Polariton excitation rates from time dependent dielectrics

S Bugler-Lamb and S A R Horsley

Department of Physics and Astronomy, University of Exeter, Stocker Road, Exeter, EX4 4QL, UK

E-mail: slb235@exeter.ac.uk and s.horsley@exeter.ac.uk

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Abstract

We apply time dependent perturbation theory to a general class of problems involving time dependent dielectric media in cases where dissipation cannot be ignored. The theory is used to derive polariton excitation rates in three illustrative cases, including that of a travelling Gaussian dependent dielectric media in cases where dissipation cannot be ignored. The theory is used to find that the excitation rate is increased when the wave-vector and frequency of each polariton in the pair is close to satisfying either the dispersion relation for electromagnetic waves, or is close to a material resonance. We thus predict that the time dependence of a medium can excite a pair of polaritons, only one of which may appear as a propagating electromagnetic wave.

Keywords: time dependent media, perturbation theory, macroscopic QED

(Some figures may appear in colour only in the online journal)

1. Introduction

For a large class of problems, the interaction between the electromagnetic field and magneto-dielectric materials is modelled by the use of a permittivity $\epsilon$ and permeability $\mu$. To consider a broad range of frequencies of radiation, it becomes necessary to account for the dissipation of the field due to its coupling to matter. However it has proven difficult to construct a theory that accounts for the effects of dispersion and dissipation in systems where quantum mechanical behaviour is important; e.g. the Casimir effect [1, 2], quantum friction [4–6], or even analogues of Hawking radiation [7–11]. Although a great deal of progress can be made through applying the fluctuation–dissipation theorem [3], the absence of a Hamiltonian operator (for instance) can make it difficult to resolve conflicting predictions (such as those concerning quantum friction [5, 12, 13]).

During recent years, much progress has been made towards a rigorous quantisation scheme—macroscopic QED—that accounts for the effects of dispersion and dissipation, see for example [14, 15, 17–21] and references therein. In this work we shall apply the theory given in [19], which is a generalisation of the Huttner–Barnett theory [15] and uses a bath of simple harmonic oscillators to account for the energy lost from the electromagnetic field. The usual frequency domain constitutive equations $D(\omega) = \epsilon_0 \epsilon(\omega) E(\omega)$ and $H(\omega) = B(\omega)/\mu_0 \mu(\omega)$ are recovered from the equations of motion, with the relative permittivity $\epsilon(\omega)$ and permeability $\mu(\omega)$ obeying the Kramers–Kronig relations [3]. One of the significant consequences of this theory (which was also pointed out earlier by Huttner and Barnett [15]), is that the excitations of the electromagnetic field and medium are strongly coupled so that one must work in terms of polaritons instead of photons. Unlike photons these quanta have no set relationship between their frequency $\omega$ and wave-vector $k$.

We note that although the theory was initially developed for isotropic stationary media, it can be generalised to bi-anisotropic and moving media [20, 21].

In this work we apply macroscopic QED to problems where the material is time dependent. This time dependence could either be due to motion, or the fact that the dielectric function is explicitly time dependent (perhaps a nonlinear response to an applied field). Our motivation is to understand how the theoretical description of dynamic Casimir-type effects must be changed when the frequencies of interest are close to a material resonance. We start by developing the extensions to macroscopic QED necessary to describe such time dependent media, and then apply time dependent perturbation theory to find the excitation of the medium and field when initially prepared in the ground state. Although such
2. Lagrangian of electromagnetism in a time dependent, moving dielectric

We now construct the classical theory of the electromagnetic field interacting with a moving or time dependent material exhibiting dispersion and dissipation. The Lagrangian of [19] is extended in a similar fashion to [20, 21], accounting for non-relativistic motion and an arbitrary time dependence of the permittivity profile. The Lagrangian density is written in terms of dynamical variables of the electromagnetic field, the vector potential $\mathbf{A}$ and the scalar potential $\phi$, related to the electric and magnetic field strengths by $\mathbf{E} = - \nabla \phi - \partial_t \mathbf{A}$ and $\mathbf{B} = \nabla \times \mathbf{A}$ respectively. The dissipation of electromagnetic energy is provided by the coupling of the electromagnetic field to a field of simple harmonic oscillators $\mathbf{X}_\omega(x, t)$ present throughout space, with every possible natural frequency $\omega$.

2.1. Lagrangian density

The Lagrangian density for our system is

$$\mathcal{L} = \frac{\varepsilon_0}{2} (\mathbf{E}^2 - c^2 \mathbf{B}^2) + \mathbf{E} \cdot \int_0^\infty d\omega \, \alpha(\omega, x, t) \mathbf{X}_\omega + \mathbf{B} \cdot \int_0^\infty d\omega \, \alpha_B(\omega, x, t) \mathbf{X}_\omega + \frac{1}{2} \int_0^\infty d\omega \left[ (\mathbf{X}_\omega \cdot (\mathbf{v}(x, t) \cdot \nabla)) \mathbf{X}_\omega \right]^2 - \omega^2 \mathbf{X}_\omega^2$$

(1)

which is valid to first order in the local velocity of the material $\mathbf{v}(x, t)$ and assumes for simplicity that $\mu = 1$. The coupling constant $\alpha_B(\omega, x, t)$ is given in terms of $\alpha$ by $\alpha_B(\omega, x, t) = - \mathbf{v}(x, t) \times \alpha(\omega, x, t) \mathbf{B}$. To motivate (1) note that for isotropic media the local polarisation of the medium is related to the reservoir of simple harmonic oscillators by the formula [19–21]

$$\mathbf{P}(x, t) = \int_0^\infty d\omega \, \alpha(\omega, x, t) \mathbf{X}_\omega(x, t)$$

(2)

with $\alpha(\omega, x, t) = \sqrt{2 \omega \varepsilon_0 \text{Im} \{ \epsilon(\omega, x, t) \}} / \pi$. When the material is in motion with local velocity $\mathbf{v}(x, t)$ this local polarisation appears partly as a magnetisation (a phenomenon we shall sometimes refer to as polarisation mixing), which to first order in velocity equals

$$\mathbf{M}(x, t) = - \mathbf{v}(x, t) \times \mathbf{P} = \int_0^\infty d\omega \, \alpha_B(\omega, x, t) \cdot \mathbf{X}_\omega.$$ (3)

It is thus clear that the first three terms of the Lagrangian density (1) equal the free space Lagrangian density $\mathcal{L}_F = (\varepsilon_0 / 2) (\mathbf{E}^2 - c^2 \mathbf{B}^2)$, minus the electric dipole interaction energy density $-\mathbf{E} \cdot \mathbf{P}$, and the magnetic dipole interaction energy density $-\mathbf{B} \cdot \mathbf{M}$. The final term is the Lagrangian density for a field of simple harmonic oscillators $\mathcal{L}_B = (1 / 2) \int_0^\infty d\omega \left[ \mathbf{X}_\omega^2 - \omega^2 \mathbf{X}_\omega^2 \right]$, with the rest frame coordinates $\mathbf{x}'$ transformed into the laboratory frame such that $t' = t$ and $\mathbf{x}' = \mathbf{x}(x, t)$

$$\frac{\partial}{\partial t'} = \frac{\partial}{\partial t} + \mathbf{v}(x, t) \cdot \nabla,$$

(4)
where \( x' = v(x, t) \). Although we have set \( \mu = 1 \), this amounts to ignoring the second oscillator bath that appears in [19–21], and can be re-introduced with no fundamental modification to our results. The coupling between the field of oscillators and the electromagnetic field is mediated via a term that is proportional to a quantity \( \alpha \) which in this case is a scalar quantity that depends on space, time, and frequency. Here we treat the time dependence of the medium as a perturbation to an otherwise isotropic, time independent background: \( \alpha(\omega, x, t) = \alpha_0(\omega, x) + \delta \alpha(\omega, x, t) \). The background permittivity of the medium \( \epsilon_0(\omega, x, t) \) is related to this time independent coupling term by the expression given in [19]: \( \alpha_0(\omega, x) = \sqrt{2\varepsilon\varepsilon_0} \text{Im} \left[ \epsilon_0(x, \omega) / \pi \right] \).

It is worth making an additional comment about the reservoir degrees of freedom \( X_\omega \). As stated above, this reservoir is a device introduced to account for the energy lost from the electromagnetic field. Yet there is more physical reality to this than merely accounting for lost energy. The reservoir exactly mimics the linear response of any medium with local permittivity \( \epsilon(\omega, x) \). In any real absorption process radiation excites a time dependent current which could be for example an optical phonon, or an exciton. The reservoir will mimic as much of the physics of this process as is included in the dispersive properties of \( \epsilon(\omega) \) (or \( \epsilon(\omega, k) \) if non-locality is included [25]). Therefore when we later come to calculate excitation rates for the coupled matter–field system, the calculated dynamics of the reservoir must be read as a proxy for the electromagnetic field exciting a real quasiparticle within the medium. We expect the approximations inherent in our use of the reservoir to describe the medium to be the same as those involved in assuming that the medium can be characterised by a linear response \( \epsilon(\omega, x) \).

### 2.2. Hamiltonian

The Hamiltonian of the system is defined in terms of the canonical momenta and the Lagrangian (see for example [27]) and is equal to

\[
H = \int d^3x \left[ \Pi_A \cdot \dot{A} + \int_0^\infty d\omega \left( \Pi_{X_\omega} \cdot \dot{X}_\omega - \mathcal{L} \right) \right],
\]

where the canonical momentum of the electromagnetic field \( \Pi_A \) and of the oscillator field \( \Pi_{X_\omega} \) are given by

\[
\Pi_A = -\epsilon_0 \mathbf{E} - \int_0^\infty d\omega \alpha(\omega, x, t) \mathbf{X}_\omega,
\]

\[
\Pi_{X_\omega} = \dot{X}_\omega + (\mathbf{v} \cdot \nabla)X_\omega.
\]

The scalar potential \( \phi \) is not a dynamical variable, and can be removed from the Hamiltonian as in [19].

Because we treat the time dependence of the dielectric as having a small effect on the total field, we divide this Hamiltonian up into two parts

\[
H = H_0 + H_1,
\]

where \( H_0 \) is the Hamiltonian of a time-independent nonmagnetic dielectric [19]

\[
H_0 = \frac{1}{2} \int d^3x \left[ \int \frac{1}{\epsilon_0} \left( \Pi_A + \int_0^\infty d\omega \alpha_b(\omega, x)X_\omega \right)^2 \right.
\]

\[
\left. + \frac{1}{\mu_0} (\nabla \times \mathbf{A})^2 + \int_0^\infty d\omega (\Pi_{X_\omega}^2 + \omega^2 \mathbf{X}_\omega^2) \right] (8)
\]

and \( H_1 \) is the interaction Hamiltonian due to the motion and time dependence of the dielectric, given by

\[
H_1 = \int d^3x \int_0^\infty d\omega \left[ -\Pi_{X_\omega} \cdot (\mathbf{v} \cdot \nabla)X_\omega + \mathbf{B} \cdot \alpha_b(\omega, x, t) \cdot \mathbf{X}_\omega + \mathbf{E} \cdot \mathbf{X}_\omega \delta \phi(\omega, x, t) \right].
\]

The interaction Hamiltonian \( H_1 \) contains three terms: the first accounts for the fact that the bath of oscillators is in motion; the second is due to the mixing of polarisations by the motion; and the third term is due to the time dependence of the permittivity, which could—for example—be due to the motion of the boundaries or simply the time-variation of the permittivity of a stationary object.

### 2.3. Classical equations of motion

Before embarking on the quantum mechanical calculation, we derive the classical equations of motion and show how the time dependence of the system can be seen as a modification of the relative permittivity. As an example, consider the case of a spatially homogeneous medium moving with an arbitrary time dependent velocity (in this case \( \delta \phi = 0 \)). Through the use of the Euler–Lagrange equations (see e.g. [27]), we obtain the equations of motion for the oscillator bath

\[
\left( \frac{\partial}{\partial t} + (\mathbf{v} \cdot \nabla) \right)^2 + \omega^2 \right] X_\omega(x, t) = \alpha_b(\omega) \mathbf{E}(x, t)
\]

and for the field

\[
\nabla \times \nabla \times \mathbf{E} + \frac{1}{c^2} \frac{\partial^2 \mathbf{P}}{\partial t^2} = -\mu_0 \frac{\partial^2 \mathbf{P}}{\partial t^2}\]

For simplicity we have ignored the polarisation mixing term \( \mathbf{B} \cdot \int_0^\infty d\omega \alpha_b(\omega, x, t) \cdot \mathbf{X}_\omega, \) concentrating on only the electric response of the material. In order to see how the time dependence of the medium affects the material parameters, we solve equations (10) and (11). The solutions to (10) are given by

\[
X_\omega(x, t) = \alpha_b(\omega) \int d^3x' \int_0^\infty dr' G_{X_\omega}(x, x', t, t') \mathbf{E}(x, t)
\]

\[
+ X_\omega^H(x, t),
\]

where the \( X_\omega^H(x, t) \) are the homogeneous solutions to (10) and are of the general form

\[
X_\omega^H(x, t) = h_a \left( x - \int_0^t v(t') dt' \right) e^{-i\omega t} + c.c.,
\]

where \( h_a \) are arbitrary functions of position and the lower
limit in the integral, \( t_0 \) is the initial time of the evolution of the system where the amplitudes are simply \( \mathbf{h}_0(x) \). In the quantum theory, these amplitudes will become the creation and annihilation operators of the quantum fields in the same way as the theory developed in [19]. The oscillator Green’s function \( G_{\omega}(x, x', t, t') \) in (12) satisfies

\[
\left[ \frac{\partial}{\partial t} + (\mathbf{v} \cdot \nabla) \right] \left[ \frac{\partial}{\partial t} + (\mathbf{v} \cdot \nabla) \right] G_{\omega}(x, x', t, t') = \delta^{(3)}(x - x') \delta(t - t')
\]

or

\[
\frac{\partial}{\partial t} \left[ \mathbf{E}(x, t) \right] = -\frac{1}{\varepsilon_0 \mu_0} \mathbf{J}(x, t),
\]

the retarded solution (zero for \( t < t' \)) to which is given by

\[
G_{\omega}(x, x', t, t') = \frac{1}{\varepsilon_0 \mu_0} \mathbf{E}(x, t) \mathbf{E}^*(x', t') \delta(t - t').
\]

Substituting (15) into (12) gives an expression for the amplitudes of the oscillator field in terms of the field \( \mathbf{E} \) and the arbitrary amplitudes \( \mathbf{h}_0 \).

\[
\mathbf{X}_\omega(x, t) = \frac{\alpha_0^2(\omega)}{\omega} \int_{-\infty}^t \mathbf{E} \left( x - \int_{t'}^{t} \mathbf{v}(t') dt' \right) + \mathbf{X}_\omega(x, t).
\]

The meaning of the integral expression is simply that the oscillator amplitudes at \( x \) no longer depend solely on the strength of the field at \( x \) (as it does in [19]) but also on the previous positions of the moving oscillator for times \( t' \) before \( t \). The motion of the material thus leads to a non-local response, due to the fact that energy dissipated from the electromagnetic field is carried away from the point where it was absorbed. The wave equation for the electric field is found through substituting expression (16) into (2), and combining the resultant expression with (11)

\[
\nabla \times \nabla \times \mathbf{E} + \frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} + \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \int d^3 x' \int_{-\infty}^t \mathbf{E}(x', t') \mathbf{E}(x', t') = -\mu_0 \mathbf{J}(x, t),
\]

where the non-local effective susceptibility of the system \( \chi \) is given by

\[
\chi(x, x', t, t') = \frac{1}{\varepsilon_0 \mu_0} \rho_0(\omega) \delta^{(3)}(x - x') \delta(t - t') \sin[\omega(t - t')].
\]

The theta function in (18) ensures that the effective susceptibility is non-zero only for times past, enforcing causality and consequently also the Kramers–Kronig relations. In the limit \( v \to 0 \), the wave equation (17) reduces to the usual wave equation in stationary dispersive media (see e.g. [3]), and (18) reduces to the local expression \( \chi(x, x', t, t') = \delta^{(3)}(x - x') \chi(t - t') \), where \( \chi(t - t') \) is the Fourier transform of \( \epsilon_0(\omega) - 1 \). The source of the electromagnetic field \( \mathbf{J}(x, t) \) that appears in the wave equation (11) depends on the arbitrary functions \( \mathbf{h}_0 \), and is given by the non-local expression\(^2\)

\[
\mathbf{J}(x, t) = \int_0^\infty d\omega \frac{\alpha_0^2(\omega)}{\omega} \frac{\partial^2}{\partial t^2} \left[ \mathbf{h}_0 \left( x - \int_{t_0}^t \mathbf{v}(t') dt' \right) e^{-i \omega t} + c.c. \right].
\]

This current has the same meaning as in [19]: the un-driven part of the motion of the bath of oscillators is the source of the electromagnetic field in an absorbing medium. In [19], the \( \mathbf{h}_0 \) are related to the amplitude of the current \( \mathbf{J} \) in a local manner. Here, the relative motion of the medium \( \mathbf{v} \) means that the current at \( x \) now depends on the value of \( \mathbf{h}_0 \) at the point \( x - \int_{t_0}^t \mathbf{v}(t') dt' \). In analogy to [19], the solution to (17) written in terms of the electromagnetic Green’s function is

\[
\mathbf{E}(x, t) = -\mu_0 \int d^3 x' \int_{-\infty}^t d^3 x'' \mathbf{G}(x, x', t, t_2) \mathbf{J}(x', t_2).
\]

where \( \mathbf{G} \) is a bi-tensor satisfying

\[
\nabla \times \nabla \times \mathbf{G} + \frac{1}{c^2} \frac{\partial^2 \mathbf{G}}{\partial t^2} + \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \int d^3 x' \int_{-\infty}^t \mathbf{E}(x', t') \mathbf{E}(x', t') \mathbf{G}(x, x', t_2) = -i \delta^{(3)}(x - x') \delta(t - t_2) \]

and \( \chi(t, t') \) is equal to (18) without the spatial delta function. In the cases treated in [19], the difficulty of finding analytic expressions for the Green function resides in the complexity of the geometry. In this case though, even in homogeneous media, the integro-differential equation above presents great difficulties due to its non-locality. Yet in a few simple cases, some progress can be made. Given the spatial homogeneity of the medium in this case, (21) can by Fourier transformed in space so that it reduces to the integral equation

\[
\mathbf{k} \times \mathbf{k} \times \mathbf{G}(k, \Omega, \Omega_2) + \frac{\Omega^2}{c^2} \mathbf{G}(k, \Omega, \Omega_2)
\]

\[
+ \frac{\Omega^2}{c^2} \int d^3 k_1 (k, \Omega, \Omega_1) \mathbf{G}(k, \Omega, \Omega_2)
\]

\[
= -2\pi i \delta(\Omega - \Omega_2),
\]

where the kernel of the integral represents the Fourier transformed susceptibility of the system and is given by

\[
\chi(k, \Omega, \Omega_1) = \int_{-\infty}^\infty d\Omega_1 \chi(\Omega, \Omega_1) e^{-i k_1 v(t') dt'} e^{i \Omega_1 t'} e^{-i \omega_1 t'}.
\]

For the case of constant velocity \( v(t) = v \), the wave equation (21) simplifies to that given in [21] (again, ignoring polarisation mixing) where the Doppler shifted frequency

\(^2\) Note that here the source term \( J \) does not have units of electric current.
\[ \Omega - v \cdot k \] appears in the argument of the susceptibility

\[
\left\{ \mathbf{k} \otimes \mathbf{k} - \left[ k^2 - \frac{\Omega^2}{c^2} \right. \left( 1 + \chi(\Omega - k \cdot v) \right) \right\}_k
\cdot G_k(\Omega, \Omega_2)
= -2\pi i \delta(\Omega - \Omega_2) I_3. \tag{24}
\]

The Green function is then equal to the inverse of the square bracketed matrix on the left times \(2\pi\omega_0^2\delta(\Omega - \Omega_2)\). In other words, the constant motion of the dielectric, gives rise to a new effective permittivity in which the frequency response is shifted by \(-k \cdot v\).

In the slightly more complicated case of time dependent oscillatory motion in the \(z\)-direction, with frequency \(\nu\), \((v(t) = z_0\nu \sin(\nu t) \mathbf{z})\), where \(z_0\) is the maximum displacement, the integral over the velocity in the susceptibility (23) becomes

\[
k \int_0^\tau v(t) \, dt' = k_z z_0 \cos(\nu t) - \cos(\nu t).
\tag{25}
\]

The susceptibility (23) can then be written as

\[
\chi(k, \Omega, \Omega_1) = 2\pi \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \frac{i^{n-m} J_n(k_z z_0) J_m(k_z z_0)\delta}{(\Omega - \Omega_1 + (m + n)\nu) \chi(\Omega_1 - n\nu)},
\tag{26}
\]

where we applied the generating function for Bessel functions [26]

\[
e^{i x \cos \theta} = \sum_{n=-\infty}^{\infty} i^n J_n(z) e^{in\theta}
\tag{27}
\]

with \(J_n\) as the Bessel functions of the first kind. Expression (26) demonstrates a coupling between the frequencies of the field arising from the oscillatory motion, occurring in discrete multiples of the oscillation frequency \(\nu\). It is interesting to note that the summand vanishes when \(k_z z_0\) equals a zero of a Bessel function, which for a fixed \(k\) and \(\Omega\) excludes the coupling between those frequencies \(\Omega_1 - (n + m)\nu\) where \(J_{n+m}(k_z z_0) = 0\). In the limiting case when the argument of the Bessel functions, \(k_z z_0\), becomes very large, the material behaves like the vacuum since the Bessel functions all tend to zero. For very small \(k_z z_0\), only the \(m = n = 0\) survives and the material behaves as if it were stationary. This occurs for very large wavelengths or in the limit of a vanishing displacement amplitude.

The above brief examination demonstrates that for most cases of interest it is difficult to find exact solutions to the equations of motion. In order to establish quantitative results in general, proceeding via perturbation theory is the approach adopted from here on.

\section*{3. Emission rates for time-dependent media}

We now quantise the classical theory outlined in the previous section, applying it to the situations illustrated in figure 1. Most of the formulae resemble the classical theory, but we must replace the classical fields with operators that obey the canonical commutation relations [29]

\[
\begin{align*}
[\hat{A}(\mathbf{x}, t), \hat{A}_i(\mathbf{x}', t)] &= i\hbar \delta^{(3)}(\mathbf{x} - \mathbf{x}') \\
&\times [\hat{X}_i(\mathbf{x}, t), \hat{X}_{i'}(\mathbf{x}', t)] = i\hbar \delta^{(3)}(\mathbf{x} - \mathbf{x}')\delta(\omega - \omega').
\end{align*}
\tag{28}
\]

We work in the interaction picture [29], where the time dependence of the operators is generated by the bare Hamiltonian \(\hat{H}_0\), given by the operator equivalent of (8) and that of the quantum state by the interaction Hamiltonian \(\hat{H}_I(t) = e^{-i\hat{H}_I t} \hat{H}_0 e^{i\hat{H}_I t}\) given by the operator equivalent of (9) where the full Hamiltonian is given by \(\hat{H} = \hat{H}_0 + \hat{H}_I\).

With the time dependence of the operators being generated by \(\hat{H}_0\), the canonical operators are expanded in terms of a set of creation and annihilation operators \(\hat{C}(\mathbf{x}, \omega), \hat{C}^\dagger(\mathbf{x}, \omega)\) as

\[
\begin{align*}
\hat{A}(\mathbf{x}, t) &= \int d^3x' \int_0^\infty d\omega' \{\hat{C}(\mathbf{x}, \omega', \mathbf{x}') \cdot \hat{C}(\mathbf{x}, \omega') e^{-i\omega't} + \text{h.c.}\}
\end{align*}
\tag{29}
\]

\[
\hat{A}_i(\mathbf{x}, t) = \int d^3x' \int_0^\infty d\omega' \{\hat{C}_i(\mathbf{x}, \omega', \mathbf{x}') \cdot \hat{C}_i(\mathbf{x}, \omega') e^{-i\omega't} + \text{h.c.}\}
\tag{30}
\]

\[
\hat{X}_i(\mathbf{x}, t) = \int d^3x' \int_0^\infty d\omega' \{\hat{C}_i(\mathbf{x}, \omega', \mathbf{x}') \cdot \hat{C}_i(\mathbf{x}, \omega') e^{-i\omega't} + \text{h.c.}\}
\tag{31}
\]

\[
\hat{X}_{i'}(\mathbf{x}, t) = \int d^3x' \int_0^\infty d\omega' \{\hat{C}_{i'}(\mathbf{x}', \omega, \mathbf{x}') \cdot \hat{C}_{i'}(\mathbf{x}', \omega, \mathbf{x}') e^{-i\omega't} + \text{h.c.}\}
\tag{32}
\]

where the expansion coefficients are identical to those given in [19], and are given for reference here in appendix A. There are also similar expansions for the \(\hat{E}\) and \(\hat{B}\) operators. The physical meaning of the operators \(\hat{C}\) and \(\hat{C}^\dagger\) is that they are the creation and annihilation operators for the normal modes of the coupled system of field plus matter (polaritons) and obey bosonic commutation relations

\[
[\hat{C}(\mathbf{x}, \omega), \hat{C}^\dagger(\mathbf{x}', \omega')] = \hbar \delta(\mathbf{x} - \mathbf{x}')\delta(\omega - \omega')
\tag{33}
\]

\[
[\hat{C}(\mathbf{x}, \omega), \hat{C}(\mathbf{x}', \omega')] = 0.
\]

The expansion given in (29) is chosen such that it diagonalises the bare Hamiltonian

\[
\hat{H}_0 = \int d^3x \int_0^\infty d\omega /\hbar \omega \hat{C}^\dagger(\mathbf{x}, \omega) \hat{C}(\mathbf{x}, \omega).
\tag{34}
\]

For more details about this diagonalization see [15–17, 19, 21].

The methods of time dependent perturbation theory are now applied to find the effect of the time dependence of the medium on the excitation of polaritons. In the absence of any time dependence, we take the system to be prepared in its ground state \(|0\rangle\) (defined as the state where \(\hat{C}_0|0\rangle = 0\)). The interaction Hamiltonian \(\hat{H}_I\) is a combination of squares of field operators, which will lead to the creation of pairs of
polaritons. We thus represent the wave function of the system as
\[
|\psi(t)\rangle = |0\rangle + \sum_{m,n} \int d^3x_1 \int d^3x_2 \int_0^\infty d\omega_1 \int_0^\infty d\omega_2 \zeta_{mn} \times (x_1, x_2, \omega_1, \omega_2, t) \hat{C}_m(x_1, \omega) \hat{C}_n(x_2, \omega)|0\rangle,
\]
(35)
where the expansion coefficient obeys \(\zeta_{mn}(x_1, x_2, \omega_1, \omega_2, t) = \zeta_{nm}(x_2, x_1, \omega_2, \omega_1, t)\), in accordance with bosonic exchange symmetry. The physical meaning of \(\zeta_{mn}\) is as the probability amplitude for the creation of a pair of current excitations in the material, located at positions \(x_1\) and \(x_2\), oscillating at frequencies \(\omega_1\) and \(\omega_2\), and pointing in the \(m\)- and \(n\)-directions. Inserting (35) into the Schrödinger equation \(\partial|\psi\rangle/\partial t = -(i/\hbar)\hat{H}|\psi\rangle\), we find the rate of change of the expansion coefficient is given by
\[
\dot{\zeta}_{mn}(x_1, x_2, \omega_1, \omega_2, t) = -\frac{i}{2\hbar}\langle 0|\hat{C}_m(x_1, \omega)\dot{\hat{C}}_n(x_2, \omega)|\hat{H}|0\rangle.
\]
(36)
Due to the form of the interaction Hamiltonian (9), the rate of change of the rank-2 tensor \(\zeta_{mn}\) separates into the sum of two parts
\[
\dot{\zeta}_{mn}(x_1, x_2, \omega_1, \omega_2, t) = \zeta_{mn}^{(a)}(x_1, x_2, \omega_1, \omega_2, t) + \zeta_{mn}^{(b)}(x_1, x_2, \omega_1, \omega_2, t).
\]
(37)
The first term \(\zeta_{mn}^{(a)}\) arises from the first two terms in the interaction Hamiltonian (9) and represents the effect of the motion of the medium, both through the movement of energy within the bath of oscillators, and the mixing of the electromagnetic field polarisation between reference frames. Note that although in general the coupling term \(\alpha_{fg}(\omega, x, t)\) depends upon the time dependence of the material \(\alpha(\omega, x, t)\) as well as the motion, in what follows we shall only treat particular cases, where \(\alpha_{fg}(\omega, x, t) = -v(x, t) \times \alpha_{fg}(x, \omega)I\) (i.e. the motion of a homogeneous medium). The second term, \(\zeta_{mn}^{(b)}\), comes from the final term in (9), and represents the effect of any time dependence in the permittivity. Evaluating (36) by substituting the expansion of the operators (29) into the interaction Hamiltonian and using the commutation relations (33), the dyadic form of the rate of change of the two parts to the expansion coefficient (37) is found to be
\[
\dot{\zeta}_{mn}^{(a)}(x_1, x_2, \omega_1, \omega_2, t) = -\frac{i}{2\hbar} e^{i(\omega_1+\omega_2)t} \int d^3x \int_0^\infty d\omega \times \left[ \hat{f}_E(x_1, x_2, \omega_1, \omega_2) \right. \\
\left. \times (\omega, x, t) \right] \hat{F}_E^\dagger(x_1, x_2, \omega_1, \omega_2) \right. \\
+ \hat{f}_E(x_1, x_2, \omega_1) \cdot \alpha_{fg}(\omega, x, t) \cdot \hat{F}_E^\dagger(x_2, x_1, \omega_1, \omega_2) \right. \\
+ 1 \leftrightarrow 2
\]
(38)
and
\[
\dot{\zeta}_{mn}^{(b)}(x_1, x_2, \omega_1, \omega_2, t) = -\frac{i}{2\hbar} e^{i(\omega_1+\omega_2)t} \int d^3x \int_0^\infty d\omega \Delta \alpha \times (\omega, x, t) \hat{f}_E(x_1, x_2, \omega_1, \omega_2) \right. \\
\left. \times (\omega, x, t) \hat{F}_E^\dagger(x_1, x_2, \omega_1, \omega_2) \right. \\
+ 1 \leftrightarrow 2
\]
(39)
where the notation ‘1 ↔ 2’ indicates the repetition of the preceding expression but with the two particles interchanged (which in the above expressions involves both taking the transpose and swapping subscripts 1 and 2). It is evident from expressions (38) and (39) that the rate of polariton excitation is in general quite different for time dependent and moving media. For instance, a time dependent permittivity profile constructed to appear as a moving material would have \(\zeta^{(a)} = 0\), whereas true motion has in general both non-zero \(\zeta^{(a)}\) and \(\zeta^{(b)}\).

In the remainder of the paper we shall evaluate the polariton excitation rates (38) and (39) for the three cases shown in figure 1. To be specific we assume the simplest case of a lossy dispersive medium, where the background dielectric function \(\varepsilon_{bg}\) is Lorentzian, with a resonant frequency \(\omega_0\)
\[
\varepsilon_{bg}(\omega) = 1 + \frac{\omega_p^2}{\omega^2 - \omega_0^2 - i\gamma\omega}
\]
(40)
and where we assume the arbitrary values \(\omega_p = 0.5\ \omega_0\), and damping constant \(\gamma = 0.1\ \omega_0\).

4. Emission due to the motion of a homogeneous medium

The first example we investigate is the polariton emission rate within an infinite homogeneous medium performing an oscillatory motion. Due to the lack of boundaries and the translational invariance, the second term in (37), \(\zeta^{(b)}\) is equal to zero, and the probability amplitude for emitting a pair of polaritons is then equal to the integral of \(\zeta^{(a)}\) over the time interval \(t \in [-T/2, T/2]\) (it is assumed that the interaction Hamiltonian is ‘turned on’ at the initial time \(-T/2\). The absolute value squared of the result divided by \(T\) gives us the average net excitation rate of polariton pairs over the time interval \(T\). Over a very long time interval \(T \to \infty\) this is given by
\[
\Gamma = \int d^3x_1 \int d^3x_2 \int_0^\infty d\omega_1 \times \int_0^\infty d\omega_2 \sum_{m,n} \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} dt \left| \zeta_{mn}^{(a)}(x_1, x_2, \omega_1, \omega_2, t) \right|^2.
\]
(41)
The emission rate given by (41) is valid even for spatially dependant velocities \(v\), as occurs e.g. for rotating bodies. However here we make the simplification that the velocity does not depend on position, in which case—as is evident from (38)—the time dependence of \(\zeta^{(a)}\) can be factored out from the spatial dependence. For example, in the case of constant velocity the time dependence of (38) is given by the factor \(\exp(i(\omega_1 + \omega_2)t)\) so that
\[
\lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} dt e^{i(\omega_1+\omega_2)t} \left| \zeta^{(a)}(\omega_1, \omega_2) \right|^2 = 2\pi\delta(\omega_1 + \omega_2).
\]
(42)
The emission rate (41) is then reduced to

$$\Gamma = 2\pi \int d^3k \int d^3x \int_0^\infty d\omega_1 \times \int_0^\infty d\omega_2 \sum_{m,n} |A_{mm}^4(x_1, x_2, \omega_1, \omega_2)|^2 \delta(\omega_1 + \omega_2),$$

where we have used $\zeta^{(a)}(x_1, x_2, \omega_1, \omega_2, \mathbf{t}) = A_{mm}^{(a)}(x_1, x_2, \omega_1, \omega_2) \exp[i(\omega_1 + \omega_2)t]$. The argument of the delta function is only zero at the one point $\omega_1 = \omega_2 = 0$, where the integrand is zero. This leaves

$$\Gamma = 0$$

as expected.

When the velocity is time dependent then the emission rate $\Gamma$ will be non-zero. Consider some time dependent velocity $v(t) = v(t)\mathbf{k}$. In order to analyse the emission rate as a function of frequency and wave-vector, we work in terms of the Fourier transform of $\zeta^{(a)}$.

$$\zeta^{(a)}(x_1, x_2, \omega_1, \omega_2, \mathbf{t}) = \int \frac{d^3k}{(2\pi)^3} \zeta^{(a)}(k, \omega_1, \omega_2, \mathbf{t}) e^{i\mathbf{k} \cdot (x_1 - x_2)} = v(t) e^{i(\omega_1 + \omega_2)t} \int \frac{d^3k}{(2\pi)^3} A(k, \omega_1, \omega_2) e^{i\mathbf{k} \cdot (x_1 - x_2)},$$

where the quantity $A(k, \omega_1, \omega_2)$ is given in appendix B and is a sum of terms depending of products of Green’s functions and combinations of the permittivity at frequencies $\omega_1$ and $\omega_2$. The amplitude $A(k, \omega_1, \omega_2)$ is proportional to the probability amplitude for exciting a pair of polaritons with wave-vectors $\mathbf{k}$ and $-\mathbf{k}$ and frequencies $\omega_1$ and $\omega_2$ from the ground state, due to the motion of the medium.

We take the particular case: $v(t) = z_0 \nu \cos(\nu t)$, where $z_0$ is the maximum displacement from the mean position. Taking the absolute value squared of (45) and integrating with respect to time, the equivalent of (42) now equals

$$\lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} dt \left| v(t) e^{i(\omega_1 + \omega_2)t} \right|^2 = \frac{\pi z_0^2 \nu^2}{2} [\delta(\omega_1 + \omega_2 - \nu) + \delta(\omega_1 + \omega_2 + \nu)]$$

the proof of which is given in appendix A. While the second term on the right of (46) fails to contribute to the emission rate (owing to $\omega_1, \omega_2$ and $\nu$ all being positive), the first term does. Combining (45) and (46) with the expression for the net rate of excitation (43) gives the emission rate per unit volume $V$

$$\Gamma = \int \frac{d^3k}{(2\pi)^3} \int_0^\infty d\omega \rho(k, \omega),$$

where the ‘spectral density’ for the emission of polariton pairs $\rho$ is equal to

$$\rho(k, \omega) = \frac{\pi z_0^2 \nu^2}{2} \theta(\nu - \omega) \sum_{m,n} |A_{mn}(k, \omega, \nu - \omega)^2| \delta(\omega_1 + \omega_2),$$

with $\theta$ the Heaviside step function. This dimensionless ‘spectral density’ depends only on the wave vector and frequency since the translational symmetry of this system guarantees momentum conservation for the two polaritons, $k_1 + k_2 = 0$, and energy conservation implies that the total energy of the pair of particles must equal that due to the motion: $h\nu = h(\omega_1 + \omega_2)$. Note that the time dependence of the velocity sets this relationship between $\omega_1$ and $\omega_2$, and that for a general motion this may be more complicated.

The excitation rate (47) evidently scales quadratically with the maximum displacement $z_0$. However, the dependence on the oscillation frequency $\nu$ is somewhat more intricate. As can be seen from the expression for $A$ given in appendix B, as well as figure 2(a), the largest contribution to the spectral density $\rho$ comes from regions where the frequency and wave vector are close to satisfying the dispersion relations for electromagnetic waves $\omega = c|\mathbf{k}|/\epsilon^*(\omega)$ and $\nu - \omega = \pm c|\mathbf{k}|/\epsilon^*(\nu - \omega)$ and where the frequency matches the resonant frequency of the dielectric $\omega_0$, and the shifted resonance at $\nu - \omega_0$. The largest emission of polaritons occurs in the region close to where the dispersion curves for radiation intersect with the material resonances. These excitations can be measured through detecting the electric field at a point inside or outside the medium. In appendix D we show how the excitation rate of such a detector is related to the quantity $\zeta_{mn}$ that appears within $\rho$.

We note that there is a term that contributes to (47)—given as $d(\omega, \nu - \omega)$ in appendix—which diverges when $\omega_1 = \nu/2$ but is zero for all other frequencies. This has been omitted from figure 2(a). Physically, this problematic term represents the emission rate for two polaritons at the same frequency from the same position. This is specific to moving media and does not recur in the remainder of the paper. It nevertheless deserves further attention and will be treated in future work.

5. Time dependent permittivity

The second contribution to the emission rate (39) $\zeta^{(b)}$ comes from the time dependence of the material response $\epsilon_{\alpha\beta}(x, t, \omega)$. For moving media, this term can be non-zero either due to moving inhomogeneities or changing boundaries. This term can also be used to model changes in the permittivity due to external forces, such as those in dynamical Casimir experiments [23, 28] or optical analogues of Hawking radiation [7].

Inserting the expansion coefficients listed in appendix A into (39), we find the rate of change of the probability amplitude for exciting a pair of polaritons is given by

$$\dot{\zeta}^{(b)}(x_1, x_2, \omega_1, \omega_2, t) = -\frac{i\mu_0 \omega_1^2 \alpha_{\beta}(\omega_1)}{4\sqrt{\omega_1^2}} \epsilon^*(\omega_1 + \omega_2) \times \left[ \mu_0 \omega_2^2 \alpha_{\beta}(\omega_2) \int_0^\infty d\omega \frac{\alpha_{\beta}(\omega)}{\omega^2 - (\omega_2 - i\omega_0)^2} \right.$$

$$\left. \times \int d^3x \frac{\delta(x_1, x_2, t)}{G'(x, x_1, \omega_1) \cdot G''(x, x_2, \omega_2)} \delta(\alpha_{\beta}(\omega_2)) + \delta(\alpha_{\beta}(\omega_2)) \right] 1 \leftrightarrow 2.$$

We now consider two particular applications of (49), firstly
where the permittivity of the medium uniformly oscillates as a function of time, and secondly where a travelling pulse-like perturbation \( \delta \alpha(x - vt, \nu) \) to the permittivity moves through the medium at a uniform velocity.

5.1. Emission from a time dependent permittivity

For a time dependent change to the permittivity of the form \( \delta \alpha(x, t) = \alpha_0 \beta(\omega) \cos(\nu t) \), the time dependence can be factored out as in the previous section, and the identity (46) can be applied. The dimensionless `spectral density` appearing in the polariton emission rate per unit volume (the analogue of (47)) is then found to be

\[
\rho(k, \omega) = \frac{\pi}{2} \int_0^\infty d\omega_2 \delta(\omega + \omega_2 - \nu) \sum_{m,n} |A_{mn}(k, \omega, \omega_2)|^2 = \frac{\pi}{2} \sum_{m,n} |A_{mn}(k, \omega, \nu - \omega)|^2 \theta(\nu - \omega), \tag{50}
\]

where

\[
\mathcal{A}(k, \omega_1, \omega_2) = -\frac{i\mu_0\alpha_0\omega_1^2\alpha_b(\omega_1)}{4\omega_2^2} \times \int_0^\infty d\omega \frac{\alpha_b(\omega)\beta(\omega)}{\omega^2 - (\omega_2 - i\omega_0)^2} G^*(k, \omega_1) \cdot G(-k, \omega_2) + \beta(\omega_2)G^*(k, \omega_1) [1 + 1 \leftrightarrow 2], \tag{51}
\]

with ‘1 \leftrightarrow 2’ again indicating an interchange of the two particles (which now involves swapping the subscripts 1 and 2, interchanging \( k \) for \(-k \) and taking the transpose). As in the case of the moving dielectric that we just discussed, the emission conserves momentum \( k_1 + k_2 = 0 \) and the energy of the pair of polaritons is taken from the oscillation: \( \hbar \nu = \hbar (\omega_1 + \omega_2) \). Equation (51) can be further simplified in the particular case where \( \beta(\omega) = \alpha_b(\omega) \) (i.e. the change in the permittivity has the same frequency dependent response as the background)

\[
\mathcal{A}(k, \omega_1, \omega_2) = -\frac{i\mu_0\alpha_0\omega_1^2\alpha_b(\omega_1)}{2\pi} \sqrt{\text{Im}[\chi_b(\omega_1)] \text{Im}[\chi_b(\omega_2)]} \frac{\omega_1^2}{c^2} \times G^*(k, \omega_1) \cdot \left[ \chi_b^*(\omega_2) \frac{\omega_2^2}{c^2} G(-k, \omega_2) + 1 \right] + 1 \leftrightarrow 2. \tag{52}
\]

As is evident from (50)—and in similarity to the time dependent velocity investigated in the previous section—the time dependence of \( \delta \alpha \) gives rise to emission only for frequencies in the range \( \omega_1 > \nu \). The emission rate per unit volume is then given by integrating (50) over wave vector and frequency. The frequency and wave vector dependence of the spectral density of polariton emission are plotted in figure 3.

Due to the lack of any preferred direction in the changing permittivity \( \delta \alpha \) the emission is isotropic, but otherwise it is in many respects similar to that of a dielectric in oscillatory motion. However, note that the emission rate is not proportional \( \nu^2 \) in this case.
5.2. Emission rate from a travelling refractive index perturbation

As a more involved example of macroscopic QED applied to time dependent media, we address the case of a medium through which a perturbation of the refractive index travels at a constant velocity \( \mathbf{v} = v \mathbf{e}_z \), producing pairs of polaritons.

Recent work \cite{7, 10, 11} has established a connection between such a process and an analogue of Hawking radiation \cite{7, 8}.

Here we do not emphasise the connection to general relativity, but rather look for a description of such an emission process that fully accounts for the effects of dispersion and dissipation. To the authors’ knowledge, previous treatments have not fully accounted for such effects. We assume that the travelling perturbation to the permittivity can be represented in terms of a function \( \omega_c = \omega(\mathbf{x}) \) taking the form

\[
\omega_c = \int d^2 \mathbf{k} \langle \mathbf{k} | e^{i \mathbf{k} \cdot \mathbf{x} - i \omega \mathbf{k} \cdot \mathbf{v}} \rangle.
\]

where the dyadic \( A \) is given by

\[
A(k_1, k_2, \omega_1, \omega_2) = \frac{i \mu_0 \alpha_b(\omega) \omega^2}{4 \sqrt{\omega^2 - \omega_0^2}} G^*(k_1, \omega_1) \cdot \beta(\omega) \alpha_b(\omega) \\
\times G^\dagger (k_2, \omega_2) + \beta(\omega) \alpha_b(\omega) + 1 \leftrightarrow 2.
\]

For simplicity, we again choose the change in permittivity to have the same dispersion as the background material \( \beta(\omega) = \alpha_b(\omega) \). Equation (55) then reduces to

\[
A(k_1, k_2, \omega_1, \omega_2) = \frac{i}{2 \pi} \sqrt{\text{Im}[\chi(\omega)] \text{Im}[\chi(\omega)]} \omega^2 G^*(k_1, \omega_1) \cdot \left[ \chi_0(\omega) \omega^2 c^2 G^\dagger (k_2, \omega_2) + 1 \right] + 1 \leftrightarrow 2.
\]

Defining the net emission rate in terms of a spectral density that is now a function of two wave-vectors and frequency

\[
\Gamma = \int_0^\infty d\omega \int \frac{d^2 k_1}{(2\pi)^2} \int \frac{d^2 k_2}{(2\pi)^2} \rho(k_1, k_2, \omega)
\]

we find \( \rho \) to be

\[
\rho(k_1, k_2, \omega) = 2 \pi \sum_{m,n} |A_{mn}(k_1, k_2, \omega, v)| \cdot (k_1 + k_2 - \omega) \cdot (k_1 + k_2 - \omega) \cdot |f(k_1 + k_2)|^2 \theta(v \cdot (k_1 + k_2) - \omega)\]
where we applied
\[
\lim_{T \to \infty} \frac{1}{T} \left| \int_{-T/2}^{T/2} dt \ e^{i(\omega_1 + \omega_2 - v \cdot (k_1 + k_2))t} \right|^2 \\
= 2\pi \delta[\omega_1 + \omega_2 - v \cdot (k_1 + k_2)].
\]  
(58)

The expression for \(\rho\) has a similar form to that given in the previous sections. However, the lack of translational symmetry (or equivalently the momentum exchanged between the polaritons and the moving perturbation) means that this 'spectral density' \(\rho\) now depends on two wave vectors \(k_1\) and \(k_2\), and it now has dimensions of volume squared.

Despite the rather dense notation, the form of (57) can be motivated from fairly simple physical considerations: the spatial distribution of the moving perturbation to the permeability \(\delta\alpha\) is determined by the function \(f(k)\), which has a corresponding Fourier spectrum \(f(K)\). The two polaritons can exchange momentum with the moving perturbation so long as the conservation law \(h \cdot K = h \cdot (k_1 + k_2)\) is satisfied. Energy conservation must also be obeyed, with the pulse containing the frequencies \(v \cdot K\) so that \(h \cdot (\omega_1 + \omega_2) = h v \cdot K\).

Combining these two conservation laws leads to the constraint \(\omega_2 = v \cdot (k_1 + k_2) - \omega_1\), with the rate of pair production being proportional to the amplitude of the relevant Fourier amplitude of the perturbation, \(|A_{mn}|^2\). The \(\theta\) function in (57) ensures that the energy of both of the polaritons is positive, while \(|A_{mn}|^2\) scales the emission rate depending on how close the frequencies \(\omega_1\) and \(\omega_2\) are to the resonances of the material (where \(\alpha_0\) is large), and how close the dispersion relations \(|k_{1,2}|^2 = \epsilon^*(\omega_{1,2})\omega_{1,2}^2/c^2\) are to being fulfilled (where the Green function \(G(K, \omega)\) is large). From (56) it is once again evident that the peak pair production will occur when the dispersion relation is fulfilled or the frequency of one or both members of the pair are at a resonance of the material. In particular we note that it is possible for the moving refractive index perturbation to generate pairs of excitations where only one of the pair is a propagating electromagnetic mode.

In figure 4 we plot the dependence of \(\rho(k_1, k_2, \omega)\) as a function of both the relative angle between \(k_1\) and \(k_2\), and the magnitude of the modulus of both wave-vectors when \(k_1 = k_2 = k\) for a fixed angle \(\cos(\theta) = k_1/|k|\). We take the particular case where the moving perturbation to the permittivity takes the form of a Gaussian
\[
f(k) = f_0 \exp \left(\frac{\alpha^2}{4} - k^2\right)
\]
with the area under the function \(f(x)\) equal to \(f_0\). The spectral density for the excitation of polariton pairs (57) then becomes
\[
\rho(k_1, k_2, \omega) = 2\pi f_0^2 \sum_{m,n} A_{mn}(k_1, k_2, \omega, v \cdot (k_1 + k_2) - \omega) \exp \left(\frac{\alpha^2}{2} |k_1 + k_2|^2\right) \delta(v \cdot (k_1 + k_2) - \omega).
\]  
(59)

It is evident from (59) that there is a trade off in the spectral density between the requirement of energy conservation, and the shape of the moving perturbation. The maximum in the Fourier spectrum \(f(k_1 + k_2)\) in this case occurs when \(k_1 = -k_2\), which is where the energy of one of the polaritons \(v \cdot (k_1 + k_2) - \omega\) is negative, and thus the spectral density is zero. Meanwhile when \(k_1 = k_2\), both polaritons can have positive energy, but the exponential factor \(\exp(\sigma^2|k_1 + k_2|^2/2)\) reduces the spectral density exponentially with the magnitude of \(k_{1,2}\). Spatially sharp, rapidly moving perturbations thus have relatively large emission rates for pairs of polaritons, and these pairs tend to be emitted away from the axis of propagation. This is in qualitative agreement with the findings of e.g. [7] but here derived without the analogy to general relativity. Spatially broad perturbations correspond to large values of \(\sigma\), and thus small values of \(k_{1,2}\).

As the velocity of the perturbation drops to 0, the \(\theta\) function in (59) picks out ever larger magnitudes of \(k_{1,2}\), which are ever further from fulfilling the dispersion relation (i.e. where the Green function \(G(k, \omega)\) in (56) becomes even smaller). In this zero velocity limit the spectral density thus reduces to zero, as expected.

Figure 4(a) confirms the generic behaviour observed in the previous two sections: the emission rate spectral density contains terms that contribute significantly for frequencies and wave-vectors where one of the pairs is close to fulfilling the dispersion relation and/or the condition for resonant material response. Figure 4(b) shows the angular dependence of the spectral density of the emission, as the angle between \(k_1\) and \(k_2\) is varied for a fixed frequency and equal magnitude of the two wave-vectors. The emission is concentrated in several cone-like regions. These cones correspond to either the shifted dispersion curves or the shifted resonances seen as blurred lines in figure 4(a). Increasing the absorption within the dielectric response has the effect of further blurring the lines of 4(a) and of thickening the shells of the cone-like shapes of 4(b). This effectively means that it becomes possible to excite polaritons that lie further from the resonance of the material or the dispersion curve for electromagnetic waves. We emphasise that unlike the previous sections, the choices made in figure 4 do not show the complete radiation pattern from the moving moving perturbation and only serve to show under what conditions the emission is increased, as well as the directional distribution of the emission. It would be too lengthy to fully explain the different regimes of this effect here, and further results will be given in a future publication.

It is interesting to compare this effect with those explored in sections 4 and 5.1; in particular the time dependent permittivity of section 5.1 which would typically be called a dynamic-Casimir type effect. In cases of oscillatory motion or material time dependence, pairs of polaritons can be emitted with the total energy of the pair \(\hbar \nu\) coming from the frequency of oscillation. The emission from a moving perturbation is of exactly the same form, and from the perspective of the perturbation theory comes from the same ‘dynamic Casimir’ term (39). The only difference is that in this case the energy available for the creation of excitations instead comes from the frequencies \(v \cdot k\) that are due to the motion of the refractive index perturbation.
The final example we considered was the emission of polariton pairs from a uniformly moving perturbation to the permittivity of a medium. We found that the description of this process is not fundamentally different from the simpler case of a uniform time oscillating permittivity. The main difference is that rather than take energy $\hbar \nu$ from the oscillation of the material properties, the polaritons take energy $\hbar \nu \cdot \mathbf{k}$ from the motion of the perturbation. The wave-vector dependence of the energy makes the emission increase with spatially sharper perturbations, and typically causes the polaritons to be emitted in cones. However, the peak emission is still concentrated around the regions of frequency where one of the pair is close to a material resonance, and the other is close to fulfilling the dispersion relation for electromagnetic waves, where both are at resonances, or where both are close to fulfilling the dispersion relation. This finding that a emitted photon may be paired with an excitation of the material at a resonance may be useful for understanding some of the recent experiments on analogue Hawking emission.

6. Discussion

We analysed the necessary modifications to the Hamiltonian of macroscopic QED that must be made to describe time dependent dielectric media, applying this to calculate the polariton excitation rate in the three cases summarised in figure 1. Having found the separate modifications to the Hamiltonian to describe such media, we applied perturbation theory to calculate the emission rate of polaritons in these three cases. In general, we found that so long as the energies of both polaritons are positive, and energy and momentum are conserved then there is always some—albeit often very small—pair excitation rate $\rho$ (per unit volume of $\omega$–$\mathbf{k}$ space) that can be written as a function of frequency and wave-vector. This quantity $\rho$ becomes particularly large in certain regions of $\omega$–$\mathbf{k}$ space. These regions of enhanced emission are one of three types: (i) where both of the polaritons are close to satisfying the dispersion relation for electromagnetic waves within the material (i.e. they are both photon-like); (ii) one of the pair is close to satisfying the dispersion relation and the other is close to the resonant frequency of the material (where $\text{Im}(\epsilon)$ is very large); and (iii) where both of the pair are close to the resonant frequency of the material. Experimentally it might be of particular importance that when dissipation is important one might detect a photon produced from a dynamic Casimir type effect, the partner of which does not escape the medium because it is excited at a frequency where there is large dissipation.

Appendix A. Operator expansion coefficients

From [19], the coefficients in the expansion of the field operators in terms of the polariton creation and annihilation

**Figure 4.** (a) Logarithm of the spectral density for polariton emission $\log_{10}[\rho/|f_0|^2(\omega_0/e)^2]$ for $\mathbf{k}_0 = \mathbf{k}_2 = \mathbf{k}$ at an arbitrary angle given by $\cos(\theta) = k_0/|\mathbf{k}| = 1/\sqrt{3}$. The background dielectric function of the material $\epsilon_0$ is given by the Lorentzian response (40) with a resonance at $\omega_0$. The velocity is taken to be $v = 0.5c$ and the Gaussian pulse width as $\sigma = 0.1c/\omega_0$. The quantity $k_v$ equals $\sqrt{3}\omega_0/2v$. The spectral density is significant at frequencies in the vicinity of the resonant frequency of the material (vertical line at $\omega = \omega_0$), the shifted resonant frequencies $v(k_v + k_v') \pm \omega_0$ (A and B) and along the dispersion curve satisfying $k_v^2 - \epsilon^*(\omega_0)\omega_0^2 = 0$ (see for example line D) and the shifted dispersion curve $k_v^2 - \epsilon^*(\omega_{k_v} - \epsilon(k_v)\omega_0 - \epsilon(k_v')^2 = 0$ (see for example line C). The shaded area to the right of the line $\omega = v(k_v + k_v')$ does not contribute to the total emission rate due to the $\Theta$ function in the spectral density (59). The spectral density is greatest around intersections between dispersion curves and/or resonances of the material. The black circle points to a particular example of a region where two resonances coincide. (b) Angular dependence of the spectral density $\rho/|f_0|^2(\omega_0/e)^2$ for a fixed frequency $\omega_1 = \omega_0$ where the frequency and wave-vector magnitude are arbitrarily chosen to be $\omega = \omega_0$ and $|\mathbf{k}| = \omega_0\sqrt{3}$ respectively corresponding to the centre of the black circle in (a), for varying angle between $\mathbf{k}$ and $\mathbf{k}$. The rate is seen to be negligible for all directions except along the surface of a number of cones traced out along rings of constant inclination angle $\theta$. 

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operators are given by
\[
f_b(x, x_1, \omega_1) = -i \int_0^\infty \frac{h}{2\omega_1} \omega_1 \alpha_b(x_1, \omega_1) \times \nabla \times \nabla \times G(x, x_2, \omega_1) f_b(x, x_2, \omega_1)
\]
\[
\times \frac{\delta^{(3)}(x - x_2) \delta(\omega - \omega_2) h}{\sqrt{2\omega_2}} G(x, x_2, \omega_2) + \delta^{(3)}(x - x_2) \delta(\omega - \omega_2) h \}
\]
\[
f_e(x, x_1, \omega_1) = \int_0^\infty \frac{h}{2\omega_1} \omega_1 \alpha_b(x_1, \omega_1) G(x, x_1, \omega_1)
\]
\[
f_{11}(x, x_1, \omega_1) = -i \int_0^\infty \frac{h}{2\omega_1} \omega_1 \alpha_b(x_1, \omega_1) G(x, x_1, \omega_1) \delta^{(3)}(x - x_1)
\]
\[
f_{11}(x, x_1, \omega_1) = -i \omega_1 f_X(x, x_1, \omega_1),
\]
where $0^+$ is an infinitely small number serving to shift the coefficients off the real axis in the complex plane, the subscript $T$ denotes the transverse components, and the Green function obeys
\[
\nabla \times \nabla \times G(x, x_2, \omega_1) = -\frac{\omega_1^2}{c^2} \epsilon_0(x, \omega_1) G(x, x_2, \omega_1) = 1 \delta^{(3)}(x - x_2)
\]
with the boundary condition that the waves are outgoing at infinity (i.e. the retarded Green function). For the case of an infinite homogeneous medium with permittivity $\epsilon_0$, the solution to (A2) is given by
\[
G(x_1 - x_2, \omega_1) = \int \frac{d^3k}{(2\pi)^3} G(k, \omega_1) e^{ik(x_1 - x_2)}
\]
\[
= \int \frac{d^3k}{(2\pi)^3} \epsilon_0(\omega_1) (\omega_1/c)^2 I_1 k \cos(\omega_1 t / c^2 - k^2 x^2)
\]
where a positive imaginary part for $\text{Im}[\epsilon_0]$ picks out the retarded Green function.

**Appendix B. The probability amplitude for polariton emission in a moving, oscillating medium**

For the case of an infinite homogeneous oscillating medium of section 4, the rate of change amplitude $A(k, \omega_1, \omega_2)$ is
\[
A(k, \omega_1, \omega_2) = \frac{1}{2} \left[ B(k, \omega_1, \omega_2) + B(-k, \omega_1, \omega_2) \right],
\]
where the quantity $B(k, \omega_1, \omega_2)$ is given by
\[
B(k, \omega_1, \omega_2) = \frac{1}{k^2} \sqrt{\eta} (\omega_1 \gamma_I(\omega_1)) \alpha_0(\omega_1, \omega_2) G^*(k, \omega_1)
\]
\[
= \frac{1}{k^2} \sqrt{\eta} (\omega_1 \gamma_I(\omega_1)) \alpha_0(\omega_1, \omega_2) G^*(k, \omega_1)
\]
\[
\times \delta(\omega_1 - \omega_2) + \delta(\omega_1 - \omega_2) G^*(k, \omega_1) + \epsilon(\omega_1, \omega_2) G^*(k, \omega_1)
\]
\[
+ \epsilon(\omega_1, \omega_2) \frac{1}{k^2} (G^*(k, \omega_1) \times k) \cdot (\xi \times \eta),
\]
with the coefficients within this quantity given as
\[
a(\omega_1, \omega_2) = \frac{1}{\pi c^4} \omega_1^3 \omega_2^2 \left[ e^* (\omega_1) \left(\omega_1^2 - (\omega_2 - i\omega_2 0^+)^2 \right) \right] \]
\[
+ \frac{e^* (\omega_2)}{\omega_2^2 - (\omega_1 - i\omega_2 0^+)^2}
\]
\[
(b(\omega_1, \omega_2) = \omega_1 \pi c^2 \omega_1^2 - (\omega_2 - i\omega_2 0^+)^2
\]
\[
c(\omega_1, \omega_2) = \omega_1 \pi c^2 \omega_1^2 - (\omega_1 - i\omega_2 0^+)^2
\]
\[
d(\omega_1, \omega_2) = \omega_1 \pi c^2 \omega_1^2 - (\omega_1 - i\omega_2 0^+)^2
\]
\[
e(\omega_1, \omega_2) = \frac{1}{2} \sqrt{\epsilon_1(\omega_1) \epsilon_1(\omega_2)} \delta(\omega_1 - \omega_2)
\]
\[
e(\omega_1, \omega_2) = \frac{1}{2} \sqrt{\epsilon_1(\omega_1) \epsilon_1(\omega_2)} \delta(\omega_1 - \omega_2)
\]

**Appendix C. A proof of a delta function identity (46)**

We wish to evaluate the limit
\[
\lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} dt \ v(t) e^{i(\omega_1 + \omega_2)t} dt^2
\]
where $\nu(t) = \nu_0 \cos(\nu t) \hat{z}$. Performing the integral and taking the limit yields
\[
\lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} dt \ v(t) e^{i(\omega_1 + \omega_2)t} dt^2
\]
\[
= \frac{\nu_0}{2} (2\pi \delta(\omega_1 + \omega_2 - \nu) + 2\pi \delta(\omega_1 + \omega_2 + \nu))
\]
\[
+ 8 \lim_{T \to \infty} \cos[\nu T] - \cos[\omega_1 + \omega_2 T] \sin[(\omega_1 + \omega_2)^2 - \nu^2 T] T^2,
\]
where the first two terms are derived in the same way as (42). The third can be evaluated by considering that it will eventually be convoluted with a function of $\omega_1$ such as
\[
\int_{-\infty}^{\infty} \omega_1 f(\omega_1) \times \lim_{T \to \infty} \cos[\nu T] - \cos[\omega_1 + \omega_2 T] \sin[(\omega_1 + \omega_2)^2 - \nu^2 T] T^2.
\]
Over the the range $(-\infty, \infty)$, only a small region will contribute to the integral when $T$ is large, namely the region around $\nu = \pm(\omega_1 + \omega_2)$. This integral is therefore equivalent to the following expression
\[
\lim_{T \to \infty} \int_{-\Delta}^{\Delta} d\eta f(\nu - \omega_2 + \eta) \times \cos[\nu T] - \cos[\nu T] \sin[(\nu + \eta)^2 - \nu^2 T] T^2
\]
For large $T$, $\Delta$ will be small and as such we can treat $\eta$ as
small and make the following approximation
\[
\lim_{T \to \infty} \frac{1}{T} \left| \int_{-T/2}^{T/2} dt \, v(t) e^{-i(\omega_{\nu} + \omega_z)t} \right|^2 \\
\approx \lim_{T \to \infty} \int_{-\Delta}^{\Delta} d\eta f(\nu - \omega_z + \eta) \times \frac{\sin[\sqrt{\nu}T]}{2\nu}. \tag{C5}
\]
In the limit \( T \to \infty \), the range \( \Delta \to 0 \) and as such the area under the integral above vanishes. We can thus ignore the third term of equation (C2).

**Appendix D. Measuring the excitation rates**

Consider coupling a small two-level detector of transition frequency \( \Omega \) to the electromagnetic field, either inside or outside the material. In this appendix we establish a relationship between the expansion coefficients in (37) and the excitation rate of this detector.

Suppose the detector has energy levels \( n\hbar\Omega \) with associated raising and lowering operators \( \hat{a}^\dagger \) and \( \hat{a} \) respectively. If the detector is placed at the position \( x_0 \), the Hamiltonian gains an extra interaction term
\[
\hat{H}_I = -\int d^3x \, \hat{a} \cdot \hat{E} (x - x_0), \tag{D1}
\]
where the dipole moment operator is given by
\[
\hat{d} = \sqrt{\frac{\hbar}{2\Omega}} [e^{i\hat{a}^\dagger \hat{a}} + e^{i\hat{a} \hat{a}^\dagger}] \kappa \tag{D2}
\]
with the vector \( \kappa \) giving a direction and magnitude to the dipole moment. As in the main text the electric field is expanded as
\[
\hat{E}(x, t) = \int \frac{d^3k}{(2\pi)^3} \int_0^\infty d\omega \exp(i \mathbf{k} \cdot \mathbf{x} - i \omega t) \left\{ \hat{f}_\kappa(k, \omega) \cdot \hat{C}(k, \omega) + \hat{f}^\dagger_\kappa(k, \omega) \cdot \hat{C}^\dagger(k, \omega) \right\}.
\tag{D3}
\]
Prior to coupling to the detector we take the initial state of the system to be
\[
|\psi_0\rangle = |0\rangle_\Lambda \otimes \left( |\rangle_{0} + \sum_{\kappa} \int \frac{d^3k}{(2\pi)^3} \int_0^\infty d\omega \int_0^\infty d\omega_2 \, \zeta_{\kappa m} \times (k, \omega_1, \omega_2, T) \hat{C}_{\kappa m}^\dagger \hat{C}_{\kappa m} (k, \omega_1) |0\rangle \right)
\tag{D4}
\]
where \( \zeta_{\kappa m} \) is e.g. the expression derived in section 4, supposing that the time dependence of the medium has been driven for a fixed time \( T \). The coupling of the detector to the electromagnetic field leads to a probability amplitude for the detector to make a transition to an excited state, reducing the number of quanta in the field and medium by one
\[
|\psi(t)\rangle = |\psi_0\rangle + \sum_p \int \frac{d^3k}{(2\pi)^3} \int_0^\infty d\omega \, \xi_p \times (k, \omega, t) \hat{C}_p^\dagger (k, \omega) \hat{a} |0\rangle_\Lambda \otimes |0\rangle \tag{D5}
\]
Using the above interaction Hamiltonian to time evolve the system, the rate of change of \( \xi_p \) is found to be
\[
\frac{\partial \xi_p(k, \omega, t)}{\partial t} = -\frac{2\pi}{\Omega} e^{i\mathbf{k} \cdot \mathbf{x}} \int_0^\infty d\omega_1 \, \kappa \cdot \hat{f}_\kappa(k, \omega_1) \cdot \hat{C}_p^\dagger (k, \omega) \hat{C}_p (k, \omega_1) \tag{D6}
\]
Integrating over a time \([0, \tau]\) and using the result
\[
\lim_{\tau \to \infty} \frac{e^{i(\omega - \omega') \tau/2}}{\Omega - \omega'} \sin((\Omega - \omega)\tau/2) \sin((\Omega - \omega')\tau/2)
\]
we obtain the detector’s transition rate
\[
\Gamma = \int \frac{d^3k}{(2\pi)^3} \int_0^\infty d\omega \left| \frac{\xi_p(k, \omega, \tau)}{\tau} \right|^2
= \frac{2\pi^2}{\Omega} \int \frac{d^3k}{(2\pi)^3} \int_0^\infty d\omega \, \kappa \cdot \hat{f}_\kappa(k, \Omega) \cdot \hat{C}_p^\dagger (k, \omega) \hat{C}_p (k, \omega_1) \tag{D8}
\]
where we took the limit \( \tau \to \infty \), assuming that the ratio \( T/\tau \) does not diverge or tend to zero. Note that the terms in addition to the product of delta functions in (D7) do not contribute to (D8) in the limit \( \tau \to \infty \) because they yield terms proportional to something finite times \( 1/\tau \). It is thus clear that our model detector is excited at a rate proportional to the integral of the square of \( \zeta \) divided by \( \tau \). This square of \( \zeta \) has an almost identical form to the polariton excitation rate of this detector.

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