Exactly pairing two-dimensional charged particles using a magnetic field

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It is demonstrated that a uniform magnetic field can exactly pair the two-dimensional (2D) charged particles only for some quantized magnetic intensity values. For the particle-pair consisting of two like charges the Landau level of the center-of-mass motion is multiple degenerate that implies the shell structure. However, any particle-pair has only two non-degenerate relative levels, which are associated with the diamagnetic and paramagnetic states respectively. There exist a upper critical magnetic strength and a lower critical magnetic length across which two like charges cannot be paired. The theoretical results agree with the experimental data on the sites and widths of the integer and fractional quantum Hall plateaus in a CaAs-Al$_2$Ga$_{1-x}$As heterojunction, that gives a new explanation for the quantum Hall effect.

PACS numbers: 73.43.Cd, 05.30.Fk, 73.43.-f, 03.65.Ge

The motions of the charged particles and neutral atoms in a uniform magnetic field have been long studied, and some important experimental developments of this topic were reported successively. The corresponding theoretical investigations reveal many interesting properties concerning the quadratic Zeeman effects, Landau level, classical and quantum chaos, and the integral and fractional quantum Hall effects. All of the works are correlated with the fundamental system: a pair of charged particles interacting with a uniform magnetic field, where both the Coulomb and harmonic potentials work simultaneously. The previous works have shown that the Coulomb-harmonic system is exactly solvable only for some particular values of the harmonic oscillator frequency.

In this paper we shall extend the results to a 2D electronic gas and seek its exact eigenstates and eigenenergies, where the harmonic potential is supplied by the external magnetic field. This extension will lead to some interesting new results that demonstrate the possibility for using a magnetic field to pair the charged particles, and suggests a new theory to confirm the assertion on “the fractional quantum Hall effect must result from the condensation of the 2D electrons”.

Quantum motion of the 2D charged-particle-pair in a uniform magnetic field toward $z$ direction is dominated by the stationary-state Schrödinger equation:

$$\sum_{i=1}^{2} \left[ -\frac{\hbar^2}{2m_i} \nabla_i^2 + \frac{1}{2} m_i \omega_i^2 (x_i^2 + y_i^2) \mp \omega_i l_i + \frac{q_1 q_2}{e r} \right] \Psi = E_T \Psi, \quad \text{(1)}$$

where $E_T$ is the total energy, $m_i$ and $q_i$ are the effective mass and charge of $i$-th particle, $x_i, y_i$ and $\nabla_i^2$ are the coordinates and Laplace operators of particle $i$, $r = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ is the relative radial coordinate, $l_i = x_i p_{y_i} - y_i p_{x_i}$, denotes the z-component angular momentum of particle $i$, $\omega_i = |q_i| B / (2 m_i e)$ represents the Larmor cyclotron frequency with the light velocity $c$ and the strength $B$ of magnetic field, and $e$ is the effective dielectric constant of the background semiconductor. Here and throughout the paper the above signs of $\pm$ and $\mp$ are selected for the positive-charge system, and the below signs are taken for the negative-charge system. In order to exactly solve Eq. (1) we introduce the relative coordinate $\mathbf{r}$ and center-of-mass coordinate $\mathbf{R}$ by $\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1 = (x_2 - x_1) \mathbf{e}_x + (y_2 - y_1) \mathbf{e}_y = x \mathbf{e}_x + y \mathbf{e}_y$, $\mathbf{R} = (m_1 \mathbf{r}_1 + m_2 \mathbf{r}_2) / M = X \mathbf{e}_x + Y \mathbf{e}_y$ with $M = m_1 + m_2$ being the total mass, and $\mathbf{e}_x$ and $\mathbf{e}_y$ the unit vectors in $x$ and $y$ directions. Setting $E_T = E_c + E$ and $\Psi = \psi^{(c)}(\mathbf{R})\psi(\mathbf{r})$, under the separability condition $q_1/m_1 = q_2/m_2 (\omega_2 = \omega_1 = \omega)$, Eq. (1) is separated as the center-of-mass motion equation

$$\left[ -\frac{\hbar^2}{2M} \nabla_R^2 + \frac{1}{2} M \omega^2 R^2 \mp \omega L_z \right] \psi^{(c)} = E_c \psi^{(c)}, \quad \text{(2)}$$

and the relative motion equation

$$\left[ -\frac{\hbar^2}{2\mu} \nabla_r^2 + \frac{1}{2} \mu \omega^2 r^2 \mp \omega l_z + \frac{q_1 q_2}{r} \right] \psi = E \psi, \quad \text{(3)}$$

where $\mu = m_1 m_2 / M$ is the reduced mass, $L_z = X P_y - Y P_x$ and $l_z = x p_y - y p_x$ are the z-component angular momenta on the center-of-mass and relative coordinate frames respectively. Clearly, Eq. (3) includes the equation of a 2D hydrogen atom in a uniform magnetic field, which can be directly derived from Eq. (1) by fixing the atomic nucleus and taking the positive sign of $\mp$. Therefore, Eq. (3) can govern the particle pairs consisting of the like charges (e.g., two electrons, two holes or two protons) or the unlike charges (e.g., hydrogen atom or hydrogenic donor-electron). Noticing the Pauli’s exclusion principle, the two like charges should be in opposite spin orientation.

For two-electron (or two-hole) system with $m_1 = m_2 = m_e$ and $|q_1| = |q_2| = e$, Eq. (2) becomes the normal equation of a single charged particle with

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mass $M = 2m_e$ and charge $|q| = 2e$ in a uniform magnetic field, which has the well known solution
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\psi^c = \psi_{n,m,c}(R, \phi ,\rho) = e^{imc_\phi } - R^{2} B R^{m_\rho} / (n_c|m + 1, R^2), \quad n_c = 0, 1, 2, \cdots; \quad m_c = 0, \pm 1, \pm 2, \cdots \quad \text{for the center-of-mass energy (Landau level) } E_c/(\hbar \omega) = 2n_c + |m_c| + m_\rho = 1, 2, 3, \cdots \quad \text{Here } F(-n_c, m_c + 1, R^2) \text{ denotes the confluent hypergeometric function, } m_c \text{ is the rest mass of a free electron, and the coordinate } R \text{ has been normalized by the magnetic length } a_c = \sqrt{\hbar/(M \omega)}. \quad \text{Obviously, } \psi_{n,m,c} \text{ expresses the multiple degenerate states, since the energy } E_c/(\hbar \omega) = 2n_c + 1 \text{ corresponds to the } M_n + 1 \text{ states with } |m_c| + m_\rho = 0 \text{ for } m_\rho = 0, \pm 1, \pm 2, \cdots, \pm M_n. \quad \text{The largest magnetic quantum number } m_n \text{ and principal quantum number } n_c \text{ are limited by the mean square-radius (or area) } R_{m_c}^2 = \langle \psi_{n,m,c} | R^2 | \psi_{n,m,c} \rangle / \langle \psi_{n,m,c} | \psi_{n,m,c} \rangle \leq R_0^2 \text{ for a 2D circular system with radius } R_0. \quad \text{Taking the lowest Landau level } E_c = 1/(\hbar \omega) \text{ with } n_c = 0 \text{ as an example, this equation gives } R_{m_c} = \sqrt{\hbar/(\mu \omega)} \quad \text{Thus we have the degeneracy of the lowest level } M_0 + 1 = \max(|m_c| + 1) = R_0^2/\omega/m_\rho = 2\pi R_0^2 M_0 \hbar / (\hbar \omega) = 2\pi R_0^2 \omega B/(hc) \quad \text{is the magnetic flux quantum of the electron-pairs and } \Phi = \pi R_0^2 B \text{ denotes the magnetic flux through the sample, which equates the integral times of } \Phi. \quad \text{We are interested in exactly solving the relative motion equation (3) and using the solution to describe the properties of the particle-pairs, including the application to the integral and fractional quantum Hall effect. Adopting the normalized polar coordinate } \rho = r/\alpha_r, \quad \alpha_r = \sqrt{\hbar/(\mu \omega)} \quad \text{and the separate wavefunction 21} \quad \psi(\rho, \phi) = A e^{i m \phi - \rho^2/2 |m| \mu |u(\rho)\rangle, \quad m = 0, \pm 1, \cdots (4) \quad \text{with the normalization constant } A, \quad \text{applying Eq. (4) to Eq. (3) we arrive at the 1D dimensionless equation } \frac{d^2}{d\rho^2} + \left( \frac{2|m| + 1}{\rho} - 2 \rho \right) \frac{d}{d\rho} + \left[ \frac{2}{\rho} \left( \frac{2n_f - |m| - 1}{\rho} - \frac{1}{\rho} \right) u = 0, \quad \text{where the energy has been changed to } E^\prime = E + \hbar n \omega, \quad \text{the dimensionless constant } \sigma = 2q_1 q_2 \sqrt{\hbar/(\mu \omega)} \quad \text{expresses the importance of the Coulomb potential compared to the harmonic one, since when } q_1 = q_2 = -e, \mu = m/e/2 \text{ and } \epsilon = 1 \text{ we have } \sigma^2 = 2e^2/\hbar \omega, \quad \text{with } a_0 = \hbar^2/(m e^2) \text{ being the Bohr radius. According to the form of } u \text{ equation we expect the power-series solution 22} \quad u = \sum_{n=0}^{\infty} C_n \rho^n \quad \text{for } n = 1, 2, \cdots \quad \text{and } C_0 \text{ =constant. Inserting the series into the equation yields the algebraic equation } \sum_{n=0}^{\infty} \left( i^2 + 2|m| |n|^2 + \sigma m + 1 - 2 E^\prime/(\hbar \omega) - |m| - 1 - \frac{1}{\rho} \right) C_n = 0 \quad \text{This equation can be satisfied if and only if (if) the constants } C_n, \quad \sigma \text{ and } E^\prime \text{ obey the coefficients equations 22} \quad E^\prime/(\hbar \omega) = n + |m| + 1, \quad (n - j)(n - j + 2|m|) C_{n-j} - \sigma c_{n-j,2} + 2(j + 2) c_{n-j,2} - 2 = 0 \quad \text{for } C_0 = 1, \quad C_1 = C_{n+1} = 0, \quad j = -1, 0, 1, \cdots (n - 1). \quad \text{By using a computer we have solved this equation group for } n = 1, 2, \cdots, 30 \text{ and several simple solutions are listed as the following:} \quad n = 1, \quad \sigma^2 = (\sigma^{(1)}_{|n|})^2 = 2(2|m| + 1), \quad C_1 = \sigma^{(1)}_{|n|} 2|m| + 1; \quad n = 2, \quad (\sigma^{(l)}_{2|m|})^2 = 4(4|m| + 3), \quad C_1 = \sigma^{(l)}_{2|m|} 2|m| + 1; \quad n = 3, \quad (\sigma^{(l)}_{3|m|})^2 = 20(|m| + 1) \pm 2 \sqrt{64m^2 + 128|m| + 73}; \quad n = 4, \quad (\sigma^{(l)}_{4|m|})^2 = 50 + 40|m| \pm 6 \sqrt{16m^2 + 40|m| + 33}.(5) \quad \text{where we have set } l = 1, 2, \cdots, \pm m_{\max}, \quad l_{\max} = n \text{ for even } n \text{ and } l_{\max} = (n + 1) \text{ for odd } n, \quad \text{which is the label of the different solutions for a set of fixed quantum numbers } n \text{ and } |m|. \quad \text{In the cases } n = 3, 4, \text{ any } C_l \text{ is a complicated function of } |m|, \text{ which is not shown in Eq. (5). When } n \geq 5, \text{ the computer cannot give } C_l \text{ and } \sigma \text{ as the explicit functions of } |m|, \text{ however, we can numerically calculate them for any given } n \text{ and } |m|. \quad \text{Noticing the above-mentioned relationships between } \omega \text{ and } \sigma, \quad \text{and } B \text{ and } \omega, \quad \text{the quantized } \sigma \text{ values imply the quantization of the cyclotron frequency } \omega, \text{ magnetic strength } B \text{ and energy } E, \quad \omega = \omega^{(l)}_{n|m} = \frac{4q_1 q_2 \mu}{\hbar^2 \sigma^{(l)}_{n|m} 2}, \quad B = B^{(l)}_{n|m} = \frac{8q_1 q_2 \mu n_1 c}{\hbar^2 \sigma^{(l)}_{n|m} 2}, \quad E^{(l)}_{n|m} = E^{(l)}_{n|m} \pm m \hbar \omega^{(l)}_{n|m} = (n + |m| \pm m + 1) \hbar \omega^{(l)}_{n|m}.(6) \quad \text{The dependence of } C_l \text{ and } \rho (a_r) \text{ on the quantum numbers } n, |m|, l \text{ leads to the power-series } u = u^{(l)}_{n|m} \text{ and the relative wavefunction } \psi = \psi^{(l)}_{n|m}. \quad \text{Given Eqs. (5), (6) and the label of the wavefunctions, we find the following interesting properties:} \quad \text{a) Two like charges can be paired iff the magnetic strength is quantized as in Eq. (6). The necessity and sufficiency of the pairing condition infer that the magnetic field fitted to the electron-pairs will suppress the second pairing of four electrons, since the latter does not satisfy the pairing condition of the former. The size of the particle-pair in the states of lower quantum numbers is in the order of the relative magnetic length } a_r = \sigma^{(l)}_{n|m} \sqrt{\hbar/(\mu \omega^{(l)}_{n|m})}, \quad \text{that may approach the size of the lowly excited hydrogen atom, when the magnetic field is strong enough. For example, the magnetic strength value } B \sim 30\text{Tesla (T) corresponds to the cyclotron frequency } \omega \sim 10^{13}\text{Hz and magnetic length } a_r \sim 10^{-9}\text{m for } \mu = m/e/2. \quad \text{b) The cyclotron frequency and magnetic strength depend on } \sigma^2 \text{ rather than } \sigma. \text{ This means that for a fixed magnetic strength the electron-pairs } (\sigma > 0) \text{ and hydrogen atom } (\sigma < 0) \text{ with same } |\sigma| \text{ have the same level structure determined by the cyclotron frequency. However, the coefficients } C_l \text{ depends on } \sigma \text{ that results in the different power-series solutions } u \text{ for the two different systems. For instance, in the Coulomb repulsion case } (\sigma > 0), \text{ Eq. (5) implies the simplest power-series solution } u = u_{1|0}^{(1)} = (1 + C_1 \rho) = (1 + \sqrt{2} \rho), \quad \text{but in the Coulomb attraction case } (\sigma < 0), \text{ the simplest power-series solution is } u = u_{1|0}^{(2)} = (1 - \sqrt{2} \rho). \quad \text{Substituting}
them into Eq. (4) respectively, produce the mean radius \( r_{1|0}^{(1)} = (\psi_{1|0}^{(1)}|r|\psi_{1|0}^{(1)})/(\psi_{1|0}^{(1)}|\psi_{1|0}^{(1)}) = 1.15739(a_{1|0}^{(1)}) \) in the Coulomb repulsion state and \( r_{1|0}^{(2)} = 1.45221(a_{1|0}^{(1)}) \) in the Coulomb attraction state \( \psi_{1|0}^{(2)} \), the latter is greater than the former. This is very important that the combined action between the Coulomb repulsion and magnetic field make the particle-pair of two like charges the tight-binding quasi-particle.

c) The magnetic strength is correlated to \( |m| \) but the energy and wavefunction are correlated to \( m \), that in- ers the particle-pair under the magnetic field \( B \) being a two internal-level system with two non-degenerate relative states, the paramagnetic state (\( m > 0 \)) and diamagnetic state (\( m < 0 \)). The level difference between the two states reads \( \Delta E_{m}^{(l)} = 2m\hbar\omega_{n|m|}. \) If the particle-pair is composed of two positive charges, the sign “-” in Eq. (6) is taken that means the paramagnetic state being the ground state (\( |m| - m = 0 \)) and the diamagnetic state being the excitation one (\( |m| - m = 2|m| \)). Conversely, the diamagnetic state of the electronic particle-pair (hydrogen atom or two electrons) is the ground state and the paramagnetic state is the excitation state. Because the amplitude \( |\psi| \) depends on \( |m| \) rather than \( m \) so the para-

magnetic and diamagnetic states have the same relative probability distribution.

d) Differing from the Landau level of center-of-mass motion, the larger quantum number \( n \) corresponds to the lower cyclotron frequency and relative energy for a given \( m \). As \( n \) tends to infinity, the frequency and energy tend to zero, since the increase velocity of \( \sigma^{2} \) is much greater than that of the \( n \), consequently, the relative magnetic length \( a_{\sigma} \) approaches infinity such that the particle-pair is ionized. Similarly, for a given \( n \) and in the ground state (\( |m| - m = 0 \)) case, the larger \( |m| \) corresponds to the lower \( \omega \) and \( E \), and the infinite \( |m| \) is associated with the zero energy and unbound state, however, in the excitation state case (\( |m| - m = 2|m| \)), the limit \( \lim_{m \to \infty} E_{m|0}^{(l)} \) is equal to a constant, although \( a_{\sigma} \) still tends to infinity.

e) There are two kinds of the quantum transitions for the particle-pair. One is the transitions between the ground and excitation states determined by a fixed magnetic field, which can be operated by using a laser with the frequency \( 2|m|\omega \). Because the amplitude \( |\psi| \) only depends on \( |m| \), the transition between \( m \) state and \( -m \) state does not change the relative probability distribution of the particle-pair, but varies the direction of angular momentum. Another kind of transitions occurs between the states with different magnetic strengths. Therefore, this kind of transitions can be controlled by adjusting the magnetic strengths from one value of \( B_{n|m|}^{(l)} \) to another. These quantum transitions are probably useful for performing the quantum logic operations.

f) The parameter \( |\sigma| \) has a minimal value \( |\sigma|^{(1)}_{1|0}, \) which corresponds to the upper critical magnetic field and lower critical magnetic length across which two like charges cannot be paired. For an electron-pair we insert the pa-

rameters \( q_{1} = q_{2} = -e, \mu = m_{1}/2 = m_{c}/2, \epsilon = 1 \) into Eqs. (5) and (6), producing the largest magnetic strength \( B_{10}^{(1)} = 2m^{2}3e^{3}c/\hbar^{3} \approx 4.7 \times 10^{5}T \), which cannot be experimentally realized yet. The experimentally allowable magnetic strength \( B < 10^{5}T \) and the pairing condition of two electrons require the parameter \( \sigma^{2} \) being greater than \( 10^{3} \). However, if one adopts the modulation-doped CaAs-Al\(_{x}\)Ga\(_{1-x}\)As heterojunction and let the 2D electrons exist in GaAs at the interface between GaAs and Al\(_{x}\)Ga\(_{1-x}\)As \( B < 10^{5}T \), \( B_{10}^{(1)} \approx 12.4842T \), such magnetic field is experimentally realizable. The largest magnetic strength \( B_{10}^{(1)} \) deter-

mines the lower critical magnetic length \( a_{1|0}^{(1)} \). By im-

proving the quality of the CaAs-Al\(_{x}\)Ga\(_{1-x}\)As interface, the effective mass \( m_{1} \) and dielectric constant \( \epsilon \) can be adjusted in the regions \( m_{1} \geq 0.065m_{c} \) and \( \epsilon \geq 1 \) in a practical experiment \[11, 24\] therefore, we can obtain the upper critical magnetic strength in a great region.

We now try to extend the above-mentioned results to a 2D electronic gas. Because of the multiple degenerate states of the center-of-mass motion and the two non-degenerate relative states, we expect the existence of the shell structure of the electron-pairs. There may be \( 2(M_{1} + 1) \) electron-pairs occupy the lowest Landau level of the center-of-mass motion and two of them labelled by \( (m_{c}, m) \) and \( (m_{c}, -m) \) have the mean square-radius \( R_{m_{c}m}^{2} \) for the fixed magnetic strength \( B_{n|m|}^{(l)} \). The multiple electron-pairs in the lowest Landau level seem to behave like the condensed Bosons. According to the superconductivity theory of the Cooper pairs \[27\], the condensation of electrons is necessary for the resist-

ance vanishing approximately. Therefore, we expect the minima in the resistivity appearing at the quantized magnetic strength values of the pairing condition (6).

This expectation has been proved by the experimental data on the integral and fractional quantum Hall effect \[7, 8, 9, 10\]. In fact, rewriting the pairing condition as \( B_{n|m|}^{(l)}B_{n|m|}^{(1)} = B_{n|m|}^{(1)}/v_{n|m|}^{(l)}v_{n|m|}^{(1)} = (\sigma^{(1)}_{n|m|})^{2}/30 \) and comparing the constant \( v_{n|m|}^{(l)} \) with the filling factor \( \nu \) at which the diagonal part of the resistivity tensor vanishing or taking minimum experimentally, good agreement is found in the experimental accuracy, as in table 1. Here \( B_{17}^{(1)} \) is the magnetic strength at \( \nu = 1 \), which is dependent of the sample material and can be determined by the experiments \[7, 8, 10\]. In table 1 we show the strict agreement between the experimental \( \nu \) and theoretical \( v_{n|m|}^{(l)} \) for the experimentally strong minima of the re-

sitivity and the approximate agreement in \( 10^{-3} \) order for the experimentally weak states, containing a lot of data disappearing in the table. The small errors may be caused from the difference between the finite temperature effect in the experiments \[7, 8\] and the zero tem-

perature assumption in theory. It is worth noting that
TABLE I: Comparison between the theoretical $\nu_{n[m]}^{(i)}$ and experimental $\nu$

| $\nu$ | 2/3 | 1/3 | 1/2 | 1 | 0 | 1/4 | 1/3 | 1/2 | 1 | 0 | 1/4 | 1/3 | 1/2 | 1 | 0 |
|-------|-----|-----|-----|---|---|-----|-----|-----|---|---|-----|-----|-----|---|---|
| $n[m]$ | 5.1 | 21.0 | 1.2 | 29.0 | 3.2 | 2.0 | 1.4 | 3.4 | 6.5 | 5.5 | 3.5 | 3.5 | 1.7 | 2.9 | 1.12 |
| $\nu_{n[m]}^{(i)}$ | 0.291 | 0.295 | 0.367 | 0.387 | 0.659 | 0.715 | 0.794 | 1.324 | 1.371 |

the experimentally strong minima appear in $n = 1, 2$ states with lower relative energies. For example, the two neighbor states $\nu_{1,2}^{(1)} \approx 5/17$ and $\nu_{1,2}^{(1)} = 1/3$ have near $\sigma$ and $\omega$ values, but Eq. (6) give their relative energies as $E_{21,0}^{(1)} = 22\hbar\omega_{21,0}$ and $E_{1,2}^{(1)} = \frac{1}{2}E_{1,-2}^{(1)} \approx 2\hbar\omega_{21,0}$ respectively, the former is much greater than the latter, so the $1/3$ state is much strong compared to the $5/17$ state [10], since there are more electron-pairs to occupy the lower energy states. The stronger state is associated with the lower diagonal resistivity and wider Hall plateau in the experiments.

In conclusion, we have revealed a new pairing mechanism of the 2D charged particles, namely the combining interaction of the quantized magnetic field and Coulomb potential governs the charged particle-pairs. The exact pairing states and level structure are found that show many new and important physical properties of the system. Applying the results to the quantum Hall effect of 2D electronic gas, we obtain the places and strengths of the minimal diagonal resistivity, which are in good agreements with the well known experimental data.

At the end of the paper we must point out that the theoretical results have hinted vaguely some connections between the electronic pairing states and the superconductivity of high critical temperature ($T_c$), such as the 2D electron-pairs in the quantum Hall system behaving like the Cooper pairs in the 2D superconductive material, the property of the critical magnetic strength exceeding the experimental limit being similar to that of the upper critical field of the type II superconductor with high $T_c$, and the lower critical magnetic length to be in the order of the electronic correlation length of the high $T_c$ YBa$_2$Cu$_3$O$_7$-$\delta$ superconductor [28], and so on. Further investigating the applications of the results to the high $T_c$ superconductivity, the state preparation of the harmonically trapped ions and the quantum computation will be interesting.

Acknowledgments

This work was supported by the National Natural Science Foundation of China under Grant No. 10275023.