Generalized tournament matrices with the same principal minors

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ABSTRACT
An generalized tournament matrix $M$ is a nonnegative matrix that satisfies $M + M^T = J - I$, where $J$ is the all ones matrix and $I$ is the identity matrix. In this paper, a characterization of generalized tournament matrices with the same principal minors of orders 2, 3, and 4 is given. In particular, it is proven that the principal minors of orders 2, 3, and 4 determine the rest of the principal minors.

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1. Introduction

Let $M = (m_{ij})$ be an $n \times n$ matrix. With each non-empty subset $X \subseteq \{1, \ldots, n\}$, we associate the principal submatrix $M[X]$ of $M$ whose rows and columns are indexed by the elements of $X$. A principal minor of $M$ is the determinant of a principal submatrix of $M$. The order of a minor is $k$ if it is the determinant of a $k \times k$ submatrix. In this paper, we address the following problem.

Problem 1.1: What is the relationship between matrices with equal corresponding principal minors.

Clearly, if two matrices are diagonally similar then they have the same corresponding principal minors. Conversely, it follows from the main result of Engel and Schneider [1] that two symmetric matrices with no zeroes off the diagonal having the same principal minors of orders 1, 2 and 3 are necessarily diagonally similar. Hartfiel and Loewy [2] identified a special class of matrices in which two matrices with equal corresponding principal minors of all orders are diagonally similar up to transposition. This result was improved in Ref. [3] for skew-symmetric matrices with no zeroes off the diagonal by considering only the equality of corresponding principal minors of orders 2 and 4.
Boussaïri and Chergui [4] consider the class of skew-symmetric matrices with entries from \([-1, 0, 1]\) and such that all off-diagonal entries of the first row are nonzero. They characterize the pairs of matrices of this class that have equal corresponding principal minors of orders 2 and 4. This characterization involves a new transformation that generalizes diagonal similarity up to transposition.

A tournament matrix of order \(n\) is the adjacency matrix of some tournament. In other words, it is an \(n \times n\) \((0,1)\)-matrix \(M\) which satisfies

\[ M + M^t = J_n - I_n, \]

where \(J_n\) denotes the all ones \(n \times n\) matrix and \(I_n\) denotes the \(n \times n\) identity matrix. Boussaïri et al. [5] characterize the pairs of tournaments having the same three cycles. Clearly, two tournaments have the same three cycles if and only if their adjacency matrices have the same principal minors of order 3. This implies a characterization of tournament matrices with the same principal minors of order 3.

A generalized tournament matrix \(M = (m_{ij})\) is a nonnegative matrix that satisfies (1). By definition, \(m_{ij} = 1 - m_{ji} \in [0,1]\) for all \(i \neq j \in \{1, \ldots, n\}\). Thus, we can interpret \(m_{ij}\) as the a priori probability that player \(i\) defeats player \(j\) in a round-robin tournament [6].

In this work, we characterize the pairs of generalized tournament matrices with the same principal minors of order at most 4. We prove in particular that if two generalized tournament matrices have the same principal minors of orders at most 4, then they have the same principal minors of all orders.

2. Preliminaries and main result

Let \(T\) be a tournament with vertex set \(V\). A clan of \(T\) is a subset \(X\) of \(V\), such that for all \(a, b \in X\) and \(x \in V \setminus X\), \((a, x)\) is an arc of \(T\) if and only if \((b, x)\) is an arc of \(T\). For a subset \(Y\) of \(V\), we denote by \(\text{Inv}(T, Y)\) the tournament obtained by reversing all the arcs with both ends in \(Y\). If \(Y\) is a clan, we call this operation clan reversal. It is easy to check that clan reversal preserves three cycles. Conversely, Boussaïri et al. [5] proved that two tournaments on the same vertex set have the same three cycles if and only if one is obtained from the other by a sequence of clan reversals.

Let \(M = (m_{ij})\) be an \(n \times n\) matrix. A clan \(X\) of \(M\) is a subset of \([n] := \{1, \ldots, n\}\) such that for all \(i, j \in X\) and \(k \in [n] \setminus X\), \(m_{jk} = m_{kj}\) and \(m_{ki} = m_{ik}\). Denote by \(M[X, [n] \setminus X]\) the submatrix of \(M\) whose rows and columns are indexed by elements of \(X\) and \([n] \setminus X\), respectively. Clearly, \(X\) is a clan of \(M\) if and only if \(M[X, [n] \setminus X] = \mathbf{1} \cdot v^t\) and \(M[[n] \setminus X, X] = w \cdot \mathbf{1}^t\) for some column vectors \(v\) and \(w\). The empty set, the singletons \(\{i\}\) where \(i \in [n]\) and \([n]\) are clans called trivial. We say that \(M\) is indecomposable if all its clans are trivial, otherwise it is called decomposable. For a subset \(Y\) of \([n]\), we denote by \(\text{Inv}(M, Y)\) the matrix obtained from \(M\) by replacing the entry \(m_{ij}\) by \(m_{ji}\) for all \(i, j \in Y\). As for tournaments, if \(Y\) is a clan of \(M\), we call this operation clan reversal.

Let \(M\) be a tournament matrix and let \(T\) be its corresponding tournament. A subset \(X\) of \([n]\) is a clan of \(M\) if and only if it is a clan of \(T\). Moreover, for every \(Y \subset [n]\), the corresponding tournament of \(\text{Inv}(M, Y)\) is \(\text{Inv}(T, Y)\). As the two possible tournaments on three vertices have different determinants, we can write Theorem 2 of Ref. [5] as follows.
**Theorem 2.1:** Let $A$ and $B$ be two tournament matrices. The following assertions are equivalent:

(i) $A$ and $B$ have the same principal minors of order 3.

(ii) There exists a sequence $A_0 = A, \ldots, A_m = B$, such that $A_{i+1} = \text{Inv}(A_i, X_i)$, where $X_i$ is a clan of $A_i$ for all $i \in \{0, \ldots, m-1\}$.

This theorem solves Problem 1.1 completely in the case of tournament matrices. Another result, in relation to our work, is the following theorem of Loewy [7].

**Theorem 2.2:** Let $A$, $B$ be two $n \times n$ matrices. Suppose that $n \geq 4$, $A$ is irreducible and for every partition of $[n]$ into two subsets $X$, $Y$ with $|X| \geq 2$ and $|Y| \geq 2$, $\text{rank}(A[X, Y]) \geq 2$ or $\text{rank}(A[Y, X]) \geq 2$. If $A$ and $B$ have equal corresponding minors of all orders, then they are diagonally similar up to transposition.

Let $M$ be an $n \times n$ generalized tournament matrix and let $X$, $Y$ be a bipartition of $[n]$ with $|X| \geq 2$ and $|Y| \geq 2$. It is not hard to prove that if $\text{rank}(M[X, Y]) \leq 1$ and $\text{rank}(M[Y, X]) \leq 1$, then $X$ or $Y$ is a nontrivial clan of $A$. It follows that an indecomposable generalized tournament matrix satisfies the conditions of Theorem 2.2. Another fact is that if two generalized tournament matrices are diagonally similar, then they are equal. Then, from Theorem 2.2, we have the following proposition.

**Proposition 2.3:** Let $A$ and $B$ be two $n \times n$ generalized tournament matrices. Suppose that $n \geq 4$ and $A$ is indecomposable. If $A$ and $B$ have equal corresponding minors of all orders, then $A = B$ or $A = B^t$.

It follows from Theorem 2.1 that it is enough to consider only principal minors of orders at most 3 in the case of tournament matrices. This fact is not true for arbitrary generalized tournament matrices. Indeed, we will give in Section 4 two indecomposable $4 \times 4$ matrices which have the same principal minors of orders 2 and 3 but do not have the same determinant. Afterward, we will prove that Proposition 2.3 still holds if we consider principal minors of orders at most 4. Then, we prove the following theorem.

**Theorem 2.4:** Let $A$ and $B$ be two $n \times n$ generalized tournament matrices. The following assertions are equivalent:

(i) $A$ and $B$ have the same minors of orders at most 4.

(ii) $A$ and $B$ have the same minors of every order.

(iii) There exists a sequence $A_0 = A, \ldots, A_m = B$ of $n \times n$ generalized tournament matrices, such that for $k = 0, \ldots, m-1$, $A_{k+1} = \text{Inv}(A_k, X_k)$, where $X_k$ is a clan of $A_k$.

It is worth noting that the proof of Theorem 2.2 in Ref. [7] uses tools from linear algebra. It seems hard to prove Theorem 2.4 in a similar fashion, even for $n = 5$. We will use graph theoretic tools via a correspondence between generalized tournament matrices and weighted oriented graphs.

Let $M = (m_{ij})$ be an $n \times n$ generalized tournament matrix. For all $i \neq j \in [n]$, $m_{ij}$ is in $[0, 1]$, and $m_{ij} = m_{ji}$ if and only if $m_{ij} = 1/2$. Then, we associate to $M$ a weighted oriented
graph $\Gamma_M$ with vertex set $[n] := \{1, \ldots, n\}$, such that $(i,j)$ is an arc with weight $m_{ij}$ if and only if $m_{ij} \in (1/2, 1]$. Conversely, let $\Gamma$ be a weighted oriented graph with vertex set $[n]$ and weights in $(1/2, 1]$. We associate to $\Gamma$ a generalized tournament matrix $M = (m_{ij})$, such that if $(i,j)$ is an arc then $m_{ij}$ is equal to the weight of $(i,j)$, and $m_{ij} = m_{ji} = 1/2$ if $(i,j)$ and $(j,i)$ are not arcs of $\Gamma$.

This correspondence between generalized tournament matrices and weighted oriented graphs allows us to use some techniques from Ref. [5] in the proof of Theorem 2.4.

3. Decomposable and indecomposable weighted oriented graphs

Let $\Gamma$ be a weighted oriented graph with vertex set $V$. We write $x \xrightarrow{\alpha} y$ if $(x, y)$ is an arc of $\Gamma$ with weight $\alpha$, and $x \cdot \cdot \cdot y$ if there is no arc between $x$ and $y$. Similarly, if $X$ and $Y$ are two disjoint subsets of $V$, we write $X \xrightarrow{\alpha} Y$ if $(x, y)$ is an arc with weight $\alpha$ for every $x \in X$ and $y \in Y$. If $X = x$ we simply write $x \xrightarrow{\alpha} Y$ and $Y \xrightarrow{\alpha} x$ instead of $\{x\} \xrightarrow{\alpha} Y$ and $Y \xrightarrow{\alpha} \{x\}$. The notations $X \cdot \cdot \cdot Y, x \cdot \cdot \cdot y$ and $Y \cdot \cdot \cdot x$ are defined in the same way.

A clan of a weighted oriented graph $\Gamma$ with vertex set $V$ is a subset $X$ of $V$ such that for every $x \in V \setminus X$, either $x \cdot \cdot \cdot X, x \xrightarrow{\alpha} X$ or $X \xrightarrow{\alpha} x$ for some weight $\alpha$. The empty set, the singletons $\{x\}$ where $x \in V$ and $V$ are clans called trivial. We say that $\Gamma$ is indecomposable if all its clans are trivial, otherwise it is called decomposable. The notion of clans was introduced, under different names, for graphs, digraphs and more generally two-structures [8]. The next proposition gives some basic properties of clans.

**Proposition 3.1:** Let $\Gamma$ be a weighted oriented graph with vertex set $V$. Let $X, Y$ and $Z$ be subsets of $V$.

(i) If $X$ is a clan of $\Gamma$, then $X \cap Z$ is a clan of $\Gamma[Z]$.
(ii) If $X$ and $Y$ are clans of $\Gamma$, then $X \cap Y$ is a clan of $\Gamma$.
(iii) If $X$ and $Y$ are clans of $\Gamma$, such that $X \cap Y \neq \emptyset$, then $X \cup Y$ is a clan of $\Gamma$.
(iv) If $X$ and $Y$ are clans of $\Gamma$, such that $X \setminus Y \neq \emptyset$, then $Y \setminus X$ is a clan of $\Gamma$.
(v) If $X$ and $Y$ are clans of $\Gamma$, such that $X \cap Y = \emptyset$, then either $X \xrightarrow{\alpha} Y, Y \xrightarrow{\alpha} X$ or $X \cdot \cdot \cdot Y$ for some weight $\alpha$.

The following theorem of Ehrenfeucht and Rozenberg [9] shows that indecomposability is hereditary.

**Theorem 3.2:** Let $\Gamma$ be an indecomposable weighted oriented graph with $n \geq 5$ vertices. Then, $\Gamma$ contains an indecomposable weighted oriented graph with $n-1$ or $n-2$ vertices.

A weighted oriented graph $\Gamma$ is said to be separable if its vertex set $V$ can be partitioned into two non-empty clans, otherwise it is inseparable. If $\Gamma$ is separable, then there exists a bipartition $X, Y$ of $V$ such that $X \xrightarrow{\alpha} Y$ for some weight $\alpha$, or $X \cdot \cdot \cdot Y$. In the first case, $\Gamma$ is called $\alpha$-separable. We say that $\Gamma$ is $\alpha$-linear if its vertices can be ordered into a sequence $x_1, \ldots, x_n$ such that $x_i \xrightarrow{\alpha} x_j$ if $i < j$. The notions defined above can be extended naturally to generalized tournament matrices.

By definition, a tournament is inseparable if and only if it is irreducible. It is well known that every irreducible tournament with $n$ vertices contains an irreducible tournament with
n−1 vertices. The next theorem extends this result to weighted oriented graphs and will be used in the proof of the main theorem.

**Theorem 3.3:** Let \( \Gamma \) be an inseparable weighted oriented graph with \( n \geq 5 \) vertices. Then, \( \Gamma \) contains an inseparable weighted oriented graph with \( n−1 \) vertices.

**Proof:** Suppose that \( \Gamma \) is decomposable and let \( C \) be a nontrivial clan of \( \Gamma \). Let \( u \) be a vertex in \( C \). We will prove that \( \Gamma[V \setminus \{u\}] \) is inseparable. Suppose, for the sake of contradiction, that \( \Gamma[V \setminus \{u\}] \) is separable, and let \( X, Y \) be a bipartition of \( V \setminus \{u\} \) into two clans. Without loss of generality, we can suppose that \( X \xrightarrow{\alpha} Y \). Since \( C \neq V \), we have \( C \setminus \{u\} \neq X \cup Y \).

1. If \( C \setminus \{u\} \subseteq X \), then \( C \setminus \{u\} \xrightarrow{\alpha} Y \). As \( C \) is a clan of \( \Gamma \), \( u \xrightarrow{\alpha} Y \). Hence, \( X \cup \{u\} \xrightarrow{\alpha} Y \), which contradicts the fact that \( \Gamma \) is inseparable. Similarly, \( C \setminus \{u\} \subseteq Y \) yields a contradiction.

2. \( C \setminus \{u\} \cap X \) and \( C \setminus \{u\} \cap Y \) are non-empty. Let \( z \in V \setminus C \), we can suppose that \( z \in X \setminus C \). We have \( z \xrightarrow{\alpha} Y \), in particular \( z \xrightarrow{\alpha} Y \cap C \). Since \( C \) is a clan, \( z \xrightarrow{\alpha} C \). It follows that \( (X \setminus C) \xrightarrow{\alpha} V \setminus (X \cap C) \).

Suppose now that \( \Gamma \) is indecomposable. The result is trivial if \( \Gamma \) contains an indecomposable graph with \( n−1 \) vertices. If no such graph exists, then, by Theorem 3.2, there exist two distinct vertices \( x, y \in V \) such that \( \Gamma[V \setminus \{x, y\}] \) is indecomposable. Then, \( \Gamma[V \setminus \{x\}] \) or \( \Gamma[V \setminus \{y\}] \) is inseparable. Indeed, if \( \Gamma[V \setminus \{x\}] \) is separable, then there exists a bipartition \( X, Y \) of \( V \setminus \{x\} \) into two clans. Suppose that \( X \xrightarrow{\alpha} Y \). If \( V \setminus \{x, y\} \cap X \) and \( V \setminus \{x, y\} \cap Y \) are both non-empty, then they are a bipartition of \( V \setminus \{x, y\} \) into two clans, which contradicts the fact that \( \Gamma[V \setminus \{x, y\}] \) is indecomposable. Hence, \( V \setminus \{x, y\} \) is a clan of \( \Gamma[V \setminus \{x\}] \). Similarly, if \( \Gamma[V \setminus \{y\}] \) is separable, then \( V \setminus \{x, y\} \) is a clan of \( \Gamma[V \setminus \{y\}] \). It follows that if \( \Gamma[V \setminus \{x\}] \) and \( \Gamma[V \setminus \{y\}] \) are both separable, then \( V \setminus \{x, y\} \) is a clan of \( \Gamma \), which contradicts the assumption that \( \Gamma \) is indecomposable. \( \square \)

### 4. Indecomposable generalized tournament matrices

In this section, we improve Proposition 2.3 by showing that it is enough to consider principal minors of orders at most 4. More precisely, we prove the following result.

**Theorem 4.1:** Let \( A \) and \( B \) be two \( n \times n \) generalized tournament matrices. Suppose that \( n \geq 4 \) and \( A \) is indecomposable. If \( A \) and \( B \) have equal corresponding principal minors of orders at most 4, then \( A = B \text{ or } A = B^t \).

Let \( A = (a_{ij}) \) and \( B = (b_{ij}) \) be two \( n \times n \) generalized tournament matrices with the same principal minors of orders 2. Then, for all \( i \neq j \in [n] \), \( a_{ij} = b_{ij} \) or \( a_{ij} = 1 - b_{ij} \). It follows that the set \( \binom{[n]}{2} \) can be partitioned into three subsets.

- \( P_\neq := \{ \{i, j\} \in \binom{[n]}{2} : a_{ij} = b_{ij} \text{ and } a_{ij} \neq 1/2 \} \)
- \( P_\neq := \{ \{i, j\} \in \binom{[n]}{2} : a_{ij} = 1 - b_{ij} \text{ and } a_{ij} \neq 1/2 \} \)
- \( P_{1/2} := \{ \{i, j\} \in \binom{[n]}{2} : a_{ij} = b_{ij} = 1/2 \} \)
The equality graph and the difference graph of \( A \) and \( B \), denoted by \( \mathcal{E}(A, B) \) and \( \mathcal{D}(A, B) \), respectively, are the undirected graphs with vertex set \( V = [n] \), and whose arc sets are \( P_\leq \) and \( P_\neq \). It follows from the definition that
\[
\mathcal{E}(A, B) = \mathcal{D}(A, B').
\] (2)

In what follows, we give some information about generalized tournament matrices with the same principal minors of orders 2 and 3, via the equality and difference graphs.

**Lemma 4.2:** Let \( A = (a_{ij}) \) and \( B = (b_{ij}) \) be two \( n \times n \) generalized tournament matrices with the same principal minors of orders 2 and 3. For every \( i, j, k \in [n] \), we have

(i) if \( \{i, j\} \in P_\neq \) and \( \{i, k\}, \{j, k\} \in P_\leq \), then \( a_{ik} = a_{jk} = b_{ik} = b_{jk} \).

(ii) if \( \{i, j\} \in P_\leq \) and \( \{i, k\}, \{j, k\} \in P_\neq \), then \( a_{ik} = a_{jk} = 1 - b_{ik} = 1 - b_{jk} \).

(iii) if \( \{i, j\} \in P_\leq \) and \( \{i, k\}, \{j, k\} \notin P_\leq \), then \( a_{ik} = 1/2 \) if and only if \( a_{jk} = 1/2 \).

(iv) if \( \{i, j\} \in P_\neq \) and \( \{i, k\}, \{j, k\} \notin P_\neq \), then \( a_{ik} = 1/2 \) if and only if \( a_{jk} = 1/2 \).

**Proof:** Let \( i, j, k \in [n] \). Then, we have
\[
\det A[[i, k]] = \det B[[i, k]],
\]
\[
\det A[[j, k]] = \det B[[j, k]],
\]
\[
\det A[[i, j, k]] = \det B[[i, j, k]].
\]

It follows that
\[
a_{ik} = b_{ik} \text{ or } a_{ik} = 1 - b_{ik},
\] (3)
\[
a_{jk} = b_{jk} \text{ or } a_{jk} = 1 - b_{jk},
\] (4)
\[
a_{ik} - a_{ij}a_{ik} + a_{ij}a_{jk} - a_{ik}a_{jk} = b_{ik} - b_{ij}b_{ik} + b_{ij}b_{jk} - b_{ik}b_{jk}.
\] (5)

If \( \{i, j\} \in P_\neq \) and \( \{i, k\}, \{j, k\} \in P_\leq \), then \( a_{ij} = 1 - b_{ij}, a_{ik} = b_{ik} \) and \( a_{jk} = b_{jk} \). Using (5), we get \( a_{jk} = a_{ik} \) and then \( b_{jk} = b_{ik} \). This proves assertion (i).

To prove (iii), suppose that \( \{i, j\} \in P_\leq \), \( \{i, k\} \notin P_\leq \) and \( a_{ik} = 1/2 \). Then \( b_{ij} = a_{ij} \notin 1/2 \), \( b_{jk} = 1 - a_{jk} \) and \( b_{ik} = 1/2 \). By substituting in (5), we get \( a_{jk} = 1/2 \). Assertions (ii) and (iv) can be obtained from (i) and (ii) using (2).

**Proposition 4.3:** Let \( A = (a_{ij}) \) and \( B = (b_{ij}) \) be two \( n \times n \) generalized tournament matrices. If \( A \) and \( B \) have the same principal minors of orders 2 and 3, then the connected components of \( \mathcal{E}(A, B) \) and \( \mathcal{D}(A, B) \) are clans of \( A \) and \( B \).

**Proof:** By (2), it suffices to consider \( \mathcal{E}(A, B) \). Let \( C \) be a connected component of \( \mathcal{E}(A, B) \). If \( C = [n] \) or \( C = \{i\} \) for some \( i \in [n] \), then \( C \) is a trivial clan of \( A \) and \( B \). Otherwise, let \( i \neq j \in C \) be two adjacent vertices and let \( k \in [n] \setminus C \). Then, \( \{i, j\} \in P_\leq \) and \( \{i, k\}, \{j, k\} \notin P_\leq \).

We have to prove that \( a_{ik} = a_{jk} \) and \( b_{ik} = b_{jk} \). For this, there are two cases to consider.

(1) If \( a_{ik} = 1/2 \), then by assertion (iii) of Lemma 4.2, we have \( a_{jk} = 1/2 \), and hence \( b_{ik} = b_{jk} = 1/2 \).
(2) If \( a_{i,k} \neq 1/2 \), then \( a_{j,k} \neq 1/2 \). It follows that \( \{i, k\}, \{j, k\} \in P_\neq \). We conclude by assertion (ii) of Lemma 4.2.

Let \( A \) and \( B \) be two \( n \times n \) generalized tournament matrices with the same principal minors of orders 2 and 3. Suppose that \( A \) is indecomposable. By Proposition 4.3, \( A = B \) or \( A = B^t \) if and only if \( \mathcal{E}(A, B) \) and \( \mathcal{D}(A, B) \) are not both connected. In general, \( \mathcal{E}(A, B) \) and \( \mathcal{D}(A, B) \) can be both connected. Indeed, let \( a, b \in [0, 1] \setminus \{1/2\} \) and consider the matrix \( M_{a,b} \) defined as follows:

\[
M_{a,b} = \begin{pmatrix}
0 & a & b & b \\
1 - a & 0 & 1 - a & b \\
1 - b & a & 0 & a \\
1 - b & 1 - b & 1 - a & 0
\end{pmatrix}.
\]

It is easy to check that the matrices \( M_{a,b} \) and \( M_{1-a,b} \) have equal corresponding minors of orders 2 and 3, and that both \( \mathcal{E}(M_{a,b}, M_{1-a,b}) \) and \( \mathcal{D}(M_{a,b}, M_{1-a,b}) \) are connected. Moreover, if \( a \neq b \) and \( a \neq 1 - b \), then \( M_{a,b} \) and \( M_{1-a,b} \) are indecomposable and do not have the same determinant. This shows the necessity of the equality of principal minors of order 4 in the assumptions of Theorem 4.1. With this strengthening, we obtain the following result which implies Theorem 4.1.

**Proposition 4.4:** Let \( A \) and \( B \) be two \( n \times n \) generalized tournament matrices with the same principal minors of orders at most 4. If \( A \) is inseparable, then \( \mathcal{E}(A, B) \) or \( \mathcal{D}(A, B) \) is not connected.

We will prove this proposition by induction on \( n \). The next lemma allows us to solve the base case \( n = 4 \).

**Lemma 4.5:** Let \( A = (a_{ij}) \) and \( B = (b_{ij}) \) be two \( 4 \times 4 \) generalized tournament matrices with equal corresponding minors of orders 2 and 3. If \( \mathcal{D}(A, B) \) and \( \mathcal{E}(A, B) \) are connected, then there exists a permutation matrix \( P \) such that \( A = P M_{a,b} P^t \) and \( B = P M_{1-a,b} P^t \) where, \( a, b \in [0, 1] \setminus \{1/2\} \). Moreover, \( \det(A) = \det(B) \) if and only if \( a = b \) or \( a = 1 - b \).

**Proof:** Suppose that \( \mathcal{D}(A, B) \) and \( \mathcal{E}(A, B) \) are connected. The only possibility is that \( \mathcal{D}(A, B) \) and \( \mathcal{E}(A, B) \) are disjoint paths of length three. Then, there is a permutation matrix \( P \) such that

- the edges of \( \mathcal{D}(P^t AP, P^t BP) \) are \( \{1, 2\}, \{2, 3\} \) and \( \{3, 4\} \).
- the edges of \( \mathcal{E}(P^t AP, P^t BP) \) are \( \{1, 3\}, \{1, 4\} \) and \( \{2, 4\} \).

Let \( A' := P^t AP \) and \( B' := P^t BP \). The off-diagonal entries of \( A' = (a'_{ij}) \) and \( B' = (b'_{ij}) \) are not equal to 1/2. Moreover, we have \( a'_{12} = 1 - b'_{12}, a'_{22} = 1 - b'_{22}, a'_{34} = 1 - b'_{34}, a'_{13} = b'_{13}, a'_{14} = b'_{14} \) and \( a'_{24} = b'_{24} \). The matrices \( A' \) and \( B' \) have equal corresponding minors of orders 2 and 3. Then, by assertions (i) and (ii) of Lemma 4.2, we get \( a'_{32} = a'_{34} = a'_{13} = a'_{14} = a'_{24} \). Let \( a := a'_{12} \) and \( b := a'_{13} \). We have \( a, b \in [0, 1] \setminus \{1/2\} \), \( A' = M_{a,b} \) and \( B' = M_{1-a,b} \). Hence, \( A = PM_{a,b}P^t \) and \( B = PM_{1-a,b}P^t \). Then, \( \det(A) - \)
Let \( A = \begin{bmatrix} A_{11} & \alpha \beta^t \\ \beta \alpha^t & A[X] \end{bmatrix} \) and \( \text{Inv}(A, X) = \begin{bmatrix} A_{11} & \alpha \beta^t \\ \beta \alpha^t & A[X]^t \end{bmatrix} \),

where \( \beta = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \) and \( \alpha = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{n-|X|} \end{bmatrix} \).
\begin{align*}
\beta &= \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \\
\\text{and} \\
\alpha &= \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_{n-|X|} \end{pmatrix},
\end{align*}

where \( a_i \in [0, 1] \). By Proposition 3 of Ref. [10], \( A \) and \( \text{Inv}(A, X) \) have the same determinant. Then we have the following result.

**Lemma 5.1:** Clan inversion preserves principal minors.

It follows from this lemma that matrices obtained from a series of clan inversions have the same principal minors. This proves the implication \((iii) \Rightarrow (ii)\) of Theorem 2.4. The implication \((ii) \Rightarrow (iii)\) is trivial. The remaining of the section is devoted to proving the implication \((i) \Rightarrow (iii)\), that is, pairs of matrices with the same principal minors of orders at most 4 are obtained by a series of clan inversions.

We start by reducing the problem to the case of matrices with a common nontrivial clan.

**Proposition 5.2:** Let \( A \) and \( B \) be two \( n \times n \) generalized tournament matrices with the same principal minors of orders at most 4. Suppose that \( A \) is inseparable. If \( A \) and \( B \) have no common nontrivial clans, then \( A = B \) or \( A = B^t \).

**Proof:** Since \( A \) is inseparable, by Proposition 4.4, \( E(A, B) \) or \( D(A, B) \) is not connected. If \( D(A, B) \) is not connected, then by Proposition 4.3, its connected components are common clans of \( A \) and \( B \). If \( A \) and \( B \) have no common nontrivial clans, then the connected components of \( D(A, B) \) must be singletons and hence \( A = B \). If \( E(A, B) \) is not connected, using (2), we get \( A = B^t \). \( \blacksquare \)

**Proposition 5.3:** Let \( A \) and \( B \) be two \( n \times n \) generalized tournament matrices. If \( A \) and \( B \) are \( \alpha \)-linear for some \( \alpha > 1/2 \), then there exists a clan \( X \) of \( A \) such that \( \text{Inv}(A, X) \) and \( B \) have a common nontrivial clan.

**Proof:** Let \( \Gamma_A \) and \( \Gamma_B \) be the corresponding graphs of \( A \) and \( B \). Without loss of generality, we can suppose that for all \( i \neq j \in [n] \), \( i \overset{\alpha}{\rightarrow} j \) in \( \Gamma_A \) if \( i < j \). There exists a permutation \( \sigma \) of \([n]\), such that for all \( i \neq j \in [n], \sigma(i) \overset{\alpha}{\rightarrow} \sigma(j) \) in \( \Gamma_B \) if \( i < j \). Consider the clan \( X = \{1, \ldots, \sigma(1)\} \) of \( A \). Clearly, \( \sigma(1) \overset{\alpha}{\rightarrow} [n] \setminus \{\sigma(1)\} \) in the graph corresponding to \( \text{Inv}(A, X) \). Hence, \( \{\sigma(2), \ldots, \sigma(n)\} \) is a common nontrivial clan of \( \text{Inv}(A, X) \) and \( B \). \( \blacksquare \)

For the last case, when \( A \) is separable and there is no \( \alpha > 1/2 \) such that \( A \) and \( B \) are \( \alpha \)-linear, we need the following results.
Lemma 5.4: Let $A$ be an $n \times n$ decomposable generalized tournament matrix and let $I$ be a nontrivial clan of $A$. Let $x \in I$, then $A$ is inseparable if and only if $A[(\{n\} \setminus I) \cup \{x\}]$ is inseparable.

Proof: Suppose that $V := [n]$ can be partitioned into two clans $X$, $Y$ of $A$. If $((V \setminus I) \cup \{x\}) \cap X$ and $((V \setminus I) \cup \{x\}) \cap Y$ are non-empty, then they are a bipartition of $(V \setminus I) \cup \{x\}$ into two clans of $A[(V \setminus I) \cup \{x\}]$. Otherwise, suppose for example that $((V \setminus I) \cup \{x\}) \cap X$ is empty, then $X \subset I \setminus \{x\}$. Hence $\{x\}, V \setminus I$ is a bipartition of $(V \setminus I) \cup \{x\}$ into two clans. In both cases, $A[(V \setminus I) \cup \{x\}]$ is separable.

Conversely, let $X$, $Y$ be a bipartition of $(V \setminus I) \cup \{x\}$ into two clans of $A[(V \setminus I) \cup \{x\}]$ and assume for example that $x \in X$. Then, $X \cup I$, $Y$ is a bipartition of $V$ into two clans of $A$ and, hence, $A$ is separable. $\blacksquare$

Proposition 5.5: Let $A$ and $B$ be two $n \times n$ generalized tournament matrices with the same principal minors of orders at most 4. Then $A$ is inseparable if and only if $B$ is inseparable.

Proof: We proceed by induction on $n$. For $n = 3$, the result is trivial. Suppose that $A$ is inseparable. If $B = A$ or $B = A^t$, then $B$ is inseparable. Otherwise, by Proposition 5.2, $A$ and $B$ have a common nontrivial clan $I$. Let $x \in I$, then by Lemma 5.4, $A[(V \setminus I) \cup \{x\}]$ is inseparable and so is $B[(V \setminus I) \cup \{x\}]$ by induction hypothesis. It follows by Proposition 5.2 again that $B$ is inseparable. $\blacksquare$

Corollary 5.6: Under the assumptions of the previous lemma, for $\alpha > 1/2$, $A$ is $\alpha$-separable if and only if $B$ is $\alpha$-separable.

Clearly, if a matrix $A$ is $\alpha$-linear, then for every clan $I$ of $A$, $A[I]$ is $\alpha$-separable. Conversely, by induction on $n$, we obtain the following result.

Lemma 5.7: If there exists $\alpha > 1/2$ such that for every clan $I$ of $A$, $A[I]$ is $\alpha$-separable, then $A$ is an $\alpha$-linear.

Proposition 5.8: Let $A$ and $B$ be two $n \times n$ generalized tournament matrices with the same principal minors of orders at most 4. Suppose that $A$ is separable. If there is no $\alpha > 1/2$ such that $A$ and $B$ are $\alpha$-linear, then $A$ and $B$ have a common nontrivial clan.

Proof: Let $\Gamma_A$ and $\Gamma_B$ be the corresponding graphs of $A$ and $B$. Suppose that there exists a bipartition $X$, $Y$ of $[n]$ such that $X \cdots Y$ in $\Gamma_A$. Clearly, $X \cdots Y$ in $\Gamma_B$. As $n \geq 3$, $X$ or $Y$ is a common nontrivial clan of $A$ and $B$. Suppose now that $A$ is $\alpha$-separable for some $\alpha > 1/2$. Let $J_A$ be the set of clans $I$ of $A$ such that $A[I]$ is not $\alpha$-separable, $J_B$ is defined similarly. Assume that $A$ or $B$ is not $\alpha$-linear. Then, by Lemma 5.7, $J_A \cup J_B$ is not empty. Let $I$ be an element of $J_A \cup J_B$ with maximum cardinality and assume, for example, that $I \in J_A$. Consider the smallest clan $\bar{I}$ of $B$ containing $I$. Clearly, $B[\bar{I}]$ is not $\alpha$-separable.

Indeed, if $X$, $Y$ is a bipartition of $\bar{I}$ such that $X \searrow Y$ in $\Gamma_B$, then $I \subset X$ or $I \subset Y$ because, by Corollary 5.6, $B[I]$ is not $\alpha$-separable. This contradicts the minimality of $\bar{I}$ because $X$ and $Y$ are both clans of $B$. Then, $\bar{I} \in J_B$ and, hence, $\bar{I} = I$ by maximality of the cardinality of $I$. It follows that $I$ is a common nontrivial clan of $A$ and $B$. $\blacksquare$
Now we are able to complete the proof of Theorem 2.4. The implications (iii) ⇒ (ii) and (ii) ⇒ (i) are already proven. The proof of the implication (i) ⇒ (iii) is similar to that of Ref. [5, Theorem 2], and it will be added in order for the paper to be self-contained.

Let \( A \) and \( B \) be two \( n \times n \) generalized tournament matrices with the same principal minors of orders at most 4. We want to prove that \( B \) is obtained from \( A \) by a sequence of clan inversions. For this, we proceed by induction on \( n \). The result is trivial for \( n = 2 \). Assume that \( n \geq 4 \). There is nothing to prove if \( A = B \) or \( A = B' \). Otherwise, by Propositions 5.2, 5.3 and 5.8, we can suppose that \( A \) and \( B \) have a common nontrivial clan \( X \). Let \( x \in X \) and denote by \( U \) the set \( (V \setminus X) \cup \{x\} \). By induction hypothesis, there exist matrices \( S_0 = A[U], \ldots, S_l = B[U] \) such that \( S_{k+1} = \text{Inv}(S_k, Y_k) \), where \( Y_k \) is a clan of \( S_k \) for all \( k \in \{0, \ldots, l-1\} \). For each \( i \in \{0, \ldots, l-1\} \), the subsets \( \tilde{Y}_i \) of \( V \) is defined from \( Y_i \) as \( \tilde{Y}_i = Y \) if \( x \notin Y_i \) and \( \tilde{Y}_i = Y_i \cup X \) if \( x \in Y_i \). Now, the sequence \( (\tilde{S}_i) \) is defined by \( \tilde{S}_0 = A \) and for all \( i \in \{0, \ldots, l-1\}, \tilde{S}_{i+1} = \text{Inv}(\tilde{S}_i, \tilde{Y}_i) \). Clearly, \( \tilde{S}_m[U] = B[U], \tilde{S}_m[X] = A[X] \) or \( A[X]^t \), and since \( A[X] \) and \( B[X] \) have also the same principal minors of orders at most 4, \( \tilde{S}_m[X] \) and \( B[X] \) have the same principal minors of orders at most 4. By the induction hypothesis, there are matrices \( R_0 = \tilde{S}_m[X], \ldots, R_p = B[X] \) such that \( R_{i+1} = \text{Inv}(R_i, Z_i) \), where \( Z_i \) is a clan of \( R_i \). By considering \( \tilde{R}_0 = A'' \) and for all \( i \in \{0, \ldots, p-1\}, \tilde{R}_{i+1} = \text{Inv}(\tilde{R}_i, Z_i) \), it is obtained that \( \tilde{R}_p = B \).

6. Remarks and questions

(1) Let \( T \) be a tournament with vertex set \( V \). We can associate to \( T \) the three-uniform hypergraph \( \mathcal{H}_T \) with vertex set \( V \) whose hyperedges are the three subsets of \( V \) that induce three cycles in \( T \). We call this hypergraph the C3-structure of \( T \). Clearly, not every three-uniform hypergraph arises as the C3-structure of some tournament. Linial and Morgenstern [11] asked if the C3-structure of tournaments can be recognized in polynomial time. Some progress on this problem has been made in Ref. [12].

As the determinant of a tournament on three vertices is 1 if it is a three cycle and 0 otherwise, Linial and Morgenstern’s problem can be stated matricially as follows. Does there exist a polynomial time algorithm that decides if a vector \( P \in \{0, 1\}^{|T|} \) arises as the principal minors of order 3 of a tournament matrix. This problem can be generalized naturally to generalized tournament matrices.

**Problem 6.1:** Is there a polynomial time algorithm that decides whether a collection \( (P_\alpha)_{\alpha \in \mathbb{Z}^{|T|}}, 2 \leq |\alpha| \leq 4 \) of real numbers arises as the principal minors of orders 2, 3 and 4 of a generalized tournament matrix?

(2) Let \( n \geq 4 \) be an integer. Denote by \( GT_n \) the set of all \( n \times n \) generalized tournament matrices and by \( PM_n \) the set of collections \( (P_\alpha)_{\alpha \in [n]}, 2 \leq |\alpha| \leq 4 \) of real numbers that arise as the principal minors of orders 2, 3 and 4 of \( n \times n \) generalized tournament matrices. Let \( \phi : GT_n \to PM_n \) the map which associates to each generalized tournament matrix the collection of its principal minors of orders 2, 3 and 4. By Theorem 2.4, the determinant of a generalized tournament matrix is determined by the principal minors of orders at most 4. Hence, there exists a unique map \( \psi : PM_n \to \mathbb{R} \), such that \( \psi \circ \phi(M) = \det(M) \), for every \( n \times n \) generalized tournament matrix \( M \). That is, the following diagram is commutative.
We can ask if the map $\psi$ can be found explicitly, that is if the determinant of an $n \times n$ generalized tournament matrix can be expressed in terms of its principal minors of orders at most 4.

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