Two-loop Euler-Heisenberg effective actions from charged open strings

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Abstract

We present the multiloop partition function of open bosonic string theory in the presence of a constant gauge field strength, and discuss its low-energy limit. The result is written in terms of twisted determinants and differentials on higher-genus Riemann surfaces, for which we provide an explicit representation in the Schottky parametrization. In the field theory limit, we recover from the string formula the two-loop Euler-Heisenberg effective action for adjoint scalars minimally coupled to the background gauge field.

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1 Introduction

The dynamics of charged particles in a constant electromagnetic background has been a focus of considerable theoretical interest since the early days of quantum field theory. To recall just two important contributions, Euler and Heisenberg [1] computed the one-loop QED effective action by integrating out fermion fields, and later Schwinger [2] derived the probability of pair creation in a constant electric field, focusing on the absorptive part of the one-loop calculation (see [3] for a recent review and a detailed list of references). A similar problem can be studied in string theory, and also in this context it has provided several important insights. Open strings, in particular, represent a natural generalization of charged particles since they couple, through their endpoints, to a gauge field. Bachas and Porrati [4] generalized Schwinger’s computation to the case of open (super)strings, and later the same technique was applied to study the T-dual case of moving D-branes [5] and to a finite temperature environment [6].

In this paper we will focus on the case of open bosonic strings and we will study the multiloop partition function in the presence of a constant Yang-Mills field strength $F$. We will then use the string formula in the low-energy limit to recover the Euler-Heisenberg effective action at one and two loops, considering specifically the coupling of the gauge field to adjoint scalars. Even for the bosonic string, the explicit formulation of this partition function requires some new input, beyond the known results of multiloop perturbative string theory. From the mathematical point of view, the open string diagram is represented as usual by a Riemann surface with $g + 1$ boundaries, however the presence of an external field $F$ introduces twisted boundary conditions along some of the boundaries. As a consequence, the basic geometric building blocks of the string amplitude, such as the determinant of the Laplace operator on the Riemann surface, are deformed by $F$. The necessary ingredients to derive the multiloop partition function for charged open strings were assembled in [7, 8], developing earlier studies [9, 10]. We note also that twisted boundary conditions in the open string channel correspond to cuts along homology cycles for closed strings. This suggests that the present formalism might have broader applications: the appearance of cuts on the Riemann surface representing a string amplitude, in fact, is a generic feature of the Ramond-Neveu-Schwarz formalism. The Ramond sector of the superstring, for instance, has a square-root cut for fermion fields, while closed strings on orbifolds have $n$-fold cuts in the twisted sector. Thus it is not surprising to
see the same mathematical objects appearing in different contexts (compare, for instance, [8, 11, 12]).

Even if the mathematical formulation of the string effective action is written in the language of two-dimensional Riemannian geometry, its physical content is very close to the quantum field theory counterpart. In fact it is expected, and we will explicitly show, that the two results should be precisely connected in the field theory limit, when the typical length of the open string $\sqrt{\alpha'}$ is sent to zero.

The idea to construct a precise mapping between the low energy behaviour of string theory and the corresponding field theory Feynman diagrams is very old [13], and was applied to effective actions in [14]; it received renewed interest when it was noted [15] that the organization of tree-level gluon amplitudes suggested by string theory is an efficient computational tool for high energy processes in QCD. Subsequently, the methods to perform the field theory limit of string amplitudes at loop level were pioneered by Bern and Kosower [16], who studied one-loop amplitudes in gauge theories. String-inspired techniques were then used to obtain novel results, relevant to collider phenomenology [17, 18]. Later, the correspondence between string diagrams and ordinary Feynman diagrams was made more precise, and it was shown that one-loop Yang-Mills amplitudes could be recovered from open bosonic string theory on a diagram by diagram basis [19], and even completely off-shell [20]. While at one loop these techniques have been very successful for Yang-Mills theory, and have also been applied to gravity [21, 22], extension to two loops for gauge bosons has proven difficult [23, 24]. It has however been shown that one can tune the field theory limit of bosonic strings to reproduce Feynman diagrams in scalar theories, at one and two loops [25], and with both cubic and quartic vertices [26, 27].

In the second part of this paper we will make use some of these techniques to study the low energy regime of the charged string partition function at one and two loops. We will take the simplest field theory limit [26, 27], isolating the contribution of the string ground state (the tachyon). As expected, the Euler-Heisenberg effective action will arise from the string formula in the scaling limit in which the dimensionful physical gauge field is kept constant as $\alpha' \to 0$.

The structure of the paper is the following. In Section 2 we begin by recalling the computation of vacuum diagrams for ordinary bosonic strings, and then move on to study the modifications due to the presence of a constant gauge field strength. The crucial point is that when $F$ is constant the
world-sheet equations of motion of the string are unchanged and only the boundary conditions are modified. A powerful method to take into account these twisted boundary conditions is the boundary state formalism, which describes *closed* strings propagating between the various boundaries. To get our final result, we must then perform the modular transformation exchanging $a$ and $b$ homology cycles, leading to an expression for the amplitude in the open string channel, as described pictorially in Fig. 1. In order to be self-contained and to keep track of all normalizations, we give in Section 3 the derivation of the two-loop Euler-Heisenberg effective action in the appropriate field theory. Finally, in Section 4 we perform the field theory limit described in [26, 27] on the $g = 1, 2$ string partition function, and we show that it precisely reproduces the results of Section 3. The final Section discusses further applications of our results and some possible developments.
2 Higher-genus effective actions for bosonic strings

2.1 Partition functions without external fields

The partition function of closed bosonic string theory at 2 and 3 loops was first derived in [28, 29], using previous mathematical results on modular forms. This derivation, however, is not directly generalizable beyond genus \( g = 3 \). In fact, only for \( g \leq 3 \) one can choose to parametrize the moduli space of inequivalent Riemann surfaces by using the elements of the period matrix \( \tau_{\mu\nu} \), with \( \mu \leq \nu \), as independent parameters. An alternative derivation of the bosonic partition function was presented in [30], where the operator approach was used. The final expression is automatically written in the Schottky parametrization of the Riemann surface, so that it is valid for any \( g \), in a framework which however has the drawback of blurring the modular properties of the results.

As usual, one can derive the open string partition function from the closed string result by requiring that the period matrix be compatible with the involution [31, 32] that squeezes a closed genus-\( g \) surface into an open one. In our case we can obtain the disk with \( g \) holes by restricting the period matrix to be purely imaginary. In the Schottky parametrization the genus-\( g \) partition function for the open bosonic string, with Neumann boundary conditions, is given by

\[
Z(g) = \int \frac{1}{dV_{abc}} \prod_{\mu=1}^{g} dk_{\mu}d\eta_{\mu}d\xi_{\mu} \frac{(1 - k_{\mu})^2}{k_{\mu}^2(\xi_{\mu} - \eta_{\mu})^2} \left[ \det (\textnormal{Im} \tau) \right]^{-\frac{d}{2}} \times \prod_{\alpha} \left[ \frac{\prod_{n=2}^{\infty}(1 - k_{\alpha}^n)^2}{\prod_{n=1}^{\infty}(1 - k_{\alpha}^n)^{d}} \right].
\]

(2.1)

Here \( d \) is the space-time dimension, \( d = 26 \) for the ordinary bosonic string. Moduli space is parametrized by the multipliers \( k_{\mu} \) and by the fixed points \( \xi_{\mu} \) and \( \eta_{\mu} \) of the \( g \) projective transformations forming the basis for the Schottky group at genus \( g \). We refer to the Appendices of [7] for a short explanation of the Schottky parametrization and for all the conventions and notations we use in this regard. Here we only note that the primed product in (2.1) is over primary classes of the Schottky group, characterized by elements with multipliers \( k_{\alpha} \). The factor \( dV_{abc} \) accounts for the volume of the projective
group which leaves the measure invariant, corresponding to the freedom to fix arbitrarily three fixed points. At two loops, we will take advantage of this freedom by setting \( \eta = 0, \xi_1 = \infty \) and \( \xi_2 = 1 \).

Eq. (2.1) can be compared, for \( g = 2 \), with the results of [28, 29],

\[
Z(2) = \int d\tau_{11}d\tau_{12}d\tau_{22} [\det (\Im \tau)]^{-d/2} \frac{1}{\pi^{12} \chi_{10}(\tau)} ,
\]

(2.2)

where \( \chi_{10} \) is the unique modular form of weight 10 with no zeros, and is equal to the product of the squares of the ten even \( \theta \) functions at genus \( g = 2 \), \( \chi_{10} = \prod_{m \text{ even}} \theta_m^2(\tau) \). Note that the three moduli appearing in (2.1) for \( g = 2 \) (\( k_1, k_2 \) and \( \eta \equiv \eta_2 \)) are related to the elements of the period matrix. In particular, in the limit where the Riemann surface degenerates into a graph (\( k_i \to 0 \)), we have

\[
2\pi i \tau_{11} = \log k_1 + O(k) , \quad 2\pi i \tau_{22} = \log k_2 + O(k) , \quad 2\pi i \tau_{12} = \log \eta + O(k) .
\]

(2.3)

The two expressions (2.1) and (2.2) were shown to agree perturbatively (when expanded for small \( k_i \)) in [33,34]. In order to see that they are exactly equivalent, one can check that both of them follow from the same ‘first principles’ formula [35],

\[
Z(g) = \int \prod_{a=1}^{3g-3} dm_a \frac{\det \langle \mu_j^{(a)} | \phi_k \rangle}{\sqrt{\det \langle \phi_j | \phi_k \rangle}} \det'(\partial^\dagger \partial) [\det (\Im \tau)]^{-d/2} Z_1^{-d} ,
\]

(2.4)

which can be derived directly from Polyakov path integral. Here \( \mu_j^{(a)} \) is a system of Beltrami differentials related to the moduli \( m_a \); \( \partial \) is the operator appearing in the ghost Lagrangian and acting on a \((b, c)\) system of weight \((2, -1)\); \( \phi_j \) is a basis of periodic and holomorphic differentials of weight 2; finally, \( Z_1 \) is related to the partition function of a single chiral boson, for which one can find explicit expression in Eq. (7.3) of [36], or one can use the Schottky parametrization, where \( Z_1 = \prod_{a} (1 - k_a^2) \). To prove the equivalence of (2.1) and (2.2), the basic idea is to make two different choices for the moduli \( m_a \), and thus for the bases of Beltrami and quadratic differentials in (2.4), and show that they give rise to (2.2) and (2.1) respectively. The two results must then be equal, since (2.4) does not depend on the particular choice made for the moduli or for the differentials. A few steps of this proof are summarized in Appendix A.
So far, we have described the partition function by using the open string point of view, where the world-sheet looks like a disk with \( g \) holes. In this case the moduli \( \{ \tau_{\mu \nu} \} \) (or \( \{ k_i, \eta \} \)) are directly related to the lengths of the various strips representing the open strings propagating in the string diagram. An alternative approach is to adopt a closed string description for the same amplitude. In this picture the world-sheet has the shape of a disk glued to \( g \) cylinders, whose boundaries are described by boundary states (for a review see [37, 38]). In this channel the partition functions reads \([41]\)

\[
Z^c(g) = \int dV_{abc} \prod_{\mu=1}^g \left[ \frac{dq_\mu}{q_\mu^2} \frac{d^2 \eta_\mu^c (1 - q_\mu)^2}{(\eta_\mu^c - \bar{\eta}_\mu^c)^2} \right] \prod_{n=1}^{\infty} \left( \frac{1 - q_\alpha^n}{1 - q_\alpha^n} \right)^{-d} \prod_{n=2}^{\infty} \left( 1 - q_\alpha^n \right)^2,
\]

where \( dV_{abc} \) again signals that we have to fix three real variables among the \( \eta \)'s. The superscript \( c \) is a reminder of the fact that the parameters appearing in (2.5) are appropriate for describing closed string exchanges among the various boundaries. Notice the absence of factors of \( \det (\text{Im} \, \tau) \) in this formulation.

At the level of the Schottky parametrization the modular transformation connecting the closed string (2.5) and the open string (2.1) channels is rather complicated. In fact, the relation between the multipliers \( q \) and \( k \) is non-analytic, since \( 2\pi i \tau_{11}^c \sim \log q_1 \) and \( 2\pi i \tau_{11} \sim \log k_1 \), while the open and closed string period matrices are connected by means of the usual modular transformation \( \tau^c = -\tau^{-1} \). In order to transform (2.5) into (2.1), one must first rewrite the integrand in terms of geometrical objects with simple modular properties, such as \( \theta \)-functions. Then it is possible to perform the modular transformation by using the known transformation properties of these functions, as done in [7, 8]. On the other hand, of course, the modular transformation \( \tau^c = -\tau^{-1} \) can be directly performed on Eq. (2.2): the result is again the same expression, now written as a function of \( \tau^c \), but again without any factor of \( \det (\text{Im} \, \tau) \).

### 2.2 Open strings in a constant background field

The results reviewed in the previous section are appropriate for open strings with Neumann boundary conditions along all boundaries. We will now outline the derivation of the partition function for charged open strings, \( i.e. \) open...
strings with mixed boundary conditions

\[
\left[ \partial_\sigma X^i + i \partial_\tau X^j F_j^{i(A)} \right]_{\sigma=0} = 0 ,
\]  

(2.6)

where \( F \) is a constant gauge field strength and \( A = 0, \ldots, g \) labels the boundary on which the boundary condition (2.6) is imposed (\( A = 0 \) being the external boundary). \( F \) can always be put in a block-diagonal form, so for the sake of simplicity we will take the space-time indices to be in the plane \( i, j = 1, 2 \). For charged strings at least one of the differences \( F^{(\mu)} - F^{(0)} \), for \( \mu = 1, \ldots, g \), is nonvanishing.

A direct computation of the charged partition function in the open string channel is difficult, mainly because the string coordinates have a non-integer mode expansion. It is possible to sew with a propagator two charged states of the 3-string vertex, but the result [42] is rather complicated and it is difficult to proceed and build multiloop diagrams. Here we will recall the derivation presented in [8], where an alternative approach was followed. The idea is to compute the string diagram in the closed string channel by using boundary states satisfying

\[
(\partial_\tau X^1 + i \tan(\pi \epsilon^A) \partial_\sigma X^2)_{\tau=0} |B\rangle_{F_A} = 0 ,
\]

(\partial_\tau X^2 - i \tan(\pi \epsilon^A) \partial_\sigma X^1)_{\tau=0} |B\rangle_{F_A} = 0 .

(2.7)

These boundary conditions are just a rewriting of those in Eq. (2.6) after the exchange \( \tau \leftrightarrow -\sigma \) and with the convention \( F_{12}^{(A)} = -F_{21}^{(A)} = \tan(\pi \epsilon^A) \). With the change of coordinates \( X^\pm = \frac{1}{\sqrt{2}} (X^1 \pm iX^2) \), the constraints (2.7) become diagonal and the computation of vacuum diagrams is almost identical to that of [41]. The only difference is that the matrices \( S^A \) appearing there\(^\dagger\) in the boundary state contain, in the plane \( \{X^1, X^2\} \), some \( \epsilon \)-dependent phases, instead of having all elements equal to \( \pm 1 \). To be specific, we take the external boundary to have Neumann boundary conditions \( (F^{(0)} = 0) \), so that \( \tilde{\epsilon} \) is a vector with \( g \) components, denoted by \( \epsilon_\mu \), encoding the values of the gauge field on the remaining \( g \) boundaries. The explicit form of the matrices \( S_\mu \) appearing in the boundary state is then \( S_\mu = \text{diag}\{e^{2\pi i \epsilon_\mu}, e^{-2\pi i \epsilon_\mu}\} \). It is not difficult to follow these phases through the computation. One verifies that their effect is simply to modify the contribution of the oscillators in the

\(^\dagger\)These matrices are denoted by \( S \) in Ref. [41]; here we label them \( S \) to distinguish them from the Schottky group generators introduced below.
charged plane \( \{X^1, X^2\} \) to the partition function. The result is of the form

\[
Z_F^c(g) = \left( \prod_{\mu=1}^g \frac{1}{\cos \pi \epsilon_\mu} \right) \int [dZ]_g^c \ R_g(q_\alpha, \vec{\epsilon}) ,
\]

where \([dZ]_g^c\) is the integrand in Eq. (2.5), representing the \( F = 0 \) result, while the \( \vec{\epsilon} \) dependence is encoded in the factor

\[
R_g(q_\alpha, \vec{\epsilon}) = \frac{\prod_{\alpha} \prod_{n=1}^{\infty} (1 - q_\alpha^n)^2}{\prod_{\alpha} \prod_{n=1}^{\infty} \left( 1 - e^{-2\pi i \vec{\epsilon} \cdot \vec{N}_\alpha q_\alpha^n} \right) \left( 1 - e^{2\pi i \vec{\epsilon} \cdot \vec{N}_\alpha q_\alpha^n} \right)} .
\]

Here \( \vec{N}_\alpha \) is a vector with \( g \) integer entries: the \( \mu \)th entry counts how many times the Schottky generator \( S_\mu \) enters in the element of the Schottky group \( T_\alpha \), whose multiplier is \( q_\alpha \) (for example \( S_\mu \) contributes 1, while \( (S_\mu)^{-1} \) contributes \(-1\)). The factors of \( 1/\cos(\pi \epsilon) \) in Eq. (2.8), finally, are nothing but a rewriting of the Born-Infeld contribution to the boundary state normalization (see for instance [38]). For \( g = 1 \) Eq. (2.8) agrees with the results of [39, 40], as one can see by using \( \zeta \)-function regularization to rewrite \( 1/\cos(\pi \epsilon) \) as \( \prod_{n=1}^{\infty} (1 + F^2)^{-1} \).

Now we would like to perform the modular transformation \( \tau^c = -\tau^{-1} \) on (2.8), in order to write the effective action in the presence of a nonvanishing \( F \) in the open string channel. We already know from the previous section that \([dZ]_g^c\) transforms into the integrand of Eq. (2.1), which we denote by \([dZ]_g\). Thus we only need to transform the factor \( R_g(q_\alpha, \vec{\epsilon}) \), which contains the dependence on the external field \( F \), and to write it in terms of the open strings variables \( k_\mu, \eta_\mu \) and \( \xi_\mu \). To do this, we follow the same approach discussed in the previous section. First, we rewrite the products over the Schottky group in terms of geometrical objects with simple transformation properties under the modular group, like \( \theta \) functions, differentials and the prime form; then, we perform the modular transformation; as a last step, we go back to the Schottky parametrization, which is the most appropriate for performing the low energy limit. The technical tool needed in this derivation is the higher-genus generalization of the Jacobi formulae expressing \( \theta \) functions as products. These formulae can be derived by exploiting bosonization identities in two dimensions, as done in [43–45]. Details are given in [7, 8], where the presence of the twists \( \vec{\epsilon} \) is also taken into account. The results of Refs. [7, 8] can be written as

\[
R_g(q_\alpha, \vec{\epsilon}) = (e^{2\pi i \epsilon_\alpha} - 1) \ R_g(k_\alpha, \vec{\epsilon} \cdot \tau) \ e^{-i\pi \vec{\epsilon} \cdot \tau \cdot \vec{\epsilon}} \frac{\det (\tau)}{\det (\tau \vec{\epsilon})} ,
\]

(2.10)
where we have assumed $\epsilon_g \neq 0$. The only new object appearing in Eq. \((2.10)\) is the matrix $\tau_\eps$. As suggested by the notation, it is an $\epsilon$-dependent generalization of the period matrix. Recall that the matrix elements of $\tau$ are the periods along the $b$ cycles of the normalized Abelian differentials

$$\frac{1}{2\pi i} \int_{b_\nu} \omega_\mu = \tau_{\nu\mu} , \quad \frac{1}{2\pi i} \int_{a_\nu} \omega_\mu = \delta_{\nu\mu} . \quad (2.11)$$

Similarly, $\tau_\eps$ can be expressed in terms of the periods of twisted meromorphic differentials (known as Prym differentials). In our case the twists are along the $a$ cycles and are fixed by $\epsilon$, so they depend on the external gauge field. An explicit expression for the matrix $\tau_\eps$ was derived in \([7, 8]\). Begin by defining $g \epsilon$-dependent differentials, as

$$\zeta_\mu^\eps(z) = \sum_\alpha^{(\mu)} e^{2\pi i (\epsilon \cdot N_\alpha + \epsilon_\mu)} \left[ \frac{1}{z - T_\alpha(\eta_\mu)} - \frac{1}{z - T_\alpha(\xi_\mu)} \right]$$

$$+ (1 - e^{2\pi i \epsilon_\mu}) \sum_\alpha e^{2\pi i \epsilon \cdot N_\alpha} \left[ \frac{1}{z - T_\alpha(z_0)} - \frac{1}{z - T_\alpha(a_\mu^\alpha)} \right] . \quad (2.12)$$

The first sum runs over all elements of the Schottky group, except those whose rightmost generator is $S_\nu^\pm 1$, while the second sum is unrestricted; furthermore, in the second line, $a_\mu^\alpha = \eta_\mu$ if $T_\alpha$ is of the form $T_\alpha = T_\beta S_\nu^l \eta_\mu$, with $l \geq 1$, while $a_\mu^\alpha = \xi_\mu$ otherwise. These differentials are characterized by the following features: first, they are twisted along the $b$ cycles, i.e. they obey $\zeta_\mu^\eps(S_\nu(z)) dS_\nu(z) = \exp(2\pi i \epsilon_\nu) \zeta_\mu^\eps(z) dz$; next, they are holomorphic everywhere except at the arbitrarily chosen point $z = z_0$, where they have a single pole with residue $(1 - e^{2\pi i \epsilon_\mu})$; finally, they reduce, in the $\epsilon \to 0$ limit, to the usual Abelian differentials normalized as in \((2.11)\). There are $g$ independent differentials satisfying these requirements in agreement with Riemann-Roch theorem. The matrix $\tau_\eps$ is defined as the usual period matrix, with the abelian differentials substituted by Prym differentials, where however the twist is placed along the $a$ cycles \([8]\), in order to take into account the modular transformation to the open string configuration. Explicitly,

$$(\tau_\eps)_{\nu\mu} = \frac{1}{2\pi i} \int_{S_\nu(u)} dz \left[ \zeta_\mu^\eps(z) e^{2\pi i \epsilon \cdot \Delta z} \right] ; \quad (\nu \neq g) ; \quad (\tau_\eps)_{g\mu} = e^{2\pi i (\epsilon \cdot \tau)_\mu} - 1 , \quad (2.13)$$
where \( \vec{\Delta}_z \) is the vector of Riemann constants (or Riemann class) whose definition in the Schottky parametrization is

\[
\Delta_z = \frac{1}{2\pi i} \left[ -\frac{1}{2} \log k_\mu + i\pi + \sum_{\nu=1}^{g} \sum_{\alpha} \left( \xi_\nu - T_\alpha(\eta_\mu) \right) \left( \frac{\xi_\nu - T_\alpha(\xi_\mu)}{\xi_\nu - T_\alpha(\eta_\mu)} \right) \right].
\]

(2.14)

Here the sum over \( T_\alpha \) excludes those with a power of \( S_\nu \) to the left and those with a power of \( S_\mu \) to the right; moreover, the identity is excluded for \( \mu = \nu \).

A few comments are now in order. First of all, notice that the matrix elements of \( \tau_\vec{\epsilon} \) actually do not depend on the base point \( w \), as one can easily check by taking a derivative of (2.13) with respect to \( w \) and using the periodicity of the integrand along the \( b \) cycles. Next, we remark that the asymmetry of the last line of \( \tau_\vec{\epsilon} \) is just apparent. The integrals of the twisted differentials along the \( b \) cycles, in fact, are not independent: a linear combination of them is fixed by the value of the residue at the pole, as one can see by integrating them along the cycle \( \prod_{\alpha} a_\mu b_\mu a_\mu^{-1} b_\mu^{-1} \). Using this linear dependence one gets

\[
(\tau_\vec{\epsilon})_{g\mu} = e^{-\frac{2\pi i}{g-1} \vec{\epsilon} \cdot \vec{\Delta}_z_0} \sum_{\nu=1}^{g} e^{2\pi i \epsilon_\nu} - \frac{1}{2\pi i} \int_{b_\nu} \left[ e^{\frac{2\pi i}{g-1} \vec{\epsilon} \cdot \vec{\Delta}_z} \right] \frac{d\zeta_\vec{\epsilon} \cdot \tau_\vec{\mu}(z) e^{2\pi i \epsilon_\nu}}{e^{\frac{2\pi i}{g-1} \vec{\epsilon} \cdot \vec{\Delta}_z}}. \quad (2.15)
\]

As a consequence, if desired, one could replace Eq. (2.10) with the more symmetric expression

\[
R_g(q_\alpha, \vec{\epsilon}) = R_g(k_\alpha, \vec{\epsilon} \cdot \tau) e^{-i\pi \vec{\epsilon} \cdot \left( \tau \vec{\epsilon} - \frac{1}{g-1} \vec{\Delta}_z_0 \right)} \frac{\det(\tau)}{\det(\tau_\vec{\epsilon})}, \quad (2.16)
\]

where

\[
(\tau_\vec{\epsilon})_{\nu\mu} = \frac{1}{2\pi i} \int_{w} d\zeta_\vec{\epsilon} \cdot \left[ e^{\frac{2\pi i}{g-1} \vec{\epsilon} \cdot \vec{\Delta}_z} \right] \frac{d\zeta_\vec{\epsilon} \cdot \tau_\vec{\mu}(z)}{e^{\frac{2\pi i}{g-1} \vec{\epsilon} \cdot \vec{\Delta}_z}}, \quad \mu, \nu = 1, \ldots, g. \quad (2.17)
\]

From the computational point of view, the formulation of Eq. (2.13) is somewhat easier to use, and we will adopt it in what follows. The final observation is that the dependence of \( \tau_\vec{\epsilon} \) on the position of the pole, \( z_0 \), disappears when one takes the determinant. In fact, as discussed in [7], it is possible to rewrite \( \det(\tau_\vec{\epsilon}) \) as the determinant of a \((g-1) \times (g-1)\) matrix containing linear combinations of \( \zeta_\vec{\mu} \) in which the pole has cancelled. These linear combinations can then be interpreted as holomorphic Prym differentials with periodicities.
fixed by $\vec{e}$. As the Riemann-Roch theorem dictates, there are only $g-1$ holomorphic Prym differentials. Using the formulation of Eq. (2.17), on the other hand, one finds that $\det(\tau_\tau)$ has a dependence on $z_0$ which cancels the explicit dependence arising in Eq. (2.16) through the Riemann class $\Delta_{z_0}$.

We are now in a position to write a completely explicit expression for the string effective action in the open string channel. It is

$$Z_F(g) = \left( e^{2\pi i g - \frac{1}{g}} \prod_{\mu=1}^{g} \cos \pi \epsilon_{\mu} \right) \int [dZ]_g \left[ e^{-i\pi \vec{e} \cdot \tau \cdot \hat{\epsilon}} \frac{\det(\tau)}{\det(\tau_\tau)} \mathcal{R}_g(k_\alpha, \vec{e} \cdot \tau) \right]. \quad (2.18)$$

This expression can be easily generalized to the case where the gauge field strength is non-trivial also in other space-time directions. It is sufficient to introduce a different $\vec{e}$ for each charged plane, and add in (2.18) other $\vec{e}$-dependent factors exactly like those in parenthesis. If the field is of electric type, one has to use an imaginary $\vec{e}$, or simply substitute $\vec{e} \rightarrow i \vec{e}$ in all the formulae presented in this section. Eq. (2.18) will be our starting point in taking the low-energy limit in Section 4.

3 Euler-Heisenberg effective action for scalar fields

3.1 Adjoint scalars in a constant background field

To make direct contact with the string formalism, let us consider a $U(N)$ gauge field $A_\mu$, represented as a hermitean $N \times N$ matrix $A_\mu = \sum_a A^a_\mu T_a$, where $T_a$ ($a = 0, \ldots, N^2 - 1$) are $U(N)$ generators satisfying

$$[T_a, T_b] = if_{abc} T^c; \quad \text{Tr} (T_a T_b) = \frac{1}{2} \delta_{ab}. \quad (3.1)$$

We will treat $A_\mu$ as a classical background gauge field coupled to a quantum massive scalar fields, also in the adjoint representation, $\varphi = \sum_a \varphi^a T_a$. The covariant derivative then acts on the matrix $\Phi$ as

$$D_\mu \Phi = \partial_\mu \Phi + i[\Phi, A_\mu], \quad (3.2)$$

where we absorbed a factor of the gauge coupling in the normalization of the field $A_\mu$. The lagrangian we will consider (with the normalizations of [26])
The gauge field configuration corresponding to a single charged brane in the string picture is a diagonal $A_\mu$ matrix, with all eigenvalues vanishing except one, which we take to be the last one, $(A_\mu)_{AB} = A_\mu \delta_{A,N} \delta_{B,N}$. In order to have a constant field strength $F_{\mu\nu}$ corresponding to a constant chromomagnetic field in the $x_3$ direction, we then pick $A_\mu = B x_1 g_{\mu2}$. This choice of background breaks the symmetry in color space, so that the matter “multiplet” will have both neutral and charged components with respect to $A_\mu$.

One can write
\[
\Phi(x) = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{2} \Pi(x) & \xi(x) \\ \xi^\dagger(x) & \sigma(x) \end{pmatrix},
\] (3.4)
where $\Pi$ is a hermitean $(N-1) \times (N-1)$ matrix representing a field in the adjoint representation of $U(N-1)$, $\xi$ is complex vector in the fundamental representation of $U(N-1)$, while $\sigma$ is a singlet real field. With the normalizations given by (3.4), all fields are canonically normalized and the lagrangian (3.3) becomes
\[
\mathcal{L} = \text{Tr} \left[ D_\mu \Phi D^\mu \Phi - m^2 \Phi^2 + \frac{2}{3} \lambda \Phi^3 \right].
\] (3.3)

The presence of the external field affects only the propagation of the charged field $\xi$, so the effective action in the background of $A_\mu$ can be determined with the ordinary Feynman rules, upon replacing the free $\xi$ propagator with the one computed in the chosen background. The form of the propagator of a scalar field in a constant electromagnetic background has been known for a long time [2, 46], and we give a derivation suitable for our purposes in Appendix B. The result for the coordinate-space propagator, expressed as a Schwinger parameter integral, is given by
\[
G_\xi(x, y) = \frac{1}{(4\pi)^{d/2}} e^{-\frac{i}{2}B(x_1+y_1)(x_2-y_2)} \int_0^\infty dt \frac{e^{-tm^2}}{t^{-d/2+1}} \frac{B}{\sinh(Bt)} \times (3.7)
\]
\[
\exp\left[\frac{1}{4t} \left( (x_0 - y_0)^2 - (x_\perp - y_\perp)^2 \right) - \frac{B}{4 \tanh(Bt)} \left( (x_1 - y_1)^2 - (x_2 - y_2)^2 \right) \right],
\]

where we denote by \( x_\perp \) the \((d - 3)\)-dimensional position vector orthogonal to the \((x_1, x_2)\) plane. Eq. (3.7) can be used directly to construct vacuum diagrams, as described below.

### 3.2 Vacuum diagrams

Vacuum diagrams contributing to the effective action for the field \( \xi \) can be computed in an intuitively appealing way using the coordinate-space representation of the charged propagator, Eq. (3.7). At one loop, for example, the only contributing diagram is a \( \xi \) loop without interaction vertices. Recalling that at one loop the vacuum diagram is not directly related to the effective action, but rather to its derivative with respect to the mass, one can simply write

\[
W^{(1)}_{\xi}(m, B) = - \int dm^2 \int d^d x G_{\xi}(x, x),
\]

(3.8)

where the overall factor of \( 1/2 \) related the square root arising from the semiclassical Gaussian integration has been cancelled by the fact that we are dealing with a complex field, involving two real degrees of freedom. Using Eq. (3.7), one immediately obtains the well known result [47]

\[
W^{(1)}_{\xi}(m, B) = V_d \frac{N - 1}{(4\pi)^{d/2}} \int_0^\infty dt e^{-tm^2} t^{-d/2} \frac{B}{\sinh(Bt)},
\]

(3.9)

where \( V_d \) is the volume of space-time and the factor \( N - 1 \) simply counts the components of \( \xi \) circulating in the loop. Notice that the integration with respect to the mass squared in (3.8) simply adds a factor of \( 1/t \) to the powers contained in the propagator (3.7). Notice also that here and below we will be concerned only with bare diagrams, and we will not discuss the inclusion of renormalization counterterms.

At two loops, the effective action is given directly by the sum of one-particle-irreducible vacuum diagrams. The nontrivial contribution in this case comes from diagrams involving a \( \xi \) loop, shown in Fig. (2). To compute these diagrams, we need the Feynman rules for the \( \xi - \Pi \) and the \( \xi - \sigma \) vertices, derived from the lagrangian in Eq. (3.5). Parametrizing the matrix field \( \Pi \) as \( \Pi = \sum_a \pi_a t^a \), with \( t_a \) the \( U(N - 1) \) generators in the fundamental representation, the \( \xi - \Pi \) vertex is simply given by \( i\lambda(t^a)_{BC} \). Similarly, the
Figure 2: Two-loop irreducible diagrams with charged propagators.

$\xi - \sigma$ vertex is given by $i\lambda\delta_{BC}$. Taking into account our normalization of group generators, and a symmetry factor equal to $1/2$, one finds that the first diagram in Fig. 2 contributes

$$W_{\xi\Pi}^{(2)}(m, B) = -\frac{\lambda^2(N-1)^2}{4} \int d^d x d^d y G_\xi(x, y) G_\xi(y, x) G_{\Pi}(x, y),$$

where $G_{\Pi}$ is the ordinary free scalar propagator. A short calculation gives

$$W_{\xi\Pi}^{(2)}(m, B) = -iV_d \frac{\lambda^2(N-1)^2}{4} \int_0^\infty dt_1 dt_2 dt_3 e^{-m^2(t_1 + t_2 + t_3)} \Delta_0^{-\frac{d}{2}+1} \Delta_B^{-1},$$

where we have defined

$$\Delta_B = \frac{1}{B^2} \sinh(B t_2) \sinh(B t_3) + \frac{t_1}{B} \sinh[B(t_2 + t_3)] ,$$

while $\Delta_0 = \lim_{B \to 0} \Delta_B = t_1 t_2 + t_1 t_3 + t_2 t_3$. Here we labelled by $t_2$ and $t_3$ the Schwinger parameters associated with the $\xi$ propagators. The second diagram in Fig. 2 is identical in form to the first one, with the same symmetry factor. The only change is the color factor, so that the resulting contribution to the effective action is identical to Eq. (3.11), with the replacement $(N-1)^2 \to N-1$.

Given our Feynman rules for vertices and propagators, it is of course straightforward to compute two-loop reducible vacuum diagrams as well. Although these diagrams do not contribute to the effective action, it is interesting to study them, since they can also be derived from string theory, as shown in Section 4. Specifically, we give here the results for the two diagrams depicted in Fig. 3, characterized by the fact they have just one charged propagator.
The Feynman rule for the \( \Pi^3 \) vertex with our conventions is just \( i\lambda d_{abc} \), with \( d_{abc} \) the symmetric \( U(N-1) \) structure constants, while the \( \sigma^3 \) vertex is \( \sqrt{2i}\lambda \). The calculation of the first diagram in Fig. (3) yields then

\[
W_{\xi \Pi}^{(2,\text{red})}(m, B) = -\frac{\lambda^2(N-1)^2}{2} \int d^4x d^4y G_\xi(x, x) G_\Pi(x, y) G_\Pi(y, y) = -i V_d \frac{\lambda^2}{(4\pi)^d} \frac{(N-1)^2}{2} \int_0^\infty dt_1 dt_2 dt_3 e^{-m^2(t_1+t_2+t_3)} \times (t_1 t_2)^{-d/2} \frac{Bt_2}{\sinh(Bt_2)},
\]

where use was made of the fact that the symmetry factor also in this case is 1/2. Again, the second diagram of Fig. (3) differs from Eq. (3.13) only in the substitution \( (N-1)^2 \to N-1 \). Similar results can be derived for the two reducible diagrams with two charged propagators.

4 The low-energy limit

We now wish to consider the low-energy limit of the string partition function in Eq. (2.18), at genus \( g = 1, 2 \), in order to isolate the contribution of charged scalars circulating in the loops. As we shall see, although for the bosonic string these scalars are tachyons, it is possible to recover exactly the expressions derived with field theory methods in Section 3, on a diagram by diagram basis and for arbitrary space-time dimension \( d \).

The basic idea of the field theory limit for a string amplitude or effective action is to trade the moduli describing the shape of the Riemann surface for dimensionful quantities, measuring the size of various sections of the string diagram in units of \( \alpha' \). The logarithms of the multipliers of Schottky transformations, for example, are associated with the length of the corresponding loops by setting \( \log(k_\mu) = -T_\mu/\alpha' \), where \( T_\mu \) is the sum of the Schwinger...
parameters associated with the propagators forming the $\mu$th loop. It is also straightforward to identify the contributions of states belonging to different mass levels of the string circulating in a given loop: the operator formalism, in fact, shows that each mass level corresponds to a given power of the multiplier in a Taylor expansion of the integrand for small $k_{\mu}$. For the bosonic string, this expansion starts with $k_{\mu}^{-2}$, a sign of the tachyonic instability. This singularity can however be readily regularized by recalling that the tachyon mass squared is $m^2 = -1/\alpha'$ and setting

$$\frac{dk_{\mu}}{k_{\mu}^2} = -\frac{1}{\alpha'} \exp \left( \frac{T_{\mu}}{\alpha'} \right) dT_{\mu} = -\frac{1}{\alpha'} \exp \left( -m^2 T_{\mu} \right) dT_{\mu}. \tag{4.1}$$

The external field is a source of further powers of $\alpha'$: in fact, the field $F_{12}^{(\mu)} = \tan(\pi \epsilon_{\mu})$ introduced in Section 2.2 is dimensionless, while we want to take the low energy limit keeping fixed the physical, dimensionful field $B$. Below, we will always concentrate on the case in which only one boundary is charged, setting $\epsilon_g \equiv \epsilon \neq 0$. The field theory limit is then defined by $\tan(\pi \epsilon) = 2\pi \alpha' B$, which implies $\epsilon = 2\alpha' B + O(\alpha'^3)$. Finally, one must introduce in Eq. (2.18) the appropriate overall factor, consistent with unitarity and containing the appropriate power of the string coupling. To this end, we follow the conventions of [19] and normalize the string diagrams with an overall constant $C_g$, given by (4.2). The string coupling $g_s$ must also be matched with the scalar self-coupling $\lambda$, which can be done by computing a simple tree-level amplitude from the Lagrangian (3.3) and comparing with the result obtained from string theory, as done in [26]. The results are

$$C_g = \frac{1}{(2\pi)^d} g_s^{2g-2} (2\alpha')^{-d/2}, \quad g_s = \frac{1}{4} \lambda (2\alpha')^{(6-d)/4}. \tag{4.2}$$

To illustrate the procedure, we begin by deriving from Eq. (2.18) the one-loop effective action, Eq. (3.9).

### 4.1 One loop

It is straightforward to write down an explicit expression for the partition function in Eq. (2.18) evaluated at one loop. In this case, the period matrix is just a number, simply related to the multiplier $k$ by $2\pi i \tau = \log k$. Next, observe that for $g = 1$ there are no holomorphic Prym differentials, so that the matrix $\tau_\epsilon$ is just the number given by the second expression in Eq. (2.13),

$$\tau_\epsilon = e^{2\pi i \epsilon} - 1 = k^\epsilon - 1. \tag{4.3}$$
The ratio $R$ defined in Eq. (2.9) also simplifies considerably, since at one loop there is only one primary class in the Schottky group (the one represented by the single generator $S(z)$, which can be taken to act as $S(z) = k z$ by fixing the overall projective invariance). One finds then

$$R_1(k, \epsilon \tau) = \prod_{n=1}^{\infty} \frac{(1 - k^n)^2}{(1 - k^{n-\epsilon})(1 - k^{n+\epsilon})}. \quad (4.4)$$

At one loop, in order to get the effective action, one must introduce an additional factor of $(-\log k)^{-1}$ in the integration measure, exactly as was done in the field theory computation. Notice that the factor of $1/2$ present in the definition of the 1-loop effective action cancels against the contributions related to the two possible orientations of the open strings. Putting together all the ingredients we find

$$Z_F(1) = i C_1 \frac{\tan (\pi \epsilon)}{\pi} \int_0^1 dk \frac{k^{\epsilon(1-\epsilon)/2}}{k^2 - 1} \left( -\frac{\log k}{2\pi} \right)^{\frac{d}{2}} \prod_{n=1}^{\infty} \frac{(1 - k^n)^{4-d}}{(1 - k^{n-\epsilon})(1 - k^{n+\epsilon})}, \quad (4.5)$$

which is the results of Ref. [4], for the magnetic case. Recovering Eq. (3.9) is now straightforward: one must change variables according to $k = \exp(-t/\alpha')$, as noted above; then, substituting Eqs. (4.1) and (4.2) and setting $\tan(\pi \epsilon) = 2\pi \alpha' B$, one observes that the overall power of $\alpha'$ cancels, a necessary condition for the field theory limit to be well defined. Expanding in powers of $k$ and retaining only the leading power (all subleading powers are now exponentially suppressed as $\alpha' \rightarrow 0$) one finally recovers Eq. (3.9), with the exact normalization factor, except for the ‘color’ factor $N-1$. This factor is easily understood in terms of the $D$-brane picture of the string calculation: as shown in Fig. (1), strings contributing to Eq. (2.18) stretch between a charged $D$-brane and $N - 1$ neutral $D$-branes, building up the fundamental representation of $U(N)$, broken to $U(N - 1)$ by the choice of background. At one loop, there is only one string, and thus $N - 1$ possible attachments to the neutral branes. Eq. (3.9) is thus reproduced exactly, including the correct dependence on the space-time dimension $d$.

### 4.2 Two loops

The field theory limit of string amplitudes appropriate to recover Feynman diagrams for adjoint scalars at two loops was studied in detail in Ref. [26,27].
Here we briefly summarize the general features of the method, and then focus on the application to the new quantity arising in the presence of an external field, the matrix $\tau_{\vec{\epsilon}}$.

At two loops, vacuum diagrams with cubic vertices have two possible topologies, depicted in Fig. (2) and in Fig. (3). As discussed in [26], and having fixed projective invariance as described in Section 2.1, a change of variables appropriate to isolate the corner of moduli space associated with the irreducible diagram in Fig. (2) is given by

$$k_1 = \exp\left(-\frac{t_1 + t_3}{\alpha'}\right), \quad k_2 = \exp\left(-\frac{t_2 + t_3}{\alpha'}\right), \quad \eta = \exp\left(-\frac{t_3}{\alpha'}\right), \quad (4.6)$$

where $t_i$ are the Schwinger parameters associated with the three propagators in the diagram. Referring to the open string diagram in Fig. (1), this choice of variables corresponds to the assignment of proper times $t_2$ and $t_3$ to the loop with a charged boundary, with $t_3$ associated with the propagator shared with the other loop. The integration region in moduli space is over all inequivalent surfaces. In the field theory limit it is determined [25] by requiring that the surface be non-singular, and by taking into account the symmetry under the exchange $k_1 \leftrightarrow k_2$ (part of the ‘residual modular group’ [48]). In terms of the field theory parameters introduced in (4.6) one finds simply $0 < t_3 < t_2 < t_1 < \infty$.

The reducible diagram in Fig. (3), on the other hand, arises from the other singular corner of moduli space, which can be parametrized by picking

$$k_1 = \exp\left(-\frac{t_1}{\alpha'}\right), \quad k_2 = \exp\left(-\frac{t_2}{\alpha'}\right), \quad \eta = 1 - \exp\left(-\frac{t_3}{\alpha'}\right), \quad (4.7)$$

with the integration region, in the field theory limit, given by $0 < t_2 < t_1 < \infty$ and $0 < t_3 < \infty$.

The expansion in powers of $k_\mu$ of quantities expressed in terms of infinite series or products over the Schottky group, such as the period matrix or the differentials $\zeta_{\vec{\epsilon}}$, is greatly simplified by the fact that higher-order Schottky transformations, such as, say $S^n_\mu$, contribute to projective invariant quantities terms of order $k^n_\mu$. Since here we are interested in the lowest order in the expansion, in principle we can then discard all contributions from Schottky transformations other than the identity. In the presence of an external field, however, this is not quite true, since the matrix elements of $\tau_{\vec{\epsilon}}$ are expressed as integrals over the $b$-cycles of the surface, and thus they can receive leading
order contributions even from terms in the integrand arising from first order Schottky transformations. To illustrate this fact, consider, say, the term in the series defining $\zeta_2^\tau(z)$ which involves the transformation $S_1$. One can write

$$
\frac{1}{z - S_1(\eta)} - \frac{1}{z - S_1(1)} = \frac{d}{dz} \log \left[ \frac{S_1^{-1}(z) - \eta}{S_1^{-1}(z) - 1} \right],
$$

(4.8)

where we have introduced and arbitrary point $x_0$, and we made use of the projective invariance of harmonic ratios such as the one appearing as argument of the logarithm. Clearly, upon performing a definite integration with a limit of the form $S_1(w)$ as in Eq. (2.13), Eq. (4.8) can give a contribution of order zero in the multipliers.

Let us now focus on the main new ingredient in Eq. (2.18), the determinant of the twisted period matrix $\tau_\epsilon$. Using the definition (2.13), it can readily be written as

$$
\det (\tau_\epsilon) = \frac{1}{2\pi i} \int_y^{S_1(y)} e^{2\pi i \epsilon \Delta_s} \left[ (e^{2\pi i (\epsilon \tau)_2} - 1) \zeta_1^{\epsilon \tau}(z) - (e^{2\pi i (\epsilon \tau)_1} - 1) \zeta_2^{\epsilon \tau}(z) \right],
$$

(4.9)

where $y$ is an arbitrary point of the surface. For the $\zeta_\mu^{\epsilon \tau}$ differentials, we can use Eq. (2.12), retaining in the sums only the terms arising from $T_\alpha = 1$ and $T_\alpha = S_1$, as discussed above. The result, with generic fixed points $\eta_\mu$ and $\xi_\mu$, is

$$
\zeta_1^{\epsilon \tau}(z) \simeq \frac{e^{2\pi i (\epsilon \tau)_1}}{z - \eta_1} - \frac{1}{z - \xi_1} - \left( 1 - e^{2\pi i (\epsilon \tau)_1} \right) \frac{e^{2\pi i (\epsilon \tau)_1}}{z - S_1(\eta_1)},
$$

$$
\zeta_2^{\epsilon \tau}(z) \simeq \frac{e^{2\pi i (\epsilon \tau)_2}}{z - \eta_2} - \frac{1}{z - \xi_2} \frac{e^{2\pi i (\epsilon \tau)_1}}{z - S_1(\xi_2)} + \frac{e^{2\pi i (\epsilon \tau)_1 + (\epsilon \tau)_2}}{z - S_1(\eta_2)},
$$

(4.10)

where we ignored all $z_0$-dependent terms, since they cancel out in the determinant (4.9). Eq. (4.10) can be simplified by making use of the fact that $S_1(\eta_1) = \eta_1$, and further by implementing our choice of fixed points, $\eta_1 = 0$, $\xi_1 = \infty$, $\xi_2 = 1$, which implies that the explicit form of $S_1$ is just $S_1(z) = k_1 z$. At this point we must also specify the diagram we are considering: we begin with the irreducible contribution of Fig. (2). Inserting $\epsilon = (0, \epsilon)$, and substituting our parametrization for this diagram, Eq. (4.10), we find

$$
\zeta_1^{\epsilon \tau}(z) \simeq \frac{1}{z^4} e^{-4Bt_3},
$$
The last ingredient to be expanded is the Riemann class (2.14). At $g = 2$, retaining again only the contributions of $T_a = \{1, S_1\}$ and using Eq. (4.6) we find

\[
\exp \left[ 2\pi i \vec{e} \cdot \Delta_2 \right] \simeq e^{B(t_2+t_3)} \left[ \frac{z-1}{z-\eta} \right]^{k_1\eta} e^{\eta z \cdot \Delta_2} \left[ \frac{z-k_1\eta}{z-\eta z-k_1\eta} \right]^{\epsilon} e^{i\pi \epsilon}.
\] (4.12)

The expression for the determinant of $\tau_\epsilon$ is then

\[
det (\tau_\epsilon) \simeq \frac{e^{i\pi \epsilon}}{2\pi i \epsilon} e^{B(t_2+t_3)} \left[ \frac{1-k_1\eta}{1-k_1} \right]^{k_1\eta} \int_\eta \frac{dz}{z} \left( \frac{z-1}{z-\eta z-k_1\eta} \right)^{\epsilon} e^{-4Bt_3} \left[ \frac{1}{z} - \frac{1}{z-1} \right] + \left( \frac{e^{-2B(t_2+t_3)}}{z-\eta} - \frac{e^{-2Bt_3}}{z-k_1} \right) \left( \frac{e^{-2B(t_2+2t_3)}}{z-k_1\eta} \right),
\] (4.13)

where we picked $y = \eta$, so that both limits of integration coincide with branching points of the integrand. One observes that $det (\tau_\epsilon)$ is a linear combination of integrals of the form

\[
I(a) \equiv - \int_\eta^{\eta} \frac{dz}{z-a} \left( \frac{z-1}{z-\eta z-k_1\eta} \right)^{\epsilon} = e^{-i\pi \epsilon} I_1(a) + I_2(a),
\] (4.14)

where $a$ can take the values $\{0, k_1\eta, k_1, \eta, 1\}$. Since in the limit we are considering one has $0 < k_1\eta \ll k_1 \ll \eta \ll 1$, the integrand has a cut for $z < k_1$, which we have made explicit by defining $I_1(a)$ as the integral ranging between $k_1\eta$ and $k_1$, while $I_2(a)$ ranges between $k_1$ an $\eta$. The phase multiplying $I_1(a)$ depends on the choice of Riemann sheet beyond the branch point $z = k_1$, and does not affect the final result in the limit $\alpha' \to 0$.

To proceed, we need to know to what order in the $\alpha'$ expansions these integrals need to be computed. Inspection of Eqs. (2.18), (4.2) and (4.6) shows that we only need the keep singular terms, which behave as $\alpha'^{-1}$. With this accuracy one finds

\[
I_1(1) \simeq I_1(\eta) \simeq 0; \quad I_1(k_1) \simeq I_1(k_1\eta) \simeq \frac{1}{2\alpha' B} e^{2Bt_3};
\]

\[
I_1(0) \simeq e^{2Bt_3} \frac{1-e^{2Bt_3}}{2\alpha' B},
\] (4.15)
for the various $I_1$ integrals, while the results for $I_2$ are

$$I_2(k_1 \eta) \simeq I_2(0) \simeq -\frac{e^{2Bt_3}}{\alpha'} t_1; \quad I_2(\eta) \simeq -\frac{1}{2\alpha' B} e^{2Bt_3};$$

$$I_2(1) \simeq 0; \quad I_2(k_1) \simeq -\frac{e^{2Bt_3}}{\alpha'} \left( \frac{1}{2B} + t_1 \right).$$

(4.16)

Substituting these results into Eq. (4.13), and keeping only the leading power in $\alpha'$, we find finally

$$\det (\tau_c) = \frac{1}{i\pi \alpha'} \left[ t_1 \sinh [B(t_2 + t_3)] + \frac{\sinh (Bt_2) \sinh (Bt_3)}{B} \right] + \mathcal{O} \left( (\alpha')^0 \right),$$

(4.17)

precisely the same structure arising in field theory, as seen by comparing to Eq. (3.11). With a straightforward computation of the measure $dZ_2$ in this limit, noting that the ratio $R_2$ contributes just a factor of unity, and assembling all normalization factors, the string-derived result for the effective action is

$$W_{\text{st}}^{(2)}(m, B) = V_4 \left( \frac{\lambda^2}{4\pi} \right)^d \frac{(N - 1)^2}{2} \int_0^\infty dt_1 \int_0^{t_1} dt_2 \int_0^{t_2} dt_3 \times e^{-m^2(t_1 + t_2 + t_3) \Delta_{0}^{-d/2+1} \Delta_{B}^{-1}}.$$  \hspace{1cm} (4.18)

This result maps exactly onto (3.11). The color factor $(N - 1)^2$ arises in string theory because at two loops we have two strings attached at one end to the charged brane, while the two free ends can choose between $N - 1$ neutral branes. Further, to get the complete result we must add to (4.18) the two contributions corresponding to charging the other two boundaries of the double annulus: they symmetrize the integrand, so that one can complete the integration region including a factor of $1/2^2$, reproducing the exact normalization. Finally, the missing factor of $i$ in (4.18) is due to the fact that string amplitudes are normalized to give directly the $T$-matrix element [19], while Feynman diagrams give the $S$-matrix element. Once again the result is correct for arbitrary $d^5$. Notice that the string calculation described in Section 2.2 corresponds only to the first diagram in Fig. (2): to get a singlet field $\sigma$ propagating in the string picture we would need a string with both ends on the charged brane, a configuration corresponding to having two charged boundaries with the same value of $\epsilon$.  

\footnote{For a justification of the fact that the $d$ dependence is correct even if $d \neq 26$, see [20].}
To conclude, we briefly show that one can also recover exactly the reducible diagrams in Fig. (3). Again, the string computation in the chosen configuration will yield only the diagram involving the propagation of Π fields (the first in Fig. (3)), but a very similar calculation would reproduce the σ diagram as well.

The calculation in this case is simplified by the fact that the ordinary period matrix $\tau$ becomes diagonal in the limit defined by Eq. (4.7), as $\alpha' \to 0$. As a consequence, the off-diagonal matrix element $(\tau_\epsilon)_{21}$ also vanishes, and thus $\text{det} \tau_\epsilon$ involves only $\zeta^\epsilon_1$, which is simpler in the field theory limit, as shown by Eq. (4.11). A short calculation yields

$$
\text{det} \ (\tau_\epsilon) = \frac{1}{i\pi} \sinh (Bt_2) \frac{t_1}{\alpha'} + \mathcal{O} \left( (\alpha')^0 \right), \quad (4.19)
$$

Inserting all normalization factors we find then

$$
W_{\text{st}}^{(2,\text{red})}(m, B) = V_d \frac{\lambda^2}{(4\pi)^d} \frac{(N-1)^2}{2} \int_0^\infty dt_1 \int_0^{t_1} dt_2 \int_0^{\infty} dt_3 e^{-m^2(t_1+t_2+t_3)}
\times \ (t_1 t_2)^{-d/2+1} \frac{B}{t_1 \sinh(Bt_2)}, \quad (4.20)
$$

which has the right structure to reproduce Eq. (3.13). It is interesting to observe how the combinatoric factors arising from the string match those computed in field theory. Here again we could let any one of the three boundaries of the double annulus be the charged one, however when the charged boundary is the external one the corresponding field theory diagram is different: the charged fields $\xi$ are now propagating in both loops, so that the propagator joining them must be a σ propagator. If we want to recover the first diagram in Fig. (3), we must add to Eq. (4.20) only the configuration with $t_1 \leftrightarrow t_2$; this symmetrizes the integrand, so that the integration region can be extended to match Eq. (3.13), with precisely the required overall factor. This completes our proof that all two-loop vacuum diagrams for adjoint scalars in a constant background field can be precisely recovered from bosonic string theory: the remaining diagrams, involving the coupling of $\xi$ with $\sigma$, can be similarly reproduced starting with the appropriate modification of Eq. (2.18).
5 Conclusions

Bosonic open strings in the presence of a constant gauge field strength provide an interesting system to study. On the technical side they display the general features of more complicated models, in particular the presence of cuts along some cycles of the Riemann surface representing the multiloop string amplitude. One can then use this simpler system to study objects like twisted determinants and differentials that will appear also in various other contexts. On the other hand, charged open strings are physically a very interesting system, primarily because they are directly related to perturbative gauge theories. As an example of this relation, we have shown that the Euler-Heisenberg effective action for a gauge field is naturally encoded in the string result.

A direct generalization of the computation presented in this paper would be to derive the Euler-Heisenberg effective action for pure Yang-Mills theory. In order to do this, there is no need to modify the string construction and the starting point is always Eq. (2.18). What needs to be changed is the definition of the field theory limit, in order to isolate the contributions to the loop integrals of the first excited state in the spectrum of the open bosonic string, which is a massless vector. In practice, this means that the expansion in multipliers of the various geometrical objects appearing in Eq. (2.18) has to be pushed one order higher. Hopefully, this computation will provide a simpler setup where it might be possible to see explicitly Yang-Mills Feynman diagrams arising from the field theory limit of string amplitudes beyond one loop. Another interesting aspect is to study the relation between our computations and the world-line formalism (see [49] for a recent detailed review and references). The world-line formalism has already been applied successfully to the study of Euler-Heisenberg effective action at one loop and beyond, see for instance [50–52].

At the string level, it would be very interesting to extend our computation to the case of superstrings and complete the multiloop extension of the result of [4]. On the other hand, the supersymmetric setup can be T-dualized to get the multi-body interaction among D-branes in type II theories. There are many results available on this problem derived by using different approaches, like 11D supergravity and M(atrix) theory. It would clearly be interesting to see whether an exact string computation can bring some further insight into this problem.

Let us conclude with a general comment on the two different formu-
lations we gave for the charged open string partition function, Eq. (2.8) and Eq. (2.18). Even if the two formulae are exactly equivalent at the string level, the gauge theory results can be derived only by performing the field theory limit of the second equation. The reason is clear: the expansion of Eq. (2.8) in the multipliers $q$ organizes the result by separating the contributions to the string diagram coming from the exchange of closed string states between D-branes; in the open string channel, on the other hand, one uses Eq. (2.18), where different powers of $k$ are related to the propagation of open string states. Thus the first formulation is useful if one wants to study at low energies a gravitational theory, while the second formulation is the one we used to derive the gauge theory results. In general, the modular transformation that maps the two equations into each other mixes all terms of the $q$ and $k$ expansions, so that the two low-energy results are completely unrelated. In special cases, however, it may happen that the series in $q$ reduces to a single term, which then has to match what is found in the $k$ expansion of the open string formula. In these cases, then, the gauge theory result can be directly derived from the closed string (or “gravitational”) description (see [53] for an explicit discussion of some such examples). It would be very interesting to see whether a similar situation can occur also at two loops, though clearly this is not the case for bosonic strings. If an example of this type were to exist also for $g = 2$, it could help to gain a better understanding of the perturbative aspects of the gauge/gravity correspondence.

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A Relating different representations of the two-loop partition function

For completeness, we give here a brief discussion of the derivation of (2.1) and (2.2) from (2.4). The steps connecting (2.4) and (2.1) are carefully explained by Roland in [54]. The quadratic differentials used are those derived in [55] and their overlap with the Beltrami differentials related to the moduli $k_i$ and $\eta$ is explicitly computed in the Schottky parametrization. Actually this derivation is valid beyond two loops and Roland obtains from (2.4) the general expression of [30]. As a remark, notice that the formulae in [54] do not contain explicitly the normalization factor $\sqrt{\det \langle \phi_j | \phi_k \rangle}$. This is simply because the sewing procedure selects a particular form for the quadratic differentials and the resulting determinants, written in the Schottky parametrization [7, 55], already include this normalization.

The connection between (2.4) and (2.2), on the other hand, has been explicitly studied in Section 7.1 of [36]. The basic idea is to choose as basis for the $\phi$’s simply the set of products of the usual Abelian differentials $\omega^\mu$. We will denote this particular basis as $\Omega_1(z) = (\omega_1(z))^2$, $\Omega_2(z) = (\omega_2(z))^2$ and $\Omega_3(z) = \omega_1(z)\omega_2(z)$. The important property of this choice is that it naturally parametrizes the variations of the period matrix under changes of the 2-dimensional metric $g$ (see for instance [9]). In fact, for example, $\delta \tau_{ij}/\delta g^{zz} \sim \omega_i(z)\omega_j(z)$. From this fact, we can see that $\delta \tau_{ij}/\delta m^a = \langle \mu^a | \delta \tau_{ij}/\delta g^{zz} \rangle \sim \langle \mu^a | \omega_i(z)\omega_j(z) \rangle$, where $\mu^a$ is the Beltrami differential associated to the modulus $m^a$. By identifying the elements of the period matrix with the moduli $m^a$, we immediately see that the overlap $\langle \tau_{ij} | \Omega_I(z) \rangle$ is simply proportional to the identity matrix. Then, by using the bosonization equivalence on a surface of genus $g = 2$, one can reexpress the fermionic determinant in terms of $\theta$-functions and other geometrical objects, as

$$\frac{\det'(\partial^I \partial)}{\sqrt{\det(\Omega_I | \Omega_k)}} = \frac{\prod_{I<J} E(z_I, z_J) \prod_{I=1}^3 \sigma(z_I)^3}{Z_1 \det \left( \Omega_I(z_J) \right)} \theta \left( \frac{3 \bar{\Delta} - \sum_{I=1}^3 \bar{J}(z_I) | \tau \right), \tag{A.1}$$

where the prime form $E(z, w)$, the function $\sigma(z)$, the Riemann class $\bar{\Delta}$ and the Jacobi map $\bar{J}$ are defined as in Ref. [7]. Note that although the right hand side of Eq. (A.1) appears to depend on $g + 1 = 3$ coordinates $z_I$ on the Riemann surface, the equality with the left hand side shows that this dependence actually cancels out in the ratio.
By using Eq. (A.1) inside (2.4) one obtains Eq. (7.2) of [36] and then the expression (2.2) originally proposed in [28, 29]. This proves the equivalence between (2.2) and (2.1).

B Scalar propagator in a background field

In our chosen gauge, \( A_\mu = B x_1 g_\mu 2 \), the propagator for the charged field \( \xi \) in \( d \) space-time dimensions must satisfy the equation

\[
(\Box + m^2 + 2i B x_1 \partial_2 + B^2 x_1^2) G_\xi (x, y) = -i \delta^d (x - y). \tag{B.2}
\]

This equation can be easily solved by Fourier transforming with respect to all coordinates \( x_\mu \) except \( x_1 \). To keep our notation simple, we will denote Fourier transforms with the same symbol as the original function. The partial Fourier transform of \( G_\xi (x, y) \) obeys the equation

\[
[\partial_1^2 + k_0^2 - |k_\perp|^2 - m^2 - (B x_1 + k_2)^2] G_\xi (k_0, k_\perp; x_1, y_1) = i \delta^d (x_1 - y_1). \tag{B.3}
\]

One recognizes in the \( x_1 \)-dependent terms of the kinetic operator the quantum mechanical hamiltonian of a harmonic oscillator with mass \( M = 1/2 \) and angular frequency \( \Omega = 2 |B| \), oscillating around the point \( x_1 = -k_2/B \). Exploiting the completeness of the eigenfunctions of this hamiltonian one is immediately lead to the representation

\[
G_\xi (k_0, k_\perp; x_1, y_1) = \sum_{n=0}^{\infty} \psi^*_n \left(y_1 + \frac{k_2}{B}\right) \psi_n \left(x_1 + \frac{k_2}{B}\right) \times \frac{i}{k_0^2 - k_\perp^2 - m^2 - (2n + 1) |B|}, \tag{B.4}
\]

where

\[
\psi_n (z) = \sqrt{\frac{1}{2^n n!} \frac{|B|}{\pi}} H_n \left( \sqrt{|B|} z \right) \exp \left(-|B| z^2 / 2 \right) \tag{B.5}
\]

are the appropriate eigenfunctions, with \( H_n \) the Hermite polynomials. It is now straightforward to Fourier transform also with respect to the remaining coordinates \( x_1 \) ad \( y_1 \), recalling that the momentum space eigenfunctions of the harmonic oscillator are again given by Hermite polynomials. Taking into
account the fact that momentum is not conserved in the $x_1$ direction, define then the momentum space propagator as

$$G_\xi(k_0, k_\perp; k_1, k'_1) = \int dx_1 dy_1 e^{-i(k_1 x_1 - k'_1 y_1)} G_\xi(k_0, k_\perp; x_1, y_1), \quad (B.6)$$

to find explicitly

$$G_\xi(k_0, k_\perp; k_1, k'_1) = \frac{1}{\pi |B|} \exp \left[ -i \frac{k_2}{B} (k_1 - k'_1) \right] \exp \left[ - \frac{k_1^2 + k'_1^2}{2|B|} \right] \quad (B.7)$$

$$\times \sum_{n=0}^{\infty} \frac{1}{2^n n!} H_n \left( \frac{k_1}{\sqrt{|B|}} \right) H_n \left( \frac{k'_1}{\sqrt{|B|}} \right) \frac{2\pi i}{k_0^2 - k_1^2 - m^2 - (2n + 1)|B|}.$$
Eq. (B.10) back to coordinate space. This last calculation is straightforward, and leads to Eq. (3.7). Notice that the phase factor in the first line of Eq. (3.7) is the expected gauge link between point $x$ and point $y$. In fact with our choice of gauge field

$$\exp \left[ -\frac{i}{2} B (x_1 + y_1)(x_2 - y_2) \right] = \exp \left[ -i \int_x^y A_\mu(z) dz^\mu \right], \quad (B.12)$$

where the integral is performed along the straight line joining points $x$ and $y$. The terms in Eq. (B.12) which depend on both $x$ and $y$ are gauge-invariant and can be assembled into a factor of the form $\exp \left( -i \mathcal{F}_{\mu\nu} x^\mu y^\nu / 2 \right)$. Factors depending only on $x$ or on $y$, on the other hand, can be removed with an appropriate gauge transformation. For example, choosing as gauge function $f(x_\mu) \equiv -x_1 x_2 B / 2$ one moves to the "symmetric" gauge,

$$A_1'(x) = -\frac{x_2}{2} B; \quad A_2'(x) = \frac{x_1}{2} B, \quad (B.13)$$

where only the gauge-invariant contribution to Eq. (B.12) survives. In this gauge, in coordinate space, one can make a direct comparison with the results of Refs. [57–59], recently discussed in [3], finding complete agreement.

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