Precision Measuring of Velocities via the Relativistic Doppler effect

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ABSTRACT

Just as the ordinary Doppler effect serves as a tool to measure radial velocities of celestial objects, so can the relativistic Doppler effect be implemented to measure a combination of radial and transverse velocities by using recent improvements in observing techniques. A key element that makes a further use of this combination feasible is the periodicity in changes of the orbital velocity direction for the source. Two cases are considered: (i) a binary star; and (ii) a solitary star with the planetary companion. It is shown that, in case (i), several precision Doppler measurements employing the gas absorption cell technique would determine both the total orbital velocity and the inclination angle of the binary orbit disentangled from the peculiar velocity of the system. The necessary condition for that is the measured, at least with a modest precision, proper motion and distance to the system.

Key words: Relativity – techniques: radial velocities – stars: kinematics – binaries: general – globular clusters: dynamics

1 1. INTRODUCTION

Measuring transverse (tangential) velocities has always been one of the most important and, at the same time, the least advanced issues in astronomy. Available methods include measurement of either annual parallaxes or proper motions, which is possible only for nearby objects. Furthermore, the method of proper motions requires knowledge of the distance to the object. This Letter shows that, just as the ordinary Doppler effect serves as a tool to measure radial velocities of celestial objects, the relativistic Doppler effect can be implemented to measure a combination of radial and transverse velocities by using the rapidly improving technique of precision Doppler measurements. This combination, as I show here, makes it possible to derive the inclination angle of a binary star provided that both the distance to the binary and its proper motion are measurable.

2 2. THE METHOD

The frequency of a spectral line measured at the Earth and corrected for the Earth’s rotation is supposed to be associated with the geocentric coordinate system. It is necessary to introduce at least two more reference frames: the frame of the source barycentre, $S'$, and the frame of the Solar system barycentre, $S''$. Let the motion of the source relative to $S'$ be described by a (generally variable) velocity $\vec{\beta} \equiv \vec{v}/c$, $S'$ moves relative to $S''$ with a (constant in both direction and
time) velocity $\vec{b} = \vec{V}/c$, and the Earth moves relative to $S''$ with a velocity $\vec{\beta}_E$. The relativistic Doppler equation that relates the observed frequency of a spectral line, $\nu$, to the emitted frequency, $\nu_0$, is given by successive application of the usual 4-wavevector transformation (Landau & Lifshitz 1951) between all the reference frames involved:

$$\frac{\nu}{\nu_0} = \frac{\nu'}{\nu''} \frac{\nu''}{\nu_0} = \sqrt{\frac{1 - \beta^2}{1 - \beta_r}} \sqrt{\frac{1 - b^2}{1 - b_r}} \frac{1 - \beta \cos \theta}{\beta} \equiv BCE, \quad (1)$$

where $B$ absorbs the source’s internal velocities $\beta$ and $\beta_r$, $C$ absorbs the motion of the source barycentre with velocity $\vec{b} = \vec{V}/c$, $E$ absorbs the Earth’s velocity $\vec{\beta}_E$, and $\theta$ is the angle in $S''$ between $\vec{\beta}_E$ and the direction to the source.

At first sight, the very structure of equation (1) – an isolation of different types of velocity strictly within the corresponding multipliers, i.e. factorization – makes it impossible to determine the total and radial velocities separately. However, special circumstances, such as periodic changes of the direction of the source’s velocity, improve the situation.

Below, I consider two particular cases: (i) the source as a binary star and (ii) the source as a star with the planetary companion.

### 2.1 2.1. Case (i): a binary

Let us consider a binary in a circular orbit with inclination angle $i$. In this case, the change of the radial velocity $\beta_r$ is periodic while the total velocity $\beta = |\vec{\beta}|$ is constant. Then $B$ changes in a periodic fashion as well. Let $\beta_r$ reach its maximum, $\beta_{r, max}$, at an instant $t_1$ so that $B$ has a maximum $B_{max}$ at that instant too. In half a period (the instant $t_2$), $\beta_r = -\beta_{r, max}$ so that $B$ reaches its minimum $B_{min}$. Evidently,

$$B_{max}^{-1} + B_{min}^{-1} = \frac{2}{\sqrt{1 - \beta^2}}, \quad (2)$$

i.e. the dependence on $\beta_{r, max}$ is cancelled out and the result depends on $\beta^2$ only. Once $\beta^2$ is found, $\beta_{r, max}$ can be derived from

$$B_{min}^{-1} - B_{max}^{-1} = \frac{2\beta_{r, max}}{\sqrt{1 - \beta^2}}. \quad (3)$$

It is easy to see that

$$\beta_{r, max} = \beta \sin i. \quad (4)$$

Therefore equations (2) and (4) yield

$$\sin i = \frac{B_{max} - B_{min}^{-1}}{B_{max} + B_{min}^{-1}} \frac{1}{\beta}, \quad (5)$$

where

$$\beta = \sqrt{1 - \left(\frac{2B_{max}B_{min}}{B_{max} + B_{min}}\right)^2}. \quad (6)$$

Thus both the total velocity $\beta$ and the inclination angle $i$ are derivable from the precision Doppler measurements.

In practical implementation of the above approach, we assume that the measured frequency $\nu$ is corrected for annual motion of Earth (term $E$). Motion of the visible star in the binary (with variable velocity $\vec{\beta}$) and motion of the
binary barycentre (with constant velocity $\vec{b}$) result in a (variable) total Doppler shift

$$\frac{\nu_0}{\tilde{\nu}} = \frac{1 - \beta_r}{\sqrt{1 - \beta_r^2}} \frac{1 - b_r}{\sqrt{1 - b_r^2}} \approx (1 - \beta_r) \left( 1 - b_r + \frac{1}{2} b_r^2 + \frac{1}{2} \beta_r^2 \right). \quad (7)$$

The instants when $\nu_0/\tilde{\nu}$ reaches its maximum and minimum enable us to determine the instant when $\beta_r = 0$. At that instant, observations bring us an approximate value of $b_r$ (by neglecting the higher order terms $\frac{1}{2} b_r^2 + \frac{1}{2} \beta_r^2$). Measurements of the binary’s proper motion and distance yield the transverse velocity of the binary $b_t$ and thus enable one to evaluate $b^2 = b_r^2 + b_t^2$. The difference between $\nu_0/\tilde{\nu}$ (measured at the instant when $\beta_r = 0$) and $\left( 1 - b_r + \frac{1}{2} b_r^2 \right)$ yields an approximate value of $\frac{1}{2} \beta_r^2$. Obviously, $\beta^2$ is kept constant in orbit. By applying the iteration method, the procedure is repeated until it converges to give accurate values of $b_r$. As soon as $b_r$, $b^2$, and $\beta^2$ are found, the measured change of $\nu_0/\tilde{\nu}$ in orbit yields $|\beta_r|$, the modulus of $\beta_r$. Finally, knowledge of both $|\beta_r| = \beta \sin i$ and $\beta^2$ allows us to evaluate $\sin i$.

However, if $\beta \ll b$, which would correspond to wide binaries or a small mass of the companion, practical implementation of the above approach is very difficult, as illustrated by case (ii).

### 2.2. 2.2. Case (ii): a single star with a planetary companion

Since the motion of a star that has a planetary companion orbiting around it is only slightly disturbed by the planet, this case is reduced to case (i) of a binary with $\beta \ll b$. In the frame $S''$ associated with the barycenter of the Solar System $b$ is the speed of the source, $b_r$ is its radial component, and $b_t$ is the transverse component. Although $\beta \ll b$, the disentangling of $\beta_r$ and $b_r$ does not make any problem as long as the terms linear in $\beta$ and $b$ are only involved. However, in this case, as opposite to case (i), determination of the inclination angle of the planetary orbit turns out to be in practice very difficult. The reason is that inaccuracy in measuring $b_t$, the transverse component of $b$, is currently too large. To see this, it is worth discussing the issue of accuracy in more detail.

To account for the contribution of the second-order terms that enter the expansion of the relativistic Doppler formula into a series in $v/c$, precision of Doppler measurements of spectral line shifts (translated into the precision of radial velocity measurements) should be no less than

$$\delta v \approx \frac{1}{2} \frac{v}{c}$$

- $\approx 67 \text{ m/s if } v = 200 \text{ km/s (halo stars)}$,
- $\approx 17 \text{ m/s if } v = 100 \text{ km/s (bulge stars)}$,
- $\approx 1.5 \text{ m/s if } v = 30 \text{ km/s (disk stars)}$. \quad (8)

Remarkably, accounting for the relativistic Doppler effect appears to be within the limits of modern astronomical techniques (see Sec. 3), even when the measured velocity does not exceed 30 km/s.
In contrast to the radial velocity, the tangential velocity currently cannot be measured with a similar precision. However, since $v_t$ enters the relativistic Doppler equation only as a second and higher order terms, accuracy in measuring proper motion and distance to the star are allowed to be rather modest. For instance, if an inaccuracy in the inferred $v_t$ does not exceed, say, 10%, as a product of combined uncertainties in measuring the distance to and proper motion of the star, a similar inaccuracy is translated into the value of total velocity, provided that the radial velocity measurements are as accurate as to obey equation (8). Although this would still make possible to evaluate the inclination angle of a binary as described by case (i), it fails for a planetary companion because the current inaccuracy in measuring $b_t$ exceeds the value of $\beta$ that has to be derived. For instance, even if $\delta V_t = 1$ km s$^{-1}$, the planetary companion induces $v$ typically not exceeding 0.1 km s$^{-1}$. Nonetheless, since the "signal" (the value of $v_t$) changes periodically, and the "noise" (the value of $V_t$) is kept constant, an extraction of that signal from the noise seems to be although a difficult but in principle solvable problem.

3. PRECISION VELOCITY MEASUREMENTS

As can be seen from equation (8), the proposed method requires high precision in velocity measurements as well as stability of the instrumental system. Any seasonal instrumental effects must be carefully eliminated. An idea of the appropriate reference frame for measuring velocities, namely passing the starlight through an absorbing gas prior to its entrance into the spectrograph, was proposed long ago (Griffin & Griffin 1973) but only recently has been developed to a level sufficient to provide high enough precision. The modern technique employs a stabilized gaseous iodine ($I_2$) absorption cell as a wavelength standard (Libbrecht 1988; Marcy & Butler 1992; Cochran & Hatzes 1994; Kürster et al. 1994). The $I_2$ cell used in the wavelength region 5000-6000 Å, in combination with a high resolving power $R$ provides a velocity precision proportional to $R^{-3/2}(S/N)^{-1}$, where $S/N$ is the signal-to-noise ratio (Hatzes & Cochran 1994). The long-term velocity precision achievable with the current techniques is 4-7 m/s (Kürster et al. 1994) to 3 m/s or even better (Butler et al. 1996). For discussion of conditions that must be controlled in order to achieve such a high precision (changes in dispersion and the spectrograph point spread function, careful determination of the topocentric velocity relative to the Solar system barycentre, etc.), see e.g. Marcy & Butler (1992). Other effects resulting in radial velocity changes, which have been revealed and studied during programmes of searches for extra-solar planets, include the variability of K giants (e.g. Hatzes & Cochran 1994), variability among non-Cepheid stars in the instability strip (Butler 1992), and rapid oscillations of Ap stars (Hatzes & Kürster 1994).

Recently, the stabilized gas absorption cell technique has been implemented by several groups (Marcy & Butler 1992; Cochran & Hatzes 1994; Kürster et al. 1994) in a search for Jupiter-like planets. This procedure is commonly named ‘precision radial velocity measurements’, although, as is clear from equation (1), it is the changes in the total velocity (i.e. including the transverse velocity) that are actually being measured as soon as the accuracy of Doppler measurements reaches the level given by equation (8).
4 4. DISCUSSION

A central point of this letter is that if the accuracy of precision Doppler measurements is as high as to comply with equation (8), a combination (7) of linear and quadratic in velocity terms, and not just radial velocity alone, is actually measured. If the radial velocity changes periodically (the source is a binary star), the binary parameters, including the inclination angle of the orbit, can in principle be disentangled. In practice, this task might be not easy. For instance, in an extreme case of a contact binary consisting of solar-mass stars \((a \approx 1.5 \, R_\odot, \, P \approx 0.2 d)\), the velocity vector changes in \(\Delta t_{\text{sec}}\) seconds by \(\Delta v = (v^2/a)\Delta t \approx 0.14\Delta t_{\text{sec}}\) m/s, i.e. by 1.4 m/s every 10 sec, which requires a very short exposure time to make measurements.

Even if the source is a solitary star, precision Doppler measurements separated in time can bring important dynamical information, especially if the star is located in a dense stellar field such as a globular cluster. Indeed, differentiation of equation (1) gives the frequency of a spectral line (corrected for annual motion of Earth) measured with time interval \(\Delta t\) to be shifted by

\[
\frac{\Delta \tilde{\nu}}{\tilde{\nu}} = \left( \frac{\tilde{V}^2}{c} + \frac{V^2}{ct} + \frac{V\tilde{V}}{c^2} \right) \Delta t, \tag{9}
\]

where \(\tilde{V}\) is acceleration and \(d\) is distance to the source. A similar equation is used in pulsar timing (e.g. Blandford et al. 1993), where it is written in terms of pulsar period and its derivative. In a globular cluster with the core radius \(r_c = 0.1 \, \text{pc}\) and star density within the core \(\rho_c = 3 \cdot 10^6 \, M_\odot/\text{pc}^3\), the dynamical acceleration in the core is \(\tilde{V} = 4/3\pi G\rho_c r_c = 1.7 \cdot 10^{-5} \, \text{cm s}^{-2}\). By neglecting the 2nd and 3rd terms in equation (9), one finds that in \(\Delta t_{\text{yr}}\) years the star’s velocity would change by \(\Delta V = 5.4\Delta t_{\text{yr}}\) m/s, i.e. by 5 m/s in a year of precision Doppler measurements. Therefore, they seem appropriate as a tool to measure gravitational field in the cores of globular clusters – an opportunity that so far seemed only possible for a few millisecond pulsars.

The prospects for application of the I\(_2\) absorption cell technique (Sec. 3) seem to be very promising. Combining the echelle spectrograph with the I\(_2\) cell allows for routine observations of stars of magnitude \(V = 13\) (Marcy & Butler 1992). If the effects of all systematic errors can be reduced so that the velocity precision is limited only to the photon statistics, then the velocity precision improves as \(N^{-1/2}\), \(N\) being the number of detected photons. Fourier analysis makes it possible to extract the embedded periodicities with amplitudes of about 1 m/s (Brown 1990). This, as is seen from equation (8), enables one to measure the relativistic correction to the ordinary Doppler effect even when peculiar velocities of stars are \(\sim 30\) km/s or less (disk stars).

Summarizing, this Letter demonstrates in principle a possibility to disentangle, for binary stars, the first and second order velocity terms of the relativistic Doppler effect and, as a result, to extract the inclination angle of the orbit (by neglecting gravitational fields). This could be done by making use of recent substantial improvements in the technique of precision Doppler measurements combined with proper motion data. However, much further work is needed to relax the underlying explicit and implicit assumptions and to evaluate the associated corrections as well.
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