Abstract

We consider a set of interwoven harmonic oscillators where the acceleration of a given oscillator is determined by the position of its nearest neighbor. We show that this problem of $N$ non-local oscillators with periodic boundary conditions leads to a $2N$-th order initial value problem. We discuss the numerical solution of this using a non-polynomial spline method. A very precise numerical method that minimizes the error can be developed, which we test for a few examples of driving forces.

Keywords: Non-local coupled harmonic oscillators, non-polynomial spline; consistency relations; end conditions; $N$th-order Initial Value Problem.

1 Introduction

Coupled harmonic oscillators are a standard paradigm in many engineering, physical, chemical or biological systems. The basic ingredient of the harmonic oscillator dynamics is the fact that the acceleration of each oscillator is proportional to its position, with a proportionality constant given by the negative of the frequency squared. In addition, for non-equilibrium situations there are time-dependent driving forces on each oscillator.

In this paper we are interested in a fundamental modification of this approach. We assume that the acceleration of each oscillator is proportional to the position of its neighboring oscillator. By this, of course, a strongly coupled structure is introduced, and a kind of non-local dynamics, since the acceleration of a given oscillator is determined by a position variable elsewhere. Such a dynamics is motivated for very strongly coupled individual systems. For example in [1, 2, 3] strongly coupled oscillator systems are studied, which degenerate to our type of dynamics in the infinite coupling limit. In this limit individual local oscillator behavior is influenced in a hierarchical way by nearest neighbors. We assume a driven non-equilibrium situation where each oscillator is also driven by individual time-dependent driving forces.

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We will show that the above dynamics, for \( N \) oscillators with periodic boundary conditions, leads to a high-order initial value problem, indeed of order \( 2N \). Thus rather large derivatives will become relevant if there are just a few oscillators coupled in this way. These types of initial value problems require effective numerical methods, which depend on the number \( N \). Useful in this context are non-polynomial spline methods. We will describe the optimum way to solve this system numerically for a given set of driving forces. In fact, we will show that there is an optimum method with minimum error, which is in particular useful if highest precision numerical results are required. As main examples, we will deal with the cases \( N = 2 \) and \( N = 3 \).

2 Nonlocal coupling of driven harmonic oscillators

To illustrate the idea we start with two uncoupled driven harmonic oscillators:

\[
\begin{align*}
\ddot{y}_1 + \omega_1^2 y_1 &= g_1(t) \\
\ddot{y}_2 + \omega_2^2 y_2 &= g_2(t)
\end{align*}
\]  

\( \omega_i \) is the frequency of oscillator \( i \), and \( g_i(t) \) is a local driving force on oscillator \( i \). As mentioned in the introduction, the basic idea is to consider a modified dynamics where the position \( y_1 \) of the first oscillator determines the acceleration force of the second one, and vice versa. That is to say, instead of the above trivial (uncoupled) dynamics we consider the following coupled dynamics:

\[
\begin{align*}
\ddot{y}_1 + \omega_1^2 y_2 &= g_1(t) \\
\ddot{y}_2 + \omega_2^2 y_1 &= g_2(t)
\end{align*}
\]  

Differentiating eq. (2.4) twice we get

\[
y_2^{(4)} + \omega_2^2 y_1^{(2)} = g_2^{(2)}(t)
\]  

where \( y^{(k)} \) denotes the \( k \)-th derivative with respect to time \( t \). Eliminating in this equation \( y_1^{(2)} \) using eq. (2.3) we get

\[
y_2^{(4)} + \omega_2^2 (g_1(t) - \omega_1^2 y_2) = g_2^{(2)}(t)
\]  

This is equivalent to a 4-th order equation of the form

\[
y_2^{(4)} + f(t)y_2(t) = g(t)
\]  

where

\[
f(t) = -\omega_1^2 \omega_2^2
\]  

and

\[
g(t) = g_2^{(2)}(t) - \omega_2^2 g_1(t).
\]  

Together with the initial position and velocity of the two oscillators, this leads to an initial value problem of 4th-order, for which we will describe the optimum numerical method in section 3.
But let us here first extend the problem, by considering $N = 3$ non-local oscillators, and later an arbitrary number $N$. The non-locally coupled dynamics for 3 oscillators reads

\begin{align}
\dot{y}_1 + \omega_1^2 y_2 &= g_1(t) \quad (2.10) \\
\dot{y}_2 + \omega_2^2 y_3 &= g_2(t) \\
\dot{y}_3 + \omega_3^2 y_1 &= g_3(t) \quad (2.12)
\end{align}

Differentiating eq. (2.12) twice we get

\begin{equation}
y_3^{(4)} + \omega_3^2 y_1^{(2)} = g_3^{(2)}(t). \quad (2.13)
\end{equation}

Eliminating in this equation $y_1^{(2)}$ using eq. (2.10) we get

\begin{equation}
y_3^{(4)} + \omega_3^2(g_1(t) - \omega_1^2 y_2) = g_3^{(2)}(t) \quad (2.14)
\end{equation}

or

\begin{equation}
y_3^{(4)} - \omega_1^2 \omega_3 y_2 = g_3^{(2)}(t) - \omega_3^2 g_1(t). \quad (2.15)
\end{equation}

Differentiating twice this leads to

\begin{equation}
y_3^{(6)} - \omega_1^2 \omega_3^2 y_2 = g_3^{(4)}(t) - \omega_3^2 g_1^{(2)}(t) \quad (2.16)
\end{equation}

and using (2.11) to eliminate $y_2^{(2)}$ we get

\begin{equation}
y_3^{(6)} + \omega_1^2 \omega_2^2 \omega_3^2 y_3 = g_3^{(4)}(t) - \omega_3^2 g_1^{(2)}(t) + \omega_2^2 \omega_1^2 g_2(t). \quad (2.17)
\end{equation}

Apparently, this can be written in the form

\begin{equation}
y_3^{(6)} + f(t)y_3(t) = g(t) \quad (2.18)
\end{equation}

where

\begin{equation}
f(t) = +\omega_1^2 \omega_2^2 \omega_3 
\end{equation}

and

\begin{equation}
g(t) = g_3^{(4)}(t) - \omega_3^2 g_1^{(2)}(t) + \omega_2^2 \omega_1^2 g_2(t). \quad (2.20)
\end{equation}

It is obvious how to generalize this problem to $N$ oscillators. In this case one obtains a $2N$-th order initial value problem of the form

\begin{equation}
y_N^{(2N)} + f(t)y_N = g(t) \quad (2.21)
\end{equation}

where

\begin{equation}
f(t) = (-1)^{N+1} \prod_{i=1}^{N} \omega_i^2 
\end{equation}

and $g(t)$ is a sum of various derivatives of local driving forces $g_i(t)$ weighted with frequencies.

The optimum numerical way to solve these high-order initial value problems depends on $N$ in a nontrivial way. In the following, we allow for general time-dependent functions $f(t)$ and deal in detail with the cases $N = 2$ and $N = 3$. 

3
3 N=2: Optimized numerical solution of the 4th order initial value problem

3.1 The initial value problem

The case of two oscillators \((N = 2)\) leads to the following fourth order initial value problem

\[
\begin{align*}
y^{(4)}(t) + f(t)y(t) &= g(t), \quad t \in [a, b], \\
y(a) &= u_0, \\
y^{(1)}(a) &= u_1, \\
y^{(2)}(a) &= u_2,
\end{align*}
\]

(3.23)

where the \(u_i(i = 0, 1, 2, 3)\) are finite real constants while the functions \(f(t)\) and \(g(t)\) are continuous on \([a, b]\). To simplify the notation, we have written \(y(t)\) instead of \(y_2(t)\).

Of course, from an engineering point we are interested in four initial values given by initial position and velocity of oscillator 1 and 2. These are related to the constants \(u_i\) by

\[
\begin{align*}
y_2(a) &= u_0 \\
y_2(a) &= u_1 \\
y_1(a) &= \frac{g_2(a) - u_2}{\omega_2^2} \\
y_1(a) &= \frac{\dot{g}_2(a) - u_3}{\omega_2^2}
\end{align*}
\]

(3.24-3.27)

We have reduced the 4-dimensional problem of the space-space structure of the two coupled nonlocal oscillators to a 1-dimensional initial value problem of 4-th order, for which we can apply very precise numerical methods, as described in the following subsection. While for our nonlocal oscillator example \(f(t) = \text{const} = -\omega_1^2\omega_2^2\), the numerical method developed in the following is applicable general time-dependent functions \(f(t)\), as long as they are continuous.

3.2 Nonpolynomial Spline Method

To develop the spline approximation to the problem (3.23), the interval \([a, b]\) is divided into \(n\) equal subintervals, using the grid points \(t_i = a + ih; i = 0, 1, \ldots, n\), where \(h = (b - a)/n\). Consider the following restriction \(S_i\) of the solution to each subinterval \([t_i, t_{i+1}]\), \(i = 0, 1, \ldots, n - 1\),

\[
S_i(t) = a_i \cos \omega(t - t_i) + b_i \sin \omega(t - t_i) + c_i(t - t_i)^3 + d_i(t - t_i)^2 + e_i(t - t_i) + p_i
\]

(3.28)

Let

\[
\begin{align*}
y_i &= S_i(t_i) \\
N_i &= S_i^{(4)}(t_i), \\
M_i &= S_i^{(2)}(t_i), \\
\end{align*}
\]

(3.29)

Following [4] and postulating that at the end points of the intervals the 1st and 3rd derivatives are continuous, one derives the following consistency relation between the values of
splines and their fourth order derivatives at border points:

\[
(\alpha h^4 N_{i-4} + \beta h^4 N_{i-3} + \gamma h^4 N_{i-2} + \beta h^4 N_{i-1} + \alpha h^4 N_i)
= [y_{i-4} - 4y_{i-3} + 6y_{i-2} - 4y_{i-1} + y_i]; \quad i = 4, 5, \ldots, n,
\]

where

\[
\alpha = \left(\frac{1}{6\theta \sin \theta} - \frac{1}{\theta^3 \sin \theta} + \frac{1}{\theta^4}\right), \quad \beta = \left(\frac{2(1 + \cos \theta)}{\theta^3 \sin \theta} - \frac{(\cos \theta - 2)}{3\theta} - \frac{4}{\theta^4}\right)
\]

and

\[
\gamma = \left(-\frac{2(1 + 2 \cos \theta)}{\theta^3 \sin \theta} + \frac{(1 - 4 \cos \theta)}{3\theta \sin \theta} + \frac{6}{\theta^4}\right).
\]

Here \(\theta = \omega h\) is an arbitrary parameter. The relation (3.30) forms a system of \(n - 3\) linear equations in the \(n\) unknowns \((y_i, i = 1, 2, \ldots, n)\), while \(N_i\) is taken from IVP (3.23) to be equal to \(-f_iy_i + g_i, i = 0, 1, \ldots, n\).

Following [5], three equations (end conditions) are determined to find the complete solution of \(y_i\)'s appearing in eq. (3.30), as given below:

\[
N_0 + N_4 = \frac{1}{h^4} \left[ \frac{-220}{9} y_0 + 40y_1 - 20y_2 + \frac{40}{9} y_3 - \frac{40}{3} h y_0^{(1)} - \frac{4}{3} h^4 y_0^{(4)} \right],
\]

\[
N_1 + N_5 = \frac{1}{h^4} \left[ \frac{18336}{575} y_1 - \frac{22992}{575} y_2 + \frac{4656}{575} y_3 + \frac{2736}{575} h y_0^{(1)}
+ \frac{15864}{575} h^2 y_0^{(2)} + \frac{6648}{575} h^3 y_0^{(3)} \right],
\]

and

\[
N_2 + N_6 = \frac{1}{h^4} \left[ \frac{8157}{865} y_2 - \frac{11424}{865} y_3 + \frac{3267}{865} y_4 + \frac{978}{173} h y_0^{(1)} + \frac{8958}{865} h^2 y_0^{(2)}
+ \frac{5684}{865} h^3 y_0^{(3)} \right].
\]

The local truncation errors associated with the linear equations (3.30) − (3.33) and (3.30) are calculated as

\[
\hat{t}_i = \begin{cases} 
-\frac{47}{9} h^6 y^{(6)}(t_1) + O(h^7), & i = 1, \\
-\frac{796}{575} h^6 y^{(6)}(t_2) + O(h^7), & i = 2, \\
-\frac{18336}{12975} h^6 y^{(6)}(t_3) + O(h^7), & i = 3, \\
\frac{1}{6}(-1 + 24 \alpha + 6 \beta) h^6 y^{(6)}(t_i) + O(h^7), & i = 4, 5, \ldots, n
\end{cases}
\]

and

\[
\|\hat{T}\| = ch^6 R_1 = O(h^6), \quad R_1 = \max_{t \in [a, b]} |y^{(6)}(t)|,
\]

where \(c\) is a constant which depends only on the values of \(\alpha\) and \(\beta\) and is independent of \(h\).
Let us mention that the solution obtained using the system of linear equations (3.31) and (3.30) is second order convergent. But if \( \alpha, \beta \) and \( \gamma \) are taken such that \( \alpha = -\frac{1}{1720}, \beta = \frac{31}{180}, \gamma = \frac{79}{120} \) then the order of the truncation error in eq. (3.30) is \( O(h^{10}) \) and the order of convergence can then be improved up to sixth order, using this method of improved order of end conditions.

The improved end conditions with truncation error of order \( O(h^{10}) \) are

\[
N_0 + \frac{433268}{2081} N_1 + \frac{330342}{2081} N_2 - \frac{16892}{2081} N_3 + N_4 \\
= \frac{1}{h^4} \left[ \begin{array}{c}
-68397280 \cdot y_0 + \frac{13366080}{2081} y_1 - \frac{7408800}{2081} y_2 + \frac{14781700}{2081} y_3 - \frac{10427200}{6243} h y_0^{(1)} + \frac{743680}{2081} h^2 y_0^{(2)} \\
+ \frac{259840}{2081} h^3 y_0^{(3)}
\end{array} \right], \tag{3.36}
\]

and

\[
N_1 - \frac{15609620732}{158360705} N_2 - \frac{40456201386}{158360705} N_3 - \frac{690708092}{158360705} N_4 + N_5 \\
= \frac{1}{h^4} \left[ \begin{array}{c}
180155114496 \cdot y_1 - \frac{34072682352}{31672141} y_2 + \frac{210168798336}{31672141} y_3 - \frac{49597629480}{31672141} y_4 + \frac{69181575120}{31672141} h y_0^{(1)} \\
+ \frac{42396452784}{31672141} h^2 y_0^{(2)} + \frac{7557647328}{31672141} h^3 y_0^{(3)}
\end{array} \right], \tag{3.37}
\]

The truncation errors of the corresponding equations are

\[
\tilde{t}_i = \begin{cases}
-0.3034 h^{10} y^{(10)}(t_1) + O(h^{11}), & i = 1, \\
-1.4034 h^{10} y^{(10)}(t_2) + O(h^{11}), & i = 2, \\
-1.0163 h^{10} y^{(10)}(t_3) + O(h^{11}), & i = 3, \\
\frac{1}{30240} (-17 + 5376 \alpha + 84 \beta) h^{10} y^{(10)}(t_i) + O(h^{11}), & i = 4, 5, \ldots, n
\end{cases} \tag{3.39}
\]

and

\[
\| \tilde{T} \| = c h^{10} R_2 = O(h^{10}), \quad R_2 = \max_{t \in [a, b]} |y^{(10)}(t)|, \tag{3.40}
\]

where \( c \) is a constant which depends only on the values of \( \alpha \) and \( \beta \) and is independent of \( h \).

To illustrate the powerfulness of the method, two analytically solvable examples are discussed in the following:

### 3.3 Examples

**Example 1**
Consider the following IVP

\[
\begin{align*}
    y^{(4)}(t) - y(t) &= 4\cos(t), \quad t \in [-1, 1], \\
    y(-1) &= -2\sin(1), \\
    y'(1) &= -2\cos(1) + \sin(1), \\
    y''(-1) &= -2\cos(1) + 2\sin(1), \\
    y'''(-1) &= -2\cos(1) - 3\sin(1).
\end{align*}
\]

(3.41)

The analytic solution of the above problem is

\[ y(t) = (1 - t) \sin(t). \]

The observed maximum errors (in absolute values) associated with \( y_i \), for the problem (3.41), corresponding to the different values of \( \alpha, \beta \) and \( \gamma \), are summarized in Table 1. It is confirmed from Table 1 that if \( h \) is reduced by factor 1/2, then \( \|E\| \) is reduced by a factor 1/4, which indicates that the method gives second-order results.

Table 1: Maximum absolute errors for problem (3.41) in \( y_i \).

| \( n \) | \( \alpha = 0, \beta = 0 \) | \( \alpha = 1/2, \beta = 1/2 \) | \( \alpha = 1/6, \beta = 1/6 \) |
|------|-----------------|-----------------|-----------------|
| \( \gamma = 1 \) | 6.74 \times 10^{-1} | 3.6 \times 10^0 | 1.73 \times 10^0 |
| \( \gamma = -1 \) | 5.77 \times 10^{-2} | 7.3 \times 10^{-1} | 2.22 \times 10^{-1} |
| \( \gamma = 1/3 \) | 3.3 \times 10^{-3} | 4.5 \times 10^{-2} | 1.3 \times 10^{-2} |
| \( \gamma = 1/3 \) | 1.48 \times 10^{-4} | 2.1 \times 10^{-3} | 5.93 \times 10^{-4} |
The observed maximum errors (absolute values) associated with $y_i$ for the problem (3.41), corresponding to the use of improved end conditions, are summarized in Table 2. A significant improvement of precision is obtained.

### Table 2: Maximum absolute errors for problem (3.41) in $y_i$.  

| $n$ | $|y(t_i) - y_i|$ |
|-----|-----------------|
| 6   | $1.7 \times 10^{-3}$ |
| 12  | $1.17 \times 10^{-5}$ |
| 24  | $7.19 \times 10^{-8}$ |
| 48  | $7.72 \times 10^{-11}$ |

**Example 2**

Consider the following IVP

$$
\begin{align*}
    y^{(4)}(t) + ty(t) &= -e^t(8 + 7t + t^3), \\
    y(0) &= 0, \\
    y^{(2)}(0) &= 0, \\
    y^{(3)}(0) &= -3.
\end{align*}
$$

(3.42)

The corresponding analytic solution is now

$$
y(t) = t(1 - t) e^t.
$$

The observed maximum errors for different values of $\alpha$, $\beta$ and $\gamma$ are summarized in Table 3. It is confirmed from Table 3 that if $h$ is reduced by factor $1/2$, then $\|E\|$ is reduced by a factor $1/4$, which indicates that the method gives second-order results.

The observed maximum errors using improved improved end conditions are summarized in Table 4.

### 4 N=3: Optimized numerical solution of the 6th order initial value problem

#### 4.1 Non-polynomial spline solution for N=3

A similar optimized numerical method can be developed for the case $N = 3$. In this case the IVP reads
Table 3: Maximum absolute errors for problem (3.42) in $y_i$.  

| $n$ | $\alpha = 0, \beta = 0$ | $\alpha = 1/2, \beta = 1/2$ | $\alpha = 1/6, \beta = 1/6$ |
|-----|-------------------------|-----------------------------|-----------------------------|
|     | $\gamma = 1$            | $\gamma = -1$               | $\gamma = 1/3$              |
| 6   | $1.14 \times 10^{-1}$   | $2.31 \times 10^{-2}$       | $6.86 \times 10^{-2}$       |
| 12  | $1.14 \times 10^{-2}$   | $1.55 \times 10^{-2}$       | $2.4 \times 10^{-3}$        |
| 24  | $1.4 \times 10^{-3}$    | $4.8 \times 10^{-3}$        | $6.40 \times 10^{-4}$       |
| 48  | $2.18 \times 10^{-4}$   | $1.3 \times 10^{-3}$        | $2.87 \times 10^{-4}$       |

Table 4: Maximum absolute errors for problem (3.42) in $y_i$.  

| $n$ | $|y(t_i) - y_i|$ |
|-----|-----------------|
| 6   | $2.53 \times 10^{-5}$ |
| 12  | $1.53 \times 10^{-7}$ |
| 24  | $1.06 \times 10^{-9}$ |
| 48  | $1.09 \times 10^{-10}$ |
\[
\begin{align*}
y^{(6)}(t) + f(t)y(t) &= g(t), \quad t \in [a, b], \\
y(a) &= u_0, \quad y^{(1)}(a) = u_1, \\
y^{(2)}(a) &= u_2, \quad y^{(3)}(a) = u_3, \\
y^{(4)}(a) &= u_4, \quad y^{(5)}(a) = u_5,
\end{align*}
\]

Again the interval \([a, b]\) is divided into \(n\) equal subintervals, using the grid points \(t_i = a + ih\) \((i = 0, 1, \ldots, n)\), where \(h = (b - a)/n\).

Again we consider the restrictions \(S_i\) of the solution to each subinterval \([t_i, t_{i+1}]\), \(i = 0, 1, \ldots, n - 1,\)

\[
S_i(t) = a_i \cos \omega(t - t_i) + b_i \sin \omega(t - t_i) + c_i(t - t_i)^5 + d_i(t - t_i)^4 + e_i(t - t_i)^3 + q_i(t - t_i)^2 + r_i(t - t_i) + v_i.
\]

and define

\[
\begin{align*}
y_i &= S_i(t_i), \\
L_i &= S^{(6)}_i(t_i), \\
M_i &= S^{(2)}_i(t_i), \\
N_i &= S^{(4)}_i(t_i),
\end{align*}
\]

\(i = 0, 1, \ldots, n.\)

We denote by \(y(t)\) the exact solution of the IVP \((4.43)\) and \(y_i\) is the approximation to \(y(t_i)\), obtained by the spline \(S(t_i)\). From continuity of the first, third and fifth derivatives at the border points, \(i.e.\) \(S^{(\mu)}_{i-1}(t_i) = S^{(\mu)}_i(t_i), \mu = 1, 3\) and 5, one gets

\[
\begin{align*}
&h^6 L_{i-6} \left( \theta - \sin \theta \right) \frac{\theta^3 \sin \theta}{\theta^6 \sin \theta} + \frac{1}{60 \theta^3 \sin \theta} + \frac{1}{120 \theta \sin \theta} \\
+h^6 L_{i-5} \left( \frac{6}{\theta^6} - \frac{2(\cos \theta + 2)}{\theta^3 \sin \theta} + \frac{\cos \theta - 1}{3 \theta^3 \sin \theta} - \frac{3 \theta - 13}{60 \theta \sin \theta} \right) \\
+h^6 L_{i-4} \left( \frac{8 \cos \theta + 7}{\theta^3 \sin \theta} - \frac{15}{\theta^6} + \frac{4 \cos \theta + 5}{3 \theta^3 \sin \theta} - \frac{33 \cos \theta - 13}{30 \theta \sin \theta} \right) \\
+h^6 L_{i-3} \left( \frac{20}{\theta^6} - \frac{2(6 \cos \theta + 1)}{\theta^3 \sin \theta} - \frac{2(3 \cos \theta + 1)}{3 \theta^3 \sin \theta} - \frac{33 \cos \theta - 13}{30 \theta \sin \theta} \right) \right) \\
+h^6 L_{i-2} \left( \frac{8 \cos \theta + 7}{\theta^3 \sin \theta} - \frac{15}{\theta^6} + \frac{4 \cos \theta + 5}{3 \theta^3 \sin \theta} - \frac{33 \cos \theta - 13}{30 \theta \sin \theta} \right) \\
+h^6 L_{i-1} \left( \frac{6}{\theta^6} - \frac{2(\cos \theta + 2)}{\theta^3 \sin \theta} + \frac{\cos \theta - 1}{3 \theta^3 \sin \theta} - \frac{3 \theta - 13}{60 \theta \sin \theta} \right) \\
+h^6 L_i \left( \theta - \sin \theta \right) \frac{\theta^3 \sin \theta}{\theta^6 \sin \theta} - \frac{1}{60 \theta^3 \sin \theta} + \frac{1}{120 \theta \sin \theta},
\end{align*}
\]

\(i = 6, 7, \ldots, n,\)

which can further be written as

\[
(\alpha h^6 L_{i-6} + \beta h^6 L_{i-5} + \gamma h^6 L_{i-4} + \delta h^6 L_{i-3} + \epsilon h^6 L_{i-2} + \theta h^6 L_{i-1} + \alpha h^6 L_i)
\]

\[
= \left[ y_{i-6} - 6y_{i-5} + 15y_{i-4} - 20y_{i-3} + 15y_{i-2} - 6y_{i-1} + y_i \right];
\]

\(i = 6, 7, \ldots, n,\)

\[
(4.46)
\]

where

\[
\alpha = \left( \frac{\theta - \sin \theta}{\theta^6 \sin \theta} - \frac{1}{6 \theta^3 \sin \theta} + \frac{1}{12 \theta \sin \theta} \right),
\]

\[
\beta = \left( \frac{6}{\theta^6} - \frac{2(\cos \theta + 2)}{\theta^3 \sin \theta} + \frac{\cos \theta - 1}{3 \theta^3 \sin \theta} - \frac{3 \theta - 13}{60 \theta \sin \theta} \right),
\]

\[
\gamma = \left( \frac{8 \cos \theta + 7}{\theta^3 \sin \theta} - \frac{15}{\theta^6} + \frac{4 \cos \theta + 5}{3 \theta^3 \sin \theta} - \frac{33 \cos \theta - 13}{30 \theta \sin \theta} \right).
\]
and
\[
\delta = \begin{pmatrix}
\frac{20}{\theta^6} & \frac{2(6 \cos \theta + 4)}{\theta^5 \sin \theta} & \frac{2(3 \cos \theta + 1)}{3 \theta^3 \sin \theta} & \frac{33 \cos \theta - 13}{30 \theta^2 \sin \theta}
\end{pmatrix}.
\]

Here \( \theta = \omega h \). The relation (4.47) forms a system of \( n-5 \) linear equations in the \( n \) unknowns \( (y_i, \ i = 1, 2, \ldots, n) \), while \( L_i \) is taken from IVP (3.23) to be equal to \(-f_i y_i + g_i\), \( i = 0, 1, \ldots, n \).

Following [6], five equations (end conditions) are determined to find the complete solution of \( y_i s \) appearing in eq. (4.47), as given below:

\[
L_0 + L_4 = \frac{1}{h^6} \left[ \begin{array}{c}
\frac{2905}{12} y_0 - 336 y_1 + 126 y_2 - \frac{112}{3} y_3 + \frac{21}{4} y_4 + 175 h y_0^{(1)} \\
+ 42 h^2 y_0^{(2)} - \frac{4}{5} h^6 y_0^{(6)}
\end{array} \right],
\]

(4.48)

\[
L_1 + L_5 = \frac{1}{h^6} \left[ \begin{array}{c}
\frac{797790}{21983} y_1 - \frac{1660890}{21983} y_2 + \frac{1299060}{21983} y_3 - \frac{523110}{21983} y_4 \\
+ \frac{87150}{21983} y_5 + \frac{283500}{21983} h y_0^{(1)} + \frac{172620}{21983} h^2 y_0^{(2)} - \frac{40167}{21983} h^6 y_1^{(6)}
\end{array} \right],
\]

(4.49)

\[
L_2 + L_6 = \frac{1}{h^6} \left[ \begin{array}{c}
\frac{22267}{50887} y_0 - \frac{108239440}{22267} y_3 + \frac{1103910}{22267} y_4 - \frac{446800}{22267} y_5 + \frac{5949805}{1803627} y_6 \\
+ \frac{675200}{85887} h y_0^{(1)} + \frac{700180}{66081} h^2 y_0^{(2)} + \frac{851440}{200403} h^3 y_0^{(3)}
\end{array} \right],
\]

(4.50)

\[
L_3 + L_7 = \frac{1}{h^6} \left[ \begin{array}{c}
\frac{12961750}{2016477} y_0 + \frac{44149995}{1568371} y_1 - \frac{23862240}{1568371} y_4 + \frac{77684300}{4705113} h^2 y_0^{(2)} - \frac{15282605}{4705113} h^3 y_0^{(3)}
\end{array} \right],
\]

(4.51)

and

\[
L_4 + L_8 = \frac{1}{h^6} \left[ \begin{array}{c}
\frac{49567095}{12837314} y_1 - \frac{34289280}{6418657} y_5 + \frac{19011465}{12837314} y_6 + \frac{2182545}{916951} h y_0^{(1)} \\
+ \frac{59244435}{6418657} h^2 y_0^{(2)} + \frac{115282605}{6418657} h^3 y_0^{(3)}
\end{array} \right].
\]

(4.52)

The local truncation errors associated with the linear equations (4.48) – (4.52) and (4.47) are calculated, as

\[
\tilde{t}_i = \begin{cases}
-4.75 h^8 y^{(8)}(t_1) + O(h^9), & i = 1, \\
-5.0467 h^8 y^{(8)}(t_2) + O(h^9), & i = 2, \\
-5.9909 h^8 y^{(8)}(t_3) + O(h^9), & i = 3, \\
-12.3201 h^8 y^{(8)}(t_4) + O(h^9), & i = 4, \\
-23.7869 h^8 y^{(8)}(t_5) + O(h^9), & i = 5, \\
-1 + 2\alpha + 2\beta + 2\gamma + \delta h^6 y^{(6)}(t_i) & (4.53)
\end{cases}
\]

\[
\tilde{t}_i = \begin{cases}
\frac{1}{2}(-1 + 36\alpha + 16\beta + 4\gamma) h^8 y^{(8)}(t_i) + O(h^9), & i = 6, 7, \ldots, n.
\end{cases}
\]

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To make the truncation errors of the system (4.47) of order $h^8$, $\alpha$, $\beta$, $\gamma$ and $\delta$ are taken such that $\alpha + \beta + \gamma + \delta = \frac{1}{2}$ and then

$$\|T\| = ch^8 R_3 = O(h^8), \quad R_3 = \max_{t \in [a, b]} |y^{(8)}(t)|,$$

(4.54)

where $c$ is a constant.

The solution obtained using the system of linear equations (4.48) – (4.52) and (4.47) in general is second order convergent. Again, however, the order of accuracy of the method can be improved significantly to $h^8$. The local truncation error of the system (4.47) can be expressed in the following form

$$\tilde{t}_i = \begin{cases} 
(-1 + 2\alpha + 2\beta + 2\gamma + \delta)h^6 y^{(6)}(t_i) + \frac{1}{4}(-1 + 36\alpha + 16\beta + 4\gamma)h^8 y^{(8)}(t_i) \\
+ \frac{1}{240}(-7 + 1620\alpha + 320\beta + 20\gamma)h^{10} y^{(10)}(t_i) \\
+ \frac{1}{120960}(-13 + 39366\alpha + 1536\beta + 6\gamma)h^{12} y^{(12)}(t_i) \\
+ \frac{1}{159667200}(-651 + 5196312\alpha + 90112\beta + 88\gamma)h^{14} y^{(14)}(t_i) \\
+ O(h^{16}),
\end{cases}$$

(4.55)
i = 6, 7, \ldots, n,

Thus, the order of the truncation error $\tilde{t}_i$ can be improved to be of order $h^{14}$ and correspondingly the order of method can be improved up to $h^8$, if $\alpha = \frac{1}{30240}$, $\beta = 41/5040$, $\gamma = 2189/10080$, $\delta = 4153/159667200$. For other choices of the parameters (not listed here), one can make the method to be of order $h^2$, $h^4$, $h^6$, respectively. Results corresponding to the order $h^2$, $h^4$, $h^6$ and $h^8$ are described in the following section.

4.2 Test of the method with analytically solvable examples

Example 3

Consider the IVP

$$\begin{align*}
y^{(6)}(t) - y(t) &= -6e^t, \quad 0 \leq t \leq 1 \\
y(0) &= 1, \quad y^{(1)}(0) = 0, \\
y^{(2)}(0) &= -1, \quad y^{(3)}(0) = -2, \\
y^{(4)}(0) &= -3, \quad y^{(5)}(0) = -4.
\end{align*}$$

(4.56)

The analytic solution of this IVP (4.56) is

$$y(t) = (1 - t) e^t.$$

The observed maximum errors are summarized in Table 5. It is confirmed from Table 5 that if $h$ is reduced by factor 1/2, then $\|E\|$ is reduced by a factor 1/4, which indicates that the method gives second-order results.

The observed maximum errors (in absolute values) associated with $y_i$, for the problem (4.56), corresponding to different orders of method are summarized in Table 6.
Table 5: Maximum absolute errors for problem (4.56) in \( y_i \).

| \( n \) | \( \alpha = 1/120, \beta = 15/120 \) | \( \alpha = \frac{1}{720}, \beta = \frac{1}{36} \) | \( \alpha = \frac{1}{5040}, \beta = \frac{6}{5040} \) |
|---|---|---|---|
| \( \gamma = 1/4, \ \delta = 28/120 \) | \( 7.98 \times 10^{-4} \) | \( 9.13 \times 10^{-4} \) | \( 9.51 \times 10^{-4} \) |
| \( 8 \) | \( 7.50 \times 10^{-5} \) | \( 9.64 \times 10^{-5} \) | \( 1.03 \times 10^{-4} \) |
| \( 16 \) | \( 5.45 \times 10^{-6} \) | \( 1.02 \times 10^{-5} \) | \( 1.18 \times 10^{-5} \) |
| \( 32 \) | \( 1.28 \times 10^{-7} \) | \( 9.42 \times 10^{-7} \) | \( 1.37 \times 10^{-6} \) |
| \( 64 \) | \( 1.10 \times 10^{-6} \) | \( 8.99 \times 10^{-9} \) | \( 4.80 \times 10^{-7} \) |

Table 6: Maximum absolute errors for problem (4.56) in \( y_i \).

| \( n \) | \( O(h^4) \) | \( O(h^6) \) | \( O(h^8) \) |
|---|---|---|---|
| \( 8 \) | \( 4.04 \times 10^{-5} \) | \( 2.07 \times 10^{-1} \) | \( 2.13 \times 10^{-1} \) |
| \( 16 \) | \( 1.10 \times 10^{-6} \) | \( 8.99 \times 10^{-9} \) | \( 4.80 \times 10^{-7} \) |
Example 4

The second example is

\[
\begin{align*}
\dot{y}^{(6)}(t) + y(t) &= 6(2t \cos(t) + 5 \sin(t)), \quad -1 \leq t \leq 1 \\
y(-1) &= 0, \quad \dot{y}(1)(-1) = 2 \sin(1), \\
\ddot{y}^{(2)}(-1) &= -4 \cos(1) - 2 \sin(1), \quad \dot{y}^{(3)}(-1) = 6 \cos(1) - 6 \sin(1), \\
\dddot{y}^{(4)}(-1) &= 8 \cos(1) + 12 \sin(1), \quad \ddot{y}^{(5)}(-1) = -20 \cos(1) + 10 \sin(1).
\end{align*}
\]

(4.57)

with analytic solution

\[ y(t) = (t^2 - 1) \sin(t). \]

The observed maximum errors (in absolute values) associated with \( y_i \), for the system (4.57), corresponding to the different values of \( \alpha, \beta, \gamma \) and \( \delta \) are summarized in Table 7. Again it is confirmed from Table 7 that if \( h \) is reduced by factor 1/2, then \( ||E|| \) is reduced by a factor 1/4, which indicates that the method gives second-order results.

The observed maximum errors corresponding to the different orders of the method are summarized in Table 8.

Table 7: Maximum absolute errors for problem (4.57) in \( y_i \).

| \( n \) | \( \alpha = 1/120, \beta = 15/120 \) | \( \alpha = 1/20, \beta = 1/60 \) | \( \alpha = 1/5040, \beta = 6/504 \) |
|---|---|---|---|
| \( \gamma = 1/4, \delta = 28/120 \) | \( 7.35 \times 10^{-2} \) | \( 9.64 \times 10^{-2} \) | \( 1.03 \times 10^{-1} \) |
| 16 | \( 1.01 \times 10^{-2} \) | \( 1.62 \times 10^{-2} \) | \( 1.82 \times 10^{-2} \) |
| 32 | \( 4.51 \times 10^{-4} \) | \( 2.0 \times 10^{-3} \) | \( 2.5 \times 10^{-3} \) |
| 64 | \( 1.98 \times 10^{-4} \) | \( 1.79 \times 10^{-4} \) | \( 3.05 \times 10^{-4} \) |
| 128 | \( 1.98 \times 10^{-4} \) | \( 1.79 \times 10^{-4} \) | \( 3.05 \times 10^{-4} \) |

5 Conclusion and Outlook

In this paper we started from a set of \( N \) nonlocal coupled harmonic oscillators, each driven by a driving force. We showed that this leads to an \( 2N \)-th order initial value problem (IVP) in a single variable. In a sense this gives ‘physical meaning’ to high-order IVP in one variable, which so far have mainly been looked at without any physical interpretation. Engineering applications include strongly coupled oscillator problems where the state of a local oscillator is strongly influenced by the position of the nearest neighbor.
Table 8: Maximum absolute errors for problem (4.57) in $y_i$.

| $n$ | $O(h^4)$     | $O(h^6)$     | $O(h^8)$     |
|-----|--------------|--------------|--------------|
| 8   | $2.31 \times 10^{-2}$ | $2.87 \times 10^{-1}$ | $2.98 \times 10^{-1}$ |
| 16  | $8.6 \times 10^{-3}$     | $7.98 \times 10^{-5}$  | $9.93 \times 10^{-8}$  |

By implementing improved end conditions, a very precise numerical method could be developed to solve this system numerically. In fact, we believe it is one of the most precise methods known in the field. Apparently our results are relevant to find very precise numerical solution schemes for higher-dimensional differential equations. We showed that a transformation of an $N$-dimensional 2nd order differential equation to a 1-dimensional differential equation of order $2N$ can be highly advantageous from a numerical point of view. One can implement improved end conditions that allow for a significant reduction of the error. After the $2N$-th order IVP has been solved very precisely, the solution can be translated back into the original physical setting of $N$ nonlocal oscillators.

While we have explicitly worked out the cases $N = 2$ and $N = 3$, in principle our method can be extended to higher values of $N$, though the complexity of the formulas used to minimize the truncation error increases rapidly.

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