Quantum speed limit time in a magnetic resonance

E. A. Ivchenko

Institute for Theoretical Physics, National Science Center “Institute of Physics and Technology”, 1, Akademicheskaya str., 61108 Kharkov, Ukraine

(Dated: April 18, 2022)

A visualization for dynamics of a qudit spin vector in a time-dependent magnetic field is realized by means of mapping a solution for a spin vector on the three-dimensional spherical curve (vector hodograph). The obtained results obviously display the quantum interference of precessional and nutational effects on the spin vector in the magnetic resonance. For any spin the bottom bounds of the quantum speed limit time (QSL) are found. It is shown that the bottom bound goes down when using multilevel spin systems. Under certain conditions the non-nil minimal time, which is necessary to achieve the orthogonal state from the initial one, is attained at spin S=2. An estimation of the product of two and three standard deviations of the spin components are presented. We discuss the dynamics of the mutual uncertainty, conditional uncertainty and conditional variance in terms of spin standard deviations. The study can find practical applications in the magnetic resonance, 3D visualization of computational data and in designing of optimized information processing devices for quantum computation and communication.

PACS numbers: 87.63.L,82.56.-b,03.67.-a

Keywords: 3D visualization, spin resonance, quantum speed limit time, Mandelstam-Tamm bound

INTRODUCTION

The magnetic resonance realization in a continuous mode depends on the kind of magnetic field modulations. Let’s consider the spin dynamics in the alternating field

\[ \vec{h}(t) = (h_1 \text{cn}(\omega t|k), h_2 \text{sn}(\omega t|k), H \text{dn}(\omega t|k)) \],

where \( \text{cn}, \text{sn}, \text{dn} \) are the Jacobi elliptic functions, \( \omega \) is the field frequency. Such field modulation under the changing of the elliptic modulus \( k \) from 0 to 1 describes the whole class of field forms from trigonometric to exponentially impulse ones (\( k = 1 \)). The elliptic functions \( \text{cn}(\omega t|k) \) and \( \text{sn}(\omega t|k) \) have a real period \( 4K/\omega \), while the function \( \text{dn}(\omega t|k) \) has a period of half a duration. Here \( K \) is a full elliptic integral of the first kind. At \( k \neq 0 \) and \( h_1 = h_2 = h \) we call such field consistent (cf). At \( k = 0 \) and \( h_1 = h_2 = h \) it is a circularly polarized magnetic field, and at \( k = 0 \) and \( h_1 = h, h_2 = 0 \) it is a linearly polarized field.

In this Letter we will map the solution of the equations of motion for a spin qudit vector on the three-dimensional oriented spherical curve and show quantum interference of the precession and nutation to use these results for finding quantum speed limits (QSLs).

We find QSLs in the magnetic resonance as fundamental bounds at minimum time which is necessary for a quantum system to evolve into a different state. The first studies are the Mandelstam-Tamm, Margolus-Levitin-Toffoli bounds, which have led to numerous extensions. QSLs have many applications, for instance, in a quantum cryptography, quantum control, quantum computation, communication, quantum thermodynamics and quantum metrology.

MASTER EQUATION

We will restrict ourselves to a closed quantum system, so that our initial state \( \rho_0 \) undergoes a unitary evolution and use the explicit model (1) at \( h_1 = h_2 = h \). The Liouville - Neumann equation for the density matrix \( \rho \), describing the qudit dynamics looks like

\[ \partial_t \rho = -i[\hat{H}, \rho], \rho(t=0) = \rho_0. \]

The Hamiltonian of a magnetic qudit (spin S particle) which is in the external consistent magnetic field \( h(t) \) is equal to \( \hat{H} = \mu C_1 h_1(t) \), where \( h_1(t) \) are the Cartesian components of the magnetic field in the frequency units, \( \mu \) is the qudit magnetic moment; \( C_1, C_2, C_3 \) are the matrix representation of components of the qudit spin. We use units chosen so, that the magnetic moment \( \mu \) equals 1 and \( h = 1 \).

The solution of the equation (2) looks like

\[ \rho = U(t) \rho_0 U(t)^\dagger, \]

where

\[ U = e^{-it\hat{h}}, \hat{h} = \alpha \hat{H}. \alpha^{-1} - \alpha \partial_t \alpha^{-1}. \]

\( \alpha = \text{diag}(f(S), f(-S)) \) is the \( 2S+1 \times 2S+1 \) diagonal matrix, in which \( f(S) = \text{cn}(S \omega t|k) + \text{sn}(S \omega t|k) + \text{dn}(S \omega t|k) \) are the solutions of quantum speed limit time in a magnetic resonance.

It is convenient to rewrite Eq. (3) in the form of orthogonal Hermitian matrices \( C_\nu \):
in which, here and further, we imply the summation on repetitive Greek coefficients from zero to $(2S + 1)^2 - 1$ and on Latin ones from one to three. The coherence Bloch vector $R_\phi$ is widely used in the theory of the magnetic resonance and characterizes the qudit behavior. In the case of the consistent field, as the expansion of the Rabi model, the solution (with the initial matrix elements $\rho_{1,1}(t = 0) = 1$ and other ones are equal to zero) at resonance $\omega = H$ for the spin components of qubit or qutrit is the following:

$$\vec{R} = r_B \sin(\omega t |k|) \sin ht, - \cos(\omega t |k|) \sin ht, \cos ht), \quad (5)$$

where $r_B = \sqrt{3S/(S + 1)}$ is the Bloch sphere radius. For higher spins, the direct calculation looks the same in Eq. [9] only if $k = 0$.

It will be useful to map the solution for the qudit spin vector on the geometrical model \[23\]. We parameterize a unit polarization vector by spherical angles $(p_1, p_2, p_3) = (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta)$. Thus, the parameter $\varphi (0 \leq \varphi \leq 2\pi)$ becomes a precession angle of the end of the vector $\vec{\rho}$ on a sphere, referred to as an apex, and the angle $\theta (0 \leq \theta \leq \pi)$ characterizes the nutation. And $\theta = 0$ corresponds to the north pole on the sphere.

In the resonance case the nutation angular velocity has the piecewise-constant time dependence and the precession one is constant \[3\]:

$$\theta' (\delta = 0) = h \text{sgn} (\sin ht), \quad \varphi' (\delta = 0) = \omega \quad (6)$$

with the period $T = 2\pi/h$.

In the consistent Rabi field \[1\] at $k \neq 0$, at the resonance, the angular velocities depend on time and are equal to \[2\]:

$$\theta_{cf}' (\delta = 0) = h \text{sgn} (\sin ht), \quad \varphi_{cf}' (\delta = 0) = \omega \sin (\omega t |k|) \quad (7)$$

**MINIMUM TIME FOR THE EVOLUTION TO THE ORTHOGONAL QUANTUM STATE**

The variance of the qudit energy is read

$$(\Delta E(t))^2 = \text{Tr} \rho H^2 - (\text{Tr} \rho \dot{H})^2, \quad (8)$$

where $\Delta E(t)$ is a standard deviation. The determination of the hodograph length $s$ for pure states \[9\] is

$$s = 2 \int_0^t \Delta E(t') dt' \quad (9)$$

and should be co-ordinated with the length on the sphere

$$l = \int_0^t v dt, \quad (10)$$

where the scalar of the vector velocity $v = r_B \sqrt{p_1^2 + p_2^2 + p_3^2}$ \[24\], the prime is used to denote the derivative with respect to time, $\tau' = \partial_\tau$. Having compared $s$ and $l$ from \[9\] and \[10\], we receive the formula

$$r_B \sqrt{p_1^2 + p_2^2 + p_3^2} = \sqrt{4p (\text{Tr} \rho \dot{H}^2 - (\text{Tr} \rho H)^2)}, \quad (11)$$

from which it follows, that the velocity of the apex is proportional to the standard deviation of the qudit energy. In this formula $p = 1$ only for spin $S = 1/2$ as the Fubini-Study metric is determined up to a numerical factor \[13, 22, 24\]. The introduction of the parameter $p$ is a speed normalization. With the increase of spin $S$ the Bloch sphere radius grows, but it is useful to note that $\lim_{S \to \infty} r_B = \sqrt{3}$, and the parameter $p = \frac{1}{2(S + 1)}$ decreases.

The circulation on the closed path at $k = 0$ is proportional to the qudit energy variance, as it appears from the formulas \[9\], \[10\].

$$C = \oint \vec{R}' \cdot d\vec{R} = 4p \int_0^T (\Delta E(t))^2 dt = \frac{3\pi S}{S + 1} (2h + \frac{H^2}{h}), \quad (12)$$

where $T$ is the time of passage of the closed contour $\Gamma$. At the fixed operating parameters $h$ and $H$ with the growth of spin $S$ the circulation grows.

In the resonant case $\forall \omega = H$ the distance from the north pole to the south pole on the Bloch sphere during $\tau = \pi/h$ equals

$$s = r_B \int_0^\tau \sqrt{h^2 + H^2 \sin^2 \theta} \sqrt{2} dt \geq \pi r_B. \quad (13)$$

As $h \to \infty$, hence

$$\lim_{h \to \infty} \int_0^\tau \sqrt{h^2 + H^2 \sin^2 \theta} \sqrt{2} dt = \pi, \quad (14)$$

the bottom bound is obtained in Eq. \[14\]. In other words, in a wide interval of the consistent field forms ($\forall k \in [0, 1]$ and for all resonance cases $\omega = H$) for a qubit and a qutrit the universal value $\pi$ is established.

At $k = 0$ and for the higher spins $S \geq 3/2$ the same result follows as

$$\lim_{h \to \infty} \int_0^\tau \sqrt{h^2 + H^2 \sin^2 \theta} \sqrt{2} dt = \lim_{h \to \infty} \frac{E(\pi, -H^2/h^2)}{\pi} = \pi, \quad (15)$$

where $E(\phi|m) = \int_0^\phi (1 - m \sin^2 \nu)^{1/2} d\nu$ is an elliptic integral of the second kind. Thus, the minimum distance $\pi r_B$ is the universal value because it is independent of field parameters in Eqs. \(14, 15\). This distance is nonzero even though $\tau \to 0$ as $h \to \infty$.

The evolution of the initial qudit spin vector with the doubly stochastic matrix and the diagonal time-independent Hamiltonian $\eta C_3 \underbrace{[3]}{[3]}$ geometrically presented on the Bloch sphere is a precession in the equatorial plane: $(R_1, R_2, R_3) = r_B (\cos \eta t, \sin \eta t, 0)$, $\varphi'(t) = \eta$, where $\eta$ is a positive constant. This geodesic
line has the length $\pi r_b$ during half-time $\pi/\eta$. In this case from Eq. (11) follows the formula $p = \frac{3r_b^2}{4S(S+1)}$. The straightforward calculation shows that as well as in case of the variable magnetic field, the Bloch sphere radius $r_b$ is finite for qudit spins: $(S, r_b, p) = (1/2, 1, 1), (1, 1.15, 1/2), (3/2, 1.22, 0.3), (2, 1.26, 0.2), ...(4, 1.32, 0.06), ... (10, 1.86, 0.02), ...$ Therefore at the infinitesimal time the apex passes the infinitesimal distance in this model. The distance for large spins, incompatible with the distance on the sphere Eq. (10), was used in (15). Eq. (15)).

We have from (11)

$$\int_0^\pi \Delta E(t)dt \geq \frac{\pi r_B}{\sqrt{4p}} = \frac{\sqrt{S}}{2} \pi, \quad (16)$$

Hence the Mandelstam-Tamm relation [8, 27] between the averaged on time $\tau$ standard deviation energy $\Delta E_\tau = \tau^{-1} \int_0^\tau \Delta E(t)dt$ during $\tau = \pi/h$ becomes

$$\Delta E_\tau \tau \geq \frac{\sqrt{S}}{2} \pi. \quad (17)$$

The bottom bound $\tau_{QSL}$ is determined by the formula

$$\tau \geq \frac{\pi^2 \sqrt{S}}{2\hbar E(-\hbar^2/h^2)} = \tau_{QSL}. \quad (18)$$

At $H \ll \hbar$ the bottom bound goes to zero for $\forall S$; at $H \gg \hbar$ the non-nil bottom bound $\lim_{h \to 0} \tau_{QSL} = \frac{\pi\sqrt{S}}{2\hbar}$ is attained. 

The distance from the north pole on the Bloch sphere to the nearest orthogonal state is more or equal $\pi r_B/(2S)$ in a qudit during $\tau_1 = \pi/(2S\hbar)$, because at a resonance the nutation speed has the piecewise-constant time dependence [6, 17]. At $\hbar \to \infty$ the value is $\pi/(2S)$ as it follows from (15). The bottom bound in this case goes down in comparison with a qubit.

The relation between the time-averaged standard deviation energy $\Delta E_\tau = \tau^{-1} \int_0^\tau \Delta E(t)dt$ during $\tau_1$ is

$$\Delta E_\tau \tau \geq \frac{\pi}{2\sqrt{2S}}. \quad (19)$$

The equation (19) directly leads to

$$\tau_1 \geq \frac{\pi^2}{2\hbar(2S)^{3/2}E(\pi/(2S), -\hbar^2/h^2)} = \tau_{QSL}, \quad (20)$$

from which it is obvious that the bottom bound goes down at the use of multilevel spin systems. At $H \ll \hbar$ the bottom bound goes to zero for $\forall S$; at $H \gg \hbar$ the non-nil bottom bound $\lim_{h \to 0} \tau_{QSL} = \frac{\pi^2}{2\hbar(2S)^{3/2}((-1)^{S/2}(1-\cos(\pi/(2S)))^2+2r[x]})$ is attained. The function $r[x]$ gives the integer closest to $x$. The ratio $(\lim_{h \to 0} \tau_{QSL}/\tau_{QSL}, S)$ is

Figure 1. The top view of the parametrical plot $(\Delta S_1(t), \Delta S_2(t), \Delta S_3(t))$ of the closed space curve (22) on the positive 1/8 part of the sphere for a spin 1 with the parameters of the circularly polarised field $\omega = H = 1, \ h = 2$.

(1, 1/2), (2, 1), (9/4, 3/2), (2.343, 2), ..., (2.467, $S \gg 1$). Hence, there is a possibility to use multilevel systems for a transition time reduction between orthogonal states.

The quantum speed limit time is defined with respect to a given class of Hamiltonians and the minimal time depends on the driving parameters. Our results coincide qualitatively with the results [28] for the whole class of spin Hamiltonians. We have shown that the QSL determination requires the knowledge of the system state during the evolution.

Dynamics of spin standard deviations

Figure 2. Dynamics of $S_3$, curvature $k$, torsion of curve $\kappa$, velocity modulus of an apex $V$ for a spin 1 with the parameters as in Fig. 1.

The real and non-negative eigenvalues of the $3 \times 3$ covariance matrix

$$\text{Cov}(S_i, S_k) = \frac{1}{2} \text{Tr} [\rho (C_i C_k + C_k C_i) - \text{Tr} [\rho C_i] \text{Tr} [\rho C_k] \quad (21)$$

equal $\lambda_1 = \lambda_2 = \frac{S}{2}, \lambda_3 = 0$ and the invariants of the tensor are $\lambda_1 + \lambda_2, \lambda_1 \lambda_2, \lambda_3$.

Hence, the sum of the variances [29, 30] of the spin
components $S_i$ in the magnetic resonance in the circularly polarized magnetic field is conserved for any $t$ during the evolution:

$$(\Delta S_1)^2 + (\Delta S_2)^2 + (\Delta S_3)^2 = S, \quad (22)$$

where $$(\Delta S_1, \Delta S_2, \Delta S_3) = \frac{1}{2}\sqrt{S/2} (\sqrt{3 + \cos 2ht + 2\sin^2 ht \cos 2\omega t},$$

$$\sqrt{3 + \cos 2ht - 2\sin^2 ht \cos 2\omega t}, 2|\sin ht|).$$ In the consistent field the relation (22) is fulfilled only for a qubit and a qutrit.

Due to $\Delta S_1(t) \geq 0, \Delta S_2(t) \geq 0, \Delta S_3(t) \geq 0$ and quantum restriction (22), the curve lies on 1/8 part of the sphere. This curve is the closed one with period $T_c = 2\pi m/h = \pi l/\omega$, where $m, l$ are integers. The oriented spherical curve is characterized by the curvature $k$, torsion of the curve $\kappa$, apex speed $V$, and length of the path $s$. In Fig. 2 it is shown that the speed $V$ is minimum, when $S_2$ and $k$ change a sign and $k$ has a local minimum. $S_3$ is extreme, when $V$ is maximal, $k$ is minimum, $\kappa$ has a local maximum/locall minimum. The quantitative characteristic of the curve at the coordinates $\Delta S_1, \Delta S_2, \Delta S_3$ in Fig. 1 represents in details the resonant evolution in Fig. 2.

It is possible to apply a known bilateral inequality (between the harmonic, geometrical and arithmetic averages) to the estimation of the product of two $\Delta S_i \Delta S_k, (i \neq k = 1, 2, 3)$ and three standard deviations $\Delta S_1 \Delta S_2 \Delta S_3$. The dynamics of the spin standard deviations and their products is presented in Fig. 1, 3.

In Fig. 3 the plots of the harmonicous, geometrical and arithmetic averages are presented. As it can be seen in Fig. 3 for both $\Delta S_1 \Delta S_2$ and $\Delta S_1 \Delta S_3, \Delta S_2 \Delta S_3$ the inequalities practically transform locally into the equalities when the equation (22) represents the closed curve on the sphere.

In terms of the standard spin deviations and new notions [31]: it is possible to describe the dynamics of the magnetic resonance with help of the mutual uncertainty $(M) M(S_i : S_k) = \Delta S_i + \Delta S_k - \Delta(S_i + S_k)$, the conditional uncertainty $(\Delta) \Delta(S_i|S_k) = \Delta(S_i + S_k) - \Delta S_k$ and the conditional variance (Var) $\text{Var}(S_i|S_k) = \text{Var}(S_i + S_k) - \text{Var} S_k$

From Fig. 2, 4, 5 it is seen that $M, \Delta, \text{Var}$ are extreme when $S_3$ changes sign. For the spin $S \geq 1$ we obtain $M \geq 0, \text{Var} \geq 0$ and $\Delta$ is sign-changing. The conditional variance equals zero in the exact resonance ($S_3 = -1$). These new characteristics specify the description of the magnetic resonance. The description of the magnetic resonance both in geometrical terms of Figs. 1, 2 and in terms of conditional uncertainty and variance Figs. 4, 5 mutually complement each other.

**CONCLUSION**

At resonance in the circularly polarization field the Bloch sphere radius is limited by value $\sqrt{3}$ when $S \to \infty$. The Fubini-Study measure in the finite-dimensional spin space is specified. The universal length value is found both for $S = 1/2, 1$ in the coordinated magnetic field and for any $S$ in the circularly polarization field. For any spin the bottom bounds QSLs are found. With the spin growth the transition time between levels decreases. The minimal time goes to zero at $H \ll h$. The dynamics of the spin standard deviations has been presented. An experimental confirmation of our results can be implemented at $H \gg h$ using a NMR setup [32, 33].
ACKNOWLEDGMENT

The author is thankful to Yu. L. Bolotin for useful discussion.

[1] E. A. Ivanchenko, Physica B 358, 308 (2005).
[2] E. A. Ivanchenko, Low Temp. Phys. 31, 577 (2005).
[3] M. Abramovitz and I. A. Stegun, Handbook of Mathematical Functions, edited by I. A. Stegun (Dover, New York, 1968) p. 832.
[4] I. I. Rabi, Phys. Rev. 51, 652 (1937).
[5] A. Bambini and P. R. Berman, Phys. Rev. A 23, 2496 (1981).
[6] F. Bloch and A. Siegert, Phys. Rev. 57, 522 (1940).
[7] E. A. Ivanchenko, arXiv:1311.2537v2 [quant-ph] 11 Nov (2013).
[8] L. Mandelstam and I. Tamm, J. Phys. USSR 9, 249 (1945).
[9] J. Anandan and Y. Aharonov, Phys. Rev. Lett. 65, 1697 (1990).
[10] L. Vaidman, Am. J. Phys. 60, 182 (1992).
[11] N. Margolus and L. B. Levitin, Physica D 120, 188 (1998).
[12] L. B. Levitin and T. Toffoli, Phys. Rev. Lett. 103, 160502 (2009).
[13] D. P. Pires, M. Cianciaruso, L. C. Celeri, G. Adesso, and D. O. Soares-Pinto, Phys. Rev. X 6, 021031 (2016).
[14] S. N. Molotkov and S. S. Nazin, JETP Lett. 63, 924 (1996).
[15] T. Caneva, M. Murphy, T. Calarco, R. Fazio, S. Montangero, V. Giovannetti, and G. E. Santoro, Phys. Rev. Lett. 103, 240501 (2009).
[16] T.-M. Zhang, R.-B. Wu, F.-H. Zhang, T.-J. Tarn, and G.-L. Long, IEEE TRANSACTIONS ON CONTROL SYSTEMS TECHNOLOGY 23, 2015 (2015).
[17] I. L. Markov, Nature 512, 147 (2014).
[18] L. Heaney and V. Vedral, Phys. Rev. Lett. 103, 200502 (2009).
[19] S. Deffner and E. Lutz, Phys. Rev. Lett. 105, 170 (2010).
[20] A. W. Chin, S. F. Huelga, and M. B. Plenio, Phys. Rev. Lett. 109, 233601 (2012).
[21] E. A. Ivanchenko, J. Math. Phys. 50, 042704 (2009).
[22] E. A. Ivanchenko, Int. J. Quant. Inf. 10, 1250068 (2012).
[23] R. P. Feynman, J. F. L. Vernon, and R. W. Hellwarth, Journal of applied physics 28, 49 (1957).
[24] Y. A. Aminov, Differential geometry and topology of curves, edited by Mistishenko (Nauka, Moskow, 1987) p. 160.
[25] in The mathematical encyclopaedia, Vol. 5, edited by I. M. Vinogradov (The Soviet encyclopaedia, Moskow, 1985) p. 1 247.
[26] V. V. Dodonov, O. V. Man’ko, V. I. Man’ko, and A. Wnensche, Phys.Scripta 59, 81 (1999).
[27] S. Deffner and E. Lutz, Journal of Physics A: Mathematical and Theoretical 46, 335302 (2013).
[28] G. C. Hegerfeldt, Phys. Rev. A 90, 032110 (2014).
[29] L. Macone and A. K. Pati, Phys. Rev. Lett. 113, 260401 (2014).
[30] B. Chen and S. M. Fei, Scientific Reports 5, 14238 (2015).
[31] S. Sazim, S. Adhikari, A. K. Pati, and P. Agrawal, arXiv:1702.0756v2 [quant-ph] (2017).
[32] J. B. Miller, B. H. Suits, and A. N. Garroway, Journal of mag. resonance 151, 228 (2001).
[33] W. Ma, B. Chen, Y. Liu, M. Wang, X. Ye, F. Kong, F. Shi, S.-M. Fei, and J. Du, Phys. Rev. A 95, 042334 (2017).