ABOUT MULTIPLECTIES AND APPLICATIONS TO BEZOUT NUMBERS

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ABSTRACT. Let \((A, m, k)\) denote a local Noetherian ring and \(q\) an ideal such that \(\ell_A(M/qM) < \infty\) for a finitely generated \(A\)-module \(M\). Let \(\bar{a} = a_1, \ldots, a_d\) denote a system of parameters of \(M\) such that \(a_i \in q^{c_i} \setminus q^{c_i+1}\) for \(i = 1, \ldots, d\). It follows that \(\chi := e_0(\bar{a}; M) - c \cdot e_0(q; A) \geq 0\), where \(c = c_1 \cdot \ldots \cdot c_d\).

The main results of the report are a discussion when \(\chi = 0\) resp. to describe the value of \(\chi\) in some particular cases. Applications concern results on the multiplicity \(e_0(\bar{a}; M)\) and applications to Bezout numbers.

Dedicated to Winfried Bruns on the occasion of his 70th birthday.

1. INTRODUCTION

Let \((A, m, k)\) denote a local ring. Let \(q \subseteq A\) be an \(m\)-primary ideal and \(\bar{a} = a_1, \ldots, a_d\) be a system of parameters in \(A\) such that \(a_i \in q^{c_i}, i = 1, \ldots, d\), with \(c_i > 0\). The main interest in the present report is a comparison of the multiplicities \(e_0(\bar{a}; A)\) and \(e_0(q; A)\).

Let \(M\) be a finitely generated \(A\)-module. Note that for an ideal \(q \subseteq A\) such that the length \(\ell_A(M/qM)\) is finite, the multiplicity \(e_0(q; M)\) is defined as the leading term of the Hilbert-Samuel polynomial

\[
\ell_A(M/q^{n+1}M) = \sum_{i=0}^{d} e_i(q; M) \binom{n + d - i}{d - i} \quad \text{for } n \gg 0, \text{ with } d = \dim_A M
\]

(see for instance [15], [9] for all the details or more general [12] for generalizations to filtered modules).

With the previous assumption, clearly \(e_0(\bar{a}; A) \geq e_0(q; A)\). We will discuss this relation in more detail. First let us recall some known results:

(a) For \(q = \bar{a} A\) we get \(e_0(\bar{a} A; A) = n \cdot e_0(\bar{a}; A)\), where \(\bar{a} A = a_1^{n_1}, \ldots, a_d^{n_d}\) and \(n = n_1 \cdot \ldots \cdot n_d\) for some \(n_1, \ldots, n_d \in \mathbb{N}^d\).

(b) If \(\bar{a}\) is a minimal reduction of \(q\), then \(e_0(\bar{a}; A) = e_0(q; A)\). The converse is true (see [11]) provided \(A\) is formally equidimensional.

(c) It follows that \(e_0(\bar{a}; A) \geq c \cdot e_0(q; A)\), where \(c = c_1 \cdot \ldots \cdot c_d\) with the above notation.

For the proof of (a) we refer to [1]. The first part of (b) is well-known, while the converse is an outstanding result of Rees (see [11]). The claim in (c) is easy to prove (see [2] for details). The main goal of the present report is a discussion of the difference \(\chi := e_0(\bar{a}; A) - c \cdot e_0(q; A) \geq 0\) of (c) in various situations, its vanishing resp. a simplified proof of some known results.

The importance of the understanding of \(\chi\) has to do with Bezout’s Theorem in the plane. Let \(C = V(F), D = V(G) \subseteq \mathbb{P}_k^2\) be an algebraically closed field, be two curves in the projective plane without a common component. Then

\[
\sum_{P \in C \cap D} \mu(P; C, D) = \deg C \cdot \deg D,
\]

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where \( \mu(P; C, D) \) denotes the local intersection multiplicity of \( P \) in \( C \cap D \). In the case of \( P \) is the origin, it follows that \( \mu(P; C, D) = e_0(f, g; A) \), where \( A = k[x, y]_{(x, y)} \) and \( f, g \) denote the equations in \( A \). Since \( C, D \) have no component in common, \( \{f, g\} \) forms as system of parameters in \( A \). Then

\[
e_0(f, g; A) \geq c \cdot d \cdot e_0(m; A) = c \cdot d,
\]

since \( e_0(m; A) = 1 \). Here \( c, d \) denote the initial degree of \( f, g \) respectively. This estimate is well-known (see for instance \([4]\) or \([5]\)) proved by resultants or Puiseux expansions. Moreover equality holds if and only if \( C, D \) intersect transversally in the origin. In other words, \( f^*, g^* \), the initial forms of \( f, g \) in the form ring \( G_A(m) \cong k[X, Y] \) are a homogeneous system of parameters.

Here we shall provide another argument with extensions to arbitrary local rings. Let \( A \) denote a local ring with \( a = a_1, \ldots, a_d \in A \) a system of parameters. Put \( c = c_1 \cdot \cdots \cdot c_d \) for \( a_i \in m^{c_i} \setminus m^{c_i+1} \).

In his paper (see \([10]\)) the author claimed that \( e_0(q; A) = c \cdot e_0(m; A) \) if and only if the sequence of initial elements \( a_1^*, \ldots, a_d^* \in G_A(m) \) forms a regular sequence. This is not true as the following example shows.

**Example 1.1.** Let \( k \) denote a field and \( A = k[[t^4, t^5, t^{11}]] \subset k[[t]] \), where \( t \) is an indeterminate over \( k \). Then \( A \) is a one-dimensional domain and therefore a Cohen-Macaulay ring with \( A \simeq k[[X, Y, Z]]/(X^4 - YZ, Y^3 - XZ, Z^2 - X^3Y^2) \). Clearly, the residue class \( a = x \) of \( X \) is a parameter with \( a \in m \setminus m^2 \), so that \( c = 1 \).

Furthermore, by easy calculations it follows that \( e_0(a, A) = \ell_A(A/aA) = 4 \) and \( e_0(m, A) = 4 \). So, the equation \( e_0(a, A) = c \cdot e_0(m, A) \) holds, while \( G_A(m) = k[X, Y, Z]/(XZ, YZ, Y^4, Z^2) \) is not a Cohen-Macaulay ring (see \([2, \text{Section } 3]\) for the details).

In Section 2 we start with some preliminaries and Koszul complexes. In Section 3 we derive some new complexes from certain Koszul complexes important for the study of multiplicities. Euler characteristics are the feature of Section 4. As an application we derive a short argument for computing certain multiplicities as Euler characteristics of Koszul complexes (originally done by Auslander and Buchsbaum (see \([1]\)) and Serre (see \([13]\)) by spectral sequence arguments). In Section 5 we study the equality \( e_0(q; M) = c_1 \cdot \cdots \cdot c_d \cdot e_0(q; M) \). Under some additional regularity condition on the sequence of initial forms \( a_1^*, \ldots, a_d^* \in G_A(q) \) we estimate the difference \( \ell_A(M/q^nM) = c_1 \cdot \cdots \cdot c_d \cdot e_0(q; M) \). As an application we get a bound of the local Bezout intersection numbers of two curves in the projective plane without common component.

### 2. Preliminaries

First let us fix the notations we will use in the following. For the basics on \( \mathbb{N} \)-graded structures we refer e.g. to \([6]\).

**Notation 2.1.** (A) We denote by \( A \) a commutative Noetherian ring with \( 0 \neq 1 \). For an ideal we write \( q \subset A \). An \( A \)-module is denoted by \( M \). Mostly we consider \( M \) as finitely generated.

(B) We consider the Rees and form rings of \( A \) with respect to \( q \) by

\[
R_A(q) = \oplus_{n \geq 0}q^nT^n \subseteq A[T] \quad \text{and} \quad G_A(q) = \oplus_{n \geq 0}q^n/q^{n+1}.
\]

Here \( T \) denotes an indeterminate over \( A \). Both rings are naturally \( \mathbb{N} \)-graded. For an \( A \)-module \( M \) we define the Rees and form modules in the corresponding way by

\[
R_M(q) = \oplus_{n \geq 0}q^nM T^n \subseteq M[T] \quad \text{and} \quad G_M(q) = \oplus_{n \geq 0}q^nM/q^{n+1}M.
\]

Note that \( R_M(q) \) is a graded \( R_A(q) \)-module and \( G_M(q) \) is a graded \( G_A(q) \)-module. Note that \( R_A(q) \) and \( G_A(q) \) are both Noetherian rings. In case \( M \) is a finitely generated \( A \)-module then \( R_M(q) \) resp. \( G_M(q) \) is finitely generated over \( R_A(q) \) resp. \( G_A(q) \).
(C) There are the following two short exact sequences of graded modules

\[ 0 \to R_M(q)_+ [1] \to R_M(q) \to G_M(q) \to 0 \]

\[ 0 \to R_M(q)_+ \to R_M(q) \to M \to 0, \]

where \( R_M(q)_+ = \oplus_{n=0} q^n M^n \).

(D) Let \( m \in M \) and \( m \in q^e M \setminus q^{e+1} M \). Then we define \( m^* := m + q^{e+1} M \in [G_M(q)]_e \). If \( m \in \cap_{n \geq 1} q^n M \), then we write \( m^* = 0 \). \( m^* \) is called the initial element of \( m \) in \( G_M(q) \) and \( e \) is called the initial degree of \( m \). Here \([X]_n, n \in \mathbb{Z}\), denotes the \( n \)-th graded component of an \( \mathbb{N} \)-graded module \( X \).

For these and related results we refer to [6] and [14]. Another feature for the investigations will be the use of Koszul complexes.

**Remark 2.2. (Koszul complex.)** (A) Let \( \underline{a} = a_1, \ldots, a_t \) denote a system of elements of the ring \( A \). The Koszul complex \( K_\bullet(a; A) \) is defined as follows: Let \( F \) denote a free \( A \)-module with basis \( e_1, \ldots, e_t \). Then \( K_i(a; A) = \bigwedge^i F \) for \( i = 1, \ldots, t \). A basis of \( K_i(a; A) \) is given by the wedge products \( e_{j_1} \wedge \cdots \wedge e_{j_i} \) for \( 1 \leq j_1 < \cdots < j_i \leq t \). The boundary homomorphism \( K_i(a; A) \to K_{i-1}(a; A) \) is defined by

\[ d_{j_1 \cdots j_i} : e_{j_1} \wedge \cdots \wedge e_{j_i} \mapsto \sum_{k=1}^i (-1)^{k+1} a_{j_k} e_{j_1} \wedge \cdots \wedge \widehat{e}_{j_k} \wedge \cdots \wedge e_{j_i} \]

on the free generators \( e_{j_1} \wedge \cdots \wedge e_{j_i} \).

(B) Another way of the construction of \( K_\bullet(a; A) \) is inductively by the mapping cone. To this end let \( X \) denote a complex of \( A \)-modules. Let \( a \in A \) denote an element of \( A \). The multiplication by \( a \) on each \( A \)-module \( X_i, i \in \mathbb{Z} \), induces a morphism of complexes \( m_a : X \to X \). We define \( K_\bullet(a; X) \) as the mapping cone \( \text{Mc}(m_a) \). Then we define inductively

\[ K_\bullet(a_1, \ldots, a_t; A) = K_\bullet(a_t, K_\bullet(a_1, \ldots, a_{t-1}; A)). \]

It is easily seen that

\[ K_\bullet(a; A) \cong K_\bullet(a_{t}; A) \otimes_A \cdots \otimes_A K_\bullet(a_1; A). \]

Therefore it follows that \( K_\bullet(a; A) \cong K_\bullet(a_\sigma; A), \) where \( a_\sigma = a_{\sigma(1)}, \ldots, a_{\sigma(t)} \) with a permutation \( \sigma \) on \( t \) letters. For an \( A \)-complex \( X \) we define \( K_\bullet(a; X) = K_\bullet(a; A) \otimes_A X \). We write \( H_i(a; X), i \in \mathbb{Z} \), for the \( i \)-th homology of \( K_\bullet(a; X) \). A short exact sequence of \( A \)-complexes \( 0 \to X' \to X \to X'' \to 0 \) induces a long exact homology sequence for the Koszul homology

\[ \cdots \to H_i(a; X') \to H_i(a; X) \to H_i(a; X'') \to H_{i-1}(a; X') \to \cdots. \]

Let \( a \) as above a system of \( t \) elements in \( A \) and \( b \in A \). Then the mapping cone construction provides a long exact homology sequence

\[ \cdots \to H_i(a; X) \to H_i(a; X) \to H_i(a, b; X) \to H_{i-1}(a; X) \to H_{i-1}(a; X) \to \cdots, \]

where the homomorphism \( H_i(a; X) \to H_i(a; X) \) is multiplication by \((-1)^i b \). Moreover, \( aH_i(a; X) = 0 \) for all \( i \in \mathbb{Z} \).

For the proof of the last statement, we recall the following well-known argument.

**Lemma 2.3.** Let \( X \) denote a complex of \( A \)-modules. Let \( a \in A \) denote an element. Then \( aH_i(a, X) = 0 \) for all \( i \in \mathbb{Z} \).
3. THE CONSTRUCTION OF COMPLEXES

First we fix notations for this section. As above let $A$ denote a commutative Noetherian ring and $q \subseteq A$. Let $a_1, \ldots, a_t$ denote a system of elements of $A$. Suppose that $a_i \in q^{c_i}$ for some integers $c_i \in \mathbb{N}$ for $i = 1, \ldots, t$. Let $M$ denote a finitely generated $A$-module. We define two complexes here, see also [8] for more detail.

**Notation 3.1.** Let $n$ denote an integer. We define a complex $K_\bullet(a, q; M; n)$ in the following way:

(a) For $0 \leq i < t$ put $K_i(a, q; M; n) = \oplus_{1 \leq j_1 < \cdots < j_i \leq t} q^{n-c_{j_1} - \cdots - c_{j_i}} M$ and $K_i(a, q; M; n) = 0$ for $i > t$ or $i < 0$.

(b) The boundary map $K_i(a, q; M; n) \to K_{i-1}(a, q; M; n)$ is defined by maps on each of the direct summands $q^{n-c_{j_1} - \cdots - c_{j_i}} M$. On $q^{n-c_{j_1} - \cdots - c_{j_i}} M$ it is the map given by $d_{j_1 \cdots j_i} \otimes 1_M$ restricted to $q^{n-c_{j_1} - \cdots - c_{j_i}} M$, where $d_{j_1 \cdots j_i}$ denotes the homomorphism as defined in 2.2.

It is clear that the image of the map is contained in $\oplus_{1 \leq j_1 < \cdots < j_{i-1} \leq t} q^{n-c_{j_1} - \cdots - c_{j_{i-1}}} M$. Clearly it is a boundary homomorphism. By the construction it follows that $K_\bullet(a, q; M; n)$ is a subcomplex of the Koszul complex $K_\bullet(a; M)$ for each $n \in \mathbb{N}$.

Another way for the construction is the following.

**Remark 3.2.** Let $R_A(q)$ and $R_M(q)$ denote the Rees ring and the Rees module. For $a_i, i = 1, \ldots, t$, we consider $a_i T^{c_i} \in [R_A(q)]_{c_i}$. Then we have the system $a T^{c_i} = a_1 T^{c_1}, \ldots, a_t T^{c_t}$ of elements of $R_A(q)$. Note that $\deg a_i T^{c_i} = c_i, i = 1, \ldots, t$. Then we may consider the Koszul complex $K_\bullet(a T^{c_i}; R_M(q))$. This is a complex of graded $R_A(q)$-modules. It is easily seen that the degree $n$-component $[K_\bullet(a T^{c_i}; R_M(q))]_n$ of $K_\bullet(a T^{c_i}; R_M(q))$ is the complex $K_\bullet(a, q; M; n)$ as introduced in 3.1. We write $H_i(a, q; M; n)$ for the $i$-th homology of $K_\bullet(a, q; M; n)$ for $i \in \mathbb{Z}$.

We come now to the definition of second complex.

**Definition 3.3.** With the previous notation we define $L_\bullet(a, q; M; n)$ the quotient of the embedding $K_\bullet(a, q; M; n) \to K_\bullet(a; M)$. That is there is a short exact sequence of complexes

$$0 \to K_\bullet(a, q; M; n) \to K_\bullet(a; M) \to L_\bullet(a, q; M; n) \to 0.$$

Note that $L_i(a, q; M; n) \cong \oplus_{1 \leq j_1 < \cdots < j_i \leq t} M/q^{n-c_{j_1} - \cdots - c_{j_i}} M$. The boundary maps are those induced by the Koszul complex. We write $L_i(a, q; M; n)$ for the $i$-th homology of $L_\bullet(a, q; M; n)$ and any $i \in \mathbb{Z}$.

For a construction by mapping cones we need the following technical result. For a morphism $f : X \to Y$ we write $C(f)$ for the mapping cone of $f$.

**Lemma 3.4.** With the previous notation let $b \in q^d$ an element. The multiplication map by $b$ induces the following morphisms

$$m_b(K) : K_\bullet(a, q; M; n-d) \to K_\bullet(a, q; M; n)$$

and $m_b(L) : L_\bullet(a, q; M; n-d) \to L_\bullet(a, q; M; n)$ of complexes. They induce isomorphism of complexes

$$C(m_b(K)) \cong K_\bullet(a, b, q; M; n)$$

and $C(m_b(L)) \cong L_\bullet(a, b, q; M; n)$.

**Proof.** The proof follows easily by the structure of the complexes and the mapping cone construction.

We begin with a few properties of the previous complexes.

**Theorem 3.5.** Let $a = a_1, \ldots, a_t$ denote a system of elements of $A$, $q \subseteq A$ an ideal and $M$ a finitely generated $A$-module. Let $n \in \mathbb{N}$ denote an integer.

(a) $H_i(a, q; M; n) \cong H_i(a_{\sigma}, q; M; n)$ and $L_i(a, q; M; n) \cong L_i(a_{\sigma}, q; M; n)$ for all $i \in \mathbb{Z}$ and any $\sigma$, a permutation on $t$ letters.
Lemma 4.2. Let \( A \), \( B \), and \( C \) be finitely generated \( A \)-modules. Let \( a : A \to B \) and \( b : B \to C \). Then \( a \circ b : A \to C \) is a homomorphism if and only if \( a \) is surjective and \( b \) is injective.

Proof. The statement in (a) follows by virtue of the short exact sequence of complexes in 3.3 and the long exact homology sequence. Note that the homology of Koszul complexes is isomorphic under permutations.

The claim in (c) is a consequence of (b) since the homology modules \( H_i(A, q, M; n) \) and \( L_i(A, q, M; n) \) are finitely generated \( A \)-modules.

For the proof of (b) we follow the mapping cone construction of 3.4 with the arguments of 2.3. To this end let \( K_n = K(A, q, M; n) \) and \( C = C(m_{k}(K)_{n}) = K(A, b, q, M; n) \). Then there is a short exact sequence of complexes

\[
0 \to K_n \to C \to K_{n-d}[-1] \to 0.
\]

The differential \( \partial_{i} \) on \( (x, y) \in C_{i} = (K_{n-d})_{i-1} \oplus (K)_{i} \) is given by

\[
\partial_{i}(x, y) = (d_{i-1}(x), d_{i}(y) + (-1)^{i-1}b(x)).
\]

Suppose that \( \partial_{i}(x, y) = 0 \), i.e., \( d_{i-1}(x) = 0 \) and \( d_{i}(y) = (-1)^{i}b(x) \). Then

\[
(y, 0) \in (K_{n-d})_{i} \oplus (K)_{i+1} = C_{i+1}
\]

because \( (K)_{i} \subseteq (K_{n-d})_{i} \) and therefore \( \partial_{i+1}((-1)^{i}y, 0) = b(x, y) \). That is \( bH_{i}(C) = 0 \) for all \( i \in \mathbb{Z} \).

In order to show the claim in (b) we use the previous argument. So let us consider \( K(A, q, M; n) = C(m_{k}(K(A, q, M; n))) \), where \( a_{i} = a_{1}, \ldots, a_{k-1} \). The previous argument shows \( a_{i}H_{i}(A, q, M; n) = 0 \) for all \( i \in \mathbb{Z} \). By view of (a) this finishes the proof in the case of \( H_{i}(A, q, M; n) \).

For the proof of \( \partial H_{i}(A, q, M; n) = 0 \) we follow the same arguments. Instead of the injection \( (K)_{i} \subseteq (K_{n-d})_{i} \) we use the surjection \( (L)_{i} \to (L_{n-d})_{i} \), where \( L_{n} = L(A, q, M; n) \). We skip the details here.

4. Euler characteristics

Let \( A \) denote a commutative ring. Let \( X \) denote a complex of \( A \)-modules.

**Definition 4.1.** Let \( X : 0 \to X_{n} \to \cdots \to X_{1} \to X_{0} \to 0 \) denote a bounded complex of \( A \)-modules. Suppose that \( H_{i}(X), i = 0, 1, \ldots, n, \) is an \( A \)-module of finite length. Then

\[
\chi_{A}(X) = \sum_{i=0}^{n} (-1)^{i} \ell_{A}(H_{i}(X))
\]

is called the Euler characteristic of \( X \).

We collect a few well known facts about Euler characteristics.

**Lemma 4.2.** Let \( A \) denote a Noetherian commutative ring.

(a) Let \( 0 \to X' \to X \to X'' \to 0 \) denote a short exact sequence of complexes such that all the homology modules are of finite length. Then \( \chi_{A}(X) = \chi_{A}(X') + \chi_{A}(X'') \).

(b) Suppose \( X : 0 \to X_{n} \to \cdots \to X_{1} \to X_{0} \to 0 \) is a bounded complex such that \( X_{i}, i = 0, 1, \ldots, n, \) is of finite length. Then \( \chi_{A}(X) = \sum_{i=0}^{n} (-1)^{i} \ell_{A}(X_{i}) \).

Proof. The statement in (a) follows by the long exact cohomology sequence derived by \( 0 \to X' \to X \to X'' \to 0 \). The second statement might be proved by induction on \( n \), the length of the complex \( X \).

As an application we get the following result about multiplicities, originally shown by [11] and [13].

**Proposition 4.3.** Let \( (A, m) \) be a local ring and \( a_{1}, \ldots, a_{d} \in m \) a system of parameters for \( M \), a finitely generated \( A \)-module. Then

\[
\chi_{A}(\mathfrak{a}; M) = c_{0}(\mathfrak{a}; M),
\]

where \( \chi_{A}(\mathfrak{a}; M) = \chi_{A}(K(A, \mathfrak{a}, M)) \) and \( c_{0}(\mathfrak{a}; M) \) denotes the Hilbert-Samuel multiplicity.
Proof. Let \( q = a_1, \ldots, a_d \) be the system of parameters and \( \alpha A = q \). We choose \( c_i = 1, i = 1, \ldots, d \).

Then the short exact sequence of \( 3.3 \) has the following form

\[
0 \to K_\bullet(a, q; M; n) \to K_\bullet(q; M) \to L_\bullet(a, q; M; n) \to 0.
\]

All of the three complexes have homology modules of finite length and therefore \( \chi_A(a; M) = \chi_A(K_\bullet(a, q; M; n)) + \chi_A(L_\bullet(a, q; M; n)) \) for all \( n \in \mathbb{N} \).

First we show that \( \chi_A(K_\bullet(a, q; M; n)) = 0 \) for all \( n \gg 0 \). To this end recall that \( H_i(a, q; M; n) = [H_i(aT; R_M(a))]_n \). We know that \( aT(H_i(aT; R_M(a))) = 0 \) for all \( i = 0, \ldots, d \). Therefore \( H_i(aT; R_M(a)) \) is a finitely generated module over \( R_A(a)/aT R_A(q) = A/aA \). This implies that \( [H_i(aT; R_M(a))]_n = H_i(a, q; M; n) = 0 \) for all \( n \gg 0 \). That is \( \chi_A(K_\bullet(a, q; M; n)) = 0 \) for all \( n \gg 0 \).

By view of Lemma 4.2 (b) we get \( \chi_A(L_\bullet(a, q; M; n)) = \sum_{i=0}^d (-1)^i \ell_A(M/a^{n-i} M) \). For \( n \gg 0 \) the length \( \ell_A(M/a^n M) \) is given by the Hilbert polynomial \( e_0(a; M) (d+n) + \ldots + e_d(a; M) \). Therefore, it follows that \( \chi_A(L_\bullet(a, q; M; n)) = e_0(a; M) \). This completes the argument.

The more general situation of a system of parameters \( a = a_1, \ldots, a_d \) of a finitely generated \( A \)-module \( M \) of a local ring \((A, m)\) and an ideal \( q \supset a \) with \( a_i \in q^n, i = 1, \ldots, d \) is investigated in the following.

**Proposition 4.4.** With the previous notation we have the equality

\[
e_0(a; M) = e_1 \cdot \ldots \cdot e_d \cdot e_0(q; M) + \chi_A(K_\bullet(a, q; M; n))
\]

for all \( n \gg 0 \). In particular, for all \( n \gg 0 \) the Euler characteristic \( \chi_A(K_\bullet(a, q; M; n)) \) is a constant.

**Proof.** The proof follows by the inspection of the short exact sequence of complexes

\[
0 \to K_\bullet(a, q; M; n) \to K_\bullet(q; M) \to L_\bullet(a, q; M; n) \to 0
\]

of \( 3.3 \). By view of Proposition 4.3 we have \( e_0(a; M) \) for the Euler characteristic of the complex in the middle. For the Euler characteristic on the right we get (see \( 4.2 \) (b))

\[
\chi_A(L_\bullet(a, q; M; n)) = \sum_{i=0}^d (-1)^i \sum_{1 \leq j_1 < \ldots < j_i \leq d} \ell_A(M/q^{n-e_{j_1}-\ldots-e_{j_i}} M),
\]

which gives the first summand in the above formula (see also \( [2] \) for the details in the case of \( M = A \)). This finally proves the claim.

For several reasons it would be interesting to have an answer to the following problem.

**Problem 4.5.** With the notation of Proposition 4.4 it would be of some interest to give an interpretation of \( \chi_A(a, q; M) := \chi_A(K_\bullet(a, q; M; n)) \) for large \( n \gg 0 \) independently of \( n \). By a slight modification of an argument given in \( [2] \) it follows that \( \chi_A(a, q, M) \geq 0 \).

First we shall use the previous results in order to prove a few formulas for the multiplicity. The following Theorem provides a simplified proof of some of the main results of Auslander and Buchsbaum (see \( [1] \)), originally proved by the use of spectral sequences.

**Theorem 4.6.** Let \((A, m)\) denote a local ring with \( a = a_1, \ldots, a_d, d = \dim_A M, \) a system of parameters for a finitely generated \( A \)-module \( M \).

(a) \( e_0(a; M) = e_0(a'; M/aM) \) of \( a = a_1 \) and \( a' = a_2, \ldots, a_d \), where \( a = a_1 \) and \( a' = a_2, \ldots, a_d \).

(b) \( e_0(a; M) = e_0(a; a' M) + e_0(a; b; a' M) \), where \( a_1 = a \) and \( a' = a_2, \ldots, a_d \).

(c) \( e_0(a; M) = n \cdot e_0(a; M) \), where \( a^2 = a_1^2, \ldots, a_d^2 \) for \( n = (n_1, \ldots, n_d) \in \mathbb{N}^d \) and \( n = n_1 \cdot \ldots \cdot n_d \).
Proof. We start with the comparison of the Koszul complexes \( K_\bullet(b; M) \) and \( K_\bullet(ab; M) \). This leads to the following commutative diagram with exact rows

\[
\begin{array}{cccc}
0 & 0 & M & 0 \\
\downarrow b & \downarrow ab & \downarrow \alpha & \\
0 & 0 & a : M & M/aM \to 0 \\
\end{array}
\]

where the columns are the Koszul complexes. That is, we have an exact sequence of complexes

\[
0 \to 0 : M \to K_\bullet(b; M) \to K_\bullet(ab; M) \to M/aM \to 0.
\]

We tensor now by \( K_\bullet(a'; A) \), a bounded complex of free \( A \)-modules. By the definition it induces an exact sequence of complexes

\[
0 \to K_\bullet(a'; 0 : M a) \to K_\bullet(b, a'; M) \to K_\bullet(ab, a'; M) \to K_\bullet(a'; M/aM) \to 0.
\]

Now we inspect the previous sequence in the case \( b = 1 \). Then it follows that \( K_\bullet(1, a'; M) \) is exact. By virtue of the Euler characteristic and by 4.3, the statement in (a) follows.

Next we look at the case of a general \( b \). With the previous result (a) it implies that

\[
e_0(ab, a'; M) = e_0(a, a'; M) + e_0(b, a'; M)
\]

which proves (b).

By induction and permutability of the sequence this yields the statement in (c).

It is noteworthy to say that \( e_0(a'; 0 : M a) \neq 0 \) if and only if \( \dim A 0 : M a = \dim A M - 1 \).

5. On an equality

As before let \((A, m)\) denote a local ring with \( M \) a finitely generated \( A \)-module. Let \( q \) denote an ideal of \( A \) such that \( \ell_A(M/qM) < \infty \). For a system of parameters \( a = a_1, \ldots, a_d, d = \dim_A M \), of \( M \) suppose that \( a_i \in q^n \setminus q^{n+1}, i = 1, \ldots, d \). Besides of the Rees module \( R_M(q) \) we need the following.

**Definition 5.1.** With the previous notation we investigate the ring \( \mathfrak{R} = A[a_1T^{c_1}, \ldots, a_dT^{c_d}] \subset A[T] \). Note that \( \mathfrak{R} \) is a graded \( A \)-algebra. Because of \( a_i q^n \subseteq q^{n+c_i}, i = 1, \ldots, d \) it follows that the Rees module \( R_M(q) \) is an \( \mathfrak{R} \)-module.

In the following we shall explore when \( R_M(q) \) is a finitely generated \( \mathfrak{R} \)-module.

**Lemma 5.2.** With the previous notation the following conditions are equivalent:

(i) \( R_M(q) \) is a finitely generated \( \mathfrak{R} \)-module.

(ii) There is a positive integer \( k \) such that \( q^n M = \sum_{i=1}^{d} a_i q^{n-c_i} M \) for all \( n > k \).

(iii) The initial forms \( a_1^*, \ldots, a_d^* \) are a system of parameters of \( G_M(q) \).

**Proof.** First we prove (i) \( \implies \) (ii). If \( R_M(q) \) is finitely generated over \( \mathfrak{R} \), then

\[
R_M(q) = \mathfrak{R}(m_1T^{d_1}, \ldots, m_rT^{d_r}) \text{ for } m_iT^{d_i} \in R_M(q)_{d_i}.
\]

Let \( k = \max\{d_1, \ldots, d_r\} \) and \( n > k \). Let \( m \in q^n M \) and therefore

\[
mT^n = \sum_{j=1}^{r} \left( \sum_{(\alpha_j) \in \mathcal{A}_n} r^{(j)}(a_1T^{c_1})^{\alpha_j,1} \cdots (a_dT^{c_d})^{\alpha_j,d} \right) m_jT^{d_j}, \text{ with } r^{(j)} \in A
\]
and \( d_j + \sum c_i \alpha_{j,i} \geq n \). The last inequality implies
\[
\sum c_i \alpha_{j,i} \geq n - d_j \geq n - k > 0.
\]
Therefore for each \( j \) there is an \( i \) such that \( \alpha_{j,i} > 0 \) and \( mT^n \in \sum_{i=1}^d a_i q^{n-c_i} M^T \), which proves the statement in (ii).

For the converse we claim \( R_M(q) = \mathfrak{R}(M, qMT, \ldots, q^k M^k) \). Let \( mT^n \) with \( m \in q^n M \). If \( n \leq k \) then clearly \( mT^n \in \mathfrak{R}(M, qMT, \ldots, q^k M^k) \). If \( n > k \), then \( m = \sum_{i=1}^d a_i m_i \), \( m_i \in q^{n-c_i} M \). By induction \( m_i T^{n-c_i} \in \mathfrak{R}(M, qMT, \ldots, q^k M^k) \) and therefore \( mT^n = \sum_{i=1}^d a_i T^{c_i} m_i T^{n-c_i} \in \mathfrak{R}(M, qMT, \ldots, q^k M^k) \), which finishes the argument.

For the equivalence of (ii) and (iii) recall that \( a_i^* \) is a homogeneous system of parameters if and only if \([G_M(q)/(a_1^*, \ldots, a_d^*)G_M(q)]_n = 0 \) for all \( n > k \). This is equivalent to
\[
q^n M = \sum_{i=1}^d a_i q^{n-c_i} M + q^{n+1} M \text{ for all } n > k.
\]
By Nakayama Lemma this is equivalent to the condition in (ii).

For the following we define \( c = c_1 \cdots c_d \) and \( e_i = c/c_i, i = 1, \ldots, d \). Then \( a^c = a_1^{e_1}, \ldots, a_d^{e_d} \) is a system of parameters of \( M \) and \( a_i^{e_i} \in q^c \). With these notation we get the following commutative diagram
\[
\begin{array}{c}
R_M(q^c) \\ \cup \\
\mathfrak{S} \\ \cup \\
R_M(q)
\end{array}
\]
where \( \mathfrak{S} = A[a_1^{c_1} T^{c_1}, \ldots, a_d^{c_d} T^{c_d}] \). It is easily seen that \( \mathfrak{S} \subset \mathfrak{R} \) is a finitely generated extension since it is integral. Note that \( a_i^{c_i} T^{c_i} = (a_i T^{c_i})^{c_i}, i = 1, \ldots, d \).

**Corollary 5.3.** With the previous notation the following conditions are equivalent:

(i) The initial forms \( a_1^*, \ldots, a_d^* \) are a system of parameters of \( G_M(q) \).

(ii) There is an integer \( k > 0 \) such that \( q^k M = \sum_{i=1}^d a_i^{c_i} q^{n-c_i} M \) for all \( n > k \).

(iii) The initial forms \( (a_1^*)^*, \ldots, (a_d^*)^* \) in \( G_A(q^c) \) are a system of parameters in \( G_M(q^c) \).

**Proof.** We have the isomorphism \( R_M(q^c) \cong M[q^c T^c] =: \mathfrak{N} \) and
\[
R_M(q) \cong \mathfrak{M} \oplus q^c T^c \mathfrak{M} \oplus \cdots \oplus q^{c-1} T^{c-1} \mathfrak{M}.
\]
Whence \( \mathfrak{S} \subset R_M(q^c) \) is a finitely generated extension if and only if \( \mathfrak{S} \subset R_M(q) \) is a finitely generated extension. Then Artin-Rees lemma yields the equivalence of (i) and (ii) by view of 5.2.

The equivalence of (ii) and (iii) follows by Lemma 5.2.

**Corollary 5.4.** With the previous notation suppose that \( a_1^*, \ldots, a_d^* \) forms a system of parameters in \( G_M(q) \). Then \( e_0(\mathfrak{S}; M) = c_1 \cdots c_d \cdot e_0(q; M) \) and therefore \( \chi_A(K_0(\mathfrak{S}, q, M; n)) = 0 \) for \( n \gg 0 \).

**Proof.** By view of 5.3 we get \( (q^c)^n M = \sum_{i=1}^d a_i^{c_i} (q^c)^{n-1} M \) and \( (q^c)^l M \subset q^{c-1} M \subset q^c M \) for some \( l \in N \). Moreover it follows that \( (q^c)^{n+k} M \subset (q^c)^{l+k} M \) for all \( k \geq l \). Then
\[
\ell_A(M/(q^c)^{n+k} M) \geq \ell_A(M/(q^c)^{l+k} M) \geq \ell_A(M/(q^c)^{k} M),
\]
which implies that \( e_0(q^c; M) = e_0(q^c, M) \). Because of \( e_0(q^c; M) = c_d \cdot e_0(q; M) \) as easily seen and \( e_0(q^c, M) = e_1 \cdot \cdots \cdot e_d \cdot e_0(q; M) \) (see 4.6). This finishes the proof.

The previous result is a generalization of [2, Theorem 5.1] to the situation of finitely generated \( A \)-modules. In the case of a formally equidimensional ring the converse is also true (see [2, Theorem 5.2]).
6. The subregular case

As a consequence of the definition of \( \mathcal{L}_\bullet(a, q, M; n) \) we get the following equality

\[
c_1 \cdot \ldots \cdot c_d \cdot e_0(q; M) = \sum_{i=1}^{d} (-1)^i \ell_A(L_i(a, q, M; n)) \text{ for all } n \gg 0.
\]

We have \( L_0(a, q, M; n) \equiv M/(q, q^n)M \). The other homology modules are difficult to describe. For the vanishing of some of them in relation to the existence of \( G_M(q) \)-regular sequences we refer to [7].

In the previous section we have shown that \( e_0(a; M) = c \cdot e_0(q; M) \) provided \( a_1^*, \ldots, a_d^* \) forms a system of parameters of \( G_M(q) \). In the next, we consider the case that \( (a_1^*, \ldots, a_d^*)G_A(q) \) contains a \( G_M(q) \)-regular sequence of length \( d - 1 \).

We start with the behavior of \( \chi(a, q, M; n) \) by passing to a certain element. To this end, let \( a' = a_2, \ldots, a_d, a = a_1 \) and \( c_1 = f \).

**Lemma 6.1.** Suppose that \( a^* \) is \( G_M(q) \) regular. Then \( \chi(a, q, M; n) = \chi(a', q, M/\mathfrak{a}M; n) \) for all \( n \in \mathbb{Z} \).

**Proof.** If \( a^* \) is \( G_M(q) \)-regular, then \( 0 :_M a = 0 \) and \( q^nM :_M a = q^{n-f}M \) for all \( n \geq 0 \). So it follows that \( 0 :_{R_M(q)} aT^f = 0 \) and \( R_M(q)/(aT^f)R_M(q) \cong R_{M/\mathfrak{a}M}(q) \).

Moreover, there is the following diagram with exact rows

\[
\begin{array}{ccc}
0 & \to & 0 \\
\downarrow & & \downarrow \\
0 & \to & R_{M}(q)(-f) \to R_{M}(q)(-f) \to 0 \\
\downarrow & & \downarrow \\
0 & \to & R_{M}(q)(-f) \to aT^f \to R_{M}(q) \to R_{M/\mathfrak{a}M}(q) \to 0 \\
\downarrow & & \downarrow \\
0 & \to & 0 \\
\end{array}
\]

where the columns are the Koszul complexes \( K_\bullet(1; R_{M}(q))(-f) \) and \( K_\bullet(aT^f; R_{M}(q)) \) resp. That is, we have a short exact sequence of complexes

\[
0 \to K_\bullet(1; R_{M}(q))(-f) \xrightarrow{aT^f} K_\bullet(aT^f; R_{M}(q)) \to R_{M/\mathfrak{a}M}(q) \to 0.
\]

By tensoring with \( K_\bullet(aT^{\mathfrak{a}'}; R_A(q)) \) it provides a short exact sequence of complexes

\[
0 \to K_\bullet(1, aT^{\mathfrak{a}'}; R_{M}(q))(-f) \xrightarrow{aT^f} K_\bullet(aT^{\mathfrak{a}'}; R_{M}(q)) \to K_\bullet(aT^{\mathfrak{a}'}; R_{M/\mathfrak{a}M}(q)) \to 0.
\]

Since the first Koszul complex in the previous sequence is exact (see 2.3), the claim follows by the definition.

The following result is a particular case of [15, VIII, Lemma 3] resp. to [12, Lemma 1.6], a generalization to filtered modules.

**Lemma 6.2.** Let \( q \subset A \) denote an ideal such that \( \ell_A(M/qM) < \infty \).

(a) Suppose that \( a^* \) is \( G_M(q) \) regular. Then \( f \cdot e_0(q; M) = e_0(q; M/\mathfrak{a}M) \).

(b) Let \( \dim_A M = 1 \) and \( a \in q^n \) be a parameter of \( M \). Then

\[
f \cdot e_0(q; M) = \ell_A(M/\mathfrak{a}M) - \ell_A(q^nM :_M a/q^{n-f}M)
\]

for all \( n \gg 0 \). In particular \( \ell_A(q^nM :_M a/q^{n-f}M) \) is a constant for all \( n \gg 0 \).
Proof. The statements follow by counting the length in the exact sequence
\[ 0 \to q^nM :_M a/q^{n-1}M \to M/q^{n-1}M \xrightarrow{a} M/q^nM \to M/(a, q^n)M \to 0 \]
for all \( n \in \mathbb{N} \). If \( a^* \) is \( G_M(q) \) regular, then \( q^nM :_M a = q^{n-1}M \) for all \( n \in \mathbb{N} \). If \( \dim_A M = 1 \) and \( a \in q^f \) is a parameter of \( M \), then \( \ell_A(M/q^{n-1}M) = \ell_A(M/q^nM) = f \cdot e_0(q; M) \) for all \( n \gg 0 \). Moreover \( q^nM \subset aM \) for all \( n \gg 0 \). For details see [15, VIII, Lemma 3] resp. [12, Lemma 1.6].

Now we are prepared for the main situation of this section.

**Corollary 6.3.** With the previous notation assume that \( a_1^*, \ldots, a_{d-1}^* \) is a \( G_M(q) \)-regular sequence. Then
\[ c_1 \cdot \ldots \cdot c_d \cdot e_0(q; M) = \ell_A(M/qM) - \ell_A((a_1^*, q^n)M :_M a_d/(a_1^*, q^{n-c_d})M) \]
for all \( n \gg 0 \) where \( a_1^* = a_1, \ldots, a_{d-1}^* \).

**Proof.** By an iterative application of 6.2 (a) it follows \( c_1 \cdot \ldots \cdot c_{d-1} \cdot e_0(q; M) = e_0(q; M/a'M) \). By view of 6.2 (b) it implies \( c_d \cdot e_0(q; M/a'M) = \ell_A(M/aM) - \ell_A((a_1^*, q^n)M :_M a_d/(a_1^*, q^{n-c_d})M) \). Putting both together yields the claim.

It is a problem to give an interpretation of the constant \( \ell_A((a_1^*, q^n)M :_M a_d/(a_1^*, q^{n-c_d})M) \) in intrinsic data of the module \( M \). As a partial result in this direction we get the following.

**Lemma 6.4.** With the notation and assumption of 6.3 we have the following inequality
\[ \ell_A(M/qM) \geq c \cdot e_0(q; M) + \mathfrak{r}, \]
where \( \mathfrak{r} = \ell_A([\text{Ext}^{d-1}_{G_A(q)}(G_A(q)/(a^*_d), G_M(q))]_{n-s-1}) \) is a constant for \( n \gg 0 \) and \( s \) denotes \( s = c_1 + \ldots + c_d \).

**Proof.** There is an injection of
\[ \mathfrak{X} := [(a^*_d)G_M(q) : a_d^*/(a^*_d)G_M(q)]_{n-c_d-1} \cong ((a_1^*, q^n)M :_M a_d) \cap (a_1^*, q^{n-c_d-1})M/(a_1^*, q^{n-c_d})M \]
into \( (a_1^*, q^n)M :_M a_d/(a_1^*, q^{n-c_d})M \). Moreover, there is an isomorphism
\[ \mathfrak{X} \cong [\text{Hom}_{G_A(q)}(G_A(q)/(a^*_d), G_M(q)/(a^*_d)G_M(q))]_{n-c_d-1}. \]
Since \( a^*_d \) is a \( G_M(q) \)-regular sequence of length \( d - 1 \), it follows that
\[ \mathfrak{X} \cong [\text{Ext}^{d-1}_{G_A(q)}(G_A(q)/(a^*_d), G_M(q))]_{n-s-1}, \]
which proves the claim.

We conclude with a geometric application on the local Bezout numbers.

**Example 6.5.** Let \( C = V(F), D = V(G) \subset \mathbb{P}^2_K \) be two curves in the projective plane without a common component. Let \( \mu(P; C, D) \) denote the local intersection multiplicity of \( P \) in \( C \cap D \). In the case of \( P \) is the origin, it follows that \( \mu(P; C, D) = \ell_A(A/(f, g)) \), where \( A = \mathbb{k}[x, y]_{(x, y)} \) and \( f, g \) denote the equations in \( A \). Since \( C, D \) have no component in common, \( \{f, g\} \) forms as system of parameters in \( A \). Let \( m \) denote the maximal ideal of \( A \). Then \( B := \mathbb{k}[X, Y] = G_A(m) \) and \( 1 = e_0(m; A) \). We distinguish two cases:

1. \( C \) and \( D \) intersect transversally in the origin. Then \( f^*, g^* \) form a homogeneous system of parameters in \( B \) and therefore
   \[ \ell_A(A/(f, g)) = c \cdot d, \]
   where \( c, d \) denote the initial degree of \( f, g \) resp.
2. Suppose that \( C \) and \( D \) do not intersect transversally. Then
   \[ \ell_A(A/(f, g)) \geq c \cdot d + t, \]
   where \( t \) denotes the number of common tangents of \( f \) and \( g \) at the origin counted with multiplicities.
Proof. First note that $A$ is a Cohen-Macaulay ring and therefore $e_0(f, g; A) = \ell_A(A/(f, g))$. Then the equality in the first case is a consequence of 5.4. To this end note that $f^*, g^*$ forms a system of parameters in $B$ provided $C$ and $D$ intersect transversally in the origin.

For the second case we use 6.4. To this end we have to show that $\pi = t$. We put $\mathfrak{q}_n = f^*B :_B g^*/f^*B$. Since $f^*, g^*$ are not relatively prime, we write $f^* = h \cdot f', g^* = h \cdot g'$ with homogeneous polynomials $f', g', h \in B$, where $f', g'$ are relatively prime. Then
\[\mathfrak{q}_n = f'hB :_B g'h/f'hB \cong f'B/f'BhB \cong B/hB[-\deg f']\]
and $\dim_k \mathfrak{q}_n = \deg h$ for all $n \gg 0$. Since $\deg h$ is the number of common tangents counted with multiplicities, this confirms the second case.

The second case was also proved by Bydžovský (see [3]). Note that this is an improvement of the corresponding result in [5] where it is shown that $\ell_A(A/(f, g)) \geq c \cdot d + 1$ in case there is a common tangent.

A further discussion of the difference $\ell_A(A/(f, g)) - c \cdot d - t \geq 0$ is given in [2]. There is also another approach by blowing-ups.

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