Spinning particle orbits around a black hole in an expanding background

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Abstract
We investigate analytically and numerically the orbits of spinning particles around black holes in the post Newtonian limit and in the presence of cosmic expansion. We show that orbits that are circular in the absence of spin, get deformed when the orbiting particle has spin. We show that the origin of this deformation is twofold: (a) the background expansion rate which induces an attractive (repulsive) interaction due to the cosmic background fluid when the expansion is decelerating (accelerating) and (b) a spin–orbit interaction which can be attractive or repulsive depending on the relative orientation between spin and orbital angular momentum and on the expansion rate.

Keywords: geodesics, spinning particle orbits, spin–orbit coupling, black hole spacetime with expansion

(Some figures may appear in colour only in the online journal)

1. Introduction

Even though most astrophysical bodies have spins and evolve in an expanding cosmological background, their motion is described well by ignoring the cosmic expansion and under the nonspinning test particle approximation for large distances from a central massive body and for relatively low spin values [1]. These approximations however become less accurate for large values of the spin and/or when the mass of the cosmic fluid inside the particle orbit becomes comparable to the mass of the central massive object. For such systems new types of interactions appear which are proportional to the time derivatives of the cosmic scale factor and the spin of the orbiting particle. For example, phantom dark energy models can lead to dissociation of all bound systems in the context of a Big-Rip future singularity [2–5]. Also, the spin-curvature interaction [6] can modify the motion of the test particles in black hole spacetimes [7–11] due to spin–spin or spin–orbit couplings [12–14], or make the motion chaotic.
Thus modifying significantly the orbit of the test body leading to the emission of characteristic forms of gravitational waves [18–21]. Such interactions have been investigated previously for nonspinning test particles in an expanding background around a massive body (McVittie background [22]) and it was shown that accelerating cosmic expansion can lead to dissociation of bound systems in the presence of phantom dark energy with equation of state parameter $w < -1$ [2–4]. In the absence of expansion but in the presence of spin for the test particles it has been shown that spin–orbit and spin–spin interactions in a Kerr spacetime can lead to deformations of circular orbits for large spin values [14]. In view of these facts, the following interesting questions emerge

1. Are there circular orbit deformations for spinning test particles embedded in the post Newtonian limit of McVittie background (Schwarzschild metric embedded in an expanding background)? Such deformations could be anticipated due to the coupling of the particle spin with its orbital angular momentum.

2. What is the nature of such deformation and how do the corresponding deformations depend on the orientation of the spin with respect to the angular momentum?

3. How do these deformations depend on the nature of the background expansion?

These questions are addressed in the present analysis.

The structure of this paper is the following: in the next section we briefly review the Mathisson–Papapetrou (MP) equations [23] and the common supplementary conditions, we introduce the McVittie background corresponding to a black hole embedded in an expanding background and its post Newtonian limit. In section 3 we discuss the conserved quantities of a spinning test particle in a given spherically symmetric metric in an expanded background, we consider the post Newtonian limit of McVittie metric and construct the geodesic equations of a spinning particle using the MP equations. We also solve these equations numerically and identify the deformation of orbits due to the presence of test particle spin. We identify the dependence of this deformation on the relative orientation between spin and orbital angular momentum of the spinning test particle. Finally, in section 4 we summarize, discuss the implications of our results and identify possible future extensions of our analysis.

2. The equations of motion of a spinning particle. The MP equations

Consider a massive spinning test particle, in MP’s model [23, 24]. The equations of motion of a spinning particle originally derived from Papapetrou (1951) and later on reformulated by Dixon [25, 26] can be extracted through the corresponding Hamiltonian [27, 28] or through the extremization of the corresponding action [29], whose variation is [30]

$$\delta L = -p^\mu \delta v_\mu - \frac{1}{2} S^{\mu \nu} \delta \Omega_{\mu \nu}$$

(2.1)

where $v^\mu = \frac{dx^\mu}{d\tau}$ is the four-velocity of the test particle tangent to the orbit $x^\mu = x^\mu(\tau)$, $\tau$ is the proper time across the worldline, $x^\mu(\tau)$, $p^\mu$ is its four-momentum and $S^{\mu \nu}$ are the components of the antisymmetric spin tensor. Also, $\Omega_{\mu \nu} = \eta^{\mu}_{\nu} e^I_{\mu} \frac{Dg_{IJ}}{d\tau}$ is an antisymmetric tensor, $\eta^{\mu}_{\nu} = e^I_{\mu} e^J_{\nu} g_{IJ}$ and $e^I_{\mu}$ is a tetrad attached to each point of the worldline.

The MP equations are of the form [30–33]:

$$\frac{Dp^\mu}{d\tau} \equiv \frac{dp^\mu}{d\tau} + \Gamma^{\mu}_{\lambda \nu} v^\lambda p^\nu = -\frac{1}{2} S^{\mu \nu} \kappa_{\nu}$$

(2.2)
\[
\frac{DS_{\mu\nu}}{d\tau} \equiv \frac{dS_{\mu\nu}}{d\tau} + \Gamma^\mu_{\lambda \rho} v^\lambda S^{\rho \nu} + \Gamma^\nu_{\lambda \rho} v^\lambda S^{\mu \rho} = p^\mu v^\nu - p^\nu v^\mu. 
\] (2.3)

The dynamical equations imply, spin–orbit coupling, i.e. spin couples to the velocity of the orbiting spinning particle, thus deforming the geodesic. Therefore the spin force deforms the geodesic.

The spin tensor keeps track of the intrinsic angular momentum associated with a spinning particle. The term in the rhs of equation (2.2) shows an interaction between the curvature of the spacetime and the spin of the particle. Due to the coupling between curvature and spin, the four-momentum is not always parallel to the \( v^\mu \). This may be seen by multiplying equation (2.3) with \( v^\nu \). Then, leads to
\[
p^\mu = m v^\mu - v^\nu \frac{DS_{\mu\nu}}{d\tau}
\] (2.4)

where \( m = -p^\mu v_\mu \) is the rest mass of the particle with respect to \( v_\mu \).

Since \( \tau \) is the proper time, the condition \( v_\mu v^\mu = -1 \) applies. The measure of the four-momentum
\[
p_\mu p^{\mu} = -\mu^2
\] (2.5)

provides the ‘total’ or ‘effective’ [8] rest mass \( \mu \) (\( p^\mu = \mu u^\mu \)) with respect to \( p^\mu \) where \( u^\mu \) is the ‘dynamical four-velocity’ and is equal to \( m \), only if \( v^\mu \) coincides with the four-velocity \( u^\mu \) (\( u^\mu = v^\mu \)). In the linear approximation of the spin \( p^\mu \) and \( v^\mu \) are parallel. Generally, since \( u^\mu \neq v^\mu \) which means that \( \frac{DS_{\mu\nu}}{d\tau} \neq 0 \) (see equation (2.4)) a spinning particle does not follow the geodesics of the spacetime (the rhs of equation (2.2) is non zero, since \( S^{\mu\nu} \neq 0 \)). Therefore its motion is generalized on a world line rather than geodesics.

In the context of the MP equations the multipole moments of the particle higher than a spin dipole are ignored [34]. This is the spin-dipole approximation, because the particle is described as a mass monopole and spin dipole [35]. The equations in quadratic order of spin have also been derived [36]. The MP equations can also get generalized in order to describe a test spinning particle in modified theories of gravity [37].

The MP equations (2.2) and (2.3) have been discussed by many authors and solutions have been presented. These solutions refer mainly to Schwarzschild background spacetime [21, 38–44], to Kerr spacetime [21, 45–52], to de Sitter spacetime [53–56] and to FRW spacetime [57] for chargeless or charged test spinning particles [58, 59]. The evolution of spinning particles in spacetimes with torsion has also been investigated [60, 61].

Equations (2.2) and (2.3) are the equations of motion for a spinning body which reduce to the familiar geodesic equations when the spin tensor \( S^{\mu\nu} \) vanishes. However, they do not form a complete set of equations and we need further equations to close the system [62]. The problem of the unclosed set of equations in (2.2) and (2.3) can be physically understood by the requirement that the particle must have a finite size which does not make the choice of the reference worldline uniquely defined 3. The additional conditions used are the spin supplementary conditions (SSC) [63]. When we choose a SSC, we define the evolution of the test body in a unique worldline \( x^\mu(\tau) \) and we fix the center of mass (corresponds to the centre where the mass dipole vanishes), which is usually called centroid. The centroid is a single reference point inside the body, with respect to which the spin is measured [64].

There are several SSC but two of them are more commonly used

- The P condition (Mathisson–Pirani) [65]

3 https://d-nb.info/1098374932/34
so that the spin four-vector is perpendicular to the four-velocity and implies that $\frac{dS}{\nu} = 0$ [66]. It does not provide a unique choice of representative worldline, as it is dependent on the observer’s velocity and therewith on the initial conditions. It is often referred to as the proper centre of mass [63].

- The T condition (Tulczyjew–Dixon) [67]

\[ p_\mu S^{\mu\nu} = 0 \]  

so that the spin four-vector is perpendicular to the four-momentum and implies that $\frac{dm}{\nu} = 0$ [62]. This condition is physically correct, since the trajectory of the extended body is determined by the position of the center of mass of the body itself [68]. This constraint is a consequence of the theory, i.e. the Tulczyjew constraint can be derived from the Lagrangian theory [69] and restricts the spin tensor to generate rotations only.

Analytic discussions and thorough reviews on different choices about the SSCs may be found in [70–72]. Generally, different SSC are not equivalent since every SSC defines a different centroid for the system. The author of [1] point out that the difference between the two conditions (2.6) and (2.7) is third order in the spin, so results for physically realistic spin values, are unaffected. In what follows we use the T condition, which defines the centre of mass of the particle in the rest frame of the central gravitating body.

The McVittie metric describes an expanding cosmological background with strong gravity, such as a spacetime near a black hole or a neutron star. In a $(t, r, \theta, \phi)$ coordinate system McVittie [73] found a solution given by the equation (see equation (29) of [73] with $G = c = 1$)

\[ ds^2 = -(1 - \frac{m(t)}{2r})(1 + \frac{m(t)}{2r})^{-2}dt^2 + (1 + \frac{m(t)}{2r})^2a^2(t)(dr^2 + r^2d\Omega^2) \]  

(2.8)

where $d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2$.

The component $G'_t$ of the Einstein tensor is

\[ G'_t = \frac{8(2r + m)}{a(2r - m)^3}(\dot{a}m + a\dot{m}). \]  

(2.9)

Imposing the ‘no-accretion’ condition $G'_t = 0$ (there is no flux of relativistic mass across the equatorial surface [73]) we find that $\frac{\dot{r}}{r} = -\frac{\dot{m}}{m}$ or $m = \frac{m_0}{1 + m_0}$, where $m_0$ is a constant of integration and is identified with the mass of the central body at the origin [74]. The curvature of space is here assumed to be asymptotically zero.

At any instant of time $t_1$ the observer’s coordinate for measuring distance from the origin is $\tilde{r} = ra(t_1)$. If we write $M = m(t_1)a(t_1)$, the metric (2.8) becomes

\[ ds^2 = -(1 - \frac{M}{2r})^2dt^2 + (1 + \frac{M}{2r})^4(dr^2 + r^2d\Omega^2). \]  

(2.10)

In the weak field limit we have $\frac{M}{2r} \ll 1$, ie
which is the Newtonian limit of Schwarzschild’s spacetime.

Setting \( r = a(t) \rho \) and \( R_s = 2M \) the metric (2.11) reads

\[
ds^2 = -(1 - \frac{R_s}{a \rho}) dt^2 + a^2(1 + \frac{R_s}{a \rho})(d\rho^2 + \rho^2 d\Omega^2). \tag{2.12}
\]

For a static background (\( a = 1 \)) the metric (2.12) becomes the Schwarzschild metric in isotropic coordinates (the spacelike slices are as close as possible to Euclidean) as expected [75], while for \( R_s = 0 \) becomes the FRW metric in spherical coordinates.

The ‘areal’ radius [76] of the metric (2.12) is equal to the square root of the modulus of the coefficient of the angular part \( d\Omega^2 \) of the metric, namely

\[
R(t, \rho) = (1 + \frac{R_s}{a \rho})^{1/2} a \rho \tag{2.13}
\]

and the corresponding modulus of angular momentum, which is a constant of motion for a spinless particle, defined as

\[
\mathcal{L} = R^2(t, \rho) \dot{\phi}. \tag{2.14}
\]

3. Spinning particle in McVittie spacetime-Post Newtonian limit

3.1. The MP equations in an expanding Universe

We consider the case where the spinning particle orbits on the equatorial plane, which means that \( \theta = \pi/2 \). Also, on the equatorial plane valid \( v^2 \equiv v^\theta = 0 \) and \( \rho^\theta = 0 \) since \( p^\mu = \frac{\dot{x}^\mu}{m} \). The metric (2.12) is independent of the \( \phi \) coordinate, therefore admits a \( \phi \)-Killing vector e.g. \( \xi^\mu = (0, 0, 0, 1) \) which gives

\[
J_z = p_\mu \xi^\mu - \frac{1}{2} \xi_{\mu\nu} S^{\mu\nu} \tag{3.1}
\]

or

\[
J_z = p_\phi = \frac{1}{2} g_{\phi\mu\nu} S^{\mu\nu} \tag{3.2}
\]

where \( J_z \) is the \( z \) component of the angular momentum, which is a conserved quantity of the motion of a spinning particle. This constant of motion exists independently of the choice of the supplementary condition and reflects the symmetry of the background spacetime.

The spin tensor has six independent components but since we demand equatorial planar motion, the particle must have angular momentum only in \( z \) axis (\( J_z \neq 0 \)). The conditions \( J_x = 0, J_y = 0 \) and \( p^\theta = 0 \) (necessary conditions for motion in the equatorial plane) require that \( S^\theta = 0 \) and \( S^{\phi\phi} = 0 \). Also, the absence of acceleration perpendicular to the equatorial plane implies that \( S^{\theta} = 0 \) [57]. Thus, planar motion requires alignment of the spin with the orbital angular momentum and the motion characterized only by three independent spin components. With these assumptions, the spin tensor becomes a vector and the formulation
will be simpler. From the T condition (2.7) we derive the spin components \( S_{03} \) and \( S_{13} \) in terms of \( S_{01} \) as

\[
S_{03} = \frac{p_1}{p_3} S_{01}, \\
S_{13} = \frac{p_0}{p_3} S_{01}.
\] (3.3)

In order to complete the system of equations (2.2) and (2.3) we have to add two more equations, corresponding to conserved quantities in the context of the T condition. The first is the dynamical mass \( \mu \) [77] with respect to the four-momentum \( p^\mu \) which defined through equation (2.5) and the second is the particle’s total spin \( s \) which is defined as the positive root of

\[
s^2 = \frac{1}{2} S_{\mu \nu} S^{\mu \nu}.
\] (3.4)

The first derivative of \( s^2 \) with respect to \( \tau \) is

\[
\dot{s}^2 = \frac{1 - \xi}{\rho^2 (p^3)^2} \mu^2
\] (3.5)

where

\[
\xi \equiv \frac{R_s}{\rho} \ll 1.
\] (3.6)

Using the equations (2.5) and (3.5) we define the parameter \( \Omega^2 \) as the ratio

\[
\Omega^2 \equiv \frac{s^2}{\mu^2} = \left( \frac{S_{01}^2}{\rho^2 (p^3)^2} \right) (1 - \xi)
\] (3.7)

which is a constant of motion, since \( \mu \) and \( s \) are conserved quantities. From equation (3.7) it is easy to calculate the spin component \( S_{01} \)

\[
S_{01} = \frac{\rho \Omega}{\sqrt{1 - \xi}} \rho^3
\] (3.8)

Thus, from equations (3.3) and (3.8) the non zero spin components in our consideration are

\[
S_{01} = \frac{\rho \Omega}{\sqrt{1 - \xi}} p^3, \\
S_{03} = -\frac{p_1}{\rho^2 p^3} S_{01}, \\
S_{13} = -\frac{(1 - 2 \xi) \rho^3}{a^2 \rho^2 p^3} S_{01}.
\] (3.9)

Using now the post Newtonian limit of McVittie metric (2.12), starting from the MP equation (2.2) and setting the index \( \mu = 1 \) it is straightforward to derive the radial geodesic equation for the spinning particle. We replace the distance \( \rho \) as \( \rho = r/a \) and the corresponding derivatives with respect to \( t, \rho = d\rho/dt \) and \( \dot{\rho} = d^2 \rho/dt^2 \). Also, we ignore terms of order \( (R_s)^2 \) (post Newtonian limit) and the final result is

\[
\ddot{r} - \frac{\ddot{a}}{a} r - r \dot{\phi}^2 = -r \Omega \dot{\phi} \left( \frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} \right) + \frac{R_s}{2} \left( \frac{1}{r^2} - \dot{\phi}^2 - \frac{\dot{a}^2}{a^2} + \frac{j^2}{r^2} - \frac{3 \Omega \dot{\phi}}{r^2} \right).
\] (3.10)
Similarly, from the MP equation (2.2) and setting the index $\mu = 3 = \phi$ we obtain

$$\frac{d(r^2 \dot{\phi})}{dt} = r \dot{\phi} R + \Omega r^2 \left( \frac{\dot{r}}{r} - \frac{\dot{a}}{a} \right) \left( \frac{\ddot{a}}{a} - \frac{a^2}{a^2} \right)$$  \hspace{1cm} (3.11)

which would lead to orbital angular momentum conservation in the absence of spin ($\Omega = 0$). Indeed, the first derivative of equation (2.14) with respect to time must be zero and gives the equation (3.11) for a spinless particle [76].

Now, we introduce the rescaling through the variables $\bar{t} \equiv t/R$, $\bar{r} \equiv r/R$, and $\Omega_s \equiv \Omega/R = \frac{a}{mR}$ and from now on we omit the bar. The radial equation (3.10) leads to

$$\ddot{r} - \frac{\ddot{a}}{a} r - r \dot{\phi}^2 = -r \Omega_s \dot{\phi} \left( \frac{a}{a} - \frac{\dot{a}^2}{a^2} + \frac{3}{2r^2} \right) + \frac{1}{2} \left( -r^2 - \dot{\phi}^2 - \frac{\ddot{a}^2}{a^2} + \frac{\ddot{r}^2}{r^2} \right).$$  \hspace{1cm} (3.12)

In the same way, the azimuthal equation (3.11) leads to

$$\frac{d(r^2 \dot{\phi})}{dt} = r \dot{\phi} + \Omega_s r^2 \left( \frac{\dot{r}}{r} - \frac{\dot{a}}{a} \right) \left( \frac{\ddot{a}}{a} - \frac{a^2}{a^2} \right).$$  \hspace{1cm} (3.13)

Equations (3.12) and (3.13) are the main results of the present analysis. They generalize the geodesic equation of non-spinning particles in the post-Newtonian limit of McVittie metric and they reduce to those equations for $\Omega_s = 0$. It is straightforward to solve numerically the system (3.12) and (3.13) and we implement such solutions in what follows. The following comments can be made on equations (3.12) and (3.13):

- It is clear from equation (3.13) that the orbital angular momentum is not conserved due to the presence of the spin angular momentum. What is actually conserved is the $z$ component of the total angular momentum $J_z$ which is expressed through equation (3.2) in terms of the angular and the spin angular momenta.

- The driving force term proportional to $\Omega_s$ and $\dot{\phi}$ in the radial geodesic equation (3.12) has the form of a spin–orbit coupling and changes sign when the spin angular momentum reverses its direction with respect to the orbital angular momentum which is proportional to $\dot{\phi}$. This term is responsible for the deformation of the circular orbits and induces the well known chaotic behavior [78] of the spinning particle orbits in the absence of background expansion.

In what follows we solve the geodesic equations (3.12) and (3.13) for different forms of the expansion (static, accelerating, decelerating and constant) of the cosmological background and various values of the magnitude of the spin $s$ and consequently of the dimensionless parameter $\Omega_s$. We set $r(t_i) = 0$ ($t_i = 1$ is the initial time of the simulation) and $\dot{\phi}(t_i)$ so that $\dot{r}(t_i) = 0$ corresponding to an initially circular orbit. We present analytically this issue in the appendix. Also, we normalize the scale factor setting $a(1) = 1$ and we set the particle at initial distance $r_i = 6$ from the black hole.

3.2. Numerical solutions

For a static universe ($a(t) = 1$) equation (3.12) reduces to

$$\ddot{r} = r \dot{\phi}^2 + \frac{1}{2} \left( -\frac{1}{r^2} - \dot{\phi}^2 + \frac{\ddot{r}}{r^2} - \frac{3 \Omega_s \dot{\phi}}{r^2} \right).$$  \hspace{1cm} (3.14)
while the equation (3.13) becomes

$$\frac{d(r^2\dot{\phi})}{dr} = \dot{r} \ddot{\phi} + (2r - 1)\dot{r} = 0. \tag{3.15}$$

The effect of the spin–orbit coupling force is demonstrated in figures 1 and 2 where we show the circular orbits disrupted due to the spin–orbit coupling. For \( \Omega_s \dot{\phi} > 0 \) (see figure 1) the spin–orbit coupling force is attractive, since the term \(-\frac{3\Omega_s \dot{\phi}}{2r}\) in equation (3.12) is negative and the circular orbits (for a spinless particle) are deformed inward. The orbit of the motion of the particle remains bounded if the radius of the orbit is larger than \( 3R_s \). This is the well known effect of the ‘innermost stable circular orbit’ (ISCO) \([50, 79, 80]\). It is defined as the smallest circular orbit in which a test particle can stably orbit a massive object \([81]\). Since \( r_{\text{ISCO}} = 3R_s \) for a spinless central body in Schwarzschild spacetime, it is obvious that only black holes have innermost radius outside their surface.

This minimum allowed radius for bounded motion corresponds to a critical value of the dimensionless parameter \( \Omega_s = 0.6 \) (left panel). Generally, in the presence of spin the orbits are bounded between a minimum and a maximum radius \( (R_s = 6) \). As the spin increases \( (\Omega_s > 0.6) \), at some time the orbit’s radius becomes less than \( 3R_s \) and the particle gets captured by the black hole (right panel). For non-spinning particle \( (\Omega_s = 0) \) the circular orbits shown in right panel remain undisrupted.

For \( \Omega_s \dot{\phi} < 0 \) (see figure 2) the spin–orbit coupling force is repulsive, since the term \(-\frac{3\Omega_s \dot{\phi}}{2r}\) in equation (3.12) is positive and the circular orbits (for \( s = 0 \)) are deformed outward. The orbits of the motion of the spinning particle in all cases are bounded between a minimum \( (R_s = 6) \) and a maximum radius.

In the presence of a decelerating expansion with \( a(t) \sim t^{2/3} \) the orbits (solutions of equations (3.12) and (3.13)) are shown in figure 3 for clockwise and counterclockwise rotation and initial conditions that would lead to a circular orbit in the absence of spin and expansion. In this case the effects of the expansion combined with the effects of the spin lead to rapid dissociation of the system or capture by the black hole. The result depends on the magnitude of the attractive and repulsive terms in equation (3.12). Some orbits of the spinning particles for this case are shown in figure 3.

In left panel of figure 3 the initial rotation is clockwise, since \( \dot{\phi}(1) < 0 \). In this case, the term \(-\frac{3\Omega_s \dot{\phi}}{2r}\) in equation (3.12) is repulsive and even if the cosmological background is decelerating, for large enough values of spin, such as \( \Omega_s = 1 \) the particle rapidly gets deflected to an unbounded orbit. However, for small values of spin, such as \( \Omega_s = 0.1 \) or \( \Omega_s = 0.5 \) the decelerating background dominates and at some time the particle gets captured by the black hole.

Similar results are shown in the right panel of figure 3, where the initial rotation of the particle is counterclockwise. In this case, the term \(-\frac{3\Omega_s \dot{\phi}}{2r}\) in equation (3.12) which describes the spin–orbit interaction is attractive. For small values of the dimensionless parameter \( \Omega_s \), such as \( \Omega_s = 0.1 \) the spinning particle approaches the black hole and when the radius of the orbit becomes less than \( 3R_s \), the particle gets captured by the strong gravity of the central body. However, when the spin takes larger values such as \( \Omega_s = 0.5 \) or \( \Omega_s = 1 \) the particle gets deflected to an unbounded orbit, despite of the initially attractive effective force induced on the spinning particle. The expansion effects lead to dissociation of the initially bound system.

Now, we consider the effects of a de Sitter background expansion of the form

$$a(t) = e^{Ht} \tag{3.16}$$
Figure 1. Spinning particle orbits in a static universe. The circular orbits that would be present for a non-spinning particle get disrupted due to the spin–orbit coupling in the presence of spin. For $\Omega_s \dot{\phi} > 0$ the spin–orbit coupling force is attractive and the circular orbits are deformed inward. The left panel (where $\Omega_s = 0.6$) corresponds to maximum (critical) value of $\Omega_s$, for which the particle remains bounded. The innermost stable circular orbit (ISCO) is $3R_s$. When $\Omega_s > 0.6$, at some time the radius of the orbit becomes less than $3R_s$ and the particle is captured by the black hole (right panel). For non-spinning particle ($\Omega_s = 0$) the circular orbits shown in right panel remain undisrupted.

Figure 2. Same as figure 1 but the spinning particle orbits in the opposite direction. The circular orbits that would be present for a non-spinning particle get disrupted due to the spin–orbit coupling in the presence of spin. For $\Omega_s \dot{\phi} < 0$ the spin–orbit coupling force is repulsive and the circular orbits are deformed outward. For non-spinning particle ($\Omega_s = 0$) the circular orbits shown in both panels remain undisrupted. Notice that the $\Omega_s = 0$ circular orbit, which corresponds to the absence of spin, is an inner bound for clockwise rotation. In any case the particle remains bound.
where \( H = \sqrt{\frac{\Lambda}{3}} \) and \( \Lambda \) is the cosmological constant in dimensionless form. We solve the system of equations (3.12) and (3.13) with the same initial conditions (circular orbit in the absence of spin and expansion). We set the cosmological constant equal to \( \Lambda = \Lambda R_s^2 = 3 \times 10^{-2} \) \([82]\) and we present the trajectories of the particle in figure 4. We also show the corresponding orbit of a spinless particle in a static universe, in order to observe the deviation of each orbit from the circular.

Setting a mass value of a typical black hole as \( M = 10M_\odot = 2 \times 10^{31} \text{ Kg} \), we conclude that the dimensionless value \( \Lambda R_s^2 = 0.03 \) corresponds to \( \Lambda \simeq 3 \times 10^{-6} \text{ s}^{-2} \) or \( \Lambda \simeq 1.3 \times 10^{-42} \text{ GeV}^2 \). Due to this normalization, orbit disturbances are much larger than the realistic form corresponding to a realistic cosmological setup.

In left panel of figure 4 the initial rotation is clockwise, since \( \dot{\phi}(1) < 0 \). In this case, the term \( -\frac{3\Omega_s}{2}r^2 \) in equation (3.12) is positive and induces repulsion. In this case the repulsive effects of the accelerating cosmic expansion are amplified by the effects of the spin.

For initial counterclockwise rotation (right panel in figure 4) the term \(-\frac{3\Omega_s}{2r^2}\) in radial equation is negative and induces attraction. However, for spinless particle or small values of spin and consequently of the parameter \( \Omega_s \), such as \( \Omega_s = 10 \), the accelerating cosmological background dominates and the particles get deflected to unbounded orbit. On the contrary, when the spin of the particle is large, such as \( \Omega_s = 100 \), the attractive term \(-\frac{3\Omega_s}{2r^2}\) in radial equation dominates the expansion and the spinning particle gets captured by the black hole.

A crucial question of our analysis is which are the cosmological time intervals after which the effects of the expansion would become apparent. The answer can be easily obtained on dimensional grounds by equating the dimensionless parameters relevant for gravitational attraction \( (M/r) \) and background expansion \( H_0 \Delta t \) where \( H_0 \) is the Hubble parameter \( \dot{a}/a \) at the present time and \( \Delta t \) is the required time interval for the expansion effects to be observable. By equating these two parameters we find that the required time interval after which the cosmological expansion effects would become apparent on the trajectories is
where we have set $G = 1$. The time interval $\Delta t$ can be easily derived in S.I. as $\Delta t \approx \frac{GM}{H_0 r^2}$ and in table 1 we give some estimates of the cosmological time intervals for a typical black hole, the solar system, a typical galaxy and a typical cluster of galaxies. The time intervals are in years, since we have consider that $\frac{1}{H_0} \approx 1.4 \times 10^{10}$ years (the approximate age of the Universe).

The MP equations have also been generalized to the case of modified theories of gravity, in which the matter energy-momentum tensor is not conserved. In modified gravity theories the Schwarzschild metric gets modified and so does the weak field limit, as we can see e.g. from equation (32) of [83], which state to $f(R)$ theories ($G = 1$)

$$\chi(r) = 1 - \frac{2M}{r} + \frac{(1 + f'(R_0))Q^2}{r^2} - \frac{R_0}{12} r^2. \quad (3.18)$$

Here, $Q = rV(r)$ is the charge of a black hole, $V(r)$ the potential and $R_0$ the curvature of the spacetime which we consider constant. An analysis along the line of the derivation of the McVittie metric for general relativity (as discussed in [5]) could generalize this metric to the case of $f(R)$ theories and also lead to the derivation of its Newtonian limit (the generalization of equation (2.12)). Alternatively one could directly include the scale factor $a(t)$ as a new factor along with the radial coordinate in equation (32) of [83] and then take the Newtonian limit showing that it is a good approximation of the dynamical field equations for $f(R)$ gravity. This task is beyond the scope of the present analysis but it should be straightforward to implement in a future extension of our analysis.

Figure 4. Same as figure 3, but the scale factor is of the form $a(t) = e^{\sqrt{\Lambda t}}$ (de Sitter universe) with $\Lambda = \Lambda R_s^2$. Notice the strong repulsive effects on the trajectories of the spinning/spinless particle for initial clockwise rotation (left panel) due to accelerating background expansion. The term $-\frac{3\Omega_s^2}{2r}$ in radial equation (3.12) induces repulsion (left panel). However, for initial counterclockwise rotation (right panel) and extremely large spin, the particle captured by the black hole since the term $-\frac{3\Omega_s^2}{2r}$ in radial equation induces attraction and dominates.
4. Conclusions

We have constructed and solved numerically the MP equations in the post Newtonian limit of McVittie background thus obtaining the orbits of spinning particles close to a massive object in an expanding cosmological background. We have identified the effects of a spin–orbit coupling which can be repulsive or attractive depending on the relative orientation between spin and orbital angular momentum. A static universe (no expansion) was shown to lead to disrupted spinning particle orbits which are not closed and are confined between a maximum and a minimum radius. This range increases with the value of the spin. As expected for the spin values, for which the radius of the motion of the particle becomes less that $3R_s$, the particle is captured by the black hole. This result is in agreement with previous studies that have indicated the presence of such behavior of the orbits [31].

Interesting extensions of our analysis include the construction and solution of the MP equations for the strong field regime of the McVittie metric, or the consideration of different SSC like the P condition.

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Appendix

In the present analysis we have focused on the distortion of orbits that would be circular in the absence of expansion and spin. In order to solve the system of equations (3.12) and (3.13) we have assumed that initially the test particle has zero radial velocity ($\dot{r}(t_i = 1) = 0$) and zero radial acceleration ($\ddot{r}(t_i = 1) = 0$). The initial value of the derivative $\dot{\phi}(1)$ can derived through the geodesic equation (3.12). We set $a(t_i = 1) = 1$ and initial position for the particle $r_i = 6$ in units of $R_s$. Assuming a static Universe with $a(t) = 1$ we compute the initial angular momentum from equation (3.12). We set all the time derivatives of the scale factor equal to zero and thus we arrive at the following quadratic equation

$$r_i^2(2r_i - 1)(\dot{\phi}(1))^2 - 3\Omega_s\dot{\phi}(1) - 1 = 0. \quad (A.1)$$

Setting $\Omega_s = 0$, we obtain

$$\dot{\phi}(1) = \pm \frac{\sqrt{T}}{66} \simeq \pm 5 \times 10^{-2}. \quad (A.2)$$

Table 1. In this table we present estimations for some cosmological structures for the required time interval $\Delta t \simeq \frac{1}{H_0}\frac{\Omega_M}{\Omega_s}$ after which the cosmological expansion effects would become apparent on the trajectories. For the Hubble rate we have set $H_0^{-1} \simeq 1.4 \times 10^{10}$ years. In the case of black hole we have consider the distance $r = 6R_s$, as in the present work.

| Structure          | Distance $r$ (m) | Mass $M$ (Kg) | $\Delta t$ (years) |
|-------------------|----------------|--------------|---------------------|
| Solar system      | $5 \times 10^{12}$ | $2 \times 10^{30}$ | $\sim 4 \times 10^0$ |
| Typical galaxy    | $9 \times 10^{20}$ | $2 \times 10^{41}$ | $\sim 3 \times 10^3$ |
| Cluster of galaxies | $3 \times 10^{22}$ | $2 \times 10^{45}$ | $\sim 7 \times 10^7$ |
| Black hole        | $2 \times 10^5$   | $2 \times 10^{31}$ | $\sim 1 \times 10^9$ |
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