thus obtain a path from \( C \) to \( C' \) through simple reflections, each of which fixes \( \overline{C \cap C'} \), and hence so does their product. \[\square\]

We now finish the proof of Lemma 2.7, following [HN12, §3.3] very closely:

**Proof.** (ii): Let \( \tau \in \text{Tran}(w, w') \) and consider \( C' := \tau(C) \), so that \( w' = wC \). We induct on \( \text{dim} V_\Theta \); the base case is that \( \Theta \) is an empty sequence, but then \( \mathcal{F} = \mathcal{F}_\Theta \subseteq \mathcal{F}^{w} \) implies that \( w \) is the identity element.

By Lemma 2.6(i) and (ii), we may factorise \( w = xy \) and (the connected component in \( V \setminus \mathcal{F}_V \) of) \( C \) is in good position for the sequence \( \Theta' \) for \( \delta_x y \) in \( \langle \delta_x \rangle \times W_{V_1} \), and similarly for \( C' \) and any other Weyl chamber in its connected component for \( V \setminus \mathcal{F}_V \). Let \( \tau \) be an element in \( \tilde{W}_{V_1} \) such that \( \tau(C) \) lies in the same component as \( C' \) in \( V \setminus \mathcal{F}_V \). From the induction hypothesis for the sequence \( \Theta' \) for the element \( \delta_x y \) in \( \langle \delta_x \rangle \times \tilde{W}_{V_1} \), we then obtain a preimage for \( \tau \) in

\[ \text{Tran}^{\overline{\Theta'}}(\delta_x y, \tau \delta_x y^{-1}) \rightarrow \text{Tran}(\delta_x y, \tau \delta_x y^{-1}). \]

Since this preimage is a sequence of elements in \( \tilde{W}_{V_1} \) (thus fixing \( V_1 \)) and each of them commute with \( \delta_x \) in the same way as with \( x \), it also yields a preimage for \( \tau \) in

\[ \text{Tran}^{\overline{\Theta'}}(w, \tau w^{-1}) \rightarrow \text{Tran}(w, \tau w^{-1}), \]

21
and similarly for $\Tran^*(w, tw\tau^{-1})$.

Hence we may assume that $C$ and $C'$ lie in the same connected component of $V \setminus \delta V_1$. By Proposition 2.28 there exists a sequence of Weyl chambers $C = C_0, C_1, \ldots, C_n = C'$ in the same component of $V \setminus \delta V_1$, such that $w_{C_i}$ lies in $O_i^{\delta}$ and that each pair $C_i, C_{i+1}$ is $V_1$-adjacent. By induction on the length of this sequence, we may assume that $n = 1$. Consider the hyperplane $\mathfrak{h}' := \mathfrak{h} \cap V_1$, where $\mathfrak{h} \in \delta V_1 \setminus \delta V_1$ is a root hyperplane separating $C$ and $C'$. Since $C \cap C'$ spans $\mathfrak{h}'$, this intersection $C \cap C'$ contains a regular point $v$ of $\mathfrak{h}'$.

Case 1: $w(\mathfrak{h}') \neq \mathfrak{h}'$. From Proposition 2.29 it follows that the “minimal path” from $C$ to $C'$ yields a sequence of Weyl chambers $C = C_0, \ldots, C_m = C'$ in the same component of $V \setminus \delta V_1$, where each consecutive pair of Weyl chambers is adjacent and $V_1$-adjacent. By [HN12, Proposition 2.5] the lengths of the corresponding elements is the same, so that the simple reflections corresponding to this sequence of Weyl chambers yields a sequence in $\Tran^w(w, w')$ mapping to $\tau$. A suitable element in $\Tran^{\delta^w}(w, w')$ is obtained from Lemma 2.27, using Lemma 2.7(i).

Case 2: $w(\mathfrak{h}') = \mathfrak{h}'$ and $\dim V_1 \geq 2$, so $\dim \mathfrak{h}' \geq 1$. We use Lemma 2.6 to split $w = xy = \delta_xy$ for $\langle \delta_x \rangle \ltimes \tilde{W}_{\mathfrak{h}'}$, then since $\mathfrak{h}' \neq 0$ the sequence $O'$ is “smaller” again. By Proposition 2.29 the translation $\tau \in \tilde{W}$ fixes $\mathfrak{h}'$, so it lies in $\tilde{W}_{\mathfrak{h}'}$. As before, we use the induction hypothesis to obtain an element in $\Tran^w(w, w')$ or $\Tran^{\delta^w}(w, w')$ mapping to $\tau$.

Case 3: $w(\mathfrak{h}') = \mathfrak{h}'$ and $\dim V_1 = 1$. Thus the eigenvalue $\lambda_1$ corresponding to $V_1$ is either 1 or $-1$, and the claim follows from Lemma 2.27 again.

Unless $w$ is an involution, we cannot (as is well-known) always find a sequence of Weyl chambers such that the length of the corresponding element strictly increases or decreases:

Example 2.30. Consider $w = s_2s_3s_4s_1s_2s_3$ in type $A_4$, which is not of minimal length in its conjugacy class and has eigenvalues $e^{\pm 2\pi i/5}$ and $e^{\pm 4\pi i/5}$. Let $\theta$ be all of its eigenvalues, so $V_{\theta} = V$ and $V_{\theta}^\perp = V_{2/5}^w$. The latter is spanned by the vectors

$$\varphi \omega_1 - \varphi \omega_2 + \omega_3 \quad \text{and} \quad \omega_1 - \omega_3 + \varphi \omega_4,$$

where $\varphi := (1 + \sqrt{5})/2$ denotes the golden ratio. Its intersection with the closure of the dominant Weyl chamber is the zero vector, but one can calculate that

$$\ell(s_iws_i) \geq \ell(w)$$

for any simple reflection $s_i$.

Nor does there always exist a sequence of Weyl chambers containing a regular point of $V_{\theta}$:

Example 2.31. Consider $w = w_0s_1s_3s_4$ in type $B_4$, which has eigenvalues $\pm 1$ and $e^{\pm 2\pi i/3}$. Let $\theta$ be all of them except 1, so $V_{\theta} = V_{\omega_3}$. Then $V_{\theta}$ is the hyperplane of $s_4$, so $s_1ws_1$ yields another element whose corresponding chamber $s_1(C)$ touches $V_{\theta}$, and this is the only other element reachable from $w$ by such simple shifts. As the eigenspace for $-1$ is the line through the fundamental weight $\omega_3$, neither the dominant Weyl chamber $C$ nor $s_1(C)$ contains a regular point of $V_{\theta}^\perp$, which is the span of $\alpha_1$ and $\alpha_2$.

Proof of Proposition B. (ii): Let $w$ be an element of $O_{\theta}^{\max/min}$. From part (i) we obtain an element $w'$ in $O_{\pm}^{\theta} \subseteq O_{\theta}$ such that $w \cong w'$, so as $w$ has maximal/minimal length it follows that $w'$ also has maximal/minimal length. It then follows from Lemma 2.7(i) that

$$\ell(O_{\theta}^{\max/min}) = \ell(w) = \ell(w') = \ell(O_{\pm}^{\theta}) = 2 \sum_{i=1}^m \theta_i |\delta F_{i-1} \setminus \delta F_i|$$

as claimed.

(iii): By using the second or third statement in (i) of the proposition and maximality/minimality, we may use either strong conjugations or cyclic shifts to move both $w$ and $w'$ to $O_{\pm}^{\theta}$. The claim then follows from 2.7(ii).
3 Minimally dominant elements

In this section we study standard parabolic subgroups of twisted finite Coxeter groups, and prove the main Lemma and Proposition C. We will need

**Lemma 3.1.** Let \( w \) be an element of a twisted finite Coxeter group and let \( v \) be a vector in the intersection of \( V_w \) with the closure of the dominant Weyl chamber. Then \( v \) is a regular point for \( V_w \) if and only if

\[
(\beta, v) \begin{cases} 
> 0 & \text{if } \beta \text{ is a positive root and is not fixed by } w, \\
= 0 & \text{if } \beta \text{ is fixed by } w, \\
< 0 & \text{if } \beta \text{ is a negative root and is not fixed by } w.
\end{cases}
\]  

(3.1)

**Proof.** \( \Rightarrow \): As \( v \) lies in the closure of the dominant Weyl chamber, equation (3.1) follows when it is shown that \((\beta, v) = 0\) if and only if \( \beta \) is fixed by \( w \). But due to the orthogonal decomposition \( V = V_w \oplus V^w \), a regular point \( v \) satisfies

\[
(\beta, v) = 0 \quad \text{if and only if} \quad (\beta, V_w) = 0 \quad \text{if and only if} \quad \beta \in V^w \quad \text{if and only if} \quad \beta \text{ is fixed by } w.
\]

\( \Leftarrow \): If \( v \) is not regular, then there exists a root \( \beta \) not in \((V_w)^\perp = V^w\) such that \((\beta, v) = 0\). \( \square \)

3.1 Isotropy subgroups

We begin this subsection with a description of isotropy subgroups, then analyse intersections of the form \( V^w \cap \Omega \) and finish by proving the main Lemma.

**Proposition 3.2.** Let \( W = \Omega \ltimes \tilde{W} \) be a twisted finite Coxeter group, let \( V \) be its reflection representation with dominant Weyl chamber \( C \) and let \( V' \) be a subset of \( V \).

(i) The isotropy subgroup \( W_V \) is a parabolic subgroup of \( W \). More precisely, it is conjugate to a standard parabolic subgroup of the form \( \tilde{\Omega}' \ltimes \tilde{W}' \), where \( \tilde{W}' \) is a standard parabolic subgroup of \( \tilde{W} \) and \( \tilde{\Omega}' \) is the subgroup of \( \Omega \) preserving a certain partition of the simple reflections not in \( \tilde{W}' \).

(ii) If the closure \( \overline{C} \) contains a regular point of the span of \( V' \), then \( W_{V'} \) is a standard parabolic subgroup.

(iii) If it does not, then there exists a simple reflection fixing \( V' \cap \overline{C} \) but not \( V' \).

**Proof.** (i), (ii): We replace \( V' \) by its span to make it convex, and subsequently pick a regular point \( v \) inside of it. After conjugating \( W_{V'} \), we may assume that this point lies in the closure of the dominant Weyl chamber. Now let \( \delta \tilde{w} \) be an element in \( W_{V'} \subseteq W = \Omega \ltimes \tilde{W} \) fixing \( v \). As \( \delta \) preserves \( \overline{C} \) and \( \overline{C} \) is a fundamental domain for the action of \( \tilde{W} \), any point in \( \overline{C} \) fixed by \( w \) must be fixed by both \( \delta \) and \( \tilde{w} \). Thus \( \tilde{w} \) lies in the subgroup of \( \tilde{W} \) fixing \( v \); it is well-known that this is a standard parabolic subgroup whose indices correspond to the zero coordinates of \( v \) in the basis of fundamental weights. Since \( v \) is fixed by \( \delta \), that set of coordinates must be preserved by \( \delta \). Since it also acts as a permutation on the remaining coordinates, the claim follows.

(iii): Again replace \( V' \) by its span, and note that \( W_{V'} \subseteq W_{V' \cap \overline{C}} \). If the span of \( V' \cap \overline{C} \) does not contain a regular point of \( V' \) then it there must be some reflection in \( W_{V'} \) that is not in \( W_{V' \cap \overline{C}} \), so this inclusion is proper. Since by (ii) the subgroup \( W_{V' \cap \overline{C}} \) is a standard parabolic, it is generated by the simple reflections it contains, so the claim follows. \( \square \)

**Corollary 3.3.** Let \( w \) be an element of a twisted finite Coxeter group.

(i) If \( w \) has an anisotropic braiding sequence of eigenspaces, then it is dominant.

(ii) If \( w \) is dominant, then it is firmly convex.
Proof. (i): Let $\Theta$ be this sequence, so by assumption the closure of the dominant Weyl chamber contains an open subset $U$ of $V_\Theta$. The subgroup fixing it is a standard parabolic, or in other words, the intersection of the hyperplanes in $H_\Theta$, is some facet of the dominant Weyl chamber. Then we can find an open neighbourhood $U'$ of $U$ inside of this facet, and by shrinking it we may assume that it lies in the closure of the dominant Weyl chamber. From $H_{V_w} \subseteq H_\Theta \subseteq H_{V_w} = H_{V_w}$, it follows that $V_w$ is also contained in this facet, so as $V_\Theta \subseteq V_w$ it follows that $V_w \cap U'$ is an open subset of $V_w$.

(ii): The subspace $V_w$ is spanned by its intersection with the closure of the dominant Weyl chamber $C$. By the previous lemma the reflections whose hyperplanes contain $V_w$ are the reflections of a standard parabolic subgroup, so equivalent to the roots that are perpendicular to $V_w$ (which as in the proof of Lemma 3.1 are precisely the set of roots $R^w$ of $w$) form a standard parabolic subgroup.

Now pick a regular vector $v$ in $V_w \cap C$ and let $\beta$ be a positive root in $R^w$ which is not fixed by $w$. Then $w^i(\beta)$ is also a positive root not fixed by $w$ for any integer $i$, so $(w^i(\beta), v) > 0$ by Lemma 3.1. By summing these equations for $i$ in $\{1, \ldots, \text{ord}(w)\}$ it follows that

$$\sum_{i=1}^{\text{ord}(w)} w^i(\beta), v = \sum_{i=1}^{\text{ord}(w)} (w^i(\beta), v) > 0,$$

but as the vector $\sum_{i=1}^{\text{ord}(w)} w^i(\beta)$ is fixed by $w$ it should be orthogonal to $V_w$ and hence to $v$. □

Notation 3.4. For any element $w$ of a twisted finite Coxeter group and subset $V'$ of its reflection representation, we write $(V')^w := V_w \cap V'$ for the fixed points of $w$ that lie in $V'$. In particular, $(\overline{C}^w) := V_w \cap \overline{C}$ denotes the intersection of $V^w$ with the closure of the dominant Weyl chamber $C$.

Corollary 3.5. Let $W$ be a twisted finite Coxeter group and pick an element $w = \delta \tilde{w}$ in $W$. Consider the subgroup $W_w$ of $W$ consisting of elements fixing $C^w$.

(i) This is a standard parabolic subgroup containing $\delta$ and $\tilde{w}$.

(ii) Moreover, $w$ is an elliptic element of this subgroup if and only if $\overline{C}^w$ contains a regular point of $V^w$.

Proof. (i): Immediately follows from part (ii) of the previous proposition.

(ii): By definition (and convexity) the cone $\overline{C}^w$ contains a regular point of $V^w$ if and only if the set of reflections fixing $\overline{C}^w$ coincides with the set of reflections fixing $V^w$, if and only if $W_w$ is the subgroup of $W$ of elements fixing $V^w$ (by part (iii)). But since $w$ lies in $W_w$, that is true if and only if $w$ is elliptic in $W_w$. □

Proposition 3.6. Let $W = \Omega \times \tilde{W}$ be a twisted finite Coxeter group and pick an element $w = \delta \tilde{w}$.

(i) There is an orthogonal decomposition $V^w = V_{W_w}^w \oplus (V_{W_w}^{\perp})^\delta$.

(ii) $(V_{W_w}^{\perp})^\delta$ is spanned by the vectors

$$\left\{ \sum_{k=1}^{\text{ord}(\delta)} \delta^k(\omega_i) : s_i \notin W_w \right\}. \quad (3.2)$$

(iii) The intersection $\overline{C}^w$ spans $(V_{W_w}^{\perp})^\delta$.

Proof. (i): As $w$ lies in $W_w$, the action of $w$ preserves $V_{W_w}^w$ and thus its orthogonal complement $V_{W_w}^{\perp}$, so that $V^w = V_{W_w}^w \oplus (V_{W_w}^{\perp})^w$. As $\tilde{w}$ lies in $W_w$ we have $V_{\tilde{w}} \subseteq V_{W_w}^w$, so $V_{W_w}^{\perp} \subseteq V^w$ and then $(V_{W_w}^{\perp})^w = (V_{W_w}^{\perp})^\delta$.

(ii): This follows because $V_{W_w}^{\perp}$ is spanned by $\{\omega_i : s_i \notin W_w\}$.

(iii): By construction we have $\overline{C}^w \subseteq V^w \cap V_{W_w}^{\perp} = (V_{W_w}^{\perp})^\delta$, and then the claim follows from the explicit description of $(V_{W_w}^{\perp})^\delta$ in (ii); those basis vectors lie inside of $\overline{C}$. □

Lemma 3.7. Let $w = \delta \tilde{w}$ be an element of a twisted finite Coxeter group $W$, let $s_i$ be a simple reflection and consider the conjugate $w' := s_i w s_i$. Then
\[
\mathcal{C}^w = \begin{cases} \subseteq \mathcal{C}^{w'} & \text{if } s_i \in W_w, \\
\text{cone on } \mathcal{C}^{w'} \text{ and } \sum_{j=1}^{\text{ord}(\delta)} \delta^j(\omega_i) & \text{if } s_i \notin W_w. 
\end{cases}
\]

**Proof.** Case 1: If \( s_i \) lies inside of \( W_w \) then it fixes \( \mathcal{C}^w \), and then so does \( w' = s_iw_s \).

Case 2: Since \( \delta \) lies in \( W_w \) and \( s_i \) does not, neither does \( \delta^k(s_i) \) for any integer \( k \). It follows from the previous proposition that \( \sum_{k=1}^{\text{ord}(\delta)} \delta^k(\omega_i) \in (V_{W_w}^\perp)^\delta \subseteq V^w \). Since \( (V_{W_w}^\perp)^\delta \cap V_{W_w}^w \subseteq V_{W_w}^\perp \cap V_{W_w} = \emptyset \), we may shift each element of \( V_{W_w}^w \subseteq V^w \) by this one to obtain an \( s_i \)-invariant element, yielding a linear (but not orthogonal) decomposition

\[
V^w = V_{W_w}^{w,i} \oplus (V_{W_w}^\perp)^\delta.
\]

Now consider an element \( v = (v_0, v_1) \) of

\[
s_i(V^w) = V_{W_w}^{w,i} + s_i((V_{W_w}^\perp)^\delta).
\]

By construction, the expansion of \( v_0 = \sum_{j=1}^n c_j \omega_j \) in the basis of fundamental weights satisfies \( c_j = 0 \). If \( c_j = 0 \) for all \( s_j \) in \( W_w \) then it lies in \( V_{W_w}^\perp \) which implies \( v_0 = 0 \). So if \( \sum_{k=1}^{\text{ord}(\delta)} \delta^k(\omega_i) \) is not orthogonal then by multiplying with \(-1\) we can make them all positive, and then by adding to \( v \) suitable multiples of the basis elements of \( (V_{W_w}^\perp)^\delta \) that are given in (3.2) we obtain an element in \( V^w \) all of whose coordinates \( c_j \) are positive, so it lies in \( V^w \cap \mathcal{C} \) but since \( s_j' \in W_w \) this is a contradiction.

Now remove \( \sum_{k=1}^{\text{ord}(\delta)} \delta^k(\omega_i) \) from the set of vectors (3.2) and call the resulting span \( (V_{W_w}^\perp)^\delta \), so

\[
(V_{W_w}^\perp)^\delta = (V_{W_w}^\perp)^\delta \oplus \mathbb{R}\left( \sum_{k=1}^{\text{ord}(\delta)} \delta^k(\omega_i) \right)
\]

and thus

\[
s_i((V_{W_w}^\perp)^\delta) = (V_{W_w}^\perp)^\delta \oplus \mathbb{R}\left( s_i \sum_{k=1}^{\text{ord}(\delta)} \delta^k(\omega_i) \right).
\]

If \( \omega_i \) occurs \( n > 0 \) times in \( \sum_{k=1}^{\text{ord}(\delta)} \delta^k(\omega_i) \) then

\[
s_i\left( \sum_{k=1}^{\text{ord}(\delta)} \delta^k(\omega_i) \right) = \sum_{k=1}^{\text{ord}(\delta)} \delta^k(\omega_i) - n\alpha_i
\]

has a negative \( \omega_i \)-coefficient, and the rest is positive. No other basis elements has an nontrivial \( \omega_i \)-coefficient however; thus if \( v_0 \neq 0 \) then \( v \) does not lie in \( \mathcal{C} \), and hence

\[
V^w \cap \mathcal{C} = s_i(V^w) \cap \mathcal{C} = s_i((V_{W_w}^\perp)^\delta) \cap \mathcal{C}.
\]

And as \( s_i \) does not fix the vector \( \sum_{k=1}^{\text{ord}(\delta)} \delta^k(\omega_i) \) it now follows that

\[
V^w \cap \mathcal{C} = s_i((V_{W_w}^\perp)^\delta) \cap \mathcal{C} = (V_{W_w}^\perp)^\delta \cap \mathcal{C},
\]

and the claim follows. \( \square \)

**Lemma 3.8.** Let \( w \) be an element of a twisted finite Coxeter group.

(i) \( \text{Set } w' := s_iw_s \text{ for some simple reflection } s_i. \text{ If } V^w \cap \mathcal{C} \text{ is not contained in } V^w \cap \mathcal{C}, \text{ then } \ell(w) > \ell(w'). \)
In particular, if \( w \) has minimal length, then the closure of the dominant Weyl chamber contains a regular point of its fixed space \( V^w \).

This implies that an element of minimal length in its conjugacy class lies in a standard parabolic subgroup of lower rank if and only if it is not elliptic.\(^8\)

(ii) If the closure of the dominant Weyl chamber contains a nonzero point of \( V_w \) (e.g. \( w \) is a nontrivial dominant element), then \( w \) does not lie inside a standard parabolic subgroup of \( W \) which has lower rank than \( W \).

In particular, if furthermore \( w \) lies in the untwisted part \( \tilde{W} \) then its length can be recovered from its Alexander polynomial (see [Mala] or [Tri21, Theorem 6]) and
\[
\ell(w) \geq \text{rank}(W),
\]
with equality if and only if \( w \) is a Coxeter element of minimal length.\(^9\)

Proof. (i): The previous lemma implies that \( s_i \not\in W_w \). Thus
\[
\ell(s_iw) = \ell(w) + 1 = \ell(ws_i).
\]
If \( s_iw = ws_i \) then we obtain \( w = s_iws_i = w' \) which contradicts the assumptions. Hence \( s_iw \not= ws_i \), so from [Bou68, Exercice VI.2.23] we conclude that \( \ell(w) > \ell(w') \).

As we’ve shown in Lemma 2.7(i), we can always find a conjugate \( w' \) of \( w \) with \( \overline{w} \rightarrow \overline{w'} \), such that \( \overline{C} \) contains a regular point of \( V^{w'} \). Thus if \( \overline{C} \) does not contain a regular point of \( V^w \), the length must drop at least once in the corresponding sequence. By say Corollary 3.5(i) it now follows that an element of minimal length lies in a proper parabolic subgroup if and only if it is not elliptic.

(ii): Suppose that \( w \) does lie in a standard parabolic subgroup of lower rank, then \( V_w \) lies in the span of the corresponding simple roots, so any nontrivial element \( v \) in the intersection of \( V_w \) with the closure of the dominant Weyl chamber \( C \) decomposes as a sum of strictly positive scalar multiples of a proper subset of the simple roots. (Here we’re using that the dominant Weyl chamber lies inside of its dual cone, i.e. that each fundamental weight can be expressed as a positive (rational) sum of simple roots.) Then as the Coxeter-Dynkin diagram of \( W \) is connected, there exists a root \( \alpha_j \) in this proper subset and a root \( \alpha_k \) outside of it, such that \( (\alpha_j, \alpha_k) < 0 \). Then for this vector \( v = \sum_{i=1}^{rk} c_i \alpha_i \) we have
\[
(v, \alpha_k) = \sum_{i=1}^{rk} c_i (\alpha_i, \alpha_k) \leq c_j (\alpha_j, \alpha_k) < 0
\]
but then \( v \) does not lie in \( \overline{C} \), which is a contradiction.

If \( w \) lies in the untwisted part \( \tilde{W} \) it now follows that each simple reflection must occur at least once in a reduced decomposition of \( w \), and if they do each occur once then \( w \) is a Coxeter element.

Example 3.9. Consider the reflections in the roots \( \alpha_{12} \) and \( \alpha_{1112} \) in type \( G_2 \). For both elements we have \( V_w \cap \overline{C} = \{0\} \), but neither of them lies in a standard parabolic subgroup.

Example 3.10. Consider the elements \( w = s_1us_1 \) and \( w' = s_3ws_3 \) in the conjugacy class of \( u = s_2s_3s_4 \) in the Weyl group \( W \) of type \( A_4 \). Then \( V_u \) is spanned by \( \alpha_2, \alpha_3, \alpha_4 \), and then
\[
V_w = s_1(V_u) = \langle \alpha_{12}, \alpha_3, \alpha_4 \rangle = \langle \omega_2 - \omega_3, -\omega_2 + 2\omega_3 - \omega_4, -\omega_3 + 2\omega_4 \rangle = \langle \omega_2, \omega_3, \omega_4 \rangle
\]
\[
V_{w'} = s_3(V_u) = \langle \omega_2, \omega_3 - \alpha_3, \alpha_4 \rangle = \langle \omega_2, \omega_2 - \omega_3 + \omega_4, \omega_4 \rangle = \langle \omega_2, \omega_3, \omega_4 \rangle
\]

\(^8\)The untwisted version is the main lemma of [GM97] which is deduced there from extensive case-by-case computations, whilst Howlett also gave a proof of the untwisted statement using the parabolic Burnside ring [GP00, Proposition 3.1.12]; his proof extends, ours is entirely different.

\(^9\)This statement can be used to simplify Michel’s proof [Wil18] of Kamgarpour’s inequality [Kam15].
implies that both \( w \) and \( w' \) are dominant. Since

\[
\ell(w') > \ell(w) = 5 = \text{rank}(W) + 1
\]

only \( w \) is minimally dominant, but the conjugate \( s_1 w' s_1 = s_3 u s_3 \) lies in a standard parabolic subgroup and is hence not dominant.

The analogous statement for (ii) is false for elliptic elements when the subspace of fixed points is replaced by the eigenspace whose eigenvalue has minimal argument:

**Example 3.11.** Consider the Coxeter element \( s_1 s_2 s_3 \) in type \( A_3 \). This subspace is its Coxeter plane, which does not pass through the dominant Weyl chamber.

**Proof of the main Lemma.** (i): For the first equality, note that \( O^\Theta \subseteq O^{\text{dom}} \), where \( \Theta \) is the set of all eigenspaces (simply remove the eigenspaces with eigenvalue 1 at the end of the sequence), but also \( O^\Theta \subseteq O^\Theta_{\text{max}} = O_{\text{max}} \) by Proposition B(ii). The rest follows from part (i) of the previous lemma.

(ii): Follows from part (ii) and (iii) of Proposition B.

(iii): Follows from equation (1.2) and from part (ii) of the previous lemma.

\[\square\]

### 3.2 Bruhat cells

In this subsection we study the relationship between minimally dominant elements and Bruhat cells, outlining a possible approach to the main Conjecture.

Lusztig used a character formula for Iwahori-Hecke algebra basis elements (which is useful computationally, see also [Mic15, §6-§7]) to show that his constructions do not depend on the conjugacy class of the braid \( b_w \), when working over an algebraically closed field [Lus11a, §1.2]. According to Proposition A(ii), this conjugacy class can be obtained from strong conjugations and cyclic shifts; for the latter, it is not necessary to make any assumptions on the base:

**Proposition 3.12.** Let \( C \) be a conjugacy class of a twisted reductive group \( G \) with Borel subgroup \( B \) containing a maximal torus. Let \( w \) and \( w' \) be elements of its Weyl group lying in the same cyclic shift class. Then for every conjugacy class \( C \) in \( G \) we have

\[
C \cap BwB \neq \emptyset \quad \text{if and only if} \quad C \cap Bw'B \neq \emptyset.
\]

**Proof.** Let \( N_+ := [B, B] \) be the unipotent radical of \( B \), let \( N_- \) be the unipotent radical of the opposite Borel subgroup and then set \( N_z := N_+ \cap z^{-1} N_- z \) for any element \( z \) in this (twisted) Weyl group. By induction, it suffices to consider the case where \( w = xy \) and \( w' = yx \) are both reduced decompositions. If an element \( bxyb' \) of \( BxyB \) lies in the conjugacy class \( C \), then so does the element \( yb'b \) which furthermore lies inside of

\[
B_y B x B = B_y B x N_w = B_y N_y x N_x = B w' N w = B w' B.
\]

The following statement was partially inspired by constructions in [Sev19, p. 336-337].

**Lemma 3.13.** Let \( w \) be a minimally dominant element of a twisted finite Coxeter group with dominant Weyl chamber \( C \). Then there exists a Weyl chamber \( C' \) such that

(i) \( w \) has minimal length with respect to \( C' \).

(ii) \( V_w \) is generated by the roots of a standard parabolic subroot system in \( C' \).

(iii) The positive roots for \( C \) inside \( V_w \) coincide with the positive roots for \( C' \) in \( V_w \).

We’ve already seen in Lemma 3.8(i) that (ii) follows from (i), but we will explicitly use it here in proving (i):
Proof. Given an arbitrary Weyl chamber $\tilde{C}$, we denote by $\ell_{\tilde{C}}(\cdot)$ the corresponding length function. Let $v_0$ be a regular element in $V^w$ and $v_1$ be a regular element in $V_w \cap \overline{C}$. Since $w$ is dominant, $v_1$ is also a regular element of $V_w$. In particular, $v_0 + \varepsilon v_1$ is regular for $\varepsilon > 0$ sufficiently small, defining a Weyl chamber $C'$ satisfying the condition of (iii). By construction, the closure $\overline{C}$ contains a regular point of $V^w$, so the isotropy subgroup fixing it is a standard parabolic subgroup $W'$, satisfying $V_{W'} \subseteq V_w = (V^w)^\perp$. As this subgroup also contains $w$, this inclusion must be an equality. Thus (ii) holds, which implies that $w$ lies inside of this standard parabolic subgroup. Then $\ell_{C'}(w) = |\mathcal{R}_w \cap V_w|$, for the inversion set $\mathcal{R}_w$ defined with respect to either $C$ or $C'$.

Let $O$ denote the $\tilde{W}$-orbit of $w$. According to Proposition B(ii) there is an inclusion $O^{\text{dom}} \subseteq O_{\text{min}}$, so by (iii) of this proposition we can cyclic shift $w$ to an element $w'$ in $O^{\text{dom}}$; applying the same reasoning, we obtain a Weyl chamber $C''$ satisfying (ii) and (iii) for this element. But with respect to $C''$ the element $w'$ lies in $O_-$ and hence in $O_{\text{min}}$ by Proposition B(ii) again. Then from Corollary 4.23 we deduce that

$$\ell_{C'}(w) = |\mathcal{R}_w \cap V_w| = |\mathcal{R}_{w'} \cap V_{w'}| = \ell_{C''}(w')$$

so (i) follows for $w$ as well. \qed

Example 3.14. Consider the conjugate of $s_1s_2s_3s_4s_4s_2s_1$ of $s_1s_2s_3$ in type $A_4$. It is dominant but not minimally dominant, and there is no Weyl chamber such that the conditions of the lemma hold.

Corollary 3.15. Let $w$ be a minimally dominant element in a twisted finite Coxeter group. Then there exists a sequence of simple reflections $s_{i_0}, \ldots, s_{i_{n-1}}$ such that upon setting $w_{j+1} := s_i w_j s_i$ for $0 \leq j \leq n - 1$, the properties

$$\ell(w_j) \geq \ell(w_{j+1}) \quad \text{and} \quad \alpha_i \notin V_{w_j}$$

hold and $w_n$ has minimal length.

Proof. Take any regular point $v_1$ of $V_w$ lying in $\overline{C}$ and regular point $v_0$ of $V^w$ such that $\varepsilon v_0 + v_1$ lies in $C$ and is regular for $\varepsilon > 0$ sufficiently small. The gradient flow explicitly given in Proposition 2.22 now defines a straight line from the half-line through $\varepsilon v_0 + v_1$ to the half-line through $v_0 + v_1$, and hence from $C$ to the chamber $C'$ constructed in the previous proof. The vectors along this path then yield a continuous family of hyperplanes where the positive roots of $V_w$ remain positive, i.e. condition (iii) must be true at each of the chambers along this straight path. But this implies that along this path, $\alpha_i \notin V_{w_j}$ is always satisfied. The condition $\ell(w_j) \geq \ell(w_{j+1})$ follows from Corollary 2.20(ii). \qed

Conjecture 3.16. Let $G$ be a twisted reductive group over an algebraically closed field, with Borel subgroup $B$. Let $w$ be an element of the corresponding Weyl group and $s_i$ a simple reflection such that $\alpha_i \notin V_w$ and set $w' := s_i w s_i$. Then for every conjugacy class $C$ in $G$ we have

$$C \cap BwB \neq \emptyset \quad \text{if and only if} \quad C \cap Bw' B \neq \emptyset. \quad (3.3)$$

As shown in [EG04, Example 3.6], this fails when the underlying field is not algebraically closed. Note that by [EG04, Proposition 3.4], it suffices to prove $\Rightarrow$ with $\ell(w) > \ell(w')$. Assuming this conjecture, one obtains

Corollary 3.17. Let $w$ be minimally dominant element and $w \rightarrow w'$ for some element $w'$. Then (3.3) holds.

Proof. By say Proposition B(i) there exists a minimal length element $w''$ such that $w' \rightarrow w''$, and thus $w'' \rightarrow w' \rightarrow w$. From [EG04, Proposition 3.4] it then follows that

$$C \cap Bw''B \neq \emptyset \quad \Rightarrow \quad C \cap Bw'B \neq \emptyset \quad \Rightarrow \quad C \cap BwB \neq \emptyset.$$

Now let $w'''$ denote the minimal length element constructed from $w$ in Corollary 3.15. Then the previous conjecture coupled with strong conjugation invariance yields

$$C \cap BwB \neq \emptyset \quad \Rightarrow \quad C \cap Bw'''B \neq \emptyset \quad \Rightarrow \quad C \cap Bw''B \neq \emptyset. \qed

28
And that would yield the main Conjecture:

**Proof.** By Proposition B(i) there exists a minimal length element \( w'' \) such that \( w' \overset{\sim}{\rightarrow} w'' \). The previous corollary and strong conjugation invariance then yield

\[ C \cap BwB \neq \emptyset \iff C \cap Bw''B \neq \emptyset \iff C \cap Bw'B \neq \emptyset. \]

### 3.3 Involutions

In this subsection we prove Proposition C. My primary contribution here is generalising known results about conjugacy classes to orbits of standard parabolic groups, and in the new characterisation given in part (ii)(d).

**Notation 3.18.** Given an element \( w \) of a twisted finite Coxeter group, we let \( \mathcal{R}^w_1 \) denote the set of roots on which it acts as \( -1 \). Given a subset \( J \) of simple reflections, we denote by \( W_J \) the standard parabolic subgroup generated by \( J \) and by \( J^{\mathcal{R}_+} \) the corresponding set of positive roots.

The explicit description for part (ii)(c) of Proposition C is:

Let \( J \) denote the set of simple reflections such that \( W' = W_J \) and consider the subset

\[ J' := \{ s_j \in J : s_jws_j = w \text{ and } \ell(w) \leq \ell(ws_j) \} \subseteq J. \]

Decompose \( w \) into \( w = w'w'', \) where \( w'' \) denotes the identity/longest element of \( W_J \) and \( w'' \) is the maximal/minimal left coset representative for \( w \) in \( W_J \). Then

\[ J^{-1}R_+ \cap J^{-1}R_+ \cap J^{-1}R_+ = J^{-1}R_+ \cap J^{-1}R_+ (\text{resp. } J^{-1}R_+ \cap J^{-1}R_+ = J^{-1}R_+ \cap J^{-1}R_+), \]

the element \( w'' \) is an involution, a maximal/minimal length double coset representative in \( \tilde{W}_J \times \tilde{W}_J \) and permutes the simple roots of \( J^{\mathcal{R}_+} \).

For part (ii)(d) we will need

**Proposition 3.19.** Let \( V \) be a vector space over a field with a half-space \( H \subseteq V \) and a nontrivial subspace \( V' \subseteq V \). Let \( X \) be a subset of the dual vector space \( V^* \). Then either there exists in \( X \) a functional \( \alpha \) which is nonpositive and nontrivial on some orthogonal basis for \( V' \) lying in \( H \), or there exists a nontrivial vector in \( V' \cap H \) on which all elements of \( X \) evaluate nonnegatively.

**Proof.** Let \( h \) be the hyperplane bordering \( H \). Since \( h \) has codimension 1 the intersection \( V' \cap h \) has dimension \( \dim(V') - 1 \) or \( \dim(V') \). Thus there exists an orthogonal basis \( \{ v'_i \} \) for \( V' \) consisting of \( \dim(V') - 1 \) vectors lying in \( h \) and another vector \( v' \) lying in \( H \). If the values \( \alpha(v') \) are nonnegative for each \( \alpha \in X \), we are done. Otherwise, at least one of them is negative. Multiplying the other basis vectors of \( V' \) (which lie in the linear space \( h \)) by \(-1\) if this \( \alpha \) evaluates positively on them, we obtain an orthogonal basis for \( V' \) lying in \( H \) on which \( \alpha \) always evaluates nonpositively.

The main proof of Hart-Rowley in [PR04] can be rephrased as

**Lemma 3.20 ([PR04]).** Let \( w \) be an involution of a twisted Coxeter group and let \( s_j \) be a simple reflection. Suppose that \( \alpha_j \notin R_w \cup R_w \) (resp. \( \alpha_j \in R_w \setminus R_w^{\mathcal{R}_w} \)). Then

\[ \ell(s_jws_j) \geq \ell(w). \]

**Proof.** Using that \( w \) is an involution, we deduce from \( \alpha_j \notin R_w \) (resp. \( \alpha_j \in R_w \)) that

\[ \ell(s_jw) = \ell(ws_j) \geq \ell(w). \]

From \( \alpha_j \notin R_w \cup R_w \) (resp. \( \alpha_j \in R_w \setminus R_w^{\mathcal{R}_w} \)) it follows that \( \alpha_j \notin R_{s_jw} \) (resp. \( \alpha_j \in R_{s_jw} \)), and hence

\[ \ell(s_jws_j) \geq \ell(s_jw) \geq \ell(w). \]
For the implications \((b) \Rightarrow (c) \Rightarrow (a)\) we build upon a proof of Howlett [GP00, Proposition 3.2.10], just as [He07, Lemma 3.6] did for the twisted case.\(^{10}\) The latter however only considered the standard parabolic subgroup \(W\). Richardson originally obtained his result through proving the minimal case (for \(\bar{W}' = \bar{W}\)) of (d) \(\Rightarrow (c)\) [Ric82]; we will not follow his approach.

**Proof of Proposition C.** (ii), \((a) \Rightarrow (b)\): If the statement in \((b)\) is false, then the previous lemma implies that \(\ell(s_jws_j) \gg \ell(w)\) for some \(s_j\) in \(\bar{W}'\).

(ii), \((b) \Rightarrow (c)\): From \(w^2_{J'} = \id\) and the properties of \(J'\) it follows that

\[
\id = w^2 = w^J w_{J'} w = w^J w_{J'} w w_{J'} = w^J w_{J'} w w_{J'} = (w^J)^2,
\]

so that we may also write \(w = w_{J'} w^J\). Given a simple reflection \(s_{J'} \in J\), then from the definition of \(w_{J'}\) it follows that know that \(w_J s_{J'} w_{J'} = s_{J'}\) for another simple reflection \(s_{J'} \in J\). Thus

\[
w^J s_{J'} w^J = w^J w_{J'}, s_{J'} w_{J'} w^J = w s_{J'} w = s_{J'},
\]

which proves the claim on permuting simple roots.

Suppose now that \(w^J\) is not a maximal/minimal double coset representative for \(\bar{W}_J \times \bar{W}_{J'}\), so there exists a simple reflection \(s_J\) in \(J\) such that \(\ell(s_J w^J) \gtrless \ell(w_{J'})\). As \(w^J\) is a maximal/minimal left coset representative for \(\bar{W}_{J'}\), this inequality implies that \(s_J w^J\) is as well. Then

\[
\ell(s_J w) = \ell(s_J w^J w_{J'}) = \ell(s_J w_{J'}) + \ell(w_{J'}) \gtrless \ell(w_{J'}) + \ell(w_{J'}) = \ell(w)
\]

and as \(w\) is an involution it also follows that \(\ell(w_{J'}) \gtrless \ell(w)\). The element \(s_J w\) lies in the coset \(s_J w^J \bar{W}_{J'}\), and if \(s_J \in J'\) then \(s_J w = w_{J'}\) which would imply that it also lies in the coset \(w^J \bar{W}_{J'}\). But as both \(s_J w^J\) and \(w^J\) are maximal/minimal coset representatives, this is a contradiction. Hence \(s_J w \neq w s_J\), which by [Bou68, Exercice VI.2.23] implies that \(\ell(s_J w_{J'}) \gtrless \ell(w_{J'})\). We conclude that \(\alpha_j\) does not lie in \(\mathcal{R}_w \cup \mathcal{R}_w^\perp\) (resp. lies in \(\mathcal{R}_w \setminus \mathcal{R}_w^\perp\)), which contradicts the assumption of \((b)\).

(ii), \((c) \Rightarrow (a)\): Pick any element \(x\) in \(\bar{W}_J\) and factorise it as \(x^J x_{J'}\) as before, so \(x_{J'}\) lies in \(\bar{W}_{J'}\) and \(x^J\) is a maximal/minimal left coset representative. By definition of \(J'\), we have

\[
x w_{J'} x^{-1} = x^J (x_{J'} w x_{J'}^{-1}) (x^J)^{-1} = x^J w (x^J)^{-1}.
\]

As \(x^J\) lies in \(\bar{W}_J\) it then follows that

\[
\ell(x w_{J'} x^{-1}) = \ell(x^J w (x^J)^{-1}) \leq \ell(x^J) + \ell(w_{J'}) = \ell(x^J) + \ell(w_{J'}) = \ell(x^J) = \ell(w),
\]

and hence \(w\) is of maximal/minimal length in its \(\bar{W}_J\)-orbit.

(ii), \((b) \Leftrightarrow (d)\): By (a twisted analogue of) [Bou68, Exercice V.4.2(d)], we may assume that \(W\) is finite. Let \(v\) be a regular point of the dominant Weyl chamber \(C\), so that the halfplane

\[
H_v := \{ x \in V : (x, v) \geq 0 \}
\]

separates the positive roots from the negative roots. Let \(v_w\) denote a regular point of \(V_w^{\dag} \cap C\).

\((d) \Rightarrow (b)\): \(v_w\) is also a regular point of \(V_w^\perp\). For any root \(\beta\) we have

\[
(w(\beta), v_w) = (\beta, w^{-1}(v_w)) = (\beta, v_w).
\]

In the \(-\)-case, as the nontrivial eigenvalues of an involution are all \(-1\) and \(V_w\) thus coincides with \(V_{-1}w\), we deduce from Lemma 3.1 that for any positive root \(\beta\), the root \(w(\beta)\) is negative if and only if \(\beta\) is not fixed by \(w\). In the \(+\)-case, since \(v_w\) lies in the dominant Weyl chamber this equation implies that \(w(\beta)\) is positive.

\(^{10}\)The assumption \(\delta^2 = \id\) is missing there; see Example 3.22.
unless \((w(\beta), v_w) = 0\). But by regularity this implies that \(\beta\) is orthogonal to \(V^w_1\), so as \(V = V^w_1 \oplus V^w_{-1}\) this means that \(\beta\) lies in \(V^w_{-1}\).

(b) \(\Rightarrow\) (d): On the other hand, suppose that \(v_w\) is not a regular point of \(V^w_{\pm 1}\). Let \(I_w\) denote the set of indices corresponding to hyperplanes \(h_i\) of simple reflections \(s_i\) not containing \(V^w_{\pm 1}\). We split it \(I_w = J_w \cup J'_w\) into the subset of indices \(J_w\) such that \(J_w\) doesn’t contain \(\overline{C} \cap V^w_{\pm 1}\), and the subset \(J'_w\) of indices for which it does; according to Proposition 3.2(iii) the subset \(J'_w\) is now nonempty. Consider the basis \(\{w_i\}_{i=1}^\infty\) of fundamental weights \((\omega_j)_{j=1}^\infty\), then evidently for each \(j\) in \(J_w\) there exists an vector in \(\overline{C} \cap V^w_{\pm 1}\) that has a nontrivial \((\text{and thus positive})\ \omega_j\)-component. Taking for each \(j\) such an element and summing them, we obtain an element \(w'_w = \sum_{j \in J_w} c_j \omega_j\) lying in \(\overline{C} \cap V^w_{\pm 1}\). We let \(V''\) denote the orthogonal complement inside \(V^w_{\pm 1}\) to the span of \(\overline{C} \cap V^w_{\pm 1}\).

Suppose that there exists a nontrivial vector \(v'\) in \(V''\) on which all the \(\alpha_j\) for \(j \in J'_w\) evaluate nonnegatively. By adding a sufficiently large positive multiple of \(v'_w\) to \(v'\), we obtain an element whose remaining coordinates in this basis are positive. But by construction this element lies in the span of \(\overline{C} \cap V^w_{\pm 1}\), and hence so does \(v'\) which is a contradiction.

The previous proposition then yields an orthogonal basis \(\{v'_{i\prime}\}_{i=1}^{\dim V'}\) for \(V''\) lying in \(\mathbb{H}_v\), and a particular \(j \in J'_w\) such that \(\alpha_j\) evaluates and nonpositively on all of these basis elements, and negatively on at least one of them. Denote the projection of \(\alpha_j\) to \(V^w_{\pm 1}\) by \(\alpha_j^\perp\). As \(V = V^w_1 \oplus V^w_{-1}\) and \(\alpha_j\) projects to zero in the orthogonal complement of \(V''\) in \(V^w_{\pm 1}\) (which is spanned by \(V^w_{\pm 1} \cap \overline{C}\)), we have

\[
w(\alpha_j) = \pm \alpha_j^\perp + \sum_{i=1}^{\dim V'} w((\alpha_j, v'_{i\prime}))v'_{i\prime} = \pm \alpha_j^\perp + \sum_{i=1}^{\dim V'} (\alpha_j, v'_{i\prime})v'_{i\prime} = \pm \alpha_j + 2 \sum_{i=1}^{\dim V'} (\pm \alpha_j, v'_{i\prime})v'_{i\prime}.
\]

Since \(\alpha_j\) and \(v'_{i\prime}\) all lie in \(\mathbb{H}_v\) and \((-\alpha_j, v'_{i\prime}) \geq 0\), so does \(\pm w(\alpha_j)\), so \(w(\alpha_j)\) is a positive/negative root. As the inequality is strict for at least one \(i\), \(\alpha_j\) does not lie in \(V^w_{\pm 1}\), so the assumption of (b) does not hold.

(i): If \(w\) does not have maximal/minimal length, then by (ii) it does not satisfy equation (1.3). From the lemma we deduce that \(w \not\rightarrow s_j w s_j\) for some \(s_j\) in \(W'\), and then the claim follows from induction.

**Definition 3.21.** Consider the element \(s_1 w_3 s_1 = w_3 s_1 s_3\) in type \(A_3\). This is an involution of the form \(w_5 w_4\), but as it is not of maximal length (and does not fix any roots) it is not dominant.

**Example 3.22.** Consider the element \(w := \delta w_5\) in type \(D_4\), where \(\delta\) is the diagram automorphism

\[1 \mapsto 3 \mapsto 4 \mapsto 1.\]

As \(\delta^{-1} w_5 \delta = w_5 = w_5^{-1}\) (for any twist) it follows that \(\delta^2 w = 1\), but \(\ell(s_1 w s_1) < \ell(w)\) so \(w\) is not of minimal length in its conjugacy class (another example is \(\delta s_4 s_3 s_1 = s_1 s_3 s_3 s_1\)). Further conjugating \(s_1 w s_1\) with \(s_2\) and then \(s_3\), we find

\[w \not\rightarrow \delta s_3 s_2 s_3 s_4 s_1 s_2 s_1\]

which has minimal length, but is not of the form \(\delta w_j\) (which is what is the last requirement in (3.4) specialises to when \(W' = W\)).

**Proposition 3.23.** Orbits for standard parabolic subgroups of involutions in a twisted finite Coxeter group can be algorithmically classified in terms of its Coxeter-Dynkin diagram.

**Proof.** Let \(J\) be such that the standard parabolic subgroup \(W'\) equals \(\tilde{W}_J\). By Proposition C, we need to understand when elements of the minimal case in (3.4) are conjugate under \(\tilde{W}_J\). So we may assume that they are the form \(w' = w'_{J'} w_{J'}\) and \(w'' = w''_{J''} w_{J''}\), where \(w_{J'}\) and \(w_{J''}\) are the longest elements of standard parabolic subgroups for certain \(J', J'' \subseteq J\). By (3.4) the element \(w''_{J''}\) (resp. \(w''_{J''}\)) is a double coset representative; in particular, they are minimal right coset representatives for \(\tilde{W}_J\). If \(x w' x^{-1} = w''\) for some \(x\) in \(\tilde{W}_J\), then as

\[x w' x^{-1} = (w' x w^{-1} (w'_{J''})^{-1}) w''\]
and \( w"xw'^1(ww'')^{-1} \) lies in \( \tilde{W}_J \), it follows by uniqueness of minimal coset representatives that \( w' = w"' \).

By assumption the projection of the \(-1\) eigenspace to \( V_{w'} \) is given by \( V_{\tilde{W}_J} \) and \( V_{\tilde{W}_J''} \) respectively. As \( x \) only acts on \( V_{w'} \), it thus yields a bijection between the roots of \( \tilde{W}_J \) to the roots of \( \tilde{W}_J'' \). It has to map the simple roots of \( \tilde{W}_J \) to a fundamental system in \( \tilde{W}_J'' \), so after modifying \( x \) by an element of \( \tilde{W}_J'' \) (which commutes with \( w' = w"' \)) we may assume that it maps the simple roots of \( \tilde{W}_J \) to the simple roots of \( \tilde{W}_J'' \). But then \( x \) is in particular a “\( \tilde{W} \)-equivalence”, and those are described by Richardson’s algorithm [Ric82].

\[\]

4 Powers of reduced braids

Finally, in this section we prove Proposition A, Proposition D and the main Theorem. Throughout we will employ the language of Garside categories wherever possible, but I have tried to set up the notation for readers who are only familiar with braid monoids:

**Definition 4.1.** Let \( \tilde{B}^+ \) be a right-cancellative category with Garside family \( \tilde{W} \) [DDG+15, Definition III.1.31]. Given a monoid \( \Omega \) of endomorphisms of \( \tilde{B}^+ \) preserving \( \tilde{W} \), we may interpret the semidirect product \( B^+ := \Omega \ltimes \tilde{B}^+ \) as a category: the objects are those of \( \tilde{B}^+ \), whereas the source of an arbitrary morphism \( b = \delta \tilde{b} := (\delta, \tilde{b}) \) is given by that of \( \tilde{b} \), and its target is the image of the target of \( b \) under \( \delta \); we will call this a twisted locally Garside category. We will say that a morphism \( b \) of \( B^+ \) is reduced if it lies in \( \Omega \ltimes \tilde{W} \).

When \( B^+ \) is small and right-Noetherian, then so is \( B^+ \); then there exists a right-length (or right-height) function \( \ell := \ell_{B^+} \) on the morphisms of \( B^+ \) [DDG+15, Proposition II.2.47], which extends the right-length function of \( \tilde{B}^+ \) via \( \ell_{B^+}(b) = \ell_{\tilde{B}^+}(b) = \ell_{B^+}(b) \).

**Remark 4.2.** One can reinterpret \( B^+ \) as a locally Garside category with Garside family \( \Omega \ltimes \tilde{W} \) (or just \( \tilde{W} \) if the endomorphisms in \( \Omega \) are all invertible); since the semidirect product \( \Omega \ltimes \tilde{W} \) does not strictly speaking yield a Coxeter group when \( \tilde{W} \) is itself a Coxeter group, we will not do so.

As the relations of Coxeter groups are homogeneous, twisted braid monoids are examples of small, Noetherian twisted locally Garside categories with one object and with finite lengths [DDG+15, Proposition II.2.32]; many of the statements that we need for their Deligne-Garside normal forms generalise immediately:

4.1 Stability of the Deligne-Garside normal form

Roughly speaking, the right Deligne-Garside normal form of a morphism in such categories can be obtained by starting with any decomposition into reduced morphisms, and subsequently trying to make the rightmost factors as large as possible:

**Definition/Theorem 4.3.** Let \( B^+ := \Omega \ltimes \tilde{B}^+ \) be a twisted locally Garside category. Then every morphism \( b \) in \( B^+ \) admits an (essentially unique [DDG+15, Proposition III.1.25]) (right) Deligne-Garside (or right greedy) normal form

\[ b = \delta b_m \cdots b_1, \]

where \( \delta \in \Omega \), and \( b_m, \ldots, b_1 \in \tilde{B}^+ \) are reduced morphisms such that if some morphism \( b' \) right-divides \( b_{i+1} b_i \) for some \( 1 \leq i \leq m - 1 \), then it right-divides \( b_i \) [DDG+15, Proposition IV.1.20]. We denote the left-complement to \( DGN(b) \) in \( DGN(b) \) by

\[ DGN(b) := \delta DGN(b) \cdots DG_{i+1}(b) \in B^+. \]

In the language of twisted Coxeter groups, this means that we can uniquely decompose any element \( b \) of its twisted braid monoid as

\[ b = \delta b_w \cdots b_w, \]

32
where $\delta \in \Omega$ and $w_m, \ldots, w_1 \in \hat{W}$, such that if $\ell(w_i s_j) = \ell(w_{i+1}) - 1$ for some simple reflection $s_j$ and $1 \leq i \leq m - 1$ then also $\ell(s_j w_i) = \ell(w_i) - 1$. In this context, by $\text{DG}_i(b)$ we will either mean the element $w_i$ of the group $\hat{W}$ or the reduced braid $b_{w_i}$ in $\text{Br}^+_\hat{W}$.

**Notation 4.4.** We write $b \geq b'$ for $b, b' \in \text{Br}^+$ whenever we can decompose $b = b''b' := b' \circ b'$ for some $b'' \in \text{Br}^+$. If this morphism $b''$ is not invertible, then we may also write $b > b'$.

**Proposition 4.5.** Let $\text{Br}^+ := \Omega \ltimes \text{Br}^+$ be a twisted locally Garside category.

(i) If $b_k, \ldots, b_1$ are elements of $\text{Br}^+$ such that each consecutive pair $b_{i+1}b_i$ is in Deligne-Garside normal form, then so is their product $b_k \cdots b_1$.

(ii) Assume that $\text{Br}^+$ is small and right-Noetherian and let $b, b'$ be two morphisms such that $b \geq b'$. Then $\ell(b) \geq \ell(b')$, with inequality if and only if $b > b'$.

(iii) If $\hat{W}$ is a Coxeter group and $v, w \in \hat{W}$ are elements such that $w \geq v$, then the product $b_v b_{w-1}$ is in Deligne-Garside normal form.

For convenience, the head function $\text{DG}(\cdot) := \text{DG}_1(\cdot)$ is always assumed to be sharp [DDG+15, Definition IV.1.42]; this means that

**Lemma 4.6 ([DDG+15, Proposition IV.1.50]).** For any pair of composable morphisms $b, b'$ in a twisted locally Garside category, we have $\text{DG}(bb') = \text{DG}((\text{DG}(b)b')$.

It can also be used to prove certain stability properties:

**Corollary 4.7.** Moreover, for any $i \geq 1$ we have

(i) $\text{DG}_{i+1}(bb') = \text{DG}_{i+1}(\text{DG}_{i+1}(b)b')$, so in particular

(ii) if $b$ is an endomorphism then $\text{DG}_{i+2}(b^d) \geq \text{DG}_{i+2}(b^d)$ for any $d \geq 1$.

**Proof.** (i): We induct on $i$. If it’s true $< i$, then from the case $i - 1$ we deduce

$$\text{DG}_{i+1}(bb') \text{DG}_{i+1}(b) = bb'$$

which implies that

$$\text{DG}_{i+1}(bb') = \text{DG}_{i+1}(b) \text{DG}_{i+1}(b).$$

Similarly, the case $i - 1$ yields

$$\text{DG}_{i+1}(\text{DG}_{i+1}(b)b') \text{DG}_{i+1}(b) = \text{DG}_{i+1}(b)b'$$

and thus

$$\text{DG}_{i+1}(\text{DG}_{i+1}(b)b') = \text{DG}_{i+1}(b) \text{DG}_{i+1}(b).$$

33
As $DG_i(b) = DG(DG_{> i-1}(b))$, we combine these two identities with the lemma to obtain
\[
DG_i(bb') = DG(DG_{> i-1}(bb')) \\
= DG\left(DG_{> i-1}(b)DG_{> i-1}(DG_{i\geq}(b)b')\right) \\
= DG\left(DG(DG_{> i-1}(b))DG_{> i-1}(DG_{i\geq}(b)b')\right) \\
= DG\left(DG_i(b)DG_{> i-1}(DG_{i\geq}(b)b')\right) \\
= DG\left(DG_{> i-1}(DG_{i\geq}(b)b')\right) = DG_i(DG_{i\geq}(b)b').
\]

We now combine this identity with the induction hypothesis to find
\[
DG_{i\geq}(bb') = DG_i(bb')DG_{i\geq}(bb') \\
= DG_i(bb')DG_{i\geq}(DG_{i\geq}(b)b') \\
= DG_i(DG_{i\geq}(b)b')DG_{i\geq}(DG_{i\geq}(b)b') = DG_{i\geq}(DG_{i\geq}(b)b'). \quad \qed
\]

(ii): From (i) we obtain
\[
DG_{i\geq}(b^{d-1})b \geq DG_{i\geq}(DG_{i\geq}(b^{d-1})b) = DG_{i\geq}(b^d).
\]

**Corollary 4.8.** Let $b$ be an endomorphism of a twisted locally Garside category and let $i, d \geq 1$ be integers such that
\[
DG_{i\geq}(b^{d+1}) = DG_{i\geq}(b^d),
\]
then for any integer $d' \geq d$ we have
\[
DG_{i\geq}(b^{d'}) = DG_{i\geq}(b^{d}).
\]

**Proof.** By induction on $d' > d + 1$. If the claim holds for all integers between $d$ and $d'$, then (i) yields
\[
DG_{i\geq}(b^{d'}) = DG_{i\geq}(DG_{i\geq}(b^{d-1})b) = DG_{i\geq}(DG_{i\geq}(b^{d-2})b) = DG_{i\geq}(b^{d-1}) = DG_{i\geq}(b^d). \quad \qed
\]

**Example 4.9.** Consider $w = s_2 s_1 s_4 s_3 s_2 s_1$ in type $A_4$. Then
\[
\ell(DG(b_w)) = \ell(w) = 6, \quad \ell(DG(s^2_w)) = \ell(s_1 w) = 7, \quad \ell(DG(s^3_w)) = \ell(w_o) = 10.
\]

**Example 4.10.** Consider $w = w_o s_1$ in type $A_3$. By induction on $i \geq 0$ one finds
\[
\begin{align*}
DGN(b_w^{4i}) &= b_{2312}^{2i} b_{w_o}^{2i}, \\
DGN(b_w^{4i+1}) &= b_{2312}^{2i} b_w b_{w_o}, \\
DGN(b_w^{4i+2}) &= b_{2312}^{2i+1} b_{w_o}^{2i+1}, \\
DGN(b_w^{4i+3}) &= b_{2312}^{2i+1} b_{w-o} b_{w-o} b_{w-o}^{2i+1}.
\end{align*}
\]

**Corollary 4.11.** Let $Br^+ := \Omega \times Br^+$ be a twisted locally Garside category and let $b = \delta b$ be an endomorphism in $Br^+$ such that $DG(b^2) = b$; in other words, $b$ is reduced and $\delta^{-1}(b)b$ is in Deligne-Garside normal form.

(i) For any integer $d \geq 0$ we have
\[
DGN(b^d) = \delta^{-d} \delta^{-d} \ldots \delta^{-1}(b) \delta^{-1}(b) \delta^{-1}(b).
\]

(ii) and for any other composable element $b'$ in $B^+$ we have
\[
b' b' \geq DGN_{i\geq}(b^d b').
\]
Definition 4.14. Let \( b \) be an element of its braid monoid \( B^r \). For any sequence of roots \( \mathfrak{A} = (\ldots, \beta_2, \beta_1) \) we obtain new one by setting
\[
b(\mathfrak{A}) := w_b(\mathfrak{A}) := (\ldots, w_b(\beta_2), w_b(\beta_1)), \quad b^{-1}(\mathfrak{A}) := w_b^{-1}(\mathfrak{A}),
\]
and similarly for a set \( \mathfrak{B} \). In particular, for each element \( w \) in \( W \) we have \( b_w(\mathfrak{B}) = w(\mathfrak{B}) \) and \( b_w^{-1}(\mathfrak{B}) = w^{-1}(\mathfrak{B}) \).

We define a simple decomposition of \( b \) to be a decomposition of the form
\[
b = \delta b_i_1 \cdots b_i_n,
\]
and given one we consider the (right) (root) inversion sequence
\[
\mathfrak{R}_b := (s_{i_1} \cdots s_{i_{n-1}} (\alpha_{i_1}), \ldots, s_{i_1} (\alpha_{i_2}), \alpha_{i_1}), \quad \text{for } n \geq 2,
\]
and call the underlying set \( \mathfrak{R}_b \) its (right) (root) inversion set.\(^{11}\)

Unlike \( \mathfrak{R}_b \), the roots in \( \mathfrak{R}_{b_w} \) are distinct and positive:

Proposition 4.15. Let \( W \) be a twisted Coxeter group and pick an element \( w \) in \( W \).

(i) There is an identity of sets of roots
\[
\mathfrak{R}_{b_w} = \mathfrak{R}_w := w^{-1}(-\mathfrak{R}_+) \cap \mathfrak{R}_+.
\]

(ii) For a pair of elements \( x, y \) in \( W \) we have
\[
\mathfrak{R}_{xy} = \{ \beta \in y^{-1}(\mathfrak{R}_x) \cup \mathfrak{R}_y : -\beta \notin y^{-1}(\mathfrak{R}_x) \cup \mathfrak{R}_y \}.
\]

In particular,
\[
\ell(xy) = \ell(x) - \ell(y) \quad \text{if and only if} \quad \mathfrak{R}_y \subseteq \mathfrak{R}_{x-1},
\]
\[
\ell(xy) = \ell(y) - \ell(x) \quad \text{if and only if} \quad \mathfrak{R}_x \subseteq \mathfrak{R}_{y-1},
\]
\[
\ell(xy) = \ell(x) + \ell(y) \quad \text{if and only if} \quad \mathfrak{R}_y \subseteq \mathfrak{R}_{xy}.
\]

\(^{11}\)Independently, a similar construction appears in [STW15, §2.9.3].
(iii) We have \( w(\mathcal{R}_w) = -\mathcal{R}_w \) if and only if \( w^2 \in \Omega \).

**Proof.** (i): Since twists do not affect these sets (after moving them to the left), this follows from [Bou68, Corollaire VI.17.2].

(iii): The implication \( \Rightarrow \) follows from (4.2), as that formula yields \( \mathcal{R}_{w^2} = \emptyset \) but then \( w^2 \in \Omega \). For \( \Leftarrow \) since the cardinalities of these sets agree, it suffices to show \( \subseteq \): if \( \beta \) lies in \( \mathcal{R}_w \), then \(-w(\beta)\) is positive and \(-ww(\beta) = -\delta(\beta)\) is negative, so \(-w(\beta)\) lies in \( \mathcal{R}_w \).

**Definition 4.16.** Let \( \mathcal{R} \) be a root system and pick a subset of roots \( \mathcal{R} = \{\beta_1, \ldots, \beta_1\} \). We will say that it is **convex** if for any pair of roots \( \beta_i, \beta_j \) in \( \mathcal{R} \) such that \( c_i\beta_i + c_j\beta_j = \beta_k \) for some \( c_i, c_j \in \mathbb{R} \) is also a root, then \( \beta_k \) lies in \( \mathcal{R} \).

We will say that an ordering \( \beta_i > \cdots > \beta_1 \) of the roots in \( \mathcal{R} \) is **convex** if for such a triple of positive roots \( \beta_i, \beta_j, \beta_k \) with \( i > j \) we have \( \beta_i > \beta_k > \beta_j \).

**Proposition 4.17.** (i) Let \( w \) be an element of a twisted Coxeter group \( W \). Then there is a map

\[
\{\text{reduced decompositions of } w\} \rightarrow \{\text{convex orderings on its inversion set } \mathcal{R}_w\}
\]

sending

\[
\delta s_{i_1} \cdots s_{i_k} \mapsto (s_{i_1} \cdots s_{i_{m-1}}(\alpha_{i_m})), \ldots, s_{i_1}(\alpha_{i_2}), \alpha_{i_1})
\]

When \( W \) is finite and \( w \) has maximal length, this is a bijection.

(ii) Let \( \mathcal{R} \) be a rank 2 root system. Then there are only two convex orderings on the set of positive roots, and they differ from each other by inverting; these correspond to the two reduced decompositions of the corresponding longest element.

**Proof.** (i): This is essentially due (independently) to Zhelobenko [Zhe87, §3], Dyer [Dye93, Proposition 2.13] and Papi [Pap94, p. 663].

(ii): This was observed in [Zhe87, §4] and [Pap94, p. 664] in case-by-case calculations for rank 2 Weyl groups. For a case-free proof: any convex ordering needs to start and end with a simple root, since all of the other positive roots lie inside of their convex cone. There are only two such roots in the rank 2 case, and by convexity each choice yields a unique ordering, which are the same after inverting one of them.

**Lemma 4.18.** Let \( W \) be a twisted Coxeter group and let \( b \) be an element of its braid monoid \( \mathcal{B} \).

(i) Let \( b' \) be another element of the braid monoid \( \mathcal{B} \). By choosing simple decompositions of \( b' \) and \( b \), their product yields one for \( b'b \) (after moving twists) and thus a concatenation of sequences

\[
\mathcal{R}_{w'b} = (b^{-1}(\mathcal{R}_{w'}), \mathcal{R}_b)
\]

(ii) Given a simple decomposition of \( b \) with corresponding root ordering \( \mathcal{R}_b \), applying a braid move to the decomposition corresponds to inverting the corresponding section of roots. Thus whilst the ordering of the sequence \( \mathcal{R}_b \) depends on the choice of simple decomposition of the braid \( b \), the underlying set \( \mathcal{R}_b \) does not.

In particular, for any other pair of elements \( b', b'' \) in \( \mathcal{B} \) we have an inclusion of sets

\[
\mathcal{R}_{b'} \subseteq b(\mathcal{R}_{b''}b)
\]

(iii) Furthermore, we have \( b = b_w \) for some \( w \) in \( W \) if and only if the set \( \mathcal{R}_b \) consists entirely of positive roots.
(iv) Suppose we can decompose \( b = b_x b_y \) for some \( x \) and \( y \) in \( W \). Then for any choice of decomposition of \( b \), all roots in the sequence \( \mathfrak{R}_i \) are distinct.

In particular, if we can decompose this same braid as \( b = b' b_x \) for some \( x \) in \( W \) and \( b' \) in \( \mathfrak{R}_+ \) then

\[
\mathfrak{R}_y = x(\mathfrak{R}_y \setminus \mathfrak{R}_x).
\]

**Proof.** (i): This follows from the explicit construction (4.1).

(ii): Let \( b''', b', b \) be sequences of simple elements of \( \mathfrak{R}_+ \), where \( b' = (\cdots, b_j, b_i) \) is one “side” of a braid relation, so by part (i) there is the concatenation

\[
\mathfrak{R}_{b'''} = ((b'b)^{-1}(\mathfrak{R}_y), b^{-1}(\mathfrak{R}_y), \mathfrak{R}_y).
\]

The product of the elements in \( b' \) equals the lift of the Coxeter group element \( w = \cdots s_j s_i \), which is the longest element of a rank 2 root system. Proposition 4.15(i) and part (ii) of the previous proposition then imply that \( \mathfrak{R}_{b'''} \) obtained from the other side of the braid relation is the inversion of \( \mathfrak{R}_{b'} \), and then the same holds for \( b^{-1}(\mathfrak{R}_{b'''} b_i), b^{-1}(\mathfrak{R}_{b'''} b_i) \).

(iii): The implication \( \Rightarrow \) follows from Proposition 4.15(i). For \( \Leftarrow \), choose a simple decomposition \( b = \delta b_i \cdots b_{i_1} \). Inducting on \( l \) and using \( \mathfrak{R}_b \subseteq \mathfrak{R}_+ \), one can deduce from say (4.5) that the expression

\[
w := \delta s_{i_1} \cdots s_{i_l}
\]

is reduced, so that

\[
b_w = \delta b_i \cdots b_{i_1} = b.
\]

(iv): Choosing decompositions for \( x \) and \( y \) yields the root sequence

\[
\mathfrak{R}_b = (x^{-1}(\mathfrak{R}_x), \mathfrak{R}_y),
\]

but if a root \( \beta \) lies in the intersection \( x^{-1}(\mathfrak{R}_x) \cap \mathfrak{R}_y \) then \( y(\beta) \) lies in \( \mathfrak{R}_x \cap \mathfrak{R}_+ = \emptyset \).

**Lemma 4.19.** Let \( \mathfrak{R}_+ \) be a small, right-Noetherian locally Garside category and pick an endomorphism \( b \) in \( \mathfrak{R}_+ \).

(i) Suppose that for some \( i \geq 1 \) and each \( 1 \leq j \leq i \), certain integers \( d_j \) are given the property that \( \ell(DG_j(b^d)) \leq d_j \) (e.g. \( d_j \) is the length of a Garside map), and set

\[
d := i - \ell(DG_{i \geq i}(b^d)) + \sum_{j=1}^i d_j
\]

then \( DG_{i \geq i}(b^d) \) has “stabilised”, i.e. for any \( d' \geq d \) we have

\[
DG_{i \geq i}(b^d') = DG_{i \geq i}(b^d).
\]

(ii) If \( \mathfrak{R}_+ \) is the braid monoid of a twisted Coxeter group, then there is an inclusion

\[
\mathfrak{R}_{DG_i(b)} \subseteq DG_{i-1}(b) \cdots DG_1(b)(\mathfrak{R}_b) \cap \mathfrak{R}_+.
\]

**Proof.** (i): By the pigeonhole principle, for some \( 0 \leq k \leq d \) we must have

\[
\ell(DG_{i \geq i}(b^{i+k+1})) = \ell(DG_{i \geq i}(b^{i+k})),
\]

and then the claim follows from Proposition 4.5(ii) and Corollary 4.8.

(ii): Follows by induction from Lemma 4.18(iii).
Proof of Proposition D. Pick a reduced decomposition for \( w \). By induction on \( d \geq 1 \), Lemma 4.18(iv) furnishes an identity of concatenated sequences

\[
\mathcal{R}_{\beta w} = (w^{-1}(\mathcal{R}_{\beta w}^{d-1}), \mathcal{R}_w) = (w^{1-d}(\mathcal{R}_w), \ldots, w^{-2}(\mathcal{R}_w), w^{-1}(\mathcal{R}_w), \mathcal{R}_w).
\]

As we have an inclusion of sets

\[
w^{-k}(\mathcal{R}_w) \subseteq w^{-k}(\mathcal{R}\setminus\mathcal{R}_{st}^w) = \mathcal{R}\setminus\mathcal{R}_{st}^w
\]

for any integer \( k \in \mathbb{Z} \), we now also have an inclusion \( \mathcal{R}_{\beta w} \subseteq \mathcal{R}\setminus\mathcal{R}_{st}^w \).

The final claim in (i) then also follows from Lemma 4.18(iv), as linearity yields an equality of cardinalities

\[
|x(\mathcal{R}\setminus\mathcal{R}_{st}^w) \cap \mathcal{R}_+| = |\mathcal{R}_+\setminus\mathcal{R}_{st}^w|
\]

for any element \( x \) in \( W \), and the rest of this proposition follows from the previous lemma as

\[
\ell(DG_{i\geq}(\beta_w)) = \ell(b_w^i) = i \ell(w).
\]

Lemma 4.20. Let \( \beta \) be a positive root in a (not necessarily spherical or finite) root system. Then there exists a sequence \((c_1\alpha_{i_1}, c_2\alpha_{i_2}, \ldots, c_m\alpha_{i_m})\) with each scalar \( c_i \in \mathbb{R}_{>0} \) so that the partial sums

\[
c_i \alpha_{i_1}, \quad c_i \alpha_{i_1} + c_i \alpha_{i_2}, \quad \ldots, \quad \sum_{j=1}^{m} c_j \alpha_{i_j} = \beta
\]

(4.6)

are all positive roots.

Proof. We may assume that \( \beta \) is not simple and follow the proof of [Bou68, Proposition VI.1.19]: If \((\beta, \alpha_i) \leq 0 \) for each \( \alpha_i \), then by linearity also \((\beta, \beta) \leq 0 \) which is a contradiction. Thus we have \((\beta, \alpha_i) > 0 \) for some \( i \), which implies that \( \beta - c_\alpha_i \) is a root for some \( c_\alpha_i \in \mathbb{R}_{>0} \). As \( \beta \) is not simple this root still has some positive coefficients when decomposed into simple roots, and hence it must be positive.

Papi showed that a subset of positive roots \( \mathcal{R} \subseteq \mathcal{R}_+ \) is of the form \( \mathcal{R}_w \) if and only if both \( \mathcal{R} \) and \( \mathcal{R}_+ \setminus \mathcal{R} \) are convex [Pap94].

Proposition 4.21. Let \( \mathcal{R}_+ \) be a system of positive roots of a root system \( \mathcal{R} \). A subset of roots \( \mathcal{L} \subseteq \mathcal{R} \) is a standard parabolic subsystem if and only if it is invariant under multiplication by \(-1\) and both \( \mathcal{L} \) and the positive complement \( \mathcal{R}_+ \setminus \mathcal{L} \) are convex.

Hence, for any automorphism \( w \) of \( \mathcal{R} \) we have

\[
\text{\( w \) is convex} \quad \text{iff} \quad \mathcal{R}_+\setminus\mathcal{R}_{st}^w \text{ is convex} \quad \text{iff} \quad \mathcal{R}_+\setminus\mathcal{R}_{st}^v = \mathcal{R}_v \quad \text{for some} \quad v \text{ in } W,
\]

where \( W \) denotes the associated Coxeter group. In particular, the equation

\[
\mathcal{R}_{DG(b_w')} = \mathcal{R}_+\setminus\mathcal{R}_{st}^w \quad (\text{resp.} \quad \mathcal{R}_{DG(b_w')} = \mathcal{R}_+\setminus\mathcal{R}_{st}^w)
\]

generalising the braid equation \((\ast)\) to arbitrary elements, always fail when \( w \) is not convex (resp. firmly convex).

Proof. Suppose that \( \mathcal{L} \) is not a parabolic subsystem, then as it is convex under multiplication by \(-1\) it follows that for some positive root \( \beta \) in \( \mathcal{L} \) there exists a simple root \( \alpha_n \) in its support which is not in \( \mathcal{L} \). The previous lemma yields a sequence \((c_1\alpha_{i_1}, c_2\alpha_{i_2}, \ldots, c_m\alpha_{i_m})\) so that the partial sums (4.6) are all roots. Consider those \( k \) such that at least one of \( \sum_{j=1}^{k-1} c_j \alpha_{i_j} \) or \( \alpha_{i_k} \) is not in \( \mathcal{L} \); as \( \alpha_{i_k} \) occurs in the sequence, this set is nonempty. As the final root \( \beta \) itself lies in \( \mathcal{L} \), downward induction and finiteness yields that for at least one such pair of roots \((\beta_{i-1}, \beta_i)\) their sum \( \beta_{i-1} + c_i \beta_i \) does lie in \( \mathcal{L} \). But as \( \mathcal{R}_+\setminus\mathcal{L} \) is convex it is not possible that both roots of this pair lie in \( \mathcal{R}_+\setminus\mathcal{L} \), so at least one of them lies in \( \mathcal{L} \), say \( \beta_1 \). But as \( \mathcal{L} \) is convex and convex under multiplication by \(-1\), the equation

\[
c_i \beta_i = (\beta_{i-1} + c_i \beta_i) + (-\beta_{i-1})
\]

implies that \( \beta_i \) also lies in \( \mathcal{L} \), yielding a contradiction.

For the final claim, note that if \( \mathcal{R}_w \neq \mathcal{R}_{st}^w \), then the previous proposition implies that the second equation fails. 

\[\square\]
The first sentence of part (i) of Proposition A follows from

**Lemma 4.22.** Let $x$ and $w$ be elements of a twisted Coxeter group and consider the conjugate $w' := xwx^{-1}$. If this conjugation

(i) is a cyclic shift then $x(\mathcal{R}_{st}^w \cap \mathcal{R}_+) = \mathcal{R}_{st}^{w'} \cap \mathcal{R}_+$, and if

(ii) it is a strong conjugation then $\mathcal{R}_{st}^w \cap \mathcal{R}_x$ is stable under the action of $w$.

Hence in either case we have $x(\mathcal{R}_{st}^w) = \mathcal{R}_{st}^{w'}$.

**Proof.** Let $\beta$ be a root in $\mathcal{R}_{st}^w \cap \mathcal{R}_+$ and assume the intersection of its $w$-orbit with $\mathcal{R}_x$ is empty (resp. is the entire orbit). Then

$$(w')^i(x(\beta)) = xw^i(\beta)$$

is a positive (resp. negative) root for all integers $i$ in $\mathbb{Z}$, so $\beta$ lies in $\mathcal{R}_{st}^{w'} \cap \mathcal{R}_+$ (resp. $\mathcal{R}_{st}^{w'} \cap \mathcal{R}_-$).

(i): The cyclic shift implies reduced decompositions $w = yx$ and $w' = xy$, where we set $y := wx^{-1}$, or similarly $w = x^{-1}y$ and $w' = yx^{-1}$ with $y = xw$. By taking inverses and using $\mathcal{R}_{st}^{w^{-1}} = \mathcal{R}_{st}^{w}$, the second case reduces to the first case. Hence it suffices to show the inclusion $x(\mathcal{R}_{st}^w \cap \mathcal{R}_+) \subseteq \mathcal{R}_{st}^{w'} \cap \mathcal{R}_+$. The reduced decomposition yields

$$\mathcal{R}_x \subseteq \mathcal{R}_w \subseteq \mathcal{R}_+ \setminus \mathcal{R}_{st}^{w'},$$

so the previous paragraph implies the claim.

(ii): By assumption either $wx$ or $wx^{-1}$ is reduced. In the former case, we have $\ell(wx^{-1}) = \ell(wx) - \ell(x^{-1})$ which by (4.3) implies that $\mathcal{R}_x \subseteq \mathcal{R}_{st}^w$. In the latter case, $\ell(xw^{-1}) = \ell(x) - \ell(wx^{-1})$ which by (4.4) implies $\mathcal{R}_x \subseteq \mathcal{R}_{st}^{w^{-1}}$. Now suppose that $\beta$ lies in $\mathcal{R}_{st}^w \cap \mathcal{R}_x = \mathcal{R}_{st}^{-w} \cap \mathcal{R}_x$, so $w(\beta)$ and $w^{-1}(\beta)$ are still positive. In the first case, we find $xw(\beta) < 0$ so $w(\beta) \in \mathcal{R}_x$ and in the second case we find $xw^{-1}(\beta) < 0$ so $w^{-1}(\beta) \in \mathcal{R}_x$.

Since $w$ has finite order either yields an inclusion $w(\mathcal{R}_{st}^w \cap \mathcal{R}_x) \subseteq \mathcal{R}_{st}^{w'} \cap \mathcal{R}_x$ which must be an equality.

Hence in either case, the conditions of the first paragraph are met, and the final claim follows. 

For the comment following Proposition A(i), recall that the number of fixed roots in a conjugacy class is constant.

**Corollary 4.23.** Let $w$ and $w'$ be two elements of a twisted finite Coxeter group such that $w \sim w'$. Then

$$|\mathcal{R}_w \cap V_w| = |\mathcal{R}_{w'} \cap V_{w'}|.$$ 

**Proof.** By assumption there exists an element $\tau$ in the Coxeter group such that $\tau w \tau^{-1} = w'$, which yields $\tau(V_w) = V_{w'}$. By the lemma we have $\tau(\mathcal{R}_{st}^w) = \mathcal{R}_{st}^{w'}$, so this means that $\tau$ bijectively maps the nonstable orbits of $w$ in $V_w$ to nonstable orbits of $w'$ in $V_{w'}$. As each non-stable orbit contributes 1 to the cardinality $|\mathcal{R}_{st}^w|$, the claim follows.

For the proof of the main Theorem we will use

**Lemma 4.24.** Let $w$ be an element of a twisted finite Coxeter group $W$ and let $w'$ be a cyclic shift, so there are reduced decompositions $w = yx$ and $w' = xy$ for some elements $x$ and $y$ in $W$. Assume that $w$ and $w'$ are convex, and set $x' := \text{pb}(w')^{-1}x\text{pb}(w)$ for some $x$. Then there are braid identities

(i) $b_{\text{pb}(w')}x' = x'b_{\text{pb}(w)}$ and

(ii) $b_{\text{pb}(w')^{-1}}x' = x'b_{\text{pb}(w)^{-1}}$.

**Proof.** (i): From Lemma 4.22(i) it follows that

$$\text{pb}(w)x^{-1}(\mathcal{R}_{st}^w \cap \mathcal{R}_+) = \text{pb}(w)(\mathcal{R}_{st}^w \cap \mathcal{R}_+) \subseteq \mathcal{R}_+.$$
This implies that $R_{\text{pb}(w)x^{-1}} \subseteq R_+ \setminus R_{x^{-1}}$, which means that $\text{pb}(w') \geq \text{pb}(w)x^{-1}$ in the weak left Bruhat-Chevalley order. As $\text{pb}(w) \geq w \geq x$, that is equivalent to $b_{\text{pb}(w')b_x} \geq b_{\text{pb}(w)}$.

(ii): Note that

$$R_{\text{pb}(w)x^{-1}} = R_- \setminus w_x(R_{x^{-1}} \cap -R_+)$$

and similarly for $\text{pb}(w')^{-1}$. Using Lemma 4.22(i) we deduce that

$$x'(w_x(R_{x^{-1}} \cap -R_+)) = w_xw_x'x(R_{x^{-1}} \cap R_+) = w_xw_x'(R_{x^{-1}} \cap R_+) = w_x(R_{x^{-1}} \cap -R_+) \subseteq R_+,$

which combines with the previous identity to $\text{pb}(w)^{-1} \geq x'$. Similarly,

$$\text{pb}(w)^{-1}(x')^{-1}(w_x(R_{x^{-1}} \cap -R_+)) = \text{pb}(w)^{-1}w_x(R_{x^{-1}} \cap -R_+) = R_{x^{-1}} \cap R_+ \subseteq R_+$$

yields $\text{pb}(w')^{-1} \geq (\text{pb}(w)^{-1}(x')^{-1})^{-1}$, and thus $b_{\text{pb}(w)^{-1}}b_{x'} = b_xb_{\text{pb}(w)^{-1}}$.

4.3 Mixed shifts and braid conjugation

In this subsection we prove part (ii) of Proposition A; it won’t be used elsewhere in this paper.

Definition 4.25. Recall the notions of Definition 4.1. When we omit the adjective locally, the category is also left-cancellative, and the Garside family is bounded by a Garside map $b_{w_x}$ [DDG+15, Proposition V.1.20]. For every reduced morphism $b_{w_x}$ there exists a reduced morphism $\partial(b_{w_x})$ such that $b_{w_x} = \partial(b_{w_x})b_{w_x}$ [DDG+15, Proposition IV.1.23], which is unique by cancellativity. The square of $\partial(\cdot)$ extends to a unique endofunctor $\partial^2(\cdot)$ of $\mathcal{B}_r$, satisfying $b_{w_x}b = \partial^2(b)b_{w_x}$ for any morphism $b$ [DDG+15, Corollary V.1.34]; we will assume that it is invertible, and then so is $\partial(\cdot)$ [DDG+15, Proposition V.2.17]. One then inductively defines for $m \geq 0$ the iterates $b_{w_x}^m : \text{ob}(\mathcal{B}_r) \to \mathcal{B}_r$ by

$$b_{w_x}^0(\cdot) := \text{id}(\cdot), \quad b_{w_x}^m(\cdot) := \partial^2(b_{w_x}^{m-1}(\cdot))b_{w_x}(\cdot), \quad b_{w_x}^{-m}(\cdot) := b_{w_x}^{m}(\partial^{-2m}(\cdot))^{-1},$$

so that $b_{w_x}^mbb_{w_x}^{-m} = \partial^{2m}(b)$ for any $m \in \mathbb{Z}$ [DDG+15, Lemma V.1.41].

The enveloping groupoid of a twisted locally Garside category $\mathcal{B}_r^+$ is isomorphic to the semidirect product $\mathcal{B} := \Omega \rtimes \mathcal{B}_r$, where $\mathcal{B}_r$ is the enveloping groupoid of $\mathcal{B}^+$; we call this the twisted locally Garside groupoid corresponding to $\mathcal{B}_r^+$. A Garside category is Ore [DDG+15, Proposition V.2.32], from which it follows that $\mathcal{B}_r^+$ embeds into $\mathcal{B}_r$. This is a groupoid of left and of right fractions [DDG+15, Proposition II.3.11] and it admits right-lcms [DDG+15, Proposition V.2.35], which implies that we can remove common factors in a left fraction. Then decomposing their complements to this lcm into normal forms, one obtains the symmetric normal form [DDG+15, Corollary III.2.21]

$$b = \delta b'_1 \cdots b'_i(b''_i)^{-1} \cdots (b''_j)^{-1},$$

which is essentially unique [DDG+15, Proposition III.2.16]. Rewriting each of the negative terms as a positive term times a power of $b_{w_x}$ and then moving all of these powers to the right yields

$$b = b'(b'')^{-1} = \delta b'_1 \cdots b'_i \partial^{-1}(b''_i) \cdots \partial^{1-2j}(b''_j)b_{w_x}^j,$$

which after removing the final factor is the usual normal form of $bb_{w_x}^j \in \mathcal{B}_r^+$ [DDG+15, Proposition V.3.35].

Lemma 4.26. Let $\mathcal{B}_r^+$ be a twisted locally Garside category, and assume that it embeds into its groupoid $\mathcal{B}_r$.

(i) If $b, b', b''$ are composable morphisms of $\mathcal{B}_r^+$ such that the morphism $bb'(b'')^{-1}$ of $\mathcal{B}_r$ still lies in $\mathcal{B}_r^+$, then $DG(bb') \geq DG(b'')$. In particular, if $bb'$ is in right Deligne-Garside normal form and $b''$ is reduced, then $b' \geq b''$.  


(ii) Now suppose that \( b_w \) is a reduced endomorphism of \( \text{Br}^+ \), and \( b \) is any composable morphism of \( \text{Br}^+ \) such that the conjugate \( bb_w b^{-1} \) also lies in \( \text{Br}^+ \), and consider the reduced morphism \( b_u := \text{DG}(bb_w b^{-1}) \). Then \( b_u b_w b^{-1} \) is also reduced.

**Proof.** (i): Setting \( b'' := bb'(b')^{-1} \in \text{Br}^+ \), we have an identity \( bb' = b''b'' \) inside \( \text{Br}^+ \), which implies

\[
\text{DG}(bb') = \text{DG}(b''b'') \geq \text{DG}(b'').
\]

Under the additional assumptions, we now deduce

\[
b' = \text{DG}(bb') \geq \text{DG}(b'') \geq b''.
\]

(ii): Since \( b = bb_w b^{-1} \geq \text{DG}(bb_w) b^{-1} = b_u \), we may write \( b = b'b_u \) for some morphism \( b' \) in \( \text{Br}^+ \). Then from the assumption it follows that

\[
bb_w b^{-1} = bb_w b^{-1} b'
\]

also lies in \( \text{Br}^+ \), so by (i) we have \( b_u b_w = \text{DG}(bb_w) b^{-1} \geq b_u \). Thus we may split \( b_u b_w = b'' b_u \) with \( b'' \) in \( \text{Br}^+ \), and since \( b_u b_w \) is reduced so is \( b'' = b_u b_w b^{-1} \).

**Example 4.27.** Consider a simple reflection \( v := s_1 \) and the longest element \( w := w_0 \) in type \( A_2 \). Then \( b_1 b_2 b_1 b_2 = b_1 b_1 b_2 \) lies in \( \text{Br}^+ \), but it is not reduced.

The following “convexity” property goes back to Garside’s solution of the conjugacy problem via summit sets [Gar69, Lemma 12] and their refinement by super summit sets [EM94, Corollary 4.2]; a similar statement for Garside groups appears in [KL10, Lemma 3.2]:

**Lemma 4.28.** Let \( \text{Br}^+ \) be a twisted Garside category. Suppose \( b', b'' \) are endomorphisms in its Garside groupoid \( \text{Br} \), such that \( b'' \geq b', b'' \geq b'' \) for some integers \( m \) and \( n \) and \( bb'' b^{-1} = b'' \) for some morphism \( b \) in \( \text{Br} \). Let \( \text{DG}(b) \) be the right-most term appearing in its symmetric normal form (4.7). Then also

\[
b'' \geq \text{DG}(b') \text{DG}(b) = b''.
\]

**Proof.** Considering the negative factors in (4.7), set \( \tilde{b} := bb_w b^{-1} \in \text{Br}^+ \) and \( \tilde{b}' := b^{-1} b'' b' b_w \). Then \( \tilde{b} \in \text{Br}^+ \) and \( \tilde{b}'(\tilde{b})^{-1} = b'' \), and from \( \partial^2(b_w) = \partial(\text{id}) = b_w \), it follows that \( b'' \geq b' \geq b'' \) still holds. By assumption we may write \( b'' = \tilde{b}' b_w b'' \) with \( b'' \) in \( \text{Br}^+ \). Then

\[
\text{DG}(\tilde{b}) \tilde{b}' \geq \text{DG}(\tilde{b}) \text{DG}(\tilde{b}) = \text{DG}(\tilde{b}') \text{DG}(\tilde{b}) = \text{DG}(\tilde{b})
\]

which yields the second inequality, for \( \tilde{b} \) and \( \tilde{b}' \). Taking inverses, we have \( b'(\tilde{b})^{-1} b^{-1} = (b'')^{-1} \) with \( (b'')^{-1} \geq b'' \). The previous paragraph therefore furnishes that

\[
\text{DG}(\tilde{b}) (\tilde{b}^{-1}) \text{DG}(\tilde{b})^{-1} \geq b''
\]

so taking inverses again now yields the first inequality for \( \tilde{b} \) and \( \tilde{b}' \).

But from (4.8) we obtain \( b_w^{-1} \text{DG}(\tilde{b}) b_w^{ij} = \text{DG}(b) \), so

\[
b_w^{-1} (\text{DG}(\tilde{b}) b_w^{ij} \text{DG}(\tilde{b})^{-1}) b_w^{ij} = (b_w^{-1} \text{DG}(\tilde{b}) b_w^{ij}) (b_w^{-1} \text{DG}(\tilde{b})^{-1} b_w^{ij}) = \text{DG}(b) b' \text{DG}(b)^{-1}
\]

and then using \( \partial^2(b_w) = b_w \), again the claim for \( \text{DG}(b') b' \text{DG}(b) \) also follows. □
Corollary 4.29. If the endomorphisms \( b' = b_w \) and \( b'' = b b_w b^{-1} \) are both reduced, then so is
\[
DG(b) b_w DG(b)^{-1}.
\]

The following statement was originally employed in a different approach to proving part (ii) of Proposition A; it won’t be used but might be of interest in itself.

Proposition 4.30. Let \( b_v = \delta b_v \) and \( b_w \) be reduced endomorphisms of a twisted Garside category satisfying \( DG(b_v^2) = b_v \). If the element
\[
b_v^k b_w b_v^{-k}
\]
is reduced (and thus equal to \( b_{w^k w^{-k}} \) in the case of twisted Coxeter groups) for some integer \( k \geq 1 \), then it is also reduced for \( k = 1 \).

Proof. Follows immediately from the previous corollary and Corollary 4.11(i). \( \square \)

In the case where \( v \) is an involution of a twisted Coxeter group (which is locally Garside but not in general Garside), we can give an entirely different proof:

Proof 2. As \( v \) is an involution and \( b_v^k b_w = b_w b_v^k \) for some \( w' \) in \( W \), Corollary 4.11(ii) implies that
\[
b_v b_w \geq DG(b_v^k b_w) = DG(b_w b_v^k) \geq b_v,
\]
so \( b_v b_w = bb_v \) for some \( b \) in \( Br^+ \). According to Lemma 4.18(iv) we have
\[
\mathcal{R}_b = v\left( (w^{-1}(\mathcal{R}_v) \cup \mathcal{R}_w) \setminus \mathcal{R}_v \right),
\]
and according to Lemma 4.18(ii) it suffices to show that \( \mathcal{R}_b \) consists of positive roots.

We may assume that \( k > 1 \); again using that \( v \) is an involution, Proposition 4.15(ii) and Lemma 4.18(i) yield
\[
\mathcal{R}_{b_v^k} = \mathcal{R}_v \cup -\mathcal{R}_v
\]
and therefore an inclusion of sets of roots
\[
w^{-1}(\mathcal{R}_v) \cup \mathcal{R}_w \subseteq w^{-1}(\mathcal{R}_v) \cup w^{-1}(-\mathcal{R}_v) \cup \mathcal{R}_w = \mathcal{R}_{b_v^k b_w} = \mathcal{R}_{b_v} b_v^k = v^k(\mathcal{R}_{w'}) \cup \mathcal{R}_v \cup -\mathcal{R}_v,
\]
and hence
\[
\mathcal{R}_b \subseteq v^{k+1}(\mathcal{R}_{w'}) \cup -v(\mathcal{R}_v) = v^{k+1}(\mathcal{R}_{w'}) \cup \mathcal{R}_v.
\]
Thus if a root of \( \mathcal{R}_b \) is negative, it lies in \( v^{k+1}(\mathcal{R}_{w'}) \). If \( k \) is odd then \( v^{k+1}(\mathcal{R}_{w'}) = \mathcal{R}_{w'} \) consists of positive roots, and if it is even then it lies in \( v^{k+1}(\mathcal{R}_{w'}) = v(\mathcal{R}_{w'}) \). But then
\[
\emptyset \neq v(\mathcal{R}_b) \cap \mathcal{R}_{w'} \subseteq v(\mathcal{R}_b) \cap \mathcal{R}_v = \left( (w^{-1}(\mathcal{R}_v) \cup \mathcal{R}_w) \setminus \mathcal{R}_v \right) \cap \mathcal{R}_v = \emptyset,
\]
which is a contradiction. \( \square \)

The converse is false:

Example 4.31. Consider \( b = b_2 \) and \( b' = b_{12} \) in type \( A_2 \). Then \( bb'b^{-1} = b_{21} \), but \( b^2 b'b^{-2} = b_2 b_{21} b_2^{-1} \).

And it may fail when \( v \) is not an involution:

Example 4.32. Consider \( b = b_{231} \) and \( b' = b_{2321} \) in type \( A_3 \). Then \( bb'b^{-1} = b_{23} b_{13} \), but \( b^2 b'b^{-2} = b_{123} \).

For twisted finite Coxeter groups, the second proof is slightly weaker:

Example 4.33. Consider \( v = s_3 s_4 s_1 s_2 s_3 \) in type \( A_4 \). Then \( DG(b_v^2) = b_v \), but \( v \) is not an involution.

We now generalise Definition 1.15 to the Garside setting:
Notation 4.34. Mirroring $\geq$, we write $b' \leq b$ if there exists a morphism $b''$ such that $b' \circ b'' = b$.

Definition 4.35. Let $\Br^+ := \Omega \ltimes \tilde{\Br}^+$ be a cancellative twisted locally Garside category and pick two reduced endomorphisms $b_w$ and $b_{w'}$ in $\Br^+$.

(i) If there exists a sequence of reduced morphisms $b_{w'} = b_{w_{n+1}}, \ldots, b_{w_0} = b_w$ in $\Br^+$ and reduced morphisms $b_{r_1}, \ldots, b_{r_n}$ in $\tilde{\Br}^+$ such that

$$b_{r_i}^\perp b_{w_i} = b_{w_{i+1}} b_{r_i}^\perp$$

and $b_{r_i}^\perp b_{w_i}$ is reduced

or

$$b_{w_i} b_{r_i} = b_{r_i} b_{w_{i+1}}$$

and $b_{w_i} b_{r_i}$ is reduced

for each $0 \leq i \leq n$, then we denote these strong conjugations by $b_w \sim b_{w'}$.

(ii) If instead

$$b_{r_i} b_{w_i} = b_{w_{i+1}} b_{r_i}$$

and $b_{w_i} b_{r_i} \geq b_{r_i}$

or

$$b_{w_i} b_{r_i} = b_{r_i} b_{w_{i+1}}$$

and $b_{r_i} b_{w} \leq b_{w_i}$

for each $0 \leq i \leq n$, then we denote these cyclic shifts by $b_w \sim b_{w'}$.

(iii) If this sequence is a combination of these, then we denote these mixed shifts by $b_w \sim b_{w'}$.

(iv) Analogously, we define transporters $\text{Tran}^\times_{\Br^+}(b_w, b_{w'})$ for parabolic subcategories $\Br^{+'} \subseteq \tilde{\Br}^+$ [DDG+15, §VII.1.4] and $* \in \{+, -, \times\}$.

Part (ii) of Proposition A is then a special case of

Proposition 4.36. Let $\Br^+$ be a twisted Garside category and pick two reduced endomorphisms $b_w$ and $b_{w'}$. Then for any parabolic subcategory $\Br^{+'} \subseteq \tilde{\Br}^+$, the projection map

$$\text{Tran}^\times_{\Br^+}(b_w, b_{w'}) \longrightarrow \text{Tran}^\times_{\Br^{+'}}(b_w, b_{w'}) := \{b \in \Br^{+'} : bb_w = b_w b\}$$

surjects.

Proof. By the previous lemma and proposition, the claim follows inductively if we prove the statement when $b = b_w$ is a reduced morphism $b_w$ in $\Br^{+'}$. For this we induct on the length of $b_w$; now consider $b_w := \text{DG}(b_{w}, b_w) b_{w}^{-1}$. If $b_w$ is the identity then $\text{DG}(b_{w}, b_w) = b_w$, so Lemma 4.26(i) implies that $b_w$ is a cyclic shift. Otherwise, we may decompose $b_w = b_{w'} b_{u}$ for a reduced $b_{w'}$ lying in $\Br^{+'}$, as it is parabolic. Then by Lemma 4.26(ii) $b_w$ is a strong conjugation and we applying the induction hypothesis to the shorter morphism $b_{w'}$ to finish.

Employing the induction hypothesis in the final step was necessary:

Example 4.37. Consider $v := w := b_{12}$ in type $A_2$. Then $\text{DGN}(b_{v}, b_{w}) = b_1 b_{212}$ but

$$\text{DGN}(b_1 b_2 b_{w} b_{2}^{-1}) = \text{DGN}(b_1 b_{21}) = b_1 b_{21}.$$  

We won’t be using the following statement; in the sequel it is shown to preserve convexity [Malb, Corollary 2.19].

Proposition 4.38. Let $b$ be an endomorphism in a twisted locally Garside category. Consecutively conjugating $b$ by $\text{DG}(b^{d+1}) \text{DG}(b^{d})^{-1}$ for $d \geq 0$, we obtain a sequence

$$b, \quad \text{DG}(b^2) b \text{DG}(b^2)^{-1}, \quad \text{DG}(b^3) b \text{DG}(b^3)^{-1}, \quad \ldots,$$

of cyclic shifts.

Proof. Follows from (ii) of Corollary 4.7.
4.4 From braiding sequences to braid powers

He-Nie proved that for elements \( w \) with a decreasing complete sequence of eigenspaces, the braid \( b_{w}^{\text{ord}(w)} \) can be explicitly decomposed in terms of the longest elements of the standard parabolic subgroups corresponding to the filtration \( F_{i} \) [HN12, Theorem 5.3]. This subsection begins with generalising this to a statement for elements with a braiding sequence of eigenspaces; subsequently specialising it to Sevostyanov’s elements, it will imply that they satisfy the braid equation (*). After that, we prove the main Theorem.

**Notation 4.39.** For any element \( w \) of a twisted finite Coxeter group, we write \( b_{\pm} := b_{w^{-1}}b_{w} \).

The following is well-known, but I believe the usual proof for the conclusion in (ii) involves embedding \( \text{Br}^{+} \) into the corresponding braid group and invoking (i); this embedding can be avoided:

**Proposition 4.40.** Let \( W \) be a twisted finite Coxeter group.

(i) Let \( \delta \) be a twist of \( \tilde{W} \). Any decomposition of a maximal length element \( \delta w_{0} = xy \) into elements \( x, y \) in \( W \) is a reduced decomposition, e.g.

\[
(w_{0}w^{-1})w = w_{0} = (w_{0}ww_{0})(w_{0}w^{-1}).
\]

In particular, \( b_{0}^{2}w_{0} \) is central in \( \text{Br}^{+} \).

(ii) Let \( W_{1} \) be a standard parabolic subgroup of \( W \) and denote its longest untwisted element by \( w_{1} \). Then for any element \( w \) in \( W_{1} \) there are reduced decompositions

\[
(w_{0}w_{1})w = w_{0}w_{1}w = (w_{0}w_{1}ww_{0})(w_{0}w_{1})
\]

and

\[
(w_{1}w_{0})(w_{0}w_{1}ww_{0}) = w_{1}w_{0} = w(w_{0}).
\]

In particular, the braid \( b_{\pm w_{1}} \) of \( \text{Br}^{+} \) centralises the standard parabolic submonoid \( \text{Br}_{W_{1}}^{+} \subseteq \text{Br}^{+} \).

**Proof.** (i): The first claim follows from equation (4.5) and

\[
xy(\mathcal{R}_{y}) = \delta w_{0}(\mathcal{R}_{y}) \subseteq \mathcal{R}_{+}.
\]

The two given decompositions of \( w_{0} \) then furnish

\[
b_{w_{0}}b_{w} = b_{w_{0}ww_{0}}b_{w_{0}^{-1}}b_{w} = b_{w_{0}ww_{0}}b_{w_{0}}.
\]

(ii): From \( w \in W_{1} \) it follows that \( w(\mathcal{R}_{w}) \subseteq -\mathcal{R}_{w_{1}} \). The first reduced decomposition \( (w_{0}w_{1})w \) then follows from equation (4.5) and

\[
w_{0}w_{1}w(\mathcal{R}_{w}) \subseteq w_{0}w_{1}(\mathcal{R}_{w_{1}}) = w_{0}(\mathcal{R}_{w_{1}}) \subseteq \mathcal{R}_{+}.
\]

The second reduced decomposition of \( w_{0}w_{1}w \) follows similarly, using that \( w_{1}w \in W_{1} \) implies \( w_{1}w(\mathcal{R}_{w_{1}w_{1}}) \subseteq \mathcal{R}_{+} \). The second claim follows by taking inverses.

Combining these two statements then furnishes

\[
b_{\pm w_{1}w_{0}}b_{w} = b_{w_{1}w_{0}}b_{w_{1}w_{0}}b_{w_{1}w_{0}} = b_{w_{1}w_{0}}b_{w_{1}w_{0}}b_{w_{1}w_{0}} = b_{w_{1}w_{0}}b_{w_{1}w_{0}}b_{w_{1}w_{0}}.
\]

**Definition 4.41.** Let \( \Theta = (V_{m}, \ldots, V_{1}) \) be a sequence of eigenspaces of an element of a twisted finite Coxeter group, which is in good position with respect to the dominant Weyl chamber. Let \( \text{arg}(\Theta) := \{ \theta_{1}, \ldots, \theta_{1} \} \) denote the set of normalised positive arguments as before. Then we inductively define for \( 0 \leq i \leq m - 1 \) the sequence \( \Theta^{i} \) and nonnegative rational

\[
d_{i} := \min \{ \theta_{j} : \theta_{j} \in \text{arg}(\Theta^{i}) \} - d^{i}_{i-1},
\]

(4.9)
be a braiding sequence of eigenspaces. Unwinding the definitions, one finds that the sequence is anisotropic if and only if \( w_i < j_k \) such that \( i_k \) = \( j_k \), start with \( i_k \) = \( j_k \) and consecutively go downward, i.e., we set

\[
\vartheta_{i-1} = w_i' w_{i+1}' \cdots w_n'
\]

for some \( 0 \leq i_0 < \cdots < i_n \leq m-1 \), which is reduced by the same reasoning as in Proposition 4.40(ii) (allowing that some of these elements equal the identity, or simply removing redundant eigenspaces). Moreover, we can obtain a reduced decomposition for \( \vartheta_i \) from this one by omitting factors \( w_j' \), \( w_k' \) for some \( j_1 \leq \cdots \leq j_l \). For each \( j_k \) we inductively conjugate \( w_{j_k}' \) by all the \( w_i' \) with \( i' < j_k \); define \( k' \) such that \( i_k' = j_k \), start with \( i_k = j_k = \vartheta_{i-1} \geq \vartheta_i \).

In particular, \( \vartheta_0 = w_n w_m \) and if some eigenspace \( V_j \) in \( \Theta \) is redundant, then \( w_j' = \text{id} \).

**Proposition 4.42.** For any sequence of eigenspaces of length \( m \) and each integer \( 1 \leq i \leq m-1 \), these elements satisfy \( \vartheta_{i-1} \geq \vartheta_i \).

**Proof.** By construction, we have a decomposition

\[
\vartheta_{i-1} = w_i' w_{i+1}' \cdots w_n'
\]

which inductively yields a reduced decomposition

\[
\vartheta_{i-1} = (w_i' w_{i+1}' \cdots w_{i-1}' w_j') \cdots w_n'
\]

Remark 4.43. Let \( w \) be a nontrivial element of a twisted finite Coxeter group and let \( \Theta \) be a braiding sequence of eigenspaces. Unwinding the definitions, one finds that the sequence is anisotropic if and only if \( d_0' > 0 \) and \( \text{pb}(w) = \vartheta_0 \). Moreover \( d_0' > 0 \) and \( \text{pb}(w) = \vartheta_0 \).

**Proposition 4.44.** Let \( W = \Omega \ltimes \bar{W} \) be a twisted finite Coxeter group with reflection representation \( V \). Suppose \( w = \delta \bar{w} \) is an element of \( W \) with a braiding sequence of eigenspaces \( \Theta \). Let \( d \geq 0 \) denote any integer such that for each \( \vartheta_i \) we have \( d \vartheta_i \in \mathbb{N}_0 \) (e.g., \( d = \text{ord}(w) \)). Let \( d_i' \) and \( \vartheta_i \) be as in Definition 4.41, and set \( d_i := d \cdot d_i' \) for \( 0 \leq i \leq m-1 \). Then

\[
\text{DGN}(b_w^d) = \delta^d b_{d-1} \cdots b_d b_0 ,
\]

and if moreover \( d \) is even then

\[
\text{DGN}(b_w^{d/2}) = \delta^{d/2} b_{d-1}^2 \cdots b_d^2 b_0 ,
\]

where \( n \) is a half-integer we mean

\[
b_{d/2} :\vartheta_i := b_{d/2} b_{d/2}^{-1/2} .
\]
We follow the proof of [HN12, Theorem 5.3]:

Proof. We induct on the rank of $\hat{W}$ (or the dimension of $V_\Theta$); the base case essentially follows from Lemma 2.6(iii). As before we let $W_1$ denote the standard parabolic subgroup of elements in $W$ fixing $V_1$, where $\Theta = (\ldots, V_1)$. Since $V_1$ is nontrivial, the rank of $W_1$ is smaller than the rank of $W$. By Lemma 2.6(i), we have $w = xy$ for $y \in W_1$ and $x \in W$ an element whose action preserves the set of simple roots of $W_1$. From Lemma 2.6(ii), it follows that $x \in W_1$ has a braiding sequence of eigenspaces $\Theta'$ constructed out of $\Theta$. If we relabel the integers (4.9) corresponding to the sequence $\Theta'$ as $\tilde{d}_1, \ldots, \tilde{d}'_{m-1}$ and the elements on the right-hand side of (4.10) as $\hat{\vartheta}_1, \ldots, \hat{\vartheta}_{m-1},$ set $\hat{d}_i := d_i \cdot \tilde{d}_i$ and write $k := \min\{\vartheta_i \in \arg(\Theta') : \vartheta_i \geq \vartheta_i\} \geq 0$, then $k$ is the largest integer such that

$$k := \theta_1 - \sum_{i=1}^{k} \tilde{d}_i \geq 0.$$ 

These are related to $d_i$ and $\vartheta_i$ by

$$(d_i, \vartheta_i) = \begin{cases} (\tilde{d}_{i+1}, w_\vartheta w_1 \hat{\vartheta}_{i+1}) & \text{if } i < k, \\ (\hat{d}_{i+1}, \hat{\vartheta}_{i+1}) & \text{if } \tilde{k} = 0, i \geq k, \\ (\hat{d}_i, w_\vartheta w_1 \hat{\vartheta}_{i+1}) & \text{if } \tilde{k} \neq 0 \text{ and } i = k, \\ (\tilde{d}_i - \tilde{k}, \hat{\vartheta}_i) & \text{if } \tilde{k} \neq 0 \text{ and } i = k + 1, \\
(\tilde{d}_i, \hat{\vartheta}_i) & \text{if } \tilde{k} \neq 0 \text{ and } i > k + 1. \end{cases}$$

The induction hypothesis on $W'$ yields

$$(\delta_x b_y)^d = \delta_x^d b_{\hat{\vartheta}_1} \tilde{d}_{m-1} \cdots b_{\hat{\vartheta}_1} \tilde{d}_1.$$ 

By [HN12, Lemma 5.2] (the first condition in the second sentence is satisfied because $x$ is a minimal coset representative), Proposition 4.40(ii) and the submonoid identification $B_{W'} \subset B^+$ it now follows that

$$b_w^d = b_w^d b_{x_1 \cdots x_{d-1}} \cdots b_{x_1 \cdots x_{d-1}} b_y$$

$$= b_w^d (\delta_x^{-d}(\delta_x b_y)^d)$$

$$= (\delta_x b_{\hat{\vartheta}_1} \tilde{d}_{m-1} \cdots b_{\hat{\vartheta}_1} \tilde{d}_1)(\delta_x b_{\hat{\vartheta}_1} \tilde{d}_{m-1} \cdots b_{\hat{\vartheta}_1} \tilde{d}_1)$$

$$= \delta_x^d b_{\hat{\vartheta}_1} \tilde{d}_{m-1} \cdots b_{\hat{\vartheta}_1} \tilde{d}_1 b_{\hat{\vartheta}_1} \tilde{d}_{m-1} \cdots b_{\hat{\vartheta}_1} \tilde{d}_1 b_{\hat{\vartheta}_1} \tilde{d}_{m-1} \cdots b_{\hat{\vartheta}_1} \tilde{d}_1$$

$$= \delta_x^d b_{\hat{\vartheta}_1} \tilde{d}_{m-1} \cdots b_{\hat{\vartheta}_1} \tilde{d}_1.$$ 

From Propositions 4.5 and 4.42, it follows that the final expression is in Deligne-Garside normal form.

The claim for $d$ even is proven analogously.

Corollary 4.45. In particular, if the eigenspaces of $\Theta$ are not redundant then

$$\text{DG}(b_w^d) = \begin{cases} b_{pb(w)} & \text{if } \Theta \text{ is anisotropic,} \\
\hat{b}_1 & \text{if } \Theta \text{ is not anisotropic,} \end{cases}$$

and similarly for $\text{DG}(b_{d/2}^d)$ when $d$ is even.

Proof. This follows immediately by combining the previous two statements.

We now consider several specialisations of this proposition:
Corollary 4.46. Suppose furthermore that the sequence $\Theta$ is decreasing. Then

$$\text{DGN}(b_w^d) = \delta^d b_{\pm w_{m-1} w_m} \cdots b_{\pm w_{2} w_1} b_{\pm w_1}$$

and if moreover $d$ is even then

$$\text{DGN}(b_w^{d/2}) = \delta^{d/2} b_{\pm w_{m-1} w_m} \cdots b_{\pm w_{2} w_1} b_{\pm w_1}.$$

If instead $\Theta$ is increasing, then

$$\text{DGN}(b_w^d) = \delta^d b_{\pm w_{1} w_2} \cdots b_{\pm w_{m-1} w_m} b_{\pm w_1}$$

and if moreover $d$ is even then

$$\text{DGN}(b_w^{d/2}) = \delta^{d/2} b_{\pm w_{1} w_2} \cdots b_{\pm w_{m-1} w_m} b_{\pm w_1}.$$

In particular, all of these braids centralise the submonoid $Br^+_W$, where $k \geq 0$ is the largest integer such that $W_k$ is nontrivial.

Proof. Decreasing implies that $d_i = w_i w_m$ and $d_i = d(\theta_{i+1} - \theta_i)$ (briefly setting $\theta_0 := 0$); the increasing case is similar. The final claim then follows from Proposition 4.40(ii) in the case when $w_m$ is nontrivial, and from the centrality of $b_w^0$, in $Br^+_W$ otherwise. \square

Example 4.47. Consider the elliptic element $w = s_2 s_3 s_1 s_2 s_3 s_2 s_1$ in type $B_3$. Then by setting $\Theta = (V_{-1}', V_1')$ we obtain an anisotropic $V_w$-admissible sequence of eigenspaces such that the dominant Weyl chamber is in good position, and as $d := \text{ord}(w) = 4$ is even we find

$$b_w^2 = b_{12321} b_{w_2}.$$

It has been conjectured that if braids $b, b'$ in $Br^+$ satisfy $b^d = (b')^d$ for some integer $d \geq 1$, then $b$ and $b'$ are conjugate. In type $A$ this was proven by employing the Nielsen-Thurston classification [GM03], which can be rephrased in terms of the conjugation action on the set of proper parabolic subgroups:

Definition 4.48 ([Par06, §4]). Let $W$ be a twisted finite Coxeter group. One defines the notion of (standard) parabolic subgroups of the corresponding untwisted braid group $Br_W$, analogous to those of $W$, and considers the set of proper, nontrivial parabolic subgroups of $Br_W$; elements of $Br$ act on it by conjugating. An element $b$ of $Br$ is then said to be pseudo-Anosov if its action on this set does not have a finite orbit.

Finally, we deduce the main Theorem:

Proof of Theorem: (i): This is a rephrasing of Corollary 4.45.

(ii): By induction it suffices to prove the claim for a single cyclic shift between two convex elements, so there are reduced decompositions $w = yr$ to $w' = xy$. Part (i) of Lemma 4.24 yields

$$b_w^{d+1} = b_y b_w^d b_x = b_y b_{pb(w')b_x} = b_y b_{pb(w)}$$

for some elements $b', b''$ in $B^+$. The upper bound on $d$ is obtained from Proposition D(ii).

(iii): The quasiregular eigenspace of a quasiregular element furnishes a braiding sequence of length 1 for some Weyl chamber; after conjugating, the claim then follows from Proposition 4.44 for certain minimally dominant elements in its conjugacy class. More generally, suppose we are given a cyclic shift from $w = yr$ to a quasiregular element $w' = xy$ satisfying this braid equation. Part (i) and (ii) of Lemma 4.24 yield

$$i_{w'}^{\text{ord}(w)} = (b_x^{-1} b_{w'} b_x)^{\text{ord}(w)} = b_x^{-1} b_{w'} b_x = b_x^{-1} b_{pb(w')^{-1}} b_{pb(w')} b_x = b_x^{-1} b_{pb(w')^{-1}} b_x b_{pb(w)} = b_{pb(w')^{-1}} b_{pb(w)}.$$
For the converse, we may assume that \( w \) is nontrivial; from part (iii) of the main Lemma and the braid equation we obtain that
\[
\text{ord}(w) \cdot \ell(\mathcal{O}_{\text{dom}}) \geq \text{ord}(w) \cdot \frac{|\mathcal{R}| - \ell_f(w)}{\text{ord}(w)} = 2\ell(\text{pb}(w)) = \text{ord}(w) \cdot \ell(w)
\]
with equality if and only if \( w \) is quasiregular, so \( \ell(\mathcal{O}_{\text{dom}}) \geq \ell(w) \) with equality if and only if \( w \) is quasiregular. But it was shown by Broué, Digne and Michel that an element satisfying this braid equation is quasiregular [DM14, Lemma 8.4(iii)].

(iv): If an element \( w \) lies inside of a proper parabolic subgroup, then all roots outside of the corresponding standard parabolic subroot system are stable. Since the highest root does not lie in a standard parabolic subroot system, it follows that if \( w \) is convex then all roots are stable and hence \( w \) must be trivial.

Combined with part (i) of Lemma 3.8, this also yields \((b) \Rightarrow (a)\). From the definition of firmly convex and convex one deduces \((c) \Rightarrow (b)\), and finally \((a) \Rightarrow (c)\) follows from part (ii) as those elements are minimally dominant.

(v): By choosing a suitable Weyl chamber, it follows that within \( \mathcal{O} \) there exist elements with an decreasing braiding sequence of eigenspaces; depending on choice, they have maximal length, minimal length or are minimally dominant. According to Corollary 4.46, the element \( b_w^{\text{ord}(w)} \) centralises a nontrivial “standard parabolic” submonoid of \( \mathcal{B}_r \), but then it also centralises the corresponding standard parabolic subgroup of \( \mathcal{B}_r \).

Since all minimal length, maximal length and minimally dominant elements are in the same strong conjugacy or cyclic shift class, their Artin-Tits braids are conjugate as braids, and hence each of these elements centralises a nontrivial parabolic subgroup of \( \mathcal{B}_r \).

\[ \Box \]

Equation (1.5) cannot be sharpened to \( d \geq \text{ord}(w) \):

**Example 4.49.** Consider the element \( w = xyx^{-1} \) in the conjugacy class of \( y = s_5s_4s_3s_2s_1s_6s_4s_3s_2s_1 \) in type \( D_6 \), where we conjugate by the element \( x = s_2s_3s_4 \). As \( y \) is elliptic and has order 6 the same is true for \( w \), but
\[
\text{DG}(b_w^6) = w_o > w_2s_2 = \text{DG}(b_w^6).
\]

And even when a lower power \( d \) initially works, cyclic shifts can increase it:

**Example 4.50.** Consider the Coxeter elements \( w = s_3s_2s_1 \) and \( w' = s_2s_3s_1 \) in type \( A_3 \). Then
\[
\text{DG}(b_w^3) = \text{DG}(b_w^2) = w_o,
\]
but \( \text{DG}(b_w^2) \neq w_o \).

**Example 4.51.** Consider \( w = s_2s_1s_2s_1 \) in type \( A_3 \). This is a regular element which is neither elliptic nor of minimal length, but \( b_w^{\text{ord}(w)} = b_{w_o}^2 \).

Convexity and firm convexity are not necessarily preserved under cyclic shifts:

**Example 4.52.** The elements \( s_1s_2s_3s_2 \) and \( s_2s_1s_2s_3 \) in type \( B_3 \) are conjugate by a cyclic shift, but they fix different roots; the first element is convex and satisfies \( \mathcal{R}_{\text{DG}(b_w^d)} = \mathcal{R}_+ \backslash \mathcal{R}^w \) for \( d \) large, the second is not convex and hence by Proposition 4.21 cannot satisfy this equation. The same is true for the non-elliptic elements \( s_4s_1s_2s_3s_2s_1 \) and \( s_1s_2s_1s_3s_4s_3 \) in type \( A_4 \).

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