ON THE LOG CANONICAL RING WITH KODAIRA DIMENSION TWO

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Abstract. We prove that the log canonical ring of a projective log canonical pair with Kodaira dimension two is finitely generated.

1. Introduction

In this paper, we prove the following main result:

Theorem 1.1 (Main Theorem). Let $(X, \Delta)$ be a projective log canonical pair such that $\Delta$ is a $\mathbb{Q}$-divisor. Assume that $\kappa(X, K_X + \Delta) = 2$. Then the log canonical ring

$$R(X, \Delta) = \bigoplus_{m \geq 0} H^0(X, \mathcal{O}_X(|m(K_X + \Delta)|))$$

is a finitely generated $\mathbb{C}$-algebra.

This is a special case of the well-known finite generation conjecture for log canonical pairs (cf. [13, Conjecture 1.1]). In [13], Fujino and the author listed some recent progress on the finite generation conjecture, and proved it under the assumptions that $(X, \Delta)$ is plt and $\kappa(X, K_X + \Delta) = 2$. Their proof used the lc-trivial fibration (in the sense of [11]) to deal with fibrations by using a Hodge theoretic approach rather than by running the relative minimal model program as in [2]. We will follow the basic idea in [13] and prove the main result by using a more general kind of connectedness lemma (Subsection 2.2) and the slc-trivial fibration theory (Section 3).

This paper can be viewed as a continuation of [13]. Some of the notation and proofs here are the same as those in that paper, so we recommend the interested readers to read [13] as a warm-up.

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We work over $\mathbb{C}$, the complex number field, throughout this paper. We also freely use the basic notation of the minimal model program as in [6] and [7]. A variety means a reduced separated scheme of finite type over $\mathbb{C}$. In this paper, we do not use $\mathbb{R}$-divisors. We only use $\mathbb{Q}$-divisors.

2. Preliminaries

In this section, we prepare some results needed in our proof of the main theorem. For the notation and conventions of this paper, we refer to [13, Preliminaries].

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2.1. Variation of mixed Hodge structures. Let $S$ be a path connected and locally 1-connected topological space. A local system on $S$ is a locally constant sheaf $V$ of $\mathbb{Q}$-vector spaces on $S$ (cf. [20, Lemma B.34]). In particular, a constant system is a constant sheaf $V$. One example of the local systems is the so-called variation of (mixed) Hodge structures. We follow the notation and definitions in [9] and recommend the interested readers to read [9] for more details.

The following theorem is taken out from [9, Theorem 7.1]. It shows that for any graded polarizable variation of $\mathbb{Q}$-mixed Hodge structures given in this paper, the $\mathcal{O}_D$-module $V$ is directly defined as $\mathcal{O}_D \otimes V$. It follows that $\alpha : V \to V := \mathcal{O}_D \otimes V$ is given by simply tensoring $\mathcal{O}_D$ and thus induces trivially an identification $\mathcal{O}_D \otimes V \simeq V$ of $\mathcal{O}_D$-modules. We can omit the morphism $\alpha$ since there is no danger of confusion.

Theorem 2.1 ([9, Theorem 7.1]). Let $(V, T)$ be a simple normal crossing pair such that $T$ is reduced, and $f : V \to W$ a projective surjective morphism onto a smooth variety $W$. Assume that every stratum of $(V, T)$ dominates $W$. Let $\Sigma$ be a simple normal crossing divisor on $W$ such that every stratum of $(V, T)$ is smooth over $W^* = W \setminus \Sigma$. Put $V^* = f^{-1}(W^*)$, $T^* = T|_{V^*}$. Let $\iota : V^* \setminus T^* \to V^*$ be the natural open immersion. Then the local system $V_k := R^k(f|_{V^*})_* \mathcal{L}(\iota|_{\mathcal{L}(V^*) \setminus T^*})$ underlies a graded polarizable admissible variation of $\mathbb{Q}$-mixed Hodge structure $V = ((V, W), (V, W, F), \text{id})$ on $W^*$ for every $k$. Note that $V_k := \mathcal{O}_{W^*} \otimes V_k$.

2.2. Connectedness lemma. [13, Section 4] showed a kind of connectedness lemma for plt pairs. But it is not sufficient if we try to deal with finite generation conjecture for lc pairs. For our purposes, we need a more general kind of connectedness lemma as follows. Note that it is also a special case of adjunction formula for quasi-log canonical pairs (cf. [7, Theorem 6.3.5]).

Lemma 2.2 (Connectedness). Let $f : V \to W$ be a surjective morphism from a smooth projective variety $V$ onto a normal projective variety $W$. Let $B_V$ be a $\mathbb{Q}$-divisor on $V$ such that $K_V + B_V \sim_{\mathbb{Q}, f} 0$, $(V, B_V)$ sub lc, and $\text{Supp } B_V$ a simple normal crossing divisor.

Assume that the natural map

$$\mathcal{O}_W \to f_*\mathcal{O}_V([-B_V^{<1}])$$

is an isomorphism. Let $Z$ be a union of some images of stratum of $B_V^{\leq 1}$ such that $Z \subseteq W$. Let $S$ be the union of strata of $B_V^{\leq 1}$ mapping into $Z$. Assume that $S$ is a union of irreducible components of $B_V^{\leq 1}$. Put $K_S + B_S = (K_V + B_V)|_S$ by adjunction. Then $(S, B_S)$ is sub slc and the natural map

$$\mathcal{O}_Z \to g_*\mathcal{O}_S([-B_S^{\leq 1}])$$

is an isomorphism, where $g := f|_S$. In particular, $S$ is connected if $Z$ is connected.

Proof. The proof is very similar to [13, Lemma 4.1 and Corollary 4.2]. We can easily check that $(S, B_S)$ is sub slc by adjunction. Consider the following short exact sequence

$$0 \to \mathcal{O}_V([-B_V^{<1}]) - S) \to \mathcal{O}_V([-B_V^{\leq 1}]) \to \mathcal{O}_S([-B_S^{\leq 1}]) \to 0.$$ 

Note that $B_V^{\leq 1}|_S = B_S^{\leq 1}$ holds. By [7, Theorem 5.6.3] and our assumptions of $Z$ and $S$, no lc stratum of $(V, \{B_V\} + B_V^{<1} - S)$ are mapped into $Z$ by $f$. By the same proof of [7, Theorem 6.3.5 (i)], the natural map $\mathcal{O}_Z \to g_*\mathcal{O}_S([-B_S^{\leq 1}])$ is an isomorphism. In particular, the natural map $\mathcal{O}_Z \to g_*\mathcal{O}_S$ is an isomorphism. This implies that $S$ is connected if $Z$ is connected.

The following corollary allows us to remove those strata of $B_V^{<1}$ in $S$ which are not dominant onto $Z$ when $Z$ is normal.
Corollary 2.3. Notation as in Lemma [2,2]. Assume further that $Z$ is irreducible and normal. Let $S'$ be the union of irreducible components of $B_{V}^{-1}$ dominant onto $Z$. Put $K_{S'} + B_{S'} = (K_{V} + B_{V})|_{S'}$ by adjunction. Then $(S', B_{S'})$ is also sub slc and the natural map

$$\mathcal{O}_{Z} \to g'_*\mathcal{O}_{S'}(\lceil -(B_{S'}^{\leq 1}) \rceil)$$

is an isomorphism, where $g' = f|_{S'}$. In particular, $S'$ is connected.

Proof. Consider the following commutative diagram:

$$
\begin{array}{ccc}
S' & \xrightarrow{\iota} & S \\
g' \downarrow & & \downarrow g \\
\tilde{Z} & \xrightarrow{p} & Z
\end{array}
$$

where $\iota : S' \to S$ is the natural closed immersion and $p$ is the normalization by [12, Claim 1]. Since $Z$ is normal, $p$ is an isomorphism. Then the rest of the proof is exactly the same as [12, Claim 2]. \qed

2.3. MMP for projective dlt surfaces. Finally, we give a special case of the minimal model program for projective dlt surfaces. It plays the same role in this paper as [13, Lemma 6.1] in [13, Theorem 1.2]. The proof is exactly the same as [13, Lemma 6.1], so we omit it here.

Lemma 2.4. Let $(X, B)$ be a projective dlt surface such that $B$ is a $\mathbb{Q}$-divisor and let $M$ be a nef $\mathbb{Q}$-divisor on $X$. Assume that $K_X + B + M$ is big. Then we can run the minimal model program with respect to $K_X + B + M$ and get a sequence of extremal contraction morphisms

$$(X, B + M) =: (X_0, B_0 + M_0) \xrightarrow{\varphi_0} \cdots \xrightarrow{\varphi_{k-1}} (X_k, B_k + M_k) =: (X^*, B^* + M^*)$$

with the following properties:

(i) each $\varphi_i$ is a $(K_{X_i} + B_i + M_i)$-negative extremal birational contraction morphism,

(ii) $K_{X_i+1} = \varphi_i K_{X_i}$, $B_{i+1} = \varphi_i B_i$, and $M_{i+1} = \varphi_i M_i$ for every $i$,

(iii) $M_i$ is nef for every $i$, and

(iv) $K_{X^*} + B^* + M^*$ is nef and big.

3. ON slc-trivial fibrations

Recently, Fujino generalized the klt-trivial fibration in [11] and the lc-trivial fibration in [11] to the so-called slc-trivial fibration in [8], where using some deep results of theory of variations of mixed Hodge structures on cohomology with compact support. For more details about slc-trivial fibrations, see [8] and [10].

Let $f : V \to W$ be a projective surjective morphism from a projective simple normal crossing variety $V$ onto a normal projective variety $W$ such that every stratum of $V$ is dominant onto $W$ and $f_*\mathcal{O}_V = \mathcal{O}_W$. Let $B_V$ be a $\mathbb{Q}$-divisor on $V$ such that $(V, B_V)$ is sub slc and Supp $B_V$ a simple normal crossing divisor. Put

$$B_W := \sum_{P} (1 - b_P)P,$$

where $P$ runs over prime divisors on $W$ and

$$b_P := \max \{ t \in \mathbb{Q} \mid (V, B_V + tf^*P) \text{ is sub slc over the generic point of } P \}.$$
It is easy to see that $B_W$ is a well-defined $\mathbb{Q}$-divisor on $W$ (cf. [8 4.5]). We call $B_W$ the discriminant $\mathbb{Q}$-divisor of $f: (V, B_V) \to W$. We assume that the natural map

$$\mathcal{O}_W \to f_*\mathcal{O}_V([-B_V^{1}])$$

is an isomorphism. Then the same as [13, Lemma 5.1], we immediately get that $B_W$ is a boundary $\mathbb{Q}$-divisor on $W$.

From now on, we assume that $K_V + B_V \sim_{\mathbb{Q}, f} 0$. Let $b = \min\{m \in \mathbb{Z}_{>0} \mid m(K_F + B_F) \sim 0\}$ where $F$ is a general fiber of $f$ and $K_F + B_F = (K_V + B_V)|_F$. Then we can take a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor $D$ on $W$ and a rational function $\varphi \in \Gamma(V, K_V)$ (see [8 Section 6]) such that

$$K_V + B_V + \frac{1}{b}(\varphi) = f^*D.$$

Then put

$$M_W := D - K_W - B_W,$$

where $K_W$ is the canonical divisor of $W$. We call $M_W$ the moduli $\mathbb{Q}$-divisor of $K_V + B_V + \frac{1}{b}(\varphi) = f^*D$. Under above assumptions and definitions, such a morphism $f: (V, B_V) \to (W, D)$ is a kind of (basic) slc-trivial fibrations defined in [8 Definition 4.1] and we call

$$D = K_W + B_W + M_W$$

the structure decomposition. Note that $D$ is uniquely determined by $\varphi$ once $K_V, K_W$ and $B_V$ are fixed (cf. [18 (2.6.i)], [14, Proposition 4.2] or [1, Remark 2.5]); thus so is $M_W$. Note also that $\varphi$ can be viewed as a b-divisor in the sense of [1 1.2 and Example 1.1 (2)] or [8 Definition 2.12].

Based on the theory of slc-trivial fibrations, we get a useful corollary from [10] which is a generalization of [1 Theorem 0.1] and [1 Theorem 1.4].

**Theorem 3.1** ([10 Corollary 1.4]). Notation as above. If $\dim W = 1$, then the moduli $\mathbb{Q}$-divisor $M_W$ is semi-ample.

The same as [13 Corollary 5.4], we immediately get that

**Corollary 3.2.** Notation as above. If $\dim W = 1$ and $D$ is nef, then $D$ is semi-ample.

The following lemma seems to be a simple fact hidden behind the proof of Theorem 3.1. But it will play a very key role in this paper. It shows that if the moduli part of an slc-trivial fibration is numerically trivial, then this moduli part defines a local system coming from the variation of (mixed) Hodge structures, and the difference between the moduli part and the local system is given by the rational section $\varphi$. This property makes the moduli parts possible to be glued together in the non-normal cases.

**Lemma 3.3.** Notation as above. If $\dim W = 1$ and $M_W \equiv 0$, then there exists a positive integer $k$ such that $\mathcal{O}_W(kM_W) \cdot (\sqrt[k]{\varphi^k}) = \mathcal{O}_W$.

**Proof.** We can assume that the morphism $f: (V, B_V) \to W$ satisfies the following conditions (a)–(g). They are nothing but the conditions stated in [8 Proposition 6.3] and [10 Section 5]:

(a) $W$ is a smooth curve and $V$ is a projective simple normal crossing variety.
(b) $\Sigma_W$ and $\Sigma_V$ are simple normal crossing divisors on $W$ and $V$ respectively.
(c) $f$ is a projective surjective morphism.
(d) $B_V$ and $B_W$, $M_W$ are supported by $\Sigma_V$ and $\Sigma_W$ respectively.
(e) every stratum of $(V, \Sigma_V^b)$ is smooth over $W^* := W \setminus \Sigma_W$.
(f) $f^{-1}(\Sigma_W) \subset \Sigma_V$, $f(\Sigma_W^b) \subset \Sigma_W$.
(g) $(B_V^b)^{1}$ is Cartier.
By [8 Lemma 7.3, Theorem 8.1], there exists a finite surjective morphism $\pi: C \to W$ (unipotent reduction) and a Cartier divisor $M_C$ such that $M_C = \pi^* M_W$. Note that there is also an induced (pre-basic) slc-trivial fibration (see [8 4.3]) $f': (V', B_{V'}) \to (C, \pi^* D)$ with

$$K_{V'} + B_{V'} + \frac{1}{b}(\phi') = f'^*(\pi^* D).$$

where $\phi'$ is the pullback of $\phi$. By the proof of [10 Theorem 1.3] (see also [11 Lemma 5.2]), $\mathcal{O}_C(M_C) \cdot (\sqrt[n]{\phi'})|_{C^*}$ is a direct summand of $F^0 \text{Gr}_i^W((\mathcal{V}', \phi'))$ where $C^* = \pi^{-1}(W^*)$, and $\text{Gr}_i^W((\mathcal{V}', \phi'))$ is a polarizable variation of $\mathbb{Q}$-Hodge structures. By definitions and Theorem 2.1,

$$\text{Gr}_i^W((\mathcal{V}', \phi')) = \mathcal{O}_{C^*} \otimes \mathbb{V}$$

where $\mathbb{V}$ is a local system on $C^*$. Note that the induced filtration $F^0(\mathbb{V})$ is not necessary a local subsystem of $\mathbb{V}$. But by [8 Proposition 6.3] and the assumption that $M_C = \pi^* M_W \equiv 0$, there is an induced identification:

$$\mathcal{O}_C(M_C) \cdot (\sqrt[n]{\phi'})|_{C^*} = \mathcal{O}_{C^*} \otimes \mathcal{M}$$

where $\mathcal{M} \subset \mathbb{V}$ is a local subsystem of rank one by [10 Lemma 4.8]. Then by [3 Corollaire (4.2.8) (iii) b)], there is a positive integer $t$ such that $\mathcal{M} \otimes t$ is a constant system and

$$\mathcal{O}_C(tM_C) \cdot (\sqrt[n]{\phi'})|_{C^*} = \mathcal{O}_{C^*} \otimes \mathcal{M} \otimes t.$$ 

Therefore, we can take a canonical extension such that

$$\mathcal{O}_C(tM_C) \cdot (\sqrt[n]{\phi'}) = \mathcal{O}_C \otimes \mathcal{M} \otimes t$$

by [9 Theorem 7.1] (see also [16 Lemma 1], [19 Theorem 1] or [17 Theorem 2.6]). That is, $tM_C + t(\sqrt[n]{\phi'}) = 0$ by viewing as Cartier divisors. By pushing forward, we have that

$$t \cdot \deg \pi \cdot (M_W + (\sqrt[n]{\phi})) = 0.$$

Let $k = t \cdot \deg \pi$. Then $\mathcal{O}_W(kM_W) \cdot (\sqrt[n]{\phi}) = \mathcal{O}_W$. \hfill \Box

**Remark 3.4.** Note that on $W$, we can show that $\mathcal{O}_W(M_W)(\sqrt[n]{\phi})|_{W^*}$ defines a local subsystem by the same proof of [10 Theorem 1.3]. Then by [3 Corollaire (4.2.8) (iii) b)], there is a positive integer $k$ such that $\mathcal{O}_W(kM_W) \cdot (\sqrt[n]{\phi})|_{W^*}$ is a constant system. This $k$ coincides with that $k$ in Lemma 3.3.

Now we are ready to prove the following corollary.

**Corollary 3.5.** Notation as above. If $\dim W = 2$, $(W, B_W)$ is dlt, and $D$ is nef and big, then $D$ is semi-ample.

**Proof.** Let $C = B_W^{\leq 1}$ and assume that $C = \sum C_i$ is connected for simplicity. Note that $D = K_W + B_W + M_W$ where $B_W = C + B_W^{\leq 1}$ is a boundary $\mathbb{Q}$-divisor and $M_W$ is nef by [8 Theorem 1.2]. Then $2D - (K_W + B_W) = D + M_W$ is nef and big. Therefore, to prove that $D$ is semi-ample, it suffices to prove that $D|_C$ is semi-ample by Kawamata–Shokurov basepoint-free theorem (cf. [13 Lemma 4.3]). Let $C = A + B$ where $A = \sum_{D(C) > 0} C_i$ and $B = \sum_{D(C) > 0} C_j$. Then $D|_A$ is numerically trivial on $A$ and $D|_B$ is ample on $B$. If we can prove that $D|_A$ is $\mathbb{Q}$-linearly trivial, then $D|_C$ is semi-ample by [15 Lemma 2.16]. Note that $D|_A$ is numerically trivial is equivalent to

$$(K_W + A + B + B_W^{\leq 1} + M_W) \cdot A = 0.$$
This implies that \( 2p_a(A) - 2 = \deg K_A = (K_W + A) \cdot A \leq 0 \). If \( p_a(A) = 0 \), then it is obvious that \( D|_A \) is \( \mathbb{Q} \)-linearly trivial. Thus we assume that \( p_a(A) = 1 \). It follows that

\[
(K_W + A) \cdot A = B \cdot A = B_{W}^{1} \cdot A = M_{W} \cdot A = 0.
\]

We can see that \( A \) does not intersect \( B \) by \( B \cdot A = 0 \). Since we assume that \( C \) is connected at the beginning, it follows that \( B = 0 \) in this case. That is, \( C = A \). Moreover, \( (K_W + C) \cdot C = (K_W + A) \cdot A = 0 \) implies that \( C \) is either a smooth elliptic curve or a nodal rational curve or a cycle of smooth rational curves (taking analytic dlt pairs into consideration). When \( C \) is a smooth elliptic curve, \( D|_C \) is \( \mathbb{Q} \)-linearly trivial by Corollary 3.2. When \( C \) is a nodal rational curve and \( P \) is the nodal point, we blow up \( W \) at point \( P \) and denote it as \( \pi: W' \to W \). Note that \( W' \) is smooth at around \( \pi^{-1}(P) \) since \( W \) is smooth at the nodal point \( P \). Let \( B_{W'} \) be the \( \mathbb{Q} \)-divisor such that \( K_{W'} + B_{W'} = \pi^{*}(K_W + B_W) \), \( D' = \pi^{*}D \) and \( M_{W'} = \pi^{*}M_W \). Then it is easy to see that \( C' = B_{W}^{1} \) is a cycle of two smooth rational curves and \( D' \) is semi-ample if and only if \( D \) is semi-ample. Thus we reduce the case to that \( C \) is a cycle of smooth rational curves. Then \( M_{W'} \cdot A = M_{W} \cdot C = 0 \) implies that \( M_{W'} \cdot C_i = 0 \) for every \( i \) since \( M_{W} \) is nef. Let \( S \) be the union of strata of \( B_{W}^{1} \) mapping into \( C \). By further resolutions, we can assume that \( S \) is a union of irreducible components of \( B_{W}^{1} \) (cf. \([7, \text{Proposition 6.3.1}]\)). By Lemma 2.2, the natural map \( \mathcal{O}_C \to g_i^{*}\mathcal{O}_S([-B_{S}^{1}]) \) is an isomorphism, where \( K_S + B_S = (K_V + B_V)|_S \) and \( g = f|_S \). Similarly, let \( S_i \) be the union of irreducible components of \( B_{V}^{1} \) dominant onto (not only mapping into) \( C_i \) for every \( i \). By Lemma 2.2 and Corollary 2.3, \( \mathcal{O}_{C_i} \to g_{i*}\mathcal{O}_{S_i}([-B_{S_i}^{1}]) \) is an isomorphism where \( K_{S_i} + B_{S_i} = (K_V + B_V)|_{S_i} \) and \( g_i = f|_{S_i} \). Then by adjunction,

\[
K_S + B_S + \frac{1}{b}(\varphi)|_S = (K_V + B_V + \frac{1}{b}(\varphi))|_S = g^{*}(D|_C)
\]

and

\[
K_{S_i} + B_{S_i} + \frac{1}{b}(\varphi)|_{S_i} = (K_V + B_V + \frac{1}{b}(\varphi))|_{S_i} = g_{i*}^{*}(D|_{C_i}).
\]

Note that the number \( b_i := \min\{m \in \mathbb{Z}_{>0} | m(K_{F_i} + B_{F_i}) \sim 0 \} \) is a factor of \( b \) where \( F_i \) is the general fiber of \( g_i \) for every \( i \). That is, there exists a positive integer \( s_i \) such that \( b = s_i b_i \) for every \( i \). Then the morphism \( g_i: (S_i, B_{S_i}) \to C_i \) satisfies our definition of slc-trivial fibrations with \( K_{S_i} + B_{S_i} + \frac{1}{b_i}(\varphi)|_{S_i} = g_{i*}^{*}(D|_{C_i}). \) and

\[
D|_{C_i} = K_{C_i} + B_{C_i} + M_{C_i}.
\]

By Lemma 3.3 there exists a positive integer \( k \) (not depending on \( i \)) such that

\[
\mathcal{O}_{C_i}(kM_{C_i}) \cdot (\sqrt[k]{\varphi}|_{C_i}) = \mathcal{O}_{C_i}.
\]

By adjunction, we have

\[
D|_{C_i} = (K_W + C + M_W)|_{C_i} = K_{C_i} + (C - C_i)|_{C_i} + M_W|_{C_i}.
\]

Comparing (3.1) and (3.2), it is easy to get that \( B_{C_i} = (C - C_i)|_{C_i} \) consists of two reduced points on \( C_i \) and \( M_{C_i} = M_W|_{C_i} \). Therefore,

\[
(\mathcal{O}_{C}(kM_W) \cdot (\sqrt[k]{\varphi}|_{C_i}))|_{C_i} = \mathcal{O}_{C_i}(kM_{C_i}) \cdot (\sqrt[k]{\varphi}|_{C_i}) = \mathcal{O}_{C_i}.
\]

Since the right hand side is the structure sheaf for every \( i \), we can glue them together and get \( \mathcal{O}_{C} \) exactly. That is, \( \mathcal{O}_{C}(kM_W) \cdot (\sqrt[k]{\varphi}|_{C}) = \mathcal{O}_{C} \). Then \( \mathcal{O}_{C}(M_W) \sim_{\mathbb{Q}} \mathcal{O}_{C} \) and thus \( M_W|_{C} \sim_{\mathbb{Q}} 0 \). Therefore,

\[
D|_{C} = K_{C} + M_{W}|_{C} \sim M_{W}|_{C} \sim_{\mathbb{Q}} 0,
\]

and this is what we want. \( \square \)
Remark 3.6. In fact, we showed that if dim\(W = 2\), \((W, B_W)\) is dlt, \(D\) is nef and there is some number \(a > 0\) such that \(aD - (K_W + B_W)\) is nef and big, then \(D\) is semi-ample. The proof is without any change.

proof of the main theorem. By the same proof of [13, Theorem 1.2], we can reduce to prove that the ring \(R(Y, D)\) is finitely generated for an slc-trivial fibration \(f: (V, B_V) \to (Y, D)\) where \(D = K_Y + B_Y + M_Y\), \((Y, B_Y)\) is dlt and \(D\) is big. By Lemma 2.4, we can further assume that \(D\) is nef. Then our conclusion follows from Corollary 3.5. \(\Box\)

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