Fluctuation Conductivity in Insulator-Superconductor Transitions with Dissipation

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We analyze the fluctuation conductivity near the critical point in a 2D Josephson junction array shunted by an Ohmic resistor. We find that at the Gaussian level, the conductivity acquires a logarithmic dependence on $T/m$ ($m$, the inverse correlation length) when the dissipation is sufficiently weak. In the renormalized classical regime, this logarithmic dependence gives rise to a leveling of the resistivity at low to intermediate temperatures when fluctuations are included. We show, however, that this trend does not persist to $T = 0$ at which point the resistivity vanishes.

In two dimensional superconductors, the resistivity vanishes not at the mean-field temperature, $T_{BCS}$, at which the pair amplitude is established but at a lower temperature, $T_c$, where global phase coherence obtains \[\psi(\text{field}) = 1 \\text{on average}.\] The prominence of phase fluctuations in 2D is primarily responsible for the discrepancy between the on-set of pairing and the subsequent thermodynamic phase transition to a state with zero resistance. In 2D the zero-resistance state possesses algebraic long-range order \[\psi(\text{field}) = 1 \\text{on average}.\] Above $T_c$ but below $T_{BCS}$, the system is in a para-coherent state characterised by incoherent motion of the Cooper pairs. In this regime, phase fluctuations give rise to the conductivity which at the mean-field level is of the Aslamazov-Larkin form \[\psi(\text{field}) = 1 \\text{on average}.\]

We are concerned in this work with the role of dissipation on the conductivity in the regime where phase fluctuations dominate. While thermally-excited vortices are the most common dissipative mechanism in 2D superconductors in the presence of a magnetic field, we specialize to the zero field case and consider here the case of shunting an array of Josephson junctions with an Ohmic resistor \[\psi(\text{field}) = 1 \\text{on average}.\] Resistively-shunted Josephson junction arrays \[\psi(\text{field}) = 1 \\text{on average}.\] in 2D represent an idealization of an important class of insulator-superconductor systems, namely granular superconductors \[\psi(\text{field}) = 1 \\text{on average}.\] in which the role of Coulomb interactions and dissipation is paramount. At $T = 0$, such systems undergo an insulator-superconductor transition (IST) when the Josephson coupling energy exceeds the Coulomb charging energy. Dissipation suppresses quantum fluctuations and hence favours the superconducting state. However, in the para-coherent regime, dissipation can also slow the motion of Cooper pairs. Naive considerations suggest that the fluctuation conductivity should decrease by an amount proportional to the inverse of the strength of the dissipation. In fact, Mason and Kapitulnik \[\psi(\text{field}) = 1 \\text{on average}.\] have recently suggested that dissipation in the presence of disorder can give rise to a leveling-off of the resistivity at $T = 0$—that is, a “metallic state”.

We calculate here the role dissipation and fluctuation effects play in the fluctuation conductivity in 2D Josephson junction arrays. Within a Gaussian theory, we show that the standard \[\psi(\text{field}) = 1 \\text{on average}.\] form for the fluctuation conductivity does not hold when the strength of the dissipation is less than the gap proportional to the inverse correlation length. In this regime, we find that quite generally the conductivity scales logarithmically with the strength of the gap. While calculations of the fluctuation conductivity have been performed previously \[\psi(\text{field}) = 1 \\text{on average}.\] none have revealed the regime with the logarithmic behavior at finite temperature. We then analyze the effect of the quartic term on the fluctuation conductivity in the $N = \infty$ limit. In a strictly $D = 2$ system, the $N \to \infty$ limit results in ordering only at $T = 0$. We show that in the renormalized classical regime, the logarithmic temperature dependence is transformed into a constant resistivity at low to intermediate temperatures. However, at sufficiently low temperatures, the resistivity is shown to vanish exponentially.

To formulate the conductivity, we replace the microscopic Hamiltonian for an array of Josephson junctions with an effective Ginsburg-Landau theory. The coarse-graining approximation of Doniach \[\psi(\text{field}) = 1 \\text{on average}.\] offers a straightforward way of obtaining the appropriate Landau theory. To this end, we introduce the complex order parameter $\psi(\text{field}) = 1 \\text{on average}.\$ whose expectation value is proportional to $\langle \exp(i\phi) \rangle$, where $\phi$ is the phase of a particular junction. An effective Landau theory is valid if $\psi$ is sufficiently small as is the case in the vicinity of the onset of global phase coherence. The minimal Ginsburg-Landau theory \[\psi(\text{field}) = 1 \\text{on average}.\] required to model quantum fluctuations and dissipation near the zero-resistance quantum critical point is the Gaussian free energy functional,

\[
F[\psi] = \int d^2r \int d\tau \left\{ \left[ \left( \nabla + \frac{ie^*}{\hbar} A(r, \tau) \right) \psi^*(r, \tau) \right] \cdot \left[ \left( \nabla - \frac{ie^*}{\hbar} A(r, \tau) \right) \psi(r, \tau) \right] + \kappa^2 |\partial_r \psi(r, \tau)|^2 + m^2 |\psi(r, \tau)|^2 \right\} + L_{\text{dis}} \tag{1}
\]

where $A(r, \tau)$ is the vector potential, $e^* = 2e$, $m^2$ is proportional to the inverse correlation length, and $\kappa$ and
\[ \sigma_{\alpha\beta}(i\omega_n, \mathbf{q}) = -\frac{\hbar}{\omega_n} \int d^2r \int d\tau \frac{\delta^2 \ln Z}{\delta A_\alpha(r, \tau) \delta A_\beta(0)} e^{i\mathbf{q} \cdot \mathbf{r} + i\omega_n \tau} \]

is related to the Fourier transform of the second variation of the partition function \( Z \) with respect to the vector potential. The partition function is defined in the usual way as the functional integral of \( \exp(-F[\psi]) \).

In the zero wavevector limit, the conductivity to 1-loop order [20] can be rewritten as

\[ \sigma(i\omega_n) = \frac{2(e^*)^2}{\hbar \omega_n} \frac{2k^2}{(2\pi)^2} \int dq d\tau \frac{\delta^2 \ln Z}{\delta A_\alpha(\mathbf{r}, \tau) \delta A_\beta(0)} e^{i\mathbf{q} \cdot \mathbf{r} + i\omega_n \tau} G(\mathbf{k}, \omega_m) \]

\[ (G(\mathbf{k}, \omega_m) - G(\mathbf{k}, \omega_m + \omega_n)), \quad (2) \]

where \( G(\mathbf{k}, \omega_n) \) is the standard Matsubara Green function. To obtain the conductivity for real frequencies, we must analytically continue to real frequencies. As van Otterlo, et. al. [20] have pointed out, inclusion of dissipation changes the analytical properties of the Matsubara sums. We define the retarded and advanced Green functions, \( G^R(z) = (k^2 + m^2 - \kappa^2 z^2 - i\kappa z)^{-1} \) and \( G^A(z) = (k^2 + m^2 - \kappa^2 z^2 + i\kappa z)^{-1} \). Upon choosing the appropriate contour, we find that

\[ \sigma(\omega) = \frac{(e^*)^2}{2\pi \hbar \omega} \int_0^\infty k^3 dk \int_{-\infty}^\infty \coth \frac{z}{2\tau} \left[ (G^R(z) - G^A(z)) \right] \]

\[ [G^R(z) + G^A(z) - G^R(z + \omega) - G^A(z - \omega)]. \quad (3) \]

We are particularly interested in the limit of zero frequency. In this limit, the product of Green functions simplifies significantly leading to

\[ \sigma(\omega = 0) = \frac{(e^*)^2}{2\pi \hbar} \int_0^\infty k^3 dk \int_{-\infty}^\infty \frac{dx}{\sinh^2 x} \]

\[ \frac{8\eta^2 T^2 x^2}{(e_k^2 - 4T^2 \kappa^2 x^2 + 4T^2 \eta^2 x^2)^2} \]

\[ (4) \]

as our working expression for the zero-frequency conductivity. In this expression, \( e_k^2 = k^2 + m^2 \).

From the analytical structure of the integrand in Eq. (4), it is clear that distinctly different regimes arise when 1) \( m \ll \kappa T \) and 2) \( m \gg \kappa T \). In each of these cases, we explore the limiting form for the conductivity as the magnitude of the dissipation varies. In our treatment, we explicitly assume that \( \eta/\kappa \ll 1 \). In the limit of \( \eta = 0 \), the pole in Eq. (4) makes the conductivity infinite at least to 1-loop order as obtained previously by van Otterlo, et. al. [20] and Damle and Sachdev [21]. Two loop and higher order corrections regularize the static conductivity even when \( \eta = 0 \), however. Nonetheless, as the dissipation is turned on, the divergent 1-loop conductivity becomes finite. In the present treatment, we address only the form of the static conductivity at lowest order. We focus first on the region characterized by \( m \ll \kappa T \). The two subregimes of interest are a) \( \eta/\kappa \ll m \) and b) \( m \ll \eta/\kappa \). Only the latter regime has been considered previously [9]. It is in the former regime that the new logarithmic behaviour emerges. To obtain this result, we note that the main contribution from the integral in Eq. (4) arises from small \( x \) and the minima of the denominator of the integrand which occur at \( x^\pm = 1/4\kappa^2 T^2[\kappa^2 - \eta^2/2T^2] \). The latter two roots are real only if \( m > \eta/\kappa \). If we work in the limit in which \( m \gg \eta/\kappa \), we expand the integral about \( x^\pm \) and perform the \( x \) integration

\[ \sigma = \frac{2e^2\kappa^2 T}{\hbar \eta} \int_0^\infty \frac{\xi^3 d\xi}{(\xi^2 + \tilde{m}^2) \sinh^2 \sqrt{\xi^2 + \tilde{m}^2}} \]

\[ (5) \]

by introducing the new variables, \( \xi = k/2\kappa T \) and \( \tilde{m} = m/2\kappa T \). In the limit that \( \tilde{m} \ll 1 \), we can approximate \( \sinh^2 \sqrt{\xi^2 + \tilde{m}^2} \) as \( \xi^2 + \tilde{m}^2 \). This approximation is valid if \( \xi^2 \ll 1 \) or equivalently an ultraviolet cutoff is introduced on the momentum of order \( 2\kappa T \). Introducing this cutoff into the integral in Eq. (5) leads immediately to the logarithmic form for the fluctuation conductivity:

\[ \sigma = \frac{2e^2 \pi \kappa T}{\hbar \eta} \ln \frac{\kappa T}{m} \eta/\kappa \ll m. \quad (6) \]

In the regime that \( m \ll \eta/\kappa \), the original integral in the conductivity is dominated by the minimum at \( x = 0 \). Performing the integrations in this regime results in the more standard form [3] for the conductivity:

\[ \sigma = \frac{e^2 \eta T}{\hbar m^2} \ll \eta/\kappa \ll m. \quad (7) \]

Should the transition to the superconducting state occur at the finite temperature \( T_c \) and the effective field theory close to the transition is given by Eq. (4) with \( m^2 \propto (T - T_c) \), we find that the fluctuation conductivity is consistent with the standard Aslamazov-Larkin [3] form when the dissipation is of intermediate strength. However, in the weak dissipation limit, that is, weak in comparison to the inverse correlation length, the standard \( 1/(T - T_c) \) is replaced by the logarithmic behavior sufficiently close to \( T_c \).

While the logarithmic conductivity follows naturally from an expansion in powers of \( \eta/(\kappa m) \) in the conductivity, this appears to be the first time this result has been derived. Consider the regime \( m \gg \kappa T \). Fluctuations are suppressed in this regime. In this case, \( x^\pm \gg 1 \) and its subsequent contribution to the integral is exponentially small. Thus, it will compete with the contribution from the vicinity of \( x = 0 \). Evaluating Eq. (4) and retaining both contributions results in the total conductivity:
\[
\sigma = \frac{4e^2}{\pi h} \left[ \frac{\pi \kappa^2 T}{\eta} e^{-m/\kappa T} + \left( \frac{\pi \eta T}{3m^2} \right)^2 \right] m \gg \kappa T. \tag{8}
\]

The exponentially-decaying term is in agreement with the result of van Otterlo, et. al. [23], though their coefficient is a factor of 8 smaller.

We now go beyond the quadratic theory we have constructed. It is expedient to generalize to an N-component vector field, \(\psi_\mu(\omega, \mathbf{k})\) and introduce the quartic term

\[
\frac{U}{2N^2} \sum_{\omega_1, \omega_2, \omega_3, \omega_4} \delta_{\omega_1 \cdots \omega_4, 0} \delta_{\mathbf{k}_1 \cdots \mathbf{k}_4} \int \frac{d\mathbf{k}_1 \cdots d\mathbf{k}_4}{(2\pi)^d} \psi_\mu(\omega_1, \mathbf{k}_1) \psi_\nu(\omega_2, \mathbf{k}_2) \overleftrightarrow{\psi_\rho}(\omega_3, \mathbf{k}_3) \psi_\lambda(\omega_4, \mathbf{k}_4) \tag{9}
\]

into our free-energy functional. Because the phase ordering transition is no longer determined by the vanishing of the coefficient of the quadratic term at \(k = \omega = 0\), it is expedient then to redefine \(\delta\) as the bare temperature independent coefficient of the quadratic term.

We then introduce the auxiliary field, \(\lambda(\omega, \mathbf{k})\), to decouple the quartic term by means of a Hubbard-Stratonovich transformation. Formally, the auxiliary field, \(\lambda(\omega, \mathbf{k})\) is related to the correlation function, \(\langle \psi_\nu(\omega, \mathbf{k}) \psi^*_{\alpha}(\omega, \mathbf{k}) \rangle\). Consequently, if we invoke the saddle-point approximation, \(\lambda(\omega, \mathbf{k}) = \sqrt{\beta} \delta_{\omega, 0} \delta_{\mathbf{k}, 0}\), in the \(N \to \infty\) limit, the following self-consistency equation for the gap \(m^2 = \delta + \lambda\) where

\[
\lambda = \frac{U}{\beta} \sum_{\omega_n} \int \frac{d^2 k}{(2\pi)^2} \frac{1}{m^2 + m^2 + \frac{\kappa^2 \omega_n^2}{\eta}} \tag{10}
\]

must be solved. Let us define \(a = \eta/4\pi \kappa^2 T\), \(b = \sqrt{\eta^2/4\kappa^2 - m^2}/2\pi \kappa T\), and \(c = \Lambda/2\pi \kappa T\). Upon performing the sum over boson frequencies and subsequent \(\text{k}\)-integration by introducing the relativistic hard cutoff, \(\Lambda\), in Eq. (10), we obtain

\[
m^2 = \delta - \frac{UT}{2\pi} \ln \frac{\Gamma(a + ic) \Gamma(a - ic) \Lambda}{\Gamma(a + b) \Gamma(a - b) m} \tag{11}
\]

as the self-consistency condition for the effective inverse correlation length. Here \(\Gamma(z)\) is the Gamma function. While Sachdev, Chubukov and Sokol [22] have obtained a similar expression, the form derived here is more suitable for the asymptotic analysis of the three critical regimes of interest.

Expansion of the gamma functions when \(T\) is the smallest parameter reveals that the behaviour of \(m^2\) depends strongly on the sign of the quantity \(\Delta = \delta + U(\Lambda + O(\eta/\kappa))/4\pi \kappa\). At the quantum critical point, \(\Delta = 0\) and consequently \(m^2 = 0\). At any finite temperature, the form of \(m^2\), which enters directly into the conductivity, is determined by the relative strength of three parameters: \(\kappa T\), \(\Delta\), and \(\eta/\kappa\). Consequently, the following regimes arise as depicted in Fig. (1).

\[\text{FIG. 1. Heuristic phase diagram as a function of temperature } T \text{ and } \Delta. \text{ The quantum critical point corresponds to } \Delta = 0. \text{ Regions I, II, and III correspond to 1) quantum disordered, 2) quantum critical, and 3) renormalized classical. The dotted line demarcates the region in which dissipation dominates the physics of the correlation length.}\]

a) Quantum disordered (insulator): Two conditions (see Fig. (1)) demarcate this region: \(\Delta > 0 \text{ and } \Delta \gg \kappa T\). Expansion of Eq. (11) in this limit indicates for \(\Delta \gg \eta/\kappa\), \(m = 4\pi \Delta U/(\eta U)\) and for \(\eta/\kappa \gg \Delta\), \(m = \sqrt{4\pi^2 \eta \Delta/(\eta U \ln (\eta U / \kappa T))}\). We used the fact that from microscopic considerations of the XY model, \(\kappa / U \approx O(1)\) close to the transition point. Substitution of these results into Eq. (9) leads to the corresponding conductivity in the quantum disordered regime. b) Renormalized Classical: This regime is characterized by \(\Delta < 0\) but \(\Delta \gg \kappa T\) as shown in Fig. (1). The solution to Eq. (11) in this limit results in the following forms for the gap parameter \(m\):

\[
m = \begin{cases} 
\kappa T \exp \left( -\frac{2\pi |\Delta|}{U} \right) & |\Delta| \gg \kappa T \gg \eta/\kappa \\
\sqrt{\frac{\kappa T}{|\Delta|}} \exp \left( -\frac{2\pi |\Delta|}{U} \right) & |\Delta|, \eta/\kappa \gg \kappa T 
\end{cases} \tag{12}
\]

In the first sub-regime, \(m\) can be both smaller than or greater than \(\eta/\kappa\). Should \(m\) be greater than \(\eta/\kappa\), then the logarithmic form for the conductivity applies. However, because of the exponential dependence of \(m\), the resultant conductivity

\[
\sigma = \frac{4e^2 \pi \kappa^2 |\Delta|}{h \eta U} \eta/\kappa < m \tag{13}
\]

is independent of temperature. Because \(\eta/\kappa\) is small, the prefactor of \(e^2/h\) in the conductivity is large. This result is truly remarkable in light of the experiments by several groups [15,16,23] that have reported a leveling-off of the conductivity in 2D thin film superconductors at low temperatures. Note that the conductivity as well as the width of the region over which \(\sigma\) is constant increases as the magnitude of the dissipation decreases. Hence, the model presented here is in principle compatible with the low values of the resistivity once the leveling-off ensues. Ultimately Mason and Kapitulnik [18] predict that dissipation leads to a “metallic” state at \(T = 0\) in field-tuned
IST’s. However, within the framework presented here, a true “metallic” state does not exist at \( T = 0 \). We find, in fact, that at sufficiently low temperatures such that \( m < \eta/\kappa \), the second of Eqs. \((12)\) apply and the conductivity

\[
\sigma = \begin{cases} 
(e^2/h)(\eta/\kappa^2T) \exp \left( \frac{4\pi|m|}{\kappa T} \right) & \kappa T > \eta/\kappa \\
(e^2/h) \exp \left( \frac{4\pi|m|}{\kappa T} \right) & \kappa T < \eta/\kappa
\end{cases}
\]  

(diverges exponentially as \( T \to 0 \)) as is expected on the superconducting side. Nonetheless, the existing experiments cannot rule out the possibility that the resistivity ultimately vanishes at zero temperature.

c) Quantum critical region. In this regime, \( \kappa T \gg |\Delta| \).

We consider for simplicity the transition point, \( \Delta = 0 \). From Eq. \((11)\), it follows that \( m = \sqrt{2\pi\eta T}f_1(\kappa^2T/\eta) \), where \( f_1 \) is a numerically calculable function. We find that in the limiting case \( \kappa T \ll \eta/\kappa \), \( m = \Theta\kappa T \) with \( \Theta = 2\ln(\sqrt{5} + 1)/2 \). In general, the conductivity is given by \( \sigma = (e^2/h)f_2(\kappa^2T/\eta, \kappa/\kappa, U) \), with \( f_2 \) a universal function. To calculate \( f_2 \), we investigate Eq. \((3)\). For \( \kappa T \gg \eta/\kappa \), the main contribution to the integral over \( x \) comes from the vicinity of \( x = \mp \frac{1}{\kappa} \), leading to a conductivity of the form, \( \sigma = 2K_1(e^2/h)(\kappa^2T/\eta) \) with \( K_1 = \int_0^\infty \frac{t^3dt}{(t^2 + \Theta^2/4)\sinh^2(t^2 + \Theta^2/4)} = 0.0149 \) \((15)\).

For \( \kappa T \ll \eta/\kappa \), it is convenient to perform the \( t \)-integration in Eq. \((3)\) first. We obtain for the conductivity, \( \sigma = (2K_2/\pi)(e^2/h) \) where

\[
K_2 = \int_0^\infty \frac{dt}{\sinh^2t} \left[ 1 - \frac{\pi A}{t} \left( \frac{\pi}{2} - \arctan \frac{\pi A}{t} \right) \right] = 0.0076.
\]

We see that in the limit \( T \to 0 \), the conductivity acquires a universal value independent of the dissipation. The independence of the critical resistance on \( \eta \) is a consequence of hyperuniversality \((22)\). One should note, though, that the extremely small numerical prefactor, is likely to be an artifact of this model.

Our analysis of the role of dissipation in the vicinity of the quantum critical point has revealed a key new feature, namely the leveling of the resistivity at sufficiently low temperatures. To determine if this leveling is experimentally relevant, we evaluate the resistivity by inverting Eq. \((13)\). As shown previously \((13)\), \( U/\kappa = 7/2 \) and \( \kappa = 1/4E_c = 0.3K^{-1} \) close to the quantum critical point. Here \( E_c \) is the charging energy. Consequently, the level resistivity reduces to \( R = R_o(7/2\pi)(\eta/\kappa) / \| \Delta \| \), where \( R_o = \hbar/4e^2 = 6.4\Omega \). Typical values for the leveled resistance are \( R = 10\Omega \) \((17)\) which would require the ratio \( \eta/\kappa |\Delta| = .01 \). For \( |\Delta| \) in the range \([0.01, 1]\), we find that the corresponding value of the dissipation ranges between \( 10^{-5} < \eta/\kappa < 10^{-4} \) which is certainly in the weak dissipation regime as required by the present analysis. From the condition \( m \approx \eta/\kappa \), we find from Eq. \((13)\) that the leveling behavior ceases at \( T \sim O(10mK) \). Hence, we find that for realistic parameters, our analysis predicts a level resistance that is consistent with experimental trends.

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