ASPECTS OF TOEPLITZ DETERMINANTS

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Abstract. We review the asymptotic behavior of a class of Toeplitz (as well as related Hankel and Toeplitz + Hankel) determinants which arise in integrable models and other contexts. We discuss Szegő, Fisher-Hartwig asymptotics, and how a transition between them is related to the Painlevé V equation. Certain Toeplitz and Hankel determinants reduce, in certain double-scaling limits, to Fredholm determinants which appear in the theory of group representations, in random matrices, random permutations and partitions. The connection to Toeplitz determinants helps to evaluate the asymptotics of related Fredholm determinants in situations of interest, and we review the corresponding results.

1. Introduction

Let \( f(z) \) be a function integrable over the unit circle \( C \) with Fourier coefficients

\[
 f_j = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) e^{-ij\theta} d\theta, \quad j = 0, \pm 1, \pm 2, \ldots
\]

Then the \( n \)-dimensional Toeplitz determinant of a Toeplitz matrix with symbol \( f(z) \) is given by

\[
 D_n(f) = \det(f_{j-k})_{j,k=0}^{n-1}.
\]

Substituting here the expressions for the Fourier coefficients, and using formulae for Vandermonde determinants, one obtains another useful representation:

\[
 D_n(f) = \frac{1}{(2\pi)^n n!} \int_0^{2\pi} \cdots \int_0^{2\pi} \prod_{1 \leq j < k \leq n} |e^{i\theta_j} - e^{i\theta_k}|^2 \prod_{j=1}^{n} f(e^{i\theta_j}) d\theta_j.
\]

Toeplitz determinants are closely related to the polynomials orthogonal with weight \( f(z) \) on the unit circle. Namely, if \( D_k(f) \neq 0 \), \( k = k_0, k_0 + 1, \ldots \), for some \( k_0 \geq 0 \), then the polynomials \( \phi_k(z) = \chi_k z^k + \cdots \), \( \widetilde{\phi}_k(z) = \chi_k z^k + \cdots \) of degree \( k \), \( k = k_0, k_0 + 1, \ldots \), satisfying

\[
 \frac{1}{2\pi} \int_0^{2\pi} \phi_k(z) z^{-j} f(z) d\theta = \chi_k^{-1} \delta_{jk}, \quad \frac{1}{2\pi} \int_0^{2\pi} \widetilde{\phi}_k(z^{-1}) z^j f(z) d\theta = \chi_k^{-1} \delta_{jk},
\]

exist and \( \chi_k = \sqrt{D_k/D_{k+1}} \), where, by convention, \( D_0 \equiv 1 \). We see from (1.2) that if \( f(z) \) is positive on \( C \), we have \( D_n(f) > 0 \) for all \( n \), and therefore in this case we can set \( k_0 = 0 \).

A Toeplitz determinant can be represented as a Fredholm determinant of an integral operator acting on \( L^2(C) \) which belongs to the special class of so-called integrable operators [48]. It can also be written in a different way in terms of a Fredholm determinant of an operator now acting on \( \ell^2(n, n+1, \ldots) \) [62, 27, 30]. (Note that the symbols \( f(z) \) considered in [62, 27, 30] are assumed to be sufficiently smooth.) Another useful property of many
$D_n(f)$’s encountered in applications is the existence of simple differential identities relating the determinant to orthogonal polynomials evaluated at a few special points. The precise form of such identities depends on the given $f(z)$.

It turns out that the above properties play a key role in making Toeplitz determinants amenable to a detailed asymptotic analysis, in particular, by Riemann-Hilbert-Problem methods.

In this paper, we will review some asymptotic results on $D_n(f)$ (and related Hankel, Toeplitz+Hankel, and Fredholm determinants) and briefly mention their applications in integrable models, random matrices, random permutations, group representation theory, and also in various conjectures on Riemann’s $\zeta$ and Dirichlet’s $L$-functions. This review is based, to a large extent, on the recent work of the author with T. Claeys, P. Deift, A. Its, and J. Vasilevska. For other aspects of Toeplitz determinants not mentioned here, the reader is referred to [32, 34, 33, 67, 47, 49, 50] for properties of Toeplitz matrices and determinants, to [34, 108, 109, 110, 89, 64, 98, 19, 91, 90, 105, 72, 69, 14] for generalizations to the continuous, higher-dimensional, and the block-Toeplitz cases (with relations to stationary determinantal processes, integrable models, and entanglement entropy), and to [88, 1, 20] for connections with multiple orthogonal polynomials.

The paper consists of three parts: in Section 2, the simplest asymptotics with $f(z)$ fixed and $n \to \infty$ are considered; in Section 3, the symbol $f(z)$ is allowed to depend on $n$ in ways which describe a transition between different asymptotic regimes arising in Section 2; in Section 4, the symbol $f(z)$ also depends on $n$, but in such a way that in the limit $n \to \infty$, Toeplitz determinants turn into certain Fredholm determinants which are important for random matrices and random permutations. Following standard practice, we refer to the large $n$ asymptotics of Sections 3 and 4 as double-scaling limits.

2. ASYMPTOTICS FOR A FIXED SYMBOL

We assume in this section that $f(z)$ does not depend on the size of the determinant $n$. We are interested in the asymptotics of $D_n(f)$ as $n \to \infty$.

The following result is basic.

**Theorem 2.1** (Strong Szegő limit theorem). Let $f(z)$ be non-zero on $C$, $\ln f(z) \in L^1(C)$, and suppose that the sum

$$S(f) = \sum_{k=-\infty}^{\infty} |k||\ln f)_k|^2, \quad (\ln f)_k = \frac{1}{2\pi} \int_0^{2\pi} \ln f(e^{i\theta})e^{-ik\theta}d\theta,$$

converges. Then

$$\ln D_n(t) = n(\ln f)_0 + \sum_{k=1}^{\infty} k(\ln f)_k(\ln f)_{-k} + o(1), \quad \text{as } n \to \infty.$$
function in the model can be written as a Toeplitz determinant $D_n(f)$, where $n$ denotes the distance between the spins. For temperatures less than critical ($T < T_c$), the symbol of this Toeplitz determinant has an analytic logarithm in a neighborhood of the unit circle and, moreover, $(\ln f)_0 = 0$. Therefore, Szegő’s theorem can be applied, and one concludes that $D_n(f)$ tends to a constant as $n \to \infty$. Thus the correlation does not decay as the distance increases, which indicates the presence of a long-range order, and hence, a magnetization. As $T \nearrow T_c$, however, 2 singularities of $f(z)$ approach the unit circle at $z = 1$, and, at $T = T_c$ merge into a single singularity on $C$; namely, a jump-type singularity at $z = 1$ (see, e.g., [94]). For $f(z)$ with such a singularity, the sum (2.1) diverges, and Theorem 2.1 can no longer be applied. In fact, it turns out [94] that in this case $D_n(f)$ decays as $n^{-1/4}$ and, therefore, there exists no long-range order. For correlation functions arising in other situations, such as, e.g., the so-called emptiness formation probability in the XY spin chain in a magnetic field [40, 61], one obtains Toeplitz determinants with both jump-type and root-type singularities, and in the most general situation one is led to consider symbols of the form:

$$f(z) = e^{V(z)} \sum_{j=0}^{m} \beta_j \prod_{j=0}^{m} |z - z_j|^{2\alpha_j} g_{j, \beta_j}(z) z_j^{-\beta_j}, \quad z = e^{i\theta}, \quad \theta \in [0, 2\pi),$$

for some $m \geq 0$, where

$$z_j = e^{i\theta_j}, \quad j = 0, \ldots, m, \quad 0 = \theta_0 < \theta_1 < \cdots < \theta_m < 2\pi;$$

$$g_{j, \beta_j}(z) = \begin{cases} e^{i\pi \beta_j} & 0 \leq \arg z < \theta_j \\ e^{-i\pi \beta_j} & \theta_j \leq \arg z < 2\pi \end{cases},$$

$$\Re \alpha_j > -1/2, \quad \beta_j \in \mathbb{C}, \quad j = 0, \ldots, m,$$

and $V(e^{i\theta})$ is a sufficiently smooth function on the unit circle (see below) with Fourier coefficients

$$V_k = \frac{1}{2\pi} \int_0^{2\pi} V(e^{i\theta}) e^{-ki\theta} d\theta.$$

The canonical Wiener-Hopf factorization of $e^{V(z)}$ is given by

$$e^{V(z)} = b_+(z) e^{V_0} b_-(z), \quad b_+(z) = e^{\sum_{k=1}^{\infty} V_k z^k}, \quad b_-(z) = e^{\sum_{k=-\infty}^{-1} V_k z^k}.$$

The condition (2.6) on $\alpha_j$ ensures the integrability of $f$. Note that the size of the jump at $z_j$ is determined by the parameter $\beta_j$, and the root-type singularity, by $\alpha_j$. We assume that $z_j, j = 1, \ldots, m$, are genuine singular points, i.e., either $\alpha_j \neq 0$ or $\beta_j \neq 0$. However, the absence of a singularity at $z = 1$, i.e. the case $\alpha_0 = \beta_0 = 0$, is allowed. Singularities of type (2.3) are known as Fisher-Hartwig singularities because of the work [58] where the authors summarized a variety of applications of Toeplitz determinants with such symbols and presented a conjecture about the asymptotic form of $D_n(f)$ in this case. Due to the subsequent efforts of many workers, we have the following description of the asymptotics.

Define the seminorm

$$|||\beta||| = \max_{j,k} |\Re \beta_j - \Re \beta_k|,$$

where the indices $j,k = 0$ are omitted if $z = 1$ is not a singular point, i.e. if $\alpha_0 = \beta_0 = 0$. Note that in the case of a single singularity, we always have $|||\beta||| = 0$. 


First, consider the situation when $|||\beta||| < 1$ is strictly less than 1.

**Theorem 2.2.** Let $f(z)$ be defined in (2.3), $|||\beta||| < 1$, $\Re \alpha_j > -1/2$, $\alpha_j \pm \beta_j \neq -1, -2, \ldots$ for $j, k = 0, 1, \ldots, m$, and $V(z)$ satisfies the smoothness conditions (2.11), (2.12) below. Then as $n \to \infty$,

\begin{equation}
D_n(f) = \exp \left[ nV_0 + \sum_{k=1}^{m} kV_k V_{-k} \right] \prod_{j=0}^{m} \left[ b_+(z_j)^{-\alpha_j + \beta_j} b_-(z_j)^{-\alpha_j - \beta_j} \right.
\end{equation}

\begin{align*}
&\times n^{\sum_{j=0}^{m} (\alpha_j^2 - \beta_j^2)} \prod_{0 \leq j < k \leq m} |z_j - z_k|^{2(\beta_j \beta_k - \alpha_j \alpha_k)} \left( \frac{z_k}{z_j e^{i\pi}} \right)^{-\alpha_j \beta_k - \alpha_k \beta_j} \\
&\left. \times \prod_{j=0}^{m} \frac{G(1 + \alpha_j + \beta_j) G(1 + \alpha_j - \beta_j)}{G(1 + 2\alpha_j)} (1 + o(1)) \right],
\end{align*}

where $G(x)$ is Barnes' $G$-function [10]. The double product over $j < k$ is set to 1 if $m = 0$. The branches in (2.10) are determined as follows: $b_\pm(z_j)^{-\alpha_j \pm \beta_j} = \exp\{(-\alpha_j \pm \beta_j) \sum_{k=1}^{\infty} V_{\pm k} z^{\pm k}\}, (z_k z_j^{-1} e^{-i\pi})^{\alpha_j \beta_k - \alpha_k \beta_j} = \exp\{i(\theta_k - \theta_j - \pi)(\alpha_j \beta_k - \alpha_k \beta_j)\}.

Note that since $G(-k) = 0$, $k = 0, 1, \ldots$, formula (2.10) no longer represents the leading asymptotics if $\alpha_j + \beta_j$ or $\alpha_j - \beta_j$ is a negative integer for some $j$. Such degenerate cases can be handled by carrying the analysis to higher order, but we present no further details here.

The smoothness condition on $V(z)$ assumed in Theorem 2.2 is that

\begin{equation}
\sum_{k=-\infty}^{\infty} |k|^s |V_k| < \infty
\end{equation}

holds for some $s$ such that

\begin{equation}
s > \frac{1 + \sum_{j=0}^{m} [ (3\alpha_j)^2 + (\Re \beta_j)^2 ]}{1 - |||\beta|||}.
\end{equation}

Note that the condition $|||\beta||| < 1$ is important here.

The Barnes' $G$-function first appeared in asymptotic Toeplitz theory in the work of Lenard [92]. Theorem 2.2 was proved by Widom [104] in the case when $\Re \alpha_j > -1/2$, and all $\beta_j = 0$, and with a stronger condition on $V(z)$. In [12], Basor extended the result to $\Re \alpha_j > -1/2$, $\Re \beta_j = 0$, and in [12], to $\alpha_j = 0$, $|\Re \beta_j| < 1/2$. In [31], Böttcher and Silbermann established the result in the case that $|\Re \alpha_j| < 1/2$, $|\Re \beta_j| < 1/2$. In [53], Ehrhardt proved the theorem for the full range of parameters, namely $\Re \alpha_j > -1/2$, $|||\beta||| < 1$, and for $C^\infty$ functions $V(z)$. These results were established by operator-theory methods (see [53] for a review of these and other related results including an extension to $\Re \alpha < -1/2$, $2\alpha \neq -1, -2, \ldots$, when $f$ is replaced by a suitable distribution). In [44], the authors reprove the theorem by Riemann-Hilbert-Problem methods, and relax the smoothness conditions on $V(z)$ to (2.11), (2.12).

Consider now the general case of Fisher-Hartwig symbols $f(z)$ with the restriction $|||\beta||| < 1$ removed. Note first that $f(z)$ has several representations of type (2.3) with different sets of parameters $\beta_j$. Namely, if each $\beta_j$ in (2.3) such that either $\beta_j \neq 0$ or $\alpha_j \neq 0$ is replaced by $\beta_j = \beta_j + n_j$, where $n_j$ are integers subject to the condition $\sum_{j=0}^{n} n_j = 0$, then the resulting
function $f(z; n_0, \ldots, n_m)$ is related to $f(z)$ in the following way:

$$f(z; n_0, \ldots, n_m) = \prod_{j=0}^{m} z_j^{-n_j} f(z),$$

i.e., it differs from $f(z)$ only by a constant. Each $f(z; n_0, \ldots, n_m)$ so obtained is called a FH-representation of the symbol. Denote by $\mathcal{M}$ the (finite) set of FH-representations for which $\sum_{j=0}^{m} (\Re \hat{\beta}_j)^2$ is minimal. There exists a simple procedure (see [43]) to solve this discrete variational problem and to construct $\mathcal{M}$ explicitly. One can show that there is always a FH-representation with $|||\hat{\beta}||| \leq 1$, and we have the following 2 mutually exclusive possibilities:

- If there exists a FH-representation such that $|||\hat{\beta}||| < 1$ then it turns out that this FH-representation is the single element of $\mathcal{M}$. In particular, if $|||\beta||| < 1$, the set $\mathcal{M}$ consists of a single element corresponding to all $n_j = 0$, and Theorem 2.3 below reduces to Theorem 2.2.
- If there exists a FH-representation such that $|||\hat{\beta}||| = 1$ then $\mathcal{M}$ consists of several (at least 2) elements.

The set $\mathcal{M}$ is called non-degenerate if it contains no representations for which $\alpha_j + \hat{\beta}_j$ or $\alpha_j - \hat{\beta}_j$ is a negative integer for some $j$. The general result is as follows.

**Theorem 2.3.** Let $f(z)$ be given in (2.3), $V(z)$ satisfy the condition (2.11) above for some sufficiently large $s$ (depending only on $\alpha_j$, $\beta_j$), and $\Re \alpha_j > -1/2$, $\beta_j \in \mathbb{C}$, $j = 0, 1, \ldots, m$. Let $\mathcal{M}$ be non-degenerate. Then, as $n \to \infty$,

$$D_n(f) = \sum \left[ \prod_{j=0}^{m} z_j^{n_j} \right]^n \mathcal{R}(f(z; n_0, \ldots, n_m))(1 + o(1)), \tag{2.13}$$

where the sum is over all FH-representations in $\mathcal{M}$. Each $\mathcal{R}(f(z; n_0, \ldots, n_m))$ stands for the right-hand side of the formula (2.10), without the error term, corresponding to $f(z; n_0, \ldots, n_m)$.

An explicit lower bound on $s$ (depending on $\beta_j$, $\alpha_j$) similar to (2.12) is given in [43].

This theorem was conjectured by Basor and Tracy [18] and proved in [43].

Hankel and Toeplitz+Hankel determinants are also of interest. Let $w(x)$ be an integrable function on a subset $J$ of $\mathbb{R}$. Then the Hankel determinant with symbol $w(x)$ supported on $J$ is given by

$$D_n^H(w(x)) = \det \left( \int x^{j+k} w(x) dx \right)_{j,k=0}^{n-1}. \tag{2.14}$$

When $J$ is a finite interval – we then set $J = [-1, 1]$ without loss of generality – Hankel determinants are related to Toeplitz determinants by the following formulae [43], involving the orthogonal polynomials (1.3):

$$w(x) = \frac{f(e^{i\theta})}{\sin \theta}, \quad x = \cos \theta, \quad x \in [-1, 1]; \tag{2.15}$$

$$[D_n^H(w(x))]^2 = \frac{\pi^{2n}}{4(n-1)^2} \frac{(\chi_{2n} + \phi_{2n}(0))^2}{\phi_{2n}(1)\phi_{2n}(-1)} D_{2n}(f(z)). \tag{2.16}$$
A particularly interesting class of Toeplitz+Hankel determinants appearing in the theory of classical groups and its applications to random matrices and statistical mechanics (see, e.g., [7, 60, 81]) is defined as follows for even $f(e^{i\vartheta}) = f(e^{-i\vartheta})$ (for even $f$ the matrices involved are symmetric):

\begin{equation}
\det(f_{j-k} + f_{j+k})_{j,k=0}^{n-1}, \quad \det(f_{j-k} - f_{j+k+2})_{j,k=0}^{n-1}, \quad \det(f_{j-k} \pm f_{j+k+1})_{j,k=0}^{n-1}.
\end{equation}

They are related to Hankel determinants with symbols on $[-1, 1]$ by the expressions

\begin{align}
\det(f_{j-k} + f_{j+k})_{j,k=0}^{n-1} &= \frac{2^{n^2-2n+2}}{\pi^n} D_n^H(f(e^{i\vartheta(x)})/\sqrt{1-x^2}), \\
\det(f_{j-k} - f_{j+k+2})_{j,k=0}^{n-1} &= \frac{2^{n^2}}{\pi^n} D_n^H(f(e^{i\vartheta(x)})\sqrt{1-x^2}), \\
\det(f_{j-k} + f_{j+k+1})_{j,k=0}^{n-1} &= \frac{2^{n^2-n}}{\pi^n} D_n^H(f(e^{i\vartheta(x)})\sqrt{1+x}), \\
\det(f_{j-k} - f_{j+k+1})_{j,k=0}^{n-1} &= \frac{2^{n^2-n}}{\pi^n} D_n^H(f(e^{i\vartheta(x)})\sqrt{1-x}).
\end{align}

Asymptotic formulae for Hankel and Toeplitz+Hankel determinants with Fisher-Hartwig singularities, whose derivation was based on the above theorems for Toeplitz determinants, an asymptotic Riemann-Hilbert-Problem analysis of the polynomials (1.3), and the above relations, are presented in [43]. For other asymptotic results, see [16, 17, 8, 13, 36].

For related asymptotic results on an important class (see Section 4) of Toeplitz determinants when the symbol is supported on an arc of the unit circle, see [107, 87, 85, 46].

Theorem 2.3 and asymptotic formulae for Hankel and Toeplitz+Hankel determinants find applications, e.g., for correlation functions in the XY spin chain in a magnetic field mentioned above, in the theory of the impenetrable Bose gas [60, 96], in random matrix conjectures for average values of Riemann’s $\zeta$-function, and Dirichlet’s $L$-functions [82, 66, 35].

For a more detailed discussion of the material presented in this section so far, the reader is referred to [43].

A related area of interest is the asymptotic analysis of Hankel determinants whose symbol has Fisher-Hartwig singularities and is supported on the whole real line, or the half-line. In particular, in the Gaussian Unitary Ensemble of random matrix theory, the correlation function of products of powers of the absolute values of the characteristic polynomial is precisely such a Hankel determinant $D_n^H(w)$: namely, the symbol is supported on $\mathbb{R}$ and given by $w(x) = \exp(-x^2) \prod_{j=1}^{m} |x - \mu_j|^{2\alpha_j}, \mu_j \in \mathbb{R}, \Re \alpha_j > -1/2$. This determinant is also related to the 1-dimensional impenetrable Bose gas and conjectures for mean values of Riemann’s $\zeta$-function on the critical line. For a discussion of the results in this area, see [86, 70, 71]. For analysis of some other Hankel determinants appearing in random matrix models, see [21, 55]. For a recent application of Hankel determinants in the six-vertex model see [74, 22, 23, 24, 25].

Note that the importance of Fisher-Hartwig singularities appears to stem from the following feature. The asymptotics of the orthogonal polynomials at the location of such a singularity are described by the confluent hypergeometric function [43]. The two independent parameters of such functions are related to the parameters $\alpha$ and $\beta$ of the singularity (for $\beta = 0$ confluent hypergeometric functions reduce to Bessel functions). The location of
the singularity corresponds to the single finite branch point of the confluent hypergeometric functions. As hypergeometric functions (which depend on 3 parameters) have 2 finite branch points, we would not be able to confine ourselves to hypergeometric functions if we wanted to consider a singular point which generalizes Fisher-Hartwig in some essential way. Roughly speaking, a Fisher-Hartwig singularity is the most general hypergeometric singular point. Modifications, of course, are possible: e.g., the end points of the interval $[-1, 1]$ can be regarded as modified Fisher-Hartwig singularities for a Hankel determinant with symbol on $[-1, 1]$ as discussed above.

3. Transition Asymptotics

A natural question to ask is how the transition between various asymptotic regimes of the previous section occurs. Consider once again the 2-spin correlation function for the 2-dimensional Ising model discussed above, which is a Toeplitz determinant. As $T \nearrow T_c$ a transition between the Szegő asymptotics and the Fisher-Hartwig asymptotics takes place. It was first investigated in [112, 93, 101], and the authors found that if $T \to T_c$ and $n \to \infty$ in such a way that $x \equiv (T_c - T)n$ is fixed, then the determinant is given in terms of Painlevé III (reducible to Painlevé V) functions. This transition corresponds to the emergence of one Fisher-Hartwig singularity with $\alpha = 0$, $\beta = -1/2$ at $z_0 = 1$. The condition that $x \equiv (T_c - T)n$ is fixed was removed in [38] where uniform asymptotics were obtained for any $\alpha$, $\Re \alpha > -1/2$, $\beta \in \mathbb{C}$ in terms of Painlevé V functions. Namely, consider the following symbol

\begin{equation}
(3.1) \quad f_t(z) = (z - e^t)^{\alpha + \beta}(z - e^{-t})^{\alpha - \beta}z^{\alpha + \beta}e^{-i\pi(\alpha + \beta)}e^{V(z)}, \quad \alpha \pm \beta \neq -1, -2, \ldots
\end{equation}

where $t \geq 0$ is sufficiently small (in the above example of the Ising model, $t = \text{const}(T_c - T)$), $V(z)$ is analytic in a neighborhood of $C$, and $\alpha, \beta \in \mathbb{C}$ with $\Re \alpha > -\frac{1}{2}$. The singularities of the symbol are at the points $e^{\pm t}$. If $t = 0$ the symbol possesses a Fisher-Hartwig singularity at $z = 0$ and Theorem 2.2 applies to $D_n(f_0)$. If $t > 0$ then $f_t(z)$ is analytic in a neighborhood of $C$, and Szegő’s Theorem 2.1 applies. We have [38]

**Theorem 3.1.** Let $\alpha, \beta \in \mathbb{C}$ with $\Re \alpha > -\frac{1}{2}$ and let $s_\delta$ denote a sector $-\pi/2 + \delta < \arg x < \pi/2 - \delta$, $0 < \delta < \pi/2$. Let $f_t$ be given by (3.1) and consider the Toeplitz determinants $D_n(f_t)$ defined by (1.1) corresponding to this symbol. There exists a finite set $\{x_1, \ldots, x_k\} \subset s_\delta$ (with $k = k(\alpha, \beta)$ and $x_j = x_j(\alpha, \beta) \neq 0$) such that there holds the following expansion as $n \to \infty$ with the error term uniform for $0 < t < t_0$ (with $t_0$ sufficiently small) as long as $2nt$ remains bounded away from the set $\{x_1, \ldots, x_k\}$:

\begin{equation}
(3.2) \quad \ln D_n(f_t) = nV_0 + (\alpha + \beta)nt + \sum_{k=1}^{\infty} k \left[ V_k - (\alpha + \beta)\frac{e^{-tk}}{k} \right] \left[ V_{-k} - (\alpha - \beta)\frac{e^{tk}}{k} \right] + \ln \frac{G(1 + \alpha + \beta)G(1 + \alpha - \beta)}{G(1 + 2\alpha)} + \Omega(2nt) + o(1),
\end{equation}

where $G(z)$ is Barnes’ $G$-function, and

\begin{equation}
(3.3) \quad \Omega(2nt) = \int_{0}^{2nt} \frac{\sigma(x) - \alpha^2 + \beta^2}{x} dx + (\alpha^2 - \beta^2) \ln 2nt.
\end{equation}
The function $\sigma(x)$ is a particular solution to the Jimbo-Miwa-Okamoto $\sigma$-form [76, 77] of the Painlevé V equation
\begin{equation}
\left( x \frac{d^2 \sigma}{dx^2} \right)^2 = \left( \sigma - x \frac{d \sigma}{dx} + 2 \left( \frac{d \sigma}{dx} \right)^2 + 2 \alpha \frac{d \sigma}{dx} \right)^2 - 4 \left( \frac{d \sigma}{dx} + \alpha + \beta \right) \left( \frac{d \sigma}{dx} + \alpha - \beta \right).
\end{equation}

This solution has the following asymptotics for $x > 0$:
\begin{equation}
\sigma(x) = \begin{cases} 
\alpha^2 - \beta^2 + \frac{\alpha^2 - \beta^2}{2 \alpha} (x - x^{1+2\alpha} C(\alpha, \beta)) (1 + O(x)), & x \to 0, \quad 2\alpha \notin \mathbb{Z} \\
\alpha^2 - \beta^2 + O(x) + O(x^{1+2\alpha}) + O(x^{1+2\alpha} \ln x), & x \to 0, \quad 2\alpha \in \mathbb{Z} \\
x^{-1+2\alpha} e^{-x} \frac{-1}{\Gamma(\alpha - \beta) \Gamma(\alpha + \beta)} (1 + O \left( \frac{1}{x} \right)), & x \to +\infty,
\end{cases}
\end{equation}
with
\begin{equation}
C(\alpha, \beta) = \frac{\Gamma(1 + \alpha + \beta) \Gamma(1 + \alpha - \beta)}{\Gamma(1 - \alpha + \beta) \Gamma(1 - \alpha - \beta)} \frac{1}{\Gamma(1 + 2\alpha)^2 (1 + 2\alpha)}.
\end{equation}

where $\Gamma(z)$ is Euler’s $\Gamma$-function. The path of integration in (3.3) is such as to avoid the set $\{x_1, \ldots, x_k\}$ and is contained within the sector $s_\delta$.

Note that (3.4) is the $\sigma$-form of the Painlevé V equation
\begin{equation}
u_{xx} = \left( \frac{1}{2u} + \frac{1}{u-1} \right) u_x^2 - \frac{1}{x} u_x + \frac{(u-1)^2}{x^2} \left( Au + B \right) + Cu + D \frac{u(u+1)}{u-1},
\end{equation}
with the parameters $A, B, C, D$ given by
\begin{equation}
A = \frac{1}{2} (\alpha - \beta)^2, \quad B = -\frac{1}{2} (\alpha + \beta)^2, \quad C = 1 + 2\beta, \quad D = -\frac{1}{2}.
\end{equation}

The points $x_j$ refer to possible poles of $\sigma(x)$. In the case when $\alpha$ is real and $\beta$ is purely imaginary, one can show [38] that $\sigma(x)$ is real analytic for $x > 0$, and the path of integration can therefore be chosen along the real axis.

If one takes the limit as $t \to 0$ on the r.h.s. of (3.2), one obtains the correct form of the appropriate Fisher-Hartwig asymptotics as given by Theorem 2.2.

Since the asymptotics of $D_n(f)$ are known both at $t = 0$ and $t = t_0$, one obtains an amusing identity for the Painlevé function $\sigma(x)$:
\begin{equation}
\Omega(+\infty) = -\ln \frac{G(1 + \alpha + \beta) G(1 + \alpha - \beta)}{G(1 + 2\alpha)}.
\end{equation}

Methods used in [38] to prove Theorem 3.1 can be adapted to describe other transition regimes, e.g., two singularities approaching each other along the unit circle, or emergence of an arc on which $f(z) = 0$. These situations arise for other correlation functions in integrable models [61] and appear in the application of random matrix theory to the theory of $L$-functions [35].

4. Asymptotics for Fredholm determinants

We now consider another type of double-scaling limit for Toeplitz determinants which yields interesting Fredholm determinants and, after certain analysis, allows us to obtain asymptotics of the latter. Note that this approach combined with Riemann-Hilbert-Problem techniques allows us to obtain the full asymptotics of these Fredholm determinants including
the multiplicative constants which resisted other methods: see [84] for a short review of the approach and [85, 41, 42, 9, 46] for details. For a review of other applications of Riemann-Hilbert problems to Toeplitz and Fredholm determinants see [48]. For analysis of some other Fredholm determinants, see [111, 83, 37] and references in Introduction.

Let \( f(e^{i\theta}; n) = 1 \) on the arc \( 2s/n \leq \theta \leq 2\pi - 2s/n, \) \( 0 < s < n, \) and \( f(z; n) = 0 \) on the rest of the unit circle. Then the Fourier coefficients are \( f_0 = 1 - 2s/(n\pi), f_j = -\frac{\sin(2sj/n)}{\pi j}, j \neq 0. \)

In the limit of growing \( n \) and, accordingly, a closing arc,

\[
\lim_{n \to \infty} D_n(f(z; n)) = \det(I - K_{\text{sine}}^{(s)}),
\]

where \( K_{\text{sine}}^{(s)} \) is the trace-class operator on \( L^2(-s, s) \) with kernel

\[
K_{\text{sine}}(x, y) = \frac{\sin(x - y)}{\pi(x - y)}.
\]

In the Gaussian Unitary Ensemble of random matrix theory (and many other random matrix ensembles [45, 56]), the Fredholm sine-kernel determinant \( \det(I - K_{\text{sine}}^{(s)}) \) describes, in the bulk scaling limit, the probability that an interval of length \( 2s \) contains no eigenvalues. Of interest are the asymptotics of \( \det(I - K_{\text{sine}}^{(s)}) \) when \( s \) is large.

**Theorem 4.1.** Let \( K_{\text{sine}}^{(s)} \) be the operator acting on \( L^2(-s, s), s > 0, \) with kernel (4.2). Then as \( s \to +\infty, \)

\[
\det(I - K_{\text{sine}}^{(s)}) = c_{\text{sine}} s^{-\frac{1}{2}} \exp \left( -\frac{s^2}{2} \right) \left[ 1 + O(s^{-1}) \right], \quad c_{\text{sine}} = 2^{1/12} e^{\zeta'(1)},
\]

and \( \zeta'(x) \) is the derivative of Riemann’s zeta function.

We note that \( s(d/ds) \ln \det(I - K_{\text{sine}}^{(s)}) \) satisfies [75, 40] a form of the Painlevé V equation. In particular, this fact enables one to reconstruct the full asymptotic series of the logarithmic derivative in the inverse powers of \( s \) from the first few terms provided the existence of such asymptotic expansion is established. However, the multiplicative constant \( c_{\text{sine}} \) is not determined this way.

Theorem 4.1 was conjectured by Dyson [51] who used, in particular, (4.1) and an earlier result of Widom on Toeplitz determinants with a symbol which vanishes on a fixed arc of the unit circle [107]. The leading asymptotic term was proved by Widom [106], and the lower-order terms apart from \( c_{\text{sine}}, \) in other words the expansion for the derivative \( (d/ds) \ln \det(I - K_{\text{sine}}^{(s)}) \), by Deift, Its, and Zhou [40] using Riemann-Hilbert methods. Application of a more detailed Riemann-Hilbert analysis to Toeplitz determinants allowed the authors in [85, 41] to extend the result of Widom [107] to varying arcs, and the relation (4.1) then produced the asymptotics of \( \det(I - K_{\text{sine}}^{(s)}) \) including \( c_{\text{sine}}, \) which completed the proof of Theorem 4.1. An alternative proof of the theorem was given independently by Ehrhardt [52] who used (different) methods of operator theory.

Recall that \( f(z; n) \) was defined above on the arc whose end-points converge to \( z = 1 \) as \( n \to \infty. \) We now modify the definition of \( f(z; n) \) by placing a Fisher-Hartwig singularity at \( z = 1. \) Namely, consider the symbol \( F(z; n) = |z - 1|^{2\alpha} z^\beta e^{-i\pi\beta}, \) \( z = e^{i\theta}, \) on the arc \( 2s/n \leq \theta \leq 2\pi - 2s/n, 0 < s < n, \) and \( F(z; n) = 0 \) on the rest of the unit circle. We then
of the Painlevé V equation. If we set

\[ K(\alpha, \beta, s) = \text{trace-class operator on } L^2(-s, s) \] with kernel

\[ \phi(a, c, z) = 1 + \sum_{n=1}^{\infty} \frac{a(a + 1) \cdots (a + n - 1)}{c(c + 1) \cdots (c + n - 1)} z^n. \]

The kernel \( K(\alpha, \beta, s) \) appears in the representation theory of the infinite-dimensional unitary group [29, 26], and the logarithmic derivative \( (d/ds) \ln \det(I - K(\alpha, \beta, s)) \) is related to a solution of the Painlevé V equation. If we set \( \alpha = \beta = 0 \), the kernel reduces to the sinc-kernel (4.2).

**Theorem 4.2.** Let \( K(\alpha, \beta, s) \) be the operator acting on \( L^2(-s, s) \), \( s > 0 \), with kernel (4.5). Then as \( s \to +\infty \),

\[ \det(I - K(\alpha, \beta, s)) = \frac{\sqrt{\pi} \Gamma(1/2) \Gamma(1 + 2\alpha)}{2^{2\alpha^2} \Gamma(1 + \alpha + \beta) \Gamma(1 + \alpha - \beta)} s^{-1 - \alpha^2 + \beta^2} \exp \left( -\frac{s^2}{2} + 2\alpha s \right) \left[ 1 + O(s^{-1}) \right], \]

where \( G(x) \) is Barnes’ \( G \)-function.

This theorem was proved in [46] using the relation (4.4) and a Riemann-Hilbert analysis. The theorem reduces to Theorem 4.1 if \( \alpha = \beta = 0 \) (recall that \( 2 \ln G(1/2) = (1/12) \ln 2 - \ln \sqrt{\pi} + 3\zeta'(1) \)).

A particular case of the determinant \( D_n(F(z; n)) \) with \( \beta = 0 \) is related, via a Hankel determinant and the formula (2.16), to the following Bessel-kernel determinant \( \det(I - K_{Bessel}(a, s)) \) on \( (0, s) \), where the kernel

\[ K_{Bessel}(a, s) = \frac{\sqrt{y} J_a(\sqrt{x}) J_a(\sqrt{y}) - \sqrt{x} J_a(\sqrt{y}) J_a(\sqrt{x})}{2(x - y)}, \]

and \( J_a(x) \) is Bessel function. In the Jacobi Unitary Ensemble of random matrix theory (and many other ensembles with a so-called hard edge), the Bessel-kernel determinant \( \det(I - K_{Bessel}(a, s)) \) describes, in the (left) edge scaling limit, the probability that the interval \( (0, s) \) contains no eigenvalues. In other words, it describes the distribution of the extreme (smallest) eigenvalue.
Theorem 4.3. Let $K_{Bessel}^{(a,s)}$ be the operator acting on $L^2(0,s)$, $s > 0$, with kernel (4.8) where $\Re a > -1$. Then as $s \to +\infty$,
\begin{equation}
\det(I - K_{Bessel}^{(a,s)}) = c_{Bessel}(a)s^{-a^2/4} \exp \left( -\frac{s}{4} + a\sqrt{s} \right) \left[ 1 + O(s^{-1/2}) \right], \quad c_{Bessel}(a) = \frac{G(1+a)}{(2\pi)^{a/2}}.
\end{equation}

These asymptotics were conjectured by Tracy and Widom [102] and proved in [46]. An alternative proof of the particular case $|\Re a| < 1$ is given in [54] by methods of operator theory.

Finally, we consider the case of the so-called Airy kernel. Let $w(x) = e^{-4xn}$ be supported on $J = [0, 1 + s(2n)^{-2/3}]$ for a fixed $s \in \mathbb{R}$, and let $D_n^H(w)$ be the corresponding Hankel determinant. Then
\begin{equation}
\lim_{n \to \infty} D_n^H \left( 1 + \frac{s}{(2n)^{2/3}} \right) = \det \left( I - K_{Airy}^{(s)} \right),
\end{equation}
where $K_{Airy}^{(s)}$ is the trace-class operator on $L^2(s, +\infty)$ with kernel
\begin{equation}
K_{Airy}^{(s)}(x, y) = \frac{\text{Ai}(x)\text{Ai}'(y) - \text{Ai}(y)\text{Ai}'(x)}{x - y}.
\end{equation}
Here $\text{Ai}(x)$ is the Airy function (see, e.g., [3]).

In the Gaussian Unitary Ensemble of random matrix theory (and many other ensembles with a so-called soft edge), the Airy-kernel determinant $\det(I - K_{Airy}^{(s)})$ describes, in the (right) edge scaling limit, the probability that the interval $(s, +\infty)$ contains no eigenvalues. In other words, it describes the distribution of the extreme (largest) eigenvalue.

Theorem 4.4. Let $K_{Airy}^{(s)}$ be the operator acting on $L^2(s, +\infty)$, $s \in \mathbb{R}$, with kernel (4.11). Then as $s \to -\infty$,
\begin{equation}
F_{TW}(s) = \det(I - K_{Airy}^{(s)}) = c_{Airy}|s|^{-1/8} \exp \left( -\frac{|s|^3}{12} \right) \left[ 1 + O(|s|^{-3/2}) \right], \quad c_{Airy} = 2^{1/24}e^{c'(-1)},
\end{equation}

where $F_{TW}(s)$ is known as the Tracy-Widom distribution. Tracy and Widom showed that [103]
\begin{equation}
F_{TW}(s) = \exp \left\{ -\int_s^\infty (x-s)u^2(x)dx \right\},
\end{equation}
where $u(x)$ is the Hastings-McLeod solution of the Painlevé II equation
\begin{equation}
u''(x) = xu(x) + 2u^3(x),
\end{equation}
specified by the following asymptotic condition:
\begin{equation}
u(x) \sim \text{Ai}(x) \quad \text{as} \quad x \to +\infty.
\end{equation}
The asymptotics of the logarithmic derivative $(d/ds)\ln F_{TW}(s)$ follow, up to a constant (which is in fact zero), from (4.15) and the known asymptotics of the Hastings-McLeod solution at $-\infty$. The constant $c_{airy}$ (as well as $c_{Bessel}$ above) was conjectured by Tracy and Widom using numerical computations and an analogy with the Dyson formula (4.3). The full proof of Theorem 4.4 was given in [42] using (4.10).
The determinant \( \det(\mathbf{I} - K^s_{\text{Airy}}) \) also describes the distribution of the longest increasing subsequence of random permutations. Namely, let \( \pi = i_1i_2\ldots i_N \) be a permutation in the group \( S_N \) of permutations of \( 1, 2, \ldots, N \). Then a subsequence \( i_{k_1}, i_{k_2}, \ldots i_{k_r}, k_1 < k_2 < \cdots < k_r \) of \( \pi \) is called an increasing subsequence of length \( r \) if \( i_{k_1} < i_{k_2} < \cdots < i_{k_r} \). Let \( \ell_N(\pi) \) denote the length of a longest increasing subsequence of \( \pi \) and let \( S_N \) have the uniform probability distribution. Then \( \ell_N(\pi) \) is a random variable, and

\[
F_{TW}(s) = \lim_{N \to \infty} \operatorname{Prob} \{ \pi \in S_N : (\ell_N(\pi) - 2\sqrt{N})N^{-1/6} \leq s \}
\]

This result was obtained by Baik, Deift, and Johansson [6] from the double-scaling limit \( N \to \infty, n \leq N \sim \lambda \), of the Toeplitz determinant \( \Delta_{n,\lambda} = D_n(\exp\{\sqrt{\lambda}(z + z^{-1})\}) \). As shown earlier by Gessel [63], this determinant is precisely the following generating function:

\[
\Delta_{n,\lambda} = \sum_{N=0}^{\infty} u_n(N) \frac{\lambda^N}{N!^2}, \quad u_n(N) = \#(\text{permutations } \pi \text{ in } S_N \text{ with } \ell_N(\pi) \leq n).
\]

An alternative proof of Theorem 4.4 based on this determinant was given in [9].

By the Robinson-Schensted-Knuth correspondence (see, e.g., [2]) a permutation \( \pi \) is related to a pair of Young tableaux (of the same shape) of integer plane partitions of \( N \). The number \( \ell_N(\pi) \) is the length of the first row of the related tableaux.

We also note that random permutations are related to last passage percolation and random vicious walks [59, 4, 5].

There are many related results and extensions of the above results on random partitions and permutations, which, in particular, involve asymptotic analysis of special Toeplitz, Hankel, and Toeplitz+Hankel determinants. This large and growing research area has many connections to geometry, group representation theory, and integrable models. For details and a selection of results, see [6, 78, 79, 28, 95, 7, 8, 73, 57] and references therein.

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