STRING ORIENTATIONS OF SIMPLICIAL HOMOLOGY MANIFOLDS

JACK MORAVA

Abstract. Simplicial homology manifolds are proposed as an interesting class of geometric objects, more general than topological manifolds but still quite tractable, in which questions about the microstructure of space-time can be naturally formulated. Their string orientations are classified by $H^3$ with coefficients in an extension of the usual group of D-brane charges, by cobordism classes of homology three-spheres with trivial Rokhlin invariant.

[This note grew out of the May 08 AIM workshop on algebraic topology and physics. Thanks to Hisham Sati for organizing that meeting, and to the participants for enduring multiple fragmentary presentations of these ideas. I owe M. Ando, D. Christensen, M. Furuta, N. Kitchloo, A. Ranicki, R. Stern, L. Taylor and B. Williams particular thanks for help (without their bearing any responsibilities for remaining howlers).]

1. Notation and background

1.1 In homotopy-theoretic terms, a string structure on a smooth manifold $M$ is a lift

\[
\begin{array}{ccc}
& & BO \langle 8 \rangle \\
& \downarrow & \\
M & \to & BO
\end{array}
\]

defined by the map classifying the stable tangent bundle of $M$; where the 7-connected cover $BO \langle 8 \rangle$ of $BO$ is the fiber

\[
BO \langle 8 \rangle \to BO \to (BO)_{(7)}
\]

of the map to its Postnikov approximation [27] having homotopy groups concentrated in degrees seven and below. Alternately:

\[
(BO)_{\langle 8 \rangle} = B(O_{\langle 7 \rangle})
\]

is the classifying space for the topological group

\[
O_{\langle 7 \rangle} \to O \to O_{\langle 6 \rangle}
\]

Date: 31 Oct 2009.

2000 Mathematics Subject Classification. Primary 57R55, Secondary 81T30.

The author was supported in part by AIM and the NSF.
whose homotopy groups agree with those of $O$ in degrees greater than or
equal to seven.

A string structure can thus be interpreted as a ‘reduction’ of the structure
group of the stable tangent bundle of $M$, from $O$ to $O\langle 7 \rangle$, just as a spin
structure is similarly a ‘reduction’ of that structure group to

$$\text{Spin} = O\langle 2 \rangle \to O \to \mathbb{Z}_2 \times H(\mathbb{Z}_2, 1).$$

An oriented manifold $M$ admits a spin structure iff the Stiefel-Whitney class

$w_2 = 0$; similarly, the existence of a string structure requires the vanishing
of a version $p_1/2$ of the Pontrjagin class defined for spin manifolds. The set
of spin structures on $M$ admits a transitive free action of $H^1(M, \mathbb{Z}_2)$, and by
essentially the same argument the set of string structures is an $H^3(X, \mathbb{Z})$-
torsor. The associated twisted $K$-groups of $X$ are natural repositories [7]
for Ramond-Ramond $D$-brane charges.

The interest in string orientations comes from quantum field theory,
where they were recognized as necessary to define an analog of the Dirac
operator on the space $LM$ of free loops on $M$, but mathematical interest
[1] in their properties goes back to the early 70’s [16]. There is a parallel
interest in the representation of three-dimensional cohomology classes in
local geometric terms, analogous [6] to the description of two-dimensional
classes by complex line bundles.

1.2 If $G$ is a connected, simply-connected simple Lie group, then

$$\pi_3(G) = \mathbb{Z}$$

by a classical theorem of Bott. A theorem of Kuiper says that the group
$\text{Gl}(\mathbb{H})$ of invertible bounded operators on an (infinite-dimensional) Hilbert
space is contractible; the projective general linear group

$$\text{PGL}(\mathbb{H}) = \text{Gl}(\mathbb{H})/\mathbb{C}^\times \sim BT$$

is therefore an Eilenberg-MacLane space of type $H(\mathbb{Z}, 2)$, from which it
follows that the classifying space for $\text{PGL}(\mathbb{H})$-bundles is an EM space of type
$H(\mathbb{Z}, 3)$. The associated bundle

$$1 \to \text{PGL}(\mathbb{H}) \to G\langle 4 \rangle \to G \to 1$$

is an extension of topological groups [28]. The following construction is due
to Kitchloo [18, appendix]:

A level one projective representation of the loop group of $G$ on $\mathbb{H}$ defines
a homomorphism to $\text{PGL}(\mathbb{H})$ which pulls $\text{Gl}(\mathbb{H})$ back to a version of the
universal central extension of $LG$. Let $\mathcal{A}$ denote the topological Tits building
of $LG$, modeled by the contractible space of connections on a trivial $G$-
bundle over a circle; the pointed loops act freely on it, and the holonomy
map makes it a principal $\Omega G$ bundle. The diagonal action of $LG$ on

$$\mathcal{A} \times_{\Omega G} \text{PGL}(\mathbb{H}) := G\langle 4 \rangle$$
factors through an action of $G$ on $G\langle 4 \rangle$ lifting the action of $G$ on itself by conjugation.

Note that because $\text{PGL}(\mathbb{H})$ is the group of automorphisms of $\text{Gl}(\mathbb{H})$, the cohomology group

$$H^1(X, \text{PGL}(\mathbb{H})) \cong H^1(X, H(\mathbb{Z}, 2)) \cong H^3(X, \mathbb{Z})$$

can be interpreted as a Brauer group of bundles of $C^*$-algebras (up to Morita equivalence) over $X$. A refinement of this argument [21] represents the Brauer group of bundles of graded $C^*$-algebras over $X$ by the generalized Eilenberg-MacLane space

$$H(\mathbb{Z}_2, 1) \times H(\mathbb{Z}, 3),$$

which suggests interpreting $O\langle 7 \rangle = O\langle 4 \rangle$ in terms of a bundle of such algebras over $(\text{SO})$.

2. Simplicial homology manifolds

2.1 I also want to thank Kitchloo for observing that in low dimensions, the map

$$\text{PL}/O \to B\text{O} \to B\text{PL}$$
is almost an equivalence: the homotopy groups of the fiber are the Kervaire-Milnor groups of differentiable structures on spheres, which (aside from the smooth 4D Poincaré conjecture . . . ) are trivial below dimension seven. This implies that a string structure on a smooth manifold is the same as a smoothing of a topological manifold endowed with a lift

$$B\text{Top}\langle 8 \rangle \leftarrow B\text{Top} \leftarrow M$$
of the map classifying its tangent topological block bundle. The theorem

$$\text{Top}/\text{PL} \sim H(\mathbb{Z}_2, 3)$$
of Kirby and Siebenman [24] seems also to point in this direction.

2.2 The classification of string structures on geometric objects more general than smooth manifolds is accessible nowadays, thanks to many researcher-years of deep work related to the Hauptvermutung, suggesting that questions like ‘Who ordered the differentiable structure’ may not be out of reach. I will summarize some background from Ranicki’s elegant account [22], but in some cases I’ll use terminology from [15]:

**Definition:** A space $X$ is a $d$-dimensional homology manifold iff for any $x \in X$,

$$H_*(X, X - \{x\}; \mathbb{Z}) \cong H_*(S^{d-1}; \mathbb{Z});$$
but a simplicial homology manifold is a simplicial complex $K$ such that for any $k$-simplex $\sigma \in K$, 

$$H_*(\text{link}_K(\sigma); \mathbb{Z}) \cong H_*(S^{d-k-1}; \mathbb{Z}).$$

The polyhedron $|K|$ of $K$ is a homology manifold iff $K$ is a simplicial homology manifold. A manifold homology resolution $f : M \to X$ of a space $X$ is a compact topological manifold $M$ together with a surjective map $f$ with acyclic point inverses.

The element $\kappa_4(K) \in H^k(|K|, \Theta_{k-1})$ Poincaré dual to the cycle 

$$\sum_{|\sigma|=d-k} |\text{link}_K(\sigma)| \cdot \sigma \in H_{d-k}(|K|, \Theta_{k-1})$$

[29 p. 63-65] with coefficients in the group of simplicial homology spheres (up to cobordism through PL homology cylinders) is trivial unless $k = 4$: for $\Theta := \Theta_3$ is the only nontrivial coefficient group. [It is not finitely generated [12, 15, 25].] There is a block bundle theory [14, 15] for simplicial homology manifolds, resulting in a fibration 

$$B_{\text{PL}} \to B_{\text{HL}} \to H(\Theta, 4)$$

of classifying spaces.

**Theorem** [cf. [22 §5]]: A simplicial homology manifold $K$ of dimension $\geq 5$ admits a PL manifold homology resolution iff 

$$\kappa_4(K) = 0.$$ 

The resolutions themselves are classified by maps to $H(\Theta, 3)$ [8].

**2.3** Using this technology, the question motivating this note can be formulated as the problem of understanding the map 

$$B(\text{Top}(7)) = B(\text{PL}(7)) \to B_{\text{HL}}.$$ 

Its fiber can be decomposed as 

$$\text{PL}/\text{Top}(7) \to \text{HL}/\text{PL}(7) \to \text{HL}/\text{PL} = H(\Theta, 3);$$ 

the fibration 

$$\text{PL}/\text{Top}(7) \to \text{Top}/\text{Top}(7) = H(\mathbb{Z}_2, 1) \times H(\mathbb{Z}, 3) \to \text{Top}/\text{PL} = H(\mathbb{Z}_2, 3)$$ 

shows that $\text{PL}/\text{Top}(7)$ is a three-stage Postnikov system, with homotopy group $\mathbb{Z}$ in degree three and $\mathbb{Z}_2$ in degrees one and two. The group $\pi_*(\text{HL}/\text{PL}(7))$ is therefore $\mathbb{Z}_2$ in degree one and zero in degree two, while in degree three there is an exact sequence 

$$0 \to \mathbb{Z} \to \pi_3(\text{HL}/\text{Top}(7)) \to \Theta \to \mathbb{Z}_2 \to 0.$$ 

The map on the right can be identified with Rokhlin’s invariant 

$$\Sigma \mapsto \rho(\Sigma) := \frac{1}{8} \text{signature } (W) \bmod (2) : \Theta \to \mathbb{Z}_2.
of a homology three-sphere $\Sigma$ (where $W$ is a spin four-manifold with $\partial W = \Sigma$). Let $\Theta_0$ be the kernel of $\rho$.

**Corollary:** When $K$ is a smoothable ($\chi(K) = 0$) simplicial homology string manifold of dimension $\geq 5$, PL manifold structures on a homology resolution are classified by elements of $H^3(|K|, \tilde{\Theta})$; where

$$\tilde{\Theta} := \mathbb{Z} \oplus \Theta_0 \cong \pi_3(\text{HL}/\text{Top}(7)) .$$

**Proof:** The exact sequence above splits, because the infinite cyclic group maps isomorphically to the three-dimensional homotopy group of Top. This suggests, among other things, the existence of a combinatorial formula [2] for its Pontrjagin class. When that class vanishes, lifts of the classifying map from $|K|$ to $B\text{HL}$ to a map from a homology resolution $X$ are classified by maps to $\text{HL}/\text{Top}(7)$. □

**2.4 The inclusion of the fiber in**

$$H(\Theta, 3) \to B\text{PL} \to B\text{HL}$$

is trivial at odd primes: $B\text{PL} \cong B\mathbb{Z} \otimes [30]$, but $K$-theory is blind to Eilenberg-MacLane spaces $H(A, n)$ for $n > 2$. At the prime two, there are still open questions. In particular, it is not known if $\rho$ splits: this is equivalent to the conjecture that all topological manifolds of dim $> 4$ are simplicial complexes.

Freed [10] identifies the classifying space of the Picard category of $\mathbb{Z}_2$-graded complex lines as a two-stage Postnikov system. Its associated infinite-loop spectrum

$$\mathbb{L}_+ \longrightarrow H\mathbb{Z}_2 \longrightarrow \beta S^2 \Delta \longrightarrow \Sigma^3 H\mathbb{Z}$$

is the truncation to positive dimensions

$$(\Sigma^2 \tilde{I})_{<0} \longrightarrow \Sigma^2 \tilde{I} \longrightarrow \mathbb{L}_\pm$$

of the double suspension of the Anderson dual [17 appendix B, 18] of the sphere spectrum, which is characterized by a short exact sequence

$$0 \to \text{Ext}(E_{s-1}, \mathbb{Z}) \to \tilde{I}^*(E) \to \text{Hom}(E_s, \mathbb{Z}) \to 0$$

associated to a spectrum $E$. This same small Postnikov system appears as the base

$$F/\text{PL} \cong \Sigma^{4s} H\mathbb{Z} \times \Sigma^{4s+2} H\mathbb{Z}_2 \times \Sigma^2 \mathbb{L}_\pm (* > 1)$$

of the (two-localization) of the infinite-loop space classifying piecewise-linear structures on a Poincaré-duality space [19]. These observations can then be assembled into the diagram

\[
\begin{array}{ccc}
\Sigma^2 H\mathbb{Z}_2 = \text{Top}/\text{PL} & \longrightarrow & \Sigma^2 \tilde{I} \\
\Sigma^3 H\Theta & \longrightarrow & \text{HL}/\text{PL} & \xrightarrow{\kappa} & F/\text{PL} & \longrightarrow & \Sigma^2 \mathbb{L}_\pm .
\end{array}
\]
On the other hand, the Rokhlin homomorphism defines an exact sequence
\[ \cdots \to \tilde{I}^2(H\Theta_0) \to \tilde{I}^1(HZ_2) \to \tilde{I}^1(H\Theta) \to \cdots ; \]
but \( \tilde{I}^2(H\Theta_0) = 0 \), so (the nonzero invariant defined by) \( \rho \in \tilde{I}^1(HZ_2) \) maps to (the nonzero invariant defined by) \( \kappa \in \tilde{I}^1(H\Theta) \): which defines a homomorphism from \( B\Theta \) to \( \mathbb{L}_\pm \), and can thus be interpreted as a topological field theory mapping the category with one object, and 3D homology cylinders as morphisms, to the category of \( \mathbb{Z}_2 \)-graded complex lines.

This can be regarded as a lift of Rokhlin’s invariant, regarded as a topological field theory taking values in the Picard category of real lines. It suggests the interest of super-Chern-Simons theories [11 §9] defined on simplicial homology spin manifolds.

3. ÜBER DIE HYPOTHESEN, WELCHE ZU GRUNDE DER PHYSIK LIEGEN

Following [6 §VII], it is tempting to interpret the elements of \( \mathbb{Z} \) in \( \tilde{\Theta} \) as topological twists ‘in the large’ (or at infinity), and elements of \( \Theta_0 \) as twists ‘in the small’. Physics has a tradition of concern (cf eg Wheeler) with the possibility that the microstructure of the Universe might in some way be non-Euclidean. This seems legitimate: experiments in very high-energy physics probe the topology of space-time at very short distance, and it is conceivable that at very fine scales physical space might be described by some kind of quantum foam model [23], possibly involving ensembles with varying topology.

These ideas have a big literature, but interested researchers seem not to be very aware of the long history of interest in analogous questions among topologists. In particular, the extended homology cobordism group \( \tilde{\Theta} \) seems to capture rather precisely the idea that the space-time ‘bubbles’ in which very-short-distance interactions occur - I’m thinking of the way Feynman diagrams are often drawn - might be bounded by non-standard spheres.

If physics starts by hypothesizing the existence of a simplicial homology manifold structure\(^1\) on some (say, ten or eleven-dimensional) space-time \( K \), then the vanishing of \( w_2 \) decides the existence of a spin structure and the vanishing of \( \varpi(K) \) decides the existence of a PL resolution. When \( p_1/2 = 0 \), resolutions of \( |K| \) admit string structures; this is quite like the standard situation. However, the possible twists of that string resolution (related to \( B \)-fields [26], Vafa’s discrete torsion [2 §1.6] and perhaps more generally to \( D \)-brane charges [5 §4.4]) lie in \( H^3(|K|, \tilde{\Theta}) \), which is much bigger than the usual group of gerbes over \( |K| \). There may even be ‘experimental’ evidence for the physical relevance of twisting by homology three-spheres, in that deep results about the structure of such manifolds are derived by scattering Yang-Mills bosons off them: ie, from Donaldson theory [9].

\(^1\)conceptually similar to a causal structure
As for exotic homology manifolds [4], hypotheses non fingo: in part because they, unlike simplicial homology manifolds, seem to lack the clocks and measuring rods that play the role of rulers and compasses in classical geometry. This is probably a lack of imagination on my part; a deeper concern is that major questions in 4D geometric topology remain open. Here I want only to make the point that simplicial homology manifolds are understood well enough to test against the models of contemporary physics.

REFERENCES

1. M. Ando, MJ Hopkins, N. Strickland, Elliptic spectra, the Witten genus and the theorem of the cube, Invent. Math. 146 (2001) 595–687
2. ——, C. French, Discrete torsion for the supersingular orbifold sigma genus, available at arXiv:math/0308056
3. D. Biss, The homotopy type of the matroid Grassmannian, Ann. of Math. 158 (2003) 929–952.
4. J. Bryant, S. Ferry, W. Mio, S. Weinberger, Topology of homology manifolds, Ann. of Math. 143 (1996) 435–467
5. J. Brodzki, V. Mathai, J. Rosenberg, RJ Szabo, D-branes, KK-theory and duality on noncommutative spaces, available at arXiv:0709.2128
6. JL Brylinski, Loop spaces, characteristic classes and geometric quantization, Progress in Mathematics 107, Birkhäuser (1993)
7. AL Carey, BL Wang, Thom isomorphism and pushforward map in twisted K-theory, available at arXiv:math/0507414v4
8. AL Edmonds, R. Stern, Resolutions of homology manifolds: a classification theorem. J. London Math. Soc. 11 (1975) 474–480.
9. R. Fintushel, R. Stern, Instanton homology of Seifert fibred homology three spheres, Proc. London Math. Soc. 61 (1990) 109–137
10. D. Freed, Pions and generalized cohomology, available at arXiv:hep-th/0607134
11. ——, M. Hopkins, J. Lurie, C. Teleman, Topological quantum field theories from compact Lie groups, available at arXiv:0905.0731
12. Y. Fukumoto, M. Furuta, M. Ue, W-invariants and Neumann-Siebenmann invariants for Seifert homology 3-spheres, Topology Appl. 116 (2001) 333–369.
13. M. Furuta, Homology cobordism group of homology 3-spheres, Invent. Math. 100 (1990) 339–355
14. DE Galewski, R. Stern, The relationship between homology and topological manifolds via homology transversality, Invent. Math. 39 (1977) 277–292
15. ——, —— Classification of simplicial triangulations of topological manifolds. Ann. of Math. 111 (1980) 1–34
16. V. Giambalvo, On (8)-cobordism, Illinois J. Math. 15 (1971) 533–541
17. M. Hopkins, I. Singer, Quadratic functions in geometry, topology, and M-theory, available at arXiv:math/0211216
18. N. Kitchloo, J. Morava, Thom prospectra for loopgroup representations, available at arXiv:math/0405414
19. K. Knapp, Anderson duality in K-theory and Im(J)-theory, K-Theory 18 (1999) 137–150.
20. I. Madsen, RJ Milgram, The classifying spaces for surgery and cobordism of manifolds, Annals of Math Studies 92, Princeton (1979)
21. EM Parker, The Brauer group of graded continuous trace C*-algebras, Trans. AMS 308 (1988) 115–132
22. AA Ranicki, On the Hauptvermutung, p. 3-26 of The Hauptvermutung Book, available at http://www.maths.ed.ac.uk/~aar/books/haupt.pdf
23. C. Rovelli, *Quantum gravity*, Cambridge Monographs on Mathematical Physics, CUP (2004)
24. YB Rudyak, Piecewise linear structures on topological manifolds, available at [arXiv:math/0105047](http://arxiv.org/abs/math/0105047)
25. N. Saveliev, Fukumoto-Furuta invariants of plumbed homology 3-spheres, Pacific J. Math. 205 (2002) 465–490
26. GB Segal, Topological structures in string theory, in *Topological methods in the physical sciences*, R. Soc. Lond. Philos. Trans. Ser. A Math. Phys. Eng. Sci. 359 (2001) 1389–1398
27. RM Seymour, Some functional constructions on $G$-spaces. Bull. London Math. Soc. 15 (1983) 353 - 359
28. S. Stolz, P. Teichner. What is an elliptic object? in *Topology, geometry and quantum field theory*, 247–343, LMS Lecture Notes 308, CUP (2004)
29. D. Sullivan, Geometric periodicity and the invariants of manifolds, in *Manifolds–Amsterdam 1970*: Springer LNM 197 (1971)
30. ——, *Geometric Topology: Localization, Periodicity and Galois Symmetry*; 1970 MIT notes, edited by Andrew Ranicki. K-Monographs in Mathematics, Vol. 8, Springer (2005), available at [http://www.maths.ed.ac.uk/~aar/books/gtop.pdf](http://www.maths.ed.ac.uk/~aar/books/gtop.pdf)

DEPARTMENT OF MATHEMATICS, JOHNS HOPKINS UNIVERSITY, BALTIMORE, MARYLAND 21218

E-mail address: jack@math.jhu.edu