TOPOLOGICAL CONSTRUCTIONS OF TENSOR FIELDS ON
MODULI SPACES

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ABSTRACT. We show how topology of a space may lead to tensor fields on (the
smooth part of) moduli spaces of the fundamental group.

1. INTRODUCTION

Moduli spaces of the fundamental groups of surfaces carry beautiful geometric
structures, in particular, Poisson brackets, see [AB], [Wo], [FR]. These brackets
were described by Goldman [Go1], [Go2] in terms of a Lie bracket in the mod-
ule of loops in the surface, see [AKKN1], [AKKN2], [Ka], [LS] for recent work on
Goldman’s bracket. Here we extend this line of study. Our starting point is the
work of van den Bergh [VdB] and Crawley-Boevey [Cb] who derive from any alge-
bra $A$ and an integer $n \geq 1$ the (commutative) coordinate algebra $A_n$ of the affine
scheme $\text{Rep}_n(A)$ of $n$-dimensional representations of $A$. These authors also define
a subalgebra $A^t_n \subset A_n$ which - under appropriate assumptions - is the coordinate
algebra of the affine quotient scheme $\text{Rep}_n(A)/\text{GL}_n$. We view the latter affine
scheme as the moduli space of $n$-dimensional representations of $A$. Inspired by the
interpretation of vector fields on a smooth manifold as derivations of the algebra of
smooth functions on this manifold, we can define vector fields on $\text{Rep}_n(A)/\text{GL}_n$ as
derivations of the algebra $A^t_n$. More generally, for any integers $m, n \geq 1$, we
can define $m$-tensor fields on $\text{Rep}_n(A)/\text{GL}_n$ as $m$-linear forms $(A^t_n)^m \to A^t_n$ (or
$(A^t_n)^{\otimes m} \to A^t_n$) which are derivations in all $m$ variables. Despite a purely algebraic
formulation, this approach may lead to smooth tensor fields on the smooth parts of
moduli spaces, see [MT2]. To construct $m$-linear forms in $A^t_n$ which are derivations
in all variables, we use a method inspired by the work of Crawley-Boevey [CB] on
Poisson structures. Namely, we set $\tilde{A} = A/[A, A]$ and derive such $m$-linear forms in
$A^t_n$ from $m$-linear forms $\tilde{A}^m \to \tilde{A}$ satisfying certain assumptions. We call $m$-linear
forms in $\tilde{A}$ satisfying these assumptions $m$-braces in $A$.

Our main aim is a construction of braces in the group algebras of the fundamental
groups of topological spaces. We give two such constructions. First, consider a
topological space $X$ and let $A$ be the group algebra of $\pi_1(X)$. A gate in $X$ is a
path-connected subspace $C \subset X$ such that all loops in $C$ are contractible in $X$
and $C$ has a cylinder neighborhood $C \times [-1, 1]$ in $X$. We show that a gate in $X$ gives
rise to an $m$-brace in $A$ for all $m \geq 1$. This “gate brace” induces $m$-tensor fields
on the moduli spaces of $\pi_1(X)$ for all $m \geq 1$. For example, if $X$ is a surface with
boundary, then any properly embedded segment in $X$ is a gate; so, it determines
an $m$-brace in $A$ and an $m$-tensor field on the moduli space $\text{Rep}_n(A)/\text{GL}_n$ for all
$m, n \geq 1$. 

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Our second construction of braces applies to so-called quasi-surfaces which we introduce here as generalizations of the usual surfaces with boundary. A quasi-surface, $X$, is obtained by gluing a surface $\Sigma$ to an arbitrary topological space along a finite set of disjoint segments in $\partial \Sigma$. These segments give rise to gates in $X$ which split $X$ into the surface part (a copy of $\Sigma$) and the singular part (the rest). By the above, each gate induces an $m$-brace in the group algebra $A$ of $\pi_1(X)$ for all $m \geq 1$. For oriented $\Sigma$, we use intersections of loops to define a skew-symmetric “intersection 2-brace” in $A$ generalizing the Goldman bracket of surfaces. Our main result is a Jacobi-type identity relating the intersection 2-brace to the gate 3-braces. This generalizes to quasi-surfaces the Jacobi identity for the Goldman bracket of surfaces. We also define intersection pairings in 1-homology of quasi-surfaces generalizing the usual intersection pairings in 1-homology of surfaces.

Any surface $\Sigma$ with boundary may be viewed as a quasi-surface in multiple ways determined by a choice of disjoint properly embedded segments in $\Sigma$ splitting $\Sigma$ into the “surface part” and the “singular part”. For oriented $\Sigma$, each such splitting determines a 2-brace in the group algebra $A$ of $\pi_1(\Sigma)$ and the induced pairings $\{A_n^\Sigma \times A_n^\Sigma \to A_n^\Sigma\}_{n \geq 1}$. By the above, these brace and pairings satisfy Jacobi-type identities involving the 3-braces associated with the segments in question.

The first part of the paper (Sections 2–5) presents our algebraic methods and the second part (Sections 6–10) is devoted to topological constructions.

This work was supported by the NSF grant DMS-1664358.

2. Preliminaries

We briefly recall representation schemes and trace algebras following [VdB], [Cb]. Then we discuss derivations in algebras.

2.1. Representation schemes. Throughout the paper we fix a commutative base ring $R$. By a module we mean an $R$-module and by an algebra we mean (unless explicitly stated to the contrary) an associative $R$-algebra with unit. We associate with every algebra $A$ and an integer $n \geq 1$ an affine scheme $\text{Rep}_n(A)$, the $n$-th representation scheme of $A$. For any commutative algebra $S$, the set of $S$-valued points of $\text{Rep}_n(A)$ is the set of algebra homomorphisms $A \to \text{Mat}_n(S)$. The coordinate ring, $A_n$, of $\text{Rep}_n(A)$ is generated (over $R$) by the symbols $x_{ij}$ with $x \in A$ and $i, j \in \{1, 2, \ldots, n\}$. These generators commute and satisfy the following relations: $1_{ij} = \delta_{ij}$ for all $i, j$, where $\delta_{ij}$ is the Kronecker delta; for all $x, y \in A$, $r \in R$, and $i, j \in \{1, 2, \ldots, n\}$,

$$(rx)_{ij} = rx_{ij}, \quad (x + y)_{ij} = x_{ij} + y_{ij} \quad \text{and} \quad (xy)_{ij} = \sum_{l=1}^{n} x_{il} y_{lj}.$$ 

The function on the set of $S$-valued points of $\text{Rep}_n(A)$ determined by $x_{ij}$ assigns to a homomorphism $f : A \to \text{Mat}_n(S)$ the $(i, j)$-entry of the matrix $f(x)$. That these functions satisfy the relations above is straightforward.

The action of the group $G = GL_n(R)$ on $\text{Hom}(A, \text{Mat}_n(S))$ by conjugations induces an action of $G$ on the commutative algebra $A_n$ for all $n$. Explicitly, for
g = (g_{kl}) \in G and any x \in A, i, j \in \{1, \ldots, n\} we have
\[ g \cdot x_{ij} = \sum_{k=1}^{n} \sum_{l=1}^{n} g_{ik}(g^{-1})_{lj}x_{kl}. \]
The set of invariant elements \( A^G_n = \{ a \in A_n \mid Ga = a \} \) is a subalgebra of \( A_n \). This is the coordinate algebra of the affine quotient scheme \( \text{Rep}_n(A)/G \) which we view as the “moduli space” of \( n \)-dimensional representations of \( A \).

2.2. The module \( \hat{A} \) and the trace. Given an algebra \( A \), let \( A' = [A, A] \) be the submodule of \( A \) spanned by the commutators \( xy - yx \) with \( x, y \in A \). The quotient module \( \hat{A} = A/A' \) is the zeroth Hochschild homology of \( A \). Now, for any integer \( n \geq 1 \), the linear map \( A \to A_n, x \mapsto \sum_{i=1}^{n} x_{ii} \) is called the trace and denoted \( \text{tr} \). The trace annihilates all the commutators in \( A \) and therefore \( \text{tr}(A') = 0 \). Thus, the trace induces a linear map \( \hat{A} \to A_n \) also denoted \( \text{tr} \).

The subalgebra of \( A_n \) generated by \( \text{tr}(A) = \text{tr}(\hat{A}) \) is called the \( n \)-th trace algebra of \( A \) and is denoted \( A^t_n \). A direct computation shows that \( \text{tr}(A) \subset A^t_n \) and therefore \( A^t_n \subset A^G_n \). If the ground ring \( R \) is an algebraically closed field of characteristic zero and \( A \) is a finitely generated algebra, then a theorem of Le Bruyn and Procesi \cite{LB} implies that \( A^t_n = A^G_n \) so that \( A^t_n \) is the coordinate algebra of \( \text{Rep}_n(A)/G \).

2.3. Derivations. A derivation of an algebra \( A \) is a linear map \( d : A \to A \) such that \( d(xy) = d(x)y + xd(y) \) for all \( x, y \in A \). We denote by \( \text{Der}(A) \) the module of derivations of \( A \). Given \( d_1, d_2 \in \text{Der}(A) \), the commutator \( [d_1, d_2] = d_1 \circ d_2 - d_2 \circ d_1 \) is a derivation of \( A \). This defines a Lie bracket \([\cdot, \cdot]\) in \( \text{Der}(A) \).

Any derivation \( d : A \to A \) carries \( A' = [A, A] \) into itself as
\[ d(xy - yx) = d(x)y - yd(x) + xd(y) - d(y)x \]
for \( x, y \in A \). Therefore \( d \) induces a linear endomorphism of \( \hat{A} = A/A' \). A linear endomorphism of \( \hat{A} \) is a weak derivation if it is induced by a derivation \( A \to A \).

By \cite{G} Lemma 4.4, for any derivation \( d : A \to A \) and any integer \( n \geq 1 \), there is a unique derivation \( \tilde{d} : A_n \to A_n \) such that \( \tilde{d}(a_{ij}) = (d(a))_{ij} \) for all \( a \in A, i, j \in \{1, \ldots, n\} \). Indeed, this formula defines \( \tilde{d} \) on the generators of the algebra \( A_n \); the compatibility with the defining relations is straightforward. Clearly, \( \tilde{d}(\text{tr}(a)) = \text{tr}(d(a)) \) for all \( a \in A \). Therefore \( \tilde{d}(A^t_n) \subset A^t_n \) and the restriction of \( \tilde{d} \) to \( A^t_n \) is a derivation of the algebra \( A^t_n \).

2.4. Remark. If \( C^\infty(M) \) is the algebra of smooth \( \mathbb{R} \)-valued functions on a smooth manifold \( M \), then each smooth vector field \( v \) on \( M \) induces a derivation \( d_v \) of \( C^\infty(M) \) carrying a function \( f \in C^\infty(M) \) to the function \( df(v) : M \to \mathbb{R} \). The map \( v \mapsto d_v \) defines a Lie algebra isomorphism from the Lie algebra of smooth vector field on \( M \) (with the Jacobi-Lie bracket) onto \( \text{Der}(C^\infty(M)) \). Given an algebra \( A \) and an integer \( n \geq 1 \), these results suggest to view the derivations of the trace algebra \( A^t_n \) as vector fields on (the smooth part of) the affine quotient scheme \( \text{Rep}_n(A)/G \). More generally, tensor fields on this affine scheme may be defined as maps \( \{(A^t_n)^m \to A^t_n\}_{m \geq 1} \) which are derivations in all \( m \) variables. Here for a set \( E \) and an integer \( m \geq 1 \), we let \( E^m \) be the direct product of \( m \) copies of \( E \).

3. Braces and brackets

We define and study braces.
3.1. **Braces.** For an integer $m \geq 1$, an $m$-brace in an algebra $A$ is a mapping $\mu : (\hat{A})^m \to \hat{A}$ which is a weak derivation in all $m$ variables: for any $1 \leq j \leq m$ and $x_1, \ldots, x_j-1, x_{j+1}, \ldots, x_m \in \hat{A}$, the map

$$\hat{A} \to \hat{A}, \ x \mapsto \mu(x_1, \ldots, x_j-1, x, x_{j+1}, \ldots, x_m)$$

is a weak derivation. In particular, $\mu$ has to be linear in all variables. For $m = 1$, an $m$-brace in $A$ is just a weak derivation $\hat{A} \to \hat{A}$.

If $A$ is a commutative algebra, then $A' = 0$, $\hat{A} = A$, and an $m$-brace in $A$ is a mapping $\mu : A^m \to A$ which is a derivation in all variables: for any $1 \leq j \leq m$ and any $x_1, \ldots, x_j-1, x_{j+1}, \ldots, x_m \in A$, the map

$$A \to A, \ x \mapsto \mu(x_1, \ldots, x_j-1, x, x_{j+1}, \ldots, x_m)$$

is a derivation.

The following lemma - inspired by W. Crawley-Boevey [Cb] - is our main tool producing braces in the trace algebras.

**Lemma 3.1.** For any integers $m, n \geq 1$ and any $m$-brace $\mu$ in an algebra $A$, there is a unique $m$-brace $\mu_n$ in the algebra $A_n^m$ such that the trace $\tr : \hat{A} \to A_n^m$ carries $\mu$ to $\mu_n$ that is for all $x_1, \ldots, x_m \in \hat{A}$, we have

$$(3.1.1) \quad \mu_n(\tr(x_1), \ldots, \tr(x_m)) = \tr(\mu(x_1, \ldots, x_m)).$$

**Proof.** The uniqueness of $\mu_n$ is clear as $\tr(A)$ generates the algebra $A_n^m$. We first prove the existence of $\mu_n$ for $m = 1$. We need to show that given a weak derivation $\mu : \hat{A} \to \hat{A}$, there is a derivation $\mu_n : A_n^1 \to A_n^1$ such that $\mu_n(\tr(x)) = \tr(\mu(x))$ for all $x \in A$. Pick a derivation $d : A \to A$ inducing $\mu$. By Section 2.3, the induced derivation $\tilde{d} : A_n \to A_n$ restricts to a derivation $\mu_n : A_n^1 \to A_n^1$ of the algebra $A_n^1$. The map $\mu_n$ satisfies the conditions of the lemma.

Suppose now that $m \geq 2$. Since the algebra $A_n^1$ is generated by the set $\tr(\hat{A}) \subset A_n^1$, every $y \in A_n^1$ has a (non-unique) finite expansion

$$(3.1.2) \quad y = \sum c_{x_1^y \ldots x_r^y} \tr(x_1^y) \cdots \tr(x_r^y)$$

where the sum is over some finite sequences $x_1^y, \ldots, x_r^y \in \hat{A}$ and the coefficients $c_{x_1^y \ldots x_r^y}$ are in $R$. Pick any $y_1, \ldots, y_m \in A_n^1$ and for $j = 1, \ldots, m$ pick an expansion $E_j$ of $y_j$ as in (3.1.2). If an $m$-brace $\mu_n : (A_n^m)^m \to A_n^1$ satisfies the conditions of the lemma, then using $E_1, \ldots, E_m$, the Leibnitz rule and (3.1.1), we obtain that $\mu_n(y_1, \ldots, y_m) = F(E_1, \ldots, E_m)$ where $F(E_1, \ldots, E_m)$ is a sum of products determined by the summands on the right-hand sides of the expansions $E_1, \ldots, E_m$. Each product involves factors of 3 types:

(I) the coefficients $c \in R$ appearing in the summands in question;

(II) the traces $\tr(x_j^y) \in A_n^1$ where $j = 1, \ldots, m$ and $i$ runs over the indices $1, \ldots, r = r_j$ determined by the summand of $E_j$ except one of these indices, say, $i_j$;

(III) the factor $\tr(\mu(x_{i_j}^y))$.

We claim that the element $F = F(E_1, \ldots, E_m)$ of $A_n^1$ does not depend on the choice of the expansion $E_m$ of $y_m$. It is easy to reduce this claim to its special case where $y_j = \tr(x_j)$ for $j = 1, \ldots, m - 1$ and $x_j \in \hat{A}$. (It is understood that we keep $E_m$ and use the formula $y_j = \tr(x_j)$ as the expansion $E_j$ for $j \leq m - 1$). Consider the projection $p : A \to \hat{A} = A/A'$. By the assumptions of the lemma, there is a derivation $d : A \to A$ (possibly, depending on $x_1, \ldots, x_{m-1}$) such that

$$\mu(x_1, \ldots, x_{m-1}, p(a)) = p(d(a)) \in \hat{A} \quad \text{for all } a \in A.$$
For any $x \in \hat{A}$ and $a \in p^{-1}(x) \subset A$, we have the following equalities in $A'_n$:
\[
\text{tr}(\mu(x_1,\ldots,x_{m-1},x)) = \text{tr}(\mu(x_1,\ldots,x_{m-1},p(a)))
\]
\[
= \text{tr}(p(d(a))) = \text{tr}(d(a)) = \sum_{i=1}^{n} (d(a))_{ii} = \sum_{i=1}^{n} \tilde{d}(a_{ii}) = \tilde{d}(\text{tr}(a)) = \tilde{d}(\text{tr}(x)).
\]
Using this formula to compute all factors of type (III) above, we easily deduce that $F = d(y)$. Since $\tilde{d}(y)$ does not depend on the choice of $E_m$, neither does $F$.

Coming back to arbitrary $y_1,\ldots,y_m \in A'_n$, we similarly prove that the element $F = F(E_1,\ldots,E_m)$ of $A'_n$ does not depend on the choice of the expansion $E_i$ for all $i = 1,\ldots,m$. In other words, $F$ depends only on $y_1,\ldots,y_m$. We take $F$ as $\mu_n(y_1,\ldots,y_m)$. The resulting map $\mu_n : (A'_n)^m \to A'_n$ is easily seen to be an $m$-brace in $A'_n$ and to satisfy (3.1.1).

3.2. Brackets. Given an integer $m \geq 1$, an $m$-bracket in a module $M$ is a map $\mu : M^m \to M$ which is linear in every variable. For $\varepsilon \in R$, we say that $\mu$ is $\varepsilon$-symmetric if for all $x_1,\ldots,x_m \in M$,
\[
\mu(x_1,\ldots,x_{m-1},x_m) = \varepsilon \mu(x_m,x_1,\ldots,x_{m-1}).
\]
If $\varepsilon = +1$, then $\varepsilon$-symmetric brackets are said to be cyclically symmetric. If $\varepsilon = -1$, then $\varepsilon$-symmetric brackets are said to be skew-symmetric.

Lemma 3.2. Given an algebra $A$, integers $m,n \geq 1, \varepsilon \in R$, and an $\varepsilon$-symmetric $m$-bracket $\mu$ in $A$ which is a weak derivation in the $m$-th variable, there is a unique $m$-brace $\mu_n$ in the algebra $A'_n$ such that the trace $\text{tr} : \hat{A} \to A'_n$ carries $\mu$ to $\mu_n$. The brace $\mu_n$ is $\varepsilon$-symmetric.

Proof. Since $\mu$ is $\varepsilon$-symmetric and is a weak derivation in one variable, it is a weak derivation in all variables. Thus, $\mu$ is a brace. By Lemma 3.1 there is a unique $m$-brace $\mu_n$ in $A'_n$ such that the trace carries $\mu$ to $\mu_n$. The $\varepsilon$-symmetry of $\mu$ implies that $\mu_n$ is $\varepsilon$-symmetric.

4. Braces in group algebras

In this section, $A = R[\pi]$ is the group algebra of a group $\pi$. We construct braces in $A$ starting from Fox derivatives in $A$.

4.1. Computation of $\hat{A}$. By definition, the module $A' \subset A$ is generated by the set $\{uw - vu \mid u, v \in A\}$. Since $\pi \subset A$ generates $A$, the module $A'$ is generated by the set $\{uw - vu \mid u, v \in \pi\}$. Since $uw = u(vu)u^{-1}$ for $u, v \in \pi$, the module $A'$ is generated by the set $\{uwu^{-1} - u \mid u, w \in \pi\}$. Thus, $\hat{A} = A/A' = R\tilde{\pi}$ is the free module whose basis $\tilde{\pi}$ is the set of conjugacy classes of elements of $\pi$.

4.2. Fox derivatives. A (left) Fox derivative in $A$ is a linear map $\partial : A \to A$ such that $\partial(xy) = \partial(x) + x\partial(y)$ for all $x, y \in \pi \subset A$. For any $x, y \in A$, we have then $\partial(xy) = \partial(x) \text{aug}(y) + x\partial(y)$ where $\text{aug} : A \to R$ is the linear map carrying all elements of $\pi$ to $1 \in R$. For $x \in \pi$, we can uniquely expand $\partial(x) = \sum_{a \in \pi} (x/a)\partial a$ where $(x/a)\partial \in R$ is non-zero for a finite set of $a$. Consider the map
\[
\pi \to A, \quad x \mapsto \sum_{a \in \pi} (x/a)\partial a^{-1}xa
\]
and denote its linear extension $A \to A$ by $\Delta_{\partial}$.

Lemma 4.1. $\Delta_{\partial}(A') = 0$. 
Proof. It suffices to prove that $\Delta_\partial(xy-xy) = 0$ for any $x, y \in \pi$. We have

$$\partial(xy) = \partial(x) + x\partial(y) = \sum_{a \in \pi} ((x/a)\partial a + (y/a)\partial xa).$$

Therefore, by the definition of $\Delta_\partial$,

$$\Delta_\partial(xy) = \sum_{a \in \pi} ((x/a)\partial a^{-1}xya + (y/a)\partial (xa)^{-1}xy(xa))$$

$$= \sum_{a \in \pi} ((x/a)\partial a^{-1}xya + (y/a)\partial a^{-1}yxa).$$

The latter expression is invariant under the permutation $x \leftrightarrow y$. So, $\Delta_\partial(xy) = \Delta_\partial(xy) + \Delta_\partial(xy - yx) = 0$. \hfill \Box

The linear map $\tilde{A} = A/A' \to A$ induced by $\Delta_\partial : A \to A$ is denoted by $\tilde{\Delta}_\partial$.

**Theorem 4.2.** Let $p : A \to \tilde{A}$ be the projection. For any $m \geq 1$ and any Fox derivatives $\partial_1, \ldots, \partial_m : A \to A$, the map $\mu^m : \tilde{A}^m \to A$ defined by

$$\mu^m(x_1, \ldots, x_m) = p(\tilde{\Delta}_{\partial_1}(x_1) \cdots \tilde{\Delta}_{\partial_m}(x_m))$$

for $x_1, \ldots, x_m \in \tilde{A}$ is an $m$-brace in $A$.

Proof. We need to prove that $\mu^m$ is a weak derivation in all variables, i.e., for any $i = 1, \ldots, m$ and $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_m \in \tilde{A}$, the map

$$\tilde{A} \to \tilde{A}, \ x \mapsto \mu^m(x_1, \ldots, x_{i-1}, x, x_{i+1}, \ldots, x_m)$$

is induced by a derivation in $A$. Set

$$G = \tilde{\Delta}_{\partial_1}(x_1) \cdots \tilde{\Delta}_{\partial_{i-1}}(x_{i-1}) \in A \quad \text{and} \quad H = \tilde{\Delta}_{\partial_{i+1}}(x_{i+1}) \cdots \tilde{\Delta}_{\partial_m}(x_{im}) \in A.$$

For $x \in \pi$, we expand $\partial_i(x) = \sum_{a \in \pi} (x/a)a$ with $(x/a) = (x/a)\partial_i$. Then

$$\mu^m(x_1, \ldots, x_{i-1}, x, x_{i+1}, \ldots, x_m) = p(G \tilde{\Delta}_\partial(x)H)$$

$$= p(G \sum_{a \in \pi} (x/a)a^{-1}xaH) = p(G \sum_{a \in \pi} (x/a)Ga^{-1}xaH) = p(G \sum_{a \in \pi} (x/a)HaGa^{-1}x)$$

where we use that $p(Ga^{-1}xaH) = p(aHaGa^{-1}x)$. Thus, the map (4.2.2) is induced by the linear map $A \to A$ carrying any $x \in \pi$ to $\sum_{a \in \pi} (x/a)aHaGa^{-1}x$. It remains to prove that for any $F \in A$, the linear map $d = d_F : A \to A$ carrying any $x \in \pi$ to $\sum_{a \in \pi} (x/a)aFa^{-1}x$ is a derivation. Indeed, for $x, y \in \pi$, we have

$$d(x) = \sum_{a \in \pi} (x/a)aFa^{-1}x \quad \text{and} \quad d(y) = \sum_{a \in \pi} (y/a)aFa^{-1}y.$$ 

Also,

$$\partial_i(xy) = \partial_i(x) + x\partial_i(y) = \sum_{a \in \pi} ((x/a)a + (y/a)xa)$$

and so

$$d(xy) = \sum_{a \in \pi} ((x/a)aFa^{-1}xy + (y/a)xaF(xa)^{-1}xy) = d(x)y + xd(y).$$

Thus, $d$ is a derivation in $A$. This completes the proof of the theorem. \hfill \Box
For $m = 1$, Theorem 4.2 may be rephrased by saying that for any Fox derivative $\partial$ in $A$, the linear map $\Delta : A \to A$ induced by $\Delta \partial : A \to A$ is also induced by a derivation $\partial = d_\partial : A \to A$. This derivation carries any $x \in \pi$ to $\sum_{a \in \pi} (x/a)_\partial x = \text{aug}(\partial(x))x$. In contrast to $\Delta \partial$, the derivation $d_\partial$ may not annihilate $A'$.

Combining Theorem 4.2 with Lemma 3.1 we obtain the following.

**Corollary 4.3.** For any integers $m, n \geq 1$ and Fox derivatives $\partial_1, \ldots, \partial_m$ in $A$, there is a unique $m$-brace $\mu^m_n$ in $A^t$ such that for all $x_1, \ldots, x_m \in A$, we have

$$\mu^m_n(tr(x_1), \ldots, tr(x_m)) = tr(\Delta \partial_1(x_1) \cdots \Delta \partial_m(x_m)).$$

If $\partial_1 = \cdots = \partial_m$, then the $m$-braces $\mu^m$ and $\mu^m_n$ are cyclically symmetric. This follows from the identities $p(xy) = p(yx)$ and $tr(xy) = tr(yx)$ for all $x, y \in A$.

**4.3. Equivalence of Fox derivatives.** Given a Fox derivative $\partial$ in $A$ and any $g \in \pi$, the linear map $A \to A, x \mapsto (\partial \cdot g) x$ is also a Fox derivative denoted $\partial \cdot g$. We say that two Fox derivatives $\partial, \partial'$ in $A$ are equivalent if there is $g \in \pi$ such that $\partial' = \partial \cdot g$. This is indeed an equivalence relation. Moreover, equivalent Fox derivatives induce the same braces in $A$ and $A^t$. This follows from the identities

$$\Delta \partial \cdot g(x) = g^{-1}\Delta \partial(x) g, \quad p(g^{-1}xg) = p(x), \quad tr(g^{-1}xg) = tr(x)$$

for all $x \in A$ and $g \in \pi$.

5. **Quasi-Lie brackets andbrace algebras**

We define quasi-Lie brackets and brace algebras.

**5.1. Quasi-Lie brackets.** A quasi-Lie pair of brackets in a module $M$ is a pair formed by a skew-symmetric 2-bracket $[-, -]$ in $M$ and a cyclically symmetric 3-bracket $[-, -]$ in $M$ such that for any $x, y, z \in M$, we have

$$[[x, y], z] + [[y, z], x] + [[z, x], y] = [x, y, z] - [y, x, z].$$

Here the left-hand side is the usual Jacobiator of the 2-bracket. Both sides of (5.1.1) are cyclically symmetric. We call (5.1.1) the quasi-Jacobi identity. For the zero 3-bracket, we recover the standard Jacobi identity.

**5.2. Examples.** 1. Any bilinear pairing $M^2 \to M, (x, y) \mapsto xy$ induces a quasi-Lie pair of brackets in $M$ with the 2-bracket $[x, y] = xy - yx$ and the 3-bracket

$$[x, y, z] = (xy)z + (yz)x + (zx)y - x(yz) - y(zx) - z(xy)$$

for $x, y, z \in M$.

2. For a quasi-Lie pair of brackets $[-, -], [-, -]$ in a module $M$ and a 3-bracket $b$ in $M$ invariant under all permutations of the variables, the pair $[-, -], [-, -], + b$ is also a quasi-Lie pair.

**5.3. Brace algebras.** A brace algebra is an algebra $A$ endowed with a quasi-Lie pair of brackets in the module $A$ such that both these brackets are braces in $A$ in the sense of Section 3.1. For example, a commutative brace algebra with zero 3-bracket is a Poisson algebra in the usual sense.

A brace homomorphism from a brace algebra $A$ to a brace algebra $B$ is a bracket-preserving linear map $f : A \to B$. Thus, $f$ should satisfy $[f(x), f(y)] = f([x, y])$ and $[f(x), f(y), f(z)] = f([x, y, z])$ for all $x, y, z \in A$. 
Lemma 5.1. Let $A$ be a commutative algebra carrying a skew-symmetric 2-brace $[-,-]$ and a cyclically symmetric 3-brace $[-,-,-]$. If \((5.1.1)\) holds for all elements of a generating set of $A$, then it holds for all elements of $A$.

Proof. Let $L(x,y,z)$ and $R(x,y,z)$ be respectively the left and the right hand-sides of \((5.1.1)\). Since $L(x,y,z)$ and $R(x,y,z)$ are linear in $x,y,z$ and cyclically symmetric, it suffices to verify the following: if \((5.1.1)\) holds for the triples $x,y,z \in A$ and $x,y,z \in A$, then it holds for the triple $x,y,z$. Since $[-,-]$ is a brace, 

\[
[x,y],z] = z[[x,y],t] + [[x,y],z]t,
\]

\[
[y,zt],x] = [z[y,t],x] + [[y,z],x]t,
\]

\[
= z[[y,t],x] + [z,x][y,t] + [y,z][t,x] + [[y,z],x]t.
\]

Similarly,

\[
[zt,y] = z[[t,x],y] + [z,y][t,x] + [z,x][t,y] + [[z,x],y]t.
\]

Adding these three expansions and using the skew-symmetry of $[-,-]$, we get

\[
L(x,y,z) = zL(x,y,t) + L(x,y,z)t.
\]

Thus, $L$ satisfies the Leibnitz rule in the last variable. Since the bracket $[-,-,-]$ also satisfies this rule, so does $R(x,y,z) = [x,y,z] - [y,x,z]$. Consequently, if \((5.1.1)\) holds for the triples $x,y,z$ and $x,y,z$, then it holds for the triple $x,y,z$. \hfill \Box

Recall the trace algebras $\{A_n\}_{n \geq 1}$ associated with any algebra $A$.

Theorem 5.2. For any brace algebra $A$ and integer $n \geq 1$, there is a unique brace algebra structure on $A_n$ such that $tr : \hat{A} \rightarrow A_n$ is a brace homomorphism.

Proof. Let $[-,-]$ and $[-,-,-]$ be the brackets in $\hat{A}$ forming a quasi-Lie pair. By Lemma 3.2, there are unique braces $[-,-]^t$ and $[-,-,-]^t$ in $A_n$ such that

\[
[tr(x),tr(y)]^t = tr([x,y]) \quad \text{and} \quad [tr(x),tr(y),tr(z)]^t = tr([x,y,z])
\]

for all $x,y,z \in \hat{A}$. Then \((5.1.1)\) holds for all elements of the set $tr(\hat{A}) \subset A_n$. Since this set generates $A_n$, Lemma 5.1 implies that \((5.1.1)\) holds for all elements of $A_n$. Also, since $[-,-]$ is $(-1)$-symmetric and $[-,-,-]$ is 1-symmetric, so are the braces $[-,-]^t$ and $[-,-,-]^t$. Thus, these braces form a quasi-Lie pair. This turns $A_n$ into a brace algebra satisfying the conditions of the theorem. \hfill \Box

5.4. Remark. A bracket in a module is fully symmetric if it is invariant under all permutations of the variables. A quasi-Lie pair of brackets in a module $M$ gives rise to a fully symmetric 3-bracket $s : M^3 \rightarrow M$ by

\[
(5.4.1) \quad s(x,y,z) = 2[x,y,z] - [[x,y],z] - [[y,z],x] - [[z,x],y]
\]

for any $x,y,z \in M$. The cyclic symmetry of $s$ is obvious and the invariance of $s(x,y,z)$ under the permutation $x \leftrightarrow y$ follows from \((5.1.1)\). Conversely, if $2$ is invertible in $R$, then we can recover the 3-bracket from $[-,-]$ and $s$ via \((5.4.1)\). Formula \((5.1.1)\) follows then from the identity $s(x,y,z) = s(y,x,z)$. This establishes a bijective correspondence between quasi-Lie pairs of brackets in $M$ and pairs (a skew-symmetric 2-bracket in $M$, a fully symmetric 3-bracket in $M$).

6. Topological gates

We define gates in topological spaces and show how they give rise to braces.
6.1. **Gates.** A cylinder neighborhood of a subset \( C \) of a topological space \( X \) is a pair consisting of a closed set \( U \subset X \) with \( C \subset U \) and a homeomorphism \( U \cong C \times [-1, 1] \) carrying \( C \) onto \( C \times \{0\} \) and carrying \( \text{Int}(U) \) onto \( C \times (-1, 1) \). Note that then \( C \subset \text{Int}(U) \) and \( C \) is closed in \( X \). A gate in \( X \) is a path-connected subspace \( C \subset X \) endowed with a cylinder neighborhood in \( X \) and such that all loops in \( C \) are contractible in \( X \). An example of a gate is provided by a simply connected codimension 1 proper submanifold \( C \) of a manifold together with a suitable homeomorphism of a closed neighborhood of \( C \) onto \( C \times [-1, 1] \).

For the rest of this section, we fix a path-connected topological space \( X \), a gate \( C \subset X \), and its cylinder neighborhood \( U \subset X \) which we identify with \( C \times [-1, 1] \) so that \( C = C \times \{0\} \). Pick a point \( * \in X \setminus U \) and set \( \pi = \pi_1(X, *) \).

6.2. **Gate derivatives.** Here we associate with the gate \( C \) an equivalence class of Fox derivatives in the algebra \( A = R[\pi] \). We start with preliminaries on (continuous) paths. Let \( q : X \to S^1 = \{ z \in \mathbb{C} | |z| = 1 \} \) be the map which carries \( C \times \{t\} \subset U \) to \( \exp(\pi it) \in S^1 \) for all \( t \in [-1, 1] \) and carries \( X \setminus U \) to \( -1 \in S^1 \). We say that a path \( a : [0, 1] \to X \) is transversal to \( C \) if \( a(0), a(1) \in X \setminus C \) and the map \( qa : [0, 1] \to S^1 \) restricted to \( (0, 1) \) is transversal to \( 1 \in S^1 \). Then \( a^{-1}(C) = (qa)^{-1}(1) \) is a finite subset of \( (0, 1) \). For a path \( a \), we denote the inverse path by \( \overline{a} \). A path \( a : [0, 1] \to X \) is a loop based in \( * \) if \( a(0) = a(1) = * \). Such a loop \( a \) represents an element of \( \pi \) denoted \( [a] \).

Pick a path \( \gamma : [0, 1] \to X \) such that \( \gamma(0) = * \) and \( \gamma(1) \in C \). Consider a loop \( a : [0, 1] \to X \) based in \( * \) and transversal to \( C \). For \( t \in a^{-1}(C) \subset (0, 1) \), we let \( a_t^{-1}(C) \) be the path in \( X \) obtained as the product of the path \( a|_{[0,t]} \) with any path \( \beta \in C \) from \( a(t) \) to \( \gamma(1) \), and finally with \( \overline{\gamma} \). Then \( a_t^{-1}(C) \) a loop based in \( * \). Since all loops in \( C \) are contractible in \( X \), the homotopy class \( [a_t^{-1}(C)] \) of \( a_t^{-1}(C) \) does not depend on the choice of \( \beta \). Set \( \varepsilon_t(a) = 1 \) if at \( a(t) \in C \) the loop \( a_t^{-1}(C) \) crosses \( C \) upwards (i.e., from \( C \times [-1, 0] \) to \( C \times (0, 1) \)), and \( \varepsilon_t(a) = -1 \) otherwise. Set

\[
(6.2.1) \quad \partial_C^\varepsilon(a) = \sum_{t \in a^{-1}(C)} \varepsilon_t(a) [a_t^{-1}(C)] \in R[\pi] = A.
\]

**Lemma 6.1.** Formula (6.2.1) defines a map \( \pi \to A \) whose linear extension \( A \to A \), denoted \( \partial_C^* \), is a Fox derivative. If \( \gamma' : [0, 1] \to X \) is another path from \( * \) to \( C \), then the Fox derivatives \( \partial_C^\varepsilon \) and \( \partial_C^\gamma' \) are equivalent in the sense of Section 4.3.

**Proof.** It is clear that all elements of \( \pi \) can be represented by loops based in \( * \) and transversal to \( C \). We claim that if two such loops \( a, a' \) are homotopic, then \( \partial_C^\varepsilon(a) = \partial_C^\varepsilon(a') \). There is a homotopy \( (a_u)_{u \in [0, 1]} \) from \( a = a_0 \) to \( a' = a_1 \) such that the loop \( a_u \) is based in \( * \) and transversal to \( C \) except for a finite set of \( u \in (0, 1) \) near which the homotopy pushes a branch of \( a_u \) across \( C \) creating or destroying a pair of transversal crossings with \( C \). It is easy to see that the contributions of these two crossings to \( \partial_C^\varepsilon(a_u) \) cancel each other. Therefore, Formula (6.2.1) yields a well-defined map \( \pi \to A \) which extends by linearity to a map \( \partial_C^\varepsilon : A \to A \).

If \( a, b \) are loops in \( X \) based in \( * \) and transversal to \( C \), then so is their product, and it follows directly from the definitions that \( \partial_C^\varepsilon(ab) = \partial_C^\varepsilon(a) + a\partial_C^\varepsilon(b) \). Consequently, \( \partial_C^\varepsilon \) is a Fox derivative in \( A \).

Given two paths \( \gamma, \gamma' \) from \( * \) to \( C \), we let \( g \in \pi \) be the homotopy class of the loop obtained as the product of \( \gamma \) with a path in \( C \) from \( \gamma(1) \in C \) to \( \gamma'(1) \in C \),
and with $\gamma$. It is easy to see that $\partial_C' = \partial_C \cdot g$. Thus, the Fox derivatives $\partial_C'$ and $\partial_C^\ast$ are equivalent. \hfill $\square$

6.3. Gate braces. Let $L = L(X)$ be the set of free homotopy classes of loops in $X$ and let $RL$ be the free module with basis $L$. The map $\pi \to L$ carrying the homotopy classes of loops to their free homotopy classes induces a bijection $\bar{\pi} \approx L$ where $\bar{\pi}$ is the set of conjugacy classes of elements of $\pi$. By Section 4.1 $A = A'/R = R\bar{\pi}$ so that we can identify $A$ with $RL$.

By Lemma 6.1 a sequence of $m \geq 1$ gates in $X \setminus \{\ast\}$ (not necessarily disjoint or distinct) determines a sequence of $m \geq 1$ equivalence classes of Fox derivatives in $A$. By Section 4 the latter induces $m$-braces in the algebras $A$ and $\{A_n\}_{n \geq 1}$. In particular, the sequence of $m \geq 1$ copies of a gate $C \subset X$ determines a cyclically symmetric $m$-brace $\mu^m_C : A^m \to \check{A}$ in $A$. We compute $\mu^m_C$ in geometric terms as follows. Consider $m$ loops $a_1, \ldots, a_m : [0, 1] \to X$ based in $\ast$ and transversal to $C$. Pick a point $\ast \in C$. For $i = 1, \ldots, m$ and $t \in \gamma_i^{-1}(C) \subset (0, 1)$, let $a_{i,t}$ be the loop based in $\ast$ and obtained as the product of a path $\beta_{i,t}$ in $C$ from $\ast$ to $a_i(t) \in C$, the loop $a_i|_{[t,1]} a_i|_{[0,t]}$ based in $a_i(t)$, and the path $\beta_{i,t}$. In the next lemma, the free homotopy class of a loop $b$ in $X$ is denoted $\langle b \rangle$.

Lemma 6.2. Under the assumptions above,

\begin{equation}
(6.3.1) \quad \mu^m_C(\langle a_1 \rangle, \ldots, \langle a_m \rangle) = \sum_{t_i \in a_i^{-1}(C), \ldots, t_m \in a_m^{-1}(C)} \prod_{i=1}^m \varepsilon_{t_i}(a_i) \langle \prod_{i=1}^m a_{i,t_i} \rangle \in RL.
\end{equation}

Proof. We claim that both sides of (6.3.1) are preserved when the loop $a_1$ is replaced by the loop $a'_1 = ba_1\bar{b}$ for a path $\bar{b} : [0, 1] \to X$ transversal to $C$ and such that $\bar{b}(1) = a_1(0)$. The invariance of the left-hand side is obvious since $\langle a'_1 \rangle = \langle a_1 \rangle$. We prove the invariance of the right-hand side which we denote by $\sigma(a_1, \ldots, a_m)$. If the path $b$ misses $C$, then the claim is obvious because the loops $a_1$ and $a'_1$ meet $C$ in the same points which contribute the same to $\sigma(a_1, \ldots, a_m)$ and $\sigma(a'_1, \ldots, a_m)$. Otherwise, the path $b$ expands as a product of a finite number of paths each of which is transversal to $C$ and intersects $C$ in one point. Thus, it suffices to prove our claim in the case where $b$ intersects $C$ in one point, say, $c$. Then $a'_1 = ba_1\bar{b}$ meets $C$ at the crossings of $a_1$ with $C$ and two additional crossings at the point $c$ which is traversed first by $b$ and then by $\bar{b}$. Let $0 < u < w < 1$ be the corresponding values of the parameter, so that $a'_1(u) = c = a'_1(w)$ are respectively the first and the last crossings of $a'_1$ with $C$. It is easy to see that $\varepsilon_u(a'_1) = -\varepsilon_w(a'_1)$ and the corresponding loops $a'_{1,u}$ and $a'_{1,w}$ are homotopic. Therefore the terms of $\sigma(a'_1, \ldots, a_m)$ associated with $t_1 = u$ and $t_1 = w$ cancel each other, while the remaining terms yield $\sigma(a_1, \ldots, a_m)$.

Thus, $\sigma(a_1, \ldots, a_m) = \sigma(a'_1, \ldots, a_m)$.

Replacing $a_1$ by $ba_1\bar{b}$ for a path $b$ transversal to $C$ and running from $\ast$ to $a_1(0)$, we obtain a loop transversal to $C$ and based in $\ast$. By the previous paragraph, it suffices to prove (6.3.1) with this new loop instead of $a_1$. By a similar argument, it suffices to prove (6.3.1) in the case where all the loops $a_1, \ldots, a_m$ are based in $\ast$.

Now, pick a path $\gamma : [0, 1] \to X$ from $\gamma(0) = \ast$ to $\gamma(1) = \ast \in C$. By the definition of the Fox derivative $\partial = \partial_C^\ast$, for any $i = 1, \ldots, m$,

$$\partial(\langle a_i \rangle) = \sum_{t \in a_i^{-1}(C)} \varepsilon_t(a_i) [\eta_{i,t}] \in R[\pi] = A$$
where \( \eta_{i,t} = a_i |_{[0,t]} \beta_{i,t} \gamma \). Therefore

\[
\Delta_\partial([a_i]) = \sum_{t \in a_i^{-1}(C)} \varepsilon_t(a_i) [\eta_{i,t} a_i \eta_{i,t}] = \sum_{t \in a_i^{-1}(C)} \varepsilon_t(a_i) [\gamma \beta_{i,t} a_i |_{[0,t]} a_i a_i |_{[0,t]} \beta_{i,t} \gamma]
\]

\[
= \sum_{t \in a_i^{-1}(C)} \varepsilon_t(a_i) [\gamma \beta_{i,t} a_i |_{[1,t]} a_i |_{[0,t]} \beta_{i,t} \gamma] = \sum_{t \in a_i^{-1}(C)} \varepsilon_t(a_i) [\gamma a_i \beta_{i,t} \gamma].
\]

Setting \( x_i = \langle a_i \rangle \in \pi \) for \( i = 1, \ldots, m \) and substituting in Formula (4.2.1) the above expression for \( \Delta_\partial(x_i) = \Delta_\partial([a_i]) \), we obtain a formula equivalent to (6.3.1). \( \square \)

6.4. The dual map. The gate \( C \subset X \) determines a linear map \( v_C : H_1(X; R) \to R \) “dual” to \( C \). This map carries the homology class \( [a] \in H_1(X; R) \) of a loop \( a : [0,1] \to X \) transversal to \( C \) to \( \sum_{t \in a^{-1}(C)} \varepsilon_t(a) \). It is clear that the sum of the coefficients of the expression (6.3.1) is equal to \( \prod_{i=1}^m v_C([a_i]) \).

7. Quasi-surfaces

7.1. Generalities. By a surface we mean a smooth 2-dimensional manifold with boundary. A quasi-surface is a topological space \( X \) obtained by gluing a surface \( \Sigma \) to a topological space \( Y \) along a continuous map \( f : \alpha \to Y \) where \( \alpha \subset \partial \Sigma \) is a union of a finite number of disjoint segments in \( \partial \Sigma \). Note that \( Y \subset X \) and \( X \setminus Y = \Sigma \setminus \alpha \). Here we impose no conditions on \( Y \) and do not require \( \Sigma \) to be compact or connected or maximal among surfaces in \( X \).

The quasi-surface \( X \) has path-connected components of 3 types: (i) components of \( \Sigma \) disjoint from \( \alpha \); (ii) path-connected components of \( Y \) disjoint from \( f(\alpha) \subset Y \); (iii) path-connected components of \( X \) meeting both \( \Sigma \) and \( Y \). For components of type (i) our results below are standard in the topology of surfaces. For components of type (ii), all our operations are identically zero. The novelty of this work concerns the components of type (iii).

7.2. Examples. In the following examples, \( \Sigma \) is a surface.

1. When \( \alpha \) is a family of \( m \geq 1 \) disjoint segments in \( \partial \Sigma \), the unique map from \( \alpha \) to a 1-point space determines a quasi-surface. For \( m = 1 \), it is a copy of \( \Sigma \). As a consequence, any surface with non-void boundary is a quasi-surface.

2. Given \( m \geq 1 \) disjoint finite subsets of \( \partial \Sigma \), we obtain a quasi-surface by collapsing each of these subsets into a point. Here \( Y \) is an \( m \)-point set with discrete topology and \( \alpha \) is a small closed neighborhood in \( \partial \Sigma \) of the union of our finite sets.

3. Given \( m \geq 1 \) disjoint segments \( \alpha_1, \ldots, \alpha_m \) in \( \partial \Sigma \) and \( m \) points \( y_1, \ldots, y_m \) in a topological space \( Y \), we obtain a quasi-surface by gluing \( \Sigma \) to \( Y \) along the map carrying \( \alpha_k \) to \( y_k \) for \( k = 1, \ldots, m \).

4. Let \( \Sigma_0 \) be a surface with boundary and let \( \alpha \subset \Sigma_0 \) be a union of a finite number of disjoint proper embedded segments in \( \Sigma_0 \). Suppose that \( \alpha \) splits \( \Sigma_0 \) into two subsurfaces (possibly disconnected) \( \Sigma \subset \Sigma_0 \) and \( \Sigma' \subset \Sigma_0 \) so that \( \alpha = \Sigma \cap \Sigma' = \partial \Sigma \cap \partial \Sigma' \). Then \( \Sigma_0 \) is homeomorphic to the quasi-surface determined by the tuple \( (\Sigma, \alpha \subset \partial \Sigma, Y = \Sigma', f) \) where \( f : \alpha \to \Sigma' \) is the inclusion.

7.3. Conventions. Fix for the rest of the paper a tuple \( X, Y, \Sigma, \alpha, f \) as in Section 7.1. We assume that \( X \) is path-connected, \( \alpha \neq \emptyset \), and \( \Sigma \) is oriented. We will identify a closed neighborhood of \( \alpha \) in \( \Sigma \) with \( \alpha \times [-2,1] \) so that \( \alpha = \alpha \times \{-2\} \) and

\[
\partial \Sigma \cap (\alpha \times [-2,1]) = (\alpha \times \{-2\}) \cup (\partial \alpha \times [-2,1]).
\]
We will often use the surface

$$\Sigma' = \Sigma \setminus (\alpha \times [-2, 0]) \subset \Sigma \setminus \alpha \subset X$$

which is a copy of $\Sigma$ embedded in $X$. It is called the surface part of $X$. We provide $\Sigma'$ with the orientation induced from that of $\Sigma$.

For $k \in \pi_0(\alpha)$, denote by $\alpha_k^\circ$ the corresponding component of $\alpha \subset \partial \Sigma$. Set

$$\alpha_k = \alpha_k^\circ \times \{0\} \subset \partial \Sigma' \subset X.$$  

Clearly, $\alpha_k$ is an embedded segment in $X$. Endowing $\alpha_k$ with the cylinder neighborhood $\alpha_k^\circ \times [-1, 1] \subset \Sigma \setminus \alpha \subset X$ we turn $\alpha_k$ into a gate in $X$ in the sense of Section 6.1. This is the $k$-th gate of $X$. The gates $\{\alpha_k\}_k$ split $X$ into the surface part $\Sigma'$ and the singular part which is the mapping cylinder of the gluing map $f : \alpha \to Y$. All paths from a point of $\Sigma'$ to a point of $X \setminus \Sigma'$ have to cross a gate.

7.4. Gate orientations. A gate orientation of $X$ is an orientation of all the gates $\{\alpha_k\}_k$ of $X$. Gate orientations of $X$ canonically correspond to orientations of the 1-manifold $\alpha \subset \partial \Sigma$. Given a gate orientation $\omega$ of $X$ and points $p, q \in \alpha_k$, we say that $p$ lies on the $\omega$-left of $q$ and $q$ lies on the $\omega$-right of $p$ if $p \neq q$ and the $\omega$-orientation of $\alpha_k$ leads from $p$ to $q$. We write then $p <_\omega q$ or $q >_\omega p$. We set $\varepsilon(\omega, k) = +1$ if the $\omega$-orientation of $\alpha_k$ is compatible with the orientation of $\Sigma$, i.e., if the pair $(\alpha_k \subset \partial \Sigma', \text{a vector directed inside } \Sigma')$ is positively oriented in $\Sigma$. Otherwise, $\varepsilon(\omega, k) = -1$. Also, we let $k\omega$ be the gate orientation obtained from $\omega$ by inverting the direction of $\alpha_k$ while keeping the directions of the other gates. We let $\nabla$ denote the gate orientation of $X$ opposite to $\omega$ on all gates.

7.5. Generic loops. In the rest of the paper by a loop in $X$ we mean a circular loop, i.e., a continuous map $\alpha : S^1 \to X$. The intersection of the set $a(S^1)$ with the $k$-th gate $\alpha_k \subset X$ is denoted $a \cap \alpha_k$. A generic loop $a$ in $X$ is a loop in $X$ such that (i) all branches of $a$ in $\Sigma'$ are smooth immersions meeting $\partial \Sigma'$ transversely at a finite set of points lying in the interior of the gates, and (ii) all self-intersections of $a$ in $\Sigma'$ are double transversal intersections lying in $\text{Int}(\Sigma') = \Sigma' \setminus \partial \Sigma'$. The set of self-intersections in $\Sigma'$ (= double points) of a generic loop $a$ is denoted by $\# a$. This set is finite and lies in $\text{Int}(\Sigma')$.

A generic loop $a$ in $X$ never traverses a point of a gate $\alpha_k$ more than once, and the set $a \cap \alpha_k$ is finite. The sign $\varepsilon_p(a)$ of $a$ at a point $p \in a \cap \alpha_k$ is $+1$ if $a$ goes near $p$ from $X \setminus \Sigma'$ to $\text{Int}(\Sigma')$ and $-1$ otherwise.

We define six local moves $L_0 - L_5$ on a generic loop $a$ in $X$ keeping its free homotopy class. The move $L_0$ is a deformation of $a$ in the class of generic loops. This move preserves the number $\# a$. The moves $L_1 - L_3$ modify $a$ in a small disk in $\text{Int}(\Sigma')$ and are modeled on the Reidemeister moves on knot diagrams (with over/under-data dropped). The move $L_1$ adds a small curl to $a$ and increases $\# a$ by 1. The move $L_2$ pushes a branch of $a$ across another branch of $a$ increasing $\# a$ by 2. The move $L_3$ pushes a branch of $a$ across a double point of $a$ keeping $\# a$. The move $L_4$ pushes a branch of $a$ across a gate keeping $\# a$. The move $L_5$ pushes a double point of $a$ across a gate decreasing $\# a$ by 1. Graphically, the moves $L_4, L_5$ are similar to $L_2, L_3$. We call the moves $L_0 - L_5$ and their inverses loop moves. It is clear that generic loops in $X$ are freely homotopic if and only if they can be related by a finite sequence of loop moves.
A finite family of loops in $X$ is generic if all these loops are generic and all their mutual crossings in $\Sigma'$ are double transversal intersections in $\text{Int}(\Sigma')$. In particular, these loops can not meet at the gates. We will use the following notation. For generic loops $a, b$ in $X$, consider the set of triples

$$T(a, b) = \{(k, p, q) \mid k \in \pi_0(\alpha), p \in a \cap \alpha_k, q \in b \cap \alpha_k, p \neq q\}.$$ 

Given a gate orientation $\omega$ of $X$, we define a set $T_\omega(a, b) \subset T(a, b)$ by

$$T_\omega(a, b) = \{(k, p, q) \in T(a, b) \mid q <_\omega p\}.$$

Clearly,

$$T(a, b) \setminus T_\omega(a, b) = \{(k, p, q) \in T(a, b) \mid p <_\omega q\} = \overline{T_\omega(a, b)}.$$

8. Homological intersection forms

As a prelude to more sophisticated operations, we define here intersection forms in 1-homology of $X$.

8.1. First homological intersection form. Given a gate orientation $\omega$ of $X$, we define a bilinear form

$$(8.1.1) \quad \omega : H_1(X; \mathbb{R}) \times H_1(X; \mathbb{R}) \to \mathbb{R}$$

called the first homological intersection form of $X$. The idea is to properly position the loops near the gates and then to count intersections of the loops in the surface part $\Sigma' \subset X$ of $X$ with signs. We say that an (ordered) pair of loops $a, b$ in $X$ is $\omega$-admissible if this pair is generic and $T_\omega(a, b) = \emptyset$ so that the crossings of $a$ with every gate lie on the $\omega$-left of the crossings of $b$ with this gate. Taking a generic pair of loops $a, b$ in $X$ and pushing the branches of $a$ crossing the gates to the $\omega$-left and pushing the branches of $b$ crossing the gates to the $\omega$-right, we obtain an $\omega$-admissible pair of loops (possibly, with more crossings than the initial pair). Thus, any pair of loops in $X$ may be deformed into an $\omega$-admissible pair.

For a generic pair of loops $a, b$ in $X$, the set of crossings of $a$ with $b$ in $\Sigma'$ is a finite subset of $\text{Int}(\Sigma')$ denoted $a \cap b$. For a point $r \in a \cap b$, set $\varepsilon_r(a, b) = 1$ if the (positive) tangent vectors of $a$ and $b$ at $r$ form an $\omega$-positive basis in the tangent space of $\Sigma'$ at $r$ and set $\varepsilon_r(a, b) = -1$ otherwise.

Lemma 8.1. For any $\omega$-admissible pair $a, b$ of loops in $X$, the “crossing number”

$$(8.1.2) \quad a \cdot_\omega b = \sum_{r \in a \cap b} \varepsilon_r(a, b) \in \mathbb{R}$$

depends only on the homology classes of $a, b$ in $H_1(X; \mathbb{R})$. The formula $(a, b) \mapsto a \cdot_\omega b$ defines a bilinear form $(\mathcal{S}_1(1))$.

Proof. For each $k \in \pi_0(\alpha)$, one endpoint of the gate $\alpha_k$ lies on the $\omega$-left of the other endpoint. Pick disjoint closed segments $\alpha_k^- \subset \alpha_k$ and $\alpha_k^+ \subset \alpha_k$ containing these two endpoints respectively. Clearly, $p <_\omega q$ for all $p \in \alpha_k^-$ and $q \in \alpha_k^+$. We say that a loop in $X$ is $\omega$-left (respectively, $\omega$-right) if it is generic and meets the gates of $X$ only at points of $\cup_k \alpha_k^-$ (respectively, of $\cup_k \alpha_k^+$). Given an $\omega$-admissible pair of loops $a, b$ in $X$, we can push the branches of $a$ crossing the gates to the left and push the branches of $b$ crossing the gates to the right without creating or destroying intersections between $a$ and $b$. Consequently, $a$ is homotopic (in fact, isotopic) to an $\omega$-left loop $a'$ and $b$ is homotopic to an $\omega$-right loop $b'$ such that $a \cdot_\omega b = a' \cdot_\omega b'$. Since $\alpha_k^-$ is a deformation retract of $\alpha_k$ for all $k$, any $\omega$-left loops...
homotopic in \( X \) are homotopic in the class of \( \omega \)-left loops in \( X \). Similarly, any \( \omega \)-right loops homotopic in \( X \) are homotopic in the class of \( \omega \)-right loops in \( X \). Such homotopies of \( a', b' \) obviously preserve \( a', b' \). Therefore the number \( a \cdot \omega b = a' \cdot \omega b' \) depends only on the \( \omega \) homotopy classes of \( a, b \) in \( X \). Moreover, since this number linearly depends on both loops \( a \) and \( b \), it depends only on their homology classes. This implies the claim of the lemma. \( \square \)

We emphasize that the crossings of loops in \( X \setminus \Sigma' \) do not contribute to the crossing number. For loops in the surface \( \Sigma' \subset X \), the crossing number is the usual homological intersection number. The crossing numbers of loops in \( X \setminus \Sigma' \) with arbitrary loops in \( X \) are equal to zero.

For an \( \omega \)-admissible pair \( a, b \) of loops in \( X \), the pair \( b, a \) is \( \overline{\omega} \)-admissible. Using these pairs to compute \( a \cdot \omega b \) and \( b \cdot \omega a \), we obtain two sums which differ only in the signs of the terms. Hence, for any \( x, y \in H_1(X; R) \),

\[
x \cdot \omega y = -y \cdot \overline{\omega} x.
\]

**Lemma 8.2.** For any homology classes \( x, y \in H_1(X; R) \) represented by a generic pair of loops \( a, b \) in \( X \),

\[
x \cdot \omega y = \sum_{r \in a \cap b} \varepsilon_r(a, b) + \sum_{(k, p, q) \in T_\omega(a, b)} \varepsilon(\omega, k) \varepsilon_p(a) \varepsilon_q(b).
\]

**Proof.** Consider an \( \omega \)-admissible pair of loops \( a', b' \) obtained from \( a, b \) by pushing the branches of \( a \) meeting the gates of the branches of \( b \) meeting the gates. This modifies \( a \) in a small neighborhood of the gates; we can assume that \( a', b' \) have the same intersections in \( \Sigma' \) as \( a, b \) plus one additional intersection \( r = r(k, p, q) \in \Sigma' \) for each triple \( (k, p, q) \in T_\omega(a, b) \). Observe that \( \varepsilon_r(a', b') = \varepsilon(\omega, k) \varepsilon_p(a) \varepsilon_q(b) \). Consequently,

\[
x \cdot \omega y = \sum_{r \in a \cap b} \varepsilon_r(a', b') = \sum_{r \in a \cap b} \varepsilon_r(a, b) + \sum_{(k, p, q) \in T_\omega(a, b)} \varepsilon(\omega, k) \varepsilon_p(a) \varepsilon_q(b).
\]

\( \square \)

Formula \((8.1.4)\) generalizes \((8.1.2)\) because \( T_\omega(a, b) = \emptyset \) for an \( \omega \)-admissible pair of loops \( a, b \). We describe next the dependence of \( \varepsilon \) on \( \omega \). We will use the linear map \( v_k : H_1(X; R) \to R \) “dual” to the \( k \)-th gate. This map carries the homology class of any generic loop \( a \) to \( \sum_{p \in a \cap \alpha_k} \varepsilon_p(a) \). In the notation of Section \( 6.2 \) \( v_k = v_{\alpha_k} \).

**Theorem 8.3.** For any \( x, y \in H_1(X; R) \) and \( k_0 \in \pi_0(\alpha) \),

\[
x \cdot k_0 \omega y = x \cdot \omega y - \varepsilon(\omega, k_0) v_{k_0}(x) v_{k_0}(y).
\]

**Proof.** Pick an \( \omega \)-admissible pair of loops \( a, b \) representing respectively \( x, y \). We compute \( x \cdot \omega y = a \cdot \omega b \) from the definition and compute \( x \cdot k_0 \omega y = a \cdot k_0 \omega b \) from Lemma 8.2. The resulting expressions differ in the sum associated with \( T_{k_0 \omega}(a, b) \). Since the pair \( (a, b) \) is \( \omega \)-admissible, the set \( T_{k_0 \omega}(a, b) \) consists of all triples \( (k_0, p, q), p \in a \cap \alpha_{k_0}, q \in b \cap \alpha_{k_0} \). Therefore

\[
x \cdot k_0 \omega y = x \cdot \omega y + \sum_{p \in a \cap \alpha_{k_0}, q \in b \cap \alpha_{k_0}} \varepsilon(k_0 \omega, k_0) \varepsilon_p(a) \varepsilon_q(b)
\]

\[\begin{align*}
&= x \cdot \omega y - \varepsilon(\omega, k_0) v_{k_0}(x) v_{k_0}(y).
\end{align*}\]
8.2. **Second homological intersection forms.** Pick a gate orientation $\omega$ of $X$ and define a skew-symmetric bilinear form $i_X : H_1(X; R) \times H_1(X; R) \to R$ by

\[ i_X(x, y) = x \cdot \omega y - y \cdot \omega x \tag{8.2.1} \]

for $x, y \in H_1(X; R)$. This form does not depend on $\omega$ because, by (8.1.5),

\[ x \cdot k\omega y - y \cdot k\omega x = x \cdot \omega y - y \cdot \omega x \]

for any $x, y \in H_1(X; R)$ and $k \in \pi_0(\alpha)$. We call $i_X$ the second homological intersection form of $X$. Both the first and the second homological intersection forms generalize the standard intersection form in 1-homology of a surface. Indeed, the value of the form (8.1.1) (respectively, (8.2.1)) on any pair of homology classes of loops in $\Sigma' \subset X$ is equal to the usual intersection number of these loops in $\Sigma'$ (respectively, twice this number).

**Theorem 8.4.** For any gate orientation $\omega$ and any $x, y \in H_1(X; R)$, we have

\[ 2x \cdot \omega y = i_X(x, y) + \sum_{k \in \pi_0(\alpha)} \varepsilon(\omega, k) v_k(x) v_k(y) \tag{8.2.2} \]

**Proof.** Applying (8.1.5) consecutively to all elements of $\pi_0(\alpha)$, we get

\[ x \cdot \omega y = x \cdot \omega y - \sum_{k \in \pi_0(\alpha)} \varepsilon(\omega, k) v_k(x) v_k(y). \]

Substituting $x \cdot \omega y = -y \cdot \omega x$, we get

\[ x \cdot \omega y + y \cdot \omega x = \sum_{k \in \pi_0(\alpha)} \varepsilon(\omega, k) v_k(x) v_k(y). \]

This formula and the equality $x \cdot \omega y - y \cdot \omega x = i_X(x, y)$ imply (8.2.2). \qed

Formula (8.2.2) shows that if $1/2 \in R$, then $\cdot \omega$ is a sum of $(1/2)i_X$ and terms associated with the gates.

8.3. **Remark.** For $R = \mathbb{Z}/2\mathbb{Z}$, the definitions in this section and below do not depend on the orientation of $\Sigma$ and extend to non-orientable quasi-surfaces.

9. **Homotopy intersection forms**

We define homotopy intersection forms of $X$ refining the homological forms above. In this section and below, $\mathcal{L} = \mathcal{L}(X)$ is the set of free homotopy classes of loops in $X$ and $R\mathcal{L}$ is the free $R$-module with basis $\mathcal{L}$. By Sections 4 and 8, for each $m \geq 1$, the gate $\alpha_k \subset X$ determines a cyclically symmetric $m$-bracket in $R\mathcal{L}$. It is denoted $\mu^m_k$.

9.1. **First homotopy intersection form.** Pick a gate orientation $\omega$ of $X$. Any pair $x, y \in \mathcal{L}$ can be represented by an $\omega$-admissible pair of loops $a, b$ in $X$, cf. Section 8.1. For a point $r \in a^{-1}b$, consider the loops $a_r, b_r$ which are reparametrizations of $a, b$, respectively, starting and ending in $r$. Consider the product loop $a_rb_r$ based in $r$ and set

\[ x \cdot \omega y = \sum_{r \in a^{-1}b} \varepsilon_r(a, b) \langle a_rb_r \rangle \in R\mathcal{L} \tag{9.1.1} \]
where for a loop $c$ in $X$, we let $\langle c \rangle \in L$ be its free homotopy class. The sum on the right-hand side of (9.1.1) is an algebraic sum of all possible ways to graft $a$ and $b$. This sum is preserved under all loop moves on $a, b$ keeping this pair $\omega$-admissible. Hence, $x \bullet \omega y$ does not depend on the choice of $a, b$ in the homotopy classes $x, y$. Extending the map $(x, y) \mapsto x \bullet \omega y$ by bilinearity, we obtain a bilinear pairing
\[ (9.1.2) \quad \bullet : RL \times RL \to RL. \]
We call this pairing the first homotopy intersection form of $X$. The proof of Formula (8.1.3) applies here and shows that for any $x, y \in RL$,
\[ (9.1.3) \quad x \bullet \omega y = -y \bullet \omega x. \]
For a generic (non-double) point $p$ of a generic loop $a$, we let $a_p$ be the loop which starts at $p$ and goes along $a$ until coming back to $p$. Having two generic loops $a, b$ and points $p \in a \cap \alpha_k, q \in q \cap \alpha_k$ on the same gate, we can multiply the loops $a_p, b_q$ using an arbitrary path in $\alpha_k$ connecting their base points $p, q$. The resulting loop determines a well-defined element of $L$ denoted $\langle a_p b_q \rangle$.

Lemma 9.1. Let $x, y \in L$ be represented by a generic pair of loops $a, b$. Then
\[ (9.1.4) \quad x \bullet \omega y = \sum_{r \in a \cap b} \varepsilon_r(a, b)(a_r b_r) + \sum_{(k, p, q) \in T_\omega(a, b)} \varepsilon(\omega, k) \varepsilon_p(a) \varepsilon_q(b)(a_p b_q). \]

The proof repeats the proof of Lemma 8.2 with obvious modifications. If $a \cap b = \emptyset$, then (9.1.4) simplifies to
\[ (9.1.5) \quad x \bullet \omega y = \sum_{(k, p, q) \in T_\omega(a, b)} \varepsilon(\omega, k) \varepsilon_p(a) \varepsilon_q(b)(a_p b_q). \]

Theorem 9.2. For any $x, y \in RL$ and $k \in \pi_0(\alpha)$,
\[ (9.1.6) \quad x \bullet \omega y = x \bullet \omega y - \varepsilon(\omega, k) \mu_k^2(x, y). \]

Proof. It suffices to handle the case $x, y \in L$. Then proceed as in the proof of Corollary 10.5.17 replacing $\cdot$ by $\bullet$ and using Formula (8.1.1) to compute $\mu_k^2 = \mu_{\alpha_k}^2$.

Applying (9.1.6) consecutively to all $k \in \pi_0(\alpha)$ and using (9.1.3), we get

Corollary 9.3. For any $x, y \in RL$,
\[ (9.1.7) \quad x \bullet \omega y + y \bullet \omega x = \sum_{k \in \pi_0(\alpha)} \varepsilon(\omega, k) \mu_k^2(x, y). \]

9.2. Second homotopy intersection form. We define a 2-bracket $[\cdot, \cdot]$ in $RL$ by $[x, y] = x \bullet \omega y - y \bullet \omega x$ for all $x, y \in RL$. This skew-symmetric bracket does not depend on $\omega$ because, by (9.1.6),
\[ x \bullet \omega y - y \bullet \omega x = x \bullet \omega y - y \bullet \omega x \]
for all $k \in \pi_0(\alpha)$. (Here we use the symmetry of the brackets $\{\mu_k\}_k$.) We call the 2-bracket $[\cdot, \cdot]$ the second homotopy intersection form of $X$. Both the first and the second homotopy intersection forms generalize Goldman’s [Go1], [Go2] bracket: the value of $\bullet$ (respectively, $[\cdot, \cdot]$) on any pair of free homotopy classes of loops in $\Sigma' \subset X$ is equal to their Goldman’s bracket (respectively, twice this bracket).
Theorem 9.1 allows us to compute \([x, y]\) for \(x, y \in L\) from any generic pair of loops \(a, b\) representing \(x, y\) and any gate orientation \(\omega\) of \(X\). Namely,

\[
\tag{9.2.1}
[x, y] = 2 \sum_{r \in a \cap b} \varepsilon_r(a, b) \langle a_r b_r \rangle + \sum_{(k, p, q) \in T(a, b)} \delta_\omega(p, q) \varepsilon_k(a) \varepsilon_p(b) \langle a_p b_q \rangle
\]

where \(\delta_\omega(p, q) = 1\) if \(p \succ \omega q\) and \(\delta_\omega(p, q) = -1\) if \(p < \omega q\). Note also the identity

\[
2x \bullet_\omega y = [x, y] + \sum_{k \in \pi_0(\alpha)} \varepsilon(\omega, k) \mu_k^2(x, y)
\]

which can be easily deduced from (9.1.7). Consequently, if \(1/2 \in R\), then the form \(\bullet_\omega\) expands as a sum of \((1/2)[-,-]\) and terms associated with the gates.

9.3. Remark. Other algebraic operations associated with surfaces may be extended to quasi-surfaces. This includes algebraic intersections of loops (see [Tu1]), Lie cobrackets (see [Tu2], [Ha]), double brackets and generalized Dehn twists (see [MT1]), and quasi-Poisson structures on the representation spaces (see [MT2]). In a sequel to this paper, the author plans to discuss natural cobrackets appearing in the study of quasi-surfaces.

10. Main theorem

10.1. Statement. We state our main result concerning the quasi-surface \(X\). Let \(\pi = \pi_1(X,*)\) with \(* \in Y \subset X\). Consider the group algebra \(A = R[\pi]\) and the second homotopy intersection form \([-,-] : \hat{A} \times \hat{A} \rightarrow \hat{A} \) in \(\hat{A} = RL\). The next theorem computes the Jacobiator of this 2-form via the 3-bracket

\[
\mu = \sum_{k \in \pi_0(\alpha)} \mu_k^2 : (\hat{A})^3 \rightarrow \hat{A}.
\]

This theorem shows that the failure of the intersection form to satisfy the Jacobi identity is entirely due to the presence of the gates.

Theorem 10.1. The brackets \([-,-]\) and \(\mu\) are braces in \(A\) forming a quasi-Lie pair.

This theorem can be rephrased by saying that the pair \([-,-], \mu\) turns \(A\) into a brace algebra. Combining Theorems 10.1 and 5.2, we conclude that for all \(n \geq 1\), the \(n\)-th trace algebra \(A_n\) of \(A\) carries a unique structure of a brace algebra such that the trace \(\text{tr} : \hat{A} \rightarrow A_n\) is a brace homomorphism.

By Example 7.2.1 (case \(m = 1\)), every surface \(\Sigma\) is a quasi-surface with a single gate which separates the surface part (a copy of \(\Sigma\)) from a cone over a segment in \(\partial \Sigma\). Here \(\mu = 0\) since each loop in this quasi-surface may be deformed into the complement of the gate. Theorem 10.1 yields then the usual Jacobi relation for \([-,-]\). On the other hand, in Example 7.2.4 it may well happen that \(\mu \neq 0\).

The proof of Theorem 10.1 occupies the rest of the section.
10.2. **Proof of Theorem [10.1] beginning.** Throughout the proof we fix a gate orientation \( \omega \) of \( X \). By Section 6.3 the brackets \( \{ \mu^k \}_k \) in \( \hat{A} \) are cyclically symmetric braces in \( A \) (independent of \( \omega \)). Therefore, so is their sum \( \mu \). The skew-symmetry of the 2-bracket \([-,-]\) in \( \hat{A} \) is obvious. We now prove that this 2-bracket is a brace in \( A \). Since it is skew-symmetric, it suffices to prove that \([-,-]\) is a weak derivation in the second variable. Pick any \( x \in \mathcal{L} = \hat{\pi} \), \( y \in \pi \) and represent the pair \( x, y \) by an \( \omega \)-admissible pair of loops \( a, b \) in \( X \) where \( b \) is based in \( \ast \). For each point \( r \in a \cap b \), consider the loop \( a_r \) obtained by reparametrization of \( a \) so that its starts and ends in \( r \). We have \( b = b^-_r b^+_r \) where \( b^-_r \) is the path in \( X \) going from \( \ast \) to \( r \) along \( b \) and \( b^+_r \) is the path in \( X \) going from \( r \) to \( \ast \) along \( b \). The product path \( b^-_r a_r b^+_r \) in \( X \) is a loop based in \( \ast \); consider its homotopy class \([b^-_r a_r b^+_r] \in \pi \). Recall the crossing sign \( \varepsilon_r(a,b) = \pm 1 \) and set

\[
(x \bullet \omega y) = \sum_{r \in a \cap b} \varepsilon_r(a,b) [b^-_r a_r b^+_r] \in A = R[\pi].
\]

The sum on the right-hand side is an algebraic sum of all possible ways to graft \( a \) to \( b \). (For surfaces, the pairing \( [\cdot,\cdot] \) was first introduced by Kawazumi and Kuno [KK1], [KK2].) It is easy to see from the definitions that (i) the expression \( x \bullet \omega y \) depends only on \( x, y \) and does not depend on the choice of \( a, b \) and (ii) the linear extension \( A \to A \) of the map \( y \mapsto x \bullet \omega y \) is a derivation of the algebra \( A \). Next, denote the projection \( A \to \hat{A} \) by \( p \) and note that for each \( y \in \pi \subseteq A \), its image \( p(y) \in \hat{\pi} \subseteq \hat{A} \) is the conjugacy class of \( y \) in \( \pi \). Comparing Formulas (9.1.1) and (10.2.1), we obtain that \( p(x \bullet \omega y) = x \bullet \omega p(y) \). By (9.1.3),

\[
[x, p(y)] = x \bullet \omega p(y) - p(y) \bullet \omega x = x \bullet \omega p(y) + x \bullet p(y)
\]

\[
= p(x \bullet \omega y) + p(x \bullet p(y) = p(x \bullet \omega y + x \bullet p(y).
\]

Consequently, the linear endomorphism \([x, -]\) of \( \hat{A} \) is induced by the linear endomorphism \( y \mapsto x \bullet \omega y + x \bullet p(y) \). Since the maps \( y \mapsto x \bullet \omega y \) and \( y \mapsto x \bullet p(y) \) are derivations of \( A \), so is their sum \( y \mapsto x \bullet \omega y + x \bullet p(y) \). This implies that the bracket \([-,-]\) is a weak derivation in the second variable and is a brace.

It remains to verify the Jacobi-type identity (5.1.1). This is done in the next three subsections.

10.3. **Preliminaries on simple loops.** We say that a finite family of loops in the quasi-surface \( X \) is simple if these loops meet the gates of \( X \) transversely and have no mutual crossings or self-crossings in the surface part \( \Sigma' \subset X \) of \( X \). A simple family of loops is generic.

**Lemma 10.2.** Any finite family of loops in \( X \) can be deformed in \( X \) into a simple family of loops.

**Proof.** Consider first a single loop in \( X \). Since \( X \) is path-connected and contains a gate, we can deform our loop into a generic loop \( a \) which meets a gate at least once. If the set \#\( a \) of double points of \( a \) in \( \text{Int}(\Sigma') \) is empty, then we are done. Otherwise, pick a point \( r \in \#a \). Starting at \( r = r_0 \) and moving along \( a \) (in the given direction of \( a \)), we meet several double points \( r_1, \ldots, r_n \in \#a \) with \( n \geq 0 \) and then come to a point \( p \in a \cap \alpha_k \) of a certain gate \( \alpha_k \). The segment \( b \) of \( a \) connecting \( r_n \) to \( p \) is embedded in \( \Sigma' \) and meets \#\( a \) only at its endpoint \( r_n \). Let \( c \) be the branch of \( a \) transversal to \( b \) at \( r_n. \) Push the branch \( c \) towards \( p \) along \( b \) while keeping \( c \) and \( b \) transversal and eventually push \( c \) across \( \alpha_k \) at \( p \). This transformation of \( a \) decreases
card(#a) by 1 and increases card(a ∩ αk) by 2. Continuing by induction, we deform our loop into a generic loop without self-intersections in Σ'. If the original family of loops contains ≥ 2 loops, then we first deform it into a generic family of loops which all meet some gates. Then pushing branches at crossings and self-crossings as above, we deform the latter family into a simple family of loops.

10.4. Preliminaries on sign functions. We define two functions used in the proof. The first function, δ, is defined on the set {±1} = {−1, 1} by δ(1) = 1 and δ(−1) = 0. The second function, also denoted δ, is defined on the set of all triples ε, ε', ε'' ∈ {±1} by the formula

\[
\delta(\epsilon, \epsilon', \epsilon'') = \epsilon \epsilon' \delta(\epsilon'') + \epsilon \epsilon'' \delta(\epsilon') + \epsilon' \epsilon'' (1 - \delta(\epsilon)).
\]

This function is invariant under all permutations of ε, ε', ε'' as easily follows from the identity 2δ(ε) = ε + 1 for all ε ∈ {±1}. The same identity implies another useful equality: for all ε, ε', ε'' ∈ {±1}, we have

\[
\delta(\epsilon, \epsilon', \epsilon'') - \epsilon \epsilon' \epsilon'' = \epsilon \epsilon' \delta(\epsilon'') + \epsilon \epsilon'' (1 - \delta(\epsilon')) + \epsilon' \epsilon'' (1 - \delta(\epsilon)).
\]

10.5. Proof of [5.1.1]. Any (possibly, non-associative) algebra \( \mathcal{R} \) carries the bracket [x, y] = xy − yx. For x, y, z ∈ \( \mathcal{R} \), set

\[
P(x, y, z) = (xy)z + (yz)x + (zx)y + x(yz) + y(xz) ∈ \mathcal{R}.
\]

A direct computation shows that

\[
[[x, y], z] + [[y, z], x] + [[z, x], y] = P(x, y, z) - P(y, x, z).
\]

We apply these observations to the algebra \( \tilde{\mathcal{A}} = \mathcal{R} \mathcal{L} \) with multiplication \( \cdot = \cdot_\omega \). In view of (10.5.1), the identity (5.1.1) is equivalent to the identity

\[
P(x, y, z) - P(y, x, z) = \mu(x, y, z) - \mu(y, x, z)
\]

for all \( x, y, z ∈ \tilde{\mathcal{A}} \). Since both sides are linear in \( x, y, z \), it suffices to handle the case \( x, y, z ∈ \mathcal{L} \). Set

\[
u_\omega(x, y, z) = (x \cdot_\omega y) \cdot_\omega z + (y \cdot_\omega z) \cdot_\omega x + (z \cdot_\omega x) \cdot_\omega y.
\]

Formula (9.1.3) implies that

\[
u_\omega(x, y, z) = z \cdot_\omega (y \cdot_\omega x) + x \cdot_\omega (z \cdot_\omega y) + y \cdot_\omega (x \cdot_\omega z).
\]

Thus,

\[
P(x, y, z) = \nu_\omega(x, y, z) + \nu_\omega(x, y, z).
\]

In our computations, we represent \( x, y, z \) by loops \( a, b, c \) in X, respectively.

If the loops \( a, b, c \) lie in \( \Sigma' \subset X \) then Goldman's results imply that \( \nu_\omega(x, y, z) = 0 \) for all \( \omega \) so that \( P(x, y, z) = 0 \). It is also clear that \( \mu(x, y, z) = 0 \), and (10.5.2) follows. For completeness, we check the identity \( \nu_\omega(x, y, z) = 0 \) in this case (it is also included in the general case treated below). Deforming if necessary \( a, b, c \), we can assume that the triple \( \{a, b, c\} \) is generic in the sense of Section 7.4. Then \( x \cdot_\omega y \) is computed by (9.1.1). To compute \( (x \cdot_\omega y) \cdot_\omega z \), consider all intersections of the loop c with the loops \( \{a_r b_r\}_{r ∈ a ∩ b} \). At such an intersection, say s, the loop c meets either a or b. Thus, \( (x \cdot_\omega y) \cdot_\omega z = \sigma(a, b, c) + \tau(a, b, c) \) where

\[
\sigma(a, b, c) = \sum_{r ∈ a ∩ b} \sum_{s ∈ a ∩ c} \epsilon_r(a, b) \epsilon_s(a, c) \langle b ∘_r a ∘_s c \rangle,
\]

\[
\tau(a, b, c) = \sum_{r ∈ a ∩ b} \sum_{s ∈ a ∩ c} \epsilon_r(a, b) \epsilon_s(a, c) \langle b ∘_r a ∘_s c \rangle.
\]
\[ \tau(a, b, c) = \sum_{r \in a \cap b} \sum_{s \in b \cap c} \epsilon_r(a, b) \epsilon_s(b, c) \langle a \circ_r b \circ_s c \rangle. \]

Here \( b \circ_r a \circ_s c \) is the loop obtained by grafting the loops \( b_s, s_c \) to \( a \) at the points \( r, s \). More precisely, this loop goes along \( b \) starting and ending in \( r \), then along \( a \) from \( r \) to \( s \), then along \( c \) starting and ending in \( s \), and finally returns along \( a \) to \( r \).

Note that the inclusions \( r \in a \cap b, s \in a \cap c \) ensure that \( r \neq s \) so that the loop \( b \circ_r a \circ_s c \) is well-defined. The loop \( a \circ_r b \circ_s c \) is defined similarly grafting the loops \( a_r, s_c \) to \( b \) at the points \( r, s \). Therefore

\[ u_\omega(x, y, z) = \sigma(a, b, c) + \tau(a, b, c) + \sigma(b, c, a) + \tau(b, c, a) + \sigma(c, a, b) + \tau(c, a, b). \]

Consider now the general case where the loops \( a, b, c \) do not necessarily lie in \( \Sigma' \). By Lemma \[10.2\] deforming \( a, b, c \) in \( X \), we can ensure that this triple of loops is simple. By \[10.1.5\],

\[ x \bullet_\omega y = \sum_{(k, p, q) \in T_\omega(a, b)} \epsilon(\omega, k) \epsilon_p(a) \epsilon_q(b) (a_p \gamma_{p, q} b_q \gamma_{p, q}^{-1}) \]

where \( a_p, b_q \) are loops reparametrizing \( a, b \) and based respectively in \( p, q \) while \( \gamma_{p, q} \) is a path connecting \( p \) and \( q \) in \( \alpha_k \). We deform the loop \( a_p \gamma_{p, q} b_q \gamma_{p, q}^{-1} \) by slightly pushing its subpaths \( \gamma_{p, q}^{\pm 1} \) “behind the gate”, i.e., into \( X \setminus \Sigma' \). (The endpoints \( p, q \) of these subpaths are pushed into \( X \setminus \Sigma' \) along \( a, b \), respectively.) The resulting loop is denoted by \( a \circ_{p, q} b \). Thus, \[10.5.6\]

\[ x \bullet_\omega y = \sum_{(k, p, q) \in T_\omega(a, b)} \epsilon(\omega, k) \epsilon_p(a) \epsilon_q(b) (a \circ_{p, q} b). \]

Note that the loop \( a \circ_{p, q} b \) is simple; moreover, the pair formed by this loop and \( c \) is simple. Applying \[10.1.5\] again, we get

\[ (a \circ_{p, q} b) \bullet_\omega z = \sum_{(l, s, t) \in T_\omega(a \circ_{p, q} b, c)} \epsilon(\omega, l) \epsilon_s(a \circ_{p, q} b) \epsilon_t(c) ((a \circ_{p, q} b) \circ c). \]

For any \( l \in \pi_0(\alpha) \), the set \( (a \circ_{p, q} b) \cap \alpha_l \) is a disjoint union of the sets \( a \cap \alpha_l \) and \( b \cap \alpha_l \). Therefore \( T_\omega(a \circ_{p, q} b, c) = T_\omega(a, c) \coprod T_\omega(b, c) \). By \[10.5.7\],

\[ (a \circ_{p, q} b) \bullet_\omega z = \varphi_\omega(k, p, q) + \psi_\omega(k, p, q) \]

where

\[ \varphi_\omega(k, p, q) = \sum_{(l, s, t) \in T_\omega(a, c)} \epsilon(\omega, l) \epsilon_s(a) \epsilon_t(c) ((a \circ_{p, q} b) \circ c) \]

and

\[ \psi_\omega(k, p, q) = \sum_{(l, s, t) \in T_\omega(b, c)} \epsilon(\omega, l) \epsilon_s(b) \epsilon_t(c) ((a \circ_{p, q} b) \circ c). \]

Combining \[10.5.6\] with \[10.5.3\], we obtain

\[ (x \bullet_\omega y) \bullet_\omega z = \sum_{(k, p, q) \in T_\omega(a, b)} \epsilon(\omega, k) \epsilon_p(a) \epsilon_q(b) (\varphi_\omega(k, p, q) + \psi_\omega(k, p, q)). \]

To compute the latter sum, we rewrite \( \varphi_\omega(k, p, q) \) as follows. For \( s \in a \cap \alpha_l \), the homotopy class \( \langle (a \circ_{p, q} b) \circ c \rangle \) is represented by the loop \( b \circ_{p, q} a \circ_{s, t} c \) obtained by grafting \( b \) and \( c \) to \( a \) via a path in \( \alpha_k \) from \( p \in a \cap \alpha_k \) to \( q \in b \cap \alpha_k \) and a path
in $\alpha_l$ from $s \in a \cap \alpha_l$ to $t \in c \cap \alpha_l$. To give a precise description of this loop, we separate two cases.

Case 1: $p \neq s$ so that the points $p, q, s, t$ are pairwise distinct (possibly, $k = l$). In this case the loop $b \circ_{p,q} a \circ_{s,t} c$ starts at $p$ and goes: along the gate $\alpha_k$ to $q$, then along the full loop $b$ back to $q$, then along $\alpha_k$ back to $p$, then along $a$ to $s$, then along the gate $\alpha_l$ to $t$, then along the full loop $c$ back to $t$, then along $\alpha_l$ back to $s$, and finally along $a$ back to $p$.

Case 2: $p = s$. Then $k = l$ and $p, q, t$ are three distinct points on the gate $\alpha_k$. If $\varepsilon_p(a) = +1$, then the loop $b \circ_{p,q} a \circ_{s,t} c$ starts at $p$ and goes: along $\alpha_k$ to $q$, then along the full loop $b$ back to $q$, then along $\alpha_k$ to $t$, then along the full loop $c$ back to $t$, then along $\alpha_k$ to $p$, and finally along the full loop $a$ back to $p$. If $\varepsilon_p(a) = -1$, then the loop $b \circ_{p,q} a \circ_{s,t} c$ starts at $p$ and goes: along $\alpha_k$ to $q$, then along the full loop $b$ back to $q$, then along $\alpha_k$ to $p$, then along the full loop $a$ back to $p$, then along $\alpha_k$ to $t$, then along the full loop $c$ back to $t$, and finally along $\alpha_k$ to $p$.

In both cases,

$$\varphi_\omega(k, p, q) = \sum_{(l, s, t) \in T_\omega(a, c)} \varepsilon(\omega, l) \varepsilon_s(a) \varepsilon_t(c)(b \circ_{p,q} a \circ_{s,t} c).$$

We call the summands corresponding to the triples $(l, s, t) \in T_\omega(a, c)$ with $p \neq s$ the 4-tuple terms. The summands with $p = s$ (and $k = l$) are called 3-tuple terms.

Similarly, for $s \in b \cap \alpha_l$, the homotopy class $[(a \circ_{p,q} b)_{s} \alpha_l]$ is represented by the loop $a \circ_{p,q} b \circ_{s,t} c$ obtained by grafting $a$ and $c$ to $b$ via a path in $\alpha_k$ from $p \in a \cap \alpha_k$ to $q \in b \cap \alpha_k$ and a path in $\alpha_l$ from $s \in b \cap \alpha_l$ to $t \in c \cap \alpha_l$. A precise description of this loop also includes two cases determined by whether or not $q = s$; we leave the details to the reader. Thus,

$$\psi_\omega(k, p, q) = \sum_{(l, s, t) \in T_\omega(b, c)} \varepsilon(\omega, l) \varepsilon_s(b) \varepsilon_t(c)(a \circ_{p,q} b \circ_{s,t} c).$$

We call the summands corresponding to the triples $(l, s, t) \in T_\omega(b, c)$ with $q \neq s$ the 4-tuple terms. The summands with $q = s$ (and $l = k$) are called 3-tuple terms.

Substituting these expressions for $\varphi, \psi$ in (10.5.3), we expand $(x \bullet_\omega y) \bullet_\omega z$ as a linear combination of 4-tuple and 3-tuple terms. Then Formula (10.5.3) yields such an expansion of $u_\omega(x, y, z)$ and Formula (10.5.5) yields such an expansion of $P(x, y, z)$. The (total) contribution of the 4-tuple terms to $P(x, y, z)$ is denoted by $P_4(x, y, z)$, and the (total) contribution of the 3-tuple terms to $P(x, y, z)$ is denoted by $P_3(x, y, z)$. We stress that $P(x, y, z) = P_4(x, y, z) + P_3(x, y, z)$.

We prove next that $P_4(x, y, z) = P_4(y, x, z)$. Note first that each point $p, q, s, t$ in a 4-tuple (or a 3-tuple) term is traversed by exactly one of the loops $a, b, c$. We will write $\varepsilon_p, \varepsilon_q, \varepsilon_s, \varepsilon_t$ for the corresponding signs $\pm 1$. For example, $\varepsilon_p = \varepsilon_p(a), \varepsilon_q = \varepsilon_q(b)$, etc. Also set $\varepsilon(\omega, k, l) = \varepsilon(\omega, k) \varepsilon(\omega, l)$. In this notation, the contribution of 4-tuple terms to $(x \bullet_\omega y) \bullet_\omega z$ is equal to $\varphi^x_{\omega, y; z} + \psi^x_{\omega, y; z}$ where

$$\varphi^x_{\omega, y; z} = \sum_{(k, p, q) \in T_\omega(a, b)} \sum_{(l, s, t) \in T_\omega(a, c), p \neq s} \varepsilon(\omega, k, l) \varepsilon_p \varepsilon_q \varepsilon_s \varepsilon_t(b \circ_{p,q} a \circ_{s,t} c)$$
and

\[ \psi_{\omega}^{x,y,z} = \sum_{(k, p, q) \in T_\omega(a, b)} \varepsilon(\omega, k, l) \varepsilon_p \varepsilon_q \varepsilon_s \varepsilon_t \langle a \circ_{p,q} b \circ_{s,t} c \rangle. \]

To compute \( \varphi_{\omega}^{y,z,x} \) and \( \psi_{\omega}^{y,z,x} \), we cyclically permute \( x, y, z \) and \( a, b, c \) in the formulas above via \( a \rightarrow b \rightarrow c \rightarrow a \). It is convenient to simultaneously permute the indices \( k, l \) and permute the labels \( p, q, s, t \) via \( p \rightarrow s \rightarrow q \rightarrow t \rightarrow p \). This gives

\[ \varphi_{\omega}^{y,z,x} = \sum_{(l, s, t) \in T_\omega(b, c)} \varepsilon(\omega, k, l) \varepsilon_p \varepsilon_q \varepsilon_s \varepsilon_t \langle c \circ_{s,t} b \circ_{q,p} a \rangle \]

and

\[ \psi_{\omega}^{y,z,x} = \sum_{(l, s, t) \in T_\omega(b, c)} \varepsilon(\omega, k, l) \varepsilon_p \varepsilon_q \varepsilon_s \varepsilon_t \langle b \circ_{s,t} c \circ_{q,p} a \rangle. \]

Applying the same permutations again, we get

\[ \varphi_{\omega}^{x,y,z} = \sum_{(k, p, q) \in T_\omega(c, a)} \varepsilon(\omega, k, l) \varepsilon_p \varepsilon_q \varepsilon_s \varepsilon_t \langle a \circ_{q,p} c \circ_{t,s} b \rangle. \]

To compute \( \psi_{\omega}^{x,y,z} \), we apply to \( \psi_{\omega}^{y,z,x} \) the permutation \( a \rightarrow b \rightarrow c \rightarrow a \) and the following permutation of the indices: \( p \rightarrow q \rightarrow p, s \rightarrow t \rightarrow s, k \rightarrow k, l \rightarrow l. \) Thus,

\[ \psi_{\omega}^{x,y,z} = \sum_{(l, s, t) \in T_\omega(c, a)} \varepsilon(\omega, k, l) \varepsilon_p \varepsilon_q \varepsilon_s \varepsilon_t \langle c \circ_{t,s} a \circ_{p,q} b \rangle. \]

We conclude that the contribution of 4-tuple terms to \( u_\omega(x, y, z) \) is equal to

\[ \varphi_{\omega}^{x,y,z} + \psi_{\omega}^{x,y,z} + \varphi_{\omega}^{y,z,x} + \psi_{\omega}^{y,z,x} + \varphi_{\omega}^{z,x,y} + \psi_{\omega}^{z,x,y} = \Delta_{\omega}^{x,y,z} + \Delta_{\omega}^{y,z,x} + \Delta_{\omega}^{z,x,y} \]

for

\[ \Delta_{\omega}^{x,y,z} = \psi_{\omega}^{x,y,z} + \varphi_{\omega}^{x,y,z}, \quad \Delta_{\omega}^{y,z,x} = \psi_{\omega}^{y,z,x} + \varphi_{\omega}^{y,z,x}, \quad \Delta_{\omega}^{z,x,y} = \psi_{\omega}^{z,x,y} + \varphi_{\omega}^{z,x,y}. \]

Then

\[ P_\delta(x, y, z) = (\Delta_{\omega}^{x,y,z} + \Delta_{\omega}^{x,y,z}) + (\Delta_{\omega}^{y,z,x} + \Delta_{\omega}^{y,z,x}) + (\Delta_{\omega}^{z,x,y} + \Delta_{\omega}^{z,x,y}). \]

We now compute all the \( \Delta \)'s. Comparing the expansions of \( \psi_{\omega}^{x,y,z} \) and \( \varphi_{\omega}^{y,z,x} \) above, we observe that their summands are defined by the same formula; here we use the obvious fact that the loops \( a \circ_{p,q} b \circ_{s,t} c \) and \( c \circ_{s,t} b \circ_{q,p} a \) are freely homotopic provided \( q \neq s \). The summation in these two expansions goes over complementary sets of indices as the inclusion \( (k, q, p) \in T_\omega(b, a) \) holds if and only if \( (k, p, q) \in T(a, b) \setminus T_\omega(a, b) \). Therefore

\[ \Delta_{\omega}^{x,y,z} = \sum_{(k, p, q) \in T(a, b)} \varepsilon(\omega, k, l) \varepsilon_p \varepsilon_q \varepsilon_s \varepsilon_t \langle a \circ_{p,q} b \circ_{s,t} c \rangle. \]
As a consequence, the permutation of $x, y \in \mathcal{V}$ using that

$$\Delta = \sum_{(l, s, t) \in T(b, c)} \varepsilon (\omega, k, l) \varepsilon _{p_q} \varepsilon _{s_t} (b \circ _{s_t} c \circ _{p_q} a)$$

and

$$\Delta _{x, y} = \sum_{(k, q, p) \in T(a, c), q \neq t} \varepsilon (\omega, k, l) \varepsilon _{p_q} \varepsilon _{s_t} (b \circ _{p_q} a \circ _{s_t} c).$$

Using that $\varepsilon (\omega, k, l) = \varepsilon (\mathcal{W}, k, l)$, we deduce from these expressions that

$$\Delta _{x, y} + \Delta _{y, z} = \sum_{(k, p, q) \in T(a, b)} \varepsilon (\omega, k, l) \varepsilon _{p_q} \varepsilon _{s} (a \circ _{p_q} b \circ _{s} c),$$

$$\Delta _{y, z} + \Delta _{z, x} = \sum_{(l, s, t) \in T(b, c)} \varepsilon (\omega, k, l) \varepsilon _{p_q} \varepsilon _{s_t} (b \circ _{p_q} c \circ _{s_t} a),$$

$$\Delta _{z, x} + \Delta _{x, y} = \sum_{(k, q, p) \in T(c, a), q \neq t} \varepsilon (\omega, k, l) \varepsilon _{p_q} \varepsilon _{s_t} (b \circ _{p_q} a \circ _{s_t} c).$$

As a consequence, the permutation of $x, y$ keeps $\Delta _{y, z} + \Delta _{z, x}$ and transforms $\Delta _{x, y} + \Delta _{y, z}$ and $\Delta _{z, x} + \Delta _{x, y}$ into each other. Hence, $P_3(x, y, z) = P_4(y, x, z)$ and so

$$P(x, y, z) - P(y, x, z) = P_3(x, y, z) - P_3(y, x, z)$$

where $P_3(x, y, z)$ is the sum of the 3-tuple terms in the expansion of $P(x, y, z)$. Therefore to prove (10.5.12) we need to check that

$$P_3(x, y, z) - P_3(y, x, z) = \mu (x, y, z) - \mu (y, x, z).$$

Observe that each 3-tuple term in the expansion of $P(x, y, z)$ is associated with an element $k = l$ of $\pi _0 (\alpha )$ and pairwise distinct points $p \in a \cap \alpha _k, q \in b \cap \alpha _k, t \in c \cap \alpha _k$. For each such triple of points, set

$$j(p, q, t) = \delta (\varepsilon _{p_q}, \varepsilon _{s_t}) \langle a_pb_qc_t \rangle + (\delta (\varepsilon _{p_q}, \varepsilon _{s_t}) - \varepsilon _{p_q} \varepsilon _{s_t}) (a_pc_tc_q)$$

and

$$j'(p, q, t) = \delta (\varepsilon _{p_q}, \varepsilon _{s_t}) \langle a_pb_qc_t \rangle + (\delta (\varepsilon _{p_q}, \varepsilon _{s_t}) - \varepsilon _{p_q} \varepsilon _{s_t}) (a_pb_qc_t).$$

Here $\delta$ is the function of 3 signs defined in Section 10.3. The loop $a_pb_qc_t$ is the product of three loops $a_p, b_q, c_t$ formed via connecting their base points $p, q, t$ by arbitrary paths in $\alpha _k$. (In other words, we treat $\alpha _k$ as a big base point for these loops.) The loop $a_pc_tc_q$ is defined similarly. Note that by the cyclic symmetry of free homotopy classes of loops, we have

$$\langle a_pb_qc_t \rangle = \langle b_qc_tc_p \rangle = \langle c_tc_pb_q \rangle$$

and

$$\langle a_pc_tc_q \rangle = \langle b_qc_tc_p \rangle = \langle c_tc_pb_q \rangle.$$

Let $P_{3, k}(x, y, z)$ be the sum of 3-tuple terms associated with $k \in \pi _0 (\alpha )$. Clearly,

$$P_3(x, y, z) = \sum_{k \in \pi _0 (\alpha )} P_{3, k}(x, y, z).$$
We prove below that for all \( k \),

\[
(10.5.16) \quad P_{3,k}(x, y, z) = \sum_{p, q, t} j(p, q, t).
\]

In this and similar sums \( p, q, t \) run respectively over the sets \( a \cap \alpha_k, b \cap \alpha_k, c \cap \alpha_k \).

We first explain that this formula implies (10.5.12). Indeed, using the invariance \( P(10.5.16) \),

\[
\text{We prove below that for all } k \in \pi_0(\alpha), \text{ we get (10.5.12).}
\]

It remains to prove (10.5.16). Fix \( k \in \pi_0(\alpha) \). Observe that Formulas (10.5.9)–(10.5.11) simplify for 3-tuple terms. Indeed, \( \varepsilon(\omega, k) \varepsilon(\omega, l) = +1 \) as \( k = l \). Also, if \( s = p \), then \( \varepsilon_s \varepsilon_p = +1 \); if \( s = q \), then \( \varepsilon_s \varepsilon_q = +1 \). By the computations above, the contribution of the 3-tuple terms (with fixed \( k \)) to \( (x \cdot_y y) \cdot_z z \) is equal to \( \Phi_{x,y,z}^{x,y,z} + \Psi_{x,y,z}^{x,y,z} \).

\[
\Phi_{x,y,z}^{x,y,z} = \sum_{q < \omega_p, t < \omega_p} \varepsilon_q \varepsilon_t \langle b \circ_{p,q} a \circ_{p,t} c \rangle,
\]

\[
\Psi_{x,y,z}^{x,y,z} = \sum_{t < q < \omega_p} \varepsilon_p \varepsilon_t \langle a \circ_{p,q} b \circ_{q,t} c \rangle.
\]

It is understood that the sum runs over \( p \in a \cap \alpha_k, q \in b \cap \alpha_k, t \in c \cap \alpha_k \) satisfying the indicated inequalities. The description of the loop \( b \circ_{p,q} a \circ_{p,t} c \) above shows that it is freely homotopic to \( a_p b_q c_t \) if \( \varepsilon_p = 1 \) and to \( a_p c_t b_q \) if \( \varepsilon_p = -1 \). Thus, \( \langle b \circ_{p,q} a \circ_{p,t} c \rangle = \delta(\varepsilon_p) \langle a_p b_q c_t \rangle + (1 - \delta(\varepsilon_p)) \langle a_p c_t b_q \rangle \)

and

\[
(10.5.17) \quad \Phi_{x,y,z}^{x,y,z} = \sum_{q < \omega_p, t < \omega_p} \varepsilon_q \varepsilon_t \left( \delta(\varepsilon_p) \langle a_p b_q c_t \rangle + (1 - \delta(\varepsilon_p)) \langle a_p c_t b_q \rangle \right).
\]

Similarly,

\[
(10.5.18) \quad \Psi_{x,y,z}^{x,y,z} = \sum_{t < q < \omega_p} \varepsilon_p \varepsilon_t \left( \delta(\varepsilon_q) \langle a_p c_t b_q \rangle + (1 - \delta(\varepsilon_q)) \langle a_p b_q c_t \rangle \right).
\]

Cyclically permuting \( (x, y, z), (a, b, c), (p, q, t) \), we get

\[
(10.5.19) \quad \Phi_{x,y,z}^{x,y,z} = \sum_{t < q, p < \omega_q} \varepsilon_t \varepsilon_q \left( \delta(\varepsilon_q) \langle a_p b_q c_t \rangle + (1 - \delta(\varepsilon_q)) \langle a_p c_t b_q \rangle \right),
\]

\[
(10.5.20) \quad \Psi_{x,y,z}^{x,y,z} = \sum_{p < \omega_t, q < \omega_t} \varepsilon_p \varepsilon_q \left( \delta(\varepsilon_t) \langle a_p c_t b_q \rangle + (1 - \delta(\varepsilon_t)) \langle a_p b_q c_t \rangle \right),
\]

\[
(10.5.21) \quad \Phi_{x,y,z}^{x,y,z} = \sum_{p < \omega_t, q < \omega_t} \varepsilon_p \varepsilon_q \left( \delta(\varepsilon_t) \langle a_p b_q c_t \rangle + (1 - \delta(\varepsilon_t)) \langle a_p c_t b_q \rangle \right),
\]
The contribution of 3-tuple terms (with given $k$) to $u_\omega(x, y, z)$ is the sum of 6 terms \(\Psi_{\omega, k}^{x, z, y}\). Then, by \(10.5.22\), $P_{3,k}(x, y, z)$ is the sum of these 6 terms and similar terms obtained by replacing $\omega$ with $\overline{\omega}$. Under this replacement, the only change on the right-hand sides of Formulas \(10.5.17\) - \(10.5.22\) concerns the summation domain. For example, the summation domain in \(10.5.22\) changes from the set of triples $p, q, t$ such that $q < \omega, p, t < \omega$ to the set of triples $p, q, t$ such that $q < \overline{\omega}, p, t < \overline{\omega}$. The latter condition may be rewritten as $p < \omega, q, p < \omega, t$.

For $p \in \alpha \cap \alpha_k, q \in \beta \cap \alpha_k, t \in \gamma \cap \alpha_k$, consider the 12 terms as above and pick their $(p, q, t)$-summands (some of the $(p, q, t)$-summands may be zero). We claim that the sum of these 12 summands is equal to $j(p, q, t)$ for all $p, q, t$. This clearly implies Formula \(10.5.16\). To prove our claim, we consider possible positions of the points $p, q, t$ on $\alpha_k$. Replacing, if necessary, $\omega$ by $\overline{\omega}$, we can assume that $p < \omega$. This leaves us with 3 cases: (a) $t < \omega p$; (b) $p < \omega t < \omega q$, and (c) $q < \omega t$.

In Case (a), only the $(p, q, t)$-summands of $\Phi_{\omega, k}^{y, z, x}, \Phi_{\overline{\omega}, k}^{x, z, y}$, and $\Psi_{\omega, k}^{x, z, y}$ may be non-zero and their sum is

\[
\varepsilon_t \varepsilon_p \delta(\varepsilon_q) (a_p b_q c_t) + \varepsilon_t \varepsilon_p (1 - \delta(\varepsilon_q)) (a_p c_t b_q) + \varepsilon_p \varepsilon_q (1 - \delta(\varepsilon_t)) (a_p b_q c_t)
\]

where the last equality follows from \(10.4.1\) and \(10.4.2\). In Case (b), only the $(p, q, t)$-summands of $\Phi_{\omega, k}^{y, z, x}, \Phi_{\overline{\omega}, k}^{x, z, y}$, and $\Phi_{\overline{\omega}, k}^{y, z, x}$ may be non-zero and their sum is equal to

\[
\varepsilon_t \varepsilon_p \delta(\varepsilon_q) (a_p b_q c_t) + \varepsilon_t \varepsilon_p (1 - \delta(\varepsilon_q)) (a_p c_t b_q) + \varepsilon_p \varepsilon_q (1 - \delta(\varepsilon_t)) (a_p c_t b_q) + \varepsilon_p \varepsilon_q (1 - \delta(\varepsilon_t)) (a_p c_t b_q) = j(p, q, t).
\]

In Case (c), only the $(p, q, t)$-summands of $\Phi_{\omega, k}^{x, z, y}, \Phi_{\overline{\omega}, k}^{x, z, y}$, and $\Psi_{\omega, k}^{x, z, y}$ may be non-zero and their sum is equal to

\[
\varepsilon_t \varepsilon_p \delta(\varepsilon_q) (a_p b_q c_t) + \varepsilon_t \varepsilon_p (1 - \delta(\varepsilon_q)) (a_p c_t b_q) + \varepsilon_q \varepsilon_t (1 - \delta(\varepsilon_p)) (a_p c_t b_q) + \varepsilon_p \varepsilon_t (1 - \delta(\varepsilon_p)) (a_p b_q c_t) = j(p, q, t).
\]

This proves the claim above and completes the proof of the theorem.

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