The Music of the Aetherwave - B-mode Polarization in Einstein-Aether Theory

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We study how the dynamical vector degree of freedom in modified gravity affects the CMB B-mode polarization in terms of the Einstein-aether theory. In this theory, vector perturbations can be generated from inflation, which can grow on superhorizon scales in the subsequent epochs and thereby leaves imprints on the CMB B-mode polarization. We derive the linear perturbation equations in a covariant formalism, and compute the CMB B-mode polarization using the CAMB code modified so as to incorporate the effect of the aether vector field. We find that the amplitude of the B-mode signal from the aether field can surpass the contribution from the inflationary gravitational waves for a viable range of model parameters. We also give an analytic argument explaining the shape of the spectrum based on the tight coupling approximation.

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I. INTRODUCTION

It is widely believed that the present Universe is dominated by the two dark components: cold dark matter and dark energy. Dark matter plays an essential role in explaining galaxy rotation curves and in structure formation, while dark energy is presumably responsible for the current cosmic acceleration. The presence of the dark components is thus perceived through the gravitational interaction, having not yet identified what they really are. It is therefore legitimate to think of these major mysteries of today’s cosmology as a mystery of gravity. This motivates us to explore long-distance modification of the gravitational law, asking to what extent general relativity (GR) is correct on cosmological scales.

Modification of gravity is most commonly made by adding an extra scalar degree of freedom à la Brans-Dicke gravity \([1]\). In recent years, various refined models of scalar-tensor gravity have been proposed which would be an alternative to material dark energy while being consistent with solar-system constraints. They include chameleon \(f(R)\) \([2, 3]\) and Galileon \([4, 5]\) theories, and have been tested against cosmological observations \([6]\). It is also possible to modify the spin-2 sector as in massive gravity \([7]\) and bi-gravity theories \([8]\).

In this paper, we are going to consider a hypothetical vector degree of freedom of gravity. Specifically, we shall focus on the Einstein-aether (EA) theory proposed by Jacobson and Mattingly \([9]\), in which a fixed norm vector field with a Lorentz-violating vacuum expectation value takes part in the gravitational interaction. The effect of the aether on the cosmological background was clarified in \([10]\). The scalar cosmological perturbations in the EA theory and their impact on the cosmic microwave background (CMB) temperature anisotropy have been studied in \([11, 13]\). Recently, Amendariz-Picon et al. \([15]\) performed a comprehensive analysis on cosmological perturbations in the EA theory. See also Refs. \([14, 19]\) for the other aspects of the EA theory, such as spherically symmetric solutions and compact objects. Interestingly, it was recently pointed out that the healthy extension \([20]\) of Horava’s quantum theory of gravity \([21]\) reduces to a special case of the EA theory at low energies \([22]\). Cosmological perturbations in the healthy extension of Horava gravity were studied in \([22]\).

The purpose of the present paper is to clarify the impact of the aether vector field on the CMB polarization. The CMB polarization arises from all the three types of cosmological perturbations, i.e., scalar, vector, and tensor perturbations. Among them, as pointed out by \([24]\), the vector perturbations most effectively generate the B-mode polarization. However, the effect of vector perturbations has been less investigated because the vector mode decays unless sourced, e.g., by topological defects \([25, 26]\), while the scalar and tensor modes are certainly generated from inflation \([27]\). Other possible ways of seeding vector perturbations include the neutrino anisotropic stress \([28]\), the second-order effect \([29, 32]\), and primordial magnetic fields generated somehow \([34]\). Modifying the vector sector of gravity offers a yet another possibility of producing vector perturbations, leaving a unique signature in the CMB polarization due to nontrivial dynamics of the aether field.

The paper is organized as follows. In the next section we introduce the EA theory and the basic equations. In Sec. III, we describe the dynamics of vector perturbation in the EA theory using the covariant approach, in order to incorporate the aether vector field into the CAMB code. We then specify the initial conditions for the perturbation evolution in Sec. IV. Our numerical results are presented in Sec. V. In Sec. VI, we examine the spectrum shape in an analytic approach using the tight coupling approximation. We draw our conclusions in Sec. VII. We
II. EINSTEIN-AETHER THEORY

The action of the EA theory is given by [13]
\[
S = \frac{M_P^2}{2} \int d^4x \sqrt{-g} [\mathcal{R} + \mathcal{L}_A] + S_m,
\]
where
\[
\mathcal{L}_A = -[c_1 \nabla_a A^b \nabla^a A_b + c_2 (\nabla_b A^b)^2 + c_3 \nabla_a A^b \nabla_b A^a \\
+ c_4 A^a \nabla_a \nabla_c A_c \nabla_b A_b] + \lambda (A_b A^b - 1)
\] (2)
is the Lagrangian of the aether field and \(S_m\) is the action of ordinary matter. It is assumed that the aether is not coupled to the matter field directly.

Variation with respect to the metric leads to the Einstein equations
\[
\mathcal{R}_{ab} - \frac{1}{2} g_{ab} \mathcal{R} = T_{ab} + \kappa \tau_{ab},
\]
where \(\mathcal{R}_{ab}\) is the Ricci tensor, \(\kappa = M_P^{-2}\),
\[
T_{ab} := \frac{1}{2} \mathcal{L}_A g_{ab} - \frac{\delta \mathcal{L}_A}{\delta g^{ab}}
\]
is the energy-momentum tensor of the aether, and \(\tau_{ab}\) is the energy momentum tensor of ordinary matter. Explicitly, we have
\[
T_{ab} = c_1 \left[ (\nabla_A \nabla_b A_c) - (\nabla_c A_b) \nabla^c A_a \right] \\
+ c_2 \nabla_c \left[ A^e \nabla_c (A_b A_e) - (\nabla_c A_a) A_b - A_a (\nabla_b A_c) \right] \\
+ c_3 \nabla_c \left[ A^e \nabla_c (A_b A_e) - (\nabla_c A_b) A_a + A_a (\nabla_b A_c) \right] \\
- c_4 A^a A^b (\nabla_c A_a) (\nabla_d A_b) \\
- c_5 A_a A_b A^a \nabla_d A_b - 2 A^a A^b (\nabla_d A_a) A_b \right] \\
+ \frac{1}{2} \mathcal{L}_A g_{ab} + \lambda A_a A_b.
\]
(5)

Ordinary matter includes photons, baryons, etc., so that we write \(\tau_{ab} = \sum_i \tau_{ab}^{(i)}\), where \(i\) labels different components. Note that \(\mathcal{L}_A\) is taken to be a functional of \(A^a\) rather than \(A_b\). Variation with respect to \(A^a\) yields the equation of motion for the aether:
\[
c_1 \square A_a + c_2 \nabla_a \nabla_b A^b + c_3 \nabla_b \nabla_a A^b \\
+ c_4 \left[ \nabla_b (A^b A^c \nabla_c A_a) - A^b (\nabla_b A_c) \nabla_a A^c \right] = -\lambda A_a.
\]
(6)

Finally, variation with respect to the Lagrange multiplier \(\lambda\) gives the fixed norm constraint
\[
A_a A^a = 1.
\]
(7)

It is convenient to use the following abbreviations:
\[
c_{13} = c_1 + c_3, \quad c_{14} = c_1 + c_4, \\
\alpha = c_1 + 3c_2 + c_3, \quad c_{123} = c_1 + c_2 + c_3
\]
(8)

III. COVARIANT APPROACH

To describe background cosmology and the evolution of vector perturbations, we employ the covariant equations obtained by the method of \(3 + 1\) decomposition. We begin with splitting physical quantities with respect to observer’s 4-velocity \(u^a\). Following the usual procedure, the projection tensor is defined as \(h_{ab} := g_{ab} - u_a u_b\), and the covariant spatial derivative \(D_a\) acting on a tensor \(T_{cdef}^{(v)}\) is defined as \(D^a T^{(v)}_{cdef} := h^a b \cdots h^d e \cdots \nabla^a T_{cdef}\).

The energy-momentum tensors of ordinary matter and the aether field are decomposed respectively as
\[
\tau_{ab}^{(i)} = \rho^{(i)} u_a u_b - p^{(i)} h_{ab} + 2 \mathcal{q}^{(i)}_{(a} u_{b)} + \pi_{ab}^{(i)},
\]
(9)
\[
T_{ab} = \tilde{\rho} u_a u_b - \tilde{p} h_{ab} + 2 \tilde{q}_{(a} u_{b)} + \tilde{\pi}_{ab},
\]
(10)
while \(\nabla_a u_b\) is decomposed as
\[
\nabla_a u_b = \frac{1}{3} \partial_b h_{ab} + \sigma_{ab} + \omega_{ab} - u_a \dot{u}_b.
\]
(11)

Here, \(\sigma_{ab} := D(a u_b) - (1/3) \nabla_c u^c h_{ab}\) is the shear tensor, \(\omega_{ab} := D(a u_b)\) is the vorticity, \(\theta := \nabla_a u^a\) is the expansion, and the overdot denotes time derivative := \(u^a \nabla_a\). The expansion may be written as \(\theta = 3 \tilde{S}/S\), where \(S\) is the averaged scale factor.

The conservation equations for the matter energy-momentum tensor imply
\[
\dot{\rho} + \theta (\rho + p) + D^a q_a = 0,
\]
(12)
\[
\dot{q}_a + \frac{4}{3} \theta q_a + (\rho + p) \dot{u}_a - D_ap + D^b \pi_{ab} = 0,
\]
(13)
where \(\rho = \sum_i \rho^{(i)}\), \(p = \sum_i p^{(i)}\), \(q_a = \sum_i q_a^{(i)}\), and \(\pi_{ab} = \sum_i \pi_{ab}^{(i)}\). We also have the corresponding conservation equations for the aether:
\[
\dot{\tilde{\rho}} + \theta (\tilde{\rho} + \tilde{p}) + D^a \tilde{q}_a = 0,
\]
(14)
\[
\dot{\tilde{q}}_a + \frac{4}{3} \theta \tilde{q}_a + (\tilde{\rho} + \tilde{p}) \dot{u}_a - D_a \tilde{p} + D^b \tilde{\pi}_{ab} = 0.
\]
(15)

Let us first consider the homogeneous and isotropic limit. In this limit, the constraint \(h_{ab} = \delta_{ab}\) implies \(A^a = u^a\). Substituting this to the equation of motion \(\square\), one finds
\[
\lambda = \frac{c_{13}}{3} \theta^2 - c_2 \dot{\theta}.
\]
(16)

The energy density and pressure of the aether are then given by
\[
\tilde{\rho} = c_2 \left( \dot{\theta} + \dot{\theta}^2 \right) + \lambda - \frac{\alpha}{6} \dot{\theta}^2,
\]
(17)
\[
\tilde{p} = \frac{\alpha}{6} \left( 2 \dot{\theta} + \dot{\theta}^2 \right).
\]
(18)

It can be seen that \(\tilde{\rho}\) and \(\tilde{p}\) are expressed in terms of the expansion \(\theta\), and they are the same as what appear in
the left hand side of the Einstein equations. This means that the effect of the aether on the background evolution is just to renormalize the gravitational constant: \( \kappa \rightarrow \tilde{\kappa} := (1 - \alpha/2)^{-1}\kappa \) [10]. The background equations are thus given by

\[
\mathcal{H}^2 = \frac{\tilde{\kappa}}{3} S^2 \rho, \\
\mathcal{H}' = -\frac{\tilde{\kappa}}{6} S^2 (\rho + 3p),
\]

(19) (20)

where we have introduced the comoving Hubble parameter, \( \mathcal{H} := S\theta/3 \), and the derivative with respect to the conformal time, \( \tau := S u^a \nabla_a \).

In order for the background Friedmann equation (19) to have a solution, the condition

\[
\alpha < 2
\]

(21)

must be satisfied for positive matter energy density and positive \( \kappa \). On the other hand, the effective gravitational constant on small scales \( \kappa_N \) is also different from the bare one: \( \kappa_N = (1 + c_{14}/2)^{-1}\kappa \) [10]. The difference between these two effective gravitational constants is constrained by nucleosynthesis as \( |1 - \tilde{\kappa}/\kappa_N| < 10\% \). In terms of the aether parameters, this constraint is roughly expressed as

\[
c_{14} + \alpha \lesssim 0.2.
\]

(22)

Note that this is trivially satisfied if we consider the special case \( \alpha = -c_{14} \). Actually, this is the special combination for evading the existing observational constraints on \( c_1 \) (see Appendix A), and we will often use this case later.

Having studied the background effect of the aether, we then move on to the dynamics of vector perturbations. As employed in [28], we choose \( u_a \) to be hypersurface orthogonal, so that \( \text{curl} u = 0 \Rightarrow \dot{u}_b = 0 \) at linear order. This simplifies the following analysis.

At linear order, the aether field can be written as

\[
A_b = u_b + D_b V^{(s)} + V_b,
\]

(23)

where \( V^{(s)} \) and \( V_b \) are first order quantities. \( V^{(s)} \) corresponds to a scalar perturbation which we do not consider in this paper, while \( V_b \) a vector perturbation that satisfies \( D_b V^b = 0 \). The fixed norm constraint \( 1 \) leads to \( u_b V^b = 0 \), i.e., \( V^b \) is a spatial vector. Since \( \nabla_a V_b = D_b V_a - (1/3)\theta V_a u_b + u_a V_b \), we have

\[
\nabla_a A_b = \frac{1}{3} \theta h_{ab} + \sigma_{ab} + D_a V_b - \frac{1}{3} \theta V^b u_a + u_a V_b.
\]

(24)

Substituting this to Eq. (23), we find, up to first order,

\[
c_{13} \left[ \frac{\theta^2}{3} u_a - D_b \sigma_{ab} + \frac{1}{3} (\dot{\theta} + \theta^2) V_a \right] - c_1 D^2 V_a
\]

\[
- c_3 D_b D_a V^b - c_2 \theta u_a - c_{14} \left( \chi_a + \frac{2}{3} \theta \chi_a \right)
\]

\[
= \lambda u_a + \lambda V_a, \tag{25}
\]

(25)

where

\[
\chi_a = \dot{V}_a + \frac{1}{3} \theta V_a. \tag{26}
\]

Since we are interested in vector perturbations, we have dropped in the above the scalar perturbation \( D_b \theta \). Multiplying \( u^a \) gives rise to the background equation which we have already derived. Multiplying \( h_b \), we obtain

\[
c_{14} \left( \dot{\chi}_a + \frac{2}{3} \theta \chi_a \right) =
\]

\[
- \left( c_{13} D^b \sigma_{ab} + c_1 D^2 V_a + c_3 D_b D_a V^b \right) + \frac{\alpha}{3} \theta V_a, \tag{27}
\]

(27)

where we used Eq. (10). At linear order scalar-type quantity \( \tilde{\rho} \) and \( \tilde{\theta} \) have the same expression as in the background. The heat-flux vector \( \tilde{q}_a \) and the anisotropic stress \( \tilde{\pi}_{ab} \) of the aether are given respectively by

\[
\tilde{q}_a = c_{14} \left( \dot{\chi}_a + \frac{2}{3} \theta \chi_a \right) + \frac{c_1 - c_3}{2} (D^2 V_a - D_b D_a V^b)
\]

\[
- \frac{c_{13}}{3} (\dot{\theta} + \theta^2) V_a + \lambda V_a,
\]

(28)

\[
= -c_{13} D^b \left[ \sigma_{ab} + D_{(a} V_b) \right],
\]

(28)

\[
\tilde{\pi}_{ab} = c_{13} \left\{ \sigma_{ab} + \theta \sigma_{ab} + \left[ D_{(a} V_b) \right]' + \theta D_{(a} V_b \right\} , \tag{29}
\]

(29)

where we used Eq. (10) to remove \( \lambda \). One can check that the momentum conservation equation (15) is automatically satisfied.

Combining Eq. (28) with the momentum constraint equation,

\[
D^b \sigma_{ab} = \kappa q_a + \tilde{q}_a, \tag{30}
\]

we obtain

\[
D^b \sigma_{ab} = \frac{1}{1 + c_{13}} \left[ \kappa q_a - c_{13} D^b D_{(a} V_b \right]. \tag{31}
\]

(31)

Note in passing that \( D^b D_a V_b = 0 \) at linear order.

To proceed further, it is convenient to introduce the transverse eigenfunctions. The definitions and the basic properties of the eigenfunctions are presented in [28]. In terms of the eigenfunctions the vector perturbations can be expanded as

\[
V_a = \sum Q_{a}^\pm, \quad \sigma_{ab} = \sum \frac{k}{\sqrt{2}} \sigma Q_{ab}^\pm,
\]

\[
q_{a}^{(i)} = \sum q_i Q_{a}^\pm, \quad \pi_{ab}^{(i)} = \sum \Pi_i Q_{ab}^\pm, \tag{32}
\]

(32)

where \( k \) is the eigenvalue. In terms of the harmonic coefficients, our perturbation equations are written as

\[
q' + 4H q + \frac{k}{2} \Pi = 0, \tag{33}
\]

\[
k^2 \sigma = \frac{1}{1 + c_{13}} \left( 2\kappa S^2 q - c_{13} k^2 V \right), \tag{34}
\]

\[
c_{14} \left[ V'' + 2H V' + (H^2 + H') V \right] + \alpha (H^2 - H') V + r_k k^2 V = -\frac{c_{13}}{2} k^2 \sigma, \tag{35}
\]

(35)
where \( q := \sum_i q_i \) and \( \Pi := \sum_i \Pi_i \).

Equation (33) shows that the fluctuation of the aether obeys the wave equation which is similar to the evolution equation for cosmological tensor perturbations. The crucial difference is the effective mass term which is dependent on the expansion rate \( \mathcal{H} \) and the model parameters. The fluctuation of the aether is related to \( \sigma \) via the momentum constraint, which in turn translates to the magnetic Weyl tensor \( H_{ab} \) via \( H_{ab} = \text{curl} \sigma_{ab} \), and thus produces perturbations of geometry. If we neglect the matter contents, the above equations reduce to those derived in [13].

The velocity \( v_i \) of each fluid component is given by \( v_i = q_i / (\rho^{(i)} + p^{(i)}) \). The baryons are coupled to photons via Thomson scattering. The baryon velocity \( v_b \) obeys

\[
v'_b + \mathcal{H}v_b = -\frac{\rho_b}{\rho_b} S_n e \sigma_{T} \left( \frac{4}{3} v_b - I_1 \right),
\]

with \( I_1 = 4v_\gamma/3 \).

We now replicate the photon multipole equations and the polarization multipole equations for vectors presented in [28, 33, 34]. The photon multipole equations for vectors are

\[
I'_\ell + k \frac{\ell}{2\ell + 1} \left( \frac{\ell + 2}{\ell + 1} I_{\ell+1} - I_{\ell-1} \right) = -S n e \sigma_{T} \left( I_\ell - \frac{4}{3} \delta_{11} v_b - \frac{2}{15} \zeta \delta_{\ell2} \right) + \frac{8}{15} k \sigma \delta_{\ell2},
\]

where \( \zeta := 3I_2/4 - 9E_2/2 \) and \( I_2 = \Pi_\gamma/\rho_\gamma \), while the polarization multipole equations for vectors are

\[
E^{\pm}_{\ell} + \frac{(\ell + 3)(\ell + 2)(\ell - 1)}{(\ell + 1)^3(2\ell + 1)} k E^{\pm}_{\ell+1} - \frac{\ell}{2\ell + 1} k E^{\pm}_{\ell-1} - \frac{2}{\ell(\ell + 1)} k B^{\pm}_{\ell-1} = -S n e \sigma_{T} \left( E^{\pm}_{\ell} - \frac{2}{15} \zeta \delta_{\ell2} \right),
\]

\[
B^{\pm}_{\ell} + \frac{(\ell + 3)(\ell + 2)(\ell - 1)}{(\ell + 1)^3(2\ell + 1)} k B^{\pm}_{\ell+1} - \frac{\ell}{2\ell + 1} k B^{\pm}_{\ell-1} + \frac{2}{\ell(\ell + 1)} k E^{\pm}_{\ell-1} = -S n e \sigma_{T} B^{\pm}_{\ell},
\]

where \( E_\ell \) and \( B_\ell \) are moments of the E and B polarization.\(^1\) The integral solutions to these equations are given in [28, 33]. We will only use the solution for \( B_\ell \) in the following discussion:

\[
B_\ell(\eta_0) = -\frac{\ell + 1}{\ell + 1} \int^{\eta_0} d\eta \, \tilde{\tau} e^{-\tilde{\tau}} \Psi_\ell(k(\eta_0 - \eta)) \zeta, \quad (40)
\]

where \( \tilde{\Psi}_\ell(x) := \ell x \tilde{e}_\ell(x)/x \) corresponding to the projection function \( \tilde{\beta}_\ell^{(1)} \) in the total angular momentum approach [24], and \( \tilde{\tau} \) represents the optical depth: \( \tau := \int^{\eta_0} d\eta S_n e \sigma_{T} \). The neutrino multipole equations are of the form (37) without the Thomson scattering terms.

IV. INITIAL CONDITIONS

We solve the relevant set of equations at early times in order to clarify the initial conditions. This is done by a series expansion in terms of the conformal time \( \eta \), following [28] but now taking into account the presence of the aether.

First, by invoking the tight coupling approximation for the baryons and photons it is easy to obtain

\[
v_{\gamma} \simeq v_b \simeq \frac{v_0}{1 + R}, \quad (41)
\]

The neutrino multipole equations read

\[
I_{1}^{(\nu)} + k \frac{I_2^{(\nu)}}{2} = 0, \quad I_2^{(\nu)} - 2k \frac{I_1^{(\nu)}}{5} = \frac{8}{15} k \sigma, \quad (46)
\]

\(^1\) Here we have corrected the typo found in [24].
where $I_1^{(v)} = 4\nu_v/3$ and $I_2^{(v)} = \Pi_v/\rho_v$. We use the multipole equations \[40\] and the momentum constraint \[41\] to get the following early time solution:

$$
\sigma = B_k \left( 1 - \frac{15}{2} \frac{\omega \eta}{4R_\nu^* + 15} \right)
- \frac{\nu^*}{\nu^* + 4R_\nu^* + 1 + c_{13}} A_k \eta^{-v}, 
$$

(47)

$$
\nu_\gamma = \frac{B_k}{4} \frac{4R_\nu^* + 5}{R_\gamma^*} \left( 1 - \frac{3R_\nu}{4R_\gamma} \omega \eta \right),
$$

(48)

$$
\nu_\nu = \frac{B_k}{4} \frac{4R_\nu^* + 5}{R_\nu^*} \frac{1 + 3R_\nu}{15 + 4R_\nu} \omega \eta,
$$

(49)

$$
\Pi_\nu = \frac{2B_k k \eta}{3} \frac{1 + 3R_\nu}{15 + 4R_\nu} \omega \eta
- \frac{8}{15(1 + \nu)} \frac{\nu^*}{\nu^* + 4R_\nu^* + 1 + c_{13}} A_k \eta^{1 + \nu}. 
$$

(50)

In the above we defined

$$
R_\nu^* := \frac{1 - \alpha/2}{1 + c_{13}} R_\nu,
\nu^* := \frac{5}{2} (1 + \nu)(2 + \nu),
$$

(51)

where $R_\nu$s are defined in the same way as in \[28\]: $R_\nu := \Omega_\nu/\Omega_R$, $R_\gamma = \Omega_\gamma/\Omega_R$, and $R_\theta = \Omega_\theta/\Omega_m$. The mode associated with $B_k$ is identified as the regular vector mode in the presence of the neutrino anisotropic stress \[28\]. Since $B_k$ may be fixed independently of the effect of the aether $A_k$, we discard this mode for clarity and focus on the initial condition with $A_k \neq 0$ and $B_k = 0$.

Once the inflation model and the subsequent reheating history are specified, one can determine the primordial spectrum of the vector perturbation and hence $A_k$. During inflation with $\epsilon := 1 - H'/H^2 = \text{const}$, one finds, on superhorizon scales, that \[12\]

$$
V \sim \frac{1}{M_{\text{Pl}}} \frac{(-\eta)^{1/2}}{a} (-k \eta)^{(n_v - 3)/2},
$$

(52)

where

$$
n_v := 3 - \sqrt{1 - \frac{\alpha}{c_{14}} \frac{4\epsilon}{(1 - \epsilon)^2}}.
$$

(53)

For $-1 \leq \alpha/c_{14} \leq 0$, we have $2 - 2\epsilon/(1 - \epsilon) \leq n_v \leq 2$. At the end of inflation, $\eta = \eta_e$, we have the estimate

$$
k^{3/2}V \sim \frac{H}{M_{\text{Pl}}} \left( \frac{k}{k_e} \right)^{n_v/2},
$$

(54)

where $H$ is the inflationary Hubble scale and $k_e^{-1}$ corresponds to the horizon scale at $\eta = \eta_e$. The factor $k^{n_v/2}$ reflects the fact that $V$ decays during inflation if $n > 0$. The amplitude may further change from \[54\] during the reheating stage, and hence the primordial amplitude depends also on the detailed history of reheating. In our actual calculation, we simply assume that

$$
A_k = A_0 k^{(n_v - 3)/2},
$$

(55)

\[\text{FIG. 1: CMB B-mode polarization and temperature anisotropy power spectra in the EA theory. For comparison, those from the tensor perturbation in standard GR are also plotted in the case of the tensor-to-scalar ratio $r = 0.1$. In this figure, $c_1 = -0.2, c_{13} = -0.3, c_{14} = -\alpha = -0.2$, and dimensionless primordial power spectra are $P_T \propto k^{\nu_m}$ and $P_T \propto k^3$.}\]

where $A_0$ is a constant.

As was already derived in Eq. \[43\], $V$ grows on superhorizon scales, $V \sim \eta^\nu$, in the radiation-dominated stage. In the matter-dominated stage it turns out that the superhorizon behavior is given by $V \sim \eta^{\nu_m}$ with

$$
\nu_m := -3 + \frac{1 - 24\alpha/c_{14}}{2}.
$$

(56)

Since $-1 \leq \nu_m \leq 1$, $V$ may grow or decay in the matter-dominated stage, depending on $\alpha/c_{14}$. Therefore, although the amplitude \[54\] (or $A_0$) may be tiny as a consequence of the decay of $V$ during inflation, in the subsequent radiation- and matter-dominated stages $V$ can be amplified on superhorizon scales, leading to an observationally relevant aether perturbation. The special case $\alpha = -c_{14}$ is an example of such situations, for which $\nu = \nu_m = 1$.

### V. NUMERICAL RESULTS

Using all the ingredients derived above and the CAMB code \[37\] modified so as to incorporate the presence of the aether, we have completed the numerical calculation for the B-mode polarization power spectra in the EA theory. An example of our numerical results is presented in the Fig. 1. For comparison, we show contributions from inflationary gravitational waves in GR, assuming that the tensor-to-scalar ratio is given by $r = 0.1$. In fact, the aether modifies the behavior of the tensor modes as well. However, the effects of the aether on the tensor modes are just to shift the location of the peak and to change...
FIG. 2: CMB B-mode polarization and temperature anisotropy power spectra in the EA theory. For comparison, those from the tensor perturbation in standard GR are also plotted in the case of the tensor-to-scalar ratio $r = 0.1$. In this figure, $c_1 = -0.019, c_{13} = -0.03, c_{14} = -0.0128$, and dimensionless primordial power spectra are $P_V \propto k^{n_V}$ and $P_T \propto k^0$.

the absolute amplitude of the primordial spectrum, and they are very small for small values of $c_i$. The amplitude $A_0$ is adjusted so that the low-$\ell$ TT spectrum from the vector perturbation has the same magnitude as this primordial tensor contribution. We see in this case that the BB spectrum in the EA theory is larger than that from primordial tensor modes at $\ell \gtrsim 100$, and hence the B-mode is potentially detectable in future CMB observations aiming to detect $r = \mathcal{O}(0.1) - \mathcal{O}(0.01)$ [38]. In plotting Fig. 1 we chose $\alpha = -c_{14}$. In this case, it has been discussed in [15] that the TT power spectrum has roughly the same shape as the one from the inflationary gravitational waves. As one can see, Fig. 1 shows the same scalings for the two TT spectra.

Using a more realistic parameter set evading all the existing constraints (see Appendix A), we plot the B-mode spectrum in Fig. 2. From this we conclude that, for a viable range of the model parameters, the B-mode from the vector perturbation is potentially detectable in future CMB probes even if its amplitude at the end of inflation is very small. Note that the amplification of $V$ after inflation is determined basically by the ratio between $\alpha$ and $c_{14}$. This means that, even if the model parameters $c_i$ are too small to discriminate the EA theory from GR with the other observational and experimental tests, the CMB B-mode polarization could be a powerful probe for the aether field.

We show the evolution of each variable in a normal plot (Fig. 3) and in a log plot (Fig. 4). From these figures, we find that compared with the aether perturbation $V$ and the shear $\sigma$, matter components are negligibly small especially at early times. It can be seen from Fig. 3 that the growth rate of $V$ on superhorizon scales is given by $V \propto S$ for $\alpha = -c_{14}$. This confirms the early time solution derived in the previous section.

VI. ANALYTIC ESTIMATES

In this section, let us try to understand the shape of the B-mode angular power spectrum $C_{BB}^\ell$ in the EA theory in an analytic way. The following discussion is similar to the one introduced in [39] for the B-mode spectrum from the inflationary gravitational wave.

The starting point is the integral solution for the B-field which was already introduced in Eq. (40):

$$B_{\ell}(\eta_0) = -\frac{\ell + 1}{\ell - 1} \int^{\eta_0} d\eta e^{-\tau_\Psi} \Psi_{\ell}[k(\eta_0 - \eta)] \zeta,$$

Using the approximation for the visibility function,
\[ \tau e^{-\tau} \sim \delta(\eta - \eta_R), \] we see that it is important to know $\zeta$ at the last scattering surface for determining the $B$-mode polarization.

Now, we expand the multipole equations \[37\], \[38\] and \[39\] in terms of $k/\tau$. Neglecting $I_0$, $I_\ell$ ($\ell \geq 3$), and $E_\ell$ ($\ell \neq 2$), we have

\[ I'_1 + \frac{k}{2} I_2 = -\dot{\tau} \left( I_1 - \frac{4}{3} v_0 \right), \]
\[ I'_2 - \frac{2k}{3} I_1 = -\dot{\tau} \left( I_2 - \frac{2}{15} \zeta \right) + \frac{8}{15} k \sigma, \]
\[ E'_2 = -\dot{\tau} \left( E_2 - \frac{2}{15} \zeta \right). \]

At leading order, Eq. \[59\] reduces to

\[ E_2 = \frac{2}{15} \zeta \Leftrightarrow E_2 = \frac{1}{16} I_2. \]

Substituting this into Eq. \[58\] and neglecting a higher order term, we find

\[ I_2 = \frac{32 k}{75} \left( I_1 + \frac{4}{3} \sigma \right) \]

and

\[ \zeta = \frac{4}{15} \frac{k}{\tau} \left( I_1 + \frac{4}{3} \sigma \right). \]

The initial conditions we have adopted in Sec. IV imply that at the early times $I_1 = 4v_0/3 = q_0/\rho_0$ and $v_0$ can be neglected compared with $\sigma$. Then, we have the relation:

\[ \zeta = \frac{4}{15} \frac{k}{\tau} \sigma. \]

Under the same approximation, $\sigma$ and the fluctuation in the aether field $V$ are related as

\[ \sigma = -\frac{c_{13}}{1 + c_{13}} V. \]

The right-hand side of the evolution equation for the aether field \[34\] is expressed by the aether field itself and this equation becomes the closed form,

\[ V'' + 2HV' + c_v^2 k^2 V + \left[ \left( 1 + \frac{\alpha}{c_{14}} \right) \mathcal{H}^2 - \left( 1 - \frac{\alpha}{c_{14}} \right) \mathcal{H}' \right] V = 0, \]

where the sound speed $c_v$ is given by

\[ c_v^2 = \frac{c_1}{c_{14}} \left[ 1 - \frac{c_{13}^2}{2c_1(1 + c_{13})} \right]. \]

Equation \[63\] tells us that on superhorizon scales ($c_v k\eta \ll 1$) $V$ is proportional to $\eta''$ (or, equivalently, $S''$) in the radiation-dominated stage and to $\eta''_{\ast}$ (or, equivalently, $S''_{\ast}/2$) in the matter-dominated stage, while on subhorizon scales ($c_v k\eta \gg 1$) $V$ decays as $S^{-1}$ with oscillations $\cos(kc_v\eta)$.

Bearing these facts in mind, we can derive the wavenumber dependence of $V$ at recombination $\eta_{\text{rec}}$. Superhorizon modes evolve in the same way and keep their dependence on $k$, $A_k$, until horizon crossing. Therefore, the superhorizon modes at $\eta_{\text{rec}}$ ($k < 1/c_v\eta_{\text{rec}}$) keep the primordial $k$-dependence, $A_k$. After horizon crossing, $V$ decays as $S^{-1}$, so that $V(\eta_{\text{rec}})/V(\eta_*) \propto S(\eta_*)$, where $\eta_* := 1/c_v k$. The modes with $1/c_v\eta_{\text{rec}} < k < 1/c_v\eta_{\text{eq}}$ reenter the horizon in the matter-dominated stage, where $\eta_{\text{eq}}$ refers to the radiation-matter equality time. For these modes, we have $S(\eta_*) \propto \eta_*^2 \propto k^{-2}$ and $V(\eta_*) \propto \eta_*^{\nu_m} \propto k^{-\nu_m}$, and hence $V(\eta_{\text{rec}}) \propto k^{-2-\nu_m} A_k$. Similarly, the modes with $1/c_v\eta_{\text{rec}} < k$ reenter the horizon in the radiation-dominated stage and so $V(\eta_{\text{rec}}) \propto k^{-1-\nu} A_k$.

In summary, the wavenumber dependence of $V(\eta_{\text{rec}})$ is:

\[ V(\eta_{\text{rec}}) \propto \begin{cases} A_k & (k < 1/c_v \eta_{\text{rec}}), \\ k^{-2-\nu} A_k & (1/c_v \eta_{\text{rec}} < k < 1/c_v \eta_{\text{eq}}), \\ k^{-1-\nu} A_k & (1/c_v \eta_{\text{eq}} < k). \end{cases} \]

In the above argument we ignored the subhorizon oscillation of $V$.

In all the above calculations, we have neglected the matter velocities $v_i$ compared with $\sigma$. Our numerical results in Fig. 3 justify the approximation. The amplitudes of $V$ and $\sigma$ are large enough to neglect $v_i$ long before the recombination epoch. In Fig. 4, we see that the evolution of $\sigma$ tracks that of $V$ and the ratio is approximately equal to $-c_{13}/(1 + c_{13})$ until the neutrino velocity $v_0$ becomes comparable. (In Fig. 4 $c_{13} = -0.3$, so that the ratio is $3/7$.)

We are now in position to derive the simple scaling relation for $C_{\ell}^{BB}$ in an analytic way. The CMB $B$-mode power spectrum is roughly expressed as

\[ C_{\ell}^{BB} \sim \int d \ln k \mathcal{P}_V(k) B_{\ell} B_{\ell'}, \]

where $\mathcal{P}_V(k)$ is the dimensionless primordial power spectrum for $V$. From the above discussion, we see

\[ B_{\ell}(\eta_0) \sim \int_{\eta_0}^{\eta_{\text{rec}}} d\eta \delta(\eta - \eta_{\text{rec}}) \frac{\ell j_{\ell}[k(\eta_0 - \eta)]}{k(\eta_0 - \eta)} \frac{k V}{\mathcal{A}_k} \frac{c_{13}}{1 + c_{13}} \]
\[ \sim \frac{\ell j_{\ell}[k(\eta_0 - \eta_{\text{rec}})]}{k(\eta_0 - \eta_{\text{rec}})} \frac{V(\eta_{\text{rec}})}{\mathcal{A}_k} \frac{c_{13}}{1 + c_{13}}. \]

Using the fact that the projection factor $\ell j_{\ell}(x)/x$ has a peak at $\ell \sim x$, the power spectrum reduces to

\[ C_{\ell}^{BB} \sim \left( \frac{V}{\mathcal{A}_k} \right)^2 k^{2+n_v} \left[ \Psi_{\ell}(k(\eta_0 - \eta_{\text{rec}})) \right]^2 d \ln k. \]

For example, taking $n_v = 1$, the above integral can be
evaluated to give the scaling

\[ \ell (\ell + 1) C^{BB}_\ell \propto \begin{cases} \ell^3 & (\ell < \ell_{\text{rec}}), \\ \ell^{-1-2\nu} & (\ell_{\text{rec}} < \ell < \ell_{\text{eq}}), \\ \ell^{1-2\nu} & (\ell_{\text{eq}} < \ell < \ell_{\Delta_{\text{hrec}}}), \end{cases} \]  

(71)

showing a peak at \( \ell_{\text{peak}} \sim \ell_{\text{rec}} \), where we have defined \( \ell_{\text{rec}} := (\eta_0 - \eta_{\text{rec}})/c_s\eta_{\text{rec}} \) and \( \ell_{\text{eq}} := (\eta_0 - \eta_{\text{eq}})/c_s\eta_{\text{eq}} \). \( \Delta_{\text{hrec}} \) represents the scale over which the phase-damping effect shows up due to the width of the last-scattering surface. On scales \( \ell > \ell_{\Delta_{\text{hrec}}} \), the above approximation is no longer justified. In deriving the scaling we used the following integral formula for the Bessel function,

\[ \int d\ln k k^{2+m}\Psi_1(k) = \ell^2 \sqrt{\pi} \frac{\Gamma(1-m/2)}{4\Gamma(3/2-m/2)} \frac{\Gamma(l+m/2)}{\Gamma(l+2-m/2)}. \]  

(72)

The scaling behavior can indeed be seen in Fig. 5 for the illustrative case \( \alpha = -c_{14} \) \((\nu = \nu_m = 1)\), though the scaling in the range \( \ell_{\text{eq}} < \ell < \ell_{\Delta_{\text{hrec}}} \) is not clearly seen due to the phase-damping effect [39]. Actually, in order to evaluate the shape at the smallest scales correctly, we must take into account more complicated physics such as neutrino anisotropic stresses and the Silk-damping effect. We would emphasize that our numerical calculations incorporate all of these effects. For instance, in Fig. 5 we can confirm the oscillatory behavior at \( \ell > \ell_{\text{peak}} \) arising from the subhorizon oscillation of \( V \). (In Fig 6, the B-mode spectrum from tensor perturbations in GR is also plotted for comparison.)

We can also gain an understanding of how the shape of the angular power spectrum depends on the model parameters. From Fig. 6 one can confirm the following three things: (i) Since the angular power spectrum on the largest scales \( \ell < \ell_{\text{rec}} \) depends only on the primordial spectrum, the plotted examples show the same scaling at this scale. This means that the observation of the polarization at the largest scale can determine the primordial spectral index; (ii) The peak position is inversely proportional to the sound velocity of the aether vector perturbation \( c_v \). Actually, if we find the peak of the observed B-mode spectrum, the information is directly converted into the exact value of \( c_v \) in case the other cosmological parameters are already determined; (iii) The difference of the small scale scaling arises due to the difference of the growth rate of \( V \) on superhorizon scales, which can be seen in Eq. (71).

VII. SUMMARY

In this paper, we have considered the Einstein-aether theory in which gravity is modified through the additional vector degree of freedom (the aether), and studied possible signatures of the aether field in the CMB polarization. In standard GR, vector cosmological perturbations simply decay without sources, and hence they are less relevant to observations. In the Einstein-aether theory, however, the vector modes are dynamical, so that they could leave imprints on the CMB signals in a way similar to the inflationary gravitational waves. Using the CAMB code modified to incorporate the aether vector perturbation, we have computed the CMB B-mode polarization power spectrum. We have found that the amplitude of the B-mode polarization from the aether vector can be larger than that from inflationary gravitational waves with \( r = O(0.1) \) on small angular scales, which would be measurable with near future CMB probes.

Moreover, we have found that, for a set of parameters
c_i evading all the existing observational and experimental constraints, the B-mode polarization spectrum indeed shows distinguishable features from that from inflationary gravitational waves in GR. Thus, the B-mode polarization spectrum would potentially be a crucial test for the Einstein-aether theory.

We performed analytical calculations, by which we clarified the physical origins of the shape of the B-mode power spectrum. The scaling relations derived in an analytical way turned out to be consistent with the numerical results.

In this paper, we have simply assumed the primordial amplitude of the vector perturbation which is generated anyway in the presence of the aether field. The primordial spectrum actually depends upon the underlying inflation model and the reheating process. It would be interesting to explore whether the B-mode signal is measurable or not assuming concrete inflation models. We will report elsewhere the evolution of vector perturbation during the whole history of the Universe starting from inflation in the EA theory.

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Appendix A: Observational constraint

In this appendix, we summarize the existing observational constraints on the aether parameters $c_1, \cdots, c_4$, following [13, 40], [40].

1. Post-Newtonian limits

Parametrized post-Newtonian (PPN) parameters in the EA theory have already been analyzed in [41, 42]. Two PPN parameters, the Eddington-Robertson-Schiff parameters $\beta$ and $\gamma$, are identical to those in pure GR [41]. The Whitehead parameter, $\xi$, which characterizes a peculiar sort of three-body interaction vanishes in the EA theory [42], and the five energy-momentum conservation parameters $\alpha_3$ and $\zeta_{1,2,3,4}$ vanish because the theory is derived from a Lagrangian.

The aether defines a preferred frame, and its effect is encoded in the remained PPN parameters $\alpha_1$ and $\alpha_2$. The exact values of $\alpha_1$ and $\alpha_2$ are found in [42]:

$$\alpha_1 = \frac{-8(c_3^2 + c_1 c_4)}{2c_1 - c_1^2 + c_3^2}, \quad \alpha_2 = \frac{\alpha_1}{2} - \frac{(2c_{13} - c_{14})(\alpha + c_{14})}{c_{13} c_{23}(2 - c_{14})}. \quad (A1)$$

The easiest way to pass the stringent observational constraints, $\alpha_1 \lesssim 10^{-4}$ and $\alpha_2 \lesssim 4 \times 10^{-7}$ [13], is to set $\alpha_{1,2}$ exactly to zero by imposing the conditions

$$c_2 = \frac{-2c_1^2 - c_1 c_3 + c_3^2}{3c_1}, \quad (A3)$$
$$c_4 = \frac{c_3}{c_1}. \quad (A4)$$

which is possible since EA theory has four free parameters $c_i (i = 1 - 4)$. Under these conditions, all the PPN parameters in the EA theory coincide with those of GR.

2. Stability of each perturbation mode

Linear perturbations around the flat or FRW metric have been studied in [13, 44]. They concluded that quantum and classical stabilities of tensors (spin-2 modes), vectors (spin-1), and scalars (spin-0) constrain the aether parameters to be

$$c_{13} > -1, \quad (A5)$$
$$-2 \leq c_{14} < 0, \quad c_{23} < 0, \quad (A6)$$
$$2c_1 \leq c_{13}^2(1 + c_{13}). \quad (A7)$$

3. Radiation damping and strong self-field effects

The radiation damping rate in the weak field limit was first calculated in [45], and then strong field effects are included in [40]. They found that in general the damping rate in the EA theory is different from that of GR. However, in the special case with $\alpha_1 = \alpha_2 = 0$, it is identical provided that the quadrupole coefficient of the radiation damping rate,

$$A = \left(1 + \frac{c_{14}}{2}\right) \left[\frac{1}{c_t} - \frac{2c_{14}c_{13}}{(2c_1 + c_{13}c_-)^2} c_v - \frac{c_{14}}{6(2 + c_{14})} \left(3 + \frac{2c_2 - \alpha_1}{2(2c_1 - c_{14})}\right) \frac{1}{c_s}\right], \quad (A8)$$

is equal to one. Here we defined the new parameters

$$c_{t}^2 := \frac{1}{1 + c_{13}}, \quad (A9)$$
$$c_{s}^2 := \frac{2 + c_{14})c_{13}}{(1 + c_{13})(2 - \alpha)c_{14}}, \quad (A10)$$

which correspond to the velocity of tensor modes and scalar mode, respectively, and $c_- := c_1 - c_3$.

Foster [10] derived several constraints by considering strong field effects. All of them reduce to the condition $|c_i| \lesssim \mathcal{O}(0.1)$ with $\alpha_1 = \alpha_2 = 0$ under the current observational uncertainties.
4. Cherenkov Radiation

If the sound speed of each perturbation mode were smaller than the speed of light, then the ultra-high-energy particles would loose their energies into the mode in a similar way to Cherenkov radiation. Using this fact, Elliott et al. \[4\] derived limits on the aether parameters. Since the above conditions together with the PPN vanishing conditions \[A3\] and \[A4\], respectively. The connection between superluminality and violation of causality has been under debates \[48–52\]. Since the above conditions together with the PPN vanishing conditions \[A3\] and \[A4\] are equivalent to the stability condition of scalar and vector modes, we here allow for the superluminal propagation in the EA theory.

5. Scalar mode constraint

Armendariz-Picon et al. \[12\] have formulated the evolution equations for perturbations around the FRW background metric and calculated CMB temperature anisotropy spectrum. They found mainly two constraints on the aether parameters:

- In order for the scalar isocurvature mode not to grow on superhorizon scales,

\[
\alpha \leq -c_{14} \tag{A14}
\]

must be satisfied.

- In order not to have too large an anisotropic stress,

\[
|c_{13}| \lesssim 1 \tag{A15}
\]

must be satisfied.

Appendix B: Allowed parameter values

If one imposes that the PPN parameters coincide exactly with those in GR, one has two constraints derived from \(\alpha_1 = \alpha_2 = 0\). We are now left with two free parameters, for which we use \(c_{13} = c_1 + c_3\) and \(c_- = c_1 - c_3\).

Interestingly, the special combination \(\alpha = -c_{14}\) is automatically satisfied, and we have

\[
\alpha = -c_{14} = -2 \frac{c_{13} c_-}{c_{13} + c_-}. \tag{B1}
\]

Imposing that the perturbations are stable and superluminal, we can restrict the two parameters within the range as

\[
-1 \leq c_{13} \leq 0, \tag{B2}
\]

\[
\frac{c_{14}}{3(1 + c_{13})} \leq c_- \leq 0. \tag{B3}
\]

For the parameters satisfying the above constraints, we have safely a cosmological solution (see Eq. \(21\)), and we do not have a growing isocurvature mode (see Eq. \(A14\)).

Imposing further that \(\mathcal{A} = 1\), one can evade the constraint from radiation damping rate. Thus, we are finally left with a single parameter, say, \(c_{13}\), within the range \(\mathcal{A}\). Taking \(|c_3| \lesssim O(0.1)\), we choose to use

\[
c_1 = -0.019, \ c_2 = 0.014, \ c_3 = -0.011, \ c_4 = 0.0063, \ c_6 = 1.241, \ \alpha = -c_{14} = 0.0128 \tag{B4}
\]

in Sec. V of the main text.

[1] Y. Fujii, K. Maeda, “The scalar-tensor theory of gravitation,” Cambridge, USA: Univ. Pr. (2003) 240 p.
[2] D. F. Mota and J. D. Barrow, Phys. Lett. B 581, 141 (2004) [arXiv:astro-ph/0306047]; J. Khoury and A. Weltman, Phys. Rev. Lett. 93, 171104 (2004) [arXiv:astro-ph/0309300]; J. Khoury and A. Weltman, Phys. Rev. D 69, 044026 (2004) [arXiv:astro-ph/0309411].
[3] A. A. Starobinsky, JETP Lett. 86, 157-163 (2007). [arXiv:0706.2041 [astro-ph]]; W. Hu, I. Sawicki, Phys. Rev. D76, 064004 (2007). [arXiv:0705.1158 [astro-ph]]; S. A. Appleby, R. A. Battye, Phys. Lett. B654, 7-12 (2007). [arXiv:0705.3199 [astro-ph]]; S. Nojiri and S. D. Odintsov, Phys. Rev. D 77, 026007 (2008) [arXiv:0710.1738 [hep-th]].
[4] A. Nicolis, R. Rattazzi and E. Trincherini, Phys. Rev. D 79, 064036 (2009) [arXiv:0811.2197 [hep-th]]; C. Def- kayet, G. Esposito-Farese and A. Vikman, Phys. Rev. D 79, 084003 (2009) [arXiv:0901.1314 [hep-th]]; C. Def-fayet, S. Deser and G. Esposito-Farese, Phys. Rev. D 80, 064015 (2009) [arXiv:0906.1967 [gr-qc]].
[5] N. Chow and J. Khoury, Phys. Rev. D 80, 024037 (2009) [arXiv:0905.1325 [hep-th]]; F. P. Silva and K. Koyama, Phys. Rev. D 80, 123101 (2009) [arXiv:0909.4528 [astro-ph.CO]]; T. Kobayashi, H. Tashiro and D. Suzuki, Phys. Rev. D 81, 063513 (2010) [arXiv:0912.4641 [astro-ph.CO]]; T. Kobayashi, Phys. Rev. D 81, 103533 (2010) [arXiv:1003.3281 [astro-ph.CO]]; R. Gannouji and M. Sami, Phys. Rev. D 82, 024011 (2010) [arXiv:1004.2808 [gr-qc]]; A. De Felice, S. Tsujikawa, JCAP 1007, 024 (2010). [arXiv:1005.0863 [astro-ph.CO]]; A. De Felice, S. Mukohyama, S. Tsujikawa, Phys. Rev. D82, 023524 (2010). [arXiv:1006.0281 [astro-ph.CO]]; A. De Felice, S. Tsujikawa, Phys. Rev. Lett. 105, 111301 (2010). [arXiv:1007.2700 [astro-ph.CO]]; A. Ali, R. Gannouji, M. Sami, Phys. Rev. D82, 103015 [arXiv:1008.0060 [astro-ph]].
attella and A. A. Starobinsky, JCAP 0802, 016 (2008) [arXiv:0711.4212 [astro-ph]].

[53] M. L. Graesser, A. Jenkins and M. B. Wise, Phys. Lett. B 613 (2005) 5 [arXiv:hep-th/0501223].