Existence and regularity of weak solutions for a thermoelectric model

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Abstract
This paper concerns a time-independent thermoelectric model with two different boundary conditions. The model is a nonlinear coupled system of the Maxwell equations and an elliptic equation. By analysing carefully the nonlinear structure of the equations, and with the help of the De Giorgi–Nash estimate for elliptic equations, we obtain weak solutions in Lipschitz domains for general boundary data. Using Campanato’s method, we obtain regularity results for the weak solutions.

Keywords: thermoelectric model, Maxwell system, elliptic equation, existence, regularity, div-curl system, Campanato space

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1. Introduction

1.1. The system

This paper is devoted to the study of the existence, regularity and uniqueness of weak solutions of the following

\[
\begin{aligned}
\nabla \times [\sigma(u)^{-1}\nabla \times \mathbf{H}] &= 0, \quad \nabla \cdot \mathbf{H} = 0 \quad \text{in } \Omega, \\
-\Delta u &= \sigma(u)^{-1}|\nabla \times \mathbf{H}|^2 \quad \text{in } \Omega, \\
u = u^0, \quad \nu \cdot \mathbf{H} &= 0, \quad \nu \times [\sigma(u)^{-1}\nabla \times \mathbf{H}] = \nu \times \mathbf{E}^0 \quad \text{on } \partial \Omega,
\end{aligned}
\]

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and
\[
\begin{align*}
\nabla \times [\sigma(u)^{-1} \nabla \times \mathbf{H}] &= 0, & \nabla \cdot \mathbf{H} &= 0 & \text{in } \Omega, \\
-\Delta u &= \sigma(u)^{-1} |\nabla \times \mathbf{H}|^2 & \text{in } \Omega, \\
u \times \mathbf{H} &= 0 & \text{on } \partial \Omega.
\end{align*}
\] (1.2)

Here \( \Omega \) is a bounded domain in \( \mathbb{R}^3 \) with a Lipschitz boundary \( \partial \Omega \), \( u \) is a scalar function and \( \mathbf{H} \) is a vector field, \( \nu \) is the unit outer normal vector on \( \partial \Omega \), \( \sigma \) is a continuous scalar function bounded both from above and away from zero, and \( \mathbf{E}^0 \) and \( \mathbf{H}^0 \) are given vector fields. In (1.1) it is natural to assume that \( \nabla \times \mathbf{E}^0 = 0 \). Let us emphasise that the boundary condition for \( \mathbf{H} \) in (1.1) is the actual electric boundary condition (namely the boundary condition of prescribing the tangential component of the electric field \( \mathbf{E} \), and it follows from the Maxwell equations and the Ohm’s law \( \mathbf{E} = \sigma(u)^{-1} \nabla \times \mathbf{H} \)), and the boundary condition for \( \mathbf{H} \) in (1.2) prescribes the tangential boundary condition for the magnetic field.

Systems (1.1) and (1.2) are the time-independent version of the thermoelectric model derived in [27], which describes electromagnetism in a medium with the electrical conductivity \( \sigma \) depending on the temperature \( u \), i.e. \( \sigma = \sigma(u) \). Assuming that the electric current \( \mathbf{J} \) and the electric field \( \mathbf{E} \) obey Ohm’s law \( \mathbf{J} = \sigma(u) \mathbf{E} \), and taking the Joule heating \( \mathbf{J} \cdot \mathbf{E} = \sigma(u) |\mathbf{E}|^2 \) as heat source, Yin derived the equation for the temperature \( u \) as follows:
\[
\frac{\partial u}{\partial t} - \Delta u = \sigma(u) |\mathbf{E}|^2.
\]

Yin combined this equation with the Maxwell equations
\[
\begin{align*}
\nabla \cdot \mathbf{B} &= 0, \\
\nabla \times \mathbf{H} &= \frac{4\pi \sigma(u)}{c} \mathbf{E}, \\
\nabla \times \mathbf{E} &= -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t},
\end{align*}
\]
where \( \mathbf{B} \) represents the magnetic induction, \( \mathbf{H} \) represents the magnetic field, and \( c \) is the speed of light. Assuming that the magnetic induction \( \mathbf{B} \) equals \( \mu \mathbf{H} \), where the magnetic permeability \( \mu \) is constant, and normalizing the constants in the equations, Yin derived the following model:
\[
\begin{align*}
\frac{\partial \mathbf{H}}{\partial t} + \nabla \times [\sigma(u)^{-1} \nabla \times \mathbf{H}] &= 0, \\
\nabla \cdot \mathbf{H} &= 0, \\
\frac{\partial u}{\partial t} - \Delta u &= \sigma(u)^{-1} |\nabla \times \mathbf{H}|^2.
\end{align*}
\] (1.3)

For more details of the derivation and analysis of results see [27, 28].

In this paper we consider the steady state of (1.3):
\[
\begin{align*}
\nabla \times [\sigma(u)^{-1} \nabla \times \mathbf{H}] &= 0, \\
\nabla \cdot \mathbf{H} &= 0, \\
-\Delta u &= \sigma(u)^{-1} |\nabla \times \mathbf{H}|^2,
\end{align*}
\]
and we shall establish the existence and regularity of the weak solutions under natural assumptions. We hope our mathematical results be helpful for the application of this model in physics and engineering and for computations.

If the domain is simply connected, then there exists a potential function \( \varphi \) such that
\[
\sigma(u)^{-1} \nabla \times \mathbf{H} = \nabla \varphi,
\]
and the above system is reduced to
\[
\begin{align*}
\nabla \cdot (\sigma(u) \nabla \varphi) &= 0, \\
-\Delta u &= \sigma(u) |\nabla \varphi|^2.
\end{align*}
\]

This simplified model was used to analyse the Joule heating of electrically conducting media; see [30–32] and the references therein. See also [7, 10] and the references therein for the use of this model in the thermistor problem with a current limiting device. However if the domain is multiply connected, then such a potential function does not exist, and such a reduction is not possible.

Recently, under the condition of small boundary data, Pan [21] obtained existence of classical solutions of (1.2):

(i) if \( \Omega \) is a simply connected domain, then (1.2) has classical solutions if \( u^0 \) and \( \nu \cdot \nabla \times H^0 \) are small ([21, theorem 4.8]);

(ii) if \( \Omega \) is a multiply connected domain, then (1.2) has classical solutions if \( u^0 \) and \( H^0 \) are small and \( \nu \cdot \nabla \times H^0 = 0 \) ([21, theorem 4.9]).

The main purpose of this paper is to prove the existence of weak solutions of (1.1) and (1.2) for general domains. We shall obtain the existence results for Lipschitz domains and without the extra condition of small boundary data. We shall also study regularity of weak solutions of (1.1) and (1.2). Since in the present case \( u \) is the temperature, its boundedness is essential for the models to be physically meaningful. Fortunately, this is a simple corollary of regularity results.

Systems (1.1) and (1.2) are also interesting to us for their special type of nonlinear structure. Since the only difference between systems (1.1) and (1.2) is the boundary condition for \( H \), we illustrate this point on system (1.1). Due to the quadratic nonlinearity in \( \nabla \times H \) in the second equation of (1.1), the problem of regularity of the weak solutions is non-trivial. In fact, if \( (u, H) \) is a weak solution of (1.1), then \( u \) can be viewed as a weak solution of the Laplace equation with the right-hand term \( \sigma(u)^{-1} |\nabla \times H|^2 \in L^1(\Omega) \):

\[
-\Delta u = \sigma(u)^{-1} |\nabla \times H|^2 \quad \text{in } \Omega, \quad u = u^0 \quad \text{on } \partial \Omega.
\] (1.4)

It is well-known that regularity of the Laplace equation with an \( L^1 \) right-hand term is a complicated problem. In order to obtain higher regularity of a weak solution \( (u, H) \) of (1.1), we shall first improve the regularity of \( H \). With the assumption \( \nabla \times E^0 = 0 \) and using the first equation of (1.1), we can write

\[
\sigma(u)^{-1} \nabla \times H - E^0 = \nabla \varphi + h,
\] (1.5)

for some \( \varphi \in H^1_0(\Omega) \) and \( h \in \mathbb{H}_D(\Omega) \), where \( \mathbb{H}_D(\Omega) \) is the space of the harmonic Dirichlet fields (see section 2). Noting that \( \nabla \cdot (\nabla \times H) = 0 \), we derive that \( \varphi \) is a weak solution of a linear problem with measurable coefficient \( \sigma(u) \):

\[
\begin{cases}
\nabla \cdot (\sigma(u)(\nabla \varphi + h + E^0)) = 0 & \text{in } \Omega, \\
\varphi = 0 & \text{on } \partial \Omega.
\end{cases}
\] (1.6)

Since \( E^0 \in L^q(\Omega, \mathbb{R}^3) \) for some \( q > 3 \), we can use Campanato’s method to get \( \nabla \varphi \in L^{2+\mu}(\Omega) \) for some \( \mu > 1 \), from which we derive \( \nabla \times H \in L^{2+\mu}(\Omega) \) and \( \varphi \in C^{0,\alpha}(\overline{\Omega}) \) for some \( \alpha \in (0, 1) \). Here \( L^{2+\mu} \) denotes the Campanato space. Then we write the right-hand term in (1.4) in the form (see (4.22))

\[
\sigma(u)^{-1} |\nabla \times H|^2 = \nabla \cdot [(\varphi + \varphi^0)\nabla \times H] + (h + h_1) \cdot \nabla \times H,
\]
where $\varphi^0 \in H^1(\Omega)$ with $\int_\Omega \varphi^0 dx = 0$, $h_1 \in H_\nu(\Omega)$ (the space of the harmonic Neumann fields), and both $(\varphi + \varphi^0) \nabla \times H$ and $(h + h_1) \cdot \nabla \times H$ belong to $L^2(\Omega)$. So we can apply lemma 2.4 to improve the regularity of $u$.

Let us mention that we will also re-write the equation for $u$ in various forms for other purposes; see for instance (3.14) and (4.23).

Regarding the existence results of (1.2), we mention that a corresponding problem with the boundary condition for $H$ replaced by a full Dirichlet boundary condition $H = H^0$ on $\partial \Omega$ has been studied by several authors. Yin [28] studied the steady states of (1.3) under the full Dirichlet boundary condition for $H$ but without the divergence-free condition for $H$:

$$\begin{cases}
\nabla \times [\sigma(u)^{-1} \nabla \times H] = 0 & \text{in } \Omega, \\
-\Delta u = \sigma(u)^{-1} |\nabla \times H|^2 & \text{in } \Omega, \\
u = u^0, \quad H = H^0 & \text{on } \partial \Omega.
\end{cases} \tag{1.7}$$

Among other results, Yin [28, theorem 5.4] claimed the existence of a weak solution $(u, H) \in W^{1,q}(\Omega) \times H^1(\Omega, \mathbb{R}^3)$ for the system (1.7) with $1 < q < 3/2$. See also [29] for the study of a more general system. Under both the full Dirichlet boundary condition and the divergence-free condition for $H$, Kang and Kim [15, 16] proved that the weak solutions of (1.7) are globally H"older continuous. Hong et al [13] studied this system in the settings of differential forms in higher dimensions and obtained partial regularity of the weak solutions.

For the magnetic field $H$ in Maxwell equations, it is more natural to consider the type of prescribing the normal or tangential boundary condition, rather than prescribing the full value of $H$ (see for instance [6, 9, 22]). Moreover, due to the different boundary conditions on $H$, the existence results for problem (1.2) and (1.7) are quite different; see [21, section IV.G].

1.2. Main results

Assume the function $\sigma(s)$ satisfies the following condition:

$$\sigma \text{ is continuous on } \mathbb{R}, \quad \sigma_1 \leq \sigma(s) \leq \sigma_2 \quad \text{for all } s \in \mathbb{R}, \tag{1.8}$$

where $\sigma_1 \leq \sigma_2$ are two positive constants. The notation of spaces used in the following theorems will be given in section 2.

**Theorem 1.1.** Let $\Omega$ be a bounded Lipschitz domain in $\mathbb{R}^3$. Assume the function $\sigma(s)$ satisfies (1.8), $u^0 \in H^1(\Omega)$, and $E^0 \in L^q(\Omega, \mathbb{R}^3)$ for some $q > 3$ with $\nabla \times E^0 = 0$ in $\Omega$. Then (1.1) has a weak solution $(u, H) \in H^1(\Omega) \times [H_0(\text{div}0, \Omega) \cap H(\text{curl}, \Omega)]$. Furthermore we have

(i) If $\Omega$ is of class $C^2$ and $E^0 \in W^{1,q}(\Omega)$, then $(u, H) \in W^{1,q}(\Omega) \times W^{1,q}(\Omega, \mathbb{R}^3)$;

(ii) If $\Omega$ is of class $C^{\alpha}$ and simply connected, $\sigma \in C^{1,\alpha}_{\text{loc}}(\mathbb{R})$ with $\alpha \in (0, 1)$, and if

$$(u^0, E^0) \in C^{2,\alpha}(\overline{\Omega}) \times C^{1,\alpha}(\overline{\Omega}, \mathbb{R}^3), \tag{1.9}$$

then $(u, H) \in C^{2,\alpha}(\overline{\Omega}) \times C^{2,\alpha}(\overline{\Omega}, \mathbb{R}^3)$.

(iii) Assume that $\Omega$ is of class $C^2$, the function $\sigma$ is Lipschitz on $\mathbb{R}$, i.e. there exists a positive constant $L$ such that

$$|\sigma(s) - \sigma(t)| \leq L|s - t| \text{ for any } s, t \in \mathbb{R}, \tag{1.10}$$

and assume $u^0 \in W^{1,q}(\Omega)$. Then there exists $\eta > 0$ such that if

$$\|E^0\|_{L^q(\Omega)} < \eta, \tag{1.11}$$
then the weak solution of (1.1) in the space \( H^1(\Omega) \times [H_0(\text{div}, \Omega) \cap H(\text{curl}, \Omega) \cap \mathbb{H}_D(\Omega)^\perp] \) is unique.

Theorem 1.1 will be proved through theorems 3.3, 4.2, 4.4 and 5.2.

**Theorem 1.2.** Let \( \Omega \) be a bounded Lipschitz domain in \( \mathbb{R}^3 \). Assume the function \( \sigma \) satisfies (1.8), \( u^0 \in H^1(\Omega) \), and \( H^0 \in W^{1,q}(\Omega, \mathbb{R}^3) \) for some \( q > 3 \). Then (1.2) has a weak solution \((u, \mathbf{H}) \in H^1(\Omega) \times [H(\text{div}, 0) \cap H(\text{curl}, \Omega)] \). Furthermore we have

(i) If \( \Omega \) is of class \( C^2 \) and \( u^0 \in W^{1,q}(\Omega) \), then \((u, \mathbf{H}) \in W^{1,q}(\Omega) \times W^{1,q}(\Omega, \mathbb{R}^3) \);  

(ii) If \( \Omega \) is of class \( C^{3,\alpha} \) and without holes, \( \sigma \in C^{3,\alpha}_{\text{loc}}(\mathbb{R}) \) with \( \alpha \in (0,1) \), and if \( (u^0, \mathbf{H}^0) \in C^{2,\alpha}(\overline{\Omega}) \times C^{2,\alpha}(\overline{\Omega}, \mathbb{R}^3) \),  

then \((u, \mathbf{H}) \in C^{2,\alpha}(\overline{\Omega}) \times C^{2,\alpha}(\overline{\Omega}, \mathbb{R}^3) \).

(iii) Assume that \( \Omega \) is of class \( C^2 \), \( \sigma \) satisfies (1.10), and \( u^0 \in W^{1,q}(\Omega) \). Then there exists \( \eta > 0 \) such that if

\[
\| \nabla \times \mathbf{H}^0 \|_{L^3(\Omega)} < \eta,
\]

then the weak solution of (1.2) in the space \( H^1(\Omega) \times [H(\text{div}, 0) \cap H(\text{curl}, \Omega) \cap (H^0 + \mathbb{H}_D(\Omega)^\perp)] \) is unique.

Theorem 1.2 will be proved in section 6.

This paper is organised as follows. In section 2 we collect some preliminary results that will be needed in the later sections. In section 3 we prove the existence of weak solutions for system (1.1) by using Schauder’s fixed point theorem. Regularity of the weak solutions is given in section 4. Uniqueness under the condition of small boundary data is proved in section 5. In section 6 we discuss system (1.2).

## 2. Preliminaries

Let \( \Omega \) be a bounded Lipschitz domain in \( \mathbb{R}^3 \). Let \( \mathcal{D}(\Omega, \mathbb{R}^3) \) denote the space of 3D vector-valued functions that are infinitely differentiable and have compact supports in \( \Omega \), and \( \mathcal{D}^\prime(\Omega, \mathbb{R}^n) \) denote its dual space. We use \( L^p(\Omega) \), \( W^k,p(\Omega) \) and \( C^{k,\alpha}(\overline{\Omega}) \) to denote the usual Lebesgue spaces, Sobolev spaces and Hölder spaces for scalar functions, and use \( L^p(\Omega, \mathbb{R}^3) \), \( W^k,p(\Omega, \mathbb{R}^3) \) and \( C^{k,\alpha}(\overline{\Omega}, \mathbb{R}^3) \) to denote the corresponding spaces of vector fields. However we use the same notation to denote both the norm of scalar functions and that of vector fields in the corresponding spaces. For instance, we write \( \| \phi \|_{L^p(\Omega)} \) for \( \phi \in L^p(\Omega) \) and write \( \| \mathbf{u} \|_{L^p(\Omega)} \) for \( \mathbf{u} \in L^p(\Omega, \mathbb{R}^3) \).

In the study of problems (1.1) and (1.2), the topology of the domain \( \Omega \) plays important roles. The domain topology is well represented by the spaces of harmonic Neumann fields \( \mathbb{H}_N(\Omega) \) and of harmonic Dirichlet fields \( \mathbb{H}_D(\Omega)^\perp \):

\[
\mathbb{H}_N(\Omega) = \{ \mathbf{u} \in L^2(\Omega, \mathbb{R}^3) : \nabla \times \mathbf{u} = 0, \ \nabla \cdot \mathbf{u} = 0 \text{ in } \Omega, \ \nu \cdot \mathbf{u} = 0 \text{ on } \partial \Omega \}, \quad \mathbb{H}_D(\Omega) = \{ \mathbf{u} \in L^2(\Omega, \mathbb{R}^3) : \nabla \times \mathbf{u} = 0, \ \nabla \cdot \mathbf{u} = 0 \text{ in } \Omega, \ \nu \times \mathbf{u} = 0 \text{ on } \partial \Omega \}.
\]

The dimension of \( \mathbb{H}_N(\Omega) \) is equal to the number of ‘handles’ of \( \Omega \), and the dimension of \( \mathbb{H}_D(\Omega) \) is equal to the number of holes in \( \Omega \). In particular, if \( \Omega \) is simply connected, i.e. if \( \Omega \)

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5One may also denote \( \mathbb{H}_N(\Omega) \) with \( \mathbb{H}_1(\Omega) \), and denote \( \mathbb{H}_D(\Omega) \) with \( \mathbb{H}_2(\Omega) \).
has no 'handles', then \( H_N(\Omega) = \{0\} \); and if \( \Omega \) has no holes, then \( H_D(\Omega) = \{0\} \). The harmonic Neumann or Dirichlet fields enjoy good regularities. In fact, for \( C^{1,1} \) domains we have \( H_N(\Omega), H_D(\Omega) \subset W^{1,p}(\Omega, \mathbb{R}^3) \) for any \( 1 < p < \infty \); see [2, corollarys 4.1 and 4.2]. Thus by Morrey embedding, we have \( H_N(\Omega), H_D(\Omega) \subset C^{0,1-3/p}(\Omega, \mathbb{R}^3) \) for any \( 3 < p < \infty \). Furthermore, if \( \Omega \) is of class \( C^{\theta} \), where \( r \geq 2 \) and \( 0 < \theta < 1 \), then \( H_N(\Omega), H_D(\Omega) \subset C^{r-1,0}(\Omega, \mathbb{R}^3) \); see [9, pp 219–222].

We also use the following notation:

\[
\begin{align*}
H(\text{div} 0, \Omega) &= \{ u \in L^2(\Omega, \mathbb{R}^3) : \nabla \cdot u = 0 \text{ in } \Omega \}, \\
H_0(\text{div} 0, \Omega) &= \{ u \in H(\text{div} 0, \Omega) : \nu \cdot u = 0 \text{ on } \partial \Omega \}, \\
H(\text{curl} , \Omega) &= \{ u \in L^2(\Omega, \mathbb{R}^3) : \nabla \times u \in L^2(\Omega, \mathbb{R}^3) \}, \\
H_0(\text{curl} , \Omega) &= \{ u \in H(\text{curl} , \Omega) : \nu \times u = 0 \text{ on } \partial \Omega \}.
\end{align*}
\]

For \( 1 < p < \infty \) we denote

\[ H^p(\Omega, \text{curl} , \text{div}) = \{ u \in L^p(\Omega, \mathbb{R}^3) : \nabla \times u \in L^p(\Omega, \mathbb{R}^3), \nabla \cdot u \in L^p(\Omega) \}. \]

If \( X(\Omega) \) denotes a space of scalar functions defined on \( \Omega \), then we write

\[ \tilde{X}(\Omega) = \{ \phi \in X(\Omega) : \int_{\Omega} \phi(x) \, dx = 0 \}. \]

Recall the Helmholtz–Weyl decompositions of \( L^2(\Omega, \mathbb{R}^3) \) on Lipschitz domain (see [5, theorem 5.3] or [23, (1.19) on p 156]):

\[
\begin{align*}
L^2(\Omega, \mathbb{R}^3) &= [\nabla \times H(\text{curl} , \Omega)] \oplus \nabla H^1_0(\Omega) \oplus H_D(\Omega), \\
L^2(\Omega, \mathbb{R}^3) &= [\nabla \times H_0(\text{curl} , \Omega)] \oplus \nabla H^1(\Omega) \oplus H_N(\Omega).
\end{align*}
\]

As a direct corollary, we have the following decompositions for curl-free vector fields, which will be used frequently in this paper:

**Lemma 2.1.** Assume \( \Omega \) is a bounded Lipschitz domain in \( \mathbb{R}^3 \), and \( u \in L^2(\Omega, \mathbb{R}^3) \) with \( \nabla \times u = 0 \) in \( \Omega \). Then the following conclusions are true.

(i) \( u \) can be decomposed in the form \( u = \nabla \varphi + \mathbf{h} \), where \( \varphi \in H^1(\Omega) \) and \( \mathbf{h} \in H_N(\Omega) \).

(ii) If furthermore \( \nu \times u = 0 \) on \( \partial \Omega \) in the sense of trace, then \( u \) can be decomposed in the form \( u = \nabla \varphi + \mathbf{h} \), where \( \varphi \in H^1(\Omega) \) and \( \mathbf{h} \in H_D(\Omega) \).

A regularity result for div-curl system is also an important ingredient in our proof of the existence of weak solutions for (1.1) and (1.2). We use \( H^{\alpha,p}(\Omega, \mathbb{R}^3) \) to denote fractional-order Sobolev space.

**Lemma 2.2 ([19, Theorem 11.2]).** For any bounded Lipschitz domain \( \Omega \) in \( \mathbb{R}^3 \), there exists \( \epsilon = \epsilon(\Omega) > 0 \) with the following significance. Let \( p \in (2 - \epsilon, 2 + \epsilon) \). Set \( s = 1/p \) if \( p \geq 2 \) and \( s = 1 - 1/p \) if \( p \leq 2 \). Assume that \( u \in H^s(\Omega, \text{curl} , \text{div}) \) is such that either \( \nu \cdot u \in L^p(\partial \Omega) \) or \( \nu \times u \in L^p(\partial \Omega, \mathbb{R}^3) \). Then

\[ u \in \bigcap_{\mu > 0} H^{-\mu,p}(\Omega, \mathbb{R}^3), \]

and, for each \( \mu > 0 \), there exists \( C = C(\mu, p, \Omega) \) such that
\[ \|u\|_{H^{0,p,\mu}(\Omega)} \leq C(\|u\|_{L^p(\Omega)} + \|\nabla \cdot u\|_{L^p(\Omega)} + \|\nabla \times u\|_{L^p(\Omega)}) + \min\{\|\nu \cdot u\|_{L^p(\partial\Omega)}, \|\nu \times u\|_{L^p(\partial\Omega)}\}. \]

When \( p = 2 \), the above inequality remains true for \( \mu = 0 \).

We use \( L^{2,\mu}(\Omega) \) to denote a Campanato space, which consists of scalar functions satisfying
\[ \|u\|_{L^{2,\mu}(\Omega)} = \left( \|u\|_{L^2(\Omega)}^2 + \sup_{0 < r < \infty} r^{-\mu} \int_{\Omega_r(x_0)} |u - u_{x_0,r}|^2 \, dx \right)^{1/2} < \infty, \]
where
\[ \Omega_r(x_0) = \Omega \cap B_r(x_0), \quad u_{x_0,r} = \frac{1}{|\Omega_r(x_0)|} \int_{\Omega_r(x_0)} u(x) \, dx. \]

Campanato spaces play a key role in our proof of regularity of weak solutions for (1.1) and (1.2). Below we list some properties for Campanato spaces, which can be found in [24, theorem 1.17, lemma 1.19, theorem 1.40].

**Lemma 2.3.** Assume \( \Omega \) is a bounded \( C^1 \) domain in \( \mathbb{R}^n \).

(i) Let \( 0 \leq \mu < n \). Then the mapping
\[ u \mapsto \left( \sup_{0 < r < \infty} r^{-\mu} \int_{\Omega_r(x_0)} u^2 \, dx \right)^{1/2} \]
defines an equivalent norm on \( L^{2,\mu}(\Omega) \). Hence \( L^\infty(\Omega) \) is a space of multipliers for \( L^{2,\mu}(\Omega) \). That is to say, for any \( u \in L^{2,\mu}(\Omega) \) and any \( v \in L^\infty(\Omega) \), we have
\[ \|uv\|_{L^{2,\mu}(\Omega)} \leq C(n, \mu, \Omega) \|u\|_{L^{2,\mu}(\Omega)} \|v\|_{L^\infty(\Omega)}. \]

(ii) Let \( n < \mu \leq n + 2 \). Then \( L^{2,\mu}(\Omega) \) is isomorphic to \( C^{0,\delta}(\overline{\Omega}) \) for \( \delta = (\mu - n)/2 \).

(iii) Let \( 0 \leq \mu < n \). If \( u \in H^1(\Omega) \) and \( \nabla u \in L^{2,\mu}(\Omega) \), then \( u \in L^{2,2+\mu}(\Omega) \) with
\[ \|u\|_{L^{2,2+\mu}(\Omega)} \leq C(n, \mu, \Omega)(\|u\|_{L^2(\Omega)} + \|\nabla u\|_{L^{2,\mu}(\Omega)}). \]

(iv) We have the following embedding:
\[ L^{2,\lambda}(\Omega) \hookrightarrow L^{2,\mu}(\Omega) \quad \text{if} \quad 0 \leq \lambda < \mu \leq n + 2, \]
\[ L^p(\Omega) \hookrightarrow L^{2,\mu}(\Omega) \quad \text{if} \quad p > 2, \quad \mu = n(p - 2)/p. \]

The \( L^{2,\mu} \) regularity of first derivatives for the Dirichlet problem
\[ \nabla \cdot (A\nabla u) = f + \nabla \cdot F \quad \text{in} \ \Omega, \quad u = u^0 \quad \text{on} \ \partial\Omega, \quad (2.1) \]
and for the Neumann problem
\[ \nabla \cdot (A\nabla u) = \nabla \cdot F \quad \text{in} \ \Omega, \quad \nu \cdot (A\nabla u) = \nu \cdot F \quad \text{on} \ \partial\Omega, \quad (2.2) \]
can be derived by Campanato’s method; see [24, theorem 2.19].
Lemma 2.4. \(\) Let \( n \geq 3 \) and \( \Omega \) be a bounded \( C^1 \) domain in \( \mathbb{R}^n \). Suppose the matrix-valued function \( A \) satisfies
\[
\lambda |\xi|^2 \leq \langle A \xi, \xi \rangle \leq \Lambda |\xi|^2, \quad \forall \xi \in \mathbb{R}^n,
\]
where \( 0 < \lambda \leq \Lambda < \infty \). There exist constants \( C > 0 \) and \( \delta \in (0, 1) \), both depending only on \( \Omega, \lambda, \Lambda \), such that for any \( 0 < \mu < n - 2 + 2\delta \), if
\[
f \in L^{2(\mu-2)^+}(\Omega), \quad F \in L^{2\mu}(\Omega), \quad u^0 \in H^1(\Omega), \quad \nabla u^0 \in L^{2\mu}(\Omega),
\]
and if \( u \in H^1(\Omega) \) is a weak solution of (2.1), then \( \nabla u \in L^{2\mu}(\Omega) \), and we have the estimate
\[
\|\nabla u\|_{L^{2\mu}(\Omega)} \leq C \left\{ \|u\|_{H^1(\Omega)} + \|f\|_{L^{2(\mu-2)^+}(\Omega)} + \|F\|_{L^{2\mu}(\Omega)} + \|\nabla u^0\|_{L^{2\mu}(\Omega)} \right\}.
\]

Lemma 2.5. \(\) Assume \( \Omega \) and \( A \) satisfy the conditions in lemma 2.4. There exist constants \( C > 0 \) and \( \delta \in (0, 1) \), both depending only on \( \Omega, \lambda, \Lambda \), such that for \( 0 < \mu < n - 2 + 2\delta \), if \( F \in L^{2\mu}(\Omega) \), and if \( u \in H^1(\Omega) \) is a weak solution of (2.2), then \( \nabla u \in L^{2\mu}(\Omega) \), and we have the estimate
\[
\|\nabla u\|_{L^{2\mu}(\Omega)} \leq C \left\{ \|u\|_{H^1(\Omega)} + \|F\|_{L^{2\mu}(\Omega)} \right\}.
\]

3. Existence of weak solutions

Definition 3.1. \(\) We say that \((u, H) \in H^1(\Omega) \times [H_0(div\Omega) \cap H(curl, \Omega)]\) is a weak solution of (1.1) if \( u = u^0 \) on \( \partial\Omega \) in the sense of trace, and if
\[
\begin{align*}
\int_{\Omega} \nabla u \cdot \nabla v \, dx &= \int_{\Omega} \sigma(u)^{-1} \nabla \times H \times v \, dx, \\
\int_{\Omega} \sigma(u)^{-1} \nabla \times H \cdot \nabla \times w \, dx &= \int_{\Omega} E^0 \cdot \nabla \times w \, dx,
\end{align*}
\quad \forall v \in H^1_0(\Omega) \cap L^{\infty}(\Omega),
\quad \forall w \in H_0(div\Omega) \cap H(curl, \Omega).
\]

Proof of the existence result in theorem 3.3 needs the following lemma, which will also be needed in the proof of theorem 4.2 in the next section.

Lemma 3.2. \(\) Let \( \Omega \) be a bounded Lipschitz domain in \( \mathbb{R}^3 \). Assume that the function \( \sigma \) satisfies (1.8), \( u^0 \in H^1(\Omega) \), and \( E^0 \in L^q(\Omega, \mathbb{R}^3) \) for some \( q > 3 \) with \( \nabla \times E^0 = 0 \) in \( \Omega \). For any given \( w \in L^2(\Omega) \), the following system
\[
\begin{align*}
\nabla \times [\sigma(w)^{-1} \nabla \times H] &= 0, \quad \nabla \cdot H = 0 \quad \text{in } \Omega, \\
-\Delta u &= \sigma(w)^{-1} \nabla \times H^2 \quad \text{in } \Omega, \\
uu &= u^0, \quad \nu \cdot H = 0, \quad \nu \times [\sigma(w)^{-1} \nabla \times H] = \nu \times E^0 \quad \text{on } \partial\Omega,
\end{align*}
\]
has a unique weak solution \((u_w, H_w) \in H^1(\Omega) \times [H_0(div\Omega) \cap H(curl, \Omega) \cap H(curl, \Omega)\] with the estimates
\[
\begin{align*}
\|\nabla \times H_w\|_{L^2(\Omega)} &\leq \sigma_2 \|E^0\|_{L^2(\Omega)}, \\
\|u_w\|_{L^2(\Omega)} &\leq C(\Omega, q, \sigma_1, \sigma_2) \left\{ \|E^0\|_{L^2(\Omega)} \|E^0\|_{L^2(\Omega)} + \|u^0\|_{H^1(\Omega)} \right\}.
\end{align*}
\]
Proof.

Step 1. For any given \( w \in L^2(\Omega) \), let \( \mathbf{H}_w \in H_0(\text{div} 0, \Omega) \cap H(\text{curl}, \Omega) \cap \mathbb{H}_X(\Omega)^\perp \) be the unique weak solution of the following system

\[
\begin{aligned}
\nabla \times [\sigma(w)^{-1} \nabla \times \mathbf{H}_w] &= 0, & \nabla \cdot \mathbf{H}_w &= 0 \quad &\text{in } \Omega, \\
\nu \cdot \mathbf{H}_w &= 0, & \nu \times [\sigma(w)^{-1} \nabla \times \mathbf{H}_w] &= \nu \times \mathbf{E}^0 \quad &\text{on } \partial \Omega.
\end{aligned}
\]  

(3.3)

Existence of a unique weak solution \( \mathbf{H}_w \) of (3.3) can be proved by using the Lax–Milgram theorem, with the help of Poincaré-type inequality

\[
\|v\|_{L^2(\Omega)} \leq C(\Omega) \|\nabla \times v\|_{L^2(\Omega)} \quad \forall v \in H_0(\text{div} 0, \Omega) \cap H(\text{curl}, \Omega) \cap \mathbb{H}_X(\Omega)^\perp,
\]

which is a consequence of a compact embedding theorem in Lipschitz domains established in [23]; see also [5] for weak Lipschitz domains and mixed boundary conditions. Taking \( \mathbf{H}_w \) as a test function for (3.3) and using condition (1.8), we obtain the estimate (3.1).

Step 2. For the weak solution \( \mathbf{H}_w \) of (3.3) obtained above, we have

\[
\nabla \times [\sigma(w)^{-1} \nabla \times \mathbf{H}_w - \mathbf{E}^0] = 0 \quad \text{in } \Omega, \quad \nu \times [\sigma(w)^{-1} \nabla \times \mathbf{H}_w - \mathbf{E}^0] = 0 \quad \text{on } \partial \Omega.
\]

It follows from lemma 2.1 that there exist \( \varphi_w \in H^1_0(\Omega) \) and \( \mathbf{h}_w \in \mathbb{H}_D(\Omega) \) such that

\[
\sigma(w)^{-1} \nabla \times \mathbf{H}_w - \mathbf{E}^0 = \nabla \varphi_w + \mathbf{h}_w,
\]

(3.4)

and \( \varphi_w \) satisfies the equation

\[
\begin{aligned}
\Delta \varphi_w &= \nabla \cdot [\sigma(w)^{-1} \nabla \times \mathbf{H}_w - \mathbf{E}^0] \quad &\text{in } \Omega, \\
\varphi_w &= 0 \quad &\text{on } \partial \Omega.
\end{aligned}
\]

(3.5)

We show that there exists \( C_1 = C_1(\Omega, q, \sigma_1, \sigma_2) > 0 \) such that

\[
\|\varphi_w\|_{L^\infty(\Omega)} \leq C_1 \|\mathbf{E}^0\|_{L^2(\Omega)}.
\]

(3.6)

To prove (3.6), we note that the vector field \( \mathbf{h}_w \) in (3.4) can be written as follows:

\[
\mathbf{h}_w = \sum_{j=1}^m c_j \mathbf{e}_j, \quad c_j = \int_\Omega \mathbf{e}_j \cdot (\sigma(w)^{-1} \nabla \times \mathbf{H}_w - \mathbf{E}^0) \, dx, \quad j = 1, \cdots, m,
\]

(3.7)

where \( \{\mathbf{e}_1, \cdots, \mathbf{e}_m\} \) is an orthonormal basis of \( \mathbb{H}_D(\Omega) \) with respect to the \( L^2 \) norm. (3.7) can be verified by using the \( L^2 \) orthogonality of \( \nabla \varphi_w \) and \( \mathbf{h}_w \). By lemma 2.2, we get \( \mathbf{h}_w \in H^1(\Omega, \mathbb{R}^3) \). Hence the Sobolev embedding implies that \( \mathbf{h}_w \in L^3(\Omega, \mathbb{R}^3) \). Let \( \epsilon \) be the constant in lemma 2.2. We choose \( p \in (2, \min\{2 + \epsilon, 3\}) \). By lemma 2.2 again, we obtain

\[
\mathbf{h}_w \in H^{1-\mu \frac{2}{p}}(\Omega), \quad \text{where } \mu = \frac{p - 2}{2p}.
\]

(3.8)

Via Sobolev embedding, we get \( \mathbf{h}_w \in L^{\frac{2p}{2-\mu}}(\Omega, \mathbb{R}^3) \) with the estimate

\[
\|\mathbf{h}_w\|_{L^{\frac{2p}{2-\mu}}(\Omega)} \leq C(p, \Omega) \|\mathbf{h}_w\|_{L^3(\Omega)}.
\]

(3.9)

On the other hand, for any \( \zeta \in H^1(\Omega) \), we have

\[
\|\zeta\|_{L^\infty(\Omega)} \leq C(\Omega) \|\zeta\|_{L^2(\Omega)}.
\]

(3.10)

where the constant \( C(\sigma) > 0 \) is independent of \( \sigma \). Thus, we have

\[
\|\varphi_w\|_{L^\infty(\Omega)} \leq C_1 \|\mathbf{E}^0\|_{L^2(\Omega)}/\|\mathbf{h}_w\|_{L^{\frac{2p}{2-\mu}}(\Omega)}.
\]

(3.11)

Combining (3.6), (3.10), and (3.11), we finish the proof.\[\]
\[
\int \sigma(w)(\nabla \varphi_w + h_w + E^0) \cdot \nabla \zeta \, dx = \int \Omega (\nabla \times H_w) \cdot \nabla \zeta \, dx = 0. \tag{3.9}
\]

Hence \( \varphi_w \) is also a solution of the following equation:
\[
\begin{cases}
\nabla \cdot [\sigma(w)(\nabla \varphi_w + h_w + E^0)] = 0 & \text{in } \Omega, \\
\varphi_w = 0 & \text{on } \partial \Omega. 
\end{cases} \tag{3.10}
\]

By applying [12, theorem 8.16] to (3.10) we obtain
\[
\| \varphi_w \|_{L^\infty(\Omega)} \leq C(\Omega, q, \sigma_1, \sigma_2) \| h_w + E^0 \|_{L^q(\Omega)}, \quad \text{where } q_1 = \min\{q, \frac{6p}{p+2}\} > 3. \tag{3.11}
\]

Then (3.6) follows from (3.1), (3.7), (3.8) and (3.11).

Furthermore, by applying [12, theorem 8.29] instead of [12, theorem 8.16], we see that there exist constants \( \alpha = \alpha(\Omega, q, \sigma_1, \sigma_2) \in (0, 1) \) and \( C = C(\Omega, q, \sigma_1, \sigma_2) \) such that
\[
\| \varphi_w \|_{C^{\alpha,\alpha}(\Omega)} \leq C\| E^0 \|_{L^q(\Omega)}. \tag{3.12}
\]

**Step 3.** For the \( w \) and \( H_w \) given above, we show that the following equation has a unique \( H^1 \) weak solution \( u_w \):
\[
-\Delta u_w = \sigma(w)^{-1} |\nabla \times H_w|^2 \quad \text{in } \Omega, \quad u_w = u^0 \quad \text{on } \partial \Omega. \tag{3.13}
\]

To prove this, note that from (3.4) we have
\[
\sigma(w)(\nabla \varphi_w + h_w + E^0) \cdot \nabla \varphi_w = \nabla \cdot [\varphi_w \sigma(w)(\nabla \varphi_w + h_w + E^0)].
\]

This equality can also be verified by taking \( \zeta = \varphi_w \varphi \) in (3.9), with an arbitrary \( \varphi \in \mathcal{D}(\Omega) \). Hence
\[
\sigma(w)^{-1} |\nabla \times H_w|^2 = \sigma(w) |\nabla \varphi_w + h_w + E^0|^2
\]
\[
= \nabla \cdot [\varphi_w \sigma(w)(\nabla \varphi_w + h_w + E^0)] + \sigma(w)(\nabla \varphi_w + h_w + E^0) \cdot (h_w + E^0)
\]
\[
= \nabla \cdot (\varphi_w \nabla \times H_w) + (h_w + E^0) \cdot \nabla \times H_w,
\]

from which we see that \( \sigma(w)^{-1} |\nabla \times H_w|^2 \in H^{-1}(\Omega) \). Therefore, via the Lax–Milgram theorem, the Dirichlet problem (3.13) has a unique weak solution \( u_w \in H^1(\Omega) \).

To prove (3.2), write (3.13) in the following form
\[
\begin{cases}
-\Delta(u_w - u^0) = \nabla \cdot (\varphi_w \nabla \times H_w + \nabla u^0) + (h_w + E^0) \cdot \nabla \times H_w & \text{in } \Omega, \\
u_w - u^0 = 0 & \text{on } \partial \Omega.
\end{cases} \tag{3.14}
\]

Taking \( u_w - u^0 \) as a test function, we get
\[
\| \nabla(u_w - u^0) \|_{L^2(\Omega)} \leq \| \varphi_w \nabla \times H_w + \nabla u^0 \|_{L^2(\Omega)} + C(\Omega)(\| h_w + E^0 \|_{L^q(\Omega)}) \| \nabla \times H_w \|_{L^q(\Omega)}
\]
\[
\leq C(\Omega) (\| \varphi_w \|_{L^\infty(\Omega)} + \| h_w + E^0 \|_{L^q(\Omega)}) \| \nabla \times H_w \|_{L^q(\Omega)} + \| \nabla u^0 \|_{L^2(\Omega)}.
\]

From this, (3.1) and (3.6), and via the Poincaré inequality, we find that
\[ \|u_w\|_{L^2(\Omega)} \leq \|u_w - u^0\|_{L^2(\Omega)} + \|u^0\|_{L^2(\Omega)} \]
\[ \leq C \left( \|E^0\|_{L^2(\Omega)} \|E^0\|_{L^2(\Omega)} + \|u^0\|_{W^1(\Omega)} \right), \]

where the constant \( C \) depends only on \( \Omega, q, \sigma_1, \sigma_2 \). This gives the estimate (3.2).

\[ \square \]

**Theorem 3.3.** Let \( \Omega \) be a bounded Lipschitz domain in \( \mathbb{R}^3 \). Assume that the function \( \sigma \) satisfies (1.8), \( u^0 \in H^1(\Omega) \), and \( E^0 \in L^q(\Omega, \mathbb{R}^3) \) for some \( q > 3 \) with \( \nabla \times E^0 = 0 \) in \( \Omega \). Then (1.1) has a weak solution \( (u, H) \in H^1(\Omega) \times [H_0(\text{div}, \Omega) \cap H(\text{curl}, \Omega)] \).

**Proof.** We define an operator \( T : L^2(\Omega) \rightarrow L^2(\Omega) \) as follows. Given \( w \in L^2(\Omega) \), we define \( H_w, \varphi_w \) and \( u_w \) by the solution of (3.3), (3.5) and (3.13) successively, and then define \( T(w) = u_w \). Recall that the estimates of \( \|\varphi_w\|_{C_{\alpha}(\overline{\Omega})} \) and \( \|u_w\|_{L^2(\Omega)} \) (see (3.12) and (3.2)) do not depend on the choice of \( w \in L^2(\Omega) \), and these uniform (in \( w \)) estimates will be crucial in the proof of the existence of a fixed point of \( T \). Denote the right-hand side of the inequality (3.2) with \( K \) and let
\[ D = \{ w \in L^2(\Omega) : \|w\|_{L^2(\Omega)} \leq K \}. \]

Clearly, \( D \) is convex and closed in \( L^2(\Omega) \), and \( T \) maps \( D \) into itself.

We show that \( T \) is continuous from \( L^2(\Omega) \) to \( H^1(\Omega) \). Suppose \( w_k \rightarrow w_0 \) in \( L^2(\Omega) \) as \( k \rightarrow \infty \). With \( H_k, \varphi_k, u_k \) and \( H_0, \varphi_0, u_0 \) denote the solutions \( H_{w_k}, \varphi_{w_k}, u_{w_k} \) and \( H_{w_0}, \varphi_{w_0}, u_{w_0} \) obtained by setting \( w = w_k \) and \( w = w_0 \) in the equations (3.3), (3.5) and (3.13), respectively. Then we obtain
\[ \begin{cases} \nabla \times (\sigma(w_k)^{-1}\nabla \times (H_k - H_0)) = \nabla \times (\sigma(w_0)^{-1} - \sigma(w_k)^{-1}) \nabla \times H_0 & \text{in } \Omega, \\ \nabla \cdot (H_k - H_0) = 0 & \text{in } \Omega, \\ \nu \cdot (H_k - H_0) = 0 & \text{on } \partial \Omega, \\ \nu \times (\sigma(w_k)^{-1}\nabla \times (H_k - H_0)) = \nu \times (\sigma(w_0)^{-1} - \sigma(w_k)^{-1}) \nabla \times H_0 & \text{on } \partial \Omega. \end{cases} \]
(3.15)

Given the Lebesgue’s dominated convergence theorem we have
\[ \| (\sigma(w_0)^{-1} - \sigma(w_k)^{-1}) \nabla \times H_0 \|_{L^2(\Omega)} \rightarrow 0 \quad \text{as } k \rightarrow \infty. \]

Then with the \( L^2 \) estimate of (3.15) and using condition (1.8) we find that
\[ \| \nabla \times (H_k - H_0) \|_{L^2(\Omega)} \leq \sigma_2 \| (\sigma(w_0)^{-1} - \sigma(w_k)^{-1}) \nabla \times H_0 \|_{L^2(\Omega)} \rightarrow 0 \quad \text{(3.16)} \]

Thus we have
\[ \| \sigma(w_0)^{-1}\nabla \times H_k - \sigma(w_0)^{-1}\nabla \times H_0 \|_{L^2(\Omega)} \]
\[ \leq \| \sigma(w_k)^{-1}\nabla \times (H_k - H_0) \|_{L^2(\Omega)} + \| (\sigma(w_0)^{-1} - \sigma(w_k)^{-1}) \nabla \times H_0 \|_{L^2(\Omega)} \]
\[ \leq \left( \frac{\sigma_2}{\sigma_1} + 1 \right) \| (\sigma(w_k)^{-1} - \sigma(w_0)^{-1}) \nabla \times H_0 \|_{L^2(\Omega)} \rightarrow 0. \]
(3.17)
As in (3.4) we have
\[
\sigma(w_k)^{-1}\nabla \times H_k - E^0 = \nabla \varphi_k + h_k, \quad \sigma(w_0)^{-1}\nabla \times H_0 - E^0 = \nabla \varphi_0 + h_0.
\]
Using formula (3.7) for the representations of \( h_k \) and \( h_0 \), and using (3.17) we obtain
\[
\|h_k - h_0\|_{L^2(\Omega)} = \left\| \sum_{j=1}^{m} \left\{ \int_{\Omega} e_j \cdot [\sigma(w_k)^{-1}\nabla \times H_k - \sigma(w_0)^{-1}\nabla \times H_0] \, dx \right\} e_j \right\|_{L^2(\Omega)}
\leq \|\sigma(w_k)^{-1}\nabla \times H_k - \sigma(w_0)^{-1}\nabla \times H_0\|_{L^2(\Omega)} \sum_{j=1}^{m} \|e_j\|_{L^2(\Omega)} \|e_j\|_{L^2(\Omega)}
\leq C(\Omega)\|\sigma(w_k)^{-1}\nabla \times H_k - \sigma(w_0)^{-1}\nabla \times H_0\|_{L^2(\Omega)} \to 0. \tag{3.18}
\]
It follows that
\[
\|\nabla \varphi_k - \nabla \varphi_0\|_{L^2(\Omega)} = \|\sigma(w_k)^{-1}\nabla \times H_k - h_k - \sigma(w_0)^{-1}\nabla \times H_0 + h_0\|_{L^2(\Omega)}
\leq \|\sigma(w_k)^{-1}\nabla \times H_k - \sigma(w_0)^{-1}\nabla \times H_0\|_{L^2(\Omega)} + \|h_k - h_0\|_{L^2(\Omega)} \to 0.
\]
Therefore, via the Poincaré inequality, we have \( \|\varphi_k - \varphi_0\|_{L^2(\Omega)} \to 0 \). With the estimate (3.12) we see that \( \|\varphi_k\|_{C^{0,\alpha}(\Omega)} \) is bounded uniformly in \( k \). From the Arzela–Ascoli theorem we know that for any sequence \( k_j \to \infty \) there exist a subsequence \( \{\varphi_{k_j}\} \subset C^0(\Omega) \) and \( \varphi^* \in C^0(\Omega) \) such that \( \varphi_{k_j} \to \varphi^* \) in \( C^0(\Omega) \) as \( k_j \to \infty \). Given the above convergence in \( L^2(\Omega) \), we obtain \( \varphi^* = \varphi_0 \). Thanks to the uniqueness of \( \varphi^* \), we have
\[
\|\varphi_k - \varphi_0\|_{C^{0,\alpha}(\Omega)} \to 0 \quad \text{as} \quad k \to \infty. \tag{3.19}
\]
By subtraction of the equations for \( u_k \) and \( u_0 \), we get an equation for \( u_k - u_0 \) in \( \Omega \):
\[
-\Delta(u_k - u_0) = \nabla \cdot (\varphi_k \nabla \times H_k - \varphi_0 \nabla \times H_0) + (h_k + E^0) \cdot \nabla \times H_k - (h_0 + E^0) \cdot \nabla \times H_0.
\]
We re-collect the two terms in the right side as follows:
\[
\varphi_k \nabla \times H_k - \varphi_0 \nabla \times H_0 = \varphi_k (\nabla \times H_k - \nabla \times H_0) + (\varphi_k - \varphi_0) \nabla \times H_0,
\]
and
\[
(h_k + E^0) \cdot \nabla \times H_k - (h_0 + E^0) \cdot \nabla \times H_0 = (h_k + E^0) \cdot (\nabla \times H_k - \nabla \times H_0) + (h_k - h_0) \cdot \nabla \times H_0.
\]
Recall that \( u_k - u_0 = 0 \) on \( \partial \Omega \). By applying the \( H^1 \) estimate of Laplace equation to the above equation for \( u_k - u_0 \) we have
\[
\|\nabla u_k - \nabla u_0\|_{L^2(\Omega)} \leq \|\varphi_k \nabla \times H_k - \varphi_0 \nabla \times H_0\|_{L^2(\Omega)}
+ C\|\nabla u_k + E^0\|_{L^2(\Omega)} \cdot (h_k + E^0) \cdot \nabla \times H_0\|_{L^{\infty}(\Omega)}
\leq \|\varphi_k \|_{L^{\infty}(\Omega)} \|\nabla \times (H_k - H_0)\|_{L^2(\Omega)} + \|\varphi_k - \varphi_0\|_{L^{\infty}(\Omega)} \|\nabla \times H_0\|_{L^2(\Omega)}
+ C\|E^0\|_{L^2(\Omega)} \|\nabla \times (H_k - H_0)\|_{L^2(\Omega)} + \|h_k - h_0\|_{L^2(\Omega)} \|\nabla \times H_0\|_{L^2(\Omega)} \to 0,
\]

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where $C$ depends on $\Omega$. Here we have used (3.16), (3.18) and (3.19). Hence via the Poincaré inequality we find $u_k \to u_0$ in $H^1(\Omega)$ as $k \to \infty$. So $T$ is continuous from $L^2(\Omega)$ to $H^1(\Omega)$.

Finally, since the embedding $H^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact, $T$ is compact on $D$. By applying Schauder’s fixed point theorem we conclude that $T$ has a fixed point $u \in D$. Since $T$ maps $D$ into $H^1(\Omega)$ we know that $u = T(u) \in H^1(\Omega)$. Let $\mathbf{H} \in H_0^q(\text{curl}, \Omega) \cap H(\text{div}, \Omega) \cap H^1_w(\Omega)^\perp$ be the solution of (3.3) with $w$ replaced by $u$. Then $(u, \mathbf{H})$ is a weak solution of (1.1).

\[ \square \]

4. Regularity of weak solutions

4.1. Higher integrability of derivatives

In theorem 3.3 we get a weak solution $(u, \mathbf{H})$ to (1.1). Under the assumption that $\Omega$ is of class $C^2$, we can show that actually $\mathbf{H} \in W^{1,p}(\Omega, \mathbb{R}^3)$ whenever $\mathbf{E}^0 \in L^p(\Omega, \mathbb{R}^3)$, where $p$ is either slightly larger than $2$ (see proposition 4.1) or $p > 3$ (see theorem 4.2).

**Proposition 4.1.** Assume $\Omega$ is a bounded domain in $\mathbb{R}^3$ with a $C^2$ boundary, and the function $\sigma$ satisfies (1.8). Let $(u, \mathbf{H}) \in H^1(\Omega) \times [H_0^q(\text{div}, \Omega) \cap H(\text{curl}, \Omega)]$ be a weak solution of (1.1) corresponding to the boundary datum $(u^0, \mathbf{E}^0)$. Then there exists a constant $p_0 > 2$, which depends only on $\Omega, \sigma_1, \sigma_2$, such that the following conclusions hold.

(i) If $\mathbf{E}^0 \in L^p(\Omega, \mathbb{R}^3)$ with $2 < p < p_0$, then $u \in W^{2,p/2}_0(\Omega), \mathbf{H} \in W^{1,p}(\Omega, \mathbb{R}^3)$ and

$$
\|\mathbf{H}\|_{W^{1,p}(\Omega)} \leq C_1 \{\|\mathbf{H}\|_{L^2(\Omega)} + \|\mathbf{E}^0\|_{L^p(\Omega)}\}. 
$$

(ii) If furthermore $u^0 \in W^{2,p/2}(\Omega)$, then $u \in W^{2,p/2}(\Omega)$, and we have

$$
\|u\|_{W^{2,p/2}(\Omega)} \leq C_2 \{\|\mathbf{E}^0\|_{L^p(\Omega)}^2 + \|u^0\|_{W^{2,p/2}(\Omega)}\}. 
$$

In the above, $C_1, C_2$ depend only on $\Omega, p, \sigma_1, \sigma_2$.

**Proof.** We first mention that if $(u, \mathbf{H})$ is a weak solution of (1.1), then for any $\mathbf{h} \in \mathbb{H}_N(\Omega)$, $(u, \mathbf{H} + \mathbf{h})$ is also a weak solution of (1.1). Since $\mathbb{H}_N(\Omega)$ is of finite dimension, we can always choose $\mathbf{h} \in \mathbb{H}_N(\Omega)$ such that $\mathbf{H} + \mathbf{h} \in \mathbb{H}_N(\Omega)^\perp$.

**Step 1.** Since

$$
\nabla \times [\sigma(u)^{-1} \nabla \times \mathbf{H} - \mathbf{E}^0] = 0 \quad \text{in} \ \Omega,
$$

$$
\nu \times [\sigma(u)^{-1} \nabla \times \mathbf{H} - \mathbf{E}^0] = 0 \quad \text{on} \ \partial \Omega,
$$

by lemma 2.1, there exist $\varphi \in H^1_0(\Omega)$ and $\mathbf{h} \in \mathbb{H}_D(\Omega)$ such that (1.5) holds. We write

$$
\mathbf{h} = \sum_{j=1}^m c_j \mathbf{e}_j, \quad c_j = \int_{\Omega} \mathbf{e}_j \cdot (\sigma(u)^{-1} \nabla \times \mathbf{H} - \mathbf{E}^0) \, dx, \quad j = 1, \ldots, m.
$$

Since $\Omega$ is of class $C^2$, we have $\mathbb{H}_D(\Omega) \subset C^0(\overline{\Omega}, \mathbb{R}^3)$ and

$$
\|\mathbf{h}\|_{L^p(\Omega)} \leq C_3 \|\mathbf{E}^0\|_{L^2(\Omega)},
$$

(4.3)

where $C_3$ depends on $\Omega, p, \sigma_1, \sigma_2$. 



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Step 2. With a similar derivation used for (3.10), we see that $\varphi$ is a weak solution of
\[
\begin{cases}
\nabla \cdot [\sigma(u)(\nabla \varphi + h + E^0)] = 0 & \text{in } \Omega, \\
\varphi = 0 & \text{on } \partial \Omega.
\end{cases}
\]
(4.4)
We show that there exists $p_0 > 2$ which depends only on $\Omega, \sigma_1, \sigma_2$ but is independent of the solution, such that for any $2 < p < p_0$ and for any weak solution $\varphi$ of (4.4), it holds that
\[
\|\nabla \varphi\|_{L^p(\Omega)} \leq C_4\|E^0\|_{L^p(\Omega)},
\]
where $C_4$ depends on $\Omega, p, \sigma_1, \sigma_2$.
In fact, by Meyers’ estimate of higher integrability of gradient (see [18]), there exists $p_0 > 2$, which depends only on $\Omega, \sigma_1, \sigma_2$, but it is independent of the solution $\varphi$ of (4.4), such that $\varphi \in W^{1,p}(\Omega)$ for any $2 < p < p_0$. Moreover, we have the estimate
\[
\|\nabla \varphi\|_{L^p(\Omega)} \leq C(\Omega, p, \sigma_1, \sigma_2)\{\|h\|_{L^p(\Omega)} + \|E^0\|_{L^p(\Omega)}\}.
\]
From this and using (4.3) we get (4.5).

Step 3. We prove $H \in W^{1,p}(\Omega, \mathbb{R}^3)$.
First, from (4.3) and (4.5) we see that $\nabla \times H \in L^p(\Omega, \mathbb{R}^3)$ and
\[
\|\nabla \times H\|_{L^p(\Omega)} \leq C_5\|E^0\|_{L^p(\Omega)},
\]
where $C_5$ depends on $\Omega, p, \sigma_1, \sigma_2$.
Next, we show $H \in L^p(\Omega, \mathbb{R}^3)$. We prove this by a duality method, which has been used in the proof of [26, lemma 3.1]. Given $F \in C_c^\infty(\Omega, \mathbb{R}^3)$ and $1 < r < \infty$, by Helmholtz–Weyl decomposition (see [2, theorem 6.1] or [17, theorem 2.1]), there exists $w \in W^{1,r}(\Omega, \mathbb{R}^3)$ with $\nu \times w = 0$ on $\partial \Omega$, $\chi \in W^{1,r}(\Omega)$, and $z \in H^1_N(\Omega)$ such that
\[
F = \nabla \times w + \nabla \chi + z.
\]
Moreover, the triplet $(w, \chi, z)$ satisfies the estimate
\[
\|w\|_{W^{1,r}(\Omega)} + \|\chi\|_{W^{1,r}(\Omega)} + \|z\|_{L^r(\Omega)} \leq C(\Omega, r)\|F\|_{L^r(\Omega)}.
\]
(4.7)
Using (1.5) we see that $H$ satisfies the following div-curl system
\[
\begin{cases}
\nabla \times H = \sigma(u)(\nabla \varphi + h + E^0), & \text{in } \Omega, \\
\nu \cdot H = 0 & \text{on } \partial \Omega,
\end{cases}
\]
(4.8)
where $\sigma(u)(\nabla \varphi + h + E^0) \in L^p(\Omega, \mathbb{R}^3)$. Since $\nu \times w = 0$ on $\partial \Omega$, $\nabla \cdot H = 0$ in $\Omega$ and $\nu \cdot H = 0$ on $\partial \Omega$, we have
\[
\begin{align*}
\int_{\Omega} H \cdot (\nabla \times w) \, dx &= \int_{\Omega} (\nabla \times H) \cdot w \, dx + \int_{\partial \Omega} (\nu \times w) \cdot H \, dS = \int_{\Omega} \nabla \times H \cdot w \, dx, \\
\int_{\Omega} H \cdot \nabla \chi \, dx &= \int_{\partial \Omega} (\nu \cdot H) \chi \, dS - \int_{\Omega} (\nabla \cdot H) \chi \, dx = 0.
\end{align*}
\]
Using the above two equalities and (4.7), we get
\[
\int_{\Omega} H \cdot F \, dx = \int_{\Omega} H \cdot (\nabla \times w + \nabla \chi + z) \, dx
\]
\[
= \int_{\Omega} (\nabla \times H) \cdot w + H \cdot z \, dx
\]
\[
\leq C(\Omega, p)\{\|\nabla \times H\|_{L^r(\Omega)} + \|H\|_{L^r(\Omega)}\}\|F\|_{L^{r'}(\Omega)},
\]
from which we obtain $H \in L^p(\Omega, \mathbb{R}^3)$ with the estimate
\[
\|H\|_{L^p(\Omega)} \leq C(\Omega, p) \left\{ \|H\|_{L^2(\Omega)} + \|\nabla \times H\|_{L^p(\Omega)} \right\}.
\] (4.9)

With $H \in L^p(\Omega, \mathbb{R}^3)$ in hand, we can apply the $L^p$ regularity theory for the div-curl systems (see [1, theorem 2.2] and [2, theorem 3.5]; see also [25] and [17]) to (4.8), and conclude that $H \in W^{1,p}(\Omega, \mathbb{R}^3)$. Since $\text{div} H = 0$ in $\Omega$ and $\nu \cdot H = 0$ on $\partial \Omega$, we have
\[
\|H\|_{W^{1,p}(\Omega)} \leq C(\Omega, p) \left\{ \|H\|_{L^p(\Omega)} + \|\nabla \times H\|_{L^p(\Omega)} \right\}.
\]

From this, (4.6) and (4.9), we get (4.1).

If we choose $H \in H^1(\Omega)^{\perp}$, then, $\text{div} H = 0$ in $\Omega$, and $\nu \cdot H = 0$ on $\partial \Omega$, so we can use the following Poincaré-type inequality:
\[
\|H\|_{L^2(\Omega)} \leq C(\Omega) \|\nabla \times H\|_{L^2(\Omega)}.
\]

From this and (3.1), by increasing the constant $C_1$ if necessary, we can remove the term $\|H\|_{L^2(\Omega)}$ in the right side of (4.1).

**Step 4.** Finally, using (1.8) we see that $\sigma(u)^{-1} |\nabla \times H|^2 \in L^{p/2}(\Omega)$. By applying elliptic regularity theory to the Laplace equation (1.4), we see that $u \in W^{2,p/2}_{\text{loc}}(\Omega)$. If furthermore $u^0 \in W^{2,p/2}_{\text{loc}}(\Omega)$, then we have $u \in W^{2,p/2}(\Omega)$, and
\[
\|u\|_{W^{2,p/2}(\Omega)} \leq C(\Omega, p) \left\{ \|\sigma(u)^{-1} |\nabla \times H|^2\|_{L^{p/2}(\Omega)} + \|u^0\|_{W^{2,p/2}(\Omega)} \right\}.
\]

From this and (4.6) we get (4.2).

Now we show that if $(u^0, E^0)$ satisfies
\[
(u^0, E^0) \in W^{1,q}(\Omega) \times L^q(\Omega, \mathbb{R}^3) \quad \text{for some } q > 3,
\] (4.10)
then the weak solution of (1.1) has $W^{1,q}$ regularity.

**Theorem 4.2.** Assume that $\Omega$ is a bounded domain in $\mathbb{R}^3$ with a $C^2$ boundary, the function $\sigma$ satisfies (1.8), and $(u^0, E^0)$ satisfies (4.10). Let $(u, H) \in H^1(\Omega) \times [H_0(\text{div} 0, \Omega) \cap H(\text{curl}, \Omega)]$ be a weak solution of (1.1). Then we have the following conclusions.

(i) $u \in C^{\delta, \mu - 1/2}(\Omega)$ for all $1 < \mu < 1 + 2 \min\{\delta, 1 - 3/q\}$, where $\delta = \delta(\Omega, \sigma_1, \sigma_2) \in (0, 1)$, and there exists $C_1 = C_1(\Omega, \mu, q, \sigma_1, \sigma_2) > 0$ such that
\[
\|u\|_{C^{\delta, \mu - 1/2}(\Omega)} \leq C_1 \left\{ \|E^0\|_{L^2(\Omega)} + \|u^0\|_{W^{1,q}(\Omega)} \right\}.
\] (4.11)

(ii) $(u, H) \in W^{1,q}(\Omega) \times W^{1,q}(\Omega, \mathbb{R}^3)$, and
\[
\|H\|_{W^{1,q}(\Omega)} \leq C_2 \left\{ \|H\|_{L^2(\Omega)} + \|E^0\|_{L^2(\Omega)} \right\},
\] (4.12)
where $C_2$ depends on $\Omega, q, \sigma_1, \sigma_2$ and the VMO modulus of continuity of $\sigma(u)$. If furthermore we choose $H \in H^1(\Omega)^{\perp}$, then the term $\|H\|_{L^2(\Omega)}$ in the right side of (4.12) can be removed.

(iii) Assume furthermore that the function $\sigma$ satisfies (1.10). Then we have the estimate
\[
\|\nabla \times H\|_{L^2(\Omega)} \leq C_3 \|E^0\|_{L^2(\Omega)},
\] (4.13)
where the constant $C_3$ depends only on $\Omega, q, \sigma_1, \sigma_2, L$, and the $W^{1,q}(\Omega)$ norm of $u^0$ and the $L^q(\Omega)$ norm of $E^0$.

\[\text{Then by Morrey embedding theorem, } (u, H) \in C^{\delta, 1/3 - \mu /q}(\Omega) \times C^{\delta, 1/3 - \mu /q}(\Omega, \mathbb{R}^3)\]
Proof. The key point in the proof is to establish first the Hölder continuity of $u$.

Step 1. Let $(u, H)$ be a weak solution of (1.1). As in the proof of theorem 3.3, we have the equality (1.5), where $h \in H_0^1(\Omega)$, $H, \varphi$ and $u$ are solutions of (3.3), (3.10) and (3.13), with $w$ replaced by $u$, respectively. By lemma 3.2, we have the $L^2$ estimate for $\nabla \times H$:

$$\|\nabla \times H\|_{L^2(\Omega)} \leq \sigma_2 \|E^0\|_{L^2(\Omega)},$$

and the $L^2$ estimate for $u$:

$$\|u\|_{L^2(\Omega)} \leq C_4 \left( \|E^0\|_{L^2(\Omega)} \|E^0\|_{L^2(\Omega)} + \|u^0\|_{H^1(\Omega)} \right),$$

where $C_4$ depends on $\Omega, \sigma_1, \sigma_2$.

Next we show the inequality

$$\|\varphi\|_{C^{0,\alpha}(\Omega)} + \|\nabla \varphi\|_{L^{2,\alpha}(\Omega)} \leq C_5 \|E^0\|_{L^2(\Omega)},$$

where $1 < \mu < 1 + 2 \min \{\delta, 1 - 3/q\}$, the constants $\delta = \delta(\Omega, \sigma_1, \sigma_2) \in (0, 1)$, and $C_5 = C_5(\Omega, \mu, \sigma_1, \sigma_2)$.

To prove this conclusion, we apply lemma 2.4 to (3.10) and conclude that there exists $\delta = \delta(\Omega, \sigma_1, \sigma_2) \in (0, 1)$, such that the following estimate holds for all $0 < \mu < 1 + 2\delta$:

$$\|\nabla \varphi\|_{L^{2,\mu}(\Omega)} \leq C(\|\varphi\|_{H^1(\Omega)} + \|\sigma(u)(h + E^0)\|_{L^{2,\mu}(\Omega)}) \leq C(\|E^0\|_{L^2(\Omega)} + \|h\|_{L^{2,\mu}(\Omega)} + \|E^0\|_{L^{2,\mu}(\Omega)}),$$

(4.16)

Here we have used (3.1) and the standard $H^1$ estimate for (3.5), and the constant $C$ depends on $\Omega, \mu, \sigma_1, \sigma_2$.

Then by lemma 2.3 we obtain

$$\|\varphi\|_{L^{2,\mu+2}(\Omega)} \leq C(\Omega, \mu) \left( \|\varphi\|_{L^2(\Omega)} + \|\nabla \varphi\|_{L^{2,\mu}(\Omega)} \right) \leq C_6 \left( \|E^0\|_{L^2(\Omega)} + \|h\|_{L^{2,\mu}(\Omega)} + \|E^0\|_{L^{2,\mu}(\Omega)} \right),$$

(4.17)

where the constant $C_6$ depends on $\Omega, \mu, \sigma_1, \sigma_2$.

Since $\Omega$ is of class $C^2$, $H_0^1(\Omega) \subset C^0(\Omega, \mathbb{R}^3)$. Recalling that $h$ is in the form of (3.7), we get

$$\|h\|_{L^\infty(\Omega)} \leq C(\Omega, \sigma_1, \sigma_2) \|E^0\|_{L^2(\Omega)},$$

(4.18)

Using the embedding of $L^\prime(\Omega)$ into a Campanato space, we have

$$L^\prime(\Omega) \subseteq L^{2,3-6/r}(\Omega) \subseteq L^{2,\mu}(\Omega) \quad \text{for } r > 2, \ 0 < \mu \leq 3 - 6/r.$$

So we get

$$\|h\|_{L^{2,\mu}(\Omega)} + \|E^0\|_{L^{2,\mu}(\Omega)} \leq C(\Omega, \mu, q) \left( \|h\|_{L^2(\Omega)} + \|E^0\|_{L^2(\Omega)} \right) \leq C(\Omega, \mu, q) \left( \|h\|_{L^\infty(\Omega)} + \|E^0\|_{L^\infty(\Omega)} \right),$$

(4.19)

where $1 < \mu < 1 + 2 \min \{\delta, 1 - 3/q\}$. Then by lemma 2.3, inequalities (4.16)–(4.19) we get

(4.15).

Step 2. Since $\nabla \times H = \sigma(u)(\nabla \varphi + h + E^0)$, by (4.15), (4.19) and (4.18) we get

$$\|\nabla \times H\|_{L^{2,\mu}(\Omega)} \leq C(\Omega, \mu, q, \sigma_1, \sigma_2) \|E^0\|_{L^2(\Omega)}.$$

(4.20)

Since $\nabla \times E^0 = 0$ in $\Omega$, by lemma 2.1, there exist $\varphi^0 \in H^1(\Omega)$ and $h_1 \in H^1(\Omega)$ such that $E^0 = \nabla \varphi^0 + h_1$. Here $\varphi^0$ satisfies that $\Delta \varphi^0 = \nabla \cdot E^0$ in $\Omega$ and $\nu \cdot \nabla \varphi^0 = \nu \cdot E^0$ on $\partial \Omega$. Hence $\varphi^0 \in W^{1,q}(\Omega)$. Similar to (4.18), we also have the $L^\infty$ estimate for $h_1$.
\[ \| \mathbf{h}_1 \|_{L^\infty(\Omega)} \leq C(\Omega)\|E^0\|_{L^2(\Omega)}. \] (4.21)

Since
\[
\sigma(u)^{-1}\| \nabla \times \mathbf{H} \|^2 = (\nabla \varphi + \mathbf{h} + E^0) \cdot \nabla \times \mathbf{H} = [\nabla (\varphi + \varphi^0) + \mathbf{h} + h_1] \cdot \nabla \times \mathbf{H}
\]
\[ = \nabla \cdot [(\varphi + \varphi^0)\nabla \times \mathbf{H}] + (\mathbf{h} + h_1) \cdot \nabla \times \mathbf{H}, \]
we can write the equation for \( u \) in the following form:
\[ -\Delta u = \nabla \cdot [(\varphi + \varphi^0)\nabla \times \mathbf{H}] + (\mathbf{h} + h_1) \cdot \nabla \times \mathbf{H} \quad \text{in} \; \Omega, \quad u = u^0 \quad \text{on} \; \partial \Omega. \] (4.22)

By applying lemma 2.4 to the above equation, and using lemma 2.3 we obtain
\[
\| \nabla u \|_{L^{2n}(\Omega)} \leq C \{ \| u \|_{H^1(\Omega)} + \| (\mathbf{h} + h_1) \cdot \nabla \times \mathbf{H} \|_{L^2(\Omega)} + \| \varphi + \varphi^0 \|_{L^2(\Omega)} \| \nabla \times \mathbf{H} \|_{L^2(\Omega)}
\]
\[ \leq C_7 \{ \| \mathbf{h} + h_1 \|_{L^\infty(\Omega)} + \| \nabla \times \mathbf{H} \|_{L^2(\Omega)} + \| \varphi + \varphi^0 \|_{L^2(\Omega)} \| \nabla \times \mathbf{H} \|_{L^2(\Omega)}
\]
\[ + \| E^0 \|_{L^2(\Omega)} + \| u^0 \|_{W^{1,q}(\Omega)} + \| \nabla u^0 \|_{L^2(\Omega)} \}, \]
where \( C_7 \) depends on \( \Omega, \mu, q, \sigma_1, \sigma_2 \). Then by lemma 2.3, inequalities (4.14), (4.15), (4.18), (4.20) and (4.21), we get
\[
\| u \|_{C^{0,(n-1)/2}(\overline{\Omega})} \leq C \| u \|_{L^2(\Omega)} \leq C \{ \| u \|_{L^2(\Omega)} + \| \nabla u \|_{L^2(\Omega)} \}
\]
\[ \leq C_8 \{ \| E^0 \|_{L^2(\Omega)}^2 + \| u^0 \|_{W^{1,q}(\Omega)}^2 \}, \]
where \( C_8 \) depends on \( \Omega, \mu, q, \sigma_1, \sigma_2 \).

**Step 3.** Now \( u \) is continuous on \( \Omega \). From the continuity of the function \( \sigma \) we see that \( \sigma(u) \) is continuous, hence \( \sigma(u) \in \text{VMO}(\Omega) \). Therefore we can apply [3, theorem 1] to the Dirichlet problem (4.4), and get \( \varphi \in W^{1,q}(\Omega) \) with the estimate
\[
\| \nabla \varphi \|_{L^q(\Omega)} \leq C_9 \| \sigma(u)(\mathbf{h} + E^0) \|_{L^2(\Omega)},
\]
where \( C_9 \) depends on \( \Omega, q, \sigma_1, \sigma_2 \) and the VMO modulus of continuity of \( \sigma(u) \). Hence
\[ \nabla \times \mathbf{H} = \sigma(u)(\nabla \varphi + \mathbf{h} + E^0) \in L^2(\Omega, \mathbb{R}^3). \]

Then, in the same way as in Step 3 of the proof of proposition 4.1, we conclude that \( \mathbf{H} \in W^{1,q}(\Omega, \mathbb{R}^3) \), and we also have the estimate
\[
\| \mathbf{H} \|_{W^{1,q}(\Omega)} \leq C_{10} \{ \| \mathbf{H} \|_{L^2(\Omega)} + \| E^0 \|_{L^2(\Omega)} \},
\]
where the constant \( C_{10} \), differently from the case of proposition 4.1, depends not only \( \Omega, q, \sigma_1, \sigma_2 \), but also on the VMO modulus of continuity of \( \sigma(u) \).

Next we re-write the equation for \( u \) in the following form
\[ -\Delta u = \nabla \cdot (\mathbf{H} \times \sigma(u)^{-1}\nabla \times \mathbf{H}) \quad \text{in} \; \Omega, \quad u = u^0 \quad \text{on} \; \partial \Omega. \] (4.23)

Since \( q > 3 \), from the Sobolev embedding theorem we see that \( \mathbf{H} \in C^0(\overline{\Omega}, \mathbb{R}^3) \). Hence
\[ \nabla \cdot (\mathbf{H} \times \sigma(u)^{-1}\nabla \times \mathbf{H}) \in W^{-1,q}(\Omega). \]

From elliptic regularity theory, we obtain \( u \in W^{1,q}(\Omega) \). We can also obtain an estimate of \( \| u \|_{W^{1,q}(\Omega)} \), with the constant depending also on the VMO modulus of continuity of \( \sigma(u) \).
Step 4. Assume \( \sigma \) satisfies (1.10). Then \( \sigma(u) \in C^{0, (\mu - 1)/2}(\overline{\Omega}) \). By applying [24, theorem 3.16 (iv)] to the Dirichlet problem (4.4), we obtain
\[
\| \nabla \varphi \|_{L^2(\Omega)} \leq C \| \sigma(u)(h + E^0) \|_{L^2(\Omega)} \leq C_{11} \| E^0 \|_{L^2(\Omega)},
\]
where \( C_{11} \) depends on \( \Omega, q, \sigma_1, \sigma_2, \) and also on the bound of \( \| \sigma(u) \|_{C^{0, (\mu - 1)/2}(\overline{\Omega})} \), which can be estimated as follows:
\[
\| \sigma(u) \|_{C^{0, (\mu - 1)/2}(\overline{\Omega})} \leq \sigma_2 + L \| u \|_{C^{0, (\mu - 1)/2}(\Omega)} \leq \sigma_2 + LC_8 \left\{ \| E^0 \|_{L^2(\Omega)}^2 + \| u^0 \|_{W^{1,4}(\Omega)}^2 \right\}.
\]
So \( C_{11} \) depends only on \( \Omega, q, \sigma_1, \sigma_2, L \) and the norms of \( E^0 \) and \( u^0 \) appearing in the above inequality. Thus we get (iii).

Remark 4.3. Let \( \| E^0 \|_{L^2(\Omega)} \leq 1 \). Then
\[
\| \sigma(u) \|_{C^{0, (\mu - 1)/2}(\overline{\Omega})} \leq \sigma_2 + LC_8 \left\{ 1 + \| u^0 \|_{W^{1,4}(\Omega)}^2 \right\}.
\]
Hence the constant \( C_3 \) in (4.13) depends only on \( \Omega, q, \sigma_1, \sigma_2, L, \| u^0 \|_{W^{1,4}(\Omega)} \). This point will play an important role in the proof of small boundary data uniqueness in section 5.

4.2. Hölder continuity of derivatives

Next we prove Hölder continuity of derivatives of \( u \) and \( H \) as the domain and boundary data allow.

Theorem 4.4. Let \( k \geq 0 \) be an integer, and \( \alpha \in (0, 1) \). Let \( \Omega \) be a bounded and simply connected \( C^{k+2, \alpha} \) domain in \( \mathbb{R}^3 \), and \( \sigma \) satisfy (1.8). Let \( (u, H) \in H^k(\Omega) \times [H_0(\text{div, } \Omega) \cap H(\text{curl, } \Omega)] \) be a weak solution of (1.1). Assume in addition that
\[
(u^0, E^0) \in C^{k+1, \alpha}(\overline{\Omega}) \times C^{k, \alpha}(\overline{\Omega}, \mathbb{R}^3), \quad \sigma \in C^{0, \alpha}_{\text{loc}}(\mathbb{R}),
\]
then we have \( (u, H) \in C^{k+1, \alpha}(\overline{\Omega}) \times C^{k, \alpha}(\overline{\Omega}, \mathbb{R}^3) \).

Proof. We give the proof for \( k = 0, 1 \). Then by induction we get the conclusion for \( k \geq 2 \).

Step 1. We start with the case where \( k = 0 \). Suppose \( (u^0, E^0) \in C^{1, \alpha}(\overline{\Omega}) \times C^{0, \alpha}(\overline{\Omega}, \mathbb{R}^3) \) and \( \sigma \in C^{0, \alpha}_{\text{loc}}(\mathbb{R}) \). From theorem 4.2 we obtain \( \sigma(u) \in C^{0, \beta}(\overline{\Omega}) \), where \( \beta = (\mu - 1)\alpha/2 \). By applying elliptic regularity theory to the equation (4.4), we conclude that \( \varphi \in C^{1, \beta}(\overline{\Omega}) \). Noting the regularity of the elements in \( H_0(\text{div}, \Omega) \), using the identity
\[
\sigma(u)^{-1} \| \nabla \times H \|^2 = \sigma(u) \| \nabla \varphi + h + E^0 \|^2 \in C^{0, \beta}(\overline{\Omega}),
\]
and by applying elliptic regularity theory to Laplace equation (1.4), we conclude that \( u \in C^{1, \alpha}(\overline{\Omega}) \). This implies that \( \sigma(u) \in C^{1, \alpha}(\overline{\Omega}) \). By applying elliptic regularity theory to the equation (4.4) again, we conclude that \( \varphi \in C^{1, \alpha}(\overline{\Omega}) \). Since \( \nabla \times H = \sigma(u)(\nabla \varphi + h + E^0) \in C^{0, \alpha}(\overline{\Omega}, \mathbb{R}^3) \), using the Hölder regularity of the div-curl system (4.8) given by [4, proposition 2.1], we obtain \( H \in C^{1, \alpha}(\overline{\Omega}, \mathbb{R}^3) \).

Step 2. Now we consider the case where \( k = 1 \). Suppose \( (u^0, E^0) \in C^{2, \alpha}(\overline{\Omega}) \times C^{1, \alpha}(\overline{\Omega}, \mathbb{R}^3) \) and \( \sigma \in C^{1, \alpha}_{\text{loc}}(\mathbb{R}) \). By applying the Hölder regularity of the Laplace equation to (1.4) we derive
$u \in C^{2,\alpha}(\Omega)$. Using (4.4) we obtain $\varphi \in C^{2,\alpha}(\Omega)$. Since $\sigma(u)\nabla \varphi \in C^{1,\alpha}(\Omega)$, by applying [4, proposition 2.1] to (4.8) we have $H \in C^{2,\alpha}(\Omega, \mathbb{R}^3)$.

Here we mention that the local Schauder theory has been given by Kang and Kim; see [15, theorem 3.2 and remark 3.3] or [16, remark 5.10].

5. Uniqueness under small boundary data

In this section we establish uniqueness results under small boundary data. We assume that the function $\sigma$ satisfies (1.10). Let $S(n, p, \Omega)$ be the best constant for Sobolev inequality

$$S(n, p, \Omega) \|v\|_{L^{2n/(n-2)}(\Omega)} \leq \|\nabla v\|_{L^2(\Omega)} \text{ for any } v \in H^1_0(\Omega).$$

It is well-known that if $p = n > 2$, then $S(n, p, \Omega)$ does not depend on $\Omega$. So we denote $S(3, 3, \Omega)$ with $S(3)$.

In order to prove the uniqueness result for (1.1), we establish the following lemma, which is similar to [14, theorem 5].

Lemma 5.1. Let $\Omega$ be a bounded domain in $\mathbb{R}^3$, and $\sigma$ satisfy (1.8) and (1.10). Let $\kappa$ be a positive constant satisfying

$$\kappa < \frac{S(3)\sigma_1}{\sqrt{(2\sigma_2/\sigma_1 + 1)L}}.$$

Then (1.1) has at most one weak solution lying in the following set

$$\{ (u, H) \in H^1(\Omega) \times [H_0(\text{div}0, \Omega) \cap H(\text{curl}, \Omega) \cap \mathbb{H}_N(\Omega)^{-1}) : \|\nabla \times H\|_{L^2(\Omega)} \leq \kappa \}.$$

Proof. Let $(u_1, H_1)$ and $(u_2, H_2)$ be two weak solutions in the above set. Set $v = u_1 - u_2$ and $B = H_1 - H_2$. Then $v$ and $B$ satisfy

$$-\Delta v = \sigma(u_1)^{-1}\nabla \times H_1|^2 - \sigma(u_2)^{-1}\nabla \times H_2|^2 \text{ in } \Omega, \quad v = 0 \quad \text{on } \partial \Omega,$$

and

$$\nabla \times \sigma(u_1)^{-1}\nabla \times B = \nabla \times [(\sigma(u_2)^{-1} - \sigma(u_1)^{-1})\nabla \times H_2], \quad \nabla \cdot B = 0 \quad \text{in } \Omega,$$

$$\nu \times \sigma(u_1)^{-1}\nabla \times B = \nu \times [(\sigma(u_2)^{-1} - \sigma(u_1)^{-1})\nabla \times H_2] \quad \text{on } \partial \Omega.$$

Then we have the equalities

$$\int_{\Omega} |\nabla v|^2 \, dx = \int_{\Omega} (\sigma(u_1)^{-1}|\nabla \times H_1|^2 - \sigma(u_2)^{-1}|\nabla \times H_2|^2)v \, dx, \quad (5.1)$$

$$\int_{\Omega} \sigma(u_1)^{-1}|\nabla \times B|^2 \, dx = \int_{\Omega} (\sigma(u_2)^{-1} - \sigma(u_1)^{-1})\nabla \times H_2 \cdot \nabla \times B \, dx. \quad (5.2)$$

From (5.2) and using the conditions (1.8) and (1.10), we have

$$\sigma_2^{-1}\|\nabla \times B\|_{L^2(\Omega)} \leq \sigma_1^{-2}L\|v\|_{L^2(\Omega)} \|\nabla \times H_2\|_{L^2(\Omega)} \leq \sigma_1^{-2}L\|v\|_{L^2(\Omega)} \|\nabla \times H_2\|_{L^2(\Omega)} \leq \sigma_1^{-2}L^2\|v\|_{L^2(\Omega)}.$$

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Using (5.1), by Sobolev inequality and Hölder inequality, it follows that
\[ S(3)^2 \|v\|_{L^2(\Omega)}^2 \leq \|\nabla v\|_{L^2(\Omega)}^2 \]
\[ \leq \|\sigma(u_1)^{-1}\nabla (H_1) - \sigma(u_2)^{-1}\nabla (H_2)\|_{L^{5/3}(\Omega)} \|v\|_{L^6(\Omega)} \]
\[ \leq \|\sigma(u_1)^{-1}\nabla (B) \cdot [\nabla (H_1 + H_2)]\|_{L^{5/3}(\Omega)} \|v\|_{L^6(\Omega)} \]
\[ + \|\sigma(u_1)^{-1} - \sigma(u_2)^{-1}\|\nabla (H_2)\|_{L^{5/3}(\Omega)} \|v\|_{L^6(\Omega)}. \tag{5.4} \]

Based on Hölder inequality and the conditions (1.8), (1.10), we get
\[ \|\sigma(u_1)^{-1}\nabla (B) \cdot [\nabla (H_1 + H_2)]\|_{L^{5/3}(\Omega)} \leq 2\sigma_1^{-1}\kappa \|\nabla (B)\|_{L^2(\Omega)}, \tag{5.5} \]
and
\[ \|\sigma(u_1)^{-1} - \sigma(u_2)^{-1}\|\nabla (H_2)\|_{L^{5/3}(\Omega)} \leq \sigma_1^{-2}L \|v\|_{L^6(\Omega)} \|\nabla (H_2)\|_{L^{5/3}(\Omega)} \]
\[ \leq \sigma_1^{-2}L \|v\|_{L^6(\Omega)} \|\nabla (H_2)\|_{L^6(\Omega)} \leq \sigma_1^{-2}L \kappa^2 \|v\|_{L^6(\Omega)}. \tag{5.6} \]

Combining the above four inequalities, we obtain
\[ S(3)^2 \|v\|_{L^2(\Omega)}^2 \leq \sigma_1^{-2} \left( \frac{2\sigma_2}{\sigma_1} + 1 \right) L \kappa^2 \|v\|_{L^6(\Omega)}^2. \]

Hence, if \( \sigma_1^{-2}(2\sigma_2/\sigma_1 + 1) L \kappa^2 < S(3)^2 \), then \( v = 0 \). Consequently, (5.3) implies that \( \nabla \times B = 0 \) in \( \Omega \). Since \( \nabla \cdot B = 0 \) in \( \Omega \), \( \nu \cdot B = 0 \) on \( \partial \Omega \) and \( B \in H^1(\Omega)^\perp \), we derive \( B = 0 \).

Now we prove uniqueness of weak solutions of (1.1) under small boundary data condition.

**Theorem 5.2.** Let \( \Omega \) be a bounded domain in \( \mathbb{R}^3 \) with a \( C^2 \) boundary and \( \sigma \) satisfy (1.8) and (1.10). Assume \( (u^0, E^0) \) satisfies (4.10). Then there exists \( \eta > 0 \) such that if (1.11) holds for this \( \eta \), then (1.1) has a unique weak solution in the space \( H^1(\Omega) \times \{H_0(\text{div}, \Omega) \cap H(\text{curl}, \Omega) \cap H^1(\Omega)^\perp\} \).

**Proof.** Theorem 3.3 proves the existence of at least one weak solution \( (\tilde{u}, \tilde{H}) \in H^1(\Omega) \times \{H_0(\text{div}, \Omega) \cap H(\text{curl}, \Omega) \} \). Furthermore, we can choose \( \tilde{H} \in H^1(\Omega)^\perp \). Then we show the uniqueness.

We first assume (1.11) holds with \( \eta = 1 \). Let \( (u, H) \) be any possible weak solution of (1.1). By remark 4.3, we conclude that the constant \( C_3 \) in (4.13) depends only on \( \Omega, q, \sigma_1, \sigma_2, L, \)
\[ \|u^0\|_{W^{1,\infty}(\Omega)} \]
Hence
\[ \|\nabla \times H\|_{L^2(\Omega)} \leq C_3(\Omega, q) \|\nabla \times H\|_{L^2(\Omega)} \leq C_3 \|E^0\|_{L^6(\Omega)}, \]
where \( C_3 \) depends only on \( \Omega, q, \sigma_1, \sigma_2, L, \|u^0\|_{W^{1,\infty}(\Omega)} \), and is independent of the solution. Let
\[ \eta = \min \left\{ 1, \frac{S(3)\sigma_1}{2C_\ast \sqrt{(2\sigma_2/\sigma_1 + 1)L}} \right\}. \]

If \( E^0 \) satisfies (1.11) for this \( \eta \), then it holds that
\[ \| \nabla \times H \|_{L^1(\Omega)} \leq \frac{S(3)\sigma_1}{2\sqrt{(2\sigma_2/\sigma_1 + 1)L}}, \]
and hence uniqueness follows from lemma 5.1.

6. Tangential boundary condition

In this section, we establish the existence, regularity and uniqueness of weak solutions of system (1.2).

**Definition 6.1.** We say that \((u, H) \in H^1(\Omega) \times [H(\text{div0}, \Omega) \cap H(\text{curl}, \Omega)]\) is a weak solution of (1.2) if \( u = u_0 \) and \( \nu \times H = \nu \times H^0 \) on \( \partial \Omega \) in the sense of trace, and if it holds that
\[ \int_{\Omega} \nabla u \cdot \nabla v \, dx = \int_{\Omega} \sigma(u)^{-1} |\nabla \times H|^2 \, v \, dx, \quad \forall v \in H^0_0(\Omega) \cap L^\infty(\Omega), \]
\[ \int_{\Omega} \sigma(u)^{-1} \nabla \times H \cdot \nabla \times w \, dx = 0, \quad \forall w \in H(\text{div0}, \Omega) \cap H(\text{curl}, \Omega). \]

**Proposition 6.2 (Existence of weak solutions).** Let \( \Omega \) be a bounded Lipschitz domain in \( \mathbb{R}^3 \). Assume the function \( \sigma \) satisfies (1.8), \( u_0 \in H^1(\Omega) \) and \( H^0 \in W^{1,q}(\Omega, \mathbb{R}^3) \) for some \( q > 3 \). Then (1.2) has a weak solution \((u, H) \in H^1(\Omega) \times [H(\text{div0}, \Omega) \cap H(\text{curl}, \Omega)]\).

**Proof.**

**Step 1.** For any given \( w \in L^2(\Omega) \), let \( H_w \in H(\text{div0}, \Omega) \cap H(\text{curl}, \Omega) \) be a weak solution of the system
\[ \begin{align*}
\nabla \times [\sigma(w)^{-1}\nabla \times H_w] &= 0, & \nabla \cdot H_w &= 0 \quad & \text{in } \Omega, \\
\nu \times H_w &= \nu \times H^0 \quad & \text{on } \partial \Omega.
\end{align*} \]

Let \( \phi \in H^0_0(\Omega) \) be such that \( \Delta \phi = \text{div} H^0 \) in \( \Omega \) and \( \phi = 0 \) on \( \partial \Omega \). Taking \( H_w - (H^0 - \nabla \phi) \) as a test function for (6.1), we have the following \( L^2 \) estimate:
\[ \| \nabla \times H_w \|_{L^2(\Omega)} \leq \frac{\sigma_2}{\sigma_1} \| \nabla \times H^0 \|_{L^2(\Omega)}. \]

**Step 2.** Since \( \nabla \times [\sigma(w)^{-1}\nabla \times H_w] = 0 \) in \( \Omega \), by lemma 2.1, there exist \( \varphi_w \in H^1(\Omega) \) and \( h_w \in H^N(\Omega) \) such that
\[ \sigma(w)^{-1} \nabla \times H_w = \nabla \varphi_w + h_w. \]

Let \( v_1, \ldots, v_N \) be an orthonormal basis of \( H^N(\Omega) \) with respect to the \( L^2 \) norm. We can write
\[ h_w = \sum_{j=1}^{N} c_j v_j, \quad c_j = \int_{\Omega} v_j \cdot [\sigma(w)^{-1} \nabla \times H_w] \, dx, \quad j = 1, \ldots, N. \]
From this and (6.2) we get
\[ |c_j| \leq C_1 \| \nabla \times H^0 \|_{L^2(\Omega)}, \quad j = 1, \cdots, N, \tag{6.4} \]
where \( C_1 \) depends on \( \Omega, \sigma_1, \sigma_2 \).

It is not difficult to see that \( \varphi_\omega \) satisfies the following equation:
\[ \begin{cases} 
    \nabla \cdot [\sigma(w)(\nabla \varphi_\omega + h_\omega)] = 0 & \text{in } \Omega, \\
    \nu \cdot [\sigma(w)(\nabla \varphi_\omega + h_\omega)] = \nu \cdot \nabla \times H^0 & \text{on } \partial \Omega. 
\end{cases} \tag{6.5} \]

By applying the De Giorgi–Nash estimate for elliptic equations with Neumann boundary condition (see [20, proposition 3.6]) to (6.5), we see that there exist \( \alpha = \alpha(\Omega, q, \sigma_1, \sigma_2) \in (0, 1) \) and \( C_2 = C_2(\Omega, q, \sigma_1, \sigma_2) \) such that
\[ \| \varphi_\omega \|_{C^{\alpha}(\overline{\Omega})} \leq C_2 \| \nabla \times H^0 \|_{L^2(\Omega)}. \tag{6.6} \]

**Step 3.** For \( w \) and \( H_\omega \) given above, we look for a solution \( u_\omega \) of (3.13). Using the decomposition (6.3) we have
\[ \sigma(w)^{-1} |\nabla \times H_\omega|^2 = (\nabla \varphi_\omega + h_\omega) \cdot \nabla \times H_\omega. \]

So we can use the Lax–Milgram theorem to conclude that (3.13) has a unique weak solution \( u_\omega \in H^1(\Omega) \). Moreover, we have the following estimate:
\[ \| u_\omega \|_{L^2(\Omega)} \leq C_3 \{ \| \nabla \times H^0 \|_{L^2(\Omega)} \| \nabla \times H^0 \|_{L^2(\Omega)} + \| u^0 \|_{W^1(\Omega)} \}, \tag{6.7} \]
where \( C_3 \) depends on \( \Omega, q, \sigma_1, \sigma_2 \).

**Step 4.** For any given function \( w \in L^2(\Omega) \), let \( u_\omega \) be the solution of (3.13), where \( H_\omega \) is the solution of (6.1) associated with \( w \). We define \( T(w) = u_\omega \). Then \( T \) is a map from \( L^2(\Omega) \) to \( L^2(\Omega) \). Having the estimates (6.6) and (6.7), we can apply Schauder’s fixed point theorem to get a solution of (1.2). The details are similar to the counterpart in the proof of theorem 3.3, and are hence omitted. \( \square \)

**Proposition 6.3 (Regularity of weak solutions).** Assume that \( \Omega \) is a bounded \( C^2 \) domain in \( \mathbb{R}^3 \) and \( \sigma \) satisfies (1.8). Let \( (u, H) \) be an \( H^1(\Omega) \times [H(\text{div}, \Omega) \cap H(\text{curl}, \Omega)] \) weak solution of (1.2). Then we have the following conclusions.

(i) There exists \( p_0 > 2 \) such that if \( (u^0, H^0) \in W^{2p/2}(\Omega) \times W^{1p}(\Omega, \mathbb{R}^3) \) with \( 2 < p < p_0 \), then \( (u, H) \in W^{2p/2}(\Omega) \times W^{1p}(\Omega, \mathbb{R}^3) \).

(ii) Assume \( (u^0, H^0) \in W^{1,q}(\Omega) \times W^{1,q}(\Omega, \mathbb{R}^3) \) for some \( q > 3 \). Then there exists \( \delta = \delta(\Omega, \sigma_1, \sigma_2) \in (0, 1) \) such that for any \( 1 < \mu < 1 + 2 \min\{\delta, 1 - 3/q\} \), we have \( u \in C^{(\mu-1)/2}(\overline{\Omega}) \) with the estimate
\[ \| u \|_{C^{(\mu-1)/2}(\overline{\Omega})} \leq C(\Omega, \mu, q, \sigma_1, \sigma_2)(\| \nabla \times H^0 \|^2_{L^2(\Omega)} + \| u^0 \|_{W^{1,q}(\Omega)}). \]

Moreover, we have \( (u, H) \in W^{1,q}(\Omega) \times W^{1,q}(\Omega, \mathbb{R}^3) \).

(iii) Assume \( \partial \Omega \) is connected (hence \( \mathbb{H}^1_0(\Omega) = \{0\} \)). Let \( \alpha \in (0, 1) \).

(a) If \( \Omega \) is of class \( C^{2,\alpha} \), \( \sigma \in C^{2,\alpha}(\mathbb{R}) \), and \( (u^0, H^0) \in C^{1,\alpha}(\overline{\Omega}) \times C^{1,\alpha}(\overline{\Omega}, \mathbb{R}^3) \), then \( (u, H) \in C^{1,\alpha}(\overline{\Omega}) \times C^{1,\alpha}(\overline{\Omega}, \mathbb{R}^3) \).
(b) If $\Omega$ is of class $C^{3,\alpha}$, $\sigma \in C^{1,\alpha}_{\text{loc}}(\mathbb{R})$, and $(u^0, H^0) \in C^{2,\alpha}(\overline{\Omega}) \times C^{2,\alpha}(\overline{\Omega}, \mathbb{R}^3)$, then $(u, H) \in C^{2,\alpha}(\overline{\Omega}) \times C^{2,\alpha}(\overline{\Omega}, \mathbb{R}^3)$.

**Proof.** Conclusion (i) follows from lemma 2.1 and Meyers’ estimate ([11, theorem 2]).

Now we prove (ii). Since $\nabla \times [\sigma(u)^{-1}\nabla \times H] = 0$ in $\Omega$, by lemma 2.1, there exist $\varphi \in \tilde{H}^1(\Omega)$ and $h \in H_N(\Omega)$ such that $\sigma(w)^{-1}\nabla \times H = \nabla \varphi + h$, where $\varphi$ solves the equation

\[
\begin{aligned}
\nabla \cdot [\sigma(u) (\nabla \varphi + h)] &= 0 \quad \text{in } \Omega, \\
\nu \cdot [\sigma(u)(\nabla \varphi + h)] &= \nu \cdot \nabla \times H^0 \quad \text{on } \partial \Omega.
\end{aligned}
\]

Then there exists $\delta = \delta(\Omega, \sigma_1, \sigma_2) \in (0, 1)$ such that for any $1 < \mu < 1 + 2\min\{\delta, 1 - 3/q\}$, it holds that

$$
\|\varphi\|_{C^{0,\mu-1/2}(\overline{\Omega})} + \|\nabla \varphi\|_{L^2(\Omega)} \leq C(\Omega, \mu, q, \sigma_1, \sigma_2) \|\nabla \times H^0\|_{L^2(\Omega)}.
$$

Since

$$
\sigma(u)^{-1}\nabla \times H^2 = \nabla \cdot [\varphi \nabla \times H^0] + h \cdot \nabla \times H^0,
$$

we can write the equation for $u$ in the following form:

$$
-\Delta u = \nabla \cdot [\varphi \nabla \times H^0] + h \cdot \nabla \times H^0 \quad \text{in } \Omega, \quad u = u^0 \quad \text{on } \partial \Omega.
$$

Similarly to the proof of theorem 4.2, we obtain

$$
\|u\|_{C^{0,\mu-1/2}(\overline{\Omega})} \leq C(\|\nabla \times H^0\|_{L^2(\Omega)}^2 + \|u^0\|_{W^{1,q}(\Omega)}).$

The rest of proof is similar to the counterpart for (1.1), and is hence omitted. $\square$

Similarly to theorem 5.2, we also have small boundary data uniqueness for (1.2).

**Proposition 6.4.** Let $\Omega$ be a bounded domain in $\mathbb{R}^3$ with a $C^2$ boundary, and $\sigma$ satisfy (1.8) and (1.10). Assume $(u^0, H^0) \in W^{1,q}(\Omega) \times W^{1,q}(\Omega, \mathbb{R}^3)$ for some $q > 3$. If $\|\nabla \times H^0\|_{L^q(\Omega)}$ is sufficiently small, then the solution of (1.2) in the space $H^1(\Omega) \times [H(\text{div}, \Omega) \cap H(\text{curl}, \Omega) \cap (H^0 + H^D(\Omega) \cap H^D(\Omega) \cap H^D(\Omega))$ is unique.

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