Möbius invariant metrics on the space of knots

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Abstract
We give a condition for a function to produce a Möbius invariant weighted inner product on the tangent space of the space of knots, and show that some kind of Möbius invariant knot energies can produce Möbius invariant and parametrization invariant weighted inner products. They would give a natural way to study the evolution of knots in the framework of Möbius geometry.

Keywords Möbius energy · Knot space · Regularized potential

Mathematics Subject Classification (2000) 57M25 · 53A30 · 57M25

1 Introduction
The motivation of endowing a Möbius invariant metric to the space of knots originated from the study of the energy $E$ of knots (16) which was given in [15]. The purpose of introducing the energy is to produce a representative configuration for each knot type as an energy minimizer in the knot type. It turns out to be the beginning of so-called geometric knot theory. Freedman et al. [7] gave the gradient of $E$. The regularity of $E$-critical knots has been studied by He [9] and later by researchers of analysis including Simon Blatt, Philipp Reiter, Armin Schikorra, Aya Ishizeki, Takeyuki Nagasawa, Alexandra Gilsbach, Heiko von der Mosel, and Nicole Vorderobermeier [2–4,8,10,11,18], where the gradient is the first step of the study.

One of the important properties of the energy $E$ is the Möbius invariance, which is the reason why $E + 4$ (17) is sometimes called “Möbius energy”. It was proved by Freedman et al. [7], and using it they showed that there is an energy minimizer in each prime knot type.

Although the energy $E$ is Möbius invariant, the gradient is not. Freedman, He and Wang made a comment “Oded Schramm observed that if $\gamma$ is not already a round circle, there exists a Möbius-invariant inner product $\langle \cdot, \cdot \rangle$ on the tangent bundle of $\mathbb{R}^3$ restricted to $\gamma$”
In this paper we give a necessary and sufficient condition for a weight function to produce a Möbius invariant inner product on the tangent spaces of the space of knots which are not round circles. In particular, we will be concerned with a weight function that is independent of the parametrization of knots since we are interested in the shapes of knots i.e. the images of the embeddings. Our first example of a weight function is the cube of the regularization of $r^{-2}$-potential, which is the integrand of the energy $E$. The construction can be generalized to obtain weight functions from Möbius invariant 2-forms that are integrands of Möbius invariant energies of knots.

Our method would provide a natural way to study the evolution of knots with respect to a Möbius invariant functional in the quotient space of the shapes of knots modulo the Möbius group action.

2 Möbius invariance condition for weight functions

We use the following notation in this paper. Let $S^1$ be $\mathbb{R}/\mathbb{Z}, f: S^1 \hookrightarrow \mathbb{R}^3$ a knot, $(\cdot, \cdot)$ the standard inner product, and $T$ a Möbius transformation of $\mathbb{R}^3 \cup \{\infty\}$. Let $\mathcal{K}$ be the space of knots and $T_f \mathcal{K}$ the tangent space, $\mathcal{K} = \{f: S^1 \hookrightarrow \mathbb{R}^3 : f \text{ is injective and } |f'(t)| \neq 0 \ (\forall t)\}$, $T_f \mathcal{K} = \{u: S^1 \hookrightarrow \mathbb{R}^3 : u(t) \perp f'(t) \ (\forall t \in S^1)\}$.

We will choose a suitable differentiability for $f$ in $\mathcal{K}$ and $u$ in $T_f \mathcal{K}$ according to the functional we study. For example, if we study the energy $E$ then we can use a Sobolev space $H^3(\mathbb{R}/\mathbb{Z}, \mathbb{R}^3)$ after [2] so that both $E$ and the gradient of $E$ are well-defined and a solution of the Euler–Lagrange equation exists.

The $L^2$ metric on $T_f \mathcal{K}$ is given by

$$(u, v)_f = \int_{S^1} (u(t), v(t)) |f'(t)| dt \quad (u, v \in T_f \mathcal{K}).$$

We say that a functional $e$ on $\mathcal{K}$ has a gradient $G_e$ if for any $f \in \mathcal{K}, G_e(f) \in T_f \mathcal{K}$ satisfies

$$\frac{d}{d\varepsilon} e(f + \varepsilon u) \bigg|_{\varepsilon=0} = (u, G_e(f))_f$$

for any $u \in T_f \mathcal{K}$.

We say that a functional $e$ on $\mathcal{K}$ is Möbius invariant if $e(T \circ f) = e(f)$ holds for any $f \in \mathcal{K}$ and for any Möbius transformation $T$ such that $T(f(S^1))$ is compact.

**Proposition 1** Suppose a functional $e$ on $\mathcal{K}$ is Möbius invariant and has a gradient $G_e$ in $T_f \mathcal{K}$. Then round circles are critical with respect to $e$, namely $G_e$ vanishes at round circles.

**Proof** Let $f_0: S^1 \rightarrow \mathbb{R}^2 \subset \mathbb{R}^3$ be a unit circle with center the origin and let $u \in T_{f_0} \mathcal{K}$. Put $f_{\varepsilon} = f_0 + \varepsilon u$ for small $|\varepsilon|$. Let $I$ be an inversion in a unit sphere with center the origin and let $R$ be a reflection in $\mathbb{R}^2 \subset \mathbb{R}^3$. Put $f_{\tilde{\varepsilon}} = R \circ I \circ f_{\varepsilon}$. We first show

$$\frac{\partial}{\partial \varepsilon} f_{\varepsilon}(t) \bigg|_{\varepsilon=0} = -\frac{\partial}{\partial \varepsilon} f_{\tilde{\varepsilon}}(t) \bigg|_{\varepsilon=0} \quad (\forall t \in S^1).$$

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Let $\Pi_t$ be a plane through the origin and $f_0(t)$ which is perpendicular to $f'_0(t)$. It contains both $f_ε(t)$ and $\tilde{f}_ε(t)$. Let $(x(ε), z(ε))$ and $(\tilde{x}(ε), \tilde{z}(ε))$ be the coordinates of $f_ε(t)$ and $\tilde{f}_ε(t)$ in $\Pi_t$ with respect to the axes given by $0f_0(t)$ and the $z$-axis. Then

$$(\tilde{x}(ε), \tilde{z}(ε)) = \left(\frac{x(ε)}{x(ε)^2 + z(ε)^2}, -\frac{z(ε)}{x(ε)^2 + z(ε)^2}\right),$$

and therefore, substituting $x(0) = 1$ and $z(0) = 0$ we obtain $(\tilde{x}'(0), \tilde{z}'(0)) = -(x'(0), z'(0))$, which means (3).

As a result, using $\tilde{f}_0 = f_0$ we have

$$\frac{d}{dε} e(\tilde{f}_ε) \bigg|_{ε=0} = \int_{S^1} \left( \frac{∂}{∂ε} \tilde{f}_ε(t) \bigg|_{ε=0}, G_e(\tilde{f}_0)(t) \right) |\tilde{f}_0'(t)| \ dt$$

$$= \int_{S^1} \left( -\frac{∂}{∂ε} f_ε(t) \bigg|_{ε=0}, G_e(f_0)(t) \right) |f_0'(t)| \ dt$$

$$= -\frac{d}{dε} e(f_ε) \bigg|_{ε=0}.$$

Since $e(\tilde{f}_ε) = e(f_ε)$ by the Möbius invariance of $e$, the above implies

$$\frac{d}{dε} e(f_ε) \bigg|_{ε=0} = 0,$$

hence

$$0 = \frac{d}{dε} e(f_0 + εu) \bigg|_{ε=0} = \int_{S^1} (u(t), G_e(f_0)(t)) |f_0'(t)| \ dt \ (∀u ∈ T_{f_0}K),$$

which implies $G_e(f_0) \equiv 0$.  

Even if a functional is Möbius invariant, the gradient is not, namely, $T_s G_e(f) \neq G_e(T_0 f)$. This is because

$$(T_s u, T_s v)_{T_0 f} \neq (u, v)_f$$

in general. For example, when $T$ is a homothety by a factor $k > 0$ we have $(T_s u, T_s v)_{T_0 f} = k^3(u, v)_f$.

Let us look for a weighted inner product that is compatible with Möbius transformations and that is independent of the parametrization of knots. Proposition 1 implies that we can restrict ourselves to the weight functions defined on $K$ which implies $Ge$. Circular knots. Our weight function $\Phi$ should satisfy the following conditions:

(i) $\Phi$ is positive on $K^o \times S^1$, and $\Phi(f, t)$ is continuous in $t ∈ S^1$ for each $f ∈ K^o$.

(ii) $\Phi$ is parametrization independent, i.e., for any diffeomorphism $ρ$ of $S^1$, $\Phi(f \circ ρ, t) = \Phi(f, ρ(t))$ for any $t ∈ S^1$.

(iii) The $\Phi$-weighted inner product

$$\langle u, v \rangle_\phi = \int_{S^1} (u(t), v(t)) \Phi(f, t) |f'(t)| \ dt \ (u, v ∈ T_fK)$$

is Möbius invariant, i.e.,

$$\langle T_s u, T_s v \rangle_\phi = \langle u, v \rangle_\phi \ (∀u, v ∈ T_fK).$$
We recall some basics of Möbius geometry. Let $T$ be a Möbius transformation of $\mathbb{R}^3 \cup \{\infty\}$. For a point $p$ in $\mathbb{R}^3$ put
\[
|T'(p)| = |\text{det } DT(p)|^{1/6},
\]
where $DT$ is the Jacobian matrix of $T$. In particular, if $T$ is a homothety by $k > 0$ then $|T'(p)| = \sqrt{k}$, and if $T$ is an inversion in a sphere of radius $r$ with center $C$ then
\[
|T'(p)| = \frac{r}{|C - p|}.
\]
Note that we have
\[
|T(p) - T(q)| = |T'(p)| \left| T'(q) \right| |p - q| \quad (p, q \in \mathbb{R}^3),
\]
\[
|(T \circ \gamma)'(t)| = \left| T'(\gamma(t)) \right|^2 |\gamma'(t)| \quad (\gamma : J \to \mathbb{R}^3, J \text{ is an interval}).
\]
We first investigate without assuming the parametrization independence of $\Phi$.

**Theorem 1**
1. Suppose $\Phi : K^0 \times S^1 \to \mathbb{R}$ satisfies the condition (i) above. Then the $\Phi$-weighted inner product (4) is Möbius invariant i.e. (5) holds if and only if $\Phi$ satisfies
\[
\Phi(T \circ f, t) = |T'(f(t))|^{-6} \Phi(f, t)
\]
for any $f$ in $K$, $t$ in $S^1$, and for any Möbius transformation $T$ such that $T(f(S^1))$ is compact.
2. Suppose $\Phi$ satisfies the condition (i) and (9). Suppose a functional $e$ on $K$ is Möbius invariant and has a gradient $G_e$ in $T_f K$. Define the $\Phi^{-1}$-weighted gradient $G_e^\Phi(f)$ in $T_f K$ by $G_e^\Phi(f)(t) = (\Phi(f, t))^{-1} G_e(f)(t)$ for a non-circular knot $f$ in $K^0$ and $G_e^\Phi(f_0) = 0$ for a round circle $f_0$. Then for any $f$ in $K$
\[
\frac{d}{d\varepsilon} e(f + \varepsilon u) \big|_{\varepsilon = 0} = \{u, G_e^\Phi(f)\}_\Phi \quad (u \in T_f K).
\]
Namely, $G_e^\Phi$ is a Möbius invariant gradient of $e$ with respect to $\Phi$-weighted inner product.

**Proof**
1. The formula (8) implies
\[
(T_u u(t), T_v v(t)) = \frac{1}{2} 
\left[ \left| T_u (u + v)(t) \right|^2 - \left| T_u u(t) \right|^2 - \left| T_v v(t) \right|^2 \right] 
\]
\[
= \frac{1}{2} \left| T'(f(t)) \right|^4 \left[ \left| (u + v)(t) \right|^2 - \left| u(t) \right|^2 - \left| v(t) \right|^2 \right] 
\]
\[
= \left| T'(f(t)) \right|^4 (u(t), v(t)),
\]
\[
\left| (T \circ f)'(t) \right| = \left| T'(f(t)) \right|^2 \left| f'(t) \right|.
\]
It follows that
\[
\langle T_u u, T_v v \rangle_\Phi = \int_{S^1} \left( T_u u(t), T_v v(t) \right) \Phi(T \circ f, t) \left| (T \circ f)'(t) \right| dt
\]
\[
= \int_{S^1} (u(t), v(t)) \left| T'(f(t)) \right|^6 \Phi(T \circ f, t) \left| f'(t) \right| dt.
\]
Therefore, $\langle T_u u, T_v v \rangle_\Phi = \langle u, v \rangle_\Phi$ for any $u$ and $v$ in $T_f K^0$ if and only if $\left| T'(f(t)) \right|^6 \Phi(T \circ f, t) = \Phi(f, t)$ for all $t$. 

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2. The equality (10) follows from (2), (1), (4), and the definition of $G_e^\Phi$. The equality (11) follows from the uniqueness of the gradient as follows. First note that since a Möbius transformation $T$ is conformal, $T^*(G_e^\Phi(f)(t))$ is perpendicular to $T^*f'(t)$, and hence $T^*(G_e^\Phi(f)) \in T_{T^*f}K$. By Proposition 1 we may assume that $f$ and $T \circ f$ are not round circles. Let $u \in T_fK$. The equality (10) and the Möbius invariance of $\Phi$-weighted inner product imply

$$\left. \frac{d}{d\varepsilon} e(f + \varepsilon u) \right|_{\varepsilon=0} = \langle u, G_e^\Phi(f) \rangle_\Phi = \left. \langle T_*u, T_*(G_e^\Phi(f)) \rangle_\Phi \right|_{\varepsilon=0}.$$  (12)

On the other hand, since $e(f + \varepsilon u) = e(T \circ (f + \varepsilon u))$ by the Möbius invariance of $e$ and since

$$\left. \frac{d}{d\varepsilon} (T \circ (f + \varepsilon u))(t) \right|_{\varepsilon=0} = T_*u(t),$$

we have

$$\left. \frac{d}{d\varepsilon} e(f + \varepsilon u) \right|_{\varepsilon=0} = \left. \frac{d}{d\varepsilon} e(T \circ (f + \varepsilon u)) \right|_{\varepsilon=0}
= \left. \frac{d}{d\varepsilon} e(T \circ f + \varepsilon T_*u) \right|_{\varepsilon=0}
= \langle T_*u, G_e^\Phi(T \circ f) \rangle_\Phi,$$  (13)

where the last equality follows from (10). As $u$ is arbitrary, from (12) and (13) we have $G_e^\Phi(T \circ f) = T_*(G_e^\Phi(f))$. \hfill \Box

Remark 1 The Eq. (8) implies that if we put $\Phi_0(f, t) = |f'(t)|^{-3}$ then it satisfies the condition (9), hence

$$\langle u, v \rangle_{\Phi_0} = \int_{S^1} \frac{(u(t), v(t))}{|f'(t)|^2} dt$$

is Möbius invariant, although $\Phi_0$ is not parametrization independent.

Suppose $\Phi : K^\circ \times S^1 \to \mathbb{R}$ satisfies the condition (i) and (9). Then a functional $E_{\Phi^{1/3}}$ on $K^\circ$ given by

$$E_{\Phi^{1/3}}(f) = \int_{S^1} 3/\Phi(f, t) |f'(t)| dt$$

is Möbius invariant. From a viewpoint of geometric knot theory, it would be desirable if $\Phi$ is parametrization independent, $\Phi$ can be continuously extended to a non-negative function on $K \times S^1$, and the global minimum of $E_{\Phi^{1/3}}$ is given by round circles. The examples we introduce in the following two sections enjoy the above properties.

3 Weight function given by a potential of the Möbius energy

We give examples of $\Phi$ that satisfy the conditions (i), (ii) and (iii) in what follows. First example is the simplest one, and it seems the most appropriate to be applied to the study the energy $E$. 

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Definition 1 ([15]) Define the regularized $r^{-2}$-potential of a knot $f \in \mathcal{K}$ at a point $f(s)$ by

$$V(f, s) = \lim_{\varepsilon \downarrow 0} \left( \int_{d_f(s, t) \geq \varepsilon} \frac{|f'(t)| dt}{|f(t) - f(s)|^2} - \frac{2}{\varepsilon} \right),$$

where $d_f(s, t)$ is the arc-length between $f(s)$ and $f(t)$ along the knot $f(S^1)$.

Obviously it is parametrization independent, i.e., $V(f \circ \rho, s) = V(f, \rho(s))$ for any diffeomorphism $\rho$ of $S^1$.

We introduce so-called “wasted length argument” by Doyle and Schramm reported in [12] (with a tiny modification), which is an epochal observation in the Möbius geometric study of knots. Let $I$ be an inversion in a unit sphere with center $f(s)$. Let $\tilde{f}_s = I \circ f$ be an inverted open knot. Then (8) implies that the integrand of $V(f, s)$,

$$\frac{|f'(t)| dt}{|f(t) - f(s)|^2}$$

is a length element of the inverted open knot. This observation implies

$$V(f, s) = \lim_{s_1 \to s, s_2 \to s} \left( d_{\tilde{f}_s}(s_1, s_2) - |\tilde{f}_s(s_1) - \tilde{f}_s(s_2)| \right),$$

where $f(s_1)$ and $f(s_2)$ approach $f(s)$ from opposite sides. In fact, when $d_f(s, s_1) = d_f(s, s_2) = \varepsilon$ the above equality follows from the fact that $|\tilde{f}_s(s_1) - \tilde{f}_s(s_2)| = 2/\varepsilon + O(\varepsilon)$.

Remark $d_{\tilde{f}_s}(s_1, s_2)$ is the arc-length of $\tilde{f}_s(S^1 \setminus (s_1, s_2))$. Thus $V(f, s)$ can be interpreted as the difference of the lengths of the open knot inverted at $f(s)$ and the asymptotic line. Therefore, $V(f, s)$ is non-negative and is equal to 0 if and only if the image of $\tilde{f}_s$ is a line, which occurs if and only if $f(S^1)$ is a round circle.

This argument yields the cosine formula of $V$;

$$V(f, s) = \int_{S^1} \frac{1 - \cos \theta_f(s, t)}{|f(s) - f(t)|^2} |f'(t)| dt,$$

where $\theta_f(s, t)$ ($0 \leq \theta_f \leq \pi$) is the angle between two circles, both through $f(s)$ and $f(t)$, one is tangent to the knot at $f(s)$ ad the other at $f(t)$. We call $\theta_f$ the conformal angle. Since $\theta_f$ is defined by using only circles, tangency and angle, it is Möbius invariant.

The conformal angle $\theta_f$ is of the order of $|s - t|^2$ near the diagonal set $\Delta = \{(t, t) : t \in S^1\}$ [13]. Therefore, the integrand of (15) can be extended to a continuous function on $S^1 \times S^1$, which implies that $V(f, s)$ is continuous in $s$.

Summarizing it up;

Proposition 2 The potential $V(f, s)$ is continuous in $s$, and $V(f, s) \geq 0$ for any $f$ in $\mathcal{K}$ and for any $s$ in $S^1$, where the equality holds if and only if $f(S^1)$ is a round circle.

The energy $E$ [15] is given by

$$E(f) = \int_{S^1} V(f, s) |f'(s)| ds$$

$$= \lim_{\varepsilon \downarrow 0} \left( \int_{S^1 \times S^1} \frac{|f'(s)||f'(t)| ds dt}{|f(t) - f(s)|^2} - \frac{2L(f(S^1))}{\varepsilon} \right),$$

where $L(f(S^1))$ is the length of the knot. Then

$$E(f) + 4 = \int_{S^1 \times S^1} \left( \frac{1}{|f(t) - f(s)|^2} - \frac{1}{d_f(s, t)^2} \right) |f'(s)||f'(t)| ds dt \quad (17)$$
The right hand side is called the Möbius energy of a knot $f$.

**Lemma 1** Let $T$ be a Möbius transformation such that $T(f(S^1))$ is compact. Then we have

$$V(T \circ f, s) = \frac{2}{T'(f(s))} V(f, s).$$

The lemma follows directly from (15) since (7), (8) and the Möbius invariance of the conformal angle implies

$$\frac{1 - \cos \theta_{T \circ f}(s, t)}{|(T \circ f)(s) - (T \circ f)(t)|^2} \frac{|(T \circ f)'(t)|}{|T'(f(s))|^2 |f(s) - f(t)|^2} |f'(t)|.$$

We introduce an alternative proof using analytic continuation based on the idea of Brylinski [5]. Another proof by Hadamard regularization (14) will be given in Section 5.

**Proof** Define the local Brylinski’s beta function $B_{T \circ f, s}^{\text{loc}}(z)$ ($z \in \mathbb{C}$) by the meromorphic regularization of

$$\int_{S^1} |f(t) - f(s)|^z |f'(t)| \, dt,$$

which can be carried out as follows. First note that the integral, considered as a function of $z$, is well-defined if $\Re z > -1$ and is a holomorphic function of $z$ there. We can extend the domain by analytic continuation to the whole complex plane to obtain a meromorphic function only with (possible) simple poles at negative odd integers, $z = -1, -3, \ldots$. Then the regularized $r^{-2}$-potential satisfies $V(f, s) = B_{T \circ f, s}^{\text{loc}}(-2)$ [5]. Then by (7) and (8),

$$B_{T \circ f, s}^{\text{loc}}(z) = \int_{S^1} |T(f(t)) - T(f(s))|^z |(T \circ f)'(t)| \, dt$$

$$= \int_{S^1} (|T'(f(t))| |T'(f(s))| |f(t) - f(s)|^z |T'(f(t))|^2 |f'(t)|) \, dt$$

$$= |T'(f(s))|^z \int_{S^1} |T'(f(t))|^z |f(t) - f(s)|^z |f'(t)| \, dt.$$

Substituting $z = -2$ we obtain

$$V(T \circ f, s) = B_{T \circ f, s}^{\text{loc}}(-2) = |T'(f(s))|^{-2} B_{T \circ f, s}^{\text{loc}}(-2) = |T'(f(s))|^{-2} V(f, s).$$

**Corollary 1** The cube of the regularized $r^{-2}$-potential, $V^3$, satisfies the conditions (i), (ii) and (iii). Therefore,

$$\langle u, v \rangle_{V^3} = \int_{S^1} (u(t), v(t)) V(f, t)^3 |f'(t)| \, dt$$

is a Möbius invariant and parametrization independent inner product, and if $e$ is a Möbius invariant functional on $K$ with a gradient $G_e$ then $G_e^{V^3} = (V(f))^3 G_e(f)$ ($f \in K^0$) is a Möbius invariant and parametrization independent gradient with respect to $\langle \cdot, \cdot \rangle_{V^3}$. 
4 Generalization

Note that (15) induces the cosine formula of the energy $E$ by Doyle and Schramm:

$$E(f) = \int_{S^1 \times S^1} \frac{1 - \cos \theta_f(s, t)}{|f(s) - f(t)|^2} |f''(s)||f'(t)| \, ds dt,$$

where $\theta_f$ is the conformal angle. Since

$$\frac{|f''(s)||f'(t)|}{|f(s) - f(t)|^2}$$

is Möbius invariant by (7) and (8), so is the integrand of (19).

The construction of $\Phi$ by the potential $V$ can be generalized as follows.

**Theorem 2** Let $\psi$ be a function from $\mathcal{K} \times \left((S^1 \times S^1) \setminus \Delta\right)$ to $\mathbb{R}$, where $\Delta$ is the diagonal set $\Delta = \{(t, t) : t \in S^1\}$. Suppose $\psi(f, s, \cdot) |f'(\cdot)|$ is integrable on $S^1$ for any $f \in \mathcal{K}$ and for any $s \in S^1$. Put

$$\Psi(f, s) = \int_{S^1} \psi(f, s, t) |f'(t)| \, dt.$$

Suppose $\Psi$ is positive on $\mathcal{K} \times S^1$ and $\Psi(f, s)$ is continuous in $s$ for any $f \in \mathcal{K}$.

Then, if the 2-form $\psi(f, s, t) |f''(s)||f'(t)| \, ds dt$ on $S^1 \times S^1 \setminus \Delta$ is Möbius invariant, i.e.

$$\psi(T \circ f, s, t) |(T \circ f)'(s)||T \circ f'(t)| = \psi(f, s, t) |f'(s)||f'(t)|$$

for any Möbius transformation $T$ such that $T(f(S^1))$ is compact, then $(\Psi)^3$ satisfies the condition (9) for $\Phi$ in Theorem 1 (1).

**Proof** By (8), (21) implies

$$\Psi(T \circ f, s) |T'(f(s))|^2 = \Psi(f, s).$$

In the above scheme, the weight function studied in the previous section is given by

$$\psi(f, s, t) = \frac{1 - \cos \theta_f}{|f(s) - f(t)|^2}.$$

Since (20) is Möbius invariant, if $\mu(\theta)$ is a continuous function on $[0, \pi]$ that is positive on $(0, \pi)$ and is of the order of $\theta$ (to be precise, $O(\theta^{1/2+\varepsilon})$ for some $\varepsilon > 0$) near $\theta = 0$, then

$$\psi(f, s, t) = \frac{\mu(\theta_f(s, t))}{|f(s) - f(t)|^2}$$

satisfies the condition in Theorem 2. Furthermore, $\Psi$ is parametrization independent.

Let us give examples of $\mu(\theta)$ which have been studied in geometric knot theory to produce knot energies by

$$E_{\mu}(f) = \int_{S^1 \times S^1} \mu(\theta_f(s, t)) \frac{|f''(s)||f'(t)|}{|f(s) - f(t)|^2} \, ds dt$$

(cf. Section 2 of [12]).

1. As we have seen, the energy $E$ (16) is given by $\mu(\theta) = 1 - \cos \theta$.
2. The absolute sine energy (Section 4 of [12]) or the absolute imaginary cross-ratio energy (Section 5 of [13]) is given by $\mu(\theta) = \sin \theta$. Remark that our convention $0 \leq \theta_f \leq \pi$ implies $\sin \theta_f \geq 0$. 

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3. The measure of acyclicity (Section 6 of [13]) is given by \( \mu(\theta) = (\pi/4)(\theta - \theta \sin \theta) \) (Proposition 6.13 of [13]).

Note that \( \psi \) of the form (22) depends only on the first-order information of the knot \( f \). Conversely, if a Möbius invariant function \( \psi(f, s, t) \) depends only on the first-order information of the knot \( f \), then \( \psi \) should be expressed in terms of (20) and the conformal angle \( \theta_f \) [13, page 234]. In fact, the 2-form \( |f(s) - f(t)|^{-2} |f'(s)| |f'(t)| \, ds \, dt \) and \( \theta_f \) are the absolute value and the argument of the infinitesimal cross ratio of the knot [13], which is the cross ratio of the four points \( f(s), f(s + ds), f(t) \) and \( f(t + dt) \), where the four points are considered as complex numbers through a stereographic projection from a sphere through these four points to a plane, which is identified with \( \mathbb{C} \). Now the above assertion follows from the fact that the conjugacy class of the cross ratio is essentially the only Möbius invariant of the set of (ordered) four points in \( \mathbb{R}^3 \).

5 Proof of Möbius invariance by Hadamard regularization

The gradient of the energy \( E, G_E \), was given in [7]. Corollary 1 shows that \( G_E^\psi(f) = (V(f))^{-3} G_E(f) \) is Möbius invariant. In this section we give an outline of the proof of it by Hadamard regularization.

Hadamard regularization is a method to obtain a finite value from a divergent integral. Suppose \( \int_X \omega \) diverges on \( \Delta \subset X \). Integrate \( \omega \) on the complement of an \( \varepsilon \) neighbourhood of \( \Delta \), expand the resulting value in a (Laurent) series of \( \varepsilon \), and finally take the constant term, which is called Hadamard’s finite part, denoted by \( \text{Pf.} \int_X \omega. \) It is a kind of generalization of Cauchy’s principal value.

Let us first show Lemma 1 by Hadamard regularization. It is enough to show it when \( T \) is an inversion \( I \) in a unit sphere with center the origin and \( f(S^1) \) does not pass through the origin. Since

\[
V(f, s) = \text{Pf.} \int_{S^1} \frac{1}{|f(t) - f(s)|^2} |f'(t)| \, dt,
\]

we have

\[
V(I \circ f, s) = \text{Pf.} \int_{S^1} \frac{|f(s)|^2 |f(t)|^2}{|f(t) - f(s)|^2} \frac{|f'(t)|}{|f(t)|^2} \, dt = |f(s)|^2 V(f, s),
\]

which proves (18) since \( |f'(f(s))| = |f(s)|^{-1} \) by (6).

Let \( P_{f'(s)^\perp} : \mathbb{R}^3 \rightarrow \mathbb{R}^3 \) be the orthogonal projection to \( \left(\text{Span}(f'(s))\right)^\perp \). Then \( G_E(f)(s) \) is given by

\[
2 \text{p.v.} \int_{S^1} \left\{ 2 \frac{P_{f'(s)^\perp}(f(t) - f(s))}{|f(t) - f(s)|^2} - \frac{1}{|f'(s)|} \frac{d}{ds} \left( \frac{f'(s)}{|f'(s)|} \right) \right\} \frac{|f'(t)|}{|f(t) - f(s)|^2} \, dt,
\]

where p.v. means Cauchy’s principal value integral [7]. Note that

\[
\frac{1}{|f'(s)|} \frac{d}{ds} \left( \frac{f'(s)}{|f'(s)|} \right) = \frac{f''(s)}{|f'(s)|^2} - \frac{(f'(s), f''(s))}{|f'(s)|^4} f'(s),
\]

and that it is perpendicular to \( f'(s) \).

In the formulae in what follows we fix \( s \) in \( S^1 \) and may omit \( s \) from \( f(s), f'(s), \) and \( f''(s) \) for the sake of visible clarity when the formulae are complicated.
Using Bouquet’s formula that gives the expansion of \( f(t) \) with respect to the Frenet frame at \( f(s) \) in a series of \( t-s \), which can be deduced from Frenet-Serret formulas, we have

\[
\int _{d_{f(t,s)} \geq \varepsilon } \frac{f(t)-f(s)}{|f(t)-f(s)|^4} |f''(t)| \, dt = \frac{1}{\varepsilon} \left( \frac{f''}{|f'|^2} - \frac{(f', f'')}{|f'|^4} f' \right) + O(1),
\]

which implies

\[
Pf. \int _{S^1} \frac{f(t)-f(s)}{|f(t)-f(s)|^4} |f''(t)| \, dt
\]

\[
= \lim _{\varepsilon \to 0^+} \left[ \int _{d_{f(t,s)} \geq \varepsilon } \frac{f(t)-f(s)}{|f(t)-f(s)|^4} |f''(t)| \, dt - \frac{1}{\varepsilon} \left( \frac{f''}{|f'|^2} - \frac{(f', f'')}{|f'|^4} f' \right) \right].
\]

Together with

\[
\int _{d_{f(t,s)} \geq \varepsilon } \frac{|f''(t)|}{|f(t)-f(s)|^2} \, dt = V(f, s) + \frac{2}{\varepsilon} + O(\varepsilon),
\]

we have

\[
Pf. \int _{S^1} \frac{f(t)-f(s)}{|f(t)-f(s)|^4} |f''(t)| \, dt
\]

\[
= \int _{S^1} \left[ \frac{f(t)-f(s)}{|f(t)-f(s)|^4} |f''(t)| - \frac{|f''(t)|}{2|f(t)-f(s)|^2} \left( \frac{f''}{|f'|^2} - \frac{(f', f'')}{|f'|^4} f' \right) \right] \, dt
\]

\[+ \frac{1}{2} V(f, s) \left( \frac{f''}{|f'|^2} - \frac{(f', f'')}{|f'|^4} f' \right).
\]

Taking the image of the above under \( P_{f'(s)} \) and comparing it with (23), one obtain

\[
G_E(f)(s) = 2P_{f'(s)} \left[ 2 Pf. \int _{S^1} \frac{f(t)-f(s)}{|f(t)-f(s)|^4} |f''(t)| \, dt - \frac{V(f, s)}{|f'(s)|^2} f''(s) \right].
\]

Put

\[
u_1(f, s) = 2 Pf. \int _{S^1} \frac{f(t)-f(s)}{|f(t)-f(s)|^4} |f''(t)| \, dt,
\]

\[
u_2(f, s) = - \frac{V(f, s)}{|f'(s)|^2} f''(s),
\]

\[
u_3(f, s) = - \left( u_1(f, s), \frac{f'(s)}{|f'(s)|} \right) \frac{f''(s)}{|f'(s)|},
\]

\[
u_4(f, s) = - \left( u_2(f, s), \frac{f'(s)}{|f'(s)|} \right) \frac{f''(s)}{|f'(s)|}.
\]

Then

\[
G_E(f)(s) = 2(\nu_1(f, s) + \nu_2(f, s) + \nu_3(f, s) + \nu_4(f, s)).
\]

**Lemma 2** Let \( u \) be a map from \( K \times S^1 \) to \( \mathbb{R}^3 \) that maps a pair \((f, s)\) to \( u(f, s) \) in \( T_{f(s)} \mathbb{R}^3 \cong \mathbb{R}^3 \). Then \( V^{-3} u \) is Möbius invariant, i.e.

\[
V(T \circ f, s)^{-3} u(T \circ f, s) = T_{u} \left( V(f, s)^{-3} u(f, s) \right)
\]

for any \( f \) in \( K \), \( s \) in \( S^1 \), and a Möbius transformation \( T \) such that \( T(f(S^1)) \) is compact if and only if the following conditions are satisfied for any \( f \) and \( s \).
1. For any translation $p$ of $\mathbb{R}^3$, $u(p \circ f, s) = u(f, s)$ holds.
2. For any homothety by $k > 0$, $u(kf, s) = k^{-2}u(f, s)$ holds.
3. Let $I$ be an inversion in a unit sphere with center the origin. If $f(S^1)$ does not pass through the origin then
   \[ u(I \circ f, s) - |f(s)|^4u(f, s) + 2|f(s)|^2(u(f, s), f(s)) f(s) \] holds.

Let the left hand side of (30) be denoted by $J(u, f, s)$.

**Proof** First note that for $v \in T_f(s) \mathbb{R}^3$ we have
\[ I_v(v) = \frac{u}{|f|^2} - 2\frac{(v, f)}{|f|^4} f. \]
where we omitted $(s)$ from $f(s)$ as before. Put $\tilde{f} = I \circ f$. Then by (18) we have
\[ \frac{u(\tilde{f}, s)}{V(\tilde{f}, s)} - I_v \left( \frac{u(f, s)}{V(f, s)} \right) = \frac{1}{V(f, s)^2} \left( \frac{u(\tilde{f}, s)}{|f|^6} - \frac{u(f, s)}{|f|^2} + 2 \frac{(u(f, s), f)}{|f|^4} f \right), \]
which implies (30).

\[ \square \]

**Lemma 3** The following equalities hold, where we omit $(s)$ from $f(s)$ and $f'(s)$.
\[ J(u_1, f, s) = -2V(f, s)|f|^2 f, \]
\[ J(u_2, f, s) = 2V(f, s)|f|^2 f - 8 \frac{V(f, s)(f, f')^2}{|f'|^2} f + 4 \frac{V(f, s)|f|^2(f, f')}{|f'|^2} f', \]
\[ J(u_3, f, s) = 4 \frac{V(f, s)(f, f')^2}{|f'|^2} f - 2 \frac{V(f, s)|f|^2(f, f')}{|f'|^2} f', \]
\[ J(u_4, f, s) = 4 \frac{V(f, s)(f, f')^2}{|f'|^2} f - 2 \frac{V(f, s)|f|^2(f, f')}{|f'|^2} f'. \]

**Proof** Let us prove (31). The proof of the other formulae is automatic. Since
\[ u_1(\tilde{f}, s) = 2 \text{Pf}. \int_{S^1} \frac{f(t)}{|f(t)|^4} - \frac{f(s)}{|f(s)|^4} \cdot \frac{|f'(t)|}{|f(t)|^2} \, dt \]
\[ = 2 \text{Pf}. \int_{S^1} \frac{|f(s)|^4 f(t) - |f(s)|^2 f(t)^2 f(s)}{|f(t) - f(s)|^4} \cdot |f'(t)| \, dt \]
\[ = |f(s)|^4 u_1(f, s) + 2 |f(s)|^2 \left( \text{Pf}. \int_{S^1} \frac{|f(s)|^2 - |f(t)|^2}{|f(t) - f(s)|^4} |f'(t)| \, dt \right) f(s), \]
we have
\[ J(u_1, f, s) = 2 |f(s)|^2 \left( \text{Pf}. \int_{S^1} \frac{|f(s)|^2 - |f(t)|^2 + 2(f(t) - f(s), f(s)) f(t)}{|f(t) - f(s)|^4} |f'(t)| \, dt \right) f(s) \]
\[ = 2 |f(s)|^2 \left( \text{Pf}. \int_{S^1} \frac{-|f'(t)|}{|f(t) - f(s)|^2} \, dt \right) f(s) \]
\[ = -2 |f(s)|^2 V(f, s) f(s). \]
\[ \square \]
Now (29), Lemmas 2 and 3 imply

**Corollary 2** The $V^{-3}$-weighted gradient $G_{E}^{V^3} (f) = (V (f))^{-3} G_E (f)$ is Möbius invariant.

### 6 Remarks and open problems

We like to close the article with remarks and open problems.

1. The argument for the case when $\Phi = V^3$ can be generalized to the space of embedded odd dimensional closed submanifolds in $\mathbb{R}^N$ since the potential and the energy have natural generalization. The difficulty in even dimensional case is explained in Subsection 3.3 of [17]. Unfortunately, for the moment the author does not know whether an analogue of Proposition 2, i.e. the positivity of the potential for non-spherical submanifolds holds or not.

2. If we forget about the positivity condition for $\Phi$, we can use a “local” function as $\Phi$ as follows. There is a notion of conformal arc-length $\rho$ given by

$$d\rho = \sqrt{\kappa'^2 + \kappa^2 \tau^2} \, ds,$$

where $s$ is the usual arc-length parameter of the knot $f(S^1)$, $\kappa$ and $\tau$ are the curvature and torsion, and $\kappa'$ is the derivative with respect to the arc-length, $\kappa' = d\kappa / ds$ [6]. Since the conformal arc-length $\rho$ is invariant under Möbius transformations, if we denote the arc-length parameter of $T(f(S^1))$ by $\tilde{s}$, then (8) implies

$$\frac{d\tilde{s}}{d\rho} = \left| T'(f(t)) \right|^2 \frac{ds}{d\rho}.$$

Therefore, if we put

$$\Phi_c (f, t) = \left( \frac{d\rho}{ds} \right)^3 = \left( \left( \frac{d\kappa}{ds} \right)^2 + \kappa^2 \tau^2 \right)^{3/4} (s \text{ is the arc-length}),$$

then $\Phi_c$ satisfies the condition (9). But $\Phi_c$ is not necessarily positive; it vanishes on a circular or linear arc of a knot.

3. Similarly, among the Möbius invariant functionals for knots, the absolute value of writhe [1] and the renormalized measure of circles linking a knot [16] do not fit our scheme since the writhe vanishes for planar curves and the latter may be negative.

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