Compactification, and beyond, of composition operators on Hardy spaces by weights

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Abstract. We study when multiplication by a weight can turn a non-compact composition operator on $H^2$ into a compact operator, and when it can be in Schatten classes. The $q$-summing case in $H^p$ is considered. We also study when this multiplication can turn a compact composition operator into a non-compact one.

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1 Introduction

Let $\varphi: \mathbb{D} \to \mathbb{D}$ be an analytic self-map and $C_\varphi: H^2 \to H^2$ be the associated composition operator $f \mapsto f \circ \varphi$. For $w \in H^2$, the multiplication operator $M_w$ is defined formally by $f \mapsto wf$ and the weighted composition operator by $f \mapsto w(f \circ \varphi)$. It is known (see [5] for instance) that twisting $C_\varphi$ by some $M_w$ can improve its compactness properties, and even its membership in Schatten classes $S_p$, or the decay of its approximation numbers ([7, Theorem 2.3]).

In this note, we study, in a rather qualitative way, the following problem: given a symbol $\varphi$, when can we find a non-trivial $w \in H^2$ such that $M_w$ has a smoothing effect on $C_\varphi$, namely when is $M_wC_\varphi$ compact if $C_\varphi$ was not? Or the other way round: when can we find $w$ such that $M_wC_\varphi$ is not compact if $C_\varphi$ was?

In [13, Proposition 2.4], it is proved that for $M_wC_\varphi$ to be compact for some $w \in H^2$ ($w \neq 0$), it is necessary that:

(1.1) $m(\{|\varphi^*| = 1\}) = 0$,

where $m$ is the normalized Lebesgue measure on $\mathbb{T}$ and $\varphi^*$ the boundary values function of $\varphi$. On the other hand, in order that $M_wC_\varphi$ be Hilbert-Schmidt for
some $w \in H^2$, $w \neq 0$, it is sufficient that:

\[(1.2) \quad \int_T \log(1 - |\varphi^*|) \, dm > -\infty \]

([13, Proposition 2.5]). Note that (1.1) means that $\varphi$ is not an exposed point of the unit ball of $H^\infty$ ([1]), and that (1.2) means that it is not an extreme point of this unit ball ([4, Theorem 7.9]).

There is a gap between these two conditions. The purpose of this work to fill this gap in several respects, this filling explaining in passing the initial gap.

In Section 3, we show that condition (1.1) is necessary and sufficient to have a compact weighted composition operator. We also give examples showing how small approximation numbers we can obtain. In Section 4, we show that condition (1.2) is necessary and sufficient to get a Hilbert-Schmidt weighted composition operator, and we show that it is also necessary and sufficient for getting a weighted composition operator in some, or all, Schatten classes. In Section 5, we consider the case of $H^p$ spaces and study the nuclearity and the summing properties of the weighted composition operators. In Section 6 we show that a composition operator can become non-compact by weighting it if and only if the image of the symbol touches the boundary of the unit disk.

2 Notation

Let $\mathbb{D}$ be the open unit disk. The Hardy space $H^p$, $1 \leq p < \infty$, is the space of analytic functions $f: \mathbb{D} \to \mathbb{C}$ such that:

$$\|f\|_p^p := \sup_{0 < r < 1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{it})|^p \, dt < \infty.$$  

Such functions have non-tangential limits $f^*(e^{it})$ almost everywhere on $T = \partial \mathbb{D}$ and we have:

$$\|f\|_p^p = \frac{1}{2\pi} \int_0^{2\pi} |f^*(e^{it})|^p \, dt.$$  

For $p = 2$, $H^2$ is equivalently the space of analytic functions in $\mathbb{D}$ that can be written $f(z) = \sum_{n=0}^{\infty} c_n z^n$ with $\|f\|_2^2 = \sum_{n=0}^{\infty} |c_n|^2 < \infty$. In the sequel, we write simply $\| \cdot \|_2 = \| \cdot \|$.

Any analytic self-map $\varphi: \mathbb{D} \to \mathbb{D}$ induces a bounded operator $C_\varphi: H^p \to H^p$, called the composition operator of symbol $\varphi$.

For $w \in H^p$, the multiplication operator $M_w$ is defined, formally, by $M_w f = w f$, and the weighted composition operator $M_w C_\varphi$ by $(M_w C_\varphi)(f) = w (f \circ \varphi)$. Note that to get $M_w C_\varphi: H^p \to H^p$, it is necessary to have $w \in H^p$ since $(M_w C_\varphi)(\mathbb{I}) = w$. Throughout this paper it will be assumed that $w \in H^p$, and that $w \neq 0$. This membership is not sufficient in general; however $w \in H^\infty$ is sufficient (but not necessary!), since $H^\infty$ is the set of multipliers of $H^p$. Note that we may consider the bounded operator $M_w C_\varphi$, even if $M_w$ is not bounded.
Except in Section 5, we work only with the Hilbert space $H^2$. For convenience, we will adopt in this paper the following terminology.

**Definition 2.1.** We say that the symbol $\varphi$ is:

- compactifiable if $M_wC_\varphi$ is compact for some $w \in H^2$ with $w \neq 0$;
- decompactifiable if $M_wC_\varphi$ is bounded but not compact for some $w \in H^2$.

For $\xi \in \mathbb{T} = \partial \mathbb{D}$ and $0 < h < 1$, the Carleson window $W(\xi, h)$ is defined as:

$$W(\xi, h) = \{ z \in \mathbb{D}; 1 - h \leq |z| \text{ and } |\arg(z\xi)| \leq \pi h \}.$$  (2.1)

If $\mu$ is a positive measure on $\overline{\mathbb{D}}$, the Carleson function of $\mu$ is:

$$\rho_\mu(h) = \sup_{\xi \in \mathbb{T}} \mu(W(\xi, h)).$$  (2.2)

The measure $\mu$ is called a Carleson measure when $\rho_\mu(h) = O(h)$, and a vanishing Carleson measure when $\rho_\mu(h) = o(h)$. By the Carleson embedding theorem, this is equivalent to say that the canonical inclusion $J_\mu: H^2 \rightarrow L^2(\mu)$ is respectively bounded or compact.

It is convenient to coin the Hastings-Luecking box $\tilde{W}(\xi, h) \subseteq W(\xi, h)$ defined by:

$$\tilde{W}(\xi, h) = \{ z \in \mathbb{D}; 1 - h \leq |z| < 1 - h/2 \text{ and } -\pi h < \arg(z\xi) \leq \pi h \}.$$  (2.3)

We denote $m$ the Haar measure (normalized Lebesgue measure) of $\mathbb{T}$. For a symbol $\varphi$, $m_{\varphi} = \varphi^*(m)$ is the pull-back measure of $m$ by $\varphi^*: \mathbb{T} \rightarrow \mathbb{C}$, the (almost everywhere defined) radial limit function associated with $\varphi$:

$$\varphi^*(\xi) = \lim_{r \rightarrow 1^{-}} \varphi(r \xi).$$  (2.4)

By definition $m_{\varphi}(B) = m[\varphi^{*-1}(B)]$ for all Borel sets $B \subseteq \overline{\mathbb{D}}$. This measure $m_{\varphi}$ is always a Carleson measure, due to the Littlewood subordination principle.

The Carleson function of $\varphi$ is that of $m_{\varphi}$ and is denoted $\rho_{\varphi}$:

$$\rho_{\varphi}(h) = \sup_{\xi \in \mathbb{T}} m(\varphi^{*-1}[W(\xi, h)]).$$  (2.5)

When the composition operator $C_\varphi$ is compact on $H^2$, we have $|\varphi^*| < 1$ a.e., and $m_{\varphi}$ is supported by $\mathbb{D}$. Moreover, $m_{\varphi}$ is then a vanishing Carleson measure.

Recall that a compact operator $T$ between separable Hilbert spaces $H_1$ and $H_2$ is in the Schatten class $S_p = S_p(H_1, H_2)$, $p > 0$, if $\sum_{n \geq 0} |s_n(T)|^p < \infty$, where $(s_n(T))$ is the sequence of singular numbers of $T$, i.e. the eigenvalues, arranged in non-increasing order, of $|T| = \sqrt{T^*T}$. For $p = 2$, $S_2(H_1, H_2)$ is the Hilbert-Schmidt class. Let us also recall that, for $p \geq 2$, we have $T \in S_p$ if and
only if \(\sum_n \|Te_n\|^p < \infty\) for every orthonormal basis \((e_n)\) of \(H_1\), and, for \(p \leq 2\), we have \(T \in S_p\) if and only if \(\sum_n \|Te_n\|^p < \infty\) for some orthonormal basis \((e_n)\) of \(H_1\) (see [6] for instance). It follows that if \(S, T : H_1 \to H_2\) are two compact operators such that \(\|Sx\| \leq \|Tx\|\) for all \(x \in H_1\), then, for all \(p > 0\), \(T \in S_p\) implies \(S \in S_p\).

We recall Luecking’s theorem ([14]).

**Theorem 2.2 (Luecking’s theorem).** Let \(\mu\) be a positive Borel measure on \(\mathbb{D}\). Then the canonical inclusion \(J_\mu : H^2 \to L^2(\mu)\) is in the Schatten class \(S_p\), \(p > 0\), if and only if:

\[
\sum_{n=0}^{\infty} \sum_{j=0}^{2^n-1} [2^n \mu(\tilde{W}_{n,j})]^{p/2} < \infty,
\]

where \(\tilde{W}_{n,j} = \tilde{W}(e^{2jn\pi/2^n}, 2^{-n})\).

Let us point out that the above condition can be replaced by the following variant ([9, Proposition 3.3]):

\[
\sum_{n=0}^{\infty} \sum_{j=0}^{2^n-1} [2^n \mu(W_{n,j})]^{p/2} < \infty,
\]

where \(W_{n,j} = W(e^{2jn\pi/2^n}, 2^{-n})\).

As usual, the notation \(A \lesssim B\) means that \(A \leq cB\) for some positive constant \(c\), and \(A \approx B\) means that \(A \lesssim B\) and \(B \lesssim A\).

### 3 Compactification

**Theorem 3.1.** An analytic self-map \(\varphi : \mathbb{D} \to \mathbb{D}\) is compactifiable if and only if \(m(\{|\varphi^*| = 1\}) = 0\).

**Proof.** The necessary part is proved in [13, Proposition 2.4]. Let us recall the easy proof of this fact.

Indeed, suppose that \(T = M_w C_\varphi\) is compact and that \(|\varphi^*| = 1\) on \(E\), with \(m(E) > 0\). Since \((z^n)_n\) converges weakly to 0 in \(H^2\) and since \(T(z^n) = w \varphi^n\), we should have:

\[
\int_E |w^*|^2 \, dm = \int_E |w^*|^2 |\varphi^*|^{2n} \, dm \leq \int_T |w^*|^2 |\varphi^*|^{2n} \, dm = ||T(z^n)||^2 \underset{n \to \infty}{\rightarrow} 0;
\]

but this would imply that \(w\) is null a.e. on \(E\) and hence \(w \equiv 0\) (see [4], Theorem 2.2), which was excluded.

Let us now prove the sufficient condition.

Assume that \(m(\{|\varphi^*| = 1\}) = 0\) holds. Given \(w \in H^2\), we can write:

\[
\|M_w C_\varphi(f)\|^2 = \int_T |w^*|^2 |f \circ \varphi^*|^2 \, dm = \int_D |f|^2 \, d\nu,
\]
where \( \nu = \nu_w = \phi^*([w^*]^2m) \), that is \( \nu(A) = \int_{\phi^* - 1(A)} |w^*|^2 \, dm \). By the Carleson embedding theorem (see [2, page 129]), a necessary and sufficient condition for the operator \( M_w C_{\phi^*} : H^2 \to H^2 \) to be compact is that \( \nu \) is a vanishing Carleson measure for \( H^2 \). We now produce a suitable \( w \), \( w \not\equiv 0 \).

Let:

\[
\Gamma_h = \{ z : 1 - h \leq |z| < 1 \}
\]

and set:

\[
F_n = \phi^*^{-1}(\Gamma_{2^{-n}}) \quad \text{and} \quad c_n = m(F_n).
\]

Our assumption implies that \( c_n \to 0 \) as \( n \to \infty \). We can hence find an increasing sequence \( (k_n)_{n \geq 1} \) of integers such that:

\[
\sum_{n=1}^{\infty} c_{k_n} \log n < \infty.
\]

Let \( \phi_n : T \to \mathbb{R}^+ \) be defined as:

\[
\phi_n = \begin{cases} 
\frac{1}{n} & \text{on } F_{k_n}, \\
1 & \text{on } T \setminus F_{k_n}.
\end{cases}
\]

Let \( w_n \) be the associated outer function, satisfying \( |w_n^*| = \phi_n \), namely \( w_n = \exp(-\psi_n) \), with:

\[
\psi_n(z) = \int_T \frac{1 + z e^{-it}}{1 - z e^{-it}} \log \frac{1}{\phi_n(t)} \, dm(t) = \log n \int_{F_{k_n}} \frac{1 + z e^{-it}}{1 - z e^{-it}} \, dm(t).
\]

Observe that \( \Re \psi_n(z) = \log n \int_{F_{k_n}} P_z(t) \, dm(t) \), where \( P_z(t) = \frac{1 - |z|^2}{1 - z e^{-it}} \) is the Poisson kernel, so that \( \Re \psi_n(z) \geq 0 \) and \( |w_n(z)| \leq 1 \). Moreover \( |w_n^*| = \frac{1}{n} \) on \( F_{k_n} \).

The condition (3.2) ensures that the infinite product \( w = \prod_n w_n \) converges uniformly on compact subsets of \( D \), and defines a function \( w \in H^\infty \), bounded by 1 and without zeros. Indeed, since \( \Re \psi_n \geq 0 \), we see that:

\[
|1 - w_n(z)| \leq |\psi_n(z)| \leq \log n \int_{F_{k_n}} \frac{1 + |z|}{1 - |z|} \, dm(t) = (c_{k_n} \log n) \frac{1 + |z|}{1 - |z|};
\]

subsequently, the series \( \sum (1 - w_n) \) converges normally on compact subsets of \( D \), and the infinite product \( \prod w_n \) converges uniformly on compact subsets of \( D \), as claimed.

The weighted composition operator \( M_w C_{\phi^*} \) is bounded since \( w \in H^\infty \).

Let finally \( 0 < h < 2^{-k_1} \) and \( n = n(h) \) such that \( 2^{-k_n + 1} \leq h < 2^{-k_n} \). Let \( \xi \in T \). Then \( W(\xi, h) \subseteq \Gamma_h \), so that:

\[
\phi^*^{-1}[W(\xi, h)] \subseteq \phi^*^{-1}(\Gamma_h) \subseteq \phi^*^{-1}(\Gamma_{2^{-k_n}}) = F_{k_n}.
\]
As a consequence, $|w^*(u)| \leq |w^*_n(u)| \leq \frac{1}{n}$ for all $u \in \varphi^{-1}[W(\xi, h)]$, and:

$$
\nu[W(\xi, h)] = \int_{\varphi^{-1}[W(\xi, h)]} |w^*|^2 \, dm \leq \frac{1}{n^2} m_\varphi[W(\xi, h)] \leq \frac{1}{n^2} Ch,
$$

because we know (see [2, page 129]) that $m_\varphi$ is a Carleson measure. This ends the proof, since $n = n(h)$ tends to $\infty$ when $h$ goes to 0.

**Remark.** The previous argument can be sometimes quantified, and the degree of compactness of $M_w C_\varphi$ specified (even if there are limitations, as shown by the forthcoming Theorem 4.1).

**Theorem 3.2.** For each $\gamma$ with $0 < \gamma < 1/2$, there exist a non-compact composition operator $C_\varphi: H^2 \to H^2$ and a weight $w \in H^\infty$ such that, for some constant $b > 0$, we have:

$$a_n(M_w C_\varphi) \lesssim \exp(-bn^\gamma).$$

In particular $M_w C_\varphi$ belongs to all Schatten classes $S_p(H^2)$, $p > 0$.

For the proof, we recall the following simple result.

**Proposition 3.3.** Let $\nu$ be a vanishing Carleson measure on $\mathbb{D}$. Then:

$$a_n(J_\nu) \lesssim \inf_{0 < h < 1} \left( e^{-nh} + \sup_{0 \leq t \leq h} \frac{\rho_\nu(t)}{t} \right),$$

where $J_\nu : H^2 \to L^2(\nu)$ is the canonical inclusion.

In particular, if $w \in H^\infty$ and $\varphi$ is a symbol, we have:

$$a_n(M_w C_\varphi) \lesssim \inf_{0 < h < 1} \left( e^{-nh} + \sup_{0 \leq t \leq h} \frac{\rho_\nu(t)}{t} \right),$$

where $\nu = \varphi^*(|w^*|^2 m)$ is the pull-back measure of $|w^*|^2 m$ by $\varphi^*$.

For the proof of Proposition 3.3, we refer to [12, Theorem 5.1], where the result is given only for composition operators, but working exactly the same for inclusions, except only that we have to replace the quantity $\sqrt{\rho_\nu(h)/h}$ by $\sup_{0 \leq t \leq h} \sqrt{\rho_\nu(t)/t}$. For the special case, just use that $\|J_\nu f\| = \|(M_w C_\varphi) f\|$ for all $f \in H^2$, so there exist two contractions $U : L^2(\nu) \to H^2$ and $V : H^2 \to H^2$ such that $(M_w C_\varphi) = U J_\nu$ and $J_\nu = V (M_w C_\varphi)$, and hence $a_n(M_w C_\varphi) = a_n(J_\nu)$.

**Proof of Theorem 3.2.** We use a construction made in [9, Section 3.2].

Let $1 < \beta \leq 2$ and:

$$u(t) = |\sin(t/2)|^{\beta}.$$

There is an analytic function $U : \mathbb{D} \to \Pi^+ = \{ \Re z > 0 \}$ whose boundary values are:

$$U^\star(e^{it}) = u(t) + i \mathcal{H} u(t),$$

with
where $\mathcal{H}$ is the Hilbert transform. The symbol $\varphi$ is defined, for $z \in \mathbb{D}$, as:

\begin{equation}
\varphi(z) = \exp\left( -U(z) \right).
\end{equation}

By [9, Lemma 3.6 and Lemma 4.3], the composition operator $C_\varphi : H^2 \to H^2$ is not compact.

Moreover, since $|\varphi^*(e^{it})| = \exp\left( -|\sin(t/2)|^\beta \right)$, we have:

$$|\varphi^*(e^{it})| \geq 1 - h \iff |t| \leq \left( \log \frac{1}{1-h} \right)^{1/\beta} \approx h^{1/\beta};$$

so, if $\Gamma_h$ is the annulus $\{ z : 1 - h \leq |z| < 1 \}$, and we set:

$$F_k = \varphi^*-1(\Gamma_{2^{-k}}),$$

we have:

$$c_k := m(F_k) \approx 2^{-k/\beta}.$$

Now, let $\delta_k = \exp(-2^{k/\beta} / k^2)$. We slightly modify the example of Theorem 3.1 as follows:

$$\phi_k = \left\{ \begin{array}{ll}
\delta_k & \text{on } F_k, \\
1 & \text{on } T \setminus F_k.
\end{array} \right.$$

Then, the series $\sum_{k \geq 1} c_k \log(1/\delta_k)$ converges since $c_k \log(1/\delta_k) \leq 1/k^2$. As in the proof of Theorem 3.1, we can define an outer function $w$ such that $|w^*| = \prod_{k \geq 1} \phi_k$. The same computation gives us, for any Carleson window $W(\xi, t)$ and for $\nu = \varphi^*(|w^*|^2 m)$:

$$\nu[|W(\xi, t)|] \lesssim \delta_j^2 t, \quad \text{for } 2^{-j-1} \leq t < 2^{-j}.$$

Let $0 < h < 1$ arbitrary.

There exists an integer $l \geq 0$ such that $2^{-l-1} \leq h < 2^{-l}$. Then for $0 < t \leq h$, we have $2^{-j-1} \leq t < 2^{-j}$ for some $j \geq l$; hence:

$$\frac{\rho_\nu(t)}{t} \lesssim \delta_j^2 \leq \delta_l^2.$$

Therefore Proposition 3.3 gives:

$$a_n(M_w C_\varphi) \lesssim \inf_{l \in \mathbb{N}} (e^{-n2^{-l}} + \delta_l) \lesssim \inf_{l \geq 0} \left( \exp(-n2^{-l}) + \exp(-2^{l/\beta}/l^2) \right).$$

The choice $l = \left\lfloor \frac{\beta}{(\beta+1) \log 2} \log n \right\rfloor$ gives, for some $b > 0$:

$$a_n(M_w C_\varphi) \lesssim \exp \left( -b n^{1/(\beta+1)}/(\log n)^2 \right).$$

Now, if $0 < \gamma < 1/2$, we take $\beta$ such that $1 < \beta < 1 - 1/\gamma$ and $\beta \leq 2$. We obtain, with another $b > 0$:

$$a_n(M_w C_\varphi) \lesssim \exp(-b n^{\gamma}),$$

as claimed. \qed
Remark 1. For $\beta < 1$, since we have $m_\varphi(\Gamma_h) \approx h^{1/\beta}$, the composition operator $C_\varphi$ is already compact. When $\beta = 1$, we have $m_\varphi(\Gamma_h) \approx h$, but it can be checked that nevertheless $C_\varphi$ is compact and $\rho_\varphi(h) = O(h/\log(1/h))$ (see [8, Remark 3, page 3117]). Without doing that, we can use [9, Theorem 4.1] (which is an improvement of [8, Theorem 4.1]): there exists a compact composition operator with symbol $\tilde{\varphi}$ such that $|\tilde{\varphi}^{\ast}| = |\varphi^{\ast}|$; therefore $m_{\tilde{\varphi}}(\Gamma_h) = m_\varphi(\Gamma_h) \approx h$.

For $\beta = 1$, the above proof only gives:

$$a_n(M_w C_\varphi) \gtrsim \exp \left( -b n^{1/2}/(\log n)^2 \right).$$

Though in this case $C_\varphi$ was already compact, that nevertheless allows to improve the compactness.

Remark 2. The case $\beta = 2$ corresponds to the simple symbol $\varphi(z) = \frac{1+z}{2}$.

Indeed, we only used in our construction the modulus of the symbol and for this $\varphi$, we have $|\varphi^{\ast}(e^{it})| = |\cos(t/2)| \approx 1 - t^2/8 \approx \exp \left( -|\sin(t/2\sqrt{2})|^2 \right)$.

We get the following result.

Theorem 3.4. Let $\varphi(z) = \frac{1+z}{2}$. For each decreasing sequence $(\varepsilon_k)$ of positive numbers such that $(\delta_k) = (2^{k/2}\varepsilon_{2k})$ is decreasing, there exist a weight $w \in H^\infty$ and a positive constant $b$ such that:

$$a_n(M_w C_\varphi) \gtrsim \exp \left( -b n^{1/3}\varepsilon_n \right).$$

Proof. We only have to modify the proof of Theorem 3.2: we replace $F_k$ by:

$$F_k = \varphi^{-1}(\Gamma_{4^k/3})$$

so $c_k = m(F_k) \approx 2^{-k/3}$, and we replace $\delta_k = \exp(-2^{k/3}/k^2) = \exp(-2^{k/2}/k^2)$ by:

$$\delta_k = \exp(-2^{k/3}\varepsilon_{2k}),$$

where $(\varepsilon_k)_k$ is a given decreasing sequence of positive integers such that $(\delta_k)$ is decreasing. Note that, since $(\delta_k)$ is decreasing, we have $\varepsilon_{2k} \lesssim 2^{-k/2}$, so $\sum_k \varepsilon_{2k} < \infty$. We get:

$$a_n(M_w C_\varphi) \lesssim \inf_{l \geq 0} \left( e^{-n^{2l+1}} + e^{-2l/3}\varepsilon_{2l} \right),$$

and, with $l = \lceil \log n/\log 2 \rceil$, we get, since $\varepsilon_n \leq \varepsilon_{2l}$, for some $b > 0$:

$$a_n(M_w C_\varphi) \lesssim \exp \left( -b n^{1/3}\varepsilon_n \right).$$

For example, with $\varepsilon_k = 1/(\log k)^2$, we get $a_n(M_w C_\varphi) \lesssim e^{(-b n^{1/3}/(\log n)^2)}$.

Theorem 3.4 improves a result of [7, Theorem 2.3], where for this symbol and a given $\alpha > 0$, weights $w$ are obtained such that:

$$a_n(M_w C_\varphi) \lesssim \left( \frac{\log n}{n} \right)^\alpha.$$
4 Hilbert-Schmidt and Schatten regularizations

We begin with a characterization of the symbols that can give a Hilbert-Schmidt weighted composition operator.

**Theorem 4.1.** An analytic self-map \( \varphi : \mathbb{D} \to \mathbb{D} \) can induce a Hilbert-Schmidt weighted composition operator \( M_w C_\varphi \), for some weight \( w \in H^2 \), if and only if:

\[
\int_T \log \left( \frac{1}{1 - |\varphi^*|} \right) \, dm < +\infty.
\]

**Proof.** That the condition is sufficient is proved in [13, Proposition 2.5]. For sake of completeness, we recall the argument.

The hypothesis implies that there exists an outer function \( w \) on \( \mathbb{D} \) such that

\[
|w^*|^2 = 1 - |\varphi^*|.
\]

Then, writing \( T = M_w C_\varphi \), we have:

\[
\sum_{n=0}^{\infty} \|T(z^n)\|^2 = \sum_{n=0}^{\infty} \int_T (1 - |\varphi^*|)|\varphi^*|^{2n} \, dm = \int_T \frac{1}{1 + |\varphi^*|} \, dm < +\infty,
\]

and \( T \) is Hilbert-Schmidt, as claimed.

Let us prove the necessity of the condition.

If \( w \in H^2 \) exists such that \( M_w C_\varphi : H^2 \to H^2 \) is Hilbert-Schmidt, we have in particular \( |\varphi^*| < 1 \) \( m \)-almost everywhere, by the easy part of Theorem 3.1.

Since \( M_w C_\varphi \) is Hilbert-Schmidt, we have:

\[
\sum_{n=0}^{\infty} \|w \varphi^n\|^2 = \sum_{n=0}^{\infty} \|(M_w C_\varphi)(z^n)\|^2 < \infty,
\]

i.e.:

\[
\int_T |\varphi^*|^2 \frac{1}{1 - |\varphi^*|^2} \, dm < \infty.
\]

The following lemma, with \( u = |w^*|^2 \), \( v = 1 - |\varphi^*|^2 \) and \( \alpha = 1 \), then shows that \( \int_T \log \frac{1}{1 - |\varphi^*|^2} \, dm < \infty \). In fact, since \( w \in H^2 \) and \( w \neq 0 \), Jensen’s inequality tells that the first condition of that lemma is satisfied. \( \square \)

**Lemma 4.2.** Let \((\Omega, \nu)\) be a measure space and \( u, v : \Omega \to (0,1] \) measurable functions such that, for some \( \alpha > 0 \):

\[
\int_\Omega |\log u| \, d\nu < \infty \quad \text{and} \quad \int_\Omega w^{-\alpha} \, d\nu < \infty.
\]

Then \( \int_\Omega |\log v| \, d\nu < \infty \).

**Proof.** If we set \( g = v^{-\alpha} \) and \( f = w^{-\alpha} \), we have:

\[
0 \leq \log g = \log f + \log \frac{1}{u} \leq \log^+ f + |\log u| \leq f + |\log u|.
\]
By hypothesis, $f$ (which is positive) and $|\log u|$ are integrable; hence $\log g$ is integrable and:

$$\int_T |\log v| \, dv < \infty.$$  

In Theorem 4.1 we showed that for the outer function $w$ such that $|w^*|^2 = 1 - |\varphi^*|$, the weighted composition operator $M_wC_\varphi$ is Hilbert-Schmidt. For this weight, we cannot expect better in general, as said by the following theorem.

**Theorem 4.3.** There exist a symbol $\varphi$ satisfying $\int_T \log(1 - |\varphi^*|) \, dm > -\infty$ such that, if $w$ is any outer function satisfying $|w^*|^2 = 1 - |\varphi^*|$, the weighted composition operator $M_wC_\varphi$ is Hilbert-Schmidt, but $M_wC_\varphi \notin S_p$, for all $p < 2$.

**Proof.** Let, for $|t| \leq \pi$:

$$u(t) = 1 - \exp(-e^{1/|t|}).$$

We have $0 < 1 - \exp(-e^{1/\pi}) \leq u(t) \leq 1$; hence $\int_0^\pi \log u(t) \, dt > -\infty$; therefore there is an outer function $\varphi \in H^\infty$ such that $|\varphi^*(e^{it})| = u(t)$.

Moreover, we also have $\int_T \log(1 - |\varphi^*|) \, dm = \int_0^\pi \log (1 - u(t)) \, dt > -\infty$.

Hence if $w$ is an outer function such that $|w^*|^2 = 1 - |\varphi^*|$, the weighted composition operator $M_wC_\varphi$ is Hilbert-Schmidt.

We are going to show that $M_wC_\varphi$ does not belong to any Schatten class for $p < 2$.

For that, we use Theorem 2.2. The weighted composition operator $M_wC_\varphi$ can be viewed as an inclusion $J_\nu : H^2 \to L^2(\nu)$, where $\nu = \varphi^*(|w^*|^2 m)$. Here, we also have $d\nu(z) = (1 - |z|) \, dm(z)$.

Since $p < 2$, we have:

$$\sum_{j=0}^{2^n-1} [2^n \nu(\tilde{W}_{n,j})]^{p/2} \geq \left( \sum_{j=0}^{2^n-1} 2^n \nu(\tilde{W}_{n,j}) \right)^{p/2} = [2^n \nu(\tilde{\Gamma}_{2^n})]^{p/2},$$

where $\tilde{\Gamma}_h = \{ z \in \mathbb{D} : 1 - h \leq |z| \leq 1 - h/2 \}$.

But $\nu(\tilde{\Gamma}_{2^{-n}}) \approx 2^{-n} m_\varphi(\tilde{\Gamma}_{2^{-n}})$ and

$$m_\varphi(\tilde{\Gamma}_h) \approx \frac{1}{(\log 1/h)(\log \log 1/h)^2}.$$  

In fact, we have $\varphi^*(e^{it}) \in \tilde{\Gamma}_h$ if and only if $h/2 \leq \exp(-e^{1/|t|}) \leq h$, which is equivalent to:

$$\frac{1}{\log \log 2/h} \leq |t| \leq \frac{1}{\log \log 1/h},$$

and:

$$\frac{1}{\log \log 1/h} - \frac{1}{\log \log 2/h} \approx \frac{1}{(\log \log 1/h)^2} \log \left( 1 + \frac{\log 2}{\log 1/h} \right) \approx \frac{1}{(\log 1/h)(\log \log 1/h)^2}.$$  

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Hence:
\[ 2^n \nu(n) \gtrsim \frac{1}{n (\log n)^2} \]
and we obtain:
\[ +\infty \sum_{n=0}^{2^n-1} |2^n \nu(n)\|^p/2 \gtrsim \sum_{n=0}^{+\infty} \frac{1}{nP/2 (\log n)^p} = \infty, \]
since \( p/2 < 1 \). Luecking’s theorem tells that \( M_w C_{\varphi} \notin S_p \).

If Theorem 4.3 does not allow to have better than Hilbert-Schmidt with the same weight, an improvement is possible by taking another weight.

**Theorem 4.4.** Assume that the composition operator \( C_{\varphi} \) can induce a Hilbert-Schmidt weighted composition operator. Then there exists another weight \( w \in H^2 \) such that \( M_w C_{\varphi} \in S_p \) for every \( p < 2 \).

**Proof.** By Theorem 4.1, we have
\[
\int_D \log \frac{1}{1-|z|} \ dm_{\varphi}(z) < \infty.
\]
 Take an integer \( K > 1/p \) and let \( w_K \) be an outer function such that
\[
|w_K^*| = (1 - |\varphi^*|)^K.
\]
We point out that
\[
\|w_K^*(\varphi^*)^n\|_{L^\infty(T)} \leq \sup_{t \in (0,1)} (1-t)^K t^n \lesssim \frac{1}{n^K}.
\]
Hence we have, for some positive constant \( C \) (depending on \( K \) but not on \( n \)): \[
\|(M_{w_K} C_{\varphi})(z^n)\|^2 = \int_T |w_K^*|^2 |\varphi^*|^{2n} \ dm \leq \frac{C}{n^{2K}}.
\]
It follows that \( \|(M_{w_K} C_{\varphi})(z^n)\|^p \leq C^{p/2} / n^{Kp} \) and hence
\[
\sum_{n=1}^{+\infty} \|(M_{w_K} C_{\varphi})(z^n)\|^p < \infty,
\]
since \( Kp > 1 \).

Now, by the du Bois-Reymond lemma, there exists a measurable function \( g: [0,1] \to \mathbb{R}_+ \) such that \( g(t) \to 1 \) and \( \int_0^1 g(|z|) \log \frac{1}{|z|} \ dm_{\varphi}(z) < \infty \). So there is an outer function \( w \) such that \( |w^*| = (1 - |\varphi^*|)|g| \varphi^* | \). Since \( g(t) \to 1 \), we have \( g(t) \geq K \) for \( t \) close enough to 1 and it follows that \( |w^*| \lesssim |w_K^*| \) (up to a constant depending on \( K \) only). Hence \( \|(M_w C_{\varphi})f\| \lesssim \|(M_{w_K} C_{\varphi})f\| \) for all \( f \in H^2 \), and \( M_w C_{\varphi} \in S_p \) since \( M_{w_K} C_{\varphi} \in S_p \). \( \square \)
Theorem 4.5. For every \( p < \infty \), if \( M_w C_\varphi \in S_p \) for some weight \( w \), then there exists another weight \( \tilde{w} \) for which \( M_{\tilde{w}} C_\varphi \) is Hilbert-Schmidt.

Proof. For \( p \leq 2 \), this is obvious, with the same weight, since \( S_p \subseteq S_2 \). So we assume \( p > 2 \). We have \( \sum_{n=0}^{\infty} \| (M_w C_\varphi) (z^n) \|_p < \infty \), i.e.:

\[
\sum_{n=0}^{\infty} \left( \int_T |w^n|^2 |\varphi^n|^2 \, dm \right)^p < \infty.
\]

When \( \sum_{n=0}^{\infty} |c_n|^p < \infty \), the Hölder inequality implies that, for \( \beta > 1/q \) (\( q \) is the conjugate exponent of \( p \)), we have:

\[
\sum_{n=0}^{\infty} \frac{1}{n^p} |c_n| \leq \left( \sum_{n=0}^{\infty} \frac{1}{n^{q\beta}} \right)^{1/q} \left( \sum_{n=0}^{\infty} |c_n|^p \right)^{1/p} < \infty.
\]

Now,

\[
(1 - |\varphi^*|^2)^{-\beta} = \sum_{n=0}^{\infty} \left( \frac{-\beta}{n} \right) (-1)^n |\varphi^*|^{2n},
\]

and, by the Stirling formula \((-\beta/n)(-1)^n \approx n^{\beta-1}\). Hence if we take \( \beta \) such that \( 1/q < \beta < 1 \) and set \( \alpha = 1 - \beta \), we have \( \alpha > 0 \) and:

\[
\int_T |w^*|^2 (1 - |\varphi^*|^2)^{-\alpha} \, dm \approx \sum_{n=0}^{\infty} \frac{1}{n^\beta} \int_T |w^*|^2 |\varphi^*|^{2n} \, dm < \infty.
\]

It follows from Lemma 4.2 that \( \int_T \log(1 - |\varphi^*|^2) \, dm < \infty \), and then, from Theorem 4.1, that there is a weight \( \tilde{w} \) for which \( M_{\tilde{w}} C_\varphi \) is Hilbert Schmidt.

Let us put together Theorem 4.1, Theorem 4.4 and Theorem 4.5.

Theorem 4.6. For any symbol \( \varphi \), the following assertions are equivalent:

1) there is a weight \( w \), with \( w \in H^2 \), such that \( M_w C_\varphi \) is Hilbert-Schmidt;

2) there is a weight \( \tilde{w} \), with \( \tilde{w} \in H^\infty \), such that \( M_{\tilde{w}} C_\varphi \in S_p \) for all \( p > 0 \);

3) there exist \( p < \infty \) and a weight \( w_p \), with \( w_p \in H^\infty \), such that \( M_{w_p} C_\varphi \in S_p \);

4) \( \int_T \log \frac{1}{1 - |\varphi^*|} \, dm < \infty \).

As a consequence, we see that in general, the condition \( m(\{|\varphi| = 1\}) = 0 \) cannot give better than a compactification.

Theorem 4.7. There exists a compactifiable symbol \( \varphi \), i.e. \( m(\{|\varphi^*| = 1\}) = 0 \), such that, whatever the weight \( w \), \( M_w C_\varphi \) is not in any Schatten class \( S_p \), with \( p < \infty \).
Proof. It suffices to find a symbol $\varphi$ such that $m(\{|\varphi^*| = 1\}) = 0$ but such that
$$\int_T \log \frac{1}{1-|\varphi^*|} \, dm = \infty,$$
i.e. an element of the unit ball of $H^\infty$ that is an extreme point of that unit ball but not an exposed point. If we set $u(t) = 1 - e^{-1/|t|}$ for $|t| \leq \pi$, then $0 < 1 - e^{-1/\pi} \leq u(t) \leq 1$, so $\int_{|t|\leq\pi} \log u(t) \, dt > -\infty$, so there exists an outer function $\varphi \in H^\infty$ such that $|\varphi^*(e^{it})| = u(t)$. Clearly, this function works. 

5 Weighted composition operators on $H^p$

In this section we assume that $1 \leq p < +\infty$. We are interested here in finding a characterization of the symbols that can give a weighted composition operator belonging to some specific ideal of operators. In particular, we focus on the ideal of nuclear operators and the ideal of absolutely summing operators.

First let us recall

- An operator $T: X \to Y$ between Banach spaces $X$ and $Y$ is nuclear if there are elements $y_n \in Y$ and linear forms $x_n^* \in X^*$ with $\sum_{n=0}^{\infty} \|x_n^*\| \|y_n\| < \infty$ such that $Tx = \sum_{n=0}^{\infty} x_n^*(x) y_n$ for all $x \in X$.

- An operator $T: X \to Y$ between Banach spaces $X$ and $Y$ is $r$-summing, $1 \leq r < \infty$, if there is a positive constant $C$ such that:
$$\left( \sum_{k=1}^{n} \|Tx_k\|^r \right)^{1/r} \leq C \sup_{x^* \in B_{X^*}} \left( \sum_{k=1}^{n} |\langle x_k, x^* \rangle|^r \right)^{1/r}$$
for all finite sequence $(x_1, \ldots, x_n)$ in $X$.

The main result of this section is

Theorem 5.1. Let $\varphi: \mathbb{D} \to \mathbb{D}$ be a symbol.
The following assertions are equivalent.

(1) There exists a weight $w$ such that $M_w C_\varphi: H^p \to H^p$ is a nuclear operator for every $p \geq 1$.

(2) There exists a weight $w$ such that $M_w C_\varphi: H^p \to H^p$ is 1-summing for every $p \geq 1$ (and hence is $r$-summing for every $r \geq 1$).

(3) There exists a weight $w$ such that $M_w C_\varphi: H^p \to H^p$ is $r$-summing for some $r \geq 1$ and some $p \geq 1$.

(4) $\int_T \log \frac{1}{1-|\varphi^*|} \, dm < \infty$.

Proof. Clearly (1) implies (2), which implies (3).

The weighted composition operator $(M_w C_\varphi)$ can be viewed as the Carleson embedding $J_{\nu_p}: H^p \to L^p(\nu_p)$ where $\nu_p = \varphi^* (|w^*|^p m)$ is a finite measure on $\mathbb{D}$.  

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Assume (3). Then $J_{\nu_p}$ is actually $r$-summing on $H^s$ where $s = \min(2, p)$ thanks to [11, Theorem 8.4]. By [11, Proposition 2.3, 1)], we have:

$$
\int_T \frac{|w^*|^p}{(1 - |\varphi^*|)^{s/2}} \, dm = \int_B \frac{d\nu_p(z)}{(1 - |z|)^{s/2}} < \infty.
$$

By Lemma [12] that implies that $\int_T \log \frac{1}{1 - |\varphi^*|} \, dm < \infty$ and (4) is satisfied.

Now assume that (4) is satisfied. For every $f \in H^p$, we denote by $\hat{f}(n)$ its $n^{th}$ Taylor coefficient. We point out that the functional $f \in H^p \mapsto \hat{f}(n)$ has norm 1. Then, for any operator $T: H^p \to Y$ satisfying $\sum_{n=0}^{\infty} \|Te_n\| < \infty$ where $e_n(z) = z^n$, it is easy to check that $T$ is a nuclear operator.

Our assumption implies that there exists an outer function $w$ such that $|w^*| = (1 - |\varphi^*|)^2$ a.e. and we already pointed out that $\|w^*|\varphi^*|n\|_{L^\infty(T)} \leq \frac{C}{n^2}$, for some constant $C > 0$.

Hence:

$$
\|(M_wC_{\varphi})(e_n)\|_p = \left( \int_T |w^*|^p |\varphi^*|^pn \, dm \right)^{\frac{1}{p}} \leq \frac{C}{n^2}.
$$

We get that $\sum_n \|(M_wC_{\varphi})(e_n)\|_p < +\infty$ and hence that $(M_wC_{\varphi})$ is a nuclear operator.

6 Decomplexification

6.1 An initial example

We refer to [15, page 27] (see also [10]) for the definition of the lens map $\lambda_\theta$ of parameter $\theta$, $0 < \theta < 1$.

We saw in [17, Theorem 4.1] that multiplication by a second symbol $w$ can improve the degree of compactness of a composition operator $C_{\varphi}$. For example, if $\varphi = \lambda_\theta$, which satisfies [10, Theorem 2.1]):

$$
e^{-b_1 \sqrt{n}} \lesssim a_n(C_{\lambda_\theta}) \lesssim e^{-b_2 \sqrt{n}}
$$

(implying in particular that $C_{\lambda_\theta}$ is in all Schatten classes $S_p(H^2)$, $p > 0$), we exhibited functions $w \in H^\infty$ such that:

$$
e^{-b'_1 n/\log n} \lesssim a_n(M_wC_{\varphi}) \lesssim e^{-b'_2 n/\log n}.
$$

We wish to prove here that, conversely, multiplication by $w$ can in some sense “decompactify” $C_{\varphi}$ while keeping it bounded. We shall begin with an explicit example.

**Theorem 6.1.** Let $\lambda_\theta$ be a lens map, $0 < \theta < 1$, and let $w(z) = (1 - \lambda_\theta(z))^a$ where $a = \frac{1}{2}(1 - \frac{1}{\theta}) < 0$. Then $w \in H^2$ and the weighted composition operator $M_wC_{\lambda_\theta}$ is bounded but not compact on $H^2$, though $C_{\lambda_\theta}$ is in all Schatten classes $S_p(H^2)$, $p > 0$. 

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Proof. We first observe that $w \in H^2$ since $|1 - \lambda_\theta^*(\xi)| \approx |1 - \xi|\theta$ when $\xi \in T$ (see Lemma 2.5) and $2a\theta = \theta - 1 > -1$. Let now $f \in H^2$. Then we have, formally:

$$\|M_w C_w(f)\|^2 = \int_T |1 - \lambda_\theta^*(\xi)|^{2a} |f \circ \lambda_\theta^*(\xi)|^2 dm(\xi) = \int_{\mathbb{B}} |f(u)|^2 d\mu(u),$$

where:

$$d\mu = |1 - u|^{2a} dm_{\lambda_\theta}(u),$$

with $m_{\lambda_\theta} = \lambda_\theta^*(m)$.

It is sufficient to prove that $\mu$ is a Carleson measure, but not a vanishing one, for $H^2$. We can restrict ourselves to the Carleson windows $W(1,h)$ centered at 1.

We know (Lemma 2.5) that, for some constants $C > c > 0$, depending on $\theta$, we have $c|t|\theta \leq 1 - |\lambda_\theta^*(e^{it})| \leq C|t|\theta$ and $|\arg(\lambda_\theta^*(e^{it}))| \leq C\pi|t|\theta$; it follows easily that $m_{\lambda_\theta}[W(1,h)] \approx h^{1/\theta}$. Hence:

$$\mu[W(1,h)] = \sum_{n=0}^\infty \mu[W(1,2^{-n}h) \setminus W(1,2^{-n-1}h)]$$

$$\approx \sum_{n=0}^\infty (2^{-n}h)^{2a} m_{\lambda_\theta}[W(1,2^{-n}h) \setminus W(1,2^{-n-1}h)]$$

$$\leq \sum_{n=0}^\infty (2^{-n}h)^{2a} (2^{-n}h)^{1/\theta} \approx h \sum_{n=0}^\infty 2^{-n} = 2h$$

(since $2a + 1/\theta = 1$), proving that $\mu$ is a Carleson measure.

On the other hand, if we consider the modified Hastings-Luecking windows:

$$\tilde{W}(1,h) = \{ z \in \mathbb{D}; (c/2C) h \leq 1 - |z| \leq h \quad \text{and} \quad |\arg(z)| \leq \pi h \},$$

we have $m_{\lambda_\theta}(\tilde{W}(1,h)) \gtrsim h^{1/\theta}$, because if $(h/2C)^{1/\theta} \leq |t| \leq (h/C)^{1/\theta}$, we have $1 - |\lambda_\theta^*(e^{it})| \leq C |t|\theta \leq h$, $1 - |\lambda_\theta^*(e^{it})| \geq c |t|\theta \geq (c/2C) h$ and $|\arg(\lambda_\theta^*(e^{it}))| \leq C\pi|t|\theta \leq \pi h$, so $\lambda_\theta^*(e^{it}) \in \tilde{W}(1,h)$. It follows that:

$$\mu[W(1,h)] \geq \mu(\tilde{W}(1,h)) \gtrsim h^{2a} m_{\lambda_\theta}(\tilde{W}(1,h)) \gtrsim h^{2a} h^{1/\theta} = h,$$

so $\mu$ is not a vanishing Carleson measure. \qed

6.2 The general case

We now turn to the general case, with a less explicit construction, under the following form.
Theorem 6.2. An analytic self-map \( \varphi : \mathbb{D} \to \mathbb{D} \) is decompactifiable if and only if \( \| \varphi \|_\infty = 1 \).

Proof. First assume that \( \| \varphi \|_\infty < 1 \). Let \( w \in H^2 \) and \( (f_n) \) a weakly null sequence in \( H^2 \); this implies that \( f_n \to \varphi \) uniformly on compact subsets of \( \mathbb{D} \), so that \( \| f_n \circ \varphi \|_\infty \to 0 \). But then:

\[
\| M_w C_\varphi (f_n) \|_2 \leq \| w \|_2 \| f_n \circ \varphi \|_\infty \to 0 .
\]

This shows that \( M_w C_\varphi \) is compact for any \( w \in H^2 \).

Now, assume that \( \| \varphi \|_\infty = 1 \). We are going to show that \( \varphi \) is decompactifiable.

We need to find a weight \( w \in H^2 \) such that the finite (since \( w \in H^2 \)) measure \( \nu = \varphi^* (|w^*|^2 \, m) \), namely:

\[
\nu(A) = \int_{\varphi^{-1}(A)} |w^*|^2 \, dm
\]

is Carleson (ensuring that \( M_w C_\varphi : H^2 \to H^2 \) is bounded), but not vanishing Carleson (implying that \( M_w C_\varphi : H^2 \to H^2 \) is not compact).

If \( C_\varphi \) is not compact, it suffices to take \( w = 1 \).

We now assume that \( C_\varphi \) is compact. Then \( m(\{ |\varphi^*| = 1 \}) = 0 \).

This fact and the hypothesis \( \| \varphi \|_\infty = 1 \) clearly imply that \( m_\varphi (\Gamma_n) > 0 \) for each \( n \), where \( \Gamma_n \) is the annulus \( \{ z \in \mathbb{D} ; 1 - 2^{-n} \leq |z| < 1 \} \). If we set:

\[
C_l = \{ z \in \mathbb{D} ; 1 - 2^{-l} \leq |z| < 1 - 2^{-l-1} \},
\]

we have \( \Gamma_n = \bigcup_{l \geq n} C_l \), so that \( m_\varphi (C_l) > 0 \) for some \( l \geq n \). We can therefore find an increasing sequence \( (k_n) \) of integers such that \( m_\varphi (C_{k_n}) > 0 \) for each \( n \). Splitting in the natural way \( C_{k_n} \) into \( 2^{k_n} \) Hastings-Luecking boxes, we can find a sequence \( (\xi_n) \) of points of \( \mathbb{T} \) such that, with \( \tilde{W}_{k_n} = \tilde{W} (\xi_n, 2^{-k_n}) \):

\[
m_\varphi (\tilde{W}_{k_n}) > 0 .
\]

We define our weight \( w \) as an outer function \( w \in H^2 \) with boundary values \( w^* \). Let

\[
u(A) = \int_{\varphi^{-1}(A)} |w^*|^2 \, dm
\]

Then \( u \geq 1 \), so \( \log u \geq 0 \), and:

\[
0 \leq \int_{\mathbb{T}} \log u \, dm \leq \int_{\mathbb{T}} (u - 1) \, dm = \sum_{n=1}^{\infty} 2^{-k_n} \leq 1 < \infty ;
\]

Hence \( \log u \in L^1(\mathbb{T}) \) and there is an outer function \( w \in H^2 \) such that \( |w^*|^2 = u \) (see [14, page 24]).
Now, if $\nu = \varphi^* (|w|^2 m) = \varphi^*(um)$, we have:

$$\nu(A) = m_\varphi(A) + \sum_{n=1}^{\infty} \frac{2^{-kn}}{m_\varphi(W_{kn})} m_\varphi(A \cap \bar{W}_{kn}),$$

and $\nu$ is not a vanishing Carleson measure since, with $W_{kn} = W(\xi_n, 2^{-kn})$:

$$\nu(W_{kn}) \geq 2^{-kn} \frac{m_\varphi(W_{kn} \cap W_{kn})}{m_\varphi(W_{kn})} = 2^{-kn}.$$

Let now $W = W(\xi, h)$ be an arbitrary Carleson window. Without loss of generality, we can assume $h = 2^{-N}$ for some positive integer $N$, and we observe that if $z \in W \cap \bar{W}_{kn}$, then $1 - 2^{-N} \leq |z| \leq 1 - 2^{-kn-1}$, implying $k_n \geq N - 1$. Hence $W \cap \bar{W}_{kn} = \emptyset$ for $k_n < N - 1$ and:

$$\nu(W) = m_\varphi(W) + \sum_{k_n \geq N-1} 2^{-kn} \frac{m_\varphi(W_{kn} \cap W)}{m_\varphi(W_{kn})} \leq m_\varphi(W) + \sum_{k_n \geq N-1} 2^{-kn} \leq m_\varphi(W) + \sum_{l \geq N-1} 2^{-l} = m_\varphi(W) + 4h.$$

Since $C_\varphi$ is bounded, $m_\varphi$ is a Carleson measure and $m_\varphi(W) = O(h)$; therefore $\nu(W) = O(h)$ and hence $\nu$ is a Carleson measure. This shows that $C_\varphi$ is decompactified by $M_w$ and that completes the proof. \qed

**Remark.** For $p \geq 1$, if we set $\bar{w} = w^{2/p}$, then $w \in H^p$ and the same proof shows that the weighted composition operator $M_\bar{w} C_\varphi : H^p \to H^p$ is bounded but not compact.

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