Dynamic Co-Quantile Regression

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Abstract

The popular systemic risk measure CoVaR (conditional Value-at-Risk) is widely used in economics and finance. Formally, it is defined as an (extreme) quantile of one variable (e.g., losses in the financial system) conditional on some other variable (e.g., losses in a bank’s shares) being in distress and, hence, measures the spillover of risks. In this article, we propose a dynamic “Co-Quantile Regression”, which jointly models VaR and CoVaR semiparametrically. We propose a two-step M-estimator drawing on recently proposed bivariate scoring functions for the pair (VaR, CoVaR). Among others, this allows for the estimation of joint dynamic forecasting models for (VaR, CoVaR). We prove the asymptotic normality of the proposed estimator and simulations illustrate its good finite-sample properties. We apply our co-quantile regression to correct the statistical inference in the existing literature on CoVaR, and to generate CoVaR forecasts for real financial data, which are shown to be superior to existing methods.

Keywords: CoVaR, Estimation, Forecasting, Modeling, Systemic Risk

JEL classification: C14 (Semiparametric and Nonparametric Methods); C22 (Time-Series Models); C58 (Financial Econometrics); G17 (Financial Forecasting and Simulation); G32 (Financial Risk and Risk Management)

1 Motivation

Since the introduction of the Value-at-Risk (VaR), risk forecasts have become a key input in financial decision making (Jorion, 2006). For instance, VaR and Expected Shortfall (ES) forecasts are now routinely used for setting capital requirements of financial institutions under the Basel framework (Liu and Stentoft, 2021). Consequently, a huge literature on forecasting VaR and ES has emerged (see, e.g., McNeil and Frey, 2000; Engle and Manganelli, 2004; Massacci, 2017; Patton et al., 2019). By definition, these measures are primarily designed to assess the riskiness of banks in isolation.

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Thus, these measures are well-suited to address microprudential objectives in banking regulation, that is, to limit risk taking of individual institutions.

However, in the aftermath of the financial crisis of 2007–09, macroprudential objectives have gained importance on the regulatory agenda (Acharya et al., 2012, 2017). While also attempting to limit risk taking of individual financial institutions, the macroprudential approach also takes into account the commonality of risk exposures among banks. For instance, a measure of interconnectedness of banks is now used under the Basel framework of the Basel Committee on Banking Supervision (2019, SCO40) to determine the global systemically important banks (G-SIBs), which are subjected to higher capital requirements. This new focus calls for systemic risk measures to accurately measure the interlinkages for which VaR and ES are unsuitable. Hence, a plethora of systemic risk measures is available by now (Giesecke and Kim, 2011; Chen et al., 2013; Adrian and Brunnermeier, 2016; Acharya et al., 2017).

One of the most popular systemic risk measures is the conditional VaR (CoVaR) of Adrian and Brunnermeier (2016). It is defined as a quantile of a financial loss (e.g., of an entire market), given a reference asset (e.g., a systemically important bank) is in distress, where the latter is measured as an exceedance of its VaR. Yet, while forecasting models and the asymptotic properties of forecasts are well explored for VaR and ES (Chan et al., 2007; Gao and Song, 2008; Wang and Zhao, 2016; Hoga, 2019), the models for CoVaR forecasting are hitherto rather ad-hoc and little is known about the consistency of the forecasts. In particular, DCC–GARCH models of Engle (2002) are popular due to their ability to accurately forecast conditional variance-covariance matrices (Laurent et al., 2012; Caporin and McAleer, 2014). However, as Francq and Zakoïan (2016, p. 620) point out, “[n]o formally established asymptotic results exist for the full estimation of the DCC […] models”; see also Darolles et al. (2018). This renders their use in forecasting systemic risk measures questionable. Moreover, there are additional problems associated with their use in systemic risk forecasting that stem from the non-uniqueness of the decomposition of the variance-covariance matrix (see Section 2.5). In sum, there is a need for multivariate models that can deliver accurate systemic risk forecasts with strong theoretical underpinnings.

The first main contribution of this paper is to introduce such models for the CoVaR, which we call dynamic co-quantile regression (CoQR) models. In the spirit of Engle and Manganelli (2004) and Patton et al. (2019), we model the pair (VaR, CoVaR) depending on external covariates, past financial losses, and lagged (VaR, CoVaR). Our models are semiparametric in the sense that the quantities of interest (VaR and CoVaR) are modeled parametrically, yet no assumptions are placed on the remaining conditional distribution. Among others, our general model class nests so-called dynamic CoCAViaR models, extending the classical CAViaR models of Engle and Manganelli (2004) to joint (VaR, CoVaR) models. It also covers predictive co-quantile regressions in the spirit of Adrian and Brunnermeier (2016).
There are new challenges in modeling systemic risk vis-à-vis modeling univariate quantities such as VaR (Engle and Manganelli, 2004) and ES (Patton et al., 2019). Modeling VaR or (VaR, ES) requires to specify the univariate dynamics only. Yet, since CoVaR measures the interlinkage of bivariate random variables, it becomes necessary to model the co-movements as well. We explore different models for doing so, and identify several competitive performers in our empirical application.

Our second main contribution is to propose an estimator for the model parameters and derive its large sample properties. The main technical hurdle to overcome in developing asymptotic theory is that—unlike VaR and (VaR, ES)—the pair (VaR, CoVaR) fails to be elicitable, such that no real-valued scoring function exists that is uniquely minimized by the true report (Fissler and Hoga, 2021). This renders standard M-estimation, adopted by Engle and Manganelli (2004), Dimitriadis and Bayer (2019) and Patton et al. (2019), infeasible (Dimitriadis et al., 2021, Theorem 2.5). Instead, we exploit the multi-objective elicitation of (VaR, CoVaR) (Fissler and Hoga, 2021). This property suggests a two-step M-estimator. In the first step, the score in the VaR component is minimized, and then the CoVaR score is minimized in the second step. We show consistency and asymptotic normality of our proposed two-step M-estimator and propose valid inference based on consistent estimation of the asymptotic variance-covariance matrix. Our proofs show how to deal with non-smooth objective functions in the context of two-step M-estimation for dynamic models (based on past model values). We speculate that our proof strategy may also be used elsewhere and, therefore, may be of independent interest. We provide the open source R package CoQR (Dimitriadis and Hoga, 2022) that contains the implementation of our two-step M-estimator together with inference tools for the flexible class of dynamic CoQR models.

We stress that our theory can be used to carry out valid inference in predictive co-quantile regressions, i.e., predictive regressions for (VaR, CoVaR). Hitherto, researchers—such as Brunnermeier et al. (2020)—have mainly built on Adrian and Brunnermeier (2016), who propose a two-step quantile regression (QR) approach to estimate CoVaR regressions. However, Adrian and Brunnermeier (2016) did not provide asymptotic theory for their estimators, relying instead on a naive approach, where the second stage standard errors ignore the first step estimation. Drawing on our theoretical results, we can show that such naive two-step regressions lead to invalid inference. Yet, applying our results in a predictive CoQR framework leads to valid inference. We view this as one of the most important implications of our asymptotic normality result. Our CoQR theory may also be seen as an extension of classical QR, due to Koenker and Bassett (1978). Thus, our work also contributes to the strand of the literature that aims to extend the scope of QR, such as unconditional QR (Firpo et al., 2009), high-dimensional QR (Belloni and Chernozhukov, 2011) or vector QR (Carlier et al., 2016).

In simulations, we confirm the good finite-sample properties of our estimator. We do so for
both predictive CoQR models and dynamic CoCAViaR time series models. For the former class of models, the confidence intervals for the parameters derived from our theory tend to be more accurate than those for the latter. While correct inference is essential in predictive CoQR models as illustrated in Section 4.1, the main use of our CoCAViaR models is forecasting, such that inference for the model parameters is of lesser importance.

Similarly as the simulations, our empirical application also considers both predictive CoQR and dynamic CoCAViaR time series models. Our first application revisits the predictive CoVaR regression of Adrian and Brunnermeier (2016, Section III (C–D)). By virtue of our asymptotic theory, valid standard errors can now be computed to assess the statistical significance of several macroeconomic and financial variables in predicting systemic risk. For instance, the TED spread as a measure of liquidity risk is significant, as is the realized volatility of the S&P 500. However, the return of the S&P 500 does not appear to be a significant predictor of systemic risk in the US. We speculate that our CoQR methodology will be useful in many more contexts than the assessment of systemic risk. For instance, in economics, Adrian et al. (2019, 2021) have recently drawn attention to tail risks and their interconnections by popularizing the Growth-at-Risk, which is simply the VaR of GDP growth. Hence, the CoQR approach developed in this paper could be relevant for studying interconnections of macroeconomic risks.

Our second empirical application compares CoVaR forecasts issued from our CoCAViaR models with those from benchmark DCC–GARCH models. It does so for the four most systemically risky US banks according to Financial Stability Board (2021), whose impact on a broader market index is assessed by CoVaR. Our various CoCAViaR model specifications tend to outperform the DCC–GARCH benchmarks, particularly in the CoVaR forecasts but also in the VaR component. Often, these differences in predictive ability are also statistically significant, as judged by the Diebold and Mariano (1995)-type comparative backtest of Fissler and Hoga (2021). The superiority of our proposals may be explained by two reasons. First, our CoCAViaR specifications are specifically tailored to model the quantities of interest (VaR and CoVaR), whereas multivariate GARCH processes focus on modeling the complete predictive distribution. Second, our estimation technique is tailored to provide an accurate description of the (VaR, CoVaR) evolution and, hence, is not too strongly influenced by center-of-the-distribution observations.

The rest of the paper is structured as follows. Section 2 formally introduces CoVaR, our modeling framework and the appertaining parameter estimator. It also gives large sample results for our estimator, and points out the difficulties of DCC–GARCH models to predict CoVaR. We illustrate the finite-sample properties of our estimator in Section 3. Section 4 presents two empirical applications. The final Section 5 concludes. All proofs are relegated to the online Supplementary Material.
2 Modeling and Estimation

2.1 The CoVaR

Throughout the paper, we consider a sample of size \( n \in \mathbb{N} \) of the bivariate series \( \{(X_t, Y_t')\}_{t \in \mathbb{N}} \). Specifically, \( Y_t \) stands for the log-losses of interest (e.g., system-wide losses in the financial system) and \( X_t \) are the log-losses of some reference position (e.g., the losses of a bank’s shares). Here, log-losses are simply the negated log-returns. Let \( \mathcal{F}_t \) denote some time-\( t \) information set to be specified below. Define \( \text{VaR}_{t,\beta} = \text{VaR}_\beta(F_{X_t|\mathcal{F}_{t-1}}) \), where \( \text{VaR}_\beta(F) = F^{-1}(\beta) \) denotes the \( \beta \)-quantile of the (absolutely continuous) distribution \( F \) and \( F_{X_t|\mathcal{F}_{t-1}} \) denotes the conditional distribution of \( X_t \) given \( \mathcal{F}_{t-1} \). The stress event that is considered in the definition of the CoVaR is that the loss of the reference position exceeds its VaR, i.e., \( \{X_t \geq \text{VaR}_{t,\beta}\} \). With our orientation of \( X_t \) denoting financial losses, we commonly consider values for \( \beta \) close to one. We define \( \text{CoVaR}_{t,\alpha|\beta} = \text{CoVaR}_{\alpha|\beta}(F(X_t,Y_t')|\mathcal{F}_{t-1}) \), where

\[
\text{CoVaR}_{\alpha|\beta}(F_{X,Y}) = \text{VaR}_\alpha(F_{Y|X \geq \text{VaR}_\beta(F_X)})
\]

for a joint distribution function \( F_{X,Y} \) with marginals \( F_X \) and \( F_Y \). For \( \beta = 0 \), we simply have \( \text{CoVaR}_{t,\alpha|\beta} = \text{VaR}_\alpha(F_{Y|\mathcal{F}_{t-1}}) \), and if \( \beta = \alpha \) we simply write \( \text{CoVaR}_{t,\alpha} = \text{CoVaR}_{t,\alpha|\alpha} \). Note that our definition of CoVaR deviates from the original one of Adrian and Brunnermeier (2016), who used \( \{X_t = \text{VaR}_{t,\beta}\} \) as the stress event. This latter choice is problematic because it often has probability zero and it does not fully incorporate all tail events of \( X_t \). Thus, we follow Girardi and Tolga Ergün (2013) and Nolde and Zhang (2020) in using the above alternative definition.

2.2 Dynamic Models for the VaR and CoVaR

In the spirit of Engle and Manganelli (2004) and Patton et al. (2019), we consider semiparametric models for the pair (VaR, CoVaR) of the general form

\[
\begin{pmatrix}
\nu_t(\theta^v) \\
\sigma_t(\theta^c)
\end{pmatrix} = \begin{pmatrix}
\nu_t((X_{t-1}, Y_{t-1})', Z_{t-1}, \ldots, (X_1, Y_1)', Z_1; \theta^v) \\
\sigma_t((X_{t-1}, Y_{t-1})', Z_{t-1}, \ldots, (X_1, Y_1)', Z_1; \theta^c)
\end{pmatrix}, \quad t \in \mathbb{N},
\]

(1)

where \( \nu_1(\theta^v) \) and \( \sigma_1(\theta^c) \) are initial values, \( Z_{t-1} \) is some (possibly multivariate) exogenous covariate vector and \( \mathcal{F}_{t-1} = \sigma((X_{t-1}, Y_{t-1})', Z_{t-1}, \ldots, (X_1, Y_1)', Z_1) \). Throughout the paper, we assume that the model in (1) is correctly specified in the sense that there exist true parameter values \( \theta^v_0 \in \Theta^v \subset \mathbb{R}^p \) and \( \theta^c_0 \in \Theta^c \subset \mathbb{R}^q \) such that

\[
\begin{pmatrix}
\text{VaR}_{t,\beta} \\
\text{CoVaR}_{t,\alpha|\beta}
\end{pmatrix} = \begin{pmatrix}
\nu_t(\theta^v_0) \\
\sigma_t(\theta^c_0)
\end{pmatrix}
\]

(2)

almost surely (a.s.).

Our model is semiparametric in the sense that—while the model dynamics are governed by
the parameters $\theta^v$ and $\theta^c$—we impose no additional distributional assumption on the innovations driving the evolution of $(X_t, Y_t)'$. As the model in (1) can depend on the full history of lagged values of $X_t$ and $Y_t$, our setup allows for the inclusion of lagged (VaR, CoVaR); see Example 1 below for details. We further mention that $Z_{t-1}$ denote (often macroeconomic or financial) state variables that are predictive for systemic risk. The inclusion of such variables seems to be empirically relevant (Adrian and Brunnermeier, 2016; Brunnermeier et al., 2020). However, formal tests of their statistical significance are not available so far in the literature, due to a lack of asymptotic theory. It is one of the main aims of this paper to provide such a theory and, hence, to enable formal hypothesis testing. Importantly, we consider here the case of separated parameters, where the parameters are exclusive to the VaR and CoVaR model. This is vital for our two-step M-estimator introduced below.

**Example 1.** The general formulation of our model class in (1) allows for dynamic models, where (VaR, CoVaR) may depend on their lags. E.g., it nests models of the form

$$
\left(
\begin{array}{c}
v_t(\theta^v) \\
c_t(\theta^c)
\end{array}
\right) = \omega + A \left(\begin{array}{c} X_{t-1} \\ |Y_{t-1}|
\end{array}\right) + B \left(\begin{array}{c} v_{t-1}(\theta^v) \\ c_{t-1}(\theta^c)
\end{array}\right)
$$

(3)

with parameters $\omega \in \mathbb{R}^2$, $A, B \in \mathbb{R}^{2 \times 2}$ that can be collected in $\theta^v$ and $\theta^c$. Borrowing terminology from Engle and Manganelli (2004), we call models of the form (3) *Symmetric Absolute Value (SAV) CoCAViaR* models. We later restrict the upper-right element of $B$ to be zero (i.e., $B_{12} = 0$) to facilitate two-step M-estimation.

The models of Example 1 are correctly specified under a type of Jeantheau’s (1998) extended constant conditional correlation (ECCC) process, that has been studied extensively in the literature (He and Teräsvirta, 2004; Conrad and Karanasos, 2010):

$$
\left(\begin{array}{c}
X_t \\
Y_t
\end{array}\right) = \left(\begin{array}{c}
\sigma_{X,t} \varepsilon_{X,t} \\
\sigma_{Y,t} \varepsilon_{Y,t}
\end{array}\right), \quad \left(\begin{array}{c}
\sigma_{X,t} \\
\sigma_{Y,t}
\end{array}\right) = \tilde{\omega} + \tilde{A} \left(\begin{array}{c} X_{t-1} \\ |Y_{t-1}|
\end{array}\right) + \tilde{B} \left(\begin{array}{c} \sigma_{X,t-1} \\ \sigma_{Y,t-1}
\end{array}\right),
$$

(4)

where $\tilde{\omega} \in \mathbb{R}^2$, $\tilde{A}, \tilde{B} \in \mathbb{R}^{2 \times 2}$. In (4), the $F_{t-1}$-measurable conditional volatilities $\sigma_{X,t}$ and $\sigma_{Y,t}$ are independent of the independent, identically distributed (i.i.d.) innovations $(\varepsilon_{X,t}, \varepsilon_{Y,t})'$, i.i.d. $F(0, \Sigma)$, where $F$ denotes a generic absolutely continuous, bivariate distribution with zero mean and covariance matrix $\Sigma = \left(\begin{array}{cc} 1 & \rho \\ \rho & 1 \end{array}\right)$, $\rho \in (-1, 1)$. We deviate from Jeantheau (1998) by using absolute (instead of squared) returns as drivers of volatility, since these have more predictive content for volatility (Forsberg and Ghysels, 2007). Model (4) implies the (VaR, CoVaR) dynamics in (3), where the true model parameters $\omega_0, A_0, B_0$ of the model in (3) arise as transformations of $\tilde{\omega}, \tilde{A}, \tilde{B}$.\(^1\)

\(^1\)Multiplying the rows of the volatility equation in (4) by the true VaR and CoVaR of the innovations respectively gives that $\theta^v_0 = (\omega_0, A_{11,0}, A_{12,0}, B_{11,0}, B_{12,0})'$ = $(v_0 \tilde{v}_1, c_0 \tilde{c}_1, \tilde{\omega}1, \tilde{\omega}2, \tilde{c}_1, \tilde{c}_2, \tilde{B}_{11}, \tilde{B}_{12}, \tilde{B}_{21}, \tilde{B}_{22})'$ and $\theta^c_0 = (\omega_2, A_{21,0}, A_{22,0}, B_{21,0}, B_{22,0})'$ = $(c_0 \varepsilon_{X,t}, \varepsilon_{Y,t})'$, where $v_0$ is the $\alpha$-VaR of $\varepsilon_{X,t}$ and $c_0$ the $\alpha$-$\beta$-CoVaR of the pair $(\varepsilon_{X,t}, \varepsilon_{Y,t})'$, whose analytical form can be obtained from Mainik and Schaanning (2014,
In our simulations in Section 3, we use (a restricted version of) model (4) to generate data to assess how well our two-step M-estimator performs. In our empirical application, we consider various SAV-CoCAViaR model candidates based on zero restrictions in $A$ and $B$ as well as generalizations to so-called “asymmetric slope” models based on the positive and negative components of $X_t$ and $Y_t$ instead of on their absolute values.

The model in (1) also generalizes classical (time-series) quantile regressions based on lagged values of $X_t$, $Y_t$ and the covariate vector $Z_t$. Purely cross-sectional (co-)quantile regressions can be obtained by letting $t$ index cross-sectional units, and by shifting the index of the $Z$ variables. Finally, the “VAR for VaR” models (of dimension two) of White et al. (2015) can be seen to be nested by choosing $\beta = 0$ such that the CoVaR simply becomes the VaR of $Y_t$.

### 2.3 Parameter Estimation

In the following, we introduce estimators of the unknown parameters $\theta_0^u$ and $\theta_0^c$. As we consider M-estimation in this paper, we require a to-be-minimized objective (or also: scoring) function. However, as pointed out in the Motivation, there is no real-valued scoring function associated with the pair (VaR, CoVaR). Dimitriadis et al. (2021, Theorem 2.5) show that the existence of such a (strictly consistent) scoring function is a necessary condition for consistent M-estimation of semiparametric models. To overcome this drawback, Fissler and Hoga (2021) propose to consider a $\mathbb{R}^2$-valued scoring function in the closely related context of forecast evaluation. To be able to compare forecasts, $\mathbb{R}^2$ has to be equipped with an order, and Fissler and Hoga (2021) show that the lexicographic order is suitable for that purpose. Specifically, they show that (under some regularity conditions) the expectation of the $\mathbb{R}^2$-valued scoring function

$$ S\left( \left( \begin{array}{c} \nu \\ c \end{array} \right), \left( \begin{array}{c} X \\ Y \end{array} \right) \right) = \left( S_{\text{VaR}}(\nu, X), S_{\text{CoVaR}}(c, Y) \right) = \left( \begin{array}{c} \mathbb{I}\{x \leq \nu\} - \beta[v - \nu] \\ \mathbb{I}\{c \leq \nu\} - \alpha[c - \nu] \end{array} \right) $$

is minimized by the true VaR and CoVaR with respect to the lexicographic order. That is, for all $\nu, c \in \mathbb{R}$,

$$ \mathbb{E} \left[ S\left( \left( \begin{array}{c} \text{VaR}_\beta(F_X) \\ \text{CoVaR}_{\alpha\beta}(F_{X,Y}) \end{array} \right), \left( \begin{array}{c} X \\ Y \end{array} \right) \right) \right] \preceq_{\text{lex}} \mathbb{E} \left[ S\left( \left( \begin{array}{c} \nu \\ c \end{array} \right), \left( \begin{array}{c} X \\ Y \end{array} \right) \right) \right], $$

where $(x_1, x_2) \preceq_{\text{lex}} (y_1, y_2)$ if $x_1 < y_1$ or ($x_1 = y_1$ and $x_2 \leq y_2$). Note that $S_{\text{VaR}}(\cdot, \cdot)$ in (5) is the standard tick-loss function known from quantile regression. Clearly, $S_{\text{CoVaR}}(\cdot, \cdot)$ is similar in structure to $S_{\text{VaR}}(\cdot, \cdot)$, with the only difference being the indicator $\mathbb{I}\{x > v\}$ that restricts the evaluation to observations with VaR exceedances in the first component. In the related literature, these scoring functions are often also called loss functions (Gneiting, 2011). To avoid confusion

Theorem 3.1 (b)).
with financial "losses", we adhere to the term *scoring function*.

The definition of the lexicographic order suggests the following two-step M-estimator of $\theta^v_0$ and $\theta^c_0$. In the first step, we use

$$\hat{\theta}^v_n = \arg\min_{\theta^v \in \Theta^v} \frac{1}{n} \sum_{t=1}^{n} S^{\text{VaR}}(v_t(\theta^v), X_t),$$

that is, our estimate minimizes the average empirical score in the first component for some VaR parameter space $\Theta^v \subset \mathbb{R}^p$. For the VaR model $v_t(\cdot)$, this is the quantile regression estimator of Engle and Manganelli (2004).

With the estimate $\hat{\theta}^v_n$ at hand, the lexicographic order then suggests to minimize the average empirical score in the second component via

$$\hat{\theta}^c_n = \arg\min_{\theta^c \in \Theta^c} \frac{1}{n} \sum_{t=1}^{n} S^{\text{CoVaR}}(c_t(\hat{\theta}^v_n), (X_t, Y_t)'),$$

for some CoVaR parameter space $\Theta^c \subset \mathbb{R}^q$. For this two-step estimator to be feasible, the requirement that the VaR evolution does not depend on the CoVaR model is essential. Section 2.4.2 shows that the presence of $\hat{\theta}^v_n$ impacts the asymptotic variance of $\hat{\theta}^c_n$. Of course, this is usually the case for two-step estimators (Newey and McFadden, 1994, Section 6).

2.4 Asymptotic Properties of the Estimators

2.4.1 Consistency

We first show consistency of our two-step M-estimators $\hat{\theta}^v_n$ and $\hat{\theta}^c_n$. To do so, we introduce several regularity conditions in Assumption 1, where $K < \infty$ is some large positive constant, and $\|x\|$ denotes the Euclidean norm when $x$ is a vector, and the Frobenius norm when $x$ is matrix-valued.

The joint cumulative distribution function (c.d.f.) of $(X_t, Y_t)'$ $\mid \mathcal{F}_{t-1}$ is denoted by $F_t(\cdot, \cdot)$, and its Lebesgue density (which we assume exists) by $f_t(\cdot, \cdot)$. Similarly, $F^{W}(\cdot) (f^{W}(\cdot))$ denotes the distribution (density) function of $W_t \mid \mathcal{F}_{t-1}$ for $W \in \{X, Y\}$. For sufficiently smooth functions $\mathbb{R}^p \ni \theta \mapsto f(\theta) \in \mathbb{R}$, we denote the $(p \times 1)$-gradient by $\nabla f(\theta)$, its transpose by $\nabla' f(\theta)$ and the $(p \times p)$-Hessian by $\nabla^2 f(\theta)$.

**Assumption 1.**

(i). The models $v_t(\cdot)$ and $c_t(\cdot)$ are correctly specified in the sense of (2). Furthermore, $\mathbb{P}\{v_t(\theta^v) = v_t(\theta^v_0)\} = 1$ for all $t \in \mathbb{N}$ implies that $\theta^v = \theta^v_0$, and $\mathbb{P}\{c_t(\theta^c) = c_t(\theta^c_0)\} = 1$ for all $t \in \mathbb{N}$ implies that $\theta^c = \theta^c_0$.

(ii). $\{(X_t, Y_t, Z_t')\}_{t \in \mathbb{N}}$ is strictly stationary.

(iii). The model $v_t(\cdot)$ does not depend on $c_{t-1}(\cdot), c_{t-2}(\cdot), \ldots$. 


(iv). $F_t(\cdot, \cdot)$ belongs to a class of distributions on $\mathbb{R}^2$ that is continuous and possesses finite first moments (componentwise). Moreover, $f_t(\cdot, \cdot)$ satisfies $\sup_{x \in \mathbb{R}} \int_{-\infty}^{\infty} |y| f_t(x, y) \, dy \leq K$ and $f_t^X(\cdot) \leq K$.

(v). $\Theta = \Theta^v \times \Theta^c$ is compact, where $\Theta^v \subset \mathbb{R}^p$ and $\Theta^c \subset \mathbb{R}^q$.

(vi). $\{ S\left((v_t(\theta^v), c_t(\theta^c))^\prime, (X_t, Y_t)^\prime \right) \}_{t \in \mathbb{N}} \text{ obeys the uniform law of large numbers (ULLN in } (\theta^v, \theta^c)' \in \Theta.$

(vii). For all $t \in \mathbb{N}$, $v_t(\theta^v)$ and $c_t(\theta^c)$ are $\mathcal{F}_{t-1}$-measurable and a.s. continuous in $\theta^v$ and $\theta^c$.

(viii). For all $t \in \mathbb{N}$, $v_t(\theta^v)$ is continuously differentiable on int$(\theta^v)$.

(ix). There exists a neighborhood of $\theta^v_0$, such that $\| \nabla v_t(\theta^v) \| \leq V_1(\mathcal{F}_{t-1})$ for all elements $\theta^v$ of that neighborhood. Furthermore, $|v_t(\theta^v)| \leq V(\mathcal{F}_{t-1})$ for all $\theta^v \in \Theta^v$, and $|c_t(\theta^c)| \leq C(\mathcal{F}_{t-1})$ for all $\theta^c \in \Theta^c$.

(x). $E[V(\mathcal{F}_{t-1})] \leq K$, $E[V_1(\mathcal{F}_{t-1})] \leq K$, $E[V_1(\mathcal{F}_{t-1})C(\mathcal{F}_{t-1})] \leq K$.

In Assumption 1, item (i) ensures identification of the true parameters and item (ii) is a standard stationarity condition. Item (iii) renders our two-step estimation strategy feasible by ensuring that the CoVaR dynamics (which are only estimated in the second step) do not interfere with consistent estimation of the VaR model in the first step. Item (iv) ensures—among other things—strict (multi-objective) consistency of the scoring function given in (5); see Fissler and Hoga (2021, Theorem 4.2). Compactness in (v) is a standard requirement in extremum estimation; see Newey and McFadden (1994). Assumption 1 (vi) is also a standard condition, imposed for instance by Engle and Manganelli (2004, C6) and Patton et al. (2019, Assumption 1 (A)). It can be verified using, e.g., Lemma 2.8 in Newey and McFadden (1994) or Theorem 21.9 in Davidson (1994). The final items (vii)–(x) are smoothness conditions on the model.

Our first main theoretical result establishes the consistency of $\hat{\theta}_n^v$ and $\hat{\theta}_n^c$:

**Theorem 1.** Suppose Assumption 1 holds. Then, as $n \to \infty$, $\hat{\theta}_n^v \xrightarrow{P} \theta^v_0$ and $\hat{\theta}_n^c \xrightarrow{P} \theta^c_0$.

The proof of Theorem 1 proceeds by verifying the conditions of Theorem 2.1 in Newey and McFadden (1994), and can be found in Appendix A. The result that $\hat{\theta}_n^v \xrightarrow{P} \theta^v_0$ is essentially a version of Theorem 1 in Engle and Manganelli (2004); the only difference being that our regularity conditions are slightly more involved, since we also show that $\hat{\theta}_n^c \xrightarrow{P} \theta^c_0$.

### 2.4.2 Asymptotic Normality

To show asymptotic normality of our estimators, we have to impose additional regularity conditions on the model. We denote the partial derivative of $F_t(\cdot, \cdot)$ with respect to the $i$-th argument by $\partial_i F_t(\cdot, \cdot)$ for $i = 1, 2$. 

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[Caption: This page contains a detailed explanation of the statistical and econometric concepts related to model estimation and asymptotic normality. It includes conditions for the consistency of estimators and a theorem that establishes the convergence of these estimators to their true values.]
Assumption 2.

(i). \( \theta_0^w \in \text{int}(\Theta^w) \) and \( \theta_0^c \in \text{int}(\Theta^c) \), where \( \text{int}(\cdot) \) denotes the interior of a set.

(ii). \( v_t(\cdot) \) and \( c_t(\cdot) \) are a.s. twice continuously differentiable on \( \text{int}(\Theta^w) \) and \( \text{int}(\Theta^c) \), respectively.

(iii). There exists a neighborhood of \( \theta_0^w \), such that \( \|\nabla^2 v_t(\theta^w)\| \leq V_2(F_{t-1}), \|\nabla^2 v_t(\theta^w) - \nabla^2 v_t(\tau^w)\| \leq V_3(F_{t-1})\|\theta^w - \tau^w\| \) for all elements \( \theta^w, \tau^w \) of that neighborhood.

(iv). There exists a neighborhood of \( \theta_0^c \), such that \( \|\nabla c_t(\theta^c)\| \leq C_1(F_{t-1}), \|\nabla^2 c_t(\theta^c)\| \leq C_2(F_{t-1}), \|\nabla^2 c_t(\theta^c) - \nabla^2 c_t(\tau^c)\| \leq C_3(F_{t-1})\|\theta^c - \tau^c\| \) for all elements \( \theta^c, \tau^c \) of that neighborhood.

(v). There exists some \( \epsilon > 0 \), such that \( \mathbb{E}(V_{3+1}^w(F_{t-1})) \leq K, \mathbb{E}(V_{2(3+1)/2}^w(F_{t-1})) \leq K, \mathbb{E}(V_3(F_{t-1})) \leq K, \mathbb{E}(C_{2(3+1)/2}^w(F_{t-1})) \leq K, \mathbb{E}(C_3(F_{t-1})) \leq K \).

(vi). \( |f_t^X(x) - f_t^X(x')| \leq K|x - x'|, \ |f_t^Y(y) - f_t^Y(y')| \leq K|y - y'|, \ |\partial_1 F_t(x, y) - \partial_1 F_t(x', y)| \leq K|x - x'|, \ |\partial_2 F_t(x, y) - \partial_2 F_t(x, y')| \leq K|y - y'| \) for all \( x, x', y, y' \) in the support of \( F_t(\cdot, \cdot) \).

(vii). \( f_t^Y(\cdot) \leq K, f_t(\cdot, \cdot) \leq K, \ |\partial_i F_t(\cdot, \cdot)| \leq K \text{ for } i = 1, 2 \).

(viii). The matrices

\[
\Lambda = \mathbb{E} \left[ f_t^X(v_t(\theta_0^w)) \nabla v_t(\theta_0^w) \right] \in \mathbb{R}^{p \times p}, \\
\Lambda_{(1)} = \mathbb{E} \left\{ \nabla c_t(\theta_0^c) \nabla' c_t(\theta_0^c) f_t^Y(c_t(\theta_0^c)) \right\} \in \mathbb{R}^{q \times q}, \\
V = \beta(1 - \beta) \mathbb{E} \left\{ \nabla v_t(\theta_0^w) \nabla' v_t(\theta_0^w) \right\} \in \mathbb{R}^{p \times p}, \\
C^w = \alpha(1 - \alpha)(1 - \beta) \mathbb{E} \left\{ \nabla c_t(\theta_0^c) \nabla' c_t(\theta_0^c) \right\} \in \mathbb{R}^{q \times q}
\]

are positive definite, and

\[
\Lambda_{(2)} = \mathbb{E} \left\{ \nabla c_t(\theta_0^c) \nabla' v_t(\theta_0^w) \right\} \alpha f_t^X(v_t(\theta_0^w)) \right\} \in \mathbb{R}^{q \times p}
\]

has full rank.

(ix). \( \sup_{\theta^w \in \Theta^w} \sum_{t=1}^n 1\{X_t = v_t(\theta^w)\} = O(1) \) and \( \sup_{\theta^c \in \Theta^c} \sum_{t=1}^n 1\{Y_t = c_t(\theta^c)\} = O(1) \) a.s., as \( n \to \infty \).

(x). \( \left\{ (X_t, Y_t, Z_t, v_t(\theta_0^w), \nabla' v_t(\theta_0^w), c_t(\theta_0^c), \nabla' c_t(\theta_0^c))' \right\}_{t \in \mathbb{N}} \) is \( \alpha \)-mixing with mixing coefficients \( \alpha(\cdot) \) satisfying \( \sum_{m=1}^{\infty} \alpha((\hat{q} - 2)/\hat{q}(m)) \) for some \( \hat{q} > 2 \).

Item (i) of Assumption 2 is essential for asymptotic normality of extremum estimators, where examples of non-normal estimators can easily be constructed when the true parameter is on the boundary (Newey and McFadden, 1994, p. 2144). Items (ii)–(v) are again smoothness conditions on the model that are similar in spirit to those of Engle and Manganelli (2004) and Patton et al. (2019). Item (vi) is a (uniform) Lipschitz condition on derivatives of the joint conditional c.d.f. and item (vii)
provides convenient bounds on the densities and partial derivatives. The positive definiteness of $\Lambda$ and $\Lambda(1)$ in item (viii) is a sufficient condition for the existence of a unique minimum of the expected population loss with respect to the lexicographic order. Item (ix) bounds the number of exact equalities of financial losses and (VaR, CoVaR) model values; see also Patton et al. (2019). Finally, item (x) is a standard mixing condition that ensures validity of a suitable central limit theorem.

**Theorem 2.** Suppose Assumptions 1 and 2 hold. Then, as $n \to \infty$, 

$$\sqrt{n}(\hat{\theta}_v^n - \theta_v^0) \xrightarrow{d} N(0, \Lambda^{-1}VA^{-1}),$$

$$\sqrt{n}(\hat{\theta}_c^n - \theta_c^0) \xrightarrow{d} N(0, \Gamma C \Gamma'),$$

where $\Gamma = \begin{pmatrix} A_{(1)}^{-1}A_{(2)}A^{-1} & -A_{(1)}^{-1} \\ -A_{(1)}^{-1} & \Lambda_{(2)}^{-1} \Lambda_{(1)}^{-1} \end{pmatrix} \in \mathbb{R}^{q \times (p+q)}$ and $C = \begin{pmatrix} V & 0 \\ 0 & C^* \end{pmatrix} \in \mathbb{R}^{(p+q) \times (p+q)}$.

The proof of Theorem 2 is in Appendix B. Similarly as in Engle and Manganelli (2004) and Patton et al. (2019), the key step in the proof is to apply Lemma A.1 of Weiss (1991). However, since we consider a two-step estimator, our arguments necessarily extend those of the aforementioned authors, who only consider one-step estimators. Specifically, in showing asymptotic normality of $\hat{\theta}_c^n$ one has to take into account the fact that it depends on the first-step estimate $\hat{\theta}_v^n$. This complicates the technical treatment and, in fact, increases the asymptotic variance relative to the case where $\theta_v^0$ is known. In more detail, our proof shows that when the true value of $\theta_v^0$ were known in the second step, the asymptotic variance of $\hat{\theta}_c^n$ would be given by $A_{(1)}^{-1}C^*A_{(1)}^{-1}$. Comparing this with the asymptotic variance $\Gamma C \Gamma'$ from Theorem 2 demonstrates that, as expected, the first-stage estimation has an asymptotic effect on $\hat{\theta}_c^n$. More precisely, the first-stage estimation increases the asymptotic variance (in terms of the Loewner order), because

$$\Gamma C \Gamma' - A_{(1)}^{-1}C^*A_{(1)}^{-1} = A_{(1)}^{-1}A_{(2)}A^{-1}V(A_{(1)}^{-1}A_{(2)}A^{-1})'$$

is positive semi-definite. This is akin to two-step GMM-estimation, where uncorrelated first- and second-step moment conditions imply an increased variance of the second-step estimator; see Newey and McFadden (1994, Eq. (6.9)). In our context of two-step M-estimation, the (a.s.) derivatives of the two components of the scoring functions in (5) play the role of these moment conditions. This uncorrelatedness is the reason for the off-diagonal zero blocks in the matrix $C$. See in particular the proof of Theorem 2 in Appendix B.2.

**Remark 1.** It is easy to see from the proof of Theorem 2 that joint normality of the parameter estimators also holds. Specifically, as $n \to \infty$,

$$\sqrt{n}\left(\hat{\theta}_v^n - \theta_v^0, \hat{\theta}_c^n - \theta_c^0\right) \xrightarrow{d} N\left(0, \bar{\Gamma} C \bar{\Gamma}'\right),$$

where $\bar{\Gamma} = \begin{pmatrix} -A_{(1)}^{-1} & 0 \\ A_{(1)}^{-1}A_{(2)}A^{-1} & -A_{(1)}^{-1} \end{pmatrix}$.
This result may be useful when testing joint restrictions on the parameters, such as the significance of some (macroeconomic or finance) variables in explaining VaR and CoVaR dynamics.

2.4.3 Consistent Estimation of the Asymptotic Variance

To draw inference on the model parameters, we require consistent estimates of the asymptotic variances appearing in the limiting distributions of Theorem 2. For the VaR parameters, these are well-explored. For instance, Engle and Manganelli (2004) and Patton et al. (2019) estimate $\mathbf{V}$ and $\mathbf{A}$ via

$$
\hat{V}_n = \frac{1}{n} \sum_{t=1}^{n} \beta (1 - \beta) \nabla v_t(\hat{\theta}^v_n) \nabla v^*_t(\hat{\theta}^v_n),
$$

$$
\hat{A}_n = \frac{1}{n} \sum_{t=1}^{n} (2\hat{b}_{n,x})^{-1} \{ |X_t - v_t(\hat{\theta}^v_n)| < \hat{b}_{n,x} \} \nabla v_t(\hat{\theta}^v_n) \nabla' v_t(\hat{\theta}^v_n),
$$

respectively, where $\hat{b}_{n,x} = o_p(1)$ is a (possibly stochastic) bandwidth. As we are mainly interested in the CoVaR parameters, we have to estimate $\mathbf{F}$ and $\mathbf{C}$. For the latter matrix, we only need to estimate $\mathbf{C}^*$ via

$$
\hat{C}^*_n = \frac{1}{n} \sum_{t=1}^{n} \alpha (1 - \alpha) (1 - \beta) \nabla c_t(\hat{\theta}^c_n) \nabla' c_t(\hat{\theta}^c_n).
$$

The estimators for $\mathbf{A}_{(1)}$ and $\mathbf{A}_{(2)}$ are more involved and are given by

$$
\hat{A}_{n,(1)} = \frac{1}{n} \sum_{t=1}^{n} \nabla c_t(\hat{\theta}^c_n) \nabla'^* c_t(\hat{\theta}^c_n) (2\hat{b}_{n,y})^{-1} \left[ \mathbb{I} \left\{ |Y_t - c_t(\hat{\theta}^c_n)| < \hat{b}_{n,y} \right\} - \mathbb{I} \left\{ |X_t - v_t(\hat{\theta}^v_n), |Y_t - c_t(\hat{\theta}^c_n)| < \hat{b}_{n,y} \right\} \right],
$$

$$
\hat{A}_{n,(2)} = \frac{1}{n} \sum_{t=1}^{n} \nabla c_t(\hat{\theta}^c_n) \nabla' v_t(\hat{\theta}^v_n) (2\hat{b}_{n,x})^{-1} \left[ \alpha \mathbb{I} \left\{ |X_t - v_t(\hat{\theta}^v_n)| < \hat{b}_{n,x} \right\} - \mathbb{I} \left\{ |X_t - v_t(\hat{\theta}^v_n)| < \hat{b}_{n,x}, Y_t \leq c_t(\hat{\theta}^c_n) \right\} \right],
$$

where $\hat{b}_{n,y} = o_p(1)$ is another (possibly stochastic) bandwidth. Footnote 2 reports the data-dependent bandwidth choices we use in the empirical parts of the paper. These estimators rely on kernel density estimates of the derivatives of the conditional c.d.f., and use a rectangular kernel. However, at the expense of some additional technicality other kernels may also be considered.

Assumption 3.

(i) It holds for $\hat{b}_{n,z}$ (z $\in \{x, y\}$) that $\hat{b}_{n,z}/b_{n,z} \xrightarrow{p} 1$, where the non-stochastic $b_{n,z} > 0$ satisfies $b_{n,z} = o(1)$ and $b_{n,z}^{-1} = o(n^{1/2})$, as $n \to \infty$.

(ii) \( \frac{1}{n} \sum_{t=1}^{n} f_t^X (v_t(\theta^v_0)) \nabla v_t(\theta^v_0) \nabla^v v_t(\theta^v_0) - \mathbf{A} \xrightarrow{p} \mathbf{0}. \)

(iii) \( \frac{1}{n} \sum_{t=1}^{n} \nabla c_t(\theta^c_0) \nabla' c_t(\theta^c_0) \left[ f_t^X (c_t(\theta^c_0)) - \partial_2 F_t(v_t(\theta^v_0), c_t(\theta^c_0)) \right] - \mathbf{A}_{(1)} \xrightarrow{p} \mathbf{0}. \)

(iv) \( \frac{1}{n} \sum_{t=1}^{n} \nabla c_t(\theta^c_0) \nabla' v_t(\theta^v_0) \left[ \alpha f_t^X (v_t(\theta^v_0)) - \partial_1 F_t(v_t(\theta^v_0), c_t(\theta^c_0)) \right] - \mathbf{A}_{(2)} \xrightarrow{p} \mathbf{0}. \)

(v) \( \mathbb{E}[C^4_t(F_{t-1})] \leq K, \mathbb{E}[V^4_t(F_{t-1})] \leq K. \)
Theorem 3, which is proven in Appendix C, shows consistency of our asymptotic variance estimators.

**Theorem 3.** Suppose Assumptions 1–3 hold. Then, as \( n \to \infty \), \( \hat{V}_n \overset{p}{\to} V, \hat{C}_n^* \overset{p}{\to} C^* \), \( \hat{\Lambda}_n \overset{p}{\to} \Lambda \), \( \hat{\Lambda}_{n,(1)} \overset{p}{\to} \Lambda_{(1)} \) and \( \hat{\Lambda}_{n,(2)} \overset{p}{\to} \Lambda_{(2)} \).

### 2.5 CoVaR Forecasting with Multivariate GARCH Models

This section points out the difficulties of multivariate volatility models in computing CoVaR. By doing so, it sheds some light on the advantages of our modeling approach.

**Example 2.** Suppose that financial losses are generated by \((X_t, Y_t)' = \Sigma_t \varepsilon_t\) for some positive-definite, \(\mathcal{F}_{t-1}\)-measurable \(\Sigma_t\) and \(\varepsilon_t \overset{i.i.d.}{\sim} F(0, I)\), independent of \(\mathcal{F}_{t-1}\), where \(F\) is some generic distribution. Then, the conditional variance-covariance matrix is \(\text{Var}\left((X_t, Y_t)' \mid \mathcal{F}_{t-1}\right) = \Sigma_t \Sigma_t' =: H_t\).

Much like univariate GARCH processes model the conditional variance, multivariate GARCH models—such as the DCC–GARCH of Engle (2002) and the corrected DCC–GARCH of Aielli (2013)—directly model the conditional variance-covariance matrix \(H_t\). While for many applications—e.g., portfolio construction as in Hautsch et al. (2015)—\(H_t\) is the object of interest, CoVaR forecasting explicitly requires \(\Sigma_t\) to be specified. See Appendix D for details on how CoVaR forecasts can be extracted from a DCC–GARCH model.

However, for a given \(H_t\), there exist infinitely many choices of \(\Sigma_t\) satisfying the decomposition \(\Sigma_t \Sigma_t' = H_t\), such as the symmetric square root implied by the eigenvalue decomposition \((\Sigma_t^*)\) or the lower triangular matrix of the Cholesky decomposition \((\Sigma_t^l)\). The problem is that each possibility, while implying the same variance-covariance dynamics \(H_t\), implies different values for the CoVaR. E.g., consider

\[
H_t = \begin{pmatrix} 10 & 6 \\ 6 & 10 \end{pmatrix}, \quad \Sigma_t^* = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}, \quad \Sigma_t^l = \begin{pmatrix} 3.16 & 0 \\ 1.89 & 2.52 \end{pmatrix},
\]

and \(\varepsilon_t = (\varepsilon_{t,X}, \varepsilon_{t,Y})'\) with standardized \(t_5\)-distributed \(\varepsilon_{t,X}\) and \(\varepsilon_{t,Y}\) that are independent of each other. Then, for the model \(\Sigma_t^s \varepsilon_t (\Sigma_t^l \varepsilon_t)\) based on the symmetric (lower triangular) decomposition, we obtain CoVaR\(_{t,0.95}\) = 9.76... (CoVaR\(_{t,0.95}\) = 9.10...). Thus, CoVaR forecasts depend intimately on the decomposition of \(H_t\), on which typical multivariate GARCH models stay silent. Thus, in principle, any GARCH-type model for \(H_t\) may be consistent with an infinite number of CoVaR forecasts (depending on \(\Sigma_t\)), thus imposing an unsatisfactory ambiguity when applied to CoVaR forecasting.

We mention that the issue raised in Example 2 has not kept researchers from using standard multivariate volatility models for systemic risk forecasting. For instance, Girardi and Tolga Ergün (2013) and Nolde and Zhang (2020) use Engle’s (2002) DCC–GARCH model for CoVaR forecasting.
Girardi and Tolga Ergün (2013) leave open which $\Sigma_t$ they use (symmetric, lower triangular etc.). Nolde and Zhang (2020) explicitly use a lower triangular $\Sigma_t$. Specifically, Nolde and Zhang (2020) model the loss of the broader market $X_t$ and that of some individual unit $Y_t$, i.e., the opposite scenario as we consider in our empirical applications. They do so using the structural model
\[
\begin{pmatrix}
X_t \\
Y_t
\end{pmatrix} = \begin{pmatrix}
\sigma_{X,t} & 0 \\
\sigma_{Y,t}\rho_{XY,t} & \sigma_{Y,t}\sqrt{1 - \rho_{XY,t}^2}
\end{pmatrix} \begin{pmatrix}
\varepsilon_{X,t} \\
\varepsilon_{Y,t}
\end{pmatrix} = \begin{pmatrix}
\sigma_{X,t}\varepsilon_{X,t} \\
\sigma_{Y,t}\rho_{XY,t}\varepsilon_{X,t} + \sigma_{Y,t}\sqrt{1 - \rho_{XY,t}^2}\varepsilon_{Y,t}
\end{pmatrix}
\]
based on the lower-triangular matrix from the Cholesky decomposition of the variance-covariance matrix $H_t = \left( \begin{array}{cc}
\sigma_{X,t}^2 & \sigma_{X,t}\sigma_{Y,t}\rho_{XY,t} \\
\sigma_{X,t}\sigma_{Y,t}\rho_{XY,t} & \sigma_{Y,t}^2
\end{array} \right)$. In such a setting it may be acceptable to assume that the individual shock $\varepsilon_{Y,t}$ does not impact the market losses, as the individual institution is very small relative to the market. However, when comparing systemic risk contributions across a handful of business units of comparable size (e.g., different trading desks), such an assumption may be untenable, and a different modeling approach may be needed. Yet, which one exactly may be difficult to say. Our approach sidesteps these difficulties by directly modeling CoVaR (and VaR).

3 Simulations

Replication material for the simulations and the applications in Section 4 is available on Github under https://github.com/TimoDimi/replication_CoQR. It draws on the corresponding open source package CoQR (Dimitriadis and Hoga, 2022) implemented in R (R Core Team, 2022).

3.1 CoQR Estimation

We first consider a time series regression as a data-generating process (DGP). For $t = 1, \ldots, n$, the covariates $Z_t = (Z_{1,t}, Z_{2,t})'$ follow the autoregressive structure
\[
Z_{1,t} = 0.5 + 0.3\exp(W_t) \quad \text{with} \quad W_t = 0.5W_{t-1} + \varepsilon_{W,t},
\]
\[
Z_{2,t} = 0.8Z_{2,t-1} + \varepsilon_{Z,t},
\]
where $\varepsilon_{Z,t}, \varepsilon_{W,t} \overset{i.i.d.}{\sim} N(0,1)$, independently of each other. We simulate from the heteroskedastic process
\[
(X_t, Y_t)' = (\gamma_1, \gamma_1)' + (\gamma_2, \gamma_2)'Z_{1,t} + (\gamma_3, \gamma_3)'Z_{2,t} + (\gamma_4 + \gamma_5 Z_{1,t})V_t, \quad t = 1, \ldots, n.
\] (6)
The multivariate $t$-distributed innovations $V_t = (V_{1,t}, V_{2,t})' \overset{i.i.d.}{\sim} t_8(0, \Sigma = (1 1))$ (such that $\text{Corr}(V_{1,t}, V_{2,t}) = 0.5$) are independent of the $\{\varepsilon_{Z,t}\}$ and $\{\varepsilon_{W,t}\}$. For simplicity, our DGP in (6) implies that $X_t$ and $Y_t$ are driven by the same factors. Only the (heteroskedastic) shock differs
between the variables. The DGP results in the linear (VaR, CoVaR) model

\[ v_t(\theta^v) = \theta^v_1 + \theta^v_2 Z_{1t} + \theta^v_3 Z_{2t} \quad \text{and} \quad c_t(\theta^c) = \theta^c_1 + \theta^c_2 Z_{1t} + \theta^c_3 Z_{2t}. \]

The true parameter values \( \theta^v_0 = (\theta^v_{1,0}, \theta^v_{2,0}, \theta^v_{3,0})' \) and \( \theta^c_0 = (\theta^c_{1,0}, \theta^c_{2,0}, \theta^c_{3,0})' \) are given by

\[ \theta^v_0 = (\gamma_1 + \gamma_4 q(\beta), \gamma_2 + \gamma_5 q(\beta), \gamma_3)' \quad \text{and} \quad \theta^c_0 = (\gamma_1 + \gamma_4 c(\alpha|\beta), \gamma_2 + \gamma_5 c(\alpha|\beta), \gamma_3)', \]

where \( c(\alpha|\beta) \) is the \( \alpha|\beta \)-CoVaR of the distribution \( t_8(0, \Sigma) \), and \( q(\beta) \) is the \( \beta \)-quantile of its first component. We choose \( \gamma = (\gamma_1, \ldots, \gamma_5)' = (1, 1.5, 2, 0.25, 0.5)' \) and estimate a correctly specified six-parameter (VaR, CoVaR) regression model based on an intercept and the covariates \( Z_{1,t} \) and \( Z_{2,t} \).

Table 1 shows simulation results for the estimated model parameters and their associated standard deviations. It is based on \( M = 5000 \) Monte Carlo replications of the DGP in (6) for the choices \( \alpha = \beta \in \{0.9, 0.95\} \) and sample sizes of \( n \in \{500, 1000, 2000, 4000\} \). The asymptotic variance-covariance matrices are estimated as detailed in Section 2.4.3. Our (stochastic) bandwidth choices follow Koenker (2005) and Machado and Santos Silva (2013). A formal description of the table columns is given in the table caption.

We find that all six parameters are estimated consistently as the (average and median) biases and the standard deviations are decreasing with the sample size. The biases are smaller for the VaR than for the CoVaR parameters. This is unsurprising as the CoVaR is roughly the \( \alpha \)-quantile of the approximately \((1 - \beta) n\) observations which are beyond the estimated VaR, such that the effective sample size is reduced. Furthermore, the estimation is more accurate for smaller (more central) probability levels \( \alpha, \beta \). The standard errors obtained by estimating the asymptotic variance-covariance approximate the empirical counterpart, substantiating numerical evidence for Theorem 3. This also results in confidence interval coverage rates close to the nominal level of 95%. Again, we see that the results in terms of coverage are more accurate for the VaR parameters and for more central probability levels.
| VaR   | $\theta_1^\nu$   | $\theta_2^\nu$   | $\theta_3^\nu$   |
|-------|------------------|------------------|------------------|
| $\alpha, \beta$ | n | Bias | M Bias | $\hat{\sigma}_{\text{emp}}$ | $\hat{\sigma}_{\text{asy}}$ | CI | Bias | M Bias | $\hat{\sigma}_{\text{emp}}$ | $\hat{\sigma}_{\text{asy}}$ | CI | Bias | M Bias | $\hat{\sigma}_{\text{emp}}$ | $\hat{\sigma}_{\text{asy}}$ | CI |
| 0.90  | 500 | 0.0463 | 0.0380 | 0.925 | 0.871 | 0.93 | -0.0287 | -0.0334 | 0.604 | 0.570 | 0.92 | -0.0005 | 0.0000 | 0.177 | 0.169 | 0.93 |
|       | 1000 | 0.0076 | -0.0068 | 0.650 | 0.625 | 0.94 | -0.0067 | -0.0042 | 0.428 | 0.411 | 0.93 | 0.0016 | 0.0009 | 0.125 | 0.121 | 0.94 |
|       | 2000 | 0.0124 | 0.0038 | 0.463 | 0.451 | 0.94 | -0.0069 | -0.0056 | 0.304 | 0.297 | 0.93 | -0.0010 | -0.0011 | 0.090 | 0.086 | 0.93 |
|       | 4000 | -0.0014 | -0.0025 | 0.318 | 0.319 | 0.95 | 0.0017 | 0.0016 | 0.209 | 0.209 | 0.95 | -0.0007 | -0.0008 | 0.062 | 0.061 | 0.95 |
| 0.95  | 500 | 0.1214 | 0.1052 | 1.283 | 1.138 | 0.91 | -0.0692 | -0.0913 | 0.835 | 0.735 | 0.89 | -0.0022 | -0.0091 | 0.250 | 0.221 | 0.91 |
|       | 1000 | 0.0565 | 0.0447 | 0.900 | 0.829 | 0.91 | -0.0335 | -0.0442 | 0.592 | 0.544 | 0.90 | 0.0036 | 0.0019 | 0.171 | 0.160 | 0.93 |
|       | 2000 | 0.0281 | 0.0226 | 0.645 | 0.605 | 0.92 | -0.0137 | -0.0161 | 0.423 | 0.395 | 0.92 | 0.0023 | 0.0018 | 0.120 | 0.117 | 0.93 |
|       | 4000 | 0.0308 | 0.0297 | 0.441 | 0.435 | 0.94 | -0.0186 | -0.0222 | 0.291 | 0.285 | 0.93 | -0.0012 | -0.0016 | 0.086 | 0.083 | 0.94 |

| CoVaR | $\theta_1^\nu$   | $\theta_2^\nu$   | $\theta_3^\nu$   |
|-------|------------------|------------------|------------------|
| $\alpha, \beta$ | n | Bias | M Bias | $\hat{\sigma}_{\text{emp}}$ | $\hat{\sigma}_{\text{asy}}$ | CI | Bias | M Bias | $\hat{\sigma}_{\text{emp}}$ | $\hat{\sigma}_{\text{asy}}$ | CI | Bias | M Bias | $\hat{\sigma}_{\text{emp}}$ | $\hat{\sigma}_{\text{asy}}$ | CI |
| 0.90  | 500 | 0.2525 | 0.1761 | 1.824 | 1.600 | 0.92 | -0.1550 | -0.1974 | 1.163 | 0.992 | 0.88 | 0.0041 | 0.0009 | 0.336 | 0.304 | 0.94 |
|       | 1000 | 0.1405 | 0.1015 | 1.193 | 1.084 | 0.92 | -0.0827 | -0.0989 | 0.769 | 0.698 | 0.90 | -0.0035 | -0.0032 | 0.238 | 0.207 | 0.92 |
|       | 2000 | 0.1086 | 0.0714 | 0.870 | 0.767 | 0.92 | -0.0679 | -0.0619 | 0.562 | 0.499 | 0.90 | -0.0027 | 0.0008 | 0.163 | 0.148 | 0.93 |
|       | 4000 | 0.0281 | 0.0101 | 0.610 | 0.550 | 0.92 | -0.0162 | -0.0220 | 0.394 | 0.363 | 0.92 | 0.0004 | 0.0000 | 0.115 | 0.106 | 0.93 |
| 0.95  | 500 | 1.0392 | 0.7198 | 3.786 | 2.892 | 0.90 | -0.7368 | -0.7989 | 2.317 | 1.698 | 0.80 | -0.0164 | -0.0208 | 0.754 | 0.530 | 0.90 |
|       | 1000 | 0.5559 | 0.4383 | 2.675 | 2.025 | 0.88 | -0.3299 | -0.4009 | 1.705 | 1.237 | 0.82 | 0.0045 | 0.0092 | 0.516 | 0.378 | 0.89 |
|       | 2000 | 0.2931 | 0.1922 | 1.839 | 1.444 | 0.88 | -0.1668 | -0.2099 | 1.174 | 0.919 | 0.84 | 0.0005 | 0.0024 | 0.344 | 0.274 | 0.89 |
|       | 4000 | 0.1546 | 0.1134 | 1.258 | 1.062 | 0.89 | -0.0908 | -0.1077 | 0.820 | 0.687 | 0.87 | 0.0026 | 0.0029 | 0.245 | 0.203 | 0.89 |

Table 1: Simulation results for the six parameter CoQR based on the DGP in (6) and $M = 5000$ simulation replications. The columns “Bias” show the average bias and the columns “M Bias” the median bias of the parameter estimates. The columns “$\hat{\sigma}_{\text{emp}}$” report the empirical standard deviation of the parameter estimates, “$\hat{\sigma}_{\text{asy}}$” the median of the estimated standard deviations, and the columns “CI” show the coverage rates of 95%-confidence intervals.
| VaR | $\omega_1$ | $A_{11}$ | $B_{11}$ |
|-----|-----------|----------|----------|
| $\alpha,\beta$ | $n$ | Bias | M Bias | $\hat{\sigma}_{\text{emp}}$ | $\hat{\sigma}_{\text{asy}}$ | CI | Bias | M Bias | $\hat{\sigma}_{\text{emp}}$ | $\hat{\sigma}_{\text{asy}}$ | CI |
| 0.90 | 500 | 0.0258 | 0.0029 | 0.080 | 0.128 | 0.96 | 0.0132 | -0.0004 | 0.109 | 0.143 | 0.95 | -0.0828 | -0.0143 | 0.256 | 0.378 | 0.95 |
| | 1000 | 0.0126 | 0.0008 | 0.050 | 0.094 | 0.98 | 0.0079 | -0.0004 | 0.080 | 0.100 | 0.94 | -0.0412 | -0.0089 | 0.167 | 0.278 | 0.97 |
| | 2000 | 0.0063 | 0.0004 | 0.031 | 0.067 | 0.99 | 0.0049 | 0.0017 | 0.055 | 0.072 | 0.96 | -0.0208 | -0.0049 | 0.107 | 0.200 | 0.98 |
| | 4000 | 0.0029 | 0.0004 | 0.019 | 0.047 | 1.00 | 0.0030 | 0.0009 | 0.039 | 0.051 | 0.96 | -0.0101 | -0.0042 | 0.067 | 0.140 | 0.99 |
| 0.95 | 500 | 0.0355 | 0.0044 | 0.106 | 0.179 | 0.96 | 0.0217 | 0.0026 | 0.148 | 0.195 | 0.94 | -0.0861 | -0.0217 | 0.253 | 0.396 | 0.94 |
| | 1000 | 0.0200 | 0.0022 | 0.074 | 0.130 | 0.98 | 0.0118 | -0.0004 | 0.108 | 0.137 | 0.93 | -0.0479 | -0.0099 | 0.181 | 0.286 | 0.96 |
| | 2000 | 0.0086 | 0.0004 | 0.041 | 0.092 | 0.99 | 0.0050 | 0.0002 | 0.075 | 0.098 | 0.95 | -0.0207 | -0.0044 | 0.106 | 0.205 | 0.98 |
| | 4000 | 0.0040 | 0.0001 | 0.026 | 0.065 | 1.00 | 0.0030 | -0.0004 | 0.055 | 0.069 | 0.95 | -0.0099 | -0.0019 | 0.070 | 0.145 | 0.99 |

| CoVaR | $\omega_2$ | $A_{22}$ | $B_{22}$ |
|------|-----------|----------|----------|
| $\alpha,\beta$ | $n$ | Bias | M Bias | $\hat{\sigma}_{\text{emp}}$ | $\hat{\sigma}_{\text{asy}}$ | CI | Bias | M Bias | $\hat{\sigma}_{\text{emp}}$ | $\hat{\sigma}_{\text{asy}}$ | CI |
| 0.90 | 500 | 0.1766 | 0.0644 | 0.241 | 0.254 | 0.82 | 0.0573 | 0.0388 | 0.459 | 0.520 | 0.89 | -0.5344 | -0.2664 | 0.660 | 0.727 | 0.79 |
| | 1000 | 0.1326 | 0.0255 | 0.218 | 0.206 | 0.85 | 0.0567 | 0.0289 | 0.301 | 0.351 | 0.90 | -0.4001 | -0.1266 | 0.606 | 0.598 | 0.82 |
| | 2000 | 0.0896 | 0.0096 | 0.183 | 0.159 | 0.89 | 0.0425 | 0.0225 | 0.206 | 0.246 | 0.91 | -0.2708 | -0.0563 | 0.512 | 0.463 | 0.86 |
| | 4000 | 0.0417 | 0.0006 | 0.124 | 0.122 | 0.93 | 0.0285 | 0.0114 | 0.137 | 0.176 | 0.93 | -0.1292 | -0.0140 | 0.360 | 0.359 | 0.92 |
| 0.95 | 500 | 0.2651 | 0.1578 | 0.337 | 0.383 | 0.82 | 0.0809 | 0.0384 | 0.958 | 0.982 | 0.87 | -0.6146 | -0.4355 | 0.666 | 0.833 | 0.78 |
| | 1000 | 0.2403 | 0.0956 | 0.330 | 0.330 | 0.82 | 0.0925 | 0.0371 | 0.661 | 0.689 | 0.86 | -0.5325 | -0.2868 | 0.650 | 0.709 | 0.79 |
| | 2000 | 0.1942 | 0.0400 | 0.307 | 0.274 | 0.83 | 0.0769 | 0.0400 | 0.454 | 0.505 | 0.87 | -0.4271 | -0.1475 | 0.615 | 0.586 | 0.81 |
| | 4000 | 0.1380 | 0.0165 | 0.265 | 0.228 | 0.86 | 0.0635 | 0.0235 | 0.306 | 0.361 | 0.89 | -0.3042 | -0.0666 | 0.545 | 0.480 | 0.84 |

Table 2: Simulation results for the six parameter CoCAViaR model based on the CCC–GARCH model in (4) and $M = 5000$ simulation replications. The columns “Bias” show the average bias and the columns “M Bias” the median bias of the parameter estimates. The columns “$\hat{\sigma}_{\text{emp}}$” report the empirical standard deviation of the parameter estimates, “$\hat{\sigma}_{\text{asy}}$” the median of the estimated standard deviations, and the columns “CI” show the coverage rates of 95%-confidence intervals.
3.2 Estimation of CoCAViaR Models

Here, we consider estimation of a dynamic SAV CoCAViaR model given in (3). For this, we simulate \(\{(X_t, Y_t)^\prime\}_{t=1,...,n}\) from the absolute value ECCC model in (4). We choose the parameter values \(\hat{\omega} = (0.04, 0.02)^\prime\), \(\hat{A} = \begin{pmatrix} 0.1 & 0 \\ 0 & 0.15 \end{pmatrix}\), \(\hat{B} = \begin{pmatrix} 0.8 & 0 \\ 0 & 0.75 \end{pmatrix}\) and let \(F\) be the multivariate (marginally standardized) \(t\)-distribution with \(\nu = 8\) degrees of freedom and a residual correlation of \(\rho = 0.5\). The off-diagonal zero-restrictions in \(\tilde{A}\) and \(\tilde{B}\) result in (more or less) the classic CCC–GARCH model of Bollerslev (1990). Notice that \(\hat{B}_{12} = 0\) is essential for our two-step M-estimator. We estimate the SAV-diag CoCAViaR model, which arises for diagonal \(A\) and \(B\) in (3). (Table 4 below provides a complete nomenclature of CoCAViaR models considered in this paper.) Therefore, the to-be-estimated parameters are \(\theta^v = (\omega_1, A_{11}, B_{11})^\prime\) and \(\theta^c = (\omega_2, A_{22}, B_{22})^\prime\), whose true values can be obtained as in footnote 1.

Table 2 shows simulation results for the dynamic CoCAViaR model based on \(M = 5000\) replications in the same format as Table 1. The columns reporting the (average and median) bias and the standard deviations confirm the consistency of the estimator. In general, the VaR parameters have a smaller empirical bias than the CoVaR parameters, which is not surprising given that CoVaR is further out in the tail and, hence, subject to larger estimation uncertainty. Furthermore, the average bias is often larger than the median bias indicating that the empirical distributions of the parameter estimates are still subject to some skewness or outliers. Notice that even for the largest sample size of \(n = 4000\) for our choice of \(\alpha = \beta = 0.95\), the CoVaR model is essentially estimated as a 95%-quantile based on an effective sample size of only \(\tilde{n} = (1 - \beta)n = 200\) observations, which is an inherently difficult task. We further see that sample sizes of around 2000 days are required to reliably estimate the models for these extreme levels. This is especially true for the CoVaR parameters.

The results on the estimated standard deviations and the confidence interval coverage rates show that asymptotic variance-covariance estimation is a very difficult task for (Co)CAViaR models. The empirical standard errors are somewhat overestimated for the VaR parameters in Table 2, whereas they are interestingly more accurate for the CoVaR parameters. While the confidence intervals for the VaR parameters are rather conservative, the ones for CoVaR display some undercoverage for...
the extreme probability levels of $\alpha = \beta = 0.95$, and exhibit almost correct coverage for $\alpha = \beta = 0.9$. While this shows the need for future research on improving the estimation accuracy of the asymptotic variance-covariance matrix for dynamic (Co)CAViaR models, their main purpose lies in prediction and, hence, inference is less important than for the CoQR example of Section 3.1.

### 4 Empirical Applications

#### 4.1 Predictive CoVaR Regression

In their section III (C–D), Adrian and Brunnermeier (2016) investigate the predictive content of seven macroeconomic and financial variables for systemic risk. We revisit their application for a subset of five of these variables. We leave out the real estate sector return and the change in the three-month Treasury bill rate, mainly to decrease the number of parameters to estimate. Another reason for excluding the real estate returns is that the financial crisis of 2007–09 originated from the real estate sector (Mishkin, 2011). Of course, knowing from which source a financial crisis will originate, necessarily entails high predictive content of a variable related to that source. Therefore, including real estate sector returns may be seen as introducing hind-sight bias in the analysis.

The main differences of our analysis to theirs are as follows. First, we consider daily data instead of monthly data. We do so to have a reasonably large sample size for estimating the CoQR. Second, we use the generalized definition of CoVaR from Section 2.1. Third, instead of considering the whole universe of publicly traded US financial institutions, for illustration purposes we only focus on JPMorgan Chase as the G-SIB that is deemed most systemically risky by the Financial Stability Board (2021). Fourth, we consider a different time frame in our analysis to also cover the COVID-19 crash of March 2020.

The quantities of interest are $\text{VaR}_t$, $\beta$ and $\text{CoVaR}_{t, \alpha | \beta}$ defined in terms of the following variables. The $Y_t$ denote log-losses on the S&P 500 Financials and the $X_t$ denote log-losses on JPMorgan Chase shares, such that $\text{CoVaR}_{t, \alpha | \beta}$ measures the risk in the financial system conditional on JPMorgan

| Covariate                          | VaR Estimate | VaR Std. Err. | VaR p-value | CoVaR Estimate | CoVaR Std. Err. | CoVaR p-value |
|------------------------------------|--------------|---------------|-------------|----------------|-----------------|---------------|
| (Intercept)                        | 0.0126       | 0.0031        | 0.000       | 0.0053         | 0.0076          | 0.482         |
| Spread                             | 0.0025       | 0.0008        | 0.001       | 0.0023         | 0.0027          | 0.378         |
| TED Spread                         | 0.0266       | 0.0065        | 0.000       | 0.0631         | 0.0093          | 0.000         |
| Change Credit Spread               | -0.0311      | 0.0275        | 0.257       | 0.1923         | 0.0924          | 0.037         |
| S&P 500 Return                     | -0.1507      | 0.0693        | 0.030       | -0.1372        | 0.4911          | 0.780         |
| S&P 500 RV                         | 0.7151       | 0.1959        | 0.000       | 3.0606         | 1.1472          | 0.008         |

Table 3: Predictive (VaR, CoVaR) regression results for $Y_t$ being the log-losses of the S&P 500 Financials and $X_t$ the log-losses of JPMorgan Chase. The probability levels are $\alpha = \beta = 0.95$. 

...
Chase being in distress. We fit the linear model

\[
\begin{pmatrix}
v_t(\theta^v) \\
c_t(\theta^c)
\end{pmatrix} = \begin{pmatrix}
Z_{t-1}^v \\
Z_{t-1}^c
\end{pmatrix},
\]

where \( \theta^v, \theta^c \in \mathbb{R}^6 \) and the covariate vector \( Z_{t-1} = (1, Z_{t-1,1}, \ldots, Z_{t-1,5})' \) contains an intercept, the spread between 10-year Treasury bills and 3-month Treasury bills (Spread), the spread between the 3-month LIBOR and 3-month Treasury bills (TED Spread), the change in spread between Moody’s Baa-rated bonds and 10-year Treasury bill rate (Change Credit Spread), the market return of the S&P 500 (S&P 500 Return), and the 5-minute realized volatility of the S&P 500 as a measure of equity volatility (S&P 500 RV). All data are taken from the financial data provider Refinitiv, except for the S&P 500 RV, which is downloaded from the “realized library” of Heber et al. (2009). We let \( \alpha = \beta = 0.95 \) and estimate the model based on a sample from January 5, 2000 until December 31, 2021, yielding a total of \( n = 5418 \) trading days.

Table 3 displays the CoQR results for the above model. Four out of the five estimated slope coefficients for the VaR model, and three out of five for the CoVaR model are statistically significant at the 5%-level. As pointed out above, the standard error computations for the CoVaR parameters correct for the fact that the VaR parameters are estimated. Hence, our theory allows to draw statistically sound conclusions on the drivers of systemic risk.

### 4.2 CoVaR Forecasting with CoCAViaR Models

For the empirical forecasting application, we use daily close-to-close log-losses from January 4, 2000 until December 31, 2021 with a total of \( n = 5535 \) trading days, obtained from the financial data provider Refinitiv. We use data for the S&P 500 index as our \( Y_t \), and for \( X_t \) the S&P 500 Financials (SPF) that represents the financial sector of the S&P 500, Bank of America (BAC), Citigroup (C), Goldman Sachs (GS) and JPMorgan Chase (JPM). The latter institutions are the four systemically most important US banks according to the Financial Stability Board (2021). With these variables, \( \text{CoVaR}_{t,\alpha|\beta} \) measures the spillover risk of the financial system to the overall economy. We focus on the probability levels \( \alpha = \beta = 0.95 \) and estimate all models using a rolling window with estimation samples of length 3000 days. To reduce the computational burden, we only re-estimate the models every 100 days.

We consider 12 forecasting models in total. The first three candidate models are from the general class of SAV CoCAViaR models given in (3). The acronym SAV indicates that the driving forces of the models are the absolute values of the log-losses, \( |X_{t-1}| \) and \( |Y_{t-1}| \). The top panel of Table 4 summarizes which covariates are included in each of the employed SAV CoCAViaR model specifications. The suffix “diag” in the first model indicates that the off-diagonal elements of the parameter matrices \( A \) and \( B \) are set to zero. Similarly, “full” indicates that the full specification
Table 4: In this table the symbol • indicates which covariates are used in the six CoCAViaR specifications employed in the empirical application. All models additionally contain intercept terms.

Table 5 reports the estimated parameters together with their standard errors for the six CoCAViaR model specifications. The results are based on the initial estimation window of 3000 observations. The autoregressive coefficients are between 0.8 and 0.97, and highly significant. The other coefficients are all barely significant, but their direction is very reasonable. The “cross-terms” of (3) is used (only restricting $B_{12} = 0$ to facilitate two-step M-estimation) and the suffix “fullA” indicates that the full matrix $A$ is considered, while $B$ is restricted to be a diagonal matrix.

Along the lines of Engle and Manganelli (2004), we extend the CoCAViaR model class by using signed values of $X_t$ and $Y_t$ to the Asymmetric Slope (AS) CoCAViaR models,

$$
\begin{pmatrix}
  v_t(\theta^v) \\
  c_t(\theta^c)
\end{pmatrix} = \omega + A^+ \begin{pmatrix} X_{t-1}^+ \\ Y_{t-1}^+ \end{pmatrix} + A^- \begin{pmatrix} X_{t-1}^- \\ Y_{t-1}^- \end{pmatrix} + B \begin{pmatrix} v_{t-1}(\theta^v) \\ c_{t-1}(\theta^c) \end{pmatrix},
$$

(7)

where $\omega \in \mathbb{R}^2$, $A^+, A^-, B \in \mathbb{R}^{2 \times 2}$ are collected in the parameter vectors $\theta^v$ and $\theta^c$. Here, we define $x^+ = \max(x, 0)$ and $x^- = -\min(x, 0)$ for $x \in \mathbb{R}$. Parameter equalities in $A^+$ and $A^-$ can be used to generate absolute values $|X_{t-1}|$ and $|Y_{t-1}|$ in (7). Intuitively, the positive values of $X_{t-1}$ and $Y_{t-1}$ (i.e., financial losses in our orientation) are expected to contribute more to the future VaR and CoVaR than their negative values. This is much like large losses often have more predictive content for volatility than equally large gains in GJR–GARCH models of Glosten et al. (1993). The three model suffixes “pos”, “signs” and “mixed” in the bottom panel of Table 4 imply that for the first only the positive components are included, for the second both positive and negative components are used, and the latter one includes a mix of positive, negative and absolute value losses (see Table 4 for details).
Table 5: CoCAViaR model parameter estimates based on the first estimation window consisting of 3000 trading days starting on January 4, 2000 until December 5, 2011. Estimated standard errors are given in parentheses below the estimates.

| CoCAViaR Model | \( v_t(\cdot) \) | \(|X_{t-1}|\) | \(X^+_{t-1}\) | \(X^-_{t-1}\) | \(|Y_{t-1}|\) | \(Y^+_{t-1}\) | \(Y^-_{t-1}\) | \(v_{t-1}(\cdot)\) | \(c_{t-1}(\cdot)\) |
|----------------|----------------|--------------|--------------|--------------|--------------|--------------|--------------|--------------|--------------|
| SAV-diag       | 0.019          | 0.096        | 0.073        | 0.947        | (0.169)      | (0.073)      | 0.947        | (0.056)      |
|                | 0.060          | 0.751        | 0.592        | 0.834        | (0.213)      | (0.079)      | 0.834        | (0.163)      |
| SAV-fullA      | 0.022          | 0.096        | 0.079        | 0.939        | (0.213)      | (0.079)      | 0.939        | (0.079)      |
|                | 0.171          | 0.106        | 0.197        | 0.798        | (0.974)      | (0.197)      | 0.798        | (0.304)      |
| SAV-full       | 0.022          | 0.096        | 0.079        | 0.939        | (0.213)      | (0.079)      | 0.939        | (0.079)      |
|                | 0.148          | 0.073        | 0.136        | 0.827        | (0.856)      | (0.136)      | 0.827        | (0.280)      |
| AS-pos         | 0.011          | 0.091        | 0.078        | 0.965        | (0.176)      | (0.078)      | 0.965        | (0.059)      |
|                | 0.149          | 0.028        | 0.420        | 0.880        | (0.569)      | (0.028)      | 0.880        | (0.160)      |
| AS-signs       | 0.023          | 0.132        | 0.069        | 0.950        | (0.218)      | (0.132)      | 0.950        | (0.074)      |
|                | 0.138          | 0.112        | 0.149        | 0.909        | (0.430)      | (0.112)      | 0.909        | (0.150)      |
| AS-mixed       | 0.027          | 0.152        | 0.215        | 0.955        | (0.326)      | (0.152)      | 0.955        | (0.121)      |
|                | 0.228          | 0.110        | 0.079        | 0.811        | (0.687)      | (0.110)      | 0.811        | (0.149)      |

Table 5: CoCAViaR model parameter estimates based on the first estimation window consisting of 3000 trading days starting on January 4, 2000 until December 5, 2011. Estimated standard errors are given in parentheses below the estimates.

seem to be less important in general. From the SAV-full and AS-mixed models, we infer that \(|X_{t-1}|\) is more important in \(c_t(\cdot)\) than \(|Y_{t-1}|\) is for \(v_t(\cdot)\). Furthermore, in the SAV-full model, the lagged \(v_{t-1}(\cdot)\) has no significant influence on \(c_t(\cdot)\), which is also reflected by the model’s poor forecasting behavior; see Table 6 below. Consistent with the idea of Glosten et al. (1993) that losses have a larger impact on volatility than do equally large gains, we also find that losses are more important than gains in predicting systemic risk in the AS models; especially \(Y^+_{t-1}\) and \(Y^-_{t-1}\) in the CoVaR equation of the AS-mixed model.

Figure 1 illustrates the rolling window forecasts from the SAV-fullA CoCAViaR model for the evaluation window ranging from December 6, 2011 until December 31, 2021. The upper panel shows the log-losses of JPMorgan Chase—the systemically most relevant bank according to the Financial Stability Board (2021)—together with its VaR forecasts. Losses exceeding the VaR forecasts are highlighted in black, which correspond to days with an (out-of-sample) stress event \(\{X_t \geq \widehat{\text{VaR}}_{t,\beta}\}\) in the definition of the CoVaR in Section 2.1. The lower panel shows the log-losses of the S&P 500 together with the model’s CoVaR forecasts. There, the log-losses of days with a VaR exceedance
Figure 1: Out-of-sample VaR and CoVaR forecasts from the CoCAViaR-SAV-fullA model where JPMorgan Chase’s log-losses are used for $X_t$ and the S&P 500 for $Y_t$. Log-losses on days with a VaR exceedance of JPMorgan Chase are displayed in black in both panels.

(of JPMorgan Chase) are displayed in black, such that the CoVaR forecasts can be interpreted as $\alpha = 95\%$ quantile forecasts among these days with VaR exceedances.

As competitors for our six CoCAViaR models, we use six different DCC–GARCH specifications. DCC–GARCH models have attained benchmark status, because of their accurate variance-covariance matrix predictions (Laurent et al., 2012; Caporin and McAleer, 2014). Particularly, we use three DCC–GARCH model specifications, containing two standard DCC–GARCH(1,1) models with multivariate Gaussian and $t$-distributed innovations respectively and a DCC specification based on a univariate GJR–GARCH(1,1) model. All models are estimated by maximum likelihood. Following Section 2.5, the VaR and CoVaR forecasts are obtained by combining all three model specifications with a symmetric and a Cholesky decomposition of the forecasted variance-covariance matrices, yielding six sets of forecasts. We refer to Appendix D for details on the CoVaR computations.

The multi-objective elicitability of (VaR, CoVaR) complicates inference on the predictive ability
Figure 2: This figure graphically illustrates the one and a half-sided forecast comparison tests for the VaR and CoVaR of Fissler and Hoga (2021) based on their multi-objective scoring function in (5) for a significance level of 10%. We use log-losses of JPMorgan Chase for $X_t$ and of the S&P 500 for $Y_t$. The respective CoCAViaR models are given in the captions of the two plots and are compared against the baseline “DCC-t-Chol” model.

of forecasts, since the scoring function is bivariate; see (5). We follow Fissler and Hoga (2021) and their “one and a half-sided” tests together with their extended traffic light system that we illustrate exemplarily in Figure 2 for two models. These tests require to fix a baseline model, and we arbitrarily choose the “DCC-t-Chol” model for all evaluations. Such a baseline model is necessary as classical extensions to multi-model forecast comparison methods as, e.g., the model confidence set of Hansen et al. (2011) are not available in the case of bivariate (multi-objective) scoring functions.

Multi-objective elicitability implies that the scores of two competing sequences of CoVaR forecasts can only be compared if their underlying VaR forecasts perform equally well. To obtain a reasonable finite-sample counterpart, Fissler and Hoga (2021) interpret this as an insignificant score difference in the standard Diebold and Mariano (1995) test for the VaR forecasts. Hence, the null that the expected VaR score differences (calculated based on the first component in (5)) are equal to zero cannot be rejected. The red zone in Figure 2 indicates that the baseline VaR is significantly superior, and the comparison model is rejected without consideration of its CoVaR forecasts. The grey zone indicates the reverse, while the three remaining zones in the intermediary corridor imply insignificant score differences of the VaR forecasts. Here, the orange zone implies that the baseline CoVaR is significantly superior (i.e., the CoVaR score differences based on the second component of (5) are smaller than zero in expectation), the green zone that the alternative model is superior.
and the yellow zone represents insignificant score differences. As the baseline is the “DCC-t-Chol” model, a comparison with our CoCAViaR models should ideally yield results in the green zone indicates (which indicates superior CoVaR forecasts and comparable VaR forecasts) or in the grey zone (which implies superior VaR forecasts).

Table 6 presents results on the forecast performance of all 12 models. For the VaR, we report its average VaR score (multiplied by 10 for better readability) using the first component of (5), its rank among the different models, and the “hits” as the percentage of days where the loss is larger than the VaR forecast, \( X_t \geq \widehat{\text{Var}}_{t,\beta} \). For the CoVaR, we report the average CoVaR score using the second component (5) (multiplied by 1000 for better readability), the corresponding rank, and the CoVaR hits defined as the percentage of days where \( Y_t \geq \widehat{\text{CoVar}}_{t,\alpha|\beta} \) among all days with a VaR hit, i.e., the \( t \) where \( X_t \geq \widehat{\text{Var}}_{t,\beta} \). The VaR forecast should ideally be exceeded with probability \( 1 - \beta = 5\% \), and on those days the CoVaR forecast should be exceeded with probability \( 1 - \alpha = 5\% \).

The last two columns of Table 6 report results for the previously described one and a half-sided test of Fissler and Hoga (2021). There we use “DCC-t-Chol” as the baseline model, and report the test’s \( p \)-value together with the resulting zone of their extended traffic light system. For each considered asset \( X_t \), we sort the table rows (i.e., the models) according to their CoVaR loss, as the CoVaR forecasts are of main interest in this section.

Overall, we find a superior forecasting performance of the CoCAViaR models for all five employed assets for \( X_t \) compared to the DCC models. While the rankings of the VaR scores vary over the different assets, their superior forecasting performance is more substantial for the CoVaR. This is supported by CoVaR hits (corresponding to unconditional forecast calibration) much closer to the nominal level of 5\% than the DCC models, whose hit frequencies are almost all above 10\%. While none of the CoCAViaR models are significantly outperformed by being in the red or orange zones, some significantly outperform the baseline DCC model and are located in the green and grey zones.

The favorable performance of our CoCAViaR models may be explained as follows. First, DCC–GARCH forecasts of CoVaR intimately rely on the somewhat arbitrary decomposition of the conditional variance-covariance matrix, which is a problem that is sidestepped by our modeling approach; see Section 2.5. Second, our CoCAViaR specifications directly model the quantities of interest (VaR and CoVaR), whereas multivariate GARCH processes model the whole predictive density. Third, our estimators are tailored to accurately capture the (VaR, CoVaR) evolution and, hence, do not place undue emphasis on center-of-the-distribution observations. In contrast, estimators of DCC–GARCH models may trade off a better fit in the body of the distribution for a worse fit in the tails.

Among the CoCAViaR models, we find a better performance of the AS than the SAV models for the VaR, but a relatively comparable performance for CoVaR forecasting. A reason for this
| $X_t$ | model | Co VA | Co VAR | inference |
|-------|-------|-------|---------|-----------|
|       |       | VaR   | CoVaR   |            |            |
|       |       | score | rank    | hits | score | rank | hits | zone | p-value |
| BAC   | CoCAViaR-SAV-fullA | 2.060 | 6 | 4.5 | 5.913 | 1 | 7.9 | green | 0.01 |
|       | CoCAViaR-AS-pos | 2.062 | 7 | 4.9 | 6.793 | 2 | 6.5 | yellow | 0.11 |
|       | CoCAViaR-SAV-diag | 2.096 | 12 | 4.7 | 6.977 | 3 | 7.6 | yellow | 0.18 |
|       | CoCAViaR-AS-mixed | 2.041 | 1 | 4.9 | 7.019 | 4 | 6.4 | grey | 0.01 |
|       | CoCAViaR-AS-signs | 2.043 | 4 | 5.0 | 7.228 | 5 | 7.1 | grey | 0.01 |
|       | CoCAViaR-SAV-full | 2.060 | 6 | 4.5 | 7.609 | 6 | 8.8 | yellow | 0.30 |
|       | CoCAViaR-AS-pos | 2.062 | 7 | 4.9 | 7.400 | 2 | 6.5 | yellow | 0.30 |
|       | CoCAViaR-SAV-diag | 2.096 | 12 | 4.7 | 7.687 | 3 | 7.6 | yellow | 0.18 |
|       | CoCAViaR-AS-mixed | 2.041 | 1 | 4.9 | 7.719 | 4 | 6.4 | grey | 0.01 |
|       | CoCAViaR-AS-signs | 2.043 | 4 | 5.0 | 7.928 | 5 | 7.1 | grey | 0.01 |
|       | CoCAViaR-SAV-full | 2.060 | 6 | 4.5 | 8.609 | 6 | 8.8 | yellow | 0.30 |
|       | CoCAViaR-AS-pos | 2.062 | 7 | 4.9 | 8.800 | 2 | 6.5 | yellow | 0.30 |
|       | CoCAViaR-SAV-diag | 2.096 | 12 | 4.7 | 9.087 | 3 | 7.6 | yellow | 0.18 |
|       | CoCAViaR-AS-mixed | 2.041 | 1 | 4.9 | 9.199 | 4 | 6.4 | grey | 0.01 |
|       | CoCAViaR-AS-signs | 2.043 | 4 | 5.0 | 9.428 | 5 | 7.1 | grey | 0.01 |
|       | CoCAViaR-SAV-full | 2.060 | 6 | 4.5 | 10.609 | 6 | 8.8 | yellow | 0.30 |
|       | CoCAViaR-AS-pos | 2.062 | 7 | 4.9 | 10.800 | 2 | 6.5 | yellow | 0.30 |
|       | CoCAViaR-SAV-diag | 2.096 | 12 | 4.7 | 11.087 | 3 | 7.6 | yellow | 0.18 |
|       | CoCAViaR-AS-mixed | 2.041 | 1 | 4.9 | 11.199 | 4 | 6.4 | grey | 0.01 |
|       | CoCAViaR-AS-signs | 2.043 | 4 | 5.0 | 11.428 | 5 | 7.1 | grey | 0.01 |
|       | CoCAViaR-SAV-full | 2.060 | 6 | 4.5 | 12.609 | 6 | 8.8 | yellow | 0.30 |
|       | CoCAViaR-AS-pos | 2.062 | 7 | 4.9 | 12.800 | 2 | 6.5 | yellow | 0.30 |
|       | CoCAViaR-SAV-diag | 2.096 | 12 | 4.7 | 13.087 | 3 | 7.6 | yellow | 0.18 |
|       | CoCAViaR-AS-mixed | 2.041 | 1 | 4.9 | 13.199 | 4 | 6.4 | grey | 0.01 |
|       | CoCAViaR-AS-signs | 2.043 | 4 | 5.0 | 13.428 | 5 | 7.1 | grey | 0.01 |
|       | CoCAViaR-SAV-full | 2.060 | 6 | 4.5 | 14.609 | 6 | 8.8 | yellow | 0.30 |
|       | CoCAViaR-AS-pos | 2.062 | 7 | 4.9 | 14.800 | 2 | 6.5 | yellow | 0.30 |
|       | CoCAViaR-SAV-diag | 2.096 | 12 | 4.7 | 15.087 | 3 | 7.6 | yellow | 0.18 |
|       | CoCAViaR-AS-mixed | 2.041 | 1 | 4.9 | 15.199 | 4 | 6.4 | grey | 0.01 |
|       | CoCAViaR-AS-signs | 2.043 | 4 | 5.0 | 15.428 | 5 | 7.1 | grey | 0.01 |
|       | CoCAViaR-SAV-full | 2.060 | 6 | 4.5 | 16.609 | 6 | 8.8 | yellow | 0.30 |
|       | CoCAViaR-AS-pos | 2.062 | 7 | 4.9 | 16.800 | 2 | 6.5 | yellow | 0.30 |
|       | CoCAViaR-SAV-diag | 2.096 | 12 | 4.7 | 17.087 | 3 | 7.6 | yellow | 0.18 |
|       | CoCAViaR-AS-mixed | 2.041 | 1 | 4.9 | 17.199 | 4 | 6.4 | grey | 0.01 |
|       | CoCAViaR-AS-signs | 2.043 | 4 | 5.0 | 17.428 | 5 | 7.1 | grey | 0.01 |
|       | CoCAViaR-SAV-full | 2.060 | 6 | 4.5 | 18.609 | 6 | 8.8 | yellow | 0.30 |

Table 6: VaR and CoVAR forecasting results for $Y_t$ equaling S&P 500 losses and various choices of $X_t$. Details on the table columns are given in the main text.
may be that the additional VaR parameters in the AS models are estimated with more precision and, hence, the predictive content of the positive/negative parts emerges more clearly than for CoVaR, where the effective sample size is much reduced. It is further noteworthy that the SAV-full CoCAViaR model, that includes the lagged VaR in the CoVaR model equation performs relatively poorly. This might be caused by a high co-linearity of the VaR and CoVaR models, and also shows that the restriction $B_{12} = 0$ which we have to impose for our two-step M-estimator is likely to be unproblematic in practice.

5 Conclusion

Our first main contribution is to propose a flexible, semiparametric approach for modeling (VaR, CoVaR) jointly. To the extent that we only model (VaR, CoVaR) and not the full predictive distribution, we ‘let the tails speak for themselves’ (DuMouchel, 1983). As we find in an empirical application on the systemic riskiness of US financial institutions, this yields models that improve upon benchmark DCC–GARCH models in terms of predictive accuracy.

Our second main contribution is to study the large sample properties of our proposed estimators of the model parameters. We illustrate the importance of this by revisiting the application of Adrian and Brunnermeier (2016), whose two-step quantile regression estimator ignores the estimation uncertainty of the VaR model in fitting the CoVaR parameters. Applying our theory corrects for this fact and, therefore, allows to draw statistically valid conclusions on the predictive content of several macroeconomic and financial variables for systemic risk.

We expect our modeling framework to have applications beyond the ones considered here, for instance in studying macroeconomic tail risks and their interconnections. Much like Brownlees and Souza (2021) compared the predictive accuracy of Growth-at-Risk model, our model may be used as a competitor in Co-Growth-at-Risk comparisons.

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Supplementary Material

In this Supplementary Material, we prove Theorems 1–3 in Sections A–C, respectively. We also give some details on the computation of VaR and CoVaR from DCC–GARCH models in Section D.

We use the following notational conventions throughout this appendix. The probability space that we work on is \((Ω, F, P)\). We denote by \(C > 0\) a large positive constant that may change from line to line. If not specified otherwise, all convergences are to be understood with respect to \(n → \infty\). We also write \(E_{t-1}[\cdot] = E[\cdot | F_{t-1}]\) and \(P_{t-1}\{\cdot\} = P\{\cdot | F_{t-1}\}\) for short. We exploit without further mention that the Frobenius norm is submultiplicative, i.e., that \(\|AB\| ≤ \|A\| \cdot \|B\|\) for conformable matrices \(A\) and \(B\).

A Proof of Theorem 1

Proof of Theorem 1: We prove both convergences by verifying conditions (i)–(iv) of Theorem 2.1 in Newey and McFadden (1994).

First, we prove \(\tilde{θ}^v_n \xrightarrow{P} θ^v_0\). For condition (i), we have to show that \(Q^v_0(\theta^v) = E\left[\frac{1}{n} \sum_{t=1}^n S^{\text{VaR}}(v_t(\theta^v), X_t)\right]\) is uniquely minimized at \(θ^v = θ^v_0\). By Assumption 1 (iv) and from Fissler and Hoga (2021, Theorem 4.4) it follows that \(E_{t-1}[S^{V\text{aR}}(\cdot, X_t)]\) is uniquely minimized at \(\text{VaR}_β(X_t | F_{t-1})\), which equals \(v_t(θ^v_0)\) under correct specification. By Assumptions 1 (i) and (vii), \(θ^v_0\) is then the unique minimizer of \(E_{t-1}[S^{\text{VaR}}(v_t(\cdot), X_t)]\). By the law of the iterated expectations (LIE) we have that

\[
E\left[S^{\text{VaR}}(v_t(\theta^v), X_t)\right] = E\left(E_{t-1}\left[S^{\text{VaR}}(v_t(\theta^v), X_t)\right]\right),
\]

which implies that \(θ^v_0\) is also the unique minimizer of \(Q^v_0(\cdot)\), as desired.

The compactness requirement of condition (ii) of Theorem 2.1 in Newey and McFadden (1994) is immediate from Assumption 1 (v). Condition (iii), i.e., the continuity of \(θ^v \mapsto Q^v_0(θ^v)\), holds because for any \(θ^v ∈ Θ^v\),

\[
\lim_{θ \to θ^v} Q^v_0(θ) = \frac{1}{n} \sum_{t=1}^n E\left[S^{\text{VaR}}(v_t(θ), X_t)\right] = \frac{1}{n} \sum_{t=1}^n E\left(S^{\text{VaR}}(v_t(θ^v), X_t)\right) = Q^v_0(θ^v),
\]

where the second step follows from Assumption 1 (iv) and the dominated convergence theorem.
condition (iv) follows directly from our Assumption 1 (vi). Thus, \( \theta^v \mapsto Q_0^v(\theta^v) \) is continuous. Note that we may apply the DCT, because \( S^{\text{VaR}}(v_t(\theta), X_t) \) is dominated by

\[
|S^{\text{VaR}}(v_t(\theta), X_t)| \leq |X_t - v_t(\theta)| \leq |X_t| + |v_t(\theta)| \leq |X_t| + V(J_{t-1}),
\]

where the final term is integrable due to Assumption 1 (iv) and (x). The ULLN required by Newey and McFadden (1994). Condition (i) for Assumption 1 (v). Condition (iii), i.e., the continuity of \( DCT \), because follows from similar arguments used in the proof of \( S \).

Now, we prove \( \hat{\theta}_n^v \xrightarrow{p} \theta_0^v \). To this end, we again verify conditions (i)–(iv) of Theorem 2.1 in Newey and McFadden (1994). Condition (i) for

\[
Q_0^c(\theta^v) = \mathbb{E}\left[\frac{1}{n} \sum_{t=1}^{n} S^{\text{CoVaR}}((v_t(\theta_0^v), c_t(\theta^c))', (X_t, Y_t)')\right]
\]

follows from similar arguments used in the proof of \( \hat{\theta}_n^v \xrightarrow{p} \theta_0^v \). Condition (ii) is in force, due to Assumption 1 (v). Condition (iii), i.e., the continuity of \( \theta^c \mapsto Q_0^v(\theta^v) \), follows because for any \( \theta^c \in \Theta^c \),

\[
\lim_{\theta \to \theta^c} Q_0^v(\theta) = \frac{1}{n} \sum_{t=1}^{n} \mathbb{E}\left[ S^{\text{CoVaR}}((v_t(\theta_0^v), c_t(\theta^c))', (X_t, Y_t)')\right] = Q_0^v(\theta^v),
\]

where the second step follows from Assumption 1 (iv) and the DCT, and the third step from the continuity of the map \( c \mapsto S^{\text{CoVaR}}((v, c)', (x, y)') \) together with the continuity of \( c_t(\cdot) \) (by Assumption 1 (vii)). Thus, \( \theta^c \mapsto Q_0^v(\theta^v) \) is continuous. Note that we may apply the DCT, because \( S^{\text{CoVaR}}((v_t(\theta_0^v), c_t(\theta))', (X_t, Y_t)') \) is dominated by

\[
|S^{\text{CoVaR}}((v_t(\theta_0^v), c_t(\theta^c))', (X_t, Y_t)')| \leq |Y_t - c_t(\theta)| \leq |Y_t| + |c_t(\theta)| \leq |Y_t| + V(J_{t-1}),
\]

where the final term is integrable due to Assumption 1 (iv) and (x). For condition (iv), let

\[
\hat{Q}_n^c(\theta^v) = \frac{1}{n} \sum_{t=1}^{n} S^{\text{CoVaR}}((v_t(\hat{\theta}_n^v), c_t(\theta^c))', (X_t, Y_t)'),
\]

where we absorb the fact that \( \hat{\theta}_n^v \) is estimated into \( \hat{Q}_n^c \). We then have to show that

\[
\sup_{\theta^c \in \Theta^c} \left| \hat{Q}_n^c(\theta^v) - Q_0^c(\theta^v) \right| = \sup_{\theta^c \in \Theta^c} \left| \frac{1}{n} \sum_{t=1}^{n} S^{\text{CoVaR}}((v_t(\theta_0^v), c_t(\theta^c))', (X_t, Y_t)') - \mathbb{E}\left[ S^{\text{CoVaR}}((v_t(\theta_0^v), c_t(\theta^c))', (X_t, Y_t)') \right] \right|
\]
is $o_p(1)$. The term in the second line can be bounded by

$$A_{1n} + B_{1n} :=$$

$$\sup_{\theta \in \Theta^c} \left\{ \frac{1}{n} \sum_{t=1}^n S \left( (v_t(\tilde{\theta}_n^{(0)}), c_t(\theta^c))', (X_t, Y_t)' \right) - E \left[ S \left( (v_t(\tilde{\theta}_n^{(0)}), c_t(\theta^c))', (X_t, Y_t)' \right) \right] \right\}$$

$$+ \sup_{\theta \in \Theta^c} \left\{ \frac{1}{n} \sum_{t=1}^n E \left[ S \left( (v_t(\tilde{\theta}_n^{(0)}), c_t(\theta^c))', (X_t, Y_t)' \right) - E \left[ S \left( (v_t(\tilde{\theta}_n^{(0)}), c_t(\theta^c))', (X_t, Y_t)' \right) \right] \right\}.$$  

We show in turn that $A_{1n} = o_p(1)$ and $B_{1n} = o_p(1)$. By the ULLN from Assumption 1 (vi) it holds that

$$A_{1n} \leq \sup_{(\theta^v, \theta^c) \in \Theta} \left\{ \frac{1}{n} \sum_{t=1}^n S \left( (v_t(\theta^v), c_t(\theta^c))', (X_t, Y_t)' \right) - E \left[ S \left( (v_t(\theta^0), c_t(\theta^c))', (X_t, Y_t)' \right) \right] \right\}$$

$$= o_p(1).$$

It remains to show $B_{1n} = o_p(1)$. For any $\varepsilon > 0$ and $\delta > 0$ (with $\delta$ chosen sufficiently small, such that Assumption 1 (ix) is satisfied) it holds that

$$P\{ B_{1n} > \varepsilon \} \leq P\left\{ B_{1n} > \varepsilon, \| \tilde{\theta}_n^{(0)} - \theta_0^0 \| \leq \delta \right\} + P\left\{ \| \tilde{\theta}_n^{(0)} - \theta_0^0 \| > \delta \right\}$$

$$\leq P\left\{ \sup_{\theta \in \Theta^c} \left\{ \frac{1}{n} \sum_{t=1}^n E \left[ S \left( (v_t(\theta^v), c_t(\theta^c))', (X_t, Y_t)' \right) - E \left[ S \left( (v_t(\theta_0^0), c_t(\theta^c))', (X_t, Y_t)' \right) \right] \right\} \right\}$$

$$\leq o(1). \quad (A.1)$$

Write

$$E \left[ S \left( (v_t(\theta^v), c_t(\theta^c))', (X_t, Y_t)' \right) \right] - E \left[ S \left( (v_t(\theta_0^0), c_t(\theta^c))', (X_t, Y_t)' \right) \right]$$

$$= E \left[ \left( 1_{\{X_t > v_t(\theta^v)\}} - 1_{\{X_t > v_t(\theta_0^0)\}} \right) \left( 1_{\{Y_t > c_t(\theta^c)\}} - \alpha \right) \left( c_t(\theta^c) - Y_t \right) \right]$$

$$\leq E \left[ \left| 1_{\{X_t > v_t(\theta^v)\}} - 1_{\{X_t > v_t(\theta_0^0)\}} \right| \cdot \left| c_t(\theta^c) \right| \right] + E \left[ \left| 1_{\{X_t > v_t(\theta^v)\}} - 1_{\{X_t > v_t(\theta_0^0)\}} \right| \cdot |Y_t| \right]$$

$$=: B_{11t} + B_{12t}.$$ 

For $B_{11t}$, we obtain using the LIE (in the first step), Assumption 1 (iv) (in the third step), the mean value theorem (in the fourth step) and Assumptions 1 (ix)-(x) (in the fifth and sixth step) that

$$B_{11t} = E \left\{ \left| c_t(\theta^c) \right| E_{t-1} \left[ \left| 1_{\{X_t > v_t(\theta^v)\}} - 1_{\{X_t > v_t(\theta_0^0)\}} \right| \right] \right\}$$

$$\leq E \left\{ C(F_{t-1}) \left| \int_{v_t(\theta^c)}^{v_t(\theta_0^0)} f_t^X(x) \, dx \right| \right\}.$$
where \( \theta^* \) is some mean value between \( \theta_0^v \) and \( \theta^v \). Hence, choosing \( \delta > 0 \) sufficiently small, we can ensure that \( \sup_{\theta^v \in \Theta \atop \|\theta^v - \theta_0^v\| \leq \delta} B_{11t} < \varepsilon/2 \).

For \( B_{12t} \), we obtain that for some \( \theta^* \) between \( \theta^v \) and \( \theta_0^v \) (which may change from line to line)

\[
B_{12t} = \mathbb{E} \left\{ E_{t-1} \left[ I_{\{X_t > v_t(\theta^v)\}} - I_{\{X_t > v_t(\theta_0^v)\}} \right] Y_t \right\}
\]

\[
= \mathbb{E} \left\{ \int_{-\infty}^{\infty} \text{sgn} \left( v_t(\theta^v) - v_t(\theta_0^v) \right) \left[ \int_{v_t(\theta_0^v)}^{v_t(\theta^v)} y f_t(x,y) \, dx \, dy \right] \right\}
\]

\[
= \mathbb{E} \left\{ \int_{-\infty}^{\infty} \left[ \int_{v_t(\theta_0^v)}^{v_t(\theta^v)} |y| f_t(x,y) \, dx \, dy \right] \right\}
\]

\[
= \mathbb{E} \left\{ \sup_{x \in \mathbb{R}} \int_{-\infty}^{\infty} |y| f_t(x,y) \, dy \left[ v_t(\theta^v) - v_t(\theta_0^v) \right] \right\}
\]

\[
\leq \mathbb{E} \left\{ KV_t(\mathcal{F}_{t-1}) \|\theta^v - \theta_0^v\| \right\}
\]

\[
\leq C \delta,
\]

where the penultimate step follows from Assumption 1 (iv). Once again, this allows us to conclude that \( \sup_{\theta^v \in \Theta \atop \|\theta^v - \theta_0^v\| \leq \delta} B_{12t} < \varepsilon/2 \) by choosing \( \delta > 0 \) sufficiently small. Therefore, for a suitable choice of \( \delta \), the first right-hand side term in (A.1) can be shown to equal zero, such that \( B_{11n} = o_p(1) \) follows. This establishes condition (iv). The desired result that \( \hat{\theta}^v_n \overset{P}{\to} \theta^c_0 \) now follows from Theorem 2.1 in Newey and McFadden (1994).

\[\square\]

**B Proof of Theorem 2**

We split the proof of Theorem 2 into two parts. In Section B.1, we prove the asymptotic normality of \( \hat{\theta}^v_n \), and in Section B.2 that of \( \hat{\theta}^c_n \).
B.1 Asymptotic Normality of the VaR Parameter Estimator

The proof follows closely that of Theorem 2 in Engle and Manganelli (2004), although some of our assumptions differ from theirs. Our main motivation for detailing the proof is that some of the subsequent results are needed to prove the asymptotic normality of $\hat{\theta}_n$ in Section B.2.

Before giving the formal proof, we collect some results that will be used in the sequel. For $v \neq x$, it holds that

$$\frac{\partial}{\partial v} S^{\text{VaR}}(v, x) = [\mathbb{1}_{\{x \leq v\}} - \beta].$$

Thus, by the chain rule, it holds a.s. that

$$v_t(\theta_v) := \frac{\partial}{\partial \theta_v} S^{\text{VaR}}(v_t(\theta_v), X_t) = \nabla v_t(\theta_v) \mathbb{1}_{\{X_t \leq v_t(\theta_v)\}} - \beta.$$  \hspace{1cm} (B.1)

This implies by the LIE and Assumption 1 (vii) that

$$\mathbb{E}[v_t(\theta_v)] = \mathbb{E}\left\{ \nabla v_t(\theta_v) \left[ F_t^X(v_t(\theta_v)) - \beta \right] \right\}.$$  

Finally, Assumption 2 and the dominated convergence theorem allow us to interchange differentiation and expectation to yield that

$$\frac{\partial}{\partial \theta_v} \mathbb{E}[v_t(\theta_v)] = \mathbb{E}\left\{ \nabla^2 v_t(\theta_v) \left[ F_t^X(v_t(\theta_v)) - \beta \right] + \nabla v_t(\theta_v) \nabla' v_t(\theta_v) f_t^X(v_t(\theta_v)) \right\}. \hspace{1cm} (B.2)$$

Evaluating this quantity at the true parameters gives

$$\Lambda = \left. \frac{\partial}{\partial \theta_v} \mathbb{E}[v_t(\theta_v)] \right|_{\theta_v = \theta_0} = \mathbb{E}\left\{ \nabla v_t(\theta_0^v) \nabla' v_t(\theta_0^v) f_t^X(v_t(\theta_0^v)) \right\}. \hspace{1cm} (B.3)$$

By virtue of Assumption 1 (ii), $\Lambda$ does not depend on $t$.

Similarly as in Engle and Manganelli (2004) and Patton et al. (2019), the key step in the proof is to apply Lemma A.1 in Weiss (1991). For this we require some preliminary results. Define

$$\lambda_n(\theta_v) = \frac{1}{n} \sum_{t=1}^n \mathbb{E}[v_t(\theta_v)],$$

$$\Lambda_n(\theta^*) = \frac{1}{n} \sum_{t=1}^n \left. \frac{\partial}{\partial \theta_v} \mathbb{E}[v_t(\theta_v)] \right|_{\theta_v = \theta^*},$$

and note that $\Lambda = \Lambda_n(\theta_0^v)$.

**Lemma V.1.** Suppose Assumptions 1 and 2 hold. Then, as $n \to \infty$,

$$\sqrt{n}(\hat{\theta}_n - \theta_0^v) = \left[ A^{-1} + o_P(1) \right] \left[ -\frac{1}{\sqrt{n}} \sum_{t=1}^n v_t(\theta_0^v) + o_P(1) \right].$$

**Proof:** The mean value theorem (MVT) implies that for all $i = 1, \ldots, p$,

$$\lambda_n^{(i)}(\hat{\theta}_n^v) = \lambda_n^{(i)}(\theta_0^v) + A_n^{(i, \cdot)}(\theta^*_v)(\hat{\theta}_n^v - \theta_0^v),$$
where \( \lambda_n(\cdot) = (\lambda_n^{(1)}(\cdot), \ldots, \lambda_n^{(p)}(\cdot))' \), \( A_n^{(i)}(\cdot) \) denotes the \( i \)-th row of \( A_n(\cdot) \) and \( \theta^*_n \) lies on the line connecting \( \theta^*_0 \) and \( \hat{\theta}_n^v \). To economize on notation, we shall slightly abuse notation (here and elsewhere) by writing this as

\[
\lambda_n(\hat{\theta}_n^v) = \lambda_n(\theta^*_0) + A_n(\theta^*)(\hat{\theta}_n^v - \theta^*_0) \tag{B.4}
\]

for some \( \theta^* \) between \( \hat{\theta}_n^v \) and \( \theta^*_0 \); keeping in mind that the value of \( \theta^* \) is in fact different from row to row in \( A_n(\theta^*) \). However, this does not change any of the subsequent arguments. Interpreted verbatim, (B.4) would be an instance of what Feng et al. (2013) call the non-existent mean value theorem, which is widely applied in statistics; e.g., by Engle and Manganelli (2004) and Patton et al. (2019).

We have that

\[
\lambda_n(\theta^*_0) = 1/n \sum_{t=1}^n \mathbb{E}\left\{ \nabla v_t(\theta^*_0) \left[ F_t^X(v_t(\theta^*_0)) - \beta \right] \right\} = 0, \tag{B.5}
\]

since \( F_t^X(v_t(\theta^*_0)) = \beta \) by correct specification. Plugging this into (B.4) gives

\[
\lambda_n(\hat{\theta}_n^v) = \Lambda_n(\theta^*)(\hat{\theta}_n^v - \theta^*_0).
\]

To establish the claim, we therefore only have to show that

(i) \( \Lambda_n^{-1}(\theta^*) = \Lambda^{-1} + o_p(1) \);

(ii) \( \sqrt{n}\lambda_n(\hat{\theta}_n^v) = -\frac{1}{\sqrt{n}} \sum_{t=1}^n v_t(\theta^*_0) + o_p(1) \).

Claim (i) is verified in Lemma V.2. For this, note that since \( \theta^* \) is a mean value between \( \hat{\theta}_n^v \) and \( \theta^*_0 \), and \( \hat{\theta}_n^v \xrightarrow{p} \theta^*_0 \) (from Theorem 1), it also follows that \( \theta^* \xrightarrow{p} \theta^*_0 \).

To prove (ii), by Lemma A.1 in Weiss (1991), we only have to show that

(ii.a) conditions (N1)–(N5) in the notation of Weiss (1991) hold;

(ii.b) \( \frac{1}{\sqrt{n}} \sum_{t=1}^n v_t(\hat{\theta}_n^v) = o_p(1) \);  

(ii.c) \( \hat{\theta}_n^v \xrightarrow{p} \theta^*_0 \).

For (ii.a), note that (N1) is immediate and (N2) follows from (B.5). The mixing condition (N5) is implied by our Assumption 2 (x). Condition (N3) is verified in Lemmas V.4–V.6 below and the remaining condition (N4) in Lemma V.7. The result in (ii.b) follows from Lemma V.3 and (ii.c) follows from Theorem 1. In sum, the desired result follows.

\[\square\]

Lemma V.2. Suppose Assumptions 1 and 2 hold. Then, as \( n \to \infty \), \( \Lambda_n^{-1}(\theta^*) \xrightarrow{p} \Lambda^{-1} \) for any \( \theta^* \) with \( \theta^* \xrightarrow{p} \theta^*_0 \).

Proof: We first show that \( \|A_n(\tau) - A_n(\theta)\| \leq C\|\tau - \theta\| \) for all \( \tau, \theta \in \mathcal{N}(\theta^*_0) \), where \( \mathcal{N}(\theta^*_0) \) is some neighborhood of \( \theta^*_0 \) such that Assumptions 1 (ix) and 2 (iii) holds.
Use (B.2) to write

\[ \| A_n(\tau) - A_n(\theta) \| = \frac{1}{n} \sum_{t=1}^{n} \mathbb{E} \left\{ \nabla^2 v_t(\tau) \left[ F_t^X(v_t(\tau)) - \beta \right] - \nabla^2 v_t(\theta) \left[ F_t^X(v_t(\theta)) - \beta \right] \right. \\

\vspace{0.5cm}

\left. + \nabla v_t(\tau) \nabla' v_t(\tau) f_t^X(v_t(\tau)) - \nabla v_t(\theta) \nabla' v_t(\theta) f_t^X(v_t(\theta)) \right\} \| \]

\[ \leq \frac{1}{n} \sum_{t=1}^{n} \mathbb{E} \left\| \nabla^2 v_t(\tau) \left[ F_t^X(v_t(\tau)) - F_t^X(v_t(\theta)) \right] \right\| + \beta \mathbb{E} \left\| \nabla^2 v_t(\tau) - \nabla^2 v_t(\theta) \right\| \\

\vspace{0.5cm}

\left. + \mathbb{E} \left\| \nabla v_t(\tau) \nabla' v_t(\tau) f_t^X(v_t(\tau)) - \nabla v_t(\theta) \nabla' v_t(\theta) f_t^X(v_t(\theta)) \right\| . \right \} \tag{B.6} \]

By Assumption 2 (iii), we have that

\[ \beta \mathbb{E} \left\| \nabla^2 v_t(\tau) - \nabla^2 v_t(\theta) \right\| \leq \beta \mathbb{E} \left[ V_3(\mathcal{F}_{l-1}) \right] \| \tau - \theta \| \leq C \| \tau - \theta \|. \tag{B.7} \]

The other two terms in (B.6) can be dealt with as follows. First, using a mean value expansion around \( \theta_0^* \) and Assumption 2, we obtain that

\[ \mathbb{E} \left\| \nabla^2 v_t(\tau) \left[ F_t^X(v_t(\tau)) - F_t^X(v_t(\theta)) \right] \right\| \leq \mathbb{E} \left\| V_2(\mathcal{F}_{l-1}) f_t^X(v_t(\theta^*)) \nabla v_t(\theta^*)(\tau - \theta) \right\| \\

\vspace{0.5cm}

\left. \leq K \mathbb{E} \left[ V_1(\mathcal{F}_{l-1}) V_2(\mathcal{F}_{l-1}) \right] \| \tau - \theta \| \\

\vspace{0.5cm}

\leq K \left\{ \mathbb{E} \left[ V_3^3(\mathcal{F}_{l-1}) \right] \right\} \frac{1}{3} \left\{ \mathbb{E} \left[ V_2^{3/2}(\mathcal{F}_{l-1}) \right] \right\} \frac{2}{3} \| \tau - \theta \| \\

\vspace{0.5cm}

\leq C \| \tau - \theta \|. \right \} \tag{B.8} \]

where \( \theta^* \) is some value on the line connecting \( \tau \) and \( \theta \), and the penultimate step uses Hölder’s inequality.

Second, using a mean value expansion around \( \theta \) to obtain for some \( \theta^* \) between \( \tau \) and \( \theta \) (where \( \theta^* \) may vary from line to line) that

\[ \mathbb{E} \left\| \nabla v_t(\tau) \nabla' v_t(\tau) f_t^X(v_t(\tau)) - \nabla v_t(\theta) \nabla' v_t(\theta) f_t^X(v_t(\theta)) \right\| \\

\vspace{0.5cm}

\left. = \mathbb{E} \left[ \nabla v_t(\tau) \nabla' v_t(\tau) f_t^X(v_t(\tau)) - \nabla v_t(\theta) \nabla' v_t(\theta) f_t^X(v_t(\theta)) \right. \right. \\

\vspace{0.5cm}

\left. + \nabla v_t(\theta) \nabla' v_t(\theta) f_t^X(v_t(\theta)) - \nabla v_t(\theta) \nabla' v_t(\theta) f_t^X(v_t(\theta)) \right. \\

\vspace{0.5cm}

\left. + \nabla v_t(\theta) \nabla' v_t(\theta) \{ f_t^X(v_t(\tau)) - f_t^X(v_t(\theta)) \} \right| \right\| \\

\vspace{0.5cm}

\leq \mathbb{E} \left[ K V_2(\mathcal{F}_{l-1}) V_1(\mathcal{F}_{l-1}) + K V_1(\mathcal{F}_{l-1}) V_2(\mathcal{F}_{l-1}) + K V_3^3(\mathcal{F}_{l-1}) \right] \| \tau - \theta \|
\[
\begin{aligned}
&\leq K \left\{ 2 \left\{ \mathbb{E} \left[ V_1^3(F_{t-1}) \right] \right\}^{1/3} \left\{ \mathbb{E} \left[ V_2^{3/2}(F_{t-1}) \right] \right\}^{2/3} + \left\{ \mathbb{E} \left[ V_1^3(F_{t-1}) \right] \right\} \right\} \| \tau - \theta \|
&\leq C \| \tau - \theta \|,
\end{aligned}
\]
(B.9)

where we used Assumption 2.

Plugging (B.7)–(B.9) into (B.6), we obtain that
\[
\| A_n(\tau) - A_n(\theta) \| \leq C \| \tau - \theta \|. 
\]
(B.10)

Using this, we obtain for \( \delta > 0 \) with \( \{ \| \theta - \theta_{0}^v \| \leq \delta \} \subset \mathcal{N}(\theta_{0}^v) \) that
\[
\begin{aligned}
P\left\{ \| A_n(\theta^*) - A \| > \varepsilon \right\} &= P\left\{ \| A_n(\theta^*) - A_n(\theta_{0}^v) \| > \varepsilon \right\} \\
&\leq P\left\{ \| A_n(\theta^*) - A_n(\theta_{0}^v) \| > \varepsilon, \| \theta^* - \theta_{0}^v \| \leq \delta \right\} + P\left\{ \| \theta^* - \theta_{0}^v \| > \delta \right\} \\
&\leq P\left\{ \sup_{\| \theta - \theta_{0}^v \| \leq \delta} \| A_n(\theta) - A_n(\theta_{0}^v) \| > \varepsilon \right\} + o(1) \\
&\leq P\left\{ \sup_{\| \theta - \theta_{0}^v \| \leq \delta} C \| \theta - \theta_{0}^v \| > \varepsilon \right\} + o(1) \\
&= 0 + o(1),
\end{aligned}
\]

where the final line additionally requires \( \delta < \varepsilon/C \). This shows that \( A_n(\theta^*) \xrightarrow{P} A \). By Assumption 2 (viii) and continuity of \( A_n(\cdot) \) (recall (B.10)), \( A_n(\theta^v) \) is non-singular in a neighborhood of \( \theta_{0}^v \). Therefore, the continuous mapping theorem (CMT) applied to \( A_n(\theta^*) \xrightarrow{P} A \) implies that \( A_n^{-1}(\theta^*) \xrightarrow{P} A^{-1} \), as desired.

\[\square\]

Lemma V.3. Suppose Assumptions 1 and 2 hold. Then, as \( n \to \infty \),
\[
\frac{1}{\sqrt{n}} \sum_{t=1}^{n} v_t(\hat{\theta}_n^v) = o_P(1).
\]

Proof: Recall from Assumption 1 (v) that \( \Theta^v \subset \mathbb{R}^p \), such that \( \theta^v \) is a \( p \)-dimensional parameter vector. Let \( e_1, \ldots, e_p \) denote the standard basis of \( \mathbb{R}^p \). Define
\[
S_{j,n}^\text{VaR}(a) := \frac{1}{\sqrt{n}} \sum_{t=1}^{n} S^\text{VaR}(v_t(\hat{\theta}_n^v + ae_j), X_t), \quad j = 1, \ldots, p,
\]
where \( a \in \mathbb{R} \). Let \( G_{j,n}(a) \) be the right partial derivative of \( S_{j,n}^\text{VaR}(a) \), such that (see (B.1))
\[
G_{j,n}(a) = \frac{1}{\sqrt{n}} \sum_{t=1}^{n} \nabla_j v_t(\hat{\theta}_n^v + ae_j) \left[ \mathbb{I}_{\{ X_t \leq v_t(\hat{\theta}_n^v + ae_j) \}} - \beta \right],
\]
where \( \nabla_j v_t(\cdot) \) is the \( j \)-th component of \( \nabla v_t(\cdot) \). Then, \( G_{j,n}(0) = \lim_{\xi \downarrow 0} G_{j,n}(\xi) \) is the right partial
derivative of

\[ S_{n}^{VaR}(\theta^v) := \frac{1}{\sqrt{n}} \sum_{t=1}^{n} S^{VaR}(v_t(\theta^v), X_t) \]

at \( \hat{\theta}^v \) in the direction \( \theta^v \), where \( \theta^v = (\theta^v_1, \ldots, \theta^v_p)' \). Correspondingly, \( \lim_{\xi \to 0} G_{j,n}(\xi) \) is the left partial derivative. Because \( S_{n}^{VaR}() \) achieves its minimum at \( \hat{\theta}^v \), the left derivative must be non-positive and the right derivative must be non-negative. Thus,

\[ |G_{j,n}(0)| \leq G_{j,n}(\xi) - G_{j,n}(-\xi) = \frac{1}{\sqrt{n}} \sum_{t=1}^{n} \nabla_j v_t(\hat{\theta}^v + \xi e_j) \mathbb{1}_{\{X_t \leq v_t(\hat{\theta}^v + \xi e_j)\}} - \beta \]

By continuity of \( \nabla v_t(\cdot) \) (see Assumption 2 (ii)) it follows upon letting \( \xi \to 0 \) that

\[ |G_{j,n}(0)| \leq \frac{1}{\sqrt{n}} \sum_{t=1}^{n} |\nabla_j v_t(\hat{\theta}^v)| \mathbb{1}_{\{X_t = v_t(\hat{\theta}^v)\}} \leq \frac{1}{\sqrt{n}} \left[ \max_{t=1,\ldots,n} V_1(\mathcal{F}_{t-1}) \right] \sum_{t=1}^{n} \mathbb{1}_{\{X_t = v_t(\hat{\theta}^v)\}}. \quad (B.11) \]

We have that, by subadditivity, Markov’s inequality and Assumption 2 (v),

\[ \mathbb{P}\left\{ \frac{1}{n^{1/2}} \max_{t=1,\ldots,n} V_1(\mathcal{F}_{t-1}) > \varepsilon \right\} \leq \sum_{t=1}^{n} \mathbb{P}\left\{ V_1(\mathcal{F}_{t-1}) > \varepsilon n^{1/2} \right\} \leq \sum_{t=1}^{n} \varepsilon n^{-3/2} \mathbb{E}[V_1^3(\mathcal{F}_{t-1})] = O(n^{-1/2}) = o(1). \]

Combining this with Assumption 2 (ix), we obtain from (B.11) that

\[ |G_{j,n}(0)| \overset{a.s.}{\to} o_P(1) = o_P(1). \]

As this holds for every \( j = 1, \ldots, p \), we get that

\[ \frac{1}{\sqrt{n}} \sum_{t=1}^{n} v_t(\hat{\theta}^v) = \frac{1}{\sqrt{n}} \sum_{t=1}^{n} \nabla_j v_t(\hat{\theta}^v) \mathbb{1}_{\{X_t \leq v_t(\hat{\theta}^v)\}} - \beta = o_P(1), \]

which is just the conclusion. \( \square \)

**Lemma V.4.** Suppose Assumptions 1 and 2 hold. Then, condition (N3) (i) of Weiss (1991) holds, i.e.,

\[ \| \lambda_n(\theta) \| \geq a \| \theta - \theta^v_0 \| \quad \text{for} \quad \| \theta - \theta^v_0 \| \leq d_0 \]
for sufficiently large $n$ and some $a > 0$ and $d_0 > 0$.

**Proof:** Choose $d_0 > 0$ sufficiently small, such that $\{\theta \in \Theta^o : \|\theta - \theta_0^o\| < d_0\}$ is a subset of the neighborhoods of Assumptions 1 (ix) and 2 (iii). The MVT and (B.5) imply that

$$
\lambda_n(\theta) = \lambda_n(\theta_0^o) + \Lambda_n(\theta^*) (\theta - \theta_0^o)
$$

$$
= \Lambda_n(\theta^*)(\theta - \theta_0^o)
$$

for some $\theta^*$ between $\theta$ and $\theta_0^o$. (Again, to be precise we should allow for the mean value $\theta^*$ to vary across rows of $\Lambda_n(\theta^*)$; however, for the subsequent argument this does not matter.) Use (B.2) to write

$$
\Lambda_n(\theta^*) = \frac{1}{n} \sum_{t=1}^{n} \mathbb{E} \left\{ \nabla^2 v_t(\theta^*) \left[ F_t^X(v_t(\theta^*)) - \beta \right] + \nabla v_t(\theta^*) \nabla' v_t(\theta^*) f_t^X(v_t(\theta^*)) \right\}.
$$

Recall that

$$
\Lambda = \Lambda_n(\theta_0^o) = \frac{1}{n} \sum_{t=1}^{n} \mathbb{E} \left[ \nabla v_t(\theta_0^o) \nabla' v_t(\theta_0^o) f_t^X(v_t(\theta_0^o)) \right].
$$

We first show that

$$
\Lambda_n(\theta^*) = \Lambda + O(\|\theta^* - \theta_0^o\|)
$$

by decomposing $\|\Lambda_n(\theta^*) - \Lambda\|$ into two terms, each bounded by a $O(\|\theta^* - \theta_0^o\|)$-term. We have the following decomposition:

$$
\|\Lambda_n(\theta^*) - \Lambda\| = \left\| \frac{1}{n} \sum_{t=1}^{n} \mathbb{E} \left\{ \nabla^2 v_t(\theta^*) \left[ F_t^X(v_t(\theta^*)) - \beta \right] \right\} + \mathbb{E} \left[ \nabla v_t(\theta^*) \nabla' v_t(\theta^*) f_t^X(v_t(\theta^*)) - \nabla v_t(\theta_0^o) \nabla' v_t(\theta_0^o) f_t^X(v_t(\theta_0^o)) \right] \right\|.
$$

**First term:** Following similar steps as for (B.8), we obtain that

$$
\left\| \mathbb{E} \left\{ \nabla^2 v_t(\theta^*) \left[ F_t^X(v_t(\theta^*)) - \beta \right] \right\} \right\| \leq C\|\theta^* - \theta_0^o\|.
$$

**Second term:** Again following similar steps as for (B.9), we get that

$$
\left\| \frac{1}{n} \sum_{t=1}^{n} \mathbb{E} \left[ \nabla v_t(\theta^*) \nabla' v_t(\theta^*) f_t^X(v_t(\theta^*)) - \nabla v_t(\theta_0^o) \nabla' v_t(\theta_0^o) f_t^X(v_t(\theta_0^o)) \right] \right\| \leq \frac{1}{n} \sum_{t=1}^{n} \mathbb{E} \left\| \nabla v_t(\theta^*) \nabla' v_t(\theta^*) f_t^X(v_t(\theta^*)) - \nabla v_t(\theta_0^o) \nabla' v_t(\theta_0^o) f_t^X(v_t(\theta_0^o)) \right\| \leq C\|\theta^* - \theta_0^o\|.
$$

Thus, we have shown that $\|\Lambda_n(\theta^*) - \Lambda\| \leq C\|\theta^* - \theta_0^o\|$ for some $C > 0$ large enough. By Assump-
tion 2 (viii), $A$ has eigenvalues bounded below by some constant $a > 0$. Therefore,

\[
\|\lambda_n(\theta)\| = \|A_n(\theta^*)(\theta - \theta_0^v)\|
\]

\[
= \|A(\theta - \theta_0^v) - [A - A_n(\theta^*)](\theta - \theta_0^v)\|
\]

\[
\geq \|A(\theta - \theta_0^v)\| - \| [A - A_n(\theta^*)] (\theta - \theta_0^v) \|
\]

\[
\geq \left( a - C\|\theta - \theta_0^v\| \right) \|\theta - \theta_0^v\|
\]

where the second-to-last inequality holds by the triangle inequality. The conclusion follows upon choosing $d_0$ small enough, such that $a - C\|\theta - \theta_0^v\| > 0$. 

\[\square\]

**Lemma V.5.** Suppose Assumptions 1 and 2 hold, and define

\[
\mu_t(\theta, d) = \sup_{\|\tau - \theta\| \leq d} \|v_t(\tau) - v_t(\theta)\|
\]

Then, condition (N3) (ii) of Weiss (1991) holds, i.e.,

\[
E[\mu_t(\theta, d)] \leq bd \quad \text{for} \quad \|\theta - \theta_0^v\| + d \leq d_0,
\]

for sufficiently large $n$ and some strictly positive $b, d, d_0$.

**Proof:** Choose $d_0 > 0$ sufficiently small, such that $\{\theta \in \Theta^v : \|\theta - \theta_0^v\| < d_0\}$ is a subset of the neighborhoods of Assumptions 1 (ix) and 2 (iii). Recalling the definition of $v_t(\theta^v)$ from (B.1), we decompose

\[
\mu_t(\theta, d) = \sup_{\|\tau - \theta\| \leq d} \left\| \nabla v_t(\tau) 1_{\{X_t \leq v_t(\tau)\}} - \nabla v_t(\theta) 1_{\{X_t \leq v_t(\theta)\}} \right\|
\]

\[
\leq \sup_{\|\tau - \theta\| \leq d} \left\| \nabla v_t(\tau) 1_{\{X_t \leq v_t(\tau)\}} - \nabla v_t(\theta) 1_{\{X_t \leq v_t(\theta)\}} \right\| + \beta \sup_{\|\tau - \theta\| \leq d} \left\| \nabla v_t(\tau) - \nabla v_t(\theta) \right\|
\]

\[
=: \mu_t^{(1)}(\theta, d) + \mu_t^{(2)}(\theta, d).
\]

Define the $\mathcal{F}_{t-1}$-measurable quantities

\[
\mathcal{I} := \arg\min_{\|\tau - \theta\| \leq d} v_t(\tau),
\]

\[
\mathcal{F} := \arg\max_{\|\tau - \theta\| \leq d} v_t(\tau),
\]

which exist by continuity of $v_t(\cdot)$.

We first consider $\mu_t^{(1)}(\theta, d)$. To take the indicators out of the supremum, we distinguish two cases:

**Case 1:** $X_t \leq v_t(\theta)$

We further distinguish two cases (a)–(b):
(a) If $X_t < v_t(\tau)$, then both indicators are one, such that
\[
\mu_t^{(1)}(\theta, d) = \sup_{\|\tau - \theta\| \leq d} \| \nabla v_t(\tau) - \nabla v_t(\theta) \|.
\]

(b) If $v_t(\tau) \leq X_t \leq v_t(\bar{\tau})$, then
\[
\mu_t^{(1)}(\theta, d) = \max \left\{ \sup_{\|\tau - \theta\| \leq d, X_t \leq v_t(\tau)} \| \nabla v_t(\tau) - \nabla v_t(\theta) \|, \| \nabla v_t(\theta) \| \right\} \leq \sup_{\|\tau - \theta\| \leq d} \| \nabla v_t(\tau) - \nabla v_t(\theta) \| + \| \nabla v_t(\theta) \|. \tag{B.12}
\]
(Note that the third case that $X_t > v_t(\bar{\tau})$ cannot occur, because already $X_t \leq v_t(\theta).$)

**Case 2: $X_t > v_t(\theta)$**

\[
\mu_t^{(1)}(\theta, d) = I_{\{X_t \leq v_t(\tau)\}} \sup_{\|\tau - \theta\| \leq d, X_t \leq v_t(\tau)} \| \nabla v_t(\tau) \| \leq I_{\{X_t \leq v_t(\tau)\}} \sup_{\|\tau - \theta\| \leq d} \| \nabla v_t(\tau) \|. \tag{B.13}
\]

Before combining the two cases, note that our assumptions $\| \tau - \theta \| \leq d$ and $\| \theta - \theta_0^t \| + d \leq d_0$ together imply that $\theta$ and $\tau$ are in a $d_0$-neighborhood of $\theta_0^t$. (For $\theta$ this is immediate, and for $\tau$ this follows from $\| \tau - \theta_0^t \| = \| \tau - \theta + \theta - \theta_0^t \| \leq \| \tau - \theta \| + \| \theta - \theta_0^t \| \leq d + (d_0 - d) = d_0.$) Hence, for the final terms in (B.12) and (B.13), we have
\[
\| \nabla v_t(\theta) \| \leq \sup_{\|\tau - \theta\| \leq d} \| \nabla v_t(\tau) \| \leq \sup_{\|\theta - \theta_0^t\| \leq d_0} \| \nabla v_t(\theta) \|.
\]
Therefore, combining the results from Cases 1 and 2,
\[
\mu_t^{(1)}(\theta, d) \leq \left[ I_{\{v_t(\tau) \leq X_t \leq v_t(\theta)\}} + I_{\{v_t(\theta) < X_t \leq v_t(\tau)\}} \right] \sup_{\|\theta - \theta_0^t\| \leq d_0} \| \nabla v_t(\theta) \| + \sup_{\|\tau - \theta\| \leq d} \| \nabla v_t(\tau) - \nabla v_t(\theta) \|. \tag{B.14}
\]
By Assumptions 1 (iv) and (ix) we have
\[
\mathbb{E}_{t-1} \left[ I_{\{v_t(\tau) \leq X_t \leq v_t(\theta)\}} \right] = \int_{v_t(\tau)}^{v_t(\theta)} f_t^X(x) dx \leq K |v_t(\theta) - v_t(\tau)| = K \| \nabla v_t(\theta^*) (\theta - \tau) \| \leq K V_1(F_{t-1}) \| \theta - \tau \| \leq K V_1(F_{t-1}) d, \tag{B.15}
\]
and, similarly,
\[ \mathbb{E}_{t-1} \left[ \mathbb{I}_{\{ v_t(\theta) < X_t \leq v_t(\tau) \}} \right] \leq K V_1(\mathcal{F}_{t-1}) d. \]  
(B.16)

Moreover, we have by the MVT that for some \( \theta^* \) on the line connecting \( \tau \) and \( \theta \),
\[ \sup_{\| \tau - \theta \| \leq d} \| \nabla v_t(\tau) - \nabla v_t(\theta) \| = \sup_{\| \tau - \theta \| \leq d} \| \nabla^2 v_t(\theta^*)(\tau - \theta) \| \]
\[ \leq V_2(\mathcal{F}_{t-1}) \| \tau - \theta \| \leq V_2(\mathcal{F}_{t-1}) d. \]  
(B.17)

Therefore,
\[ \mathbb{E}[\mu_t^{(1)}(\theta, d)] \leq \mathbb{E}[K V_1^2(\mathcal{F}_{t-1})] d + \mathbb{E}[V_2(\mathcal{F}_{t-1})] d \leq C d. \]

By arguments leading to (B.17), we also have that
\[ \mathbb{E}[\mu_t^{(2)}(\theta, d)] \leq \beta \mathbb{E}[V_2(\mathcal{F}_{t-1})] d \leq C d. \]

Overall,
\[ \mathbb{E}[\mu_t(\theta, d)] \leq \mathbb{E}[\mu_t^{(1)}(\theta, d)] + \mathbb{E}[\mu_t^{(2)}(\theta, d)] \leq b d \]
for some suitable \( b > 0 \), as desired. \( \square \)

**Lemma V.6.** Suppose Assumptions 1 and 2 hold. Then, condition (N3) (iii) of Weiss (1991) holds, i.e.,
\[ \mathbb{E}[\mu_t^q(\theta, d)] \leq c d \quad \text{for} \quad \| \theta - \theta^0 \| + d \leq d_0, \]
for sufficiently large \( n \) and some \( c > 0, d \geq 0, d_0 > 0 \) and \( q > 2 \).

**Proof:** The arguments in this proof are similar to that of Lemma V.5. We again pick \( d_0 > 0 \) sufficiently small, such that \( \{ \theta \in \Theta^0 : \| \theta - \theta^0 \| < d_0 \} \) is a subset of the neighborhoods of Assumptions 1 (ix) and 2 (iii). We also work with \( \mu_t^{(i)}(\theta, d) \) \( i = 1, 2 \), \( \tau \) and \( \overline{\tau} \) as defined in the proof of Lemma V.5.

By the \( c_r \)-inequality (e.g., Davidson, 1994, Theorem 9.28) and the fact that \( \mu_t(\theta, d) \leq \mu_t^{(1)}(\theta, d) + \mu_t^{(2)}(\theta, d) \), we get that
\[ \mathbb{E}[\mu_t(\theta, d)]^q \leq 2^{q-1} \sum_{i=1}^2 \mathbb{E}[\mu_t^{(i)}(\theta, d)]^q. \]

Hence, to prove the claim, it suffices to show that \( \mathbb{E}[\mu_t^{(i)}(\theta, d)]^{2+i} \leq c d \) for \( i > 0 \) (from Assumption 2 (v)) and some \( c > 0 \).

**First term:** Following the same arguments that led to (B.14), we obtain that
\[ [\mu_t^{(1)}(\theta, d)]^{2+i} \leq \left[ \mathbb{I}_{\{ v_t(\tau) \leq X_t \leq v_t(\theta) \}} + \mathbb{I}_{\{ v_t(\theta) < X_t \leq v_t(\tau) \}} \right] \]
\[ \times \left( \sup_{\| \theta - \theta^0 \| \leq d_0} \| \nabla v_t(\tau) \| \right)^{2+i} + \left( \sup_{\| \tau - \theta \| \leq d} \| \nabla v_t(\tau) - \nabla v_t(\theta) \| \right)^{2+i}. \]
By the LIE,
\[
\mathbb{E}\left[ \mu_t^{(1)}(\theta, d) \right]^{2+\delta} \leq \mathbb{E}\left[ \mathbb{E}_{t-1}\left[ \mathbb{I}\left\{ \nu_t(\tau) \leq X_t \leq \nu_t(\theta) \right\} + \mathbb{I}\left\{ \nu_t(\theta) < X_t \leq \nu_t(\tau) \right\} \right] \right. \\
\times \left( \sup_{\|\theta - \theta_0\| \leq d_0} \|\nabla \nu_t(\theta)\| \right)^{2+\epsilon} + \mathbb{E}\left[ \sup_{\|\tau - \theta\| \leq d} \|\nabla \nu_t(\tau) - \nabla \nu_t(\theta)\| \right]^{2+\epsilon}.
\]
(B.18)

From (B.15)–(B.16),
\[
\mathbb{E}_{t-1}\left[ \mathbb{I}\left\{ \nu_t(\tau) \leq X_t \leq \nu_t(\theta) \right\} + \mathbb{I}\left\{ \nu_t(\theta) < X_t \leq \nu_t(\tau) \right\} \right] \leq 3K V_1(F_{t-1}) d.
\]

From Assumption 1 (ix),
\[
\left( \sup_{\|\theta - \theta_0\| \leq d_0} \|\nabla \nu_t(\theta)\| \right)^{2+\epsilon} \leq V_1^{2+i}(F_{t-1}),
\]
and from (B.17)
\[
\left( \sup_{\|\tau - \theta\| \leq d} \|\nabla \nu_t(\tau) - \nabla \nu_t(\theta)\| \right)^{2+\epsilon} = \sup_{\|\tau - \theta\| \leq d} \|\nabla \nu_t(\tau) - \nabla \nu_t(\theta)\| \\
\times \left( \sup_{\|\tau - \theta\| \leq d} \|\nabla \nu_t(\theta) - \nabla \nu_t(\theta)\| \right)^{1+\epsilon}
\]
\[
\leq V_2(F_{t-1}) d \times 2V_1^{1+i}(F_{t-1}).
\]
(B.19)

Inserting the above three relations into (B.18), we obtain (using Assumption 2 (v)) that
\[
\mathbb{E}\left[ \mu_t^{(1)}(\theta, d) \right]^{2+\epsilon} \leq \mathbb{E}\left[ 3K V_1(F_{t-1}) d \times V_1^{2+i}(F_{t-1}) \right] + \mathbb{E}\left[ V_2(F_{t-1}) d \times 2V_1^{1+i}(F_{t-1}) \right]
\]
\[
\leq 3K \mathbb{E}\left[ V_1^{3+i}(F_{t-1}) \right] d + 2\left\{ \mathbb{E}\left[ V_1^{3+i}(F_{t-1}) \right] \right\}^{(1+i)/(3+i)} \left\{ \mathbb{E}\left[ V_2^{(3+i)/2}(F_{t-1}) \right] \right\}^{2/(3+i)} d
\]
\[
\leq C d,
\]
where the penultimate step follows from Hölder’s inequality.

**Second term:** By definition of \( \mu_t^{(2)}(\theta, d) \) and exploiting (B.19), we have that
\[
\mathbb{E}\left[ \mu_t^{(2)}(\theta, d) \right]^{2+\epsilon} = \beta^{2+\epsilon} \mathbb{E}\left[ \sup_{\|\tau - \theta\| \leq d} \|\nabla \nu_t(\tau) - \nabla \nu_t(\theta)\| \right]^{2+\epsilon}
\]
\[
\leq \beta^{2+\epsilon} \mathbb{E}\left[ V_2(F_{t-1}) d \times 2V_1^{1+i}(F_{t-1}) \right]
\]
\[
\leq 2\beta^{2+\epsilon} \left\{ \mathbb{E}\left[ V_1^{3+i}(F_{t-1}) \right] \right\}^{(1+i)/(3+i)} \left\{ \mathbb{E}\left[ V_2^{(3+i)/2}(F_{t-1}) \right] \right\}^{2/(3+i)} d
\]
\[
\leq C d.
\]

Combining the results for the first and second term, the conclusion follows. \( \square \)
**Lemma V.7.** Suppose Assumptions 1 and 2 hold. Then,
\[
E\|v_t(\theta_0^v)\|^{2+t} \leq C \quad \text{for all } t.
\]
In particular, condition (N4) of Weiss (1991) holds.

**Proof:** The claim follows easily from the definition of \(v_t(\cdot)\) in (B.1), which implies that
\[
E\|v_t(\theta_0^v)\|^{2+t} \leq E\|\nabla v_t(\theta^v)\|^{2+t} \leq E[V_t^{2+t}(F_{t-1})] \leq K
\]
by Assumption 1 (ix) and Assumption 2 (v).

**Lemma V.8.** Suppose Assumptions 1 and 2 hold. Then, as \(n \to \infty\),
\[
n^{-1/2} \sum_{t=1}^n v_t(\theta_0^v) \xrightarrow{d} N(0, V),
\]
where \(V = E[v_t(\theta_0^v)v_t'(\theta_0^v)] = \beta(1 - \beta)E[\nabla v_t(\theta_0^v)\nabla' v_t(\theta_0^v)]\).

**Proof:** By a Cramér–Wold device, it suffices to show the univariate convergence
\[
n^{-1/2} \sum_{t=1}^n \lambda' v_t(\theta_0^v) \xrightarrow{d} N(0, \lambda' V \lambda), \quad n \to \infty,
\]
for any conformable \(\lambda\) with \(\lambda' \lambda = 1\). This, however, follows easily from a standard central limit theorem for mixing random variables (e.g., White, 2001, Theorem 5.20), since \(E[\lambda' v_t(\theta_0^v)] = 0\), Lemma V.7 holds and the \(\{\lambda' v_t(\theta_0^v)\}_{t \in \mathbb{N}}\) are stationary and \(\alpha\)-mixing of size \(-\tilde{q}/(\tilde{q} - 2)\) with
\[
\text{Var}(\lambda' v_t(\theta_0^v)) = \lambda' \text{Var}(v_t(\theta_0^v)) \lambda = \lambda' E[v_t(\theta_0^v)v_t'(\theta_0^v)] \lambda = \lambda' V \lambda > 0.
\]
Note that
\[
V = E[v_t(\theta_0^v)v_t'(\theta_0^v)]
\]
\[
= E\left[\nabla v_t(\theta_0^v)\nabla' v_t(\theta_0^v)(1_{\{X_t \leq v_t(\theta_0^v)\}} - \beta)^2\right]
\]
\[
= E\left\{\nabla v_t(\theta_0^v)\nabla' v_t(\theta_0^v)E_{t-1}\left[(1_{\{X_t \leq v_t(\theta_0^v)\}} - \beta)^2\right]\right\}
\]
\[
= E\left\{\nabla v_t(\theta_0^v)\nabla' v_t(\theta_0^v)E_{t-1}\left[(1_{\{X_t \leq v_t(\theta_0^v)\}} - 2\beta 1_{\{X_t \leq v_t(\theta_0^v)\}} + \beta^2)\right]\right\}
\]
\[
= E\left\{\nabla v_t(\theta_0^v)\nabla' v_t(\theta_0^v)\left[(\beta - 2\beta^2 + \beta^2)\right]\right\}
\]
\[
= \beta(1 - \beta)E[\nabla v_t(\theta_0^v)\nabla' v_t(\theta_0^v)] \quad \text{(B.20)}
\]
is positive definite by Assumption 2 (viii). This concludes the proof.
Proof of Theorem 2: In this first part of the proof, we show asymptotic normality of \( \hat{\theta}_n^u \). From Lemma V.1, we have the expansion
\[
\sqrt{n}(\hat{\theta}_n^u - \theta_0^u) = \left[ A^{-1} + o_P(1) \right] \left[ -\frac{1}{\sqrt{n}} \sum_{t=1}^n v_t(\theta_0^u) + o_P(1) \right].
\]
From this, Slutsky’s theorem and Lemma V.8, we then obtain that, as \( n \to \infty \),
\[
\sqrt{n}(\hat{\theta}_n^u - \theta_0^u) \overset{d}{\to} N(0, A^{-1}V A^{-1}),
\]
which is the claimed result.

B.2 Asymptotic Normality of the CoVaR Parameter Estimator

The proof of the asymptotic normality of \( \hat{\theta}_n^u \) requires some further preliminary notation and lemmas. To see the analogy to the proof of the asymptotic normality of \( \hat{\theta}_n^u \) clearer, we label the lemmas as Lemma C.1, C.2, . . . .

For \( c \neq y \), it holds that
\[
\frac{\partial}{\partial c} S^\text{CoVaR}((v, c'), (x, y')) = \mathbb{1}_{\{x > v\}} \left[ \mathbb{1}_{\{y \leq c\}} - \alpha \right].
\]
Thus, by the chain rule, it a.s. holds that
\[
c_t(\theta^c, \theta^u) := \frac{\partial}{\partial \theta} S^\text{CoVaR}((v_t(\theta^u), c_t(\theta^c)), (X_t, Y_t)') = \mathbb{1}_{\{X_t > v_t(\theta^u)\}} \nabla c_t(\theta^c) \left[ \mathbb{1}_{\{Y_t \leq c_t(\theta^c)\}} - \alpha \right].
\]
(B.21)

This implies by the LIE and Assumption 1 (vii) that, with \( \mathbb{P}_{t-1}\{\cdot\} = \mathbb{P}\{\cdot \mid \mathcal{F}_{t-1}\} \),
\[
\mathbb{E}[c_t(\theta^c, \theta^u)] = \mathbb{E} \left\{ \nabla c_t(\theta^c) \left[ \mathbb{P}_{t-1}\{X_t > v_t(\theta^u), Y_t \leq c_t(\theta^c)\} - \alpha \mathbb{P}_{t-1}\{X_t > v_t(\theta^u)\} \right] \right\} = \mathbb{E} \left\{ \nabla c_t(\theta^c) \left[ F_t^V(c_t(\theta^c)) - F_t(v_t(\theta^u), c_t(\theta^c)) - \alpha \{1 - F_t^X(v_t(\theta^u))\} \right] \right\}.
\]

Finally, Assumptions 1–2 and the dominated convergence theorem allow us to interchange differentiation and expectation to yield that
\[
\frac{\partial}{\partial \theta^c} \mathbb{E}[c_t(\theta^c, \theta^u)] = \mathbb{E} \left\{ \nabla^2 c_t(\theta^c) \left[ F_t^V(c_t(\theta^c)) - F_t(v_t(\theta^u), c_t(\theta^c)) - \alpha \{1 - F_t^X(v_t(\theta^u))\} \right] \right\}
+ \mathbb{E} \left\{ \nabla c_t(\theta^c) \nabla c_t(\theta^c) \left[ f_t^V(c_t(\theta^c)) - \partial_2 F_t(v_t(\theta^u), c_t(\theta^c)) \right] \right\},
\]
(B.22)
\[
\frac{\partial}{\partial \theta^u} \mathbb{E}[c_t(\theta^c, \theta^u)] = \mathbb{E} \left\{ \nabla c_t(\theta^c) \nabla v_t(\theta^u) \left[ \alpha f_t^X(v_t(\theta^u)) - \partial_1 F_t(v_t(\theta^u), c_t(\theta^c)) \right] \right\},
\]
(B.23)
where \( \partial_i F_t(\cdot, \cdot) \) denotes the partial derivative with respect to the \( i \)-th component \( (i = 1, 2) \). Evalu-
ating these quantities at the true parameters gives

\[ \Lambda_{(1)} = \frac{\partial}{\partial \theta^v} \mathbb{E}[c_t(\theta^c, \theta^v)] \bigg| \theta^c = \theta^c_0, \theta^v = \theta^v_0 = \mathbb{E} \left\{ \nabla c_t(\theta^c_0) \nabla' c_t(\theta^v_0) \left[ f_t^y(\theta^c_0) \right] - \partial_2 F_t(v_t(\theta^v_0), c_t(\theta^c_0)) \right\}, \]

\[ \Lambda_{(2)} = \frac{\partial}{\partial \theta^v} \mathbb{E}[c_t(\theta^c, \theta^v)] \bigg| \theta^c = \theta^c_0, \theta^v = \theta^v_0 = \mathbb{E} \left\{ \nabla c_t(\theta^c_0) \nabla' v_t(\theta^v_0) \left[ \alpha f_t^x(v_t(\theta^v_0)) \right] - \partial_1 F_t(v_t(\theta^v_0), c_t(\theta^c_0)) \right\}. \]

By virtue of Assumption 1 (ii), \( \Lambda_{(1)} \) and \( \Lambda_{(2)} \) do not depend on \( t \).

The crucial step in the proof is once again to apply Lemma A.1 in Weiss (1991). For this we require some preliminary results. Define

\[ \lambda_n(\theta^c, \theta^v) = \frac{1}{n} \sum_{t=1}^{n} \mathbb{E}[c_t(\theta^c, \theta^v)], \]

\[ A_{n,(1)}(\theta^c, \theta^v) = \frac{1}{n} \sum_{t=1}^{n} \frac{\partial}{\partial \theta^c} \mathbb{E}[c_t(\theta^c, \theta^v)] \bigg| \theta^c = \theta^c_0, \theta^v = \theta^v, \]

\[ A_{n,(2)}(\theta^c, \theta^v) = \frac{1}{n} \sum_{t=1}^{n} \frac{\partial}{\partial \theta^v} \mathbb{E}[c_t(\theta^c, \theta^v)] \bigg| \theta^c = \theta^c_0, \theta^v = \theta^v, \]

and note that \( \Lambda_{(1)} = A_{n,(1)}(\theta^c_0, \theta^v) \) and \( \Lambda_{(2)} = A_{n,(2)}(\theta^c_0, \theta^v) \) by stationarity.

Lemma C.1. Suppose Assumptions 1 and 2 hold. Then, as \( n \to \infty \),

\[ \sqrt{n}(\hat{\theta}^c_n - \theta^c_0) = \left[ \Lambda_{(1)}^{-1} A_{(2)}^{-1} + o_p(1) \right] \left[ \frac{1}{\sqrt{n}} \sum_{t=1}^{n} v_t(\theta^v_0) + o_p(1) \right] \]

\[ - \left[ \Lambda_{(1)}^{-1} + o_p(1) \right] \left[ \frac{1}{\sqrt{n}} \sum_{t=1}^{n} c_t(\hat{\theta}^c_{0,n}, \hat{\theta}^v_n) + o_p(1) \right], \]

where \( \theta^c_{0,n} \) is defined in the proof below.

Proof: The MVT and \( \lambda_n(\theta^c_0, \theta^v) = 0 \) imply that

\[ \lambda_n(\hat{\theta}^c_n, \hat{\theta}^v_n) = \lambda_n(\theta^c_0, \theta^v) + A_{n,(1)}(\theta^c, \theta^v)(\hat{\theta}^c_n - \theta^c_0) \]

\[ = \lambda_n(\theta^c_0, \theta^v) + A_{n,2}(\theta^c_0, \theta^v)(\hat{\theta}^v_n - \theta^v_0) + A_{n,1}(\theta^c, \hat{\theta}^v_n)(\hat{\theta}^c_n - \theta^c_0) \]

\[ = A_{n,2}(\theta^c_0, \theta^v)(\hat{\theta}^v_n - \theta^v_0) + A_{n,1}(\theta^c, \hat{\theta}^v_n)(\hat{\theta}^c_n - \theta^c_0) \]

for some \( \theta^c (\theta^v) \) on the line connecting \( \hat{\theta}^c_n \) and \( \theta^c_0 \) \( (\hat{\theta}^v_n \) and \( \theta^v_0 \). (Recall our convention to abuse notation slightly when applying the mean value theorem to multivariate functions. Again, the argument goes through componentwise, yet the notation would be more involved.) Thus, if the inverse exists,

\[ \sqrt{n}(\hat{\theta}^c_n - \theta^c_0) = A_{n,(1)}^{-1}(\theta^c, \hat{\theta}^v_n) \sqrt{n} \lambda_n(\hat{\theta}^c_n, \hat{\theta}^v_n) - A_{n,(1)}^{-1}(\theta^c, \hat{\theta}^v_n) A_{n,2}(\theta^c_0, \theta^v) \sqrt{n}(\hat{\theta}^v_n - \theta^v_0) \]

\[ = A_{n,(1)}^{-1}(\theta^c, \hat{\theta}^v_n) \sqrt{n} \lambda_n(\hat{\theta}^c_n, \hat{\theta}^v_n) + A_{n,(1)}^{-1}(\theta^c, \hat{\theta}^v_n) A_{n,2}(\theta^c_0, \theta^v) \sqrt{n}(\hat{\theta}^v_n - \theta^v_0) \]
(iii.b) \(1 \sum_{t=1}^{n} c_t(\theta_{0,n}^{c}, \hat{\theta}_n^{v}) = o_p(1)\); 

(iii.c) \(\hat{\theta}_n^{c} - \theta_{0,n}^{c} \xrightarrow{p} 0\). 

For (iii.a), note again that (N1) is immediate, as is the mixing condition (N5) (see Assumption 2 (x)). Establishing (N2) requires some more work than in Lemma V.1. To do so, we have to show that for each \(n\) there exists some \(\theta_{0,n}^{c} \in \Theta^c\) such that \(\lambda_n(\theta_{0,n}^{c}, \hat{\theta}_n^{v}) = 0\). Fix \(n \in \mathbb{N}\) and recall from the LIE that \(\lambda_n(\theta_{0}^{c}, \theta_{0}^{v}) = 0\). Next, we want to apply the implicit function theorem. This is possible because of the continuity of \(A_{n,(2)}(\cdot, \cdot)\) (see also the proof of Lemma C.2), the invertibility of 

\[ A_{(2)} = A_{n,(2)}(\theta_{0}^{c}, \theta_{0}^{v}) = \frac{1}{n} \sum_{t=1}^{n} \frac{\partial}{\partial \theta^v} \mathbb{E}[c_t(\theta^c, \theta^v)]\bigg|_{\theta^c = \theta_{0}^{c}, \theta^v = \theta_{0}^{v}} \] 

(by Assumption 2 (viii)) and the continuous differentiability of 

\[ \lambda_n(\cdot, \cdot) = \frac{1}{n} \sum_{t=1}^{n} \mathbb{E}[c_t(\cdot, \cdot)] \]

from (B.22), (B.23) and Assumption 2 (ii) and (vi). An application of the implicit function theorem now implies that there exists a neighborhood \(\mathcal{N}(\theta_{0}^{c}) \subset \Theta^c\) of the interior point \(\theta_{0}^{v}\), such that 

\[ \lambda_n(\theta_{n}^{c}(\theta^v), \theta^v) = 0 \]

holds for all \(\theta^v \in \mathcal{N}(\theta_{0}^{c})\) and some (unique) continuously differentiable function \(\theta_{n}^{c}(\cdot)\) satisfying \(\theta_{n}^{c}(\theta_{0}^{c}) = \theta_{0}^{c}\). By continuity of \(\theta_{n}^{c}(\cdot)\) this ensures that we can choose \(\mathcal{N}(\theta_{0}^{c})\) such that the image of the map \(\mathcal{N}(\theta_{0}^{c}) \ni \theta \mapsto \theta_{n}^{c}(\theta)\) is in \(\Theta^c\) (using also the fact that \(\theta_{0}^{c}\) is in the interior of \(\Theta^c\)).
conclude that on the set \( \{ \| \hat{\theta}_n^v - \theta_0^v \| < \varepsilon_0 \} \) with \( \varepsilon_0 > 0 \) chosen such that \( \hat{\theta}_n^v \in \mathcal{N}(\theta_0^v) \), there exists \( \theta_{0,n}^c \in \Theta^c \) (viz. \( \theta_{0,n}^c = \theta_n^v(\hat{\theta}_n^v) \)) with
\[
\lambda_n(\theta_{0,n}^c, \hat{\theta}_n^v) = 0.
\]
(B.24)

From the above it also follows that \( \theta_{0,n}^c \xrightarrow{p} \theta_0^c \).

Without further mention, in the following we work on the set \( \{ \| \hat{\theta}_n^v - \theta_0^v \| < \varepsilon_0 \} \) such that the existence of \( \theta_{0,n}^c \) satisfying (B.24) is guaranteed. This is possible, because
\[
\mathbb{P}\{ \sqrt{n}(\hat{\theta}_n^v - \theta_0^v) \leq \cdots \} = \mathbb{P}\{ \sqrt{n}(\hat{\theta}_n^v - \theta_0^v) \leq \cdots \}
\]
where we used Theorem 1 in the final step. Thus, it is on the set \( \{ \| \hat{\theta}_n^v - \theta_0^v \| < \varepsilon_0 \} \) that we verify (N3)–(N4) for proving asymptotic normality of \( \sqrt{n}(\hat{\theta}_n^v - \theta_0^v) \). Specifically, condition (N3) is verified in Lemmas C.4–C.6 below and the remaining condition (N4) in Lemma C.7. This verifies (iii.a).

The result in (iii.b) follows from Lemma C.3 and (iii.c) follows from Theorem 1 in combination with \( \theta_{0,n}^c \xrightarrow{p} \theta_0^c \). In sum, the desired result follows.

**Lemma C.2.** Suppose Assumptions 1 and 2 hold. Then, as \( n \to \infty \),
\[
\mathbb{A}_{n,1}(\theta^*, \hat{\theta}_n^v) \xrightarrow{p} \mathbb{A}_{1}(\theta^*) \quad \text{for any } \theta^* \xrightarrow{p} \theta_0^c \quad \text{and} \quad \hat{\theta}_n^v \xrightarrow{p} \theta_0^v,
\]
\[
\mathbb{A}_{n,2}(\theta_0^c, \theta^*) \xrightarrow{p} \mathbb{A}_{2}(\theta^*) \quad \text{for any } \theta^* \xrightarrow{p} \theta_0^v.
\]

**Proof:** We begin with the second statement. Our first goal is to show that
\[
\left\| \mathbb{A}_{n,2}(\theta_0^c, \theta^*) - \mathbb{A}_{2}(\theta^*) \right\| \leq C \left\| \theta^* - \theta_0^v \right\|.
\]

Exploit (B.23) to write
\[
\left\| \mathbb{A}_{n,2}(\theta_0^c, \theta^*) - \mathbb{A}_{2}(\theta^*) \right\| = \left\| \mathbb{A}_{n,2}(\theta_0^c, \theta^*) - \mathbb{A}_{n,2}(\theta_0^c, \theta_0^v) \right\|
\]
\[
= \left\| \frac{1}{n} \sum_{t=1}^{n} \mathbb{E} \left\{ \nabla c_t(\theta_0^v) \nabla' v_t(\theta^*) \left[ \alpha f_t^X(v_t(\theta^*)) - \partial_2 F_t(v_t(\theta^*), c_t(\theta_0^v)) \right] \right\} \right.
\]
\[
- \left. \mathbb{E} \left\{ \nabla c_t(\theta_0^c) \nabla' v_t(\theta_0^v) \left[ \alpha f_t^X(v_t(\theta_0^v)) - \partial_2 F_t(v_t(\theta_0^v), c_t(\theta_0^v)) \right] \right\} \right\|
\]
\[
\leq \frac{\alpha}{n} \sum_{t=1}^{n} \mathbb{E} \left\| \nabla c_t(\theta_0^c) \nabla' v_t(\theta^*) f_t^X(v_t(\theta^*)) - \nabla c_t(\theta_0^c) \nabla' v_t(\theta_0^v) f_t^X(v_t(\theta_0^v)) \right\|
\]
\[
+ \frac{1}{n} \sum_{t=1}^{n} \mathbb{E} \left\| \nabla c_t(\theta_0^c) \nabla' v_t(\theta^*) \partial_1 F_t(v_t(\theta^*), c_t(\theta_0^v)) - \nabla c_t(\theta_0^c) \nabla' v_t(\theta_0^v) \partial_1 F_t(v_t(\theta_0^v), c_t(\theta_0^v)) \right\|
\]
\[
=: A_{2n} + B_{2n}.
\]

We only consider \( A_{2n} \), because \( B_{2n} \) can be dealt with similarly. A mean value expansion around
\( \theta_0^n \) implies

\[
A_{2n} = \frac{\alpha}{n} \sum_{t=1}^{n} \mathbb{E} \left[ \nabla c_t(\theta_0^n) \left[ \nabla' v_t(\theta^*) - \nabla' v_t(\theta_0^n) \right] f_t^X(v_t(\theta^*)) \right. \\
+ \nabla c_t(\theta_0^n) \nabla' v_t(\theta_0^n) \left[ f_t^X(v_t(\theta^*)) - f_t^X(v_t(\theta_0^n)) \right] \\
\left. \leq \frac{\alpha}{n} \sum_{t=1}^{n} \mathbb{E} \left[ \nabla c_t(\theta_0^n) \left[ \theta^* - \theta_0^n \right]' \left[ \nabla^2 v_t(\theta^{**}) \right]' f_t^X(v_t(\theta^*)) \right] \\
+ K \mathbb{E} \left[ \nabla c_t(\theta_0^n) \nabla' v_t(\theta_0^n) \right] \left[ v_t(\theta^*) - v_t(\theta_0^n) \right] \right] \\
\leq \frac{\alpha}{n} \sum_{t=1}^{n} K \mathbb{E} \left[ C_1(\mathcal{F}_{t-1}) V_2(\mathcal{F}_{t-1}) \right] \left\| \theta^* - \theta_0^n \right\| \\
+ K \mathbb{E} \left[ C_1(\mathcal{F}_{t-1}) V_2^2(\mathcal{F}_{t-1}) \right] \left\| \theta^* - \theta_0^n \right\| \\
\leq \frac{C}{n} \sum_{t=1}^{n} \left\{ \mathbb{E} \left[ C_1^3(\mathcal{F}_{t-1}) \right] \right\}^{1/3} \left\{ \mathbb{E} \left[ V_2^3(\mathcal{F}_{t-1}) \right] \right\}^{2/3} \left\| \theta^* - \theta_0^n \right\| \\
+ \left\{ \mathbb{E} \left[ C_1^3(\mathcal{F}_{t-1}) \right] \right\}^{1/3} \left\{ \mathbb{E} \left[ V_2^3(\mathcal{F}_{t-1}) \right] \right\}^{2/3} \left\| \theta^* - \theta_0^n \right\| \\
\leq C \left\| \theta^* - \theta_0^n \right\|,
\]

where \( \theta^{**} \) is some mean value between \( \theta^* \) and \( \theta_0^n \). Using identical arguments, we may show that

\[
B_{2n} \leq C \left\| \theta^* - \theta_0^n \right\|.
\]

Hence,

\[
\left\| A_{n,(2)}(\theta_0^n, \theta^*) - A_{(2)} \right\| \leq C \left\| \theta^* - \theta_0^n \right\| = o_p(1) \tag{B.25}
\]

by the assumption that \( \theta^* \xrightarrow{p} \theta_0^n \). The second statement of the lemma follows.

To prove the first claim, decompose

\[
\left\| A_{(1)} - A_{n,(1)}(\theta^*, \hat{\theta}_n^n) \right\| \leq \left\| A_{(1)} - A_{n,(1)}(\theta_0^n, \hat{\theta}_n^n) \right\| + \left\| A_{n,(1)}(\theta_0^n, \hat{\theta}_n^n) - A_{n,(1)}(\theta^*, \hat{\theta}_n^n) \right\| \\
= : A_{3n} + B_{3n}. \tag{B.26}
\]

First, consider \( A_{3n} \). Use (B.22) to write

\[
A_{3n} = \left\| A_{(1)} - A_{n,(1)}(\theta_0^n, \hat{\theta}_n^n) \right\| - \left\| A_{n,(1)}(\theta_0^n, \hat{\theta}_n^n) - A_{n,(1)}(\theta_0^n, \hat{\theta}_n^n) \right\| \\
= \left\{ \frac{1}{n} \sum_{t=1}^{n} \mathbb{E} \left[ \nabla c_t(\theta_0^n) \nabla' c_t(\theta_0^n) \left[ f_t^Y(c_t(\theta_0^n)) - \partial_2 F_t(v_t(\theta_0^n), c_t(\theta_0^n)) \right] \right. \\
- \mathbb{E} \left. \left[ \nabla c_t(\theta_0^n) \nabla' c_t(\theta_0^n) \left[ f_t^Y(c_t(\theta_0^n)) - \partial_2 F_t(v_t(\theta_0^n), c_t(\theta_0^n)) \right] \right] \right. \\
+ \mathbb{E} \left. \left[ \nabla^2 c_t(\theta_0^n) \left[ F_t^Y(c_t(\theta_0^n)) - F_t(v_t(\theta_0^n), c_t(\theta_0^n)) - \alpha \{ 1 - F_t^X(v_t(\theta_0^n)) \} \right] \right] \right\}
\]

20
where $\theta^*$ is some value on the line connecting $\theta^*_n$ and $\hat{\theta}_n$, and the penultimate step follows from Hölder's inequality.

First term ($\Delta_0$): A mean value expansion around $\theta^*_n$ gives

Second term ($\Delta_1$): A mean value expansion around $\theta^*_n$ gives

where $\theta^*$ is some value on the line connecting $\theta^*_n$ and $\hat{\theta}_n$, and the penultimate step follows from Hölder's inequality:

Third term ($\Delta_2$): A mean value expansion around $\theta^*_n$ gives

$\|\theta - \theta^*\|_{\infty} \leq C\|\theta^*_n - \theta^*\|_1$.
\[
\leq K \mathbb{E}\left[C_2(F_{t-1}) V_1(F_{t-1}) \right] \left\| \widehat{\theta}_n^v - \theta_0^v \right\|
\]
\[
\leq K \left\{ \mathbb{E}\left[C_2^{3/2}(F_{t-1}) \right] \right\}^{2/3} \left\{ \mathbb{E}\left[V_1^3(F_{t-1}) \right] \right\}^{1/3} \left\| \widehat{\theta}_n^v - \theta_0^v \right\|
\]
\[
\leq C \left\| \widehat{\theta}_n^v - \theta_0^v \right\|
\]

where \( \theta^* \) is some value on the line connecting \( \theta_0^v \) and \( \widehat{\theta}_n^v \), and the penultimate step follows from Hölder’s inequality.

Combining the results for the different terms, we conclude that

\[
A_{3n} = \left\| A_{1(1)} - A_{n,(1)}(\theta_0^v, \widehat{\theta}_n^v) \right\| \leq C \left\| \widehat{\theta}_n^v - \theta_0^v \right\|. \quad \text{(B.27)}
\]

Now, consider \( B_{3n} \). For this, write

\[
\Lambda_{n,(1)}(\theta^*, \widehat{\theta}_n^v) - \Lambda_{n,(1)}(\theta_0^v, \widehat{\theta}_n^v)
\]
\[
= \frac{1}{n} \sum_{t=1}^{n} \mathbb{E}\left\{ \nabla^2 c_t(\theta^*) \left[ F_t^Y(c_t(\theta^*)) - F_t(v_t(\widehat{\theta}_n^v), c_t(\theta^*)) - \alpha \{ 1 - F_t^X(v_t(\widehat{\theta}_n^v)) \} \right] \right\}
\]
\[
- \frac{1}{n} \sum_{t=1}^{n} \mathbb{E}\left\{ \nabla^2 c_t(\theta_0^v) \left[ F_t^Y(c_t(\theta_0^v)) - F_t(v_t(\widehat{\theta}_n^v), c_t(\theta_0^v)) - \alpha \{ 1 - F_t^X(v_t(\widehat{\theta}_n^v)) \} \right] \right\}
\]
\[
+ \frac{1}{n} \sum_{t=1}^{n} \mathbb{E}\left\{ \nabla c_t(\theta^*) \nabla' c_t(\theta^*) f_t^Y(c_t(\theta^*)) \right\}
\]
\[
- \frac{1}{n} \sum_{t=1}^{n} \mathbb{E}\left\{ \nabla c_t(\theta_0^v) \nabla' c_t(\theta_0^v) f_t^Y(c_t(\theta_0^v)) \right\}
\]
\[
- \frac{1}{n} \sum_{t=1}^{n} \mathbb{E}\left\{ \nabla c_t(\theta^*) \nabla' c_t(\theta^*) \partial_2 F_t(v_t(\widehat{\theta}_n^v), c_t(\theta^*)) \right\}
\]
\[
+ \frac{1}{n} \sum_{t=1}^{n} \mathbb{E}\left\{ \nabla c_t(\theta_0^v) \nabla' c_t(\theta_0^v) \partial_2 F_t(v_t(\widehat{\theta}_n^v), c_t(\theta_0^v)) \right\}.
\]

**First term** \((B_{3n})\): Using a mean value expansion around \( \theta_0^v \), (B.24) and Assumption 2, we obtain that

\[
\left\| \frac{1}{n} \sum_{t=1}^{n} \mathbb{E}\left\{ \nabla^2 c_t(\theta^*) \left[ F_t^Y(c_t(\theta^*)) - F_t(v_t(\widehat{\theta}_n^v), c_t(\theta^*)) - \alpha \{ 1 - F_t^X(v_t(\widehat{\theta}_n^v)) \} \right] \right\} \right\|
\]
\[
- \frac{1}{n} \sum_{t=1}^{n} \mathbb{E}\left\{ \nabla^2 c_t(\theta_0^v) \left[ F_t^Y(c_t(\theta_0^v)) - F_t(v_t(\widehat{\theta}_n^v), c_t(\theta_0^v)) - \alpha \{ 1 - F_t^X(v_t(\widehat{\theta}_n^v)) \} \right] \right\} \right\|
\]
\[
\leq \frac{1}{n} \sum_{t=1}^{n} \left\| \nabla^2 c_t(\theta^*) F_t^Y(c_t(\theta^*)) - \nabla^2 c_t(\theta_0^v) F_t^Y(c_t(\theta_0^v)) \right\|
\]
\[
+ \frac{1}{n} \sum_{t=1}^{n} \left\| \nabla^2 c_t(\theta^*) F_t(v_t(\widehat{\theta}_n^v), c_t(\theta^*)) - \nabla^2 c_t(\theta_0^v) F_t(v_t(\widehat{\theta}_n^v), c_t(\theta_0^v)) \right\|
\]
\[
+ \frac{\alpha}{n} \sum_{t=1}^{n} \mathbb{E}\left\| \nabla^2 c_t(\theta^*) - \nabla^2 c_t(\theta_0^c) \right\| \left\{ 1 - F_{t,X}^n(v_t(\hat{\theta}_n^v)) \right\} \\
\leq \frac{1}{n} \sum_{t=1}^{n} \mathbb{E}\left\| \nabla^2 c_t(\theta^*) - \nabla^2 c_t(\theta_0^c) \right\| F_t^Y(c_t(\theta^*)) \\
+ \frac{1}{n} \sum_{t=1}^{n} \mathbb{E}\left\| \nabla^2 c_t(\theta_0^c) \left[ F_t^Y(c_t(\theta^*)) - F_t^Y(c_t(\theta_0^c)) \right] \right\| \\
+ \frac{1}{n} \sum_{t=1}^{n} \mathbb{E}\left\| \nabla^2 c_t(\theta_0^c) \left[ F_t(v_t(\hat{\theta}_n^v), c_t) - F_t(v_t(\hat{\theta}_n^v), c_t(\theta_0^c)) \right] \right\| \\
+ \frac{\alpha}{n} \sum_{t=1}^{n} \mathbb{E}[C_3(F_{t-1})] \left\| \theta^* - \theta_0^c \right\| \\
\leq \frac{1}{n} \sum_{t=1}^{n} \mathbb{E}[C_3(F_{t-1})] \left\| \theta^* - \theta_0^c \right\| \\
+ \frac{1}{n} \sum_{t=1}^{n} \mathbb{E}\left[ \nabla^2 c_t(\theta_0^c) \nabla' c_t(\theta^{**}) f_t^Y(c_t(\theta^{**})) \right] \left\| \theta^* - \theta_0^c \right\| \\
+ \frac{1}{n} \sum_{t=1}^{n} \mathbb{E}[C_3(F_{t-1})] \left\| \theta^* - \theta_0^c \right\| \\
+ \frac{1}{n} \sum_{t=1}^{n} \mathbb{E}\left[ \nabla^2 c_t(\theta_0^c) \nabla' c_t(\theta^{**}) \partial_2 F_t(v_t(\hat{\theta}_n^v), c_t(\theta^{**})) \right] \left\| \theta^* - \theta_0^c \right\| \\
+ \frac{\alpha}{n} \sum_{t=1}^{n} \mathbb{E}[C_3(F_{t-1})] \left\| \theta^* - \theta_0^c \right\| \\
\leq \frac{1}{n} \sum_{t=1}^{n} \mathbb{E}[C_3(F_{t-1})] \left\| \theta^* - \theta_0^c \right\| \\
+ \frac{K}{n} \sum_{t=1}^{n} \mathbb{E}[C_2(F_{t-1})C_1(F_{t-1})] \left\| \theta^* - \theta_0^c \right\| \\
+ \frac{1}{n} \sum_{t=1}^{n} \mathbb{E}[C_3(F_{t-1})] \left\| \theta^* - \theta_0^c \right\| \\
+ \frac{K}{n} \sum_{t=1}^{n} \mathbb{E}[C_2(F_{t-1})C_1(F_{t-1})] \left\| \theta^* - \theta_0^c \right\| \\
+ \frac{\alpha}{n} \sum_{t=1}^{n} \mathbb{E}[C_3(F_{t-1})] \left\| \theta^* - \theta_0^c \right\| \\
\leq \frac{2 + \alpha}{n} \sum_{t=1}^{n} \mathbb{E}[C_3(F_{t-1})] \left\| \theta^* - \theta_0^c \right\| + \frac{2K}{n} \sum_{t=1}^{n} \left\{ \mathbb{E}[C_2^{3/2}(F_{t-1})] \right\}^{2/3} \left\{ \mathbb{E}[C_1^3(F_{t-1})] \right\}^{1/3} \left\| \theta^* - \theta_0^c \right\| \\
\leq C \left\| \theta^* - \theta_0^c \right\|,
\]
where $\theta^{**}$ is some value between $\theta^*$ and $\theta^*_0$ that may change from line to line.

**Second term $(B_{3n})$:** This term is dealt with similarly as the second term in the proof of Lemma V.4. Expand

$$
\left\| \frac{1}{n} \sum_{t=1}^{n} \mathbb{E} \left[ \nabla c_t(\theta^*) \nabla' c_t(\theta^*) f_t^Y(c_t(\theta^*)) - \nabla c_t(\theta^*_0) \nabla' c_t(\theta^*_0) f_t^Y(c_t(\theta^*_0)) \right] \right\| 
$$

$$
= \left\| \frac{1}{n} \sum_{t=1}^{n} \mathbb{E} \left[ \nabla c_t(\theta^*) \nabla' c_t(\theta^*) f_t^Y(c_t(\theta^*)) - \nabla c_t(\theta^*_0) \nabla' c_t(\theta^*_0) f_t^Y(c_t(\theta^*_0)) \right. 
+ \nabla c_t(\theta^*_0) \nabla' c_t(\theta^*) f_t^Y(c_t(\theta^*)) - \nabla c_t(\theta^*_0) \nabla' c_t(\theta^*_0) f_t^Y(c_t(\theta^*_0)) 
+ \left. \nabla c_t(\theta^*_0) \nabla' c_t(\theta^*_0) f_t^Y(c_t(\theta^*_0)) - \nabla c_t(\theta^*_0) \nabla' c_t(\theta^*_0) f_t^Y(c_t(\theta^*_0)) \right] \right\| 
$$

$$
= \left\| \frac{1}{n} \sum_{t=1}^{n} \mathbb{E} \left[ \nabla^2 c_t(\theta^{**})(\theta^* - \theta^*_0) \nabla' c_t(\theta^*) f_t^Y(c_t(\theta^*)) 
+ \nabla c_t(\theta^*_0)(\theta^* - \theta^*_0)'[\nabla^2 c_t(\theta^{**})]' f_t^Y(c_t(\theta^*)) 
+ \nabla c_t(\theta^*_0) \nabla' c_t(\theta^*_0) \{ f_t^Y(c_t(\theta^*)) - f_t^Y(c_t(\theta^*_0)) \} \right] \right\| 
$$

$$
\leq \frac{1}{n} \sum_{t=1}^{n} \mathbb{E} \left[ KC_2(\mathcal{F}_{t-1})C_1(\mathcal{F}_{t-1}) + KC_1(\mathcal{F}_{t-1})C_2(\mathcal{F}_{t-1}) + KC_1^3(\mathcal{F}_{t-1}) \right] \left\| \theta^* - \theta^*_0 \right\| 
$$

$$
\leq \frac{1}{n} \sum_{t=1}^{n} \left\{ 2K \left\{ \mathbb{E}[C_2^3(\mathcal{F}_{t-1})] \right\}^{1/3} \left\{ \mathbb{E}[C_2^3(\mathcal{F}_{t-1})] \right\}^{2/3} + K \mathbb{E}[C_3^3(\mathcal{F}_{t-1})] \right\} \left\| \theta^* - \theta^*_0 \right\| 
$$

$$
\leq C \left\| \theta^* - \theta^*_0 \right\|, 
$$

where $\theta^{**}$ is some mean value between $\theta^*$ and $\theta^*_0$ that may be different in different places.

**Third term $(B_{3n})$:** The third term can be dealt with similarly as the second term to show that

$$
\left\| \frac{1}{n} \sum_{t=1}^{n} \mathbb{E} \left[ \nabla c_t(\theta^*) \nabla' c_t(\theta^*) \partial_2 F_t(v_t(\hat{\theta}_n), c_t(\theta^*)) - \nabla c_t(\theta^*_0) \nabla' c_t(\theta^*_0) \partial_2 F_t(v_t(\hat{\theta}_n), c_t(\theta^*_0)) \right] \right\| 
$$

$$
\leq C \left\| \theta^* - \theta^*_0 \right\|. 
$$

Thus, we have shown that

$$
B_{3n} = \left\| A_{n,(1)}(\theta^*, \hat{\theta}_n) - A_{n,(1)}(\theta^*_0, \hat{\theta}_n) \right\| \leq C \left\| \theta^* - \theta^*_0 \right\|. \quad (B.28)
$$

Plugging (B.27) and (B.28) into (B.26) gives

$$
\left\| A_{(1)} - A_{n,(1)}(\theta^*, \hat{\theta}_n) \right\| \leq C \left\| \hat{\theta}_n - \theta^*_0 \right\| + C \left\| \theta^* - \theta^*_0 \right\|. 
$$

Using consistency of the parameter estimators, proves that $A_{n,(1)}(\theta^*, \hat{\theta}_n) \xrightarrow{p} A_{(1)}$. By positive definiteness of $A_{(1)} = A_{n,(1)}(\theta^*_0, \theta^*_0)$ and continuity of $A_{n,(1)}(\cdot, \cdot)$, $A_{n,(1)}^{-1}(\theta^*, \hat{\theta}_n) \xrightarrow{p} A_{(1)}^{-1}$ follows
Lemma C.3. Suppose Assumptions 1 and 2 hold. Then, as \( n \to \infty \),

\[
\frac{1}{\sqrt{n}} \sum_{t=1}^{n} c_t(\hat{\theta}_n, \hat{\theta}_n^v) = o_p(1).
\]

**Proof:** Recall from Assumption 1 (v) that \( \Theta^c \subseteq \mathbb{R}^q \), such that \( \Theta^c \) is a \( q \)-dimensional parameter vector. Let \( e_1, \ldots, e_q \) denote the standard basis of \( \mathbb{R}^q \). Define

\[
S_{\text{CoVaR}}(a) := \frac{1}{\sqrt{n}} \sum_{t=1}^{n} S^\text{CoVaR}\left( (v_t(\hat{\theta}_n^v), c_t(\hat{\theta}_n^c + ae_j))', (X_t, Y_t)' \right), \quad j = 1, \ldots, q,
\]

where \( a \in \mathbb{R} \). Let \( G_{j,n}(a) \) be the right partial derivative of \( S_{\text{CoVaR}}^n(a) \), such that (see (B.21))

\[
G_{j,n}(a) = \frac{1}{\sqrt{n}} \sum_{t=1}^{n} \mathbb{I}_{\{X_t > v_t(\hat{\theta}_n^v)\}} \nabla_j c_t(\hat{\theta}_n^c + ae_j)[\mathbb{I}_{\{Y_t \leq c_t(\hat{\theta}_n^c + ae_j)\}} - \alpha],
\]

where \( \nabla_j c_t(\cdot) \) is the \( j \)-th component of \( \nabla c_t(\cdot) \). Then, \( G_{j,n}(0) = \lim_{\xi \to 0} G_{j,n}(\xi) \) is the right partial derivative of

\[
S^\text{CoVaR}(\Theta^c) = \frac{1}{\sqrt{n}} \sum_{t=1}^{n} S^\text{CoVaR}(v_t(\hat{\theta}_n^v), c_t(\Theta^c))', (X_t, Y_t)'
\]

at \( \hat{\theta}_n^c \) in the direction \( \Theta_j^c \), where \( \Theta^c = (\Theta_1^c, \ldots, \Theta_q^c)' \). Correspondingly, \( \lim_{\xi \to 0} G_{j,n}(-\xi) \) is the left partial derivative. Because \( S_n^\text{CoVaR}(\cdot) \) achieves its minimum at \( \hat{\theta}_n^c \), the left derivative must be non-positive and the right derivative must be non-negative. Thus,

\[
|G_{j,n}(0)| \leq G_{j,n}(\xi) - G_{j,n}(-\xi)
\]

\[
= \frac{1}{\sqrt{n}} \sum_{t=1}^{n} \mathbb{I}_{\{X_t > v_t(\hat{\theta}_n^v)\}} \nabla_j c_t(\hat{\theta}_n^c + \xi e_j)[\mathbb{I}_{\{Y_t \leq c_t(\hat{\theta}_n^c + \xi e_j)\}} - \alpha]
\]

\[
- \frac{1}{\sqrt{n}} \sum_{t=1}^{n} \mathbb{I}_{\{X_t > v_t(\hat{\theta}_n^v)\}} \nabla_j c_t(\hat{\theta}_n^c - \xi e_j)[\mathbb{I}_{\{Y_t \leq c_t(\hat{\theta}_n^c - \xi e_j)\}} - \alpha].
\]

By continuity of \( \nabla c_t(\cdot) \) (see Assumption 2 (ii)) it follows upon letting \( \xi \to 0 \) that

\[
|G_{j,n}(0)| \leq \frac{1}{\sqrt{n}} \sum_{t=1}^{n} \mathbb{I}_{\{X_t > v_t(\hat{\theta}_n^v)\}} |\nabla_j c_t(\hat{\theta}_n^c)| \mathbb{I}_{\{Y_t = c_t(\hat{\theta}_n^c)\}}
\]

\[
\leq \frac{1}{\sqrt{n}} \sum_{t=1}^{n} |\nabla_j c_t(\hat{\theta}_n^c)| \mathbb{I}_{\{Y_t = c_t(\hat{\theta}_n^c)\}}
\]

\[
\leq \frac{1}{\sqrt{n}} \left[ \max_{t=1,\ldots,n} C_1(F_{t-1}) \right] \sum_{t=1}^{n} \mathbb{I}_{\{Y_t = c_t(\hat{\theta}_n^c)\}}. \tag{B.29}
\]
From subadditivity, Markov’s inequality and Assumption 2 (v),

\[
P\left\{ n^{-1/2} \max_{t=1,\ldots,n} C_1(F_{t-1}) > \varepsilon \right\} \leq \sum_{t=1}^n P\left\{ C_1(F_{t-1}) > \varepsilon n^{1/2} \right\} \\
\leq \sum_{t=1}^n \varepsilon^{-3} n^{-3/2} E[C^3(F_{t-1})] \\
= O(n^{-1/2}) = o(1).
\]

Combining this with Assumption 2 (ix), we obtain from (B.29) that

\[
|G_{j,n}(0)| \xrightarrow{a.s.} o_P(1) = o(1).
\]

As this holds for every \( j = 1, \ldots, q \), we get that

\[
\frac{1}{\sqrt{n}} \sum_{t=1}^n c_t(\hat{\theta}_n, \hat{\theta}_n^c) \overset{(B.21)}{=} \frac{1}{\sqrt{n}} \sum_{t=1}^n \mathbb{1}_{\{X_t > v_t(\hat{\theta}_n)\}} \nabla c_t(\hat{\theta}_n^c) [\mathbb{1}_{\{v_t(\hat{\theta}_n) \leq c_t(\hat{\theta}_n)\}} - \alpha] = o_P(1),
\]

which is just the conclusion. \( \square \)

**Lemma C.4.** Suppose Assumptions 1 and 2 hold. Then, condition (N3) (i) of Weiss (1991) holds, i.e.,

\[
\|\lambda_n(\theta, \hat{\theta}_n^c)\| \geq a\|\theta - \theta_{0,n}^c\| \quad \text{for} \quad \|\theta - \theta_{0,n}^c\| \leq d_0
\]

for sufficiently large \( n \) and some \( a > 0 \) and \( d_0 > 0 \).

**Proof:** Since we implicitly work on the set \( \{\|\hat{\theta}_n^c - \theta_{0,n}^c\| < \varepsilon_0\} \), \( \theta_{0,n}^c \) satisfying (B.24) exists. A mean value expansion around \( \theta_{0,n}^c \) together with (B.24) imply that

\[
\lambda_n(\theta, \hat{\theta}_n^c) = \lambda_n(\theta_{0,n}^c, \hat{\theta}_n^c) + \Lambda_{n,(1)}(\theta^*, \hat{\theta}_n^c)(\theta - \theta_{0,n}^c) \\
= \Lambda_{n,(1)}(\theta^*, \hat{\theta}_n^c)(\theta - \theta_{0,n}^c)
\]

for some \( \theta^* \) on the line connecting \( \theta_{0,n}^c \) and \( \theta \). Recall from stationarity that

\[
\Lambda_{(1)} = \Lambda_{n,(1)}(\theta_0^c, \theta_0^c) = \mathbb{E}\left\{ \nabla c_t(\theta_0^c) \nabla' c_t(\theta_0^c) \left[ f_t^Y(c_t(\theta_0^c)) - \partial_2 F_t(v_t(\theta_0^c), c_t(\theta_0^c)) \right] \right\}.
\]

By Assumption 2 (viii), \( \Lambda_{(1)} \) has eigenvalues bounded below by some \( a > 0 \), such that

\[
\|\lambda_n(\theta, \hat{\theta}_n^c)\| = \|\Lambda_{n,(1)}(\theta^*, \hat{\theta}_n^c)(\theta - \theta_{0,n}^c)\| \\
= \|\Lambda_{(1)}(\theta - \theta_{0,n}^c) - [\Lambda_{(1)} - \Lambda_{n,(1)}(\theta^*, \hat{\theta}_n^c)](\theta - \theta_{0,n}^c)\| \\
\geq \|\Lambda_{(1)}(\theta - \theta_{0,n}^c)\| - \| [\Lambda_{(1)} - \Lambda_{n,(1)}(\theta^*, \hat{\theta}_n^c)](\theta - \theta_{0,n}^c)\| \\
\geq a\|\theta - \theta_{0,n}^c\| - \|\Lambda_{(1)} - \Lambda_{n,(1)}(\theta^*, \hat{\theta}_n^c)\| \cdot \|\theta - \theta_{0,n}^c\|. \quad (B.30)
\]
Our goal in the following is to show that for sufficiently large $n$,

$$\| A(1) - A_{n,(1)}(\theta^*, \hat{\theta}_n^v) \| \leq c$$ (B.31)

for some $c \in (0, a)$, because then the lemma is established. We show (B.31) by decomposing the left-hand side into the three terms

$$\| A(1) - A_{n,(1)}(\theta^*, \hat{\theta}_n^v) \|
= \| [A(1) - A_{n,(1)}(\theta^*_0, \hat{\theta}_n^v)] + [A_{n,(1)}(\theta^*_0, \hat{\theta}_n^v) - A_{n,(1)}(\theta^*_{0,n}, \hat{\theta}_n^v)] + [A_{n,(1)}(\theta^*_{0,n}, \hat{\theta}_n^v) - A_{n,(1)}(\theta^*, \hat{\theta}_n^v)] \|
\leq \| A(1) - A_{n,(1)}(\theta^*_0, \hat{\theta}_n^v) \| + \| A_{n,(1)}(\theta^*_0, \hat{\theta}_n^v) - A_{n,(1)}(\theta^*_{0,n}, \hat{\theta}_n^v) \| + \| A_{n,(1)}(\theta^*_{0,n}, \hat{\theta}_n^v) - A_n^{(1)}(\theta^*, \hat{\theta}_n^v) \|
=: A_{4n} + B_{4n} + C_{4n}. \quad \text{(B.32)}$$

The term $A_{4n}$ is equal to $A_{3n}$ from the proof of Lemma C.2, where it is shown in (B.27) that

$$A_{4n} = A_{3n} \leq C \| \hat{\theta}_n^v - \theta^*_0 \|$$

Since we work on the set $\{ \| \hat{\theta}_n^v - \theta^*_0 \| < \varepsilon_0 \}$ with arbitrarily small $\varepsilon_0 > 0$, we can choose $\varepsilon_0 > 0$ such that $A_{4n}$ is arbitrarily small.

Now, consider $C_{4n}$. Reasoning similarly as for (B.28) in the proof of Lemma C.2, we obtain that

$$C_{4n} = \| A_{n,(1)}(\theta^*, \hat{\theta}_n^v) - A_{n,(1)}(\theta^*_{0,n}, \hat{\theta}_n^v) \| \leq C \| \theta^* - \theta^*_{0,n} \|,$$

which may be made arbitrarily small by a suitable choice of $d_0 > 0$.

Similarly, we may show that

$$B_{4n} = \| A_n^{(1)}(\theta^*_0, \hat{\theta}_n^v) - A_n^{(1)}(\theta^*_{0,n}, \hat{\theta}_n^v) \| \leq C \| \theta^*_0 - \theta^*_0 \|$$

which again may be made arbitrarily small because—by definition of $\theta^*_0$—we have that for some mean value $\theta^*$,

$$\| \theta^*_0 - \theta^*_0 \| = \| \theta^*_{0,n}(\theta^*_0) - \theta^*_{0,n}(\hat{\theta}_n^v) \|
= \| \nabla \theta^*_{0,n}(\theta^*)(\theta^*_0 - \hat{\theta}_n^v) \|
\leq C \| \theta^*_0 - \hat{\theta}_n^v \|. \quad \text{(B.33)}$$

Hence, by working on the set $\{ \| \hat{\theta}_n^v - \theta^*_0 \| < \varepsilon_0 \}$, $B_{4n}$ can be made arbitrarily small by a suitable choice of $\varepsilon_0 > 0$.

In total, we have shown that $A_{4n} + B_{4n} + C_{4n}$ can be made arbitrarily small, which—by (B.32)—establishes (B.31). Combining this with (B.30), the conclusion follows. \qed
The remainder of the proof is almost identical to that of Lemma V.5. One merely has to replace \( c \) of Assumption 2 (iv). Recalling the definition of suitable choice of \( d \) where we used (B.33) and the fact that we work on the set \( \theta \) neighborhood of Assumption 2 (iv). This is possible because for all \( \theta \in \mathcal{N}(\theta_{0,n}^c) \), we have that

\[
\| \theta - \theta_0^c \| = \left\| \theta - \theta_{0,n}^c + \theta_{0,n}^c - \theta_0^c \right\| \\
\leq \| \theta - \theta_{0,n}^c \| + \| \theta_{0,n}^c - \theta_0^c \| \\
\leq d_0 + C \| \theta_{0,n}^c - \theta_0^c \| \\
\leq d_0 + C \varepsilon_0,
\]

where we used (B.33) and the fact that we work on the set \( \{ \| \hat{\theta}_n^\nu - \theta_0^c \| < \varepsilon_0 \} \). Therefore, by a suitable choice of \( d_0 \) and \( \varepsilon_0 > 0 \), we may ensure that any \( \theta \in \mathcal{N}(\theta_{0,n}^c) \) is in the neighborhood of \( \theta_0^c \) of Assumption 2 (iv). Recalling the definition of \( c_t(\theta_0^c, \theta_0^c) \) from (B.21), we decompose

\[
\mu_t(\theta, d) = \sup_{\| \tau - \theta \| \leq d} \left\| \mathbb{I}_{\{X_t > v_t(\hat{\theta}_n^c)\}} \nabla c_t(\tau) \left[ \mathbb{I}_{\{Y_t \leq c_t(\tau)\}} - \alpha \right] - \mathbb{I}_{\{X_t > v_t(\hat{\theta}_n^c)\}} \nabla c_t(\theta) \left[ \mathbb{I}_{\{Y_t \leq c_t(\theta)\}} - \alpha \right] \right\| \\
= \sup_{\| \tau - \theta \| \leq d} \left\| \mathbb{I}_{\{X_t > v_t(\hat{\theta}_n^c)\}} \left[ \nabla c_t(\tau) \mathbb{I}_{\{Y_t \leq c_t(\tau)\}} - \nabla c_t(\theta) \mathbb{I}_{\{Y_t \leq c_t(\theta)\}} \right] \right\| \\
+ \alpha \sup_{\| \tau - \theta \| \leq d} \left\| \nabla c_t(\tau) \mathbb{I}_{\{Y_t \leq c_t(\tau)\}} - \nabla c_t(\theta) \right\| \\
=: \mu_t^{(1)}(\theta, d) + \mu_t^{(2)}(\theta, d).
\]

The remainder of the proof is almost identical to that of Lemma V.5. One merely has to replace \( \theta_0^c \) by \( \theta_{0,n}^c \), \( v_t(\cdot) \) by \( c_t(\cdot) \), and \( X_t \) by \( Y_t \). We nonetheless carry out the steps here to demonstrate where in the proof the various conditions of Assumption 2 come into play.

Define the \( \mathcal{F}_{t-1} \)-measurable quantities

\[
\tau := \arg \min_{\| \tau - \theta \| \leq d} c_t(\tau), \\
\overline{\tau} := \arg \max_{\| \tau - \theta \| \leq d} c_t(\tau).
\]
which exist by continuity of \( c_t(\cdot) \).

We first consider \( \mu_t^{(1)}(\theta, d) \). To take the indicators out of the supremum, we distinguish two cases:

**Case 1:** \( Y_t \leq c_t(\theta) \)

We further distinguish two cases (a)–(b).

(a) If \( Y_t < c_t(\tau) \), then both indicators are one, such that

\[
\mu_t^{(1)}(\theta, d) = \sup_{\|\tau - \theta\| \leq d} \left\| \nabla c_t(\tau) - \nabla c_t(\theta) \right\|.
\]

(b) If \( c_t(\tau) \leq Y_t \leq c_t(\bar{\tau}) \), then

\[
\mu_t^{(1)}(\theta, d) = \max \left\{ \sup_{\|\tau - \theta\| \leq d, Y_t \leq c_t(\bar{\tau})} \left\| \nabla c_t(\tau) - \nabla c_t(\theta) \right\|, \|\nabla c_t(\theta)\| \right\}
\]

\[
\leq \sup_{\|\tau - \theta\| \leq d} \left\| \nabla c_t(\tau) - \nabla c_t(\theta) \right\| + \|\nabla c_t(\theta)\|. \tag{B.34}
\]

(Note that the third case that \( Y_t > c_t(\bar{\tau}) \) cannot occur, because already \( Y_t \leq c_t(\theta) \).)

**Case 2:** \( Y_t > c_t(\theta) \)

\[
\mu_t^{(1)}(\theta, d) = 1_{\{Y_t \leq c_t(\bar{\tau})\}} \sup_{\|\tau - \theta\| \leq d, Y_t \leq c_t(\bar{\tau})} \left\| \nabla c_t(\tau) \right\|
\]

\[
\leq 1_{\{Y_t \leq c_t(\bar{\tau})\}} \sup_{\|\tau - \theta\| \leq d} \left\| \nabla c_t(\tau) \right\|. \tag{B.35}
\]

Before combining the two cases, note that our assumptions \( \|\tau - \theta\| \leq d \) and \( \|\theta - \theta_{0,n}^6\| + d \leq d_0 \) together imply that \( \theta \) and \( \tau \) are in a \( d_0 \)-neighborhood of \( \theta_{0,n}^6 \). (For \( \theta \) this is immediate, and for \( \tau \) this follows from \( \|\tau - \theta_{0,n}^6\| = \|\tau - \theta + \theta - \theta_{0,n}^6\| \leq \|\tau - \theta\| + \|\theta - \theta_{0,n}^6\| \leq d + (d_0 - d) = d_0 \).)

Hence, for the final terms in (B.34) and (B.35), we have

\[
\left\| \nabla c_t(\theta) \right\| \leq \sup_{\|\tau - \theta\| \leq d} \left\| \nabla c_t(\tau) \right\| \leq \sup_{\|\theta - \theta_{0,n}^6\| \leq d_0} \left\| \nabla c_t(\theta) \right\|.
\]

Therefore, combining the results from Cases 1 and 2,

\[
\mu_t^{(1)}(\theta, d) \leq \left[ 1_{\{c_t(\bar{\tau}) \leq Y_t \leq c_t(\theta)\}} + 1_{\{c_t(\theta) < Y_t \leq c_t(\tau)\}} \right] \sup_{\|\theta - \theta_{0,n}^6\| \leq d} \left\| \nabla c_t(\theta) \right\|
\]

\[
+ \sup_{\|\tau - \theta\| \leq d} \left\| \nabla c_t(\tau) - \nabla c_t(\theta) \right\|.
\]

By Assumption 2 (vii) we have

\[
E_{t-1} \left[ 1_{\{c_t(\bar{\tau}) \leq Y_t \leq c_t(\theta)\}} \right] = \int_{c_t(\bar{\tau})}^{c_t(\theta)} f^Y_r(x) dx
\]

29
\[ \leq K|c_t(\theta) - c_t(\tau)| = K|\nabla c_t(\theta^*)(\theta - \tau)| \]
\[ \leq KC_1(F_{t-1})\|\theta - \tau\| \leq KC_1(F_{t-1})d, \]

and, similarly,
\[ \mathbb{E}_{t-1}\left[ \mathbbm{1}_{\{c_t(\theta) < Y_t \leq c_t(\tau)\}} \right] \leq KC_1(F_{t-1})d. \]

Moreover, we have by the MVT that for some \( \theta^* \) on the line connecting \( \tau \) and \( \theta \),
\[ \sup_{\|\tau - \theta\| \leq d} \|\nabla c_t(\tau) - \nabla c_t(\theta)\| = \sup_{\|\tau - \theta\| \leq d} \|\nabla^2 c_t(\theta^*)(\tau - \theta)\| \]
\[ \leq C_2(F_{t-1})\|\tau - \theta\| \leq C_2(F_{t-1})d. \] (B.36)

Therefore,
\[ \mathbb{E}\left[ \mu^{(1)}_t(\theta, d) \right] \leq \mathbb{E}\left[ KC_2^2(F_{t-1}) \right]d + \mathbb{E}\left[ C_2(F_{t-1}) \right]d \leq Cd. \]

By arguments leading to (B.36), we also have that
\[ \mathbb{E}\left[ \mu^{(2)}_t(\theta, d) \right] \leq \beta \mathbb{E}\left[ C_2(F_{t-1}) \right]d \leq Cd. \]

Overall,
\[ \mathbb{E}\left[ \mu_t(\theta, d) \right] \leq \mathbb{E}\left[ \mu^{(1)}_t(\theta, d) \right] + \mathbb{E}\left[ \mu^{(2)}_t(\theta, d) \right] \leq bd \]
for some suitable \( b > 0 \), as desired.

Lemma C.6. Suppose Assumptions 1 and 2 hold. Then, condition (N3) (iii) of Weiss (1991) holds, i.e.,
\[ \mathbb{E}\left[ \mu^{(2)}_t(\theta, d) \right] \leq \beta \mathbb{E}\left[ C_2(F_{t-1}) \right]d \leq Cd. \]

Proof: Mutatis mutandis the proof follows the same lines as that of Lemma V.6 and, hence, is omitted.

Lemma C.7. Suppose Assumptions 1 and 2 hold. Then,
\[ \mathbb{E}\left[ \left\| c_t(\theta_0^n, \hat{\theta}_n^v) \right\|^{2^{2+1}} \right] \leq C \text{ for all } t. \]

In particular, condition (N4) of Weiss (1991) holds.

Proof: Note that from (B.33),
\[ \left\| \theta_0^v - \theta_0^n \right\| \leq C\left\| \theta_0^v - \hat{\theta}_n^v \right\|, \]
such that—on the set \( \left\{ \left\| \hat{\theta}_n^v - \theta_0^v \right\| < \varepsilon_0 \right\} \) —\( \theta_0^n \) is in the neighborhood of \( \theta_0^v \) of Assumption 2 (iv) for a suitable small \( \varepsilon_0 \). Therefore, from Assumption 2 (iv)–(v) and (B.21), we deduce that
\[ \mathbb{E}\left[ \left\| c_t(\theta_0^n, \hat{\theta}_n^v) \right\|^{2^{2+1}} \right] \leq \mathbb{E}\left[ \left\| \nabla c_t(\theta_0^n) \right\|^{2^{2+1}} \right] \leq \mathbb{E}\left[ C_1^{2+1}(F_{t-1}) \right] \leq K. \]
This ends the proof.

**Lemma C.8.** Suppose Assumptions 1 and 2 hold. Then, as $n \to \infty$,

$$n^{-1/2} \sum_{t=1}^{n} c_t(\theta_{0,n}^c, \tilde{\theta}_n^v) \xrightarrow{d} N(0, C^*),$$

where $C^* = \mathbb{E}[c_t(\theta_0^c, \theta_0^v)c_t'(\theta_0^c, \theta_0^v)] = \alpha(1 - \alpha)(1 - \beta)\mathbb{E}[\nabla c_t(\theta_0^c)\nabla' c_t(\theta_0^c)].$

**Proof:** Similarly as in Lemma C.7 it may be shown that

$$\mathbb{E}\|c_t(\theta_0^c, \theta_0^v)\|^{2 + t} \leq C \quad \text{for all } t. \quad (B.37)$$

Hence, following similar steps used in the proof of Lemma V.8, we can show that, as $n \to \infty$,

$$n^{-1/2} \sum_{t=1}^{n} c_t(\theta_{0,n}^c, \tilde{\theta}_n^v) \xrightarrow{d} N(0, C^*),$$

where, by the LIE and the definition of CoVaR,

$$C^* = \mathbb{E}[c_t(\theta_0^c, \theta_0^v)c_t'(\theta_0^c, \theta_0^v)]$$

$$= \mathbb{E} \left[ \nabla c_t(\theta_0^c)\nabla' c_t(\theta_0^c) \mathbb{I}_{\{X_t > v_t(\theta_0^v)\}} \left( \mathbb{I}_{\{Y_t \leq c_t(\theta_0^v)\}} \right)^2 \right]$$

$$= \mathbb{E} \left\{ \nabla c_t(\theta_0^c)\nabla' c_t(\theta_0^c) \mathbb{P}_{t-1} \left[ X_t > v_t(\theta_0^v), Y_t \leq c_t(\theta_0^v) \right] \right\}$$

$$= \mathbb{E} \left\{ \nabla c_t(\theta_0^c)\nabla' c_t(\theta_0^c) \left[ \mathbb{P}_{t-1} \left\{ Y_t \leq c_t(\theta_0^v) \mid X_t > v_t(\theta_0^v) \right\} (1 - \beta) \right. \right.$$

$$- 2\alpha \mathbb{P}_{t-1} \left\{ Y_t \leq c_t(\theta_0^v) \mid X_t > v_t(\theta_0^v) \right\} (1 - \beta) + \alpha^2 (1 - \beta) \right\}$$

$$= \mathbb{E} \left\{ \nabla c_t(\theta_0^c)\nabla' c_t(\theta_0^c) \left[ \alpha(1 - \beta) - 2\alpha^2 (1 - \beta) + \alpha^2 (1 - \beta) \right] \right\}$$

$$= \alpha(1 - \alpha)(1 - \beta)\mathbb{E}[\nabla c_t(\theta_0^c)\nabla' c_t(\theta_0^c)] \quad (B.38)$$

is positive definite by Assumption 2 (viii). Thus, to establish the lemma, we only have to show that

$$o_p(1) = \frac{1}{\sqrt{n}} \sum_{t=1}^{n} \left[ c_t(\theta_0^c, \theta_0^v) - c_t(\theta_{0,n}^c, \tilde{\theta}_n^v) \right]$$

$$= \frac{1}{\sqrt{n}} \sum_{t=1}^{n} \left\{ c_t(\theta_0^c, \theta_0^v) - \mathbb{E}[c_t(\theta_0^c, \theta_0^v)] \right\} - \left\{ c_t(\theta_{0,n}^c, \tilde{\theta}_n^v) - \mathbb{E}[c_t(\theta_{0,n}^c, \tilde{\theta}_n^v)] \right\}$$
+ \frac{1}{\sqrt{n}} \sum_{t=1}^{n} \left\{ c_t(\theta^*_0, \hat{\theta}^*_n) - E[c_t(\theta^*_0, \hat{\theta}^*_n)] \right\} - \left\{ c_t(\theta^*_{0,n}, \hat{\theta}^*_n) - E[c_t(\theta^*_{0,n}, \hat{\theta}^*_n)] \right\} \\
= A_{5n} + B_{5n},

where we used that \( E[c_t(\theta^*_0, \theta^*_0)] = 0 \) and \( 0 = \lambda_n(\theta^*_0, \hat{\theta}^*_n) = (1/n) \sum_{t=1}^{n} E[c_t(\theta^*_{0}, \hat{\theta}^*_n)]. \)

Our first goal is to show that \( B_{5n} = o_p(1) \). To task this, we first prove that

\[
\sqrt{n} \| \theta^*_{0,n} - \theta^*_0 \| = o_p(1). 
\]

(B.39)

Recall from the definition of the continuously differentiable function \( \theta^*_c(\cdot) \) above (B.24) that \( \theta^*_{0,n} - \theta^*_0 = \theta^*_n(\hat{\theta}^*_n) - \theta^*_n(\theta^*_0) \). Then, with \( \theta^*_c \) some mean value between \( \theta^*_0 \) and \( \hat{\theta}^*_n \),

\[
\theta^*_{0,n} - \theta^*_0 = \nabla \theta^*_c(\theta^*_c)(\hat{\theta}^*_n - \theta^*_0) \\
= O_p(1/\sqrt{n})
\]

by \( \sqrt{n} \)-consistency of \( \hat{\theta}^*_n \) from Theorem 2 and the boundedness (uniformly in \( n \)) of the continuous function \( \nabla \theta^*_c(\cdot) \) on compact intervals. Thus, (B.39) follows.

Now, we can exploit (B.39) to prove \( B_{5n} = o_p(1) \). Recall from the proof of Lemma C.1 that conditions (N1)–(N5) of Weiss (1991) are satisfied. Therefore, Lemma A.1 of Weiss (1991) implies

\[
\sup_{\| \theta - \theta^*_0 \| \leq d_0} \left\| \frac{1}{\sqrt{n}} \sum_{t=1}^{n} \left\{ c_t(\theta, \hat{\theta}^*_n) - E[c_t(\theta, \hat{\theta}^*_n)] \right\} - \left\{ c_t(\theta^*_{0,n}, \hat{\theta}^*_n) - E[c_t(\theta^*_{0,n}, \hat{\theta}^*_n)] \right\} \right\| = o_p(1).
\]

(B.40)

We can bound the term in the denominator (using a mean value expansion around \( \theta^*_0 \) and (B.22)) as follows:

\[
\left\| \frac{1}{n} \sum_{t=1}^{n} E[c_t(\theta, \hat{\theta}^*_n)] \right\| = \left\| \frac{1}{n} \sum_{t=1}^{n} E[c_t(\theta^*_0, \hat{\theta}^*_n)] + \frac{1}{n} \sum_{t=1}^{n} \frac{\partial}{\partial \theta^*} E[c_t(\theta^*, \theta^*)]_{\theta^* = \theta^*_0} \right\| \\
= \left\| \frac{1}{n} \sum_{t=1}^{n} E\left\{ \nabla^2 c_t(\theta^*) \left[ F_t^Y \left( c_t(\theta^*) \right) - F_t \left( v_t(\hat{\theta}^*_n), c_t(\theta^*) \right) \right] - \alpha \left\{ 1 - F_t^X \left( v_t(\theta^*_0) \right) \right\} \right\} (\theta - \theta^*_0) \right\| \\
+ \frac{1}{n} \sum_{t=1}^{n} E\left\{ \nabla c_t(\theta^*) \right\} \left( \frac{\partial}{\partial \theta^*} E[c_t(\theta^*, \theta^*)]_{\theta^* = \theta^*_0} \right\} (\theta - \theta^*_0) \right\| \\
\leq \frac{1}{n} \sum_{t=1}^{n} 3 E \left[ C_2(F_t-1) \right] \| \theta - \theta^*_0 \| + \frac{1}{n} \sum_{t=1}^{n} 2 K E \left[ C_2^2(F_t-1) \right] \| \theta - \theta^*_0 \| \right\| \\
\leq C_0 d_0,
\]

(B.41)

where we also used Assumption 2 in the final two steps. For any \( \varepsilon > 0 \), write

\[
P\left\{ \| B_{5n} \| > \varepsilon \right\} \leq P\left\{ \| B_{5n} \| > \varepsilon, \| \theta^*_0 - \theta^*_0 \| \leq C/\sqrt{n} \right\} + P\left\{ \| \theta^*_0 - \theta^*_0 \| > C/\sqrt{n} \right\}
\]

32
By (B.39), for any $\delta > 0$ we may choose $C > 0$ sufficiently large, such that $\mathbb{P}\{\|\theta^n_0 - \hat{\theta}^{n}_{0,n}\| > C/\sqrt{n}\} < \delta/2$. Therefore, $B_{5n} = o_p(1)$ follows, if we can show that the first term on the right-hand side of the above display is also bounded by $\delta/2$. To task this, write

$$\mathbb{P}\{\|B_{5n}\| > \varepsilon, \|\theta^n_0 - \hat{\theta}^{n}_{0,n}\| \leq C/\sqrt{n}\} \leq \mathbb{P}\left\{ \sup_{\|\theta - \theta^n_{0,n}\| \leq C/\sqrt{n}} \left\| \frac{1}{\sqrt{n}} \sum_{t=1}^{n} \left\{ c_t(\theta, \hat{\theta}^{n}_{0,n}) - \mathbb{E}[c_t(\theta, \hat{\theta}^{n}_{0,n})] \right\} - \left\{ c_t(\theta^n_{0,n}, \hat{\theta}^{n}_{0,n}) - \mathbb{E}[c_t(\theta^n_{0,n}, \hat{\theta}^{n}_{0,n})] \right\} \right\| > \varepsilon \right\} \leq \mathbb{P}\left\{ o_p(1) \times \left[ 1 + \sqrt{n} C_0(C/\sqrt{n}) \right] > \varepsilon \right\} \leq \delta/2$$

for sufficiently large $n$, where the penultimate step follows from (B.40) and (B.41). This proves $B_{5n} = o_p(1)$.

It remains to show that $A_{5n} = o_p(1)$. To do so, we again verify conditions (N1)–(N5) of Weiss (1991) for

$$\lambda_n(\theta^v) := \frac{1}{n} \sum_{t=1}^{n} \mathbb{E}[c_t(\theta^n_0, \theta^v)],$$

$$\mu_t(\theta, d) := \sup_{\|\tau - \theta\| \leq d} \left\| c_t(\theta^n_0, \tau) - c_t(\theta^n_0, \theta) \right\|.$$  

To promote flow, we do this in Section B.3. Lemma A.1 of Weiss (1991) then implies that

$$\sup_{\|\theta - \theta^n_{0}\| \leq d_0} \left\| \frac{1}{\sqrt{n}} \sum_{t=1}^{n} \left\{ c_t(\theta^n_{0}, \theta) - \mathbb{E}[c_t(\theta^n_{0}, \theta)] \right\} - \left\{ c_t(\theta^n_{0}, \theta^n_{0}) - \mathbb{E}[c_t(\theta^n_{0}, \theta^n_{0})] \right\} \right\| = o_p(1).$$

We can bound the term in the denominator (using a mean value expansion around $\theta^n_{0}$ and (B.23)) as follows:

$$\left\| \frac{1}{n} \sum_{t=1}^{n} \mathbb{E}[c_t(\theta^n_{0}, \theta)] \right\| = \left\| \frac{1}{n} \sum_{t=1}^{n} \mathbb{E}[c_t(\theta^n_{0}, \theta^v)] + \frac{1}{n} \sum_{t=1}^{n} \frac{\partial}{\partial \theta^v} \mathbb{E}[c_t(\theta^n_{0}, \theta^v)]_{\theta^n_{0} = \theta^n_{0}} (\theta - \theta^n_{0}) \right\| = \left\| \frac{1}{n} \sum_{t=1}^{n} \left\{ \nabla c_t(\theta^n_{0}) \nabla v_t(\theta^v) \left[ \alpha f_t^X(v_t(\theta^v)) - \partial F_t(v_t(\theta^v), c_t(\theta^n_{0})) \right] \right\} (\theta - \theta^n_{0}) \right\| \leq \frac{1}{n} \sum_{t=1}^{n} (\alpha K + K) \mathbb{E}[V_1(F_{t-1})C_1(F_{t-1})] \left\| \theta - \theta^n_{0} \right\|$$
that, as $n \to \infty$, and similarly, $E$ is positive definite by Assumption 2 (viii). For the above display recall (B.20) and (B.38) and note where we also used Assumption 2 in the final two steps. Now, $A_{5n} = \alpha_P(1)$ follows by similar steps used below (B.41) (combined with the fact that $\sqrt{n}\|\hat{\theta}_n - \theta_0^n\| = o_P(1)$ holds instead of (B.39)).

Proof of Theorem 2 (continued): We can now show asymptotic normality of $\hat{\theta}_n^n$. From Lemma C.1, we have the decomposition

$$\sqrt{n}(\hat{\theta}_n^n - \theta_0^n) = [A^{-1}(1)A^{-1}A^{-1} + o_P(1)]$$

$$- \left[A^{-1}(1) + o_P(1)\right] \left[\frac{1}{\sqrt{n}} \sum_{t=1}^{n} v_t(\theta_0^n) + o_P(1)\right]$$

$$= \left(A_{(1)}^{-1}A(2)A^{-1} + o_P(1)\right) \left(A_{(1)}^{-1} + o_P(1)\right) \left[\frac{1}{\sqrt{n}} \sum_{t=1}^{n} c_t(\theta_{0,n}^c, \hat{\theta}^n) + o_P(1)\right].$$

From simple adaptations of the proofs of Lemmas V.8 and C.8, we have that, as $n \to \infty$,

$$n^{-1/2} \sum_{t=1}^{n} \left(\frac{v_t(\theta_0^n)}{c_t(\theta_{0,n}^c, \hat{\theta}^n)}\right) \xrightarrow{d} N(0, C),$$

where

$$C = \begin{pmatrix}
E[v_t(\theta_0^n)v_t'(\theta_0^n)] & E[v_t(\theta_0^n)c_t'(\theta_0^n, \theta_0^n)] \\
E[c_t(\theta_0^n, \theta_0^n)v_t'(\theta_0^n)] & E[c_t(\theta_0^n, \theta_0^n)c_t'(\theta_0^n, \theta_0^n)]
\end{pmatrix} = \begin{pmatrix}
V & 0 \\
0 & C^*
\end{pmatrix}$$

is positive definite by Assumption 2 (viii). For the above display recall (B.20) and (B.38) and note that

$$E[c_t(\theta_0^n, \theta_0^n)v_t'(\theta_0^n)] = E\left[\nabla c_t(\theta_0^n)\nabla' v_t(\theta_0^n)\mathbb{I}_{\{X_t > v_t(\theta_0^n)\}}(\mathbb{I}_{\{Y_t \leq c_t(\theta_0^n)\}} - \beta)(\mathbb{I}_{\{Y_t \leq c_t(\theta_0^n)\}} - \alpha)\right]$$

$$= -\beta E\left\{\nabla c_t(\theta_0^n)\nabla' v_t(\theta_0^n)\mathbb{I}_{\{X_t > v_t(\theta_0^n)\}}(\mathbb{I}_{\{Y_t \leq c_t(\theta_0^n)\}} - \alpha)\right\}$$

$$= -\beta E\left\{\nabla c_t(\theta_0^n)\nabla' v_t(\theta_0^n)\mathbb{I}_{\{X_t > v_t(\theta_0^n)\}}(1 - \beta)\right\}$$

$$= -\beta E\left\{\nabla c_t(\theta_0^n)\nabla' v_t(\theta_0^n)\mathbb{I}_{\{Y_t \leq c_t(\theta_0^n)\}}(1 - \beta)\right\}$$

$$= -\beta E\left\{\nabla c_t(\theta_0^n)\nabla' v_t(\theta_0^n)\mathbb{I}_{\{Y_t \leq c_t(\theta_0^n)\}}(1 - \beta)\right\} = 0$$

and, similarly, $E[v_t(\theta_0^n)c_t'(\theta_0^n, \theta_0^n)] = 0$. We then obtain from the continuous mapping theorem that, as $n \to \infty$,

$$\sqrt{n}(\hat{\theta}_n^n - \theta_0^n) \xrightarrow{d} N(0, \Gamma C \Gamma'),$$

34
where

\[ \Gamma = \begin{pmatrix} A_{(1)}^{-1}A_{(2)}A_{(1)}^{-1} & -A_{(1)}^{-1} \end{pmatrix} \]

This, however, is just the conclusion. \[ \square \]

### B.3 Supplementary Proofs

In this section, we verify conditions (N1)–(N5) of Weiss (1991) for

\[ \lambda_n(\theta^v) := \frac{1}{n} \sum_{t=1}^{n} \mathbb{E}[c_t(\theta_0^c, \theta^v)], \]

\[ \mu_t(\theta, d) := \sup_{\|\tau - \theta\| \leq d} \| c_t(\theta_0^c, \tau) - c_t(\theta_0^c, \theta) \|. \]

The measurability condition of (N1) is immediate, as is the mixing condition of (N5) (due to Assumption 2 (x)). Condition (N2) follows from correct specification of the VaR and CoVaR model such that

\[ \lambda_n(\theta_0^v) = \frac{1}{n} \sum_{t=1}^{n} \mathbb{E}[c_t(\theta_0^c, \theta_0^v)] = 0. \quad (B.42) \]

Next, condition (N3) is verified by Lemmas 1–3, and condition (N4) by Lemma 4.

**Lemma 1.** Suppose Assumptions 1 and 2 hold. Then, condition (N3) (i) of Weiss (1991) holds, i.e.,

\[ \| \lambda_n(\theta) \| \geq a \| \theta - \theta_0^v \| \quad \text{for } \| \theta - \theta_0^v \| \leq d_0 \]

for sufficiently large \( n \) and some \( a > 0 \) and \( d_0 > 0 \).

**Proof:** A mean value expansion around \( \theta_0^v \) together with (B.42) imply

\[ \lambda_n(\theta) = \lambda_n(\theta_0^v) + A_{n,(2)}(\theta_0^c, \theta^v)(\theta - \theta_0^v) \]

\[ = A_{n,(2)}(\theta_0^c, \theta^v)(\theta - \theta_0^v) \]

for some \( \theta^v \) on the line connecting \( \theta_0^v \) and \( \theta \). (Again, this is an instance of the non-existent mean value theorem.) By Assumption 2 (viii), \( \| A_{(2)} \| = \sqrt{\sum_{i=1}^{q} \sum_{j=1}^{p} |A_{(2),ij}|^2} \) is bounded below by some \( a > 0 \), such that

\[ \| \lambda_n(\theta) \| \geq \| A_{n,(2)}(\theta_0^c, \theta^v)(\theta - \theta_0^v) \| \]

\[ \geq \| A_{(2)}(\theta - \theta_0^v) \| - \| A_{(2)}(\theta_0^c, \theta^v)(\theta - \theta_0^v) \| \]

\[ \geq a \| \theta - \theta_0^v \| - \| A_{(2)} - A_{n,(2)}(\theta_0^c, \theta^v) \| \cdot \| \theta - \theta_0^v \|. \]

By Lemma C.2, in particular equation (B.25), we have that

\[ \| A_{(2)} - A_{n,(2)}(\theta_0^c, \theta^v) \| \leq C \| \theta^v - \theta_0^c \| \leq C d_0 \]

35
for arbitrarily small $d_0$. In particular, choosing $d_0 > 0$ such that $Cd_0 < a$, the desired conclusion follows.

\[ \square \]

**Lemma 2.** Suppose Assumptions 1 and 2 hold, and define

$$\mu_t(\theta, d) = \sup_{\|\tau-\theta\| \leq d} \left\| c_t(\theta_0^c, \tau) - c_t(\theta_0^c, \theta) \right\|. $$

Then, condition (N3) (ii) of Weiss (1991) holds, i.e.,

$$\mathbb{E}[\mu_t(\theta, d)] \leq bd$$

for sufficiently large $n$ and some strictly positive $b, d, d_0$.

**Proof:** Choose $d_0 > 0$ sufficiently small, such that $\{ \theta \in \Theta^c : \|\theta - \theta_0^c\| < d_0 \}$ is a subset of the neighborhoods of Assumptions 1 (ix) and 2 (iii). Recalling the definition of $c_t(\theta_0^c, \theta^c)$ from (B.21), we decompose

$$\mu_t(\theta, d) = \sup_{\|\tau-\theta\| \leq d} \left\| \left[ \mathbb{1}_{\{X_t > v_t(\tau)\}} - \mathbb{1}_{\{X_t > v_t(\theta)\}} \right] \nabla c_t(\theta_0^c) \left[ \mathbb{1}_{\{Y_t \leq c_t(\theta_0^c)\}} - \alpha \right] \right\|. $$

The remainder of the proof is similar to that of Lemma V.5 and Lemma C.5.

Define the $\mathcal{F}_{t-1}$-measurable quantities

$$\tau := \arg \min_{\|\tau-\theta\| \leq d} v_t(\tau),$$

$$\varphi := \arg \max_{\|\tau-\theta\| \leq d} v_t(\tau),$$

which exist by continuity of $v_t(\cdot)$.

To take the indicators out of the supremum, we again consider two cases:

**Case 1:** $X_t \leq v_t(\theta)$

$$\mu_t(\theta, d) = \mathbb{1}_{\{X_t > v_t(\tau)\}} \sup_{\|\tau-\theta\| \leq d} \left\| \nabla c_t(\theta_0^c) \left[ \mathbb{1}_{\{Y_t \leq c_t(\theta_0^c)\}} - \alpha \right] \right\|. $$

**Case 2:** $X_t > v_t(\theta)$

We further distinguish two cases (a)–(b).

(a) If $X_t > v_t(\varphi)$, then $\mu_t(\theta, d) = 0$.

(b) If $v_t(\tau) \leq X_t \leq v_t(\varphi)$, then

$$\mu_t(\theta, d) = \left\| \nabla c_t(\theta_0^c) \left[ \mathbb{1}_{\{Y_t \leq c_t(\theta_0^c)\}} - \alpha \right] \right\|. $$

(Note that the third case that $X_t < v_t(\tau)$ cannot occur, because already $X_t > v_t(\theta)$.)
Before combining the two cases, note that—similarly as below (B.13)—\(\theta\) and \(\tau\) are in a \(d_0\)-neighborhood of \(\theta_0^v\). Therefore, combining the results from Cases 1 and 2,

\[
\mu_t(\theta, d) \leq \left[ \mathbb{1}_{\{v_t(\tau) < X_t \leq v_t(\theta)\}} + \mathbb{1}_{\{v_t(\theta) < X_t \leq v_t(\tau)\}} \right] \| \nabla c_t(\theta^v_0) \| \mathbb{1}_{\{Y_t \leq c_t(\theta^v_0)\}} - \alpha \bigg].
\]  

(B.43)

Then, using the LIE and (B.15)–(B.16),

\[
\mathbb{E}[\mu_t(\theta, d)] \leq \mathbb{E}\left\{ \mathbb{E}_{t-1} \left[ \mathbb{1}_{\{v_t(\tau) < X_t \leq v_t(\theta)\}} + \mathbb{1}_{\{v_t(\theta) < X_t \leq v_t(\tau)\}} \right] \| \nabla c_t(\theta^v_0) \| \right\}
\leq \mathbb{E}\left\{ 2KV_1(F_{t-1})C_1(F_{t-1})d \right\}
\leq 2K \left\{ \mathbb{E}[V^2_1(F_{t-1})] \right\}^{1/2} \left\{ \mathbb{E}[C^2_1(F_{t-1})] \right\}^{1/2} d
\leq bd
\]

for some sufficiently large \(b > 0\).

\[\square\]

**Lemma 3.** Suppose Assumptions 1 and 2 hold. Then, condition (N3) (iii) of Weiss (1991) holds, i.e.,

\[
\mathbb{E}\left[ \mu_t^q(\theta, d) \right] \leq cd \quad \text{for } \|\theta - \theta^v_0\| + d \leq d_0,
\]

for sufficiently large \(n\) and some \(c > 0\), \(d \geq 0\), \(d_0 > 0\) and \(q > 2\).

**Proof:** As in the proof of Lemma 2, we again pick \(d_0 > 0\) sufficiently small, such that \(\{\theta \in \Theta^v : \|\theta - \theta^v_0\| < d_0\}\) is a subset of the neighborhoods of Assumptions 1 (ix) and 2 (iii). We also work with \(\tau\) and \(\bar{\tau}\) as defined in the proof of Lemma 2. For \(\epsilon > 0\) from Assumption 2 (v), we get from (B.43), (B.15)–(B.16) and the LIE that

\[
\mathbb{E}[\mu_t^{2+\epsilon}(\theta, d) \leq \mathbb{E}\left\{ \mathbb{E}_{t-1} \left[ \mathbb{1}_{\{v_t(\tau) < X_t \leq v_t(\theta)\}} + \mathbb{1}_{\{v_t(\theta) < X_t \leq v_t(\tau)\}} \right] \| \nabla c_t(\theta^v_0) \|^{2+\epsilon} \right\}
\leq \mathbb{E}\left\{ 2KV_1(F_{t-1})C_1^{2+\epsilon}(F_{t-1})d \right\}
\leq 2K \left\{ \mathbb{E}[V^{3+\epsilon}_1(F_{t-1})] \right\}^{1/(3+\epsilon)} \left\{ \mathbb{E}[C^{3+\epsilon}_1(F_{t-1})] \right\}^{(2+\epsilon)/(3+\epsilon)} d
\leq cd
\]

for some sufficiently large \(c > 0\).

\[\square\]

**Lemma 4.** Suppose Assumptions 1 and 2 hold. Then, condition (N4) of Weiss (1991) holds, i.e.,

\[
\mathbb{E}\left\| c_t(\theta^v_0, \theta^v_0) \right\|^2 \leq C \quad \text{for all } t.
\]

**Proof:** The bound follows immediately from (B.37).

\[\square\]
C Proof of Theorem 3

Proof of Theorem 3: The two convergences \( \hat{V}_n \xrightarrow{p} V \) and \( \hat{A}_n \xrightarrow{p} A \) are shown similarly as in Engle and Manganelli (2004, Theorem 3). Similarly as the proof of \( \hat{C}_n \xrightarrow{p} C^* \), the proof of \( \hat{C}_n^{*} \xrightarrow{p} C^* \) is standard and, hence, omitted. It remains to show that \( \hat{A}_{n,(1)} \xrightarrow{p} A_{(1)} \) and \( \hat{A}_{n,(2)} \xrightarrow{p} A_{(2)} \). Since the proofs are very similar, we only show the latter convergence, that is,

\[
\frac{1}{n} \sum_{t=1}^{n} \nabla c_t(\hat{\theta}_n^*) \nabla v_t(\hat{\theta}_n^*)(2\hat{b}_{n,x})^{-1} \left[ \alpha \mathbb{1}_{\{X_t-v_t(\hat{\theta}_n^*)<\hat{b}_{n,x}\}} - \mathbb{1}_{\{X_t-v_t(\hat{\theta}_n^*)<\hat{b}_{n,x}, Y_t \leq c_t(\hat{\theta}_n^*)\}} \right]
- \mathbb{E} \left[ \nabla c_t(\theta_0^*) \nabla v_t(\theta_0^*) \alpha f_t^Y(v_t(\theta_0^*), c_t(\theta_0^*)) \right] = o_P(1).
\]

Here, we only prove that

\[
\hat{D}_n - D_0 := \frac{1}{n} \sum_{t=1}^{n} \nabla c_t(\hat{\theta}_n^*) \nabla v_t(\hat{\theta}_n^*)(2\hat{b}_{n,x})^{-1} \mathbb{1}_{\{X_t-v_t(\hat{\theta}_n^*)<\hat{b}_{n,x}, Y_t \leq c_t(\hat{\theta}_n^*)\}}
- \mathbb{E} \left[ \nabla c_t(\theta_0^*) \nabla v_t(\theta_0^*) \alpha f_t^Y(v_t(\theta_0^*), c_t(\theta_0^*)) \right] = o_P(1), \quad (C.1)
\]

because

\[
\frac{1}{n} \sum_{t=1}^{n} \nabla c_t(\hat{\theta}_n^*) \nabla v_t(\hat{\theta}_n^*)(2\hat{b}_{n,x})^{-1} \alpha \mathbb{1}_{\{X_t-v_t(\hat{\theta}_n^*)<\hat{b}_{n,x}\}}
- \mathbb{E} \left[ \nabla c_t(\theta_0^*) \nabla v_t(\theta_0^*) \alpha f_t^Y(v_t(\theta_0^*)) \right] = o_P(1)
\]

can be shown similarly. Define

\[
\bar{D}_n := \frac{1}{n} \sum_{t=1}^{n} \nabla c_t(\theta_0^*) \nabla v_t(\theta_0^*)(2b_{n,x})^{-1} \mathbb{1}_{\{X_t-v_t(\theta_0^*)<b_{n,x}, Y_t \leq c_t(\theta_0^*)\}}.
\]

Then, to establish (C.1) it suffices to show that

\[
\hat{D}_n - \bar{D}_n = o_P(1), \quad (C.2)
\]

\[
\hat{D}_n - D_0 = o_P(1). \quad (C.3)
\]

We first prove (C.2). Observe that

\[
\left\| \hat{D}_n - \bar{D}_n \right\| = \frac{b_{n,x}}{b_{n,x}} \left\| (2b_{n,x}n)^{-1} \sum_{t=1}^{n} \left[ \mathbb{1}_{\{X_t-v_t(\theta_0^*)<b_{n,x}, Y_t \leq c_t(\theta_0^*)\}} - \mathbb{1}_{\{X_t-v_t(\theta_0^*)<b_{n,x}, Y_t \leq c_t(\theta_0^*)\}} \right] \nabla c_t(\hat{\theta}_n^*) \nabla v_t(\hat{\theta}_n^*) \right.
+ \mathbb{1}_{\{X_t-v_t(\theta_0^*)<b_{n,x}, Y_t \leq c_t(\theta_0^*)\}} \left[ \nabla c_t(\hat{\theta}_n^*) - \nabla c_t(\theta_0^*) \right] \nabla v_t(\hat{\theta}_n^*)
+ \mathbb{1}_{\{X_t-v_t(\theta_0^*)<b_{n,x}, Y_t \leq c_t(\theta_0^*)\}} \nabla c_t(\theta_0^*) \left[ \nabla v_t(\hat{\theta}_n^*) - \nabla v_t(\theta_0^*) \right]
+ \frac{b_{n,x} - \hat{b}_{n,x}}{b_{n,x}} \mathbb{1}_{\{X_t-v_t(\theta_0^*)<b_{n,x}, Y_t \leq c_t(\theta_0^*)\}} \nabla c_t(\theta_0^*) \nabla v_t(\theta_0^*) \Bigg] \right\|.
\]

38
Write the indicators in the second line of the above display as

\[
\begin{align*}
&\mathbbm{1}\{X_t-v_t(\hat{\theta}_n^v)<\hat{b}_{n,x}, \ Y_t\leq c_t(\hat{\theta}_n^v)\} - \mathbbm{1}\{X_t-v_t(\theta_0^v)<b_{n,x}, \ Y_t\leq c_t(\theta_0^v)\} \\
&= \left[ \mathbbm{1}\{X_t-v_t(\hat{\theta}_n^v)<\hat{b}_{n,x}, \ Y_t\leq c_t(\hat{\theta}_n^v)\} - \mathbbm{1}\{X_t-v_t(\theta_0^v)<b_{n,x}, \ Y_t\leq c_t(\theta_0^v)\} \right] \\
&\quad + \left[ \mathbbm{1}\{X_t-v_t(\theta_0^v)<b_{n,x}, \ Y_t\leq c_t(\theta_0^v)\} - \mathbbm{1}\{X_t-v_t(\theta_0^v)<b_{n,x}, \ Y_t\leq c_t(\theta_0^v)\} \right] \\
&=: A_{2t} + B_{2t}.
\end{align*}
\]  

(C.5)

For \(A_{2t}\), note that for any set \(A \in \mathcal{F}\),

\[
\left| \mathbbm{1}\{X_t-v_t(\hat{\theta}_n^v)<\hat{b}_{n,x}, \ A\} - \mathbbm{1}\{X_t-v_t(\theta_0^v)<b_{n,x}, \ A\} \right| \\
\leq \mathbbm{1}\{X_t-v_t(\hat{\theta}_n^v)-\hat{b}_{n,x}<v_t(\theta_0^v)-v_t(\theta_0^v)|+\hat{b}_{n,x}-b_{n,x}|, \ A\} + \mathbbm{1}\{X_t-v_t(\theta_0^v)+b_{n,x}|v_t(\hat{\theta}_n^v)-v_t(\theta_0^v)|+\hat{b}_{n,x}-b_{n,x}|, \ A\}.
\]  

(C.6)

To see this, note that the difference of the indicators in the first line of (C.6) can only be non-zero if either \(|X_t-v_t(\hat{\theta}_n^v)| < \hat{b}_{n,x}\) and \(|X_t-v_t(\theta_0^v)| \geq b_{n,x}\) or \(|X_t-v_t(\hat{\theta}_n^v)| \geq \hat{b}_{n,x}\) and \(|X_t-v_t(\theta_0^v)| < b_{n,x}\). Consider the former case; the latter case can be dealt with similarly. It holds that

\[
X_t - v_t(\hat{\theta}_n^v) < \hat{b}_{n,x} \\
\iff -\hat{b}_{n,x} < X_t - v_t(\hat{\theta}_n^v) < \hat{b}_{n,x} \\
\iff v_t(\hat{\theta}_n^v) - v_t(\theta_0^v) - \hat{b}_{n,x} < X_t - v_t(\theta_0^v) - b_{n,x} < v_t(\hat{\theta}_n^v) - v_t(\theta_0^v) + \hat{b}_{n,x} - b_{n,x} \\
\iff v_t(\hat{\theta}_n^v) - v_t(\theta_0^v) - \hat{b}_{n,x} + b_{n,x} < X_t - v_t(\theta_0^v) + b_{n,x} < v_t(\hat{\theta}_n^v) - v_t(\theta_0^v) + \hat{b}_{n,x} + b_{n,x}
\]  

(C.7)

and

\[
X_t - v_t(\theta_0^v) \geq b_{n,x} \iff X_t - v_t(\theta_0^v) \geq b_{n,x} \quad \text{or} \quad X_t - v_t(\theta_0^v) \leq -b_{n,x}.
\]  

(C.9)

The inequality \(X_t - v_t(\theta_0^v) \geq b_{n,x}\) from (C.9) together with the second inequality of (C.7) yields

\[
0 \leq X_t - v_t(\theta_0^v) - b_{n,x} < v_t(\hat{\theta}_n^v) - v_t(\theta_0^v) + \hat{b}_{n,x} - b_{n,x},
\]

which in turn implies that

\[
|X_t - v_t(\hat{\theta}_n^v) - b_{n,x}| < |v_t(\hat{\theta}_n^v) - v_t(\theta_0^v)| + |\hat{b}_{n,x} - b_{n,x}|.
\]

Therefore, when \(X_t - v_t(\theta_0^v) \geq b_{n,x}\),

\[
\left| \mathbbm{1}\{X_t-v_t(\hat{\theta}_n^v)<\hat{b}_{n,x}, \ A\} - \mathbbm{1}\{X_t-v_t(\theta_0^v)<b_{n,x}, \ A\} \right| \\
\leq \mathbbm{1}\{X_t-v_t(\theta_0^v)-b_{n,x}|v_t(\hat{\theta}_n^v)-v_t(\theta_0^v)|+b_{n,x}-b_{n,x}|, \ A\}.
\]  

(C.10)

But also, the inequality \(X_t - v_t(\theta_0^v) \leq -b_{n,x}\) from (C.9) together with the first inequality of (C.8)
yields
\[ v_t(\hat{\theta}_n^\nu) - v_t(\theta_0^\nu) + b_{n,x} - \hat{b}_{n,x} < X_t - v_t(\theta_0^\nu) + b_{n,x} \leq 0, \]
which in turn implies that
\[ |X_t - v_t(\theta_0^\nu) + b_{n,x}| < |v_t(\theta_n^\nu) - v_t(\theta_0^\nu)| + |\hat{b}_{n,x} - b_{n,x}|. \]
Therefore, when \( X_t - v_t(\theta_0^\nu) \leq -b_{n,x}, \)
\[ \left| \mathbb{I}\left\{ |X_t - v_t(\hat{\theta}_n^\nu)| < b_{n,x}, \ A \right\} - \mathbb{I}\left\{ |X_t - v_t(\theta_0^\nu)| < b_{n,x}, \ A \right\} \right| \leq \mathbb{I}\left\{ |X_t - v_t(\theta_0^\nu) + b_{n,x}| < |v_t(\theta_n^\nu) - v_t(\theta_0^\nu)| + |\hat{b}_{n,x} - b_{n,x}|, \ A \right\}. \tag{C.11} \]
Combining (C.10) and (C.11) gives (C.6).

For \( B_{2t}, \) note that for any set \( A \in \mathcal{F}, \)
\[ \left| \mathbb{I}\left\{ A, \ Y_t \leq c_t(\theta_0^\nu) \right\} - \mathbb{I}\left\{ A, \ Y_t \leq c_t(\hat{\theta}_n^\nu) \right\} \right| \leq \mathbb{I}\left\{ A, |Y_t - c_t(\theta_0^\nu)| \leq |c_t(\hat{\theta}_n^\nu) - c_t(\theta_0^\nu)| \right\}. \tag{C.12} \]
To see this, note that the difference of the indicators on the left-hand side can only be non-zero when \( \text{either } Y_t \leq c_t(\theta_0^\nu) \) and \( Y_t > c_t(\theta_0^\nu) \) or \( Y_t > c_t(\hat{\theta}_n^\nu) \) and \( Y_t \leq c_t(\theta_0^\nu) \). Again, we only consider the former case, where
\[ Y_t \leq c_t(\theta_0^\nu) \quad \text{and} \quad Y_t > c_t(\theta_0^\nu) \quad \iff \quad c_t(\theta_0^\nu) < Y_t \leq c_t(\hat{\theta}_n^\nu), \]
\[ \iff \quad 0 < Y_t - c_t(\theta_0^\nu) \leq c_t(\hat{\theta}_n^\nu) - c_t(\theta_0^\nu), \]
\[ \iff \quad |Y_t - c_t(\theta_0^\nu)| \leq |c_t(\hat{\theta}_n^\nu) - c_t(\theta_0^\nu)|. \]

Exploiting (C.6) and (C.12) for \( A_{2t} \) and \( B_{2t}, \) we get from (C.5) that
\[ \left| \mathbb{I}\left\{ |X_t - v_t(\hat{\theta}_n^\nu)| < b_{n,x}, \ Y_t \leq c_t(\theta_0^\nu) \right\} \right| - \mathbb{I}\left\{ |X_t - v_t(\theta_0^\nu)| < b_{n,x}, \ Y_t \leq c_t(\theta_0^\nu) \right\} \]
\[ \leq \mathbb{I}\left\{ |X_t - v_t(\theta_0^\nu) + b_{n,x}| < |v_t(\theta_n^\nu) - v_t(\theta_0^\nu)| + |\hat{b}_{n,x} - b_{n,x}|, \ Y_t \leq c_t(\theta_0^\nu) \right\} \]
\[ + \mathbb{I}\left\{ |X_t - v_t(\theta_0^\nu) + b_{n,x}| < |v_t(\theta_n^\nu) - v_t(\theta_0^\nu)| + |\hat{b}_{n,x} - b_{n,x}|, \ Y_t \leq c_t(\hat{\theta}_n^\nu) \right\} \]
\[ + \mathbb{I}\left\{ |X_t - v_t(\hat{\theta}_n^\nu)| < b_{n,x}, \ Y_t \leq c_t(\theta_0^\nu) \leq |c_t(\hat{\theta}_n^\nu) - c_t(\theta_0^\nu)| \right\} \]
\[ \nabla c_t(\theta_0^\nu) \nabla' v_t(\theta_n^\nu) \]
\[ + \mathbb{I}\left\{ |X_t - v_t(\hat{\theta}_n^\nu)| < b_{n,x}, \ Y_t \leq c_t(\theta_0^\nu) \right\} \nabla^2 c_t(\theta^\nu) [\hat{\theta}_n^\nu - \theta_0^\nu] \nabla' v_t(\theta_n^\nu) \]
\[ \leq \frac{b_{n,x}}{b_{n,x}} \left( 2b_{n,x} n \right)^{-1} \sum_{t=1}^{n} \mathbb{I}\left\{ |X_t - v_t(\theta_0^\nu) + b_{n,x}| < |v_t(\theta_n^\nu) - v_t(\theta_0^\nu)| + |\hat{b}_{n,x} - b_{n,x}|, \ Y_t \leq c_t(\theta_0^\nu) \right\} \]
\[ + \mathbb{I}\left\{ |X_t - v_t(\theta_0^\nu) + b_{n,x}| < |v_t(\theta_n^\nu) - v_t(\theta_0^\nu)| + |\hat{b}_{n,x} - b_{n,x}|, \ Y_t \leq c_t(\theta_0^\nu) \right\} \]
\[ + \mathbb{I}\left\{ |X_t - v_t(\hat{\theta}_n^\nu)| < b_{n,x}, \ Y_t \leq c_t(\theta_0^\nu) \leq |c_t(\hat{\theta}_n^\nu) - c_t(\theta_0^\nu)| \right\} \]
\[ \nabla c_t(\theta_0^\nu) \nabla' v_t(\theta_n^\nu) \]
\[ + \mathbb{I}\left\{ |X_t - v_t(\hat{\theta}_n^\nu)| < b_{n,x}, \ Y_t \leq c_t(\theta_0^\nu) \right\} \nabla^2 c_t(\theta^\nu) [\hat{\theta}_n^\nu - \theta_0^\nu] \nabla' v_t(\theta_n^\nu). \]
Therefore, it suffices to consider

\[ \theta^* \]

occur with probability approaching 1, as \( n \to \infty \). Theorem 2 and Assumption 3 (i) then imply that for any \( \theta^* \) is some mean value that may differ from line to line. We show in turn that

\[ b_{n,x} \leq \frac{\theta_n - \theta_0}{\theta_n - \theta_0} \leq d \quad \text{and} \quad b_{n,x} \leq \frac{\theta_n - \theta_0}{\theta_n - \theta_0} \leq d \quad \text{and} \quad \frac{b_{n,x} - b_{n,x}}{b_{n,x}} \leq d \]

occur with probability approaching 1, as \( n \to \infty \). This means that \( \mathbb{P}\{E_n^c\} \leq \varepsilon/2 \) for sufficiently large \( n \), where

\[ E_n := \left\{ b_{n,x}^{-1} \left| \theta_n - \theta_0 \right| < d, \ b_{n,x}^{-1} \left| \theta_n - \theta_0 \right| < d, \ b_{n,x}^{-1} \left| \theta_n - \theta_0 \right| < d \right\} \]

Therefore, it suffices to consider \( A_{6n}, B_{6n}, C_{6n} \) and \( D_{6n} \) on this set, because (e.g.) \( \mathbb{P}\{A_{6n} > \varepsilon, E_n\} + \mathbb{P}\{A_{6n} > \varepsilon, E_n^c\} = \mathbb{P}\{A_{6n} > \varepsilon \} + o(1) \).

We first consider \( A_{6n} \). On the set \( E_n \), we may bound \( A_{6n} \) as follows:

\[ A_{6n} \leq \frac{2b_{n,x}}{2b_{n,x}n} \sum_{t=1}^{n} \left\{ \left| X_t - v_t(\theta_n^0) - b_{n,x} \right| \left| \nabla v_t(\theta^*) \right| \left| \theta_n - \theta_0 \right| + \left| b_{n,x} - b_{n,x} \right|, \ Y_t \leq c_t(\theta_n) \right\} \]

\[ + \left\{ \left| X_t - v_t(\theta_n^0) + b_{n,x} \right| \left| \nabla v_t(\theta^*) \right| \left| \theta_n - \theta_0 \right| + \left| b_{n,x} - b_{n,x} \right|, \ Y_t \leq c_t(\theta_n) \right\} \]

\[ + \left\{ \left| X_t - v_t(\theta_n^0) \right| \left| \nabla v_t(\theta^*) \right| \left| \theta_n - \theta_0 \right| + \left| b_{n,x} - b_{n,x} \right|, \ Y_t \leq c_t(\theta_n) \right\} \]

\[ \leq \frac{2b_{n,x}}{2b_{n,x}n} \sum_{t=1}^{n} \left\{ \left| X_t - v_t(\theta_n^0) - b_{n,x} \right| \left| \nabla v_t(\theta^*) \right| \left| \theta_n - \theta_0 \right| + \left| b_{n,x} - b_{n,x} \right|, \ Y_t \leq C_{\theta(\theta_n)} \right\} \]

\[ + \left\{ \left| X_t - v_t(\theta_n^0) + b_{n,x} \right| \left| \nabla v_t(\theta^*) \right| \left| \theta_n - \theta_0 \right| + \left| b_{n,x} - b_{n,x} \right|, \ Y_t \leq C_{\theta(\theta_n)} \right\} \]

\[ =: \frac{b_{n,x}}{b_{n,x}} \left[ A_{6n} + B_{6n} + C_{6n} + D_{6n} \right], \quad (C.13) \]
We show that each of these terms is \( o_p(1) \). Use the LIE to obtain that

\[
\mathbb{E}[A_{61n}] \leq (2b_{n,x}n)^{-1} \sum_{i=1}^{n} \mathbb{E} \left\{ C_1(F_{i-1}) V_1(F_{i-1}) \mathbb{E}_{t-1} \left[ \mathbb{1} \left\{ |X_{t-v_t(\theta_0^v)}| < b_{n,x}, |Y_{i-c_t(\theta_0^v)}| \leq C_1(F_{i-1})b_{n,x} \right\} \right] \right\}.
\]

Write

\[
\mathbb{E}_{t-1} \left[ \mathbb{1} \left\{ |X_{t-v_t(\theta_0^v)}| < [V_1(F_{i-1}) + 1]b_{n,x}, Y_i \leq C(F_{i-1}) \right\} \right] = \mathbb{P}_{t-1} \left\{ -[V_1(F_{i-1}) + 1]b_{n,x} < X_t - v_t(\theta_0^v) - b_{n,x} < [V_1(F_{i-1}) + 1]b_{n,x}, Y_i \leq C(F_{i-1}) \right\}
\]

\[
= F_1(b_{n,x} + v_t(\theta_0^v) + [V_1(F_{i-1}) + 1]b_{n,x}, C(F_{i-1})) - F_1(b_{n,x} + v_t(\theta_0^v) - [V_1(F_{i-1}) + 1]b_{n,x}, C(F_{i-1}))
\]

\[
\leq \sup_{x \in \mathbb{R}} \left[ \partial_1 F_1(x, C(F_{i-1})) \right] 2[V_1(F_{i-1}) + 1] b_{n,x}
\]

\[
\leq 2K [V_1(F_{i-1}) + 1] b_{n,x}.
\]

Hence, also using Assumption 3 (v),

\[
0 \leq \mathbb{E}[A_{61n}] \leq (2b_{n,x}n)^{-1} \sum_{i=1}^{n} \mathbb{E} \left[ C_1(F_{i-1}) V_1(F_{i-1}) 2K [V_1(F_{i-1}) + 1] b_{n,x} \right]
\]

\[
= Kd \frac{1}{n} \sum_{i=1}^{n} \mathbb{E} \left[ C_1(F_{i-1}) V_1^2(F_{i-1}) \right]
\]

\[
\leq C d.
\]

Since \( d > 0 \) can be chosen arbitrarily small, Markov’s inequality implies that \( A_{61n} = o_p(1) \).

The proof that \( A_{62n} = o_p(1) \) is almost identical and, hence, omitted.

It remains to show \( A_{63n} = o_p(1) \). Again, we use the LIE to get that

\[
\mathbb{E}[A_{63n}] \leq (2b_{n,x}n)^{-1} \sum_{i=1}^{n} \mathbb{E} \left\{ C_1(F_{i-1}) V_1(F_{i-1}) \mathbb{E}_{t-1} \left[ \mathbb{1} \left\{ |X_{t-v_t(\theta_0^v)}| < b_{n,x}, |Y_{i-c_t(\theta_0^v)}| \leq C_1(F_{i-1})b_{n,x} \right\} \right] \right\}.
\]

Use Assumption 1 (vii) to write

\[
\mathbb{E}_{t-1} \left[ \mathbb{1} \left\{ |X_{t-v_t(\theta_0^v)}| < b_{n,x}, |Y_{i-c_t(\theta_0^v)}| \leq C_1(F_{i-1})b_{n,x} \right\} \right] = \int_{c_t(\theta_0^v) - C_1(F_{i-1})b_{n,x}}^{c_t(\theta_0^v) + C_1(F_{i-1})b_{n,x}} \int_{v_t(\theta_0^v) - b_{n,x}}^{v_t(\theta_0^v) + b_{n,x}} f_t(x, y) \, dx \, dy
\]

\[
\leq K^2 2b_{n,x} 2C_1(F_{i-1})b_{n,x}.
\]
Hence,

\[ 0 \leq \mathbb{E}[A_{63n}] \leq (2b_{n,x}n)^{-1} \sum_{t=1}^{n} \mathbb{E}[C_1(F_{t-1}) V_1(F_{t-1}) 2b_{n,x} 2C_1(F_{t-1}) db_{n,x}] \]

\[ = 2Kb_{n,x} d \left( \sum_{t=1}^{n} \mathbb{E}[C_1^2(F_{t-1}) V_1(F_{t-1})] \right) \leq Cb_{n,x} d = o(1)d = o(1). \]

Markov’s inequality then implies that \( A_{63n} = o_P(1) \). Overall, we obtain that \( A_{6n} = o_P(1) \).

On the set \( E_n \), we may bound \( B_{6n} \) and \( C_{6n} \) as follows:

\[ 0 \leq B_{6n} \leq (2b_{n,x}n)^{-1} \sum_{t=1}^{n} \mathbb{E}[C_2(F_{t-1}) V_1(F_{t-1}) db_{n,x}, |X_t - v_t(\theta_0^n)| < b_{n,x}, Y_t \leq \epsilon_t(\theta_0^n)] \]

\[ 0 \leq C_{6n} \leq (2b_{n,x}n)^{-1} \sum_{t=1}^{n} \mathbb{E}[C_1(F_{t-1}) V_2(F_{t-1}) db_{n,x}, |X_t - v_t(\theta_0^n)| < b_{n,x}, Y_t \leq \epsilon_t(\theta_0^n)] \]

such that

\[ 0 \leq \mathbb{E}[B_{6n}] \leq (2b_{n,x}n)^{-1} \sum_{t=1}^{n} \mathbb{E}[C_2(F_{t-1}) V_1(F_{t-1}) db_{n,x}] \leq C d, \]

\[ 0 \leq \mathbb{E}[C_{6n}] \leq (2b_{n,x}n)^{-1} \sum_{t=1}^{n} \mathbb{E}[C_1(F_{t-1}) V_2(F_{t-1}) db_{n,x}] \leq C d. \]

Again, since \( d > 0 \) can be chosen arbitrarily small, Markov’s inequality gives \( B_{6n} = o_P(1) \) and \( C_{6n} = o_P(1) \).

It remains to show that \( D_{6n} = o_P(1) \). Write

\[ D_{6n} = (2b_{n,x}n)^{-1} \sum_{t=1}^{n} \frac{[\hat{b}_{n,x} - b_{n,x}]}{b_{n,x}} \mathbb{1} \{ |X_t - v_t(\theta_0^n)| < b_{n,x}, Y_t \leq \epsilon_t(\theta_0^n) \} C_1(F_{t-1}) V_1(F_{t-1}) \]

\[ \leq o_P(1)(2b_{n,x}n)^{-1} \sum_{t=1}^{n} \mathbb{1} \{ |X_t - v_t(\theta_0^n)| < b_{n,x}, Y_t \leq C(F_{t-1}) \} C_1(F_{t-1}) V_1(F_{t-1}). \]

Since

\[ \mathbb{E}_{t-1} \left[ \mathbb{1} \{ |X_t - v_t(\theta_0^n)| < b_{n,x}, Y_t \leq C(F_{t-1}) \} \right] \]

\[ = \mathbb{P}_{t-1} \{ v_t(\theta_0^n) - b_{n,x} < X_t < v_t(\theta_0^n) + b_{n,x}, Y_t \leq C(F_{t-1}) \} \]

\[ = F_t(v_t(\theta_0^n) + b_{n,x}, C(F_{t-1})) - F_t(v_t(\theta_0^n) - b_{n,x}, C(F_{t-1})) \]

\[ \leq \sup_{x \in \mathbb{R}} |\partial F_t(x, C(F_{t-1}))| 2b_{n,x} \]

\[ \leq C b_{n,x}, \]

43
we obtain from the LIE that

\[ 0 \leq \mathbb{E}\left[ (2b_{n,x})^{-n} \sum_{t=1}^{n} \mathbb{I}\left\{ |X_t - v_t(\theta_0^u)| < b_{n,x}, Y_t \leq C(F_{t-1}) \right\} C_1(F_{t-1})V_1(F_{t-1}) \right] \]

\[ = (2b_{n,x})^{-n} \sum_{t=1}^{n} \mathbb{E}\left\{ C_1(F_{t-1})V_1(F_{t-1}) \mathbb{E}_{t-1}\left[ \mathbb{I}\left\{ |X_t - v_t(\theta_0^u)| < b_{n,x}, Y_t \leq C(F_{t-1}) \right\} \right] \right\} \]

\[ \leq (2b_{n,x})^{-n} \sum_{t=1}^{n} Cb_{n,x} \sqrt{\mathbb{E}[C_1^2(F_{t-1})]} \mathbb{E}[V_1^2(F_{t-1})] \leq C. \]

Thus, \( D_{6n} = \sigma \mathbb{P}(1) \sigma \mathbb{P}(1) = \sigma \mathbb{P}(1). \) We therefore obtain (C.2) from (C.13).

It remains to show (C.3), i.e., \( \bar{D}_n - D_0 = \sigma \mathbb{P}(1). \) Write this as

\[ \bar{D}_n - D_0 = (2b_{n,x})^{-n} \sum_{t=1}^{n} \left\{ \nabla c_t(\theta_0^u) \nabla' v_t(\theta_0^u) \left[ \mathbb{I}\left\{ |X_t - v_t(\theta_0^u)| < b_{n,x}, Y_t \leq c_t(\theta_0^u) \right\} \right] \right. \]

\[ \left. - \mathbb{E}_{t-1}\left[ \mathbb{I}\left\{ |X_t - v_t(\theta_0^u)| < b_{n,x}, Y_t \leq c_t(\theta_0^u) \right\} \right] \right\} \]

\[ + \frac{1}{n} \sum_{t=1}^{n} \left\{ \nabla c_t(\theta_0^u) \nabla' v_t(\theta_0^u) \left[ (2b_{n,x})^{-1} \mathbb{E}_{t-1}\left[ \mathbb{I}\left\{ |X_t - v_t(\theta_0^u)| < b_{n,x}, Y_t \leq c_t(\theta_0^u) \right\} \right] \right. \right. \]

\[ \left. - \partial_1 F_t(v_t(\theta_0^u), c_t(\theta_0^u)) \right\} \right\} \]

\[ + \frac{1}{n} \sum_{t=1}^{n} \left\{ \nabla c_t(\theta_0^u) \nabla' v_t(\theta_0^u) \partial_1 F_t(v_t(\theta_0^u), c_t(\theta_0^u)) \right. \]

\[ \left. - \mathbb{E}\left[ \nabla c_t(\theta_0^u) \nabla' v_t(\theta_0^u) \partial_1 F_t(v_t(\theta_0^u), c_t(\theta_0^u)) \right] \right\} \]

\[ =: A_{7n} + B_{7n} + C_{7n}. \]

Next, we show that each of these terms vanishes in probability. Observe that \( \mathbb{E}[A_{7n}] = 0 \) and for any \((i,j)\)-th element of \( A_{7n} \), denoted by \( A_{7n,ij} \) we obtain

\[ \text{Var}(A_{7n,ij}) = \mathbb{E}\left[ A_{7n,ij}^2 \right] \]

\[ = (2b_{n,x}n)^{-2} \mathbb{E}\left\{ \sum_{t=1}^{n} \nabla_i c_t(\theta_0^u) \nabla_j v_t(\theta_0^u) \left[ \mathbb{I}\left\{ |X_t - v_t(\theta_0^u)| < b_{n,x}, Y_t \leq c_t(\theta_0^u) \right\} \right] \right. \]

\[ \left. - \mathbb{E}_{t-1}\left[ \mathbb{I}\left\{ |X_t - v_t(\theta_0^u)| < b_{n,x}, Y_t \leq c_t(\theta_0^u) \right\} \right] \right\}^2 \]

\[ = (2b_{n,x}n)^{-2} \sum_{t=1}^{n} \mathbb{E}\left\{ \nabla_i c_t(\theta_0^u) \nabla_j v_t(\theta_0^u) \right\}^2 \left\{ \mathbb{I}\left\{ |X_t - v_t(\theta_0^u)| < b_{n,x}, Y_t \leq c_t(\theta_0^u) \right\} \right. \]

\[ \left. - \mathbb{E}_{t-1}\left[ \mathbb{I}\left\{ |X_t - v_t(\theta_0^u)| < b_{n,x}, Y_t \leq c_t(\theta_0^u) \right\} \right] \right\}^2 \]

\[ \leq (2b_{n,x}n)^{-2} \sum_{t=1}^{n} \mathbb{E}\left[ C_1^2(F_{t-1})V_1^2(F_{t-1}) \right] \]

44
\[ \leq (2b_{n,x}n)^{-2} \sum_{t=1}^{n} \left\{ \mathbb{E}[C_1^4(F_{t-1})] \right\}^{1/2} \left\{ \mathbb{E}[V_1^4(F_{t-1})] \right\}^{1/2} \]

\[ \leq C b_{n,x}^{-2} n^{-1} = o(1), \]

where we used for the third equality that all cross-products are zero by the LIE. That \( A_{7n} = o_p(1) \) now follows from Chebyshev’s inequality.

For \( B_{7n} \), we obtain that

\[ \mathbb{E}\|B_{7n}\| \leq \frac{1}{n} \sum_{t=1}^{n} \mathbb{E}\left\| \nabla c_t(\theta_0) \nabla' v_t(\theta_0) \right\| (2b_{n,x})^{-1} |E_{t-1}| \left\{ 1 \{ |X_t - v_t(\theta_0^e)| < b_{n,x}, Y_t \leq c_t(\theta_0^e) \} \right\} \]

\[ \leq \frac{1}{n} \sum_{t=1}^{n} \mathbb{E}\left[ C_1(F_{t-1}) V_1(F_{t-1}) \right] (2b_{n,x})^{-1} |E_{t-1}| \left\{ 1 \{ |X_t - v_t(\theta_0^e)| < b_{n,x}, Y_t \leq c_t(\theta_0^e) \} \right\} \]

\[ \leq \frac{1}{n} \sum_{t=1}^{n} \mathbb{E}\left[ C_1(F_{t-1}) V_1(F_{t-1}) \right] (2b_{n,x})^{-1} \left\{ F_t(v_t(\theta_0^e) + b_{n,x}, c_t(\theta_0^e)) - F_t(v_t(\theta_0^e) - b_{n,x}, c_t(\theta_0^e)) \right\} \]

\[ \leq \frac{1}{n} \sum_{t=1}^{n} \mathbb{E}\left[ C_1(F_{t-1}) V_1(F_{t-1}) \right] \sup_{x \in [v_t(\theta_0^e) - b_{n,x}, v_t(\theta_0^e) + b_{n,x}]} (2b_{n,x})^{-1} \partial_1 F_t(x, c_t(\theta_0^e)) 2b_{n,x} \]

\[ \leq \frac{1}{n} \sum_{t=1}^{n} \mathbb{E}\left[ C_1(F_{t-1}) V_1(F_{t-1}) \right] \sup_{x \in [v_t(\theta_0^e) - b_{n,x}, v_t(\theta_0^e) + b_{n,x}]} K |x - v_t(\theta_0^e)| \]

\[ \leq \frac{1}{n} \sum_{t=1}^{n} Kb_{n,x} \mathbb{E}\left[ C_1(F_{t-1}) V_1(F_{t-1}) \right] \]

\[ \leq C b_{n,x} = o(1), \]

such that Markov’s inequality implies \( B_{7n} = o_p(1) \).

Finally, Assumption 3 (iv) implies that \( C_{7n} = o_p(1) \).

Thus, (C.3) follows, ending the proof.

**D Computation of Risk MeasureForecasts**

**D.1 Computation of Risk Measure Forecasts for lower triangular decomposition**

Consider the standard specification of \( \Sigma_t \) in DCC–GARCH models as the lower-triangular matrix from the Cholesky decomposition of the variance-covariance matrix \( H_t = (\sigma_{XX,t}'(\sigma_{XY,t}(\sigma_{XY,t}', \sigma_{XY,t}^2)' \}};
Here, we consider a generic 

\[
\begin{pmatrix}
X_t \\
Y_t
\end{pmatrix} = \Sigma_t \begin{pmatrix}
\varepsilon_{X,t} \\
\varepsilon_{Y,t}
\end{pmatrix} = \begin{pmatrix}
\sigma_{X,t} & 0 \\
\sigma_{Y,t} \rho_{XY,t} & \sigma_{Y,t} \sqrt{1 - \rho_{XY,t}^2}
\end{pmatrix} \begin{pmatrix}
\varepsilon_{X,t} \\
\varepsilon_{Y,t}
\end{pmatrix}, \tag{D.1}
\]

where \(\rho_{XY,t} = \sigma_{XY,t}/(\sigma_{X,t} \sigma_{Y,t})\) is the conditional correlation. Under (D.1), we have that \(\text{VaR}_{t,\beta} = \sigma_{X,t} F_X^{-1}(\beta),\) where \(F_X^{-1}(\cdot)\) denotes the quantile function of the \(\varepsilon_{X,t}\). In practice, \(F_X^{-1}(\beta)\) can be estimated as the empirical \(\beta\)-quantile of the standardized residuals to yield forecasts \(\hat{\text{VaR}}_{t,\beta}\).

CoVaR can be forecast from the following relation

\[
\alpha = P_{t-1} \{ Y_t \leq \text{CoVaR}_{t,\alpha|\beta} | X_t \geq \text{VaR}_{t,\beta} \} = P_{t-1} \{ \sigma_{Y,t} \rho_{XY,t} \varepsilon_{X,t} + \sigma_{Y,t} (1 - \rho_{XY,t}^2)^{1/2} \varepsilon_{Y,t} \leq \text{CoVaR}_{t,\alpha|\beta} | X_t \geq \text{VaR}_{t,\beta} \}.
\]

From this relation, it is obvious that CoVaR\(_{t,\alpha|\beta}\) can be computed as the \(\alpha\)-quantile of the distribution of \(\sigma_{Y,t} \rho_{XY,t} \varepsilon_{X,t} + \sigma_{Y,t} (1 - \rho_{XY,t}^2)^{1/2} \varepsilon_{Y,t} \mid X_t \geq \text{VaR}_{t,\beta}\), where \((\varepsilon_{X,t}, \varepsilon_{Y,t})\) is a generic element of the sequence \((\varepsilon_{X,i}, \varepsilon_{Y,i})\)'\(i \in \mathcal{I} = \{ t = 1, \ldots, n : X_t \geq \text{VaR}_{t,\beta} \}\).

This describes how VaR and CoVaR forecasts are obtained from DCC–GARCH models in the empirical application.

### D.2 Computation of Risk Measure Forecasts for general decomposition

Here, we consider a generic

\[
\Sigma_t = \begin{pmatrix}
\sigma_{X,t} & \sigma_{XY,t} \\
\sigma_{XY,t} & \sigma_{Y,t}
\end{pmatrix}
\]

satisfying \(\Sigma_t \Sigma_t' = H_t\). The leading special case is the symmetric square root

\[
\Sigma_t = \frac{1}{\tau} \begin{pmatrix}
\sigma_{X,t}^2 + s & \sigma_{XY,t} \\
\sigma_{XY,t} & \sigma_{Y,t}^2 + s
\end{pmatrix},
\]

where \(s = \sigma_{X,t} \sigma_{Y,t} (1 - \rho_{XY,t}^2)^{1/2}\) and \(\tau = (\sigma_{X,t}^2 + \sigma_{Y,t}^2 + 2s)^{1/2}\) with \(\rho_{XY,t} = \sigma_{XY,t}/(\sigma_{X,t} \sigma_{Y,t})\) again denoting the correlation coefficient.

Under (D.2), it holds in the model \((X_t, Y_t)' = \Sigma_t (\varepsilon_{X,t}, \varepsilon_{Y,t})'\) that \(X_t = \tilde{\sigma}_{X,t} \varepsilon_{X,t} + \tilde{\sigma}_{XY,t} \varepsilon_{Y,t}\), such that \(\text{VaR}_{t,\beta}\) is the \(\beta\)-quantile of the distribution of \(\tilde{\sigma}_{X,t} \varepsilon_{X} + \tilde{\sigma}_{XY,t} \varepsilon_{Y}\), where \((\varepsilon_{X,t}, \varepsilon_{Y,t})'\) is a generic element of the sequence \((\varepsilon_{X,i}, \varepsilon_{Y,i})'\). Thus, \(\text{VaR}_{t,\beta}\) may be estimated as \(\hat{\text{VaR}}_{t,\beta}\), i.e., the empirical \(\beta\)-quantile of

\[
\tilde{\sigma}_{X,t} \varepsilon_{X,i} + \tilde{\sigma}_{XY,t} \varepsilon_{Y,i}, \quad i = 1, \ldots, n.
\]
For general $\Sigma_t$ from (D.2), CoVaR solves

$$\alpha = P_{t-1}\{ Y_t \leq \text{CoVaR}_{t,\alpha|\beta} \mid X_t \geq \text{VaR}_{t,\beta} \}$$

$$= P_{t-1}\{ \tilde{\sigma}_{X,Y,t} \varepsilon_{X,t} + \tilde{\sigma}_{Y,t} \varepsilon_{Y,t} \leq \text{CoVaR}_{t,\alpha|\beta} \mid X_t \geq \text{VaR}_{t,\beta} \}. $$

Thus, CoVaR$_{t,\alpha|\beta}$ can be estimated as the empirical $\alpha$-quantile of

$$\tilde{\sigma}_{X,Y,t} \varepsilon_{X,i} + \tilde{\sigma}_{Y,t} \varepsilon_{Y,i}, \quad i \in \mathcal{I} = \{ t = 1, \ldots, n : X_t \geq \hat{\text{VaR}}_{t,\beta} \}. $$