The Notion of Convexity and Concavity on Wiener Space

D. Feyel and A. S. Üstünel

Abstract—We define, in the frame of an abstract Wiener space, the notions of convexity and of concavity for the equivalence classes of random variables. As application we show that some important inequalities of the finite dimensional case have their natural counterparts in this setting.

1 Introduction

On an infinite dimensional vector space $W$ the notion of convex or concave function is well-known. Assume now that this space is equipped with a probability measure. Suppose that there are two measurable functions on this vector space, say $F$ and $G$ such that $F = G$ almost surely. If $F$ is a convex function, then from the probabilistic point of view, we would like to say that $G$ is also convex. However this is false; since in general the underlying probability measure is not (quasi) invariant under the translations by the elements of the vector space. If $W$ contains a dense subspace $H$ such that $w \to w + h$ ($h \in H$) induces a measure which is equivalent to the initial measure or absolutely continuous with respect to it, then we can define a notion of “$H$–convexity” or “$H$–concavity in the direction of $H$. Of course these properties are inherited by the corresponding equivalence classes, hence they are particularly useful for the probabilistic calculations.

The notion of $H$-convexity has been used in [17] to study the absolute continuity of the image of the Wiener measure under the monotone shifts. In this paper we study further properties of such functions and some additional ones in the frame of an abstract Wiener space, namely $H$-convex, $H$-concave, log $H$-concave and log $H$-convex Wiener functions, where $H$ denotes the associated Cameron-Martin space. In particular we extend some finite dimensional results of [12] and [3] to this setting and prove that some finite dimensional convexity-concavity inequalities have their counterparts in infinite dimensions.
2 Preliminaries

In the sequel \((W, H, \mu)\) denotes an abstract Wiener space, i.e., \(H\) is a separable Hilbert space, called the Cameron-Martin space. It is identified with its continuous dual. \(W\) is a Banach or a Fréchet space into which \(H\) is injected continuously and densely. \(\mu\) is the standard cylindrical Gaussian measure on \(H\) which is concentrated in \(W\) as a Radon probability measure. In the classical case we have either

\[ W = C_0([0,1]) \text{ or } W = C_0(\mathbb{R}_+) \]

respectively.

Let \(X\) be a separable Hilbert space and \(a\) be an \(X\)-valued (smooth) polynomial on \(W\):

\[ a(w) = \sum_{i=1}^{m} \eta_i(\langle h_1, w \rangle, \ldots, \langle h_n, w \rangle) x_i , \]

with \(x_i \in X\), \(h_i \in W^*\) and \(\eta_i \in C^\infty_b(\mathbb{R}^n)\). The Gross-Sobolev derivative of \(a\) is defined as

\[ \nabla a(w) = \sum_{i=1}^{m} \sum_{j=1}^{n} \partial_j \eta_i(\langle h_1, w \rangle, \ldots, \langle h_n, w \rangle)x_i \otimes \tilde{h}_j , \]

where \(\tilde{h}\) denotes the image of \(h \in W^*\) in \(H\) under the canonical injection \(W^* \hookrightarrow H\) (in the sequel we shall omit this notational detail and write \(h\) instead of \(\tilde{h}\) when there is no ambiguity). The derivatives of higher orders \(\nabla^k a(w)\) are defined recursively. Thanks to the Cameron-Martin theorem, all these operators are closable on all the \(L^p\)-spaces and the Sobolev spaces \(\mathbb{D}_{p,k}(X), p > 1, \ k \in \mathbb{N}\) can be defined as the completion of \(X\)-valued smooth polynomials with respect to the norm:

\[ \| a \|_{p,k} = \sum_{i=0}^{k} \| \nabla^i a \|_{L^p(\mu, X \otimes H^\otimes i)} . \]

From the Meyer inequalities (cf., for instance [15]), it is known that the \((p,k)\)-norm, defined above, is equivalent to the following norm

\[ \| (I + L)^{k/2} a \|_{L^p(\mu, X)} \]
where $L$ is the Ornstein-Uhlenbeck operator on $W$ (cf. [15]) and we denote these two norms with the same notation. Since $L$ is a positive, self adjoint operator, we can also define the norms, via spectral theorem, for $k \in \mathbb{R}$. It is easy to see that the spaces with negative differentiability index describe the dual spaces of the positively indexed Sobolev spaces. We denote by $\mathbb{D}(X)$ the intersection of the Sobolev spaces $\{ \mathbb{D}_{p,k}(X); \ p > 1, \ k \in \mathbb{N} \}$, equipped with the intersection (i.e., projective limit) topology. The continuous dual of $\mathbb{D}(X)$ is denoted by $\mathbb{D}'(X)$ and in case $X = \mathbb{R}$ we write simply $\mathbb{D}(\mathbb{R})$, $\mathbb{D}'(\mathbb{R})$ respectively. Consequently, for any $p > 1$, $k \in \mathbb{R}$, $\nabla : \mathbb{D}_{p,k}(\mathbb{X}) \rightarrow \mathbb{D}_{p,k-1}(\mathbb{X} \otimes \mathbb{H})$ continuously, where $\mathbb{X} \otimes \mathbb{H}$ denotes the completed Hilbert-Schmidt tensor product of $\mathbb{X}$ and $\mathbb{H}$. Therefore $\delta = \nabla^*$ is a continuous operator from $\mathbb{D}_{p,k}(\mathbb{X} \otimes \mathbb{H})$ into $\mathbb{D}_{p,k-1}(\mathbb{X})$ for any $p > 1$, $k \in \mathbb{R}$. We call $\delta$ the divergence operator on $W$. Let us remark that from these properties, $\delta$ and $\nabla$ extend continuously as operators from $\mathbb{D}'(\mathbb{X} \otimes \mathbb{H})$ to $\mathbb{D}'(\mathbb{X})$ and from $\mathbb{D}'(\mathbb{X})$ to $\mathbb{D}'(\mathbb{X} \otimes \mathbb{H})$ respectively. Let us recall that, in the case of classical Wiener space, $\delta$ coincides with the Itô stochastic integral on the adapted processes. We recall that, if $F$ is in $\mathbb{D}_{p,1}(\mathbb{H})$ for some $p > 1$, then almost surely, $\nabla F$ is an Hilbert-Schmidt operator on $\mathbb{H}$, and if $F$ is an $\mathbb{H}$-valued polynomial, then $\delta F$ can be written as

$$
\delta F = \sum_{i=1}^{\infty} \left[ (F, e_i)_{\mathbb{H}} \delta e_i - \left( \nabla(F, e_i)_{\mathbb{H}}, e_i \right)_{\mathbb{H}} \right],
$$

where $(e_i, i \in \mathbb{N})$ is any complete orthonormal basis in $\mathbb{H}$.

In the sequel we shall use the notion of second quantization of bounded operators on $\mathbb{H}$; although this is a well-known subject, we give a brief outline below for the reader’s convenience (cf. [1], [5], [13]). Assume that $A : \mathbb{H} \rightarrow \mathbb{H}$ is a bounded, linear operator, then it has a unique, $\mu$-measurable (i.e., measurable with respect to the $\mu$-completion of $\mathcal{B}(W)$) extension, denoted by $\tilde{A}$, as a linear map on $W$ (cf.[1, 5]). Assume in particular that $\|A\| \leq 1$ and define $S = (I_H - A^*A)^{1/2}$, $T = (I_H - AA^*)^{1/2}$ and $U : \mathbb{H} \times \mathbb{H} \rightarrow \mathbb{H} \times \mathbb{H}$ as $U(h, k) = (Ah +Tk, -Sh + A^*k)$. $U$ is then a unitary operator on $\mathbb{H} \times \mathbb{H}$, hence its $\mu \times \mu$-measurable linear extension to $W \times W$ preserves the Wiener measure $\mu \times \mu$ (this is called the rotation associated to $U$, cf. [19], Chapter VIII). Using this observation, one can define the second quantization of $A$ via the generalized Mehler formula as

$$
\Gamma(A)f(w) = \int_{W} f(\tilde{A}^*w + \tilde{S}y) \mu(dy),
$$

which happens to be a Markovian contraction on $L^p(\mu)$ for any $p \geq 1$. $\Gamma(A)$
can be calculated explicitly for the Wick exponentials as

$$\Gamma(A) \exp \{\delta h - 1/2|h|^2_H\} = \exp \{\delta Ah - 1/2|Ah|^2_H\} \quad (h \in H).$$

This identity implies that $\Gamma(AB) = \Gamma(A)\Gamma(B)$ and that for any sequence $(A_n, n \in \mathbb{N})$ of operators whose norms are bounded by one, $\Gamma(A_n)$ converges strongly to $\Gamma(A)$ if $\lim_n A_n = A$ in the strong operator topology. A particular case of interest is when we take $A = e^{-t}I_H$, then $\Gamma(e^{-t}I_H)$ equals to the Ornstein-Uhlenbeck semigroup $P_t$. Also if $\pi$ is the orthogonal projection of $H$ onto a closed vector subspace $K$, then $\Gamma(\pi)$ is the conditional expectation with respect to the sigma field generated by $\{\delta k, k \in K\}$.

3 \quad $H$-convexity and its properties

Let us give the notion of $H$-convexity on the Wiener space $W$:

**Definition 3.1** Let $F : W \to \mathbb{R} \cup \{\infty\}$ be a measurable function. It is called $H$-convex if for any $h, k \in H$, $\alpha \in [0, 1]$

$$F(w + \alpha h + (1 - \alpha)k) \leq \alpha F(w + h) + (1 - \alpha)F(w + k) \quad (1)$$

almost surely.

**Remarks:**

- This definition is more general than the one given in [17, 19] since $F$ may be infinite on a set of positive measure.
- Note that the negligible set on which the relation (1) fails may depend on the choice of $h, k$ and of $\alpha$.
- If $G : W \to \mathbb{R}$ is a measurable convex function, then it is necessarily $H$-convex.
- To conclude the $H$-convexity, it suffices to verify the relation (1) for $k = -h$ and $\alpha = 1/2$.

The following properties of $H$-convex Wiener functionals have been proved in [17, 18, 19]:

**Theorem 3.1**

1. If $(F_n, n \in \mathbb{N})$ is a sequence of $H$-convex functionals converging in probability, then the limit is also $H$-convex.
2. If \( F \in L^p(\mu) \ (p > 1) \) is \( H \)-convex if and only if \( \nabla^2 F \) is positive and symmetric Hilbert-Schmidt operator valued distribution on \( W \).

3. If \( F \in L^1(\mu) \) is \( H \)-convex, then \( P_t F \) is also \( H \)-convex for any \( t \geq 0 \), where \( P_t \) is the Ornstein-Uhlenbeck semi-group on \( W \).

The following result is immediate from Theorem 3.1:

**Corollary 3.1**

\( F \in \bigcup_{p>1} L^p(\mu) \) is \( H \)-convex if and only if

\[
E \left[ \varphi \left( \nabla^2 F(w), h \otimes h \right)_2 \right] \geq 0
\]

for any \( h \in H \) and \( \varphi \in \mathbb{D}_+ \), where \((\cdot, \cdot)_2\) denotes the scalar product for the Hilbert-Schmidt operators on \( H \).

We have also

**Corollary 3.2**

If \( F \in L^p(\mu), p > 1 \), is \( H \)-convex and if \( E[\nabla^2 F] = 0 \), then \( F \) is of the form

\[
F = E[F] + \delta(E[\nabla F]).
\]

**Proof:** Let \( (P_t, t \geq 0) \) denote the Ornstein-Uhlenbeck semigroup, \( P_t F \) is again \( H \)-convex and Sobolev differentiable. Moreover \( \nabla^2 P_t F = e^{-2t} P_t \nabla^2 F \).

Hence \( E[\nabla^2 P_t F] = 0 \), and the positivity of \( \nabla^2 P_t F \) implies that \( \nabla^2 P_t F = 0 \) almost surely, hence \( \nabla^2 F = 0 \). This implies that \( F \) is in the first two Wiener chaos.

**Remark:** It may be worth-while to note that the random variable which represents the share price of the Black and Scholes model in financial mathematics (cf.[10]) is \( H \)-convex.

We shall need also the concept of \( C \)-convex functionals:

**Definition 3.2** Let \( (e_i, i \in \mathbb{N}) \subset W^* \) be any complete, orthonormal basis of \( H \). For \( w \in W \), define \( w_n = \sum_{i=1}^{n} \delta e_i(w) e_i \) and \( w_n^\perp = w - w_n \), then a Wiener functional \( f : W \to \mathbb{R} \) is called \( C \)-convex if, for any such basis \( (e_i, i \in \mathbb{N}) \), for almost all \( w_n^\perp \), the partial map

\[
w_n \to f(w_n^\perp + w_n)
\]

has a modification which is convex on the space \( \text{span}\{e_1, \ldots, e_n\} \simeq \mathbb{R}^n \).
Remark: It follows from Corollary 3.1 that, if $f$ is $H$-convex and in some $L^p(\mu)$ ($p > 1$), then it is $C$-convex. We shall prove that this is also true without any integrability hypothesis.

We begin with the following lemma whose proof is obvious:

Lemma 3.1
If $f$ is $C$-convex then it is $H$-convex.

In order to prove the validity of the converse of Lemma 3.1 we need some technical results from the harmonic analysis on finite dimensional Euclidean spaces that we shall state as separate lemmas:

Lemma 3.2
Let $B \in \mathcal{B}(\mathbb{R}^n)$ be a set of positive Lebesgue measure. Then $B + B$ contains a non-empty open set.

Proof: Let $\phi(x) = 1_B \ast 1_B(x)$, where “$\ast$” denotes the convolution of functions with respect to the Lebesgue measure. Then $\phi$ is a non-negative, continuous function, hence the set $O = \{x \in \mathbb{R}^n : \phi(x) > 0\}$ is an open set. Since $B$ has positive measure, $\phi$ can not be identically zero, hence $O$ is non-empty. Besides, if $x \in O$, then the set of $y \in \mathbb{R}^n$ such that $y \in B$ and $x - y \in B$ has positive Lebesgue measure, otherwise $\phi(x)$ would have been null. Consequently $O \subset B + B$.

The following lemma gives a more precise statement than Lemma 3.2:

Lemma 3.3
Let $B \in \mathcal{B}(\mathbb{R}^n)$ be a set of positive Lebesgue measure and assume that $A \subset \mathbb{R}^n \times \mathbb{R}^n$ with $B \times B = A$ almost surely with respect to the Lebesgue measure of $\mathbb{R}^n \times \mathbb{R}^n$. Then the set $\{x + y : (x, y) \in A\}$ contains almost surely an open subset of $\mathbb{R}^n$.

Proof: It follows from an obvious change of variables that

$1_A(y, x - y) = 1_B(y)1_B(x - y)$

almost surely, hence

$\int_{\mathbb{R}^n} 1_A(y, x - y)dy = \phi(x)$

almost surely, where $\phi(x) = 1_B \ast 1_B(x)$. Consequently, for almost all $x \in \mathbb{R}^n$ such that $\phi(x) > 0$, one has $(y, x - y) \in A$, this means that

$\{x \in \mathbb{R}^n : \phi(x) > 0\} \subset \{u + v : (u, v) \in A\}$
almost surely.

The following lemma is particularly important for the sequel:

**Lemma 3.4**

Let \( f : \mathbb{R}^n \to \mathbb{R}_+ \cup \{\infty\} \) be a Borel function which is finite on a set of positive Lebesgue measure. Assume that, for any \( u \in \mathbb{R}^n \),

\[
f(x) \leq \frac{1}{2} [f(x + u) + f(x - u)]
\]

\( dx \)-almost surely (the negligible set on which the inequality (2) fails may depend on \( u \)). Then there exists a non-empty, open convex subset \( U \) of \( \mathbb{R}^n \) such that \( f \) is locally essentially bounded on \( U \). Moreover let \( D \) be the set consisting of \( x \in \mathbb{R}^n \) such that any neighbourhood of \( x \in D \) contains a Borel set of positive Lebesgue measure on which \( f \) is finite, then \( D \subset \overline{U} \), in particular \( f = \infty \) almost surely on the complement of \( U \).

**Proof:** From the theorem of Fubini, the inequality (2) implies that

\[
2f \left( \frac{x + y}{2} \right) \leq f(x) + f(y)
\]

\( dx \times dy \)-almost surely. Let \( B \in \mathcal{B}(\mathbb{R}^n) \) be a set of positive Lebesgue measure on which \( f \) is bounded by some constant \( M > 0 \). Then from Lemma 3.2, \( B + B \) contains an open set \( O \). Let \( A \) be the set consisting of the elements of \( B \times B \) for which the inequality (3) holds. Then \( A = B \times B \) almost surely, hence from Lemma 3.3, the set \( \Gamma = \{ x + y : (x, y) \in A \} \) contains almost surely the open set \( O \). Hence for almost all \( z \in \frac{1}{2}O \), \( 2z \) belongs to the set \( \Gamma \), consequently \( z = \frac{1}{2}(x + y) \), with \( (x, y) \in A \). This implies, from (3), that \( f(z) \leq M \). Consequently \( f \) is essentially bounded on the open set \( \frac{1}{2}\Gamma \).

Let now \( U \) be set of points which have neighbourhoods on which \( f \) is essentially bounded. Clearly \( U \) is open and non-empty by what we have shown above. Let \( S \) and \( T \) be two balls of radius \( \rho \), on which \( f \) is bounded by some \( M > 0 \). Assume that they are centered at the points \( a \) and \( b \) respectively. Let \( u = \frac{1}{2}(b - a) \), then for almost all \( x \in \frac{1}{2}(S + T) \), \( x + u \in T \) and \( x - u \in S \), hence, from the inequality (2) \( f(x) \leq M \), which shows that \( f \) is essentially bounded on the set \( \frac{1}{2}(S + T) \) and this proves the convexity of \( U \).

To prove the last claim, let \( x \) be any element of \( D \) and let \( V \) be any neighbourhood of \( x \); without loss of generality, we may assume that \( V \) is convex. Then there exists a Borel set \( B \subset V \) of positive measure on which \( f \) is bounded, hence from the first part of the proof, there exists an open
neighbourhood $O \subset B + B$ such that $f$ is essentially bounded on $\frac{1}{2}O \subset \frac{1}{2}(V + V) \subset V$, hence $\frac{1}{2}O \subset U$. Consequently $V \cap U \neq \emptyset$, and this implies that $x$ is in the closure of $U$, i.e. $D \subset \overline{U}$. The fact that $f = \infty$ almost surely on the complement of $\overline{U}$ is obvious from the definition of $D$.

**Theorem 3.2**

Let $g : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ be a measurable mapping such that, for almost all $u \in \mathbb{R}^n$,

$$g(u + \alpha x + \beta y) \leq \alpha g(u + x) + \beta g(u + y)$$

for any $\alpha, \beta \in [0, 1]$ with $\alpha + \beta = 1$ and for any $x, y \in \mathbb{R}^n$, where the negligible set on which the relation (4) fails may depend on the choice of $x, y$ and of $\alpha$. Then $g$ has a modification $g'$ which is a convex function.

**Proof:** Assume first that $g$ is positive, then with the notations of Lemma 3.4, define $g' = g$ on the open, convex set $U$ and as $g' = \infty$ on $U^c$. From the relation (4), $g'$ is a distribution on $U$ whose second derivative is positive, hence it is convex on $U$, hence it is convex on the whole space $\mathbb{R}^n$. Moreover we have $\{g' \neq g\} \subset \partial U$ and $\partial U$ has zero Lebesgue measure, consequently $g = g'$ almost surely. For general $g$, define $f_\epsilon = e^{\epsilon g}$ ($\epsilon > 0$), then, from what is proven above, $f_\epsilon$ has a modification $f'_\epsilon$ which is convex (with the same fixed open and convex set $U$), hence $\limsup_{\epsilon \to 0} \frac{f'_\epsilon - 1}{\epsilon} = g'$ is also convex and $g = g'$ almost surely.

**Theorem 3.3**

A Wiener functional $F : W \to \mathbb{R} \cup \{\infty\}$ is $\mathcal{H}$-convex if and only if it is $\mathcal{C}$-convex.

**Proof:** We have already proven the sufficiency. To prove the necessity, with the notations of Definition 3.2, $\mathcal{H}$-convexity implies that $h \mapsto F(w_n^+ + w_n + h)$ satisfies the hypothesis of Theorem 3.2 when $h$ runs in any $n$-dimensional Euclidean subspace of $H$, hence the partial mapping $w_n \mapsto F(w_n^+ + w_n)$ has a modification which is convex on the vector space spanned by $\{e_1, \ldots, e_n\}$.

4 Log $\mathcal{H}$-concave and $\mathcal{C}$-log concave Wiener functionals

**Definition 4.1** Let $F$ be a measurable mapping from $W$ into $\mathbb{R}_+$ with $\mu\{F > 0\} > 0$. 

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1. \( F \) is called log \( H \)-concave, if for any \( h, k \in H, \alpha \in [0, 1] \), one has
\[
F(w + \alpha h + (1 - \alpha)k) \geq F(w + h)\alpha F(w + k)^{1-\alpha}
\] (5)
aalmost surely, where the negligible set on which the relation (5) fails may depend on \( h, k \) and on \( \alpha \).

2. We shall say that \( F \) is \( C \)-log concave, if for any complete, orthonormal basis \((e_i, i \in \mathbb{N}) \subset W^* \) of \( H \), the partial map \( w_n \rightarrow F(w_n^\perp + w_n) \) is log-concave (cf. Definition 3.2 for the notation), up to a modification, on \( \text{span}\{e_1, \ldots, e_n\} \sim \mathbb{R}^n \).

Let us remark immediately that if \( F = G \) almost surely then \( G \) is also log \( H \)-concave. Moreover, any limit in probability of log \( H \)-concave random variables is again log \( H \)-concave. We shall prove below some less immediate properties. Let us begin with the following observation which is a direct consequence of Theorem 3.3:

**Remark:** \( F \) is log \( H \)-concave if and only if \( -\log F \) is \( H \)-convex (which may be infinity with a positive probability), hence if and only if \( F \) is \( C \)-log concave.

**Theorem 4.1**
Suppose that \((W_i, H_i, \mu_i), i = 1, 2, \) are two abstract Wiener spaces. Consider \((W_1 \times W_2, H_1 \times H_1, \mu_1 \times \mu_2)\) as an abstract Wiener space. Assume that \( F : W_1 \times W_2 \rightarrow \mathbb{R}_+ \) is log \( H_1 \times H_2 \)-concave. Then the map
\[
w_2 \rightarrow \int_{W_1} F(w_1, w_2) \, d\mu_1(w_1)
\]
is log \( H_2 \)-concave.

**Proof:** If \( F \) is log \( H \times H \)-concave, so is also \( F \wedge c \) (\( c \in \mathbb{R}_+ \)), hence we may suppose without loss of generality that \( F \) is bounded. Let \((e_i, i \in \mathbb{N})\) be a complete, orthonormal basis in \( H_2 \). It suffices to prove that
\[
E_1[F](w_2 + \alpha h + \beta l) \geq (E_1[F](w_2 + h))^\alpha (E_1[F](w_2 + l))^{\beta}
\]
almost surely, for any \( h, l \in \text{span}\{e_1, \ldots, e_k\}, \alpha, \beta \in [0, 1] \) with \( \alpha + \beta = 1 \), where \( E_1 \) denotes the expectation with respect to \( \mu_1 \). Let \((P_n, n \in \mathbb{N})\) be a sequence of orthogonal projections of finite rank on \( H_1 \) increasing to the identity map of it. Denote by \( \mu_1^n \) the image of \( \mu_1 \) under the map \( w_1 \rightarrow P_n w_1 \) and by \( \mu_1^{n,\perp} \) the image of \( \mu_1 \) under \( w_1 \rightarrow w_1 - P_n w_1 \). We have, from the martingale convergence theorem,
\[
\int_{W_1} F(w_1, w_2) \, d\mu_1(w_1) = \lim_n \int_{W_1} F(w_1^{n,\perp} + w_1^n, w_2) \, d\mu_1^n(w_1^n)
\]
almost surely. Let \((Q_n, n \in \mathbb{N})\) be a sequence of orthogonal projections of finite rank on \(H_2\) increasing to the identity, corresponding to the basis \((e_n, n \in \mathbb{N})\). Let \(w_2^k = Q_k w_2\) and \(w_2^k = w_2 - w_2^k\). Write
\[
F(w_1, w_2) = F(w_1^n + w_1^k, w_2^n + w_2^k) \\
= F_{w_1^n, w_2^n}(w_1, w_2).
\]
From the hypothesis
\[(w_1^n, w_2^n) \rightarrow F_{w_1^n, w_2^n}(w_1^n, w_2^n)\]
has a log concave modification on the \((n + k)\)-dimensional Euclidean space. From the theorem of Prékopa (cf.[12]), it follows that
\[
w_2^k \rightarrow \int F_{w_1^n + w_2^n}(w_1^n, w_2^n) d\mu_n(w_1^n)
\]
is log concave on \(\mathbb{R}^k\) for any \(k \in \mathbb{N}\) (upto a modification), hence
\[
w_2 \rightarrow \int F(w_1^n + w_1^n, w_2^n) d\mu(w_1^n)
\]
is log \(H_2\)-concave for any \(n \in \mathbb{N}\), then the proof follows by passing to the limit with respect to \(n\). \(\square\)

**Theorem 4.2**
Let \(A : H \rightarrow H\) be a linear operator with \(\|A\| \leq 1\), denote by \(\Gamma(A)\) its second quantization as explained in the preliminaries. If \(F : W \rightarrow \mathbb{R}_+\) is a log \(H\)-concave Wiener functional, then \(\Gamma(A)F\) is also log \(H\)-concave.

**Proof:** Replacing \(F\) by \(F \wedge c = \min(F, c),\ c > 0\), we may suppose that \(F\) is bounded. It is easy to see that the mapping
\[
(w, y) \rightarrow F(\tilde{A} w + \tilde{S} y)
\]
is log \(H \times H\)-concave on \(W \times W\). In fact, for any \(\alpha + \beta = 1,\ h, k, u, v \in H,\) one has
\[
F(\tilde{A} w + \tilde{S} y + \alpha(A^* h + Sk) + \beta(A^* u + Sv)) \\
\geq F(\tilde{A} w + \tilde{S} y + A^* h + Sk)^\alpha F(\tilde{A} w + \tilde{S} y + A^* u + Sv)^\beta,
\]
d\(\mu \times d\mu\)-almost surely. Let us recall that, since the image of \(\mu \times \mu\) under the map \((w, y) \rightarrow \tilde{A} w + \tilde{S} y\) is \(\mu\), the terms in the inequality (6) are defined without ambiguity. Hence
\[
\Gamma(A)F(w) = \int_W F(\tilde{A} w + \tilde{S} y) \mu(dy)
\]
is log \(H\)-concave on \(W\) from Theorem 4.1. \(\square\)
**Corollary 4.1**

Let $F : W \to \mathbb{R}_+$ be a log $H$-concave functional. Assume that $K$ is any closed vector subspace of $H$ and denote by $V(K)$ the sigma algebra generated by $\{\delta_k, k \in K\}$. Then the conditional expectation of $F$ with respect to $V(K)$, i.e., $E[F | V(K)]$ is again log $H$-concave.

**Proof:** The proof follows from Theorem 4.2 as soon as we remark that $\Gamma (\pi_K) F = E[F | V(K)]$, where $\pi_K$ denotes the orthogonal projection associated to $K$. \hfill \Box

**Corollary 4.2**

Let $F$ be log $H$-concave. If $P_t$ denotes the Ornstein-Uhlenbeck semigroup on $W$, then $w \to P_tF(w)$ is log $H$-concave.

**Proof:** Since $P_t = \Gamma (e^{-t}I_H)$, the proof follows from Theorem 4.2. \hfill \Box

Here is an important application of these results:

**Theorem 4.3**

Assume that $F : W \to \mathbb{R} \cup \{\infty\}$ is an $H$-convex Wiener functional, then $F$ has a modification $F'$ which is a Borel measurable convex function on $W$. Any log $H$-concave functional $G$ has a modification $G'$ which is Borel measurable and log-concave on $W$.

**Proof:** Assume first that $F$ is positive, let $G = \exp -F$, then $G$ is a positive, bounded $C$-log concave function. Define $G_n$ as

$$G_n = E[P_{1/n}G | V_n],$$

where $V_n$ is the sigma algebra generated by $\{\delta e_1, \ldots, \delta e_n\}$, and $(\epsilon_i, i \in \mathbb{N}) \subset W^*$ is a complete orthonormal basis of $H$. Since $P_{1/n}E[G | V_n] = E[P_{1/n}G | V_n]$, the positivity improving property of the Ornstein-Uhlenbeck semigroup implies that $G_n$ is almost surely strictly positive (even quasi-surely). As we have attained the finite dimensional case, $G_n$ has a modification $G'_n$ which is continuous on $W$ and, from Corollary 4.1 and Corollary 4.2, it satisfies

$$G'_n (w + ah + bk) \geq G'_n (w + h)^a G'_n (w + k)^b$$

almost surely, for any $h, k \in H$ and $a + b = 1$. The continuity of $G'_n$ implies that the relation (7) holds for any $h, k \in H$, $w \in W$ and $a \in [0, 1]$. Hence $G'_n$ is log-concave on $W$ and this implies that $- \log G'_n$ is convex on $W$. Define $F' = \lim \sup_n ( - \log G'_n)$, then $F'$ is convex and Borel measurable on $W$ and $F = F'$ almost surely.
For general $F$, define $f_\varepsilon = e^{\varepsilon F}$, then from above, there exists a modification of $f_\varepsilon$, say $f'_\varepsilon$ which is convex and Borel measurable on $W$. To complete the proof it suffices to define $F'$ as

$$F' = \limsup_{\varepsilon \to 0} \frac{f'_\varepsilon - 1}{\varepsilon}.$$

The rest is now obvious.

Under the light of Theorem 4.3, the following definition is natural:

**Definition 4.2** A Wiener functional $F : W \to \mathbb{R} \cup \{\infty\}$ will be called almost surely convex if it has a modification $F'$ which is convex and Borel measurable on $W$. Similarly, a non-negative functional $G$ will be called almost surely log-concave if it has a modification $G'$ which is log-concave on $W$.

The following proposition summarizes the main results of this section:

**Theorem 4.4**
Assume that $F : W \to \mathbb{R} \cup \{\infty\}$ is a Wiener functional such that

$$\mu\{F < \infty\} > 0.$$

Then the following are equivalent:

1. $F$ is $H$-convex,
2. $F$ is $C$-convex,
3. $F$ is almost surely convex.

Similarly, for $G : W \to \mathbb{R}_+$, with $\mu\{G > 0\} > 0$, the following properties are equivalent:

1. $G$ is log $H$-concave,
2. $G$ is log $C$-concave,
3. $G$ is almost surely log-concave.

The notion of a convex set can be extended as

**Definition 4.3** Any measurable subset $A$ of $W$ will be called $H$-convex if its indicator function $1_A$ is log $H$-concave.
**Remark:** Evidently any measurable convex subset of $W$ is $H$-convex. Moreover, if $A = A'$ almost surely and if $A$ is $H$-convex, then $A'$ is also $H$-convex.

**Remark:** If $\phi$ is an $H$-convex Wiener functional, then the set $$\{w \in W : \phi(w) \leq t\}$$ is $H$-convex for any $t \in \mathbb{R}$.

We have the following result about the characterization of the $H$-convex sets:

**Theorem 4.5**
Assume that $A$ is an $H$-convex set, then there exists a convex set $A'$, which is Borel measurable such that $A = A'$ almost surely.

**Proof:** Since, by definition, $1_A$ is a log $H$-concave Wiener functional, from Theorem 4.3, there exists a log-concave Wiener functional $f_A$ such that $f_A = 1_A$ almost surely. It suffices to define $A'$ as the set $$A' = \{w \in W : f_A(w) \geq 1\}.$$ \qed

**Example:** Assume that $A$ is an $H$-convex subset of $W$ of positive measure. Define $p_A$ as $$p_A(w) = \inf \{|h|_H : h \in (A - w) \cap H\}.$$ Then $p_A$ is $H$-convex, hence almost surely convex (and $H$-Lipschitz c.f. [19]). Moreover, the $\{w : p_A(w) \leq \alpha\}$ is an $H$-convex set for any $\alpha \in \mathbb{R}_+$. 

5 Extensions and some applications

**Definition 5.1** Let $(e_i, i \in \mathbb{N})$ be any complete orthonormal basis of $H$. We shall denote, as before, by $w_n = \sum_{i=1}^n \delta e_i(w) e_i$ and $w_n^\perp = w - w_n$. Assume now that $F : W \to \mathbb{R} \cup \{\infty\}$ is a measurable mapping with $\mu\{F < \infty\} > 0$.

1. We say that it is $a$-convex ($a \in \mathbb{R}$), if the partial map $$w_n \to \frac{a}{2}|w_n|^2 + F(w_n^\perp + w_n)$$ is almost surely convex for any $n \geq 1$, where $|w_n|$ is the Euclidean norm of $w_n$.
2. We call $G$ a-log-concave if
\[ w_n \rightarrow \exp \left\{ -\frac{a}{2}|w_n|^2 \right\} G(w_n^+ + w_n) \]
is almost surely log-concave for any $n \in \mathbb{N}$.

**Remark:** $G$ is a-log-concave if and only if $-\log G$ is a-convex.

The following theorem gives a practical method to verify $a$-convexity or log-concavity:

**Theorem 5.1**
Let $F : W \rightarrow \mathbb{R} \cup \{\infty\}$ be a measurable map such that $\mu\{F < \infty\} > 0$. Define the map $F_a$ on $H \times W$ as
\[ F_a(h, w + h) = \frac{a}{2}|h|^2_H + F(h + w) . \]
Then $F$ is $a$-convex if and only if, for any $h, k \in H$ and $\alpha, \beta \in [0, 1]$ with $\alpha + \beta = 1$, one has
\[ F_a(\alpha h + \beta k, w + \alpha h + \beta k) \leq \alpha F_a(h, w + h) + \beta F_a(k, w + k) \tag{8} \]
$\mu$-almost surely, where the negligible set on which the inequality (8) fails may depend on the choice of $h, k$ and of $\alpha$.

Similarly a measurable mapping $G : W \rightarrow \mathbb{R}_+$ is a-log-concave if and only if the map defined by
\[ G_a(h, w + h) = \exp \left\{ -\frac{a}{2}|h|^2_H \right\} G(h + w) \]
satisfies the inequality
\[ G_a(\alpha h + \beta k, w + \alpha h + \beta k) \geq G_a(h, w + h)^\alpha G_a(k, w + k)^\beta , \tag{9} \]
$\mu$-almost surely, where the negligible set on which the inequality (9) fails may depend on the choice of $h, k$ and of $\alpha$.

**Proof:** Let us denote by $h_n$ its projection on the vector space spanned by $\{e_1, \ldots, e_n\}$, i.e. $h_n = \sum_{i \leq n} (h, e_i) e_i$. Then, from Theorem 4.4, $F$ is $a$-convex if and only if the map
\[ h_n \rightarrow \frac{a}{2} \left[ |w_n|^2 + 2(w_n, h_n) + |h_n|^2 \right] + F(h + h_n) \]
satisfies a convexity inequality like (8). Besides the term $|w_n|^2$ being kept constant in this operation, it can be removed from the both sides of the inequality. Similarly, since $h_n \to (w_n, h_n)$ is being affine, it also cancels from the both sides of this inequality. Hence $a$-convexity is equivalent to

$$F_a(\alpha h_n + \beta k_n, w + \alpha h_n + \beta k_n) \leq \alpha F_a(h_n, w + h_n) + \beta F_a(k_n, w + k_n)$$

where $k_n$ is defined as $h_n$ from a $k \in H$.

The second part of the theorem is obvious since $G$ is $a$-log-concave if and only if $-\log G$ is $a$-convex.

Corollary 5.1
1. Let $\hat{L}^0(\mu)$ be the space of the $\mu$-equivalence classes of $\mathbb{R} \cup \{\infty\}$-valued random variables regarded as a topological semi-group under addition and convergence in probability. Then $F \in \hat{L}^0(\mu)$ is $\beta$-convex if and only if the mapping

$$h \to \frac{\beta}{2}|h|_H^2 + F(w + h)$$

is a convex and continuous mapping from $H$ into $\hat{L}^0(\mu)$.

2. $F \in L^p(\mu), p > 1$ is $\beta$-convex if and only if

$$E \left[ ((\beta I_H + \nabla^2 F)h, h)_{H} \phi \right] \geq 0$$

for any $\phi \in \mathcal{D}$ positive and $h \in H$, where $\nabla^2 F$ is to be understood in the sense of the distributions $\mathcal{D}'$.

Example: Note for instance that $\sin \delta h$ with $|h|_H = 1$, is a 1-convex and that $\exp(\sin \delta h)$ is 1-log-concave.

The following result is a direct consequence of Prekopa’s theorem:

Proposition 5.1
Let $G$ be an $a$-log concave Wiener functional, $a \in [0, 1]$, and assume that $V$ is any sigma algebra generated by the elements of the first Wiener chaos. Then $E[G|V]$ is again $a$-log-concave.

Proof: From Corollary 5.1, it suffices to prove the case $V$ is generated by
\{\delta e_1, \ldots, \delta e_k\}$, where $(e_n, n \in \mathbb{N})$ is an orthonormal basis of $H$. Let
\[
\begin{align*}
  w_k &= \sum_{i \leq k} \delta e_i(w)e_i \\
  z_k &= w - w_k \\
  z_{k,n} &= \sum_{i=k+1}^{k+n} \delta e_i(w)e_i
\end{align*}
\]
and let $z_{k,n}^\perp = z_k - z_{k,n}$. Then we have
\[
E[G|V] = \int G(z_k + w_k) d\mu(z_k) = \lim_{n \to \infty} \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} G(z_{k,n}^\perp + z_{k,n} + w_k) e^{-\frac{|z_{k,n}^\perp|^2}{2}} dz_{k,n}.
\]
Since
\[
(z_{k,n}, w_k) \to \exp\left\{-\frac{1}{2}(|w_k|^2 + |z_{n,k}|^2)\right\} G(z_{k,n}^\perp + z_{k,n} + w_k)
\]
is almost surely log-concave, the proof follows from Prekopa’s theorem (cf. [12]).

The following theorem extends Theorem 4.2:

**Theorem 5.2**

Let $G$ be an $a$-log-concave Wiener functional, where $a \in [0, 1)$. Then $\Gamma(A)G$ is a-log-concave, where $A \in L(H, H)$ (i.e. the space of bounded linear operators on $H$) with $\|A\| \leq 1$. In particular $P_tG$ is a-log-concave for any $t \geq 0$, where $(P_t, t \geq 0)$ denotes the Ornstein-Uhlenbeck semi-group on $W$.

**Proof:** Let $(e_i, i \in \mathbb{N})$ be a complete, orthonormal basis of $H$, denote by $\pi_n$ the orthogonal projection from $H$ onto the linear space spanned by $\{e_1, \ldots, e_n\}$ and by $V_n$ the sigma algebra generated by $\{\delta e_1, \ldots, \delta e_n\}$. From Proposition 5.1 and from the fact that $\Gamma(\pi_n A \pi_n) \to \Gamma(A)$ in the strong operator topology as $n$ tends to infinity, it suffices to prove the theorem when $W = \mathbb{R}^n$. We may then assume that $G$ is bounded and of compact support. Define $F$ as
\[
G(x) = F(x) e^{\frac{1}{2} |x|^2} = F(x) \int_{\mathbb{R}^n} e^{\sqrt{a}(x, \xi)} d\mu(\xi).
\]
From the hypothesis, \( F \) is almost surely log-concave. Then, using the notations explained in Section 2:

\[
e^{-\frac{|x|^2}{2}} \Gamma(A) G(x)
\]

\[
= \int \int F(A^* x + Sy) \exp \left\{ -\frac{|x|^2}{2} + \sqrt{a} (A^* x + Sy, \xi) \right\} d\mu(y) d\mu(\xi)
\]

\[
= (2\pi)^{-n} \int \int F(A^* x + Sy) \exp \left\{ -\frac{\Theta(x, y, \xi)}{2} \right\} dy d\xi,
\]

where

\[
\Theta(x, y, \xi) = a|x|^2 - 2\sqrt{a} (A^* x + Sy, \xi) + |y|^2 + |\xi|^2
\]

\[
= |\sqrt{ax} - A\xi|^2 + |\sqrt{ay} - S\xi|^2 + (1 - a)|y|^2,
\]

which is a convex function of \((x, y, \xi)\). Hence the proof follows from Prékopa’s theorem (cf.[12]).

The following proposition extends a well-known finite dimensional inequality (cf.[7]):

**Proposition 5.2**

Assume that \( f \) and \( g \) are \( H \)-convex Wiener functionals such that \( f \in L^p(\mu) \) and \( g \in L^q(\mu) \) with \( p > 1, \frac{1}{p} - \frac{1}{q} = 1 - q^{-1} \). Then

\[
E[f g] \geq E[f] E[g] + (E[\nabla f], E[\nabla g])_H. \tag{10}
\]

**Proof:** Define the smooth and convex functions \( f_n \) and \( g_n \) on \( W \) by

\[
P_{1/n} f = f_n, \quad P_{1/n} g = g_n.
\]

Using the fact that \( P_t = e^{-tL} \), where \( L \) is the number operator \( L = \delta \circ \nabla \) and the commutation relation \( \nabla P_t = e^{-t} P_t \nabla \), for any \( 0 \leq t \leq T \), we have

\[
E[P_{T-t} f_n g_n] = E[P_T f_n g_n] + \int_0^t E[L P_{T-s} f_n g_n] ds
\]

\[
= E[P_T f_n g_n] + \int_0^t e^{-(T-s)} E[(P_{T-s} \nabla f_n, \nabla g_n)_H] ds
\]

\[
= E[P_T f_n g_n] + \int_0^t e^{-(T-s)} E[(P_T \nabla f_n, \nabla g_n)_H] ds
\]

\[
+ e^{-2T} \int_0^t \int_0^s e^{s+t} E[(P_{T-\tau} \nabla^2 f_n, \nabla^2 g_n)_2] d\tau ds
\]

\[
\geq E[P_T f_n g_n] + E[(P_T \nabla f_n, \nabla g_n)_H] e^{-T} (e^t - 1) \tag{11}
\]

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where \((\cdot, \cdot)_2\) denotes the Hilbert-Schmidt scalar product and the inequality (11) follows from the convexity of \(f_n\) and \(g_n\). In fact their convexity implies that \(P_t \nabla^2 f_n\) and \(\nabla^2 g_n\) are positive operators, hence their Hilbert-Schmidt tensor product is positive. Letting \(T = t\) in the above inequality we have

\[
E[f_n g_n] \geq E[P_T f_n g_n] + (1 - e^{-T}) E[(P_T \nabla f_n, \nabla g_n)]_H. \tag{12}
\]

Letting \(T \to \infty\) in (12), we obtain, by the ergodicity of \((P_t, t \geq 0)\), the claimed inequality for \(f_n\) and \(g_n\). It suffices then to take the limit of this inequality as \(n\) tends to infinity.

**Proposition 5.3**

Let \(G\) be a (positive) \(\gamma\)-log-concave Wiener functional with \(\gamma \in [0, 1]\). Then the map \(h \to E[G(w + h)]\) is a log-concave mapping on \(H\). In particular, if \(G\) is symmetric, i.e., if \(G(w) = G(-w)\), then

\[
E[G(w + h)] \leq E[G].
\]

**Proof:** Without loss of generality, we may suppose that \(G\) is bounded. Using the usual notations, we have, for any \(h\) in any finite dimensional subspace \(L\) of \(H\),

\[
E[G(w + h)] = \lim_n \frac{1}{(2\pi)^{n/2}} \int_{W_n} G(w_n^+ + w_n + h) \exp \left\{ -\frac{|w_n|^2}{2} \right\} dw_n,
\]

from the hypothesis, the integrand is almost surely log-concave on \(W_n \times L\), from Prekopa’s theorem, the integral is log-concave on \(L\), hence the limit is also log-concave. Since \(L\) is arbitrary, the first part of the proof follows. To prove the second part, let \(g(h) = E[G(w + h)]\), then, from the log-concavity of \(g\) and symmetry of \(G\), we have

\[
E[G] = g(0) = g\left(1/2(h) + 1/2(-h)\right) \geq g(h)^{1/2}g(-h)^{1/2} = g(h) = E[G(w + h)].
\]

**Remark:** In fact, with a little bit more attention, we can see that the map \(h \to \exp\left\{\frac{1}{2}(1 - \gamma)|h|_H^2\right\}E[G(w + h)]\) is log-concave on \(H\).

We have the following immediate corollary:
Corollary 5.2
Assume that $A \subset W$ is an $H$-convex and symmetric set. Then we have
$$\mu(A + h) \leq \mu(A),$$
for any $h \in H$.

**Proof:** Since $1_A$ is log $H$-concave, the proof follows from Proposition 5.3. □

Proposition 5.4
Let $F \in L^p(\mu)$ be a positive log $H$-convex function. Then for any $u \in D_{q,2}(H)$, we have
$$EF\left[(\delta u - EF[\delta u])^2\right] \geq EF\left[u_H^2 + 2\delta(\nabla u) + \text{trace}(\nabla u \cdot \nabla u)\right],$$
where $EF$ denotes the mathematical expectation with respect to the probability defined as
$$\frac{F}{EF} d\mu.$$

**Proof:** Let $F_\tau$ be $P_\tau F$, where $(P_\tau, \tau \in \mathbb{R}_+)$ denotes the Ornstein-Uhlenbeck semi-group. $F_\tau$ has a modification, denoted again by the same letter, such that the mapping $h \mapsto F_\tau(w + h)$ is real-analytic on $H$ for all $w \in W$ (cf. [19]). Suppose first also that $\|\nabla u\|_2 \in L^\infty(\mu, H \otimes H)$ where $\|\cdot\|_2$ denotes the Hilbert-Schmidt norm. Then, for any $r > 1$, there exists some $t_r > 0$ such that, for any $0 \leq t < t_r$, the image of the Wiener measure under $w \mapsto w + tu(w)$ is equivalent to $\mu$ with the Radon-Nikodym density $L_t \in L^r(\mu)$. Hence $w \mapsto F_\tau(w + tu(w))$ is a well-defined mapping on $W$ and it is in some $L^r(\mu)$ for small $t > 0$ (cf. [19], Chapter 3 and Lemma B.8.8). Besides $t \mapsto F(w + tu(w))$ is log convex on $\mathbb{R}$ since $F_\tau$ is log $H$-convex. Consequently $t \mapsto EF[F_\tau(w + tu(w))]$ is log convex and strictly positive. Then the second derivative of its logarithm at $t = 0$ should be positive. This implies immediately the claimed inequality for $\nabla u$ bounded. We then pass to the limit with respect to $u$ in $D_{q,2}(H)$ and then let $\tau \to 0$ to complete the proof. □

6 Poincaré and logarithmic Sobolev inequalities
The following theorem extends the Poincaré- Brascamp-Lieb inequality (cf.[15]):
Theorem 6.1
Assume that $F$ is a Wiener functional in $\cup_{p>1} \mathbb{D}_{p,2}$ with $e^{-F} \in L^1(\mu)$ and assume also that there exists a constant $\epsilon > 0$ such that

$$((I_H + \nabla^2 F)h, h)_H \geq \epsilon |h|_H^2$$  \hspace{1cm} (13)

almost surely, for any $h \in H$, i.e. $F$ is $(1 - \epsilon)$-convex. Let us denote by $\nu_F$ the probability measure on $(W, \mathcal{B}(W))$ defined by

$$d\nu_F = \exp \{-F - \log E[e^{-F}]\} \, d\mu.$$  

Then for any smooth cylindrical Wiener functional $\phi$, we have

$$\int_W |\phi - E_{\nu_F}[\phi]|^2 \, d\nu_F \leq \int_W ((I_H + \nabla^2 F)^{-1} \nabla \phi, \nabla \phi)_H \, d\nu_F.$$  \hspace{1cm} (14)

In particular, if $F$ is an $H$-convex Wiener functional, then the condition (13) is satisfied with $\epsilon = 1$.

**Proof:** Assume first that $W = \mathbb{R}^n$ and that $F$ is a smooth function on $\mathbb{R}^n$ satisfying the inequality (13) in this setting. Assume also for the typographical facility that $E[e^{-F}] = 1$. For any smooth function on $\mathbb{R}^n$, we have

$$\int_{\mathbb{R}^n} |\phi - E_{\nu_F}[\phi]|^2 \, d\nu_F = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-F(x) - |x|^2/2} |\phi(x) - E_F[\phi]|^2 \, dx.$$  \hspace{1cm} (15)

The function $G(x) = F(x) + \frac{1}{2} |x|^2$ is a strictly convex smooth function. Hence Brascamp-Lieb inequality (cf.[3]) implies that:

$$\int_{\mathbb{R}^n} |\phi - E_{\nu_F}[\phi]|^2 \, d\nu_F \leq \int_{\mathbb{R}^n} \left( (\text{Hess } G(x))^{-1} \nabla \phi(x), \nabla \phi(x) \right)_{\mathbb{R}^n} \, d\nu_F(x)$$

$$= \int_{\mathbb{R}^n} \left( (I_{\mathbb{R}^n} + \nabla^2 F)^{-1} \nabla \phi, \nabla \phi \right)_{\mathbb{R}^n} \, d\nu_F.$$  

To prove the general case we proceed by approximation as before: indeed let $(\epsilon_i, i \in \mathbb{N})$ be a complete, orthonormal basis of $H$, denote by $V_n$ the sigma algebra generated by $\{\delta e_1, \ldots, \delta e_n\}$. Define $F_n$ as to be $E[P_{1/n} F|V_n]$, where $P_{1/n}$ is the Ornstein-Uhlenbeck semigroup at $t = 1/n$. Then from the martingale convergence theorem and the fact that $V_n$ is a smooth sigma algebra, the sequence $(F_n, n \in \mathbb{N})$ converges to $F$ in some $\mathbb{D}_{p,2}$. Moreover $F_n$ satisfies the hypothesis (with a better constant in the inequality (13)) since $\nabla^2 F_n = e^{-2/n} E[Q_n^2 \nabla^2 F|V_n]$, where $Q_n$ denotes the orthogonal projection.
onto the vector space spanned by \( \{e_1, \ldots, e_n\} \). Besides \( F_n \) can be represented as \( F_n = \theta(\delta e_1, \ldots, \delta e_n) \), where \( \theta \) is a smooth function on \( \mathbb{R}^n \) satisfying
\[
((I_{\mathbb{R}^n} + \nabla^2 \theta(x))y, y)_{\mathbb{R}^n} \geq \varepsilon |y|^2_{\mathbb{R}^n},
\]
for any \( x, y \in \mathbb{R}^n \). Let \( w_n = \tilde{Q}_n(w) = \sum_{i \leq n}(\delta e_i)e_i \), \( W_n = P_n(W) \) and \( W_n^\perp = (I_W - \tilde{Q}_n)(W) \) as before. Let us denote by \( \nu_n \) the probability measure corresponding to \( F_n \). Let us also denote by \( V_n^\perp \) the sigma algebra generated by \( \{\delta e_k, k > n\} \). Using the finite dimensional result that we have derived, the Fubini theorem and the inequality \( 2|ab| \leq \kappa a^2 + \frac{1}{\kappa}b^2 \), for any \( \kappa > 0 \), we obtain
\[
E_{\nu_n} \left[ (|\phi - E_{\nu_n}[\phi]|)^2 \right] 
= \int_{W_n \times W_n^\perp} e^{-F_n(w_n)}|\phi(w_n + w_n^\perp) - E_{\nu_n}[\phi]|^2 d\mu_n(w_n)d\mu_n^\perp(w_n^\perp) 
\leq (1 + \kappa) \int_W e^{-F_n'}|\phi - E[e^{-F_n'}\phi|V_n^\perp]|^2 d\mu 
+ \left( 1 + \frac{1}{\kappa} \right) \int_W e^{-F_n'}|E[e^{-F_n'}\phi|V_n^\perp] - E_{\nu_n}[\phi]|^2 d\mu 
\leq (1 + \kappa)E_{\nu_n} \left[ ((I_H + \nabla^2 F_n)^{-1}\nabla \phi, \nabla \phi)_H \right] 
+ \left( 1 + \frac{1}{\kappa} \right) \int_W e^{-F_n'}|E[e^{-F_n'}\phi|V_n^\perp] - E_{\nu_n}[\phi]|^2 d\mu, \tag{16}
\]
where \( F_n' \) denotes \( F_n - \log E[e^{-F_n}] \). Since \( V_n \) and \( V_n^\perp \) are independent sigma algebras, we have
\[
|E[e^{-F_n'}\phi|V_n^\perp]| = \frac{1}{E[e^{-F_n}]|E[e^{-F_n'}\phi|V_n^\perp]|} 
\leq \frac{1}{E[e^{-F_n}]E[e^{-F_n'}|V_n^\perp]|}||\phi||_\infty 
= ||\phi||_\infty,
\]
hence, using the triangle inequality and the dominated convergence theorem, we realize that the last term in (16) converges to zero as \( n \) tends to infinity. Since the sequence of operator valued random variables \((I_H + \nabla^2 F_n)^{-1}, n \in \mathbb{N}\) is essentially bounded in the strong operator norm, we can pass to the limit on both sides and this gives the claimed inequality with a factor \( 1 + \kappa \), since \( \kappa > 0 \) is arbitrary, the proof is completed.

\[ \square \]

**Remark:** Let \( T : W \to W \) be a shift defined as \( T(w) = w + u(w) \), where \( u : W \to H \) is a measurable map satisfying \( (u(w + h) - u(w), h)_H \geq -\varepsilon |h|^2 \).  

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In [17] and in [19], Chapter 6, we have studied such transformations, called \( \epsilon \)-monotone shifts. Here the hypothesis of Theorem 6.1 says that the shift \( T = I_W + \nabla F \) is \( \epsilon \)-monotone.

The Sobolev regularity hypothesis can be omitted if we are after a Poincaré inequality with another constant:

**Theorem 6.2**

Assume that \( F \in \bigcup_{p>1} L^p(\mu) \) with \( E[e^{-F}] \) is finite and that, for some constant \( \epsilon > 0 \),

\[
E \left[ \left( (I_H + \nabla^2 F)h, h \right)_H \psi \right] \geq \epsilon |h|_H^2 E[\psi],
\]

for any \( h \in H \) and positive test function \( \psi \in \mathbb{D} \), where \( \nabla^2 F \) denotes the second order derivative in the sense of the distributions. Then we have

\[
E_{\nu_F} \left[ |\phi - E_F[\phi]|^2 \right] \leq \frac{1}{\epsilon} E_{\nu_F} \left[ |\nabla \phi|_H^2 \right] \tag{17}
\]

for any cylindrical Wiener functional \( \phi \). In particular, if \( F \) is \( H \)-convex, then we can take \( \epsilon = 1 \).

**Proof:** Let \( F_t \) be defined as \( P_t F \), where \( P_t \) denotes the Ornstein-Uhlenbeck semigroup. Then \( F_t \) satisfies the hypothesis of Theorem 6.1, hence we have

\[
E_{\nu_{F_t}} \left[ |\phi - E_{F_t}[\phi]|^2 \right] \leq \frac{1}{\epsilon} E_{\nu_{F_t}} \left[ |\nabla \phi|_H^2 \right]
\]

for any \( t > 0 \). The claim follows when we take the limits of both sides as \( t \to 0 \). \( \square \)

**Example:** Let \( F(w) = \|w\| + \frac{1}{2} \sin(\delta h) \) with \( |h|_H \leq 1 \), where \( \| \cdot \| \) denotes the norm of the Banach space \( W \). Then in general \( F \) is not in \( \bigcup_{p>1} \mathbb{D}_{p,2} \), however the Poincaré inequality (17) holds with \( \epsilon = 1/2 \).

**Theorem 6.3**

Assume that \( F \) is a Wiener functional in \( \bigcup_{p>1} \mathbb{D}_{p,2} \) with \( E[\exp -F] < \infty \). Assume that there exists a constant \( \epsilon > 0 \) such that

\[
\left( (I_H + \nabla^2 F)h, h \right)_H \geq \epsilon |h|_H^2 \tag{18}
\]

almost surely, for any \( h \in H \). Let us denote by \( \nu_F \) the probability measure on \( (W, \mathcal{B}(W)) \) defined by

\[
d\nu_F = \exp \{ -F - \log E[e^{-F}] \} d\mu.
\]

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Then for any smooth cylindrical Wiener functional $\phi$, we have
\[
E_{\nu_F} \left[ \phi^2 \left\{ \log \phi^2 - \log \|\phi\|_{L^2(\nu_F)}^2 \right\} \right] \leq \frac{2}{\epsilon} E_{\nu_F} \left[ |\nabla \phi|_H^2 \right].
\] (19)

In particular, if $F$ is an $H$-convex Wiener functional, then the condition (18) is satisfied with $\epsilon = 1$.

**Proof:** We shall proceed as in the proof of Theorem 6.1. Assume then that $W = \mathbb{R}^n$ and that $F$ is a smooth function satisfying the inequality (18) in this frame. In this case it is immediate to see that function $G(x) = \frac{1}{2} |x|^2 + F(x)$ satisfies the Bakry-Emery condition (cf. [2], [4]), which is known as a sufficient condition for the inequality (19). For the infinite dimensional case we define as in the proof of Theorem 6.1, $F'_n, \nu_n, V_n, V_n^\perp$. Then, denoting by $E_n$ the expectation with respect to the probability $\exp\{-F'_n\}d\mu$, where $F'_n = F_n - \log E[G_e^{-F_n}]$, we have
\[
E_n \left[ \phi^2 \left\{ \log \phi^2 - \log \|\phi\|_{L^2(\nu_F)}^2 \right\} \right] = E_n \left[ \phi^2 \left\{ \log \phi^2 - \log E[e^{-F'_n}\phi^2|V_n^\perp}] \right\} 
+ E_n \left[ \phi^2 \left\{ \log E[e^{-F'_n}\phi^2|V_n^\perp] - \log E_n[\phi^2] \right\} \right] \leq \frac{2}{\epsilon} E_n \left[ |\nabla \phi|_H^2 \right] + E_n \left[ \phi^2 \left\{ \log E[e^{-F'_n}\phi^2|V_n^\perp] - \log E_n[\phi^2] \right\} \right],
\] (20)
where we have used, as in the proof of Theorem 6.1, the finite dimensional log-Sobolev inequality to obtain the inequality (20). Since in the above inequalities everything is squared, we can assume that $\phi$ is positive, and adding a constant $\kappa > 0$, we can also replace $\phi$ with $\phi_\kappa = \phi + \kappa$. Again by the independence of $V_n$ and $V_n^\perp$, we can pass to the limit with respect to $n$ in the inequality (20) for $\phi = \phi_\kappa$ to obtain
\[
E_{\nu_F} \left[ \phi_\kappa^2 \left\{ \log \phi_\kappa^2 - \log \|\phi_\kappa\|_{L^2(\nu_F)}^2 \right\} \right] \leq \frac{2}{\epsilon} E_{\nu_F} \left[ |\nabla \phi_\kappa|_H^2 \right].
\]
To complete the proof it suffices to pass to the limit as $\kappa \to 0$. \qed

The following theorem fully extends Theorem 6.3 and it is useful for the applications:

**Theorem 6.4**

Assume that $G$ is a (positive) $\gamma$-log-concave Wiener functional for some $\gamma \in [0, 1)$ with $\mathbb{E}[G] < \infty$. Let us denote by $E_G[\cdot]$ the expectation with respect to the probability measure defined by
\[
dv_G = \frac{G}{\mathbb{E}[G]}d\mu.
\]
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Then we have

\[ E_G \left[ \phi^2 \left( \log \phi^2 - \log E_G[\phi^2] \right) \right] \leq \frac{2}{1 - \gamma} E_G[|\nabla \phi|^2_H], \tag{21} \]

for any cylindrical Wiener functional \( \phi \).

**Proof:** Since \( G \wedge c, c > 0, \) is again \( \gamma \)-log-concave, we may suppose without loss of generality that \( G \) is bounded. Let now \( (e_i, i \in \mathbb{N}) \) be a complete, orthonormal basis for \( H \), denote by \( V_n \) the sigma algebra generated by \( \{\delta e_1, \ldots, \delta e_n\} \). Define \( G_n \) as to be \( E[P_{1/n}G|V_n] \). From Proposition 5.1 and Theorem 5.2, \( G_n \) is again a \( \gamma \)-log-concave, strictly positive Wiener functional. It can be represented as

\[ G_n(w) = g_n(\delta e_1, \ldots, \delta e_n) \]

and due to the Sobolev embedding theorem, after a modification on a set of zero Lebesgue measure, we can assume that \( g_n \) is a smooth function on \( \mathbb{R}^n \). Since it is strictly positive, it is of the form \( e^{-f_n} \), where \( f_n \) is a smooth, \( \gamma \)-convex function. It follows then from Theorem 6.3 that the inequality (21) holds when we replace \( G \) by \( G_n \), then the proof follows by taking the limits of both sides as \( n \to \infty \).

**Example:** Assume that \( A \) is a measurable subset of \( W \) and let \( H \) be a measurable Wiener functional with values in \( \mathbb{R} \cup \{\infty\} \). If \( G \) defined by \( G = 1_A H \) is \( \gamma \)-log-concave with \( \gamma \in [0, 1) \), then the hypothesis of Theorem 6.4 are satisfied.

**Definition 6.1** Let \( T \in \mathbb{D}' \) be a positive distribution. We say that it is a-\( \gamma \)-log-concave if \( P_T \) is an a-\( \gamma \)-log-concave Wiener functional. If \( a = 0 \), then we call \( T \) simply log-concave.

**Remark:** It is well-known that (cf. for example [15]), to any positive distribution on \( W \), it corresponds a positive Radon measure \( \nu_T \) such that

\[ <T, \phi> = \int_W \tilde{\phi}(w) d\nu_T(w) \]

for any \( \phi \in \mathbb{D} \), where \( \tilde{\phi} \) represents a quasi-continuous version of \( \phi \).
Example: Let \((w_t, t \in [0, 1])\) be the one-dimensional Wiener process and denote by \(p_t\) the heat kernel on \(\mathbb{R}\). Then the distribution defined as \(\varepsilon_0(w_1) = \lim_{\tau \to 0} p_\tau(w_1)\) is log-concave, where \(\varepsilon_0\) denotes the Dirac measure at zero.

The following result is a Corollary of Theorem 6.4:

**Theorem 6.5**
Assume that \(T \in \mathcal{D}'\) is a positive, \(\beta\)-log-concave distribution with \(\beta \in [0, 1)\). Let \(\gamma\) be the probability Radon measure defined by

\[
\gamma = \frac{\nu_T}{\langle T, 1 \rangle}.
\]

Then we have

\[
E_{\gamma} \left[ \phi^2 \left\{ \log \phi^2 - \log E_{\gamma} [\phi^2] \right\} \right] \leq \frac{2}{1 - \beta} E_{\gamma} [\|\nabla \phi\|^2_H],
\]

for any smooth cylindrical function \(\phi : W \to \mathbb{R}\).

Here is an application of this result:

**Proposition 6.1**
Let \(F\) be a Wiener functional in \(\mathcal{D}_{r,2}\) for some \(r > 1\). Suppose that it is \(p\)-non-degenerate in the sense that

\[
\delta \left\{ \frac{\|\nabla F\|^2_H}{|F|^2} \right\} \in L^p(\mu)
\]

for any \(\phi \in \mathcal{D}\), for some \(p > 1\). Assume furthermore that, for some \(x_0 \in \mathbb{R}\) and \(a \in [0, 1)\),

\[
(F - x_0)\nabla^2 F + \nabla F \otimes \nabla F \geq -aI_H
\]

almost surely. Then we have

\[
E \left[ \phi^2 \left\{ \log \phi^2 - \log E [\phi^2|F = x_0] \right\} \mid F = x_0 \right] \leq \frac{2}{1 - a} E \left[ \|\nabla \phi\|^2_H \mid F = x_0 \right]
\]

for any smooth cylindrical \(\phi\).

**Proof:** Note that the non-degeneracy hypothesis (23) implies the existence of a continuous density of the law of \(F\) with respect to the Lebesgue measure (cf. [11] and the references there). Moreover it implies also the fact that

\[
\lim_{\tau \to 0} p_\tau(F - x_0) = \varepsilon_{x_0}(F),
\]
in $\mathbf{D}'$, where $\varepsilon_{x_0}$ denotes the Dirac measure at $x_0$ and $p_t$ is the heat kernel on $\mathbb{R}$. The inequality (24) implies that the distribution defined by
\[ \phi \rightarrow E[\phi|F = x_0] = \frac{\langle \varepsilon_{x_0}(F), \phi \rangle}{\langle \varepsilon_{x_0}(F), 1 \rangle} \]
is $a$-log-concave, hence the conclusion follows from Theorem 6.5.

7 Change of variables formula and log-Sobolev inequality

In this section we shall derive a different kind of logarithmic Sobolev inequality using the change of variables formula for the monotone shifts studied in [17] and in more detail in [19]. An analogous approach to derive log-Sobolev-type inequalities using the Girsanov theorem has been employed in [16].

**Theorem 7.1**

Suppose that $F \in L^p(\mu)$, for some $p > 1$, is an $a$-convex Wiener functional, $a \in [0, 1)$ with $E[F] = 0$. Assume that
\[ E \left[ \exp \left\{ c \| \nabla^2 L^{-1} F \|_2^2 \right\} \right] < \infty, \]for some
\[ c > \frac{2 + (1 - a)}{2(1 - a)}, \]
where $\| \cdot \|_2$ denotes the Hilbert-Schmidt norm on $H \otimes H$ and $L^{-1} F = \int_{\mathbb{R}_+} P_t F dt$. Denote by $\nu$ the probability measure defined by
\[ d\nu = \Lambda d\mu, \]
where
\[ \Lambda = \text{det}_2(I_H + \nabla^2 L^{-1} F) \exp \left\{ -F - \frac{1}{2} \| \nabla L^{-1} F \|_H^2 \right\} \]
and $\text{det}_2(I_H + \nabla^2 L^{-1} F)$ denotes the modified Carleman-Fredholm determinant. Then we have
\[ E_\nu \left[ f^2 \log \left( \frac{f^2}{\| f \|_{L^2(\nu)}^2} \right) \right] \leq 2E_\nu \left[ \| (I_H + \nabla^2 L^{-1} F)^{-1} \nabla f \|_H^2 \right] \] (26)
and
\[ E_\nu \| f - E_\nu[f] \|^2 \leq E_\nu \left[ \| (I_H + \nabla^2 L^{-1} F)^{-1} \nabla f \|^2_H \right] \] (27)
for any smooth, cylindrical $f$. 

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Proof: Let \( F_n = E[P_{1/n}F|V_n] \), where \( V_n \) is the sigma algebra generated by \( \{\delta e_1, \ldots, \delta e_n\} \) and let \((e_n, n \in \mathbb{N})\) be a complete, orthonormal basis of \( H \). Define \( \xi_n \) by \( \nabla L^{-1}F_n \), then \( \xi_n \) is \((1-a)\)-strongly monotone (cf. [19] or [17]) and smooth. Consequently, the shift \( T_n : W \rightarrow W \), defined by \( T_n(w) = w + \xi_n(w) \) is a bijection of \( W \) (cf. [19], Corollary 6.4.1), whose inverse is of the form \( S_n = I_W + \eta_n \), where \( \eta_n(w) = g_n(\delta e_1, \ldots, \delta e_n) \) such that \( g_n : \mathbb{R}^n \rightarrow \mathbb{R}^n \) is a smooth function. Moreover the images of \( \mu \) under \( T_n \) and \( S_n \), denoted by \( T_n^*\mu \) and \( S_n^*\mu \) respectively, are equivalent to \( \mu \) and we have

\[
\frac{dS_n^*\mu}{d\mu} = \Lambda_n
\]

\[
\frac{dT_n^*\mu}{d\mu} = L_n
\]

where

\[
\Lambda_n = \det_2(I_H + \nabla \xi_n) \exp \left\{ -\delta \xi_n - \frac{1}{2} |\xi_n|^2_H \right\}
\]

\[
L_n = \det_2(I_H + \nabla \eta_n) \exp \left\{ -\delta \eta_n - \frac{1}{2} |\eta_n|^2_H \right\}.
\]

The hypothesis (25) implies the uniform integrability of the densities \((\Lambda_n, n \geq 1)\) and \((L_n, n \geq 1)\) (cf. [17, 19]). For any probability \( P \) on \((W, \mathcal{B}(W))\) and any positive, measurable function \( f \), define \( \mathcal{H}_P(f) \) as

\[
\mathcal{H}_P(f) = f(\log f - \log E_P[f]).
\]

(28)

Using the logarithmic Sobolev inequality of L. Gross for \( \mu \) (cf. [6]) and the relation

\[
(I_H + \nabla \eta_n) \circ T_n = (I_H + \nabla \xi_n)^{-1},
\]

we have

\[
E[\Lambda_n \mathcal{H}_{\Lambda_n} df^2] = E[\mathcal{H}_\mu(f^2 \circ S_n)]
\]

\[
\leq 2E[\nabla(f \circ S_n)^2_H]
\]

\[
= 2E[(I_H + \nabla \eta_n) \nabla f \circ S_n]^2_H
\]

\[
= 2E[\Lambda_n^2 (I_H + \nabla \xi_n)^{-1} \nabla f^2_H].
\]

(29)

It follows by the \( a \)-convexity of \( F \) that

\[
\|(I_H + \nabla \xi_n)^{-1}\| \leq \frac{1}{1-a}
\]

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almost surely for any \( n \geq 1 \), where \( \| \cdot \| \) denotes the operator norm. Since the sequence \( (\Lambda_n, n \in \mathbb{N}) \) is uniformly integrable, the limit of (29) exists in \( L^1(\mu) \) and the proof of (26) follows. The proof of the inequality (27) is now trivial.

**Corollary 7.1**

Assume that \( F \) satisfies the hypothesis of Theorem 7.1. Let \( Z \) be the functional defined by

\[
Z = \det_2(I_H + \nabla^2 L^{-1} F) \exp \frac{1}{2} |\nabla L^{-1} F|_H^2
\]

and assume that \( Z, Z^{-1} \in L^\infty(\mu) \). Then we have

\[
E \left[ e^{-F} f^2 \log \left\{ \frac{f^2}{E[e^{-F} f^2]} \right\} \right] \leq 2KE \left[ e^{-F} \left| (I_H + \nabla^2 L^{-1} F)^{-1} \nabla f^2_H \right| \right] \tag{30}
\]

and

\[
E \left[ e^{-F} |f - E[e^{-F} f]|^2 \right] \leq KE \left[ e^{-F} \left| (I_H + \nabla^2 L^{-1} F)^{-1} \nabla f^2_H \right| \right] \tag{31}
\]

for any smooth, cylindrical \( f \), where \( K = \| Z \|_{L^\infty(\mu)} \| Z^{-1} \|_{L^\infty(\mu)} \).

**Proof:** Using the identity remarked by Holley and Stroock (cf. [8], p.1183)

\[
E_P [\mathcal{H}_P(f^2)] = \inf_{x>0} E_P \left[ f^2 \log \left( \frac{f^2}{x} \right) - (f^2 - x) \right],
\]

where \( P \) is an arbitrary probability measure, and \( \mathcal{H} \) is defined by the relation (28), we see that the inequality (30) follows from Theorem 7.1 and the inequality (31) is trivial.

**Remark:** If \( F \) is \( H \)-convex, then \( \det_2(I_H + \nabla^2 L^{-1} F) \geq 1 \) almost surely. Hence in this case it suffices to assume that \( \det_2(I_H + \nabla^2 L^{-1} F) \in L^\infty(\mu) \) and that \( |\nabla L^{-1} F|_H \in L^\infty(\mu) \).

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