ON TORIC CALABI-YAU HYPERSURFACES FIBERED BY WEIGHTED K3 HYPERSURFACES

JOSHUA P. MULLET

ABSTRACT. In response to a question of Reid, we find all anti-canonical Calabi-Yau hypersurfaces $X$ in toric weighted $\mathbb{P}^3$-bundles over $\mathbb{P}^1$ where the general fiber of $X$ over $\mathbb{P}^1$ is a weighted K3 hypersurface. This gives a direct generalization of Reid’s discovery of the 95 families of weighted K3 hypersurfaces in [17]. We also treat the case where $X$ is fibered over $\mathbb{P}^2$ with general fiber a genus one curve in a weighted projective plane.

1. Introduction

In light of the successes of the minimal model program in the classification of threefolds, Corti and Reid promulgate in [2] a program of explicit birational study of threefolds. Simply put, the interest is to obtain explicit equations for threefolds as hypersurfaces or complete intersections in simple varieties such as projective spaces or weighted projective spaces, and then focus study to these explicit examples, regarding them as birational models. An early example of such an explicit result is Reid’s discovery of the 95 families of K3 hypersurfaces in weighted projective 3-space (see [17]).

Miles Reid posed to the author the problem of classifying Calabi-Yau threefolds fibered over the projective line where the general fiber is one of the 95 weighted K3 hypersurfaces mentioned in the previous paragraph. Since the fibers are to be weighted hypersurfaces, a natural place to look for such threefolds is in weighted projective bundles. Furthermore, if we are interested in being explicit, we should look for these threefolds in some “simple” varieties. For us, those simple varieties are toric varieties. We are thus naturally led to the problem of finding K3-fibered Calabi-Yau threefolds in toric weighted projective bundles over $\mathbb{P}^1$. In this paper, we show how to find all such threefolds and thereby obtain a direct generalization of Reid’s discovery of the 95 families of weighted K3 hypersurfaces.

The contents of this paper are as follows. In Section 2 we fix some definitions and notation regarding the theory of toric varieties that we use throughout the paper. In Section 3 we construct weighted projective space bundles as toric varieties. Weighted projective space bundles can also be obtained by taking Proj of a sheaf of weighted polynomial algebras. In Section 4 we describe the Proj construction and compare it to the construction using toric varieties. In Section 5 we describe the intersection theory of weighted $\mathbb{P}^1$-bundles over $\mathbb{P}^1$ in terms of the well-known intersection theory on ordinary $\mathbb{P}^1$-bundles. We will need this description in the
proof of Theorem 8.5. In Section 6 we consider linear systems on toric varieties and, more specifically, on weighted projective spaces and bundles. Section 7 is a technical section in which we show that the condition of well-formedness (Definition 2.2), which is a condition imposed on hypersurfaces to make the adjunction formula work as expected, is automatic in the case of quasi-smooth hypersurfaces (Definition 2.1). In Section 8 we prove our main result, Theorem 8.5, in which we give necessary and sufficient conditions for a toric weighted projective bundle over \( \mathbb{P}^1 \) to admit a K3-fibered anti-canonical Calabi-Yau hypersurface, thus providing an answer to Reid’s original question. Section 9 contains a statement of the analogous result for elliptically fibered Calabi-Yau threefolds over \( \mathbb{P}^2 \) whose general fiber is a genus one curve in a weighted \( \mathbb{P}^3 \). The complete list of all our Calabi-Yau varieties is quite long, but the appendix contains a sampling of data, and we refer the reader to \([16]\) for all the data and details of the calculation.

We should mention that Masanori Kobayashi considered the problem in the non-weighted case \([13]\) and without the explicit use of toric varieties. Our work is inspired by his, and we use some of his notation in Section 6.

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2. Conventions and notation

We work over \( \mathbb{C} \), the field of complex numbers, and we follow standard conventions regarding the theory of toric varieties as described, for example, in \([7]\) or \([5]\). If \( X \) is a toric variety, we will denote its associated lattices as \( N_X \) and \( M_X := \text{Hom}_\mathbb{Z}(N_X, \mathbb{Z}) \), and if \( N \) is a lattice we define \( N_\mathbb{R} := N \otimes_\mathbb{Z} \mathbb{R} \). We always assume that our toric varieties have the property that all their maximal cones have dimension equal to the rank of the lattice used to define them.

We will need the notions of quasi-smoothness and well-formedness, which we define below for subvarieties of toric varieties. Let \( X_\Sigma \) be a simplicial toric variety associated to a fan \( \Sigma \), and let \( S_\Sigma \) be its homogeneous coordinate ring with irrelevant ideal \( B \subseteq S_\Sigma \) (see \([3]\)). According to \([3]\) Theorem 2.1, \( X_\Sigma \) is isomorphic to the geometric quotient

\[
(Spec \, S_\Sigma - Z(B))/G
\]

where

\[
G = \text{Hom}_\mathbb{Z}(A^1 X_\Sigma, \mathbb{C}^*).
\]

Definition 2.1. Let

\[
\text{Spec } S_\Sigma - Z(B) \xrightarrow{q} X_\Sigma
\]

be the quotient map. We say that a closed subvariety \( Y \subseteq X_\Sigma \) is quasi-smooth if \( q^{-1}Y \) is nonsingular.
**Definition 2.2.** Let \( X_\Sigma \) be a simplicial toric variety, and let \( \text{sing} \ X_\Sigma \) denote the singular locus of \( X_\Sigma \). We say that a closed subvariety \( Y \subseteq X_\Sigma \) is well-formed if the codimension of \((\text{sing} \ X_\Sigma) \cap Y\) is at least two in \( Y\).

**Remark 2.3.** If \( Y \subseteq X_\Sigma \) is quasi-smooth, then \( Y \) is an orbifold.

3. **Weighted Projective Space Bundles as Toric Varieties**

In this section we consider the problem of constructing weighted projective bundles as toric varieties. We first recall how to construct the weighted projective space \( \mathbb{P}(a_0, \ldots, a_n) \) as a toric variety.

**Construction 3.1.** Let \((a_0, \ldots, a_n)\) be a list of positive integers such that
\[
\gcd(a_0, \ldots, \overset{\wedge}{a_i}, \ldots, a_n) = 1 \tag{3.1}
\]
where \(\overset{\wedge}{a_i}\) means that we omit \(a_i\). Then form the weighted projective space
\[
\mathbb{P}(a_0, \ldots, a_n) := \text{Proj} \mathbb{C}[X_0, \ldots, X_n]
\]
where \(\deg X_i = a_i\). (See [6, 12] for generalities about weighted projective space.)

As described in [7], we may write \(\mathbb{P}(a_0, \ldots, a_n)\) as a toric variety as follows. Define the lattice \(N\) via the following exact sequence:
\[
\begin{array}{c}
0 \rightarrow \mathbb{Z} \rightarrow \mathbb{Z}^{n+1} \rightarrow N \rightarrow 0
\end{array}
\]
and define the fan \(\Sigma \subseteq N^*_\mathbb{R}\) to consist of cones spanned by the images of proper subsets of the standard basis for \(\mathbb{Z}^{n+1}\). Then
\[
\mathbb{P}(a_0, \ldots, a_n) \cong X_\Sigma.
\]
Note that if we take \(a_i = 1\) for all \(0 \leq i \leq n\), we obtain a description of (ordinary) projective space as a toric variety.

We come now to the notion of fibration for toric varieties that we will use (cf. [7, Exercise, p. 41]).

**Definition 3.2.** By a locally trivial toric fibration we mean a fibration
\[
\begin{array}{ccc}
F & \xrightarrow{i} & X \\
\pi \downarrow & & \downarrow \pi' \\
& & B
\end{array}
\]
in which \(F, X,\) and \(B\) are toric varieties associated to fans \(\Sigma_F, \Sigma_X, \Sigma_B\) and lattices \(N_F, N_X, N_B\) respectively. We further require that the fibration trivialize over affine open toric subvarieties of \(B\), that \(F\) is the fiber over the identity element of the torus for \(B\), and that the maps \(i\) and \(\pi\) are induced by maps of fans \(i'\) and \(\pi'\) respectively.
Remark 3.3. We may be imprecise and say that $X$ is a “fibration” with fiber $F$, or a “toric $F$-bundle.”

Remark 3.4. Note that the maps of lattices in Definition 3.2 form an exact sequence

$$0 \rightarrow N_F \xrightarrow{\iota'} N_X \xrightarrow{\pi'} N_B \rightarrow 0.$$ 

Construction 3.5. Let $n$ and $k$ be positive integers, let $(a_0, \ldots, a_n)$ be a list of positive integers satisfying (3.1), and let $(d_0, \ldots, d_n)$ be an arbitrary list of integers. Define a lattice $N$ via the following exact sequence:

$$0 \rightarrow \mathbb{Z}^2 \xrightarrow{(a_0 - d_0 \ 0 \ \cdots \ \cdots \ a_n - d_n \ 0 \ 1)} \mathbb{Z}^{n+1} \oplus \mathbb{Z}^{k+1} \rightarrow N \rightarrow 0. \quad (3.3)$$

Now let $\{e_0, \ldots, e_n\}$ denote the standard basis for $\mathbb{Z}^{n+1}$, and let $\{f_0, \ldots, f_k\}$ denote the standard basis for $\mathbb{Z}^{k+1}$. We let $\Sigma \subseteq N_{\mathbb{R}}$ be the fan that consists of cones spanned by proper subsets of the standard bases that contain neither all of the $e_i$ nor all of the $f_i$.

Proposition 3.6. Let $\Sigma$ be the fan constructed in Construction 3.5, and let $X_{\Sigma}$ be the associated toric variety. Then $X_{\Sigma}$ is a locally trivial toric fibration over $\mathbb{P}^k$ (in the sense of Definition 3.2) over $\mathbb{P}^k$ with fiber $\mathbb{P}(a_0, \ldots, a_n)$.

Proof. Let $N_{X_{\Sigma}}$ be the lattice for $X_{\Sigma}$, let $N_{\mathbb{P}^k}$ be the lattice for $\mathbb{P}^k$, and let $N_{\mathbb{P}(a_0, \ldots, a_n)}$ be the lattice for $\mathbb{P}(a_0, \ldots, a_n)$; here we regard $\mathbb{P}^k$ and $\mathbb{P}(a_0, \ldots, a_n)$ as toric varieties according to Construction 3.1. We have the following commutative diagram whose rows and columns are exact.

$$0 \rightarrow \mathbb{Z} \xrightarrow{a_0 \ \cdots \ a_n} \mathbb{Z}^{n+1} \rightarrow N_{\mathbb{P}(a_0, \ldots, a_n)} \rightarrow 0$$

Now let $\{e_0, \ldots, e_n\}$ denote the standard basis for $\mathbb{Z}^{n+1}$, and let $\{f_0, \ldots, f_k\}$ denote the standard basis for $\mathbb{Z}^{k+1}$. We let $\Sigma \subseteq N_{\mathbb{R}}$ be the fan that consists of cones spanned by proper subsets of the standard bases that contain neither all of the $e_i$ nor all of the $f_i$.

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Proof. Let $N_{X_{\Sigma}}$ be the lattice for $X_{\Sigma}$, let $N_{\mathbb{P}^k}$ be the lattice for $\mathbb{P}^k$, and let $N_{\mathbb{P}(a_0, \ldots, a_n)}$ be the lattice for $\mathbb{P}(a_0, \ldots, a_n)$; here we regard $\mathbb{P}^k$ and $\mathbb{P}(a_0, \ldots, a_n)$ as toric varieties according to Construction 3.1. We have the following commutative diagram whose rows and columns are exact.

$$0 \rightarrow \mathbb{Z}^2 \xrightarrow{(a_0 - d_0 \ 0 \ \cdots \ \cdots \ a_n - d_n \ 0 \ 1)} \mathbb{Z}^{n+1} \oplus \mathbb{Z}^{k+1} \rightarrow N \rightarrow 0. \quad (3.3)$$

Now let $\{e_0, \ldots, e_n\}$ denote the standard basis for $\mathbb{Z}^{n+1}$, and let $\{f_0, \ldots, f_k\}$ denote the standard basis for $\mathbb{Z}^{k+1}$. We let $\Sigma \subseteq N_{\mathbb{R}}$ be the fan that consists of cones spanned by proper subsets of the standard bases that contain neither all of the $e_i$ nor all of the $f_i$.
The right column of (3.4) gives the required maps of lattices. Furthermore, since the middle column is just inclusion of the first factor followed by projection onto the second factor, we can canonically extend the maps of lattices to maps of fans. These maps of fans give the required maps of toric varieties.

\[
\mathbb{P}(a_0, \ldots, a_n) \xrightarrow{i} X_\Sigma \\
\pi \downarrow \\
\mathbb{P}^k
\]

(3.5)

It remains to show that (3.5) trivializes over affine open toric subvarieties of \( \mathbb{P}^k \), and for this it suffices to consider only those open toric subvarieties of \( \mathbb{P}^k \) that correspond to maximal cones in the fan for \( \mathbb{P}^k \). Let \( U_i \) be the open toric subvariety corresponding to the cone spanned by the images of all but the \( i \)-th standard basis vector of \( \mathbb{Z}^{k+1} \). Next observe that for all \( 0 \leq i \leq k \) we have an isomorphism of exact sequences

\[
0 \rightarrow \mathbb{Z}^2 \xrightarrow{0} \mathbb{Z}^{n+1} \oplus \mathbb{Z}^{k+1} \xrightarrow{R} N \rightarrow 0
\]

where \( M \) is an \((n+1) \times (k+1)\) matrix whose only nonzero column lies in the same column as the \( i \)-th basis vector in \( I_{k+1} \) and is equal to

\[
\begin{pmatrix}
d_0 \\ \vdots \\ d_n
\end{pmatrix}
\]

Therefore, if we consider the open subvariety of \( X_\Sigma \) that corresponds to cones that do not contain the ray corresponding to the \( i \)-th standard basis vector of \( \mathbb{Z}^{k+1} \), the map of exact sequences above yields an isomorphism of fans for \( \pi^{-1} U_i \) and \( \mathbb{P}(a_0, \ldots, a_n) \times U_i \) where \( \pi \) is the structure map in (3.5).

Our next result shows that all toric weighted projective space bundles are of the form described in Construction 3.5.
Theorem 3.7. Let
\[ \mathbb{P}(a_0, \ldots, a_n) \xrightarrow{i} \tilde{\mathbb{P}} \xrightarrow{\pi} \mathbb{P}^k \] (3.6)
be a locally trivial toric fibration as in Definition 3.2, where we regard \( \mathbb{P}^k \) as a toric variety according to Construction 3.1. Then there exist integers \( d_0, \ldots, d_n \) such that (3.6) is isomorphic to the fibration constructed in Construction 3.5.

Proof. Let \( N_{\tilde{\mathbb{P}}} \) be the lattice for \( \tilde{\mathbb{P}} \). Let \( \rho_0, \ldots, \rho_n \) be the rays of the fan for \( \mathbb{P}(a_0, \ldots, a_n) \), and let \( \tau_0, \ldots, \tau_k \) be the rays of the fan for \( \mathbb{P}^k \). Following standard convention, we write \( v_{\rho_i} \) and \( v_{\tau_i} \) for the primitive vectors in the rays \( \rho_i \) and \( \tau_i \). Since the fibration is locally trivial, there exist unique rays \( \tilde{\tau}_0, \ldots, \tilde{\tau}_k \) in \( N_{\tilde{\mathbb{P}}} \) such that \( \pi'(v_{\tilde{\tau}_i}) = v_{\tau_i} \) for \( 0 \leq i \leq k \) where \( \pi' \) is the map of fans corresponding to the map \( \pi \). Let \( \{e_0, \ldots, e_n\} \) be the standard basis for \( \mathbb{Z}^{n+1} \) and let \( \{f_0, \ldots, f_k\} \) be the standard basis for \( \mathbb{Z}^{k+1} \). We define a map
\[ F : \mathbb{Z}^{n+1} \oplus \mathbb{Z}^{k+1} \longrightarrow N_{\tilde{\mathbb{P}}} \]
by setting
\[ F(e_i, f_j) := i'(v_{\rho_i}) + v_{\tilde{\tau}_j} \]
where \( i' \) is the map of fans corresponding to the map \( i \) in (3.6). We easily check that \( F \) is surjective, and we have an exact sequence,
\[ 0 \longrightarrow \mathbb{Z}^2 \longrightarrow \mathbb{Z}^{n+1} \oplus \mathbb{Z}^{k+1} \xrightarrow{F} N_{\tilde{\mathbb{P}}} \longrightarrow \mathbb{Z}^2 \] (3.7)
Since the fibration is locally trivial, we may use the fact ([7, Exercise, p. 22]) that the fan for the product of two toric varieties is isomorphic to the product of their fans to check that the fan \( \Sigma_{\tilde{\mathbb{P}}} \) is as described in Construction 3.5 but now with respect to the map \( F \).

It remains to show that the kernel of (3.7) is isomorphic to the kernel in (3.3). From (3.2) we see that \( \sum a_i F(e_i) = 0 \). Furthermore, since \( \sum v_{\tau_i} = 0 \), we can apply Remark 3.3 to find
\[ F \left( \sum \tilde{f}_i \right) = F \left( \sum d_j e_j \right) \]
for some integers \( d_0, \ldots, d_n \). Therefore, (3.7) fits into a commutative diagram
\[
\begin{array}{c}
\mathbb{Z}^2 \\
\phi \\
\end{array} 
\begin{array}{c}
\mathbb{Z}^2 \\
| \\
\downarrow \\
0 \\
\end{array} 
\begin{array}{c}
\mathbb{Z}^{n+1} \oplus \mathbb{Z}^{k+1} \\
\rightarrow \\
N_{\tilde{\mathbb{P}}} \\
\rightarrow \\
0 \\
\end{array} 
\]
but by (3.1) the columns in the matrix are primitive, therefore \( \phi \) is an isomorphism as required. \( \square \)
Having characterized all toric $\mathbb{P}(a_0, \ldots, a_n)$-bundles over $\mathbb{P}^k$, we conclude this section by finding their homogeneous coordinate rings and irrelevant ideals \((\text{X})\). The description given in Construction 3.5 makes this particularly easy. Recall that for a toric variety $X$ there exists an exact sequence

$$
0 \longrightarrow M \longrightarrow \mathbb{Z}^{\Sigma(1)} \longrightarrow A^1 X \longrightarrow 0,
$$

where $\Sigma(1)$ denotes the set of one-dimensional cones in the fan, and where $A^1 X$ denotes the Chow group of codimension one cycles modulo rational equivalence.

We begin with a proposition in which we compute the exact sequence \((3.8)\) for toric weighted projective bundles.

**Proposition 3.8.** Let $(a_0, \ldots, a_n)$ be a sequence of positive integers satisfying \((3.1)\), and let $\bar{\mathbb{P}}$ be a locally trivial toric fibration over $\mathbb{P}^k$ with fiber $\mathbb{P}(a_0, \ldots, a_n)$. There is an isomorphism of exact sequences:

$$
0 \longrightarrow M_\bar{\mathbb{P}} \longrightarrow \text{Hom}(\mathbb{Z}^{n+1} \oplus \mathbb{Z}^{k+1}, \mathbb{Z}) \overset{T_{\text{tr}}}{\longrightarrow} \text{Hom}(\mathbb{Z}^2, \mathbb{Z}) \longrightarrow 0
$$

$$
0 \longrightarrow M_\bar{\mathbb{P}} \longrightarrow \mathbb{Z}^{\Sigma(1)} \longrightarrow A^1(\bar{\mathbb{P}}) \longrightarrow 0,
$$

where $T$ is the matrix in Construction 3.5 and the bottom row is the standard exact sequence \((3.8)\).

**Proof.** Since the rays of the fan for $\bar{\mathbb{P}}$ are given by the images of standard basis vectors under the cokernel of the map $T$, the result will follow immediately provided we show that the images of these vectors are primitive vectors in $N_\bar{\mathbb{P}}$. Let $(e_0, \ldots, e_n)$ be the standard basis for $\mathbb{Z}^{n+1}$ and let $(f_0, \ldots, f_k)$ be the standard basis for $\mathbb{Z}^{k+1}$.

Without loss of generality, we consider the case in which the image of $e_0$ or $f_0$ is not primitive in $N_\bar{\mathbb{P}}$. Let

$$
r_1 = \begin{pmatrix}
a_0 \\
\vdots \\
a_n \\
0
\end{pmatrix} \quad \text{and} \quad r_2 = \begin{pmatrix}
-d_0 \\
\vdots \\
-d_n \\
1
\end{pmatrix}
$$

be the columns of the matrix $T$. Since the image of $e_0$ is not primitive in $N_\bar{\mathbb{P}}$ or the image of $f_0$ is not, there exists a vector $v \in \mathbb{Z}^{n+1} \oplus \mathbb{Z}^{k+1}$ such that

$$
e_0 + nv \in \text{Span}\{r_1, r_2\} \quad \text{or} \quad f_0 + nv \in \text{Span}\{r_1, r_2\},
$$

for some integer $n$ such that $|n| > 1$.

In case the image of $e_0$ is not primitive in $N_\bar{\mathbb{P}}$ then

$$
e_0 + nv = c_1 r_1 + c_2 r_2
$$

for some integers $c_1$ and $c_2$. Note that $n$ does not divide $\gcd(c_1, c_2)$. If we consider the components of $nv$, we find that $n \mid c_2$ and hence $n \mid c_1 a_i$ for all $i \neq 0$, and this contradicts \((3.1)\). Thus $n = 1$ and $v$ is primitive.
In case the image of $f_0$ is not primitive in $N_{\mathbb{P}}$ then
\[ f_0 + nv = c_1 r_1 + c_2 r_2 \]
for some integers $c_1$ and $c_2$. Considering the components of $nv$ we find that $n \mid c_2$ and $n \mid (c_2 - 1)$, hence $n = 1$, and again we have a contradiction. \qed

We are now ready to write down the homogeneous coordinate ring and irrelevant ideal $B$ for a toric $\mathbb{P}(a_0, \ldots, a_n)$-bundle over $\mathbb{P}^k$.

**Corollary 3.9.** Let $(a_0, \ldots, a_n)$ be a sequence of positive integers satisfying (3.1), and let $\mathbb{P}$ be a locally trivial toric fibration over $\mathbb{P}^k$ with fiber $\mathbb{P}(a_0, \ldots, a_n)$. Then the homogeneous coordinate ring of $\tilde{\mathbb{P}}$ is isomorphic to the $\mathbb{Z}^2$-graded polynomial ring
\[ \mathbb{C}[X_0, \ldots, X_n, S_0, \ldots, S_k] \]
where
\[ \deg X_i = (a_i, -d_i) \quad \text{and} \]
\[ \deg S_j = (0, 1). \]
Furthermore, the irrelevant ideal $B \subseteq S_X$ is given by
\[ B = (X_0 S_0, \ldots, X_n S_0, X_0 S_1, \ldots, X_n S_1, \ldots, X_0 S_k, \ldots, X_n S_k). \]

**Proof.** This follows immediately from Theorem 3.7, Proposition 3.8, and the construction in [3] of the homogeneous coordinate ring of a toric variety. \qed

4. **Weighted projective space bundles as Proj of weighted symmetric algebras**

In this section we view weighted projective space bundles over projective space as Proj of weighted symmetric algebras of sums of invertible sheaves and show that they are isomorphic to the toric varieties in Construction 3.5. For convenience, we adopt the following definition.

**Definition 4.1.** Let $X$ be a scheme, and let $(a_0, \ldots, a_n)$ be a sequence of positive integers. We define a **weighted locally free sheaf** with weights $(a_0, \ldots, a_n)$ to be a locally free sheaf of $\mathcal{O}_X$-modules $\mathcal{E}$ together with an ordered decomposition $\mathcal{E} \cong \mathcal{E}_0 \oplus \cdots \oplus \mathcal{E}_n$ such that $\mathcal{E}_i$ is an invertible sheaf and such that the direct sum is to be interpreted as a graded sheaf with $\mathcal{E}_i$ placed in degree $a_i$ for $0 \leq i \leq n$. The weights will be used in Definition 4.3 when we form symmetric algebras of weighted locally free sheaves.

**Remark 4.2.** It certainly is possible not to require that the locally free sheaves $\mathcal{E}_i$ in the decomposition in Definition 4.1 have rank one, but we will restrict to this case as this is case that relates to toric varieties.

**Definition 4.3.** Let $X$ be a scheme. Given a weighted locally free sheaf $\mathcal{E}$ with weights $(a_0, \ldots, a_n)$ let $\mathcal{S}$ denote the weighted symmetric algebra of $\mathcal{E}$ where we insist that $\mathcal{E}_i$ have homogeneous degree $a_i$ in $\mathcal{S}$. We define the **weighted projective bundle** associated to $\mathcal{E}$ to be the $X$-scheme
\[ \tilde{\mathbb{P}}(\mathcal{E}) := \text{Proj} \mathcal{S} \longrightarrow X. \]
We have the following two basic lemmas for weighted projective bundles.

**Lemma 4.4.** Let $X$ be a variety over $\mathbb{C}$, and let $E$ be a weighted locally free sheaf on $X$ with weights $(a_0, \ldots, a_n)$. Then the weighted projective bundle $\tilde{\mathbb{P}}(E)$ is a locally trivial fiber bundle over $X$ with fiber the weighted projective space $\mathbb{P}(a_0, \ldots, a_n)$.

**Proof.** This follows from the construction of Proj of a sheaf of graded algebras as described, for example, in [10, §3]. □

**Lemma 4.5.** Let $X$ be a scheme, let $E$ be a weighted locally free sheaf on $X$ with weights $(a_0, \ldots, a_n)$, and let $L$ be an invertible sheaf of $O_X$-modules. Then there is a canonical $X$-isomorphism $\tilde{\mathbb{P}}(E) \cong \tilde{\mathbb{P}}(\bigoplus_{i=0}^{n} E_i \otimes L \otimes a_i)$.

**Proof.** This is just the application to our special case of [10, Proposition 3.1.8 (iii)]. □

We also state the relative version of [12, Lemma 5.7].

**Lemma 4.6.** Let $(a_0, \ldots, a_n)$ be a sequence of positive integers having no common factor, let $X$ be a scheme, and let $E = \bigoplus_{i=0}^{n} E_i$ be a weighted locally free sheaf of $O_X$-modules with weights $(a_0, \ldots, a_n)$. Let $q$ be a positive integer such that $q \mid a_i$ for $i > 0$, and define a new weighted locally free sheaf $E'$ with weights $(a_0, \frac{a_1}{q}, \ldots, \frac{a_n}{q})$ via $E' := \bigoplus_{i=0}^{n} E_i \otimes \bigoplus_{i=1}^{n} E_i$.

Then $\tilde{\mathbb{P}}(E)$ isomorphic to $\tilde{\mathbb{P}}(E')$.

We conclude this section by relating the two notions of weighted projective bundles that we have described. We point out that Proposition 4.7 is a generalization of the exercise [7, Exercise p. 42].

**Proposition 4.7.** Let $(a_0, \ldots, a_n)$ be a sequence of positive integers satisfying (3.1), and let $d_0, \ldots, d_n$ be integers. Let $E$ denote the weighted locally free sheaf $E := O(d_0) \oplus \cdots \oplus O(d_n)$ on $\mathbb{P}^k$ with weights $(a_0, \ldots, a_n)$. Then $\tilde{\mathbb{P}}(E)$ is isomorphic to $\tilde{\mathbb{P}}$, the toric weighted projective bundle from Construction 3.5.

**Proof.** We first fix some notation. Write $\mathbb{P}^k = \text{Proj} \mathbb{C}[S_0, \ldots, S_k]$ and let $U_i = \text{Spec} \mathbb{C}[S_0/S_i, \ldots, S_k/S_i]$. Let $\{X_i\}_{i=0}^{n}$ be global coordinates on $\tilde{\mathbb{P}}(E)$; this means that $\tilde{\mathbb{P}}(E)|_{U_i} \cong \text{Proj} \mathbb{C} \left[ \frac{S_0}{S_i}, \ldots, \frac{S_k}{S_i}, S_i^{d_0}X_0, \ldots, S_i^{d_n}X_n \right]$. 


where \( \deg S^d_j X_j = a_j \). We find for \( 0 \leq i \leq k \) that \( \overline{\mathbb{P}(E)}|_{U_i} \) is isomorphic to the toric variety \( A^k \times \mathbb{P}(a_0, \ldots, a_n) \). Let \( U := \bigcap_i U_i \). Distinguishing \( S_0 \), we find

\[
\overline{\mathbb{P}(E)}|_{U} \cong \text{Proj} \left[ \left( \frac{S_1}{S_0} \right)^{\pm 1}, \ldots, \left( \frac{S_k}{S_0} \right)^{\pm 1} \right] [S^d_0 X_0, \ldots, S^d_n X_n]
\]

where \( \deg S^d_j X_j = a_j \). Distinguishing \( S_0 \) amounts to choosing a basis for the torus \( T^k \) in \( \mathbb{P}^k \) and thereby identifying the lattice \( M_{\mathbb{P}^k} \) with \( \mathbb{Z}^k \). Similarly, the coordinates \( S^d_i X_i \) correspond to a basis for \( \mathbb{Z}^{n+1} \) in the context of the exact sequence (3.2). Therefore, the inclusion

\[
\overline{\mathbb{P}(E)}|_{U} \to \overline{\mathbb{P}(E)}|_{U_i}
\]

induces an isomorphism

\[
M_{U_i} \oplus \mathbb{Z}^{n+1} \xrightarrow{\phi_i} M_{T^k} \oplus \mathbb{Z}^{n+1},
\]

which is the identity if \( i = 0 \) and if \( i > 0 \) is given by the matrix

\[
\begin{pmatrix}
0 & 1 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \\
0 & 0 & \cdots & 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
-1 & -1 & \cdots & -1 & -1 & \cdots & -1 & d_0 & d_1 & \cdots & d_n \\
0 & 0 & \cdots & 0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \\
0 & 0 & \cdots & 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \\
0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 \\
\end{pmatrix}
\]

(4.1)

where

\[
(-1 -1 \cdots -1 -1 \cdots -1 d_0 d_1 \cdots d_n)
\]

is the \( i \)-th row. Taking the transpose of \( \phi_i \) and composing with the map in (3.2) we obtain isomorphisms

\[
N_{T^k} \oplus N_{\mathbb{P}(a_0, \ldots, a_n)} \xrightarrow{\psi_i} N_{U_i} \oplus N_{\mathbb{P}(a_0, \ldots, a_n)}. \]

We put

\[
N_{\overline{\mathbb{P}(E)}} := N_{T^k} \oplus N_{\mathbb{P}(a_0, \ldots, a_n)},
\]

and the rays of the fan are the inverse images under the isomorphisms \( \psi_i \) of the rays of the fans for \( U_i \times \mathbb{P}(a_0, \ldots, a_n) \). Observe that the inverse of the matrix (4.1) is
given by
\[
\begin{pmatrix}
-1 & -1 & \ldots & -1 & -1 & \ldots & -1 & d_0 & d_1 & \ldots & d_n \\
1 & 0 & \ldots & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 \\
0 & 1 & \ldots & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & 1 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 \\
0 & 0 & \ldots & 0 & 0 & 1 & \ldots & 0 & 0 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & 0 & 0 & \ldots & 1 & 0 & 0 & \ldots & 0 \\
0 & 0 & \ldots & 0 & 0 & \ldots & 0 & 1 & 0 & \ldots & 0 \\
0 & 0 & \ldots & 0 & 0 & \ldots & 0 & 0 & 1 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 1 \\
\end{pmatrix}
\]
\[(4.2)\]
where the $i$-th column is the column whose only nonzero entry is $-1$. Therefore the set of rays is the set of images in $N_{\mathbb{P}(E)}$ of the set of row vectors of the matrices (4.2) for all $i$. They are given by the row vectors (as opposed to the column vectors) because the matrix for the map $\psi_i$ is the transpose of the matrix (4.2). It is now straightforward to check the relations among this set of row vectors and verify that the fan for $\mathbb{P}(E)$ is isomorphic to the fan described in Construction 3.5. 

5. Intersection numbers on weighted $\mathbb{P}^1$-bundles over $\mathbb{P}^1$ 

Let $a_0$ and $a_1$ be integers that are relatively prime. In this section we consider divisors on $\mathbb{P}(a_0, a_1)$-bundles over $\mathbb{P}^1$. By Lemma 4.6 such bundles are isomorphic to non-weighted $\mathbb{P}^1$-bundles over $\mathbb{P}^1$, and we check how divisors transform under this isomorphism so that we may apply the formula (5.2) to obtain intersection numbers on $\mathbb{P}(a_0, a_1)$. In order to fix terminology and notation, we recall below the intersection theory of $\mathbb{P}^1$-bundles over $\mathbb{P}^1$. See [8] for details.

Example 5.1. Let $a$ and $b$ be integers, and form the locally free sheaf
\[ E := \mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(b) \]
over $\mathbb{P}^1$. Let $H \in A^1(\mathbb{P}(E))$ be the class of a fiber over $\mathbb{P}^1$, and let $L \in A^1(\mathbb{P}(E))$ be the divisor class corresponding to $\mathcal{O}_{\mathbb{P}(E)}(1)$. Then the Chow ring of $\mathbb{P}(E)$ is given by
\[
A^*\mathbb{P}(E) \cong \frac{\mathbb{Z}[H, L]}{(H^2, L^2 - (a + b)HL)}.
\]
(5.1)

Let $D_1$ be a divisor of type $(d_1, e_1)$, that is $D_1 \sim d_1L + e_1H$, and let $D_2$ be a divisor of type $(d_2, e_2)$. Then we find from (5.1), that the intersection number $D_1 \cdot D_2$ is given by
\[
D_1 \cdot D_2 = \deg(D_1D_2) = (a + b)d_1d_2 + d_1e_2 + d_2e_1.
\]
(5.2)

The calculation in (5.2) uses the fact that the cycle class $HL$ is the class of a point in $A^2\mathbb{P}(E)$. 

\[
\]
Let $d_0$ and $d_1$ be integers, and let $\mathcal{E} := \mathcal{O}_{\mathbb{P}^1}(d_0) \oplus \mathcal{O}_{\mathbb{P}^1}(d_1)$ be a weighted locally free sheaf with relatively prime weights $(a_0, a_1)$. Let $X_0$ and $X_1$ be global coordinates on $\mathbb{P}(\mathcal{E})$ as in the proof of Proposition 4.6 and let $S_0$ and $S_1$ be coordinates on the base $\mathbb{P}^1$. Each coordinate $X_i$ defines an effective divisor $Z(X_i)$ on $\mathbb{P}(\mathcal{E})$ by taking Proj of the surjective map of weighted symmetric algebras induced by the surjective map of sheaves

$$\mathcal{E} \longrightarrow \mathcal{E}/\mathcal{O}_{\mathbb{P}^1}(d_i).$$

The coordinates $S_i$ give divisors that are the pullbacks of divisors from the base $\mathbb{P}^1$. From Lemma 4.6 we know that

$$\mathbb{P}(\mathcal{E}) \cong \mathbb{P}(\mathcal{E}'),$$

where $\mathcal{E}' = \mathcal{O}_{\mathbb{P}^1}(a_1d_0) \oplus \mathcal{O}_{\mathbb{P}^1}(a_0d_1)$. The next lemma shows how to realize a divisor given by global coordinates on $\mathbb{P}(\mathcal{E})$ as a divisor given by global coordinates on $\mathbb{P}(\mathcal{E})$. For any real number $r$, the symbol $[r]$ denotes the smallest integer greater than or equal to $r$.

**Lemma 5.2.** Let $\mathcal{E} = \mathcal{O}_{\mathbb{P}^1}(d_0) \oplus \mathcal{O}_{\mathbb{P}^1}(d_1)$ be a weighted locally free sheaf with relatively prime weights $(a_0, a_1)$. Let $D = Z(S_0^{e_0}S_1^{e_1}X_0^{f_0}X_1^{f_1})$ be an effective divisor on $\mathbb{P}(\mathcal{E})$ defined by global coordinates. Let $D'$ be the corresponding divisor on the isomorphic variety $\mathbb{P}(\mathcal{E}')$ where

$$\mathcal{E}' = \mathcal{O}_{\mathbb{P}^1}(a_1d_0) \oplus \mathcal{O}_{\mathbb{P}^1}(a_0d_1).$$

Then $D' = Z(S_0^{e_0}S_1^{e_1}X_0^{f'_0}X_1^{f'_1})$ where for $i = 0$ or $i = 1$ and $j \neq i$ the integer $f'_i$ is given by

$$f'_i = \left\lceil \frac{f_i}{a_j} \right\rceil.$$

Furthermore, the divisor $D'$ has type

$$\left( \left\lfloor \frac{f_0}{a_1} \right\rfloor + \left\lfloor \frac{f_1}{a_0} \right\rfloor, e_0 + e_1 - \left\lceil \frac{f_0}{a_1} \right\rceil a_1d_0 - \left\lceil \frac{f_1}{a_0} \right\rceil a_0d_1 \right).$$

**Proof.** The isomorphism of Lemma 4.6 can be realized by taking Proj of the degree $a_0a_1$ inclusion of sheaves of graded algebras

$$\text{Sym}_{\mathcal{O}_{\mathbb{P}^1}} \mathcal{E}' \xrightarrow{h} \text{Sym}_{\mathcal{O}_{\mathbb{P}^1}} \mathcal{E},$$

where we have decorated the second “Sym” to emphasize that it is a weighted symmetric algebra. The divisor $D$ is given by a subsheaf of graded ideals

$$\mathcal{I}_0 \cdot \mathcal{J} \subseteq \text{Sym}_{\mathcal{O}_{\mathbb{P}^1}} \mathcal{E},$$

where $\mathcal{I}_0$ comes from the $S_1$ and is generated in degree 0, and where the degree $d$ part of $\mathcal{J}$ is given by

$$\mathcal{J}_d = \left( \bigoplus_{k,l \geq 0 \atop a_0(f_0+k)+a_1(f_1+l)=d} \mathcal{O}_{\mathbb{P}^1}((f_0+k)d_0 + (f_1+l)d_1) \cdot X_0^{f_0+k}X_1^{f_1+l} \right).$$
Now, the divisor $D'$ is defined by the sheaf of graded ideals
\[ h^{-1}(I_0 \cdot J) \subseteq \text{Sym}_{\mathcal{O}_P} \mathcal{L}', \]
and the first assertion follows easily from the description of $J_d$ above and the fact that $a_0$ and $a_1$ are relatively prime.

The second assertion follows from the fact that on any projective bundle
\[ \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(d_0) \oplus \mathcal{O}_{\mathbb{P}^1}(d_1)), \]
the global coordinate $X_i$ defines a divisor of type $(1, -d_i)$, and the global coordinate $S_j$ defines a divisor of type $(0, 1)$. This may be checked directly, and it also follows from our description of the bundle as a toric variety. □

6. LINEAR SYSTEMS

Before we address the question of finding anti-canonical hypersurfaces in weighted projective space bundles, we give some results regarding linear systems that we will need in what follows. These results are generalizations of the standard results for linear systems on nonsingular varieties as discussed, for example, in [11] or [9].

Let $X$ be a normal, projective variety, and let $\mathcal{L}$ be a coherent sheaf of $\mathcal{O}_X$-modules. Recall that $\mathcal{L}$ is reflexive if the canonical map
\[ \mathcal{L} \rightarrow \mathcal{L}^* \rightarrow \mathcal{L}^{**} \]
is an isomorphism, where we define
\[ \mathcal{L}^* := \mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X). \]
Furthermore, if $\mathcal{L}$ is a reflexive subsheaf of the function field $k(X)$, then we say that $\mathcal{L}$ is divisorial. See [17] for details about divisorial sheaves. The main point is that to any Weil divisor $D$ on $X$, we may associate a divisorial sheaf $\mathcal{O}_X(D)$ and this assignment induces a one-to-one correspondence between the set of Weil divisors on $X$ modulo linear equivalence and the set of isomorphism classes of divisorial sheaves of $\mathcal{O}_X$-modules. The following fact is straightforward to check.

**Lemma 6.1.** Let $D$ be a Weil divisor on $X$. The set of divisors that are linearly equivalent to $D$ is in one-to-one correspondence with the set of closed points of the projective space
\[ \mathbb{P}H^0(X, \mathcal{O}_X(D)). \]

**Definition 6.2.** Let $X$ be a normal projective variety, and let $D$ be a Weil divisor on $X$. The complete linear system, denoted $|D|$, is the set of all Weil divisors that are linearly equivalent to $D$. A linear system is a linear subspace of a complete linear system, where we regard the complete linear system as having the structure of a projective space guaranteed by Fact 6.11.

**Definition 6.3.** The base locus of a linear system $L$ is the set-theoretic intersection of the supports of the members of $L$.

**Definition 6.4.** Let $L \subseteq |D|$ be a linear system for some Weil divisor $D$ on $X$. We denote by $V_L$ the associated linear subspace of $H^0(X, \mathcal{O}_X(D))$. 
Now suppose that $X_\Sigma$ is a simplicial toric variety associated to some fan $\Sigma \subseteq N_\mathbb{R}$ for some lattice $N$. Let $S$ be the homogeneous coordinate ring of $X_\Sigma$. Recall that $S$ is an $A^1(X_\Sigma)$-graded polynomial ring. Let $D$ be a Weil divisor on $X$. From [3, Proposition 1.1] we find that

$$H^0(X_\Sigma, \mathcal{O}_{X_\Sigma}(D)) \cong S_{[D]},$$

where $[D]$ denotes the class of the divisor $D$ in $A^1(X_\Sigma)$, and $S_{[D]}$ denotes the degree $[D]$ part of the graded ring $S$. One easily checks that the zero scheme (see [3]) of a homogeneous form $F \in S_{[D]}$ is a divisor on $X$ in the class of $D$ and, hence, that linear systems may be described as linear subspaces of $S_{[D]}$. We illustrate the case for weighted projective bundles in the following example.

**Example 6.5.** Any complete linear system $|D|$ on $\tilde{\mathbb{P}}(E)$ may be written as the zero locus of forms

$$\sum \phi_{e_0 \cdots e_n}(S_0, \ldots, S_k)X_0^{e_0} \cdots X_n^{e_n} \subseteq S_{[D]}$$

where the exponents $e_i$ satisfy $\sum a_ie_i = C$ for some integer $C$, and $\phi_{e_0 \cdots e_n}$ is any form in $H^0(\mathbb{P}^k, \mathcal{O}(\eta))$ for some integer-valued function $\eta$ that depends on the exponents $e_i$. In case $|D| = -K$, the function $\eta$ is given by

$$\eta(e_0, \ldots, e_n) := \sum d_i(e_i - 1) + k + 1.$$  

We now check that the property of a linear system on a toric variety having a quasi-smooth member is an open condition.

**Lemma 6.6.** Let $X$ be a simplicial toric variety that is proper over $\mathbb{C}$, and let $L$ be a nonempty linear system on $X$. Then there exists a quasi-smooth member of $L$ if and only if the general member of $L$ is quasi-smooth.

**Proof.** One direction is clear. Since $L$ is nonempty and the general member is quasi-smooth, it must be the case that $L$ contains a quasi-smooth member.

We now establish the converse. Assume that $L$ contains a quasi-smooth member. Consider the closed incidence subvariety $Z_{\text{sing}} \subseteq L \times X$ given by

$$Z_{\text{sing}} := \{[D] \times p \mid p \in \text{sing } D\},$$

where $\text{sing } D$ denotes the locus where $D$ fails to be quasi-smooth. Let $\pi_L$ be the projection of $Z_{\text{sing}}$ onto $L$. The result now follows from the properness of $\pi_L$. □

One of the main tools in the analysis of linear systems is Bertini’s Theorem. We check that a version of Bertini’s Theorem holds that addresses the quasi-smoothness of members of linear systems on toric varieties.

**Proposition 6.7.** Let $X$ be a simplicial toric variety that is proper over $\mathbb{C}$, and let $L$ be a linear system on $X$. Then the general member of $L$ is quasi-smooth away from the base locus of $L$.

**Proof.** Let $S$ be the homogeneous coordinate ring of $X$, and let

$$\text{Spec } S - Z(B) \xrightarrow{q} X$$
be the quotient map (see Section 2). Since quasi-smoothness of a divisor on $X$ is determined by the nonsingularity of a hypersurface on $\text{Spec } S - Z(B)$, the argument given in the proof of Bertini’s Theorem in [9, p. 137] applies in our case to establish the proposition. □

It will be convenient to have the notion of the restriction of a linear system to the fiber of a weighted projective space bundle.

**Definition 6.8.** Let $X$ be a simplicial toric variety, let $j : Y \hookrightarrow X$ be a well-formed quasi-smooth subvariety, and let $L \subseteq |D|$ be a linear system on $X$ for some Weil divisor $D$. Since $Y$ is well-formed, the sheaf $j^*\mathcal{O}_X(D)$ is divisorial. We define the **restricted linear system**, denoted $L|_Y$, to be the linear system associated to the image of $V_L$ under the natural $\mathbb{C}$-linear map $H^0(X, \mathcal{O}_X(D)) \rightarrow H^0(Y, j^*\mathcal{O}_X(D))$.

**Example 6.9.** Let $|D|$ be a complete linear system on a weighted projective space bundle, and let $F \cong \mathbb{P}(a_0, \ldots, a_n)$ be a fiber of the bundle. It follows easily from the description given in Example 6.5 that the restricted linear system $|D||_F$ is independent of the choice of fiber $F$ when regarded as a linear system on $\mathbb{P}(a_0, \ldots, a_n)$.

**Proposition 6.10.** Let $|D|$ be a complete linear system on $\widetilde{\mathbb{P}}(\mathcal{E})$ over $\mathbb{P}^k$ for some divisor $|D|$, and let $F \cong \mathbb{P}(a_0, \ldots, a_n)$ be a fiber. If the general member of $|D|$ is quasi-smooth, then so is the general member of the restricted linear system $L := |D||_F$.

**Proof.** Suppose, to the contrary, that every member of $L$ fails to be quasi-smooth. We will show that the same is true for $|D|$. Let $G$ be the equation for a member of $|D|$. Restricting $G$ to any fiber, the assumption that every member of $L$ fails to be quasi-smooth implies that the system of equations

$$G = \frac{\partial G}{\partial X_0} = \cdots = \frac{\partial G}{\partial X_n} = 0$$

has a nontrivial solution $P$. Furthermore, by Proposition 6.7, we may assume that $(X_0(P), \ldots, X_n(P)) \in \text{Bs } L$. We claim that $P$ is also a solution to the system of equations

$$\frac{\partial G}{\partial S_0} = \cdots = \frac{\partial G}{\partial S_k} = 0.$$  

Indeed, restricting the equations above to the fiber $F_P$ that contains $P$ yields a system of $(k + 1)$ equations, each of which defines a member of $L$. But $P|_{F_P} \in \text{Bs } L$ by assumption, and the claim follows. □

To conclude this section we state a result from [12] that addresses the problem of determining whether a linear system on $\mathbb{P}(a_0, \ldots, a_3)$ has a quasi-smooth member. In [12] the result is stated only for complete linear systems, but the result is a corollary of [12, 8.1 Theorem] whose proof applies more generally to linear systems whose associated vector space of homogeneous forms is generated by monomials.

**Definition 6.11.** Let $L$ be a linear system on weighted projective space. If the associated vector space of forms is generated by monomials, then we say that $L$ is a **monomial linear system**.
Proposition 6.12 ([12, 8.5 Corollary]). Let \((a_0, \ldots, a_3)\) be a sequence of positive integers satisfying (3.1), and let \(L\) be a monomial linear system on \(P(a_0, \ldots, a_3)\) of degree \(d\) where \(d > a_i\) for all \(0 \leq i \leq 3\). Then the general member of \(L\) is quasi-smooth if and only if the following two conditions hold.

1. For all \(0 \leq i \leq 3\) there exists a monomial in \(V_L\) that does not involve \(X_i\).
2. For all \(0 \leq i \leq 3\) there exists a monomial in \(V_L\) of the form \(X_i^{p_1}X_i^{q_2}X_{e_1}\) for some \(0 \leq e_1 \leq 3\).
3. For all \(0 \leq i < j \leq 3\)
   a) there exists a monomial in \(V_L\) of the form \(X_i^{p_1}X_j^{q_2}\) or
   b) there exist monomials in \(V_L\) of the form \(X_i^{p_1}X_j^{q_2}X_{e_1}\) and \(X_i^{p_2}X_j^{q_2}X_{e_2}\) for distinct \(e_1\).

7. Well-formedness and anti-canonical hypersurfaces

This section is concerned with the condition of well-formedness as given in Definition 2.2. In Section 8, we will apply the adjunction formula to a quasi-smooth anti-canonical hypersurface in a weighted projective bundle in order to show that it is a Calabi-Yau variety. For the adjunction formula to work as expected, we will need to know that the hypersurface in question is well-formed. As we now establish, it turns out that well-formedness is automatic for quasi-smooth anti-canonical hypersurfaces. In [12] Iano-Fletcher considered the problem of well-formedness of hypersurfaces in weighted projective space, and we take up this case first.

In order to obtain information about the well-formedness of hypersurfaces in weighted projective space, we need to understand the singular locus of weighted projective space. All the singularities are finite cyclic quotient singularities, and we recall notation for such singularities below.

Definition 7.1. Let \(a_1, \ldots, a_n\) be integers, let \(r\) be a positive integer, and let \(x_1, \ldots, x_n\) be coordinates on \(\mathbb{A}^n\). Let \(Z_r\) act on \(\mathbb{A}^n\) via

\[x_i \mapsto \varepsilon^{a_i}x_i\quad\text{for all } i,\]  

where \(\varepsilon\) is a fixed primitive \(r\)-th root of unity. Let \(X\) be an algebraic variety over \(\mathbb{C}\). A singularity \(Q \in X\) is a (quotient) singularity of type \(\frac{1}{r}(a_1, \ldots, a_n)\) if \((X, Q)\) is locally analytically isomorphic to the quotient \((\mathbb{A}^n/Z_r, O)\) under the action defined in (7.1), where \(O\) denotes the image of the origin under the quotient map.

Lemma 7.2 ([12, §5.15]). Let \((a_0, \ldots, a_n)\) be a sequence of integers satisfying (3.1). Let \(Z_{i_1 \cdots i_d} \subseteq \mathbb{P}(a_0, \ldots, a_n)\) be the closed subvariety defined by

\[X_{i_1} = \cdots = X_{i_d} = 0\]

for distinct \(i_1, \ldots, i_d\). The general point of \(Z_{i_1 \cdots i_d}\) is locally analytically isomorphic to \((O, Q) \in \mathbb{A}^{n-d} \times Y\) where \(Q \in Y\) is a singularity of type

\[\frac{1}{h_{i_1 \cdots i_d}}(a_{i_1} \cdots, a_{i_d}),\]

and where \(h_{i_1 \cdots i_d}\) is the greatest common divisor of the complement of \((a_{i_1}, \ldots, a_{i_d})\) in \((a_0, \ldots, a_d)\).
The next lemma is a generalization of \([12, 6.10]\), in which it is stated (without proof) only for complete linear systems.

**Lemma 7.3** (\([12, \S 6.10]\)). Let \((a_0, \ldots, a_n)\) be a sequence of positive integers satisfying (3.1), and let \(L\) be a monomial linear system of degree \(d\) on \(\mathbb{P}(a_0, \ldots, a_n)\). The general member of \(L\) is well-formed if and only if for all \(0 \leq i < j \leq n\)

1. there exists a monomial \(M \in V_L\) such that \(M \notin (X_i, X_j)\), or
2. we have \(h_{ij} = 1\) where \(h_{ij}\) is as defined in the statement of Lemma 7.2.

**Proof.** First suppose that the general member \(L\) is well-formed and that condition 1 fails to hold for some \(0 \leq i < j \leq n\). We must show that condition 2 holds. The failure of condition 1 to hold means precisely that the subvariety \(Z_{ij}\) of \(\mathbb{P}(a_0, \ldots, a_n)\) defined by \(X_i = X_j = 0\) is contained in the base locus of \(L\). But if the general member of \(L\) is well-formed it must be the case that the general point of \(Z_{ij}\) is nonsingular. According to Lemma 7.2 we must have \(h_{ij} = 1\) and this is condition 2.

Now assume that condition 1 holds for all \(0 \leq i < j \leq n\). Then the codimension of the base locus of \(L\) is greater than two, and it follows that the general member of \(L\) is well-formed. Suppose that condition one fails to hold for some \(0 \leq i < j \leq n\), but condition two does hold. Then the subvariety \(Z_{ij}\) defined in the previous paragraph is in the base locus, but its general point is nonsingular. Again, it follows that the general member of \(L\) is well-formed.

**Proposition 7.4.** Let \((a_0, \ldots, a_3)\) be a sequence of positive integers satisfying (3.1). Let \(L \subseteq \left| -K_{\mathbb{P}(a_0, \ldots, a_3)} \right|\) be a monomial linear system. If the general member of \(L\) is quasi-smooth, then the general member of \(L\) is well-formed.

**Proof.** Suppose that condition 1 of Lemma 7.3 fails to hold for some \(0 \leq i < j \leq 3\). This means that the base locus of \(L\) has an irreducible component given by \(X_i = X_j = 0\). Let \({i', j'}\) denote the complement of \({i, j}\) in the set \({0, \ldots, 3}\). Then according to Proposition 6.12 there exist monomials in \(V_L\) of the form \(X_{e_1}^{p_1} X_{e_2}^{p_2} X_{e_1}^{p_3} X_{e_2}^{p_4}\) for distinct \(e_1\) and \(e_2\) such that \({e_1, e_2}\) = \({i, j}\) In particular, this means that

\[
p_1 a_{i'} + p_2 a_{j'} + a_{e_1} = \sum_{l=0}^{3} a_l.
\]

Since we are assuming that (3.1) holds, the equation above implies that

\[
\gcd(a_{i'}, a_{j'}) = 1
\]

as required to satisfy condition 2 of Lemma 7.3. Therefore, the general member of \(L\) is well-formed.

**Corollary 7.5.** Let \(\tilde{\mathbb{P}}(E)\) be a weighted \(\mathbb{P}(a_0, \ldots, a_3)\)-bundle over \(\mathbb{P}^1\). If the general member of \(\left| -K_{\tilde{\mathbb{P}}(E)} \right|\) is quasi-smooth, then it is also well-formed.

**Proof.** If the general member of \(\left| -K_{\tilde{\mathbb{P}}(E)} \right|\) is not well-formed, then the same is true for the linear system \(L\) on \(\mathbb{P}(a_0, \ldots, a_3)\) obtained by restricting \(\left| -K_{\tilde{\mathbb{P}}(E)} \right|\) to a fiber. The result now follows easily from Proposition 6.10 and Proposition 7.4
8. QUASI-SMOOTH AND WELL-FORMED ANTI-CANONICAL HYPERSURFACES

We come now to our main result, in which we classify all \( \mathbb{P}(a_0,\ldots,a_3) \)-bundles over \( \mathbb{P}^1 \) whose anti-canonical linear systems have a quasi-smooth member. We first check that quasi-smooth anti-canonical hypersurfaces in weighted projective bundles over \( \mathbb{P}^k \) are necessarily Calabi-Yau varieties fibered by smaller dimensional Calabi-Yau varieties. Due to the various notions in the literature, we include here the definition of a Calabi-Yau variety that we use.

**Definition 8.1** ([4, Definition 1.4.1]). Let \( X \) be a normal variety that is proper over \( \mathbb{C} \) and has at worst canonical singularities. We say that \( X \) is a **Calabi-Yau variety** if

\[
H^1(X, \mathcal{O}_X) = \cdots = H^{\dim X-1}(X, \mathcal{O}_X) = 0.
\]

In order to conclude that quasi-smooth anti-canonical hypersurfaces in weighted projective space bundles are fibered by smaller dimensional Calabi-Yau varieties, we must first check that quasi-smooth anti-canonical hypersurfaces of weighted projective space are themselves Calabi-Yau varieties.

**Proposition 8.2.** Let \((a_0,\ldots,a_n)\) be a sequence of integers satisfying (3.1). A quasi-smooth and well-formed member

\[
X \in | -K_{\mathbb{P}(a_0,\ldots,a_n)}|
\]

is a Calabi-Yau variety.

**Proof.** First we show that the dualizing sheaf \( \omega_X \) is trivial. For any well-formed, quasi-smooth hypersurface \( Z_d \) of degree \( d \) in \( \mathbb{P}(a_0,\ldots,a_n) \) we have

\[
\omega_{Z_d} \cong \mathcal{O}_{Z_d} \left( d - \sum a_i \right)
\]

(see [6, Theorem 3.3.4] or [12, §6.14]). Since \( X \) is anti-canonical, it has degree \( \sum a_i \), and it follows that \( \omega_X \cong \mathcal{O}_X \). Since \( X \) is quasi-smooth, it is an orbifold, and a Gorenstein orbifold necessarily has canonical singularities (see [17, Proposition 1.7]). Finally, the necessary vanishing of cohomology follows from the known cohomology of hypersurfaces in weighted projective space. (See [6, Section 3.4.3] or [12, 7.1 Lemma]).

**Theorem 8.3.** Let \( \overline{\mathbb{P}}(\mathcal{E}) \) be a weighted projective bundle over \( \mathbb{P}^k \). A quasi-smooth and well-formed member of \( | -K_{\overline{\mathbb{P}}(\mathcal{E})} | \) is a Calabi-Yau variety whose general fiber over \( \mathbb{P}^k \) is a weighted Calabi-Yau hypersurface.

**Proof.** Let \( X \in | -K_{\overline{\mathbb{P}}(\mathcal{E})} | \) be quasi-smooth and well-formed. Note that if \( X \) is quasi-smooth, it is automatically irreducible (this follows since \( X \) is transverse to a fiber). We first show that \( K_X = 0 \). Let \( X_0 = \overline{\mathbb{P}}(\mathcal{E})_{sm} \cap X \) where \( \overline{\mathbb{P}}(\mathcal{E})_{sm} \) denotes the nonsingular locus of \( \overline{\mathbb{P}}(\mathcal{E}) \), and let \( i : X_0 \to X \) be the inclusion map. Since \( X \) is well-formed, it follows that \( X_0 \) is smooth and that the codimension of \( X \setminus X_0 \) in
The adjunction formula applied to $X_0$ and $\tilde{\mathbb{P}}(E)_{sm}$ yields
\[
\mathcal{O}_X(K_X) \cong i_* \mathcal{O}_{X_0}(K_{X_0}) \\
\cong i_* \left( (\mathcal{O}_{\tilde{\mathbb{P}}(E)_{sm}}(K_{\tilde{\mathbb{P}}(E)_{sm}}) \otimes \mathcal{O}_{\tilde{\mathbb{P}}(E)_{sm}}(X_0) \otimes \mathcal{O}_{X_0} \right) \\
\cong i_* \mathcal{O}_{X_0} \\
\cong \mathcal{O}_X.
\]
The last isomorphism follows from the fact that both sheaves are isomorphic to the divisorial sheaf associated to the zero divisor. The dualizing sheaf of $X$ is given by $\mathcal{O}_X(K_X) \cong \mathcal{O}_X$, so $X$ is Gorenstein, and a Gorenstein orbifold necessarily has canonical singularities (see [17, Proposition 1.7]).

We next show the necessary vanishing of cohomology. We use the following exact sequence.
\[
0 \longrightarrow \mathcal{O}_{\tilde{\mathbb{P}}(E)}(K_{\tilde{\mathbb{P}}(E)}) \longrightarrow \mathcal{O}_{\tilde{\mathbb{P}}(E)} \longrightarrow \mathcal{O}_X \longrightarrow 0 \quad (8.1)
\]
Taking the long exact sequence in cohomology, we find it suffices to show that $H^i(\tilde{\mathbb{P}}(E), \mathcal{O}_{\tilde{\mathbb{P}}(E)}) = 0$ for $1 \leq i \leq k + n - 2$ and $H^j(\tilde{\mathbb{P}}(E), \mathcal{O}_{\tilde{\mathbb{P}}(E)}(K_{\tilde{\mathbb{P}}(E)})) = 0$ for $2 \leq j \leq k + n - 1$. By Serre duality we have
\[
H^j(\tilde{\mathbb{P}}(E), \mathcal{O}_{\tilde{\mathbb{P}}(E)}(K_{\tilde{\mathbb{P}}(E)})) \cong H^{k+n-j}(\tilde{\mathbb{P}}(E), \mathcal{O}_{\tilde{\mathbb{P}}(E)})^*,
\]
and for any toric variety $Z$, we have $H^i(Z, \mathcal{O}_Z) = 0$ for $i > 0$ ([5, Corollary 7.4]); this completes the proof that $X$ is a Calabi-Yau variety.

For the second assertion, note that generic smoothness over $\mathbb{P}^k$ of $q^{-1}(X)$, where $q$ is the quotient map in Definition 2.1, implies that the general fiber of $X \longrightarrow \mathbb{P}^k$ is quasi-smooth. Therefore, the general fiber is a well-formed anti-canonical quasi-smooth hypersurface in $\mathbb{P}(a_0, \ldots, a_n)$, whence a weighted Calabi-Yau hypersurface by Proposition 8.2.

It remains to find necessary and sufficient conditions for the anti-canonical system of a weighted projective bundle to have a quasi-smooth member. We begin with a lemma that allows us to assume that the weighted sheaf from which we form the weighted projective bundle has a certain normalized form.

**Lemma 8.4.** Every weighted projective space bundle over $\mathbb{P}^k$ is isomorphic to $\tilde{\mathbb{P}}(E)$, where $E$ is a weighted locally free sheaf of rank $n + 1$ with weights $(a_0, \ldots, a_n)$ satisfying (3.1) and such that
\[
E \cong \bigoplus_{i=0}^{n} \mathcal{O}_{\mathbb{P}^k}(d_i)
\]
where
\[
0 \leq d_0 \leq d_1 \leq \cdots \leq d_n, \quad (8.2)
\]
and there exists an $0 \leq i_0 \leq n$ such that
\[
d_{i_0} < a_{i_0}. \quad (8.3)
\]
Proof. Let $\mathcal{E}$ be a weighted locally free sheaf with weights $(a_0, \ldots, a_n)$. Since we are only considering weighted locally free sheaves that are sums of invertible sheaves (see Definition 4.1), we may assume that

$$\mathcal{E} \cong \bigoplus_{i=0}^{n} \mathcal{O}_{\mathbb{P}^k}(e_i)$$

for some integers $e_0, \ldots, e_n$ and that $\mathcal{E}$ is weighted by $(a_0, \ldots, a_n)$. Given any integer $l$ we find from Lemma 4.5 that

$$\widetilde{P}(\mathcal{E}) \cong \widetilde{P}(\mathcal{E}')$$

where $\mathcal{E}' = \bigoplus_{i=0}^{n} \mathcal{O}_{\mathbb{P}^k}(e_i + la_i)$.

Let $\lambda$ be the smallest integer such that $e_i + \lambda a_i \geq 0$ for all $0 \leq i \leq n$, define

$$d_i := e_i + \lambda a_i,$$

and renumber so that $d_0 \leq d_1 \leq \cdots \leq d_n$. If $d_i \geq a_i$ for all $0 \leq i \leq n$, then we have $e_i + (\lambda - 1)a_i \geq 0$, but this contradicts the minimality of the integer $\lambda$. Therefore $d_0, \ldots, d_n$ give the required integers. \[ \square \]

We are now in position to prove our classification result for anti-canonical systems on $\mathbb{P}(a_0, \ldots, a_3)$-bundles over $\mathbb{P}^1$.

**Theorem 8.5.** Let $(d_0, \ldots, d_3)$ be a sequence of integers satisfying (8.2) and (8.3), and let $(a_0, \ldots, a_3)$ be a sequence of positive integers satisfying (3.1). Let $\mathcal{E}$ be the weighted locally free sheaf

$$\mathcal{E} := \mathcal{O}_{\mathbb{P}^1}(d_0) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(d_3)$$

with weights $(a_0, \ldots, a_3)$. Let $L$ denote the linear system on $\mathbb{P}(a_0, \ldots, a_3)$ obtained by restricting $|−K_{\mathbb{P}(\mathcal{E})}|$ to a fiber, let $V_L$ be its associated vector space of homogeneous forms, and let $\eta$ be defined via (6.3). Then the linear system $|−K_{\mathbb{P}(\mathcal{E})}|$ has a well-formed quasi-smooth member if and only if the following conditions hold.

1. The general member of the linear system $L$ is quasi-smooth.
2. For all $0 \leq i \leq 3$
   (a) there exists a monomial of the form $X_i^p$ in $V_L$, or
   (b) there exists an integer $j \neq i$ and a monomial $M \in V_L$ of the form $X_j X_i^p$ such that $\eta(M) = 0$, or
   (c) there exist distinct integers $j_1 \neq i$ and $j_2 \neq i$ and two monomials $M_1$ and $M_2$ of the form $X_{j_1} X_i^p$ and $X_{j_2} X_i^p$ respectively such that $\eta(M_1) > 0$ and $\eta(M_2) > 0$.
3. For all $0 \leq i < j \leq 3$
   (a) there exists a monomial of the form $X_i^p X_j^q \notin V_L$, or
   (b) there exist monomials of the form
   $$N_1 = X_i^{p_1} X_j^{p_2} X_{e_1} \in V_L$$
   and
   $$N_2 = X_i^{p_2} X_j^{p_3} X_{e_2} \in V_L$$
toric calabi-yau hypersurfaces 21

in $V_L$ such that $e_1$ and $e_2$ are distinct and such that

\[
\left( \left\lfloor \frac{p_1}{a_j} \right\rfloor + \left\lceil \frac{q_1}{a_i} \right\rceil \right) \eta(N_2) + \left( \left\lfloor \frac{p_2}{a_j} \right\rfloor + \left\lceil \frac{q_2}{a_i} \right\rceil \right) \eta(N_1) + \\
\left( \left\lceil \frac{p_1}{a_j} \right\rceil \left\lfloor \frac{p_2}{a_j} \right\rfloor + \left\lceil \frac{q_1}{a_i} \right\rceil \left\lfloor \frac{q_2}{a_i} \right\rceil \right) (a_i d_j - a_j d_i) = 0. \quad (8.4)
\]

Proof. Recall that if the general member of $|-K_{\mathbb{P}(E)}|$ is quasi-smooth, then it is automatically well-formed by Corollary 7.5. Now suppose that the general member of $|-K_{\mathbb{P}(E)}|$ is quasi-smooth. From Proposition 6.10 we see that condition 1 holds. We must show that conditions 2 and 3 hold. We proceed according to the codimension of the base locus of $|-K_{\mathbb{P}(E)}|$. The equations that define the base locus are given by monomials in the variables $X_0, \ldots, X_3$. If the base locus is empty, then conditions 2.(a) and 3.(a) hold. Also, there can be no fixed component because then $L$ would have a fixed component, and this would contradict the fact that condition 1 holds. This leaves two possibilities: either the base locus of $|-K_{\mathbb{P}(E)}|$ has codimension three or it has codimension two.

We first consider the case in which

\[
\text{codim Bs} |-K_{\mathbb{P}(E)}| = 3,
\]

and we suppose that $C$ is a component of the base locus given by

\[
X_{j_1} = X_{j_2} = X_{j_3} = 0
\]

for

\[
0 \leq j_1 < j_2 < j_3 \leq 3.
\]

Let $F$ denote the equation for a general member of $|-K_{\mathbb{P}(E)}|$. Since $Z(F)$ is quasi-smooth we find, in particular, that the system of equations

\[
\frac{\partial F}{\partial X_{j_1}} = \frac{\partial F}{\partial X_{j_2}} = \frac{\partial F}{\partial X_{j_3}} = 0 \quad (8.5)
\]

has no nontrivial solution on $C$. Restricting to $C$, the equations in (8.5) become

\[
\phi_{j_1} X_i^{p_1} = \phi_{j_2} X_i^{p_2} = \phi_{j_3} X_i^{p_3} = 0, \quad (8.6)
\]

for $i \notin \{j_1, j_2, j_3\}$ and for forms

\[
\phi_{ji} \in H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(\eta(X_{ji}, X_i^{p_i}))).
\]

At least one of the forms must be nonzero otherwise there could be no quasi-smooth member. If only one form is nonzero, it must have degree zero, otherwise it would have a zero. This is condition 2.(b). If at least two of the $\phi_{ji}$ are nonzero, and neither has degree zero, then condition 2.(c) holds. Finally, since the dimension of the base locus is zero, it follows that condition 3.(a) holds.

Now assume that

\[
\text{codim Bs} |-K_{\mathbb{P}(E)}| = 2.
\]
If the base locus has a component of dimension zero, then we are in the situation of the previous paragraph. Let \( C \) be a component of the base locus given by \( X_{e_1} = X_{e_2} = 0 \) and let \( \{i, j\} \) denote the complement of \( \{e_1, e_2\} \) in \( \{0, 1, 2, 3\} \). Observe that

\[
C \cong \mathbb{P}(O_{\mathbb{P}^1}(d_i) \oplus O_{\mathbb{P}^1}(d_j))
\]

with weights \((a_i, a_j)\). Furthermore, we claim that \( \gcd(a_i, a_j) = 1 \). Indeed, since condition one is satisfied, Proposition 6.12 ensures the existence of a monomial of the form \( X_{p_1}^i X_{p_2}^j X_{e_1} \). Hence we have

\[
p_1 a_i + p_2 a_j + a_{e_1} = \sum a_i,
\]

from which the claim follows since the weights \((a_0, \ldots, a_3)\) satisfy (3.1). Therefore, it follows from Lemma 5.2 that

\[
C \cong \mathbb{P}(O_{\mathbb{P}^1}(a_j d_i) \oplus O_{\mathbb{P}^1}(a_i d_j)).
\]

Arguing as before we find that the quasi-smoothness of the general member \( Z(F) \) implies that the system of equations

\[
\frac{\partial F}{\partial X_{e_1}} = \frac{\partial F}{\partial X_{e_2}} = 0
\]

has no nontrivial solution on \( C \). If we restrict the partial derivatives (8.7) to \( C \) we obtain the equation for two divisors

\[
D_{e_1} := Z \left( \frac{\partial F}{\partial X_{e_1}} \right) \bigg|_C \quad \text{and} \quad D_{e_2} := Z \left( \frac{\partial F}{\partial X_{e_2}} \right) \bigg|_C
\]

on \( C \) that meet transversally. Therefore, the existence of a nontrivial solution on \( C \) to (8.7) is equivalent to the intersection number \( D_{e_1} \cdot D_{e_2} \) not vanishing. Furthermore, since condition 1 is satisfied, Proposition 6.12 guarantees the existence of monomials of the form \( N_1 \) and \( N_2 \) in \( V_L \), and we must check that (8.4) is satisfied. The presence of the monomials \( N_1 \) and \( N_2 \) in \( V_L \) allows us to use Lemma 5.2 to determine the type of the two divisors on \( C \). Referring to Example 5.1 for notation and terminology, we find that

\[
D_{e_1} \text{ has type } \left( \left[ \frac{p_1}{a_j} \right], \eta(N_1) - \left[ \frac{q_1}{a_j} \right] a_j d_i - \left[ \frac{q_1}{a_j} \right] a_i d_j \right)
\]

and

\[
D_{e_2} \text{ has type } \left( \left[ \frac{p_2}{a_j} \right], \eta(N_2) - \left[ \frac{q_2}{a_j} \right] a_j d_i - \left[ \frac{q_2}{a_j} \right] a_i d_j \right),
\]

and the intersection number \( D_{e_1} \cdot D_{e_2} \) simplifies to the left-hand side of (8.4).

Now suppose that conditions 1-3 hold. We will show that the general member is quasi-smooth. It suffices to proceed according to irreducible components present in the base locus of \( |-K_{\mathbb{P}(E)}| \). If the base locus is empty, then Proposition 6.7 guarantees the existence of a quasi-smooth member. Condition 1 prevents there from being a fixed component in the base locus. Now suppose that the base locus has an irreducible component \( C \) of codimension 2 given by \( \{X_{e_1} = X_{e_2} = 0\} \). Then we must check that it cannot be the case that all the members fail to be quasi-smooth on \( C \). As described in the previous paragraph, condition 3.(b) prevents this from
happening. Note that we also need condition 1 to ensure that the two divisors $D_{e_1}$ and $D_{e_2}$ from the previous paragraph intersect transversally. It remains to check the case in which the base locus has an irreducible component $C$ of codimension three. Again, as described in the previous paragraph, conditions 2.(b) and 2.(c) ensure that not every member fails to be quasi-smooth on $C$. This completes the proof.

\[ \square \]

**Corollary 8.6.** With the notation and assumptions of Theorem 8.5, the general member of the linear system $| - K_{\mathbb{P}(E)}|$ is quasi-smooth and well-formed if and only if the following conditions hold.

1. For all $0 \leq i \leq 3$ there exists a monomial in $V_L$ that does not involve $X_i$.
2. For all $0 \leq i \leq 3$
   a. there exists a monomial of the form $X_i^p$ in $V_L$, or
   b. there exists an integer $j \neq i$ and a monomial $M \in V_L$ of the form $X_j X_i^p$ such that $\eta(M) = 0$, or
   c. there exist distinct integers $j_1 \neq i$ and $j_2 \neq i$ and two monomials $M_1$ and $M_2$ of the form $X_{j_1} X_i^p$ and $X_{j_2} X_i^p$ respectively such that $\eta(M_1) > 0$ and $\eta(M_2) > 0$.
3. For all $0 \leq i < j \leq 3$
   a. there exists a monomial of the form $X_i^p X_j^q \in V_L$ or
   b. there exist monomials of the form
      \[ N_1 = X_i^{p_1} X_j^{q_1} X_{e_1} \in V_L \]
      \[ N_2 = X_i^{p_2} X_j^{q_2} X_{e_2} \in V_L \]
      in $V_L$ such that $e_1$ and $e_2$ are distinct and such that
      \[
      \left( \left\lfloor \frac{p_1}{a_j} \right\rfloor + \left\lfloor \frac{q_1}{a_i} \right\rfloor \right) \eta(N_2) + \left( \left\lfloor \frac{p_2}{a_j} \right\rfloor + \left\lfloor \frac{q_2}{a_i} \right\rfloor \right) \eta(N_1) + \left( \left\lfloor \frac{p_1}{a_j} \right\rfloor \left\lfloor \frac{p_2}{a_j} \right\rfloor + \left\lfloor \frac{q_1}{a_i} \right\rfloor \left\lfloor \frac{q_2}{a_i} \right\rfloor \right) (a_i d_j - a_j d_i) = 0. \quad (8.8)
      \]

**Proof.** This follows immediately from Theorem 8.5 and Proposition 6.12. \[ \square \]

**9. Elliptically Fibered Calabi-Yau Threefolds**

Using techniques similar to those used in the proof of Theorem 8.5, we can prove the analogous result for the case of Calabi-Yau threefolds fibered over $\mathbb{P}^2$ whose general fiber is a genus one curve in $\mathbb{P}(1,1,1)$, $\mathbb{P}(1,1,2)$, or $\mathbb{P}(1,2,3)$. The statement of the full result follows. It turns out that these Calabi-Yau threefolds are nonsingular. See [16, Section 4.4] for details.

**Theorem 9.1.** Let $(d_0, d_1, d_2)$ be a sequence of integers that satisfy (8.2) and (8.3), and let $(a_0, a_1, a_2)$ be a sequence of positive integers satisfying (3.1). Let $\mathcal{E}$ be the weighted locally free sheaf

\[ \mathcal{E} := \mathcal{O}_{\mathbb{P}^2}(d_0) \oplus \mathcal{O}_{\mathbb{P}^2}(d_1) \oplus \mathcal{O}_{\mathbb{P}^2}(d_2) \]
with weights \((a_0, a_1, a_2)\). Let \(L\) denote the linear system on \(\mathbb{P}(a_0, a_1, a_2)\) obtained by restricting \(|-K_{\mathbb{P}(E)}|\) to a fiber, and let \(\eta\) be defined via (6.3). Then the linear system \(|-K_{\mathbb{P}(E)}|\) has a well-formed nonsingular member if and only if the following conditions hold.

1. For all \(i \in \{0, 1, 2\}\) there exists a monomial in \(V_L\) that does not involve \(X_i\).
2. For all \(i \in \{0, 1, 2\}\) either
   a. there exists a monomial of the form \(X_i^p\) in \(V_L\) or
   b. there exists an integer \(j \neq i\) and a monomial \(M\) of the form \(X_i^pX_j\) such that \(\eta(M) = 0\).

There are 92 families of elliptically fibered Calabi-Yau threefolds over \(\mathbb{P}^2\) arising as members of the anti-canonical linear system of \(\mathbb{P}(a_0, a_1, a_2)\)-bundle over \(\mathbb{P}^2\). We list these families in the appendix.

10. Final remarks

It is natural to ask to what extent the Calabi-Yau threefolds we have found are new. From the point of view of mirror symmetry, they are not new in the sense that they can be related to Batyrev’s mirror symmetry construction (see [1]). Although the varieties \(\tilde{\mathbb{P}}(E)\) are not, in general, reflexive (i.e. Fano and Gorenstein), their Calabi-Yau hypersurfaces have crepant resolutions that are anti-canonical hypersurfaces in reflexive toric varieties, and these have all been classified in [14]. The existence of these resolutions is established in [16] by checking that the Newton polyhedra of all toric weighted projective bundles admitting quasi-smooth anti-canonical hypersurfaces are reflexive.

The varieties are interesting, however, for other reasons. For example, in [15] fiber structures of toric Calabi-Yau hypersurfaces are considered in the reflexive case. It can be tricky to find fiber structures from that point of view, and as far as the author is aware, no such explicit description of weighted K3-fibered Calabi-Yau threefolds as given here has appeared before, and it was not necessary to have reflexive toric varieties to write them down. Additionally, our result gives a direct generalization of Reid’s discovery of the “famous 95” families of weighted K3 hypersurfaces, and is thus interesting from the point of view of explicit birational geometry.

Appendix A. Some data in the K3 case

We give here a partial list of K3-fibered Calabi-Yau hypersurfaces in toric weighted projective bundles over \(\mathbb{P}^1\). The list is in the form of a hash table in Macaulay 2. (Note: The raw Macaulay 2 output has been altered by inserting line-breaks.) Each entry in the hash table consists of a key and an ASCII arrow “\(=\)" followed by a value. Each key is a list of four integers, and each value is a list of lists of integers. If \{\(a, b, c, d\)\} is a key and \{\(e, f, g, h\)\} is a member of the list of values for \{\(a, b, c, d\)\}, then the general member of the linear system \(|-K_{\mathbb{P}(E)}|\) is quasi-smooth where \(E\) is the weighted locally free sheaf

\[O_{\mathbb{P}^1}(e) \oplus O_{\mathbb{P}^1}(f) \oplus O_{\mathbb{P}^1}(g) \oplus O_{\mathbb{P}^1}(h)\]
weighted by \((a, b, c, d)\). The complete list of all 3,723 families and details of the calculation appear in [16].

\[
\begin{align*}
\{1, 1, 1, 1\} & \Rightarrow \{0, 0, 0, 0\}, \{0, 0, 0, 1\}, \{0, 0, 0, 2\}, \{0, 0, 1, 1\}, \\
& \{0, 0, 1, 2\}, \{0, 1, 1, 1\}, \{0, 1, 1, 2\}, \{0, 1, 1, 3\}, \\
& \{0, 1, 1, 4\}\}
\end{align*}
\]
\[
\begin{align*}
\{1, 1, 1, 2\} & \Rightarrow \{0, 0, 0, 0\}, \{0, 0, 0, 1\}, \{0, 0, 0, 2\}, \{0, 0, 1, 1\}, \\
& \{0, 0, 2, 2\}, \{0, 1, 1, 1\}, \{0, 1, 1, 2\}, \{0, 1, 1, 3\}, \\
& \{0, 1, 1, 4\}, \{0, 1, 1, 5\}, \{1, 1, 1, 1\}, \{0, 0, 1, 0\}, \\
& \{0, 0, 2, 0\}, \{0, 1, 2, 1\}, \{0, 1, 3, 1\}, \{1, 1, 2, 1\}, \\
& \{0, 1, 1, 0\}, \{0, 2, 2, 0\}, \{1, 2, 2, 1\}, \{1, 2, 3, 1\}, \\
& \{1, 2, 4, 1\}, \{1, 0, 0, 0\}, \{2, 0, 0, 0\}, \{1, 0, 1, 0\}, \\
& \{2, 0, 2, 0\}, \{1, 1, 1, 0\}, \{2, 1, 1, 0\}, \{3, 1, 1, 0\}, \\
& \{4, 1, 1, 0\}, \{5, 1, 1, 0\}, \{2, 1, 1, 1\}, \{2, 1, 2, 1\}, \\
& \{3, 1, 2, 1\}\}
\end{align*}
\]
\[
\begin{align*}
\{1, 1, 1, 3\} & \Rightarrow \{0, 0, 0, 0\}, \{0, 0, 0, 1\}, \{0, 0, 0, 2\}, \{0, 0, 1, 1\}, \\
& \{0, 0, 2, 2\}, \{0, 1, 1, 1\}, \{0, 1, 1, 2\}, \{0, 1, 1, 3\}, \\
& \{0, 1, 1, 4\}, \{0, 1, 1, 5\}, \{0, 1, 1, 6\}, \{1, 1, 1, 1\}, \\
& \{1, 1, 1, 2\}, \{1, 1, 2, 2\}, \{0, 0, 1, 0\}, \{0, 0, 2, 0\}, \\
& \{0, 1, 2, 1\}, \{0, 1, 1, 0\}, \{1, 0, 0, 0\}, \{2, 0, 0, 0\}, \\
& \{1, 0, 1, 0\}\}
\end{align*}
\]
\[
\begin{align*}
\{1, 1, 2, 2\} & \Rightarrow \{0, 0, 0, 0\}, \{0, 0, 0, 1\}, \{0, 0, 0, 2\}, \{0, 0, 1, 1\}, \\
& \{0, 0, 2, 2\}, \{0, 1, 1, 1\}, \{0, 1, 1, 2\}, \{0, 1, 1, 3\}, \\
& \{0, 1, 1, 4\}, \{0, 1, 1, 5\}, \{1, 1, 2, 1\}, \{1, 2, 2, 1\}, \\
& \{1, 2, 1, 1\}, \{1, 0, 0, 0\}, \{0, 2, 0, 2\}, \{1, 2, 1, 3\}, \\
& \{1, 0, 1, 0\}, \{1, 1, 2, 2\}, \{1, 2, 1, 2\}, \{1, 2, 2, 1\}, \\
& \{1, 2, 1, 4\}, \{0, 0, 1, 0\}, \{0, 2, 0, 2\}, \{1, 1, 1, 0\}, \\
& \{2, 0, 2, 0\}, \{1, 1, 1, 0\}, \{1, 1, 2, 0\}, \{1, 1, 3, 0\}, \\
& \{1, 1, 4, 0\}, \{1, 1, 5, 0\}, \{1, 1, 6, 0\}, \{1, 1, 2, 1\}, \\
& \{2, 1, 2, 1\}, \{2, 1, 3, 1\}, \{2, 1, 4, 1\}\}
\end{align*}
\]
\[
\begin{align*}
\{1, 1, 2, 3\} & \Rightarrow \{0, 0, 0, 0\}, \{0, 0, 0, 1\}, \{0, 0, 0, 2\}, \{0, 0, 1, 1\}, \\
& \{0, 0, 2, 2\}, \{0, 1, 1, 1\}, \{0, 1, 1, 2\}, \{0, 1, 1, 3\}, \\
& \{0, 1, 1, 4\}, \{1, 1, 1, 1\}, \{1, 1, 1, 2\}, \{1, 1, 1, 3\}, \\
& \{1, 1, 2, 2\}, \{1, 2, 2, 2\}, \{1, 2, 1, 2\}, \{0, 0, 1, 0\}, \\
& \{0, 0, 2, 0\}, \{0, 1, 2, 1\}, \{0, 1, 3, 1\}, \{1, 1, 1, 1\}, \\
& \{1, 1, 3, 2\}, \{1, 2, 3, 2\}, \{0, 1, 0, 1\}, \{0, 2, 0, 2\}, \\
& \{1, 2, 1, 2\}, \{1, 2, 1, 3\}, \{1, 2, 1, 4\}, \{1, 2, 1, 5\}, \\
& \{1, 2, 1, 6\}, \{1, 3, 1, 3\}, \{0, 1, 0, 0\}, \{0, 2, 0, 0\}, \\
& \{0, 2, 1, 1\}, \{1, 2, 1, 1\}, \{1, 4, 1, 3\}, \{2, 3, 2, 2\}, \\
& \{2, 4, 2, 2\}, \{0, 1, 0, 0\}, \{0, 2, 0, 0\}, \{1, 2, 2, 1\}, \\
& \{1, 3, 3, 2\}, \{1, 3, 2, 1\}, \{1, 4, 2, 1\}, \{1, 5, 2, 1\}, \\
& \{1, 4, 3, 2\}, \{1, 0, 0, 0\}, \{2, 0, 0, 0\}, \{1, 0, 0, 1\}, \\
& \{2, 0, 0, 2\}, \{1, 1, 0, 1\}, \{2, 1, 0, 1\}, \{3, 1, 0, 1\}, \\
& \{2, 1, 1, 1\}, \{2, 1, 1, 2\}, \{3, 1, 1, 3\}, \{4, 1, 1, 3\}, \\
& \{2, 2, 1, 2\}, \{3, 2, 1, 2\}, \{3, 2, 2, 2\}, \{4, 2, 2, 2\}, \\
& \{1, 1, 0, 0\}, \{2, 2, 0, 0\}, \{3, 2, 1, 1\}, \{1, 0, 1, 0\}, \\
& \{2, 0, 2, 0\}, \{1, 1, 1, 0\}, \{2, 1, 1, 0\}, \{3, 1, 1, 0\}, \\
& \{4, 1, 1, 0\}, \{2, 1, 2, 1\}, \{3, 1, 2, 1\}, \{4, 1, 2, 1\}, \\
& \{5, 1, 2, 1\}, \{1, 1, 0, 2\}, \{1, 1, 0, 3\}, \{1, 1, 0, 4\}, \\
& \{1, 1, 0, 5\}, \{1, 1, 0, 6\}, \{1, 1, 0, 7\}, \{2, 1, 1, 3\}, \\
& \{2, 1, 1, 4\}, \{2, 1, 1, 5\}\}
\end{align*}
\]
Appendix B. Data in the elliptic case

In the section we give the complete list of the elliptically fibered Calabi-Yau threefolds discussed in Section 9. This list is to be interpreted in a manner similar to the list in Appendix A.

\{1, 1, 1\} \Rightarrow \\{\{0, 0, 0\}, \{0, 0, 1\}, \{0, 0, 2\}, \{0, 0, 3\}, \{0, 1, 1\}, \{0, 1, 2\}, \{0, 1, 3\}, \{0, 2, 3\}\}
\{1, 1, 2\} \Rightarrow \\{\{0, 0, 0\}, \{0, 0, 1\}, \{0, 0, 2\}, \{0, 0, 3\}, \{0, 1, 1\}, \{0, 1, 2\}, \{0, 1, 3\}, \{0, 2, 3\}, \{0, 3, 3\}, \{0, 3, 4\}, \{0, 3, 5\}, \{0, 3, 6\}, \{0, 3, 7\}, \{0, 3, 8\}, \{0, 3, 9\}, \{0, 3, 10\}, \{0, 3, 11\}, \{0, 3, 12\}, \{1, 1, 1\}, \{0, 1, 0\}, \{0, 2, 0\}, \{0, 3, 0\}, \{0, 2, 1\}, \{0, 3, 1\}, \{0, 3, 2\}, \{0, 4, 3\}, \{0, 5, 3\}, \{0, 6, 3\}, \{1, 2, 1\}, \{1, 3, 1\}, \{1, 1, 0\}, \{1, 2, 0\}, \{2, 2, 1\}\}
\{1, 2, 3\} \Rightarrow \\{\{0, 0, 0\}, \{0, 0, 1\}, \{0, 0, 2\}, \{0, 0, 3\}, \{0, 1, 1\}, \{0, 1, 2\}, \{0, 1, 3\}, \{0, 2, 3\}, \{0, 3, 3\}, \{0, 3, 4\}, \{0, 3, 5\}, \{0, 3, 6\}, \{0, 3, 7\}, \{0, 3, 8\}, \{0, 3, 9\}, \{0, 3, 10\}, \{0, 3, 11\}, \{0, 3, 12\}, \{1, 1, 1\}, \{1, 2, 2\}, \{1, 3, 2\}, \{1, 1, 0\}, \{1, 0, 2\}, \{2, 1, 2\}, \{2, 1, 3\}, \{1, 0, 0\}, \{2, 0, 0\}, \{3, 0, 0\}, \{2, 0, 1\}, \{2, 1, 1\}, \{3, 1, 1\}, \{3, 1, 2\}, \{3, 2, 2\}, \{1, 1, 0\}, \{1, 2, 0\}, \{2, 2, 1\}, \{2, 1, 0\}\}

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Department of Mathematics, The Ohio State University, Columbus, OH 43210

E-mail address: mullet@math.ohio-state.edu