Asymptotically Lifshitz black hole solutions in $F(R)$ gravity

S. H. Hendi$^{1,2*}$, B. Eslam Panah$^{3,4†}$ and C. Corda$^{5‡}$

$^1$ Physics Department and Biruni Observatory, College of Sciences, Shiraz University, Shiraz 71454, Iran
$^2$ Research Institute for Astrophysics and Astronomy of Maragha (RIAAM), P.O. Box 55134-441, Maragha, Iran
$^3$ Department of Physics, University of Tabriz, Tabriz 51664, Iran
$^4$ Young Research Club, Islamic Azad University, Talesh Branch, Talesh, Iran
$^5$ Institute for Theoretical Physics and Advanced Mathematics Einstein-Galilei, Via Santa Gonda 14, 59100 Prato, Italy

We consider a class of spherically symmetric spacetime to obtain some interesting solutions in $F(R)$ gravity without matter field (pure gravity). We investigate the geometry of the solutions and find that there is an essential singularity at the origin. In addition, we show that there is an analogy between obtained solutions with the black holes of Einstein-$\Lambda$-power-Maxwell-invariant theory. Furthermore, we find that these solutions are equivalent to the asymptotically Lifshitz black holes. Also, we calculate $d^2F/dR^2$ to examine the Dolgov-Kawasaki stability criterion.

I. INTRODUCTION

The nature of current accelerated expansion of the universe is one of the mysterious and interesting topics in cosmology [1]. In order to interpret this expansion, some various candidates have been proposed by many authors, such as cosmological constant idea [2], dark energy models [3], (exotic substance with large negative pressure $P_{DE} \simeq -\rho_{DE}$ ($\rho_{DE}$ is the dark energy density)) and modified gravities like Lovelock gravity [4], brane world cosmology [5], scalar-tensor theories [6] and also the so-called $F(R)$ gravity theories [7–12].

On the other hand, amongst the nonlinear modifications of Einstein gravity, $F(R)$ models seem to provide an interpretation to dark energy, the hierarchy problem [13], the four cosmological phases [14], the power law early-time inflation [15, 16], late-time cosmic accelerated expansion [16–19], singularity problem arising in the strong gravity regime [20–24], rotation curves of spiral galaxies [25] and detection of gravitational waves [26]. In addition, when one considers $F(R)$ theory...
as a modification of general relativity, it is quite natural to ask about black hole existence in this theory. Furthermore, it is notable that viable modifications in gravity should pass all sorts of tests from the large scale structure of the galaxy and cluster dynamics to the solar system tests. Hence, many different models of \( F(R) \) theory with various motivations have been proposed (see [11] and references therein for more details).

Moreover, regarding Einstein gravity, one can find a large number of exact spherically symmetric solutions. One of important works in the modified theories of gravity is obtaining the static spherically symmetric solutions, a requirement usually referred to as spatial isotropy and time independence. Therefore, the most widely explored exact solutions in \( F(R) \) gravity are the spherically symmetric solutions. Spherically symmetric solutions of \( F(R) \) gravity have been studied before [27]. Furthermore, some of interesting black objects with various geometry have been investigated in \( F(R) \) gravity, for e.g., static and rotating black hole/string solutions, magnetic string and so on [28, 29]. Also, the black hole/brane solutions with a nonlinear Maxwell source in \( F(R) \) gravity has been investigated in [30]. Charged spherically symmetric black hole solutions have been obtained in Refs. [10, 31] and creation of the electric charge and cosmological constant from pure gravity, simultaneously, have been extracted in [11].

Recently, Sebestiani and Zerbini [32] have considered a class of interesting static spherically symmetric spacetime in the form

\[
\text{ds}^2 = - \left( \frac{1}{\alpha} \right)^q g(r) dt^2 + \frac{dr^2}{g(r)} + r^2 d\Omega^2_k
\]

with a model of \( F(R) \) gravity. In this work, we would like to consider this metric and obtain the exact spherically symmetric black hole solutions for general value of \( q \) and discuss about the behavior of these solutions. We will consider \( F(R) = R + f(R) \) theory, namely pure gravity \((T_{\alpha\beta} = 0)\), with a constant curvature scalar. One of our purposes is finding the four-dimensional charged black hole solutions with cosmological constant in pure gravity similar to solutions of Einstein-\( \Lambda \)-power Maxwell invariant gravity [33]. In addition, we will investigate the behavior of these solutions and also discuss about Dolgov-Kawasaki (DK) stability [34].

The outline of our paper is as follows. In section II we obtain a static spherically symmetric solutions of a class of \( F(R) \) model and some special values of free parameter, \( q \), will be discussed in appendix. In section III we compare obtained solutions with asymptotically Lifshitz black holes. We terminate our paper by a conclusion.
II. EXACT BLACK HOLE SOLUTIONS

Considering the field equation of 4-dimensional pure $F(R)$ gravity with the following form \[7–12\]

\[
R_{\mu\nu}F_R - \nabla_\mu \nabla_\nu F_R + \left( \Box F_R - \frac{1}{2} F(R) \right) g_{\mu\nu} = 0, \tag{1}
\]

where $R_{\mu\nu}$ is the Ricci tensor, $F_R \equiv dF(R)/dR$ and $F(R)$ is an arbitrary function of Ricci scalar $R$. Here, we want to obtain the static solutions of Eq. (1) with positive, negative and zero curvature horizons. For this purpose, we assume that the metric has the following form \[32\]

\[
ds^2 = -\left( \frac{r}{r_0} \right)^q g(r) dt^2 + \frac{dr^2}{g(r)} + r^2 d\Omega_k^2 \tag{2}
\]

where $q$ is an arbitrary constant and

\[
d\Omega_k^2 = \begin{cases} 
  d\theta^2 + \sin^2 \theta d\phi^2 & k = 1 \\
  d\theta^2 + \sinh^2 \theta d\phi^2 & k = -1 \\
  d\theta^2 + d\phi^2 & k = 0 
\end{cases} \tag{3}
\]

To find the function $g(r)$, We use the components of Eq. (1) with well-known $F(R) = R - \lambda \exp(-\xi R) + \kappa R^n$ model. This special model and some of its properties have been investigated in Refs. \[11, 12\]. Using Eq. (1) with metric (2), one can obtain a general solution in the following form

\[
g(r) = \frac{4k}{q^2 + 2q + 4} + Ar^2 + Cr^{\Gamma_1+\Gamma_2} + Dr^{\Gamma_1-\Gamma_2}, \tag{4}
\]

where $A, C$ and $D$ are integration constants and

\[
\Gamma_1 = \frac{-3q}{4} - \frac{3}{2}, \tag{5}
\]

\[
\Gamma_2 = \frac{\sqrt{q^2 + 20q + 4}}{4}, \tag{6}
\]

Since we know that the cosmological constant may arises from various models of pure $F(R)$ gravity (see \[11\] and references therein), one may set $A = -\Lambda/3$ to obtain a consistent solutions for $q = C = D = 0$. In order to satisfy all components of the field equation (1), we should set the parameters of the $F(R)$ model to satisfy the following equations

\[
\Lambda \Psi_1 \left( \xi \lambda + e^{\xi R} \right) + 6n\kappa R^n e^{\xi R} = 0, \tag{7}
\]

\[
e^{\xi R} (\kappa R^n \Psi_2 - 4\Lambda \Psi_3) + \lambda (\xi \Lambda \Psi_4 + 6\Psi_1) = 0, \tag{8}
\]
where
\[
\begin{align*}
\Psi_1 &= q^2 + 8q + 24, \\
\Psi_2 &= 6q^2(n - 1) + 24(n - 2)(q + 3), \\
\Psi_3 &= q^3 + 11q^2 + 48q + 72, \\
\Psi_4 &= q^4 + 12q^3 + 68q^2 + 192q + 288.
\end{align*}
\]

Solving Eqs. (7) and (8) for \( \lambda \) and \( \kappa \) with arbitrary \( \xi \) lead to the following solutions
\[
\begin{align*}
\lambda &= \frac{R(n - 1)e^{\xi R}}{n + \xi R}, \\
\kappa &= -\frac{(1 + \xi R) R^{1-n}}{n + \xi R}.
\end{align*}
\]

After some cumbersome manipulation we obtain
\[
\begin{align*}
R &= \frac{\Lambda \Psi_1}{6}, \\
\lim_{r \to 0} R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} &= \infty, \\
\lim_{r \to \infty} R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} &= \left[ \frac{q^4}{4} + 2q^3 + 8q^2 + 16q + 24 \right] \frac{\Lambda^2}{9},
\end{align*}
\]
which confirm that there is a singularity at \( r = 0 \). The position of the black hole event horizon is determined as a largest root of \( g(r) = 0 \). Eqs. (11) and (13) show that the curvature scalars are asymptotically finite and related to the cosmological constant. But we cannot redefine a unique effective cosmological constant for whole curvature scalars with an arbitrary \( q \). Therefore, we could not state these solutions are asymptotically (a)ds with an effective cosmological constant. Nevertheless, the asymptotic behavior of the solutions are near to (a)ds solutions.

Here, we give a brief discussion about the stability of the mentioned \( F(R) \) model. It was showed that there is no stable ground state for \( F(R) \) models if \( F(R) \neq 0 \) and \( F_R = 0 \) [35]. For the obtained solutions, we find that \( F(R) = F_R = 0 \). Furthermore, it has been shown that \( F_{RR} = d^2 F/dR^2 \) is related to the the effective mass of the dynamical field of the Ricci scalar [34]. Therefore, the positive effective mass, a requirement usually referred to as the DK stability criterion, leads to the stable dynamical field [36]. In addition, One can discuss other instability criteria of the asymptotically Lifshitz spacetimes considered in [37, 38]. In Ref. [37], the authors have employed the initial value problem to investigate the stability of such spacetimes. We will investigate other instability criteria for the future works and in this paper, we consider DK stability. In order to
check the DK stability criterion, we should calculate the second derivative of the $F(R)$ function with respect to the Ricci scalar for this specific model

$$F_{RR} = \frac{-R(n-1)\left(\xi^2 + nR^{-2}(1 + \xi R)\right)}{n + \xi R},$$

(14)

The positivity of $F_{RR}$ depends on the model parameters. In order to study the stability, we plot $F_{RR}$ in Fig. 1. This figure shows that one may obtain stable solution for special values of $\xi$ for negative cosmological constant. In other words, we can set free parameters to obtain positive $F_{RR}$. It is notable to mention that since Eq. (11) is a trivial relation for obtained solutions, we are not allowed to use the conformal transformation to discuss about the dynamic of the solutions.

### III. ASYMPTOTICALLY LIFSHITZ SOLUTIONS:

In recent years, it has been discussed about duality of gravity and nonrelativistic scale invariant theories. It may be considered a scale invariant fixed point that do not exhibit Galilean symmetry with the metric of the corresponding gravity duals [39]. These metrics exhibit an anisotropic scale-invariant field theories in which time and space scale differently

$$t \rightarrow \lambda^zt,$$

$$\mathbf{x} \rightarrow \lambda\mathbf{x},$$

which is characterized by the so-called dynamical exponent $z$. One may obtain the standard scaling behavior of conformal invariant systems for $z = 1$. In other words, for $z = 1$, it corresponds to
relativistic invariance (the scaling is isotropic). Also for arbitrary values of \(z\), one can say that the system has Lifshitz scaling. In some literature a special solution was found, which corresponds to the asymptotically Lifshitz black hole with various dynamical exponents [40].

Now, let us compare the presented solutions with asymptotically Lifshitz black holes. In order to accomplish this goal, we consider the following asymptotically Lifshitz ansatz

\[
\text{ds}^2 = -\left(\frac{r^2}{l^2}\right)^z h(r)dt^2 + \frac{l^2dr^2}{r^2h(r)} + r^2d\Omega_k^2.
\]

It is easy to show that Eq. (2) is asymptotically Lifshitz spacetime provided one apply the following transformation

\[
\begin{align*}
    r_0 &\rightarrow l, \\
    g(r) &\rightarrow \frac{r^2}{l^2}h(r), \\
    q &\rightarrow 2z - 2.
\end{align*}
\]

In other word, asymptotically Lifshitz metric (15) (with the mentioned \(F(R)\) model) leads to the following metric function

\[
h(r) = \frac{kl^2}{(z^2 - z + 1)r^2} - \frac{A}{3} + C_l^2t^{\Gamma_1}r^{2-\Gamma_2} + D_l^2t^{\Gamma_1-\Gamma_2},
\]

where

\[
\begin{align*}
    \Gamma_1 &= -\frac{3z}{2}, \\
    \Gamma_2 &= \frac{\sqrt{z^2 + 8z - 8}}{2}.
\end{align*}
\]

It is notable that the generic solution Eqs. (15) and (17) reduces to the solution of obtained in Ref. [41], for \(C = D = 0\) and \(z = 2\).

**IV. CONCLUSIONS**

In this paper, we have considered a new class of static spherically symmetric spacetime with a special model of \(F(R)\) gravity. We have obtained some interesting solutions for different values of free parameter, \(q\), and studied the geometrical properties of the solutions.

We have investigated the analogy between obtained solutions with 4-dimensional solutions of Einstein-\(\Lambda\) gravity in the presence of a nonlinear source, namely power Maxwell invariant (where \(s\) denotes its nonlinearity parameter). In other words, one can find that the integral constant \(C\) may
interpreted as \((\text{charge})^{2s}\) provided \(s = 1/2 - (\Gamma_1 + \Gamma_2)^{-1}\). In addition, for \(s = 1/2 - (\Gamma_1 - \Gamma_2)^{-1}\), one may interpreted \(D\) as \((\text{charge})^{2s}\).

Furthermore, we have calculated Kretschmann scalar and found that there is a curvature singularity at \(r = 0\). We have investigated the behavior of the solutions for large value of \(r\) (\(r \rightarrow \infty\)) and showed that the asymptotic behavior of the solutions is neither flat nor (a)dS. Also, we have examined the DK stability criterion and showed that one may obtain a stable theory provided the parameters of the \(F(R)\) model are chosen suitably.

Finally, we should remark that obtained solutions are interesting because of the following three main reasons: (i) both the geometry of the horizon(s) and asymptotic behavior of the solutions depend on the values of free parameter \(q\), (ii) starting from pure \(F(R)\) gravity and extract the solutions of Einstein-\(A\)-power Maxwell invariant theory, (iii) obtained solutions are, exactly, the same as asymptotically Lifshitz black holes, provided the transformation between them are chosen suitably.

V. APPENDIX

Now, we would like to discuss about the geometry of the spacetime with various values of metric parameters.

A. First type: \(C = D = 0\)

Considering the first two terms of Eq. (4), one can obtain interesting spacetimes with some special values of \(q\).

1. Case I: \(q = 0\)

In this trivial case, the Ricci and Kretschmann scalars are \(4\Lambda\) and \(\frac{8\Lambda^2}{3}\), respectively, and therefore the solution is asymptotically (a)dS.

2. Case II: \(q = -2\)

Inserting \(q = -2\) in Eq. (4), one can obtain

\[
ds^2 = -\left(\frac{r_0}{r}\right)^2 \left(k - \frac{\Lambda}{3} r^2\right) dt^2 + \frac{dr^2}{\left(k - \frac{\Lambda}{3} r^2\right)} + r^2 d\Omega_k^2, \tag{20}\]
with the following Ricci and Kretschmann scalars

\[ R = 2\Lambda, \]
\[ R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} = \frac{72k^2}{3r^4} + \frac{4\Lambda^2}{3}, \]

where confirm that the spacetime is asymptotically near to (a)dS which its curvature scalars are the half of their counterpart in (a)dS spacetime.

3. Case II: \( q = -4 \)

One may consider \( q = -4 \) in Eq. (4) to obtain

\[ ds^2 = -\left(\frac{r_0}{r}\right)^4 \left(\frac{k - \Lambda r^2}{3}\right)dt^2 + \frac{3dr^2}{k - \Lambda r^2} + r^2d\Omega_k^2, \]

where the Ricci and Kretschmann scalars are

\[ R = \frac{4\Lambda}{3}, \]
\[ R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} = \frac{8\Lambda^2}{3} - \frac{64\Lambda k}{9r^2} + \frac{64k^2}{3r^4} \]

This solution is near to asymptotically (a)dS spacetime, but as one may find, obtained scalars are different delicately. Now, we investigate the solutions with nonzero \( C \) and \( D \).

B. Second type: \( C \neq 0 \& D \neq 0 \)

In order to have a real solution with nonzero \( C \) or \( D \), we should consider \( q \leq -10 - 4\sqrt{6} \approx -19.8 \) or \( q \geq -10 + 4\sqrt{6} \approx -0.2 \) (in other words: \( q \notin (-10 - 4\sqrt{6}, -10 + 4\sqrt{6}) \)). In general, considering the metric (2) with \( g(r) = Ar^m \), one can show that the curvature scalars diverge at \( r = 0 \) with arbitrary \( m \). In addition, curvature scalars diverge at spatial infinity for \( m > 2 \). To avoid non-physical singularity, we restrict our discussions for \( m \leq 2 \) and hence we only consider \( q \geq -10 + 4\sqrt{6} \) branch (for a discussion about the essential singularity at infinity, we refer the reader to Ref. [42]).

Now, we discuss about some specific values of allowed \( q \). At first, we consider the trivial case \( (q = 0) \) and then we investigate the boundary values of \( q \) and after that we analyze some interesting values of it which lead to integer number for the numerator of Eq. (6).

1. Case I: \( q = 0 \)

This trivial case leads to
\[ \Gamma_1 = -3/2, \]
\[ \Gamma_2 = 1/2, \]

with trivial RN solution as follows

\[ ds^2 = -g(r)dt^2 + \frac{dr^2}{g(r)} + r^2d\Omega^2_k, \]  \hfill (26)

where

\[ g(r) = k - \frac{\Lambda r^2}{3} - \frac{M}{r} + \frac{Q^2}{r^2}, \]  \hfill (27)

where we redefined \( C = -M \) and \( D = Q^2 \). This solution indicates that one may obtain cosmological constant as well as electric charge of Maxwell field from pure gravity which has studied before \[11\].

2. Case II: \( q = -10 + 4\sqrt{6} \)

For this value of \( q \), we encounter with vanishing \( \Gamma_2 \) and \( \Gamma_1 = (6 - 3\sqrt{6}) \simeq -1.35 \), and so the solution is

\[ g(r) \approx \frac{4k}{3.64} - \frac{\Lambda r^2}{3} + \frac{(C + D)}{r^{1.35}}, \]  \hfill (28)

where the second term is dominant for large \( r \). Calculating the Ricci and Kretschmann scalars show that they diverge at the origin and they are finite for \( r \neq 0 \). It is easy to show that the presented solutions may be interpreted as black hole solutions with two horizons, extreme black hole and naked singularity provided the parameters of the solutions are chosen suitably (see Fig. 2 for more details). Considering \( k = 0 \), it is easy to show that the mentioned \( g(r) \) is the same as that in the Einstein-power Maxwell invariant gravity \[33\] with vanishing mass when the nonlinearity parameter is chosen \( s = \frac{3\sqrt{6} - 4}{6(\sqrt{6} - 2)} \).

3. Case III: \( q = 1 \)

One may consider another interesting value of \( q \) to obtain

\[ \Gamma_1 = \frac{-9}{4}, \]
\[ \Gamma_2 = \frac{5}{4}. \]
FIG. 2: $g(r)$ (Eq. (28)) versus $r$ for $k = -1$, $D = 0.1$, $\Lambda = -1$ and $C = 0.1$ (bold line), $C = 0.7$ (solid line) and $C = 2$ (dashed line).

where Eq. (4) simplified to

$$g(r) = \frac{4k}{7} - \frac{\Lambda r^2}{3} + \frac{C}{r} + \frac{D}{r^{5/2}}. \quad (29)$$

Despite of the first constant term in Eq. (29), this solution is near to asymptotically (a)dS charged solution of nonlinear Maxwell gravity [33] with $s = 11/14$ (for $k = 0$, they are exactly the same). Thus, one may conclude that $C$ and $D$ are related to the mass and charge of the spacetime, respectively. In other words, we can extract the electric charge and cosmological constant form pure gravity, simultaneously. One may calculate the Ricci and Kretschmann scalars to achieve

$$R = \frac{11}{2} \Lambda, \quad (30)$$

$$R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} \bigg|_{\text{Large } r} = \frac{67\Lambda^2}{12} - \frac{2k\Lambda}{7r^2} + O(r^{-4}), \quad (31)$$

$$R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} \bigg|_{\text{Small } r} = \frac{141D^2}{r^{11}} + \frac{22CD}{r^{17/2}} - \frac{104kD}{r^{15/2}} + \frac{6C^2}{r^6} + O(r^{-11/2}). \quad (32)$$

Equations (30)-(32) show that there is an essential singularity at the origin and this solution is, asymptotically, near to (a)dS spacetime.

In order to investigate the geometry of the function $g(r)$, we plot it versus $r$ in Fig. 3. This figure shows that the geometry of the horizons are near to those in charged black holes.
FIG. 3: $g(r)$ (Eq. (29)) versus $r$ for $\Lambda = -1$, $D = 1$, $k = 1$, and $C = -0.1$ (dashed line), $C = -1.86$ (solid line), and $C = -3$ (bold line).

4. Case IV: $q = 4$

For $q = 4$, and therefore $\Gamma_1 = -9/2$ and $\Gamma_2 = 10/4$, Eq. (4) reduces to

$$g(r) = \frac{k}{7} - \frac{\Lambda r^2}{3} + \frac{C}{r^2} + \frac{D}{r^4}. \quad (33)$$

Let us first consider $D = 0$. In this case Eq. (33) may be interpreted as approximately asymptotically (a)dS charged solution with vanishing mass and therefore $C$ is related to electric charge.

In addition, one may consider $C = 0$ with nonvanishing $D$ to obtain approximately asymptotically (a)dS solution that for $k = 0$, this solution is the same one obtained in Ref. [33] when the nonlinearity parameter is equal to $9/14$. We can obtain the Ricci and Kretschmann scalars in the following manner

$$R = 12\Lambda, \quad (34)$$

$$R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} = \frac{136\Lambda^2}{3} - \frac{64k\Lambda}{7r^2} + \frac{32kC}{7r^6} - \frac{24C^2}{r^8} - \frac{176\Lambda D}{r^9} + \frac{48kD}{7r^{11}} - \frac{32kC}{7r^{13}} + \frac{444D^2}{r^{18}}. \quad (35)$$

Therefore as we expected, there is a curvature singularity at $r = 0$, and asymptotic behavior of the spacetime is near to (a)dS.

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[1] S. Perlmutter et al., Astrophys. J. 517, 565 (1999);
    S. Perlmutter, M. S. Turner and M. White, Phys. Rev. Lett. 83, 670 (1999);
    A. G. Riess et al., Astrophys. J. 607, 665 (2004).
[2] T. Padmanabhan, Phys. Rept. 380, 235 (2003);
    J. A. Frieman, M. S. Turner and D. Huterer, Ann. Rev. Astron. Astroph. 46, 385 (2008).
[3] E. J. Copeland, M. Sami and S. Tsujikawa, Int. J. Mod. Phys. D 15, 1753 (2006);
    M. Li, X. D. Li, S. Wang, Y. Wang, Commun. Theor. Phys. 56, 525 (2011).
[4] D. Lovelock, J. Math. Phys. 12, 498 (1971);
    D. Lovelock, J. Math. Phys. 13, 874 (1972);
    N. Deruelle and L. Farina-Busto, Phys. Rev. D 41, 3696 (1990);
    S. H. Hendi and M. H. Dehghani, Phys. Lett. B 666, 116 (2008);
    M. H. Dehghani, N. Alinejadi, S. H. Hendi, Phys. Rev. D 77, 104025 (2008);
    M. H. Dehghani, N. Bostani, S. H. Hendi, Phys. Rev. D 78, 064031 (2008);
    S. H. Hendi, S. Panahiyan and H. Mohammadpour, Eur. Phys. J. C 72, 2184 (2012).
[5] L. A. Gergely. Phys. Rev. D 74, 024002 (2006);
    M. Demetrian, Gen. Relativ. Gravit. 38, 953 (2006);
    P. Brax and C. van de Bruck, Class. Quantum Gravit. 20, R201 (2003);
[6] C. Brans and R. H. Dicke, Phys. Rev. 124, 925 (1961);
    T. P. Sotiriou, Class. Quantum Gravit. 23, 5117 (2006);
    K. I. Maeda and Y. Fujii, Phys. Rev. D 79, 084026 (2009);
    R. G. Cai and Y. S. Myung, Phys. Rev. D 56, 3466 (1997);
    M. H. Dehghani, J. Pakravan and S. H. Hendi, Phys. Rev. D 74, 104014 (2006);
    S. H. Hendi, J. Math. Phys. 49, 082501 (2008);
    S. H. Hendi and R. Katebi, Eur. Phys. J. C 72, 2235 (2012).
[7] M. Akbar and R. G. Cai, Phys. Lett. B 648, 243 (2007);
    J. C. C. de Souza and V. Faraoni, Class. Quantum Gravit. 24, 3637 (2007);
    K. Atazadeh, M. Farhoudi and H. R. Sepangi, Phys. Lett. B 660, 275 (2008);
    K. Bamba and S. D. Odintsov, JCAP 0804, 024 (2008);
    C. Corda and H. J. Mosquera Cuesta, Europhys. Lett. 86, 20004 (2009).
[8] G. Cognola, E. Elizalde, S. Nojiri, S.D. Odintsov, L. Sebastiani and S. Zerbini, Phys. Rev. D 77, 046009
(2008).

[9] T. P. Sotiriou and V. Faraoni, Rev. Mod. Phys. 82, 451 (2010);
    S. Nojiri and S.D. Odintsov, Rev. Mod. Phys. 82, 451 (2010);
    V. Faraoni, [arXiv:0810.2602];
    N. Straumann, [arXiv:0809.5148].

[10] S. H. Hendi, Phys. Lett. B 690, 220 (2010).

[11] S. H. Hendi, B. E. Panah and S. M. Mousavi, Gen. Relativ. Gravit. 44, 835 (2012).

[12] S. H. Hendi, R. B. Mann, N. Riazi and B. E. Panah, Phys. Rev. D 86, 104034 (2012).

[13] G. Cognola, E. Elizalde, S. Nojiri, S. D. Odintsov and S. Zerbini, Phys. Rev. D 73, 084007 (2006).

[14] S. Nojiri and S. D. Odintsov, Phys. Rev. D 74, 086005 (2006);
    S. Nojiri and S. D. Odintsov, Phys. Rev. D 78, 046006 (2008).

[15] A. A. Starobinsky, Phys. Lett. B 91, 99 (1980).

[16] K. Bamba and S. D. Odintsov, JCAP 0804, 024 (2008).

[17] S. M. Carroll, V. Duvvuri, M. Trodden and M. S. Turner, Phys. Rev. D 70, 043528 (2004).

[18] A. de la Cruz-Dombriz and A. Dobado, Phys. Rev. D 74, 087501 (2006).

[19] S. Fay, R. Tavakol and S. Tsujikawa, Phys. Rev. D 75, 063509 (2007).

[20] M. C. B. Abdalla, S. Nojiri and S. D. Odintsov, Class. Quantum Gravit. 22, L35 (2005).

[21] F. Briscese, E. Elizalde, S. Nojiri and S. D. Odintsov, Phys. Lett. B 646, 105 (2007).

[22] S. Nojiri and S. D. Odintsov, Phys. Rev. D 78, 046006 (2008).

[23] K. Bamba, S. Nojiri and S. D. Odintsov, JCAP 0810, 045 (2008).

[24] T. Kobayashi and K. I. Maeda, Phys. Rev. D 79, 024009 (2009).

[25] S. Capozziello, V.F. Cardone, A. Troisi, JCAP 0608, 001 (2006);
    S. Capozziello, V.F. Cardone, A. Troisi, Mon. Not. Roy. Astron. Soc. 375, 1423 (2007);
    C. F. Martins and P. Salucci, Mon. Not. Roy. Astron. Soc. 381, 1103 (2007).

[26] C. Corda, Astropart. Phys. 30, 209 (2008);
    C. Corda, Int. J. Mod. Phys. D 18, 2275 (2009);
    C. Corda, Gen. Relativ. Gravit. 42, 1323 (2010); Erratum-ibid. 42, 1335 (2010);
    C. Corda, Astropart. Phys. 34, 412 (2011).

[27] T. Multamaki and I. Vilja, Phys. Rev. D 76, 064021 (2007);
    S. Capozziello, A. Stabile and A. Troisi, Class. Quantum Gravit. 25, 085004 (2008);
    L. Hollenstein and F. S. N. Lobo, Phys. Rev. D 78, 124007 (2008);
    A. Azadi, D. Momeni and M. Nouri-Zonoz: Phys. Lett. B 670, 210 (2008);
    D. Momeni and H. Gholizade, Int. J. Mod. Phys. D 18, 1 (2009).

[28] S. H. Mazharimousavi, M. Halilsoy and T. Tahamtan, Eur. Phys. J. C 72, 1851 (2012).

[29] A. Larranaga, Pramana J. Phys. 78, 697 (2012);
    J. A. R. Cembranos, A. de la Cruz-Dombriz and P. Jimeno Romero, [arXiv:1109.4519].

[30] A. Sheykhi, Phys. Rev. D 86, 024013 (2012);
A. Sheykhi and S. H. Hendi, Phys. Rev. D 87, 084015 (2013).

[31] S. H. Mazharimousavi, M. Halilsoy, T. Tahamtan, [arXiv:1110.0094].

[32] L. Sebastiani and S. Zerbini, Eur. Phys. J. C 71, 1591 (2011).

[33] S. H. Hendi, Eur. Phys. J. C 69, 281 (2010).

[34] A. D. Dolgov and M. Kawasaki, Phys. Lett. B 573, 1 (2003); S. Nojiri and S. D. Odintsov, Phys. Rev. D 68, 123512 (2003); V. Faraoni, Phys. Rev. D 74, 104017 (2006); I. Sawicki and W. Hu, Phys. Rev. D 75, 127502 (2007).

[35] H. J. Schmidt, Astron. Nachr. 308, 183 (1987).

[36] V. Faraoni, Phys. Rev. D 74, 104017 (2006); V. Faraoni, Phys. Rev. D 76, 127501(2007); O. Bertolami and M. C. Sequeira, Phys. Rev. D 79, 104010 (2009).

[37] K. Copsey and R. Mann, JHEP 03, 039 (2011).

[38] D. Saez-Gomez, Phys. Rev. D 83, 064040 (2011); B. Cuadros-Melgar, J. de Oliveira and C. E. Pellicer, Phys. Rev. D 85, 024014 (2012); B. Hsu and E. Fradkin, Phys. Rev. B 87, 085102 (2013).

[39] S. Kachru, X. Liu and M. Mulligan, Phys. Rev. D 78, 106005 (2008).

[40] D. W. Pang, [arXiv:0905.2678]; D. W. Pang, JHEP 10, 031 (2009); G. Bertoldi, B. A. Burrington and A. Peet, [arXiv:0905.3183]; S. R. Das and G. Murthy, Phys. Rev. D 80, 065006 (2009); G. Bertoldi, B. A. Burrington and A. W. Peet, Phys. Rev. D 80, 126004 (2009).

[41] R. B. Mann, [arXiv:0905.1136].

[42] B. Elsner, Annales de l’Institut Fourier 49, 303 (1999); J. W. Moffat, Eur. Phys. J. Plus 126, 43 (2011).