Abstract. The early history of the universe might be described by a topological phase followed by a standard second phase of Einstein gravity. To study this scenario in its full generality, we consider a four-manifold of Euclidean signature in the topological phase, which shares a common boundary with a corresponding manifold of Lorentzian signature in the Einstein phase. We find that the boundary should have vanishing extrinsic curvature, whereas the manifold in the topological phase should have zero Euler number. In addition, we show that the second phase must be characterized by an initial vanishing Weyl tensor and that the standard cosmological flatness problem is not automatically solved unless a conformal invariant boundary term is added. We also characterize the scalar perturbations in the standard Einstein phase. We show that they must contain an initial non-vanishing shear component inherited from the topological phase and we estimate the non-Gaussian parameters. Finally, we argue that the topological early universe cosmology shares common features of previous ideas, such as the so-called Weyl curvature hypothesis, the universe’s creation out of nothing and the no-boundary proposal.
1 Introduction and Conclusions

It has been recently proposed that the early phase of the universe may not be governed by
Einstein gravity, but rather described by a topological phase [1]. According to this idea, partially
motivated by string theory and duality symmetries, the matter content of the universe could
be quite different from the one we explore today. For example, in string gas cosmology [2] the
matter of the universe is described by momentum modes which are very heavy when the radius
of the universe is at the string scale. On the other hand, winding modes (normally superheavy
at large scales) are light at that scale, and they replace the momentum modes for the matter
content driving the dynamics. The new idea proposed in Ref. [1] is that, viewed from our current
perspective, the early phase of the universe is described by a topological phase during which the
gravitational degrees of freedom are absent as there are no metric fluctuations. In this phase,
Einstein gravity is replaced by Witten’s topological gravity [3] on a 4D Riemannian manifold of
Euclidean signature. This theory is a kind of conformal gravity with a local fermionic symmetry
of BRST type. It does not have local degrees of freedom since all degrees of freedom in this
topological phase (phase I) can be gauged away due to the BRST invariance. Local physics
emerges due to anomalies. The local diffeomorphism invariance is broken down to Poincaré
symmetry in the second, non-topological Einstein phase (phase II). The latter is described by
a 4D pseudo-Riemannian manifold of Lorentzian signature. This theory is a kind of conformal gravity with a local fermionic symmetry
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emerges due to anomalies. The local diffeomorphism invariance is broken down to Poincaré
symmetry in the second, non-topological Einstein phase (phase II). The latter is described by
a 4D pseudo-Riemannian manifold of Lorentzian signature with dynamics determined by the
Einstein equations. The two phases are sewed together at a common boundary hypersurface B,
much like the continuation from the Euclidean to Lorentzian regime in the no-boundary proposal
[4] and the creation of the universe from nothing [5, 6]. From this point of view, the topological
phase provides initial conditions on the Cauchy hypersurface at some time $t = 0$ for the evolution
of the universe under Einstein equations in phase II.

In this paper we elaborate on the topological early universe cosmology, by characterizing the
physical conditions emerging in the present setup at the boundary. Our results can be summarized
as follows.

1. Phase I may be described by a Riemannian manifold of Euclidean signature $\mathcal{M}_I$ and self-dual
Weyl tensor, and phase II by a pseudo-Riemannian manifold $\mathcal{M}_{II}$ of Lorentzian signature.
The two phases have a common boundary $B$. The manifold $B$ is a codimension one totally
geodesic submanifold (vanishing extrinsic curvature). The Euler number of $M_I$ vanishes, while the scalar curvature of $B$ is non-negative.

2. The standard cosmological horizon problem can be solved, but the flatness problem requires non-trivial dynamics on the boundary hypersurface.

3. The initial value of Weyl tensor in phase II vanishes.

4. Scalar perturbations are created due to the breaking of conformal symmetry by the trace anomaly and there are no tensor perturbations. However, the matching at the boundary requires a non-vanishing initial shear in phase II. The amount of non-Gaussianity of the scalar perturbations in the squeezed limit \[ f_{NL} = \mathcal{O}(1) \] as far as the three-point correlator is concerned.

5. The topological early universe cosmology is related one way or the other with previous approaches and in particular with the “Weyl curvature hypothesis” \[8–10\], the universe’s creation out of nothing \[5, 6\] and the no-boundary proposal \[4\].

The structure of the paper is the following: in section 2, we describe the two-phase model. In section 3 we discuss the cosmological problems in the present framework. In section 4 we consider cosmological perturbations and in section 5 we compare the two-phase model with previous proposals.

## 2 The two-phase gravity model

We will assume as in Ref. \[1\], that gravity appears in two phases and is described by the geometry of a 4D manifold $M$. We will denote the manifold in phase I as $M_I$ and the manifold in phase II as $M_{II}$. Therefore, if $B$ is the common boundary of $M_I$ and $M_{II}$, we may write

\[
M = M_I \cup M_{II}, \\
\partial M_I = \partial M_{II} = B.
\]

In particular, the manifold $M_I$ has a Riemannian metric of $(+, +, +, +)$ signature, whereas $M_{II}$ has a Lorentzian metric with signature $(-, +, +, +)$. The induced metric $\gamma_{ij}$ ($i, j = 1, 2, 3$) either from the $M_I$ or the $M_{II}$ side agree on their common boundary $B$

\[
\gamma_{ij}\big|_{\partial M_I} = \gamma_{ij}\big|_{\partial M_{II}}.
\]

The gravitational dynamics in $M_{II}$ is governed by the Einstein equations

\[
G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = M_p^{-2}T_{\mu\nu}, \quad \mu, \nu = 0, 1, 2, 3,
\]

where $M_p$ is the Planck mass and $T_{\mu\nu}$ is the energy-momentum tensor. On the other hand, the theory in $M_I$ is assumed to be topological as advocated in Ref. \[1\], and in particular, it is Witten’s topological gravity. The action of the pure gravitational part of the latter is the conformal Weyl square theory

\[
S_{\text{top}} = \int_{M_I} d^4x \sqrt{g} \left( \frac{1}{2g_W} W_{\mu\nu\rho\sigma} W^{\mu\nu\rho\sigma} + \frac{1}{2g_E^4} E_4 \right),
\]

where $W_{\mu\nu\rho\sigma}$ is the Weyl tensor,

\[
E_4 = R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 4R_{\mu\nu} R^{\mu\nu} + R^2
\]
Figure 1: The two manifolds $\mathcal{M}_I$ and $\mathcal{M}_{II}$ are connected through their common boundary $B$. $\mathcal{M}_I$ and $\mathcal{M}_{II}$ are endowed with Riemannian and Lorentzian metrics, respectively. The time coordinate $t_{II}$ in $\mathcal{M}_{II}$ is the Wick rotation of $t_I$ in $\mathcal{M}_I$. The geometry is of the “no-boundary” type.

is the Euler density and $g_W$ and $g_E$ are the corresponding couplings. Usually, the $E_4$ term is not written as it is a topological term, but we keep it as the theory is topological anyway. The full action is given by

$$S_I = S_{\text{top}} + S_{\text{KS}}, \quad (2.6)$$

where

$$S_{\text{KS}} = \int_{\mathcal{M}_I} d^4x \sqrt{-g} \left\{ \tau \left( cW_{\mu\nu\rho\sigma}W^{\mu\nu\rho\sigma} - aE_4 \right) - 4a \partial_\mu \tau \partial_\nu \tau \left( G^{\mu\nu} - g^{\mu\nu} \Box \tau + \frac{1}{2} \partial^\mu \tau \partial^\nu \tau \right) \right\} \quad (2.7)$$

is the trace-anomaly action [11, 12] and the scalar $\tau$ is the dilaton. The full action (2.6) contains second derivatives of the metric and therefore it is a higher-derivative theory. Such theories are classically unstable and quantum mechanically have indefinite metric Hilbert space with ghosts. Indeed, the fourth order equations of motion of Weyl conformal gravity give rise not only to the ordinary massless graviton but also to helicity $(\pm 2, \pm 1, 0)$ ghost-like states [13], forming all together a helicity $\pm 2$ dipole ghost [14–19]. In addition, there exists an ordinary massless helicity $\pm 1$ vector in the spectrum. Therefore, despite of its improved UV properties, it is questionable if the theory (2.4) can make sense at all as a consistent gravity theory since a way to eliminate the ghosts from conformal gravity is lacking until to now. Supersymmetry cannot help here since it just adds extra fermionic ghost degrees of freedom. However, Witten implemented a BRST symmetry to the theory by adding new bosonic and fermionic degrees of freedom which eliminates not only the ghosts but all degrees of freedom altogether. In other words, in Witten’s topological theory there are no propagating degrees of freedom.

Fields in $\mathcal{M}_I$ are in $SO(4) = SU(2) \otimes SU(2)$ representations. In a two-component spinor notation, a field $\Psi_{A_1,\ldots,A_n,\dot{A}_1,\ldots,\dot{A}_n}$ has spin $(m/2, n/2)$, and an $n$-index tensor $\Psi_{\mu_1,\ldots,\mu_n}$ can be written as $\Psi_{A_1,\ldots,A_n,\dot{A}_1,\ldots,\dot{A}_n}$. The latter can be decomposed in irreducible representations by appropriate symmetrizations and antisymmetrization of its indices. The field content of Witten’s
topological gravity contains the metric (vierbien) \( e_{\mu A A} \) and two additional bosonic fields \( B_{A \dot{A}} \) and \( C_{A \dot{A}} \) in addition to the fermions \( \lambda_{A \dot{A}} \), \( \psi_{A B \dot{A} \dot{B}} \) and \( \chi_{A B C D} \). The corresponding fermionic BRST shifts are

\[
\delta \lambda_{A \dot{A}} = \epsilon \partial^\mu D_\mu \lambda_{A \dot{A}} - \frac{i}{2} \epsilon \left( \lambda_{A \dot{C}} B^{\dot{C}} \dot{A} + \lambda_{A \dot{C}} B^{\dot{C}} A \right) + \frac{1}{2} \epsilon \lambda_{A \dot{A}} S_{A \dot{A}},
\]

\[
\delta \psi_{A B \dot{A} \dot{B}} = \frac{1}{2} \epsilon \left( e_{\mu A A} D^\mu C_{\dot{B} \dot{B}} + e_{\mu B \dot{A}} D^\mu C_{A \dot{B}} + e_{\mu A B} D^\mu C_{\dot{B} \dot{A}} + e_{\mu B B} D^\mu C_{A \dot{A}} \right),
\]

\[
\delta \chi_{A B C D} = i W_{A B C D},
\]

(2.8)

where \( k \) is the conformal dimension of \( \lambda \), \( \Lambda_{A B} \) and \( S \) are appropriate quadratic expressions of the bosonic fields [3], and \( W_{A B C D} \) is the spin \((0, 2)\) content of the Weyl tensor, i.e., its self dual part.

Note that the Weyl tensor \( W_{\mu \nu \rho \sigma} \) in \( M_I \) can be decomposed into a self-dual and an anti-self dual part as

\[
W_{\mu \nu \rho \sigma} = W_{+ \mu \nu \rho \sigma} + W_{- \mu \nu \rho \sigma}.
\]

\[
W_{+ \mu \nu \rho \sigma} = \frac{1}{2} \left( W_{\mu \nu \rho \sigma} \pm \ast W_{\mu \nu \rho \sigma} \right),
\]

(2.9)

where

\[
\ast W_{\mu \nu \rho \sigma} = \frac{1}{2} \epsilon_{\nu \rho \sigma} \lambda_{\kappa \lambda \rho \sigma} \quad \text{in} \quad M_I.
\]

In two-component notation we have

\[
W_{+ \mu \nu \rho \sigma} = W_{ABCD} \epsilon_{\dot{A} \dot{B}} \epsilon_{\dot{C} \dot{D}},
\]

\[
W_{- \mu \nu \rho \sigma} = W_{\dot{A} \dot{B} \dot{C} \dot{D}} \epsilon_{AB} \epsilon_{CD},
\]

(2.10)

where \( \epsilon_{AB} \) is the \( SU(2) \) invariant totally antisymmetric tensor.

In \( M_{II} \) we have a similar decomposition of the Weyl tensor

\[
W_{\mu \nu \rho \sigma} = W_{+ \mu \nu \rho \sigma} + W_{- \mu \nu \rho \sigma},
\]

\[
W_{+ \mu \nu \rho \sigma} = \frac{1}{2} \left( W_{\kappa \lambda \rho \sigma} \mp i \ast W_{\mu \nu \rho \sigma} \right),
\]

(2.11)

where \( \ast W_{\mu \nu \rho \sigma} = \frac{i}{2} \epsilon_{\nu \rho \sigma} \lambda_{\kappa \lambda \rho \sigma} \) in \( M_{II} \).

In an obvious two-component \( SL(2, C) \) notation we have

\[
W_{+ \mu \nu \rho \sigma} = \Psi_{\alpha \beta \gamma \delta} \epsilon_{\alpha \beta} \epsilon_{\gamma \delta},
\]

\[
W_{- \mu \nu \rho \sigma} = \Psi_{\alpha \beta \gamma \delta} \epsilon_{\dot{\alpha} \dot{\beta} \epsilon_{\gamma \delta}},
\]

(2.12)

where now \((\alpha, \beta, \cdots = 1, 2)\) are \( SL(2, C) \) indices.

Let us note that due to the fact that there are no degrees of freedom in the topological phase I, there are no gravitational field equations. Classical configurations are still determined by the critical points of the action (2.4). These can be found by recalling that in \( M_I \), the inequality [20]

\[
\left( W_{\mu \nu \rho \sigma} \pm \ast W_{\mu \nu \rho \sigma} \right) \left( W_{\mu \nu \rho \sigma} \pm \ast W_{\mu \nu \rho \sigma} \right) \geq 0,
\]

(2.13)

leads to

\[
S_I = \frac{1}{g_{\text{W}}} \int_{M_I} d^4 x \sqrt{-g} W_{\mu \nu \rho \sigma} W_{\mu \nu \rho \sigma} \geq \frac{48 \pi^2}{g_{\text{W}}^2} |\tau_R|,
\]

(2.14)
where
\[
\tau_R = \frac{1}{48\pi^2} \int_{M_I} d^4x \sqrt{g} W_{\mu\nu\rho\sigma} *W^{\mu\nu\rho\sigma}
\] (2.17)
is the Hirzebruch signature (for a compact manifold). Therefore, the topological action is minimized for self dual configurations
\[
W_{\mu\nu\rho\sigma} = W_{ABCD} \varepsilon_{AB} \varepsilon_{CD} = 0,
\] (2.18)
or anti-self dual configurations
\[
W^{\mu\nu\rho\sigma} = W_{ABCD} \varepsilon_{AB} \varepsilon_{CD} = 0.
\] (2.19)
Such configurations are annihilated by the BRST charge since for example
\[
\delta \chi_{ABCD} = i W_{ABCD} = 0 \quad (2.20)
on anti-self dual geometry [1]. We will now assume that the boundary \(B\) is not special in the sense that nothing happens there apart from the change of signature. In particular, there are no discontinuities along the boundary \(B\). Let us then see what are the consequences of such an assumption.

### 2.1 Condition on the extrinsic curvature

Standard Einstein equations (2.3) hold in the non-distributional sense in the whole of \(M_{II}\) including its boundary. That is, all geometric quantities are continuous at \(B\) and in particular, the second fundamental form (extrinsic curvature) should also be continuous across \(B\). Viewed from phase I, i.e., from \(M_I\), the second fundamental form is
\[
K_{\mu\nu} = \nabla_{\mu} n_{\nu},
\] (2.21)
where \(n^\mu\) is the unit normal vector to the boundary \(B\). On the other hand, from the \(M_{II}\) point of view we find
\[
K_{\mu\nu} = i \nabla_{\mu} n_{\nu},
\] (2.22)
since we have to Wick rotate going from \(M_I\) to \(M_{II}\). Since the extrinsic curvature is continuous on \(B\), the only way both (2.21) and (2.22) to hold is
\[
K_{\mu\nu} \bigg|_B = 0. \quad (2.23)
\]
In other words, the common boundary of \(M_I\) and \(M_{II}\) has vanishing extrinsic curvature and therefore is a totally geodesic submanifold as in the no-boundary proposal [34].¹ This also specifies the scalar curvature of \(B\). Indeed, from Einstein equations (2.3), the Hamiltonian constraint is
\[
3R + K^2 - K_{ij} K^{ij} = 2M_p^{-2} n^\mu n^\nu T_{\mu\nu} \quad \text{in } M_{II}
\] (2.24)
and therefore, due to (2.23), we get that
\[
3R \bigg|_B = 2M_p^{-2} \rho \bigg|_B, \quad (2.25)
\]
where \(\rho = n^\mu n^\nu T_{\mu\nu}\) is the energy density. Since \(n^\mu\) is timelike, the weak energy condition \(n^\mu n^\nu T_{\mu\nu} \geq 0\) gives
\[
3R \bigg|_B \geq 0. \quad (2.26)
\]
Therefore, the induced metric on \(B\) has non-negative scalar curvature. This result should be kept in mind when discussing the flatness problem.

Since (2.25) is nothing else than the Hamiltonian constraint, it provides initial data for Einstein equations. However, these initial data should not be arbitrary but should describe a totally geodesic hypersurface of the Riemannian space \(M_I\), where \(M_I\) and \(M_{II}\) are glued in an at least \(C^{(2)}\) way.

¹A hypersurface like \(B\) with vanishing extrinsic curvature is also called “moment of time symmetry” [21].
2.2 Condition on the Weyl tensor

We have seen that geometries with vanishing self dual (or the anti-self dual) part of the Weyl tensor are critical points of Witten’s topological gravity. Therefore, at the boundary $B$ of $\mathcal{M}_I$ we will have by continuity

$$W_{ABCD} = 0 \quad \text{in} \quad \mathcal{M}_I, \quad \implies \quad W_{ABCD} \big|_{B=\partial \mathcal{M}_I} = 0 \quad (2.27)$$

as well. The same condition should hold from the $\mathcal{M}_{II}$ side (phase II) i.e.,

$$W_{ABCD} \big|_{B=\partial \mathcal{M}_{II}} = 0. \quad (2.28)$$

Since we have to Wick rotate from $\mathcal{M}_I$ to $\mathcal{M}_{II}$, we have that

$$W_{\mu\nu\rho\sigma}^- = \frac{1}{2} \left( W_{\mu\nu\rho\sigma} + *W_{\mu\nu\rho\sigma} \right) \quad \text{in} \quad \mathcal{M}_I,$$

$$W_{\mu\nu\rho\sigma}^- = \frac{1}{2} \left( W_{\mu\nu\rho\sigma} + i *W_{\mu\nu\rho\sigma} \right) \quad \text{in} \quad \mathcal{M}_{II}. \quad (2.29)$$

The only way to match on their common boundary is the full Weyl tensor to vanish, i.e.,

$$W_{\mu\nu\rho\sigma} \big|_B = 0. \quad (2.30)$$

Let us note that the above condition (2.30) can also be written as initial data for the electric and magnetic part of the Weyl tensor, which will be useful below, as follows. The boundary hypersurface $B$ is spacelike with timelike normal $n^\mu$. We can decompose the Weyl tensor into its electric and magnetic parts with respect to $n^\mu$ as

$$E_{\mu\nu} = W_{\mu\sigma\nu\rho} n^\rho n^\sigma, \quad (2.31)$$

$$B_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\rho\sigma\lambda} W_{\nu\delta\rho\lambda} n^\rho n^\sigma. \quad (2.32)$$

In terms of the projection $h_{\mu\nu}$ orthogonal to $n^\mu$

$$h_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu, \quad (2.33)$$

Eqs.(2.31) and (2.32) are expressed as

$$E_{\mu\nu} = -3 R_{\mu\nu} + K K_{\mu\nu} - K_{\mu\lambda} K_{\nu}^\lambda + \frac{1}{2} h^\rho_{\mu} h^\sigma_{\nu} R_{\rho\sigma} + \left( \frac{1}{2} h^{\rho\sigma} R_{\rho\sigma} - \frac{1}{3} R \right) h_{\mu\nu}, \quad (2.34)$$

$$B_{\mu\nu} = D_\rho K_{\mu(\sigma}^\lambda \epsilon_{\nu)} \sigma^\rho \rho n_\kappa, \quad (2.35)$$

where $D_\rho$ is the covariant derivative with respect to the induced metric $h_{\mu\nu}$. The vanishing of the Weyl tensor on $B$ then implies

$$E_{\mu\nu} \big|_B = B_{\mu\nu} \big|_B = 0. \quad (2.36)$$

Since the extrinsic curvature of $B$ vanishes according to Eq.(2.23), the magnetic part of the Weyl tensor is automatically zero. On the other hand, the vanishing of the electric part gives that the boundary $B$ is a space of constant curvature [22].

There is a second condition related to the Weyl tensor, namely, the Bach tensor should vanish in $\mathcal{M}_I$. Indeed, the variation of (2.4) leads to field equations of conformal gravity

$$B_{\mu\nu} = \left( \nabla^\rho \nabla^\sigma + \frac{1}{2} R^\rho\sigma \right) W_{\mu\rho\sigma} = 0, \quad (2.37)$$
where $B_{\mu\nu}$ is the Bach tensor. If the Weyl tensor is self dual or anti-self dual, then the Bach tensor vanishes identically. Indeed, in two-component notation the Bach tensor is written as

$$B_{\mu\nu} = 2\left(\nabla^C_A\nabla^D_B + \Phi^{CD}_{AB}\right)W_{ABCD}$$

$$= 2\left(\nabla^C_A\nabla^D_B + \Phi^{CD}_{AB}\right)W_{ABCD}, \quad (2.38)$$

where $\Phi^{AB}_{\dot{A}\dot{B}} = -\frac{1}{2}(R_{\mu\nu} - \frac{1}{4}Rg_{\mu\nu})$ is the Ricci spinor. Clearly $B_{\mu\nu} = 0$ for self dual ($W_{\dot{A}\dot{B}\dot{C}\dot{D}} = 0$), or anti-self dual ($W_{ABCD} = 0$) Weyl. Therefore, the Bach tensor should vanish in $\mathcal{M}_I$ and by continuity, it should vanish also on the boundary $B$, i.e.,

$$B_{\mu\nu} |_B = 0. \quad (2.39)$$

The vanishing of the Bach tensor on the boundary $B$ should also be true for observers in $\mathcal{M}_{II}$.

3 The cosmological problems in the topological early universe cosmology

Here we will consider the usual cosmological problems in view of the pre-existence of a topological phase of gravity. In particular, we will examine the horizon and the flatness problem.

3.1 Horizon Problem

A serious shortcoming of the standard big bang cosmology is the horizon problem, which is associated to the puzzling homogeneity of the observed universe across patches which had never been in causal contact since the onset of the cosmological evolution. In the present scenario, all regions of the universe have emerged from a single phase I. As can be seen from Fig. 1, two past light cones, although have never been in contact in $\mathcal{M}_I$, have emerged, through the boundary $B$, from the common phase I of the Riemannian $\mathcal{M}_I$ space. Therefore, their homogeneity is the result of the same universal initial conditions set from phase I.

This is also related by the fact that the initial data for the cosmological evolution in phase II in $\mathcal{M}_{II}$, as we have seen, is the vanishing of the Weyl tensor and the extrinsic curvature on the common boundary $B$.

Let us recall that it is generally accepted that an initial vanishing Weyl tensor, a hypothesis that is usually referred to as “the Weyl curvature hypothesis”, suffices to explain the isotropy and homogeneity of the universe [8–10]. In fact, it has been conjectured that an initially zero Weyl tensor necessary implies an FRW cosmology. Although a general proof is lacking, this conjecture has been proved for a universe filled with a perfect fluid with equation of state $P = w\rho$ for $0 < w \leq 1$ [22–26].

3.2 Flatness

Although it seems that the horizon problem and the associated homogeneity and isotropy of the universe can easily be explained withing the present set up, this is not automatically the case for the flatness problem. Indeed, as we have seen above, the conditions Eqs. (2.23) and (2.30) lead to a constant curvature 3D boundary $B$, see equation (2.26). This poses a threat to the solution to the flatness problem as both possibilities $k = 0$ and $k = 1$ of the spatial geometry of the FRW universe are allowed.

One may hope that a flat $B (k = 0)$ can be selected by adding an appropriate theory on $B$. Such a boundary theory should be conformal as we want the conformal invariance to be broken only by anomalies. Luckily, such a theory exists and is a 3D conformally invariant version of conventional gravity of Chern-Simons type with action [27–29]

$$S_{HW} = \int_B \epsilon^{ijk} \left\{ \omega_{ia}(\partial_j\omega_k^a - \partial_k\omega_j^a) + \frac{2}{3}\epsilon^{abc}\omega_{ia}\omega_{jb}\omega_{kc} \right\}, \quad (3.1)$$
where $\omega^a_i$ is the spin connection. As it has been proven in [29], 3D conformal gravity described by (3.1) is classically equivalent to a Chern-Simons theory for the conformal group $SO(3,2)$. Variation of (3.1) gives

$$\delta S_{HW} = \int_B \varepsilon^{ijk} R_{ij}^{ab} \delta \omega_{kab}. \quad (3.2)$$

In the usual treatment of the gravitational Chern-Simons action (3.1), one trades the variation of the spin connection with that of the vierbeins. This leads to the equation of motion

$$C_{ijk} = \nabla_k W_{ij} - \nabla_j W_{ki}.$$ \hspace{1cm} (3.3)

where $W_{ij} = R_{ij} - \frac{1}{4} R g_{ij}$, i.e., to the vanishing of the 3D Cotton tensor $C_{ijk}$. Therefore, one is tempted to conclude that $B$ should be only conformally flat (due to the vanishing of the Cotton tensor), but not necessarily flat ($k = 0$). However, in our case, $B$ is the boundary of both $\mathcal{M}_I$ and $\mathcal{M}_{II}$ so that variations of the vierbein vanishes

$$\delta e^a_i \big|_B = 0. \quad (3.4)$$

This means that we should treat the spin connection $\omega^{ab}_i$ as independent field as in Palatini formulation, so that its equation of motion is

$$R_{ij}^{ab} = 0.$$ \hspace{1cm} (3.5)

Therefore we see that the addition of the conformal invariant 3D action Eq. (3.1) selects a flat boundary $B$ solving the flatness problem.

### 4 Cosmological Perturbations

We would like now to characterize the allowed curvature perturbation in $\mathcal{M}_{II}$ given the initial data (2.30) and (2.23). In other words, we are looking for perturbations that have vanishing Weyl tensor and extrinsic curvature on $B$ , i.e., perturbations $\delta g_{\mu\nu}$ such that\footnote{The boundary term (3.1) does not contribute to the perturbations at the boundary.}

$$\delta E_{ij} \big|_B = 0, \quad \delta B_{ij} \big|_B = 0 \quad (4.1)$$

and

$$\delta K_{ij} \big|_B = 0.$$ \hspace{1cm} (4.2)

A perturbed FRW universe is written in conformal Poisson gauge as

$$ds^2 = a(\eta)^2 \left[ -(1 + 2\Phi) d\eta^2 + 2\omega_id\eta dx^i + \left[ (1 - 2\Psi) \gamma_{ij} + h_{ij} \right] dx^i dx^j \right], \quad (4.3)$$

where $\omega_i$ is transverse, $h_{ij}$ is transverse and traceless and $\gamma_{ij}$ is the metric with curvature $k = \pm 1, 0$. Then we find that the associated non-vanishing perturbations of the electric and magnetic parts of the Weyl tensor are [30]

- **Scalar perturbations:**

  $$\delta E_{ij}^S = \frac{1}{2} \left( \nabla_i \nabla_j - \frac{1}{3} \gamma_{ij} \nabla^2 \right) (\Phi + \Psi), \quad (4.4)$$

  $$\delta B_{ij}^S = 0.$$ \hspace{1cm} (4.5)
• Vector perturbations:
\[
\delta E^V_{ij} = \frac{1}{2} \nabla (i \omega' j),
\]
\[
\delta B^V_{ij} = \frac{1}{2} \epsilon^{mn} \left( \nabla_j \nabla_m \omega_n - \frac{1}{2} \gamma_{jm} \nabla^2 \omega_n \right).
\]

• Tensor perturbations:
\[
\delta E^T_{ij} = -\frac{1}{2} h''_{ij} - \frac{1}{2} \nabla^2 h_{ij} + kh_{ij},
\]
\[
\delta B^T_{ij} = -\epsilon^{mn} \nabla_m h'_{nj}.
\]

It is clear that there are no non-trivial tensor and vector perturbations since the conditions (4.1) specifies the initial second and first derivatives of \( h_{ij} \) and \( \omega_i \), respectively, on \( B = 0 \):
\[
0 = \left( -\frac{1}{2} h''_{ij} - \frac{1}{2} \nabla^2 h_{ij} + kh_{ij} \right) \bigg|_{B},
\]
\[
0 = \nabla (i \omega' j) \bigg|_{B},
\]
so that the corresponding Cauchy problem has no solution. However, there are scalar perturbations which should satisfy the initial condition
\[
(\Phi + \Psi) \bigg|_{B} = 0,
\]
as can be seen from Eq. (4.4). On the other hand, it follows from the perturbed Einstein equations that
\[
\Phi - \Psi = -8\pi G \delta \sigma,
\]
where \( \delta \sigma \) is the anisotropic stress. Therefore, from Eqs. (4.12) and (4.13) we find that there must exist an initial anisotropic stress given by
\[
\delta \sigma \bigg|_{B} = \frac{1}{4\pi G} \Psi \bigg|_{B}.
\]
However, since shear viscosity is redshifted like \( a^{-6} \), an initial \( \delta \sigma \) promptly decays. It remains though to explain its origin from phase I. One concludes that only scalar perturbations are possible in the FRW background in the two-phase gravity. Next, we will determine the power spectrum of scalar perturbations.

4.1 Scalar Perturbations

We are interested now in scalar perturbations in \( M_I \). Such perturbations will provide initial conditions for the two-point correlators at the Cauchy surface \( B \) and determine therefore the spectrum of curvature perturbations in the \( M_{II} \) universe. As long as the theory in \( M_I \) is conformal, conformal transformations of the metric
\[
\delta g_{\mu\nu} = 2h g_{\mu\nu}
\]
do not change the theory. Therefore, the conformal mode corresponding to the conformal perturbation
\[
g_{\mu\nu} = (1 - 2h) \bar{g}_{\mu\nu}, \quad \tau = \bar{\tau} + h,
\]
where $\delta g_{\mu\nu}$ is the background metric, is not physical. However, when conformal invariance is broken, the conformal mode becomes physical (dilaton). In particular, if conformal invariance is broken due to an anomaly, the trace of the energy-momentum tensor is given by

$$T^\mu_\mu = \frac{c}{1920\pi^2} W^\mu_{\nu\rho\sigma} W^{\nu\rho\sigma} - \frac{a}{5760\pi^2} E_4, \quad \text{(4.17)}$$

normalized such that a real scalar contribute with $a = c = 1$ to the anomaly. The response of the effective action under (4.16) is

$$\delta h S = \int d^4x \sqrt{g} h \left( \frac{c}{1920\pi^2} W^\mu_{\nu\rho\sigma} W^{\nu\rho\sigma} - \frac{a}{5760\pi^2} E_4 \right). \quad \text{(4.18)}$$

Then, it is straightforward to verify that

$$S_h = -\frac{a}{720\pi^2} \int d^4x \sqrt{\bar{g}} \left\{ \bar{G}^{\mu\nu} \partial_\mu h \partial_\nu h + \bar{R}^{\mu\nu} h \partial_\mu h \partial_\nu h + \partial_\mu h \partial_\nu h \partial^\alpha h \partial^\beta h + \mathcal{O}(h^5) \right\}, \quad \text{(4.19)}$$

to fourth order where $\bar{G}^{\mu\nu}$ ($\bar{R}^{\mu\nu}$) the background Einstein (Ricci) tensor. Therefore, the conformal mode is dynamical now. Its two-point correlator can be found by using the Ward identity for scale invariance. The latter is written as

$$\langle T^\mu_\mu(x) O_1(x_1) \cdots O_n(x_n) \rangle = -\sum_{i=1}^n \delta^{(4)}(x - x_i) \Delta_i \langle O_1(x_1) \cdots O_i(x_i) \cdots O_n(x_n) \rangle, \quad \text{(4.20)}$$

where $\Delta_i$ is the dimension of $O_i$. In particular, for $O_i = h$, the Ward identity (4.20) for the two-point correlator is written as

$$\langle T^\mu_\mu(x) h(y) h(0) \rangle = -\Delta \delta^{(4)}(x - y) \langle h(y) h(0) \rangle - \Delta \delta^{(4)}(x) \langle h(y) h(0) \rangle. \quad \text{(4.21)}$$

Therefore, after integration we get that

$$2\Delta \langle h(y) h(0) \rangle = -\int_{\mathcal{M}_1} d^4x \sqrt{g} \langle T^\mu_\mu(x) h(y) h(0) \rangle \approx \int_{\mathcal{M}_1} d^4x \sqrt{g} \langle T^\mu_\mu(x) \rangle \langle h(y) h(0) \rangle, \quad \text{(4.22)}$$

where the approximation $\langle T^\mu_\mu(x) h(y) h(0) \rangle \approx \langle T^\mu_\mu(x) \rangle \langle h(y) h(0) \rangle$ has been used. Using the integrated conformal anomaly

$$\int_{\mathcal{M}_1} d^4x \sqrt{g} \langle T^\mu_\mu(x) \rangle = \frac{c}{240} \langle I_W \rangle - \frac{a}{180} \chi_R, \quad \text{(4.23)}$$

where $\chi_R$ is given by

$$\chi_R = \frac{1}{32\pi^2} \int_{\mathcal{M}_1} E_4, \quad \text{(4.24)}$$

and $I_W$ denotes the integral

$$I_W = \frac{1}{8\pi^2} \int_{\mathcal{M}_1} W^2_{\mu\nu\rho\sigma} \sqrt{g} d^4x, \quad \text{(4.25)}$$

we find that the scaling dimension of $h$ turns out to be

$$\Delta \approx \frac{a}{360} \chi_R - \frac{c}{480} \langle I_W \rangle. \quad \text{(4.26)}$$
Note that $\chi_R$ is the Euler number for a compact 4D manifolds. If there are boundaries, then the Euler number gets boundary corrections [31] (and so does the integrated conformal anomaly [32, 33]) so that

$$\chi_R = \frac{1}{32\pi^2} \int_{\mathcal{M}_1} E_4 - \frac{1}{4\pi^2} \int_{\partial \mathcal{M}_1} X,$$

(4.27)

where $X$ is

$$X = K^{\mu\nu} n^\lambda n^\kappa R_{\mu\kappa\lambda\nu} - K^{\mu\nu} R_{\mu\nu} - Kr_{\mu\nu} n^\mu n^\nu$$

$$+ \frac{1}{2} K R - \frac{1}{3} K^3 + KK^{\mu\nu} K_{\mu\nu} + \frac{2}{3} K^{\mu\kappa} K_{\kappa\lambda} K^\lambda. (4.28)$$

In the present case, since the extrinsic curvature $K_{\mu\nu}$ vanishes, we have $X = 0$ and therefore, there are no boundary corrections to the Euler number. In other words, the Euler number of $\mathcal{M}_1$ with boundary the totally geodesic codimension-one space $B$, is given just by (4.24).

An estimate of the term $\langle I_W \rangle$ gives

$$\langle I_W \rangle \sim a_W \Lambda^4_{UV} L^4_I,$$

(4.29)

where $\Lambda_{UV}$ is the UV cutoff in $\mathcal{M}_1$, $L_I$ is a characteristic scale in $\mathcal{M}_1$ (curvature scale) and $a_W = \hat{g}_{II}/8\pi$. Indeed, based on dimensional grounds, $(W^2_{\mu\nu\rho\sigma}) \sim a_W \Lambda^4_{UV}$ and therefore, we expect (4.29) to hold.

Note that we may use the doubling trick to construct a new compact manifold out of $\mathcal{M}_1$ if its boundary is only $B$. The rule is the following: since the extrinsic curvature of $\mathcal{M}_1$ vanishes, we may construct the manifold $2\mathcal{M}_1$ consisting of two copies $\mathcal{M}_1^+$ and $\mathcal{M}_1^-$ joined across $B$. Since $K_{ij} = 0$ on $B$, the induced metric on $2\mathcal{M}_1$ from the metrics on $\mathcal{M}_1^\pm$ is at least $C^1$. The manifold $2\mathcal{M}_1$ is a compact manifold without boundary, it has $\chi_R(2\mathcal{M}_1) = 2\chi_R(\mathcal{M}_1)$ and signature $\tau_R(2\mathcal{M}_1) = 0$ [34]. In addition, $\mathcal{M}_1$ admits a reflection map $\theta$ interchanging $\mathcal{M}_1^+$ and $\mathcal{M}_1^-$ while leaving $B$ invariant. In this case, the metric close to the boundary $B$ of the form

$$ds^2_I \approx dt^2_I + g_{ij}(x,t^2_I)dx^i dx^j,$$

(4.30)

where the reflection map is realized by $t_1 \to -t_1$. Note also that this form of the metric close to $B$ allows the Wick rotation $t_1 \to i\Pi$. Then the boundary $B$ is just the fixed point of the reflection map $\theta$ (acting here as $t_1 \to -t_1$).

The two-point function turns out to be

$$\langle h(x)h(0) \rangle \sim \frac{720\pi^2 L^2_I}{a} \frac{1}{|x|^{2\Delta}},$$

(4.31)

and therefore, we have at the boundary $B$

$$\langle h(x)h(0) \rangle \big|_B = \langle h(0,\vec{x})h(0,\vec{0}) \rangle \sim \frac{720\pi^2 L^2_I}{a} \frac{1}{|\vec{x}|^{2\Delta}}.$$

(4.32)

This is the initial condition for the scalar perturbation in $\mathcal{M}_{II}$. The factor $L^2_I/a$ originates from the coupling of the conformal mode in (4.19). In particular, $h|_B = \Psi|_B$ and, after Fourier transforming, we get that the initial condition for the two-point correlator in $\mathcal{M}_{II}$ should be

$$|\Psi_k|^2 \big|_B \sim \frac{L^2_I M^2_p}{a} k^{-3+2\Delta},$$

(4.33)

since $\Psi$ and $h$ are differently normalized as their kinetic terms are multiplied with $M^2_p$ and $L^2_I$, respectively. This leads to a spectral index

$$n_s = 1 + 2\Delta = 1 + \frac{a}{180} \chi_R - \frac{c}{240} a_W \Lambda^4_{UV} L^4_I$$

(4.34)
and an amplitude

\[ A \sim \frac{L^2 M_p^2}{a}. \]  

(4.35)

We encounter now a problem related to the spectral index in Eq. (4.34). Consider the simplest most symmetric case of an \( S^4 \) of radius \( L \) as a candidate for \( 2M_1 \) and the standard metric on \( S^4 \)

\[ ds^2 = L^2 d\tau^2 + L^2 \cos^2 \tau d\Omega_3, \quad -\frac{\pi}{2} \leq \tau \leq \frac{\pi}{2}, \]  

(4.36)

where \( d\Omega_3 \) is the metric of the unit three-sphere and we have shifted the azimuthal angle \( \tau \) by \(-\pi/2\). Then, the equatorial \( \tau = 0 \) has vanishing extrinsic curvature \( K_{ij} = 0 \) and therefore, it is the boundary \( B \) which will be continued into a Lorentzian FRW \( \mathcal{M}_\text{II} \) with \( k = 1 \). However, we find that in this case, \( \chi_R = 1 \), and therefore we will have a ridiculously large blue spectrum as follows from (4.34) and (4.35). Now, in order to get an amplitude \( A \sim 10^{-10} \), we should have \( a \sim 10^{10} L^2 M_p^2 \). In addition, we need

\[ -\frac{a}{180} \chi_R \approx 0.03 \]  

(4.37)

for a red spectrum with \( n_s \approx 0.97 \). However, according to Hopf conjecture, the Euler number of a compact manifold is no-negative, so that

\[ \chi_R(\mathcal{M}_1) = \frac{1}{2} \chi_R(2\mathcal{M}_1) = 1 - b_1 + \frac{1}{2} b_2 \geq 0 \]  

(4.38)

where \( b_1, b_2 \) are the Betti numbers of \( 2\mathcal{M}_1 \) [34]. Therefore, we end up in general in an enormous blue spectrum for scalar perturbations. The only way to avoid this is the manifolds \( \mathcal{M}_1 \) to have vanishing Euler number

\[ \chi_R = 0. \]  

(4.39)

In this case, the spectral index is given by

\[ n_s = 1 + 2\Delta = 1 - \frac{c}{480} \alpha_W \Lambda^4_{UV} L^4, \]  

(4.40)

which may have the correct value for appropriate values of the parameters [1].

Compact manifolds with \( \chi_R = 0 \) admit codimension-one foliations [35]. In addition, the leaves of the foliation are totally geodesic (vanishing extrinsic curvature) if \( \tau_R = 0 \) as well [36]. In other words, compact manifolds satisfying (4.39) can be cut along appropriate leaf \( B \) and continued to \( \mathcal{M}_\text{II} \) with Lorentzian signature. However, \( \mathcal{M}_1 \) should also be self-dual. Examples of such spaces include \( T^4 \) and \( S^1 \otimes S^3 \) and connected sums of these which are (trivially) self-dual and satisfy (4.39). The boundary in this case can be either \( T^3 \) or \( S^1 \otimes S^2 \). The latter case however, is excluded as it leads to a not flat boundary, as required by Eq.(3.5).

Non-compact manifolds with \( \chi_R = 0 \) can be constructed. For example, let us assume that the metric of \( 2\mathcal{M}_1 \) is of the form

\[ ds^2 = d\tau^2 + a^2(\tau)\delta_{ij}(x)dx^i dx^j, \quad -\tau_0 \leq \tau \leq \tau_0, \]  

(4.41)

where \( \tau_0 \) is a real constant. It is conformally flat, and therefore trivially self-dual (vanishing Weyl tensor). The boundary \( B \) is the hypersurface at \( \tau = 0 \). Its extrinsic curvature is

\[ K_{ij} = a'(0)\delta_{ij} \]

and therefore, \( B \) is totally geodesic if \( a'(0) = 0 \). Its Euler number \( \chi_R \) turns out to be

\[ \chi_R = \frac{V}{4\pi^2} a'(\tau)^3, \]  

(4.42)
Then, $\chi_R = 0$ is satisfied for $\alpha'(-\tau_0) = 0$.

One can also estimate the non-linear parameters of the three- and four-point functions of the comoving curvature perturbation $\zeta = 5\Psi/3$ [1]. In the squeezed limit of the three-point function $(k_1 \ll k_2 \sim k_3)$, we have (the prime indicates we do not write the Dirac delta for the momentum conservation and the $(2\pi)^3$ factors) [7]

$$\left\langle \frac{\zeta_{k_1} \zeta_{k_2} \zeta_{k_3}}{2} \right\rangle' = \frac{12}{5} f_{NL} P_{k_1}^N P_{k_2}^N P_{k_3}^N,$$

(4.43)

where $P_k^N$ is the power spectrum deduced by Eq. (4.33). On the other hand, for the four-point function in the collapsed limit ($k_{12} = k_1 + k_2 \approx 0$), we have

$$\left\langle \frac{\zeta_{k_1} \zeta_{k_2} \zeta_{k_3} \zeta_{k_4}}{2} \right\rangle' = 4 f_{NL} P_{k_1}^N P_{k_2}^N P_{k_3}^N P_{k_{12}}^N,$$

(4.44)

whereas, in the squeezed limit ($k_4 \ll k_1, k_2, k_3$),

$$\left\langle \frac{\zeta_{k_1} \zeta_{k_2} \zeta_{k_3} \zeta_{k_4}}{2} \right\rangle' = \left( 2\tau_{NL} + \frac{54}{25} g_{NL} \right) P_{k_4}^N \left( P_{k_1}^N P_{k_2}^N + 2 \text{ perm.} \right).$$

(4.45)

From the action (4.19), it can easily be verified that the three- and four-point functions scale as

$$\left\langle \frac{\zeta_{k_1} \zeta_{k_2} \zeta_{k_3}}{2} \right\rangle' \sim \frac{L_4^4 M_p^4}{a^2}, \quad \left\langle \frac{\zeta_{k_1} \zeta_{k_2} \zeta_{k_3} \zeta_{k_4}}{2} \right\rangle' \sim \frac{L_5^8 M_p^8}{a^3}.$$

(4.46)

By using that $P_k^N \sim L_2^2 M_p^2/a$, we find

$$f_{NL} \sim O(1), \quad \tau_{NL} \sim g_{NL} \sim L_2^4 M_p^4.$$

(4.47)

Therefore, since the IR scale $L_I > M_p^{-1}$, the Suyama-Yamaguchi inequality [37, 38]

$$\tau_{NL} \geq \left( \frac{6}{5} f_{NL} \right)^2,$$

(4.48)

is satisfied for this class of models.

To summarise, we have found that the Riemannian manifold $M_I$ satisfy the following conditions:

- It has self-dual Weyl tensor (vacuum of the topological theory on $M_I$).
- It accommodates a codimension-one totally geodesic submanifold $B$ (zero extrinsic curvature).
- It admits a reflection map (at least locally close to $B$) such that the Wick rotation to the Lorentzian $M_{II}$ is possible.
- It has vanishing Euler number ($\chi_R = 0$) in order to have a red spectrum of scalar perturbations.

Importantly, we have seen that the space of manifolds $M_I$ satisfying the above criteria is not empty.

## 5 Relation with other proposals

The two-phase gravity is related also to other proposals for the early history of the universe. In particular, it is related to the Weyl curvature hypothesis, the no-boundary program and the out-of-nothing creation of the universe. We will examine here the relation of the two-phase model to the aforementioned proposals.
5.1 Weyl curvature hypothesis

The Weyl curvature hypothesis has been proposed by R. Penrose motivated by the second law of thermodynamics [8–10]. According to it, since the entropy of the universe increases with time it had a tiny value in the past. Penrose noticed an apparent paradox related with this. The best evidence for Big Bang arises from observations of the CMB. The latter follows Planck’s law to extraordinary precision, leading to the unavoidable conclusion that the early universe was in thermal equilibrium. But thermal equilibrium represents maximum randomness and therefore it corresponds to maximum entropy. How such a picture could be correct given the fact that the universe has started in a low entropy state? Penrose answers this puzzle by noticing that the high entropy of the CMB is related to the matter content of the universe only and not to gravity. In other words, the extraordinary uniformity of the early universe is attributed just to matter while gravitational degrees of freedom, although potentially available, have not been excited. The moment gravitational degrees of freedom are activated, the entropy starts to increase due to the clumping of the initially distribution of matter.

But how gravitational degree of freedom can be inactivated? According to Penrose, this may be implemented by assuming a principle which he named “the Weyl curvature hypothesis”. Let us recall that curvature is not completely specified in Einstein gravity. Indeed, matter distribution can only determine the Ricci curvature once the energy-momentum tensor is known. The rest piece of Riemann curvature is just the Weyl tensor $W_{\mu\nu\rho\sigma}$ which is not specified by Einstein equations. Weyl tensor describes pure gravitational dof and the Weyl curvature hypothesis asserts that this tensor vanishes at the initial Bing-Bang singularity, i.e.,

$$W_{\mu\nu\rho\sigma}|_{\text{Big-Bang}} = 0.$$  \hspace{1cm} (5.1)

Clearly, this is just our (2.30), when the boundary $B$ is identified with the moment of Big Bang. This of course should be expected as phase I has no gravitational degrees of freedom and therefore, necessarily, the Weyl tensor should vanish on any hypersurface that separates phase I with a second phase where Einstein equations hold. In other words, the fact that phase I is topological means that when continued to phase II, the Weyl curvature hypothesis should valid. However, in the case of Penrose, the hypersurface $B$ is just a part of an infinite sequence of universes, the “aeons”, making up the Cyclic Conformal Cosmology [39], according to which the universe undergoes repeated expansion cycles, named aeons, such that each one starting from its own Big-Bang, ending in a a stage of accelerated expansion and continues indefinitely.

In the two-phase proposal on the other side, there are no infinitely many past universes. The latter are replaced with a single topological phase where geometry is Riemannian and there are no gravitational degrees of freedom. In this respect it seems similar to the out of nothing creation of the universe which we will now turn.

5.2 Creatio Ex Nihilo

Cosmological spacetimes, even inflationary ones, are past-incomplete under general assumptions. Therefore, such spacetimes should have a past boundary where initial conditions should be defined. Clearly, a question that should be asked in this case, is what determines these initial conditions. Here, following [1] we have consider the case in which the universe had two phases: a topological phase I where gravity is not dynamical and the geometry is Riemannian and a dynamical phase Einstein II, where gravity is propagating and the geometry is pseudo-Riemannian specified by Einstein equations. However, motivated by quantum cosmology, it may be that the universe can spontaneously nucleated out of nothing [5, 6]. This is an old idea and it is based on non-perturbative tunneling where one is looking for a “bounce” solution of the classical euclidean equations. This is a solution which approaches the putative vacuum state asymptotically, at infinity. A particular bounce is the de Sitter instanton [34], which has metric

$$ds^2 = d\tau^2 + \frac{1}{H^2} \cos^2 (H \tau) \left( \frac{dr^2}{1 - r^2} + r^2 d\Omega_2^2 \right),$$  \hspace{1cm} (5.2)
and describes the round four-sphere $S^4$. At $\tau = 0$ we have a plane of symmetry and we can rotate $\tau \rightarrow it$ where the geometry turns out to be that of standard de Sitter with metric

$$\begin{equation}
\begin{aligned}
ds^2 &= -dt^2 + \frac{1}{H^2} \cosh^2 (Ht) \left( \frac{dr^2}{1-r^2} + r^2 d\Omega^2 \right).
\end{aligned}
\end{equation}$$

The picture looks similar with that of fig 1, where $M_I$ is $S^4$ and $M_{II}$ is de Sitter joined together at the maximal (totally geodesic) $B = S^3$ at $\tau = 0$. However, the interpretation is different. Although (5.2) indeed bounces at the classical turning point, there is no any putative vacuum state to approach asymptotically at infinity since $S^4$ is a compact space and $\tau$ is bounded in the range $-\pi/H < \tau < \pi/H$. From this perspective, the de Sitter spacetime (5.3) appears out of nothing, since there is no classical space, time or matter from which de Sitter emerges. However, in the two phase gravity model we advocate here, $S^4$ describes phase I and de Sitter phase II. So, the out of nothing creation of the universe is not a bounce in the two-phase model but rather a Riemannian manifold (here $S^4$) sewing with a pseudo-Riemannian one (here de Sitter) along a hypersurface of vanishing extrinsic curvature (here $S^3$). In this respect, the model we are advocating here fits better to the no-boundary proposal which we now briefly present below.

### 5.3 No-boundary proposal

The Hartle-Hawking no-boundary proposal [4] provides initial conditions of the universe in the sense that it assigns weighted probabilities among all possible cosmological evolutions of our universe. Similarly with the Weyl curvature hypothesis and the out of nothing creation of the universe, the Big Bang singularity is replaced with a smoothed geometry. The no-boundary geometry is constructed by gluing two sectors described by a Riemannian manifold with Euclidean signature and a pseudo-Riemannian one with Lorentzian signature which are solutions of the Euclidean and Lorentzian Einstein field equation, respectively. Then in the presence of other matter fields like scalars, there are complex solutions that interpolate between the two sectors. Among these possible complex solutions, particularly important are solutions describing real tunneling metrics. The latter describes transitions from a purely Euclidean sector to a Lorentzian one. Semiclassically, such transitions are described by a Euclidean instanton glued to a Lorentzian solution representing a bubble of true vacuum expanding at the speed of light. Such gravitational instantons have the form of Fig.1 and provide initial conditions for its Lorentzian counterpart, similarly to the present two-phase gravity model we discuss here. In addition, these instantons exhibit discontinued metrics since the metric changes signature in the two regions, and therefore, the Lorentzian sector of the tunneling solutions provides background spacetimes for description of the dynamics of the late universe.

### Acknowledgments

We would like to thank C. Vafa for correspondence and J. Sonner for discussions. A.R. is supported by the Swiss National Science Foundation (SNSF), project *The Non-Gaussian Universe and Cosmological Symmetries*, project number: 200020-178787.

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