RADIAL SYMMETRY FOR LOGARITHMIC CHOQUARD EQUATION INVOLVING A GENERALIZED TEMPERED FRACTIONAL $p$-LAPLACIAN

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Abstract. In this paper, we investigate radial symmetry and monotonicity of positive solutions to a logarithmic Choquard equation involving a generalized nonlinear tempered fractional $p$-Laplacian operator by applying the direct method of moving planes. We first introduce a new kind of tempered fractional $p$-Laplacian ($\left(-\Delta - \lambda\right)_p^\alpha$) based on tempered fractional Laplacian ($\left(\Delta + \lambda\right)^{\beta/2}$), which was originally defined in [3] by Deng et.al [Boundary problems for the fractional and tempered fractional operators, Multiscale Model. Simul., 16(1)(2018),125-149]. Then we discuss the decay of solutions at infinity and narrow region principle, which play a key role in obtaining the main result by the process of moving planes.

1. Introduction. In recent years, the theory of fractional calculus is found to be of great utility in diverse streams due to its growing applications in the mathematical modeling of various physical processes, owing to its notable linkage with memory and fractal nature of the concerned phenomenon or process. Examples include a fractional-order biological population model with carrying capacity [12], a new non-integer model for convective straight fins with temperature-dependent thermal conductivity associated with Caputo-Fabrizio fractional derivative [6], a fractional extension of the vibration equation for large membranes [7], etc.

In 1996, Bertoin [1] interpreted the fractional Laplacian as an infinitesimal generator for a stable Lévy diffusion process. The scaling limit of Lévy flight with the $\beta$-stable Lévy process, generated by the fractional Laplacian ($\Delta)^{\beta/2}$. In order to make the Lévy flight a more suitable physical model, the concept of the tempered Lévy flight was introduced. The scaling limit of the tempered Lévy flight is called the tempered Lévy process, which is generated by the tempered fractional Laplacian

$\Delta^{\alpha/2}$

2020 Mathematics Subject Classification. 35A01, 35A30, 35B09.

Key words and phrases. Generalized tempered fractional $p$-Laplacian, direct method of moving planes, Choquard equations, logarithmic nonlinearity, radial symmetry and monotonicity.

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\((\Delta + \lambda)^{\beta/2}\) physically introduced and mathematically defined in [3] as
\[
(\Delta + \lambda)^{\beta/2}u(x) = -C_{n,\beta}\lambda PV \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{e^{|x-y|}|x-y|^{n+\beta}} dy,
\]
where \(PV\) stands for the Cauchy principal value, \(\lambda\) is a sufficiently small positive constant, \(C_{n,\beta,\lambda} = \frac{\Gamma(\frac{n}{\beta})}{2\pi^{n/2}n!\Gamma\left(\frac{n-\beta}{\beta}\right)} (\beta \in (0, 2))\) and \(\Gamma(t) = \int_0^\infty e^{-s} s^{t-1} ds\) is the Gamma function. Let us now dwell some works on the tempered fractional Laplacian operator. In [19], Zhang, Deng and Karniadakis developed numerical methods for the tempered fractional Laplacian in the Riesz basis Galerkin framework. Zhang, Deng and Fan [18] designed the finite difference schemes for the tempered fractional Laplacian in the Riesz basis Galerkin framework. Zhang, Deng and Karniadakis developed numerical methods for the tempered fractional Laplacian equation with the generalized Dirichlet type boundary condition. In [5] Duo and Fan [18] designed the finite difference schemes for the tempered fractional Laplacian with the generalized Dirichlet type boundary condition. In [13], Sun, Nie and Deng obtained the finite difference discretization for the two dimensional tempered fractional ordinary differential equation. In [8], Li, Deng and Zhao discussed the properties of the time tempered fractional derivative and presented the Jacobi-predictor-corrector algorithm for the tempered fractional ordinary differential equation. In [13], Sun, Nie and Deng obtained the finite difference discretization for the two dimensional tempered fractional Laplacian \((\Delta + \lambda)^{\beta/2}\). In addition, recent contributions in tempered fractional differential equations are available to interested readers in [10], [14].

Motivated by aforementioned work, we generalize a kind of tempered fractional \(p\)-Laplacian defined by
\[
(\lambda f)^p_m(x) = C_{n,p} PV \int_{\mathbb{R}^n} \frac{|m(x) - m(y)|^{p-2}[m(x) - m(y)]}{e^{|m(x-y)|}|x-y|^{n+sp}} dy,
\]
where \(PV\) stands for the Cauchy principal value, \(\lambda\) is a sufficiently small positive constant and \(f\) is nondecreasing with respect to \(|x-y|\). As a special case, when \(p = 2\) and \(f\) is an identity map, the generalized nonlinear tempered fractional \(p\)-Laplacian becomes the tempered fractional Laplacian \((\Delta + \lambda)^{\beta/2}\). As another special case, \((\lambda f)^p_m(x)\) turns into the fractional \(p\)-Laplacian \((\lambda)^p_m(x)\). Furthermore, \((\lambda)^p_m(x)\) takes the form of the well-known fractional Laplacian \((\Delta)^p_m(x)\) when \(p = 2\).

In view of the nonlocal nature of fractional Laplacian, conventional methods are no longer effective for the problems involving this operator. To surmount this difficulty, Caffarelli and Silvestre introduced the extension method to reduce the nonlocal problem into a local one in higher dimensions, which proved to be a key tool to deal with nonlocal problems. Another way to overcome the nonlocal fractional Laplacian relies on the integral equations method.

However, there are certain operators, where the extension method and integral equation method fail to work. For such operators, a direct method of moving planes, introduced by Jarohs-Weth-Chen-Li-Li. Later this method was effectively used to tackle a series of nonlocal nonlinear problems. The direct method of moving planes was also efficiently used to study the radial symmetry and monotonicity of the solutions of fractional Laplacian or fractional \(p\)-Laplacian equations and systems [2, 15, 16, 17, 20].

In relation to the study of many problems in physics, Choquard equation received considerable attention [4, 9, 11]. Moroz and Schaftingen [9] established regularity and positivity of the groundstates for the following nonlinear Choquard or Choquard-Pekar equation
\[
-\Delta v + v = (I_\alpha * |v|^p)|v|^{p-2}v, \text{ in } \mathbb{R}^n.
\]
They have also shown that all positive groundstates are radially symmetric and monotone decaying about some point.

In [4], the authors obtained the regularity, existence, nonexistence, symmetry as well as decays properties for a class of nonlinear Schrödinger equations with a generalized Choquard type nonlinearity and fractional diffusion

\[ (-\Delta)^s u + wu = (\kappa_\alpha * |u|^p)|u|^{p-2} u, \quad u \in H^s(\mathbb{R}^n). \]

Ma and Zhang [11] studied the radial symmetry of positive solutions for Choquard equation with fractional p-Laplacian:

\[
\begin{cases}
(\Delta)^{s}u = C_{n,t}(|x|^{2t-n} * u^{q})u^{q-1} & \text{in } \mathbb{R}^n, \\
\quad \quad u > 0 & \text{on } \mathbb{R}^n.
\end{cases}
\]

In a recent paper [20], Zhang and Hou studied the radial symmetry of standing waves for the following nonlinear fractional p-Laplacian equation involving logarithmic nonlinearity:

\[
i \frac{\partial \Phi}{\partial t}(x, t) - (\Delta)^{s}_p \Phi(x, t) + |\Phi(x, t)|^{m-1} \Phi(x, t) \log(\|\Phi(x, t)\|^h + 1) = 0 \text{ in } \mathbb{R}^n \times \mathbb{R}_+.
\]

Enlightened by the above works, the purpose of this paper is to apply the direct method of moving planes to study the following Choquard equation involving a generalized nonlinear tempered fractional p-Laplacian:

\[
\begin{cases}
(\Delta - \lambda f)^s m(x) = C_{n,t}(|x|^{2t-n} \times |\ln(m(x) + 1)|^{q})|\ln(m(x) + 1)|^{q-1} & x \in \mathbb{R}^n, \\
m(x) > 0 & x \in \mathbb{R}^n,
\end{cases}
\]

where, \(0 < s, t < 1, \quad 2 < p < \infty, \quad p - 1 < q < \infty\) and \(n \geq 2\).

As far as we know, unlike the numerical method, this is the first attempt to use analytic methods to study the radial symmetry of solutions for the equation involving tempered fractional p-Laplacian.

For the integral (1), we require that

\[ m \in C_{loc}^{1,1} \cap L_{sp} \]

with

\[ L_{sp} = \{ m \in L_{loc}^{p-1} | \int_{\mathbb{R}^n} \frac{1 + m(x)}{1 + |x|^{n+sp}} dx < \infty \}. \]

We define

\[ n(x) = C_{n,t}(|x|^{2t-n} \times |\ln(m(x) + 1)|^{q}) = C_{n,t} \int_{\mathbb{R}^n} \frac{|\ln(m(y) + 1)|^{q}}{|x - y|^{n-2t}} dy, \]

since \(\frac{C_{n,t}}{|x - y|^{n-2t}}\) is the Green’s function associated with fractional Laplacian \((-\Delta)^t\) in \(\mathbb{R}^n\). Therefore we have

\[ (-\Delta)^t n(x) = |\ln(m(x) + 1)|^{q} \text{ in } \mathbb{R}^n. \]

So, equation (2) is transformed to the following system:

\[
\begin{cases}
(\Delta - \lambda f)^s m(x) = n(x)|\ln(m(x) + 1)|^{q-1} & x \in \mathbb{R}^n, \\
(\Delta)^t n(x) = |\ln(m(x) + 1)|^{q} & x \in \mathbb{R}^n, \\
m(x) > 0, \quad n(x) > 0 & x \in \mathbb{R}^n.
\end{cases}
\]

Now we study equation (3) to obtain the radial symmetry (a.e., \(m(x) = m(|x|)\)) of the positive solution of (2).
2. Preliminary. Before proceeding for the main results, let us fix our terminology. We define
\[ T_\alpha = \{ x \in \mathbb{R}^n | x_1 = \alpha, \text{ for some } \alpha \in \mathbb{R} \} \]
being the moving planes,
\[ \Sigma_\alpha = \{ x \in \mathbb{R}^n | x_1 < \alpha \} \]
being the region to the left of \( T_\alpha \), and
\[ x^\alpha = (2\alpha - x_1, x_2, \ldots, x_n) \]
being the reflection of \( x \) about \( T_\alpha \). Furthermore, we denote
\[ m_\alpha(x) = m(x^\alpha), \quad n_\alpha(x) = n(x^\alpha), \]
\[ M_\alpha(x) = m(x^\alpha) - m(x), \quad N_\alpha(x) = n(x^\alpha) - n(x), \]
\[ \hat{\Sigma}_\lambda = \{ x | x^\lambda \in \Sigma_\lambda \}. \]

**Theorem 2.1.** Let \( p > q-1 \) and \((m, n)\) be a pair of positive solutions of (3), where, \( m \in C^{1,1}_{loc} \cap L_\infty \), \( n \in C^{1,1}_{loc} \cap L_\infty \) are such that
\[ m(x) \sim \frac{1}{e^{|x|^p}}, \quad n(x) \sim \frac{1}{e^{|x|^q}} \tag{4} \]
for sufficiently large \(|x|\) and \( \gamma \), \( \theta \) satisfy
\[ f(|x|) < \frac{1}{\lambda} \min \{|x|^p + (q - p - 2)|x|^\gamma, (q - p - 1)|x|^\gamma\}. \tag{5} \]
Then \( m(x) \) must be radially symmetric. In addition, it is monotonically decreasing with respect to some point in \( \mathbb{R}^n \).

The following two lemmas play a pivotal role in proving Theorem 2.1 by the method of moving planes.

**Lemma 2.2.** (Decay at infinity).
Let \( \Omega \subset \mathbb{R}^n \) be unbounded, and that \( m \in L_\infty \cap C^{1,1}_{loc}(\Omega) \) and \( n \in L_\infty \cap C^{1,1}_{loc}(\Omega) \) are lower semi-continuous on \( \Omega \) satisfying
\[ \begin{cases}
(-\Delta - \lambda f)^p m_\alpha(x) - (-\Delta - \lambda f)_p^s m(x) - C_1(x)M_\alpha(x) - C_2(x)N_\alpha(x) \geq 0, & x \in \Omega, \\
(-\Delta)^i N_\alpha(x) - C_3(x)M_\alpha(x) \geq 0, & x \in \Omega, \\
M_\alpha(x) \geq 0, \quad N_\alpha(x) \geq 0, & x \in \Sigma_\lambda \setminus \Omega,
\end{cases} \tag{6} \]
where \( C_i(x) > 0 \) \((i = 1, 2, 3)\) and
\[ C_1(x) \sim \frac{1}{e^{|x|^p + (q-3)|x|^\gamma}}, \quad C_2(x) \sim \frac{1}{e^{(q-1)|x|^\gamma}}, \quad C_3(x) \sim \frac{1}{e^{(q-2)|x|^\gamma}}, \tag{7} \]
for sufficiently large \(|x|\) and \( \gamma \), \( \theta \) satisfy the conditions of Theorem 2.1. Then there is \( R_0 > 0 \) such that, if
\[ M_\alpha(x^0) = \min_{\Omega} M_\alpha < 0, \quad N_\alpha(x^1) = \min_{\Omega} N_\alpha < 0, \tag{8} \]
then at least one of \( x^0 \) and \( x^1 \) satisfies the following condition:
\[ |x| \leq R_0. \tag{9} \]
Lemma 2.3. (Narrow region principle). Let $\Omega$ be bounded in $\Sigma$, such that it is contained in $\{x | \lambda - \delta < x_1 < \lambda\}$ with small $\delta$. Assume that $m \in L_{sp} \cap C_{loc}^{1,1}(\Omega)$ and $n \in L_{2t} \cap C_{loc}^{1,1}(\Omega)$ are lower semi-continuous on $\Omega$ satisfying

\[
\begin{cases}
(-\Delta - \lambda_f)^{n}m(x) - (-\Delta - \lambda_f)^{n}m(x) - C_1(x)M_\alpha(x) - C_2(x)N_\alpha(x) \geq 0, & x \in \Omega, \\
(-\Delta)^{n}N_\alpha(x) - C_3(x)M_\alpha(x) \geq 0, & x \in \Omega,
\end{cases}
\]

where $C_1(x), C_2(x), C_3(x)$ are bounded from below in $\Omega$. If there exists $y^0 \in \Sigma$ such that $M_\alpha(y^0) > 0$, then

\[
M_\alpha(x), N_\alpha(x) \geq 0, \quad x \in \Omega
\]

for $\delta$ being small enough. Furthermore, if $M_\alpha(x) = 0$ or $N_\alpha(x) = 0$ at some point in $\Omega$, then

\[
M_\alpha(x) = N_\alpha(x) \equiv 0 \text{ almost everywhere in } \mathbb{R}^n.
\]

When the narrow region $\Omega$ is unbounded, the above conclusions still hold provided that

\[
\lim_{|x| \to \infty} M_\alpha(x), N_\alpha(x) \geq 0.
\]

3. The proof of lemmas.

3.1. Proof of Lemma 2.1 (Decay at infinity).

Suppose (9) is violated; according to a similar derivation procedure employed in Theorem 1.3 [11], we obtain the inequalities:

\[
(-\Delta)^{n}N_\alpha(x^1) \leq \frac{C}{|x^1|^2}N_\alpha(x^1) < 0,
\]

\[
M_\alpha(x^1) < 0
\]

and

\[
N_\alpha(x^1) \geq CC_3|x^1|^2M_\alpha(x^1).
\]

In view of (7), (14) and the lower semi-continuity of $u$ on $\bar{\Omega}$, we can show that there exists $x^0 \in \Omega$ such that

\[
M_\alpha(x^0) = \min_{\Omega} M_\alpha < 0.
\]

For the sake of brevity, let $G(t) = |t|^{p-2}t$, then $G'(t) = (p-1)|t|^{p-2} \geq 0$.

\[
\begin{align*}
(-\Delta - \lambda_f)^{n}m_\alpha(x^0) - (-\Delta - \lambda_f)^{n}m(x^0) \\
= C_{n,sp}PV \int_{\mathbb{R}^n} \frac{G(m_\alpha(x^0) - m_\alpha(y)) - G(m(x^0) - m(y))}{e^{\lambda f(|x^0-y|)|x^0-y|^{n+sp}}} dy \\
= C_{n,sp}PV \int_{\mathbb{R}^n} \frac{G(m_\alpha(x^0) - m_\alpha(y)) - G(m(x^0) - m(y))}{e^{\lambda f(|x^0-y|)|x^0-y|^{n+sp}}} dy \\
+ C_{n,sp}PV \int_{\mathbb{R}^n} \frac{G(m_\alpha(x^0) - m(y)) - G(m(x^0) - m(y))}{e^{\lambda f(|x^0-y|)|x^0-y|^{n+sp}}} dy
\end{align*}
\]
\[ C_{n, sp} PV \int_{\Sigma} \left[ \frac{1}{e^{\lambda f(|x^0 - y^0|)|x^0 - y^0|^{n+sp}}} - \frac{1}{e^{\lambda f(|x^0 - y^0|)|x^0 - y^0|^{n+sp}} - \frac{1}{e^{\lambda f(|x^0 - y^0|)|x^0 - y^0|^{n+sp}}} \right] \times \left[ G(m_\alpha(x^0) - m_\alpha(y)) - G(m(x^0) - m(y)) \right] dy \]
\[ + C_{n, sp} PV \int_{\Sigma} \frac{G(m_\alpha(x^0) - m_\alpha(y)) - G(m(x^0) - m_\alpha(y)) + \hat{G}(y) dy}{e^{\lambda f(|x^0 - y^0|)|x^0 - y^0|^{n+sp}}} \leq C_{n, sp} \int_{\Sigma} \frac{G'(\xi(y)) + G'(\eta(y))}{e^{\lambda f(|x^0 - y^0|)|x^0 - y^0|^{n+sp}}} dy, \]

where \( \hat{G}(y) = [G(m_\alpha(x^0) - m(y)) - G(m(x^0) - m(y))], \xi(y) \in (m_\alpha(x^0) - m(y), m(x^0) - m_\alpha(y)) \) and \( \eta(y) \in (m_\alpha(x^0) - m(y), m(x^0) - m(y)) \). Let \( R = |x^0| \). Fix a point \( x_R \in \Sigma \) satisfying

\[ B_R(x_R) \subset \Sigma \text{ and } |x_R| = MR. \]

Here \( A \) is a sufficiently large number such that, for any \( y \in B_R(R) \), we have

\[ m(y) \leq \frac{C}{A^2 R^2} \leq \frac{c}{R^2} \leq m(x^0), \quad (17) \]

which is ensured by (7). According to

\[ C_{n, sp} M_\alpha(x^0) \int_{\Sigma} \frac{G'(\xi(y)) + G'(\eta(y))}{e^{\lambda f(|x^0 - y^0|)|x^0 - y^0|^{n+sp}}} dy \]
\[ \leq C_{n, sp} M_\alpha(x^0) \int_{B_R(x_R)} \frac{G'(\eta(y))}{e^{\lambda f(|x^0 - y^0|)|x^0 - y^0|^{n+sp}}} dy \]
\[ \leq CM_\alpha(x^0) \int_{B_R(x_R)} \frac{m(p-2)(x^0)}{e^{\lambda f(|x^0 - y^0|)|x^0 - y^0|^{n+sp}}} dy \]
\[ \leq C \frac{M_\alpha(x^0)}{e^{(p-2)|x_R^2|}} \int_{B_R(x_R)} \frac{1}{e^{\lambda f(|x^0 - y^0|)|x^0 - y^0|^{n+sp}}} dy \]
\[ \leq C \frac{M_\alpha(x^0)}{e^{(p-2)|x_R^2|}} \int_{B_R(x_R)} \frac{1}{e^{\lambda f(|x^0 - y^0|)|x^0 - y^0|^{n+sp}}} dy \]

we arrive at

\[ (-\Delta - \lambda f)_p^* m_\alpha(x^0) - (-\Delta - \lambda f)_p^* m(x^0) \leq C \frac{M_\alpha(x^0)}{|x^0|^{sp} e^{(p-2)|x^0|^2 + \lambda f(|x^0|)}}, \quad (18) \]

Considering (5), (6), (7), (15), (18) and \( C_2(x), C_3(x) > 0 \), we can get

\[ 0 \leq (-\Delta - \lambda f)_p^* m_\alpha(x^0) - (-\Delta - \lambda f)_p^* m(x^0) - C_1(x) M_\alpha(x^0) - C_2(x^0) N_\alpha(x^0) \]
\[ \leq C \frac{M_\alpha(x^0)}{|x^0|^{sp} e^{(p-2)|x^0|^2 + \lambda f(|x^0|)}} - C_1(x) M_\alpha(x^0) - C_2(x^0) N_\alpha(x^1) \]
Proof of Lemma 2.3 (Narrow region principle).

3.2. Proof of Lemma 2.3 (Narrow region principle).
In general, we assume that there exists \( x^1 \in \Omega \) such that

\[ N_\alpha(x^1) = \min_\Omega N_\alpha < 0. \]

Otherwise, the arguments similar to the ones employed in the last proof leads to a contradiction that there exists \( x^0 \in \Omega \) such that

\[ M_\alpha(x^0) = \min_\Omega M_\alpha < 0. \]

It is easy to compute that

\[ (-\Delta)^t N_\alpha(x^1) \leq C_{n,t} \int_\Sigma \frac{2N_\alpha(x^1)}{|x^1 - y|^\alpha n + 2t} dy. \]  \hfill (19)

According to a similar derivation procedure used in the proof of Theorem 1.4 in [11], we have

\[ \int_\Sigma \frac{1}{|x^1 - y|^\alpha n + 2t} dy \sim C \frac{1}{\delta^{2t}}. \]  \hfill (20)

Combining (10), (19) with (20) yields

\[ C_3(x^1) M_\alpha(x^1) \leq (-\Delta)^t N_\alpha(x^1) \leq \frac{C N_\alpha(x^1)}{\delta^{2t}} < 0. \]  \hfill (21)

In view of \( C_3(x) > 0 \), (21) reveals that

\[ M_\alpha(x^1) < 0. \]

Hence, there exists \( x^0 \in \Omega \) such that

\[ M_\alpha(x^0) = \min_\Omega M_\alpha < 0. \]

Similar to (16), One can obtain

\[ (-\Delta - \lambda_f)_p^* m_\alpha(x^0) - (-\Delta - \lambda_f)_p^* m(x^0) \]

\[ = C_{n,sp} PV \int_\Sigma \left[ G(m_\alpha(x^0) - m_\alpha(y)) - G(m(x^0) - m_\alpha(y)) \right] \chi_{\Sigma} \frac{dy}{e^{\lambda f(|x^0 - y|^\alpha)|x^0 - y|^\alpha n + sp}} \]

\[ + C_{n,sp} PV \int_\Sigma \left[ \frac{1}{e^{\lambda f(|x^0 - y|^\alpha)|x^0 - y|^\alpha n + sp}} - \frac{1}{e^{\lambda f(|x^0 - y|^\alpha)|x^0 - y|^\alpha n + sp}} \right] \times \left[ G(m_\alpha(x^0) - m_\alpha(y)) - G(m(x^0) - m_\alpha(y)) \right] dy \]

\[ = C_{n,sp} PV (J_1 + J_2). \]  \hfill (22)
According to (22), (23) and (27), we get
\[ \tau \] where
\[ \lambda, \] together with continuity of \( \tau \).

For \( J_2 \), we define \( \delta_{x^0} = \text{dist}\{T_{x^0}, x^0\} \) and find that \( \delta_{x^0} = \alpha - x_1^0 \). By the direct calculation, we obtain
\[
\frac{1}{e^{\lambda f(|x^0-y^0|)|x^0-y^0|}} - \frac{1}{e^{\lambda f(|y^0-y^0|)|x^0-y^0|}}
\]
\[ = \frac{2(\lambda |x^0 - \tau| f'(|x^0 - \tau|) + n + sp)(\lambda - y_1)}{e^{\lambda f(|x^0-\tau|)|x^0-\tau|}}, \] where \( \tau \) is between \( y \) and \( y^0 \). In consequence, we get
\[
J_2 = \int \left[ \frac{1}{e^{\lambda f(|x^0-y|)|x^0-y|}} - \frac{1}{e^{\lambda f(|y^0-y|)|x^0-y|}} \right] \times [G(m_{a}(x^0) - m_{a}(y)) - G(m(x^0) - m(y))] dy
\]
\[ = \delta_{x^0} \int \frac{2(\lambda |x^0 - \tau| f'(|x^0 - \tau|) + n + sp)(\lambda - y_1)}{e^{\lambda f(|x^0-\tau|)|x^0-\tau|}} \times [G(m_{a}(x^0) - m_{a}(y)) - G(m(x^0) - m(y))] dy
\]
\[ = \delta_{x^0} S(x^0). \]

Then we can obtain that there exists a constant \( a \) such that
\[ S(x^0) = -a, \] as \( \delta_{x^0} \to 0, \]
which, together with continuity of \( S(x^0) \) with respect to \( x^0 \), leads to
\[ J_2 \leq -\frac{a}{2} \delta_{x^0}. \] (26)

According to (22), (23) and (27), we get
\[ (-\Delta - \lambda f)^*m_{a}(x^0) - (-\Delta - \lambda f)^*m(x^0) \leq -C\delta_{x^0}. \] (27)

For \( x^* = (\lambda, (x^0)^*) \in T_{x^0} \),
\[ 0 = M_{a}(x^*) = M_{a}(x^0) + \nabla M_{a}(x^0)(x^* - x^0) + o(|x^* - x^0|), \]
due to
\[ \nabla M_{a}(x^0) = 0. \]

So we have
\[ M_{a}(x^0) = o(1)\delta_{x^0}, \] for sufficiently small \( \delta_{x^0} \).

(28)

In view of (10), (21), (27), and (28), we get for sufficiently small \( \delta \) that
\[ 0 \leq (-\Delta - \lambda f)^*m_{a}(x^0) - (-\Delta - \lambda f)^*m(x^0) - C_1(x^0)M_{a}(x^0) - C_2(x^0)N_{a}(x^0) \]
\[ \leq C(-1 + o(1) - \delta^2 \delta_{x^0}) < 0, \]
which is a contradiction. Hence (11) holds. For the proof of (12), we refer the reader to the details of Theorem 1.4 in [11]. This completes the proof of Lemma 2.3 (Narrow region principle).
4. Radial symmetry of positive solutions. In this part, we show the radial symmetry of positive solutions for (2) by using the direct method of moving planes. It will be accomplished in two steps with the aid of Lemmas 2.2 and 2.3.

Step 1. We establish the following inequality.

\[ M_\alpha(x), N_\alpha(x) \geq 0, \forall x \in \Sigma_\lambda, \text{ when } \alpha \text{ is negative enough.} \tag{29} \]

Suppose that (29) does not hold and there exists \( x^0 \in \Sigma_\alpha \) such that

\[ M_\alpha(x^0) = \min_{\Sigma_\alpha} M_\alpha < 0. \]

By applying the process used to obtain (16)-(18), we obtain

\[ (-\Delta - \lambda f)_p m_\alpha(x^0) - (-\Delta - \lambda f)_p m(x^0) \leq C \frac{M_\alpha(x^0)}{|x^0|^{p-1} \gamma + \lambda f(|x^0|)}. \tag{30} \]

Next, we show that the following relation holds true:

\[ N_\alpha(x^0) < 0. \tag{31} \]

Otherwise, according to (3) and (4), we get for sufficiently negative \( \alpha \) that

\[
\begin{align*}
(-\Delta - \lambda f)_p m_\alpha(x^0) - (-\Delta - \lambda f)_p m(x^0) \\
= n_\alpha(x^0)\ln(m_\alpha(x^0) + 1)]^{q-1} - n(x^0)\ln(m(x^0) + 1)]^{q-1} \\
\geq n(x^0)\ln(m(x^0) + 1)]^{q-1} - n(x^0)\ln(m(x^0) + 1)]^{q-1} \\
\geq \frac{C}{e^{1/|x^0|^\gamma} m_\alpha(x^0)} \left[ \ln(m(x^0) + 1) \right]^{q-1} - n(x^0)\ln(m(x^0) + 1)]^{q-1} \\
\geq \frac{C q}{e^{1/|x^0|^\gamma} m_\alpha(x^0)} \xi(x^0)^{q-1} - n(x^0)\ln(m(x^0) + 1)]^{q-1} \\
\geq \frac{C q}{e^{1/|x^0|^\gamma} m_\alpha(x^0)} \xi(x^0)^{q-2}(x^0) M_\alpha(x^0) \\
\geq \frac{C q}{e^{1/|x^0|^\gamma} m(x^0)} \xi(x^0)^{q-2}(x^0) M_\alpha(x^0),
\end{align*}
\]

where \( \xi(x) \in (m_\alpha(x^0), m(x^0)) \). By the condition (5) assumed in Theorem (2.1), we observe that (30) contradicts (32). Hence, (31) holds true. Therefore, there exists \( x^1 \in \Sigma_\alpha \) such that

\[ N_\alpha(x^1) = \min_{\Sigma_\alpha} N_\alpha < 0. \]

Next, we verify that

\[ M_\alpha(x^1) < 0. \tag{33} \]

On the contrary, let (33) be false. Then, analogue to (13), we get

\[ (-\Delta)^t N_\alpha(x^1) \leq \frac{C}{|x^1|^{2t}} N_\alpha(x^1) < 0, \]
Hence, by the proof of Lemma (2.2), one of 
\[ M \text{ and } H \]
where \( \zeta \in (m(x), m(x^1)) \), which contradicts (33).
Hence we obtain
\[
\alpha, \quad N
\]
\( \zeta \in (m(x), m(x^1)) \), which contradicts (33).
Hence we obtain
\[
(\Delta - \lambda f)^p_m \alpha(x^0) - (\Delta - \lambda f)^p m(x^0)
\]
Thus (29) holds true and the first step is complete.

Step 2. We can move \( T \) to the right until it reaches its limit as long as (29) holds.
Setting
\[ \alpha_0 = \sup \{ \alpha | M_\mu(x) \geq 0, N_\mu(x) \geq 0, \forall x \in \Sigma_\mu, \mu \leq \alpha \}, \]
we prove that
\[ M_{\alpha_0}(x) = N_{\alpha_0}(x) \equiv 0, \quad x \in \Sigma_{\alpha_0} \quad (37) \]
or
\[ M_{\alpha_0}(x), N_{\alpha_0}(x) > 0, \quad x \in \Sigma_{\alpha_0}. \]
Without loss generality, we assume that there exists \( \varphi \in \Sigma_{\alpha_0} \) satisfying
\[ M_{\alpha_0}(\varphi) = \min_{\Sigma_{\alpha_0}} M_{\alpha_0} = 0, \]
and
\[ (-\Delta)^f N_{\alpha}(x^1) \geq qm^{q-2}(x^1)M_{\alpha}(x^1). \quad (35) \]
For sufficiently negative \( \alpha \), let
\[ C_1(x^0) = \frac{Cq}{e^{x^0} + (q-3)|x^0|^q}, \quad C_2(x^0) = \left[ \frac{\ln(m(x^0) + 1)}{q-1} \right]^{q-1} \approx \frac{1}{e^{(q-1)|x^0|^q}}, \quad C_3(x^1) = \left[ \frac{\ln(m(x^1) + 1)}{q-1} \right]^{q-1} \approx \frac{1}{e^{(q-1)|x^0|^q}}. \]

Hence, by the proof of Lemma (2.2), one of \( M_{\alpha}(x) \) and \( N_{\alpha}(x) \) must be nonnegative in \( \Sigma_{\alpha} \) when \( \alpha \) is sufficiently negative. In general, we suppose that
\[ M_{\alpha}(x) > 0, \quad x \in \Sigma_{\alpha}. \quad (36) \]
If \( N_{\alpha}(x) < 0 \) at some point in \( \Sigma_{\alpha} \), then there exists \( x^1 \in \Sigma_{\alpha} \) such that
\[ N_{\alpha}(x^1) = \min_{\Sigma_{\alpha}} N_{\alpha} < 0. \]
From the foregoing steps, we obtain the contradiction:
\[ 0 > \frac{CN_{\alpha}(x^1)}{|x|^2} \geq (-\Delta)^f N_{\alpha}(x^1) \geq qm^{q-2}(x^1)M_{\alpha}(x^1) \geq 0. \]
Thus (29) holds true and the first step is complete.

Step 2. We can move \( T \) to the right until it reaches its limit as long as (29) holds.
Setting
\[ \alpha_0 = \sup \{ \alpha | M_\mu(x) \geq 0, N_\mu(x) \geq 0, \forall x \in \Sigma_\mu, \mu \leq \alpha \}, \]
we prove that
\[ M_{\alpha_0}(x) = N_{\alpha_0}(x) \equiv 0, \quad x \in \Sigma_{\alpha_0} \quad (37) \]
or
\[ M_{\alpha_0}(x), N_{\alpha_0}(x) > 0, \quad x \in \Sigma_{\alpha_0}. \]
Without loss generality, we assume that there exists \( \varphi \in \Sigma_{\alpha_0} \) satisfying
\[ M_{\alpha_0}(\varphi) = \min_{\Sigma_{\alpha_0}} M_{\alpha_0} = 0, \]
that is, 
\[ M_{\alpha_0}(x) \equiv 0, \ x \in \Sigma_{\alpha_0}. \]

Otherwise, following the calculations used to obtain (16), we get
\[
(-\Delta - \lambda_f)^s_p m_\alpha(\varphi) - (-\Delta - \lambda_f)^s_p m(\varphi) \\
= C_{n,sp} \text{PV} \int_{\mathbb{R}^n} \frac{G(m_\alpha(\varphi) - m_\alpha(y)) - G(m(\varphi) - m(y))}{e^{\lambda f(|\varphi - y|)|\varphi - y|^{n+sp}}} \, dy \\
= C_{n,sp} \text{PV} \int_{\mathbb{R}^n} \frac{G(m(\varphi) - m_\alpha(y)) - G(m(\varphi) - m(y))}{e^{\lambda f(|\varphi - y|)|\varphi - y|^{n+sp}}} \, dy \\
= C_{n,sp} \text{PV} \int_{\Sigma_{\alpha_0}} \left[ \frac{1}{e^{\lambda f(|\varphi - y|)|\varphi - y|^{n+sp}}} - \frac{1}{e^{\lambda f(|\varphi - y'|)|\varphi - y'|^{n+sp}}} \right] \times \\
\times [G(m(\varphi) - m_\alpha(y)) - G(m(\varphi) - m(y))] \, dy \\
< 0.
\]

(38)

On the other side of the shield, we find that
\[
(-\Delta - \lambda_f)^s_p m_\alpha(\varphi) - (-\Delta - \lambda_f)^s_p m(\varphi) \\
= n_\alpha(\varphi)[\ln(n_\alpha(\varphi) + 1)]^{q-1} - n(\varphi)[\ln(n(\varphi) + 1)]^{q-1} \\
= N_{\alpha_0}(\varphi)[\ln(n_\alpha(\varphi) + 1)]^{q-1} \geq 0,
\]

which contradicts (38). In view of the anti-symmetry of \( M_{\alpha_0} \), we have
\[ M_{\alpha_0}(x) \equiv 0, \ x \in \mathbb{R}^n. \]

In a similar manner, we can obtain
\[ N_{\alpha_0}(x) \equiv 0, \ x \in \mathbb{R}^n. \]

If (37) is false, then
\[ M_{\alpha_0}(x), \ N_{\alpha_0}(x) > 0, \ x \in \Sigma_{\alpha_0}. \]  

Next, we show that there exists \( i > 0 \) such that, for any \( \alpha \in (\alpha_0, \alpha_0 + i) \),
\[ M_\alpha(x), \ N_\alpha(x) \geq 0, \ x \in \Sigma_\alpha \]
holds. From (40), there exist \( K_\delta, R_0 > 0 \) such that
\[ M_{\alpha_0}(x), \ N_{\alpha_0}(x) \geq K_\delta > 0, \ x \in \Sigma_{\alpha_0 - \delta} \cap B_{R_0}(0). \]

In view of the continuity of \( M_\alpha(x), \ N_\alpha(x) \) about \( \alpha \), there is \( i > 0 \) such that, for \( \alpha \in (\alpha_0, \alpha_0 + i) \),
\[ M_{\alpha_0}(x), \ N_{\alpha_0}(x) \geq 0, \ x \in \Sigma_{\alpha_0 - \delta} \cap B_{R_0}(0) \]
holds. By Theorem 2.2, if
\[ M_\alpha(x^0) = \min_{\Sigma_\alpha} M_\alpha < 0, \ N_\alpha(x^1) = \min_{\Sigma_\alpha} N_\alpha < 0, \]
then there is large enough \( R_0 \) such that, one of the \( x^0 \) and \( x^1 \) must be in the interior of \( B_{R_0}(0) \). Assuming that \( |x^0| < R_0 \), we get
\[ x^0 \in (\Sigma_\alpha \setminus \Sigma_{\alpha - \delta}) \cap B_{R_0}(0). \]  

(42)
Now we verify that $x^1$ also satisfies (42). On the contrary, $x^1 \in \Sigma_\alpha \cap B^c_{R_0}(0)$ and it follows by (3), (22) and (28) that
\begin{equation}
0 > \frac{C}{|x^1|^{2q}} N_\alpha(x^1) \geq (-\Delta)^{q} N_\alpha(x^1) \geq \frac{q[\ln(\zeta(x) + 1)]^{q-1}}{\zeta(x) + 1} M_\alpha(x^0),
\end{equation}
\begin{equation}
-C \delta_x \geq (-\Delta - \lambda_f)^p m_\alpha(x^0) - (-\Delta - \lambda_f)^p m(x^0)
= \ln(m_\alpha(x^0) + 1)^{q-1} N_\alpha(x^0) + \frac{(q-1)n(x^0)[\ln(\sigma(x) + 1)]^{q-2}}{\sigma(x) + 1} M_\alpha(x^0)
\geq \ln(m(x^0) + 1)^{q-1} N_\alpha(x^1) + \frac{(q-1)n(x^0)[\ln(\sigma(x) + 1)]^{q-2}}{\sigma(x) + 1} M_\alpha(x^0),
\end{equation}
where $\delta$ is sufficiently small, $\zeta(x) \in (m_\alpha(x^1), m(x^0))$, $\sigma(x) \in (m_\alpha(x^0), m(x^0))$. Combining (4) with (44), for sufficiently small $\delta$, $\iota$, we have
\begin{align}
1 \leq -\frac{C}{\delta_x} \left\{ \ln(m(x^0) + 1)^{q-1} N_\alpha(x^1) + \frac{(q-1)n(x^0)[\ln(\sigma(x) + 1)]^{q-2}}{\sigma(x) + 1} M_\alpha(x^0) \right\}
& \leq -\frac{C}{\delta_x} \left\{ q|x^1|^{2q} \ln(\zeta(x) + 1)^{q-1} M_\alpha(x^0) + \frac{(q-1)n(x^0)[\ln(\sigma(x) + 1)]^{q-2}}{\sigma(x) + 1} M_\alpha(x^0) \right\}
& = -\frac{C}{\delta_x} M_\alpha(x^0) \left\{ q|x^1|^{2q} \ln(\zeta(x) + 1)^{q-1} M_\alpha(x^0) + \frac{(q-1)n(x^0)[\ln(\sigma(x) + 1)]^{q-2}}{\sigma(x) + 1} M_\alpha(x^0) \right\}
& \leq C \circ (1) \left\{ \frac{|x^1|^{2q}}{\sigma(x) + 1} + \frac{(q-1)n(x^0)[\ln(\sigma(x) + 1)]^{q-2}}{\sigma(x) + 1} \right\},
\end{align}
where $R_0$ is large enough. Since $|x^1| > R_0$ and $x^0 \in (\Sigma_\alpha \setminus \Sigma_{\alpha - \delta}) \cap B_{R_0}(0)$, so $\frac{|x^1|^{2q}}{\sigma(x) + 1}$ is bounded, which means that (45) does not hold true for $\delta$, $\iota$ being small enough. This contradiction leads to
\begin{equation}
x^1 \in (\Sigma_\alpha \setminus \Sigma_{\alpha - \delta}) \cap B_{R_0}(0).
\end{equation}
Then, by (43), (44) and Lemma 2.3, we have
\begin{equation}
M_\alpha(x), \ N_\alpha(x) \geq 0, \ x \in (\Sigma_\alpha \setminus \Sigma_{\alpha - \delta}) \cap B_{R_0}(0)
\end{equation}
for sufficiently small $\delta$, $\iota$. Thus (41) holds, which contradicts the definition of $\alpha_0$. Hence, we must have
\begin{equation}
M_{\alpha_0}(x) = N_{\alpha_0}(x) \equiv 0, \ x \in \Sigma_{\alpha_0}.
\end{equation}
As the direction of $x_1$ is arbitrary, therefore we obtain the radial symmetry of the positive solutions of (2). Moreover, the monotonic decreasing property follows from (29). This finishes the proof.

5. Conclusions. We investigated radial symmetry and monotonicity of positive solutions to a logarithmic Choquard equation by introducing a new kind of generalized nonlinear tempered fractional $p$-Laplacian operator of the form $(-\Delta - \lambda_f)^p$. Our approach is analytic and is based on the direct method of moving planes. It is imperative to note that tempered fractional derivatives play a significant role in modeling the natural phenomena and much of the known material on the topic relies on numerical techniques. Since the work presented in this paper is of analytic nature, so it will add versatility to the related literature.
Acknowledgments. This work is supported by NSFC (No.12001344), Science and Technology Innovation Project of Shanxi Normal University (No.2019XSY025), the Graduate Innovation Program of Shanxi, China (Nos.2019SY301 and 2020SY337). All authors equally contributed this manuscript.

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Received May 2020; revised July 2020.

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