Theoretical and numerical study of the decay in a viscoelastic Bresse System

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Abstract

In this paper, we consider a one-dimensional finite-memory Bresse system with homogeneous Dirichlet-Neumann-Neumann boundary conditions. We prove some general decay results for the energy associated with the system in the case of equal and non-equal speeds of wave propagation under appropriate conditions on the relaxation function. In addition, we show by giving an example that in the case of equal speeds of wave propagation and for certain polynomially decaying relaxation functions, our result gives an optimal decay rate in the sense that the decay rate of the system is exactly the same as that of the relaxation function considered.

1 Introduction

Bresse system is a mathematical model that describes the vibration of a planar, linear shearable curved beam. The model was first derived by Bresse [6] and it consists of three coupled wave equations given by

\[\begin{align*}
\rho_1 \varphi_{tt} - k_1 (\varphi_x + \psi + lw)_x - lk_3 (w_x - \varphi) + F_1 &= 0 \quad \text{in } (0, L) \times (0, \infty), \\
\rho_2 \psi_{tt} - k_2 \psi_{xx} + k_1 (\varphi_x + \psi + lw) + F_2 &= 0 \quad \text{in } (0, L) \times (0, \infty), \\
\rho_1 w_{tt} - k_3 (w_x - \varphi)_x + lk_1 (\varphi_x + \psi + lw) + F_3 &= 0 \quad \text{in } (0, L) \times (0, \infty),
\end{align*}\]

(1.1)

where \(\varphi, \psi, w\) represent the vertical displacement, the shear angle, and the longitudinal displacement, respectively; \(\rho_1, \rho_2, k_1, k_2, k_3, l\) are positive parameters and \(F_1, F_2, F_3\) are external forces.

A lot of results dealing with well-posedness and asymptotic behaviour of the above system have been published. We start with the work of Santos et al. [28] from 2010, where they studied...
the Bresse system with Dirichlet-Dirichlet-Dirichlet boundary conditions and linear frictional
damping acting on each equation, that is,

\[(F_1, F_2, F_3) = (\gamma_1 \varphi_t, \gamma_2 \psi_t, \gamma_3 w_t),\]  \tag{1.2}

where \(\gamma_1, \gamma_2, \gamma_3 > 0\). They established an exponential decay rate for the system using spectral
theory approach developed by Z. Liu and S. Zheng in [18]. They also gave a numerical scheme
using finite difference method to illustrate their theoretical result. Soriano
et al. [29] used the
method developed by Lasiecka and Tataru in [16] and proved a uniform decay rate for the
same system with a nonlinear frictional damping acting on the second equation and locally
distributed nonlinear damping acting on the other equations. Precisely, the external forces are
given by

\[(F_1, F_2, F_3) = (\alpha(x)g_1(\varphi_t), g_2(\psi_t), \gamma(x)g_3(w_t))\]

with \(\alpha, \gamma \in L^\infty(0,L)\) and the \(g_i\)'s are continuous and monotone increasing functions. The
results of [28] and [29] were established without imposing any restriction on the speeds of wave
propagation given by

\[s_1 = \sqrt{\frac{k_1}{\rho_1}}, \quad s_2 = \sqrt{\frac{k_2}{\rho_2}}, \quad \text{and} \quad s_3 = \sqrt{\frac{k_3}{\rho_3}}.\]  \tag{1.3}

Alves et al. [4] used the semigroup and spectral theory to obtain the exponential stability of
the Bresse system with three controls at the boundary.

In the presence of dissipating terms in only one or two of the equations in system (1.1), the
decay rates of the energy associated to the system depend totally on the speeds of the wave
propagation. As illustrated in [2], Alabau-Boussouira et al. studied (1.1) with linear frictional
damping acting on the second equation; that is, they used (1.2), with \(\gamma_1 = \gamma_3 = 0\) and \(\gamma_2 > 0\)
and showed that the system is exponentially stable if and only if it has equal speeds of wave
propagation,

\[\frac{k_1}{\rho_1} = \frac{k_2}{\rho_2} = \frac{k_3}{\rho_3}.\]  \tag{1.4}

As mentioned by many authors [2, 3], relation (1.4) is physically unrealistic. In the case of
non-equal speeds of wave propagation, they proved polynomial stability with rates which can be
improved with the regularity of the initial data. Fatori and Monteiro [8] improved this result
in the case of non-equal speeds of wave propagation by proving optimal decay rate. Soriano
et al. [30] established the same exponential stability result as in [2] by replacing the frictional
damping with indefinite one; that is, they replaced \(\gamma_2\) in [2] with a function \(a : (0, L) \rightarrow \mathbb{R}\)
such that \(\bar{a} = \frac{1}{L} \int_0^L a(x)dx > 0\) and \(\|a - \bar{a}\|_{L^2(0,L)}\) is small enough. Wehbe and Youcef [31]
inspected the situation of two locally distributed dampings acting on the last two equations;
that is,

\[(F_1, F_2, F_3) = (0, a_1(x)\psi_t, a_2(x)w_t),\]

where \(a_i : (0, L) \rightarrow \mathbb{R}\) are non-negative functions which can take value zero on some part of the
interval \((0, L)\). By using the frequency domain and the multiplier method, they proved that the
system is exponentially stable if and only if \(s_1 = s_2\). When \(s_1 \neq s_2\) they established a polynomial
decay rate which can be improved with the regularity of the initial data. The same result was
established by Alves et al. in [3], in the case of non-equal speeds of wave propagation, they used the recent result of Borichev and Tomilov in [5] to show that the solution is polynomially stable with optimal decay rate.

Concerning the dissipation via heat effect, we mention the work of Liu and Rao [17] where the following system

\[
\begin{align*}
\rho_1 \phi_{tt} - k_1 (\phi_x + \psi + lw)_x - l k_3 (w_x - \varphi) + l \gamma \chi &= 0 \quad \text{in } (0, L) \times (0, \infty), \\
\rho_2 \psi_{tt} - k_2 \psi_{xx} + k_1 (\phi_x + \psi + lw) + \gamma \theta_x &= 0 \quad \text{in } (0, L) \times (0, \infty), \\
\rho_1 w_{tt} - k_3 (w_x - \varphi)_x + l k_1 (\phi_x + \psi + lw) + \gamma \chi_t &= 0 \quad \text{in } (0, L) \times (0, \infty), \\
\rho_3 \theta_t - \theta_{xx} + \gamma \psi_{xt} &= 0 \quad \text{in } (0, L) \times (0, \infty), \\
\rho_3 \chi_t - \chi_{xx} + \gamma (w_x - l \varphi)_t &= 0 \quad \text{in } (0, L) \times (0, \infty),
\end{align*}
\]

with boundary and initial conditions was considered. They showed that the exponential stability of the system is equivalent to the validity of the identity (1.4). In the case where (1.4) does not hold, they established a polynomial-type decay rate. Fatori and Muñoz Rivera [9] obtained a similar result as in [17] for the thermoelastic Bresse system (1.5) when the fifth equation is omitted. They also showed that the polynomial decay rate is optimal in the case of non-equal speeds of wave propagation. Filippo Dell’Oro [7] gave a detail stability analysis of the thermoelastic Bresse-Gurtin-Pipkin system of the form:

\[
\begin{align*}
\rho_1 \phi_{tt} - k_1 (\phi_x + \psi + lw)_x - l k_0 (w_x - \varphi) &= 0 \quad \text{in } (0, L) \times (0, \infty), \\
\rho_2 \psi_{tt} - k_2 \psi_{xx} + k_1 (\phi_x + \psi + lw) + \gamma \theta_x &= 0 \quad \text{in } (0, L) \times (0, \infty), \\
\rho_1 w_{tt} - k_0 (w_x - \varphi)_x + l k_1 (\phi_x + \psi + lw) &= 0 \quad \text{in } (0, L) \times (0, \infty), \\
\rho_3 \theta_t - k_1 \int_0^\infty g(s)\theta_{xx}(t-s)ds + \gamma \psi_{xt} &= 0 \quad \text{in } (0, L) \times (0, \infty),
\end{align*}
\]

where \( g \) is a bounded convex integrable function on \([0, \infty)\) satisfying

\[
\int_0^\infty g(s)ds = 1,
\]

and there exists a non-increasing absolutely continuous function \( \mu : (0, \infty) \rightarrow [0, \infty) \) such that

\[
\mu(0) = \lim_{s \to 0} \mu(s) \in (0, \infty), \quad g(s) = \int_s^\infty \mu(\tau)d\tau, \quad \forall s \in [0, \infty)
\]

and

\[
\mu'(s) + \nu \mu(s) \leq 0 \quad \text{for some } \nu > 0 \quad \text{and } a.e. \ s \in (0, \infty).
\]

By introducing a new stability number of the form

\[
\chi_g = \left( \frac{\rho_1}{\rho_3 k} - \frac{1}{g(0) k_1} \right) \left( \frac{\rho_1}{k} - \frac{\rho_2}{b} \right) - \frac{1}{g(0) k \rho_3 b k'},
\]

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he proved that the semigroup generated by (1.6) is exponentially stable if and only if
\[ \chi_g = 0 \quad \text{and} \quad k = k_0. \]

As a special case, he showed that his stability result gave the stability characterization of Bresse systems with Fourier, Maxwell-Cataneo and Coleman-Gurtin thermal dissipation. The reader is referred to [1, 10, 15, 22, 23, 24, 25, 26] and the references therein for more recent results on thermoelastic Bresse system.

There are few results that dealt with stabilization of Bresse system via infinite memory. We begin with the work of Guesmia and Kafini [12] in 2015. They studied the following system

\[
\begin{align*}
\rho_1 \varphi_{tt} - k_1 (\varphi_x + \psi + lw_x) - lk_3 (w_x - l \varphi) + \int_0^\infty g_1 (s) \varphi_{xx} (x, t - s) ds &= 0, \\
\rho_2 \psi_{tt} - k_2 \psi_{xx} + k_1 (\varphi_x + \psi + lw) + \int_0^\infty g_2 (s) \psi_{xx} (x, t - s) ds &= 0, \\
\rho_1 w_{tt} - lk_3 (w_x - l \varphi)_x + lk_1 (\varphi_x + \psi + lw) + \int_0^\infty g_3 (s) w_{xx} (x, t - s) ds &= 0,
\end{align*}
\]

\[(1.7)\]

where \((x,t) \in (0,L) \times \mathbb{R}_+\), \(g_i : \mathbb{R}_+ \rightarrow \mathbb{R}_+\) are differentiable non-increasing and integrable functions, and \(L, l, \rho_i, k_i\) are positive constants. They proved the well-posedness and the asymptotic stability of (1.7). Later, Guesmia and Kirane [13] used two infinite memories to obtain the same stability result of [12] under the following conditions on the speeds of wave propagation:

\[
k_1 \rho_1 = k_2 \rho_2 \quad \text{in case } g_1 = 0, \quad k_1 \rho_1 = k_2 \rho_2 \quad \text{in case } g_2 = 0, \quad k_1 \rho_1 = k_3 \rho_3 \quad \text{in case } g_3 = 0.
\]

Santos et al. [27] discussed the Bresse system with only one infinite memory acting on the shear angle displacement equation. Precisely, they studied problem (1.7) with

\[
g_1 = g_3 = 0 \quad \text{and} \quad g_2 \quad \text{satisfying} : \quad -\alpha_1 g_2 (t) \leq g_2' (t) \leq -\alpha_2 g_2 (t), \quad \forall t \geq 0,
\]

for some \(\alpha_1, \alpha_2 > 0\). They showed that the solution of the system decays exponentially to zero if and only if (1.4) holds, otherwise a polynomial stability of the system with an optimal decay rate of type \(t^{-1/2}\) was obtained. Recently, Guesmia [11] analysed the asymptotic stability of Bresse system with one infinite memory in the longitudinal displacement.

To the best of our knowledge, there is no result in the literature that deals with the stability of Bresse system via viscoelastic damping of finite memory-type. In
There exists a non-increasing differentiable function $g$ such that

$$g(0) > 0 \quad \text{and} \quad k_2 - \int_0^{+\infty} g(s) ds > 0.$$  

(A2) There exists a non-increasing differentiable function $\xi : [0, \infty) \longrightarrow (0, \infty)$ and a constant $p$, with $1 \leq p < \frac{3}{2}$, such that

$$g'(t) \leq -\xi(t)g^p(t), \quad \forall t \geq 0.$$  

In this paper we will discuss the decay property of the following finite memory-type Bresse system:

$$\left\{ \begin{array}{ll}
\rho_1 \varphi_{tt} - k_1 (\varphi_x + \psi + lw)_x - lk_3 (w_x - l\varphi) = 0, & \text{in } (0, L) \times (0, +\infty), \\
\rho_2 \psi_{tt} - k_2 \psi_{xx} + k_1 (\varphi_x + \psi + lw) + \int_0^t g(t - s) \psi_{xx}(s) ds = 0, & \text{in } (0, L) \times (0, +\infty), \\
\rho_1 w_{tt} - k_3 (w_x - l\varphi)_x + lk_1 (\varphi_x + \psi + lw) = 0, & \text{in } (0, L) \times (0, +\infty), \\
\varphi(0, t) = \varphi(L, t) = \psi_x(0, t) = \psi_x(L, t) = w_x(0, t) = w_x(L, t) = 0, & \text{for } t \geq 0, \\
\varphi(x, 0) = \varphi_0(x), \quad \varphi_t(x, 0) = \varphi_1(x), & \text{for } x \in (0, L), \\
\psi(x, 0) = \psi_0(x), \quad \psi_t(x, 0) = \psi_1(x), & \text{for } x \in (0, L), \\
w(x, 0) = w_0(x), \quad w_t(x, 0) = w_1(x), & \text{for } x \in (0, L), \\
\end{array} \right.$$  

(P)

where $l$, $k_1$, $k_2$, $k_3$, $\rho_1$, $\rho_2$ are positive constants, $\varphi_0$, $\varphi_1$, $\psi_0$, $\psi_1$, $w_0$, $w_1$ are given data and $g$ is a relaxation function satisfying some conditions to be specified in the next section. Our problem is motivated by the following classical Bresse system

$$\begin{align*}
\rho_1 \varphi_{tt} - S_x - lN &= 0 \text{ in } (0, L) \times (0, \infty), \\
\rho_2 \psi_{tt} - M_x + S &= 0 \text{ in } (0, L) \times (0, \infty), \\
\rho_1 w_{tt} - N_x - lS &= 0 \text{ in } (0, L) \times (0, \infty),
\end{align*}$$

where $t$ and $x$ represent the time and space variables, respectively, and $N$, $S$ and $M$ denote the axial force, the shear force and the bending moment given by

$$S = k_1 (\varphi_x + \psi + w), \quad M = k_2 \psi_x - \int_0^t g(t - s) \psi_x(s, \cdot) ds, \quad N = k_3 (w_x - \varphi).$$

We will prove, under a smallness condition on $l$, generalized energy decay results for the system in the case of equal and different speeds of wave propagation. This paper is organized as follows: in Section 2, we state some preliminary results. In Section 3, we state and prove some technical lemmas. The statement and proof of our main results are given in Sections 4 and 5, while in Section 6 we present some numerical illustrations to validate our results. Through out this work we use $c$ to represent a generic positive constant, independent of $t$ but may depend on the initial data.

2 Preliminaries

In this section, we introduce our assumptions, present some useful lemmas and state the existence theorem.

**Assumptions:** We assume that the relaxation function $g$ satisfies the following hypotheses:

(A1) $g : [0, \infty) \longrightarrow [0, \infty)$ is a non-increasing differentiable function such that

$$g(0) > 0 \quad \text{and} \quad k_2 - \int_0^{+\infty} g(s) ds > 0.$$  

(A2) There exists a non-increasing differentiable function $\xi : [0, \infty) \longrightarrow (0, \infty)$ and a constant $p$, with $1 \leq p < \frac{3}{2}$, such that

$$g'(t) \leq -\xi(t)g^p(t), \quad \forall t \geq 0.$$  

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Lemma 2.1. Assume that $g$ satisfies hypotheses (A1) and (A2). Then,
\[
\int_0^{+\infty} \xi(t)g^{1-\sigma}(t)dt < +\infty, \quad \forall 0 < \sigma < 2 - p.
\]
Proof. From (A1), we have
\[
\lim_{t \to +\infty} g(t) = 0.
\]
Using (A2), we have
\[
\int_0^{+\infty} \xi(t)g^{1-\sigma}(t)dt = \int_0^{+\infty} \xi(t)g^\sigma(t)dt - \int_0^{+\infty} g'(t)g^{1-\sigma-p}(t)dt < +\infty,
\]
since $\sigma < 2 - p$.

Now, integrating both sides of the second and third equations in (P) over $(0, L)$ and using
the boundary conditions, we get
\[
\begin{align*}
\frac{d^2}{dt^2} \int_0^L \psi(x,t)dx + \frac{k_1}{\rho_2} \int_0^L \psi(x,t)dx + \frac{lk_1}{\rho_2} \int_0^L w(x,t)dx &= 0 \quad \forall t \geq 0 \quad (2.1) \\
\frac{d^2}{dt^2} \int_0^L w(x,t)dx + \frac{l^2k_1}{\rho_1} \int_0^L \psi(x,t)dx + \frac{l k_1}{\rho_1} \int_0^L w(x,t)dx &= 0 \quad \forall t \geq 0. \quad (2.2)
\end{align*}
\]
Solving these ODEs simultaneously yields
\[
\int_0^L \psi(x,t)dx = a_1 \cos(a_0t) + a_2 \sin(a_0t) + a_3t + a_4 \quad (2.3)
\]
and
\[
\int_0^L w(x,t)dx = \frac{a_1}{l} \left( \frac{\rho_2a_0^2}{k_1} - 1 \right) \cos(a_0t) + \frac{a_2}{l} \left( \frac{\rho_2a_0^2}{k_1} - 1 \right) \sin(a_0t) - \frac{a_3}{l} t - \frac{a_4}{l}, \quad (2.4)
\]
where
\[
\begin{align*}
a_0 &= \sqrt{\frac{k_1}{\rho_2} + \frac{l^2 k_1}{\rho_1}} \\
a_1 &= \frac{k_1}{\rho_2 a_0^2} \int_0^L \psi_0(x)dx + \frac{lk_1}{\rho_2 a_0^2} \int_0^L w_0(x)dx, \\
a_2 &= \frac{k_1}{\rho_2 a_0^3} \int_0^L \psi_1(x)dx + \frac{lk_1}{\rho_2 a_0^3} \int_0^L w_1(x)dx, \\
a_3 &= \left( 1 - \frac{k_1}{\rho_2 a_0^2} \right) \int_0^L \psi_1(x)dx - \frac{lk_1}{\rho_2 a_0^2} \int_0^L w_1(x)dx, \\
a_4 &= \left( 1 - \frac{k_1}{\rho_2 a_0^2} \right) \int_0^L \psi_0(x)dx + \frac{lk_1}{\rho_2 a_0^2} \int_0^L w_0(x)dx.
\end{align*}
\]
Therefore, we perform the following change of variables
\[
\tilde{\psi} = \psi - \frac{1}{L} \left(a_1 \cos(a_0 t) + a_2 \sin(a_0 t) + a_3 t + a_4\right)
\]
\[
\tilde{w} = w - \frac{1}{L} \left[\frac{a_1}{l} \left(\frac{\rho_2 a_0^2}{k_1} - 1\right) \cos(a_0 t) + \frac{a_2}{l} \left(\frac{\rho_2 a_0^2}{k_1} - 1\right) \sin(a_0 t) - \frac{a_3}{l} t - \frac{a_4}{l}\right]
\]
to get
\[
\int_0^L \tilde{\psi}(x,t) dx = \int_0^L \tilde{w}(x,t) dx = 0, \quad \forall t \geq 0.
\]
Furthermore, \((\varphi, \tilde{\psi}, \tilde{w})\) satisfies the equations and the boundary conditions in \((P)\) with the initial data
\[
\tilde{\psi}_0 = \psi_0 - \frac{1}{L} (a_1 + a_4), \quad \tilde{\psi}_1 = \psi_1 - \frac{1}{L} (a_0 a_2 + a_3)
\]
\[
\tilde{w}_0 = w_0 - \frac{1}{L} \left[\frac{a_1}{l} \left(\frac{\rho_2 a_0^2}{k_1} - 1\right) - \frac{a_3}{l}\right], \quad \tilde{w}_1 = w_1 - \frac{1}{L} \left[\frac{a_2 a_0}{l} \left(\frac{\rho_2 a_0^2}{k_1} - 1\right) - \frac{a_3}{l}\right].
\]
From now on, we work with \(\tilde{\psi}, \tilde{w}\) and, respectively, write \(\psi, w\) for convenience. We also introduce the following spaces,
\[
L^2_*(0,L) := \left\{w \in L^2(0,L) : \int_0^L w(x) dx = 0\right\}, \quad H^1_*(0,L) := H^1(0,L) \cap L^2_*(0,L),
\]
and
\[
H^2_*(0,L) := \left\{w \in H^2(0,L) : w_x(0) = w_x(L) = 0\right\}.
\]
Then, Poincaré’s inequality is applicable to the elements of \(H^1_*(0,L)\), that is,
\[
\exists c_0 > 0 \text{ such that } \int_0^L v^2 dx \leq c_0 \int_0^L v_x^2 dx \quad \forall v \in H^1_*(0,L). \tag{2.5}
\]
For completeness, we state, without proof the global existence and regularity result which can be established by repeating the steps of the proof of the existence result in [21].

**Theorem 2.1.** Let \((\varphi_0, \varphi_1) \in H^1_0(0,L) \times L^2(0,L)\) and \((\psi_0, \psi_1), (w_0, w_1) \in H^1_*(0,L) \times L^2_*(0,L)\) be given. Assume that \(g\) satisfies hypothesis (A1). Then, the problem \((P)\) has a unique global (weak) solution
\[
\varphi \in C(\mathbb{R}_+; H^1_0(0,L)) \cap C^1(\mathbb{R}_+; L^2(0,L)), \quad \psi, w \in C(\mathbb{R}_+; H^1_*(0,L)) \cap C^1(\mathbb{R}_+; L^2_*(0,L)).
\]
Moreover, if
\[
(\varphi_0, \varphi_1) \in (H^2(0,L) \cap H^1_0(0,L)) \times H^1_0(0,L)
\]
and
\[
(\psi_0, \psi_1), (w_0, w_1) \in (H^2_*(0,L) \cap H^1_*(0,L)) \times H^1_*(0,L),
\]
then
\[
\varphi \in C(\mathbb{R}_+; H^2(0,L) \cap H^1_0(0,L)) \cap C^1(\mathbb{R}_+; H^1_0(0,L)) \cap C^2(\mathbb{R}_+; L^2(0,L)),
\]
and
\[
\psi, w \in C(\mathbb{R}_+; H^2_*(0,L) \cap H^1_*(0,L)) \cap C^1(\mathbb{R}_+; H^1_*(0,L)) \cap C^2(\mathbb{R}_+; L^2(0,L)).
\]
Now, we introduce the energy functional

\[
E(t) := \frac{1}{2} \int_0^L \left[ \rho_1 \varphi_t^2 + \rho_2 \psi_t^2 + \rho_1 w_t^2 + \left( k_2 - \int_0^t g(s) ds \right) \psi_x^2 \\
+ k_3 (w_x - \dot{\varphi})^2 + k_1 (\varphi_x + \psi + lw)^2 \right] dx + \frac{1}{2} (g \circ \psi_x)(t), \quad \forall t \geq 0,
\]

(2.6)

where for any \( v \in L^2_{loc}([0, +\infty); L^2(0, L)) \),

\[
(g \circ v)(t) := \int_0^L \int_0^t g(t-s)(v(t) - v(s))^2 ds dx.
\]

By multiplying the equations in (P) by \( \varphi_t, \psi_t, w_t \), respectively, integrating over \((0, L)\) and exploiting the boundary conditions we have the following lemma.

**Lemma 2.2.** Let \((\varphi, \psi, w)\) be the weak solution of (P). Then,

\[
E'(t) = -\frac{1}{2} \int_0^L \psi_x^2 dx + \frac{1}{2} (g' \circ \psi_x)(t) \leq 0, \quad \forall t \geq 0.
\]

(2.7)

From the Cauchy-Schwarz and Poicaré’s inequalities we have the following lemma.

**Lemma 2.3** ([19]). There exists a constant \( c > 0 \) such that for any \( v \in L^2_{loc}(\mathbb{R}^+; H^1_v(0, L)) \), we have

\[
\int_0^L \left( \int_0^t g(t-s)(v(t) - v(s)) ds \right)^2 dx \leq c(g \circ v_x)(t), \quad \forall t \geq 0.
\]

**Lemma 2.4** ([19]). Assume that conditions (A1) and (A2) hold and let \((\varphi, \psi, w)\) be the weak solution of (P). Then, for any \( 0 < \sigma < 1 \), we have

\[
g \circ \psi_x \leq c \left[ \int_0^t g^{1-\sigma}(s) ds \right]^\frac{p-1}{p+\sigma-1} (g^\sigma \circ \psi_x)^\frac{1}{p+\sigma-1}.
\]

For \( \sigma = \frac{1}{2} \), we obtain the following inequality

\[
g \circ \psi_x \leq c \left( \int_0^t g^{1/2}(s) ds \right)^{2p-2} (g^p \circ \psi_x)^{\frac{1}{p+1}}.
\]

(2.8)

**Corollary 2.1.** Assume that \( g \) satisfies (A1), (A2) and \((\varphi, \psi, w)\) is the weak solution of (P). Then,

\[
\xi(t)(g \circ \psi_x)(t) \leq c(-E'(t))^{\frac{1}{2p-1}}, \quad \forall t \geq 0.
\]

**Proof.** Multiplying both sides of the inequality (2.8) by \( \xi(t) \) and using Lemmas 2.1 and 2.2, we get

\[
\xi(t)(g \circ \psi_x)(t) \leq c \xi^{\frac{2p-2}{2p-1}}(t) \left( \int_0^t g^{1/2}(s) ds \right)^{\frac{2p-2}{2p-1}} (\xi g^p \circ \psi_x)^{\frac{1}{2p-1}}(t)
\]

\[
\leq c \left( \int_0^t \xi(s) g^{1/2}(s) ds \right)^{\frac{2p-2}{2p-1}} (-g' \circ \psi_x)^{\frac{1}{2p-1}} \leq c(-E'(t))^{\frac{1}{2p-1}}.
\]
Lemma 2.5 (Jensen’s inequality). Let $G : [a, b] \rightarrow \mathbb{R}$ be a concave function. Assume that the functions $f : \Omega \rightarrow [a, b]$ and $h : \Omega \rightarrow \mathbb{R}$ are integrable such that $h(x) \geq 0$, for any $x \in \Omega$ and $\int_{\Omega} h(x)dx = k > 0$. Then,

$$\frac{1}{k} \int_{\Omega} G(f(x))h(x)dx \leq G\left(\frac{1}{k} \int_{\Omega} f(x)h(x)dx\right).$$

In particular, for $G(y) = y^{\frac{1}{p}}$, $y \geq 0$, $p > 1$, we have

$$\frac{1}{k} \int_{\Omega} f^{1/p}(x)h(x)dx \leq \left(\frac{1}{k} \int_{\Omega} f(x)h(x)dx\right)^{1/p}.$$

3 Technical Lemmas

In this section, we state and prove some lemmas needed to establish our main results. All the computations are done for regular solutions but they still hold for weak and strong solutions by a density argument.

Lemma 3.1. Assume that conditions (A1) and (A2) hold. Then, the functional $I_1$ defined by

$$I_1(t) := -\rho_2 \int_{0}^{L} \psi_{t} \int_{0}^{t} g(t-s)(\psi(t)-\psi(s))dsdx$$

satisfies, along the solution of $(P)$, the estimates

$$I_1(t) \leq -\rho_2 \left( \int_{0}^{t} g(s)ds - \delta \right) \int_{0}^{L} \psi_{t}^2 dx + \delta \int_{0}^{L} (\varphi_{x} + \psi + lw)^2 dx + c\delta \int_{0}^{L} \psi_{x}^2 dx + \frac{c}{\delta} (g \circ \psi_{x} - g' \circ \psi_{x}), \quad \forall \delta > 0. \quad (3.1)$$

Proof. Differentiating $I_1$, using equations in $(P)$ and integrating by parts, we get

$$I_1'(t) = -\rho_2 \int_{0}^{L} \psi_{t} \int_{0}^{t} g'(t-s)(\psi(t)-\psi(s))dsdx - \rho_2 \left( \int_{0}^{t} g(s)ds \right) \int_{0}^{L} \psi_{t}^2 dx + k_2 \int_{0}^{L} \psi_{x} \int_{0}^{t} g(t-s)(\psi_{x}(t)-\psi_{x}(s))dsdx$$

$$+k_1 \int_{0}^{L} (\varphi_{x} + \psi + lw) \int_{0}^{t} g(t-s)(\psi(t)-\psi(s))dsdx - \int_{0}^{L} \left( \int_{0}^{t} g(t-s)\psi_{x}(s)ds \right) \left( \int_{0}^{t} g(t-s)(\psi_{x}(t)-\psi_{x}(s))ds \right) dx.$$

Next, we estimate the terms on the right-hand side of the above equation.

Using Young’s inequality and Lemma 2.3 for $(-g')$, we obtain, for any $\delta > 0$,

$$-\rho_2 \int_{0}^{L} \psi_{t} \int_{0}^{t} g'(t-s)(\psi(t)-\psi(s))dsdx \leq \delta \rho_2 \int_{0}^{L} \psi_{t}^2 dx - \frac{c}{\delta} (g' \circ \psi_{x}).$$
Similarly, we have
\[ k_2 \int_0^L \psi_x \int_0^t g(t-s)(\psi_x(t) - \psi_x(s))ds dx \leq \delta \int_0^L \psi_x^2 + \frac{c}{\delta} (g \circ \psi_x), \]
\[ k_1 \int_0^L (\varphi_x + \psi + lw) \int_0^t g(t-s)(\psi(t) - \psi(s))ds dx \leq k_1 \delta \int_0^L (\varphi_x + \psi + lw)^2 dx + \frac{c}{\delta} (g \circ \psi_x), \]
and
\[ - \int_0^L \left( \int_0^t g(t-s)\psi_x(s)ds \right) \left( \int_0^t g(t-s)(\psi_x(t) - \psi_x(s))ds \right) dx \leq c \delta \int_0^L \psi_x^2 dx + c \left( \delta + \frac{1}{\delta} \right) (g \circ \psi_x). \]

A combination of these estimates gives the desired result.

\[ \square \]

**Lemma 3.2.** Assume that the hypotheses (A1) and (A2) hold. Then, for any \( \varepsilon_0, \delta_1 > 0 \), the functional \( I_2 \) defined by
\[ I_2(t) := -\rho_1 k_3 \int_0^L (w_x - l\varphi) \int_0^x w_1(y,t)dy dx - \rho_1 k_1 \int_0^x (\varphi_x + \psi + lw)(y,t)dy dx \]
satisfies, along the solution of \( (P) \), the estimate
\[ I_2'(t) \leq k_1^2 \int_0^L (\varphi_x + \psi + lw)^2 dx - k_3^2 \int_0^L (w_x - l\varphi)^2 dx + \frac{c}{\varepsilon_0} \int_0^L \psi_x^2 dx + \left( \varepsilon_0 - \rho_1 k_1 + \frac{l \rho_1 |k_3 - k_1| \delta_1}{2} \right) \int_0^L \psi_x^2 dx \]
\[ + \rho_1 \left( k_3 + \frac{c_0 l |k_3 - k_1|}{2 \delta_1} \right) \int_0^L w_1^2 dx. \]

**Proof.** Differentiation of \( I_2 \), using equations in \( (P) \) and integration by parts yield
\[ I_2 = \rho_1 k_3 \int_0^L w_1^2 dx + l \rho_1 k_3 \int_0^L \psi_x \int_0^x w_1(y,t)dy dx - k_3^2 \int_0^L (w_x - l\varphi)^2 dx \]
\[ + k_1^2 \int_0^L (\varphi_x + \psi + lw)^2 dx - \rho_1 k_1 \int_0^L \varphi_x^2 dx - \rho_1 k_1 \int_0^x (\varphi_x + \psi + lw)(y,t)dy dx. \]

Using Young’s inequality, we get, for any \( \varepsilon_0, \delta_1 > 0 \),
\[ I_2 \leq k_1^2 \int_0^L (\varphi_x + \psi + lw)^2 dx - k_3^2 \int_0^L (w_x - l\varphi)^2 dx + \frac{c}{\varepsilon_0} \int_0^L \psi_x^2 dx \]
\[ + \left( \varepsilon_0 - \rho_1 k_1 + \frac{l \rho_1 |k_3 - k_1| \delta_1}{2} \right) \int_0^L \psi_x^2 dx + \rho_1 \left( k_3 + \frac{c_0 l |k_3 - k_1|}{2 \delta_1} \right) \int_0^L w_1^2 dx. \]

\[ \square \]
**Lemma 3.3.** Under the conditions (A1) and (A2), the functional $I_3$ defined by

$$I_3(t) := -\rho_1 \int_0^L (\varphi_x + \psi + lw)w_t dx - \frac{k_3 \rho_1}{k_1} \int_0^L (w_x - l\varphi)\varphi_t dx$$

satisfies, along the solution of (P) and for any $\varepsilon_0 > 0$, the estimate

$$I_3'(t) \leq l k_1 \int_0^L (\varphi_x + \psi + lw)^2 dx - \frac{\frac{k_3^2}{k_1}}{\varepsilon_0} \int_0^L (w_x - l\varphi)^2 dx + \frac{c}{\varepsilon_0} \int_0^L \psi_t^2 dx$$

$$+ \frac{l \rho_1 k_3}{k_1} \int_0^L \varphi_t^2 dx + (\varepsilon_0 - l \rho_1) \int_0^L w_t^2 dx + \rho_1 \left( \frac{k_3}{k_1} - 1 \right) \int_0^L \varphi_x w_t dx. \quad (3.3)$$

**Proof.** Differentiating $I_3$, using equations in (P) and integrating by parts, we have

$$I_3 = -\rho_1 \int_0^L \psi_t w_t dx - l \rho_1 \int_0^L w_t^2 dx + l k_1 \int_0^L (\varphi_x + \psi + lw)^2 dx$$

$$+ \frac{l \rho_1 k_3}{k_1} \int_0^L \varphi_t^2 dx + (\varepsilon_0 - l \rho_1) \int_0^L w_t^2 dx + \rho_1 \left( \frac{k_3}{k_1} - 1 \right) \int_0^L \varphi_x w_t dx.$$ 

Use of Young’s inequality for the first term in the right-hand side gives (3.3). \qed

**Lemma 3.4.** Assume that conditions (A1) and (A2) hold. Then for any $\delta > 0$, the functional $I_4$ defined by

$$I_4(t) := -\int_0^L (\rho_1 \varphi \varphi_t + \rho_2 \psi \psi_t + \rho_1 \psi w_t) dx$$

satisfies, along the solution of (P), the estimate

$$I_4'(t) \leq -\int_0^L (\rho_1 \varphi_t^2 + \rho_2 \psi_t^2 + \rho_1 \psi w_t^2) dx + k_1 \int_0^L (\varphi_x + \psi + lw)^2 dx$$

$$+ k_3 \int_0^L (w_x - l\varphi)^2 dx + \left( \frac{k_2}{\delta} - \int_0^t g(s) ds \right) \int_0^L \psi_t^2 dx + \frac{c}{\delta} (g \circ \psi_x). \quad (3.4)$$

**Proof.** Differentiation of $I_4$, using equations of (P) gives

$$I_4'(t) = -\int_0^L (\rho_1 \varphi_t^2 + \rho_2 \psi_t^2 + \rho_1 \psi w_t^2) dx + k_1 \int_0^L (\varphi_x + \psi + lw)^2 dx + k_3 \int_0^L (w_x - l\varphi)^2 dx$$

$$+ \left( k_2 - \int_0^t g(s) ds \right) \int_0^L \psi_x^2 dx - \int_0^L \psi_x \int_0^t g(t - s) (\varphi_x(t) - \varphi_x(s))ds dx.$$ 

Repeating the above computations yields the desired result. \qed

**Lemma 3.5.** Assume that conditions (A1) and (A2) hold. Then for any $\delta, \delta > 0$, the functional $I_5$ defined by

$$I_5(t) := -\rho_2 \int_0^L \psi_x \int_0^x \psi_t(y, t) dy dx$$

satisfies, along the solution of (P), the estimate

$$I_5'(t) \leq \rho_2 \int_0^L \psi_t^2 dx + \left( k_1 \frac{1}{2 \delta_2} + \int_0^t g(s) ds + \delta - k_2 \right) \int_0^L \psi_x^2 dx$$

$$+ \frac{c \epsilon k_1 \delta_2}{2} \int_0^L (\varphi_x + \psi + lw)^2 dx + \frac{c}{\delta} (g \circ \psi_x). \quad (3.5)$$
Proof. Using equations of \((P)\) and repeating similar computations as above, we arrive at

\[
I_5'(t) = \rho_2 \int_0^L \psi_t^2 dx - k_2 \int_0^L \psi_s^2 dx + k_1 \int_0^L \psi_x \int_0^x (\varphi_x + \psi + l \omega)(y,t) dy dx \\
+ \int_0^L \psi_x \int_0^t g(t - s) \psi_x(s) ds dx \\
\leq \rho_2 \int_0^L \psi_t^2 dx + \left( \frac{k_1}{2\delta_2} + \int_0^t g(s) ds + k_2 \right) \int_0^L \psi_s^2 dx \\
+ \frac{k_1 \delta_2}{2} \int_0^L \left( \int_0^x (\varphi_x + \psi + l \omega)(y,t) dy \right)^2 dx + \frac{c}{\delta}(g \circ \psi_x).
\]

Poincaré’s inequality for the third term yields (3.5).

**Lemma 3.6.** Assume that the hypotheses (A1) and (A2) hold. Then, for any \(\varepsilon_0, \varepsilon_1, \varepsilon_2, \delta > 0\), the functional \(I_6\) defined by

\[
I_6(t) := \rho_2 \int_0^L \psi_t(\varphi_x + \psi + l \omega) dx + \frac{b \rho_1}{k_1} \int_0^L \varphi_t \psi_x dx - \frac{\rho_1}{k_1} \int_0^t \varphi_t \int_0^t g(t - s) \psi_x(s) ds dx
\]
satisfies, along the solution of \((P)\), the estimate

\[
I_6'(t) \leq -k_1 \int_0^L (\varphi_x + \psi + l \omega)^2 dx + \left( \frac{lk_2 k_3 \varepsilon_1}{2k_1} + \frac{lk_3 \varepsilon_2}{2k_1} \right) \int_0^L (w_x - t \varphi)^2 dx \\
+ \delta \int_0^L \psi_t^2 dx + \left( \frac{lk_2 k_3}{2k_1 \varepsilon_1} + \frac{lk_3}{2k_1 \varepsilon_2} \right) \int_0^t g(s) ds + \frac{c}{\delta} g(t) \int_0^L \psi_x^2 dx \\
+ \varepsilon_0 \int_0^L \psi_t^2 dx + \frac{c}{\varepsilon_0} \int_0^L \psi_t^2 dx + \frac{c}{\delta}(g \circ \psi_x - g' \circ \psi_x) + \left( \frac{k_2 \rho_1}{k_1} - \rho_2 \right) \int_0^L \varphi_t \psi_x dx.
\]

**Proof.** Use of equations of \((P)\) and integration by parts lead to

\[
I_6'(t) = -k_1 \int_0^L (\varphi_x + \psi + l \omega)^2 dx + \rho_2 \int_0^L \psi_t^2 dx + \rho_2 \int_0^L \psi_t w_t dx \\
+ \frac{lk_2 k_3}{k_1} \int_0^L (w_x - t \varphi) \psi_x dx - \frac{lk_3}{k_1} \int_0^L (w_x - t \varphi) \int_0^t g(t - s) \psi_x(s) ds dx \\
- \frac{\rho_1}{k_1} g(t) \int_0^L \varphi_t \psi_x dx + \frac{\rho_1}{k_1} \int_0^L \varphi_t \int_0^t g'(t - s) (\psi_x(t) - \psi_x(s)) ds dx \\
+ \left( \frac{k_2 \rho_1}{k_1} - \rho_2 \right) \int_0^L \varphi_x \psi_x dt dx.
\]

Next, we estimate the terms in the right-hand side of the above equation.

Exploiting Young’s inequality, we get

\[
l \rho_2 \int_0^L \psi_t w_t dx \leq \varepsilon_0 \int_0^L w_t^2 dx + \frac{c}{\varepsilon_0} \int_0^L \psi_t^2 dx, \quad \forall \varepsilon_0 > 0.
\]
Using Young’s inequality and Lemma 2.8, we obtain, for any $\varepsilon_1, \varepsilon_2, \delta > 0$,
\[
\frac{lk_2k_3}{k_1} \int_0^L (w_x - l\varphi)\psi_x dx - \frac{lk_3}{k_1} \int_0^L (w_x - l\varphi) \int_0^t g(t-s)\psi_x(s) ds dx
\]
\[= \frac{lk_3}{k_1} \left( k_2 - \int_0^t g(s) ds \right) \int_0^L (w_x - l\varphi)\psi_x dx
\]
\[+ \frac{lk_3}{k_1} \int_0^L (w_x - l\varphi) \int_0^t g(t-s)(\psi_x(t) - \psi_x(s)) ds dx
\]
\[\leq \left( \frac{lk_2k_3\varepsilon_1}{2k_1} + \frac{lk_3\varepsilon_2}{2k_1} \int_0^t g(s) ds + \delta \right) \int_0^L (w_x - l\varphi)^2 dx
\]
\[+ \left( \frac{lk_2k_3}{2k_1\varepsilon_1} + \frac{lk_3}{2k_1\varepsilon_2} \int_0^t g(s) ds \right) \int_0^L \psi_x^2 dx + \frac{c}{\delta} (g \circ \psi_x)
\]
and
\[
- \frac{\rho_1}{k_1} g(t) \int_0^L \varphi t \psi_x dx + \frac{\rho_1}{k_1} \int_0^L \varphi t \int_0^t g'(t-s)(\psi_x(t) - \psi_x(s)) ds dx
\]
\[\leq \delta \int_0^L \varphi_t^2 dx + \frac{c}{\delta} g(t) \int_0^L \psi_x^2 dx - \frac{c}{\delta} (g' \circ \psi_x)
\]
A combination of these estimates gives the desired result.

\[\square\]

4 General Decay Rates for Equal Speeds of Wave Propagation

In this section, we state and prove a general decay result under equal speeds of wave propagation condition. The exponential and polynomial decay results are only special cases.

Theorem 4.1. Let $(\varphi_0, \varphi_1) \in H^1_0(0, L) \times L^2(0, L)$ and $(\psi_0, \psi_1), (w_0, w_1) \in H^1(0, L) \times L^2(0, L)$. Assume that (A1) and (A2) hold and that
\[
\frac{k_1}{\rho_1} = \frac{k_2}{\rho_2} \quad \text{and} \quad k_1 = k_3. \tag{4.1}
\]
Then for $l$ small enough and for any $t_0 > 0$, the solution of (P) satisfies, for $t > t_0$,
\[
E(t) \leq C \exp \left( -\lambda \int_{t_0}^t \xi(s) ds \right), \quad \text{for} \ p = 1, \tag{4.2}
\]
and
\[
E(t) \leq C \left( \frac{1}{1 + \int_{t_0}^t \xi^{2p-1}(s) ds} \right)^{\frac{1}{2p-2}}, \quad \text{for} \ 1 < p < \frac{3}{2}, \tag{4.3}
\]
where $C > 0$ is a constant independent of $t$ but may depend on the initial data and $\lambda > 0$ is a constant independent of both $t$ and the initial data. Moreover, if
\[
\int_{t_0}^{+\infty} \left( \frac{1}{1 + \int_{t_0}^t \xi^{2p-1}(s) ds} \right)^{\frac{1}{2p-2}} dt < +\infty, \quad \text{for} \ 1 < p < \frac{3}{2}, \tag{4.4}
\]
then
\[ E(t) \leq C \left( \frac{1}{1 + \int_0^t \xi^p(s)ds} \right)^{\frac{1}{p-1}}, \quad \text{for } 1 < p < \frac{3}{2}. \] (4.5)

**Remark 4.1.** Inequalities (4.3) and (4.4) together give

\[ \int_0^{+\infty} E(t)dt < +\infty. \]

**Remark 4.2.** The smallness condition on \( l \) makes the Bresse system close to Timoshenko system and, hence, inherits some of its stability properties.

**Proof of Theorem 4.1.** Define a functional \( \mathcal{L} \) by

\[ \mathcal{L} := NE + \sum_{j=1}^6 N_j I_j, \]

where \( N, N_j > 0 \) for \( j = 1, 2, \ldots, 6 \) with \( N_3 = N_6 = 1 \). Then from (3.1) – (3.6) we have

\[
\mathcal{L}'(t) \leq \left[ -\rho_1(k_1N_2 + N_4) + \frac{l\rho_1|k_3 - k_1|\delta_1N_2}{2} + \frac{l\rho_1k_3}{k_1} + \varepsilon_0N_2 + \delta \right] \int_0^L \varphi_x^2dx \\
+ \left[ -\rho_2 \left( N_1 \int_0^t g(s)ds + N_4 - N_5 \right) + \rho_2\delta N_1 + \frac{c}{\varepsilon_0}(1 + N_2) \right] \int_0^L \psi_t^2dx \\
+ \left[ -l\rho_1 + \rho_1(k_3N_2 - N_4) + \frac{c\rho_1|k_3 - k_1|N_2}{2\delta_1} + \varepsilon_0 \right] \int_0^L w_t^2dx \\
+ \left[ (N_5 - N_4) \int_0^t g(s)ds + k_2(N_4 - N_5) + \frac{k_1N_5}{2\delta_2} + \frac{lk_2k_3}{2k_1\varepsilon_1} \right] \int_0^L \psi_t^2dx \\
+ \left[ -\frac{lk_3^2}{k_1} - k_3(k_3N_2 - N_4) + \frac{lk_2k_3\varepsilon_1}{2k_1} + \frac{lk_2k_3\varepsilon_2}{2k_1} \right] \int_0^t g(s)ds + \delta \int_0^L (w_x - l\varphi)^2dx \\
+ \left[ -k_1 \left( 1 - k_1N_2 - l - N_4 - \frac{c_0\delta_2N_5}{2} \right) + \delta N_1 \right] \int_0^L (\varphi_x + \psi + lw)^2dx \\
+ \frac{c}{\delta}(1 + N_1 + N_4 + N_5)g \circ \psi_x - \frac{c}{\delta}(1 + N_1)g' \circ \psi_x + NE'(t) \\
+ \left( \frac{k_2\rho_1}{k_1} - \rho_2 \right) \int_0^L \varphi \psi_{xx}dx + \rho_1 \left( \frac{k_3}{k_1} - 1 \right) \int_0^L \varphi_xw_tdx.
\]

By setting \( \delta_1 = 1 \), \( N_4 = k_3N_2 \), \( N_5 = 4k_3N_2 \), \( \delta_2 = \frac{k_1}{k_2 - \rho_0} \), \( \varepsilon_1 = \frac{k_2}{k_1^2} \), and \( \varepsilon_2 = \frac{k_3}{2\rho_0} \), where \( g_0 = \int_0^\infty g(s)ds \), we arrive at...
\[\mathcal{L}'(t) \leq -\rho_1 \left[ (k_1 + k_3) N_2 - l \left( \frac{|k_3 - k_1|}{2} N_2 + \frac{k_3}{k_1} \right) \right] \int_0^L \varphi^2_t dx
- \rho_2 \left( N_1 \int_0^L g(s) ds - 3k_3 N_2 - \frac{l}{\rho_1 \rho_2} \right) \int_0^L \psi^2_t dx - \frac{l k_3^2}{4 k_1} \int_0^L (w_x - l \varphi)^2 dx
- \left[ (k_2 - g_0) k_3 N_2 - \frac{l}{k_1} \left( \frac{k_2^2}{2} + g_0^2 \right) \right] \int_0^L \psi^2_2 dx - \frac{l k_3^2}{4 k_1} \int_0^L (w_x - l \varphi)^2 dx
- k_1 \left[ 1 - \left( k_1 + k_3 + \frac{2 c_0 k_1 k_3}{k_2 - g_0} \right) N_2 - l \right] \int_0^L (\varphi_x + \psi + l w)^2 dx
+ \left( \frac{k_3}{k_1} - 1 \right) \int_0^L \varphi_x w_t dx.
\]

Now, we set \( \varepsilon_0 = \frac{l \rho_1}{2(1+N_2)} \), to get

\[\mathcal{L}'(t) \leq -\rho_1 \left[ (k_1 + k_3) N_2 - l \left( \frac{1}{2} + \frac{k_3}{k_1} \right) \right] \int_0^L \varphi^2_t dx
- \rho_2 \left( N_1 \int_0^L g(s) ds - 3k_3 N_2 - \frac{c(1 + N_2)^2}{l \rho_1 \rho_2} \right) \int_0^L \psi^2_t dx - \frac{l k_3^2}{4 k_1} \int_0^L (w_x - l \varphi)^2 dx
- \frac{l \rho_1}{2} \left( 1 - c_0 |k_3 - k_1| N_2 \right) \int_0^L w^2_t dx - \frac{l k_3^2}{4 k_1} \int_0^L (w_x - l \varphi)^2 dx
- k_1 \left[ 1 - \left( k_1 + k_3 + \frac{2 c_0 k_1 k_3}{k_2 - g_0} \right) N_2 - l \right] \int_0^L (\varphi_x + \psi + l w)^2 dx
+ \frac{c}{\delta} \left( 1 + N_1 \right) \int_0^L \psi^2_2 dx + \frac{c}{\delta} \left( 1 + N_1 \right) \int_0^L \psi^2_2 dx + \frac{c}{\delta} \left( 1 + N_1 \right) \int_0^L \psi^2_2 dx
+ \left( \frac{k_3}{k_1} - 1 \right) \int_0^L \varphi_x w_t dx.
\]

Fix \( t_0 > 0 \) and choose \( N_2 \) so small that

\[1 - c_0 |k_3 - k_1| N_2 > 0 \quad \text{and} \quad 1 - \left( k_1 + k_3 + \frac{2 c_0 k_1 k_3}{k_2 - g_0} \right) N_2 > 0.
\]

Next, we select \( l \) small enough so that

\[ (k_1 + k_3) N_2 - l \left( \frac{1}{2} + \frac{k_3}{k_1} \right) > 0, \quad (k_2 - g_0) k_3 N_2 - \frac{l k_3^2}{k_1} + g_0 \]
and
\[ 1 - \left( k_1 + k_3 + \frac{2c_0k_1k_3}{k_2 - g_0} \right) N_2 - l > 0. \]

After that, we pick \( N_1 \) very large so that
\[ N_1 \int_0^{t_0} g(s) ds - 3k_3N_2 - \frac{c(1 + N_2)^2}{t_0\rho_1\rho_2} > 0. \]

Therefore, we have
\[
\mathcal{L}'(t) \leq -(\beta - c\delta)E(t) + \left( N - \frac{c}{\delta} \right) E'(t) + \frac{c}{\delta} g \circ \psi_x \\
+ \left( \frac{k_2\rho_1}{k_1} - \rho_2 \right) \int_0^L \varphi_t \psi_t dx + \rho_1 \left( \frac{k_3}{k_1} - 1 \right) \int_0^L \varphi_x w_t dx,
\]
for some \( \beta > 0 \). At this point, we take \( \delta < \frac{\beta}{c} \). Consequently, we obtain, for some \( k > 0 \),
\[
\mathcal{L}'(t) \leq -kE(t) + (N - c)E'(t) + c(g \circ \psi_x) \\
+ \left( \frac{k_2\rho_1}{k_1} - \rho_2 \right) \int_0^L \varphi_t \psi_t dx + \rho_1 \left( \frac{k_3}{k_1} - 1 \right) \int_0^L \varphi_x w_t dx, \quad \forall t \geq t_0. \quad (4.6)
\]

Finally, we choose \( N \) so large that \( N > c \) and \( \mathcal{L} \sim E \), therefore we have, \( \forall t \geq t_0 \),
\[
\mathcal{L}'(t) \leq -kE(t) + c(g \circ \psi_x) + \left( \frac{k_2\rho_1}{k_1} - \rho_2 \right) \int_0^L \varphi_t \psi_t dx + \rho_1 \left( \frac{k_3}{k_1} - 1 \right) \int_0^L \varphi_x w_t dx. \quad (4.7)
\]

Note that from this point, the proof goes similarly as in [20]. But we will continue for the sake of completeness.

By recalling (4.1) and multiplying both sides of (4.7) by \( \xi(t) \) and using Corollary 2.1, we arrive at
\[
\xi(t)\mathcal{L}'(t) \leq -k\xi(t)E(t) + c\xi(t)(g \circ \psi_x)(t) \leq -k\xi(t)E(t) + c(-E'(t))^\frac{1}{p-1}, \quad \forall t \geq t_0. \quad (4.8)
\]

For \( p = 1 \), it follows from non-increasing property of \( \xi \) and (4.8) that
\[
(\xi(t)\mathcal{L}(t) + cE(t))' \leq \xi(t)\mathcal{L}'(t) + cE'(t) \leq -k\xi(t)E(t), \quad \forall t \geq t_0.
\]

Using the fact that \( \mathcal{F} = \xi\mathcal{L} + cE \sim E \), there exists a \( \lambda > 0 \) such that
\[
\mathcal{F}'(t) \leq -\lambda\xi(t)\mathcal{F}(t), \quad \forall t \geq t_0.
\]

A simple integration over \( (t, t_0) \) leads to
\[
E(t) \leq C \exp \left( -\lambda \int_{t_0}^t \xi(s) ds \right), \quad \forall t \geq t_0.
\]

For \( 1 < p < \frac{3}{2} \), we multiply both sides of (4.8) by \( (\xi E)^\alpha(t) \), with \( \alpha = 2p - 2 \), to obtain
\[
\xi^{\alpha+1}(t)E^\alpha(t)\mathcal{L}'(t) \leq -k(\xi E)^{\alpha+1}(t) + c(\xi E)^\alpha(t)(-E'(t))^\frac{1}{p-1}.
\]
Applying Young’s inequality with \( q = \frac{\alpha + 1}{\alpha} \) and \( q' = \alpha + 1 \), we get

\[
\xi^{\alpha+1}(t)E^{\alpha}(t)L'(t) \leq -(k - c\gamma)(\xi E)^{\alpha+1}(t) - c_x E'(t), \quad \forall \gamma > 0.
\]

We choose \( \gamma \) such that \( \lambda_1 := k - c\gamma > 0 \) and use the non-increasing property of \( \xi \) and \( E \), to have

\[
(\xi^{\alpha+1}E^{\alpha}L)'(t) \leq \xi^{\alpha+1}(t)E^{\alpha}(t)L'(t) \leq -\lambda_1(\xi E)^{\alpha+1}(t) - cE'(t),
\]

this entails that

\[
(\xi^{\alpha+1}E^{\alpha}L + cE)'(t) \leq -\lambda_1(\xi E)^{\alpha+1}(t).
\]

Let \( F = \xi^{\alpha+1}E^{\alpha}L + cE \sim E \), then

\[
F'(t) \leq -\lambda \xi^{\alpha+1}(t)F^{\alpha+1}(t),
\]

for some \( \lambda > 0 \). Integration over \((t_0, t)\) gives

\[
E(t) \leq C \left( \frac{1}{1 + \int_{t_0}^{t} \xi^{2p-1}(s)ds} \right)^{\frac{1}{2p-2}}, \quad \forall t \geq t_0.
\]

This establishes (4.3).

To prove (4.5), we treat (4.8) as follows

\[
\xi(t)L'(t) \leq -k\xi(t)E(t) + c\xi(t)(g \circ \psi_x)(t)
\]

\[
\leq -k\xi(t)E(t) + c \frac{\eta(t)}{\eta(t)} \int_{0}^{t} \left( \xi^p(s)g^p(s) \right)^{\frac{1}{p}} \| \psi_x(t) - \psi_x(t-s) \|_2^2 ds,
\]

for any \( t \geq t_0 \), where

\[
\eta(t) = \int_{0}^{t} \| \psi_x(t) - \psi_x(t-s) \|_2^2 ds \leq 2 \int_{0}^{t} (\| \psi_x(t) \|_2^2 + \| \psi_x(t-s) \|_2^2) ds
\]

\[
\leq 4 \int_{0}^{t} (E(t) + E(t-s)) ds \leq 8 \int_{0}^{t} E(t-s) ds = 8 \int_{0}^{t} E(s) ds
\]

\[
\leq 8 \int_{0}^{\infty} E(s) ds < +\infty,
\]

by Remark 4.1. Applying Jensen’s inequality to the second term in the right-hand side of (4.9), with \( G(y) = y^{\frac{1}{p}}, y > 0, f(s) = \xi^p(s)g^p(s) \) and \( h(s) = \| \psi_x(t) - \psi_x(t-s) \|_2^2 \), we obtain

\[
\xi(t)L'(t) \leq -k\xi(t)E(t) + c\eta(t) \left( \frac{1}{\eta(t)} \int_{0}^{t} \xi^p(s)g^p(s) \| \psi_x(t) - \psi_x(t-s) \|_2^2 ds \right)^{\frac{1}{p}},
\]

where we assume that \( \eta(t) > 0 \), otherwise we get, from (4.7),

\[
E(t) \leq C \exp(-kt), \quad \forall t \geq t_0.
\]
Therefore,

\[
\xi(t)\mathcal{L}'(t) \leq -k\xi(t)E(t) + c\eta^{\frac{p-1}{p}}(t)\left(\xi^{p-1}(0)\int_0^t \xi(s)g^p(s)\|\psi_x(t) - \psi_x(t-s)\|^2ds\right)^\frac{1}{p}
\]

\[
\leq -k\xi(t)E(t) + c(-g' \circ \psi_x)\frac{1}{t}(t) \leq -k\xi(t)E(t) + (E'(t))^{\frac{1}{p}}.
\]

Multiplying both sides of the above inequality by \((\xi E)\alpha(t)\), for \(\alpha = p - 1\), and repeating the above computations, we arrive at

\[
E(t) \leq C\left(\frac{1}{1 + \int_{t_0}^t \xi^p(s)ds}\right)^\frac{1}{p-1}, \quad \forall t > t_0,
\]

which establishes (4.5).

\[\square\]

**Example 4.1.** Let \(g(t) = \frac{a}{(1+t)^q}\) with \(q > 2\), and \(a > 0\) is to be chosen so that (A1) is satisfied. Then

\[
g'(t) = -a_0 \left(\frac{a}{(1+t)^q}\right)^{q+1} = -\xi(t)g^p(t),
\]

with \(\xi(t) = a_0 = \frac{q}{a^{1/q}}\) and \(p = \frac{q+1}{q} < \frac{3}{2}\), we have, for any fixed \(t_0 > 0\),

\[
\int_{t_0}^{+\infty} \left(\frac{1}{1 + \int_{t_0}^t \xi^2(t)ds}\right)^{\frac{1}{p-2}} dt = \int_{t_0}^{+\infty} \left(\frac{1}{1 + c(t-t_0)}\right)^{\frac{1}{p-2}} dt < +\infty.
\]

Therefore, inequality (4.5) entails that there exists \(C > 0\) such that

\[
E(t) \leq C\left(\frac{1}{1 + \int_{t_0}^t \xi^p(s)ds}\right)^\frac{1}{p-1} = \frac{c}{(1+t)^q},
\]

with the optimal decay rate \(q\). For more examples, see [20].

### 5 General Decay Rate for Different Speeds of Wave Propagation

In this section, we state and prove a generalized decay result in the case of non-equal speeds of wave propagation. We start by differentiating both sides of the differential equations in \((P)\) with respect to \(t\) and use the fact that

\[
\frac{\partial}{\partial t}\left[\int_0^t g(t-s)\psi_{xx}(s)ds\right] = \frac{\partial}{\partial t}\left[\int_0^t g(s)\psi_{xx}(t-s)ds\right] = g(t)\psi_{xx}(0) + \int_0^t g(s)\psi_{xxt}(t-s)ds
\]

\[
= \int_0^t g(t-s)\psi_{xxt}(s)ds + g(t)\psi_{0xx},
\]
to obtain the following system

\[
\begin{align*}
\rho_1 \varphi_{ttt} - k_1 (\varphi_{xt} + \psi_t + lw_t)_x - lk_3 (w_{xt} - l\varphi_t) &= 0, \\
\rho_2 \psi_{ttt} - k_2 \psi_{xxt} + k_1 (\varphi_{xt} + \psi_t + lw_t) + \int_0^t g(t - s) \psi_{xxt}(s) ds + g(t) \psi_{0xx} &= 0, \\
\rho_1 w_{ttt} - k_3 (w_{xt} - l\varphi_t)_x + lk_1 (\varphi_{xt} + \psi_t + lw_t) &= 0.
\end{align*}
\]  

\( (P) \)

The energy functional associated to \((P)\) is given by

\[
E^*(t) := \frac{1}{2} \int_0^L \left[ \rho_1 \varphi_{xt}^2 + \rho_2 \psi_{xt}^2 + \rho_1 w_{xt}^2 + \left( k_2 - \int_0^t g(s) ds \right) \psi_{xt}^2 \\
+ k_3 (w_{xt} - l\varphi_t)^2 + k_1 (\varphi_{xt} + \psi_t + lw_t)^2 \right] dx + \frac{1}{2} (g \circ \psi)(t), \quad \forall t \geq 0,
\]  

\( (5.1) \)

Using similar arguments as in [14, Lemma 3.11] we have the following result.

**Lemma 5.1.** Let \((\varphi, \psi, w)\) be the strong solution of \((P)\). Then, the energy of \((P)\) satisfies, for all \(t \geq 0\),

\[
E^*_t = -\frac{1}{2} g(t) \int_0^L \psi_{xt}^2 dx + \frac{1}{2} (g' \circ \psi_{xt}) - g(t) \int_0^L \psi_{xt} \psi_{0xx} dx
\]  

\( (5.2) \)

and

\[
E^*_t \leq c \left( E^*_0 + \int_0^L \psi_{0xx}^2 dx \right). \tag{5.3}
\]

**Lemma 5.2.** Assume that hypotheses \((A1)\) and \((A2)\) hold and let \((\varphi, \psi, w)\) be the strong solution of \((P)\). Then, for any \(0 < \sigma < 1\), we have

\[
g \circ \psi_{xt} \leq c \left( E^*_0 + \int_0^L \psi_{0xx}^2 dx \right) \int_0^t g^{1-\sigma}(s) ds \right]^{\frac{p}{p-1}} (g^p \circ \psi_{xt})^{\frac{1}{p-1}}.
\]  

\( (5.4) \)

In particular, for \(\sigma = \frac{1}{2}\), we get the following inequality

\[
g \circ \psi_{xt} \leq c \left( \int_0^t g^{1/2}(s) ds \right)^{\frac{2p-2}{2p-1}} (g^p \circ \psi_{xt})^{\frac{1}{2p-1}}.
\]  

Proof. By setting \(r = \frac{p + \sigma - 1}{p - 1}\) and \(q = \frac{(p - 1)(1 - \sigma)}{p + \sigma - 1}\), we have \(\frac{r}{r - 1} = \frac{p + \sigma - 1}{\sigma}\) and
\[ 1 - q = \frac{\sigma p}{p + \sigma - 1}. \] Then exploiting Hölder’s inequality and (5.3), we obtain

\[
g \circ \psi_{xt} = \int_0^L \int_0^t g(t-s)(\psi_{xt}(t) - \psi_{xt}(s))^2 dsdx
\]

\[
= \int_0^L \int_0^t \left[ g^q(t-s)(\psi_{xt}(t) - \psi_{xt}(s))^2 \right] g^{1-q}(t-s)(\psi_{xt}(t) - \psi_{xt}(s))^{2q-2} dsdx
\]

\[
\leq \left[ \int_0^L \int_0^t g^{\sigma}(t-s)(\psi_{xt}(t) - \psi_{xt}(s))^2 dsdx \right]^{\frac{1}{\sigma}}
\times \left[ \int_0^L \int_0^t g^{(1-q)r}(t-s)(\psi_{xt}(t) - \psi_{xt}(s))^2 dsdx \right]^{\frac{r-1}{r}}
\]

\[
\leq \left[ \int_0^L \int_0^t g^{1-\sigma}(t-s)(\psi_{xt}(t) - \psi_{xt}(s))^2 dsdx \right]^{\frac{\rho-1}{\rho+\sigma - 1}} (g^\circ \psi_{xt})^{\frac{\alpha}{\rho+\sigma - 1}}
\leq \left[ 2 \int_0^L \int_0^t g^{1-\sigma}(\psi_{xt}^2(t) + \psi_{xt}^2(t-s)) dsdx \right]^{\frac{\rho-1}{\rho+\sigma - 1}} (g^\circ \psi_{xt})^{\frac{\alpha}{\rho+\sigma - 1}}
\leq \left[ \frac{4}{k_2 - g_0} \int_0^t g^{1-\sigma}(E_*(t) + E_*(t-s)) ds \right]^{\frac{\rho-1}{\rho+\sigma - 1}} (g^\circ \psi_{xt})^{\frac{\alpha}{\rho+\sigma - 1}}
\leq \left[ c \left( E_*(0) + \int_0^L \psi_{0xx}^2 dx \right) \right]^{\frac{\rho-1}{\rho+\sigma - 1}} (g^\circ \psi_{xt})^{\frac{\alpha}{\rho+\sigma - 1}}.
\]

For \( \sigma = \frac{1}{2} \), we get (5.4). This completes the proof. \( \square \)

**Corollary 5.1.** Assume that conditions (A1) and (A2) hold and let \((\varphi, \psi, w)\) be the strong solution of \((P)\). Then,

\[
\xi(t)(g \circ \psi_{xt})(t) \leq c \left( -E_*(t) + c_1 g(t) \right)^{\frac{1}{\rho-1}}, \quad \forall t \geq 0,
\]

for some positive constant \(c_1\).

**Proof.** From equation (5.2) and inequality (5.3) we have

\[
0 \leq -g' \circ \psi_{xt} = -2E'_*(t) - g(t) \int_0^L \psi_{xt}^2 dx - 2g(t) \int_0^L \psi_{tt} \psi_{0xx} dx
\]

\[
\leq -2E'_*(t) - 2g(t) \int_0^L \psi_{tt} \psi_{0xx} dx
\]

\[
\leq -2E'_*(t) + g(t) \int_0^L (\psi_{tt}^2 + \psi_{0xx}^2) dx
\]

\[
\leq -2E'_*(t) + g(t) \left( \frac{2}{\rho_1} E_*(t) + \int_0^L \psi_{0xx}^2 dx \right)
\]

\[
\leq c \left( -E'_*(t) + c_1 g(t) \right), \quad \forall t \geq 0.
\]

(5.5)
for some positive constant $c_1$. Multiplication of both sides of (5.4) by $\xi(t)$ and use of Lemma 2.1 and inequality (5.5) give

$$\xi(t)(g \circ \psi_{xt})(t) \leq c \left( \xi(t) \int_0^t g^{1/2}(s) ds \right)^{\frac{2p-2}{2(p-1)}} (\xi g^p \circ \psi_{xt}) \frac{1}{p-1}(t) \leq c \left( \int_0^t \xi(s) g^{1/2}(s) ds \right)^{\frac{2p-2}{2(p-1)}} (g' \circ \psi_{xt}) \frac{1}{p-1}(t) \leq c (-E'(t) + c_1 g(t)) \frac{1}{p-1}.$$

\[\square\]

Now we estimate the third term in the right-hand side of (4.7) as in [14].

**Lemma 5.3.** Let $(\varphi, \psi, w)$ be the strong solution of $(P)$. Then, for any $\varepsilon > 0$, we have

$$\left( \frac{\rho_1 k_2}{k_1} - \rho_2 \right) \int_0^L \varphi_t \psi_{xt} dx \leq \varepsilon E(t) + \frac{c}{\varepsilon} (g \circ \psi_{xt} - E'(t) + g(t)), \quad \forall t \geq t_0. \quad (5.6)$$

**Proof.**

$$\left( \frac{\rho_1 k_2}{k_1} - \rho_2 \right) \int_0^L \varphi_t \psi_{xt} dx = \left( \frac{\rho_1 k_2}{k_1} - \rho_2 \right) \int_0^L g(s) ds \int_0^t \varphi_t \int_0^t g(t-s)(\psi_{xt}(t) - \psi_{xt}(s)) ds dx + \left( \frac{\rho_1 k_2}{k_1} - \rho_2 \right) \int_0^L g(s) ds \int_0^t \varphi_t \int_0^t g(t-s) \psi_{xt}(s) ds dx. \quad (5.7)$$

By observing that $\int_0^t g(s) ds \geq \int_0^{t_0} g(s) ds$, for all $t \geq t_0$ and exploiting Young’s inequality and Lemma 2.3 (for $\psi_{xt}$), we get, for $\varepsilon > 0$ and $t \geq t_0$,

$$\left( \frac{\rho_1 k_2}{k_1} - \rho_2 \right) \int_0^L \varphi_t \int_0^t g(t-s)(\psi_{xt}(t) - \psi_{xt}(s)) ds dx \leq \frac{\varepsilon}{4} \rho_1 \int_0^L \varphi_t^2 dx + \frac{c}{\varepsilon} (g \circ \psi_{xt}).$$

On the other hand, by integration by parts and using Lemma 2.3 (for $-g'$ and $\psi_x$) and the fact that $E$ is non-increasing, we get

$$\left( \frac{\rho_1 k_2}{k_1} - \rho_2 \right) \int_0^t g(s) ds \int_0^L \varphi_t \int_0^t g(t-s) \psi_{xt}(s) ds dx$$

$$= \left( \frac{\rho_1 k_2}{k_1} - \rho_2 \right) \int_0^L \varphi_t \left( g(0) \psi_x - g(t) \psi_{ox} + \int_0^t g'(t-s) \psi_x(s) ds \right) dx$$

$$= \left( \frac{\rho_1 k_2}{k_1} - \rho_2 \right) \int_0^L \varphi_t \left( g(t)(\psi_x - \psi_{ox}) - \int_0^t g'(t-s)(\psi_x(t) - \psi_x(s)) ds \right) dx$$

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\[ \leq \frac{\varepsilon}{4} \rho_1 \int_0^L \varphi^2 dx + \frac{c}{\varepsilon} g(t) \int_0^L (\psi_x^2 + \psi_{xx}^2) dx - \frac{c}{\varepsilon} g' \circ \psi_x \]
\[ \leq \frac{\varepsilon}{4} \rho_1 \int_0^L \varphi^2 dx + \frac{c}{\varepsilon} E(0) g(t) - \frac{c}{\varepsilon} g' \circ \psi_x. \]

Inserting the last two inequalities in (5.7), we get (5.6).

**Theorem 5.1.** Let 
\[ (\varphi_0, \varphi_1) \in (H^2(0, L) \cap H^1_0(0, L)) \times H^1_0(0, L) \]
and 
\[ (\psi_0, \psi_1), (w_0, w_1) \in (H^2(0, L) \cap H^1_0(0, L)) \times H^1_0(0, L). \]
Assume that conditions (A1), (A2) hold and that 
\[ \frac{\rho_1}{k_1} \neq \frac{\rho_2}{k_2} \quad \text{and} \quad k_1 = k_3. \]
Then for \( l \) small enough and for any \( t_0 > 0 \), there exists a positive constant \( C \) that may depend on the initial data but independent of \( t \), for which the strong solution of (P) satisfies, for \( t > t_0 \),
\[ E(t) \leq C \left( \frac{1}{\int_0^t \xi^{2p-1}(s) ds} \right)^{\frac{1}{2p-1}}, \quad \text{for} \quad 1 \leq p < \frac{3}{2}. \]  

**Proof.** Repeating the steps of the proof of Theorem 4.1 up to inequality (4.6), then inserting (5.6) into (4.6) we obtain
\[ \mathcal{L}'(t) \leq -(k - \varepsilon)E(t) + \left[ N - c \left( 1 + \frac{1}{\varepsilon} \right) \right] E'(t) + cg \circ \psi_x + \frac{c}{\varepsilon} g \circ \psi_{xt} + \frac{c}{\varepsilon} g(t), \quad \forall t \geq t_0. \]
Now we choose \( \varepsilon \) so small that \( k - \varepsilon > 0 \), and then pick \( N > c \left( 1 + \frac{1}{\varepsilon} \right) \) to get
\[ \mathcal{L}'(t) \leq -k_0 E(t) + c(g \circ \psi_x + g \circ \psi_{xt}) + cg(t), \quad \forall t \geq t_0, \]
for some \( k_0 > 0 \). We then multiply both sides of the above inequality by \( \xi(t) \) and use Corollaries 2.1 and 5.1 to get
\[ \xi(t) \mathcal{L}'(t) \leq -k_0 \xi(t) E(t) + c \xi(t) (g \circ \psi_x + g \circ \psi_{xt}) + c \xi(t) g(t) \]
\[ \leq -k_0 \xi(t) E(t) + c \left[ \left( -E'(t) \right)^{\frac{1}{2p-1}} + \left( -E'(t) + c_1 g(t) \right)^{\frac{1}{2p-1}} \right] + c \xi(t) g(t). \]
Next, we set \( \alpha = 2p - 2 \), then multiply both sides of the above inequality by \( (\xi E)^\alpha(t) \) and exploit Young’s inequality, with \( q = \frac{\alpha + 1}{\alpha} \) and \( q' = \alpha + 1 \), to obtain
\[ (\xi^{\alpha+1} E^\alpha(t) \mathcal{L}'(t) \leq -(k_0 - c\gamma)(\xi E)^{\alpha+1}(t) - cE'(t) - cE'_x(t) + c_1 g(t) + c \xi^{\alpha+1}(t) E^\alpha(t) g(t), \quad \forall \gamma > 0. \]
We choose \( \gamma > 0 \) so small such that \( \lambda_2 := k_0 - c\gamma > 0 \) and use the non-increasing property of \( \xi \) and \( g \) to get
\[ (\xi^{\alpha+1} E^\alpha \mathcal{L} + cE + cE'_a)'(t) \leq -\lambda_2 (\xi E)^{\alpha+1}(t) + c \xi^{\alpha+1}(t) E^\alpha(t) g(t) + c_1 g(t), \]
which implies that
\[ \lambda_2(\xi E)^{\alpha+1}(t) \leq - (\xi^{\alpha+1} E^\alpha L + cE + cE_*)'(t) + c\xi^{\alpha+1}(t)E^\alpha(t)g(t) + c_1g(t). \]

Then integration over \((t_0, t)\) together with the non-increasing property of \(E\) and \(\xi\), and the hypothesis \((A1)\) yield, for \(t \geq t_0\),
\[
\lambda_2 E^{\alpha+1}(t) \int_{t_0}^t \xi^{\alpha+1}(s)ds \leq \lambda_2 \int_{t_0}^t (\xi E)^{\alpha+1}(s)ds \leq -(\xi^{\alpha+1} E^\alpha L + cE + cE_*)(0) + \int_0^L \psi_0xx^2 dx \\
+ (\xi^{\alpha+1} E^\alpha L + cE + cE_*)(0) + \int_0^t g(s)ds \\
+ (c\xi^{\alpha+1}(0)E^\alpha(0) + c_1) \int_{t_0}^t g(s)ds.
\]

Therefore, we get
\[
E(t) \leq C \left( \frac{1}{\int_{t_0}^t \xi^{2p-1}(s)ds} \right)^{\frac{1}{p-1}}, \quad \forall t > t_0.
\]

This completes the proof of the Theorem 5.1. \(\square\)

**Example 5.1.** Let \(g(t) = e^{-at}\), where \(a > 0\). Then \(g'(t) = -\xi(t)g(t)\) with \(\xi(t) = a\). It follows from (5.8) that for any fixed \(t_0 > 0\), there exists \(C > 0\) such that
\[
E(t) \leq \frac{C}{t - t_0}, \quad \forall t > t_0.
\]

**Example 5.2.** Consider the same function \(g\) as in Example 4.1 and write \(g'\) as in Example 4.1. Then it follows from (5.8) that for any fixed \(t_0 > 0\), there exists \(C > 0\) such that
\[
E(t) \leq C \left( \frac{1}{\int_{t_0}^t \xi^{2p-1}(s)ds} \right)^{\frac{1}{p-1}} = \frac{c}{(1 + t)^{\frac{p}{2}}} \quad \forall t > t_0.
\]

For more examples, see [14].

### 6 Full discrete problem

In this section, we introduce a scheme for the problem based on \(P_1\)-finite element method in space and implicit Euler scheme for time discretization. Then we draw graphs for the discredited energy showing it’s decay in both cases, polynomial and exponential. Finally, we implement the approximation of the solutions \(\varphi, \psi\) and \(w\) in 3D and their cross section at \(x = 0.5\).
6.1 Finite element setup

We denote by \((\Gamma_h)_h\) a partition of \(\Omega\) which fulfills the following conditions:

1. \(\Gamma_h = \{R \subset \bar{\Omega}; R\; \text{is closed in} \; \Omega\};\)
2. \(\forall(R, R') \in \Gamma_h \times \Gamma_h; |R| = |R'|\), where their intersection is either empty or an end point;
3. \(\bar{\Omega} = \bigcup_{R \in \Gamma_h} R.\)

We define the uniform partition of \(\Omega\) as \(0 = x_0 < x_1 < \cdots < x_s\) and denote the length of \((x_j, x_{j+1})\) as \(h = \frac{x_{j+1} - x_j}{s}\). Now for time discretization, denote by \(\Delta t = \frac{T}{N}\) the step time, where \(T\) is the total time and \(N\) is a positive integer. Finally we define the discrete finite element space by

\[
s_h = \{u_h \in H^1(0, L) : \forall R \in \Gamma_h; u_h|_R \in P_k(R)\},
\]

where \(P_k(R)\) denotes the space of restrictions of \(R\) of polynomials with one variable and of order less than or equal to \(k\).

Now we introduce the scheme and the discrete energy by using implicit Euler scheme

\[
\begin{aligned}
\left\{ \begin{array}{l}
\frac{\partial}{\partial t}(\Phi^n_h - \Phi^{n-1}_h, \bar{\varphi}_h) + k_1(\varphi^n_{h,x} + \psi^n_h + lw^n_h) - l \varphi^n_{h,x} - l \varphi^n_h, \bar{\varphi}_h = 0 \\
\frac{\partial}{\partial t}(\Psi^n_h - \Psi^{n-1}_h, \bar{\psi}_h) + k_2(\psi^n_{h,x} + \bar{\psi}_h) + k_1(\varphi^n_{h,x} + \psi^n_h + lw^n_h, \bar{\psi}_h)
\end{array} \right.
\end{aligned}
\]

\[-\Delta t \sum_{m=1}^{n} g(t_{n-m})(\psi^n_m, \bar{\psi}_h, x) = 0\]

\[
\frac{\partial}{\partial t}(W^n_h - W^{n-1}_h, \bar{w}_h) + k_3(w^n_{h,x} - l \varphi^n_{h,x}, \bar{w}_h, x) + k_1 l(\varphi^n_{h,x} + \psi^n_h + lw^n_h, \bar{w}_h) = 0
\]

where \(t_j = j \Delta t\) and

\[
E^n = \rho_1 ||\Phi^n||^2 + \rho_1 ||W^n_h||^2 + \rho_2 ||\Psi^n_h||^2 + k_1 ||\varphi^n_{h,x} + \psi^n_h + lw^n_h||^2
\]

\[
+ k_3 ||w^n_{h,x} - l \varphi^n_{h,x}||^2 + k_2 ||\psi^n_{h,x}||^2 - \left( \int_0^{t_n} g(t)dt \right) ||\psi^n_{h,x}||^2
\]

\[
+ \frac{1}{2} \Delta t \int_0^L \sum_{m=1}^{n} g(t_{n-m})(\psi^n_m - \psi^m_{h,x})^2 dx
\]

6.2 Numerical Experiments

By using the following data

\[k_1 = k_2 = k_3 = 1, \quad \rho_1 = \rho_2 = 0.1, \quad \Delta t = 0.012, \quad h = 0.024, \quad T = 7.4\; \text{and} \; g(x) = e^{-3x};\]

we draw the solutions \(\varphi, \psi, \) and \(w\) in 3D (see Figures 1, 2 and 3, respectively) and their cross section at \(x = 0.5\) (see Figures 4, 5 and 6, respectively).

For the energy we have two cases, taking the conditions of equal and non-equal speeds of wave propagation.

If \(\frac{k_3}{k_2} = \frac{\rho_1}{\rho_2}\) and \(k_1 = k_3\) we obtain an exponential decay by using the same data taken for the solutions as shown in the following Figures 7 – 10.
Figure 1: The evolution in time and space of $\varphi$

Figure 2: The evolution in time and space of $\psi$

Figure 3: The evolution in time and space of $w$
Figure 4: The evolution in time of $\varphi$ at $x = 0.5$.

Figure 5: The evolution in time of $\psi$ at $x = 0.5$. 
Figure 6: The evolution in time of $w$ at $x = 0.5$

Figure 7: The evolution in time of $E^n$
Figure 8: The evolution in time of $\ln(E^n)$ that shows the exponential decay

Figure 9: The evolution in time of $\ln(E^n)$ with its regression line
Figure 10: The evolution in time of $\ln(E^n)/t$

Figure 11: The evolution in time of $E^n$
Figure 12: The variation of $-ln(E^n)$ with respect to $ln(t)$

Figure 13: The variation of $-ln(E^n)$ with respect to $ln(t)$ with it’s regression line
If \( \frac{k_2}{k_1} \neq \frac{\rho_1}{\rho_2} \) and \( k_1 \neq k_3 \) we obtain a polynomial decay by taking the following data \( k_1 = 5, k_2 = k_3 = 1, \rho_1 = 0.02, \rho_2 = 0.1, \Delta t = 0.03125, h = 0.0625 \), and total time \( T = 16.4 \) with 
\[ g(x) = \frac{1}{(x + 1)^2} \]
as shown in the following Figures 11–13.

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