Enhanced Numerov Method for the Numerical Solution of Second Order Initial Value Problems

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Abstract. Numerov method is a multistep numerical method that is used in solving second order differential equations. In this work, we apply this method as a Boundary Value Method (BVM) for the numerical approximation of both linear and nonlinear second order initial value problems. This is achieved by constructing the Numerov method via interpolation and collocation process while utilizing data at off-step points and implementing it as a BVM. On comparing the results obtained from the solved problems, it shows that the method is accurate with high level of convergence to their exact forms and performs better than results from literature.

Keywords: Hybrid BVM; Linear Multistep Method; Initial Value Problem; Boundary Value Method

1. Introduction
Due to the needs of mathematical modelization of real-life problems, the search for a better numerical method, which can handle different problems, remain active research for numerical analysts [1-6]. In this work, we derive a new scheme based on a Linear Multistep Method (LMM) called the Numerov method. This method is a numerical method used in approximating second order differential equations. Lots of studies have been done with this method [7, 8].

Our focus is to develop a new scheme called Hybrid Boundary Value Method (HyBVM). The development will be achieved by collocating and interpolating the LMM (mentioned above) at both step and off-step points. We then implement this scheme as a Boundary Value Method (BVM) to solve second order initial value problems of the form:

\[
\begin{align*}
\frac{d^2 y}{dx^2} &= f(x, y, y') \\
y(0) &= \alpha_0, \quad y'(0) = \beta_0
\end{align*}
\]  

(1)

where \(\alpha_0, \beta_0\) are constants and \(f\) is a continuous function which satisfies the conditions for existence and uniqueness of solutions, which are guaranteed by the theorem of Henrici in [9] for Initial value problems.
In deriving this method, we will be adopting the two step Numerov method, which is a LMM of the form:

\[
y_{n+k} + \sum_{i=0}^{k-1} \alpha_i y_{n+i} = h^2 \sum_{i=0}^{k} \beta_i f_{n+i}
\]  \hspace{1cm} (2)

where \( h \) is the step-size and \( \alpha, \beta \) are constants.

Different hybrid formulas based on LMM have been derived by authors [10–14], since they are more flexible the way they have been used and also possess small error constants. The BVMs were introduced to overcome some of the limitations encountered by the LMMs; they are better in approximating solutions. Several BVMs have been developed and discussed fully in literature [15–28].

The remaining sections of this study are structured as follows: In section two, the derivation and the specification of the method are discussed. In the third section, the results are presented with examples on both linear and nonlinear Initial Value Problems (IVPs). Section four offers a discussion of the results. The concluding remark is given in the last section.

2. Derivation of Methods [28]

In this section, the aim is to derive a two-step LMM of the form:

\[
y_{n+k} + \sum_{i=0}^{k-1} \alpha_i y_{n+i} = h^2 \sum_{i=0}^{k} \beta_i f_{n+i} + h^2 \sum_{i=0}^{k} \beta_i f_{n+i}
\]  \hspace{1cm} (3)

And also the derivative formula of the form

\[
y'_{n+k} + \sum_{i=0}^{k-1} \alpha'_i y'_{n+i} = h^2 \sum_{i=0}^{k} \beta'_i f_{n+i} + h^2 \sum_{i=0}^{k} \beta'_i f_{n+i}
\]  \hspace{1cm} (4)

using the same continuous scheme where \( \alpha_k = 1, \beta_k \neq 0 \) and \( \alpha_0, \beta_0 \) do not both vanish.

We start the process of derivation by seeking to approximate the analytical solution \( y(x) \) by a continuous method \( Y(x) \) with its second derivative of the form:

\[
Y(x) = \sum_{i=0}^{a+b-1} \beta_i P_i(x)
\]  \hspace{1cm} (5)

\[
Y''(x) = \sum_{i=0}^{a+b-1} \beta_i P_i''(x)
\]  \hspace{1cm} (6)

where \( a, b \) are the number of interpolation and collocation points, \( P_i(x) \) are the polynomial basis of degree \( a+b-1 \). A \( k \)-step multistep collocation method is then constructed from:

\[
Y(x) = V^T (M^{-1}) P(x)
\]  \hspace{1cm} (7)

where

\[
P(x) = [P_0(x), P_1(x), P_2(x), \ldots, P_{a+b-1}(x)]
\]  \hspace{1cm} (8)
\[ V = \begin{bmatrix} y_0, \ldots, y_{n+r-1}, f_n, \ldots, f_{n+b-1} \end{bmatrix} \]  

(9)

\[
M = \begin{pmatrix}
P_0(x_n) & P_1(x_n) & \cdots & P_{a+b-1}(x_n) \\
\vdots & \vdots & \ddots & \vdots \\
P_0(x_{n+a-1}) & P_1(x_{n+a-1}) & \cdots & P_{a+b-1}(x_{n+a-1}) \\
P_0'(x_n) & P_1'(x_n) & \cdots & P_{a+b-1}'(x_n) \\
\vdots & \vdots & \ddots & \vdots \\
P_0''(x_{n+b-1}) & P_1''(x_{n+b-1}) & \cdots & P_{a+b-1}''(x_{n+b-1})
\end{pmatrix}
\]

(10)

Which results into a continuous LMM

\[
Y(x) = \sum_{i=0}^{a-1} \alpha_i(x) y_{n+i} + h^2 \sum_{i=0}^{b-1} \beta_i(x) f_{n+i}
\]

(11)

where \( \alpha_i(x), \beta_i(x) \) are continuous coefficients to be determined. This is then used to generate the discrete LMMs of the form (3) and derivative formulas of the form (4) and other additional methods. These equations are then applied simultaneously to solve (1) above.

### 2.1. Specification of the Methods [28]

In this section, we specify the derived method.

Consider the case \( k = 2 \) with the specification \( a = 2 \) and \( b = 5 \) using (5) and (6) we have the following polynomials of degree \( a+b-1 \):

\[
Y(x) = \sum_{i=0}^{6} \beta_i P_i(x)
\]

(12)

\[
Y''(x) = \sum_{i=0}^{6} \beta_i P_i''(x)
\]

(13)

which will yield the following vectors and \( 7 \times 7 \) collocation/interpolation matrix:

\[
P = \begin{bmatrix} 1, x, x^2, x^3, x^4, x^5, x^6 \end{bmatrix}
\]

(14)

\[
V = \begin{bmatrix} y_0, y_1, f_0, f_1, f_2, f_3, f_4 \end{bmatrix}
\]

(15)
\[
M = \begin{pmatrix}
1 & x_0 & x_0^2 & x_0^3 & x_0^4 & x_0^5 & x_0^6 \\
1 & x_1 & x_1^2 & x_1^3 & x_1^4 & x_1^5 & x_1^6 \\
0 & 0 & 2 & 6x_0 & 12x_0^2 & 20x_0^3 & 30x_0^4 \\
0 & 0 & 2 & 6x_1 & 12x_1^2 & 20x_1^3 & 30x_1^4 \\
0 & 0 & 2 & 6x_2 & 12x_2^2 & 20x_2^3 & 30x_2^4 \\
0 & 0 & 2 & 6x_3 & 12x_3^2 & 20x_3^3 & 30x_3^4
\end{pmatrix}
\]

These are then substituted into the equation below:

\[
Y(x) = V^T \left( M^{-1} \right) P(x)
\]  \hspace{1cm} (17)

which results into a continuous LMM

\[
Y(x) = \frac{(h-x+x_0)y_0 + (x-x_0)y_1 - f_0}{h} \frac{y_0}{360h^4} (x-x_0)(h-x+x_0) \\
\frac{53h^4 + 123h^2(x-x_0)^2 - 52h(x-x_0)^3}{h} \\
8h(x-x_0)^4 + 127h^3(-x+x_0) \\
- \frac{f_2}{45h^4} \left(18h^5(x-x_0) - 60h^3(x-x_0)^2 + 65h^2(x-x_0)^3\right) \\
-27h(x-x_0)^5 + 4(x-x_0)^6 \\
+ \frac{f_1}{60h^3} \left(5h^2(x-x_0) - 60h^3(x-x_0)^2 + 95h^2(x-x_0)^3\right) \\
-48h(x-x_0)^5 + 8(x-x_0)^6 \\
- \frac{f_2}{45h^4} \left(2h^5(x-x_0) - 20h^3(x-x_0)^2 + 35h^2(x-x_0)^3\right) \\
-21h(x-x_0)^5 + 4(x-x_0)^6 \\
+ \frac{f_2}{360h^4} \left(3h^3(x-x_0) - 30h^3(x-x_0)^2 + 55h^2(x-x_0)^3\right) \\
+ 36h(x-x_0)^5 + 8(x-x_0)^6
\]\n
\hspace{1cm} (18)

The main method (19) is then obtained by evaluating (18) at \(x_{n+2}\):

\[
y_{n+2} - 2y_{n+1} + y_n = \frac{h^2}{60} \left[ f_n + 26f_{n+1} + f_{n+2} + 16 \left(f_{n+2} + f_{n+3}\right) \right]
\]  \hspace{1cm} (19)

which is used together with the following initial methods:

\[
y_{\frac{1}{2}} - \frac{1}{2} y_0 - \frac{1}{2} y_1 = \frac{h^2}{1920} \left[ -19f_0 - 14f_1 + f_2 - 204f_4 - 204f_5 \right]
\]  \hspace{1cm} (20)
\[ y_{1} + \frac{1}{2} y_{0} - \frac{1}{2} y_{1} = \frac{h^{2}}{1920} \left[ 17 f_{0} + 402 f_{1} - 3 f_{2} + 252 f_{4} + 52 f_{6} \right] \]  
(21)

\[ y_{2} + \frac{1}{2} y_{0} - \frac{1}{2} y_{1} = \frac{h^{2}}{1920} \left[ 17 f_{0} + 402 f_{1} - 3 f_{2} + 252 f_{4} + 52 f_{6} \right] \]  
(22)

and with the following derivative formulas:

\[ h y_{1} + y_{0} - y_{1} = h^{2} \left[ \frac{13 f_{0}}{480} - \frac{f_{1}}{10} - \frac{7 f_{2}}{1440} + \frac{7 f_{4}}{144} + \frac{7 f_{6}}{240} \right] \]  
(23)

\[ h y_{2} + y_{0} - y_{1} = h^{2} \left[ \frac{f_{0}}{72} + \frac{13 f_{1}}{60} + \frac{f_{2}}{360} + \frac{13 f_{4}}{45} - \frac{f_{6}}{45} \right] \]  
(24)

\[ h y_{3} + y_{0} - y_{1} = h^{2} \left[ \frac{31 f_{0}}{1440} + \frac{8 f_{1}}{15} + \frac{f_{2}}{90} - \frac{19 f_{4}}{80} - \frac{157 f_{6}}{720} \right] \]  
(25)

\[ h y_{n+2} + y_{n} - y_{n+1} = h^{2} \left[ \frac{f_{n}}{120} + \frac{7 f_{n+1}}{20} + \frac{59 f_{n+2}}{360} + \frac{14 f_{n+4}}{45} + \frac{2 f_{n+6}}{3} \right] \]  
(26)

3. Numerical Examples

In this section, we apply the main method and additional method derived in the previous section to two (2) second order initial value problems. The obtained results are compared with their exact solutions and also with results from [28]. These are shown in the graphs (Figure 1 and 2) and Table 1 and 2.

**Problem 3.1: Consider the linear second order IVP [28]:**

\[ \frac{d^{2}y}{dx^{2}} - 4 \frac{dy}{dx} + 8 y = x^{3}, \quad x \in (0, 1) \]

with initial conditions:

\[ y(0) = 2, \quad y'(0) = 4 \]

with exact solution:

\[ y(x) = e^{2x} \left[ 2 \cos 2x - \frac{3}{64} \sin 2x \right] + \frac{3}{32} x + \frac{3}{16} x^{2} + \frac{1}{8} x^{3} \]
Figure 1: Solution of Problem 3.1 computed with HyBVM ($k=2$, $h=0.03$)

Table 1: Absolute errors for Problem 3.1 ($h=0.1$)

| x    | SSM($k=4$) | BVM($k=4$) | BVM($k=5$) | HyBVM($k=2$) |
|------|------------|------------|------------|--------------|
| 0.0  | 0.00000    | 0.00000    | 0.00000    | 0.00000      |
| 0.1  | 5.11×10^{-6} | 6.13×10^{-7} | 8.14×10^{-8} | 7.14×10^{-8} |
| 0.2  | 1.50×10^{-5} | 1.85×10^{-6} | 2.44×10^{-7} | 1.75×10^{-7} |
| 0.3  | 2.79×10^{-5} | 3.42×10^{-6} | 4.55×10^{-7} | 2.94×10^{-7} |
| 0.4  | 4.29×10^{-5} | 5.55×10^{-6} | 7.29×10^{-7} | 4.02×10^{-7} |
| 0.5  | 6.70×10^{-5} | 8.39×10^{-6} | 1.06×10^{-6} | 4.72×10^{-7} |
| 0.6  | 1.03×10^{-4} | 1.23×10^{-5} | 1.45×10^{-6} | 4.66×10^{-7} |
| 0.7  | 1.45×10^{-4} | 1.74×10^{-5} | 1.93×10^{-6} | 3.36×10^{-7} |
| 0.8  | 1.91×10^{-4} | 2.35×10^{-5} | 2.47×10^{-6} | 2.24×10^{-8} |
| 0.9  | 2.40×10^{-4} | 3.09×10^{-5} | 3.08×10^{-6} | 5.39×10^{-7} |
| 1.0  | 2.95×10^{-4} | 3.86×10^{-5} | 4.06×10^{-6} | 1.42×10^{-6} |

Problem 3.2: Consider the nonlinear second order IVP [28]:

$$\frac{d^2 y}{dx^2} - x \left( \frac{dy}{dx} \right)^2 = 0, \quad x \in (0, 1)$$
with initial conditions:
\[ y(0) = 1, \quad y'(0) = \frac{1}{2} \]

with exact solution:
\[ y(x) = 1 + \frac{1}{2} \ln \left[ \frac{2 + x}{2 - x} \right] \]

**Figure 2:** Solution of Problem 3.2 computed with HyBVM \((k=2, h = 0.05)\)

| \( x \)   | SSM\((k=4)\)  | BVM\((k=4)\)  | BVM\((k=4)\)  | HyBVM\((k=2)\)  |
|----------|----------------|----------------|----------------|------------------|
| 0.1      | \( 7.51 \times 10^{-9} \) | \( 6.67 \times 10^{-9} \) | \( 1.03 \times 10^{-9} \) | \( 1.18 \times 10^{-10} \) |
| 0.2      | \( 1.80 \times 10^{-8} \) | \( 1.60 \times 10^{-8} \) | \( 2.51 \times 10^{-9} \) | \( 2.37 \times 10^{-10} \) |
| 0.3      | \( 2.88 \times 10^{-8} \) | \( 2.55 \times 10^{-8} \) | \( 3.97 \times 10^{-9} \) | \( 3.56 \times 10^{-10} \) |
| 0.4      | \( 3.65 \times 10^{-8} \) | \( 3.23 \times 10^{-8} \) | \( 5.71 \times 10^{-9} \) | \( 4.70 \times 10^{-10} \) |
| 0.5      | \( 7.05 \times 10^{-8} \) | \( 4.45 \times 10^{-8} \) | \( 7.62 \times 10^{-9} \) | \( 5.77 \times 10^{-10} \) |
| 0.6      | \( 1.20 \times 10^{-7} \) | \( 4.82 \times 10^{-8} \) | \( 9.31 \times 10^{-9} \) | \( 6.64 \times 10^{-10} \) |
| 0.7      | \( 1.73 \times 10^{-7} \) | \( 6.27 \times 10^{-8} \) | \( 1.27 \times 10^{-8} \) | \( 7.11 \times 10^{-10} \) |
| 0.8      | \( 2.14 \times 10^{-7} \) | \( 6.09 \times 10^{-8} \) | \( 1.26 \times 10^{-8} \) | \( 6.73 \times 10^{-10} \) |
| 0.9      | \( 5.82 \times 10^{-7} \) | \( 7.42 \times 10^{-8} \) | \( 2.00 \times 10^{-8} \) | \( 4.51 \times 10^{-10} \) |
| 1.0      | \( 1.15 \times 10^{-6} \) | \( 5.55 \times 10^{-8} \) | \( 3.46 \times 10^{-8} \) | \( 1.73 \times 10^{-10} \) |
4. Discussion of Result

In this work, a new method referred to as HyBVM has been applied to two second-order Initial Value Problems (linear and nonlinear). Figure 1 and Figure 2 show the comparison between the approximate solutions obtained from the two cases and their exact solutions.

Tables I and II also show the absolute errors from Boundary Value Method (BVM \( k = 4, 5 \)) and Self Starting Method (SSM) obtained in [28] and also the absolute error from our proposed method for problems 3.1 and 3.2, respectively. It was observed that the HyBVM performed better than these other methods for the two cases.

5. Conclusion

In this paper, we have extended the Numerov method by applying and implementing them as Boundary Value Method (BVM). This was achieved by constructing the Numerov method via collocation and interpolation procedure while utilizing data at off-step point. We call the new method: Hybrid BVM (HyBVM). This new scheme was applied to two second order initial value problems, and the numerical tests confirmed that it of high accuracy when compared to the one in literature.

Acknowledgments

The authors are thankful to Covenant University for sponsoring this research.

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