$L^2$-transverse conformal Killing forms on complete foliated manifolds

Seoung Dal Jung$^1$ · Huili Liu$^2$

Received: 6 October 2016 / Accepted: 19 March 2017 / Published online: 12 May 2017
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Abstract In this article, we study the $L^2$-transverse conformal Killing forms on complete foliated Riemannian manifolds and prove some vanishing theorems. Also, we study the same problems on Kähler foliations with a complete bundle-like metric.

Keywords Transverse Killing form · Transverse conformal Killing form · Mean curvature form

Mathematics Subject Classification 53C12 · 53C27 · 57R30

1 Introduction

Let $(M, g_M, \mathcal{F})$ be a foliated Riemannian manifold with a foliation $\mathcal{F}$ of codimension $q$ and a bundle-like metric $g_M$ with respect to $\mathcal{F}$. A transverse conformal Killing field is a normal field with a flow preserving the conformal class of the transverse metric. As a generalization of a transverse conformal Killing field, we define the transverse conformal Killing $r$-forms $\phi$ as follows: for any vector field $X$ normal to the foliation,

$$\nabla_X \phi = \frac{1}{r+1} i(X) d\phi - \frac{1}{q-r+1} X^b \wedge \delta_T \phi,$$

where $X^b$ is the $g_M$-dual 1-form of $X$. For the definition of $\delta_T$, see Sect. 2. The transverse conformal Killing form $\phi$ with $\delta_T \phi = 0$ is called transverse Killing form, which is a generalization of a transverse Killing field. There are many researches working on such fields.
Moreover, transverse (conformal) Killing forms on Riemannian manifolds are just (conformal) Killing forms, which were studied by many authors [14, 16, 20, 21]. Recently, the transverse Killing and conformal Killing forms were studied in [4, 5, 7, 8]. In particular, the non-existence of the transverse Killing and conformal Killing forms on compact foliated Riemannian manifolds was studied by Jung and Richardson [8]. And the properties of such forms on Kähler foliations were studied by Jung and Jung (for the transverse Killing forms) in 2012 [7] and by Jung (for the transverse conformal Killing forms) in 2015 [5], respectively.

Jung [4] studied the non-existence of the $L^2$-transverse Killing forms on complete foliated Riemannian manifolds. In this paper, we study the $L^2$-transverse conformal Killing forms for Riemannian and Kähler foliations with a bundle-like metric. In fact, we prove the following theorem.

**Theorem A** (cf. Theorem 3.6) Let $(M, g_M, F)$ be a complete foliated Riemannian manifold, all of whose leaves are compact. Assume that the basic part of the mean curvature form $\kappa_B$ is bounded and coclosed. If the curvature endomorphism $F$ is nonpositive, then every $L^2$-transverse conformal Killing $r$-form $(1 \leq r \leq q-1)$ is parallel.

**Theorem B** (cf. Corollary 3.7) Let $(M, g_M, F)$ be as in Theorem A. Assume that $\kappa_B$ is bounded and coclosed. If the curvature endomorphism $F$ is nonpositive and either negative at some point or $\text{Vol}(M) = \infty$, then every $L^2$-transverse conformal Killing $r$-form $(1 \leq r \leq q-1)$ is trivial.

Since the transverse conformal Killing field is a dual vector field of the transverse conformal Killing 1-form, we have the following theorem.

**Theorem C** (cf. Corollary 3.9) Let $(M, g_M, F)$ be as in Theorem B. Assume that $\kappa_B$ is bounded and coclosed. If the transverse Ricci curvature is nonpositive and either negative at some point or $\text{Vol}(M) = \infty$, then every $L^2$-transverse conformal Killing field is trivial.

Remark Theorem C was proved by Yorozu [23] when $F$ is a point foliation, and by Aoki and Yorozu [2], by Nishikawa and Tondeur [18] when $F$ is a minimal foliation.

Let $J$ be the extension of complex structure $J$ to basic forms on Kähler foliations. For details, see (4.3) in Sect. 4. Then we have the following theorem.

**Theorem D** (cf. Theorem 4.12) Let $(M, g_M, F, J)$ be a complete Riemannian manifold with a Kähler foliation of codimension $q = 2m > 4$, all of whose leaves are compact. If $F$ is minimal, then for an $L^2$-transverse conformal Killing $r$-form $(2 \leq r \leq q-2)$ $\phi$, $J\phi$ is parallel.

### 2 Transverse conformal Killing forms

Let $(M, g_M, F)$ be a $(p + q)$-dimensional Riemannian manifold with a foliation $F$ of codimension $q$ and a bundle-like metric $g_M$ with respect to $F$. Then there exists an exact sequence of vector bundles

$$0 \longrightarrow TF \longrightarrow TM \longrightarrow Q \longrightarrow 0,$$

where $TF$ is the tangent bundle and $Q = TM/TF$ is the normal bundle of $F$. The metric $g_M$ determines an orthogonal decomposition $TM = TF \oplus TF^\perp$, identifying $Q$ with $TF^\perp$ and inducing a metric $g_Q$ on $Q$. Let $\Omega_B^s(F)$ be the space of all basic forms on $M$, i.e.,

$$\Omega_B^s(F) = \{ \phi \in \Omega^s(M) \mid i(X)\phi = 0, \ i(X)d\phi = 0, \ \forall X \in TF \}.$$
Then $\Omega^*(M) = \Omega_B^*(\mathcal{F}) \oplus \Omega_B^{*-1}(\mathcal{F})$ [1]. Let $\nabla$ be the transverse Levi–Civita connection on $Q$ [11], which is extended to $\Omega_B^*(\mathcal{F})$. The exterior differential $d$ on the de Rham complex $\Omega^*(M)$ restricts a differential $d_B : \Omega_B^r(\mathcal{F}) \to \Omega_B^{r+1}(\mathcal{F})$. Let $\kappa \in Q^*$ be the mean curvature form of $\mathcal{F}$. It is well-known that the basic part $\kappa_B$ of $\kappa$ is closed [1]. We now recall the star operator $* : \Omega_B^r(\mathcal{F}) \to \Omega_B^{2-r}(\mathcal{F})$, which is defined by [19,22]

$$\tilde{\phi} = (-1)^{p(q-r)} * (\phi \wedge \chi_{\mathcal{F}}), \quad \forall \phi \in \Omega_B^r(\mathcal{F}),$$

where $\chi_{\mathcal{F}}$ is the characteristic form of $\mathcal{F}$ and $*$ is the Hodge star operator associated to $g_M$. Then, for any $\phi, \psi \in \Omega_B^r(\mathcal{F})$, it is well-known that $\phi \wedge \tilde{\psi} = \psi \wedge \tilde{\phi}$ and $\tilde{\phi}^2 = (-1)^{r(q-r)} \phi$ [19]. Let $v$ be the transversal volume form, i.e., $*v = \chi_{\mathcal{F}}$. The pointwise inner product $\langle \cdot , \cdot \rangle$ on $\Lambda^r Q^*$ is defined uniquely by

$$\langle \phi, \psi \rangle_v = \phi \wedge \tilde{\psi}. \quad (2.4)$$

The global inner product $\ll \cdot , \cdot \gg_B$ on $L^2 \Omega_B^r(\mathcal{F})$ is given by

$$\ll \phi, \psi \gg_B = \int_M \langle \phi, \psi \rangle \mu_M,$$  

where $\mu_M = v \wedge \chi_{\mathcal{F}}$ is the volume form with respect to $g_M$. With respect to this scalar product, the formal adjoint operator $d_B : \Omega_B^r(\mathcal{F}) \to \Omega_B^{r+1}(\mathcal{F})$ of $d_B$ is given by

$$d_B \phi = (-1)^{q(r+1)+1} \tilde{\phi} \nabla^* \phi = \delta_T \phi + (-1)^{q(r+1)} \kappa_B \wedge \tilde{\phi},$$

where $d_T = d - \kappa_B$ and $\delta_T = (-1)^{q(r+1)+1} \tilde{\phi} \nabla^* \phi$ is the formal adjoint operator of $d_T$ with respect to this scalar product. Trivially, for any basic form $\phi \in L^2 \Omega_B^r(\mathcal{F})$,

$$i \left( \kappa_B^* \right) \phi = (-1)^{q(r+1)} (\kappa_B \wedge \tilde{\phi}),$$

where $\kappa_B^*$ is a $g_Q$-dual vector to $\kappa$ [8]. The basic Laplacian $\Delta_B$ is given by $\Delta_B = d_B \delta_B + \delta_B d_B$. Let $\{E_a\}(a = 1, \cdots, q)$ be a local orthonormal basic frame on $Q$. We define $\nabla^*_u \nabla^*_r : \Omega_B^r(\mathcal{F}) \to \Omega_B^2(\mathcal{F})$ by

$$\nabla^*_u \nabla^*_r \phi = - \sum_a \nabla^2_{E_a, E_a} \phi + \nabla^*_r \phi, \quad \phi \in \Omega_B^r(\mathcal{F}),$$

where $\nabla^2_{X,Y} = \nabla_X \nabla_Y - \nabla_{\nabla_X Y}$ for any $X, Y \in TM$ and $\nabla^M$ is the Levi–Civita connection on $M$. Then the operator $\nabla^*_u \nabla^*_r$ is positive definite and formally self-adjoint on $L^2 \Omega_B^r(\mathcal{F})$ [3]. We define the bundle map $A_Y : \Lambda^r Q^* \to \Lambda^r Q^*$ for any $Y \in TM$ [12] by

$$A_Y \phi = \theta(Y) \phi - \nabla_Y \phi,$$  

where $\theta(Y)$ is the transverse Lie derivative. For any vector field $X \in T \mathcal{F}$, $\theta(X) \phi = \nabla_X \phi$ [13] and so $A_X \phi = 0$. Now we define the curvature endomorphism $F : \Omega_B^r(\mathcal{F}) \to \Omega_B^2(\mathcal{F})$ by

$$F(\phi) = \sum_{a,b} \theta^a \wedge i(E_b) R^\nabla(E_b, E_a) \phi,$$  

where $R^\nabla$ is the curvature tensor with respect to $\nabla$ and $\theta^a$ is a dual 1-form to $E_a$. Note that if $\phi$ is a basic 1-form, then $F(\phi)^\nabla = \text{Ric}^Q(\phi^\nabla)$, where $\text{Ric}^Q$ is the transverse Ricci curvature of $\mathcal{F}$. Now we recall the generalized Weitzenböck formula.
Theorem 2.1 [3] On a Riemannian foliation $\mathcal{F}$, we have that for any $\phi \in \Omega^r_B(\mathcal{F})$,
\[
\Delta_B \phi = \nabla^*_\mathcal{H} \nabla \phi + F(\phi) + A_{\kappa_B^\sharp} \phi.
\]
Since $\frac{1}{2} \Delta_B |\phi|^2 = \langle \nabla^*_\mathcal{H} \nabla \phi, \phi \rangle - |\nabla \phi|^2$, from Theorem 2.1, we obtain that for any $\phi \in \Omega^r_B(\mathcal{F})$ [6, Corollary 3.2],
\[
\frac{1}{2} \Delta_B |\phi|^2 = \langle \Delta_B \phi, \phi \rangle - \langle F(\phi), \phi \rangle - \langle A_{\kappa_B^\sharp} \phi, \phi \rangle.
\]
(2.11)

Definition 2.2 A basic $r$-form $\phi \in \Omega^r_B(\mathcal{F})$ is called a transverse conformal Killing $r$-form if for any normal vector field $X \in T\mathcal{F}^\perp$,
\[
\nabla_X \phi = \frac{1}{r + 1} i(X) d_B \phi - \frac{1}{r^* + 1} X^b \wedge \delta_T \phi,
\]
where $r^* = q - r$ and $X^b$ is the $g_B$-dual 1-form of $X$. In addition, if the basic $r$-form $\phi$ satisfies $\delta_T \phi = 0$, it is called a transverse Killing $r$-form.

Note that a transverse conformal Killing 1-form (resp. transverse Killing 1-form) is a $g_Q$-dual form of a transverse conformal Killing field (resp. transverse Killing field) [8].

Proposition 2.3 [8] Let $\phi$ be a transverse conformal Killing $r$-form. Then
\[
F(\phi) = \frac{r}{r + 1} \delta_T d_B \phi + \frac{r^*}{r^* + 1} d_B \delta_T \phi,
\]
(2.12)
\[
\nabla^*_\mathcal{H} \nabla \phi = \frac{1}{r + 1} \delta_B d_B \phi + \frac{1}{r^* + 1} d_T \delta_T \phi.
\]
(2.13)

Theorem 2.4 [8] Any basic $r$-form $\phi$ is a transverse conformal Killing $r$-form if and only if $\tilde{\phi}$ is a transverse conformal Killing $(q - r)$-form.

Now we recall the generalized maximum principle on a complete foliated Riemannian manifold, that is, Riemannian foliation with a complete bundle-like metric.

Theorem 2.5 [10] Let $(M, g_M, \mathcal{F})$ be a complete foliated Riemannian manifold, all of whose leaves are compact. Assume that $\kappa_B$ is bounded and coclosed. Then a nonnegative basic function $f$ such that $(\Delta_B - \kappa_B^\sharp) f \leq 0$ with $\int_M f^p < \infty$ (for some $p > 1$) is constant.

3 Vanishing theorems on Riemannian foliations

Now we recall the vanishing theorem on a compact foliated Riemannian manifold.

Theorem 3.1 [8] Let $(M, g_M, \mathcal{F})$ be a compact foliated Riemannian manifold with a foliation $\mathcal{F}$ of codimension $q$ and a bundle-like metric $g_M$ such that $\delta_B \kappa_B = 0$. Suppose $F$ is nonpositive and negative at some point. Then, for any $1 \leq r \leq q - 1$, there are no non-trivial transverse conformal Killing $r$-forms on $M$.

In this section, we study some vanishing theorems of the $L^2$-transverse conformal Killing forms on complete foliated Riemannian manifolds. The basic form $\phi$ is said to be $L^2$-basic form if $\phi \in L^2 \Omega^*_B(\mathcal{F})$, i.e., $\|\phi\|^2_B < \infty$.  

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Let \((M, g_M, \mathcal{F})\) be a complete foliated Riemannian manifold, all of whose leaves are compact. We consider a smooth function \(\mu\) on \(\mathbb{R}\) satisfying

\[
\begin{align*}
&\text{(i) } 0 \leq \mu(t) \leq 1 \text{ on } \mathbb{R}, \\
&\text{(ii) } \mu(t) = 1 \text{ for } t \leq 1, \\
&\text{(iii) } \mu(t) = 0 \text{ for } t \geq 2.
\end{align*}
\]

Now, we fix a point \(x_0 \in M\). For each point \(y \in M\), we denote by \(\rho(y)\) the distance between leaves through \(x_0\) and \(y\). For any real number \(l > 0\), we define a Lipschitz continuous function \(\omega_l\) on \(M\) by

\[
\omega_l(y) = \mu(\rho(y)/l).
\]

Trivially, \(\omega_l\) is a basic function. Let \(B(l) = \{y \in M | \rho(y) \leq l\}\). Then \(0 \leq \omega_l(y) \leq 1\) for any \(y \in M\), \(\text{supp } \omega_l \subset B(2l)\), \(\omega_l(y) = 1\) for any \(y \in B(l)\), \(\lim_{l \to \infty} \omega_l = 1\) and \(|d_B \omega_l| \leq C/l\) almost everywhere on \(M\), where \(C\) is a positive constant independent of \(l \geq 2\). Hence \(\omega_l \psi\) has compact support for any basic form \(\psi \in \Omega^*_B(\mathcal{F})\) and \(\omega_l \psi \to \psi\) (strongly) when \(l \to \infty\).

**Lemma 3.2** [15] For any \(\phi \in \Omega^*_B(\mathcal{F})\), there exists a number \(A\) depending only on \(\mu\), such that

\[
\|d_B \omega_l \wedge \phi\|^2_{B(2l)} \leq \frac{A^2}{l^2} \|\phi\|^2_{B(2l)},
\]

\[
\|d_B \omega_l \otimes \phi\|^2_{B(2l)} \leq \frac{A^2}{l^2} \|\phi\|^2_{B(2l)},
\]

where \(\|\phi\|^2_{B(2l)} = \int_{B(2l)} \langle \phi, \phi \rangle_{\mu_M}\).

**Lemma 3.3** Let \((M, g_M, \mathcal{F})\) be a complete foliated Riemannian manifold, all of whose leaves are compact. Assume that \(\kappa_B\) is bounded and coclosed. Then an \(L^2\)-transverse conformal Killing \(r\)-form satisfies

\[
\frac{1}{r + 1} \lim_{l \to \infty} \ll i \left(\kappa^B_B\right) d_B \phi, \omega_l^2 \phi \gg_{B(2l)} = \frac{1}{r + 1} \lim_{l \to \infty} \ll \kappa_B \wedge \delta_T \phi, \omega_l^2 \phi \gg_{B(2l)}.
\]

**Proof** Let \(\phi\) be an \(L^2\)-transverse conformal Killing \(r\)-form. By a direct calculation, we have

\[
\kappa^B_B(\langle \omega_l \phi \rangle^2) = 2 \left\langle \nabla^B_{\kappa_B} \omega_l \phi, \omega_l \phi \right\rangle = 2 \langle \omega_l \kappa_B, d_B \omega_l \rangle \langle \phi, \omega_l^2 \phi \rangle + 2 \left\langle \nabla^B_{\kappa_B} \phi, \omega_l^2 \phi \right\rangle.
\]

Since \(\delta_B \kappa_B = 0\), \(\int_{B(2l)} \kappa^B_B(\langle \omega_l \phi \rangle^2) = 0\). Hence we have

\[
0 = \int_{B(2l)} \langle \omega_l \kappa_B, d_B \omega_l \rangle \langle \phi, \omega_l^2 \phi \rangle + \int_{B(2l)} \left\langle \nabla^B_{\kappa_B} \phi, \omega_l^2 \phi \right\rangle.
\]

Since \(|d \omega_l| < C/l\) and \(|\kappa_B| < \infty\), \(\lim_{l \to \infty} \int_M \langle \omega_l \kappa_B, d_B \omega_l \rangle \langle \phi, \omega_l^2 \phi \rangle = 0\). Hence

\[
\lim_{l \to \infty} \int_{B(2l)} \left\langle \nabla^B_{\kappa_B} \phi, \omega_l^2 \phi \right\rangle = 0. \quad (3.1)
\]

From Definition 2.2, the proof follows.

**Lemma 3.4** Let \((M, g_M, \mathcal{F})\) be as in Lemma 3.3. If \(\phi\) is an \(L^2\)-basic form, then

\[
\lim_{l \to \infty} \ll \omega_l d_B \phi, d_B \omega_l \wedge \phi \gg_{B(2l)} \geq -A_1 \|d_B \phi\|^2_{B}, \quad (3.2)
\]

\[
\lim_{l \to \infty} \ll \omega_l \delta_T \phi, i (\nabla \omega_l) \phi \gg_{B(2l)} \geq -A_2 \|\delta_T \phi\|^2_{B}, \quad (3.3)
\]

\[
\lim_{l \to \infty} \ll \kappa_B \wedge \delta_T \phi, \omega_l^2 \phi \gg_{B(2l)} \geq -A_3 \|\delta_T \phi\|^2_{B} - \frac{1}{4A_3} \left\| i \left(\kappa^B_B\right) \phi \right\|^2_{B}. \quad (3.4)
\]
for any positive real numbers \(A_1, A_2\) and \(A_3\).

**Proof** Let \(\phi\) be an \(L^2\)-basic form. From Lemma 3.2 and the Schwartz inequality, we have

\[
2 \left| \ll \omega_l d_B \phi, d_B \omega_l \land \phi \gg B(2l) \right| \leq \varepsilon_1 \|\omega_l d_B \phi\|^2_{B(2l)} + \frac{q A^2}{r^2 \varepsilon_1} \|\phi\|^2_{B(2l)}
\]

for a positive real number \(\varepsilon_1\). If we let \(l \to \infty\), then (3.2) is proved. Similarly, by using the equality \(|d_B \omega_l|^2 |\phi|^2 = |i(\nabla \omega_l) \phi|^2 + |d_B \omega_l \land \phi|^2\), we have

\[
2 \left| \ll \omega_l \delta_T \phi, i(\nabla \omega_l) \phi \gg B(2l) \right| \leq \varepsilon_2 \|\omega_l \delta_T \phi\|^2_{B(2l)} + \frac{C^2 - q A^2}{r^2 \varepsilon_2} \|\phi\|^2_{B(2l)}
\]

for a positive real number \(\varepsilon_2\). If we let \(l \to \infty\), then (3.3) is proved. From the Schwartz’s inequality, the proof of (3.4) is trivial. \(\square\)

From Lemma 3.3 and Lemma 3.4, we have the following proposition.

**Proposition 3.5** Let \((M, g_M, F)\) be as in Lemma 3.3. Assume that \(\kappa_B\) is bounded and coclosed. Then

\[
\limsup_{l \to \infty} \ll F(\phi), \omega_l^2 \phi \gg B(2l) \geq \frac{r}{r + 1} \|\omega_l d_B \phi\|^2_{B(2l)} + \frac{r^*}{r^* + 1} \|\omega_l \delta_T \phi\|^2_{B(2l)}
\]

\[
- \frac{q - 2 r}{4 (r^* + 1) A_3} \|i \left(\kappa_B^* \right) \phi\|^2_{B(2l)} + \frac{r^*}{r^* + 1} \ll \omega_l \delta_T \phi, i(\nabla \omega_l) \phi \gg B(2l)
\]

for any positive real numbers \(A_1, A_2\) and \(A_3\).

**Proof** Let \(\phi\) be an \(L^2\)-transverse conformal Killing \(r\)-form. Since \(\delta_T (\omega_l^2 \phi) = \omega_l^2 \delta_T \phi - 2 \omega_l i(\nabla \omega_l) \phi\), from (2.12) we have

\[
\ll F(\phi), \omega_l^2 \phi \gg B(2l) = \frac{r}{r + 1} \|\omega_l d_B \phi\|^2_{B(2l)} + \frac{r^*}{r^* + 1} \|\omega_l \delta_T \phi\|^2_{B(2l)}
\]

\[
+ \frac{2 r}{r + 1} \ll \omega_l d_B \phi, d_B \omega_l \land \phi \gg B(2l) - \frac{2 r^*}{r^* + 1} \ll \omega_l \delta_T \phi, i(\nabla \omega_l) \phi \gg B(2l)
\]

\[
- \frac{r}{r + 1} \ll i \left(\kappa_B^* \right) d_B \phi, \omega_l^2 \phi \gg B(2l) + \frac{r^*}{r^* + 1} \ll \kappa_B \land \delta_T \phi, \omega_l^2 \phi \gg B(2l)
\]

From Lemma 3.3, we have

\[
\limsup_{l \to \infty} \ll F(\phi), \omega_l^2 \phi \gg B(2l) = \frac{r}{r + 1} \|d_B \phi\|^2_{B(2l)} + \frac{r^*}{r^* + 1} \|\delta_T \phi\|^2_{B(2l)}
\]

\[
+ \frac{2 r}{r + 1} \limsup_{l \to \infty} \ll \omega_l d_B \phi, d_B \omega_l \land \phi \gg B(2l)
\]

\[
- \frac{2 r^*}{r^* + 1} \limsup_{l \to \infty} \ll \omega_l \delta_T \phi, i(\nabla \omega_l) \phi \gg B(2l)
\]

\[
+ \frac{q - 2 r}{r^* + 1} \limsup_{l \to \infty} \ll \kappa_B \land \delta_T \phi, \omega_l^2 \phi \gg B(2l)
\]

From Lemma 3.4, the proof is completed. \(\square\)
Hence we have the following theorem.

**Theorem 3.6** Let \((M, g_M, \mathcal{F})\) be as in Lemma 3.3. Assume that \(\kappa_B\) is bounded and coclosed. If the curvature endomorphism \(F\) is nonpositive, then every \(L^2\)-transverse conformal Killing \(r\)-form \((1 \leq r \leq q - 1)\) is parallel.

**Proof** Let \(\phi\) be an \(L^2\)-transverse conformal Killing \(r\)-form.

(i) In case of \(r \geq \frac{q}{2}\). From Proposition 3.5, we have

\[
\limsup_{l \to \infty} \ll F(\phi), \omega^2_l \phi \gg_B(2l) \geq \frac{r}{r+1}(1-2A_1)\|d_B\phi\|^2_B + \frac{1}{r^*+1}\{r^*(1-2A_2) - A_3(q-2r)\}\|\delta_T\phi\|^2_B.
\]

If we choose \(0 < A_1 < \frac{1}{2}\) and \(0 < A_2 < \frac{1}{2}\), then

\[
\limsup_{l \to \infty} \ll F(\phi), \omega^2_l \phi \gg_B(2l) \geq 0. \tag{3.5}
\]

Since \(F\) is nonpositive, from (3.5) we have

\[
\limsup_{l \to \infty} \ll F(\phi), \omega^2_l \phi \gg_B(2l) = 0. \tag{3.6}
\]

Hence from Proposition 3.5, we have \(d_B\phi = 0\), \(\delta_T\phi = 0\) and \(i(\kappa^B_\#)\phi = 0\). Also from (2.13), we have

\[
\nabla^*_T \nabla_T \phi = 0. \tag{3.7}
\]

By multiplying \(\omega^2_l \phi\) and by the Schwarz’s inequality, we have that, for a real number \(\epsilon > 0\),

\[
0 = \ll \nabla^*_T \nabla_T \phi, \omega^2_l \phi \gg_B(2l) = \|\omega_l \nabla_T \phi\|^2_B(2l) + \ll \nabla_T \phi, 2\omega_l d_B \omega_l \otimes \phi \gg_B(2l) \geq (1-\epsilon)\|\omega_l \nabla_T \phi\|^2_B(2l) - \frac{1}{\epsilon}\|d_B \omega_l \otimes \phi\|^2_B(2l).
\]

From Lemma 3.2, we have

\[
0 \geq (1-\epsilon)\|\omega_l \nabla_T \phi\|^2_B(2l) - \frac{q A^2}{l^2 \epsilon}\|\phi\|^2_B(2l).
\]

If we let \(l \to \infty\), then \((1-\epsilon)\|\nabla_T \phi\|^2_B \leq 0\). So if we choose \(\epsilon < 1\), then

\[
\|\nabla_T \phi\|^2_B = 0. \tag{3.8}
\]

Hence \(\nabla_T \phi = 0\), that is, \(\phi\) is parallel.

(ii) In case of \(r \leq \frac{q}{2}\). From Theorem 2.4, \(\bar{\#}\phi\) is also \(L^2\)-transverse conformal Killing \(r^*\)-form. Since \(r^* \geq \frac{q}{2}\), from (i), \(\bar{\#}\phi\) is parallel. Since \(\bar{\nabla}_T \bar{\#}\phi = \bar{\#} \nabla_T \phi\) and \(\bar{\#}\) is an isometry, \(\phi\) is parallel. Hence any \(L^2\)-transverse conformal Killing \(r\)-form is parallel. From (i) and (ii), the proof is completed.

**Corollary 3.7** Let \((M, g_M, \mathcal{F})\) be as in Lemma 3.3. Assume that \(\kappa_B\) is bounded and coclosed. If \(F\) is nonpositive and either negative at some point or \(\text{Vol}(M) = \infty\), then every \(L^2\)-transverse conformal Killing \(r\)-form \((1 \leq r \leq q - 1)\) is trivial.
Proof Let $\phi$ be an $L^2$-transverse conformal Killing form. Since $F$ is nonpositive, from Theorem 3.6, $\phi$ is parallel, and so $F(\phi) = 0$. Hence the negativity of $F$ means that $\phi$ is trivial. Now, we consider Vol$(M) = \infty$. From (2.11) and Theorem 2.5, $|\phi|^2$ is constant. Hence $\int_M |\phi|^2 < \infty$ and Vol$(M) = \infty$ yield that $\phi$ is trivial. 
\[ \Box \]

Remark 3.8 (cf. [4]) Let $(M, g_M, \mathcal{F})$ be as in Lemma 3.3. Assume that $\kappa_B$ is bounded and coclosed. If $F$ is nonpositive and either negative at some point or Vol$(M) = \infty$, then every $L^2$-transverse Killing $r$-form $(1 \leq r \leq q - 1)$ is trivial.

Corollary 3.9 Let $(M, g_M, \mathcal{F})$ be as in Lemma 3.3. Assume that $\kappa_B$ is bounded and coclosed. If the transverse Ricci curvature is nonpositive and either negative at some point or Vol$(M) = \infty$, then every $L^2$-transverse conformal Killing field is trivial.

Remark 3.10 (1) When $\mathcal{F}$ is a foliation by points, Corollary 3.9 was given in [23]. (2) When $\mathcal{F}$ is minimal, Corollary 3.9 was proved in [2] and [18]. So Corollary 3.9 is a generalization of the results in [2, 18] to the non-minimal case.

4 The properties on Kähler foliations

Let $(M, g_M, \mathcal{F}, J)$ be a Riemannian manifold with a Kähler foliation $\mathcal{F}$ of codimension $q = 2m$ and a bundle-like metric $g_M$ [5, 17]. Namely, there is a holonomy invariant almost complex structure $J : Q \rightarrow Q$ with respect to which $g_Q$ is Hermitian, i.e., $g_Q(JX, JY) = g_Q(X, Y)$ for any $X, Y \in Q$ and $\nabla J = 0$. Note that for any $X, Y \in TM$,

$$\Omega(X, Y) = g_Q(\pi(X), J\pi(Y))$$

defines a basic 2-form $\Omega$, which is closed as consequence of $\nabla g_Q = 0$ and $\nabla J = 0$. Now, we define the operators $L : \Omega^r_B(\mathcal{F}) \rightarrow \Omega^{r+2}_B(\mathcal{F})$ and $\Lambda : \Omega^r_B(\mathcal{F}) \rightarrow \Omega^{r-2}_B(\mathcal{F})$ respectively by Jung and Jung [9]

$$L(\phi) = \epsilon(\Omega)\phi, \quad \Lambda(\phi) = i(\Omega)\phi,$$

where $\epsilon(\Omega)\phi = \Omega \wedge \phi$ and $i(\Omega) = -\frac{1}{2} \sum_{a=1}^{2m} i(J E_a)i(E_a)$. Trivially, for any basic forms $\phi \in \Omega^r_B(\mathcal{F})$ and $\psi \in \Omega^{r+2}_B(\mathcal{F})$, $(L(\phi), \psi) = (\phi, \Lambda(\psi))$. Moreover, for any basic $r$-form $\phi$, $[\Lambda, L]\phi = \frac{1}{2}(q - 2r)\phi$.

Also, we define the operator $\tilde{J} : \Omega^r_B(\mathcal{F}) \rightarrow \Omega^r_B(\mathcal{F})$ by

$$\tilde{J}(\phi) = \sum_{a=1}^{2m} J\theta^a \wedge i(E_a)\phi.$$  

For any basic 1-form $\phi$, $J\phi(X) = -\phi(JX)$. Hence $\tilde{J}\phi = J\phi$. So $\tilde{J}$ is an extension of the complex structure $J$ to basic forms. From now on, if we have no confusion, we write $\tilde{J} \equiv J$.

Lemma 4.1 [5] On a Kähler foliation $(\mathcal{F}, J)$, we have

$$[J, L] = [J, \Lambda] = [F, J] = [F, \Lambda] = 0.$$ 

In particular, if $\mathcal{F}$ is minimal, then

$$[J, \Delta_B] = [\Lambda, \Delta_B] = 0.$$
Lemma 4.3 [5] On a Kähler foliation $(\mathcal{F}, J)$, a transverse conformal Killing $r$-form $\phi$ satisfies

$$ F(J\phi) = 0. $$

Now, we recall the operators $d_B^c : \Omega^r_B(\mathcal{F}) \to \Omega^{r+1}_B(\mathcal{F})$ and $\delta^c_B : \Omega^r_B(\mathcal{F}) \to \Omega^{r-1}_B(\mathcal{F})$, which are given by Jung and Jung [9]

$$ d^c_B \phi = \sum_{a=1}^{2m} J\theta^a \wedge \nabla E_a \phi, \quad (4.4) $$

$$ \delta^c_B \phi = -\sum_{a=1}^{2m} i(JE_a)\nabla E_a \phi + i \left( J\kappa^a_B \right) \phi. \quad (4.5) $$

Trivially, $\delta^c_B$ is a formal adjoint of $d^c_B$ and $(\delta^c_B)^2 = (d^c_B)^2 = 0$ [9]. Also, we define two operators $d^c_T$ and $\delta^c_T$ by

$$ d^c_T = d^c_B - \epsilon (J\kappa_B), \quad \delta^c_T = \delta^c_B - i \left( J\kappa^a_B \right). \quad (4.6) $$

If $\mathcal{F}$ is minimal, then $d^c_B = d^c_T$ and $\delta^c_B = \delta^c_T$. Then we have the following lemma.

Lemma 4.4 [5] On a Kähler foliation $(\mathcal{F}, J)$, a transverse conformal Killing $r$-form $\phi$ satisfies

$$ (rr^*-r-2) d^c_B \phi = (r^*+1) d_B J\phi - 2(r+1) \delta_T L\phi, \quad (4.8) $$

$$ (rr^*-r^*-2) \delta^c_T \phi = (r+1) \delta_T J\phi + 2(r^*+1) d_B \Lambda\phi. \quad (4.9) $$

Lemma 4.5 [5] Let $(M, g_M, J, \mathcal{F})$ be a Kähler foliation. Then a transverse conformal Killing $r$-form $\phi$ satisfies

$$ a_1 \delta_T d_B J\phi + a_2 d_B \delta_T J\phi + a_3 \left( \kappa^a_B \right) \Lambda \phi = 0, $$

where

$$ a_1 = (r^*+1)(rr^*-r^*-r)(rr^*-r^*-2), \quad a_2 = (r^*+1)(rr^*-r^*-r)(rr^*-r^*-2), \quad a_3 = 2(r-r^*)(rr^*-r^*-2). $$

Note that $rr^*-r^*-r = 0$ for some $1 \leq r \leq q-1$ if and only if $q = 4$. Hence we have the following.

Lemma 4.6 [5] Let $(M, g_M, \mathcal{F})$ be a minimal Kähler foliation of codimension $q(\neq 4)$. Then for a transverse conformal Killing $r$-form $2 \leq r \leq q - 2 \phi$, $b_1 b_2 b_3 \delta_B d_B J\Lambda\phi = (1 - b_1 b_3) \delta_B d^c_B \Lambda\phi + b_3 (1 - b_1) \delta_B d_B \Lambda\phi, \quad (4.10) $

$$ b_2 d_B \delta_B J\Lambda\phi + b_2 (1 - b_1) \delta_B d_B J\Lambda\phi = (b_1 b_2 - 1) \delta_B d_B \Lambda\phi + b_2 (b_1 - 1) \delta_B d^c_B \Lambda\phi, \quad (4.11) $$

where

$$ b_1 = \frac{r^*(r+1)}{r^* - r^* - 2}, \quad b_2 = \frac{r^* + 1}{(r^*+1)(r^*-1)} \quad \text{and} \quad b_3 = \frac{r^* + 1}{(r^*+1)(r^*-1)}. $$

From now on, let $(M, g_M, \mathcal{F}, J)$ be a complete Kähler foliation, i.e., a Kähler foliation with a complete bundle-like metric.
**Theorem 4.7** Let \((M, g_M, \mathcal{F}, J)\) be a complete Kähler foliation of codimension \(q = 2m\), all of whose leaves are compact. Assume that \(\kappa_B\) is bounded and coclosed. Let \(\phi \in \Omega^m_B(\mathcal{F})\) be an \(L^2\)-transverse conformal Killing \(\frac{q}{2}\)-form. If \(q \neq 4\), then \(J\phi \in \Omega^m_B(\mathcal{F})\) is parallel. In addition, if \(q = 4\) and \(\mathcal{F}\) is minimal, then \(J\phi \in \Omega^2_B(\mathcal{F})\) is parallel.

**Proof** The proof is a process similar to the one in [5, Theorem 5.6]. Let \(\phi \in \Omega^1_B(\mathcal{F})\) be an \(L^2\)-transverse conformal Killing \(m\)-form. By Lemma 4.5, \(a_1 = a_2 = m(m+1)^3(m-2)^2\) and \(a_3 = 0\).

(i) In the case of \(m \neq 2\), i.e., \(q \neq 4\), we have \(\phi \neq 0\). That is,

\[ d_B\delta_T J\phi + \delta_T d_B J\phi = 0. \]

Equivalently, we have

\[ \Delta_B J\phi = \theta \left( \kappa_B^\phi \right) J\phi. \]

(4.12)

Hence, by the scalar Weitzenböck formula (2.11) and Proposition 4.2, we have

\[ \frac{1}{2} \left( \Delta_B - \kappa_B^\phi \right) |J\phi|^2 = -|\nabla_{\nabla} J\phi|^2 \leq 0. \]

(4.13)

From the generalized maximum principle (Theorem 2.5), \(|J\phi|\) is constant. Again, from (4.13), we have

\[ \nabla_{\nabla} J\phi = 0, \]

(4.14)

which implies that \(J\phi\) is parallel.

(ii) In the case of \(m = 2\), i.e., \(q = 4\), from Lemma 4.4, we have

\[ d_B J\phi = 2\delta_T L\phi, \quad \delta_T J\phi = -2d_B \Lambda\phi. \]

(4.15)

From (4.15), we have

\[ \delta_B d_B J\phi = 2\delta_B \delta_T L\phi, \quad d_B \delta_T J\phi = 0. \]

Hence if \(\mathcal{F}\) is minimal, then \(\Delta_B J\phi = 0\). From Theorem 2.1 and Proposition 4.2, we have

\[ \nabla^\nabla \nabla_{\nabla} J\phi = 0. \]

(4.16)

By multiplying (4.16) by \(\omega^2 \phi\) and by integrating, we have

\[ \|\omega\nabla_{\nabla} J\phi\|_{B(2\mathcal{F})}^2 + 2 \ll \omega\nabla_{\nabla} J\phi, d_B \omega\phi \otimes J\phi \gg B(2\mathcal{F}) = 0. \]

(4.17)

By the Schwarz inequality and Lemma 3.2, we have

\[ \lim_{l \to \infty} \ll \omega l \nabla_{\nabla} J\phi, d_B \omega l \otimes J\phi \gg B(2\mathcal{F}) = 0. \]

Hence from (4.17), we have \(\|\nabla_{\nabla} J\phi\|_{B}^2 = 0\), i.e., \(J\phi\) is parallel. Consequently, from (i) and (ii), the proof is completed.

**Corollary 4.8** Let \((M, g_M, \mathcal{F}, J)\) be as in Theorem 4.7, and suppose that \(\mathcal{F}\) is minimal. Then for any \(L^2\)-transverse conformal Killing \(\frac{q}{2}\)-form, \(J\phi\) is parallel.
Lemma 4.9  Let \((M, g_M, \mathcal{F}, J)\) be as in Theorem 4.7, suppose that \(\mathcal{F}\) is minimal. Then for any \(L^2\)-basic form \(\phi\),
\[
\begin{align*}
\lim_{l \to \infty} & \ll \omega_l^2 \delta_B d_B J \lambda \phi, \delta_B^c d_B \lambda \phi \gg \gg_B(2l) = 0, \quad (4.18) \\
\lim_{l \to \infty} & \ll \omega_l^2 \delta_B d_B^c \lambda \phi, \delta_B^c d_B \lambda \phi \gg \gg_B(2l) = 0, \quad (4.19) \\
\lim_{l \to \infty} & \ll \omega_l^2 \delta_B \{d_B J \lambda \phi + d_B^c \lambda \phi\}, d_B \delta_B J \Lambda \phi \gg \gg_B(2l) = 0. \quad (4.20)
\end{align*}
\]

Proof  Note that for any \(\phi \in L^2 \mathcal{O}'_0(\mathcal{F})\), we have
\[
\begin{align*}
\omega_l^2 \delta_B d_B J \lambda \phi &= \delta_B \left[ d_B \left( \omega_l^2 J \lambda \phi \right) - 2 \omega_l |d_B \omega_l| \wedge J \lambda \phi \right] + 2 \omega_l i \left( \nabla \omega_l \right) d_B J \lambda \phi, \quad (4.21) \\
\omega_l^2 \delta_B d_B^c \lambda \phi &= \delta_B \left[ d_B^c \left( \omega_l^2 \lambda \phi \right) - 2 \omega_l J (d_B \omega_l) \wedge \lambda \phi \right] + 2 \omega_l i \left( \nabla \omega_l \right) d_B^c \lambda \phi. \quad (4.22)
\end{align*}
\]

Since \(\mathcal{F}\) is minimal, from Lemma 4.3 and (4.21), we have
\[
\ll \omega_l^2 \delta_B d_B J \lambda \phi, \delta_B^c d_B \lambda \phi \gg \gg_B(2l) = -2 \ll \omega_l d_B J \lambda \phi, d_B \lambda \phi \wedge d_B^c \lambda \phi \gg \gg_B(2l) . \quad (4.23)
\]

From (4.23), by using the Schwarz inequality and Lemma 3.2, we have
\[
\left| \ll \omega_l^2 \delta_B d_B J \lambda \phi, d_B \delta_B^c \lambda \phi \gg \gg_B(2l) \right| \leq \epsilon \left\| \omega_l d_B J \lambda \phi \right\|^2_B(2l) + \frac{1}{\epsilon} \left\| d_B \omega_l \right\|^2_B + \frac{q A^2}{\epsilon l^2} \left\| d_B \delta_B^c \lambda \phi \right\|^2_B(2l)
\]
for any positive real number \(\epsilon\). If we let \(l \to \infty\), then
\[
\lim_{l \to \infty} \left| \ll \omega_l^2 \delta_B d_B J \lambda \phi, \delta_B^c d_B \lambda \phi \gg \gg_B(2l) \right| \leq \epsilon \left\| d_B J \lambda \phi \right\|^2_B.
\]

Since \(\epsilon\) is arbitrary, the proof of (4.18) is completed. By using (4.22), the proof of (4.19) is similarly completed. Similarly, the proof of (4.20) follows. \(\square\)

Theorem 4.10  Let \((M, g_M, \mathcal{F}, J)\) be as in Theorem 4.7, and suppose that \(\mathcal{F}\) is minimal. Then for any \(L^2\)-transverse conformal Killing \(r\)-form \((2 \leq r \leq q - 2)\), \(J \Lambda \phi\) is basic-harmonic.

Proof  (i) In the case of \(q \neq 4\). From (4.10), we have
\[
\begin{align*}
b_1 b_3 & \ll \omega_l^2 \delta_B d_B J \lambda \phi, \delta_B^c d_B \lambda \phi \gg \gg_B(2l) = (1 - b_1 b_3) \ll \omega_l^2 \delta_B d_B^c \Lambda \phi, \delta_B^c d_B \Lambda \phi \gg \gg_B(2l) \\
& \quad + b_3 (1 - b_1) \left\| \omega_l \delta_B^c d_B \Lambda \phi \right\|^2_B(2l). \quad (4.24)
\end{align*}
\]
Since \(b_1 \neq 1\) and \(b_3 \neq 0\), from Lemma 4.9, if we let \(l \to \infty\), then
\[
\left\| \delta_B^c d_B \Lambda \phi \right\|^2_B = 0, \quad \text{i.e., } \delta_B^c d_B \Lambda \phi = 0. \quad (4.25)
\]
Hence from (4.10), we have
\[
b_2 d_B \delta_B J \Lambda \phi = b_2 (b_1 - 1) \delta_B \left\{ d_B J \Lambda \phi + d_B^c \Lambda \phi \right\}. \quad (4.26)
\]
By multiplying \(\omega_l^2 d_B \delta_B J \phi\) in (4.26) and by integrating, we have
\[
b_2 \left\| \omega_l d_B \delta_B J \Lambda \phi \right\|^2_B = b_2 (b_1 - 1) \ll \omega_l^2 \delta_B \left\{ d_B J \Lambda \phi + d_B^c \Lambda \phi \right\}, d_B \delta_B J \Lambda \phi \gg \gg_B(2l) .
\]
If we let \(l \to \infty\), then from (4.20),
\[
\left\| d_B \delta_B J \Lambda \phi \right\|^2_B = 0, \quad \text{i.e., } d_B \delta_B J \Lambda \phi = 0. \quad (4.27)
\]
Again from Lemma 4.6, since $b_2(1 - b_1) \neq 0$, we have
\[ b_1b_3\delta_Bd_B\Lambda = (1 - b_2d_2)\delta_Bd_B\Lambda \quad \text{(4.28)} \]
\[ \delta_Bd_B\Lambda = -\delta_Bd_B\Lambda. \quad \text{(4.29)} \]
Since $b_1b_3 \neq b_2b_2 - 1$, from (4.27) and (4.28), we have
\[ \delta_Bd_B\Lambda = 0 \quad \text{and} \quad \delta_Bd_B\Lambda = 0. \quad \text{(4.30)} \]
From (4.27) and (4.30), we have
\[ \Delta_B\Lambda = 0. \quad \text{(4.31)} \]
That is, $\Lambda$ is basic-harmonic. (ii) In the case of $q = 4$, since $\mathcal{F}$ is minimal, from Theorem 4.7, $\phi \in \Omega_B^1(\mathcal{F})$ is parallel and so basic-harmonic, i.e., $\Delta_B\phi = 0$. Hence from Lemma 4.1, $\Delta_B\Lambda = 0$, i.e., $\Lambda$ is a basic-harmonic 2-form. Hence from (i) and (ii), the proof is completed.

**Corollary 4.11** Let $(M, g_M, \mathcal{F}, J)$ be as in Theorem 4.7, and suppose that $\mathcal{F}$ is minimal. Then for any $L^2$-transverse conformal Killing $r$-form $(2 \leq r \leq q - 2)$, $\Lambda$ is parallel.

**Proof** From Lemma 4.1 and Proposition 4.2, we have $F(J\Lambda) = \Lambda F(J\phi) = 0$. Hence by the generalized Weitzenböck formula (Theorem 2.1) and Theorem 4.10, we have
\[ \nabla^*_{t_1}\nabla_{t_2}\Lambda = 0. \quad \text{(4.32)} \]
By multiplying $\omega_i^2\Lambda$ in (4.32) and by integrating, we have
\[ \|\omega_i\nabla_{t_1}\Lambda\|^2_{B(2l)} + 2 \ll \omega_i\nabla_{t_1}\Lambda \frac{d_B\omega_i \otimes J\Lambda \phi}{\mathcal{B}(2l)} = 0. \quad \text{(4.33)} \]
By the Schwarz inequality and Lemma 3.3, we have
\[ \lim_{l \to \infty} \ll \omega_i \nabla_{t_1}\Lambda \frac{d_B\omega_i \otimes J\Lambda \phi}{\mathcal{B}(2l)} = 0. \quad \text{(4.34)} \]
Hence from (4.33) and (4.34), if we let $l \to \infty$, then
\[ \nabla_{t_1}\Lambda = 0. \quad \text{(4.35)} \]
That is, $\Lambda$ is parallel. \[\Box \]

**Theorem 4.12** Let $(M, g_M, J, \mathcal{F})$ be as in Theorem 4.7, and suppose that $\mathcal{F}$ is minimal. Then for any $L^2$-transverse conformal Killing $r$-form $(2 \leq r \leq q - 2)$, $\phi$ is parallel.

**Proof** The proof is a process similar to the one in [5, Theorem 5.12]. That is, let $\phi$ be a $L^2$-transverse conformal Killing $r$-form. Then $\bar{\phi}$ is also a $L^2$-transverse conformal Killing $(q-r)$-form. Hence by Corollary 4.11, $\Lambda$ is parallel. Since $[\nabla_{t_1}, \bar{\phi}] = 0$, $[J, \bar{\phi}] = 0$ and $L^\Lambda = \bar{\phi}\Lambda$, it follows that $\bar{\phi}J\Lambda \phi = \pm LJ \phi$ is parallel. Note that $(m-r)J \phi = [\Lambda, L]J \phi$. Since $[L, \nabla_{t_1}] = [\Lambda, \nabla_{t_1}] = 0$ and $J\Lambda \phi$ is parallel, $(m-r)\nabla_{t_1}\phi = 0$. So if $r \neq m$, then $\phi$ is parallel. For $r = m$, from Corollary 4.8, $\phi$ is parallel. So the proof is completed. \[\Box \]

For the point foliation, we have the following corollary for Kähler manifolds, which seems to be new as far as we know.

**Corollary 4.13** Let $(M, g_M, J)$ be a complete Kähler manifold of dimension $n = 2m$. Then for any $L^2$-conformal Killing $r$-form $(2 \leq r \leq n - 2)$, $\phi$ is parallel.
Acknowledgements The authors would like to thank the referee for the valuable suggestions and the comments. The first author was supported by the National Research Foundation of Korea (NRF) Grant funded by the Korea government (MSIP) (NRF-2015R1A2A2A01003491) and the second author was supported by NSFC (No. 11371080).

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