Quasitriangular chiral WZW model in a nutshell

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Abstract

We give the bare-bone description of the quasitriangular chiral WZW model for the particular choice of the Lu-Weinstein-Soibelman Drinfeld double of the affine Kac-Moody group. The symplectic structure of the model and its Poisson-Lie symmetry are completely characterized by two $r$-matrices with spectral parameter. One of them is ordinary and trigonometric and characterizes the $q$-current algebra. The other is dynamical and elliptic (in fact Felder’s one) and characterizes the braiding of $q$-primary fields.
1. Introduction. In reference [9], we have constructed a one-parameter deformation of the standard WZW model. It is the theory possessing a huge Poisson-Lie symmetry generated by left and right $q$-deformed current algebra. The goal of this paper is to offer the simplest possible description of the $q$-deformed chiral WZW theory for the particular choice of the affine Lu-Weinstein-Soibelman double. We address those readers who want to obtain the first acquaintance with the $q$-deformed WZW model by working with a representative simple example. Nothing will be derived or proved here and only those results will be presented which do not require any preliminary understanding of the Poisson-Lie world. In particular, we do not wish to bother the reader with more general choices of the Drinfeld double neither with the natural origin of the model from the symplectic reduction of a simpler system living on the centrally extended double.

2. Language. The classical actions of dynamical systems enjoying the Poisson-Lie symmetry look typically forbiddingly complicated when written in some of standard parametrizations of simple Lie groups. This is the case already for toy systems with few degrees of freedom, not even speaking about field theories. It is therefore necessary to develop a language suited for effective dealing with such models.

We can describe a classical dynamical system in more or less three different languages:

a) By defining a classical action on a space of fields. This way is best suited for the path integral quantization.

b) By identifying a symplectic manifold and the Hamiltonian function on it. This is good for the geometric quantization.

c) By picking up a representative set of (coordinate) functions and Hamiltonian and defining their mutual Poisson brackets. This is the starting point for the canonical quantization.

We stress that at the classical level all three languages are fully equivalent though in studying some particular feature of the system one of them can turn out to be more convenient than the others.

The standard WZW model [13] was invented in the language a). The language b) was then extensively used for the description of the (finite) Poisson-Lie symmetry of the chiral WZW model [8]. Finally, the language c) has been
also developed in papers devoted to the canonical quantization of the (chiral) WZW model [1, 3, 4, 2].

The quasitriangular WZW model [9] was conceived by thinking and working in the language b). However, the most explicit (by physicist’s taste) and economical description of this theory can be offered in the language c). This is what we shall do here. The language a) in the quasitriangular case in principle also exists. Indeed, the interrelation between a) and b) is well-known. We recall that for a couple (symplectic form $\omega$, Hamiltonian $H$) we can immediately write the first order classical action of the form

$$S = \int (d^{-1} \omega - H dt).$$

(1)

The reader will agree, however, that the formula (1) contains no additional insights with respect to b). Moreover, as we have already mentioned, the coordinate description of (1) is awful. We do not therefore expect usefulness of the path integral method in the quasitriangular WZW story.

3. Review of the standard chiral WZW model. Now we first review the definition of the standard WZW model [3] in the language c). Consider a compact, simple, connected and simply connected group $G$. Recall that the Weyl alcove $A_+$ is the fundamental domain of the action of the Weyl group on the maximal torus of $G$. Now the points of the phase space $P$ of the standard chiral WZW model are the maps $m : \mathbb{R} \to G$, fulfilling the monodromy condition

$$m(\sigma + 2\pi) = m(\sigma) M.$$

Here the monodromy $M$ sits in the Weyl alcove $A_+$ and it is convenient to parametrize it as

$$M = \exp (-2\pi i a^\mu H^\mu).$$

In other words, $a^\mu$ are coordinates on the alcove $A_+$ corresponding to the choice of the orthonormal basis $H^\mu$ in the Cartan subalgebra of $\text{Lie}(G)$.

The following matrix Poisson bracket (written in some representation of $G$) completely characterizes the symplectic structure of the standard non-deformed chiral WZW model [3]:

$$\{m(\sigma) \otimes m(\sigma')\}_{WZ} = (m(\sigma) \otimes m(\sigma')) B_0(a^\mu, \sigma - \sigma'),$$

(2)
where

\[ B_0(\alpha^\mu, \sigma) = -\frac{\pi}{\kappa} \left[ \eta(\sigma)(H^\mu \otimes H^\mu) - i \sum_{\alpha} |\alpha|^2 \exp \left( i\pi \eta(\sigma)(\alpha, H^\mu) a^\mu \right) \frac{\sin \left( \frac{\pi \langle \alpha, H^\mu \rangle}{\alpha^\mu} \right)}{\sin \left( \frac{\pi \langle \alpha, H^\mu \rangle}{\alpha^\mu} \right)} E^\alpha \otimes E^{-\alpha} \right]. \]

Here \( \kappa \) is the level and \( \eta(\sigma) \) is the function defined by

\[ \eta(\sigma) = 2\left[ \frac{\sigma}{2\pi} \right] + 1, \]

with \( \left[ \sigma/2\pi \right] \) being the largest integer less than or equal to \( \frac{\sigma}{2\pi} \).

An important \( \text{Lie}(G) \)-valued observable is the chiral Kac-Moody current

\[ j = \kappa m m^{-1}. \quad (3) \]

It generates the hamiltonian action of (the central extension of) the loop group \( LG \) on the phase space \( P \). This is reflected in the following matrix Poisson brackets which follow from (2) and (3):

\[ \{m(\sigma) \otimes j(\sigma')\}_{WZ} = 2\pi C \delta(\sigma - \sigma')(m(\sigma) \otimes 1), \quad (4) \]

\[ \{j(\sigma) \otimes j(\sigma')\}_{WZ} = \pi \delta(\sigma - \sigma')\left[ C, j(\sigma) \otimes 1 - 1 \otimes j(\sigma')\right] + \kappa 2\pi C \partial_\sigma \delta(\sigma - \sigma'), \quad (5) \]

where \( C \) is the Casimir element defined by

\[ C = \sum_\mu H^\mu \otimes H^\mu + \sum_{\alpha > 0} \frac{|\alpha|^2}{2} (E^{-\alpha} \otimes E^\alpha + E^\alpha \otimes E^{-\alpha}). \quad (6) \]

The Poisson bracket (5) corresponds to the non-deformed current algebra and (4) can be interpreted as a statement, that \( m(\sigma) \) is the Kac-Moody primary

\footnote{Our conventions for the normalization of the step generators \( E^\alpha \) of \( \text{Lie}(G) \) are as follows

\[ [H^\mu, E^\alpha] = \langle \alpha, H^\mu \rangle E^\alpha, \quad (E^\alpha)^\dagger = E^{-\alpha}; \]

\[ [E^\alpha, E^{-\alpha}] = \alpha^\vee, \quad [\alpha^\vee, E^{\pm \alpha}] = \pm 2 E^{\pm \alpha}, \quad (E^\alpha, E^{-\alpha})_{\varphi^C} = \frac{2}{|\alpha|^2}. \]

The element \( \alpha^\vee \) in the Cartan subalgebra of \( \text{Lie}(G^C) \) is called the coroot of the root \( \alpha \) and it is given by the formula

\[ \alpha^\vee = \frac{2}{|\alpha|^2} \langle \alpha, H^\mu \rangle H^\mu. \]}

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field. Finally, the relation (3) can be viewed as the classical version of the Knizhnik-Zamolodchikov equation [10].

The Hamiltonian of the non-deformed chiral theory is given by the Sugawara formula:

\[ H^{WZ} = -\frac{1}{2\kappa} (\kappa \partial_\sigma mm^{-1}, \kappa \partial_\sigma mm^{-1})_0^G. \]

It leads to the following simple time evolution in the phase space:

\[ [m(\sigma)](\tau) = m(\sigma - \tau). \]

4. q-current algebra. The concept of the \( q \)-deformation of the current algebra (5) was apparently first introduced in [11] who have worked out the complex case. The detailed discussion of the real case can be found in [9]. Here we shall need only the classical (Poisson bracket) story, which is based on the concept of a meromorphic classical \( r \)-matrix \( \hat{r}(\sigma) \in \text{Lie}(G) \otimes \text{Lie}(G) \) fulfilling the ordinary classical Yang-Baxter equation with spectral parameter, i.e.

\[
[r^{12}(\sigma_1 - \sigma_2), r^{13}(\sigma_1 - \sigma_3) + r^{23}(\sigma_2 - \sigma_3)] + [r^{13}(\sigma_1 - \sigma_3), r^{23}(\sigma_2 - \sigma_3)] = 0. \quad (7)
\]

The \( q \)-current \( L(\sigma) \) is a (hermitian in real case) matrix taking values in some representation of \( G^C \) and whose Poisson brackets are given by the formula

\[
\{L(\sigma) \otimes L(\sigma')\} = (L(\sigma) \otimes L(\sigma')) \varepsilon \hat{r}(\sigma - \sigma') + \varepsilon \hat{r}(\sigma - \sigma')(L(\sigma) \otimes L(\sigma')) \\
- (1 \otimes L(\sigma')) \varepsilon \hat{r}(\sigma - \sigma' + 2i\varepsilon \kappa)(L(\sigma) \otimes 1) - (L(\sigma) \otimes 1) \varepsilon \hat{r}(\sigma - \sigma' - 2i\varepsilon \kappa)(1 \otimes L(\sigma')). \quad (8)
\]

Here \( \kappa \) is the level and \( \varepsilon \) the deformation parameter related to \( q \) as \( q = e^{\varepsilon}. \)

Everywhere in this paper, \( \hat{r}(\sigma) \) will denote the following concrete trigonometric solution of the YB equation (7):

\[ \hat{r}(\sigma) = r + C\cotg \frac{1}{2} \sigma, \]

where \( C \) is given by (6) and \( r \) by

\[ r = \sum_{\alpha > 0} \frac{i|\alpha|^2}{2} (E^{-\alpha} \otimes E^\alpha - E^\alpha \otimes E^{-\alpha}). \]
5. Felder’s elliptic dynamical $r$-matrix. Consider the following $r$-matrix $B_\varepsilon(a^\mu, \sigma) \in \text{Lie}(G) \otimes \text{Lie}(G)$ (meromorphically) depending also on the coordinates $a^\mu$ of the alcove $\mathcal{A}_+$:

$$B_\varepsilon(a^\mu, \sigma) = -\frac{i}{\kappa} \rho \left( \frac{i \sigma}{2 \kappa \varepsilon}, \frac{i \pi}{\kappa \varepsilon} \right) H^\mu \otimes H^\mu - \frac{i}{\kappa} \sum_\alpha |\alpha|^2 \sigma^{a^\mu(a, H^\mu)} \left( \frac{i \sigma}{2 \kappa \varepsilon}, \frac{i \pi}{\kappa \varepsilon} \right) E^\alpha \otimes E^{-\alpha}.$$ 

The elliptic functions $\rho(z, \tau), \sigma_w(z, \tau)$ are defined as (cf. [6, 7, 5])

$$\sigma_w(z, \tau) = \frac{\theta_1(w - z, \tau) \theta_1'(0, \tau)}{\theta_1(w, \tau) \theta_1(z, \tau)}, \quad \rho(z, \tau) = \frac{\theta_1'(z, \tau)}{\theta_1(z, \tau)}.$$ 

Note that $\theta_1(z, \tau)$ is the Jacobi theta function\footnote{We have $\theta_1(z, \tau) = \vartheta_1(\pi z, \tau)$ with $\vartheta_1$ in [12].}

$$\theta_1(z, \tau) = -\sum_{j=-\infty}^{\infty} e^{\pi i (j+\frac{1}{2})^2 \tau + 2 \pi i (j+\frac{1}{2})(z+\frac{1}{2})},$$

the prime ‘ means the derivative with respect to the first argument $z$ and the argument $\tau$ (the modular parameter) is a nonzero complex number such that $\text{Im} \tau > 0$.

It is straightforward to check (it helps to use [12]), that $B_\varepsilon(a^\mu, \sigma)$ verifies the following relations

$$[B_\varepsilon, 1 \otimes H^\mu + H^\mu \otimes 1] = [B_\varepsilon^{12}, (H^\mu)^1 + (H^\mu)^2] = 0;$$

$$[B_\varepsilon^{12}(\sigma_1 - \sigma_2), B_\varepsilon^{13}(\sigma_1 - \sigma_3) + B_\varepsilon^{23}(\sigma_2 - \sigma_3)] + [B_\varepsilon^{13}(\sigma_1 - \sigma_3), B_\varepsilon^{23}(\sigma_2 - \sigma_3)] +$$

$$+ \frac{i}{\kappa} \left( \frac{\partial}{\partial a^\mu} B_\varepsilon^{12} \right) (H^\mu)^3 + \frac{i}{\kappa} \left( \frac{\partial}{\partial a^\mu} B_\varepsilon^{23} \right) (H^\mu)^1 + \frac{i}{\kappa} \left( \frac{\partial}{\partial a^\mu} B_\varepsilon^{31} \right) (H^\mu)^2 = 0.\quad (10)$$

Here (10) is called the dynamical Yang-Baxter equation with spectral parameter [3, 4].

6. The quasitriangular chiral WZW model. The phase space of the quasitriangular chiral WZW model is $P$; i.e. it is the same as that of its
non-deformed counterpart. The symplectic structure of the deformed model is completely characterized by the following matrix Poisson bracket:

\[ \{ m(\sigma) \otimes m(\sigma') \}_W = (m(\sigma) \otimes m(\sigma')) B\varepsilon(a^\mu, \sigma - \sigma') + \varepsilon \hat{r}(\sigma - \sigma')(m(\sigma) \otimes m(\sigma')), \]

(11)

where all notations have been already explained before. The properties of the elliptic functions (9) imply that

\[ \lim_{\varepsilon \to 0} B\varepsilon(a^\mu, \sigma) = B_0(a^\mu, \sigma). \]

This fact gives immediately the correct limit \( q \to 1 \) (cf. Eq. (2)).

The \( q \)-current \( L(\sigma) \) is given by the classical version of the \( q \)-KZ equation:

\[ L(\sigma) = m(\sigma + i\kappa\varepsilon)m^{-1}(\sigma - i\kappa\varepsilon). \]

(12)

From (11) and (12), it follows

\[ \{ m(\sigma) \otimes L(\sigma') \}_W = \]

\[ = \varepsilon \hat{r}(\sigma - \sigma' - i\varepsilon\kappa)(m(\sigma) \otimes L(\sigma')) - (1 \otimes L(\sigma'))\varepsilon \hat{r}(\sigma - \sigma' + i\varepsilon\kappa)(m(\sigma) \otimes 1) \]

(13)

and

\[ \{ L(\sigma) \otimes L(\sigma') \}_W = (L(\sigma) \otimes L(\sigma'))\varepsilon \hat{r}(\sigma - \sigma') + \varepsilon \hat{r}(\sigma - \sigma')(L(\sigma) \otimes L(\sigma')) \]

\[ - (1 \otimes L(\sigma'))\varepsilon \hat{r}(\sigma - \sigma' + 2i\varepsilon\kappa)(L(\sigma) \otimes 1) - (L(\sigma) \otimes 1)\varepsilon \hat{r}(\sigma - \sigma' - 2i\varepsilon\kappa)(1 \otimes L(\sigma')). \]

(14)

The relation (13) can be interpreted that \( m(\sigma) \) is the \( q \)-primary field and (14) is nothing but the defining relation (8) of the \( q \)-current algebra.

A few words about the \( q \to 1 \) limit: From the classical \( q \)-KZ equation (12), we derive

\[ L(\sigma) = 1 + 2i\varepsilon\kappa\partial_\sigma mm^{-1} + O(\varepsilon^2) = 1 + 2i\varepsilon j(\sigma) + O(\varepsilon^2). \]

Inserting this into (13) and (14), we obtain in the lowest order in \( \varepsilon \) the desired relations (4) and (5):

\[ \{ m(\sigma) \otimes j(\sigma') \}_W = 2\pi C\delta(\sigma - \sigma')(m(\sigma) \otimes 1); \]

\[ \{ j(\sigma) \otimes j(\sigma') \}_W = \pi \delta(\sigma - \sigma')[C, j(\sigma) \otimes 1 - 1 \otimes j(\sigma')] + 2\pi\kappa C\partial_\sigma\delta(\sigma - \sigma'). \]
It turns out that the flow

\[ [m(\sigma)](\tau) = m(\sigma - \tau) \]

on \( P \) is Hamiltonian also for the \( q \)-deformed symplectic structure (11). Its generator \( H^{qWZ} \) is the Hamiltonian of the quasitriangular chiral WZW model. The explicit formula for it is given by the relation (1.20) in [9]. We do not list it here in order not to break the basic promise expressed in the introduction: reading of this paper did not require any preliminary knowledge of the Poisson-Lie world.

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