JOINT MOMENTS OF A CHARACTERISTIC POLYNOMIAL AND ITS DERIVATIVE FOR THE CIRCULAR $\beta$-ENSEMBLE

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The problem of calculating the scaled limit of the joint moments of the characteristic polynomial, and the derivative of the characteristic polynomial, for matrices from the unitary group with Haar measure first arose in studies relating to the Riemann zeta function in the thesis of Hughes. Subsequently, Winn showed that these joint moments can equivalently be written as the moments for the distribution of the trace in the Cauchy unitary ensemble and furthermore that they relate to certain hypergeometric functions based on Schur polynomials, which enabled explicit computations. We give a $\beta$-generalisation of these results, where now the role of the Schur polynomials is played by the Jack polynomials. This leads to an explicit evaluation of the scaled moments and the trace distribution for all $\beta > 0$, subject to the constraint that a particular parameter therein is equal to a nonnegative integer. Consideration is also given to the calculation of the moments of the singular statistic $\sum_{j=1}^{N} 1/x_j$ for the Jacobi $\beta$-ensemble.

1. Introduction

1A. The particular circular $\beta$-ensemble joint moments and statement of their scaled large $N$ form.

With motivations in the application of random matrix theory to the study of statistical properties of the Riemann zeta function in accordance with the hypothesis of Keating and Snaith [35], Hughes [34] considered the quantity

$$F_N(s, h) = \int_U \left| V_U(\theta) \right|_{\theta=0}^{2(s-h)} \left| \frac{d}{d\theta} V_U(\theta) \right|_{\theta=0}^{2h} d\mu_N(U), \quad -\frac{1}{2} < h < s + \frac{1}{2}. \quad (1-1)$$

In (1-1), $U \in U(N)$, the group of $N \times N$ unitary matrices, which are to be chosen with Haar measure. The latter is indicated in (1 -1) by the symbol $d\mu_N(U)$, which furthermore is to be normalised so that its integration over $U$ gives unity. With $\{e^{i\theta_j}\}_{j=1}^{N}$ denoting the eigenvalues of $U$, the function $V_U(\theta)$ appearing in (1-1) is specified by

$$V_U(\theta) = e^{iN(\theta+\pi)/2-i\sum_{j=1}^{N} \theta_j/2} \prod_{j=1}^{N} (1-e^{-i(\theta-\theta_j)})^2; \quad (1-2)$$

note in particular that (1-2) is real for $\theta$ real, and its absolute value is equal to the absolute value of the characteristic polynomial of $U$.

The integrand of (1-1) is a function of the eigenvalues of $U$ only. Since the work of Weyl [46] — see the review [18] for historical context — it has been known that the Haar measure $d\mu_N(U)$, upon changing

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variables to the eigenvalues and eigenvectors, decomposes to an eigenvalue-dependent factor, and an eigenvector-dependent factor. This allows the latter to be integrated over, reducing (1-1) to read

$$F_N(s, h) = \frac{1}{N!(2\pi)^N} \int_0^{2\pi} d\theta_1 \cdots \int_0^{2\pi} d\theta_N |V_U(\theta)|_{\theta=0}^{2(s-h)} \left| \frac{d}{d\theta} V_U(\theta) \right|_{\theta=0}^{2h} \prod_{1 \leq j < k \leq N} |e^{i\theta_j} - e^{i\theta_j}|^2. \quad (1-3)$$

There is a natural generalisation of (1-3) obtained by replacing the power of 2 in the product over \( j < k \) by the power of \( \beta \),

$$F_{N,\beta}(s, h) = \frac{1}{C_{N,\beta}} \int_0^{2\pi} d\theta_1 \cdots \int_0^{2\pi} d\theta_N |V_U(\theta)|_{\theta=0}^{2(s-h)} \left| \frac{d}{d\theta} V_U(\theta) \right|_{\theta=0}^{2h} \prod_{1 \leq j < k \leq N} |e^{i\theta_j} - e^{i\theta_j}|^{\beta}, \quad (1-4)$$

where

$$C_{N,\beta} = (2\pi)^N \frac{\Gamma(\beta N/2 + 1)}{(\Gamma(1 + \beta/2))^N}. \quad (1-5)$$

When multiplied by \( 1/C_{N,\beta} \), the product involving \( \beta \) in (1-4) for \( \beta = 1 \) and 4 is the eigenvalue probability density function (PDF) for particular random matrix ensembles constructed out of unitary matrices \( U \) with Haar measure. For \( \beta = 1 \), the construction consists of forming \( U^T U \); for \( \beta = 4 \), with \( U \) of size \( 2N \times 2N \), and \( Z_{2N} \) the \( 2N \times 2N \) antisymmetric tridiagonal with entries \(-1\) (respectively, 1) on the leading upper (respectively, lower) subdiagonal, the construction is to form \( U^D U \), where \( U^D = Z_{2N}^{-1} U^T Z_{2N} \), and to consider only distinct eigenvalues (in this case the spectrum is doubly degenerate). These two ensembles, together with the Haar distributed unitary matrices themselves, are the three circular ensembles of Dyson [19]. Many years later it was shown in [36] how to construct certain random unitary Hessenberg matrices which give rise to the eigenvalue PDF implied by (1-3) for general \( \beta > 0 \). We remark that the parameter \( \beta \) is often referred to as the Dyson index.

Our primary interest in the present paper is to extend a number of identities associated with (1-3), which culminate in an explicit determination of the scaled large \( N \) limit for \( s \) a positive integer and continuous \( h \), to the general \( \beta > 0 \) case (1-4). The known identities can be found in the work of Winn [47], as can the determination of the large \( N \) limit, provided \( h \) is restricted to nonnegative integer or half-integer values. The very recent works [5; 3] give a viewpoint which allows the methods in [47] to be adapted to continuous values of \( h \). Applying this to the present setting, our results in the general \( \beta > 0 \) case similarly allow for the large \( N \) form of (1-4) for continuous \( h \) to be determined.

**Theorem 1.1.** With \( s \) a nonnegative integer, let \( \kappa = (\kappa_1, \ldots, \kappa_s) \) denote a partition of \( s \) parts. Define the generalised Pochhammer symbol by (2-11) below. Let \( C^{(\alpha)}_\kappa(x_1, \ldots, x_s) \) denote the Jack polynomial indexed by the partition \( \kappa \) and depending on the parameter \( \alpha \), normalised so that \( C^{(\alpha)}_\kappa((1)^s) := C^{(\alpha)}_\kappa(x_1, \ldots, x_s)|_{x_1=\cdots=x_s=1} \) is given by (2-35) below. For \( \beta > 0 \) and with

$$-\frac{1}{2} < \text{Re} \, h < s + \frac{1}{2} \quad (1-6)$$
At the value for $\beta$ independent of $F$ formula extends to all $s$ for results known from the earlier works \[7; 35\]. The second line of (1-7) is in fact a rational function of $\langle|\rangle$ equal to unity. The resulting formula then corresponds to the leading asymptotics of the moments $|\kappa|$ of the Riemann zeta function to averages over $U$ of the Riemann zeta function motivating the study of (1-1), the value using ideas extending those in \[47\]. In the setting of joint moments of the Hardy function in the theory Here the explicit values have been determined from closed forms of $2$-ensemble. Then (1-7) simplifies to read
\begin{equation}
(2) \text{ In the case } s = 1, \quad C_\kappa((1)^s) = 1 \text{ and the generalised Pochhammer symbols reduce to their classical counterparts. Then (1-7) simplifies to read}
\end{equation}
\begin{equation}
\lim_{N \to \infty} \frac{1}{N^{2s/\beta + 2h}} F_{N, \beta}(s, h) = \frac{1}{2^{2h}} \prod_{j=1}^{s} \frac{\Gamma(2j/\beta)}{\Gamma(2s + j)/\beta) \times 1 \cos \pi h \sum_{\kappa} \frac{(-2h)_{|\kappa|}}{|\kappa|!} [4s/\beta]^{(\beta/2)} (\frac{4}{\beta} \kappa) C_\kappa(1/2) \langle(1)^s \rangle. \quad (1-7)
\end{equation}

Remark 1.2. (1) In the case $s = 0$ the sum over $\kappa$ is empty, and (1-7) reduces to
\begin{equation}
\lim_{N \to \infty} \frac{1}{N^{2h}} F_{N, \beta}(s, h)|_{s=0} = \frac{1}{2^{2h}} \frac{1}{\cos \pi h}, \quad -\frac{1}{2} < h < \frac{1}{2}, \quad (1-8)
\end{equation}
independent of $\beta$.

(2) In the case $s = 1, \quad C_\kappa((1)^s) = 1$ and the generalised Pochhammer symbols reduce to their classical counterparts. Then (1-7) simplifies to read
\begin{equation}
\lim_{N \to \infty} \frac{1}{N^{2h+2}} F_{N, \beta}(s, h)|_{s=1} = \frac{1}{2^{2h}} \frac{\Gamma(2/\beta)}{\Gamma(4/\beta)} \frac{1}{\cos \pi h} F_1(-2h; 4/\beta; 4/\beta), \quad -\frac{1}{2} < h < \frac{3}{2}. \quad (1-9)
\end{equation}

At $h = \frac{1}{2}$ this is to be computed as a limit to this value, which gives
\begin{equation}
\lim_{N \to \infty} \frac{1}{N^{2h+2}} F_{N, \beta}(s, h)|_{s=1, h=1/2} = \frac{\Gamma(2/\beta)}{\Gamma(4/\beta)} \left( \frac{1}{\pi} - \frac{1}{2\pi} \frac{1}{1 + \beta/4} 2F_2(1, 1; 3, 4/\beta + 2; 4/\beta) \right)
= \begin{cases} 
(3e^4 - 103)/(768\pi) & \text{if } \beta = 1, \\
(e^2 - 5)/(4\pi) & \text{if } \beta = 2, \\
(e - 2)/\sqrt{\pi} & \text{if } \beta = 4.
\end{cases} \quad (1-10)
\end{equation}

Here the explicit values have been determined from closed forms of $2F_2$ found using computer algebra. The value for $\beta = 2$ was first derived in this context by Winn \[47\], and most recently by Assiotis et al. \[5\] using ideas extending those in \[47\]. In the setting of joint moments of the Hardy function in the theory of the Riemann zeta function motivating the study of (1-1), the value $(e^2 - 5)/(4\pi)$ was earlier known from the work of Conrey and Ghosh \[12\]. Combining this, and the Keating–Snaith hypothesis relating averages of the Riemann zeta function to averages over $U(N)$, the statement of (1-10) with $\beta = 2$ first appeared as a conjecture in the thesis of Hughes \[34\].

(3) For $h$ a nonnegative integer in (1-7) the sum over partitions can be restricted to partitions of size $|\kappa| \leq 2h$, and so terminates. In the case $h = 0$ the sum is empty and is to be interpreted as equal to unity. The resulting formula then corresponds to the leading asymptotics of the moments $\langle|V_U(\theta)|_{\theta=0}^{2s} \rangle$, where the average is with respect to the circular $\beta$-ensemble, and is consistent with results known from the earlier works \[7; 35\]. The second line of (1-7) is in fact a rational function of $s$ for $h$ a nonnegative integer. We show in Remark 3.6 below that the validity of this rational function formula extends to all $s > h = \frac{1}{2}$.

As a secondary point, discussed in Section 4, we identify a structurally very similar class of moments to $F_{N, \beta}(s, h)$ for the Jacobi $\beta$-ensemble, although now parametrised by a single exponent $h$, rather than
two exponents \((s, h)\) as in \((1-4)\). These are specified by

\[
H_N^{(J)}(h) = \frac{1}{S_N(a, b, \beta / 2)} \int_0^1 du_1 \cdots \int_0^1 du_N \left| \frac{d}{dt} \log R(t) \right|_{t=0}^{h} \prod_{i=1}^{N} u_i^a (1 - u_i)^b \prod_{1 \leq j < k \leq N} |u_k - u_j|^\beta,
\]

where

\[
R(t) := \prod_{l=1}^{N} (t - u_l)
\]

and the normalisation, which is referred to as the Selberg integral, has the gamma function evaluation

\[
S_N(\lambda_1, \lambda_2, \lambda) = \prod_{j=0}^{N-1} \frac{\Gamma(\lambda_1 + 1 + j \lambda)\Gamma(\lambda_2 + 1 + j \lambda)\Gamma(1 + (j + 1)\lambda)}{\Gamma(\lambda_1 + \lambda_2 + 2 + (N + j - 1)\lambda)\Gamma(1 + \lambda)}.
\]

This work culminates in the evaluation of the scaled limit for \(h\) a positive integer; see Corollary 4.3.

### 1B. Relationship to a Trace Distribution for the Cauchy \(\beta\)-ensemble and Consequences.

After recalling \((1-2)\) we see that the integrand in \((1-3)\) with \(h = 0\), under the stereographic projection mapping 
\[
\cot(\theta_j/2) = x_j \quad \text{for} \quad j = 1, \ldots, N
\]
from the unit circle to the real line, transforms \([20, \text{Equation (3.123)}]\) according to

\[
\prod_{l=1}^{N} |1 - e^{i \theta_l}|^{2s} |\Delta_N(e^{i \theta})|^{\beta} \ d\theta = 2^{\alpha_0 - 2s} \prod_{j=1}^{N} \frac{1}{(1 + x_j^2)^{\alpha_1}} |\Delta_N(x)|^{\beta} \ dx,
\]

where, with \(x = \{x_j\}_{j=1}^{N}, \Delta_N(x) := \prod_{1 \leq j < k \leq N} (x_k - x_j),\) and

\[
\alpha_0 = \beta N(N - 1)/2 + N(1 + 2s) - 2h \quad \text{and} \quad \alpha_1 = \beta (N - 1)/2 + 1 + s.
\]

The PDF corresponding to the right-hand side of \((1-14)\) is known in random matrix theory as the symmetric Cauchy \(\beta\)-ensemble.

We will see in Proposition 2.1 below that \(F_{N, \beta}(s, h)\) has the interpretation as the moment \(\langle |T|^h \rangle\) of the trace statistic

\[
T = \sum_{j=1}^{N} x_j
\]

for the symmetric Cauchy \(\beta\)-ensemble. In fact in deducing Theorem 1.1 we will make use of knowledge of the explicit form of the scaled PDF of the trace distribution in this ensemble, \(P_{\infty, s}^{(Cy)}(T)\), say.

**Theorem 1.3.** Let the PDF of the trace statistic \((1-16)\) for the symmetric Cauchy \(\beta\)-ensemble corresponding to the right-hand side of \((1-14)\) be denoted by \(P_{N, s}^{(Cy)}(T)\), and define

\[
P_{\infty, s}^{(Cy)}(T) = \lim_{N \to \infty} N P_{N, s}^{(Cy)}(NT).
\]
For \( s \in \mathbb{Z}_{\geq 0} \) we have

\[
P_{\infty,s}^{(\text{Cy})}(T) = \frac{1}{\pi} \Re \int_0^\infty e^{-t(1+iT)} F_1^{(\beta/2)}(\_; 4s/\beta; (4t/\beta)^s) \, dt. \tag{1-18}
\]

Here \( F_1^{(\beta/2)} \) is a particular hypergeometric function based on Jack polynomials as defined by (2-10) below, and where the notation \((4t/\beta)^s\) indicates that each argument \( x_i \) for \( i = 1, \ldots, s \) is equal to \( 4t/\beta \).

It turns out that knowledge of \( P_{\infty,s}^{(\text{Cy})}(T) \) is of independent interest. To see why, we first make note of the fact that for \( \beta = 1, 2 \) and 4 the right-hand side of (1-14) is proportional to the eigenvalue PDF on the space of Hermitian matrices \{\( H \)\} with real \( (\beta = 1) \), complex \( (\beta = 2) \) and real quaternion \( (\beta = 4) \) entries chosen according to a distribution with PDF proportional to

\[
(\det(\mathbb{I}_N + H^2))^{-\beta(N-1)/2 - 1 - s}; \tag{1-19}
\]

see, e.g., [20, §2.5]. Matrices from this ensemble have the special property that the submatrix obtained by deleting \( n \) (where \( n < N \)) rows and \( n \) columns also has the distribution implied by (1-19), but with \( N \) replaced by \( N - n \) [11; 33]. This property underlies the Cauchy ensemble being an invariant measure—referred to as a Hua–Pickrell measure—on the space of infinite Hermitian matrices with the action of \( U(\infty) \) by conjugation; see [10] for \( \beta = 2 \) and [2] for \( \beta = 1, 4 \). The latter reference shows how a reformulation in terms of the Dixon–Anderson conditional PDF (see [20, Proposition 4.2.1]) extends this to general \( \beta > 0 \).

On the other hand, a classification of measures which are ergodic under this action—the so-called extremal measures—has been given in the work of Pickrell [42] and Olshanski and Vershik [39] for \( \beta = 2 \), and extended to general \( \beta > 0 \) by Assiotis and Najnudel [2], in terms of real parameters \( \{\alpha_i^+\}_{i=1}^\infty, \{\alpha_i^-\}_{i=1}^\infty, \gamma_1, \gamma_2\}. \) Moreover, an arbitrary invariant measure with respect to the action can, by use of the Fourier transform, be decomposed into ergodic components. The task taken up by Borodin and Olshanski [10] was to specify the ergodic components, and thus the set of real parameters, in the case of the \( \beta = 2 \) symmetric Cauchy ensemble. This was successfully carried out for the parameters \( \{\alpha_i^\pm\}_{i=1}^\infty \). Later Qiu [43] showed that \( \gamma_2 = 0 \), and that \( \gamma_1 \) has the distribution \( P_{\infty,s}^{(\text{Cy})}(T)|_{\beta=2} \), i.e., is the limiting trace statistic for the \( \beta = 2 \) symmetric Cauchy ensemble. This latter characterisation was extended to general \( \beta > 0 \) in [2]. Hence for the parameter \( s \) in (1-19) a nonnegative integer, the result of Theorem 1.3 has the interpretation of specifying the distribution of the real parameter \( \gamma_1 \) for general \( \beta > 0 \).

1C. Overview of the strategy of Winn in relation to \( \mathcal{F}_N(s, h) \), and proposed \( \beta \) modification. The first identity of interest from [47] gives that

\[
\mathcal{F}_N(s, h) = \frac{2^{N^2+2sN-2h}}{(2\pi)^N N!} \int_{\mathbb{R}^N} \prod_{j=1}^N \frac{1}{(1 + x_j^2)^{N+s}} |x_1 + \cdots + x_N|^{2h} (\Delta_N(x))^2 \, dx, \tag{1-20}
\]

where \( \Delta_N(x) \) is as in (1-14) and \( dx = dx_1 \cdots dx_N \). This is extended to the general \( \beta > 0 \) case in Section 2A. As noted in relation to (1-14), with \( h = 0 \) the integrand is familiar in random matrix theory as specifying the PDF for the symmetric Cauchy ensemble with Dyson index \( \beta = 2 \); see, e.g., [20, §2.5].
In the case of $h$ a nonnegative integer, the exponential generating function for the right-hand side of (1-20) is the multiple integral

$$
\mathcal{H}^{(Cy)}_N(s; t) := \int_{\mathbb{R}^N} \prod_{j=1}^N \frac{e^{itx_j}}{(1 + x_j^2)^{N+s}} (\Delta_N(x))^2 \, dx.
$$

(1-21)

Equivalently, up to normalisation this quantity is the characteristic function for the distribution of the trace statistic (1-16). In [47] Winn derives a transformation identity for (1-21), where it was expressed as $e^{-N|t|}$ times a multiple integral which for $s$ a nonnegative integer is polynomial of degree $sN$. In the general $\beta$ case we don’t know how to derive such an identity directly. However, for $\beta$ even, using the residue calculus, we exhibit in Proposition 2.3 that the structure of $e^{-N|t|}$ times a polynomial of degree $sN$ persists. This simple analytic structure in the case $s$ a nonnegative integer enables further calculation.

Winn proceeds to show that the multiple integral specifying the polynomial can be identified with a certain generalised hypergeometric function based on Jack polynomials (see Section 2B below for the definitions and notation), showing in particular that for $s$ a nonnegative integer

$$
\mathcal{H}^{(Cy)}_N(s; t) \propto e^{-N|t|} \, _1F_1(-s, -2s; (2|t|)^N),
$$

(1-22)

and furthermore making explicit the proportionality constant. Using insights into averages over the Cauchy ensemble from the recent work [24] we show in Proposition 2.5 how to derive (1-22) directly. Most importantly this method extends to general $\beta > 0$, allowing for the derivation of the general $\beta > 0$ analogue of (1-22).

The generalised hypergeometric function $_1F_1$ in (1-22) is a function of $N$ variables. By interchanging the roles of rows and columns in the underlying sum over partitions which defines this function, Winn obtains [47, Equation (6.5)] the transformation identity

$$
_1F_1(-s, -2s; (2|t|)^N) = _1F_1(-N, 2s; (-2|t|)^N).
$$

(1-23)

Most significant is that the generalised hypergeometric function on the right-hand side is a function of $s$ variables, not $N$. After substituting in (1-22) this functional form underlies Winn’s analysis, given in [47, §6.2], of the large $N$ limiting form (1-7) in the case $\beta = 2$, and with the assumption that $h$ is an integer or half-integer within the required range (1-6). As noted in [47], the integer case results are the same as those derived by Dehaye [16; 17], using symmetric-function/representation-theoretic ideas closely related to the special function in (1-22); for recent extensions of the study of the characteristic polynomial for $U(N)$ along these lines, see [8; 44]. In our Proposition 2.8 we give the $\beta$ generalisation of (1-23) relevant to providing the $\beta$ generalisation of (1-22).

The recent paper [5] places emphasis on the limiting scaled PDF for the trace statistic (1-16), as defined by (1-17). However the technical details relied not on the generalised hypergeometric formula (1-23) directly, but rather a determinant evaluation in terms of Laguerre polynomials for $_1F_1$ in (1-22) as being proportional [21; 47] to

$$
\det[L_{N+s-1,j-k}^{2s-1}(2|t|)]_{j,k=0,...,s-1}.
$$

(1-24)
With \(I_v(z)\) denoting the modified Bessel function, this gives
\[
\lim_{N \to \infty} \frac{1}{C_N} \mathcal{H}_N^{(C_0)}(s; t/2N) = (-1)^s(s-1)/2 \frac{G(2s + 1)}{G(s+1)^2} \frac{e^{-|t|/2}}{|t|^{s/2}} \det[I_{j+k+1}(2\sqrt{|t|})]_{j,k=0,\ldots,s-1}. \tag{1-25}
\]
where \(\widetilde{C}_N = \widetilde{C}_N(s, \beta)\) is chosen so that the left-hand side is unity for \(t = 0\). Taking its inverse Fourier transform gives the limiting distribution implied by (1-21). Associated moments can now be obtained by a direct evaluation. We remark that other essential uses of (1-25) in the present context were already known from the earlier work of Conrey, Rubinstein and Snaith [13], and used subsequently in [5; 3; 6; 9; 28].

Working with the finite \(N\) formula for the characteristic function based on \(N\) generalisation of (1-23) we are able to evaluate the limiting scaled PDF for the trace statistic as the inverse Fourier transform of the corresponding characteristic function, which is our Theorem 1.3.

Working with the generalised hypergeometric function expression for the finite \(N\) trace distribution, as implied by taking the inverse Fourier transformation of the formula for the characteristic function based on (1-23), a finite summation formula for the moments in the range (1-6) can be obtained. This is given in Corollary 3.5. Its large \(N\) form is readily computed, and implies the result of Theorem 1.1.

**Remark 1.4.** For general \(s\) the structure of (1-21) is more complex, as can already be seen from the case \(N = 1\). Thus it is well known (see the recent work [31] for references and an independent derivation) that then (1-21) — identified up to scaling of \(t\) and name of the parameter \(s\) as the characteristic function of Student’s \(t\)-distribution — is proportional to \(|t|^{s+1/2}K_{s+1/2}(|t|)\), where \(K_{s+1/2}\) denotes the modified Bessel function of the second kind of order \(s + \frac{1}{2}\). This has the analytic structure of \(e^{-|t|}\) times a linear combination of a power series in \(|t|\) and \(|t|^{2s+1}\) times a power series in \(t\). Also, the latter involves a logarithm for \(s + \frac{1}{2}\) a nonnegative integer. For \(s\) a nonnegative integer the former power series terminates to a polynomial of degree \(s\), and the scalar multiplying the functional form of \(|t|^{2s+1}\) times a power series in \(t\) vanishes. Further insight into the analytic structure present for continuous \(s\) can be obtained by revisiting the known [27] Painlevé nonlinear differential equation characterising the scaled limit of \(\mathcal{H}_N^{(C_0)}(s; t)\) evaluated by (1-25). This same differential equation has recently been shown to hold for continuous \(s\) [5]. Results from [28] can be used to specify the small \(t\) boundary condition for continuous \(s\). This is done in the Appendix, which reveals a so-called Puiseux-type series expansion involving in general an infinite series of fractional powers \(|t|^{p(2s+1)}\) (for \(p = 1, 2, \ldots\), each times a power series in \(|t|\).

2. Transforming \(F_{N,\beta}(s, h)\)

2A. Generalising (1-20). Starting from the definition (1-4), we seek an identity which generalises (1-20).

**Proposition 2.1.** Let \(C_{N,\beta}\) be given by (1-5), and make use of the notation \(\Delta_N(x)\) as specified below (1-14). We have
\[
F_{N,\beta}(s, h) = \frac{2^{\alpha_0}}{C_{N,\beta}} \int_{-\infty}^\infty dx_1 \cdots \int_{-\infty}^\infty dx_N \frac{1}{(1 + x_j^{\alpha_1})} |x_1 + \cdots + x_N|^{2h} |\Delta_N(x)|^\beta, \tag{2-1}
\]
where
\[
\alpha_0 = \beta N(N - 1)/2 + N(1 + 2s) - 2h \quad \text{and} \quad \alpha_1 = \beta(N - 1)/2 + 1 + s. \tag{2-2}
\]
Proof. It turns out that in (2-1) the Dyson index $\beta$ does not play an essential role, and so we can follow the proof of [47, Proposition 3.4] in the case $\beta = 2$. The first step is to note from the definitions that

$$\frac{d}{d\theta} \log V_U(\theta) \Big|_{\theta = 0} = -\frac{1}{2} \sum_{j=1}^{N} \cot \frac{\theta_j}{2}.$$  

Substituting in (1-4) gives

$$F_{N,\beta}(s, h) = \frac{1}{C_{N,\beta}} \frac{1}{2^h} \int_{0}^{2\pi} \cdots \int_{0}^{2\pi} d\theta_1 \cdots d\theta_N \left| 1 - e^{i\theta_j} \right|^{2s} \sum_{j=1}^{N} \cot \frac{\theta_j}{2} \left| \Delta_N(e^{i\theta}) \right|^\beta. \quad (2-3)$$

The second and final step is to use the familiar fact in random matrix theory that under the mapping $x_j = \cot(\theta_j/2)$ (for $j = 1, \ldots, N$), which geometrically can be viewed as a stereographic projection from the unit circle to the real line, the integrand in (2-3) with $h = 0$ transforms according to (1-14). The stated identity (2-1) follows. 

Remark 2.2. (1) The generalisation of the circular $\beta$-ensemble on the left-hand side of (1-14) is given the name circular Jacobi $\beta$-ensemble in random matrix theory; see [20, §3.9]. For a recent study see [29].

(2) The Cauchy ensemble average of $\sum_{j=1}^{N} |x_j|^{2h}$, rather than $|x_1 + \cdots + x_N|^{2h}$ as in (2-1), has been the subject of the recent works [4] (for $\beta = 2$) and [24] (for $\beta = 1, 2$ and 4).

2B. Generalising (1-22). A generalisation of the PDF corresponding to the right-hand side of (1-14), is

$$\frac{1}{\mathcal{N}_{N}^{(Cy)}} \prod_{j=1}^{N} \frac{1}{(1 + i x_j)^{\beta(N-1)/2 + 1 + s_+} (1 - i x_j)^{\beta(N-1)/2 + 1 + s_-} \left| \Delta_N(x) \right|^\beta}, \quad (2-4)$$

where

$$\mathcal{N}_{N}^{(Cy)} = \mathcal{N}_{N}^{(Cy)}(s_+, s_-, \beta)$$

is the normalisation, which is well defined for $\text{Re}(s_+ + s_-) > -1$. We will refer to this as the nonsymmetric Cauchy $\beta$-ensemble, which with $s_+ = s_- = s$ specialises to the symmetric Cauchy $\beta$-ensemble corresponding to the right-hand side of (1-14). Note that only in the case that $s_-$ is equal to the complex conjugate of $s_+$ can this be interpreted as a PDF, as otherwise it may not be real and nonnegative. However this does not affect us here. We remark too that in the symmetric case the normalisation is evaluated in terms of gamma functions [20, §4.7.1] according to

$$\mathcal{N}_{N}^{(Cy)} = 2^{-\beta N(N-1)/2 - 2N s} \pi^N M_N(s, s, \beta/2), \quad (2-5)$$

where

$$M_N(a, b, \lambda) = \prod_{j=0}^{N-1} \frac{\Gamma(\lambda j + a + b + 1)\Gamma(\lambda(j + 1) + 1)}{\Gamma(\lambda j + a + 1)\Gamma(\lambda j + b + 1)\Gamma(1 + \lambda)}. \quad (2-6)$$
For the symmetric Cauchy $\beta$-ensemble consider the trace statistic (1-16) and let $P_{N,s}^{(C)}(T)$ denote its PDF. From the definitions,

$$m_{N,s}^{(C)}(h) := \int_{-\infty}^{\infty} |T|^{2h} P_{N,s}^{(C)}(T) \, dT$$

$$= \frac{1}{N_N^{(C)}} \int_{-\infty}^{\infty} dx_1 \cdots \int_{-\infty}^{\infty} dx_N \, |x_1 + \cdots + x_N|^{2h} \prod_{j=1}^{N} \frac{1}{(1 + x_j^2)^{\beta(N-1)/2}} \, |\Delta_N(x)|^\beta,$$  

which we see from the behaviour of the integrand for $|x_j| \to \infty$ as decaying at a rate proportional to $|x_j|^{-2(1+s)}$, is well defined for the parameter range (1-6) in keeping with the condition associated with (1-1). Thus, recalling too (2-1), knowledge of $P_{N,s}^{(C)}(T)$ is sufficient to compute $F_{N,\beta}(s, h)$, where the average is with respect to the PDF (2-4) with $s_+ = s_- = s$.

More tractable than studying $P_{N,s}(T)$ directly is to study its characteristic function (Fourier transform), $\widehat{P}_{N,s}(t)$, say, specified by

$$\widehat{P}_{N,s}^{(C)}(t) = \left( \prod_{j=1}^{N} e^{it x_j} \right)^{(C)}_s,$$  

where the average is with respect to the PDF (2-4) with $s_+ = s_- = s$. In fact it is advantageous to generalise (2-8) so that the average is with respect to the nonsymmetric Cauchy weight (2-4) with $s_+$ and $s_-$ independent and thus to consider

$$\widehat{P}_{N,s_+,s_-}^{(C)}(t) = \left( \prod_{j=1}^{N} e^{it x_j} \right)^{(C)}_{s_+, s_-}.$$

Our immediate aim is to extend to the general $\beta > 0$ case the $\beta = 2$ transformation formula (1-22) due to Winn [47], involving hypergeometric functions based on Jack polynomials; for an account of the latter see [20, Chapters 12 and 13].

Such hypergeometric functions are functions of several variables, $\{x_j\}_{j=1}^{m}$, say, as well as a parameter $\alpha$. They can be defined by the series

$$\ _pF_q^{(\alpha)}(a_1, \ldots, a_p; b_1, \ldots, b_q; x_1, \ldots, x_m) := \sum_{\kappa} \frac{1}{|\kappa|!} [a_1]^{(\alpha)}_\kappa \cdots [a_p]^{(\alpha)}_\kappa C^{(\alpha)}_\kappa (x_1, \ldots, x_m).$$  

In this expression the sum is over all partitions $\kappa_1 \geq \kappa_2 \geq \cdots \geq \kappa_m \geq 0$. Conventionally this sum is performed in order of increasing $|\kappa| := \sum_{j=1}^{m} \kappa_j$. The symbol $[u]^{(\alpha)}_\kappa$ denotes the generalised Pochhammer symbol and is defined by

$$[a]^{(\alpha)}_\kappa = \prod_{j=1}^{m} \left( a - \frac{1}{\alpha} (j - 1) \right)_{\kappa_j}, \quad (a)_k = a(a + 1) \cdots (a + k - 1) = \frac{\Gamma(a + k)}{\Gamma(a)}.$$

The functions $C^{(\alpha)}_\kappa (x_1, \ldots, x_m)$ are polynomials proportional to the Jack symmetric polynomial (see, e.g., [20, §12.6]). As such they are homogeneous symmetric polynomials of degree $|\kappa|$. For $m = 1$,
\( C_κ(α) (x) = x^{κ_1} \), \([α]_κ(α) = (α)_k \) and (2-10) reduces to the classical definition of \( qF_p \) in one variable. As with \( qF_p \) in one variable, \( qF_p(α) \) exhibits the confluence property

\[
\lim_{κ_p \to \infty} p F_p(α) \left( a_1, \ldots, a_p; b_1, \ldots, b_q; \frac{x_1}{a_p}, \ldots, \frac{x_m}{a_p} \right) = p_1 F_q(α) \left( a_1, \ldots, a_p-1; b_1, \ldots, b_q; x_1, \ldots, x_m \right). \quad (2-12)
\]

It is well known that hypergeometric functions based on Jack polynomials are naturally related to the Jacobi \( β \)-ensemble from random matrix theory; see, e.g., [20, Chapter 13]. The Jacobi \( β \)-ensemble conventionally refers to the eigenvalue PDF supported on \( x_i \in (0, 1) \) (for \( l = 1, \ldots, N \)) proportional to

\[
\prod_{l=1}^N x_l^{λ_1} (1 - x_l)^{λ_2} |Δ_N(α)|^β, \quad (2-13)
\]

where it is required that \( λ_1, λ_2 > -1 \) for this to be normalisable. In the case \( N = 1 \), (2-13) reduces to the PDF for the beta distribution \( B[λ_1 + 1, λ_2 + 1] \) in probability theory and statistics, specified by the functional form \( x^{λ_1} (1 - x)^{λ_2} / B(λ_1 + 1, λ_2 + 1) \), where

\[
B(a, b) = \frac{Γ(a)Γ(b)}{Γ(a + b)} \quad (2-14)
\]

refers to the Euler beta function in usual notation. It is a classical result that this single variable PDF relates to the Gauss hypergeometric function — that is, the case \( p = 2, q = 1, m = 1 \) of (2-10) — via the integral representation

\[
\frac{1}{B(λ_1 + 1, λ_2 + 1)} \int_0^1 x^{λ_1} (1 - x)^{λ_2} (1 - tx)^{-r} \, dx = 2 F_1(r, λ_1 + 1; λ_1 + λ_2 + 2; t). \quad (2-15)
\]

Similarly in the multivariable case, the way (2-13) relates to the hypergeometric functions based on Jack polynomials is through particular integral representations.

To make use of these known results for the Jacobi \( β \)-ensemble in the problem of analysing the Cauchy \( β \)-ensemble quantity (2-8), we require theory from the recent work [24] which relates averages in the two ensembles. To present this theory, with the PDF for the nonsymmetric Cauchy \( β \)-ensemble specified by (2-4), denote the ensemble average of \( f = f(x_1, \ldots, x_N) \) by \( ⟨f⟩_{Cy}^{Cy} \). Similarly, with the PDF for the Jacobi \( β \)-ensemble specified by the normalised version of (2-13), denote the corresponding average of \( f \) by \( ⟨f⟩_{J(λ_1, λ_2)}^{J(λ_1, λ_2)} \). For both classes of averages the dependence on \( N \) and \( β \) is implicit. With this notation, we have from [24, Proposition 2.1] that for \( f \) a multivariable polynomial

\[
⟨(f(1 + ix_1, \ldots, 1 + ix_N))⟩_{Cy}^{Cy} = ⟨(f(2y_1, \ldots, 2y_N))⟩_{J(λ_1, λ_2)}^{J(λ_1, λ_2)} \quad (2-16)
\]

An essential point regarding the interpretation of this identity is that both sides are to be understood in the sense of analytic continuation in \( s_+ \) and \( s_- \).

Before considering the general \( N \) case, it is instructive to first specialise to \( N = 1 \). With \( f(x) = x^p \), where \( p ∈ Z ≥ 0 \), (2-16) then reads

\[
⟨(1 + ix)^p⟩_{Cy}^{Cy} = ⟨(2x)^p⟩_{J(λ_1, λ_2)}^{J(λ_1, λ_2)} \quad (2-17)
\]
Note that the left-hand side is well defined for

$$0 \leq p \leq \text{Re}(s_+ + s_-) - 1.$$  \hspace{1cm} (2-18)

We want to use knowledge of the functional form of the right-hand side of (2-17) to compute the left-hand side in the range (2-18). By definition, the exponential generating function of the moments specifying the right-hand side of (2-17) is given by

$$\langle e^{tx} \rangle_{J \lambda_1=-1-s_+}^{\lambda_2=-1-s_-} = \frac{1}{B(-s_+, -s_-)} \int_0^1 x^{-s_+ - 1}(1 - x)^{-s_- - 1} e^{2tx} \, dx.$$ \hspace{1cm} (2-19)

This is well defined as presented for \( \text{Re}(s_+) \), \( \text{Re}(s_-) < 0 \), and defined more generally by analytic continuation. The integral on the right-hand side of (2-19) is a confluent limit of (2-15) (replace \( t \) by \( -2t/r \) and take \( r \to \infty \)), telling us that

$$\frac{1}{B(-s_+, -s_-)} \int_0^1 x^{-s_+ - 1}(1 - x)^{-s_- - 1} e^{2tx} \, dx = _1F_1(-s_+, -s_+ - s_-; 2t).$$ \hspace{1cm} (2-20)

Equating like powers of \( t \), making use of (2-10) with \( p = q = m = 1 \) on the right-hand side, we can read off the moments, which are given in terms of particular Pochhammer symbols as defined in (2-11). Use now of (2-17) gives the sought evaluation

$$\langle (1 + i x)^p \rangle_{s_+, s_-}^{(C y)} = 2^p \frac{(-s_+)^p}{(-s_+ - s_-)^p}.$$ \hspace{1cm} (2-21)

Note from the definition of the Pochhammer symbols that the right-hand side is indeed well defined in the parameter range specified by (2-18).

The utility of (2-21) in calculating the \( N = 1 \) cases of (2-8) and (2-9) shows itself if we restrict to \( s \in \mathbb{Z}_{\geq 0} \) and \( s_+ \in \mathbb{Z}_{\geq 0} \) respectively. For definiteness, consider (2-9) with \( s_+ \in \mathbb{Z}_{\geq 0} \), and require too that \( t > 0 \). The significance of restricting to \( s_+ \in \mathbb{Z}_{\geq 0} \) is that then the integrand is analytic in the upper half-plane, apart from a pole of order \( 1 + s_+ \) at \( z = i \). And with \( t > 0 \) the integrand decays exponentially fast along the arc \( R e^{i\theta} \), where \( 0 < \theta < \pi \), as \( R \to \infty \). Closing the contour in the upper half-plane, the residue calculus implies

$$\langle e^{t(1+ix)} \rangle_{s_+, s_-}^{(C y)} = q_{s_+}(t; s_-),$$ \hspace{1cm} (2-22)

for some polynomial \( q_{s_+} \) of degree \( s_+ \) in \( t \). We know the moments obtained by expanding the left-hand side in \( t \) are all well defined in the parameter range (2-18). Requiring then that \( s_+ < \text{Re}(s_+ + s_-) \), the polynomial \( q_{s_+} \) is fully determined by the moments. Reading from (2-21) we conclude

$$q_{s_+}(t; s_-) = \sum_{p=0}^{s_+} \frac{(-s_+)^p}{p!(s_+ - s_-)^p} (2t)^p.$$ \hspace{1cm} (2-23)

We seek to extend (2-22) and (2-23) for \( N > 1 \) and general \( \beta > 0 \). The extension of (2-22) will be deduced first.
**Proposition 2.3.** Suppose that \( t > 0, s_+ \in \mathbb{Z}_{\geq 0}, \) \( \Re(s_+ + s_-) \geq 0 \) and \( \beta \) is even. We have

\[
(e^{it(x_1 + \cdots + x_N)}(Cy))_{s_+, s_-}^{(Cy)} = e^{-Nt} Q_{s_+, N}(t; s_-),
\]

(2-24)

where \( Q_{s_+, N}(t; s_-) \) is a polynomial in \( t \) of degree \( s_+ N \). Moreover, with

\[
\Re(s_-) \geq (N - 1)s_+,
\]

(2-25)

we have

\[
Q_{s_+, N}(t; s_-) = \sum_{p=0}^{s_+ N} \frac{t^p}{p!} \left( \frac{i}{N} \sum_{i=1}^N x_i + N \right)^p_{s_+, s_-}. \tag{2-26}
\]

**Proof.** In the case \( \beta \) even we can expand \(|\Delta_N(x)|^{\beta}\) as a multivariable polynomial in \( \{x_j\}_{j=1}^N \), homogeneous of degree \( \beta N(N - 1)/2 \). The integrand is an analytic function in each \( x_j \) in the upper half-plane, except for a pole of degree \( \alpha_+ \), where \( \alpha_\pm := \beta(N - 1)/2 + 1 + s_\pm \), at \( x_j = i \) for each \( j \). Hence, by the residue theorem, the integral defining the average is proportional to

\[
e^{-Nt} \frac{\partial^{\alpha_+ - 1}}{\partial x_1^{\alpha_+ - 1}} \cdots \frac{\partial^{\alpha_+ - 1}}{\partial x_N^{\alpha_+ - 1}} \prod_{l=1}^N \frac{e^{it(x_l - i)}}{(1 - i x_l)^{\alpha_-}} (\Delta_N(x))^{\beta} \bigg|_{x_1 = \cdots = x_N = i}. \tag{2-27}
\]

We see immediately the structure (2-24), although it remains to determine the degree of the polynomial.

From the meaning of the factor \( (\Delta_N(x))^{\beta} \), we see that (2-27) vanishes unless exactly \( \beta N(N - 1)/2 \) derivative operations are applied to this factor. As a consequence, a maximum of \( s_+ N \) derivative operations can act on the factor \( \prod_{l=1}^N e^{it(x_l - i)} \), telling us that the highest power of \( t \) in \( Q_{s_+, N}(t) \) is \( s_+ N \), which is the required result.

To derive (2-26), note that the assumption (2-25) implies each of the averages in (2-26) are well defined, thus implying that it is a consequence of (2-24), multiplied through by \( e^{Nt} \).

\[\square\]

**Remark 2.4.** The analytic structure of an exponential times a polynomial has been present, and played an important role, in a number of recent studies of distributions in random matrix theory [22; 25; 37].

The task now is to evaluate the averages in (2-26) which determine the polynomial \( Q_{s_+, N}(t; s_-) \).

**Proposition 2.5.** With \( s_+ \in \mathbb{Z}_{\geq 0} \) and condition (2-25) we have

\[
Q_{s_+, N}(t; s_-) = {}_1 F_1^{(2/\beta)}(-s_+, -(s_+ + s_-); (2t)^N), \tag{2-28}
\]

where \( (u)^N \) denotes the argument \( u \) repeated \( N \) times, and \( {}_1 F_1^{(2/\beta)} \) denotes the series implied by (2-10). Moreover, the condition (2-25) can be weakened to

\[
\Re(s_+ + s_-) \geq 0, \tag{2-29}
\]

provided we interpret \( {}_1 F_1^{(2/\beta)} \) in (2-28) as terminating when the largest part of the partition \( \kappa \) is greater than \( s_+ \).
Proof. A fundamental result in Jack polynomial theory is an integration formula associated with Macdonald, Kadell and Kaneko (see [26]),

$$\frac{1}{S_N(\lambda_1, \lambda_2; 1/\alpha)} \int_0^1 dt_1 \cdots \int_0^1 dt_N \prod_{i=1}^N t_i^{\lambda_1}(1-t_i)^{\lambda_2} C_k^{(\alpha)}(t_1, \ldots, t_N) \prod_{j<k} |t_j - t_k|^{2/\alpha}$$

$$= C_k^{(\alpha)}((1)^N) \frac{[\lambda_1 + (N-1)/\alpha + 1]^{(\alpha)}}{[\lambda_1 + \lambda_2 + 2(N-1)/\alpha + 2]^{(\alpha)}}. \quad (2-30)$$

Here the normalisation $S_N$ is the Selberg integral, which, as already remarked, has the gamma function evaluation (1-13). The polynomials $\{C_k^{(\alpha)}\}$ are the Jack polynomials in the same normalisation as in (2-10). The value of $C_k^{(\alpha)}((1)^N)$ is specified in Remark 2.6 below.

To be able to make use of (2-30), it needs to be possible to expand the polynomial $(i \sum_{i=1}^N x_i + N)^p$ in (2-26) in terms of $\{C_k^{(\alpha)}\}$. In fact we have the simple formula (see [20, Equation (12.135)])

$$(x_1 + \cdots + x_N)^p = \sum_{|\kappa| = p} C_k^{(\alpha)}(x_1, \ldots, x_N). \quad (2-31)$$

It follows that

$$(x_1 + \cdots + x_N)^p \frac{(\lambda_1, \lambda_2)}{\lambda_1 = -\beta(N-1)/2-1-s_+} \lambda_2 = -\beta(N-1)/2-1-s_- = \sum_{|\kappa| = p} C_k^{(2/\beta)}((1)^N) \frac{[-s_+ + k]^{(2/\beta)}}{[-s_+ - s_-]^{(2/\beta)}}. \quad (2-32)$$

where the right-hand side is be understood as providing an analytic continuation in $s_+$ and $s_-$ of the left-hand side. Upon application of (2-16) we obtain, for the moments in (2-26),

$$(\left\langle \left( \sum_{i=1}^N x_i + N \right)^p \right\rangle_{C_\nu}}_{s_+, s_-} = 2^{|\nu|} \sum_{|\kappa| = p} C_k^{(2/\beta)}((1)^N) \frac{[-s_+ + k]^{(2/\beta)}}{[-s_+ - s_-]^{(2/\beta)}}. \quad (2-33)$$

Note that the denominator in (2-33) is well defined by assumption (2-25) for all partitions with $\kappa_1 \leq s_+$ since $p \leq s_+ N$. Substituting in (2-26) and comparing with the series definition of $\text{I}_F^{(2/\beta)}$ as implied by (2-10) gives (2-28).

With the condition (2-25),

$$\text{I}_F^{(2/\beta)}(-s_+, -(s_+ + s_-); (2t)^N) = \sum_{p=0}^{s_+ N} \sum_{|\kappa| = p} C_k^{(2/\beta)}((2t)^N) \frac{[-s_+ + k]^{(2/\beta)}}{[-s_+ - s_-]^{(2/\beta)}}. \quad (2-34)$$

Substituting (2-28) in (2-24), which we know is valid for $s_+ \in \mathbb{Z}_{\geq 0}$ and $\text{Re}(s_+ + s_-) \geq 0$, we see that (2-34) remains valid while it is an analytic function of $s_-$, that is, for the range (2-29). \qed

Remark 2.6. The value of $C_k^{(\beta/2)}((1)^\nu)$ is known explicitly from Jack polynomial theory (see, e.g., [20, Chapter 12]). It involves the diagram of the partition $\kappa$; see, e.g., [20, Definition 10.1.3]. On this, recall the diagram records each nonzero part $k_i$ as row $i$ of $k_i$ boxes, drawn flush left starting from the first column. Consider a square $(i, j)$ in the diagram. The arm length $a(i, j)$ is defined as the number of boxes in the row to the right and the co-arm length $a'(i, j)$ is the number in the row to the left. The leg
Moreover, which is based on Jack polynomials in variables, permits a transformation to involve a hypergeometric function based on Jack polynomials in variables. The advantage of this latter form is that is well suited to taking the scaled large limit.

The simplest case is when \( n = 1 \), and we have
\[
C_\kappa^{(\alpha)}((1)^n)_{|n=1} = 1
\]
for all \( \kappa = (\kappa_1, 0, \ldots, 0) \) in keeping with \( C_\kappa^{(\alpha)}(x) = x^{\kappa_1} \), as already noted below (2-11).

**Corollary 2.7.** Suppose that \( t > 0 \) and \( s \in \mathbb{Z}_{\geq 0} \). For all \( \beta > 0 \) we have
\[
\langle e^{it(x_1 + \cdots + x_N)} \rangle_\kappa^{(CY)} = \lim_{\epsilon \to 0} e^{-Nt} \sum_{p=0}^{s} \sum_{\kappa_1 \leq s} \frac{2t}{} |\kappa|! \left[ C_\kappa^{(\beta)}((1)^N) \right] \left[ s \right] \left[ 2\beta \right].
\]
Moreover, replacing \( t \) by \(|t|\) on the right-hand side allows the condition \( t > 0 \) to be replaced by \( t \in \mathbb{R} \).

**Proof:** Substituting (2-28) in (2-24) and recalling (2-34) gives (2-39) valid for \( \beta \) even. To remove the latter restriction, note from (2-10), (2-11) and (2-35) that each term on the right-hand side of (2-39) is a bounded analytic function of \( \beta \) for \( \text{Re}(\beta) > 0 \). On the left-hand side, the function being averaged — a complex exponential — has modulus equal to 1, and so it is immediate that this side is similarly a bounded analytic function of \( \beta \) for \( \text{Re}(\beta) > 0 \). Application of Carlson’s theorem (see, e.g., [20, Proposition 4.1.4]), as is familiar in the theory of the Selberg integral [19; 45], then allows the validity of (2-39) for general \( \beta > 0 \) to be concluded from knowledge of its validity for positive even \( \beta \).

The facts that the left-hand side is an even function of \( t \) from its definition (thus the PDF proportional to the right-hand side of (1-14) for the symmetric Cauchy \( \beta \)-ensemble is unchanged by the mapping \( x_j \mapsto -x_j \) for each \( j = 1, \ldots, N \)) and that both sides equal unity for \( t = 0 \) show that replacing \( t \) by \(|t|\) in the right-hand side of (2-39) gives an equation valid for all \( t \in \mathbb{R} \). \( \square \)

**2C. Generalising (1-24).** The generalised hypergeometric function on the right-hand side of (2-39), which is based on Jack polynomials in \( N \) variables, permits a transformation to involve a hypergeometric function based on Jack polynomials in \( s \) variables. The advantage of this latter form is that is well suited to taking the scaled large \( N \) limit.
Proposition 2.8. Let \( s \in \mathbb{Z}_{\geq 0} \). We have
\[
\lim_{\epsilon \to 0} f_1^{(2/\beta)}(-s, -2s - \epsilon; (2t)^N) = f_1^{(\beta/2)}(-N; 4s/\beta; (-4t/\beta)^s).
\]
(2-40)
Consequently,
\[
\hat{P}_N^{(Cy)}(t) = e^{-N|t|} f_1^{(\beta/2)}(-N; 4s/\beta; (-4|t|/\beta)^s)
\]
and
\[
\hat{P}_\infty^{(Cy)}(t) := \lim_{N \to \infty} \hat{P}_N^{(Cy)}(t/N) = e^{-|t|} f_1^{(\beta/2)}(\_; 4s/\beta; (4|t|/\beta)^s).
\]
(2-41) (2-42)
Proof. For \( s \in \mathbb{Z}_{\geq 0} \) and general \( s_1 \not\in \mathbb{Z}_{\geq 0} \), analogously to the second line in (2-39) we have
\[
f_1^{(2/\beta)}(-s, -s_1; (2t)^N) = \sum_{p=0}^{sN} \sum_{|\kappa|=p} \frac{(2t)^{|\kappa|} C_k^{(2/\beta)}((1)^N)}{|\kappa|!} \frac{(-N)^{|\kappa|}}{[2s_1/\beta]^{(\beta/2)}}.
\]
(2-43)
Hence in the diagram picture of a partition the sum of \( \kappa \) is over partitions with no more than \( N \) rows and \( s \) columns. Now replace each \( \kappa \) by its conjugate \( \kappa' \), which by definition interchanges rows and columns. The sum now is over partitions with diagram consisting of no more that \( s \) rows and \( N \) columns. Moreover, we have the general relations between quantities relevant to the summand on the left-hand side of (2-43)
\[
[a]_\kappa^{(\alpha)} = (-\alpha)^{-|\kappa|} \alpha \alpha a^{(1/\alpha)}|_{\alpha \to 1/\alpha},
\]
(2-44)
\[
d'_\kappa = \alpha^{|\kappa|} h_\kappa \big|_{\alpha \to 1/\alpha}, \quad h_\kappa = \alpha^{|\kappa|} d'_\kappa \big|_{\alpha \to 1/\alpha},
\]
(2-45)
where the relevance of (2-45) is seen from (2-35). Thus we obtain the transformation identity
\[
f_1^{(2/\beta)}(-s, -s_1; (2t)^N) = \sum_{p=0}^{sN} \sum_{|\kappa|=p} \frac{(2t)^{|\kappa|} C_k^{(2/\beta)}((1)^N)}{|\kappa|!} \frac{(-N)^{|\kappa|}}{[2s_1/\beta]^{(\beta/2)}}
\]
\[
= f_1^{(\beta/2)}(-N, 2s_1/\beta; (-4t/\beta)^s),
\]
(2-46)
from which (2-40) follows. Substituting (2-40) in the first line of (2-39) gives (2-41). Now replacing \( t \) by \( t/N \) and taking the limit, making use of (2-12) on the right-hand side of (2-40), gives (2-42).

Remark 2.9. (1) Suppose that \( s = 1 \). The hypergeometric functions based on Jack polynomials in Proposition 2.8 all reduce to their classical counterparts. For example, from (2-41),
\[
\hat{P}_N^{(Cy)}(t)|_{s=1} = e^{-N|t|} f_1(-N; 4/\beta; -4t/\beta).
\]
(2-47)
A check is to take the limit \( \beta \to 0 \), since it follows from the definitions that for general \( s \) such that the left-hand side is well defined we must have
\[
\hat{P}_N^{(Cy)}(t)|_{\beta=0} = (\hat{P}_N^{(Cy)}(t)|_{N=1})^N.
\]
(2-48)
Since for \( \beta \to 0 \) the right-hand side of (2-47) reduces to
\[
e^{-N|t|} f_0(-N; -t) = (e^{-|t|}(1 + |t|))^N,
\]
consistency with (2-48) is indeed observed.
(2) In words, the definition (2-8) of \( b_{PN}^{(s)}(t) \) is the characteristic function for the distribution of a linear statistic \( \sum_{j=1}^{N} x_j \) in the symmetric Cauchy \( \beta \)-ensemble, or equivalently from the proof of Proposition 2.1, of the linear statistic \( \sum_{j=1}^{N} \cot \theta_j / 2 \) for the circular Jacobi \( \beta \)-ensemble. From the latter, the linear statistic is thus singular, as it diverges for eigenvalues at \( \theta = 0 \). As such it exhibits anomalous large \( N \) behaviour relative to the characteristic function for a smooth linear statistic \( A = \sum_{j=1}^{N} a(x_j) \). Denoting the characteristic function by \( b_{PN}^{(s)}(t; A) \), in the smooth case (see, e.g., [40]),

\[
\lim_{N \to \infty} b_{PN}^{(s)}(t/N; A) = e^{it\mu_A} \quad \text{for} \quad \mu_A = \lim_{N \to \infty} \frac{1}{N} \langle A \rangle.
\]  

(2-49)

Hence the scaled moments beyond the first are all trivial, just being powers of the mean. For the present singular statistic, the scaled (even integer) moments are no longer simply related, as we see from (2-42).

3. Proof of Theorems 1.1 and 1.3

3A. Proof of Theorem 1.3. Writing the scaled PDF \( N P_{N,s}^{(Cy)}(NT) \) in (1-17) as the inverse Fourier transform of \( \hat{P}_{N,s}^{(Cy)}(t/N; A) \) we see that

\[
P_{\infty,s}^{(Cy)}(T) = \frac{1}{2\pi} \lim_{N \to \infty} \int_{-\infty}^{\infty} \hat{P}_{N,s}^{(Cy)}(t/N)e^{-itT} dt.
\]  

(3-1)

Application of Levy’s continuity theorem permits the limit to be interchanged with the integral. Evaluating the limit inside the integral using (2-42) gives (1-18). Note that the existence of the limit in (1-17) for general \( s \geq 0 \) follows from results in [2] on the generalised Hua–Pickrell measure; recall the discussion below Theorem 1.3.

3B. Evaluating the integral (1-18) as a sum. Upon integrating (1-18) term-by-term particularly simple functional forms result in the cases \( s = 0 \) and \( s = 1 \).

**Proposition 3.1.** We have

\[
P_{\infty,s}^{(Cy)}(T)|_{s=0} = \frac{1}{\pi} \frac{1}{1 + T^2} \quad \text{and} \quad P_{\infty,s}^{(Cy)}(T)|_{s=1} = \frac{1}{\pi} \Re \frac{1}{1 + iT} \, _1F_1 \left( 1; 4/\beta; \frac{4/\beta}{1 + iT} \right).
\]  

(3-2) \quad (3-3)

**Proof.** The case \( s = 0 \) is immediate since the function \( \, _0F_1^{(\beta/2)} \) in (1-18) is identically equal to one. The facts that all the terms in \( \, _0F_1^{(\beta/2)} \) are positive for \( s \geq 1 \) and that the integral is absolutely convergent (this is immediate since the large \( t \) leading behaviour of \( \, _0F_1^{(\beta/2)} \) is of order \( e^{4s(t/\beta)^{1/2}} \), as follows from the integral form [20, Equation (13.27)]; see in particular the discussion above [20, Equation (13.52)]) justify the interchange of the sum and integral for \( s \in \mathbb{Z}^+ \), and in particular for \( s = 1 \), when \( \, _0F_1^{(\beta/2)} \) reduces to its classical counterpart. \( \square \)

**Remark 3.2.** (1) The fact that the limiting distribution in the case \( s = 0 \) is the Cauchy distribution for all \( \beta > 0 \) can already be seen from (2-39). Thus with \( s = 0 \) this result tells us that \( \hat{P}_{N,s}^{(Cy)}(t/N) = e^{-|t|} \), implying \( \sum_{j=1}^{N} x_j / N \) has a Cauchy distribution independent of both \( \beta \) and \( N \).
(2) For the classical values \( \beta = 1, 2 \) and 4 (at least), the right-hand side of (3-3) can be simplified. Thus
\[
P^{(\text{Cy})}_{\infty,s} (T) |_{s=1, \beta=1} = \frac{-39 + 3T^2}{32\pi} + \frac{3}{32\pi} e^{4/(1+T^2)} \left( (1 - T^2) \cos \left( \frac{4T}{1+T^2} \right) + 2T \sin \left( \frac{4T}{1+T^2} \right) \right),
\]
\[
P^{(\text{Cy})}_{\infty,s} (T) |_{s=1, \beta=2} = \frac{1}{2\pi} \left( -1 + e^{2/(1+T^2)} \cos \left( \frac{2T}{1+T^2} \right) \right),
\]
\[
P^{(\text{Cy})}_{\infty,s} (T) |_{s=1, \beta=4} = \frac{e^{1/(1+T^2)}}{\pi (1 + T^2)} \left( \cos \left( \frac{T}{1+T^2} \right) - T \sin \left( \frac{T}{1+T^2} \right) \right).
\]
In the case \( \beta = 2 \) this agrees with a result reported in [5]. As implied by (3-3), independent of \( \beta \), the leading large \( T \) form exhibited by each of the above is
\[
P^{(\text{Cy})}_{\infty,s} (T) |_{s=1} = \frac{1}{(1 + T^2)^2} \left( 1 + O \left( \frac{1}{1+T^2} \right) \right).
\] (3-4)

Term-by-term integration in (1-18) also evaluates \( P^{(\text{Cy})}_{\infty,s} (T) \) for general integers \( s \geq 2 \), but the resulting expression cannot be recognised as a stand-alone special function.

**Proposition 3.3.** Let \( \kappa = (\kappa_1, \ldots, \kappa_s) \) be a partition of \( s \) parts. For all integers \( s \geq 1 \) we have
\[
P^{(\text{Cy})}_{\infty,s} (T) = \frac{1}{\pi} \text{Re} \left( \frac{1}{1 + iT} \sum_{\kappa} \frac{1}{[4s/\beta]^{(\beta/2)} \kappa} \left( \frac{4}{\beta (1 + iT)} \right)^{|\kappa|} C^{(\beta/2)}_{\kappa} ((1)^s) \right).
\] (3-5)

**Proof:** Justification of the interchange of the integral with the sum has already been given in the proof of Proposition 3.1 in the case \( s = 1 \). \( \square \)

**Remark 3.4.** Taking the inverse Fourier transform of (2-41) and integrating term-by-term (the validity of this is immediate since the sum is finite) shows that for finite \( N \)
\[
P^{(\text{Cy})}_{N,s} (T) = \frac{1}{\pi} \text{Re} \left( \frac{1}{N + iT} \sum_{\kappa} \left[ \frac{-N}{[4s/\beta]^{(\beta/2)} \kappa} \right] \left( \frac{4}{\beta (N + iT)} \right)^{|\kappa|} C^{(\beta/2)}_{\kappa} ((1)^s) \right).
\] (3-6)

**3C. Proof of Theorem 1.1.** Define the moments \( m^{(\text{Cy})}_{N,s} (h) \) associated with \( P^{(\text{Cy})}_{N,s} (T) \) by (2-7). Substitution of (3-5) together with knowledge of the integral evaluation [32, Entry 3.194.3]
\[
\int_0^\infty \frac{x^{2h}}{(1 - ix)^{k+1}} dx = -i \pi e^{i\pi h} \frac{(-2h)_k}{k! \sin 2\pi h} \quad \text{for } k > 0 \text{ and } -\frac{1}{2} < \text{Re} \, h < \frac{1}{2} k,
\] (3-7)
provides for the evaluation of these moments as a finite sum.

**Corollary 3.5.** Let \( \kappa = (\kappa_1, \ldots, \kappa_s) \) be a partition of \( s \) parts. For the parameter range (1-6), and with \( s \in \mathbb{Z}^+ \), we have
\[
m^{(\text{Cy})}_{N,s} (h) = \frac{N^{2h}}{\cos \pi h} \sum_{\kappa} \frac{(-N)^{(\beta/2)}_{[\kappa]}}{N^{[\kappa]}} \frac{(-2h)_{|\kappa|}}{\beta^{(\beta/2)}} \left( \frac{4}{\beta} \right)^{|\kappa|} C^{(\beta/2)}_{\kappa} ((1)^s).
\] (3-8)

This expression has a removable singularity at \( h \) half an odd positive integer, and is to be understood in the sense of analytic continuation at these points.
Proof. Suppose first that $-\frac{1}{2} < \text{Re} h < \frac{1}{2}$. Since each term in (3-5) decays at least as fast as $1/|T|^2$ as $|T| \to \infty$, and furthermore the sum in finite, we can interchange the order of the integral and the sum, and the formula (3-8) results. This formula extends to the full range (1-6) by analytic continuation. Thus the analyticity of $m^{(Cy)}_{N,s}(h)$ in this range follows from the discussion below (2-7), while the right-hand side is in fact analytic throughout all of $\text{Re}(h) > -\frac{1}{2}$, with the points $h \in \mathbb{Z}_{\geq 0}$ being removable singularities. □

We are now in a position to prove Theorem 1.1. According to (2-1) and (2-7), using the notation therein, we have

$$F_{N,\beta}^{(Cy)}(s, h) = 2^{a_0} \frac{\mathcal{N}_{N}^{(Cy)}}{C_{N,\beta}} m^{(Cy)}_{N,s}(h).$$

(3-9)

In the case $s$ a nonnegative integer, we can make the simplification

$$2^{a_0} \frac{\mathcal{N}_{N}^{(Cy)}}{C_{N,\beta}} = \frac{1}{2^{2h}} \prod_{j=1}^{s} \frac{\Gamma(2(s+j)/\beta + N)}{\Gamma(2j/\beta + N)} \frac{\Gamma(2j/\beta)}{\Gamma(2(s+j)/\beta)}.$$

(3-10)

For large $N$ the product over the ratio of the gamma functions involving $N$ has the leading form $N^{2s^2/\beta}$, as follows from the standard fact that for any fixed $u$ the leading large $N$ form of $\Gamma(N+u)/\Gamma(N)$ is equal to $N^u$. We use this in (3-10) and substitute the resulting expression in (3-9), telling us that

$$\lim_{N \to \infty} \frac{1}{N^{2s^2/\beta + 2h}} F_{N,\beta}^{(Cy)}(s, h) = \frac{1}{2^{2h}} \prod_{j=1}^{s} \frac{\Gamma(2j/\beta)}{\Gamma(2(s+j)/\beta)} \lim_{N \to \infty} \frac{1}{N^{2h}} m^{(Cy)}_{N,s}(h).$$

(3-11)

In the sum of (3-8) for $m^{(Cy)}_{N,s}(h)$, from the definition (2-11) we see that we have the bound

$$|[-N]^{(\beta/2)/N}| \leq r_1^{[\kappa]}$$

(3-12)

for some $r_1 > 1$ independent of $N$. In addition, as a consequence of (2-31) and the fact that $C_{\kappa}^{(\alpha)}((1)^s)$ is nonnegative, we have the bound

$$|C_{\kappa}^{(\alpha)}((1)^s)| \leq s^{[\kappa]}.$$

(3-13)

Substituting (3-12) and (3-13) in the sum in (3-8) shows that the absolute value of the latter is bounded by the convergent sum (since for large $|\kappa|$ the leading asymptotic of $(-2h)|\kappa|/|\kappa|!$ is $|\kappa|^{-2h-1}/\Gamma(-2h)$, the definition (2-11), along with Stirling’s formula, shows that the decay of the summand is to leading order like the reciprocal of a gamma function in each part $\kappa_i$, implying convergence)

$$\sum_{\kappa} \frac{|(-2h)|\kappa|}{|\kappa|!|4s/\beta|^{(\beta/2)}_{k}} r_2^{[\kappa]}$$

(3-14)

for some $r_2 > 0$ and independent of $N$. Consequently, by dominated convergence, the $N \to \infty$ limit of the sum in (3-8) can be taken term-by-term, and we obtain the limit formula

$$\lim_{N \to \infty} \frac{1}{N^{2h}} m^{(Cy)}_{N,s}(h) = \frac{1}{\cos \pi h} \sum_{\kappa} \frac{(-2h)|\kappa|}{|\kappa|!|4s/\beta|^{(\beta/2)}_{k}} \left(\frac{4}{\beta}\right)^{|\kappa|} C_{\kappa}^{(\beta/2)}((1)^s).$$

(3-15)

Substituting this in (3-11) gives (1-7).
Note that this argument also shows that in the range (1-6),
\[
m^{(Cy)}_{\infty,s}(h) := \int_{-\infty}^{\infty} |T|^{2h} P^{(Cy)}_{\infty,s}(T) \, dT = \lim_{N \to \infty} \frac{1}{N^{2h}} m^{(Cy)}_{N,s}(h),
\]
as is suggested by the scaling in (1-17), and thus \(m^{(Cy)}_{\infty,s}(h)\) has the evaluation (3-13).

**Remark 3.6.** For a given \(h \in \mathbb{Z}_{\geq 0}\), the sum in (3-15) terminates. The explicit evaluation of \(C_k^{(\beta/2)}(1^+)\) implied by (2-35) then tells us that the right-hand side of (3-15) is a rational function of \(s\). For example, with \(h = 1\), we compute
\[
m^{(Cy)}_{\infty,s}(h) |_{h=1} = \frac{1}{(2s-1)(4s/\beta + 1)},
\]
where use has been made of (3-16) on the left-hand side. The derivation of (3-15) generally required \(s\) to be a nonnegative integer such that \(\text{Re } h < s + \frac{1}{2}\) and thus in (3-17) that \(s \in \mathbb{Z}^+\). Yet (3-17) is well defined for all real \(s > \frac{1}{2}\), suggesting that it in fact holds in that domain. For \(\beta = 2\), this statement, along with its generalisation for all \(h \in \mathbb{Z}_{\geq 0}\), has been proved true in the recent work [3].

For general \(\beta > 0\), we can also establish the validity of the right-hand side of (3-15) for a fixed \(h \in \mathbb{Z}^+\) and continuous \(s > h - \frac{1}{2}\). We simply observe that in the evaluation formula (3-8) for \(m^{(Cy)}_{N,s}(h)\), both the left-hand side (by considering its definition (2-7)) and the right-hand side (using the fact that it is a finite sum, and inspecting the analytic form of each term as a function of \(s\)) are well defined in the complex plane for \(\text{Re } s > h - \frac{1}{2}\) and, moreover, are bounded in this half-space. Application of Carlson’s theorem as used in the proof of Corollary 2.7 then tells us that the evaluation formula (3-8) is, with \(h \in \mathbb{Z}^+\) fixed, value for continuous \(s > h - \frac{1}{2}\). Being a finite sum with some number of terms independent of \(N\), the scaled limit required by (3-16) can be taken term-by-term, giving the formula as implied by (3-15) in the setting \(h \in \mathbb{Z}^+\) fixed, but now valid for continuous \(s > h - \frac{1}{2}\).

### 4. The average (1-11)

Up to normalisation, (2-13) is the PDF specifying the Jacobi \(\beta\)-ensemble. Denote an average with respect to this ensemble by \((\cdot)^{\langle J \rangle}_{(\lambda_1, \lambda_2, \beta)}\). We see that \(\mathcal{H}^{(J)}_{N}(h)\) as defined by (1-11) can be written as a Jacobi \(\beta\)-ensemble average according to
\[
\mathcal{H}^{(J)}_{N}(h) = \left( \sum_{j=1}^{N} \frac{1}{u_j} \right)^{\langle J \rangle}_{(a,b,\beta)}.
\]
This shows that \(\mathcal{H}^{(J)}_{N}(h)\) is proportional to the \(h\)-th moment of the singular statistic \(\sum_{j=1}^{N} 1/u_j\) for the Jacobi \(\beta\)-ensemble.

A variant is to change variables \(x_l \mapsto \frac{1}{2}(1 - \cos \theta_l)\) in (2-13) to obtain the density
\[
\frac{1}{\bar{S}_N(a,b,\beta/2)} \prod_{l=1}^{N} (1 - \cos \theta_l)^{a+1/2} (1 + \cos \theta_l)^{b+1/2} \prod_{1 \leq j < k \leq N} |\cos \theta_k - \cos \theta_j|^{\beta},
\]
for appropriate normalisation $\tilde{S}(a, b, \beta)$ simply related to the Selberg integral. We have that

$$H_N^{(j)}(h) \propto 2^h \left( \sum_{j=1}^{N} \frac{1}{1-\cos \theta_j} \right)^h_{(a,b,\beta)}.$$  \hfill (4-3)

The interest in this variant is that for $\beta = 2$ the PDF (4-2) includes for particular $a$ and $b$ the eigenvalue PDF of the classical orthogonal and unitary symplectic matrix groups with Haar measure; see, e.g., [20, Proposition 3.7.1]. Therefore $H_N^{(j)}(h)$ is proportional to the $2h$-th moment of the singular statistic $\sum_{j=1}^{N} 1/(1-\cos \theta_j)$; the analysis of the moments for a related singular statistic of these classical groups has recently been the subject of the study [1].

The characteristic function for the singular statistic relating to (4-1) is

$$\tilde{P}_N^{(j)}(t) := \exp \left( it \sum_{j=1}^{N} \frac{1}{u_j} \right)_{(a,b,\beta)}.$$  \hfill (4-4)

To this author’s knowledge, this quantity, specialised to $\beta = 1$, first arose in a study of Davis [15] into the distribution of Hotelling’s generalised $T_0^2$ statistic. Davis makes use of matrix differential equations to encode recursive properties of the power series expansion of $\tilde{P}_N^{(j)}(t)$ and the corresponding distribution function. At a practical level for applications in statistics, this leads to an efficient and accurate computational scheme to numerically evaluate the PDF of the distribution for fixed values of the parameters, including $N$. However it does not give information on the large $N$ scaled limit.

In the study of (2-8), enabling progress in analysing the large $N$ scaled limit at the level of the underlying distribution function has been the analytic structure of (2-24) in the case $s$ a positive integer. Consideration of the $N = 1$ case, when after a change of variables the problem reduces to computing the characteristic function of the univariate $F$-distribution [41], shows that no such property holds for (4-4) (here the role of $s$ might be expected to be played by $a$), so computation of the explicit form of the limiting scaled distribution of $\sum_{j=1}^{N} 1/u_j$ seems intractable. On the other hand, some progress is possible at the level of moments, allowing (4-1) to be computed for $h$ a nonnegative integer such that it is well defined. An explicit formula for the mean is in fact already known from previous literature. Thus we have [30, Equations (33), (34), (36) with $k = \lambda_1 = 1$] (see also [38])

$$\left\{ \sum_{j=1}^{N} \frac{1}{u_j} \right\}^{(j)}_{(a,b,\beta)} = N \frac{a + b + 1 + \beta(N - 1)/2}{\tilde{a}}.$$  \hfill (4-5)

This can be generalised, allowing for the computation of the scaled large $N$ limit.

**Proposition 4.1.** Let $p \in \mathbb{Z}^+$ with the further constraint $p < a + 1$. Let $\ell(\kappa)$ denote the number of nonzero parts in $\kappa$. We have

$$\left\{ \left( \sum_{j=1}^{N} \frac{1}{u_j} \right)^p \right\}^{(j)}_{(a,b,\beta)} = \sum_{|\kappa| = p} C_{\kappa}^{(2/\beta)}((1)^N) \prod_{j=0}^{\ell(\kappa)-1} \frac{\Gamma(a + \beta j/2 + 1 - \kappa_j)}{\Gamma(a + \beta j/2 + 1)} \frac{\Gamma(a + \beta(N + j - 1)/2 + 2 - \kappa_j)}{\Gamma(a + \beta(N + j - 1)/2 + 2)}.$$  \hfill (4-6)
Consequently, in the notation of (2-35) with $\alpha = 2/\beta$, 

$$
\lim_{N \to \infty} \frac{1}{N^{2p}} \left( \prod_{j=1}^{N} \frac{1}{u_j} \right)^p_{(a,b,\beta)} = \sum_{|\kappa|=p} |\kappa|! \frac{1}{d_{\kappa}^k h_{\kappa}} \prod_{j=0}^{\ell(\kappa)-1} \frac{\Gamma(a+\beta j/2 + 1 - \kappa)}{\Gamma(a+\beta j/2 + 1)}. 
$$

(4-7)

Proof. Let $p$ be a nonnegative integer. According to (2-31) we have 

$$
\left( \prod_{j=1}^{N} \frac{1}{u_j} \right)^p_{(a,b,\beta)} = \sum_{|\kappa|=p} \left( C_k^{(2/\beta)} \left( \frac{1}{u_1}, \ldots, \frac{1}{u_N} \right) \right)^{(J)}_{(a,b,\beta)}.
$$

(4-8)

We know [20, Exercises 12.1 q.1 & q.2] that for $q \in \mathbb{Z}^+$ such that $q \geq \kappa_1$, 

$$
C_k^{(2/\beta)} \left( \frac{1}{u_1}, \ldots, \frac{1}{u_N} \right) = C_{-\kappa^R}^{(2/\beta)}(u_1, \ldots, u_N) = \prod_{l=1}^{N} u_l^{-q} C_{(q)N-\kappa^R}(u_1, \ldots, u_N). 
$$

(4-9)

In (4-9), $-\kappa^R$ denotes the partition $\kappa$, including the zero parts, with order reversed and each part multiplied by $-1$. So, for example, if $N = 4$ and $\kappa = (2, 1, 0, 0)$ then $-\kappa^R = (0, 0, -1, -2)$ and $(q)^{N-\kappa^R} = (q, q, q-1, q-2)$.

According to (2-30), 

$$
\left( \prod_{l=1}^{N} u_l^{-q} C_{(q)N-\kappa^R}(u_1, \ldots, u_N) \right)^{(J)}_{(a,b,\beta)} = \frac{S_N(a-q, b, \beta)}{S_N(a, b, \beta)} C_k^{(2/\beta)}((1)^N) \frac{[a-q+\beta(N-1)/2 +1]^{(2/\beta)}_{(q)N-\kappa^R}}{[a-q+b+\beta(N-1)+2]^{(2/\beta)}_{(q)N-\kappa^R}}. 
$$

(4-10)

Here use has been made of the fact that $C_{-\kappa^R}^{(2/\beta)}((1)^N) = C_k^{(2/\beta)}((1)^N)$, as follows from (4-9). From the definition (2-11) we can check that 

$$
\prod_{j=0}^{N-1} \frac{\Gamma(a-q+1+j\beta/2)}{\Gamma(a+1+j\beta/2)} [a-q+b+\beta(N-1)+2]^{(2/\beta)}_{(q)N-\kappa^R} = \prod_{j=0}^{\ell(\kappa)-1} \frac{\Gamma(a+\beta j/2 + 1 - \kappa^R)}{\Gamma(a+\beta j/2 + 1)}. 
$$

(4-11)

The relevance of the first ratio of gamma functions comes from making use of (1-13) in relation to the ratio of Selberg integrals in (4-10). Using this as written, and also with $a \mapsto a+\beta(N-1)/2 +1$, and noting too that $\kappa^R_{N-j} = \kappa_j$ we see that (4-11) reduces to (4-6).

Using the standard asymptotic formula for the ratio of gamma functions, and making use too of (2-35) and (2-36), we see that (4-7) is a direct consequence of (4-6). 

\quad \Box

Remark 4.2. (1) The averages $\left( \prod_{j=1}^{N} u_j \right)^p_{(a,b,\beta)}$, which in words are the moments of the trace in the Jacobi $\beta$-ensemble, have been studied in the recent work [23], and the analogue of (4-6) has been given. In addition, it has been shown that these moments satisfy a matrix recurrence with respect to $p$, where the matrices involved are of size $(N+1) \times (N+1)$. For the averages (4-6), the results in [15] imply an analogous matrix recurrence, at least for the case $\beta = 1$. 

(2) Denote the left-hand side of (4-7) by $\mu^{(J)}_\infty(p)$, and set $\alpha = 2/\beta$. Evaluation of the right-hand side shows

$$\mu^{(J)}_\infty(p)|_{p=1} = \frac{1}{\alpha a} \quad \text{and} \quad \mu^{(J)}_\infty(p)|_{p=2} = \frac{1}{\alpha^2(\alpha+1)} \frac{1}{\tilde{a}(a-1)} + \frac{1}{\alpha(\alpha+1)} \frac{1}{\tilde{a}(a+1/\alpha)}.$$  \hspace{1cm} (4-12)

In the case $\beta = 2$ (and thus $\alpha = 1$), these formulas are consistent with known results [14, Equation (3.2)]. It is furthermore the case that for $\beta = 2$ the exponential generating function for the sequence of cumulants $\{\mu^{(J)}_\infty(p)\}|_{p=1}$ corresponding to the moments satisfies a nonlinear differential equation [14; 48].

(3) Let $\langle \cdot \rangle_{(a,\beta)}^{(L)}$ denote an average with respect to the Laguerre $\beta$-ensemble specified by the PDF proportional to $N\prod_{j=1}^N x_j^a e^{-x_j/|\Delta_N(x)|^\beta}$, where $x_j \in \mathbb{R}^+$ for $j = 1, \ldots, N$, and define

$$\mu^{(L)}_\infty(p) := \lim_{N \to \infty} \frac{1}{N^p} \left( \left( \sum_{j=1}^N u_j \right)^p \right)_{(a,\beta)}.$$  \hspace{1cm} (4-14)

The method of proof of Proposition 4.1 can be adapted to show

$$\mu^{(L)}_\infty(p) = \mu^{(J)}_\infty(p).$$

This is to be expected, as the singular linear statistic $\sum_{j=1}^N 1/u_j$ probes the neighbourhood of the origin, which in both ensembles is characterised by the same factor $\prod_{i=1}^N u_i^a |\Delta_N(a)\|^\beta$ in the corresponding PDFs. The scaling $1/N^{2p}$ in (4-7), in contrast to the scaling $1/N^p$ in (4-14), is due to the typical spacing between eigenvalues in the neighbourhood of the origin being of order $1/N^2$ (respectively, $1/N$) for the Jacobi (respectively, Laguerre) ensemble.

The result (4-7) from Proposition 4.1 allows for the scaled limit of (1-11) with $h$ a nonnegative integer to be read off.

**Corollary 4.3.** Define $\mathcal{H}^{(J)}(h)$ by (1-11). In the notation of Proposition 4.1, and with $h$ a nonnegative integer such that $h < a + 1$, we have

$$\lim_{N \to \infty} \frac{1}{N^{2h}} \mathcal{H}^{(J)}_N(h) = \sum_{|\kappa| = h} |\kappa|! \frac{1}{d'h_\kappa} \prod_{j=0}^{\ell(\kappa)-1} \frac{\Gamma(\alpha + \beta j/2 + 1 - \kappa_j)}{\Gamma(\alpha + \beta j/2 + 1)}.$$  \hspace{1cm} (4-15)

**Appendix**

For $\beta = 2$, when the exponential generating function (1-21) relates to determinantal structures such as (1-24) and (1-25), there are known characterisations in terms of Painlevé transcendent; see, e.g., [20, Chapters 8 and 9] for an account of results of this type in random matrix theory. Here we want to use this characterisation to exhibit the small $t$ analytic structure for continuous $s$. Fundamental to our exact calculations for positive integer $s$ has been that this structure is given by an exponential times a
power series in \(t\). For \(\beta = 2\), we will see that for continuous \(s\), an in general infinite number of fractional powers are also present.

The \(s \in \mathbb{Z}^+\) determinantal evaluation (1-25) of the exponential generating function (1-21) was first identified in the work [13]. Soon after, in [28], it was observed that up to an exponential factor, this same expression appears as a \(\tau\)-function in the Hamiltonian theory of Painlevé III′ known from [27]. Thus an alternative evaluation of (1-25) is, under the assumption \(s \in \mathbb{Z}_{\geq 0}\).

\[
\lim_{N \to \infty} \frac{1}{C_N} H_N^{(Cy)}(s; t/2N) = e^{t/2} \tilde{E}_s^{\text{hard}}(t) \quad \text{with} \quad \tilde{E}_s^{\text{hard}}(t) := \exp \left( -\int_0^{4t} (\sigma_{III}(x) + s^2) \frac{dx}{x} \right),
\]
where \(\sigma_{III}(x) = \sigma_{III}(x; s)\) satisfies the particular \(\sigma\)-Painlevé III′ equation

\[
(x \sigma_{III}''')^2 + \sigma_{III}'(4\sigma_{III} - 1)(\sigma_{III} - x\sigma_{III}') - \frac{1}{16} s^2 = 0,
\]
subject to the small \(x\) boundary condition

\[
\sigma_{III}(x) = -s^2 + \frac{1}{8} x + O(x^2), \quad s \in \mathbb{Z}_{\geq 0}.
\]

In [28] it was shown that \(\tilde{E}_s^{\text{hard}}(t)\) also results as a scaling limit of the Laguerre unitary ensemble (\(\beta = 2\) case of (4-13))

\[
\tilde{\mathcal{E}}_{N,s}(t) \propto \left( \prod_{l=1}^N (1 - \chi_{(0,t)}^{(l)})(x_l - t)^s \right)^{(L)}_{(s, \beta = 2)},
\]
where \(\chi_{(0,t)}^{(l)} = 1\) for \(x_l \in (0, t)\) and \(\chi_{(0,t)}^{(l)} = 0\) otherwise, and the proportionality is such that both sides equal unity when \(t = 0\). Thus

\[
\tilde{E}_s^{\text{hard}}(t) = \lim_{N \to \infty} \tilde{E}_{N,s}(t/N).
\]

From the work of Winn [47] we know that (A-4) is a transformed version of the exponential generating function (1-21) valid for continuous \(s\). For continuous \(s\), the quantity \(\tilde{E}_{N,s}(t)\) was identified as a \(\tau\)-function in the Hamiltonian theory of Painlevé V in [27], and its scaled limit was shown formally to give rise to the same \(\sigma\)-Painlevé III′ equation as for the \(s \in \mathbb{Z}_{\geq 0}\) case (A-2) and to the same functional form (A-1). Very recently a rigorous demonstration of the validity of (A-1) for continuous \(s\) has been given in [5].

To make use of the knowledge that (A-1) remains valid for \(s\) continuous, for our aim of determining the analytic form \(\tilde{E}_s^{\text{hard}}(t)\), requires that the boundary condition (A-3) of the \(\sigma\)-Painlevé III′ equation (A-3) be appropriately extended. This problem was solved in [28], by the strategy of starting with (A-4) rewritten as a certain Hankel determinant with elements given in terms of the hypergeometric \(_1F_1\) function, then expanding the latter for small \(t\), which allows the small \(t\) form of \(\tilde{E}_{N,s}(t)\) to be systematically determined. The small \(t\) form of \(\tilde{E}_s^{\text{hard}}(t)\) then follows by taking the limit term-by-term. With the latter expressed in terms of \(\sigma_{III}(x)\) according to (A-1) this procedure shows [28, Equation (4.2)] with \(s \mapsto x, \mu = a = s, \xi = 1\)

\[
\sigma_{III}(x) = (-s^2 + \frac{1}{8} x + O(x^2)) + c_s x^{2s+1} (1 + O(x)) + O(x^{2(2s+1)}),
\]
where

\[
c_s := \left( \lim_{\epsilon \to 0} \frac{\sin \pi s}{\sin(2\pi s + \epsilon)} \right) \frac{(\Gamma(s+1))^2}{\Gamma(2(s+1))\Gamma^2(2s+1)}.
\]
The limit in (A-7) diverges for \(2s\) half an odd positive integer. In these cases \(\log x\) terms appear in the functional form (A-6).

Substituting (A-6) in the \(\sigma\)-Painlevé III’ equation (A-3) reveals that about the origin the transcendent \(\sigma_{\III'}(x)\), for \(2s\) not half an odd positive integer, exhibits the Puiseux-type series solution

\[
\sigma_{\III'}(x) = \sum_{j=0}^{\infty} x^{j(2s+1)} g_j(x),
\]

where each \(g_j(x)\) is a power series in \(x\). Substituting in (A-1) gives that this same analytic structure persists for \(E_x^{\text{hard}}(t)\). Moreover, in the case that \(s\) is a nonnegative integer, we know from (A-3) that only the term \(j = 0\) in the sum contributes, showing how this case is special.

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