ON THE FORMALIST ACCOUNT OF AN ARBITRARY FIRST-ORDER THEORY

STEPHEN BOYCE

Abstract. This paper examines the metatheory of the formalist account of an arbitrary first-order theory. The paper considers whether the metatheory can be expressed (using Tarskian semantics) in a model of a first-order theory that, roughly speaking, contains a proper axiom (schema) corresponding to a set-theoretic axiom (schema) of subsets. The hypothesis is reduced to absurdity.

1. THE INFORMAL SEMANTIC METATHEORY OF FIRST-ORDER LOGIC

This paper examines the formalist account for first-order logic and first-order theories more generally. For brevity, I characterise the formalist account as the view that the formation and inference rules of first-order theories generally (including the various predicate calculi) may be described in purely syntactical terms. That is, the formation and inference rules may be described in statements that refer only to uninterpreted primitive symbols, sequences of such symbols, sequences of sequences such symbols and so on; clearly this does not imply, of itself, that the collection of theorems of such systems may be characterised in purely syntactical terms. In discussing first-order logic below I have in mind Mendelson's [6] presentation, though may aim is to exploit only features of this account shared by any orthodox presentation. In brief then I consider in this paper whether the metatheory of first-order logic, viewed as a collection of propositions, is expressed in a model of any first-order theory, which, roughly speaking, contains an axiom schema that corresponds to an axiom schema of subsets. A precise definition of the intended class of first-order theories is presented below, however I will firstly present a more precise definition of the notion of an arbitrary first-order theory (adapting [4] and [6]).

Definition 1. The arbitrary first-order theory $K$ shall be a definite but unspecified value of the following function $\phi$; for $L$ a first-order language (see e.g. [6] § 2.2) and $\kappa$ a possible empty set of $L$ well-formed formulas (the proper axioms of $K$), the value of $\phi(L, \kappa)$ shall be the smallest set that: includes every formula in $\kappa$ and every well-formed $L$ formula that is a first-order logical truth, that is closed with respect to the relation of $C$ following from $A$ and $B$ by either the rule of modus ponens or generalisation.

In the discussion that follows, I take the following five key propositions to form part of the informal semantic metatheory of an arbitrary first-order theory.

Proposition 1. If $K$ is any first-order theory then by the formalist account of first-order logic the following metatheoretical sentences express true propositions [6].
Soundness: The \( \mathcal{L} \) formulas in \( K \) (the \( K \) theorems) are all true under every interpretation of \( \mathcal{L} \) that is a model of \( K \);

Consistency: If \( \mathcal{B} \) (any \( \mathcal{L} \) formula) is in \( K \) then either every \( \mathcal{L} \) formula is in \( K \) or the negation of \( \mathcal{B} \) is not in \( K \);

Semantic completeness: Every well-formed \( \mathcal{L} \) formula that is logically valid is a \( K \) theorem (and its theoremhood formally provable within a first-order calculus using only explicitly given logical axioms and rules of inference);

Syntactic incompleteness: If \( K \) contains Peano arithmetic and is recursively axiomatised, then either (i) there exists a well-formed \( \mathcal{L} \) formula such that neither this formula nor its negation is in \( K \), or (ii) every well-formed \( \mathcal{L} \) formula is in \( K \);

Semi-decidability: If there is an \( \mathcal{L} \) formula that is not in \( K \) then there is no mechanical or recursive algorithm for determining whether an arbitrary \( \mathcal{L} \) formula is or is not a \( K \) theorem; if however the set of Gödel numbers associated with the formulas in \( \kappa \) (and \( \mathcal{L} \)) is recursive, then there exists a mechanical / recursive test that confirms that \( \mathcal{B} \) (an arbitrary \( K \) theorem) is indeed a theorem.

Before examining the question of whether this metatheory can be made precise in the indicated manner, I make some additional comments on the symbolism and terminology used below. A reader who is not interested in pedantic distinctions should be able to skip this subsection without losing essential details.

1.1. A note on symbolism and terminology. For brevity, I use a primitive notion of 'expressing a proposition' throughout this paper, though it should be clear that this idea can be made precise in a thoroughly orthodox (Tarskian) way. For example, if under the intended interpretation of some first-order language, the binary predicate \( 'A_2' \) is assigned the relation of equality, and the individual constants \( 'a_1', 'a_2' \) are assigned respectively to the individuals \( a, b \), then the proposition that \( a \) equals \( b \) is expressed in this language (under this interpretation) by the formula \( 'A_2(a_1, a_2)' \). In the discussion below, the notion of a first-order language and an arbitrary first-order theory are used in the sense indicated above; that is, the notions are used in an orthodox / formalist sense that corresponds to the usage of, for example, [6]. In the following discussion it is important to avoid confusion between use of a formula (under some interpretation) and mention of the formula (viewed as an uninterpreted sequence of signs). Similarly, it is important to distinguish metatheoretical expressions from expressions of some object language under discussion. To reduced the risk of confusion I always rely on the intelligence of the reader, though I sometimes also use quotation symbols or corners for quasi-quotat. To illustrate, suppose that \( '\mathcal{B}(x_1)' \), is used in a context as a metalinguistic name for an object language expression \( '(x_1 = x_1)' \). Then \( '(\forall x_1)\mathcal{B}(x_1)' \) is a metalinguistic name for the object language expression \( '(\forall x_1)(x_1 = x_1)' \), while \( '\mathcal{B}(0)' \) names \( '0 = 0' \).

2. Expressing the metatheory in a first-order theory of domains

The question at hand is whether the informal semantic metatheory of an arbitrary first-order theory \( K \) is expressed in a model of some first-order theory that has a proper axiom (schema) corresponding essentially to a set-theoretic axiom (schema) of subsets (or subclasses). To be more precise, I consider below whether
the metatheory might be 'expressed under a $\mathcal{T}$-model $\mathcal{M}$' of such a theory in the following sense.

**Definition 2.** Let $\mathcal{T}$ be a first-order theory in language $L$ with (primitive or defined) predicates $\mathcal{D}(x)$ and $\mathcal{U}(x,y)$. The semantic metatheory of first-order logic is 'expressed under the $\mathcal{T}$-model $\mathcal{M}$' if (for $\mathcal{M}$ a model of $\mathcal{T}$):

A: $\mathcal{D}(x)$ is true under $\mathcal{M}$ (or satisfied under an assignment $s^*$ of objects in the domain $\mathcal{D}$ of $\mathcal{M}$ to $L$ terms) if and only if: the object assigned to the $L$ term $x$ under $\mathcal{M}$ or at the assignment $s^*$ (i.e. $(x)^M$ or $s^*(x)$ respectively) - is the domain of an interpretation of some first-order language;

B: Similarly, $\mathcal{U}(x,y)$ is true under $\mathcal{M}$ (or satisfied at an assignment $s^*$) if and only if: $\mathcal{D}(x)$ is true under $\mathcal{M}$ (or satisfied at $s^*$) and the object $(y)^M$ (or $s^*(y)$) is an object in the domain $(x)^M$ (or $s^*(x)$ respectively);

C: Every $L$ formula that is an instance of the following schema is a $\mathcal{T}$ theorem (where $\mathcal{B}(z)$ is any first-order condition on $z$, c.f. [3]):

\[
\mathcal{D}(x) \Rightarrow (\exists y)[\mathcal{D}(y) \land (\forall z)[\mathcal{U}(y,z) \Leftrightarrow (\mathcal{U}(x,z) \land \mathcal{B}(z))]]
\]

D: There exist well-formed $L$ formulas $\mathcal{B}_1, \ldots, \mathcal{B}_5$ such that (under $\mathcal{M}$): (1) $\mathcal{B}_1, \ldots, \mathcal{B}_5$ (respectively) express the five key metatheoretical propositions concerning an arbitrary first-order theory $\mathcal{K}$ (mentioned at Proposition [4]); and (2) $\mathcal{B}_1, \ldots, \mathcal{B}_5$ are all true under $\mathcal{M}$.

To illustrate Definition 2 suppose that the metatheory of first-order logic is expressed under a model $\mathcal{M}$ of ZFC set theory in which the individuals are sets and the symbol for the membership relation is assigned to this relation itself (defined on this domain); then if domains are sets, the metatheory would clearly be expressed under this ZFC-model $\mathcal{M}$. (The illustration tacitly assumes, per impossible by most accounts, that there exists a set of all sets.) The main result of this section may now be stated as follows.

**Proposition 2.** If the metatheory of first-order logic is expressed under the $\mathcal{T}$-model $\mathcal{M}$ (in the sense of Definition 2), then the domain $\mathcal{D}$ of $\mathcal{M}$ contains an object $\mathcal{E}$ that is a member of itself if and only if it is not a member of itself.

**Proof.** For the proof of Proposition 2 I note that, by Definition 2 $\mathcal{T}$ includes formulas which, under the intended interpretation $\mathcal{M}$, quantify over every domain of interpretation of any first-order language. Since $L$, the language of $\mathcal{T}$, is a first-order language, the assumed Tarskian semantics (§2.2) thus imply that the domain $\mathcal{D}$ of $\mathcal{M}$ contains every domain of interpretation of any first-order language (including of course $\mathcal{D}$ itself). Let $s$ then be a denumerable sequence of elements of $\mathcal{D}$ such that $s^*(x_1)$, the first element of $s$, is $\mathcal{D}$. Choosing $\mathcal{D}(z) \land \neg[\mathcal{U}(z,z)] \land \mathcal{B}(z)$ for $\mathcal{B}(z)$, [2.1] yields:

\[
(2.2) \quad |_M \mathcal{D}(x_1) \Rightarrow (\exists y)[\mathcal{D}(y) \land (\forall z)[\mathcal{U}(y,z) \Leftrightarrow (\mathcal{U}(x_1,z) \land \mathcal{D}(z) \land \neg[\mathcal{U}(z,z)])]]
\]

[2.1] implies that the consequent of [2.2] is satisfied at $s$; that is, there exists a sequence $s'$ which differs from $s$ in at most the the $j$th place (taking $y$ to be $x_j$ without loss of generality) which satisfies the consequent of [2.2] (To avoid any clash of variables, we may if necessary take $'x_j'$ to be the least $L$ variable that does not occur in [2.2] substitute $'x_j'$ for all occurrences of $y$ bound by the existential quantifier at the start of the consequent of [2.2] and focus on the formula that thus results.) Let
\[\mathcal{E} \text{ then be the object } s'(x_j) \text{ in } \mathfrak{D}. \text{ By hypothesis we have that the following is satisfied at } s':\]

(2.3) \[\mathcal{D}(x_j) \land (\forall z)[\mathcal{U}(x_j, z) \Leftrightarrow (\mathcal{U}(x_1, z) \land \mathcal{D}(z) \land \neg\{\mathcal{U}(z, z)\})]\]

After conjunction elimination, and instantiating \(z\) at \(x_j\), this implies that the following is satisfied at \(s'\):

(2.4) \[\mathcal{U}(x_j, x_j) \Leftrightarrow (\mathcal{U}(x_1, x_j) \land \mathcal{D}(x_j) \land \neg\{\mathcal{U}(x_j, x_j)\})\]

Consider then whether \(\mathcal{U}(x_j, x_j)\) is or is not satisfied at \(s'\). The above implies that \(s'\) satisfies the following:

(2.5) \[\mathcal{U}(x_j, x_j) \Leftrightarrow \neg\{\mathcal{U}(x_j, x_j)\}\]

The demonstration in brief is as follows:

1. 'Every instance of a tautology is true for any interpretation' (\[6\] §2.2:VII), hence the following is satisfied at \(s'\):

   (2.6) \[
   [\mathcal{U}(x_j, x_j) \Leftrightarrow (\mathcal{U}(x_1, x_j) \land \mathcal{D}(x_j) \land \neg\{\mathcal{U}(x_j, x_j)\})] \Rightarrow \\
   [\mathcal{U}(x_1, x_j) \land \mathcal{D}(x_j)] \Rightarrow (\mathcal{U}(x_j, x_j) \Leftrightarrow \neg\{\mathcal{U}(x_j, x_j)\})
   
   \]

2. \(s'(x_j)\) is, by hypothesis, an object in \(\mathfrak{D}\) such that both \(\mathcal{U}(x_1, x_j)\) and \(\mathcal{D}(x_j)\) are satisfied at \(s'\);

3. Thus conjunction introduction and two applications of modus ponens yield that \(s'\) satisfies \(2.5\).

In short, the hypothesis that the metatheory of first-order logic is expressed under the \(\mathcal{T}\)-model \(\mathfrak{M}\) implies that the domain \(\mathfrak{D}\) of \(\mathfrak{M}\) contains an object \(s'(x_j)\) such that \(\mathcal{U}(x_j, x_j)\) is satisfied at this object if and only if it is not.

### 3. Conclusions

The above result is closely related to the demonstration in [2] (Proposition 2.1) that the metatheory of the pure predicate calculus cannot be expressed in a model of NBG set theory. The above demonstration differs however not simply in focusing on the case of an arbitrary first-order theory - [2] Proposition 2.1 might be rewritten in such terms - but in showing that the difficulty is not essentially linked with some weakness in the NBG distinction between proper classes and sets. The standard solution to so called 'semantic paradoxes', denying the existence of the hypothesised entity ([3]: Chapter One), cannot be applied in the above case without abandoning the claim that the metatheory in question can be made precise in the required way. (The above paradox might be described as 'logical', in terms of the traditional classification, if one accepts that the key semantic concepts involved - of an interpretation, truth under an interpretation and so on - can be made precise in set-theoretic terms (see e.g. [5]). The paradox however challenges the idea that the notions may be made precise in this way when the aim is to express the semantic metatheory of first-order logic in a model of either ZFC or NBG or some equivalent first-order set theory.)

While the result does not instill confidence in the formalist program it falls short of a definitive refutation. A more serious objection is raised by the argument that the correctness of classical arithmetic implies that the metatheory of the formalist account of arithmetic is false [1]. (I note in passing that in the proof of [1] Corollary 4.2 the claim that 'the proper axioms of S and \(S_{\omega_2}\) are the same' should read that they 'have the same form'; the exhibited proof, with some minor elaboration
to accommodate this point, holds.) This result, combined with the observation that *Principia*’s first-order logic and arithmetic avoid such problems [1], effectively refutes the claim that formalism provides a viable account of classical logic.

**References**

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**University of Sydney**