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VORTICITY AND STREAM FUNCTION FORMULATIONS FOR THE 2D NAVIER-STOKES EQUATIONS IN A BOUNDED DOMAIN

JULIEN LEQUEURRE AND ALEXANDRE MUNNIER

Abstract. The main purpose of this work is to provide a Hilbertian functional framework for the analysis of the planar Navier-Stokes (NS) equations either in vorticity or in stream function formulation. The fluid is assumed to occupy a bounded possibly multiply connected domain. The velocity field satisfies either homogeneous (no-slip boundary conditions) or prescribed Dirichlet boundary conditions. We prove that the analysis of the 2D Navier-Stokes equations can be carried out in terms of the so-called nonprimitive variables only (vorticity field and stream function) without resorting to the classical NS theory (stated in primitive variables, i.e. velocity and pressure fields). Both approaches (in primitive and nonprimitive variables) are shown to be equivalent for weak (Leray) and strong (Kato) solutions. Explicit, Bernoulli-like formulas are derived and allow recovering the pressure field from the vorticity fields or the stream function. In the last section, the functional framework described earlier leads to a simplified rephrasing of the vorticity dynamics, as introduced by Maekawa in [52]. At this level of regularity, the vorticity equation splits into a coupling between a parabolic and an elliptic equation corresponding respectively to the non-harmonic and harmonic parts of the vorticity equation. By exploiting this structure it is possible to prove new existence and uniqueness results, as well as the exponential decay of the palinstrophy (that is, loosely speaking, the $H^1$ norm of the vorticity) for large time, an estimate which was not known so far.

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1. Introduction

The NS equations stated in primitive variables (velocity and pressure) have been received much attention since the pioneering work of Leray [44] [46] [45]. Strong solutions were shown to exist in 2D by Lions and Prodi [50] and Lions [47]. Henceforth, we will refer for instance to the books of Lions [48, Chap. 1, Section 6], Ladyzhenskaya [42] and Temam [63] for the main results that we shall need on this topic.

In 2D, the vorticity equation provides an attractive alternative model to the classical NS equations for describing the dynamics of a viscous, incompressible fluid. Thus it exhibits many advantages: It is a nice advection-diffusion scalar equation while the classical NS system, although parabolic as well, is a coupling between an unknown vector field (the velocity) and an unknown scalar field (the pressure). However the lack of natural and simple boundary conditions for the vorticity field makes the analysis of the vortex dynamics troublesome and explains why the problem has been addressed mainly so far in the case where the fluid occupies the whole space. In this configuration, a proof of existence and uniqueness for the corresponding Cauchy problem assuming the initial data to be integrable and twice continuously differentiable was first provided by McGrath [54]. Existence results were extended independently by Cottet [12] and Giga et al. [25] to the case where the initial data is a finite measure. These authors proved that uniqueness also holds when the atomic part of the initial vorticity is sufficiently small; see also [34]. For initial data in $L^1(\mathbb{R}^2)$, the Cauchy problem was proved to be well posed by Ben-Artzi [5], and Brézis [9]. Then, Gallay and Wayne [22] and Gallagher et al. [17] proved the uniqueness of the solution for an initial vorticity that is a large Dirac mass. Finally, Gallagher and Gallay [18] succeeded in removing the smallness assumption on the atomic part of the initial measure and shown that the Cauchy problem is globally well-posed for any initial data in $\mathcal{M}(\mathbb{R}^2)$.

As explained in [20], Chap. 11, §2.7, the vorticity equation (still set in the whole space) provides an interesting line of attack to study the large time behavior of the NS equations. This idea was exploited for instance by Giga and Kambe [24], Carpio [10], Gallay et al. [21] [20] [22] [19] and Kukavica and Reis [40].

Among the quoted authors above, some of them, such as McGrath [54] and Ben-Artzi [5] were actually interested in studying the convergence of solutions to the NS equations towards solutions of the Euler equations when the viscosity vanishes. This is a very challenging problem, well understood in the absence of solid walls (that is, when the fluid fills the whole space) and for which the vorticity equation plays a role of...
paramount importance. In the introduction of the chapter “Boundary Layer Theory” in the book [59], Lighthill argues that To explain convincingly the existence of boundary layers, and, also to show what consequences of flow separation (including matters of such practical importance as the effect of trailing vortex wakes) may be expected, arguments concerning vorticity are needed. More recently, Chemin in [11] claims The key quantity for understanding 2D incompressible fluids is the vorticity. There exists a burgeoning literature treating the problem of vanishing viscosity limit and we refer to the recently-released book [26, Chap. 15] for a comprehensive list of references. When the fluid is partially of totally confined, the analysis of the vanishing viscosity limits turns into a more involved problem due to the formation of a boundary layer. In this case, the vorticity equation still plays a crucial role: In [33], Kato gives a necessary and sufficient condition for the vanishing viscosity limit to hold and this condition is shown by Kelliher [35, 36, 38] to be equivalent to the formation of a vortex sheet on the boundary of the fluid domain.

In the presence of walls, the derivation of suitable boundary conditions for the vorticity was also of prime importance for the design of numerical schemes. A review of these conditions (and more generally on stream-vorticity based numeral schemes), can be found in [23, 29, 14] and [56]. However, it has been actually well known since the work of Quatarpelle and co-workers [57, 31, 32, 16, 3, 8], that the vorticity does not satisfy pointwise conditions on the boundary but rather a non local or integral condition which reads:

\[(1.1) \quad \text{for all } h \in \mathcal{H}, \quad \int_{\mathcal{F}} \omega h \, dx = 0,\]

where \(\mathcal{F}\) is the domain of the fluid and \(\mathcal{H}\) the closed subspace of the harmonic functions in \(L^2(\mathcal{F})\) (see also [6, Lemma 1.2]). Anderson [1] and more recently Maekawa [52] propose nonlinear boundary conditions that will be shown to be equivalent (see Section 8) to:

\[(1.2) \quad \text{for all } h \in \mathcal{H}, \quad \int_{\mathcal{F}} (-\nu \Delta \omega + u \cdot \nabla \omega) h \, dx = 0,\]

where \(\nu > 0\) is the kinematic viscosity and \(u\) the velocity field deduced from \(\omega\) via the Biot-Savart law. Providing that \(\omega\) is a solution to the classical vorticity equation, Equality (1.2) is nothing but the time derivative of (1.1).

Starting from (1.1), the aim of this paper is to provide a Hilbertian functional framework allowing the analysis of the 2D vorticity equation in a bounded multiply connected domain. The analysis is wished to be self-contained, without recourse to classical results on the NS equations in primitive variables. We shall prove that the analysis can equivalently be carried out at the level of the stream function. Homogeneous and nonhomogeneous boundary conditions for the velocity field will be considered and explicit formulas for the pressure will be derived. In the last section, new estimates (in particular for the palinstrophy) will be established.

2. General settings

The planar domain \(\mathcal{F}\) occupied by the fluid is assumed to be open, bounded and path-connected. We assume furthermore that its boundary \(\Sigma\) can be decomposed into a disjoint union of \(C^{1,1}\) Jordan curves:

\[(2.1) \quad \Sigma = \left( \bigcup_{k=1}^{N} \Sigma_{k}^{-} \right) \cup \Sigma^{+}.\]

The curves \(\Sigma_{k}^{-}\) for \(k \in \{1, \ldots, N\}\) are the inner boundaries of \(\mathcal{F}\) while \(\Sigma^{+}\) is the outer boundary. On \(\Sigma\) we denote by \(n\) the unit normal vector directed toward the exterior of the fluid and by \(\tau\) the unit tangent vector oriented in such a way that \(\tau^\perp = n\) (see Fig. 1). Here and subsequently in the paper, for every \(x = (x_1, x_2) \in \mathbb{R}^2\), the notation \(x^\perp\) is used to represent the vector \((-x_2, x_1)\).

Let now \(T\) be a positive real number and define the space-time cylinder \(\mathcal{F}_T = (0, T) \times \mathcal{F}\), whose lateral boundary is \(\Sigma_T = (0, T) \times \Sigma\). The velocity of the fluid is supposed to be prescribed, equal on \(\Sigma_T\) to some vector field \(b\) satisfying the compatibility condition:

\[(2.2) \quad \int_{\Sigma} b \cdot n \, ds = 0 \quad \text{on } (0, T).\]
The density and the dynamic viscosity of the fluid, denoted respectively by \( \rho \) and \( \mu \), are both positive constants. The flow is governed by the Navier-Stokes equations. Introducing \( u \) the Eulerian velocity field and \( \pi \) the (static) pressure field, the equations read:

\[
\begin{align*}
\partial_t u + \omega u^\perp - \nu \Delta u + \nabla \left( p + \frac{1}{2} |u|^2 \right) &= f \quad \text{in } \mathcal{F}_T, \\
\nabla \cdot u &= 0 \quad \text{in } \mathcal{F}_T, \\
u \frac{\partial u}{\partial n} &= b \quad \text{on } \Sigma_T, \\
u \frac{\partial u}{\partial t}(0) &= u_i \quad \text{in } \mathcal{F}_T.
\end{align*}
\]

In this system \( \nu = \mu/\rho \) is the kinematic viscosity, \( \frac{1}{2} \rho |u|^2 \) is the dynamic pressure, \( p = \pi/\rho \), \( f \) is a body force, \( u_i \) is a given initial condition and \( \omega \) the vorticity field defined as the curl of \( u \), namely:

\[
\omega = \nabla^\perp \cdot u \quad \text{in } \mathcal{F}_T.
\]

2.1. The NS system in nonprimitive variables. The Helmholtz-Weyl decomposition of the velocity field (see [27, Theorem 3.2]) leads to the existence, at every moment \( t \), of a potential function \( \varphi(t, \cdot) \) and a stream function \( \psi(t, \cdot) \) such that:

\[
u \frac{\partial \varphi}{\partial n}(t, \cdot) = b(t, \cdot) \cdot n \quad \text{on } \Sigma.
\]

The potential function (also referred to as Kirchhoff potential) depends only on the boundary conditions satisfied by the velocity field of the fluid. It is defined at every moment \( t \) as the solution (unique up to an additive constant) of the Neumann problem:

\[
\begin{align*}
\Delta \varphi(t, \cdot) &= 0 \quad \text{in } \mathcal{F} \quad \text{and} \quad \frac{\partial \varphi}{\partial n}(t, \cdot) = b(t, \cdot) \cdot n \quad \text{on } \Sigma.
\end{align*}
\]

The stream function \( \psi \) in (2.5) vanishes on \( \Sigma^+ \) and is constant on every connected component \( \Sigma^-_j \) \((j = 1, \ldots, N)\) of the inner boundary \( \Sigma^- \). Moreover, it satisfies:

\[
\begin{align*}
\Delta \psi(t, \cdot) = \omega(t, \cdot) \quad \text{in } \mathcal{F} \quad \text{and} \quad \frac{\partial \psi}{\partial n}(t, \cdot) = - \left[ b(t, \cdot) - \nabla \varphi(t, \cdot) \right] \cdot \tau \quad \text{on } \Sigma \quad \text{for all } t > 0.
\end{align*}
\]

Forming, at any moment, the scalar product in \( L^2(\mathcal{F}) \) (the bold font notation \( L^2(\mathcal{F}) \) stands for \( L^2(\mathcal{F}; \mathbb{R}^2) \)) of (2.3a) with \( \nabla^\perp \theta \) where \( \theta \) is a test function that vanishes on \( \Sigma^+ \) and is constant on every \( \Sigma^-_j \), we obtain (up to an integration by parts):

\[
\begin{align*}
\int_{\mathcal{F}} \nabla \partial_t \psi \cdot \nabla \theta \, dx + \int_{\mathcal{F}} \omega u \cdot \nabla \theta \, dx - \nu \int_{\mathcal{F}} \nabla \omega \cdot \nabla \theta \, dx &= \int_{\mathcal{F}} \nabla \psi_f \cdot \nabla \theta \, dx \quad \text{on } (0, T).
\end{align*}
\]

In this equality, the force field \( f(t, \cdot) \) has been decomposed according to the Helmholtz-Weyl theorem:

\[
f(t, \cdot) = \nabla \varphi_f(t, \cdot) + \nabla^\perp \psi_f(t, \cdot) \quad \text{for all } t > 0.
\]
Integrating by parts again the terms in (2.8), we end up with the system:

\[
\begin{align*}
\frac{\partial \omega}{\partial t} + u \cdot \nabla \omega - \nu \Delta \omega &= f_V \quad \text{in } F_T \quad (2.9a) \\
- \frac{\partial}{\partial t} \left( \int_{\Sigma_k^b} b \cdot \tau \, ds \right) + \int_{\Sigma_k^b} \omega (b \cdot n) \, ds - \nu \int_{\Sigma_k^b} \frac{\partial \omega}{\partial n} \, ds &= \int_{\Sigma_k^b} \frac{\partial \psi_f}{\partial n} \, ds \quad \text{on } (0, T), \quad k = 1, \ldots, N, \quad (2.9b) \\
\omega(0) &= \omega_i \quad \text{in } F, \quad (2.9c)
\end{align*}
\]

where \( f_V = \Delta \psi_f \) and the initial condition \( \omega_i \) is the curl of \( u_i \) in (2.3d). To be closed, System (2.9) has to be supplemented with the identities (2.5), (2.6) and (2.7).

**Remark 2.1.** The \( N \) equations (2.9b) (that will be termed “Lamb’s fluxes conditions” in the sequel) cannot be derived from (2.9a) (this is well explained in [32, Remark 3.2]). They control the mean amount of vorticity produced on the inner boundaries. Such relations can be traced back to Lamb in [43, Art. 328a] (see also [65] for more recent references), where in a two-dimensional viscous flow the change of circulation along any curve is given by:

\[
\frac{D \Gamma}{D t} = \nu \int \frac{\partial \omega}{\partial n} \, ds.
\]

At this point, the lack of boundary conditions for \( \omega \) might indicate that System (2.9) is unlikely to be solved. Indeed, seeking for an a priori enstrophy estimate (enstrophy is the square of the \( L^2(F) \) norm of the vorticity), we multiply (2.9a) by \( \omega \) and integrate over \( F \), but shortly get stuck with the term:

\[
(2.10) \quad \int_F \Delta \omega \omega \, dx,
\]

that cannot be integrated by parts. The other sticking point is that the boundary value problem (2.7) permitting the reconstruction of the stream function from the vorticity is overdetermined since the stream function \( \psi \) has to satisfy both Dirichlet and Neumann boundary conditions on \( \Sigma \). All these observations are well known.

### 2.2. Some leading ideas

Before going into details, we wish to give some insights on how the aforementioned difficulties can be circumvented. To simplify, we shall focus for the time being on the case of homogeneous boundary conditions (i.e. \( b = 0 \)) and of a simply connected fluid domain (i.e. \( \Sigma = \Sigma^+ \)). The latter assumption leads to the disappearance of the equations (2.9b) in the system.

The first elementary observation, that can be traced back to Quartapelle and Valz-Gris in [57], is that a function \( \omega \) defined in \( F \) is the Laplacian of some function \( \psi \) if and only if the following equality holds for every harmonic function \( h \):

\[
\int_F \omega h \, dx = \int_{\Sigma} \left( \frac{\partial \psi}{\partial n} \bigg|_{\Sigma^+} - \Lambda_{DN} \psi \bigg|_{\Sigma} \right) h \bigg|_{\Sigma} \, ds,
\]

where the notation \( \Lambda_{DN} \) stands to the Dirichlet-to-Neumann operator. Introducing \( \mathcal{H} \), the closed subspace of the harmonic functions in \( L^2(F) \), we deduce from this assertion that:

\[
(2.11) \quad \Delta \mathcal{H}^0_0(F) = \mathcal{H}^\perp \quad \text{in } L^2(F).
\]

We denote by \( V_0 \) the closed space \( \mathcal{H}^\perp \) and decompose the space \( L^2(F) \) into the orthogonal sum

\[
(2.12) \quad L^2(F) = V_0 \oplus \mathcal{H}.
\]

This orthogonality condition satisfied by the vorticity plays the role of boundary conditions classically expected when dealing with a parabolic type equation like (2.9a). The authors in [57] and in [31] do not elaborate on this idea and instead of deriving an autonomous functional framework for the analysis of the vorticity equation (2.9a), System (2.9) is supplemented with the identity:

\[
\omega(t, \cdot) = \Delta \psi(t, \cdot) \quad \text{in } F \quad \text{for all } t \in (0, T),
\]

and some function spaces for the stream function are introduced. However, as it will be explained later on, the dynamics of the flow can be dealt with with any one of the nonprimitive variable alone (vorticity or stream function) by introducing the appropriate functional framework.
Let us go back to the splitting (2.12). The orthogonal projection onto $\mathcal{F}$ in $L^2(\mathcal{F})$ is usually referred to as the harmonic Bergman projection and has been received much attention so far. The Bergman projection, as well as the orthogonal projection onto $V_0$, denoted by $P$ in the sequel, enjoys some useful properties (see for instance [4], [9] and references therein). In particular, $P$ maps continuously $H^k(\mathcal{F})$ onto $H^k(\mathcal{F})$ for every nonnegative integer $k$, providing that $\Sigma$ is of class $C^{k+1,1}$. This leads us to define the spaces $V_1 = PH^1_0(\mathcal{F})$, which is therefore a subspace of $H^1(\mathcal{F})$. We denote by $P_1$ the restriction to $H^0_0(\mathcal{F})$ of the projection $P$. A quite surprising result is that $P_1 : H^0_0(\mathcal{F}) \to V_1$ is invertible and we denote by $Q_1$ its inverse. The operator $Q_1$ will be proved to be the orthogonal projector onto $H^1_0(\mathcal{F})$ in $H^1(\mathcal{F})$ for the semi-norm $\|\nabla \cdot \|_{L^2(\mathcal{F})}$. The space $V_1$ is next equipped with the scalar product

$$(\omega_1, \omega_2)_{V_1} = (\nabla Q_1 \omega_1, \nabla Q_1 \omega_2)_{L^2(\mathcal{F})}, \quad \omega_1, \omega_2 \in V_1,$$

and the corresponding norm is shown to be equivalent to the usual norm of $H^1(\mathcal{F})$. Since the inclusion $H^1_0(\mathcal{F}) \subset L^2(\mathcal{F})$ is continuous, dense and compact, we can draw the same conclusion for the inclusion $V_1 \subset V_0$. Identifying $V_0$ with its dual space by means of Riesz Theorem and denoting by $V_{-1}$ the dual space of $V_1$, we end up with a so-called Gelfand triple of Hilbert spaces (see for instance [7, Chap. 14]):

$$V_1 \subset V_0 \subset V_{-1},$$

where $V_0$ is the pivot space. With these settings, it is classical to introduce first the isometric operator $A^V_1 : V_1 \to V_{-1}$ defined by the relation:

$$(A^V_1 \omega_1, \omega_2)_{V_{-1}, V_1} = (\omega_1, \omega_2)_{V_1}, \quad \text{for all } \omega_1, \omega_2 \in V_1,$$

and next the space $V_2$ as the preimage of $V_0$ by $A^V_1$. The space $V_2$ is a Hilbert space as well, once equipped with the scalar product

$$(\omega_1, \omega_2)_{V_2} = (A^V_2 \omega_1, A^V_2 \omega_2)_{V_0}, \quad \text{for all } \omega_1, \omega_2 \in V_2,$$

and the inclusion $V_2 \subset V_1$ is continuous dense and compact. We denote by $A^V_2$ the restriction of $A^V_1$ to $V_2$ and classical results on Gelfand triples assert that the operator $A^V_2$ is an isometry from $V_2$ onto $V_0$. The crucial observation for our purpose is that, providing that $\Sigma$ is of class $C^{1,1}$:

$$V_2 = \left\{ \omega \in H^2(\mathcal{F}) \cap V_1 : \frac{\partial \omega}{\partial n}|_{\Sigma} = \Lambda_{DN} \omega|_{\Sigma} \right\}, \quad \text{and} \quad A^V_2 \omega = -\Delta \omega \quad \text{for every } \omega \in V_2.$$

In particular, every vorticity in $V_2$ has zero mean flux through the boundary $\Sigma$. Denoting by $V_{k+2}$ the preimage of $V_k$ by $A^V_k$ for every integer $k \geq 1$, we define by induction a chain of embedded Hilbert spaces $V_k$ whose dual spaces are denoted by $V_{-k}$. Each one of the following inclusion is continuous dense and compact:

$$\ldots \subset V_{k+1} \subset V_k \subset V_{k-1} \subset \ldots \subset V_1 \subset V_0 \subset V_{-1} \subset \ldots \subset V_{-k+1} \subset V_{-k} \subset V_{-k-1} \subset \ldots$$

We define as well isometries $A^V_k : V_k \to V_{k-2}$ for all the integers $k$. This construction is made precise in Appendix A. It supplies a suitable functional framework to deal with the linearized vorticity equation. Thus, we shall prove in the sequel that for every $T > 0$, every integer $k$, every $f_V \in L^2(0, T; V_{k-1})$ and every $\omega^i$ in $V_k$ there exists a unique solution

$$(2.13a) \quad \omega \in H^1(0, T; V_{k-1}) \cap C([0, T]; V_k) \cap L^2(0, T; V_{k+1}),$$

to the Cauchy problem:

$$(2.13b) \quad \partial_t \omega + \nu A^V_{k+1} \omega = f_V \quad \text{in } \mathcal{F}_T,$$

$$\omega(0) = \omega^i \quad \text{in } \mathcal{F}.$$  

Let us go back to the problem of enstrophy estimate where we got stuck with the term (2.10). At the level of regularity corresponding to $k = 0$ in (2.13) for instance, we obtain:

$$(2.14) \quad \frac{1}{2} \frac{d}{dt} \|\omega\|_{V_0}^2 + \nu \|\omega\|_{V_1}^2 = \langle f_V, \omega \rangle_{V_1, V_{-1}} \quad \text{on } (0, T).$$

By definition $\|\omega\|_{V_0} = \|\omega\|_{L^2(\mathcal{F})}^2$, which is the expected quantity but the second term in the left hand side is $\|\omega\|_{V_1}^2 = \|\nabla Q_1 \omega\|_{L^2(\mathcal{F})}^2$, whereas one would naively expect $\|\nabla \omega\|_{L^2(\mathcal{F})}^2$. We recall that $Q_1$ is the orthogonal projection onto $H^1_0(\mathcal{F})$. So now, instead of multiplying (2.9a) by $\omega$, let multiply this equation by $Q_1 \omega$, whose
trace vanishes on $\Sigma$, and integrate over $\mathcal{F}$. The term (2.10) is replaced by a quantity that can now be integrated by parts. Thus:

$$
\int_{\mathcal{F}} \Delta \omega Q_1 \omega \, dx = - (\nabla \omega, \nabla Q_1 \omega)_{L^2(\mathcal{F})} = - \| \nabla Q_1 \omega \|^2_{L^2(\mathcal{F})} = - \| \omega \|^2_{V_1}.
$$

On the other hand, regarding the first term in (2.9a), we still have (at least formally):

$$
\int_{\mathcal{F}} \partial_t \omega Q_1 \omega \, dx = \int_{\mathcal{F}} \partial_t \omega \omega \, dx = \frac{1}{2} \frac{d}{dt} \int_{\mathcal{F}} |\omega|^2 \, dx,
$$

because $\omega$ is orthogonal in $L^2(\mathcal{F})$ to the harmonic functions and $Q_1 \omega$ and $\omega$ differ only up to an harmonic function. To sum up, in the enstrophy estimate, the natural dissipative term is not $\| \nabla \omega \|^2_{L^2(\mathcal{F})}$ but $\| \nabla Q_1 \omega \|^2_{L^2(\mathcal{F})}$.

Notice that, since $Q_1$ is the orthogonal projector onto $H_0^1(\mathcal{F})$:

$$
\| \nabla Q_1 \omega \|^2_{L^2(\mathcal{F})} \leq \| \nabla \omega \|^2_{L^2(\mathcal{F})} \quad \text{for all } \omega \in H^1(\mathcal{F}).
$$

Defining the lowest eigenvalue of $A^\dagger$ by means of a Rayleigh quotient:

$$
\lambda_\mathcal{F} = \min_{\omega \in V_1} \frac{\| \nabla \omega \|^2_{V_1}}{\| \omega \|^2_{V_1}} = \min_{\omega \in V_1} \frac{\| \nabla Q_1 \omega \|^2_{L^2(\mathcal{F})}}{\| \omega \|^2_{L^2(\mathcal{F})}}
$$

the following Poincaré-type estimate holds true:

$$
\lambda_\mathcal{F} \| \omega \|^2_{V_1} \leq \| \omega \|^2_{V_1} \quad \text{for all } \omega \in V_1,
$$

and classically leads with (2.14) (assuming that $f_{\nu} = 0$ to simplify) and Grönwall’s inequality to the estimate:

$$
\| \omega(t) \|_{V_0} \leq \| \omega^0 \|_{V_0} e^{-\nu \lambda_\mathcal{F} t}, \quad t \geq 0,
$$

where the constant $\lambda_\mathcal{F}$ is optimal. This constant governing the exponential decay of the solution is actually the same at any level of regularity. Thus, the solution to (2.13) (with $\beta = 0$) satisfies for every integer $k$:

$$
\| \omega(t) \|_{V_k} \leq \| \omega^0 \|_{V_k} e^{-\nu \lambda_\mathcal{F} t}, \quad t \geq 0.
$$

**Remark 2.2.** Kato’s criteria for the existence of the vanishing viscosity limit in [63] and rephrased in terms of the vorticity by Kelliher in [66] will be shown to be equivalent to the convergence of $\omega^\nu$ toward $\omega$ in the space $V_{-1}$ ($\omega^\nu$ stands for the vorticity of NS equations with vorticity $\nu$ and $\omega$ is the vorticity of Euler equations). Some care should be taken with the space $V_{-1}$ because it is not a distribution space, what may result in some mistakes or misunderstandings (we refer here to the very instructive paper of Simon [61]).

As mentioned earlier, the analysis of the dynamics of the flow can as well be carried out in terms of the sole stream function. It suffices to introduce the function spaces $S_0 = H_0^1(\mathcal{F})$ and $S_1 = H_0^3(\mathcal{F})$. The inclusion $S_1 \subset S_0$ being continuous dense and compact, the configuration $S_1 \subset S_0 \subset S_{-1}$ (with $S_{-1}$ the dual space of $S_1$) is a Gelfand triple where $S_0$ is the pivot space. We proceed as for the vorticity spaces and define a chain of embedded Hilbert spaces $S_k$ and related isometries $A_k^S : S_k \rightarrow S_{k-2}$ for every integer $k$ (we refer again to Appendix A for the details). In particular, providing that $\Sigma$ is of class $C^{2,1}$, we will verify that:

$$
S_2 = H^3(\mathcal{F}) \cap H_0^3(\mathcal{F}) \quad \text{and} \quad A_2^S \psi = -Q_1 \Delta \psi \quad \text{for all } \psi \in S_2.
$$

The counterpart of the Cauchy problem (2.13), restated in terms of the stream function is:

$$
\begin{align*}
\partial_t \psi + \nu A_{k+1}^S \psi &= f_S \quad \text{in } \mathcal{F} \cap \mathcal{F}_T, \\
\psi(0) &= \psi^0 \quad \text{in } \mathcal{F}.
\end{align*}
$$

For every $T > 0$, every integer $k$, every $f_S \in L^2(0,T;S_{k-1})$ and every $\psi^0 \in S_k$, this problem admits a unique solution:

$$
\psi \in H^1(0,T;S_{k-1}) \cap C([0,T];S_k) \cap L^2(0,T;S_{k+1}),
$$

which satisfies in addition the exponential decay estimate (assuming that $f_S = 0$ to simplify):

$$
\| \psi(t) \|_{S_k} \leq \| \psi^0 \|_{S_k} e^{-\nu \lambda_\mathcal{F} t} \quad \text{for all } t \geq 0.
$$

The constant $\lambda_\mathcal{F}$ is defined in (2.15) and is therefore the same as the one governing the exponential decay of the enstrophy.
The solution to problem (2.13) can easily be deduced from the solution to problem (2.16) and vice versa. Indeed, for every integer $k$, the operator:
\[
\Delta_k : \psi \in S_{k+1} \mapsto \Delta \psi \in V_k,
\]
can be shown to be an isometry. Thus, let be given $T > 0$ and consider
\begin{itemize}
  \item $\omega \in H^1(0,T;V_{k-1}) \cap C([0,T];V_k) \cap L^2(0,T;V_{k+1})$ the unique solution to Problem (2.13) for some integer $k$, some initial condition $\omega^i \in V_k$ and some source term $f_\omega \in L^2(0,T;V_{k-1})$;
  \item $\psi \in H^1(0,T;S_{k-1}) \cap C([0,T];S_k) \cap L^2(0,T;S_{k+1})$ the unique solution to Problem (2.16) for some integer $k'$, some initial condition $\psi^i \in S_k$ and some source term $f_S \in L^2(0,T;S_{k-1})$.
\end{itemize}
Providing that $k' = k + 1$, we claim that both following assertions are equivalent:
\begin{enumerate}
  \item $\omega = \Delta_k \psi$;
  \item $\omega^i = \Delta_k \psi^i$ and $f_\omega = \Delta_k f_S$.
\end{enumerate}
If we take for granted that the operators $P_k : S_{k-1} \to V_k$ and $Q_k : V_k \to S_{k-1}$ can be defined at any level of regularity in such a way that $P_k$ extend $P_{k'}$ if $k \leq k'$ and $Q_k = P_{k'}^{-1}$, we can show that the diagram in Fig. 2 commutes and all the operators are isometries.

\begin{figure}[h]
\centering
\includegraphics{diagram}
\caption{The top row contains the spaces $V_k$ for the vorticity fields while the bottom row contains the stream function spaces $S_k$. The operators $A_k^V$ and $A_k^S$ appears in the Cauchy problems (2.13) and (2.16) respectively. The operators $\Delta_k$ link isometrically the stream functions to the corresponding vorticity fields.}
\end{figure}

To accurately state the equivalence result between Problems (2.13) (Stokes problem in vorticity variable), (2.16) (Stokes problem in stream function variable) and the evolution homogeneous Stokes equations in primitive variables, it is worth recalling the functional framework for the Stokes equations by introducing the spaces:
\begin{align}
J_0 &= \{ u \in L^2(\mathcal{F}) : \nabla \cdot u = 0 \text{ in } \mathcal{F} \text{ and } u|_{\Sigma} \cdot n = 0 \}, \\
J_1 &= \{ u \in H^1(\mathcal{F}) : \nabla \cdot u = 0 \text{ in } \mathcal{F} \text{ and } u|_{\Sigma} = 0 \},
\end{align}
whose scalar products are respectively:
\begin{align}
(u_1, u_2)_{J_0} &= \int_{\mathcal{F}} u_1 \cdot u_2 \, dx \quad \text{for all } u_1, u_2 \in J_0, \\
(u_1, u_2)_{J_1} &= \int_{\mathcal{F}} \nabla u_1 \cdot \nabla u_2 \, dx \quad \text{for all } u_1, u_2 \in J_1.
\end{align}
The inclusion $J_1 \subset J_0$ being continuous dense and compact, from the Gelfand triple $J_1 \subset J_0 \subset J_{-1}$ we can define a chain of embedded Hilbert spaces $J_k$ and isometries $A_k^V : J_k \to J_{k-2}$ for every integer $k$. Providing that $\Sigma$ is of class $C^{1,1}$, it can be shown in particular that:
\[
J_2 = J_1 \cap H^2(\mathcal{F}) \quad \text{and} \quad A_2^V = -\Pi \Delta,
\]
where $\Pi : L^2(\mathcal{F}) \to J_0$ is the Leray projector. For every $T > 0$, every integer $k$, every $f_\Pi \in L^2(0,T;J_{k-1})$ and every $u^i$ in $J_k$, it is well known that there exists a unique solution
\[
u \in H^1(0,T;J_{k-1}) \cap C([0,T];J_k) \cap L^2(0,T;J_{k+1}),
\]
to the Cauchy problem:

\[(2.18a)\quad \partial_t u + \nu A_{k+1}^I u = f_I \quad \text{in } \mathcal{F}_T\]

\[(2.18b)\quad u(0) = u^I \quad \text{in } \mathcal{F}.\]

The operator:

\[\nabla_k^\perp : \psi \in S_k \mapsto \nabla_k^\perp \psi \in J_{k-1},\]

will be proved to be an isometry for every integer \(k\). It allows us to link Problem (2.18) to the equivalent problems (2.13) and (2.16). More precisely, let be given \(T > 0\) and consider

- \(u \in H^1(0; T; J_{k-1}) \cap \mathcal{C}([0, T]; J_k) \cap L^2(0; T; J_{k+1})\) the unique solution to Problem (2.13) for some integer \(k\), some initial condition \(u^I \in J_k\) and some source term \(f_I \in L^2(0; T; J_{k+1})\);
- \(\psi \in H^1(0; T; S_{k'-1}) \cap \mathcal{C}([0, T]; S_{k'}) \cap L^2(0; T; S_{k'+1})\) the unique solution to Problem (2.16) for some integer \(k'\), some initial condition \(\psi^I \in S_{k'}\) and some source term \(f_S \in L^2(0; T; S_{k'+1})\).

Providing that \(k' = k\), we claim that both following assertions are equivalent:

1. \(u = \nabla_k^\perp \psi^I\);
2. \(u^I = \nabla_k^\perp \psi^I\) and \(f_I = \nabla_k^\perp f_S\).

To conclude this short presentation of the main ideas that will be further elaborated in this paper, it is worth noticing that, contrary to what happens with primitive variables, the case where \(\mathcal{F}\) is multiply connected is notably more involved than the simply connected case. The same observation could still be came across in the articles of Glowinski and Pironneau [28] and Guermond and Quartapelle [31].

2.3. Organization of the paper. The next section is devoted to the study of the Stokes operator in non-primitive variables (namely the operators \(A_k^V\) and \(A_k^S\) mentioned in the preceding section). The expression of the Biot-Savart law is also provided. Then, in Section 4, lifting operators (for both the vorticity field and the stream function) are defined. They are required in Section 5 for the analysis of the evolution Stokes problem (in nonprimitive variables) with nonhomogeneous boundary conditions. The NS equations in nonprimitive variables is dealt with in Section 6 where weak and strong solutions are addressed. Explicit formulas to recover the pressure from the vorticity or the stream function are supplied in Section 7. The existence and uniqueness of more regular vorticity solutions is examined in Section 8. In this section we also prove the exponential decay of the palinstrophy (i.e. of the quantity \(\|\nabla \omega\|_{L^2(\mathcal{F})}\)) when time grows. In Section 9 we conclude with providing some insights on upcoming generalization results for coupled fluid-structure systems.

3. Stokes operator

3.1. Function spaces. Let \(\Sigma_0\) stands for either \(\Sigma^+\) or \(\Sigma^-\) for some \(j \in \{1, \ldots, N\}\). Providing that \(\Sigma_0\) is of class \(C^{k-1}\) (\(k\) being a nonnegative integer), it makes sense to consider the boundary Sobolev space \(H^{k+\frac{1}{2}}(\Sigma_0)\) and its dual space \(H^{-k-\frac{1}{2}}(\Sigma_0)\). Using \(L^2(\Sigma_0)\) as pivot space, we shall use a boundary integral notation in place of the duality pairing all along this paper. More precisely, we adopt the following convention of notation:

\[(3.1)\quad \langle g_1, g_2 \rangle_{H^{-k-\frac{1}{2}}(\Sigma_0), H^{k+\frac{1}{2}}(\Sigma_0)} = \int_{\Sigma_0} g_1 g_2 \mathrm{d}s \quad \text{for all } g_1 \in H^{-k-\frac{1}{2}}(\Sigma_0) \text{ and } g_2 \in H^{k+\frac{1}{2}}(\Sigma_0).\]

In particular, following this rule:

\[(3.2a)\quad S_0 = \{ \psi \in H^1(\mathcal{F}) : \psi|_{\Sigma^+} = 0 \quad \text{and} \quad \psi|_{\Sigma^-} = c_j, \quad c_j \in \mathbb{R}, \quad j = 1, \ldots, N \},
\]

\[(3.2b)\quad S_1 = \left\{ \psi \in S_0 \cap H^2(\mathcal{F}) : \frac{\partial \psi}{\partial n}_{|\Sigma} = 0 \right\}.\]

Fundamental function spaces. For every nonnegative integer \(k\), we denote by \(H^k(\mathcal{F})\) the classical Sobolev spaces of index \(k\) and we define the Hilbert spaces:

\[(3.3a)\quad F_0 = \{ \psi \in H^1(\mathcal{F}) : \psi|_{\Sigma^+} = 0 \quad \text{and} \quad \psi|_{\Sigma^-} = c_j, \quad c_j \in \mathbb{R}, \quad j = 1, \ldots, N \},
\]

\[(3.3b)\quad F_1 = \left\{ \psi \in S_0 \cap H^2(\mathcal{F}) : \frac{\partial \psi}{\partial n}_{|\Sigma} = 0 \right\}.\]
provided with the scalar products:
\[ (\psi_1, \psi_2)_{S_0} = (\nabla \psi_1, \nabla \psi_2)_{L^2(F)} \text{ for all } \psi_1, \psi_2 \in S_0, \]
\[ (\psi_1, \psi_2)_{S_1} = (\Delta \psi_1, \Delta \psi_2)_{L^2(F)} \text{ for all } \psi_1, \psi_2 \in S_1. \]
The norm \( \| \cdot \|_{S_0} \) is equivalent in \( S_0 \) to the usual norm of \( H^1(F) \). For every \( j = 1, \ldots, N \), we define the continuous linear form \( \text{Tr}_j : \psi \in S_0 \mapsto \psi|_{\Sigma_j} \in \mathbb{R} \) and the function \( \xi_j \) as the unique solution in \( S_0 \) to the variational problem:
\[ (\xi_j, \theta)_{S_0} + \text{Tr}_j \theta = 0 \text{ for all } \theta \in S_0. \]
The functions \( \xi_j \) are harmonic in \( F \) and obey the mean fluxes conditions:
\[ \int_{\Sigma_j} \frac{\partial \xi_j}{\partial n} \, ds = -\delta_j^k \text{ for } k = 1, \ldots, N, \]
where \( \delta_j^k \) is the Kronecker symbol. We denote by \( F_S \) the \( N \) dimensional subspace of \( S_0 \) spanned by the functions \( \xi_j \) (\( j = 1, \ldots, N \)) that will account for the fluxes of the stream functions through the inner boundaries. Notice that the Gram matrix \( ([\xi_j, \xi_k]_{S_0})_{1 \leq j, k \leq N} \) is invertible and equal to the matrix of the traces \( (-\text{Tr}_k \xi_j)_{1 \leq j, k \leq N} \).

Therefore, by means of a Gram-Schmidt process, we can derive from the free family \( \{\xi_j, j = 1, \ldots, N,\} \), an orthonormal family in \( S_0 \), denoted by \( \{\hat{\xi}_j, j = 1, \ldots, N\} \). The space \( S_0 \) admits the following orthogonal decomposition:
\[ S_0 = H^1_0(F) \oplus F_S. \]
In \( S_1 \), the norm \( \| \cdot \|_{S_1} \) is equivalent to the usual norm of \( H^2(F) \). For every \( j = 1, \ldots, N \), we denote by \( \chi_j \) the unique solution in \( S_1 \) such that:
\[ \Delta^2 \chi_j = 0 \text{ in } F \text{ and } \int_{\Sigma_k} \frac{\partial \Delta \chi_j}{\partial n} \, ds = -\delta_j^k \text{ for } k = 1, \ldots, N, \]
where the normal derivative of \( \Delta \chi_j \) is in \( H^{-\frac{1}{2}}(\Sigma_k) \) (see the convention of notation \((3.1)\)). We denote by \( B_S \) the \( N \) dimensional subspace of \( S_1 \) spanned by the functions \( \chi_j \). The Gram matrix \( ([\chi_j, \chi_k]_{S_1})_{1 \leq j, k \leq N} \) being invertible and equal to the matrix of traces \( (\text{Tr}_j \chi_k)_{1 \leq j, k \leq N} \), we infer that:
\[ S_1 = H^1_0(F) \oplus B_S. \]
In \( L^2(F) \), we denote by \( \mathfrak{H} \) the closed subspace of the harmonic functions with zero mean flux through every connected part \( \Sigma_j \) of the inner boundary (\( j = 1, \ldots, N \)), namely:
\[ \mathfrak{H} = \{ h \in L^2(F) : \Delta h = 0 \text{ in } D(F) \text{ and } (h, \Delta \chi_j)_{L^2(F)} = 0, \text{ for } j = 1, \ldots, N \}. \]
The space \( L^2(F) \) admits the orthogonal decomposition:
\[ L^2(F) = \mathfrak{H} \oplus V_0 \text{ where } V_0 = \mathfrak{H}^\perp. \]
It results from the following lemma that the space \( V_0 \) is the natural function space for the vorticity field.

**Lemma 3.1.** The operator \( \Delta_0 : \psi \in S_1 \mapsto \Delta \psi \in V_0 \) is an isometry.

**Proof.** Let \( \omega \) be in \( V_0 \) and denote by \( \psi_0 \) the unique function in \( H^1_0(F) \cap H^2(F) \) satisfying \( \Delta \psi_0 = \omega \). On the other hand, using the rule of notation \((3.1)\), the function
\[ h_0 = h - \sum_{j=1}^N \left( \int_{\Sigma_j} \frac{\partial \xi_j}{\partial n} h \, ds \right) \xi_j, \]
is in the space \( \mathfrak{H}^\perp = \mathfrak{H} \cap H^1(F) \) providing that \( h \) is a harmonic function in \( H^1(F) \). It follows that:
\[ (\omega, h_0)_{L^2(F)} = (\Delta \psi_0, h_0)_{L^2(F)} = \int_{\Sigma} \frac{\partial \psi_0}{\partial n} h_0 \, ds = \int_{\Sigma} \left[ \frac{\partial \psi_0}{\partial n} \sum_{j=1}^N \left( \int_{\Sigma_j} \frac{\partial \xi_j}{\partial n} \xi_j \, ds \right) \frac{\partial \xi_j}{\partial n} \right] h \, ds = 0. \]
Since every element in $H^2(\Sigma)$ can be achieved as the trace of a harmonic function in $H^1(\mathcal{F})$, the equality above entails that:

$$\frac{\partial \psi_0}{\partial n} = \sum_{j=1}^{N} \left( \int_{\Sigma} \frac{\partial \psi_0}{\partial n} \hat{\xi}_j \, ds \right) \frac{\partial \hat{\xi}_j}{\partial n} \quad \text{in } H^{-\frac{1}{2}}(\Sigma).$$

We are done by noticing now that the function:

$$\psi = \psi_0 - \sum_{j=1}^{N} \left( \int_{\Sigma} \frac{\partial \psi_0}{\partial n} \hat{\xi}_j \, ds \right) \hat{\xi}_j,$$

is in $S_1$ and solves $\Delta \psi = \omega$. Uniqueness being straightforward, the proof is then complete. \qed

The Bergman projection and its inverse. Considering the orthogonal splitting (3.8) of $L^2(\mathcal{F})$, we denote by $\mathcal{P}$ the orthogonal projection from $L^2(\mathcal{F})$ onto $V_0$ while the notation $\mathcal{P}^\perp$ will stand for the orthogonal projection onto $\mathcal{S}$. When the domain $\mathcal{F}$ is simply connected, the operator $\mathcal{P}^\perp$ is referred to as the harmonic Bergman projection and has been extensively studied (see for instance [4], [62] and references therein). The projector $\mathcal{P}$ (and also $\mathcal{P}^\perp$ which we are less interested in) enjoys the following property:

**Lemma 3.2.** Assume that $\Sigma$ is of class $C^{k+1,1}$ for some nonnegative integer $k$, then $\mathcal{P}$ (and $\mathcal{P}^\perp$) maps $H^k(\mathcal{F})$ into $H^k(\mathcal{F})$ and $\mathcal{P}$, seen as an operator from $H^k(\mathcal{F})$ into $H^k(\mathcal{F})$, is bounded.

**Proof.** Let $u$ be in $L^2(\mathcal{F})$. The proof consists in verifying that

$$\mathcal{P}u = \Delta w_0 + \sum_{k=1}^{N} (\Delta \chi_k, u)_{L^2(\mathcal{F})} \Delta \chi_k,$$

where the functions $w_0$ belongs to $H^2_0(\mathcal{F})$ and satisfies the variational formulation:

$$\langle \Delta w_0, \Delta \theta_0 \rangle_{L^2(\mathcal{F})} = (u, \Delta \theta_0)_{L^2(\mathcal{F})}, \quad \text{for all } \theta_0 \in H^2_0(\mathcal{F}).$$

The conclusion of the Lemma will follow according to elliptic regularity results for the biharmonic operator stated for instance in [27, Theorem 1.11]. By definition:

$$\mathcal{P}u = \arg\min \left\{ \frac{1}{2} \int_{\mathcal{F}} |v - u|^2 \, dx : v \in V_0 \right\}.$$  

According to Lemma 3.1 there exists a unique $w \in S_1$ such that $\mathcal{P}u = \Delta w$ and:

$$w = \arg\min \left\{ \frac{1}{2} \int_{\mathcal{F}} |\Delta \theta - u|^2 \, dx : \theta \in S_1 \right\}.$$  

Owing to the orthogonal decomposition (3.6), the function $w$ can be decomposed as:

$$w = w_0 + \sum_{k=1}^{N} \alpha_k \chi_k,$$

where $w_0 \in H^2_0(\mathcal{F})$ and $(\alpha_1, \ldots, \alpha_N) \in \mathbb{R}^N$ are such that:

$$(w_0, \alpha_1, \ldots, \alpha_N) = \arg\min \left\{ \frac{1}{2} \int_{\mathcal{F}} |\Delta \theta_0 + \sum_{k=1}^{N} \beta_k \Delta \chi_k - u|^2 \, dx : (\theta_0, \beta_1, \ldots, \beta_N) \in H^2_0(\mathcal{F}) \times \mathbb{R}^N \right\}.$$  

It follows that $w_0$ solves indeed the variational problem (3.9) and $\alpha_k = (\Delta \chi_k, u)_{L^2(\mathcal{F})}$ for every $k = 1, \ldots, N$. \qed

**Remark 3.3.** The following observations are in order:

1. The harmonic Bergman projection is quite demanding in terms of boundary regularity, and one may wonder if the assumption on the regularity of $\Sigma$ is optimal in the statement of Lemma 3.2. Focusing on the case $k = 0$, the definition of the space $\mathcal{S}$ requires defining the flux of harmonic functions through the connected parts of $\Sigma^-$. The normal derivative of harmonic functions in $L^2(\mathcal{F})$ can be defined as elements of $H^{-\frac{1}{2}}(\Sigma)$. However, it requires the boundary to be $C^{1,1}$ (see [30, page 54]), which is the default level of regularity assumed for the domain $\mathcal{F}$ throughout this article.
(2) For every $u \in L^2(F)$, the function $P^1 u$ belongs to $\mathcal{S}$ and therefore admits a normal trace on every $\Sigma_j^-$ ($j = 1, \ldots, N$) in $H^{-\frac{1}{2}}(\Sigma_j^-)$. We deduce that, when $u$ belongs to $H^2(F)$, the fluxes of $u$ across the parts $\Sigma_j^-$ ($j = 1, \ldots, N$) of the boundary are conserved by the projection $P$, namely:

$$\int_{\Sigma_j^-} \frac{\partial P u}{\partial n} \, ds = \int_{\Sigma_j^-} \frac{\partial u}{\partial n} \, ds \quad \text{for all } j = 1, \ldots, N,$$

where $\partial P u / \partial n$ belongs to $H^{-\frac{1}{2}+k}(\Sigma_j^-)$ providing that $\Sigma_j^-$ is of class $C^{k+1,1}$ for $k = 0, 1, 2$.

Let us define now the operator:

$$(3.10) \quad Q : u \in H^1(F) \mapsto Q u = \text{argmin} \left\{ \int_F |\nabla \theta - \nabla u|^2 \, dx : \theta \in S_0 \right\} \in S_0.$$

The variational formulation corresponding to the minimization problem reads:

$$(3.11) \quad (\nabla Q u, \nabla \theta)_{L^2(F)} = (\nabla u, \nabla \theta)_{L^2(F)} \quad \text{for all } \theta \in S_0,$$

what means that $Q u$ is the unique function in $S_0$ that satisfies $\Delta Q u = \Delta u$ in $F$. Denoting $Q^\perp = \text{Id} - Q$, this entails that $Q^\perp u$ is harmonic and choosing $\chi_k$ ($k = 1, \ldots, N$) as test function in (3.11) and integrating by parts, we obtain:

$$(Q^\perp u, \Delta \chi_k)_{L^2(F)} = 0, \quad k = 1, \ldots, N,$$

whence we deduce that $Q^\perp u$ lies in $\mathcal{S}_1^1$. Besides, there exists real coefficients $\alpha_j$ such that the function:

$$u_0 = Q u - \sum_{j=1}^N \alpha_j \chi_j,$$

belongs to $H^1_0(F)$ because the Gram matrix $\left( (\Delta_j \chi_j)_{L^2(F)} \right)_{1 \leq j, k \leq N}$ is invertible and equal to the matrix of traces $(T_j \chi_k)_{1 \leq j, k \leq N}$. It follows that for every $h \in \mathcal{S}_1^1$:

$$(3.12) \quad (\nabla Q u, \nabla h)_{L^2(F)} = (\nabla u_0, \nabla h)_{L^2(F)} - \sum_{j=1}^N \alpha_j (h, \Delta \chi_j)_{L^2(F)} = 0.$$

So the operators $P$ and $Q$ are both orthogonal projections whose kernels are harmonic functions ($\mathcal{S}$ for $P$ and $\mathcal{S}_1^1$ for $Q$) but for different scalar products. They are tightly related, as expressed in the next lemma, the statement of which requires introducing a new function space. Thus, we define $V_1$ as the image of $S_0$ by $P$ and we denote by $P_1$ the restriction of $P$ to $S_0$. It is elementary to verify that

$$P_1 : S_0 \rightarrow V_1$$

is one-to-one. We denote by $Q_1$ the inverse of $P_1$. The space $V_1$ is then provided with the image topology, namely with the scalar product:

$$(\omega_1, \omega_2)_{V_1} = (Q_1 \omega_1, Q_1 \omega_2)_{S_0} = (\nabla Q_1 \omega_1, \nabla Q_1 \omega_2)_{L^2(F)} \quad \text{for all } \omega_1, \omega_2 \in V_1.$$

Observe that since $H^1(F) = S_0 \oplus \mathcal{S}_1^1$, we have also $V_1 = \text{PH}^1(F)$.

**Lemma 3.4.** If $\Sigma$ is of class $C^{2,1}$, $V_1$ is a subspace of $H^1(F)$, $Q_1$ is the restriction of $Q$ to $V_1$ and the topology of $V_1$ is equivalent to the topology of $H^1(F)$.

**Proof.** According to Lemma 3.2 if $\Sigma$ is of class $C^{2,1}$, the space $V_1$ is a subspace of $H^1(F)$ and $P$ is bounded from $S_0$ onto $V_1$ (seen as subspace of $H^1(F)$). The operator $P_1 : S_0 \rightarrow V_1$ being bounded and invertible, it is an isomorphism according to the bounded inverse theorem. Moreover, for every $\psi$ be in $S_0$:

$$Q P_1 \psi = Q (\psi + (P_1 \psi - \psi)) = Q \psi + Q (P_1 \psi - \psi) = Q \psi = \psi,$$

since the function $P_1 \psi - \psi$ is in $\mathcal{S}_1^1$. The proof is now complete. \hfill \Box

**Remark 3.5.** When $\Sigma$ is only of class $C^{1,1}$, $V_1$ is a subspace of $S_0 + \mathcal{S}_1^1$. In particular, every element of $V_1$ has a trace on $\Sigma$ in the space $H^{-\frac{1}{2}}(\Sigma)$. This trace is in $H^\frac{1}{2}(\Sigma)$ when $\Sigma$ is of class $C^{2,1}$. 


Further scalar products. For every nonnegative integer \( k \), we define \( \mathcal{S}^k = \mathcal{S} \cap H^k(\mathcal{F}) \). Assuming that \( \Sigma \) is of class \( C^{k-1,1} \), the space \( \mathcal{S}^k \) is provided with the scalar product:

\[
(h_1, h_2)_{\mathcal{S}^k} = (h_1|\Sigma, h_2|\Sigma)_{H^{k-\frac{1}{2}}(\Sigma)}
\]

for all \( h_1, h_2 \in \mathcal{S}^k \).

It would sometimes come in handy to provide \( H^1(\mathcal{F}) \) with a scalar product that turns \( Q \) into an orthogonal projection. To do that, it suffices to define:

\[
(\mathcal{U}_1, \mathcal{U}_2)_{H^1_1} = (Q\mathcal{U}_1, Q\mathcal{U}_2)_{\mathcal{S}_0} + (Q^\perp \mathcal{U}_1, Q^\perp \mathcal{U}_2)_{\mathcal{S}_1}, \quad \text{for all } \mathcal{U}_1, \mathcal{U}_2 \in H^1(\mathcal{F}).
\]

Similarly, the scalar product:

\[
(\mathcal{U}_1, \mathcal{U}_2)_{H^1_2} = (P\mathcal{U}_1, P\mathcal{U}_2)_{\mathcal{V}_1} + (P^\perp \mathcal{U}_1, P^\perp \mathcal{U}_2)_{\mathcal{S}_1}, \quad \text{for all } \mathcal{U}_1, \mathcal{U}_2 \in H^1(\mathcal{F}),
\]

turns the direct sum \( H^1(\mathcal{F}) = V_1 \oplus \mathcal{S}^1 \) into an orthogonal sum.

### 3.2. Stokes operator in nonprimitive variables

The inclusion \( S_1 \subset S_0 \) is clearly continuous dense and compact. Identifying the Hilbert space \( S_0 \) with its dual and denoting by \( S_{-1} \) the dual space of \( S_1 \), we obtain the Gelfand triple:

\[
S_1 \subset S_0 \subset S_{-1}.
\]

Following the lines of Appendix A, we can define (with obvious notation) a family of embedded Hilbert spaces \( \{ S_k, k \in \mathbb{Z} \} \), a family of isometries \( \{ A^S_k : S_k \to S_{k-2}, k \in \mathbb{Z} \} \) and a positive constant:

\[
\lambda^S_k = \min_{\psi \neq 0} \frac{\| \psi \|^2_{S_k}}{\| \Delta \psi \|^2_{S_0}}.
\]

**Lemma 3.6.** The space \( S_2 \) is equal to \( H^3(\mathcal{F}) \cap S_1 \) providing that \( \Sigma \) is of class \( C^{2,1} \). For \( k \geq 2 \), the expressions of the operator \( A^S_k \) is:

\[
A^S_k : \psi \in S_k \mapsto -Q_1 \Delta \psi \in S_{k-2}.
\]

If \( \Sigma \) is of class \( C^{k,1} \), then \( S_k \) is a subspace of \( H^{k+1}(\mathcal{F}) \) and the norm in \( S_k \) is equivalent to the classical norm of \( H^{k+1}(\mathcal{F}) \).

**Proof.** We recall that \( A^S_1 \) is the operator \( \psi \in S_1 \mapsto (\psi, \cdot)_{S_1} \in S_{-1} \). The space \( S_2 \) is defined as the preimage of \( S_0 \) by \( A^S_1 \), namely:

\[
S_2 = \{ \psi \in S_1 : (\psi, \cdot)_{S_1} = (f, \cdot)_{S_0} \text{ in } S_{-1} \text{ for some } f \in S_0 \}.
\]

Upon an integration by parts and according to Lemma 3.1, one easily obtains that the identity \( (\psi, \cdot)_{S_1} = (f, \cdot)_{S_0} \) in \( S_{-1} \) is equivalent to the equality \(-P_3 \Delta \psi = Pf \) in \( V_0 \). Invoking Lemma 3.1 again, we deduce, on the one hand, that \( P_3 \Delta \psi = \Delta \psi \). Under the assumption on the regularity of the boundary \( \Sigma \), the equality \( -\Delta \psi = Pf \) where \( f \) and hence also \( Pf \) is in \( H^1(\mathcal{F}) \) entails that \( \psi \) belongs to \( H^3(\mathcal{F}) \). On the other hand, since \( f \) belongs to \( S_0 \), \( Pf = P_1 f \). Applying then the operator \( Q_1 \) to both sides of the identity \(-\Delta \psi = P_1 f \), we end up with the equality \(-Q_1 \Delta \psi = f \) and \( \lambda^S_k \) is proven for \( k = 2 \). The expressions for \( k > 2 \) follow from the general settings of Appendix A. Then, by induction on \( k \), invoking classical elliptic regularity results, one proves the inclusion \( S_k \subset H^{k+1}(\mathcal{F}) \) and the equivalence of the norms. \( \square \)

We straightforwardly deduce that, by definition of the space \( V_1 \), the inclusion \( V_1 \subset V_0 \) enjoys the same properties as the inclusion \( S_1 \subset S_0 \), namely it is continuous dense and compact. We consider then the Gelfand triple:

\[
V_1 \subset V_0 \subset V_{-1},
\]

in which \( V_0 \) is the pivot space and \( V_{-1} \) is the dual space of \( V_1 \). As beforehand, we define a family of embedded Hilbert spaces \( \{ V_k, k \in \mathbb{Z} \} \), the corresponding family of isometries \( \{ A^V_k : V_k \to V_{k-2}, k \in \mathbb{Z} \} \) and the positive constant:

\[
\lambda^V_k = \min_{\omega \neq 0} \frac{\| \omega \|^2_{V_k}}{\| \omega \|^2_{V_0}}.
\]

**Remark 3.7.**

1. As already mentioned earlier, the space \( V_{-1} \) is clearly not a distributions space.
The guiding principle that the vorticity should be $L^2$-orthogonal to harmonic functions is somehow still verified in a weak sense in $V_{-1}$. Indeed, $A_1^V$ being an isometry, every element $\omega$ of $V_{-1}$ is equal to some $A_1^V \omega'$ with $\omega' \in V_1$ and:

$$\langle \omega, \cdot \rangle_{V_{-1}, V_1} = \langle A_1^V \omega', \cdot \rangle_{V_{-1}, V_1} = (\nabla Q_1 \omega', \nabla Q_1 \cdot)_{L^2(F)}.$$ 

Identifying the duality pairing with the scalar product in $L^2(F)$ (i.e. the scalar product of the pivot space $V_0$), we obtain that formally "$(\omega, h)_{L^2(F)} = 0$" for every $h \in \mathcal{K}_1$.

For the analysis of the spaces $V_k$ and their relations with the spaces $S_k$, it is worth introducing at this point an additional Gelfand triple, that will come in handy later on. Thus, denote by $Z_0$ the space $L^2(F)$ (equipped with the usual scalar product) and by $Z_1$ the space $S_0$. The configuration:

$$Z_1 \subset Z_0 \subset Z_{-1},$$

is obviously a Gelfand triple in which $Z_0$ is the pivot space and $Z_{-1}$ the dual space of $Z_1$. As usual, following Appendix A we define a family of embedded Hilbert spaces $\{Z_k, k \in \mathbb{Z}\}$, and a family of isometries $\{A_k^Z : Z_k \rightarrow Z_{k-2}, k \in \mathbb{Z}\}$. Focusing on the case $k = 2$, a simple integration by parts leads to:

**Lemma 3.8.** The expressions of the space $Z_2$ and of the operator $A_2^Z$ are respectively:

$$Z_2 = \left\{ \psi \in H^2(F) \cap S_0 : \int_{\Sigma_j} \frac{\partial \psi}{\partial n} \, ds = 0, \quad j = 1, \ldots, N \right\} \quad \text{and} \quad A_2^Z : \psi \in Z_2 \mapsto -\Delta \psi \in Z_0.$$

The space of biharmonic functions in $L^2(F)$ with zero mean flux through the inner boundaries is denoted by $\mathcal{B}$, namely:

$$\mathcal{B} = \left\{ \theta \in L^2(F) : \Delta \theta \in \mathcal{K}_1 \text{ and } \int_{\Sigma_j} \frac{\partial \theta}{\partial n} \, ds = 0, \quad j = 1, \ldots, N \right\}.$$

Since $\Delta \theta$ belongs to $L^2(F)$, the trace of $\theta$ on $\Sigma$ is well defined and belongs to $H^{-\frac{1}{2}}(\Sigma)$ and the trace of the normal derivative is in $H^{-\frac{3}{2}}(\Sigma)$. On the other hand, since $\Delta \theta$ belongs to $\mathcal{K}_1$, its trace on $\Sigma$ is in $H^{-\frac{1}{2}}(\Sigma)$ while its normal trace is in $H^{-\frac{3}{2}}(\Sigma)$.

The space $S_1$ being a closed subspace of $Z_2$ it admits an orthogonal complement denoted by $\mathcal{B}_S$:

$$Z_2 = S_1 \oplus \mathcal{B}_S.$$

An integration by parts and classical elliptic regularity results allow to deduce that:

$$\mathcal{B}_S = S_0 \cap \mathcal{B},$$

and that $\mathcal{B}_S \subset H^2(F)$.

**Lemma 3.9.** The operator $A_2^Z$ is an isometry from $S_1$ onto $V_0$ and also an isometry from $\mathcal{B}_S$ onto $\mathcal{K}_1$, i.e. the operator $A_2^Z$ is block-diagonal with respect to the following decompositions of the spaces:

$$A_2^Z : S_1 \oplus \mathcal{B}_S \rightarrow V_0 \oplus \mathcal{K}_1.$$

The operators $A_1^V$ and $A_2^Z$ and the operators $A_1^Z$ and $A_2^Z$ are connected via the identities:

$$A_1^V = Q_1^* A_2^Z Q_1 \quad \text{and} \quad A_2^Z = A_2^Z Q_1 \quad \text{in } V_2,$$

where the operator $Q_1^*$ is the adjoint of $Q_1$.

**Proof.** The first claim of the lemma is a direct consequence of Lemma 3.1. By definition, for every $\omega \in V_1$:

$$A_1^V \omega = (\nabla Q_1 \omega, \nabla Q_1 \cdot)_{L^2(F)} = Q_1^* A_2^Z Q_1 \omega,$$

and the first identity in (3.23) is proved. Addressing the latter, notice that for every $\omega \in V_2$, the function $w = A_2^Z \omega$ is the unique element in $V_0$ such that:

$$(w, v)_{V_0} = (\omega, v)_{V_1}, \quad \text{for all } v \in V_1,$$
which can be rewritten as:

\[(w, v)_{L^2(F)} = (\nabla Q_1 \omega, \nabla Q_1 v)_{L^2(F)}, \quad \text{for all} \ v \in V_1.\]

But the functions \(v\) and \(Q_1 v\) differ only up to an element of \(\mathcal{H}\), and since \(w\) belongs to \(V_0 = \mathcal{H}^1\), it follows that:

\[(w, v)_{L^2(F)} = (w, Q_1 v)_{L^2(F)}, \quad \text{for all} \ v \in V_1.\]

Finally, since \(S_0 = Z_1\) and \(Q_1 : V_1 \to S_0\) is an isometry, the function \(w\) satisfies:

\[(w, z)_{L^2(F)} = (\nabla Q_1 \omega, \nabla z)_{L^2(F)}, \quad \text{for all} \ z \in Z_1,\]

which means that \(w = A_1^2 Q_1 \omega\) and completes the proof.

We can now go back to the study of the vorticity spaces \(V_k\) and the related operators \(A_k^V\). Starting with the case \(k = 2\), we claim:

**Lemma 3.10.** The space \(V_2\) is equal to \(P_1 S_1\), or equivalently:

\[(3.24a) \quad V_2 = \left\{ \omega \in PH^2(F) : \frac{\partial Q_1 \omega}{\partial n} \bigg|_{\Sigma} = 0 \right\}.\]

Moreover, the expression of the operator \(A_k^V\) is:

\[(3.24b) \quad A_1^V : \omega \in V_2 \mapsto -\Delta \omega \in V_0.\]

**Proof.** The second formula in (3.23) yields the following identity between function spaces:

\[A_1^V V_2 = A_1^2 Q_1 V_2\]

and then, since \(A_1^V V_2 = V_0:\)

\[(A_1^2)^{-1} V_0 = Q_1 V_2.\]

Invoking the first point of Lemma 3.9, we deduce first that \(Q_1 V_2 = S_1\) and then, applying the operator \(P_1\) to both sides of the identity, that \(V_2 = P_1 S_1\). Using again the second formula in (3.23) together with the expression of \(A_1^2\) given in (3.19), we obtain the expression (3.24b) of the operator \(A_1^V\).

**Remark 3.11.** According to Lemma 3.2, if \(\Sigma\) is of class \(C^{3,1}\) then \(V_2\) is a subspace of \(H^2(F)\). If \(\Sigma\) is of class \(C^{2,1}\), \(V_2\) is a subspace of \(H^1(F)\) and the functions in \(V_2\) can be given a trace in \(H^{\frac{3}{2}}(\Sigma)\) and a normal trace in \(H^{-\frac{3}{2}}(\Sigma)\). Finally, if \(\Sigma\) is only of class \(C^{1,1}\), the trace still exists in \(H^{-\frac{3}{2}}(\Sigma)\) and the normal trace in \(H^{-\frac{3}{2}}(\Sigma)\). We use the fact that every function in \(V_2\) is by definition the sum of a function in \(H^2(F)\) with a harmonic function in \(L^2(F)\).

For the ease of the reader, we can still state the following lemma which is a straightforward consequence of (3.24b) and the general settings of Appendix A.

**Lemma 3.12.** For every positive integer \(k\), the expression of the operators \(A_k^V\) are:

\[A_1^V : u \in V_1 \mapsto (u, \cdot)_{V_1} \in V_{-1} \quad \text{and} \quad A_k^V : u \in V_k \mapsto (-\Delta) u \in V_{k-2} \quad \text{for} \ k \geq 2.\]

For nonnegative integers \(k\), \(V_k\) is a subspace of \(H^k(F)\) providing that \(\Sigma\) is of class \(C^{k+1,1}\) and the norm in \(V_k\) is equivalent to the classical norm of \(H^k(F)\). For nonpositive indices, the operators are defined by duality as follows:

\[A_k^V : u \in V_{-k} \mapsto (u, (-\Delta) \cdot)_{V_{-k}} \in V_{-k-2}, \quad (k \geq 0).\]

The next result states that the chain of embedded spaces for the stream function \(\{S_k, k \in \mathbb{Z}\}\) is globally isometric to the chain of embedded spaces for the vorticity \(\{V_k, k \in \mathbb{Z}\}\), the isometries being, loosely speaking, the operators \(P\) and \(Q\). So far, we have proven that \(P_1 S_1 = V_2\) and \(P_1 S_0 = V_1\). To generalized these relations to every integer \(k\), we need to extend the operators \(P_1\) and \(Q_1\).

**Lemma 3.13.** For every positive integer \(k\), the following inclusions hold:

\[P_1 S_{k-1} \subset V_k \quad \text{and} \quad Q_1 V_k \subset S_{k-1}.\]
Considering this lemma as granted, it makes sense to define for every positive integer $k$ the operators:

\begin{equation}
(3.25) \quad P_k : u \in S_{k-1} \mapsto P_1 u \in V_k \quad \text{and} \quad Q_k : u \in V_k \mapsto Q_1 u \in S_{k-1}.
\end{equation}

Then, we define also by induction, for every $k \geq 0$:

\begin{equation}
(3.26a) \quad P_{-k} = A_{k+2}^V P_{k+2} (A_{k+1}^S)^{-1} : S_{k-1} \to V_{-k},
\end{equation}

and

\begin{equation}
(3.26b) \quad Q_{-k} = A_{k+1}^S Q_{k+2} (A_{k+2}^V)^{-1} : V_{-k} \to S_{k-1}.
\end{equation}

**Theorem 3.14.** For every integer $k$, the operators $P_k$ and $Q_k$ defined in (3.25) and (3.26) are inverse isometries (i.e. $P_k Q_k = \text{Id}$ and $Q_k P_k = \text{Id}$). Moreover, formulas (3.26) can be generalized to every integer $k$:

\begin{equation}
(3.27) \quad A_k^V P_k = P_{k-2} A_k^S \quad \text{and} \quad A_{k-1}^S Q_k = Q_{k-2} A_k^V,
\end{equation}

and for every pair of indices $k', k$ such that $k' \leq k$:

\begin{equation}
(3.28) \quad P_{k'} = P_k \quad \text{in} \quad S_{k-1} \quad \text{and} \quad Q_{k'} = Q_k \quad \text{in} \quad V_k.
\end{equation}

**Remark 3.15.** By definition, $P$ is a projection in $L^2(F)$ and $Q$ a projection in $H^1(F)$. The theorem tells us that formulas (3.26) allow extending these projectors to larger spaces.

The theorem ensures also that to every stream function $\psi$ in some space $S_{k-1}$, it can be associated a vorticity field $\omega = P_k \psi$ in $V_k$. The vorticity $\omega$ has the same regularity as $\psi$ and is obviously not the Laplacian of $\psi$.

The lemma and the theorem are proved at once:

*Proof of Lemma 3.13 and Theorem 3.14.* Denoting by $Q_2$ the restriction of $Q_1$ to $V_2$, the second formula in (3.23) can be rewritten as $A^V_2 = A^S_2 Q_2$. Since the operators $A^V_2$ and $A^S_2$ are both isometries, this property is also shared by $Q_2$ and its inverse, which is denoted by $P_2$. We have now at our disposal two pairs of isometries $(P_1, Q_1)$ and $(P_2, Q_2)$ corresponding to two successive indices in the chain of embedded spaces. Furthermore, $P_2$ is the restriction of $P_1$ to $V_2$. This fits within the framework of Subsection A.2. We define first $P_k$ and $Q_k$ (for the indices $k \neq 1, 2$) by induction with formulas (3.27) and we apply Lemma A.6. We obtain that the operators $P_k$ and $Q_k$ are indeed isometries from $S_{k-1}$ onto $V_k$ and from $V_k$ onto $S_{k-1}$ respectively. Lemma A.6 also ensures that $P_k = P_{k'}$ in $V_k$ and $Q_k = Q_{k'}$ in $S_k$ for indices $k' \leq k$, whence we deduce that $P_k$ and $Q_k$ for $k \geq 1$ can be equivalently defined by (3.25). Next, since $P_1$ and $Q_1$ are reciprocal isometries and $P_k$ and $Q_k$ are just restrictions of $P_1$ and $Q_1$, then $P_k$ and $Q_k$ are reciprocal isometries as well. We draw the same conclusion for nonpositive indices using formulas (3.27) and complete the proof.

**Corollary 3.16.** The constant $\lambda^S_F$ defined in (3.15) and the constant $\lambda^V_F$ defined in (3.18) are equal. We denote simply by $\lambda_F$ their common value.

Notice also that since $S_1 \subset Z_2$ and $S_0 = Z_1$, we have:

\begin{equation}
(3.29) \quad \lambda^Z_F = \min_{\psi \in Z_2} \frac{\|\psi\|^2_{Z_2}}{\|\psi\|^2_{Z_1}} \leq \min_{\psi \in S_0} \frac{\|\psi\|^2_{S_0}}{\|\psi\|^2_{S_0}} = \lambda_F.
\end{equation}

**Remark 3.15** points out that, for a given stream function $\psi$ in some space $S_{k-1}$, the function $\omega = P_k \psi$ is not the (physical) vorticity corresponding to the velocity field $\nabla^\perp \psi$. We shall now define the operators $\Delta_k$ that associates the stream function to its corresponding vorticity field.

**Definition 3.17.** For every integer $k$, the operator $\Delta_k : S_{k+1} \to V_k$ is defined equivalently (according to (3.27)) by:

\begin{equation}
(3.30) \quad \text{Either} \quad \Delta_k = -A^V_{k+2} P_{k+2} \quad \text{or} \quad \Delta_k = -P_k A^S_{k+1}.
\end{equation}

The main properties of the operators $\Delta_k$ are summarized in the following lemma:

**Lemma 3.18.** The following assertions hold:

1. For every integer $k$, the operator $\Delta_k$ is an isometry.
2. For every pair of integers $k, k'$ such that $k' \leq k$, $\Delta_k = \Delta_{k'}$ in $S_{k+1}$. 

Proof. We recall that the operators $P_k$ and $A^V_k$ are isometries, what yields the first point of the lemma. The second point is a consequence of (3.28) and the similar general property (A.8) satisfied by the operators $A^V_k$. The third point is a direct consequence of Lemma 3.12. □

For any integer $k$, the operators $Q_k$ and $-\Delta_{-k}$ can be shown to be somehow adjoint:

**Lemma 3.19.** For negative indices, the operators $\Delta_k$ and $Q_k$ satisfy the adjointness relations below:

\[
\begin{align*}
Q_{-k} = -\Delta'_k &: \omega \in V_{-k} \mapsto -\langle \omega, \Delta_k \rangle_{V_{-k}, V_k} \in S_{-k-1} & (k \geq 0), \\
\Delta_{-k} = -Q'_k &: \psi \in S_{-k+1} \mapsto -\langle \psi, Q_k \rangle_{S_{-k+1}, S_{k-1}} \in V_{-k} & (k \geq 1).
\end{align*}
\]

Proof. We prove (3.31a) by induction, the proof of (3.31b) being similar. According to (3.27), $Q_0A^V_2 = A^S_2Q_2$ what means, recalling (A.3) that:

\[
Q_0A^V_2 \omega = (Q_2\omega, \cdot)_{S_1} = (\omega, P_2 \cdot)_{V_2} = (A^V_2 \omega, A^S_2 \cdot)_{V_0} = (A^V_2 \omega, (-\Delta_0) \cdot)_{V_0},
\]

where we have used the fact that the operators $P_2$ and $A^V_2$ are isometries. This proves (3.31a) at the step $k = 0$.

According to the definition (3.26b) with $k = 1$, $Q_{-1} = A^S_0Q_1(A^V_1)^{-1}$. Equivalently stated, for every $\omega \in V_1$:

\[
Q_{-1}A^V_1 \omega = A^S_0Q_1 \omega = (Q_1 \omega, A^S_2 \cdot)_{S_0} \quad \text{for all } \omega \in V_1,
\]

where we have used the general relation (A.6). The operator $P_1$ being an isometry:

\[
(Q_1 \omega, A^S_2 \cdot)_{S_0} = (P_1Q_1 \omega, P_1A^S_2 \cdot)_{V_1} = -\langle \omega, \Delta_1 \cdot \rangle_{V_1} = -\langle A^V_1 \omega, \Delta_1 \cdot \rangle_{V_{-1}, V_1},
\]

and (3.31a) is then proved for $k = 1$.

Let us assume that (3.31a) holds true at the step $k - 2$ for some integer $k \geq 2$. According to (3.27), $Q_{-k}A^V_{k+2} = A^S_{k+1}Q_{-k+2}$ whence, recalling the definition (A.7):

\[
Q_{-k}A^V_{k+2} = (Q_{-k+2} \omega, A^S_{k+1} \cdot)_{S_{-k+1}, S_{k-1}} \quad \text{for all } \omega \in V_{k+2},
\]

Using the induction hypothesis for the operator $Q_{-k+2}$, it comes:

\[
Q_{-k}A^V_{k+2} = -\langle \omega, \Delta_{k-2}A^S_{k+1} \rangle_{V_{k+2}, V_{k-2}} = -\langle \omega, A^V_k \Delta_k \cdot \rangle_{V_{k+2}, V_{k-2}} \quad \text{for all } \omega \in V_{k+2},
\]

where the latter identity results from (3.30). Keeping in mind (A.7), we have indeed proved (3.31a) at the step $k$ and complete the proof. □

### 3.3. Biot-Savart operator

With a slight abuse of terminology, the inverse of the operator $\Delta_k$ denoted by $N_k$ will be referred to as the Biot-Savart operator. Quite surprisingly, the expression of this operator is independent from the fluid domain $\mathcal{F}$. We recall that the fundamental solution of the Laplacian is the function:

\[
\mathcal{G}(x) = \frac{1}{2\pi} \ln |x|, \quad x \in \mathbb{R}^2 \setminus \{0\}.
\]

In the rest of the paper, we will denote generically by $c$ the real constants that should arise in the estimates. The value of the constant may change from line to line. The parameters the constant should depend on is indicated in subscript.

**Theorem 3.20.** For every nonnegative index $k$, the Biot-Savart operator $N_k = (\Delta_k)^{-1}$ is simply the Newtonian potential defined by:

\[
N_k : \omega \in V_k \mapsto N\omega \in S_{k+1},
\]

where $N\omega = \mathcal{G} * \omega$ ($\omega$ is extended by 0 outside $\mathcal{F}$).

Proof. Let $\omega$ be in $V_0$ (extended by 0 outside $\mathcal{F}$) and denote by $\psi$ the Newtonian potential $N\omega$. According to classical properties of the Newtonian potential, $\Delta \psi = \omega$ in $\mathbb{R}^2$ and therefore it suffices to verify that $\psi$ is constant on every connected part of the boundary $\Sigma$ and that its normal derivative vanishes. Define the
constant $\delta = 2 \max\{|x - y| : x \in \Sigma^-, y \in \Sigma^+\}$. Let $j$ be in $\{1, \ldots, N\}$ and let $q$ be in $L^2(\Sigma^-)$. Notice now that there exists a positive constant $c_{\Sigma^-}$ such that:

$$
\int_{\Sigma^-} \int_{\mathbb{R}^2} |\mathcal{G}(x - y)\omega(y)q(x)| dy\, dx = \int_{\Sigma^-} \int_{B(0,\delta)} |\mathcal{G}(z)\omega(x - z)q(x)| dz\, dx \\
\leq c_{\Sigma^-} \|\omega\|_{L^2(\mathbb{R}^2)} \|q\|_{L^2(\Sigma^-)}.
$$

We are then allowed to apply Fubini’s theorem, which yields:

$$
\int_{\mathbb{R}^2} \mathcal{G}(x - y)q(x) dy = \int_{\Sigma^-} \left( \int_{\mathbb{R}^2} \mathcal{G}(x - y)\omega(y) dy \right) q(x) dx = 0,
$$

and this identity can be rewritten as:

$$(3.33) \quad \int_{\Sigma^-} \Delta\psi(S_jq) dx + \int_{\Sigma^-} \psi q ds = 0,$$

where $S_jq$ is the single layer potential of density $q$ supported on the boundary $\Sigma_j^-$, that is:

$$S_jq(x) = -\int_{\Sigma_j^-} \mathcal{G}(x - y)q(y) dy \quad \text{for all } x \in \mathbb{R}^2 \setminus \Sigma_j^-.$$

The simple layer potential $S_jq$ is harmonic in $\mathbb{R}^2 \setminus \Sigma_j^-$ (we refer to the book of McLean [55] for details about layer potentials) and we denote respectively by $S_j^+q$ and $S_j^-q$ the restriction of $S_jq$ to the unbounded and bounded connected components of $\mathbb{R}^2 \setminus \Sigma_j^-$. The functions $S_j^+q$ and $S_j^-q$ share the same trace on $\Sigma_j^-$ and $q$ is the jump of the normal derivative across the boundary $\Sigma_j^-$:

$$(3.34) \quad q = \frac{\partial}{\partial n} S_j^+q - \frac{\partial}{\partial n} S_j^-q \quad \text{on } \Sigma_j^-.$$

From the obvious equality

$$
\int_{\Sigma_j^-} \frac{\partial}{\partial n} S_j^-q ds = 0,
$$

we deduce that the harmonic function $S_j^+q$ has zero mean flux through the boundary $\Sigma_j^-$ if and only if

$$
\int_{\Sigma_j^-} q ds = 0.
$$

On the other hand, for indices $k \neq j$, we have also:

$$
\int_{\Sigma_k^-} \frac{\partial}{\partial n} S_j^+q ds = 0,
$$

because the normal derivative of $S_j^+q$ is continuous across $\Sigma_k^-$ and $S_j^+q$ is harmonic inside $\Sigma_k^-$. From (3.33), we infer that for every $\omega \in V_0$ and every $q \in L^2(\Sigma_j^-)$ with zero mean value:

$$
\int_{\Sigma_j^-} \psi q ds = 0,
$$

and then that $\psi$ is constant on $\Sigma_j^-$. For every $q$ in $L^2(\Sigma^+)$, the corresponding single layer potential $S_0q$ supported on $\Sigma^+$ has zero mean flux through every inner boundary $\Sigma_k^-$ (for $k \in \{1, \ldots, N\}$) and we deduce from (3.33) again that the trace of $\psi$ is nul on $\Sigma^+$. It follows that $\psi$ is in the space $S_0$.

Let now $h$ be a harmonic function in $\mathcal{F}$ and assume that the normal derivative of $h$ belongs to $L^2(\Sigma)$. Then there exists $q_0 \in L^2(\Sigma^+)$ and $q_j \in L^2(\Sigma_j^-)$ ($j = 1, \ldots, N$) such that:

$$h = S_0q_0 + \sum_{j=1}^N S_jq_j \quad \text{in } \mathcal{F}.$$
Using again (3.33) and the fact that \( \psi \) is nul on \( \Sigma^+ \) and constant on \( \Sigma^- \), we deduce that:
\[
\int_{\mathcal{F}} \Delta \psi \, dx + \int_{\Sigma^-} \psi \frac{\partial h}{\partial n} \, ds = 0,
\]
and therefore, integrating by parts, that:
\[
\int_{\Sigma} \frac{\partial \psi}{\partial n} h \, ds = 0.
\]
This last equality being true for every harmonic function \( h \), it follows that the normal derivative of \( \psi \) is nul on \( \Sigma \) and therefore that \( \psi \) belongs to \( S_1 \). The proof is now completed. \( \square \)

The expression (3.32) when \( \mathcal{F} = \mathbb{R}^2 \) can be found in the book \([53, \S 2.1]\). For the Euler equations in a domain with holes, the Biot-Savart operator (in the sense considered above, that is the operator allowing recovering the stream function) is given by (see \([51]\) for a proof):
\[
(3.35) \quad N^E \omega(x) = \int_{\mathcal{F}} \mathcal{H}(x,y) \omega(y) \, dy + \sum_{j=1}^{N} (I_j - \alpha_j(\omega)) \xi_j(x) \quad \text{for all } x \in \mathcal{F}.
\]
In this identity:

(1) \( \mathcal{H} : \mathcal{F} \times \mathcal{F} \to \mathbb{R} \) is the Green’s function of the domain \( \mathcal{F} \). It is defined by:
\[
\mathcal{H}(x,y) = \mathcal{G}(x-y) - \mathcal{H}(x,y) \quad \text{for all } (x,y) \in \mathcal{F} \times \mathcal{F} \text{ s.t. } x \neq y,
\]
where, for every \( x \) in \( \mathcal{F} \), the function \( \mathcal{H}(x,\cdot) \) is harmonic in \( \mathcal{F} \) and satisfies
\[
\mathcal{H}(x,\cdot) = \mathcal{G}(x-\cdot) \quad \text{on } \Sigma.
\]

(2) The real constants \( \alpha_j(\omega) \) are given by:
\[
\begin{bmatrix}
\alpha_1(\omega) \\
\vdots \\
\alpha_N(\omega)
\end{bmatrix} = \begin{bmatrix}
(\xi_1, \xi_1)_{S_0} & \cdots & (\xi_1, \xi_N)_{S_0} \\
\vdots & \ddots & \vdots \\
(\xi_N, \xi_1)_{S_0} & \cdots & (\xi_N, \xi_N)_{S_0}
\end{bmatrix}^{-1} \begin{bmatrix}
\int_{\mathcal{F}} \omega(y) \xi_1(y) \, dy \\
\vdots \\
\int_{\mathcal{F}} \omega(y) \xi_N(y) \, dy
\end{bmatrix},
\]

where we recall the the functions \( \xi_j \) (\( j = 1, \ldots, N \)) are defined in Section 3.1.

(3) For every \( j \), the scalar \( I_j \) is the circulation of the fluid around the inner boundary \( \Sigma_j^- \).

Notice that for the Euler equations in a multiply connected domain, both the vorticity and the circulation are necessary to recover the stream function.

In case the vorticity is in \( V_0 \) and in the absence of circulation, then the Biot-Savart operator for NS equations and the Biot-Savart operator for Euler equations give the same stream function:

**Proposition 3.21.** Let \( \omega \) be in \( V_0 \) and assume that the flow is such that \( I_j = 0 \) for every \( j = 1, \ldots, N \). Then \( N_0 \omega \) (defined in (3.32)) and \( N^E \omega \) (defined in (3.35)) are equal.

The proof relies on the following lemma in which we denote simply by \( S \) the simple layer potential supported on the whole boundary \( \Sigma \).

**Lemma 3.22.** For every function \( h \in \mathcal{F}_S \) (i.e. \( h \) harmonic in \( \mathcal{F} \) and \( h \) belongs to \( S_0 \)):
\[
\left( S \frac{\partial h}{\partial n} \right)(x) = h(x) \quad \text{for all } x \in \mathcal{F}.
\]

**Proof.** Let \( h \) be in \( S_0 \). Basic results of potential theory ensures that there exists a unique \( p \in H^{-\frac{1}{2}}(\Sigma) \) such that \( Sp(x) = h(x) \) for every \( x \in \mathcal{F} \). The single layer potential \( Sp \) is harmonic in \( \mathbb{R}^2 \setminus \Sigma \) and belongs to \( H^1_{loc}(\mathbb{R}^2) \) (which means that the trace of the function matches on both sides of the boundary \( \Sigma \)). Since the trace of \( h \) is equal to 0 on \( \Sigma^+ \), the single layer potential vanishes identically on the unbounded connected component of \( \mathbb{R}^2 \setminus \Sigma^+ \). For similar reasons, \( Sp \) is constant inside \( \Sigma_j^- \) for every \( j = 1, \ldots, N \). According to the jump formula (3.34), we obtain that \( p = \partial h/\partial n \) on \( \Sigma \) and the proof is completed. \( \square \)

We can move on to the:
Proof of Proposition 3.21. For every \( x \in \mathcal{F} \), the function:

\[
\mathcal{H}_0(x, \cdot) = \mathcal{H}(x, \cdot) - \sum_{j=1}^N (\nabla \mathcal{H}(x, \cdot), \nabla \hat{\xi}_j)_{L^2(\mathcal{F})} \hat{\xi}_j,
\]

belongs to \( \delta \). Integrating by parts the terms in the sum, we obtain for every \( j = 1, \ldots, N \):

\[
(\nabla \mathcal{H}(x, \cdot), \nabla \hat{\xi}_j)_{L^2(\mathcal{F})} = \int_{\Sigma} \mathcal{H}(x, y) \frac{\partial \hat{\xi}_j}{\partial n}(y) \, ds_y = -\left( \xi \frac{\partial \hat{\xi}_j}{\partial n} \right)(x) = -\hat{\xi}_j(x),
\]

according to Lemma 3.22. Let now \( \omega \) be in \( V_0 \). Then, providing that \( \Gamma_j = 0 \) for every \( j = 1, \ldots, N \):

\[
N^\mathcal{E}_\omega(x) = N_0\omega(x) - \int_{\mathcal{F}} \mathcal{H}_0(x, y) \omega(y) \, dy + \sum_{j=1}^N \hat{\alpha}_j(\omega) \hat{\xi}_j(x) - \sum_{j=1}^N \alpha_j(\omega) \xi_j(x) \quad \text{for all } x \in \mathcal{F},
\]

where, for every \( j = 1, \ldots, N \):

\[
\hat{\alpha}_j(\omega) = \int_{\mathcal{F}} \omega(y) \hat{\xi}_j(y) \, dy.
\]

The second term in the right hand side of (3.36) vanishes by definition of \( V_0 \) and both last terms cancel out since they stand for the same linear application expressed in two different bases of \( \mathbb{F}_S \).

It remains now to link the spaces \( S_k \) for the stream functions to the spaces \( J_k \) for the velocity fields. We recall the definitions (2.17) of the spaces \( J_k \) and \( \mathbf{J}_1 \). For every other integers \( k \), the spaces \( J_k \) are classically defined from the Gelfand triple \( J_1 \subset J_0 \subset J_{-1} \), as well as the isometries \( A^1_k : J_k \to J_{k-2} \). The following Lemma can be found in [31].

Lemma 3.23. The operators \( \nabla_0^\perp : \psi \in S_0 \mapsto \nabla^\perp \psi \in J_0 \) and \( \nabla_1^\perp : \psi \in S_1 \mapsto \nabla^\perp \psi \in J_1 \) are well defined and are isometries.

Applying the abstract results of Section A.2 we deduce:

Lemma 3.24. For every index \( k \), it can be defined an isometry:

\[
\nabla_k^\perp : S_k \mapsto J_k,
\]

such that, for every pair of indices \( k \leq k' \), \( \nabla_k^\perp = \nabla_{k'}^\perp \) in \( S_{k'} \) and Diagram 3 commutes.

![Diagram 3](image)

Figure 3. The top row contains the function \( J_k \) for the velocity field and the bottom row contains the spaces \( S_k \) for the stream functions. All the operators are isometries.

Remark 3.25. Let be given a sequence \( (\psi_n)_n \) in \( S_0 \) and \( \bar{\psi} \in S_0 \). Define the corresponding velocity fields \( u_n = \nabla_0^\perp \psi_n \) and \( \bar{u} = \nabla_0^\perp \bar{\psi} \) and the vorticity fields \( \omega_n = \Delta_{-1} \psi_n \) and \( \bar{\omega} = \Delta_{-1} \bar{\psi} \). Then, the following assertions are equivalent:

\[
\begin{align*}
\psi_n &\rightrightarrows \bar{\psi} \quad \text{in } S_0, \\
u_n &\rightrightarrows \bar{u} \quad \text{in } \mathbf{J}_0, \\
\omega_n &\rightrightarrows \bar{\omega} \quad \text{in } V_{-1}.
\end{align*}
\]
Let a time $T > 0$ be given and suppose now that $(\psi_n)_n$ is a sequence in $L^\infty([0,T];S_0)$ and that $\bar{\psi}$ lies in $L^\infty([0,T];S_0)$. Then the velocity fields $u_n$ and $\bar{u}$ belong to $L^\infty([0,T];\mathbf{J}_0)$ and the vorticity fields $\omega_n$ and $\bar{\omega}$ are in $L^\infty([0,T];V_{-1})$. In the context of vanishing vorticity limit, assume that $\bar{u}$ is a solution to the Euler equations and that $u_n$ is a solution to the NS equations with a viscosity that tends to zero along with $n$. Following Kelliher [30], the vanishing viscosity limit holds when $u_n \rightarrow \bar{u}$ in $\mathbf{J}_0$, uniformly on $[0,T]$. According to (3.37), this conditions is then equivalent to either $\psi_n \rightarrow \bar{\psi}$ in $S_0$, uniformly on $[0,T]$ or to $\omega_n \rightarrow \bar{\omega}$ in $V_{-1}$, uniformly on $[0,T]$; see also Remark 2.2.

Most of the material elaborated so far in this section is summarized in the commutative diagram of Fig. 4, which contains the main operators and their relations.

![Figure 4](image)

**Figure 4.** The top row contains the function spaces $V_k$ for the vorticity fields while the bottom row contains the spaces $S_k$ for the stream functions. The operators $A^V_k$ and $A^S_k$ are Stokes operators (see the Cauchy problems (5.3) and (5.2) in the next section). The operators $\Delta_k$ link the stream functions to the corresponding vorticity fields.

### 3.4. A simple example: The unit disk.

In this subsection, we assume that $\mathcal{F}$ is the unit disk and we aim at computing the spectrum of the operator $A^V_2$ (i.e. the operator $A^V_2$ seen as an unbounded operator of domain $V_2$ in $V_0$; see (A.11)).

All the harmonic functions in $\mathcal{F}$ are equal to the real part of a holomorphic function in $\mathcal{F}$. The holomorphic functions can be expanded as power series with convergence radius equal to 1. It follows that a function $\omega \in L^2(\mathcal{F})$ belongs to $V_0 = \mathcal{B}^1$ if and only if, for every nonnegative integer $k$:

$$
\text{Re} \left( \int_{\mathcal{F}} \omega(z) z^k \, |z| \, d|z| \right) = 0 \quad \text{and} \quad \text{Im} \left( \int_{\mathcal{F}} \omega(z) z^k \, |z| \, d|z| \right) = 0.
$$

Using the method of separation of variables in polar coordinates, we find first that a function $\omega(r, \theta) = \rho(r)\Theta(\theta)$ is in $V_0$ when:

$$
\left( \int_0^1 \rho(r) r^{k+1} \, dr \right) \left( \int_0^{2\pi} \Theta(\theta) e^{ik\theta} \, d\theta \right) = 0 \quad \text{for all } k \in \mathbb{N}.
$$

Then, providing that $-\Delta \omega = \lambda \omega$ in $\mathcal{F}$ for some positive real number $\lambda$, we deduce the expression of the function $\omega$, namely:

$$(r, \theta) \mapsto \rho_k(r) \cos(k\theta) \quad \text{or} \quad (r, \theta) \mapsto \rho_k(r) \sin(k\theta)$$

for some nonnegative integer $k$. The function $\rho_k$ solves the differential equation in $(0, 1)$:

$$
(3.38a) \quad \rho_k''(r) + \frac{1}{r} \rho_k'(r) + \left( \lambda - \frac{k^2}{r^2} \right) \rho_k(r) = 0 \quad r \in (0, 1),
$$

and satisfies:

$$
(3.38b) \quad \int_0^1 \rho_k(r) r^{k+1} \, dr = 0.
$$

The solution of (3.38a) (regular at $r = 0$) is $\rho_k(r) = J_k(\sqrt{\lambda} r)$ where $J_k$ is the Bessel function of the first kind. Multiplying the equation (3.38a) by $r^{k+1}$ and integrating over the interval $(0, 1)$, we show that (3.38b) is equivalent to:

$$
\sqrt{\lambda} k J_k'(\sqrt{\lambda}) - k J_k(\sqrt{\lambda}) = 0.
$$
Using the identity \( J'_k(r) = k J_k(r) / r - J_{k+1}(r) \), the condition above can be rewritten as:

\[
J_{k+1}(\sqrt{\lambda}) = 0.
\]

We denote by \( \alpha_k^j \) (for every integers \( j, k \geq 1 \)) the \( j \)-th zero of the Bessel function \( J_k \) and we set

\[
\lambda_k^j = (\alpha_k^j)^2 \quad \text{for all} \quad k \geq 0 \quad \text{and} \quad j \geq 1.
\]

**Proposition 3.26.** The eigenvalues of \( A_k^j \) (and then also of \( A_k^1, A_k^2 \) and \( A_k^2 \) for every index \( k \), since they all have the same spectrum) are the real positive numbers \( \lambda_k^j \) (\( k \geq 0, \ j \geq 1 \)). The eigenspaces corresponding to \( \lambda_0^j \) (\( j \geq 1 \)) are of dimension 1, spanned by the eigenfunctions:

\[
(r, \theta) \mapsto J_0(\sqrt{\lambda_0^j} r).
\]

The eigenspaces of the other eigenvalues \( \lambda_k^j \) (for \( k \geq 1 \)) are of dimension 2, spanned by the eigenfunctions:

\[
(r, \theta) \mapsto J_k(\sqrt{\lambda_k^j} r) \cos(k\theta) \quad \text{and} \quad (r, \theta) \mapsto J_k(\sqrt{\lambda_k^j} r) \sin(k\theta).
\]

**Proof.** By construction, the functions defined in (3.40) are indeed eigenfunctions of \( A_k^j \). To prove that every eigenfunction of this operator is of the form (3.40), it suffices to follow the lines of the proof of [13, §8.1.1d.] for the Dirichlet operator in the unit disk.

We recover the spectrum of the Stokes operator as computed for instance in [37]. □

### 4. Lifting operators of the boundary data

#### 4.1. Lifting operators for the stream functions.

Considering (2.3c), the velocity field \( u \) solution to the NS equations in primitive variables is assumed to satisfy Dirichlet boundary conditions on \( \Sigma \), the trace of \( u \) on \( \Sigma \) being denoted by \( b \). Classically, this constraint is dealt with by means of a lifting operator. We refer to [58] and references therein for a quite comprehensive survey on this topic. In nonprimitive variables, as already mentioned earlier in (2.5), (2.6) and (2.7), the Dirichlet conditions for \( u \) translate into Neumann boundary conditions for both the potential and the stream function, namely:

\[
\frac{\partial \varphi}{\partial n} = b \cdot n \quad \text{and} \quad \frac{\partial \psi}{\partial n} = \frac{\partial \varphi}{\partial \tau} - b \cdot \tau \quad \text{on} \ \Sigma.
\]

Around every inner boundaries \( \Sigma_j^- \), the circulation of the fluid is classically defined by:

\[
\Gamma_j = \int_{\Sigma_j^-} b \cdot \tau \, ds = - \int_{\Sigma_j^-} \frac{\partial \psi}{\partial n} \, ds \quad (j = 1, \ldots, N).
\]

This being reminded, identities (4.1a) and (4.1b) suggest that instead of the field \( b \), the prescribed data on the boundary shall rather be given at every moment under the form of a triple \( (g_n, g_\tau, \Gamma) \) where \( g_n \) and \( g_\tau \) are scalar functions defined on \( \Sigma \) and \( \Gamma = (\Gamma_1, \ldots, \Gamma_N) \) is a vector in \( \mathbb{R}^N \) in such a way that:

\[
b = g_n n + \left( g_\tau - \sum_{j=1}^N \Gamma_j \frac{\partial \xi_j}{\partial n} \right) \tau \quad \text{with} \quad \int_\Sigma g_n \, ds = 0 \quad \text{and} \quad \int_{\Sigma_j^-} g_\tau \, ds = 0 \quad (j = 1, \ldots, N).
\]

We recall that \( n \) and \( \tau \) stand respectively for the unit outer normal and unit tangent vectors to \( \Sigma \). The definition of suitable function spaces for \( g_n \) and \( g_\tau \) requires introducing the following indices used to make precise the regularity of the boundary \( \Sigma \). Thus, for every integer \( k \), we define:

\[
I_1(k) = \left| k - \frac{1}{2} \right| - \frac{1}{2}, \quad J_1(k) = \left| k - \frac{1}{2} \right| + \frac{1}{2} = \max\{I_1(k-1), I_1(k+1)\},
\]

\[
I_2(k) = |k-1| + 1, \quad J_2(k) = ||k|-1| + 2 = \begin{cases} \max\{I_2(k-1), I_2(k+1)\} & \text{if} \ k \geq 0 \\ I_2(k+1) & \text{if} \ k \leq -1, \end{cases}
\]

and we can now state:
Lemma 4.2. Assume that $B$ is a map. Regarding now the second identity in (4.1a), we seek a lifting operator valued in the kernel of the $B$

Definition 4.3. Let $k$ be an integer. Assuming that $\Sigma$ is of class $C^{1(k),1}$, it makes sense to define:

(4.3a) \[ G^n_k = \left\{ g \in H^{k-\frac{1}{2}}(\Sigma) : \int_{\Sigma} g \, ds = 0 \right\} \text{ if } k \geq -1 \quad \text{and} \quad G^n_{k-1} = \{ 0 \} \text{ otherwise} \]

(4.3b) \[ G^n_0 = \left\{ g \in H^{k-\frac{1}{2}}(\Sigma) : \int_{\Sigma_j} g \, ds = 0, \quad j = 1, \ldots, N \right\}, \]

where the boundary integrals are understood according to the rule of notation (3.1).

The only purpose of setting $G^n_k = G^n_{k-1}$ when $k \leq -2$ in (4.3a) is to simplify the statement of the next results.

The problem of lifting the normal component $g_n$ by the harmonic Kirchhoff potential function is addressed in the lemma below, where, for every nonnegative integer $k$:

(4.4) \[ \mathcal{S}^k_k = \left\{ \varphi \in H^k(\mathcal{F}) : \Delta \varphi = 0 \text{ in } \mathcal{D}'(\mathcal{F}), \int_{\mathcal{F}} \varphi \, dx = 0 \text{ and } \int_{\Sigma} \frac{\partial \varphi}{\partial n} = 0 \right\}. \]

Lemma 4.2. Assume that $\Sigma$ is of class $C^{k,1}$ for some integer $k \geq -1$ and that $g_n$ belongs to $G^n_k$. Then the operator

(4.5) \[ L^n_k : g_n \in G^n_k \mapsto \varphi \in \mathcal{S}^{k+1}_k \quad \text{where} \quad \frac{\partial \varphi}{\partial n} = g_n, \]

is well defined and bounded. The operator

(4.6) \[ T_k : g_n \in G^n_k \mapsto \frac{\partial \varphi}{\partial n} \bigg|_{\Sigma} \in G^n_k, \]

is bounded as well. Moreover, as for the definition of $G^n_k$, the definition of $T_k$ is extended to integers $k \leq -2$ by setting $T_k = T_{-1}$.

Proof. Let us only consider the weakest case, i.e. $k = -1$. We introduce the Hilbert space $E$ and its scalar product whose corresponding norm is equivalent in $E$ to the usual norm of $H^2(\mathcal{F})$:

\[ E = \left\{ \theta \in H^2(\mathcal{F}) : \frac{\partial \theta}{\partial n} \bigg|_{\Sigma} = 0, \int_{\Sigma} \theta \, ds = 0 \right\}, \quad (\theta_1, \theta_2)_E = \int_{\mathcal{F}} \Delta \theta_1 \Delta \theta_2 \, dx. \]

According to Riesz representation Theorem, for every $g_n \in G^n_{-1}$, there exists a unique $\theta_g \in E$ such that:

\[ (\theta, \theta_g)_E = -\int_{\Sigma} g_n \theta \, ds \quad \text{for all } \theta \in E. \]

One easily verifies that the function $\varphi = \Delta \theta_g$ is in $L^2(\mathcal{F})$ and satisfies $\int_{\mathcal{F}} \varphi \, dx = 0$ and $\frac{\partial \varphi}{\partial n} = g_n$ in $H^{-\frac{1}{2}}(\Sigma)$. The rest of the lemma being either classical or obvious, the proof is complete.

The operator $T_k$ is the tangential differential operator composed with the classical Neumann-to-Dirichlet map. Regarding now the second identity in (4.1a), we seek a lifting operator valued in the kernel of the operator $Q\Delta$, that is the kernel of the Stokes operator for the stream function (see Lemma 3.6). Loosely speaking (disregarding regularity issues), this kernel is $\mathfrak{B}_S$, the space of the biharmonic stream functions defined in (3.22).

Definition 4.3. Let $k$ be an integer and assume that $\Sigma$ is of class $C^{1(k),1}$. The space of biharmonic functions $\mathfrak{B}^b_k$ and the lifting operator $L^b_k : G^n_k \to \mathfrak{B}^b_k$ are defined differently, depending upon the sign of $k$:

1. When $k \geq 1$, $\mathfrak{B}^b_k = \mathfrak{B}_S \cap H^{k+1}(\mathcal{F})$ (and hence $\mathfrak{B}^b_k$ is simply equal to $\mathfrak{B}_S$ defined in (3.22)) and for any $g_r \in G^n_k$, $L^b_k g_r$ is the unique stream function $\psi$ in $\mathfrak{B}^b_k$ satisfying the Neumann boundary condition:

\[ \frac{\partial \psi}{\partial n} \bigg|_{\Sigma} = g_r \quad \text{on } \Sigma. \]
Remark 4.4. \(1\) For \(k \leq 0\), the operator \(L_k^*\) is well defined according to Lemma 3.2 and Lemma 3.6, under the regularity assumption on the boundary \(\Sigma\) of Definition 4.1. However, further regularity is needed to define the lifting operator, namely \(C^{I_2(k),1}\).

For every pair of integers \((k', k)\), both positive or both nonpositive, the inequality \(k' \geq k\) entails the inclusion \(\mathfrak{B}_S^{k'} \subset \mathfrak{B}_S^k\). We shall prove that the inclusion \(\mathfrak{B}_S^{k'} \subset \mathfrak{B}_S^k\) still holds and that the diagram on Fig. 5 commutes. Notice that \(L_k^*\) is clearly invertible when \(k\) is positive. The question of invertibility for nonpositive indices \(k\), or more precisely of injectivity (since surjectivity is obvious) is not clear. This amounts to determine whether the traces of the functions of \(V_{-k+1}\) are dense in \(H^{-k+\frac{1}{2}}(\Sigma)\).

\[
\begin{array}{c}
G_{k'}^* \subset G_k^* \\
\downarrow L_{k'}^* \quad \downarrow L_k^* \\
\mathfrak{B}_S^{k'} \subset \mathfrak{B}_S^k
\end{array}
\]

Figure 5. The diagram commutes for any pair of integers \((k, k')\) such that \(k' \geq k\).

Lemma 4.5. The operator \(L_k^*\) is bounded for every integer \(k\) and is an isomorphism when \(k\) is positive. For every pair of integers \((k', k)\) such that \(k' \geq k\), the restriction of \(L_k^*\) to \(G_{k'}^*\) is equal to \(L_{k'}^*\) (providing that \(\Sigma\) is of class \(C^{\max\{I_2(k),I_2(k')\},1}\)).

Proof. The boundedness is a consequence of Lemma 3.2, Lemma 3.6 and the continuity of the trace operator. Let \(\Sigma\) be of class \(C^{2,1}\), \(g_{r}\) belong to \(G_{1}^*\) and introduce the stream function \(\psi = L_k^* g_{r}\). Considering \(\psi \in \mathfrak{B}_S^k\) as an element of \(S_0\) identified with its dual space, we get:

\[
(\psi, \theta)_{S_0} = (\nabla \psi, \nabla P_1 \theta)_{L^2(\mathcal{F})} = \int_{\Sigma} (P_1 \theta) g_{r} \, ds - (\Delta \psi, P_1 \theta)_{L^2(\mathcal{F})} \quad \text{for all } \theta \in S_0,
\]

where the last term vanishes because \(\Delta \psi\) belongs to \(\mathfrak{S}\). This proves that \(L_k^* = L_k^*\) in \(G_{1}^*\). The other cases derive straightforwardly and the proof is complete.

We can gather Lemma 4.2 and Definition 4.3 in order to define a lifting operator taking into account the circulation of the fluid around the fixed obstacles. In view of \((4.1a)\) and \((4.1b)\), we are led to set:

Definition 4.6. Let \(k\) be any integer and assume that \(\Sigma\) is of class \(C^{I_2(k),1}\) and that the triple \((g_n, g_{r}, \Gamma)\) is in \(G^n_k \times G_r^* \times \mathbb{R}^N\) with \(\Gamma = (\Gamma_1, \ldots, \Gamma_N)\). We define the operator:

\[
L_k^S(g_n, g_{r}, \Gamma) = L_k^S(T_k g_n - g_{r}) + \sum_{j=1}^{N} \Gamma_j \xi_j,
\]

which is valued in the space

\[
S_k^b = \mathfrak{B}_S^k \oplus F_S.
\]

We can address the case of time dependent spaces:
Definition 4.7. Let $T$ be a positive real number, $k$ be an integer and assume that $\Sigma$ is of class $C^{1(k)}$ (the expression of $J_1(T)$ is given in (4.2)). We begin by introducing the spaces:

$$G_k^\Sigma(T) = L^2(0,T;G_k^{n+1}) \cap C([0,T];G_k^n) \cap H^1(0,T;G_k^{n-1})$$

and also:

$$G_k^\Sigma(T) = L^2(0,T;G_k^{n+1}) \cap C([0,T];G_k^n) \cap H^1(0,T;G_k^{n-1}),$$

(4.10)

Assuming that $\Sigma$ is of class $C^{12(k)}$ (with $J_2(k)$ defined in (4.2)) the operator $L^2_{k+1}$ maps to space $G_k(T)$ into the space:

$$S_k^b(T) = \begin{cases} H^1(0,T;S_k^{b-1}) \cap C([0,T];S_k^b) \cap L^2(0,T;S_k^{b+1}) & \text{if } k \geq 0, \\
L^2(0,T;S_k^{b+1}) & \text{if } k \leq -1. \end{cases}$$

(4.11)

As a direct consequence of Lemmas 4.2 and 4.5 we can state:

Lemma 4.8. Let $k$ be any integer and assume that $\Sigma$ is of class $C^{12(k)}$. Then the lifting operator for the stream function:

$$L^2_{k+1} : G_k^\Sigma \times G_k^{\infty} \times \mathbb{R}^N \rightarrow S_k^b,$$

is well defined and is bounded. Moreover, if $k$ and $k'$ are two integers such that $k' \leq k$, then $L^2_{k'} = L^2_k$ in $G_k^\Sigma \times G_k^{\infty} \times \mathbb{R}^N$. It follows that for every positive real number $T$ and every integer $k$, providing that $\Sigma$ is of class $C^{12(k)}$, the operator:

$$L^2_{k+1} : G_k(T) \rightarrow S_k^b(T),$$

is well defined and bounded as well, the bound being uniform with respect to $T$. 

4.2. Additional function spaces. We aim now at building a lifting operator valued in vorticity spaces (i.e. we aim at giving the counterpart of Definitions 4.6-4.7 and Lemma 4.8 for the vorticity). We recall that, for every positive integer $k$, the lifting operator $L^2_k$ is valued in $S_k^b$. For nonpositive integers $k$, $S_k^b$ is a subspace of $S_k$ and therefore, the corresponding vorticity space is simply $V_{k-1}^b = \Delta_{k-1}S_k^b$. However, when $k$ is positive, $S_k^b \cap S_k = \{0\}$. A somehow naive approach would consist in taking simply the Laplacian of $S_k^b$ but one easily verifies that $\Delta S_k^b \subset \mathcal{F}$ and $\mathcal{F}$ is in no space $V_j$ for any integer $j$. This difficulty is circumvented by noticing that $S_k^b \subset S_0$ (still considering positive integers $k$). So $V_0^b = \Delta_{-1}S_0^b$ (with $\Delta_{-1}$ defined in (3.31b)) seems to be a good candidate for our purpose, an idea we are now going to elaborate on. More precisely, for every integer $k$, $S_k^b$ is a subspace of $S_k$ defined by:

$$\tilde{S}_k = S_0 \cap H^{k+1}(\mathcal{F}) \quad \text{if } k \geq 1 \quad \text{and} \quad \tilde{S}_k = S_k \quad \text{if } k \leq 0. \quad (4.12)$$

The corresponding vorticity space is therefore in the image of $\tilde{S}_k$ (seen as a subspace of $S_0$) by $\Delta_{-1}$ if $k \geq 1$ and by $\Delta_{-1}$ if $k \leq 0$ (see Fig. 3). Thus we define:

$$\tilde{V}_k = \Delta_{-1}\tilde{S}_{k+1} \quad \text{if } k \geq 0 \quad \text{and} \quad \tilde{V}_k = \Delta_k\tilde{S}_{k+1} = V_k \quad \text{if } k \leq -1. \quad (4.13)$$

It is crucial to understand that, no matter how regular the functions are, the spaces $\tilde{V}_k$ are always dual spaces (for every integer $k$). They are subspaces of $V_{-1}$. We will show that $V_{-1}$ is the space of largest index that contains in some sense the harmonic functions. We shall focus our analysis on the pairs $(\tilde{S}_1,\tilde{V}_1)$, $(\tilde{S}_2,\tilde{V}_1)$ and $(\tilde{S}_3,\tilde{V}_2)$ only, the other cases being of less importance as it will becomes clear in the next section.

The pair $(\tilde{S}_1,\tilde{V}_0)$. The space $\tilde{S}_1$ is provided with the scalar product:

$$\langle \tilde{\psi}_1, \tilde{\psi}_2 \rangle_{\tilde{S}_1} = (\Delta \tilde{\psi}_1, \Delta \tilde{\psi}_2)_{L^2(\mathcal{F})} + \Gamma(\tilde{\psi}_1) \cdot \Gamma(\tilde{\psi}_2), \quad \text{for all } \tilde{\psi}_1, \tilde{\psi}_2 \in \tilde{S}_1, \quad (4.14)$$

where, for every $\theta \in H^2(\mathcal{F})$:

$$\Gamma(\theta) = (\Gamma_1(\theta), \ldots, \Gamma_N(\theta))^t \in \mathbb{R}^N \quad \text{with} \quad \Gamma_j(\theta) = -\int_{\Sigma_j} \frac{\partial \theta}{\partial n} \, ds, \quad (j = 1, \ldots, N).$$

Lemma 4.9. The space $\tilde{S}_1$ enjoys the following properties:
(1) The norm induced by the scalar product (4.14) is equivalent in \( \tilde{S}_1 \) to the usual norm of \( H^2(\mathcal{F}) \).

(2) The space \( \tilde{S}_1 \) admits the following orthogonal decompositions:

\[
\tilde{S}_1 = Z_2 \oplus F_S = S_1 \oplus \mathcal{B}_S \oplus F_S,
\]

where we recall that the expression of the space \( \mathcal{B}_S \) is given in (3.22) and that the finite dimensional space \( F_S \) is spanned by the functions \( \xi_j \) (\( j = 1, \ldots, N \)).

The orthogonal decomposition (4.15) can be given a physical meaning: The subspace \( S_1 \) contains the stream functions accounting for the circulation of the fluid around the inner boundaries \( \Sigma \), the space \( F_S \) contains the harmonic stream functions which solve stationary Stokes problems (with zero circulation though). Finally, the space \( F_S \) contains the harmonic stream functions accounting for the circulation of the fluid around the inner boundaries \( \Sigma_j \) (\( j = 1, \ldots, N \)).

**Proof of Lemma 4.9.** The equivalence of the norms derives from classical elliptic regularity results. On the other hand, according to (3.21):

\[
Z_2 = S_1 \oplus \mathcal{B}_S \subset \tilde{S}_1.
\]

Applying the fundamental homomorphism theorem to the surjective operator \((-\Delta) : \tilde{S}_1 \rightarrow Z_0 \) whose kernel is the space \( F_S \), we next obtain that:

\[
\tilde{S}_1 = Z_2 \oplus F_S.
\]

Finally, combining (4.16) and (4.17) yields (4.15) after verifying that the direct sum is orthogonal for the scalar product (4.14). The proof is then completed.

Let us determine now the corresponding decomposition for the vorticity space \( \bar{\Omega}_0 = \Delta_{-1} \hat{S}_1 \) which is a subspace of the dual space \( V_{-1} \) (see Fig. 4). We shall prove in particular that \( \bar{\Omega}_0 \) contains \( V_0 \) whose expression is (seen as a subspace of \( V_{-1} \)):

\[
V_0 = \{ (\omega, Q_1)_{L^2(\mathcal{F})} : \omega \in \mathcal{S}_1 \}.
\]

Notice that in (4.18), one would expect merely the term \((\omega, \cdot)_{L^2(\mathcal{F})}\) in place of \((\omega, Q_1, \cdot)_{L^2(\mathcal{F})}\), but both linear forms are equal in \( V_1 \). We define below two additional subspaces of \( V_{-1} \):

\[
\mathcal{S}_V = \{ (\omega, Q_1)_{L^2(\mathcal{F})} : \omega \in \mathcal{S}_1 \} \quad \text{and} \quad L^2_v = \{ (\omega, Q_1)_{L^2(\mathcal{F})} : \omega \in L^2(\mathcal{F}) \}.
\]

The space \( \mathcal{S}_V \) contains in some sense the harmonic vorticity field. Finally, we introduce the finite dimensional subspace of \( V_{-1} \):

\[
F^*_V = \text{span} \{ \zeta_j, j = 1, \ldots, N \},
\]

where, for every \( j = 1, \ldots, N \):

\[
\langle \zeta_j, \omega \rangle_{V_{-1}, V_1} = -\langle \nabla \xi_j, \nabla Q_1 \omega \rangle_{L^2(\mathcal{F})} = -\int_{\Sigma_j} \frac{\partial \xi_j}{\partial n} Q_1 \omega \, ds \quad \text{for all} \ \omega \in V_1.
\]

We can now state:

**Theorem 4.10.** The space \( \bar{\Omega}_0 \) can be decomposed as follows:

\[
\bar{\Omega}_0 = L^2_v \oplus F^*_V = V_0 \oplus V_0^b \quad \text{where} \quad V_0^b = \mathcal{S}_V \oplus F^*_V.
\]

The direct sum above is orthogonal for the scalar product defined, for every \( \bar{\omega}_1 \) and \( \bar{\omega}_2 \) in \( \bar{\Omega}_0 \) by:

\[
\langle \omega_1, \omega_2 \rangle_{\bar{\Omega}_0} = (\omega_1, \omega_2)_{L^2(\mathcal{F})} + \sum_{j=1}^N \alpha_{1,j} \alpha_{2,j},
\]

where, for \( k = 1, 2, \omega_k = (\omega_k, Q_1, \cdot)_{L^2(\mathcal{F})} + \zeta^k \) with \( \omega_k \in L^2(\mathcal{F}) \) and \( \zeta^k = \sum_{j=1}^N \alpha_{k,j} \zeta_j \) in \( F^*_V \) (\( \alpha_{k,j} \in \mathbb{R} \) for \( j = 1, \ldots, N \)).

Moreover, the restriction of \( \Delta_{-1} \) to \( \tilde{S}_1 \), denoted by \( \Delta_0 \), is an isometry from \( \tilde{S}_1 \) onto \( \bar{\Omega}_0 \) (see Fig. 6).

**Remark 4.11.** Let us emphasize that:
(1) In the decomposition \( \bar{\omega} = (\omega, Q_1)_{L^2(F)} + \zeta \) of every \( \bar{\omega} \) of \( \bar{V}_0 \), the term \( \omega \) which belongs to \( L^2(F) \) will be referred to as the regular part of \( \bar{\omega} \) while \( \zeta \) will stand for the singular part.

(2) Loosely speaking, the space \( \bar{V}_0 \) consists in functions in \( L^2(F) \) and measures \( \zeta_j \) (\( j = 1, \ldots, N \)) supported on the boundaries \( \Sigma_j \) (notice again that \( F_\Sigma \) is not a distributions space). This can be somehow understood from a physical point of view by observing that \(-\zeta_j \) is the vorticity corresponding to the harmonic stream function \( \xi_j \) which accounts for the circulation of the fluid around \( \Sigma_j \). Hence the vorticity is a measure supported on the boundary of the obstacle. In connection with this topic, wondering how is vorticity imparted to the fluid when a stream flow past an obstacle, Lighthill answers in [59] that the solid boundary is a distributed source of vorticity (just as, in some flows, it may be a distributed source of heat).

(3) The fact that \( \bar{\Delta}_0 \) is an isometry asserts in particular that to any given vorticity in \( \bar{V}_0 \) corresponds a unique stream function \( \psi \) in \( S_1 \) that can be uniquely decomposed as \( \psi = \psi_S + \psi_C \) where \( \nabla \perp \psi = 0 \) on the boundary \( \Sigma, \nabla \perp \psi_S \) solves a stationary Stokes system and \( \psi_C \) is harmonic in \( F \) and accounts for the circulation of the fluid around the boundaries \( \Sigma_j \) (\( j = 1, \ldots, N \)).

The rest of this subsection is dedicated to the proof of Theorem 4.10. In order to determine the image of \( \bar{S}_1 \) by the operator \( \Delta_{-1} \), the factorization \( \Delta_{-1} = -A_1^T P_1 \) suggests to determine first the expression of the space \( P_1 \bar{S}_1 \). This space is provided with the scalar product:

\[
(\omega_1, \omega_2)_{P_1 \bar{S}_1} = (\Delta \omega_1, \Delta \omega_2)_{L^2(F)} + \Gamma(\omega_1) \cdot \Gamma(\omega_2), \quad \text{for all } \omega_1, \omega_2 \in P_1 \bar{S}_1,
\]

and we denote by \( \bar{P}_2 \) the restriction of \( P_1 \) to \( \bar{S}_1 \).

**Remark 4.12.** According to Lemma 3.3, when \( \Sigma \) is of class \( C^{3,1} \), the space \( P_1 \bar{S}_1 \) is simply equal to \( V_1 \cap H^2(F) \). When \( \Sigma \) is less regular Remark 4.11 applies replacing \( V_2 \) with \( P_1 \bar{S}_1 \).

The decomposition (4.15) leads to introducing the spaces:

\[
B_V = P_1 B_S \quad \text{and} \quad F_V = P_1 F_S.
\]

Nothing more than \( B_V = B \cap V_0 \) (where \( B \) is defined in (3.20)) can be said on the space \( B_V \). The space \( F_V \) however can be bound to the space \( B_S \) spanned by the functions \( \chi_j \) (\( j = 1, \ldots, N \)) defined in (3.5) (see the definition below the identity (3.6)).

**Lemma 4.13.**

1. The space \( B_S \) is a subspace of \( S_2 \) and \( F_V = \Delta_1 B_S \).

2. The space \( P_1 \bar{S}_1 \) admits the following orthogonal decomposition:

\[
P_1 \bar{S}_1 = V_2 \oplus \frac{1}{\perp} B_V \oplus F_V.
\]

Moreover, the operator \( \bar{P}_2 \) is an isometry from \( \bar{S}_1 \) onto \( P_1 \bar{S}_1 \) (see Fig. [9]).

**Proof.** For every \( \theta \in D \), an integration by parts yields:

\[
(\chi_j, \theta)_{S_1} = -(Q_1 \Omega_j, \theta)_{S_0} \quad \text{for all } j = 1, \ldots, N,
\]

where \( \Omega_j = \Delta \chi_j \), because \( \chi_j \) is of class \( C^\infty \) in the support of \( \theta \). On the other hand, for every pair of indices \( j, k \in \{1, \ldots, N\} \), we have also:

\[
(\chi_j, \chi_k)_{S_1} = \int_{\Sigma} Q_1 \Omega_j \frac{\partial \chi_k}{\partial n} \, ds - (Q_1 \Omega_j, \chi_k)_{S_0} = - (Q_1 \Omega_j, \chi_k)_{S_0},
\]

where we have used the rule of notation (3.1) (as being harmonic in \( L^2(F) \), the trace of \( \Omega_j \) on \( \Sigma \) is well defined in \( H^{-\frac{1}{2}}(\Sigma) \)). Since the space \( D(F) \oplus B_S \) is dense in \( S_1 \) according to the decomposition (3.6), we deduce from the identities (4.26) that \( B_S \subset S_2 \) (recall the \( S_2 \) is the preimage of \( S_0 \) by \( A^2_0 \)).

Notice now that the functions \( P_1 \xi_j \) (\( j = 1, \ldots, N \)) belong to \( V_1 \), are harmonic in \( F \) and according to the second point of Remark 3.3 they satisfy the same fluxes conditions (3.3b) as the functions \( \xi_j \). All these properties are also shared by the functions \( \Omega_j \) whence we deduce first that:

\[
P_1 \xi_j = \Omega_j \quad (j = 1, \ldots, N),
\]
and then that $\mathcal{F}_V = \Delta_1 \mathcal{B}_S$, which is the first point of the lemma. The second point can easily be deduced from (4.15) and the second occurrence of Remark 3.3.

Yet it remains to apply the operator $A^{
abla}_1$ to the equality (4.25) in order to get the expression of $\bar{V}_0 = \Delta_{-1} \bar{S}_1$. Since $A^{
abla}_1$ is an isometry, the decomposition (4.22) is a direct consequence of the decomposition (4.25) and the following lemma, where the spaces $\mathcal{S}_V$ and $\mathcal{F}_V$ are defined respectively in (4.19) and (4.20).

**Lemma 4.14.** The following equalities hold:

\begin{equation}
A^\nabla_1 \mathcal{B}_V = \mathcal{S}_V \quad \text{and} \quad A^\nabla_1 \mathcal{F}_V = \mathcal{F}_V^*.
\end{equation}

**Proof.** By definition, every element of $\mathcal{B}_V$ can be written $P_1 \psi$ for some $\psi \in \mathcal{B}_S$. The definitions of the operator $A^\nabla_1$ and of the scalar product in $V_1$ lead to:

$$\langle A^\nabla_1 P_1 \psi, \theta \rangle_{V_1} = (P_1 \psi, \theta)_{V_1} = (\psi, Q_1 \theta)_{Z_1}, \quad \text{for all } \theta \in V_1.$$ 

But $\mathcal{B}_S$ is a subspace of $Z_2$ according to (3.21) and $Q_1 V_1 = Z_1$. It follows that:

$$\langle \psi, Q_1 \theta \rangle_{Z_1} = (A^\nabla_2 \psi, Q_1 \theta)_{Z_0} = (\omega, Q_1 \theta)_{L^2(\mathcal{F})}, \quad \text{for all } \theta \in V_1,$$

where $\omega = A^\nabla_2 \psi$ belongs to $\mathcal{S}$ according to Lemma 3.9. The first equality in (4.28) being proven, let us address the latter. For every $j = 1, \ldots, N$, some elementary algebra yields:

$$\langle A^\nabla_1 P_1 \xi_j, \omega \rangle_{V_1} = (P_1 \xi_j, \nabla Q_1 \omega)_{L^2(\mathcal{F})} = (\xi_j, Q_1 \omega)_{S_0}, \quad \text{for all } \omega \in V_1.$$ 

Comparing with (4.21), we obtain indeed that $A^\nabla_1 P_1 \xi_j = -\xi_j$ and recalling the definition of $\mathcal{F}_V$ given in (4.4), we are done with both identities in (4.28) and the proof is completed.

As we did for $\bar{S}_1$ and $P_1 \bar{S}_1$, the space $\bar{V}_0$ can be provided with a norm stronger than the one of the amiant space $V_{-1}$, namely the norm which derives from the scalar product (4.23). One easily verifies that the direct sum (4.22) is indeed orthogonal for this scalar product. Furthermore, the operator $\bar{A}^\nabla_1 : P_1 \bar{S}_1 \to \bar{V}_0$ which is the restriction of $A^\nabla_1$ to $P_1 \bar{S}_1$ is an isometry. Since $\bar{\Delta}_0 = -\bar{A}^\nabla_1 P_2$ and the operators $\bar{A}^\nabla_1_1$ and $P_2$ are both isometries, we can draw the same conclusion for $\bar{\Delta}_0$. The proof of the theorem is now completed.

![Figure 6. Some function spaces and isometric operators appearing in the statement of Theorem 4.10 and its proof. As usual, the top row contains the vorticity spaces while the bottom row contains the spaces for the stream functions.](image)

The pair $(\bar{S}_2, \bar{V}_1)$. We assume that $\Sigma$ is of class $C^{2,1}$ and we consider the spaces:

\begin{equation}
\bar{S}_2 = S_0 \cap H^3(\mathcal{F}) \quad \text{and} \quad \bar{V}_1 = \Delta_{-1} \bar{S}_2.
\end{equation}

The analysis of these spaces being very similar to those of $\bar{S}_1$ and $\bar{V}_0$, we shall skip the details and focus on the main results.
Lemma 4.15. The spaces $\bar{S}_2$ and $P_1\bar{S}_2$ admit respectively the following orthogonal decompositions:
\begin{equation}
\bar{S}_2 = S_2 \oplus B_2^3 \oplus F_S \quad \text{and} \quad P_1\bar{S}_2 = V_3 \oplus B_3^1 \oplus F_V,
\end{equation}
where $B_2^3 = B_3 \cap H^3(F)$ was introduced in Definition 4.3 and $B_3^1 = P_1 B_2^3$. The spaces $\bar{S}_2$ and $P_1\bar{S}_2$ are provided with the same scalar product, namely:
\begin{align*}
(\psi_1,\psi_2)_{\bar{S}_2} &= (\Delta \psi_1, \Delta \psi_2)^{V}_{H^1} + \Gamma(\psi_1) \cdot \Gamma(\psi_2) \quad \text{for all } \psi_1, \psi_2 \in \bar{S}_2, \\
(\omega_1, \omega_2)_{P_1\bar{S}_2} &= (\Delta \omega_1, \Delta \omega_2)^{V}_{H^1} + \Gamma(\omega_1) \cdot \Gamma(\omega_2) \quad \text{for all } \omega_1, \omega_2 \in P_1\bar{S}_2,
\end{align*}
the scalar product $(\cdot, \cdot)^{V}_{H^1}$ being defined in (3.13b).

Finally, the operator $P_3$ defined as the restriction of $P_1$ to $\bar{S}_2$ is an isometry from $\bar{S}_2$ onto $P_1\bar{S}_2$ (see Fig. 7).

We turn now our attention to $\bar{V}_1 = \Delta_{-1} \bar{S}_2$, which is a subspace of the dual space $V_{-1}$. As a subspace of $V_{-1}$ the spaces $V_1$ is identified with $\{ (\omega, Q_{1\cdot})_{L^2(F)} : \omega \in V_1 \}$ and we define as well:
\begin{equation}
\bar{S}_V^1 = \{ (\omega, Q_{1\cdot})_{L^2(F)} : \omega \in \bar{S}_1 \},
\end{equation}
where we recall that $\bar{S}_1 = \bar{S} \cap H^1(F)$ (defined in Subsection 3.1). Finally, in the same way, we introduce:
\begin{equation}
H_V^1 = \{ (\omega, Q_{1\cdot})_{L^2(F)} : \omega \in H^1(F) \},
\end{equation}
that can be compared with the space $L_V^1$ defined in (4.19).

Theorem 4.16. The space $\bar{V}_1$ is a subspace of $V_{-1}$ which can be decomposed as follows:
\begin{equation}
\bar{V}_1 = H_V^1 \oplus F^*_V = V_1 \oplus V_1^b \quad \text{with} \quad V_1^b = \bar{S}_V^1 \oplus F^*_V.
\end{equation}
It is provided with the scalar product, defined for every $\bar{\omega}_1, \bar{\omega}_2 \in \bar{V}_1$ by:
\begin{equation}
(\bar{\omega}_1, \bar{\omega}_2)_{\bar{V}_1} = (\omega_1, \omega_2)^{V}_{H^1} + \sum_{j=1}^N \alpha_{1,j} \alpha_{2,j},
\end{equation}
where, for $k = 1, 2$, $\bar{\omega}_k = (\omega_k, Q_{1\cdot})_{L^2(F)} + \zeta^k$ with $\omega_k \in H^1(F)$ and $\zeta^k = \sum_{j=1}^N \alpha_{k,j} \zeta_j$ in $F_V^*$ ($\alpha_{k,j} \in \mathbb{R}$ for $j = 1, \ldots, N$).

Finally, the operator $\Delta_1$ which is the restriction of $\Delta_{-1}$ to $\bar{S}_2$ is an isometry from $\bar{S}_2$ onto $\bar{V}_1$ (see Fig. 7).

Remark 4.17. As in Remark 4.13 in the decomposition $\bar{\omega} = (\omega, Q_{1\cdot})_{L^2(F)} + \zeta$ of every vorticity field $\bar{\omega}$ in $V_1$, the term $\omega$ (belonging to $H^1(F)$) will be called the regular part of $\bar{\omega}$ and $\zeta$, the singular part.

These results can be summarized in the commutative diagram on Fig. 7 where the operator $\bar{A}_3^V$ defined as the restriction of $A_3^V$ to the space $P_1\bar{S}_2$ is an isometry from $P_1\bar{S}_2$ onto $V_1$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure7.png}
\caption{Some function spaces and isometric operators appearing in the statement of Lemma 4.15 and Theorem 4.16. This diagram is worth being compared with the diagrams on Fig. 5 and Fig. 6. In particular, the following inclusions hold: $S_2 \subset \bar{S}_1 \subset S_0$ and $\bar{V}_1 \subset V_0 \subset V_{-1}$.}
\end{figure}
The pair $\bar{S}_3, \bar{V}_2)$. The decompositions of $\bar{V}_0$ and $\bar{V}_1$ rested mainly on the simple equalities $L^2(\mathcal{F}) = V_0 \oplus \bar{S}_1$ and $H^1(\mathcal{F}) = V_1 \oplus \bar{S}^1$. However, $H^2(\mathcal{F})$ is not equal to $V_2 \oplus \bar{S}^2$. Indeed, according to Fig. 6, the decompositions is more complex, namely:

$$H^2(\mathcal{F}) = V_2 \oplus \mathcal{B}_V \oplus F_V \oplus \bar{S}^2.$$  

We are led to define:

$$V_2 = H^2_V \oplus F_V^* = V_2 \oplus \mathcal{B}_V \oplus F_V \oplus V_b^h \text{ with } H^2_V = V_2 \oplus \mathcal{B}_V \oplus F_V \oplus \bar{S}_V^2 \text{ and } V_b^h = \bar{S}_V^2 \oplus F_V^*.$$  

This direct sum is orthogonal once the following decompositions hold:

$$V_2 = H^2_V \oplus F_V^* = V_2 \oplus \mathcal{B}_V \oplus F_V \oplus V_b^h \text{ with } H^2_V = V_2 \oplus \mathcal{B}_V \oplus F_V \oplus \bar{S}_V^2 \text{ and } V_b^h = \bar{S}_V^2 \oplus F_V^*.$$  

for every $\bar{V}_2$ such that $\bar{V}_k = (\omega, Q_1)_{L^2(\mathcal{F})} + (P^\perp \omega_1, P^\perp \omega_2)_{\bar{S}^2} + \Gamma(\omega_1) \cdot \Gamma(\omega_2) + \sum_{j=1}^N \alpha_{1,j} \alpha_{2,j},$  

for every $\bar{V}_2$ in $V_2$ such that $\bar{V}_k = (\omega, Q_1)_{L^2(\mathcal{F})} + \sum_{j=1}^N \alpha_{1,j} \alpha_{2,j}$ with $\omega_k \in H^2(\mathcal{F})$ and $\alpha_{k,j} \in \mathbb{R}$ for $k = 1, 2$.  

The decompositions (4.32) will play an important role in Section 8 and in particular the fact that $\bar{V}_2$ is not equal to $V_2 \oplus V_2^p$.  

We do not need to enter into the details of the decomposition of $\bar{S}_3$. Let us just make precise the norm this space is equipped with, namely:

$$(\psi_1, \psi_2)_{\bar{S}_3} = (\Delta^2 \psi_1, \Delta^2 \psi_2)_{L^2(\mathcal{F})} + (P^\perp \psi_1, P^\perp \psi_2)_{\bar{S}^2} + \Gamma(\Delta \psi_1) \cdot \Gamma(\Delta \psi_2) + \Gamma(\psi_1) \cdot \Gamma(\psi_2),$$  

for every $\psi_1, \psi_2$ in $\bar{S}_3$. As usual, we denote by $\bar{\Delta}_2$ the restriction of $\Delta_{-1}$ to $\bar{S}_3$ and we let is to the reader to verify that:

**Lemma 4.18.** The operator $\bar{\Delta}_2$ is an isometry from $\bar{S}_3$ onto $\bar{V}_2$.

Notice that obviously $\bar{S}_3$ contains the space $S_b^1$.

4.3. Lifting operators for the vorticity field. The expressions of the lifting operators for the vorticity field derive straightforwardly from Fig. 6 and Fig. 7. Following the lines of Definition 4.6 and recalling that the indices $I_2(k)$ and $J_2(k)$ are defined in (4.22), we can write:

**Definition 4.19.** Let $k$ be an integer such that $k \leq 2$ and assume that $\Sigma$ is of class $C^{4(k+1),1}$. For every triple $(g_n, g_{\tau}, \Gamma)$ in $G_k \times G_{k+1} \times \mathbb{R}^N$ with $\Gamma = (\Gamma_1, \ldots, \Gamma_N)$ we define:

$$(4.33a) \quad L^V_k (g_n, g_{\tau}, \Gamma) = \bar{\Delta}_k L^V_{k+1} (g_n, g_{\tau}, \Gamma) = \bar{\Delta}_k L^V_{k+1} (T_{k+1}g_n - g_{\tau}) + \sum_{j=1}^N \Gamma_j \zeta_j \quad \text{if } k = 0, 1, 2,$$

$$(4.33b) \quad L^V_k (g_n, g_{\tau}, \Gamma) = \bar{\Delta}_k L^V_{k+1} (g_n, g_{\tau}, \Gamma) \quad \text{if } k \leq -1.$$  

The operator $L^V_k$ is valued in the space $V_b^b$ defined by:

$$V_b^b = \bar{\Delta}_k S_{k+1}^b \oplus \mathcal{B}_V^* \quad \text{if } k = 0, 1, 2 \quad \text{and} \quad V_b^b = \bar{\Delta}_k S_{k+1}^b \subset V_k \quad \text{if } k \leq -1.$$  

Let $T$ be a positive real number, $k$ be an integer such that $k \leq 1$ and assume that $\Sigma$ is of class $C^{4(k+1),1}$. The operator $L^V_{k+1}$ maps the space $G_{k+1} (T)$ (defined in 4.10) into the space:

$$(4.34) \quad V_b^b (T) = \left\{ H^2(0, T; V_{k+1}^b) \cap C([0, T]; V_b^b) \right\} \cap L^2(0, T; V_{k+1}^b) \quad \text{if } k = -1, 0, 1,$$

$$L^2(0, T; V_{k+1}^b) \quad \text{if } k = -2.$$  

As a direct consequence of Lemmas 4.8 we are allowed to claim:
Lemma 4.20. Let $k$ be an integer such that $k \leq 2$ and assume that $\Sigma$ is of class $C^{2(k+1),1}$. Then the lifting operator for the vorticity:

$$L_k^V : G_{k+1}^n \times G_{k+1}^T \times \mathbb{R}^N \rightarrow V_k^b,$$

is well defined and is bounded. Moreover if $k$ and $k'$ are two integers such that $k' \leq k \leq 2$, then $L_k^V = L_k^{V'}$ in $G_{k+1}^n \times G_{k+1}^T \times \mathbb{R}^N$. It follows that for every positive real number $T$ and every $k \leq 1$, providing that $\Sigma$ is of class $C^{2(k+1),1}$, the operator:

$$L_{k+1}^V : G_{k+1}(T) \rightarrow V_k^b(T),$$

is well defined and bounded as well, the bound being uniform with respect to $T$.

This lemma makes precise the expression of the vorticity corresponding to any prescribed boundary Dirichlet conditions for the velocity field on $\Sigma$.

Definition 4.19 and Lemma 4.20 justify the lengthy construction of the the spaces $\hat{V}_k$ ($k = 0, 1, 2$) carried out in Subsection 4.2. As already mentioned, the naive approach consisting in taking the Laplacian of a lifting stream function does not result in the correct result, first because the correct vorticity (in both cases of Fig. 6 and Fig. 7) belongs actually to dual spaces, the expressions of which requires the construction of the spaces $V_k$ and $\hat{V}_k$ and second because the circulation would vanish at the vorticity level.

5. Evolution Stokes problem in nonprimitive variables

The evolution Stokes problem, stated in the original primitive variables $(u, p)$, reads:

\begin{align}
(5.1a) & \quad \partial_t u - \nu \Delta u + \nabla \left( \frac{p}{\rho} \right) = f \quad \text{in } \mathcal{F}_T \\
(5.1b) & \quad \nabla \cdot u = 0 \quad \text{in } \mathcal{F}_T \\
(5.1c) & \quad u = b \quad \text{on } \Sigma_T \\
(5.1d) & \quad u(0) = u^i \quad \text{in } \mathcal{F},
\end{align}

where the source term $f$, the boundary data $b$ and the initial data $u^i$ are prescribed. We recall that the constant $\nu > 0$ is the kinematic viscosity of the fluid.

5.1. Homogeneous boundary conditions. We have at our disposal all the material allowing to deal with the evolution Stokes problem in terms of both the vorticity field and the stream function.

Definition 5.1. in terms of the stream function, the evolution Stokes problem (called $\psi-$Stokes problem) can be stated as follows: Let $k$ be any integer, $T$ be a positive real number, $\psi^i$ be in $S_k$ and $f_S$ be an element of $L^2(0, T; S_{k-1})$. The Cauchy problem for the stream function with homogeneous boundary conditions, at regularity level $k$, reads:

\begin{align}
(5.2a) & \quad \partial_t \psi + \nu A_{k+1}^S \psi = f_S \quad \text{in } \mathcal{F}_T, \\
(5.2b) & \quad \psi(0) = \psi^i \quad \text{in } \mathcal{F}.
\end{align}

Problem (5.2) can be rephrased in terms of the vorticity field: Let $k$ be any integer, $T$ be a positive real number, $\omega^i$ be in $V_k$ and $f_V$ be an element of $L^2(0, T; V_{k-1})$. The Cauchy problem for the vorticity field, called $\omega-$Stokes problem, at regularity level $k$ reads:

\begin{align}
(5.3a) & \quad \partial_t \omega + \nu A_{k+1}^V \omega = f_V \quad \text{in } \mathcal{F}_T, \\
(5.3b) & \quad \omega(0) = \omega^i \quad \text{in } \mathcal{F}.
\end{align}

For every integer $k$, we introduce the function spaces:

\begin{align}
(5.4a) & \quad S_k(T) = H^1(0, T; S_{k-1}) \cap C([0, T]; S_k) \cap L^2(0, T; S_{k+1}), \\
(5.4b) & \quad V_k(T) = H^1(0, T; V_{k-1}) \cap C([0, T]; V_k) \cap L^2(0, T; V_{k+1}).
\end{align}

Invoking for instance [39] Theorem 4.1 or simply Proposition [A.10] (we felt somewhat uncomfortable with quoting general results on semigroups in Banach spaces in such a simple case for which everything can be shown “by hand”; see the short subsection [A.3], we claim:
Proposition 5.2. For every integer \( k \), every \( T > 0 \), every \( \psi \in S_k \) and every \( f \in L^2(0, T; S_{k-1}) \), there exists a unique solution \( \psi \) to problem (5.2) in the space \( S_k(T) \). Moreover, there exists a real positive constant \( c_{\psi} \) (depending on \( \psi \) but uniform in \( F \), \( k \) and \( T \)) such that:

\[
\|\psi\|_{S_k(T)} \leq c_{\psi} \left( \|\psi\|_{S_k} + \|f\|_{L^2(0, T; S_{k-1})} \right). \tag{5.5a}
\]

For every integer \( k \), every \( T > 0 \), every \( \omega \in V_k \) and every \( f \in L^2(0, T; V_{k-1}) \), there exists a unique solution \( \omega \) to problem (5.3) in the space \( V_k(T) \). Moreover, the following estimate holds with the same constant \( c_{\omega} \) as in (5.5a):

\[
\|\omega\|_{V_k(T)} \leq c_{\omega} \left( \|\omega\|_{V_k} + \|f\|_{L^2(0, T; V_{k-1})} \right). \tag{5.5b}
\]

The solutions \( \psi \) and \( \omega \) to problems (5.2) and (5.3) respectively, satisfy the following exponential decay estimates:

Lemma 5.3. Let \( \psi \) be a solution to the Cauchy problem (5.2) in the space \( S_k(T) \) for some integer \( k \), some source term \( f \in L^2(0, T; S_{k-1}) \) and some initial condition \( \psi^0 \in S_k \). Then, the following estimate holds:

\[
\|\psi(t)\|_{S_k} \leq e^{-[\nu(1-\varepsilon)\lambda_T]t} \left[ \|\psi^0\|_{S_k}^2 + \frac{1}{2\nu\varepsilon} \|f\|_{L^2(0, T; S_{k-1})}^2 \right] \frac{1}{2} \quad \text{for all } t \in [0, T] \text{ and } \varepsilon \in (0, 1), \tag{5.6a}
\]

where \( \lambda_T > 0 \) is the constant defined in Corollary 5.1.

If \( f = 0 \), we can choose \( \varepsilon = 0 \) in (5.6a).

Let \( \omega \) be a solution to the Cauchy problem (5.3) in the space \( V_k(T) \) for some integer \( k \), some source term \( f \in L^2(0, T; V_{k-1}) \) and some initial condition \( \omega^0 \in V_k \). Then, the following estimate holds:

\[
\|\omega(t)\|_{V_k} \leq e^{-[\nu(1-\varepsilon)\lambda_T]t} \left[ \|\omega^0\|_{V_k}^2 + \frac{1}{2\nu\varepsilon} \|f\|_{L^2(0, T; V_{k-1})}^2 \right] \frac{1}{2} \quad \text{for all } t \in [0, T] \text{ and } \varepsilon \in (0, 1). \tag{5.6b}
\]

If \( f = 0 \), we can choose \( \varepsilon = 0 \) in (5.6b).

We can easily connect problems (5.2) and (5.3) by means of either the operators \( P_k \) and \( Q_k \) or with the operator \( \Delta_k \). The proof is straightforward, resting on the commutative diagrams of Fig. 3 and Fig. 4.

Theorem 5.4. Let \( k \) and \( k' \) be two integers and \( T > 0 \) a positive real number. Let \( \psi \) be the solution in \( S_k(T) \) to Problem (5.2) with source term \( f \in L^2(0, T; S_{k-1}) \) and initial condition \( \psi^0 \in S_k \). Let \( \omega \) be the solution in \( V_k(T) \) to Problem (5.3) with source term \( f \in L^2(0, T; V_{k-1}) \) and initial condition \( \omega^0 \in V_k \).

If \( k' = k + 1 \), then the following assertions are equivalent:

1. \( \omega^0 = P_{k+1} \psi \) and for a.e. \( t \in (0, T) \), \( f(t) = P_k f_S(t) \).
2. For a.e. \( t \in (0, T) \), \( \omega(t) = P_{k+2} \psi(t) \).

If \( k' = k - 1 \) then the following assertions are equivalent:

1. \( \omega^0 = \Delta_{k-1} \psi \) and for a.e. \( t \in (0, T) \), \( f(t) = \Delta_{k-2} f_S(t) \).
2. For a.e. \( t \in (0, T) \), \( \omega(t) = \Delta_k \psi(t) \).

In addition to the data already introduced, let

\[
u \in H^1(0, T; J_{k-1}) \cap C([0, T]; J_k) \cap L^2(0, T; J_{k+1})
\]

be the solution to Problem (2.18) for some \( f_1 \in L^2(0, T; J_{k-1}) \) and \( u^1 \) in \( J_k \). The following assertions are equivalent:

1. \( u^1 = \nabla^1_k \psi \) and for a.e. \( t \in (0, T) \), \( f(t) = \nabla_{k-1}^1 f_S(t) \).
2. For a.e. \( t \in (0, T) \), \( u(t) = \nabla_{k+1}^1 \psi(t) \).

5.2. Nonhomogeneous boundary conditions. Following the definition of \( S_k(T) \) and \( V_k(T) \), we introduce for every real positive number \( T \) and every integer \( k \leq 2 \):

\[
S_k(T) = H^1(0, T; S_{k-1}) \cap C([0, T]; S_k) \cap L^2(0, T; S_{k+1}), \tag{5.7a}
\]

where we recall that the spaces \( S_k \) are defined in (4.12). The counterpart stated in terms of the vorticity is, for every integer \( k \leq 1 \), the space (see Fig. 6 and Fig. 7):

\[
V_k(T) = H^1(0, T; V_{k-1}) \cap C([0, T]; V_k) \cap L^2(0, T; V_{k+1}), \tag{5.7b}
\]
where the spaces $V_k$ are defined in (4.13).

**Definition 5.5.** Let a positive real number $T$, an integer $k \leq 1$, a source term $f_S \in L^2(0,T;S_{k-1})$, an initial data $\psi^0 \in \bar{S}_k$ and a triple $(g_n,g_r,\Gamma) \in G_k(T)$ be given. Define $\psi^0_k = \psi^0 - L^S_{k+1}(g_n(0),g_r(0),\Gamma(0))$ when $k = 0,1$ and $\psi^0_1 = \psi^0$ when $k \leq -1$. Finally, assume that $\Sigma$ is of class $C^{j_2(k),1}$ and that the following compatibility condition holds:

\[
\psi^0 \in S_k \quad \text{if } k = 1.
\]

We say that a function $\psi \in \bar{S}_k(T)$ is solution of the evolution $\psi-$Stokes problem satisfying the Dirichlet boundary conditions on $\Sigma_T$ as described in (4.1) by the triple $(g_n,g_r,\Gamma)$ if:

1. When $k = 0$ or $k = 1$: There exists $\psi_0 \in S_k(T)$ solution to the homogeneous $\psi-$Stokes Cauchy problem

\[
\begin{align}
\partial_t \psi_0 + \nu A^S_{k+1} \psi_0 &= -\partial_t L^S_{k+1}(g_n,g_r,\Gamma) + f_S \quad &\text{in } F_T, \\
\psi_0(0) &= \psi^0 \quad &\text{in } F,
\end{align}
\]

such that $\psi = \psi_0 + L^S_{k+1}(g_n,g_r,\Gamma)$.

2. When $k \leq -1$: The function $\psi$ is the solution to the Cauchy problem

\[
\begin{align}
\partial_t \psi + \nu A^S_{k+1} \psi &= \nu A^S_{k+1} L^S_{k+1}(g_n,g_r,\Gamma) + f_S \quad &\text{in } F_T, \\
\psi(0) &= \psi^0 \quad &\text{in } F.
\end{align}
\]

The case $k = 2$ is more involved and will be treated in Section 8. The difference of definition depending on the level of regularity $k$ is worth some additional explanation. Before that, combining Proposition 5.2 and Lemma 4.20 we are allowed to claim:

**Proposition 5.6.** Every $\psi-$Stokes problem as stated in Definition 5.5 admits a unique solution. Moreover, there exists a positive constant $c_{[k,F,N]}$ uniform in $T$ such that the solution $\psi \in \bar{S}_k(T)$ satisfies the estimate:

\[
\|\psi\|_{S_k(T)} \leq c_{[k,F,N]} \left[\|\psi^0\|_{\bar{S}_k} + \|f_S\|_{L^2(0,T;S_{k-1})} + \|(g_n,g_r,\Gamma)\|_{G_k(T)}^2\right]^{\frac{1}{2}}.
\]

The consistency of Definition 5.5 is asserted by the following results:

**Proposition 5.7.** Let a positive real number $T$, an integer $k$, a source term $f_S \in L^2(0,T;S_{k-1})$, an initial data $\psi^0 \in \bar{S}_k$ and a triple $(g_n,g_r,\Gamma) \in G_k(T)$ be given as in Definition 5.5. Denote by $\psi^k$ the solution whose existence and uniqueness in the space $\bar{S}_k(T)$ are asserted in Proposition 5.6.

Let $k'$ be any integer lower than $k$, and all other data remaining equal, denote by $\psi^{k'}$ the corresponding solution in $\bar{S}_{k'}(T)$. Then $\psi^k = \psi^{k'}$.

**Proof.** The proposition is obvious when $k$ and $k'$ are both nonnegative or when $k$ and $k'$ are both negative, so let us focus on the case $k = 0$ and $k' = -1$ and compare the solutions $\psi^0$ and $\psi^{-1}$.

By definition, the function $\psi^0$ solves the Cauchy problem:

\[
\begin{align}
\partial_t (\psi^0 - L^S_1(g_n,g_r,\Gamma)) + \nu A^S_1 (\psi^0 - L^S_1(g_n,g_r,\Gamma)) &= -\partial_t L^S_1(g_n,g_r,\Gamma) + f_S \quad &\text{in } F_T, \\
\psi^0(0) &= \psi^0 \quad &\text{in } F.
\end{align}
\]

Since the operator $A^S_0$ extends the operator $A^S_1$ to $S_0$, the function $\psi^0$ belongs to $L^2(0,T;\bar{S}_1) \subset L^2(0,T;S_0)$ and the lifting operator $L^S_1$ is valued in $S^1_0$, which is a subspace of $S_0$, we are allowed to write that:

\[
A^S_1 (\psi^0 - L^S_1(g_n,g_r,\Gamma)) = A^S_0 \psi^0 - A^S_0 L^S_1(g_n,g_r,\Gamma).
\]

It follows that $\psi^0$ solves as well the Cauchy problem:

\[
\begin{align}
\partial_t \psi^0 + \nu A^S_0 \psi^0 &= \nu A^S_0 L^S_1(g_n,g_r,\Gamma) + f_S \quad &\text{in } F_T, \\
\psi^0(0) &= \psi^0 \quad &\text{in } F,
\end{align}
\]

a solution of which is $\psi^{-1}$. The proof is now completed.
The definition of weak solutions (i.e. for negative integers \( k \)) with nonhomogeneous boundary conditions given in Definition 5.5 can be rephrased by means of the duality method (or transposition method; see Definition 5.8 and references therein).

**Proposition 5.8.** Let data be given as in Definition 5.5 and assume that \( k \) is a negative integer and \( f_S = 0 \). Denote by \( \psi \) the unique solution to the corresponding Cauchy nonhomogeneous \( \psi \)–Stokes problem (5.10). Then for every \( \vartheta \in L^2(0, T; S_{-k-1}) \) and \( \theta \in S_{-k}(T) \) solution to the backward Cauchy problem:

\[
\begin{align*}
-\partial_t \theta + \nu A^S_{-k+1} \theta &= \vartheta & \text{in } F_T, \\
\theta(T) &= 0 & \text{in } \mathcal{F},
\end{align*}
\]

the following identity holds:

\[
\int_0^T \langle \psi, \vartheta \rangle_{S_{k+1}, S_{-k-1}} dt = \int_0^T \langle \psi^i, \theta(0) \rangle_{S_{k+1}, S_{-k-1}} - \nu \int_0^T \langle \Delta_{-k} \theta \rangle_{S_{k+1}, S_{-k-1}} dt,
\]

where (see identities (4.1)):

\[
b_r = T_{k+1} g_n - g_r + \sum_{j=1}^N \Gamma_j \frac{\partial \xi_j}{\partial n} |_{\Sigma}.
\]

**Proof.** Equation (5.10a) holds in \( L^2(0, T; S_{-k-1}) \), which is the dual space of \( L^2(0, T; S_{k+1}) \). Forming the duality pairing of (5.10a) with \( \theta \) yields:

\[
\int_0^T \langle \partial_t \psi, \vartheta \rangle_{S_{k+1}, S_{-k-1}} dt + \nu \int_0^T \langle A^S_{k+1} \psi, \theta \rangle_{S_{k+1}, S_{-k-1}} dt = \nu \int_0^T \langle A^S_{k+1} L^S_{k+1} (g_n, g_r, \Gamma), \theta \rangle_{S_{k+1}, S_{-k-1}} dt.
\]

Integrating by parts and using the definition of the operator \( A^S_{k+1} \), we obtain for the term in the left hand side:

\[
\int_0^T \langle \partial_t \psi, \vartheta \rangle_{S_{k+1}, S_{-k-1}} dt + \nu \int_0^T \langle A^S_{k+1} \psi, \theta \rangle_{S_{k+1}, S_{-k-1}} dt = -\langle \psi^i, \theta(0) \rangle_{S_{k+1}, S_{-k-1}} + \int_0^T \langle \psi, -\partial_t \theta + A^S_{-k+1} \theta \rangle_{S_{k+1}, S_{-k-1}} dt.
\]

The right hand side term is dealt with as follows:

\[
\int_0^T \langle A^S_{k+1} L^S_{k+1} (g_n, g_r, \Gamma), \theta \rangle_{S_{k+1}, S_{-k-1}} dt = \int_0^T \langle L^S_{k+1} (g_n, g_r, \Gamma), A^S_{-k+1} \theta \rangle_{S_{k+1}, S_{-k-1}} dt,
\]

where, by definition (see (4.8)):

\[
\int_0^T \langle L^S_{k+1} (g_n, g_r, \Gamma), A^S_{-k+1} \theta \rangle_{S_{k+1}, S_{-k-1}} dt
\]

\[
= \int_0^T \langle L^S_{k+1} (T_{k+1} g_n - g_r) + \sum_{j=1}^N \Gamma_j \xi_j A^S_{-k+1} \theta \rangle_{S_{k+1}, S_{-k-1}} dt.
\]

Since the index \( k \) is negative, \( A^S_{-k+1} \theta \) belongs to \( S_0 \). It follows that for every \( j \in \{1, \ldots, N\} \):

\[
\langle \xi_j, A^S_{-k+1} \theta \rangle_{S_{k+1}, S_{-k-1}} = -\langle \nabla \xi_j, \nabla \Delta_{-k} \theta \rangle_{L^2(\mathcal{F})} = -\int_{\Sigma} \frac{\partial \xi_j}{\partial n} \Delta_{-k} \theta ds.
\]

Recalling the definition (4.7) of the operator \( L^S_{k+1} \) and the factorization (3.30) of \( \Delta_k \) and then gathering (5.13), (5.14) and (5.15), we obtain indeed (5.12) and complete the proof.

Definition 5.5 and Propositions 5.6, 5.7 and 5.8 can be restated in terms of the vorticity field.
Definition 5.9. Let a positive real number $T$, a nonnegative integer $k$, a source term $f_V \in L^2(0,T;V_{k-1})$, an initial data $\omega^i \in \bar{V}_k$ and a triple $(g_n,g_r,\Gamma) \in G_{k+1}(T)$ be given. Define $\omega^\beta_0 = \omega^i - L^V_{k+1}(g_n(0),g_r(0),\Gamma(0))$ when $k = -1,0$ and $\omega^\beta_0 = \omega^i$ when $k \leq -2$. Finally, assume that $\Sigma$ is of class $C^{(k+1),1}$ and that the following compatibility condition holds:

\begin{equation}
(5.16) \quad \omega^i_0 \in V_k \quad \text{if } k = 0.
\end{equation}

We say that a function $\omega \in \bar{V}_k(T)$ is solution of the evolution $\omega-$Stokes problem satisfying the Dirichlet boundary conditions on $\Sigma^T$ as described in (4.1) by the triple $(g_n,g_r,\Gamma)$ if:

1. When $k = -1$ or $k = 0$: There exists $\omega_0 \in \bar{V}_k(T)$ solution to the homogeneous $\omega-$Stokes Cauchy problem

\begin{align}
(5.17a) & \quad \partial_t \omega_0 + \nu A^V_{k+1} \omega_0 = -\partial_t L^V_{k+1}(g_n,g_r,\Gamma) + f_V \quad \text{in } \mathcal{F}_T, \\
(5.17b) & \quad \omega_0(0) = \omega^i_0 \quad \text{in } \mathcal{F},
\end{align}

such that $\omega = \omega_0 + L^V_{k+1}(g_n,g_r,\Gamma)$.

2. When $k \leq -2$: The function $\omega$ is the solution to the Cauchy problem:

\begin{align}
(5.18a) & \quad \partial_t \omega + \nu A^V_{k+1} \omega = \nu A^V_{k+1} L^V_{k+1}(g_n,g_r,\Gamma) + f_V \quad \text{in } \mathcal{F}_T, \\
(5.18b) & \quad \omega(0) = \omega^i_0 \quad \text{in } \mathcal{F}.
\end{align}

Proposition 5.10. Every $\omega-$Stokes problem as stated in Definition 5.9 admits a unique solution. Moreover, there exists a positive constant $c_{[k,F,U]}$ (uniform in $T$) such that the solution $\omega \in \bar{V}_k(T)$ satisfies the estimate:

\begin{equation}
(5.19) \quad \|\omega\|_{\bar{V}_k(T)} \leq c_{[k,F,U]} \left[\|\omega^i_0\|^2_{V_k} + \|f_V\|^2_{L^2(0,T;V_{k-1})} + \|(g_n,g_r,\Gamma)\|^2_{G_{k+1}(T)}\right].
\end{equation}

Proposition 5.11. Let a positive real number $T$, an integer $k$, a source term $f_V \in L^2(0,T;V_{k-1})$, an initial data $\omega^i \in \bar{V}_k$ and a triple $(g_n,g_r,\Gamma) \in G_{k+1}(T)$ be given as in Definition 5.9. Denote by $\omega^k$ the solution whose existence and uniqueness in the space $\bar{V}_k(T)$ are asserted in Proposition 5.10. Let $k'$ be any integer lower than $k$ and, all other data remaining equal, denote by $\omega^{k'}$ the corresponding solution in $\bar{V}_{k'}(T)$. Then $\omega^k = \omega^{k'}$.

Proposition 5.12. Let data be given as in Definition 5.9 with $k \leq -2$ and $f_V = 0$. Denote by $\omega$ the unique solution to the corresponding Cauchy nonhomogeneous $\omega-$Stokes problem (5.18). Then for every $\theta \in L^2(0,T;V_{k-1})$ and $\theta \in V_{k}(T)$ solution to the backward Cauchy problem:

\begin{align}
-\partial_t \theta + \nu A^V_{k+1} \theta &= \theta \quad \text{in } \mathcal{F}_T, \\
\theta(T) &= 0 \quad \text{in } \mathcal{F},
\end{align}

the following identity holds:

\begin{equation}
(5.20) \quad \int_0^T \langle \omega, \theta \rangle_{V_k, V_{k-1}} \, dt = \langle \omega^i, \theta(0) \rangle_{V_k, V_{k-1}} - \nu \int_0^T \langle A^V_{k+1} \theta \rangle_{\Sigma}, b_r \rangle_{H^{-\frac{1}{2}}(\Sigma), H^{\frac{3}{2}}(\Sigma)} \, dt,
\end{equation}

where (see identities (4.1)):

\begin{equation}
(5.21) \quad b_r = T_{k+2} g_n - g_r + \sum_{j=1}^{N} F_j \frac{\partial \xi_j}{\partial n} \bigg| \Sigma.
\end{equation}

Proof. We form the duality pairing of (5.18a) in $L^2(0,T;V_{k-1})$ with $\theta$ in $L^2(0,T;V_{k-1})$, then the proof follows mainly the lines of the proof of Proposition 5.8. Let us focus on the right hand side term only, namely:

\begin{equation}
(5.22) \quad \int_0^T \langle A^V_{k+1} L^V_{k+1}(g_n,g_r,\Gamma), \theta \rangle_{V_{k-1}, V_{k-1}} \, ds.
\end{equation}

According to (A.7), the duality pairing can be turned into:

\begin{equation}
(5.23) \quad \langle A^V_{k+1} L^V_{k+1}(g_n,g_r,\Gamma), \theta \rangle_{V_{k-1}, V_{k-1}} = \langle L^V_{k+1}(g_n,g_r,\Gamma), A^V_{k+1} \theta \rangle_{V_{k+1}, V_{k-1}}.
\end{equation}
Then, using the definition (4.33b) of $L^v_{k+1}$ and the second formula in Lemma 3.19 we obtain:

$$\langle L^v_{k+1}(g_n, g_T, \Gamma), A^{v}_{k+1}\theta \rangle_{V_{k+1}, V_{k-1}} = \langle L^S_{k+2}(g_n, g_T, \Gamma), Q_{k-1}A^{v}_{k+1}\theta \rangle_{S_{k+2}, S_{k-2}}.$$

Resting on formula (3.27), the last term is proven to be equal to:

$$\langle L^S_{k+2}(g_n, g_T, \Gamma), A^{S}_{k}Q_{k-1}\theta \rangle_{S_{k+2}, S_{k-2}},$$

and therefore it is very much alike the left hand side in (6.14). The proof is then completed after noticing that $-\Delta_{k-1}Q_{k-1} = A^{v}_{k+1}$ (see for instance Fig. 4).

\[\square\]

6. NAVIER-STOKES EQUATIONS IN NONPRIMITIVE VARIABLES

6.1. Estimates for the nonlinear advection term. Following our rules of notation, we define for every positive integer $k$ and every positive time $T$, the time dependent space for the Kirchhoff potential:

(6.1) $\bar{\delta}^k(T) = H^1(0, T; \bar{\delta}^{k-1}_T) \subset C([0, T]; \bar{\delta}^k_T) \cap L^2(0, T; \bar{\delta}^{k+1}_T),$

where we recall that the spaces $\bar{\delta}^k_T$ were defined in (4.4). Then, we aim at establishing some (very classical) estimates for the nonlinear advection term of the Navier-Stokes equations. Denoting by $\dot{u}$ a smooth velocity field defined in $\mathcal{F}$, an integration par parts yields the equality:

$$(\nabla \dot{u}, \nabla^\perp \theta)_{L^2(\mathcal{F})} = (D^2\dot{u} \theta, u^\perp)_{L^2(\mathcal{F})} \quad \text{for all } \theta \in S_1.$$

The main estimates satisfied by the right hand side term are summarized in the lemma below:

**Lemma 6.1.** Let a stream function $\tilde{\psi}$ be in $\tilde{S}_1$ (this space being defined in (4.15)) and a Kirchhoff potential $\varphi$ be in $\bar{\delta}^k_T$ (defined in (4.4)). Then the linear form:

(6.2) $A^S_1(\tilde{\psi}, \varphi): \theta \in S_1 \rightarrow -(D^2\theta \nabla \tilde{\psi}, \nabla^\perp \varphi)_{L^2(\mathcal{F})} + (D^2\theta \nabla^\perp \varphi, \nabla^\perp \tilde{\psi})_{L^2(\mathcal{F})} - (D^2\theta \nabla \varphi, \nabla \tilde{\psi})_{L^2(\mathcal{F})} \in \mathbb{R},$

is well defined and bounded. Moreover, there exists a positive constant $c_{[\mathcal{F}, \varphi]}$ such that:

1. For every $\theta \in S_1$:

(6.3a) $\|A^S_1(\tilde{\psi} + \theta, \varphi, \theta)_{S_{-1}, S_1}\| \leq \frac{\nu}{2} \|\theta\|_{S_1}^2 + c_{[\mathcal{F}, \varphi]} \left(\|\tilde{\psi}\|_{S_1}^2 \|\theta\|_{S_0}^2 + \|\varphi\|_{\bar{\delta}^k_T}^2 \|\varphi\|_{\bar{\delta}^k_T}^2\right) \|\theta\|_{S_0}^2$

2. For every pair $(\theta_1, \theta_2) \in S_1 \times S_1$:

(6.3b) $\|A^S_1(\tilde{\psi} + \theta_2, \varphi) - A^S_1(\tilde{\psi} + \theta_1, \varphi, \theta)_{S_{-1}, S_1}\| \leq \frac{\nu}{2} \|\theta_1\|_{S_1}^2$

where $\theta = \theta_2 - \theta_1$.

If for some positive real number $T$, $\tilde{\psi}$ belongs to $C([0, T]; S_0) \cap L^2(0, T; \tilde{S}_1)$ and $\varphi$ to $C([0, T]; \bar{\delta}^k_T) \cap L^2(0, T; \bar{\delta}^{k+1}_T)$ then $A^S_1(\tilde{\psi}, \varphi)$ is in $L^2(0, T; S_1)$ and there exists a positive constant $c_{[\mathcal{F}]}$ such that:

(6.3c) $\|A^S_1(\tilde{\psi}, \varphi)\|_{L^2(0, T; S_1)} \leq c_{[\mathcal{F}]} \left(\|\tilde{\psi}\|_{C([0, T]; S_0)} \|\varphi\|_{L^2(0, T; \bar{\delta}^k_T)} + \|\varphi\|_{C([0, T]; \bar{\delta}^{k+1}_T)} \|\varphi\|_{L^2(0, T; \bar{\delta}^{k+1}_T)}\right).$

**Proof.** Consider the first term in the right hand side of (6.2), Hölder’s inequality yields:

(6.4a) $|\langle D^2\theta \nabla \tilde{\psi}, \nabla^\perp \varphi \rangle_{L^2(\mathcal{F})}| \leq \|\nabla \tilde{\psi}\|_{L^2(\mathcal{F})} \|\theta\|_{S_1} \quad \text{for all } \theta \in S_1.$

Then Sobolev embedding Theorem followed by an interpolation inequality between the spaces $L^2(\mathcal{F})$ and $H^1(\mathcal{F})$ leads to:

$$\|u\|_{L^2(\mathcal{F})} \leq c_{[\mathcal{F}]} \|u\|_{H^2(\mathcal{F})} \leq c_{[\mathcal{F}]} \|u\|_{L^2(\mathcal{F})} \|u\|_{H^1(\mathcal{F})}^{\frac{1}{2}} \quad \text{for all } u \in H^1(\mathcal{F}).$$

Combining this inequality with Lemma 4.9 (equivalence of the norms in $\tilde{S}_1$ and $H^2(\mathcal{F})$), we obtain first:

$$\|\nabla \tilde{\psi}\|_{L^2(\mathcal{F})} \leq c_{[\mathcal{F}]} \|\tilde{\psi}\|_{S_0}^{\frac{1}{2}} \|\tilde{\psi}\|_{\tilde{S}_1}^{\frac{1}{2}}.$$
Finally, for every pair \( (\tilde{\psi}, \varphi) \),\( \theta \in S_1 \).

Based on the same arguments, it is then straightforward to prove the existence of a positive constant \( c_F \) such that:

\[
|\langle A^S_1(\tilde{\psi}, \varphi), \theta \rangle_{S_1, S_1}| \leq c_F \left( \|\tilde{\psi}\|_{S_1} \|\varphi\|_{S_1} + \|\bar{\varphi}\|_{S_1} \|\bar{\varphi}\|_{S_1} \right) \|\theta\|_{S_1} \quad \text{for all } \theta \in S_1.
\]

This shows that the linear form \( A^S_1(\tilde{\psi}, \varphi) \) is indeed bounded and satisfies:

\[
\|A^S_1(\tilde{\psi}, \varphi)\|_{S_1} \leq c_F \left( \|\tilde{\psi}\|_{S_1} \|\varphi\|_{S_1} + \|\bar{\varphi}\|_{S_1} \|\bar{\varphi}\|_{S_1} \right).
\]

Let us move on to the estimate (6.3a). Some of the terms vanishing after an integration by parts, we end up with the following equality:

\[
\langle A^S_1(\tilde{\psi} + \theta), \varphi \rangle_{S_1} = -\langle D^2\theta \nabla \tilde{\psi}, \nabla \tilde{\psi} \rangle_{L^2} + (D^2\theta \nabla \tilde{\psi}, \nabla \tilde{\psi})_{L^2} - (D^2\theta \nabla \varphi, \nabla \tilde{\psi})_{L^2}.
\]

Addressing the first term in the right hand side, let us start over from the inequality (6.4b) to which we apply Young’s inequality:

\[
|\langle D^2\theta \nabla \tilde{\psi}, \nabla \tilde{\psi} \rangle_{L^2}| \leq \frac{\nu}{8} \|\theta\|^2_{S_1} + c_F \|\tilde{\psi}\|^2_{S_0} \|\tilde{\psi}\|^2_{S_0}.
\]

The four remaining terms in the right hand side of (6.6) can be handled the same way and summing the resulting estimates yields (6.3a).

With the notation of the occurrence of Lemma 2, some elementary algebra leads to:

\[
\langle A^S_1(\tilde{\psi} + \theta_2), \varphi \rangle = -\langle D^2\theta \nabla (\tilde{\psi} + \theta_1), \nabla \psi \rangle_{L^2} + (D^2\theta \nabla \varphi, \nabla \tilde{\psi})_{L^2},
\]

and proceeding as for (6.3a) we quickly obtain (6.3b).

Finally (6.3c) derives straightforwardly from (6.5) and the proof is completed.

In case the stream function \( \bar{\psi} \) is more regular, the nonlinear term satisfies better estimates:

\[
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\]

Lemma 6.2. Let a stream function \( \bar{\psi} \) be in \( \bar{S}_2 \) (see (4.30) for a definition) and a Kirchhoff potential \( \varphi \) be in \( \bar{S}_K^2 \) (see (4.4)). Then the linear form \( A^S_1(\tilde{\psi}, \varphi) \) defined in (6.2) extends to a continuous linear form in \( S_0 \) whose expression is:

\[
A^S_0(\tilde{\psi}, \varphi) : \theta \in S_0 \mapsto (\Delta \tilde{\psi}(\nabla \varphi + \nabla \tilde{\psi}), \nabla \theta)_{L^2} \in \mathbb{R}.
\]

Moreover, there exists a positive constant \( c_F \) such that:

\[
\|A^S_0(\tilde{\psi}, \varphi)\|_{L^2} \leq c_F \left( \|\tilde{\psi}\|^2_{S_0} + \|\varphi\|^2_{S_0} \right)^{1/2} \|\theta\|^2_{S_0}.
\]

Let \( T \) be a positive real number and let \( \tilde{\psi} \) be in \( S_1(T) \), \( \varphi \) be in \( S_K^2(T) \) and \( \theta \) in \( S_1(T) \) (these spaces being defined respectively in (5.7a), (5.1) and (5.4a)). Then \( A^S_0(\tilde{\psi} + \phi, \varphi) \) belongs to \( L^2(0, T; S_0) \) and:

\[
\|A^S_0(\tilde{\psi} + \phi, \varphi)\|_{L^2} \leq c_F T^{1/2} \left( \|\tilde{\psi}\|^2_{S_1(T)} + \|\varphi\|^2_{S_K^2(T)} + \|\theta\|^2_{S_1(T)} \right).
\]

Finally, for every pair \( (\phi, \theta) \in S_1(T) \times S_1(T) \):

\[
\|A^S_0(\tilde{\psi} + \phi, \varphi)\|_{L^2} \leq c_F T^{1/2} \left( \|\tilde{\psi}\|^2_{S_1(T)} + \|\phi\|^2_{S_1(T)} + \|\theta\|^2_{S_1(T)} \right)^{1/2} \|\phi\|_{S_1(T)}.
\]

Proof. Assume that \( \theta \) belongs to \( S_1 \). Then, integrating by parts, we obtain:

\[
(\Delta \tilde{\psi}(\nabla \psi + \nabla \varphi), \nabla \theta)_{L^2} = -\langle D^2\varphi \nabla \tilde{\psi}, \nabla \theta \rangle_{L^2} - (D^2\varphi \nabla \tilde{\psi}, \nabla \tilde{\psi})_{L^2} - (D^2\varphi \nabla \varphi, \nabla \theta)_{L^2}.
\]

Integrating by parts again the first term in the right hand side, it comes:

\[
(D^2\varphi \nabla \tilde{\psi}, \nabla \theta)_{L^2} = -(\Delta \varphi \psi, \nabla \theta)_{L^2} - (D^2\varphi \nabla \varphi, \nabla \theta)_{L^2}.
\]
While the last term can be rewritten as follows:

\[(6.10b) \quad (D^2 \tilde{\psi} \nabla \varphi, \nabla \varphi)_{L^2(\mathbb{F})} = (\nabla (\nabla \theta \cdot \nabla \tilde{\psi}), \nabla \varphi)_{L^2(\mathbb{F})} - (D^2 \theta \nabla \tilde{\psi}, \nabla \varphi)_{L^2(\mathbb{F})},\]

and the first term in the right hand side vanishes. Gathering (6.10a) and (6.10b) yields:

\[(6.11) \quad (D^2 \varphi \nabla \tilde{\psi}, \nabla \varphi)_{L^2(\mathbb{F})} = -((\Delta \theta \nabla \varphi, \nabla \tilde{\psi})_{L^2(\mathbb{F})} + (D^2 \theta \nabla \tilde{\psi}, \nabla \varphi)_{L^2(\mathbb{F})} = -((D^2 \theta \nabla \tilde{\psi}, \nabla \varphi)_{L^2(\mathbb{F})}.

Replacing this expression in (6.9), we recover indeed the definition (6.2) of $A^1_T(\tilde{\psi}, \varphi)$.

Applying Hölder’s inequality yields:

\[\|\Delta \tilde{\psi}(\nabla \tilde{\psi} + \nabla \varphi)\|_{L^2(\mathbb{F})}^2 \leq c_F \|\Delta \tilde{\psi}\|_{L^2(\mathbb{F})}^{\frac{2}{3}} \|\Delta \tilde{\psi}\|_{L^2(\mathbb{F})}^{\frac{2}{3}} (\|\nabla \tilde{\psi}\|_{L^2(\mathbb{F})} + \|\nabla \varphi\|_{L^2(\mathbb{F})})^2,
\]

and then, Sobolev embedding theorem leads straightforwardly to (6.8).

From (6.8a), we deduce that:

\[\|A^S_0(\tilde{\psi} + \theta, \varphi)\|_{S_0} \leq c_F \left(\|\tilde{\psi}\|_{S_1}^2 + \|\varphi\|_{S_1}^2 + \|\theta\|_{S_1}^2\right) \|\tilde{\psi}\|_{S_2}^2 + \|\theta\|_{S_2}^2,
\]

and therefore, in particular:

\[\|A^S_0(\tilde{\psi} + \theta, \varphi)\|_{L^2(0,T;S_0)} \leq c_F \left(\|\tilde{\psi}\|_{S_1(0,T)}^2 + \|\varphi\|_{S_1(0,T)}^2 + \|\theta\|_{S_1(0,T)}^2\right).
\]

The estimate (6.8c) follows with Hölder’s inequality. The last inequality (6.8c) is proved the same way. □

**Remark 6.3.**

1. Denoting $u = \nabla \tilde{\psi} + \nabla \varphi$ in the statement of the lemma, it is classical to verify that:

\[(6.12) \quad (A^S_0(\tilde{\psi}, \varphi), \theta)_{S_0} = (\nabla uu, \nabla \theta)_{L^2(\mathbb{F})} \quad \text{for all} \theta \in S_0.
\]

2. In this lemma, the assumption $\varphi \in S^2_K(T)$ is too strong. Indeed, the Kirchhoff potential is not required to belong to $L^2(0,T;S^2_K)$ and one can verify that $\varphi \in C([0,T];S^2_K)$ would be sufficient. However, the hypothesis $\varphi \in S^2_K(T)$ will be necessary later on.

### 6.2. Weak solutions.

**Definition 6.4.** Let a positive real number $T$, a source term $f_S \in L^2(0,T;S_{-1})$, an initial data $\psi^i \in S_0$ and a triple $(g_n, g_r, \Gamma) \in G_0(T)$ be given. Define $\psi^i_0 = \psi^i - L^0_0(g_n(0), g_r(0), \Gamma(0))$ and assume that $\Sigma$ is of class $C^{3,1}$.

We say that a stream function $\psi \in S_0(T)$ is a weak (or Leray) solution to the $\psi$–Navier-Stokes equations satisfying the Dirichlet boundary conditions on $\Sigma_T$ as described in (4.11) by the triple $(g_n, g_r, \Gamma)$ if $\psi = \psi_B + \psi_L + \psi_A$ where:

1. The function $\psi_B$ accounts for the boundary conditions. It belongs to $S^2_0(T)$ defined in (4.11) and is equal to $L^0_0(g_n, g_r, \Gamma)$;
2. The function $\psi_L$ accounts for the source term and the initial condition. It is defined as the unique solution in $S_0(T)$ of the homogeneous (linear) $\psi$–Stokes Cauchy problem

\[
\begin{align*}
(6.13a) & \quad \partial_t \psi_L + \nu A^S_1 \psi_L = -\partial_t \psi_B + f_S \quad \text{in} \mathcal{F}_T, \\
(6.13b) & \quad \psi_L(0) = \psi^i_0 \quad \text{in} \mathcal{F},
\end{align*}
\]

3. The function $\psi_A$ accounts for the nonlinear advection term. It belongs to the space $S_0(T)$ and solves the nonlinear Cauchy problem:

\[
\begin{align*}
(6.14a) & \quad \partial_t \psi_A + \nu A^S_1 \psi_A = -A^S_1(\psi_B + \psi_L + \psi_A, \varphi) \quad \text{in} \mathcal{F}_T, \\
(6.14b) & \quad \psi_A(0) = 0 \quad \text{in} \mathcal{F},
\end{align*}
\]

where $\varphi = L^0_1 g_n$ is the Kirchhoff potential that belongs to $S^1_K(T)$.

**Theorem 6.5.** For any set of data as described in Definition 6.4, there exists a unique (weak) solution in $S_0(T)$ to the $\psi$–Navier-Stokes equations.
Proof. The existence and uniqueness of \( \psi_{\ell} \) is asserted by Proposition 5.6 and Lemma 6.1, being granted, the proof of existence and uniqueness of the function \( \psi_A \) is quite similar to the proof [68, Chap. 1, Section 6]. Let us focus on the main differences and omit some details.

Denote by \( \tilde{\psi} \) the function in \( S_0(T) \) equal to the sum \( \psi_b + \psi_{\ell} \) (where the functions \( \psi_b \) and \( \psi_{\ell} \) are given) and notice that, according to Proposition 5.6 and Lemma 4.2,

\[
\begin{align*}
(6.15a) & \quad \| \tilde{\psi} \|_{S_0(T)} \leq C_{[F, \nu]} \left[ \| \psi_b \|_{S_0}^2 + \| f_s \|_{L^2(0, T; S_{-1})}^2 + \langle (g_n, g_{\tau}, \Gamma) \rangle_{G_0(T)}^2 \right]^{\frac{1}{2}} \\
(6.15b) & \quad \| \varphi \|_{\tilde{S}_k(T)} \leq C_F \| g_n \|_{G_0(T)} \leq C_F \| (g_n, g_{\tau}, \Gamma) \|_{G_0(T)}.
\end{align*}
\]

Then, for every positive integer \( m \), introduce \( \mathbb{S}_1^m \), the finite dimensional subspace of \( S_1 \) spanned by the \( m \)-th first eigenvalues of \( \mathcal{A}_1^S \) (loosely speaking, this operator is equal to \( \mathcal{A}_1^S \) seen as an unbounded operator in \( S_{-1} \) of domain \( S_1 \); see (A.1)). Denote by \( \Pi_m \) the orthogonal projector from \( S_1 \) onto \( \mathbb{S}_1^m \) and by \( \Pi_m^* \) its adjoint for the duality pairing \( S_{-1} \times S_1 \). Finally, let \( \psi_A^m \) be the unique solution in \( \mathbb{S}_1^m \) of the Cauchy problem:

\[
\begin{align*}
(6.16a) & \quad \partial_t \psi_A^m + \nu \mathcal{A}_1^S \psi_A^m = -\Pi_m^* \mathcal{A}_1^S \tilde{\psi} + \psi_A^m \varphi \quad \text{in } F_T, \\
(6.16b) & \quad \psi_A^m (0) = 0 \quad \text{in } F.
\end{align*}
\]

The existence and uniqueness of \( \psi_A^m \in C^1 \left( [0, T_m]; S_1 \right) \) on a time interval \( (0, T_m) \) is guaranteed by Cauchy-Lipschitz Theorem. Forming now for any \( s \in (0, T_m) \) the duality pairing of equation (6.14a) set in \( S_{-1} \) with \( \psi_A^m (s) \) in \( S_1 \) and using the estimate [6.34] for the nonlinear term, we obtain:

\[
(6.17) \quad \frac{d}{dt} \| \psi_A^m (s) \|_{S_0}^2 + \nu \| \psi_A^m (s) \|_{S_1}^2 \leq \Phi(s) - \frac{1}{2} \| \varphi(s) \|_{S_0}^2 + \frac{1}{2} \| \varphi(s) \|_{S_1}^2.
\]

One easily verifies that \( \Phi \) belongs to \( L^1 \left( 0, T \right) \) and that, according to the estimates (6.15) above:

\[
(6.18) \quad \| \Phi \|_{L^1 \left( 0, T \right)} \leq C_{[F, \nu]} \left[ \| \psi_b \|_{S_0}^2 + \| f_s \|_{L^2 \left( 0, T; S_{-1} \right)}^2 + \langle (g_n, g_{\tau}, \Gamma) \rangle_{G_0(T)}^2 \right].
\]

Then, integrating (6.17) over \( (0, t) \) for any \( t \in (0, T_m) \) and introducing the constant \( \lambda_F \) defined in (5.29) yields the estimate:

\[
\| \psi_A^m (t) \|_{S_0}^2 + \int_0^t \left( \nu \lambda_F - \Phi(s) \right) \| \psi_A^m (s) \|_{S_0}^2 \, ds \leq \int_0^t \Phi(s) \, ds \quad \text{for all } t \in (0, T),
\]

which, with Grönwall’s inequality, leads to the estimate below, uniform in \( t \) according to (6.18):

\[
(6.19a) \quad \| \psi_A^m (t) \|_{S_0} \leq \left[ \Phi \|_\mathcal{L}^1 \left( 0, T \right) \right]^{\frac{1}{2}} \| \Phi \|_{L^1 \left( 0, T \right)}^{\frac{1}{2}} \leq C_{[F, \nu]} \| \psi_b \|_{S_0} \| f_s \|_{L^2 \left( 0, T; S_{-1} \right)} \langle (g_n, g_{\tau}, \Gamma) \rangle_{G_0(T)}.
\]

We deduce that \( T_m \) can be chosen equal to \( T \). Going back to inequality (6.17), integrating it again over the time interval \( (0, T) \) and using the estimate (6.19a), we get another estimate uniform in \( m \):

\[
(6.19b) \quad \| \psi_A^m \|_{L^2 \left( 0, T; S_{-1} \right)} \leq C_{[F, \nu]} \| \psi_b \|_{S_0} \| f_s \|_{L^2 \left( 0, T; S_{-1} \right)} \langle (g_n, g_{\tau}, \Gamma) \rangle_{G_0(T)}.
\]

From identity (6.16a), we deduce now that:

\[
\| \partial_t \psi_A^m \|_{L^2 \left( 0, T; S_{-1} \right)} \leq \nu \| \psi_A^m \|_{L^2 \left( 0, T; S_1 \right)} + \| A \psi_A^m \varphi \|_{L^2 \left( 0, T; S_{-1} \right)}.
\]

Combining (6.3c), (6.19a) and (6.19b) allows us to deduce that:

\[
(6.19c) \quad \| \partial_t \psi_A^m \|_{L^2 \left( 0, T; S_{-1} \right)} \leq C_{[F, \nu]} \| \psi_b \|_{S_0} \| f_s \|_{L^2 \left( 0, T; S_{-1} \right)} \langle (g_n, g_{\tau}, \Gamma) \rangle_{G_0(T)}.
\]

It follows from the estimates (6.19) that the sequence \( \left\{ \psi_A^m \right\}_{m \geq 1} \) remains in a ball of \( S_0(T) \), centered at the origin and whose radius depends only on \( F, \nu \) and the norms of the data \( \| \psi_0 \|_{S_0}, \| f_s \|_{L^2 \left( 0, T; S_{-1} \right)} \) and \( \| (g_n, g_{\tau}, \Gamma) \|_{G_0(T)} \). The existence of a solution as limit of a subsequence of \( \left\{ \psi_A^m \right\}_{m \geq 1} \) is next obtained, following exactly the lines of the proof [68, Chap. 1, Section 6].
Let us address now the uniqueness of the solution. We denote by $\Psi_A$ the difference $\psi_A^2 - \psi_A^1$ between two solutions to the Cauchy problem (6.14) and this function satisfies:

$$\partial_t \Psi_A + \nu A^S_1 \Psi_A = -A^S_1(\bar{\psi} + \psi_A^2, \varphi) + A^S_1(\bar{\psi} + \psi_A^1, \varphi) \quad \text{ in } F_T.$$ 

Forming, for a.e. $t \in (0, T)$, the duality pairing of this identity set in $S_{-1}$ with $\Psi_A(t) \in S_1$ and using the inequality (6.3b) results in the estimate:

$$\frac{d}{dt} \|\Psi_A(t)\|_{S_0}^2 + \left[ \nu \lambda F - c_{[F, v]} \left( \|\bar{\psi}\|_{S_1}^2 \|\bar{\psi}\|_{S_0}^2 + \|\varphi\|_{S_K}^2 \|\varphi\|_{S_K}^2 + \|\psi_A^1\|_{S_1}^2 \|\psi_A^1\|_{S_0}^2 \right) \right] \|\Psi_A(t)\|_{S_0}^2 \leq 0.$$ 

The conclusion follows with Grönwall’s inequality, keeping in mind that inequalities (6.19a) and (6.19b) hold for $\psi_A^1$ as well. The proof is now completed.

**Definition 6.6.** Let a positive real number $T$, a source term $f_V \in L^2(0, T; V_{-2})$, an initial data $\omega^0 \in V_{-1}$ and a triple $(g_n, g_r, \Gamma) \in G_0(T)$ be given. Define $\omega^1 = \omega - L^0_{\omega, \Gamma}(g_n(0), g_r(0), \Gamma(0))$ and assume that $\Sigma$ is of class $C^{3,1}$.

We say that a vorticity function $\omega \in \tilde{V}_{-1}(T)$ is a (weak) solution to the $\omega-$Navier-Stokes equations satisfying the Dirichlet boundary conditions on $\Sigma_T$ as described in (4.1) by the triple $(g_n, g_r, \Gamma)$ if $\omega = \omega_b + \omega_t + \omega_A$ where:

1. The function $\omega_b$ accounts for the boundary conditions. It belongs to $V^2_{-1}(T)$ (defined in (4.34)) and is equal to $L^0_{\omega, \Gamma}(g_n, g_r, \Gamma)$.
2. The function $\omega_t$ accounts for the source term and the initial condition. It is defined as the unique solution in $V_{-1}(T)$ of the homogeneous (linear) $\omega-$Stokes Cauchy problem

\begin{align*}
\partial_t \omega_t + \nu A^V_0 \omega_t &= -\partial_t \omega_b + f_V \quad \text{in } F_T, \\
\omega_t(0) &= \omega^1_0 \quad \text{in } F.
\end{align*}

3. The function $\omega_A$ accounts for the nonlinear advection term. It belongs to the space $V_{-1}(T)$ and solves the nonlinear Cauchy problem:

\begin{align*}
\partial_t \omega_A + \nu A^V_0 \omega_A &= -A^V_1(\omega_b + \omega_t + \omega_A, \varphi) \quad \text{in } F_T, \\
\omega_A(0) &= 0 \quad \text{in } F,
\end{align*}

where $\varphi = L^1_{\omega} g_n$ is the Kirchhoff potential that belongs to $S^1_K(T)$.

**Theorem 6.7.** With any set of data as described in Definition 6.6 there exists a unique solution $\omega$ to the $\omega-$Navier-stokes equations. Moreover, if $\psi$ is the unique solution to the $\psi-$Navier-Stokes equations as defined in Definition 6.4 and $\psi^1 = \Delta_{-1} \psi^1$, $f_V = \Delta_{-2} f_S$, all the other data being equal, then $\omega = \Delta_0 \psi$.

**Proof.** It suffices to apply the operator $\Delta_{-2}$ to (6.13) and (6.14) to obtain (6.20) and (6.21), because, according to the commutative diagram of Fig. 4, $\Delta_{-2} A^S_1 = A^V_1 \Delta_0$. 

In case of homogeneous boundary conditions, we recover the exponential decay estimates as stated in Lemma 5.3 for the $\psi-$Stokes Cauchy problem, namely:

**Corollary 6.8.** Assume that $\psi$ is solution to the $\psi-$Navier-Stokes equations with homogeneous boundary conditions (i.e. $\psi_b = 0$ in Definition 6.4). Then the exponential decay (5.6a) holds true with $k = 0$.

Similarly, if $\omega$ is solution to the $\omega-$Navier-Stokes equations with homogeneous boundary conditions (i.e. $\omega_b = 0$ in Definition 6.6), then the exponential decay (5.6b) holds true with $k = -1$.
Proof of Theorem 6.11. The proof is based on a fixed point argument. The existence and uniqueness of \( \bar{\psi} \) and \( \psi \) is contained, for every inequality to complete the proof.

\[
(6.22)
\]

Remark 6.9. In Definition 6.6, the initial condition \( \omega^1 \) can be taken in the dual space \( V_{-1} \). This space contains, for every \( 1 < p < 2 \):

\[
L^p_V = \{ (\omega, Q_1)_{L^2(F)} : \omega \in L^p(\Omega) \},
\]

which can be identified with \( L^p(F) \). The space \( V_{-1} \) contains also, for every Lipschitz curve \( \mathcal{C} \) included in \( F \) and for every \( q \in H^{-\frac{1}{2}}(\mathcal{C}) \) what can be identified as a vorticity filament:

\[
\omega_q : \theta \in V_1 \mapsto \int_\mathcal{C} q Q_1 \theta ds.
\]

6.3. Strong solutions.

Definition 6.10. Let a positive real number \( T \), a source term \( f_S \in L^2(0,T;S_0) \), an initial data \( \psi^i \in \bar{S}_1 \) and a triple \( (g_n, g_r, \Gamma) \in G_1(T) \) be given. Define \( \psi^1_0 = \psi^i - L^S_1(g_0(0), g_r(0), \Gamma(0)) \) and assume that \( \Sigma \) is of class \( C^{2,1} \) and that the compatibility condition:

\[
(6.23a)
\]

\[
(6.23b)
\]

(3) The function \( \psi_A \) accounts for the nonlinear advection term. It belongs to the space \( S_1(T) \) and solves the nonlinear Cauchy problem:

\[
(6.24a)
\]

\[
(6.24b)
\]

where \( \varphi = L_2^n g_n \) is the Kirchhoff potential that belongs to \( S_1_2(\mathcal{K}) \).

Theorem 6.11. With any set of data as described in Definition 6.10 there exists a unique (strong) solution in \( S_1(T) \) to the \( \psi \)-Navier-Stokes equations.

Proof of Theorem 6.11. The proof is based on a fixed point argument. The existence and uniqueness of \( \psi_b \) and \( \psi_t \) being granted, denote by \( \tilde{\psi} \) the sum \( \psi_b + \psi_t \) that belongs to \( \bar{S}_1(T) \) and introduce the constant:

\[
R_0 = \left[ \| \psi^1_0 \|^2_{\bar{S}_1} + \| f_S \|^2_{L^2(0,T;S_0)} + \| g_0 \|^2_{G_1(T)} \right]^{\frac{1}{2}}.
\]

Then, define three maps:

(1) \( X_T : L^2(0,T;S_0) \rightarrow S_1(T) \) where, for every \( f \in L^2(0,T;S_0) \), \( \theta = X_T f \) is the unique solution in \( S_1(T) \) to the Cauchy problem:

\[
(6.22)
\]
advection term is defined for every \( \Sigma_T \) may not happen and therefore
\( \omega \) be written in the most common form, namely as the advection term
\( \nabla \psi \). Notice that even at this level of regularity, the nonlinear term of the vorticity equation cannot
be extended. The time of existence
\( \Theta \) and by Grönwall’s inequality, we conclude that
\( \| Z_T f \|_{L^2(0,T,S_0)} \leq c_{[f,\omega]} T \frac{R_0}{\| f \|_{L^2(0,T,S_0)}} \),
\( \| Z_T f_2 - Z_T f_1 \|_{L^2(0,T,S_0)} \leq c_{[f,\omega]} T \frac{R_0}{\| f_1 \|_{L^2(0,T,S_0)} + \| f_2 \|_{L^2(0,T,S_0)}} \| f_2 - f_1 \|_{L^2(0,T,S_0)} \).

Let now \( R \) be equal to \( 2R_0 \). Then, for every \( f, f_1, f_2 \) in \( B_R \), the ball of center 0 and radius \( R \) in \( L^2(0,T,S_0) \):
\( \| Z_T f \|_{L^2(0,T,S_0)} \leq c_{[f,\omega]} T \frac{R^2}{R}, \)
\( \| Z_T f_2 - Z_T f_1 \|_{L^2(0,T,S_0)} \leq c_{[f,\omega]} T \frac{R^2}{R} \| f_2 - f_1 \|_{L^2(0,T,S_0)} \).

Thus, for \( T_0 = 1/(4c_{[f,\omega]} R_0)^{10} \), the mapping \( Z_{T_0} \) is a contraction from \( B_{R_0} \) into itself. Banach fixed point
Theorem asserts that \( Z_{T_0} \) admits a unique fixed point whose image by \( X_{T_0} \) yields a solution \( \psi_A \) to the Cauchy problem
(6.24) on \((0,T_0)\). Let \( (0,T^*) \) be the larger time interval to which the solution \( \psi = \psi_A + \psi_r + \psi_b \) can be extended. The time of existence \( T_0 \) depending only on \( R_0 \), standard arguments ensure that the following alternative holds:

\[
\text{Either } T^* = T \text{ or } \lim_{t \to T^*} \| \psi(t) \|_{S_1} = +\infty.
\]

As being a weak solution, estimates (6.19a) and (6.19b) hold for \( \psi_A \), namely \( \| \psi_A \|_{C(0,T;S_0)} \) and \( \| \psi_A \|_{L^2(0,T;S_1)} \) are bounded. On the other hand, forming the scalar product of equation (6.24a) with \( A_2^\sigma \psi_A \) in \( S_0 \), we obtain:
\( \frac{1}{2} \frac{d}{dt} \| \psi_A \|_{S_1}^2 + \nu \| \psi_A \|_{S_2}^2 = -(A_0(\psi,\varphi),A_2^\sigma \psi_A)_{S_0} \) on \( (0,T^*) \).

Considering the nonlinear term in the right hand side, Hölder’s inequality yields:
\( \| (A_0(\psi,\varphi),A_2^\sigma \psi_A)_{S_0} \| \leq \| A_0 \|_{L^1(\mathcal{F})} \| \nabla^\perp \psi + \nabla \varphi \|_{L^1(\mathcal{F})} \| \psi_A \|_{S_2} \),
whence we deduce, proceeding as in the proof of Lemma 6.1 that:
\( \| (A_0 \psi, A_2^\sigma \psi_A)_{S_0} \| \leq \left[ \frac{\nu}{2} + c_{[f,\omega]} \Theta_1 \right] \| \psi_A \|_{S_2}^2 + c_{[f,\omega]} \Theta_2 \) on \( (0,T^*) \),
with \( \Theta_1 = \| \psi_A \|_{S_1} \| \psi_b \|_{S_2} + \| \psi_r \|_{S_2} + \| \varphi \|_{S_2} \) and \( \Theta_2 = \| \psi_A \|_{S_1} \| \psi_b \|_{S_2} + \| \psi_r \|_{S_2} + \| \varphi \|_{S_2} \). The functions \( \Theta_1 \) and \( \Theta_2 \) both belong to \( L^1(0,T) \). It follows that:
\( \frac{d}{dt} \| \psi_A \|_{S_2}^2 + (\nu - c_{[f,\omega]} \Theta_1) \| \psi_A \|_{S_2}^2 = c_{[f,\omega]} \Theta_2 \) on \( (0,T^*) \),
and by Grönwall’s inequality, we conclude that \( \| \psi_A \|_{S_2} \) is bounded on \( [0,T^*) \). The latter occurrence in (6.25) may not happen and therefore \( T^* = T \).

Definition 6.10 and Theorem 6.11 can easily be rephrased in terms of the vorticity field. The nonlinear
advection term is defined for every \( \omega \in V_1 \) and \( \varphi \in S_2^\perp \) as an element of \( V_{-1} \) by:
\( A_0^\sigma(\omega,\varphi) = \Delta_{-1} A_0^\sigma(\Delta_{-1}^{-1} \omega, \varphi) = - (\omega(\nabla^\perp \psi + \nabla \varphi), \nabla Q_1)_{L^2(\mathcal{F})} \),
the latter identity being deduced from (3.31b). In (6.27), \( \omega \) stands for the regular part of \( \tilde{\omega} \) (see Remark 4.17) and \( \tilde{\omega} = \Delta_{-1}^{-1} \tilde{\omega} \).

**Remark 6.12.** Notice that even at this level of regularity, the nonlinear term of the vorticity equation cannot
be written in the most common form, namely as the advection term \( \mathbf{u} \cdot \nabla \omega \) (see Section 4).

**Definition 6.13.** Let a positive real number \( T \), a source term \( f_V \in L^2(0,T;V_{-1}) \), an initial data \( \omega^i \in V_0 \) and a triple \( (g_n, g_r, \Gamma) \in G_1(T) \) be given. Define \( \omega_0^i = \omega^i - L_0^\sigma \) \((g_n(0), g_r(0), \Gamma(0)) \) and assume that \( \Sigma \) is of class \( C^{2,1} \) and that the compatibility condition \( \omega_0^i \in V_0 \) is satisfied.

We say that a vorticity function \( \omega \in V_0(T) \) is a strong (or Kato) solution to the \( \omega \)-Navier-Stokes equations
satisfying the Dirichlet boundary conditions on \( \Sigma_T \) as described in (4.1) by the triple \( (g_n, g_r, \Gamma) \) if \( \omega = \omega_b + \omega_r + \omega_A \) where:
(1) The function $\omega_b$ accounts for the boundary conditions. It lies in $V_0(T)$ and is equal to $L^1_1(g_n, g_r, \Gamma)$;

(2) The function $\omega_\ell$ accounts for the source term and the initial condition. It is defined as the unique solution in $V_0(T)$ of the homogeneous (linear) $\omega-$Stokes Cauchy problem

$$\begin{align*}
\partial_t \omega_\ell + \nu A^V_1 \omega_\ell &= -\partial_t \omega_b + f_V \quad \text{in } F_T, \\
\omega_\ell(0) &= \omega_\ell^0 \quad \text{in } F.
\end{align*}$$

(3) The function $\omega_A$ accounts for the nonlinear advection term. It belongs to the space $V_0(T)$ and solves the nonlinear Cauchy problem:

$$\begin{align*}
\partial_t \omega_A + \nu A^V_1 \omega_A &= -A^V_1(\omega_b + \omega_\ell + \omega_A, \varphi) \quad \text{in } F_T, \\
\omega_A(0) &= 0 \quad \text{in } F,
\end{align*}$$

where $\varphi = L^2_0 g_n$ is the Kirchhoff potential that belongs to $S_0^2(T)$.

The counterpart of Theorem 6.11 reads:

**Theorem 6.14.** For any set of data as described in Definition 6.13 there exists a unique (strong) solution in $V_0(T)$ to the $\omega-$Navier-Stokes equations. Moreover, if $\psi$ is the unique solution to the $\psi-$Navier-Stokes equations as defined in Definition 6.10 and $\omega = \Delta_1 \psi$, $f_V = \Delta_{-1} f_S$, all the other data being equal, then $\omega = \Delta_1 \psi$.

Once again, we point out that Equations (6.28a) and (6.29a) are set in $L^2(0, T; V_{-1})$ where $V_{-1}$ is not a distribution space. As very well explained in [31], this may be the cause of numerous mistakes and misunderstandings. Inspired by Guermond and Quartapelle in [32], let us elaborate a “distribution-based” reformulation of Systems (6.28)-(6.29). Any solution $\omega$ to the $\omega-$NS equations can be decomposed into:

$$\omega = \omega_A + \omega_\ell + \omega_b$$

where $\omega_b = \omega_b^\beta + \zeta_b$ with $\zeta_b = \sum_{j=1}^N \Gamma_j \zeta_j$. In these sums, $\omega_A$ and $\omega_\ell$ belong to $V_0(T)$, $\omega_b^\beta$ is in $H^1(0, T; V_{-1}) \cap C([0, T], L^2_0) \cap L^2(0, T; H^1_V)$ and $\Gamma_j \in H^1(0, T)$ for every $j = 1, \ldots, N$. In the splitting (6.30a) $\zeta_b$ is identified as the singular part of $\omega$ while the “regular part” is:

$$\omega_r = \omega_A + \omega_\ell + \omega_b^\beta.$$

Recalling the decomposition (3.4) of the space $S_0$, namely:

$$S_0 = H^1_0(\mathcal{F}) \oplus F_S,$$

where the finite dimensional space $F_S$ is spanned by the functions $\xi_j$ ($j = 1, \ldots, N$), we deduce that:

$$V_1 = P_1 H^1_0(\mathcal{F}) \oplus F_V.$$

From any source term $f_V \in L^2(0, T; V_{-1})$, we define $f'_V \in L^2(0, T; H^{-1}(\mathcal{F}))$ by setting:

$$f'_V = \langle f_V, P_1 \rangle_{V_{-1}, V_1}.$$

**Theorem 6.15.** Let $\omega$ be a solution to the $\omega-$NS equations as described in Definition 6.13 and introduce $\omega_r$, $\zeta_b$ and $f'_V$ as explained in the relations (6.30). Then $\omega_r$ obeys the equation:

$$\begin{align*}
\partial_t \omega_r - \nu \Delta \omega_r + \nabla \cdot \left[ \omega_r(\nabla^{\perp} \psi + \nabla \varphi) \right] &= f'_V \quad \text{in } L^2(0, T; H^{-1}(\mathcal{F})), \\
\text{and for every } j = 1, \ldots, N: \\
\Gamma'_j + \nu (\nabla \omega_r, \nabla \xi_j)_{L^2(\mathcal{F})} - (\omega_r(\nabla^{\perp} \psi + \nabla \varphi), \nabla \xi_j)_{L^2(\mathcal{F})} &= \langle f_V, P_1 \xi_j \rangle_{V_{-1}, V_1} \quad \text{in } L^2(0, T).
\end{align*}$$

**Proof.** By definition of a strong solution to the $\omega-$NS equations, the following equality holds for every $\theta \in V_1$:

$$\begin{align*}
\frac{d}{dt} \langle \omega_r, \theta \rangle_{V_{-1}, V_1} + \sum_{j=1}^N \Gamma'_j \langle \zeta_j, \theta \rangle_{V_{-1}, V_1} + \nu \langle \omega_r, \theta \rangle_{V_{-1}, V_1} - (\omega_r(\nabla^{\perp} \psi + \nabla \varphi), \nabla Q_1 \theta)_{L^2(\mathcal{F})} &= \langle f_V, \theta \rangle_{V_{-1}, V_1} \quad \text{on } (0, T).
\end{align*}$$
Notice now that
\[ \langle \omega, \theta \rangle_{V^{-1}, V_1} = (\omega_A + \omega_L, \theta)_{V_0} + (\omega_0^\theta, Q_1 \theta)_{L^2(F)} = (\omega, Q_1 \theta)_{L^2(F)}. \]
Choosing the test function \( \theta \) in \( P_1 H^1_0(F) \), we obtain (6.31a) and choosing \( \theta \) in \( F_{V} \) leads to (6.31b). \( \square \)

Apart from the nonlinear advection term, formulation (6.31) is quite similar to System (2.9) displayed at the beginning of this paper. In Section 8, we shall seek more regular solutions to the \( \omega \)–NS system in order to obtain Identity (6.31a) satisfied in \( L^2(0, T; L^2(F)) \).

7. The Pressure

The purpose of this section is to explain how the pressure can be recovered from the stream function or the vorticity field, i.e. to derive Bernoulli-like formulas for the \( \psi \)–NS equations. In the literature, the existence of the pressure field is usually deduced from the Helmholtz-Weyl decomposition and no expression is supplied.

7.1. Hilbertian framework for the velocity field. The following Lebesgue spaces shall enter the definition of the pressure:

\[ L^2_m = \{ f \in L^2(F) : \int_F f \, dx = 0 \} \quad \text{and} \quad S_m = S \cap L^2_m, \]
as well as the Sobolev spaces below:

\[ H^1_m = H^1(F) \cap L^2_m \quad \text{and} \quad H^2_m = \{ f \in H^2(F) \cap L^2_m : \frac{\partial f}{\partial n} \bigg|_{\Sigma} = 0 \}. \]
The last two spaces are provided with the norms:

\[ (f_1, f_2)_{H^1_m} = (\nabla f_1, \nabla f_2)_{L^2(F)} \quad \text{for all } f_1, f_2 \in H^1_m, \]
and

\[ (f_1, f_2)_{H^2_m} = (\Delta f_1, \Delta f_2)_{L^2(F)} \quad \text{for all } f_1, f_2 \in H^2_m. \]
We recall that the lifting operators \( L_k^\pm \) (for every integer \( k \)) were introduced in Definition 4.3.

**Definition 7.1.** For every \( f \in L^2_m \) we denote by \( \Theta_f \) the unique function in \( H^2_m \) satisfying:

\[ \Delta \Theta_f = f \quad \text{in } F, \]
and we denote by \( \Psi_f \) the unique preimage of \( f \) in \( Z_2 \) by the operator \( L_2^\pm \) (see Lemma 3.8). Then, we define the operator \( H : L^2_m \rightarrow L^2_m \) by:

\[ Hf = \Delta L_1^\pm \frac{\partial \Theta_f}{\partial \tau} \bigg|_{\Sigma} \quad \text{for all } f \in L^2_m. \]

It is worth noticing the obvious equality:

\[ \Psi_{Hf} = L_1^\pm \frac{\partial \Theta_f}{\partial \tau} \bigg|_{\Sigma} \quad \text{for all } f \in L^2_m. \]

The operator \( H \) will come in handy for defining the pressure from the stream function. The main properties of \( H \) are summarized in the following lemma:

**Lemma 7.2.** The operator \( H \) is bounded, \( \text{Im} \, H = S_m \) and \( \ker H = V_0 \), what entails that \( H \) is an isomorphism from \( S_m \) onto \( S_m \).

Denoting classically by \( H^* \) the adjoint of \( H \), we deduce that \( \text{Im} \, H^* = S_m \), \( \ker H^* = V_0 \) and \( H^* \) is an isomorphism from \( S_m \) onto itself. Furthermore, for every \( f \in S_m \), the function \( H^* f \) is the harmonic conjugate of \( f \) i.e. the unique function in \( S_m \) such that the complex function

\[ z = (x_1 + ix_2) \mapsto f(x_1, x_2) + i (H^* f)(x_1, x_2) \]
is holomorphic in \( F \).
Proof. The boundedness results from elliptic regularity results for Θ_f and from the boundedness of the operator L^1 (we recall that by default Σ is assumed to be at least of class C^{1,1}). By construction, Η is valued in Σ_m so let a function h be given in Σ_m. According to Lemma [3.9] Ψ_h belongs to $\mathcal{B}_S$ and:

$$\int_{\Sigma^+} \frac{\partial \Psi_h}{\partial n} \, ds = \int_{\mathcal{F}} h \, dx - \sum_{j=1}^{N} \int_{\Sigma_j^-} \frac{\partial \Psi_h}{\partial n} \, ds = 0.$$ 

We can then define $g \in H^2(\Sigma)$ such that $\partial g / \partial \tau = \partial \Psi_h / \partial n$ on $\Sigma$. We denote now by $\theta_h$ the biharmonic function in $H^2(\mathcal{F})$ such that $\partial^2 \theta_h / \partial n = 0$ and $\theta_h = g$ on $\Sigma$. One easily verifies that $\Delta \theta_h$ belongs to $L^2_m$ and $H \Delta \theta_h = h$. This proves that $\text{Im} \, \mathcal{H} = \Sigma_m$.

According to Lemma [1.5] the operator $L^1$ is an isomorphism from $G^1_1$ onto $\mathcal{B}_S$. Therefore, if $Hf = 0$ for some $f$, then $\Theta_f$ is in $S_1$ and hence $f = \Delta \Theta_f$ is in $V_0$, which means that indeed $\ker \mathcal{H} = V_0$.

Let now $h$ be in $\Sigma_m$ and $f$ be in $L^2_m$. Then:

$$(h, Hf)_{L^2(\mathcal{F})} = (h, \Delta \Psi_h f)_{L^2(\mathcal{F})} = \int_{\Sigma} h \frac{\partial \Psi_h f}{\partial n} \, ds = \int_{\Sigma} h \frac{\partial \Theta_f}{\partial \tau} \, ds = - \int_{\Sigma} \frac{\partial h}{\partial \tau} \Theta_f \, ds.$$ 

Introducing $\bar{h}$ the harmonic conjugate of $h$, we deduce that:

$$(h, Hf)_{L^2(\mathcal{F})} = - \int_{\Sigma} \frac{\partial \bar{h}}{\partial n} \Theta_f \, ds = (\bar{h}, \Delta \Theta_f)_{L^2(\mathcal{F})} = (\bar{h}, f)_{L^2(\mathcal{F})},$$

and the proof is completed. □

We turn now our attention to the Gelfand triple:

$$H_1 \subset H_0 \subset H^{-1},$$

where $H_1 = H^1_0(\mathcal{F})$, $H_0 = L^2(\mathcal{F})$ is the pivot space and $H^{-1} = H^{-1}(\mathcal{F})$ is the dual space of $H_1$. The space $H_1$ is provided with its usual scalar product, namely:

$$(u, v)_{H_1} = \int_{\mathcal{F}} \nabla u : \nabla v \, dx \quad \text{for all } u, v \in H_1.$$ 

Theorem 7.3. For every $u \in H_1$ there exists a unique triple $(\psi, \phi, h) \in S_1 \times S_1 \times \Sigma_m$ such that $h$ is the harmonic Bergman projection of the divergence of $u$ and

$$u = \nabla^\perp \psi + \nabla^\perp \psi_{Hh} + \nabla \Theta_h + \nabla \phi \quad \text{in } \mathcal{F}.$$ 

It follows that the divergence and the curl of $u$ are given respectively by:

$$\nabla \cdot u = \Delta \phi + h \quad \text{and} \quad \nabla^\perp \cdot u = \Delta \psi + Hh \quad \text{in } \mathcal{F},$$

and these decompositions in $L^2(\mathcal{F})$ of $\nabla \cdot u$ and $\nabla^\perp \cdot u$ agree with the orthogonal decomposition $V_0 \oplus \Sigma$ of the space $L^2(\mathcal{F})$.

Finally, let $u_1, u_2$ be in $H_1$ and denote $\psi_1, \psi_2, \phi_1, \phi_2$ the functions in $S_1$ and $h_1, h_2$ the functions in $\Sigma_m$ such that:

$$u_k = \nabla^\perp \psi_k + \nabla^\perp \psi_{Hh_k} + \nabla \Theta_{h_k} + \nabla \phi_k \quad \text{in } \mathcal{F} \quad (k = 1, 2).$$

Then, the scalar product [7.5] can be expanded as follows:

$$(u_1, u_2)_{H_1} = (\nabla \cdot u_1, \nabla \cdot u_2)_{L^2(\mathcal{F})} + (\nabla^\perp \cdot u_1, \nabla^\perp \cdot u_2)_{L^2(\mathcal{F})}$$

$$= (\Delta \psi_1, \Delta \psi_2)_{L^2(\mathcal{F})} + (Hh_1, Hh_2)_{L^2(\mathcal{F})} + (h_1, h_2)_{L^2(\mathcal{F})} + (\Delta \phi_1, \Delta \phi_2)_{L^2(\mathcal{F})}.$$ 

Proof. Let $u$ be given and decompose the $L^2$ functions $\nabla \cdot u$ and $\nabla \cdot u$ respectively into the sums $\omega + \omega_h$ and $\delta + h$ with $\omega, \delta \in V_0$ and $\omega_h, h \in \Sigma$. Since:

$$\int_{\mathcal{F}} h \, dx = \int_{\mathcal{F}} (\delta + h) \, dx = \int_{\Sigma} u \cdot n \, ds = 0,$$

the harmonic function $h$ is actually in $\Sigma_m$. In the same way:

$$\int_{\mathcal{F}} \omega_h \, dx = \int_{\mathcal{F}} (\omega + \omega_h) \, dx = - \int_{\Sigma} u \cdot \tau \, ds = 0,$$

and we conclude.

$$\int_{\mathcal{F}} h \, dx = \int_{\mathcal{F}} (\delta + h) \, dx = \int_{\Sigma} u \cdot n \, ds = 0,$$

the harmonic function $h$ is actually in $\Sigma_m$. In the same way:

$$\int_{\mathcal{F}} \omega_h \, dx = \int_{\mathcal{F}} (\omega + \omega_h) \, dx = - \int_{\Sigma} u \cdot \tau \, ds = 0,$$

and we conclude.
and $\omega_h$ is in $S_m$ as well. Define now $\phi$ and $\psi$ in $S_1$ such that $\Delta \phi = \delta$ and $\Delta \psi = \omega$. One easily verifies that the vector field:

$$v = u - \left[ \nabla^\perp \psi + \nabla^\perp \psi H_h + \nabla \Theta_h + \nabla \phi \right] \quad \text{in } F,$$

is in $H_1$ and that $\nabla \cdot v = 0$. On the other hand $\nabla^\perp \cdot v = \omega_h - H_h$, which means in particular that $\nabla^\perp \cdot v \in S_m$. This entails that $v = 0$. Indeed, according to Helmholtz-Weyl decomposition (see [27, Theorem 3.2]), there exists $\Phi \in H^1_m$ and $\Psi \in S_0$ such that:

$$v = \nabla^\perp \Psi + \nabla \Phi \quad \text{in } F.$$

But $\Phi = 0$ since $\nabla \cdot v = 0$ and $\Psi$ belongs to $S_1$ according to the boundary conditions and the regularity of $v$. It follows that $\Delta \Psi \in V_0$ but as observed earlier, $\Delta \Psi = \nabla^\perp \cdot v \in S_m$, which implies that $\Psi = 0$. This proves the existence and uniqueness of the decomposition (7.6).

Assume now that $u_1$ and $u_2$ are in $D(F) = \mathcal{D}(F; \mathbb{R}^2)$ and introduce their decompositions as in (7.8). Integrating by parts, we obtain:

$$(u_1, u_2)_{H_1} = -(\Delta u_1, u_2)_{L^2(F)}$$

$$(7.10)$$

$$= -\langle \nabla^\perp \Delta \psi_1 + \nabla^\perp H h_1, u_2 \rangle_{D'(F), D(F)} - \langle \nabla h_1 + \nabla \Delta \phi_1, u_2 \rangle_{D'(F), D(F)}.$$

We switch to the duality pairing in the second equality because although $u_1$ is smooth, this does not guaranty that every term in the decomposition (7.6) is also smooth (notice that invoking elliptic regularity results would require the boundary $\Sigma$ to be smoother than $C^{1,1}$). The former term in the right hand side of (7.10) yields:

$$\langle \nabla^\perp \Delta \psi_1 + \nabla^\perp H h_1, u_2 \rangle_{D'(F), D(F)} = (\Delta \psi_1 + \nabla^\perp H h_1, u_2)_{L^2(F)}$$

$$= (\Delta \psi_1, \Delta \psi_2)_{L^2(F)} = (H h_1, H h_2)_{L^2(F)},$$

while the latter leads to:

$$\langle \nabla h_1 + \nabla \Delta \phi_1, u_2 \rangle_{D'(F), D(F)} = -(h_1 + \nabla \Delta \phi_1, \nabla \cdot u_2)_{L^2(F)} = -(h_1, h_2)_{L^2(F)} = (\Delta \phi_1, \Delta \phi_2)_{L^2(F)}.$$

The equality (7.9) follows by density of $D(F)$ into $H_1$ and the proof is complete. \(\square\)

**Remark 7.4.**

1. The decomposition (7.6) differs from the one in [27, Theorem 3.3] where $u \in H_1$ is decomposed into

$$(7.11)$$

$$u = \nabla^\perp \psi + (-\Delta_D)^{-1} \nabla p \quad \text{in } F,$$

with $\psi \in S_1$ and a potential $p$ in $L^2_m$. The operator $(-\Delta_D)^{-1}$ obviously stands for the inverse of the Laplacian operator with homogeneous boundary conditions. The stream function $\psi$ is the same in (7.6) and (7.11).

2. In [2, Theorem 3] or [39, Theorem 2.1], every vector field $u \in L^2(F)$ is shown to admit the decomposition:

$$(7.12)$$

$$u = \nabla^\perp \psi + \nabla^\perp h + \nabla p \quad \text{in } F,$$

with $\psi \in H^1_0(F)$, $p \in H^1_m$ and $h \in \mathcal{F}_S$. This expression is used by Mackawa in [52] to derive necessary and sufficient conditions for $u$ to be in $J_1$.

3. Identity (7.9) is a trivial version of Friedrich’s second inequalities; see [27, Lemma 2.5 and Remark 2.7] and also for instance [11]. However, it can also be readily deduced from (7.9) that there exists a constant $c_F$ such that for every $u \in H_1$:

$$\|u\|_{H_1} \leq c_F \|
abla^\perp \omega\|_{L^2(F)} + \|
abla \delta\|_{L^2(F)} \quad \text{or} \quad \|u\|_{H_1} \leq c_F \|
abla^\perp \omega\|_{L^2(F)} + \|
abla \delta\|_{L^2(F)},$$

where $\omega = \nabla^\perp \cdot u$ and $\delta = \nabla \cdot u$. It means that the $H_1$-norm of a vector field is controlled by:

(a) Either the harmonic Bergman projection of the curl and the divergence of this vector field in $L^2$;

(b) or by the curl and the harmonic Bergman projection of the divergence of this vector field in $L^2$.

We were not able to find this result in the literature.
(4) The decomposition \([7.6]\) of Theorem \(7.3\) allows to deduce a necessary and sufficient condition for the following overdetermined div-curl problem to be well-posed: There exists a unique \(u \in H_1\) such that:

\[
\nabla \cdot u = \delta \quad \text{and} \quad \nabla^\perp \cdot u = \omega \quad \text{in } F,
\]

with \(\delta\) and \(\omega\) in \(L^2(F)\) if and only if \(P^\perp \omega\) is the harmonic conjugate of \(P^\perp \delta\). This result seems to be new as well.

As shown in \([86]\), the definition of the pressure for the Navier-Stokes equations (in classical velocity-pressure formulation) is not possible for a source term \(f_2\) in \(L^2(0,T;J_{-1})\), what means in nonprimitive variables, for \(f_S \in L^2(0,T;S_{-1})\) (see Fig. 3). The definition of the pressure requires the source term to be in \(L^2(0,T;H_{-1})\).

To be more specific, we need to elaborate on the structure of the dual space \(H_{-1}\).

**Proposition 7.5.** For every linear form \(f_H\) in \(H_{-1}\), there exists a unique pair \((\delta_H, \omega_H) \in L^2_m \times V_0\) such that:

\[
\langle f_H, u \rangle_{H_{-1}, H_1} = (\delta_H, \nabla \cdot u)_{L^2(F)} + (\omega_H, \nabla^\perp \cdot u)_{L^2(F)} \quad \text{for all } u \in H_1.
\]

If \(f_H\) belongs to \(H_0\), then:

\[
\delta_H = -\phi_H - H^* \psi_H \quad \text{and} \quad \omega_H = -P_1 \psi_H,
\]

where \(f_H = \nabla \phi_H + \nabla^\perp \psi_H\) is the Helmholtz-Weyl decomposition of \(f_H\). The identities \((7.13a)\) can easily be inverted:

\[
\phi_H = -\delta_H + H^* \omega_H \quad \text{and} \quad \psi_H = -Q_1 \omega_H.
\]

**Proof.** Let \(f_H\) be in \(H_{-1}\). According to Riesz representation Theorem and Theorem \([7.3]\) there exists \(\phi, \psi \in S_1\) and \(h \in S_m\) such that:

\[
\langle f_H, u \rangle_{H_{-1}, H_1} = \langle \Delta \phi + (Id + H^* h), \nabla \cdot u \rangle_{L^2(F)} + \langle \Delta \psi, \nabla^\perp \cdot u \rangle_{L^2(F)} \quad \text{for all } u \in H_1.
\]

It suffices to set \(\delta_H = \Delta \phi + (Id + H^* h)\) and \(\omega_H = \Delta \psi\).

Assume now that \(f_H\) lies in \(H_0\) and denote by \(\phi_H \in H^*_m\) and \(\psi_H \in S_0\) the functions entering the Helmholtz-Weyl decomposition of \(f_H\), i.e.

\[
f_H = \nabla \phi_H + \nabla^\perp \psi_H \quad \text{in } F.
\]

By definition of \(H_0\) as pivot space:

\[
\langle f_H, u \rangle_{H_{-1}, H_1} = \langle f_H, u \rangle_{L^2(F)} = -\langle \phi_H, \nabla \cdot u \rangle_{L^2(F)} - \langle \psi_H, \nabla^\perp \cdot u \rangle_{L^2(F)} \quad \text{for all } u \in H_1.
\]

The orthogonal decomposition \(L^2_m(F) = V_0 \perp S_m\) leads to \(\nabla^\perp \cdot u = P^\perp_0 \nabla^\perp \cdot u + P_0 \nabla^\perp \cdot u\). But according to Theorem \([7.3]\) \(P^\perp_0 \nabla^\perp \cdot u = H \nabla \cdot u\), whence we deduce that:

\[
\langle f_H, u \rangle_{H_{-1}, H_1} = -\langle \phi_H + H^* \psi_H, \nabla \cdot u \rangle_{L^2(F)} - \langle P_1 \psi_H, \nabla^\perp \cdot u \rangle_{L^2(F)},
\]

and the proof is complete.

**Remark 7.6.** It is now easy to verify that if \(f_H \in H_{-1}\) and \(f_S \in S_{-1}\) are two linear forms such that:

\[
\langle f_H, \nabla^\perp \psi \rangle_{H_{-1}, H_1} = \langle f_S, \psi \rangle_{S_{-1}, S_1} \quad \text{for all } \psi \in S_1,
\]

then \(P f_S = \omega_H\) where \(\omega_H \in V_0\) and \(\delta_H \in L^2_m\) are defined from \(f_H\) in Proposition \([7.3]\). We shall prove that the pressure depends only upon \(\delta_H\) and therefore is actually independent of the source term \(f_S\).

**7.2. Weak solutions.** When the equation is nonlinear, the operator \(H\) is not sufficient to define the pressure. Thus, for every \(u \in L^4(F)\), define \(\pi_\phi[u], \pi_\psi[u] \in L^2_m\) by means of Riesz representation Theorem as:

\[
\pi_\phi[u, f]_{L^2_m} = -(D^2 \Theta_f u, u)_{L^2(F)}
\]

\[
\pi_\psi[u, f]_{L^2_m} = (D^2 \Psi_f u, u^\perp)_{L^2(F)} \quad \text{for all } f \in L^2_m.
\]
Definition 7.7. Let $T$ be a positive real number, $\psi$ be a function in $\bar{S}_0(T)$, $\varphi$ be in $S^1_1(T)$ (this space is defined in (6.1)) and $\delta_H$ be in $L^2(0,T;L^2_m)$. Then introduce the velocity field $u = \nabla^\perp \psi + \nabla \varphi$ and for a.e. $t \in (0,T)$ define $p_r(t)$ by:

(7.15a) \[ p_r(t) = -\partial_t \varphi(t) + \pi_\theta[u(t)] + H^\perp \left[ \nu \omega(t) - \pi_\psi[u(t)] \right] - \delta_H(t), \]

where $\omega(t) = \Delta \psi(t)$ (i.e. $\omega(t)$ is the regular part of $\bar{\Delta}_0 \psi(t)$, see Remark 4.14). The pressure $p$ corresponding to these data is obtained by summing $p_r$, called the regular part of the pressure, and a singular part $p_s$:

(7.15b) \[ p = p_r + p_s \quad \text{with} \quad p_s = -\partial_t H^* \psi. \]

The proof of the lemma below is obvious:

Lemma 7.8. The function $p_r$ belongs to $L^2(0,T;L^2_m)$ and the mapping

$$(\psi, \varphi, \delta_H) \in S_0(T) \times S^1_1(T) \times L^2(0,T;L^2_m) \mapsto p_r \in L^2(0,T;L^2_m),$$

is continuous. The function $p_s$ lies in $W^{-1,\infty}(0,T;L^2_m)$ and the mapping $\psi \in S_0(T) \mapsto p_s \in W^{-1,\infty}(0,T;L^2_m)$ is continuous.

We can now state the main result of this subsection:

Theorem 7.9. Let $T$ be a positive real number and let $\psi \in \bar{S}_0(T)$ be a weak solution to the $\psi$–Navier-Stokes equations as defined in Definition 6.4, and whose source term is recalled to be denoted by $f^\perp$. Let $\varphi \in S^1_1(T)$ be the Kirchhoff potential also introduced in Definition 6.4. Finally, let $f_H$ be in $L^2(0,T;H^2)$ such that (see Remark 6.6):

$$(f_H, \nabla^\perp \varphi)_{H^{-1},H^1} = (f^\perp, \theta)_{S^{-1},S_1} \quad \text{for all } \theta \in S_1.$$

According to Proposition 7.3, to the linear form $f_H$ can be associated a pair $(\delta_H, \omega_H) \in L^2(0,T;L^2_m) \times L^2(0,T;V_0)$.

Denote now by $u$ the vector field $\nabla \phi + \nabla^\perp \psi$ and by $p$ the pressure defined from $\psi$, $\varphi$ and $\delta_H$ as explained in Definition 7.7. Then the pair $(u, p)$ is a weak (Leray) solution to the Navier-Stokes equations, namely, for every $w$ in $H_1$:

(7.16) \[ \frac{d}{dt}(u,w)_{L^2(F)} - \nu \int_F \nabla u : \nabla w \, dx = (p, \nabla \cdot w)_{L^2(F)} = (f_H, w)_{H^{-1},H^1} \quad \text{on } (0,T). \]

Proof. Remind that $\psi$ satisfies, for every $\theta \in S_1$:

(7.17) \[ \frac{d}{dt}(\nabla \psi, \nabla \varphi)_{L^2(F)} + (D^2 \varphi \psi, w)_{L^2(F)} = (f_H, \nabla^\perp \varphi)_{H^{-1},H^1} \quad \text{on } (0,T). \]

Let $u_\theta = \nabla^\perp \psi + \nabla \varphi$, $u_0 = u - u_\theta = \nabla^\perp \psi + \nabla \psi_A$ (see Definition 6.10) and $\omega_\theta = \Delta \psi_\theta$. Then, for every $w \in D(F)$:

(7.18a) \[ \int_F \nabla u_\theta : \nabla w \, dx = -(\Delta u_\theta, w)_{D^*(F),D(F)} = -(\nabla^\perp \omega_\theta, w)_{L^2(F)} = (\omega, \nabla^\perp \cdot w)_{L^2(F)}, \]

and this result extends by density to every $w \in H_1$. According to Theorem 7.3 we can decompose $w$ into

(7.18b) \[ w = \nabla^\perp \theta + \nabla \psi_h + \nabla \theta_h + \nabla \phi \quad \text{in } F, \]

with $(\theta, \phi, h) \in S_1 \times S_1 \times S_m$ and it follows that $\nabla^\perp \cdot w = \Delta \theta + Hh$. Since $\omega_\theta \in S_1$, we infer that:

(7.18c) \[ (\omega, \nabla^\perp \cdot w)_{L^2(F)} = (\omega, Hh)_{L^2(F)} = (H^* \omega_\theta, h)_{L^2(F)} = (H^* \omega, \nabla \cdot w)_{L^2(F)}, \]

because $H^* \omega = H^* \omega_\theta$. On the other hand, since $u_\theta$ belongs to $H_1$, according to (7.9), it follows that:

(7.18d) \[ \int_F \nabla u_\theta : \nabla w \, dx = (\Delta (\psi_A + \psi_\theta), \Delta \theta)_{L^2(F)} = (\Delta \psi, \Delta \theta)_{L^2(F)}, \]

the latter equality resulting from the orthogonality property $(\Delta \psi_\theta, \Delta \theta)_{L^2(F)} = 0$. Gathering now the identities (7.18), we obtain:

(7.19) \[ \nu \int_F \nabla u : \nabla w \, dx = \nu (\Delta \psi, \Delta \theta)_{L^2(F)} + (\nabla^\perp \omega, \nabla \cdot w)_{L^2(F)}. \]
Invoking again the decomposition (7.18b), we get:

\[(u, w)_{L^2(F)} = (\nabla \psi, \nabla \theta)_{L^2(F)} - (H^* \psi + \varphi, \nabla \cdot w)_{L^2(F)},\]

and also:

\[(\nabla w, u)_{L^2(F)} = (D^2 \theta u, u_{\perp})_{L^2(F)} + (D^2 \Phi_{H} u, u_{\perp})_{L^2(F)} + (D^2 (\Theta_h + \phi) u, u)_{L^2(F)}.
\]

But notice that \(H \in H(\nabla \cdot w)\) and \(\Theta_h + \phi = \Theta_{\nabla \cdot w}\) (both functions share the same boundary conditions and the same Laplacian). Using the notation (7.14b), we are then allowed to rewrite the above equality as:

\[(\nabla w, u)_{L^2(F)} = (D^2 \theta u, u_{\perp})_{L^2(F)} + (H^* \pi \Psi [u], \nabla \cdot w)_{L^2(F)} - (\pi \theta [u], \nabla \cdot w)_{L^2(F)}.
\]

Finally, considering the source term:

\[(f_H, w)_{H_{-1}, H_{-1}} = (f_H, \nabla \cdot \theta)_{H_{-1}, H_{-1}} + (f_H, w - \nabla \cdot \theta)_{H_{-1}, H_{-1}} = (f_{\Psi}, \Theta)_{H_{-1}, H_{-1}} + (\delta_{H}, \nabla \cdot w)_{L^2(F)}.
\]

Summing now the time derivative of (7.20) with (7.19) and subtracting (7.21) and the term \((p, \nabla \cdot w)_{L^2(F)}\), we obtain (7.22), taking into account (7.17). The resulting equality is therefore (7.16) and the proof is completed.

7.3. Strong solutions. For every \(u \in H^2(F)\), define \(\Phi[u]\) as the unique element in \(H^1_{m} \) such that:

\[(\Phi[u], \theta)_{H^1_{m}} = - (\nabla uu, \nabla \theta)_{L^2(F)} \quad \text{for all } \theta \in H^1_{m}.
\]

**Definition 7.10.** Let \(T\) be a positive time, \(\psi\) be a function in \(S_1(T)\), \(\varphi\) be in \(S^2_0(T)\) and \(\phi_{H}\) (accounting for the source term) be in \(L^2(0, T; H^1_{m})\). Then introduce the velocity field \(u = \nabla \psi + \nabla \varphi\). For a.e. \(t \in (0, T)\) define \(p(t)\) by:

\[p(t) = -\partial_t \psi(t) + \Phi[u(t)] + \nu H^* Q^\perp_1 \omega(t) + \phi_{H}(t),\]

where \(\omega(t) = \Delta \psi(t)\) (i.e. \(\omega(t)\) is the regular part of \(\Lambda_1 \psi(t)\), see Remark 4.17).

**Proposition 7.11.** The function \(p\) belongs to \(L^2(0, T; H^1_{m})\) and the mapping

\[(\psi, \varphi, \phi_{H}) \in S_1(T) \times S^2_0(T) \times L^2(0, T; S_0) \mapsto p \in L^2(0, T; H^1_{m}),\]

is continuous.

Moreover, if \(\psi\) is a solution to the \(\psi\)–NS equations as described in Definition 6.10 with Kirchhoff potential \(\varphi\) and source term \(f_s \in L^2(0, T; S_0)\), then \(p\) defined in (7.24) from the triple \((\psi, \varphi, \phi_{H})\) is equal to the pressure of Definition 7.7 computed from the triple \((\psi, \varphi, \delta_{H})\) with \(\delta_{H} = -\phi_{H} - H^* f_s\).

**Proof.** The continuity of the mapping being obvious, let us verify the claim that \(p\) in (7.24) matches the expression given in (7.15b). For a.e. \(t \in (0, T)\), \(\partial_t \psi\) is in \(S_0\) and since \(\psi\) is a strong solution to the \(\psi\)–NS equations it follows that:

\[\partial_t H^* \psi = H^* \partial_t \psi = H^* (-\nu A^S_2(\psi_A + \psi_\epsilon) - A^S_0(\psi, \varphi) + f_s).
\]

On the one hand, according to the expression (3.16) of \(A^S_2\):

\[H^* A^S_2(\psi_A + \psi_\epsilon) = -H^* Q_1(\omega_A + \omega_\epsilon) = -H^* P^\perp_1 Q_1(\omega_A + \omega_\epsilon) = H^* Q^\perp_1(\omega_A + \omega_\epsilon),\]

because \(H^* = H^* P^\perp_1\) and \(P^\perp_1 Q_1 = (\text{Id} - P_1) Q_1 = -Q^\perp_1\). On the other hand, for every \(f \in L^2_{m}\):

\[(H^* A^S_0(\psi, \varphi), f)_{L^2(F)} = (A^S_0(\psi, \varphi), \Delta \Psi_{H.f})_{L^2(F)} = -(A^S_0(\psi, \varphi), \Psi_{H.f})_{S_0} = -(\nabla uu, \nabla^\perp \Psi_{H.f})_{L^2(F)},\]

the latter equality resulting from (6.12). Summing up, we obtain that for every \(f \in L^2_{m}\) and a.e. \(t \in (0, T)\):

\[(7.25) \quad (\pi_\psi [u](t))^T + H^* [uu(t) - \pi_\psi [u(t)]] - H^* [\psi(t), f]_{L^2_{m}} = -(D^2 \Theta_f(t)u(t), u(t))_{L^2_{m}} - \nu (H^* Q^\perp_1(\omega_A + \omega_\epsilon(t), f)_{L^2_{m}} - (H^* f_s, f)_{L^2_{m}}.
\]

The two first terms in the right hand side can be rewritten as:

\[(D^2 \Theta_f(t)u(t), u(t))_{L^2_{m}} + (D^2 \Psi_{H.f}(t)u(t), u(t))_{L^2_{m}} = (\nabla (\nabla \Theta_f + \nabla \Psi_{H.f}(t)u(t), u(t))_{L^2_{m}},\]

and the resulting quantity can now be integrated by parts:

\[(\nabla (\nabla \Theta_f + \nabla \Psi_{H.f}(t))u(t), u(t))_{L^2_{m}} = -(\nabla u(t), \nabla \Theta_f(t) + \nabla \Psi_{H.f}(t))_{L^2_{m}},\]
Turning our attention to the two last terms in the right hand side of (7.25), we observe that $\mathbb{H}^* \omega(t) = H^* \omega_b(t) = H^* Q_\perp \omega_b(t)$ according to the properties of $H^*$ stated in Lemma 7.2 and the fact that $\omega_b$ is the harmonic part of $\omega = \omega_A + \omega_T + \omega_b$. We have now proved that:

$$(\pi_\omega[u(t)] + H^* [\nu \omega(t) - \pi_\omega[u(t)]] - \partial_t H^* \psi(t) ; f)_{L_2^2} = (\nabla u(t) u(t), \nabla \Theta(t))_{L^2(\mathcal{F})} + \nu (H^* Q_\perp \omega(t), f)_{L_2^2} - (H^* f_S , f)_{L_2^2}.$$ 

Recalling the definition (7.23) of $\Phi[u(t)]$, we can integrate by parts the first term in the right hand side:

$$(\nabla u(t) u(t), \nabla \Theta(t))_{L^2(\mathcal{F})} = - (\nabla \Phi[u(t)] ; (\partial_t u(t), \nabla \Theta(t))_{L^2(\mathcal{F})} = (\Phi[u(t)] , f)_{L_2^2},$$

and thus complete the proof. □

**Theorem 7.12.** Let $T$ be a positive real number and let $\psi \in \mathcal{S}_1(T)$ be a strong solution to the $\psi-$Navier-Stokes equations as defined in Definition 6.10. Let $\varphi \in \mathcal{S}_2^K(T)$ be the Kirchhoff potential also introduced in Definition 6.10.

Denote now by $u$ the vector field $\nabla \varphi + \nabla \psi$ and by $p$ the pressure defined from $\psi, \varphi$ and $\phi_H = - H^* f_S$ as explained in Definition 7.11. Then the pair $(u, p)$ is a strong (Kato) solution to the Navier-Stokes equations.

Proof. The stream function $\psi$ is also a weak solution to the $\psi-$NS equations in the sense of Definition 6.4. According to Proposition 7.11, the pressure $p$ is a “weak” pressure in the sense of Definition 7.7 as well. It follows from Theorem 7.9 that $(u, p)$ is a weak solution to the NS equations in primitive variables. From the regularity of $u$ and $p$ we are allowed to deduce that $(u, p)$ is indeed a strong solution to the NS equations in primitive variables. □

### 8. More regular vorticity solutions

So far and even for strong solutions as described in the preceding subsection, the regularity of the functions does not allow writing the vorticity equation in the most common form (2.9), that is, loosely speaking, as an advection-diffusion equation set in $L^2(F)$ (see Remark 6.12). To achieve this level of regularity, a first guess would be to seek solutions in $V_1(T)$, in which case, the operator $(-\Delta)$ should be replaced by $\nu \Delta$. However, since the nonlinear advection term $u \cdot \nabla \omega$ does not belong to $V_0$ in general, we are inclined to conclude that this approach leads to a dead end. We shall prove that the solution should rather be looked for in the space $\bar{V}_1(T)$. We recall that the spaces $V_2, \bar{V}_1$ and $\bar{V}_0$ are all of them subspaces of $V_{-1}$. They are defined in Subsection 4.2.

Before addressing the $\omega-$NS equations, we begin as usual with Stokes problems. All along this Section, we assume that the boundary $\Sigma$ is of class $C^{3,1}$.

#### 8.1. Regular Stokes vorticity solutions

For every positive real number $T$, we aim to define solutions to Stokes problems belonging to:

$$\bar{V}_1(T) = H^1(0, T; \bar{V}_0) \cap C([0, T]; \bar{V}_1) \cap L^2(0, T; \bar{V}_2).$$

We recall that $\bar{V}_2 \subset \bar{V}_1 \subset \bar{V}_0 \subset V_{-1}$ (see Subsection 4.2). So, let a triple $(g_n, g_r, \Gamma)$ be given in $G_2(T)$ and define $\omega_b$ in $V_1^b(T)$ by:

$$\omega_b(t) = 2\nu \left( g_n(t), g_r(t) ; f(t) \right) = (g_n(t), g_r(t))_{L^2(\mathcal{F})} + \sum_{j=1}^{N} \Gamma_j(t) \zeta_j \quad \text{for a.e. } t \in (0, T),$$

where $\omega_b^\beta(t)$ belongs to $\mathcal{S}_c^b$ (the operator $L^b$ is defined in (4.33) and maps continuously $G_2(T)$ into the space $V_1^b(T)$ defined in (4.34)). The source term $f_V$ is expected to belong to $L^2(0, T; \bar{V}_0)$ and the decomposition (4.32) of the space $\bar{V}_0$ leads to the splitting:

$$f_V(t) = f_V^0(t) + f_V^\beta(t) + \sum_{j=1}^{N} \alpha^\beta_j(t) \zeta_j \quad \text{for a.e. } t \in (0, T),$$

with $f_0(t)$ is in $V_0; f_\beta(t)$ in $\mathcal{S}_c$ and $\alpha^\beta_j(t)$ in $\mathbb{R}$ for every $j = 1, \ldots, N$. Similarly, taking now into account the decomposition (4.32) of $\bar{V}_2$, we seek the total vorticity in the form:

$$\omega(t) = \omega_1(t) + \omega_2(t) \quad \text{with } \omega_1(t) = \omega_0(t) + \omega_2(t) + \sum_{j=1}^{N} \beta_j(t) \Omega_j \quad \text{for a.e. } t \in (0, T),$$

where $\omega_2(t)$ is in $V_2$ and $\sum_{j=1}^{N} \beta_j(t) \Omega_j$ is the advection term. The advection velocity $u(t)$ is $\omega_1(t)$.
where the function \( \omega_0(t) \) belongs to \( V_2 \), \( \omega_3(t) \) in \( \mathfrak{B}_0^2 \) and \( \beta_j(t) \) in \( \mathbb{R} \) for every \( j = 1, \ldots, N \). They are the unknowns of the problem. The function \( \omega_1 \) is supposed to satisfy in particular Equation (5.17) for \( k = 0 \), namely:

\[
\begin{align*}
(8.4) \quad \partial_t \omega + \nu A^V_1 \omega_1 &= f_V - \partial_t \omega_b \quad \text{in } \mathcal{F}_T,
\end{align*}
\]

this equality being set in \( L^2(0, T; V_{-1}) \). We want this equation to be satisfied in the slightly more regular space \( L^2(0, T; \bar{V}_0) \). Thus, the operator \( A^V_1 \) turns into the operator \( \bar{A}^V \) (defined right above Proposition 4.18). Keeping in mind the decomposition (4.22) of \( \bar{V}_0 \) we apply successively to Equation (8.4), the orthogonal projections onto the spaces \( \bar{V}_0 \), \( \mathcal{F}_V \) and \( \bar{F}_V \) respectively to obtain the system:

\[
\begin{align*}
(8.5a) \quad \partial_t \omega_0 + \nu A^V_2 \omega_0 &= f^0_V - \partial_t \omega_b - \sum_{j=1}^N \beta_j^j \Omega_j \quad \text{in } \mathcal{F}_T, \\
(8.5b) \quad \nu A^V_2 \omega_B &= f^0_V - \partial_t \omega_b^0 \quad \text{in } \mathcal{F}_T, \\
(8.5c) \quad \nu \beta_j = \Gamma_j^j - \alpha^V_j \quad \text{in } (0, T) \quad \text{for every } j = 1, \ldots, N.
\end{align*}
\]

A solution can be worked out by taking the time derivatives of the equations (8.5b) and (8.5c). Thus, one gets the expressions of \( \partial_t \omega_b \) and \( \partial_t \beta_j \) that can be used in (8.5a). This leads us to the following statement:

**Proposition 8.1.** Let \( T \) be a positive real number, \( (g_n, g_\tau, \Gamma) \) be a triple in

\[
\begin{align*}
(8.6a) \quad G_r(T) &= G_2(T) \cap C^1([0, T]; G_0^n \times G_0^\tau \times \mathbb{R}^N) \cap \mathcal{H}^2(0, T; G^\alpha_1 \times G^\tau_1 \times \mathbb{R}^N) \\
(8.6b) \quad &= \left\{(g_n, g_\tau, \Gamma) \in G_2(T) : (\partial_t g_n, \partial_t g_\tau, \partial_t \Gamma) \in G_0(T) \right\},
\end{align*}
\]

and \( f_V \) be a source term in

\[
F_r(T) = \left\{ f_V \in L^2(0, T; \bar{V}_0) : f_V = f^0_V + f^\beta_V + \sum_{j=1}^N \alpha^j V \right\},
\]

Let \( \omega_b \) be defined from the boundary data as in (8.1), let \( \omega^\gamma \) be an initial data in \( \bar{V}_1 \) satisfying the compatibility condition \( \omega^\gamma - \omega_b(0) \in V_1 \) and let:

\[
\omega^\gamma = \omega^\gamma_0 - \frac{1}{\nu} (A^V_1)^{-1} (f^\beta_V - \partial_t \omega_b^0)(0) - \frac{1}{\nu} \sum_{j=1}^N (\Gamma^j_0 - \alpha^j_0) \Omega_j.
\]

Then \( \omega^\gamma_0 \) belongs to \( \bar{V}_1 \) and there exists a unique solution \( \omega_0 \in V_1(T) \) to the Cauchy problem:

\[
\begin{align*}
(8.8a) \quad \partial_t \omega_0 + \nu A^V_2 \omega_0 &= f^0_V - \frac{1}{\nu} (A^V_0)^{-1} (f^\beta_V - \partial_t \omega_b^0) - \frac{1}{\nu} \sum_{j=1}^N (\Gamma^j_0 - \alpha^j_0) \Omega_j \quad \text{in } \mathcal{F}_T, \\
(8.8b) \quad \omega_0(0) &= \omega^\gamma_0 \quad \text{in } \mathcal{F}.
\end{align*}
\]

The vorticity function:

\[
\omega = \omega_0 + \frac{1}{\nu} (A^V_2)^{-1} (f^\beta_V - \partial_t \omega_b^0) + \frac{1}{\nu} \sum_{j=1}^N (\Gamma^j_0 - \alpha^j_0) \Omega_j + \omega_b,
\]

belongs to \( \bar{V}_1(T) \) and solves System (8.5). It will be called a regular vorticity solution to the \( \omega \)-Stokes equations.

It is worth noticing that:

1. System (8.5) is no longer a simple parabolic system but rather a coupled parabolic-elliptic system.
2. The regularity assumptions (8.6) entail that the functions \( f^\beta_V \) and \( \partial_t \omega_b^0 \) both belong to \( C([0, T]; V_{-1}) \) and therefore that the equality (8.7) at the initial time makes sense.
3. Under the hypotheses of the Proposition, the function \( \omega_1 \) can be defined as in (8.4). One easily verifies that \( \omega_1 \) solves (8.4).
4. In the definition (8.6c) of the space \( F_r(T) \), the regularities of the harmonic and nonharmonic parts of the source term are different.

**Proof.** The proof is straightforward: The right-hand side of equation (8.8a) clearly belongs to \( L^2(0, T; \bar{V}_0) \) and hence it suffices to apply Proposition 5.10. \( \square \)
From Proposition 6.10 and Equality (8.9), we deduce:

**Corollary 8.2.** The spaces $G_r(T)$ and $F_r(T)$ being equipped with their natural topologies, there exists a positive constant $c_{r,F}$ such that, for every regular vorticity solution to a $\omega-$Stokes problem as defined in Proposition 8.1, the estimate below holds true:

$$\|\omega\|_{V_1(T)} \leq c_{r,F} [\|\omega^1\|_{V_1} + \|(g_n, g_r, \Gamma)\|_{G_r(T)} + \|f_V\|_{F_r(T)}]^{1/2}. \tag{8.10}$$

8.2. Regular Navier-Stokes vorticity solutions.

Setting up the system of equations. To begin with, let us recall the expression (2.27) of the advection term in the NS equation (strong vorticity version). For every $\omega \in V_1$ and every Kirchhoff potential $\varphi \in \mathcal{S}_K$:

$$A_V^0(\omega, \varphi) = -(\omega_r(\nabla^\perp \psi + \nabla \varphi), \nabla Q_1)_{L^2(F)},$$

where $\omega_r$ is the orthogonal projection of $\omega$ on $H^1_0$ (i.e. $\omega_r$ is the regular part of $\omega$; see Remark 4.17) and the stream function $\psi = (-\Delta_1)^{-1} \omega$ belongs to $\mathcal{S}_2$ (see Fig. 7). Assuming now more regularity, namely that $\omega$ is in $V_2$ and $\varphi$ in $\mathcal{S}_K$, an integration by parts yields:

$$(A_V^0(\omega, \varphi), \theta)_{V_1} = -\int_{\Sigma} \omega_r \frac{\partial \varphi}{\partial n} Q_1 \theta \, ds + ((\nabla^\perp \psi + \nabla \varphi) \cdot \nabla \omega_r, Q_1 \theta)_{L^2(F)}$$

for all $\theta \in V_1$,

which leads us to define:

**Definition 8.3.** For every vorticity $\omega$ in $V_2$, we denote by $\psi = \bar{\Delta}^{-1} \omega$ the corresponding stream function and by $(\omega_r, Q_1)_{L^2(F)}$ the orthogonal projection of $\omega$ on $H^1_0$ (i.e. $\omega_r$ is the regular part of $\omega$). For every Kirchhoff potential $\varphi$ in $\mathcal{S}_K$, we define:

$$\gamma_j(\omega, \varphi) = -\int_{\Sigma} \omega_r \frac{\partial \varphi}{\partial n} \, ds \quad (j = 1, \ldots, N) \quad \text{and} \quad A_V^0(\omega, \varphi) = (\nabla^\perp \psi + \nabla \varphi) \cdot \nabla \omega_r \in L^2(F), \tag{8.11a}$$

and the linear form on $V_0$:

$$\tilde{A}_V^0(\omega, \varphi) = \sum_{j=1}^N \gamma_j(\omega, \varphi) \zeta_j + (A_V^0(\omega, \varphi), Q_1)_{L^2(F)}. \tag{8.11b}$$

Let boundary data $(g_n, g_r, \Gamma)$ and a source term $f_V$ be given as in the preceding subsection. Taking now into account the nonlinear advection term, System (8.5) can be rewritten as follows:

$$\begin{align*}
\partial_t \omega_r + \nu A_V^0(\omega_r) &= f_V^0 - P A_V^0(\omega, \varphi) - \partial_t \omega_3b - \sum_{j=1}^N \beta_j^0 \Omega_j \quad \text{in} \ F_T \tag{8.12a} \\
\nu \tilde{A}_V^0(\omega_3b) &= f_V^0 - P^0 A_V^0(\omega, \varphi) - \partial_t \omega_3b \quad \text{in} \ F_T \tag{8.12b} \\
\nu \beta_j &= I_j^r - \alpha_j \quad \text{in} \ (0, T) \text{ for every } j = 1, \ldots, N. \tag{8.12c}
\end{align*}$$

This formulation allows recovering the formulation (2.9) given at the beginning of the paper and that can be rewritten with the notation of this Section:

$$\begin{align*}
\partial_t \omega_r + u \cdot \nabla \omega_r - \nu \Delta \omega_r &= f_V \quad \text{in} \ F_T \tag{8.13a} \\
-\Gamma_j^r + \int_{\Sigma} \omega_r g_n \, ds - \nu \int_{\Sigma} \frac{\partial \omega_r}{\partial n} \, ds &= -\alpha_j \quad \text{on} \ (0, T), \quad j = 1, \ldots, N, \tag{8.13b}
\end{align*}$$

with $u = \nabla^\perp \psi + \nabla \varphi$ and $\psi = \bar{\Delta}^{-1} \omega$. Thus, decomposing $\omega$ in $V_2$ as in (8.3), the regular part of the vorticity $\omega_r$ is given by:

$$\omega_r = \omega_0 + \omega_3b + \sum_{j=1}^N \beta_j \Omega_j + \omega_3^0$$

and $\omega = \omega_r + \sum_{j=1}^N \Gamma_j \zeta_j$.

Summing (8.12a) and (8.12b) gives (8.13a) and (8.13b) is a rephrasing of (8.12c). Indeed, since $\omega_0$, $\omega_3b$ and $\omega_3^0$ have zero mean flux through the inner boundaries, we have for every $k = 1, \ldots, N$:

$$\int_{\Sigma} \frac{\partial \omega_r}{\partial n} \, ds = \sum_{j=1}^N \beta_j \int_{\Sigma} \frac{\partial \Omega_j}{\partial n} \, ds = -\beta_k,$$
assuming that an orthonormal family. Besides this observation, the condition (8.14) can be rephrased in a simpler way: 

\[ \xi_j \in 39, \text{Theorem 3.20} \]

where the family \( (8.15) \) is equivalent to the condition: 

\[ \partial_n (-\Delta) \xi_j \]

where the functions \( \tilde{q}_j \) are a free family in \( F_S \) chosen in such a way that \( \tilde{q}_j = c_j \delta_j^L \) on \( \Sigma_1^* \) with \( c_j \) a normalizing constant ensuring that \( \| \nabla \tilde{q}_j \|_{L^2(F)} = 1 \). In equality (8.14), \( (-\Delta_D)^{-1} \) obviously stands for the inverse of the Laplacian operator with homogeneous Dirichlet boundary conditions on \( \Sigma \). It seems however that the family \( \{ \tilde{q}_j, j = 1, \ldots, N \} \) should be replaced by an orthonormal family (such as the one we have denoted by \( \{ \xi_j, j = 1, \ldots, N \} \)). This remark holds earlier as well, in [52, Theorem 2.1], the proof of which amounts to quoting [39, Theorem 3.20] where the family \( \{ \tilde{q}_j, j = 1, \ldots, N \} \) (with different notation though) is indeed an orthonormal family. Besides this observation, the condition (8.14) can be rephrased in a simpler way:

**Proposition 8.4.** Assuming that \( \omega \) belongs to \( H^1(F) \), condition (8.14) (replacing the functions \( \tilde{q}_j \) by the functions \( \xi_j \)) is equivalent to the condition:

\[ \psi = (-\Delta_D)^{-1} \omega + \sum_{j=1}^N \left( (-\Delta_D)^{-1} \nabla \omega, \nabla \xi_j \right)_{L^2(F)} \xi_j \quad \text{in} \ F. \]

Then \( \psi \) belongs in particular to \( \tilde{S}_1 \) and \( -\Delta \psi = \omega \) in \( F \). Condition (8.14) means that \( \psi \) is in \( S_1 \) and hence that \( \omega \) is in \( V_0 \). Reciprocally, let \( \omega \) be in \( V_0 \cap H^1(F) \) and denote by \( \psi \) the stream function in \( S_1 \) such that \( -\Delta \psi = \omega \). In that case, \( \nabla \psi = (-\Delta_D)^{-1} \nabla \omega \) in \( F \). Decomposing \( \psi \) according to the orthogonal decomposition of the space \( S_0 = H^1_0(F) \oplus F_S \), we obtain exactly the right hand side of (8.16) (and indeed the family \( \{ \xi_j, j = 1, \ldots, N \} \) has to be an orthonormal family at this stage). Therefore \( \psi = \psi \) and (8.14) holds.

Further in [52], the dynamics for the vorticity is claimed to be governed by the system of equations:

\[
\begin{align*}
\partial_t \omega - \nu \Delta \omega + u \cdot \nabla \omega &= 0 \quad \text{in} \ F_T, \\
\omega(0) &= \omega_0 \in V_0,
\end{align*}
\]

with \( u = \nabla \psi \), the stream function being given by the Biot-Savart law (8.16). The classical evolution equation (8.17a) is supplemented with an initial condition:

\[
\omega(0) = \omega_0 \in V_0
\]

and a boundary condition on \( \Sigma_T \) (once again it again seems that the functions \( \tilde{q}_j \) in Maekawa’s paper have to be replaced by the functions \( \xi_j \)):

\[
\nu \left( \frac{\partial \omega}{\partial n} - \Lambda_{DN} \omega + \sum_{j=1}^N \left( \nabla \omega, \nabla \xi_j \right)_{L^2(F)} \frac{\partial \xi_j}{\partial n} \right) = \frac{\partial}{\partial n} \left( (-\Delta_D)^{-1} \nabla \omega \right) + \sum_{j=1}^N (\omega u, \nabla \xi_j)_{L^2(F)} \frac{\partial \xi_j}{\partial n}.
\]

Notice that in the case of a simply connected domain, this condition was already mentioned by Weinan and Jian-Guo in [14], borrowed from an earlier article of Anderson [1].

**Proposition 8.5.** Condition (8.17c) is equivalent for every \( \omega \) solving (8.17a) to:

\[
\partial_t \omega(t) \in V_0 \quad \text{for a.e.} \ t \in (0,T).
\]

**Proof.** As already mentioned in the proof of Lemma 3.1 for every function \( h \) harmonic in \( F \), the function

\[
h_0 = h - \frac{1}{\Sigma} \left( \int \frac{\partial \xi_j}{\partial n} h \, ds \right) \xi_j,
\]

is harmonic and satisfies the integral condition:

\[
\frac{\partial}{\partial n} (-\Delta_D) h + \sum_{j=1}^N \left( (-\Delta_D) \xi_j, \nabla h \right)_{L^2(F)} \xi_j = \frac{\partial}{\partial n} \left( (-\Delta_D)^{-1} \nabla h \right) - \sum_{j=1}^N (\xi_j, h)_{L^2(F)} \xi_j.
\]
is in $\mathcal{D}$. For a.e. $t \in (0, T)$, let us form the scalar product in $L^2(\mathcal{F})$ of (8.17a) with $h_0$. Integrating by parts, we obtain on the one hand:

$$\nu(\Delta \omega, h_0)_{L^2(\mathcal{F})} = \nu \int_{\Sigma} \left\{ - \frac{\partial}{\partial n} (\Delta \omega) + \Lambda_{DN} \omega + \sum_{j=1}^{N} (\nabla \hat{\xi}_j, \nabla \hat{\xi}_j)_{L^2(\mathcal{F})} \frac{\partial \hat{\xi}_j}{\partial n} \right\} h \, ds,$$

and on the other hand, considering the advection term:

$$(u \cdot \nabla \omega, h_0)_{L^2(\mathcal{F})} = \int_{\Sigma} \left\{ - \frac{\partial}{\partial n} (\Delta \omega) - (\Delta_L)^{-1} (u \cdot \nabla \omega) + \sum_{j=1}^{N} (\omega u, \nabla \hat{\xi}_j)_{L^2(\mathcal{F})} \frac{\partial \hat{\xi}_j}{\partial n} \right\} h \, ds.$$

This shows that (8.17c) is indeed equivalent to $(\partial_t \omega(t), h_0)_{L^2(\mathcal{F})} = 0$ for a.e. $t \in (0, T)$ and completes the proof.

Summarizing Propositions 8.4 and 8.5, Maekawa’s System (8.17) turns out to be equivalent to:

(8.18a) $\partial_t \omega - \nu \Delta \omega + u \cdot \nabla \omega = 0$ in $\mathcal{F}_T$,

(8.18b) $\omega(0) = \omega_0 \in V_0$ in $\mathcal{F}$,

(8.18c) $\partial_t \omega \in V_0$ on $(0, T)$.

This seems to contradict the claim of [52, Theorem 2.3] (namely, the equivalence of System (8.17) with the classical NS equations in primitive variables) in a multiply connected domain because Lamb’s fluxes conditions (8.18a) (see [32, Remark 3.2]) on the inner boundaries:

$$\int_{\Sigma_j} \frac{\partial \omega}{\partial n} \, ds = 0 \quad \text{for every } j = 1, \ldots, N,$$

are missing and cannot be figured out from System (8.18) (this is explained in [32, Remark 3.2]). Notice however that the equivalence holds in the particular case of a simply connected fluid domain.

Existence and uniqueness of a global solution. We shall now study the existence of solutions to System (8.12) (or equivalently (8.13)). For simplicity purpose, we restrict our analysis to the case where there is no source term and to homogeneous boundary conditions for the velocity field. The system we consider reads therefore as follows:

(8.19a) $\partial_t \omega_0 + \nu A^V_2 \omega_0 = -P A^V (\omega) - \partial_t \omega_B$ in $\mathcal{F}_T$

(8.19b) $\nu \hat{A}^V_2 \omega_B = -P \hat{A}^V (\omega)$ in $\mathcal{F}_T$,

(8.19c) $\omega(0) = \omega^i$ in $\mathcal{F}$,

with $\omega^i \in V_1$, $\omega = \omega_0 + \omega_B \in V_2 \oplus B^2_V$, $A^V (\omega) = \nabla^\perp \psi \cdot \nabla \omega$ and $\psi = \bar{\Delta}_2^{-1} \omega$. System (8.19) can be rephrased as a more standard Cauchy problem whose unknown is $\omega_0$ (the coupling condition (8.19b) cannot be got rid of though since $\omega$ still appears in the nonlinear advection term):

(8.20a) $\partial_t \omega_0 + \nu A^V_2 \omega_0 = -P A^V (\omega) + \frac{1}{\nu} (A^V_0)^{-1} \partial_t (P \hat{A}^V (\omega))$ in $\mathcal{F}_T$

(8.20b) $\omega_0(0) = \omega^i + \frac{1}{\nu} (A^V_0)^{-1} P \hat{A}^V (\omega^i)$ in $\mathcal{F}$.

The solution $\omega$ to System (8.19) will be looked for in the space:

$$\Omega(T) = [L^2(0, T; H^2_V) \cap H^1(0, T; V_0)] \cap C([0, T]; V_1) \cap C^1([0, T]; V_{-1}).$$

Theorem 8.6. For every positive time $T$ and every initial data $\omega^i \in V_1$, System (8.19) admits a unique solution $\omega$ in $\Omega(T)$. Moreover this solution satisfies the exponential decay estimate:

$$\|\omega(t)\|_{V_1} \leq c_{\mathcal{F}, \nu, \omega} e^{-\frac{\nu}{2} \lambda_T t} \text{ for all } t \in (0, T),$$

where we emphasis that the constant $c_{\mathcal{F}, \nu, \omega}$ does not depend on $T$.

Remark 8.7. It is worth noticing that:
(1) In [52], the author shows local in time existence for the same system, considering the particular case where the fluid domain \( F \) is a half-plane.

(2) The quantity \( \| \nabla \omega \|_{L^2(F)}^2 \) is sometimes called the palinstrophy. The palinstrophy being controlled by \( \| \omega \|_{V_1}^2 \), estimates \( \| \omega \|_{V_1}^2 \) asserts that the palinstrophy is exponentially decreasing as time grows. This result was not known so far and may play an important role in turbulence theory.

We begin with establishing the \textit{a priori} estimate \( (8.21) \).

**Lemma 8.8.** For every initial condition \( \omega^0 \in V_1 \), there exists a positive constant \( c_{[F,\nu,\omega]} \) such that for every positive time \( T \), any solution \( \omega \) to System \( (8.19) \) in \( \Omega(T) \) satisfies estimate \( (8.21) \).

**Proof.** The proof is divided in several steps:

**First step:** As being a weak solution to the \( \omega-\text{NS} \) equations, we can apply Corollary [6.8] which provides us with the following estimates, satisfied for every \( t \) in \((0,T)\):

\[
(8.22a) \quad \| \omega(t) \|_{V_1} \leq \| \omega \|_{V_1} e^{-\nu \lambda_1 t} \quad \text{and} \quad \int_0^t \| \omega(s) \|_{V_0}^2 \, ds \leq \frac{1}{2\nu} \| \omega \|_{V_1}^2.
\]

Arguing that \( \omega \) is also a strong solution to the \( \omega-\text{NS} \) equation, we obtain that:

\[
\frac{1}{2} \frac{d}{dt} \| \omega(t) \|_{V_0}^2 + \nu \| \omega \|_{V_1}^2 \leq c_{F} \| \omega \|_{V_1}^2 \| \omega \|_{V_0} \| \omega \|_{V_0} \| \omega \|_{V_1}^2 \leq \frac{\nu}{2} \| \omega \|_{V_0}^2 + \frac{c_{F}}{\nu} \| \omega \|_{V_1}^2 \| \omega \|_{V_0}^4,
\]

that is

\[
\frac{d}{dt} \| \omega(t) \|_{V_0}^2 + \nu \| \omega \|_{V_1}^2 \leq c_{F} \| \omega \|_{V_1}^2 \| \omega \|_{V_0}^4,
\]

which leads us to the estimates:

\[
(8.22b) \quad \| \omega(t) \|_{V_0} \leq \| \omega \|_{V_0} e^{-\nu \lambda_1 t} \quad \text{with} \quad E_{[F,\nu,\omega]} = \exp \left( \frac{c_{F}}{\nu} \| \omega \|_{V_1}^2 \right),
\]

and also:

\[
(8.22c) \quad \int_0^t \| \omega(s) \|_{V_0}^2 \, ds \leq \frac{1}{\nu} \| \omega \|_{V_0}^2 \left[ 1 + \frac{c_{F}}{\nu} \| \omega \|_{V_1}^4 \right].
\]

**Second step:** We need now to estimate \( \| \omega(t) \|_{V_1} \) in term of \( \| \omega_0(t) \|_{V_1} \) and \( \| \omega(t) \|_{V_2} \) in term of \( \| \omega_0(t) \|_{V_2} \) (and possibly some lower order terms).

Starting from the expression \( (8.19b) \) and forming for a.e. \( t \) in \((0,T)\) the duality pairing with \( \omega_B(t) \), we obtain:

\[
(8.23) \quad \nu \| \omega_B(t) \|_{V_1}^2 \leq \langle P^{-1} \Lambda_1^V(\omega(t)), \omega_B(t) \rangle_{V_1} \leq c_{F}\|P^{-1} \Lambda_1^V(\omega(t))\|_{L^1(V_1)} \| \omega_B(t) \|_{V_1}.
\]

Introducing the stream function \( \psi = \Delta^{-1} \omega \), we have for every \( \theta \in V_1 \):

\[
\langle P^{-1} \Lambda_1^V(\omega(t)), \theta \rangle_{V_1} = \langle P^{-1} (\nabla \psi \cdot \nabla \omega(t)), Q_1 \theta \rangle_{L^2(F)} = -\langle \nabla \psi \cdot \nabla \omega(t), Q_1 \theta \rangle_{L^2(F)} = \langle \omega \nabla \psi, \nabla Q_1 \theta \rangle_{L^2(F)},
\]

the latter expression resting on the equalities \( P^{-1} Q_1 = (\text{Id} - P) Q_1 = Q_1 - \text{Id} = -Q_1^+ \). We can deduce first that:

\[
(8.24) \quad \| P^{-1} \Lambda_1^V(\omega(t)) \|_{V_1} \leq c_{F}\| \omega(t) \|_{L^1(V_1)} \| \nabla \psi(t) \|_{L^1(F)} \| \omega(t) \|_{V_1} \| \omega(t) \|_{V_2}.
\]

and next, combining the inequality above with \( (8.23) \), that:

\[
\| \omega_B(t) \|_{V_1} \leq \frac{c_{F}}{\nu} \| \omega(t) \|_{V_1} \| \omega(t) \|_{V_1} \leq \frac{c_{F}}{\nu} \| \omega(t) \|_{V_1} \| \omega(t) \|_{V_1} \| \omega(t) \|_{V_0}.
\]

Since \( \| \omega(t) \|_{V_1} \leq \| \omega_B(t) \|_{V_1} + \| \omega_0(t) \|_{V_1} \), we obtain, using Young’s inequality:

\[
(8.25a) \quad \| \omega(t) \|_{V_1} \leq c \| \omega_0(t) \|_{V_1} + \frac{c_{F}}{\nu} \| \omega(t) \|_{V_1} \| \omega(t) \|_{V_0}.
\]

We are done with the term \( \| \omega(t) \|_{V_1} \), so let us turn our attention to \( \| \omega(t) \|_{V_2} \). Forming, for a.e. \( t \) in \((0,T)\), the scalar product of \( \overset{\circ}{\Lambda}_1^V \omega_B(t) \) in \( V_0 \), we obtain (using Hölder's inequality followed by interpolation inequalities):

\[
\nu \| \omega_B(t) \|_{V_2}^2 = \langle \nabla \psi \cdot \nabla \omega(t), \Delta \omega_B(t) \rangle_{L^2(F)} \leq c_{F} \| \omega(t) \|_{V_1} \| \omega(t) \|_{V_0} \| \omega(t) \|_{V_1} \| \omega(t) \|_{V_0} \| \omega_B(t) \|_{V_2},
\]
that is to say:
\[ \|\omega_2(t)\|_{V_2} \leq \frac{c_F}{\nu} \|\omega(t)\|_{V_{-1}}^{\frac{1}{2}} \|\omega(t)\|_{V_0}^{\frac{1}{2}} \|\omega(t)\|_{V_1} \|\omega(t)\|_{V_2}^{\frac{1}{2}}. \]

But \( \|\omega(t)\|_{V_2}^2 = \|\omega_0(t)\|_{V_2}^2 + \|\omega_3(t)\|_{V_2}^2 \) which, by Young's inequality yields:

\[ (8.25b) \quad \|\omega(t)\|_{V_2} \leq c \|\omega_0(t)\|_{V_2} + \frac{c_F}{\nu} \|\omega(t)\|_{V_{-1}} \|\omega(t)\|_{V_0} \|\omega(t)\|_{V_1}. \]

Our goal for this step is now achieved.

**Third step:** We form for a.e. \( t \in (0, T) \), the scalar product of \( A_Y^0 \omega_0(t) \) in \( V_0 \) to obtain:

\[ (8.26) \quad \frac{1}{2} \frac{d}{dt} \|\omega(t)\|_{V_1}^2 + \nu \|\omega_0(t)\|_{V_2}^2 = -(P A_Y^0(\omega)(t), A_Y^0 \omega_0(t))_{V_0} + \frac{1}{\nu} (\|A_Y^0\|^{-1} [\partial_t P^A Y^0(\omega)(t)], A_Y^0 \omega_0(t))_{V_0}. \]

Both terms in the right hand side have to be estimated, this task being easier for the first one than for the second one. Indeed, the first term can be rewritten as:

\[ (P A_Y^0(\omega)(t), A_Y^0 \omega_0(t))_{V_0} = (V \cdot \nabla \omega(t), \Delta \omega(t))_{L^2(F)}, \]

where we recall that the definition of \( \|\omega(t)\|_{V_{-1}} \) is given in (6.27). This equality provides us with the inequality:

\[ \|\omega(t)\|_{V_{-1}} \leq c_f \|\omega(t)\|_{V_0}^{\frac{1}{2}} \|\omega(t)\|_{V_1} \|\omega(t)\|_{V_2}^{\frac{1}{2}} \|\omega(t)\|_{V_2}^{\frac{1}{2}} \|\omega_0(t)\|_{V_2}. \]

and we are now done with the first nonlinear term.

**Fourth step:** The second term in the right hand side of \( (8.26) \) can be turned into:

\[ (\|A_Y^0\|^{-1} [\partial_t P^A Y^0(\omega)(t)], A_Y^0 \omega_0(t))_{V_0} = (\partial_t P^A Y^0(\omega)(t), A_Y^0 \omega_0(t))_{V_{-2}} = (\partial_t P^A Y^0(\omega)(t), \omega_0(t))_{V_{-2}}, \]

and therefore:

\[ (8.28) \quad \|\omega(t)\|_{V_{-2}} \leq c \|\nabla \psi(\omega(t))\|_{L^4(F)} \|\nabla \partial_t \psi(\omega(t))\|_{L^4(F)} \leq c_f \|\omega(t)\|_{V_0}^{\frac{1}{2}} \|\partial_t \omega(t)\|_{V_0}^{\frac{1}{2}} \|\partial_t \omega(t)\|_{V_0}^{\frac{1}{2}}, \]

and we need now to estimate both terms involving a time derivative. As being a strong solution to the \( \omega-\)NS equation, \( \omega(t) \) satisfies for a.e. \( t \) in \( (0, T) \) the identity below, set in \( V_{-1} \):

\[ \partial_t \omega(t) = -\nu A_Y^0 \omega(t) - A_Y^0 (\omega(t), 0), \]

where we recall that the definition of \( A_Y^0 \) is given in (6.27). This equality provides us with the inequality:

\[ \|\partial_t \omega(t)\|_{V_{-1}} \leq \nu \|\omega(t)\|_{V_1} + \|A_Y^0 (\omega(t), 0)\|_{V_{-1}}. \]

Resting on the definition \( (6.27) \), we next easily obtain that:

\[ \|A_Y^0(\omega(t), 0)\|_{V_{-1}} \leq \|\nabla \psi(\omega(t))\|_{L^4(F)} \|\omega(t)\|_{L^4(F)} \leq c_f \|\omega(t)\|_{V_0} \|\omega(t)\|_{V_1}^{\frac{1}{2}}, \]

and therefore:

\[ (8.30a) \quad \|\partial_t \omega(t)\|_{V_{-1}} \leq \nu \|\omega(t)\|_{V_1} + c_f \|\omega(t)\|_{V_0} \|\omega(t)\|_{V_1}^{\frac{1}{2}}. \]

On the other hand, since by hypothesis \( \omega \) is a solution on \( (0, T) \) to System \( (8.19) \), it satisfies for a.e. \( t \in (0, T) \):

\[ \partial_t \omega(t) = \nu \Delta \omega(t) - \nabla^2 \psi(t) \cdot \nabla \omega(t) \]

in \( V_0 \), whence we deduce that:

\[ \|\partial_t \omega(t)\|_{V_0} \leq \nu \|\omega(t)\|_{V_2} + c_f \|\omega(t)\|_{V_0} \|\omega(t)\|_{V_1} \|\omega(t)\|_{V_2}^{\frac{1}{2}}. \]
Once combined with (8.25b), this estimate becomes:

\[(8.30b) \quad \|\partial_t \omega(t)\|_{V_0} \leq c \nu \|\omega_0\|_{V_2} + \frac{c_F}{\nu} \|\omega(t)\|_{V^{-1}} \|\omega(t)\|_{V_1} + c_F \|\omega(t)\|_{V_1} \|\omega(t)\|_{V_2} + \frac{c_F}{\nu} \|\omega(t)\|_{V_1} \|\omega(t)\|_{V_2} \|
\]

Gathering now (8.28) and identities (8.30) we finally obtain the following estimate for the second nonlinear term in (8.26):

\[(8.31) \quad |(A_Y^\nu)(t) + \partial_t P \Delta Y^\nu(t)| \leq c_F \|\omega(t)\|_{V_1} \|\omega(t)\|_{V_2} + \frac{c_F}{\nu} \|\omega(t)\|_{V_1} \|\omega(t)\|_{V_2} + \frac{1}{\nu} \|\omega(t)\|_{V_1} \|\omega(t)\|_{V_2} \|
\]

Both terms in the right hand side of (8.26) have now be estimated. Let us collect all the estimates obtained so far and move on to the next step consisting in applying Grönwall’s inequality.

**Fifth step:** Combining (8.27) and (8.31) with (8.26) and using craftily Young’s inequality several times, we deduce that for a.e. \(t \in (0, T)\):

\[(8.32) \quad \frac{1}{2} \frac{d}{dt} \|\omega(t)\|_{V_1}^2 + \nu \|\omega_0(t)\|_{V_2}^2 \leq \frac{\nu}{2} \|\omega(t)\|_{V_2}^2 + \frac{c_F}{\nu} \|\omega(t)\|_{V_1} \|\omega(t)\|_{V_2} + \frac{c_F}{\nu} \|\omega(t)\|_{V_1} \|\omega(t)\|_{V_2}.
\]

Applying Grönwall’s inequality to (8.32), we obtain:

\[(8.33) \quad e^{\nu \lambda_F t} \|\omega(t)\|_{V_1}^2 \leq \|\omega_0\|_{V_1}^2 + \int_0^t \left[ \frac{c_F}{\nu} \|\omega(s)\|_{V_1} \|\omega(s)\|_{V_0} + \frac{c_F}{\nu} \|\omega(s)\|_{V_1} \|\omega(s)\|_{V_0} \right] e^{\nu \lambda_F s} \, ds,
\]

where, according to (8.20b), the initial data \(\omega_0\) is defined by:

\[\omega_0 = \omega + \frac{1}{\nu} \left( A_Y^{\nu} \right)^{-1} P \Delta Y^{\nu}(\omega).
\]

With (8.24), we deduce that:

\[(8.34a) \quad \|\omega_0\|_{V_1} \leq \|\omega\|_{V_1} + \frac{c_F}{\nu} \|\omega\|_{V_1} \|\omega\|_{V_2} \leq c \|\omega\|_{V_1} + \frac{c_F}{\nu} \|\omega\|_{V_1} \|\omega\|_{V_2}.
\]

The second term in the right-hand side of (8.33) can be estimated using the estimates (8.22) as follows:

\[(8.34b) \quad \int_0^t \left[ \frac{c_F}{\nu} \|\omega(s)\|_{V_1} \|\omega(s)\|_{V_0} e^{\nu \lambda_F s} \right. \left. \right] \, ds \leq \frac{c_F}{\nu} \|\omega(s)\|_{V_1} \|\omega(s)\|_{V_0} e^{\nu \lambda_F s} \, ds \leq \frac{c_F}{\nu} \|\omega(s)\|_{V_1} \|\omega(s)\|_{V_0} \left[ 1 + \frac{c_F}{\nu} \|\omega(s)\|_{V_1} \|\omega(s)\|_{V_0} \right] ;
\]

and

\[(8.34c) \quad \int_0^t \frac{c_F}{\nu} \|\omega(s)\|_{V_1} \|\omega(s)\|_{V_0} \, ds \leq \frac{c_F}{\nu} \|\omega(s)\|_{V_1} \|\omega(s)\|_{V_0} \left[ 1 + \frac{c_F}{\nu} \|\omega(s)\|_{V_1} \|\omega(s)\|_{V_0} \right] .
\]

We finally obtain, gathering (8.33) and inequalities (8.34) (using again Young’s inequality):

\[(8.35) \quad \|\omega(t)\|_{V_1} \leq c_F \left[ \|\omega\|_{V_1} + \frac{1}{\nu} \|\omega\|_{V_1} \|\omega\|_{V_2} \right] \left[ 1 + \frac{1}{\nu} \|\omega\|_{V_1} \|\omega\|_{V_2} \right] e^{-\nu \lambda_F t},
\]

what, with (8.25a) and estimates (8.22), completes the proof of the lemma.
The rest of the section is devoted to the proof of the theorem, which is classical and based again on a fixed point argument. Let us fix $T > 0$ and $\omega^i \in V_1$ and introduce the spaces:

$$\Omega(T, \omega^i) = \{ \omega \in \Omega(T) : \omega(0) = \omega^i \},$$

$$\Psi(T) = [L^2(0, T; \tilde{S}_1) \cap H^1(0, T; S_1)] \cap [C([0, T]; S_2) \cap C^1([0, T]; S_0)],$$

and

$$F(T) = \{ f_V \in L^2(0, T; L^2_{\overline{\Omega}}) \cap C([0, T]; V_{-1}) : P^1 f_V \in H^1(0, T; V_{-2}) \}.$$ 

Then define the mapping $X_T : f_V \in F(T) \mapsto \omega \in \Omega(T, \omega^i)$ where $\omega = \omega_0 + \omega_3$ is the solution to the $\omega$-Stokes problem:

$$\begin{align*}
\partial_t \omega_0 + \nu A_2^\omega \omega_0 &= P f_V - \partial_t \omega_3 \quad \text{in } \mathcal{F}_T, \\
\nu A_2^\omega \omega_3 &= P^1 f_V \quad \text{in } \mathcal{F}_T, \\
\omega(0) &= \omega^i \quad \text{in } \mathcal{F}_T.
\end{align*}$$

and $Y_T : \omega \in \Omega(T, \omega^i) \mapsto A^\omega_T (\omega) \in F(T)$ where $A^\omega_T (\omega)$ is defined in (8.11a) (with $\varphi = 0$ since, as already mentioned, we consider only homogeneous boundary conditions).

**Lemma 8.9.** The mapping $X_T$ is well-defined and there exists a positive constant $\epsilon_{[\mathcal{F}_i, \nu]}$ such that:

$$\begin{align*}
\|X_T(f_V)\|_{\Omega(T)} &\leq \epsilon_{[\mathcal{F}_i, \nu]} \left[ \|\omega\|_{V_1}^2 + \|f_V\|_{F(T)}^2 \right]^{\frac{1}{2}} \quad \text{for all } f_V \in F(T). \\
\|Y_T(\omega_2) - Y_T(\omega_1)\|_{F(T)} &\leq \epsilon_{[\mathcal{F}_i, \nu]} T^{\frac{1}{2}} \left[ \|\omega_1\|_{\Omega(T)}^2 + \|\omega_2\|_{\Omega(T)}^2 \right]^{\frac{1}{2}} \|\omega_2 - \omega_1\|_{\Omega(T)},
\end{align*}$$

**Proof.** The mapping $X_T$ is well-defined from the space $F_1(T)$ (defined in (8.6c)) into $\bar{V}_1(T)$ according to Proposition 8.1 and following Proposition 8.1, there exists a positive constant $\epsilon_{[\mathcal{F}_i, \nu]}$ such that:

$$\|\omega\|_{\bar{V}_1(T)} \leq \epsilon_{[\mathcal{F}_i, \nu]} \left[ \|\omega\|_{V_1}^2 + \|f_V\|_{L^2(0, T; L^2_{\overline{\Omega}})}^2 + \|\partial_t f_V\|_{L^2(0, T; V_{-2})}^2 \right]^{\frac{1}{2}}.$$

However, comparing with Proposition 8.1, the source term $f_V$ is assumed herein to satisfy the extra hypothesis $f_V \in C([0, T]; V_{-1})$ (and not only $P^1 f_V \in C([0, T]; V_{-1})$). We recall that every solution to the $\omega$-Stokes problem (8.36) satisfies also:

$$\partial_t \omega = -A^\omega_T \omega + f_V \quad \text{in } \mathcal{F}_T.$$

Since $\omega$ belongs in particular to $C([0, T], V_1)$, we infer that $\partial_t \omega$ is in $C([0, T]; V_{-1})$ and finally that there exists a constant $\epsilon_{[\mathcal{F}_i, \nu]}$ such that (8.37a) holds.

For every $\theta \in V_1$, we have by definition:

$$\begin{align*}
\langle A^\omega_T (\omega), \theta \rangle_{V_{-1}, V_1} &= (\nabla \cdot \psi, \nabla \omega, Q_1 \theta)_{L^2(\mathcal{F})} = -(\omega \nabla \psi, \nabla (Q_1 \theta))_{L^2(\mathcal{F})}, \\
\langle P_1 A^\omega_T (\omega), \theta \rangle_{V_{-1}, V_1} &= (P_1 (\nabla \cdot \psi, \nabla \omega), Q_1 \theta)_{L^2(\mathcal{F})} = -(\nabla \cdot \psi, \nabla \omega, Q_1 \theta)_{L^2(\mathcal{F})},
\end{align*}$$

the latter expression resting on the equalities $P^1 Q_1 = (\text{Id} - P)Q_1 = Q_1 - \text{Id} = -Q_1$. Assuming now that $\theta$ belongs to $V_2$, the right hand side in (8.38b) can be integrated by parts twice to obtain:

$$\nabla \cdot \psi, \nabla \omega, Q_1 \theta)_{L^2(\mathcal{F})} = (D^2(Q_1 \theta) \nabla \psi, \nabla \psi)_{L^2(\mathcal{F})},$$

whence it can be deduced in particular that:

$$\begin{align*}
\langle \partial_t (P_1 A^\omega_T (\omega)), \theta \rangle_{V_{-2}, V_2} &= (D^2(Q_1 \theta) \nabla \partial_t \psi, \nabla \psi)_{L^2(\mathcal{F})} + (D^2(Q_1 \theta) \nabla \psi, \nabla \partial_t \psi)_{L^2(\mathcal{F})},
\end{align*}$$

the same arguments as those used in the proof of Equality (6.8b) yield:

$$\|\nabla \cdot \psi, \nabla \omega, Q_1 \theta)_{L^2(0, T; L^2(\mathcal{F}))} \leq \epsilon_{[\mathcal{F}_i, \nu]} T^{\frac{1}{2}} \|\psi\|_{C([0, T]; V_1)} \|\partial_t \psi\|_{C([0, T]; S_1)} \|\partial_t \psi\|_{L^2(0, T; H^2_{\overline{\Omega}})},$$

which entails that:

$$\|A^\omega_T (\omega_2) - A^\omega_T (\omega_1)\|_{L^2(0, T; L^2_{\overline{\Omega}})} \leq \epsilon_{[\mathcal{F}_i, \nu]} T^{\frac{1}{2}} \left[ \|\omega_1\|_{\Omega(T)}^2 + \|\omega_2\|_{\Omega(T)}^2 \right]^{\frac{1}{2}} \|\omega_2 - \omega_1\|_{\Omega(T)}.$$
Considering now the expression (8.38a), we first easily obtain:

\[(8.40a) \| \omega_2 \nabla \psi_2 - \omega_1 \nabla \psi_1 \|_{L^1(T)} \leq c_f \| \omega_2 - \omega_1 \|_{L^1(T)}^\frac{1}{2} \| \omega_2 - \omega_1 \|_{L^1(T)}^\frac{3}{2} \|
abla \psi_2 \|_{L^1(T)} + \| \omega_1 \|_{L^1(T)} \|
abla (\psi_2 - \psi_1) \|_{L^1(T)}.
\]

On the one hand, since \( \omega_1 \) and \( \omega_2 \) share the same initial value, we are allowed to write that:

\[(8.40b) \| \omega_2 - \omega_1 \|_{C([0,T];V_0)} \leq T^\frac{1}{2} \| \partial_t \omega_2 - \partial_t \omega_1 \|_{L^2(0,T;V_0)}.
\]

On the other hand, Sobolev embedding theorem ensures that:

\[(8.40c) \| \omega_2 - \omega_1 \|_{L^1(T)} \leq c_f \| \omega_2 - \omega_1 \|_{C([0,T];V_1)}
\]

\[(8.40d) \| \nabla \psi_2 \|_{C([0,T];L^2(T))} \leq c_f \| \omega_2 \|_{C([0,T];V_0)}
\]

The second term in the right hand side of (8.40) is estimated in a similar manner, thus:

\[(8.40e) \| \nabla (\psi_2 - \psi_1) \|_{C([0,T];L^2(T))} \leq c_f T^\frac{1}{2} \|
abla \partial_t \omega_2 - \nabla \partial_t \omega_1 \|_{L^2(0,T;V_0)}.
\]

Assuming that \( T < 1 \), the estimates (8.40) give rise to:

\[(8.41) \| A^V_1 (\omega_2) - A^V_1 (\omega_1) \|_{C([0,T];V_{-1})} \leq c_f T^\frac{1}{2} \left( \| \omega_1 \|_{L^1(T)}^2 + \| \omega_2 \|_{L^1(T)}^2 \right)^\frac{1}{2} \| \omega_2 - \omega_1 \|_{L^1(T)}.
\]

We turn now our attention to the right hand side of (8.38c). On the one hand, we obtain that:

\[(8.42a) \| \nabla (\partial_t \psi_2 - \partial_t \psi_1) \|_{L^2(0,T;L^2(T))} \leq c_f T^\frac{1}{2} \| \partial_t \omega_2 - \partial_t \omega_1 \|_{C([0,T];V_{-1})} \| \omega_2 \|_{C([0,T];V_0)} \| \partial_t \omega_2 - \partial_t \omega_1 \|_{L^2(0,T;V_0)}.
\]

On the other hand, using again (8.40b):

\[(8.42b) \| \nabla \partial_t \psi_1 || \nabla (\psi_2 - \psi_1) ||_{L^2(0,T;L^2(T))} \leq c_f T^\frac{1}{2} \| \partial_t \omega_1 \|_{L^2(0,T;V_0)} \| \partial_t \omega_2 - \partial_t \omega_1 \|_{L^2(0,T;V_0)}.
\]

Providing again that \( T < 1 \), both estimates (8.42) yield:

\[(8.43) \| \partial_t (A^V_1 (\omega_2)) - \partial_t (A^V_1 (\omega_1)) \|_{L^2(0,T;V_{-2})} \leq c_f T^\frac{1}{2} \left( \| \omega_1 \|_{L^1(T)}^2 + \| \omega_2 \|_{L^1(T)}^2 \right)^\frac{1}{2} \| \omega_2 - \omega_1 \|_{L^1(T)}.
\]

Estimate (8.37b) derives now straightforwardly from (8.39), (8.41) and (8.43). This completes the proof. \( \square \)

**Proof of Theorem 8.6** Define the mapping \( Z_T : f_V \in F(T) \mapsto Y_T \circ X_T (f_V) \in F(T) \) et let \( f_V^* = Z_T (0) \). Then, according to the estimates of Lemma 8.9,

\[(8.49) \| Z_T (f_V) - f_V^* \|_{F(T)} \leq c_{[F,\nu]} T^\frac{1}{2} \left( \| f_V \|_{F(T)}^2 + \| \omega_1^0 \|_{V_1}^2 \right)
\]

for every \( f_V^1, f_V^2 \) in \( F(T) \) and \( T < 1 \). For every \( R > 0 \), there exists a time \( T^* < 1 \) (depending only on \( F, \nu, \| \omega_1^0 \|_{V_1} \) and \( R \) such that \( Z_T^* \) is a contraction from \( B(f_V^1, R) \subset F(T^*) \) into \( B(f_V^2, R) \). From Banach fixed point theorem, the mapping \( Z_T^* \) admits a unique fixed point in \( B(f_V^1, R) \), the image of which by the mapping \( X_T^* \) is a solution to System (8.19) on \([0,T^*] \). We conclude that \( T^* \) can be chosen arbitrarily large following the lines of the proof of Theorem 6.11 using the estimate of Lemma 8.8. Finally, every solution is also a strong solution in the sense of Definition 6.13 which was proved to be unique. \( \square \)

9. **Concluding remarks**

By introducing a suitable functional framework, the 2D vorticity equation has been shown to be not a classical parabolic equation but rather a parabolic-elliptic coupling. Indeed, applying the harmonic Bergman projection to the equation \( \partial_t \omega - \nu \Delta \omega + u \cdot \nabla \omega = 0 \) leads to its splitting into, on the one hand, an evolution diffusion-advection equation for the non-harmonic part of \( \omega \) (equation (8.36a)) and on the other hand a (steady) elliptic equation for the remaining harmonic part (equation (8.36b)). By exploiting this structure of the equation, we were able to prove the exponential decay of the palinstrophy for large time, a result which was not known so far. In this work, it is worth noticing the surprising role played by the circulation in this context, circulation being well known for entering the analysis of perfect fluids but usually less came across in
the context of viscous fluids. The other point that deserves to be highlighted is the simple form taken by the Biot-Savart operator, described in Theorem [3.20]

In a forthcoming work, we shall apply our method to fluid-structure problems by considering a set of disks, pinned at their centers but free to rotate, immersed in a viscous fluid. The equations governing the coupled fluid-rotating disks system can be stated in terms of the vorticity of the fluid and the angular velocities of the disks only. The analysis of these equations will obviously be carried out in nonprimitive variables.

APPENDIX A. GELFAND TRIPLE

A.1. General settings. Let \( H_1 \) and \( H_0 \) be two Hilbert spaces. Their scalar products are denoted respectively by \( \langle \cdot , \cdot \rangle_1 \) and \( \langle \cdot , \cdot \rangle_0 \) and their norms by \( \| \cdot \|_1 \) and \( \| \cdot \|_0 \). We assume that:

\[
(A.1) \quad H_1 \subset H_0,
\]

where the inclusion is continuous and dense. Applying Riesz representation theorem, the space \( H_0 \) is identified with its dual \( H_0' \). It means that, for every \( u \in H_0 \), the linear form \( \langle \cdot , u \rangle_0 \) is identified with \( u \). The space \( H_0 \) is usually referred to as the pivot space. Therefore, the space \( H_1 \) cannot be identified with its dual \( H_{-1} \) but with a subspace of \( H_0 \). Thus, the configuration

\[
(A.2) \quad H_1 \subset H_0 \subset H_{-1},
\]

is called (with a slight abuse of terminology) Gelfand triple. The inclusions are both continuous and dense.

We define the operator

\[
(A.3) \quad A_1 : H_1 \rightarrow H_{-1}, \quad A_1u = (u, \cdot)_1, \quad \text{for all } u \in H_1,
\]

and it can be readily verified that \( A_1 \) is an isometry. Then, we define the space \( H_2 = A_1^{-1}H_0 \) and the operator \( A_2 : H_2 \rightarrow H_0 \) by setting, for every \( u \in H_2 \):

\[
(A.4) \quad (A_2u, \cdot)_0 = A_1u \quad \text{in } H_{-1}.
\]

We equip the space \( H_2 \) with the scalar product:

\[
(u,v)_2 = (A_2u, A_2v)_0, \quad \text{for all } u, v \in H_2,
\]

and the corresponding norm \( \| \cdot \|_2 \).

Lemma A.1. The space \( H_2 \) is a Hilbert space, the operator \( A_2 \) is an isometry and the inclusion \( H_2 \subset H_1 \) is continuous and dense. It entails that the inclusion \( H_{-1} \subset H_{-2} \), where \( H_{-2} \) stands for the dual space of \( H_{-2} \), is continuous and dense as well.

Proof. The estimate below is satisfied by every \( u \in H_2 \):

\[
(A.5) \quad \|u\|_0 \leq c\|A_2u\|_0.
\]

Indeed, the continuity of the inclusion \( (A.1) \) yields \( \|u\|_0 \leq c\|u\|_1 \). Then, the definitions of both the operator \( A_2 \) and the space \( H_2 \) lead to the identity \( \|u\|_2^2 = (A_2u, u)_0 \). Applying Cauchy-Schwarz inequality, we obtain \( (A.5) \).

Let assume that \((u_n)_{n \geq 0}\) is Cauchy sequence in \( H_2 \), or equivalently that \((A_2u_n)_{n \geq 0}\) is a Cauchy sequence in \( H_0 \), and denote by \( v^* \) the limit of \((A_2u_n)_{n \geq 0}\) in \( H_0 \). According to \( (A.5) \), we deduce that \((u_n)_{n \geq 0}\) is a Cauchy sequence in \( H_0 \) as well. The equality:

\[
(A_2u_n - A_2u_m, u_n - u_m)_0 = \|u_n - u_m\|_1^2,
\]

available for every pair of indices \( n \) and \( m \), entails that the sequence \((u_n)_{n \geq 0}\) is also a Cauchy sequence in \( H_1 \). We denote by \( u^* \) its limit in this space. Letting \( n \) goes to \( \infty \) in the identity:

\[
(A_2u_n, \cdot)_0 = A_1u_n \quad \text{in } H_{-1},
\]

we obtain:

\[
(u^*, \cdot)_0 = A_1u^*,
\]

and therefore \( u^* \) belongs to \( H_2 \). This proves that \( H_2 \) is complete and hence is a Hilbert space.

Let now \( v \) be in \( H_2 \) in \( H_1 \). There exists \( u \in H_2 \) such that \( A_2u = v \) and:

\[
\|v\|_2^2 = (A_2u, v)_0 = (u, v)_1 = 0.
\]
It follows that $H^2 \subset H^1$ and therefore $H_2$ is dense in $H_1$.

The continuity of the inclusion $H_2 \subset H_1$ results from the identity:
\[ \|a\|_1^2 = (A_2a, a)_0, \quad \text{for all } a \in H_2, \]
combined with Cauchy-Schwarz inequality:
\[ (A_2a, a)_0 \leq \|A_2a\|_0 \|a\|_0, \quad \text{for all } a \in H_2, \]
and estimate (A.5).

Finally, the operator $A_2$ is onto by definition and it is also injective because the identity $A_2a = 0$ for some $a \in H_2$ leads to $(A_2a, a)_0 = \|a\|^2_0 = 0$. The proof of the lemma is now completed.

Let us define the operator $A_0 : H_0 \to H_{-2}$ by:
\[ (A.6) \quad A_0 : u \in H_0 \mapsto (A_{-2}, u)_0 \in H_{-2}. \]

**Lemma A.2.** The operator $A_0$ is an isometry.

**Proof.** The operator $A_0$ is injective. Indeed, the identity $A_0a = 0$ for some $a \in H_0$ entails that $(A_2A^{-1}_2a, a)_0 = \|a\|^2_0 = 0$. The operator $A_0$ is also onto: Any element of $H_{-2}$ can be written, according to Riesz theorem, as $(\cdot, v)_2 = (A_2, A_2v)_0$ for some $v \in H_2$, and hence it is equal to $A_0a$ with $a = A_1v \in H_0$.

Finally, the operator $A_0$ is also an isometry since we have, for every $u \in H_0$:
\[ \|A_0u\|_{-2} = \sup_{v \neq 0} \frac{|(A_2v, u)_0|}{\|v\|_2} = \sup_{v \neq 0} \frac{|(A_2v, u)_0|}{\|A_2v\|_0} = \sup_{w \neq 0} \frac{|(w, u)_0|}{\|w\|_0} = \|u\|_0, \]
and the proof is completed. 

So far, we have proved that in the chain of inclusions:
\[ H_2 \subset H_1 \subset H_0 \subset H_{-1} \subset H_{-2}, \]
every inclusion is continuous and dense and that the operators $A_k : H_k \to H_{k-2}$ for $k = 0, 1, 2$ are isometries.

By induction, we can next define $H_{k+2} = A_{k+1}^{-1}H_k$ for every positive integer $k$. The operator
\[ A_{k+2} : H_{k+2} \to H_k \]
is defined from the operator $A_{k+1}$ by setting $A_{k+2}a = A_{k+1}a$ for every $a \in H_{k+2}$. The spaces $H_{k+2}$ are Hilbert spaces once equipped with the scalar products:
\[ (u, v)_{k+2} = (A_{k+2}u, A_{k+2}v)_{H_k}, \quad \text{for all } u, v \in H_{k+2}. \]

For every $k \geq 1$, the dual space of $H_k$ is denoted by $H_{-k}$ and we introduce the operator
\[ A_{-k} : H_{-k} \to H_{-k-2}, \]
defined by duality as follows:
\[ (A.7) \quad A_{-k}u = (u, A_{k+2})_{-k,k} \in H_{-k-2}, \quad \text{for all } u \in H_{-k}. \]
It can be readily verified that the Hilbert spaces $H_k$ ($k \in \mathbb{Z}$) satisfy:
\[ \ldots \subset H_{k+1} \subset H_k \subset H_{k-1} \subset \ldots \subset H_1 \subset H_0 \subset H_{-1} \subset \ldots \subset H_{-k+1} \subset H_{-k} \subset H_{-k-1} \subset \ldots \]
each inclusion being continuous and dense. Furthermore, for every integer $k$, the operator:
\[ A_k : H_k \to H_{k-2}, \]
is an isometry.

**Lemma A.3.** For every integers $n, n'$ such that $n' \leq n$ and for every $u \in H_n$, the following equality holds:
\[ (A.8) \quad A_nu = A_{n'}u. \]
Proof. For \( n' \geq 0 \), the property \([A.8]\) is obvious.

On the other hand, let \( 0 \leq k' \leq k \) be given and assume that \( u \in H_{-k'} \subset H_{-k} \). The definition of \( A_{-k}u \) leads to:

\[
A_{-k}u = \langle u, A_{k+2}\rangle_{-k,k} \in H_{-k-2}.
\]

But \( A_{k+2} = A_{k'+2} \) in \( H_{k+2} \) and therefore:

\[
\langle u, A_{k+2}\rangle_{-k,k} = \langle u, A_{k'+2}\rangle_{-k',k'} = A_{-k'}u.
\]

The proof is now complete. \( \square \)

Lemma A.4. For every integer \( k \), the following identity hold:

\[
(A.9) \quad (A_{k+1}u, v)_{k-1} = (u, v)_k, \quad \text{for all } u \in H_{k+1}, \quad \text{for all } v \in H_k.
\]

Proof. The proof is by induction on \( k \). The equality \([A.9]\) is true for \( k = 1 \) according to the definition \([A.4]\) of \( A_2 \). Let us assume that \([A.9]\) is true for some integer \( k \). By definition, if \( z \) belongs to \( H_{k+2} \), then \( A_{k+1}z \) belongs to \( H_k \). Replacing \( v \) by \( A_{k+1}z \) in \([A.9]\), we obtain:

\[
(A_{k+1}u, A_{k+1}z)_{k-1} = (u, A_{k+1}z)_k, \quad \text{for all } u \in H_{k+1}, \quad \text{for all } z \in H_{k+2},
\]

that is to say, reorganizing the terms:

\[
(A_{k+2}z, u)_k = (z, u)_{k+1}, \quad \text{for all } u \in H_{k+1}, \quad \text{for all } z \in H_{k+2},
\]

and therefore, formula \([A.9]\) is true replacing \( k \) by \( k + 1 \). Let us verify that it is also true for \( k - 1 \). Thus, we have:

\[
(A_{k+1}u, v)_{k-1} = (u, v)_k = (A_{k}u, A_{k}v)_{k-2}, \quad \text{for all } u \in H_{k+1}, \quad \text{for all } v \in H_k.
\]

But \( A_{k+1} \) is an isometry from \( H_{k+1} \) onto \( H_{k-1} \) and \( A_{k} = A_{k+1} \) in \( H_{k+1} \), then:

\[
(u, v)_{k-1} = (u, v)_k = (w, A_{k}v)_{k-2}, \quad \text{for all } w \in H_{k-1}, \quad \text{for all } v \in H_k.
\]

The proof is now complete. \( \square \)

Lemma A.5. Let \( k \) be an integer, \( w \) be in \( H_{k-1} \) and \( u \) be in \( H_k \) such that:

\[
(w, v)_{k-1} = (u, v)_k, \quad \text{for all } v \in H_k.
\]

Then \( u \in H_{k+1} \) and \( w = A_{k+1}u \).

Proof. Let \( \tilde{u} = A_{k+1}^{-1}w \). Then \( (\tilde{u} - u, v)_k = 0 \) for every \( v \in H_k \) and therefore \( u = \tilde{u} \). \( \square \)

A.2. Isometric chain of embedded Hilbert spaces. Let \( \{H_k, k \in \mathbb{Z}\} \) and \( \{\hat{H}_k, k \in \mathbb{Z}\} \) be two families of embedded Hilbert spaces build from Gelfand triples. We assume that \( H_0 \) and \( \hat{H}_0 \) are not necessary the pivot spaces. As usual, for every integer \( k \), there exist isometries \( A_k : H_k \to H_{k-2} \) and \( \hat{A}_k : \hat{H}_k \to \hat{H}_{k-2} \) such that \( A_k = A_{k-1} \) in \( H_k \) and \( \hat{A}_k = \hat{A}_{k-1} \) in \( \hat{H}_k \).

We assume furthermore that there exist isometries \( p_0 : H_0 \to \hat{H}_0 \) and \( p_1 : H_1 \to \hat{H}_1 \) such that \( p_1 = p_0 \) in \( H_1 \). For every integer \( k \geq 2 \), we define by induction \( p_k = \hat{A}_k^{-1}p_{k-2}\hat{A}_k \) and for every \( k \geq 1 \), we set \( p_{-k-2} = \hat{A}_{-k}p_{-k}\hat{A}_k^{-1} \).

Lemma A.6. For every pair of integers \( k \) and \( k' \) such that \( k' \leq k \):

\[
(A.10) \quad p_{k'} = p_k \quad \text{in } H_k.
\]

Moreover, for every integer \( k \), the operator \( p_k : H_k \to \hat{H}_k \) is an isometry.

Proof. Since \( A_k \) and \( \hat{A}_k \) are isometries for every integer \( k \), we can draw the same conclusion for \( p_k \) providing that \( p_{k-2} \) is an isometry as well. The conclusion follows by induction for every \( k \geq 0 \). The same reasoning allows proving that \( p_{-k} \) is also an isometry for every \( k \geq 1 \).

It remains to verify that the equalities \([A.10]\) are true. Assume that for some index \( k \geq 0 \), \( p_k = p_{k+1} \) in \( H_{k+1} \). So, from the identity:

\[
(A_{k+2}u, v)_{H_k} = (u, v)_{H_{k+1}}, \quad \text{for all } u \in H_{k+2}, \quad \text{for all } v \in H_{k+1},
\]
we deduce that:

\[(p_k A_k + p v) H_k = (p_{k+1} u, p_{k+1} v) H_{k+1}, \quad \text{for all } u \in H_{k+2}, \text{ for all } v \in H_{k+1}.\]

From the definition of \( p_{k+2} \), we deduce that \( p_k A_{k+2} = A_{k+2} p_{k+2} \), whence, denoting \( v = p_k v = p_{k+1} v \):

\[(A_{k+2} p_{k+2} u, v) H_k = (p_{k+1} u, v) H_{k+1}, \quad \text{for all } u \in H_{k+2}, \text{ for all } v \in H_{k+1}.\]

This equality entails first that \( \hat{\theta}_n \) share the same eigenfunctions, denoted by \( \theta_k \), and hence strongly convergent in \( H_0 \). The conclusion follows by induction and the cases \( k \leq 0 \) are treated similarly. \( \square \)

A.3. **Semigroup.** We assume that the inclusion \([A.1]\) is in addition compact. In that case, we claim:

**Lemma A.7.** For every integer \( k \), the inclusion \( H_{k+1} \subset H_k \) is compact.

**Proof.** We address the case \( k = 1 \). Assume that the sequence \((u_n)_{n \geq 0}\) is weakly convergent toward 0 in \( H_2 \). We assume that the inclusion \((A.1)\) is in addition compact. In that case, we claim:

\[(A_2 u_n, A_2)_{n \to 0} = (u_n v, v)_{n \to 0} \quad \text{as } n \to +\infty \quad \text{for all } v \in H_2,
\]

and therefore, that:

\[A_2 u_n, w)_{n \to 0} = (u_n v, v)_{n \to 0} \quad \text{as } n \to +\infty \quad \text{for all } w \in H_0.
\]

That is, \((A_2 u_n)_{n \geq 0}\) is weakly convergent toward 0 in \( H_0 \). On the other hand, since \( H_2 \) is continuously included into \( H_1 \), the sequence \((u_n)_{n \geq 0}\) is also weakly convergent toward 0 in \( H_1 \) and hence strongly convergent in \( H_0 \). It follows that:

\[\|u_n\|^2 = (A_2 u_n, u_n)_{n \to 0} \quad \text{as } n \to +\infty.
\]

Since the operators \( A_k \) were proved to be isometries for every \( k \), the other cases follows and the proof is completed. \( \square \)

For every integer \( k \), we define the unbounded operators \( A_k \) of domain \( D(A_k) = H_{k+2} \) in \( H_k \) by:

\[(A_{k+1} x, A_k x) = (A_{k+1} x, x) \quad \text{for all } x \in D(A_k).
\]

**Proposition A.8.** For every integer \( k \), the operator \( A_k \) is self-adjoint with compact inverse. All the operators \( A_k \) share the same spectrum that consists in a sequence \((\lambda_n)_{n \geq 1}\) of positive eigenvalues that tends to \(+\infty\). All the operators \( A_k \) share also the same eigenfunctions, denoted by \( e_n \) \((n \geq 1)\) and for every nonnegative integer \( n \):

\[e_n \in H_{\infty} \quad \text{with} \quad H_{\infty} = \bigcap_{p \geq 0} H_p.
\]

The eigenfunctions are chosen to form an orthogonal Riesz basis in every \( H_k \) and they are scaled to be of unit norm in \( H_0 \).

The spaces \( H_k \) are isometric to the spaces:

\[\ell_k = \left\{ u = (u_n)_{n \geq 1} \in \mathbb{R}^N : \sum_{n \geq 1} \lambda_n^k u_n^2 < +\infty \right\},
\]

provided with the scalar product:

\[\langle u, v \rangle_{\ell_k} = \sum_{n \geq 1} \lambda_n^k u_n v_n \quad \text{for all } u = (u_n)_{n \geq 1} \text{ and } v = (v_n)_{n \geq 1} \text{ in } \ell_k,
\]

the isometries being obviously:

\[T_k : u \in H_k \mapsto ((u, e_n)_{n \geq 1} \in \ell_k \quad \text{with inverse} \quad T_k^{-1} : u = (u_n)_{n \geq 1} \in \ell_k \mapsto \sum_{n=1}^{+\infty} u_n e_n \in H_k.
\]

In \( \ell_k \) we define the strongly continuous semigroup of contraction \((T_k(t))_{t \geq 0}\) by:

\[T_k(t) = \left( e^{-\lambda_n t} u_n \right)_{n \geq 0} \quad \text{for all } t > 0 \text{ and } u = (u_n)_{n \geq 0} \in \ell_k.
\]
This semigroup admits the operator $B_k = I_k A_k I_k^{-1}$ as infinitesimal generator. We deduce that the semigroup $(S_k(t))_{t \geq 0}$ defined by:

$$S_k(t) = T_k^{-1}T(t)I_k,$$

is a strongly continuous semigroup of contraction in $H_k$ with infinitesimal generator $A_k$. It is a simple exercise to verify that:

**Lemma A.9.**

1. For every $u \in \ell_k$ and for every positive real number $T$:

$$\mathbb{T}_k(\cdot)u \in H^1(0, T; \ell_{k-1}) \cap C([0, T]; \ell_k) \cap L^2(0, T; \ell_{k+1}).$$

2. Let $v$ be in $L^2(0, T; \ell_{k-1})$ and define $w(s) = \int_0^t T_k(t - s)v(s)\,ds$ for every $t \in (0, T)$. Then:

$$w \in H^1(0, T; \ell_{k-1}) \cap C([0, T]; \ell_k) \cap L^2(0, T; \ell_{k+1}).$$

Considering, for any integer $k$, any time $T > 0$ and any initial data $u^i \in H_k$ the Cauchy problem:

(A.12a) $\quad \partial_t u + A_{k+1} u = f$ on $(0, T)$

(A.12b) $\quad u(0) = u^i$ in $H_k$,

where the source term $f$ is given in $L^2(0, T; H_{k-1})$, we deduce:

**Proposition A.10.** The Cauchy problem (A.12) admits a unique solution in the space:

$$H^1(0, T; H_{k-1}) \cap C([0, T]; H_k) \cap L^2(0, T; H_{k+1}),$$

and this solution is given by:

$$u(t) = S_k(t)u^i + \int_0^t S_k(t - s)f(s)\,ds \quad \text{for all } t \in [0, T].$$

**Remark A.11.**

1. The chain of embedded spaces $H_k$ and semigroup $(S_k(t))_{t \geq 0}$ fit with the notion of Sobolev towers as described in [15 §II.2.C].

2. The semigroups $(S_k(t))_{t \geq 0}$ are called diagonalizable semigroups; see [64 §2.6].

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