Electromagnetic corrections to the hadronic phase shifts in low energy $\pi^+ p$ elastic scattering

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Abstract

We calculate for the $s$-, $p_{1/2}$- and $p_{3/2}$-waves the electromagnetic corrections which must be subtracted from the nuclear phase shifts obtained from the analysis of low energy $\pi^+ p$ elastic scattering data, in order to obtain hadronic phase shifts. The calculation uses relativised Schrödinger equations containing the sum of an electromagnetic potential and an effective hadronic potential. We compare our results with those of previous calculations and estimate the uncertainties in the corrections.

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1 Introduction

The aim of this paper is to present a careful recalculation of the electromagnetic corrections which have to be applied in the analysis of low energy $\pi^+ p$ elastic scattering data in order to recover the hadronic phase shifts.

For these electromagnetic effects in $\pi N$ scattering ($\pi^+ p$ elastic and $\pi^- p$ elastic and charge exchange) two different approaches were developed: the dispersion
theory method and a method using potentials for the hadronic and electromagnetic interactions in a relativised Schrödinger equation (RSE for short). By this we mean an extension of the Schrödinger equation to the relativistic domain.

The dispersion theory approach was initiated by Sauter [1,2]. It was then extended by the NORDITA group. The final results for their corrections are given in Tromborg et al. [3]. In Ref.[4] the same authors present a detailed exposition of the underlying ideas and point out their limitations.

The potential theory approach was initiated by Oades and Rasche [5,6] and applied in detail by Zimmermann [7,8] to an analysis of the then existing data. His final results appear in the Landolt-Börnstein tables on $\pi N$ scattering [9].

Both methods have their shortcomings. In the dispersion theory approach the only contribution from $t$- and $u$-channel exchange which is taken into account is that from $u$-channel nucleon exchange. The rest of the unphysical contributions are omitted. Ref.[4] states: ‘We are not able to determine or even estimate this term, which is of course a serious drawback of our method’. For this reason, Ref.[4] makes it clear that the results of an analysis of data for the three measured reactions which uses the corrections given there cannot be used to draw any conclusion about isospin symmetry violation. It seems that the fact that a dispersion theory method cannot give absolute values of the electromagnetic corrections has not been sufficiently recognised. This is an unsatisfactory situation, particularly for the analysis of the large amount of data at low energies.

In the potential theory approach of Refs.[7,8] hadronic potentials were constructed via RSEs to reproduce the hadronic phase shifts. These potentials then were used in RSEs, containing also the electromagnetic potential, to calculate the electromagnetic corrections. The idea is that by this addition of an effective short range hadronic potential and the long range electromagnetic potential we can estimate the corrections reliably. The weak point here is the use of a hadronic potential in a RSE. But it should be noted that

(i) the hadronic potentials are used only to calculate the corrections, not to calculate the hadronic quantities themselves;
(ii) the potential model is the only method that can give a well defined separation between hadronic quantities and their electromagnetic corrections.

It was indeed possible (Ref.[8]) to analyse the best experimental data of that time in an isospin invariant way, using energy dependent hadronic potentials. The main reasons for recalculting the electromagnetic corrections in the potential model approach are the following.

(i) The results need to be extended to lower energies, where there is now a large amount of new data obtained at pion factories.
(ii) The slight effect which a change of the hadronic input has on the electromagnetic corrections needs to be controlled. This has not been done by the NORDITA group.

(iii) The effect of some fine details of the electromagnetic interaction, which might become important at the present precision of the experimental data, needs to be checked. This also is not contained in the NORDITA results.

(iv) Instead of the energy dependent potentials used in Refs.[7,8] we want to use energy independent hadronic potentials to calculate the electromagnetic corrections for pion laboratory kinetic energy $T_\pi \leq 100$ MeV.

In this paper we treat only the single channel problem of $\pi^+ p$ elastic scattering. It enables us to describe the formalism carefully and to provide a survey of the relative importance of the contributions to the corrections. In the next paper we extend the formalism to the two-channel ($\pi^- p, \pi^0 n$) case and give the results for the electromagnetic corrections which need to be applied in the analysis of low energy $\pi^- p$ elastic and charge exchange scattering data.

In Section 2 we give a detailed description of our model for the electromagnetic corrections to $\pi^+ p$ scattering. In Section 3 we explain the method of evaluation of the corrections in this model and in Section 4 we give the final numerical results for these corrections.

2 Detailed description of the model

For the purpose of calculating the electromagnetic corrections we describe low energy $\pi^+ p$ scattering by means of individual potentials for each of the partial waves which are used in relativised Schrödinger equations (RSEs). We therefore develop the model by motivating the construction of the RSEs and discussing the components of the potentials that appear in them.

The point charge Coulomb potential is

$$V^{pc}(r) = \frac{\alpha}{r},$$

(1)

where $r$ is the distance between $\pi^+$ and $p$ in the c.m. system. When the potential (1) is used in the nonrelativistic Schrödinger equation, one obtains the nonrelativistic point charge Coulomb amplitude

$$f^{pc}_{NR} = \frac{2\alpha m_e}{t} \exp \{2i(\sigma_0)_{NR} - i\eta \ln (\sin^2 \frac{1}{2}\theta)\},$$

(2)
where
\[
(\sigma_0)_{NR} = \arg \Gamma(1 + i\eta),
\]
\[
\eta = \frac{\alpha m_c}{q_c},
\]
\[
t = -2q_c^2(1 - \cos \theta),
\]
\(\theta\) being the c.m. scattering angle and \(m_c\) the reduced mass of the \(\pi^+ p\) system:
\[
m_c = \frac{m_p \mu_c}{m_p + \mu_c},
\]
with \(m_p, \mu_c\) the masses of the proton and charged pion respectively. The quantity \(q_c\) is the c.m. momentum of the \(\pi^+ p\) system. The Born approximation to the point charge Coulomb amplitude is given by the expression (2) without the phase factor.

To make the relativistic generalisation of these nonrelativistic results, we start from the one photon exchange (1 \(\gamma E\)) no-flip and spin-flip amplitudes \(f_{1\gamma E}, g_{1\gamma E}\), modified by the pion and proton form factors \(F^\pi, F^p_1, F^p_2\) respectively, and separate them into three parts:

\[
f_{1\gamma E} = f^{pc}_{1\gamma E} + f^{ext}_{1\gamma E} + f^{rel}_{1\gamma E},
\]
\[
g_{1\gamma E} = g^{pc}_{1\gamma E} + g^{ext}_{1\gamma E} + g^{rel}_{1\gamma E},
\]
where
\[
f^{pc}_{1\gamma E} = \frac{2\alpha m_c f_c}{t}, \quad g^{pc}_{1\gamma E} = 0,
\]
\[
f^{ext}_{1\gamma E} = \frac{2\alpha m_c f_c}{t} (F^\pi F^p_1 - 1), \quad g^{ext}_{1\gamma E} = 0,
\]
\[
f^{rel}_{1\gamma E} = \frac{\alpha}{2W} \left\{ \frac{W + m_p}{E + m_p} F^p_1 + 2(W - m_p + \frac{t}{4(E + m_p)}) F^p_2 \right\} F^\pi,
\]
\[
g^{rel}_{1\gamma E} = \frac{i\alpha}{2W \tan (\frac{1}{2} \theta)} \left\{ \frac{W + m_p}{E + m_p} F^p_1 + 2(W + \frac{t}{4(E + m_p)}) F^p_2 \right\} F^\pi
\]
and
\[
f_c = \frac{W^2 - m_p^2 - \mu_c^2}{2m_c W}, \quad q_c^2 = \frac{[W^2 - (m_p - \mu_c)^2][W^2 - (m_p + \mu_c)^2]}{4W^2},
\]
\[ E = (m_p^2 + q_c^2)^{1/2}. \]

The quantity \( W \) is the total energy in the c.m. frame and the results in Eqs. (6)-(10) are fully relativistic.

We now note that the relativistic \( 1\gamma E \) point charge amplitude \( f_{1\gamma E}^{pc} \) in Eq.(6) is the Born approximation to \( f_{NR}^{pc} \) multiplied by the factor \( f_c \). It is thus the Born approximation to the amplitude obtained from the RSE

\[ \{ \nabla^2 + q_c^2 - 2m_c f_c V_{pc}(r) \} \psi(r) = 0. \quad (11) \]

The inclusion of \( f_c \) in Eq.(11) gives the unambiguous relativistic generalisation of the nonrelativistic potential term \( 2m_c V_{pc} \). The factor \( f_c = 1 \) when \( W = \mu_c + m_p \) and increases as \( W \) increases. The RSE (11) is obtained in a more rigorous way by following the standard route from the Bethe-Salpeter equation, making a three-dimensional reduction, using only the leading part of the \( 1\gamma E \) diagram contribution and converting to coordinate space. This is the method used in Ref.[10]; the factor \( f_c \) is just the quantity \( a \) in Eqs.(2) and (4) of that paper.

The full amplitude \( f_{REL}^{pc} \) obtained from the RSE (11) is

\[ f_{REL}^{pc} = \frac{2\alpha m_c f_c}{t} \exp \{ 2i\sigma_0 - i\eta f_c \ln (\sin^2 \frac{1}{2}\theta) \}, \quad (12) \]

where

\[ \sigma_l = \arg \Gamma(l + 1 + i\eta f_c). \quad (13) \]

The expression (12) is the point charge Coulomb amplitude to all orders in \( \eta \). It agrees with the results in Eqs.(2) and (6) in the appropriate limits and generalises those results.

From the RSE (11) one obtains the radial RSEs for individual partial waves:

\[ \left( \frac{d^2}{dr^2} - \frac{l(l + 1)}{r^2} + q_c^2 - 2m_c f_c V_{pc}(r) \right) u_l(r) = 0. \quad (14) \]

The wavefunction regular at \( r = 0 \) has the asymptotic behaviour as \( r \to \infty \)

\[ \sin(q_c r - \eta f_c \ln(2q_c r) - l\pi/2 + \sigma_l), \]

where \( \sigma_l \) is given by Eq.(13). We shall now proceed to add other potentials to
\( V^{pc}, \) so that the wavefunction regular at \( r = 0 \) behaves as \( r \to \infty \) like
\[
\sin(q_c r - \eta f_c \ln(2q_c r) - l\pi/2 + \sigma_l + \delta_{l\pm}).
\]

The notation \( \delta_{l\pm} \) takes account of the fact that the extra phase shifts, as well as the added potentials, in general depend on the orbital angular momentum \( l \) and on the total angular momentum \( j = l \pm \frac{1}{2} \). We now take the sum over all partial waves to obtain the full no-flip and spin-flip amplitudes \( f, g \) respectively and use the identity
\[
e^{2i(\sigma + \delta)} - 1 = (e^{2i\sigma} - 1) + e^{2i\sigma}(e^{2id} - 1).
\]

Furthermore, as is customary practice, we remove the phase factor \( e^{2i\sigma_0} \) from both amplitudes; this does not affect any observable. The result is
\[
\begin{align*}
  f &= f^{pc} + \sum_{l=0}^{\infty} e^{2i(\sigma_l - \sigma_0)} \{(l + 1)f_{l+} + lf_{l-}\}P_l, \\
  g &= i \sum_{l=1}^{\infty} e^{2i(\sigma_l - \sigma_0)} (f_{l+} - f_{l-})P_l^1,
\end{align*}
\]
with
\[
\begin{align*}
  f_{l\pm} &= \exp\left(2i\delta_{l\pm}\right) - 1 \over 2iq_c, \\
  f^{pc} &= {2\alpha m_e f_c \over t} \exp\{-i\eta f_c \ln(\sin^2{1 \over 2}\theta)\}.
\end{align*}
\]

The potentials other than \( V^{pc} \) are sufficiently well behaved as \( r \to \infty \) for the sums over partial waves to be convergent.

Two photon exchange contributions give a very small correction to the \( 1\gamma E \) amplitudes and can be neglected at the level of accuracy required for the present calculations. However, even though vacuum polarisation is of higher order in \( \alpha \), it needs to be taken into account because it is of extremely long range compared with the hadronic interaction. For vacuum polarisation we used the standard Uehling potential for point charges given by Durand [11]:
\[
V^{vp}(r) = V^{pc}(r)I(2m_e r)2\alpha/3\pi,
\]
where \( m_e \) is the electron mass. The explicit integral representation of \( I \) can be taken from Ref.[11]. Since \( V^{vp} \) is a potential of electromagnetic origin, we assume (as is done in the treatment of \( pp \) scattering) that, when \( V^{vp} \) is
added to $V^{pc}$ in the RSEs (14), the factor $f_c$ should multiply it as well. The tiny effect of the extended charge distributions on $V^{vp}$ can be neglected. The vacuum polarisation amplitude $f^{vp}$ is given with sufficient accuracy for the analysis of current $\pi^+p$ data by the lowest order expression in Eq.(12.2) of Ref.[11]:

$$f^{vp} = -\frac{\alpha \eta f_c}{3\pi q_c} (1 - \cos \theta)^{-1} F(\cos \theta),$$

(20)

where

$$F(\cos \theta) = -\frac{5}{3} + X + (1 - \frac{1}{2} X)(1 + X)^{1/2} \ln \left\{ \frac{(1 + X)^{1/2} + 1}{(1 + X)^{1/2} - 1} \right\},$$

$$X = -\frac{4m_e^2}{t}.$$

The next step is to explicitly separate from the phase shifts $\delta_l$ the very small phase shifts $\sigma^{ext}_l, \sigma^{rel}_l$ and $\sigma^{vp}_l$. The partial-wave projections of the amplitudes given in Eqs.(7)-(9) and (20) lead to the results

$$\sigma^{ext}_l = \alpha m_c f_c q_c \int_{-1}^{+1} dz \frac{t^{-1} P_l(z)(F^p_1 F^\pi - 1),}{},$$

(21)

$$\sigma^{rel}_l = -\frac{\alpha q_c m_p}{2W} \int_{-1}^{+1} dz \frac{P_l(z)F^p_2 F^\pi \pm \frac{\alpha q_c}{4W(l \pm 1/2 + 1/2)} \times}{},$$

$$\int_{-1}^{+1} dz (P'_l(z) + P'_{l \pm 1}(z))(\frac{W + m_p}{E + m_p} F^p_1 + (W + \frac{t}{4(E + m_p)})2F^p_2) F^\pi, $$

(22)

$$\sigma^{vp}_l = -\frac{\alpha \eta f_c}{3\pi} \int_0^{1} dy (1 + \frac{1}{2} y)^{1/2} y^{-1} Q_l(1 + \nu y^{-1}),$$

(23)

where $\nu = 2m^2_e/q^2_c$ and $Q_l$ is the Legendre function of the second kind. The expression in Eq.(23) comes from Eq.(8.2) of Ref.[11]. For $l = 0, ..., 3$ this result was checked by direct integration of the RSEs, integrating to 1000 fm because of the very long range of $V^{vp}$. The expressions (21)-(23) are completely sufficient for these very small phase shifts.

In order to calculate the electromagnetic corrections (which we do for $l = 0, 1$) it is necessary to construct potentials $V^{ext}_{0,1}$ and $V^{rel}_{0+,1 \pm}$ which reproduce the
phase shifts $\sigma_{0,1}^{\text{ext}}$ and $\sigma_{0+,1\pm}^{\text{rel}}$ with good accuracy up to $T_\pi = 100$ MeV. The phase shifts themselves are calculated from Eqs.(21) and (22) with the dipole form factors

$$F_1^p(t) = (1 - t/\Lambda_p^2)^{-2},$$

$$F_2^p(t) = \frac{\kappa_p}{2m_p} F_1^p(t),$$

$$F_\pi(t) = (1 - t/\Lambda_\pi^2)^{-2},$$

where $\Lambda_p = 805$ MeV, $\Lambda_\pi = 1040$ MeV. The parameters $\Lambda_p$ and $\Lambda_\pi$ are chosen to correspond to the measured charge radii of the proton and charged pion, which can be found in Refs.[12,13]. All numerical constants not given explicitly (e.g. the anomalous magnetic moment $\kappa_p$ of the proton) are taken from Ref.[14]. The radial RSEs (14) are integrated outwards from the origin, with $V^\text{pc}$ replaced by $V^{\text{ext}}$ or $V^{\text{rel}}$ and the factor $f_c$ included.

Even though dipole form factors are used in calculating $\sigma_{0,1}^{\text{ext}}$, we found that it is quite sufficient for reproducing these phase shifts and calculating the electromagnetic corrections to use the simple potential for gaussian charge distributions

$$V^{\text{ext}}(r) = \alpha/r\{\text{erf}(r/c) - 1\},$$

where

$$c^2 = 2/3\{<r^2>_p + <r^2>_\pi\}.$$  

Since $\sigma_{1}^{\text{ext}}$ in Eq.(21) contains $m_c f_c = (W^2 - m_p^2 - \mu_c^2)/2W$, it is the potential term $2m_c f_c V^{\text{ext}}$, with $V^{\text{ext}}$ given by Eq.(24), that reproduces $\sigma_{1}^{\text{ext}}$ satisfactorily.

For $\sigma_{1\pm}^{\text{rel}}$ the numerator $(W^2 - m_p^2 - \mu_c^2)$ of $m_c f_c$ does not appear explicitly in Eq.(22). The potential term can be written as $2m_c V^{\text{rel}}_{l\pm}$ or as $2m_c f_c V^{\text{rel}}_{l\pm}$, with $V^{\text{rel}}_{l\pm}$ energy independent in each case, and the phase shifts $\sigma_{0\pm}^{\text{rel}}$ and $\sigma_{1\pm}^{\text{rel}}$ satisfactorily fitted. We checked that this makes an insignificant difference to the contribution of $V^{\text{rel}}_{l\pm}$ to the electromagnetic corrections. For convenience we therefore took the term in the form $2m_c f_c V^{\text{rel}}_{l\pm}$. The phase shift $\sigma_{0\pm}^{\text{rel}}$ is proportional to $q_c$ near threshold and can be reproduced very well for $T_\pi < 100$ MeV by means of a very short range energy independent potential $V^{\text{rel}}_{0\pm}$. We constructed $V^{\text{rel}}_{1+}$ and $V^{\text{rel}}_{1-}$ by adding together a very short range potential and a potential which has the $r^{-3}$ behaviour for large $r$ characteristic of a spin-orbit potential and is modified for small $r$ to take account of the charge distributions. These three potentials are plotted in Fig. 1.
If we now construct the total electromagnetic potentials

\[ V_{l\pm}^{em} = V_{l\pm}^{pc} + V_{l\pm}^{ext} + V_{l\pm}^{rel} + V_{l\pm}^{vp} \]  

and insert them in the radial RSEs of the form given in Eq.(14), with \( V_{l\pm}^{pc} \) replaced by \( V_{l\pm}^{em} \), the wavefunction regular at \( r = 0 \) has the asymptotic behaviour as \( r \to \infty \)

\[ \sin(q_c r - \eta_f c \ln(2q_c r) - l\pi/2 + \sigma_l + \sigma_l^{ext} + \sigma_l^{rel} + \sigma_l^{vp}) \].

Here we have neglected higher order contributions in \( \alpha \) to the total electromagnetic phase shifts. The distances at which the asymptotic behaviour is reached are determined by \( V_{l\pm}^{vp} \). Since the potentials \( V_{l\pm}^{rel} \) have an \( r^{-3} \) tail one might think that they require integration of the appropriate RSEs to the largest distance. It turns out however that one can stop the integration for \( V_{l\pm}^{rel} \) at a smaller distance than for \( V_{l\pm}^{vp} \).
We now return to Eqs.(15)-(17) and write

\[ \delta_{\pm} = \sigma^{ext}_{l} + \sigma^{rel}_{l \pm} + \sigma^{vp}_{l} + \delta^{n}_{l \pm}. \]  

(26)

The residual phase shifts \( \delta^{n}_{l \pm} \), which arise because of the hadronic interaction, are usually called nuclear phase shifts, in contrast to the hadronic phase shifts to be defined later. We now use the identity

\[ e^{2i(\sigma_{l} - \sigma_{0})}(e^{2i\delta_{\pm}} - 1) = e^{2i(\sigma_{l} - \sigma_{0})}\{e^{2i(\sigma^{ext}_{l} + \sigma^{rel}_{l \pm} + \sigma^{vp}_{l})} - 1\} + e^{2i\Sigma_{l \pm}}(e^{2i\delta^{n}_{l \pm}} - 1), \]  

(27)

where

\[ \Sigma_{l \pm} = (\sigma_{l} - \sigma_{0}) + \sigma^{ext}_{l} + \sigma^{rel}_{l \pm} + \sigma^{vp}_{l} \]  

(28)

and

\[ \sigma_{l} - \sigma_{0} = \sum_{n=1}^{l} \arctan(\eta_{f_{c}} n), \]  

(29)

using Eq.(13). The separation in Eq.(27) is made because the phase shifts \( \sigma^{rel}_{l \pm} \) and \( \sigma^{vp}_{l} \) drop off slowly as \( l \) increases and all partial waves must be taken into account. It is convenient to treat \( \sigma^{ext}_{l} \) in the same way as \( \sigma^{rel}_{l \pm} \) and \( \sigma^{vp}_{l} \) and to take the sums indicated in Eqs.(15) and (16) over all values of \( l \). If we work consistently to the lowest order in \( \alpha \) (\( \alpha \) for \( ext \) and \( rel \), \( \alpha \eta \) for \( vp \)), we can replace

\[ e^{2i(\sigma_{l} - \sigma_{0})}(e^{2i(\sigma^{ext}_{l} + \sigma^{rel}_{l \pm} + \sigma^{vp}_{l})} - 1) \]

by

\[ (e^{2i\sigma^{ext}_{l}} - 1) + (e^{2i\sigma^{rel}_{l \pm}} - 1) + (e^{2i\sigma^{vp}_{l}} - 1). \]

The sum over all partial waves then yields

\[ f = f^{pc} + f^{ext}_{1} + f^{rel}_{1} + f^{vp} + \sum_{l=0}^{\infty}\{(l + 1)e^{2i\Sigma_{l +}} f^{n}_{l +} + le^{2i\Sigma_{l -}} f^{n}_{l -}\} P_{l}, \]  

(30)

\[ g = g^{rel}_{1} + i \sum_{l=1}^{\infty}(e^{2i\Sigma_{l +}} f^{n}_{l +} - e^{2i\Sigma_{l -}} f^{n}_{l -}) P_{l}^{1}, \]  

(31)

with

\[ f^{n}_{l \pm} = \frac{e^{2i\delta^{n}_{l \pm}} - 1}{2i\eta_{c}}. \]  

(32)
The electromagnetic amplitudes are defined in Eqs.(7)-(9), (18) and (20). The electromagnetic phase shifts $\Sigma_{l\pm}$ are given in Eqs.(28), (29) and (21)-(23).

We want to emphasise that the expressions given in Eqs.(30) and (31) are completely sufficient for the phase shift analysis of the present experimental data for $T_\pi \leq 100$ MeV. The lowest energy at which we shall give results is $T_\pi = 10$ MeV, for which $\eta f_c = 0.0203$. Present experiments do not go to an energy as low as this, and the measurements are at angles sufficiently away from the forward direction that the lowest order approximation to $f^{\pi p}$ given in Eq.(20) is adequate.

For small angles and energies, the leading correction to the lowest order approximation to $f^{\pi p}$ is the imaginary part of the quantity $a^{(1)}_{\pi p}/q_c$, where $a^{(1)}_{\pi p}$ is given in Eq. (18.2) of Ref. [11]. The magnitude of the ratio of this correction to the lowest order approximation increases as the scattering angle decreases. For $T_\pi = 10, 15$ and 20 MeV the ratio reaches 10% at values of $\theta$ very close to $3^\circ$, $2^\circ$ and $1^\circ$ respectively. At $T_\pi = 15$ MeV, where $\pi^+p$ and $\pi^-p$ elastic scattering are currently being measured, the magnitude of this ratio is 6.6% at $\theta = 5^\circ$ and it might just be necessary to take the correction into account at such a small angle.

The final step is to introduce the hadronic interaction via effective hadronic potentials $V^{h\pm}_{l\pm}$ (in the presence of the electromagnetic interaction) which are added to $V^{em\pm}_{l\pm}$:

$$V_{l\pm} = V^{em\pm}_{l\pm} + V^{h\pm}_{l\pm}.$$  \hspace{1cm} (33)

Eq.(33) implies that the hadronic potential term has the form $2m_c f_c V^{h\pm}_{l\pm}$. With $V^{h\pm}_{l\pm}$ chosen to be energy independent, this means that the term has the specific energy dependence of the factor $f_c$. It is equally possible to take the hadronic potential term as $2m_c V^{h\pm}_{l\pm}$, with of course a different energy independent potential. For the $\pi^+p$ case we checked that the values of the electromagnetic corrections are practically independent of the energy dependence of the hadronic potential term that is chosen. As we shall see in the next paper, this is not the case for the two-channel $\pi^-p$ situation; the choice of the energy dependence of the hadronic potential term has significant consequences for the values of the electromagnetic corrections. We shall defer to that paper a full discussion of this delicate point of principle; for the present it is sufficient to note that, for the $\pi^+p$ corrections, what energy dependence is chosen for the hadronic potential term is not important. In the calculation of the corrections given in Section 4 we use RSEs with the form

$$\left( \frac{d^2}{dr^2} - \frac{l(l+1)}{r^2} + q^2_c - 2m_c f_c V^{h\pm}_{l\pm}(r) \right) u_{l\pm}(r) = 0,$$  \hspace{1cm} (34)
with $V_{l\pm}$ given by Eqs. (33) and (25).

The solutions of the RSEs (34) regular at $r = 0$ have the asymptotic behaviour as $r \to \infty$

$$\sin(q_c r - \eta f_c \ln(2q_c r) - l\pi/2 + \sigma_l + \sigma_l^{ext} + \sigma_l^{rel} + \sigma_l^{vp} + \delta^n_{l\pm}).$$

The hadronic phase shifts $\delta^{h}_{l\pm}$ are defined by the asymptotic behaviour as $r \to \infty$

$$\sin(q_c r - l\pi/2 + \delta^{h}_{l\pm})$$

of the solutions regular at $r = 0$ of the RSEs in which $V_{l\pm}$ in Eq.(34) is replaced by $V_{l\pm}^h$. The electromagnetic corrections $C_{l\pm}$ are then

$$C_{l\pm} = \delta^{h}_{l\pm} - \delta^{h}_{l\pm}. \quad (35)$$

They arise from the interplay of the potentials $V_{l\pm}^h$ and $V_{l\pm}^{em}$. The quantity $C_{l\pm}$ can, again neglecting terms of higher order in $\alpha$, be split into four parts:

$$C_{l\pm} = C_{l\pm}^{pc} + C_{l\pm}^{ext} + C_{l\pm}^{rel} + C_{l\pm}^{vp}, \quad (36)$$

where each of the parts arises from the interplay of $V_{l\pm}^h$ and the corresponding part of $V_{l\pm}^{em}$. We emphasize again that the $V^h$, as well as $V^{ext}$ and $V^{rel}$, are needed only for the calculation of the electromagnetic corrections. Once these corrections are determined, the phase shift analysis, using Eqs.(30)-(32) and (35), determines hadronic phase shifts without any further reference to potentials. We proceed in the next section to describe the construction of the hadronic potentials and the evaluation of the corrections.

### 3 Method of evaluation of the corrections

Numerically we evaluated the electromagnetic corrections $C_{l\pm}$ only for the three lowest partial waves ($0+$, $1\pm$). This was done in an iterative procedure of which each step consisted of three stages:

a) from values of the hadronic phase shifts $\delta^{h}_{l\pm}$ we determined hadronic potentials $V_{l\pm}^h$ which reproduced these $\delta^{h}_{l\pm}$;

b) using these $V_{l\pm}^h$ we calculated values for $C_{l\pm}$;

c) with these values of $C_{l\pm}$ we determined via a phase-shift analysis (PSA) new values for the $\delta^{h}_{l\pm}$.

With these new values of $\delta^{h}_{l\pm}$ we started again at stage a). The starting phase
shifts for the whole procedure are of no great importance; in practice we used the output from the analysis of Arndt et al. [15]. The iterative procedure was continued until the $C_{l\pm}$, and therefore the $\delta_{l\pm}^h$, achieved stable values; three iteration steps were found to be sufficient. We now describe the three stages in more detail.

a) For each $V_{l\pm}^h$ we used a parametric form containing a range parameter which was fixed at 1 fm. This form contains three further parameters for each partial wave. For a fixed set of these parameters we inserted $V_{l\pm}^h$ instead of $V_{l\pm}$ in Eq.(34) and by numerical integration obtained the solutions $u_{l\pm}^h(r)$ regular at $r=0$. Outside the range of $V_{l\pm}^h$ the asymptotic form as $r \to \infty$ of the $u_{l\pm}^h(r)$,

$$u_{l\pm}^h(r) \sim a_{l\pm}^h j_l(q_cr) + b_{l\pm}^h n_l(q_cr),$$

(37)

gave the values of $\delta_{l\pm}^h$ generated by the $V_{l\pm}^h$ via the equation

$$\tan \delta_{l\pm}^h = \frac{b_{l\pm}^h}{a_{l\pm}^h}.$$  

(38)

The three parameters in each $V_{l\pm}^h$ were then varied in order to get the best possible fit to the values of $\delta_{l\pm}^h$ up to $T_\pi = 100$ MeV. The potentials constructed in this way reproduced these values within their experimental errors. In Fig. 2 we show the potentials $V_{0+}^h$ and $V_{1\pm}^h$ for the final step of the iteration. The analytical form and further details can be found in Ref.[16].

b) Using the $V_{l\pm}^h$ constructed in a) the electromagnetic corrections were evaluated in the following way. We integrated Eq.(34) with the full potentials $V_{l\pm}$ and obtained the solutions $u_{l\pm}$ regular at $r=0$, integrating outwards to a distance where $V_{l\pm}^h + V_{l\pm}^{ext} + V_{l\pm}^{rel} + V_{l\pm}^{vp}$ is negligible compared to $V_{l\pm}^{pc}$. In this region the asymptotic form of the $u_{l\pm}$,

$$u_{l\pm}(r) \sim a_{l\pm} F_l(\eta_f, q_cr) + b_{l\pm} G_l(\eta_f, q_cr),$$

(39)

where $F_l$ and $G_l$ are the standard point charge Coulomb wavefunctions, yielded the nuclear phase shifts $\delta_{l\pm}^n$ via the equation

$$\tan(\sigma_{l\pm}^{ext} + \sigma_{l\pm}^{rel} + \sigma_{l\pm}^{vp} + \delta_{l\pm}^n) = \frac{b_{l\pm}}{a_{l\pm}}.$$  

(40)

Eq.(35) then gave the corrections $C_{l\pm}$ for this step of the iteration procedure, the $\delta_{l\pm}^h$ being of course the hadronic phase shifts generated by the potentials $V_{l\pm}^h$. Note that the use of the asymptotic forms (37) and (39) instead of those
given just before Eq.(35) enables the numerical integration to go to smaller distances.

Though the PSA of the $\pi^+ p$ data is not the topic of this paper we see that because of the iterative procedure we have described, the PSA goes hand in hand with the evaluation of the electromagnetic corrections. We therefore describe briefly how the PSA is done; a full description will be given in a separate paper.

c) The PSA uses Eqs.(30) and (31) with the sums over $l$ taken to $l = 3$. For $l = 2, 3$ the small and poorly known nuclear phase shifts $\delta_n^{l\pm}$ need to be taken from a PSA at higher energies and then extrapolated to the lower energies with which we are concerned. Taking the electromagnetic corrections $C_{l\pm}$ as calculated in stage b), the PSA determined the three hadronic phase shifts $\delta_h^{0+}, \delta_h^{1\pm}$. Parametric forms for these phase shifts were used and the parameters varied to obtain the best fit to the data. The parametric forms are simple low energy expansions in the pion c.m. kinetic energy. Three parameters were used for the s-wave and two for the non-resonant p-wave. For the resonant phase shift $\delta^h_{1+}$,
A parametrised background term was added to a Breit-Wigner resonant term with an energy dependent width. The hadronic potentials constructed in a) can be considered as an auxiliary alternative parametrisation of the hadronic phase shifts which is used only for the calculation of the electromagnetic corrections.

A very important observation of a conceptual nature needs to be made. We have called the quantities $\delta_{lh}^h$ hadronic phase shifts, but they are obtained from a phase shift analysis which contains the physical masses of $\pi^+$ and $p$. It is therefore clear that the $\delta_{lh}^h$ cannot be considered to be strictly hadronic quantities. Such quantities relate to a situation in which the electromagnetic interaction is switched off ($\alpha = 0$) and therefore need to be obtained using the hadronic masses of $\pi^+$ and $p$. These hadronic masses are not the same as their physical masses (indeed it is universally accepted that the hadronic mass of $\pi^+$ is very close to the physical mass of $\pi^0$). In the same sense we have noted that the quantities $V_{lh}^h$ are effective hadronic potentials in the presence of the electromagnetic interaction. They would be different in the complete absence of all electromagnetic interactions. Therefore in our present work, as in all previous work known to us, we avoid speculation about the strictly hadronic situation and give quantities that we call hadronic phase shifts and electromagnetic corrections which, though they have a precise definition within the framework of our potential model, are not the full corrections that would give truly hadronic phase shifts by subtraction from the nuclear phase shifts. Any attempt to completely purge the experimental data of all electromagnetic effects would be very speculative. Our aim is the more modest one of taking account of those electromagnetic effects that may be calculated with reasonable confidence.

4 Numerical results for the corrections

The final results for the three electromagnetic corrections $C_{0+}$, $C_{1\pm}$ are given in Table 1 from 10 to 100 MeV pion lab kinetic energy. Estimates of the uncertainties in the values of these corrections are also given in the table. In making these estimates it is necessary to distinguish between a) uncertainties coming from applying our model and b) uncertainties coming from the choice of the model itself.

One source of a) is the uncertainty in the hadronic phase shifts, which comes from the PSA of the experimental data (experimental errors in the data and uncertainty in the $d$- and $f$-wave phase shifts used as input). The uncertainties in the hadronic phase shifts will be given in full in the separate paper on the PSA that was mentioned earlier. The resulting uncertainties in the corrections were thoroughly studied by keeping track of the corrections obtained
Table 1
Values in degrees of the electromagnetic corrections $C_{0+}$, $C_{1-}$ and $C_{1+}$ as functions of the pion lab kinetic energy $T_\pi$ (in MeV).

| $T_\pi$ | $C_{0+}$   | $C_{1+}$   | $C_{1-}$   |
|-------|------------|------------|------------|
| 10    | 0.081± 0.003 | -0.023± 0.000 | 0.005± 0.000 |
| 15    | 0.083± 0.003 | -0.034± 0.000 | 0.007± 0.000 |
| 20    | 0.085± 0.004 | -0.047± 0.001 | 0.009± 0.000 |
| 25    | 0.088± 0.005 | -0.060± 0.001 | 0.010± 0.000 |
| 30    | 0.090± 0.005 | -0.074± 0.002 | 0.012± 0.000 |
| 35    | 0.093± 0.006 | -0.090± 0.002 | 0.013± 0.001 |
| 40    | 0.096± 0.006 | -0.106± 0.003 | 0.014± 0.001 |
| 45    | 0.099± 0.007 | -0.125± 0.003 | 0.015± 0.001 |
| 50    | 0.101± 0.007 | -0.145± 0.004 | 0.016± 0.001 |
| 55    | 0.104± 0.008 | -0.168± 0.005 | 0.017± 0.002 |
| 60    | 0.107± 0.009 | -0.194± 0.006 | 0.018± 0.002 |
| 65    | 0.109± 0.010 | -0.223± 0.008 | 0.018± 0.002 |
| 70    | 0.111± 0.010 | -0.255± 0.011 | 0.019± 0.002 |
| 75    | 0.114± 0.011 | -0.291± 0.016 | 0.019± 0.003 |
| 80    | 0.115± 0.012 | -0.332± 0.023 | 0.019± 0.003 |
| 85    | 0.118± 0.014 | -0.378± 0.032 | 0.019± 0.004 |
| 90    | 0.119± 0.015 | -0.429± 0.045 | 0.019± 0.004 |
| 95    | 0.121± 0.016 | -0.485± 0.062 | 0.019± 0.004 |
| 100   | 0.123± 0.018 | -0.547± 0.088 | 0.020± 0.005 |

using a variety of input values for the hadronic phase shifts (in particular for the successive iteration steps described in Section 3). These uncertainties are substantially smaller than those coming from the use of the particular form of the parametrised hadronic potentials, which are also of type a). For the final numerical calculations we fixed the range parameter in the potentials at 1 fm. Its value can be varied from 0.8 fm to 1.2 fm without any significant change in the quality of the fits to the hadronic phase shifts. In this way we obtained an estimate of the uncertainties coming from the fact that some of the fine details of the potentials may have been missed.

The estimated uncertainties given in Table 1 were obtained by combining the two sources of type a) described in the previous paragraph. For $C_{0+}$ and $C_{1-}$ these uncertainties are much smaller than the errors on the hadronic phase
Table 2

Values in degrees of the various contributions to the electromagnetic corrections $C_{0+}$ as functions of the pion lab kinetic energy $T_\pi$ (in MeV).

| $T_\pi$ | $C_{0+}^{pc}$ | $C_{0+}^{ext}$ | $C_{0+}^{rel}$ | $C_{0+}^{vp}$ |
|--------|---------------|----------------|----------------|--------------|
| 10     | 0.093         | -0.008         | -0.002         | 0.000        |
| 20     | 0.102         | -0.013         | -0.004         | 0.000        |
| 30     | 0.113         | -0.018         | -0.005         | 0.000        |
| 40     | 0.123         | -0.022         | -0.005         | 0.000        |
| 50     | 0.134         | -0.027         | -0.006         | 0.000        |
| 60     | 0.145         | -0.031         | -0.007         | 0.000        |
| 70     | 0.155         | -0.036         | -0.008         | 0.000        |
| 80     | 0.165         | -0.041         | -0.009         | 0.000        |
| 90     | 0.175         | -0.046         | -0.010         | 0.000        |
| 100    | 0.183         | -0.051         | -0.010         | 0.000        |

shifts that arise from the PSA itself. For example, at 100 MeV the errors on $\delta_{0+}$ and $\delta_{1-}$ are 0.15° and 0.09°, compared with the uncertainties in the corrections of 0.018° and 0.005° respectively. The situation is different for $C_{1+}$, where the error on $\delta_{1+}$ at 100 MeV is 0.05°, while the correction is $-0.547°$ and its uncertainty is 0.088°. The correction itself is very large ($\delta_{1+}$ itself is 21.27° at this energy) and the uncertainty in the correction is somewhat larger than the error arising from the PSA.

The corrections in Table 1 are intended for use in future PSAs of experimental data. Such PSAs need to use Eqs (30)-(32), (28) and (35) as well as higher order corrections to $f_{\pi}$ and $\sigma_{\pi}$ for experiments at very low energies and angles, as discussed in Section 2. For the output phase shift $\delta_{1+}$ the uncertainty in the correction $C_{1+}$ needs to be taken into account as well as the statistical error arising from the PSA itself.

Uncertainties of type b) arise from the choice of the RSEs (34), with the hadronic potential term having the specific energy dependence of the factor $f_{\pi}$. We have already remarked that the choice of the energy dependence of this term has no significant effect on the corrections. The RSEs themselves are the only relativistic equations that are well suited to a two-body problem. It is not meaningful to give an uncertainty arising from this source; all one can do is to compare the results obtained using our present model with those from alternative models, as we do in a moment.

In Tables 2 and 3 we give (at a smaller set of energies) the various contributions to the corrections (Eq.(11)). These components are additive to a very good
Table 3
Values in degrees of the various contributions to the electromagnetic corrections $C_{1-}$ and $C_{1+}$ as functions of the pion lab kinetic energy $T_\pi$ (in MeV).

| $T_\pi$ | $C_{1+}^{pc}$ | $C_{1+}^{ext}$ | $C_{1+}^{rel}$ | $C_{1+}^{vp}$ |
|--------|----------------|----------------|----------------|--------------|
| 10     | -0.023         | 0.000          | 0.000          | -0.000       |
| 20     | -0.047         | 0.001          | -0.000         | -0.000       |
| 30     | -0.074         | 0.001          | -0.001         | -0.001       |
| 40     | -0.104         | 0.002          | -0.004         | -0.001       |
| 50     | -0.141         | 0.004          | -0.008         | -0.001       |
| 60     | -0.184         | 0.007          | -0.016         | -0.001       |
| 70     | -0.236         | 0.011          | -0.029         | -0.002       |
| 80     | -0.300         | 0.017          | -0.048         | -0.002       |
| 90     | -0.376         | 0.025          | -0.077         | -0.003       |
| 100    | -0.464         | 0.036          | -0.119         | -0.003       |

| $T_\pi$ | $C_{1-}^{pc}$ | $C_{1-}^{ext}$ | $C_{1-}^{rel}$ | $C_{1-}^{vp}$ |
|--------|----------------|----------------|----------------|--------------|
| 10     | 0.004          | -0.000         | 0.000          | 0.000        |
| 20     | 0.008          | -0.000         | 0.000          | 0.000        |
| 30     | 0.011          | -0.000         | 0.000          | 0.000        |
| 40     | 0.014          | -0.000         | 0.000          | 0.000        |
| 50     | 0.016          | -0.000         | 0.000          | 0.000        |
| 60     | 0.018          | -0.000         | -0.000         | 0.000        |
| 70     | 0.019          | -0.000         | -0.001         | 0.000        |
| 80     | 0.021          | -0.001         | -0.001         | 0.000        |
| 90     | 0.022          | -0.001         | -0.002         | 0.000        |
| 100    | 0.024          | -0.001         | -0.004         | 0.000        |

approximation and tiny differences between the sums of the numbers in these tables and the complete corrections in Table 1 are due to higher order effects. The corrections are clearly dominated by the component $C^{pc}$ arising from the interplay between the hadronic and point charge Coulomb potentials. The only other significant components are $C_{0+}^{ext}$ and $C_{1+}^{rel}$ near $T_\pi = 100$ MeV.

Our results are compared with those of the dispersion theory approach used by NORDITA[3] in Fig.3. They have not included the vacuum polarisation contribution but we see from Tables 2 and 3 that it is negligible for each of the partial waves at the energies we consider. The results of Zimmermann [8],
Fig. 3. Values in degrees of the electromagnetic corrections $C_{0+}$, $C_{1+}$ and $C_{1-}$ from our present calculation (solid curves), from NORDITA [3] (circles) and Zimmermann [8] (triangles).

who uses a slightly different version of the potential model, are also shown in Fig. 3. We have added our values of the relativistic corrections to the results of Zimmermann. We have indicated in Fig.3 the uncertainties in our corrections at 100 MeV, as given in Table 1. No errors are given in Refs. [3] and [8] for the corrections presented there. For $C_{0+}$ the results are in very good agreement, the differences being considerably smaller than the error on $\delta_{0+}^{h}$ arising from the PSA. For $C_{1+}$ our results agree quite well with those of Zimmermann, which indicates the stability of the results obtained with the potential model. However, there is a systematic difference from the results of NORDITA, the discrepancy increasing with energy. At 100 MeV the difference is about 2.5 times our estimated uncertainty in $C_{1+}$ and over 4 times the error in $\delta_{1+}^{h}$ arising from the PSA. For $C_{1-}$ our results agree with those of NORDITA at low energies, but at 100 MeV the discrepancy is roughly twice the error in $\delta_{1-}^{h}$. However, since $\delta_{1-}^{h}$ is very small, the corrections are of minor importance for the results of the PSA.
As we discussed in Section 1, the calculation of electromagnetic corrections using dispersion relations omits what could be important medium range effects due to $t$-and $u$-channel exchanges, which the potential model includes quite reliably. The differences of our results from those of NORDITA are probably due to such medium range effects. In particular the stronger energy dependence of the NORDITA results for $C_{1+}$ shown in Fig. 3 is likely to be a medium range effect, perhaps due to their omission of $t$-channel $\pi\pi$ ($T = 0, J = 0$) exchange. We therefore claim a higher degree of reliability for our present calculation of the electromagnetic corrections, compared with that of NORDITA. At the same time it is pleasing that the corrections in the $\pi^+p$ case are very reliably known and are to a large extent independent of the model used for their calculation.

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