STANLEY DECOMPOSITIONS IN LOCALIZED POLYNOMIAL RINGS

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Abstract. We introduce the concept of Stanley decompositions in the localized polynomial ring $S_f$ where $f$ is a product of variables, and we show that the Stanley depth does not decrease upon localization. Furthermore it is shown that for monomial ideals $J \subset I \subset S_f$ the number of Stanley spaces in a Stanley decomposition of $I/J$ is an invariant of $I/J$. For the proof of this result we introduce Hilbert series for $\mathbb{Z}^n$-graded $K$-vector spaces.

Introduction

Let $K$ be a field, $S = K[x_1, \ldots, x_n]$ be the polynomial ring in $n$ variables over $K$, $A \subset \{1, 2, \ldots, n\}$ and $f = \prod_{j \in A} x_j$. Then $S_f$ is a $\mathbb{Z}^n$-graded $K$-algebra whose $K$-basis consists of the monomials $x_1^{a_1}x_2^{a_2} \cdots x_n^{a_n}$ with $a_j \in \mathbb{Z}$ and $a_j \geq 0$ if $j \notin A$.

Standard algebraic operations like reduction module regular elements or restricted localizations have been considered in the papers [5] and [6]. To better understand localization of Stanley decompositions we define in this paper Stanley decompositions of $S_f$-modules of the type $I/J$ where $J \subset I \subset S_f$ are monomial ideals in $S_f$. To do this we first have to extend the definition of Stanley spaces as it is given for $\mathbb{N}^n$-graded $K$-algebras. Here the $K$-subspace of the form $uK[Z] \subset I/J$, where $u$ is a monomial in $I \setminus J$ and $Z \subset \{x_1, \ldots, x_n\} \cup \{x_j^{-1}: j \in A\}$ such that $\{x_j, x_j^{-1}\} \not\subseteq Z$ for all $j \in A$, is called Stanley space if $uK[Z]$ is a free $K[Z]$-module of $I/J$. The dimension of the Stanley space $uK[Z]$ is defined to be $|Z|$. As in the classical case we define a Stanley decomposition of $I/J$ as a finite direct sum of Stanley spaces $\mathcal{D} : I/J = \bigoplus_{i=1}^{r} u_iK[Z_i]$, and set $sdepth(\mathcal{D}) = \min\{|Z_i| : 1 \leq i \leq r\}$ and $sdepth I/J = \max\{sdepth(\mathcal{D}) : \mathcal{D} \text{ is a Stanley decomposition of } I/J\}$.

Our more general definition of Stanley spaces is required for $S_f$, because otherwise it would be impossible to obtain a Stanley decomposition of $S_f$ when $f \neq 1$. Actually we show in Proposition 2.1 that $S_f$ has a nice canonical Stanley decomposition with Stanley spaces as defined above. Indeed, we have $S_f = \bigoplus_{L \subseteq A} \bigoplus_{l \in L} x_l^{-1}K[Z_L]$, where $Z_L = \{x_l^{-1} \mid l \in L\} \cup \{x_l \mid l \notin L\}$.

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and $f_L = \prod_{i \in L} x_i$. In this decomposition all Stanley spaces are of dimension $n$ and the number of summands is $2^{|A|}$. One may ask if there exist Stanley decompositions of $S_f$ with less summands. We will show that this is not possible. Indeed, as a generalization of a result in [3] we show in Theorem 6.5 that number of Stanley spaces of maximal dimension in a Stanley decomposition of an $S_f$-module of the form $I/J$ is independent of the particular Stanley decomposition of $I/J$ that of $(I/J)_f$. We note that this fact we introduce in Section 6 a modified Hilbert function for $\mathbb{Z}^n$-graded finitely generated $K$-vector spaces $M$ with the property that $\dim_K M_a < \infty$ for all $a \in \mathbb{Z}^n$.

For such modules we set $H(M, d) = \sum_{a \in \mathbb{Z}^n, \ |a| = d} \dim_K M_a$ and call the formal power series $H_M(t) = \sum_{d \geq 0} H(M, d) t^d$ the Hilbert series of $M$. Here we use the notation $|a| = \sum_{i=1}^n |a_i|$ for $a = (a_1, a_2, \ldots, a_n)$. Our Hilbert series does not as well behave as the usual Hilbert series with respect to shifts. Nevertheless Proposition 6.2 implies that for any monomial $u$ in $S_f$ one has $H_{uS_f}(t) = t^{\deg(u')} (1 + t)^{|A|} / (1 - t)^n$, where $u'$ is obtained from $u$ by removing the unit factors in $u$. Quite generally we show that, just as for the usual Hilbert series, one has $H_{I/J}(t) = P(t) / (1 - t)^d$ where $d$ is the Krull dimension of $I/J$ and $P(1) > 0$.

Any monomial ideal $I \subset S_f$ is obtained by localization from a monomial ideal in $S$. Therefore, for $J \subset I \subset S$ it is natural to compare the Stanley depth of $I/J$ with that of $(I/J)_f$. Theorem 3.1 states that $\text{sdepth}(I/J) \leq \text{sdepth}(I/J)_f$, and we give examples that show that this inequality can be strict. In Theorem 5.1 we prove a similar result for the so-called fdepth which is define via filtrations. Theorem 5.1 is related to a theorem proved by the first author in [4], where it is shown that $\text{sdepth}(S'/I_{x_1} \cap S') \geq \text{sdepth}(S'/I - 1)$, where $S' = K[x_2, \ldots, x_n]$.

Let $J \subset I \subset S$ be monomial ideals, and consider the polynomial ring $S[t]$ over $S$. Herzog, Vladoiu and Zheng in [2] proved that $\text{sdepth}(I/J)[t] = \text{sdepth}(I/J) + 1$. In Theorem 4.1 we see that a similar relation holds between the Stanley depth of $I/J$ and $\text{sdepth}(I/J)[t, t^{-1}]$ for monomial ideals $J \subset I \subset S_f$. It implies that Stanley depth of $I/J$, where $J \subset I \subset S_f$ monomial ideals, also if $f \neq 0$ can be computed.

1. Monomial ideals in $S_f$

Let $K$ be a field and $S = K[x_1, x_2, \ldots, x_n]$ be the polynomial ring in $n$ variables over $K$. We fix a subset $A \subset \{1, \ldots, n\}$ and set

$$f = \prod_{j \in A} x_j.$$

As usual, the localization of $S$ with respect to multiplicative set $\{1, f, f^2, \ldots\}$ is denoted by $S_f$. We note that $S_f = K[x_1, x_2, \ldots, x_n, x_{j}^{-1} : j \in A]$.

The monomials in $S_f$ are the element $x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$ with $a_j \in \mathbb{Z}$ if $j \in A$ and $a_j \in \mathbb{N}$ if $j \notin A$. The set $\text{Mon}(S_f)$ of monomials is a $K$-basis of $S_f$. An ideal $I \subset S_f$ is called a monomial ideal if it is generated by monomials. A minimal set of monomial generators of $I$ is uniquely determined, and is denoted $G(I)$.

Any element $g \in S_f$ is a unique $K$-linear combination of monomials.

$$g = \sum a_u u,$$

with $a_u \in K$ and $u \in \text{Mon}(S_f)$. 


We call the set 
\[ \text{Supp}(g) = \{ u \in \text{Mon}(S_f) : a_u \neq 0 \} \]
the support of \( g \).

For monomial \( u = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n} \), we set \( \text{supp}(u) = \{ i \in [n] : a_i \neq 0 \} \), \( \text{supp}_+(u) = \{ i \in [n] : a_i > 0 \} \) and \( \text{supp}_-(u) = \{ i \in [n] : a_i < 0 \} \), and call these sets support, positive support, and negative support of \( u \), respectively.

Just as for monomial ideals in the polynomial ring one shows that the set of monomials \( \text{Mon}(I) \) belonging to \( I \) is a \( K \)-basis of \( I \). We obviously have

**Proposition 1.1.** Let \( I \subset S_f \) be an ideal. The following are equivalent.

(a) \( I \subset S_f \) is a monomial ideal.
(b) There exists a monomial ideal \( I' \subset S \) such that \( I = I'S_f \).

If the equivalent conditions hold, then \( I' \) can be chosen such that \( \text{supp}(u) \subset [n] \setminus A \) for all \( u \in G(I') \). The monomial ideal \( I' \) satisfying this extra condition is uniquely determined. Indeed, \( I' = I \cap S \).

Let \( J \subset I \subset S_f \) be monomial ideals. Observe that the residue classes of the monomials belonging to \( I \setminus J \) form a \( K \)-basis of the residue class \( I/J \). Therefore we may identify the classes of \( I/J \) with the monomials in \( I \setminus J \).

### 2. Stanley decomposition of \( I/J \)

Let \( J \subset I \subset S_f \) be monomial ideals and \( f = \prod_{j \in A} x_j \). A Stanley space of \( I/J \) is a free \( K[Z] \)-submodule \( uK[Z] \) of \( I/J \), where \( u \) is a monomial of \( I/J \) and

\[ Z \subset \{ x_1, \ldots, x_n \} \cup \{ x_j^{-1} : j \in A \}, \]

satisfying the condition that \( \{ x_j, x_j^{-1} \} \notin Z \) for all \( j \in A \).

A Stanley decomposition of \( I/J \) is a finite direct sum of Stanley spaces

\[ D : I/J = \bigoplus_{i=1}^r u_i K[Z_i]. \]

We set

\[ \text{sdepth} D = \min\{|Z_i| : 1 \leq i \leq r\}, \]

and

\[ \text{sdepth}(I/J) = \max\{ \text{sdepth} D : D \text{ is a Stanley decomposition of } I/J \}. \]

This number is called the Stanley depth of \( I/J \).

**Proposition 2.1.** The ring \( S_f \) admits the following canonical Stanley decomposition

\[ D : S_f = \bigoplus_{L \subseteq A} f_L^{-1} K[Z_L], \]

where \( Z_L = \{ x_l^{-1} | l \in L \} \cup \{ x_l | l \notin L \} \) and \( f_L = \prod_{l \in L} x_l \).
Proof. Consider a monomial \( h = x_1^{a_1} \cdots x_n^{a_n} \) of \( S_f \). We choose

\[
L = \{1 \leq l \leq n : a_l < 0\} \subset A.
\]

Then

\[
h = f_L^{-1}(x_1^{a_1} \cdots x_n^{a_n} f_L) = f_L^{-1}x_1^{b_1} \cdots x_n^{b_n},
\]

where \( b_l = a_l \in \mathbb{N} \) if \( l \notin L \) and \( b_l = a_l + 1 \in \mathbb{Z} \) if \( l \in L \), and hence \( h \in f_L^{-1}K[Z_L] \).

To show that this sum is direct, it is enough to prove any two distinct Stanley spaces in the decomposition of \( I/J \) have no monomial in common, since the multigraded components of \( I/J \) are of dimension \( \leq 1 \).

Let \( L, L' \subset A \) with \( L \neq L' \). Then \( f_L^{-1} \neq f_{L'}^{-1} \). Suppose that there is a monomial \( u \in I/J \) such that

\[
u \in f_L^{-1}K[Z_L] \cap f_{L'}^{-1}K[Z_{L'}],
\]

that is

\[
u = f_L^{-1}v = f_{L'}^{-1}v'
\]

for some monomials \( v \in K[Z_L], v' \in K[Z_{L'}] \). Since is a contradiction, because

\[L = \text{supp}(u) = L',\]

as follows from the above equation.

\[\square\]

Example 2.2. Let \( S = K[x, y, z] \) and \( f = yz \). Then according to Lemma 2.1, \( S_f = K[x, y, z] \oplus y^{-1}K[x, y^{-1}, z] \oplus z^{-1}K[x, y, z^{-1}] \oplus y^{-1}z^{-1}K[x, y^{-1}, z^{-1}] \) is a Stanley decomposition of \( S_f \).

As a consequence of Proposition 2.1, we have

Corollary 2.3. \( \text{sdepth}(S_f) = n. \)

3. Localization of Stanley decompositions

Let \( J \subset I \subset S \) be monomial ideals and we set \( f = \prod_{j \in A} x_j \). The next result shows how a Stanley decomposition of \( I/J \) localizes.

Theorem 3.1. Let \( D : I/J = \bigoplus_{i=1}^r u_iK[Z_i] \) be a Stanley decomposition of \( I/J \). Then

\[
D_f : (I/J)_f = \bigoplus_{i \in C} ( \bigoplus_{L \in A} u_i f_L^{-1}K[Z_i^L] ) \]

is a Stanley decomposition of \( (I/J)_f \), where

\[C = \{i : Z_A \subset Z_i\} \]

and \( Z_i^L = \{x_i^{-1} \mid l \in L, x_i \in Z_i\} \cup \{x_i \mid l \notin L, x_i \in Z_i\} \).

In particular, we have

\[\text{sdepth}(I/J) \leq \text{sdepth}(I/J)_f.\]

Proof. Let \( u \in (I/J)_f \) be a nonzero monomial. We claim that

\[
u \in \bigoplus_{i \in C} ( \bigoplus_{L \in A} u_i f_L^{-1}K[Z_i^L] ).
\]

Since \( u \in (I/J)_f \) is nonzero it follows that \( uf^a \in I \setminus J \) for all \( a \gg 0 \). Hence there exists an integer \( i \) such that \( uf^a \in u_i K[Z_i] \) for infinitely many \( a > 0 \). This
implies that \( Z_A \subset Z_i \), that is, \( i \in C \). So, \( u f^a \in u_i K[Z_i] \), that is, \( u f^a = u_i v \) where \( v = x_1^{a_1} \cdots x_n^{a_n} \in \text{Mon}(Z_i) \). Then \( u = u_i v f^{-a} = u_i x_1^{b_1} \cdots x_n^{b_n} \), where

\[
\begin{align*}
b_i &= \begin{cases} 
a_i - a, & \text{if } i \in A \\
a_i, & \text{otherwise.}
\end{cases}
\end{align*}
\]

Say \( L = \text{supp}_-(x_1^{b_1} \cdots x_n^{b_n}) \). Note that \( L \subset A \). Hence \( u \in u_i f_L^{-1} K[Z_i] \).

In order to prove other inclusion, consider a monomial \( w \in \sum_{i \in C} ( \sum_{L \subseteq A} u_i K[Z_i^L]) \).

This implies that there exists an \( i \in C \) and \( L \subset A \) such that \( w \in u_i K[Z_i^L] \). Since \( Z_A \subset Z_i \), we have \( w f^a \in u_i K[Z_i] \) for \( a \gg 0 \). It follows that \( w f^a \in I \), so \( w \in I_f \). On the other hand, \( w \notin J_f \). Otherwise \( w f^a \in J \) for \( a \gg 0 \), which is a contradiction, since \( w f^a \in u_i K[Z_i] \) for \( a \gg 0 \).

Now we prove that the sum is direct. Let \( L, H \subset A \) and

\[
u \in u_i f_L^{-1} K[Z_i^L] \cap u_j f_H^{-1} K[Z_i^H]
\]

be a monomial such that \( i \neq j \) or \( L \neq H \). Since \( Z_A \subset Z_i, Z_j \), we have for \( a \gg 0 \) that

\[
u f^a \in u_i K[Z_i] \cap u_j K[Z_j].
\]

If \( i \neq j \), then \( u f^a = 0 \) in \( I/J \) implies \( u = 0 \) in \( (I/J)_f \). If \( i = j \) and \( L \neq H \) then \( u = u_i f_L^{-1} v = u_i f_H^{-1} v' \) for some monomials \( v \in K[Z_i^L] \) and \( v' \in K[Z_i^H] \). Since \( \text{supp}_-(u_i f_L^{-1} v) = L \) and \( \text{supp}_-(u_i f_H^{-1} v') = H \), so \( \text{supp}_-(u) = L = H \), which is a contradiction.

The next examples show that in the inequality on Theorem 3.1 we may have equality or strict inequality.

**Example 3.2.** Let \( J = (xy, yz) \subset I = (y) \subset S = K[x, y, z] \) be ideals, \( D : I/J = yK[y] \) is a Stanley decomposition of \( I/J \). Thus \( \text{sdepth} D = 1 \). Localizing with the monomial \( f = y \), we obtain that \( D_f : (I/J)_f = yK[y] \oplus K[y^{-1}] \) is a Stanley decomposition of \( (I/J)_f \) and \( \text{sdepth} D_f = 1 \).

**Example 3.3.** Let \( J = (x^2 y, x^2 y^2) \subset I = (x, y) \subset S = K[x, y] \) be ideals, \( D : I/J = xy K[y] \oplus x^2 K[x] \oplus x^2 y K \) is a Stanley decomposition of \( I/J \). Thus \( \text{sdepth} D = 0 \). Localizing with the monomial \( f = x \), \( D_f : (I/J)_f = x^2 K[x] \oplus x K[x^{-1}] \) is a Stanley decomposition of \( (I/J)_f \) and \( \text{sdepth} D_f = 1 \).

The following example shows that a Stanley decomposition of \( I/J \) which gives the Stanley depth of \( I/J \) may localize to a Stanley decomposition whose Stanley depth is smaller than the Stanley depth of the localization of \( I/J \).

**Example 3.4.** Let \( I = (x, y, z) \subset S = K[x, y, z] \) be the graded maximal ideal of \( S \) \( D : I = xK[x, y] \oplus yK[y, z] \oplus zK[x, z] \oplus xyzK[x, y, z] \) is a Stanley decomposition of \( I \). Thus \( \text{sdepth} D = 2 \) which is also the Stanley depth of \( I \). Localizing with the monomial \( f = x \), we get the Stanley decomposition \( D_f \) of \( I_f \) which is

\[
xK[x, y] \oplus K[x^{-1}, y] \oplus zK[x, z] \oplus zx^{-1} K[x^{-1}, z] \oplus xyzK[x, y, z] \oplus yzK[x^{-1}, y, z].
\]

Thus \( \text{sdepth} D_f = 2 \). However \( I_f = K[x^\pm 1, y, z] \), and hence \( \text{sdepth} I_f = 3 \).
4. STANLEY DECOMPOSITIONS AND POLYNOMIAL EXTENSIONS

Herzog et al. in [2, Lemma 3.6] proved that the Stanley depth of the module increases by one in a polynomial ring extension by one variable. We generalize this result to localized rings.

**Theorem 4.1.** Let \( J \subset I \subset S_f \) be monomial ideals. Then

\[
\text{sdepth}(I/J)[t] = \text{sdepth}(I/J)[t, t^{-1}] = \text{sdepth} I/J + 1.
\]

**Proof.** Let \( \mathcal{D} : I/J = \bigoplus_{i=1}^r v_i K[Z_i] \) be a Stanley decomposition of \( I/J \). Then we obtain the Stanley decompositions

\[
(I/J)[t] = \bigoplus_{i=1}^r v_i K[Z_i][t] = \bigoplus_{i=1}^r v_i K[Z_i, t]
\]

\[
((I/J)[t])_t = \bigoplus_{i=1}^r (v_i K[Z_i, t] \oplus v_i t^{-1} K[Z_i, t^{-1}]).
\]

This implies that

\[
1 + \text{sdepth} I/J \leq \text{sdepth}(I/J)[t, t^{-1}], \text{sdepth}(I/J)[t].
\]

Let \( \mathcal{D'} : (I/J)[t, t^{-1}] = \bigoplus_{i=1}^r v_i K[W_i] \) be a Stanley decomposition of \( (I/J)[t, t^{-1}] \). Next we show that

\[
I/J = \bigoplus_{i\in[r]} v_i K[W_i] \cap S_f,
\]

and that each \( v_i K[W_i] \cap S_f = u_i K[W_i \setminus \{t, t^{-1}\}] \) for a suitable monomial \( u_i \in S_f \), provided \( v_i K[W_i] \cap S_f \neq 0 \).

Since the direct sum \( \mathcal{D'} \) commutes with taking the intersection with \( S_f \) and since \( (I/J)[t, t^{-1}] \cap S_f = I/J \), the desired decomposition of \( I/J \) follows.

Suppose \( v_i K[W_i] \cap S_f \neq 0 \). Then there exist monomials \( v \in S_f \) and \( w \in K[W_i] \) such that \( v = v_i w \). We may assume that \( v_i \) does not contain \( t \) as a factor, because \( t^{-1} \) must be a factor of \( w \) which implies that \( t^{-1} \in W_i \). Thus we may replace \( v_i \) by the monomial which is obtained from \( v_i K \) by removing the power of \( t \) which appears in \( v_i \). Similarly we may assume \( t^{-1} \) is not a factor of \( v_i \). Then it follows that \( v_i K[W_i] \cap S_f \) consists of all monomials \( v_i w \) with \( w \in K[W_i] \) where neither \( t \) nor \( t^{-1} \) is a factor of \( w \). In other words, \( w \in K[W_i \setminus \{t, t^{-1}\}] \).

From these considerations we conclude that \( 1 + \text{sdepth} I/J \geq \text{sdepth}(I/J)[t, t^{-1}] \). In the same way one proves the inequality \( 1 + \text{sdepth} I/J \geq \text{sdepth}(I/J)[t] \). This yields the desired conclusions. \( \square \)

An immediate consequence of Theorem 4.1 is the following

**Corollary 4.2.**

\[
\text{sdepth}(I/J)[t_1^{\pm 1}, \ldots, t_r^{\pm 1}] = \text{sdepth} I/J + r.
\]
Let $J \subset I \subset S_f$ be monomial ideals, and $S' = K[x_i : i \notin A] \subset S$. Then there exist monomial ideals $J' \subset I' \subset S'$ such that $J'S_f = J$ and $I'S_f = I$. We have

$$I'/J'[x_i^{\pm 1} : i \in A] = I/J.$$  

Hence $\text{sdepth} I'/J' + |A| = \text{sdepth} I/J$, by Corollary 1.2. Since the Stanley depth of $I'/J'$ can be computed in finite number of steps by the method given by Herzog, Vladoiu and Zheng [2], the Stanley depth of $I/J$ can be computed as well in a finite number of steps.

5. Fdepth and localization

Let $J \subset I \subset S$ be monomial ideals of $S$. A chain $F : J = J_0 \subset J_1 \subset \ldots \subset J_r = I$ of monomial ideals is called a prime filtration of $I/J$, if $J_i/J_{i-1} \cong S/P_i(-a_i)$ where $a_i \in \mathbb{Z}^n$ and where each $P_i$ is a monomial prime ideal. We call the set of prime ideals \{P_1, \ldots, P_r\} the support of $F$ and denote it by $\text{Supp}(F)$. In [2], Herzog, Vladoiu and Zheng define $\text{fdepth}$ of $I/J$ as follows:

$$\text{fdepth}(F) = \min \{\dim S/P : P \in \text{Supp}(F)\}$$

and

$$\text{fdepth} I/J = \max \{\text{fdepth} F : F \text{ is a prime filtration of } I/J\}.$$ 

These definitions can be immediately extended to monomial ideals $J \subset I \subset S_f$. We then get

**Theorem 5.1.**

$$\text{fdepth} I/J \leq \text{fdepth}(I/J)_f.$$  

*Proof.* Let $F : J = J_0 \subset J_1 \subset \ldots \subset J_r = I$ be a prime filtration of $I/J$ with $\text{Supp}(F) = \{P_1, \ldots, P_r\}$. Then we obtain a filtration $F_f : J_f = (J_0)_f \subset (J_1)_f \subset \ldots \subset (J_r)_f = I_f$ with factors $S_f/P_iS_f$. Thus if we drop the ideals $(J_i)_f$ for which $S_f/P_iS_f = 0$ we obtain a prime filtration of $(I/J)_f$. Since $\dim S_f/P_iS_f = \dim S/P_i$, whenever $\dim S_f/P_iS_f \neq 0$, it follows that $\text{fdepth} F_f \geq \text{fdepth} F$. This implies the desired conclusion. \qed

6. Hilbert series of multigraded $K$-vector spaces

Let $J \subset I \subset S_f$ be monomial ideals. In this section we want to show that the number of maximal Stanley spaces in any Stanley decomposition of $I/J$ is an invariant of $I/J$. To prove this we introduce a new kind of Hilbert series.

Note that the localized ring $S_f$ is naturally $\mathbb{Z}^n$-graded with $\mathbb{Z}^n$-graded components

$$(S_f)_a = \begin{cases} 
Kx^a, & a_i \geq 0 \text{ if } i \notin \text{supp}(f); \\
0, & \text{otherwise.}
\end{cases}$$

Let $M = \bigoplus_{a \in \mathbb{Z}^n} M_a$ be $\mathbb{Z}^n$-graded $K$-vector space with $\dim_K M_a < \infty$ for all $a \in \mathbb{Z}^n$. Then for all $d \in \mathbb{N}$ we define

$$M_d = \bigoplus_{|a|=d} M_a \text{ where } |a| = \sum_{j=1}^r |a_j|,$$
and set \( H(M, d) = \text{dim}_k M_d \). We call the function \( H(M, -): \mathbb{N} \to \mathbb{N} \) the Hilbert function, and the series \( H_M(t) = \sum_{d \geq 0} H(M, d)t^d \) the Hilbert series of \( M \).

We consider an example to illustrate our definition. Figure 1 displays the ideal \( I = (x^3, x^2y, y^2) \subset S = K[x, y] \). The grey area represents the monomial \( K \)-vector space spanned by the monomials in \( I \). The slanted lines represent \( S_d = \bigoplus_{|a|=d} S_a \).

**Figure 1.**

The \( S \)-modules \( I(a) \) with \( a = (4, 3) \) is an \( S \)-submodule of \( S_f \) where \( f = xy \). In Figure 2, the grey area displays \( I(a) \). For all \( d \in \mathbb{N} \), the boundaries of the squares of diagonal length 2\( d \), represent \( \bigoplus_{|a|=d} (S_f)_a \).

In this example, if we consider \( I(a) \) as a graded \( S \)-module in the usual sense, then \( I(a)_4 = I_{11} \), and hence \( \text{dim}_K I(a)_4 = 12 \). If we apply our new definition of \( M_d \) to \( I(a) \), then \( \text{dim}_K I(a)_4 = 15 \), as can be seen in the picture.

In case that \( f = 1 \), and \( M \) is a finitely generated \( \mathbb{Z}^n \)-graded \( S \)-module with the property that the multidegree of all generators of \( M \) belong to \( \mathbb{N}^n \), then our definition of the Hilbert function coincides with the usual definition. For properties of classical Hilbert series we refer to the book [1].

If we would define \( H(M, d) \) in the usual way as \( \text{dim}_K \left( \bigoplus_{a \in \mathbb{N}^n} \sum_{a_i = d} M_a \right) \), then this number would be in general infinite. On the other hand, our definition has the drawback, that the components \( (S_f)_d \) don’t define a grading of \( S_f \). In other words, we do not have in general that \( (S_f)_{d_1} (S_f)_{d_2} \subset (S_f)_{d_1 + d_2} \) for all \( d_1 \) and \( d_2 \). Nevertheless, we shall prove that \( H(I/J, d) \) is a polynomial function for \( d \gg 0 \).

In the case that \( J \subset I \subset S_f \) are monomial ideals, then the Hilbert function of \( I/J \) is given by

\[
H(I/J, d) = |\{a \in \mathbb{Z}^n \mid x^a \in I \setminus J \text{ and } |a| = d\}|.
\]

Let \( u = x_1^{a_1} \cdots x_n^{a_n} \in \text{Mon}(S_{[n]}) \). We set \(|u| = x_1^{|a_1|} \cdots x_n^{|a_n|}|. Then \(|u| \in \text{Mon}(S)\).
Figure 2.

Let \( S_{[n]} = K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] \) and \( K[Z] \subset S_{[n]} \) be a Stanley space. We set
\[
\overline{Z} = \{ x_i : x_i \in Z \text{ or } x_i^{-1} \in Z \}.
\]

**Lemma 6.1.** Let \( v \in \text{Mon}(S_{[n]}) \) such that
\[
\text{supp}_+(v) \cap \{ i : x_i^{-1} \in Z \} = \emptyset
\]
and
\[
\text{supp}_-(v) \cap \{ i : x_i \in Z \} = \emptyset.
\]
Then \( vK[Z] \cong |v|K[Z] \), and for each \( d \in \mathbb{N} \) we have
\[
(vK[Z])_d \cong (|v|K[Z])_d.
\]

**Proof.** Since \( v \in \text{Mon}(S_{[n]}) \) such that \( \text{supp}_+(v) \cap \{ i : x_i^{-1} \in Z \} = \emptyset \) and \( \text{supp}_-(v) \cap \{ i : x_i \in Z \} = \emptyset \), it follows that \( |vw| = |v||w| \) for all \( w \in K[Z] \). Therefore the map \( u \mapsto |u| \) induces for each \( d \) an isomorphism \( (vK[Z])_d \cong (|v|K[Z])_d \) of \( K \)-vector spaces. □

**Proposition 6.2.** Let \( T_A = K[x_i^{\pm 1} : i \in A, x_{i_1}, \ldots, x_{i_t}] \subset S_{[n]} \), where \( \{i_1, \ldots, i_t \} \subset [n] \setminus A \). Let \( u \in \text{Mon}(S_{[n]}) \), \( u = x_1^{a_1}x_2^{a_2} \cdots x_n^{a_n} \) such that \( a_i \geq 0 \) for \( i \notin A \). We set \( u' = \prod_{i \notin A} x_i^{a_i} \). Then
\[
H_{uT_A}(t) = H_{u'T_A}(t) = t^{\deg(u')}H_{T_A}(t) = t^{\deg(u')(1 + t)^{|A|}} \frac{(1 + t)^{|A|}}{(1 - t)^d},
\]
where \( d = \dim T_A \).

**Proof.** Since \( uT_A = u'T_A \), it follows that \( H_{uT_A}(t) = H_{u'T_A}(t) \). In order to prove the second equality, we use the Stanley decomposition \( T_A = \bigoplus_{L \subset A} f_L^{-1}K[Z_L] \) where
$f_L = \prod_{i \in L} x_i$ and $Z_L = \{x_i^{-1} : i \in L\} \cup \{x_i : i \in A \setminus L\} \cup \{x_{i_1}, \ldots, x_{i_r}\}$, from which we obtain the Stanley decomposition

$$u'T_A = \bigoplus_{L \subseteq A} u'f_L^{-1}K[Z_L].$$

Applying Lemma 6.1, we obtain from this decomposition the following identities

$$H_{u'T_A}(t) = \sum_{k=0}^{|A|} \sum_{L \subseteq A, |L|=k} H_{u'f_L^{-1}K[Z_L]}(t) = \sum_{k=0}^{|A|} H_{u'f_L^{-1}K[Z_L]}(t)$$

$$= \sum_{k=0}^{|A|} \frac{t^{k+\deg(u')}}{(1-t)^d} = t^{\deg(u')} \sum_{k=0}^{|A|} \binom{|A|}{k} \frac{t^k}{(1-t)^d}$$

$$= t^{\deg(u')} \frac{(1+t)^{|A|}}{(1-t)^d} = t^{\deg(u')} H_{T_A}(t),$$

since $T_A = \bigoplus_{L \subseteq A} f_L^{-1}K[Z_L]$ and $H_{f_L^{-1}K[Z_L]} = \frac{(1+t)^{|A|}}{(1-t)^d}$ by Lemma 6.1, we get that

$$H_{T_A}(t) = \sum_{L \subseteq A} H_{f_L^{-1}K[Z_L]}(t) = \frac{(1+t)^{|A|}}{(1-t)^d}. \quad \square$$

It follows from the next result that the Hilbert series of finitely generated $\mathbb{Z}^n$-graded $S_f$-module $I/J$ can be written as a rational function with denominator $(1-t)^d$, where $d = \dim I/J$.

**Theorem 6.3.** Let $A \subseteq [n], f = \prod_{i \in A} x_i$, and $J \subseteq I \subseteq S_f$ be monomial ideals. Then $I/J$ admits a filtration

$$0 = M_0 \subset M_1 \subset \ldots \subset M_r = I/J$$

of $\mathbb{Z}^n$-graded $S_f$-submodules of $I/J$ such that for each $i$ we have

$$M_{i+1}/M_i \cong (S_f/P_i)(-a_i),$$

where $\{P_1, \ldots, P_r\}$ is a set of $\mathbb{Z}^n$-graded prime ideals of $S_f$ containing all minimal prime ideals of $I/J$, and where $a_i \in \mathbb{N}^n$ with $a_i(j) = 0$ if $j \in A$. Moreover,

$$H_{I/J}(t) = \sum_{i=1}^r H_{S_f/P_i(-a_i)}(t) \quad \text{and} \quad H_{S_f/P_i(-a_i)}(t) = \frac{Q_i(t)(1+t)^{|A|}}{(1-t)^{d_i}},$$

where $d_i = \dim S_f/P_i$ and $Q_i(t)$ is a polynomial with $Q_i(1) = 1$. In particular, $H_{I/J}(t) = Q(t)/(1-t)^d$ with $Q(1) = k \cdot 2^{|A|}$, where

$$k = \{|i : \dim S_f/P_i = d|\} \quad \text{and} \quad d = \dim I/J.$$

**Proof.** Since $J \subseteq I \subseteq S_f$ monomial ideals, we may assume that $\supp(u) \subseteq [n] \setminus A$ for all $u \in G(I) \cup G(J)$, see Proposition 11.1. Let $S' = K[x_i : i \notin A] \subseteq S$ be the polynomial ring over $K$. Then there exist monomial ideals $J' \subseteq I' \subseteq S'$ such that $J = J'S_f$, $I = I'S_f$. Consider a prime filtration

$$J' = I_0 \subset I_1 \subset \ldots \subset I_r = I'$$

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of $I'/J'$ where $I_{i+1}/I_i \cong (S'/p_i)(-a_i)$, where $p_i$ is a monomial prime ideal $a_i \in \mathbb{N}^n$ with $a_i(j) = 0$ if $j \in A$. It follows that

$$J = J'S_f = I_0S_f \subset I_1S_f \ldots \subset I_rS_f = I$$

is a prime filtration of $I/J$ with $I_{i+1}S_f/I_iS_f \cong S_f/P_i(-a_i)$ where $p_iS_f = P_i$. Because of the additivity of our Hilbert function, it suffices to show that

$$H_{S_f/P_i(-a_i)}(t) = Q_i(t)H_{S_f/P_i}(t)$$

for some polynomial $Q_i(t)$ with $Q_i(1) \neq 0$. But this follows from Proposition 6.2. The rest of the statements are obvious. □

For the proof of our main result, we need the following

**Lemma 6.4.** Let $uK[Z]$ be a Stanley space of $S_f$. Then $H_{uK[Z]}(t) = Q(t)/(1 - t)^m$ where $Q(1) = 1$ and $m = |Z|$.

**Proof.** Let $u = x_1^{a_1} \cdots x_n^{a_n}$, and let $\mathcal{I}$ be the set of indices $i$ for which either $a_i > 0$ and $x_i \in Z$, or $a_i < 0$ and $x_i \in Z$. Then we set $r = \sum_{i \in \mathcal{I}} |a_i|$, and prove the lemma by induction on $s = \min\{r, m\}$. If $m = 0$, then the assertion is trivial, and if $r = 0$, then the result follows from Lemma 6.1.

Now let $s > 0$. Then we may assume that $a_1 > 0$ and $x_1 \in Z$. In this case we have the following direct sum of $K$-vector spaces

$$uK[Z] = uK[Z \setminus \{x_1^{-1}\}] \oplus vK[Z],$$

where $v = x_1^{a_1-1} \cdots x_n^{a_n}$. In the first summand $m$ is smaller and in the second $r$ is smaller than the corresponding numbers for $uK[Z]$. Thus we may apply our induction hypothesis according to which there exist polynomials $Q_1(t)$ and $Q_2(t)$ with $Q_1(1) = 1$, and such that

$$H_{uK[Z \setminus \{x_1^{-1}\}]}(t) = Q_1(t)/(1 - t)^{m-1} \quad \text{and} \quad H_{vK[Z]}(t) = Q_2(t)/(1 - t)^m.$$

It follows that $H_{uK[Z]}(t) = Q(t)/(1 - t)^m$ with $Q(t) = Q_1(t)(1 - t) + Q_2(t)$. Since $Q(1) = 1$, the proof is completed. □

**Theorem 6.5.** Let $J \subset I \subset S_f$ be monomial ideals, $\mathcal{D} : I/J = \bigoplus_{i=1}^r u_iK[Z_i]$ a Stanley decomposition of $I/J$ and $d = \max\{|Z_i| : i \in 1, 2, \ldots, r\}$. Then $H_{I/J}(t) = P_{I/J}(t)/(1 - t)^d$ with $P_{I/J}(1) = \{|i: |Z_i| = d\}$.

**Proof.** We have

$$H_{I/J}(t) = \sum_{i=1}^r H_{u_iK[Z_i]}(t)$$

By Lemma 6.4, for all $i \in \{1, 2, \ldots, r\}$ we obtain that $H_{u_iK[Z_i]}(t) = \frac{Q_i(t)}{(1 - t)^{|Z_i|}}$, where $Q_i(1) = 1$. Thus

$$H_{I/J}(t) = \sum_{i=1}^r \frac{Q_i(t)}{(1 - t)^{|Z_i|}} = \frac{P_{I/J}(t)}{(1 - t)^d}$$

where, $P_{I/J}(t) = \sum_{i=1}^r (1 - t)^{d-|Z_i|}Q_i(t)$. It follows that $P_{I/J}(1) = \{|i: |Z_i| = d\}$.

□
This Theorem implies that the number of Stanley spaces of maximal dimension in a Stanley decomposition of $I/J$ is independent of the special Stanley decomposition of $I/J$.

Proposition 2.1 and Theorem 6.5 yield the following result.

**Corollary 6.6.** The number of Stanley spaces of maximal dimension in any Stanley decomposition of $S_f$ is equal to $2^{|A|}$.

We conclude our paper with the following concrete example.

**Example 6.7.** Let $S = K[x, y, z]$ and $f = z$. Then $S_f = K[x, y, z^{±1}]$. Let $J = (x^2) \subset I = (x, y^2) \subset S_f$ be monomial ideals in $S_f$. A Stanley decomposition of $I/J$ is $xK[y, z] \oplus xz^{-1}K[y] \oplus xz^{-2}K[y, z^{-1}] \oplus y^2K[y, z] \oplus y^2z^{-1}K[y, z^{-1}]$. Thus in any other Stanley decomposition of $I/J$ the number of maximal Stanley spaces is 4. Calculating the Hilbert function of $I/J$ by using this Stanley decomposition we find that $H_{I/J}(t) = \frac{t + 2t^2 + t^3}{(1-t)^2}$. Thus $P(t) = t + 2t^2 + t^3$, and $P(1) = 4$, as expected.

**References**

[1] W. Bruns, J. Herzog, *Cohen-Macaulay rings*, Revised Edition, Cambridge, 1996.
[2] J. Herzog, M. Vladoiu, X. Zheng, How to compute the Stanley depth of a monomial ideal. J. Alg. in press.
[3] A. Soleyman Jahan, Prime filtrations of monomial ideals and polarizations. J. Alg. **312**(2)(2007), 1011–1032.
[4] S. Nasir, Stanley decompositions and localization, Bull. Math. Soc. Sc. Math. Roumanie, **51**(99), no. 2, (2008), 151-158 (see www.rms.unibuc.ro/bulletin).
[5] A. Rauf, Stanley decompositions, pretty clean filtrations and reductions modulo regular elements, Bull. Math. Soc. Sc. Math. Roumanie, **50**(98), no. 4, (2007), 347-354 (see www.rms.unibuc.ro/bulletin).
[6] A. Rauf, Depth and Stanley depth of multigraded modules, *Comm. Alg.*, **38** (2010), 773–784.

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