Light-curve modelling for mutual transits

András Pál\footnote{Konkoly Observatory of the Hungarian Academy of Sciences, Konkoly These Miklós út 15-17, H-1121 Budapest, Hungary} \footnote{Department of Astronomy, Loránd Eötvös University, Pázmány P. st. 1/A, H-1117 Budapest, Hungary}

1 INTRODUCTION

With the advent of the space missions Corot and Kepler (see Barge et al. 2008; Borucki et al. 2009), and as the result of numerous successful ground-based surveys, nearly 200 transiting extrasolar planets are known to date. Furthermore, the number of candidates awaiting for confirmation of planetary properties exceeds 1000 in magnitude (Borucki et al. 2011). These systems give us a unique perspective on various studies, because most of the planetary and orbital parameters can be obtained without any ambiguity. Transiting planetary companions are also known in multiple stellar systems (Doyle et al. 2011). Moreover, both systems of transiting planets and eclipsing binaries provide substantial information about stars, from which the absolute physical properties can be easily obtained completely independently of other methods; therefore such studies are essential to confirm stellar evolution models.

In the case of multiple transiting planetary systems (e.g. Holman 2010), triple or hierarchical stellar systems or circumbinary planetary systems (Doyle et al. 2011), planetary systems around one component of a binary or systems with planetary companions (exomoons, see e.g. Szabó et al. 2006; Kipping 2009; Simon, Szabó & Szatmáry 2009), there is a chance to observe mutual eclipsing or transiting events when (at least) three of the bodies are aligned along the line of sight. Moreover, if planet and/or stellar formation prefers coplanar orbits, the chance is even higher. Conversely, observing mutual transit events yields additional information about the orbital characteristics of the whole system. Recently, Sato & Asada (2009) and Ragozzine & Holman (2010) analysed these effects and their qualitative influence on extrasolar moons and multiple planetary systems. In this paper we discuss how the photometric measurements are affected due to such mutual transiting or eclipsing events, by giving an algorithm that models the light curves of such phenomena. Our method described here is capable of computing light-curve models for an arbitrary number of eclipsing or transiting bodies and for all well-known limb-darkening models without any restrictions on the projected diameters of the active components.

The structure of this paper is as follows. Section 2 describes the algorithm and the formulae needed to evaluate the fluxes or light-curve points for events with multiple transiting, eclipsing or occulting companions. In Section 3 we briefly discuss the qualitative properties of systems in which mutual transits may occur and demonstrate how information gathered from such mutual events can be exploited in order to constrain orbital alignments in such a transiting planetary system. Finally, Section 4 summarizes the key points and results of this paper.

2 THE LIGHT-CURVE MODEL

In this section we briefly describe the methods used to compute the light-curve models for multiple transiting objects. Recently, Kipping (2011) published an algorithm that is capable of estimating the observed flux when two bodies transit their host star simultaneously. However, that method works only when one of these bodies is very small (i.e. assuming a homogeneous flux density beyond a very small disc). Here we demonstrate an alternative algorithm that is significantly more concise and can be treated as an extension of the approach by Kipping (2011) in several ways. First, the presented method is capable of incorporating more than two transiting bodies. Space-borne missions like Kepler are expected to detect both extrasolar moons (Szabó et al. 2006; Kipping 2009) through different methods (e.g. detecting timing variations or via photometry) and
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2.1 Net of arcs

In principle, the projected stellar, planetary or lunar discs are characterized by the centre coordinates \(x_0, y_0\) and the radius \(r\). An arc on one of these circles is quantified by the additional parameters \(\varphi_{\ell (0)}\) and \(\Delta \varphi_{\ell}\), where \(\varphi_{\ell (0)}\) is the position angle between the reference axis \((x +)\) and the beginning of the arc while \(\Delta \varphi\) is the length of the arc in radians. All of the arcs in this model are oriented in a counter-clockwise (i.e. prograde or positive) direction. In the following, the circles and arcs are indexed by \(k, \ell\), respectively. Obviously, for the \(k\)th circle,

\[
\sum_{\ell} \Delta \varphi_{k,\ell} = 2\pi \tag{1}
\]

and

\[
\varphi_{k,\ell + 1}^{(0)} = \varphi_{k,\ell}^{(0)} + \Delta \varphi_{k,\ell}. \tag{2}
\]

Completely disjoint circles, or circles for which the edge does not intersect others, have only one arc, which is used to represent the circle itself. Namely, \(\{\ell\} = \{1\}\) and \(\Delta \varphi_{k,1} = 2\pi\), while the value of \(\varphi_{k,1}^{(0)}\) can be arbitrary.

The net of arcs is built iteratively. If the new circle is disjoint then only one arc is placed with \(\Delta \varphi_{k,1} = 2\pi\), otherwise the two position angles for the two intersection points are computed using known trigonometric relations and the appropriate arcs are split into two or three smaller ones. After obtaining this set of arcs, the topology is also generated, namely by checking for each arc which circles contain this arc inside them. Let us denote this subset of circles related to the \(\ell\)th arc of the \(k\)th circle by \(C_{k,\ell}\). Note that this set might be empty or might even contain all circle indices with the exception of \(k\). See Fig. 1 for one particular example of five intersecting circles.

![Figure 1. A complex configuration of five circles. The arcs are denoted by \(k{:}\{C_{k,\ell}\}\), where \(k\) and \(k'\) are the indices of the circles and the set \(C_{k,\ell}\) is a list of circle indices the given arc passes through. The thick arcs mark the boundary of the region in circle \#1 that is disjoint from the other circles. It is easy to see that these boundary arcs are labelled by either \(1{:}\{\}\) or \(k{:}\{1\}\) (where \(2 \leq k\)).](image)

2.2 Surface brightness and vector fields

The surface brightness of the host star can be modelled by various limb-darkening laws. Recalling Green’s theorem over \(\mathbb{R}^2\), we can write

\[
\int_S (D \wedge f) \, dA = \int_{\partial S} f \cdot d\mathbf{r}. \tag{3}
\]

Here \(S \subset \mathbb{R}^2\) and \(\partial S\) is the boundary of \(S\). These are two and one-dimensional manifolds on \(\mathbb{R}^2\), on which the standard measures are \(dA\) and \(d\mathbf{r}\), respectively. This equation can also be viewed as a somewhat special case of Stokes’ theorem, known for the curl operator in three-dimensional space. The term \(D \wedge f\) denotes the exterior derivative of \(f\), which can be written in terms of vector components as

\[
D \wedge f = \frac{\partial f_y}{\partial x} - \frac{\partial f_x}{\partial y}. \tag{4}
\]

For our problem discussed in this paper we can apply the above equation (3) as follows. First, we have to find a function \(f \equiv (f_x, f_y)\), the exterior derivative of which is the given stellar surface brightness density. We note that, due to the Young theorem, this function is ambiguous, since we can add an arbitrary scalar gradient to \(f\) for which addition does not change the exterior derivative. Therefore, it is recommended to find a function \(f\) that is antisymmetric in its two arguments, making the computation of the integral on the right-hand side of equation (3) convenient. For instance, a homogeneous surface can be modelled with the function

\[
f_1 = \left( \frac{f_x}{f_y} \right) = \frac{1}{2} \begin{pmatrix} -y \\ +x \end{pmatrix}. \tag{5}
\]

The exterior derivative of this function is unity and there is no preferred direction or position angle in the vector field described by \(f_1\).

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1 See e.g. [http://www.astro.keele.ac.uk/jkt/codes/jktld.html](http://www.astro.keele.ac.uk/jkt/codes/jktld.html).
2.3 Integration on the arcs

Let us consider a set arcs, the union of which is the boundary $a \subseteq \partial S$. The arc $a$ corresponding to the circle centred at $(x_a, y_a)$ with a radius of $r_a$ and parametrized via the position angle $\varphi$ implies the measure

$$dr = \left( -r \sin \varphi, +r \cos \varphi \right) d\varphi.$$  

Thus, the total flux $F$ coming from the area $S$ is then computed as

$$F = \int_{a \subseteq \partial S} f_a(x, y) \cos \varphi - f_a(x, y) \sin \varphi \, r_a \, d\varphi,$$  

where, for a more compact notation, we define $x \equiv x_a + r_a \cos \varphi$ and $y \equiv y_a + r_a \sin \varphi$. Note that the integration limits $\varphi^{(1)}$ and $\varphi^{(2)}$ are not necessarily $\varphi_a^{(0)}$ and $\varphi_a^{(0)} + \Delta \varphi_a$, because the direction of the $\int f(x, y) \, d\varphi$ integral must be positive in all cases. If we use $\varphi^{(1)} = \varphi_a^{(0)}$ and $\varphi^{(2)} = \varphi_a^{(0)} + \Delta \varphi_a$, we have to multiply the integrand by $\pm 1$, depending on whether the right-hand-directed arc $a$ points inside or outside the area $S$ (see Figs 1 and 2 for examples and further explanation).

2.4 Some surface-brightness functions

Now, we compute the integrals behind the sum of equation (7) for various surface-brightness functions. Let us consider a star with projected disc centre located at $(0, 0)$ and a radius of unity. The domain of the functions of interest is the unit circle, i.e. $x^2 + y^2 \leq 1$.

2.4.1 Homogeneous surface

As we have seen earlier (see e.g. equation 5), the vector field $f = (−\frac{1}{2}y, +\frac{1}{2}x)$ has a curl of unity. For a given arc $a \equiv 0$, the integrals behind the sum of equation (7) can be written as

$$F_0 = \int_{\varphi_1}^{\varphi_2} \frac{1}{2} (x_0 + r \cos \varphi) r \cos \varphi + \frac{1}{2} (y_0 + r \sin \varphi) r \sin \varphi$$

$$= \int_{\varphi_1}^{\varphi_2} \frac{1}{2} \varphi \cos \varphi + y_0 \sin \varphi + \frac{1}{2} r^2$$

$$= \frac{1}{2} r (\varphi_2 - \varphi_1) + \frac{1}{2} r x_0 (\sin \varphi_2 - \sin \varphi_1)$$

$$+ \frac{1}{2} r y_0 (\cos \varphi_1 - \cos \varphi_2).$$

Note that the value of $F_0$ does depend on the actual choice for $f$, i.e. it would be different if we added a gradient to the vector field $f$. However, $\sum_a F_a$ in equation (7) would not be altered after such an addition of a gradient field. By summing the results yielded by equation (8) for the arcs $\{a\}$, we can easily reproduce the results in Section 2 and 3 and fig. 5 of Kipping (2011).

2.4.2 Polynomial intensities

Various limb-darkening models contain terms that can be quantified as polynomial functions of the $(x, y)$ centroid coordinates (for instance, the quadratic limb-darkening law). In addition, any analytical limb-darkening profiles can be well-approximated by polynomial functions, therefore it is worth computing the terms in equation (7) for such cases.

Without any restrictions, let us consider the term $x^0 y^q$. Due to the linearity of the integral and summation, if the surface intensity can be described by polynomials then computing the integral in equation (7) for the above terms is sufficient. First, let us define

$$M_{pq} := (x_0 + r \cos \varphi)^p (y_0 + r \sin \varphi)^q$$

and introduce $c = \cos \varphi$ and $s = \sin \varphi$ for simplicity. Thus, in the expansion of equation (7), we should compute expressions like $\int M_{pq} c$ or $\int M_{pq} s$. Here we give a set of recurrence relations with which these indefinite integrals can be evaluated. It is easy to show that

$$\int M_{pq} = x_0 \int M_{p-1,q} + r \int M_{p-1,q} c, \quad \text{or}$$

$$\int M_{pq} = y_0 \int M_{p,q-1} + r \int M_{p,q-1} s.$$  

For the terms $\int M_{pq} c$ and $\int M_{pq} s$, we can write

$$(1 + p + q) \int M_{pq} c = +M_{pq} s + rp \int M_{p-1,q} c + x_0 p \int M_{p-1,q} s + y_0 q \int M_{p,q-1} c,$$  

$$(1 + p + q) \int M_{pq} s = -M_{pq} c + rq \int M_{p,q-1} c + x_0 p \int M_{p,q-1} s + y_0 q \int M_{p,q-1} s.$$  

In order to bootstrap these sets of recurrence relations, we only have to use the following:

$$M_{00} = 1,$$  

$$\int M_{00} = \int 1 = \text{id}.$$  

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Figure 2. Progress of a mutual transit, caused by two relatively large companions. The host star is the largest fixed circle, while the two companions are the smaller circles moving from top to bottom. The arc boundaries of the area(s) on which the surface intensity is integrated are denoted by thick lines. Thin solid lines mark where the integrals in equation (7) are directed in a prograde (counterclockwise) direction while thin dashed lines mark where the integrals are directed in a retrograde (clockwise) direction. Thin lines mark the other arcs, which are irrelevant as regards the integration. Note that even in the topologically complex cases the number of relevant boundary arcs is three or less (with the exception of the sixth frame, where the number of relevant arcs is six).
\[
\int M_{00}\mathrm{c} = +M_{00}\mathrm{x},
\]
\[
\int M_{00}\mathrm{x} = -M_{00}\mathrm{c}.
\]

However, for some cases we might compute these integrals a little more easily. For instance, the surface density \(x^2 + y^2\) can be integrated as the exterior derivative of
\[
f = \left( -\frac{1}{2}x^2 y - \frac{1}{6} y^3, +\frac{1}{6} x^3 + \frac{1}{2} x y^2 \right).
\]
Hence, the primitive integral in equation (7) for this \(f\) will be
\[
F[\varphi] = \frac{r}{48} \left[ 24 \left( x^2 + y^3 \right) r + 12 r^3 \right] \varphi - 4y_0 \left( 6x_0^2 + 2x_0^3 + 9r^2 \right) \cos \varphi + 4x_0 \left( 2x_0^2 + 6y_0^3 + 9r^2 \right) \sin \varphi - 4r^2 \left( y_0 \cos(3\varphi) + x_0 \sin(3\varphi) \right)
\]
\[
- 24x_0 y_0 r \cos(2\varphi) - r^3 \sin(4\varphi)\right].
\]

2.4.3 Linear limb darkening

The well known formula for the linear limb-darkening law gives us a surface flux density that can be written in the form
\[
I(x, y) = 1 - c(1 - \mu),
\]
where \(\mu = \sqrt{1 - x^2 - y^2}\) and \(c\) denotes the linear limb-darkening coefficient.Performing integrals of this function will yield a linear combination of integrating a constant (see Section 2.4.1) and integrating the function \(\sqrt{1 - x^2 - y^2}\). Thus, here we derive a function of which the exterior derivative is \(\sqrt{1 - x^2 - y^2}\) as well as computing the indefinite integral of that is going to appear in equation (7). First of all, we note that this function is defined only in the domain bounded by the circle \(x^2 + y^2 = 1\).

Let us search the function \(f\), the exterior derivative of which is \(\sqrt{1 - x^2 - y^2} = \sqrt{1 - r^2}\), in the form \(f = ( -y f(r^2)/2, +x f(r^2)/2)\). For simplicity, we introduce \(r^2 = x^2 + y^2\). It is easy to show that the expansion of equation (4) for this function yields an ordinary differential equation (ODE) for \(f(\cdot)\) that is
\[
f'(\xi) + \xi f'(\xi) = \sqrt{1 - \xi},
\]
where \(\xi = r^2\). One of the solutions of this ODE is
\[
2 \left[ 1 - (1 - \xi)^{1/2} \right],
\]
This solution behaves analytically at \(\xi = 0\), i.e. at the centre of the stellar disc. Therefore, the function \(f\) can be written as
\[
f = \left( f_1, f_2 \right) = \left( 1 - (1 - x^2 - y^2)^{1/2} / (1 + x) \right).
\]
By substituting this \(x = (f_1, f_2)\) into equation (7), one finds that the primitive integral
\[
F[\varphi] = \frac{1}{3} \int \left[ \frac{1}{R^2 + 2r \rho \cos \varphi} \left( r + \rho \cos \varphi \right) r \, d\varphi \right]
\]
should be computed, if we parametrize the arc (on which \(f\) is integrated) in a similar manner to earlier. The constants are \(\rho^2 = x_0^2 + y_0^2\), \(R^2 = r^2 + \rho^2\), and \(\varphi^*\) is defined to be \(\rho \cos \varphi^*\), equal to \(x_0 \cos \varphi + y_0 \sin \varphi\). The primitive integral in equation (23) can be computed analytically and this computation yields a function that contains elementary functions as well as elliptic integrals. As a first step, let us introduce the constants
\[
q_2 = r^2 + \rho^2 + 2r \rho \cos \varphi^*,
\]
\[
s_2 = (r + \rho)^2,
\]
\[
d_2 = (r - \rho)^2,
\]
\[
Q = \frac{1}{\sqrt{r \rho}},
\]
\[
s_E = 2 \cos(\varphi^*/2) \sqrt{r \rho / 1 - d_2},
\]
\[
k_E = \frac{1}{2} \sqrt{1 - d_2 / r \rho},
\]
\[
n_E = -\frac{1 - d_2}{d_2},
\]
\[
\hat{F} = F(s_E; k_E),
\]
\[
\hat{E} = E(s_E; k_E),
\]
\[
\hat{P} = \Pi(s_E; n_E, k_E).
\]
Here \(F(\cdot; \cdot), E(\cdot; \cdot)\) and \(\Pi(\cdot; \cdot; \cdot)\) denote the incomplete elliptic integrals of the first, second and third kind, respectively.

One may note some similarities between the terms and the equations in Mandel & Agol (2002) or Pál (2008). Although the formulae above include incomplete elliptic integrals, the actual evaluation of these does not require longer computation time than the complete ones. Both types of elliptic integrals are computed via the Carlson symmetric forms (Carlson & Gustafson 1993), for which computation is very fast and robust algorithms are available in the literature (Press et al. 1992; Carlson 1994).

It should also be noted that the evaluation of equation (34) might be done with caution in some cases where the values of the variables or constants defined in equations (24)–(33) introduce singularities in some of the terms. These values correspond to cases in which the arc endpoints are at the edge of the bounding circle at \(x^2 + y^2 = 1\) and/or when the arc intersects the origin (i.e. if \(x = y = 0\)). However, these singularities yield more simple formulae in general.
For instance, in the case of $\rho = 0$ (i.e. when the bounding circle and the arc are concentric), equation (34) becomes simply

$$F(\psi) = \frac{1}{3} \left[ 1 - (1 - r^2)^{3/2} \right] \psi.$$  

(35)

In particular, when the arc is part of the bounding circle itself (which is a frequent case: see e.g. the thick solid lines in Fig. 2), i.e. when $r = 1$ and $\rho = 0$, then $F(\psi)$ is merely $\psi/3$. All in all, these cases should be treated carefully during a practical implementation.

2.4.4 Quadratic limb darkening

The quadratic limb-darkening stellar profile is characterized by the surface flux density $I = 1 - c_1(1 - \mu) - c_2(1 - \mu)^2$. Since $\mu = \sqrt{1 - x^2 - y^2}$, by expanding this equation we obtain a constant term with a value of $1 - c_1 - 2c_2$, a polynomial term $x^2 + y^2$ with a coefficient $+c_2$ and a term that is proportional to $\mu$ and has a coefficient $c_1 + 2c_2$. Hence, the formulae in the previous three subsections (2.4.1, 2.4.2 and 2.4.3) can be applied accordingly to evaluate the final apparent fluxes in the case of a quadratic limb-darkening law.

3 ORBITAL INCLINATIONS

Mutual transits occur when at least two bodies (which can be, for instance, two planets or a planet and its moon) transit the host star simultaneously and their projections also overlap. Due to this overlap, the observed flux coming from the host star is larger than if we considered (naively) the case in which the flux decreases from each body independently. In Fig. 2 we display a series of images that clearly show this effect. In the previous section we deduced the algorithms and mathematical formulae that are needed for the computation of the total observed flux for arbitrary geometry and for various limb-darkening models.

As a demonstration, in Fig. 3 we display two simulated light curves with nearly the same orbital geometry. The planet-to-size ratios for the two companions are $R_1/R_2 = 0.13$ and $R_2/R_1 = 0.10$, while the orbital parameters are the following: $a_1/R_1 = 4.3000, b_1 = 0.35, n_1 = 2.0 \text{d}^{-1}, a_2/R_2 = 9.5952, b_2 = 0.22, n_2 = 0.6 \text{d}^{-1}, \Delta \Omega = 18^\circ$ and both of the planets have a circular orbit. Here $n_1$ denotes the orbital angular frequency: it is $n_1 = 2\pi/P_1$, where $P_1$ is the orbital period. $b_1$ is the impact parameter of the transit, $a_1/R_1$ is the normalized semimajor axis (in units of stellar radii) and $\Omega_1 - \Omega_2$ is the difference in the orbital ascending nodes (note that the reference plane here is the plane of the sky). The mid-transit time of the inner planet is $E_1 = 0.02 \text{d}$ while the outer planet has $E_2 = -0.06 \text{d}$ in the left panel of Fig. 3 and $E_2 = 0.00 \text{d}$ in the right panel. This difference between the mid-transit times yields a mutual transit in the latter case (see the flux excess in Fig. 3 at $t \approx 0.02, \ldots, 0.08 \text{d}$), while there is no overlap between the apparent planetary discs in the former case.

It can easily be seen that the time evolution of the flux excess yielded by the mutual transit\(^2\) has similar qualitative properties to the normal transits: namely it has a mid-time, a peak and a duration. The larger the flux-excess peak, the larger the overlapping area. At first glance, the only quantity for which an observation of a mutual transit yields additional constraints is the difference in $\Delta \Omega$, the difference between the orbital ascending nodes. Qualitatively, the longer the duration of this flux excess, the smaller the absolute value of $\Delta \Omega$.\(^3\) However, the depth and the exact time of the mutual event defines the impact parameters more precisely. This is rather relevant when one or both of the impact parameters are relatively small: the uncertainty in $b^2$ does not depend strongly on the actual value of $b$ (see e.g. Carter et al. 2008; Pál 2008) and thus the uncertainty in $b$ will be rather large for small values of $b$, due to the relation $\Delta b = (2b)^{-1} \Delta(b^2)$. For instance, the analysis of the light curves shown in Fig. 3 yields the following. If no mutual transit occurs (left panel), the best-fitting values for $b_1$ will be $b_1 = 0.323 \pm 0.012$ and $b_2 = 0.215 \pm 0.016$, while if we can observe the mutual transit we obtain $b_1 = 0.351 \pm 0.003$ and $b_2 = 0.223 \pm 0.007$: for the node difference we obtained $\Delta \Omega = 17.4 \pm 0.5^\circ$. For this demonstration of light-curve analysis, we employed an improved Markov Chain Monte–Carlo algorithm as implemented in the 1fit utility (Pál 2009).

Of course, if the difference in the nodes $\Delta \Omega = \Omega_1 - \Omega_2$ is known, we can also compute the mutual inclination $i_m$ of the orbits, using the well-known relation

$$\cos i_m = \cos i_1 \cos i_2 + \sin i_1 \sin i_2 \cos \Delta \Omega.$$  

(36)

\(^2\)Here we treat this ‘flux excess’ relative to the flux level that would exist if we neglected the effect of overlapping and simply calculated the yield of the two components independently.

\(^3\)Imagine two completely retrograde orbits: in this case, the relative speed of the transiting planets is highest and thus the duration of overlapping will be smallest.

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It should also be mentioned that the analysis of mutual transits resolves the ambiguity between values of $\pm \Delta \Omega$. Also, of course, the precise analysis of mutual transits should involve the gravitational interactions between the companions (see e.g. Pál 2010), especially when data are available on a time-scale on which the perturbations are not negligible (in contrast to the demonstration presented here).

4 DISCUSSION

In this paper we investigated the possibilities for computing apparent stellar fluxes in multiple or hierarchical stellar and/or planetary systems during simultaneous transits or occultations. The algorithm presented is capable of deriving these fluxes for an arbitrary number of bodies that are actively parts of the transiting or eclipsing event. This method can then be applied to various analyses of complex astrophysical systems, including multiple transiting planetary systems, hierarchical stellar systems with planetary companions and extrasolar moons.

Currently, the algorithm is implemented in ANSI C, in the form of a plug-in module for the program lfit, and is available from http://szofi.elte.hu/~apal/utils/astro/mttr/. This module features functions named mttrXy(), where $X$ denotes the number of transiting bodies and $y$ can be ‘u’, ‘l’ or ‘q’ for uniform flux density, linear limb darkening and quadratic limb darkening. Evidently these functions have $3X + y$ parameters, where $y$ is 0, 1 or 2 for ‘u’, ‘l’ or ‘q’, respectively. The current version of this module does not compute the parametric derivatives of the functions analytically but emulates them using numerical approximations for the lfit utility. Since both the parametric derivatives of the arcs (with respect to the circle centre coordinates and radii) and the parametric derivatives of equation (7) can be computed analytically, the composition of these two would give us the required derivatives.

As a demonstration, we applied this method to obtain mutual inclinations of orbits in multiple transiting planetary systems. The analysis presented here clearly shows that observing a mutual transit not only yields an accurate value for the ascending node difference but also results in a more precise value for the impact parameters and therefore the orbital inclinations.

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