Abstract. We study the structure algebra $Z$ of the stable moment graph for the case of the affine root system $A_1$. The structure algebra $Z$ is an algebra over a symmetric algebra and in particular, it is a module over a symmetric algebra. We study this module structure on $Z$ and we construct a basis. By “setting $c$ equal to zero” in $Z$, we obtain the module $Z_{c=0}$. This module can be described in terms of the finite root system $A_1$ and we show that it is determined by a set of certain divisibility relations. These relations can be regarded as a generalization of ordinary moment graph relations that define sections of sheaves on moment graphs, and because of this we call them higher-order congruence relations.

1. INTRODUCTION

In 1979 Kazhdan and Lusztig introduced a special basis of the Hecke algebra attached to a Coxeter system (cf. [KL79]). Entries of the transition matrix between the standard basis and this special basis are given by the Kazhdan-Lusztig polynomials. In the same article a character formula for simple highest weight modules over a complex semisimple Lie algebra was conjectured. This character formula was expressed in terms of the Kazhdan-Lusztig polynomials associated with the Weyl group of a semisimple Lie algebra. The Kazhdan-Lusztig polynomials together with the Kazhdan-Lusztig conjecture, which was proved independently by Beilinson and Bernstein (cf. [BB81]) and by Brylinski and Kashiwara (cf. [BK81]), constitute the foundations of geometric representation theory.

From a Coxeter group, a standard parabolic subgroup and a sign Deodhar constructed parabolic (spherical or anti-spherical—depending on the sign) modules over the Hecke algebra (cf. [Deo87]). More precisely, following the notation of [Soe97], let $(W, S)$ be a Coxeter system and $H$ its Hecke algebra over the ring $L = \mathbb{Z}[v, v^{-1}]$ of Laurent polynomials in one variable with integer coefficients. Let $I \subset S$ be a subset of the set of simple reflections, $W_I = \langle I \rangle$ the parabolic subgroup corresponding to $I$ and $H_I$ the Hecke algebra of $(W_I, I)$. For $u \in \{-v, v^{-1}\}$, $\mathcal{L} = \mathcal{L}(u)$ can be endowed with a structure of an $H_I$-bimodule. After induction two right $H$-modules are obtained: the spherical module $\mathcal{M} = \mathcal{L}(v^{-1}) \otimes_{H_I} H$ and the anti-spherical module $\mathcal{N} = \mathcal{L}(-v) \otimes_{H_I} H$ (the terms spherical and anti-spherical are taken from [LV]). Parabolic (spherical or anti-spherical) Kazhdan-Lusztig polynomials appear as elements of the
transition matrix between the standard basis and the Kazhdan-Lusztig basis of the parabolic (spherical or anti-spherical) Hecke module and they can be regarded as a generalization of the ordinary Kazhdan-Lusztig polynomials (when $I = \emptyset$, i.e. the parabolic subgroup $W_I$ is trivial, the parabolic Kazhdan-Lusztig polynomials coincide with the ordinary Kazhdan-Lusztig polynomials).

The Kazhdan-Lusztig polynomials possess fascinating properties, which are not apparent from their definition. Let us mention non-negativity of their coefficients, which was proved in full generality by Elias and Williamson (cf. [EW14]). The same property was established for the parabolic Kazhdan-Lusztig polynomials in [LW].

Another distinguished property, proved by Lusztig in [Lus80], is the stabilization property of the Kazhdan-Lusztig polynomials. It can be stated in terms of the parabolic Kazhdan-Lusztig polynomials corresponding to the spherical module $M$ ($m$-polynomials in Soergel’s notation, cf. [Soe97], Theorem 6.1.): far enough inside the fundamental (Weyl) chamber the $m$-polynomials stabilize. In this stabilization phenomenon the generic Kazhdan-Lusztig polynomials ($q$-polynomials in Soergel’s notation) make their appearance.

Via sheaves on moment graphs (these notions were introduced in [GKM98] and [BM01]) the stabilization property of the Kazhdan-Lusztig polynomials is lifted to the categorical level. In [Lan15] the moment graph analogue of this stabilization property—the stable moment graph—is defined. It is an oriented graph whose vertices are given by the alcoves in the fundamental (Weyl) chamber and whose edges are labeled with affine coroots. The stable moment graph arises as a subgraph of the parabolic Bruhat graph and with the help of the stable moment graph the stabilization property is established in the categorical framework of sheaves on moment graphs: the stalks of indecomposable Braden-MacPherson sheaves (canonical sheaves in [BM01], which are constructed to compensate for failure of flabbiness of the structure sheaf) on finite intervals of the parabolic Bruhat graph (far enough inside the fundamental (Weyl) chamber) stabilize as well (cf. [Lan15]). In addition, in characteristic 0, graded ranks of the stalks of indecomposable Braden-MacPherson sheaves (for the definition of the graded rank and connection to the Kazhdan-Lusztig polynomials, the reader may refer to [Fie10]) on the stable moment graph are given by the generic Kazhdan-Lusztig polynomials—the previously mentioned $q$-polynomials that, besides appearing in the stabilization phenomenon of the parabolic spherical polynomials, also appear as coefficients (up to a factor $\pm 1$ and an involution) in the expansion with respect to the standard basis of the periodic Hecke module (for more details on the periodic Hecke module, the reader may consult [Lus80] and [Soe97]) of certain self-dual elements of a completion of the periodic Hecke module (cf. [Kat85], [Soe97], Theorem 6.4.2.).

Also via sheaves on moment graphs in [Fie11] Fiebig constructs a categorification of the affine Hecke algebra. The category in this categorification is a category of special modules over the affine structure algebra that can be interpreted as a category of sheaves on moment graphs and even as Soergel’s
category of bimodules associated with a reflection faithful representation of a Coxeter system (the latter interpretation is established in [Fie08b] and Soergel’s construction can be found in [Soe07]). Furthermore, Fiebig’s combinatorial categorification of the affine Hecke algebra is generalized to the case of the (parabolic) spherical Hecke module in [Lan14].

Let us now observe another categorification of the Hecke algebra that is representation-theoretic in its nature—category $\mathcal{O}$ of Bernstein, Gelfand and Gelfand. In [Fie08a] a link between (equivariant) representation theory and theory of sheaves on moment graphs is formed: if a moment graph is associated with a non-critical block $\mathcal{O}_\Lambda$ of the equivariant category $\mathcal{O}$ over a symmetrizable Kac-Moody algebra, then a certain subcategory of the category of sheaves on this associated moment graph is equivalent (as an exact category) to the subcategory of $\mathcal{O}_\Lambda$ of modules that admit a finite (equivariant) Verma flag. Therefore, the stable moment graph should conjecturally yield information on stabilization phenomena for non-critical singular blocks of the (equivariant) category $\mathcal{O}$ for affine Kac-Moody algebras.

Let us add a few words on the structure algebra of the moment graph associated with a root system. Let $k$ be a field and $S$ the symmetric algebra over $k$ associated with the coroot lattice. The structure algebra is a commutative algebra over $S$. If $k = \mathbb{C}$, the structure algebra of the moment graph associated with a non-critical block of the equivariant category $\mathcal{O}$ over a symmetrizable Kac-Moody algebra is isomorphic to the categorical center of that non-critical block (cf. [Fie08a]). If $\text{char}(k) = 0$, the structure algebra of the moment graph associated with a complex equivariantly formal variety with a torus action is isomorphic to the torus equivariant cohomology with coefficients in $k$ of that equivariantly formal variety (cf. [GKM98]).

In this article we study the structure algebra $Z$ of the stable moment graph for the case of the affine root system $A_1$ (the subgeneric case). Let us mention that the obtained results apply to an arbitrary affine root system—the studied phenomena occur in all root directions. In the forthcoming article we will study the structure algebra of a certain (translated) subgraph of the stable moment graph, first for the case of the affine root system $A_2$ and then for the general case. In particular, the structure algebra $Z$ is a module over the polynomial algebra $S$ (Subsection 3.2) and we examine this $S$-module structure of $Z$. We construct an $S$-basis (of “linear-algebraic type”) of $Z$. Then we regard the same notions for the case of the finite root system $A_1$ (Subsection 4.1): the associated moment graph $\mathcal{G}^{\text{fin}}$, the corresponding symmetric algebra $S^{\text{fin}}$ and the structure algebra $Z^{\text{fin}}$. After “setting $c$ equal to zero” ($c$ is a central element of the affine Kac-Moody algebra $\hat{sl}_2$) in the structure algebra $Z$, we obtain the $S^{\text{fin}}$-module $Z_{c=0}$. We observe that this $S^{\text{fin}}$-module is contained in $\prod_{j \in \mathbb{Z}^2_0} Z^{\text{fin}}$ (we could say that it is “locally finite”) and our main result (Theorem 4.3) is that the module $Z_{c=0}$ is determined by (quite surprising) divisibility relations, which we call higher-order congruence relations.
1.1. Contents. In Section 2 we collect the basic structural results on affine Kac-Moody algebras (e.g. affine roots, the affine Weyl group, alcoves) with the purpose of setting the notation.

In Section 3 we consider affine Bruhat graphs (Definition 3.2) with special emphasis placed on the parabolic Bruhat graph (Subsection 3.1) and its subgraph that contains only stable edges—edges whose labels are invariant under translation by an element of the finite coroot lattice (Proposition 3.1), i.e. the stable moment graph (Definition 3.3). Next we introduce sheaves on moment graphs and give an example of the most natural sheaf on a moment graph: the structure sheaf $\mathcal{Z}$ (Example 3.3). Then we define sections of sheaves on moment graphs and we end Section 3 with the definition of the structure algebra $\mathcal{Z}$ of a moment graph (Subsection 3.3).

In Section 4 first we construct a basis of the $S$-module $\mathcal{Z}$ of the stable moment graph in the subgeneric case (Propositions 4.1 and 4.2). Then we define the $S_{\text{fin}}$-module $\mathcal{Z}_c=0$ by “setting $c$ equal to zero” in the $S$-module $\mathcal{Z}$ (Subsection 4.1) and we establish the main result of this article: Theorem 4.3

\[ \mathcal{Z}_{c=0} = \left\{ (a_j) \in \prod_{j \in \mathbb{Z}_{\geq 0}} \mathcal{Z}_{\text{fin}}^{\mathbb{Z}} : \sum_{j=0}^{m} (-1)^j \binom{m}{j} a_j \in ((-\alpha)^m, \alpha^m)\mathcal{Z}_{\text{fin}}^{\mathbb{Z}} \forall m \in \mathbb{Z}_{\geq 0} \right\}. \]

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2. PRELIMINARIES

2.1. Affine Kac-Moody algebras. Let $\mathfrak{g}$ be a finite-dimensional simple complex Lie algebra and let $\hat{\mathfrak{g}}$ be the corresponding affine Kac-Moody algebra. As a vector space, $\hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d$, where $c$ is a central element and $d$ is a derivation operator and the Lie bracket on $\hat{\mathfrak{g}}$ is determined by:

\[
[x \otimes t^m, y \otimes t^n] = [x, y] \otimes t^{m+n} + m\delta_{m,-n}(x, y)c,
\]

\[
[c, \hat{\mathfrak{g}}] = \{0\},
\]

\[
[d, x \otimes t^n] = nx \otimes t^n,
\]

for $x, y \in \mathfrak{g}$, $m, n \in \mathbb{Z}$ and $\langle \cdot, \cdot \rangle: \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ denotes the Killing form on $\mathfrak{g}$.

Let $\mathfrak{h} \subset \mathfrak{g}$ be a Cartan subalgebra. Then $\hat{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}c \oplus \mathbb{C}d$ denotes the corresponding affine Cartan subalgebra of $\hat{\mathfrak{g}}$ and its dual is $\mathfrak{h}^* = \mathfrak{h}^* \oplus \mathbb{C}\delta \oplus \mathbb{C}\Lambda_0$, where if $\langle \cdot, \cdot \rangle: \mathfrak{h}^* \times \mathfrak{h} \to \mathbb{C}$ is the natural pairing, we have:

\[
\langle \delta, \mathfrak{h} \oplus \mathbb{C}c \rangle = \{0\},
\]

\[
\langle \delta, d \rangle = 1,
\]

\[
\langle \Lambda_0, \mathfrak{h} \oplus \mathbb{C}d \rangle = \{0\},
\]

\[
\langle \Lambda_0, c \rangle = 1.
\]
We regard $\mathfrak{h}^*$ as a subspace of $\hat{\mathfrak{h}}^*$ by letting each $\lambda \in \mathfrak{h}^*$ act trivially on $C_c \oplus C_d$.

The Killing form $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$ can be extended to a symmetric non-degenerate bilinear form on $\hat{\mathfrak{g}}$. It is determined by the following:

\[
\begin{align*}
(x \otimes t^m, y \otimes t^n) &= \delta_{m,-n} \langle x, y \rangle, \\
(c, \mathfrak{g} \otimes C[\mathfrak{t}, t^{-1}] \oplus Cc) &= \{0\}, \\
(d, \mathfrak{g} \otimes C[\mathfrak{t}, t^{-1}] \oplus Cd) &= \{0\}, \\
(c, d) &= 1
\end{align*}
\]

for $x, y \in \mathfrak{g}$ and $m, n \in \mathbb{Z}$. This form induces a non-degenerate bilinear form on the Cartan subalgebra $\mathfrak{h}$ and consequently establishes an isomorphism $\hat{\mathfrak{h}} \cong \hat{\mathfrak{h}}^*$.

We also denote by $\langle \cdot, \cdot \rangle$ the induced symmetric non-degenerate bilinear form on the dual $\hat{\mathfrak{h}}^*$. Let us observe that from the definitions it follows directly that the isomorphism $\hat{\mathfrak{h}} \cong \hat{\mathfrak{h}}^*$ identifies $c$ with $\delta$, i.e. for any $\lambda \in \hat{\mathfrak{h}}^*$ we have

\[
\langle \lambda, c \rangle = \langle \delta, \lambda \rangle.
\]

In particular, $\langle \delta, \beta \rangle = 0$ for any $\beta \in \hat{\Delta}$ (Definition (1)).

2.2. Affine roots. Let $\Delta \subset \mathfrak{h}^*$ be the set of roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$. The set $\hat{\Delta} \subset \hat{\mathfrak{h}}^*$ of roots of $\hat{\mathfrak{g}}$ with respect to $\hat{\mathfrak{h}}$ is

\[
\hat{\Delta} = \{ \alpha + n\delta \mid \alpha \in \Delta, n \in \mathbb{Z} \} \cup \{ n\delta \mid n \in \mathbb{Z}, n \neq 0 \}.
\]

(1)

The subsets

\[
\begin{align*}
\hat{\Delta}^{\text{re}} &= \{ \alpha + n\delta \mid \alpha \in \Delta, n \in \mathbb{Z} \}, \\
\hat{\Delta}^{\text{im}} &= \{ n\delta \mid n \in \mathbb{Z}, n \neq 0 \}
\end{align*}
\]

are called the sets of real roots and of imaginary roots, respectively.

We denote by $\Delta_+ \subset \Delta$ the subset of chosen positive (finite) roots. Then the set of positive affine roots is

\[
\hat{\Delta}_+ = \{ \alpha + n\delta \mid \alpha \in \Delta, n \geq 1 \} \cup \Delta_+ \cup \{ n\delta \mid n \geq 1 \},
\]

while the set of positive real roots is given by

\[
\hat{\Delta}_+^{\text{re}} = \{ \alpha + n\delta \mid \alpha \in \Delta, n \geq 1 \} \cup \Delta_+.
\]

Furthermore, if $\Pi$ is the set of simple (finite) roots and $\theta \in \Delta^+$ is the highest root, then

\[
\hat{\Pi} = \Pi \cup \{ -\theta + \delta \}
\]

is the set of simple affine roots.
2.3. The Weyl group and its set of alcoves. For every real root \( \alpha + n\delta \) it holds that \((\alpha + n\delta, \alpha + n\delta) = (\alpha, \alpha) \neq 0\), therefore we can define the reflection

\[
s_{\alpha + n\delta}: \widehat{\mathfrak{h}}^* \to \widehat{\mathfrak{h}}^*
\]

\[
\lambda \mapsto \lambda - 2 \frac{(\lambda, \alpha + n\delta)}{(\alpha, \alpha)} (\alpha + n\delta).
\]

Notice that the reflection \( s_{\alpha + n\delta} \) fixes pointwise the hyperplane \((\cdot, \alpha + n\delta) = 0\) and maps \( \alpha + n\delta \) to \( -\alpha - n\delta \). Using the isomorphism \( \widehat{\mathfrak{h}} \to \widehat{\mathfrak{h}}^* \) we may identify the affine coroot \((\alpha + n\delta)^\vee\) with \(\frac{2(\alpha + n\delta)}{(\alpha, \alpha)}\) and then the reflection \( s_{\alpha + n\delta} \) is given by \( s_{\alpha + n\delta}(\lambda) = \lambda - (\lambda, (\alpha + n\delta)^\vee)(\alpha + n\delta) \).

We denote by \( \widehat{W} \subset GL(\widehat{\mathfrak{h}}^*) \) the affine Weyl group, namely the subgroup generated by the reflections \( s_\beta \) for \( \beta \in \widehat{\Delta}_{re} \). The subgroup \( W \subset \widehat{W} \) generated by the reflections \( s_\beta \) with \( \beta \in \Delta_+ \) can be identified with the Weyl group of \( g \).

Let us denote by \( \widehat{\mathfrak{h}}_\mathbb{R} = \text{span} \mathbb{R} \Pi \). We set \( \widehat{\mathfrak{h}}^* \mathbb{R} = \widehat{\mathfrak{h}}^* \mathbb{R} \oplus \mathbb{R}\delta \oplus \mathbb{R}\Lambda_0 \). Now we will consider another realization of \( \widehat{\mathfrak{h}}^* \mathbb{R} \) as a group of affine transformations of \( \widehat{\mathfrak{h}}^* \mathbb{R} \). In order to do that, we identify \( \widehat{\mathfrak{h}}^*_\mathbb{R} \) with the affine space \( \widehat{\mathfrak{h}}^*_1 \) mod \( \mathbb{R}\delta \) where

\[
\widehat{\mathfrak{h}}^*_1 = \{ \lambda \in \widehat{\mathfrak{h}}^*_\mathbb{R} \mid \langle \lambda, c \rangle = 1 \}
\]

(cf. [Kac90, §6.6.]). Then \( \widehat{W} \) acts on \( \lambda \in \widehat{\mathfrak{h}}^*_\mathbb{R} \) by

\[
s_{\alpha + n\delta}(\lambda) = s_\alpha(\lambda) - 2 \frac{n}{(\alpha, \alpha)} \alpha \text{ mod } \mathbb{R}\delta = s_\alpha(\lambda) - na^\vee \text{ mod } \mathbb{R}\delta.
\]

We denote by \( Q^\vee \) the coroot lattice of \( g \) and by \( T_\mu \) the translation by \( \mu \in Q^\vee \), i.e. \( T_\mu(\lambda) = \lambda + \mu \) for \( \lambda \in \widehat{\mathfrak{h}}^*_\mathbb{R} \), so (2) can be written as

\[
s_{\alpha + n\delta}(\lambda) = T_{-na^\vee} s_\alpha(\lambda) \quad \text{for } \lambda \in \widehat{\mathfrak{h}}^*_\mathbb{R},
\]

i.e. \( s_{\alpha + n\delta} = T_{-na^\vee} s_\alpha \). The group of translations by an element of the coroot lattice is a normal subgroup of the affine Weyl group and it holds that \( \widehat{W} = \mathcal{W} \ltimes Q^\vee \).

Let us denote by

\[
H_{\alpha,n} := \{ \lambda \in \widehat{\mathfrak{h}}^*_\mathbb{R} \mid \langle \lambda, \alpha \rangle = -n \}
\]

and note that the affine reflection \( s_{\alpha + n\delta} \) fixes \( H_{\alpha,n} \) pointwise. The connected components of

\[
\widehat{\mathfrak{h}}^*_\mathbb{R} \backslash \bigcup_{\alpha + n\delta \in \widehat{\Delta}^*_+} H_{\alpha,n}
\]
are called alcoves and we denote by $\mathcal{A}$ the set of all alcoves.

The fundamental (Weyl) chamber is

$$C^+ := \{ \lambda \in h_\mathbb{R}^* \mid \langle \lambda, \alpha^\vee \rangle > 0 \ \forall \alpha \in \Pi \}$$

and an element $\lambda \in C^+$ is called dominant weight. Let $\mathcal{A}^+ = \{ A \in \mathcal{A} \mid A \subset C^+ \}$ be the set of all alcoves contained in $C^+$.

We denote by $X := \{ \lambda \in h_\mathbb{R}^* \mid \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z} \ \forall \alpha \in \Delta \}$ the (finite) integral weight lattice and by $\hat{X} := X \oplus \mathbb{Z}\delta$ the affine integral weight lattice. The latter contains the affine root lattice $\mathbb{Z}\Delta^\text{aff}$. Furthermore, we denote by $X^\vee := \{ \lambda \in h_\mathbb{R}^* \mid (\lambda, \alpha) \in \mathbb{Z} \ \forall \alpha \in \Delta \}$ the (finite) integral coweight lattice, i.e. (under the identification of $h_\mathbb{R}$ and $h_\mathbb{R}^*$)

$$X^\vee := \{ \lambda \in h_\mathbb{R}^* \mid (\lambda, \theta) < 1 \}$$

(cf. [Hum90, §4.3]).

There is a one-to-one correspondence between $\hat{W}$ and $\mathcal{A}$. Let $A^+$ be the unique alcove in $\mathcal{A}^+$ whose closure contains the zero vector. $A^+$ is called the fundamental alcove and we have

$$A^+ = \{ \lambda \in h_\mathbb{R}^* \mid 0 < (\lambda, \alpha) < 1 \ \forall \alpha \in \Delta_+ \} = \{ \lambda \in h_\mathbb{R}^* \mid 0 < (\lambda, \alpha) \ \forall \alpha \in \Pi, \ (\lambda, \theta) < 1 \}$$

(cf. [Hum90, §4.3]).

Since the left action (given by (2)) of the affine Weyl group $\hat{W}$ on $\mathcal{A}$ is simply transitive (cf. [Hum90, §4.5]), the bijection between $\hat{W}$ and $\mathcal{A}$ is given by

$$w \mapsto wA^+. \quad (3)$$

Notice that each reflection $s \in \hat{S}$ fixes pointwise exactly one wall of $A^+$. That wall is called $s$-wall of $A^+$. In general, every $A \in \mathcal{A}$ has one and only one wall in the $\hat{W}$-orbit of the $s$-wall of $A^+$ and this is called $s$-wall of $A$.

We can also define a right action of the affine Weyl group $\hat{W}$ on $\mathcal{A}$ using the right action of $\hat{W}$ on itself via right multiplication. We define the action of an arbitrary generator of the group: for each alcove $A$ let $As$ be the unique alcove having in common with $A$ the $s$-wall, i.e. if $A = wA^+$, $w \in \hat{W}$, then the alcove $As$ is defined by $As := wsA^+$, for $s \in \hat{S}$ (cf. [Soe97]).

Finally, let us observe that the Bruhat order on $\hat{W}$ induces a partial order on $\mathcal{A}$: for $A, B \in \mathcal{A}$ where $A = xA^+$, $B = yA^+$, $x, y \in \hat{W}$

$$A \leq B \iff x \leq y.$$ 

It is again called Bruhat order.

### 3. MOMENT GRAPHS

In the first part of this section we introduce the notion of a moment graph over a lattice and then we consider closely particular moment graphs that are important for the representation theory of affine Kac-Moody algebras. In the second part we define sheaves on moment graphs and sections of sheaves on moment graphs.
Definition 3.1. (cf. [Fie16]) Let \( Y \cong \mathbb{Z}^r \) be a lattice of finite rank. A moment graph over the lattice \( Y \) is the datum \( \mathcal{G} = (V, \mathcal{E}, \leq, l) \) where:

(i) \( (V, \mathcal{E}) \) is a directed graph without loops or multiple edges,

(ii) \( \leq \) is a partial order on \( V \) such that if \( x, y \in V \) and \( E : x \rightarrow y \), then \( x \leq y \),

(iii) \( l : \mathcal{E} \rightarrow Y \setminus \{0\} \) is a map called the labeling.

Using the notation of the previous section, for any subset \( I \) of \( \hat{S} \) we define now the affine Bruhat graph \( \hat{G}^I \) in the following way.

Definition 3.2. (cf. [Fie11]) \( \hat{G}^I \) is the moment graph over the affine coroot lattice \( \hat{\mathcal{Q}}^\vee = Q^\vee \oplus \mathbb{Z} \) given by

(i) \( V := \hat{W}^I \) where \( \hat{W}^I \) is the set of minimal length representatives of the left cosets of \( \langle I \rangle \) in \( \hat{W} \),

(ii) \( \leq \) is the Bruhat order on \( \hat{W} \) and \( \mathcal{E} := \{ x \rightarrow y \mid x < y, \exists \beta \in \hat{\Delta}^\vee, \exists w \in \langle I \rangle \text{ such that } y = s_\beta x w \} \),

(iii) \( l(x \rightarrow s_\beta x w) := \beta^\vee \).

If \( I = \emptyset \), the corresponding affine Bruhat graph \( \hat{G}^\emptyset \) we denote by \( \hat{G}^{\text{reg}} \) and we call it the regular Bruhat graph of \( \hat{g} \). If \( I = \mathcal{S} := \{ s_\beta \mid \beta \in \Pi \} \), we denote by \( \hat{G}^{\text{par}} := \hat{G}^S \) the parabolic Bruhat graph of \( \hat{g} \).

Example 3.1. Let \( \hat{g} = \hat{\mathfrak{sl}}_2 \). Then the set of positive real roots is

\[ \hat{\Delta}^\vee_+ = \{ \pm \alpha + n\delta \mid n \geq 1 \} \cup \{ \alpha \} \]

where \( \alpha \) is the unique positive root of \( \mathfrak{sl}_2 \) with the property \( (\alpha, \alpha) = 2 \). Simple affine reflections are

\[ s_0 := s_{-\theta + \delta} = s_{-\alpha + \delta} = T_\alpha s_\alpha \text{ and } s_1 := s_\alpha, \]

so we have

\[ T_\alpha = s_0 s_1, \]

\[ s_{-\alpha + n\delta} = T_{n\alpha} s_\alpha = (s_0 s_1)^n s_1 = (s_0 s_1)^{n-1} s_0, \]

\[ s_{\alpha + n\delta} = T_{-n\alpha} s_\alpha = (s_1 s_0)^n s_1 \]

for \( n \in \mathbb{Z}_{>0} \). Using the isomorphism \( \hat{\mathfrak{h}} \rightarrow \hat{\mathfrak{h}}^\vee \) we obtain

\[ (\pm \alpha + n\delta)^\vee = \pm \alpha^\vee + \frac{2n}{(\alpha, \alpha)} c = \pm \alpha + nc. \]

Therefore a subgraph of the corresponding regular Bruhat graph is as in Figure.
3.1. Parabolic Bruhat graph. Let us observe that we can approach the set of vertices of the parabolic Bruhat graph $\hat{G}^{\text{par}}$ in two ways:

(i) via the identification with the finite coroot lattice $Q^\vee$ or

(ii) via the identification with the set $\mathcal{A}^+$ of alcoves contained in the fundamental chamber $C^+$.

Let us consider the left action (given by (2)) of the affine Weyl group $\hat{W}$ on $h^*_R$. The orbit of 0 is the finite coroot lattice $Q^\vee$, while the stabilizer subgroup of $\hat{W}$ with respect to 0 is the finite Weyl group $W$. The orbit-stabilizer theorem then gives a bijection between $\hat{W}/W$ and $Q^\vee$ and for any two minimal length
representatives \(x,y\) of the left cosets of \(W\) in \(\hat{W}\) we have
\[
\exists \beta \in \hat{\Delta}^\text{re}_+, \exists w \in W \text{ such that } y = s_\beta x w \iff \\
\exists \beta \in \hat{\Delta}^\text{re}_+ \text{ such that } y(0) = s_\beta x(0).
\]

Now observe that the map \(xW \mapsto Wx^{-1}\) defines a bijection between the left cosets of \(W\). In particular, we obtain the bijection between the set of minimal length representatives of the left cosets of \(W\) and the set of minimal length representatives of the right cosets of \(W\). The latter is in bijection with the set \(A^+\) of alcoves contained in the fundamental chamber \(C^+\) via (3). Furthermore, for any two minimal length representatives \(x^{-1}, y^{-1}\) of the left cosets of \(W\) in \(\hat{W}\) we have
\[
\exists \beta \in \hat{\Delta}^\text{re}_+, \exists w^{-1} \in W \text{ such that } y^{-1} = s_\beta x^{-1} w^{-1} \iff \\
\exists \beta \in \hat{\Delta}^\text{re}_+, \exists w \in W \text{ such that } y = wx s_\beta,
\]
therefore the vertices \(x A^+, y A^+ \in A^+\) are adjacent if and only if there exist \(\beta \in \hat{\Delta}^\text{re}_+\) and \(w \in W\) such that \(y = wxs_\beta\).

**Example 3.2.** Let \(\hat{\mathfrak{g}} = \hat{\mathfrak{sl}}_2\). Then by (i) the set of vertices of \(\hat{G}_\text{par}\) is \(\mathbb{Z} \alpha^\vee = \mathbb{Z} \alpha\) and we have
\[
s_{-\alpha + (n + n')\delta}(n\alpha) = s_{\alpha - (n + n')\delta}(n\alpha) = T_{(n + n')\alpha} s_\alpha(n\alpha) = \\
= -n\alpha + (n + n')\alpha = n'\alpha, \quad n, n' \in \mathbb{Z}.
\]

It follows that \(\hat{G}_\text{par}\) (considered as an undirected graph) is a complete graph and the labeling is given by
\[
l(n\alpha - n'\alpha) = \begin{cases} 
-\alpha + (n + n')c & \text{if } n + n' > 0 \\
\alpha - (n + n')c & \text{if } n + n' \leq 0.
\end{cases}
\]

Regarding the direction of the edges, notice that
\[
n\alpha = (s_0 s_1)^{n-1} s_0(0) \quad \text{for } n > 0
\]
and
\[
n\alpha = (s_1 s_0)^{n|}(0) \quad \text{for } n \leq 0,
\]
so we have
\[
n\alpha < n'\alpha \iff |n| < |n'| \quad \forall \quad n' = -n < 0.
\]
Hence the interval \([n\alpha, (-n - 2)\alpha], n > 0\) of the parabolic Bruhat graph \(\hat{G}_\text{par}\) is as in Figure 2.

Let \(\rho = \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha^\vee\). Finite intervals of the parabolic Bruhat graph \(\hat{G}_\text{par}\) far enough in the fundamental chamber \(C^+\) have the following property.
Proposition 3.1. (cf. [Lan15]) Let $A, B \in A^+$. There exists an integer $n_0 = n_0(A, B)$ such that for any $\lambda \in X^+ \cap (C^+ + n\rho)$, $n \geq n_0$ and for any $D, E \in [A + \lambda, B + \lambda]$ there is an edge $D - E$ in $\hat{G}^{par}$ if and only if

(i) either $E = s_\beta D$ for some $\beta \in \hat{\Delta}^+$

(ii) or $E = T_{k\alpha^\vee} D$ for some $k \in \mathbb{Z}, k \neq 0$ and $\alpha \in \Delta_+$. 

Remark. Let $D = xA^+$ and $E = yA^+$ for some $x, y \in \hat{W}$. If $E = s_\beta D$ for some $\beta \in \hat{\Delta}^+$, i.e. $yA^+ = s_\beta xA^+$, then $y = s_\beta x$ because the left action of $\hat{W}$ on $A$ is simply transitive. Therefore we obtain

$$x^{-1}y = x^{-1}s_\beta x = s_\gamma$$

for some $\gamma \in \hat{\Delta}^+$ (cf. [Hum90 §4.2]). It follows that $y = xs_\gamma$ and since the right action of $\hat{W}$ on $A$ is given by $2.3$ we have

$$Ds_\gamma = xs_\gamma A^+ = yA^+ = E.$$ 

The edges in (i) are called stable and the edges in (ii) are called non-stable. It holds that the labels of stable edges are invariant under translation by $\mu \in Q^\vee$, but that does not hold for the labels of non-stable edges (Figure 3).

Now the stable moment graph is defined as follows.
Definition 3.3. (cf. [Lan15]) The stable moment graph \( \hat{G}^{\text{stab}} = (\mathcal{V}, \mathcal{E}, \leq, l) \) is the moment graph over the affine coroot lattice \( \hat{Q}^\vee \) given by

1. \( \mathcal{V} := A^+ \),
2. \( \leq \) is the Bruhat order on \( A \) and \( \mathcal{E} := \{ xA^+ \to yA^+ \mid x \leq y, \exists \beta \in \hat{\Delta}^\text{re}_+ \text{ such that } y = xs_\beta \} \),
3. \( l(xA^+ \to xs_\beta A^+) := \beta^\vee \).

3.2. Sheaves on moment graphs. Let \( \mathcal{G} \) be a moment graph defined over the lattice \( Y \) and \( k \) a field. Let us denote by \( Y_k = Y \otimes_k k \) the \( k \)-vector space spanned by \( Y \) and by \( S = S(Y_k) \) the symmetric algebra of \( Y_k \), which is a polynomial algebra (over \( k \)) of rank \( \dim_k Y_k \). We double the standard grading on that polynomial algebra, i.e. we set \( S_2 = Y_k \) and we consider all \( S \)-modules to be finitely generated and graded.

Definition 3.4. (cf. [BM01])

A \( k \)-sheaf on \( \mathcal{G} \) is a triple \( \mathcal{F} = (\{ \mathcal{F}^x \}, \{ \mathcal{F}^E \}, \{ \rho_{x,E} \}) \) where

1. \( \mathcal{F}^x \) is an \( S \)-module for each vertex \( x \in \mathcal{V} \),
2. \( \mathcal{F}^E \) is an \( S \)-module such that \( l(E) \mathcal{F}^E = \{0\} \) for each edge \( E \in \mathcal{E} \).
(iii) \( \rho_{x,E} : \mathcal{F}^x \to \mathcal{F}^E \) is a homomorphism of \( S \)-modules defined whenever the vertex \( x \) and the edge \( E \) are incident.

**Definition 3.5.** (cf. [Fie16])

Let \( \mathcal{F}_1 = (\{ \mathcal{F}_1^x \}, \{ \mathcal{F}_1^E \}, \{ \rho_{x,E}^1 \}) \) and \( \mathcal{F}_2 = (\{ \mathcal{F}_2^x \}, \{ \mathcal{F}_2^E \}, \{ \rho_{x,E}^2 \}) \) be two sheaves on \( \mathcal{G} \). A morphism \( f : \mathcal{F}_1 \to \mathcal{F}_2 \) is a family of homomorphisms of \( S \)-modules \( f^x : \mathcal{F}_1^x \to \mathcal{F}_2^x \) and \( f^E : \mathcal{F}_1^E \to \mathcal{F}_2^E \) for all \( x \in \mathcal{V} \) and \( E \in \mathcal{E} \) such that if the vertex \( x \) and the edge \( E \) are incident, the diagram

\[
\begin{array}{ccc}
\mathcal{F}_1^x & \xrightarrow{\rho_{x,E}^1} & \mathcal{F}_1^E \\
\downarrow f^x & & \downarrow f^E \\
\mathcal{F}_2^x & \xrightarrow{\rho_{x,E}^2} & \mathcal{F}_2^E
\end{array}
\]

commutes.

Therefore we obtain the category of \( k \)-sheaves on \( \mathcal{G} \), which we denote by \( Sh_k(\mathcal{G}) \).

**Example 3.3.** The **structure sheaf** on \( \mathcal{G} \) is an object \( \mathcal{L} \) in \( Sh_k(\mathcal{G}) \) given by

(i) \( \mathcal{L}^x := S \) for each vertex \( x \in \mathcal{V} \),

(ii) \( \mathcal{L}^E := S/l(E)S \) for each edge \( E \in \mathcal{E} \),

(iii) \( \rho_{x,E} : S \to S/l(E)S \) is the canonical surjection, whenever the vertex \( x \) and the edge \( E \) are incident.

### 3.3. Sections of sheaves.

Let \( \mathcal{F} \) be a sheaf on \( \mathcal{G} \). For \( \mathcal{I} \subseteq \mathcal{V} \) the set of **local sections** of \( \mathcal{F} \) over \( \mathcal{I} \) is

\[
\Gamma(\mathcal{I}, \mathcal{F}) := \left\{ (f_x) \in \prod_{x \in \mathcal{I}} \mathcal{F}^x \mid \rho_{x,E}(f_x) = \rho_{y,E}(f_y) \text{ for each edge } E : x \to y \right\}
\]

such that \( x, y \in \mathcal{I} \).

We denote by \( \Gamma(\mathcal{F}) := \Gamma(\mathcal{V}, \mathcal{F}) \) the set of **global sections** of \( \mathcal{F} \).

Let \( \mathcal{I}_1 \) and \( \mathcal{I}_2 \) be two subsets of the set of vertices \( \mathcal{V} \) such that \( \mathcal{I}_2 \subseteq \mathcal{I}_1 \). Then the projection

\[
\prod_{x \in \mathcal{I}_1} \mathcal{F}^x \to \prod_{x \in \mathcal{I}_2} \mathcal{F}^x
\]

gives a restriction morphism

\[
\Gamma(\mathcal{I}_1, \mathcal{F}) \to \Gamma(\mathcal{I}_2, \mathcal{F}).
\]
The structure algebra \( \mathcal{Z} \) of \( \mathcal{G} \) is the set of global sections of the structure sheaf \( \mathcal{Z} \), i.e.

\[
\mathcal{Z} := \Gamma(\mathcal{Z}) = \left\{ (z_x) \in \prod_{x \in \mathcal{V}} S \mid \text{for each edge } E : x - y, z_x - z_y = l(E)s \text{ for some } s \in S \right\}.
\]

\( \mathcal{Z} \) is indeed an \( S \)-algebra: the addition and the multiplication are given componentwise and \( S \) acts on \( \mathcal{Z} \) via the diagonal action. Remark that for any sheaf \( \mathcal{F} \) the set \( \Gamma(\mathcal{F}) \) has a structure of a \( \mathcal{Z} \)-module.

4. THE STRUCTURE ALGEBRA OF THE STABLE MOMENT GRAPH IN THE SUBGENERIC CASE

Let \( \widehat{\mathfrak{g}} = \widehat{\mathfrak{sl}}_2 \). Then the (coroot) lattice \( \mathcal{Y} \) is \( \mathbb{Z}\alpha \oplus \mathbb{Z}c \) (Example 3.1) and the corresponding symmetric algebra \( S \) over the field \( k = \mathbb{C} \) is \( k[\alpha, c] = \mathbb{C}[\alpha, c] \). Let \( \mathcal{Z} = \mathcal{Z}(\widehat{\mathcal{G}}^{\text{stab}}_{>0\alpha}) \) be the structure algebra of the moment graph \( \widehat{\mathcal{G}}^{\text{stab}}_{>0\alpha} \) (Figure 3). Let us remark that omitting the vertex \( 0\alpha \) corresponds to being far enough inside the fundamental (Weyl) chamber, away from its walls.

We explore now the \( S \)-module structure of the structure algebra \( \mathcal{Z} \). The first step in that direction consists of constructing perhaps the most natural (from a linear algebra perspective) elements \( u_n, n \in \mathbb{Z}_{\geq 0} \) and \( v_n, n \in \mathbb{Z}_{>0} \) of the structure algebra \( \mathcal{Z} \) (Proposition 4.1). Afterwards, we will show that these elements form a basis of the \( S \)-module \( \mathcal{Z} \) (Proposition 4.2).

We use the following conventions: a product with only one factor evaluates to that factor, while a product with no factors, i.e. an empty product evaluates to 1.

**Proposition 4.1.** Let \( u_n, n \in \mathbb{Z}_{\geq 0} \) and \( v_n, n \in \mathbb{Z}_{>0} \) be defined by

\[
(u_n)_{j\alpha} := \begin{cases} 
0 & \text{if } 0 < |j| \leq n \\
\prod_{l=1}^{n} (-\alpha + lc) & \text{if } j = n + 1 \\
\prod_{l=1}^{n} (\alpha + lc) & \text{if } j = -n - 1 \\
k_{j-n-1,n} \cdot \prod_{l=j-n}^{j-1} (-\alpha + lc) & \text{if } j \geq n + 2 \\
k_{|j-|n-1,n} \cdot \prod_{l=|j|-n}^{|j|-1} (\alpha + lc) & \text{if } j \leq -n - 2
\end{cases}
\]

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where \( k_{i,n}, i \in \mathbb{Z}_{>0} \) are elements of the field \( k = \mathbb{C} \) given by

\[
k_{i,n} = \binom{n+i}{n}, \quad i \in \mathbb{Z}_{>0}
\]  

(4)

and

\[
(v_n)_{j_\alpha} := \begin{cases}
0 & \text{if } 0 < |j| \leq n - 1 \\
0 & \text{if } j = n \\
\prod_{l=0}^{n-1} (\alpha + lc) & \text{if } j = -n \\
(a_{j-n,n}\alpha + (j-n)b_{j-n,n}\epsilon) \cdot \prod_{l=j-n+1}^{j-1} (-\alpha + lc) & \text{if } j \geq n + 1 \\
b_{j|-n,n} \cdot \prod_{l=|j|-n}^{|j|-1} (\alpha + lc) & \text{if } j \leq -n - 1
\end{cases}
\]

where \( a_{i,n} \) and \( b_{i,n}, i \in \mathbb{Z}_{>0} \) are elements of the field \( k = \mathbb{C} \) given by

\[
a_{i,n} = \binom{n+i-1}{n}(n-1),
\]

(5)

\[
b_{i,n} = \binom{n+i-1}{n} \frac{i-(i-1)n}{i}, \quad i \in \mathbb{Z}_{>0}.
\]

Then in that way defined elements \( u_n, n \in \mathbb{Z}_{>0} \) and \( v_n, n \in \mathbb{Z}_{>0} \) of the direct product

\[
\prod_{j_{\alpha} \in V(\hat{G}_{>0}^{stab})} S
\]

belong to the structure algebra \( Z \) of the moment graph \( \hat{G}_{>0}^{stab} \).

**Remark.** One can think of \( u_n \) as

\[
\begin{pmatrix}
\vdots & \vdots \\
k_{3,n} \cdot \prod_{l=1}^{n+3} (-\alpha + lc) & k_{3,n} \cdot \prod_{l=1}^{n+3} (\alpha + lc) \\
k_{2,n} \cdot \prod_{l=2}^{n+2} (-\alpha + lc) & k_{2,n} \cdot \prod_{l=2}^{n+2} (\alpha + lc) \\
k_{1,n} \cdot \prod_{l=2}^{n} (-\alpha + lc) & k_{1,n} \cdot \prod_{l=2}^{n} (\alpha + lc) \\
\prod_{l=1}^{n} (-\alpha + lc) & \prod_{l=1}^{n} (\alpha + lc) \\
0 & 0 \\
\vdots & \vdots \\
0 & 0 \\
0 & 0
\end{pmatrix}
\]

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where the first \( n \) \( (n \geq 0) \) rows are zero and \( k_{i,n}, i \in \mathbb{Z}_{>0} \) are elements of the field \( k = \mathbb{C} \) given by
\[
k_{i,n} = \binom{n + i}{n}, \quad i \in \mathbb{Z}_{>0}
\]
and \( v_n \) as
\[
v_n = \begin{pmatrix}
\vdots \\
(a_{3,n} \alpha + 3b_{3,n} c) \cdot \prod_{l=4}^{n+1} (-\alpha + lc) & b_{3,n} \cdot \prod_{l=3}^{n+1} (\alpha + lc) \\
(a_{2,n} \alpha + 2b_{2,n} c) \cdot \prod_{l=3}^{n} (-\alpha + lc) & b_{2,n} \cdot \prod_{l=2}^{n} (\alpha + lc) \\
(a_{1,n} \alpha + b_{1,n} c) \cdot \prod_{l=2}^{n-1} (-\alpha + lc) & b_{1,n} \cdot \prod_{l=1}^{n-1} (\alpha + lc) \\
0 & \prod_{l=0}^{n-1} (\alpha + lc) \\
0 & \vdots \\
0 & \vdots \\
0 & 0 \\
0 & 0
\end{pmatrix}
\]
where the first \( n - 1 \) \( (n \geq 1) \) rows are zero and \( a_{i,n}, b_{i,n}, i \in \mathbb{Z}_{>0} \) are elements of the field \( k = \mathbb{C} \) given by
\[
a_{i,n} = \binom{n + i - 1}{n} (n - 1),
\]
\[
b_{i,n} = \binom{n + i - 1}{n} \frac{i - (i - 1)n}{i}, \quad i \in \mathbb{Z}_{>0}.
\]
This very convenient notation will be used throughout the rest of the article.

**Proof.** First we prove that \( u_n, n \in \mathbb{Z}_{>0} \) are elements of the structure algebra \( \mathcal{Z} \). Let \( n, i, i' \in \mathbb{Z}_{>0}, i \neq i' \) (if \( i = i' \), then the corresponding structure algebra condition is obviously satisfied, i.e.
\[
\alpha \mid k_{i,n} \cdot \prod_{l=i+1}^{i+n} (-\alpha + lc) - k_{i,n} \cdot \prod_{l=i+1}^{i+n} (\alpha + lc),
\]
where we set \( k_{0,n} := 1 \). If \( i > i' \), we need to prove that
\[
\alpha - (i' - i)c = \alpha + (i - i')c \mid k_{i,n} \cdot \prod_{l=i+1}^{i+n} (\alpha + lc) - k_{i',n} \cdot \prod_{l=i'+1}^{i'+n} (-\alpha + lc)
\]
(this structure algebra condition corresponds to the following edge: Figure 4).
Since the polynomial $\alpha + (i - i')c$ is irreducible, it is enough to show that

$$k_{i,n} \cdot \prod_{l=i+1}^{i+n} (\alpha + lc) - k_{i',n} \cdot \prod_{l=i'+1}^{i'+n} (-\alpha + lc) \bigg|_{\alpha = -(i-i')c} = 0,$$

i.e.

$$k_{i,n} \cdot \prod_{l=i+1}^{i+n} ((i' - i)c + lc) - k_{i',n} \cdot \prod_{l=i'+1}^{i'+n} ((i - i')c + lc) = 0. \quad (6)$$

We have

$$k_{i,n} \cdot \prod_{l=i+1}^{i+n} ((i' - i)c + lc) = \binom{n+i}{n} (i'+1)(i'+2) \cdots (i'+n)c^n =$$

$$= \binom{n+i}{n} (n+1)(n+2) \cdots (n+c)c^n.$$

Furthermore, we have

$$k_{i',n} \cdot \prod_{l=i'+1}^{i'+n} ((i - i')c + lc) = \binom{n+i'}{n} (i+1)(i+2) \cdots (i+n)c^n =$$

$$= \binom{n+i'}{n} (n+i)(n+i) \cdots (n+i)c^n.$$

This proves (6).

Now if $i < i'$, we need to prove that

$$-\alpha + (i' - i)c \bigg| k_{i',n} \cdot \prod_{l=i'+1}^{i'+n} (-\alpha + lc) - k_{i,n} \cdot \prod_{l=i+1}^{i+n} (\alpha + lc),$$

which is equivalent to

$$\alpha - (i' - i)c \bigg| k_{i,n} \cdot \prod_{l=i+1}^{i+n} (\alpha + lc) - k_{i',n} \cdot \prod_{l=i'+1}^{i'+n} (-\alpha + lc),$$

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but that we have already proved. With this we have proved that $u_n \in \mathbb{Z}$ for every $n \in \mathbb{Z}_{\geq 0}$.

Now we prove that $v_n, n \in \mathbb{Z}_{\geq 0}$ are also elements of the structure algebra $\mathcal{Z}$. Let $n \in \mathbb{Z}_{\geq 0}$. First, let $i' \in \mathbb{Z}_{\geq 0}$. We need to prove that

\[-\alpha + i'c \mid (a_{i',n}\alpha + i'b_{i',n}c) \cdot \prod_{l=i'+1}^{i'+n-1} (-\alpha + lc) - \prod_{l=0}^{n-1} (\alpha + lc)\]

(this structure algebra condition corresponds to the following edge: Figure 5).

**Figure 5:** Another edge of the stable moment graph of $\hat{s}_{12}$

Because the polynomial $-\alpha + i'c$ is also irreducible, it suffices to show that

\[
(a_{i',n}\alpha + i'b_{i',n}c) \cdot \prod_{l=i'+1}^{i'+n-1} (-\alpha + lc) - \prod_{l=0}^{n-1} (\alpha + lc) = 0, \quad \alpha = i'c
\]

i.e.

\[
(a_{i',n}i'c + i'b_{i',n}c) \cdot \prod_{l=i'+1}^{i'+n-1} (-i'c + lc) - \prod_{l=0}^{n-1} (i'c + lc) = 0. \quad (7)
\]

We have

\[
(a_{i',n}i'c + i'b_{i',n}c) \cdot \prod_{l=i'+1}^{i'+n-1} (-i'c + lc) = i'(a_{i',n} + b_{i',n})(n-1)!c^n =
\]

\[
= i' \left( \binom{n+i'-1}{n} (n-1) + \binom{n+i'-1}{n} \frac{i'-(i'-1)n}{i'} \right) (n-1)!c^n =
\]

\[
= i' \left( \binom{n+i'-1}{n} (n-1) + \frac{i'-(i'-1)n}{i'} \right) (n-1)!c^n =
\]

\[
= i' \binom{n+i'-1}{n} \frac{i'(n-1)+i'-(i'-1)n}{i'} (n-1)!c^n =
\]

\[
= i' \binom{n+i'-1}{n} \frac{n}{i'} (n-1)!c^n = \left( \frac{n+i'-1}{n} \right) n!c^n.
\]
Then we have
\[
\prod_{l=0}^{n-1} (i'c + lc) = i'(i' + 1)(i' + 2) \cdots (i' + n - 1)c^n = \\
\left(\frac{i' + n - 1}{n}\right)n!c^n.
\]
Thus (7) is proved.

Next, let \(i, i' \in \mathbb{Z}_{>0}, i \neq i'\) (if \(i = i'\), then
\[
\alpha \mid (a_{i,n} \alpha + ib_{i,n}c) \cdot \prod_{l=i+1}^{i+n-1} (-\alpha + lc) - b_{i,n} \cdot \prod_{l=i}^{i+n-1} (\alpha + lc),
\]

i.e. the corresponding structure algebra condition is satisfied). If \(i' > i\), we need to prove that
\[
-\alpha + (i' - i)c \mid (a_{i',n} \alpha + i'b_{i',n}c) \cdot \prod_{l=i'+1}^{i'+n-1} ((i' - i)c + lc) - b_{i,n} \cdot \prod_{l=i}^{i+n-1} ((i' - i)c + lc).
\]
Again it is enough to show that
\[
(a_{i',n}(i' - i)c + i'b_{i',n}c) \cdot \prod_{l=i'+1}^{i'+n-1} ((i - i')c + lc) - b_{i,n} \cdot \prod_{l=i}^{i+n-1} ((i' - i)c + lc) = 0. \quad (8)
\]
We have
\[
(a_{i',n}(i' - i)c + i'b_{i',n}c) \cdot \prod_{l=i'+1}^{i'+n-1} ((i - i')c + lc) = \\
\left(\binom{n + i'}{n} - n(i' - i) + i' \left(\binom{n + i'}{n} - (i' - 1)n\right) \frac{i' - (i' - 1)n}{i'}\right) \prod_{l=1}^{n-1} (i + l) \cdot c^n = \\
\left(\binom{n + i'}{n} - ni + i + n\right) \prod_{l=1}^{n-1} (i + l) \cdot c^n = \\
\left(\binom{n + i'}{n} - (i - 1)n\right) \prod_{l=1}^{n-1} (i + l) \cdot c^n
\]
and

\[ b_{i,n} \cdot \prod_{l=i}^{i+n-1} ((i'-i)c + lc) = \]

\[ = (n + i - 1) \frac{i - (i - 1)n}{n} i'(i' + 1) \cdots (i' + n - 1) c^n = \]

\[ = (n + i - 1)(n + i - 2) \cdots i \frac{i - (i - 1)n}{n} i'(i' + 1) \cdots (i' + n - 1) c^n = \]

\[ = \left( \prod_{l=1}^{n-1} (i + l) \right) \cdot \frac{(i - (i - 1)n)(n + i' - 1)}{n} c^n, \]

so we have proved (8).

Last, if \( i' < i \), we need to prove that

\[ \alpha + (i - i')c \mid b_{i,n} \cdot \prod_{l=i}^{i+n-1} (\alpha + lc) - (a_{i',n}\alpha + i'c_{i',n}c) \cdot \prod_{l=i'}^{i'+n-1} (-\alpha + lc) \]

or equivalently

\[ -\alpha + (i' - i)c \mid (a_{i',n}\alpha + i'c_{i',n}c) \cdot \prod_{l=i'}^{i'+n-1} (-\alpha + lc) - b_{i,n} \cdot \prod_{l=i}^{i+n-1} (\alpha + lc), \]

which has been previously proved. With this the proof has been completed.  

\[ \square \]

**Proposition 4.2.** The elements \( u_n, n \in \mathbb{Z}_{\geq 0} \) and \( v_n, n \in \mathbb{Z}_{>0} \) of the structure algebra \( \mathcal{Z} \) of the moment graph \( \hat{G}^{\text{stab}}_{>0 \alpha} \) form a basis of the \( S \)-module \( \mathcal{Z} \).

**Proof.** Let us consider an arbitrary global section \( w \) of the following form:

\[ w = \begin{pmatrix} \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ f & g & 0 & 0 \\ 0 & 0 & \vdots & \vdots \\ 0 & 0 & \vdots & \vdots \\ 0 & 0 & \vdots & \vdots \end{pmatrix} \]

where the first \( n \) (for some \( n \in \mathbb{Z}_{\geq 0} \)) rows are zero and \( f \) and \( g \) are some elements of the algebra \( S = k[\alpha, c] = \mathbb{C}[\alpha, c] \). Since \( \prod_{j \in \mathcal{V}(\hat{G}^{\text{stab}}_{>0 \alpha})} S \) is a direct product of graded algebras and the section conditions imply that

\[ \prod_{l=1}^{n} (-\alpha + lc) \mid f \quad \text{and} \quad \prod_{l=1}^{n} (\alpha + lc) \mid g, \]

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we may assume without loss of generality that \( f \) and \( g \) (and all other components) are homogeneous polynomials of degree \( m \), for some non-negative integer \( m \geq n \).

If \( m = n \), then

\[
\begin{align*}
f &= s \cdot \prod_{l=1}^{n} (-\alpha + lc), \\
g &= s \cdot \prod_{l=1}^{n} (\alpha + lc),
\end{align*}
\]

for some \( s \in k = \mathbb{C} \) as

\[
\prod_{l=1}^{n} (-\alpha + lc) \mid f, \\
\prod_{l=1}^{n} (\alpha + lc) \mid g, \\
\alpha \mid g - f.
\]

After subtraction we obtain the section

\[
w - su_n = \begin{pmatrix}
\vdots & \vdots \\
\tilde{f} & \tilde{g} \\
0 & 0 \\
\vdots & \vdots \\
0 & 0 \\
0 & 0
\end{pmatrix}
\]

where the first \( n + 1 \) rows are zero and \( \tilde{f} \) and \( \tilde{g} \) are some homogeneous polynomials of degree \( m = n \). Because \( \prod_{l=1}^{n} (-\alpha + lc) \mid \tilde{f} \) and \( \prod_{l=1}^{n} (\alpha + lc) \mid \tilde{g} \), the polynomials \( \tilde{f} \) and \( \tilde{g} \) must be equal to zero. The same is true for all other components, i.e. \( w = su_n \).

Now if \( m > n \), because we again have the following:

\[
\prod_{l=1}^{n} (-\alpha + lc) \mid f, \\
\prod_{l=1}^{n} (\alpha + lc) \mid g, \\
\alpha \mid g - f,
\]
it follows that
\[
g \cdot \prod_{l=1}^{n} (-\alpha + lc) - f \cdot \prod_{l=1}^{n} (\alpha + lc) = \frac{g}{\prod_{l=1}^{n} (\alpha + lc) \cdot \prod_{l=1}^{n} (-\alpha + lc)} - \frac{f}{\prod_{l=1}^{n} (-\alpha + lc)}
\]
is a polynomial in \( S \), i.e.
\[
\prod_{l=1}^{n} (\alpha + lc) \cdot \prod_{l=1}^{n} (-\alpha + lc) \mid g \cdot \prod_{l=1}^{n} (-\alpha + lc) - f \cdot \prod_{l=1}^{n} (\alpha + lc)
\]
and
\[
\alpha \mid g \cdot \prod_{l=1}^{n} (-\alpha + lc) - f \cdot \prod_{l=1}^{n} (\alpha + lc).
\]
Therefore we can define the following homogeneous polynomials:
\[
x := \frac{f}{\prod_{l=1}^{n} (-\alpha + lc)}
\]
and
\[
y := \frac{g \cdot \prod_{l=1}^{n} (-\alpha + lc) - f \cdot \prod_{l=1}^{n} (\alpha + lc)}{\alpha \cdot \prod_{l=1}^{n} (\alpha + lc) \cdot \prod_{l=1}^{n} (-\alpha + lc)} = \frac{1}{\alpha} \left( \frac{g}{\prod_{l=1}^{n} (\alpha + lc)} - \frac{f}{\prod_{l=1}^{n} (-\alpha + lc)} \right).
\]
Now we have
\[
x u_n + y v_{n+1} =
\]
\[
= \left( \begin{array}{ccc}
\vdots & \vdots \\
\prod_{l=1}^{n} (-\alpha + lc) & \prod_{l=1}^{n} (\alpha + lc) \\
0 & 0 \\
\vdots & \vdots \\
0 & 0 \\
0 & 0
\end{array} \right)
+ y \left( \begin{array}{ccc}
\vdots & \vdots \\
0 & \prod_{l=0}^{n} (\alpha + lc) \\
0 & 0 \\
\vdots & \vdots \\
0 & 0 \\
0 & 0
\end{array} \right)
= \left( \begin{array}{ccc}
\vdots & \vdots \\
f & g \\
0 & 0 \\
\vdots & \vdots \\
0 & 0 \\
0 & 0
\end{array} \right).
\]
If the obtained section differs from the section \( w \), then we apply the same procedure to the section \( w - (x u_n + y v_{n+1}) \). We stop this procedure when we obtain the section that has \( m \) zero rows, i.e. the degree of the components and the number of zero rows coincide and that brings us to the above case.

Hence we have shown that the set \( \{ u_n \mid n \in \mathbb{Z}_{\geq 0} \} \cup \{ v_n \mid n \in \mathbb{Z}_{> 0} \} \) generates the \( S \)-module \( \mathcal{Z} \). Since the linear independence is obvious, we conclude that the set \( \{ u_n \mid n \in \mathbb{Z}_{\geq 0} \} \cup \{ v_n \mid n \in \mathbb{Z}_{> 0} \} \) is indeed a basis of the \( S \)-module \( \mathcal{Z} \).
4.1. “Setting \( c \) equal to zero”. We denote by \( \mathcal{G}^{\text{fin}} \) the moment graph associated with the finite root system \( A_1 \), i.e. the root system \( \{ \alpha, -\alpha \} \) (cf. \cite{Fie16}). This is the graph with two vertices that are connected by one edge labeled with \( \alpha \).

Then we denote by \( S^{\text{fin}} = k[\alpha] = \mathbb{C}[\alpha] \) the corresponding symmetric algebra and by \( \mathcal{Z}^{\text{fin}} \) the structure algebra of \( \mathcal{G}^{\text{fin}} \), i.e.

\[
\mathcal{Z}^{\text{fin}} = \left\{ (z_1, z_2) \mid z_1, z_2 \in S^{\text{fin}}, \alpha \mid z_1 - z_2 \right\}.
\]

We define now the \( S^{\text{fin}} \)-module \( \mathcal{Z}_{c=0} \) by “setting \( c \) equal to zero” in the \( S \)-module \( \mathcal{Z} \) in the following way.

For \( f \in S = k[\alpha, c] = \mathbb{C}[\alpha, c] \) we define \( \overline{f} := f|_{c=0} \in S^{\text{fin}} = k[\alpha] = \mathbb{C}[\alpha] \).

Then \( \mathcal{Z}_{c=0} := \left\{ (z_k\alpha)_{k \in \mathbb{Z}\setminus\{0\}} \mid (z_k\alpha)_{k \in \mathbb{Z}\setminus\{0\}} \in \mathcal{Z} \right\} \).

The elements \( \overline{u}_n := ((u_n)_{k \in \mathbb{Z}\setminus\{0\}})_{n \in \mathbb{Z}_{\geq 0}} \) and \( \overline{v}_n := ((v_n)_{k \in \mathbb{Z}\setminus\{0\}})_{n \in \mathbb{Z}_{> 0}} \) form a basis of the \( S^{\text{fin}} \)-module \( \mathcal{Z}_{c=0} \).

Indeed, because \( u_n, n \in \mathbb{Z}_{\geq 0} \) and \( v_n, n \in \mathbb{Z}_{> 0} \) are generators of the \( S \)-module \( \mathcal{Z} \), \( \overline{u}_n, n \in \mathbb{Z}_{\geq 0} \) and \( \overline{v}_n, n \in \mathbb{Z}_{> 0} \) are generators of the \( S^{\text{fin}} \)-module \( \mathcal{Z}_{c=0} \). Furthermore, observe that “the row echelon form” of \( u_n, n \in \mathbb{Z}_{\geq 0} \) and \( v_n, n \in \mathbb{Z}_{> 0} \) is preserved under “setting \( c \) equal to zero”, which means that \( \overline{u}_n, n \in \mathbb{Z}_{\geq 0} \) and \( \overline{v}_n, n \in \mathbb{Z}_{> 0} \) are of the same form. For this reason, \( \overline{u}_n, n \in \mathbb{Z}_{\geq 0} \) and \( \overline{v}_n, n \in \mathbb{Z}_{> 0} \) are \( S^{\text{fin}} \)-linearly independent in \( \mathcal{Z}_{c=0} \).

Let \( \overline{w} = (\overline{w}_k\alpha)_{k \in \mathbb{Z}\setminus\{0\}} \) be an arbitrary element of \( \mathcal{Z}_{c=0} \), i.e.

\[
\overline{w} = \begin{pmatrix}
... \\
\overline{w}_{k\alpha} & \overline{w}_{-k\alpha} \\
... \\
\overline{w}_{2\alpha} & \overline{w}_{-2\alpha} \\
... \\
\overline{w}_\alpha & \overline{w}_{-\alpha}
\end{pmatrix}.
\]

We regard the rows of \( \overline{w} \) as ordered pairs:

\[
\mathbf{a}_j := (\overline{w}_{(j+1)\alpha}, \overline{w}_{-(j-1)\alpha}), \ j \in \mathbb{Z}_{\geq 0}.
\]

Observe that \( \mathbf{a}_j \in \mathcal{Z}^{\text{fin}} \) for all non-negative integers \( j \) as \( \overline{w}_{(j+1)\alpha} - \overline{w}_{-(j-1)\alpha} \) is divisible by \( \alpha \). Now we have the following theorem.

**Theorem 4.3.**

\[
\mathcal{Z}_{c=0} = \left\{ (\mathbf{a}_j) \in \prod_{j \in \mathbb{Z}_{\geq 0}} \mathcal{Z}^{\text{fin}} \mid \sum_{j=0}^{m} (-1)^j \binom{m}{j} \mathbf{a}_j \in ((-\alpha)^m, \alpha^m) \mathcal{Z}^{\text{fin}} \ \forall m \in \mathbb{Z}_{\geq 0} \right\}.
\]

(9)
Proof. In order to prove that $Z_{c=0}$ is a subset of the set given on the right-hand side, it is enough to show that the basis elements $\overline{u}_n$, $n \in \mathbb{Z}_{\geq 0}$ and $\overline{v}_n$, $n \in \mathbb{Z}_{> 0}$ of the $S^{\text{fin}}$-module $Z_{c=0}$ satisfy divisibility relations.

Let

$$\overline{u}_n = \begin{pmatrix} \vdots & \vdots \\ k_{3,n}(-\alpha)^n & k_{3,n}\alpha^n \\ k_{2,n}(-\alpha)^n & k_{2,n}\alpha^n \\ k_{1,n}(-\alpha)^n & k_{1,n}\alpha^n \\ (-\alpha)^n & \alpha^n \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

be a basis element of the $S^{\text{fin}}$-module $Z_{c=0}$, where the first $n$ rows are zero for an arbitrary $n \in \mathbb{Z}_{\geq 0}$ and $k_{i,n}, i \in \mathbb{Z}_{> 0}$ are as in (4).

If $m \leq n$, then the corresponding divisibility relation is obviously satisfied.

Therefore let us assume that $m = n + k$ for some $k \in \mathbb{Z}_{> 0}$. We want to show that $((-\alpha)^{n+k}, \alpha^{n+k})$ divides

$$\sum_{j=0}^{n+k} (-1)^j \binom{n+k}{j} a_j.$$

The sum above equals to

$$\sum_{j=n}^{n+k} (-1)^j \binom{n+k}{j} \binom{j}{n} ((-\alpha)^n, \alpha^n),$$

so we need to prove that

$$\sum_{j=n}^{n+k} (-1)^j \binom{n+k}{j} \binom{j}{n} = 0. \quad (10)$$

We have

$$\sum_{l=0}^{k} (-1)^{n+l} \binom{n+k}{n+l} \binom{n+l}{n} = \sum_{l=0}^{k} (-1)^{n+l} \binom{n+k}{n} \binom{k}{l} =$$

$$= (-1)^n \binom{n+k}{n} \sum_{l=0}^{k} (-1)^l \binom{k}{l} = 0,$$

which proves (10) (here we have used the following binomial coefficient identity:

$$\binom{n}{m} \binom{m}{k} = \binom{n}{k} \binom{n-k}{m-k}, \quad 0 \leq k \leq m \leq n,$$

which is sometimes called in the literature the subset-of-a-subset identity).
Now let
\[
\overline{v}_n = \begin{pmatrix}
  \vdots \\
- a_{3,n} (-\alpha)^n & b_{3,n} \alpha^n \\
- a_{2,n} (-\alpha)^n & b_{2,n} \alpha^n \\
- a_{1,n} (-\alpha)^n & b_{1,n} \alpha^n \\
0 & \alpha^n \\
0 & 0 \\
\vdots \\
0 & 0 \\
0 & 0
\end{pmatrix}
\]
be a basis element of the $S_{\text{fin}}$-module $Z_{c=0}$, where the first $n - 1$ rows are zero for an arbitrary $n \in \mathbb{Z}_{>0}$ and $a_{i,n}, b_{i,n}, i \in \mathbb{Z}_{>0}$ are as in [5].

If $m \leq n - 2$, then the corresponding divisibility relation is trivially satisfied.

If $m = n - 1$, then we have
\[
\sum_{j=0}^{n-1} (-1)^j \binom{n-1}{j} a_j = (-1)^{n-1} (0, \alpha^n) = (0, (-1)^{n-1} \alpha ((-\alpha)^{n-1}, \alpha^{n-1}).
\]

If $m = n$, we have
\[
\sum_{j=0}^{n} (-1)^j \binom{n}{j} a_j = (-1)^{n-1} \binom{n}{n-1} (0, \alpha^n) + (-1)^n (-a_{1,n} (-\alpha)^n, b_{1,n} \alpha^n) =
\]
\[
= (0, (-1)^{n-1} n \alpha^n) + ((-1)^{n-1} (n - 1) (-\alpha)^n, (-1)^n \alpha^n) = 
\]
\[
\left( (\alpha)^n, (\alpha)^n \right).
\]

Now let $m = n + k$ for some $k \in \mathbb{Z}_{>0}$. Then
\[
\sum_{j=0}^{n+k} (-1)^j \binom{n+k}{j} a_j =
\]
\[
= (-1)^{n-1} \binom{n+k}{n-1} (0, \alpha^n) + \sum_{j=n}^{n+k} (-1)^j \binom{n+k}{j} (-a_{j-n+1,n} (-\alpha)^n, b_{j-n+1,n} \alpha^n),
\]

hence we need to show that
\[
(-1)^{n-1} \binom{n+k}{n-1} (0, \alpha^n) + \sum_{j=n}^{n+k} (-1)^j \binom{n+k}{j} (-a_{j-n+1,n} (-\alpha)^n, b_{j-n+1,n} \alpha^n) = 0,
\]

i.e.
\[
\sum_{j=n}^{n+k} (-1)^j \binom{n+k}{j} a_{j-n+1,n} = 0 \quad (11)
\]

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and

\[ (-1)^{n-1} \binom{n+k}{n-1} + \sum_{j=n}^{n+k} (-1)^j \binom{n+k}{j} b_{j-n+1,n} = 0. \]  \hspace{1cm} (12)

The left-hand side of (11) equals

\[
\sum_{j=n}^{n+k} (-1)^j \binom{n+k}{j} (n-1) = 
\]

\[
= (n-1) \sum_{l=0}^{k} (-1)^{n+l} \binom{n+k}{n+l} \binom{n}{l} = 
\]

\[
= (-1)^n (n-1) \binom{n+k}{n} \sum_{l=0}^{k} (-1)^l \binom{k}{l} = 0.
\]

Therefore (11) is proved.

Then on the left-hand side of (12) we have the following:

\[
(-1)^{n-1} \binom{n+k}{n-1} + \sum_{j=n}^{n+k} (-1)^j \binom{n+k}{j} \binom{j}{n} \frac{(j-n+1)-(j-n)n}{j-n+1} =
\]

\[
= (-1)^{n-1} \binom{n+k}{n-1} + \sum_{j=n}^{n+k} (-1)^j \binom{n+k}{j} \binom{j}{n} \left(1 - \frac{j-n}{j-n+1} \frac{n}{n}\right) =
\]

\[
= (-1)^{n-1} \binom{n+k}{n-1} + \sum_{l_1=0}^{k} (-1)^{n+l_1} \binom{n+k}{n+l_1} \binom{n+l_1}{n} -
\]

\[
- \sum_{l_2=0}^{k} (-1)^{n+l_2} \binom{n+k}{n+l_2} \binom{n+l_2}{n} \frac{l_2}{l_2+1} n =
\]

\[
= (-1)^{n-1} \binom{n+k}{n-1} + (-1)^n \binom{n+k}{n} \sum_{l_1=0}^{k} (-1)^l_1 \binom{k}{l_1} -
\]

\[
- (-1)^n \binom{n+k}{n} \sum_{l_2=0}^{k} (-1)^l_2 \binom{k}{l_2} \frac{l_2}{l_2+1}.
\]  \hspace{1cm} (13)

Therefore (11) is proved.

Then on the left-hand side of (12) we have the following:
Now we calculate the sum (∗).

\[
\sum_{l_2=0}^{k} (-1)^{l_2} \binom{k}{l_2} \frac{l_2}{l_2 + 1} = \sum_{l_2=0}^{k} (-1)^{l_2} \binom{k + 1}{l_2 + 1} \frac{l_2}{k + 1} =
\]

\[
= -\frac{1}{k + 1} \sum_{l_2=0}^{k} (-1)^{l_2} \binom{k + 1}{l_2 + 1} l_2
\]

\[
= -\frac{1}{k + 1} \sum_{l_2=0}^{k} (-1)^{l_2} \binom{k + 1}{l_2} l_2
\]

The sum (∗∗∗) is equal to

\[
\sum_{l_3=1}^{k+1} (-1)^{l_3} \binom{k + 1}{l_3} l_3 = (k + 1) \sum_{l_4=0}^{k} (-1)^{l_4+1} \binom{k}{l_4} =
\]

\[
= -(k + 1) \sum_{l_4=0}^{k} (-1)^{l_4} \binom{k}{l_4} = 0
\]

and the sum (∗∗∗) is equal to

\[
\sum_{l_3=0}^{k+1} (-1)^{l_3} \binom{k + 1}{l_3} - 1 = -1.
\]

Therefore the sum (∗) equals \( -\frac{1}{k + 1} \) and we have in (13) the following:

\[
(-1)^{n-1} \binom{n + k}{n - 1} + (-1)^n n \binom{n + k}{n} \frac{1}{k + 1} =
\]

\[
= (-1)^{n-1} \binom{n + k}{n - 1} + (-1)^n \binom{n + k}{n - 1} =
\]

\[
= \left( \binom{n + k}{n - 1} (-1)^{n-1} + (-1)^n \right) = 0.
\]

Thus we have proved (12).

Finally, to prove that the \( S^{\text{fin}} \)-module given on the right-hand side of (9) is included in \( Z_{c=0} \), we will show that the basis elements \( \overline{u}_n, n \in \mathbb{Z}_{\geq 0} \) and \( \overline{v}_n, n \in \mathbb{Z}_{> 0} \) of \( Z_{c=0} \) generate that \( S^{\text{fin}} \)-module (by performing a procedure similar to Gaussian elimination).
For $n \geq 1$, because of the gradation on the direct product $\prod_{\alpha \in \hat{V}_{\alpha}^{\text{stab}}} S$ (and on the direct product $\prod_{\alpha \in \hat{V}_{\alpha}^{\text{stab}}} S_{\text{fin}}$), we may assume without loss of generality that an arbitrary element of that $S_{\text{fin}}$-module is of the following form:

$$
\begin{bmatrix}
\vdots & \vdots \\
x_2 \alpha^n & y_2 \alpha^n \\
x_1 \alpha^n & y_1 \alpha^n \\
x_0 \alpha^n & y_0 \alpha^n \\
\end{bmatrix}
$$

where $x_i, y_i \in k = \mathbb{C}, i \in \mathbb{Z}_{\geq 0}$. We have the following:

$$
\begin{bmatrix}
\vdots & \vdots \\
x_2 \alpha^n & y_2 \alpha^n \\
x_1 \alpha^n & y_1 \alpha^n \\
x_0 \alpha^n & y_0 \alpha^n \\
\end{bmatrix} - x_0 \alpha^n \begin{bmatrix}
1 & 1 \\
1 & 1 \\
1 & 1 \\
\end{bmatrix}
=_{m_0}
\begin{bmatrix}
\vdots & \vdots \\
x_2 - x_0 \alpha^n & (y_2 - x_0) \alpha^n \\
x_1 - x_0 \alpha^n & (y_1 - x_0) \alpha^n \\
0 & (y_0 - x_0) \alpha^n \\
\end{bmatrix}
$$

$$
\begin{bmatrix}
\vdots & \vdots \\
x_2 - x_0 \alpha^n & (y_2 - x_0) \alpha^n \\
x_1 - x_0 \alpha^n & (y_1 - x_0) \alpha^n \\
0 & (y_0 - x_0) \alpha^n \\
\end{bmatrix} - (y_0 - x_0) \alpha^{n-1} \begin{bmatrix}
0 & \alpha \\
0 & \alpha \\
0 & \alpha \\
\end{bmatrix}
=_{m_1}
\begin{bmatrix}
\vdots & \vdots \\
x_2 - x_0 \alpha^n & (y_2 - y_0) \alpha^n \\
x_1 - x_0 \alpha^n & (y_1 - y_0) \alpha^n \\
0 & 0 \\
\end{bmatrix}
$$

The procedure stops at:

$$
\begin{bmatrix}
\vdots & \vdots \\
z_2 \alpha^n & z_3 \alpha^n \\
0 & z_1 \alpha^n \\
0 & 0 \\
\end{bmatrix}
$$

where the first $n$ rows are zero and $z_i \in k = \mathbb{C}, i \in \mathbb{Z}_{\geq 0}$.

Since $((-\alpha)^n, \alpha^n) \left| \sum_{j=0}^{n} (-1)^j \binom{n}{j} a_j \right.$, it follows that

$$
(-1)^n (0, z_1 \alpha^n) = (u, u) \left((-\alpha)^n, \alpha^n \right)_{\in \mathbb{Z}_{\text{fin}}^n}
$$

for some $u \in k = \mathbb{C}$.
Therefore $z_1$ is equal to zero.

By applying the divisibility relations for $m \geq n + 1$, we conclude that all rows are zero.

For $n = 0$, let
\[
\begin{pmatrix}
  \vdots & \vdots \\
  c_2 & c_2 \\
  c_1 & c_1 \\
  c_0 & c_0
\end{pmatrix}, \quad c_i \in k = \mathbb{C}, \ i \in \mathbb{Z}_{\geq 0}
\]
be an arbitrary element of the $S^{\text{fin}}$-module on the right-hand side of (9).

Then
\[
\begin{pmatrix}
  \vdots & \vdots \\
  c_2 & c_2 \\
  c_1 & c_1 \\
  c_0 & c_0
\end{pmatrix} - c_0 \begin{pmatrix}
  \vdots & \vdots \\
  1 & 1 \\
  1 & 1 \\
  1 & 1
\end{pmatrix} = \begin{pmatrix}
  \vdots & \vdots \\
  c_2 - c_0 & c_2 - c_0 \\
  c_1 - c_0 & c_1 - c_0 \\
  0 & 0
\end{pmatrix}
\]
and again by applying the divisibility relations for $m \geq 1$, we obtain
\[
c_i = c_0 \quad \forall i \in \mathbb{Z}_{>0}.
\]

This completes the proof.

\[\square\]

Remark. The structure algebra $Z$ is an $S$-algebra and the $S^{\text{fin}}$-module $Z_{c=0}$ inherits the algebra structure. Therefore $Z_{c=0}$ is an $S^{\text{fin}}$-algebra, where the multiplication is given componentwise. This algebra structure on $Z_{c=0}$ implies that higher-order congruence relations from Theorem 4.3 are closed under multiplication. This property is far from obvious.

REFERENCES

[BB81] A. Beilinson and J. Bernstein, Localisation de $\mathfrak{g}$-modules, *C. R. Acad. Sci. Paris Sér. I Math.* **292** (1981), no. 1, 15–18.

[BM01] T. Braden and R. MacPherson, From moment graphs to intersection cohomology, *Math. Ann.* **321** (2001), no. 3, 533–551.

[BK81] J-L. Brylinski and M. Kashiwara, Kazhdan-Lusztig conjecture and holonomic systems, *Invent. Math.* **64** (1981), no. 3, 387–410.

[Deo87] V. V. Deodhar, On some geometric aspects of Bruhat orderings II. The parabolic analogue of Kazhdan-Lusztig polynomials, *J. Algebra* **111** (1987), no. 2, 483–506.

[EW14] B. Elias and G. Williamson, The Hodge theory of Soergel bimodules, *Ann. of Math.* **180** (2014), no. 3, 1089–1136.
[Fie08a] P. Fiebig, Sheaves on moment graphs and a localization of Verma flags, *Adv. in Math.* 217 (2008), no. 2, 683–712.

[Fie08b] P. Fiebig, The combinatorics of Coxeter categories, *Trans. Amer. Math. Soc.* 360 (2008), no. 8, 4211–4233.

[Fie10] P. Fiebig, Lusztig’s conjecture as a moment graph problem, *Bull. London Math. Soc.* 42 (2010), no. 6, 957–972.

[Fie11] P. Fiebig, Sheaves on affine Schubert varieties, modular representations and Lusztig’s conjecture, *J. Amer. Math. Soc.* 24 (2011), no. 1, 133–181.

[Fie16] P. Fiebig, Moment graphs in representation theory and geometry, *Schubert Calculus–Osaka 2012, Advanced Studies in Pure Mathematics* 71 (2016), 75–96.

[GKM98] M. Goresky, R. Kottwitz, and R. MacPherson, Equivariant cohomology, Koszul duality, and the localization theorem, *Invent. Math.* 131 (1998), no. 1, 25–83.

[Hum90] J. E. Humphreys, *Reflection groups and Coxeter groups*, Cambridge University Press, Cambridge, 1990.

[Kac90] V. G. Kac, *Infinite dimensional Lie algebras*, Third edition, Cambridge University Press, Cambridge, 1990.

[Kat85] S. Kato, On the Kazhdan-Lusztig polynomials for affine Weyl groups, *Adv. in Math.* 55 (1985), no. 2, 103–130.

[KL79] D. Kazhdan and G. Lusztig, Representations of Coxeter groups and Hecke algebras, *Invent. Math.* 53 (1979), no. 2, 165–184.

[Lan14] M. Lanini, Categorification of a parabolic Hecke module via sheaves on moment graphs, *Pac. J. Math.* 271 (2014), no. 2, 415–444.

[Lan15] M. Lanini, On the stable moment graph of an affine Kac-Moody algebra, *Trans. Amer. Math. Soc.* 367 (2015), no. 6, 4111–4156.

[LW] N. Libedinsky and G. Williamson, The anti-spherical category, preprint, [arXiv:1702.00459](https://arxiv.org/abs/1702.00459).

[Lus80] G. Lusztig, Hecke algebras and Jantzen’s generic decomposition patterns, *Adv. in Math.* 37 (1980), no. 2, 121–164.

[Soc97] W. Soergel, Kazhdan-Lusztig polynomials and a combinatorics for tilting modules, *Represent. Theory* 1 (1997), 83–114 (electronic).

[Soc07] W. Soergel, Kazhdan-Lusztig-Polynome und unzerlegbare Bimoduln über Polynomringen, *J. Inst. Math. Jussieu* 6 (2007), no. 3, 501–525.
