On the odderon mechanism for transverse single spin asymmetry in the Wandzura-Wilczek approximation

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(Dated: October 20, 2022)

We compute the transverse single spin asymmetry in forward $p^\uparrow p \to hX$ and $p^\uparrow A \to hX$ collisions from the odderon mechanism originally suggested by Kovchegov and Sievert [1]. Working in the hybrid approach of the Color Glass Condensate effective theory we firstly identify the relevant collinear parton distribution function (PDF) of the transversely polarized proton $p^\uparrow$ as the intrinsic twist-3 $g_T(x)$ distribution. We further argue that the complete polarized cross section also contains contributions from the kinematical and the dynamical twist-3 PDFs, in addition to the intrinsic twist-3 PDF. By restricting to the Wandzura-Wilczek approximation, where the dynamical twist-3 PDFs are dropped, we find that the odderon contribution to the polarized cross section for inclusive hadron production is exactly zero at the next-to-leading order in the strong coupling.

I. INTRODUCTION AND MOTIVATION

Transverse single spin asymmetry (SSA) [2–5] is a phenomena associated with azimuthally asymmetric particle production in collisions involving a transversely polarized proton $p^\uparrow$. SSA is characterized by a sine modulation $P_{h\perp} \times S_\perp = P_{h\perp} S_\perp \sin(\phi_h - \phi_S)$. Here $P_{h\perp}$ is the transverse momentum of the produced hadron and $S_\perp$ is the spin of the transversely polarized proton. Decades of dedicated measurements have demonstrated its persistence even at the highest collision energies, with the SSA being largest in the forward region of the produced particle, typically a hadron. This is so across different collision systems such as $ep^\uparrow$ and $p^\uparrow p$ but also most recently for $p^\uparrow A$ collisions [6, 7].

On the theory front, it is known that a presence of the phase in the cross section is crucial to generate SSA. In the forward region, where the momentum fraction $x$ in the target is small, one naturally expects the phenomena of gluon saturation [8–11] to play an important role in determining SSA [1, 12–19]. In this work we are revisiting the computation by Kovchegov and Sievert [1], where they used the Color Glass Condensate (CGC) effective theory [8–11] for gluon saturation, to suggest a new mechanism for SSA. The special property of this mechanism is in supplying the phase by the odderon distribution [20–22]

$$O(x_\perp, y_\perp) \equiv \frac{1}{2N_c} \text{tr} \left\langle V(x_\perp)V^\dagger(y_\perp) - V(y_\perp)V^\dagger(x_\perp) \right\rangle,$$

(1)

that is, the imaginary part of the dipole distribution $\text{tr} \left\langle V(x_\perp)V^\dagger(y_\perp) \right\rangle / N_c$. Here $V(x_\perp)$ is a fundamental Wilson line with $\langle \ldots \rangle$ denoting the color average. This “odderon mechanism”, as we will refer to it in this work, leads to a substantial $A$ suppression of SSA, $\sim A^{-7/6}$ parametrically [1].

In the following Sec. II we take as a starting point the polarized cross section in the hybrid approach [14–17] with a transversely polarized proton described by the collinear twist-3 PDFs and the dense target (a nuclei or a proton in forward collisions) are given in terms of Wilson line correlators. To set up our

II. GENERAL REMARKS

In the hybrid approach a dilute projectile proton is described using collinear PDFs, while the distributions of the dense target (nuclei, or a proton in forward collisions) are given in terms of Wilson line correlators. To set up our
notations we first write down the unpolarized $pA \rightarrow hX$ cross section in terms of the familiar twist-2 PDFs and fragmentation functions (FFs). For convenience, this is given in the following way

$$E_h \frac{d\sigma}{d^3 P_h} = \frac{1}{2(2\pi)^3} \int \frac{dz_h}{z_h^2} D(z_h) \int dx_p \left\{ \frac{1}{2} f(x_p) \text{Tr} \left[ \frac{P_p}{z_h} S^{(0)}(p_1) \right] + \frac{1}{2} G(x_p) (-g_\perp^{\alpha\beta} S^{(0)}_{\alpha\beta}(p_1)) \right\},$$

(2)

where we have separated out the the twist-2 hadron FF $D(z_h)$ and the twist-2 PDF in the quark (gluon) $f(x_p)$ ($G(x_p)$) initiated channel. The proton and the nucleus move along the light-cone with momenta in the center-of-mass frame where we have separated out the twist-2 hadron FF produced hadron $h$. The first part of (3) is arising from the contribution arising from twist-3 FFs [27, 28] has been computed in up to twist-3 in the polarized proton PDF. We will be restricting here to the usual twist-2 FF, $g$ part we have the ETQS distributions $S^{(1)}(x_pP_p, x_p'P_p)$ into the definition of $S^{(0)}(p_1)$.

A. Polarized cross section

In order to generalize to collisions with a transversely polarized proton, the polarized cross section, $d\Delta \sigma$, is computed up to twist-3 in the polarized proton PDF. We will be restricting here to the usual twist-2 FF, $D(z_h)$ – the particular contribution arising from twist-3 FFs [27, 28] has been computed in $p^1 A$ [17]. We are also not considering various pole contributions to $d\Delta \sigma$, see [16]. Our starting point is a separate (non-pole) contribution that has already been discussed in SIDIS [29, 30]. Adapting to the $p^1 A$ computation, we have the following gauge invariant all-order expression

$$E_h \frac{d\Delta \sigma}{d^3 P_h} = \frac{1}{2(2\pi)^3} \int \frac{dz_h}{z_h^2} D(z_h) \left\{ \frac{M_N}{2} \int dx_p g_T(x_p) \text{Tr} \left[ \gamma_5 S^{(0)}_{\perp}(p_1) \right] \right\}$$

$$+ \frac{M_N}{4} \int dx_p g^{(1)}_{1T}(x_p) \text{Tr} \left[ \gamma_5 \gamma_5^{\alpha\beta} S^{(0)}_{\perp}(k_1) \right]$$

$$+ \frac{M_N}{4} \int dx_p dx'_p \text{Tr} \left[ \left( \gamma_5 \gamma_5^{\alpha\beta} G_F(x_p, x'_p) \right) S^{(1)}_{\perp}(x_pP_p, x'_pP_p) \right] \right\},$$

(3)

The first part of (3) is arising from the $g_T(x_p)$ distribution function. This is also referred to as an intrinsic (i.e. $\sim \langle P_p S_{\perp} | \bar{\psi} \gamma_5 \psi | P_p S_{\perp} \rangle$) contribution. The second part is a kinematical ($\langle P_p S_{\perp} | \bar{\psi} \gamma_5 \gamma_5^{\alpha\beta} \gamma_5 \psi | P_p S_{\perp} \rangle$) part we have the ETQS distributions $G_F(x_p, x'_p)$ and $G_F(x_p, x_p)$. Note the appearance of the same two-parton scattering kernel $S^{(0)}(p_1)$ as in (2). To compute the cross section we also need its finite-$k_{1\perp}$ variant $S^{(0)}(k_1)$, as well as $S^{(1)}_{\perp}(x_pP_p, x'_pP_p)$, which is a three-parton scattering kernel containing an additional gluon from the polarized proton.

We should appreciate the appearance of the $k_{1\perp}$-derivative as a consequence of performing the computation up to twist-3 but also due to the connection of $\partial S^{(0)}(k_1)/\partial k_{1\perp}$ with $S^{(1)}_{\perp}(x_pP_p, x'_pP_p)$ through the Ward identity for the gluon from the proton [29, 30].

We point out here that the computation in [1] is on the parton level, taking transversely polarized spinors $u(p, S_{\perp})$ for the initial quark. Thanks to the decomposition $u(p, S_{\perp})u(p, S_{\perp}) = (1 + \gamma_5 S_{\perp})/2 [33]$ only the $\gamma_5 S_{\perp}$ Dirac structure is relevant for the polarized quark in the current context (the remaining $S_{\perp}$-dependent term eventually gets interpreted as the transversity PDF, but this does not contribute in what follows). Thus the computation in [1] clearly corresponds to the term in (3) that is proportional to $g_T(x_p)$, where one naturally replaces the quark mass $m$ by the nucleon mass $M_N$.

The above introduced distributions satisfy the QCD equation of motion identity [31, 32]

$$x g_T(x) = g^{(1)}_{1T}(x) - \frac{1}{2} \int dx' \frac{G_F(x, x') + \tilde{G}_F(x, x')}{x - x'}.$$

(4)

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1 Here $P_h$ is the center-of-mass momentum per nucleon.

2 We are using the convention $\epsilon_{0123} = +1 = -\epsilon_{0123}$ and $\gamma_5 = i\gamma_1\gamma_2\gamma_3$.

3 Instead of $g^{(1)}_{1T}(x)$ sometimes a function $\tilde{g}(x)$ is used [31], with the relation $\tilde{g}(x) = -2g^{(1)}_{1T}(x)$ [32].
The $g_T(x)$ distribution satisfies another important relation connecting it to the twist-2 helicity PDF $\Delta q(x)$ [31, 32] thus revealing that $g_T(x)$ itself has a twist-2 piece

$$g_T(x) = \int_x^1 \frac{dx'}{x'} \Delta q(x') + (\text{genuine twist } - 3).$$  \hspace{1cm} (5)

The remainder is given in terms of the ETQS functions, see [31, 32] for the explicit expression. Our computation will be based on the WW approximation, that is, taking into account only the $g_T(x_p)$ and the $g^{(1)}_T(x_p)$ contributions to $d\Delta \sigma$ from (3). Furthermore, in the WW approximation $g^{(1)}_T(x)$ is fixed through (4) as $g^{(1)}_T(x) \approx x g_T(x)$ and so (3) takes the following compact form

$$E_h \frac{d\Delta \sigma}{d^3 p_h} \simeq \frac{1}{2(2\pi)^3} \frac{M_N}{x_h} \int \frac{dz_h}{z_h} D(z_h) \int dx_p x_p g_T(x_p) \left( S_\lambda^\pi \frac{\partial}{\partial k_{i\perp}^\pi} \text{tr}[\gamma_5 \Gamma_{\text{1}} S_0^{(0)}(k_1)] \right)_{k_1=p_i},$$  \hspace{1cm} (6)

that we will take as the starting point of our explicit computations below.

The analogous expression for the gluon initiated channel is adapted from (17) and (25) in [34] (see also (32) in [35]) to read

$$E_h \frac{d\Delta \sigma}{d^3 p_h} = \frac{1}{2(2\pi)^3} \frac{M_N}{x_h} \int \frac{dz_h}{z_h} D(z_h) \left[ iM_N \int dx_p g_{3T}(x_p) \frac{1}{p_1} \epsilon^{\alpha \beta S_\omega} S^{(0)a'\beta'}(p_1) \omega_{\alpha'} \omega_{\beta'} \right]$$

$$- iM_N \int dx_p \frac{1}{x_p} \tilde{g}(x_p) \left( \frac{\partial S^{(0)}(k_1)}{\partial k_1^\perp} \right)_{k_1=p_i}$$

$$- \frac{1}{2} \int \frac{dx_p dx_p'}{x_p x_p'} \frac{M_F^{a'\beta'}(x_p, x_p') S^{(1)a'\beta'}(x_p P_p, x_p' P_p)}{x_p' - x_p} \omega_{\alpha'} \omega_{\beta'} \omega_{\gamma} \gamma,$$  \hspace{1cm} (7)

where $\omega_{\alpha} = g_{\alpha \beta} - \bar{n}_\alpha n_\beta$. In the first line we have the intrinsic contribution with $G_{3T}(x_p)$ being the gluonic counterpart of $g_T(x_p)$. In the second line $\tilde{g}(x_p)$ is the gluonic kinematical function, see [36, 37] for the definition, and $M_F^{a'\beta'}(x_p, x_p')$ is the three-gluon correlator. In the WW approximation $G_{3T}(x)$ becomes related to the gluon helicity PDF $\Delta G(x)$ as [36]

$$G_{3T}(x) \simeq \frac{1}{2} \int_x^1 \frac{dx'}{x'} \Delta G(x'),$$  \hspace{1cm} (8)

while $\tilde{g}(x) \simeq x^2 G_{3T}(x)$ [36]. The WW truncation then amounts to the first two lines of (7).

B. A recap of the leading order inclusive hadron production

The leading order (LO) amplitude for inclusive hadron production from the $q(k_1) \to q(l)$ channel in the $n \cdot A = A^+ = 0$ gauge is simply given as [38]

$$M = \gamma^+ \int_{x_\perp} e^{i(q_\perp - k_1\perp) \cdot x_\perp} [V(x_\perp) - 1].$$  \hspace{1cm} (9)

Here $V(x_\perp) = \mathcal{P} \exp \left[ ig \int_{-\infty}^\infty dx^+ A_\perp^+(x^+ t^a) \right]$ is the fundamental Wilson line with $A_\perp^+(x)$ being the classical field of the target and we use $\int_{x_\perp} \equiv \int d^2 x_\perp$. In (9) we have omitted the overall light-cone delta function, $(2\pi)\delta (k_1^+ - q^+)$, as well as the initial and final state spinors, thus leaving a matrix in spinor (and color) space. From $M$ we obtain the leading order result for $S^{(0)}(k_1)$ as

$$S^{(0)}(k_1) \equiv \frac{1}{2P_p^+} \frac{1}{N_c} \langle \mathcal{M} \Gamma M \rangle (2\pi)\delta (k_1^+ - q^+)$$

$$= (2\pi)\delta (k_1^+ - q^+) \frac{x_p}{2k_1^+} \gamma^+ d\gamma^+ \int_{x_\perp x_\perp'} S(x_\perp, x_\perp') e^{i(q_\perp - k_1\perp) \cdot (x_\perp - x_\perp')},$$  \hspace{1cm} (10)
where $1/2P_p^+$ is the flux factor, $1/N_c$ is coming from averaging over the color of the initial state quark and

$$S(x_\perp, x'_\perp) \equiv \frac{1}{N_c} \text{tr} \left\langle V(x_\perp)V^\dagger(x'_\perp) \right\rangle,$$

(11)
is the color averaged dipole distribution. Here and in the following we are suppressing the dependence of the nuclear distributions on the momentum fraction $x_A = k^+_2/P^+_A$, where $k_2$ is the partonic momenta from the nuclei. At the LO we have the momentum conservation $k_1 + k_2 = q$. We readily conclude that $\text{tr} \left[ \gamma_5 \not{k}_1 S^{(0)}(k_1) \right] \sim \text{tr} \left[ \gamma_5 \not{k}_1 \gamma^+ g^+ \right] \sim \epsilon^+ qk_1 = 0$ and so (6) vanishes at the LO.

The analogous expressions in the $g(k_1) \to g(k_g)$ channel are [39]

$$\mathcal{M} = (-2k_1^+) \int_{x_\perp} \epsilon^{i(k_g, -k_1)} x_\perp [U(x_\perp) - 1],$$

(12)

and

$$S^{(0)}_{\alpha\beta}(k_1) = \frac{1}{2P_p^+} \frac{1}{N_c^2 - 1} (M^\dagger M) d^{\alpha\beta}(k_g)
= (2\pi) \delta(k_1^+ - k_g^+) (2k_1^+)^2 d^{\alpha\beta}(k_g) \int_{x_\perp} S_A(x_\perp, x'_\perp) \epsilon^{i(k_g, -k_1)} (x_\perp - x'_\perp),$$

(13)

where

$$d^{\alpha\beta}(k) = -g^{\alpha\beta} + \frac{n^\alpha k^\beta + n^\beta k^\alpha}{k^+},$$

(14)
is the gluon polarization tensor. Note that $d^{\alpha\beta}(p_1) = -g^{\alpha\beta}_\perp$. $U(x_\perp)$ is the adjoint Wilson line and

$$S_A(x_\perp, x'_\perp) \equiv \frac{1}{N_c^2 - 1} \text{tr} \left\langle U(x_\perp)U^\dagger(x'_\perp) \right\rangle,$$

(15)
is the adjoint dipole distribution. The contribution from this channel also vanishes at the LO simply due to the realness of the adjoint Wilson line.

### III. NLO INCLUSIVE HADRON PRODUCTION IN $p^+A \to hX$: THE $q \to qg$ CHANNEL

At the NLO we have in general the $q \to qg$, $g \to qg$ and $g \to gg$ channels. In this Section we compute the $q \to qg$ channel (together with the accompanying virtual contribution), while the $g \to qg$ and the $g \to gg$ channels are discussed separately in Secs. IV and V, respectively.

#### A. $q \to q$: real contribution

We consider the $q(k_1) \to q(q)g(k_g)$ partonic channel where in the final state a real gluon with momentum $k_g$ gets radiated in addition to the quark with momentum $q$. Here $k_2$ is set by momentum conservation at NLO: $k_1 + k_2 = q + k_g$. We will focus on the case where the quark fragments into a final state hadron and we integrate over the (untagged) gluon phase space to compute the inclusive hadron cross section according to (6). The main quantity to compute is $S^{(0)}(k_1)$ which takes the following form

$$S^{(0)}(k_1) = \frac{1}{2P_p^+} \int \frac{d^3k_g}{(2\pi)^32E_g N_c} \left\langle \mathcal{M}^{\mu}\not{q}\mathcal{M}^{\nu} \right\rangle d_{\mu\nu}(k_g)(2\pi)^3\delta(k_1^+ - q^+ - k_g^+),$$

(16)
where \( \int_{k_{\perp}} \equiv \int \frac{d^2k_{\perp}}{(2\pi)^2} \) and similar for other transverse momentum integrations. Using the quark and gluon propagators in the CGC background \([40-42]\) we can compute the following amplitudes

\[
\mathcal{M}_1^\mu = -ig \gamma^\mu \frac{q + \bar{k}_g}{(q + k_g)^2 + i\epsilon} \gamma^+ \int_{x_{\perp}} e^{ik_{2\perp} \cdot x_{\perp}} [V(x_{\perp}) - 1],
\]

\[
\mathcal{M}_2^\mu = -ig \gamma^+ \frac{\bar{k}_1 - \bar{k}_g}{(k_1 - k_g)^2 + i\epsilon} \gamma^\mu \int_{x_{\perp}} e^{ik_{2\perp} \cdot x_{\perp}} [V(x_{\perp}) - 1] t^a,
\]

\[
\mathcal{M}_3^\mu = ig(2k_g^+ \gamma^\nu \frac{d^\mu\nu(k_1 - q)}{(k_1 - q)^2 + i\epsilon} \epsilon^\mu \epsilon^\nu (k_1 + k - q) + i\epsilon) \int_{x_{\perp}} e^{ik_{2\perp} \cdot x_{\perp}} [U^a b(x_{\perp}) - \delta^{ab}],
\]

\[
\mathcal{M}_4^\mu = -g(2k_g^+) \int_{k_1} \int_{-\infty}^{\infty} \frac{dk^-}{2\pi} \gamma^+ \frac{q - \bar{k}}{(q - \bar{k})^2 + i\epsilon} \gamma^\nu \frac{d^\nu\mu(k_1 + k - q)}{(k_1 + k - q)^2 + i\epsilon} \int_{x_{\perp}} e^{ik_{2\perp} \cdot x_{\perp}} e^{i(k_{2\perp} - k_{\perp} \cdot y_{\perp})} [V(x_{\perp}) - 1] t^b [U^a b(y_{\perp}) - \delta^{ab}],
\]

(17)

with the total amplitude \( \mathcal{M}^\mu = \sum_{k=1}^{4} \mathcal{M}_k^\mu \). We find that the result (17) agrees with \([43]\) except for the overall sign and the adjoint indices in the eikonal gluon vertex that enters \( \mathcal{M}_3^\mu \) and \( \mathcal{M}_4^\mu \). In the special case when the final quark and gluon are on-shell\(^4\), \( \mathcal{M}^\mu \) takes the following simple form

\[
\mathcal{M}^\mu = -ig \int_{k_{\perp}} \int_{x_{\perp} y_{\perp}} e^{ik_{2\perp} \cdot x_{\perp}} e^{i(k_{2\perp} - k_{\perp} \cdot y_{\perp})} \left[ T^\mu_q t^a V(x_{\perp}) + T^\mu_{qq}(k_{\perp}) V(x_{\perp}) t^b U^a b(y_{\perp}) \right],
\]

(18)

where

\[
T^\mu_q = \gamma^\mu \frac{q + \bar{k}_g}{(q + k_g)^2} \gamma^+,
\]

(19)

and

\[
T^\mu_{qq}(k_{\perp}) = -\gamma^+ \frac{q - \bar{k}}{(q - \bar{k})^2} \gamma^\mu d^\nu\mu(k_1 + k - q).
\]

(20)

Here, we evaluated the \( k^- \) integral in favor of \((k_1 + k - q)^2 + i\epsilon = 0\) so that

\[
(q - k)^2 = -\frac{1}{k_1^+ k_g^+} [q^+ k_1^+ + k_1^+ (k_{\perp} - q_{\perp})]^2.
\]

(21)

Inserting (17) into (16) we find

\[
S^{(0)}(k_{\perp}) = \frac{q^+ g^2 C_F}{P^+ 4q^+ k_g^+} \int_{k_{\perp}} \int_{x_{\perp} y_{\perp}} e^{ik_{2\perp} \cdot x_{\perp}} e^{i(k_{2\perp} - k_{\perp} \cdot y_{\perp})} s_{\perp}(x_{\perp}, x_{\perp}) S_{\perp}(y_{\perp}) U^a b(y_{\perp})
\]

\[
\times \left[ S(x_{\perp}, x'_{\perp}) T^\mu_q \eta T^\mu_q S_{\perp}(x'_{\perp}, x_{\perp}, y_{\perp}) \eta T^\mu_{qq}(k_{\perp}) + S_{\perp}(x'_{\perp}, x_{\perp}, y'_{\perp}) T^\mu_{qq}(k_{\perp}) \eta T^\mu_q + S_{\perp}(x'_{\perp}, x_{\perp}, y'_{\perp}) T^\mu_{qq}(k_{\perp}) \eta T^\mu_{qq}(k_{\perp}) \right],
\]

(22)

where \( S(x_{\perp}, x'_{\perp}) \) is the dipole defined in (11), and the additional distributions are given as

\[
S_{\perp}(x_{\perp}, x_{\perp}, y_{\perp}) = \frac{1}{C_F N_c} \left< \text{tr} \left( V(x_{\perp}) t^a V(x_{\perp}) \right) U^a b(y_{\perp}) \right>,
\]

\[
S_{\perp}(x'_{\perp}, x_{\perp}, y'_{\perp}, x_{\perp}, y_{\perp}) = \frac{1}{C_F N_c} \left< \text{tr} \left( V(x_{\perp}) V(x_{\perp}) t^a t^b \right) U^a b(y_{\perp}) U^a b(y_{\perp}) \right>.
\]

(23)

We now show that the first and the fourth term in the square brackets in (22) do not contribute to the polarized cross section when integrated over the gluon momenta. This is intuitively clear as the SSA must come from interferences

\(^4\) Without loss of generality, the initial quark can be taken as on-shell even at finite \( k_{1\perp} \), that is, \( k_1^2 = 0 \). Eq. (6) is not affected due to the \( \partial/\partial k_{1\perp}^2 \) derivative.
of different amplitudes, that are given by the second and the third term (c.f., Fig. 1, while the first and the fourth term are squares of amplitudes. The analogous structure can also be identified in the computation of [1]. Consider the first term in (22), where in the context of (6) we have

$$\text{tr} \left[ \gamma_5 k_1 T^\mu_\perp q^T_\perp \right].$$

(24)

We now apply C-parity transformation where $C = i \gamma^0 \gamma^2$ and easily deduce that

$$d_{\mu\nu} (k_g) \text{tr} \left[ \gamma_5 k_1 T^\mu_\perp q^T_\perp \right] = d_{\mu\nu} (k_g) \text{tr} \left[ \gamma_5 k_1 T^\mu_\perp q^T_\perp \right]^T = d_{\mu\nu} (k_g) \text{tr} \left[ C(\gamma_5 k_1 T^\mu_\perp q^T_\perp)^T C^{-1} \right]$$

(25)

since $d_{\mu\nu} (k_g)$ is symmetric. Therefore, the first term vanishes under the trace by C-parity. As for the fourth term we first note that it is in fact independent of $k_{g\perp}$. This seems almost trivial as $k_{g\perp}$ never enters (20). However, there is a potential $k_{g\perp}$ dependence in $d_{\mu\nu} (k_g)$ given through $d^{-+} (k_g) = 2k_g^- / k_g^+ = k_{g\perp}^2 / k_g^+$. These two terms must couple to $T^+_{qg}(k^+_\perp) q^T_{qg}(k^\perp_\perp)$ and $T^0_{qg}(k^0_\perp) q^T_{qg}(k^\perp_\perp)$, respectively. But, $T^+_{qg}(k^\perp_\perp) \sim d^{-+} (k_1 + k - q) = 0$ so there is no $k_{g\perp}$ dependence in the fourth term after all. In order to be able to integrate the fourth term with respect to $k_{g\perp}$ we consider the following observation. Defining first

$$z \equiv \frac{k_g^+}{q^+ + k_g^+}, \quad \xi \equiv 1 - z,$$

(26)

as the momentum fraction of the recoiling gluon, the parton momentum fraction in the projectile is $x_p = q^+ / (\xi P_p^+)$.

For the target we have

$$x_A = \frac{q^-}{P_A} \left( 1 + \frac{k^-_d}{q^-} - \frac{k^-_1}{q^-} \right) = \frac{q^-}{P_A} \left[ 1 + \frac{z (k_{1\perp} + k_{2\perp} - q_\perp)^2}{q_\perp^2} - \frac{z}{q_\perp^2} k_{1\perp}^2 \right].$$

(27)

Typically in CGC computations, one ignores the dependence of $x_A$ on $k_{1\perp}$ and $k_{2\perp}$, see e.g. [44], and uses an approximate formula $x_A \approx q^- / (\xi P_A^-)$. The argument is that large values of $k_{1\perp}$ should be exponentially suppressed in the cross section due to the nature of the CGC distributions, and the derivative with respect to $k_{1\perp}$, see (6), should be $\alpha_s$ suppressed via small-$x$ evolutions [45–48]. The former is implicit in [1] where the computation is based on the initial condition model for the target gluon distributions. With this approximation, the only $k_{g\perp}$ dependence is in the phases and this leads to

$$\int_{k_{g\perp}} e^{i k_{g\perp} (y_\perp - y'_\perp)} = \delta (y_\perp - y'_\perp),$$

(28)

in which case for $y'_\perp \to y_\perp$ we have $S_{ggg}(x'_\perp, y'_\perp, x_\perp, y_\perp) \to S(x_\perp, x'_\perp)$ which is independent of $y_\perp$. This allows us to perform the $y_\perp$ integration, yielding

$$\int_{y_\perp} e^{i(-k_\perp + k'_\perp) y_\perp} = (2\pi)^2 \delta (k_\perp - k'_\perp).$$

(29)
The Dirac trace from the fourth term becomes 
\[ \text{tr} \left[ \gamma_\parallel \tilde{T}_q \mu \left( k_\perp \right) g \tilde{T}_{qg} \mu \left( k_\perp \right) \right] \]
which vanishes by C-parity. Therefore, only the second and the third terms (corresponding to the interference diagram from Fig. 1) in (22) are left and we have
\[
\text{tr} \left[ \gamma_\parallel \tilde{T}_q \mu \left( k_\perp \right) \right] = \frac{q^+}{P_p} g^2 C_F \int \frac{d^2k_\perp}{2} \int \frac{d^2k_\perp'}{2} \int \frac{d^2k_\perp''}{2} C \left[ e^{ik_\perp \cdot x - i k_\perp' \cdot x'} e^{-ik_\perp'' \cdot x} \right]
\]
\[
\left[ - S_{qqg}(x'_1, x_1, y'_1) \mathcal{H}(k'_1, k_1) + S_{qqg}(x'_1, x_1, y_1') \mathcal{H}(k'_1, k_1) \right],
\]
where
\[
\mathcal{H}(k'_1, k_1) \equiv \frac{1}{4g^2 + k_9^2} d_{\mu'}(k_9) \text{Tr} \left[ \gamma_\parallel \tilde{T}_q \mu \left( k_\perp \right) \right]
\]
In (30) we have used \( \mathcal{H}(k'_1, k_1) = - \mathcal{H}(k_1, k'_1) \) which is due to the appearance of \( \gamma_\parallel \). With the help of the \( SU(N_c) \) identity \( U^{ab}(x_\perp) = 2 \text{tr} \left( t^a V(x_\perp) t^b V^\dagger(x_\perp) \right) \) and taking the large \( N_c \) limit we have
\[
S_{qqg}(x'_1, x_1, y'_1) \simeq \frac{1}{2C_F N_c} \left( N_c^2 S(y'_1, x'_1) S(x_1, y'_1) - S(x_1, x'_1) \right)
\]
The second (dipole) term in (32) drops out when combined with the hard factors in (30). To see this, note that the \( y_1 \) and the \( y'_1 \) integrations in (30) result in \( \delta \)-functions that yield \( k'_1 = k_1 = k_2 \). This gives \( - \mathcal{H}(k_2, k_1) + \mathcal{H}(k_2, k_1) = 0 \).

Now we split the dipole into its real and imaginary parts
\[
S(x_1, y_1) = T(x_1, y_1) + i \mathcal{O}(x_1, y_1),
\]
where
\[
T(x_1, y_1) = \frac{1}{2} \left( S(x_1, y_1) + S(y_1, x_1) \right)
\]
\[
\mathcal{O}(x_1, y_1) = \frac{1}{2i} \left( S(x_1, y_1) - S(y_1, x_1) \right),
\]
are the pomeron and the odderon distributions, respectively. We also replace the primed and unprimed transverse coordinate and momenta labels in the first term in (30), namely: \( k'_1 \rightarrow k_1, x'_1 \rightarrow x_1, y'_1 \rightarrow y_1 \). By compensating for the reversed sign in the exponentials with \( x_1 \rightarrow -x_1 \), and using overall invariance under reflections for the distributions in the unpolarized target, we obtain
\[
\text{tr} \left[ \gamma_\parallel \tilde{T}_q \mu \left( k_\perp \right) \right] = i g^2 N_c \left( \frac{q^+}{P_p} \int \frac{d^2k_\perp}{2} \int \frac{d^2k_\perp'}{2} \int \frac{d^2k_\perp''}{2} C \left[ e^{ik_\perp \cdot x - i k_\perp' \cdot x'} e^{-ik_\perp'' \cdot x} \right]
\]
\[
\times \left[ T(x_1, y_1) \mathcal{O}(x'_1, y'_1) - \mathcal{O}(x_1, y_1) T(x'_1, y'_1) \right] \mathcal{H}(k'_1, k_1).
\]
Here we have used the following symmetry properties \( T(y_1, x_1) = T(x_1, y_1) \) and \( \mathcal{O}(y_1, x_1) = -\mathcal{O}(x_1, y_1) \) which follow from (34). We have also passed from \( k_g \) to \( k_\perp \) integration. This result clearly demonstrates that the polarized cross section is proportional to the odderon operator.

The Dirac trace is easy to calculate and we find
\[
\mathcal{H}(k_1, k_1) = 4i(\bar{z} + 1) \frac{v_{11} \times v_{21}}{v_{11} \cdot v_{21}},
\]
where
\[
v_{11} \times v_{21} \equiv e^{-\epsilon_1} v_{11} = v_{11} \sin(\phi_1 - \phi_2)
\]
\[
v_{11} \equiv \bar{z} q_1 - \bar{z} k_1 = q_1 - \bar{z} k_1 - \bar{z} k_{21},
\]
\[
v_{21} \equiv q_1 - \bar{z} k_1 - k_1.
\]
Eqs. (35) and (36) represent the main results of this section. The vectors \( v_{11}, v_{21} \) reflect the collinear gluon radiations so that when \( v_{11} \rightarrow 0 (v_{21} \rightarrow 0) \) the radiated gluon would be collinear to the final (initial) state quark. Note, however, that in (36) both of these limits are completely finite, meaning that the usual collinear divergences one encounters in the NLO computations of an unpolarized cross section for inclusive hadron production, see for example [44], are absent in this particular computation of the polarized cross section. In addition, when \( z \rightarrow 0 (\bar{z} \rightarrow 1) \), i.e., when the
radiated gluon is collinear to the nucleus \((k^+_g \to 0\) and so \(k^- \to \infty\) effectively), the hard factor is also finite. In fact, a close inspection reveals that the resulting cross section is zero in this limit. Namely, when \(z \to 1\) there is a symmetry in the hard factor such that by interchanging \(k_\perp \leftrightarrow k_{2\perp}\) so that \(v_{1\perp} \leftrightarrow v_{2\perp}\) the hard factor picks up a sign \(\mathcal{H} \to -\mathcal{H}\). On the other hand the soft part in (35) is even under such a transformation and so the overall cross section is zero in this limit. In the case of the NLO unpolarized cross section the \(z \to 0\) divergence recovers a part of the small-\(x\) evolution of the nuclear wavefunction [44].

We reflect here also on the computation in [1] that takes into account only the \(g_T(x_p)\) contribution (on the parton level) in (6). The resulting hard factor associated with \(g_T(x)\) is found to be

\[
\mathcal{H}^{(g_T)}(k_\perp) = \frac{1}{4q^+k^+_g}d_{\mu\nu}(k_g) \text{tr} \left[ \gamma_5 \not{k}_\perp T^\mu_\perp g T^\nu_g(k_\perp) \right]_{k_1\perp=0} = 4i\varepsilon \frac{\hat{v}_{1\perp} \times S_\perp}{\hat{v}_{1\perp} \cdot \hat{v}_{2\perp}},
\]

with \(\hat{v}_{1\perp}\) and \(\hat{v}_{2\perp}\) obtained from \(v_{1\perp}\) and \(v_{2\perp}\) by setting \(k_{1\perp} = 0\), see (37). It is important to observe that, while the final state collinear divergence \((\hat{v}_{1\perp} \to 0)\) is absent, the hard factor has a divergence when the radiated gluon is collinear to the initial state proton \((v_{2\perp} \to 0)\). This divergence is also present in [1], as can be seen from their Eq. (15) by setting the quark mass \(m \to 0\) [5]. In hindsight, this means that the result in [1] must be incomplete in the sense that the lowest order computation should be free from any divergences. That is, by taking into account also the \(g_{1T}^{(1)}(x)\) part of the full cross section (3), as per the WW approximation (6), we indeed find that the initial state collinear divergence is cancelled between the \(g_T(x)\) and the \(g_{1T}^{(1)}(x)\) parts, resulting in a finite hard factor (36). A similar conclusion was also reached in a collinear framework in SIDIS, see [30] and [35] where the \(g_T(x)\) contribution to the cross section contained an initial state collinear divergence, that gets exactly cancelled with the collinear divergence in the \(g_{1T}^{(1)}(x)\) part.

B. Proof that the real contribution in the \(q \to q\) channel vanishes

We now argue that in fact (35) is exactly zero. Before performing an explicit computation we can appreciate it in an intuitive way as follows. In general for the polarized cross section to be non-zero we need two vectors: the transverse momentum of the final state and the spin so that we can form the familiar cross product \(q_\perp \times S_\perp\). In case of (35), we have \(q_\perp\), while we can think of \(k_{2\perp}\) as a proxy for the spin, thanks to the derivative \(S_\perp \partial / \partial k_\perp\). However, owing to the particular form of the hard factor (36), the two vectors \(q_\perp\) and \(k_{1\perp}\) enter the cross section only through the linear combination \(q_{1\perp} \equiv q_\perp - \hat{z}k_{1\perp}\) (the soft part of the cross section is independent of \(k_{1\perp}\)). Thus the final result depends only on a single vector, \(q_{1\perp}\), and therefore must be zero.

To see the above statement explicitly we start by switching to the coordinates

\[
\begin{align*}
 r_\perp &= x_\perp - y_\perp, & b_\perp &= \frac{x_\perp + y_\perp}{2}, \\
 r'_\perp &= y_\perp - x'_\perp, & b'_\perp &= \frac{y_\perp + x'_\perp}{2},
\end{align*}
\]

to obtain

\[
\text{tr} \left[ \gamma_5 \not{k}_1 S_\perp^{(0)}(k_1) \right] = ig^2 N_c q^+ P_T \int_{k_{2\perp}, k_1} \int_{r_\perp, b_\perp} e^{ik_\perp \cdot r_\perp} e^{ik_{2\perp} \cdot r'_\perp} \times \left[ \mathcal{P}(r_\perp, b_\perp) \mathcal{O}(r'_\perp, b'_\perp) - \mathcal{O}(r_\perp, b_\perp) \mathcal{P}(r'_\perp, b'_\perp) \right] \mathcal{H}(k_\perp, k_{1\perp}).
\]

Note that not all transverse coordinates in (40) are independent: we have the following relation for \(b'_\perp\)

\[
b'_\perp = b_\perp - \frac{1}{2}(r_\perp + r'_\perp).
\]

This is an important point because \(\mathcal{O}(r_\perp, -b_\perp) = -\mathcal{O}(r_\perp, b_\perp)\) and so an integral over \(b_\perp\) would superficially vanish simply via \(b_\perp \to -b_\perp\). Next, in order to de-convolve the transverse integrals in (40) we Fourier transform the distributions as

\[
\mathcal{P}(r_\perp, b_\perp) = \int_{\kappa, \Delta} e^{-i\kappa_\perp \cdot r_\perp} e^{-i\Delta_\perp \cdot b_\perp} \mathcal{P}(\kappa, \Delta),
\]

5 One should be careful here in first factoring out one power of \(m\) in (15) in [1], as per the definition of \(g_T(x)\).
and similarly for $\mathcal{O}(r_\perp, b_\perp)$. In terms of the Fourier-transformed distributions, (40) becomes

$$\text{tr}[\gamma_5 F_1 S^{(0)}(k_1)] = ig^2 N_c q^+ \int_{\kappa_\perp \Delta_\perp} \left[ \mathcal{P}(k_\perp, \Delta_\perp) \mathcal{O}^+(k'_\perp, \Delta'_\perp) - \mathcal{O}(k'_\perp, \Delta'_\perp) \mathcal{P}(k_\perp, \Delta_\perp) \right] \mathcal{H}(k_\perp, k_\perp),$$

(43)

where $k_\perp = k_\perp + \frac{1}{2} \Delta_\perp$, $k_{2\perp} = k'_{\perp} + \frac{1}{2} \Delta'_{\perp}$ and $\Delta'_\perp = -\Delta_\perp$.

The key quantity to consider in (43) is the integral over the angular variables. While the pomeron carries no angular dependence, the odderon has the following modulation $\mathcal{O}(k_\perp, \Delta_\perp) \propto (k_\perp \cdot \Delta_\perp)$ (this is simply the momentum space counterpart of the more familiar $\mathcal{O}(r_\perp, b_\perp) \sim (r_\perp \cdot b_\perp)$ modulation), see e. g. [49–51]. Focusing on the first part in (43) we start from the following expression,

$$\int_0^{2\pi} \frac{d\phi_\perp}{2\pi} \int_0^{2\pi} \frac{d\phi_\perp'}{2\pi} \int_0^{2\pi} \frac{d\phi_\perp''}{2\pi} (\phi_\perp, \phi_\perp', \phi_\perp'') \frac{1}{v_{1\perp} v_{2\perp}} (v_{1\perp} \times v_{2\perp}).$$

(44)

Introducing $\delta_{1\perp} = q_{1\perp} - \Delta_\perp/2$, $\delta_{2\perp} = q_{1\perp} - \Delta_\perp/2$, we have

$$v_{1\perp} \times v_{2\perp} = \delta_{1\perp} \times \delta_{2\perp} - \delta_{1\perp} \times \delta_{2\perp} - \delta_{1\perp} \times \delta_{2\perp} + \delta_{1\perp} \times \delta_{2\perp},$$

(45)

and

$$v_{1\perp} \times v_{2\perp} = \delta_{1\perp} \times \delta_{2\perp} - \delta_{1\perp} \times \delta_{2\perp} - \delta_{1\perp} \times \delta_{2\perp} + \delta_{1\perp} \times \delta_{2\perp}.$$

(46)

Now we compute the integrals over $\phi_\perp$ and $\phi_\perp'$. From the first term in (45) we obtain

$$\int_0^{2\pi} \frac{d\phi_\perp}{2\pi} \int_0^{2\pi} \frac{d\phi_\perp'}{2\pi} (\delta_{1\perp} \times \delta_{2\perp}) = - \frac{1}{4} \frac{z}{\delta_{1\perp} \times \delta_{2\perp}} = -\frac{1}{4} \frac{z}{\delta_{1\perp} \times \delta_{2\perp}} (q_{1\perp} \times \Delta_\perp) \frac{1}{|\delta_{1\perp} \times \delta_{2\perp}|}.$$

(47)

where we used $\delta_{1\perp} \times \delta_{2\perp} = -z(q_{1\perp} \times \Delta_\perp)/2$. Inserting now the definition of $\delta_{1\perp}$ we will in general have an expression of the type

$$\int_0^{2\pi} \frac{d\phi_\perp}{2\pi} \sin(\phi_{q_1} - \phi_\Delta) f(\cos(\phi_{q_1} - \phi_\Delta)) = \int_{\phi_{q_1}}^{\phi_{q_1} - 2\pi} \frac{d\phi}{2\pi} \sin f(\cos \phi),$$

(48)

but this is simply zero as

$$\int_{\phi_{q_1}}^{\phi_{q_1} - 2\pi} \frac{d\phi}{2\pi} \sin f(\cos \phi) = -\int_{\phi_{q_1}}^{\phi_{q_1} - 2\pi} d(\cos \phi) f(\cos \phi) = F(\cos \phi)|_{\phi_{q_1}}^{\phi_{q_1} - 2\pi} = 0,$$

(49)

where $F(\cos \phi)$ is a primitive function of $f(\cos \phi)$. By a completely analogous computation we can show that each of the remaining three pieces in (45) is also zero and thus conclude that the complete real contribution in the $q \rightarrow q$ channel vanishes.

C. $q \rightarrow q$: virtual contribution

For the virtual correction to the $q \rightarrow q$ channel we have the following amplitude

$$\mathcal{M} = g^2 \int \frac{dk_2^+}{(2\pi)} \int_{k_{1\perp} k_{2\perp}} e^{ik_{1\perp} x_{1\perp} + \frac{k_{1\perp}}{2} x_{1\perp} k_{1\perp} - k_{1\perp}} \mathcal{O}^{ab}(x_\perp) \mathcal{T}_{q_{1},b} + V(x_\perp) t^a t^b \mathcal{T}_{q_{1},2} + t^a V(x_\perp) t^b U^{ab}(y_\perp) \mathcal{T}_{q_{q}},$$

(50)

where

$$\mathcal{T}_{q_{1},1} = i \int \frac{dk_{1\perp}}{(2\pi)} \gamma^\mu \left( q - k_{1\perp} g_2 \right)^2 + \bar{q}^\nu \left( q - k_{2\perp} g_1 \right)^2 + \bar{q}^\mu \gamma^\nu d_{\mu\nu}(k_{1\perp}),$$

$$\mathcal{T}_{q_{1},2} = i \int \frac{dk_{2\perp}}{(2\pi)} \gamma^\mu \left( k_{1\perp} - k_{2\perp} g_1 \right)^2 + \bar{q}^\nu \left( k_{2\perp} - k_{1\perp} g_2 \right)^2 + \bar{q}^\mu \gamma^\nu d_{\mu\nu}(k_{2\perp}),$$

$$\mathcal{T}_{q_{q}} = i \int \frac{dk_{2\perp}}{(2\pi)} \gamma^\mu \left( q - k_{2\perp} g_1 \right)^2 + \bar{q}^\nu \left( q - k_{1\perp} g_2 \right)^2 + \bar{q}^\mu \gamma^\nu d_{\mu\nu}(k_{1\perp} + k_{2\perp} - q) + \bar{q}^\mu \gamma^\nu d_{\mu\nu}(k_{1\perp} + k_{2\perp} - q) \frac{1}{k_{1\perp} + k_{2\perp} + \bar{q}^\mu \gamma^\nu d_{\mu\nu}(k_{1\perp} + k_{2\perp} - q)}.$$
Above, in the first line of $\mathcal{T}_{qg}$ we have evaluated the $k^-$ integration in favor of the singularity at $(k_1 + k + q - q)^2 + i\epsilon = 0$ so that

\[(q - k - k_g)^2 = -\frac{1}{k_1 + k_g} [k_1^2 (k_{g\perp} + k_{1\perp} + k_g - q) - k_g^2 k_{1\perp}]^2. \tag{52}\]

To get the NLO virtual contribution to $S^{(0)}(k_1)$ we combine the virtual amplitude (50) with the LO amplitude (9) and find

\[S^{(0)}(k_1) = (2\pi)\delta(k_1^+ - q^+)C_F g^2 \frac{1}{2P_p^+} \int \frac{dk_g^+}{(2\pi)} \int_\vec{x}\vec{y} e^{ik_1\cdot \vec{x}} e^{ik_2\cdot \vec{y}} \epsilon^{i(k_1 - k_2 - k_g - q')\cdot \vec{y}} \times \left[ S_q(\vec{x}_1, \vec{x}_\perp) \gamma^+ g T_q + S_q(\vec{x}_1, \vec{x}_\perp) T_g \gamma^+ + S_{qgq}(\vec{x}_1, \vec{x}_\perp, \vec{y}_\perp) \gamma^+ g T_{qg}(k_1) + S_{qgq}(\vec{x}_1, \vec{x}_\perp, \vec{y}_\perp) T_{qg}(k_1) \right], \tag{53}\]

where now $\vec{k}_2 = \vec{q} - \vec{k}_{1\perp}$ and $\mathcal{T}_q = \mathcal{T}_{q,1} + \mathcal{T}_{q,2}$. Analogous to the case of real production, the terms in (53) that are proportional to the dipole operator will not contribute as a consequence of $C$-parity. This includes the entirety of the second line and the dipole pieces of the third line according to (32). Repeating further the steps of the calculation used for real production we find

\[\text{tr} \left[ \gamma_5 f_1 S^{(0)}(k_1) \right] = i(2\pi)\delta(k_1^+ - q^+)N_c g^2 \frac{2}{P_p^+} \int \frac{dk_g^+}{(2\pi)} \int_\vec{x}\vec{y} e^{ik_1\cdot \vec{x}} e^{ik_2\cdot \vec{y}} \epsilon^{i(k_1 - k_2 - q')\cdot \vec{x}} \times \left[ \mathcal{P}(\vec{x}, \vec{y}) \mathcal{O}(\vec{x}', \vec{y}') - \mathcal{O}(\vec{x}, \vec{y}) \mathcal{P}(\vec{x}', \vec{y}') \right] \mathcal{H}(k_{1\perp}, k_{1\perp}) \right]. \tag{54}\]

See Fig. 2 for the diagram corresponding to this remaining contribution. Here now $\mathcal{H}(k_{1\perp}, k_{1\perp})$ is

\[\mathcal{H}(k_{1\perp}, k_{1\perp}) = \frac{1}{2q^+} \text{tr} \left[ \gamma_5 f_1 \gamma^+ q T_{qg}(k_1) \right] = -4i(\bar{y} + 1) \frac{v_{1\perp} \times v_{2\perp}}{v_{1\perp}^2 v_{2\perp}^2}, \tag{55}\]

where we have introduced $y \equiv k_g^+/k_1^+ = k_g^+/q^+ = z/\bar{z}$ and $v_{1\perp}$ ($v_{2\perp}$) are associated with final (initial) state collinear configurations explicitly given as $v_{1\perp} \equiv y q_{1\perp} - k_{g\perp}$, $v_{2\perp} \equiv k_{1\perp} + \bar{y} k_{1\perp} + k_{g\perp} - q_{\perp}$. To compute (55) we have evaluated the $k_g^-$ integral in $\mathcal{T}_{qg}(k_1)$ in favor of the singularity $k_g^2 + i\epsilon = 0$. Proceeding with the $k_{g\perp}$ loop integral we pass from the variable $k_{g\perp}$ to $v_{2\perp}$ and write

\[\int_{k_{g\perp}} \frac{v_{1\perp} \times v_{2\perp}}{v_{1\perp}^2 v_{2\perp}^2} = -\int_{v_{2\perp}} \frac{v_{1\perp} \times v_{2\perp}}{(v_{1\perp} + v_{2\perp})^2 v_{2\perp}^2}, \tag{56}\]

where $v_{\perp} \equiv \bar{y} q_{\perp} - y k_{1\perp} - k_{\perp}$. But this contains an angular integral that is precisely of the form (49) and therefore vanishes.
D. \( q \rightarrow g \)

In this case we only have the real diagram with gluon fragmenting into a final state hadron. The expression for \( S^{(0)}(k_1) \) takes the same form as \( (22) \), with the only difference being that now we are integrating over \( q_\perp \) (the momenta of the untagged quark) instead of over \( k_\perp \),

\[
S^{(0)}(k_1) = \frac{k_\perp^+}{P^+_p} \frac{g^2C_F}{(2\pi)^32E_pN_c^2-1} \frac{1}{1} \text{Tr}(M_\alpha \gamma \bar{M}_\beta \gamma) \\
= \frac{g^2T_R}{P^+_p (2\pi)^32E_pN_c^2-1} \int \int c^{ik_\perp \cdot x_\perp e^{i(k_\perp \cdot k_\perp')} y_\perp} \cdot \text{tr} \left[ T_{g,\alpha} g q T_{g,\beta} \right] + S_{qqg}(x_\perp, y_\perp, x_\perp, y_\perp) \frac{1}{1} \text{Tr}(M_\alpha \gamma \bar{M}_\beta \gamma) \\
= \left\{ S_A(x_\perp, x_\perp') \text{tr} \left[ T_{g,\alpha} g q T_{g,\beta} \right] \right\} + \left\{ S_{qqg}(x_\perp, y_\perp, x_\perp, y_\perp) \text{tr} \left[ T_{g,\alpha} g q T_{g,\beta} \right] \right\},
\]

The first term again vanishes due to \( \alpha \beta \) parity. To show that last term vanishes, we first need to make following replacements: \( k_\perp - q_\perp \rightarrow k_\perp \) and \( k_\perp' - q_\perp \rightarrow k_\perp' \) for the \( k_\perp \) and \( k_\perp' \) integrals which makes \( T_{gq}(k_\perp) \) independent of \( q_\perp \). Additionally, since \( g \) is sandwiched between two \( \gamma^+ \) matrices it does not give a \( q_\perp \) contribution. Therefore, the respective hard factor does not depend on \( q_\perp \), and the only \( q_\perp \) dependence in the last term appears in the exponential. From here we take the analogous steps as in the \( q \rightarrow q \) channel. First performing the \( q_\perp \) integration

\[
S_{qqg}(x_\perp, y_\perp, x_\perp, y_\perp) \frac{1}{1} \text{Tr}(M_\alpha \gamma \bar{M}_\beta \gamma) \\
= \delta(2)(x_\perp - x_\perp'), \quad (58)
\]

The odderon mechanism in the WW approximation does not contribute to SSA at NLO. Together with the result from Sec. III B and Sec. III C this completes the statement that in the \( q \rightarrow g \) channel the odderon mechanism in the WW approximation does not contribute to SSA at NLO.

IV. THE \( g \rightarrow q\bar{q} \) CHANNEL

In this channel we label the momenta as \( g(k_1) \rightarrow q(q')\bar{q}(p) \). The NLO amplitude can be written as

\[
M^\alpha = (\gamma^+) \int k_\perp \int x_\perp y_\perp c^{ik_\perp \cdot x_\perp e^{i(k_\perp \cdot k_\perp')} y_\perp} \left[ T_{g,\alpha} g q T_{g,\beta} \gamma \bar{M}_\beta \gamma \right], \quad (59)
\]

where

\[
T_{g,\alpha} = 2k_\perp^+ \gamma^+ \frac{d^{\alpha\beta}(q + p)}{(q + p)^2}
\]

and

\[
T_{qg}(k_\perp) = \frac{1}{2p^+ \gamma^+} \frac{q - \bar{k}}{(q - k)^2} \gamma^+ \frac{q - \bar{k} - k_\perp}{(q - k)^2}. \quad (61)
\]

In the above expressions, similar to the discussion in Sec. III A, \( k^+ \) is obtained by picking up the pole from the condition \((q - k - k_1)^2 + i\epsilon = 0\). The above results can be shown to agree with the corresponding amplitude in Eq. (38) in [52] used for unpolarized \( pA \) collisions after taking the collinear limit for the gluon from the proton. Using (59) we calculate \( S^{(0)}_{\alpha\beta}(k_1) \) as

\[
S^{(0)}_{\alpha\beta}(k_1) = \frac{1}{2P^+_p} \int \frac{d^3p}{(2\pi)^32E_pN_c^2-1} \frac{1}{1} \text{Tr}(M_\alpha \gamma \bar{M}_\beta \gamma) \\
= \frac{q^+}{P^+_p (2\pi)^32E_pN_c^2-1} \int \int c^{ik_\perp \cdot x_\perp e^{i(k_\perp \cdot k_\perp')} y_\perp} \cdot \text{tr} \left[ T_{g,\alpha} g q T_{g,\beta} \right] + S_{qqg}(x_\perp, y_\perp, x_\perp, y_\perp) \text{tr} \left[ T_{g,\alpha} g q T_{g,\beta} \right] \\
= \left\{ S_A(x_\perp, x_\perp') \text{tr} \left[ T_{g,\alpha} g q T_{g,\beta} \right] \right\} + \left\{ S_{qqg}(x_\perp, y_\perp, x_\perp, y_\perp) \text{tr} \left[ T_{g,\alpha} g q T_{g,\beta} \right] \right\}. \quad (62)
\]
After some inspection, this can be also re-written in a more convenient form as in Fig. 3. According to the WW truncation of (7) these require the evaluation of the following hard factors

\[ S_{qqq}(x'_\perp, x_\perp, y'_\perp, y_\perp) \equiv \frac{1}{C_F N_c} \text{tr} \left( V^\dagger(x_\perp)V(x_\perp) \epsilon^\alpha V^\dagger(y_\perp)V(y'_\perp) \epsilon^\alpha \right), \]

is an additional gluon distribution of the target. Similar to the findings in Sec. III A, the first and the fourth term in (62) vanish. For the first term this can be argued from C-parity on the Dirac trace (another way is simply from the fact that the adjoint dipole \( S_A(x_\perp, x_\perp') \) is real). For the fourth term the key point is that the hard factor does not depend on \( p_\perp \) (\( p \) is sandwiched between \( \gamma^+ \) and so the \( p_\perp \) dependence drops out). Then, the fourth term does not contribute by the same steps used in Sec. III A. This leaves the interference term in (62) that is represented graphically in Fig. 3. According to the WW truncation of (7) these require the evaluation of the following hard factors

\[
\mathcal{H}(G_{st})(k_\perp) = \frac{1}{4q^+ p^+ p^+_1} \beta^\alpha \bar{S}_\perp \omega_{\alpha\gamma} \omega^\gamma \beta \text{tr} \left[ T_g' \gamma_5 T_{q\bar{q}}(k_\perp) \right], \\
\mathcal{H}((\bar{q}\gamma^\lambda)(k_\perp, k_\perp)) = \frac{1}{4q^+ p^+} \left( g^\lambda_{\perp} \epsilon_{\alpha-} + g^\lambda_{\perp} \epsilon_{\beta-} + S_\perp \right) \text{tr} \left[ T_{g,\alpha} \gamma_5 T_{q\bar{q},\beta}(k_\perp) \right].
\]

We find

\[
\mathcal{H}(G_{st})(k_\perp) = 4z \bar{z}(z - \bar{z}) \hat{v}_1 \perp \times \bar{S}_2 \perp \hat{v}_2 \perp, \\
\mathcal{H}((\bar{q}\gamma^\lambda)(k_\perp, k_\perp)) = \frac{4}{v_1 \perp v_2 \perp} \left\{ - \left[ (z^2 + \bar{z}^2)(v_2 \perp \times \bar{S}_1 \perp) + z \bar{z}(z - \bar{z})(k_1 \perp \times S_\perp) \right] v_1 \perp \right. \\
+ \left. (z^2 + \bar{z}^2)(v_1 \perp \times S_\perp)v_2 \perp \\
+ z \bar{z}(z - \bar{z})(v_1 \perp \times S_\perp) k_1 \perp \right\},
\]

where now \( v_1 \perp \equiv zq_\perp - \bar{z}p_\perp = q_\perp - \bar{z}k_1 \perp - \bar{z}k_2 \perp - q_\perp + \bar{z}k_1 \perp + k_\perp, \) with \( z \equiv p^+/k_1^+ \) the momentum fraction of the recoiling antiquark. By \( \hat{v}_1 \perp (v_2 \perp) \) in (65) we again denote \( v_1 \perp (v_2 \perp) \) at \( k_1 \perp = 0. \) According to the WW truncation of the polarized cross section we also need to take a derivative of \( \mathcal{H}(\bar{q}\gamma^\lambda)(k_\perp, k_\perp) \) with respect to \( k_1 \perp \) (and sum over \( \lambda \)), c.f., 2nd line in (7). After this, we could proceed with the angular integrals as in Sec. III B, but this time the expressions would involve \( S_\perp. \) A simpler way to proceed is to first combine the 1st and the 2nd line in (7) leading to

\[
\left[ \left. \frac{\partial}{\partial k_1^\perp} \mathcal{H}(\bar{q}\gamma^\lambda)(k_\perp, k_\perp) \right|_{k_1^\perp = p_1} \right] = -\frac{4 z^2 \bar{z}^2}{\hat{v}_1 \perp \hat{v}_2 \perp} \times \left[ (\hat{v}_1 \perp \hat{v}_2 \perp + 2(\hat{v}_1 \perp \cdot \hat{v}_2 \perp)\hat{v}_2 \perp)(\hat{v}_1 \perp \times \bar{S}_1 \perp) - (\hat{v}_1 \perp \hat{v}_2 \perp + 2(\hat{v}_1 \perp \cdot \hat{v}_2 \perp)\hat{v}_2 \perp)(\hat{v}_2 \perp \times S_\perp) \right].
\]

After some inspection, this can be also re-written in a more convenient form as

\[
\left[ \left. \frac{\partial}{\partial k_1^\perp} \mathcal{H}(\bar{q}\gamma^\lambda)(k_\perp, k_\perp) \right|_{k_1^\perp = p_1} \right] = \left[ S^\lambda_{\perp} \frac{\partial}{\partial k_1^\perp} \left( 4(z^2 + \bar{z}^2) \frac{v_1 \perp \times v_2 \perp}{v_1 \perp v_2 \perp} \right) \right]_{k_1^\perp = p_1},
\]
where \( S^\lambda \) is now factored out and the effective hard factor inside the brackets in (68) now has finite \( k_{1\perp} \) through \( v_{1\perp} \) and \( v_{2\perp} \). To prove the equivalence of (67) and (68) we have used the Schouten identity \( \tilde{v}_{1\perp}(\tilde{v}_{2\perp} \times S_{\perp}) + \tilde{v}_{2\perp}(S_{\perp} \times \tilde{v}_{1\perp}) + S_{\perp}(\tilde{v}_{1\perp} \times \tilde{v}_{2\perp}) = 0 \). Eq. (68) reveals that the general structure of the \( g \to q \) hard factor is the same as in the \( q \to q \) channel, see (36). Therefore, by following the same logic as in Sec. III B we conclude that the corresponding polarized cross section in the \( g \to q\bar{q} \) channel also vanishes.

V. THE \( g \to gg \) CHANNEL

In the case of the \( g \to gg \) channel there is a great simplification due to the fact that the purely gluonic contributions involve only adjoint Wilson lines, which are real, and therefore the odderon mechanism is absent. The only exception is the quark loop correction to the tree-level \( g(k_1) \to g(k_g) \) amplitude, see Fig. 4 which we compute below. Discarding immediately the dipole pieces, \( S^{(0)}_{\alpha\beta}(k_1) \) takes the following form

\[
S^{(0)}_{\alpha\beta}(k_1) = \frac{1}{2P_p} \int_{q^+} \int_{q_{1\perp}} \int_{q_{2\perp}} \int_{x_{1\perp}} \int_{y_{1\perp}} \int_{y'_{1\perp}} e^{ik_{1\perp}x_{1\perp}} e^{ik_{2\perp}y_{1\perp}} e^{-ik_{2\perp}y'_{1\perp}} \times \left[ S_{qq}(x_{1\perp}, y_{1\perp}, x'_{1\perp})(-2k_1^+)d_{\alpha\mu}(k_g)T_{q\bar{q},\beta}(k_{1\perp}) + S_{qq}(y'_{1\perp}, x'_{1\perp}, x_{1\perp})(-2k_1^+)d_{\alpha\mu}(k_g)T_{q\bar{q},\beta}(k'_{1\perp}) \right].
\]

(69)

where \( q \) is the quark loop momentum and

\[
T_{q\bar{q}}(k_{1\perp}) = \frac{1}{2q^+} \int_{q^-} \text{tr} \left[ \frac{\hat{g} - \hat{k}_g}{(q - k_g)^2 + i\epsilon} \gamma^\mu \frac{\hat{g} - \hat{k}_g}{q^2 + i\epsilon} \gamma^\nu (\hat{g} - \hat{k}_g + \hat{k}_1 + \hat{k}) \gamma^\beta \frac{\hat{g} - \hat{k}_g + \hat{k}}{(q - k_g + k)^2} \gamma^+ \right].
\]

(70)

Similar to the computation in Sec. IV, and according to (6), we are to evaluate the following combination

\[
\mathcal{H}^{(g\gamma T)}(k_{1\perp}) - \left[ \frac{\partial}{\partial k_1^\perp} \mathcal{H}^{(g\gamma)}(k_{1\perp}, k_{1\perp}) \right]_{k_1 = p_1},
\]

(71)

where we define

\[
\mathcal{H}^{(g\gamma T)}(k_{1\perp}) = \frac{1}{(2q^+)^2 p_1^2} e_{\alpha} e_{\gamma} S_{\alpha\gamma} (-2k_1^+)d_{\alpha\mu}(k_g)T_{q\bar{q},\beta}(k_{1\perp}),
\]

\[
\mathcal{H}^{(g\gamma)}(k_{1\perp}, k_{1\perp}) = \frac{1}{(2q^+)^2} \left( g_{\perp}^{\beta\alpha} e_{\alpha n} S_{n} - g_{\perp}^{\alpha\beta} e_{\beta n} S_{n} \right) (-2k_1^+)d_{\alpha\nu}(k_g)T_{q\bar{q},\beta}(k_{1\perp}).
\]

(72)

A direct computation leads to

\[
\mathcal{H}^{(g\gamma T)}(k_{1\perp}) - \left[ \frac{\partial}{\partial k_1^\perp} \mathcal{H}^{(g\gamma)}(k_{1\perp}, k_{1\perp}) \right]_{k_1 = p_1} = \left[ S_{\parallel} \frac{\partial}{\partial k_1^\perp} \left( 4(y^2 + \hat{y}^2) \frac{v_{1\perp} \times v_{2\perp}}{v_{1\perp}^2 \cdot v_{2\perp}^2} \right) \right]_{k_1 = p_1},
\]

(73)

where now \( v_{1\perp} \equiv -q_{1\perp} + k_g - \hat{y}k_{1\perp} - k_{1\perp} \) and \( v_{2\perp} \equiv q_{1\perp} - yk_{1\perp} \). But this is completely analogous to the result for the loop correction in Sec. III C and so \( S^{(0)}_{\alpha\beta}(k_1) \) vanishes after the \( q_{1\perp} \) integral.
VI. CONCLUSIONS AND OUTLINE

We have revisited the odderon mechanism for SSA in $p^5A$ originally suggested in [1] at the quark level. At the hadron level this mechanism would involve the $g_T(x)$ distribution. We have considered the WW truncation of the full twist-3 polarized cross section and argued that in addition to $g_T(x)$ we also need to take into account the $g_T^{(1)}(x)$ for a consistent computation. Our main finding is that under this truncation the polarized cross section vanishes exactly up to NLO for all possible partonic channels.

It is natural to consider whether any of the above assumptions can be relaxed so that a non-zero contribution to SSA from the odderon mechanism may be found after all. One option is to go beyond the WW approximation, namely including the ETQS pieces in (6). Note the difference from the more conventional pole calculus – here one needs to pick up the principal value of internal propagators so that the general functional forms of the ETQS functions would be required. Alternatively, one can consider the twist-3 FF mechanism, where we pick up the real part of the twist-3 FFs with the phase provided by the odderon. Once more, this in contrast to the conventional computations where the phase is supplied by the imaginary part of twist-3 FFs. Given that the current global fits constrain only the imaginary part of the twist-3 FFs [53, 54], the phenomenological implications of this alternative would be worth exploring.

Another possibility would be to retain the WW approximation but compute the hard factor up to NNLO. While of course only an explicit computation can reveal whether the odderon appears at NNLO, we mention here a competing mechanism that is already known to appear at NNLO. The basic premise is very simple: at higher orders it is an imaginary part of the loop amplitude that can supply the phase. A specific NNLO contribution illustrating this is given in Fig. 5, where the crosses denote cut propagators. Physically, the initial $q \rightarrow qg$ splitting occurs inside the target nucleus in the amplitude. The $qg$ system subsequently rescatters with a $t$-channel quark into the final state providing a phase with respect to the amplitude on the opposite side of the final state cut. Such final state rescattering is sometimes referred to as the lensing mechanism and was considered in [19]. In fact, this idea [55] is closely related to the very first estimate of SSA in perturbative QCD [56]. The computation in [19] was in the quark-diquark model. As a future work it would be important to consider this in the hybrid approach.

ACKNOWLEDGMENTS

S. B. thanks Yoshitaka Hatta for suggesting to work on the odderon mechanism for SSA. We thank Yoshitaka Hatta and Yuri Kovchegov for useful comments on the manuscript. S. B., A. K. and E. A. V. are supported by the Croatian Science Foundation (HRZZ) no. 5332 (UIP-2019-04).

[1] Y. V. Kovchegov and M. D. Sievert, Phys. Rev. D 86, 034028 (2012), 1201.5890, [Erratum: Phys.Rev.D 86, 079906 (2012)].
[2] V. Barone, A. Drago, and P. G. Ratcliffe, Phys. Rept. 359, 1 (2002), hep-ph/0104283.
[3] U. D’Alesio and F. Murgia, Prog. Part. Nucl. Phys. 61, 394 (2008), 0712.4328.
[4] D. Pitonyak, Int. J. Mod. Phys. A 31, 1630049 (2016), 1608.05353.
