The Gauss Map and 2+1 Gravity

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Abstract

We prove that the Gauss map of a surface of constant mean curvature embedded in Minkowski space is harmonic. This fact will then be used to study 2+1 gravity for surfaces of genus higher than one. By considering the energy of the Gauss map, a canonical transform between the ADM reduced variables and holonomy variables can be constructed. This allows one to solve (in principle) for the evolution in the ADM variables without having to explicitly solve the constraints first.

1 Preamble

Given a surface embedded in Euclidean space, one defines the Gauss map in the following way: at any point on the surface, one can construct a (unique) normal vector of unit length. A unit vector can be viewed as a point on the sphere. Thus we have a mapping from the embedded surface into the sphere. A result of Gauss states that, if the surface is minimal (and hence has zero mean curvature), this map is conformal. A later result of Ruh and Vilms [1] states that if the map has constant (not necessarily zero) mean curvature, the Gauss map is a harmonic map. These results are extremely useful in that they allow one to apply the vast body of knowledge about conformal and harmonic maps between surfaces to the study of constant mean curvature surfaces. As an example of the power of these results, witness the formulas of Enneper, Weierstrass, and Kenmotsu which allow one to write the parametric equation for a general surface of constant mean curvature as an integral.
To apply these results to general relativity in three dimensions, we make the following two observations. First, the theorems of Gauss, Ruh, and Vilms apply to surfaces embedded in Minkowski space as well as to surfaces in Euclidean space. This will be proven explicitly later. Second, in three dimensions, the Einstein equations imply that spacetime is flat. The reason for this is that, in three dimensions (and only in three dimensions!), the entire Riemann tensor is uniquely determined by the Einstein tensor. Einstein’s equations in vacuo imply the vanishing of the Einstein tensor, and hence the vanishing of the entire Riemann tensor. Therefore the resulting spacetime is flat and locally isometric to Minkowski space, allowing us to apply the aforementioned theorems to surfaces of constant mean curvature embedded in these spacetimes.

But why would anyone want to consider surfaces of constant mean curvature embedded in a spacetime in the first place? The reasons go back to York’s extrinsic time program and have nothing to do with Gauss maps whatsoever. In York’s program, one considers slicing the spacetime into spacelike slices of constant mean curvature and using the mean curvature of the slice as time. The reason York chose this slicing condition is that it allows one to solve the Hamiltonian and momentum constraints of general relativity separately instead of having to consider a nonlinear coupled system of partial differential equations. Moreover, experience with actual spacetimes has shown that constant mean curvature slicing is a good slicing in the sense that constant mean curvature slices usually foliate the whole spacetime, are unique, and do not develop singularities unrelated to the singularities of the spacetime. In contrast, Gaussian normal slicing, for example, is a bad slicing since, even in Minkowski space, Gaussian normal slices usually develop caustics after a finite lapse of time.

York’s extrinsic time proposal can be made more exact and explicit by Moincief’s reduction of the Einstein equations. In this approach, one solves the constraints, substitutes the result back into the action, and rearranges the result to obtain a Hamiltonian system with no constraints. The configuration space of the resulting system turns out to be Teichmüller space, the space of equivalence classes of spatial metrics under diffeomorphisms and conformal (Weyl) transforms. Time is identified with mean curvature, the momenta are transverse traceless tensor fields (long known to mathematicians as a realization of the cotangent space to Teichmüller space), and the Hamiltonian is equal to the area of the spatial slice expressed as a function of the coordinates and their conjugate momenta.

Considering this reduction approach reveals a remarkable fact about the Gauss map. The fact that the Gauss map is harmonic means that it only depends on the conformal class of the spatial geometry. Thus the Gauss map, in a sense, automatically performs the reduction to Teichmüller space! Contrast this with the usual situation where one has to put in the Teichmüller parameters by hand by writing the metric as the product of a conformal factor and a metric of constant negative curvature. This fact is all the more remarkable when one considers that a) The unit hyperboloid in Minkowski space has constant
negative curvature. b) Eells, Earle, and Sampson proved that one can represent Teichmüller space by metrics of constant negative curvature by using harmonic maps! To add to the amazement, one finds that the energy of the Gauss map has a simple expression in terms of dynamical quantities associated with the reduced Hamiltonian system. These miraculous facts will be exploited extensively in a later section to obtain information about the spacetimes.

2 The Gauss Map in Minkowski Space

In this section we will prove the theorems of Gauss, Ruh, and Vilms for spacelike surfaces in Minkowski space. Since these theorems are local, we can make life simple by using coordinate systems which can always be defined in a small enough neighborhood of a given point but which do not necessarily extend to coordinate systems on the whole of space and/or spacetime.

First we will demonstrate the result of Gauss that the Gauss map of a surface of vanishing mean curvature is conformal. To do so we first introduce Cartesian coordinates on spacetime and isothermal coordinates on space. That is to say, we have spacetime coordinates \( t, x, y \) such that \( ds^2 = -dt^2 + dx^2 + dy^2 \) and spatial coordinates \( \xi, \eta \) such that the induced spatial metric has the form \( d\sigma^2 = e^{2\lambda}(d\xi^2 + d\eta^2) \). The unit normal to the surface is given by taking the (Minkowski space) cross product of two tangents to the surface and normalizing to unit length:

\[
n_{\alpha} = \frac{\varepsilon_{\alpha\beta\gamma} x^\beta_{,\xi} x^\gamma_{,\eta}}{\sqrt{x^\alpha_{,\xi} x^\beta_{,\eta} x^\gamma_{,\beta,\eta} - (x^\alpha_{,\xi} x^\alpha_{,\eta})^2}} \quad (1)
\]

The fact that the metric \( e^{2\lambda}(d\xi^2 + d\eta^2) \) is induced from the Minkowski metric implies the following two equations:

\[
x^\alpha_{,\xi} x^\alpha_{,\eta} = x^\alpha_{,\eta} x^\alpha_{,\xi} \quad (2)
\]

\[
x^\alpha_{,\xi} x^\alpha_{,\eta} = 0 \quad (3)
\]

Using the above two equations allows us to simplify the denominator in the equation for the unit normal:

\[
n_{\alpha} = \frac{\varepsilon_{\alpha\beta\gamma} x^\beta_{,\xi} x^\gamma_{,\eta}}{x^\beta_{,\xi} x^\beta_{,\eta}} = \frac{\varepsilon_{\alpha\beta\gamma} x^\beta_{,\xi} x^\gamma_{,\eta}}{x^\beta_{,\eta} x^\beta_{,\eta}} \quad (4)
\]

Now consider the quantity \( H^\alpha := (\partial_{\xi} + \partial_{\eta})x^\alpha \). By differentiating equations (2) and (3) with respect to \( \xi \) and \( \eta \), we see that \( H^\alpha x_{,\alpha,\xi} = H^\alpha x_{,\alpha,\eta} = 0 \) Thus \( H^\alpha \) is

4 these coordinates will also be sometimes referred to as \( x^0, x^1, \) and \( x^2 \). In this article I will follow the convention that greek indices denote spacetime indices \( \alpha, \beta, \ldots = 0, 1, 2 \), latin indices denote spatial indices \( a, b, \ldots = 1, 2 \), and indices inside parentheses such as \( e^{(\alpha)} \) denote frame indices.
normal to the surface. I claim that the length of $H^\alpha$ is proportional to the mean curvature. To see this, let us Lorentz transform our coordinates such that $x = \xi$ and $y = \eta$ to first order at some given point P. Then it is easy to compute the mean curvature at P by differentiating equation (4):

$$\tau = g^{ab}n_a; b = e^{-2\lambda}(n_x, \xi + n_y, \eta) = e^{-2\lambda}(-t, \xi - t, \eta) = -e^{-2\lambda}H^0$$  (5)

Thus $H^\alpha = -e^{2\lambda}\tau n^\alpha$. In particular, if our surface has vanishing mean curvature, $H^\alpha = 0$

To show that vanishing mean curvature implies conformality of the Gauss map, introduce a complex parameter $z := \xi + i\eta$ and the quantities $w^\alpha := x^\alpha_z$. The pair of real equations (2) and (3) can be combined into the complex equation

$$g_{\alpha\beta}w^\alpha w^\beta = 0$$  (6)

This complex equation can be seen as defining a curve in 3-dimensional complex projective space which is conformal to the unit hyperboloid in Minkowski space. With this identification, $w^\alpha(z)$ is the Gauss map. In this complex notation, $H^\alpha = x^\alpha_{z\bar{z}} = w^\alpha_{\bar{z}}$. Thus, when our surface has vanishing mean curvature, $w^\alpha_{\bar{z}} = 0$, which means that the Gauss map is holomorphic, hence conformal.

Q.E.D.

**Theorem.** The Gauss map of a spacelike surface embedded in Minkowski space is conformal.

Now we will consider a surface of constant mean curvature embedded in Minkowski space and prove the theorem of Ruh and Vilms which states that its Gauss map is harmonic. Whereas in the previous section Cartesian coordinates proved convenient, Gaussian normal coordinates will be useful in this section. Let $\gamma_{ab}$ be the spatial metric (Unlike last section, the exact form of $\gamma_{ab}$ is irrelevant in this section). Then the metric on spacetime is given by $-dt^2 + \gamma_{ab}dx^adx^b$ and the surface of constant mean curvature is given by setting $t=0$. Introduce an orthonormal frame of parallel vectors $e^{(\alpha)}_\beta$. By parallel I mean that $e^{(\alpha)}_{\beta\gamma} = 0$ i.e. that the frame would be the coordinate frame for Cartesian coordinates. By definition of normal coordinates, the unit normal vector is $n^\alpha = (1, 0, 0)$ and hence the Gauss map is given by $n^{(\alpha)}(x, y) := e^{(\alpha)}_{\beta}n_\beta(x, y)$.

To show that the gauss map is harmonic, we will directly compute the variation of the energy and show that it is stationary for the gauss map. Putting the usual metric on the unit hyperboloid, we write down the harmonic map energy for the Gauss map as

$$E = \int \eta_{\alpha\beta}n^{(\alpha)}_\alpha n^{(\beta)}_\beta \gamma^{\alpha\beta\gamma}^{-1/2}d^2x$$  (7)

where $\eta_{\alpha\beta} := diag(-1, 1, 1)$. The variation of the energy is given by

$$\delta E = 2\int \eta_{\alpha\beta}\delta n^{(\alpha)}_\alpha n^{(\beta)}_\beta \gamma^{\alpha\beta\gamma}^{-1/2}d^2x$$  (8)

\[2\] Very soon we will apply the theorem being proven here to the case where the spacetime is
Now, the quantity \( \delta n^{(\alpha)} \) is only defined on the surface and we are free to define it off of the slice in any consistent way. For simplicity, we will define it by setting \( \frac{\partial \delta n^{(\alpha)}}{\partial t} = 0 \). Then we will not change the value of \( \delta E \) if we replace the ordinary derivatives by covariant derivatives and \( \gamma \) by \( g \).

\[
\delta E = 2 \int \eta_{\alpha\beta} \delta n^{(\alpha)}_{,\gamma} n^{(\beta)}_{,\delta} g^{\gamma \delta} g^{-1/2} d^2 x \tag{9}
\]

Now we may integrate by parts

\[
\delta E = -2 \int \eta_{\alpha\beta} \delta n^{(\alpha)}_{,\gamma} n^{(\beta)}_{,\delta} g^{\gamma \delta} g^{-1/2} d^2 x \tag{10}
\]

Now we note that \( n_{\gamma;\alpha} = 0, n_{\partial;\alpha} = 0 \) in Gaussian normal coordinates, hence we obtain

\[
\eta_{\alpha\beta} \delta n^{(\alpha)}_{,\gamma} n^{(\beta)}_{,\delta} g^{\gamma \delta} g^{-1/2} d^2 x \tag{11}
\]

\[
\delta E = -2 \int \eta_{\alpha\beta} \delta n^{(\alpha)}_{,\gamma} n^{(\beta)}_{,\delta} g^{\gamma \delta} g^{-1/2} d^2 x \tag{12}
\]

Substituting into equation (10),

\[
\delta E = -2 \int \eta_{\alpha\beta} \delta n^{(\alpha)}_{,\gamma} n^{(\beta)}_{,\delta} g^{\gamma \delta} g^{-1/2} d^2 x \tag{13}
\]

Remembering that \( \frac{\partial \delta n^{(\alpha)}}{\partial t} = 0 \), we may finally simplify the integral to

\[
\delta E = -2 \int \eta_{\alpha\beta} \delta n^{(\alpha)}_{,\gamma} n^{(\beta)}_{,\delta} g^{\gamma \delta} g^{-1/2} d^2 x \tag{14}
\]

Thus, if \( \tau \) is constant on the surface, \( \delta E = 0 \), and hence the Gauss map is harmonic. Q.E.D.

**Theorem.** The Gauss map of a spacelike surface of constant mean curvature embedded in Minkowski space is harmonic.

### 3 Relation to the Reduced Hamiltonian System

In this section we will relate the different objects which appeared in the Gauss map to objects appearing in the Hamiltonian reduction. First, by considering the holonomies of our spacetime, we will show that the target space for the gauss map can be thought of as a closed Riemann surface whose Teichmüller parameters are given by certain constants of the motion. This will allow us to only locally Minkowski. However, this does not invalidate the current proof since, in varying an action, it is sufficient to consider variations \( \delta n^{(\alpha)} \) which are supported inside an arbitrarily small neighborhood. The neighborhood can be picked small enough that the coordinates and frame used here can be defined.

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apply various theorems on harmonic maps between Riemann surfaces. Next we will compute the energy of the harmonic map and show how it may be related to the area of the slice, which plays the role of Hamiltonian in the reduced formalism. In all cases, the genus of the spatial manifold is assumed to be higher than 1.

First, we consider the holonomies of the spacetime. Since the spacetime is flat, parallel transport along curves which are continuously deformable into each other will be independent of the choice of curve. Thus, given an element \([c] \in \pi_1(P)\), i.e. an equivalence class of non-contractible loops passing through the point \(P\), we can define the holonomy associated with \([c]\) to be the matrix \(H^\alpha_\beta([c])\) which describes parallel transport around a loop in \([c]\). Given a vector \(v^\mu\) at the point \(P\), the vector goes into the vector \(H^\mu_\nu([c])v_\nu\) upon parallel translation about any loop in \([c]\). The holonomies form a representation of the fundamental group in the natural way: given two curves \([c_1]\) and \([c_2]\), the holonomy associated with \([c_1 \circ c_2]\) is \(H^\mu_\nu([c_1])H^\nu_\xi([c_2])\).

To relate this to the Gauss map, consider the action of the holonomies on the normal vector. Thinking of the unit vector as point on the unit hyperboloid, the effect of the holonomy is a motion of the hyperboloid. In the usual negative curvature metric on the hyperboloid, these motions are isometries. Discrete groups of such motions have been studied extensively under the names of Fuchsian groups and the reader is referred to [3] for details. All that we need to know for the present is that we can quotient the unit hyperboloid by the group of holonomies and that this results in a nonsingular surface of constant negative curvature of the same genus as our original surface as long as the holonomy group is not degenerate. As we shall see when we consider holonomies in more detail, nondegeneracy will generically be true, so assuming nondegeneracy is justified.

As a map from the slice of constant mean curvature to the unit hyperboloid, the Gauss map is multiple-valued because the unit normal changes by a holonomy matrix every time one traverses a closed loop. However, if one instead considers the Gauss map as a map into the surface gotten by quotienting the hyperboloid as above, then the resulting map is single-valued; the holonomies and the Fuchsian matrixes cancel out. Since harmonicity is a local property, the new Gauss map into this surface is also a harmonic map.

The conformal class of the target surface does not change with time. To see this, first note that the trace of a holonomy is invariant under changing the base point on the loop, and hence, under arbitrary smooth deformations of the loop. In particular, we may push the loop forward in time, hence the trace is a constant of the motion. Second, the Fuchsian group giving our surface can be determined uniquely up to conjugation by specifying the traces of its elements. In fact, counting degrees of freedom shows that it is enough to give \(6g-6\) traces to specify the group for a genus \(g\) surface.

Thus, evolution may be viewed as the evolution of a harmonic map between two Riemann surfaces of the same genus. To understand this evolution better,
we can make use of the many theorems on harmonic maps between Riemann surfaces. For the present we will consider three theorems. For discussion and proofs refer to [4]

**Theorem 1.** For any map \( f : \Sigma_1 \to \Sigma_2 \), \( \text{Area}(f(\Sigma_1)) \leq E(f) \) with equality iff \( f \) is conformal.

**Theorem 2.** Suppose that \( \Sigma_1 \) and \( \Sigma_2 \) are compact surfaces without boundary, and that \( h : \Sigma_1 \to \Sigma_2 \) is a diffeomorphism. Then there exists a harmonic diffeomorphism \( u : \Sigma_1 \to \Sigma_2 \) isotopic to \( h \). Furthermore, \( u \) is of least energy among all diffeomorphisms isotopic to \( h \).

**Theorem 3.** Let \( \Sigma_1 \) and \( \Sigma_2 \) be Riemann surfaces of the same genus and let \( R(\Sigma_2) < 0 \). Then any map of degree one is uniquely homotopic to a harmonic map which is a diffeomorphism.

These theorems show us that our map is uniquely determined by the geometry of the slice and that it is well-behaved. Since this map is unique and the target space is fixed, the energy of this map is a well-defined function of the geometry on the source space. It is obvious from the definition that \( E \) cannot depend on the conformal factor in the geometry on the source space. Moreover, if one makes a diffeomorphism of the source geometry \( \gamma_{ab}'(y(x)) = \gamma_{cd}(x) \frac{\partial y^c}{\partial x^a} \frac{\partial y^d}{\partial x^b} \), the transformed map \( u'(\mu)(y) = u(\mu)(y(x)) \) will be the unique solution to the harmonic map equation with \( \gamma_{ab}' \) and will have the same energy as \( u \). Therefore \( E \) can be considered as a function of the Teichmüller parameters of the source space for a fixed target space.

Seeing that \( E \) can be thought of as a function of the Teichmüller parameters of the spatial slice and the target space metric which is parameterized by the holonomies, one wonders whether the value of \( E \) has any interpretation. To answer this question, let us calculate. As previously, introduce Gaussian normal coordinates:

\[
E = \int \eta_{\mu\nu} n^{(\mu)}_a n^{(\nu)}_b \gamma^{ab} \gamma^{1/2} d^2 x \tag{15}
\]

\[
= \int \eta_{\mu\nu} n^{(\mu)}_a n^{(\nu)}_b g^{ab} g^{1/2} d^2 x \tag{16}
\]

\[
= \int \eta_{\mu\nu} \epsilon^{(\mu)c} n^{(\nu)d} g^{ab} g^{1/2} d^2 x \tag{17}
\]

\[
= \int g^{cd} n^{a}_{c:a} n^{b}_{d:b} g^{ab} g^{1/2} d^2 x \tag{18}
\]

\[
= \int \gamma^{ab} \gamma^{cd} K_{ac} K_{bd} \gamma^{1/2} d^2 x \tag{19}
\]

Now recall the definitions of the gravitational momentum and the Hamiltonian constraint:

\[
\pi^{ab} = \gamma^{1/2} (\gamma^{ab} K^{c} - K^{ab}) \tag{20}
\]
\[ \gamma^{-1/2}(\pi^a \pi_{ab} - (\pi^a_a)^2) = \gamma^{1/2}R \]  \hspace{1cm} (21)

Combining and integrating we obtain

\[ \gamma^{1/2}K^{ab}K_{ab} = \gamma^{1/2}((K_c^c)^2 + R) \]  \hspace{1cm} (22)

\[ E = A\tau^2 + 4\pi\chi \]  \hspace{1cm} (23)

Thus one sees that the energy can be written in terms of the area of the surface and the mean curvature. When one considers that the area is the reduced Hamiltonian, the above formula gains in significance. To properly appreciate it, we must first consider some facts about canonical transforms and constants of the motion.

## 4 Canonical Transforms and Constants of the Motion

In the last section, we considered the holonomies associated with parallel transport. As it turns out, there is another way to introduce holonomies into 2+1 gravity. This approach, which was found by Witten, provide holonomies as path ordered integrals of matrices constructed from the frame and spin connection. Like the holonomies considered above, they are constants of the motion. In fact they provide a complete set of constants of the motion, allowing one to unambiguously specify any 2+1 spacetime by its holonomies.

Let us recall the definitions and properties of these holonomies. Pick an orthonormal frame \( e^{(\alpha)}_\beta \) for the spacetime and construct its spin connection \( \omega^{(\alpha)}_{(\beta)\mu} \). Then the holonomy associated with the loop C is:

\[ w_r(C) = \text{tr} \ P \ exp(\int \omega^{(\alpha)}_{(\beta)\mu} J^\beta_{\alpha} + e^{(\alpha)}_{\mu} P_{\alpha} dx^\mu) \]  \hspace{1cm} (24)

The subscript “r” in \( w_r \) refers to a representation of ISO(2,1) and J refers to the generator of boosts/rotations in that representation and P refers to the generator of translations. Since the space of solutions for relativity in 2+1 dimensions is 12g+12 dimensional, not all of these holonomies can be independent functions. In fact it is enough to consider just two representations and 6g-6 loops. Two representations which do the job are given below.

In the first representation, J and P are given by:

\[ J^0 = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad J^1 = \frac{1}{2} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad J^2 = \frac{1}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \]  \hspace{1cm} (25)

\[ P_1 = P_2 = P_3 = 0 \]  \hspace{1cm} (26)

In this representation, \( \text{tr} \) denotes the usual sum of the two diagonal elements of the matrix.
The second representation is 4 dimensional. The matrices that represent $J$ and $P$ are given in block form by

$$J_\alpha^\beta = \begin{pmatrix} j_\alpha^\beta & 0 \\ 0 & j_\alpha^\beta \end{pmatrix}, \quad P_\alpha = \epsilon_{\alpha\beta\gamma} \begin{pmatrix} 0 & j_\beta^\gamma \\ 0 & 0 \end{pmatrix} \tag{27}$$

$j_\alpha^\beta$ denotes the matrices $J_\alpha^\beta$ in the first representation, i.e. $J_\beta^\alpha$ last paragraph. In this representation traces are taken by the rule $\text{tr} M = M_{31} + M_{42} + M_{13} + M_{24}$. From now on $w_1(C)$ will refer to the holonomy in the first representation and $w_2(C)$ will refer to that in the second. It turns out that the holonomies in the first representation, $w_1(C)$, equal the holonomies employed in the last section. To see this, recall the result of Waelbroeck [7] that any 2+1 vacuum spacetime can be realized as the quotient of Minkowski space by an appropriate discrete subgroup of ISO(2,1). Then, given a closed curve $C$ in the spacetime, it can be considered as coming from an open curve $\tilde{C}$ in Minkowski space whose endpoints are identified by the action of an element $T$ of ISO(2,1). Now, it is clear that the holonomy of $C$ in the sense of the last section is simply the boost part of $T$. To compute the holonomy in the sense of Witten, let us assume that $\tilde{C}$ is a helicoid. (We lose no generality since the holonomies are invariant under deformations.) Namely, let us write $T = e^{j+\tilde{t}}$ where $j$ is a boost generator and $\tilde{t}$ is a translation generator. Then we assume that $C(s) = e^{(j+\tilde{t})s}P_0$ where $P_0$ is the initial endpoint of $C$. To compute $w_1(C)$, we introduce a framing in the following way: Consider the plane through $P$ normal to $C$. Introduce vectors $e_a^{(0)}$ and $e_a^{(1)}$ which form a Cartesian frame for this plane. Let the vector $e_a^{(2)}$ be the normal to this plane. Any point in spacetime not on the plane can be mapped into a point on the plane by a matrix $e^{(j+\tilde{t})}\tilde{t}$ for some suitable $\tilde{t}$. Extend the frame off of the surface by using the action of the matrix $e^{(j+\tilde{t})}\tilde{t}$. Then we have a frame in which $e_a^{(2)}$ is tangent to the curve $C$ and the other two vectors are orthogonal to it. Computing the connection is quite easy if we introduce Cartesian coordinates in the spacetime, for then we have $\omega_{\beta\mu}^{(a)} = e_{(\beta)}^{(a)}e_{\mu}^{(a)}$. For the purpose of computing the holonomies, it is enough to know the connection along the curve for a direction tangent to the curve. If we pick our cartesian coordinates at some point $P$ on the curve such that $e_{\beta}^{(a)} = \delta_\beta^a$ at $P$, then we can readily see that $\omega_{(\beta)2}^{(a)}J_\beta^\alpha = m$. This means that $w_1(C) = \text{tr} P e^{J_\alpha^\beta} = \text{tr} e^m$. Thus the two holonomies agree.

Having shown that the holonomies we considered in the last section are the same as Witten's holonomies, we are now in a position to use the results about those holonomies given by Martin [6]. According to Martin, the Poisson brackets of two $w_1$'s is zero. Let us denote the Teichmüller parameters of the unit hyperboloid quotiented by the holonomies by $Q^A$. (Capital latin indices will run form 1...6g−6 and indicate quantities that have something to do with

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3 Essentially this representation is the same as the representation given in [1], but without the funny infinitessimals which square to zero.
Teichmüller space.) The $Q^A$'s can be expressed as functions of the $w_1$'s, and hence, Poisson commute. Since the $w_1$'s and the $w_2$'s together form a complete set of observables, we can form functions $P_A(w_1, w_2)$ which are canonically conjugate to the $Q^A$'s.

Thus we have two sets of canonical variables on our phase space: the $q^A$ and $p_A$ coming from the Hamiltonian reduction and the $Q^A$ and $P_A$ coming from the holonomies. Moreover, the $Q^A$ and $P_A$ are constants of the motion. This is a standard situation in Hamiltonian mechanics and is described by a (time-dependant) generating function. Recall that, if $h$ is the hamiltonian in some canonical variables $q^i, p_i, H$ is the Hamiltonian in terms of some other canonical variables $Q^i, P_i$, and the transform between the two sets of variables is governed by a generating function $F(q^i, Q^i, t)$, we then have the equations:

$$p_i = \frac{\partial F}{\partial q^i} \quad P_i = \frac{\partial F}{\partial Q^i}$$  \hspace{1cm} (28)

$$H(q^i, Q^i, t) = h(q^i, Q^i, t) + \frac{\partial F}{\partial t}(q^i, Q^i, t)$$  \hspace{1cm} (29)

In our case, $H$ is zero since the $Q^i$ and $P_i$ are constants of the motion and $h$ is equal to the area of the constant mean curvature slice. We need $h$ expressed in terms of $q^i$ and $Q^i$. But this exactly what we obtained in equation (23) in terms of the energy of the Gauss map! Thus we have

$$h(q^i, Q^i) = A(q^i, Q^i) = \tau^{-2}(E(q^i, Q^i) - 4\pi \chi)$$  \hspace{1cm} (30)

Comparison with equation (29) allows us to read off the generating function $F$

$$F(q^i, Q^i) = A(q^i, Q^i) = \frac{1}{2\tau^3}(E(q^i, Q^i) - 4\pi \chi)$$  \hspace{1cm} (31)

which allows us to solve (in principle) for the relation between the ADM reduced variables and the holonomies.

5 Solving for $E(q,Q)$ in Principle

Having made the claim that one can solve for the relation between $q$ and $Q$ in principle several times, it is time to make good on that claim by showing a method that works in principle although appears intractable in practise. To begin, we need the harmonic map whose energy we are to obtain. To do so we could make use of the following fact: If $f : M \to N$ is a harmonic map and the image of $f$ lies on a submanifold $N_1 \subset N$ (with induced metric), then $f : M \to N_1$ is a harmonic map. In our case $N$ is Minkowski space and $N_1$ is the unit hyperboloid. For a function into Minkowski space to be as harmonic map means that each of the three coordinate functions satisfies the Laplace equation, i.e. be harmonic in the old sense of the term. For a function on a
surface (in this case our spatial slice) to be harmonic is equivalent to demanding that it be the real part of an analytic function. Thus we are led to consider the automorphic functions on our surface. In particular, we want to find three such functions \( \alpha, \beta, \gamma \) such that 1) Their real parts lie on the unit hyperboloid 2) They are multiple valued and, when one goes around some non-contractible loop on the double torus, the corresponding point on the hyperboloid is the one coming from acting on the original point with an element of \( \Gamma_Q \). Since automorphic functions, like the elliptic functions they generalize, are determined by their singularities, we can arrange these conditions by looking at what happens at the singularities (which will be branch points). Then to find the energy one would need to integrate the harmonic map energy over the surface.

There is nothing impossible about the above plan; it is just tedious. Since the final result would be a complicated surface integral which almost certainly could not be done in closed form, it is doubtful how much of the plan is worth carrying out. However it strongly indicates one important thing: the reduced constant-mean-curvature Hamiltonian is certainly not as nice an object in the higher-genus case as on the torus. It probably cannot be written in closed form and is not simply the Weil-Peterson line element as one might have hoped from simple analogy with the torus.

6 Conclusions

We have succeeded in obtaining the reduced Hamiltonian for constant mean curvature slicing without solving the constraints. The only catch is that it is given in terms of the function \( E(q^A, Q^A) \). This is some complicated function and, presumably, the only way to get at it is by solving for the unique harmonic map between two Riemann surfaces. Thus, from a practical point of view, all we did was replace one quasilinear elliptic equation (the Lichnerowitz equation) with another quasilinear elliptic equation (the harmonic map equation). As far as getting an exact solution to the problem, we seem to have gained little or nothing.

However, I consider the real value of equation (31) to be the way in which it unites three subjects in a natural way. The quasilinear elliptic equation we are obliged to consider is not just any old nonlinear equation, but one that has been studied extensively over the past three decades by both mathematicians (as the harmonic map equation) and physicists (as the nonlinear \( \sigma \)-model). Thus it allows one to apply results about that subject to the study of 2+1 gravity. Also, the holonomies of Witten have appeared in this study and been related to the ADM variables in a canonical way.

Even if the function \( E(q, Q) \) cannot be gotten at in any useful form, one might be able to say something about its limiting values. One might, for example, be able to compute it asymptotically in the limiting case where the Teichmüller parameters \( q \) are describing a double torus pinching off into a pair
of single tori. Then, one could make concrete calculations regarding this kind of topology change.

In the conclusion to his paper [3], Moncrief said that the solution to the higher genus case looked rather remote. It still seems so. However, this Gauss map construction brings us slightly closer and allows us to glimpse some general features of that solution.

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