THE DIRICHLET PROBLEM FOR THE MINIMAL SURFACE EQUATION IN Sol₃, WITH POSSIBLE INFINITE BOUNDARY DATA

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Abstract. In this paper, we study the Dirichlet problem for the minimal surface equation in Sol₃ with possible infinite boundary data, where Sol₃ is the non-abelian solvable 3-dimensional Lie group equipped with its usual left-invariant metric that makes it into a model space for one of the eight Thurston geometries. Our main result is a Jenkins-Serrin type theorem which establishes necessary and sufficient conditions for the existence and uniqueness of certain minimal Killing graphs with a non-unitary Killing vector field in Sol₃.

1. Introduction

In [7], Jenkins and Serrin considered bounded domains \( \Omega \subset \mathbb{R}^2 \), with \( \partial \Omega \) composed of straight line segments and convex arcs. They found necessary and sufficient conditions on the lengths of the sides of inscribed polygons, which guarantee the existence of a minimal graph over \( \Omega \), taking certain prescribed values (in \( \mathbb{R} \cup \{ \pm \infty \} \)) on the components of \( \partial \Omega \). Perhaps the simplest example is \( \Omega \) with a geodesic triangle with boundary data zero on two sides and \( +\infty \) on the third side. The conditions of Jenkins-Serrin reduce to the triangle inequality here and the solution exists. It was discovered by Scherk in 1835. This also works on a parallelogram with sides of equal length, with data \( +\infty \) on opposite two sides and \( -\infty \) on the other two sides. This solution was also found by Scherk.

In recent years there has been much activity on this Dirichlet problem in \( M^2 \times \mathbb{R} \) where \( M \) is a two dimensional Riemannian manifold (see [2, 13, 14, 18]). When \( M \) is the hyperbolic plane \( \mathbb{H}^2 \), there are non-compact domains for which this problem has been solved, and interesting applications have been obtained (see [2, 4, 9]). In the previous cases, authors considered the Killing graphs where the Killing vector field is unitary.

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The purpose of this paper is to consider the problem of type Jenkins-Serrin on bounded domains and some unbounded domains in $Sol_3$ which is a three-dimensional homogeneous Riemannian manifold can be viewed as $\mathbb{R}^3$ endowed with the Riemannian metric
\[ ds^2 = e^{2x_3}dx_1^2 + e^{-2x_3}dx_2^2 + dx_3^2. \]

The change of coordinates
\[ x := x_2, \quad y := e^{x_3}, \quad t := x_1, \]

turns this model into $Sol_3 = \{(x, y, t) \in \mathbb{R}^3 : y \geq 0\}$ with the Riemannian metric
\[ ds^2 = \frac{dx^2 + dy^2}{y^2} + y^2dt^2. \]

By using the Poincaré half-plane model, $Sol_3$ has the form of a warped product $Sol_3 = \mathbb{H}^2 \times_y \mathbb{R}$.

For every function $u$ of class $C^2$ defined on the domain $\Omega \subset \mathbb{H}^2$, we denote by $Gr(u) = \{(p, t) \in Sol_3 : p \in \Omega, t = u(p)\}$ a surface in $Sol_3$ and is called $\partial_t$-graph of $u$. $Gr(u)$ is a minimal surface if and only if $u$ satisfies the equation (see Proposition 2.5)
\[ \mathcal{H}u := \text{div} \left( \frac{y^2 \nabla u}{\sqrt{1 + y^2 \| \nabla u \|^2}} \right) = 0. \]

We will consider the case that the boundary $\partial \Omega$ is composed of the families of "convex" arcs $\{A_i\}$, $\{B_j\}$ and $\{C_k\}$. We give necessary and sufficient conditions on the geometry of the domain $\Omega$ which assure the existence of a minimal solution $u$ defined in $\Omega$ and $u$ assumes the value $+\infty$ on each $A_i$, $-\infty$ on each $B_j$ and prescribed continuous data on each $C_k$.

We see that $\partial_t$ is Killing and normal to the plane $\mathbb{H}^2$. A special point of the problem is that the vector field $\partial_t$ is not unitary. The important point to note here is that when $\gamma$ is a curve in $\mathbb{H}^2$, if $\gamma$ is a geodesic of $\mathbb{H}^2$, the surface $\gamma \times \mathbb{R}$ is no longer minimal in this warped product Riemannian manifold $Sol_3$. Instead of this, $\gamma \times \mathbb{R}$ is minimal in $Sol_3$ if and only if $\gamma$ is an Euclidean geodesic (see Corollary 2.2). Hence, these Euclidean geodesics will play an important role in our problem. Moreover, because of the non-unitary field $\partial_t$, we don’t use the hyperbolic length to state our problem. In $M^2 \times \mathbb{R}$ the length of a compact curve $\gamma \subset M^2$ is just the area of $\gamma \times [0, 1]$ in which we are interested. However, for a curve $\gamma \in \mathbb{H}^2$, the area calculated in $Sol_3$ of $\gamma \times [0, 1]$ is the Euclidean length of $\gamma$ (see Proposition 2.3).

The problem of type Jenkins-Serrin is also solved for some unbounded domains. The main idea in [2] is to approximate an unbounded domain $\Omega$ by a sequence bounded domain $\Omega_n$ by cutting $\Omega$ with horocycles.
In our case, we use the Euclidean geodesics, Euclidean length instead of the geodesics and the hyperbolic length, so we can’t use the horocycle of \( \mathbb{H}^2 \) to consider the problem de type Jenkins-Serrin on an unbounded domain. However, we can generalize the previous result for some unbounded domains by defining the flux for the non-compact arcs instead of using the horocycles. Our main result (Theorem 6.1) may be stated as follows.

**Theorem.** Let \( \Omega \) be a Scherk domain in \( \mathbb{H}^2 \) with the families of Euclidean geodesic arcs \( \{ A_i \} \), \( \{ B_i \} \) and of Euclidean mean convex arcs \( \{ C_i \} \).

(i) If the family \( \{ C_i \} \) is non-empty, there exists a solution to the Dirichlet problem on \( \Omega \) if and only if

\[
2a_{\text{euc}}(P) < \ell_{\text{euc}}(P), \quad 2b_{\text{euc}}(P) < \ell_{\text{euc}}(P)
\]

for every Euclidean polygonal domain inscribed in \( \Omega \). Moreover, such a solution is unique if it exists.

(ii) If the family \( \{ C_i \} \) is empty, there exists a solution to the Dirichlet problem on \( \Omega \) if and only if

\[
a_{\text{euc}}(P) = b_{\text{euc}}(P)
\]

when \( P = \Omega \) and the inequalities in (i) hold for all other Euclidean polygonal domains inscribed in \( \Omega \). Such a solution is unique up to an additive constant, if it exists.

We will have similar result for the Dirichlet problem for the minimal surface equation in \( \text{Sol}_3 \) with respect to \( \partial_x \)-graph. In the case of \( \partial_y \)-graph (\( \partial_y \) is not a Killing vector field), Ana Menezes solved on some "small" squares in the \((x, t)\)-plane with data \(+\infty\) on opposite two sides and \(-\infty\) on the other two sides (see [12, Theorem 2]).

We have organized the contents as follows: In Section 2 we will review some of the standard facts on \( \text{Sol}_3 \) and establish minimal surface equations. Section 3 will prove the maximum principle for the minimal surface equations, show the existence of solutions. A local Scherk surface in \( \text{Sol}_3 \) will be constructed in section 4. Sections 5 will be devoted to proving the monotone convergence theorem and describing the divergence set. Our main results are stated and proved in Section 6.

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2. Preliminaries

2.1. A model of $\text{Sol}_3$. The three-dimensional homogeneous Riemannian manifold $\text{Sol}_3$ can be viewed as $\mathbb{R}^3$ endowed with the Riemannian metric

$$ds^2 = e^{2x_3}dx_1^2 + e^{-2x_3}dx_2^2 + dx_3^2$$

where $(x_1, x_2, x_3)$ are canonical coordinates of $\mathbb{R}^3$ (see for instance [3] and the references given there for more details). The space $\text{Sol}_3$ has a Lie group structure with respect to which the above metric is left-invariant. The group structure is given by the multiplication

$$(x_1, x_2, x_3) \cdot (y_1, y_2, y_3) = (x_1 + e^{-x_3}y_1, x_2 + e^{x_3}y_2, x_3 + y_3).$$

In this paper, we don’t use the Lie group structure. The change of coordinates $x := x_2, \quad y := e^{x_3}, \quad t := x_1,$ turns this model into $\text{Sol}_3 = \{(x, y, t) \in \mathbb{R}^3 : y \geq 0\}$ with the Riemannian metric

$$ds^2 = \frac{dx^2 + dy^2}{y^2} + y^2dt^2.$$ 

In the present paper, the model used for the hyperbolic plane is the Poincaré half-plane, that is,

$$\mathbb{H}^2 = \{(x, y) \in \mathbb{R}^2 : y > 0\}$$

endowed with the Riemannian metric $\frac{dx^2 + dy^2}{y^2}$. Hence $\text{Sol}_3$ has the form of a warped product $\text{Sol}_3 = \mathbb{H}^2 \times_y \mathbb{R}$. From (2.2) we have

$$\|\partial_x\| = \|\partial_y\| = \frac{1}{y}, \quad \|\partial_t\| = y, \quad (\partial_x, \partial_y) = (\partial_x, \partial_t) = (\partial_y, \partial_t) = 0.$$

Hence $\left\{ y\partial_x, y\partial_y, \frac{1}{y}\partial_t \right\}$ is an orthonormal frame of $\text{Sol}_3$. Translations along the $t$-axis

$$T_h : \text{Sol}_3 \rightarrow \text{Sol}_3, \quad (x, y, t) \mapsto (x, y, t + h)$$

are isometries. Therefore the vertical vector field $\partial_t$ is a Killing vector field. Note that $\partial_t$ is not unitary.

Let us denote by $\nabla$ the Riemannian connexion of $\text{Sol}_3$ and by $\nabla$ the one in $\mathbb{H}^2$. By using Koszul’s formula

$$2 \langle \nabla_X Y, Z \rangle = X \langle Y, Z \rangle + Y \langle Z, X \rangle - Z \langle X, Y \rangle$$

$$- \langle [X, Y], Z \rangle - \langle [Y, Z], X \rangle + \langle [Z, X], Y \rangle$$
for any vector field $X,Y,Z$ of $\text{Sol}_3$, we obtain

\begin{align}
\nabla_\partial_x \partial_x &= \frac{1}{y} \partial_y, \quad \nabla_\partial_x \partial_y = \nabla_\partial_y \partial_x = -\frac{1}{y} \partial_x, \quad \nabla_\partial_y \partial_y = -\frac{1}{y} \partial_y, \\
\nabla_\partial_y \partial_t &= -y^3 \partial_y, \quad \nabla_\partial_x \partial_t = \nabla_\partial_t \partial_x = \frac{1}{y} \partial_t, \\
\nabla_\partial_t \partial_t &= -y \partial_y.
\end{align}

(2.6) \quad (2.7) \quad (2.8)

Hence, the surfaces $\{ t = \text{const} \}$ and $\{ x = \text{const} \}$ are the totally geodesic surfaces in $\text{Sol}_3$ (Note that a totally geodesic submanifold $\Sigma \subset M$ is characterized by the fact that $\nabla_X Y$ is a tangent vector field of $\Sigma$ for all tangent vector fields $X,Y$ of $\Sigma$, where $\nabla$ is the Riemannian connexion of $M$). The surfaces $\{ y = \text{const} \}$ are minimal, are not totally geodesic surfaces and are isometric to $\mathbb{R}^2$.

2.2. Euclidean geodesic. Firstly, note that the vertical lines $\{ p \} \times \mathbb{R} \subset \text{Sol}_3$ with $p = (x,y) \in \mathbb{H}^2$ aren’t geodesics in $\text{Sol}_3$. Indeed, let $p = (x,y)$ be a point of $\mathbb{H}^2$. A unit speed parametrization of $\gamma := \{ p \} \times \mathbb{R}$ is $\gamma : \mathbb{R} \to \text{Sol}_3, \ t \mapsto (x,y,\frac{t}{y})$. One has $\gamma' = \frac{1}{y} \partial_t$. Thus $\frac{d}{dt} \gamma' = \nabla_{\gamma'} \partial_t \left( \frac{1}{y} \partial_t \right) = -y \partial_y$. Since $\frac{d}{dt} \gamma' \neq 0$, $\{ p \} \times \mathbb{R}$ is not a geodesic in $\text{Sol}_3$.

Proposition 2.1. Let $\gamma$ be a curve in $\mathbb{H}^2$. Then the mean curvature vector $\vec{H}_\gamma \times \mathbb{R}$ in $\text{Sol}_3$ is

$$\vec{H}_\gamma \times \mathbb{R} = y^2 \vec{\kappa}_{\text{euc}},$$

where $\vec{\kappa}_{\text{euc}}$ is Euclidean mean curvature vector of $\gamma$ in $\mathbb{H}^2$.

Proof. We first compute $\vec{H}_\gamma \times \mathbb{R}$. Without loss of generality we can assume that $\gamma$ is a unit speed curve. So $\{ \frac{1}{y} \partial_t, \gamma' \}$ is an orthonormal frame of $\gamma \times \mathbb{R}$. The mean curvature vector of $\gamma \times \mathbb{R}$ is by definition

$$\vec{H}_{\gamma \times \mathbb{R}} = \left( \nabla_{\frac{1}{y} \partial_t} \left( \frac{1}{y} \partial_t \right) + \nabla_{\gamma'} \gamma' \right)^\perp$$

$$= (-y \partial_y + \nabla_{\gamma'} \gamma')^\perp$$

$$= -y \partial_y + \vec{\kappa},$$

where $\vec{\kappa}$ is the mean curvature vector of $\gamma$ in $\mathbb{H}^2$.

We now compute the Euclidean mean curvature vector $\vec{\kappa}_{\text{euc}}$ of $\gamma$ in $\mathbb{H}^2$. By Koszul’s formula (2.5)

$$\nabla_{\text{euc}} Y = \nabla Y + \frac{1}{y} \left( (Xy)Y + (Yy)X - \langle X,Y \rangle \nabla y \right)$$

(2.10)
where $\nabla_{\text{euc}}$ is the Riemannian connexion of $\mathbb{H}^2$ with respect to the Euclidean metric and $X, Y$ are tangent vector fields of $\mathbb{H}^2$. Hence

$$\left((\nabla_{\text{euc}})_X Y\right)^\perp = \left(\nabla_X Y\right)^\perp - \frac{1}{y} \langle X, Y \rangle \left(\nabla y\right)^\perp$$

where $X, Y$ are tangent vector fields of $\gamma$. Since $\gamma$ is a unit speed curvature, $\|\gamma'\| = 1$ and $\left\|\frac{\gamma'}{y}\right\|_{\text{euc}} = 1$. By (2.11)

$$\kappa_{\text{euc}} = \left(\nabla_{\text{euc}} \frac{\gamma'}{y}\right)^\perp = \left(\nabla \frac{\gamma'}{y}\right)^\perp - \frac{1}{y} \left\langle \frac{\gamma'}{y}, \frac{\gamma'}{y} \right\rangle \left(\nabla y\right)^\perp = \frac{1}{y^2} \kappa - \frac{1}{y} \partial_y^\perp$$

Hence

$$y^2 \kappa_{\text{euc}} = \kappa - y \partial_y^\perp.$$

Combining this equality with (2.9), we complete the proof. \hfill $\Box$

Let us mention two important consequences of the proposition.

**Corollary 2.2.** Let $\gamma$ be a curve in $\mathbb{H}^2$ and $\Omega$ be a domain in $\mathbb{H}^2$ with $\partial \Omega \in C^2$. Then

1. $\gamma \times \mathbb{R}$ is a minimal surface in $\text{Sol}_3$ if and only if $\gamma$ is an Euclidean geodesic in $\mathbb{H}^2$. However, these Euclidean geodesics need not have constant speed parametrization.
2. $\Omega \times \mathbb{R}$ is a mean convex set in $\text{Sol}_3$ if and only if $\Omega$ is an Euclidean mean convex in $\mathbb{H}^2$.

**Proposition 2.3.** Let $\gamma$ be a curve in $\mathbb{H}^2$. Then the area calculated in $\text{Sol}_3$ of $\gamma \times [0, 1]$ is

$$\mathcal{A}(\gamma \times [0, 1]) = \ell_{\text{euc}}(\gamma),$$

where $\ell_{\text{euc}}(\gamma)$ is the Euclidean length of $\gamma$.

**Proof.** Let us first compute the area of $\gamma \times [0, 1]$. The surface $\gamma \times [0, 1]$ in $\text{Sol}_3$ is defined by

$$\gamma \times [0, 1]: [0, 1] \times [0, 1] \to \text{Sol}_3, \quad (t_1, t_2) \mapsto (\gamma(t_1), t_2).$$
We have by definition
\[
\mathcal{A}(\gamma \times [0, 1]) = \int_{[0,1] \times [0,1]} \| (\gamma \times [0, 1])_{t_1} \times (\gamma \times [0, 1])_{t_2} \| \, dt_1 \, dt_2 \\
= \int_0^1 \int_0^1 \| \gamma'(t_1) \| y(\gamma(t_1)) \, dt_1 \, dt_2 \\
= \int_0^1 \| \gamma'(t_1) \| y(\gamma(t_1)) \, dt_1 = \int_\gamma y \, ds.
\]
The Euclidean length of \( \gamma \) is by definition
\[
\ell_{\text{euc}}(\gamma) = \int_\gamma ds = \int_\gamma y \, ds.
\]
Combining these equalities we conclude that
\[
\mathcal{A}(\gamma \times [0, 1]) = \int_\gamma y \, ds = \ell_{\text{euc}}(\gamma).
\]
This establishes the formula. \( \square \)

The ideal boundary of \( \mathbb{H}^2 \) is by definition
\[
\partial_{\infty} \mathbb{H}^2 = \{ (x, y) \in \mathbb{R}^2 : y = 0 \} \cup \{ \infty \}.
\]
The point \( \infty \) of \( \partial_{\infty} \mathbb{H}^2 \) is specified in our model of \( \text{Sol}_3 \) and we make the distinction with points in \( \{ y = 0 \} \).

**Definition 2.4.** A point \( p \in \partial_{\infty} \mathbb{H}^2 \) is called *removable* (resp. *essential*) if \( p \in \{ (x, y) \in \mathbb{R}^2 : y = 0 \} \) (resp. \( p = \infty \)).

**2.3. The minimal surface equations.** Let \( \Omega \) be a domain in \( \mathbb{H}^2 \) and \( u \) be a \( C^2 \)-function on \( \Omega \). Using the previous model for \( \mathbb{H}^2 \), we can consider the surface \( \text{Gr}(u) \) in \( \text{Sol}_3 \) parametrized by
\[
(x, y) \mapsto (x, y, u(x, y)).
\]
Such a surface is called the vertical Killing graph of \( u \), it is transverse to the Killing vector field \( \partial_t \) and any integral curve of \( \partial_t \) intersect at most once the surface. The upward unit normal to \( \text{Gr}(u) \) is given by
\[
N = N_u = -\frac{y \nabla u + \frac{1}{y} \partial_t}{\sqrt{1 + y^2 \| \nabla u \|^2}},
\]
where \( \nabla \) is the hyperbolic gradient operator and \( \| - \| \) is the hyperbolic norm. Indeed, \( \text{Gr}(u) = \Phi^{-1}(0) \), where the function \( \Phi : \text{Sol}_3 \to \mathbb{R} \) is defined by \( \Phi(x, y, t) = t - u(x, y) \). So, \( \nabla \Phi \) is a normal vector field to \( \text{Gr}(u) \). Moreover, since \( \nabla t = \frac{1}{y^2} \partial_t \) and \( \langle \nabla u, \partial_t \rangle = 0 \), we have
\[
\nabla \Phi = \nabla t - \nabla u = \frac{1}{y^2} \partial_t - \nabla u, \quad \| \nabla \Phi \|^2 = \frac{1}{y^2} + \| \nabla u \|^2.
\]
This establishes the formula (2.12).

Denote
\[ W = W_u := \sqrt{1 + y^2 \|\nabla u\|^2}, \quad X_u := \frac{y\nabla u}{W}. \]

It follows that
\[ N = -X_u + \frac{1}{yW} \partial_t. \]

In the sequel, we will use this unit normal vector to compute the mean curvature of a Killing graph.

**Proposition 2.5.** Let \( \Omega \) be a domain in \( \mathbb{H}^2 \) and \( u \) be a \( C^2 \)-function on \( \Omega \).

The mean curvature \( H \) of the Killing graph of \( u \) satisfies:
\[ 2yH = \text{div} \left( \frac{y^2 \nabla u}{W} \right), \]
with \( \text{div} \) the divergence operator in the hyperbolic metric, and after expanding all terms:
\[ 2H = \frac{y^3}{W^3} \left( (1 + y^4 u_y^2) u_{xx} - 2y^4 u_x u_y u_{xy} + (1 + y^4 u_x^2) u_{yy} + 2y^4 u_y + 2y^4 u_{x} \right). \]

**Proof.** We extend the vector field \( N \) to the whole \( \Omega \times \mathbb{R} \) by using the expression given in (2.12). The mean curvature of the Killing graph \( \text{Gr}(u) \) of \( u \) is then given by \( 2H = \text{div}_{\text{Gr}(u)}(-N) \).

Since \( \partial_t \) is a Killing vector field, we have
\[ 2H = \text{div}_{\text{Sol}_{3}}(N) = \text{div}_{\text{Sol}_{3}}(X_u) - \text{div}_{\text{Sol}_{3}} \left( \frac{1}{yW} \partial_t \right). \]

Let us compute
\[ \text{div}_{\text{Sol}_{3}} \left( \frac{1}{yW} \partial_t \right) = \left< \nabla \frac{1}{yW}, \partial_t \right> + \frac{1}{yW} \text{div}_{\text{Sol}_{3}}(\partial_t) = 0 + 0 = 0, \]
\[ \text{div}_{\text{Sol}_{3}}(X_u) = \text{div}(X_u) + \left< \nabla \frac{1}{y} \partial_t, X_u, \frac{1}{y} \partial_t \right>. \]

Moreover
\[ \left< \nabla \frac{1}{y} \partial_t, X_u, \frac{1}{y} \partial_t \right> = \frac{1}{y^2} \left< \nabla \partial_t, X_u, \partial_t \right> = -\frac{1}{y^2} \left< X_u, \nabla \partial_t, \partial_t \right>, \]
\[ \nabla \partial_t = -y^3 \partial_y = -y\nabla y. \]

Combining these equalities we deduce that
\[ 2H = \text{div}(X_u) + \frac{1}{y} \left< X_u, \nabla y \right>. \]

It follows that
\[ 2yH = y \text{div}(X_u) + \left< X_u, \nabla y \right> = \text{div}(yX_u) = \text{div} \left( \frac{y^2 \nabla u}{W} \right). \]
This is the formula (2.15). Expanding (2.15) yields

\[2H = \frac{1}{y} \text{div} \left( \frac{y^2 \nabla u}{W} \right) = \frac{1}{y} \text{div} \left( \frac{y^4 u_x}{W} \partial_x + \frac{y^4 u_y}{W} \partial_y \right)\]

\[= \frac{1}{y} \cdot y^2 \left( \frac{\partial}{\partial x} \left( \frac{1}{y^2} \frac{y^4 u_x}{W} \right) + \frac{\partial}{\partial y} \left( \frac{1}{y^2} \frac{y^4 u_y}{W} \right) \right)\]

\[= \frac{y^3}{W^3} \left( (1 + y^4 u_x^2) u_{xx} - 2y^4 u_x u_y u_{xy} + (1 + y^4 u_y^2) u_{yy} + 2 \frac{u_y}{y} \right)\].

This completes the proof. \(\square\)

Thus the minimal surface equation for a function \(u\) can be written

(2.17) \[\mathfrak{M}u := \text{div}(yX_u) = 0,\]

(2.18) \[(1 + y^4 u_y^2) u_{xx} - 2y^4 u_x u_y u_{xy} + (1 + y^4 u_x^2) u_{yy} + 2 \frac{u_y}{y} = 0.\]

A function \(u \in C^2(\Omega)\) is said to be a minimal solution on \(\Omega\) mean that \(u\) satisfies \(\mathfrak{M}u = 0\) on this domain.

3. Maximum principle, Gradient estimate and Existence theorem

3.1. Maximum principle. A basic tool for obtaining the results of this work is the maximum principle for differences of minimal solutions.

Firstly, by applying the proof of [5, Theorem 10.1] we have

**Proposition 3.1** (Maximum principle). Let \(u_1, u_2\) be two \(C^2\)-functions on a domain \(\Omega \subset \mathbb{H}^2\). Suppose \(u_1\) and \(u_2\) satisfy \(\mathfrak{M}u_1 \geq \mathfrak{M}u_2\). Then \(u_2 - u_1\) cannot have an interior minimum unless \(u_2 - u_1\) is a constant.

It follows from this proposition that if \(u_1, u_2\) are functions of class \(C^2\) on a bounded domain \(\Omega \subset \mathbb{H}^2\) such that \(\mathfrak{M}u_1 \geq \mathfrak{M}u_2\), and \(\lim \inf(u_2 - u_1) \geq 0\) for any approach to the boundary \(\partial \Omega\) of \(\Omega\), then we have \(u_2 \geq u_1\) in \(\Omega\).

Indeed, assume the contrary that \(\{x \in \Omega : u_2(x) < u_1(x)\}\) is not empty. Since \(\lim \inf(u_2 - u_1) \geq 0\) for any approach to the boundary \(\partial \Omega\) and \(\Omega\) is bounded, \(u_2 - u_1\) has an interior minimum in \(\Omega\). By Proposition 3.1, \(u_2 - u_1\) is constant, a contradiction.

The following result (Theorem 3.3) is a remarkable strengthening of this situation.

In what follows, for a subset \(\Omega\) of \(\mathbb{H}^2\), we will denote by \(\partial_\infty \Omega\) the boundary of \(\Omega\) in \(\mathbb{H}_2 \cup \partial_\infty \mathbb{H}^2\).

**Definition 3.2.** A domain \(\Omega \subset \mathbb{H}^2\) is called admissible if its boundary \(\partial_\infty \Omega\) is composed of a finite number of open Euclidean convex arcs \(C_i\) in \(\mathbb{H}^2\) together with their endpoints. The endpoints of the arcs \(C_i\) are called vertices of \(\Omega\) and those in \(\partial_\infty \mathbb{H}^2\) are called ideal vertices of \(\Omega\). Assume in addition that,
the ideal vertices of this domain are removable points at infinity (see Figure 3.1).

**Proposition 3.3** (General maximum principle). Let $\Omega \subset \mathbb{H}^2$ be an admissible domain. Let $u_1, u_2$ be two minimal solutions on $\Omega$. Suppose that $\lim \sup (u_1 - u_2) \leq 0$ for any approach to the boundary of $\Omega$ except for its vertices. Then $u_1 \leq u_2$.

We should remark that this result is similar to the general maximum principle stated by Spruck [16, General Maximum Principle, page 3] (resp. Hauswirth-Rosenberg-Spruck [6, Theorem 2.2]) for constant mean curvature surfaces in $\mathbb{R}^2 \times \mathbb{R}$ (resp. in $\mathbb{H}^2 \times \mathbb{R}$ and $\mathbb{S}^2 \times \mathbb{R}$) in the case of the bounded domain $\Omega$ and by Collin-Rosenberg [2, Theorem 2] for minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$ in the case of the unbounded domain $\Omega$.

**Proof.** Assume the contrary, that the set $\{u_1 > u_2\}$ is non-empty.

Let $N$ and $\varepsilon$ be positive constants, with $N$ large and $\varepsilon$ small. Define

$$
\varphi = [u_1 - u_2 - \varepsilon]_0^N = \begin{cases} 
0 & \text{if } u_1 - u_2 - \varepsilon \leq 0, \\
 u_1 - u_2 - \varepsilon & \text{if } 0 < u_1 - u_2 - \varepsilon < N, \\
 N & \text{if } u_1 - u_2 - \varepsilon \geq N.
\end{cases}
$$

Then $\varphi$ is a continuous piecewise differentiable function in $\Omega$ satisfying $0 \leq \varphi < N$. Moreover $\nabla \varphi = \nabla u_1 - \nabla u_2$ in the set where $\varepsilon < u_1 - u_2 < N + \varepsilon$, and $\nabla \varphi = 0$ almost everywhere in the complement of this set. For each ideal
vertex $p$ of $\Omega$, we consider a sequence of nested ideal geodesics $H_{p,n}$ converging to $p$. By nested we mean that if $H_{p,n}$ is the component of $\mathbb{H}^2 \setminus H_{p,n}$ containing $p$ on its ideal boundary, then $H_{p,n+1} \subset H_{p,n}$. Assume $\overline{H}_{p_1,n} \cap \overline{H}_{p_2,n} = \emptyset$ for every different ideal vertices $p_1, p_2$ of $\Omega$. Define

$$\Omega_n = \Omega \setminus \left( \bigcup_{p \in E_1} \overline{D}_n(p) \cup \bigcup_{p \in E_2} \overline{H}_{p,n} \right), \quad \Gamma_1 = \partial \Omega_n \cap \partial \Omega, \quad \Gamma_2 = \partial \Omega_n \setminus \Gamma_1,$$

where $E_1$ (resp. $E_2$) is the set of vertices in $\mathbb{H}^2$ (resp. vertices at $\partial_{\infty} \mathbb{H}^2$) of $\Omega$ (see Figure 3.2).

It follows from definition that

(3.1) $\varphi = 0$ on a neighborhood of $\Gamma_1$, \quad $\ell_{\text{euc}}(\Gamma_2) \to 0$ as $n \to \infty$.

![Figure 3.2. The domain $\Omega_n$](image)

Define

$$J_n = \int_{\partial \Omega_n} \varphi y \langle X_{u_1} - X_{u_2}, \nu \rangle \, ds$$

where $\nu$ is the exterior normal to $\partial \Omega_n$, $W_{u_i} = \sqrt{1 + y^2 \|\nabla u_i\|^2}$ and $X_{u_i} = \frac{y \nabla u_i}{W_{u_i}}$, $i = 1, 2$.

**Assertion 3.1.** \hspace{1em} (i) $J_n \geq 0$ with equality if and only if $\nabla u_1 = \nabla u_2$ on the set $\{x \in \Omega_n : \varepsilon < u_1 - u_2 < N\}$.

(ii) $J_n$ is increasing as $n \to \infty$. 
Proof. By Divergence theorem, we have

$$J_n = \int_{\Omega_n} \text{div} \left( \varphi y (X_{u_1} - X_{u_2}) \right) \, dA$$

$$= \int_{\Omega_n} (y \nabla \varphi, X_{u_1} - X_{u_2}) \, dA + \int_{\Omega_n} \varphi \text{div}(y X_{u_1} - y X_{u_2}) \, dA.$$ 

By our assumptions,

$$\varphi \text{div}(y X_{u_1} - y X_{u_2}) = \varphi (M u_1 - M u_2) = 0.$$ 

Moreover, by formula (3.3) of Lemma 3.4

$$\langle y \nabla \varphi, X_{u_1} - X_{u_2} \rangle = \langle y \nabla u_1 - y \nabla u_2, \frac{y \nabla u_1}{W_{u_1}} - \frac{y \nabla u_2}{W_{u_2}} \rangle \geq 0.$$ 

and equality if and only if $y \nabla u_1 = y \nabla u_2$. Then

$$J_n = \int_{\Omega_n} \text{div} \left( \varphi y (X_{u_1} - X_{u_2}) \right) \, dA = \int_{\Omega_n} \langle y \nabla \varphi, X_{u_1} - X_{u_2} \rangle \, dA \geq 0$$

and $J_n = 0$ if $\nabla u_1 = \nabla u_2$. Since $\Omega_n$ is an increasing domain, $J_n$ is increasing. This proves the assertion.  

Assertion 3.2. $J_n = o(1)$ as $n \to \infty$.  

Proof. We have

$$J_n = \int_{\Gamma_1} \varphi y \langle X_{u_1} - X_{u_2}, \nu \rangle \, ds + \int_{\Gamma_2} \varphi y \langle X_{u_1} - X_{u_2}, \nu \rangle \, ds.$$ 

By (3.1), and $\|X_{u_i}\| \leq 1$, $i = 1, 2$; $0 \leq \varphi \leq N$, we have

$$\int_{\Gamma_1} \varphi y \langle X_{u_1} - X_{u_2}, \nu \rangle \, ds = 0$$

and

$$\left| \int_{\Gamma_2} \varphi y \langle X_{u_1} - X_{u_2}, \nu \rangle \, ds \right| = \left| \int_{\Gamma_2} \varphi \langle X_{u_1} - X_{u_2}, \nu \rangle \, ds_{\text{euc}} \right| \leq 2 N |s_{\text{euc}}(\Gamma_2)| = o(1) \quad \text{as } n \to \infty.$$ 

Assertion is then proved.  

It follows from the previous assertions that $\nabla u_1 = \nabla u_2$ on the set $\{ \varepsilon < u_1 - u_2 < N \}$. Since $\varepsilon > 0$ and $N$ are arbitrary, $\nabla u_1 = \nabla u_2$ whenever $u_1 > u_2$. So $u_1 = u_2 + c, (c > 0)$ in any nontrivial component of the set $\{ u_1 > u_2 \}$. Then the maximum principle (Theorem 3.1) ensures $u_1 = u_2 + c$ in $\Omega$ and by assumptions of the theorem, the constant must be nonpositive, a contradiction.  

□
Lemma 3.4. Let \( v_1, v_2 \) be two vectors in a finite dimensional Euclidean space. Then
\[
(v_1 - v_2) \left( \frac{v_1}{W_1} - \frac{v_2}{W_2} \right) = \frac{W_1 + W_2}{2} \left( \left\| \frac{v_1}{W_1} - \frac{v_2}{W_2} \right\|^2 + \left( \frac{1}{W_1} - \frac{1}{W_2} \right)^2 \right)
\]
where \( W_i = \sqrt{1 + \|v_i\|^2} \). In particular,
\[
(v_1 - v_2) \left( \frac{v_1}{W_1} - \frac{v_2}{W_2} \right) \geq \left\| \frac{v_1}{W_1} - \frac{v_2}{W_2} \right\|^2, \quad (v_1 - v_2) \left( \frac{v_1}{W_1} - \frac{v_2}{W_2} \right) \geq 0
\]
with equality at a point if and only if \( v_1 = v_2 \).

Proof. Let us compute
\[
(v_1 - v_2) \left( \frac{v_1}{W_1} - \frac{v_2}{W_2} \right) = \frac{\|v_1\|^2}{W_1} + \frac{\|v_2\|^2}{W_2} - \langle v_1, v_2 \rangle \left( \frac{1}{W_1} + \frac{1}{W_2} \right)
= W_1 - \frac{1}{W_1} + W_2 - \frac{1}{W_2} - \langle v_1, v_2 \rangle \left( \frac{1}{W_1} + \frac{1}{W_2} \right)
= (W_1 + W_2) \left( 1 - \frac{\langle v_1, v_2 \rangle}{W_1 W_2} - \frac{1}{W_1 W_2} \right)
= (W_1 + W_2) \left( \frac{1}{2} \left\| \frac{v_1}{W_1} - \frac{v_2}{W_2} \right\|^2 + \frac{1}{2W_1^2} + \frac{1}{2W_2^2} \right)
\]
This proves the lemma. \( \square \)

3.2. Gradient estimate. An important result concerning minimal solutions is a gradient estimate.

Theorem 3.5 (Interior gradient estimate). Let \( u \) be a nonnegative minimal solution on \( \Omega = B_R(p) \subset \mathbb{H}^2 \). Then there exists a constant \( C \) that depends only on \( p, R \) such that
\[
\| \nabla u(p) \| \leq f \left( \frac{u(p)}{R} \right), \quad f(t) = e^{C(1+t^2)}.
\]

The proof of this result is similar to the one of the gradient estimate proved by Spruck [17, Theorem 1.1] and Mazet [10, Proposition 16].

Before beginning the proof, let us make some preliminary computation.
Lemma 3.6. Let $u$ be a minimal solution on a domain $\Omega \subset \mathbb{H}^2$. Denote by $\Sigma$ the graph of $u$. Then

$$\nabla_\Sigma u = \frac{1}{y^2} \partial_t^\top, \quad \|\nabla_\Sigma u\|^2 = \frac{1}{y^2} \left( 1 - \frac{1}{W^2} \right) \quad \text{and} \quad \Delta_\Sigma u = \frac{2 \langle \partial_y, N \rangle}{W}$$

where the subscript $\Sigma$ signifies that we compute the object in the Riemannian metric of the surface $\Sigma$.

Proof. We have

$$\nabla_\Sigma u = \nabla_\Sigma t = (\nabla t)^\top = \frac{1}{y^2} \partial_t^\top.$$  

$$\partial_t^\top = \partial_t - \langle \partial_t, N \rangle N = \partial_t - \frac{y}{W} N.$$  

It follows that

$$\|\nabla_\Sigma u\|^2 = \frac{1}{y^4} \|\partial_t\|^2 = \frac{1}{y^4} \left( \|\partial_t\|^2 - \frac{y^2}{W^2} \|N\|^2 \right)$$

$$= \frac{1}{y^4} \left( y^2 - \frac{y^2}{W^2} \right) = \frac{y^2}{y^2} \left( 1 - \frac{1}{W^2} \right).$$

We continue to compute $\Delta_\Sigma u$

$$\Delta_\Sigma u = \text{div}_\Sigma (\nabla_\Sigma t) = \text{div}_\Sigma (\nabla t)^\top = \text{div}_\Sigma (\nabla t) + \left\langle 2H, \nabla t \right\rangle = \text{div}_\Sigma (\nabla t).$$

Moreover

$$\text{div}_\Sigma (\nabla t) = \text{div}_\Sigma \left( \frac{1}{y^2} \partial_t \right) = \left\langle \nabla_\Sigma \frac{1}{y^2}, \partial_t \right\rangle + \frac{1}{y^2} \text{div}_\Sigma (\partial_t) = -\frac{2}{y^3} \langle \nabla_\Sigma y, \partial_t \rangle.$$  

By using $\nabla_\Sigma y = \nabla y - \langle \nabla y, N \rangle N$, $\langle \partial_t, N \rangle = \frac{y}{W}$, we obtain

$$\text{div}_\Sigma (\nabla t) = \frac{2}{y^4} \langle \nabla y, N \rangle \langle \partial_t, N \rangle = \frac{2}{Wy^2} \langle \nabla y, N \rangle.$$  

We conclude that

$$\Delta_\Sigma u = \frac{2}{Wy^2} \langle \nabla y, N \rangle = \frac{2 \langle \partial_y, N \rangle}{W}.$$  

This completes the proof of the lemma. \hfill \Box

Since $\partial_t$ is a Killing vector field and $\frac{y}{W} = \langle \partial_t, N \rangle$, then

$$\Delta_\Sigma \frac{y}{W} = -\left( \|A\|^2 + \text{Ric}(N, N) \right) \frac{y}{W}.$$  

Lemma 3.7. Let $u$ be a minimal solution on a domain $\Omega \subset \mathbb{H}^2$. Denote by $\Sigma$ the graph of $u$ on the domain $\Omega$.

For each function $\varphi : \Omega \to \mathbb{R}$ then

$$\Delta_\Sigma \varphi = \Delta \varphi - \frac{y^2}{W^2} \langle \nabla_\Sigma \varphi, \nabla u \rangle + \frac{1}{y} \left( 1 - \frac{1}{W^2} \right) \langle \nabla \varphi, \nabla y \rangle.$$
Proof. We have

\[
\Delta_S \varphi = \text{div}_S \nabla_S \varphi = \text{div}_S \nabla \varphi + 2 \left\langle \nabla \varphi, \vec{H} \right\rangle = \text{div}_S \nabla \varphi
\]

= \text{div}_{\text{Sol}_3} \nabla \varphi - \left\langle \nabla_N \nabla \varphi, N \right\rangle.

Let us evaluate the terms in the right-hand side

\[
\text{div}_{\text{Sol}_3} \nabla \varphi = \text{div} \nabla \varphi + \frac{1}{y^2} \left\langle \nabla_{\partial_t} \nabla \varphi, \partial_t \right\rangle
= \Delta \varphi - \frac{1}{y^2} \left\langle \nabla_{\partial_t} \partial_t, \nabla \varphi \right\rangle = \Delta \varphi + \frac{1}{y} \langle \nabla \varphi, \nabla y \rangle.
\]

Since \( N = -\frac{\nabla u}{W} + \frac{\partial_t}{yW} \),

\[
\langle \nabla_N \nabla \varphi, N \rangle = \langle -\frac{\nabla u}{W} \nabla \varphi, -\frac{y \nabla u}{W} \rangle + \langle \nabla_{\partial_t} \nabla \varphi, \frac{\partial_t}{yW} \rangle
= \frac{y^2}{W^2} \langle \nabla_{\nabla u} \nabla \varphi, \nabla u \rangle + \frac{1}{y^2 W^2} \langle \nabla_{\partial_t} \nabla \varphi, \partial_t \rangle
= \frac{y^2}{W^2} \langle \nabla_{\nabla u} \nabla \varphi, \nabla u \rangle + \frac{1}{yW^2} \langle \nabla \varphi, \nabla y \rangle.
\]

It follows that

\[
\Delta_S \varphi = \Delta \varphi - \frac{y^2}{W^2} \langle \nabla_{\nabla u} \nabla \varphi, \nabla u \rangle + \frac{1}{y} \left( 1 - \frac{1}{W^2} \right) \langle \nabla \varphi, \nabla y \rangle,
\]

which completes the proof. □

Let us mention an important consequence of the lemma.

**Corollary 3.8.** Let \( \Omega \subset \mathbb{H}^2 \) be a bounded domain, let \( p \) be a point of \( \Omega \). Denote by \( d = d_{\mathbb{H}^2}(\cdot, p) \) the hyperbolic distance from a point in \( \Omega \) to \( p \). Let \( u \) be a minimal solution on \( \Omega \). There exists a constant \( C = C(\Omega, y) \) (\( C \) doesn't depend on the point \( p \) and the function \( u \)) such that

\[
(3.8) \quad \sup_{\Omega} |\Delta_S d^2| \leq C.
\]

Using the above computations, we are ready to write the proof.

**Proof of Theorem 3.5.** Let us denote \( \nu := \frac{y}{W} = \langle \partial_t, N \rangle \). By definition, \( \partial_t = \partial_t^T + \nu N \).

We define an operator on \( \Sigma \)

\[
(3.9) \quad Lf := \Delta_S f - 2\nu \left\langle \nabla_{\frac{1}{\nu} \cdot}, \nabla_S f \right\rangle.
\]
We remark that the maximum principle is true for $L$. We have
\[
\Delta_{\Sigma} \frac{1}{\nu} = -\frac{1}{\nu^2} \Delta_{\Sigma} \nu + \frac{2}{\nu^3} \|\nabla_{\Sigma} \nu\|^2
\]
\[
= -\frac{1}{\nu^2} \left( -\left( \text{Ric}(N, N) + \|A\|^2 \right) \nu \right) + \frac{2}{\nu^3} \left\| -\nu^2 \nabla_{\Sigma} \frac{1}{\nu} \right\|^2
\]
\[
= \left( \text{Ric}(N, N) + \|A\|^2 \right) \frac{1}{\nu} + 2\nu \left\| \nabla_{\Sigma} \frac{1}{\nu} \right\|^2.
\]
Therefore
\[
L \frac{1}{\nu} = \Delta_{\Sigma} \frac{1}{\nu} - 2\nu \left\langle \nabla_{\Sigma} \frac{1}{\nu}, \nabla_{\Sigma} \frac{1}{\nu} \right\rangle = \left( \text{Ric}(N, N) + \|A\|^2 \right) \frac{1}{\nu} \geq -\frac{2}{\nu}
\]
since $\text{Ric}_{\text{Sols}} \geq -2$ (see [3]). Let us define $h = \frac{1}{\nu}$ where $\eta$ is a positive function.
\[
L h = L \left( \frac{1}{\nu} \right) = \Delta_{\Sigma} \left( \frac{1}{\nu} \right) - 2\nu \left\langle \nabla_{\Sigma} \frac{1}{\nu}, \nabla_{\Sigma} \left( \frac{1}{\nu} \right) \right\rangle
\]
\[
= \left( \eta \Delta_{\Sigma} \frac{1}{\nu} \right) + 2 \left\langle \nabla_{\Sigma} \eta, \nabla_{\Sigma} \frac{1}{\nu} \right\rangle + \frac{1}{\nu} \Delta_{\Sigma} \eta
\]
\[
- 2\nu \left\langle \nabla_{\Sigma} \frac{1}{\nu}, \eta \nabla_{\Sigma} \frac{1}{\nu} + \frac{1}{\nu} \nabla_{\Sigma} \eta \right\rangle
\]
\[
= \eta L \frac{1}{\nu} + \frac{1}{\nu} \Delta_{\Sigma} \eta \geq (\Delta_{\Sigma} \eta - 2\eta) \frac{1}{\nu}.
\]
We define on $\Sigma$ the function
\[
\varphi(x) = \max \left\{ -\frac{u(x)}{2u(p)} + 1 - \varepsilon - \frac{d(x)^2}{R^2}, 0 \right\}
\]
where $d = d(-, p)$. By definition,
\[
\varphi(p) = \frac{1}{2} - \varepsilon, \quad 0 \leq \varphi \leq 1 - \varepsilon, \quad \text{supp}\varphi \subset\subset \Sigma.
\]
We define $\eta = e^{K\varphi} - 1$. We calculate $\eta'(\varphi) = Ke^{K\varphi}$, $\eta''(\varphi) = K^2 e^{K\varphi}$. Let $q$ such that
\[
h(q) = \sup_{\Omega} h > 0.
\]
At the point $q$, we have
\[
\Delta_{\Sigma} \eta - 2\eta = \left( \eta'(\varphi) \Delta_{\Sigma} \varphi + \eta''(\varphi) \|\nabla_{\Sigma} \varphi\|^2 \right) - 2 \left( e^{K\varphi} - 1 \right)
\]
\[
= e^{K\varphi} \left( K^2 \|\nabla_{\Sigma} \varphi\|^2 + K \Delta_{\Sigma} \varphi - 2 \right) + 2
\]
\[
\geq e^{K\varphi} \left( K^2 \|\nabla_{\Sigma} \varphi\|^2 + K \Delta_{\Sigma} \varphi - 2 \right).
By the definition of $\varphi$,

$$
\|\nabla \Sigma \varphi \|^2 = \left\| -\frac{\nabla \Sigma u}{2u(p)} - \frac{\nabla \Sigma d^2}{2u(p)^2} \right\|^2 = \left\| -\frac{\partial_i^T}{2u(p)y^2} + \frac{2d\partial_r^T}{R^2} \right\|^2
$$

$$
= \frac{1}{4u(p)^2y^2} \left( 1 - \frac{1}{W^2} \right) + \frac{4d^2}{R^2} \left\| \partial_r^T \right\|^2 + \frac{2d}{u(p)R^2y^2} \left\langle \partial_r^T, \partial_r^T \right\rangle
$$

$$
\geq \frac{1}{4u(p)^2y^2} \left( 1 - \frac{1}{W^2} \right) + 0 - \frac{2d}{u(p)R^2y^2} \left\langle \partial_r, N \right\rangle
$$

$$
= \frac{1}{4u(p)^2y^2} \left( 1 - \frac{1}{W^2} \right) \left( \frac{8yu(p)}{R} \frac{1}{W} \right)
$$

$$
(3.10) \geq \frac{1}{4u(p)^2y^2} \left( 1 - \frac{1}{W^2} \right) \left( \frac{8yu(p)}{R} \frac{1}{W} \right)
$$

since $d \leq R$. Hence, if $\frac{1}{W} \leq \min \left\{ \frac{1}{2}, \frac{R}{32yu(p)} \right\}$, we have

$$
\|\nabla \Sigma \varphi \|^2 \geq \frac{1}{8u(p)^2y^2}.
$$

Moreover

$$
\Delta \Sigma \varphi = \frac{\Delta \Sigma u}{2u(p)} - \frac{\Delta \Sigma d^2}{R^2}
$$

$$
= -\frac{1}{2u(p)} \left( \frac{2}{Wy^2} \left\langle \nabla y, N \right\rangle - \frac{\Delta \Sigma d^2}{R^2} \right)
$$

$$
= -\frac{1}{y^2u(p)^2} \left( \frac{\left\langle \nabla y, N \right\rangle}{W} u(p) + \frac{\Delta \Sigma d^2}{R^2} u(p)^2 \right)
$$

$$
\geq -\frac{1}{y^2u(p)^2} \left( C_1 u(p) + \frac{C_2}{R^2} u(p)^2 \right).
$$

$$
(3.11)
$$

Combining $3.10$ with $3.11$ yields

$$
K^2 \|\nabla \Sigma \varphi \|^2 + K \Delta \Sigma \varphi - 2
$$

$$
\geq \frac{1}{8u(p)^2y^2} K^2 - \frac{1}{y^2u(p)^2} \left( C_1 u(p) + \frac{C_2}{R^2} u(p)^2 \right) K - 2
$$

$$
\geq \frac{1}{8u(p)^2y^2} \left( K^2 - 8 \left( C_1 u(p) + \frac{C_2}{R^2} u(p)^2 \right) K - 8C_3 u(p)^2 \right).
$$

It follows that, if $K = \left( 8C_1 + \frac{C_3}{C_1} \right) u(p) + 8\frac{C_2}{R^2} u(p)^2$, we obtain $K^2 \|\nabla \Sigma \varphi \|^2 + K \Delta \Sigma \varphi - 1 > 0$, then, $Lh > 0$. By Maximum principle applied to $L$, it implies that the maximum of $h$ can only be attained at a point $q$ where

$$
\frac{1}{W(q)} \geq \min \left\{ \frac{1}{2}, \frac{R}{32yu(p)} \right\}.
$$
\[
\left( e^{K(\frac{1}{2} - \varepsilon)} - 1 \right) \frac{1}{\nu(p)} = h(p) \leq h(q) = \left( e^{K\varphi(q)} - 1 \right) \frac{1}{\nu(q)} \leq \min \left\{ \frac{e^K - 1}{\min \left\{ \frac{y(q)}{2}, \frac{R}{32u(p)} \right\}} \right\}.
\]

Letting \( \varepsilon \) tending to 0 we get
\[
\nu(p) \geq \min \left\{ \frac{y(q)}{4}, \frac{R}{64u(p)} \right\} e^{-\frac{K}{2}}.
\]

So
\[
\| \nabla u(p) \| \leq \max \left\{ \frac{4}{y(q)} \frac{64}{R} u(p) \right\} e^{\frac{1}{2}} \left( (8C_1 + \frac{6C_2}{R}) u(p) + \frac{C_2}{R^2} u(p)^2 \right).
\]

Then
\[
\| \nabla u(p) \| \leq e^{C(1+\varepsilon)^2}
\]
for \( C = C(R) \) large enough.

\[\square\]

3.3. **Existence theorem.** In this subsection, we give a result concerning the existence of a solution of the Dirichlet problem for the minimal surface equation (2.17).

By using interior gradient estimate (Theorem 3.5), elliptic estimate, and Arzelà-Ascoli theorem, we obtain the compactness theorem as follows.

**Theorem 3.9** (Compactness theorem). Let \( \{u_n\} \) be a sequence of minimal solutions on a domain \( \Omega \subset \mathbb{H}^2 \). Suppose that \( \{u_n\} \) is uniformly bounded on compact subsets of \( \Omega \). Then there exists a subsequence of \( \{u_n\} \) converging on compact subsets of \( \Omega \) to a minimal solutions on \( \Omega \).

**Theorem 3.10.** Let \( \Omega \subset \mathbb{H}^2 \) be a bounded domain with \( \partial \Omega \in C^2 \). Suppose that \( \Omega \) is Euclidean mean convex. Let \( f \in C_0^0(\partial \Omega) \) be a continuous function. Then there exists a unique minimal solution \( u \) on \( \Omega \) such that \( u = f \) on \( \partial \Omega \).

**Proof.** The uniqueness is deduced by Maximum principle, Theorem 3.3.

**Existence:** Let \( \alpha, \beta \) be two real numbers such that \( \alpha < f(x) < \beta, x \in \partial \Omega \). Since \( \Omega \subset \mathbb{H}^2 \) is a bounded Euclidean mean convex domain, \( M^3 := \mathbb{H} \times [\alpha, \beta] \) is a manifold of dimension 3, compact, and mean convex. Define a Jordan curve \( \sigma \subset \partial M^3 \)

\[\sigma = \{(x, f(x)) : x \in \partial \Omega \}.
\]

By Theorem of Meeks-Yau (see \cite[Theorem 1]{Meeks-Yau}, \cite[Theorem 6.28]{Meeks-Yau}), there exists a minimal surface \( \Sigma \)

\[\partial \Sigma = \sigma, \quad \Sigma := \Sigma \setminus \sigma \subset \Omega \times [\alpha, \beta].\]
Then, it is sufficient to show that $\Sigma$ is a graph. Suppose the contrary, that $\Sigma$ is not a graph. There exists a point $p \in \Sigma$ such that $\partial_t|_p \in T_p\Sigma$. By Corollary 2.2 there exists a unique Euclidean geodesic $\gamma$ such that two minimal surfaces $\Sigma$ and $\gamma \times \mathbb{R}$ are tangents at $P$.

Since $\Sigma$ is not invariant by translation along $\partial_t$, both two surfaces $\Sigma$, $\gamma \times \mathbb{R}$ are not coincide. By Theorem of local description for the Intersections of minimal surfaces [1, Theorem 7.3], in a neighborhood of $P$, the intersection of $\Sigma$ and $\gamma \times \mathbb{R}$ composed of $2m$ ($m \geq 2$) arcs meeting at $P$.

If there exists a cycle $\alpha$ in $\Sigma \cap \gamma \times \mathbb{R}$, then $\alpha$ is the boundary of a minimal disk in $\Sigma$. Thus we could touch this disk at an interior point with another minimal surface $\beta \times \mathbb{R}$, where $\beta$ is an Euclidean geodesic curve of $\mathbb{H}^2$, but this can not happen by the maximum principle. So each branch of these curves leaving $p$ must go to $\partial \Sigma$ and, as $\gamma \cap \partial \Omega$ has exactly two points, at least two of the branches go to the same point of $\partial \Sigma$. This yields a compact cycle $\alpha$ in $\Sigma \cap (\gamma \times \mathbb{R})$ and, by the same previous argument, we have a contradiction. $\square$

A function $u \in C^0(\Omega)$ will be called subsolution (resp. supersolution) in $\Omega$ if for every disk $D \Subset \subset \Omega$ and every function $h$ minimal solution in $D$ satisfying $u \leq h$ (resp. $u \geq h$) on $\partial D$, we also have $u \leq h$ (resp. $u \geq h$) in $D$. We will have the following properties of $C^0(\Omega)$ subsolution.

Remark 3.11. (i) A function $u \in C^2(\Omega)$ is a subsolution if and only if $\mathcal{M}u \leq 0$.

(ii) If $u$ is subsolution in a domain $\Omega$ and if $v$ is supersolution in a bounded domain $\Omega$ with $v \geq u$ on $\partial \Omega$, then $v \geq u$ on $\Omega$. To prove the latter assertion, suppose the contrary. Then at some point $p_0 \in \Omega$ we have

$$ (u - v)(p_0) = \sup_{\Omega} (u - v) = M \geq 0 $$

and we may assume there is a disk $D = \mathbb{D}(p_0)$ such that $u - v \neq M$ on $\partial D$. Denote by $\overline{u}, \overline{v}$ the minimal solutions respectively equal to $u, v$ on $\partial D$ by Theorem 3.10 one sees that

$$ M \geq \sup_{\partial D} (\overline{u} - \overline{v}) \geq (\overline{u} - \overline{v})(p_0) \geq (u - v)(p_0) = M $$

and hence the equality holds throughout. By the maximum principle for minimal solution it follows that $\overline{u} - \overline{v} \equiv M$ in $D$ and hence $u - v = M$ on $\partial D$, which contradicts the choice of $D$.

(iii) Let $u$ be subsolution in $\Omega$ and $D$ be a disk strictly contained in $\Omega$. Denote by $\overline{u}$ the minimal solution in $D$ satisfying $\overline{u} = u$ on $\partial D$. We define in $\Omega$ the minimal solution lifting of $u$ (in $D$) by

$$ U(p) = \begin{cases} 
\overline{u}(p), & p \in D \\
u(p), & p \in \Omega \setminus D.
\end{cases} $$

(3.12)
Then the function $U$ is also subsolution in $\Omega$. Indeed, consider an arbitrary disk $D' \subset \subset \Omega$ and let $h$ be a minimal solution in $D'$ satisfying $h \geq U$ on $\partial D'$. Since $u \leq U$ in $D'$ we have $u \leq h$ in $D'$ and hence $U \leq h$ in $D' \setminus D$. Since $U$ is minimal solution in $D$, we have by the maximum principle $U \leq h$ in $D \cap D'$. Consequently $U \leq h$ in $D'$ and $U$ is subsolution in $\Omega$.

(iv) Let $u_1, u_2, \ldots, u_N$ be subsolution in $\Omega$. Then the function $u(p) = \max\{u_1(p), \ldots, u_N(p)\}$ is also subsolution in $\Omega$. This is a trivial consequence of the definition of subsolution. Corresponding results for supersolution functions are obtained by replacing $u$ by $-u$ in properties (i), (ii), (iii) and (iv).

Now let $\Omega$ be bounded domain and $f$ be a bounded function on $\partial \Omega$. A function $u \in C^0(\overline{\Omega})$ will be called a subfunction (resp. superfunction) relative to $f$ if $u$ is a subsolution (resp. supersolution) in $\Omega$ and $u \leq f$ (resp. $u \geq f$) on $\partial \Omega$. By [3.11 (ii)], every subfunction is less than or equal to every superfunction. In particular, constant functions $\leq \inf_{\Omega} f$ (resp. $\geq \sup_{\Omega} f$) are subfunctions (resp. superfunctions). Denote by $S_f$ the set of subfunctions relative to $f$.

The basic result of the Perron method is contained in the following theorem.

**Proposition 3.12.** The function $u(p) = \sup_{v \in S_f} v(p)$ is a minimal solution in $\Omega$.

**Proof.** By the maximum principle any function $v \in S_f$ satisfies $v \leq \sup_{\partial \Omega} f$, so that $u$ is well defined. Let $q$ be an arbitrary fixed point of $\Omega$. By the definition of $u$, there exists a sequence $\{v_n\} \subset S_f$ such that $v_n(q) \to u(q)$. By replacing $v_n$ with $\max\{v_n, \inf f\}$, we may assume that the sequence $\{v_n\}$ is bounded. Now choose $R$ so that the disk $D = D_R(q) \subset \subset \Omega$ and define $V_n$ to be the minimal solution lifting of $v_n$ in $D$ according to (iii). Then $V_n \in S_f, V_n(q) \to u(q)$ and by Theorem 3.11 the sequence $\{V_n\}$ contains a subsequence $\{V_{n_k}\}$ converging uniformly in any disk $D_\rho(q)$ with $\rho < R$ to a function $v$ that is minimal solution in $D$. Clearly $v \leq u$ in $D$ and $v(q) = u(q)$.

We claim now that in fact $v = u$ in $D$. For suppose $v(\overline{q}) < u(\overline{q})$ at some $\overline{q} \in D$. Then there exists a function $\overline{v} \in S_f$ such that $v(\overline{q}) < \overline{v}(\overline{q})$. Defining $w_k = \max\{\overline{v}, V_{n_k}\}$ and also the minimal solution liftings $W_k$ as in (iii), we obtain as before a subsequence of the sequence $\{W_k\}$ converging to a minimal solution function $w$ satisfying $v \leq w \leq u$ in $D$ and $v(q) = w(q) = u(q)$. But then by the maximum principle we must have $v = w$ in $D$. This contradicts the definition of $\overline{v}$ and hence $u$ is minimal solution in $\Omega$. \qed

We will show the solution that we obtained (called the Perron solution) will be the solution of the Dirichlet problem as follows.

**Theorem 3.13.** Let $\Omega$ be a bounded admissible domain with $\{C_i\}$ the open arcs of $\partial \Omega$. Let $f_i \in C^0(C_i)$ be bounded functions. Assume $C_i$ are Euclidean
mean convex to $\Omega$ then there exists a unique minimal solution $u$ on $\Omega$ such that $u = f_i$ on $C_i$ for all $i$.

**Proof.** Let a function $f$ defined on $\partial \Omega$ such that $f(p) = f_i(p)$ if $p \in C_i$. Denote by $u$ the Perron solution relative to $\mathcal{M}$ and $f$. Fix $\xi \in C_i$, for some $i$. We must prove that

$$
\lim_{p \in \Omega, p \to \xi} u(p) = f(\xi).
$$

We construct the local barrier at $\xi$ as follows. For $r > 0$ small enough, consider the domain $\Omega \cap D_r(\xi)$. We approximate $\Omega \cap D_r(\xi)$ by $C^2$ (Euclidean mean convex) domain $\Omega_\xi \subset \Omega \cap D_r(\xi)$ by rounding each corner point of $\Omega \cap B_r(\xi)$. By Theorem 3.10, there exist minimal solutions $w^\pm \in C^2(\Omega_\xi) \cap C^0(\Omega_\xi)$ on $\Omega_\xi$ such that $w^\pm(\xi) = f(\xi)$ and

$$
\begin{cases}
    w_- \leq f \leq w_+ & \text{on } \partial \Omega_\xi \cap \partial \Omega, \\
    w_- \leq \inf f \leq \sup f \leq w_+ & \text{on } \partial \Omega_\xi \cap \Omega.
\end{cases}
$$

From the definition of $u$ and the fact that every subfunction is dominated by every superfunction, we have

$$
w_- \leq u \leq w_+, \quad \text{on } \Omega_\xi,
$$

we obtain (3.13). \qed

4. A local Scherk surface in $\text{Sol}_3$ and Flux formula

**4.1. A local Scherk surface in $\text{Sol}_3$.**

**Proposition 4.1.** Let $\Omega \subset \mathbb{H}^2$ be an Euclidean mean convex quadrilateral domain whose boundary $\partial \Omega$ is composed of two Euclidean geodesic arcs $A_1, A_2$ and two Euclidean geodesic arcs $C_1, C_2$. Suppose that

$$
\ell_{\text{euc}}(A_1) + \ell_{\text{euc}}(A_2) < \ell_{\text{euc}}(C_1) + \ell_{\text{euc}}(C_2).
$$

Let $f_i$ be a positive continuous function on $C_i$, $i = 1, 2$. Then there exists a minimal solution $u$ in $\Omega$ taking $+\infty$ on $A_i$ and $f_i$ on $C_i$.

This construction was motivated by [13, Theorem 2].

**Proof.** This proof is divided into two cases.

**Case 4.1**. Case $f_1 = f_2 = 0$.

**Proof.** Let $n$ be a fixed positive number. By Theorem 3.13 there exists a minimal solution $u_n$ in $\Omega$ taking $n$ on $A_i$ and 0 on $C_i$. By General maximum principle (Theorem 3.3), $0 \leq u_n \leq u_{n+1}$.

We will prove that the sequence $\{u_n\}$ is uniformly bounded on compact subsets $K$ of $\Omega \cup C_1 \cup C_2$.

We first construct minimal annulus.
Fix $h \in \mathbb{R}, h > 0$ and let $\Gamma_i$ be the curves that are the boundary of $C_i \times [0, h]$. Let $\Sigma^h_i$ be a minimal disk with boundary $\Gamma_i$. Then

$$A(\Sigma^h_i) = A(C_i \times [0, h]) = h \cdot \ell_{\text{euc}}(C_i).$$

Consider the annulus $\mathfrak{A}$ with boundary $\Gamma_1 \cup \Gamma_2$ (see Figure 4.1):

$$\mathfrak{A} = \Omega \cup T_h(\Omega) \cup \bigcup_{i=1}^2 (A_i \times [0, h])$$

where $T_h$ is defined by (2.4). Then

$$A(\mathfrak{A}) = 2A(\Omega) + h(\ell_{\text{euc}}(A_1) + \ell_{\text{euc}}(A_2)).$$

Therefore

$$A(\mathfrak{A}) - (A(\Sigma^h_1) + A(\Sigma^h_2)) \leq 2A(\Omega) + h(\ell_{\text{euc}}(A_1) + \ell_{\text{euc}}(A_2)) - \ell_{\text{euc}}(C_1) - \ell_{\text{euc}}(C_2).$$

By the hypothesis (4.1), $A(\mathfrak{A}) - (A(\Sigma^h_1) + A(\Sigma^h_2)) < 0$ if $h \geq h_0$ where $h_0$ is sufficiently large. Hence, $A(\mathfrak{A})$ is smaller than the sum of the areas of the disks $C_i \times [0, h]$, and by the Douglas criteria [8], there exists a least area minimal annulus $\mathfrak{A}(h)$ with boundary $\Gamma_1 \cup \Gamma_2$ for all $h \geq h_0$.

**Assertion 4.1.** The annulus $\mathfrak{A}(h)$ is an upper barrier for the sequence $\{	ext{Gr}(u_n)\}$ for all $n > 0$ and $h \geq h_0$. Moreover, the vertical projections of the annulus $\mathfrak{A}(h)$ is an exhaustion for $\Omega \cup C_1 \cup C_2$.

**Proof.** For the proof we refer the reader to [13, page 271, 272] or [14, page 126, 127]. \(\diamondsuit\)
In this assertion we conclude that the sequence \( \{ u_n \} \) is uniformly bounded on compact subsets of \( \Omega \cup C_1 \cup C_2 \). By the compactness (Theorem 3.9), the sequence \( \{ u_n \} \) converges on compact subsets of \( \Omega \) to a minimal solution \( u \) on \( \Omega \) which assumes the above prescribed boundary values on \( \partial \Omega \).

**Case 4.2. General case.**

**Proof.** For every \( n > 0 \), by applying Theorem 3.13, there exists a minimal solution \( u_n \) on \( \Omega \) with boundary values

\[
    u_n|_{A_i} = n, \quad u_n|_{C_i} = \min\{ n, f_i \}.
\]

By Maximum principle (Theorem 3.3), \( u_n \leq u_{n+1} \).

**Assertion 4.2.** The sequence \( u_n \) is uniformly bounded on every compact subset \( K \) of \( \Omega \cup C_1 \cup C_2 \).

**Proof.** Denote by \( K \) a compact subset of \( \Omega \cup C_1 \cup C_2 \). Then \( \varepsilon := \text{dist}(K, A_1 \cup A_2) > 0 \). We define a subdomain \( \Omega' \) of \( \Omega \) by the formula

\[
    \Omega' = \left\{ p \in \Omega : \text{dist}(p, A_1 \cup A_2) > \frac{\varepsilon}{2} \right\}.
\]

Let us denote \( C'_i = C_i \cap \partial \Omega' \) and \( A'_1 \cup A'_2 = A' := \Omega \cap \partial \Omega' \). (See Figure 4.2). It follows from the definition that \( K \) is a compact subset of \( \Omega' \cup C'_1 \cup C'_2 \). There is, by the previous case, a minimal solution \( w \) on \( \Omega' \) which obtain the values \( +\infty \) on \( A'_i \) and 0 on \( C'_i \).

![Figure 4.2. The domain \( \Omega' \)](image)

By the general maximum principle (Theorem 3.3), we have \( 0 \leq u_n \leq w + \sum_{i=1}^{2} \max_{C'_i} f_i \) on \( \Omega' \cup C'_1 \cup C'_2 \). Since \( K \) is a compact of \( \Omega' \cup C'_1 \cup C'_2 \), \( \{ u_n \} \) is uniformly bounded on \( K \).

It follows from the previous affirmation and the compactness theorem (Theorem 3.9) that, the sequence \( \{ u_n \} \) converges on each compact subset of \( \Omega \cup C_1 \cup C_2 \) to a solution \( u \) on \( \Omega \). Moreover, we have \( u|_{C_i} = \lim_n u_n|_{C_i} = f_i \) and \( u|_{A_i} = \lim_n u_n|_{A_i} = +\infty \). This completes the proof.
Proposition 4.2. Let $\Omega \subset \mathbb{H}^2$ be a bounded domain whose boundary $\partial\Omega$ is composed of an Euclidean geodesic arc $A$ and an Euclidean convex arc $C$ with their endpoints. Then, there exists a minimal solution $u$ in $\Omega$ taking $+\infty$ on $A$ and arbitrarily positive continuous function $f$ on $C$.

Proof. For every $n > 0$, by applying Theorem 3.13 there is a minimal solution $u_n$ on $\Omega$ with boundary values

$$u_n|_A = n, \quad u_n|_C = \min\{n, f\}.$$  

By Maximum principle, Theorem 3.3 $0 \leq u_n \leq u_{n+1}$ for every $n$.

Assertion 4.3. The sequence $\{u_n\}$ is uniformly bounded on every compact subset $K$ of $\Omega \cup C$.

Proof. Denote by $K$ a compact subset of $\Omega \cup C$. Then $\varepsilon := \text{dist}(K, A) > 0$. We define a subdomain $\Omega'$ of $\Omega$ by the formula

$$\Omega' = \left\{ p \in \Omega : \text{dist}(p, A) > \frac{\varepsilon}{2} \right\}.$$

Let us denote $C' = C \cap \partial\Omega'$ and $A' = \Omega \cap \partial\Omega'$. (See Figure 4.3). It follows from the definition that $K$ is a compact subset of $\Omega' \cup C'$. By Theorem 1.1 there is a quadrilateral and a minimal solution $w$ defined on this quadrilateral that the values of $w$ on its boundary are $+\infty$ and 0 (see Figure 4.3).

By General maximum principle (Theorem 3.3), we have $0 \leq u_n \leq w + \max_{C'} f$ on $\Omega' \cup C'$. Since $K$ is a compact subset of $\Omega' \cup C'$, $\{u_n\}$ is uniformly bounded on $K$.  

\[ \square \]
It follows from the previous affirmation, the compactness theorem [3.3] and the monotonicity of the sequence \( \{u_n\} \), that the sequence \( \{u_n\} \) converges on every compact subset of \( \Omega \cup C \) to a minimal solution \( u \) on \( \Omega \). Moreover, we have \( u|_C = \lim_n u_n|_C = f \) and \( u|_A = \lim_n u_n|_A = +\infty \). This completes the proof. \( \square \)

**Lemma 4.3.** Let \( \Omega \subset \mathbb{H}^2 \) be a bounded domain whose boundary \( \partial \Omega \) is composed of an Euclidean geodesic arc \( A \) and an open Euclidean convex arc \( C \) with their endpoints. Let \( K \) be a compact subset of \( \Omega \cup C \). There exists a real number \( M = M(K) \) such that if \( u \) is a minimal solution on \( \Omega \) that satisfies \( u \geq c \) (resp. \( u \leq c \)) on \( C \) (c is some real number), then \( u \geq c - M \) (resp. \( u \leq c + M \)) on \( K \).

**Proof.** Since \( K \) is a compact set of \( \Omega \cup C \), \( \varepsilon := \text{dist}(A, K) > 0 \). Define \( \Omega' = \{ x \in \Omega : \text{dist}(x, A) > \frac{\varepsilon}{2} \} \). We have \( \partial \Omega' = A' \cup C' \) where \( A' := \partial \Omega' \cap \Omega \) is an Euclidean geodesic arc and \( C' := \partial \Omega' \cap C \) a sub-arc of \( C \). It follows from definition that \( K \) is a compact set of \( \Omega' \cup C' \).

By Proposition 4.2, there exists a minimal solution \( w \) on \( \Omega' \) such that \( w|_{A'} = +\infty \) and \( w|_{C'} = 0 \). Define \( M = \sup_K w < \infty \), by the general maximum theorem, Theorem 3.3, we have \( u \geq c - w \) (resp. \( u \leq c + w \)) on \( \Omega' \). So, \( u \geq c - M \) (resp. \( u \leq c + M \)) on \( K \). This completes the proof. \( \square \)

**Corollary 4.4** (Straight line lemma). Let \( \Omega \subset \mathbb{H}^2 \) be a domain, let \( C \subset \partial \Omega \) be an Euclidean mean convex arc (convex towards \( \Omega \)) and \( u \) be a minimal solution in \( \Omega \). If \( u \) diverges to \( +\infty \) or \( -\infty \) as one approach \( C \) within \( \Omega \), then \( C \) is an Euclidean geodesic arc.

**Proof.** Assume the contrary, that there exists a minimal solution \( u \) sur \( \Omega \) that takes the value \( +\infty \) on \( C \) where \( C \) is not an Euclidean geodesic arc.

Take an strictly Euclidean mean convex subarc \( C' \) of \( C \). Let \( \Gamma(C') \) be an Euclidean geodesic arc of \( \mathbb{H}^2 \) joining the endpoints of \( C' \). Denote by \( \Omega' \) the domain delimited by \( C' \cup \Gamma(C') \). We can choose \( C' \) such that \( \Omega' \subset \Omega \). (See Figure 1.3).

Let \( q \) be a point in \( \Omega' \). It follows from the lemma 4.3 that there exists a real number \( M = M(q) \) such that \( u \geq c - M \) for all real number \( c \), a contradiction. \( \square \)

**Theorem 4.5** (Boundary values lemma [2 page 1882]). Let \( \Omega \subset \mathbb{H}^2 \) be a domain and let \( C \) be an Euclidean mean convex arc in \( \partial \Omega \). Suppose \( \{u_n\} \) is a sequence of solutions in \( \Omega \) that converges uniformly on every compact subset of \( \Omega \) to a minimal solution \( u \). Suppose each \( u_n \in C^0(\Omega \cup C) \) and \( u_n|_C \) converges uniformly on every compact subset of \( C \) to a function \( f \) on \( C \) where \( f \) is continuous or \( f \equiv +\infty \) or \( -\infty \). Then \( u \) is continuous on \( \Omega \cup C \) and \( u|_C = f \).
Proof. It is sufficient to show that, for $p \in C$ and $M \in \mathbb{R}$ such that $f(p) > M$, there exists a neighborhood $U$ of $p$ in $\Omega \cup C$ that satisfies $u > M$ on $U$.

Let $M'$ such that $M < M' < f(p)$. Since $f$ is continuous (or $f \equiv \infty$) and $u_n|_C$ converges uniformly on every compact subset of $C$ to $f$, there is a neighborhood $C'$ of $p$ in $C$ and a positive natural number $N_0$ such that $u_n(x) > M'$ for every $x \in C'$ and for every $n \geq N_0$. Consider two cases as follows.

(i) If $C$ is strictly Euclidean mean convex in a neighborhood of $p$ in $C$. Without loss of generality, we suppose that $C'$ is strictly Euclidean mean convex.

Denote by $\Gamma(C')$ an open Euclidean geodesic arc of $H^2$ joining the endpoints of $C'$. We can choose $C'$ such that $\Gamma(C') \subset \Omega$. Denote by $\Omega'$ the domain delimited by $C' \cup \Gamma(C')$. (See Figure 4.5).

By Proposition 4.2, there exists a minimal solution $w$ on $\Omega'$ such that $w|_{C'} = M'$ and $w|_{\Gamma(C')} = -\infty$. It follows from the general maximum principle.
(Theorem 3.3), that \( u_n \geq w \) on \( \Omega' \) for every \( n \geq N_0 \). Hence we have \( u \geq w \) on \( \Omega' \). Since \( w \) is continuous, there is a neighborhood \( U \) of \( p \) in \( \Omega' \) such that \( w > M \) on \( U \). Therefore \( u > M \) on \( U \).

(ii) If the arc \( C \) contains an Euclidean geodesic segment in a neighborhood of \( p \). Without loss of generality, we suppose that \( C' \) is an Euclidean geodesic arc.

Consider a quadrilateral \( \mathcal{P} \subset \Omega \) such that \( \partial \mathcal{P} \) is composed of 4 Euclidean geodesics \( B_1, C_1, B_2, C_2 \) where \( C_1 = C' \) and \( \ell_{\text{euc}}(B_1) + \ell_{\text{euc}}(B_2) < \ell_{\text{euc}}(C_1) + \ell_{\text{euc}}(C_2) \). (See Figure 4.6).

\[ F_u(\gamma) = \int_{\gamma} \langle yX_u, \nu \rangle \, ds, \]

\textbf{Figure 4.6.} The domain \( \Omega' \) when \( C' \) is Euclidean geodesic.

Since \( u_n \) converges uniformly on each compact subset of \( \Omega \) to \( u \), \( M'' := \inf_{x \in C_2, n \geq 1} u_n(x) > -\infty \). By Proposition 4.1 there is a minimal solution \( w \) on \( \mathcal{P} \) such that \( w|_{C_1} = M', w|_{C_2} = M'' \) and \( w = -\infty \) on \( B_1 \cup B_2 \). It follows from the general maximum principle, Theorem 3.3, that \( u_n \geq w \) on \( \Omega' \) for every \( n \geq N_0 \). Hence we have \( u \geq w \) on \( \Omega' \). Since \( w \) is continuous, there exists a neighborhood \( U \) of \( p \) in \( \Omega' \) such that \( w > M \) on \( U \). Then \( u > M \) on \( U \).

4.2. Flux formula. Let \( u \) be a minimal graph on a domain \( \Omega \subset \mathbb{H}^2 \). It follows from definition that \( \text{div}(yX_u) = 0 \), where \( X_u = \frac{y\nabla u}{\sqrt{1+y^2\|
abla u\|^2}} \) is a vector field on \( \Omega \), \( \|X_u\| < 1 \).

Denote by \( \gamma \) an arc in \( \partial \Omega \cap \mathbb{H}^2 \) such that its Euclidean length \( \ell_{\text{euc}}(\gamma) \) is finite. Denote by \( \nu \) a unit normal to \( \gamma \) in \( \mathbb{H}^2 \). Then, we define the flux \( F_u(\gamma) \) of \( u \) across \( \gamma \) by
\[ F_u(\gamma) = \int_{\gamma} \langle yX_u, \nu \rangle \, ds, \]
if $\gamma \subset \Omega$, if not, we define $F_u(\gamma) = F_u(\Gamma)$, where $\Gamma$ is an arc in $\Omega$ joining the end-points of $\gamma$ such that $\ell_{\text{euc}}(\Gamma) < \infty$. Clearly, $F_u(\gamma)$ changes sign if we choose $-\nu$ in place of $\nu$. In the case $\gamma \subset \partial \Omega$, $\nu$ will always be chosen to be the outer normal to $\partial \Omega$.

**Proposition 4.6.** Let $u$ be a minimal graph on a domain $\Omega \subset \mathbb{H}^2$.

(i) For every curve $\gamma$ in $\overline{\Omega}$ that $\ell_{\text{euc}}(\gamma) < \infty$ we have $|F_u(\gamma)| \leq \ell_{\text{euc}}(\gamma)$.

(ii) For every admissible domain $\Omega'$ of $\Omega$ such that $\ell_{\text{euc}}(\partial \Omega') < \infty$, we have $F_u(\partial \Omega') = 0$.

(iii) Let $\gamma$ be a curve in $\Omega$ or an Euclidean mean convex curve in $\partial \Omega$ on which $u$ is continuous, obtains the finite value and $\ell_{\text{euc}}(\gamma) < \infty$. Then $F_u(\gamma) < \ell_{\text{euc}}(\gamma)$.

(iv) Let $\gamma \subset \partial \Omega$ be an Euclidean geodesic arc such that $u$ diverges to $+\infty$ (resp. $-\infty$) as one approaches $\gamma$ within $\Omega$, then $F_u(\gamma) = \ell_{\text{euc}}(\gamma)$ (resp. $F_u(\gamma) = -\ell_{\text{euc}}(\gamma)$).

**Proof.** (i) - Case $\gamma \subset \Omega$. Since $\|X_u\| < 1$ we have

$$|F_u(\gamma)| \leq \int_{\gamma} |\langle yX_u, \nu \rangle| \, ds \leq \int_{\gamma} y \, ds = \ell_{\text{euc}}(\gamma).$$

- Case $\gamma \not\subset \Omega$. For every positive real number $\varepsilon$, there is a curve $\Gamma \subset \Omega$ such that $\ell_{\text{euc}}(\Gamma) \leq \ell_{\text{euc}}(\gamma) + \varepsilon$ and $F_u(\gamma) = F_u(\Gamma)$. Then,

$$|F_u(\gamma)| = |F_u(\Gamma)| \leq \ell_{\text{euc}}(\gamma) + \varepsilon.$$

This proved the result.

(ii) - Case $\Omega'$ is bounded. By divergence theorem, we have

$$F_u(\partial \Omega') = \int_{\partial \Omega'} \langle yX_u, \nu \rangle \, ds = \int_{\Omega'} \text{div}(yX_u) \, dA = 0.$$

- Case $\Omega'$ is unbounded. Denote by $E$ the set of ideal vertices of $\Omega'$. For each $p \in E$, we take a net of the geodesics $H_{p,n}$ that converges to $p$. (See Figure 4.7). Let us denote by $H_{p,n}$ a domain of $\mathbb{H}^2$ delimited by $H_{p,n}$ such that the Euclidean mean convex vector of $H_{p,n}$ pointing interior. We define

$$\Omega'_n = \Omega' \setminus \bigcup_{p \in E} H_{p,n}.$$ 

These subdomains of $\Omega'$ are bounded. It follows from the previous case that $F_u(\partial \Omega'_n) = 0$. Thus we have

$$F_u(\partial \Omega') = F_u(\partial \Omega') - F_u(\partial \Omega'_n) = \sum_{p \in E} F_u(\partial \Omega' \cap H_{p,n}) - F_u(\partial \Omega'_n \setminus \partial \Omega').$$

Since $\ell_{\text{euc}}(\partial \Omega') < \infty$, we have

$$\sum_{p \in E} |F_u(\partial \Omega' \cap H_{p,n})| \leq \sum_{p \in E} \ell_{\text{euc}}(\partial \Omega' \cap H_{p,n}) \to 0 \text{ as } n \to \infty.$$
Moreover
\[ |F_u(\partial \Omega'_n \setminus \partial \Omega')| \leq \ell_{euc}(\partial \Omega'_n \setminus \partial \Omega') \leq \sum_{p \in E} \ell_{euc}(H_{p,n}) \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty. \]

This completes the proof.

(iii) It is sufficient to show that \( F_u(\gamma) < \ell_{euc}(\gamma) \) for a small arc \( \gamma \). Let \( p \in \gamma \), there exists a positive \( \varepsilon \) such that \( \mathbb{D}_\varepsilon(p) \cap (\partial \Omega \setminus \gamma) = \emptyset \). Let \( \Omega_\varepsilon(p) := \Omega \cap \mathbb{D}_\varepsilon(p) \). (See Figure 4.8).

By the general existence theorem, there is a minimal solution \( v \) on \( \Omega_\varepsilon(p) \) with \( v = u + 1 \) on \( \gamma \) and \( v = u \) on \( \partial \Omega_\varepsilon(p) \setminus \gamma \).

It follows from the lemma 3.4 that
\[
\int_{\Omega_\varepsilon(p)} \langle \nabla v - \nabla u, yX_v - yX_u \rangle \, dA > 0.
\]

Since \( u, v \) are the minimal solutions
\[
\langle \nabla v - \nabla u, yX_v - yX_u \rangle = \text{div} \left( (v - w)(yX_v - yX_u) \right).
\]

By the divergence theorem, we have
\[
0 < \int_{\partial \Omega_\varepsilon(p)} \langle (v - w)(yX_v - yX_u), v \rangle \, ds = F_v(\gamma) - F_u(\gamma).
\]

Therefore
\[
F_u(\gamma) < F_v(\gamma) \leq \ell_{euc}(\gamma),
\]
which completes the proof.

(iv) We show for the case $u$ diverges to $+\infty$ as one approaches $\gamma$ within $\Omega$. Without loss of generality, we assume that $\gamma$ is compact. We first prove that

\[
\lim_{q \to p} N_{u-u(q)}(q) = -\nu(p), \quad \forall p \in \gamma.
\]

Assume the contrary that there exists a sequence $q_n \in \Omega$, $q_n \to p$ such that

\[
\lim_{n \to \infty} N_{u-u(q_n)}(q_n) = \nu \neq -\nu(p).
\]

Since $u|_{\gamma} = +\infty$, there exists $R > 0$ that satisfies the distance $d\Sigma(Q_n, \partial \Sigma) > R$, \forall $n$, where $\Sigma = \text{Gr}(u)$ and $Q_n = (q_n, u(q_n))$. Since $\Sigma$ is stable, we deduce from Schoen's curvature estimate [15] or [1, Theorem 2.10] that

\[
\|A(q_n)\| \leq \kappa \quad \forall q_n \in \mathbb{D}^\Sigma_{R/2}(Q_n)
\]

where $A$ is the second fundamental form of $\Sigma$ and $\kappa$ is an absolute constant.

Hence, by [1, Lemma 2.4], around each $Q_n$ the surface $\Sigma$ is a graph over a disk $\mathbb{D}_r(Q_n)$ of the tangent plane at $Q_n$ of $\Sigma$ and the graph has bounded distance from the disk $\mathbb{D}_r(Q_n)$. The radius of the disk depends only on $R$, hence it is independent of $n$. So, if $q_n$ is close enough to $\gamma$, then the horizontal projection of $\mathbb{D}_r(Q_n)$ and thus of the surface $\Sigma$ is not contained in $\Omega$, contradiction.

For $\delta > 0$ sufficiently small, we take $\Omega_\delta \subset \{ q \in \Omega : d\Omega(q, \gamma) < \delta \}$ such that for each point $q \in \Omega_\delta$, there will exist a unique point $p = p(q) \in \gamma$ such that $d\Omega(q, p) = d\Omega(q, \gamma)$. By using the diffeomorphism $\Omega_\delta \to \gamma \times [0, \delta)$, \( q \mapsto (p(q), d(q, \gamma)) \), we can extend $\nu$ on $\Omega_\delta$. By (4.2),

\[
\lim_{q \in \Omega, q \to \gamma} \langle X_u(q), \nu(q) \rangle = 1.
\]

Denote by $p_1, p_2$ two end-points of $\gamma$. Define $\gamma_\epsilon = \{ q \in \Omega_\delta : d(q, \gamma) = \epsilon \}$ for every $0 < \epsilon < \delta$. Denote by $q_1, q_2$ two end-points of $\gamma_\epsilon$ such that $d(p_i, q_i) = \epsilon$. The domain $\Omega_\epsilon(p)$.
\[ F_u(\gamma) = F_u(p_1q_1) + F_u(\gamma_\varepsilon) + F_u(q_2p_2). \]

Since
\[ F_u(p_iq_i) \geq -\ell_{\text{euc}}(p_iq_i) \to 0, \quad \varepsilon \to 0, \quad i = 1, 2, \]
and by (4.3)
\[ F_u(\gamma_\varepsilon) = \int_{\gamma_\varepsilon} \langle X_u(q), \nu(q) \rangle \, ds_{\text{euc}}(q) \to \ell_{\text{euc}}(\gamma) \quad \text{as} \quad \varepsilon \to 0, \]
then
\[ F_u(\gamma) \geq \ell_{\text{euc}}(\gamma). \]
Therefore
\[ F_u(\gamma) = \ell_{\text{euc}}(\gamma). \]
This completes the proof. \(\square\)

**Proposition 4.7.** Let \(\{u_n\}\) be a sequence of minimal graphs on a fixed domain \(\Omega \subset \mathbb{H}\) which extends continuously to \(\partial \Omega\) and let \(A\) be an Euclidean geodesic arc in \(\partial \Omega\) such that \(\ell_{\text{euc}}(A) < \infty\). Then

(i) If \(\{u_n\}\) diverges uniformly to \(+\infty\) on compact sets of \(A\) and while remaining uniformly bounded on compact sets of \(\Omega\), then
\[ \lim_{n \to \infty} F_{u_n}(A) = \ell_{\text{euc}}(A). \]

(ii) If \(\{u_n\}\) diverges uniformly to \(+\infty\) on compact sets of \(\Omega\) while remaining uniformly bounded on compact sets of and \(A\), then
\[ \lim_{n \to \infty} F_{u_n}(A) = -\ell_{\text{euc}}(A). \]

5. **Monotone convergence theorem and Divergence set theorem**

5.1. **Monotone convergence theorem.**

**Theorem 5.1 (Local Harnack inequality).** Let \(u\) be a nonnegative minimal solution on \(\Omega = \mathbb{D}_{\mathcal{H}}(P) \subset \mathbb{H}^2\) and let \(Q\) be a point of \(\Omega\). There is a function \(\Phi(t, r)\) (that does not depend on the function \(u\)) such that
\[ |u(Q)| \leq \Phi(|u(P)|, d(P, Q)). \]
For each fixed \(t\), \(\Phi(t, \cdot)\) is a continuous strictly increasing function defined on a interval \([0, r(t)]\) with
\[ \Phi(t, 0) = t, \quad \lim_{r \to r(t)} \Phi(t, r) = \infty \]
where \(t \mapsto r(t)\) is a continuous strictly deacreasing function tending to zero as \(t\) tends to infinity.

**Proof.** Let \(\gamma : [0, R] \to \Omega\) be an geodesic arc that satisfies
\[ \gamma(0) = P, \quad \|\gamma'\| = 1, \quad Q \in \gamma([0, R]). \]
Define \(\dot{u} : [0, R] \to \mathbb{R}\) by condition
\[ \dot{u}(r) = u(\gamma(r)). \]
By Theorem 3.5

\[ \hat{u}'(r) \leq f \left( \frac{\hat{u}(r)}{R-r} \right). \]

For each \( t > 0 \), we define a function \( r \mapsto \Phi(t, r) \) by the conditions

\[ \frac{d\Phi}{dr}(t, r) = f \left( \frac{\Phi(t, r)}{R-r} \right), \quad \Phi(t, 0) = t. \]

Then \( \hat{u}(r) \leq \Phi(\hat{u}(0), r) \) whenever \( \Phi \) is well defined. \( \square \)

Theorem 5.2 (Dini’s monotone convergence theorem). If \( X \) is a compact topological space, and \( \{f_n\} \) is a monotonically increasing sequence (meaning \( f_n(x) \leq f_{n+1}(x) \) for all \( n \) and \( x \)) of continuous real-valued functions on \( X \) which converges pointwise to a continuous function \( f \), then the convergence is uniform. The same conclusion holds if \( \{f_n\} \) is monotonically decreasing instead of increasing.

Theorem 5.3 (Monotone convergence theorem). Let \( \{u_n\} \) be a monotone increasing sequence of minimal graphs on a domain \( \Omega \subset \mathbb{H}^2 \). There exists an open set \( U \subset \Omega \) (called the convergence set) such that \( \{u_n\} \) converges uniformly on compact subsets of \( U \) and diverges uniformly to \( +\infty \) on compact subsets of \( V := \Omega \setminus U \) (divergence set). Moreover, if \( u_n \) is bounded at a point \( p \in \Omega \), then the convergence set \( U \) is non-empty (it contains a neighborhood of \( p \)).

Proof. Let \( \{u_n\} \) be an increasing sequence. Denote by \( P \) a point in \( U := \{x \in \Omega : \sup_{n \geq 0} |u_n(x)| < \infty\} \). There is a positive number \( R \) such that

\[ \mathbb{D}_R(P) \subset \Omega, \quad C := \inf_{x \in \mathbb{D}_R(P)} u_1(x) > -\infty. \]

Let \( m := -C + \sup_{n \geq 0} u_n(P) \). The function \( \Phi \) is well defined on the interval \([0, r(m))\). Define \( \varepsilon := \min \left\{ \frac{r(m)}{2}, R \right\} \). For each \( Q \in \mathbb{D}_\varepsilon(P) \), by using the local Harnack inequality, we have

\[ -C + u_n(Q) \leq \Phi(-C + u_n(P), d(P, Q)) \leq \Phi \left( m, \frac{r(m)}{2} \right). \]

By definition, \( \mathbb{D}_\varepsilon(P) \subset U \). Then \( U \) is open. \( \square \)

5.2. Divergence set theorem.

Theorem 5.4 (Divergence set theorem \( \mathcal{V} \)). Let \( \Omega \subset \mathbb{H}^2 \) be a admissible domaine whose boundary is composed with finitely Euclidean mean convex arcs \( C_i \). Let \( \{u_n\} \) be an increasing or decreasing sequence of minimal graphs on \( \Omega \), respectively. Then, for each open arc \( C_i \), we assume that, for every \( n \), \( u_n \) extends continuously on \( C_i \) and either \( \{u_n|_{C_i}\} \) converges to a continuous
function or $\infty$ or $-\infty$, respectively. Let $V = V(\{u_n\})$ be the divergence set associated to $\{u_n\}$.

(i) The boundary of $V$ consists of the union of a set of non-intersecting interior Euclidean geodesic chords in $\Omega$ joining two points of $\partial \Omega$, together with arcs in $\partial \Omega$. Moreover, a component of $V$ cannot be an isolated point.

(ii) A component of $V$ cannot be an interior chord.

(iii) No two interior chords in $\partial V$ can have a common endpoint at a convex corner of $V$.

(iv) The endpoints of interior Euclidean geodesic chords are among the vertices of $\partial \Omega$. So the boundary of $V$ has a finite set of interior Euclidean geodesic chords in $\Omega$ joining two vertices of $\partial \Omega$.

Proof. Without loss of generality, assume that the sequence $\{u_n\}$ is increasing and the divergence set is not empty.

(i) It is clear by Lemma 4.3 and Corollary 4.4 that each arc of $\partial V$ must be Euclidean geodesic and that no vertex of $\partial V$ lies in $\Omega$, then (i) follows.

(iii) Assume the contrary that (iii) does not hold. Let $\gamma_1, \gamma_2$ be two arcs of $\partial V$ having a common endpoint $p \in \partial V$ at a convex corner. Choose two points $q_i \in \gamma_i, i = 1, 2$ such that the triangle $\triangle$ with vertices $p, q_1, q_2$ lies in $\Omega$. We can always assume that the triangle $\triangle$ is either in $U$ or in $V$. Indeed, if $\triangle \not\subset V$, we take a component $\triangle'$ of $U \cap \triangle$. Let $\gamma'_1, \gamma'_2$ be two Euclidean geodesic chords in $\Omega$ having a common endpoint $p$ such that the domain delimited by them is the smallest domain containing $\triangle'$. Then $\gamma'_1, \gamma'_2 \subset \partial V$ and $\triangle'$ is the triangle delimited by $\gamma'_1, \gamma'_2$ and $\overline{q_1q_2}$ and $\triangle' \subset U$. We can choose $\gamma'_1, \gamma'_2$ in place of $\gamma_1, \gamma_2$. By the lemma 4.7,

$$0 = F_{u_n}(\partial \triangle) = F_{u_n}(\overline{pq_1}) + F_{u_n}(\overline{pq_2}) + F_{u_n}(\overline{q_1q_2}),$$

$$\lim_{n \to \infty} F_{u_n}(\overline{pq_i}) = \begin{cases} \ell_{\text{euc}}(\overline{pq_i}) & \text{if } \triangle \subset U \\ -\ell_{\text{euc}}(\overline{pq_i}) & \text{if } \triangle \subset V \end{cases} \quad i = 1, 2.$$

On the other hand $\lim_{n \to \infty} |F_{u_n}(\overline{q_1q_2})| \leq \ell_{\text{euc}}(\overline{q_1q_2})$. Hence

$$\ell_{\text{euc}}(\overline{q_1q_2}) \geq \ell_{\text{euc}}(\overline{q_1q_2}) + \ell_{\text{euc}}(\overline{q_1q_2}),$$

a contradiction.

(ii) and (iv) are proved with analogous arguments, using Lemma 4.3 and Corollary 4.4. The details are left to the reader.

\[ \square \]

6. Jenkins-Serrin type theorem

Let $\Omega \subset \mathbb{H}^2$ be a domain whose boundary $\partial_\infty \Omega$ consists of a finite number of Euclidean geodesic arcs $A_i, B_i$, a finite number of Euclidean mean convex arcs $C_i$ (towards $\Omega$) together with their endpoints, which are called the vertices of $\Omega$. We mark the $A_i$ edges by $+\infty$ and the $B_i$ edges by $-\infty$, and assign arbitrary continuous data $f_i$ on the arcs $C_i$, respectively. Assume that no two
A_i edges and no two B_i edges meet at a convex corner. We call such a domain \( \Omega \) \textit{Scherk domain}. (See Figure 6.1.) Assume in addition that, the vertices at infinity of Scherk domain are the removable points at infinity.

**Figure 6.1. An example of Scherk domain**

An \textit{Euclidean polygonal domain} \( \mathcal{P} \) in \( \mathbb{H}^2 \) is a domain whose boundary \( \partial_{\infty} \mathcal{P} \) is composed of finitely many Euclidean geodesic arcs in \( \mathbb{H}^2 \) together with their endpoints, which are called the vertices of \( \mathcal{P} \).

An Euclidean polygonal domain \( \mathcal{P} \) is said to be inscribed in a Scherk domain \( \Omega \) if \( \mathcal{P} \subset \Omega \) and its vertices are among the vertices of \( \Omega \). We notice that a vertex may be in \( \partial_{\infty} \Omega \) and an edge may be one of the \( A_i \) or \( B_i \). (See Figure 6.2).

Given a polygonal domain \( \mathcal{P} \) inscribed in \( \Omega \), we denote by \( \ell_{\text{euc}}(\mathcal{P}) \) the Euclidean perimeter of \( \partial \mathcal{P} \), and by \( a_{\text{euc}}(\mathcal{P}) \) and \( b_{\text{euc}}(\mathcal{P}) \) the total Euclidean lengths of the edges \( A_i \) and \( B_i \) lying in \( \partial \mathcal{P} \), respectively.

Now is a good time to state and to prove the main theorem of this paper.

**Theorem 6.1.** Let \( \Omega \) be a Scherk domain in \( \mathbb{H}^2 \) with the families \( \{A_i\}, \{B_i\}, \{C_i\} \).

(i) \textit{If the family } \{C_i\} \textit{ is non-empty, there exists a solution to the Dirichlet problem on } \Omega \textit{ if and only if}

\begin{equation}
2a_{\text{euc}}(\mathcal{P}) < \ell_{\text{euc}}(\mathcal{P}), \quad 2b_{\text{euc}}(\mathcal{P}) < \ell_{\text{euc}}(\mathcal{P})
\end{equation}

for every Euclidean polygonal domain inscribed in \( \Omega \). Moreover, such a solution is unique if it exists.
(ii) If the family \( \{C_i\} \) is empty, there exists a solution to the Dirichlet problem on \( \Omega \) if and only if
\[
a_{\text{euc}}(P) = b_{\text{euc}}(P)
\]
when \( P = \Omega \) and the inequalities in (6.1) hold for all other Euclidean polygonal domains inscribed in \( \Omega \). Such a solution is unique up to an additive constant, if it exists.

This theorem is similar in spirit to that of [7, 13, 2, 14].

Proof. The uniqueness of the solution is deduced from Theorem 6.2.

Let us now prove that the conditions of theorem 6.1 are necessary for the existence. Assume that there is a minimal graph \( u \) on \( \Omega \) satisfying the Dirichlet problem. When \( \{C_i\} = \emptyset \) and \( P = \Omega \), using the proposition 4.6 we have
\[
0 = F_u(\partial P) = \sum_{A_i \subset \partial P} F_u(A_i) + \sum_{B_i \subset \partial P} F_u(B_i) = \sum_{A_i \subset \partial P} \ell_{\text{euc}}(A_i) + \sum_{B_i \subset \partial P} -\ell_{\text{euc}}(B_i) = a_{\text{euc}}(P) - b_{\text{euc}}(P),
\]
as the condition (6.2).

In the other case, \( \partial P \setminus \left( \bigcup_{A_i \subset \partial P} A_i \cup \bigcup_{B_i \subset \partial P} B_i \right) \neq \emptyset \) and \( u \) is continuous on this set. By Proposition 4.6 we have
\[ 0 = F_u(\partial P) \]
\[ = \sum_{A_i \subset \partial P} F_u(A_i) + \sum_{B_i \subset \partial P} F_u(B_i) + F_u\left(\partial P \setminus \left(\bigcup_{A_i \subset \partial P} A_i \cup \bigcup_{B_i \subset \partial P} B_i\right)\right), \]
\[ \sum_{A_i \subset \partial P} F_u(A_i) = \sum_{A_i \subset \partial P} \ell_{\text{euc}}(A_i) = a_{\text{euc}}(P), \]
\[ \sum_{B_i \subset \partial P} F_u(B_i) = \sum_{B_i \subset \partial P} -\ell_{\text{euc}}(B_i) = -b_{\text{euc}}(P) \]

and
\[ \left| F_u\left(\partial P \setminus \left(\bigcup_{A_i \subset \partial P} A_i \cup \bigcup_{B_i \subset \partial P} B_i\right)\right) \right| < \ell_{\text{euc}}\left(\partial P \setminus \left(\bigcup_{A_i \subset \partial P} A_i \cup \bigcup_{B_i \subset \partial P} B_i\right)\right) \]
\[ = \ell_{\text{euc}}(P) - a_{\text{euc}}(P) - b_{\text{euc}}(P). \]

We obtain \( |a_{\text{euc}}(P) - b_{\text{euc}}(P)| < \ell_{\text{euc}}(P) - a_{\text{euc}}(P) - b_{\text{euc}}(P). \) It follows the conditions (6.1).

Finally, we prove that the conditions of theorem 6.1 are sufficient. We distinguish the following cases:

**Case 6.1.** First case: Assume that the families \( \{A_i\} \) and \( \{B_i\} \) are both empty and the continuous functions \( f_i \) are bounded.

**Proof.** For any ideal vertex \( p \) of \( \Omega \), we take a net of geodesics \( H_{p,n} \) which converges to \( p \). Denote by \( H_{p,n} \) the domain of \( \mathbb{H}^2 \) delimited by \( H_{p,n} \) such that the Euclidean mean convex vector of \( H_{p,n} \) points interior. Let us define \( \Omega_n \) an Euclidean convex subdomain of \( \Omega \) delimited by \( \partial \Omega \setminus \bigcup_i H_{i,n} \) and by the Euclidean geodesics in \( \Omega \cap \bigcup_i H_{i,n} \) joining the points of \( \partial \Omega \cap \bigcup_i H_{i,n} \).

By Theorem 3.13 for each positive natural number \( n \), there exists a minimal solution \( u_n \) on an Euclidean polygonal domain of \( \Omega_n \) such that

\[ u_n = \begin{cases} f_i & \text{on } C_i \cap \partial \Omega_n, \\ 0 & \text{on the rest of } \partial \Omega_n. \end{cases} \]

By Maximum theorem, Theorem 3.3 the sequence \( \{u_n\} \) is uniformly bounded on \( \Omega \). By Compactness theorem, Theorem 3.9 there exists a subsequence of the sequence \( \{u_{n_k}\} \) converges to a minimal solution \( u : \Omega \to \mathbb{R} \) that obtains the values \( f_i \) on \( C_i \).

**Case 6.2.** Second case: The family \( \{B_i\} \) is empty and the functions \( f_i \) are non-negative.
Proof. There exists, by the previous step 6.1, for each \( n \), a minimal solution \( u_n \) on \( \Omega \) such that
\[
u_n = \begin{cases} n & \text{on } \bigcup_i A_i \\ \min\{n, f_i\} & \text{on } C_i \end{cases}
\]
It follows from the maximum principle, Theorem 5.3, that \( 0 \leq u_n \leq u_n + 1 \) for each \( n \).

**Assertion 6.1.** The divergence set \( \mathcal{V} = \mathcal{V}(\{u_n\}) \) is empty.

**Proof.** Assume the contrary, that \( \mathcal{V} \) is not empty. By the lemma 4.4 and Theorem 5.4, \( \mathcal{V} \) consists of a finite number of Euclidean polygonal domains inscribed in \( \Omega \). Let \( P \) be a component of \( \mathcal{V} \). By Lemmas 4.6 and 4.7, we have
\[
0 = F_{u_n}(\partial P) = \sum_i F_{u_n}(A_i \cap \partial P) + F_{u_n}\left( \partial P \setminus \bigcup_i A_i \right),
\]
\[
\left| \sum_i F_{u_n}(A_i \cap \partial P) \right| \leq \sum_i |F_{u_n}(A_i \cap \partial P)| \leq \sum_i \ell_{\text{euc}}(A_i) = a_{\text{euc}}(P),
\]
\[
l_{\text{euc}}\left( \partial P \setminus \bigcup_i A_i \right) = \ell_{\text{euc}}\left( \partial P \setminus \bigcup_i A_i \right) = -(\ell_{\text{euc}}(P) - a_{\text{euc}}(P)).
\]
We conclude that \( \ell_{\text{euc}}(P) - a_{\text{euc}}(P) \leq a_{\text{euc}}(P) \), which contradicts with the condition (6.1).

By the previous assertion, we have \( \mathcal{U}(\{u_n\}) = \Omega \). Thus \( \{u_n\} \) converges uniformly on the compact sets of \( \Omega \) to a minimal solution \( u \). By Theorem 4.5, \( u \) takes the values \(+\infty\) on \( A_i \) and \( f_i \) on \( C_i \).

**Case 6.3.** Third case: the family \( \{C_i\} \) is non-empty.

**Proof.** By the previous step, 6.1 and 6.2, there exists the minimal solutions \( u^+ \), \( u^- \) and \( u_n \) on \( \Omega \) with the following boundary values
\[
\begin{array}{c|c|c|c}
u^- & A_i & B_i & C_i \\
+\infty & 0 & \max\{f_i, 0\} \\

\begin{array}{c|c|c}
u_n & n & -n \\
0 & f_i|_{-n} & \min\{f_i, 0\}
\end{array}
\]
It follows from Theorem 6.2 that \( u^- \leq u_n \leq u^+ \) for each \( n \). By the compactness theorem, Theorem 3.9 and a diagonal process, we can extract a subsequence of \( \{u_n\} \) which converges on compact sets of \( \Omega \) to a minimal graph \( u \). Moreover, by Theorem 4.5, \( u \) takes the desired boundary conditions.

**Case 6.4.** Fourth case: The family \( \{C_i\} \) is empty.
Proof. We fix a positive natural number $n$. There exists, by Case 6.1 a
minimal solution $v_n$ on $\Omega$ that obtains the values $n$ on $A_i$ and 0 on $B_i$. It
follows from Theorem 6.2 that $0 \leq v_n \leq n$. For each $c \in (0, n)$, we define
$$E_c = \{ v_n > c \}, \quad F_c = \{ v_n < c \}.$$ Since $v_n = n$ on $A_i$, there exists a component $E_i^c$ of $E_c$ satifying $A_i \subset \partial E_i^c$. Moreover, by the maximum principle, Theorem 6.2 $E_c = \bigcup_i E_i^c$. Similarly, there exists, for each $i$, a component $F_i^c$ of $F_c$ satifying $B_i \subset \partial F_i^c$, and, we have $F_c = \bigcup_i F_i^c$. A detailed proof can be found in [2, Proof of Theorem 1].
We define
$$\mu_n = \inf \{ c \in (0, n) : \text{the set } F_c \text{ is connex} \} , \quad u_n = v_n - \mu_n.$$ By definition, $u_n$ is a minimal solution on $\Omega$ which take the values $n - \mu_n$ on $A_i$ and $-\mu_n$ on $B_i$.

Assertion 6.2. There exist two piecewise minimal solutions $u^+, u^-$ on $\Omega$ such that $u^- \leq u_n \leq u^+$ for every $n$.

Proof. There exist, by the case 6.2 the minimal solutions $u_i^\pm$ on $\Omega$ such that
$$u_i^+ = \begin{cases} \infty & \text{on } \bigcup_{i' \neq i} A_{i'}, \\ 0 & \text{on } A_i \cup \bigcup_j B_j \end{cases}, \quad u_i^- = \begin{cases} -\infty & \text{on } \bigcup_{i' \neq i} B_{i'}, \\ 0 & \text{on } B_i \cup \bigcup_j A_j \end{cases}.$$ Define
$$u^+ = \max_i u_i^+, \quad u^- = \min_i u_i^-.$$ Observe that, by definition of $\mu_n$, both $E_{\mu_n}$ and $F_{\mu_n}$ are disconnected. In particular, for every $i_1$, there exists an $i_2$ such that $E_{\mu_n}^{i_1} \cap E_{\mu_n}^{i_2} = \emptyset$ and we obtain, applying the maximum principle,
$$0 \leq u_n|_{E_{\mu_n}^{i_1}} \leq u_{i_2}^+|_{E_{\mu_n}^{i_1}}.$$ Similarly, for every $j_1$, there exists an $j_2$ such that $F_{\mu_n}^{j_1} \cap F_{\mu_n}^{j_2} = \emptyset$ and we obtain, applying the maximum principle,
$$u_{j_2}^-|_{F_{\mu_n}^{j_1}} \leq u_n|_{F_{\mu_n}^{j_1}} \leq 0.$$ It follows that $u^- \leq u_n \leq u^+$ for every $n$.

By the previous assertion and the compactness theorem, Theorem 3.9, there exists a subsequence $\{ u_{\sigma(n)} \}$ of $\{ u_n \}$ that converges on compact sets of $\Omega$ to a minimal solution $u$.

Assertion 6.3.
$$\lim_{n \to \infty} \mu_{\sigma(n)} = \infty, \quad \lim_{n \to \infty} (n - \mu_{\sigma(n)}) = \infty.$$
Proof. Assume the contrary, that there exists a subsequence \( \{ \mu_{\sigma(n)} \} \) of \( \{ \mu_{\sigma(n)} \} \) that converges to some \( \mu_\infty \). Then, by definition of \( u \), that \( u \) takes the values \( \infty \) on \( A_i \) and \( -\mu_\infty \) on \( B_i \). So, by the proof of necessity, \( 2a_{\text{euc}}(\Omega) < \ell_{\text{euc}}(\Omega) \), which contradicts with hypothesis \( 6.1 \). Then \( \lim_{n \to \infty} \mu_{\sigma(n)} = \infty \). In the same way, we can show that \( \lim_{n \to \infty} (n - \mu_{\sigma(n)}) = \infty \). Thus, by the previous assertion, we conclude \( u \) takes +\( \infty \) on \( A_i \) and \( -\infty \) on \( B_i \).

\( \heartsuit \) This completes the proof of the existence part of the theorem. \( \square \)

The remainder of this section will be devoted to the proof of the uniqueness of Theorem 6.1.

Theorem 6.2. (Maximum principle for unbounded domains with possible infinite boundary data) Let \( \Omega \subset \mathbb{H}^2 \) be a Scherk domain. Let \( u_1, u_2 \) be two solutions of type Jenkins-Serrin on \( \Omega \). If the family \( \{ C_i \} \) is non-empty, assume that \( \limsup (u_1 - u_2) \leq 0 \) when ones approach to \( \bigcup_i C_i \). If \( \{ C_i \} \) is empty, suppose that \( u_1 \leq u_2 \) at some point \( p \in \Omega \). Then in either case \( u_1 \leq u_2 \) on \( \Omega \). Proof. Assume the contrary, that the set \( \{ u_1 > u_2 \} \) is not empty.

Let \( N, \varepsilon \) be two positive constants with \( N \) large, \( \varepsilon \) small. Define a function

\[
\varphi = [u_1 - u_2 - \varepsilon]^{N-\varepsilon}_0 = \begin{cases} N - \varepsilon & u_1 - u_2 \geq N \\ u_1 - u_2 - \varepsilon & \varepsilon < u_1 - u_2 < N \\ 0 & u_1 - u_2 \leq \varepsilon. \end{cases}
\]

Then \( \varphi \) is Lipschitz and vanishes in a neighborhood of any point of \( C_i \), \( 0 \leq \varphi < N \) and \( \nabla \varphi = \nabla u_1 - \nabla u_2 \) on the set \( \{ \varepsilon < u_1 - u_2 < N \} \). Moreover, \( \nabla \varphi = 0 \) almost everywhere in the complement of this set. For each ideal vertex \( p \) of \( \Omega \), we take a net of geodesics \( H_{p,n} \) that converges to \( p \). Denote by \( H_{p,n} \) the domain of \( \mathbb{H}^2 \) delimited by \( H_{p,n} \) such that the Euclidean mean convex vector of \( H_{p,n} \) points interior. Define

\[
\Omega_{n,\delta} = \Omega \setminus \left( \overline{B}_\delta(\partial \Omega) \cup \bigcup_{p \in E_1} \overline{B}_\delta(p) \cup \bigcup_{p \in E_2} \overline{H}_{p,n} \right),
\]

\[
X_i' = \partial \Omega_{n,\delta} \cap \overline{B}_\delta(X_i), \quad (X \in \{ A, B, C \})
\]

and

\[
\Gamma = \partial \Omega_{n,\delta} \setminus \left( \bigcup_i A_i' \cup \bigcup_i B_i' \cup \bigcup_i C_i' \right),
\]

where \( E_1 \) (resp. \( E_2 \)) is the set of vertices (resp. ideal vertices) of \( \Omega \) and \( 0 < \delta = \delta(n) \ll \frac{1}{n} \). (See Figure 6.3.) Define

\[
J_n = \int_{\partial \Omega_{n,\delta}} \varphi \langle X_{u_1} - X_{u_2}, \nu \rangle \, ds,
\]
where \( \nu \) is the exterior normal to \( \partial \Omega_{n, \delta} \).

**Assertion 6.4.**

(i) \( J_n \geq 0 \), equality if and only if \( \nabla u_1 = \nabla u_2 \) on the set \( \{ x \in \Omega_{n, \delta} : \varepsilon < u_1 - u_2 < N \} \).

(ii) \( J_n \) is increasing as \( n \to \infty \).

**Proof.** By Divergence theorem, we have

\[
J_n = \int_{\Omega_{n, \delta}} \text{div} (\varphi(y(X u_1 - X u_2))) \, dA
\]

\[
= \int_{\Omega_{n, \delta}} (y \nabla \varphi, X u_1 - X u_2) \, dA + \int_{\Omega_{n, \delta}} \varphi \text{div}(yX u_1 - yX u_2) \, dA.
\]

By the hypotheses, we obtain

\[
\varphi \text{div}(yX u_1 - yX u_2) = \varphi(\Re u_1 - \Re u_2) = 0.
\]

Moreover, by Lemma 3.4

\[
\langle y \nabla \varphi, X u_1 - X u_2 \rangle = \left\langle y \nabla u_1 - y \nabla u_2, \frac{y \nabla u_1}{W_{u_1}} - \frac{y \nabla u_2}{W_{u_2}} \right\rangle \geq 0.
\]

\[\diamondsuit\]

**Assertion 6.5.** \( J_n = o(1) \) as \( n \to \infty \).
Proof. We have
\[
J_n = \sum_i \int_{A_i'} \varphi y \left( X_{u_1} - X_{u_2}, \nu \right) \, ds + \sum_i \int_{B_i'} \varphi y \left( X_{u_1} - X_{u_2}, \nu \right) \, ds \\
+ \sum_i \int_{C_i'} \varphi y \left( X_{u_1} - X_{u_2}, \nu \right) \, ds + \int_\Gamma \varphi y \left( X_{u_1} - X_{u_2}, \nu \right) \, ds.
\]

Since \( \varphi = 0 \) on a neighborhood of \( \bigcup_i C_i \), we have
\[
\sum_i \int_{C_i'} \varphi y \left( X_{u_1} - X_{u_2}, \nu \right) \, ds = 0.
\]

Moreover, since \( \|X_{u_i}\| \leq 1, i = 1, 2; \varphi \leq N \)
\[
\left| \int_\Gamma \varphi y \left( X_{u_1} - X_{u_2}, \nu \right) \, ds \right| \leq 2N \ell_{\text{euc}}(\Gamma) = o(1).
\]

By Lemma 4.6 then
\[
\int_{A_i'} \varphi y \left( X_{u_1} - X_{u_2}, \nu \right) \, ds \leq N (\ell_{\text{euc}}(A_i') - F_{u_2}(A_i'));
\]
\[
\int_{B_i'} \varphi y \left( X_{u_1} - X_{u_2}, \nu \right) \, ds \leq N (\ell_{\text{euc}}(B_i') + F_{u_1}(B_i')).
\]

For every \( i \) and \( X \in \{A, B\} \), denote by \( \Omega X \) a component of \( \Omega \setminus \Omega_{n, \delta} \) such that \( X_i' \subset \partial \Omega_i^X \) and define \( X_i'' = \partial \Omega_i^X \cap X_i \). By Lemma 4.6 we have
\[
0 = F_u(\partial \Omega_i^X) = F_u(X_i'') - F_u(X_i') + F_u(\partial \Omega_i^X \setminus (X_i' \cup X_i'')),
\]
\[
F_u(X_i'') = \begin{cases} \ell_{\text{euc}}(A_i'') = \ell_{\text{euc}}(A_i') + o(1) & \text{si } X_i = A_i, \\ \ell_{\text{euc}}(B_i'') = -\ell_{\text{euc}}(B_i') + o(1) & \text{si } X_i = B_i, \end{cases}
\]
\[
|F_u(\partial \Omega_i^X \setminus (X_i' \cup X_i''))| \leq \ell_{\text{euc}} (\partial \Omega_i^X \setminus (X_i' \cup X_i'')) = o(1),
\]

where \( u \in \{u_1, u_2\} \). So
\[
\ell_{\text{euc}}(A_i') - F_{u_2}(A_i') = o(1), \quad \ell_{\text{euc}}(B_i') + F_{u_1}(B_i') = o(1).
\]

This proves the assertion. \( \diamondsuit \)

It follows from the previous assertions that \( \nabla u_1 = \nabla u_2 \) on the set \( \{\varepsilon < u_1 - u_2 < N\} \). Since \( \varepsilon > 0 \) and \( N \) are arbitrary, \( \nabla u_1 = \nabla u_2 \) whenever \( u_1 > u_2 \).

It follows that \( u_1 = u_2 + c, (c > 0) \) in any nontrivial component of the set \( \{u_1 > u_2\} \). Then the maximum principle, Theorem 3.1, ensures \( u_1 = u_2 + c \) in \( \Omega \) and by the assumptions of the theorem, the constant must nonpositive, a contradiction. \( \square \)
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