Phantom-like GCG and the constraints of its parameters via cosmological dynamics

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Abstract

We extend the equation of state of GCG (Generalized Chaplygin Gas) to \( w < -1 \) regime and show, from the point of view of dynamics, that the parameters of GCG should be in the range of \( 0 < \alpha < 1 \). Also, dynamical analysis indicate that the phase \( w_g = -1 \) is a dynamical attractor and the equation of state of GCG approaches it from either \( w_g > -1 \) or \( w_g < -1 \) depending on the choice of its initial cosmic density parameter and the ratio of pressure to critical energy density.

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1. Introduction

CMB anisotropy\textsuperscript{1, 2, 3}, Supernovae\textsuperscript{4, 5, 6} and SDSS\textsuperscript{7} strongly indicate that our universe is spatially flat, with two thirds of the energy contents resulted from dark energy, a substance with negative pressure and can make the universe expanding in an accelerating fashion. Candidates for dark energy have been proposed as vacuum energy, quintessence\textsuperscript{8, 9, 10, 11, 12, 13, 14}, phantom\textsuperscript{15} and GCG\textsuperscript{16} which is stemmed from the Chaplygian gas\textsuperscript{17}. Present observation data constrain the the range of the equation of state of dark energy as $-1.38 < w < -0.82$\textsuperscript{18}, which indicates the possibility of dark energy with $w < -1$, debuted as Phantom\textsuperscript{15}. The realization of $w < -1$ could not be achieved by scalar field with positive kinetic energy and thus the negative kinetic energy is introduced although it violates some well known energy conditions\textsuperscript{19}. Another important consequence of Phantom is the Big rip\textsuperscript{20} or Big smash\textsuperscript{21} phase, in which the scale factor of the Universe goes to infinity at a finite cosmological time. The cosmological implications of Phantom have been widely studied\textsuperscript{22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39} and the Phantom model with Born-Infeld type Lagrangian has been proposed\textsuperscript{40} and its generalization to brane world has been done in Ref.\textsuperscript{41}.

GCG has a very simple equation of state, $p_g = -\frac{M_4(\alpha+1)}{\rho_0^\alpha}$, which yields an analytically solvable cosmological dynamics if Universe is GCG dominated. Another ambition of introducing GCG is to unify dark energy and dark matter into one equation of state, also known as quartessence\textsuperscript{42}. However, detailed numerical analysis turns out to disfavor the dark matter modelled by the GCG equation of state\textsuperscript{43}. But no observation so far rule out the possibility of GCG as dark energy. It is quite possible for our Universe to contain a dark energy component modelled by GCG as well as another baryotropic fluid component mimicked by the equation of state $p_\gamma = \gamma\rho_\gamma$. Although the cosmological GCG system is analytically solvable when GCG is dominant, it is no longer possible when another component is present. We therefore resort to the phase analysis with which one can gain many important information without solving the dynamical equations.

Previous study of GCG are focus on the case that $w_g > -1$. In fact, if we accept the notion that dark energy is modelled by a perfect fluid, then, current observations do not exclude the possibility of $w_g < -1$, instead, they even favor it\textsuperscript{44}. Thus, in the parameter space of the GCG model, one should not exclude the regime $w_g < -1$ although this range of
equation of state can not be smoothly continued from the \( w_g > -1 \) regime. In this letter, we
generalize the idea of GCG by considering the \( w_g < -1 \) regime of its equation of state. We
show that the system could reach the late time de Sitter attractor from either \( w_g > -1 \) or
\( w_g < -1 \) depending on the choice of the initial energy density and pressure. When \( w_g < -1 \),
it will behave as phantom with a late time de Sitter attractor, and therefore won’t lead to
the catastrophic big rip.

The equation of state of GCG has two free parameters, \( \alpha \) and \( M \), which could be fixed, in
principle, by fitting the model to the Supernovae or CMB data[16]. However, when dealing
with these fitting, we need to narrow down the possibilities of the range of the parameters
to facilitate the numerical analysis. This is just part of the purpose of this current work, in
which we constrain the parameter \( \alpha \) from the cosmological dynamics of the system as well
as the requirement of sound speed. Our result is in agreement with the numerical results
obtained by other authors.

2. Autonomous system

A general study the phase space system of phantom scalar field in FRW universe has been
given in Ref.[24]. For the GCG cosmological dynamical system, the corresponding equations
of motion and Einstein equations could be written as,

\[
\begin{align*}
\dot{H} &= -\frac{\kappa^2}{2}(\rho_\gamma + p_\gamma + \rho_g + p_g) \\
\dot{\rho}_\gamma &= -3H(\rho_\gamma + p_\gamma) \\
\dot{\rho}_g &= -3H(\rho_g + p_g) \\
H^2 &= \frac{\kappa^2}{3}(\rho_\gamma + \rho_g)
\end{align*}
\] (0.1)

where \( \kappa^2 = 8\pi G \), \( \rho_\gamma \) is the density of fluid with a baryotropic equation of state \( p_\gamma = (\gamma - 1)\rho_\gamma \),
where \( 0 \leq \gamma \leq 2 \) is a constant that relates to the equation of state by \( w_\gamma = \gamma - 1 \); \( \rho_g \) is the
energy density of GCG with an equation of state

\[
p_g = -\frac{M^{4(\alpha+1)}}{\rho_g^\alpha}
\] (0.2)

The over dot represents derivative with respect to cosmic time \( t \), and \( H \) is Hubble parameter.
To analyze the dynamical system, we rewrite the equations with the following dimensionless variables

\[ x = \frac{\kappa^2 \rho_g}{3H^2} \]
\[ y = \frac{\kappa^2 \rho_g}{3H^2} \]
\[ N = \ln a \quad (0.3) \]

The dynamical system will then reduce to

\[ \frac{dx}{dN} = -3(x + y) + 3x[\gamma(1 - x) + x + y] \]
\[ \frac{dy}{dN} = 3\alpha(y + y^2/x) + 3y[\gamma(1 - x) + x + y] \quad (0.4) \]

Accordingly, the Friedman equation yields

\[ \Omega_g + \Omega_\gamma = 1 \quad (0.5) \]

where \( \Omega_g \equiv x \) and \( \Omega_\gamma \equiv \frac{\kappa^2 \rho_g}{3H^2} \) are the cosmic density parameters for GCG and baryotrophic fluid respectively. The equation of state for the scalar fields could be expressed in terms of the new variables as

\[ w_g = \frac{p_g}{\rho_g} = \frac{y}{x} \quad (0.6) \]

and the sound speed is

\[ c_s^2 = -\alpha \frac{y}{x} \quad (0.7) \]

The critical points of the system are \((x, y) = (1, 0)\) and \((1, -1)\) which correspond to a matter dominated phase and vacuum energy dominated phase respectively. If we linearize the system near its critical points and then translate the system to origin, we could readily write the first order perturbation equation as

\[ \mathbf{U}' = A \cdot \mathbf{U} \quad (0.8) \]

where \( \mathbf{U} \) is a 2-column vector consist of the perturbations of \( x \) and \( y \). \( A \) is a \( 2 \times 2 \) matrix.
\begin{equation}
A = \begin{pmatrix}
-3 + 3\gamma - 6\gamma x + 6x + 3y & -3 + 3x \\
\frac{-3\alpha y^2}{x^2} - 3\gamma y + 3y & 3\alpha + \frac{6\alpha y}{x} + 3\gamma - 3\gamma x + 3x + 6y
\end{pmatrix}
\end{equation}

The stability of the critical points is determined by the eigenvalues of the matrix $A$ at the critical points. For the point $(1,0)$, the two eigenvalues are

$$
\lambda_1 = 3 - 3\gamma \\
\lambda_2 = 3\alpha
$$

So, it may be stable if $\alpha < 0$ and $\gamma \geq 1$. However, since we want the GCG behaves as dark energy, it won’t be appropriate if $(1,0)$ corresponds to a stable attractor phase. In other words, $\alpha < 0$ should not be considered in the real models. While for the critical point $(1,-1)$, the corresponding eigenvalues of matrix $A$ are

$$
\lambda_1 = -3(1 + \alpha) \\
\lambda_2 = -3\gamma
$$

It is clear that $(1,-1)$ is stable for $\alpha > -1$. This critical point corresponds to a phase that GCG is dominant ($\Omega_g = 1$) and its equation of state is $w = -1$. So, it is a late time de Sitter attractor [44]. Combine the above two constraints as well as the requirements of $\epsilon_s^2 < 1$ at the de Sitter attractor, we can readily reach the constraint for the $\alpha$ parameter should be $0 < \alpha < 1$.

In the following, we study the above dynamical system numerically. For definite, we choose the parameters as $\gamma = 1$ and $\alpha = 0.5$. The initial $x$ and $y$ are chosen as shown in Table I and the results are contained in the following Fig.1-Fig.4. From Fig.2, one can observe that for different initial $\rho_g$ and $p_g$, the equation of state $w_g$ could approach to the de Sitter phase ($w_g = -1$) from either $w_g > -1$ or $w_g < -1$. The critical case will be that the initial $x$ and $y$ are such chosen that $w_g = y/x = -1$. In other words, the initial choice of $w_g > -1$ or $< -1$ determines whether the equation of state of GCG will mimic that of quintessence or phantom. It is worth noting that if we choose the parameters so that the GCG behaves as phantom, it is no longer possible to make it behave as matter at early epoch.
TABLE I: The initial values of $x$ and $y$ in the plots Fig.1-Fig.4

| x  | 0.14 | 0.15 | 0.16 | 0.17 | 0.18 | 0.19 | 0.20 |
|----|------|------|------|------|------|------|------|
| y  | -0.19| -0.18| -0.17| -0.16| -0.15| -0.14| -0.13|

FIG. 1: The phase diagram of the GCG system in terms of $x$ and $y$ for different initial $x$ and $y$. Because the equation of state could not smoothly evolves from 0 to $w_g < -1$. However, in our setup of this paper, we have included a baryotropic fluid that could be used to mimic the matter sector of our universe and thus GCG can be considered only as dark energy.
FIG. 2: The evolution of the equation of state of GCG for different initial $x$ and $y$. The curves from bottom to top correspond to the initial conditions specified in Table I from left to right respectively.

FIG. 3: The evolution of the cosmic density parameter for matter $\Omega_{\gamma}$ and GCG $\Omega_g$ respectively at different initial $x$ and $y$. The plot indicates that the evolution of $\Omega_g$ and $\Omega_{\gamma}$ is not very sensitive to the initial condition due to the attractor property.
3. Conclusion and Discussion

In this letter, we analyze the dynamical evolution of GCG for different parameters and initial conditions. We show that different initial $x$ and $y$ will lead to different tracks ($w_g > -1$ and $w_g < -1$) for the equation of state $w_g$ to approach the de Sitter attractor phase ($w = -1$). That is to say, the GCG could mimic both quintessence and phantom during its evolution depending on the initial conditions. We also give constraint to the parameter of the model as $0 < \alpha < 1$ from the requirement of its dynamics and sound speed.

On the other hand, the existing studies of GCG and its fitting to Supernovae data focus on $w_g > -1$.[42, 45] But in fact, observations do not exclude the possibility of $w_g < -1$. So the GCG with $w_g < -1$ should also be considered and its fitting to SNeIa data will be carried out in a preparing work.

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