A linear-quadratic mean-field stochastic Stackelberg differential game with random exit time

Zhun Gou\textsuperscript{a}, Nan-jing Huang\textsuperscript{a} and Ming-hui Wang\textsuperscript{b}

\textsuperscript{a}Department of Mathematics, Sichuan University, Chengdu, People’s Republic of China; \textsuperscript{b}Department of Economic Mathematics, Southwestern University of Finance and Economics, Chengdu, People’s Republic of China

\textbf{ABSTRACT}

This paper is concerned with a new model of linear-quadratic mean-field stochastic Stackelberg differential game with one leader and two followers, in which only the leader is allowed to stop her strategy at a random time. By employing the backward induction method, the state equation is divided into two-stage equations. Then, the open-loop Stackelberg solution is obtained by using the maximum principle and the verification theorem. In a special case, with the help of Riccati equations, the open-loop Stackelberg solution is expressed as a feedback form of both the state and its mean.

\section{Introduction}

This paper focuses on a non-zero sum linear-quadratic mean-field stochastic Stackelberg differential game (LQ-MF-SSDG) with random exit time. Let us consider the following linear mean-field stochastic differential equation (MF-SDE):

\begin{equation}
\begin{aligned}
\frac{dX(t)}{dt} &= \left[a(t)X(t) + \overline{a}(t)\bar{X}(t) + b_1(t)v_1(t) + b_2(t)v_2(t) + b_0(t)v_0(t)\right] dt \\
&\quad + \left[c(t)X(t) + \overline{c}(t)\bar{X}(t)\right]dB_t, \quad t \in [0, T], \\
X(0) &= x_0
\end{aligned}
\end{equation}

in the complete probability space \((\Omega, \mathcal{F}, \mathbb{P})\) satisfying the usual hypothesis, where \(T > 0\) is a finite time duration; \(\tau_0\) is a positive random variable characterising a ‘surprising’ time at which the leader may stop her action (we call the ‘surprising’ time a random exit time); \(\tau = \tau_0 \land T\) is a random variable taking value in \([0, T]\); \(X(t)\) is the state process; \(v_0(t)\) is the control process of the leader; \(v_1(t)\) and \(v_2(t)\) are the control processes of the followers \(P_1\) and \(P_2\), respectively. Here \(\bar{X}(t)\) is the mean-field term with

\begin{equation}
\bar{X}(t) = \mathbb{E}\left[X(t)\mid \mathbb{F}_0\right] I_{t \in [0, T]} + \mathbb{E}\left[X(t)\mid \mathbb{F}_\tau\right] I_{t \in [\tau, T]}.
\end{equation}

The objective functions of the follower \(P_i\) (\(i = 1, 2\)) and the leader are assumed to be

\begin{equation}
J_i(v_1, v_2, v_0) = -\frac{1}{2} \mathbb{E}\left[\int_0^T p_i(t)X^2(t) + \overline{p}_i(t)\bar{X}^2(t) + q_i(t)v_i^2(t) dt + r_i(T)X^2(T)\mid \mathbb{F}_0\right],
\end{equation}

and

\begin{equation}
J_0(v_1, v_2, v_0) = -\frac{1}{2} \mathbb{E}\left[\int_0^T p_0(t)Y^2(t) + \overline{p}_0(t)\bar{Y}^2(t) + q_0(t)v_0^2(t) dt + r_0(\tau)Y^2(\tau)\mid \mathbb{F}_0\right],
\end{equation}

respectively, where \(Y(t)\) is the solution to the following MF-SDE with a single jump at \(t = \tau\):

\begin{equation}
\begin{aligned}
\frac{dY(t)}{dt} &= \left[a(t)Y(t) + \overline{a}(t)\bar{Y}(t) + b_1(t)v_1(t) + b_2(t)v_2(t) + b_0(t)v_0(t)\right] dt \\
&\quad + \left[c(t)Y(t) + \overline{c}(t)\bar{Y}(t)\right]dB_t, \quad t \in (0, \tau], \\
Y(0) &= x_0
\end{aligned}
\end{equation}

Here coefficients \(d_0(t)\) and \(\overline{d}_0(t)\) can be interpreted as the rating upgrades or downgrades in finance and economics. It is easy to see that \(Y(t) = X(t)\) a.s. for \(t \in (0, \tau]\) and \(Y(\tau) \neq X(\tau)\).

Usually, for any given \(v_0 \in \mathcal{V}[0, \tau]\) announced by the leader, the followers would like to find the Nash equilibrium point \((v_1^*, v_2^*, v_0^*) = (v_1^*(\cdot, v_0^(*)), v_2^*(\cdot, v_0^(*))) \in \mathcal{V}[0, \tau] \times \mathcal{V}[0, T]\) such that

\begin{equation}
\begin{aligned}
J_1(v_1^*, v_2^*, v_0) &= \text{ess sup}_{v_1 \in \mathcal{V}[0, T]} J_1(v_1, v_2^*, v_0), \\
J_2(v_1^*, v_2^*, v_0) &= \text{ess sup}_{v_2 \in \mathcal{V}[0, T]} J_2(v_1^*, v_2, v_0),
\end{aligned}
\end{equation}

where...
Then the leader expects to find $v_0^* \in \mathcal{V}[0, \tau]$ such that
\[
J_0(v_1^*, v_2^*, v_0^*) = \operatorname{ess sup}_{v_0 \in \mathcal{V}[0, \tau]} J_0(v_1^*, v_2^*, v_0).
\] (7)

Here the admissible control sets $\mathcal{V}[0, T]$ and $\mathcal{V}[0, \tau]$ are defined in the sequel. Setting $(v_1^{**}, v_2^{**}) = (v_1^*, v_0^*), (v_2^*, v_0^*)$, the problem of LQ-MF-SSDG with random exit time can be proposed as follows.

**Problem 1.1**: Find the Stackelberg solution $(v_0^*, v_1^{**}, v_2^{**})$ for the leader and the followers $\mathcal{P}_1$ and $\mathcal{P}_2$ satisfying (1)–(7).

Clearly, if $\tau = T$, then Problem 1.1 reduces to a form of LQ-MF-SSDGs considered by Wang and Zhang (2020). We note that various theoretical results, numerical algorithms and applications have been widely researched for the classical LQ-MF-SSDG by many authors (Bensoussan et al., 2017; Huang et al., 2021; Lin et al., 2019; Moon & Başar, 2018; Wang et al., 2020). Nevertheless, in some practical situations, it is necessary to consider Problem 1.1. Let us illustrate this point with the following three examples in different aspects.

Our first example is inspired by Novak et al. (2010), which is a differential game related to terrorism.

**Example 1.1**: Assume that the dynamics of the resource stock $x(t)$ can be written as
\[
\begin{align*}
\dot{x}(t) &= [rx(t) - h(u(t), v_1(t), v_2(t))] dt, \quad t \in [0, T], \\
x(0) &= x_0 > 0,
\end{align*}
\]
which depends on both the intensity of attacks from two terror organisations (the followers) $\mathcal{P}_1$ and $\mathcal{P}_2$ and the counterterror measure from the government (the leader). Here $r > 0$ is a constant, $h(u(t), v_1(t), v_2(t))$ denotes the influence of the control variables of the two players on the growth of the resource; $x_0$ denotes the initial stock of resources for terrorists; $u(t)$ is the counterterror measures of the government; $v_1(t)$ and $v_2(t)$ are the intensities of attacks from terrorists. Moreover, suppose that along a trajectory the following non-negativity constraint applies:
\[
x(t) \geq 0, \quad \text{for all } t \geq 0.
\]

In addition, let the objective functions of the government and the terror organisation $\mathcal{P}_1$ be, respectively, given by
\[
J(u, v_1, v_2) = \int_0^T h(x(t), u(t)) dt + S(x(T))
\]
and
\[
J_1(u, v_1, v_2) = \int_0^T h_i(x(t), v_1(t)) dt + S_i(x(T)).
\]
For a strategy $u$ announced by the leader, the terror organisations would like to take the strategy $(v_1^*, v_2^*) = (v_1^*, u(·)), v_2^*(·, u(·)))$ to maximise their objective functions given by
\[
\begin{align*}
J_1(u, v_1^*, v_2^*) &= \sup_{v_1(·) \geq 0} J_1(u, v_1^*, v_2^*), \\
J_2(u, v_1^*, v_2^*) &= \sup_{v_2(·) \geq 0} J_2(u, v_1^*, v_2^*).
\end{align*}
\]

Then the government would like to maximise her objective function
\[
J(u^*, v_1^*, v_2^*) = \sup_u J(u^*, v_1^*, v_2^*).
\]

When an emergency happens and is made a top priority, the government may stop her counterterror measures at a random time $t$ and deal with the emergency. In this case, the objective function of the government becomes
\[
J_0(u, v_1, v_2) = \int_0^{t \wedge T} h(x(t), u(t)) dt + y(t \wedge T),
\]
where $y(t \wedge T)$ represents the discounted value of $x(t \wedge T)$, i.e. the government suffers a loss at time $t$. When $t \in (τ, T]$, the terror organisations continue their competition for the resources and maximise their benefits. In this setting, the objective function $J_i(u, v_1, v_2)$ for $\mathcal{P}_i$ on the time interval $[0, T]$ may be discontinuous at $t = τ$. For instance,
\[
\begin{align*}
J_i(u, v_1, v_2) &= \int_0^T h_i(x(t), v_i(t)) dt + \tilde{h}_i(x(t), v_i(t)) dt + e^{-ρT} c_i x(T),
\end{align*}
\]
where $π \in \{ρ_1, h\}$ is different from $\tilde{π}$. Setting $(v_1^{**}, v_2^{**}) = (v_1^*, u^*(·), u^*(·)), (v_2^*, u^*(·)))$, the Stackelberg equilibrium point for the leader and followers can be given by $(u^*, v_1^{**}, v_2^{**})$. Obviously, $(u^*, v_1^{**}, v_2^{**})$ is the Stackelberg solution to Problem (1.1) in the deterministic case.

Our next example, the game of fishing, is motivated by Benchekroun and Van Long (2002).

**Example 1.2**: Suppose that the rate of growth of fish stock $x(t)$ satisfies
\[
\begin{align*}
\dot{x}(t) &= \left(a(t)x(t) - \sum_{k=0}^{2} u_k(t)\right) dt \\
&\quad + σ(t)x(t) dB_t, \quad t \in [0, T], \\
x(0) &= x_0 > 0,
\end{align*}
\]
Here $a(t)$ is the migration rate of the fish; $u_0(t)$ is the catch rate of the leader; $u_1(t)$ and $u_2(t)$ are catch rates of followers $\mathcal{P}_1$ and $\mathcal{P}_2$, respectively; $σ(t)$ is the volatility; $B_t$ is a standard one-dimensional Brownian motion. Usually, the feedback-type strategy can be described by $u_i(t) = b_i(t) x(t)$ ($i = 0, 1, 2$), where $b_i(t)$ represents the fishing effort. Consider following objective functions
\[
\begin{align*}
J_0(u_0, u_1, u_2) &= \mathbb{E}\left[\int_0^T e^{-rT} R_0(x(t), u_0(t)) dt + e^{-rT} S_0(x(T))\right], \\
J_1(u_0, u_1, u_2) &= \mathbb{E}\left[\int_0^T e^{-rT} R_1(x(t), u_1(t)) dt + e^{-rT} S_1(x(T))\right], \\
J_2(u_0, u_1, u_2) &= \mathbb{E}\left[\int_0^T e^{-rT} R_2(x(t), u_2(t)) dt + e^{-rT} S_2(x(T))\right].
\end{align*}
\]
for the leader and followers $P_1, P_2$, respectively. Then the leader has the chance to catch the fish first by taking the strategy $u_0(t)$ due to her good geographical advantage. Observing the fishing strategy $u_0(t)$ of the leader, the followers would like to maximise their objective functions by choosing the strategy $(u^*_1, u^*_2) = (u^*_1(\cdot, u_0(\cdot)), u^*_2(\cdot, u_0(\cdot)))$ satisfying

$$
\begin{align*}
J_1(u_0, u^*_1, u^*_2) &= \sup_{u_1} J_1(u_0, u_1, u^*_2), \\
J_2(u_0, u^*_1, u^*_2) &= \sup_{u_2} J_2(u_0, u^*_1, u_2),
\end{align*}
$$

where $u_i(t) \geq 0 (i = 1, 2)$ a.s. for all $t \geq 0$. Considering that followers $P_i$ would take catch strategy $u^*_i$, the leader would like to maximise her objective function, i.e.

$$
J_0(u^*_0, u^*_1, u^*_2) = \sup_{u_0} J_1(u_0, u^*_1, u^*_2), \quad u^*_0(t) \geq 0 \text{ a.s. for all } t \geq 0.
$$

If the leader could stop her catch and leave at $t = \tau$, then $J_0(u_0, u_1, u_2)$ becomes

$$
J_0(u_0, u_1, u_2) = \mathbb{E}\left[ \int_0^{\tau \wedge T} e^{-rt} R_0(x(t), u_0(t)) dt + e^{-r(\tau \wedge T)} S_0(x(\tau \wedge T)) \right].
$$

Putting $(u^*_0, u^*_1, u^*_2) = (u^*_0(\cdot, u^*_0(\cdot)), u^*_1(\cdot, u^*_0(\cdot)), u^*_2(\cdot, u^*_0(\cdot)))$, it is easy to see that $(u^*_0, u^*_1, u^*_2)$ is the Stackelberg solution to Problem 1.1 with no mean-field terms.

Our final example, borrowed from Wang et al. (2020), concerns with an asset-liability scheme of an insurance firm on a finite time horizon $[0, T]$, where two salesmen are hired to sell two different insurance products (personal insurance and property insurance).

**Example 1.3:** Suppose that the liability process $l(t)$ of the firm satisfies

$$
dl(t) = \left[ a(t)X(t) + \sum_{k=0}^2 b_k(t)u_k(t) \right] dt + \sigma(t)X(t) dB_t, \quad t \in [0, T].
$$

Here $X(t)$ represents the cash-balance; $u_1(t)$ and $u_2(t)$ denote premium rate for these two insurance products, respectively; $u_0(t)$ is an investment strategy of the firm, i.e. the firm withdraws capital from the cash-balance and then invests it in risky assets; $b_k(t)$ measures the impact of $u_k(t)$; $a(t)$ describes the volatility; $B_t$ is a standard one-dimensional Brownian motion. Moreover, assume that the firm owns an initial investment $X(0) = x$ and only invests in a money account with a compounded interest rate $a(t)$, i.e. the cash-balance of the firm satisfies

$$
dX(t) = a(t)X(t) dt + dl(t).
$$

Then combining (8) and (9), the cash-balance can be captured by the following MF-SDE:

$$
\begin{align*}
dX(t) &= \left[ a(t)X(t) + \pi(t)X(t) + \sum_{k=0}^2 b_k(t)u_k(t) \right] dt \\
&\quad + \sigma(t)X(t) dB_t, \quad t \in [0, T],
\end{align*}
$$

where $X(t)$ is defined by (2). The objective functions of the firm and salesmen ($P_1$ and $P_2$) are defined as follows:

$$
\begin{align*}
J_0(u_0, u_1, u_2) &= \frac{1}{2} \mathbb{E}\left[ \int_0^T k_0(t)u^2_0(t) + m_0(t) \text{Var}(X(t)) dt \\
&\quad + n_0(T) \text{Var}(X(T)) \right], \\
J_1(u_0, u_1, u_2) &= \frac{1}{2} \mathbb{E}\left[ \int_0^T k_1(t)u^2_1(t) + m_1(t) \text{Var}(X(t)) dt \\
&\quad + n_1(T) \text{Var}(X(T)) \right], \\
J_2(u_0, u_1, u_2) &= \frac{1}{2} \mathbb{E}\left[ \int_0^T k_2(t)u^2_2(t) + m_2(t) \text{Var}(X(t)) dt \\
&\quad + n_2(T) \text{Var}(X(T)) \right],
\end{align*}
$$

where $k_i(t), m_i(t)$ and $n_i(t)$ ($i = 0, 1, 2$) are weighted coefficients, $c \text{Var}(X(t)) = \overline{X^2(t)} - \overline{X^2}(t)$ is the conditional variance. After the firm announcing an investment strategy $u_0$, salesman $P_i$ ($i = 1, 2$) takes the strategy $u^*_i = u^*_i(\cdot, u^*_0(\cdot))$ by minimising $J_i(u_0, u_1, u_2)$ (or maximising $-J_i(u_0, u_1, u_2)$). Then the firm adopts strategy $u^*_0 = u^*_0(\cdot)$ by minimising $J_0(u_0, u_1, u_2)$ (or maximising $-J_0(u_0, u_1, u_2)$). If the firm could stop her investment at $t = \tau$, i.e. the firm takes the strategy $u^*_0(t) = 0, (t \in (\tau, T]]$ and salesmen $P_1$ and $P_2$ could continue their sale until time $T$, then the cost functional of the firm becomes

$$
J_0(u_0, u_1, u_2) = \frac{1}{2} \mathbb{E}\left[ \int_0^T k_0(t)u^2_0(t) + m_0(t) \text{Var}(X(t)) dt \\
&\quad + n_0(\tau \wedge T) \text{Var}(X(\tau \wedge T)) \right].
$$

Here the weighted impact coefficients $b_1(t), b_2(t), k_1(t)$ and $k_2(t)$ of salesmen may change after time $\tau$, i.e.

$$
\pi_i(t) = \pi_{i,1}(t)I_{t \in [0, \tau]} + \pi_{i,2}(t)I_{t \in (\tau, T]}, \quad \pi \in \{b, k\}, \quad i = 1, 2.
$$

Taking $u^*_0 = u^*_1(\cdot, u^*_0(\cdot)), u^*_2(\cdot, u^*_0(\cdot))$ is nothing but the Stackelberg solution to Problem 1.1.

The study of Stackelberg game with two players can be traced back to Stackelberg (2009), where one player acts as the leader (she) while the other behaves as the follower (he). After the leader announcing her strategy, the follower reacts to it by optimising his objective function. Then, the leader would like to seek a strategy to optimise her cost function based on the follower’s best response. The best strategy of the leader together with the best response of the follower is known as the Stackelberg solution. Since then, Stackelberg game has been extensively studied by many authors because of its wide application in various fields, including economics and finance, management and decision, transportation and evolutionary biology.
(see, for example, Askar (2018), Carreno and Santos (2019), Fang et al. (2017), Jiang and Liu (2018), Megahed (2019), Moon (2021b), Mu (2018), and Zou et al. (2020)).

Recently, the mean-field stochastic Stackelberg differential games (MF-SSDGs), which are described by MF-SDEs, have attracted much attention (Bensoussan et al., 2019, 2017; Huang et al., 2021b; Lin, 2019; Lin et al., 2019; Moon & Başar, 2018; Si & Wu, 2021; Wang et al., 2020). A significant feature of MF-SSDG is that not only the state variable and the controls but also their expectations are involved in the state equation and objective functions. Such a feature originates from the mean-field theory, which was developed to study the collective behaviours resulting from individuals mutual interactions in various physical and sociological dynamical systems.

On the other hand, the default risk, being closely related to default events and usually captured by the so-called default times, naturally appears in financial markets and economic models (Aksamit & Jeanblanc, 2017; Calvia & Gianin, 2020; Jeanblanc & Li, 2020). The standard approach to model the default time, is to use the theory of progressive enlargement of filtration (see, for example, Aksamit and Jeanblanc (2017), Jeanblanc et al. (2009), Kharrouri and Lim (2014), and Peng and Xu (2009). It is well known that many problems arising in finance and economics are indeed the stochastic control problems (SCPs), which leads to the works on SCPs with default times (Bachir Cherif et al., 2020; Gou et al., 2020; Pham, 2010; Shen & Siu, 2013). The theory of progressive enlargement of filtration usually requires the default times to be completely inaccessible to the reference filtration. Under suitable density assumptions, applying the dynamic principle of optimality, SCPs can be divided into several subproblems on random intervals by default times, in which each default time can be regarded as a ‘fixed’ time in the reference filtration (usually the Brownian filtration). Putting the solutions to subproblems together, the global optimal control can be obtained for SCPs with default times. It is worth to mention that Cordoni and Di Persio (2020) considered SCPs with default times which are assumed to be accessible from the reference filtration, i.e. each default time is defined as the first hitting time of a barrier for a reference system governed by stochastic differential equations (SDEs).

We note that stochastic differential games (SDGs) with default times, being closely related to the SCPs with default times, have been extensively studied in the literature. For instance, in Peng and Xu (2009), a zero-sum SDG with default risk was investigated and a saddle-point strategy was obtained. In Babich et al. (2007) and Wu et al. (2017), the models of retailer-supplier uncooperative replenishment and supply chain with one retailer and several suppliers were studied by employing stochastic Stackelberg differential games (SSDGs), respectively. Recently, by taking into account default and recovery of institutions, Élie et al. (2020) studied a mean-field game for large banking systems under some mild conditions.

Comparing Problem 1.1 with SDGs mentioned above, one of main differences is the terminal time of the leader is a random variable, while the ones of the followers are constants. The other is that the random exist time may lead to the discontinuity of Stackelberg solutions to differential games, since jumps may occur in the coefficients of the state equations and objective functions. We note that the differential games with random terminal times have been studied in the literature (see Elie et al. (2020), Ferreira et al. (2019), Marin-Solano and Shvetkova (2011), and Yeung and Petrosyans (2011) and references within). However, as far as we know the work concerned with Stackelberg differential games with different terminal times has never appeared. Moreover, the discontinuity of Nash and Stackelberg solutions to differential games has already been investigated in the literature (Dockner & Sorger, 1996; Lin, 2021; Moon, 2021b; Singh & Wsiazniema-Matyszuk, 2018), while the discontinuity of Stackelberg solutions to mean-field Stackelberg differential games only appears in the very recent work (Moon, 2021a).

To sum up, Problem 1.1 is new and has never been explored. The purpose of this paper is to find the Stackelberg solution to Problem 1.1 under some mild conditions. The main features of the current paper can be summarised as follows: (i) Problem 1.1 models some practical problems including Examples 1.1, 1.2 and 1.3; (ii) The MF-SDE, describing the state, can be divided into two-stage subequations by random exit time, in which the coefficients of subequations are allowed to be different; (iii) By employing the backward induction method, the Stackelberg solution is obtained for Problem 1.1 even though random coefficient is involved in the mean-field term; (iv) Feedback-type Stackelberg solution to Problem 1.1 is obtained in a special case.

The rest of this paper is structured as follows. In Section 2, we introduce some necessary notations and assumptions. After that in Section 3, we derive the Stackelberg solution to Problem 1.1 and find the feedback expression for the Stackelberg solution in a special case. Finally, we make some concluding remarks in Section 4.

2. Preliminaries

Focusing on Equation (1), we assume that $B_t$ is a standard one-dimensional Brownian motion, the $\sigma$-algebra $\mathcal{F} = (\mathcal{F}_t)_{t \geq 0}$ generated by $B_t$ is right-continuous and increasing, the initial value $x_0$ is an $\mathcal{F}_0$-measurable and square integrable random variable, and $\tilde{\mathcal{F}}$ is the smallest right-continuous extension in which $t_0$ becomes an $\tilde{\mathcal{F}}$-stopping time, i.e. $\tilde{\mathcal{F}}_t = \mathcal{F}_t \vee \sigma(\mathcal{F}_0 \cup \{u \in [0, s] \})$ for all $s > 0$.

Clearly, $\tau = t_0 \wedge T \in [0, T]$ is also an $\tilde{\mathcal{F}}$-stopping time. Noticing that when $t_0 > T$ a.s., the controller $v_0$ contributes nothing on the time interval $(\tau, T]$. Thus, it is natural to require that $b(t) = b_0(\mathbb{1}_{t \in [0, \tau]}$).

Moreover, the admissible control sets for Problem 1.1 have the following forms:

$$\forall [0, T] = \left\{ v : [0, T] \times \Omega \rightarrow \mathbb{R} \left| v(t) \right. \right. \text{is } \tilde{\mathcal{F}} \text{-progressively measurable with } \mathbb{E} \times \left( \int_0^T |v(t)|^2 \, dt \right)_{\tilde{\mathcal{F}}_0} < \infty \text{ a.s.} \right\} ;$$

$$\forall [0, \tau] = \left\{ v : [0, \tau] \times \Omega \rightarrow \mathbb{R} \left| v(t) \right. \right. \text{is } \tilde{\mathcal{F}} \text{-progressively measurable with } \mathbb{E} \times \left( \int_0^\tau |v(t)|^2 \, dt \right)_{\tilde{\mathcal{F}}_0} < \infty \text{ a.s.} \right\} ;$$
\[ V[r, T] = \left\{ v : [r, T] \times \Omega \to \mathbb{R} \, \left| \v(t) \right. \right\} \]
is \( \bar{\mathcal{G}} \)-progressively measurable with \( E \)
\[ \times \left[ \int_{r}^{T} |v(t)|^2 \, dt \left| \bar{\mathcal{G}}_r \right. \right] < \infty \text{ a.s.} \}

It follows from Theorem 4.1 in Gou et al. (2020) that, for \( \forall v_0 \in V[0, \tau] \) and \( \forall v_1, v_2 \in V[0, T] \), there exists a unique solution \( X(t) \in \mathcal{L}^2_\mathcal{G}(0, T; \mathbb{R}) \) to Equation (1). Here \( \mathcal{L}^2_\mathcal{G}(0, T; \mathbb{R}) \) represents the space of all \( \bar{\mathcal{G}} \)-adapted and square integrable processes. For convenience, we set \( i = 1, 2 \) and \( j = 0, 1, 2 \) throughout this paper.

We also make the following assumptions for the coefficients involved in (1), (3), (4) and (5).

**Assumption 2.1:** For \( b_i, \theta \in [a, \bar{a}, c, \bar{c}, \bar{\tau}] \) and \( \theta_i \in [b_i, \rho_i, \bar{\theta}_i, q_i, r_i] \), suppose that
\[ \begin{align*}
& b_i(t) = b_i(t) \mathbb{I}_{t \in [0, r]} + b_i(t) \mathbb{I}_{t \in (r, T]}, \\
& \theta_i(t) = \theta_i(t) \mathbb{I}_{t \in [0, r]} + \theta_i(t) \mathbb{I}_{t \in (r, T]}, \\
& \theta_{i2}(t) = \theta_{i2}(t) \mathbb{I}_{t \in [0, r]} + \theta_{i2}(t) \mathbb{I}_{t \in (r, T]}.
\end{align*} \tag{10}
\]

Moreover, assume that

- **(A)** \( a_i, \bar{a}_i, b_0, b_{1,1}, b_{1,2}, c_i, \bar{c}_i, d_0, \bar{d}_0 \) are all deterministic, uniformly bounded and real-valued functions on \( [0, T] \).
- **(B)** \( p_0, p_0, r_0, p_i, q_i, \bar{p}_i \) are all deterministic, continuous and non-negative functions on \( [0, T] \), and \( q_0, q_i, r_i \) are strictly positive.
- **(C)** \( a_{1,1}^0 b_{1,1}^0 = a_{2,2}^0 b_{2,2}^0 = b_i \).
- **(D)** (Hypothesis (H)) Every càdlàg \( \mathcal{F} \)-martingale remains an \( \bar{\mathcal{G}} \)-martingale.
- **(E)** There exists an \( \bar{\mathcal{G}} \)-predictable (respectively, \( \mathcal{F} \)-predictable) process \( \gamma^\mathcal{G} \) (respectively, \( \gamma^\mathcal{F} \)) with \( \gamma^\mathcal{G} \in \mathbb{I}_{t < r} \gamma^\mathcal{F} \) such that
\[ \begin{align*}
& A(t) = \mathbb{I}_{t < r} - \int_0^t \gamma^\mathcal{G}(s) \, ds \\
& = \mathbb{I}_{t \leq \tau} - \int_0^{t \wedge \tau} \gamma^\mathcal{F}(s) \, ds \quad (t \in [0, T]) \tag{11}
\end{align*} \]
is an \( \bar{\mathcal{G}} \)-martingale with jump time \( \tau \). The process \( \gamma^\mathcal{G} \) (respectively, \( \gamma^\mathcal{F} \)) is called the \( \bar{\mathcal{G}} \)-intensity (respectively, \( \mathcal{F} \)-intensity) of \( \tau \). Here \( \gamma^\mathcal{G} \) is upper bounded.

**Remark 2.1:** (i) Condition (10) shows that \( b_i \) and \( \theta_i \) are both \( \bar{\mathcal{G}} \)-predictable and \( \theta_i \) is \( \bar{\mathcal{G}} \)-adapted; (ii) (A) implies that for any \( (v_0, v_1, v_2) \in V[0, \tau] \times V[0, T] \times V[0, T] \), Equation (1) admits a unique solution \( X(t) \in \mathcal{L}^2_\mathcal{G}(0, T; \mathbb{R}) \); (iii) (B) ensures the concavity of the objective functions; (iv) (C) yields the well-posedness of related Riccati equations for followers; (v) (D) and (E) are classical assumptions in the theory of progressive enlargement.

### 3. Main results

In this section, we focus on finding the Stackelberg solution to Problem 1.1. Following the backward induction method (see, for example, Gou et al. (2020) and Pham (2010)), we decompose the state equation into the following two-stage equations on \( t \in [0, \tau] \) (the first stage) and \( t \in (\tau, T] \) (the second stage):
Finally combing the two-stage solutions \((v_{1,1}^*, v_{2,1}^*)\) and \((v_{0}^*, v_{1,2}^*, v_{2,2}^*)\), we can obtain the Stackelberg solution to Problem 1.1.

**Remark 3.1:** By the backward induction method, Problem 1.1 can be divided into two-stage subproblems. At the second stage, it is a stochastic Nash differential game with random initial time, which has been researched in numerous works (see Carmona and Delarue (2018a) and Carmona and Delarue (2018b) and references within). At the first stage, it is an SSDG with random terminal time (Bensoussan et al., 2016; Wang & Zhang, 2020). Compared the first-stage subproblem with the SSDG studied in Bensoussan et al. (2016), the main difference is: in the former the terminal time of the first-stage subproblem is random and a default jump is allowed in the objection function of the leader at time \(\tau\), while in the latter the terminal time is a constant and there is no jump in the objection function of the leader. Further comparisons with SSDGs in existing literatures are given in Remarks 3.4 and 3.5.

### 3.1 Nash equilibrium for followers at the second stage

In this subsection, we apply the Pontryagin-type maximum principle to find Nash equilibrium point \((v_{1,2}^*, v_{2,2}^*)\) for followers at the second stage. We restate problem (12) as follows.

Find \((v_{1,2}^*, v_{2,2}^*) \in \mathcal{V}[\tau, T] \times \mathcal{V}[\tau, T]\) such that

\[
\begin{align*}
J_{1,2}(v_{1,2}^*, v_{2,2}^*) &= \text{ess sup}_{v_{1,2} \in \mathcal{V} [\tau, T]} J_{1,2}(v_{1,2}, v_{2,2}), \\
J_{2,2}(v_{1,2}^*, v_{2,2}^*) &= \text{ess sup}_{v_{2,2} \in \mathcal{V} [\tau, T]} J_{2,2}(v_{1,2}^*, v_{2,2}).
\end{align*}
\]

For this subproblem, we define the Hamiltonian function of follower \(P_i\) at second stage by

\[
H_{i,2} = H_{i,2}(t, X_i, X_{i+1}, v_{i+1}, v_i, P_{i+1}, Q_{i+1}) = -\frac{1}{2} \left[ p_{i,2} X_{i+1}^2 + P_{i,2} X_{i+1}^2 + a_{i,2} \right] + \left[ a_{i,2} X_i + \bar{a}_{i,2} X_{i+1} + b_{i,2} v_{i+1} + b_{2,2} v_{2,2} \right] P_{i,2} + [c_{2,i,2} + \bar{c}_{2,i,2} Q_{i,2}].
\]

**Theorem 3.1:** \((v_{1,2}^*, v_{2,2}^*)\) is a Nash equilibrium point for Subproblem 3.1 if and only if

\[
(v_{1,2}^*, v_{2,2}^*) = \left( q_{i,2}^1 b_{i,2} P_{i,2} - q_{i,2}^2 b_{2,2} P_{i,2}^*, \right),
\]

where the triple \((X_{i,2}^*, P_{i,2}^*, Q_{i,2}^*)\) satisfies the following mean-field forward–backward stochastic differential equation (MF-FBSDE):

\[
\begin{align*}
\text{d}X_{i,2}^* &= \left[ a_{2,i,2} X_{i+1}^* + \bar{a}_{2,i,2} X_{i+1}^* + q_{i,2}^1 b_{i,2} P_{i,2}^* + q_{i,2}^2 b_{2,2} P_{i,2}^* \right] \text{d}t \\
&\quad + \left[ c_{2,i,2} + \bar{c}_{2,i,2} Q_{i,2}^* \right] \text{d}B_i^*, \quad t \in (\tau, T],
\end{align*}
\]

\[
\begin{align*}
\text{d}P_{i,2}^* &= \left[ p_{i,2} X_{i+1}^* + \bar{p}_{i,2} X_{i+1}^* - a_{i,2} P_{i,2}^* - \bar{a}_{i,2} P_{i,2}^* - c_{2,i,2} Q_{i,2}^* \right] \text{d}t + Q_{i,2}^* \text{d}B_i^*, \quad t \in (\tau, T],
\end{align*}
\]

\[
X_{i,2}^*(\tau) = X_i(\tau), \quad P_{i,2}^*(T) = -r_i(T) X_{i,2}^*(T).
\]

**Proof:** By the maximum principle of MF-SDEs with default times (see Gou et al. (2020)), we have

\[
0 = \frac{\partial H_{i,2}}{\partial v_{i,2}} \bigg|_{v_{i,2}=v_{i,2}^*} = -q_{i,2} b_{i,2} + b_{i,2} P_{i,2}^* \Rightarrow
v_{i,2}^* = \frac{1}{b_{i,2}} b_{i,2} P_{i,2}^*.
\]

where \(P_{i,2}^*\) is determined by (15). Now we only need to prove that \((v_{1,2}^*, v_{2,2}^*)\) given by (14) is the Nash equilibrium point for Subproblem 3.1. For \(v_{1,2}^* \in \mathcal{V}[\tau, T]\), set \(X_2 = X_2^* - X_2^*\) and \(X_2^* = v_{2,1} - v_{2,2}^*\). Then, we have

\[
\begin{align*}
\text{d}X_2 &= \left[ a_2 X_2^* + \bar{a}_2 X_2^* + b_{1,2} V_{1,2} \right] \text{d}t \\
&\quad + \left[ c_2 X_2^* + \bar{c}_2 X_2^* \right] \text{d}B_i^*, \quad t \in (\tau, T],
\end{align*}
\]

\[
X_2(\tau) = 0
\]

and

\[
\begin{align*}
J_{1,2}(v_{1,2}^*, v_{2,2}^*) &= J_{1,2}(v_{1,2}, v_{2,2}) \\
&= \frac{1}{2} \mathbb{E} \left[ \int_{\tau}^{T} (p_{1,2} X_2^2 + \bar{p}_{1,2} X_2^2 + q_{1,2} V_{1,2}^2) \text{d}t \bigg| \mathcal{F}_\tau \right] + \frac{1}{2} \mathbb{E} \left[ r_1(T) X_2^2(T) \bigg| \mathcal{F}_\tau \right] \\
&= \frac{1}{2} \mathbb{E} \left[ \int_{\tau}^{T} (p_{1,2} X_2^2 + \bar{p}_{1,2} X_2^2 + q_{1,2} V_{1,2}^2) \text{d}t \bigg| \mathcal{F}_\tau \right] + \frac{1}{2} \mathbb{E} \left[ r_1(T) X_2^2(T) \bigg| \mathcal{F}_\tau \right].
\end{align*}
\]

Applying Itô’s formula to \(X_2^* P_{i,2}^*\), it follows that

\[
\mathbb{E} \left[ r_1(T) X_2^2(T) X_2^* \bigg| \mathcal{F}_\tau \right] = -\mathbb{E} \left[ X_2^2(T) P_{i,2}^* - r_2(T) P_{i,2}^* \bigg| \mathcal{F}_\tau \right] \\
= -\mathbb{E} \left[ \int_{\tau}^{T} (p_{1,2} X_2^2 + \bar{p}_{1,2} X_2^2 + b_{1,2} P_{i,2}^* V_{1,2}^2) \text{d}t \bigg| \mathcal{F}_\tau \right] \\
= -\mathbb{E} \left[ \int_{\tau}^{T} (p_{1,2} X_2^2 + \bar{p}_{1,2} X_2^2 + q_{1,2} V_{1,2}^2) \text{d}t \bigg| \mathcal{F}_\tau \right] \\
= \frac{1}{2} \mathbb{E} \left[ r_1(T) X_2^2(T) \bigg| \mathcal{F}_\tau \right].
\]

and so

\[
J_{1,2}(v_{1,2}^*, v_{2,2}^*) - J_{1,2}(v_{1,2}, v_{2,2}^*) = \frac{1}{2} \mathbb{E} \left[ \int_{\tau}^{T} (p_{1,2} X_2^2 + \bar{p}_{1,2} X_2^2 + q_{1,2} V_{1,2}^2) \text{d}t \bigg| \mathcal{F}_\tau \right] + \frac{1}{2} \mathbb{E} \left[ r_1(T) X_2^2(T) \bigg| \mathcal{F}_\tau \right].
\]

Since \(p_{1,2} \geq 0, \bar{p}_{1,2} \geq 0, q_{1,2} > 0\), and \(r_1 > 0\) are positive, we have

\[
J_{1,2}(v_{1,2}^*, v_{2,2}^*) - J_{1,2}(v_{1,2}, v_{2,2}^*) \geq 0 \ a.s.
\]

Similarly, for all \(v_{2,2} \in \mathcal{V}[\tau, T]\), we have

\[
J_{2,2}(v_{1,2}^*, v_{2,2}^*) - J_{2,2}(v_{1,2}, v_{2,2}) \geq 0 \ a.s.
\]

Thus the proof is finished.
According to (13), before solving the Nash equilibrium point for followers at the first stage, we need to compute the optimal objective subfunctions $I_{1,1}(v_{1,1}^*, v_{2,1}^*)$ and $I_{2,2}(v_{1,2}^*, v_{2,2}^*)$. First, we focus on the feedback representation of the solutions to (15).

Observing the terminal condition of $P_{v_{1,2}^*}$, we set

$$ P_{v_{1,2}^*}(t) = \phi_{1,1}(t)X_{1}^*(t) + \phi_{1,2}(t)\bar{X}_{1}^*(t). \quad (16) $$

Applying the filtering theory to (15) (see Wang et al. (2018, 2017) and references within), one has the filtering equation

$$ \begin{align*}
    dX_{1}^* &= \left[ (a_0 + \alpha_0)X_{1}^* + \varphi_{1,1} + (a_1 + \beta_1)P_{v_{1,2}^*} \right] dt, \\
    \bar{X}_{1}(t) &= X_{1}(t).
\end{align*} $$

Computing the differential of $P_{v_{1,2}^*}$ and comparing it with (15), one can easily obtain the following system of ordinary differential equations (ODEs):}

$$ \begin{align*}
    \begin{cases}
        \phi_{1,1} + (2a_2 + c_2)\phi_{1,1} + l_2\phi_{1,1,1}\phi_{2,1} - (p_{1,2} + p_{2,2}) = 0, & t \in [\tau, T), \\
        \phi_{2,1} + (2a_2 + c_2)\phi_{2,1} + l_2\phi_{1,1,1}\phi_{2,1} + l_2\phi_{2,1} - (p_{1,2} + p_{2,2}) = 0, & t \in [\tau, T), \\
        \phi_{1,2} + (2a_2 + 2c_2)\phi_{1,2} + l_2\phi_{2,1}\phi_{1,2} + l_2\phi_{1,2} - (p_{1,2} + p_{2,2}) = 0, & t \in [\tau, T), \\
        \phi_{2,2} + (2a_2 + 2c_2)\phi_{2,2} + l_2\phi_{1,2}\phi_{2,2} - (p_{1,2} + p_{2,2}) = 0, & t \in [\tau, T), \\
        \phi_{1,1}(T) = r_1(T), & \phi_{2,1}(T) = 0, \quad \phi_{2,2}(T) = 0.
    \end{cases} \quad (17)
\end{align*} $$

Now we show the existence and uniqueness of solutions to (17).

**Lemma 3.1:** There exists a unique solution $(\phi_{1,1}, \phi_{1,2}, \phi_{2,1}, \phi_{2,2})$ to (17), which is bounded and continuous.

**Proof:** Set $\varphi_i = \phi_{1,i} + \phi_{2,i}$. Then, $\varphi_1$ satisfies the following Riccati equation:

$$ \begin{align*}
    \left\{ \begin{aligned}
        \varphi_1 + (2a_2 + c_2)\varphi_1 + l_2\varphi_1^2 - (p_{1,2} + p_{2,2}) &= 0, & t \in [\tau, T), \\
        \varphi_1(T) &= -r_1(T) - r_2(T).
    \end{aligned} \right. \quad (18)
\end{align*} $$

From the results in Yong and Zhou (1999), there exists a unique solution to (18). Therefore, (17) can be transformed into the following system of linear equations:

$$ \begin{align*}
    \begin{cases}
        \phi_{1,1} + (2a_2 + c_2)\phi_{1,1} + l_2\phi_{1,1}\phi_{2,1} - p_{1,2} = 0, & t \in [\tau, T), \\
        \phi_{2,1} + (2a_2 + c_2)\phi_{2,1} + l_2\phi_{1,1}\phi_{2,1} - p_{2,2} = 0, & t \in [\tau, T), \\
        \phi_{1,1}(T) = -r_1(T), & \phi_{2,1}(T) = -r_2(T),
    \end{cases}
\end{align*} $$

where $\phi_1$ is the unique solution to (18). From the results in Yong and Zhou (1999), the existence and uniqueness of solutions $\phi_{1,1}$ and $\phi_{2,1}$ is obtained immediately.

Similarly, $\varphi_2$ satisfies the following Riccati equation:

$$ \begin{align*}
    \left\{ \begin{aligned}
        \varphi_2 + (2a_2 + 2c_2)\varphi_2 + l_2\varphi_2^2 &+ p_1\left[ 2\overline{\theta}_2 + 2\bar{c}_2 \right] - (\overline{P}_{1,2} + \overline{P}_{2,2}) = 0, & t \in [\tau, T), \\
        \varphi_2(T) &= 0.
    \end{aligned} \right. \quad (19)
\end{align*} $$

Repeating the above arguments, we can obtain the existence and uniqueness of solutions $\varphi_{2,1}$ and $\varphi_{2,2}$.

Next, we aim to find the expression of $I_{2,2}(v_{1,2}^*, v_{2,2}^*)$. First, set

$$ P_{v_{1,2}^*}(t) = \Gamma_{i,1}(t)X_{1}^*(t) + \Gamma_{i,2}(t)\bar{X}_{1}^*(t) + \Gamma_{i,3}(t), $$

where $d\Gamma_{i,3}(t) = \Gamma_{i,4}(t)\bar{B}_t$. Then regarding $v_{i,j}^*$ as a fixed process, we see that $(\Gamma_{i,1}, \Gamma_{i,2}, \Gamma_{i,3})$ is a solution to the following system of equations:

$$ \begin{align*}
    \left\{ \begin{aligned}
        0 &= \Gamma_{i,1} + (2a_2 + 2c_2)\varphi_{1,2} + l_2\varphi_{1,2}^2 - p_{1,2}, & t \in [\tau, T), \\
        0 &= \Gamma_{i,2} + (2a_2 + 2c_2)\varphi_{2,2} + l_2\varphi_{2,2}^2 - p_{2,2}, & t \in [\tau, T), \\
        \varphi_{1,2}(T) &= -r_1(T), & \varphi_{2,2}(T) = 0, \quad \varphi_{2,2}(T) = 0.
    \end{aligned} \right. \quad (20)
\end{align*} $$

Observing that (21) is linear, we set

$$ X_{1}^*(t) = M_i(t)X_{1}(t) + N_i(t), $$

where $(M_i, N_i)$ is the solution to the following system of linear MF-SDEs:

$$ \begin{align*}
    dM_i &= \left[ (a_2 + \beta_2)M_i + (\overline{\theta}_2 + 2\phi_2)\overline{M}_i \right] dt \\
    &+ \left[ 2a_2M_i + \overline{c}_2\overline{M}_i \right] d\bar{B}_t, \quad t \in [\tau, T), \\
    dN_i &= \left[ (a_2 + \beta_2)N_i + (\overline{\theta}_2 + 2\phi_2)\overline{N}_i + b_{3-i,2}v_{3-i,2} \right] dt \\
    &+ \left[ 2a_2N_i + \overline{c}_2\overline{N}_i \right] d\bar{B}_t, \quad t \in [\tau, T), \\
    M_i(\tau) &= 1, \quad N_i(\tau) = 0.
\end{align*} $$

Applying Itô’s formula to $X_{1}^*P_{v_{1,2}^*}$ and $\Gamma_{i,3}M_i$, one has

$$ \begin{align*}
    &\mathbb{E}\left[ X_{1}^*(T)P_{v_{1,2}^*}(\tau) - X_{1}^*(T)P_{v_{1,2}^*}(\tau) \right] \\
    &= \mathbb{E}\left[ \int_{\tau}^{T} \Gamma_{i,2}(t)X_{1}^*(t) + \overline{P}_{1,2}(t)\bar{X}_{1}^*(t) + q_{i,2}(t)v_{i,2}^*(t) dt \right] \\
    &+ \mathbb{E}\left[ \int_{\tau}^{T} b_{3-i,2}(t)v_{3-i,2}(t)P_{v_{1,2}^*}(t) dt \right]
\end{align*} $$

and

$$ \begin{align*}
    &\mathbb{E}\left[ \int_{\tau}^{T} (\Gamma_{i,1}b_{3-i,2}v_{3-i,2}M_i + \Gamma_{i,2}b_{3-i,2}\overline{v}_{3-i,2}\overline{M}_i)(t) dt \right] \\
    &= -\Gamma_{i,3}(t).
\end{align*} $$
Then it follows that

\[
2J_{1,2}(v_{2,1}^*, v_{3,2}^*) - \left[ \Gamma_{1,1}(\tau)X_1^2(\tau) + \Gamma_{1,2}(\tau)X_1(\tau) + \Gamma_{1,3}(\tau)X_1(\tau) \right]
= E \left[ \int_0^T b_{3,1}(t)v_{3,1}^*(t)P_{v_{3,1}}(t) \, dt \right] \tilde{\gamma}_t
= E \left[ \int_0^T b_{3,1}(t)v_{3,1}^*(t)(\Gamma_{1,1}M_i + \Gamma_{1,2}N_i + \Gamma_{1,3}) \, dt \right] \tilde{\gamma}_t
= -\Gamma_{1,3}(\tau)X_1(\tau) + E \left[ \int_0^T b_{3,1}(t)v_{3,1}^*(t)(\Gamma_{1,1}N_i + \Gamma_{1,2}N_i + \Gamma_{1,3}) \, dt \right] \tilde{\gamma}_t.
\]

Therefore the Fréchet derivative of \(J_{1,2}(v_{1,2}^*, v_{2,2}^*)\) with respect to \(X_1(\tau)\) is

\[
\frac{\partial J_{1,2}(v_{1,2}^*, v_{2,2}^*)}{\partial X_1(\tau)} = \Gamma_{1,1}(\tau)X_1(\tau) + \Gamma_{1,2}(\tau)X_1(\tau)
+ \Gamma_{1,3} = P_{v_{1,2}}(\tau). \tag{21}
\]

According to (16) and (21), we have the following theorem.

**Theorem 3.2:** Of \(J_{1,2}(v_{1,2}^*, v_{2,2}^*)\), the Fréchet derivative with respect to \(X_1(\tau)\) is

\[
\frac{\partial J_{1,2}(v_{1,2}^*, v_{2,2}^*)}{\partial X_1(\tau)} = \varphi_{1,1}(\tau)X_1(\tau) + \psi_{1,2}(\tau)X_1(\tau).
\]

**Remark 3.2:** Because of the definition of the Nash equilibrium point, we should regard \(v_{3,2}^*\) as a given process when calculating the Fréchet derivative. By Theorem 3.8 in Chapter IX of Revuz and Yor (2009), it is easy to show that \(\varphi_{1,1}(t) < 0, \psi_{2,1}(t) < 0\) and \(M_i(t) > 0\). Therefore, the concavity of \(J_{1,2}(v_{1,2}^*, v_{3,2}^*)\) with respect to \(X_1(\tau)\) follows.

### 3.2 Nash equilibrium for followers at the first stage

In this subsection, we characterise the Nash equilibrium point \((v_{1,1}^*, v_{2,1}^*)\) for followers at the first stage. According to Theorem 3.2, we know that Subproblem (13) can be rewritten as follows.

Find \((v_{1,1}^*, v_{2,1}^*) \in \mathcal{V}[0, \tau] \times \mathcal{V}[0, \tau]\) such that

\[
J_{1,1}(v_{1,1}^*, v_{2,1}^*, v_{0}^*) = \text{ess sup}_{v_{2,1} \in \mathcal{V}[0, \tau]} \left\{ \int_0^\tau - \frac{1}{2} \left[ p_{1,1}(t)X_1^2(t) + \frac{1}{2}(\Gamma_{1,1}(\tau)X_1^2(t) + \Gamma_{1,2}(\tau)X_1(\tau) + \Gamma_{1,3}(\tau)X_1(\tau)) \right] \, dt + \frac{1}{2} \left( \Gamma_{2,1}(\tau)X_1^2(t) + \Gamma_{2,2}(\tau)X_1(\tau) + \Gamma_{2,3}(\tau) \right) \right\},
\]

\[
J_{2,1}(v_{1,1}^*, v_{2,1}^*, v_{0}^*) = \text{ess sup}_{v_{2,1} \in \mathcal{V}[0, \tau]} \left\{ \int_0^\tau - \frac{1}{2} \left[ p_{2,1}(t)X_1^2(t) + \frac{1}{2}(\Gamma_{1,1}(\tau)X_1^2(t) + \Gamma_{1,2}(\tau)X_1(\tau) + \Gamma_{1,3}(\tau)X_1(\tau)) \right] \, dt + \frac{1}{2} \left( \Gamma_{2,1}(\tau)X_1^2(t) + \Gamma_{2,2}(\tau)X_1(\tau) + \Gamma_{2,3}(\tau) \right) \right\}.
\]

Similar to Subsection 3.1, the Hamiltonian function of follower \(P_i\) at first stage is defined as

\[
H_{i,1} = H_{i,1}(t, X_1, X_1, v_{0}, v_{1,1}, v_{2,1}, P_{v_{1,1}}, Q_{v_{1,1}}),
= -\frac{1}{2} \left[ p_{1,1}X_1^2 + \frac{1}{2} (a_{1,1}X_1^2 + b_{1,1}v_{1,1} + b_{1,1}v_{2,1} + b_{1,0}v_{0}) P_{v_{1,1}} + c_{1,1}X_1 + c_{1,1}X_1 \right].
\]

**Theorem 3.3:** \((v_{1,1}^*, v_{2,1}^*)\) is a Nash equilibrium point for Subproblem 3.2 if and only if

\[
(v_{1,1}^*, v_{2,1}^*) = (q_{1,1}^{-1}b_{1,1}P_{v_{1,1}}, q_{2,1}^{-1}b_{2,1}P_{v_{2,1}}), \tag{22}
\]

where the optimal triple \((X_1^*, P_{v_{1,1}}^*, Q_{v_{1,1}}^*)\) enjoys the following MF-FBSDE:

\[
\begin{align*}
\frac{dX_1^*}{dt} &= \left[ a_{1,1}X_1^* + \bar{a}_{1,1}X_1^* + q_{1,1}^{-1}b_{1,1}P_{v_{1,1}}^* + q_{2,1}^{-1}b_{2,1}P_{v_{2,1}}^* + b_{1,0}v_{0} \right] dt + \left[ c_{1,1}X_1^* + \bar{c}_{1,1}X_1^* \right] dB_t, \quad t \in (0, \tau],
\frac{dP_{v_{1,1}}^*}{dt} &= \left[ p_{1,1}X_1^* + \bar{p}_{1,1}X_1^* - a_{1,1}P_{v_{1,1}}^* - \bar{a}_{1,1}P_{v_{1,1}}^* - c_{1,1}Q_{v_{1,1}}^* - \bar{c}_{1,1}Q_{v_{1,1}}^* \right] dt + Q_{v_{1,1}}^* dB_t, \quad t \in (0, \tau],
X_1^*(t) &= x_0, \quad P_{v_{1,1}}^*(t) = \varphi_{1,1}(\tau)X_1(t) + \psi_{1,2}(\tau)X_1(t).
\end{align*}
\]

**Proof:** Recalling Remark 3.2 and using the sufficient and necessary maximum principles of MF-SDEs with default (see Gou et al. (2020)), we have

\[
0 = \frac{\partial H_{i,1}}{\partial v_{1,1}} \bigg|_{v_{1,1} = v_{1,1}^*} = -q_{1,1}^{-1}b_{1,1}P_{v_{1,1}}^* \Rightarrow v_{1,1}^* = q_{1,1}^{-1}b_{1,1}P_{v_{1,1}}^*.
\]
where $P_{v_{11}}$ is governed by (23). Now we only need to verify that $(v_{1,1}^*, v_{2,1}^*)$ given by (22) is indeed a Nash equilibrium point for Subproblem 3.2. For $\forall v_{11} \in [0, \tau]$, set $X_1 = X_0 - X_1^*$ and $\nabla_{11} = v_{11} - v_{2,1}^*$. Then, one has

$$
\begin{aligned}
\{ \frac{dX_1}{dt} &= \left[ a_1 X_1 + \overline{a}_1 X_1 + b_{11} X_{11} \right] \frac{d\mathbb{B}_1}{d\tau}, \ t \in (0, \tau], \\
X_1(t) &= 0.
\end{aligned}
$$

Since $p_{1,1}$, $\overline{p}_{1,1}$ are non-negative, $q_{1,1}$ is positive and $\varphi_{1,1}(\tau)$, $\varphi_{2,1}(\tau) < 0$ a.s., we have

$$
J_{1,1}(v_{1,1}^*, v_{2,1}^*, v_0) - J_{1,1}(v_{1,2}, v_{2,2}, v_0)
= \frac{1}{2} \mathbb{E} \left[ \int_0^\tau \left( p_{1,1} X_1^2 + \overline{p}_{1,1} X_1^2 + q_{1,1} \nabla_{1,1} \right)(t) dt \right] + \frac{1}{2} \mathbb{E} \left[ \left( \varphi_{1,1} X_1^2 + \varphi_{2,1} X_1^2 \right)(t) dt \right] + \mathbb{E} \left[ \int_0^\tau \left( \varphi_{1,1}(t) X_1^2 + \varphi_{2,1}(t) \nabla_{1,1}(t) \right) dt \right] + \mathbb{E} \left[ X_1(t) P_{v_{1,1}}(t) \right].
$$

Applying Itô’s formula to $X_1 P_{v_{1,1}}$, it follows that

$$
\mathbb{E} \left[ X_1(t) P_{v_{1,1}}(t) \right] = -\mathbb{E} \left[ X_1(t) P_{v_{1,1}}(t) \right] + \mathbb{E} \left[ \int_0^\tau \left( p_{1,1} X_1 X_1^2 + \overline{p}_{1,1} X_1 X_1^2 + b_{11} P_{v_{1,1}} \nabla_{1,1} \right)(t) dt \right] + \mathbb{E} \left[ \int_0^\tau \left( \varphi_{1,1}(t) X_1 \nabla_{1,1} + \varphi_{2,1}(t) \nabla_{1,1} \nabla_{1,1} \right) dt \right].
$$

Therefore,

$$
J_{1,1}(v_{1,1}^*, v_{2,1}^*, v_0) - J_{1,1}(v_{1,2}, v_{2,2}, v_0) \geq 0 \ a.s..
$$

Similarly, for any $v_{2,1} \in [0, \tau]$, one has

$$
J_{2,1}(v_{1,1}^*, v_{2,1}^*, v_0) - J_{2,1}(v_{1,1}, v_{2,1}^*, v_0) \geq 0 \ a.s..
$$

Thus the proof is finished.

Next, we capture the optimal state $X_1^*(t)$ at Nash equilibrium point $(v_{1,1}^*, v_{2,1}^*)$. To this end, put

$$
P_{v_{1,1}}(t) = \psi_{1,1}(t) X_1(t) + \psi_{1,2}(t) X_1(t) + \psi_{1,0}(t),
$$

where $\psi_{1,1}$, $\psi_{1,2}$ are deterministic functions, $\psi_{1,0}$ is an $\bar{\mathcal{F}}$-adapted process such that

$$
\frac{d\psi_{1,0}(t)}{dt} = \psi_{1,3}(t) dt + \psi_{1,4}(t) d\mathbb{B}_1.
$$

Moreover, let $\psi_j = \psi_{1, j} + \psi_{2, j}$. Then, by repeating the arguments in Subsection 3.1, we can obtain the following system of ODEs for $(\psi_{1,0}, \psi_{1,1}, \psi_{1,2}, \psi_{2,1}, \psi_{2,2})$:

$$
\begin{aligned}
\psi_{1,1}(t) + (2a_1 + c_1^2) \psi_{1,1}(t) + h_0 \psi_{1,1}(t) + h_1 \psi_{1,2}(t) + h_2 \psi_{1,3}(t) = 0, & \quad t \in [0, \tau], \\
\psi_{1,2}(t) + (2a_1 + c_1^2) \psi_{1,2}(t) + h_0 \psi_{1,2}(t) + h_1 \psi_{1,1}(t) + h_2 \psi_{1,3}(t) = 0, & \quad t \in [0, \tau], \\
\psi_{2,1}(t) + (2a_1 + c_1^2) \psi_{2,1}(t) + h_0 \psi_{2,1}(t) + h_1 \psi_{1,1}(t) + h_2 \psi_{1,3}(t) = 0, & \quad t \in [0, \tau], \\
\psi_{2,2}(t) + (2a_1 + c_1^2) \psi_{2,2}(t) + h_0 \psi_{2,2}(t) + h_1 \psi_{1,1}(t) + h_2 \psi_{1,3}(t) = 0, & \quad t \in [0, \tau], \\
\psi_{1,0}(t) = \psi_{1,0}(0), & \quad t \in [0, \tau], \\
\psi_{2,0}(t) = \psi_{2,0}(0), & \quad t \in [0, \tau].
\end{aligned}
$$

As well as the following system of MF-BSDEs for $((\psi_{1,0}, \psi_{1,1}), (\psi_{2,0}, \psi_{2,1}))$:

$$
\begin{aligned}
\frac{d\psi_{1,0}}{dt} &= -\left( \psi_{1,1} b_0 + \psi_{1,2} b_0 + \psi_{1,1} a_1 + \psi_{1,2} b_1 \right) dt + \psi_{1,3} dB_1, & \quad t \in [0, \tau], \\
\frac{d\psi_{2,0}}{dt} &= -\left( \psi_{2,1} b_0 + \psi_{2,2} b_0 + \psi_{2,1} a_1 + \psi_{2,2} b_1 \right) dt + \psi_{2,3} dB_1, & \quad t \in [0, \tau], \\
\psi_{1,0}(0) = 0, & \quad \psi_{2,0}(0) = 0.
\end{aligned}
$$

**Remark 3.3:** Notice that $((\psi_{1,0}, \psi_{1,1}), (\psi_{2,0}, \psi_{2,1}))$ is only related to $t$ and $\tau$. According to the results in Aksamit and Jeanblanc (2017), we can easily show that $\mathcal{F}_\infty$ and $\tau$ are independent since the immersion condition $(D)$ holds and any Brownian martingale is continuous. Combining Lemma 2.1 in Pham (2010), we know that expression (24) is reasonable.

Similar to the proof of Lemma 3.1, we can show that (25) has a unique solution which is bounded and continuous. Concerning on the solutions to (26), we have the following theorem.

**Theorem 3.4:** For fixed $v_0 \in [0, \tau]$, there exists a unique $\bar{\mathcal{F}}$-adapted and square integrable solution $((\psi_{1,0}, \psi_{1,1}), (\psi_{2,0}, \psi_{2,1}))$ to (26).

**Proof:** By setting

$$
\alpha_1 = \begin{bmatrix} \psi_{1,0} \\ \psi_{2,0} \end{bmatrix}, \quad \beta_1 = \begin{bmatrix} \psi_{1,1} \\ \psi_{2,1} \end{bmatrix}, \quad \delta_1 = \begin{bmatrix} \psi_{1,0} b_0 \\ \psi_{2,0} b_0 \end{bmatrix}, \\
\mu_1 = \begin{bmatrix} a_1 + l_1 \psi_{1,1} \\ l_1 \psi_{2,1} \end{bmatrix},
$$

where $\psi_{1,0}, \psi_{1,1}$ are non-negative, $q_{1,1}$ is positive and $\varphi_{1,1}(\tau), \varphi_{1,2}(\tau) < 0$ a.s., we have
followingsystem:

\[
\begin{align*}
\bar{a}_1 &= \begin{bmatrix} \psi_{1.0} \\ \overline{\psi}_{2.0} \end{bmatrix}, \quad \bar{b}_1 = \begin{bmatrix} \psi_{1.4} \\ \overline{\psi}_{2.4} \end{bmatrix}, \quad \bar{\beta}_1 = \begin{bmatrix} \psi_{1.2}b_0 \\ \overline{\psi}_{2.2}b_0 \end{bmatrix}, \\
\bar{v}_1 &= \begin{bmatrix} \bar{a}_1 + l_1\psi_{1.2} \\ l_1\psi_{2.2} \end{bmatrix} \quad \bar{a}_1 + l_1\psi_{1.2}
\end{align*}
\]

it is easy to see that (26) can be rewritten as

\[
\begin{align*}
\alpha_1 &= - (\mu_1\alpha_1 + v_1\bar{a}_1 + c_1\beta_1) \\
&+ \bar{a}_1 + d\tau + \delta_1v_0 + \bar{\delta}_1v_0 \quad \delta_1 + d\tau + \delta_1v_0 + \bar{\delta}_1v_0, \\
\alpha_1(\tau) &= (0, 0)^T.
\end{align*}
\]

Since \(\mu_1, v_1, c_1, \bar{\delta}_1\) are all bounded, for any \(\tilde{\mathcal{S}}\)-adapted and square integrable process \(\Theta(t)\), it follows from Cauchy’s inequality that

\[
\mathbb{E} \left[ (\Theta(t)^2)^2 \right] \leq 0
\]

and so it is easy to check that the drift term in (27) satisfies the Lipschitz condition. Thus, the existence and uniqueness result of solutions to (26) follows.

To sum up, at the first stage, the optimal state \(X^*_1(t)\) at Nash equilibrium point \((v^*_1, v^*_2)\) is uniquely determined by the following system:

\[
\begin{align*}
\frac{dX^*_1}{dt} &= \begin{bmatrix} (d_1 + l_1\psi_1)^2 + \bar{\alpha}_1 \\
+ l_1\psi_1^* X^*_1 + (l_1, l_1)\alpha_1 + b_0v_0 \end{bmatrix} dt + \bar{a}_1 + \bar{\beta}_1 d\tau, \\
&+ \bar{a}_1 + d\tau + \delta_1v_0 + \bar{\delta}_1v_0 , \\
\alpha_1(n) &= (0, 0)^T.
\end{align*}
\]

3.3 Stackelberg solution of LQ-MF-SSDG with random exit time

In this subsection, we focus on finding the optimal strategy \(v^*_0\) for the leader so that we can obtain the Stackelberg solution by setting \(v_0 = v^*_0\) in (23).

Clearly, it follows from (5), (7), (11) and (28) that the optimal strategy for the leader can be described by the following optimal control problem for MF-SSDE with default.

**Subproblem 3.3:** Find \(v^*_0 \in \mathcal{V}[0, \tau]\) such that

\[
\begin{align*}
J_0(v^*_0, v^*_1, v^*_2) &= \text{ess sup} \int v_0 \in \mathcal{V}[0, \tau] \left\{ \frac{1}{2} \mathbb{E} \left[ \int_0^\tau P_0(t) Y^2(t) + \bar{\beta}_0(t) Y^2(t) \right] \right\}, \\
&+ q_0(t) v_0^2(t) \quad \text{dt} + r_0(\tau) Y^2(\tau) \quad \mathbb{E} \left[ \Theta_0 \right],
\end{align*}
\]

where the state process \(Y(t)\) is governed by

\[
\begin{align*}
\frac{dY}{dt} &= \begin{bmatrix} (d_1 + l_1\psi_1 + \gamma^2 d_0 Y + (\bar{\alpha}_1 + l_1\psi_2 + \gamma^2 d_0 Y) \\
+ (l_1, l_1)\alpha_1 + b_0v_0 \end{bmatrix} dt + \bar{a}_1 + \bar{\beta}_1 d\tau, \\
&+ (l_1, l_1)\alpha_1 + c_1\beta_1 + \bar{\delta}_1v_0 + \bar{\delta}_1v_0,\quad \delta_1 + \beta_1 d\tau, \\
\alpha_1 &= (0, 0)^T,\quad \alpha_1(\tau) = (0, 0)^T,
\end{align*}
\]

For Subproblem 3.3, define the Hamiltonian function of the leader as follows:

\[
H_0 = H_0(t, Y, Y, v_0, \alpha_1, P_0, Q_0, R_0, Z_0)
\]

\[
= -\frac{1}{2} \left[ P_0 Y^2 + \bar{P}_0 Y^2 + q_0 v_0^2 \right]
\]

\[
+ (P_0, (d_1 + l_1\psi_1 + \gamma^2 d_0 Y + (\bar{\alpha}_1 + l_1\psi_2 + \gamma^2 d_0 Y) Y
\]

\[
+ (l_1, l_1)\alpha_1 + b_0v_0)
\]

\[
+ (Q_0, c_1 Y + \bar{\delta}_1) + (R_0, (d_0 Y + \bar{\delta}_0 Y) Y) + \bar{\delta}_0 v_0 + \bar{\delta}_1 v_0,\quad \delta_1 + \beta_1 d\tau, \\
\alpha_1 &= (0, 0)^T.
\end{align*}
\]

\[
\begin{align*}
\frac{dP_0}{dt} &= \begin{bmatrix} P_0 Y \quad (P_0 Y) Y + \bar{\beta}_0 \quad \bar{\beta}_0 Y \quad \bar{\beta}_0 Y \end{bmatrix} dt + \bar{a}_1 + \bar{\beta}_1 d\tau, \\
&+ (l_1, l_1)\alpha_1 + b_0v_0,\quad \delta_1 + \beta_1 d\tau, \\
\alpha_1 &= (0, 0)^T,\quad \alpha_1(\tau) = (0, 0)^T.
\end{align*}
\]

where \((P_0, Q_0, R_0, Z_0)\) is the solution to the following MF-FBSDE:

\[
\begin{align*}
\frac{dP_0}{dt} &= \begin{bmatrix} P_0 Y \quad (P_0 Y) Y + \bar{\beta}_0 \quad \bar{\beta}_0 Y \quad \bar{\beta}_0 Y \end{bmatrix} dt + \bar{a}_1 + \bar{\beta}_1 d\tau, \\
&+ (l_1, l_1)\alpha_1 + b_0v_0,\quad \delta_1 + \beta_1 d\tau, \\
\alpha_1 &= (0, 0)^T,\quad \alpha_1(\tau) = (0, 0)^T.
\end{align*}
\]

Here \(\mu_1^T\) and \(v_1^T\) represent the transposed matrixes of \(\mu_1\) and \(v_1\), respectively. It is easy to show that \(v_0^*\) is an optimal control of Subproblem 3.2 if

\[
0 = \partial H_0 \partial v_0 + \mathbb{E} \left[ \nabla_{\tau_0} H_0 \right] \quad v_0 = v^*_0
\]

\[
= -q_0 v_0^2 + b_0P_0^* + \delta_1 Z_0^* + \bar{\delta}_1 \bar{Z}_0^*,\quad \nabla_{\tau_0} H_0
\]

where \(\nabla_{\tau_0}\) is the Fréchet derivative with respect to \(\tau_0\). It remains to verify that, for

\[
\begin{align*}
v_0^* &= q_0^{-1} (b_0P_0^* + \delta_1 Z_0^* + \bar{\delta}_1 \bar{Z}_0^*),
\end{align*}
\]

the following inequality holds:

\[
J_0(v^*_0, v^*_1, v^*_2) \leq J_0(v^*_0, v^*_1, v^*_2) \quad \forall v_0 \in \mathcal{V}[0, \tau].
\]

Setting \(Y = Y^* - Y, \bar{\alpha}_1 = \alpha_0^* - \alpha_1 \quad \bar{\alpha}_1 = v_0^* - v_0, \quad \text{one has}
\]

\[
J_0(v^*_0, v^*_1, v^*_2) - J_0(v^*_0, v^*_1, v^*_2)
\]

\[
= -\frac{1}{2} \mathbb{E} \left[ \int_0^\tau P_0(t) Y^2(t) + \bar{P}_0(t) Y^2(t) + q_0(t) v_0^2(t) \quad \text{dt}
\]

\[
+ (l_1, l_1)\alpha_1 + \bar{\beta}_1 d\tau, \\
\alpha_1 &= (0, 0)^T,\quad \alpha_1(\tau) = (0, 0)^T.
\end{align*}
\]
Theorem 3.5: The optimal strategy of the leader for Subproblem 3.3 is given by
\[ v^*_0 = q_0^{-1} \left( b_0 P_{v^*_0} + \langle \delta_1, Z_{v^*_0} \rangle + \langle \bar{\alpha}, Z_{v^*_0} \rangle \right). \]

Substituting (29) and (31) into (30) obtains
\[
J_0(v^*_1, v^*_2, v^*_0) - J_0(v_1^*, v_2^*, v_0)
\leq \mathbb{E} \left[ \int_0^T q_0(t) v^*_0(t) \tilde{v}(t) dt + [(l_0, l_1)(t)\tilde{\alpha}_1(t)
- b_0(t)\tilde{v}(t) P_{v^*_0}(t) + \langle \delta_1(t), Z_{v^*_0}(t) \rangle \tilde{v}(t)
- \langle \bar{\alpha}_1(t), (l_1, l_1)^T P_{v^*_0}(t) \rangle dt \right] \mathbb{G}_0
\]
\[
= \mathbb{E} \left[ \int_0^T \left( q_0(t) v^*_0(t) - b_0(t) P_{v^*_0}(t) - \langle \delta_1(t), Z_{v^*_0}(t) \rangle \right) \tilde{v}(t) dt \right] \mathbb{G}_0
\]
\[
= 0,
\]
which is the desired result.

Therefore, summarising the above arguments, we have the following theorem.

**Theorem 3.5:** The optimal strategy of the leader for Subproblem 3.3 is given by
\[ v^*_0 = q_0^{-1} \left( b_0 P_{v^*_0} + \langle \delta_1, Z_{v^*_0} \rangle + \langle \bar{\alpha}, Z_{v^*_0} \rangle \right). \]
\[ Z_1 = \begin{bmatrix} a_1 + l_1 \psi_1 + \gamma d_0 & 0 & q_0^{-1}b_0 \delta_1 T' \\ (0,0) & q_0^{-1}b_1 \delta_1 & v_1' \\ \end{bmatrix}, \]
\[ L_2 = \begin{bmatrix} q_0^{-1}b_2 \delta_1 T' \\ (l_1, l_1) T' & 0 \end{bmatrix}, \]
\[ L_3 = \begin{bmatrix} p_0 & (0,0) \end{bmatrix}, \]
\[ L_3 = \begin{bmatrix} \bar{p}_0 & (0,0) \end{bmatrix}, \]
\[ Z_3 = \begin{bmatrix} d_0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \]
\[ Z_4 = \begin{bmatrix} d_0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \]
\[ G(\tau) = \begin{bmatrix} -r_0(\tau) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \]

It is easy to see that (33) can be rewritten as follows:
\[ \begin{align*}
\left[ \begin{array}{c}
X(t) \\
Y(t) \\
Y(t)
\end{array} \right] &= \left[ \begin{array}{c}
Z_1 Y(t) + L_1 Y(t) + L_2 P(t) \\
+ (L_1 Y(t) + Z_1 Y(t)) dA_t, t \in (0, \tau],
\end{array} \right]
\end{align*} \]
\[ \begin{align*}
\left[ \begin{array}{c}
\mathcal{P}(t) = \mathcal{M}(t) Y(t) + N(t) Y(t)
\end{array} \right]
\end{align*} \]

Then, one can obtain the following system of Riccati equations:
\[ \begin{align*}
\frac{dM}{d\tau} &= \begin{bmatrix} ML_1 + L_1^T M + c_1^T M + \gamma L_4 M L_4 \\
+ L_3 M L_3 - L_3 \\
\end{bmatrix} d\tau, t \in [0, \tau),
\end{align*} \]
\[ \begin{align*}
\frac{dN}{d\tau} &= \begin{bmatrix} N(L_1 + Z_1) + (L_1 + Z_1)^T N \\
+ N L_2 N + (2 c_1 + \tau_1^T) M \\
+ (M \bar{Z}_1 + \bar{Z}_1^T M) + \gamma (L_4 M L_4 \\
+ L_4 M L_4 + L_2 M L_4 - L_3) d\tau, t \in [0, \tau),
\end{bmatrix}
\end{align*} \]
\[ \begin{align*}
\mathcal{M}(\tau) = G(\tau), N(\tau) = 0,
\end{align*} \]

where \( M \) and \( N \) are 3 x 3 matrices. Combining (32), (34) and (35), the feedback solution for the leader can be obtained as follows:
\[ \begin{align*}
v_0 &= q_0^{-1} \left[ \begin{bmatrix} (b_0,0,0) M + (0, \delta_1^T) \\
\end{bmatrix} Y^* \\
+ \begin{bmatrix} (b_0,0,0) N + (0, \delta_1^T) \end{bmatrix} Y^*, \end{align*} \]

where \( Y^* \) is the solution to
\[ \begin{align*}
\frac{dY^*}{d\tau} &= \begin{bmatrix} (L_4 + L_2 M) Y^* + (Z_1 + L_2 N) Y^* \\
\hspace{1cm} + (c_1 Y^* + \tau_1 W^*) dA_t + (L_4 Y^* + \bar{Z}_1 Y^*) dA_t, t \in (0, \tau],
\end{bmatrix}
\end{align*} \]
\[ \begin{align*}
Y^*(0) = (x_0,0,0)^T.
\end{align*} \]

**Remark 3.5**: Combining (24) and (35), we can easily obtain the feedback expression of \((v_0^*, v_1^*, v_2^*)\), which is a solution to the first-stage subproblem. We note that in the first-stage subproblem, only one Poisson jump is allowed for the state of the leader at \( \tau = \tau. \) For some related works on SSDGs with compensated Poisson jumps, we refer the reader to Moon (2021a) and Moon (2021b).

Now, we are able to give the following theorem.

**Theorem 3.7**: When \( \gamma^* (t) = \gamma (t) \) is a deterministic function, the feedback expression of Stackelberg solution \((v_0^*, v_1^*, v_2^*)\) to Problem 1.1 is uniquely given as follows:
\[ \begin{align*}
(v_{1,1}^*, v_{1,2}^*) &= \begin{bmatrix} q_{1,1}^{-1} b_{1,1} (\psi_1 X_1^* + \psi_1 2 X_1^* + \psi_1 1,2) \\
q_{1,2}^{-1} b_{1,2} (\psi_1 X_1^* + \psi_2 2 X_2^*) \\
q_{1,2}^{-1} b_{2,2} (\psi_2 X_2^* + \psi_2 2 X_2^*) \\
\end{bmatrix},
\end{align*} \]
\[ \begin{align*}
(v_{1,2}^*, v_{2,2}^*) &= \begin{bmatrix} q_{1,2}^{-1} b_{1,2} (\psi_1 X_1^* + \psi_1 2 X_1^* + \psi_2 2 X_2^*) \\
q_{1,2}^{-1} b_{2,2} (\psi_2 X_2^* + \psi_2 2 X_2^*) \\
\end{bmatrix},
\end{align*} \]
\[ \begin{align*}
v_0 &= q_0^{-1} \left[ \begin{bmatrix} (b_0,0,0) M + (0, \delta_1^T) \end{bmatrix} Y^* \\
+ \begin{bmatrix} (b_0,0,0) N + (0, \delta_1^T) \end{bmatrix} Y^*, \end{bmatrix}
\end{align*} \]
\[ \begin{align*}
v_1^*(t) = v_{1,1}^*(t) I_{\in [0,\tau]} + v_{1,2}^*(t) I_{t \in (\tau, T]},
\end{align*} \]
\[ \begin{align*}
v_2^*(t) = v_{1,2}^*(t) I_{t \in [0,\tau]} + v_{2,2}^*(t) I_{t \in (\tau, T]},
\end{align*} \]

where \((X_1^*, X_2^*, \alpha_1^*, \gamma^*) = (\psi_{1,1}^*, \psi_{2,2}^*)^T\) is the unique solution to the following system of equations
\[ \begin{align*}
\frac{dX_1}{dt} &= \begin{bmatrix} a_1 + l_1 \psi_1 X_1 + (a_1 + l_1 \psi_2) X_1 \\
+ (l_1, l_1) \alpha_1^* + b_0 v_0 \\
\end{bmatrix} dt + \begin{bmatrix} c_1 X_1 + \tau_1 X_1^* \end{bmatrix} dA_t, t \in (0, \tau],
\end{align*} \]
\[ \begin{align*}
\frac{dX_2}{dt} &= \begin{bmatrix} a_2 + l_2 \psi_1 X_2 + (a_2 + l_2 \psi_2) X_2 \\
+ (l_2, l_2) \alpha_2^* + b_0 v_0 \\
\end{bmatrix} dt + \begin{bmatrix} c_2 X_2 + \tau_2 X_2^* \end{bmatrix} dA_t, t \in (0, \tau].
\end{align*} \]
\[ \begin{align*}
X_1^*(0) = x_0, \alpha_1^*(t) = (0,0)^T,
\end{align*} \]
\[ \begin{align*}
X_2^*(\tau) = X_2^*(\tau), Y^*(0) = (x_0,0,0)^T.
\end{align*} \]

Here \( (\psi_{1,1}, \psi_{1,2}, \psi_{2,1}, \psi_{2,2})\), \( (\psi_{1,1}, \psi_{1,2}, \psi_{2,1}, \psi_{2,2})\) and \( (M, N) \) are unique solutions to (17), (25) and (36), respectively.

**Remark 3.6**: We emphasise that the system of Riccati Equation (36) is not standard, of which the solvability is widely open. However, in some special cases (such as \( d_0 = \bar{d}_0 = b_{1,1} = b_{2,1} = 0 \)), the solvability can be obtained by some standard techniques (see Wang and Zhang (2020) for details).

We end this subsection with the following example which is a special case of Example 1.1.
Example 3.1: Suppose that in Example 1.1, the resource stock $x(t)$ is given by

$$dX(t) = \left[ \frac{1}{2} X(t) - u(t) - v_1(t) - v_2(t) \right] dt, \quad t \in (0, 1],$$

and the objection functions are given by

$$f_0(u, v_1, v_2) = -\frac{1}{2} \int_0^1 e^{-t} u^2(t) dt + e^{-\tau} Y^2(\tau),$$
$$f_1(u, v_1, v_2) = -\frac{1}{2} \int_0^1 e^{-t} v_1^2(t) dt + e^{-\tau} X^2(1),$$
$$f_2(u, v_1, v_2) = -\frac{1}{2} \int_0^1 e^{-t} v_2^2(t) dt + e^{-\tau} X^2(1).$$

Here $Y(t)$ satisfies

$$dY(t) = [Y(t) - u(t) - v_1(t) - v_2(t)] dt,$$
$$-\frac{1}{2} Y(t) d\mathbb{I}_{t \leq \tau}, \quad t \in (0, \tau],$$
$$Y(0) = 1.$$

According to Theorem 3.7, the Stackelberg solution $(u^*, v_1^{**}, v_2^{**})$ can be given by

$$u^*(t) = \begin{bmatrix} (-e^t, 0) M(t) + \left(0, \frac{1}{2t-3} \right) \right] \begin{bmatrix} X_1^{**}(t) \\ Z^*(t) \end{bmatrix}, \quad t \in [0, \tau],$$
$$= \begin{bmatrix} (-e^t, 0) M(\tau) + \left(0, \frac{1}{2\tau-3} \right) \right] \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} X_1^{**}(\tau) \\ Z^*(\tau) \end{bmatrix}, \quad t = \tau.$$

$$v_1^{**}(t) = \frac{1}{3 - 2t} X_1^{**}(t) + \frac{3}{3 - 2t} e^{-t}$$
$$= \int_0^t \frac{e^{-s}}{3 - 2s} u(s) ds, \quad t \in [0, \tau],$$
$$= \frac{1}{3 - 2t} X_1^{**}(t), \quad t \in [\tau, 1],$$

where

$$\begin{bmatrix} X_1^{**}(t) \\ Z^*(t) \end{bmatrix} = \exp \left\{ \int_0^t \left[ \begin{bmatrix} 2s + 1 \\ 4s - 6 \\ -1 \\ 2s + 1 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ -1 \\ 4s - 6 \end{bmatrix} + e^t \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} M(s) \right] ds \right\}, \quad t \in [0, \tau]$$

and

$$X_2^{**}(t) = e^{\frac{t-\tau}{3 - 2t}} \cdot \frac{3 - 2t}{3 - 2t} \cdot X_1^{**}(t), \quad t \in (\tau, 1].$$

Moreover, $M(t)$ is a $2 \times 2$-matrix satisfying the following Riccati equation:

$$0 = \frac{dM}{dt} + M \begin{bmatrix} 2t + 1 & \gamma(t) \\ 4t - 6 & 0 \end{bmatrix}, \quad M = \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix}$$

$$+ \frac{\gamma(t)}{4} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} M + \begin{bmatrix} 0 & 0 \\ 0 & e^{-t} \end{bmatrix} M + \begin{bmatrix} 0 & 0 \\ 0 & e^{-t} \end{bmatrix}, \quad t \in (0, \tau],$$

$$M(\tau) = \begin{bmatrix} -e^{-\tau} \\ 0 \\ 0 \end{bmatrix}.$$
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