Gravitational memory effects in Chern-Simons modified gravity

Shaoqi Hou\textsuperscript{a} Tao Zhu\textsuperscript{b,c} Zong-Hong Zhu\textsuperscript{a,d}

\textsuperscript{a}School of Physics and Technology, Wuhan University, Wuhan, Hubei 430072, China
\textsuperscript{b}Institute for theoretical physics and cosmology, Zhejiang University of Technology, Hangzhou, Zhejiang 310032, China
\textsuperscript{c}United Center for Gravitational Wave Physics (UCGWP), Zhejiang University of Technology, Hangzhou, Zhejiang 310032, China
\textsuperscript{d}Department of Astronomy, Beijing Normal University, Beijing 100875, China

E-mail: hou.shaoqi@whu.edu.cn, zhut05@zjut.edu.cn, zhuzh@whu.edu.cn

ABSTRACT: The gravitational memory effects of Chern-Simons modified gravity are considered in the asymptotically flat spacetime. If the Chern-Simons scalar does not directly couple with the ordinary matter fields, there are also displacement, spin and center-of-mass memory effects as in general relativity. This is because the term of the action that violates the parity invariance is linear in the scalar field but quadratic in the curvature tensor. This results in the parity violation occurring at the higher orders in the inverse luminosity radius. Although there exists the Chern-Simons scalar field, interferometers and pulsar timing arrays are incapable of detecting its polarization. So the scalar field does not induce any new memory effects that can be detected by interferometers or pulsar timing arrays. The asymptotic symmetry is group is also the extended Bondi-Metzner-Sachs group. The constraints on the memory effects excited by the tensor modes are obtained.
1 Introduction

The gravitational memory effect is an intriguing phenomenon, which often refers to the lasting change in the relative distance between test particles after the gravitational wave disappears. This effect, sometimes named displacement memory, was first discovered in general relativity nearly 50 years ago [1–4]. Recently, new memory effects were identified including spin memory [5], center-of-mass (CM) memory [6], velocity memory [7–9], and so on. The spin memory effect leads to the difference in the periods of two counter-orbiting massless particles in a circular orbit, and the CM memory effect causes the permanent change in the CM of an isolated system. Finally, the velocity memory effect is just the lasting change in the relative velocity between test particles. All of the above effects are characterized by the permanent change in certain physical quantities, and similar phenomena also occur in electromagnetism and Yang-Mills theory [10, 11]. In this work, we will focus on the memories in gravitation.

Memory effects are deeply related to the asymptotic symmetries of the spacetime. In the case of the asymptotically flat spacetime, memories usually considered take place near the (future) null infinity [12], at which null geodesics eventually arrive. The asymptotic symmetries are diffeomorphisms that, roughly speaking, preserve the geometry of the null infinity [13]. Studies have shown that the asymptotic symmetries include the supertranslation symmetries and the Lorentz symmetries [14, 15]. They together form the so-called
Bondi-Metzner-Sachs (BMS) group, which is an infinite dimensional group, generalizing the Poincaré group. Supertranslations are certain generalizations of the usual translations, while Lorentz transformations are actually the conformal transformations on a unit 2-sphere generated by the global conformal Killing vector fields. New works extended the BMS group by allowing the conformal Killing vectors to have isolated singularities on the 2-sphere [16, 17] or by replacing Lorentz transformations by all of the diffeomorphisms on the 2-sphere [18, 19]. The former gives rise to the extended BMS group, and the latter might suffer from diverging symplectic current [20], so it will not be considered in this work.

Because of the supertranslation symmetry, there are infinitely many degenerate vacuum states in the gravity sector that can be transformed into each other via the supertranslation transformations. The vacuum transition explains the displacement memory effect [21]. The displacement memory is also constrained by the flux-balance laws associated with the supertranslations. Similar constraints can be applied to spin and CM memories. In particular, flux-balance laws associated with the super-rotations constrain spin memory effect, while flux-balance laws with the super-boosts constrain CM memory effect. These flux-balance laws play important roles in the determination of the strength of the memory effect and thus its observation.

The detection of gravitational waves [22–25] not only proved the existence of gravitational waves, but also made it possible to observe the memory effect. Indeed, ground-based interferometers such as aLIGO could detect memories [26–30]. They can also be measured by pulsar timing arrays [31, 32] and Gaia mission [33]. The spin memory effect is observable by LISA [5], but the CM memory is more difficult to be detected with the current and even the planned detectors [6].

This is what has happened in general relativity. Although general relativity is a very successful theory [34], it still suffers from some problems, such as its breakdown at the singularity, the nonrenormalizability, dark matter and dark energy, et. al. To resolve at least some of these problems, there have been a plethora of modified theories proposed. Among them, Brans-Dicke theory [35] is the simplest, which contains one extra gravitational degree of freedom, the Brans-Dicke scalar field. The memory effects of this theory have been studied in refs. [36–40]. It was found out that in addition to the memories already discovered in general relativity, there also exits the one excited by the Brans-Dicke scalar, dubbed S memory [41]. These memories are also related to the asymptotic symmetries, and constrained by the corresponding flux-balance laws, although S memory is more subtle [39]. Very interestingly, S memory effect can also be used to distinguish general relativity from Brans-Dicke theory [41, 42]. One thus speculates that the study of the memory effect in other modified theories of gravity may help probe the nature of gravity.

In this work, memory effects in a different modified gravity theory, Chern-Simons gravity [43], will be studied. This theory also includes one extra gravitational degree of freedom, called the Chern-Simons scalar field, but it is a pseudo-scalar. Thus the effects of parity violation might take place. For example, in the cosmological background, the gravitational wave might experience the amplitude and the velocity birefringences as predicted in a generic parity violating theory which incorporates Chern-Simons gravity as a special
Indeed, Chern-Simons theory predicts that the left-handed and the right-handed (tensor) gravitational waves propagate with different amplitudes — the amplitude birefringence, but they both travel at the speed of light.

With the Bondi-Sachs formalism [14, 46, 47], one can obtain the metric and the scalar fields in the asymptotically flat spacetime in this theory. It is found out that the metric resembles the one in general relativity with a canonical scalar field at the lower orders in the inverse of the luminosity radius. The parity violating terms explicitly appear at the higher orders. The asymptotic symmetries are thus expected to be the same as the extended BMS symmetries in general relativity [48]. There are also the same memories induced by the tensor degrees of freedom. Since the Chern-Simons scalar field is assumed not to couple with the matter fields, interferometers and pulsar timing arrays are not capable of detecting its memory effects if they exist. Although in this work, the conserved charges and fluxes will not be determined, the constraints on memory effects can still be obtained with the equations of motion.

This work is organized in the following way. Section 2 briefly reviews Chern-Simons gravity. Section 3 focuses on the asymptotically flat spacetime of this theory. In particular, one first discusses the boundary conditions that the metric and the scalar fields should satisfy in section 3.1, so that the asymptotic solutions can be determined in section 3.2. After that, the asymptotic symmetries are obtained (section 3.3). Then, memory effects are discussed in section 4. These effects can be introduced via solving the geodesic deviation equations in section 4.1. Then, the nonradiative regions and the canonical BMS frames are defined in section 4.2, and memories are related to the vacuum transition in section 4.3. Section 4.4 presents the constraints on memory effects by integrating the equations of motion. Finally, a briefly summary 5 concludes this work. Throughout this paper, $c = 1$.

Most of the calculation was done with the help of xAct [49].

## 2 Chern-Simons theory

The action of Chern-Simons theory is [50]

$$S = \int d^4x \sqrt{-g} \left( \kappa R + \frac{a}{4} \partial_{\lambda} R_{\lambda} + \frac{b}{2} \nabla_{\alpha} \nabla^{\alpha} \vartheta - b V(\vartheta) \right) + S_m, \quad (2.1)$$

where $\kappa = 1/16\pi G$, $a$ and $b$ are all coupling constants. $S_m$ is the action for matter fields and does not depend on $\vartheta$. $V(\vartheta)$ is the potential for the Chern-Simons coupling scalar $\vartheta$. Here, one considers the special case with $V(\vartheta) = 0$ so that $\vartheta$ is massless. Then, the action acquires the shift symmetry under the addition of a constant to $\vartheta$. $\ast R_{\lambda\mu\nu\rho} = \epsilon_{\lambda\mu\nu\rho\sigma} R^{\sigma\alpha\beta\gamma}/2$ is the Hodge dual. The second term arises probably due to the gravitational anomaly of the standard model of elementary particles [51, 52], the Green-Schwarz anomaly canceling mechanism in string theory, or the scalarization of the Barbero-Immirzi parameter in the loop quantum gravity. Because of the presence of $\epsilon^{\alpha\beta\gamma\delta}$, $\vartheta$ is a pseudo-scalar in order that the action $S$ is invariant under the parity transformation. If one ignores the second term in the action, one obtains general relativity with a canonical scalar field $\sqrt{b} \vartheta$. In this work, we will not consider the matter action for simplicity. This also amounts to assume the matter fields decay sufficiently fast as the distance to the source is approaching infinity.
The equations of motion are [50]

\[ R_{ab} - \frac{1}{2} g_{ab} R + \frac{a}{\kappa} C_{ab} = \frac{1}{2\kappa} T_{ab}^{(\vartheta)}, \]  
(2.2a)

\[ \nabla_a \nabla^a \vartheta = - \frac{a}{4b} R_{abcd}^* R^{bacd}. \]  
(2.2b)

Here, \( C_{ab} \) is called the C-tensor, given by

\[ C_{ab} = (\nabla_c \vartheta)_{e}^{cde} (a \nabla_e R^b_{\phantom{b}d}) + (\nabla_c \nabla_d \vartheta)^* R_{c}^{(ab)d}, \]  
(2.3)

where \( \nabla_a \vartheta \) and \( \nabla_a \nabla_b \vartheta \) are also called the Chern-Simons velocity and acceleration, respectively [53]. \( T_{ab}^{(\vartheta)} \) is the stress energy tensor of the Chern-Simons scalar,

\[ T_{ab}^{(\vartheta)} = b \left( \nabla_a \vartheta \nabla_b \vartheta - \frac{1}{2} g_{ab} \nabla_c \vartheta \nabla^c \vartheta \right). \]  
(2.4)

Since \( a \) and \( b \) are free, one may set \( a \neq 0 \) and \( b = 0 \). Then \( \vartheta \) should be prescribed by hand, and one is now in the non-dynamical framework. When neither \( a \) nor \( b \) is zero, \( \vartheta \) has its own dynamics. This framework is called dynamical. In this paper, we work in the dynamical framework.

Chern-Simons gravity has a lot of applications in astrophysics, cosmology and so on. Therefore, it is constrained by astrophysical tests, solar system tests and cosmological observations. For example, the mass \( m_{\text{CS}} \) of the scalar field has a lower bound about \( 10^{-3} \text{ km}^{-1} \) from the observation of the torque-induced procession in the solar system [54]. The energy scale beyond which the parity violation effect is strong was found to be at least 33 meV from the binary pulsar observation [55]. For more phenomenology and constraints, please refer to ref. [50].

3 Asymptotically flat spacetimes

Roughly speaking, the asymptotically flat spacetime is the one approaching the Minkowski spacetime at the distances very far away from the source of the gravity. In the relativistic theory, there are three types of ways to approach the infinity: along timelike, spacelike or null directions. For problems involving (massless) radiation, it is useful to consider the spacetime region near the null infinity approached by null geodesics. In general relativity, one can define the so-called asymptotically flat spacetime at the null infinity using the conformal completion technique [12, 56, 57]. Or, one may also be able to impose certain asymptotic behaviors of the metric or other fields near the null infinity in a suitable coordinate system, usually, Bondi-Sachs coordinates \((u, r, x^2 = \theta, x^3 = \phi)\) [14, 17, 46],

\[ ds^2 = e^{2\beta} \frac{V}{r} du^2 - 2e^{2\beta} dudr + h_{AB}(dx^A - U^A du)(dx^B - U^B du), \]  
(3.1)

where \( \beta, V, U^A, \) and \( h_{AB} \) \((A, B = 2, 3)\) are six metric functions. One can similarly define the asymptotically flat spacetime in modified theories of gravity, for example, Brans-Dicke theory [35] as done in refs. [36–38]. Here, for simplicity, we will assign suitable asymptotic behaviors to the metric \( g_{ab} \) and the Chern-Simons scalar \( \vartheta \) in Bondi-Sachs coordinates in order to define the asymptotic flatness in Chern-Simons gravity.
3.1 Boundary conditions

In general relativity, the boundary conditions for the metric field in the asymptotically flat spacetime at null infinity are given by \[14, 17, 46\]

\[
\begin{align*}
g_{uu} &= -1 + O \left( r^{-1} \right), \\
g_{ur} &= -1 + O \left( r^{-2} \right), \\
g_{uA} &= O \left( 1 \right), \\
g_{rr} &= g_{rA} = 0, \\
h_{AB} &= r^2 \gamma_{AB} + O \left( r \right),
\end{align*}
\] (3.2)

where \( \gamma_{AB} \) is the round metric on a unit 2-sphere,

\[
\gamma_{AB} d^A d^B = d\theta^2 + \sin^2 \theta d\phi^2.
\] (3.3)

In addition, the determinant of \( h_{AB} \) is required to be

\[
det(h_{AB}) = r^4 \sin^2 \theta,
\] (3.4)

so \( r \) is the luminosity radius \[14, 46\]. In terms of the metric functions, one can find out that \[17\]

\[
\beta = O \left( r^{-2} \right), \quad V = -r + O \left( r^0 \right), \quad U^A = O \left( r^{-2} \right).
\] (3.5)

But in Chern-Simons gravity, one may propose different boundary conditions. However, by examining the Einstein’s equations (2.2a), one realizes that if one ignores the C-tensor term, one knows that the equations describe a spacetime sourced by a canonical scalar field, just as in general relativity. Since near the null infinity, the spacetime resembles the flat one, any deviation from the Minkowski metric can be treated as the small perturbation. Then for the purpose of determining the boundary conditions, it might be reasonable to ignore the C-tensor term as it represents a higher order term. Therefore, we just impose the same boundary conditions on \( g_{ab} \) as in general relativity. In addition, one also requires that \( \vartheta = \vartheta_0 + O \left( 1 \right) \). The chosen boundary conditions also imply that the parity violating effects would be of the higher orders as shown below.

Now, the asymptotic behaviors can be written down. One expands the metric functions in the following way \[48\],

\[
\begin{align*}
\beta &= \frac{\beta_2}{r} + \frac{\beta_3}{r^2} + O \left( \frac{1}{r^4} \right), \\
U^A &= \frac{U^A}{r^2} + \frac{1}{r^3} \left[ -\frac{2}{3} N^A + \frac{1}{16} \Theta^A (e_{BC} e^{BC}) + \frac{1}{2} e^{AB} \Theta^C c_{BC} \right] + \frac{\psi^A}{r^4} + O \left( \frac{1}{r^5} \right), \\
V &= -r + 2m + \frac{2M}{r} + O \left( \frac{1}{r^2} \right), \\
h_{AB} &= r^2 \gamma_{AB} + r c_{AB} + d_{AB} + \frac{e_{AB}}{r} + O \left( \frac{1}{r^2} \right).
\end{align*}
\] (3.6)

Similarly, one has

\[
\vartheta = \vartheta_0 + \frac{\vartheta_1}{r} + \frac{\vartheta_2}{r^2} + O \left( \frac{1}{r^3} \right).
\] (3.6)

In the above expansions, all the expansion coefficients are functions of \( u \) and \( x^A \), and their indices are raised and lowered by \( \gamma_{AB} \) and its inverse \( \gamma^{AB} \), respectively. The covariant
derivative $\mathcal{D}_A$ is the one compatible with $\gamma_{AB}$. Among these coefficients, $m$ and $N^A$ are called the Bondi mass and angular momentum aspects as in general relativity, respectively [14, 46]. $c_{AB}$ is the shear tensor associated with the outgoing null geodesic congruence. Because of the determinant condition (3.4), one finds out that [48]

\begin{equation}
  c_{AB} = \hat{c}_{AB}, \quad d_{AB} = \hat{d}_{AB} + \frac{1}{4} \gamma_{AB} \hat{c}_C^D \hat{c}^C_D,
\end{equation}

\begin{equation}
  \epsilon_{AB} = \hat{\epsilon}_{AB} + \frac{1}{2} \gamma_{AB} \hat{d}_C^D \hat{\epsilon}^C_D,
\end{equation}

with the hatted tenors traceless with respect to $\gamma^{AB}$.

Since the volume $\epsilon_{abcd}$ is present in the equations of motion, its component form should be discussed. So first, let $\hat{\epsilon}_{AB}$ be the volume element compatible with $\gamma_{AB}$, then one has

\begin{equation}
  \hat{\epsilon}_{\theta\phi} = \sqrt{\gamma} = \sin\theta,
\end{equation}

where $\gamma$ is the determinant of $\gamma_{AB}$. Thus,

\begin{equation}
  \epsilon_{ur\theta\phi} = \sqrt{-g} = e^{2\beta} r^2 \sqrt{\gamma}.
\end{equation}

So both $\epsilon_{abcd}$ and $\hat{\epsilon}_{AB}$ have simple asymptotic behaviors.

### 3.2 Asymptotic solutions

Given the chosen asymptotic behaviors (3.6), one can solve the equations of motion (2.2) by directly substituting (3.6) into (2.2) to determine the relations among the expansion coefficients. After some complicated algebraic manipulations, one knows that $\vartheta_0$ is constant, which can be set to zero due to the shift symmetry. Furthermore, one obtains the following results,

\begin{equation}
  \beta_2 = -\frac{1}{32} c_{ABC}^{AB} - \frac{b}{16\kappa} \vartheta_1^2, \quad \beta_3 = -\frac{b}{6\kappa} \vartheta_1 \vartheta_2,
\end{equation}

\begin{equation}
  \dot{\vartheta}_2 = -\frac{1}{2} \mathcal{S}^2 \vartheta_1,
\end{equation}

\begin{equation}
  \dot{m} = \frac{1}{4} \mathcal{D}_A \mathcal{D}_B N^{AB} - \frac{1}{8} N_{AB} N^{AB} - \frac{b}{4\kappa} N^2,
\end{equation}

\begin{equation}
  \dot{N}_A = \mathcal{D}_A m + \frac{1}{4} (\mathcal{D}_B \mathcal{D}_A \mathcal{D}^C B_{BC} - \mathcal{D}_B \mathcal{D}^B \mathcal{D}^C \mathcal{C}_A^C) + \frac{1}{4} N^{BC} \mathcal{D}_{CAB} + \frac{1}{2} c_{AB} \mathcal{D}_C N^{BC}
\end{equation}

\begin{equation}
  + \frac{b}{8\kappa} (\vartheta_1 \mathcal{D}_A N - 3N \mathcal{D}_A \vartheta_1),
\end{equation}

\begin{equation}
  \dot{\hat{d}}_{AB} = 0.
\end{equation}
where dot means to take the derivative $\partial/\partial u$, and

$$N_{AB} = \dot{c}_{AB}, \quad N = \dot{\vartheta}_1.$$  \tag{3.13}$$

The equations for $\dot{e}_{AB}$ and $\mathcal{U}^A$ are way more complicated and relegated into Appendix A.

One can find out that $c_{AB}$ and $\vartheta_1$ have no evolution equations, and all of the above equations are written in terms of them and their derivatives. In fact, they represent the physical degrees of freedom of the theory as in Brans-Dicke theory [36, 36, 37]. Since $c_{AB}$ is a symmetric and traceless rank-2 tensor on the unit 2-sphere, there are three degrees of freedom in this theory. Like in general relativity and Brans-Dicke theory [14, 36, 37, 46], the vanishing of $N_{AB}$ and $N$ means the absence of gravitational waves.

From the above expressions, one also finds out that up to the orders considered in eqs. (3.6) and ignoring equations in Appendix A, the asymptotically flat spacetime at the null infinity looks just like the one in general relativity with a canonical scalar field $\sqrt{b} \vartheta$, which couples with the metric $g_{ab}$ minimally [48]. The non-minimal coupling between $\vartheta$ and $g_{ab}$ occurs in those equations in Appendix A, and, of course, terms at even higher orders. Because of the complicated forms of the higher order equations, none of them will be presented here. Note the appearance of $\dot{e}_{AB}$ in these equations, which represents the parity violation effect.

In the end, for the purpose of reference, we present the remaining metric components,

$$g_{uu} = -1 + \frac{2m}{r} - \frac{1}{r^2} \left[ \mathcal{D}_A N^A - \frac{b}{8\kappa} (\mathcal{D}_A \vartheta_1 \mathcal{D}^A \vartheta_1 - \vartheta_1 \mathcal{D}^2 \vartheta_1) \right] + \mathcal{O} \left( \frac{1}{r^3} \right), \tag{3.14a}$$

$$g_{ur} = -1 + \frac{1}{r^2} \left( \frac{c_{ABC}c^{ABC}}{16} + \frac{b}{8\kappa} \vartheta_1^2 \right) + \frac{b}{8\kappa} \vartheta_1 \vartheta_2 + \mathcal{O} \left( \frac{1}{r^4} \right), \tag{3.14b}$$

$$g_{uA} = \frac{\mathcal{D}_B c^{BA}}{2} + \frac{1}{r} \left( \frac{2}{3} N_A - \frac{1}{8} c_{BC} \mathcal{D}_A c^{BC} \right) - \frac{1}{r^2} \left[ \mathcal{W}^A - \frac{2}{3} c_{AB} N^B \right] \tag{3.14c}$$

In these expressions, the parity violating terms explicitly appear at the $1/r^2$ order in $g_{uA}$ due to the presence of $\mathcal{W}_A$, which is given by eq. (A.1) and contains $\dot{e}_{AB}$.

### 3.3 Asymptotic symmetries

Due to the similarity discussed previously, many conclusions valid in general relativity also hold in Chern-Simons gravity. In particular, the asymptotic symmetries are actually the same. This is due to the fact that the asymptotic symmetries are diffeomorphisms that preserve the boundary conditions (3.2) and the determinant condition (3.4), and these conditions take exactly the same forms as in general relativity [17]. So let the vector field
\( \xi^a \) generate an infinitesimal diffeomorphism such that

\[
\begin{align*}
\mathcal{L}_\xi g_{rr} &= \mathcal{L}_\xi g_{rA} = 0, \\
\mathcal{L}_\xi g_{ur} &= \mathcal{O} \left( r^{-1} \right), \\
\mathcal{L}_\xi g_{uu} &= \mathcal{O} \left( r^{-1} \right), \\
\mathcal{L}_\xi g_{AB} &= \mathcal{O} \left( r^{-1} \right).
\end{align*}
\]

(3.15a)

(3.15b)

(3.15c)

(3.15d)

(3.15e)

Like in general relativity and Brans-Dicke theory \([17, 36]\), \( \xi^a \) has the following components

\[
\begin{align*}
\xi^u &= f(u, x^A), \\
\xi^A &= Y^A(u, x^A) - (\mathcal{D}_B f) \int^\infty_r e^{2\beta} g^{AB} dr', \\
\xi^r &= \frac{r}{2} (U^A \mathcal{D}_A f - \mathcal{D}_A \xi^A),
\end{align*}
\]

(3.16a)

(3.16b)

(3.16c)

where \( f \) and \( Y^A \) are arbitrary integration functions independent of \( r \). These components are obtained by evaluating eqs. (3.15a) and (3.15b). Using the asymptotic expansions of the metric functions and the scalar field, one knows that

\[
\begin{align*}
\xi^A &= Y^A - \frac{\mathcal{D}^A f}{r} + \frac{c^{AB} \mathcal{D}_B f}{2r^2} - \frac{1}{r^3} \left( \frac{c_B c_C}{16} - \frac{b}{24\kappa} \beta f^2 \right) \mathcal{D}^A f + \mathcal{O} \left( \frac{1}{r^4} \right), \\
\xi^r &= -\frac{r}{2} \psi + \frac{1}{2} \mathcal{D}^2 f - \frac{1}{2r} \left[ (\mathcal{D}_A f) \mathcal{D}_B c^{AB} + \frac{1}{2} c^{AB} \mathcal{D}_A \mathcal{D}_B f \right] + \mathcal{O} \left( \frac{1}{r^2} \right).
\end{align*}
\]

(3.17a)

(3.17b)

Again, the parity violating terms are of the higher orders. The remaining conditions imply that \( Y^A \) is actually independent of \( u \) and is a conformal Killing vector field for \( \gamma_{AB} \), i.e.,

\[
\mathcal{L}_Y \gamma_{AB} = \psi \gamma_{AB}, \quad \psi = \mathcal{D}_A Y^A.
\]

(3.18)

Finally, one also finds out that there exist an arbitrary function \( \alpha(x^A) \) independent of \( u \) such that

\[
f = \alpha + \frac{u}{2} \psi.
\]

(3.19)

These two equations take the same forms as in general relativity and Brans-Dicke theory. Therefore, one calls the diffeomorphisms generated by \( \alpha \) the supertranslations, the generalization of the familiar translations in Minkowski spacetime. In fact, if \( \alpha \) is a linear combination of \( l = 0, 1 \) spherical harmonics, it generates the usual space and time translation. A generic supertranslation is generated by a linear combination of all spherical harmonics. The transformations generated by the global conformal Killing vector fields \( Y^A \) form a group isomorphic to the Lorentz group. One can rewrite \( Y^A \) in the following way,

\[
Y^A = \mathcal{D}^A \chi + \mathcal{D}^A \mathcal{D}_B \sigma,
\]

(3.20)

with \( \chi \) and \( \sigma \) linear combinations of \( \ell = 1 \) spherical harmonics, then \( \chi \) generates boost and \( \sigma \) generates (spatial) rotation \([48]\). The semi-direct sum of the supertranslation group
and the Lorentz group is the celebrated BMS group [14, 15]. One can also choose to work in complex stereographic coordinates \((ζ, \bar{ζ})\) where \(ζ = e^{iφ} \cot(θ/2)\) and \(\bar{ζ}\) means to take the complex conjugation. Then, eq. (3.18) becomes \(∂_ζ Y \bar{ζ} = ∂_\bar{ζ} Y ζ = 0\), so \(Y ζ = Y \bar{ζ}(ζ)\) and \(Y \bar{ζ} = Y \bar{ζ}(\bar{ζ})\). One usually chooses the following bases for \(Y^A\) [58],

\[
l_n = -ζ^{n+1}∂_ζ, \quad \bar{l}_n = -\bar{ζ}^{n+1}∂_{\bar{ζ}}, \quad \tag{3.21}
\]

where \(n\) are integers, in general. A global conformal Killing vector field is a linear combination of \(l_{−1}, l_0, l_1\) and \(\bar{l}_{−1}, \bar{l}_0, \bar{l}_1\). If one allows all conformal Killing vectors, i.e., \(n\) takes all integral values, the Lorentz algebra is extended to the Virasoro algebra, and the resultant symmetry group is called the extended BMS group [48]. Superboosts and super-rotations, generalizations of boosts and rotations, are included in this group. For completeness, the supertranslation generator \(α\) is a linear combination of \(2ζ^n \bar{ζ}^n/(1 + ζ \bar{ζ})\) with \(n, \bar{n}\) both integers [17]. When \(n, \bar{n} = 0, 1\), one obtains the generators for the usual translations.

Therefore, the extended BMS transformation takes the Bondi-Sachs coordinate system into a new Bondi-Sachs coordinate system. A particular Bondi-Sachs coordinate system is called a BMS frame. Under the asymptotic symmetry transformation generated by \(ξ^a\), a BMS frame is transformed to a new one, and the metric \(g_{ab}\) and the scalar field \(ϑ\) transform according to

\[
δξ g_{ab} = L_ξ g_{ab}, \quad δξ ϑ = L_ξ ϑ, \quad \tag{3.22a}
\]

\[
δξ N_{AB} = f N_{AB} + Λ_y N_{AB}, \quad \tag{3.22b}
\]

\[
δξ m = f m + Λ_y m + 3 ψ m + \frac{1}{8} ɛ^{ABC} g_A g_B ψ + \frac{1}{4} N_{AB} g_B N_{AB}, \quad \tag{3.22c}
\]

\[
δξ N_A = f N_A + Λ_y N_A + ψ N_A + 3 m g_A f + \frac{3}{4} (g_A g_C g_B - g_B g_C g_A) g_B f + \frac{3}{4} c_{ABC} N^{BC} g_B f + \frac{b}{8κ} \left( ∂_1 N g_A f - \frac{1}{2} θ_1^2 g_A ψ \right), \quad \tag{3.22d}
\]

where the symbol \(Λ_y\) is to take the Lie derivative along the unit 2-sphere. Then, the news tensor \(N_{AB}\) and the scalar \(N\) transform according to

\[
δξ N_{AB} = f N_{AB} + Λ_y N_{AB}, \quad \tag{3.22e}
\]

\[
δξ N = f N + ψ N + Λ_y N. \quad \tag{3.22f}
\]

From the last two equations, one knows that if there are no gravitational waves in a certain region of the future null infinity in one BMS frame, i.e., \(N_{AB} = 0 = N\), these two quantities still vanish in a different BMS frame.
4 Memory effects

Memory effects general refer to the permanent change in the relative distance between two test particles after the passage of gravitational waves [1–4]. This particular phenomenon is also called the displacement memory, as there were new memory effects discovered more recently. Among them, spin memory effect and center-of-mass (CM) memory effect [5, 6] will also be considered in this work. In the following, these effects will be presented by considering the relative motion between test particles due to the presence of gravitational waves.

4.1 Geodesic deviation

Since in section 2, one assumes there is no direct interaction between \( \vartheta \) and the ordinary matter fields, the relative acceleration of two test particles is simply due to the spacetime curvature [12],

\[
T^c \nabla_c (T^b \nabla_b S^a) = -R_{abcd} T^c S^b T^d,
\]

(4.1)

where \( T^a = (d/d\tau) \) is the 4-velocity of a freely falling test particle with \( \tau \) the proper time, and \( S^a \) is the deviation vector between adjacent test particles. This is just the geodesic deviation equation, and used to detect gravitational waves by interferometers [59]. Applying this equation to the test particles near the null infinity of the spacetime considered in the previous section, one obtains the following relative acceleration [60],

\[
\dot{S}^A \approx -R_{u\hat{A}} \dot{A} S^B = \frac{\hat{c}_B^A}{2r} S^B + \mathcal{O} \left( \frac{1}{r^2} \right).
\]

(4.2)

Here, in writing down this equation, one actually sets up an orthonormal tetrad bases which contain basic vectors

\[
T^a = (\partial_u)^a + \mathcal{O}(1/r), \quad (e_r)^a = -(\partial_u)^a + (\partial_r)^a + \mathcal{O}(1/r),
\]

\[
(e_{\theta})^a = r^{-1}(\partial/\partial \theta)^a + \mathcal{O}(1/r^2), \quad (e_{\phi})^a = (r \sin \theta)^{-1}(\partial/\partial \phi)^a + \mathcal{O}(1/r^2),
\]

(4.3)

so that the indices \( \hat{A}, \hat{B} = \hat{\theta}, \hat{\phi} \). Also, the proper time \( \tau \) approaches the retarded \( u \) as the test particle is close to the null infinity [21]. From eq. (4.2), one can find out that although there are three gravitational degrees of freedom in Chern-Simons gravity, the interferometer can detect only two of them, i.e., the plus and cross polarizations. This is drastically different from the situations in other modified gravities, where each gravitational degree of freedom would excite its own polarizations that can be detected by interferometers and pulsar timing arrays [61–65]. For example, in scalar-tensor theories, there exists an extra polarization named breathing mode if the scalar field is massless [34, 66].

Using the geodesic deviation equation, one can introduce memory effects. Let us assume that there is no gravitational wave before \( u_0 \) and after \( u_f \), during which \( N\hat{A}\hat{B} = 0 \) and \( N = 0 \). Then, integrating this equation twice, one obtains,

\[
\dot{S}^A(u) \approx S^A_0 + \frac{1}{2r} \int_{u_0}^u du' \hat{c}_B^A S^B(u'),
\]

(4.4a)

\[
S^A(u) \approx S^A_0 + (u - u_0) \dot{S}^A_0 + \frac{1}{2r} \int_{u_0}^u du' \int_{u_0}^{u'} du'' \hat{c}_B^A S^B(u''),
\]

(4.4b)
where $\dot{S}_0^\hat{A}$ and $S_0^\hat{A}$ are the initial relative velocity and the initial relative displacement, respectively. Substituting eq. (4.4b) back into itself and eq. (4.4a), one knows the following total changes at the time $u > u_f$,

$$\Delta \dot{S}_A^\hat{A} \approx -\frac{\Delta c_{\hat{A}\hat{B}}}{2r} \dot{S}_0^\hat{B},$$ (4.5a)

$$\Delta S_A^\hat{A} \approx \dot{S}_A^\hat{B} \Delta u + \frac{1}{2r} \left[ \frac{c_{\hat{A}\hat{B}}(u_f) + c_{\hat{A}\hat{B}}(u_0)}{2} \Delta u - \Delta c_{\hat{A}\hat{B}} \right] \dot{S}_0^\hat{B},$$ (4.5b)

where $\Delta u = u_f - u_0$, $\Delta c_{\hat{A}\hat{B}} = c_{\hat{A}\hat{B}}(u_f) - c_{\hat{A}\hat{B}}(u_0)$ and

$$\Delta c_{\hat{A}\hat{B}} = \int_{u_0}^{u_f} c_{\hat{A}\hat{B}}(u) du.$$ (4.5c)

Equation (4.5b) takes a different form from (4.17) in [37] and (2.23) in [39] with the effect of the Brans-Dicke scalar field ignored, but they are all equivalent.

From the above equations, one realizes that as long as there exist the relative velocity and displacement initially, the final relative velocity and displacement will change permanently even after the gravitational wave disappears. This phenomenon is the memory effect. More specifically, eq. (4.5a) describes the velocity kick memory, and eq. (4.5b) is the displacement memory. In particular, the second term on the right-hand side of eq. (4.5b) is the leading displacement memory effect, the first memory effect discovered a long time ago [1–4]. Both the velocity kick and the leading displacement memories are due to the change in $c_{\hat{A}\hat{B}}$. The second line of eq. (4.5b) is the subleading displacement memory, which also depends on $\Delta C_{\hat{A}\hat{B}}$, the time integral of $c_{\hat{A}\hat{B}}$. In fact, the electric- and magnetic parts of $\Delta C_{\hat{A}\hat{B}}$ are related to the spin and CM memory effects as discussed below [37, 38, 48].

As revealed in eq. (4.2), the interferometer cannot detect the scalar gravitational wave polarization caused by $\vartheta$, so it cannot measure the memory due to $\vartheta$ even if it exists. In fact, one cannot use pulsar timing arrays to detect the scalar memories, either. In addition, the proper description of the scalar memory effect might rely on the dual formalism of scalar fields [39, 67, 68], which is beyond the scope of this work. So here, the scalar memory effect will not be discussed.

4.2 Nonradiative regions and the canonical BMS frame

As in general relativity, one may call the region of the future null infinity nonradiative if $N_{\hat{A}\hat{B}} = 0 = N$ there. In these regions, both $c_{\hat{A}\hat{B}}$ and $\vartheta_1$ are independent of $u$. The nonradiative region is invariant under the extended BMS transform. Furthermore, if the spacetime is stationary in a neighborhood of the nonradiative region, this region is specifically called stationary. In the stationary region, both $g_{ab}$ and $\vartheta$ should be independent of $u$, and there exists a canonical BMS frame.

One notices that the particular spacetime, where the memory effect is defined, is the one with a radiative region sandwiched by nonradiative regions, and one compares the relative displacement and relative velocity in the two nonradiative regions. In general relativity, the
memory effect is defined in a more restrictive spacetime, i.e., the two nonradiative regions are actually stationary \[48\]. In modified theories of gravity, this more restrictive version of memory effects might exclude interesting phenomena like the S memory in Brans-Dicke theory \[37, 41\]. Since the canonical BMS frame defined by eq. (4.12) below is very similar to the one in Brans-Dicke theory \[37\], the less restrictive definition of the memory effect is appropriate. But here, we still discuss the canonical BMS frame for completeness.

By the evolution equation (3.12a), \(4\mathcal{D}_A m - \hat{\varepsilon}_{AB} \mathcal{D}^B \eta = 0\), \(\eta \equiv \hat{\varepsilon}^{AB} \mathcal{D}_A \mathcal{D}_C C^C_{CB}\), \(\) which implies that \(\mathcal{D}_A m = \mathcal{D}^2 \eta = 0\). Therefore, \(m = m_0\) and \(\eta\) are both constant. To move forward, rewrite \(c_{AB}\) in terms of two scalar functions,

\[
c_{AB} = \left(\mathcal{D}_A \mathcal{D}_B - \frac{1}{2} \gamma_{AB} \mathcal{D}^2\right) \Phi + \hat{\varepsilon}_{C(AB)} \mathcal{D}^C \Psi, \tag{4.7}
\]

where \(\Phi\) is the electric parity part and \(\Psi\) the magnetic parity part. With this, one finds out that

\[
\eta = \frac{1}{2} \mathcal{D}^2 (\mathcal{D}^2 + 2) \Psi. \tag{4.8}
\]

This relation requires that \(\eta = 0\) and \(\Psi\) is a linear combination of \(l = 0, 1\) spherical harmonics, but this also results in the vanishing of the second term in eq. (4.7), so one can simply set \(\Psi = 0\) in the stationary region. Finally, \(c_{AB} = 0\) can be achieved by performing the BMS transformation with \(\alpha = \Phi/2\) and \(Y^A = 0\) according to eq. (3.22b). \(\vartheta_1 = \tilde{\vartheta}_1\) and \(m = m_0\) are invariant under this transformation.

Now, using the equations in Appendix A, together with eqs. (3.11c) and (3.11d), one finds out that

\[
\mathcal{D}^2 N_A + N_A = -3 \mathcal{D}^B (mc_{AB}) + w_A + w'_A. \tag{4.9}
\]

This equation takes the similar form as eq. (2.23) in ref. \[48\] with \(w_A\) also representing quadratic and cubic terms in \(c_{AB}\) and its derivatives and \(w'_A\) quadratic terms in \(\vartheta_1\) and its derivatives. In fact, the right-hand side of the above equation is zero, since \(\vartheta_1\) is constant, and \(c_{AB} = 0\) as discussed above. Just like eq. (3.20), one can also decompose

\[
N_A = \mathcal{D}_A \Upsilon + \epsilon_{AB} \mathcal{D}^B \Lambda, \tag{4.10}
\]

and the vanishing of eq. (4.9) implies

\[
(\mathcal{D}^2 + 2) \Upsilon = (\mathcal{D}^2 + 2) \Lambda = 0. \tag{4.11}
\]

So \(\Upsilon\) and \(\Lambda\) are linear combinations of the \(l = 1\) spherical harmonics. Analogously, one can also call them electric parity and magnetic parity parts, respectively. Now, perform the BMS transformation with \(\alpha = -3\Upsilon/m_0\) and \(Y^A = 0\), one can obtain a new \(N_A\) determined completely by \(\Lambda\) due to eq (3.22d). This transformation preserves \(c_{AB} = 0\), \(\vartheta_1 = \tilde{\vartheta}_1\) and \(m = m_0\).
Therefore, a canonical BMS frame is characterized by
\[
\vartheta_1 = \bar{\vartheta}_1, \quad m = m_0, \quad c_{AB} = 0, \quad N_A(x^B) = \hat{c}_{AB} \mathcal{D}^B \Lambda(x^C),
\] (4.12)
with \( \Lambda \) the linear combination of the \( l = 1 \) spherical harmonics. A different canonical frame can be obtained by performing a supertranslation with \( \alpha \) also the linear combination of the \( l = 0, 1 \) spherical harmonics.

### 4.3 Vacuum transition

Using the transformation law (3.22b), one can relate the velocity kick and the leading displacement memory effects to vacuum transitions of the gravitational system at the null infinity [21, 36], as they both depend on \( \Delta c_{AB} \). The definition of the vacuum state in gravitational systems is not trivial. Although it is easy to understand that in a vacuum state, \( N = 0 \), i.e., there is no scalar gravitational wave, it is a little more involved to determine the correct conditions for \( c_{AB} \). Here, one would impose the same conditions as in general relativity [21, 36]. So one first needs the following Newman-Penrose tetrad basis \( \{l^a, n^a, m^a, \bar{m}^a\} \) [69],

\[
l^a = (\partial_r)^a + \mathcal{O}(r^{-1}), \quad n^a = -(\partial_u)^a + \frac{1}{2}(\partial_r)^a + \mathcal{O}(r^{-1}),
\] (4.13)

\[
m^a = \frac{1}{\sqrt{2r}} [-(\partial_\theta)^a - i \csc \theta (\partial_\phi)^a] + \mathcal{O}\left(\frac{1}{r^2}\right),
\] (4.14)

and \( \bar{m}^a \) is the complex conjugate of \( m^a \). Then one requires the following Newman-Penrose variables vanish at the leading order,

\[
\Psi_4 = C_{abcd} m^a n^b n^c m^d = -\frac{r}{2} \partial_u N_{AB} \bar{m}^A \bar{m}^B + \cdots,
\] (4.15a)

\[
\Psi_3 = C_{abcd} \bar{m}^a n^b l^c m^d = -\frac{1}{2r} \bar{m}^A \mathcal{D}_B N_A^B + \cdots,
\] (4.15b)

\[
\Im \Psi_2 = \Im(C_{abcd} \bar{m}^a n^b l^c m^d) = \frac{1}{i 8r} (N_A^C \hat{c}_{BC} - \mathcal{D}_A \mathcal{D}_C \hat{c}_B^C + \mathcal{D}_B \mathcal{D}_C \hat{c}_A^C)(\bar{m}^A m^B - m^A \bar{m}^B) + \cdots,
\] (4.15c)

where dots represent higher order terms, and \( \Im \) is to take the imaginary part. Therefore, one knows that \( N_{AB} = 0 \) and

\[
\hat{c}_{AB} = \left(\mathcal{D}_A \mathcal{D}_B - \frac{1}{2} \gamma_{AB} \mathcal{D}^2\right) \Phi(x^C),
\] (4.16)

for some arbitrary function \( \Phi \) on the unit 2-sphere. Here, tilde means to evaluate \( c_{AB} \) in the vacuum state.

To find the relation between the memory effect and the vacuum transition, it is sufficient to rewrite eq. (3.22b) in the nonradiative region, and in particular, consider the supertranslation transformation generated by \( \alpha \),

\[
\delta_\alpha \hat{c}_{AB} = -2 \mathcal{D}_A \mathcal{D}_B \alpha + \gamma_{AB} \mathcal{D}^2 \alpha.
\] (4.17)
This implies that the transformed $c_{AB}$ still describes a vacuum state $\tilde{c}'_{AB}$ with $\Phi' = \Phi - 2\alpha$. Just like in general relativity and Brans-Dicke theory [21, 36], there are also infinitely many degenerate vacuum states that can be transformed to each other via supertranslations. The vacuum transition causes the change in $c_{AB}$,

$$\Delta c_{AB} = \left(D_A D_B - \frac{1}{2} \gamma_{AB} \mathcal{D}^2\right) \Delta \Phi = \delta_{\alpha} \tilde{c}_{AB},$$

(4.18)

which explains the velocity kick and the leading order displacement memories.

### 4.4 Constraints on memory effects

From the above discussion, one knows that memory effects are related to $\Delta c_{AB}$ and $\mathcal{C}_{AB}$. These two quantities are constrained by conservation laws associated with the extended BMS symmetries [37, 38, 48, 60]. This requires to determine the conserved charges and the fluxes using certain formalisms such as the one in ref. [70], which is very involved and will be done in a subsequent paper. In fact, there is a second method to constrain memory effects by properly integrating the evolution equations (3.12b) and (3.12c) [36].

First, substituting eq. (4.7) into the first $N^{AB}$ in eq. (3.12b), multiplying both sides by an arbitrary supertranslation generator $\alpha$, and integrating the resulting equation over the null infinity, one obtains

$$\oint d^2 \Omega \sigma \mathcal{D}^2 (\mathcal{D}^2 + 2) \Delta \Phi = \oint d^2 \Omega \left[ \alpha \Delta m + \int_{u_0}^{u_f} du \alpha \left( N_{AB} N^{AB} + \frac{2b}{\kappa} N^2 \right) \right],$$

(4.19)

where $d^2 \Omega = \sin \theta d\theta d\phi$. This gives the constraint on the velocity kick and leading displacement memory effects. In fact, one could guess from the form of the equation that the terms in the square brackets are proportional to the energy density of the tensor and the scalar gravitational waves. In literature, these terms are said to cause the null memory, and the one with $\Delta m$ causes the ordinary memory [71].

Second, to obtain the constraints on the subleading displacement memory, or on the spin and CM memories, one might want to modify eq. (3.12c) in the following way,

$$\partial_u \tilde{N}_A = \mathcal{D}_A m + \frac{1}{4} \hat{\epsilon}_{AB} \mathcal{D}^B \eta + \frac{1}{4} \mathcal{D}_A (\mathcal{D}_C N_{CB}) - \frac{1}{4} N_{BC} \mathcal{D}_A C^D - \frac{1}{4} \hat{\epsilon}_{AB} \mathcal{D}^B \rho$$

(4.20)

with $\tilde{N}_A = N_A - \frac{3}{32} \mathcal{D}_A (\mathcal{D}_B \mathcal{D}_C) - \frac{1}{4} \epsilon_{ABC} \mathcal{D}_C^{BC}$ and $\rho = \hat{\epsilon}^{AB} N^C_A c_{BC}$. Then it is easy to determine the constraint on the spin memory which is measured by [48]

$$\Delta S = \int_{u_0}^{u_f} du \Psi.$$

(4.21)

One can contract both sides by $\hat{\epsilon}^{AB} \mathcal{D}_B \sigma$, which is the magnetic parity of $Y^A$; refer to eq. (3.20). Then perform the integral over the null infinity to arrive at

$$\oint d^2 \Omega \sigma \mathcal{D}^2 (\mathcal{D}^2 + 2) \Delta S = - \oint d^2 \Omega \left[ \sigma \hat{\epsilon}^{AB} \mathcal{D}_A \Delta \tilde{N}_B ight.$$

$$\left. + \int_{u_0}^{u_f} du \sigma \left( \frac{1}{4} \mathcal{D}^2 \rho - \hat{\epsilon}^{AB} \mathcal{D}_A J_B \right) \right],$$

(4.22)
where $\Delta \hat{N}_A = \hat{N}_A(u_f) - \hat{N}_A(u_0)$, and
\[ J_A = \frac{1}{4} N^C_D D_A c^C_D + \frac{b}{2\kappa} N D_A \vartheta_1. \] (4.23)

Formally, $J_A$ is proportional to the angular momentum flux density of the tensor and the scalar gravitational waves. In the end, one tries to obtain the constraint on the CM memory [6]. This is a bit more complicated. One should notice that the operator $\mathcal{D}(\mathcal{D}^2 + 2)$ on the left-hand side of eq. (4.19) is linear, so one can split $\Phi$ into two parts, $\Phi = \Phi_o + \Phi_n$ so that $\Delta \Phi_o$ is caused by $\Delta m$, and $\Delta \Phi_n$ caused by the remaining parts on the right-hand side of eq. (4.19). Since eq. (4.19) comes from the evolution equation (3.12b), one may identify the following relation,
\[ \dot{m} = \frac{1}{8} \mathcal{D}^2 (\mathcal{D}^2 + 2) \dot{\Phi}_o, \] (4.24)

because $\mathcal{D}_A \mathcal{D}_B N^{AB} = \mathcal{D}^2 (\mathcal{D}^2 + 2) \dot{\Phi}_o/2$. Now, choose $Y^A = \mathcal{D}^A \chi$, i.e., the electric parity part. The infinitesimal extended BMS transformation generated by this $Y^A$ has $f = u\psi/2 = u\mathcal{D}^2 \chi/2$ according to eq. (3.19). Multiplying eq. (4.24) by $f$, contracting eq. (4.20) by $\mathcal{D}_A \chi$, and combining the results properly, one obtains
\[ \oint d^2\Omega \chi \mathcal{D}^2 (\mathcal{D}^2 + 2) \Delta K = \int_{u_0}^{u_f} \frac{d^2 \Omega}{D} \left\{ 8 \Delta \left[ u \mathcal{D}^2 m - \mathcal{D}^A \hat{N}_A \right] + \int_{u_0}^{u_f} du \left[ \mathcal{D}^2 \left( 2 c^B_A N^A_B + \frac{b}{\kappa} \partial_1 N \right) - 8 \mathcal{D}^A J_A \right] \right\}, \] (4.25)

where the CM memory is quantified by
\[ \Delta K = \int_{u_0}^{u_f} u \dot{\Phi}_o du. \] (4.26)

As a final remark, one notices that
\[ C_{AB} = \left( \mathcal{D}_A \mathcal{D}_B - \frac{1}{2} \gamma_{AB} \mathcal{D}^2 \right) \left[ u_f \Delta \Phi + \Phi(u_0) \Delta u - \Delta L - \Delta K \right] + \epsilon_{C(A} \mathcal{D}_{B)} \mathcal{D}^C \Delta S, \] (4.27)

where $\Delta L = \int_{u_0}^{u_f} u \dot{\Phi}_n du$. By putting eq. (4.24) into eq. (3.12b), multiplying the resultant equation by $u\omega(x^A)$ and performing the integral, one gets
\[ \oint d^2 \Omega \omega \mathcal{D}^2 (\mathcal{D}^2 + 2) \Delta L = \int_{u_0}^{u_f} du \oint d^2 \Omega \omega \left( N^B_A N^A_B + \frac{2b}{\kappa} N^2 \right). \] (4.28)

Therefore, in a certain sense, the constraints on the spin and the CM memories also give the constraint on the subleading displacement memory.

To summarize, here, one properly integrates the evolution equations (3.12b) and (3.12c) multiplied by generators of the extended BMS transformations, then the constraints on various memory effects are obtained. Usually, these constraints are expressed as fluxes and the changes in the conserved charges associated with the extended BMS symmetry [37, 38, 60, 72]. So in principle, one can identify those charges and fluxes in the above constraint equations. However, we will not do that here. Instead, the conserved charges and fluxes will be computed in a future work.
5 Conclusion

This work reveals the memory effects in the asymptotically flat spacetime predicted by the dynamical Chern-Simons theory. Like in general relativity and Brans-Dicke theory, the tensor gravitational degrees of freedom induce exactly the same kinds of memory effects. That is, there are displacement, spin and CM memories. The asymptotic symmetries of the spacetime are also the extended BMS symmetries, and they are related to these memories just like what happens in general relativity and Brans-Dicke theory. So the displacement memory is related to the supertranslation transformations and the vacuum transition can be used to explain this effect. It is constrained by eq. (4.19) which contains terms proportional to the energy densities of the tensor and scalar radiation, one of the conserved charges associated with the supertranslations. The spin memory is constrained by eq. (4.22) and the CM memory by eq. (4.25). Both equations contain derivatives of $J_A$ which is proportional to the angular momentum density and associated with the Superboosts and super-rotations.

Although there is one more gravitational degree of freedom – the Chern-Simons scalar field, it does not excite memory effects that can be detected by interferometers, pulsar timing arrays or Gaia mission, due to the absence of the direction interaction between it and the matter. However, it should have its own memory effects, the analysis of which should rely on some proper dual formalism that will be proposed in the future.

Besides Brans-Dicke theory and Chern-Simons gravity, there are more interesting modified theories of gravity, such as Einstein-æther theory [73, 74], Hořava-Lifshitz gravity [75] and so on. The local Lorentz invariance is broken in these theories, and gravitational waves might have superluminal speeds [63, 64]. Whether there exist memories in these theories is to be answered. Hopefully, the memory effect can be a new tool to tell the nature of gravity.

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A Some expansion coefficients

In this appendix, we will write down the equations for the remaining expansion coefficients of the metric functions. First, one has the equation for $\mathcal{W}_A$ appearing in eq. (3.6b),

$$\mathcal{W}_A = \frac{1}{2} c_{AB} N^B + \frac{3}{4} \mathcal{D}_B \hat{e}_A^B - \frac{7}{64} \mathcal{D}_B (c_A^{BC} c^{CD}) - \frac{1}{16} c_{CDE} \mathcal{D}_B c^B_A$$

$$+ \frac{b}{6\kappa} \left( \frac{3}{16} \mathcal{D}_{BC} c_A^B + \mathcal{D}_A \mathcal{D}_B \mathcal{D}_A - \frac{1}{2} \mathcal{D}_A \mathcal{D}_B \mathcal{D}_A \right)$$

$$+ \frac{a}{4\kappa} \left[ \hat{e}_A^B \mathcal{D}_C (\mathcal{D}_A N^C_B) - \mathcal{D}_A^{BC} \mathcal{D}_B (\mathcal{D}_A N^C_B) + \mathcal{D}_A^{BC} \mathcal{D}_B (\mathcal{D}_A N^C_B) - \mathcal{D}_A^{BC} \mathcal{D}_B (\mathcal{D}_A N^C_B) \right].$$

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The evolution equation for $\dot{e}_{AB}$ is

$$\dot{e}_{AB} = \frac{1}{2} m c_{AB} + \frac{1}{4} c_{ABC} c_{CDB} N_{CD} + \frac{1}{6} (2 \mathcal{D}(\mathcal{A}N_B) - \gamma_{AB} \mathcal{D}C_{N}^C)$$

$$- \frac{a}{2 \kappa} \partial_1 \dot{e}_{(ACB)N} - \frac{1}{8} \left[ \mathcal{D}_{A} \mathcal{D}_{B} - \frac{1}{2} \gamma_{AB} \mathcal{D}^2 \right] (c_{CDB})$$

$$- \frac{1}{8} \left[ \mathcal{D}_{AB} \mathcal{D}_{CD} c_{DE} c_{DE} \right]$$

$$- \frac{3}{8} \mathcal{D}_{C} (2 c_{CD} \mathcal{D}(A_{CB})D - \gamma_{AB} c_{CD} \mathcal{D}E c_{CD})$$

$$- \frac{1}{4} (c_{C}(A \mathcal{D}C \mathcal{D}B_{CD} - c_{CD} \mathcal{D}D_{ACB}C)$$

$$- \frac{3}{8} c_{AB} \mathcal{D}_{C} \mathcal{D}_{CD} c_{CD} + \frac{1}{4} \mathcal{D}_{C}(c_{ABC} c_{CD})$$

$$- \frac{1}{4} \left[ \mathcal{D}_{C} c_{A} \mathcal{D}_{C} c_{B} + \mathcal{D}_{C} c_{A} \mathcal{D}_{C} c_{C} - \mathcal{D}_{C} c_{A} \mathcal{D}_{C} c_{BD} \right]$$

$$- \frac{b}{8 \kappa} \left[ \mathcal{D}_{A} \dot{\mathcal{D}}_{B} \vartheta_{1} - \dot{\mathcal{D}}_{A} \mathcal{D}_{B} \vartheta_{1} - \frac{1}{2} \gamma_{AB} \left( \mathcal{D}_{C} \vartheta_{1} \vartheta_{1} - \vartheta_{1} \mathcal{D}^2 \vartheta_{1} \right) \right],$$

and the evolution equation for $\dot{\mathcal{W}}^A$

$$\dot{\mathcal{W}}^A = \frac{N_{A} c_{AC} c_{BC}}{3} + \frac{1}{6} (\mathcal{D}^2 N_{A} - \mathcal{D}_{B} \mathcal{D}_{A} N_{B}) + \frac{\partial u(c_{AB}^{-1} N_{B})}{2} + \frac{\partial u(c_{AB}^{-1} N_{B})}{2} - \frac{\partial u(c_{AB}^{-1} N_{B})}{2}$$

$$- \frac{a}{6 \kappa} \partial_1 \dot{e}_{A} c_{C} + \frac{1}{4} c_{AB} c_{BCD} c_{DE} \mathcal{D}_{A} \mathcal{D}_{B} \mathcal{D} \mathcal{D}_{C} \mathcal{D}_{D} c_{DE}$$

$$- \frac{1}{4} c_{BC} c_{DE} \mathcal{D}_{C} \mathcal{D}_{A} \mathcal{D}_{B} \mathcal{D} c_{DE}$$

$$- \frac{3}{64} \partial u c_{A} c_{C} c_{DE} \mathcal{D}_{A} \mathcal{D}_{B} \mathcal{D} \mathcal{D}_{C} \mathcal{D}_{D} c_{DE}$$

$$- \frac{1}{2} \mathcal{D}_{A} \mathcal{D}_{A} \vartheta_{1} + \frac{a}{6 \kappa} \left[ \dot{\mathcal{D}}_{A} \vartheta_{1} \vartheta_{1} + \frac{\dot{\mathcal{D}}_{A} \vartheta_{1} \vartheta_{1}}{2} \right]$$

$$- \frac{\dot{N}}{2} \mathcal{D}_{B} \mathcal{D}_{B} c_{C} + \frac{3 \partial u}{16} \left( \partial_{A} \mathcal{D}_{B} \mathcal{D}_{A} \mathcal{D}_{B} \right)$$

$$- \frac{b}{8 \kappa} \left[ \mathcal{D}_{A} \dot{\mathcal{D}}_{B} \vartheta_{1} - \dot{\mathcal{D}}_{A} \mathcal{D}_{B} \vartheta_{1} - \frac{1}{2} \gamma_{AB} \vartheta_{1} \vartheta_{1} - \vartheta_{1} \mathcal{D}^2 \vartheta_{1} \right],$$

where we have not substituted eqs. (A.2) and (3.11c) into the above one, otherwise the expression would be much more complicated.

Similar equations like eq. (A.1) or (A.3) have not appeared in the literature.

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