Combinatorial problems in the semiclassical approach to quantum chaotic transport

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Abstract
A semiclassical approach to the calculation of transport moments \( M_m = \text{Tr}(t^m) \), where \( t \) is the transmission matrix, was developed by this author (2012 Europhys. Lett. 98 20006) for chaotic cavities with two leads and broken time-reversal symmetry. The result is an expression for \( M_m \) as a perturbation series in \( 1/N \), where \( N \) is the total number of open channels, which is in agreement with random matrix theory predictions. The coefficients in this series were related to two open combinatorial problems. Here we expand on this work, including the solution to one of the combinatorial problems. As a by-product, we also present a conjecture relating two kinds of factorizations of permutations.

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(Some figures may appear in colour only in the online journal)

1. Introduction

Wave scattering in systems whose classical dynamics is chaotic displays universal statistics, as first observed in experiments with coherent electronic transport in ballistic quantum dots [1, 2]. We consider a chaotic cavity attached to two ideal leads having \( N_1 \) and \( N_2 \) open channels. This is described by a \( N \times N \) scattering matrix \( S \), where \( N = N_1 + N_2 \) is the total number of channels. The \( S \) matrix is always unitary, reflecting conservation of charge (if time-reversal symmetry is present, it is also symmetric). Following similar considerations in the field of quantum chaos, it was soon realized that these statistical properties are described by the theory of random matrices (RMT) [3–7]. This theory neglects all particularities a system may have and replaces \( S \) by a random unitary matrix [8] (unitary symmetric if time-reversal symmetry is present).

Let \( t \) be the \( N_1 \times N_2 \) transmission block of \( S \). The transport moments \( M_m = \text{Tr}[(t^m)^\dagger t^m] \) carry information about the scattering process (the so-called linear statistics). The \( m = 1 \) moment is called the conductance, and the \( m = 2 \) moment is related to the shot-noise. Within
RMT, the average value of $M_m$ has long been considered. Initially, only perturbative results were obtained [9, 10], valid to leading or next-to-leading order in $1/N$, and only for the first moments. In recent years the connection with the Selberg integral was properly realized [11] and this eventually allowed the calculation of all $M_m$ to be carried out for arbitrary values of $N_1$ and $N_2$, for both symmetry classes [12, 13] (see also [14]).

It has always been a central problem to derive these universal statistics from a semiclassical approximation. Quantum universality must emerge semiclassically as a result of action correlations: there must exist sets of scattering trajectories having nearly the same total action, so that they may interfere constructively. Conductance and shot-noise were considered at leading order [15, 16] and later to all orders [17, 18]. Higher moments were treated to leading order in [19] and to next-to-leading order in [20].

Recently, works by this author [21] and by Berkolaiko and Kuipers [22] have presented semiclassical calculations of all $M_m(N_1, N_2)$ valid to all orders in $1/N$. The approach developed in [21], for systems with broken time-reversal symmetry, related the transport problem to correlated periodic orbits and then arrived at two combinatorial problems that were left open. We present here the solution to one of these problems. The other one, which remains open, is to determine the number of solutions to the factorization $(12 \cdots E) = QP$ of the cyclic permutation under some conditions on the factors. Interestingly, the approach developed in [22] (which firmly establishes the equivalence of semiclassics and RMT for all symmetry classes) requires the solution of another factorization problem, involving the so-called primitive factorizations [23]. Primitive factorizations are important within the symmetric group and the unitary group [24]. In the present work, we compare our semiclassical result to RMT and to the result of [22], arriving at a conjecture relating both kinds of factorization problems.

The paper is organized as follows. In section 2 we present some combinatorial preliminaries and advance our conjecture. In section 3 we introduce the semiclassical theory of quantum chaotic transport. In section 4 we review the work summarized in [21], and solve one of the combinatorial problems left open in that paper. In section 5 we compare our theory to RMT and to the different semiclassical approach of [22]. We present some conclusions in section 6.

2. Combinatorial preliminaries and a conjecture

We shall make use of several combinatorial results, so we start with a brief revision of the ones we need. We are rather concise since these facts are widely known. We also present the conjecture that arises from our work.

A sequence of positive integers $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell$ is called a partition of $m$ if $\sum \lambda_i = m$. This is denoted by $\lambda \vdash m$. Each of these integers is called a part, and the total number of parts is called the length of the partition, $\ell(\lambda)$. It is also usual to write a partition in a frequency representation, i.e. as $\lambda = a_1 2^{a_2} \cdots$ where $a_j$ is the number of times the part $j$ appears in $\lambda$. For example, $(3, 2, 2, 1) = 12^23$ is a partition of 8.

The quantity

$$\binom{m}{\lambda} = \frac{m!}{\lambda_1! \lambda_2! \cdots a_1! a_2! \cdots}$$

(1)
counts in how many ways we can partition the set $\{1, \ldots, m\}$ into subsets having the parts of $\lambda$ as cardinalities. The number of set partitions into exactly $k$ non-empty subsets is the Stirling number of the second kind,

$$S(m, k) = \sum_{\ell(\lambda) = k} \binom{m}{\lambda}$$

(2)
These numbers satisfy
\[ \sum_{k=0}^{m} S(m, k)[x]_k = x^m, \]  
(3)
where
\[ [x]_k = x(x - 1) \cdots (x - k + 1) \]  
(4)
is the falling factorial.

Let \( \mathfrak{S}_m \) be the group of all \( m! \) permutations of \( m \) symbols. When a given permutation acts, it divides the \( m \) elements into orbits called cycles. For example, the permutation \((123)(45)(6)\) acts on the set of the first six integers and produces three cycles: \( 1 \rightarrow 2 \rightarrow 3 \rightarrow 1 \) is a cycle of length 3, \( 4 \rightarrow 5 \rightarrow 4 \) is a cycle of length 2 and 6 is a fixed point. Fixed points are often omitted when writing a permutation, so \((123)(45)(6) \equiv (123)(45)\). By convention, all cycles start with their smallest element and they are written in increasing order of the first element.

The lengths of the cycles of a permutation \( \pi \in \mathfrak{S}_m \) form a partition of \( m \) called the cycle type of \( \pi \). The cycle type of \((123)(45)(67)\) is \( 223 \). A permutation whose cycle type is \( 1^{m-2}2 \), which simply exchanges two elements, is called a transposition.

Cycle type is preserved under the action of conjugation, i.e. \( \pi \) and \( \sigma \pi \sigma^{-1} \) have the same cycle type for any \( \sigma \). Therefore, the group \( \mathfrak{S}_m \) may be divided into conjugacy classes \( C_\lambda \), each class being determined by a partition of \( m \) which is the cycle type of its elements. The number of elements in \( C_\lambda \) is
\[ |C_\lambda| = \binom{m}{\lambda} \prod_i (\lambda_i - 1)! \]  
(5)
The number of elements of \( \mathfrak{S}_m \) which have exactly \( k \) cycles, of whatever type, is the (unsigned) Stirling number of the first kind, \( s(m, k) = \sum \ell(\lambda) = k |C_\lambda| \).

These numbers satisfy
\[ \sum_{k=0}^{m} s(m, k)x^k = [x]^m, \]  
(7)
where
\[ [x]^k = x(x + 1) \cdots (x + k - 1) \]  
(8)
is the rising factorial.

Enumerating factorizations of permutations is an important problem, with connections to many other areas. A classical factorization problem is the following. Determine the number of permutation pairs \((\sigma, \tau)\), with \( \sigma \in C_\alpha \) and \( \tau \in C_\beta \), such that \( \sigma \tau = \pi \) for a fixed permutation \( \pi \in C_\lambda \). The number of solutions to this problem, denoted by \( C_\lambda^{\alpha, \beta} \), is called the connection coefficient of the permutation group. We define
\[ C_{\alpha, \beta}^{\lambda} = \sum_{\ell(\alpha) = a} \sum_{\ell(\beta) = b} C_{\alpha, \beta}^{\lambda} \]  
(9)
as the solution to the factorization problem where only the numbers of cycles of the factors are specified. For example, the only element of \( C_1^{11} \) is the identity and its factorizations are all of the kind \( 1 = \sigma \sigma^{-1} \); thus, it is clear that \( C_{2,1}^{1} = s(m, a)\delta_{a,b} \).

A semiclassical calculation of transport moments was developed by Berkolaiko and Kuipers in [22], which is different from the one discussed here and in [21]. They encounter the
following factorization problem: when \( \pi = (s_1 t_1) \cdots (s_d t_d) \) is a product of \( d \) transpositions with \( t_j > s_j \) and \( t_i \geq t_j \) for all \( k > j \), this is called a primitive factorization of \( \pi \) of depth \( d \).

Primitive factorizations were introduced in [23] and have been further discussed in [24]. The number of such factorizations for a given permutation \( \pi \) is denoted by \( p_d(\pi) \). It depends only on the cycle type of \( \pi \), so it can be denoted equivalently by \( p_d(\omega) \) if \( \pi \in \mathcal{C}_\omega \). This quantity can be obtained by means of a recurrence relation and this allows the equivalence between RMT and semiclassics to be firmly established for all symmetry classes [22].

In this paper we define a new kind of factorization problem, which plays a central role in our semiclassical approach to transport moments. In order to state it, we need some notation. Given a permutation \( \pi \), call \( P_1 \) its first cycle, i.e. the one that contains ‘1’, and let \( [P_1] \) be the set of elements of \( P_1 \). A cycle whose elements are in increasing order is called an increasing cycle. Given a set \( s \), let the restriction of a permutation \( Q \) to \( s \), denoted by \( Q|_s \), be the permutation obtained by simply erasing from the cycle representation of \( Q \) all symbols not in \( s \). For example, \((123)(45)|_{\{1,3,5\}} = (13)(5)\). The cycle type of such a permutation is determined by taking the lengths of its cycles as a partition of the cardinality of \( s \).

**Definition.** We denote by \( \mathcal{Z}(m,E,V;\omega) \) the number of solutions in the permutation group \( \mathcal{S}_E \) to the factorization \( (12 \cdots E) =QP \) of the complete increasing cycle, which satisfy the following conditions: (i) \( P \) has \( V+1 \) cycles and no fixed points; (ii) its first cycle \( P_1 \) is increasing and of size \( m \); (iii) all cycles of \( Q \) have at least one element in common with \( P_1 \); (iv) the restriction of \( Q \) to \( [P_1] \) has cycle type \( \omega \).

For example, the factors \( Q = (1432)(5) \) and \( P = (135)(24) \) provide a factorization of \((12345)\) of the kind described above, with \( V = 2, m = 3 \) and \( \omega = (2,1) \). The factors \( Q = (1)(253)(46) \) and \( P = (124)(365) \) provide another example at \( E = 6 \), in which \( V = 2, m = 3 \) and \( \omega = 1^3 \). Finally, at any value of \( E \), the factorization where \( Q = 1 \) and \( P = (12 \cdots E) \) is also of this kind, with \( V = 1, m = E \) and \( \omega = 1^E \). As counterexamples, \( Q = (1)(24)(3) \) and \( P = (12)(34) \) are not acceptable as factors at \( E = 4 \) because not all cycles of \( Q \) intersect \( P_1 \), while the factorization with \( Q = P = (132) \) is not acceptable because \( P_1 \) is not increasing.

Comparing our semiclassical expression with the one in [22], we relate our factorization problem and the primitive factorization problem through our

**Conjecture.** For every \( d \geq 0 \) and every \( \omega \vdash m \),

\[
\sum_{V=0}^{d} (-1)^{V+d} \mathcal{Z}(m,V+m+d,V;\omega) = |\mathcal{C}_\omega| p_d(\omega). \tag{10}
\]

It is rather surprising that these two factorization problems, at first sight completely different, seem to be actually closely related. It would be interesting to have a direct combinatorial proof of it.

One consequence of this conjecture is that

\[
\sum_{\omega \vdash m} \sum_{V=0}^{d} (-1)^{V+d} \mathcal{Z}(m,V+m+d,V;\omega) = S(m+d-1,m-1). \tag{11}
\]

In order to see this, note that

\[
\sum_{\omega \vdash m} |\mathcal{C}_\omega| p_d(\omega) = \sum_{\pi \in \mathcal{S}_m} p_d(\pi) =: P(m,d) \tag{12}
\]

is simply the cardinality of the set of all primitive factorizations of depth \( d \) in \( \mathcal{S}_m \). These factorizations can be categorized into two types: those which are also primitive factorizations
of depth \(d\) in \(\mathcal{S}_{m-1}\), and those which are obtained from a primitive factorization of depth \(d = 1\) in \(\mathcal{S}_m\) by appending a factor \((sm)\) at the end, for some \(s\). There are \(m - 1\) possible such factors. Therefore, the cardinality satisfies the recurrence relation

\[
P(m, d) = P(m - 1, d) + (m - 1)P(m, d - 1). \tag{13}
\]

The Stirling numbers of second kind satisfy

\[
S(m + d - 1, m - 1) = S(m + d - 2, m - 2) + (m - 1)S(m + d - 2, m - 1), \tag{14}
\]

and it is easy to see that \(P(m, 1) = S(m, m - 1)\) and \(P(2, d) = S(d + 1, 1)\). The result (11) is thus proved.

3. Semiclassical approximation to transport moments

In the semiclassical limit \(\hbar \to 0, N \to \infty\), the matrix elements of \(t\) may be approximated [25, 26] by

\[
t_{ij} \approx \frac{1}{\sqrt{T_H}} \sum_{\gamma \to a} A_{\gamma} e^{iS_{\gamma}/\hbar}, \tag{15}
\]

where the sum is over trajectories starting at incoming channel \(i\) and ending at outgoing channel \(a\). The phase \(S_{\gamma}\) of trajectory \(\gamma\) is its action and the amplitude \(A_{\gamma}\) is related to its stability. The prefactor contains the Heisenberg time \(T_H\), which equals \(N\) times the classical dwell time, i.e. the average time a particle spends in the cavity (inverse decay rate).

Expanding the trace, transport moments become

\[
M_m \approx \frac{1}{T_H^{1/2}} \prod_{j=1}^{m} \sum_{\gamma_j, o_j} A_{\gamma_j} e^{i(S_{\gamma_j} - S_{o_j})/\hbar}. \tag{16}
\]

The sum involves two sets of \(m\) trajectories, the \(\gamma\)s and the \(\sigma\)s. \(A_{\gamma} = \prod_j A_{\gamma_j}\) is a collective stability and \(S_{\gamma} = \sum_j S_{\gamma_j}\) is a collective action, and analogously for \(\sigma\). The result of (16) is in general a strongly fluctuating function of the energy, so a local energy average is introduced. When this averaging is performed in the stationary phase approximation, it selects those sets of \(\sigma\)s that have almost the same collective action as the \(\gamma\)s.

Most importantly, the structure of the trace implies that these two sets of trajectories connect the channels in a different order, and we can arrange it so that \(\gamma_j\) goes from \(i_j\) to \(o_j\), while \(\sigma_j\) goes from \(i_j\) to \(o_{j+1}\). In other words, when we consider only the labels on the channels, trajectories \(\gamma\) implement the identity permutation, while trajectories \(\sigma\) implement \(c_m = (1 \cdots m)\).

In the past ten years [27], it has been established that the way these action correlations are produced is as follows: each \(\sigma\) must follow closely a certain \(\gamma\) for a period of time, and some of them exchange partners at what is called an encounter. An \(r\)-encounter is a region where \(r\) pieces of trajectories run nearly parallel and \(r\) partners are exchanged. The two sets of trajectories are thus nearly equal, differing only in the negligible encounter regions. In particular, this implies \(A_{\gamma}A_{\gamma'}^* = |A_{\gamma}|^2\). This theory has been presented in detail in [28, 29]. We consider only systems for which the dynamics is not invariant under time-reversal. Hence, a \(\sigma\) trajectory never runs in the opposite sense with respect to a \(\gamma\) trajectory.

Every correlated pair of trajectory sets contributing to (16) may be represented by a diagram, which consists in reducing the encounters to points (vertices) and drawing the pieces of trajectory connecting them as simple edges (when in reality they can be extremely convoluted). Doing this erases the information about how exactly the partner trajectories are exchanged at the encounters. However, it is known that the contribution of a diagram does not depend on this detailed information, but only on the total numbers of edges and vertices.
The way this comes about (see [29] for details) is roughly as follows. First, a rule is used [15] that says that a sum over trajectories connecting any given channels may be replaced by a time integral,

$$\sum_{\gamma: i \rightarrow o} |A_\gamma|^2 = \int_0^\infty dT e^{-NT/T_H} = \frac{T_H}{N}. \quad (17)$$

Such a factor arises for each of the edges. On the other hand, integration over all possible action differences at encounters results that each $r$-encounter produces a factor $-N/T_H$. The total power of $T_H$ becomes 0 because adding the value of $r$ for all encounters produces $E - m$, which cancels with the factor from the edges and the denominator in (16). In the end, it can be shown that the total contribution to (16) of a diagram with $V$ vertices and $E$ edges is simply $(-1)^V N^V - E$.

It is also necessary to consider how the endpoints of a diagram may be distributed within the leads. As we shall see, a diagram may require that some channels coincide. If the number of distinct incoming and outgoing channels in a diagram are $m_1$ and $m_2$, respectively, then there are

$$[N_1]_{m_1} [N_2]_{m_2} \quad (18)$$

many ways of assigning them among the possible ones existing in the leads, where $[N]_m$ is the falling factorial. Finally, if $k$ different $\sigma$s start and end at the same channels, their identity may be exchanged without affecting the semiclassical contribution. Therefore, in such a case the contribution of the diagram must be multiplied by $k!$. Equivalently, we may define labeled diagrams as diagrams with a fixed choice of $\sigma$s, and count labeled diagrams.

### 4. Our approach

Based on what was discussed in the preceding section, we write the semiclassical expression for transport moments as

$$M_m(N_1, N_2) = \sum_{m_1, m_2} \sum_{E, V} D(m_1, m_2, E, V) \frac{(-1)^V}{N^E-V} [N_1]_{m_1} [N_2]_{m_2}, \quad (19)$$

where $D(m_1, m_2, E, V)$ is the number of labeled diagrams with $E$ edges and $V$ vertices, having $m_1$ ($m_2$) distinct channels on the incoming (outgoing) lead.

In this section we summarize the approach to the calculation of (19) that was developed in [21], following a rather different presentation. We start by addressing correlated periodic orbits. Then we discuss how they are turned into scattering diagrams. Next, we introduce our problem regarding factorizations of permutations. We then consider the combinatorics of channels and finally present the end result.

#### 4.1. Correlated periodic orbits

In [28], pairs of correlated periodic orbits were associated with factorizations of permutations. We generalize it to take into account a situation when a single periodic orbit $\alpha$ is correlated with a set of periodic orbits $\beta_1, \beta_2$, etc. In-between encounters, $\alpha$ and $\beta$ are indistinguishable, and we can label the encounter stretches in such a way that the end of stretch $j$ is followed by the beginning of stretch $j+1$. This produces the permutation $c_E$, where $E$ is the number of stretches, acting on the ‘exit-to-entrance’ space (it goes from the exit of an encounter to the entrance of another one).

The orbits behave differently inside the encounters (the ‘entrance-to-exit’ space). At any encounter, the action of $\alpha$ corresponds to the identity permutation: it takes the entrance of a
Figure 1. Schematic representation of correlated periodic orbits and how they become scattering trajectories. Orbit $\alpha$ is depicted with a solid line and orbits $\beta$ with dashed lines. Shaded regions represent the encounter which is cut open. (a) is associated to the factorization equation $(1234567) = (1)(264)(375) \cdot (125)(36)(47)$. Note that the first cycle of $P$ is increasing. (b) is associated to $(1234567) = (1537264) \cdot (152)(36)(47)$; we do not use these orbits to produce scattering diagrams because the first cycle of $P$ is not increasing. (c) is associated to $(1234567) = (153)(274)(6) \cdot (14)(25)(673)$; we also do not use these orbits because, when the first encounter is opened, one periodic orbit remains periodic.

stretch to the exit of the same stretch. On the other hand, $\beta$ acts by implementing a non-trivial permutation, which for convenience we call $P^{-1}$ (this differs from the notation in [21]). In the example shown in figure 1 we have $P^{-1} = (152)(36)(47)$. The encounters are in bijection with the cycles of $P$. Note that $P$ does not have fixed points.

The product $c_eP^{-1} \equiv Q$, acts on ‘exit-to-exit’ space, leading from the exit of an encounter to the exit of another one. Each cycle of $Q$ corresponds to one of the periodic orbits in $\beta$. Since $c_e$ is fixed, the total number of correlated pairs equals the number of solutions to the factorization equation $c_e = QP$. As we have seen, this is counted by connection coefficients.

A few examples are shown in figure 1. In figure 1(a) there are three different $\beta$s, and we have $P^{-1} = (152)(36)(47)$. The encounters are in bijection with the cycles of $P$. In figure 1(b) there is only one $\beta$ because the exchange of partners inside the 3-encounter is different from figure 1(a). In this case $Q = (1537264)$. Finally, in figure 1(c) we have simply relabeled the stretches of figure 1(a) so that the first stretch belongs to a 2-encounter.

4.2. Turning periodic orbits into scattering ones

Given $\alpha$ and $\beta$, we ‘cut open’ an $m$-encounter (as a matter of convention, we always open the encounter that contains the stretch labeled ‘1’) to produce $2m$ endpoints, $m$ of them corresponding to the ‘beginning’ of trajectories (leaving the encounter) and the other $m$ to ‘ending’ of trajectories (arriving at the encounter). We interpret them as incoming and outgoing channels, respectively. Then, the first stretch becomes $i_1$, and we use $\alpha$ to label all channels in sequence: the piece of $\alpha$ that starts in $i_j$ (and necessarily ends in $o_j$) becomes $\gamma_j$, while the piece of $\beta$ that starts in $i_j$ becomes $\sigma_j$. This produces something we have called a pre-diagram in [21]. It resembles a diagram, but there are some points about which we must be careful.

The first is that if one of the $\beta$ orbits does not participate in the encounter, it remains a periodic orbit instead of becoming a scattering trajectory. We must therefore demand that this does not happen. Second, in the resulting situation the $\sigma$ trajectories do not necessarily connect incoming to outgoing channels according to the cyclic permutation $c_m$. We must enforce some coincidences among the channels so that the permutation implemented by the $\sigma$s, call it $\pi$, becomes effectively equal to $c_m$. Finally, some different $\alpha, \beta$ can lead to the same situation when they are opened. However, when this happens the structure of the first encounter is
necessarily different. In order to avoid this over-counting, we impose that the permutation experienced by the $\sigma$s inside the first encounter must be decreasing.

In figure 1 the lightly shaded regions indicate the encounters that are opened in order to produce scattering situations. The pre-diagram which arises from figure 1(a) is shown in figure 2(a). As it is, the $\sigma$ trajectories implement the identity permutation in the channel labels. In figure 2(b) and figure 2(c) we show two possible diagrams that can be obtained from this pre-diagram. Figure 1(b) does not lead to a pre-diagram, because the permutation inside the first encounter is not decreasing. This avoids over-counting, since it would lead to the same pre-diagram as figure 1(a). Figure 1(c) also does not lead to a pre-diagram, because it has a periodic orbit which does not participate in the first encounter.

Taking into account the above discussion, we write $D(m_1, m_2, E, V)$ as

$$D(m_1, m_2, E, V) = \sum_{\pi \in S_{m_1}} \Xi(m, E, V; \pi) f(\pi, m_1, m_2).$$

(20)

Here $\Xi(m, E, V; \pi)$ is the number of pre-diagrams with $m$ channels on each side, having $E$ edges, $V$ vertices and implementing permutation $\pi$. On the other hand, $f(\pi, m_1, m_2)$ is the number of ways to convert such a pre-diagram into a labeled diagram with $m_1$ and $m_2$ distinct channels.

4.3. A new factorization problem

Since all elements of $\beta$ must take part in the first encounter, all cycles of permutation $Q$ must have at least one element in common with the first cycle of permutation $P$. To the knowledge of this author, the problem of factorizing the cycle under this condition on the factors has not been considered before.

We must also determine what is $\pi$ for a given factorization $c_E = QP$. Let $[P]$ denote the set of integers which are not fixed points of the permutation $P$. Let $P_1$ denote the first cycle of $P$, the one that contains the element ‘1’. Given a set $s$, let $P_1 \mid s$ denote the restriction of $P$ to $s$, defined in section 2.

Suppose we have correlated orbits described by the equation $c_E = QP$. The set of elements involved in the first encounter is $[P_1]$, assumed to have $m$ elements. In the example of figure 1(a) this is $\{1, 2, 5\}$. The $\gamma$ trajectories start and end at this encounter and, by construction, visit these elements in increasing order, i.e. they implement a permutation which is simply $c_E\mid[P_1]$. This is (125) in figure 1(a).

We must determine what is $\pi$, the permutation induced by $\sigma$ on those labels. First, we restrict $Q$ to the appropriate space, $Q_{[P_1]}$. In figure 1(a) this is (1)(2)(5). This acts on exit-to-exit space, i.e. it takes incoming channels to incoming channels. We multiply it by $P_1$ in
order to reverse the permutation effected inside the first encounter. The result, $Q_1|P_1$, takes incoming channels to outgoing channels. In figure 1(a) this is also (125), just like for the $\gamma$'s.

At this point, we have the permutations implemented by both $\gamma$ and $\sigma$ on the channel labels. The first is $cE|\{P_1\}$ and the second is $Q_1|P_1$. The first should be the identity, so we multiply both quantities by $cE^{-1}|\{P_1\}$ to obtain $\tilde{\pi}$:

$$\tilde{\pi} = Q_1|P_1cE^{-1}|\{P_1\}.\tag{21}$$

We can choose the permutation experienced by the $\sigma$'s inside the first encounter to be decreasing, so that $P_1$ is increasing, i.e. $P_1 = cE|\{P_1\}$. This leads to

$$\tilde{\pi} = Q_1|P_1.\tag{22}$$

In particular, the number of cycles of $\tilde{\pi}$ equals the number of individual periodic orbits in the set $\beta$. In figure 1(a), $\tilde{\pi} = (1)(2)(5)$.

Finally, note that $\tilde{\pi}$ is a permutation of $m$ symbols, but these are not the elements of the set $\{1, \ldots, m\}$ (as we have seen in the example). We must now represent $\tilde{\pi}$ as a permutation acting on this set. This is achieved by making every element of $\tilde{\pi}$ as small as possible while maintaining the relative order. That is, the element 1 remains the same while the next larger element becomes 2, etc. Denote this operation by $(\ddot{\cdot})$. For example, $((13)(475)) = (12)(354)$. The permutation $\pi$ is then $(\ddot{\pi})$.

Let $\Xi(m, E, V; \pi)$ be the number of solutions in the permutation group $S_m$ to the factorization equation $(12\cdots E) = QP$ which satisfy the following conditions: (i) $P$ has $V + 1$ cycles and no fixed points; (ii) its first cycle, $P_1$, is increasing and of size $m$; (iii) all cycles of $Q$ have at least one element in common with $P_1$; (iv) $(Q|P_1) = \pi$.

Obviously, the problem is only defined for $m \leq E$. If $m = E$, then necessarily $V = 0$ and in that case $\Xi(m, m, 0; \pi) = 1$ since the only solution is $c_m = 1 \cdot c_m$. Since $P$ has no fixed points, the largest possible value for $V$ is the integer part of $(E - m)/2$.

It seems very natural that $\Xi(m, E, V; \pi)$ should depend on $\pi$ only via its cycle type, and we have strong numerical evidence in favor of that. This is what motivates our definition 1, which appears in section 2. Assuming this is indeed true, then clearly

$$\Xi(m, E, V; \omega) = |c_\omega|\Xi(m, E, V; \pi),\tag{23}$$

if we denote by $\omega$ the cycle type of $\pi$.

### 4.4. Combinatorics of channels

A pre-diagram only becomes an actual diagram if certain coincidences exist among the channels so that $\pi$ is effectively equal to $c_m$, as required by (16). We thus consider the following problem: Given a pre-diagram with $m$ channels on both sides, in how many ways can we turn it into an acceptable labeled diagram that has $m_1$ distinct channels on the incoming lead and $m_2$ in the outgoing one? The answer to this question is the function $f(c_m, m_1, m_2)$ which we have already introduced. The calculation of this function was left as an open problem in [21]; here we present its solution.

Take the simplest case first, when $\pi = c_m$ is already the correct permutation and the pre-diagram is already a true diagram. The idea is to consider all possible permutations that can be implemented on the $\sigma$'s, and for each such permutation to determine under which conditions it can be accepted. The answer is easy: if the permutation implemented has $k$ cycles, then there must be at most $k$ distinct channels on each side. The number of permutations of $m$ symbols with $k$ cycles is equal to $s(m, k)$. The number of ways to distribute $k$ distinct numbers among $m_1$ possibilities is $S(k, m_1)$. We conclude that

$$f(c_m, m_1, m_2) = \sum_{k=1}^{m} s(m, k)S(k, m_1)S(k, m_2).\tag{24}$$
For example, take $m = 2$ and $\pi = (12)$. With a given labeling, we can have all channels different, one coincidence of the left, one on the right or coincidence on both sides. On the other hand, the only possibility in order to be able to interchange the $\sigma$s is if there are coincidences on both sides.

Now let us look at the more general situation, when $\pi \neq c_m$. Now, instead of looking at permutations of the $\sigma$s, we must consider separately two permutations, call them $\lambda$ and $\rho$, acting on the left and right channel labels, respectively. We count all possible such pairs, under the condition that they make $\pi$ effectively equal to $c_m$. This is done as follows. Given any label on the left, say $i$, the permutation $\lambda$ takes it into another label, $\lambda(i)$. Then permutation $\pi$ acts, taking it from the left lead to the right lead. At the right lead, permutation $\sigma$ must belong to the same channel.

Similarly, the right labels that belong to the same cycle $\lambda$ of $\rho$ must also belong to the same channel. If $\rho$ has $\ell$ cycles and $\rho$ has $r$ cycles, then there are $C_{\ell, r}$ solutions, where $\alpha$ is the cycle type of $c_m\pi^{-1}$.

By considering all solutions to (25) we are in fact considering all possible labelings of the $\sigma$s. To accept a certain choice of labeling, the left labels that belong to the same cycle of $\lambda$ must belong to the same channel. Similarly, the right labels that belong to the same cycle of $\rho$ must also belong to the same channel. If $\lambda$ has $\ell$ cycles and $\rho$ has $r$ cycles, there are $S(\ell, m_1)S(r, m_2)$ ways to arrange this in the leads. Therefore, in view of the above paragraph, we have

$$f(\pi, m_1, m_2) = \sum_{\ell, r=1}^{m} C_{\ell, r}S(\ell, m_1)S(r, m_2).$$

(26)

Coming back to the semiclassical expression for moments, equation (19), we see that there appears the sum

$$F(\alpha, N_1, N_2) = \sum_{m_1, m_2=1}^{m} f(\pi, m_1, m_2)[N_1]_{m_1}[N_2]_{m_2}.$$ 

(27)

According to basic properties of Stirling numbers, reviewed in section 2, this is given by

$$F(\alpha, N_1, N_2) = \sum_{\ell, r=1}^{m} C_{\ell, r}N_1^{\ell}N_2^{r}.$$ 

(28)

4.5. Final result

In terms of previously defined quantities, our semiclassical approach yields the following result for transport moments:

$$M_m = \sum_{\pi \in S_n} F(\alpha, N_1, N_2) \sum_{d=0}^{\infty} \frac{1}{N^{m+d}} \sum_{V=0}^{d} (-1)^V \Xi(m, V + m + d, V; \pi).$$ 

(29)

where $\alpha$ is the cycle type of $c_m\pi^{-1}$ and we have used $d = E - V + m$ instead of $E$. As we have seen in section 2, there are $C_{\ell, r}$ many permutations $\pi$ with the cycle type $\omega$ such that $c_m\pi^{-1}$ has the cycle type $\alpha$. Therefore, assuming (23) we can also write

$$M_m = \sum_{\alpha, \omega, m} \frac{C_{\alpha, \omega}}{|C_{\omega}|} F(\alpha, N_1, N_2) \sum_{d=0}^{\infty} \frac{1}{N^{m+d}} \sum_{V=0}^{d} (-1)^V \Xi(m, V + m + d, V; \omega).$$ 

(30)
Let us introduce
\[ C^{(m)}_{\omega,\ell,r} = \sum_{\omega \vdash m} C^{(m)}_{\omega,\ell,r} \]
which is the number of solutions to the factorization \( c_m = \rho \pi \lambda \) where \( \lambda \) has \( \ell \) cycles, \( \rho \) has \( r \) cycles and \( \pi \) has the cycle type \( \omega \). Using this quantity and the relation (28), we obtain
\[ M_m = \sum_{\ell, r} \sum_{d=0}^{\infty} \sum_{\omega \vdash m} \frac{C^{(m)}_{\omega,\ell,r}}{|C_\omega|} \left(-1\right)^d \Xi(m, V + m + d, V; \omega). \] (32)

5. Comparisons with other results

Next, we compare the above results with RMT and with the semiclassical approach of [22] in order to arrive at the conjecture about the function \( \Xi(m, V + m + d, V; \omega) \) that was stated in section 2.

5.1. Other semiclassics

The semiclassical calculation of transport moments was also considered in [22], along different lines. In particular, a function \( p_d(\pi) \) is required which is the number of primitive factorizations of depth \( d \) that exist for the permutation \( \pi \). This function is interesting on its own and is further discussed in [23, 24].

For systems with broken time-reversal symmetry, the result obtained in [22] for the unitary symmetry class reads
\[ M_m(N_1, N_2) = \sum_{\lambda, \rho \in \mathfrak{S}_m} N_1^{(\lambda)} N_2^{(\rho)} \sum_{d=0}^{\infty} \frac{1}{N^{m+d}} (-1)^d p_d(\lambda \rho), \] (33)
where \( c(\cdot) \) denotes the number of cycles of a permutation. This can be shown to be exactly equivalent to RMT. The function \( p_d(\pi) \) depends only on the cycle type of \( \pi \), which we denote by \( \omega \).

Using the quantity \( C^{(m)}_{\omega,\ell,r} \) already defined, this can be written as
\[ M_m = \sum_{\ell, r} \sum_{d=0}^{\infty} \sum_{\omega \vdash m} C^{(m)}_{\omega,\ell,r} (-1)^d p_d(\omega). \] (34)
Since this expression must agree with our result (32) for all values of \( N_1, N_2 \), we arrive at our conjecture, stated in section 2.

At the heart of the approach developed in [22] is a technique for taking as much advantage as possible from cancellations between diagrams. Our approach, embodied in the function \( \Xi(m, V, \omega) \), does not have this characteristic. The conjecture we present here therefore encodes diagram cancellations. Perhaps taking into account other degrees of cancelations might lead to other factorization problems.

5.2. RMT with \( N_2 = 1 \)

Let us consider the case when the right lead is in the extreme quantum regime \( N_2 = 1 \). Then
\[ F(\alpha, N_1, 1) = \sum_{\ell, \tau} C_{\ell,\tau}^{\alpha} N_1^\ell N_2^\tau = \sum_{\ell=1}^{m} s(\ell, \ell) N_1^\ell = [N_1]^m. \] (35)
where we used (7). Using that $\sum_\omega C_{\omega}^{(m)} = |C_\omega|$, equation (30) gives

$$M_m(N_1, 1) = |N_1|^m \sum_{d=0}^{\infty} \frac{1}{(N_1 + 1)^{m+d}} \sum_{\alpha} \sum_{\omega} (-1)^V \Xi(m, V + m + d, V; \omega).$$

(36)

Using the consequence (11) we derived from our conjecture, this becomes

$$M_m(N_1, 1) = |N_1|^m \sum_{d=0}^{\infty} \frac{1}{(N_1 + 1)^{m+d}} (-1)^d S(m + d - 1, m - 1) = \frac{|N_1|^m}{(N_1 + 1)^m}.$$  

(37)

The RMT prediction for transport moments in the absence of time-reversal symmetry is [12]

$$M_m(N_1, N_2) = \sum_{p=0}^{m-1} \frac{(-1)^p}{m!} \binom{m-1}{p} \frac{|N_1 - p|^m[N_2 - p]^m}{|N_1 + N_2 - p|^m}.$$  

(38)

When $N_2 = 1$, we may use $[1 - p]^m = m! \delta_{p,0}$ and this reduces exactly to (37).

6. Conclusions

We have derived a semiclassical expression for transport moments, valid for arbitrary numbers of channels and in agreement with random matrix theory for broken time-reversal symmetry (as far as it can be checked). Unfortunately, the combinatorial problem of determining $\Xi(m, E, V; \omega)$ is still open, so we cannot show exact agreement with RMT. However, the merit of the semiclassical approach is not in its computational efficiency, but rather as a way to identify the dynamical origins of universality.

A deep connection has been revealed between quantum chaotic scattering and the problem of factorizing permutations (see also [30]). In fact, two different factorization problems have appeared in this area, the one discussed here and the one discussed in [22] (factorizations of permutations had already appeared [28] in the semiclassical approach to closed chaotic systems). These two problems are quite different in nature, but there probably is a relationship between them, suggested by the conjecture we have put forth in the present work. This remains to be further investigated.

RMT statistics is expected to hold when the average dwell time in the cavity is much larger than the system’s Ehrenfest time. This was assumed in the present work. When these two time scales are comparable, the semiclassical approach is more complicated, but some results have been obtained [31–33]. Whether these can be extended to a complete calculation of all moments, and what kind of combinatorics is required, is still an open problem. Other interesting open problems include the treatment of tunnel barriers (see [34, 35]) or Andreev reflection (see [36, 37]). Finally, diffraction effects should be important when the number of channels is small [38], and this is yet to be considered.

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