ON GRADED QUASI-SEMIPRIME SUBMODULES OF GRADED MODULES OVER GRADED COMMUTATIVE RINGS

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Abstract. Let $G$ be a group with identity $e$. Let $R$ be a $G$-graded commutative ring and $M$ a graded $R$-module. A proper graded submodule $N$ of $M$ is called a graded semiprime submodule if whenever $r \in h(R)$, $m \in h(M)$ and $n \in Z^+$ with $r^n m \in N$, then $rm \in N$. In this paper, we introduce the concept of graded quasi-semiprime submodule as a generalization of graded semiprime submodule and show a number of results in this class. We say that a proper graded submodule $N$ of $M$ is a graded quasi-semiprime submodule if $(N :_R M)$ is a graded semiprime ideal of $R$.

1. Introduction and Preliminaries

Throughout this paper all rings are commutative with identity and all modules are unitary.

Graded semiprime submodules of graded modules over graded commutative rings, have been introduced and studied in [1, 5, 7, 12]. Also, the concept of graded semiprime ideal was introduced by Lee and Varmazyar [7] and studied in [4].

Recently, K. Al-Zoubi, R. Abu-Dawwas and I. Al-Ayyoub in [1] introduced and studied the concept of graded semi-radical of graded submodules in graded modules.

Here, we introduce the concept of graded quasi-semiprime submodules of graded modules over a commutative graded rings as a generalization of graded semiprime submodules and investigate some properties of these classes of graded submodules.

First, we recall some basic properties of graded rings and modules which will be used in the sequel. We refer to [6] and [8-10] for these basic properties and more information on graded rings and modules.

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Let $G$ be a multiplicative group and $e$ denote the identity element of $G$. A ring $R$ is called a graded ring (or $G$-graded ring) if there exist additive subgroups $R_\alpha$ of $R$ indexed by the elements $\alpha \in G$ such that $R = \bigoplus_{\alpha \in G} R_\alpha$ and $R_\alpha R_\beta \subseteq R_{\alpha\beta}$ for all $\alpha, \beta \in G$. The elements of $R_\alpha$ are called homogeneous of degree $\alpha$ and all the homogeneous elements are denoted by $h(R)$, i.e. $h(R) = \bigcup_{\alpha \in G} R_\alpha$. If $r \in R$, then $r$ can be written uniquely as $\sum_{\alpha \in G} r_\alpha$, where $r_\alpha$ is called a homogeneous component of $r$ in $R_\alpha$. Moreover, $R_e$ is a subring of $R$ and $1 \in R_e$.

Let $R = \bigoplus_{\alpha \in G} R_\alpha$ be a $G$-graded ring. An ideal $I$ of $R$ is said to be a graded ideal if $I = \bigoplus_{\alpha \in G} (I \cap R_\alpha) := \bigoplus_{\alpha \in G} I_\alpha$. Let $R = \bigoplus_{\alpha \in G} R_\alpha$ be a $G$-graded ring. A Left $R$-module $M$ is said to be a graded $R$-module (or $G$-graded $R$-module) if there exists a family of additive subgroups $\{M_\alpha\}_{\alpha \in G}$ of $M$ such that $M = \bigoplus_{\alpha \in G} M_\alpha$ and $R_\alpha M_\beta \subseteq M_\alpha \beta$ for all $\alpha, \beta \in G$. Here, $R_\alpha M_\beta$ denotes the additive subgroup of $M$ consisting of all finite sums of elements $r_\alpha m_\beta$ with $r_\alpha \in R_\alpha$ and $m_\beta \in M_\beta$. Also if an element of $M$ belongs to $\bigcup_{\alpha \in G} M_\alpha = h(M)$, then it is called a homogeneous. Note that $M_\alpha$ is an $R_e$-module for every $\alpha \in G$. So, if $I = \bigoplus_{\alpha \in G} I_\alpha$ is a graded ideal of $R$, then $I_\alpha$ is an $R_e$-module for every $\alpha \in G$. Let $R = \bigoplus_{\alpha \in G} R_\alpha$ be a $G$-graded ring. A submodule $N$ of $M$ is said to be a graded submodule of $M$ if $N = \bigoplus_{\alpha \in G} (N \cap M_\alpha) := \bigoplus_{\alpha \in G} N_\alpha$. In this case, $N_\alpha$ is called the $\alpha$-component of $N$.

Let $R$ be a $G$-graded ring, $M$ a graded $R$-module and $N$ a graded submodule of $M$. Then $(N :_R M)$ is defined as $(N :_R M) = \{r \in R : r M \subseteq N\}$. It is shown in [3, Lemma 2.1] that if $N$ is a graded submodule of $M$, then $(N :_R M)$ is a graded ideal of $R$. The annihilator of $M$ is defined as $(0 :_R M)$ and is denoted by $Ann_R(M)$.

A proper graded submodule $N$ of $M$ is said to be a graded semiprime submodule if whenever $r \in h(R)$, $m \in h(M)$ and $n \in \mathbb{Z}^+$ with $rm \in N$, then $rm \in N$, (see [5].) A proper graded ideal $I$ of $R$ is said to be graded semiprime ideal if whenever $r, s \in h(R)$ and $n \in \mathbb{Z}^+$ with $r^ns \in I$, then $rs \in I$, (see [4].)

2. Results

**Definition 2.1.** Let $R$ be a $G$-graded ring and $M$ a graded $R$-module. A proper graded submodule $N$ of $M$ is said to be a graded quasi-semiprime submodule of $M$ if $(N :_R M)$ is a graded semiprime ideal of $R$.

**Theorem 2.2.** Let $R$ be a $G$-graded ring, $M$ a graded $R$-module and $N$ a proper graded submodule of $M$. If $N$ is a graded semiprime submodule of $M$, then $N$ is a graded quasi-semiprime submodule of $M$.

**Proof.** By [1, Theorem 2.4].
The next example shows that a graded quasi-semiprime submodule is not necessarily graded semiprime submodule.

Example 2.3. Let $G = \mathbb{Z}_2$, $R = \mathbb{Z}$ be a $G$-graded ring with $R_0 = \mathbb{Z}$ and $R_1 = \{0\}$. Let $M = \mathbb{Z} \times \mathbb{Z}$ be a graded $R$-module with $M_0 = \mathbb{Z} \times \{0\}$ and $M_1 = \{0\} \times \mathbb{Z}$. Now, consider a submodule $N = 4\mathbb{Z} \times \{0\}$ of $M$. Then it is a graded submodule and $N:M_0 = \{0\}$ is a graded semiprime ideal of $R$, and so $N$ is a graded quasi-semiprime submodule of $R$. But the graded submodule $N$ is not graded semiprime submodule of $M$, since $2^2(3,0) \in N$ but $2(3,0) \not\in N$.

Example 2.4. Let $G = \mathbb{Z}_2$, $R = \mathbb{Z}$ be a $G$-graded ring with $R_0 = \mathbb{Z}$ and $R_1 = \{0\}$. Let $M = \mathbb{Z}_8$ be a $G$-graded $R$-module with $M_0 = \mathbb{Z}_8$ and $M_1 = \{0\}$. Now, consider a submodule $N = <4>$ of $M$. Then it is a graded submodule and $N:M_0 = 4\mathbb{Z}$ is not a graded semiprime ideal of $R$ since $2^21 = 4 \in 4\mathbb{Z}$ but $2 \cdot 1 = 2 \not\in 4\mathbb{Z}$. Then $N$ is not graded quasi-semiprime submodule of $M$.

Recall that a graded $R$-module $M$ is called a graded multiplication if for each graded submodule $N$ of $M$, we have $N = IM$ for some graded ideal $I$ of $R$. If $N$ is graded submodule of a graded multiplication module $M$, then $N = (N :_R M)M$.

Theorem 2.5. Let $R$ be a $G$-graded ring, $M$ a graded multiplication $R$-module and $N$ a proper graded submodule of $M$. Then $N$ is a graded quasi-semiprime submodule of $M$ if and only if $N$ is a graded semiprime submodule of $M$.

Proof. By [1, Theorem 2.5].

Theorem 2.6. Let $R$ be a $G$-graded ring, $M$ a graded multiplication $R$-module and $N$ a proper graded submodule of $M$. Then the following statements are equivalent:

(i) $N$ is a graded quasi-semiprime submodule of $M$.
(ii) If whenever $I^kM \subseteq N$, where $I$ is a graded ideal of $R$ and $k \in \mathbb{Z}^+$, then $IM \subseteq N$.

Proof. (i) $\Rightarrow$ (ii) By Theorem 2.5 and [5, Proposition 2.6]

(ii) $\Rightarrow$ (i) Let $r^ks \in (N :_R M)$ where $r, s \in h(R)$ and $k \in \mathbb{Z}^+$. So $r^k s M \subseteq N$. Let $I = (rs)$ be a graded ideal of $R$ generated by $rs$. Then $I^kM \subseteq N$. By our assumption we have $IM = (rs)M \subseteq N$. This yields that $rs \in (N :_R M)$. So $(N :_R M)$ is a graded semiprime ideal of $R$. Therefore $N$ is a graded quasi-semiprime submodule of $M$.

Recall that a proper graded ideal $I$ of a $G$-graded ring $R$ is said to be a graded prime ideal if whenever $r, s \in h(R)$ with $rs \in I$, then either
r \in I \text{ or } s \in I \text{ (see [11].) A proper graded ideal } J \text{ of } R \text{ is said to be a graded primary ideal if whenever } r, s \in h(R) \text{ with } rs \in J, \text{ then either } r \in J \text{ or } s^n \in J \text{ for some } n \in \mathbb{Z}^+. \text{ (see [11].)}$

**Theorem 2.7.** Let $R$ be a $G$-graded ring, $M$ a graded $R$-module and $N$ a graded quasi-semiprime submodule of $M$. If $(N :_RM)$ is a graded primary ideal of $R$, then $(N :_RM)$ is a graded prime ideal of $R$.

*Proof.* Suppose that $(N :_RM)$ is a graded primary ideal of $R$. Let $rs \in (N :_RM)$ and $r \notin (N :_RM)$. Then $s \in Gr((N :_RM))$ as $(N :_RM)$ is a graded primary ideal of $R$. Hence $s^k \in (N :_RM)$ for some $k \in \mathbb{Z}^+$. Since $(N :_RM)$ is a graded semiprime ideal of $R$, we have $s \in (N :_RM)$. Therefore $(N :_RM)$ is a graded prime ideal of $R$. \hfill \Box

Let $R$ be a $G$-graded ring, $M$ a graded $R$-module and $N$ a graded submodule of $M$. The graded envelope submodule $RGE_M(N)$ of $N$ in $M$ is a graded submodule of $M$ generated by the set $GE_M(N) = \{rm : r \in h(R), m \in h(M) \text{ such that } r^nm \in N \text{ for some } n \in \mathbb{Z}^+\}$ (see [2, Definition 1].)

**Theorem 2.8.** Let $R$ be a $G$-graded ring, $M$ a graded multiplication $R$-module and $N$ a proper graded submodule of $M$. Then $N$ is a graded quasi-semiprime submodule of $M$ if and only if $N = RGE_M(N)$.

*Proof.* Suppose that $N$ is a graded quasi-semiprime submodule of $M$. Then $N$ is a graded semiprime submodule of $M$ by Theorem 2.5. Clearly, $N \subseteq RGE_M(N)$. Now, let $x \in GE_M(N)$. Then $x = rm$ for some $r \in h(R), m \in h(M)$ and there exists $k \in \mathbb{Z}^+$ such that $r^km \in N$. Then $rm \in N$ as $N$ is a graded semiprime submodule of $M$. Hence $GE_M(N) \subseteq N$. This yields that $RGE_M(N) \subseteq N$. Thus $N = RGE_M(N)$. Conversely, suppose that $N = RGE_M(N)$. Let $r \in h(R), m \in h(M)$ and $k \in \mathbb{Z}^+$ such that $r^km \in N$, so by the definition of the set $GE_M(N)$ we have $rm \in GE_M(N)$. Then $rm \in N$ as $GE_M(N) \subseteq RGE_M(N) = N$, so $N$ is a graded semiprime submodule of $M$. Therefore $N$ is a graded quasi-semiprime submodule of $M$ by Theorem 2.2. \hfill \Box

Let $R$ be a $G$-graded ring and $M, M'$ be two graded $R$-modules. Let $f : M \to M'$ be an $R$-module homomorphism. Then $f$ is said to be a graded homomorphism if $f(M_\alpha) \subseteq M'_\alpha$ for all $\alpha \in G$ (see [10].)

**Theorem 2.9.** Let $R$ be a $G$-graded ring, $M, M'$ be two graded $R$-modules and $f : M \to M'$ a graded epimorphism.

(i) If $N$ is a graded quasi-semiprime submodule of $M$ such that $\ker(f) \subseteq N$, then $f(N)$ is a graded quasi-semiprime submodule of $M'$.
(ii) If $N'$ is a graded quasi-semiprime submodule of $M'$, then $f^{-1}(N')$ is a graded quasi-semiprime submodule of $M$.

**Proof.** (i) Suppose that $N$ is a graded quasi-semiprime submodule of $M$ and $\ker(f) \subseteq N$. It is easy to see that $f(N) \neq M'$. Now let $r^k s \in (f(N) :_R M')$ where $r, s \in h(R)$ and $k \in \mathbb{Z}^+$, it follows that, $r^k s M' \subseteq f(N)$. Then $r^k s M' = r^k s f(M) = f(r^k s M) \subseteq f(N)$ since $f$ is an epimorphism. This yields that $r^k s M \subseteq N$ since $\ker(f) \subseteq N$, i.e., $r^k s \in (N :_R M)$. Since $N$ is a graded quasi-semiprime submodule of $M$, we get $rs \in (N :_R M)$, i.e., $rs M \subseteq N$. Hence $f(rs M) = rs f(M) = rs M' \subseteq f(N)$, i.e., $rs \in (f(N) :_R M')$. Therefore, $f(N)$ is a graded quasi-semiprime submodule of $M'$.

(ii) Suppose that $N'$ is a graded quasi-semiprime submodule of $M'$. It is easy to see that $f^{-1}(N') \neq M$. Let $r^k s \in (f^{-1}(N') :_R M)$ where $r, s \in h(R)$ and $k \in \mathbb{Z}^+$, it follows that, $r^k s M \subseteq f^{-1}(N')$. Then $r^k s f(M) = r^k s M' \subseteq N'$, i.e., $r^k s \in (N' :_R M')$. Therefore, $f^{-1}(N')$ is a graded quasi-semiprime submodule of $M$.

**Theorem 2.10.** Let $R$ be a $G$-graded ring, $M$ a graded $R$-module and $K$ a proper graded submodule of $M$. If $N$ is a graded quasi-semiprime submodule of $M$ with $N \subseteq K$ and $(N :_R M)$ is a graded maximal ideal of $R$, then $K$ is a graded quasi-semiprime submodule of $M$.

**Proof.** Suppose that $N \subseteq K$, it follows that $(N :_R M) \subseteq (K :_R M)$. By Lemma 2.1, $(K :_R M)$ is a proper graded ideal of $R$. Then $(N :_R M) = (K :_R M)$ as $(N :_R M)$ is a graded maximal ideal of $R$. This yields that $(K :_R M)$ is a graded semiprime ideal of $R$. Therefore $K$ is a graded quasi-semiprime submodule of $M$.

**Theorem 2.11.** Let $R$ be a $G$-graded ring, $M$ a graded $R$-module and $N$ and $K$ be two graded quasi-semiprime submodules of $M$. Then $N \cap K$ is a graded quasi-semiprime submodule of $M$.

**Proof.** Let $r^k s \in (N \cap K :_R M)$ where $r, s \in h(R)$ and $k \in \mathbb{Z}^+$. This yields that $r^k s \in (N :_R M) \cap (K :_R M)$. Since $(N :_R M)$ and $(K :_R M)$ are graded semiprime ideals of $R$, we have $rs \in (N :_R M) \cap (K :_R M)$ and so $rs \in (N \cap K :_R M)$. Therefore $N \cap K$ is a graded quasi-semiprime submodule of $M$.

Let $R$ be a $G$-graded ring and $M$ be a graded $R$-module, $M$ is called a graded semiprime module if $(0)$ is a graded semiprime submodule of $M$. 
Definition 2.12. Let $R$ be a $G$-graded ring and $M$ be a graded $R$-module. Then $M$ is said to be a graded quasi-semiprime module if $\text{Ann}_R N$ is a graded semiprime ideal of $R$, for every non-zero graded submodule $N$ of $M$.

Theorem 2.13. Let $R$ be a $G$-graded ring and $M$ be a graded $R$-module. If $M$ is a graded semiprime module, then $M$ is a graded quasi-semiprime module.

Proof. Suppose that $M$ is a graded semiprime module. Then $0$ is a graded semiprime submodule of $M$. Now, Let $N$ be a non-zero graded submodule of $M$ and $r^k s \in \text{Ann}_R N$ where $r, s \in h(R)$ and $k \in \mathbb{Z}^+$. It follows that $r^k s N = 0$. Then $rs N = 0$ as $0$ is a graded semiprime submodule of $M$. Hence $rs \in \text{Ann}_R N$, it follows that $\text{Ann}_R N$ is a graded semiprime ideal of $R$. Therefore $M$ is a graded quasi-semiprime module. □

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