Persistence distributions for non gaussian markovian processes

JEAN FARAGO 1

1 Laboratoire de Physique UMR CNRS 5672 - ENS de Lyon, 69364 Lyon cedex 07 France

PACS. 05.10.-a – Computational methods in statistical physics and nonlinear dynamics.
PACS. 05.40.-a – Fluctuation phenomena, random processes, noise, and Brownian motion.

Abstract. – We propose a systematic method to derive the asymptotic behaviour of the persistence distribution, for a large class of stochastic processes described by a general Fokker-Planck equation in one dimension. Theoretical predictions are compared to simple solvable systems and to numerical calculations. The very good agreement attests the validity of this approach.

Introduction. – The concept of persistence has recently motivated a lot of works, both experimentally and theoretically (see [1] and references therein): it appeared that this simple concept – i.e. the probability \( G(t) \) that a random variable \( X(t) \) never goes above a certain value (usually its mean value) during the whole time interval \([0, t]\) – hides a real complexity, since its definition involves a complete knowledge of the process over a large time interval. Consequently, the persistence distribution gives information on the details of the process, which are not redundant with the traditional statistical tools, as for instance the correlation functions. Until now, the works was focused, besides discrete systems as spin systems [2], essentially on gaussian processes, and the efforts concentrated on the influence of the non markovian character of the process on the persistence exponent. However, it was shown recently [3] that the concept of persistence can also be physically relevant for non gaussian situations [4]. Another physical situation leads naturally to study persistence in a non gaussian context: in nonlinear physics, the combination of discreteness and nonlinearity gives rise to localisation phenomena (the “breather-modes” [5]), which are often extremely pinned where they appear: a powerful tool to characterise the lifetime of these objects is provided by the persistence behaviour of the energy density on a site. In weakly coupled systems, the evolution of the energy density can moreover be described by a non gaussian markovian process, with energy dependent drift and diffusion coefficients.

(1) There is a slight ambiguity associated with the term “non gaussian”; in [4], the process is non gaussian due to the non gaussianity of the noise; but the process can be non gaussian, despite a gaussian noise, if an external force is present and not linear.

© EDP Sciences
In this letter, we introduce a general method to compute the persistence behaviour for a large category of processes described by a one-dimensional Fokker-Planck equation. Essentially, the method deals with systems where the persistence decreases slower than an exponential law. We check the validity of our predictions by a comparison with exact results and numerical simulations.

**The method.** We consider $X(t)$ a continuous markovian stochastic process. By definition, the probability density function (pdf) $p(x,t) = \text{Prob}(X(t) = x)$ obeys the Fokker-Planck equation

$$\partial_t p = -\partial_x (A(x)p) + \frac{1}{2} \partial^2_{xx} (B(x)p)$$

(1)

where the functions $A$ and $B$ characterise the dynamical process. Once given a real value $x_0 \in \mathbb{R}$, the probability of (positive) persistence $G_+(x, t)$, i.e. the probability that a particle, which has initiated its trajectory at $x > x_0$ at time $t = 0$, has never crossed $x_0$ up to the time $t$, obeys the backward Fokker-Planck equation

$$\partial_t G_+ = A(x)\partial_x G_+ + \frac{B(x)}{2} \partial^2_{xx} G_+$$

(2)

with the boundary conditions $G_+(x > x_0, t = 0) = 1, G_+(x = x_0, t > 0) = 0$.

For the sake of clarity, we will first temporarily restrict the discussion to cases where $B(x) = 2$, and see later how to take into consideration the general case. In addition, we assume that the stationary solution of (1), $p_{st}(x) \propto \exp \int^x A(x')dx'$, is bounded but not necessarily normalisable (it means just that the potential of the force doesn’t go to $-\infty$).

The Fokker-Planck equation (2) with constant diffusion coefficient can be mapped on a Schrödinger equation, by defining $\psi_+(x,t) = G_+(x,t)\sqrt{p_{st}(x)}$ : it reads

$$\partial_t \psi_+ = \partial^2_{xx} \psi_+ - (\frac{A'}{2} + \frac{A^2}{4}) \psi_+.$$  

(3)

The effective potential $V(x)$ of this equation is defined on $[x_0, +\infty]$ by $V(x > x_0) = A'(x)/2 + A^2(x)/4$ and $V(x_0) = \infty$; the last precision ensures the boundary conditions $\psi_+(x_0, t) = 0, \forall t \geq 0$. It can be shown that the spectrum of this Schrödinger operator $[-\partial^2_{xx} + V(x)]$ is only located in $[0, +\infty]$ (due to the boundedness of $p_{st}$), and we know from quantum mechanics that it usually consists of a discrete part $\{\lambda_1 < \lambda_2 < \ldots < \lambda_p\}$ (with normalisable eigenfunctions $\psi_{\lambda_i}$), surmounted by a continuous part $[\lambda_c, +\infty]$ (with non normalisable eigenfunctions $\psi_{\lambda_c}$). The value of $\lambda_c$ is usually easy to determine (since $\lambda_c$ is more or less related to $\lim_{x \to +\infty} A^2(x)/4$).

We obtain therefore the persistence distribution $G_+$ as

$$G_+(x, t) = \frac{1}{\sqrt{p_{st}(x)}} \left\{ \sum_{i=1}^{p} e^{-\lambda_i t} \psi_{\lambda_i}(x) B_{\lambda_i} + \int_{\lambda_c}^{\infty} d\lambda e^{-\lambda t} \psi_{\lambda}(x) B_{\lambda} \right\}$$

(4)

where $B_{\lambda} = \int_{x_0}^{\infty} dx \ \psi_{\lambda}(x) \sqrt{p_{st}(x)}$

The asymptotic temporal dependence of $G_+$ can therefore exhibit three different scenarios: first, if a discrete family of eigenvalues exists for the Schrödinger operator, the tail of distribution is proportional to $\exp(-\lambda_1 t)$, and the appropriate quantity which describes the
asymptotic behaviour is the persistence time $1/\lambda_1$. Consequently, the possibility of calculating this time is simply related to the ability of estimating the lowest eigenvalue of the corresponding Schrödinger problem.

If we consider now the cases without bound states, there are two possibilities: if $\lambda_c$ is different from zero (existence of a gap), the behaviour of $G_+$ will not be simple a priori: $G_+(t) \propto f(t) \exp(-\lambda_c t)$ (for large $t$), where $f(t)$ is unknown. An example of such a situation is the case of a particle experiencing a constant force toward 0 ($A < 0$ constant). In that simple case, we have $f(t) \propto t^{-3/2}$. Our method does not deal with these cases, which we could call marginal because the force acting on the particle neither grows nor disappears at infinity.

Finally, if $\lambda$ is equal to zero ("gapless" situation), we will show that there is a general procedure for obtaining the asymptotic behaviour of the persistence, whatever the $A(x)$ under consideration. First, the formal formula

$$G_+(x,t) = \frac{1}{\sqrt{\rho_{st}(x)}} \int_0^\infty d\lambda \ e^{-\lambda t} \psi_\lambda(x) B_\lambda$$

(5)

shows that we have to determine the $\lambda \to 0$ behaviours of $B_\lambda$ and $\psi_\lambda$ in order to get the $t \to \infty$ limit. The cornerstone of our method is that it is possible to determine this behaviour, thanks to a peculiarity of the Schrödinger problem under consideration: the "$\lambda = 0$ eigenfunction" is known, because it is related to the stationary solution of the corresponding Fokker-Planck equation. The quotation are required, because this eigenfunction is highly diverging at infinity. As the "$\lambda = 0$" eigenfunction must fulfil the boundary conditions at 0, we can write it as

$$\psi_0(x) = \sqrt{\rho_{st}(x)} \int_{x_0}^x \frac{dx'}{\rho_{st}(x')}$$

(6)

This "eigenfunction" is useful, since it is reasonable to assume that despite the diverging character of $\psi_0$, there is a continuity property of $\psi_\lambda(x)$ with respect to $\lambda$, which leads to the limit $\forall x, \lim_{\lambda \to 0} \psi_\lambda(x) = \psi_0(x)$. However, there will of course not be any related property of uniform convergence.

With this assumption, which could presumably be proved using rigorous mathematical arguments, we have the typical portrait of $\psi_\lambda(x)$ for vanishing $\lambda$, over the whole range $[x_0, \infty[$: "far" from the potential region, $\psi_\lambda(t)$ must become plane waves: $\psi_\lambda(x) \to W_\lambda \cos(\sqrt{\lambda} x + \phi_\lambda)$. The phase factor $\phi_\lambda$ is assumed to have a limit $\phi_0$ when $\lambda$ goes to zero, but its precise knowledge is useless for the persistence. On the contrary, it is important to know the $\lambda$-dependence of the coefficient of proportionality $W_\lambda$. The requirement that $\psi_\lambda(x)$ must be normalised in the sense $\int_{x_0}^{\infty} dx \psi_\lambda(x) \psi_\lambda(x) = \delta(\lambda - \lambda')$ imposes that $W_\lambda = C^{|t|} \lambda^{-1/4}$.

On the other hand, in the region where the potential is substantially different from zero, the differential equation for $\psi_\lambda$ is well approximated by the equation $\psi'' - V \psi = 0$; so in this region we can expect therefore

$$\psi_\lambda(x) \sim A_\lambda \psi_0(x)$$

(7)

where, once again, the $A_\lambda$ is important to determine, by estimating the location of the crossover of the two limiting behaviours. This location is naturally characterised by the
abscissa $x_\lambda$ given by the balance

$$V(x_\lambda) \sim \lambda$$

(8)

Nevertheless this condition will only hold if the computed $x_\lambda$ is larger than a wavelength $\sim 1/\sqrt{\lambda}$ corresponding to the minimum typical length the eigenfunction needs to join the maximum of his asymptotic behaviour. In other words, if (8) gives a $x_\lambda$ which diverges more slowly than $1/\sqrt{\lambda}$, one has to choose $x_\lambda \sim 1/\sqrt{\lambda}$ instead. In some particular cases, the determination of $x_\lambda$ is easy to achieve, since $\lambda$ is small. At this point $x_\lambda$, the continuity of $\psi_\lambda$ imposes that the magnitude of the two branches must be the same: this condition,

$$A_\lambda \psi_0(x_\lambda) \sim \lambda^{-1/4},$$

(9)

gives the $\lambda$-dependence of $A_\lambda$.

The aim is therefore achieved, because $B_\lambda$ can be evaluated with the approximation

$$B_\lambda \approx A_\lambda \int_{x_0}^{x_\lambda} dx \sqrt{p_{st}(x)} \psi_0(x) + \lambda^{-1/4} \int_{x_\lambda}^{\infty} dx \sqrt{p_{st}(x)} \cos(\sqrt{\lambda}x + \phi_\lambda).$$

(10)

The relative importance of the two terms must be checked in each case. Moreover, further simplifications can be made; for instance, if $\int_{x_\lambda}^{\infty} \sqrt{p_{st}}$ is converging, the second integral of (10) is equivalent to

$$C_{st} \times \lambda^{-1/4} \int_{x_\lambda}^{\infty} dx \sqrt{p_{st}(x)}.$$

(11)

The final calculation of the persistence behaviour is achieved by the means of saddle point expansion of the integral

$$G_+(x, t) \propto \int_{0}^{\infty} d\lambda \ e^{-\lambda t} A_\lambda B_\lambda$$

(12)

around $\lambda = 0$.

Let us consider now the general case, i.e. $B(x) \neq 2$. The change of variable defined by $dy/dx = \sqrt{2/B(x)} \equiv g$ leads to a new Fokker-Planck equation for the distribution $p(y) = p(x)dy/dx$, with the new coefficients $\tilde{B}(y)/2 = 1$ and $\tilde{A}(y) = A(y)g(y) + \partial_y g/g$. As a function of $y$, we have then to compute the characteristic abscissa $y_\lambda$. It is however more convenient to compute the corresponding abscissa $x_\lambda$ in the original coordinates, i.e. the $x_\lambda$ defined as the solution of

$$\lambda \sim V(x_\lambda) = \frac{1}{2} \sqrt{\frac{B(x_\lambda)}{2}} \frac{d\tilde{A}}{dx}(x_\lambda) + \frac{1}{4} \tilde{A}^2(x_\lambda)$$

where $\tilde{A}(x) = \sqrt{\frac{2}{B(x)}} \times \left( A(x) - \frac{1}{4} \partial_x B(x) \right).$ 

(13)

(14)

The $\psi_0$ eigenfunction can also be expressed in the original $x$ coordinate as

$$\psi_0(x) = \sqrt{p_{st}(x)} \sqrt{B/2} \int_{x_0}^{x} \frac{2dx'}{B(x')p_{st}(x')}.$$

(15)
With this change of variables, the coefficient \( A_\lambda \) is given by \( A_\lambda \psi_0(x_\lambda) \sim \lambda^{-1/4} \). The same translation can be performed on \( B_\lambda \), leading to

\[
B_\lambda = A_\lambda \int_{x_0}^{x_\lambda} dx (2/B(x))^{1/4} \sqrt{p_{st}(x)} \psi_0(x)
+ C^st \times \lambda^{-1/4} \int_{x_\lambda}^{\infty} dx (2/B(x))^{1/4} \sqrt{p_{st}(x)} \cos(\sqrt{\lambda}y(x) + \phi_0) \tag{16}
\]

The presence of \( y(x) \) in the cosine is not a problem in general, because only its asymptotic behaviour is needed, which is often easily computed.

In summary, the cases where the diffusion coefficient \( B(x) \) depends on \( x \) are not a handicap to the procedure proposed in this letter.

Examples. – To confirm the above procedure, we will first consider exactly solvable examples. The simplest is probably the free brownian motion, i.e. \( A = 0, B/2 = 1 \). In that case, \( V = 0 \) and \( x_\lambda \) behaves as \( 1/\sqrt{\lambda} \) (the prescription giving \( x_\lambda = 0 \), we have to choose the “minimal divergence” \( 1/\sqrt{\lambda} \) ; moreover, \( \psi_0(x) \sim x \), and \( p_{st} = C^st \). Consequently, \( A_\lambda \sim \lambda^{1/4}, B_\lambda \sim \lambda^{-3/4} \) and the persistence goes like \( t^{-1/2} \), as the exact result states.

To check the theory on a less trivial situation, let us consider the cases \( B/2 = 1, A = -\nu x^{-1} \) with \( \nu > 0 \). The Schrödinger potential is proportional to \( x^{-2} \), and it is easy to verify that, for large \( x \), \( p_{st} \propto x^{-\nu}, \psi_0 \propto x^{\nu/2+1}, x_\lambda \propto \lambda^{-1/2}, A_\lambda \propto \lambda^{(1+\nu)/4}, B_\lambda \propto \lambda^{(\nu-3)/4}. \) This gives the result \( G_+(x,t) \propto t^{-1+\nu/2}. \) The Schrödinger problem can actually be solved exactly, because the eigenfunctions \( \psi_\lambda \) are proportional to \( \sqrt{x}(\nu+1)/2(\sqrt{\lambda}x)Y_{\nu+1/2}(\sqrt{\lambda}x) \). The exact result matches our prediction (and the limit \( \nu \to 0 \) gives the free diffusion result).

We turn now to situations for which the knowledge of the spectrum is unknown. For instance, consider the cases \( A = -\nu x^{-1}, B/2 = 1 \), with \( \nu > 0, \alpha > 0 \).

If we consider first the cases \( \alpha > 1 \), we have then (for large \( x \)) \( p_{st} \propto \exp(\nu x^{1-\alpha}/(\alpha-1)), \psi_0 \propto x, V \propto x^{2-\alpha} \) and \( x_\lambda \propto \lambda^{-1/(\alpha+1)} \sim x_\lambda \propto \lambda^{-1/2}, A_\lambda \propto \lambda^{1/4}, B_\lambda \propto \lambda^{-3/4}. \) It gives \( G(x,t) \sim t^{-1/2} \). It is interesting to note that the potential slope is inefficient to modify the persistence behaviour of the particle, and that one obtains a discontinuity of the exponent as a function of \( \alpha \) (when \( \alpha \to 1 \)).

For cases \( 0 < \alpha < 1 \), the situation is completely different : \( \psi_0(x) \sim x^{\alpha} \exp(\nu x^{1-\alpha}/2(1-\alpha)), V \sim x^{-2\alpha} \) and \( x_\lambda \sim \lambda^{-1/2}. \) It leads to \( A_\lambda \sim \lambda^{1/4} \exp(\nu(1-\alpha)/\alpha), B_\lambda \sim A_\lambda \lambda^{-(1+\alpha)/2}, \lambda_0 \sim 4(\nu/2(1-\alpha))^{1/(1-\alpha)/2}. \) Finally, the persistence is found to behave like

\[
G \sim \exp[-(t/t_0)^{(1-\alpha)/(1+\alpha)}] \times t^{-(3\alpha-1)/2(\alpha+1)}
\]

with \( t_0 \propto \nu^{2/(1+\alpha)}4^{(\alpha-1)/(\alpha+1)}/(1-\alpha) \times \left[ 2^{2\alpha} + \zeta^{\alpha-1} \right] \) (\( \zeta = (1-\alpha)/2\alpha \)). The coefficient of proportionality for \( t_0 \) must be of order 1.

To check the validity of our theory, we have numerically computed the persistence distribution for the last case \( \alpha < 1 \) : figure 5(a) shows three different curves of \( \log(G) \) versus \( \sqrt{t} \) for \( \alpha = 1/3 \) (note that in that case, the persistence is purely a stretched exponential, with exponent 1/2) and different values of \( \nu \); the inset shows the \( \nu \) dependence of the slopes of these curves, which is well predicted by the expression of \( t_0 \) given by the above formula. Panel (b) of the figure shows two others examples of \( \log(t^{(3\alpha-1)/(2(\alpha+1))}G) \) plotted as function of \( C^st \times t^{(1-\alpha)/(1+\alpha)} \), emphasizing that the linear behaviour analytically derived is fully satisfied.
As a last example, let us consider now the case of an underdamped particle in a potential well, submitted to a damping $\gamma(E)$, a priori function of its energy. Kramers [11,12] had shown that the fast angle variable can be eliminated, leading to an effective Fokker-Planck equation for its energy $E$, with $A = \frac{\omega(E)}{2\pi} [k_B T \partial E (\gamma(E) I(E)) - \gamma(E) I(E)]$ and $B = 2k_B T \frac{\omega(E)}{2\pi} \gamma(E) I(E)$ (where $I(E), \omega(E)$ are the action and the pulsation of the trajectory). If the damping vanishes rapidly enough as $E$ increases, the problem belongs to the “gapless” category. For instance, if one considers the case of a harmonic potential ($\omega = \omega_0, I(E) = \frac{2\pi E}{\omega_0}$), with a damping $\gamma(E) = E^{-\alpha}$ (for $E$ sufficiently high), one has that, if $\alpha > 1$, the Schrödinger potential goes to zero at infinity, and the persistence is asymptotically $G \sim t^{3/2\alpha - 1} \exp - (t/t_0)^{\alpha - 1}$. This class of situations is physically particularly relevant to the cases of nonlinear coupled oscillators: the frequency shift between two adjacent oscillators having different energies leads to a enormous slowing down of the diffusion of the energy [13,14], described in a mean-field model by such a vanishing damping.

Correlation functions. – It is interesting to remark that this method can be applied to the calculation of tails of correlation functions, for systems belonging to the appropriate “gapless” class. The hypothesis that a correlation function exists implies that a real equilibrium is reachable by the system; with this restriction, the correlation function of the variables is

$$< x(t)x(0) > = \int_0^\infty d\lambda \ e^{-\lambda t} \left( \int dx \ x \sqrt{p_{st}(x)} \psi_\lambda(x) \right)^2. \quad (17)$$

It is clear that an analogous derivation can be performed on that formula, in order to extract its asymptotic behaviour with, nevertheless, a slight difference in the definition of the $\psi_\lambda$: the boundary condition $\psi_\lambda(x_0) = 0$ does no longer exist, and the $\psi_\lambda$ function is now a real eigenfunction equal to $\sqrt{p_{st}}$. 

Fig. 1 – Persistence distribution numerically computed for $A(x) = -\nu x^{-\alpha}$ with $0 < \alpha < 1$. (a) $\alpha = 1/3$ and $\nu = 0.5, 1, 2$. The inset shows the $\nu$-dependence of the coefficient $t_0$ (stars) compared to the prediction $\nu^2/(1+\alpha)$ of the theory (circles). (b) $\alpha = 0.5$ (dots) and $\alpha = 0.75$ (circles) (in that panel, the values of $t_0$ are of order 0.12). The ultimate diverging tails of these curves are artefacts due to a poor statistics in these regions.
Conclusion. – In this letter, we have presented a general method to derive analytically the asymptotic behaviour of the persistence probability, for a large number of markovian processes in one dimension: we have shown that it is possible to classify markovian processes in three categories: the first is characterised by an exponential extinction of the persistence, a second one, quite marginal, whose treatment is beyond the scope of this letter, and the third, which we called “gapless”, in reference to the structure of the spectrum of an associated Schrödinger operator. For this third class, we have introduced a systematic procedure to obtain the persistence distribution at large times, and we have tested the validity of the procedure by two complementary ways: we compared the results of our method to exactly solvable models, and the theoretical predictions to numerical simulations, when the exact result is not known. Both comparisons have shown an impressive agreement.

It is interesting to note that this method could be presumably extended to the treatment of multidimensional cases, as soon as the corresponding Fokker-Planck equation can be mapped with an appropriate change of variables on a Schrödinger equation. In this extended version of the theory, relevant quantities like $x_\lambda$ would become functions of the solid angle of the parameter space, leading probably to a more complicated behaviour of the persistence. This extension could therefore be an original way to study non markovian cases, since memory effects can always be interpreted as an elimination of additional variables.

∗ ∗ ∗

I thank T.Dauxois and A.Alastuey for comments and discussions.

REFERENCES

[1] Majumdar S., Current Science, 77 (1999) 370 cond-mat/9907407.
[2] Dornic I. and Godreche C., J.Phys. A, 31 (1998) 5413.
[3] Deloublère O. and Hilhorst H.J., cond-mat/9911466.
[4] McKay R.S. and Aubry S., Nonlinearity, 7 (1994) 1623.
[5] Flach S. and Willis Ch., Physics reports, 295 (1998) 181.
[6] Gardiner C.W., Handbook of Stochastic Methods (Springer) 1997.
[7] Risken, The Fokker-Planck equation (Springer) 1996.
[8] Cohen-Tannoudji C., Diu B. and Laloe F., Mécanique quantique (Hermann) 1986.
[9] Landau L. and Lifshitz E., Mécanique quantique (Mir, Moscow) 1975.
[10] Abramowitz M. and Stegun I., Handbook of Mathematical Functions (Dover) 1972.
[11] Kramers H.A., Physica, 7 (1940) 284.
[12] Hänggi P., Talkner P. and Borkovec M., Rev. of Mod. Phys., 62 (1990) 251.
[13] Farago J., Thèse de doctorat (ENS de Lyon) 01/2000.
[14] Tshonis G.P. and Aubry S., Phys. Rev. Lett., 77 (1996) 5225.