Reconstructing matter profiles of spherically compensated cosmic regions in $\Lambda$CDM cosmology

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ABSTRACT
The absence of a physically motivated model for large scale profiles of cosmic voids limits our ability to extract valuable cosmological information from their study. In this paper, we address this problem by introducing the spherically compensated cosmic regions, named CoSpheres. Such cosmic regions are identified around local extrema in the density field and admit a unique compensation radius $R_1$ where the internal spher-ical mass is exactly compensated. Their origin is studied by extending the standard peak model and implementing the compensation condition. Since the compensation ra-adius evolves as the Universe itself, $R_1(t) \propto a(t)$, CoSpheres behave as bubble Universes with fixed comoving volume. Using the spherical collapse model, we reconstruct their profiles with a very high accuracy until $z = 0$ in N-body simulations. CoSpheres are symmetrically defined and reconstructed for both central maximum (seeding haloes and galaxies) and minimum (identified with cosmic voids). We show that the full non linear dynamics can be solved analytically around this particular compensation ra-adius, providing useful predictions for cosmology. This formalism highlights original correlations between local extremum and their large scale cosmic environment. The statistical properties of these spherically compensated cosmic regions and the pos-sibilities to constrain efficiently both cosmology and gravity will be investigated in companion papers.

Key words: cosmology: theory; large-scale structure of Universe; N-body Simula-tions; Cosmic Voids; dark energy

INTRODUCTION
One of the main purpose of modern cosmology is to under-stand the nature of Dark Energy (DE), driving the cosmic acceleration (Riess et al. 1998; Perlmutter et al. 1999; Sil-vestri & Trodden 2009; Caldwell & Kamionkowski 2009). It is not only difficult to build consistent models to under-stand this acceleration but rather to find binding limits to discriminates between them.

Large Scale Structures (LSS) offer a large panel of probes for cosmology and the nature of gravity itself. They carry informations on both primordial Universe and gravity through the cosmological evolution. These last years, cosmic structure formation have been specially studied in the frame of cosmic voids formation and statistics (Park & Lee 2007; Lavaux & Wandelt 2012; Pan et al. 2012; Hamaus et al. 2015; Cai et al. 2015; Achitouv et al. 2017; Hamaus et al. 2016; Achitouv 2017). Although cosmic void dynamics is far from being linear, these regions are safe from the highly non linear physics occurring during haloes or galaxy formation. Moreover, cosmic voids are expected to be more sensitive to the nature of DE since their local $\Omega_{DE}$ is higher than in the average Universe (Sheth & van de Weygaert 2004; Colberg et al. 2005; van de Weygaert & Platen 2011). Voids have been studied through the Alcock-Paczynski test (Sut-ter et al. 2014; Mao et al. 2016), their ellipticity (Park & Lee 2007; Lavaux & Wandelt 2010) and their abundance or shape (Cai et al. 2015; Achitouv et al. 2016). However, all these studies suffer from the lack of a fully consistent - and phys-ically motivated - model describing both the origin and the dynamical generation of such cosmic regions. Hamaus et al. (2014b) introduced an effective parametrization of density profiles using numerical simulation. Despite not being deduced from first principles, it can provide physical insights. For example, Hamaus et al. (2016) used it to model the isotropic shape of the void-density profile and have been able to isolate the sensibility to cosmological parameters through anisotropic redshift-space and Alcock-Paczynski distortions.
In this paper and through the following ones (Alimi & de Fromont 2017a; Alimi & de Fromont 2017b; de Fromont & Alimi 2017) we present a physically motivated model studying both the primordial origin and the dynamical evolution of such cosmic regions. More precisely, we generalize cosmic void study by introducing the spherically compensated cosmic regions, named thereafter CoSpheres\(^1\). These structures are defined as the large scale cosmic environment surrounding local extrema of the density field. When defined around central under densities (local minimum), these regions can be identified to cosmic voids. Interestingly, these regions can also be defined around central over dense maxima, defining the symmetric of standard voids.

In average\(^2\), the large scale environment around maxima (respectively minima) in the density field can be separated in two distinct domains: an internal over (respectively under) dense core surrounded by a large under (respectively over) dense compensation belt. Note that even if the density field around local extrema is far from being spherical, one can always define a spherical profile by averaging over angles. However, despite being intuitive, the density contrast \(\delta(x) = \rho(x)/\bar{\rho}_m - 1\) has no dynamical interpretation. Indeed, in the spherical frame, the local gravitational dynamics is driven by the integrated density contrast, or equivalently the mass contrast

\[
\Delta(r) = \frac{3}{r^3} \int_0^r u^2 \delta(u) du = \frac{m(r)}{4\pi \bar{\rho}_m r^3} - 1 \tag{1}
\]

Like for density, the large scale environment of local extrema can be splitted in an over-massive (respectively under-massive for central minima) core surrounded by an under massive (respectively over massive) area. The transition radius between these under/over massive regions defines the compensation radius, noted \(R_1\). This radius can be uniquely define for each central extrema\(^3\). In a naive spherical description, this radius separates the collapsing over massive region from the expanding under massive one. The existence of such scale is fundamentally insured by the Bianchi identities (Héhl & Mcrea 1986) which impose the mass conservation. Moreover, the compensation radius \(R_1\) follows a remarkable evolution. Indeed, since \(R_1\) encloses a sphere whose averaged density equals the background density, it evolves as the scale factor itself, i.e. \(R_1(t) \propto a(t)\). CoSpheres thus behave as bubble universes with a fixed comoving size. Hamaus et al. (2014a) introduced a similar concept of a compensation radius for voids and its use as a static cosmological ruler that follows the background expansion. However, our definition of the compensation radius differs since it is defined uniquely for each maximum (see Eq. 5). We also stress that, on the Hubble size, there should not “over-compensated” or “under-compensated” voids as a consequence of the mass conservation.

The large scale structures are originally generated by the stochastic fluctuations of the density field in the primordial Universe. Their statistical properties, including average shape and probability distribution can be computed within the Gaussian Random Field (GRF) formalism. However, it is necessary to implement the compensation constraint (the existence of a finite compensation radius \(R_1\)) and thus to extend the results of Bardeen et al. (1986). As we show in this paper, the non linear evolution of such regions is very well described by using the spherical collapse model while neither Zel’dovich nor Eulerian linear dynamics is accurate enough.

We discuss the linear scaling of the density profiles of such regions in both primordial and evolved Universe. It turns out that these large scale profiles do not scale linearly on \(R_1\), neither on shape nor amplitude. This property emerges from the fact that on scales considered here (from \(r \sim 5\) to \(r \geq 100\ h^{-1}\) Mpc), the linear matter power spectrum is far from being scale invariant. Moreover, the non linear gravitational evolution of these profiles would have broken any primordial linear scaling.

The paper is organized as follow. In Section 1 we introduce the N-body simulations on which is based our study; the DEUS simulations (see Sec. 1.1). After defining precisely CoSpheres, we study these regions in the numerical simulations for various redshift and sizes and for both central over and under densities. This leads us to discuss the stacking method used to reconstruct the corresponding average profiles.

In Section 2 we study the shape of these regions in a Gaussian primordial field. We present an extension of the usual peak formalism of BBKS (Bardeen et al. 1986). While BBKS formalism focuses on the local properties of the field around the peak (note that for us, a peak is an extremum and can be a minimum or a maximum), we extend this model to take into account its cosmic environment on large scale. We show that this environment can be fully qualified by the compensation scale \(R_1\) and the compensation density \(\delta_1 = \delta(R_1)\).

In Section 3 we study the dynamical evolution of CoSpheres. We show that the Lagrangian Spherical Collapse (SC) model (Padmanabhan 1993; Peacock 1998) is able to reproduce precisely the evolution of such regions from small scales (typically \(r \sim 5\ h^{-1}\) Mpc) to much larger scales where the dynamics becomes almost linear. However, we explicitly show that neither the Eulerian linear theory nor the Zel’dovich approximation are able to describe their evolution with a sufficient precision. Finally, we show that we are able to reproduce the full matter field surrounding both maxima (build around haloes) and minima (identifed to cosmic voids) at \(z = 0\) in numerical simulations.

## 1 COSPHERES IN THE NUMERICAL SIMULATIONS

### 1.1 N-body DEUS simulations

In this work we use the numerical simulations from the “Dark Energy Universe Simulation ” (DEUS) project. These simulations are publicly available through the “Dark Energy Universe Virtual Observatory ” DEUVO Database\(^4\). They consist of N-body simulations of Dark Matter (DM) for realistic dark energy models. For more details we refer the interested reader to dedicated sections in Alimi et al. (2010); Rasera et al. (2010); Courtin et al. (2010); Alimi et al. (2012);

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\(^1\) For Compenated Spherical regions

\(^2\) we discuss this term more precisely in this paper

\(^3\) more generally, for any random position in the density field

\(^4\) http://www.deus-consortium.org/deus-data/
Reverdy et al. (2015). These simulations have been realized with an optimized version (Alimi et al. 2012; Reverdy et al. 2015) of the adaptive mesh refinement code RAMSES based on a multigrid Poisson solver (Teyssier 2002; Guillet & Teyssier 2011) for Gaussian initial conditions generated using the Zel’dovich approximation with MGPARIFIC code (Prunet et al. 2008) and input linear power spectrum from CAMB (Lewis et al. 2000).

In this paper we focus only on the flat ΛCDM model with cosmological parameters calibrated against measurements of WMAP 5-year data (Komatsu et al. 2009) and luminosity distances to Supernova Type Ia from the UNION dataset (Kowalski et al. 2008). The reduced Hubble constant is set to $h = 0.72$ and the cosmological parameters are $\Omega_{DE} = 0.74$, $\Omega_0 = 0.044$, $n_s = 0.963$ and $\sigma_8 = 0.79$. In this paper, we used mainly two different simulations whose properties are summarized in Table 1.

If not specified, the numerical simulation used is the reference one defined with $L_{\text{box}} = 2592 \ h^{-1}\text{Mpc}$ and $n_{\text{part}} = 2048^3$ to get both a large volume and a good mass resolution (see Table 1).

1.2 Defining CoSpheres

1.2.1 Method and definition

We construct CoSpheres in numerical simulation from the position of local extrema in the density field. For central maxima and for $z = 0$, we decide to identify these positions with the center of mass of DM haloes\(^5\). Such procedure is motivated by the possibility to extend it for observational data where DM haloes could be identified with galaxy, galaxy group or galaxy cluster.

In the symmetric case of a central minima, the position is computed as local minima in the density field smoothed with a Gaussian kernel. In the reference simulation, the physical size of the original coarse grid cell is $L_{\text{grid}} = 1.26 \ h^{-1}\text{Mpc}$ before any refinement and we choose a smoothing scale of $5 \ h^{-1}\text{Mpc}$. The comparison with analytical predictions requires to smooth the matter power spectrum on the same scale.

Around each extremum of the density field, we compute the concentric mass by counting the number of particles in the sphere of radius $r$, thus imposing the spherical symmetry

$$m(r) = \sum_i m_p \Theta[r - |x_0 - x_i|]$$

where $m_p$ is the mass of each individual particle, $x_0$ the position of the extremum and $x_i$ the position of the $i$\textsuperscript{th} particle.

$\Theta(x)$ is the standard Heaviside function such as $\Theta(x) = 1$ for $x > 0$ and $0$ elsewhere. From this mass profile we define the mass contrast profile $\Delta(r)$ defined in Eq. (1). Note that the density contrast is linked to the mass contrast by

$$\delta(r) = \frac{1}{\sqrt{\pi}} \frac{\partial \Delta}{\partial \log r} + \Delta(r)$$

We now focus on the compensation property. A volume $V$ is said to be compensated if it satisfies the condition

$$\int_V \rho_m(x) d^3x = \bar{\rho}_m V$$

The spherical symmetry imposes that the field is compensated in a sphere of radius $R_1$ if it satisfies

$$m(R_1) = \frac{4\pi}{3} \bar{\rho}_m R_1^3 \Leftrightarrow \Delta(R_1) = 0$$

this last equation defines the compensation scale $R_1$ as the first radius satisfying $\Delta(R_1) = 0$. We stress that this scale is much larger than the typical scale associated to haloes such as the virialisation radius $R_{\text{vir}}$ or $R_{200}$ such as $\rho(R_{200}) = 200 \times \bar{\rho}_m$ (Ricotti et al. 2007). It is important to note that the compensation radius is defined uniquely for each structure despite the fact that the mass contrast may vanish at other radii $R_i > R_1$. For each central extremum, there is a - possibly infinite - number of radii satisfying $\Delta(R_i) = 0$; the compensation radius is defined as the smallest one. Moreover, since these regions must be compensated on the size of the Universe (no mass excess), we have also $\lim_{r \to \infty} r^3 \Delta(r) = 0$. In Fig. 1 we show various mass contrast profiles $1 + \Delta(r)$ centred on DM haloes at $z = 0$. For each profile we identify $R_1$ where the mass is exactly balanced.

We note that compensation radii are always much smaller than the size of the computing box. In the reference simulation, 70% of profiles are compensated on $R_1 \leq 50 \ h^{-1}\text{Mpc}$ whereas less than 7% of the profiles have $R_1 \geq 100 \ h^{-1}\text{Mpc}$. It means that the compensation radius could be also measurable on observational data with a sufficiently large volume survey. Moreover $R_1$ is roughly of the same order of the effective size $R_{\text{eff}}$ used in the study of cosmic voids (Platen et al. 2007; Neyrinck 2008).

\footnote{detected by using a Friend-Of-Friend algorithm with a linking length $b = 0.2$}

Figure 1. Mass contrast profile built around various haloes of mass $M_h \sim 3 \times 10^{13} \ h^{-1} \text{M}_\odot$ from the reference simulation at $z = 0$ (see Table 1). For each profile we show the compensation radius $R_1$ defined by Eq. (5) and marked with a colored dot. The majority of profiles are compensated on scale between 10 and 30$h^{-1}\text{Mpc}$. We also show a profile with a very large compensation radius $R_1 > 100$. For central minima, i.e. cosmic voids we obtain similar profiles with finite compensation radii, but in this case, both $\delta(r)$ and $\Delta(r)$ are bounded to $-1$.\[\text{Figure 1. Mass contrast profile built around various haloes of mass $M_h \sim 3 \times 10^{13} \ h^{-1} \text{M}_\odot$ from the reference simulation at $z = 0$ (see Table 1). For each profile we show the compensation radius $R_1$ defined by Eq. (5) and marked with a colored dot. The majority of profiles are compensated on scale between 10 and 30$h^{-1}\text{Mpc}$. We also show a profile with a very large compensation radius $R_1 > 100$. For central minima, i.e. cosmic voids we obtain similar profiles with finite compensation radii, but in this case, both $\delta(r)$ and $\Delta(r)$ are bounded to $-1$.} \]
Table 1. Simulations used in this paper. $m_p$ is the mass of each particle while $M_h$ corresponds to the average mass of the DM haloes selected for stacking. Each simulation is defined by its box size $L$ in $h^{-1}\text{Mpc}$ and the number of DM particles $n$. The simulation in bold is the reference simulation.

| Simulation | $L$ (h^-1 Mpc) | $n$ (Mpc^-3) | $m_p$ (h^-1 Mpc) | $M_h$ (h^-1 Mpc) |
|------------|----------------|--------------|------------------|------------------|
| L = 648, n = 1024 | 2592, n = 2048 | 5184, n = 2048 |

| Simulation | $L$ (h^-1 Mpc) | $n$ (Mpc^-3) | $m_p$ (h^-1 Mpc) | $M_h$ (h^-1 Mpc) |
|------------|----------------|--------------|------------------|------------------|
| L = 648, n = 1024 | 2592, n = 2048 | 5184, n = 2048 |

1.2.2 Average profile at $z = 0$ and stacking procedure

Every compensated region detected in numerical simulation is characterized by two distinct properties. One concerning the central extremum fully described by its height (i.e. the mass of the halo for a maxima and the central density contrast for minima). The second concerning its cosmic environment, characterized by $R_1$. Numerical simulation provides an ensemble of profiles with various heights and radii.

Due to the stochastic nature of the density field, the only physically relevant elements are obtained by computing average quantities and their dispersion. This leads to defined average spherical profiles. All along this paper, average profiles and their corresponding dispersion are built from at least 3000 single profiles. This number insures a fair statistical estimation. These profiles are built by stacking together CoSpheres with the same height and the same compensation radius $R_1 \pm dR_1$ where the radial width is $dR_1 = 1.25 h^{-1}\text{Mpc}$. This radial bin is kept constant for the whole paper. For central minima detected in the smoothed density field (see Sec. 1.2.1), we stack together profiles with the same $R_1 \pm dR_1$ without density criteria except $\delta(x_0) < 0$. In Fig. 2 we plot the distribution of their central density contrasts (whatever $R_1$). We observe that more than 99% of central contrasts are lower -0.1, beyond Poissonian fluctuations. The resulting profiles are thus averaged over all possible realization of the field with a fixed compensation radius.

On Fig. 3 we show average profiles in the reference simulation at $z = 0$ from both halo and void with a given compensation radius $R_1 = 40 h^{-1}\text{Mpc}$. In both cases we show the various radii

- the density radius $r_1$ such that $\delta(r_1) = 0$ (on this figure we have $r_1 \approx 30 h^{-1}\text{Mpc}$). It separates the over and under dense areas.

- the compensation radius $R_1$. Note that by construction it satisfies $R_1 \geq r_1$ since it encloses an over and a under dense shell (such that they compensated each other).

Error bars are computed as the standard error on the mean, i.e. $\sigma/\sqrt{n}$ where $\sigma$ is the dispersion and $n$ the number of profiles considered.

On Fig. 4, we plot the stacked average profiles for various compensation radii $R_1$ with the same central extrema. Varying $R_1$ probes the same peak in various cosmic environments.

Using these profiles we can study the simple linear scaling assumption. For the mass contrast for example, there could exist $\Delta_{\text{univ}}(r)$ such that for any $R_1$ we would have $\Delta(r, R_1) = \alpha \Delta_{\text{univ}}(\beta r)$. On Fig. 5, we plot the rescaled profiles $\Delta(r/R_1)/\Delta_{\text{max}}$ where $\Delta_{\text{max}}$ is the maximum of the mass contrast. This figure does not indicates any simple linear scaling. Despite being normalized to the same maximal amplitude, the profiles are clearly separated on small scales (for $r \leq R_1$) but also on larger scales. Furthermore, the position of the maximum changes while varying $R_1$, indicating that $R_{\text{max}} \neq R_1$. This show that it is necessary to study the shape of these regions for various compensation radii.

We must also ensure that modifying the simulation parameters do not affect the profiles. A numerical simulation is characterized by a mass and a spatial resolution (see Table 1). Since CoSpheres trace the matter distribution on large or intermediate scales (compared to the coarse grid size), average stacked matter profile result from the dynamics computed on the coarse grid without any refinement. As long as we consider scales larger than a few cells we should not observe any significant deviations for large scale field when changing the simulation parameters. In other words, the properties of CoSpheres are robust with respect to the resolution parameters of the simulation used to trace the matter field. We illustrate this point on Fig. 6 where we plot the stacked average profile for different numerical simulations but the same halo mass $M_h = 3.0 \pm 0.075 \times 10^{13} h^{-1}M_\odot$ (200 $\pm$ 5 particles per halo) and three different compensation radii $R_1$. For each $R_1$, matter profiles are indeed merged together.

1.2.3 The spherically compensated cosmic regions at higher redshift

CoSpheres are detected in numerical simulation at $z = 0$. We then follow backward in time the evolution of the mat-
Reconstructing matter profiles of CoSpheres

(a) Stacked average profile measured around haloes of mass $M_h \sim 3.0 \times 10^{13} h^{-1}$Mpc at $z = 0$ in the reference simulation. We clearly identify the central over-dense core until $r_1$ (red dot) surrounded by the compensation belt from $r = r_1$. The same occurs for the mass contrast profile (in blue), i.e. an over massive core for $r \leq R_1$ (blue square) enclosed in a large under massive region.

(b) Same as in left panel for central minima. Now the interior region $r < R_1$ is under massive while the exterior region is over massive. The compensation radius has been chosen with the same value than in the left panel.

**Figure 3.** Average mass and density contrasts. The blue line represents the mass contrast $\Delta(r)$ while the red line represents the density contrast $\delta(r)$. The density radius (red dot at $r_1 \sim 27 h^{-1}$Mpc) and the mass radius (blue square at $r = 40 h^{-1}$Mpc) can be clearly identified. On both panels, we plot a zoom of profiles around the compensation radius.

(a) Stacked average profiles around haloes with a mass $M_h \sim 3.0 \times 10^{13} h^{-1}$Mpc at $z = 0$ in the reference simulation.
(b) Same as in left panel for central under dense regions, i.e. cosmic voids

**Figure 4.** Radial average mass contrast profiles at $z = 0$. Each curve corresponds to a fixed compensation scale from 15 to 80 $h^{-1}$Mpc. We do not show the error bars on this figure since they are almost indistinguishable from the curve itself. In both cases we note that small $R_1$ are associated to strongly contrasted regions.

For each halo, the position of its progenitor is estimated from the center of mass of its particles at $z = 0$. This estimation is correct since scales probed here are much larger than the halo size (in Appendix A we show how it is possible to model a shift in the theoretical profile). For voids, i.e. central minimum, we assume that its comoving position is conserved during evolution and equals to the position measured at $z = 0$. For every redshift and profile, this primordial position is used to compute the spherical mass by counting the number of particles in concentric shells as discussed in section 1.2.1.
This indicates that CoSpheres are generated within the primordial density field at high redshift and are not generated through gravitational dynamics only. On Fig. 8, we thus show - average - CoSphere profiles for different $R_1$ at a very high redshift $z \sim 57$ in the simulation. This figure shows that these structures are originated by large scale primordial density fluctuations with the same compensation properties.

In the two following sections we will study these structures within the primordial field in the framework of GRF (see Sec. 2). The gravitational evolution of these initial profiles will be studied in (see Sec. 3).

2 ORIGIN OF COSPHERES IN GAUSSIAN RANDOM FIELD

In this section we derive the average density profile around extrema in a GRF constrained by the compensation property Eq. (5). Density and mass profiles of CoSpheres are characterized by two family of parameters: the peaks parameters as defined by (Bardeen et al. 1986) qualifying the central extrema and the environment parameters.

As was studied in Bardeen et al. (1986), a local extrema at some position $x_0$ in GRF can be parametrized by 10 independent - but correlated - parameters. A scalar $\nu$ quantifying the central height of the extrema

$$\delta(x_0) = \nu \sigma_0$$

expressed in units of the fluctuation level $\sigma_0$

$$\sigma_0 = \left[ \frac{1}{2\pi^2} \int_0^\infty k^2 P(k) dk \right]^{1/2}$$

where $P(k)$ is the linear matter power spectrum evaluated at some fixed time $t_i$ where the field can be assumed to be Gaussian (deep inside the matter-dominated era). The extremum condition imposes that the local gradient of the field $\eta$ vanishes identically, i.e.

$$\eta_i = \frac{\partial \delta(x_0)}{\partial x_i} = 0$$

The local curvature around the extremum is described by its Hessian matrix $\zeta$

$$\zeta_{ij} = \frac{\partial^2 \delta(x_0)}{\partial x_i \partial x_j}$$

Each eigenvalue of the Hessian matrix must be negative in the case of a central maximum and positive in the opposite case of a minimum (under-dense).

Let us now consider the environment parameters. The neighbourhood of the peak is here described by the compensation radius (see Eq. 5). However, providing this radius only is not sufficient to reconstruct the large variety of profiles and it is necessary to add the compensation density contrast defined on the sphere of radius $R_1$. By construction it must be of opposite sign of the central density contrast. The compensation density $\delta_1$ is thus defined once averaging over angles the density on the sphere of radius $r = R_1$

$$\delta(R_1) := \delta_1 = \nu_1 \sigma_0$$

with $\nu_1/v < 0$. Without any assumption on the symmetry, we thus need 12 independent parameters : the scalar $\nu$, three components of the $\eta$ vector, 6 independent coefficients of the...
Figure 7. Evolution of the mass contrast profile for both maxima and minima in the density field. The radial scale is in comoving $h^{-1}$Mpc and each curve corresponds to a single redshift.

Figure 8. Mass contrast profiles at high redshift $z = 57$ for different $R_1$ between 15 and 80 $h^{-1}$Mpc detected originally from haloes with the same mass $M_h = 3.0 \times 10^{13} h^{-1} M_\odot$.

$\xi$ matrix (which is real and symmetric) together with $R_1$ and the reduced compensation density $v_1$. In the following, we are going to compute the expected averaged profile in the primordial Gaussian field satisfying both the peak constraints Eq. (6), Eq. (8) and Eq. (9) and the environmental constraints Eq. (5) and Eq. (10).

2.1 Peaks in GRF

Let us recall the basic elements necessary for the derivation of average quantities in the context of GRF. Our Gaussian field is assumed to be an homogeneous and isotropic random field with zero mean. We also restrict ourselves to GRF whose statistical properties are fully determined by its power-spectrum (or spectral density) $P(k)$ i.e. the Fourier Transform of the auto-correlation function of the field, 

$$\xi(r) = \xi(|x_1 - x_2|) = \langle \delta(x_1) \delta(x_2) \rangle$$

The Gaussianity of the field $\delta(x)$ appears in the computation of the joint probability

$$dP_N = P\left[ \delta(x_1), ..., \delta(x_N) \right] d\delta(x_1) ... d\delta(x_N)$$

that the field has values in the range $[\delta(x_1), \delta(x_N) + d\delta(x_N)]$ for each position $x_i$. In this particular case of homogeneous and isotropic GRF, this probability reaches

$$dP_N = \frac{1}{\sqrt{(2\pi)^N \det M}} \exp \left[ -\frac{1}{2} \delta^T M^{-1} \delta \right] \prod_{i=1}^N d\delta_i$$

where $\delta$ is the $N$ dimensional vector $\delta_i = \delta(x_i)$ and $M$ is the $N \times N$ covariance matrix, here fully determined by the field auto-correlation

$$M_{ij} := \langle \delta_i \delta_j \rangle = \xi(|x_i - x_j|)$$

where the average operator $\langle ... \rangle$ denotes hereafter an ensemble average. The ergodic assumption identifies the ensemble average $\langle ... \rangle$ computed on all possible statistical realisation of the observable to its spatial averaging, i.e., its mean over a sufficiently large volume. The average of any operator $X$ can be written as a mean over its Fourier component $\tilde{X}$

$$\langle X \rangle := \frac{1}{2\pi^2 \sigma^2} \int_0^\infty k^2 P(k) \tilde{X}(k) dk = \frac{\int_0^\infty k^2 P(k) \tilde{X}(k) dk}{\int_0^\infty k^2 P(k) dk}$$
Furthermore, we are interested in deriving the properties of the field subject to $n$ linear constraints $C_1, \ldots, C_n$. Following Bertschinger (1987), we write each constraint $C_i$ as a convolution of the field

$$C_i[\delta] := \int W_i(\mathbf{x}_i - \mathbf{x})\delta(\mathbf{x})d\mathbf{x} = c_i$$

(16)

where $W_i$ is the corresponding window function and $c_i$ is the value of the constraint. For example, constraining the value of the field to a certain $\delta_0$ at some point $\mathbf{x}_0$ leads to $W_i = \delta_D(\mathbf{x} - \mathbf{x}_0)$ and $c_i = \delta_0$. Since the constraints are linear, their statistics is also Gaussian and the joint probability $dP[C]$ that the field satisfies these conditions is (van de Weygaert & Bertschinger 1996; Bertschinger 1987)

$$dP[C] = \frac{1}{\sqrt{(2\pi)^n \det \mathbf{Q}}} \exp \left[ -\frac{1}{2} \mathbf{C}^T \mathbf{Q}^{-1} \mathbf{C} \right]$$

(17)

where $\mathbf{Q}$ is the covariance matrix of the constraints $\mathbf{C} = \{C_1, \ldots, C_n\}$ defined as $\mathbf{Q} = \langle C_i^T C_j \rangle$. The average density profile $\langle \delta \rangle$ subject to $C$ is computed as the most probable profile given $\mathbf{C}$ and reaches

$$\langle \delta \rangle(\mathbf{x}) := \langle \delta(\mathbf{C}) \rangle = \xi(\mathbf{x})Q_{ij}^{-1}c_j$$

(18)

where $\xi(\mathbf{x})$ is the correlation function between the field and the $i$th constraint

$$\xi_i(\mathbf{x}) = \langle \delta(\mathbf{x}) C_i \rangle$$

(19)

and $Q_{ij}$ is the $(ij)$ element of the correlation matrix $\mathbf{Q}$. Bardeen et al. (1986) derived the average spherical density profile of peaks in GRF in term of the reduced height $\nu$ (see Eq. 6), its auto-correlation function $\xi(r)$ (see Eq. 11) and the curvature parameter $\kappa$ defined by

$$x = -\frac{3}{\sigma_0 \sqrt{k^2}}$$

(20)

It yields

$$\langle \delta \rangle_{\text{peaks}}(r) = \frac{\nu - x \xi(\nu)}{1 - \frac{x^2}{\sigma_0} + \frac{r^2}{2} \Delta \xi(r)}$$

(21)

with $R_* = \sqrt{3 \langle k^2 \rangle / \langle k^4 \rangle}$ and $\gamma = \langle k^2 \rangle / \sqrt{\langle k^4 \rangle}$. The various moments of $k$ are given by

$$\langle k^n \rangle := \frac{1}{2\pi^2 \sigma_0} \int_0^\infty k^{2+n}P(k)dk$$

(22)

In the next section, we extend this result by implementing the compensation conditions Eqs. (5) and Eq. (10).

### 2.2 CoSpheres in Gaussian Random Fields

In the following, we use several functions involving $r$ and $R_1$. In order to simplify the notations, we note the Fourier components as

$$W_r := \frac{\operatorname{sin}(kr) - kr \cos(kr)}{(kr)^3}$$

$$J_r := \frac{\operatorname{sin}(kr)}{kr}$$

(23)

(24)

Their calculation goes beyond the spherical approximation but we restrict here to the simplest spherical case obtained by averaging over angles

Which are respectively the Fourier transform of the top-hat and the delta functions. Note that they are linked to the spherical Bessel functions as $W_r = 3/(kr)j_1(kr)$ and $J_r = j_0(kr)$. We also denote with the “1” subscript these quantities evaluated at the particular radius $r = R_1$, i.e.

$$W_{1r} := W_r |_{r = R_1}$$

$$J_{1r} := J_r |_{r = R_1}$$

(25)

(26)

#### 2.2.1 The average density profile of CoSphere

We now derive the average matter profile being both

(i) centred on an extremum, i.e. satisfying the conditions Eqs. (6), (8) and (9)

(ii) and compensated on a finite scale $R_1$. This is implemented by the compensation constraints Eqs. (5) and Eq. (10).

The spherical peak constraints are (see Eq. 16)

$$C_{r}[\delta] := \int \delta_D [\mathbf{x} - \mathbf{x}_0] \delta(\mathbf{x})d\mathbf{x} = c_r \equiv \sigma_0 \nu$$

(27)

$$C_p[\delta] := \int \delta_D [\mathbf{x} - \mathbf{x}_0] \delta(\mathbf{x})d\mathbf{x} = c_p \equiv 0$$

(28)

$$C_4[\delta] := \int \frac{d^2}{dx^2} \delta_D [\mathbf{x} - \mathbf{x}_0] \delta(\mathbf{x})d\mathbf{x} = c_4 \equiv -c_0 \frac{x \sqrt{\langle k^4 \rangle}}{3}$$

(29)

where $\delta_D$ is the usual Dirac function. On the other hand, the environmental constraints (see Eq. (5) and Eq. (10)) can be written

$$C_{R_1}[\delta] := \int \Theta [R_1 - |\mathbf{x} - \mathbf{x}_0|] \delta(\mathbf{x})d\mathbf{x} = c_{R_1} \equiv 0$$

(30)

$$C_{V_1}[\delta] := \int \delta_D [R_1 - |\mathbf{x} - \mathbf{x}_0|] \delta(\mathbf{x})d\mathbf{x} = c_{V_1} \equiv c_0 V_1$$

(31)

where $\Theta$ is the Heaviside step function. $V_1$ must also satisfy Eq. (10) and thus satisfy $V_1/\nu < 0$. Note that in Eq. (30), the constraint value is $c_{R_1} = 0$ and only three eigenvalues for the constraints are non zero : $\nu$, $x$ and $V_1$. The original peak constraints involve the correlations

$$\xi_{\nu}(r) := \langle \delta(\mathbf{x}) C_{\nu} \rangle = \sigma_0^2 \langle J_r \rangle$$

(32)

$$\xi_x(r) := \langle \delta(\mathbf{x}) C_x \rangle = -c_0^2 \left[ \frac{k^2}{3} J_r \right]$$

(33)

The compensation constraints introduce new correlations in the computation of the average profile

$$\xi_{R_1}(r) := \langle \delta(\mathbf{x}) C_{R_1} \rangle = \sigma_0^2 \langle W_{1r} J_r \rangle$$

(34)

$$\xi_{V_1}(r) := \langle \delta(\mathbf{x}) C_{V_1} \rangle = \sigma_0^2 \langle J_r (J_r) \rangle$$

(35)

Using Eq. (18), the average profiles are linear in $\nu$, $x$ and $V_1$ while $R_1$ implicitly appears in the various radial functions, such that we can write

$$\langle \delta \rangle(\mathbf{x}) = \sigma_0 [\nu \delta_\nu(\mathbf{x}) + x \delta_x(\mathbf{x}) + V_1 \delta_{V_1}(\mathbf{x})]$$

(36)

where $\delta_\alpha$ with $\alpha = \{\nu, x, V_1\}$ are functions of $r$ and $R_1$ only. By construction, each $\delta_\alpha(r)$ must satisfy the compensation property, i.e. its integral must vanish at $r = R_1$

$$\int_0^{R_1} u^2 \delta_\alpha(u)du = 0$$

(37)
From Eq. (18) we know that each \( \delta_\alpha(r) \) is a linear combination of the four functions \( \xi_i(r) \) (see Eq. (32) to Eq. (35)). To simplify the notations, we define three intermediate functions build from the \( \xi_i \) functions and satisfying the condition Eq. (37)

\[
\begin{align*}
  f_0(r) &= \frac{\langle k^2 W_1 \rangle (J_r) - \langle W_1 \rangle \langle k^2 J_r \rangle}{\langle k^2 W_1 \rangle - \langle k^2 \rangle (W_1)} \\
  f_1(r) &= \frac{\langle W_1 \rangle (J_r) - \langle k^2 W_1 \rangle \langle W_2 \rangle \langle k^2 J_r \rangle}{\langle W_1 \rangle \langle k^2 W_1 \rangle - \langle k^2 \rangle \langle W_2 \rangle} \\
  f_2(r) &= \frac{\langle J_r \rangle (J_r) \langle k^2 W_1 \rangle - \langle J_r W_1 \rangle \langle k^2 J_r \rangle}{\langle J_r \rangle \langle k^2 W_1 \rangle - \langle k^2 \rangle \langle J_r W_1 \rangle}
\end{align*}
\]

these functions have also been normalized such that \( f_i(0) = 1 \). Each \( \delta_\alpha(r) \) is then a linear combination of these three functions. Note that \( f_i(R_1) \neq 0 \). We also introduce three parameters \( \lambda_i \) defined locally around the extremum

\[
\lambda_i := -3 \frac{\partial^2 f_i}{\partial r^2} \quad \text{for} \quad r \to 0
\]

They explicitly reach

\[
\begin{align*}
  \lambda_0 &= \frac{\langle k^2 W_1 \rangle \langle k^2 \rangle - \langle W_1 \rangle \langle k^4 \rangle}{\langle k^2 W_1 \rangle - \langle k^2 \rangle (W_1)} \\
  \lambda_1 &= \frac{\langle k^2 W_1 \rangle ^2 - \langle W_2 \rangle \langle k^4 \rangle}{\langle W_1 \rangle \langle k^2 W_1 \rangle - \langle k^2 \rangle \langle W_2 \rangle} \\
  \lambda_2 &= \frac{\langle k^2 J_1 \rangle \langle k^2 W_1 \rangle - \langle J_1 W_1 \rangle \langle k^4 \rangle}{\langle J_1 \rangle \langle k^2 W_1 \rangle - \langle k^2 \rangle \langle J_1 W_1 \rangle}
\end{align*}
\]

With these notations and a bit of algebra we obtain the \( \delta_\alpha \) functions

\[
\begin{align*}
  \delta_\alpha(r) &= f_0(r) \frac{\lambda_1 f_2^1 - \lambda_2 f_1^1}{\omega} + f_1(r) \frac{\lambda_2 f_1^2 - \lambda_0 f_1^2}{\omega} + f_2(r) \frac{\lambda_0 f_1^1 - \lambda_1 f_0^1}{\omega}
\end{align*}
\]

for \( r \to 0 \) and

\[
\begin{align*}
  \delta_x(r) &= f_0(r) \frac{f_1^1 - f_2^1}{\omega} + f_1(r) \frac{f_2^1 - f_0^1}{\omega} + f_2(r) \frac{f_0^1 - f_1^1}{\omega}
\end{align*}
\]

and

\[
\begin{align*}
  \delta_{\nu_1}(r) &= f_0(r) \frac{\lambda_0 - \lambda_1}{\omega} + f_1(r) \frac{\lambda_0 - \lambda_2}{\omega} + f_2(r) \frac{\lambda_1 - \lambda_0}{\omega}
\end{align*}
\]

where

\[
\omega = \lambda_0 (f_1^1 - f_2^1) + \lambda_1 (f_2^1 - f_0^1) + \lambda_2 (f_0^1 - f_1^1)
\]

and we used the notation \( f_i^j := f_i(R_1) \). Note that \( \lambda_i \) and \( f_i^j \) are not constant but not linear functions of \( R_1 \).

The \( \delta_\alpha \) functions satisfy the following properties around \( r = 0 \)

\[
\begin{align*}
  \delta_\alpha(r) &= 1 + \mathcal{O}(r^4) \\
  \delta_x(r) &= -\sqrt{\langle k^4 \rangle} \frac{\omega}{6} + \mathcal{O}(r^4) \\
  \delta_{\nu_1}(r) &= \mathcal{O}(r^4)
\end{align*}
\]

while in \( r = R_1 \) we have

\[
\begin{align*}
  \delta_\alpha(R_1) &= 0 \\
  \delta_x(R_1) &= 0 \\
  \delta_{\nu_1}(R_1) &= 1
\end{align*}
\]

insuring that \( \langle \delta \rangle (R_1) = \sigma_0 \nu_1 \).

### 2.2.2 The averaged mass contrast profile of CoSpheres

The average mass contrast profile \( \langle \Delta \rangle (r) \) is obtained by integrating \( \langle \delta \rangle (r) \) with Eq. (1). By linearity of the mapping \( \delta \rightarrow \Delta \), \( \langle \Delta \rangle \) takes the same shape than Eq. (36) where each \( \delta_\alpha(r) \) transforms to \( \Delta_\alpha(r) \), i.e. we have

\[
\langle \Delta \rangle (r) = \sigma_0 \left[ \nu_1 \Delta_\sigma + x \Delta_x + v_1 \Delta_{\nu_1} \right]
\]

Since each function \( \delta_\alpha \) is a linear combination of the \( f_i \), the \( \Delta_\alpha \) functions will be linear combinations of the \( F_i \) functions defined as

\[
F_i(r) := \frac{3}{r^3} \int_0^r u^2 f_i(u) du
\]

Moreover, the \( f_i \) functions only involve linear combinations of \( J_r = \sin(\omega r)/(\omega r) \). The \( F_i \) are thus obtained from \( f_i \) by the simple replacement \( J_r \rightarrow W_r \), leading to

\[
\begin{align*}
  F_0(r) &= \frac{\langle k^2 W_1 \rangle \langle W_1 \rangle - \langle W_1 \rangle \langle k^2 \rangle}{\langle k^2 W_1 \rangle - \langle k^2 \rangle (W_1)} \langle k^2 \rangle (W_1) \\
  F_1(r) &= \frac{\langle W_1 \rangle \langle k^2 W_1 \rangle - \langle W_2 \rangle \langle k^2 \rangle}{\langle W_1 \rangle \langle k^2 W_1 \rangle - \langle k^2 \rangle \langle W_2 \rangle} \langle k^2 \rangle (W_1) \\
  F_2(r) &= \frac{\langle J_1 W_1 \rangle \langle k^2 W_1 \rangle - \langle J_1 W_1 \rangle \langle k^2 \rangle}{\langle J_1 \rangle \langle k^2 W_1 \rangle - \langle k^2 \rangle \langle J_1 W_1 \rangle}
\end{align*}
\]

We can check that for \( i = \{0, 1, 2\} \) we have \( F_i(R_1) = 0 \) insuring that \( \langle \Delta \rangle (R_1) = 0 \) whatever the shape parameters. The mapping between the \( \Delta_\alpha \) functions and the \( F_i \) is given by

\[
\begin{align*}
  \Delta_\alpha(r) &= F_0(r) \frac{\lambda_1 f_2^1 - \lambda_2 f_1^1}{\omega} + F_1(r) \frac{\lambda_2 f_1^2 - \lambda_0 f_1^2}{\omega} + F_2(r) \frac{\lambda_0 f_1^1 - \lambda_1 f_0^1}{\omega}
\end{align*}
\]

while for \( x \) we have

\[
\begin{align*}
  \Delta_x(r) &= F_0(r) \frac{f_1^1 - f_2^1}{\omega} + F_1(r) \frac{f_2^1 - f_0^1}{\omega} + F_2(r) \frac{f_0^1 - f_1^1}{\omega}
\end{align*}
\]

and

\[
\begin{align*}
  \Delta_{\nu_1}(r) &= F_0(r) \frac{\lambda_0 - \lambda_1}{\omega} + F_1(r) \frac{\lambda_0 - \lambda_2}{\omega} + F_2(r) \frac{\lambda_1 - \lambda_0}{\omega}
\end{align*}
\]

The resulting mass contrast profile satisfies, for \( r \to 0 \)

\[
\begin{align*}
  \Delta_\alpha(r) &= 1 + \mathcal{O}(r^4) \\
  \Delta_x(r) &= -\sqrt{\langle k^4 \rangle} \frac{\omega}{6} + \mathcal{O}(r^4) \\
  \Delta_{\nu_1}(r) &= \mathcal{O}(r^4)
\end{align*}
\]

while in \( r = R_1 \) we have, by construction

\[
\begin{align*}
  \Delta_\alpha(R_1) &= 0 \\
  \Delta_x(R_1) &= 0 \\
  \Delta_{\nu_1}(R_1) &= 0
\end{align*}
\]
\[ \Delta'(R_1) = \frac{3}{R_1} \nu \sigma_0 \]  

where a prime denotes the derivative with respect to \( r \). On Fig. 9 we plot the averaged mass contrast Eq. (51) for \( R_1 = 20 \, h^{-1}\text{Mpc} \) in each panel we change one of the shape parameters \( \nu \) and \( \nu_1 \) to illustrate their effect. Since both \( \nu \) and \( \nu_1 \) are associated with the central peak, changing their value only affects the profile on small scales, typically \( r \ll R_1/2 \). The compensation density \( \nu_1 \) defines the structure of the profile on larger scales from \( r \sim R_1 \). This behaviour clearly illustrates that \( x \) and \( \nu_1 \) are defined on the peak while \( R_1 \) and \( \nu_1 \) qualify the large scale surrounding environment of the peak.

2.2.3 Comparison with the BBKS peak profile

The peak profiles derived by BBKS (see Eq. 21) describe the large scale environment surrounding extrema in Gaussian Field where we only provide the properties of the density field on the peak. Our calculation is thus an extension of this result including the physical properties of the large scale environment around the peak.

Our formalism allows to probe different cosmic environment for the same central extremum. This environment is defined through \( R_1 \) and the compensation density \( \delta_1 = \nu_1 \sigma_0 \). For the same central peak, we can describe a large variety of cosmic configurations while this region is completely fixed in the standard BBKS approach.

In Fig. 10 we show how it is possible to describe various environments by varying \( R_1 \) while keeping constant \( \nu \) and \( x \), i.e. the central peak. We also plot the BBKS profile, fully determined by \( x \) and \( \nu \). Small values of \( R_1 \) correspond to isolated peaks in large under dense regions while increasing the compensation scale allows to probe denser regions. The exact same symmetric case occurs for cosmic voids with the environment around the peak.

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2.3 Numerical Reconstruction of CoSpheres in GRF

At very high redshift the density field follows a Gaussian statistics. This property is inherited from the inflation phase of the young Universe. In the previous sections we derived the average profiles of spherically compensated inhomogeneities in the framework of GRF. As presented in Section 1.2.3, the simulations can be used to follow backward in time the evolution of CoSpheres detected at \( z = 0 \). Using this numerical procedure we can compare our expected Gaussian profiles (see Eq. 36) with the numerical matter field of compensated peaks at higher redshift.

Each theoretical profile is parametrized by four scalars. The compensation radius \( R_1 \) can be read directly from the profile (build at fixed \( R_1 \)). The shape parameters \( ν \), \( x \) and \( ν_1 \) are computed using a standard \( χ^2 \) method defined from the measured profile \( Δ_j \) and its error \( σ_j \) for \( r = r_j \).

We stress that this reconstruction is done on the mass contrast profile and not directly from the density profile. Indeed, computing the number of particles in concentric shells with fixed mass. This leads to the Lagrangian spherical collapse model, first introduced in Gunn & Gott (1972) and largely discussed in (Padmanabhan 1993; Peacock 1998). While it was first developed in the context of Einstein-de-Sitter cosmology, it has been extended to \( Λ \)CDM (Lahav et al. 1991) and more general models of Dark Energy (Wang & Steinhardt 1998). In this section we derive a simple formalism for the spherical collapse suited to our problem.

3 DYNAMICAL EVOLUTION OF COSPHERES

In this section we study the gravitational collapse of CoSpheres resulting from the primordial fluctuations of the matter field as studied in Section 2.

3.1 The Lagrangian Spherical Collapse

Due to the spherical symmetry of our problem, the evolution of the matter profile reduces to the dynamics of concentric shells with fixed mass. This leads to the Lagrangian spherical collapse model, first introduced in Gunn & Gott (1972) and largely discussed in (Padmanabhan 1993; Peacock 1998). While it was first developed in the context of Einstein-de-Sitter cosmology, it has been extended to \( Λ \)CDM (Lahav et al. 1991) and more general models of Dark Energy (Wang & Steinhardt 1998). In this section we derive a simple formalism for the spherical collapse suited to our problem.
As we focus in this paper on the standard ΛCDM model describing a flat Universe (K = 0) with collision-less Cold Dark Matter and a cosmological constant Λ, the homogeneous background is described by the Friedman and the Raychaudhuri equations

\[
\begin{align*}
\left( \frac{\dot{a}}{a} \right)^2 &= H_0^2 \left[ \frac{\Omega_m^0}{a^3} + 1 - \Omega_m \right] \quad (68) \\
\frac{\ddot{a}}{a} &= -\frac{H_0^2}{2} \frac{\Omega_m^0}{a^3} + 2\Omega_m - 2^2 \quad (69)
\end{align*}
\]

where a dot (e.g. \( \dot{X} \)) denotes the derivative with respect to the proper time \( t \), \( H_0 \) is the Hubble constant today and \( \Omega_m^0 = 8\pi G \rho_{m0}/(3H_0^2) \). Moreover, in the Quasi Static Limit (QSL) where the time variation of the gravitational potentials are smooth, i.e. \( \Phi \ll \Phi_0/a \) and for scales deep inside the Hubble radius \( r \ll 1/(aH) \), the first order perturbed equations reduce to the well known Poisson equation linking the local density contrast \( \delta \) to the Newtonian potential \( \Phi \)

\[ \nabla^2 \Phi = 4\pi G \rho_m \delta \quad (70) \]

Using the spherical symmetry, the Poisson equation can be integrated once to give the Newtonian acceleration

\[ \frac{\partial \Phi}{\partial r} = \frac{r 4\pi G \rho_m}{3} \Delta(r) \quad (71) \]

with \( \Delta(r) \) is the mass contrast. It thus drives the local gravitational acceleration. Note that for \( r = R_1 \) we have \( \Phi = 0 \). In the QSL, the equation of motion of each shell with a physical radius \( r = aX \) is (Peebles 1980)

\[ \ddot{r} = -\frac{a}{r} - \nabla \Phi \quad (72) \]

For each shell we define the dimensionless comoving displacement

\[ R(x_i, t) = \frac{X(t)}{x_i} \quad (73) \]

where \( x_i \) is the comoving radius of the shell at some time \( t \) and \( X \) its initial radius \( x_i = X(t_i) \). The equation of motion for each concentric shell can be simplified assuming there is no shell-crossing (we discuss this hypothesis below) insuring the mass conservation

\[ \frac{1 + \Delta}{1 + \Delta_i} = R^{-3} \quad (74) \]

where \( \Delta_i \) is the initial mass contrast profile of the Lagrangian shell \( \Delta_i = \Delta(X(t_i), t_i) \) while \( \Delta \) is the evolved mass contrast \( \Delta = \Delta(X, t) \). We also introduce the logarithmic scale factor \( \tau \) defined through

\[ \frac{d\tau}{d \log(a)} = \sqrt{\frac{\Omega_m}{2}} \quad (75) \]

For ΛCDM, assuming \( \tau(a_i) = 0 \) we have

\[ \tau(a) = \frac{\sqrt{2}}{3} \left[ \text{arctanh} \left( \Omega_m^{1/2} \right) - \text{arctanh} \left( \Omega_m^{1/2} \right) \right] \quad (76) \]

With this new parametrisation, the equation of motion for each concentric shell (see Eq. 72) reaches

\[ \frac{\partial^2 R}{\partial \tau^2} + \frac{1}{\sqrt{2} \Omega_m} \frac{\partial R}{\partial \tau} = R - \frac{1 + \Delta_i}{R^2} \quad (77) \]

\[ \text{Figure 12. Comparison between average numerical profiles and the spherical evolution of the corresponding average primordial numerical matter profile for } z = 8 \text{ to } z = 0 \text{ (see text). Points and their associated error bars are the numerical measures for the corresponding redshift while full lines are the spherical evolution of the primordial profile at } z = 8 \text{. The computation has been performed using the 2048\(^3\) particles simulation with a box size of 5184 h\(^{-1}\)Mpc in } \Lambda \text{CDM cosmology and for } R_1 = 25 h^{-1}\text{Mpc. All single profiles have been detected from central haloes of mass } M_h = 2.5 \pm 0.13 \times 10^{14} \text{h}^{-1}M_\odot \text{ at } z = 0. \]

Eq. (77) describes the non-linear gravitational evolution of each shell until shell-crossing. Even if it does not appears now, the formulation Eq. (77) can be simply extended to any cosmologies including extensions of Gravity as we will show it in forthcoming papers (Alimi & de Fromont 2017b; de Fromont & Alimi 2017). The initial conditions for this differential problem are given by

\[ R(t_i) = 1 \quad (78) \]

together with the first derivative \( \partial_t R(t_i) \). It can be estimated from the high redshift solution where the field follows the Zel’dovich dynamic (see Appendix B)

\[ \frac{\partial R}{\partial \tau}(t_i) = -\frac{2}{\Omega_{m,i}} \frac{\partial \log(D)}{3 \log(a)} \bigg|_{t_i} = -\frac{2}{\Omega_{m,i}} \frac{\Delta_i}{3} f(t_i) \quad (79) \]

where \( f(t_i) \) is the linear growth rate evaluated at the initial time \( t_i \). The non linearly evolved profile \( \Delta \) is obtained by solving numerically Eq. (77) and using Eq. (74) for any initial profile \( \Delta_i \).

3.2 Testing the spherical approximation for the evolution of CoSpheres

The validity of the spherical evolution can be tested using the numerical simulations. At \( z = 0 \) we select haloes with the same \( R_1 \). For each halo detected we apply the "backward" procedure discussed in Section 1.2.3 to build the profile of its progenitor. Stacking these primordial profiles leads to the "initial average profile". This numerical profile is then taken as an input for the spherical dynamics as studied in Section 3.1. We thus obtain
a spherically evolved profile which can be compared to the numerical one at $z = 0$.

In Fig. 12 we plot both this spherically evolved profile (from $z = 8$ to $z = 0$) and the numerical profile for $R_1 = 25$ $h^{-1}$Mpc. For all redshift, the agreement between the simulation and the spherical evolution is excellent. On small scales however, typically $r \leq 5$ $h^{-1}$Mpc, the spherical dynamics fails to almost $5\%$ to $10\%$. It is not surprising that the central over-dense core is not well reproduced by a spherical dynamics but this work focuses on much larger scales where spherical collapse provides an excellent dynamical model.

On larger scales, such accuracy is neither reachable with the Eulerian linear nor Zel’dovich dynamics. On figure Fig. 13, we plot the differences at $z = 0$ resulting from various dynamical models in the same ΛCDM cosmology. Here we used a theoretical mass profile computed from our formalism (see Sec. 2.2) with realistic shape parameters $v$, $x$, and $v_1$ (close to unity) and we evolved this profile until $z = 0$ for each model. We choose to show these differences on a void profile, i.e. central under-dense minima. The argument is exactly symmetric for central over-densities. The Eulerian linear theory (blue lines) is clearly ruled out on non linear scales, i.e. for scales typically smaller than $20$ $h^{-1}$Mpc. As expected, linear theory, spherical collapse and Zel’dovich approximations agree on linear scales. The Zel’dovich approximation reproduces the spherical dynamics with a precision of $\approx 5\%$ on the mass contrast profile on large scales but only $\approx 10\%$ on the velocity profile. On smaller scales (inside the internal zone, $r < R_1$), the Zel’dovich approximation fails to almost $30\%$. The inaccuracy of the Zel’dovich dynamics cannot be neglected in a precision cosmology era since it could be mis-interpreted as a cosmological imprint (Almi and de Fromont 2017b; de Fromont and Alimi 2017).

Spherical dynamics is no longer valid in regions where collapse occurred (the shell reaches the singularity $r = 0$) and if any shell crosses an other one (i.e. when $\partial x / \partial t = 0$). But these two limitations are not really relevant for our purpose due to the size of the considered scales. As a matter of fact, the initial radii of shells that collapse in a finite time are very small, typically of the order of the halo size (fraction order of $h^{-1}$Mpc). In the symmetric case of a central under-density, the matter field expands such that there is no possible collapse onto $r = 0$. Moreover for compensated cosmic regions with realistic values for the shape parameter $v$, $x$, and $v_1$ (close to unity), radial shell crossing always occurs deep in our future ($z < 0$). Note that the shell crossing time $t_{sc}$ of each shell can be easily computed given the initial profile. For example, in the Zel’dovich approximation it satisfies $D(t_{sc})/D(t) = 1 + 1/(\delta_1 - 2/3\Delta_1)$, where both $\delta_1$ and $\Delta_1$ are evaluated at the same initial radius $r_i$.

### 3.3 Dynamical Evolution of the matter field around the compensation radius

#### 3.3.1 Evolution of the compensation radius

As was already mentioned in Fig. 7, the compensation radius is a conserved comoving quantity. This fundamental property is clear from the theoretical point of view. For $r = R_1$ we have $\Delta_1(R_1) = 0$ and the only solution of Eq. (77) compatible with the initial conditions Eq. (78) and Eq. (79) is $\mathcal{R}(t) = 1$, leading to $R_1 \propto a$.

Physically, since the average density in the closed sphere of radius $R_1$ equals the background density, this sphere evolves exactly as a closed bubble in the Universe and is consequently convoving.

This property stands in a spherical dynamic but initial inhomogeneities are very unlikely to be spherically symmetric (Bardeen et al. 1980). Using numerical simulations, we can follow the redshift evolution of $R_1$ for every CoSphere. In Fig. 14 we plot the mean and the dispersion of the distribution $R_1(z)/R_1(0)$ as a function of redshift for both haloes and void. For the whole range of redshift, convoming $R_1$ is constant with a precision better than $2\%$ for voids and $1\%$ for haloes. There is however a clear tendency of increasing $R_1$ for voids and decreasing $R_1$ for haloes. These small evolutions result probably from primordial anisotropies. They remain sufficiently small so that the spherical approximation holds. A deeper understanding of this small evolution goes beyond the scope of this paper.

#### 3.3.2 The evolution of the compensation density $\delta_1$

The compensation density contrast $\delta_1$ defined as $\delta_1 := v_1/\rho_0$ is an Eulerian quantity, being defined at a fixed comoving position $x_1 = R_1/a$. To derive its Eulerian dynamics, we consider two points initially located at an infinitesimal distance from the compensation radius $x_{i1}^\pm := x_1 \pm \epsilon$ where $\epsilon \ll 1$. Since we consider two points in the infinitesimal range $\pm \epsilon$ we have (see Eq. B2)

$$\mathcal{R}^\pm(t) = 1 - (\pm \epsilon) \frac{\delta_1}{\chi_1} \left( \frac{\mathcal{D}(t)}{\mathcal{D}(t_i)} - 1 \right) + O(\epsilon^2)$$

where $\delta_1$ is assumed to be the value of the local density contrast in the initial conditions. The same quantity at any time $t \pm t_i$ is explicitly noted with its time dependency $\delta_1(t)$. At first order in $\epsilon$, the position of each shell $x_{1i}^\pm$ at any time $t$ reduces to

$$x_{1i}^\pm = x_1 \pm \epsilon \left( 1 - \delta_1 \frac{\mathcal{D}(t)}{\mathcal{D}(t_i)} - 1 \right) + O(\epsilon^2)$$

Using the mass conservation Eq. (74), the mass contrast for each shell is

$$\Delta_i^\pm = \frac{x_{i1}^\pm}{\chi_1} \left( \frac{\mathcal{D}(t_i)}{\mathcal{D}(t)} \right) + O(\epsilon^2)$$

Using $\delta_1(t) = R_1/3\Delta'(R_1)$ together with $\Delta'(R_1) = \lim_{\epsilon \to 0}(\Delta^\pm)/(\chi_1 - \chi^\pm)$, we get

$$\delta_1(t) = \frac{\mathcal{D}(t_i)}{\mathcal{D}(t)} \left( \frac{\mathcal{D}(t)}{\mathcal{D}(t_i)} - 1 \right) + O(\epsilon^2)$$

With the normalized linear growth factor $\tilde{D}(t) = D(t)/D(t_i)$. This solution corresponds to a one dimensional Zel’dovich dynamics (Zel’dovich 1970). However this solution is exact within the spherical collapse. Note that for an initial negative compensation density ($\delta_1 < 0$ which corresponds to a central over-density, i.e. a maximum), the asymptotic value is $-1$ and $\delta_1(t)$ does not diverges to $\to \infty$ as expected in the linear regime. The linear regime is recovered for $\delta_1[1-\tilde{D}(t)] < 1$ where Eq. (83) reduces to the usual linear relation

$$\delta_1(t) \approx \delta_1 \tilde{D}(t)$$

We stress that Eq. (84) applies only for the very particular radius $r = R_1$ and cannot be extended to every point of
The denominator of Eq. (83) vanishes, leading to $\delta(t_{sc}) \rightarrow \infty$. This divergence is only possible for positive values of $\delta_1$, i.e. for central under-density\(^8\). The divergence of the local density illustrates the apparition of caustics in the density field, due to the possible crossing of different shells at this particular radius where matter accumulates.

High values of $\nu_1$ can lead to a collapse of the surrounding over-dense belt onto the central minimum. This is known as the void-in-cloud problem (Sheth & van de Weygaert 2004). For such voids, the compensation belt is deeply affected by shell crossing and the compensation radius is no longer conserved (it decreases with time). The smallest $\nu_1$ leading to radial shell-crossing today depends on $R_g$, the Gaussian smoothing scale of the power spectrum. In $\Lambda$CDM cosmology, this critical $\nu_1$ (computed from Eq. (85)) evolves from $\nu_1 \geq 0.2$ for $R_g \rightarrow 0$ and crosses $\nu_1 = 1$ for $R_g \approx 4 h^{-1}$Mpc. This illustrates that the shell-crossing mechanism behaves differently according to the smoothing scale. In our case, the power-spectrum is smoothed on the scale equivalent to the size of the coarse grid cell of the simulation to cut the power on smaller scales (in the reference simulation we have $R_{cell} = 1.26 h^{-1}$Mpc). For this smoothing size, the shell-crossing threshold is $\nu_1 \approx 0.65$. As will be shown in Alimi & de Fromont (2017a), this value is much larger than the typical values of $\nu_1$, which are expected to be less than $\nu_1 \approx 0.03$. This insures that spherical shell-crossing is very rarely reached in voids and the void-in-cloud effect can thus be neglected, excepted for some very rare events.

Note that this is not in contradiction with the most common definition criterion for voids, namely that they are enclosed by shell-crossed boundaries (Bertschinger 1985; Sheth & van de Weygaert 2004). Indeed, a was already pointed out

\(^8\) remember that the sign of $\delta_1$ is the opposite of the sign of the central extremum.
in Sheth & van de Weygaert (2004), this shell-crossing does not appear in sufficiently smoothed profiles, which is the case for realistic CoSphere profiles. The clumpy structuration on small scales where shell crossing locally happened to form virialized structures is not relevant for spherical averaged profiles due to the large volume of radial shells. For central maximum, the shell at \( r = R_1 \) acts as a gravitational repeller, avoiding caustic formation.

3.3.3 The local velocity field

Since the Lagrangian displacement \( \mathcal{R} \) obeys a second order differential equation (see Eq. (77)), the field is fully characterized by \( \mathcal{R} \) and its first derivative. In other word, the radial peculiar velocity (linked to the time derivative of \( \mathcal{R} \)) carries a complementary information. We thus defined the velocity contrast \( \Delta_{vel} \) as

\[
\dot{r} = r H(t) [1 + \Delta_{vel}(r, t)]
\]

measuring the radial peculiar velocity in units of the Hubble flow \( r H \). This dimensionless quantity is computed in the Lagrangian formalism as

\[
\Delta_{vel}(r, t) := \frac{\beta \log R - \beta \log a}{\sqrt{\frac{2}{\Omega_m} \frac{\beta \log(R)}{\beta \log a}}}
\]

and satisfies \( \Delta_{vel}(R_1) = 0 \) during the whole evolution. In the Zel’dovich regime, mass and velocity contrast profiles are directly proportional

\[
\Delta_{vel}(r, t) = -\frac{\Delta(r, t)}{3} \times f(t)
\]

where \( f(t) \) is the linear growth rate. We also define the velocity divergence \( \delta_{vel}(r) = \nabla \cdot v/(3H) \) linked to the velocity contrast by

\[
\Delta_{vel}'(r) = \frac{3}{r} \left[ \delta_{vel}(r) - \Delta_{vel}(r) \right]
\]

Using a similar computation than in Section 3.3.2, we can compute the exact non linear evolution of \( \delta_{vel} \) around \( R_1 \)

\[
\delta_{vel}(R_1, t) = -\frac{f(t)}{3} \frac{\delta_1 \dot{D}(t)}{1 - \delta_1 (\dot{D}(t) - 1)}
\]

With the explicit expression for \( \delta_1(t) \) (see Eq. (83)), we get

\[
\frac{\delta_{vel}(R_1, t)}{\delta_1(t)} = \frac{f(t)}{3}
\]

which is the standard relation linking the velocity divergence and the density field in the linear regime. However, in the spherical collapse model, it is an exact result at any redshift for \( r = R_1 \). For other radii, the previous relation Eq. (91) is only valid in linear regime.

Eq. (91) provides an efficient way to evaluate exactly the linear growth rate using CoSpheres. We emphasize that Eq. (91) allows to measure the linear growth rate on non linear scales, it only necessitate to consider structures will small compensation radii. The measure of the linear growth rate, for example from redshift-space distortions, is beyond the scope of this paper and will be investigated in a forthcoming paper (Alimi & de Fromont 2017b).

3.4 Reconstructing profiles at \( z = 0 \)

In Section 2 we have shown that the large scale matter profile of CoSpheres can be precisely reconstructed using GRF (see Eq. 51). Theoretical profiles are parametrized by three independent shape parameters \( v, x, v_1 \) (see Sec. 2.2) in addition to the compensation radius.

In Section 3.2 we have shown that the spherical collapse model provides a good description of the gravitational evolution of these large scale profiles. Combining the initial conditions and the dynamics, we show in this section that CoSpheres can be precisely reconstructed until \( z = 0 \) with a high accuracy on a large radial domain.

At \( z = 0 \) we build the average mass contrast profiles of CoSphere in numerical simulations (see Sec. 1.2.2). For each \( R_1 \) it provides an average profile together with its dispersion (computed as the standard error on the mean). For each profile at \( z = 0 \), the reconstruction procedure consists in finding the appropriate shape parameters \( v, x \) and \( v_1 \) in GRF (see Eq. 51). Practically, we iterate over the shape parameters and minimize a standard \( \chi^2 \) at \( z = 0 \) using the spherical evolution of the GRF expected profile.

In Fig. 15 we show the reconstructed mass contrast profiles at \( z = 0 \) for \( R_1 = 20 \) and \( R_1 = 40 \, h^{-1}\text{Mpc} \). The reconstructed CoSphere reproduces the numerical profile with a very high accuracy (a deviation smaller than 1%) on a large spatial domain. Again we emphasize that the peak parameters \( v \) and \( x \) provide the description of the field around the central extremum whereas \( v_1 \) defined at \( r = R_1 \) drives the shape on larger scales (see Fig. 9). Fig. 15 shows that the reconstruction procedure works for various neighbourhoods.

Although we considered the same haloes (same mass), we probe various neighbourhoods by varying the compensation scale \( R_1 \). A large compensation radius describes a local extremum located in a huge over/under massive region. On the other hand, the same peak with a smaller \( R_1 \) corresponds to a local extremum in an small over-dense "island" isolated in a larger under-dense region.

On Fig. 16 we show the reconstruction of CoSphere profiles defined around haloes with different masses, namely \( M_h \sim 3.6 \times 10^{12} \, h^{-1}M_{\odot} \) (extracted from the simulation with 1024\(^3\) particles and a box size 648 \, h^{-1}\text{Mpc} \) and \( M_h \sim 2.5 \times 10^{14} \, h^{-1}M_{\odot} \) (extracted from the simulation with 2048\(^3\) particles and a box size 5184 \, h^{-1}\text{Mpc} \) for the same compensation radius \( R_1 = 20 \, h^{-1}\text{Mpc} \). Varying the mass of the central halo changes the amplitude of matter fluctuation, and thus the profile itself. Increasing the mass of the central halo raises the primordial peak threshold, i.e. selects peaks with higher \( v \). Since the central extremum is correlated to its surrounding environment, large \( v \) induce higher \( \delta_1 \) and thus \( v_1 \). In other words, a massive halo is more likely to sit in a deepest void than a lighter halo.

Finally, the exact same reconstruction can be done for central under-dense regions identified to cosmic voids. The measured profiles of under-dense CoSphere together with their theoretical reconstruction are shown in Fig. 17. We show the reconstruction for two different compensation radii. Here again, CoSphere profiles are well reconstructed on all scales with a very high accuracy, even in the central under-dense core \( (r < R_1/5) \). This is not surprising since cosmic voids tends to sphericity during the tri-axial expansion, unlike their over dense symmetric (Icke 1984; van de Weygaert...
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Figure 15. Reconstruction of the mass contrast profile $1 + \Delta(r)$ measured in the simulation at $z = 0$ (blue points) for haloes of mass $M_h \sim 3.0 \times 10^{13} h^{-1} M_\odot$. The red line is the CoSphere curve obtained by minimizing a standard $\chi^2$ at $z = 0$ (see text).

Figure 16. Reconstruction of CoSphere profile at $z = 0$ (blue points) from two different haloes with the same compensation radius $R_1 = 20 h^{-1} \text{Mpc}$. The red line is the theoretical curve obtained by computing the best shape parameters $\nu$, $x$ and $\nu_1$ and spherically evolved until $z = 0$ (see text). This figures illustrate the mass dependence of the profiles. Matter profiles around heavier haloes are more amplified than the same profiles build from lighter ones. Increasing the mass of the central haloes raises the primordial height $\nu = \delta(x_0)/\sigma_0$. Shape parameters are correlated to each other such that it is more likely to get higher $\nu_1$ when $\nu$ grows (Alimi & de Fromont 2017a). Heavier haloes will thus induce more amplified profiles on all scales, as it is illustrated in this figure.

4 DISCUSSION AND OUTLOOKS

The absence of a physically motivated model for understanding the large scale matter profiles of compensated cosmic regions is a major difficulty in the precision cosmology era. Extracting reliable cosmological information from such regions, and particularly from voids, requires a deep understanding of their origin and their evolution. In this paper we address this issue by generalizing void profiles and introducing CoSpheres. These regions are build explicitly from their compensation property. The particular radius $R_1$ where the matter field compensate exactly appears to be a funda-
Reconstructing matter profiles of CoSpheres

![Graphs showing reconstructed CoSphere profiles](image)

**Figure 17.** Reconstructed CoSphere profile at \( z = 0 \) around local under-dense minimum. These minima are obtained by smoothing the density field with a Gaussian kernel with \( R_e = 2 \, h^{-1}\text{Mpc} \). The red line is the theoretical curve obtained from GRF with the best fit shape parameters and evolved with a spherical dynamics (see text). It is noticeable that the reconstruction provides an excellent fit on all scales and whatever \( R_1 \) although theoretical profiles are determined by three parameters including two parameters defined around \( r = 0 \).

mental scale for both their origin and their dynamics. This comoving radius isolates closed bubble Universe with a conserved volume during the whole cosmic evolution (see Sec. 3.3.1).

When defined around central under dense minimum, these regions can be identified to cosmic void, providing a useful theoretical framework for studying both their shape and their evolution. Interestingly, these regions can be also defined around local maximum such as DM haloes. By definition, these regions must be compensated on a finite scale, hence the existence of large under dense regions surrounding over densities.

Using numerical simulations introduced in Section 1.1 we build the averaged profiles of CoSpheres by stacking together regions with the same compensation radius \( R_1 \). These numerical simulations can be used to follow backward in time the evolution of such cosmic structures (see Sec. 2.3). From these primordial numerical profiles we have shown that CoSpheres are generated from the stochastic fluctuations of the primordial field (see Section 1.2.3 and Fig. 7).

At high redshift the matter field follows a Gaussian statistics. In order to derive the matter profile of CoSpheres in GRF formalism, we have extended the results of Bardeen et al. (1986) by implementing explicitly the compensation conditions Eq. (5) and Eq. (10) (see Sec. 2). With this original compensation constraint, the spherical density (and mass) contrast profile is now parametrized by four independent - but correlated - shape parameters; \( v \) and \( x \) qualifying the central extrema (already introduced by BBKS) while \( v_1 \) and \( R_1 \) characterize the surrounding cosmic environment on larger scales. While the standard BBKS profile was determined on all scales by providing the peak parameters \( v \) and \( x \), our extension allows to probe the same central extremum in various cosmic environments. These physical configurations can be described by the additional shape parameters \( v_1 \) and \( R_1 \) (see Fig. 10). We emphasize that \( v \) and \( x \) affect the matter profile on small scales while \( v_1 \) controls the shape and the amplitude on larger scale, typically around and beyond the compensation radius.

In Section 3.2 we show that the spherical collapse model is well suited for the dynamical evolution of CoSpheres whereas neither Zel’’dovich nor linear dynamics provide satisfying accuracy. We show that the full non linear gravitational collapse can be solved analytically around \( R_1 \) where it reduces to a one dimensional Zel’’dovich dynamics. We stress that on this particular radius, the Zel’’dovich dynamics provides an exact solution for the spherical collapse and not only a dynamical approximation. In particular, this implies that the linear growth rate can be exactly estimated on this scale (see Eq. 91). This emphasize the relevance of this particular radius. The possibilities to use this property and to constrain the underlying cosmology will be discussed in detail in Alimi & de Fromont (2017b); de Fromont & Alimi (2017).

For central minimum, CoSphere can be identified to cosmic voids. Their radial profiles exhibit a characteristic elbow around \( r \sim 20 \, h^{-1}\text{Mpc} \) (see Fig. 4b and Fig. 18). This elbow is present on both density and mass contrast profile, though it is more pronounced on density profiles (red curve on Fig. 3b). This particular shape property is a characteristic of the definition of our cosmic voids and does not appear clearly in void profiles build from other algorithm (e.g. Hamaus et al. (2014b)).

This elbow is a specific feature of our stacking operation which combine profiles with the same compensation radius \( R_1 \). For other void reconstructions based on their effective radius \( R_{eff} \), this elbow may be smoothed by the stacking together profiles with various \( R_1 \). As will be discussed in Alimi & de Fromont (2017a), this elbow is the imprint of
the progressive decorrelation between the central extrema and the surrounding cosmic environment.

We stress that our work allows a common description for the formation of both cosmic void and large scale profile surrounding haloes. The efficiency of the reconstruction procedure (see Sec. 3.4) emphasizes $R_1$ as a fundamental scale carrying the memory of the primordial Universe and qualifying cosmic structures.

Finally, all the results presented in this paper assume a non biased or distorted CDM field. In realistic surveys however, we don’t have access to the full CDM field but rather to its discrete tracers as galaxies or galaxy cluster. In a N-body simulation, these tracers can be modelled from Dark Matter haloes since galaxies are more likely to form in potential wells generated by DM collapse. As a proof of concept we show in Fig. 19 the reconstructed matter profile obtained from the field traced only by DM haloes. The global shape of profiles is not changed when using the biased field and CoSpheres can still be clearly identified. The agreement between numerical profiles (blue points) and reconstructed theoretical profile (in red) is again excellent on all scales. The only modification with the previous matter profiles reduces, in a first approximation, to the introduction of a linear bias $b$ such as $\delta_{\text{haloes}} = b \times \delta_{\text{DM}}$ without affecting the shape of CoSpheres. For cosmic voids, the linearity of the bias has been studied in Pollina et al. (2017) where it was shown to be a very good approximation, whatever the tracer population (galaxies, galaxy clusters and AGN).

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We assume that the density profile centred on the real extrema \((\delta)(r_i)\) is given by Eq. (36) where \(r_i\) denotes the comoving distance from the real extrema \(x_0\), i.e. \(r_i = |x_0 - x|\). We want to evaluate the density contrast on the shell located at a radius \(r\) where \(r\) is measured from the estimated (but ‘wrong’) center \(x_c\) shifted by \(x_c := x_0 + R\). In other words we note \(r = |x_c - x|\).

Let us define \(\varphi\) such as \(R \cdot r = R \cdot r \cos(\varphi)\). Following Eq. (36), the spherical density contrast can be written as an Hankel Transform

\[
\langle \delta \rangle (r) = \int P(k) \times \tilde{\delta}(\nu, x, v, r, t, k) \frac{\sin(kr)}{kr} dk
\]

(A1)

Where \(\tilde{\delta}\) is a linear function of \(v, x, v, r, t, k\) and depends non linearly in \(k\) and \(R\) (see Eq. 36) while \((P(k))\) is the linear power-spectrum. The “reconstructed” density contrast \(\langle \delta \rangle' (r)\) around the position \(x_c\) at a radius \(r\) is thus given by averaging \(\langle \delta \rangle\) on the shifted sphere of radius \(r\) around \(x_c\)

\[
\langle \delta \rangle'(r) = \frac{1}{2\pi} \int_0^\pi \sin(\varphi) \left( \sqrt{r^2 + R^2 - 2rr \cos(\varphi)} \right) d\varphi
\]

(A2)

Using the explicit expression Eq. (A1) for \(\langle \delta \rangle\) we find

\[
\langle \delta \rangle' (r) = \int P(k) \times \tilde{\delta}(\nu, x, v, r, t, k) \frac{\sin(kr)}{kr} \frac{\sin(kR)}{kR} dk
\]

(A3)

Thus, the shifted profile \(\langle \delta \rangle'(r)\) takes exact same form than the un-shifted profile \(\langle \delta \rangle(r)\) given by Eq. (36) with an effective power-spectrum \(P_{\text{eff}}(k)\) given by

\[
P(k) \rightarrow P_{\text{eff}}(k) := P(k) \times \frac{\sin(kR)}{kR}
\]

(A4)

Of course, for \(R \to 0\) we recover the usual profile but for \(R \neq 0\), the profile implies an effective power spectrum smoothed on the shifting scale \(R\). Note also that missing the right center of mass leads to non-isotropic profiles but here we focus only the the spherically average profile. The mass contrast profile \((\Delta)\) (see Eq. 51) is affected by the exact same factor, i.e. it is written exactly as Eq. (51) but with the effective spectrum given by Eq. (A4).

APPENDIX B: HIGH REDSHIFT SOLUTION AND THE ZEL’DOVICH APPROXIMATION

The dynamical equation Eq. (77) can be solved exactly orders by orders for \(R\) (and for each radius \(r_i\)) with the series

\[
\mathcal{R}(t) = 1 + \sum_{n \geq 1} \eta_n(t) \Delta_n^0
\]

(B1)

Where each function \(\eta_n(t)\) depends only on \(t\) and \(\eta_n(t_i) = 0\). The solution Eq. (B1) is the exact solution for the Lagrangian perturbation theory in spherical coordinates which is valid until shell-crossing.

B1 High redshift solution

In the very high redshift regime \((z \gg 1)\), the initial mass contrast \(\Delta_L\) satisfies \(\Delta_L \ll 1\) for all initial radius \(r_i\) (since \(\Delta_L \sim \sigma_0\)). The 0-th order term of Eq. (B1) corresponds to the linear Eulerian theory \(\delta(x,t) \propto D(t) \delta(x,t_i)\).
Let us now consider the first order term
\[ R(t) \simeq 1 + \eta_1(t) \Delta_i + O(\Delta_i^2) \] (B2)
with \( \eta_1(t_i) = 0 \). In this regime, the right-hand term of Eq. (77) reduce to
\[ R - \frac{1 + \Delta_i}{R^2} \rightarrow \Delta_i \left( 3\eta_1(t) - 1 \right) \] (B3)
If we define \( J \) such as \( \eta_1 = (J - 1)/3 \), using Eq. (B3), it is easy to show that \( J(t) \) satisfies
\[ \frac{d^2 J}{dt^2} + 2H(t) \frac{dJ}{dt} = \frac{3}{2} H^2(t) \Omega_m(t) J(t) \] (B4)
With \( J(t_i) = 1 \), Eq. (B4) is exactly the equation solved by the linear growth factor \( D(t) \), thus, using \( J(t_i) = 1 \) we deduce that in this weak field regime, \( \eta(t) \) is given by
\[ \eta_1(t) = - \frac{1}{3} \left( \frac{D(t)}{D(t_i)} - 1 \right) \] (B5)
In this regime, the displacement field \( R \) is given by
\[ R(t) = 1 - \frac{\Delta_i}{3} \left( \frac{D(t)}{D(t_i)} - 1 \right) \] (B6)
Where the \( \Delta_i \) and thus \( R \) depend on the initial position \( r_i \).

**B2 Link with the Zel’dovich approximation**

The Zel’dovich approximation (denoted as ZA) (Zel’dovich 1970) consists into approximating the field displacement by its initial value. With our notation, it reaches
\[ \chi(q, t) = q + s(q, t) \] (B7)
where \( s(q, t) \) is the displacement field which verifies
\[ \frac{\partial^2 s}{\partial t^2} + 2H \frac{\partial s}{\partial t} = -\nabla \phi \] (B8)
and \( \phi \) is the gravitational potential that satisfies \( \Delta \phi = 4\pi G \delta(r) \). The ZA approximates the displacement field by its initial value \( s(q, t_i) = s_0(q) D(t) \) where \( D(t) \) is the linear growth factor which verifies Eq. (B4) and \( s_0 = -\frac{(2\nabla \phi(q))/(3H^2(t) \Omega_m)}{i} \).

If we define \( \chi_i \) such as \( \chi(q, t_i) = \chi_i \), then \( \chi(q, t) = \chi_i + s(q, t) - s_0(q) \). Using Eq. (71), we can write explicitly \( s_0(q) \) in the spherical approximation. It then follows that
\[ \chi = \chi_i \left( 1 - \frac{\Delta_i}{3} \left( \frac{D(t)}{D(t_i)} - 1 \right) \right) \] (B9)
which is exactly the solution obtained in Eq. (B6). This is not surprising since the ZA is by construction the first order Lagrangian perturbation theory, we re-find it here in spherical geometry.