We try to obtain Born’s principle as a result of a subquantum heat death, using classical \( \mathcal{H} \)-theorem and the definition of a proper quantum \( \mathcal{H} \)-theorem, within the framework of Bohm’s theory. We shall show the possibility of solving the problem of action-reaction asymmetry present in Bohm’s theory and the arrow of time problem in our procedure.

Key words: Born’s principle, Bohmian mechanics, action-reaction asymmetry, arrow of time, quantum \( \mathcal{H} \)-theorem

1. INTRODUCTION

Bohm’s theory is a casual interpretation of quantum mechanics that was initially introduced by de Broglie\cite{1} and then developed by David Bohm\cite{2}. This theory is claimed to be equivalent to the standard quantum theory without having the conceptual problems of the latter\cite{3}. Yet, there are some difficulties with this theory. In this theory the velocity of a particle is given by

\[
\dot{x} = \frac{\hbar}{m} \text{Im} \left( \frac{\nabla \psi}{\psi} \right) = \frac{\nabla S}{m},
\]

where \( S \) is the phase of the wave function (\( \psi = \text{Re}^{i\frac{\psi}{\hbar}} \)). The wave function \( \psi \) itself is a solution of Schrödinger equation:

\[
i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi + V(x)\psi.
\]

By substituting \( \text{Re}^{i\frac{\psi}{\hbar}} \) for \( \psi \) in (2), we shall have the following equations
\[
\frac{\partial S}{\partial t} + \frac{(\nabla S)^2}{2m} + V(x) - \frac{\hbar^2}{2m} \nabla^2 R = 0.
\]

(3)

\[
\frac{\partial R^2}{\partial t} + \nabla \cdot \left( \frac{\nabla S}{m} R^2 \right) = 0.
\]

(4)

where (3) is the classical Hamilton-Jacobi equation with an additional term, \(Q = -\frac{\hbar^2}{2m} \nabla^2 R\), which is called quantum potential. By differentiating (1) with respect to \(t\) and making use of (3), we obtain

\[m\ddot{x} = -\nabla (Q + V),\]

which shows that the \(\psi\)-wave affects particle’s motion through the \(-\nabla Q\) term. To secure the action-reaction (AR) symmetry, we expect the presence of a term corresponding to particle’s reaction on the wave, in the wave equation of motion (2). This term is not present there\([4,5]\). Some people argue that AR symmetry is a classical principle, and it is not necessarily applicable to quantum mechanics. This is not, in our view, a strong argument. Because, if one follows this line of thought, one might question the applicability of the concept of path (i.e., relation (1)) in Bohmian quantum mechanics. Then, we have to stick to the standard formulation of quantum mechanics.

Another difficulty with Bohm’s theory is Born’s statistical principle. In this theory the field \(\psi\) enters as a guiding field for the motion of particles but at same time it is required by the experimental facts to represent a probability density through \(|\psi|^2\). The question is why the probability for the ensemble of particles has to be equal to \(|\psi|^2\).

One of foundational problems in physics is that of temporal asymmetry or the existence of the arrow of time. The laws of physics do not distinguish between the two directions of time. In spite of this, there is a very obvious difference between the future and past directions of time in our universe. This means that there is an arrow of time, i.e., time flows in one direction. Recently Wootters claimed that: time asymmetry is logically prior to quantum mechanics and that:” the attempt to envision a sub-quantum theory suggests a picture of world in which time-asymmetric events are taken as fundamental. Unitary evolution would appear as a derived concept and would of course not be universal”\([6]\). Our approach exemplifies Wootters’s suggestion.

Our paper is organized as follows: After reviewing Valentini’s quantum \(H\)-theorem in section 2, we introduce our new quantum \(H\)-theorem in section 3 and, finally, we show how the AR problem and the arrow of time problem are solved by our proposal.

2. QUANTUM \(H\)-THEOREM

Recently, A.Valentini has tried to derive the relation \(\rho = |\psi|^2\) as the result of a statistical subquantum \(H\)-theorem\([7]\). He looked for a proper quantum function \(f_N\) that, like the classical N-particle distribution function, satisfies Liouville’s equation. He considered an ensemble of N-body systems which could be described by the wave function \(\Psi\) and the distribution function
Because $|\Psi|^2$ and $P$ must equalize during the assumed heat death, in general $P \neq |\Psi|^2$ and one can write

$$P(X, t) = f_q(X, t) \, |\Psi(X, t)|^2,$$

where $f_q$ measures the ratio of $P$ to $|\Psi|^2$ at the point $X (x_1 \cdots x_N)$, at time $t$. Since both $|\Psi|^2$ and $P$ satisfy the continuity equation ($|\Psi|^2$ due to its being a solution of the Schrödinger equation and $P$ by its very definition) one can easily show that $f_q$ satisfies the following equation

$$\frac{\partial f_q}{\partial t} + \dot{X} \cdot \nabla f_q = 0,$$

where $X$ denotes $x_1 \cdots x_N$ as before. Thus, Valentini defined his quantum $\mathcal{H}$-function in the following way

$$\mathcal{H}_q = \int d^3N X \, |\Psi(X, t)|^2 \, f_q(X, t) \ln f_q(X, t).$$

The only difference with the classical one is that $f_q$ is defined in the configuration space while the classical $f_N$ is defined in the phase space and $d^3N X \, d^3N P \rightarrow |\Psi(X, t)|^2 \, d^3N X$. Valentini used Ehrenfest’s coarse-graining method[8]. He claims that $\frac{d\mathcal{H}_q}{dt} \leq 0$, where $\tilde{\mathcal{H}}_q$ is the coarse-grain $\mathcal{H}$-function and the equality holds in the equilibrium state, where $\tilde{f}_q = 1$ or $\tilde{P} = |\tilde{\Psi}|^2$. Here $\tilde{P}$ and $|\tilde{\Psi}|^2$ are coarse grained forms of $P$ and $|\Psi|^2$ respectively. Valentini termed this process a subquantumic heat death. Then, he showed that if a single particle is extracted from a large system and prepared in a state with a wavefunction $\psi$, its probability density $\rho$ will be equal to $|\psi|^2$, provided that $P = |\Psi|^2$ holds for the large system. Notice that a one-body system not in quantum equilibrium can never relax to quantum equilibrium (when it is left to itself). But any one-body system extracted from a large system can do.

Here, we want to modify Valentini’s procedure so that one can directly obtain Born’s principle for a one-body system. In this case we will be forced to use Boltzmann’s procedure, and that naturally leads to a solution of the AR problem.

### 3. AN ALTERNATIVE QUANTUM $\mathcal{H}$-THEOREM

Consider an ensemble of one-body systems. Suppose that all these systems are in the $\psi$ state and that their distribution function is $\rho$. Furthermore, suppose that at $t = 0$ we have $\rho \neq |\psi|^2$; and define

$$\rho(x, t) = f_q(x, t) \, |\psi(x, t)|^2.$$

In Valentini’s procedure the complexity of systems leads to heat death, but our systems are simple (one-body) ones. Thus, if we want to have heat death, we must assume that $f_q$ satisfies a quantum Boltzmann equation.
\[
\frac{\partial f_q}{\partial t} + \mathbf{\dot{x}} \cdot \nabla f_q = J(f_q),
\]
where \(J(f_q)\) is related to the particle reaction on its associated wave. Since \(\rho\) satisfies a continuity equation, as the result of its definition, thus (5) and (6) imply that \(|\psi|^2\) does not satisfy the continuity equation any more. In fact, the continuity equation for \(|\psi|^2\) is changed to
\[
\frac{\partial |\psi|^2}{\partial t} + \nabla \cdot \left( \frac{\nabla S}{m} |\psi|^2 \right) = - \frac{J(f_q)}{f_q} |\psi|^2.
\]
This means that \(\psi\) is a solution of a nonlinear Schrödinger equation. If we want to have the quantum potential in its regular form, i.e. \((-\frac{\hbar^2}{2m} \nabla^2 \mathcal{R})\), we must choose the nonlinear term in a particular form. The proper selection is
\[
i\hbar \left( \frac{\partial}{\partial t} + g(f_q) \right) \psi = \frac{\hbar^2}{2m} \nabla^2 \psi,
\]
where \(g(f_q)\) is a real function of \(f_q\). With the substitution \(\psi = \text{Re} e^{i\mathcal{S}}\) we shall have
\[
\frac{\partial S}{\partial t} + \frac{(\nabla S)^2}{2m} - \frac{\hbar^2}{2m} \frac{\nabla^2 \mathcal{R}}{R} = 0.
\]
By comparing (10) with (7) we have \(g(f_q) = \frac{1}{2} \frac{J(f_q)}{f_q}\). Now, we have to select \(g(f_q)\) in such a way that it leads to the equality of \(\rho\) and \(|\psi|^2\) (i.e. \(f_q = 1\)). Some requirements for such a function is:

1- It must be invariant under \(t \rightarrow -t\).

2- It must change its sign for \(f_q = 1\).

If we fined systems for which subquantum heat death has not occurred[9], we shall obtain the actual form of the \(g(f_q)\) function. A proper selection is \(g(f_q) = \alpha(1 - f_q)\) where \(\alpha\) is a constant. Then, (10) gives (with \(\alpha = \frac{1}{2}\))
\[
\frac{\partial |\psi|^2}{\partial t} + \nabla \cdot \left( \frac{\nabla S}{m} |\psi|^2 \right) = (f_q - 1) |\psi|^2.
\]
Now, consider the right hand side of (11) as the source of \(|\psi|^2\) field. At any point of space where \(|\psi|^2 < \rho\) (i.e. \(f_q > 1\)), the source of \(|\psi|^2\) is positive and, therefore, \(|\psi|^2\) increases at that point. On the other hand, at any point of space where \(|\psi|^2 > \rho\) (i.e. \(f_q < 1\)), the source of \(|\psi|^2\) is negative and therefore \(|\psi|^2\) decreases at that point. The variation of \(|\psi|^2\) continues until \(|\psi|^2\)
becomes equal to $\rho$. After that, since both $|\psi|^2$ and $\rho$ evolve under the same velocity field $(\nabla S_m)$, they remain equal. To prove $\rho \to |\psi|^2$ exactly, we introduce the following $\mathcal{H}_q$ function:

$$\mathcal{H}_q = \int d^3x (\rho - |\psi|^2) \ln(\frac{\rho}{|\psi|^2}).$$  \hspace{1cm} (12)

Since $(X - Y)\ln(\frac{X}{Y}) \geq 0$ for all $X, Y \geq 0$, we have $\mathcal{H}_q \geq 0$ – the equality being relevant to the case $\rho = |\psi|^2$. If we show that for the forgoing $J(f_q)$ one has $\frac{d\mathcal{H}_q}{dt} \leq 0$, where again the equality is to relevant to $\rho = |\psi|^2$ (i.e. $f_q = 1$) state, we have shown that $|\psi|^2$ becomes equal to $\rho$ finally. We write (12) in the form

$$\mathcal{H}_q = \int d^3x |\psi|^2 (f_q - 1) \ln f_q = \int d^3x |\psi|^2 G(f_q),$$

where $G(f_q) = (f_q - 1) \ln f_q$. Now, we have for $\frac{d\mathcal{H}_q}{dt}$

$$\frac{d\mathcal{H}_q}{dt} = \int d^3x \left\{ -\nabla \cdot (\dot{\mathbf{x}} |\psi|^2 G(f_q)) \right\}$$

$$- J(f_q) \left[ \frac{G(f_q)}{f_q} - \frac{\partial G(f_q)}{\partial f_q} \right] |\psi|^2 \},$$  \hspace{1cm} (13)

where we have done an integration by parts. Passing to the limit of large volumes and dropping the surface term in (13) leads to

$$\frac{d\mathcal{H}_q}{dt} = \int d^3x \frac{J(f_q)}{f_q} \{f_q - 1 + \ln f_q\} |\psi|^2.$$ 

The quantity in $\{\}$, is negative for $f_q < 1$ and positive for $f_q > 1$. Now, since the $J(f_q)$ (i.e. $f_q(1 - f_q)$) is positive for $f_q < 1$ and negative for $f_q > 1$, the integrand is negative or zero for all values of $f_q$ and we have

$$\frac{d\mathcal{H}_q}{dt} \leq 0,$$

where the equality holds for $f_q = 1$. The existence of a quantity that decreases continuously to its minimum value at $\rho = |\psi|^2$ (i.e. $f_q = 1$) guarantees Born’s principle.

4. THE ACTION-REACTION PROBLEM

In the classical gravity, matter fixes space-time geometry and correspondingly matter’s motion is determined by the space-time Thus, the action-reaction symmetry is preserved. In fact, the nonlinearity of Einstein equations is the result of this mutual action-reaction. In the same way, mutual action-reaction between wave and particle in Bohm’s theory, leads to a nonlinear
Schrödinger equation. But nonlinearity does not necessarily mean particle reaction on the wave. Indeed nonlinear terms must contain some information about particle’s position too. We claim that the non-linear term present in (8) arises from particle’s reaction on its associated wave, and it represents information about particle’s position. Considering the statistical nature of $\rho$, the question arises as to why particle’s associated wave is affected by an ensemble of particles. This can be answered in the following ways:

1- It is natural to expect particle’s associated wave to be a function of particle’s position, satisfying a non-linear equation of the following form:

$$\mathcal{L}_{x,y,t} \Psi(x, y, t) = 0,$$

where $y$ represents particle’s position, and the subscript $\Psi$ on the operator $\mathcal{L}$ is an indication of the non-linearity of the equation with respect to $\Psi$. Consider an ensemble of particles with the same $\Psi$, but with different $y$, distributed according to $\rho$. One can look a $\psi(x, t)$, giving the average motion of the ensemble of particles instead of individual motions and satisfying the following equation:

$$\mathcal{L}_{x,\rho,t}\psi(x, t) = 0.$$

This equation is of type (8). The $\psi$ is not a linear combination of functions $\Psi(x, y, t)$. Rather, it is an average field that gives the evolution of the ensemble correctly. Of course, it may not give the correct path of individual particles. Note, $\rho$ in (8) is the reaction of the ensemble of particles on the wave that represents ensemble’s evolution. After quantum heat death and the equality of $\rho$ with $|\psi|^2$, this equation takes the form of the Schrödinger equation:

$$\mathcal{L}_{x,t}\psi(x, t) = 0.$$

This means that the AR problem is a result of the establishment of Born’s principle. In fact this is similar to the argument that Valentini presents[10] to show that the signal-locality (i.e. the absence of practical instantaneous signalling) and the uncertainty principle are valid if and only if $\rho = |\psi|^2$.

2- The information of a wave about particles position could be incomplete and this defect could be a result of the form of the interaction between the two. Thus, $\rho$ does not represent a statistical distribution for an ensemble of particles. Rather, it represents the maximum amount of information of particle’s associated wave about particle. Then, if we have an ensemble of particles having a same $\psi$ and $\rho$ we can take $\rho$ as the distribution function for the ensemble.

5. ARROW OF TIME PROBLEM

The Arrow of time problem can essentially be seen as arising from the different behavior of the time asymmetric evolution of macroscopic irreversible processes and the time symmetric
evolution of the reversible processes governing the underlying dynamics of the atomic constituents. The apparent paradox is a direct consequence of the symmetric nature of the basic laws of physics with respect to time inversion. The nonlinear Schrödinger equation that we introduced is not time reversal, although after the subquantum heat death with the vanishing of the $g(f_q)$ term, it seems time reversal. In other words, the equation (8) for one of the two directions of time leads to the common Schrödinger equation that describes real microscopic world, but for the other direction of time, the nonlinear term $g(f_q)$ grows up and (8) could not describe the real world. Now, if $\psi(x,t)$ is a solution of the common Schrödinger equation, so is $\psi^*(x,-t)$, and both $\psi(x,t)$ and $\psi^*(x,-t)$ are solutions of our nonlinear equation in the subquantum equilibrium state. But in the framework of our theory $\psi(x,t)$ and $\psi^*(x,-t)$ are not time reversal of each other. Nevertheless, we can formally relate them to each other with a time reversal transformation (ignoring the nonlinear term in subquantum equilibrium state).

6. CONCLUSIONS

We have shown that the Born’s principle can be the result of the presence of a nonlinear term in Schrödinger equation, with special characteristics. Then, we have shown that this nonlinear term can be considered as indicator of particle’s reaction on the wave. Besides, our nonlinear equation introduces an arrow of time.

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