Radiative corrections to the level width in the presence of magnetic field

D Solovyev\(^1\)\(^,\)\(^2\) and T Zalialiutdinov\(^1\)\(^,\)\(^2\)

\(^1\)Department of Physics, St. Petersburg State University, Petrodvorets, Oulianovskaya 1, 198504, St. Petersburg, Russia
\(^2\)Petersburg Nuclear Physics Institute named by B.P. Konstantinov of National Research Centre ‘Kurchatov Institut’, St. Petersburg, Gatchina 188300, Russia

E-mail: d.solovyev@spbu.ru

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Abstract

The effect of a constant magnetic field in combination with a field induced by an external thermal environment on atomic decay rates is studied. For this purpose, radiative corrections including magnetic interaction are considered for hydrogen and hydrogen-like ions with a small nuclear charge \(Z\). Corrections to the decays of the metastable state \(2S\) and the excited state \(2p\) were calculated at various magnetic field strengths suitable for the conditions of the laboratory experiments. It is found that the combination of the magnetic field and thermal environment can lead to a broadening close to the level of experimental error, which makes it necessary to take them into account in the near future.

1. Introduction

The study of radiative decay rates is an inalienable part of precision atomic spectroscopy. Accurate theoretical calculations of transition rates are closely related to the measurement of transition frequencies. Recent experimental efforts aimed at improving the accuracy of lifetime determination (see, for example, [1, 2]) have made it possible to test quantum electrodynamics (QED) approaches by comparing experimental and theoretical results in detail. The experimental uncertainty achieved in few-electron systems, at the level of one-thousandth relative value [3], makes such studies sensitive to relativistic, correlation, QED, etc effects [4–7]. Refining the lifetime values of atomic levels is of particular interest for metastable states in hydrogen and various highly charged ions (HCI), as it enables accurate measurement of the transition frequencies between highly excited states [8–11].

The importance of the decay rates of a metastable state in the precision atomic spectroscopy has been repeatedly demonstrated in the literature. For instance, the emission of hydrogen and hydrogen-like atoms under the influence of external electric and magnetic fields has been the subject of research on parity non-conservation (PNC) effects since [12–14]. Another application relates to the measurement of the Lamb shift in the hydrogen atom and HCI by resorting to emission interference [13, 15]. A separate area of research, where the lifetime of a metastable state is of particular importance, is astrophysical studies of plasma in the early universe [16, 17]. No less attention is paid to the decays of excited states with principal quantum number \(n > 2\), the total contribution of which to the ionization fraction of the primordial plasma reaches the percentage level and exceeds the accuracy of the measurements of the cosmic microwave background [18–21].

Theoretical calculations of lifetimes and partial transition rates in HCI are the subject of ongoing research [3, 5, 6, 22–24]. Theoretically predicted outcomes at the level of experimental error (about \(10^{-3} – 10^{-4}\) relative magnitude) allow us to eliminate the existing discrepancy between theory and experiment [3, 6], and open up possibilities, e.g., to find ‘nuclear clock transitions’ [25] or study QED corrections to the \(g\)-factor of the bound electron [26, 27]. Precision spectroscopic measurements necessarily include a theoretical analysis of the observed line profile width, making residual field control imperative to reduce errors, including those due to magnetic field inhomogeneity [28–30].
Accordingly, in the present work, we consider the influence of an external magnetic field on the partial transition rates and lifetimes of the excited states in hydrogen-like atomic systems. One way of considering the effect within the framework of a rigorous QED theory corresponds to a description of radiative corrections involving the photon loop and the magnetic interaction. Then their imaginary part represents a correction to the probability of spontaneous decay (see below).

The paper is organized as follows. In the next section, the lowest order radiative correction to the rates of spontaneous transitions in an external magnetic field is discussed. Then a radiative correction involving magnetic interaction for hydrogen and hydrogen-like ions placed in a thermal equilibrium environment is considered. The conclusions can be found in the last section. The derivation of the basic formulas is given in the appendices. The relativistic units \( \hbar = c = m_e = 1 \) (\( \hbar \) is the Planck constant, \( c \) is the speed of light, \( m_e \) is the electron mass and charge units \( \alpha = e^2 \) (\( \alpha \) is the fine structure constant) are used throughout the paper.

2. Radiative corrections to spontaneous transition rates and level widths in the presence of a magnetic field

It is known that the natural width of the excited state can be found as the imaginary part of the one-loop self-energy correction of bound electron [31]. Recently, this approach has been extended to estimate the imaginary part of the corrections represented by the Feynman two-loop diagrams, see [32–35]. Its application to account for the effects of the external thermal environment was discussed in [36–38]. More recently, the thermal radiative QED correction to the \( g \)-factor of the bound electron has been investigated on this basis in [39]. Following the results of [39], the leading-order radiative correction to the level width in the presence of an external magnetic field can be found by considering the imaginary part of the contributions represented by the Feynman graphs in figure 1.

In the dipole approximation, the corresponding radiative correction to the level width in the presence of a homogeneous magnetic field \( \vec{B} \) takes the form (see appendix A and [39] for details):

\[
\delta \Gamma_a = 4e^2 \sum_{n < a} \Delta E_{mn}^2 \left| \langle a|\vec{r}|n \rangle \right|^2 \left[ \langle n|\vec{\mu}\vec{B}|n \rangle - \langle a|\vec{\mu}\vec{B}|a \rangle \right],
\]

where \( \Delta E_{mn} \equiv E_n - E_m, E_n \) is the energy of the bound electron in the state \( n \), given by the set of quantum numbers \( nlsjm_j \), \( n \) is the principal quantum number, \( l \) is the orbital momentum, \( s \) is the electron spin, \( j, m_j \) are the total angular momentum and its projection, respectively, \( \langle a|\vec{r}|n \rangle \) is the dipole matrix element, and \( \vec{\mu} = -\mu_B(\vec{l} + 2\vec{s}) \) is the electron magnetic moment (\( \mu_B \) is the Bohr magneton). The summation in equation (1) runs over all bound states below an arbitrary excited atomic state \( a \) and includes the full set of quantum numbers describing the \( n \) state. In turn, the partial decay rate corresponds to the terms of the sum in equation (1) with particular value of \( n \). Angular reduction of matrix elements can be performed in a standard way according to [40]. The details of the derivation of equation (1) are given in appendix A using the adiabatic S-matrix formalism.

The matrix element in the square brackets of equation (1) for a homogeneous magnetic field constitutes a Zeeman shift:

![Figure 1. Feynman diagrams describing the QED contributions of the order of \( \alpha \) to the hyperfine structure splitting and the bound-electron \( g \)-factor. The imaginary part of these graphs represents the radiative correction to the level width. A zigzag line with a cross at the end indicates interaction with an external magnetic field. The double solid line denotes the bound electron in the Furry picture. The wavy loop is described by a photon propagator.](image-url)
Table 1. Numerical values of the radiative correction $\delta \Gamma_{n}$ (see equation (1)) for the Lyman-alpha transition rate in the hydrogen atom and some hydrogen-like ions ($Z = 2$ and $Z = 6$) are presented. The first column shows the initial state $2p_{\nu}$, and the corresponding projection $m_{\nu}$, of the total angular momentum $J_{\nu}$.

Each delineated row contains the radiative correction summed over all projections, and the following rows give partial values for a fixed initial projection $m_{\nu}$. The second column contains the values of the natural width of the level. The partial transition rates are given in the third and fourth columns for a magnetic field of 1 G and 100 G, respectively, with values in parentheses showing the error bars.

| $|J_{\nu}|$ | $\Gamma_{n}^{1G_{s}}$ s$^{-1}$ | $\delta \Gamma_{n}^{1G_{s}}$ s$^{-1}$ | $\delta \Gamma_{n}^{100G_{s}}$ s$^{-1}$ |
|-----------|-----------------|-----------------|-----------------|
| $Z = 1$   |                 |                 |                 |
| $2p_{1/2}$ | $6.2681(3) \times 10^{8}$ | $-0.5689(1)$ | $-5.6898(1) \times 10^{7}$ |
| 1/2       | —               | $-1.3354(1) \times 10^{4}$ | $-1.3411(2) \times 10^{6}$ |
| −1/2      | —               | $1.3353(1) \times 10^{4}$ | $1.3297(2) \times 10^{6}$ |
| $2p_{3/2}$ | $6.2681(1) \times 10^{8}$ | $-0.4267(1)$ | $-4.2674(1) \times 10^{3}$ |
| 3/2       | —               | $2.0031(1) \times 10^{4}$ | $2.0073(2) \times 10^{6}$ |
| −1/2      | —               | $-6.7767(2) \times 10^{3}$ | $-6.7197(1) \times 10^{3}$ |
| −3/2      | —               | $6.7666(2) \times 10^{3}$ | $6.6343(1) \times 10^{3}$ |
| $Z = 2$   |                 |                 |                 |
| $2p_{1/2}$ | $1.0029(1) \times 10^{10}$ | $-0.5689(1)$ | $-5.6896(2) \times 10^{7}$ |
| 1/2       | —               | $-5.3413(8) \times 10^{4}$ | $5.3469(6) \times 10^{6}$ |
| −1/2      | —               | $5.3411(8) \times 10^{4}$ | $5.3355(6) \times 10^{6}$ |
| $2p_{3/2}$ | $1.0029(1) \times 10^{10}$ | $-0.4267(1)$ | $-4.2673(2) \times 10^{3}$ |
| 3/2       | —               | $-8.0121(8) \times 10^{4}$ | $-8.0162(9) \times 10^{6}$ |
| −1/2      | —               | $2.6707(4) \times 10^{4}$ | $2.6749(4) \times 10^{6}$ |
| −3/2      | —               | $8.0119(8) \times 10^{4}$ | $8.0077(9) \times 10^{6}$ |
| $Z = 6$   |                 |                 |                 |
| $2p_{1/2}$ | $8.124(2) \times 10^{11}$ | $-0.5688(4)$ | $-5.6868(2) \times 10^{7}$ |
| 1/2       | —               | $-4.805(4) \times 10^{3}$ | $-4.805(6) \times 10^{7}$ |
| −1/2      | —               | $4.805(4) \times 10^{3}$ | $4.804(6) \times 10^{7}$ |
| $2p_{3/2}$ | $8.124(4) \times 10^{11}$ | $-0.4266(2)$ | $-4.2666(2) \times 10^{3}$ |
| 3/2       | —               | $-7.207(8) \times 10^{3}$ | $-7.207(9) \times 10^{7}$ |
| −1/2      | —               | $2.402(4) \times 10^{3}$ | $2.403(2) \times 10^{7}$ |
| −3/2      | —               | $7.207(8) \times 10^{3}$ | $7.206(9) \times 10^{7}$ |

$\Delta E_n = \langle n | \vec{j} \cdot \vec{B} | n \rangle = g_n \mu_B B \ m_{j_n}$.  

Here $m_{j_n}$ is the projection of the angular momentum $j_n$, $g_n$ denotes the g-factor of the state $n$.

Substituting equation (2) into equation (1) allows for the evaluation of the partial transition rate, which includes the transition between components of the fine structure specified by the projection of the excited state $m_{j_n}$. It is important to consider the Zeeman shift, equation (2), for both energies $E_n$ and $E_j$ included in equation (1). When fixing the projection $m_{j_n}$, the $\delta \Gamma_{n}$ correction will have linear and quadratic terms over the field. However, when summing over all projections $m_{j_n}$, $m_{j_n}$ will retain only the quadratic contribution in the field.

The results of numerical calculations of the partial transition rates and level widths for $2p_{1/2}$ and $2p_{3/2}$ excited states of the hydrogen atom and several hydrogen-like ions (with $Z = 2$ and $Z = 6$, where $Z$ is the nuclear charge) are presented in table 1 for the 1 G ($10^{-4}$ T) and 100 G ($10^{-2}$ T) magnetic fields. The partial transition rates for a fixed projection of the initial state, $m_{j_n}$, and the level widths (summed over the final state projection and averaged over the initial state projection) are given in s$^{-1}$. The chosen values of $B$ lie in the region of magnetic field strength typical for experiments [10, 14, 41, 42].

The physical meaning of $\delta \Gamma_{n}$, equation (1), can be understood as the interference of photons emitted by the unperturbed and Zeeman states of the atom. Consequently, this value can be negative, as seen in table 1. The total contribution $\delta \Gamma_{n}$ can be obtained from the partial transition rates by averaging over the projection $m - j_n$, i.e. summing the values listed in table 1 with the coefficient $1/(2j_n + 1)$.

Table 1 shows results for the hydrogen atom and hydrogen-like ions with low values of $Z$. This is primarily due to the use of the nonrelativistic approximation to derive the formula (1), which allows the Schrödinger
spectrum and wave functions to be used. Thus, the accuracy of the \( \delta \Gamma_{\alpha} \) calculation is limited to corrections of the order \( c(Z\alpha)^2 \) (relativistic and QED contributions, and the \( c \) coefficient needs to be calculated separately and usually well below unity, see, for example, [43, 44]). Being valid for arbitrary dipole-resolved transitions, the expression (1) also does not include higher multipoles, which are higher-order corrections to \( (Z\alpha)^2 \).

To estimate the error bars indicated in table 1 in parentheses, we calculate \( \delta \Gamma_{\alpha} \) equation (1), with Dirac energies and double the difference with purely nonrelativistic values. At the same time, the error bars for natural widths, \( \Gamma_{\alpha}^{\text{nat}} \), are obtained by direct comparison with fully relativistic calculations [44]. The following tables show only significant digits. We used known formulas for the \( g \)-factors with corrections of order up to \( \alpha^4 \) included, see [45] and references therein. This is more accurate than would otherwise be required, because of the nonrelativistic approximation employed in other parts of our calculations. In addition, it can be found that the number of digits in the values of the partial and total rates is conserved. This is a consequence of the fact that the relativistic corrections to the energy (as well as corrections to the wave function or the radiative QED corrections) are independent of the projection and, therefore, affect the total value of \( \delta \Gamma_{\alpha} \) in the next orders due to the opposite sign of the projections.

As can be seen from table 1, the obtained values of the radiative correction to the total level width \( \delta \Gamma_{\alpha} \) are almost independent of \( Z \), at least in the limit when the perturbative \( \alpha Z \) decomposition is justified. This immediately follows from the parametric estimate of equation (1): \( \Delta E_{\text{rel}} \sim m_e(\alpha Z)^2 \) and \( \langle \tau \rangle \sim 1/(m_e\alpha Z) \) in relativistic units. Summing over projections and taking into account the Zeeman shift of energies \( E_{\text{rel}} \) (see formula (2)), the factor \( \Delta E_{\text{rel}} \) is replaced by \( \Delta E_{\text{rel}} B_\alpha B \), which reduces the dependence on \( Z \) by \( \langle \tau \rangle^2 \) in the leading order. For the same reasons, summing over projections suppresses the correction to the total level width by \( \alpha^2 \), which can be found by comparison with the partial rate. In turn, for partial transition rates, this dependence is expressed by the factor \( Z^2 \) with the same estimates (there is no reduction in this case). Finally, as expected, the radiative correction (1) to the total level width is quadratic in the field and almost linear for the partial quantities. This behavior persists until \( \approx 10^4 \) G (1 T), when the quadratic contribution to the partial probabilities starts to outweigh the linear one, see figure 2 for the \( 2p_{1/2} \) state and figure 3 for the \( 2p_{3/2} \) state in the hydrogen atom.

Note that the perturbation theory for the hydrogen atom is valid up to the value of the matrix element \( \langle n|\vec{B}|\ell \rangle \approx E_n - E_e \). The latter condition corresponds to the maximum magnetic field \( B \approx 2 \times 10^8 \) G. However, a stricter constraint follows from the fact that the correction equation (1) (given that it is negative) should not exceed the value of the natural level width. Equating the absolute values of the expression (1) and the natural level width, the maximum allowable field strength is \( B \approx 3.3 \) T (33190.9 G) for the \( 2p_{1/2} \) state, which is close to the fields used in the experiments [28, 29]. In addition, when calculating the partial or total transition rates in a strong magnetic field (with a field strength of the order of \( 10^4 \) G and higher), it is necessary to take into account the hyperfine structure of atomic levels and diagrams of the next orders (with two and so on vertex parts in figure 1). Due to the proton mass involved in the nuclear interaction with the magnetic field, it can be expected that the effect is suppressed by about three orders of magnitude, and the next-order diagrams are suppressed by an additional multiplier \( \alpha^2 \). We leave a detailed study of this phenomenon for the future, restricting the field strength to the above value.

Figure 2. Behavior of the partial \( 2p_{1/2} \rightarrow 1s_{1/2} \) transition rates in the magnetic field for the hydrogen atom. The solid line corresponds to the projection of the total angular momentum \( m_{\ell} = 1/2 \), and the dotted line corresponds to \( m_{\ell} = -1/2 \).
3. Corrections to spontaneous transition rates and level widths in the presence of a magnetic field combined with the BBR field

In view of the astrophysical interests in level widths and decay rates in atoms, and the fact that the thermal environment in laboratory experiments is hardly screened, here we consider the radiative correction induced by the blackbody radiation (BBR) field, $\delta \Gamma^{\beta}_a$. The analytical derivation of the $\delta \Gamma^{\beta}_a$ correction can be found in appendix B. The only difference with the correction of equation (1) is that in the case of Quantum Electrodynamics at finite temperature for bound states, the wavy loop in figure 1 is given by the thermal photon propagator.

The final expression for the radiative correction to the level width at a finite temperature in the presence of a homogeneous magnetic field is (see equation (B8))

$$\delta \Gamma^{\beta}_a = \frac{4e^2}{3} \sum_{n} [(|n]\mu^{\beta}\bar{B}(|n) - \langle a|\mu^{\beta}\bar{B}|a\rangle)]\langle a|\bar{\pi}|n\rangle^2 \times [\beta \Delta E_{\text{fin}}j_{\beta}(\Delta E_{\text{fin}}) e^{\beta\Delta E_{\text{fin}} - 3\Delta E_{\text{fin}}j_{\beta}(\Delta E_{\text{fin}})}].$$

(3)

Here $j_{\beta}(\omega) = (\exp(\beta\omega) - 1)^{-1}$ is the BBR photon field density number (Planck distribution function), $\beta = 1/k_B T$, $k_B$ is the Boltzmann constant in relativistic units, $T$ is the radiation temperature in Kelvin. In contrast to equation (1) the summation in equation (3) extends to all states (above and below an arbitrary state $a$ under consideration), including the continuum. This immediately leads to the conclusion that the ground state has a level width stimulated by the BBR field because of possible excitation processes, see [46]. Numerical results for $\delta \Gamma^{\beta}_a$ are collected in table 2 for various low-lying states in hydrogen and hydrogen-like ions at different temperatures and magnetic field of 100 G.

First, the values listed in table 2 show the behavior of the correction equation (3) depending on the nuclear charge $Z$. In particular, $\delta \Gamma^{\beta}_a$ decreases with increasing $Z$, becoming negligible for highly charged hydrogen-like ions in accordance with the exponential suppression due to the Planck distribution function. Second, the value of $\delta \Gamma^{\beta}_a$ increases with temperature and is close to the natural width of the $2s$ level in the hydrogen atom at 3000 K, remaining considerably smaller at $Z > 1$. Third, the correction equation (3) is larger than the widths induced by the BBR, $\Gamma^{\beta}_a$, for $Z = 1, 2$. Thus, one can expect its significance for describing the recombination epoch in the early Universe [18]. In this case, the role of the external field $\bar{B}$ can be played by the primary magnetic fields [48–52].

Finally, in the presence of blackbody radiation, the width of the ground state becomes non-zero due to induced transitions to the upper energy levels (see [46]). The same situation is observed for the correction given in equation (3). As one would expect, the contribution of $\delta \Gamma^{\beta}_a$ is much smaller than the commonly known stimulated width, $\Gamma^{\beta}_a$, and becomes noticeable only at high temperatures, around 3000 K. For example, for a hydrogen atom in a 100 Gs magnetic field, the correction reaches a value of about $-1.21 \times 10^{-15}$ s$^{-1}$, whereas $\Gamma^{\beta}_a \approx 1.35 \times 10^{-8}$ s$^{-1}$. The $\delta \Gamma^{\beta}_a$ values are negligible for other hydrogen-like ions in weak fields, and decrease exponentially with $Z$.

To complete the analysis, we further consider the correction given by equation (3) as a function of $B$ at a fixed temperature of 300 K, which is attributed to laboratory experiments. In addition, we carry out calculations for partial quantities, with a fixed projection of the angular momentum of the initial state, $m_{ij}$. Table 3 shows the numerical values of low-lying states in hydrogen and hydrogen-like ions with $Z = 2$ and $Z = 6$.
Table 2. Numerical values of the radiative correction $\delta \Gamma^G_{\Delta J}$ (see equation (3)) for various low-lying states in hydrogen and some hydrogen-like ions ($Z = 2$ and $Z = 6$) at different temperatures and a magnetic field of 100 G. The first column shows the natural level width and the atomic states for which the correction (3) is calculated. The following columns show the values of correction $\delta \Gamma^G_{\Delta J}$ at a fixed temperature. Each first substring contains the values of the depopulation level widths $\Gamma^G_{\Delta J}$ stimulated by blackbody radiation (see [47] for the details), the lower substrings contain the values of equation (1). Only significant digits are left, all values are given in $10^{-3}$.

| \( \Gamma^G_{\Delta J} \) in s\(^{-1} \) | \( \Gamma^G_{\Delta J} \) in s\(^{-1} \) at 300 K | at 500 K | at 1000 K | at 3000 K |
|-----------------|-----------------|----------|-----------|----------|
| State           | \( \delta \Gamma^G_{\Delta J} \) in s\(^{-1} \) |          |           |          |
| \( Z = 1 \) (100 G) |            |          |           |          |
| \( \Gamma^G_{2s} \) = 8.229 | \( \Gamma^G_{2s} = 1.420 \times 10^{-3} \) | 2.367 \times 10^{-3} | 0.0202 | 4.701 \times 10^{-4} |
| \( 2s_{1/2} \) | \( \delta \Gamma^G_{2s} = -0.0104 \) | -0.0173 | -0.0346 | -0.1052 |
| \( \Gamma^G_{2p} \) = 6.268 \times 10^{8} | \( \Gamma^G_{2p} = 4.734 \times 10^{-6} \) | 7.891 \times 10^{-6} | 0.0329 | 7.583 \times 10^{-4} |
| \( 2p_{1/2} \) | \( \delta \Gamma^G_{2p} = -0.0041 \) | -0.0069 | -0.0138 | -0.0425 |
| \( Z = 2 \) (100 G) |            |          |           |          |
| \( \Gamma^G_{2p} \) = 526.7 | \( \Gamma^G_{2p} = 6.252 \times 10^{-3} \) | 1.042 \times 10^{-3} | 2.085 \times 10^{-3} | 6.474 \times 10^{-3} |
| \( 2s_{1/2} \) | \( \delta \Gamma^G_{2s} = -0.0026 \) | -0.0043 | -0.0086 | -0.0259 |
| \( \Gamma^G_{2p} \) = 1.003 \times 10^{10} | \( \Gamma^G_{2p} = 2.084 \times 10^{-4} \) | 3.475 \times 10^{-4} | 6.952 \times 10^{-4} | 2.439 \times 10^{-3} |
| \( 2p_{1/2} \) | \( \delta \Gamma^G_{2p} = -0.0010 \) | -0.0017 | -0.0034 | -0.0014 |
| \( Z = 6 \) (100 G) |            |          |           |          |
| \( \Gamma^G_{2p} \) = 3.839 \times 10^{3} | \( \Gamma^G_{2p} = 2.032 \times 10^{-3} \) | 3.475 \times 10^{-1} | 7.082 \times 10^{-3} | 2.152 |
| \( 2s_{1/2} \) | \( \delta \Gamma^G_{2s} = -0.0005 \) | -0.0008 | -0.0018 | -0.0057 |
| \( \Gamma^G_{2p} \) = 8.124 \times 10^{11} | \( \Gamma^G_{2p} = 6.774 \times 10^{-2} \) | 1.158 \times 10^{-1} | 0.236 \times 10^{-1} | 7.172 \times 10^{-1} |
| \( 2p_{1/2} \) | \( \delta \Gamma^G_{2p} = -0.0001 \) | -0.0002 | -0.0004 | -0.0011 |

In particular, from table 3, one can find that the correction $\delta \Gamma^G_{\Delta J}$ for a fixed projection $m_{\Delta J}$ is significant even at room temperature and a weak magnetic field for the metastable state 2s in hydrogen and hydrogen-like ions with low Z. The value of $\delta \Gamma^G_{\Delta J}(m_{\Delta J})$ can be compared with the natural level width and decay rate $\Gamma^G_{\Delta J}$ stimulated by the BBR, see table 2. For example, in a hydrogen atom in a magnetic field of approximately 1 G at room temperature, the correction equation (3) for the $m_{\Delta J} = +1/2$ is one order of magnitude larger than $\Gamma^G_{\Delta J}$ (the same for the $m_{\Delta J} = -1/2$ projection, but with the opposite sign). The values of $\delta \Gamma^G_{\Delta J}(m_{\Delta J})$ and $\Gamma^G_{\Delta J}$ are comparable for $Z = 2$ and $Z = 6$, which can be explained by parametrization over Z for these quantities.

The radiative correction, equation (3), can be compared with the leading order correction to the two-photon decay rate of the 2s state in hydrogen-like ions [53]. In particular, the results shown in tables 2 and 3 are much larger than the radiative logarithmic correction to the level width analyzed in [53]. Furthermore, using the example of the helium ion ($Z = 2$), it can be seen that the values of $\delta \Gamma^G_{\Delta J}$ and $\Gamma^G_{\Delta J}$ significantly exceed the upper limit imposed on the amplitude of parity non-conserving admixture of the 2p state decay [54]. At room temperature and a magnetic field of 1 G, the $\delta \Gamma^G_{\Delta J}$ correction is comparable to $|\delta| < 2.4 \times 10^{-5}$ [54] and increases with the field strength $B$. The values listed in table 2 can be obtained from table 3 by averaging the corresponding partial transition rates. With that, because of the strong numerical reduction of values in table 3, more significant digits are given where necessary.

The correction (3) contains the Planck distribution function, which also depends on $\Delta E_{\text{nm}}$. As in the case of $\Gamma^G_{\Delta J}$ [47], the main contribution for the states $2s_{1/2}$ and $2p_{1/2}$ comes from the Lamb shift or fine splitting for state $2p_{3/2}$ (energy difference is closer to the maximum of the Planck distribution) at low temperatures. Our calculations show that for the values obtained for these states at room temperature, it is sufficient to consider only the Lamb shift. However, to obtain the values listed in table 3, the $\Delta E_{\text{nm}}$ difference is calculated from the Dirac energies and also includes the Lamb and Zeeman shifts. The dependence on the projection $m_{\Delta J}$ in the argument of the Planck function causes the values of the partial rates with opposite projections to have substantially different weights. Thus, the difference between them becomes larger with increasing field strength.

The dependence of the partial decay rates of the 2s-state (for fixed projections $m_{\Delta J} = 1/2$ and $m_{\Delta J} = -1/2$) on magnetic field and temperature is shown in figure 4.
Figure 4. Behavior of the correction $\delta \Gamma_{2s/1s}^{s}$ to the level width at a fixed projection $m_{2s/1s}$ in the magnetic field at different temperatures in the hydrogen atom. The orange (online colored) surface corresponds to the positive value of the projection, and the blue one represents $\delta \Gamma_{2s/1s}^{s}$ for the negative projection. The values of $\delta \Gamma_{2s/1s}^{s}(m_{2s/1s})$ are given in s$^{-1}$.

Table 3. Numerical values of the radiative correction equation (3) in the hydrogen atom and some hydrogen-like ions ($Z = 2$ and $Z = 6$) as a function of the magnetic field $B$ at room temperature. The first column gives the initial state with a fixed projection, $m_{j}$. The following columns show the partial widths for the field strength given in Gauss. Only significant digits are left, all values are given in s$^{-1}$.

| State | 1 G | 10 G | 100 G | 10$^3$ G | 10$^4$ G |
|-------|-----|------|-------|----------|---------|
| $2s_{1/2}^{m_{j}=+1/2}$ | $2.081 \times 10^{-4}$ | $1.987 \times 10^{-5}$ | $1.054 \times 10^{-2}$ | $-1.937 \times 10^{-1}$ | $34.612$ |
| $2s_{1/2}^{m_{j}=-1/2}$ | $-2.101 \times 10^{-4}$ | $2.195 \times 10^{-5}$ | $-3.128 \times 10^{-2}$ | $-1.094$ | $-48.152$ |
| $2p_{1/2}^{m_{j}=+1/2}$ | $1.564 \times 10^{-4}$ | $1.327 \times 10^{-5}$ | $1.153 \times 10^{-2}$ | $6.138 \times 10^{-2}$ | $29.596$ |
| $2p_{1/2}^{m_{j}=-1/2}$ | $-1.572 \times 10^{-4}$ | $-1.610 \times 10^{-5}$ | $-1.983 \times 10^{-2}$ | $-5.716 \times 10^{-1}$ | $-34.803$ |
| $2p_{3/2}^{m_{j}=+3/2}$ | $2.349 \times 10^{-4}$ | $2.321 \times 10^{-5}$ | $2.041 \times 10^{-2}$ | $7.598 \times 10^{-2}$ | $28.727$ |
| $2p_{3/2}^{m_{j}=-3/2}$ | $7.810 \times 10^{-5}$ | $7.530 \times 10^{-6}$ | $4.729 \times 10^{-3}$ | $8.214 \times 10^{-2}$ | $24.612$ |
| $2p_{3/2}^{m_{j}=+1/2}$ | $-7.872 \times 10^{-5}$ | $-8.152 \times 10^{-6}$ | $-3.895 \times 10^{-1}$ | $6.138 \times 10^{-2}$ | $-28.256$ |
| $2p_{3/2}^{m_{j}=-1/2}$ | $-2.355 \times 10^{-4}$ | $-2.383 \times 10^{-5}$ | $-2.663 \times 10^{-2}$ | $-5.463 \times 10^{-1}$ | $-33.417$ |

| State | 1 G | 10 G | 100 G | 10$^3$ G | 10$^4$ G |
|-------|-----|------|-------|----------|---------|
| $2s_{1/2}^{m_{j}=+1/2}$ | $6.926 \times 10^{-4}$ | $6.903 \times 10^{-5}$ | $6.670 \times 10^{-2}$ | $4.344 \times 10^{-1}$ | $-8.654$ |
| $2s_{1/2}^{m_{j}=-1/2}$ | $-6.932 \times 10^{-4}$ | $-6.935 \times 10^{-5}$ | $-7.187 \times 10^{-2}$ | $-9.514 \times 10^{-1}$ | $-32.749$ |
| $2p_{1/2}^{m_{j}=+1/2}$ | $5.196 \times 10^{-4}$ | $5.186 \times 10^{-5}$ | $5.093 \times 10^{-2}$ | $4.163 \times 10^{-1}$ | $-5.997 \times 10^{-1}$ |
| $2p_{1/2}^{m_{j}=-1/2}$ | $-5.198 \times 10^{-4}$ | $-5.207 \times 10^{-5}$ | $-5.300 \times 10^{-2}$ | $-6.251 \times 10^{-1}$ | $-15.517$ |
| $2p_{3/2}^{m_{j}=+3/2}$ | $7.794 \times 10^{-4}$ | $7.787 \times 10^{-5}$ | $7.718 \times 10^{-2}$ | $7.019 \times 10^{-1}$ | $2.714 \times 10^{-2}$ |
| $2p_{3/2}^{m_{j}=-3/2}$ | $2.598 \times 10^{-4}$ | $2.591 \times 10^{-5}$ | $2.521 \times 10^{-2}$ | $1.823 \times 10^{-1}$ | $5.573 \times 10^{-1}$ |
| $2p_{3/2}^{m_{j}=+1/2}$ | $-2.599 \times 10^{-4}$ | $-2.606 \times 10^{-5}$ | $-2.676 \times 10^{-2}$ | $-3.374 \times 10^{-1}$ | $-10.333$ |
| $2p_{3/2}^{m_{j}=-1/2}$ | $-7.796 \times 10^{-4}$ | $-7.803 \times 10^{-5}$ | $-7.873 \times 10^{-2}$ | $-8.570 \times 10^{-1}$ | $-15.537$ |

| State | 1 G | 10 G | 100 G | 10$^3$ G | 10$^4$ G |
|-------|-----|------|-------|----------|---------|
| $2s_{1/2}^{m_{j}=+1/2}$ | $1.40751 \times 10^{-1}$ | $1.407$ | $14.067$ | $1.399 \times 10^{2}$ | $1.322 \times 10^{3}$ |
| $2s_{1/2}^{m_{j}=-1/2}$ | $-1.40752 \times 10^{-1}$ | $-1.408$ | $-14.084$ | $-1.416 \times 10^{2}$ | $-1.492 \times 10^{3}$ |
| $2p_{1/2}^{m_{j}=+1/2}$ | $2.93323 \times 10^{-3}$ | $2.9322 \times 10^{-2}$ | $2.931 \times 10^{-1}$ | $2.923$ | $28.376$ |
| $2p_{1/2}^{m_{j}=-1/2}$ | $-2.93234 \times 10^{-3}$ | $-2.9324 \times 10^{-2}$ | $-2.933 \times 10^{-1}$ | $-30.266$ | $-15.517$ |
| $2p_{3/2}^{m_{j}=+3/2}$ | $4.39849 \times 10^{-5}$ | $4.39843 \times 10^{-2}$ | $4.3978 \times 10^{-1}$ | $4.391$ | $43.275$ |
| $2p_{3/2}^{m_{j}=-3/2}$ | $1.46616 \times 10^{-5}$ | $1.46609 \times 10^{-2}$ | $1.4655 \times 10^{-1}$ | $1.459$ | $13.951$ |
| $2p_{3/2}^{m_{j}=+1/2}$ | $-1.46617 \times 10^{-5}$ | $-1.46624 \times 10^{-2}$ | $-1.4669 \times 10^{-1}$ | $-1.473$ | $-15.368$ |
| $2p_{3/2}^{m_{j}=-1/2}$ | $-4.39850 \times 10^{-4}$ | $-4.39857 \times 10^{-2}$ | $-4.3992 \times 10^{-1}$ | $-4.406$ | $-44.692$ |
The values of $\delta \Gamma_{2s}^{3}$ at positive projection are in the upper half-space at low temperatures, and vice versa at negative projection. As the magnetic field strength increases, the nonlinear contributions of $B$ become significant, resulting in the ‘waves’ shown in figure 4. The amplitude of the ‘wave’ also increases with temperature. The intersection of $\delta \Gamma_{2s}^{3}(m_{b} = 1/2)$ and $\delta \Gamma_{2s}^{3}(m_{b} = -1/2)$ shows that there are field strength values that screen the temperature, making the total contribution in equation (3) equal to zero.

According to the results in tables 2 and 3, the partial transition rates, as well as the total contribution, can exceed the natural width of state $2s_{1/2}$ in the presence of the BBR field. This is due to transitions to upper states induced by blackbody radiation, which are important at high temperatures. For example, the ‘pure’ thermally induced decay rate of the $2s$ metastable state is $\Gamma_{2s}^{2} = 5.981 \times 10^{6}$ s$^{-1}$ in a hydrogen atom at a temperature of 8000 K, which exceeds $\Gamma_{2s}^{2\text{nat}} = 8.229$ s$^{-1}$ by orders of magnitude. This can also be observed in figure 4 at high field and temperature values.

4. Conclusions

In this paper, we analyzed the radiative corrections to the natural level width of hydrogen and hydrogen-like ions with a small nuclear charge $Z$, which arises in the presence of a magnetic field and its combination with blackbody radiation. Without aiming for precise calculations, we used the nonrelativistic approximation to identify the main features and determine the dominant contributions, as is typical.

The paper is an evident continuation of the TQED formalism developed by us and, in particular, the previous study [39], where we discussed the real part of the corrections arising from the Feynman diagrams shown in figure 1. It is known from QED theory that the imaginary part of the one-loop self-energy correction can be associated with the natural level width [31]. Within this approach, the imaginary part of the one-loop correction with a vertex is a correction to the natural width induced by an external field. At the same time, the S-matrix approach is easily adapted to the influence of the external thermal environment, which is provided via the replacement of the ‘ordinary’ photon propagator by a thermal one (see, for example, [55]). To our knowledge, the accounting of the corrections considered here has not been carried out before and, therefore, is of particular interest for astrophysical and laboratory research in various fields referred to in the Introduction. The obtained results can be summarized as follows.

First, we analyzed the case of zero temperature and demonstrated the significance of the correction through the example of the decay rate of $\text{Ly}_{\alpha}$ in hydrogen and hydrogen-like ions. The resulting $\delta \Gamma_{\alpha}$ values, shown in table 1, reveal that even at a weak field strength of 100 Gs, the partial value (at a fixed angular momentum projection of the initial state) is two orders of magnitude smaller than the natural width, which represents a significant line broadening. The correction of equation (1) to the total level width, $\delta \Gamma_{\alpha}$, can be estimated as independent of $Z$ (see parametric estimates in the main text). The partial transition probability is proportional to $Z$, which leads to a diminishing role of $\delta \Gamma_{\alpha}$ as the nuclear charge of these ions increases, since the natural level width is proportional to $Z^4$. However, $\delta \Gamma_{\alpha}$ grows quadratically with increasing field strength, providing significant broadening at higher fields. The dependence of the decay rates of states $2p_{1/2}$ and $2p_{3/2}$ with a fixed projection in hydrogen on the magnetic field strength is illustrated in figures 2 and 3.

As the next step in our study, we discussed the effect of a homogeneous magnetic field on the decays of low-lying states in hydrogen and hydrogen-like ions placed in a thermal environment in section 3. By directly comparing the radiative correction with the known thermal-induced widths (see [47]), we conclude from table 2 that the contribution of equation (3) exceeds the corresponding stimulated width for $Z = 1,2$ at laboratory temperatures. The total radiative correction $\delta \Gamma_{2s}^{3}$ in hydrogen is approximately $10^{-2}$ s$^{-1}$ at $T = 300$ K and $B = 100$ G, whereas $\Gamma_{2s}^{3}$ is around $1.42 \times 10^{-3}$ s$^{-1}$ (see tables 2 and 3). However, as the temperature increases, $\Gamma_{2s}^{3}$ becomes dominant. Additionally, this correction provides a relative contribution of the order of $\sim 10^{-3}$ with respect to the natural width of the $2s$ level in a hydrogen atom in a magnetic field of 100 G, exceeding the relativistic corrections by one order [44]. The dependence of the partial quantities $\delta \Gamma_{2s}^{3}(m_{2s_{1/2}})$ on temperature and field strength in the hydrogen atom is shown in figure 4.

Another application of the correction given by equations (1) and (2) is that it allows one to determine the g-factor of the bound electron, as in [26, 56]. When the experimental level widths and magnetic field are well defined, a set of equations for the necessary factors $g_{e}$ and $g_{b}$ can be constructed and solved. The most problematic thing is that lifetimes are determined experimentally with low accuracy (about $10^{-3}$ relative magnitude), which is unlikely to improve significantly in the near future. Nevertheless, the g-factor of the bound electron can be found in the leading order in this manner.

Finally, we are going to emphasize the importance of the spectral line broadening considered in this paper for the determination of frequency standards as follows. The evaluation of dynamic corrections to the BBR-Stark shift significantly affects the uncertainty of optical lattice clocks. Accurate theoretical calculations use values of several electric-dipole matrix elements that should be consolidated with the experimental ones. It was found that...
the uncertainty of the Sr optical lattice clock can be further reduced by accurately measuring the lifetimes. The authors significantly reduced the corresponding errors by improving the experimental determination of lifetimes to a relative accuracy of $\sim 10^{-3}$. Since the order of magnitude for neutral atoms is essentially the same as for the hydrogen atom, the corrections discussed here can serve to further improve in this direction.

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Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

Appendix A. Basic formulas for the radiative correction to the level width in the presence of a magnetic field

One way to determine the radiative QED correction to the level width of an arbitrary state $a$ in the presence of a magnetic field corresponds to considering the imaginary part of the graphs shown in figure 1. According to the Gell-Mann - Low - Sucher adiabatic formula ([59]), the energy shift $\Delta E_a$ of the atomic state $a$ is defined as

$$\Delta E_a = \lim_{\eta \to 0} \frac{i \hbar}{2} \langle a | \hat{S}_\eta | a \rangle.$$  (A1)

The adiabatic $S$-matrix $\hat{S}_\eta$ differs from the usual $S$-matrix by the presence of the adiabatic (exponential) factor $e^{-\eta |t|}$ in each (interaction) vertex. It refers to the concept of adiabatic switching on and off the interaction.

Following the adiabatic $S$-matrix approach [39, 59–61], the third order correction to the energy level of bound state $a$ is

$$\Delta E_a^{(3)} = \lim_{\eta \to 0} \frac{i \hbar}{2} \left[3 \langle \Phi^0_a | \hat{S}^{(3)}_\eta | \Phi^0_a \rangle - 3 \langle \Phi^0_a | \hat{S}^{(2)}_\eta | \Phi^0_a \rangle \langle \Phi^0_a | \hat{S}_\eta | \Phi^0_a \rangle + \langle \Phi^0_a | \hat{S}_\eta^3 | \Phi^0_a \rangle \right].$$  (A2)

where $\Phi^0_a$ is the solution of the unperturbed Hamiltonian, $\hat{S}_\eta$ is the evolution operator:

$$\hat{S}_\eta = \mathcal{T} \left[ \exp \left( -i e \int d^4x e^{-\eta |t|} \hat{H}_1(x) \right) \right]$$

$$= 1 + \sum_{k=1}^{\infty} \frac{(-i e)^k}{k!} \int d^4x_1 \ldots \int d^4x_k \times e^{-\eta |t|} \ldots e^{-\eta |t|} \mathcal{T} \left[ \hat{H}_1(x_k) \ldots \hat{H}_1(x_1) \right].$$  (A3)

Here $\mathcal{T}[...]$ denotes the time-ordered product of interaction density $\hat{H}_1$, which is

$$\hat{H}_1(x) = \hat{j}_\mu(x) (\hat{A}_\mu(x) + \hat{A}_\mu^{\text{pert}}(x)).$$  (A4)

In equation (A4) $\hat{A}_\mu(x)$ and $\hat{A}_\mu^{\text{pert}}(x)$ denote the operators of photon field and external perturbation, respectively. The electron current operator in equation (A4) is defined as follows

$$\hat{j}_\mu(x) = -\frac{1}{2} e \left[ \hat{\psi} (x) \gamma^\mu, \hat{\psi} (x) \right],$$  (A5)

where $\hat{\psi}(x) = \bar{\psi}(x) \gamma_0$, $\hat{\psi}(x)$ is the operator of fermion field, $\psi$ and $\bar{\psi}$ are the one-electron and its Dirac conjugated wave functions, $\gamma^\mu$ is the Dirac gamma matrices and $x = (t, \vec{r})$ represents the four-space coordinate vector.

Below we deal only with one-electron atomic systems, so we exclude graphs with more than one fermion line in the initial and final states. In addition, in the perturbing potential $\hat{A}_\mu^{\text{pert}}$ we leave only the terms of the first order.

According to the adiabatic $S$-matrix formalism, the matrix elements up to $\hat{S}^{(3)}_\eta$ in different orders in $e$ are equal to the following expressions:

$$\langle \Phi^0_a | \hat{S}^{(1)}_\eta | \Phi^0_a \rangle = -i e \int d^4x \bar{\psi}_a(x) e^{-\eta |t|} A_\mu^{\text{pert}}(x) \psi_a(x),$$  (A6)
which is a first-order contribution. The second-order contribution (self-energy correction) is
\[
\langle \Phi_0 | S^{(2)}_\eta | \Phi_0 \rangle = (-i\epsilon)^2 \int d^3x_1 d^3x_2 \bar{\psi}_\mu(x_1) e^{-i|\eta|\mu_1} S(x_1, x_2) e^{-i|\eta|\mu_2} \psi_\mu(x_2).
\] (A7)

The matrix elements of the third order are given by
\[
\langle \Phi_0 | S^{(3)}_\eta | \Phi_0 \rangle_{\text{Fig.3}(a)} = (-i\epsilon)^3 \int d^3x_1 d^3x_2 d^3x_3 \times \bar{\psi}_\mu(x_1) e^{-i|\eta|\mu_1} D_{\mu_1\mu_2}^{\text{pert}}(x_1, x_2) S(x_1, x_2) e^{-i|\eta|\mu_3} \psi_\mu(x_3),
\]
and
\[
\langle \Phi_0 | S^{(3)}_\eta | \Phi_0 \rangle_{\text{Fig.3}(b)} = \langle \Phi_0 | S^{(3)}_\eta | \Phi_0 \rangle_{\text{Fig.3}(c)}
\]
\[
= (-i\epsilon)^3 \int d^3x_1 d^3x_2 d^3x_3 \times \bar{\psi}_\mu(x_2) e^{-i|\eta|\mu_1} D_{\mu_2\mu_3}^{\text{pert}}(x_1, x_2) e^{-i|\eta|\mu_2} A_{\mu_3}^{\text{pert}}(x_3) S(x_1, x_2) \psi_\mu(x_3).
\] (A8)

Here \( \psi_\mu(x) = \psi(\vec{r})_\mu e^{iE_\mu t}, \psi(\vec{r})_\mu \) is the solution of the Dirac equation for an one-electron ion with charge \( Z \) in a state with energy \( E_\mu \). The electron propagator \( S(x_1, x_2) \) is
\[
S(x_1, x_2) = \frac{i}{2\pi} \int_{-\infty}^{+\infty} d\Omega \ e^{-i|\eta|\Omega} \sum_n \frac{\psi_\mu(\vec{r}_n) \bar{\psi}_\mu(\vec{r}_n)}{\Omega - E_\mu(1 - i0)},
\] (A10)
where the sum over \( n \) in equation (A10) runs over the entire Dirac spectrum. The photon propagator in the coordinate space representation (in the Feynman gauge \([31]\)) can be reduced to
\[
D_{\mu\nu}(x_1, x_2) = \frac{g_{\mu\nu}}{2\pi tr_{12}} \int_{-\infty}^{+\infty} d\omega \ e^{i(\omega t_1 - \omega t_2)},
\] (A11)
where \( n_2 \equiv \vec{n} - \vec{n}_1, \vec{n}_1 \).

Integration over the times included in the matrix elements can be performed using the ratio:
\[
\int_{-\infty}^{+\infty} dt \ e^{i\omega t - \omega t - |\eta| t} = \frac{2\eta}{(E_\mu - E_\mu - \omega)^2 + \eta^2},
\] (A12)
Having in mind the limit \( \eta \to 0 \) one can also obtain useful relations:
\[
\int_{0}^{\infty} d\omega \frac{4\eta^2}{[(E_\mu - E_\mu - \omega)^2 + \eta^2]^2} = \frac{2\pi}{\eta},
\]
\[
\int_{0}^{\infty} d\omega \frac{4\eta^2}{[(E_\mu - E_\mu - \omega)^2 + \eta^2]^2} = \frac{2\pi(E_\mu - E_\mu)}{\eta}.
\] (A13)

Then, by substituting equations (A6)–(A9) into equation (A2), the total energy shift can be written as a sum of the vertex (figure 1 (a)) and wave-function (figures 1(b), (c)) contributions \([62]\). Performing integration over time variables and taking the limit \( \eta \to 0 + \) \([63]\), the expression for the vertex type correction reads
\[
\Delta E_{\text{vert}} = \frac{e^3}{2\pi i} \sum_{n, k} \int_{-\infty}^{+\infty} d\omega \ \frac{\langle k | 1 - \frac{\partial |n}\rangle \langle n | \alpha | n \rangle}{E_\mu - \omega - E_\mu(1 - i0)} \times \frac{\langle \gamma | A^{\text{pert}}_\nu | k \rangle}{E_\mu - \omega - E_k(1 - i0)},
\]
where \( \alpha \) denotes the vector of Dirac matrices. A similar evaluation for the wave-function part yields (where for the case \( n = a \) one should first use the Taylor series expansion at \( E_n \to E_n \), and then take the limit \( \eta \to 0 \))
\[
\Delta E_{a}^{\text{wf}} = \frac{e^3}{\pi} \int_{-\infty}^{+\infty} d\omega \sum_{\alpha = a} \frac{\langle n | \alpha | n \rangle \langle \gamma | A^{\text{pert}}_\nu | n \rangle}{E_\mu - \omega - E_\mu(1 - i0)} \times \frac{\langle k | A^{\text{pert}}_\nu | k \rangle}{E_\mu - \omega - E_k(1 - i0)} \times \frac{\langle \gamma | \alpha | n \rangle}{E_\mu - \omega - E_\mu(1 - i0)}.
\] (A14)
Here the ‘reference-state’ contribution, \( \kappa = a \), is represented by the second term in equation (A15), see \([64]\) for details.

For a specific interaction, \( e\gamma^\nu A^{\text{pert}}_\nu = \hat{\mu} \vec{B} \), where \( \vec{B} \) is the homogeneous magnetic field, and the magnetic moment operator \( \hat{\mu} \) in the nonrelativistic limit is given by \( \hat{\mu} = -\mu_0 (\vec{l} + 2\vec{s}) = -\mu_0 (\vec{j} + \vec{s}) \) (here \( \mu_0 \) is the Bohr magneton, \( \vec{l} \) and \( \vec{s} \) are the orbital momentum and electron spin operators, \( \vec{j} = \vec{l} + \vec{s} \) is the operator of total angular momentum), one can obtain \([40]\):
\[
\langle n'l's'j'm'j'|j\rangle_{nljm} = \delta_{l'n'}\delta_{j'j}\delta_{m'm'} \frac{j(j+1)(2j+1)}{2j'+1} C_{jm\rightarrow}^{jm'}(\alpha', \beta')
\]

\[
\langle n'l's'j'|j\rangle_{nljm} = \delta_{l'n'}\delta_{j'j} \times (-1)^{j'+l+s-1-m} \Pi_j \Pi_s \sqrt{s(s+1)(2s+1)} N_j
\]

(17)

The above expressions use standard notations: \(\Pi_j, \Pi_s\) = \(\sqrt{(2n+1)(2b+1)}\), \(C_{jm\rightarrow}^{jm'}\) is the Clebsch–Gordan coefficient for the composition of \(|jm\rangle\) in terms of \(|jm',jm\rangle\); in equation (17) the factor given in parentheses refers to the ordinary 3jm-symbol, and the curly braces denote the \(6j\)-symbol [40].

According to equations (16), (17), only diagonal matrix elements survive for the \(\mu B\) interaction in equations (14), (15). Thus, the first term in equation (15) vanishes, resulting in

\[
\Delta E_n = \frac{\Delta E_{na}^{\text{rr}} + \Delta E_{na}^{\text{rf}}}{2n+1} + \frac{\Delta E_{na}^{\text{rf}}}{2n+1} \sum_n \left\langle \frac{[\langle n|\tilde{\mu}\tilde{B}|n\rangle]_r - \langle a|\tilde{\mu}|a\rangle}{|E_n - \Delta|} \right\rangle^2
\]

(18)

To evaluate the expression (18), it can be simplified by the relation:

\[
\frac{1}{|E_n - \Delta|} = -\frac{\partial}{\partial E_n} \frac{1}{E_n - \Delta}
\]

(19)

Then, the integration over the poles can be performed in the same way as in [31].

Imposing \(\Ima \eta = \int_{-\infty}^{\infty} d\omega \ e^{i\omega \eta} / (\Delta E_{na} + \omega - i\Omega)\), one can find

\[
\Ima \eta \Rightarrow \frac{\pi}{2} \frac{E_n}{|E_n|} \left(1 - \frac{\Delta E_{na}}{|\Delta E_{na}|}\right) e^{i\omega} + \frac{\Delta E_{na}}{|\Delta E_{na}|} \left[\sin(x) - \sin(x) \cos(x)\right]
\]

(20)

where \(x \equiv |\Delta E_{na}| \rho_{12}, \Delta E_{na} \equiv E_n - E_a\) and the sine and cosine integrals are defined as

\[
\sin(x) = -\int_{\pi}^{\infty} \frac{\sin t}{t} dt, \quad \cos(x) = -\int_{\pi}^{\infty} \frac{\cos t}{t} dt.
\]

(21)

In the nonrelativistic approximation, when \(\Delta E_{na} \rho_{12} \approx \alpha Z \ll 1\), the series expansion of \(\Ima \eta \Rightarrow \approx \alpha Z \ll 1\) gives

\[
\Ima \eta \approx -\pi i + 2i \Delta E_{na} \eta_{12} - \pi (|\Delta E_{na}| - \Delta E_{na}) \eta_{12} + 2i \Delta E_{na} \eta_{12} \ln(\gamma |\Delta E_{na}| \eta_{12}) - \frac{\pi i}{2} (|\Delta E_{na}| \eta_{12})^2
\]

\[
\frac{1}{6} + \frac{\pi}{6} (|\Delta E_{na}| - \Delta E_{na}) \Delta E_{na} \eta_{12}^2 - \frac{i}{3} (|\Delta E_{na}| \eta_{12})^3 \ln(\gamma |\Delta E_{na}| \eta_{12}),
\]

(22)

where \(\gamma = 0.57721...\) is the Euler’s constant.

The radiative correction to the energy of an arbitrary atomic state \(\alpha\) can be obtained by substituting the imaginary part of (22) into expression (18). The resulting real contribution diverges and requires a renormalization procedure [56]. In turn, the real part of \(\Ima \eta \Rightarrow \approx \alpha Z \ll 1\) converges and leads to the imaginary part of the energy shift, which can be associated with the radiative correction to the excited state level width.

The imaginary part, \(\Im \Delta E_{na}\), in the leading order is

\[
\Im \Delta E_{na} = -\frac{e^2}{2} \sum_n \left[\langle n|\tilde{\mu}\tilde{B}|n\rangle - \langle a|\tilde{\mu}|a\rangle\right] \times \frac{\partial}{\partial E_n} |\Delta E_{na}| = \Delta E_{na}
\]

(23)

Then, using the relations \(\Delta E_{na} \Rightarrow \approx \alpha Z \ll 1\), one can find

\[
\Im \Delta E_{na} = \frac{e^2}{3} \frac{\partial}{\partial E_n} \sum_n (|\Delta E_{na}| - \Delta E_{na}) \Delta E_{na}^2 \times \langle \langle n|\tilde{\mu}|n\rangle \langle n|\tilde{\mu}|a\rangle |\langle n|\tilde{\mu}|n\rangle - \langle a|\tilde{\mu}|a\rangle\rangle.
\]

(24)

Finally, with \(\sum_n (|\Delta E_{na}| - \Delta E_{na}) = 2\sum_{m<n} \delta_{mn}\) and considering that the level width is related to the imaginary part of the radiative shift corresponding to the one-loop self-energy correction of the bound electron as

\[
-2 \Im \Delta E_{na} = \Gamma_{na},
\]

(25)
one can arrive at

$$\delta \Gamma_a = 4 \varepsilon^2 \sum_{n < a} \Delta E_{na}^2 \left\{ |\langle a | f | n \rangle|^2 | \langle n | \mu \bar{B} | n \rangle - \langle a | \mu \bar{B} | a \rangle | \right\}. \tag{A26}$$

The scalar product of two coordinate operators in equations (A24), (A26) can decomposed via spherical components as follows \((\tilde{r}_n \tilde{r}_2) = \sum_{q} \delta_{q,1} (-1)^q r_2 r_{2-q}. In the lsjm, coupling scheme, it can be evaluated with

$$\langle n | l's'j|m'|r_q|lsjmr \rangle = \delta_{l',l} \delta_{s',s} \delta_{j,j} (-1)^{l+j+q+1-m} \times \left( \begin{array}{ccc} j' & 1 & q \\ -m' & j & m \end{array} \right) \Pi_{s'} \Pi_{l'} \Pi_{m'} \left( \begin{array}{ccc} l' & j' & s \\ 1 & 0 & 0 \end{array} \right) \int_{0}^{\infty} r d r^3 R_{w}(r) R_{w}(r), \tag{A27} \right.$$  

\(R_{w}(r)\) is the radul part of the Schrödinger wave function, \(\Pi_{a} = \sqrt{(2a + 1)}\) and the usual notation for 3j-symbols are employed [40].

The calculation of the imaginary part of the correction in figure 1 corresponds exactly to the approach used in [46], where integration over the poles in the energy denominator was carried out, see equations (A17), (A18). Since the matrix element of \(\mu \bar{B}\) is real, the imaginary part of the diagrams in figure 1 is the same as for the ‘ordinary’ one-loop self-energy correction [31]. In addition, it should be noted that, unlike the real part, the imaginary part converges and does not require regularization, see [31] and [46, 65] for applications within the NRQED theory. Another way to integrate over the poles (using the Sokhotski-Plemelj theorem) will be used in the next section.

Appendix B. Basic formulas for the radiative correction to the level width in the presence of a magnetic field combined with blackbody radiation

Next, we demonstrate another possible way to obtain the radiative correction corresponding to the Feynman diagrams shown in figure 1 for graphs where the usual photon loop is replaced by a thermal one. In this case the photon propagator (A11) should be replaced by

$$D_{\beta}^{(1)}(x_1, x_2) = -\frac{s_{\beta \omega}}{\pi \omega_{12}} \int_{-\infty}^{\infty} d\omega \, n_{\beta}(\omega) \times \sin(\omega_{12}) e^{-i\omega(t_1 - t_2)}, \tag{B1}$$

where \(n_{\beta}(\omega) = (\exp(\beta \omega) - 1)^{-1}\) is the photon density number of BBR field (Planck distribution function), \(\beta = 1/k_B T, k_B\) is the Boltzmann constant in relativistic units, \(T\) is the radiation temperature in kelvin and \(\tau_{12} \equiv |\vec{r}_1 - \vec{r}_2|\). The details of the thermal photon propagator derivation in the form equation (B1) can be found in [66, 67].

Inserting equation (B1) into equations (A6)–(A9) leads to the expressions [39]:

$$\Delta E_{a}^{\text{ver,} \beta} = -\frac{e^3}{\pi} \sum_{n \neq m} \int_{0}^{\infty} d\omega \, n_{\beta}(\omega) \times \left\{ \frac{\langle am | 1 - \hat{\alpha}_{12}^* \sin(\omega_{12}) | na \rangle \langle n | \gamma^\nu A_{1}^{\text{pert}} | m \rangle}{[E_a \pm \omega - E_n(1 - i0)][E_a \pm \omega - E_m(1 - i0)]} \right\}, \tag{B2}$$

where \(\sum_{\beta}\) denotes the sum of the two contributions with ‘+’ and ‘−’ in the energy denominator. A similar evaluation for the wave function part gives

$$\Delta E_{a}^{\text{wf,} \beta} = -\frac{2e^3}{\pi} \sum_{n \neq m} \int_{0}^{\infty} d\omega \, n_{\beta}(\omega) \times \left\{ \frac{\langle am | 1 - \hat{\alpha}_{12}^* \sin(\omega_{12}) | nm \rangle \langle m | \gamma^\nu A_{1}^{\text{pert}} | a \rangle}{[E_a \pm \omega - E_n(1 - i0)][E_a \pm \omega - E_m]} - \frac{1}{2} \sum_{n} \frac{\langle am | 1 - \hat{\alpha}_{12}^* \sin(\omega_{12}) | na \rangle \langle a | \gamma^\nu A_{1}^{\text{pert}} | a \rangle}{[E_a \pm \omega - E_n(1 - i0)]^2} \right\}. \tag{B3}$$

Then, again for a specific interaction \(\mu \bar{B}\), see equations (A16), (A17), one can obtain

$$\Delta E_{a}^{\beta} = -\frac{e^3}{\pi} \sum_{n \neq m} \left\{ \langle n | \gamma^\nu A_{1}^{\text{pert}} | n \rangle - \langle a | \gamma^\nu A_{1}^{\text{pert}} | a \rangle \right\} \int_{0}^{\infty} d\omega \, n_{\beta}(\omega) \times \frac{\langle am | 1 - \hat{\alpha}_{12}^* \sin(\omega_{12}) | na \rangle}{[E_a \pm \omega - E_n(1 - i0)]^2} \tag{B4}.$$  

The frequency integration can be performed using equation (A19) followed by application of the Sokhotski-Plemelj theorem, or using a generalized expression:
\[
\frac{1}{(x + i \epsilon)^2} \rightarrow \text{P.V.} \frac{1}{x^2} + i\pi \delta'(x), \tag{B5}
\]

where \( \delta' \) in the \( \delta \)-function denotes the derivative with respect to \( x \), and P. V. corresponds to the principal value.

Since the real and imaginary parts of \( \Delta E_{\alpha} \) are determined through the energy denominator in equation (B4), only the part containing the derivative of the \( \delta \)-function can be considered to obtain a correction to the level width. This leads to

\[
\text{Im} \Delta E_{\alpha} = -e^2 \sum_n \left[ (\alpha | \bar{B} | \alpha) - (\alpha | \bar{B} | \alpha) \right] \times \frac{\partial}{\partial E_{\alpha}} n_0(|\Delta E_{\alpha}|) \langle \alpha | n \rangle \left[ 1 - \frac{\alpha_1 \alpha_2}{n_2} \right] \sin(|\Delta E_{\alpha}|n_2) |na|, \tag{B6}
\]

where, in contrast to equation (A26), the state \( n \) can be lower and higher with respect to \( \alpha \).

Again applying the nonrelativistic limit for the matrix element \( \langle \alpha | n \rangle \), one can find

\[
\left( \frac{1 - \alpha_1 \alpha_2}{n_2} \right) \sin(|\Delta E_{\alpha}|n_2) \approx \frac{-2}{3} |\Delta E_{\alpha}|^3 \langle |\bar{r}|n \rangle^2. \tag{B7}
\]

Substituting (B7) into equation (B6) and using equation (A25) we get

\[
\delta \Gamma_{\alpha} = \frac{4e^2}{3} \sum_n \left[ (\alpha | \bar{B} | \alpha) - (\alpha | \bar{B} | \alpha) \right] \langle |\bar{r}|n \rangle^2 \times [4 |\Delta E_{\alpha}|^3 n_0^2(|\Delta E_{\alpha}|) e^3 |\Delta E_{\alpha}|^2 - 3 |\Delta E_{\alpha}|^2 n_0(|\Delta E_{\alpha}|)], \tag{B8}
\]

The formula (B8) represents the main result discussed in section 3.

ORCID iDs

D Solovyev https://orcid.org/0000-0003-0634-0906

T Zalialiutdinov https://orcid.org/0000-0001-8468-0901

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