Resonant orbits for a spinning particle in Kerr spacetime

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Abstract

In the present article, we study the orbital resonance corresponds to an extended object approximated up to the dipole order term in Kerr spacetime. We start with the Mathisson-Papapetrou equations under the linear spin approximation and primarily concentrate on two particular events. First, when the orbits are nearly circular and executing a small oscillation about the equatorial plane and second, a generic trajectory confined on the equatorial plane. While in the first case, all the three fundamental frequencies, namely, radial $\Omega_r$, angular $\Omega_\theta$, azimuthal $\Omega_\phi$ can be commensurate with each others and give rise to the resonance phenomenon, the later is only accompanied with the resonance between $\Omega_r$ and $\Omega_\phi$ as we set $\theta = \pi/2$. We provide a detail derivation in locating the prograde resonant orbits in either of these cases and also study the role played by the spin of the black hole. The implications related to spin-spin interactions between the object and black hole are also demonstrated.

1 Introduction

In the strong field regime where the effects of gravity are inescapable, the motion of test particles can attribute to many nontrivial consequences such as precession, spiraling of orbits, resonance, chaos and so on [1–6]. Even if these investigations related to the trajectories of test particles are not only present day concerns rather contain with extensive literature spanned over last several years [7, 8], there are relevant studies with possible observational aspects yet to be explored [9,10]. Regarding the observational implications, there are proposed mission such as The Laser Interferometer Space Antenna (LISA) which aims to probe extreme mass ratio binaries in near future as well [11–13]. However, in any attempt to explain the phenomenon involves incomparable masses would require to grasp the theoretical structure of particle trajectories as well. While in many cases, it is appropriate to consider that the lighter companion as a point test particle and moves along a geodesic trajectory, the addition of higher order moments may introduce larger accuracy in estimating any measurable quantities. These higher order moments can be originated from the non-trivial internal structure of the body represented by its energy momentum tensor $T^{\mu\nu}$ while their dynamics can be derived from the conservation equation $\nabla_\nu T^{\mu\nu} = 0$. By expanding $T^{\mu\nu}$ about a reference point located inside the object, say $z_\mu(t)$, the zeroth order term of the conservation equation would produce the geodesic equation while the first order introduces the Mathisson-Papapetrou equations.

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accordingly [14, 15]. The next contribution would arise from the quadrupole moment and given by the Mathisson-Papapetrou-Dixon equations [16–18]. However, in the present article, we would only consider the first order correction to the geodesic equations as introduced by the nonzero spin or dipole moment of the particle $S^\mu\nu$ given as [15]

$$S^\mu\nu = \int r^\mu T^0\nu d^3x - \int r^\nu T^0\mu d^3x,$$

(1)

with both $\mu$ and $\nu$ can run from 0 to 3. The scaled energy-momentum tensor $T^{0\mu} = \sqrt{-g} T^{0\mu}$ with $g$ being the determinant of the metric, is integrated over the spacelike hypersurface represented by $l = \text{constant}$ slices. Dipole moment is computed about a reference point $z_\mu(t)$ while the distance between $z_\mu(t)$ and any given mass point is given by $r^\mu$ which contains no time-component according to the definition.

From a theoretical standpoint, the astrophysical objects are likely to have a finite size and therefore it is appropriate to consider additional moments such as dipole, quadrupole and so on. While particle with only mass monopole are named as monopole particle, the addition of dipole and quadrupole moment would introduce pole-dipole and pole-dipole-quadrupole particle respectively. However, in the present context we shall only discuss some aspects of pole-dipole or spinning particle and not consider the quadrupole moment. The motion of a spinning particle constitutes an interesting problem and studied extensively in literature [19–22]. With the pioneering work by Papapetrou in 1951 [15], the early contributions from Mathisson [14], Ehlers and Rudolph [23] and Dixon [24] are worth to mention. In recent times, there are also significant studies regarding several aspects of spinning particle and their nontrivial features [25–35]. For excellent overview on the subject, we refer our readers Refs. [36–38]. In one way, it emerges as an engaging theoretical implication with possible usefulness in explaining motion of objects with non-trivial internal structure and also pave the way for extreme mass ratio inspiral in general.

In the present article, we aim to study the resonant orbits for a spinning particle in the Kerr background and for computational convenience, we restrict our discussions on two major aspects: first, the resonance between different small oscillation frequencies correspond to nearly circular orbits close to the equatorial plane and second, the resonance between radial and azimuthal frequencies considering the trajectories are completely confined on the equatorial plane of the black hole. In the former, it is possible to encounter the resonance in-between radial $\Omega_r$, angular $\Omega_\theta$ as well as azimuthal $\Omega_\phi$ frequencies while in the later, it is only between $\Omega_r$ and $\Omega_\phi$. Out of these three frequencies, the radial and angular are known as libration frequencies, while the later $\Omega_\phi$ can be thought of as a rotational frequency. The correspondence between these frequencies would lead to the occurrence of resonance phenomenon in astrophysical scenarios. In particular, whenever the ratio of any of these frequencies become rational fraction, the orbits become resonant. For example, with the $r\phi$ resonance, the principle equation is given as $\Omega_r/\Omega_\phi = n/m$, where both $n$ and $m$ are rational number with no common divisor.

With the above motivation, let us now introduce the physical notion of resonance in astrophysical scenarios and also highlight its importance in the present context as well. Apart from astrophysical settings, the events like resonance are usually encountered in vibrations of strings and coupled oscillations [39]. For example, in case of a forced vibrating oscillator, the resonance becomes dominant whenever the natural frequency of the system becomes nearly equal to the frequency of the forced oscillation. Similar to this, the weakly coupled oscillators can also undergo resonance phases whenever their individual frequencies are comparable to each other. Typically speaking, the starting of any resonance event would indicate an invitation to the non-integrability of Hamiltonian and therefore the system slowly descends into chaos. However, for a sufficient small perturbation, the system may remain integrable and the notion of separability constant (also known as Carter constant in the black hole spacetime) may still exist [40–42]. In case of a spinning particle, the motion is completely integrable upto the linear order and therefore we
presume that the addition of spin would not introduce any chaos in the system as far as the $O(S^2)$ terms are neglected. This would allow us to obtain the resonant orbits similar to the case of geodesic trajectories as given in Ref. [5]. It is of significant interest to locate the resonant orbits as it is likely that these orbits will witness the breakdown of phase space tori for the first time if the system undergoes any chaos [43–45].

Earlier in literature, several implications related resonance activities are addressed in various aspects. Like mentioned earlier, the orbital dynamics in Kerr depends on three fundamental frequencies relating three spatial components and therefore, there could exists different types of resonant orbits. The $r\theta$ resonant orbits are studied in Ref. [5] whereas the $\theta\phi$ and $r\phi$ resonances are addressed in Ref. [46] and Ref. [47] respectively. In Refs. [46, 47], the black hole kicks are also studied for an extreme mass ratio inspiral considering the lighter companion follows a geodesic orbit. For an investigation related to the resonance in between the spin precession frequencies and orbital frequencies of a spinning particle, we refer our readers Ref. [48] for a better understanding.

The rest of the paper is organized as follows. In section-(2), the basic governing equations of a spinning particle are introduced along with various implications such as spin supplementary condition and conserved quantities. These equations are exactly solved on the equatorial plane of the Kerr black hole and the effective radial potential is also obtained for further calculations. Following this in section-(3), we introduce the machinery to obtain the fundamental frequencies and orbital resonance for any trajectory in black hole’s spacetime. In particular, we studied two different events corresponding to a spinning object: first, the small oscillation frequencies for a particle orbiting in nearly circular orbits close to the equatorial orbits and second, a general motion confined on $\theta = \pi/2 = \text{constant}$ plane. Section-(4) is devoted to explore the resonance phenomenon related to the first case while in section-(5), the $r\phi$ resonance are elaborately studied. In the latter, the approximation technique namely the Sochnev method is employed to evaluate the integrals on the equatorial plane of the black hole. We conclude the article with a brief remark in section-(6).

## 2 Motion of a spinning particle in gravitational field

The motion of a spinning particle is described by the Mathisson-Papapetrou equations

$$\frac{DP^a}{d\tau} = -\frac{1}{2}R^a_{\ bcd}U^bS^{cd}, \quad \frac{DS^{ab}}{d\tau} = P^aU^b - P^bU^a,$$

with $P^a$ and $U^a$ correspond to the four momentum and four velocity of the particle respectively, $S^{ab}$ is the spin tensor of the extended object relating the dipole moment and $R^a_{\ bcd}$ has the usual meaning of Riemann curvature tensors correspond to the background geometry. Furthermore, it should be emphasized that the above set contains total 10 equations while we have 14 (6 from the spin, 4 from the four velocity and 4 from the four momentum) unknowns to solve and therefore, additional conditions are required. These are called spin supplementary condition and in the present context, we shall employ the Tulczyjew-Dixon condition $S^{ij}P_j = 0$ to solve Mathisson-Papapetrou equations around the Kerr black hole [49].

Let us now introduce the notion of spin vector $S^i$ for simplified computations and it is related with the spin tensor as follows [32]

$$S^i = \frac{1}{\sqrt{-g}}\epsilon^{ijkl}P_jS_{kl}, \quad (3)$$

\[\text{However, as we are interested in the linear spin approximation, both the Tulczyjew-Dixon and Mathisson-Pirani condition (S^{ij}U_j = 0) would serve the same purpose as momentum and velocity start to differ only at O(S^2).}\]
where $g$ is the determinant of the background metric. For the particle confined on the equatorial plane, it is convenient to assume that the spin vector follows $S^i = (0, 0, S^θ, 0)$ which indicates that the spin is either parallel or anti-parallel to the black hole’s rotational axis.

In addition, the symmetries associated with the background geometry will give rise to the conserved quantities such as energy $E$ and momentum $J_z$. The compact expression for any conserved quantity $C$ related to a killing vector field $K^i$ can be written as

$$C = K^i P_i - \frac{1}{2} S^{ij} \mathcal{K}_{ij},$$

where the contribution from the spin can be easily spotted [50, 51]. It should also be noted that the total spin of the particle $S^2 = \frac{1}{2} S^i S_i$ and the mass $P^i P_i = -\mu^2$ are also conserved along the trajectory of the particle as it can be established from the Tulczyjew-Dixon supplementary condition. With all these machinery and assuming $P^i U_i = -\mu$, one should be able to solve the Mathisson-Papapetrou equations and also establish a relation between four velocity and four momentum of the particle. In the present context, we only state the final results corresponding to the components of four velocity of the particle while the full derivations can be found in Ref. [32]. These equations are given as

$$\left(\Sigma_s \Lambda_s U^i\right)^2 = R_s = P_s^2 - \Delta \left(\frac{\Sigma_s^2}{r^2} + \{J_z - (a + S)E\}^2\right),$$

$$\left(\Sigma_s \Lambda_s U^0\right) = a \left(1 + \frac{3S^2}{r\Sigma_s}\right) \{J_z - (a + S)E\} + \frac{r^2 + a^2}{\Delta} P_s,$$

$$\left(\Sigma_s \Lambda_s U^3\right) = \left(1 + \frac{3S^2}{r\Sigma_s}\right) \{J_z - (a + S)E\} + \frac{a}{\Delta} P_s,$$

where $a$ and $M$ are angular momentum and mass parameter of the black hole respectively and $P_s$ and $\Sigma_s$ is given as

$$P_s = E \left(r^2 + a^2 + aS + \frac{aSM}{r}\right) - \left(a + \frac{MS}{r}\right) J_z,$$

$$\Sigma_s = r^2 (1 - MS^2/r^3) ; \quad \Delta = r^2 + a^2 - 2Mr,$$

$$\Lambda_s = 1 - \frac{3MS^2r}{\Sigma_s} - \{J_z - (a + S)E\}^2.$$

For $S > 0$, the spin is parallel to the black hole spin, and anti-parallel for $S < 0$. In addition, to guarantee that the timelike constraint $U^i U_i < 0$ is always valid, we have the following condition to hold

$$r^5(1 - MS^2/r^3)^4 - 3MS^2(2 + MS^2/r^3) \{J_z - E(a + S)\}^2 > 0.$$

Needless to say that the above condition further constraint the motion of the particle and especially close to the horizon, it becomes more dominant [35].

3 Resonance phenomenon and fundamental frequencies

The general condition for resonance can be written in terms of the fundamental frequencies, such that

$$\alpha \Omega_r + \beta \Omega_\theta + \gamma \Omega_\phi = 0,$$
where \( \alpha, \beta, \) and \( \gamma \) are rational numbers without any common divisors. For a limiting case on the equatorial plane, \( \Omega_r \) and \( \Omega_\phi \) would be of the particular interest, while \( \Omega_\theta \) has no meaning. On the other hand, for circular orbits, \( \Omega_r \) is not particularly relevant and \( \Omega_\theta \) and \( \Omega_\phi \) play the key role in the orbital dynamics. In the present context, we shall be interested in following investigations

- The quasi periodic oscillations for spinning particles and the locations of resonant orbits from the condition given by Eq. (8).

- Location of resonant orbits on the equatorial plane following the condition, \( \alpha \Omega_r + \gamma \Omega_\phi = 0 \) which is the \( r\phi \) resonance.

For the first case, both the radial and angular potentials are required to obtain the oscillation frequencies. While effective radial potential is given in Eq. (5), to express the angular potential, the notion of Carter constant for spinning particle has to be used which is only valid up to the terms linear in spin. This will instantly make the present analysis valid only at \( \mathcal{O}(S) \) which is exactly the appropriate arena to describe Mathisson-Papapetrou equations. In the second case, we also introduce the linear spin limit for computational convenience.

Having described the primary objectives behind the present analysis, we shall now briefly introduce the mathematical descriptions for the fundamental frequencies in terms of the orbital parameters. We start with the nearly circular orbits hobbling around the equatorial plane and later discuss the generic elliptical orbits confined on the \( \theta = \pi/2 \) plane. For nearly circular orbits satisfying \( \theta = \pi/2 + \eta \) where \( \eta \ll \pi/2 \), the fundamental frequencies would be identical as the small oscillation frequencies. More conveniently, these orbits can be considered as a small perturbations from the stable circular orbits confined on the equatorial plane. For convenience, we may introduce the effective potential \( V_x \) and its derivative as

\[
V_x = \dot{x}^2 = \delta x^2, \quad \text{and} \quad \frac{1}{2} \frac{dV_x}{dx} = \ddot{x} = \delta \dot{x},
\]

with \( x \) being the coordinate and the dot defines a time derivative. From the above equation, it is now easy to derive the small oscillations frequency as

\[
\Omega^2 = - \frac{1}{2} \left. \frac{d^2V_x}{dx^2} \right|_{x=x_0},
\]

where \( x_0 \) is the stable point about which the particle is oscillating. We shall explicitly use this understanding in relativistic orbits which can be written in terms of the effective radial \( V(r) \) and angular \( V(\theta) \) potentials. In case of geodesic trajectories in the Kerr background, the motion is completely integrable making \( V(r) \) and \( V(\theta) \) written independently in terms of \( r \) and \( \theta \) respectively. However for spinning particles, the motion is not completely integrable and the task to determine the effective potentials is nontrivial. For the radial potential, Mathisson-Papapetrou equations can be employed and exactly solved on the equatorial plane of the black hole. To compute \( V(\theta) \), the Carter constant which is only valid up to the linear order terms in spin, can be used appropriately.

In the case of a generic elliptical orbit, we may introduce the following expressions for radial and azimuthal frequencies

\[
\Omega_r = \frac{2\pi}{T_r}, \quad \text{and} \quad \Omega_\phi = \frac{\Delta \phi}{T_r},
\]

\[5\]
with, \( T_r = 2 \int_{r_a}^{r_p} \left( \frac{dt}{dr} \right) dr \) and \( \Delta \phi = 2 \int_{r_a}^{r_p} \left( \frac{d\phi}{dr} \right) dr \), \( r_a \) and \( r_p \) are the apastron and periastron radii respectively [52,53]. In this case, the \( r\phi \) resonance would be governed by the equation

\[ m\Omega_r - n\Omega_\phi = 0, \quad (12) \]

with \( m \) and \( n \) are two integers constrained as \( n \leq m \). More conveniently, the above equation can be written as

\[ m\pi - n \int_{r_a}^{r_p} \frac{d\phi}{dr} dr = 0. \quad (13) \]

The second term in the above expression can be written as

\[ \frac{d\phi}{dr} = \left( \frac{d\phi}{d\tau} \right) \left( \frac{dr}{d\tau} \right)^{-1}, \quad (14) \]

with \( \tau \) being the proper time. It can be further simplified by introducing the orbit equations in terms of energy, angular momentum, spin and the radial coordinate \( r \). However, as it is evident that the quantity \( dr/d\tau \) is proportional to the radial potential which in fact, vanishes at the turning points \( r_a \) and \( r_p \). This constitutes a serious problem while integrating the function that blows up in the upper and lower limits. We shall use approximate technique, namely the Sochnov method, to deal with such scenarios.

## 4 Nearly circular orbits and resonance conditions

Let us now consider the nearly circular orbits for spinning particles and determine the small oscillation frequencies accordingly. For this purpose we shall assume the spin vector has the form \( S = (0, 0, S^\theta, 0) \) and the four velocity can be written as \( U^i = (U^0, 0, 0, U^3) \). In addition, we shall approximately write the frequencies up to the linear order in spin while neglecting all the higher order terms.

### 4.1 Determining fundamental frequencies \( \Omega_r, \Omega_\theta \) and \( \Omega_\phi \)

In case of the radial frequency, \( \Omega_r \), we shall derive the radial potential and use the formula given in Eq. (10). However, to compute the angular frequency \( \Omega_\theta \), the notion of Carter constant has to be invoked which is only valid up to the linear order of the spin. Finally, the azimuthal frequency can be easily derivable from the circular orbit condition on the equatorial plane of the black hole.

#### 4.1.1 Computing \( \Omega_r \)

The Mathisson-Papapetrou equations written with the Tulczyjew-Dixon supplementary condition are exactly solvable on the equatorial plane of the Kerr black hole. Similar to geodesic trajectories, the four velocity and four momentum has an unique relation and it is possible to derive the radial potential, \( V_s(r) \), in terms of conserved quantities such as energy and momentum of the particle. This is given as

\[ V_s(r) = \frac{1}{(\Sigma_s \Lambda_s)^2} \left\{ P_s^2 - \Delta \left[ \frac{\Sigma_s^2}{r^4} + \{ J_z - (a + S)E \}^2 \right] \right\}, \quad (15) \]

with the expressions for \( P_s, \Sigma_s \) and \( \Lambda_s \) already defined in Eq. (6). The above equation can be employed to determine the small oscillation radial frequency [33]

\[ \Omega_r^2 = \frac{1}{2} \frac{d^2V_s(r)}{dr^2} = \frac{M}{r^7/2} \frac{\chi}{\eta} + \frac{2GS}{\eta^3 r^{7/4}}, \quad (16) \]
where η, χ and G has the following expressions

\[
\eta = 2aM^{1/2} + (r - 3M)r^{1/2}, \quad \chi = (-3a^2 + 8a\sqrt{M}\sqrt{r} + r(r - 6M)),
\]

\[
G = \frac{3M(\sqrt{Mr} - a)}{2} \left\{ 7a^3M^{1/2} + aM^{1/2}(14M - 5r) + 5a^2r^{1/2}(r - 4M) + r^{5/2}(r - 2M) \right\}.
\]  

(17)

However, as we are interested in the frequency computed by a static observer at spatial infinity, Ω, has to be divided with the redshift factor \(U^0\) to compute \(\omega_r\). This is given by

\[
\omega_r^2 = \frac{M}{(r^{5/2} + arM^{1/2})^2} \left[ r(r - 6M) + 8a\sqrt{rM} - 3a^2 \right] - \frac{3MS(a - \sqrt{Mr})}{r^4(r^{3/2} + a\sqrt{M})^3} \left( 2a^3\sqrt{M} \right. \\
+ \left. 5a^2r^{3/2} + r^{5/2}(r - 2M) - 2ar\sqrt{M}(2M + 3r) \right).
\]  

(18)

By substituting the spin of the particle to be zero, one can establish that the small oscillation frequency for a geodesic trajectory is

\[
\omega_r^2 \bigg|_{S=0} = \frac{M}{(r^{5/2} + arM^{1/2})^2} \left[ r(r - 6M) + 8a\sqrt{rM} - 3a^2 \right],
\]  

(19)

which would vanish at the innermost stable circular orbit in the Kerr background. Similarly for extended objects with a nonzero spin, radial frequency \(\omega_r\) would also vanish at the innermost stable circular orbit and beyond that point \(\omega_r\) contains no meaning as far as the small oscillation frequencies are concerned.

4.1.2 Computing \(\Omega_\theta\):

Unlike the non-spinning particle, the motion of a spinning particle is not completely integrable and their off-equatorial trajectories cannot be expressed in terms of the radial and angular coordinates separately. However there exists an analog of the Carter constant valid only up to the linear order in spin and it is given as [54]

\[
\frac{Q}{m^2} = \left\{ \left( \Sigma (\mathcal{U}^{(0)})^2 - (\mathcal{U}^{(1)})^2 - r^2 \right) \right\} - \frac{2a\sin \theta}{\sqrt{\Sigma}} \left\{ r \left( \mathcal{U}^{(0)} S^{(1)(3)} - 2\mathcal{U}^{(3)} S^{(1)(0)} + \mathcal{U}^{(1)} S^{(3)(0)} \right) + a\cos \theta \mathcal{U}^{(3)} S^{(2)(3)} \right\} - \frac{2\sqrt{\Lambda}}{\sqrt{\Sigma}} \left\{ a\cos \theta \left( 2\mathcal{U}^{(0)} S^{(2)(3)} - \mathcal{U}^{(3)} S^{(2)(0)} + \mathcal{U}^{(2)} S^{(3)(0)} \right) - r\mathcal{U}^{(0)} S^{(1)(0)} \right\}.
\]  

(20)

where the terms containing any bracket denote a quantity projected on the tetrad frame \(e_{\mu}^{(r)}\) [30]. The explicit expressions for the tetrad components are given as

\[
e_{\mu}^{(0)} = \left\{ \frac{\sqrt{\Delta}}{\Sigma}, 0, 0, -a\sin^2 \theta \sqrt{\Sigma} \right\}, \quad e_{\mu}^{(1)} = \left\{ 0, \frac{\Sigma}{\sqrt{\Delta}}, 0, 0 \right\}, \quad e_{\mu}^{(2)} = \left\{ 0, 0, \sqrt{\Sigma}, 0 \right\}, \quad e_{\mu}^{(3)} = \left\{ -a\sin \theta \frac{\sqrt{\Sigma}}{\Sigma}, 0, 0, \frac{r^2 + a^2}{\sqrt{\Sigma}} \sin \theta \right\}.
\]  

(21)
with $\Delta = r^2 - 2Mr + a^2$ and $\Sigma = r^2 + a^2 \cos^2 \theta$. By dropping the spin dependence terms in Eq. (20), the Carter constant for the geodesic motion can be retrieved accordingly. Here it should be reminded that in the linear order spin approximation with the Carter constant defined, it is expected that the Hamiltonian would be completely integrable. This would indicate that the additional perturbation as introduced by the spinning particle will not be responsible for any non-integrability in the system and the present analysis is applicable as far as the spin remain significantly small.

With the attempt to study nearly circular orbits close to the equatorial plane, we set $U^{(1)} = 0$ and consider the spin is of the form $S = (0, 0, S^\theta, 0)$, then the above expression can be simplified as

$$\frac{Q}{m^2} = a^2 \cos^2 \theta + \Sigma \left\{ \left[ U^{(2)} \right]^2 + \left[ U^{(3)} \right]^2 \right\} - \frac{2a \sin \theta}{\sqrt{\Sigma}} \left\{ r \left( U^{(0)} S^{(1)(3)} - 2U^{(3)} S^{(1)(0)} \right) \right\} + \frac{2\sqrt{\Delta}}{\sqrt{\Sigma}} U^{(0)} S^{(1)(0)}.$$  

(22)

The above expression can be further simplified as

$$\Sigma \left\{ U^{(2)} \right\}^2 = Q - a^2 \cos^2 \theta - \Sigma \left\{ \left[ U^{(3)} \right]^2 \right\} + \frac{2a \sin \theta}{\sqrt{\Sigma}} \left\{ r \left( U^{(0)} S^{(1)(3)} - 2U^{(3)} S^{(1)(0)} \right) \right\} - \frac{2\sqrt{\Delta}}{\sqrt{\Sigma}} U^{(0)} S^{(1)(0)}.$$  

(23)

By substituting the tetrad component, we can write $U^2 = e^2_{(i)} U^{(i)}$ as the effective potential in theta direction, i.e. $\{U^2\}^2 = V_s(\theta)$ and the Eq. (23) changes accordingly

$$\Sigma^2 \left\{ U^{(2)} \right\}^2 = Q - a^2 \cos^2 \theta - \Sigma \left\{ \left[ U^{(3)} \right]^2 \right\} + \frac{2a \sin \theta}{\sqrt{\Sigma}} \left\{ r \left( U^{(0)} S^{(1)(3)} - 2U^{(3)} S^{(1)(0)} \right) \right\} - \frac{2\sqrt{\Delta}}{\sqrt{\Sigma}} U^{(0)} S^{(1)(0)}.$$  

(24)

Expanding the Carter constant as $Q = K_{\text{spin}} + (J_z - aE)^2$ and expressing the spin tensors in terms of the spin vector, the effective potential $V_s(\theta)$ can be rewritten as

$$V_s(\theta) = \Sigma^{-2} \left( K_{\text{spin}} + (J_z - aE)^2 - \Sigma \left[ U^{(3)} \right]^2 - a^2 \cos^2 \theta + \frac{2arS \sin \theta}{\sqrt{\Sigma}} \left\{ \left[ U^{(0)} \right]^2 - 2 \left[ U^{(3)} \right]^2 \right\} - 2rS \sqrt{\frac{\Delta}{\Sigma}} U^{(0)} U^{(3)} \right).$$  

(25)

the only non-vanishing spin components are $S^{(0)(1)} = -SU^{(3)}$ and $S^{(1)(3)} = SU^{(0)}$. It is now straightforward to compute the frequency $\Omega_\theta$ as given in Eq. (10)

$$\Omega_\theta^2 = \frac{J_z^2 - a^2 \left( E^2 - 1 \right)}{r^4} + \frac{aS}{r^3} \left\{ E^2 r^3 + 14Ma^2 (J_z - aE) + a^2 r \left[ 4a^2 - 5(J_z^2 - a^2 E^2) \right] + 2Mr^2 \left[ 4a^2 (E^2 - 1) - (J_z - aE)^2 \right] + r^3 \left[ J_z^2 + a^2 (4 + 6E^2) \right] \right\}.$$  

(26)

For a static observer at infinity the frequency will be red shifted and given as, $\omega_\theta^2 = \frac{\Omega_\theta^2}{\left( U^0 \right)^2}$. By substituting the expressions for energy and momentum from the circular orbit conditions on the equatorial plane [33],
we arrive at
\[ \omega_\theta^2 = \frac{M(3a^2 + r^2 - 4a\sqrt{Mr})}{(r^{5/2} + arM^{1/2})^2} + \frac{S}{r^{9/2}(r^{3/2} + aM^{1/2})^3} \left[ 18a^5M + \sqrt{Mr}^{9/2}(-7M + 2r) + a^4\sqrt{Mr}^{5/2} \right. \\
\left. (-23M + 27r) + ar^3(12M^2 + 2Mr + r^2) + a^2\sqrt{Mr}^{3/2}(4M^2 - 32Mr + 5r^2) \right. \\
\left. + a^3r(-8M^2 - 10Mr + 9r^2) \right]. \] (27)

The above expression reduces to the usual expression for a geodesic trajectory while for the non-rotating black hole, \( \omega_\theta \) become
\[ \omega_\theta \bigg|_{a=0} = \frac{M}{r^{5/2}} + \frac{\sqrt{M}S(2r - 7M)}{r^{9/2}}. \] (28)

The first term dictates the usual Keplerian contribution while the second term captures the imprints of intrinsic rotation of the extended object. Interestingly, it is possible to find a radial solution to the vanishing \( \omega_\theta \) satisfying
\[ 1 - \frac{\tilde{S}(7 - 2\tilde{r})}{\tilde{r}^{3/2}} = 0, \] (29)
while we define \( \tilde{r} = r/M \) and \( \tilde{S} = S/M \), with \( M \) being the mass of black hole. However, the above equation only satisfies for a large spin value which is not appropriate to employ in our case and we can safely state that \( \omega_\theta \) is always nonzero.

4.1.3 Computing \( \Omega_\phi \) :

For a spinning particle moving in circular orbits on the equatorial plane, angular frequency can be written as \[ \Omega_\phi^2 = \left( \frac{d\phi}{d\tau} \right)^2 = \frac{M}{\eta r^{7/2}} + \frac{2FS}{\eta^2r^{3/4}}, \] (30)
where the expressions for \( \eta \) is already defined in Eq. (17) and for \( F \), it is given as follows
\[ F = \frac{3(a^2M + M(2M - r)r + a\sqrt{Mr}) (r - 3M)}{2r^{11/4}}. \] (31)

However, the frequency as measured by a static observer at infinity will be red shifted and can be written as
\[ \omega_\phi^2 = \frac{M}{(r^{3/2} + a\sqrt{M})^2} + \frac{3M(a - \sqrt{Mr})}{\sqrt{r(a\sqrt{M} + r^{3/2})^3}}S. \] (32)

It should be emphasized that even for the \( a = 0 \) limit, the angular frequencies \( \omega_\phi \) and \( \omega_\theta \) are not identical to each other. This is because of the nonzero spin that contributes to different non-Keplerian terms even in the Schwarzschild spacetime. Unlike non-spinning particle, it produces a finite chance for resonance conditions due to the correspondence between \( \theta \) and \( \phi \) coordinates.
4.2 Determination of the resonant orbits

Having described various frequencies correspond to the small oscillation of a spinning particle, we can now demonstrate the resonant orbits depending on their spin parameters and black hole’s angular momentum. In Fig. [1], the resonant orbits are depicted as a function of the black hole’s momentum while the spin parameter takes different values. The significance of the resonant orders are also demonstrated for $r\theta$, $r\phi$ and $\theta\phi$ resonances in Fig. (1a), Fig. (1b), Fig. (1c) respectively. Given a larger order, the orbits move away from the horizon and as it is shown in Fig. (1), this is true for each of the resonance frequencies. Furthermore, it should also be emphasized that both $r\theta$ and $r\phi$ follow an identical behavior while the $\theta\phi$ consists with a stark contrast from them. While in case of the $r\theta$ and $r\phi$ resonances, the orbits begin to move closer to the event horizon as one increases the black hole’s momentum, nearly opposite phenomena happens for $\theta\phi$. This is related to the fact that a larger momentum of the black hole drags the prograde orbits close to its horizon while the opposite is true for retrograde orbits. Because of the presence of $r$ in both $r\theta$ and $r\phi$ resonances, this nature is largely influential in either of these cases. But in case of $\theta\phi$ resonance, this is no longer valid and the orbits behaves quite differently from $r\theta$ and $\theta\phi$ resonances. Apart from that, there exists another nontrivial feature corresponds to the $r\theta$ resonance which deserves further attention and clarification. Like it is shown in Fig. (1a), the resonant orbits belonged to the positive spin (solid curve in red, blue and brown colour) switch position with the negative spin as the resonance order increases accordingly. This signals that there exists a particular order of resonance where the locations of resonant orbits correspond to positive and negative spin exactly matches.

With this, we finish our discussions regarding the resonance phenomenon between the small oscillations frequencies for a spinning particle with its spin approximated up to the linear order term. In the upcoming section, we consider the $r\phi$ resonance correspond to a generic trajectory confined on the equatorial plane of Kerr black hole.

5 Motion on the equatorial plane : the $r\phi$ resonance

With the introduction to resonant orbits for spinning particle orbiting in nearly circular trajectories close to the equatorial plane, we grasp the basic mechanism to locate them as a function of the radial distance $r$. However, to address any general motion it is presumed that orbits are neither nearly circular nor close to the equatorial plane and an universal prescription is required to study them. In case of geodesics, these are well studied in literature since when Carter discovered the existence of a fourth conserved quantity and conclude that the geodesic trajectories are completely integrable in the Kerr background [40]. However, the addition of the dipole moment will introduce a perturbation in the Hamiltonian and the separability condition would no longer work. Therefore, a general orbit for a spinning particle given in terms of arbitrary $r$ and $\theta$ can not be established in any curved spacetime. Nonetheless for orbits confined on the equatorial plane, the Mathisson-Papapetrou equations can be exactly solved and thereby the resonance between the radial $\Omega_r$ and azimuthal $\Omega_\phi$ frequency can be well determined. The primary equation governing this resonance phenomena is given as $m\Omega_r - n\Omega_\phi = 0$ which further can be simplified as

$$m\pi - n \int_{r_p}^{r_\phi} \frac{d\phi}{dr} dr = 0$$

(33)
(a) The $r\theta$ resonant orbits are shown as a function of the black hole’s momentum while the spin of the particle takes different values.

(b) The above figure demonstrates the $r\phi$ resonance for different spin of the extended object.

(c) In the above figure, $\theta\phi$ resonance is depicted as a function of black hole’s momentum.

Figure 1: The above figures capture the location of resonant orbits for different spin parameters and black hole’s momentum. Distinguishing features related to various resonances are also depicted for different orders. The blue, red and brown colours are related to $1:2$, $2:3$ and $5:7$ resonance order respectively.
with $r_a$ and $r_p$ indicates the turning points within which the motion is confined. From the solutions of Mathisson-Papapetrou equations, we can express $\frac{d\phi}{dr}$ as

$$\frac{d\phi}{dr} = \frac{U^3}{U^1} = \frac{1}{\sqrt{R_s}} \left[ \left( 1 + \frac{3MS^2}{r\Sigma_s} \right) \{ J_z - (a + S)E \} + \frac{a}{\Delta} P_s \right],$$  \hspace{1cm} (34)

where $U^1$ and $U^3$ are radial and azimuthal velocity respectively and the quantities $P_s, R_s, \Sigma_s$ has already defined in Eq. (6).

In the upcoming sections, we shall explicitly employ the above expression to locate the $r\phi$ resonance orbits in both Schwarzschild and Kerr geometries. However before studying any resonance phenomena, it is advisable to notice that the radial velocity $U^1$ identically vanishes in the turning points $(r_a, r_p)$ and as a result the integral in Eq. (33) would diverge. Therefore to compute the integral we may use an approximate technique and in the present context, Sochnev method can be extremely useful [55] (we also refer Ref. [53] where this is used in case of a spinning particle). For convenience, the basic design of Sochnev method is given below.

Let us consider an irrational function given as

$$C = \sqrt{C_1C_2...C_m}$$  \hspace{1cm} (35)

and we desire to approximate it for our convenience. The first step of Sochnev method [55] dictates that the value of $C$ would fall within the upper limit $a_1$ and lower limit $b_1$ which are given by the following expressions

$$a_1 = \frac{C_1 + C_2 + ... + C_m}{2}, \quad b_1 = \frac{C_1C_2...C_m}{a_1^{m-1}}, \quad \text{and} \quad b_1 < C < a_1,$$  \hspace{1cm} (36)

while more accuracy can be achieved with next order terms. The final expression of the sequence $(a_n, b_n)$ can be written as

$$a_{n+1} = \frac{(m-1)a_n + b_n}{m}, \quad b_{n+1} = \frac{(a_n)^{m-1}b_n}{(a_{n+1})^{m-1}},$$  \hspace{1cm} (37)

and for large $n$, it is expected both $a_n$ and $b_n$ would generate extremely accurate values of $C$. However, in the present context we often neglect the higher order $n$ terms and truncate our series at $(a_1, b_1)$. Due to the nonzero spin terms the expressions become largely cumbersome to probe higher order terms with $n > 1$.

### 5.1 In the Schwarzschild black hole

Let us now describe the $r\phi$ resonance in the static and spherically symmetric Schwarzschild black hole and later on we discuss the Kerr geometry. For convenience, we start with the geodesic trajectories and then address the motion of a spinning particle.

#### 5.1.1 Geodesic limit:

Due to the spherical symmetry, geodesic trajectories in Schwarzschild spacetime are vastly simplified and easily obtainable in comparison to the rotating case. To study the resonance phenomena in the Schwarzschild background, we set $\theta = \pi/2$ without loosing any generality and obtain the radial potential as follows

$$V(r) = E^2r^4 - r^2L_z^2 + 2MrL_z^2 - (r^2 - 2Mr)r^2,$$  \hspace{1cm} (38)
where $E$ and $L_z$ are given as the conserved energy and momentum associated with the timelike and spacelike symmetries of the geometry respectively. Like mentioned earlier, at $r = r_a$ and $r = r_p$ the radial potential identically vanishes, i.e $V(r_a) = V(r_p) = 0$, and we can arrive at the expressions for $E$ and $L_z$ in terms of $r_a$ and $r_p$. These are given by

$$E^2_{\text{geo}} = \frac{(r_a + r_p)(2M - r_a)(2M - r_p)}{r_a r_p (r_a + r_p) - 2M(r_a^2 + r_p^2 + r_a r_p)}$$

and

$$L_z^2_{\text{geo}} = \frac{2Mr_a r_p^2}{r_a r_p (r_a + r_p) - 2M(r_a^2 + r_p^2 + r_a r_p)}.$$ (39)

Substituting these expression in the potential given by 38 we arrive at

$$V(r) = \frac{2M(r - r_a)(r - r_p)(r - 0)(r - r_c)\{r_a r_p - 2M(r_a + r_p)\}}{r_a r_p (r_a + r_p) - 2M(r_a^2 + r_p^2 + r_a r_p)},$$ (40)

where we have $r_c = \frac{2Mr_a r_p}{r_a r_p - 2M(r_a + r_p)}$ and the solutions follow $r_a > r_p > r_c$. From the expressions for energy and momentum given in Eq. [39], we may conclude

$$r_a r_p (r_a + r_p) > 2M(r_a^2 + r_p^2 + r_a r_p),$$ (41)

and therefore from Eq. [40], we arrive at

$$(r - r_a)(r - r_p)(r - r_c)\{r_a r_p - 2M(r_a + r_p)\} < 0.$$ (42)

Expanding Eq. [41], we finally have

$$\frac{2p^2 [p - M(3 + e^2)]}{(1 - e^2)^2} > 0,$$ (43)

and for $p > M(3 + e^2)$, it is always satisfied. On the other hand, to ensure a bound motion within $r_p < r < r_a$, we require to have $r_p > r_c$ and this would produce

$$\frac{p - 4M}{2M(1 + e)} > 1 \implies p > 2M(3 + e).$$ (44)

The above limitation on $p$ would automatically satisfies $r_a r_p - 2M(r_a + r_p) > 0$ and the Eq. (42) remain valid. Therefore, the final constraint on the semi-latus varies between $p = 6M$ (for circular orbit) to $p = 8M$ (highest eccentric orbit $e = 1$).

### 5.1.2 Spinning particle

For a non-vanishing spin of the particle, the rescaled potential can be written as

$$V_s(r) = r^4 R_s = r^4 \left[ (E^2 - 1)r^4 + 2Mr^3 - i^2 J_z^2 + 2rJ_z \left( M J_z + ES(r - 3M) \right) \right],$$ (45)

and by setting $S = 0$, we get back the potential for a geodesic. Furthermore, it should be emphasized that the above equation contains terms only linear in $S$ while the higher order terms are ignored for a convenient computation. We shall now concentrate in obtaining the expressions for energy and momentum assuming an ansatz of the form, $E_{shh} = E_{\text{geo}} + SE_s$ and $J_{shh} = L_{\text{geo}} + SJ_s$ is valid. With the prior knowledge about
In the discussions given below, we employ the Sochenev formalism that is earlier introduced in [5] and identify the linear spin approximation, this is given as

\[
J_s = \frac{\sqrt{(2M - r_a)(2M - r_p)(r_a + r_p)}}{[r_a r_p(r_a + r_p) - 2M(r_a^2 + r_p^2 + r_a r_p)]^{3/2}},
\]

\[
E_a = \frac{-M \sqrt{2Mr_a r_p}}{[r_a r_p(r_a + r_p) - 2M(r_a^2 + r_p^2 + r_a r_p)]^{3/2}}.
\]

Further employing the above relations into the potential given up to the linear order in spin, we arrive at

\[
V_s(r) = \frac{2Mr_a^5 a_0 (r - r_a)(r - r_p)}{[r_a r_p(r_a + r_p) - 2M(r_a^2 + r_p^2 + r_a r_p)]} \left[ r + \frac{b_0}{a_0} \right],
\]

where \( a_0 \) and \( b_0 \) are given by

\[
a_0 = - [r_a r_p - 2M(r_a + r_p)] \{ r_a r_p(r_a + r_p) - 2M(r_a^2 + r_p^2 + r_a r_p) \} - Sr_a r_p \{ 2M(r_a - 2M)(r_p - 2M)(r_a + r_p) \}^{1/2},
\]

\[
b_0 = 2Mr_a r_p \{ r_a r_p(r_a + r_p) - 2M(r_a^2 + r_p^2 + r_a r_p) \} - Sr_a r_p(r_a + r_p) \{ 2M(r_a - 2M)(r_p - 2M)(r_a + r_p) \}^{1/2}.
\]

It should be noted that even if the number of solutions remain identical with the geodesic case, there now exists explicit spin dependence in one of the solutions \( r_s = -b_0(a_0)^{-1} \). Finally, the potential can be compactly written in the form of

\[
V_s(s) = \frac{2Ma_0 r^4(r - 0)(r - r_a)(r - r_p)}{[r_a r_p(r_a + r_p) - 2M(r_a^2 + r_p^2 + r_a r_p)]^2},
\]

and identical to the geodesic case, we require to have \( V_s(s) > 0 \). This condition, along with the bound \( r_p < r < r_a \) would give \( r_p > r_s \) and \( a_0 < 0 \). However, either of these constraints would depend on the numerical value of the spin parameter and therefore while obtaining the resonant orbits, we will employ it explicitly.

With the above mentioned points kept in mind, we now determine \( \mathcal{U}^3(\mathcal{U}^1)^{-1} \) from Eq. (34) and in the linear spin approximation, this is given as

\[
\frac{d\phi}{dr} = \frac{(J_z - ES) \{ r_a r_p(r_a + r_p) - 2M(r_a^2 + r_p^2 + r_a r_p) \}}{[2Ma_0(r - 0)(r - r_a)(r - r_p)(r - r_s)]^{1/2}}.
\]

By substituting it in Eq. (33), we arrive at the following expression which describes the \( r\phi \) resonance for a given particle

\[
m\pi - n \int_{r_p}^{r_a} \left( \frac{d\phi}{dr} \right) dr = m\pi - nI_s = 0.
\]

In the discussions given below, we employ the Sochenev formalism that is earlier introduced in [5] and compute \( I_s \) approximately. However, for a convenient and simplified computation, we confine our discussions
only with the first order approximation, i.e, we assume $(C_1C_2)^{1/2} \approx 2^{-1}(C_1 + C_2)$ is valid throughout our calculations.

In order to obtain an approximate value for the above integral, we start with

$$\int_{r_p}^{r_a} \frac{d\phi}{dr} dr = N \int_{r_p}^{r_a} \frac{dr}{\sqrt{a_0(r-0)(r-r_a)(r-r_p)(r-r_s)}},$$  \hspace{1cm} (52)

where $N = (2M)^{-1/2}(J_z - ES) \{r_ar_p(r_a + r_p) - 2M(r_a^2 + r_p^2 + r_ar_p)\}$ and $a_0$ is already defined earlier. Let us now introduce a coordinate transformation given as $r = 2^{-1/2}(r_a + r_p)x + x(r_a - r_p)$ and with that, we arrive at

$$\int_{r_p}^{r_a} \frac{d\phi}{dr} dr = \int_{-1}^{1} \left( \frac{d\phi}{dx} \right) dx.$$  \hspace{1cm} (53)

Finally we have

$$\frac{d\phi}{dx} = \frac{N}{\{ -a_0(1-x^2)(r-0)(r-r_s) \}^{1/2}},$$

$$\frac{d\phi}{dx} = N \left\{ -a_0 \left( \frac{r_a - r_p}{2} \right)^2 \frac{(r_a + r_p - 2r_s)}{2} (1 - x^2)(1 + K_0x)(1 + K_sx) \right\}^{-1/2}.$$  \hspace{1cm} (54)

and after further simplification, we write it as follows

$$\frac{d\phi}{dx} = \left\{ -a_0 \left( \frac{r_a + r_p}{2} \right)^2 \frac{(r_a + r_p - 2r_s)}{2} (1 - x^2)(1 + K_0x)(1 + K_sx) \right\}^{-1/2}. $$  \hspace{1cm} (55)

With $<N>$ given by

$$<N> = 2N \left\{ -a_0 (r_a + r_p)(r_a + r_p - 2r_s) \right\}^{-1/2},$$  \hspace{1cm} (56)

the above can be written in a more compact form as

$$\frac{d\phi}{dx} = \frac{<N>}{\{ (1-x^2)(1 + K_0x)(1 + K_sx) \}^{1/2}},$$  \hspace{1cm} (57)

where we have, $K_0 = \frac{r_a - r_p}{r_a + r_p}$ and $K_s = \frac{r_a - r_p}{r_a + r_p - 2r_s}$. Furthermore, to ease our computations, we introduce the expression

$$\int_{-1}^{1} \frac{1}{\sqrt{1 - x^2(1 + Kx)}} dx = \frac{\pi}{\sqrt{1 - K^2}},$$  \hspace{1cm} (58)

which is only valid with $K < 1$. We can use the same technique in our case by writing $\sqrt{(1 + K_0x)(1 + K_sx)} \approx \left( 1 + \frac{K_0 + K_s}{2} \right) = 1 + Kx$ and therefore, the final expression becomes

$$\int_{-1}^{1} \frac{d\phi}{dx} dx = \frac{\pi <N>}{\sqrt{1 - K^2}}.$$  \hspace{1cm} (59)
With the above relation, the resonance condition takes the form

$$m - n \frac{<N>}{\sqrt{1 - K^2}} = 0.$$  \hspace{1cm} (61)

We shall now employ the above condition and locate the resonant orbits for a spinning particle moving around a Schwarzschild black hole. In Fig. (2), we demonstrate the same for both 2:3 (Fig. (2a)) as well as 1:2 (Fig. (2b)) resonance condition.

(a) The orbits for 2:3 resonance is given for both spinning and non-spinning particle. For a negative spin parameter of the particle, the resonant orbits appear at a larger range of $p$ while for positive spin, the opposite phenomenon appears.

(b) Orbits for 1:2 resonance are given in a Schwarzschild background. The distinguishing features for both positive and negative values of spin are given explicitly.

Figure 2: The above figure demonstrates various resonant orbits in a Schwarzschild spacetime for both spinning and non-spinning trajectories.

5.2 In the Kerr black hole

Having described the resonance phenomena in Schwarzschild background, we shall continue our discussions in the Kerr spacetime and as expected, in rotating geometry the motion of a spinning particle is additionally complicated. Therefore before delving into the details of spinning particle, we start with the geodesic case and then address the extended object in both linearized Kerr and full Kerr spacetime.

5.2.1 Geodesic limit:

To locate the resonant orbits for a geodesic, we need to employ the following relation

$$\frac{d\phi}{dr} = \frac{\ell^3}{\mu^3} = \frac{r^2 \left\{ L_z - aE + \frac{a}{\Delta} \left[ E(r^2 + a^2) - aL_z \right] \right\}}{\left\{ r^4 \left[ E(r^2 + a^2) - aL_z \right]^2 - \Delta \left[ r^6 + (L_z - aE)^2 r^4 \right] \right\}^{1/2}},$$  \hspace{1cm} (62)
in Eq. (33). With a more simplified form, the above can be written as

\[
\frac{d\phi}{dr} = \frac{\left\{ L_z - aE + \frac{a}{\Delta} \left[ E(r^2 + a^2) - aL_z \right] \right\}}{\left( E(r^2 + a^2) - aL_z \right)^2 - \Delta \left[ r^2 + (L_z - aE)^2 \right]}^{1/2},
\]

(63)

and the potential takes the form,

\[
V(r) = \left[ E(r^2 + a^2) - aL_z \right]^2 - \Delta \left[ r^2 + (L_z - aE)^2 \right].
\]

(64)

We simplify the above further and arrive at

\[
V(r) = r \left[ E^2 r^3 - (L_z^2 - a^2 E^2) r + 2M(L_z - aE)^2 - (r^2 - 2Mr + a^2) r \right] = -\alpha(r - 0)(r - r_a)(r - r_p)(r - r_1),
\]

(65)

with \( \alpha = 1 - E^2 \). Let us now express energy and momentum in terms of the \( r_a, r_p \) and \( r_1 \) and it is given as

\[
E^2 = 1 - \frac{2M}{r_a + r_p + r_1}, \quad \text{and} \quad L_z^2 = \frac{2M}{r_a + r_p + r_1} \left\{ r_a r_p + r_1(r_a + r_p) - a^2 \right\},
\]

(66)

and the solution \( r_1 \) can be evaluated from the other condition

\[
4aMEL_z = 2Ma^2E^2 + 2ML_z^2 - (1 - E^2)r_1 r_a r_p.
\]

(67)

By solving the above equation numerically for a given eccentricity \( e \) and semi-latus rectum \( p \), we obtain \( r_1 \) and therefore, determine both energy and momentum correspond to the orbit. Henceforth, for any provided \( r_a \) and \( r_p \) we can obtain the conserved quantities require to specify the geodesic trajectory. It should be reminded that Eq. (67) is a quadratic of \( r_1 \) and these two solutions represent either prograde and retrograde orbits. Like mentioned earlier, as we are interested in the prograde orbit, we would suppress the retrograde part and continue our analysis with that motivation. Nonetheless, a similar study can be carried in the case of retrograde orbits too. The next task is to compute the integral given in Eq. (33) and locate the resonant orbits for an arbitrary order and this is carried out in the Appendix for both spinning and non-spinning trajectories. In Fig. (3), we have shown the resonant orbits for a geodesic moving in a spacetime with different rotation parameters \( a \). For smaller values of \( a \), the resonant orbits lie away from the black hole horizon and the nature remains similar to the Schwarzschild case. With the increase of resonance order, i.e, \( m + n \), the orbits keep moving away from the horizon but contained with identical properties as given in Fig. (3a). However, for larger momentum, the peak seems to disappears and the Gaussian nature changes to linear curve. This is explicitly shown in Fig. (3b).

### 5.2.2 Spinning particle: In the slowly rotating Kerr black hole

With the discussions presented above for geodesic trajectories, we shall now consider the case with spinning particles and locate their resonance orbits. However before dealing with the Kerr spacetime consists with the multipole structure of arbitrary order, we first consider only the dipole current moment and approach analytically to obtain the expressions for energy and momentum. In the usual Boyer-Lindquist coordinates, the resultant metric takes the following form

\[
ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) - \frac{4Ma^2 \sin^2 \theta}{r} d\phi dt,
\]

(68)
(a) The resonant orbits are given for $a = 0.1M$. With the increase in eccentricity, the semi-latus rectum increases and reaches a maximum value at $p = p^{\text{max}}$ while further increase in eccentricity decreases $p$ as well.

Figure 3: The above figure demonstrates various resonant orbits in the Kerr spacetime for non-spinning trajectories.

with $a$ and $M$ given as the angular momentum per mass and mass of the spacetime respectively. For the above metric, we obtain the radial potential written up to the terms linear in $S$ as

$$V_s(r) = r^3 \left[ r \left\{ E^2 r^4 - J_z^2 r^2 - 2M r J_z^2 - 4aM r E J_z - (r^2 - 2Mr)^2 \right\} + S (2EJ_z r^3 - 6EM J_z r^2 + 2aM J_z^2 + 6Ma E^2 J_z) \right], \quad (69)$$

and after further simplifications, we arrive at the following expression

$$V_s(r) = r^3 \left[ r \left\{ (E^2 - 1) r^4 + 2M r^3 - r^2 J_z^2 + 2r \left( M J_z^2 + E J_z S (r - 3M) + 3a M E^2 S - 2a M E J_z \right) \right\} + 2a M S J_z^2 \right]. \quad (70)$$

We shall now assume, $E = E_{\text{bh}} + a E_s$ and $J_z = J_{\text{bh}} + a J_s$ and solve for $E_s$ and $J_s$ from the equations $V_s(r_a) = V_s(r_p) = 0$. Due to the complexity of the forms of both $E_s$ and $J_s$, they are no explicitly written in the present paper. By substituting them in Eq. (70), we can write down the potential $V_s(r)$ as

$$V_s(r) = V_s^{\text{bh}} (r) + a V_s^{\text{kerr}} (r), \quad (71)$$

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where, $V^{\text{bh}}_s(r)$ and $V^{\text{kerr}}_s(r)$ can be written as

\begin{align}
V^{\text{bh}}_s(r) &= \frac{2Mr^2(r-r_a)(r-r_p)}{\mathcal{X}^3} \left[ \mathcal{X}^2 \left( 2M \{r_a r_p + r(r_a + r_p)\} - r_a r_p (r + r_a + r_p) \right) - \mathcal{Y}^{1/2} \mathcal{X} S \right], \\
V^{\text{kerr}}_s(r) &= \frac{2M(r-r_a)(r-r_p)F(r)}{\mathcal{X}^3},
\end{align}

and $F(r)$, $\mathcal{X}$ and $\mathcal{Y}$ has the following expressions

\begin{align}
F(r) &= -2r^2 r_a r_p \mathcal{X} \mathcal{Y} (r + r_a + r_p) + r^2 \left( 2M \{r_a r_p + r(r_a + r_p)\} - r_a r_p (r + r_a + r_p) \right) \mathcal{X}^2 - \mathcal{Y}^{1/2} \mathcal{X} S, \\
\mathcal{X} &= r_a r_p (r + r_a + r_p) - 2M (r_a^2 + r_a r_p + r_p^2), \quad \mathcal{Y} = 2M (2M - r_a)(2M - r_p)(r + r_a + r_p).
\end{align}

Therefore, the complete form of the potential can be written as

\begin{align}
V_s(r) &= \frac{2Mr^3(r-r_a)(r-r_p)}{\mathcal{X}^3} \left[ F(r) + r^2 \mathcal{X}^2 \left( 2M \{r_a r_p + r(r_a + r_p)\} \right) - r^2 r_a r_p (r + r_a + r_p) \mathcal{Y}^{1/2} \mathcal{X} S \right], \\
&= \frac{2Mr^3(r-r_a)(r-r_p)}{\mathcal{X}^3} \left[ \alpha'(r-r_1)(r-r_2)(r-r_3) \right], \\
&= -\alpha r^3 (r-r_a)(r-r_p)(r-r_1)(r-r_2)(r-r_3),
\end{align}

where, $\alpha = -2Ma'/\mathcal{X}^3$. In principle, one can solve for $r_1$, $r_2$ and $r_3$ for different values of $r_a$ and $r_p$. This way, we can obtain the resonant orbits for a spinning particle while neglecting the higher spin contribution, i.e $O(a^2)$, from the black hole. In the next discussion, we consider the Kerr black hole and numerically present the resonant orbits for a particle with spin.

### 5.2.3 Spinning particle: the Kerr spacetime

In this case, the radial potential can be written in the following form

\begin{align}
V_s(r) &= r^4 \left[ E^2 r^4 - (L_z^2 - a^2 E^2) r^2 + 2Mr(L_z - aE)^2 - (r^2 - 2Mr + a^2)^2 \right] \\
&\quad + Sr^3 \left\{ 2a^2 ME^2 - 4a^2 MEJ_z + 2aMJ_z^2 + 6aE^2 Mr^2 + 2EJ_z r^3 - 6EJ_z Mr^2 \right\} \\
&= r^3 \left[ E^2 r^4 - (L_z^2 - a^2 E^2) r^2 + 2Mr(L_z - aE)^2 - (r^2 - 2Mr + a^2)^2 \right] + \\
&\quad S \left\{ 2a^2 ME^2 - 4a^2 MEJ_z + 2aMJ_z^2 + 6aE^2 Mr^2 + 2EJ_z r^3 - 6EJ_z Mr^2 \right\} \\
&= -\alpha r^3 (r-r_1)(r-r_2)(r-r_a)(r-r_p)(r-r_3),
\end{align}

and we now estimate each solutions by equating the left hand and right hand sides. Therefore, we arrive at the following set of equations

\begin{align}
\text{(75)}
\end{align}
1. From coefficient of $r^5$, we have $\alpha_s = 1 - E^2$.

2. From coefficient of $r^4$, we manage to have

$$r_1 + r_2 + r_a + r_p + r_3 = \frac{2M}{\alpha_s}.$$  \hspace{1cm} (76)

3. From coefficient of $r^3$, we have

$$a^2(E^2 - 1) - J_z^2 + 2EJ_zS = -\alpha \{r_ar_p + r_1r_2 + (r_a + r_p)(r_1 + r_2) + r_3(r_1 + r_2 + r_a + r_p)\}. \hspace{1cm} (77)$$

4. From coefficient of $r^2$, we obtain

$$2Ma^2E^2 - 4aMEJ_z + 2MJ_z^2 + 6aE^2MS - 6EMSJ_z = \alpha_s \left\{ r_ar_p, (r_1 + r_2) \left( r_ar_p + r_3(r_a + r_p) \right) \right\} + r_1r_2(r_3 + r_a + r_p). \hspace{1cm} (78)$$

5. From coefficient of $r$, we determine

$$r_2r_3r_ar_p + r_1r_3r_ar_p + r_1r_2 \{ r_ar_p + r_3(r_a + r_p) \} = 0. \hspace{1cm} (79)$$

6. From the coefficient of $r^0$, we get

$$(2ME^2a^3 - 4aMEJ_z a^2 + 2MaJ_z^3)S = r_1r_2r_3r_ar_p \alpha_s. \hspace{1cm} (80)$$

With $r_+ = r_1 + r_2$ and $r_\times = r_1r_2$, we can further compute $E$ from Eq. (76),

$$E^2 = 1 - \frac{2M}{r_+ + r_a + r_p + r_3}. \hspace{1cm} (81)$$

From Eq. (79), we can further compute $r_3$ as

$$r_3 = -\frac{r_\times r_ar_p}{r_\times (r_a + r_p) + r_ar_p r_+}. \hspace{1cm} (82)$$

To obtain $J_z$, we can introduce the expression $X_{\pm} = 2Ma^2$ × Eq. (77) ± Eq. (80) and with the + sign, we arrive at

$$J_z^2 = a(a - S)(r_+ + r_a + r_p + r_3)^{-1} \left\{ 2a^2M \left[ r_3(r_a + r_p) + r_ar_p + r_\times + r_+(r_a + r_p + r_3) \right] - 2Ma^4 - r_\times r_3r_ar_p + a^3S(r_a + r_p + r_3 + r_+ - 2M) \right\}. \hspace{1cm} (83)$$

Along with the expressions $X_-$ and Eq. (78), we can numerically solve for both $r_1$ and $r_2$ for given values of $e$ and $p$. Therefore, whenever provided with $r_a$ and $r_p$, we can compute energy and momentum corresponding to a spinning particle confined on the equatorial plane of the Kerr black hole. The next task is to determine the integral given in Eq. (33) and for a general Kerr spacetime, this is given in the Appendix.

For convenience, we carry out the calculations within the linear framework of the spin parameter, while a general treatment shall be carried out elsewhere. In Figs (4, 5) we have shown resonant orbits for different orders of resonance while the angular momentum of the Kerr black hole varies accordingly. For a particle with spin $S = 0.1M$, various resonant orbits for $a = 0.099M$ and $a = 0.9M$ are given in Fig. (4a) and Fig. (4b) respectively. The studies related to spin $S = -0.1M$ are shown in Fig. (5) for different resonance and black hole’s momentum.
Semi-latus rectum ($p$)

Figure 5: Above figure represents the resonant orbits for particles with spin $S = -0.1M$. The influence of different angular momentum of the black hole is also depicted.

(a) The resonant orbits for a spinning particle is shown in a black hole with angular momentum $a = 0.099M$.

(b) Resonant orbits are shown while the black hole has an angular momentum of $a = 0.9M$. 

Figure 4: Above figure represents the resonant orbits for particles with spin $S = 0.1M$. The influence of different angular momentum of the black hole is also depicted.
6 Concluding remarks

In the present article we have studied the trajectories of a spinning particle and discuss their orbital resonance in the Kerr background. We typically explore two particular events, first the resonance in-between small oscillation frequencies and second, $r\phi$ resonance on the equatorial plane of the Kerr black hole. We confined our discussions for a pole-dipole particle, i.e, a particle with nonzero dipole moment while all the higher order moments are set to zero. In addition, the entire study is only valid for a linearized spin parameter and any contribution appears at $O(S^2)$ and higher are ignored for convenience. The key findings of the present article can be summarized as follows.

The first part of the paper deals with the quasi periodic oscillations for an extended object around the Kerr black hole. In the presence of non-vanishing spin, the expressions for small oscillation frequencies would change and therefore, the locations of resonant orbits would also shift. To arrive at the small oscillation frequencies about the radial and angular directions, we would require the prior information about the respective radial and angular potential of the particle. While the radial potential on the equatorial plane can be easily computed as the motion of spinning particle can be exactly solved with $\theta = \pi/2$, the angular potential can be tricky to obtain. For that, we employed the expression corresponds to the Carter constant of a spinning particle and obtain the angular potential $V_s(\theta)$ for a circular orbit. The modified relations for fundamental frequencies can be further engaged to describe accretion phenomenon around a black hole spacetime which could be useful from observational perspective.

The second part deals with the $r\phi$ resonance considering the trajectories are completely confined on the equatorial plane. We locate the resonant orbits for different values of the black hole’s momentum as well as the particle’s spin. Given the angular momentum of the black hole is small, the resonant orbits has a weak dependency on the eccentricity $e$. It is explicitly demonstrated in Fig. (4a) and Fig. (5a). However, the $e$ dependency increases as one increases the value of angular momentum of the black hole as it is shown in Fig. (4b) and Fig. (5b). The nature of resonant orbits are also largely effected by the black hole rotation as depicted in Fig. (4) and Fig. (5). For a small or no rotation of the black hole (Fig. (2), Fig. (4a), Fig. (5a)), the semi-latus rectum steadily increases with the eccentricity $e$ and attains a maximum value for a particular eccentricity say $e = e_{\text{max}}$. Further increase in the eccentricity would result in the decrease of the semi-latus rectum $p$ as shown in Figs. (2, 4a, 5a). On the other hand, for black holes with larger angular momentum, $p$ steadily increases with the eccentricity $e$ and attains the highest value as $e$ approaches unity. Furthermore, the orbits shift towards the event horizon as one either increases the momentum of the black hole or decreases the order of resonance.

The immediate follow up of the present work would be consider the $r\theta$ resonance which requires a numerical framework. It would also be interesting to study the large spin effects on the resonant orbits.

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Detail calculations to obtain the resonant orbits

In the Kerr black hole, we obtain

\[ \frac{\mathcal{U}^3}{\mathcal{U}^1} = \frac{d\phi}{dr} = \frac{r^2 \{ J_z - (a + S)E \} + \frac{aP_s r^2}{\Delta}}{\left[ V_s(r) \right]^{1/2}}, \quad (84) \]

with \( V_s(r) = -\alpha_s r^3(r - r_1)(r - r_2)(r - r_a)(r - r_p)(r - r_3). \) We can split the above integration into two parts, the first one is \( G \) and second one is \( R \) and these are given as

\[ G = \frac{r^2 \{ J_z - (a + S)E \}}{\left[ V_s(r) \right]^{1/2}} = \frac{r \{ J_z - (a + S)E \}}{-\alpha_s(r - 0)(r - r_1)(r - r_2)(r - r_a)(r - r_p)(r - r_3)^{1/2}}, \]

\[ R = \frac{ar^2 P_s}{\Delta \left[ V_s(r) \right]^{1/2}} = \frac{a \left\{ Er^3 + ar(a + S - J_z) + MS(a - J_z) \right\}}{(r - r_H)(r - r_H^r) \left[ -\alpha_s(r - 0)(r - r_1)(r - r_2)(r - r_a)(r - r_p)(r - r_3)^{1/2}. \right.} \]

(85)

Furthermore, with the substitution [53]

\[ r = \frac{(r_a + r_p) + x(r_a - r_p)}{2}, \quad (86) \]

we arrive at the following expression

\[ \frac{d\phi}{dx} = \frac{r_a - r_p}{2} \frac{d\phi}{dx} = \frac{r_a - r_p}{2} (G_x + R_x). \quad (87) \]

The \( G \) becomes \( G_x \) and given as

\[ G_x = \frac{r_x \{ J_z - (a + S)E \}}{\left[ -\alpha(r_x - 0)(r_x - r_1)(r_x - r_2)(r_x - r_a)(r_x - r_p)(r_x - r_3)^{1/2}. \right.} \]

(88)

By using the fact,

\[ r_x - r_i = \frac{r_a + r_p - 2r_i}{2} (1 + K_i x), \quad (89) \]
with \( K_i = \frac{r_a - r_p}{r_a + r_p - 2r_i} \) where \( i \) runs from 0 to 3 and \( r_0 = 0 \), the above equation reads as

\[
G_x = \frac{1}{2}(r_a + r_p)(1 + K_0 x) \{ J_z - (a + S) E \} \left\{ \frac{2^{-6} a_s (1 - x^2)(1 + K_0 x)(1 + K_1 x)(1 + K_2 x)(1 + K_3 x)}{(r_a + r_p)(r_a - r_p)^2(r_a + r_p - 2r_1)(r_a + r_p - 2r_2)(r_a + r_p - 2r_3)} \right\}^{-1/2},
\]

\[
= \frac{N_k (1 + K_0 x) \{ J_z - (a + S) E \}}{\sqrt{(1 - x^2)(1 + K_0 x)(1 + K_1 x)(1 + K_2 x)(1 + K_3 x)}},
\]

(90)

In the above, we assume

\[
N_k = \frac{4(r_a + r_p)}{\sqrt[\alpha](r_a + r_p)(r_a - r_p)^2(r_a + r_p - 2r_1)(r_a + r_p - 2r_2)(r_a + r_p - 2r_3)} = \frac{2 < N_k >}{r_a - r_p},
\]

(91)

with

\[
< N_k > = \frac{2(r_a + r_p)}{\sqrt[\alpha](r_a + r_p)(r_a + r_p - 2r_1)(r_a + r_p - 2r_2)(r_a + r_p - 2r_3)}.
\]

(92)

Therefore, we can finally write

\[
G_x = \frac{2 < N_k >}{r_a - r_p} \frac{(1 + K_0 x) \{ J_z - (a + S) E \}}{\sqrt{(1 - x^2)(1 + K_0 x)(1 + K_1 x)(1 + K_2 x)(1 + K_3 x)}},
\]

(93)
The other part can be written as

\[
R_x = \frac{a \left\{ Er_x^3 + ar_x(a + S - J_z) + MS(a - J_z) \right\}}{(r_x - r^-_H)(r_x - r^-_H) \left[-\alpha_s(r_x - 0)(r_x - r_1)(r_x - r_2)(r_x - r_3)(r_x - r_4)(r_x - r_3)^{1/2} \right]}
\]

\[
= \frac{a}{E} Er_x^3 + a(a + S - J_z)r_x + MS(a - J_z) \left\{ 2^{-6}\alpha_s(1 - x^2)(1 + K_0x)(1 + K_1x)(1 + K_2x)(1 + K_3x) \right\}
\]

\[
(1 + K_0x)^3(r_a + r_p)^3 + 4a(a + S - J_z)(r_a + r_p)(1 - K_0x) + 8MS(a - J_z)
\]

\[
\frac{2}{r_a - r_p} \frac{(r_a + r_p - 2r^-_H)(r_a + r_p - 2r^-_H)(r_a + r_p - 2r^-_H)(r_a + r_p - 2r^-_H)}{(1 + K_0^+ x)(1 + K^- x) \sqrt{(1 - x^2)(1 + K_0x)(1 + K_1x)(1 + K_2x)(1 + K_3x)}}
\]

where we assume

\[
< N_k > = \frac{4a \times 4}{(r_a + r_p - 2r^-_H)(r_a + r_p - 2r^-_H)(r_a + r_p - 2r^-_H)(r_a + r_p - 2r^-_H)} \times \frac{Er_x^3 + a(a + S - J_z)r_x + MS(a - J_z)}{(1 + K_0^+ x)(1 + K^- x) \sqrt{(1 - x^2)(1 + K_0x)(1 + K_1x)(1 + K_2x)(1 + K_3x)}}
\]

The final equation can now be written as

\[
\frac{d\phi}{dx} = \frac{< N_k >}{\sqrt{(1 - x^2)(1 + K_0x)(1 + K_1x)(1 + K_2x)(1 + K_3x)}} \frac{(1 + K_0x)\{J_z - (a + S)E\}}{\sqrt{(1 - x^2)(1 + K_0x)(1 + K_1x)(1 + K_2x)}}
\]

\[
= \frac{< N_k >}{\sqrt{(1 - x^2)(1 + K_0x)(1 + K_1x)(1 + K_2x)}} \frac{(1 + K_0x)\{J_z - (a + S)E\}}{\sqrt{(1 - x^2)(1 + K_0x)(1 + K_1x)(1 + K_2x)}} + \frac{E(1 - K_0x)^3(r_a + r_p)^3 + 4a(a + S - J_z)(r_a + r_p)(1 - K_0x) + 8MS(a - J_z)}{8(1 + K_0^+ x)(1 + K^- x) \sqrt{(1 - x^2)(1 + K_0x)(1 + K_1x)(1 + K_2x)(1 + K_3x)}}
\]

By employing the fact, \( \sqrt{(1 + K_0x)(1 + K_1x)} = 1 + \frac{(K_0 + K_1)}{2} x = 1 + K_1x \) and \( \sqrt{(1 + K_2x)(1 + K_3x)} = 1 + K_2x \), the above equation can be written as

\[
\frac{d\phi}{dx} = \frac{< N_k >}{\sqrt{(1 - x^2)(1 + K_0x)(1 + K_1x)(1 + K_2x)}} \frac{(1 + K_0x)\{J_z - (a + S)E\}}{\sqrt{(1 - x^2)(1 + K_0x)(1 + K_1x)(1 + K_2x)}}
\]

\[
+ \frac{< N_k >}{\sqrt{(1 - x^2)(1 + K_0x)(1 + K_1x)(1 + K_2x)}} \frac{E(1 - K_0x)^3(r_a + r_p)^3 + 4a(a + S - J_z)(r_a + r_p)(1 - K_0x) + 8MS(a - J_z)}{8\sqrt{(1 - x^2)(1 + K_0^+ x)(1 + K^- x)(1 + K_1x)(1 + K_2x)}}
\]

(96)
We further write
\[
\frac{1}{(1 + K_1 x)(1 + K_2 x)} = \frac{1}{K_2 - K_1} \left\{ \frac{K_2}{1 + K_2 x} - \frac{K_1}{1 + K_1 x} \right\},
\]  
and also
\[
\frac{1}{(1 + K_H^+ x)(1 + K_H^- x)(1 + K_1 x)(1 + K_2 x)} = \frac{A}{1 + K_H^- x} + \frac{B}{1 + K_H^+ x} + \frac{C}{1 + K_1 x} + \frac{D}{1 + K_2 x},
\]
with \( A, B, C \) and \( D \) has the following expressions
\[
A = \frac{(K_H^+)^3}{(K_H^+ - K_H^-)(K_H^+ - K_1)(K_H^+ - K_2)}, \quad B = \frac{(K_H^-)^3}{(K_H^- - K_H^+)(K_H^- - K_1)(K_H^- - K_2)},
\]
\[
C = \frac{(K_1)^3}{(K_1 - K_H^+)(K_1 - K_H^-)(K_1 - K_2)}, \quad D = \frac{(K_2)^3}{(K_2 - K_H^+)(K_2 - K_H^-)(K_2 - K_1)}.
\]

With all the above machinery employed, we arrive at the following expression
\[
\frac{d\phi}{dx} = \begin{align*}
&<N_k> \left\{ \frac{J_z - (a + S)E}{K_2 - K_1} \right\} \left\{ \frac{K_2(1 + K_0 x)}{\sqrt{1 - x^2(1 + K_2 x)}} - \frac{K_1(1 + K_0 x)}{\sqrt{1 - x^2(1 + K_1 x)}} \right\} + \\
&\frac{<N_k>}{8} \left\{ E(r_a + r_p)^3 \left( \frac{A(1 - K_0 x)^3}{\sqrt{(1 - x^2)(1 + K_1 x)}} + \frac{B(1 - K_0 x)^3}{\sqrt{(1 - x^2)(1 + K_2 x)}} \right) + \frac{C(1 - K_0 x)^3}{\sqrt{(1 - x^2)(1 + K_1 x)}} + \frac{D(1 - K_0 x)^3}{\sqrt{(1 - x^2)(1 + K_2 x)}} \right\} + <N_k> a(a + S - J_z)(r_a + r_p) \begin{cases} \\
2
\end{cases} \\
&+ MS(a - J_z) <N_k> \left\{ \frac{A}{\sqrt{(1 - x^2)(1 + K_1 x)}} + \frac{B}{\sqrt{(1 - x^2)(1 + K_2 x)}} + \frac{C}{\sqrt{(1 - x^2)(1 + K_1 x)}} + \frac{D}{\sqrt{(1 - x^2)(1 + K_2 x)}} \right\}.
\end{align*}
\]

Now we have to do the calculation \( \int_{-1}^{1} \left( \frac{d\phi}{dx} \right) dx \) and to do so, we have used the following equations
\[
I_1(K) = \int_{-1}^{1} \frac{dx}{\sqrt{1 - x^2(1 + K x)}} = \frac{\pi}{\sqrt{1 - K^2}},
\]
\[
I_2(K) = \int_{-1}^{1} \frac{(1 + K_0 x)dx}{\sqrt{1 - x^2(1 + K x)}} = \frac{\pi}{K} \left\{ \frac{K - K_0}{\sqrt{1 - K^2}} + K_0 \right\},
\]
\[
I_3(K) = \int_{-1}^{1} \frac{(1 + K_0 x)^3 dx}{\sqrt{1 - x^2(1 + K x)}} = \frac{\pi}{K^3} \left\{ \frac{(K - K_0)^3}{\sqrt{1 - K^2}} - \left[ \frac{(K - K_0)^3 - K^2(K_0^3 + 2K)}{2} \right] \right\}.
\]
With the above informations, we can write the final equation as follows

\[
\int_{-1}^{1} d\phi dx = \langle N_k \rangle \left\{ \frac{J_z - (a + S)E}{K_2 - K_1} \left( K_2 I_2(K_2) - K_1 I_2(K_1) \right) + \right.
\]
\[
\langle N_k \rangle \left\{ \frac{1}{8} \left( E(r_a + r_p)^3 \right) \left[ A I_2(K_H^+) + B I_2(K_H^-) + C I_3(K_1) + D I_3(K_2) \right] \right. + \left. \frac{1}{2} a(a + S - J_z)(r_a + r_p) \left[ A I_2(K_H^+) + B I_2(K_H^-) + C I_2(K_1) + D I_2(K_2) \right] \right. 
\]
\[
+ MS(a - J_z) \langle N_k \rangle \left\{ A I_1(K_H^+) + B I_1(K_H^-) + C I_1(K_1) + D I_1(K_2) \right\}. \quad (103)
\]

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