Equidistribution speed towards the Green current for endomorphisms of $\mathbb{P}^k$

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Abstract

Let $f$ be a non-invertible holomorphic endomorphism of $\mathbb{P}^k$. For a hypersurface $H$ of $\mathbb{P}^k$, generic in the Zariski sense, we give an explicit speed of convergence of $f^{-n}(H)$ towards the dynamical Green $(1,1)$-current of $f$.

Key words: Green current, equidistribution speed.
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1 Introduction

Let $f$ be a holomorphic endomorphism of algebraic degree $d \geq 2$ on the complex projective space $\mathbb{P}^k$. The iterates $f^n = f \circ \cdots \circ f$ define a dynamical system on $\mathbb{P}^k$. It is well-know that, if $\omega$ denotes the normalized Fubini-Study form on $\mathbb{P}^k$ then, the sequence $d^{-n}(f^n)^*(\omega)$ converges to a positive closed current $T$ of bidegree $(1,1)$ called the Green current of $f$ (see e.g. [8]). It is a totally invariant current, whose support is the Julia set of $f$ and that exhibits interesting dynamical properties. In particular, for a generic hypersurface $H$ of degree $s$, the sequence $d^{-n}(f^n)^*[H]$ converges to $sT$ [6]. Here, $[H]$ denotes the current of integration on $H$ and the convergence is in the sense of currents. In fact, if we denote by $T^p$ the self-intersection $T \wedge \cdots \wedge T$, Dinh and Sibony proposed the following conjecture on equidistribution.

Conjecture 1.1. Let $f$ be a holomorphic endomorphism of $\mathbb{P}^k$ of algebraic degree $d \geq 2$ and $T$ its Green current. If $H$ is an analytic set of pure codimension $p$ and of degree $s$ which is generic in the Zariski sense, then the sequence $d^{-pn}(f^n)^*[H]$ converges to $sT^p$ exponentially fast.
The aim of the paper is to prove the conjecture for $p = 1$. It is a direct consequence of the following more precise result on currents. Indeed, we only have to apply the theorem to $S := s^{-1}[H]$ for hypersurfaces $H$ which does not contain any element of $\mathcal{A}_\lambda$.

**Theorem 1.2.** Let $f$, $T$ be as above and let $1 < \lambda < d$. There exists a finite family $\mathcal{A}_\lambda$ of periodic irreducible analytic sets such that if $S$ is a positive closed $(1, 1)$-current of mass 1, whose dynamical potential $u$ verifies $\|u\|_{L^1(X)} \leq C$ for all $X$ in $\mathcal{A}_{\lambda}$, then the sequence $S_n := d^{-n}(f^n)^*(S)$ converge exponentially fast to $T$. More precisely, for every $0 < \beta \leq 2$ and $\phi \in \mathcal{C}^\beta(\mathbb{P}^k)$ we get

$$|\langle S_n - T, \phi \rangle| \leq A\|\phi\|_{\mathcal{C}^\beta} \left( \frac{\lambda}{d} \right)^{n\beta/2},$$

(1.1)

where $A > 0$ depends on the constants $C$ and $\beta$ but is independent of $S$, $\phi$ and $n$.

Here, the space $L^1(X)$ is with respect to the volume form $\omega^{\dim(X)}$ on $X$ and $\mathcal{C}^\beta(\mathbb{P}^k)$ denotes the space of $(k-1, k-1)$-forms whose coefficients are of class $\mathcal{C}^\beta$, equipped with the norm induced by a fixed atlas. The dynamical potential of $S$ is the unique quasi-plurisubharmonic function $u$ such that $S = d\partial\bar{\partial} u + T$ and $\max_{\mathbb{P}^k} u = 0$. Note that $\mathcal{A}_{\lambda}$ will be explicitly constructed.

Theorem 1.2 still holds if we replace $\mathcal{A}_{\lambda}$ by an analytic subset, e.g. a finite set, which intersects all components of $\mathcal{A}_{\lambda}$.

Equidistribution problem without speed was considered in dimension 1 by Brolin [3] for polynomials and by Lyubich [17] and Freire-Lopes-Mané [14] for rational maps. They proved that for every point $a$ in $\mathbb{P}^1$, with maybe two exceptions, the preimages of $a$ by $f^n$ converge towards the equilibrium measure, which is the counterpart of the Green current in dimension 1.

In higher dimension, for $p = k$, simple convergence in Conjecture 1.1 was established by Fornæss-Sibony [12], Briend-Duval [2]. Recently in [9], Dinh and Sibony give exponential speed of convergence, which completes Conjecture 1.1 for $p = k$. The equidistribution of hypersurfaces was proved by Fornæss and Sibony for generic maps [13] and by Favre and Jonsson in dimension 2 [11]. The convergence for general endomorphisms and Zariski generic hypersurfaces was obtained by Dinh and Sibony in [10]. These papers state convergence but without speed. In other codimensions, the problem is much more delicate. However, the conjecture was solved for generic maps in [7], using the theory of super-potentials.

We partially follow the strategy developed in [13], [11] and [10], which is based on pluripotential theory together with volume estimates, i.e. a lower bound to the contraction of volume by $f$. These estimates are available
outside some exceptional sets which are treated using hypothesis on the map 
$f$ or on the current $S$.

The exceptional set $\mathcal{A}$ will be defined in Section 5. It is in general a union of periodic analytic sets possibly singular. In our proof of Theorem 1.2, it is necessary to obtain the convergence of the trace of $S_n$ to these analytic sets. So, we have to prove an analog of Theorem 1.2 where $P_k$ is replaced with an invariant analytic set. The geometry of the analytic set near singularities is the source of important technical difficulties. We will collect in Section 2 and Section 3 several versions of Lojasiewicz’s inequality which will allow us to work with singular analytic sets and also to obtain good estimates on the size of a ball under the action of $f^n$. Such estimates are crucial in order to obtain the convergence outside exceptional sets.

Theorem 1.2 can be reformulated as an $L^1$ estimate of the dynamical potential $u_n$ of $S_n$ (see Theorem 6.1). The problem is equivalent to a size control of the sublevel set $K_n = \{u_n \leq -(\lambda/d)^n\}$. Since $T$ is totally invariant, we get that $u_n = d^{-n}u \circ f^n$ and $f^n(K_n) = \{u \leq -\lambda^n\}$. The above estimate on the size of ball can be applied provide that $K_n$ is not concentrated near the exceptional sets. The last property will be obtained using several generalizations of exponential Hörmander’s estimate for plurisubharmonic functions that will be stated in Section 4. A key point in our approach is that, by reducing the domain of integration, we obtain uniform exponential estimates for non-compact families of quasi-plurisubharmonic functions.

We close this introduction by setting some notations and conventions. The symbols $\lesssim$ and $\gtrsim$ mean inequalities up to constants which only depend on $f$ or on the ambient space. To desingularize an analytic subset of $\mathbb{P}^k$, we always use a finite sequence of blow-ups of $\mathbb{P}^k$. Unless otherwise specified, the distances that we consider are naturally induced by embedding or smooth metrics for compact manifolds. For $K > 0$ and $0 < \alpha \leq 1$, we say that a function $u : X \to \mathbb{C}$ is $(K, \alpha)$-Hölder continuous if for all $x$ and $y$ in $X$, we have $|u(x) - u(y)| \leq K\text{dist}(x, y)^\alpha$. We denote by $B$ the unit ball of $\mathbb{C}^k$ and for $r > 0$, by $B_r$ the ball centered at the origin with radius $r$. In $\mathbb{P}^k$, we denote by $B(x, r)$ the ball of center $x$ and of radius $r$. And, for $X \subset \mathbb{P}^k$ an analytic subset, we denote by $B_X(x, r)$ the connected component of $B(x, r) \cap X$ which contains $x$. We call it the ball of center $x$ and of radius $r$ in $X$. It may have more than one irreducible component. Finally, for a subset $Z \subset X$, we denote by $Z_{X,t}$ or simply $Z_t$, the tubular $t$-neighborhood of $Z$ in $X$, i.e. the union of $B_X(z, t)$ for all $z$ in $Z$. A function on $X$ is call (strongly) holomorphic if it has locally a holomorphic extension to a neighborhood of the ambient space.
2 Lojasiewicz’s inequality and consequences

One of the main technical difficulties of our approach is related to singularities of analytic sets that we will handle using blow-ups along smooth varieties. In this section, we study the behavior of metric properties under blow-ups. It will allow us to establish volume and exponential estimates onto singular analytic sets.

We will frequently use the following Lojasiewicz inequalities. We refer to [1] for further details.

**Theorem 2.1.** Let $U$ be an open subset of $\mathbb{C}^k$ and let $h, g$ be subanalytic functions in $U$. If $h^{-1}(0) \subset g^{-1}(0)$, then for any compact subset $K$ of $U$, there exists a constant $N \geq 1$ such that, for all $z$ in $K$, we have

$$|h(z)| \gtrsim |g(z)|^N.$$ 

In this paper, we only use the notion of subanalytic function in the following case. Let $U$ be an open subset of $\mathbb{C}^k$ and $A \subset U$ be an analytic set. Every compact of $U$ has a neighborhood $V \subset U$ such that the function $x \mapsto \text{dist}(x, A)$ and analytic functions on $U$ are subanalytic on $V$. Moreover, the composition or the sum of two such functions is still subanalytic on $V$. In particular, we have the following property.

**Corollary 2.2.** Let $U$ be an open subset of $\mathbb{C}^k$ and let $A, B$ be analytic subsets of $U$. Then for any compact subset $K$ of $U$, there exists a constant $N \geq 1$ such that, for all $z$ in $K$, we have

$$\text{dist}(z, A) + \text{dist}(z, B) \gtrsim \text{dist}(z, A \cap B)^N.$$ 

We briefly recall the construction of blow-up that we will use later. If $U$ is an open subset of $\mathbb{C}^k$ which contains 0, the blow-up $\hat{U}$ of $U$ at 0 is the submanifold of $U \times \mathbb{P}^{k-1}$ defined by the equations $z_i w_j = z_j w_i$ for $1 \leq i, j \leq k$ where $(z_1, \ldots, z_k)$ are the coordinates of $\mathbb{C}^k$ and $[w_1 : \cdots : w_k]$ are the homogeneous coordinates of $\mathbb{P}^{k-1}$. The sets $w_i \neq 0$ define local charts on $\hat{U}$ where the canonical projection $\pi : \hat{U} \to U$, if we set for simplicity $i = 1$ and $w_1 = 1$, is given by

$$\pi(z_1, w_2, \ldots, w_k) = (z_1, z_1 w_2, \ldots, z_1 w_k).$$ 

If $V \subset \mathbb{C}^p$ is an open subset, the blow-up of $U \times V$ along $\{0\} \times V$ is defined by $\hat{U} \times V$. This is the local model of a blow-up.

Finally, if $X$ is a complex manifold, the blow-up $\hat{X}$ of $X$ along a submanifold $Y$ is obtained by sticking copies of the above model and by using
suitable atlas of $X$. The natural projection $\pi : \hat{X} \to X$ defines a biholomorphism between $\hat{X} \setminus \hat{Y}$ and $X \setminus Y$ where the set $\hat{Y} := \pi^{-1}(Y)$ is called the exceptional hypersurface. If $A$ is an analytic subset of $X$ not contained in $Y$, the strict transform of $A$ is defined as the closure of $\pi^{-1}(A \setminus Y)$.

We have the following elementary lemma.

**Lemma 2.3.** Let $\pi : \hat{U} \times V \to U \times V$ be as above and $\hat{Y}$ denote the exceptional hypersurface in $\hat{U} \times V$. Assume that $U \times V$ is bounded in $\mathbb{C}^k \times \mathbb{C}^p$. Then for all $\hat{z}, \hat{z}' \in \hat{U} \times V$, we have

$$\text{dist}(z, z') \geq \text{dist}(\hat{z}, \hat{z}')(\text{dist}(\hat{z}, \hat{Y}) + \text{dist}(\hat{z}', \hat{Y})), $$

where $z = \pi(\hat{z})$ and $z' = \pi(\hat{z}')$.

**Proof.** Since $\pi$ leaves invariant the second coordinate, considering the maximum norm on $U \times V$, the general setting is reduced to the case of blow-up of a point. Hence we can take $V = \{0\}$. The lemma is obvious if $z$ or $z'$ is equal to 0. Since $\pi$ is a biholomorphism outside $\hat{Y}$, we can assume that $0 < ||z||, ||z'|| < 1$. Moreover, up to an isometry of $\mathbb{C}^k$, we can assume that $\max |z_i| = |z_1|$ and $\max |z_i'| = |z_1'|$. Indeed, we can send $z$ and $z'$ into the plane generated by the first two coordinates and then use the rotation group of this plane. Therefore, in the chart $w_1 = 1$, we have $\hat{z} = (z_1, w_2, \ldots, w_k)$, $\hat{z}' = (z_1', w_2', \ldots, w_k')$ with $|w_i|, |w_i'| \leq 1$.

By triangle inequality, we have

$$|z_1 w_i - z_1' w_i'| \geq |z_1 w_i - z_1 w_i'| - |z_1 w_i - z_1' w_i'| \geq |z_1||w_i - w_i'| - |z_1 - z_1'|.$$

Hence, by symmetry in $z_1$ and $z_1'$ we get

$$2|z_1 w_i - z_1' w_i'| \geq (|z_1| + |z_1'|)|w_i - w_i'| - 2|z_1 - z_1'|. $$

Therefore, there is a constant $a > 0$ independent of $z$ and $z'$ such that

$$|z_1 - z_1'| + \sum_{i=2}^{k} |z_1 w_i - z_1' w_i'| \geq a(|z_1| + |z_1'|)(|z_1 - z_1'| + \sum_{i=2}^{k} |w_i - w_i'|). $$

The left-hand side corresponds to $\text{dist}(z, z')$. We also have that $\text{dist}(\hat{z}, \hat{z}') \simeq |z_1 - z_1'| + \sum_{i=2}^{k} |w_i - w_i'|$ and $\text{dist}(\hat{z}, \hat{Y}) + \text{dist}(\hat{z}', \hat{Y}) \simeq |z_1| + |z_1'|$. The result follows. \qed

A similar result holds for analytic sets. More precisely, consider an irreducible analytic subset $X$ of $\mathbb{P}^k$ of dimension $l$ and a smooth variety $Y$ contained in $X$. Let $\overline{\pi} : \overline{\mathbb{P}}^k \to \mathbb{P}^k$ be the blow-up along $Y$ and $\pi$ the restriction of $\overline{\pi}$ to the strict transform $\hat{X}$ of $X$. Denote by $\overline{Y}$ and $\hat{Y}$ the exceptional hypersurfaces in $\overline{\mathbb{P}}^k$ and in $\hat{X}$ respectively.
Lemma 2.4. There exists $N \geq 1$ such that for all $\hat{z}$ and $\hat{z}'$ in $\hat{X}$

$$\text{dist}(z, z') \geq \text{dist}(\hat{z}, \hat{z}') (\text{dist}(\hat{z}, \hat{Y}) + \text{dist}(\hat{z}', \hat{Y}))^N,$$

Proof. The previous lemma gives the inequality with $N = 1$ if we substitute $\hat{Y}$ by $\hat{Y}$. By Corollary 2.2 applied to $A = \hat{X}$ and $B = \hat{Y}$, there exists $N \geq 1$ such that $\text{dist}(\hat{x}, \hat{Y}) \geq \text{dist}(\hat{x}, \hat{Y})^N$ for all $\hat{x}$ in $\hat{X}$. The result follows.

Here is the first estimate on contraction for blow-ups.

Lemma 2.5. There exists a constant $N \geq 1$ such that for all $0 < t \leq 1/2$, if $\text{dist}(\hat{x}, \hat{Y}) > t$ and $r < t/2$, then $\pi(B_{\hat{X}}(\hat{x}, r))$ contains $B_X(\pi(\hat{x}), t^N r)$. Moreover, if $N$ is large enough then the image by $\pi$ of a ball of radius $0 < r \leq 1/2$ contains a ball of radius $r^N$ in $X$.

Proof. Let $\hat{y}$ be a point in $\hat{X}$ such that $\text{dist}(\hat{x}, \hat{y}) = r$ and set $x = \pi(\hat{x})$, $y = \pi(\hat{y})$. The assumption on $r$ gives that $\text{dist}(\hat{y}, \hat{Y}) > t/2$. Therefore, we deduce from Lemma 2.4 that

$$\text{dist}(x, y) \geq rt^N.$$ 

The first assertion follows since $t \leq 1/2$ and $\pi$ is a biholomorphism outside $\hat{Y}$.

For a general ball $B$ of radius $r$ in $\hat{X}$, we can reduce the ball in order to avoid $\hat{Y}$ and then apply the first statement. More precisely, as $\text{dim}(\hat{Y}) \leq l-1$ there is a constant $c > 0$ such that for all $\rho > 0$, $\hat{Y}$ is cover by $c\rho^{-2(l-1)}$ balls of radius $\rho$. On the other hand, by a theorem of Lelong [10, 11], the volume of a ball of radius $\rho$ in $\hat{X}$ varies between $c'\rho^2 \rho^{2l}$ and $c'\rho^2$ for some $c' > 0$. Hence, the volume of $\hat{Y}_\rho$ is of order $\rho^2$. Take $\rho = c''r^{l}$ with $c'' > 0$ small enough. By counting the volume, we see that $B$ is not contained in $\hat{Y}_\rho$. Therefore, $B \setminus \hat{Y}$ contains a ball of radius $\rho/3$. We obtain the result using the first assertion.

In the same spirit, we have the following lemma.

Lemma 2.6. Let $\hat{Z}$ be a compact manifold, $Z$ be an irreducible analytic subset of $\mathbb{P}^k$ and $\pi : \hat{Z} \to Z$ be a surjective holomorphic map. Let $A$ be an irreducible analytic subset of $Z$ and define $\hat{A} := \pi^{-1}(A)$. There exists $N \geq 1$ such that $A_{tN}$ is included in $\pi(\hat{A}_1)$ for all $t > 0$ small enough. Moreover, if $\hat{A}$ is the union on two analytic sets $\hat{A}_1, \hat{A}_2$ such that $A_2 := \pi(\hat{A}_2)$ is strictly contained in $A$ then $\pi(\hat{A}_{1,t})$ contains $A_{tN} \setminus A_{2,t}$. 

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Therefore, since \( \pi_t > 0 \) that \( 0 < c > N \) Theorem 2.1 to these functions which implies the existence of

**Proof.** Since \( \hat{A} = \hat{A}_1 \cup \hat{A}_2 \) be as above. Since \( \pi \) is holomorphic, there exists \( c > 0 \) such that \( \pi(\hat{A}_2,t) \subset A_{2,ct} \). Therefore, \( \pi(\hat{A}_1,t) \) contains \( A_{t,N} \backslash A_{2,ct} \) for \( t > 0 \) sufficiently small.

In the sequel, we will constantly use desingularization of analytic sets. The following lemma allows us to conserve integral estimates.

**Lemma 2.7.** Let \( Z \) and \( \hat{Z} \) be irreducible analytic subsets of Kähler manifolds. Let \( \pi : \hat{Z} \rightarrow Z \) be a surjective proper holomorphic map. Then, for every compact \( \hat{L} \) of \( \hat{Z} \) there exists \( q \geq 1 \) such that if \( v \) is in \( L^q_{\text{loc}}(Z) \) then \( \hat{v} := v \circ \pi \) is in \( L^1(\hat{L}) \). Moreover, there exists \( c > 0 \), depending on \( \hat{L} \), such that

\[
\|\hat{v}\|_{L^1(\hat{L})} \leq c\|v\|_{L^q(\pi(\hat{L}))}.
\]

**Proof.** Using a desingularization, we can assume that \( \hat{Z} \) is a smooth Kähler manifold with a Kähler form \( \hat{\omega} \). Denote by \( \omega, n, m \) a Kähler form on \( Z \) and the dimensions of \( Z \) and \( \hat{Z} \) respectively. Generic fibers of \( \pi \) are compact of dimension \( m - n \) and form a continuous family. It follows that the integral of \( \hat{\omega}^{m-n} \) on that fibers is a constant.

Consider \( \hat{\lambda} = \pi^*(\omega^n) \wedge \hat{\omega}^{m-n} \) on \( \hat{Z} \). The last observation implies that \( \pi_*(\hat{\lambda}) = \omega^n \) up to a constant. Therefore, if \( v \) is in \( L^q_{\text{loc}}(Z) \) then \( \hat{v} \) is in \( L^q_{\text{loc}}(\hat{Z}, \hat{\lambda}) \). Moreover, we can write \( \hat{\lambda} = h\hat{\omega}^m \) where \( h \) is a positive function. If there exists \( \tau > 0 \) such that \( h^{-\tau} \) is integrable on \( \hat{L} \) with respect to \( \hat{\omega}^m \), we obtain for \( p = 1 + \tau \) and \( q \) its conjugate that

\[
\int_{\hat{L}} |\hat{v}|\hat{\omega}^m = \int_{\hat{L}} |\hat{v}|h^{-1}\hat{\lambda} \leq \left( \int_{\hat{L}} |\hat{v}|^q\hat{\lambda} \right)^{1/q} \left( \int_{\hat{L}} h^{-p}\hat{\lambda} \right)^{1/p} \\
\leq \left( \int_{\pi(\hat{L})} |v|^q\omega^n \right)^{1/q} \left( \int_{\hat{L}} h^{-\tau}\hat{\omega}^m \right)^{1/p} \\
\leq \|v\|_{L^q(\pi(\hat{L}))}.
\]

It remains to show the existence of \( \tau \). The set \( \{h = 0\} \) is contained in the complex analytic set \( A \) where the rank of \( \pi \) is not maximal. More precisely, if \( \pi \) has maximal rank at \( z \), then we can linearize \( \pi \) in a neighborhood of \( z \). Therefore, \( \hat{\lambda} \) and \( \tilde{\omega}^m \) are comparable in that neighborhood.
Since \( \hat{L} \) is compact, the problem is local. Let \( z_0 \) in \( \hat{L} \). We can find a small chart \( U \) at \( z_0 \) and a holomorphic function \( \phi \) on \( U \) such \( A \cap U \) is contained in \( \{ \phi = 0 \} \). We can also replace \( \hat{\omega} \) and \( \omega \) by standard Euclidean forms on \( U \) and on a neighborhood of \( \pi(U) \). So, we can assume that \( h \) is analytic. By Lojasiewicz inequality, for every compact \( K \) of \( U \), there exists \( N \geq 1 \) such that \( h(z) \gtrsim |\phi(z)|^N \) for all \( z \) in \( K \). On the other hand, exponential estimate (cf. \[15\] and Section 4) applied to the plurisubharmonic function \( \text{Lojasiewicz inequality} \), we have reduced on the product \( \hat{\omega} \) for some \( \alpha > 0 \), \( \phi \) applied to \( \phi(z) \). Therefore, \( h^{-\alpha/N} \) belongs to \( L^1(K, \hat{\omega}^m) \). We obtain the desired property near \( z_0 \) by taking \( \tau \leq \alpha/N. \) \qed

Finally, the following results establish a relation between the regularity of functions on an analytic set and that of their lifts to a desingularization.

**Proposition 2.8.** Let \( Z, \hat{Z}, \pi \) be as in Lemma 2.7 and \( v \) be a function on \( Z \). Assume that the lift of \( v \) to \( \hat{Z} \) is \((K, \alpha)\)-Hölder continuous. Then, for every compact \( L \) of \( Z \), there exist constants \( 0 < \alpha' \leq 1 \) and \( a > 0 \), independent of \( v \), such that \( v \) is \((aK, \alpha')\)-Hölder continuous on \( L \).

**Proof.** Let \( \Delta \) be the diagonal of \( Z \times Z \). We still denote by \( \pi \) the map induced on the product \( \hat{Z} \times \hat{Z} \) and we set \( \hat{\Delta} = \pi^{-1}(\Delta) \). As in Lemma 2.6 by Lojasiewicz inequality, we have

\[
\text{dist}(\hat{\alpha}, \hat{\Delta})^M \lesssim \text{dist}(\pi(\hat{\alpha}), \Delta),
\]

if \( \pi(\hat{\alpha}) \in L \times L \). Therefore, if we set \( \hat{\alpha} = (\hat{x}, \hat{x}') \) and \( \pi(\hat{\alpha}) = (x, x') \), then we can rewrite this inequality as

\[
(\text{dist}(\hat{x}, \hat{x}) + \text{dist}(\hat{x}', \hat{x}'))^M \lesssim \text{dist}(x, x'),
\]

(2.1)

for some \( \hat{x}, \hat{x}' \in \hat{Z} \) such that \( \pi(\hat{x}) = \pi(\hat{x}') \).

Now, let \( v \) be as in the proposition and \( \hat{v} = v \circ \pi \) denote its lift. Taking the same notation as above we have

\[
|v(x) - v(x')| = |\hat{v}(\hat{x}) - \hat{v}(\hat{x}')| \leq |\hat{v}(\hat{x}) - \hat{v}(\hat{x})| + |\hat{v}(\hat{x}') - \hat{v}(\hat{x}')|,
\]

since \( \pi(\hat{x}) = \pi(\hat{x}') \) which implies that \( \hat{v}(\hat{x}) = \hat{v}(\hat{x}') \). Therefore, the assumption on \( \hat{v} \) implies that

\[
|v(x) - v(x')| \leq K(\text{dist}(\hat{x}, \hat{x})^\alpha + \text{dist}(\hat{x}', \hat{x}')^\alpha),
\]

and finally (2.1) gives

\[
|v(x) - v(x')| \leq aK\text{dist}(x, x')^\alpha/M,
\]

where \( a > 0 \) depends only on \( \alpha, L \) and \( \pi. \) \qed
Corollary 2.9. For every compact $L$ of $Z$, there exists $0 < \alpha \leq 1$ such that every continuous weakly holomorphic function on $Z$ is $\alpha$-Hölder continuous on $L$. Moreover, every uniformly bounded family of such functions is uniformly $\alpha$-Hölder continuous on $L$.

Proof. Recall that a continuous function on $Z$ is weakly holomorphic if it is holomorphic on the regular part of $Z$. The result is known if $Z$ is smooth with $\alpha = 1$. Therefore, in general, it is enough to apply Proposition 2.8 to a desingularization of $Z$.

In Proposition 3.4, we will need a similar result in a local setting but with a uniform control of the constants. It is the aim of the two following results.

Proposition 2.10. Let $Z, \hat{Z}, \pi$ and $L$ be as in Proposition 2.8. Let $v$ be a function defined on a ball $B_Z(y, r) \subset L$ with $0 < r \leq 1/2$. Assume that $\hat{v} = v \circ \pi$ is $(K, \alpha)$-Hölder continuous. Then, there exist constants $0 < \alpha' \leq 1, a > 0$ and $N \geq 1$, independent of $v, y$ and $r$ such that $v$ is $(aK, \alpha')$-Hölder continuous on $B_Z(y, r^N)$.

Proof. The proof is the same as that of Proposition 2.8 except that we have to check that if $x$ and $x'$ are in $B_Z(y, r^N)$ with $N \geq 1$ large enough, then $\hat{v}$ is well-defined on the points $\hat{x}$ and $\hat{x}'$ defined in (2.1). Since $\pi$ is holomorphic, we have $\text{dist}(x, \pi(\hat{x})) \lesssim \text{dist}(\hat{x}, \hat{z})$. Therefore, by (2.1) we have

$$\text{dist}(x, \pi(\hat{x})) \lesssim \text{dist}(\hat{x}, \hat{z}) + \text{dist}(\hat{x}', \hat{z}') \lesssim \text{dist}(x, x')^{1/M}.$$ 

Hence, if $N$ is large enough, then $\hat{z}$ and $\hat{z}'$ belong to $\pi^{-1}(B_Z(y, r))$. Then, the result follows as in Proposition 2.8.

Corollary 2.11. For every compact $L$ of $Z$, there are constants $0 < \alpha \leq 1, K > 0$ and $N \geq 1$ such that if $v$ is a continuous weakly holomorphic function on $B_Z(y, r) \subset L$ with $|v| \leq 1$ then $v$ is $(K/r, \alpha)$-Hölder continuous on $B_Z(y, r^N)$.

Proof. Let $\pi : \hat{Z} \to Z$ be a desingularization. Let $\hat{z}$ be in $\pi^{-1}(B_Z(y, r/2))$. Since $\pi$ is holomorphic, there is $a > 0$ such that $B_{\hat{Z}}(\hat{z}, r/a)$ is contained in $\pi^{-1}(B_Z(y, r))$. Therefore, by Cauchy’s inequality, $\hat{v}$ is $a/r$-Lipschitz on $\pi^{-1}(B_Z(y, r/2))$. Hence, the result follow from Proposition 2.10.
3 Volume estimate for endomorphisms

The multiplicities of an endomorphism \( f \) are strongly related to volume estimates which were used successfully to solve equidistribution problems. In what follows, we generalize Lojasiewicz type inequalities obtained in [13], [6] and [9] to analytic sets, possibly singular. The aim is to control the size of a ball under iterations of \( f \) in an invariant analytic set. Singularities, in particular the points where the analytic sets are not locally irreducible, lead to technical difficulties.

In this section, \( X \) always denotes an irreducible analytic set of a smooth manifold. In order to avoid some problems related to the local connectedness of analytic sets, instead of the distance induced by an embedding of \( X \), we consider the distance \( \rho \) defined by paths in \( X \). Namely, if \( x, y \in X \) then \( \rho(x, y) \) is the length of the shortest path in \( X \) between \( x \) and \( y \). These two distances on \( X \) are related by the following result (see e.g. [1]).

**Theorem 3.1.** Let \( K \) be a compact subset of \( X \). There exists a constant \( r > 0 \) such that for all \( x, y \in K \) we have

\[
\text{dist}(x, y) \leq \rho(x, y) \lesssim \text{dist}(x, y)^r.
\]

The first step to state volume estimate is the following result.

**Proposition 3.2.** Let \( \Gamma \subset \mathbb{B} \times \mathbb{B} \) and \( X \subset \mathbb{B} \) be two analytic subsets with \( X \) locally irreducible and such that the first projection \( \pi: \mathbb{B} \times \mathbb{B} \to \mathbb{B} \) defines a ramified covering of degree \( m \) from \( \Gamma \) to \( X \). There exist constants \( a > 0 \) and \( b \geq 1 \) such that if \( x, y \in X \cap \mathbb{B}_{1/2} \) then we can write

\[
\pi^{-1}(x) \cap \Gamma = \{x^1, \ldots, x^m\} \quad \text{and} \quad \pi^{-1}(y) \cap \Gamma = \{y^1, \ldots, y^m\},
\]

with \( \text{dist}(x, y) \geq a \text{dist}(x^i, y^j)^b m \). Moreover, \( a \) increases with \( m \) but is independent of \( \Gamma \) and \( b \) depends only on \( X \).

**Proof.** By Theorem 3.1 to establish the proposition, we can replace \( \text{dist}(x, y) \) by \( \rho(x, y) \) on \( X \). For \( w \in X_{\text{Reg}} \), we can define the \( j \)-th Weierstrass polynomial, \( k + 1 \leq j \leq 2k \), on \( t \in \mathbb{C} \)

\[
P_j(t, w) = \prod_{z \in \pi^{-1}(w) \cap \Gamma} (t - z_j) = \sum_{l=0}^{m} a_{j,l}(w)t^l,
\]

where \( z = ((z_1, \ldots, z_k), (z_{k+1}, \ldots, z_{2k})) \in \mathbb{B} \times \mathbb{B} \). The coefficients \( a_{j,l} \) are holomorphic on \( X_{\text{Reg}} \) and uniformly bounded by \( m! \) since \( \Gamma \subset \mathbb{B} \times \mathbb{B} \). As \( X \) is locally irreducible, they can be extended continuously to \( X \) (see e.g. [1]).
gives a continuous extension of the polynomials $P_j$ to $X$ with $P_j(z_j, \pi(z)) = 0$ if $z$ is in $\Gamma$. Moreover, by Corollary 2.9 there exists $\alpha > 0$ such that the coefficients $a_{j,l}$ are uniformly $\alpha$-Hölder continuous on $X \cap \mathbb{B}_{3/4}$ with respect to $\rho$.

We claim that there is a constant $a > 0$ such that if $x, y \in X \cap \mathbb{B}_{1/2}$ and $x' \in \mathbb{B}$ with $\tilde{x} := (x, x') \in \Gamma$ then there is $\tilde{y} \in \pi^{-1}(y) \cap \Gamma$ with

$$\rho(x, y) \geq \text{dist}(\tilde{x}, \tilde{y})^{bm},$$

where $b = 1/\alpha$. From this, the result follows exactly as in the end of the proof of [6, Lemma 4.3].

It remains to prove the claim. Fix $c > 0$ large enough and let $r = \rho(x, y)$. We can assume that $r$ is non-zero and sufficiently small, otherwise the result is obvious. Since the covering has degree $m$, we can find an integer $2 \leq l \leq 4km$ such that for all $k + 1 \leq j \leq 2k$, no root of the polynomial $P_j(t, x)$ satisfies $c(l-1)r^{1/bm} \leq |\tilde{x}_j - t| \leq c(l+1)r^{1/bm}$.

It gives a security ring over $x$ which does not intersect $\Gamma$. Using the regularity of $P_j$, it can be extend to a neighborhood of $x$. More precisely, for $\theta \in \mathbb{R}$ we define $\xi_j = \tilde{x}_j + cle^{i\theta}r^{1/bm}$ and

$$G_{j,c,\theta}(w) = c^{-m+1}P_j(\xi_j, w).$$

The choice of $l$ implies that

$$|G_{j,c,\theta}(x)| = c^{-m+1}|P_j(\xi_j, x)| \geq c^{-m+1} \prod_{z \in \pi^{-1}(x)} |\xi_j - z_j| \geq cr^\alpha.$$

Moreover, we deduce from the properties on the coefficients $a_{j,l}$ that the functions $G_{j,c,\theta}$ are uniformly $\alpha$-Hölder continuous on $X \cap \mathbb{B}_{3/4}$ with respect to $(j, c, \theta)$. Hence, if $c$ is large enough, they do not vanish on $B := \{z \in X \mid \rho(x, z) < 2r\}$ which contains $y$.

It implies that $P_j(t, w) \neq 0$ if $w$ is in $B$ and $|t - \tilde{x}_j| = lcr^{1/bm}$. Therefore, if we denote by $\Sigma$ the boundary of the polydisc of center $x'$ and of radius $lcr^{1/bm}$, we have $\Gamma \cap (B \times \Sigma) = \emptyset$. Hence, since $B$ is connected and contains $y$, by continuity there is a point $\tilde{y}$ in $\pi^{-1}(y) \cap \Gamma$ with $|\tilde{x}_j - \tilde{y}_j| \leq lcr^{1/bm}$. This completes the proof of the claim.

**Remark 3.3.** Our proof shows that if we assume that $m \leq m_0$ for a fixed $m_0$, then $a$ depends only on the Hölder constants $(K, \alpha)$ associated with $a_{j,l}$. Furthermore, if $\alpha$ is fixed then $a$ is proportional to $K$. Note that without assumption on the constants $a$ and $b$, Proposition 3.2 can be deduced from Theorem 2.7.
The control of multiplicities of the covering gives the following more precise result.

**Proposition 3.4.** Let $X$, $\Gamma$ and $\pi$ be as above. Let $Z \subset X$ be a proper analytic subset. Assume that the multiplicity at each point of $\pi^{-1}(x)$ is at most equal to $s$ if $x \in X \setminus Z$. Then, there exist constants $a > 0$, $b \geq 1$ and $N \geq 1$ such that for all $0 < t \leq 1/2$ and $x, y \in X \cap \mathbb{B}_{1/2}$ with $\text{dist}(x, Z) > t$ and $\text{dist}(y, Z) > t$, we can write

$$\pi^{-1}(x) \cap \Gamma = \{x^1, \ldots, x^m\} \text{ and } \pi^{-1}(y) \cap \Gamma = \{y^1, \ldots, y^m\},$$

with $\text{dist}(x, y) \geq at^N \text{dist}(x^i, y^i)^b$. 

**Proof.** Let $t > 0$ and $x \in X \cap \mathbb{B}_{1/2}$ with $\text{dist}(x, Z) > t$. We want to find a neighborhood $B \subset X$ of $x$ such that each component of $\Gamma \cap \pi^{-1}(B)$ defines a ramified covering of degree at most equal to $s$ over $B$.

We construct an analytic set $Y$ associate to the multiplicities of $\Gamma$. Namely, first define $Y' \subset \Gamma^{s+1}$ by $\{z \in \Gamma^{s+1} \mid \pi \circ \tau_i(z) = \pi \circ \tau_j(z), 1 \leq i, j \leq s + 1\}$, where $\tau_i, 1 \leq i \leq s + 1$, are the canonical projections of $\Gamma^{s+1}$ onto $\Gamma$. For $i \neq j$ we set $A_{i,j} = \{z \in Y' \mid \tau_i(z) = \tau_j(z)\}$. Then, $Y$ is defined by $Y' \setminus \bigcup_{i \neq j} A_{i,j}$. The map $\pi_1 = \pi \circ \tau_1 : Y \to X$ defines a ramified covering. If $x$ is generic in $X$ then a point $z \in Y$ over $x \in X$ represents a family of $(s + 1)$ distinct points in $\pi^{-1}(x) \cap \Gamma$.

If $\pi'$ denotes the second projection of $\mathbb{B} \times \mathbb{B}$ onto $\mathbb{B}$, consider the map $h : Y \to \mathbb{C}^{(s+1)^2}$ define by

$$h(z) = (\pi' \circ \tau_i(z) - \pi' \circ \tau_j(z))_{1 \leq i, j \leq s + 1}.$$ 

By construction, $h(z) = 0$ means precisely that there is a point in $\Gamma$ with multiplicity greater than $s$ over $\pi_1(z)$. It implies that $\pi_1(h^{-1}(0)) \subset Z$. Hence, Theorem 2.1 implies there is a constant $M > 0$ such that if $\pi(z) \in X \cap \mathbb{B}_{1/2}$ then

$$\|h(z)\| \gtrsim \text{dist}(z, h^{-1}(0))^d \gtrsim \text{dist}(\pi_1(z), Z)^M. \quad (3.1)$$

Let $a, b > 0$ be the constants in Proposition 3.2. As in the proof of that proposition, we use the distance function $\rho$ on $X$. Fix $\gamma > 0$ small enough and set $B = \{z \in X \mid \rho(x, z) < \gamma t^{Mb_m}\}$. For $\tilde{x} = (x, x')$ in $\Gamma$, we can choose $2 \leq l \leq 8m$ such that $\pi^{-1}(x) \cap \Gamma$ do not intersect the ring

$$r(l - 2) \leq \|\tilde{x} - w\| \leq r(l + 2),$$

where $r = (a^{-1} \gamma)^{1/bm} M$. Hence, if $H$ denotes the ball of center $x'$ and of radius $rl$, we have $\text{dist}(\pi^{-1}(x) \cap \Gamma, B \times \partial H) > r$. Therefore, by Proposition
3.2 \( \Gamma \cap B \times \partial H = \emptyset \). It assures that \( \pi \) is proper on \( \Gamma \cap B \times H \) and then defines a ramified covering.

Moreover, this covering has degree at most equal to \( s \). Otherwise, according to the radius of \( H \), we have

\[
\min_{z \in \pi^{-1}(x)} \|h(z)\| \leq (s + 1)^2 r t,
\]

which is in contradiction with (3.1) if \( \gamma \) is small enough, since \( \text{dist}(x, Z) > t \) and \( r = (a^{-1} \gamma)^{1/bm} t^{M} \).

Now, we want to apply Proposition 3.2 to this covering. But, in order to control the constants, we have to reduce \( B \). Indeed, according to Remark 3.3 the constants of that proposition depend only on the Hölder continuity of the coefficients \( a_{j,l} \) (and on the degree of the covering). These coefficients are bounded continuous weakly holomorphic functions defined on \( B \) then by Corollary 2.11 with \( L = X \cap \mathbb{B}_{3/4} \), they are \((Kt^{-M'}, \alpha)\)-Hölder continuous on \( B(x, t^{M'}) \) for some \( M' \geq 1 \) large enough, \( 0 < \alpha \leq 1 \) and \( K > 0 \) independent of \( x \) and \( t \). Therefore, after a coordinates dilation by \( t^{-M'} \) at \( x \), we can apply Proposition 3.2 with the same exponent \( b \) and the second constant proportional to some power of \( t \).

Finally, let \( y \in X \). We can assume that \( \rho(x, y) \leq t^{M'}/2 \), otherwise the proposition is obvious. Hence, the previous observation implies that

\[
\pi^{-1}(x) \cap \Gamma \cap B \times H = \{x^1, \ldots, x^s\} \quad \text{and} \quad \pi^{-1}(y) \cap \Gamma \cap B \times H = \{y^1, \ldots, y^s\},
\]

with \( \rho(x, y) \geq a't^{N}\text{dist}(x^i, y^i)^{bs} \) where \( a' > 0, b \geq 1 \) and \( N \geq 1 \) are independent of \( x \) and \( t \). More precisely, we can write \( N = M' + N' \), where the contribution in \( t^{M'} \) comes from the dilation and that in \( t^{N'} \) comes from the estimate on Hölder continuity. The construction can be applied to each component of \( \Gamma \cap \pi^{-1}(B) \). This gives the result.

We now consider the dynamical context. Assume that \( X \subset \mathbb{P}^k \) is an irreducible analytic set of dimensions \( l \) which is invariant by \( f \). Denote by \( g \) the restriction of \( f \) to \( X \). For \( x \) in \( X \), we define the local multiplicity of \( g \) at \( x \) as the maximal number of points in \( g^{-1}(z) \) which are near \( x \) for \( z \in X \) close enough to \( g(x) \). The local multiplicity is smaller than the topological degree i.e. the number of points in \( g^{-1}(x) \) for \( x \) generic in \( X \). In our case, the topological degree of \( g \) is equal to \( d^l \), see [6].

There exists a finite covering \( \{U_i\}_{i \in I} \) of \( X \) by open subsets of \( \mathbb{P}^k \) such that \( X \cap U_i \) can by decompose into locally irreducible components. Hence, we can apply Proposition 3.4 to the graph of \( g \) over each component of \( X \cap U_i \). It gives the following corollary.
Corollary 3.5. Let \( \eta > 1 \) and \( Z \subset X \) be a proper analytic subset. Assume that the local multiplicity of \( g \) is less than \( \eta \) outside \( g^{-1}(Z) \). Then there are constants \( a > 0, b \geq 1 \) and \( N \geq 1 \) such that if \( 0 < t < 1 \) and \( x, y \) are two points outside \( Z \) where \( X \) is locally irreducible, then we can write

\[
g^{-1}(x) = \{x^1, \ldots, x^d\} \quad \text{and} \quad g^{-1}(y) = \{y^1, \ldots, y^d\}
\]

with \( \text{dist}(x, y) \geq at^N \text{dist}(x^j, y^j)^{bn} \).

From this, we obtain the following size estimate for image of balls which is crucial in the proof of our main result.

Corollary 3.6. Let \( \delta > 1 \) and \( E \subset X \) be a proper analytic subset. Denote by \( E \) the preimage of \( E \) by \( g \). Assume that the local multiplicity is less than \( \delta \) outside \( \hat{E} \). There exist constants \( 0 \leq A \leq 1, b \geq 1 \) and \( N \geq 1 \) such that if \( 0 < t \leq 1/2, r < t/2 \) and \( x \in X \setminus \hat{E} \), then \( g(B_X(x,r)) \) contains a ball of radius \( At^{Nt}b^d \). Moreover, \( b \) depends only on \( X \).

Proof. Fix \( t > 0 \) and \( r < t/2 \). As in the proof of Lemma 2.5 with \( \hat{Y} = X_{\text{Sing}} \cup g^{-1}(X_{\text{Sing}}) \), possibly after replacing \( r \) by \( cr^d \) for some \( c > 0 \), we can assume that \( B_X(x,r) \) and \( g(B_X(x,r)) \) are contained in \( X_{\text{Reg}} \). The local multiplicity of \( f \) on \( X \) is bounded \( d^i \). So, there exists \( 2 \leq i \leq 4d^i \), such that the ring \( \{ \frac{r(4^i-1)}{4d^i+1} \leq \text{dist}(x,x') \leq \frac{r(4^{i+1})}{4d^i+1} \} \) contains no preimage of \( g(x) \). Thus, if \( x' \in \partial B_X(x, r \frac{4^{i+1}}{4d^i+1}) \), then

\[
\text{dist}(x', g^{-1}(g(x))) \geq \frac{r}{4d^i + 1}.
\]

Moreover, we can apply Corollary 3.5 with \( \eta = d^i \) and \( Z = \emptyset \). Hence, there exists \( a, b > 0 \) such that \( g(B_X(x, r \frac{1}{4d^i+1})) \subset X \setminus E_{a(t/2)^bd} \). Therefore, we can apply once again Corollary 3.5 with \( \eta = \delta \), \( Z = E \) and \( a(t/2)^bd \) instead of \( t \). We get, for some constants \( a' > 0 \) and \( N_0 \geq 1 \)

\[
\text{dist}(g(x'), g(x)) \geq a' \left( a \left( \frac{t}{2} \right)^{bd} \right)^{N_0} \left( \frac{r}{4d^i + 1} \right)^{bd} \delta,
\]

and, since \( g \) is an open mapping near \( x \)

\[
B_X(g(x), At^{Nt}b^d) \subset g(B_X(x,r))
\]

with \( A = \frac{g^{N_0 a'}}{2N_0 bd (4d^i + 1)^{bd}} \) and \( N = N_0 bd^i \).

\( \Box \)

Remark 3.7. When \( X \) is smooth, the ball in \( g(B_X(x,r)) \) can be chosen centered at \( g(x) \).
4 Psh functions and exponential estimates

We refer to [4] for basics on currents and plurisubharmonic (psh for short) functions. Let $T$ be a positive closed $(1,1)$-current of mass 1 on $\mathbb{P}^k$ with continuous local potentials. Let us recall briefly the associated notions of psh and weakly psh (wpsh for short) modulo $T$ functions introduced in [6].

Let $Y$ be an analytic space. A function $v : Y \to \mathbb{R} \cup \{-\infty\}$ is wpsh if it is psh on $Y_{\text{Reg}}$ and for $y$ in $Y$, we have $v(y) = \limsup v(z)$ with $z \in Y_{\text{Reg}}$ and $z \to y$. These functions coincide with psh functions if $Y$ is smooth. On compact spaces, the notion is very restrictive. However, if $X$ is an analytic subset of $\mathbb{P}^k$, we have the more flexible notion of wpsh modulo $T$ function on $X$. Locally, it is the difference of a wpsh function on $X$ and a potential of $T$. If $X$ is smooth, we say that the function is psh modulo $T$.

Note that the restriction of a psh modulo $T$ function to an analytic subset is either wpsh modulo $T$ or equal to $-\infty$ on an irreducible component. If $u$ is wpsh modulo $T$ on $X$ then $dd^c(u[X]) + T \wedge [X]$ is a positive closed current supported on $X$. On the other hand, if $S$ is a positive closed $(1,1)$-current on $\mathbb{P}^k$ with mass 1, there is a psh modulo $T$ function $u$ on $\mathbb{P}^k$, unique up to a constant, such that $S = dd^c u + T$.

These notions bring good compactness properties which permit to obtain uniform estimates. We have the following statements established in [6].

**Proposition 4.1.** Let $(u_n)$ be a sequence of wpsh modulo $T$ functions on $X$, uniformly bounded from above. Then there is a subsequence $(u_{n_i})$ satisfying one of the following properties:

- There is an irreducible component $Y$ of $X$ such that $(u_{n_i})$ converges uniformly to $-\infty$ on $Y \setminus X_{\text{Sing}}$.
- $(u_{n_i})$ converges in $L^p(X)$ to a wpsh modulo $T$ function $u$ for every $1 \leq p < +\infty$.

In the last case, $\limsup u_{n_i} \leq u$ on $X$ with equality almost everywhere.

It implies the following lemma.

**Lemma 4.2.** Let $\mathcal{G}$ be a family of psh modulo $T$ functions on $\mathbb{P}^k$ uniformly bounded from above. Assume that each irreducible component of $X$ contains an analytic subset $Y$ such that the restriction of $\mathcal{G}$ to $Y$ is bounded in $L^1(Y)$. Then, the restriction of $\mathcal{G}$ to $X$ is bounded in $L^1(X)$.

A classical result of Hörmander [15] gives a uniform bound to $\exp(-u)$ in $L^1(\mathbb{B}_{1/2})$ for $u$ in a class of psh functions in the unit ball of $\mathbb{C}^k$. Similar
estimates can be obtained for compact families of quasi-psh functions. From now, we assume that $T$ has $(K, \alpha)$-Hölder continuous local potentials, with $0 < \alpha \leq 1$ and $K > 0$. In the rest of this section, we establish exponential estimates for psh modulo $T$ functions in different situations. A key observation in our approach is that Hölder continuity allows us to work with non-compact families. In the sequel, we will apply these estimates to $T$ the Green current associate to $f$. They allow us to control the volume of some sublevel sets of potentials of currents near exceptional sets.

As a consequence of classical Hörmander’s estimate, we have the following lemma which will be of constant use. Here, $\nu$ denotes the standard volume form on $\mathbb{C}^k$ and $T$ is seen as a fixed current on the unit ball $B$ of $\mathbb{C}^k$. We assume that its admits a potential $g$ which is $(K, \alpha)$-Hölder continuous on $B$.

**Lemma 4.3.** Let $v$ be a psh modulo $T$ function in $B_t$ with $v \leq 0$ and $v(0) > -\infty$. Let $0 < s < -v(0)^{-1}$ and $t > 0$ such that $Kt^\alpha \leq s^{-1}$. There is a constant $c > 0$ independent of $v$, $s$ and $t$ such that

$$\int_{B_t/2} \exp\left(-\frac{sv}{2}\right) \nu \leq ct^{2k}. \quad (4.1)$$

**Proof.** As $v$ is psh modulo $T$, we have $v = v' - g$ with $v'$ psh. We set $\tilde{v}(z) = v'(z) - g(0) - Kt^\alpha$. Then $\tilde{v}$ is psh in $B_t$, $\tilde{v}(0) = v(0) - Kt^\alpha \geq -2s^{-1}$ and $\tilde{v} \leq v \leq 0$ because $g(z) - g(0) \leq Kt^\alpha$ on $B_t$. By [15, Theorem 4.4.5] there exists $c > 0$ such that

$$\int_{B_{1/2}} \exp\left(-\frac{s\tilde{v}(tz)}{2}\right) \nu \leq c,$$

thus, by a change of variables $z \mapsto tz$, we get

$$\int_{B_{t/2}} \exp\left(-\frac{sv}{2}\right) \nu \leq \int_{B_{t/2}} \exp\left(-\frac{s\tilde{v}}{2}\right) \nu \leq ct^{2k}.$$

For the rest of the section, $X$ always denotes an irreducible analytic subset of $\mathbb{P}^k$ of dimension $l$ and $v$ is a psh modulo $T$ function in $\mathbb{P}^k$ with $v \leq 0$. In Section 6 we will extend the previous result to the neighborhood of $X$, where the condition at 0 is replaced by an integrability condition on $X$. For this purpose, we have to control the size of sublevel sets of $v$ in $X$. This is the aim of the following global result.
Lemma 4.4. For \( X \) as above, there exists \( q_0 \geq 1 \) with the following property. For \( q > q_0 \) we set \( \epsilon = 2ltq_0/q_0 \) and take \( M > 0 \) and \( s \geq 1 \) such that \( s^{1+r}\|v\|_{L^q(X)} \leq M \). Then, there exist constants \( a, c > 0 \) independent of \( v, q \) and \( s \) such that

\[
\int_X \exp(-asv) \omega^I \leq c. \tag{4.2}
\]

If \( X \) is smooth, we can choose \( q_0 = 1 \).

Proof. First, assume that \( X \) is a compact smooth manifold with a volume form \( \eta \). Since \( X \) has dimension \( l \), for \( t > 0 \) we can cover it by balls \( (B_i)_{i \in I} \) with \( B_i := B_X(x_i, t) \) and such that \( |I| \leq c't^{-2l} \) for some \( c' > 0 \). Let \( t = s^{-1/\alpha} \).

As above, in each ball \( B(X, 2t) \) we can write \( v = v'_i - g_i \), where \( g_i \) is a local potential of \( T \). Using local charts at \( x_i \), we can identify \( B_X(x_i, 2t) \) with \( \mathbb{B}_2t \) in \( \mathbb{C}^l \). We consider \( \widetilde{v}_i(z) = s(v'_i(tz) - g_i(0)) \). These functions are psh in \( \mathbb{B}_2 \). We show that they belong to a compact family, independent of \( v \) and \( s \). Using a change of variables \( z \mapsto tz \) and Hölder’s inequality, we get

\[
\|\widetilde{v}_i\|_{L^1(\mathbb{B}_2)} \leq \int_{\mathbb{B}_2} s|v(tz)|\nu + \int_{\mathbb{B}_2} s|g_i(tz) - g_i(0)|\nu
\leq st^{-2l}\|v\|_{L^q(X)}\|\mathbb{B}_2\|^{1/p}t^{2l/p} + 2^\alpha K\|\mathbb{B}_2\|
= s^{1+r}\|v\|_{L^q(X)}\|\mathbb{B}_2\|^{1/p} + 2^\alpha K\|\mathbb{B}_2\|
\leq M\|\mathbb{B}_2\|^{1/p} + 2^\alpha K\|\mathbb{B}_2\| \leq M',
\]

where \( p \) is the conjugate of \( q \), \( \|\mathbb{B}_2\| \) is the volume of \( \mathbb{B}_2 \) and \( M' \) is a positive constant. The family \( \mathcal{U} = \{u \in PSH(\mathbb{B}_2) \mid \|u\|_{L^1(\mathbb{B}_2)} \leq M' \} \) is compact so there exists a constant \( c > 0 \) such that \( \|\exp(-au)\|_{L^1(\mathbb{B})} \) is uniformly bounded for all \( u \in \mathcal{U} \). Therefore, for \( i \in I \)

\[
\int_{B_i} \exp(-as(v'_i(z) - g_i(0)))\nu \lesssim t^{2l}.
\]

Moreover, the Hölder continuity implies that \(-sv(z) \leq K - s(v'_i(z) - g_i(x_i)) \) in \( B_i \). Hence, since \( (B_i)_{i \in I} \) is a covering of \( X \) we obtain

\[
\int_X \exp(-asv)\eta \leq \sum_{i \in I} \int_{B_i} \exp(-asv)\eta
\leq \sum_{i \in I} \int_{B_i} \exp(a(K - s(v'_i(z) - g_i(x_i))))\eta
\lesssim \sum_{i \in I} t^{2l} \leq c'.
\]
This implies the lemma if $X$ is smooth with $q_0 = 1$.

In the general case, we consider a desingularization $\pi : \hat{X} \to X$ with a volume form $\eta$ on $\hat{X}$. The map $\pi$ is surjective, then by Lemma 2.7 there exists $q_0 \geq 1$ such that

$$\|\hat{v}\|_{L^{q/q_0}(\hat{X},\eta)} \lesssim \|v\|_{L^q(X,\omega^l)}.$$ 

Moreover, $\pi^*(T)$ possesses $\alpha$-Hölder local potentials and $\hat{v} \leq 0$ is psh modulo $\pi^*(T)$. Therefore, this choice of $q_0$ allows us to apply the lemma on $\hat{X}$ and get

$$\int_{\hat{X}} \exp(-as\hat{v})\eta \leq c.$$

The result follows since

$$\int_X \exp(-asv)\omega^l = \int_{\hat{X}} \exp(-as\hat{v})\pi^*(\omega^l) \leq \|h\|_{\infty} \int_{\hat{X}} \exp(-as\hat{v})\eta,$$

where we write $\pi^*(\omega^l) = h\eta$.

The following estimate is a consequence of Lemma 4.3 and is related to the geometry of sublevel sets of psh modulo $T$ functions. In Section 6, it will establish the existence of balls where we can apply our volume estimates.

**Lemma 4.5.** For $s \geq 2$ set $F_s = \{ x \in X \mid v(x) \leq -s^{-1} \}$. There are constants $\beta, c > 0$ independent of $v$ and $s$ such that if $F_s$ contains no ball of radius $s^{-\beta}$ then

$$\int_X \exp(-sv^2)\omega^l \leq c.$$ 

**Proof.** We first consider the case where $X$ is smooth. Let $t = 4^{-1}(Ks)^{-1/\alpha}$. As in the proof of the previous lemma, we cover $X$ by balls $(B_i)_{i \in I}$ of radius $t$ with $|I| \leq c't^{-2t}$, $c' > 0$. Assume there is no ball of radius $t$ in $F_s$. Hence, for each $i \in I$ there exists $x_i$ in $B_i$ such that $v(x_i) > -s^{-1}$. The balls $B'_i$ of center $x_i$ and of radius $2t$ cover $X$. Thus

$$\int_X \exp(-sv^2)\omega^l \leq \sum_{i \in I} \int_{B'_i} \exp(-sv^2)\omega^l.$$ 

But, $s < -v(x_i)^{-1}$ and $K(4t)^{\alpha} \leq s^{-1}$ therefore we can apply Lemma 4.3 on each ball

$$\int_{B'_i} \exp(-sv^2)\omega^l \lesssim t^{2t}.$$ 

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Hence, we get
\[ \int_X \exp\left( -\frac{s v}{2} \right) \omega^I \lesssim \sum_{i \in I} t^{2i} \leq c', \]
which gives the result when \( X \) is smooth with \( \beta > 1/\alpha \) such that \( s^{-\beta} < t \).

If \( X \) is singular, we consider a desingularization \( \pi : \hat{X} \to X \). By Lemma 2.5 there exists \( N \geq 1 \) such that the image of a ball of radius \( r \) under \( \pi \) contains a ball of radius \( r^N \). Hence, if \( \beta \) is large enough, the hypothesis on \( F_s \) assures there is no ball of radius \( t \) in \( \hat{F}_s = \pi^{-1}(F_s) \). Then, we can apply the lemma to \( \hat{v} = v \circ \pi \) which is psh modulo \( \pi^*(T) \). We get
\[ \int_X \exp\left( -\frac{s v}{2} \right) \omega^I = \int_{\hat{X}} \exp\left( -\frac{s \hat{v}}{2} \right) \pi^*(\omega^I) \leq c, \]
for some \( c > 0 \), since \( \pi^*(\omega^I) \) is smooth.

\[ \square \]

5 Exceptional sets

Let \( f \) be an endomorphism of \( \mathbb{P}^k \) of algebraic degree \( d \geq 2 \). The aim of this section is to construct two families \( \mathcal{A}_\lambda \) and \( \mathcal{B}_\lambda \) of analytic sets where the iterate sequence \( f^n \) has important local multiplicities. Let \( X \subset \mathbb{P}^k \) be an irreducible invariant analytic set. Define \( \kappa_{X,n}(x) \), or simply \( \kappa_n(x) \) if no confusion is possible, as the local multiplicity of \( f^n \big|_X \) at \( x \). It is a sub-multiplicative cocycles, namely it is upper semi-continuous for the Zariski topology on \( X \), \( \min_X \kappa_n = 1 \) and for any \( m, n \geq 0 \) and \( x \in X \) we have the following sub-multiplicative relation
\[ \kappa_{n+m}(x) \leq \kappa_m(f^n(x)) \kappa_n(x). \]
The inequality may be strict when \( X \) is singular. Define
\[ \kappa_{-n}(x) := \max_{y \in (f|_X)^{-n}(x)} \kappa_n(y). \]
We recall the following theorem of Dinh [5], see also [10].

**Theorem 5.1.** The sequence of functions \( \kappa_{-n}^{1/n} \) converges pointwise to a function \( \kappa_- \). Moreover, for every \( \lambda > 1 \), the level set \( E_\lambda(X) = \{ \kappa_- \geq \lambda \} \) is a proper analytic subset of \( X \) which is invariant under \( f|_X \). In particular, \( \kappa_- \) is upper semi-continuous in the Zariski sense.

For a generic endomorphism of \( \mathbb{P}^k \), \( E_\lambda(\mathbb{P}^k) \) is empty. In this case, Theorem 1.2 is already know in all bidegrees [7]. In our proof, we will proceed by
induction, proving the the exponentially fast convergence on $X$ if it is already established on each irreducible component of $E_{\lambda}(X)$. But, even if $E_{\lambda}(X)$ is invariant, its irreducible components are periodic and not invariant in general. Therefore, if $X$ is only periodic, we define $E_{\lambda}(X)$ in the same way, replacing $f$ by $f^p$ and $\lambda$ by $\lambda^p$, where $p$ is a period of $X$. By Theorem 5.1 this definition is independent of the choice of $p$.

Fix $1 < \lambda < d$. Define the family $B_\lambda$ of exceptional sets as follows. First, we set $P \in B_\lambda$. If $X$ is in $B_\lambda$, we add to $B_\lambda$ all irreducible components of $E_{\lambda}(X)$. This family is finite and since the functions $\kappa_-$ are upper semi-continuous in the Zariski sense, there exists $1 < \delta < \lambda$ such that $B_\lambda = B_\delta$, or equivalently $E_{\lambda}(X) = E_{\delta}(X)$ if $X \in B_\lambda$. This will give us some flexibility in order to obtain estimates using an induction process.

As all elements of $B_\lambda$ are periodic, they are invariant under some iterate $f^{n_0}$. Let us remark that it is sufficient to prove Theorem 1.2 for an iterate $f^{n_0}$. Hence, we can assume that $n_0 = 1$, replacing $f$ and $\lambda$ by $f^{n_0}$ and $\lambda^{n_0}$. Dinh also proved that $\kappa_{n_1} < \delta^{n_1}$ outside $(f_{|X})^{-n_1}(E_{\lambda}(X))$ for some $n_1 \geq 1$. Once again, we can assume that $n_1 = 1$.

The second family $\mathcal{A}_\lambda$ that takes place in Theorem 1.2 is defined as the set of minimal elements for the inclusion in $B_\lambda$. This family is not empty and each element of $B_\lambda$ contains at least one element of $\mathcal{A}_\lambda$. Note that no element of $\mathcal{A}_\lambda$ is contained in another one. These analytic sets play a special role in the next section, to start induction and to obtain compactness properties. When $P_k$ is an element of $\mathcal{A}_\lambda$, it is the only element in $\mathcal{A}_\lambda$ and the exceptional set is empty. Otherwise, define the exceptional set as the union of all the elements of $\mathcal{A}_\lambda$.

## 6 Equidistribution speed

This section is devoted to the proof of Theorem 1.2. Fix an endomorphism $f$ of algebraic degree $d \geq 2$ of $\mathbb{P}^k$, and denote by $T$ its Green current. Recall that $T$ is totally invariant i.e. $d^{-1} f^* (T) = T$, and has $(K, \alpha)$-Hölder continuous local potentials for some $0 < \alpha \leq 1$, $K > 0$.

Fix $C > 0$ and $1 < \lambda < d$, and let $\mathcal{A}_\lambda$, $B_\lambda$ be as in Section 5. Define $\mathcal{F}_{\lambda}(C)$ as the family of psh modulo $T$ functions $v$ on $\mathbb{P}^k$ such that $\max_{\mathbb{P}^k} v = 0$ and $\|v\|_{L^1(X)} \leq C$ for all $X \in \mathcal{A}_\lambda$. By construction of $\mathcal{A}_\lambda$, Lemma 4.2 implies that $\mathcal{F}_{\lambda}(C)$ is compact for each $C > 0$. Moreover, if $X$ is an element of $B_\lambda$, the restriction of $\mathcal{F}_{\lambda}(C)$ to $X$ forms a family of wpsh modulo $T$ functions on $X$ which is relatively compact in $L^p(X)$ for every $1 \leq p < +\infty$.

If $S$ is a positive closed $(1,1)$-current of mass $1$, it is cohomologous to $T$. Hence, there exists a unique psh modulo $T$ function $u$ on $\mathbb{P}^k$ such that
\( S = dd^cu + T \) and \( \max_{p^k} u = 0 \). We call \( u \) the dynamical potential of \( S \).

As \( T \) is totally invariant, the dynamical potential of \( S_n = d^{-n}(f^n)^*(S) \) is \( u_n = d^{-n}u \circ f^n \).

Since \( S_n - T \) is a continuous linear operator on \( \mathcal{C}^0(\mathbb{P}^k) \) whose norm is bounded, by interpolation theory between Banach spaces we have

\[
\|S_n - T\|_{\mathcal{C}^\beta} \lesssim \|S_n - T\|_{\mathcal{C}^2}^{\beta/2},
\]

uniformly in \( S \) and \( n \), see \[18\]. Consequently, in order to prove Theorem 1.2 we can assume that \( \beta = 2 \).

Moreover, it is easy to see that \( \|dd^c u_n\|_{\mathcal{C}^2} \lesssim \|u\|_{\mathcal{C}^2} \) for \( u \) in \( \mathcal{C}^2(\mathbb{P}^k) \). Therefore,

\[
|\langle S_n - T, \phi \rangle| = |\langle dd^c u_n, \phi \rangle| = |\langle u_n, dd^c \phi \rangle| \lesssim \|\phi\|_{\mathcal{C}^2} \|u_n\|_{L^1(\mathbb{P}^k)}.
\]

Hence, Theorem 1.2 is a direct consequence of the following theorem applied to \( p = 1 \) and \( X = \mathbb{P}^k \).

**Theorem 6.1.** For each \( 1 \leq p < +\infty \) and \( X \in \mathcal{B}_\lambda \) there exists a constant \( A_{X,p} \) such that for all \( u \in \mathcal{F}_\lambda(C) \) and \( n \geq 0 \) we have

\[
\|u_n\|_{L^p(X)} \leq A_{X,p} \left( \frac{\lambda}{d} \right)^n,
\]

where \( u_n = d^{-n}u \circ f^n \).

As in Section 5 we can assume that each element of \( \mathcal{B}_\lambda \) is invariant by \( f \), and there is \( 1 < \delta < \lambda \) satisfying the following properties for all \( X \) in \( \mathcal{B}_\lambda \):

- \( E_\lambda(X) = E_\delta(X) \),
- \( \kappa_{X,1} < \delta \) outside \( \bar{E}_\lambda(X) = (f|X)^{-1}(E_\lambda(X)) \).

Let \( X \) be an element \( \mathcal{B}_\lambda \) of dimension \( l \) and \( \lambda_1 > 0 \) with \( \delta < \lambda_1 < \lambda \). Assume that Theorem 6.1 is true on each irreducible component of \( E_\lambda = E_\lambda(X) \) for \( \lambda_1 \) and all \( p \geq 1 \). To prove it on \( X \), we consider the sublevel set \( K_n = \{ x \in X \mid u_n(x) \leq -s_n \} \) for a suitable constant \( s_n \). Exponential estimates on \( \bar{E}_\lambda \) will prove that its image by \( f^i \), \( 0 \leq i \leq n \), cannot be concentrated near \( \bar{E}_\lambda \). Therefore, volume estimates will imply that \( f^n(K_n) = \{ x \in X \mid u(x) \leq -d^n s_n \} \) is large if Theorem 6.1 is false on \( X \). Hence, a good choice of \( s_n \), allowed by the gap between \( \lambda_1 \) and \( \lambda \), will give a contradiction.

We first fix some constants. In Corollary 3.6 the constant \( b \) depends only on \( X \). Then, by replacing \( f \) by \( f^n \) and \( \delta \) by \( \delta^m \) with \( b\delta^m < \delta^m < \lambda_1^2 \), we can assume that \( b = 1 \). Let \( 0 < A \leq 1, N \geq 1 \) be the other constants of Corollary 3.6. Fix \( \lambda_2, \lambda_3 > 0 \) such that
\[ \delta < \lambda_1 < \lambda_2 < \lambda_3 < \lambda, \]

- and \( q > q_0 \) large enough such that \( \lambda_1/d < (\lambda_2/d)^{1+\epsilon} \) where \( \epsilon \) and \( q_0 \) are defined in Lemma 4.4.

Multiplicities of \( f|_X \) are controlled outside \( \tilde{E}_\lambda \). By induction hypothesis, we have a control of \( u_n \) on \( E_\lambda \). We want to extend it to \( \tilde{E}_\lambda \). Let \( E \) be an irreducible component of \( E_\lambda \). The restriction of \( f \) to each component of \( (f|_X)^{-1}(E) \) is surjective onto \( E \). Therefore, we deduce from Lemma 2.7 that there exists \( q' \geq 1 \) such that

\[ \|v \circ f\|_{L^q((f|_X)^{-1}(E))} \lesssim \|v\|_{L^{q'}(E)}, \]

for all psh modulo \( T \) function \( v \) on \( \mathbb{P}^k \). Hence, by induction hypothesis, there is a constant \( M > 0 \) such that \( \|u_n\|_{L^q(E_\lambda)} \leq M(\lambda_1/d)^n \) for \( n \geq 1 \). The next step is to obtain exponential estimates in a neighborhood of \( \tilde{E}_\lambda \).

**Lemma 6.2.** There exist constants \( c, \eta \geq 1 \) and \( n_0 \geq 1 \) such that if \( n \geq n_0 \) then for all \( u \in \mathcal{F}_\lambda(C) \) we have

\[ \int_{E_{\lambda, t_n}} \exp(-(d/\lambda_2)^n u_n) \omega^j \leq c, \]

where \( t_n = (\lambda_2/d)^{n\eta} \).

**Proof.** Let \( E \) be an irreducible component of \( \tilde{E}_\lambda \) of dimension \( i \). According to the choice of \( q \), we can find \( \lambda'_2 < \lambda_2 \) such that \( \lambda_1/d < (\lambda'_2/d)^{1+\epsilon} \). Hence

\[ \|u_n\|_{L^q(E)}(d/\lambda'_2)^{(1+\epsilon)n} \leq M(\lambda_1/d)^n(d/\lambda'_2)^{(1+\epsilon)n} \leq M. \]

and by Lemma 4.4 with \( s = (d/\lambda'_2)^n \) we have

\[ \int_E \exp(-a'(d/\lambda'_2)^n u_n) \omega^j \leq c', \]

for some constants \( a', c' > 0 \). Therefore, if we set \( \rho_n = (\lambda_2/d)^n \), the volume in \( E \) of \( F_n = \{x \in E \mid u_n(x) \leq -\rho_n\} \) is smaller than \( c' \exp(-a'(\lambda_2/\lambda'_2)^n) \). In particular, \( F_n \) contains no ball of radius \( \rho_n^{1/\alpha} \) for \( n \) large enough.

If \( X \) is smooth then set \( t_n = \rho_n^{1/\alpha} \). As in Lemma 4.5, for \( n \) large enough, we can find a covering of \( E_{t_n} \) by balls with center in \( E \) and of radius \( 2t_n \) on which Lemma 4.3 holds. Hence, we get

\[ \int_{E_{t_n}} \exp(-au_n/\rho_n) \omega^j \leq c, \]

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for some $a, c > 0$. The same argument with $\lambda_2$ slightly smaller shows that we can choose $a = 1$. We conclude the proof by summing on all irreducible components of $\tilde{E}_\lambda$.

When $X$ is singular, we consider a desingularization $\pi : \tilde{X} \to X$. In order to establish the estimate near $E$, we proceed inductively as follows. Assume that there exists a triplet $(A, a, \theta)$ with $a > 0$, $\theta \geq 1$ and an analytic set $A \subset E$ such that

$$
\int_{E_\theta \setminus A_{1/\theta}} \exp(-au_n/\rho_n) \omega^j
$$

is uniformly bounded in $n \geq 0$ for $t \leq \rho_n$. Then, using the properties of the elements of $B_\lambda$ and dynamical arguments, we claim that a similar estimate holds if we substitute $(A, a, \theta)$ by some $(A', a', \theta')$ with $\dim(A') < \dim(A)$. It will give the result for $\eta$ large enough after less than $l$ steps since $\dim(E) < l$.

More precisely, let $V$ be an irreducible component of $A$ with maximal dimension. We distinguish two cases, according to whether $V$ is in $B_\lambda$ or not. In the first case, we know that for all $p \geq 1$, $\|u_n\|_{L^p(V)} \lesssim (\lambda_1/d)^n$. We set $\hat{V} := \pi^{-1}(V)$. We denote by $\hat{V}_1$ the union of all components of $\hat{V}$ which are mapped onto $V$ and by $\hat{V}_2$ the union of the other components of $\hat{V}$. Therefore, Lemma 2.7 implies that

$$
\|\hat{u}_n\|_{L^p(\hat{V}_1)} \lesssim (\lambda_1/d)^n,
$$

for all $p \geq 1$, where $\hat{u}_n = u_n \circ \pi$. Hence, the smooth version of the lemma implies that

$$
\int_{\hat{V}_1} \exp(-a'\hat{u}_n/\rho_n) \pi^*(\omega^j)
$$

is uniformly bounded for $a' > 0$ small enough. Moreover, by Lemma 2.6 there exists a constant $\theta' \geq 1$ such that $\pi(\hat{V}_1)$ contains $V_1 \setminus V_2$, where $V_2 = \pi(\hat{V}_2)$. It gives the desired result near $V$, since $\dim(V_2) < \dim(V)$.

From now, we can assume that no irreducible component of $A$ with maximal dimension belong to $B_\lambda$ (in particular $A \neq E$). Let $V$ denote the union of all irreducible components of $A$ with maximal dimension. In particular, these components are not totally invariant for $f_{|E}$, therefor there exist an analytic set $Z \subset E$ containing no component of $V$ and an integer $m \geq 1$ such that $f^m(Z) = V$. We set $Z' = Z \cap A$. The assumption on $A$ and $\theta$ implies that if $t \leq \rho_n$ then

$$
\int_{Z_\theta \setminus A_{1/\theta}} \exp(-au_n/\rho_n) \omega^j
$$

is uniformly bounded in $n \geq 0$ for $t \leq \rho_n$. Then, using the properties of the elements of $B_\lambda$ and dynamical arguments, we claim that a similar estimate holds if we substitute $(A, a, \theta)$ by some $(A', a', \theta')$ with $\dim(A') < \dim(A)$. It will give the result for $\eta$ large enough after less than $l$ steps since $\dim(E) < l$.
is bounded uniformly on $n$. By Corollary 2.2, $Z_{t\theta} \cap A_{1/\theta}$ is contained in $Z_{t\theta}^{\prime}$ for some $\theta^\prime > \theta$. So,

$$
\int_{Z_{t\theta}^{\prime} \setminus Z_{t\theta}^{\prime\prime}} \exp(-au_n/\rho_n)\omega^j
$$

is bounded uniformly on $n$. Fix a constant $B > 1$ large enough. We deduce from Corollary 3.6 applied to $P_k$ that for all $t > 0$ $f^m(Z_t)$ contains $V_{Bt}$, where $V'' = f^m(Z')$. So, we have

$$
f^m(Z_t) \supset V_{t^\theta} \setminus V_{t^\theta/\theta'} \ni Z_t \setminus Z_t^{\prime\prime},
$$

for $t > 0$ small enough and $\theta' > \theta''$ large enough. It follows that

$$
\int_{V_t} \exp(-au_n/\rho_n)\omega^j \leq \int_{Z_t} \exp(-a'u_n \frac{\lambda_{2m}^n}{\rho_n + m}) (f^m_{|X})^*(\omega^j)
$$

$$
\lesssim \int_{Z_t} \exp(-a' \frac{u_n \lambda_{2m}^n}{\rho_n + m}) \omega^j, \quad (6.1)
$$

since $(f^m_{|X})^*(\omega^j) \lesssim \omega^j$. Moreover, for $a'$ same enough the right-hand side in (6.1) is bounded uniformly on $n$ and $\dim(V') = \dim(Z') < \dim(V)$ since $Z$ contains no component of $V$. This together with the estimate outside $A$ prove the claim with $A' = V'$. \hfill \square

From now, we fix $p \geq 1$ and for $u$ in $\mathcal{F}_\lambda(C)$ denote by $\mathcal{N}(u) = \{n \geq 1 \mid \|u_n\|_{L^p(X)} \geq (\lambda/d)^n\}$ and by $\mathcal{N}$ the union of $\mathcal{N}(u)$ for all $u$. Our goal is to prove that $\mathcal{N}$ is finite, which will imply Theorem 6.1. For this purpose, we have the following result.

**Lemma 6.3.** There are constants $n_1 \geq 1$ and $\beta \geq 1$ such that if $n$ is in $\mathcal{N}(u)$ with $n \geq n_1$ then $K_n = \{x \in X \mid u_n(x) \leq -(\lambda_3/d)^n\}$ contains a ball of radius $(\lambda_3/d)^{\beta n}$.

**Proof.** Since $x^p \lesssim \exp(x)$ if $x \geq 0$, we deduce from the assumption on $\|u_n\|_{L^p(X)}$ that

$$
(\lambda/\lambda_3)^n \lesssim \left( \int_X (-d/\lambda_3)^n u_n \omega^j \right)^{1/p}
$$

$$
\lesssim \left( \int_X \exp(-d/\lambda_3)^n u_n/2) \omega^j \right)^{1/p}. \quad (6.2)
$$
On the other hand, let $\beta$ be the constant in Lemma 4.5. For $n$ sufficiently large we have $(d/\lambda_3)^n \geq 2$. Hence, Lemma 4.5 with $s = (d/\lambda_3)^n$ imply that $K_n$ has to contain a ball of radius $(\lambda_3/d)^{n\beta}$, otherwise the right-hand side of (6.2) would be bounded uniformly on $n$, which is a impossible since $\lambda_3 < \lambda$.

We can now complete the proof of the main theorem.

**End of the proof of Theorem 6.1.** If $B \subset X$ is a Borel set then $|B|$ denotes its volume with respect to the measure $\omega$. As we have already seen, the volume of a ball of radius $r$ in $X$ is larger than $c'r^{2l}$, $0 < c' \leq 1$. Therefore, observe that if $x$ is in $\tilde{E}_{\lambda,t_n/2}$ then $|\tilde{E}_{\lambda,t_n} \cap X(x,r)| = |B_x(x,r)| \geq c'(r/2)^{2l}$ for $r = t_n/2$.

From now, assume in order to obtain a contradiction that $\mathcal{N}$ is infinite. Consider $u \in \mathcal{F}_X(C)$ and $n \in \mathcal{N}(u)$ large enough. Fix also $\beta$ large enough. So, we have $(\lambda_3/d)^{3n} < t_n/4$ and

$$c \exp(-(\lambda_3/\lambda_2)^n) \leq c'(A^s (t_n/2)^{N_s^{\lambda}} (\lambda_3/d)^{3n}/2)^{2l},$$

(6.3)

where $c$ is defined in Lemma 6.2. Let $r_0 = (\lambda_3/d)^{3n}$, and for $1 \leq i \leq n$ let $r_i = A(t_n/2)^{N r_i^{\delta_i}}$. We will prove by induction that for $0 \leq i \leq n$, $f^i(K_n) = \{x \in X \mid u_{n-i}(x) \leq -\lambda_3^{3/d^{m-i}}\}$ contains a ball $B_i$ of radius $r_i$.

Since $\beta$ is large, Lemma 6.3 implies that the assertion is true for $i = 0$. Let $0 \leq i \leq n - 1$ and assume the property is true for $i$. We deduce from Lemma 6.2 that

$$\int_{\tilde{E}_{\lambda,t_n-i}} \exp(-(d/\lambda_2)^{n-i} u_{n-i}) \omega^j \leq c,$$

and in particular

$$|\tilde{E}_{\lambda,t_n} \cap B_i| \leq |\tilde{E}_{\lambda,t_n-i} \cap f^i(K_n)| < c \exp(-(\lambda_3/\lambda_2)^n \lambda_2^i),$$

since $t_n \leq t_{n-i}$. This and (6.3) imply that

$$|B_i| \geq c r_i^{2l} > 2^{2l} |\tilde{E}_{\lambda,t_n} \cap B_i|,$$

since $r_i \geq (A^s (t_n/2)^{N r_i^{\delta_i}})^{\delta_i}$ and $\delta < \lambda_2$. Consequently, the center of $B_i$ is not in $\tilde{E}_{\lambda,t_n/2}$ and by Corollary 3.6 $f(B_i) \subset f^{i+1}(K_n)$ contains a ball $B_{i+1}$ of radius $r_{i+1} = A(t_n/2)^{N r_i^{\delta_i}}$. Note that we already reduced the problem to the case where the constant $b$ in Corollary 3.6 is equal to 1.

Therefore, for all $n$ in $\mathcal{N}(u)$ sufficiently large, the volume of $f^n(K_n) = \{x \in X \mid u(x) \leq -\lambda_3^n\}$ is greater than $D^m$, with $0 < D < 1$ independent
of \( u \) and \( n \). This contradicts the inequality \( \delta < \lambda_3 \). Indeed, since \( \mathcal{F}_\lambda(C) \) is bounded in \( L^q(X) \), by Lemma 4.4 there exists \( a' > 0 \) such that

\[
\int_X \exp(-a'u)\omega^j
\]

is uniformly bounded for \( u \) in \( \mathcal{F}_\lambda(C) \).

Hence, \( \mathcal{N} \) is finite and in particular bounded by some \( n_2 \geq 1 \). We conclude using the fact that the restriction of \( \cup_{n=0}^{n_2} d^{-n}(f^n)^*(\mathcal{F}_\lambda(C)) \) to \( X \) is a relatively compact family of wpsh modulo \( T \) functions and then bounded in \( L^p(X) \). Therefore, we have

\[
\|u_n\|_{L^p(X)} \lesssim \left( \frac{\lambda}{d} \right)^n,
\]

if \( n \leq n_2 \) and thus for every \( n \geq 0 \) by the definition of \( \mathcal{N} \). □

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