Tachyons and Representations of $SO_0(2,3)$

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**Abstract**

We have previously described an embedding of the Poincaré Lie algebra into an extension of the Lie field of the group $SO_0(1,4)$, and we used this embedding to construct irreducible representations of the Poincaré group out of representations of $SO_0(1,4)$. Some $q$ generalizations of these results have been obtained by us i.e. we embed classical structures into quantum structures. Here we report on analogous findings for $SO_0(2,3)$: we express the basis elements of the Lie algebra of the Poincaré Lie group as irrational functions of the generators of the anti de Sitter group $SO_0(2,3)$, and thus obtain an embedding of the Poincaré Lie algebra into an extension of the Lie field of $SO_0(2,3)$. We have obtained some generalizations to higher dimensions. We apply our results to certain unitary, continuous series representations of $SO(2,3)$ associated with functions on $SO(2,3)/SO(1,3)$. From the embedding theorem we obtain representations of the Poincaré Lie algebra which are associated with unitary, tachyonic representations of the Poincaré Lie group, provided these representations are integrable to group representations. Such tachyonic representations were studied by Wigner in his classical 1939 paper.

**I. INTRODUCTION**

We start with a well-known deformation [1], [2], [3] of the Poincaré Lie algebra, which is defined in terms of the generators $L_{ij}$ (Lorentz generators) and $P_i$ (translation generators) by the following:

\[ L_{ij} \rightarrow L_{ij}, \quad P_i \rightarrow L_{4i}^\pm = \frac{i [Q_2, P_i]}{2 \sqrt{-\sum_{i,j=0}^{3} P_i P_j}} + P_i \]  

(1a)

where $Q = \frac{1}{2} \sum_{i,j=0}^{3} L_{ij} L^{ji}$ is the second order Casimir operator of the Lorentz subgroup. ($[ , ]$ denotes commutator.) For the plus sign this leads to the Lie algebra of the anti-de Sitter group, $SO_0(2,3)$, and the minus sign gives the commutation relations of the de Sitter group,
$SO_0(1, 4)$. Now eqns. (1b±) may be considered as algebraic equations for the translation generators $P_i$ of the Poincaré group, and we are able to solve these equations for the $P_i$. The solution to this problem for the choice of the minus sign (i.e. for the choice of eqn. (1.b−) has been given by us in [1]. Our answer expresses the basis elements of the Poincaré Lie algebra as certain irrational functions of the generators of the de Sitter group. We give in section 3 a generalization of this “anti-deformation” to higher dimensions i.e. for $SO_0(p, q)$ groups and associated higher dimensional Poincaré groups, but only for a particular class of representations, which in the four dimensional case considered in [1] corresponds to spinless representations of the Poincaré group. It is possible to obtain the general solution for eqn. (1.1+) also, and we present this solution in section 4.

We have also obtained some analogous results for $q$-deformations in lowest dimensions i.e. 2, 3 and 4. In particular, in the $n = 2$ case, we start with the Euclidean group in two dimensions $E(2)$, with generators $L_{12}$ (rotation generator) and $P_i$ ($i = 1, 2$) (translation generators), and define the following (c.f. [4]):

$$\tilde{L}_{3i} = \frac{1}{2\sqrt{q}} \left[ \left( -iL_{21} \right)^2 , P_i \right] + P_i , \quad (Y := \sqrt{P^1P^1 + P^2P^2}) . \quad (1.2)$$

Again, we have obtained the “anti-deformation” by solving eqns. (1.2) for the $P_i$. Our result expresses the translation generators of $E(2)$ as irrational functions of $U_q(so(2, 1))$ [4]. If we let $H = -2iL_{21}$, then our solution is given by:

$$P_1 = D^{-1} \left( \left[ 1 - \frac{1}{2Y} [H]_q \right] L_{31} + \frac{i[2]\sqrt{q}}{2Y} \left[ \frac{H}{2} q \right] L_{32} \right) , \quad (1.3a)$$

and

$$P_2 = D^{-1} \left( \left[ 1 - \frac{1}{2Y} [H]_q \right] L_{32} - \frac{i[2]\sqrt{q}}{2Y} \left[ \frac{H}{2} q \right] L_{31} \right) , \quad (1.3b)$$

where

$$D = - \frac{1}{4Y^2} \left[ \frac{[H]^2}{\sqrt{q}} - \left( \frac{[H]_q}{[H]} \sqrt{q} - 2Y \right)^2 \right] . \quad (1.4)$$

Furthermore

$$Y^2 = \tilde{\Delta}_q + \frac{1}{4} \quad (1.5)$$

and $\tilde{\Delta}_q$ is the Casimir element of $U_q(so(3, \mathbb{C}))$ as defined in [4]. The reader can easily verify that the $P_i$ as defined by eqns. (1.3a) and (1.3b) satisfy the defining commutation relations for the translation generators of $E(2)$, and verify that $Y^2$ satisfies eqn. (1.5).

II. $SO_0(P + 1, Q), SO_0(P + 1, Q) \times \mathbb{R}^{P+Q+1}$ AND THEIR LIE ALGEBRAS

Let $M$ denote the real vector space $\mathbb{R}^{p+q+1}$ on which the quadratic form

$$Q(\xi) = \xi_0^2 + \xi_1^2 + \ldots + \xi_p^2 - \xi_{p+1}^2 - \ldots - \xi_{p+q}^2 \quad . \quad (2.1)$$
is defined, and the associated bilinear form of signature \((p, q)\) is given by

\[
B(\xi, \eta) = \frac{1}{2} \left\{ Q(\xi + \eta) - Q(\xi) - Q(\eta) \right\}.
\] (2.2)

Denote the matrix corresponding to \(B\) by \(\beta_0\):
\[
\beta_0 = \text{diag}(1, 1, \ldots, 1, -1, -1, \ldots, -1),
\]
where the right hand side of this equation denotes a diagonal matrix with diagonal entries as shown inside the parentheses, with \(p + 1\) entries being +1 and \(q\) entries being -1.

\(SO(p + 1, q)\) is the component connected to identity of

\[
SO(p + 1, q) = \{ g \in SL(n, \mathbb{R}) \mid g^\dagger \beta_0 g = \beta_0 \}\ .
\] (2.3)

\((\dagger\) denotes transpose of a matrix and \(n = p + q + 1\).) Its Lie algebra, \(so(p + 1, q)\), is defined as:

\[
so(p + 1, q) = \{ X \in sl(n, \mathbb{R}) \mid X^\dagger \beta_0 + \beta_0 X = 0 \}\ .
\] (2.4)

A basis of \(G\) are the generators \(L_{ij}\) \((i, j = 0, \ldots, p + q, i < j)\). We define \(L_{ij} = -L_{ji}\) for \(i > j\), and the \(L_{ij}\) satisfy the following commutation relations:

\[
[L_{ij}, L_{jk}] = -e_j L_{ik}
\] (2.5)

and all other commutators vanish. The \(e_j\) are defined as follows:

\[
e_0 = e_1 = \ldots = e_p = 1, \quad e_{p+1} = e_{p+2} = \ldots = e_{p+q} = -1.
\] (2.6)

Eqn. (2.5) can also be given a well-known form with help of the \(g_{ij} = (\beta_0)_{ij}\):

\[
[L_{ij}, L_{kl}] = g_{ik} L_{jl} + g_{jl} L_{ik} - g_{il} L_{jk} - g_{jk} L_{il} ,
\] (2.7)

We wish to give explicit matrix expressions of these \(\frac{n(n - 1)}{2}\) generators in our \((\beta_0)\) presentation of the fundamental representation of \(G\). Let \(E_{ij}\) denote the \(n \times n\) matrix whose \(i, j\) entry is one, and all other entries are zero. Clearly

\[
[E_{ij}, E_{kl}] = \delta_{jk} E_{il} - \delta_{il} E_{kj} .
\] (2.8)

Then \((q \neq 0)\)

\[
L_{ij} = -(E_{i,j} - E_{j,i}) \quad (0 \leq i, j \leq p, i \neq j)
\] (2.9a)

\[
L_{n-i,n-j} = E_{n-i,n-j} - E_{n-j,n-i} \quad (1 \leq i, j \leq q, i \neq j)
\] (2.9b)

\[
L_{i,p+j} = E_{i,p+j} + E_{p+j,i} \quad (0 \leq i \leq p, 1 \leq j \leq q).
\] (2.9c)

The quadratic Casimir operator of \(SO(p + 1, q)\) is:

\[
Q_2 = \frac{1}{2} \sum_{i,j=0}^{p+q} L_{ij} L^{ji} = -\frac{1}{2} \sum_{i,j=0}^{p} L_{ij} L_{ij} -
\]

\[
-\frac{1}{2} \sum_{i,j=p+1}^{p+q} L_{ij} L_{ij} + \sum_{i=0}^{p} \sum_{j=1}^{q} L_{i,p+j} L_{i,p+j} .
\] (2.10)
The Poincaré group in $p + q + 1$ dimensions is the semi-direct product:

$$SO_0(p + 1, q) \times \mathbb{R}^{p+q+1} = \{(g, a) \mid g \in SO_0(p + 1, q), \ a \in \mathbb{R}^{p+q+1} \}$$

(2.11)

with multiplication law given by:

$$(g, a) \ (g', a') = (g \ g', a + \phi(g) \ a')$$

(2.12)

where $\phi(g)$ is the natural action of $SO_0(p + 1, q)$ on $\mathbb{R}^{p+q+1}$. Its Lie algebra is

$$\mathcal{P} = so(p + 1, q) \oplus \mathbb{R}^{p+q+1},$$

(2.13)

and a basis of $\mathcal{P}$ is given by:

$$L_{ij}, \ P_k \ (i, j, k = 0, \ldots, p + q).$$

(2.14)

The commutation relations of these basis elements are: eqns. (2.7) for the $L_{ij}$ and

$$[L_{ij}, P_k] = -g_{jk} \ P_i + g_{ik} \ P_j , \quad [P_i, P_j] = 0 .$$

(2.15)

The following central element of the enveloping algebra of $\mathcal{P}$ will play an important role in what follows:

$$P^2 = \sum_{k=0}^{p+q} P_k \ P^k = P_0^2 + P_1^2 + \ldots + P_{p+q}^2 - P_{p+q+1}^2 - \ldots - P_{p+q}^2 .$$

(2.16)

### III. REPRESENTATIONS OF $SO_0(p + 1, Q) \times \mathbb{R}^{p+q+1}$ FROM $SO_0(p + 2, Q)$ AND $SO_0(p + 1, Q + 1)$ REPRESENTATIONS

It is well-known [1], [2] that the

$$L_{p+q+1,i} = \frac{i \ [Q_2, P_i]}{2 \sqrt{\sum_{i,j=0}^{p+q+1} P_i \ P_j}} + P_i$$

(3.1±)

together with the $L_{i,j}$ of $SO_0(p+1, q)$ satisfy the defining commutation relations for the basic generators of $SO_0(p + 2, q)$ or $SO_0(p + 1, q + 1)$ for the choice (3.1+) or (3.1−), respectively. (Henceforth the plus sign of any $\pm$ refers to the $so(p + 2, q)$ case ((3.1+)) and the minus sign of any $\pm$ refers to the $so(p + 1, q + 1)$ case ((3.1−)).) We can view eqns. (3.1±) as a system of equations for commuting quantities $P_i$, and we would like to solve these equations (3.1±) for the $P_i$. We have succeeded in doing this by working in a representation $(\pi, \mathcal{H})$ of $SO_0(p + 1, q) \times \mathbb{R}^{p+q+1}$ (strongly continuous) on an Hilbert space $\mathcal{H}$ for which:

$$d\pi( P_0 \ \Delta - \sum_{i,j=1}^{p+q} P_j \ L_{0i} \ L^{ij} ) = 0$$

(3.2)
where
\[ \Delta = \frac{1}{2} \sum_{i,j=1}^{p+q} L_{ij} L^{ji} . \]

(dπ denotes the representation of \( \mathcal{P} \) obtained from \( \pi \) (by differentiation) and also its extension to \( \mathcal{E}(\mathcal{P}) \), the enveloping algebra of \( \mathcal{P} \), and also to any algebraic extension of the Lie field of \( \mathcal{P} \) (c.f. ref. (1)).) Now the quadratic Casimir operator of \( \text{so}(p+2, q) \ [\text{so}(p+1, q+1) \]
constructed out of the \( L_{p+q+1,i} \) \( [L_{p+q+1,i} \) of (3.1+) \( [3.1-] \) and \( L_{ij} \) is:

\[ C^\pm_2 = Q_2 - \sum_{i=0}^{p+q} L^\pm_{p+q+1,i} L^{\pm p+q+1,i} = Q_2 + \sum_{i=0}^{p+q} L^\pm_{p+q+1,i} L^{\pm p+q+1,i} . \]  (3.3+)

**Lemma 3.1** For representations \( (\pi, \mathcal{H}) \) for which (3.2) holds the following is true:
\[ d\pi(C^\pm_2) = -Y^2 - \left( \frac{n+2}{2} \right)^2 \cdot I = \mp d\pi(P^2) - \left( \frac{n+2}{2} \right)^2 \cdot I \],
where \( Y \) is the square root of the operator \( \pm d\pi(P^2) \), which we assume is self-adjoint, and strictly positive, so that it has a unique square root. Furthermore \( (n = p+q+1) \):

\[ d\pi(D) \ d\pi(P_0) = d\pi(\sum_{i=0}^{p+q} A_i^0 \ L^\pm_{n,i}) \]  (3.4+)

where
\[ D = \frac{(n-3)^2}{4} \left\{ Q_2 + \frac{(n-1)(n-3)}{4} \right\} + i Y (n-3) \{ Q_2 + \]
\[ \frac{(n-2)(n-3)}{4} \} - Y^2 \left\{ Q_2 - \frac{(n-3)}{2} \right\} + i Y^3 (n-2) - Y^4 , \]
and
\[ \sum_{i=0}^{p+q} A_i^0 \ L^\pm_{n,i} = \frac{i}{4} (n-3)^2 \left\{ \left( \frac{n-3}{2} \right) L^\pm_{n,0} + \sum_{i=1}^{p+q} L^{0,i} L^\pm_{n,i} \right\} Y \]
\[ - 2 \left( \frac{n-3}{2} \right) \left\{ \frac{1}{2} \left( \frac{n-3}{2} \right) L^\pm_{n,0} + \sum_{i=1}^{p+q} L^{0,i} L^\pm_{n,i} \right\} Y^2 \]
\[ + \left( \frac{n-3}{2} \right) L^\pm_{n,0} - \sum_{i=1}^{p+q} L^{0,i} L^\pm_{n,i} \right\} Y^3 - L^\pm_{n,0} Y^4 . \]

Now start with a unitary representation of \( SO_0(p+2, q) \ [SO_0(p+1, q+1) \) on an Hilbert space \( \mathcal{H} \) and use eqn. (3.4±) to construct a representation of \( \mathcal{P} \) out this representation.

Denote the (abstract) generators of \( SO_0(p+2, q) \ [SO_0(p+1, q+1) \) by \( \tilde{L}_{ij} \ (i = 0, 1, ..., p+q) \) and \( \tilde{L}^+_n, \tilde{L}^-_n \ (n = p+q+1, i = 0, 1, ..., p+q) \).

**Theorem 3.1** Let \( (\pi^\pm, \mathcal{H}) \) be a continuous, unitary representation of \( SO_0(p+2, q) \ [SO_0(p+1, q+1) \) on an Hilbert space \( \mathcal{H} \) and let \( (\pi^\pm, \mathcal{H}) \) be such that:

\[ \left( \psi, \left\{ d\pi^\pm(\tilde{C}_2) + \left( \frac{p+q}{2} \right)^2 \cdot I \right\} \psi \right) \geq 0 \ \forall \ \psi \in \mathcal{D}(d\pi(\tilde{C}_2)) , \]  (3.5±)

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where $\mathcal{D}(\ )$ denotes the domain of an operator ($>$ is for (3.5+) and $<$ is for (3.5$-$)) and

$$
d\pi^\pm \left( \tilde{L}_{n0}^\pm \tilde{\Delta} - \sum_{i,j=1}^{p+q} \tilde{L}_{nij}^\pm \tilde{L}_{oi} \tilde{L}_{ij}^\pm \right) = 0 .
$$

(3.6)

Let $Y : \mathcal{H} \rightarrow \mathcal{H}$ be a bounded, symmetric linear operator on $\mathcal{H}$, which commutes with all elements of $d\pi^+(so(p+2,q)) \{ d\pi^-(so(p+1,q+1)) \}$ and satisfies

$$
Y^2 = - \left\{ d\pi(\tilde{C}_2) + \left( \frac{p+q}{2} \right) \cdot I \right\} .
$$

(3.7$\pm$)

Further, let $\tilde{D}$ and $\sum_{i=0}^{p+q} \tilde{A}_i^i \tilde{L}_{n,i}^\pm$ be as in eqn. (3.4$\pm$), but with $\tilde{L}_{nij}$ and $\tilde{L}_{n,i}^\pm$ instead of $L_{nij}$ and $L_{n,i}^\pm$. Then there is formally a skew symmetric representation $d\pi^+(\mathcal{P})$ of $\mathcal{P} = so(p+1,q) \oplus S^p \oplus S^{p+1}$ on $\mathcal{H}$ with $d\pi^+(L_{nij}) = d\pi^+(\tilde{L}_{nij})$ ($i = 0, 1, ..., p+q$), $d\pi^+(P_0) = d\pi^+(\tilde{D})^{-1} d\pi^+(\sum_{i=0}^{p+q} \tilde{A}_i^i \tilde{L}_{n,i}^\pm)$, and $d\pi^+(P_i) = [d\pi^+(L_{00}), d\pi^+(P_0)](i = 1, 2, ..., p+q)$, provided $d\pi^+ \left( \left[ \tilde{Q}_2, \sum_{i=0}^{p+q} \tilde{A}_i^i \tilde{L}_{n,i}^\pm \right] \right) \neq 0$, $d\pi^+(\tilde{D})^{-1}$ exists on a suitable dense domain in $\mathcal{H}$, and the operators $d\pi^+(\tilde{D})^{-1} d\pi \left( \left[ \tilde{L}_{00}, \sum_{j=0}^{p+q} \tilde{A}_j^i \tilde{L}_{n,j}^\pm \right] \right)$ ($i = 1, 2, 3$) and $d\pi^+(\tilde{D})^{-1} d\pi^+(\sum_{i=0}^{p+q} \tilde{A}_i^i \tilde{L}_{n,i}^\pm)$ commute with each other.

IV. THE GENERAL SOLUTION OF THE PROBLEM IN THE SO$_0$(3,2) CASE AND TACHYONS

Now we consider the general solution of eqns. (3.1+) for $p = 0$ and $q = 3$, without imposing any conditions, such as those of eqns. (3.2) and (3.6). In order to describe the solution it is necessary to know a bit more about the structure of the enveloping algebras $\mathcal{E}(G)$ of $G = so(2,3)$ and $\mathcal{E}(\mathcal{P})$, where $\mathcal{P}$ is the Poincaré Lie algebra.

A maximal abelian subalgebra of $\mathcal{E}(G)$ is generated by the six operators:

$$
L_{12}, L_{12}^2 + L_{23}^2 + L_{31}^2 = L^2, Q_2 = L_{01}^2 + L_{02}^2 + L_{03}^2 - L^2,
$$

$$
Q_4 = (L_{12} L_{30} + L_{23} L_{10} + L_{31} L_{20})^2, C_2, C_4
$$

(4.1)

where

$$
C_2 = -L_{04}^2 + L_{01}^2 + L_{02}^2 + L_{03}^2 - L_{12}^2 - L_{23}^2 - L_{31}^2 + L_{14}^2 + L_{24}^2 + L_{34}^2
$$

(4.2)

and

$$
C_4 = -\left[ \frac{1}{4} (\lambda - \rho) \right]^2 - (L_{12} L_{30} + L_{23} L_{10} + L_{31} L_{20})^2 +
$$

$$
+ \left( \sum_{i,j,k}^{3} \epsilon_{i j k} \left\{ \frac{1}{2} L_{04} L_{jk} - L_{0j} L_{4k} \right\} (\epsilon_{i \ell m} \left\{ \frac{1}{2} L_{04} L_{\ell m} - L_{0\ell} L_{4m} \right\}) \right) .
$$

(4.3)

with $\frac{1}{4} (\lambda - \rho) = L_{12} L_{34} + L_{23} L_{14} + L_{31} L_{24}$. $-L^2$ and $-iL_{12}$ are operators for the total angular momentum squared and the third component of the total angular momentum, respectively. The center $Z(G)$ of the universal enveloping algebra $\mathcal{E}(G)$ of $SO_0(2,3)$ is generated by $C_2$ and $C_4$.  

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The center \( Z(\mathcal{P}) \) of the enveloping algebra of \( \mathcal{P} \) is generated by the following set of elements:

\[
P^2 = \sum_{k=0}^{3} P_k^k = P_0^2 - P_1^2 - P_2^2 - P_3^2 ,
\]

and

\[
W = \sum_{\mu=0}^{3} \sum_{\nu=0}^{3} \sum_{\rho=0}^{3} \left( P_\mu P_\nu L_{\mu\rho} L^{\rho\nu} - \frac{1}{2} P_\rho P_\rho L_{\mu\nu} L^{\mu\nu} \right).
\]

\(- P_\mu P_\mu\) is the operator for the square of the mass, and \( W \) is a scalar operator, which describes the spin in a relativistically invariant way.

Now let \( C_2' = - \left( C_2 + \frac{3}{2} I \right) \) and \( C_4' = - \left( C_4 - \frac{1}{4} C_2 - \frac{9}{16} I \right) \). \( I \) is the identity in \( \mathcal{E}(G) \). We have the following result:

**Theorem 4.1**

\[
P_\mu = D^{-1} A_\mu \nu L_{\nu4}
\]

with

\[
A_\mu \nu = - C_4' \delta_\mu \nu + \frac{i}{2} \left[ \left( Q_2 + \frac{1}{4} \right) \delta_\mu \nu - \frac{3}{2} L_\mu \nu - L_{\mu\rho} L^{\rho\nu} - Q_4 \epsilon_\mu \nu \rho \tau L^{\rho\tau} \right] Y
\]

\[
- \left[ \left( Q_2 + \frac{1}{4} \right) - C_2' \right] \delta_\mu \nu - L_\mu \nu - L_{\mu\rho} L^{\rho\nu} \right] Y^2 + i \left( \frac{1}{2} \delta_\mu \nu - L_\mu \nu \right) Y^3
\]

and

\[
D = \left( Q_4 + \frac{1}{4} Q_2 - C_4' + \frac{3}{16} I \right) +
\]

\[
+ i \left( Q_2 + \frac{1}{2} \right) Y - \left( Q_2 - C_2' - \frac{1}{2} \right) Y^2 + 2 i Y^3.
\]

Furthermore \( Y^2 \), which commutes with all elements of \( \mathcal{E}(G) \), satisfies the following equation

\[
Y^4 + C_2' Y^2 + C_4' = 0.
\]

The reader may readily convince himself that in spin zero case, for which \( C_4 \) and \( Q_4 \) are represented as the zero operator, eqns. (4.6) (with eqns. (4.7) and (4.8)) for \( P_0 \) (in a given representation which satisfies eqn. (3.6)), agree with the corresponding equations (3.4+) in the Lemma 3.1. when \( p = 0 \), \( q = 3 \). The main fact, which is necessary in order to convince oneself of this, is the following:

\[
d\pi^+ \left( \tilde{L}_{04} \tilde{L}^{0\nu} \tilde{L}^{4\nu} \right) = d\pi^+ \left( \tilde{Q}_2 \tilde{L}^{40} - 2 \tilde{L}_0 i \tilde{L}^{4i} \right),
\]

which is easily proved with the help of the representation condition, eqn. (3.6).

Note that \( D^{-1} \) makes sense because \( \mathcal{E}(G) \) has no zero divisors [5]. Also eqn. (4.9) is much more interesting than the corresponding equation in the spinless case of the previous section, which we obtain out of eqn. (4.9) by setting \( C_4' = 0 \). Eqn. (4.9) is a quartic equation for the operator \( Y \) involving both Casimir operators of \( SO_0(2,3) \), and thus there are many more possibilities with which to contend, e.g. de Sitter mass becoming Poincaré spin and de Sitter spin becoming Poincaré mass.
Finally, we describe a class of spinless representations of $SO_0(2,3)$ which satisfy the hypotheses of Theorem 3.1. We show that the representations of the Poincaré Lie algebra which we get from this theorem correspond to infinitesimally unitary representations with imaginary mass, i.e. they are tachyonic representations. (It seems clear to us that they should be integrable to Poincaré group representations, although we do not have a rigorous proof of this fact.)

The $SO_0(2,3)$ representations which we consider are continuous series representations occurring in the left regular representation on anti-de Sitter space. These representations and those of the continuous series occurring in the decomposition of the left regular representation of $SO_0(2,3)$ on anti-de Sitter space into irreducibles is a special case of a very beautiful achievement of 20th century mathematics, namely: the decomposition into irreducibles of the left regular representation of $SO_0(p+1,q)$ on real hyperbolic spaces. Let us briefly describe this result here.

Real hyperbolic spaces $H^{p,q}$ ($q > 0$) are:

$$H^{p,q} = \{ \xi \in \mathbb{R}^{p+q+1}|\xi_0^2 + \xi_1^2 + \xi_2^2 + \ldots + \xi_p^2 - \xi_{p+1}^2 - \ldots - \xi_{p+q}^2 = -1 \} .$$

(4.10)

The isotropy subgroup of the point $\xi^i = 0$ ($i \neq p + q$), $\xi^{p+q} = 1$ is $H = SO_0(p+1,q-1)$. Since the action of $SO_0(p+1,q)$ is transitive on $H^{p,q}$, we have that

$$H^{p,q} \cong SO_0(p+1,q)/SO_0(p+1,q-1) .$$

(4.11)

$H^{p,q}$ are semi-Riemannian spaces of constant curvature. The representation of $G = SO_0(p+1,q)$ is the left regular representation on $H^{p,q}$ i.e. $(\pi(g)f)(\xi) = f(g^{-1}\xi)$ for any $g \in G$ and $f \in L^2(G/H)$. We have on $C^\infty(G/H)$ representations $d\pi(so(p+1,q))$ and $d\pi(E(so(p+1,q))$ of the Lie algebra and the enveloping algebra, respectively. Let

$$\mathcal{H}_s = \{ f \in C^\infty(G/H)|(d\pi(C_2)f) = s^2 - \left\{ \frac{p+q-1}{2} \right\}^2 f \}$$

(4.12)

for each $s \in \mathcal{S}$. We then have: [6]

**Proposition 4.1** $d\pi(C_2)$ has a self-adjoint closure in $L^2(G/H)$. For $q > 1$ the spectrum of $d\pi(C_2) + [(p+q-1)/2]^2 \cdot I$ in $(0, \infty)$ is discrete with eigenvalue $[s + (p+q-1)/2]^2$ for each integer $s > -(p+q-1)/2$. Corresponding to each eigenvalue is an infinite dimensional eigenspace $\mathcal{H}_s$. The completion of $\mathcal{H}_s$ in the $L^2(G/H)$ norm gives an irreducible, unitary representation of $SO_0(p+1,q)$. The spectrum of $d\pi(C_2) + [(p+q-1)/2]^2 \cdot I$ in $(-\infty,0]$ is continuous with eigenvalue $[s]^2$ for each $s \in i \mathbb{R}$. The completion of $\mathcal{H}_s$ in the $L^2(G/H)$ norm is the direct sum of two irreducible, unitary representations of $SO_0(p+1,q)$. For $q = 1$ the above is also true except that there is no discrete spectrum.

We leave it to the reader to convince himself that the representations of this Proposition fulfill the hypotheses of Theorem 3.1. (Since $d\pi(C_2)$ and $d\pi(Q_2)$ are self-adjoint it is clear that $d\pi^+(\mathbb{D})$ is invertible. To prove the mutual commutativity of the translation generators one uses an integral transform given in [7].) The representations of the continuous spectrum, are the ones we use with $(3.5+)$ ($p$ of proposition replaced by $p + 1$). The representations of the continuous spectrum are also the ones we use with $(3.5-)$ ($q$ in proposition replaced by $q + 1$). For the $SO_0(2,3)$ case $(3.5+)$ we see that $d\pi^+(P^2)$ is strictly greater than zero, and thus these representations are tachyonic [8] as claimed above. We are also considering application of these ideas to higher dimensional cases [9].
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