Rigorous derivation of the primitive equations with full viscosity and full diffusion by scaled Boussinesq equations

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Abstract

The primitive equations of large-scale ocean dynamics form the fundamental model in geophysical flows. It is well-known that the primitive equations can be formally derived by hydrostatic balance. On the other hand, the mathematically rigorous derivation of the primitive equations without coupling with the temperature is also known. In this paper, we generalize the above result from the mathematical point of view. More precisely, we prove that the scaled Boussinesq equations strongly converge to the primitive equations with full viscosity and full diffusion as the aspect ratio parameter goes to zero, and the rate of convergence is of the same order as the aspect ratio parameter. The convergence result mathematically implies the high accuracy of hydrostatic balance.

AMS Subject Classification:

Key Words: Boussinesq equations; Primitive equations; Hydrostatic balance; Strong convergence

1 Introduction

In large-scale ocean dynamics, the vertical scale of ocean is much smaller than the horizontal scale, which means that we have to use the hydrostatic balance to simulate the motion of ocean in the vertical direction. Based on this fact and the high accuracy of hydrostatic balance, the primitive equations of ocean dynamics can be formally derived from the Boussinesq equations or the coupled system of Navier-Stokes equations and thermodynamic equation, see Lions-Temam-Wang [27–29] and Cao-Titi [10].

The rigorous mathematical derivation from the Navier-Stokes equations to the primitive equations was studied first by Azérad-Guillén [1] in a weak sense, then by Li-Titi [30] in a strong sense with error estimates, and finally by Furukawa et al. [16] in a strong sense but under relaxing the regularity on the initial condition. Subsequently, the small aspect ratio limit from the compressible Navier-Stokes equations to the compressible primitive equations was proved by Gao et al. [17], in which the versatile relative entropy inequality plays an crucial role. Furthermore, the strong convergence of solutions of the scaled Navier-Stokes equations to the corresponding ones of the primitive equations with only horizontal viscosity was obtained by Li-Titi-Yuan [32]. However, none of the above work has derived the complete primitive equations, i.e., the primitive equations with full viscosity and full diffusion, from the mathematical point of view. Therefore, it is of great significance to derive the primitive equations with full viscosity and full diffusion mathematically.

Let \( \Omega_\varepsilon = M \times (-\varepsilon, \varepsilon) \) be an \( \varepsilon \)-dependent domain, where \( M \) is smooth bounded domain in \( \mathbb{R}^2 \). Here, \( \varepsilon = H/L \) is called the aspect ratio, measuring the ratio of the vertical and horizontal scales of the motion, which is usually very small. Say, for large-scale ocean circulation, the ratio \( \varepsilon \sim 10^{-3} \ll 1 \).
Denote by $\nabla_h = (\partial_x, \partial_y)$ the horizontal gradient operator. Then the horizontal Laplacian operator $\Delta_h$ is given by

$$\Delta_h = \nabla_h \cdot \nabla_h = \partial_{xx} + \partial_{yy}.$$  

Let us consider the anisotropic Boussinesq equations defined on $\Omega$.

$$\begin{cases}
\partial_t u + (u \cdot \nabla)u + \nabla p - \theta \kappa = \mu_h \Delta_h u + \mu_z \partial_{zz} u, \\
\partial_t \theta + u \cdot \nabla \theta = \kappa_h \Delta \theta + \kappa_z \partial_{zz} \theta, \\
\nabla \cdot u = 0,
\end{cases}$$  

(1.1)

where the three dimensional velocity field $u = (v, w) = (v_1, v_2, w)$, the pressure $p$ and temperature $\theta$ are the unknowns. The unit vector $\kappa = (0, 0, 1)$ points to the z-direction. $\mu_h$ and $\mu_z$ represent the horizontal and vertical viscosity coefficients respectively, while $\kappa_h$ and $\kappa_z$ represent the horizontal and vertical heat diffusion coefficients respectively.

Firstly, we transform the anisotropic Boussinesq equations (1.1), defined on the $\varepsilon$-dependent domain $\Omega$, to the scaled Boussinesq equations defined on a fixed domain. To this end, we introduce the following new unknowns with subscript $\varepsilon$

$$u_\varepsilon = (v_\varepsilon, w_\varepsilon), \quad v_\varepsilon(x, y, z, t) = v(x, y, \varepsilon z, t),$$

$$w_\varepsilon(x, y, z, t) = \frac{1}{\varepsilon} w(x, y, \varepsilon z, t), \quad p_\varepsilon(x, y, z, t) = p(x, y, \varepsilon z, t),$$

$$\theta_\varepsilon(x, y, z, t) = \varepsilon \theta(x, y, \varepsilon z, t),$$

for any $(x, y, z) \in \Omega =: M \times (-1, 1)$ and for any $t \in (0, \infty)$.

Suppose that $\mu_h = \kappa_h = 1$ and $\mu_z = \kappa_z = \varepsilon^2$. Under these scalings, the anisotropic Boussinesq equations (1.1) defined on $\Omega$ can be written as the following scaled Boussinesq equations

$$\begin{cases}
\partial_t v_\varepsilon - \Delta v_\varepsilon + (\varepsilon v_\varepsilon \cdot \nabla) v_\varepsilon + w_\varepsilon \partial_z v_\varepsilon + \nabla p_\varepsilon = 0, \\
\varepsilon^2 (\partial_t w_\varepsilon - \Delta w_\varepsilon + v_\varepsilon \cdot \nabla w_\varepsilon + w_\varepsilon \partial_z w_\varepsilon) + \partial_z p_\varepsilon - \theta_\varepsilon = 0, \\
\nabla_h \cdot v_\varepsilon + \partial_z w_\varepsilon = 0, \\
\nabla_h \cdot w_\varepsilon + \partial_z \theta_\varepsilon = 0,
\end{cases}$$  

(1.2)

defined on the fixed domain $\Omega$.

Next, we supply the scaled Boussinesq equations (1.2) with the following boundary and initial conditions

$$v_\varepsilon, w_\varepsilon, p_\varepsilon \text{ and } \theta_\varepsilon \text{ are periodic in } x, y, z,$$

$$v_\varepsilon(x, w_\varepsilon, \theta_\varepsilon)|_{t=0} = (v_0, w_0, \theta_0),$$

(1.3)  

(1.4)

where $(v_0, w_0, \theta_0)$ is given. In addition, we also equip the system (1.2) with the following symmetry condition

$$v_\varepsilon, w_\varepsilon, p_\varepsilon \text{ and } \theta_\varepsilon \text{ are even, odd, even and odd with respect to } z, \text{ respectively}.$$  

(1.5)

Noting that this symmetry condition is preserved by the scaled Boussinesq equations (1.2), i.e., it holds provided that the initial data satisfies this symmetry condition. Due to this fact, throughout this paper, we always suppose that the initial data satisfies

$$v_0 \text{ and } \theta_0 \text{ are periodic in } x, y, z, \text{ and are even and odd in } z, \text{ respectively}.$$  

(1.6)

In this paper, we will not distinguish in notation between spaces of scalar and vector-valued functions, in other words, we will use the same notation to denote both a space itself and its finite product spaces. For simplicity, we denote by notation $\|\cdot\|_p$ and $\|\cdot\|_{p,M}$ the $L^p(\Omega)$ norm and $L^p(M)$ norm, respectively.
The well-posedness theory of weak solutions to the scaled Boussinesq equations (1.2) follows the proof in Lions-Temam-Wang [28]. Namely, for any initial data \((u_0, \theta_0) = (v_0, w_0, \theta_0) \in L^2(\Omega)\), with \(\nabla \cdot u_0 = 0\), we can prove that there exists a global weak solution \((v_\varepsilon, w_\varepsilon, \theta_\varepsilon)\) of the scaled Boussinesq equations (1.2) corresponding to boundary and initial conditions (1.3)-(1.4) and symmetry condition (1.5). Moreover, by the same arguments as those for the Boussinesq equations, see, e.g., Lions-Temam-Wang [28], Constantin-Foias [14] and Chae-Nam [13], we can also show that it has a unique local strong solution \((v_\varepsilon, w_\varepsilon, \theta_\varepsilon)\) for initial data \((u_0, \theta_0) = (v_0, w_0, \theta_0) \in H^1(\Omega), \text{with } \nabla \cdot u_0 = 0\). The definition of weak solutions to the scaled Boussinesq equations (1.2) is defined as follows.

**Definition 1.1.** Given \((u_0, \theta_0) = (v_0, w_0, \theta_0) \in L^2(\Omega)\), with \(\nabla \cdot u_0 = 0\). We say that a space periodic function \((v_\varepsilon, w_\varepsilon, \theta_\varepsilon)\) is a weak solution of the system (1.2) corresponding to boundary and initial conditions (1.3)-(1.4) and symmetry condition (1.5), if

(i) \((v_\varepsilon, w_\varepsilon, \theta_\varepsilon) \in C([0, \infty) \cap L^2(\Omega) \cap L^2(0, \infty); H^1(\Omega))\) and

(ii) \((v_\varepsilon, w_\varepsilon, \theta_\varepsilon)\) satisfies the following integral equality

\[
\int_0^\infty \int_\Omega \left\{ (-v_\varepsilon \cdot \partial_t \phi_h - \varepsilon^2 w_\varepsilon \partial_t \varphi_3 - \theta_\varepsilon \partial_t \psi - \theta_\varepsilon \varphi_3) \\
+ [\nabla v_\varepsilon : \nabla \phi_h + \varepsilon^2 \nabla w_\varepsilon \cdot \nabla \varphi_3 + \nabla \theta_\varepsilon \cdot \nabla \psi] \\
+ \left[ (u_\varepsilon \cdot \nabla) v_\varepsilon \cdot \varphi_h + \varepsilon^2 (u_\varepsilon \cdot \nabla w_\varepsilon) \varphi_3 + (u_\varepsilon \cdot \nabla \theta_\varepsilon) \psi \right] \right\} dxdydzdt \\
= \int_\Omega \left( (v_0 \cdot \phi_h(0) + \varepsilon^2 w_0 \varphi_3(0) + \theta_0 \psi(0)) \right) dxdydz,
\]

for any spatially periodic function \((\phi, \psi) = (\phi_h, \varphi_3, \psi)\), with \(\phi_h = (\varphi_1, \varphi_2)\), such that \(\nabla \cdot \varphi = 0\) and \((\phi, \psi) \in C_0(\Omega \times [0, \infty))\).

Taking the limit \(\varepsilon \to 0\) in system (1.2), then this system formally turns into the following primitive equations with full viscosity and full diffusion

\[
\begin{align*}
\partial_t v - \Delta v + (v \cdot \nabla) v + w \partial_z v + \nabla p = 0, \\
\partial_t \theta - \theta = 0, \\
\partial_t \psi - \nabla \eta \theta + w \partial_z \theta = 0, \\
\nabla \cdot v + \partial_z w = 0,
\end{align*}
\]

(1.7)

satisfying the same boundary and initial conditions (1.3)-(1.4) and symmetry condition (1.5) as the system (1.2). The global well-posedness of strong solutions to the primitive equations with full viscosity and full diffusion was established by Cao-Titi [10].

The primitive equations are considered as the fundamental model in geophysical flows, which is mainly used in the numerical study of weather prediction and long-time climate dynamics, see, e.g., Zeng [42], Seidov [37], Samelson-Vallis [36], Haltiner-Williams [18], Washington-Parkinson [40], Pedlosky [34], Majda [33], and Vallis [39]. The global existence of weak solutions of the primitive equations with full viscosity and full diffusion was first given by Lions-Temam-Wang [27–29], but the question of uniqueness to this mathematical model is still unknown except for some special cases [3, 24, 31, 38]. Furthermore, the existence and uniqueness of global-in-time strong solutions of this mathematical model in different setting is due to Cao-Titi [10], Kobelkov [23], Kukavica-Ziane [25, 26], Hieber-Kashiwabara [21], as well as Hieber-Hussien-Kashiwabara [20]. Subsequently, the study of the global strong solutions to the primitive equations is naturally carried out in the case of partial dissipation, see Cao-Titi [11], Fang-Han [15], and Cao-Li-Titi [5–9]. However, the inviscid primitive equations with or without rotation is known to be ill-posed in Sobolev spaces, and its smooth solutions may develop singularity in finite time, see Renardy [35], Han-Kwan and Nguyen [19], Ibrahim-Lin-Titi [22], Wong [41], and Cao et al. [4].

As can be seen from the literature, the primitive equations have been a hot topic in mathematical research for the past thirty years. However, the primitive equations they studied were all
derived from the perspective of geophysics, and only Azérad-Guillén [1], Li-Titi [30], Furukawa et al. [16] as well as Li-Titi-Yuan [32] rigorously derived the primitive equations without coupling with the temperature from the point of view of mathematics. In consequence, the aim of this paper is to derive rigorously the primitive equations with full viscosity and full diffusion, i.e., to prove that the scaled Boussinesq equations (1.2) strongly converge to the primitive equations with full viscosity and full diffusion (1.7), in which both systems satisfy the same boundary and initial conditions (1.3)-(1.4) and symmetry condition (1.5).

Throughout this paper, the initial data can be stated in two ways. The solutions of system (1.2) and system (1.7) satisfy the same symmetry condition (1.5), so does the initial data \((u_0, \theta_0) = (v_0, w_0, \theta_0)\). Owing to this fact, it follows from divergence-free condition of initial data that \(w_0\) is uniquely determined as

\[
w_0(x, y, z) = -\int_0^z \nabla_h \cdot v_0(x, y, \xi) d\xi,
\]

for any \((x, y) \in M\) and \(z \in (-1, 1)\). Hence both statements “initial data \((v_0, w_0, \theta_0)\)” and “initial data \((v_0, \theta_0)\)” are used, where \(w_0\) is uniquely determined by (1.8) for the latter case.

Now we are to state the main results of this paper. Firstly, assuming that the initial data \((v_0, \theta_0)\) is in \(H^1(\Omega)\). Using this assumption condition, it deduces from (1.8) that \((v_0, w_0, \theta_0)\) is in \(L^2(\Omega)\), which implies the system (1.2) with the boundary and initial conditions (1.3)-(1.4) and symmetry condition (1.5) has a global weak solution \((v_\varepsilon, w_\varepsilon, \theta_\varepsilon)\). For this case, we have the following strong convergence theorem.

**Theorem 1.1.** Given a periodic function \((v_0, \theta_0) \in H^1(\Omega)\) such that

\[
\int_{-1}^{1} \nabla_h \cdot v_0(x, y, z) dz = 0, \quad \int_{\Omega} v_0(x, y, z) dx dy dz = 0, \quad \text{and} \quad \int_{\Omega} \theta_0(x, y, z) dx dy dz = 0.
\]

Suppose that \((v_\varepsilon, w_\varepsilon, \theta_\varepsilon)\) is an arbitrary weak solution of the system (1.2) on the time interval \([0, \infty)\), and that \((v, w, \theta)\) is the unique strong solution of the system (1.7) on the time interval \([0, \infty)\), with the same boundary and initial conditions (1.3)-(1.4) and symmetry condition (1.5). Let \((V_\varepsilon, W_\varepsilon, \Phi_\varepsilon) := (v_\varepsilon - v, w_\varepsilon - w, \theta_\varepsilon - \theta)\).

Then, for any \(T > 0\), we have the a priori estimate

\[
\sup_{0 \leq \varepsilon \leq T} \left( \|v_\varepsilon \|_{L^2(\Omega)}^2 \right) \leq \beta_1(T),
\]

for any \(\varepsilon \in (0, 1)\), where

\[
\beta_1(T) = C e^{C(T + \alpha_5^2(T) + \alpha_6^2(T))} \left[ \alpha_5(T) + \alpha_6^2(T) + (\|v_0\|_2 + \varepsilon^2 \|w_0\|_2^2 + T \|\theta_0\|_2^2) \right]^2,
\]

and \(C\) is a positive constant that does not depend on \(\varepsilon\). As a result, we have the following strong convergences

\[
(v_\varepsilon, \varepsilon w_\varepsilon, \varepsilon \theta_\varepsilon) \rightarrow (v, \theta, 0), \quad \text{in} \quad L^\infty(0, T; L^2(\Omega)),
\]

\[
(\nabla v_\varepsilon, \varepsilon \nabla w_\varepsilon, \nabla \varepsilon \theta_\varepsilon, w_\varepsilon) \rightarrow (\nabla v, 0, \nabla \theta, w), \quad \text{in} \quad L^2(0, T; L^2(\Omega)),
\]

and the rate of convergence is of the order \(O(\varepsilon)\).

Next, we suppose that the initial data \((v_0, \theta_0)\) belongs to \(H^2(\Omega)\). Then from (1.8) it follows that \((v_0, w_0, \theta_0)\) belongs to \(H^1(\Omega)\). By the same arguments as those for the Boussinesq equations, see, e.g., Lions-Temam-Wang [28], Constantin-Foias [14] and Chae-Nam [13], there exists a unique local strong solution \((v_\varepsilon, w_\varepsilon, \theta_\varepsilon)\) to the system (1.2) with the boundary and initial conditions (1.3)-(1.4) and symmetry condition (1.5). So we denote by \(T_\ast\) the maximal existence time of the local strong solution \((v_\varepsilon, w_\varepsilon, \theta_\varepsilon)\) to the system (1.2). For this case, we also have the following strong convergence theorem, but the convergence is stronger than that in Theorem 1.1.
**Theorem 1.2.** Given a periodic function \((v_0, \theta_0) \in H^2(\Omega)\) such that

\[
\int_{-1}^{1} \nabla_h \cdot v_0(x, y, z) dz = 0, \quad \int_{\Omega} v_0(x, y, z) dx dy dz = 0, \quad \text{and} \quad \int_{\Omega} \theta_0(x, y, z) dx dy dz = 0.
\]

Suppose that \((v_\varepsilon, w_\varepsilon, \theta_\varepsilon)\) is the unique strong solution of the system \((1.2)\) on the time interval \([0, T_*]\), and that \((v, w, \theta)\) is the unique strong solution of the system \((1.7)\) on the time interval \([0, \infty)\), with the same boundary and initial conditions \((1.3)-(1.4)\) and symmetry condition \((1.5)\). Let

\[
(V_\varepsilon, W_\varepsilon, \Phi_\varepsilon) := (v_\varepsilon - v, w_\varepsilon - w, \theta_\varepsilon - \theta).
\]

Then, there is a positive constant \(\varepsilon_0\) such that, for any \(\varepsilon \in (0, \varepsilon_0)\), the strong solution \((v_\varepsilon, w_\varepsilon, \theta_\varepsilon)\) of the system \((1.2)\) exists globally in time, and the system \((5.1)-(5.4)\) (see Section 5, below) has the following estimate

\[
\sup_{0 \leq t < \infty} \left( \| (V_\varepsilon, \varepsilon W_\varepsilon, \Phi_\varepsilon) \|_{H^2}^2 (t) + \int_{0}^{\infty} \| \nabla(V_\varepsilon, \varepsilon W_\varepsilon, \Phi_\varepsilon) \|_{H^1}^2 dt \right) \leq \varepsilon^2 \left( \tilde{\beta}_1(T_*) + \tilde{\beta}_2(T_*) \right),
\]

where both \(\tilde{\beta}_1(T_*)\) and \(\tilde{\beta}_2(T_*)\) are positive constants that do not depend on \(\varepsilon\). As a result, we have the following strong convergences

\[
(v_\varepsilon, \varepsilon w_\varepsilon, \theta_\varepsilon) \to (v, 0, \theta), \quad \text{in} \quad L^\infty (0, \infty; H^1(\Omega)),
\]

\[
(\nabla v_\varepsilon, \varepsilon \nabla w_\varepsilon, \nabla \theta_\varepsilon, w_\varepsilon) \to (\nabla v, 0, \nabla \theta, w), \quad \text{in} \quad L^2 (0, \infty; H^1(\Omega)),
\]

\[
w_\varepsilon \to w, \quad \text{in} \quad L^\infty (0, \infty; L^2(\Omega)),
\]

and the rate of convergence is of the order \(O(\varepsilon)\).

**Remark 1.1.**

(i) Theorem 1.1 and 1.2 prove that the scaled Boussinesq equations \((1.2)\) strongly converge to the primitive equations with full viscosity and full diffusion \((1.7)\), which implies the high accuracy of hydrostatic balance mathematically.

(ii) The assumption \(\int_{-1}^{1} \nabla_h \cdot v_0 dz = 0\) is preserved by the hypothesis \((1.6)\) and the divergence-free condition \(\nabla \cdot v_0 = 0\). Moreover, the assumptions \(\int_{\Omega} v_0 dx dy dz = 0\) and \(\int_{\Omega} \theta_0 dx dy dz = 0\) are to ensure that \(u_\varepsilon, \theta_\varepsilon, u, \text{and} \theta\) have integral average zero, so the Poincaré inequality can be conveniently used in the proofs of Theorem 1.1 and 1.2, but the same results still hold for the general case that \(v_0\) and \(\theta_0\) have not integral average zero.

(iii) The higher order estimate for the system \((5.1)-(5.4)\) is also valid as Li-Titi \([30]\). If \((v_0, \theta_0) \in H^k\), with \(k \geq 3\), then we have the following estimate

\[
\sup_{0 \leq t < \infty} \left( \| (V_\varepsilon, \varepsilon W_\varepsilon, \Phi_\varepsilon) \|_{H^{k-1}}^2 (t) + \int_{0}^{\infty} \| \nabla(V_\varepsilon, \varepsilon W_\varepsilon, \Phi_\varepsilon) \|_{H^{k-1}}^2 dt \right) \leq \varepsilon^2 \tilde{\beta}(T_*),
\]

where \(\tilde{\beta}(T_*)\) is a positive constant that does not depend on \(\varepsilon\).

(iv) For large-scale ocean dynamics, it is well-known that we have to add diffusion-transport equation of salinity to the system \((1.1)\). However, the salinity effects are ignored in order to simplify the mathematical presentation. When this situation is taken into account, we can still show that the scaled Boussinesq equations

\[
\begin{align*}
\partial_t v_\varepsilon - \Delta v_\varepsilon + (v_\varepsilon \cdot \nabla_h) v_\varepsilon + w_\varepsilon \partial_z v_\varepsilon + \nabla_h p_\varepsilon &= 0, \\
\varepsilon^2 (\partial_t w_\varepsilon - \Delta w_\varepsilon + w_\varepsilon \cdot \nabla_h w_\varepsilon + w_\varepsilon \partial_z w_\varepsilon) + \partial_z p_\varepsilon - \theta_\varepsilon + \gamma_\varepsilon &= 0, \\
\partial_t \theta_\varepsilon - \Delta \theta_\varepsilon + v_\varepsilon \cdot \nabla_h \theta_\varepsilon + w_\varepsilon \partial_z \theta_\varepsilon &= 0, \\
\partial_t \gamma_\varepsilon - \Delta \gamma_\varepsilon + v_\varepsilon \cdot \nabla_h \gamma_\varepsilon + w_\varepsilon \partial_z \gamma_\varepsilon &= 0, \\
\nabla_h \cdot v_\varepsilon + \partial_z w_\varepsilon &= 0.
\end{align*}
\]
strongly converge to the primitive equations with full viscosity and full diffusion

\[
\begin{align*}
\partial_t v - \Delta v + (v \cdot \nabla_h)v + w\partial_z v + \nabla_h p &= 0, \\
\partial_t p - \theta + \gamma &= 0, \\
\partial_t \theta - \Delta \theta + v \cdot \nabla_h \theta + w\partial_z \theta &= 0, \\
\partial_t \gamma - \Delta \gamma + v \cdot \nabla_h \gamma + w\partial_z \gamma &= 0, \\
\nabla_h \cdot v + \partial_z w &= 0,
\end{align*}
\]

where both \( \gamma_c \) and \( \gamma \) represent the salinity.

The proofs of Theorem 1.1 and 1.2 presented in this paper basically follow the framework of the proof in Li-Titi [30], which are divided into two parts, the a priori estimates on the global strong solutions of the primitive equations with full viscosity and full diffusion (1.7) and the a priori estimates on the difference function \((V_\epsilon, W_\epsilon, \Phi_\epsilon) := (v_\epsilon - v, w_\epsilon - w, \theta_\epsilon - \theta)\). For the first part, we successively obtain \( L^2 \) estimates on \( v \) and \( \theta \), \( L^1 \) estimates on \( v \) \ and \( \theta \), \( L^2 \) estimates on \( \partial_z v \) and \( \partial_z \theta \), \( L^2 \) estimates on \( \nabla v \) \ and \( \nabla \theta \), as well as \( L^2 \) estimates on \( \Delta v \) \ and \( \Delta \theta \), but the upper bound of each estimate is a monotonically increasing function depending only on time and defined on \([0, \infty)\).

For the second part, the idea of weak-strong uniqueness is used to estimate the difference function in the case of Theorem 1.1. Since the upper bound of each a priori estimate on the global strong solutions of the system (1.7) is a function that only depends on time, it is crucial to use these estimates at the right time. Specifically, these estimates should be used in the last step, otherwise it will be extremely difficult to obtain the desired results, see Proposition 4.2. Obviously, the upper bound for the a priori estimate on the difference function is also a monotonically increasing function depending on time and defined on \([0, \infty)\). In the case of Theorem 1.2, we apply energy method to estimate the difference function. It is noted that the basic energy estimate on the difference function here is consistent with Proposition 4.2 on the maximal existence time interval of the local strong solutions to the system (1.2), see Proposition 5.1. Moreover, the upper bound for the first order energy estimate on difference function in this case is also a monotonically increasing function depending on time and defined on \([0, T_\ast)\), see Proposition 5.2. As a consequence, it is natural to use the values of the two monotonically increasing functions at the maximal existence time as the upper bounds on the basic energy estimate and the first order energy estimate, respectively, see Proposition 5.3.

The rest of this paper is organized as follows. Some preliminary lemmas that will be used in subsequent sections are collected in Section 2. In Section 3, we establish the a priori estimates on the global strong solutions of the primitive equations with full viscosity and full diffusion (1.7). The proofs of Theorem 1.1 and 1.2 are presented in Section 4 and Section 5, respectively.

2 Preliminaries

For convenience, in this section we present some Ladyzhenskaya-type inequality in three dimensions for a class of integrals without proving them, which will be frequently used throughout the paper.

**Lemma 2.1.** (see [12]) The following inequalities

\[
\int_M \left( \int_{-1}^1 \varphi(x, y, z)dz \right) \left( \int_{-1}^1 \psi(x, y, z)\phi(x, y, z)dz \right) dxdy \\
\leq C \|\varphi\|_2^{1/2} \left( \|\psi\|_2^{1/2} + \|\nabla_h \psi\|_2^{1/2} \right) \|\psi\|_2^{1/2} \left( \|\psi\|_2^{1/2} + \|\nabla_h \psi\|_2^{1/2} \right) \|\phi\|_2,
\]

\[
\int_M \left( \int_{-1}^1 \varphi(x, y, z)dz \right) \left( \int_{-1}^1 \psi(x, y, z)\phi(x, y, z)dz \right) dxdy
\]

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\[ C \| \varphi \|_{2}^{1/2} \left( \| \varphi \|_{2}^{1/2} + \| \nabla_{h} \varphi \|_{2}^{1/2} \right) \| \phi \|_{2}^{1/2} \left( \| \phi \|_{2}^{1/2} + \| \nabla_{h} \phi \|_{2}^{1/2} \right) \| \psi \|_{2}, \]

and

\[
\int_{M} \left( \int_{1}^{1} \varphi(x, y, z)dz \right) \left( \int_{1}^{1} \psi(x, y, z) \phi(x, y, z)dz \right) dxdy
\leq C \| \psi \|_{2}^{1/2} \left( \| \psi \|_{2}^{1/2} + \| \nabla_{h} \psi \|_{2}^{1/2} \right) \| \phi \|_{2}^{1/2} \left( \| \phi \|_{2}^{1/2} + \| \nabla_{h} \phi \|_{2}^{1/2} \right) \| \varphi \|_{2},
\]

hold for every \( \varphi, \psi, \phi \) such that the right-hand sides make sense and are finite, where \( C \) is a positive constant.

**Lemma 2.2.** (see \([30]\)) Let \( \varphi = (\varphi_{1}, \varphi_{2}, \varphi_{3}) \), \( \psi \) and \( \phi \) be periodic functions in \( \Omega \). Denote by \( \varphi_{h} = (\varphi_{1}, \varphi_{2}) \) the horizontal components of the function \( \varphi \). There exists a positive constant \( C \) such that the following estimate holds

\[
\left| \int_{\Omega} (\varphi \cdot \nabla \psi) \phi dx dy dz \right| \leq C \| \nabla \varphi_{h} \|_{2}^{1/2} \| \Delta \varphi_{h} \|_{2}^{1/2} \| \Delta \psi \|_{2}^{1/2} \| \phi \|_{2},
\]

provided that \( \varphi \in H^{1}(\Omega) \), with \( \nabla \cdot \varphi = 0 \) in \( \Omega \), \( \int_{\Omega} \varphi dx dy dz = 0 \), and \( \varphi_{3}|_{z=0} = 0 \), \( \nabla \psi \in H^{1}(\Omega) \) and \( \phi \in L^{2}(\Omega) \).

### 3 A priori estimates on the primitive equations

As mentioned in the introduction, the global well-posedness of strong solutions to the primitive equations with full viscosity and full diffusion (1.7) is due to Cao-Titi \([10]\). Specifically, for initial data \((v_{0}, \theta_{0}) \in H^{1}(\Omega)\) that satisfies

\[
\int_{-1}^{1} \nabla_{h} \cdot v_{0}(x, y, z)dz = 0, \text{ for all } (x, y) \in M,
\]

there exists a unique global strong solution \((v, \theta)\) to the system (1.7) with the boundary and initial conditions (1.3)-(1.4) and symmetry condition (1.5), such that \((v, \theta) \in C([0, \infty); H^{1}(\Omega)) \cap L^{2}_{loc}([0, \infty); H^{2}(\Omega))\) and \((\partial_{t} v, \partial_{t} \theta) \in L^{2}_{loc}([0, \infty); L^{2}(\Omega))\).

In this section, we carry out the proof of the first part of Theorem 1.1 and 1.2, i.e., the a priori estimates on the global strong solutions to the system (1.7). Firstly, we reformulate the system (1.7). Integrating the second equation in system (1.7) with respect to \( z \) yields

\[
p(x, y, z, t) = p_{v}(x, y, t) + \int_{0}^{z} \theta(x, y, \xi, t) d\xi,
\]

where \( p_{v}(x, y, t) \) represents unknown surface pressure as \( z = 0 \). Based on the above relation, we can reformulate the system (1.7) as

\[
\partial_{t} v - \Delta v + (v \cdot \nabla_{h}) v + w \partial_{z} v + \nabla_{h} p_{v}(x, y, t) + \nabla_{h} \int_{0}^{z} \theta(x, y, \xi, t) d\xi = 0, \quad (3.1)
\]

\[
\partial_{t} \theta - \Delta \theta + v \cdot \nabla_{h} \theta + w \partial_{z} \theta = 0, \quad (3.2)
\]

\[
\nabla_{h} \cdot v + \partial_{z} w = 0. \quad (3.3)
\]

Next, we merely need to perform several a priori estimates on the unique global strong solutions to the system (3.1)-(3.3) with initial data \((v_{0}, \theta_{0})\), corresponding to boundary and initial conditions \((v, w)\) and \( \theta \) are periodic in \( x, y, z \),

\[
(v, \theta)|_{t=0} = (v_{0}, \theta_{0}),
\]

and symmetry condition

\[
v, w \text{ and } \theta \text{ are even, odd and odd with respect to } z, \text{ respectively.}
\]
3.1 $L^2$ estimates on $v$ and $\theta$ for $H^1$ initial data

Multiplying the equation (3.2) by $\theta$, integrating over $\Omega$, and formally integrating by parts, we obtain

$$\frac{1}{2} \frac{d}{dt} \|\theta\|_2^2 + \|\nabla \theta\|_2^2 = 0,$$

where we have used the following fact that

$$\int_{\Omega} (v \cdot \nabla \theta + w \partial_z \theta) \theta dxdydz = 0.$$

Integrating the differential equation above in time between 0 to $t$ yields

$$\|\theta(t)\|_2^2 + \int_0^t \|\nabla \theta\|_2^2 ds \leq \|\theta_0\|_2^2.$$  (3.4)

Taking the dot product of the equation (3.1) with $v$ and integrating over $\Omega$, then it follows from integration by parts that

$$\frac{1}{2} \frac{d}{dt} \|v\|_2^2 + \|\nabla v\|_2^2 = \int_{\Omega} \left( \int_0^z \theta(x, y, \xi, t) d\xi \right) (\nabla_h \cdot v) dxdydz,$$

where we have used the following facts that

$$\int_{\Omega} [(v \cdot \nabla_h) v + w \partial_z v] \cdot v dxdydz = 0,$$

$$\int_{\Omega} \nabla_h p_v(x, y, t) \cdot v dxdydz = 0.$$

By virtue of the Hölder inequality we have

$$\frac{1}{2} \frac{d}{dt} \|v\|_2^2 + \|\nabla v\|_2^2 \leq 2\sqrt{2} \|\theta\|_2 \|\nabla v\|_2.$$

Using the Cauchy-Schwarz inequality, it follows from (3.4) that

$$\frac{d}{dt} \|v\|_2^2 + \|\nabla v\|_2^2 \leq 8 \|\theta_0\|_2^2.$$

We now integrate the above inequality from 0 to $t$ to obtain

$$\|v(t)\|_2^2 + \int_0^t \|\nabla v\|_2^2 ds \leq \|v_0\|_2^2 + 8t \|\theta_0\|_2^2.$$  (3.5)

Summing (3.4) and (3.5) we have

$$\|v(t)\|_2^2 + \int_0^t \|\nabla v\|_2^2 ds + \|\theta(t)\|_2^2 + \int_0^t \|\nabla \theta\|_2^2 ds \leq \alpha_1(t),$$  (3.6)

where

$$\alpha_1(t) = (8t + 1) \left( \|v_0\|_{H^1}^2 + \|\theta_0\|_{H^1}^2 \right).$$  (3.7)

3.2 $L^4$ estimates on $v$ and $\theta$ for $H^1$ initial data

Multiplying the equation (3.2) by $|\theta|^2 \theta$, integrating over $\Omega$, and formally integrating by parts, we obtain

$$\frac{1}{4} \frac{d}{dt} \|\theta\|_4^4 + \int_{\Omega} |\theta|^2 \left( |\nabla \theta|^2 + 2 |\nabla |\theta||^2 \right) dxdydz = 0,$$  (3.8)
where we have used the following fact that
\[
\int_\Omega (v \cdot \nabla h + w \partial_z \theta) |\theta|^2 |v|^2 dx dy dz = 0.
\]
Integrating the equation (3.8) in time between 0 to \(t\) yields
\[
\|\theta(t)\|^4_4 + \int_0^t \int_\Omega |\theta|^2 |\nabla \theta|^2 dx dy dz ds \leq \|\theta_0\|^4_4. \quad (3.9)
\]
Taking the dot product of the equation (3.1) with \(|v|^2 v\) and integrating over \(\Omega\), then it follows from integration by parts that
\[
\frac{1}{4} \frac{d}{dt} \|v\|^4_4 + \int_\Omega |v|^2 \left( |\nabla v|^2 + 2 |v|^2 \right) dx dy dz
\]
\[
= \int_\Omega \left( \int_0^z \theta(x, y, \xi, t) d\xi \right) (\nabla h \cdot |v|^2 v) dx dy dz
\]
\[
- \int_\Omega \nabla h p_\nu (x, y, t) \cdot |v|^2 v dx dy dz, \quad (3.10)
\]
where we have used the following fact that
\[
\int_\Omega \left( \int_0^z \theta(x, y, \xi, t) d\xi \right) (\nabla h \cdot |v|^2 v) dx dy dz = 0.
\]
Firstly, we estimate the first term on the right-hand side of (3.10). Thanks to the Hölder inequality we have
\[
\int_\Omega \left( \int_0^z \theta(x, y, \xi, t) d\xi \right) (\nabla h \cdot |v|^2 v) dx dy dz
\]
\[
\leq C \int_\Omega \left( \int_0^1 |\theta| |dz| \right) \left( \int_0^1 |v|^2 |\nabla h v| |dz| \right) dx dy
\]
\[
\leq C \int_\Omega \left( \int_0^1 |\theta| |dz| \right) \left( \int_0^1 |v|^2 |dz| \right)^{1/2} \left( \int_0^1 |v|^2 |\nabla h v|^2 |dz| \right)^{1/2} dx dy
\]
\[
\leq C \left( \int_\Omega |v|^4 dx dy dz \right)^{1/4} \left( \int_\Omega |v|^2 |\nabla v|^2 dx dy dz \right)^{1/4} \left( \int_\Omega |v|^2 |\nabla h v|^2 dx dy dz \right)^{1/2}
\]
\[
\leq C \|\theta\|_4 \|v\|_4 \left( \int_\Omega |v|^2 |\nabla v|^2 dx dy dz \right)^{1/2}. \quad (3.11)
\]
By virtue of the Young inequality, it follows that
\[
\int_\Omega \left( \int_0^z \theta(x, y, \xi, t) d\xi \right) (\nabla h \cdot |v|^2 v) dx dy dz
\]
\[
\leq C \|\theta\|_4 \|v\|_4^2 + \frac{3}{8} \int_\Omega |v|^2 |\nabla v|^2 dx dy dz
\]
\[
\leq C \|v\|_4^4 + \frac{1}{16} \|\theta\|_4^4 + \frac{3}{8} \int_\Omega |v|^2 |\nabla v|^2 dx dy dz. \quad (3.11)
\]
Next, we estimate the second term on the right-hand side of (3.10). Using the Lemma 2.1 and Poincaré inequality, we obtain
\[
- \int_\Omega \nabla h p_\nu (x, y, t) \cdot |v|^2 v dx dy dz
\]
Applying the Gronwall inequality to the above inequality, then it follows from (3.1) and integrating the resulting equation with respect to $z$ from $-1$ to 1, we can see that $p_v(x, y, t)$ satisfies the following system
\[
-\Delta_h p_v = \frac{1}{2} \int_{-1}^{1} \nabla_h \cdot \left[ (\nabla_h \cdot (v \otimes v)) + f_0 \nabla_h \theta dz \right] dz,
\]
where the condition $\int_M p_v(x, y, t) dxdy = 0$ is imposed to guarantee the uniqueness of $p_v(x, y, t)$.

By the elliptic estimates and Poincaré inequality, we have
\[
\|\nabla p_v\|_{2, M} \leq C (\|\nabla \theta\|_{2} + \|\nabla_h \cdot (v \otimes v)\|_{2}) \leq C (\|\nabla \theta\|_{2} + \|v\|_{\nabla_h v^2}) \leq C (\|\nabla \theta\|_{2} + \|v\|_{\nabla_h v^2}).
\]

(3.13)

Substituting (3.13) into (3.12) yields
\[
-\int_\Omega \nabla_h p_v (x, y, t) \cdot |v|^2 v dxdy dz 
\leq C \|v\|_{2}^{1/2} \|\nabla v\|_{2}^{1/2} (\|\nabla \theta\|_{2} + \|\nabla_h \cdot (v \otimes v)\|_{2}) \|v\|_{4} (\|v\|_{4}^{2} + \|v\|_{\nabla_h v^2})^{1/2} 
\leq C \|v\|_{2}^{1/2} \|\nabla v\|_{2}^{1/2} (\|\nabla \theta\|_{2} + \|\nabla_h \cdot (v \otimes v)\|_{2}) (\|v\|_{4}^{2} + \|v\|_{4} \|v\|_{\nabla v^2})^{1/2} 
\leq C \|v\|_{2}^{1/2} \|\nabla v\|_{2}^{1/2} (\|\nabla \theta\|_{2} + \|\nabla_h \cdot (v \otimes v)\|_{2})^{1/2} 
+ C \|v\|_{2}^{1/2} \|\nabla v\|_{2}^{1/2} (\|v\|_{4}^{2} + \|v\|_{4} \|v\|_{\nabla v^2})^{1/2},
\]
from which, it follows from the Young inequality that
\[
-\int_\Omega \nabla_h p_v (x, y, t) \cdot |v|^2 v dxdy dz 
\leq C (\|v\|_{2}^{2} \|\nabla v\|_{2} + |v\|_{2}^{2} \|\nabla v\|_{2}^{2} + |\nabla \theta\|_{2}^{2} \|v\|_{4}^{2} 
+ \frac{1}{4} (\|v\|_{2}^{2} \|\nabla v\|_{2} + \|\nabla \theta\|_{2}^{2} \|v\|_{4}^{2} + \frac{3}{8} \|v\|_{\nabla v^2}^{2}. \]

(3.14)

Combining (3.11) with (3.14), we obtain
\[
\frac{d}{dt} \|v\|_{4}^{2} + \int_\Omega |v|^2 |\nabla v|^2 dxdy dz 
\leq C (\|v\|_{2}^{2} \|\nabla v\|_{2} + |v\|_{2}^{2} \|\nabla v\|_{2}^{2} + |\nabla \theta\|_{2}^{2} + 1) \|v\|_{4}^{2} + \|v\|_{2}^{2} \|\nabla v\|_{2} + \|\nabla \theta\|_{2}^{2} + \|\theta\|_{4}^{2}.
\]

Applying the Gronwall inequality to the above inequality, then it follows from (3.6) and (3.9) that
\[
\|v(t)\|_{4}^{2} + \int_0^t \int_\Omega |v|^2 |\nabla v|^2 dxdy dz ds 
\leq \exp \left\{ C \int_0^t (\|v\|_{2}^{2} \|\nabla v\|_{2} + |v\|_{2}^{2} \|\nabla v\|_{2}^{2} + |\nabla \theta\|_{2}^{2} + 1) \right\} ds \times \left[ \|v_0\|_{4}^{2} + \int_0^t (\|v\|_{2}^{2} \|\nabla v\|_{2} + \|\nabla \theta\|_{2}^{2} + \|\theta\|_{4}^{2}) ds \right] 
\leq e^{C(t+2)(\alpha_7(t)+\alpha_1(t))} \left[ \|v_0\|_{4}^{2} + t \|\theta_0\|_{4}^{2} + (t^{1/2} + 1) \alpha_1(t) \right],
\]

(3.15)
Adding (3.9) and (3.15), and using the Lebesgue interpolation inequality, we have

\[
\|v(t)\|_4^4 + \int_0^t \int_\Omega |v|^2 |\nabla v|^2 \, dx \, dy \, dz \, ds \\
+ \|\theta(t)\|_4^4 + \int_0^t \int_\Omega |\theta|^2 |\nabla \theta|^2 \, dx \, dy \, dz \, ds \leq \alpha_2(t),
\]

where

\[
\alpha_2(t) = (t + 2)e^{C(t+2)(\alpha_1^2(t)+\alpha_1(t)+1)} \left( \|v_0\|_{H^1}^2 + \|\theta_0\|_{H^1}^2 + \alpha_1(t) \right).
\]

### 3.3 \(L^2\) estimates on \(\partial_z v\) and \(\partial_z \theta\) for \(H^1\) initial data

Taking the inner product to the equation (3.1) with \(-\partial_{zz} v\) in \(L^2(\Omega)\) and integrating by parts, we obtain

\[
\frac{1}{2} \frac{d}{dt} \|\partial_z v\|_2^2 + \|\nabla \partial_z v\|_2^2 \\
= \int_\Omega \left( (v \cdot \nabla h) v + w \partial_z v + \nabla h \int_0^t \theta(x, y, z, t) \, dt \right) \cdot \partial_z v \, dx \, dy \, dz \\
= \int_\Omega \left[ \partial_z (v \cdot \nabla h) + (\nabla h \cdot v) \partial_z v - \nabla h \theta \right] \cdot \partial_z v \, dx \, dy \, dz \\
= \int_\Omega \left[ (\nabla h \cdot \partial_z v) v \cdot \partial_z v + (\partial_z v \cdot \nabla h \cdot v) \partial_z v \cdot \partial_z v ight] \, dx \, dy \, dz \\
+ \int_\Omega \left[ -2 (v \cdot \nabla h) \partial_z v \cdot \partial_z v + \theta \nabla h \cdot \partial_z v \right] \, dx \, dy \, dz,
\]

where we have used the following fact that

\[
\int_\Omega \nabla h p_\nu(x, y, t) \cdot \partial_z v \, dx \, dy \, dz = 0.
\]

Using the Hölder inequality, then Lebesgue interpolation inequality, and finally Young inequality, the integral terms on the right-hand side can be bounded as

\[
\frac{1}{2} \frac{d}{dt} \|\partial_z v\|_2^2 + \|\nabla \partial_z v\|_2^2 \\
\leq C \int_\Omega \left( |v| \|\partial_z v\| |\nabla h \partial_z v| + |\theta| \|\nabla h \partial_z v| \right) \, dx \, dy \, dz \\
\leq C \|v\|_4 \|\partial_z v\|_4 \|\nabla h \partial_z v\|_2 + C \|\theta\|_2 \|\nabla h \partial_z v\|_2 \\
\leq C \|v\|_4 \|\partial_z v\|_2^{1/4} \|\nabla \partial_z v\|_2^{7/4} + C \|\theta\|_2 \|\nabla \partial_z v\|_2 \\
\leq C \|v\|_4 \|\partial_z v\|_2^2 + C \|\theta\|_2^2 + \frac{1}{2} \|\nabla \partial_z v\|_2^2,
\]

where the Poincaré inequality is used. This gives

\[
\frac{d}{dt} \|\partial_z v\|_2^2 + \|\nabla \partial_z v\|_2^2 \leq C \|v\|_4^2 \|\partial_z v\|_2^2 + C \|\theta\|_2^2.
\]

By virtue of the Gronwall inequality, then it follows from (3.6) and (3.16) that

\[
\|\partial_z v\|_2^2(t) + \int_0^t \|\nabla \partial_z v\|_2^2 \, ds \\
\leq \exp \left( C \int_0^t \|v\|_4^2 \, ds \right) \left( \|\partial_z v_0\|_2^2 + C \int_0^t \|\theta\|_2^2 \, ds \right) \leq \alpha_3(t),
\]

(3.18)
where
\[ \alpha_3(t) = Ce^{-\rho_2} \left( \|v_0\|^2_{H^1} + t\alpha_1(t) \right). \] (3.19)

Taking the inner product to the equation (3.2) with \(-\partial_z\theta\) in \(L^2(\Omega)\), it follows from integration by parts that
\[
\frac{1}{2} \frac{d}{dt} \|\partial_z\theta\|_2^2 + \|\nabla \partial_z\theta\|_2^2 = \int_{\Omega} (v \cdot \nabla_h \theta + w \partial_z \theta) \partial_z \theta \, dx \, dy \, dz
\]
\[
= \int_{\Omega} [-\partial_z v \cdot \nabla_h \theta + (\nabla_h \cdot v) \partial_z \theta] \partial_z \theta \, dx \, dy \, dz
\]
\[
= \int_{\Omega} [(\nabla_h \cdot \partial_z v) \theta \partial_z \theta + (\partial_z v \cdot \nabla_h \theta) \theta - 2 (v \cdot \nabla_h \partial_z \theta) \partial_z \theta] \, dx \, dy \, dz.
\]

By using the Hölder inequality, Lebesgue interpolation inequality, Sobolev embedding, and Poincaré inequality, we have
\[
\frac{1}{2} \frac{d}{dt} \|\partial_z\theta\|_2^2 + \|\nabla \partial_z\theta\|_2^2 \leq \|\theta\|_4 \|\partial_z\theta\|_4 \|\nabla_h \partial_z v\|_2
\]
\[
+ \left( \|\theta\|_4 \|\partial_z v\|_4 + \|v\|_4 \|\partial_z \theta\|_4 \right) \|\nabla_h \partial_z \theta\|_2
\]
\[
\leq C \|\theta\|_4 \|\partial_z\theta\|_2^{1/4} \|\nabla \partial_z\theta\|_2^{3/4} \|\nabla_h \partial_z v\|_2
\]
\[
+ C \|\theta\|_4 \|\partial_z v\|_2^{1/4} \|\nabla \partial_z\theta\|_2 \|\nabla_h \partial_z \theta\|_2
\]
\[
+ \|v\|_4 \|\partial_z \theta\|_4 \|\nabla_h \partial_z \theta\|_2^{1/4} \|\nabla \partial_z\theta\|_2^{1/4}
\]
\[
=: D_1 + D_2 + D_3.
\]

In order to estimate \(D_1, D_2\) and \(D_3\), we use the Young inequality to obtain
\[
D_1 := C \|\theta\|_4 \|\partial_z\theta\|_2^{1/4} \|\nabla \partial_z\theta\|_2^{1/4} \|\nabla_h \partial_z v\|_2
\]
\[
\leq C \|\theta\|_2^2 \|\nabla \partial_z\theta\|_2^2 + \frac{1}{2} \|\partial_z\theta\|_2^2 + \frac{1}{6} \|\nabla \partial_z\theta\|_2^2,
\]
\[
D_2 := C \|\theta\|_4 \|\partial_z\theta\|_2^{1/4} \|\nabla \partial_z\theta\|_2 \|\nabla_h \partial_z \theta\|_2
\]
\[
\leq C \|\theta\|_4^2 \|\partial_z\theta\|_2^2 + \frac{1}{2} \|\nabla \partial_z\theta\|_2^2 + \frac{1}{6} \|\nabla \partial_z\theta\|_2^2,
\]
and
\[
D_3 := C \|v\|_4 \|\partial_z\theta\|_2^{1/4} \|\nabla \partial_z\theta\|_2^{7/4}
\]
\[
\leq C \|v\|_4^2 \|\partial_z\theta\|_2^2 + \frac{1}{6} \|\nabla \partial_z\theta\|_2^2.
\]

Combining the estimates for \(D_1, D_2,\) and \(D_3\) yields
\[
\frac{d}{dt} \|\partial_z\theta\|_2^2 + \|\nabla \partial_z \theta\|_2^2 \leq C \left( 1 + \|v\|_4^4 \right) \|\partial_z\theta\|_2^2
\]
\[
+ C \left( \|\theta\|_4^2 \|\nabla \partial_z \theta\|_2^2 + \|\theta\|_4^2 \|\nabla_h \partial_z \theta\|_2^2 + \|\nabla_h \partial_z \theta\|_2^2 \right).
\]

Thanks to the Gronwall inequality, we deduce from (3.16) and (3.18) that
\[
\|\partial_z\theta\|_2^2 + \int_0^t \|\nabla \partial_z \theta\|_2^2 \, ds \leq \exp \left\{ C \int_0^t \left( 1 + \|v\|_4^4 \right) \, ds \right\}
\]
\[
\times \left[ \|\partial_z \theta_0\|_2^2 + C \int_0^t \left( \|\theta\|_4^2 \|\nabla \partial_z \theta\|_2^2 + \|\theta\|_4^2 \|\nabla_h \partial_z \theta\|_2^2 + \|\nabla_h \partial_z \theta\|_2^2 \right) \, ds \right] \leq \alpha_4(t), \tag{3.20}
\]
where
\[
\alpha_4(t) = Ce^{-\rho_2} \left[ \|\theta_0\|^2_{H^1} + \left( \alpha_2^{1/2}(t) + t\alpha_2(t) + 1 \right) \alpha_3(t) \right]. \tag{3.21}
\]

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3.4  $L^2$ estimates on $\nabla v$ and $\nabla \theta$ for $H^1$ initial data

Taking the dot product of the equation (3.1) with $\partial_t v - \Delta v$ and integrating over $\Omega$, then it follows from integration by parts that

$$
\frac{d}{dt}\|\nabla v\|_2^2 + \|\partial_t v\|_2^2 + \|\Delta v\|_2^2
= \int_{\Omega} \theta(x, y, \xi, t) \cdot (\Delta v - \partial_t v) \, dx dy dz
+ \int_{\Omega} [(v \cdot \nabla h) v] \cdot (\Delta v - \partial_t v) \, dx dy dz
+ \int_{\Omega} w \partial_z v \cdot (\Delta v - \partial_t v) \, dx dy dz
=: D_1 + D_2 + D_3,
$$

where we have used the following fact that

$$
\int_{\Omega} \nabla h p_v(x, y, t) \cdot (\partial_t v - \Delta v) \, dx dy dz = 0.
$$

By the Hölder inequality and Yong inequality, the first integral term $D_1$ on the right-hand side of (3.22) can be bounded as

$$
D_1 := \int_{\Omega} \left[ \nabla h \int_0^z \theta(x, y, \xi, t) \, d\xi \right] \cdot (\Delta v - \partial_t v) \, dx dy dz
\leq \int_M \left( \int_{-1}^1 |\nabla h| \, dz \right) \left( \int_{-1}^1 (|\partial_t v| + |\Delta v|) \, dz \right) \, dx dy
\leq C\|\nabla h|\|_2 (\|\partial_t v\|_2 + \|\Delta v\|_2)
\leq C\|\nabla h|\|_2^2 + \frac{1}{6} (\|\partial_t v\|_2^2 + \|\Delta v\|_2^2).
$$

To obtain an upper bound for the second integral term $D_2$ on the right-hand side of (3.22), we use the Lemma 2.1, the Poincaré inequality and Young inequality to write

$$
D_2 := \int_{\Omega} [(v \cdot \nabla h) v] \cdot (\Delta v - \partial_t v) \, dx dy dz
\leq \int_M \left( \int_{-1}^1 (|v| + |\partial_z v|) \, dz \right) \left( \int_{-1}^1 |\nabla h| (|\partial_t v| + |\Delta v|) \, dz \right) \, dx dy
\leq C \left( \|v\|_2^{1/2} \|\nabla v\|_2^{1/2} + \|\partial_z v\|_2^{1/2} \|\nabla \partial_z v\|_2^{1/2} \right) \left( \|\nabla v\|_2^{1/2} \|\Delta v\|_2^{1/2} \right)
\leq C \left( \|v\|_2^2 \|\nabla v\|_2^2 + \|\partial_z v\|_2^2 \|\nabla \partial_z v\|_2^2 \right) + \frac{1}{6} (\|\partial_t v\|_2^2 + \|\Delta v\|_2^2).
$$

Finally, it remains to deal with the last integral term $D_3$ on the right-hand side of (3.22). Note that the fact that $u$ is divergence free, a similar argument as $D_2$ yields

$$
D_3 := \int_{\Omega} w \partial_z v \cdot (\Delta v - \partial_t v) \, dx dy dz
\leq \int_M \left( \int_{-1}^1 |\nabla h| \, dz \right) \left( \int_{-1}^1 |\partial_z v| (|\partial_t v| + |\Delta v|) \, dz \right) \, dx dy
\leq C \|\partial_z v\|_2^{1/2} \|\nabla \partial_z v\|_2^{1/2} \|\nabla v\|_2^{1/2} \|\Delta v\|_2^{1/2} \left( \|\partial_t v\|_2 + \|\Delta v\|_2 \right)
\leq C \|\partial_z v\|_2^2 \|\nabla \partial_z v\|_2^2 \|\nabla v\|_2^2 + \frac{1}{6} (\|\partial_t v\|_2^2 + \|\Delta v\|_2^2).
$$
Combining the estimates for $D_1$, $D_2$ and $D_3$ we have

$$
\frac{d}{dt} \| \nabla v \|^2 + \frac{1}{2} \left( \| \partial_t v \|^2 + \| \Delta v \|^2 \right) \\
\leq C \left( \| v \| \| \nabla v \| + \| \partial_x v \| \| \nabla \partial_x v \| \right) \| \nabla v \|^2 + C \| \nabla \theta \|^2.
$$

Applying the Gronwall inequality to the above inequality, then it follows from (3.6) and (3.18) that

$$
\| \nabla v \|^2 + \int_0^t \left( \| \partial_t v \|^2 + \| \Delta v \|^2 \right) ds \\
\leq \exp \left\{ C \int_0^t \left( \| v \| \| \nabla v \| + \| \partial_x v \| \| \nabla \partial_x v \| \right) ds \right\} \\
\times \left( \| \nabla v_0 \|^2 + C \int_0^t \| \nabla \theta \|^2 ds \right) \leq \alpha_5(t),
$$

(3.23)

where

$$
\alpha_5(t) = C e^{C \left( \alpha_5(t) + \alpha_5(t) \right)} \left( \| v_0 \|_{H^1} + \alpha_1(t) \right).
$$

(3.24)

Multiplying the equation (3.2) by $\partial_t \theta - \Delta \theta$, integrating over $\Omega$, and formally integrating by parts, we obtain

$$
\frac{d}{dt} \| \nabla \theta \|^2 + \| \partial_t \theta \|^2 + \| \Delta \theta \|^2 \\
= \int_{\Omega} \langle v \cdot \nabla \theta \rangle (\Delta \theta - \partial_t \theta) \, dx dy dz \\
+ \int_{\Omega} w \partial_x \theta (\Delta \theta - \partial_t \theta) \, dx dy dz.
$$

(3.25)

With the similar argument of the last two terms on the right-hand side of (3.22), we can bound the first and second term on the right-hand side of (3.25) as

$$
\int_{\Omega} \langle v \cdot \nabla \theta \rangle (\Delta \theta - \partial_t \theta) \, dx dy dz \\
\leq \int_M \left( \int_{-1}^1 \| v \| + \| \partial_x v \| \right) \left( \int_{-1}^1 \| \nabla \theta \| \left( \| \partial_t \theta \| + |\Delta \theta| \right) \right) \, dx dy \\
\leq C \left( \| v \| \| \nabla v \| + \| \partial_x v \| \| \nabla \partial_x v \| \right) \| \nabla \theta \|^2 + \frac{1}{4} \| \partial_t \theta \|^2 + \| \Delta \theta \|^2
$$

(3.26)

and

$$
\int_{\Omega} w \partial_x \theta (\Delta \theta - \partial_t \theta) \, dx dy dz \\
\leq \int_M \left( \int_{-1}^1 \| v \| \, dz \right) \left( \int_{-1}^1 \| \partial_x \theta \| \left( \| \partial_t \theta \| + |\Delta \theta| \right) \right) \, dx dy \\
\leq C \| \partial_x \theta \| \| \nabla \partial_x \theta \| \| \nabla v \| \| \Delta v \| + \frac{1}{4} \left( \| \partial_t \theta \|^2 + \| \Delta \theta \|^2 \right),
$$

(3.27)

respectively. Substituting (3.26) and (3.27) into (3.25) we reach

$$
\frac{d}{dt} \| \nabla \theta \|^2 + \frac{1}{2} \left( \| \partial_t \theta \|^2 + \| \Delta \theta \|^2 \right)
$$
\begin{align*}
&\leq C (\|v\|_{L^2}^2 \|\nabla v\|_{L^2}^2 + \|\partial_z v\|_{L^2}^2 \|\nabla z v\|_{L^2}^2) \|\nabla \theta\|_{L^2}^2 \\
&\quad + C (\|\partial_z \theta\|_{L^2}^2 \|\nabla \partial_z \theta\|_{L^2}^2 + \|\nabla v\|_{L^2}^2 \|\Delta v\|_{L^2}^2) .
\end{align*}

By virtue of the Gronwall inequality, then it deduces from (3.6), (3.18), (3.20), and (3.23) that

\begin{align*}
\|\nabla \theta\|_{L^2}^2 + \int_0^t (\|\partial_z \theta\|_{L^2}^2 + \|\Delta \theta\|_{L^2}^2) \, ds \\
&\leq \exp \left\{ C \int_0^t (\|v\|_{L^2}^2 \|\nabla v\|_{L^2}^2 + \|\partial_z v\|_{L^2}^2 \|\nabla z v\|_{L^2}^2) \|\nabla \theta\|_{L^2}^2 \, ds \right\} \\
&\quad \times \left[ \|\nabla \theta_0\|_{L^2}^2 + C \int_0^t (\|\partial_z \theta\|_{L^2}^2 \|\nabla \partial_z \theta\|_{L^2}^2 + \|\nabla v\|_{L^2}^2 \|\Delta v\|_{L^2}^2) \, ds \right] \leq \alpha_0(t), \quad (3.28)
\end{align*}

where

\begin{align*}
\alpha_0(t) = Ce^{C(\alpha_1^3(t)+\alpha_2^2(t))} (\|\theta_0\|_{H^1}^2 + \alpha_2(t) + \alpha_3(t)). \quad (3.29)
\end{align*}

### 3.5 $L^2$ estimates on $\Delta v$ and $\Delta \theta$ for $H^2$ initial data

Taking the $L^2(\Omega)$ inner product of the equation (3.1) with $\Delta (\Delta v - \partial_t v)$ and integrating by parts, we have

\begin{align*}
\frac{d}{dt} \|\Delta v\|_{L^2}^2 + \|\nabla \partial_t v\|_{L^2}^2 + \|\nabla \Delta v\|_{L^2}^2 \\
&= \int_\Omega \nabla [(v \cdot \nabla h) v + w \partial_z v] : \nabla (\Delta v - \partial_t v) \, dxdydz \\
&\quad + \int_\Omega \nabla \left[ \int_0^\xi \nabla h \theta(x, y, \xi, t) d\xi \right] : \nabla (\Delta v - \partial_t v) \, dxdydz, \quad (3.30)
\end{align*}

where we have used the following fact that

\[ \int_\Omega \nabla_h p_v(x, y, t) \cdot \Delta (\Delta v - \partial_t v) \, dxdydz = 0. \]

For the first integral term on the right-hand side of (3.30), we use the Lemma 2.2 and Young inequality to obtain

\begin{align*}
&\int_\Omega \nabla [(v \cdot \nabla h) v + w \partial_z v] : \nabla (\Delta v - \partial_t v) \, dxdydz \\
&= \int_\Omega \nabla [(u \cdot \nabla) v] : \nabla (\Delta v - \partial_t v) \, dxdydz \\
&= \int_\Omega [\partial_h u \cdot \nabla] v + (u \cdot \nabla) \partial_t v \cdot (\partial_h \Delta v - \partial_t \partial_t v) \, dxdydz \\
&\leq C \|\nabla v\|_{L^2}^{1/2} \|\Delta v\|_{L^2}^{1/2} \|\partial_t \nabla v\|_{L^2}^{1/2} \|\partial_t \Delta v\|_{L^2}^{1/2} (\|\partial_h \partial_t v\|_{L^2}^2 + \|\partial_t \Delta v\|_{L^2}^2) \\
&\leq C \|\nabla v\|_{L^2}^2 \|\Delta v\|_{L^2}^2 + \frac{1}{4} (\|\partial_h \partial_t v\|_{L^2}^2 + \|\partial_t \Delta v\|_{L^2}^2). \quad (3.28)
\end{align*}

In order to estimate the second integral term on the right-hand side of (3.30), we split the gradient operator into two parts, $\nabla_h$ and $\partial_z$. Then using the Hölder inequality and Young inequality yields

\begin{align*}
&\int_\Omega \nabla \left[ \int_0^\xi \nabla h \theta(x, y, \xi, t) d\xi \right] : \nabla (\Delta v - \partial_t v) \, dxdydz \\
&= \int_\Omega \nabla_h \left[ \int_0^\xi \nabla h \theta(x, y, \xi, t) d\xi \right] : \nabla_h (\Delta v - \partial_t v) \, dxdydz
\end{align*}
\[ + \int_{\Omega} \nabla_h \theta \cdot (\partial_\theta \Delta v - \partial_\theta \partial_\theta v) \, dx dy dz \]
\[ = \int_{\Omega} \left[ \int_{0}^{1} \partial \nabla_h \theta(x, y, \xi, t) \, d\xi \right] \cdot (\partial_\theta \Delta v - \partial_\theta \partial_\theta v) \, dx dy dz \]
\[ + \int_{\Omega} \nabla_h \theta \cdot (\partial_\theta \Delta v - \partial_\theta \partial_\theta v) \, dx dy dz \]
\[ \leq \int_{M} \left( \int_{-1}^{1} |\partial_\theta \nabla_h \theta| \, dz \right) \left( \int_{-1}^{1} (|\partial_\theta \partial_\theta v| + |\partial_\theta \Delta v|) \, dz \right) \, dx dy \]
\[ + \int_{\Omega} (|\partial_\theta \partial_\theta v| + |\partial_\theta \Delta v|) \, dx dy dz \]
\[ \leq C (\| \nabla \theta \|_2^2 + \| \Delta \theta \|_2^2) + \frac{1}{4} (\| \nabla \partial_\theta v \|_2^2 + \| \nabla \Delta v \|_2^2) . \]

Combining the two estimates we obtain
\[ \frac{d}{dt} \| \Delta v \|_2^2 + \frac{1}{2} (\| \nabla \partial_\theta v \|_2^2 + \| \nabla \Delta v \|_2^2) \leq C (\| \nabla \theta \|_2^2 + \| \Delta \theta \|_2^2) \leq C (\| \nabla \theta \|_2^2 + \| \Delta \theta \|_2^2) . \]

Thanks to the Gronwall inequality, then it follows from (3.23) and (3.28) that
\[ \| \Delta v \|_2^2 + \int_{0}^{t} (\| \nabla \partial_\theta v \|_2^2 + \| \nabla \Delta v \|_2^2) \, ds \leq \exp \left\{ C \int_{0}^{t} (\| \nabla \theta \|_2^2 + \| \Delta \theta \|_2^2) \, ds \right\} \leq \alpha_7(t), \quad (3.31) \]
where
\[ \alpha_7(t) = C(t + 1)e^{C\alpha_7(t)} (\| v_0 \|_{H^2}^2 + \alpha_6(t)) . \quad (3.32) \]

Taking the \( L^2(\Omega) \) inner product of the equation (3.2) with \( \Delta (\Delta \theta - \partial_\theta \theta) \) and integrating by parts yield
\[ \frac{d}{dt} \| \Delta \theta \|_2^2 + \| \nabla \partial_\theta \theta \|_2^2 + \| \nabla \Delta \theta \|_2^2 \]
\[ = \int_{\Omega} \nabla (u \cdot \nabla \theta) \cdot \nabla (\Delta \theta - \partial_\theta \theta) \, dx dy dz . \]

Using the same method as the first integral term on the right-hand side of (3.30), the integral term on the right-hand side can be bounded as
\[ \int_{\Omega} \nabla (u \cdot \nabla \theta) \cdot \nabla (\Delta \theta - \partial_\theta \theta) \, dx dy dz \]
\[ = \int_{\Omega} (\partial_\theta u \cdot \nabla \theta + u \cdot \nabla \theta) \cdot (\partial_\theta \Delta \theta - \partial_\theta \partial_\theta \theta) \, dx dy dz \]
\[ \leq C \| \partial_\theta v \|_{L^2}^{1/2} \| \partial_\theta \Delta v \|_{L^2}^{1/2} \| \nabla \theta \|_{L^2}^{1/2} \| \Delta \theta \|_{L^2}^{1/2} \| \Delta \theta \|_{L^2}^{1/2} (\| \partial_\theta \partial_\theta \theta \|_2 + \| \partial_\theta \Delta \theta \|_2) \]
\[ + C \| \nabla v \|_{L^2}^{1/2} \| \Delta v \|_{L^2}^{1/2} \| \partial_\theta \nabla \theta \|_{L^2}^{1/2} \| \Delta \theta \|_{L^2}^{1/2} \| \partial_\theta \Delta \theta \|_{L^2}^{1/2} \| \partial_\theta \partial_\theta \theta \|_2 + \| \partial_\theta \Delta \theta \|_2) \]
\[ \leq C \| \nabla \theta \|_2^2 \| \Delta v \|_2^2 \| \Delta \theta \|_2^2 + \frac{1}{2} (\| \nabla \partial_\theta \theta \|_2^2 + \| \nabla \Delta \theta \|_2^2) \]
\[ + C (\| \Delta v \|_2^2 \| \Delta v \|_2^2 + \| \nabla \theta \|_2^2 \| \Delta \theta \|_2^2) . \]
from which, we have
\[
\frac{d}{dt} \|\Delta \theta\|_2^2 + \frac{1}{2} \left( \|\nabla \partial_t \theta\|_2^2 + \|\nabla \Delta \theta\|_2^2 \right) \leq C \|\nabla v\|_2^2 \|\Delta v\|_2^2 \Delta \theta\|_2^2 + C \left( \|\Delta v\|_2^2 \|\nabla \Delta v\|_2^2 + \|\nabla \theta\|_2^2 \|\Delta \theta\|_2^2 \right).
\]

By virtue of the Gronwall inequality, it follows from (3.23), (3.28) and (3.31) that
\[
\|\Delta \theta\|_2^2 + \int_0^t \left( \|\nabla \partial_t \theta\|_2^2 + \|\nabla \Delta \theta\|_2^2 \right) ds \leq \exp \left\{ C \int_0^t \|\nabla v\|_2^2 \|\Delta v\|_2^2 ds \right\} \times \left( \|\Delta \theta_0\|_2^2 + C \int_0^t \left( \|\Delta v\|_2^2 \|\nabla \Delta v\|_2^2 + \|\nabla \theta\|_2^2 \|\Delta \theta\|_2^2 \right) ds \right) \leq \alpha_8(t),
\]
where
\[
\alpha_8(t) = Ce^{C\alpha_2(t)} \left( \|\theta_0\|_{H^2}^2 + \alpha_6(t) + \alpha_7(t) \right).
\]

4 Strong convergence for $H^1$ initial data

In this section, under the assumption of the initial data $(v_0, \theta_0) \in H^1(\Omega)$, in which
\[
\int_{-1}^1 \nabla_h \cdot v_0(x, y, z) dz = 0, \text{ for all } (x, y) \in M,
\]
we prove that the scaled Boussinesq equations (1.2) strongly converge to the primitive equations with full viscosity and full diffusion (1.7) as the aspect ratio parameter $\varepsilon$ goes to zero. The proof of the first part of Theorem 1.1 was completed in Section 3, and then we only need to estimate the difference function $(V_\varepsilon, W_\varepsilon, \Phi_\varepsilon)$. Before this, we need the following proposition.

Proposition 4.1. Suppose that $(v_\varepsilon, w_\varepsilon, \theta_\varepsilon)$ is the weak solution of the system (1.2) on the time interval $[0, \infty)$, that $(v, w, \theta)$ is the strong solution of the system (1.7) on the time interval $[0, \infty)$, with the same initial data $(v_0, w_0, \theta_0)$, and that $(v_0, \theta_0) \in H^1(\Omega)$ satisfying
\[
\int_{-1}^1 \nabla_h \cdot v_0 dz = 0 \text{ and } w_0(x, y, z) = -\int_0^z \nabla_h \cdot v_0(x, y, \xi) d\xi.
\]

Then the following integral equality
\[
\left( \int_{\Omega} (v_\varepsilon \cdot v + \varepsilon^2 w_\varepsilon w + \theta_\varepsilon \theta) \, dx dy dz \right) (r) - \frac{\varepsilon^2}{2} \|w(r)\|_2^2
\]
\[
+ \int_0^r \int_{\Omega} \left( \nabla v_\varepsilon \cdot \nabla v + \varepsilon^2 \nabla w_\varepsilon \cdot \nabla w + \nabla \theta_\varepsilon \cdot \nabla \theta \right) \, dx dy dz dt
\]
\[
= \|v_0\|_2^2 + \int_0^r \int_{\Omega} \left[ -(u_\varepsilon \cdot \nabla) v_\varepsilon \cdot v - \varepsilon^2 (u_\varepsilon \cdot \nabla w_\varepsilon) w - (u_\varepsilon \cdot \nabla \theta_\varepsilon) \theta \right] \, dx dy dz dt
\]
\[
+ \frac{\varepsilon^2}{2} \|w_0\|_2^2 + \varepsilon^2 \int_0^r \int_{\Omega} \left( \int_0^z \partial_t v(x, y, \xi, t) d\xi \right) \cdot \nabla_h W_\varepsilon \, dx dy dz dt
\]
\[
+ \|\theta_0\|_2^2 + \int_0^r \int_{\Omega} (v_\varepsilon \cdot \partial_t v + \varepsilon \partial_t \theta + \varepsilon w_\varepsilon) \, dx dy dz dt
\]
holds for any $r \in [0, \infty)$.

The Proposition 4.1 is formally obtained by testing the system (1.2) with the global strong solution $(v, w, \theta)$ of the system (1.7), while the rigorous proof for this proposition is due to the similar argument in Li-Titi [30] and Bardos et al. [2]. With the help of this proposition, we can estimate the difference function $(V_\varepsilon, W_\varepsilon, \Phi_\varepsilon)$. 
Proposition 4.2. Let \((V_\varepsilon, W_\varepsilon, \Phi_\varepsilon) := (v_\varepsilon - v, w_\varepsilon - w, \theta_\varepsilon - \theta)\). Under the same assumptions as in Proposition 4.1, the following estimate
\[
\sup_{0 \leq s \leq t} \left( ||(V_\varepsilon, \varepsilon W_\varepsilon, \Phi_\varepsilon)||^2_2 \right)(s) + \int_0^t ||\nabla (V_\varepsilon, \varepsilon W_\varepsilon, \Phi_\varepsilon)||^2 ds \leq \varepsilon^2 \beta_1(t)
\]
holds for any \(t \in [0, \infty)\), where
\[
\beta_1(t) = C e^{C(t + \alpha_5(t) + \alpha_6(t))} \left[ \alpha_5(t) + \alpha_6(t) + \left( ||v_0||^2_2 + \varepsilon^2 ||w_0||^2_2 + t ||\theta_0||^2_2 \right)^2 \right],
\]
and \(C\) is a positive constant that does not depend on \(\varepsilon\).

Proof. Firstly, we multiply the first three equation in system (1.7) by \(v_\varepsilon, w_\varepsilon\) and \(\theta_\varepsilon\), respectively, integrate over \(\Omega \times (0, r)\), and then integrate by parts to obtain
\[
\int_0^r \int_\Omega (v_\varepsilon \cdot \partial_t v + \theta_\varepsilon \cdot \theta + \nabla v_\varepsilon \cdot \nabla v + \nabla \theta_\varepsilon \cdot \nabla \theta) dx dy dt = \int_0^r \int_\Omega [\theta w_\varepsilon - (u \cdot \nabla)v \cdot v_\varepsilon - (u \cdot \nabla \theta) \theta_\varepsilon] dx dy dt, \quad (4.2)
\]
noting that the resultants have been added up. Next, replacing \((v_\varepsilon, w_\varepsilon, \theta_\varepsilon)\) with \((v, w, \theta)\) a similar argument yields
\[
\frac{1}{2} \left( ||v(r)||^2_2 + ||\theta(r)||^2_2 \right) + \int_0^r \left( ||\nabla v||^2_2 + ||\nabla \theta||^2_2 \right) dt = \frac{1}{2} \left( ||v_0||^2_2 + ||\theta_0||^2_2 \right) + \int_0^r \int_\Omega \theta^2 w dx dy dt. \quad (4.3)
\]
Finally, we perform \(L^2\) estimates for the system (1.2). Multiplying the first and second equation in system (1.2) by \(v_\varepsilon\) and \(w_\varepsilon\), respectively, summing the resultants up and integrating over \(\Omega \times (0, r)\), then it follows from integration by parts that
\[
\frac{1}{2} \left( ||v_\varepsilon(r)||^2_2 + \varepsilon^2 ||w_\varepsilon(r)||^2_2 \right) + \int_0^r \left( ||\nabla v_\varepsilon||^2_2 + \varepsilon^2 ||\nabla w_\varepsilon||^2_2 \right) dt = \frac{1}{2} \left( ||v_0||^2_2 + \varepsilon^2 ||w_0||^2_2 \right) + \int_0^r \int_\Omega \varepsilon \varepsilon w dx dy dt. \quad (4.4)
\]
Multiplying the third equation in system (1.2) by \(\varepsilon \varepsilon\), integrating over \(\Omega \times (0, r)\), and integrating by parts, we have
\[
\frac{1}{2} \|\varepsilon \varepsilon_r\|_2^2 + \int_0^r \|\nabla \varepsilon \varepsilon_r\|_2^2 dt = \frac{1}{2} \|\varepsilon \varepsilon_0\|_2^2. \quad (4.5)
\]
By the Hölder inequality and Young inequality, it deduces from (4.5) that
\[
||v_\varepsilon(r)||^2_2 + \varepsilon^2 ||w_\varepsilon(r)||^2_2 + \int_0^r \left( ||\nabla v_\varepsilon||^2_2 + \varepsilon^2 ||\nabla w_\varepsilon||^2_2 \right) dt \leq C \left( ||v_0||^2_2 + \varepsilon^2 ||w_0||^2_2 + r ||\theta_0||^2_2 \right), \quad (4.6)
\]
where we have used the fact that \(u_\varepsilon\) is divergence free. Adding (4.4) and (4.5) we obtain
\[
\frac{1}{2} \left( ||v_\varepsilon(r)||^2_2 + \varepsilon^2 ||w_\varepsilon(r)||^2_2 + ||\varepsilon \varepsilon_r||^2_2 \right) + \int_0^r \left( ||\nabla v_\varepsilon||^2_2 + \varepsilon^2 ||\nabla w_\varepsilon||^2_2 + ||\nabla \varepsilon \varepsilon_r||^2_2 \right) dt = \frac{1}{2} \left( ||v_0||^2_2 + \varepsilon^2 ||w_0||^2_2 + ||\varepsilon \varepsilon_0||^2_2 \right) + \int_\Omega \varepsilon \varepsilon w dx dy dt. \quad (4.7)
\]
Now, we subtract the sum of (4.1) and (4.2) from the sum of (4.3) and (4.7) to write
\[
\frac{1}{2} \left( \| V_\varepsilon(r) \|_2^2 + \varepsilon^2 \| W_\varepsilon(r) \|_2^2 + \| \Phi_\varepsilon(r) \|_2^2 \right) \\
+ \int_0^t \left( \| \nabla V_\varepsilon \|_2^2 + \varepsilon^2 \| \nabla W_\varepsilon \|_2^2 + \| \nabla \Phi_\varepsilon \|_2^2 \right) dt \\
= \int_0^t \left[ (u_\varepsilon \cdot \nabla \theta_\varepsilon) + (u \cdot \nabla \theta) \theta_\varepsilon + \Phi_\varepsilon W_\varepsilon \right] dx dy dz dt \\
+ \int_0^t \left[ (u_\varepsilon \cdot \nabla) v_\varepsilon + (u \cdot \nabla) v \right] dx dy dz dt \\
+ \varepsilon^2 \int_0^t \int_{\Omega} \left[ - \left( \int_0^z \partial_t v(x, y, \xi, t) d\xi \right) \cdot \nabla h W_\varepsilon - \nabla w \cdot \nabla W_\varepsilon \right] dx dy dz dt \\
+ \varepsilon^2 \int_0^t \int_{\Omega} (u_\varepsilon \cdot \nabla w_\varepsilon) w dx dy dz dt =: R_1 + R_2 + R_3 + R_4. \\
\] (4.8)

In order to estimate the first integral term $R_1$ on the right-hand side of (4.8), we use divergence-free condition and integration by parts to obtain
\[
R_1 := \int_0^t \int_{\Omega} [(u_\varepsilon \cdot \nabla \theta_\varepsilon) + (u \cdot \nabla \theta) \theta_\varepsilon + \Phi_\varepsilon W_\varepsilon] dx dy dz dt \\
= \int_0^t \int_{\Omega} [(u_\varepsilon \cdot \nabla \theta_\varepsilon) - (u \cdot \nabla \theta_\varepsilon) + \Phi_\varepsilon W_\varepsilon] dx dy dz dt \\
= \int_0^t \int_{\Omega} [((u_\varepsilon - u) \cdot \nabla) \Phi_\varepsilon + \Phi_\varepsilon W_\varepsilon] dx dy dz dt \\
= \int_0^t \int_{\Omega} [(V_\varepsilon \cdot \nabla h \Phi_\varepsilon) + W_\varepsilon (\partial_\varepsilon \Phi_\varepsilon) + \Phi_\varepsilon W_\varepsilon] dx dy dz dt \\
= : R_{11} + R_{12} + R_{13}. \\
\] (4.9)

Now we need to estimate the terms $R_{11}$, $R_{12}$ and $R_{13}$ on the right-hand side of (4.9). For the first term $R_{11}$, using the Hölder inequality, Sobolev embedding and Young inequality yields
\[
R_{11} := \int_0^t \int_{\Omega} (V_\varepsilon \cdot \nabla h \Phi_\varepsilon) dx dy dz dt \\
\leq \int_0^t \| V_\varepsilon \|_3 \| \nabla h \Phi_\varepsilon \|_2 \| \theta \|_n dt \\
\leq C \int_0^t \| V_\varepsilon \|_2^{1/2} \| \nabla V_\varepsilon \|_2^{1/2} \| \nabla h \Phi_\varepsilon \|_2 \| \nabla \theta \|_2 dt \\
\leq C \int_0^t \| V_\varepsilon \|_2^{3/2} \| \nabla \theta \|_2^2 dt + \frac{1}{24} \int_0^t \left( \| \nabla V_\varepsilon \|_2^2 + \| \nabla \Phi_\varepsilon \|_2^2 \right) dt.
\]

For the second term $R_{12}$, by the Lemma 2.1, from the Hölder inequality, Sobolev embedding, Poincaré inequality and Young inequality it follows that
\[
R_{12} := \int_0^t \int_{\Omega} W_\varepsilon (\partial_\varepsilon \Phi_\varepsilon) dx dy dz dt \\
= \int_0^t \int_{\Omega} [-(\partial_\varepsilon W_\varepsilon) \Phi_\varepsilon \theta - W_\varepsilon \Phi_\varepsilon \partial_\varepsilon \theta] dx dy dz dt \\
\leq C \int_0^t \| \nabla V_\varepsilon \|_2 \| \Phi_\varepsilon \|_2^{1/2} \| \nabla \Phi_\varepsilon \|_2^{1/2} \| \nabla \theta \|_2 dt \\
+ C \int_0^t \| \nabla V_\varepsilon \|_2 \| \Phi_\varepsilon \|_2^{1/2} \| \nabla \Phi_\varepsilon \|_2^{1/2} \| \nabla \theta \|_2^{1/2} \| \Delta \theta \|_2^{1/2} dt
\]
\[ \leq C \int_0^r \| \Phi_v \|_2^2 \| \nabla \theta \|_2^2 \| \Delta \theta \|_2^2 dt + \frac{1}{24} \int_0^r (\| \nabla V_c \|_2^2 + \| \nabla \Phi_v \|_2^2) dt, \]

where divergence-free condition and integration by parts have been used. Finally, it remains to deal with the last term \( R_{13} \). Thanks to the Hölder inequality and Young inequality we reach

\[ R_{13} := \int_0^r \int_{\Omega} \Phi_v W_c dx dy dz dt \]

\[ = \int_0^r \int_{\Omega} \Phi_v \left( - \int_0^r \nabla_h \cdot V_c (x, y, \xi, t) d\xi \right) dx dy dz dt \]

\[ \leq \int_0^r \int_{\Omega} \left( \int_{-1}^1 |\Phi_v| dz \right) \left( \int_{-1}^1 |\nabla_h V_c| dz \right) dx dy dz dt \]

\[ \leq C \int_0^r \| \Phi_v \|_2^2 dt + \frac{1}{24} \int_0^r \| \nabla V_c \|_2^2 dt. \]

Combining the estimates for \( R_{11}, R_{12} \) and \( R_{13} \) we have

\[ R_1 \leq C \int_0^r \left[ \| \Phi_v \|_2^2 + \| \nabla \theta \|_2^2 \| \Delta \theta \|_2^2 \left( \| V_c \|_2^2 + \| \Phi_v \|_2^2 \right) \right] dt \]

\[ + \frac{1}{8} \int_0^r \left( \| \nabla V_c \|_2^2 + \| \nabla \Phi_v \|_2^2 \right) dt. \quad (4.10) \]

The remaining terms \( R_2, R_3 \) and \( R_4 \) on the right-hand side of (4.8) are estimated by the same method in Li-Titi [30]. These terms can be bounded as

\[ R_2 := \int_0^r \int_{\Omega} \left[ (u_\varepsilon \cdot \nabla) v_\varepsilon \cdot v + (u \cdot \nabla) v \cdot v_\varepsilon \right] dx dy dz dt \]

\[ \leq C \int_0^r \left( \| \nabla v \|_2^2 + \| \Delta v \|_2^2 \right) \| \nabla v_\varepsilon \|_2^2 \| V_c \|_2^2 dt + \frac{1}{8} \int_0^r \| \nabla v_\varepsilon \|_2^2 dt, \quad (4.11) \]

\[ R_3 := \varepsilon^2 \int_0^r \int_{\Omega} \left[ - \left( \int_{0}^{r} \partial_\varepsilon v(x, y, \xi, t) d\xi \right) \cdot \nabla_h W_\varepsilon - \nabla w \cdot \nabla W_\varepsilon \right] dx dy dz dt \]

\[ \leq C \varepsilon^2 \int_0^r \left( \| \partial_\varepsilon v \|_2^2 + \| \nabla w \|_2^2 \right) dt + \frac{1}{8} \int_0^r \varepsilon^2 \| \nabla W_\varepsilon \|_2^2 dt \]

\[ \leq C \varepsilon^2 \int_0^r \left( \| \partial_\varepsilon v \|_2^2 + \| \Delta v \|_2^2 \right) dt + \frac{1}{8} \int_0^r \varepsilon^2 \| \nabla W_\varepsilon \|_2^2 dt \quad (4.12) \]

and

\[ R_4 := \varepsilon^2 \int_0^r \int_{\Omega} (u_\varepsilon \cdot \nabla w_\varepsilon) w dx dy dz dt \]

\[ \leq C \varepsilon^2 \int_0^r \left( \| v_\varepsilon \|_2^2 \| \nabla v_\varepsilon \|_2^2 + \| \nabla v \|_2^2 \| \Delta v \|_2^2 + \varepsilon^4 \| w_\varepsilon \|_2^2 \| \nabla w_\varepsilon \|_2^2 \right) dt \]

\[ + \frac{1}{8} \int_0^r \left( \| \nabla V_c \|_2^2 + \varepsilon^2 \| \nabla W_\varepsilon \|_2^2 \right) dt, \quad (4.13) \]

respectively.

Substituting (4.10)-(4.13) into (4.8) yields

\[ g(t) := (\| V_c, \Phi_v \|_2^2 + \varepsilon^2 \| W_\varepsilon \|_2^2) (t) + \int_0^t \left( \| \nabla (V_c, \Phi_v) \|_2^2 + \varepsilon^2 \| \nabla W_\varepsilon \|_2^2 \right) ds \]

\[ \leq C \int_0^t \left[ \| \Phi_v \|_2^2 + \| \nabla \theta \|_2^2 \| \Delta \theta \|_2^2 \left( \| V_c \|_2^2 + \| \Phi_v \|_2^2 \right) \right] ds \]
we show that the scaled Boussinesq equations (5.1) strongly converge to the primitive equations (5.2) with respect to $t$ leads to
\[
G'(t) = C \left( \mu\Phi^2 + \|\nabla\theta\|_2^2 \|\Delta\theta\|_2^2 \right) \left( \|V_r\|_2^2 + \|\Phi^2\|_2 \right) \\
+ C\|\nabla v_0\|_2^2 \|\Delta v_0\|_2^2 + C\frac{1}{\varepsilon} \left( \|\partial_t v_0\|_2^2 + \|\Delta v_0\|_2^2 \right) \\
+ C\frac{1}{\varepsilon} \left( \|v_0\|_2^2 \|\nabla v_0\|_2^2 \|\Delta v_0\|_2^2 + \varepsilon^2 \|w_0\|_2^2 \|\nabla w_0\|_2^2 \right)
\]
\[
\leq C \left( 1 + \|\nabla v_0\|_2^2 \|\Delta v_0\|_2^2 + \|\nabla\theta\|_2^2 \|\Delta\theta\|_2^2 \right) G(t) + C\varepsilon^2 \left( \|\partial_t v_0\|_2^2 + \|\Delta v_0\|_2^2 \right) \\
+ C\varepsilon^2 \left( \|v_0\|_2^2 \|\nabla v_0\|_2^2 \|\Delta v_0\|_2^2 + \varepsilon^2 \|w_0\|_2^2 \|\nabla w_0\|_2^2 \right).
\]
Noting that the fact that $G(0) = 0$, and applying the Gronwall inequality to the above inequality, it follows from (3.23), (3.28) and (4.6) that
\[
g(t) \leq C\varepsilon^2 \exp \left\{ C \int_0^t \left( 1 + \|\nabla v_0\|_2^2 \|\Delta v_0\|_2^2 + \|\nabla\theta\|_2^2 \|\Delta\theta\|_2^2 \right) \|\nabla v_0\|_2^2 \|\Delta v_0\|_2^2 + \|\nabla\theta\|_2^2 \|\Delta\theta\|_2^2 \right) ds \}
\times \left[ \int_0^t \left( \|\partial_t v_0\|_2^2 + \|\Delta v_0\|_2^2 + \|v_0\|_2^2 \|\nabla v_0\|_2^2 \right) ds \right.
\left. + \int_0^t \left( \|\nabla v_0\|_2^2 \|\Delta v_0\|_2^2 + \varepsilon^2 \|w_0\|_2^2 \|\nabla w_0\|_2^2 \right) ds \right]
\leq C\varepsilon^2 e^C \left( t + \alpha_2(t) + \alpha_2(t) \right) \left[ \alpha_3(t) + \alpha_3^2(t) + \left( \|v_0\|_2^2 + \|w_0\|_2^2 + \|\theta\|_2^2 \right) \right] ^2.
\]
This completes the proof.

\[
G(t) \leq C\varepsilon^2 \exp \left\{ C \int_0^t \left( 1 + \|\nabla v_0\|_2^2 \|\Delta v_0\|_2^2 + \|\nabla\theta\|_2^2 \|\Delta\theta\|_2^2 \right) \|\nabla v_0\|_2^2 \|\Delta v_0\|_2^2 + \|\nabla\theta\|_2^2 \|\Delta\theta\|_2^2 \right) ds \}
\times \left[ \int_0^t \left( \|\partial_t v_0\|_2^2 + \|\Delta v_0\|_2^2 + \|v_0\|_2^2 \|\nabla v_0\|_2^2 \right) ds \right.
\left. + \int_0^t \left( \|\nabla v_0\|_2^2 \|\Delta v_0\|_2^2 + \varepsilon^2 \|w_0\|_2^2 \|\nabla w_0\|_2^2 \right) ds \right]
\leq C\varepsilon^2 e^C \left( t + \alpha_2(t) + \alpha_2(t) \right) \left[ \alpha_3(t) + \alpha_3^2(t) + \left( \|v_0\|_2^2 + \|w_0\|_2^2 + \|\theta\|_2^2 \right) \right] ^2.
\]

As a consequence, it is clear that the Theorem 1.1 stated in introduction follows from the Proposition 4.2.

## 5 Strong convergence for $H^2$ initial data

In this section, under the assumption of the initial data $(v_0, \theta_0) \in H^2(\Omega)$, in which
\[
\int_{\Omega} \nabla v_0 \cdot \nabla \theta_0 \, dz = 0,
\]
we show that the scaled Boussinesq equations (1.2) strongly converge to the primitive equations with full viscosity and full diffusion (1.7) as the aspect ration parameter $\varepsilon$ goes to zero. With this assumption of the initial data, there is a unique local strong solution $(v_\varepsilon, w_\varepsilon, \theta_\varepsilon)$ to the system (1.2), corresponding to the boundary and initial conditions (1.3)-(1.4) and symmetry condition (1.5). Denote by $T_\varepsilon^*$ the maximal existence time of this local strong solution. Since the proof of the first part of Theorem 1.2 was completed in Section 3, we only need to estimate the difference function $(V_\varepsilon, W_\varepsilon, \Phi_\varepsilon)$.

We subtract the system (1.7) from the system (1.2) and then lead to the following system
\[
\partial_t v_\varepsilon - \Delta v_\varepsilon + (U_\varepsilon \cdot \nabla) v_\varepsilon + (u \cdot \nabla) v_\varepsilon + (U_\varepsilon \cdot \nabla) v_\varepsilon + \nabla h P_\varepsilon = 0, \tag{5.1}
\]
\[
\varepsilon^2 (\partial_t w_\varepsilon - \Delta w_\varepsilon + U_\varepsilon \cdot \nabla w_\varepsilon + U_\varepsilon \cdot \nabla w_\varepsilon + u \cdot \nabla w_\varepsilon + \nabla h) + \partial_3 P_\varepsilon
\]
\[
- \Phi_\varepsilon + \varepsilon^2 (\partial_t w_\varepsilon - \Delta w + u \cdot \nabla w) = 0, \tag{5.2}
\]
\[
\partial_t \Phi_\varepsilon - \Delta \Phi_\varepsilon + U_\varepsilon \cdot \nabla \Phi_\varepsilon + U_\varepsilon \cdot \nabla \theta + u \cdot \nabla \Phi_\varepsilon = 0, \tag{5.3}
\]
\[
\nabla h \cdot V_\varepsilon + \partial_3 W_\varepsilon = 0, \tag{5.4}
\]
defined on $\Omega \times (0, T_\varepsilon^*)$. 

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Proposition 5.1. Suppose that \((v_0, \theta_0) \in H^2(\Omega)\), with \(\int_{-1}^{1} \nabla h \cdot v_0 dz = 0\). Then the system (5.1)-(5.4) has the following basic energy estimate

\[
\sup_{0 \leq s \leq t} \left( \| (V_\varepsilon, \varepsilon W_\varepsilon, \Phi_\varepsilon) \|_2 \right)^2 (s) + \int_0^t \| \nabla (V_\varepsilon, \varepsilon W_\varepsilon, \Phi_\varepsilon) \|^2_2 ds \leq \varepsilon^2 \beta_1(t),
\]

for any \(t \in [0, T_*]\), where

\[
\beta_1(t) = Ce^{C(t + \alpha_1(t))} \left[ \alpha_5(t) + \alpha_2(t) + \left( \| v_0 \|_2^2 + \varepsilon^2 \| W_\varepsilon \|_2^2 + t \| \theta_0 \|_2^2 \right)^2 \right],
\]

and \(C\) is a positive constant that does not depend on \(\varepsilon\).

It should be pointed out that the Proposition 5.1 is a direct consequence of Proposition 4.2. Moreover, the basic energy estimate on the system (5.1)-(5.4) can also be obtained by the energy method. The strong solution \((v_\varepsilon, w_\varepsilon, \theta_\varepsilon)\) to the system (1.2) is local, so is this basic energy estimate. Next, we perform the first order energy estimate for the system (5.1)-(5.4).

Proposition 5.2. Suppose that \((v_0, \theta_0) \in H^2(\Omega)\), with \(\int_{-1}^{1} \nabla h \cdot v_0 dz = 0\). Then there exists a positive constant \(\lambda_0\) such that the system (5.1)-(5.4) has the following first order energy estimate

\[
\sup_{0 \leq s \leq t} \left( \| \nabla (V_\varepsilon, \Phi_\varepsilon) \|^2_2 + \varepsilon^2 \| \nabla W_\varepsilon \|^2_2 \right) (s) \leq \lambda_0^2,
\]

for any \(t \in [0, T_*]\), provided that

\[
\sup_{0 \leq s \leq t} \left( \| \nabla (V_\varepsilon, \Phi_\varepsilon) \|^2_2 + \varepsilon^2 \| \nabla W_\varepsilon \|^2_2 \right) (s) \leq \lambda_0^2,
\]

where

\[
\beta_2(t) = Ce^{C[1 + \alpha_2(t) + (1 + \varepsilon) \alpha_1(t)]} \left[ \alpha_7(t) + \alpha_3(t) \right],
\]

and \(C\) is a positive constant that does not depend on \(\varepsilon\).

Proof. Taking the \(L^2(\Omega)\) inner product of the first three equation in the system (5.1)-(5.4) with \(-\Delta V_\varepsilon, -\Delta W_\varepsilon\) and \(-\Delta \Phi_\varepsilon\), respectively, then it follows from integration by parts that

\[
\frac{1}{2} \frac{d}{dt} \left( \| \nabla (V_\varepsilon, \Phi_\varepsilon) \|^2_2 + \varepsilon^2 \| \nabla W_\varepsilon \|^2_2 \right) + \| \Delta (V_\varepsilon, \Phi_\varepsilon) \|^2_2 + \varepsilon^2 \| \Delta W_\varepsilon \|^2_2
\]

\[
= \int_{\Omega} (U_\varepsilon \cdot \nabla \Phi_\varepsilon + U_\varepsilon \cdot \nabla \theta + u \cdot \nabla \Phi_\varepsilon) \Delta \Phi_\varepsilon dx dy dz + \int_{\Omega} \Phi_\varepsilon \Delta W_\varepsilon dx dy dz
\]

\[
+ \varepsilon^2 \int_{\Omega} (\partial_h w - \Delta w + u \cdot \nabla w) \Delta W_\varepsilon dx dy dz
\]

\[
+ \varepsilon^2 \int_{\Omega} (U_\varepsilon \cdot \nabla W_\varepsilon + U_\varepsilon \cdot \nabla w + u \cdot \nabla W_\varepsilon) \Delta W_\varepsilon dx dy dz
\]

\[
+ \int_{\Omega} \left[ (U_\varepsilon \cdot \nabla) V_\varepsilon + (u \cdot \nabla) V_\varepsilon + (U_\varepsilon \cdot \nabla) V_\varepsilon \right] \cdot \Delta V_\varepsilon dx dy dz
\]

\[
= R_1 + R_2 + R_3 + R_4,
\]

(5.5)

note that the resultants have been added up.

We now estimate the first term \(R_1\) on the right-hand side of (5.5), which will be split into two parts, \(R_{11}\) and \(R_{12}\). For the term \(R_{11}\), by virtue of the Lemma 2.2, it deduces from the Poincaré and Young inequality that

\[
R_{11} = \int_{\Omega} (U_\varepsilon \cdot \nabla \Phi_\varepsilon + U_\varepsilon \cdot \nabla \theta + u \cdot \nabla \Phi_\varepsilon) \Delta \Phi_\varepsilon dx dy dz
\]
and where we have used the fact that $\int_\Omega \left(\int_\Omega (V_\varepsilon \cdot \nabla \varepsilon) \cdot \nabla \varepsilon \right) dxdydz = \int_\Omega (\nabla \varepsilon \cdot \nabla V_\varepsilon) + \nabla \varepsilon \cdot \nabla V_\varepsilon dxdydz$

\[
\leq C \|\nabla V_\varepsilon\|_{L^2}^2 \|\Delta V_\varepsilon\|_{L^2}^2 \|\nabla \Phi_\varepsilon\|_{L^2}^2 \|\Delta \Phi_\varepsilon\|_{L^2}^2 \\
+ C \|\nabla V_\varepsilon\|_{L^2}^2 \|\Delta V_\varepsilon\|_{L^2}^2 \|\nabla \theta\|_{L^2}^2 \|\Delta \theta\|_{L^2}^2 \|\Delta \Phi_\varepsilon\|_{L^2}^2 \\
+ C \|\nabla V_\varepsilon\|_{L^2}^2 \|\Delta V_\varepsilon\|_{L^2}^2 \|\nabla \Phi_\varepsilon\|_{L^2}^2 \|\Delta \Phi_\varepsilon\|_{L^2}^2
\]

\[
\leq \frac{1}{20} (\|\Delta V_\varepsilon\|_{L^2}^2 + \|\Delta \Phi_\varepsilon\|_{L^2}^2) + C (\|\nabla V_\varepsilon \cdot \nabla \varepsilon\|_{L^2}^2 + \varepsilon^2 \|\nabla W_\varepsilon\|_{L^2}^2) \\
\times \left[ (\|\Delta V_\varepsilon\|_{L^2}^2 + \|\Delta \Phi_\varepsilon\|_{L^2}^2) + \|\Delta \theta\|_{L^2}^2 \|\nabla \Delta \theta\|_{L^2}^2 + \|\Delta \varepsilon\|_{L^2}^2 \|\nabla \Delta \varepsilon\|_{L^2}^2 \right].
\]

As for another term $R_{11}$, we use the Hölder inequality, Poincaré inequality and Young inequality to obtain

\[
R_{12} := \int_\Omega \Phi_\varepsilon \Delta W_\varepsilon dxdydz = - \int_\Omega \nabla \Phi_\varepsilon \cdot \nabla W_\varepsilon dxdydz
\]

\[
= \int_\Omega \left[ \nabla \Phi_\varepsilon \cdot \left( \nabla \int_0^t \nabla \Phi_\varepsilon \cdot V_\varepsilon(x,y,\xi,t) d\xi \right) + \partial_x \Phi_\varepsilon (\nabla \cdot V_\varepsilon) \right] dxdydz
\]

\[
\leq \int_\Omega \left( \int_0^1 |\nabla \Phi_\varepsilon| dz \right) \left( \int_0^1 |\nabla (\nabla \Phi_\varepsilon) dz \right) dxdydz + \int_\Omega |\partial_x \Phi_\varepsilon| \|\nabla \Phi_\varepsilon\|_{L^2} dxdydz
\]

\[
\leq C (\|\nabla \Phi_\varepsilon\|_{L^2}^2 + \|\Delta V_\varepsilon\|_{L^2}^2 + \|\nabla \Phi_\varepsilon\|_{L^2}^2)
\]

\[
\leq \frac{1}{20} (\|\Delta V_\varepsilon\|_{L^2}^2 + \|\Delta \Phi_\varepsilon\|_{L^2}^2) + C (\|\nabla V_\varepsilon \cdot \nabla \varepsilon\|_{L^2}^2 + \varepsilon^2 \|\nabla W_\varepsilon\|_{L^2}^2),
\]

where we have used the fact that $U_\varepsilon$ is divergence free. Combining the two estimates yields

\[
R_1 := \int_\Omega (U_\varepsilon \cdot \nabla \Phi_\varepsilon + U_\varepsilon \cdot \nabla \theta + u \cdot \nabla \Phi_\varepsilon) \Delta \Phi_\varepsilon dxdydz + \int_\Omega \Phi_\varepsilon \Delta W_\varepsilon dxdydz
\]

\[
\leq \frac{1}{10} (\|\Delta V_\varepsilon\|_{L^2}^2 + \|\Delta \Phi_\varepsilon\|_{L^2}^2) + C (\|\nabla V_\varepsilon \cdot \nabla \varepsilon\|_{L^2}^2 + \varepsilon^2 \|\nabla W_\varepsilon\|_{L^2}^2)
\]

\[
\times \left[ (\|\Delta V_\varepsilon\|_{L^2}^2 + \|\Delta \Phi_\varepsilon\|_{L^2}^2) + \|\Delta \theta\|_{L^2}^2 \|\nabla \Delta \theta\|_{L^2}^2 + \|\Delta \varepsilon\|_{L^2}^2 \|\nabla \Delta \varepsilon\|_{L^2}^2 \right].
\]

(5.6)

The estimates for the rest of the terms on the right-hand side of (5.5) can be found in Li-Titi [30]. These terms can be bounded as

\[
R_2 := \varepsilon^2 \int_\Omega (\partial_t w - \Delta w + u \cdot \nabla w) \Delta W_\varepsilon dxdydz
\]

\[
\leq \frac{1}{10} \varepsilon^2 \|\Delta W_\varepsilon\|_{L^2}^2 + C \varepsilon^2 (\|\Delta \varepsilon\|_{L^2}^2 \|\nabla \Delta \varepsilon\|_{L^2}^2 + \|\nabla \partial_t v\|_{L^2}^2 + \|\nabla \Delta v\|_{L^2}^2)
\]

\[
\leq \frac{1}{10} \varepsilon^2 \|\Delta W_\varepsilon\|_{L^2}^2 + C \varepsilon^2 (1 + \|\Delta \varepsilon\|_{L^2}^2) (\|\nabla \partial_t v\|_{L^2}^2 + \|\nabla \Delta v\|_{L^2}^2),
\]

(5.7)

\[
R_3 := \varepsilon^2 \int_\Omega (U_\varepsilon \cdot \nabla V_\varepsilon + U_\varepsilon \cdot \nabla w + u \cdot \nabla V_\varepsilon) \Delta W_\varepsilon dxdydz
\]

\[
\leq \frac{1}{10} (\|\Delta V_\varepsilon\|_{L^2}^2 + \varepsilon^2 \|\Delta W_\varepsilon\|_{L^2}^2) + C (\|\nabla V_\varepsilon \cdot \nabla \varepsilon\|_{L^2}^2 + \varepsilon^2 \|\nabla \varepsilon\|_{L^2}^2)
\]

\[
\times \left[ (\|\Delta V_\varepsilon\|_{L^2}^2 + \varepsilon^2 \|\Delta W_\varepsilon\|_{L^2}^2) + (1 + \varepsilon^4) \|\Delta \varepsilon\|_{L^2}^2 \|\Delta \varepsilon\|_{L^2}^2 \right],
\]

(5.8)

and

\[
R_4 := \int_\Omega [(U_\varepsilon \cdot \nabla) V_\varepsilon + (u \cdot \nabla) V_\varepsilon + (U_\varepsilon \cdot \nabla) v] \cdot \Delta V_\varepsilon dxdydz
\]

\[
\leq \frac{1}{10} \|\Delta V_\varepsilon\|_{L^2}^2 + C (\|\nabla V_\varepsilon\|_{L^2}^2 \|\Delta V_\varepsilon\|_{L^2}^2 + \|\Delta \varepsilon\|_{L^2}^2 \|\nabla \varepsilon\|_{L^2}^2)
\]

\[
\leq \frac{1}{10} \|\Delta V_\varepsilon\|_{L^2}^2 + C \|\nabla V_\varepsilon\|_{L^2}^2 (\|\Delta V_\varepsilon\|_{L^2}^2 + \|\Delta \varepsilon\|_{L^2}^2 \|\nabla \varepsilon\|_{L^2}^2),
\]

(5.9)
respectively.

By substituting (5.6)-(5.9) into (5.5), we obtain

\[
\frac{1}{2} \frac{d}{dt} \left( \| \nabla (V, \Phi) \|^2 + \varepsilon^2 \| \nabla W \|^2 \right) + \frac{3}{5} \left( \| \Delta (V, \Phi) \|^2 + \varepsilon^2 \| \Delta W \|^2 \right) \\
\leq C_0 \varepsilon^2 \left( 1 + \| \Delta v \|^2 \right) \left( \| \nabla \partial_t v \|^2 + \| \nabla \Delta v \|^2 \right) \\
+ C_0 \left( \| \nabla (V, \Phi) \|^2 + \varepsilon^2 \| \nabla W \|^2 \right) \left[ \| \Delta (V, \Phi) \|^2 + \varepsilon^2 \| \Delta W \|^2 \right] \\
+ 1 + \| \Delta \theta \|^2 \| \nabla \Delta \theta \|^2 + (1 + \varepsilon^4) \| \Delta v \|^2 \| \nabla \Delta v \|^2 .
\]

Using the assumption given by the proposition

\[
\sup_{0 \leq s \leq t} \left( \| \nabla (V, \Phi) \|^2 + \varepsilon^2 \| \nabla W \|^2 \right) (s) \leq \lambda_0^2,
\]

and choosing \( \lambda_0 = \sqrt{\frac{1}{C_0 \varepsilon^2}} \), it deduces from the above inequality that

\[
\frac{d}{dt} \left( \| \nabla (V, \Phi) \|^2 + \varepsilon^2 \| \nabla W \|^2 \right) (t) + \int_0^t \left( \| \Delta (V, \Phi) \|^2 + \varepsilon^2 \| \Delta W \|^2 \right) ds \\
\leq 2\varepsilon^2 C_0 \exp \left\{ 2C_0 \int_0^t \left[ 1 + \| \Delta \theta \|^2 \| \nabla \Delta \theta \|^2 + (1 + \varepsilon^4) \| \Delta v \|^2 \| \nabla \Delta v \|^2 \right] ds \right\} \\
\times \int_0^t \left( 1 + \| \Delta v \|^2 \right) \left( \| \nabla \partial_t v \|^2 + \| \nabla \Delta v \|^2 \right) ds \\
\leq 2\varepsilon^2 C_0 \varepsilon^2 \left[ 1 + \varepsilon^2 (t) + (1 + \varepsilon^4) \alpha_7(t) \right] \left[ \alpha_7(t) + \alpha_7^2(t) \right] .
\]

The proof is completed. \( \square \)

**Proposition 5.3.** There exists a positive constant \( \varepsilon_0 \) such that, for any \( \varepsilon \in (0, \varepsilon_0) \), the system (1.2) corresponding to the boundary and initial conditions (1.3)-(1.4) and symmetry condition (1.5) has a unique global strong solution \( (v, w, \theta) \). Furthermore, the system (5.1)-(5.4) has the following estimate

\[
\sup_{0 \leq t < \infty} \left( \| (V, \varepsilon W, \Phi) \|^2 \right) (t) + \int_0^\infty \| \nabla (V, \varepsilon W, \Phi) \|^2 dt \leq 2 \left( \tilde{\beta}_1(T_*) + \tilde{\beta}_2(T_*) \right) ,
\]

where both \( \tilde{\beta}_1(T_*) \) and \( \tilde{\beta}_2(T_*) \) are positive constants that do not depend on \( \varepsilon \).

**Proof.** Since \( T_* \) is the maximal existence time of the strong solution \( (v, w, \theta) \) to the system (1.2) corresponding to boundary and initial conditions (1.3)-(1.4) and symmetry condition (1.5), it follows from the Proposition 5.1 that

\[
\sup_{0 \leq t < T_*} \left( \| (V, \varepsilon W, \Phi) \|^2 \right) (t) + \int_0^{T_*} \| \nabla (V, \varepsilon W, \Phi) \|^2 dt \leq \varepsilon^2 \beta_1(T_*),
\]

(5.10)
where
\[ \tilde{\beta}_1(T_\star) = C_1 \varepsilon C_1 \left( T_\star + \alpha_5^2(T_\star) + \alpha_2^2(T_\star) \right) \left[ \alpha_5(T_\star) + \alpha_5^2(T_\star) + \left( \|v_0\|_2^2 + \|w_0\|_2^2 + T_\star \|\theta_0\|_2^2 \right)^2 \right], \]
and \( C_1 \) is a positive constant that does not depend on \( \varepsilon \).

Let \( \lambda_0 \) be the constant from Proposition 5.1. Define
\[ t_\star := \sup \left\{ t \in (0, T_\star) \mid \sup_{0 \leq s \leq t} \left( \|\nabla (V_{\varepsilon}, \varepsilon W_{\varepsilon}, \Phi_{\varepsilon})\|_2^2 \right) (s) \leq \lambda_0^2 \right\}. \]

Thanks to Proposition 5.2, we have the following estimate
\[ \sup_{0 \leq s \leq t} \left( \|\nabla (V_{\varepsilon}, \varepsilon W_{\varepsilon}, \Phi_{\varepsilon})\|_2^2 \right) (s) + \int_0^t \|\Delta (V_{\varepsilon}, \varepsilon W_{\varepsilon}, \Phi_{\varepsilon})\|_2^2 ds \leq \varepsilon^2 \tilde{\beta}_2(T_\star), \tag{5.11} \]
for any \( t \in [0, t_\star) \), where
\[ \tilde{\beta}_2(T_\star) = C_2 \varepsilon C_2 \left[ T_\star + \alpha_5^2(T_\star) + \alpha_2^2(T_\star) \right] \left[ \alpha_7(T_\star) + \alpha_7^2(T_\star) \right], \]
and \( C_2 \) is a positive constant that does not depend on \( \varepsilon \). Choosing \( \varepsilon_0 = \lambda_0 / \sqrt{2 \tilde{\beta}_2(T_\star)} \), it deduces from (5.11) that
\[ \sup_{0 \leq s \leq t} \left( \|\nabla (V_{\varepsilon}, \varepsilon W_{\varepsilon}, \Phi_{\varepsilon})\|_2^2 \right) (s) + \int_0^t \|\Delta (V_{\varepsilon}, \varepsilon W_{\varepsilon}, \Phi_{\varepsilon})\|_2^2 ds \leq \frac{\lambda_0^2}{2}, \]
for any \( \varepsilon \in (0, \varepsilon_0) \), and for any \( t \in [0, t_\star) \), which leads to
\[ \sup_{0 \leq t < t_\star} \left( \|\nabla (V_{\varepsilon}, \varepsilon W_{\varepsilon}, \Phi_{\varepsilon})\|_2^2 \right) (t) \leq \frac{\lambda_0^2}{2}. \tag{5.12} \]

From the definition of \( t_\star \), (5.12) implies \( t_\star = T_\star \). Therefore, the estimate (5.11) holds for \( t \in [0, T_\star) \).

We claim that \( T_\star = \infty \). If \( T_\star < \infty \), then it is clear that
\[ \limsup_{t \to T_\star} \| (v_{\varepsilon}, w_{\varepsilon}, \theta_{\varepsilon}) \|_{H^1} = \limsup_{t \to T_\star} \| (V_{\varepsilon}, \varepsilon W_{\varepsilon}, \Phi_{\varepsilon}) \|_{H^1} = \infty. \]

Otherwise, the strong solution \( (v_{\varepsilon}, w_{\varepsilon}, \theta_{\varepsilon}) \) to the system (1.2) can be extended beyond the maximal existence time \( T_\star \). However, the above result contradicts to the local well-posedness theory of strong solutions to the system (1.2) and the estimates for (5.10) and (5.11). This leads to \( T_\star = \infty \), and hence the strong solution \( (v_{\varepsilon}, w_{\varepsilon}, \theta_{\varepsilon}) \) to the system (1.2) exists globally in time. Furthermore, combining (5.10) and (5.11) yields
\[ \sup_{0 \leq t < \infty} \left( \| (V_{\varepsilon}, \varepsilon W_{\varepsilon}, \Phi_{\varepsilon}) \|_{H^1}^2 \right) (t) + \int_0^\infty \|\nabla (V_{\varepsilon}, \varepsilon W_{\varepsilon}, \Phi_{\varepsilon})\|_{H^1}^2 dt \leq \varepsilon^2 \left( \tilde{\beta}_1(T_\star) + \tilde{\beta}_2(T_\star) \right). \]

This completes the proof. \( \square \)

Finally, it is obvious that the Theorem 1.2 stated in introduction is a direct consequence of Proposition 5.3.

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