ZERO TEMPERATURE LIMIT FOR DIRECTED POLYMERS
AND INVISCID LIMIT FOR STATIONARY SOLUTIONS OF
STOCHASTIC BURGERS EQUATION

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Abstract. We consider a space-continuous and time-discrete polymer model for positive temperature and the associated zero temperature model of last passage percolation type. In our previous work, we constructed and studied infinite-volume polymer measures and one-sided infinite minimizers for the associated variational principle, and used these objects for the study of global stationary solutions of the Burgers equation with positive or zero viscosity and random kick forcing, on the entire real line.

In this paper, we prove that in the zero temperature limit, the infinite-volume polymer measures concentrate on the one-sided minimizers and that the associated global solutions of the viscous Burgers equation with random kick forcing converge to the global solutions of the inviscid equation.

1. Introduction

Various models of directed polymers in random environment along with their zero temperature counterparts of last passage percolation type have been studied actively in recent decades, see, e.g., books [dH09], [Gia07], [Com17] and multiple references therein. On finite time intervals, positive temperature polymer measures are defined as Gibbs distributions with a random walk as a free measure and Boltzmann–Gibbs weights given by the potential accumulated by random walk paths from the random environment. The corresponding zero temperature models are defined in terms of the energy minimizing paths.

The most interesting questions concern the large time behavior of the random polymer distributions and energy minimizers. In particular, it is believed that a large family of models of this kind with fast decorrelation of the stationary random potential in dimension $1 + 1$ belongs to the KPZ universality class, i.e., satisfies limit theorems under scalings with characteristic exponents $2/3$ and $1/3$ and distributional limits of Tracy–Widom type.

Another basic question concerns the infinite-volume Gibbs distributions for polymer measures and the corresponding ground states, i.e., infinite one-sided or two-sided energy minimizers that are usually called geodesics in the literature on last passage percolation (LPP) and first passage percolation (FPP). There are numerous results concerning infinite geodesics. In particular, existence-uniqueness of one-sided planar geodesics with fixed slope and certain geometric features of the joint behavior of different geodesics are known for several models, see [HN97], [HN99], [HN01], [Win02], [CP11], [CP12], [DHI4], [Bak16], [GRAS16], [RSY16], [GRS15], and a survey [AHD15]. However, results about thermodynamic limits for directed polymers are relatively new. The first explicit result of this kind known to us is [BK10], where instead of stationarity a localization condition was imposed on the random potential, so the thermodynamic limit is a random measure on paths.
with random localization radius. More recently, in [GRASY15], [GRAS16] thermodynamic limits were constructed and studied under certain conditions that were verified for an exactly solvable lattice model called log-gamma polymer, and certain weak disorder models.

The first complete set of results for a 1 + 1-dimensional model that is not exactly solvable were obtained in [BL16], where time-discrete and space-continuous polymers based on Gaussian random walks were considered. It was shown that for any positive temperature and any fixed asymptotic slope, with probability one, there exists a family of infinite-volume polymer measures satisfying DLR conditions, concentrated on one-sided infinite paths with prescribed asymptotic slope, and indexed by the endpoint. Moreover, it was shown that these infinite-volume Gibbs measures are almost surely uniquely defined and that they are limits of various kinds (point-to-point, point-to-line, point-to-distribution) of finite-volume polymer measures. It was also shown that the total variation distance between projections of different polymer measures with the same asymptotic slope on distant coordinates is asymptotically zero, so they tend to overlap and can be effectively coupled. The results crucially depend on the explicitly known form of the dependence of the free energy density, (also known as the shape function) on the asymptotic slope. Namely, the shape function is quadratic and thus has uniform curvature.

The first main goal of the present paper is to study the zero-temperature asymptotics of the infinite-volume polymer measures constructed in [BL16]. In the finite-volume setting, the asymptotic concentration of Gibbs distributions around finite volume ground states, i.e., energy minimizers, is well-known. In the infinite-volume setting, the energies of paths are infinite, but it is natural to expect that the infinite one-sided minimizers or geodesics (infinite paths whose restrictions on any finite intervals are minimizers) are relevant for this problem. The existence-uniqueness and joint behavior of one-sided minimizers for the same model was studied in [Bak16].

In the present paper, we prove that in the zero-temperature limit, with probability one, the random infinite volume polymer measures converge to delta-measures concentrated on one-sided minimizers. To the best of our knowledge, this is the first result on zero-temperature limit for infinite directed polymers to appear in literature. In a sense, given the results of [Bak16] and [BL16], it amounts to interchanging the order of zero-temperature and infinite-volume (or time horizon) limits.

Papers [Bak16] and [BL16] were, in fact, primarily motivated by the ergodic program for randomly forced Burgers equation which is a basic nonlinear evolution equation that has multiple connections to various problems from traffic modeling to the large scale structure of the Universe. It has interpretations via fluid dynamics and growth models, and we often use the fluid dynamics interpretation where the equation describes the evolution of velocity fields of moving particles. It is also tightly related to Hamilton–Jacobi–Bellman (HJB) equations and can be solved with usual HJB methods.

The viscosity parameter of the Burgers equation can be interpreted as temperature. In fact, if the viscosity is positive, the Burgers equation can, by the Hopf–Cole transform, be reduced to the linear heat equation with multiplicative potential, and thus solved with the Feynman–Kac formula that in turn can be interpreted as averaging with respect to a polymer measure. In the zero-viscosity case, the Burgers equation can be solved by a variational Hopf–Lax–Oleinik–Hamilton–Jacobi–Bellman (HLOHJB) principle that can be derived from the large deviation principle for random walk or Brownian motion, see [FW12]. As viscosity tends to
zero, the polymer measure naturally arising in the Feynman–Kac formula concentrates around paths that minimize action in the HLOHJB variational principle, in precise agreement with zero-temperature limit for finite-volume polymer measures.

The long-term dynamics of the Burgers equation with kick forcing (where a delta-type random force is applied at every integer time and there is no forcing between those kicks) in both positive and zero viscosity settings is governed by global stationary solutions whose construction and properties was given in [Bak16] and [BL16]. It turns out that for each value of the average velocity and almost every realization of the random forcing there is a uniquely compatible global solution that can be seen as a one-point attractor. This statement is often referred to as One Force — One Solution (1F1S) principle, or synchronization by noise.

The key to understanding 1F1S principle for the Burgers equation is the analysis of polymer measures or action minimizers over long time intervals. A crucial point is the construction of global solutions using the infinite volume polymer measures (in the positive viscosity case) or one-sided infinite action minimizers (for zero viscosity). Another crucial point is to make sense of differences in action (resp. free energy) of two infinite one-sided minimizers (resp. polymers). This is done rigorously through a limiting procedure leading to the notion of Busemann function.

The ergodic program for the Burgers equation has a long history. Before [Bak16] and [BL16], the ideas around 1F1S for Burgers equation (and its generalizations) with random forcing were explored first in compact setting [Sin91], [EKMS00], [HK03], [GIKP05], [Bak07], [DV15], in quasi-compact setting in [HK03], [Sin05], [Bak13], and, finally, in fully noncompact setting in [BCK14], where stationary Poissonian forcing was considered. The work in [BCK14] used ideas from [Kes93], [HN97], [HN99], [HN01], [Wü02], [CP11], [CP12]. A similar approach to global solutions based on Busemann functions for lattice models was also developed in [GRAS16], [GRS15].

In [GIKP05], the zero-viscosity limit for stationary solutions of the randomly forced Burgers equation (and other stochastic HJB equations) was obtained in the (compact) case of the circle or torus. In the present paper, we use the zero-temperature limit for infinite-volume directed polymers in order to obtain the zero-viscosity limit for stationary solutions of the Burgers equation with random kick forcing. Namely, we prove that as the viscosity vanishes the stationary solutions of the viscous Burgers equation converge to those of the inviscid one. Of course, the PDE results of [GIKP05] can also be restated in the polymer language.

We postpone the precise description of the mode of convergence of global solutions to the later sections of the paper. Here, we only want to make a comment that our results seem to be first ones on conservation of stationary solutions of a nonlinear stochastic PDE in noncompact setting under a transition to a limit. Among hard problems in this direction is the inviscid limit of the stochastic two-dimensional Navier–Stokes system (SNS). The compact case such as SNS on the 2D-torus is well understood, see [EMS01], [BKL01], [KS00], [HM06], [HM08], [HM11]. However, as the viscosity tends to zero, one needs to scale the forcing appropriately to obtain nontrivial behavior in the limit, as was realized in [Kuk04], [Kuk07], and [Kuk08]. This contradicts the Kraichnan theory of 2D turbulence whose predictions can be interpreted as existence of a nontrivial inviscid limit under viscosity-independent forcing. This discrepancy can be explained by finite size effects since the inverse cascades of Kraichnan theory are impossible in a compact domain. It would be extremely interesting to see if this contradiction gets resolved in noncompact setting. However, the only ergodic result for Navier–Stokes system in the entire space known to us is [Bak06], where under certain conditions on the decay of the noise at infinity,
a unique invariant distribution on the Le Jan–Sznitman existence-uniqueness class is constructed for SNS in \( \mathbb{R}^3 \), and this class of solutions neither allows for spatial stationarity nor survives the inviscid limit.

In the present paper, we show that in the Burgers turbulence which exhibits a lot of contraction compared to the chaotic unstable behavior typical for the true turbulence, the situation is quite nice and the expected inviscid limit holds. We also conjecture that similar results hold for more general HJB equations with convex Hamiltonians and appropriately defined polymer models.

The rest of the paper is organized as follows. The setting and minimal background from [Bak16] and [BL16] that we need to state our results are given in Sections 2 and 3. In Section 2 we introduce the relevant information on the Burgers equation, and in Section 3 we discuss polymers and action minimizers. We state our main results in Section 4. In Sections 5, we remind some basic useful results on partition functions. The proofs of the main results are given in Sections 6–8.

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2. The Setting. Burgers equation

2.1. Forward and backward Burgers equation. The one-dimensional Burgers equation describing evolution of a velocity field \( u(t, x) \), where \( t \in \mathbb{R} \) and \( x \in \mathbb{R} \) are time and space variables, is

\[
\partial_t u + u \partial_x u = \frac{\kappa}{2} \partial_{xx} u + f.
\]

Here \( f = f(t, x) \) is external forcing, and \( \kappa \geq 0 \) is the viscosity parameter. This equation, with random kick-forcing \( f \) was studied in [Bak16] for \( \kappa = 0 \) and in [BL16] for \( \kappa > 0 \). To solve the Cauchy problem for this equation up to time \( t \in \mathbb{R} \), one needs to emit action minimizers and polymers from time \( t \) into the past, and this is what was done in [Bak16] and [BL16]. However, it is slightly more natural to work with forward polymers and action minimizers, so in this paper, we change the direction of time and state our results for the following “backward” Burgers equation in 1D:

\[
- \partial_t u + u \partial_x u = \frac{\kappa}{2} \partial_{xx} u + f.
\]

For this equation, instead of the initial value problem, it is the terminal value problem that is well-posed. It is natural to solve \( \text{(2.2)} \) backward in time, and if \( s > t \), then \( u(t, \cdot) \) is uniquely defined by \( u(s, \cdot) \) and the forcing \( f \) between \( t \) and \( s \). We stress that we change the time direction in the Burgers equation just for convenience. Restating any result obtained for equation \( \text{(2.1)} \) in terms of equation \( \text{(2.2)} \) and \text{vice versa} is trivial.

The Burgers equation is tightly connected to the following (backward) Hamilton–Jacobi–Bellman (HJB) equation:

\[
- \partial_t U + \frac{(\partial_x U)^2}{2} = \frac{\kappa}{2} \partial_{xx} U + F.
\]

Namely, if \( U \) is a solution of \( \text{(2.3)} \), then \( u = \partial_x U \) solves \( \text{(2.2)} \) with \( f = \partial_x F \).

The main model that we study in this paper is the Burgers equation with kick forcing of the following form:

\[
f(t, x) = \sum_{n \in \mathbb{Z}} f_n(x) \delta_n(t).
\]
This means that the additive forcing is applied only at integer times. On each interval \((n, n+1]\) where \(n \in \mathbb{Z}\), the velocity field evolves (from time \(n+1\) to time \(n\)) according to the unforced backward Burgers equation
\[
-\partial_t u + u \partial_x u = \frac{\kappa}{2} \partial_{xx} u,
\]
and at time \(n\), the entire velocity profile \(u\) receives an instantaneous macroscopic increment equal to \(f_n\):
\[
u(n-0, x) = u(n, x) + f_n(x), \quad x \in \mathbb{R}.
\]
We assume that the potential \(F = F_{n, \omega}(x)\) of the forcing
\[
f_n(x) = F_{n, \omega}(x) = \partial_x F_{n, \omega}(x), \quad n \in \mathbb{Z}, \ x \in \mathbb{R}, \ \omega \in \Omega,
\]
is a stationary random field defined on some probability space \((\Omega, \mathcal{F}, \mathbb{P})\). We will describe all conditions that we impose on \(F\) in section 2.2. At this point, we need only the following consequence of those conditions: for every \(\omega \in \Omega\) and every \(n \in \mathbb{Z}\), the function \(F_{n, \omega}(\cdot)\) is measurable with respect to \(\omega\), continuous with respect to \(x\), and satisfies
\[
\lim_{|x| \to \infty} \frac{F_{n, \omega}(x)}{|x|} = 0.
\]

Let us now explain how to solve the backward Burgers dynamics with kick forcing, thus introducing the dynamics that we will study in this paper. The inviscid case \((\kappa = 0)\) and the viscous case \((\kappa > 0)\) will be treated separately. For the viscous case, the Burgers dynamics will be defined through the Hopf–Cole transformation and the Feynman–Kac formula. For the inviscid case, we will use a variational characterization that can be seen as the limiting case of the positive viscosity formula.

For every \(m, n \in \mathbb{Z}\) satisfying \(m < n\), we denote the set of all paths
\[
\gamma : [m, n]_\mathbb{Z} = \{m, m+1, \ldots, n\} \to \mathbb{R}
\]
by \(S_{m, n}^m\). If in addition a point \(x \in \mathbb{R}\) is given, then \(S_{x, n}^m\) denotes the set of all such paths that satisfy \(\gamma_m = x\). If \(n = \infty\), then we understand the above spaces as the spaces of one-sided semi-infinite paths. If points \(x, y \in \mathbb{R}\) are given, then \(S_{x, y}^{m, n}\) denotes the set of all such paths that satisfy \(\gamma_m = x\) and \(\gamma_n = y\).

Let \(m < n\). Given a path \(\gamma\) defined on \([m', n']_\mathbb{Z} \supseteq [m, n]_\mathbb{Z}\), its kinetic energy \(I^{m,n}(\gamma)\), potential energy \(H^{m,n}_\omega(\gamma)\) and total action \(A^{m,n}_\omega(\gamma)\) are given by
\[
I^{m,n}(\gamma) = \frac{1}{2} \sum_{k=m+1}^{n} (\gamma_k - \gamma_{k-1})^2, \quad H^{m,n}_\omega(\gamma) = \sum_{k=m+1}^{n} F_{k, \omega}(\gamma_k),
\]
\[
A^{m,n}_\omega(\gamma) = I^{m,n}(\gamma) + H^{m,n}_\omega(\gamma).
\]
Note the asymmetry in the definition of \(H^{m,n}_\omega\): we have to include \(k = n\), but exclude \(k = m\). All our results are proved for this choice of path energy, but it is straightforward to obtain their counterparts for the version of energy where the \(k = n\) is excluded and \(k = m\) is included. For the inviscid case, we can now define the random backward evolution operator on potential by
\[
[\Psi^{m,n}_{0, \omega} U](x) = \inf_{\gamma \in S_{m,n}^{m,n}} \{U(\gamma_n) + A^{m,n}_\omega(\gamma)\}, \quad x \in \mathbb{R}, \ m < n.
\]
For the viscous case, one can introduce the Hopf–Cole transformation \(\varphi\) by
\[
\varphi(t, x) = e^{-\frac{U(t, x)}{\kappa}}.
\]
An application of the discrete Feynman–Kac formula will lead to the following backward evolution operator on $\varphi$:

$$\Xi^{m,n}_{\kappa,\omega}(x) = \int_{\mathbb{R}} \hat{Z}^{m,n}_{x,y,\kappa,\omega} \varphi(y) \, dx, \quad x \in \mathbb{R}, \, m < n,$$

where

$$\hat{Z}^{m,n}_{x,y,\kappa,\omega} = \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \prod_{k=m+1}^{n} g_\kappa(x_k - x_{k-1}) e^{-F_\kappa(x_k)} \delta_x(dx_{m+1} \cdots dx_{n-1}) \delta_y(dx_n)$$

and $g_\kappa(x) = \frac{1}{\sqrt{2\pi\kappa}} e^{-x^2/2\kappa}$. With the inverse of the Hopf–Cole transform (2.11), we can define evolution on potentials by

$$\Phi^{m,n}_{\kappa,\omega}U = -\kappa \ln \Xi^{m,n}_{\kappa,\omega} e^{-U}.$$

The space of velocity potentials that we will consider will be $\mathbb{H}$, the space of all locally Lipschitz functions $W : \mathbb{R} \rightarrow \mathbb{R}$ satisfying

$$\liminf_{x \rightarrow \pm \infty} \frac{W(x)}{|x|} > -\infty.$$

We will also need a family of spaces

$$\mathbb{H}(v_-,v_+) = \left\{ W \in \mathbb{H} : \lim_{x \rightarrow \pm \infty} \frac{W(x)}{x} = v_{\pm}, \quad v_-, v_+ \in \mathbb{R} \right\}.$$  

**Lemma 2.1.** For every $\kappa \geq 0$ and any $\omega \in \Omega$, for any $l \leq n \leq m$ with $l < n < m$ and $W \in \mathbb{H}$,

1. $\Phi^{n,m}_{\kappa,\omega}W$ is well-defined and belongs to $\mathbb{H}$;
2. if $W \in \mathbb{H}(v_-,v_+)$ for some $v_-,v_+$, then $\Phi^{n,m}_{\kappa,\omega}W \in \mathbb{H}(v_-,v_+)$;
3. (cocycle property) $\Phi^{l,m}_{\kappa,\omega}W = \Phi^{n,m}_{\kappa,\omega} \Phi^{n,l}_{\kappa,\omega}W$.

We can also introduce the Burgers dynamics on the space $\mathbb{H}'$ of velocities $w$ such that for some function $W \in \mathbb{H}$ and Lebesgue almost every $x$, $w(x) = W'(x) = \partial_x W(x)$. For all $v_-, v_+ \in \mathbb{R}$, $\mathbb{H}'(v_-,v_+)$ is the space of velocity profile with well-defined one-sided averages $v_-$ and $v_+$, it consists of functions $w$ such that the potential $W$ defined by $W(x) = \int_0^x w(y) \, dy$ belongs to $\mathbb{H}(v_-,v_+)$. We will write $w_1 = \Psi^{n,0}_{\kappa,\omega} w_0$ if $w_0 = W'_0$, $w_1 = W'_1$, and $W_1 = \Phi^{n,0}_{\kappa,\omega}W_0$ for some $W_0, W_1 \in \mathbb{H}$.

### 2.2. Assumptions on the random forcing.

For simplicity, we will work on the canonical probability space $(\Omega_0, \mathcal{F}_0, P_0)$ of realizations of the potential, although other more general settings are also possible. We assume that $\Omega_0$ is the space of continuous functions $F : \mathbb{R} \times \mathbb{Z} \rightarrow \mathbb{R}$ equipped with $\mathcal{F}_0$, the completion of the Borel $\sigma$-algebra with respect to local uniform topology, and $P_0$ is a probability measure preserved by the group of shifts $(\theta^{n,x})_{(n,x) \in \mathbb{Z} \times \mathbb{R}}$ defined by

$$(\theta^{n,x} F)_m(y) = F_{n+m}(x + y), \quad (n,x),(m,y) \in \mathbb{Z} \times \mathbb{R},$$

i.e., $(F_n(x))_{(n,x) \in \mathbb{Z} \times \mathbb{R}}$ is a space-time stationary process. In this framework, $F = F_0 = \omega$, and we will use all these notations intermittently.

In addition to this, we introduce the following requirements:

**A1:** The flow $(\theta^{0,x})_{x \in \mathbb{R}}$ is ergodic. In particular, for every $n \in \mathbb{Z}$, $F_n(\cdot)$ is ergodic with respect to the spatial shifts.

**A2:** The sequence of processes $(F_n(\cdot))_{n \in \mathbb{Z}}$ is i.i.d.

**A3:** With probability 1, for all $n \in \mathbb{Z}$, $F_n(\cdot) \in C^1(\mathbb{R})$. 

(A4): For all $(n, x) \in \mathbb{Z} \times \mathbb{R}$ and all $\beta \in \mathbb{R}_+$,\[ \lambda(\beta) := \mathbb{E} e^{-\beta F_n(x)} < \infty. \]

(A5): There are $\varphi, \eta > 0$ such that for all $(n, j) \in \mathbb{Z} \times \mathbb{Z}$,\[ e^{\varphi} = \mathbb{E} e^{\eta F^\ast_{n,j}(\omega)} < \infty, \]

where

\[ F^\ast_{n,\omega}(j) = \sup\{|F_{n,\omega}(x)| : x \in [j, j+1]\}. \]

We will use these standing assumptions throughout the paper.

Stationarity and (A5) imply that (2.6) holds with probability 1 on $\Omega_0$. It will be convenient in this paper to work on a modified probability space

\[ (2.13) \quad \Omega = \{ F \in \Omega_0 : \lim_{|x| \to \infty} \frac{F_n(x)}{|x|} = 0, \quad n \in \mathbb{Z} \} \in \mathcal{F}_0. \]

of probability 1 instead of $\Omega_0$. On this set, the Burgers evolution possesses some nice properties discussed in [Bak16] and [BL16]. Moreover, $\Omega$ is invariant under space-time shifts $\theta^{n,x}$ and under Galilean space-time shear transformations $L^v$, $v \in \mathbb{R}$, defined by

\[ (L^v F)_n(x) = F_n(x + vn), \quad (n, x) \in \mathbb{Z} \times \mathbb{R}. \]

We denote the restrictions of $F_0$ and $P_0$ onto $\Omega$ by $F$ and $P$. From now on we work with the probability space $(\Omega, \mathcal{F}, P)$. Under this modification, all the distributional properties are preserved.

### 3. Directed Polymers and Minimizers

Formulas (2.8) and (2.10)–(2.11) show that the problem of long-term properties of the Burgers equation with random forcing can be approached through analysis of properties of either action minimizing paths (for the inviscid case) or Gibbs distributions on paths (for the viscous case) over long time intervals. This section summarizes the results of [Bak16] and [BL16] for both settings. We first describe properties of finite and one-sided infinite minimizers in Section 3.1, the same is done for polymers and their thermodynamic limits in Section 3.2, and finally we stress the connection to the global solutions of the Burgers equation in Section 3.3.

#### 3.1. Minimizers

For every $(m, x) \in \mathbb{Z} \times \mathbb{R}$ and every $v \in \mathbb{R}$, we denote

\[ S^{m,+\infty}_{x,*}(v) = \{ \gamma \in S^{m,+\infty}_{x,*} : \lim_{n \to \infty} \frac{\gamma_n}{n} = v \}. \]

If $\gamma \in S^{m,+\infty}_{x,*}(v)$, then we say that $\gamma$ has asymptotic slope $v$.

Let $A_{x,y} = A_{m,n}(x, y)$ denote the minimal action between $(m, x)$ and $(n, y)$, that is,

\[ A_{m,n}(x, y) = \min_{\gamma \in S^{m,n}_{x,y}} A_{m,n}(\gamma). \]

A path $\gamma \in S^{m,*}_{x,*}$ is called a (finite) minimizer if $A_{m,n}(\gamma) = A_{m,n}(\gamma_{m}, \gamma_{n})$. A path $\gamma \in S^{m,+\infty}_{x,*}$ is called a semi-infinite minimizer (or simply minimizer if it is clear from the context) if for any $n_2 > n_1 > m$, $\gamma_{n_1, n_2}$ is a minimizer, where $\gamma^{n_1, n_2}$ denotes the restriction of $\gamma$ to $[n_1, n_2]_z$.

The following theorem summarizes the results on semi-infinite minimizers established in [Bak16]. These results were established in [Bak16] for a specific random potential of shot-noise type, but it is easy to see that they hold true for any potential satisfying assumptions (A1)–(A5) under the additional requirement of finite
dependence range. It is also natural to expect that they hold for a much broader class of mixing potentials.

**Theorem 3.1** (Theorem 3.3, Lemma 9.3 in [Bak16]). Suppose that assumptions (A1)–(A5) are satisfied and $F$ has finite dependence range. Then for every $v \in \mathbb{R}$, there is a full measure set $\Omega_{v,0}$ such that the following properties hold:

1. For all $\omega \in \Omega_{v,0}$, there is an at most countable set $N = N_{\omega} \subset \mathbb{Z} \times \mathbb{R}$ such that for all $(m, x) \in \mathbb{Z} \times \mathbb{R} \setminus N$, there is a unique minimizer $\gamma^{n, +\infty}_{x,\omega}(v) \in S_{x,\omega}^{n, +\infty}(v)$.

2. (Busemann function) Let $\omega \in \Omega_{v,0}$. For $(n_1, x_1), (n_2, x_2) \in \mathbb{Z} \times \mathbb{R}$, there is sequence $N_k \uparrow +\infty$ such that the limit

$$B_v((n_1, x_1), (n_2, x_2)) = \lim_{k \to \infty} A^{n_1, N_k}_v(\gamma^{n_1}_{x_1}(v)) - A^{n_2, N_k}_v(\gamma^{n_2}_{x_2}(v))$$

exists. Here, if the semi-infinite minimizer is not unique at $(n_i, x_i)$, then $\gamma^{n_i}_{x_i}(v)$ can be any minimizer in $S_{x_i,\omega}^{n_i, +\infty}(v)$, $i = 1, 2$. Moreover, if the limit in (3.2) exists for some other sequence $(N_k')$, then it is independent of the choice of $(N_k')$. The function $B_v$ has the property that for any $(n_i, x_i) \in \mathbb{Z} \times \mathbb{R}$,

$$B_v((n_1, x_1), (n_2, x_2)) + B_v((n_2, x_2), (n_3, x_3)) = B_v((n_1, x_1), (n_3, x_3)) + B_v((n_1, x_1), (n_2, x_2)) - B_v((n_2, x_2), (n_1, x_1)).$$

3. The function $U_{v,0}(n, \cdot) = -B_v((n, \cdot), (n, 0))$ is Lipschitz, and it is differentiable at all $(n, x) \notin N$. The derivative is given by

$$u_{v,0}(n, x) := \frac{d}{dx} U_{v,0}(x) = x - (\gamma^{n, +\infty}_{x,\omega}(v))_{n+1}.$$

4. (Solution to inviscid Burgers and HJB equations) The function $B_v$ solves the following variational problem: for $m > n$ and fixed $(n_0, x_0) \in \mathbb{Z} \times \mathbb{R}$,

$$B_v((m, y), (n_0, x_0)) = \min_{y \in \mathbb{R}} \{B_v((m, y), (n_0, x_0)) + A^{m, m}_v(m, m, x, y)\}.$$

In particular, the function $u_{v,0}$ introduced in (3.4) solves the inviscid Burgers equation.

### 3.2. Polymer measures

Let $\kappa > 0$. In the context of polymer measures, this parameter plays the role of temperature. For $m, n \in \mathbb{Z}$ with $m < n$ and $x, y \in \mathbb{R}$, the point-to-point polymer measure (at temperature $\kappa$) $\mu^{m, n}_{x, y; \kappa, \omega}$ is a probability measure on $S_{x, y; \kappa, \omega}$ that has density

$$\mu^{m, n}_{x, y; \kappa, \omega}(x_m, \ldots, x_n) = \prod_{k=m+1}^{n} g_k(x_k - x_{k-1}) e^{-\kappa F_k(x_k)} \frac{Z^{m, n}_{x, y; \kappa, \omega}}{\mathcal{L}_{x, y; \kappa, \omega}},$$

with respect to $\delta_x \times \text{Leb}^{n-m-1} \times \delta_y$, where $Z^{m, n}_{x, y; \kappa, \omega}$ is defined in (2.11).

Let us introduce

$$Z^{m, n}_{x, y; \kappa, \omega} = (2\pi \kappa)^{n/2} Z^{m, n}_{x, y; \kappa, \omega} = \int_{\gamma \in S_{x, y; \kappa, \omega}^{m, n}} e^{-\kappa A^{m, m}_{\omega}(\gamma)} d\gamma$$

$$= \int e^{-\kappa \sum_{k=m+1}^{n} \left[ \frac{1}{2} (x_k - x_{k-1})^2 + F_k(x_k) \right]} \delta_x(dx_m) dx_{m+1} \ldots dx_{n-1} \delta(dx_n),$$

where $A^{m, n}_{\omega}$ is defined in (2.4). The polymer density can also be expressed as

$$\mu^{m, n}_{x, y; \kappa, \omega}(\gamma_m, \ldots, \gamma_n) = \frac{e^{-\kappa A^{m, m}_{\omega}(\gamma)}}{Z^{m, n}_{x, y; \kappa, \omega}}.$$

We often omit the $\omega$ argument in all the notations used above. We also often write $Z^{m, n}_{\kappa}(x, y)$ for $Z^{m, n}_{x, y; \kappa, \omega}$. 
We call a measure $\mu$ on $S_{x,*}^{m,n}$ a polymer measure (at temperature $\kappa$) if there is a probability measure $\nu$ on $\mathbb{R}$ such that $\mu = \mu_{x,v,\kappa}$, where

$$\mu_{x,v,\kappa}^{m,n} = \int_{\mathbb{R}} \mu_{x,v,\kappa}^{m,n}(dy).$$

We call $\nu$ the terminal measure for $\mu = \mu_{x,v,\kappa}$. It is also natural to call $\mu$ a point-to-measure polymer measure associated to $x$ and $\nu$.

A measure $\mu$ on $S_{x,*}^{m,+\infty}$ is called an infinite volume polymer measure if for any $n \geq m$ the projection of $\mu$ on $S_{x,*}^{m,n}$ is a polymer measure. This condition is equivalent to the Dobrushin–Lanford–Ruelle (DLR) condition on the measure $\mu$.

We say that the strong law of large numbers (SLLN) with slope $v$ on $\mathbb{R}$ holds for a measure $\mu$ on $S_{x,*}^{m,+\infty}$ if $\mu(S_{x,*}^{m,+\infty}(v)) = 1$.

We say that LLN with slope $v$ on $\mathbb{R}$ holds for a sequence of Borel measures $(\nu_n)$ if for all $\delta > 0$,

$$\lim_{n \to \infty} \nu_n([[v - \delta]n, (v + \delta)n]) = 1.$$

Finally, for any $(m, x) \in \mathbb{Z} \times \mathbb{R}$, we say that a measure $\mu$ on $S_{x,*}^{m,+\infty}$ satisfies LLN with slope $v$ if the sequence of its marginals $\nu_k(\cdot) = \mu(\gamma : \gamma_k \in \cdot)$ does.

We denote by $\mathcal{P}_{x,k}^{m,+\infty}(v)$ the set of all polymer measures at temperature $\kappa$ on $S_{x,*}^{m,+\infty}$ satisfying DLR with slope $v$. The set of all polymer measures at temperature $\kappa$ on $S_{x,*}^{m,+\infty}$ satisfying LLN with slope $v$ is denoted by $\mathcal{P}_{x,k}^{m,+\infty}(\nu)$. These sets are random since they depend on the realization of the environment, but we suppress the dependence on $\omega \in \Omega$ in this notation.

The following theorem summarizes the results established in [BL16] on the infinite polymer measure with given asymptotic slope.

**Theorem 3.2** (Theorems 4.2, 4.3, 11.2 in [BL16]). Suppose that assumptions (A1) are satisfied. Then, for each $v \in \mathbb{R}$ and $\kappa > 0$, there is a full measure set $\Omega_{v,\kappa} \subset \mathcal{F}$ such that

1. For all $\omega \in \Omega_{v,\kappa}$ and all $(m, x) \in \mathbb{Z} \times \mathbb{R}$, there is a unique polymer measure $\mu_{x,v,\kappa}^{m,+\infty}$ such that

$$\mathcal{P}_{x,k}^{m,+\infty}(v) = \mathcal{P}_{x,k}^{m,+\infty}(\nu) = \{\mu_{x,v,\kappa}^{m,+\infty}\}.$$

2. For all $\omega \in \Omega_{v,\kappa}$, all $(m, x) \in \mathbb{Z} \times \mathbb{R}$ and for every sequence of measures $(\nu_n)$ satisfying LLN with slope $v$, finite-dimensional distributions of $\mu_{x,v,\kappa}^{m,n}$ converge to $\mu_{x,v,\kappa}^{m,+\infty}$ in total variation.

3. For all $\omega \in \Omega_{v,\kappa}$, all $(n_1, x_1), (n_2, x_2) \in \mathbb{Z} \times \mathbb{R}$ and for every sequence $(y_N)$ with $\lim_{N \to \infty} y_N/N = v$, we have

$$\lim_{N \to \infty} \frac{Z_{x,k,N}^{n_1}}{Z_{x,k,N}^{n_2}} = G,$$

where $G = G_{v,\kappa}(\{(n_1, x_1), (n_2, x_1)\}) \in (0, \infty)$ does not depend on $(y_N)$. Moreover, the function $G$ has the property that for any $(n_1, x_1) \in \mathbb{Z} \times \mathbb{R}$,

$$G_{v,\kappa}(\{(n_1, x_1), (n_2, x_2)\})G_{v,\kappa}(\{(n_2, x_2), (n_3, x_3)\}) = G_{v,\kappa}(\{(n_1, x_1), (n_3, x_3)\}),$$

$$G_{v,\kappa}(\{(n_1, x_1), (n_2, x_2)\}) = \left[G_{v,\kappa}(\{(n_2, x_2), (n_1, x_1)\})\right]^{-1}.$$

4. For all $(m, x) \in \mathbb{Z} \times \mathbb{R}$, the finite-dimensional distributions of $\mu_{x,v,\kappa}^{m,+\infty}$ are absolutely continuous. The density of its marginal is given by

$$\mu_{x,v,\kappa}^{m,+\infty} \pi_n^{-1}(dy) = Z_{x,v,\kappa}^{n,m}G_{v,\kappa}(\{(n, y), (m, x)\}), \quad n > m.$$
where $\pi_n$ is the projection of a path $\gamma$ onto its $n$-th coordinate $\gamma_n$.

5. Let $U_{v,\kappa}(n, \cdot) = -\kappa \ln G_{v,\kappa}((n, \cdot), (n, 0))$, then

\[
(3.9) \quad u_{v,\kappa}(x, n) := \frac{d}{dx} U_{v,\kappa}(n, x) = \int (x - y) \mu_{n,x,v,\kappa}^{n+\infty} \pi_{n+1}^{-1}(dy), \quad (n, x) \in \mathbb{Z} \times \mathbb{R}.
\]

6. (Solutions to viscous Burgers, HJB, and heat equations) The function $G_{v,\kappa}$ satisfies the following relation: for $m > n$ and fixed $(n_0, x_0) \in \mathbb{Z} \times \mathbb{R}$,

\[
(3.10) \quad G_{v,\kappa}((n, x), (n_0, x_0)) = \int_{\mathbb{R}} G_{m,\kappa}^n((m, y), (n_0, x_0)).
\]

In particular, $u_{v,\kappa}(n, x)$ defined in (3.9) solves Burgers equation with viscosity $\kappa$.

3.3. Connections to global solutions of Burgers equation. We say that $u(n, x) = u_\omega(n, x)$, $(n, x) \in \mathbb{Z} \times \mathbb{R}$ is a global solution for the Burgers equation with viscosity $\kappa$ if there is a set $\Omega' \in \mathcal{F}$ with $P(\Omega') = 1$ such that for all $\omega \in \Omega'$, for all $m$ and $n$ with $m < n$, we have $\Psi_{v,\kappa}(n, \omega(x)) = \Psi_{v,\kappa}(m, \omega(x))$.

We recall the full measure sets $\Omega_{v,\kappa}$ and the functions $u_{v,\kappa}$, $v \in \mathbb{R}$, $\kappa \geq 0$ defined in Theorem 3.1 and 3.2. As we see in the previous two sections, the relations (3.5) and (3.10), together with (3.4) and (3.9) where $u_{v,\kappa}$ are defined, show that for each $\kappa \geq 0$, $u_{v,\kappa}$ is a global solution for the Burgers for the Burgers with viscosity $\kappa$. In fact, they are the only ones in a certain sense, as the following theorem states.

Theorem 3.3 (Bak16, BL16). Let $\kappa \geq 0$. For every $v \in \mathbb{R}$, the function $u_{v,\kappa}$ defined on the full measure set $\Omega_{v,\kappa}$ is a unique global stationary solution in $\mathcal{H}(v, v)$ for the Burgers equation with viscosity $\kappa$.

4. Main results

In this section, we state the main results of this paper. Our first result concerns the zero-temperature limit of infinite volume polymer measures:

Theorem 4.1. Let $v \in \mathbb{R}$. With probability one, the following holds true:

1. For all $v \in \mathbb{R}$, all $\kappa \in (0, 1)$ and all $(m, x) \in \mathbb{Z} \times \mathbb{R}$, $\mathcal{P}_{x,v,\kappa}^{m,++}(v) \neq \emptyset$.

2. Let $v \in \mathbb{R}$ and $(m, x) \in \mathbb{Z} \times \mathbb{R}$. Then the family of probability measures $(\mathcal{P}_{x,v,\kappa}^{m,++}(v))_{\kappa \in (0, 1)}$ on $S_{x,v,\kappa}^{m,++} \equiv \mathbb{R}^N$ is tight.

3. (Zero-temperature limit.) For fixed $v \in \mathbb{R}$ and $(m, x) \in \mathbb{Z} \times \mathbb{R}$, let $\mu_{\kappa} \in \mathcal{P}_{x,v,\kappa}^{m,++}(v)$, $\kappa \in (0, 1)$. Then, any limit point $\mu$ of $(\mu_{\kappa})$ as $\kappa \downarrow 0$ concentrates on semi-infinite minimizers on $S_{x,v,\kappa}^{m,++}(v)$. In particular, if $S_{x,v,\kappa}^{m,++}$ contains only one element $\gamma$, then $\mu$ is the $\delta$-measure on $\gamma$.

Given $v \in \mathbb{R}$, Theorem 3.2 says that at every fixed temperature $\kappa > 0$, there is a full measure set $\Omega_{v,\kappa}$ on which $\mathcal{P}_{x,v,\kappa}^{m,++}(v)$ contains a unique element. However, we cannot guarantee the existence of a common full measure set on which this holds for all values of $\kappa$ simultaneously. Nevertheless, in Theorem 4.1, using a compactness argument we are able to find a full measure set on which $\mathcal{P}_{x,v,\kappa}^{m,++}(v)$ is always nonempty for all $v \in \mathbb{R}$, but may potentially contain more than one element. If one considers only countably many values of temperatures, then this difficulty with common exceptional sets does not arise. This approach is used in the next result.

Let us now state our main theorem on the inviscid limit of the global solutions of Burgers equation. In addition to (A1)–(A5), in this section we also assume the potential $F$ has the property such that conclusions of Theorem 3.4 hold true (see the discussion before Theorem 3.4), so that the global solution for inviscid Burgers is unique. To state this result, we need to specify the topology in which the solutions
converge. We recall that in the kick forcing case, if \( u(n,x) \) is a solution to the Burgers equation with viscosity \( \kappa \geq 0 \), then \( x - u(n,x) \) is a monotone increasing function (see Lemma 2.1 in [Bak16] and Lemma 2.2 in [BL16]). For this reason, it is natural to consider the space \( G \) of cadlag (i.e., right-continuous with left limits) functions \( u \) such that \( \kappa = 0 \leq u(x) \) is increasing. The monotonicity allows to define \( G \)-convergence of a sequence of functions \( u_n \in G \) to a function \( u \in G \) as convergence \( u_n(x) \to u(x) \), \( n \to \infty \), for every continuity point \( x \) of \( u \). The space \( G \) was first introduced in [Bak16].

**Theorem 4.2.** Let \( v \in \mathbb{R} \) and fix a countable set \( \mathcal{D} \subset (0,1) \) that has 0 as its limit point. Then there exists a full measure set \( \Omega_v \subset \Omega_{v,0} \cap \bigcap_{\kappa \in \mathcal{D}} \Omega_{v,\kappa} \) such that the following holds true:

1. For every \( (m,x) \notin \mathcal{N} \), as \( \mathcal{D} \ni \kappa \to 0 \), \( \mu_{v,v,\kappa}^{m,+\infty} \) converge to \( \delta_{\kappa}(v) \) weakly.
2. (Inviscid limit for stationary solutions of the Burgers equation.) For every \( n \in \mathbb{Z} \), \( u_{v,\kappa}(n,\cdot) \to u_{v,0}(n,\cdot) \) in \( G \) as \( \mathcal{D} \ni \kappa \to 0 \), where \( u_{v,\kappa} \) are the global solutions defined in (5.3) for \( \kappa > 0 \) and in (5.4) for \( \kappa = 0 \).
3. (Inviscid limit for Busemann functions and global solutions of the HJB equation.) For all \( (n_1, x_1), (n_2, x_2) \in \mathbb{Z} \times \mathbb{R} \),

\[
\lim_{\mathcal{D} \ni \kappa \to 0} -\kappa \ln G_{v,\kappa}\left((n_1, x_1), (n_2, x_2)\right) = B_v((n_1, x_1), (n_2, x_2)).
\]

The proofs of the theorems in this section will be given in Section 8 after a series of auxiliary results in Sections 5–7.

## 5. Properties of the partition function

In this section, we recall useful results on minimal action and partition functions from [Bak16] and [BL16].

We begin with a lemma on the behavior of distributional properties of partition functions under shift and shear transformations of space-time. We write \( \overset{d}{=} \) to denote identity in distribution.

**Lemma 5.1** (Lemma 5.1 in [BL16]). Let \( \kappa \in (0,1] \). For any \( m, n \in \mathbb{Z} \) satisfying \( m < n \) and any points \( x, y \in \mathbb{R} \),

\[
Z_{\kappa}^{m,n+1}(x, y + \Delta) \overset{d}{=} Z_{\kappa}^{m,n}(x, y).
\]

Also, for any \( v \in \mathbb{R} \),

\[
Z_{\kappa}^{0,n}(0, vn) \overset{d}{=} e^{-\kappa^{-1} \frac{v^2}{2}} Z_{\kappa}^{0,n}(0, 0).
\]

(5.1)

It is easy to extend this lemma to obtain the following:

**Lemma 5.2.** Let \( \kappa \in (0,1] \) and \( Z_{v,\kappa}(n) = e^{\frac{1}{\kappa} \frac{v^2}{2n}} Z_{\kappa}^{0,n}(0, vn) \), \( n \in \mathbb{N}, v \in \mathbb{R} \). Then the distribution of the process \( Z_{v,\kappa}(\cdot) \) does not depend on \( v \). Also, for every \( n \in \mathbb{N} \), the process \( \tilde{Z}_{v,\kappa}(x) = e^{\frac{1}{\kappa} \frac{v^2}{2n}} Z_{\kappa}^{0,n}(0, x), x \in \mathbb{R} \), is stationary in \( x \).

The directional linear growth of \( \ln Z_{\kappa}^{m,n}(x, y) \) over long time intervals is given by the following result from Section 6 in [BL16]:

**Theorem 5.1.** There are constants \( \alpha_{0,\kappa} \in \mathbb{R} \) such that for any \( v \in \mathbb{R} \) and \( \kappa \in (0,1) \),

\[
\lim_{n \to \infty} \frac{\kappa \ln Z_{\kappa}^{0,n}(0, vn)}{n} \overset{a.s.}{=} \alpha_{\kappa}(v) := \alpha_{0,\kappa} - \frac{v^2}{2}.
\]

(5.2)
The function $\alpha_\kappa(v)$ is called the shape function or the density of free energy. The existence of the limit in (5.2) is based on the sub-additive ergodic theorem, and the quadratic form of $\alpha_\kappa(v)$ is due to (5.1).

The counterparts of Lemma 5.1 and Theorem 5.1 for the inviscid case were established in [Bak16]. Let us briefly summarize them. We recall that $A_{m,n}(x,y)$ defined in (3.1). It is easy to see that

$$\lim_{\kappa \downarrow 0} \kappa \ln Z_{m,n}^\kappa(x,y) = -A_{m,n}(x,y).$$

We have the following:

**Theorem 5.2.**

1. For any $l \in \mathbb{Z}$ and $\Delta \in \mathbb{R},$

$$A_{m+l,n+l}(x+\Delta, y+\Delta) = A_{m,n}(x,y).$$

2. For any $v \in \mathbb{R},$

$$-A_{0,n}(0, vn) = -A_{0,n}(0, 0) - \frac{v^2}{2} n.$$

3. There is a constant $\alpha_{0,0} \in \mathbb{R}$ such that for any $v \in \mathbb{R},$

$$\lim_{n \to \infty} \frac{-A_{0,n}(0, vn)_{a.s.}}{n} = \alpha_0(v) := \alpha_{0,0} - \frac{v^2}{2}.$$

It is natural to define

$$p_n(\kappa) = \begin{cases} \kappa \ln Z_{m,n}^\kappa(0,0), & \kappa \in (0,1], \\ -A_{0,n}(0,0), & \kappa = 0. \end{cases}$$

It follows from (5.3) that $p_n(\kappa)$ is continuous for $\kappa \in [0,1].$

6. **Concentration inequality for free energy**

The aim of this section is to prove a concentration inequality for the free energy $p_n(\kappa)$. In conjunction with the shape function convexity, it will help us to establish straightness estimates.

**Theorem 6.1.** There are positive constants $c_0, c_1, c_2, c_3$ such that for all $n > c_0$ and all $u \in (c_3 n^{1/2} \ln^{3/2} n, n \ln n],$

$$\mathbb{P}\{ |p_n(\kappa) - \alpha_{0,n}| \leq u, \kappa \in [0,1] \} \geq 1 - c_1 \exp\left\{ -c_2 \frac{u^2}{n \ln^2 n} \right\}.$$

Similar inequalities for fixed $\kappa$ were established in [Bak16] and [BL16]. Theorem 6.1 is a nontrivial improvement of those bounds since it estimates the probability of the intersection of fixed $\kappa$ events over all $\kappa \in [0,1].$

6.1. **A simpler concentration inequality.** The first step in proving Theorem 6.1 is to obtain a concentration of $p_n(\kappa)$ around its expectation.

**Lemma 6.1.** There are positive constants $b_0, b_1, b_2, b_3$ such that for all $n \geq b_0$, all $\kappa \in [0,1]$ and all $u \in (b_3, n \ln n],$

$$\mathbb{P}\{ |p_n(\kappa) - \mathbb{E}p_n(\kappa)| \leq u \} \geq 1 - b_1 \exp\left\{ -b_2 \frac{u^2}{n \ln^2 n} \right\}.$$

In comparison with inequalities in [Bak16] and [BL16], the important step here is choosing the constants $b_i$'s uniformly over all $\kappa \in [0,1].$ However, the event on the left-hand side is still defined for an arbitrary but fixed $\kappa \in [0,1].$ The proof of this lemma is based on some auxiliary results that we prove first.
For a Borel set \( B \) of \( \gamma \) high probability, polymer measures assign small weights to the path summing over integer 

\[
Σ^{m,n}(γ) = \left[ \sum_{j=m+1}^{n} (γ_j - γ_{j-1})^2 - \frac{(γ_n - γ_m)^2}{n - m} \right]^{1/2}.
\]

The function \( Σ^{m,n}(\cdot) \) compares the action of a path \( γ \) between time \( m \) and \( n \) to the action of the straight line connecting \( (m, γ_m) \) and \( (n, γ_n) \). It is also easy to check that \( Σ^{m,n}(\cdot) \) is invariant under space translations and shear transformations, namely, for any path \( γ \), \( Σ^{m,n}(γ) = Σ^{m,n}(θ_{−j}γ) = Σ^{m,n}(L^νγ) \) for any \( ν, x \in \mathbb{R} \).

The next lemma summarizes various estimates which reflect the idea that with high probability, polymer measures assign small weights to the path \( γ \) that has large values of \( Σ^{m,n}(γ) \). To state the lemma, we need some more notations.

Let us define the set of paths

\[
E^{m,n}_s = \left\{ γ : \frac{1}{\sqrt{n}} Σ^{m,n}(γ) ∈ [s, s + 1) \right\}, \quad s ∈ \mathbb{Z}.
\]

For a Borel set \( B \subset \mathbb{R}^{n−m−1} \), let us define

\[
Z^{m,n}_{x,y;κ}(B) = \int_{\mathbb{R} × B × \mathbb{R}} e^{-κ^{-1}A^{m,n}(x, m+1, \ldots, x_n) δ_x(dx_x)} dx_m \ldots dx_n − δ_y(dx_n).
\]

Let \( π_{m,n} \) denote the restriction of a vector or sequence onto the time interval \([m, n]\).

For a Borel set \( D \subset \mathbb{R}^ω = S_{s,*}^{−∞,∞} \), we define

\[
\mu^{m,n}_{x,y;κ}(D) = \mu^{m,n}_{x,y;κ}(π_{m,n}D), \quad Σ^{m,n}_{x,y;κ}(D) = Σ^{m,n}_κ(x, y, D) = Σ^{m,n}_{x,y;κ}μ^{m,n}_{x,y;κ}(D).
\]

**Lemma 6.2.** Let \( n ≥ 2 \). There are constants \( d_1 > 0, R_1 > 0 \) such that if \( s, s' ≥ R_1 \), then the following statements hold:

\[
\begin{align*}
(6.3) \quad & P\left\{ Z^{0,n}_{x,y;κ}(0, 1]^{n-1} > 2^{-κ^{-1}s}n, \quad x, y \in [0, 1], \quad κ \in (0, 1] \right\} ≥ 1 - e^{-d_1sn}, \\
(6.4) \quad & P\left\{ Z^{0,n}_{x,y;κ}(E_{s'}^0) ≤ 2^{-κ^{-1}s'2n-1}, \quad x, y \in [0, 1], \quad κ \in (0, 1] \right\} ≥ 1 - e^{-d_1s'n}, \\
(6.5) \quad & P\left\{ Z^{0,n}_{x,y;κ}(\bigcup_{s' ≥ s} E_{s'}^0) ≤ 2^{-κ^{-1}s2n}, \quad x, y \in [0, 1], \quad κ \in (0, 1] \right\} ≥ 1 - 3e^{-d_1s'n}, \\
(6.6) \quad & P\left\{ \mu^{0,n}_{x,y;κ}(\bigcup_{s' ≥ s} E_{s'}^0) ≤ 2^{-κ^{-1}s}n, \quad x, y \in [0, 1], \quad κ \in (0, 1] \right\} ≥ 1 - 3e^{-d_1s'n}, \\
(6.7) \quad & P\left\{ \mu^{0,n}_{x,y;κ}(γ : \frac{1}{\sqrt{n}} \max_{1 ≤ j ≤ n} |γ_j| ≥ s) ≤ 2^{-κ^{-1}s}n, \quad x, y \in [0, 1], \quad κ \in (0, 1] \right\} ≥ 1 - 3e^{-d_1s'n}.
\end{align*}
\]

**Proof:** It suffices to show (6.3) and (6.4). Then (6.5) will follow from (6.4) by summing over integer \( s ≥ s' \), and (6.6) from (6.5) and (6.4). Finally, the convexity of \( z → z^2 \) and Jensen’s inequality imply that for all \( κ ∈ S_{s,*}^{0,n} \) and all \( x, y \in [0, 1], \)

\[
\begin{align*}
[Σ^{0,n}(γ)]^2 & ≥ \frac{n}{n} \ge \frac{1}{n} \left( \sum_{j=1}^{n} |γ_j - γ_{j-1}| \right)^2 - \frac{1}{n} \\
& ≥ \frac{1}{n} \left[ 2\left( \max_{1 ≤ j ≤ n-1} |γ_j| - 1 \right) \right]^2 - \frac{1}{n}.
\end{align*}
\]

Therefore, when \( s \) is large, \( \max_{1 ≤ j ≤ n-1} |γ_j| ≥ sn \) implies \( Σ^{0,n}(γ) ≥ s√n \), so (6.7) holds.

By definition (6.2), we have

\[
Z^{0,n}_{x,y;κ}([0, 1]^{n-1}) ≥ e^{-κ^{-1}[n/2+F_{κ}^1([0, 0]])}, \quad x, y \in [0, 1],
\]

\]}
where \( F^*_n(i_1, ..., i_n) = \sum_{j=1}^n F^*_n(j, i_j) \) (see (2.12) for the definition of \( F^*_n \)). So, for all \( x, y \in [0, 1], \kappa \in (0, 1), \)

\[
\text{(6.10)} \quad \{ \omega : Z_{x,y;\kappa}^0([0, 1]^{n-1}) < 2^{-\kappa^{-1}-s_n} \} \subset \{ \omega : n(s\ln 2 - 1/2) < F^*_n(0, ..., 0) \}.
\]

By Markov inequality, we have

\[
\text{(6.9)} \quad P \left\{ F^*_n(0, ..., 0) > r \right\} \leq e^{-\eta r} E e^{\sum_{j=1}^n \eta F^*_n(j, 0)} \leq e^{-\eta r} (E e^{\eta F^*_n(0, 0)})^n.
\]

Combining (6.8) and (6.9), we obtain (6.3): for sufficiently large \( n \),

\[
\text{(6.11)} \quad \gamma \left( \sum_{j=1}^n k_j^2 \right) \leq \sqrt{n} \left( \sum_{j=1}^n \tilde{k}_j^2 \right) \leq \sqrt{(s+1)n^2 + n}.
\]

Next we turn to (6.4). In proving this, we will write \( s \) instead of \( s' \). Let us define

\[
S^n_s = \{ (i_1, ..., i_{n-1}) : \exists \gamma \in \tilde{E}^{0, n}_s, x, y \in [0, 1] \text{ s.t. } [\gamma_j] = i_j, 1 \leq j \leq n - 1 \},
\]

where \( \tilde{E}^{0, n}_s = E^*_s \cap (\bigcup_{x,y \in [0, 1]} S^{0, n}_{x,y}) \). Then we have

\[
\text{(6.10)} \quad Z_{x,y;\kappa}^0(E^{0, n}_s) \leq |S^n_{s}| e^{-\left( \frac{n}{2} + s^2 + F^*_n(\gamma) \right)} , \quad x, y \in [0, 1], \kappa \in (0, 1],
\]

where \( F^*_{w,n,s} = \max\{ F^*_w(i_1, ..., i_{n-1}, 0) : (i_1, ..., i_{n-1}) \in S^n_s \} \).

We need to estimate the size of \( S^n_s \). For \( 1 \leq j \leq n \), let us define \( k_j = \gamma_j - \gamma_{j-1} \) and \( \tilde{k}_j = [\gamma_j] - [\gamma_{j-1}] \). Clearly, \( |k_j - \tilde{k}_j| \leq 2 \). If \( \gamma \in \tilde{E}^{0, n}_s \), then the Cauchy–Schwarz inequality implies

\[
\sum_{j=1}^n |k_j| \leq \sqrt{n} \sqrt{\sum_{j=1}^n k_j^2} \leq \sqrt{(s+1)n^2 + n}.
\]

Comparing \( \sum_{j=1}^n k_j^2 \) and \( \sum_{j=1}^n \tilde{k}_j^2 \), we obtain

\[
\left| \sum_{j=1}^n k_j^2 - \sum_{j=1}^n \tilde{k}_j^2 \right| \leq \sum_{j=1}^n |k_j - \tilde{k}_j||k_j + \tilde{k}_j| \leq 2 \sum_{j=1}^n (2|k_j| + 2) \leq 8sn.
\]

Therefore, \( \gamma \in \tilde{E}^{0, n}_s \) implies that

\[
\text{(6.11)} \quad \sum_{j=1}^n \tilde{k}_j^2 \leq (s+1)^2 n + 8sn =: [r(n)]^2.
\]

The size of \( S^n_s \) is bounded by the number of \( n \)-vectors \( (\tilde{k}_0, ..., \tilde{k}_{n-1}) \) satisfying (6.11), which is then bounded by the volume of \( n \)-dimensional ball of radius \( r(n) + \frac{\sqrt{n}}{2} \). (To obtain this estimate, we consider unit cubes centered at integer points, with half diagonal lengths \( \frac{\sqrt{n}}{2} \).) Hence, when \( s \) is large,

\[
\text{(6.12)} \quad |S^n_s| \leq \frac{\pi^{n/2}}{\Gamma(n/2 + 1)} (r(n) + \frac{\sqrt{n}}{2})^n \leq \frac{\pi^{n/2}}{\Gamma(n/2 + 1)} \cdot (K_1 s \sqrt{n})^n \leq e^{(\ln s + K_2)n},
\]

where \( K_1, K_2 \) are constants and we used \( \ln \Gamma(z) = z \ln z - z + O(\ln z), z \to \infty. \)
Choosing where

Lemma 6.3.

be large.

for some constant $K$ (6.16)

Since the distribution of $F^*$ does not depend on the choice of the vector $(i_0, ..., i_{n-1})$, we obtain that for any $r > 0$,

(6.14)

Combining (6.10), (6.12), (6.13), and (6.14), we see that

Choosing $s$ large enough concludes the proof of (6.4).

Let $E^{m,n}_{\leq R_1} = \bigcup_{s \leq R_1} E^{m,n}_{s}$. The following lemma states that $Z^{0,n}_{x,y,R_1}(E^{0,n}_{\leq R_1})$ cannot be large.

Lemma 6.3. There is some constant $d$ such that for sufficiently large $t$,

\[ P\left\{ Z^{0,n}_{x,y,R_1}(E^{0,n}_{\leq R_1}) \leq e^{-\kappa^{-1}tn-1}, \ x, y \in [0,1]; \ \kappa \in (0,1) \right\} \geq 1 - e^{-dt} n. \]

Proof: We will continue using the notations from the proof of Lemma 6.2. Let us define $S^n_{\leq R_1} = \bigcup_{s \leq R_1} S^n_{s}$. Similarly to (6.12) and (6.10), we have

(6.15)

\[ |S^n_{\leq R_1}| \leq \frac{\pi n/2}{\Gamma(n/2 + 1)} (TN/2) n \leq e^{K_1 n} \]

for some constant $K_1$, and

(6.16)

\[ Z^{0,n}_{x,y,R_1}(E^{0,n}_{\leq R_1}) \leq |S^n_{\leq R_1}| e^{\kappa^{-1}F^*_{\omega,n,\leq R_1}}, \ x, y \in [0,1]; \ \kappa \in (0,1], \]

where $F^*_{\omega,n,\leq R_1} = \max\{F^*_{\omega}(i_1, ..., i_{n-1}) : (i_1, ..., i_{n-1}) \in S^n_{\leq R_1} \}$. Therefore, for $x, y \in [0,1], \ \kappa \in (0,1)$ and sufficiently large $t$,

\[ \{ \omega : Z^{0,n}_{x,y,R_1}(E^{0,n}_{\leq R_1}) > e^{\kappa^{-1}tn-1} \} \subset \{ \omega : F^*_{\omega,n,\leq R_1} > \kappa \ln |S^n_{\leq R_1}| + \kappa \} \]

\[ \subset \{ \omega : F^*_{\omega,n,\leq R_1} > tn - \kappa (\ln |S^n_{\leq R_1}| + 1) \} \]

\[ \subset \{ \omega : F^*_{\omega,n,\leq R_1} > tn/2 \}. \]

Combining this with (6.9) and (6.15), we obtain

\[ P\left\{ Z^{0,n}_{x,y,R_1}(E^{0,n}_{\leq R_1}) \leq e^{-\kappa^{-1}tn-1}, \ x, y \in [0,1]; \ \kappa \in (0,1) \right\} \geq 1 - P\{ \omega : F^*_{\omega,n,\leq R_1} > tn/2 \} \]

\[ \geq 1 - |S^n_{\leq R_1}| P\{ \omega : F^*_{\omega}(0, ..., 0) > tn/2 \} \]

\[ \geq 1 - e^{-(n/2 - K_1)n} (e^{nF^*_{\omega}(0,0)}) n. \]

Choosing $t$ large enough concludes the proof.

Combining (6.5) with $s = R_1$ and Lemma 6.3 we obtain the following lemma.
Lemma 6.4. There are constants $d_2, R_2 > 0$ such that for all $t \geq R_2$,
\[
P\{Z_{x,y; \kappa}^n \leq e^{\kappa^{-1}tn}, \ x, y \in [0, 1]; \ \kappa \in (0, 1]\} \geq 1 - e^{-d_2tn}.
\]

Also, as a consequence of (6.15), we have the following upper bound for the Lebesgue measure of $E_{\leq R_1}^n$.

Lemma 6.5. There is a constant $d_3 > 0$ such that $|E_{\leq R_1}^n| \leq e^{d_3n}$.

Using Lemma 6.4 and (6.3) of Lemma 6.2, we have estimates on all moments of the logarithm of partition functions.

Lemma 6.6. There are constants $M(p), p \in \mathbb{N}$, such that for all $\kappa \in (0, 1]$ and any Borel set $B$ satisfying $[0, 1]^{n-1} \subset B \subset \mathbb{R}^{n-1}$,
\[
\mathbb{E} |\kappa \ln Z_{0,0; \kappa}^n (B)|^p \leq M(p)n^p.
\]

Let us denote $Z_{0,0; \kappa}^n$ by $Z_{\kappa}^n$.

Lemma 6.7. There is a constant $D_1 > 0$ such that
\[
0 \leq \kappa \left( \mathbb{E} \ln Z_{\kappa}^n - \mathbb{E} \ln Z_{\kappa}(E_{\leq R_1}^n) \right) \leq D_1, \quad n \in \mathbb{N}, \ \kappa \in (0, 1].
\]

Proof: The first inequality is obvious since $Z_{\kappa}(E_{\leq R_1}^n) \leq Z_{\kappa}^n$. Let
\[
\Lambda = \{ Z_{\kappa}^n(E_{\leq R_1}^n)/Z_{\kappa}^n \leq 1 - 2^{-\kappa^{-1}R_1n} \}.
\]

By (6.6) of Lemma 6.2 $P(\Lambda) \leq 3e^{-d_1R_1n}$. By Lemma 6.6 we have
\[
\mathbb{E} |\kappa \ln Z_{\kappa}^n (E_{\leq R_1}^n)|^2 \leq M(2)n^2, \quad \mathbb{E} |\kappa \ln Z_{\kappa}^n|^2 \leq M(2)n^2.
\]

The lemma then follows from
\[
\kappa \left( \mathbb{E} \ln Z_{\kappa}^n - \mathbb{E} \ln Z_{\kappa}(E_{\leq R_1}^n) \right)
\leq - \kappa \mathbb{E} \ln \left( Z_{\kappa}^n(E_{\leq R_1}^n)/Z_{\kappa}^n \right) 1_\Lambda + \kappa \mathbb{E} (|\ln Z_{\kappa}^n| + |\ln Z_{\kappa}(E_{\leq R_1}^n)|) 1_{\Lambda}
\leq - \kappa \ln(1 - 2^{-\kappa^{-1}R_1n}) + \kappa \sqrt{2(\mathbb{E} \ln^2 Z_{\kappa}^n + \mathbb{E} \ln^2 Z_{\kappa}(E_{\leq R_1}^n))} \sqrt{P(\Lambda)}
\leq |\ln(1 - 2^{-R_1})| + \sqrt{4M(2)n^2 \cdot 3e^{-d_1R_1n}}.
\]

Let us define
\[
\tilde{\rho}_n(\kappa) = \begin{cases}
\kappa \ln Z_{\kappa}^n(E_{\leq R_1}^n), & \kappa \in (0, 1], \\
- \min\{ A_{0,0}(\gamma) : \gamma \in Z_{\kappa}^n \cap E_{\leq R_1}^n \}, & \kappa = 0.
\end{cases}
\]

Clearly, $\tilde{\rho}_n(\cdot)$ is continuous on $[0, 1]$. We recall that $p_n(\cdot)$ defined in (5.4) is also continuous on $[0, 1]$. Since Lemma 6.3 implies uniform integrability of $(p_n(\kappa))_{\kappa \in [0, 1]}$ and $(\tilde{\rho}_n(\kappa))_{\kappa \in [0, 1]}$, we immediately obtain that both $E\rho_n(\kappa)$ and $E\tilde{\rho}_n(\kappa)$ are continuous for $\kappa \in [0, 1]$. The next lemma estimates how well $\tilde{\rho}_n(\kappa)$ approximates $p_n(\kappa)$.

Lemma 6.8. If $n$ is sufficiently large, then for all $\kappa \in [0, 1]$,
\[
P \{ |p_n(\kappa) - \tilde{\rho}_n(\kappa)| \leq 1, \ \kappa \in [0, 1] \} \geq 1 - 3e^{-d_1R_1n}
\]
and
\[
|E\rho_n(\kappa) - E\tilde{\rho}_n(\kappa)| \leq D_1, \quad \kappa \in [0, 1].
\]
Proof: Due to (6.6), we have
\[
\mathbb{P}\{ |p_n(\kappa) - \tilde{p}_n(\kappa)| \leq \ln(1 - 2^{-\kappa^{-1}R_1n}), \ \kappa \in (0, 1] \}
\geq \mathbb{P}\{ 0 < n \rightarrow \kappa (\bigcup_{s' \geq R_1} E_{s'}^{\kappa}) \leq 2^{-\kappa^{-1}R_1n}, \ \kappa \in (0, 1] \} \geq 1 - 3e^{-d_1R_1n}.
\]
Then (6.17) follows from this and the continuity of \( p_n \) and \( \tilde{p}_n \) in \( \kappa \). The second inequality (6.18) follows from Lemma 6.7 and the continuity of \( \mathbb{E}p_n \) and \( \mathbb{E}\tilde{p}_n \) in \( \kappa \).

\[\Box\]

To obtain a concentration inequality for \( \tilde{p}_n(\kappa) \), we need Azuma’s inequality:

**Lemma 6.9.** Let \( (M_k)_{0 \leq k \leq N} \) be a martingale with respect to a filtration \( (\mathcal{F}_k)_{0 \leq k \leq N} \). Assume there is a constant \( c \) such that \( |M_k - M_{k-1}| \leq c, \ 1 \leq k \leq N \). Then
\[
\mathbb{P}\{ |M_N - M_0| \geq x \} \leq 2 \exp\left(\frac{-x^2}{2Nc^2}\right).
\]

To apply Azuma’s inequality, we need to introduce an appropriate martingale with bounded increments. The function \( \tilde{p}_n(\kappa) \) depends only on the potential process on \( B = \{1, \ldots, n\} \times [-R_1n, R_1n] \) since \( \pi^{1,n}E_0^{\kappa, R_1} \subset [-R_1n, R_1n]^n \), so we need an additional truncation of the potential on \( B \). Moreover, the truncation should be independent of \( \kappa \).

Let \( b > 4/\eta \), where \( \eta \) is taken from the condition (A5). For \( 1 \leq k \leq n \) and \( x \in [-R_1n, R_1n] \), define (suppressing the dependence on \( n \) for brevity)
\[
\xi_k = \max\{F_k^*(j) : j = -R_1n, -R_1n + 1, \ldots, R_1n - 1\}, \ k = 0, \ldots, n,
\]
and setting \( x_0 = x_n = 0 \),
\[
\tilde{p}_n(\kappa, \tilde{F}) = \begin{cases} 
\kappa \ln \mathbb{P}_{\pi^{1,n}E_0^{\kappa, R_1}} \prod_{j=1}^{n} g_{e}(x) - x_j - 1 \cdot e^{-\kappa^{-1}F_j(x_j)} N_0(dx_0) dx_1 \ldots dx_{n-1} N_0(dx_n), \ \kappa \in (0, 1], \\
\min_{(x_0, x_1, \ldots, x_{n-1}, x_n) \in \pi^{1,n}E_0^{\kappa, R_1}} \sum_{j=1}^{n} \left[\frac{1}{2} (x_j - x_{j-1})^2 + F_j(x_j)\right], \ \kappa = 0.
\end{cases}
\]

**Lemma 6.10.** For sufficiently large \( n \in \mathbb{N} \), the following holds true:
\[
(6.19) \quad \mathbb{E}\exp\left(\frac{\eta}{2} \xi_k \mathbb{1}_{\{\xi_k \geq b \ln n\}}\right) \leq 2,
\]
\[
(6.20) \quad \mathbb{E}\xi_k \leq 6 \ln n + 4/\eta,
\]
\[
(6.21) \quad \mathbb{P}\{|\tilde{p}_n(\kappa) - \tilde{p}_n(\kappa, \tilde{F})| \leq x, \ \kappa \in [0, 1]\} \geq 1 - 2e^{-\eta x^2/2}, \ \ x > 0,
\]
\[
(6.22) \quad \mathbb{E}|p_n(\kappa) - \mathbb{E}\tilde{p}_n(\kappa, \tilde{F})| \leq 4/\eta, \ \kappa \in [0, 1].
\]

Proof: Since \( \xi_k \) is the maximum of \( 2R_1n \) random variables with the same distribution, we have
\[
\mathbb{E}\exp\left(\frac{\eta}{2} \xi_k \mathbb{1}_{\{\xi_k \geq b \ln n\}}\right) \leq 1 + \mathbb{E}e^{\frac{\eta}{2} \xi_k} \mathbb{1}_{\{\xi_k \geq b \ln n\}} \leq 1 + \mathbb{E} \sum_{j=-R_1n}^{R_1n-1} e^{\frac{\eta}{2} F_k^*(j)} \mathbb{1}_{\{F_k^*(j) > b \ln n\}}
\leq 1 + 2R_1n e^{\frac{\eta}{2} F_k^*(0)} \mathbb{1}_{\{F_k^*(0) > b \ln n\}} \leq 1 + 2R_1n \frac{e^{\eta F_k^*(0)}}{e^{\frac{\eta}{2} \ln n}}
\leq 1 + \frac{c}{n^{b-1}},
\]
where \( c = 2R_1 \mathbb{E}e^{\eta F_k^*(0)} \) is a constant. Now (6.19) follows from \( b > 4/\eta \).
If \( x > b \ln n \), then by Markov inequality and (6.19), we have
\[
P\{\xi_k \geq x\} \leq P\{\xi_k 1_{\{\xi_k \geq b \ln n\}} \geq x\} \leq e^{-nx/2}E \exp\left(\frac{\eta}{2} \xi_k 1_{\{\xi_k \geq b \ln n\}}\right) \leq 2e^{-nx/2}
\]
for sufficiently large \( n \). This implies (6.20):
\[
E\xi_k \leq b \ln n + E\xi_k 1_{\{\xi_k \geq b \ln n\}} \leq b \ln n + \int_{b \ln n}^{\infty} P\{\xi_k \geq x\} dx \leq b \ln n + \frac{4}{\eta}.
\]
It follows from the definition of \( \tilde{p}_n(\kappa, \tilde{F}) \) that for all \( \kappa \in [0, 1] \),
\[
|\tilde{p}_n(\kappa) - \tilde{p}_n(\kappa, \tilde{F})| \leq \sum_{k=1}^{n} \xi_k 1_{\{\xi_k > b \ln n\}}.
\]
By Markov inequality, the i.i.d. property of \( (\xi_k) \) and (6.24), we have
\[
P\{|\tilde{p}_n(\kappa) - \tilde{p}_n(\kappa, \tilde{F})| \leq x, \ \kappa \in [0, 1]\} \geq 1 - P\left\{\frac{\eta}{2} \sum_{k=1}^{n} \xi_k 1_{\{\xi_k > b \ln n\}} > \frac{\eta x}{2}\right\}
\geq 1 - e^{-\eta x/2}E \exp\left(\frac{\eta}{2} \sum_{k=1}^{n} \xi_k 1_{\{\xi_k > b \ln n\}}\right)
= 1 - e^{-\eta x/2}\left(E \exp\left(\frac{\eta}{2} \xi_0 1_{\{\xi_0 > b \ln n\}}\right)\right)^n
\geq 1 - e^{-\eta x/2}(1 + c/\eta^{\eta/2} - 1)^n.
\]
Since \( b > 4/\eta \), (6.21) follows. It immediately implies
\[
|E\tilde{p}_n(\kappa) - E\tilde{p}_n(\kappa, \tilde{F})| \leq E|\tilde{p}_n(\kappa) - \tilde{p}_n(\kappa, \tilde{F})|
= \int_{0}^{\infty} P\{|\tilde{p}_n(\kappa) - \tilde{p}_n(\kappa, \tilde{F})| > x\} dx \leq 4/\eta,
\]
so (6.22) is also proved.

\[ \square \]

**Lemma 6.11.** For all \( n \in \mathbb{N}, x > 0 \) and all \( \kappa \in [0, 1] \),
\[
P\{|\tilde{p}_n(\kappa, \tilde{F}) - E\tilde{p}_n(\kappa, \tilde{F})| > x\} \leq 2 \exp\left\{-\frac{x^2}{8nb^2 \ln^2 n}\right\},
\]

**Proof:** Let us introduce the following martingale \( (M_k, \mathcal{F}_k)_{0 \leq k \leq n} \):
\[
M_k = E(\tilde{p}_n(\kappa, \tilde{F}) | \mathcal{F}_k), \quad 0 \leq k \leq n,
\]
where
\[
\mathcal{F}_0 = \{\emptyset, \Omega\}, \quad \mathcal{F}_k = \sigma(F_i(x) : 1 \leq i \leq k), \quad k = 1, \ldots, n.
\]
If we can show that \(|M_k - M_{k-1}| \leq 2b \ln n, 1 \leq k \leq n\), then the conclusion of the lemma follows immediately from Azuma’s inequality (Lemma 6.9).

For a process \( \tilde{G} \), an independent distributional copy of \( \tilde{F} \), let us define
\[
\tilde{Z}_k^n([\tilde{F}, \tilde{G}]|_k) = \int_{|x_i| \leq K_1, n} \prod_{i=1}^{k} g_\kappa(x_i - x_{i-1})e^{-\kappa^{-1} F_i(x_i)}
\cdot \prod_{i=k+1}^{n} g_\kappa(x_i - x_{i-1})e^{-\kappa^{-1} \tilde{G}_i(x_i)}\delta_0(dx_0)dx_1 \cdots dx_{n-1}\delta_0(dx_n).
\]
Denoting by $P_k$ the distribution of $\tilde{F}_k(\cdot)$, we obtain for $\kappa \in (0, 1]$,
\[
|M_k - M_{k-1}| = \kappa \left| \int \ln \tilde{Z}_n^n(\{\tilde{F}, \tilde{G}\}|k) \prod_{i=k+1}^n P_i(d\tilde{G}_i) - \int \ln \tilde{Z}_n^n(\{\tilde{F}, \tilde{G}\}|k-1) \prod_{i=k}^n P_i(d\tilde{G}_i) \right|
\leq \kappa \left| \int \ln \tilde{Z}_n^n(\{\tilde{F}, \tilde{G}\}|k) - \ln \tilde{Z}_n^n(\{\tilde{F}, \tilde{G}\}|k-1) \right| \prod_{i=k}^n P_i(d\tilde{G}_i)
\leq \int \left( \sup_{|x| \leq R_1 n} |\tilde{F}_k(x)| + \sup_{|x| \leq R_1 n} |\tilde{G}_k(x)| \right) \prod_{i=k}^n P_i(d\tilde{G}_i) \leq 2b \ln n,
\]
since $|\tilde{F}_k(x)|$ and $|\tilde{G}_k(x)|$ are bounded by $b \ln n$. By taking $\kappa \downarrow 0$ in the above inequality (or using that resampling the potential field $\{F_i(\cdot)\}$ at any given $i$ will change the optimal action by at most $2b \ln n$), we can see that $|M_k - M_{k-1}| \leq 2b \ln n$ also holds when $\kappa = 0$. This completes the proof.

We note that in lemma 6.11, we estimate the probability of an event defined for a fixed $\kappa$, since the Azuma inequality applies to a fixed martingale and cannot be immediately used for uniform concentration of a family of martingales parametrized by $\kappa$.

**Proof of Lemma 6.1** Suppose $u \in \left(3(D_1 + 4/\eta + 3), n \ln n\right]$. Then
\[
P \left\{ |p_n(\kappa) - E p_n(\kappa)| > u \right\} \leq P \left\{ |p_n(\kappa) - \tilde{p}_n(\kappa)| > 1 \right\} + P \left\{ |\tilde{p}_n(\kappa) - \tilde{p}_n(\kappa, \tilde{F})| > \frac{u}{3} \right\}
+ P \left\{ |\tilde{p}_n(\kappa, \tilde{F}) - E \tilde{p}_n(\kappa, \tilde{F})| > \frac{u}{3} \right\} + P \left\{ |E \tilde{p}_n(\kappa, \tilde{F}) - E \tilde{p}_n(\kappa)| > 4/\eta + 1 \right\}
+ P \left\{ |E \tilde{p}_n(\kappa) - E p_n(\kappa)| > D + 1 \right\}.
\]
By (6.22) and (6.18), the last two terms equal 0. The first three terms can be bounded by using (6.17), (6.21) and Lemma 6.11 respectively. Combining all these estimates together, we obtain
\[
P \left\{ |p_n(\kappa) - E p_n(\kappa)| > u \right\} \leq 3e^{-d_1 R_1 n} + 2e^{-\frac{u n}{b_1}} + 2e^{-\frac{u^2}{2b_2 n \ln n}} \leq b_1 e^{-b_2 n \frac{u^2}{n \ln^2 n}},
\]
for some constants $b_1, b_2 > 0$, where in the last inequality we use $u \leq n \ln n$. □

We also have obtained a similar concentration inequality for $\tilde{p}_n(\kappa)$ which will be used in the next section.

**Lemma 6.12.** Let $b_i$'s be the constants in Lemma 6.1. Then for all $n \geq b_0$, all $\kappa \in [0, 1]$ and all $u \in (b_3, n \ln n]$, \[
P \left\{ |\tilde{p}_n(\kappa) - E \tilde{p}_n(\kappa)| \leq u \right\} \geq 1 - b_1 \exp \left\{ -b_2 \frac{u^2}{n \ln^2 n} \right\}.
\]

**6.2. Uniform continuity of the shape function in viscosity.** To go from Lemma 6.1 to Theorem 6.1, we have to estimate the difference of $E p_n(\kappa)$ and $\alpha_{0, \kappa} n$, and to move $\kappa \in [0, 1]$ inside the events of interest. The key point is to establish the continuity of $\alpha_{0, \kappa}$ for $\kappa \in [0, 1]$. 

**Lemma 6.13.** \(1\) There is a constant $b_4$ such that for sufficiently large $n$,
\[
|E p_n(\kappa) - \alpha_{0, \kappa} n| \leq b_4 n^{1/2} \ln^2 n, \quad \kappa \in [0, 1].
\]
\(2\) $\alpha_{0, \kappa}$ is continuous $\kappa \in [0, 1]$. 

Let us derive Theorem 6.1 from 6.13 and the results from section 6.1 first.

**Proof of Theorem 6.1.** Let us define

\[ q_n(\kappa) = \tilde{p}_n(\kappa) - \kappa \ln |E_{\leq R_1}^0|, \quad \kappa \in [0, 1], \]

where \(| \cdot |\) denotes the Lebesgue measure of a set. When \(\kappa > 0\), we have

\[ q_n(\kappa) = \ln \left( \int_{E_{\leq R_1}^0} \frac{1}{|E_{\leq R_1}^0|} e^{-\kappa^{-1} A^0, n(\gamma)} \, d\gamma \right)^\kappa. \]

Therefore, by Lyapunov’s inequality, \(q_n(\kappa)\) is decreasing in \(\kappa\). Then by Lemma 6.12 for all \(n \geq b_0\), all \(\kappa \in [0, 1]\) and \(x \in [b_0, n \ln n]\),

\[ \begin{aligned}
    (6.25) \quad &\mathbb{P}\left\{ |q_n(\kappa) - \mathbb{E}q_n(\kappa)| \leq x \right\} \geq 1 - b_1 \exp \left\{ -b_2 \frac{x^2}{n \ln^2 n} \right\},
    \end{aligned} \]

For fixed \(n\), since \(q_n(\cdot)\) is a continuous decreasing function, we can find \(M\) and \(0 = \kappa_1 < \kappa_2 < \ldots < \kappa_M = 1\) such that

\[ M \leq 2n^{-1/2} \left| \mathbb{E}q_n(1) - \mathbb{E}q_n(0) \right|, \]

and

\[ |\mathbb{E}q_n(\kappa_{i+1}) - \mathbb{E}q_n(\kappa_i)| \leq n^{1/2}, \quad 1 \leq i \leq M - 1. \]

To achieve this, we can choose \(\kappa_i\) one by one, starting with \(i = 1, 2\). Define the event \(\Lambda(x) = \{|q_n(\kappa_i) - \mathbb{E}q_n(\kappa_i)| \leq x, \, 1 \leq i \leq M\}.\) Then by (6.25),

\[ \begin{aligned}
    (6.26) \quad &\mathbb{P}(\Lambda(x)) \geq 1 - M \cdot b_1 \exp \left\{ -b_2 \frac{x^2}{n \ln^2 n} \right\}, \quad x \in (b_3, n \ln n].
    \end{aligned} \]

For \(\omega \in \Lambda(x)\) and \(\kappa \in [\kappa_i, \kappa_{i+1}]\), since \(q_n(\kappa)\) and \(\mathbb{E}q_n(\kappa)\) are both monotone in \(\kappa\),

\[ |q_n(\kappa) - \mathbb{E}q_n(\kappa)| = |\tilde{p}_n(\kappa) - \mathbb{E}\tilde{p}_n(\kappa)| \]

\[ \leq |q_n(\kappa_i) - \mathbb{E}q_n(\kappa_i)| \vee |q_n(\kappa_{i+1}) - \mathbb{E}q_n(\kappa_i)| \]

\[ \leq x + |\mathbb{E}q_n(\kappa_i) - \mathbb{E}q_n(\kappa_{i+1})| \]

\[ \leq x + n^{1/2}. \]

Combined with (6.26), this implies that

\[ (6.27) \quad \mathbb{P}\left\{ |\tilde{p}_n(\kappa) - \mathbb{E}\tilde{p}_n(\kappa)| \leq x + n^{1/2}, \, \kappa \in [0, 1] \right\} \geq 1 - M \cdot b_1 \exp \left\{ -b_2 \frac{x^2}{n \ln^2 n} \right\}, \]

for all \(x \in (b_3, n \ln n]\).

By Lemma 6.1 and (6.24), we have

\[ (6.28) \quad |\mathbb{E}\tilde{p}_n(\kappa) - \alpha_{0,n}| \leq D_1 + b_3 n^{1/2} \ln^2 n, \quad \kappa \in [0, 1]. \]

This and Lemma 6.5 imply

\[ |\mathbb{E}q_n(1) - \mathbb{E}q_n(0)| \leq |\mathbb{E}\tilde{p}_n(1) - \mathbb{E}\tilde{p}_n(0)| + |E_{\leq R_1}^0| \]

\[ \leq 2(D_1 + b_3 n^{1/2} \ln^2 n) + n|\alpha_{0,1} - \alpha_{0,0}| + d_3 n \]

\[ \leq Kn. \]

Hence \(M \leq 2Kn^{1/2}\). Using this upper bound on \(M\) and (6.27), (6.28), we complete the proof.

Next we turn to the proof of Lemma 6.13.

**Lemma 6.14.** There is positive constant \(b_5\) such that for all \(\kappa \in [0, 1]\) and sufficiently large \(n\),

\[ (6.29) \quad |\mathbb{E}\tilde{p}_n(\kappa) - 2\mathbb{E}\tilde{p}_n(\kappa)| \leq b_5 n^{1/2} \ln^2 n. \]
PROOF: Since \( p_n(\cdot) \) is continuous, it suffices to show \((6.29)\) i.e.,

\[
|E\ln Z_{0,0;\kappa}^{0,2n} - 2E\ln Z_{0,0;\kappa}^{0,n}| \leq b_\gamma n^{1/2} \ln^2 n,
\]
for \( \kappa \in (0, 1) \), and then use continuity of \( E p_n(\cdot) \).

For \( R_1 \) introduced in Lemma 6.2 define

\[
B = \{ \gamma : \max_{1 \leq i \leq 2n-1} |\gamma_i| \leq 2R_1 n \},
\]

\[
C = \{ \gamma : |\gamma_n - \gamma_{n+1}| \leq R_1 \sqrt{2n}, |\gamma_n - \gamma_{n-1}| \leq R_1 \sqrt{2n} \}.
\]

Since \( E_{0,2n}^{0,2n} \subset B \cap C \), Lemma 6.7 implies that

\[
(6.30) \quad |E\ln Z_{0,0;\kappa}^{0,2n}(B \cap C) - E\ln Z_{0,0;\kappa}^{0,2n}| \leq D_1.
\]

To prove the lemma, we need to bound \( E\ln Z_{0,0;\kappa}^{0,2n}(B \cap C) \) from above and from below using \( 2E\ln Z_{0,0;\kappa}^{0,n} \) plus some error terms. First, let us deal with the lower bound. By the definition of the sets \( B \) and \( C \), we have

\[
Z_{0,0;\kappa}^{0,2n}(B \cap C) \geq Z_{0,0;\kappa}^{0,2n}(B \cap C \cap \{ \gamma_n \in (0, 1) \}).
\]

Let us now compare the action of every path \( \gamma \) in \( B \cap C \cap \{ \gamma_n \in (0, 1) \} \) to the action of the modified path \( \tilde{\gamma} \) defined by \( \tilde{\gamma}_n = 0 \) and \( \tilde{\gamma}_j = \gamma_j \) for \( j \neq n \). We recall that the action of a path was defined in (2.7). Since \( |\gamma_{n+1} - \gamma_n| \leq R_1 \sqrt{2n} \), \( |\gamma_n - \gamma_{n-1}| \leq R_1 \sqrt{2n} \), and \( |\gamma_n| \leq 1 \), we get

\[
|A_{0,2n}(\gamma) - A_{0,2n}(\tilde{\gamma})| \leq \frac{1}{2} |(\gamma_{n+1} - \gamma_n)^2 - \gamma_{n+1}^2 + (\gamma_{n-1} - \gamma_n)^2 - \gamma_{n-1}^2| + 2F^*_n,\omega(0)
\]

\[
\leq 2R_1 \sqrt{2n} + 1 + 2F^*_n,\omega(0).
\]

So, there is a constant \( K_1 > 0 \) such that

\[
(6.31) \quad Z_{0,0;\kappa}^{0,2n}(B \cap C) \geq Z_{0,0;\kappa}^{0,n}(D^-) Z_{0,0;\kappa}^{0,2n}(D^+) e^{-K_1 \sqrt{n} - 2F^*_n,\omega(0)},
\]

where

\[
D^- = \{ \gamma : |\gamma_{n-1}| \leq R_1 \sqrt{2n} + 1, |\gamma_i| \leq 2R_1 n, 1 \leq i \leq n - 1 \},
\]

\[
D^+ = \{ \gamma : |\gamma_{n+1}| \leq R_1 \sqrt{2n} + 1, |\gamma_i| \leq 2R_1 n, n + 1 \leq i \leq 2n - 1 \}.
\]

Since \( E_{\leq R_1}^{0,n} \subset D^- \) and \( E_{\leq R_1}^{0,2n} \subset D^+ \), Lemma 6.7 implies that

\[
K_1|E \ln Z_{0,0;\kappa}^{0,n}(D^-) - E \ln Z_{0,0;\kappa}^{0,n}| \leq D_1, \quad K_1|E \ln Z_{0,0;\kappa}^{0,2n}(D^+) - E \ln Z_{0,0;\kappa}^{0,2n}| \leq D_1.
\]

Combining this with \((6.31)\), we obtain

\[
K_1|E \ln Z_{0,0;\kappa}^{0,2n}(B \cap C) \geq K_1 \left( E \ln Z_{0,0;\kappa}^{0,n}(D^-) + E \ln Z_{0,0;\kappa}^{0,2n}(D^+) \right) - K_1 \sqrt{n} - 2F^*_n,\omega(0)
\]

\[
\geq K_1 \cdot 2E \ln Z_{0,0;\kappa}^{0,n} - 2D_1 - K_1 \sqrt{n} - 2F^*_n,\omega(0),
\]

where we used \( \ln Z_{0,0;\kappa}^{0,n} \leq \ln Z_{0,0;\kappa}^{0,2n} \) in the last inequality.

Next, let us turn to the upper bound. Similarly to \((6.31)\), we compare actions of generic paths in \( B \cap C \) to the actions of the modified paths with integer value at
time $n$:

$$Z_{0,\kappa}^{0,2n}(B \cap C) = \sum_{k=-2R_1n}^{2R_1n-1} Z_{0,\kappa}^{0,2n}(B \cap C \cap \{\gamma_n \in [k, k+1]\})$$

$$\leq \sum_{k=-2R_1n}^{2R_1n-1} Z_{\kappa}^{0,n}(0, k) Z_{\kappa}^{n,2n}(k, 0)e^{-\kappa^{-1}\left(K_1 \sqrt{n + 2F_{n,\omega}(k)}\right)}$$

$$\leq 4R_1n \max_k [Z_{\kappa}^{0,n}(0, k) Z_{\kappa}^{n,2n}(k, 0)]e^{-\kappa^{-1}\left(K_1 \sqrt{n + 2\max_k F_{n,\omega}(k)}\right)},$$

where the maxima are taken over $-2R_1n \leq k \leq 2R_1n - 1$. Taking logarithm and then expectation of both sides, we obtain

$$\kappa E \ln Z_{0,\kappa}^{0,2n}(B \cap C)$$

$$\leq \kappa \left(E \max_k \ln Z_{\kappa}^{0,n}(0, k) + E \max_k \ln Z_{\kappa}^{n,2n}(k, 0)\right) + \kappa \ln(4R_1n) + K_1 \sqrt{n + 2E \max_k F_{n,\omega}(k)}$$

$$\leq \max_k E \kappa \ln Z_{\kappa}^{0,n}(0, k) + E \max_k X_k + \max_k E \kappa \ln Z_{\kappa}^{n,2n}(k, 0) + \max_k Y_k + K_2(\ln n + \sqrt{n} + 1)$$

$$\leq 2\max_k E \ln Z_{0,\kappa}^{0,n} + E \left[\max_k X_k + \max_k Y_k\right] + K_2(\ln n + \sqrt{n} + 1),$$

for some constant $K_2 > 0$, where

$$X_k = \kappa \left(\ln Z_{\kappa}^{0,n}(0, k) - E \ln Z_{\kappa}^{0,n}(0, k)\right), \quad Y_k = \kappa \left(\ln Z_{\kappa}^{n,2n}(k, 0) - E \ln Z_{\kappa}^{n,2n}(k, 0)\right).$$

In the second inequality, we used (6.20) to conclude

$$E \max_{-2R_1n \leq k \leq 2R_1n-1} F_{n,\omega}(k) \leq b \ln(2n) + 4/\eta,$$

and in the third inequality, we used the fact that

$$E \ln Z_{\kappa}^{0,n}(0, k) \leq E \ln Z_{0,\kappa}^{0,n}, \quad E \ln Z_{\kappa}^{n,2n}(k, 0) \leq E \ln Z_{0,\kappa}^{n,2n} = E \ln Z_{0,\kappa}^{0,n}.$$
To bound the second term by a constant, we use Lemma 6.1
\[
P(\Lambda^c) \leq \sum_{k=-2R_1n}^{2R_1n-1} \left[ P\left( |\kappa| \ln Z^{0,n}_\kappa(0,k) - \mathbb{E} \ln Z^{0,n}_\kappa(0,k) | \geq rn^{1/2} \ln^{3/2} n \right) \right]
+ P\left( |\kappa| \ln Z^{n,2n}_\kappa(k,0) - \mathbb{E} \ln Z^{n,2n}_\kappa(k,0) | \geq rn^{1/2} \ln^{3/2} n \right)
\leq 8R_1nP\left( |\kappa| \ln Z^{n}_\kappa - \mathbb{E} \ln Z^{n}_\kappa | \geq rn^{1/2} \ln^{3/2} n \right)
\leq 8R_1nb_n \exp\{-b_2r^2 \ln n\},
\]
and choose \( r \) to ensure \( b_2r^2 > 4 \). This completes the proof. \( \square \)

We can now use the following straightforward adaptation of Lemma 4.2 of [HN01] from real argument functions to sequences:

**Lemma 6.15.** Suppose that number sequences \((a_n)\) and \((g_n)\) satisfy the following conditions: \( a_n/n \to \nu \) as \( n \to \infty \), \( a_{2n} - 2a_n \leq g_n \) for \( n \geq n_0 \) and \( \lim_{n \to \infty} g_{2n}/g_n = \psi < 2 \). Then for any \( c > 1/(2 - \psi) \) and for \( n \geq n_1 = n_1(n_0, (g_n), c) \),
\[
|a_n - \nu n| \leq cg_n.
\]

**Proof:** Let \( b_n = a_n/n \), \( h_n = g_n/(2n) \). Then \( |b_{2n} - b_n| \leq h_n \) for \( n > n_0 \) and \( \lim_{n \to \infty} h_{2n}/h_n = \psi/2 \).

Since \( \psi/2 \leq 1 - \frac{1}{2\sqrt{2}} \), there is \( N > n_0 \) such that \( h_{2m}/h_m \leq 1 - \frac{1}{2\sqrt{2}} \) for all \( m > N \).

Let us now fix \( n > N \). Then for \( k \geq 0 \) we have \( h_{2^kn} \leq \left( 1 - \frac{1}{2\sqrt{2}} \right)^k h_n \). Therefore,
\[
|b_n - b_{2^kn}| \leq \sum_{i=0}^{k-1} |b_{2^{i+1}n} - b_{2^in}| \leq \sum_{i=0}^{k-1} h_{2^in} \leq 2ch_n.
\]

We complete the proof by letting \( k \to \infty \). \( \square \)

**Proof of Lemma 6.13** Thanks to Lemma 6.14, we can apply Lemma 6.15 to \( a_n = \mathcal{E}p_n(\kappa) \), \( g_n = b_{2n}1/2 \ln^2 n \), \( \nu = \alpha_{0\kappa} \), \( \psi = \sqrt{2} \), and some fixed constant \( c > 1/(2 - \psi) \) to obtain (6.24).

The inequality (6.24) implies that \( \frac{1}{n} \mathcal{E}p_n(\kappa) \) converge to \( \alpha_{0\kappa} \) uniformly for all \( \kappa \in [0,1] \). Since for each \( n \in \mathbb{N} \), \( \frac{1}{n} \mathcal{E}p_n(\cdot) \) is continuous and decreasing, the second part follows. \( \square \)

### 7. Straightness and tightness

In this section, we modify our approach to straightness used in [BL16], obtaining estimates that serve all \( \kappa \in (0,1] \) at the same time. Also, we avoid using monotonicity, so the argument can be extended to higher dimensions.

**Theorem 7.1.** There is a full measure set \( \Omega' \) such that for every \( \omega \in \Omega' \) the following holds: if \( (m,x) \in \mathbb{Z} \times \mathbb{R} \), \( \nu' \in \mathbb{R} \), and \( 0 \leq u_0 < u_1 \), then there is a random constant
\[
n_0 = n_0[\omega, m, [x], |\nu'| + u_1, ((u_1 - u_0)^{-1})]
\]
(where \([\cdot]\) denotes the integer part) such that
\[
\mu_{m,n}^{\nu}, N \{ \gamma : |\gamma_{m+n} - \nu'n| \geq u_1n \} \leq \nu((\nu' - u_0)N, (\nu' + u_0)N|\nu| + e^{-\kappa^{-1}n^{1/2}}
\]
and
\[
\mu_{m,n}^{\nu}, N \{ \gamma : \max_{1 \leq i \leq n} |\gamma_{m+i} - \nu'i| \geq (u_1 + R_1 + 1)n \}
\leq \nu((\nu' - u_0)N, (\nu' + u_0)N|\nu| + 2e^{-\kappa^{-1}n^{1/2}}
\]
\]}
hold true for any terminal measure \( \nu \), \( (N - m)/2 \geq n \geq n_0 \), and all \( \kappa \in (0, 1) \). Here, we use \( R_1 \) that has been introduced in Lemma \( 6.3 \).

Let us begin with a corollary of Theorem \( 6.1 \).

**Lemma 7.1.** Let \( m, p, q \in \mathbb{Z} \) and \( n \in \mathbb{N} \). If \( n \) is sufficiently large, then on an event with probability at least \( 1 - e^{-n^{1/3}} \), it holds that for all \( x \in [p, p + 1], y \in [q, q + 1], \) and \( \kappa \in (0, 1), \)

\[
|\kappa \ln Z_{x,y}^{m,n} - \alpha_\kappa(n, x - y)| \leq n^{3/4},
\]

where \( \alpha_\kappa(k, z) = \alpha_\kappa(z/k) \cdot k = \alpha_0 k - \frac{z^2}{2k} \).

**Proof:** Without loss of generality we can assume \( m = 0 \) and \( p = q = 0 \). Taking \( u = n^{3/4}/2 \), by Theorem \( 6.1 \) we have that on an event \( \Lambda_1 \) with probability at least \( 1 - c_1 e^{-c_2 \frac{n^{1/2}}{\ln n}} \),

\[
(7.3) \quad |\kappa \ln Z_{0,0}^{0,n} - \alpha_0, n| \leq n^{3/4}/2, \quad \kappa \in (0, 1).
\]

We recall the constant \( R_1 \) in Lemma \( 6.2 \) and define the following modification of \( Z_{x,y}^{0,n} \):

\[
\bar{Z}_{x,y}^{0,n} = \int_{|x_1|,|x_{n-1}| \leq R_1 \sqrt{n} + 1} Z_{x_1,x_{n-1};x,y}^{1,n-1} dx_1 dx_{n-1} - \frac{1}{2\pi \cdot \kappa} \exp \left( - \kappa^{-1} \left( \frac{(x_1 - x)^2}{2} + \frac{(x_{n-1} - y)^2}{2} + F_1(x_1) + F_n(x) \right) \right).
\]

For all \( x, y \in [0, 1] \), we have

\[
|\kappa \ln \bar{Z}_{x,y}^{0,n} - \ln Z_{0,0}^{0,n}| \leq \max_{y \in [0, 1]} (|F_n(0)| + |F_n(y)|) + \max_{x,y \in [0, 1]} \frac{1}{2} |(z - x)^2 + (w - y)^2 - z^2 - w^2| \leq \max_{y \in [0, 1]} (|F_n(0)| + |F_n(y)|) + 2R_1 \sqrt{n} + 3.
\]

Using \( [5.6] \) in Lemma \( 6.2 \) and the fact that

\[
P_{x,y}^{0,n} \left( \{ \gamma : |\gamma_1| \cap |\gamma_{n-1}| > R_1 \sqrt{n} + 1 \} \right) \leq P_{x,y}^{0,n} (\bigcup_{s \geq R_1} E_{x}^{0,n}), \quad x, y \in [0, 1],
\]

we obtain that on an event \( \Lambda_2 \) with probability at least \( 1 - 3e^{-d_1 R_1 n} \),

\[
(7.5) \quad |\kappa \ln \bar{Z}_{x,y}^{0,n} - \ln Z_{0,0}^{0,n}| \leq |\kappa \ln(1 - 2 - \kappa^{-1} R_1 n)| \leq |\ln(1 - 2 - R_1)|, \quad x, y \in [0, 1], \kappa \in (0, 1).
\]

Due to assumption \( (A5) \) and Markov inequality, there is an event \( \Lambda_3 \) with probability at least \( 1 - e^{-n^{3/4}/8} \) such that

\[
(7.6) \quad \max_{x \in [0, 1]} |F_1(x)| \leq n^{3/4}/8.
\]

Also, for all \( x, y \in [0, 1], \) we have

\[
(7.7) \quad |\alpha_0, n - \alpha_\kappa(n, x - y)| = \frac{1}{2n} (x - y)^2 \leq 1.
\]

Now consider the event \( \Lambda = \Lambda_1 \cap \Lambda_2 \cap \Lambda_3 \) and combine \( [5.3], [5.4], [5.5], [6.0], \)

and \( (7.7) \) together. Then \( P(\Lambda) \geq 1 - e^{-n^{1/3}} \) and if \( \omega \in \Lambda \), then

\[
|\kappa \ln Z_{x,y}^{0,n} - \alpha_\kappa(n, x - y)| \leq \frac{n^{3/4}}{2} + 2 \cdot \frac{n^{3/4}}{8} + 2R_1 \sqrt{n} + 4 + |\ln(1 - 2 - R_1)| \leq n^{3/4}.
\]

This concludes the proof. \( \square \)
For \((m, x), (n, y) \in \mathbb{Z} \times \mathbb{R}\) with \(m < n\), we define \([(m, x), (n, y)]\) to be the constant velocity path connecting \((m, x)\) and \((n, y)\), i.e., \([(m, x), (n, y)]_k = x + \frac{k-m}{n-m} (y-x)\) for \(k \in [m, n] \mathbb{Z}\). For \((m, p), (n, q) \in \mathbb{Z} \times \mathbb{Z}\), we define the events

\[
A^{m,n}_{p,q} = \left\{ \mu^{m,n}_{x,y;\kappa} \left\{ \max_{k \in I(m,n)} |\gamma_k - [(m, p), (n, q)]_k| \leq (n-m)^{8/9} \right\} \leq e^{-\kappa^{-1}(n-m)^{1/2}}, \right.
\]
\[
x \in [p, p+1], y \in [q, q+1], \kappa \in (0, 1]\right\},
\]

where \(I(m,n) = \left[ \frac{3m+n}{4}, \frac{m+3n}{4} \right] \mathbb{Z}\), and the events

\[
B^{m,n}_{p,q} = \left\{ \mu^{m,n}_{x,y;\kappa} \left\{ \max_{k \in [m,n] \mathbb{Z}} |\gamma_k - [(m, p), (n, q)]_k| \geq R_1 n \right\} \leq 2^{-\kappa^{-1}R_1 n}, \right.
\]
\[
x \in [p, p+1], y \in [q, q+1], \kappa \in [0, 1]\right\},
\]

where \(R_1\) is introduced in Lemma 6.2. Such events \(A^{m,n}_{p,q}\) and \(B^{m,n}_{p,q}\) are measurable since for a fixed Borel set \(D \in \mathcal{S}_{\mathbb{R}}, \mu^{m,n}_{x,y;\kappa}(D)\) is continuous in \(x, y\) and \(\kappa\). Moreover, by translation and shear invariance, the probability of \(A^{m,n}_{p,q}\) and \(B^{m,n}_{p,q}\) depends only on \(n-m\).

The probability of \(B^{m,n}_{p,q}\) can be estimated using (6.7) in Lemma 6.2 The following lemma gives estimation on the probability of \(A^{m,n}_{p,q}\).

**Lemma 7.2.** For some constant \(k_1\), if \(N\) is large enough, then

\[
\mathbb{P}(A^{0,N}_{0,0}) \geq 1 - k_1 N^2 e^{-N^{1/3}}.
\]

**Proof:** By (6.7) in Lemma 6.2 there is an event \(\Lambda_1\) with \(\mathbb{P}(\Lambda_1) \geq 1 - 3e^{-d_{R_1,N}}\) on which the following holds:

\[
\mu^{0,N}_{x,y;\kappa} \left\{ \gamma : \max_{1 \leq k \leq N-1} |\gamma_k| \leq R_1 N \right\} \leq 2^{-\kappa^{-1}R_1 n}, \quad x, y \in [0, 1], \kappa \in (0, 1].
\]

Applying Lemma 6.1 with \((m, n, p, q)\) running over the set

\[
\{ (0, k, 0, l) \cap [\mathbb{N}, \mathbb{N}] : |l| \leq R_1 N \} \cup \{ (k, N - k, l, 0) : k \in \left[ \frac{N}{4}, \frac{3N}{4} \right], |l| \leq R_1 N \},
\]

we can obtain an event \(\Lambda_2\) with probability at least \(1 - C_1 N^2 e^{-N^{1/3}}\) on which the following holds for all \(x, y \in [0, 1]\):

\[
|\kappa \ln Z^{0,1}_{x,x;\kappa} - \alpha_\kappa(k, x-z)| \leq k^{3/4} \leq N^{3/4}, \quad k \in \left[ \frac{N}{4}, \frac{3N}{4} \right], |z| \leq R_1 N,
\]
\[
|\kappa \ln Z^{0,1}_{x,y;\kappa} - \alpha_\kappa(N - k, y-z)| \leq (N-k)^{3/4} \leq N^{3/4}, \quad k \in \left[ \frac{N}{4}, \frac{3N}{4} \right], |z| \leq R_1 N,
\]
\[
|\kappa \ln Z^{0,1}_{x,y;\kappa} - \alpha_\kappa(N, x-y)| \leq N^{3/4}.
\]

Using (7.11), for \(\omega \in \Lambda_2\), all \(k \in \left[ \frac{N}{4}, \frac{3N}{4} \right]\) and all \(x, y \in [0, 1]\), we have

\[
\mu^{0,N}_{x,y;\kappa} \left\{ \gamma : |\gamma_k| \in [N^{8/9}, R_1 N] \right\} \leq \exp \left( \kappa^{-1} \frac{3N^{3/4} + (x-y)^2}{2N} \right) \int_{|z| \geq N^{8/9}} \exp \left( - \kappa^{-1} \frac{(x-z)^2}{2k} + \frac{(y-z)^2}{2(N-k)} \right) dz
\]
\[
\leq \exp \left( \kappa^{-1} \frac{3N^{3/4} + 1}{2N} \right) \int_{|z| \geq N^{8/9}} \exp \left( - \kappa^{-1} \frac{2z^2}{N} \right) dz
\]
\[
\leq N^{1/9} \exp \left( - \kappa^{-1} \left[ \frac{N^{7/9} / 2}{2} - 3N^{3/4} \right] \right).
\]
where in the last inequality we use the following bound on the tail of Gaussian integral: for $a,b > 0$, 
\[
\int_{|x| \geq b} e^{-\frac{x^2}{b^2}} dx \leq \frac{a}{b} e^{-\frac{b^2}{a}}.
\]

Combining this with (10.11), we can conclude that $A_{0,0}^0$ is included in $\Lambda_1 \cup \Lambda_2$, which has probability at least $1 - C_2 N^2 e^{-N^{1/3}}$. Here, the constants $C_1$ and $C_2$ are independent of $N$. This completes the proof.

**Lemma 7.3.** Let $c > 0$, $0 < v_0 < v_1$, $v' \in \mathbb{R}$ and $m, p \in \mathbb{Z}$. Suppose $|v'| + v_1 < c$. There are constants $n_1 = n_1(|v_1 - v_0|^{-1})$ and $k_2$ such that when $n > n_1$, there is an event $\Omega^{(1)}_{c,n}(m,p)$ with probability at least $1 - k_2 n^2 e^{-n^{1/3}}$ on which the following holds: for all $N > 2n$, $\kappa \in (0, 1)$, $x \in [p, p+1]$ and for any terminal measure $\nu$,

\[
\begin{align}
\mu^{m,m+N}_{x,v'v} & \left( \gamma : \max_{1 \leq i \leq n} |\gamma_{m+i} - p - v'i| \geq (v_1 + R_1 + 1)n \right) \\
& \leq \nu \left( [p + (v' - v_0)N, p + (v' + v_0)N]^c \right) + e^{-\kappa^{-1}n^{1/2}},
\end{align}
\]

and

\[
\begin{align}
\mu^{m,m+N}_{x,v'v} & \left( \gamma : \max_{1 \leq i \leq n} |\gamma_{m+i} - p - v'i| \geq (v_1 + R_1 + 1)n \right) \\
& \leq \nu \left( [p + (v' - v_0)N, p + (v' + v_0)N]^c \right) + 2e^{-\kappa^{-1}n^{1/2}}.
\end{align}
\]

**Proof:** We will choose $\Omega^{(1)}_{c,n}(m,p) = \theta^{m,p} \Omega^{(1)}_{c,n} (\theta$ is the space-time shift), where

\[
\Omega^{(1)}_{c,n} = \left( \bigcap_{j \geq 2n} A_{0,j}^0 \right) \cap \left( \bigcap_{|q| \leq (c+1)n} B_{0,q}^0 \right).
\]

Due to (6.12) in Lemma 6.2, $P(\Omega^{(1)}_{c,n}) \geq 1 - 3e^{-d_1 R_1 n}$. This and Lemma 6.2 imply that $P(\Omega^{(1)}_{c,n}) \geq 1 - k_2 n^2 e^{-n^{1/3}}$ for some constant $k_2$.

Without loss of generality, we will assume $(m,p) = (0,0)$. In showing (7.12) and (7.13), we will also assume $v' = 0$ for simplicity. The extension to other values of $v'$ is straightforward. Let us fix a terminal measure $\nu$ and $\kappa \in (0, 1)$, $x \in [0, 1]$, $N \geq 2n$, and assume $\omega \in \Omega^{(1)}_{c,n}.

For (7.12), it suffices to show that if $n$ is large, then

\[
\mu_{v''v''}^{0,N} \left( \{ \gamma : |\gamma_N - N v_0|, |\gamma_1| \geq n v_1 \} \right) < e^{-\kappa^{-1}n^{1/2}}.
\]

Let $k$ be the unique integer such that $2^k n \leq N < 2^{k+1} n$. For $l \in [0, k]$, define

\[
i_l = \begin{cases} n \cdot 2^l, & 0 \leq l \leq k-1, \\ N, & l = k. \end{cases}
\]

Let us consider the following inequality that appears in the definition of $A_{0,0}^{0,i_l}$:

\[
\left| (0,0), (i_l, [\gamma_{i_l}]) \right|_{i_{l-1}} - \gamma_{i_{l-1}} = \left| [\gamma_{i_l} \cdot \frac{i_{l-1}}{i_l} - \gamma_{i_{l-1}} \right| \leq (i_l)^{8/9}.
\]
If a path $\gamma$ satisfies (7.15) for all $l \in [l' + 1, k]z$, then
\[
\frac{\sum_{i=l+1}^{k} \gamma_{i} - \gamma_{N}}{l' + 1} \leq \sum_{i=l+1}^{k} \left( \frac{8}{9} + 1 \right)
\]
\[
\leq n^{-1/9} \left[ \sum_{i=l' + 1}^{k-1} \left( 2^{8/9} \cdot 2^{-\frac{8}{9}(l-1)} + 2^{-\frac{8}{9}(k-1)} (k-1) + 2^{-\frac{8}{9}(k-1)} \right) \right]
\]
\[
\leq K_1 n^{-1/9}
\]
for some absolute constant $K_1$.

For $l' \in [0, k - 1]z$, let us define the set of paths
\[
\Lambda_{l'} = \{ \gamma : \text{ (7.15) holds for all } l \in [l' + 1, k]z \text{ and } |\gamma_N| < Nv_0 \}. 
\]

We also define $\Lambda_k = \{ \gamma : |\gamma_N| < Nv_0 \}$. Suppose $n \geq \left( \frac{K_1}{|v_1 - v_0|} \right)^{9}$. If a path $\gamma \in \Lambda_{l'} \setminus \Lambda_{l'}$ ($l \in [1, k]z$), then (7.16) implies $|\gamma_{i_{l'}}| < (c + 1/2)i_{l'}$. Therefore,
\[
\mu_{x, i_{l'} / k}^{0, N} (\Lambda_{l'} \setminus \Lambda_{l'-1}) = \int \nu(dz) (Z_{x, z}^{0, N})^{-1} \int_{(c+1/2)i_{l'}}^{(c+1/2)i_{l'}} dz (Z_{x, z}^{0, N})_{\Lambda_{l'} \setminus \Lambda_{l'-1}}
\]
\[
\leq \int \nu(dz) (Z_{x, z}^{0, N})^{-1} \int_{(c+1/2)i_{l'}}^{(c+1/2)i_{l'}} dz e^{-\kappa^{-1}(i_{l'})^{1/2}} (Z_{x, z}^{0, N})_{\Lambda_{l'} \setminus \Lambda_{l'-1}}
\]
\[
\leq e^{-\kappa^{-1}(i_{l'})^{1/2}}.
\]
Here, in the second inequality we used that $\omega \in \Omega_{c, n}^{(1)} \subset A_{0, |w|}^{0, i_{l'}}$ for $|w| \leq (c + 1/2)i_{l'}$.

Also, $|v_0 - v_1| > K_1 n^{-1/9}$ (which holds for large $n$) and (7.14) imply that \[
\Lambda_0 \cap \{ \gamma : |\gamma_N| > nv_1 \} = \emptyset.
\]

Combining all these estimates, we have
\[
\mu_{x, i_{l'} / k}^{0, N} (\{ \gamma : |\gamma_N| < Nv_0, |\gamma_N| \geq nv_1 \}) \leq \sum_{l' = 1}^{k} \mu_{x, i_{l'} / k}^{0, N} (\Lambda_{l'} \setminus \Lambda_{l'-1}) \leq \sum_{l' = 1}^{k} e^{-\kappa^{-1}(i_{l'})^{1/2}}
\]
\[
\leq \sum_{m = 2n}^{\infty} e^{-\kappa^{-1}m^{1/2}} \leq e^{-\kappa^{-1}n^{1/2}},
\]
which completes the proof of (7.12).

Now we turn to (7.13). Let
\[
D = \{ \gamma : \max_{1 \leq i \leq n} |\gamma_i| \geq (v_1 + R_1 + 1)n \}.
\]

If $|z| \leq v_1 n$, then
\[
\mu_{x, z}^{0, N}(D) \leq \mu_{x, z}^{0, s}(\gamma : \max_{1 \leq i \leq n} |\gamma_i - [(0, 0), (n, |z|)]_i| \geq R_1 n)
\]

For all $|z| \leq v_1 s$, since $\omega \in B_{0, n}^{0, n}$, we have $\mu_{x, z}^{0, N}(D) \leq 2^{-\kappa^{-1}R_1 n}$. Therefore,
\[
\mu_{x, z}^{0, N}(D \cap \{ |\gamma_s| \leq v_1 s \}) \leq 2^{-\kappa^{-1}R_1 n}.
\]
Then (7.13) follows from this and (7.12).

**Proof of Theorem (7.1)** The Theorem directly follows from Lemma (7.14) and the Borel–Cantelli Lemma.
8. Infinite Volume Polymer Measures and Their Zero-Temperature Limit

In this section, we will prove our main results, Theorems 4.1 and 4.2. We will show that $\Omega'$ introduced in Section 7 can be chosen as the full measure set the existence of which is claimed in Theorem 4.1, and that we can take $\hat{\Omega}$ tight if for each $\epsilon > 0$, there is a compact set $K \subset \mathbb{R}^n$ such that

$$\mu_k \pi^{-1}_{m,m+n}(K^c) < \epsilon, \quad N_k > m + n. \tag{8.1}$$

**Proof of Part 1 in Theorem 4.1**

Let $\mu_N = \mu_{x,Nv,c}$. We will show that the family of measures $(\mu_N)_{N>m}$ is tight and any limit point in the weak topology belongs to $\mathcal{P}_{x,\infty}(v)$.

Given $\epsilon > 0$, choosing $v' = v$, $u_0 = 0$, $u_1 = 1$ in Theorem 7.1, we see that if

$$(N - m)/2 \geq n \geq n_0(\omega, m, [x], [v] + 1, 1) \vee \ln^2 \epsilon,$$

then, due to (7.2),

$$\mu_N \left\{ \gamma : \max_{m \leq i \leq m+n} |\gamma_i - vi| \geq (R_1 + 2)n \right\} \leq \delta_{Nv}([Nv, Nv]^c) + 2e^{-\kappa^{-1}n^{1/2}} \leq 2\epsilon.$$

Therefore, $(\mu_N)_{N>m}$ is tight.

Suppose $\mu_N$ converge to $\mu$ weakly. To show that $\mu \in \mathcal{P}_{x,\infty}(v)$, it suffices to show that for all $\epsilon > 0$,

$$\sum_{n=1}^{\infty} \mu \pi^{-1}_{m+n}([(m+n)(v-\epsilon), (m+n)(v+\epsilon)]^c) < \infty. \tag{8.2}$$

Let us choose $v' = u$, $u_0 = \epsilon/2$ and $u_1 = \epsilon$. Then (7.1) in Theorem 7.1 implies that if $(N - m)/2 \geq n \geq n_0(\omega, m, [x], [v] + \epsilon, 2\epsilon^{-1})$, then

$$\mu_N \pi^{-1}_{m+n}([(m+n)(v-\epsilon), (m+n)(v+\epsilon)]^c) \leq \delta_{Nv}([Nv - \frac{\epsilon}{2}, Nv + \frac{\epsilon}{2}]^c) + e^{-\kappa^{-1}n^{1/2}} e^{-\kappa^{-1}n^{1/2}}.$$

Taking $N \to \infty$, by weak convergence we have

$$\mu \pi^{-1}_{m+n}([(m+n)(v-\epsilon), (m+n)(v+\epsilon)]^c) \leq e^{-\kappa^{-1}n^{1/2}}.$$

This implies (8.2) and completes the proof. \hfill \square

Now we proceed to prove the rest of Theorem 4.1 and Theorem 4.2.

Let us fix $\omega \in \Omega'$ and $(m, x) \in \mathbb{Z} \times \mathbb{R}$. Let $\mu_\kappa \in \mathcal{P}_{x,\infty}(v)$, $\kappa \in (0, 1]$. We first derive some properties for such a family $(\mu_\kappa)$.

**Lemma 8.1.** If $n > n_0(\omega, m, [x], [v] + 1, 2)$, then for all $\kappa \in (0, 1]$,

$$\mu_\kappa \left\{ \gamma : \max_{m \leq i \leq m+n} |\gamma_i - vi| \geq (R_1 + 2)n \right\} \leq 2e^{-\kappa^{-1}n^{1/2}}. \tag{8.3}$$
PROOF: Applying Theorem 7.1 with \((v', u_0, u_1) = (v, 1/2, 1)\), when \((N-m)/2 > n\) we have

\[
\mu_{\kappa} \{ \gamma : \max_{m \leq i \leq m+n} |\gamma_i - v_i| \geq (R_1 + 2)n \} = \mu_{\kappa,x,v,N,n} \{ \gamma : \max_{m \leq i \leq m+n} |\gamma_i - v_i| \geq (R_1 + 2)n \} 
\leq \mu_{\kappa,\pi^{-1}_N} ([N(1/2), N(v+1/2)]^c) + 2e^{-\kappa^{-1}n^{1/2}}.
\]

Since \(\lim_{N \to \infty} \mu_{\kappa,\pi^{-1}_N} ([N(1/2), N(v+1/2)]^c) = 0\), (8.3) follows. \(\Box\)

**Lemma 8.2.** For any \(\varepsilon > 0\) and \(\kappa \in (0,1]\), if \(n > n_0(\omega, m, [x], [v] + \varepsilon, [2\varepsilon^{-1}])\), then

\[\mu_{\kappa} \left( \left( m+n \right)(v-\varepsilon), (m+n)(v+\varepsilon) \right)^c \leq e^{-\kappa^{-1}n^{1/2}}.\]

**Proof:** The proof is similar to that of Lemma 8.1. \(\Box\)

**Lemma 8.3.** There are a constant \(c > 0\) and terminal measures \((v_{\kappa}^N)_{N>m, \kappa \in (0,1]}\) satisfying

\[\nu_{\kappa}^N([-cN, cN]^c) = 0, \quad N > m \lor 0, \quad \kappa \in (0,1],\]

such that for each \(\kappa, \mu_{\kappa}\) is the weak limit of \(\mu_{N,x,v_{\kappa}^N}^N\) as \(N \to \infty\).

**Proof:** Let us define \(\nu_{\kappa}^N\) as follows:

\[\nu_{\kappa}^N(A) = (D_{\kappa}^N)^{-1} \mu_{\kappa,\pi^{-1}_N}(A \cap B_N), \quad A \subset \mathcal{B}(\mathbb{R}),\]

where \(B_N = [N(v-1), N(v+1)]\) and \(D_N^N = \mu_{\kappa,\pi^{-1}_N}(B_N)\). For any \(n > m\) and any Borel set \(\Lambda \subset \mathcal{B}(\mathbb{R}^{n-m})\), we have

\[
|\mu_{\kappa,\pi^{-1}_N}(\Lambda) - \mu_{\kappa,x,v_{\kappa}^N,\pi^{-1}_N}(\Lambda)| \leq |\mu_{\kappa,\pi^{-1}_N}(\Lambda) - D_N^N \mu_{\kappa,x,v_{\kappa}^N,\pi^{-1}_N}(\Lambda)| + (1-D_N^N)\mu_{\kappa,x,v_{\kappa}^N,\pi^{-1}_N}(\Lambda)
\leq \nu_{\kappa}^N(B_N) + (1-D_N^N)\mu_{\kappa,x,v_{\kappa}^N,\pi^{-1}_N}(\Lambda).
\]

The right hand side goes to zero, since \(\mu_{\kappa} \in \mathcal{P}_{x,v_{\kappa}^N}(v)\) implies that

\[1-D_N^N = \nu_{\kappa}^N(B_N) = \mu_{\kappa,\pi^{-1}_N}([N(v-1), N(v+1)]^c) \to 0, \quad N \to \infty.\]

This shows that \(\mu_{\kappa}\) is the weak limit of \(\mu_{x,v_{\kappa}^N,\pi^{-1}_N}\) and completes the proof. \(\Box\)

Let us recall that the locally uniform (LU) topology on \(C(\mathbb{R}^d)\) is defined by the metric

\[d(f, g) = \sum_{k=1}^{\infty} 2^{-k} \left( 1 \land \sup_{|x| \leq k} |f(x) - g(x)| \right), \quad f, g \in C(\mathbb{R}^d).\]

Convergence in this metric (also called LU-convergence) is equivalent to uniform convergence on every compact subset of \(\mathbb{R}^d\). LU-precompactness of a family \((f_n)\) is equivalent to equicontinuity and uniform boundedness of \((f_n)\) on every compact set.

Before we continue on properties of \((\mu_{\kappa})\), let us state the following lemma, whose proof will be given at the end of this section.
Lemma 8.4. Let $\omega \in \Omega'$ and $m, n \in \mathbb{Z}$ with $m < n$. Suppose a family of probability measures $\{\nu^N_{\kappa}\}_{N>n, \kappa \in (0, 1]}$ satisfies (8.3) for some constant $c$. For $n < N$, let $f^N_{m,n,\kappa}(x, \cdot)$ be the density of $\mu_{x,\nu^N_{\kappa},\pi^{-1}_n}$, namely,

$$f^N_{m,n,\kappa}(x, y) = \int_{-cN}^{cN} \frac{Z^{m,n}_{x,y;\kappa}}{Z^{m,n}_{x,z;\kappa}} \nu^N_{\kappa}(dz).$$

Then, $\left(\kappa \log f^N_{m,n,\kappa}(\cdot, \cdot)\right)_{N>n, \kappa \in (0, 1]}$ is an LU-precompact family of continuous functions.

Lemma 8.5. Let $m \in \mathbb{Z}$. There is an LU-precompact family of continuous functions $(h_{n,\kappa}(\cdot))_{n>m}$ such that the density of $\mu_{\pi^{-1}_n}$ can be expressed as

$$d\mu_{\pi^{-1}_n} = \frac{Z^{m,n}_{x,y;\kappa}e^{-\kappa^{-1}h_{n,\kappa}(y)}}{\int_{\mathbb{R}} Z^{m,n}_{x,y';\kappa}e^{-\kappa^{-1}h_{n,\kappa}(y')} dy'}. \tag{8.6}$$

Proof: By Lemma 8.3 there are terminal measures $\nu^N_{\kappa}$ satisfying (8.5) such that $\mu_{\kappa}$ is the weak limit of $\mu_{x,\nu^N_{\kappa},\pi^{-1}_n}$. Suppose $f^N_{m,n,\kappa}(\cdot)$ is the density of $\mu_{x,\nu^N_{\kappa},\pi^{-1}_n}$, then by Lemma 8.4 $(\kappa \log f^N_{m,n,\kappa})_{N>n, \kappa \in (0, 1]}$ is LU-precompact. Therefore, for each $\kappa$, $\kappa \log f^N_{m,n,\kappa}$ converge in LU to some continuous function $-\tilde{h}_{n,\kappa}$ as $N \to \infty$, such that $e^{-\kappa^{-1}\tilde{h}_{n,\kappa}(y)}$ is the density of $\mu_{\pi^{-1}_n}$. The family of functions $(\tilde{h}_{n,\kappa})_{\kappa \in (0, 1]}$ is also LU-compact. One can then define $h_{n,\kappa}(y) = \tilde{h}_{n,\kappa}(y) - \kappa \log Z^{m,n}_{x,y;\kappa}$ and the lemma follows. \hfill $\Box$

We are now ready to prove the rest of Theorem 4.1.

Proof of parts 2 and 3 in Theorem 4.1. Part 2 follows from Lemma 8.1. Let us prove part 3. Let $(m, x) \in \mathbb{Z} \times \mathbb{R}$ and $\mu_{\kappa} \in \mathcal{P}^{m, +\infty}(\nu)$. Then Lemma 8.5 implies that, for each $n > m$, there is an LU-precompact family of continuous functions $h_{n,\kappa}(y)$ such that (8.6) holds. Suppose $\mu$ is the weak limit of $\mu_{\kappa}$ for some sequence $\kappa_k \downarrow 0$. Using a diagonal sequence argument, we see that there is a further subsequence $\kappa_k' \downarrow 0$ such that for every $n > m$, $h_{n,\kappa_k'}(y)$ converge in LU to some $h_n(y)$ as $\kappa_k' \downarrow 0$.

For $\varepsilon > 0$, let us define the set of paths

$$A^n_\varepsilon = \{\gamma \in S^{m, +\infty}_{x, x} : A^{m,n}(\gamma) - A^{m,n}(\gamma_m, \gamma_n) > \varepsilon\},$$

where $A^{n_1,n_2}(x_1, x_2)$ denotes the minimal action between $(n_1, x_1)$ and $(n_2, x_2)$. Then we have

$$\mu_{\kappa}(A^n_\varepsilon) = \frac{\int Z^{m,n}_{x,y;\kappa}(A^n_{\varepsilon})e^{-\kappa^{-1}h_{n,\kappa}(y)}}{\int Z^{m,n}_{x,y;\kappa}e^{-\kappa^{-1}h_{n,\kappa}(y)} dy}.$$

For every $\delta > 0$, there exists $L > 0$ such that $\mu_{\kappa}(B^m_L) \geq 1 - \delta$ for all $\kappa \in (0, 1]$, where $B^m_L = \{\gamma : \gamma_i \leq L, m \leq i \leq n\}$, and $B^L_L = \{\gamma : |\gamma_i| \leq L, m \leq i \leq n\}$. Also, when $\kappa_k'$ is sufficiently small, we have

$$|h_{n,\kappa_k'}(y) - h_n(y)| \leq \varepsilon/4, \quad |y| \leq L.$$
Therefore, when \( \kappa_k \) is small,
\[
\mu_{\kappa_k}^n(\Lambda^n_n) \leq \mu_{\kappa_k}^n((B_{L}^{m,n})^c) + \mu_{\kappa_k}^n(\Lambda^n_n \cap B_{L}^{m,n}) \\
\leq \delta + \frac{\int_{|y| \leq L} Z_{x,y;\kappa_k}^{m,n}(\Lambda_n \cap B_{L}^{m,n})e^{-(\kappa_k^{-1})^{-1}h_{n,k}(y)} dy}{\int_{|y| \leq L} Z_{x,y;\kappa_k}^{m,n}e^{-(\kappa_k^{-1})^{-1}h_{n,k}(y)} dy} \\
\leq \delta + e^{(\kappa_k^{-1})^{-1}c/2} \frac{\int_{|y| \leq L} Z_{x,y;\kappa_k}^{m,n}(\Lambda_n \cap B_{L}^{m,n})e^{-(\kappa_k^{-1})^{-1}h_{n}(y)} dy}{\int_{|y| \leq L} Z_{x,y;\kappa_k}^{m,n}e^{-(\kappa_k^{-1})^{-1}h_{n}(y)} dy}.
\]
(8.7)

Due to the continuous dependence of action on paths and compactness of the set \([-L, L]\), there is \( \varepsilon_1 > 0 \) such that, for each minimizer from \((m, x)\) to \((n, y), |y| \leq L\), the action of every path in the \( \varepsilon_1 \)-neighborhood of that minimizer is at most \( A^{m,n}(x, y) + \varepsilon/4 \). (Here, if \( \gamma^* \) is a path in \( S^{*,n}_* \), its \( \eta \)-neighborhood is the set \( \{ \gamma \in S^{*,n}_* : |\gamma_k - \gamma^*_k| \leq \eta, m \leq k \leq n \} \).) Therefore,
\[
Z_{x,y;\kappa_k}^{m,n} \geq \varepsilon_{1}^{-m}e^{-(\kappa_k^{-1})^{-1}(A^{m,n}(x, y) + \varepsilon/4)}, \quad |y| \leq L.
\]
(8.8)

On the other hand, one has
\[
Z_{x,y;\kappa_k}^{m,n}(\Lambda^n_n \cap B_{L}^{m,n}) \leq L^{-m}e^{-(\kappa_k^{-1})^{-1}(A^{m,n}(x, y) - \varepsilon)}, \quad |y| \leq L.
\]
(8.9)

Combining (8.7), (8.8), and (8.9) together, we have
\[
\mu_{\kappa_k}^n(\Lambda^n_n) \leq \delta + (L/\varepsilon)^{n-m}e^{-(\kappa_k^{-1})^{-1}c/4}.
\]

Since \( \Lambda^n_n \) is an open set, by weak convergence of \( \mu_{\kappa_k}^n \), we have
\[
\mu(\Lambda^n_n) \leq \liminf_{k \to \infty} \mu_{\kappa_k}^n(\Lambda^n_n) \leq \delta.
\]

Since \( \delta \) is arbitrary, we obtain \( \mu(\Lambda^n_n) = 0 \).

The fact that \( \mu(\Lambda^n_n) = 0 \) for every \( n \) and \( \varepsilon \) implies that \( \mu \) must be a measure on \( \mathcal{P}_{x,n}^{m,n,\infty} \) that concentrates on semi-infinite minimizers. To identify the slope, we use Lemma [S.2] and take \( \kappa = \kappa_k' \downarrow 0 \) in [S.4] and conclude that for \( \varepsilon > 0 \) and \( n > n_0(\omega, m, [x], [v], [v] + [\varepsilon], [2\varepsilon^{-1}]) \),
\[
\mu([m+n](v-\varepsilon), (m+n)(v+\varepsilon)]^c) = 0.
\]

This shows that \( \mu \) concentrates on the semi-infinite minimizers in \( \mathcal{P}_{x,n}^{m,n,\infty}(v) \) and completes the proof of part [3].

**Proof of Theorem 4.2** Part [I] follows from Theorem 4.1.

For any \( p \in \mathbb{Z} \), by (7.2) in Theorem 7.3 for \( (N_2 - n)/2 \geq N_1 \geq n_1(n, p) = n_0(\omega, n, p, [v] + 1, 1) \),
\[
\mu_{y,v;\kappa}^{n, N_2}([-([v] + R_1 + 2)N_1, ([v] + R_1 + 2)N_1]^c)
\leq \nu([-([v] + 1)N_2, ([v] + 1)N_2]^c) + 2e^{-\sqrt{N_2}},
\]
for every terminal measure \( \nu \), all \( \kappa \in (0, 1) \) and all \( y \in [p, p + 1) \). Taking \( \nu = \delta_{N_2v} \) and letting \( N_2 \to \infty \), we obtain
\[
\mu_{y,v;\kappa}^{n, \infty}([-([v] + R_1 + 2)N_1, ([v] + R_1 + 2)N_1]^c) \leq 2e^{-\sqrt{N_1}}, \quad y \in [p, p + 1], \ N_1 \geq n_1(n, p).
\]

Combining this estimate with (8.8), we see that \( (u_{y,v}(n, \cdot))_{\kappa \in (0, 1)} \) is uniformly bounded on compact sets.
The first part of the theorem implies that if \((n, y) \notin \mathcal{N}\), then \(\mu_{y,L}^{m,\infty}\) converges weakly to \(\delta_{y,\infty}\). Then combining (3.4), (3.9) and (8.10), we obtain that

\[
 u_{v,\kappa}(n,y) = \int_{\mathbb{R}} (z - y) \pi^{n,\infty}_{y,L} \pi^{-1}_{n+1} (dz) \to \int_{\mathbb{R}} (z - y) \delta_{y,\infty} \pi^{-1}_{n+1} (dz) = u_{v,0}(n,y)
\]

for \((n,y) \notin \mathcal{N}\). Since \(\mathcal{N}\) is at most countable, \(u_{v,\kappa}(n,\cdot)\) converges to \(u_{v,0}(n,\cdot)\) at a.e. \(y\). This implies convergence in \(\mathbb{G}\) and completes the proof of part 2.

Finally we will prove part 3. Since the functions \(G_{v,\kappa}\) and \(B_v\) satisfy the relations (3.7) and (3.8), respectively, it suffices to show the following two limits hold:

\[
 (8.11) \quad \lim_{\mathcal{D}_{<\kappa,0}} -\kappa \ln G_{v,\kappa}((m,x),(n,0)) = B_v((m,x),(n,0)), \quad n \in \mathbb{Z}, \ x \in \mathbb{R},
\]

\[
 (8.12) \quad \lim_{\mathcal{D}_{<\kappa,0}} -\kappa \ln G_{v,\kappa}((m,x),(n,0)) = B_v((m,x),(n,0)), \quad n > m, \ x \in \mathbb{R},
\]

We recall \(U_{v,\kappa}, \kappa \in [0,1]\), defined in Theorems 3.1 and 3.2. The limit (8.11) is equivalent to \(U_{v,0}(n,x) = \lim_{\kappa \downarrow 0} U_{v,\kappa}(n,x)\).

Having shown that \((u_{v,\kappa}(n,\cdot))_{\kappa \in [0,1]}\) is uniformly bounded and that \(u_{v,\kappa}(n,\cdot)\) converge to \(u_{v,0}(n,\cdot)\) a.e. as \(\kappa \downarrow 0\), we can use bounded convergence theorem to conclude that

\[
 U_{v,\kappa}(n,x) = \int_0^x u_{v,\kappa}(n,y) dy \to U_{v,0}(n,x) = \int_0^x u_{v,0}(n,y) dy, \quad \kappa \downarrow 0.
\]

This proves (8.11), and the convergence is in LU.

To prove (8.12), we fix \(n > m\) and define \(H_\kappa(x) = -\kappa \ln G_{v,\kappa}((m,x),(n,0))\), \(\kappa \in (0,1]\), and \(H_0(x) = B_v((m,x),(n,0))\). We are going to show that \((H_\kappa(\cdot))_{\kappa \in \mathcal{D}}\) is LU-precompact, and that \(\lim_{\kappa \downarrow 0} H_\kappa(x) = H_0(x)\) for \(x \notin \mathcal{N}\) (and hence for a.e. \(x\)). Then the the convergence will hold for all \(x\) and (8.12) will follow.

As a consequence of Lemma 5.4 applied to \(\nu_{N} = \delta_{v,N}\), we see that the family \((\kappa \ln Z_{v,N}^{N}/Z_{x,v,N})_{\kappa \in \mathcal{D}}, N > n\) is LU-precompact in \(C(\mathbb{R}^2)\) in the variables \(x\) and \(y\). Hence, by part 3 of Theorem 3.2 and the condition \(\omega \in \Omega_v \subset \Omega_v,\), we have that \((\kappa \ln G_{v,\kappa}((n,y),(m,0)))_{\kappa \in \mathcal{D}}\) is LU-precompact. This shows the LU-precompactness of \((H_\kappa)_{\kappa \in \mathcal{D}}\).

Using (3.10), we have

\[
 G_{v,\kappa}((m,x),(n,0)) = \int_{\mathbb{R}} Z_{x,v,n}^{m,n} e^{-\kappa^{-1} U_{v,\kappa}(n,y)} dy
\]

\[
 = \int_{\gamma \in S_{x,v}^{m,n}} e^{-\kappa^{-1} (A^{m,n}(\gamma) + U_{v,\kappa}(n,y))} d\gamma.
\]

For every \(\delta > 0\), there is \(L > 0\) such that \(\mu_{x,v,\kappa}^{m,\infty}(B_L^{m,n}) \geq 1 - \delta\) for all \(\kappa\). Then

\[
 \int_{\gamma \in S_{x,v}^{m,n} \cap B_L^{m,n}} e^{-\kappa^{-1} (A^{m,n}(\gamma) + U_{v,\kappa}(n,y))} d\gamma \geq (1 - \delta) \int_{\gamma \in S_{x,v}^{m,n}} e^{-\kappa^{-1} (A^{m,n}(\gamma) + U_{v,\kappa}(n,y))} d\gamma,
\]
which follows from
\[
\mu^{m,+∞}_x(B^{m,n}_L) = \frac{\int_{|y|\leq L} E_{x,y}^{m,n}(B^{m,n}_L) Z^{m,n} e^{-\kappa^{-1}U_{n,κ}(y)} dy}{\int_{\mathbb{R}} Z^{m,n}_{x,y} e^{-\kappa^{-1}U_{n,κ}(y)} dy} = \frac{\int_{\gamma \in S_{x,n}^{m,n} \cap B^{m,n}_L} e^{-\kappa^{-1}(A^{m,n}(γ)+U_{n,κ}(n,y))} dγ}{\int_{\gamma \in S_{x,n}^{m,n}} e^{-\kappa^{-1}(A^{m,n}(γ)+U_{n,κ}(n,y))} dγ}.
\]

Therefore,
\[
G_{v,κ}((m,x),(n,0)) \leq (1-δ)^{-1} \int_{\gamma \in S_{x,n}^{m,n} \cap B^{m,n}_L} e^{-\kappa^{-1}(A^{m,n}(γ)+U_{n,κ}(n,y))} dγ \leq (1-δ)^{-1} L^{m-n} e^{-\kappa^{-1}\inf_{|y|\leq L} (A^{m,n}(x,y)+U_{n,κ}(n,y))}.
\]

By (8.11) and (8.3),
\[
(8.13) \liminf_{κ \to 0} \inf_{|y|\leq L} \{A^{m,n}(x,y)+U_{n,κ}(y)\} \geq \inf_{|y|\leq L} \{A^{m,n}(x,y)+B_v((n,y),(n,0))\}
\]

Taking logarithm and multiplying by \(-κ\) in (8.13) and using (8.14), we obtain that
\[
\liminf_{κ \to 0} H_v(x) \geq H_0(x).
\]

Let us fix \(ε > 0\) and define
\[
y_0 = (\gamma^{m,+∞}_x(v))_n = \arg\min_{y} \{A^{m,n}(x,y)+U_{0,0}(y)\}.
\]

There is an \(ε_1\)-neighborhood of \(π_{m,n}(\gamma^{m,+∞}_x(v))\) such that for each path \(γ\) in this neighborhood,
\[
|A^{m,n}(γ) - A^{m,n}(x,y_0)| \leq ε.
\]

Also, by the continuity of \(U_{v,0}(n,\cdot)\) and the LU-convergence of \(U_{v,κ}(n,\cdot)\) to \(U_{v,0}(n,\cdot)\), there is \(ε_2 > 0\) such that when \(κ\) is small enough we have
\[
|U_{v,κ}(n,y) - U_{v,0}(n,y_0)| \leq ε
\]

for every \(|y - y_0| \leq ε_2\). Therefore,
\[
G_{v,κ}((m,x),(n,0)) \geq (ε_1 \wedge ε_2)^{-1} e^{-\kappa^{-1}} \left(A^{m,n}(x,y_0)+U_{v,0}(n,y_0)+2ε\right)
\]

This implies that
\[
\limsup_{D \ni κ \to 0} H_v(x) \leq H_0(x) + 2ε.
\]

Since \(ε\) is arbitrary, this concludes the proof. \(\square\)

In the end of this section we give the proof of the technical Lemma 8.4.

**PROOF OF LEMMA 8.4**

We define
\[
g_κ^N(x,y) = (Z_{x,y}^{m,n})^{-1} E_{m,n,κ}(x,y) = \int_{\mathbb{R}}^{cN} \frac{Z_{x,y}^{m,n}}{Z_{y,z}^{m,n,κ}} ν^N_{κ}(dz).
\]

It suffices to show that \(\{κ \ln g_κ^N(\cdot,\cdot)\}_{N > n, κ ∈ [0,1]}\) is LU-precompact.

Let us consider a compact set \(K = [p, p+1] × [-k,k]\). Denoting \(r = c + R_1 + 2\), for \(ε ∈ (0,1/2)\), let us define
\[
s_1 = \max \{n-m, n_0(ω, n, p, [c+1], 1), \frac{k}{r}, \ln^2 \frac{ε}{16}\}\]
and

\[ s_2 = \max \left\{ n_0(\omega,n,i,[c+1],1) : |i| \leq rs_1 + 1 \right\} \leq \ln^2 \frac{c}{16}, \]

where the random function \( n_0 \) is introduced in Theorem 7.1.

We will need several truncated integrals:

\[
\begin{align*}
\tilde{Z}_{n,y,z,\kappa}^{m} &= \int_{-rs_2}^{rs_2} Z_{n,y,w,\kappa}^{m+1} Z_{w,z,\kappa}^{n+1,1} dw = \int_{-rs_2}^{rs_2} e^{-\kappa^{-1}(|w-y| + F_n+1(w))} Z_{w,z,\kappa}^{n+1,1} dw, \\
\tilde{Z}_{m,n,y,z,\kappa}^{m} &= \int_{-rs_1}^{rs_1} Z_{m,n,y,w,\kappa}^{m+1} Z_{w,z,\kappa}^{n+1,1} dw, \quad n = m + 1, \\
\tilde{Z}_{m,y,z,\kappa}^{m} &= \int_{-rs_1}^{rs_1} \tilde{Z}_{m,y,z,\kappa}^{m} \tilde{Z}_{n,y,z,\kappa}^{n} dy, \\
\tilde{g}_N^N(x, y) &= \int_{-cN}^{cN} \tilde{Z}_{n,y,z,\kappa}^{m} \tilde{Z}_{m,y,z,\kappa}^{n} (dz).
\end{align*}
\]

For \( N > n \), we also define \( h_{c,\kappa}^N = \kappa \ln \tilde{g}_N^N \) and \( K = [p, p + 1] \times [-rs_1, rs_1] \supset K \). If we can show that for every \( \varepsilon > 0 \), all large \( N \), and all \( \kappa \in (0,1] \),

\[ (8.15) \quad |\kappa \ln \tilde{g}_N^N(x, y) - h_{c,\kappa}^N(x, y)| \leq \varepsilon, \quad (x, y) \in K, \]

and that \( (h_{c,\kappa}^N) \) is precompact in \( C(K) \), then the lemma will follow since, given any \( \varepsilon > 0 \), we will be able to use an \( \varepsilon \)-net for \( (h_{c,\kappa}^N) \) to construct a \( 2\varepsilon \)-net for \( (\kappa \ln \tilde{g}_N^N) \).

Let \( N > \max\{m + 2s_1, n + 2s_2\} \). If \( |y| \leq rs_1 \) and \( |z| \leq cN \), then from (7.2) with \( \varepsilon = 0 \), \( u_1 = c + 1 \), \( u_2 = c \), \( \nu = \delta_2 \) and using \( \delta_2([-cN,cN]) = 0 \), we obtain

\[ 1 - \frac{\tilde{Z}_{n,y,z,\kappa}^{m+1}}{Z_{n,y,z,\kappa}^{m+1}} \leq \mu_{y,z,\kappa}^{m+1} ([-rs_2, rs_2]^c) \leq 2e^{-\kappa^{-1}\sqrt{2\varepsilon}} \leq \frac{\varepsilon}{8}, \quad \kappa \in (0,1]. \]

Then, using the elementary inequality \( |\ln(1 + x)| \leq 2|x| \) for \( |x| \leq 1/2 \) we find

\[ (8.16) \quad e^{-\varepsilon/4} \leq \tilde{Z}_{n,y,z,\kappa}^{m+1} \leq 1. \]

Let

\[ \tilde{Z}_{m,y,z,\kappa}^{m} = \int_{-rs_1}^{rs_1} \tilde{Z}_{m,y,z,\kappa}^{m} \tilde{Z}_{m,y,z,\kappa}^{n} dy. \]

Then (8.16) implies

\[ (8.17) \quad 1 \leq \frac{\tilde{Z}_{m,y,z,\kappa}^{m+1}}{\tilde{Z}_{m,y,z,\kappa}^{m}} \leq e^{\varepsilon/4}. \]

Similarly, if \( x \in [p, p + 1] \) and \( |z| \leq cN \), by (7.2), we obtain

\[ 1 - \frac{\tilde{Z}_{m,y,z,\kappa}^{m+1}}{Z_{m,y,z,\kappa}^{m+1}} \leq \mu_{y,z,\kappa}^{m+1} ([-rs_1, rs_1]^c) + \mu_{y,z,\kappa}^{m} ([-rs_1, rs_1]^c) \leq 4e^{-\kappa^{-1}\sqrt{2\varepsilon}} \leq \varepsilon/4. \]

Therefore,

\[ (8.18) \quad e^{-\varepsilon/2} \leq \frac{\tilde{Z}_{m,y,z,\kappa}^{m+1}}{Z_{m,y,z,\kappa}^{m+1}} \leq e^{\varepsilon/2}. \]

Combining (8.10), (8.17) and (8.18) we obtain

\[ e^{-\varepsilon} \leq \tilde{g}_N^N(x, y)/g_N^N(x, y) \leq e^\varepsilon, \]

and (8.15) follows.

The next step is to show that \( (h_{c,\kappa}^N) \) is precompact. For any \( |w| \leq rs_2 \) and \( y, y' \in [-rs_1, rs_1] \), we have

\[ \left| \frac{(y - w)^2}{2} - \frac{(y' - w)^2}{2} \right| \leq r(s_1 + s_2)|y - y'|. \]
Hence, the definition of $\tilde{Z}^{n,N}_{x,z;\kappa}$ implies that

$$|\kappa \ln \tilde{Z}^{n,N}_{y;z;\kappa} - \kappa \ln \tilde{Z}^{n,N}_{y';z;\kappa}| \leq r(s_1 + s_2)|y - y'|.$$  

Similarly, for all $x, x' \in [p, p + 1]$, we have

$$|\kappa \ln \tilde{Z}^{m,N}_{x,z;\kappa} - \kappa \ln \tilde{Z}^{m,N}_{x',z;\kappa}| \leq (r s_1 + |p| + 1)|x - x'|.$$  

Combining these two inequalities we see that

$$(8.19) \quad |h^N_{\epsilon;\kappa}(x, y) - h^N_{\epsilon;\kappa}(x', y')| \leq L(|x - x'| + |y - y'|)$$

for $L = r(s_1 + s_2) + |p| + 1$. So, $h^N_{\epsilon;\kappa}$ are uniformly Lipschitz continuous and hence equicontinuous on $\bar{K}$.

It remains to show that $h^N_{\epsilon;\kappa}$ are uniformly bounded. Let

$$\bar{f}^N_{\kappa}(x, y) = \int_{cN}^{eN} \frac{Z_{x,y;z;\kappa}^{m,n} \tilde{Z}_{x,y;z;\kappa}^{n,N}}{Z_{x,z;\kappa}^{m,n}} \nu^N_{\kappa}(dz) = \exp (\kappa^{-1} h^N_{\epsilon;\kappa}(x, y)) \tilde{Z}^{m,n}_{x,y;\kappa}.$$  

For each $x \in [p, p + 1]$, we have $\int_{-rs_1}^{rs_1} \bar{f}^N_{\kappa}(x, y') dy' = 1$. Let

$$M = \sup \{|\kappa \ln Z_{x,y;z;\kappa}^{m,n} : \kappa \in (0, 1), (x, y) \in \bar{K}|.$$  

It is easy to see that $M < \infty$ a.s. Then, by (8.19) we have for $y, y' \in [-rs_1, rs_1]$,

$$e^{-\kappa^{-1} (L - 2rs_1 + M)} \bar{f}^N_{\kappa}(x, y') \leq e^{\kappa^{-1} h^N_{\epsilon;\kappa}(x, y)} \leq \bar{f}^N_{\kappa}(x, y') e^{\kappa^{-1} (L - 2rs_1 + M)}.$$  

Integrating this inequality over $y' \in [-rs_1, rs_1]$ gives us

$$e^{-\kappa^{-1} (L - 2rs_1 + M)} \leq 2rs_1 e^{\kappa^{-1} h^N_{\epsilon;\kappa}(x, y)} \leq e^{\kappa^{-1} (L - 2rs_1 + M)}.$$  

Taking the logarithm gives $|h^N_{\epsilon;\kappa}(x, y)| \leq L \cdot 2rs_1 + M + |\ln(2rs_1)|$, so $|h^N_{\epsilon;\kappa}(x, y)|$ are uniformly bounded on $\bar{K}$. \hfill \Box

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