Quantum superpositions of clockwise and counterclockwise supercurrent states in the dynamics of a rf-SQUID exposed to a quantized electromagnetic field

R. Migliore\(^1\)\(^*\) and A. Messina\(^1\)

\(^1\)INFM and MIUR, Dipartimento di Scienze Fisiche ed Astronomiche dell’Università di Palermo, via Archirafi 36, 90123 Palermo, Italy.

(Dated: March 22, 2022)

The dynamical behavior of a superconducting quantum interference device (a rf-SQUID) irradiated by a single mode quantized electromagnetic field is theoretically investigated. Treating the SQUID as a flux qubit, we analyze the dynamics of the combined system within the low lying energy Hilbert subspace both in the asymmetric and in the symmetric SQUID potential configurations. We show that the temporal evolution of the system is dominated by an oscillatory behavior characterized by more than one, generally speaking, incommensurable Rabi frequencies whose expressions are explicitly given. We find that the external parameters may fixed in such a way to realize a control on the dynamical replay of the total system which, for instance, may be forced to exhibit a periodic evolution accompanied by the occurrence of an oscillatory disappearance of entanglement between the two subsystems. We demonstrate the possibility of generating quantum maximally entangled superpositions of the two macroscopically distinguishable states describing clockwise and counterclockwise supercurrents in the loop. The experimental feasibility of our proposal is briefly discussed.

PACS numbers: 74.50.+r, 03.67.Lx, 73.23.Hk

I. INTRODUCTION

It is of great interest to understand and study the physics of Josephson junction-based devices both for testing fundamental properties of quantum mechanics, such as the superposition principle or the occurrence of entangled states [1], and for technological applications in the context of quantum information theory and quantum computing [2].

In the last decade, rapid developments in the realm of nanotechnologies have made it possible to perform a number of beautiful and sophisticated experiments at low temperature [3] bringing to light the existence in these atom-like circuits of many macroscopic quantum phenomena like energy level quantization [4], macroscopic quantum tunneling (MQT) and quantum superposition of states [5, 6]. More recently, the usefulness of investigating these solid state devices in the context of quantum communication and information theory has been fully recognized.

Superconducting Josephson devices may, in fact, be thought of as two-state systems realizing the elementary unit of quantum information, known as quantum bit or qubit [7]. Moreover, Josephson devices can be scaled up to a large number of qubits and their dynamics may be controlled by externally applied voltages and magnetic fluxes. Superconducting devices like Cooper pair boxes, Josephson junctions or SQUIDs have been thus proposed and used as basic elements for the practical realization of quantum gates and chips [8]. The relevant macroscopic degree of freedom, allowing to store and manipulate quantum information, may be the charge on the island of a Cooper pair box or the phase differences at the junction. In the first case the charging energy \( E_C \) overcomes the Josephson energy \( E_J \). Otherwise, in the opposite regime, the Josephson energy overcomes \( E_C \). It has been already experimentally demonstrated that Cooper pair boxes behave as two level systems which can be coherently controlled [9, 10, 11] and now great efforts are devoted to prove that the same can be done with flux and phase qubits [12, 13, 14]. However, successful realization of quantum algorithms critically depends on the ability to entangle quantum states of qubits. The optimum would be the realization of a tunable coupling bus. Several coupling mechanisms are possible but the natural way of coupling two or more superconducting qubits is through an intermediate resonant LC circuit, playing the role of a data bus [15, 16]. Such a resonant LC circuit, describable as a quantum harmonic oscillator, may be in principle replaced by the electromagnetic single-mode of a high-Q cavity or by a large-area current biased Josephson junction. In all these cases the situation is similar to cavity QED [16] (where cavity and atoms play the roles of the LC circuit and qubits, respectively) and to ion-trap proposals [17].

\(^*\)Electronic address: rosanna@fisica.unipa.it
It is then evidently of interest to study the interaction between a two level solid state system, like an rf-SQUID, and an external quantum system like another qubit, a tank circuit or a monochromatic radiation source. The aim of such investigations is to bring to light the occurrence of entangled states and to construct coupling schemes by which the coherent dynamics of the system may be controlled and/or manipulated.

In this paper our main scope is to study the dynamics of a flux qubit (an rf-SQUID) coupled to a single mode quantized electromagnetic field of a resonant cavity. Confining ourselves to the low-lying energy Hilbert space, we prove that the time evolution of the combined system is characterized by the occurrence of entanglement which may be controlled in terms of the strength of the coupling and the circuit parameters. In addition we show that the dynamics is dominated by an oscillatory behavior traceable back to the existence of a finite set of characteristic Rabi frequencies whose expression may be explicitly given.

The importance of conceiving experimental schemes for realizing quantum superpositions of macroscopically distinguishable states has been quite recently emphasized. The main result of this paper is that, appropriately acting upon some control parameters, it is possible to guide the rf-SQUID toward coherent maximally entangled combinations and then macroscopically distinguishable.

In section II we describe the physical system under study. Its dynamics in a reduced low-lying energy Hilbert space is studied in section III where the main results of this paper are reported. In the last section, we discuss our results and we conclude with some remarks about the feasibility of an experiment aimed at verifying our theory in the laboratory.

II. THE QUANTUM CIRCUIT

In this section we describe in detail the physical system, namely a rf-SQUID coupled to a monochromatic field of a high-Q resonant cavity and its Hamiltonian model. In figure 1 the electromagnetic single-mode cavity is represented as an LC resonator.

In subsection II A we describe the physical conditions that allow us to consider a SQUID as a two level system, that is as a flux qubit. Then we give the Hamiltonian model for the cavity field in terms of the combined system characteristics and finally, in subsection II C, we consider the inductive coupling between these two subsystems.

A. The rf-SQUID as material two level system

Let us begin by describing as usual the rf-SQUID, a superconducting loop interrupted by a Josephson junction (figure 2a), as a fictitious particle of mass $C$ and generalized coordinate $\phi$ (the magnetic flux in the loop) subjected to the washboard potential

$$U(\phi) = -E_J \cos \left(2\pi \frac{\phi}{\phi_0}\right) + \frac{(\phi - \phi_x)^2}{2L}$$

(1)

where $C \sim 10^{-15} \div 10^{-13} \text{ F}$ is the junction capacitance, $L \sim 10 \div 100 \text{ pH}$ the self-inductance of the loop, $\phi_x$ an externally applied dc flux and $\phi_0 = \frac{\hbar}{2e}$ the flux quantum. The Josephson coupling energy $E_J$ is related to the critical
supercurrent $I_C$ by $E_J = I_C \phi_0 / 2\pi$.

Taking into account both the kinetic and potential energy, it is then immediate to write the Hamiltonian of the system as follows

$$H = \frac{Q^2}{2C} - E_J \cos \left( \frac{2\pi \phi}{\phi_0} \right) + \left( \frac{\phi - \phi_x}{2L} \right)^2$$

(2)

where the charge on the junction capacitance $Q = -i\hbar \partial / \partial \phi$ and the flux $\phi$ in the loop are canonically conjugate operators satisfying the commutation rule $[\phi, Q] = i\hbar$. Under the condition $\beta_L = 2\pi L I_C / \phi_0 > 1$ the rf-SQUID is hysteretic and the potential $U(\phi)$ may have one or several relative minima. The height of barrier between minima and the number of minima depend on the parameter $\beta_L$. The form of the potential $U(\phi)$ instead can be tuned by changing the external dc magnetic flux $\phi_x$ applied to the loop to the case where the two lowest energy wells are degenerate. This case occurs for $\phi_x = \phi_0 / 2$. These two degenerate minima correspond to the clockwise and counterclockwise sense of rotation of the supercurrent in the loop. The two relative ground states are localized flux states, hereafter denoted $|\phi_0 \rangle$ and $|\phi_0 \rangle$. For $\phi_x$ not exactly coincident with $\phi_0 / 2$, $U(\phi)$ defines an asymmetric double-well potential near $\phi = \phi_0 / 2$ with barrier $V_b$ between these two lowest energy minima. At sufficiently low temperatures, transitions between neighboring flux states are dominated by macroscopic quantum tunnelling. However, for high barrier $V_b$, tunnelling does not mix the two lowest flux states with the excited states in the two wells. Thus, in the parameter regime $V_b \gg \hbar \omega_0 \gg k_B T$ ($\hbar \omega_0$ being the separation of the first excited state from the ground state in both wells), the rf-SQUID effectively behaves as a two state system [7] with reduced Hamiltonian expressible in terms of the Pauli matrices $\sigma_x$ and $\sigma_z$ as follows

$$H_S = -\frac{\hbar}{2} \Delta \sigma_x + \frac{\hbar}{2} \epsilon \sigma_z = \frac{\hbar}{2} \begin{pmatrix} \epsilon & -\Delta \\ -\Delta & -\epsilon \end{pmatrix},$$

(3)

where the basis coincides with the two localized flux states $|L\rangle$ and $|R\rangle$. Here $\hbar \epsilon(\phi_x) \equiv \hbar \epsilon = 2I_C \sqrt{6(\beta_L - 1)(\phi_x - \phi_0 / 2)}$ is the asymmetry of the double-well and $\Delta$ is the tunnelling frequency between the wells. This tunnelling frequency can be tuned by changing the height $V_b$ of the barrier which depends on the Josephson coupling energy $E_J$. Then if we replace the junction by a hysteretic dc-SQUID (Fig. 2b), behaving as a JJ with tunable critical current, $E_J$ may be manipulated by a separately control dc flux $\phi_c$. In other words, this modified device, known as the double rf-SQUID, behaves as a normal rf-SQUID whose Josephson energy $E_J$ is related to the control flux $\phi_c$ by $E_J(\phi_c) = 2I_C \cos(\pi \phi_c / \phi_0)$, $I_C$ being the critical current of each of the two JJs of the dc-SQUID.

Since in our scheme we need to control $\Delta$, in what follows, we will investigate a double rf-SQUID with two external control fluxes $\phi_x$ and $\phi_c$ when it is exposed to a monochromatic quantized electromagnetic field.

The hamiltonian of the double rf-SQUID, given also in this case by eq. (2), can be easily cast in diagonal form

$$H_S = \frac{\hbar}{2} \begin{pmatrix} -\sqrt{\epsilon^2 + \Delta^2} & 0 \\ 0 & \sqrt{\epsilon^2 + \Delta^2} \end{pmatrix}$$

(4)

in the basis formed by its eigenstates

$$|\phi \rangle = C_- |L\rangle + \frac{\epsilon + \sqrt{\epsilon^2 + \Delta^2}}{\Delta} |R\rangle$$

(5)
and

$$|+\rangle = C_+(|R\rangle + \frac{\epsilon - \sqrt{\epsilon^2 + \Delta^2}}{\Delta}|L\rangle).$$

(6)

Here $C_+ = \left[1 + \left(\frac{\pm \sqrt{\epsilon^2 + \Delta^2}}{\Delta}\right)^2\right]^{-\frac{1}{2}}$ are the normalizing factors of $|\rangle$ and $|\rangle$ respectively and $\hbar \sqrt{\epsilon^2 + \Delta^2}$ is the energy difference between their corresponding eigenvalues $E_-= -\frac{\hbar}{2} \sqrt{\epsilon^2 + \Delta^2}$ and $E_+ = \frac{\hbar}{2} \sqrt{\epsilon^2 + \Delta^2}$.

B. The quantized electromagnetic field

In this section we model the electromagnetic field of the resonant cavity as a $LC$ resonator. In this way our treatment and our results may be easily extended to the case of the coupling between the flux qubit and a real $LC$ resonator [21] or a large-area current-biased Josephson junction [13, 15]. This fact is important because, in order to verify experimentally our theoretical predictions, it is more practical to couple the qubit with one of these two systems rather than with the single-mode of a resonant cavity. We therefore start by considering the Hamiltonian of an $LC$ resonator with infinite parallel resistance on resonance and frequency $\omega_F = \frac{1}{\sqrt{L_F C_F}}$

$$H_F = \frac{Q_F^2}{2C_F} + \frac{\phi_F^2}{2L_F}$$

(7)

where $\phi_F$ and $Q_F$ play the roles of the magnetic flux and charge operators arising from the quantized electromagnetic field and satisfying the commutation rule $[\phi_F, Q_F] = i\hbar$.

As in references [18, 20] it is possible to relate the resonator operators $\phi_F$ and $Q_F$ to bosonic annihilation and creation operators $a$ and $a^\dagger$. Defining the flux $\phi_F$ and the conjugate operator $Q_F$ as

$$\phi_F = \sqrt{\frac{\hbar}{2\omega_F C_F}} (a + a^\dagger)$$

(8)

and

$$Q_F = -i\sqrt{\frac{\hbar}{2\omega_F C_F}} (a - a^\dagger)$$

(9)

and substituting these analytical expressions in eq. (7), it is immediate to cast the Hamiltonian of the resonator in the form of a standard harmonic oscillator free Hamiltonian

$$H_F = \hbar \omega_F (a^\dagger a + \frac{1}{2}).$$

(10)

Here the frequency $\omega_F$, choosing $C_F \sim 10^{-12}$ $F$ and $L_F \sim 0.1$ $nH$, belongs in the range of microwaves ($\omega_F \approx 10^{11}$ $rad \cdot s^{-1}$). The eigenfunctions of hamiltonian (10) are, of course, harmonic oscillator eigenstates $|n\rangle$ (defined by $a^\dagger a|n\rangle = n|n\rangle$) with eigenvalues $E_n = \hbar \omega_F (n + \frac{1}{2})$.

If the rf-SQUID is effectively coupled with the quantized mode of an electromagnetic high-Q cavity, we may start again from Hamiltonian (8). In order to understand the meaning of $C_F$ in this case, we assume that the rf-SQUID is located perpendicularly to the magnetic field and within a distance small compared to the radiation wavelength. In such conditions, the vector potential $A(x)$ arising from the electromagnetic field of the cavity mode is approximately uniform throughout the region of the device and in international units and in the Coulomb gauge ($\nabla \cdot A = 0$) it assumes the form:

$$A = \left(\frac{\hbar}{2\epsilon_0 \omega_F V}\right)^{\frac{1}{2}} (a - a^\dagger)$$

(11)

$u$ being the unit polarization vector, $\epsilon_0$ the vacuum dielectric constant and $V$ the quantization volume of the field mode. Thus the expression for the operator $\phi_F$ to be inserted in eq. (8) may be written down as

$$\phi_F = \oint_\gamma A \cdot dl = \sqrt{\frac{\hbar}{2\omega_F C_F}} (a + a^\dagger)$$

(12)
where the capacitive parameter $C_F$, given by the following expression

$$C_F = \varepsilon_0 V \left( \oint_{\gamma} u \cdot dl \right)^{-2},$$

depends on the field frequency, via the quantization volume $V$, and on the SQUID geometry, via the line integral which is taken across a closed circuit $\gamma$ inside the SQUID loop. Then, also in this case, exploiting eq. (12) and the properties of conjugation between $\phi_F$ and $Q_F$, it is easy to cast Hamiltonian (5) in the form of a standard harmonic oscillator free Hamiltonian (10).

C. The coupled system

In view of the assumptions made in the previous subsection, the flux qubit and the LC-resonator modelling the monochromatic field can be thought of as coupled together inductively (see figure 1) with a contribution to the total hamiltonian given by

$$H_I = \frac{2k}{L} \phi_F = B (a+a^\dagger) [-\epsilon \sigma_z + \Delta \sigma_x]$$

where the constant

$$B = \frac{k}{L} \sqrt{\frac{\hbar}{2 \omega_F C_F}} \frac{\phi_0}{\sqrt{\varepsilon^2 + \Delta^2}}$$

has the same dimension of $\hbar$ and depends on the coupling strength and the system characteristics. Here the flux linkage factor $k$ is assumed of the order of 0.01 in accordance with the current experimental values [29]. Thus the Hamiltonian describing the combined rf-SQUID-field system can be written down as

$$H = H_S + H_F + H_I,$$

where $H_S$ and $H_F$, given by eq. (4) and (10) are the Hamiltonians describing the two free subsystems.

III. THE SYSTEM DYNAMICS IN A REDUCED LOW LYING ENERGY 4 × 4 HILBERT SUBSPACE

In view of eqs. (4), (10) and (14) the Hamiltonian $H$ may be formally interpreted as that of a two-level “atom” interacting quantum mechanically with a monochromatic quantized electromagnetic field. We wish now to focuss our attention on a physical situation wherein the operating temperature is $T \approx 10 \text{ mK}$ and the field is in resonance with the transition between the lowest and the first excited state of the rf-SQUID, that is $\hbar \omega_F = \sqrt{\varepsilon^2 + \Delta^2}$.

As a consequence we drastically simplify the difficult problem of diagonalizing $H$ in its infinite dimensional Hilbert space by a truncation procedure which confine ourselves to the 4 × 4 Hilbert space spanned by the states $\{|0\sigma\rangle, |1\sigma\rangle, |0+\rangle, |1+\rangle\}$. In other words, we investigate the dynamics of the system in the low-lying energy subspace of the free Hamiltonian $H_0 = H_S + H_F$ generated by the four states, $|n\sigma\rangle$ ($n \equiv 0, 1; \sigma \equiv -, +$) which are the common eigenstates of $H_S$ and $H_F$ such that $H_S|n\sigma\rangle = E_\sigma |n\sigma\rangle$ and $H_F|n\sigma\rangle = \hbar \omega_F (n + \frac{1}{2}) |n\sigma\rangle$.

In this reduced basis and adopting the resonant condition $\omega_F = \sqrt{\varepsilon^2 + \Delta^2}$, Hamiltonian (14) can be cast in the form of the following 4 × 4 non diagonal matrix:

$$H_R = \begin{pmatrix}
0 & -B\epsilon & 0 & B\Delta \\
-B\epsilon & \hbar \omega_F & B\Delta & 0 \\
0 & B\Delta & \hbar \omega_F & B\epsilon \\
B\Delta & 0 & B\epsilon & 2\hbar \omega_F
\end{pmatrix}.$$

We emphasize that the counter-rotating terms of $H$ contribute to this truncated Hilbert space with the presence of non diagonal matrix elements of $H_R$. These matrix elements are those connecting states of the basis having a different number $n + \sigma$ of total excitations.

In the following, we wish to study the dynamics of the coupled matter-radiation system initially prepared in different physically meaningful conditions both in the asymmetric ($\epsilon \neq 0$) and symmetric ($\epsilon = 0$) cases. To this end we need to find the eigensolutions of $H_R$. 


A. The dynamics of the system in the asymmetric case

Let us consider the time evolution of the combined rf-SQUID-radiation system initially prepared in the state $|0+\rangle$, that is the field in its vacuum state $|0\rangle$ and the SQUID in the first excited state $|+\rangle = C_+ |R\rangle + \frac{\Delta}{\Delta N} |L\rangle$ which is a superposition of the localized flux states $|L\rangle$ and $|R\rangle$.

It is worth noting that the manipulation of the shape of the potential and the height of the barrier between the two lowest energy wells via the control fluxes $\phi_s$ and $\phi_c$ allows to prepare the SQUID in a prefixed initial condition \[8\]. \[29\]. At the same time, exploiting the currently available experimental techniques in CQED, it is possible to prepare the field mode in a Fock state with 0 or 1 photon \[80\]. Eigenvalues $|u_i\rangle$ ($i\equiv 1,2,3,4$) and eigenvectors $\lambda_i$ of $H_R$ in the asymmetric case may be exactly evaluated and they are explicitly given in Appendix A.

The expansion of $|0+\rangle = \exp(-i\frac{H_R t}{\hbar}) |0+\rangle$ in terms of $|u_1\rangle$, $|u_2\rangle$, $|u_3\rangle$ and $|u_4\rangle$ may be cast in the following form:

$$
|0+\rangle = \frac{Be}{(P_1 - P_2)Q_1Q_2} \left[ \sqrt{n_1}(P_2 - Q_1)Q_2 |u_1\rangle \exp(-i\lambda_1 t) - \sqrt{n_2}(P_2 + Q_1)Q_2 |u_2\rangle \exp(-i\lambda_2 t) + \sqrt{n_3}(P_1 - Q_2)Q_1 |u_3\rangle \exp(-i\lambda_3 t) + \sqrt{n_4}(P_1 + Q_2)Q_1 |u_4\rangle \exp(-i\lambda_4 t) \right].
$$

Thus $|0+\rangle$ is as in linear superposition, with different weights, of the eigenstates $|u_i\rangle$ of Hamiltonian (17).

We are interested in exploring the ability of the system to periodically come back to the initial state $|0+\rangle$ as well as to pass thought the other states $|1-\rangle$, $|0-\rangle$ and $|1+\rangle$. The structure of the previous equation and those of eqs. (A10)-(A13) makes clear that the time evolution of the system from $|0+\rangle$ involves all the four states $|0-\rangle$, $|1-\rangle$, $|0+\rangle$ and $|1+\rangle$. However, since the states $|0+\rangle$ and $|1-\rangle$ are almost degenerate in energy, in our physical situation transitions between them are more probable than transitions between the initial state $|0+\rangle$ and the states $|0-\rangle$ and $|1+\rangle$. This fact may be clearly appreciated by evaluating the time evolution of the survival probability $P_1(t)$ of the state $|0+\rangle$, that is

$$
P_1(t) = |\langle 0+|0+\rangle|^2 = S^2 \left\{ \sum_{j=1}^{4} S_j \cos \frac{Q_j}{\hbar} t + 2S_3S_4 \cos \frac{Q_2}{\hbar} t + 2[S_1S_3 + S_2S_4] \cos \frac{(Q_1 - Q_2)}{2\hbar} t + 2[S_2S_3 + S_1S_4] \cos \frac{(Q_1 + Q_2)}{2\hbar} t \right\}.
$$

where

$$
S = \frac{Be}{(P_1 - P_2)Q_1Q_2},
$$

$$
S_1 = \frac{P_1Q_2}{2Be}(Q_1 - P_2),
$$

$$
S_2 = \frac{P_1Q_2}{2Be}(Q_1 + P_2),
$$

$$
S_3 = -\frac{P_2Q_1}{2Be}(Q_2 - P_1),
$$

$$
S_4 = -\frac{P_2Q_1}{2Be}(Q_2 + P_1),
$$

It is immediate to construct explicit expressions also for the transition probabilities $P_2(t) = |\langle 1-|0+\rangle|^2$, $P_3(t) = |\langle 0-|0+\rangle|^2$ and $P_4(t) = |\langle 1+|0+\rangle|^2$ to the states $|1-\rangle$, $|0-\rangle$ and $|1+\rangle$ respectively. Their explicit analytical expressions are given in appendix B. Here we wish to underline that these transition probabilities have the same mathematical structure of eq. (19). This means that $P_2$, $P_3$ and $P_4$ like $P_1$ are given by the sum of a constant and 4 trigonometric time dependent terms with different weights and frequencies. Since these four frequencies $\omega_1 \equiv \frac{Q_2}{\hbar}$, $\omega_2 \equiv \frac{Q_1}{\hbar}$, $\omega_3 \equiv \frac{Q_1 - Q_2}{2\hbar}$ and $\omega_4 \equiv \frac{Q_1 + Q_2}{2\hbar}$ appearing in the expressions of $P_1(t)$, $P_2(t)$, $P_3(t)$ and $P_4(t)$ are in general incommensurable, we find a quasi-periodic behavior wherein a complete exact inversion of the populations between the degenerate states $|0+\rangle$ and $|1-\rangle$ never occurs. The time evolutions of $P_1$, $P_2$, $P_3$ and $P_4$ are plotted in figure 3.
A similar quasi-periodic behavior characterizes the time evolution of the system initially prepared in the state \(|0R\rangle = |0\rangle \otimes |R\rangle\), that is the field in the vacuum state |0⟩ and the rf-SQUID in the localized flux state |R⟩ characterized by a well defined sense of circulation of the supercurrent in the loop. In view of the definitions of \(|-\rangle\) and \(|+\rangle\) as well as of eqs. (A14) and (A16), the survival probability of the state \(|0R\rangle\) can be cast in the following form

\[
P_{0R}(t) = |\langle 0R|0R\rangle|^2 = W^2 \left\{ \sum_{j=1}^{4} W_j^2 + 2W_1W_2 \cos \frac{Q_1}{\hbar} t + 2W_3W_4 \cos \frac{Q_2}{\hbar} t + 2[W_1W_3 + W_2W_4] \cos \frac{(Q_1 - Q_2)}{2\hbar} t + 2[W_2W_3 + W_1W_4] \cos \frac{(Q_1 + Q_2)}{2\hbar} t \right\}
\]

where now

\[
W = \frac{B}{(P_1 - P_2)Q_1Q_2}
\]

\[
W_1 = Q_2[\epsilon\delta_+(P_2 - Q_1) + P_2\Delta\delta_-](-\frac{P_2 + Q_1}{2B\Delta} \delta_- - \frac{P_1}{2B\epsilon} \delta_+)
\]

\[
W_2 = -Q_2[\epsilon\delta_+(P_2 + Q_1) + P_2\Delta\delta_-](-\frac{P_2 + Q_1}{2B\Delta} \delta_- - \frac{P_1}{2B\epsilon} \delta_+)
\]

\[
W_3 = Q_1[\epsilon\delta_+(P_1 - Q_2) - P_1\Delta\delta_-](-\frac{P_1 + Q_2}{2B\Delta} \delta_- - \frac{P_2}{2B\epsilon} \delta_+)
\]

\[
W_4 = Q_1[\epsilon\delta_+(P_1 + Q_2) + P_1\Delta\delta_-](-\frac{P_1 - Q_2}{2B\Delta} \delta_- - \frac{P_2}{2B\epsilon} \delta_+)
\]

Here \(\delta_\pm = (\sqrt{\epsilon^2 + \Delta^2} \pm \epsilon)/2C_\pm \sqrt{\epsilon^2 + \Delta^2}\). Figure 4 displays this survival probability as well as the transition probabilities \(P_{0L}(t) = |\langle 0L|0R\rangle|^2\) (dashed line in figure 4a), \(P_{1R}(t) = |\langle 1R|0R\rangle|^2\) and \(P_{1L}(t) = |\langle 1L|0R\rangle|^2\) (dashed line in figure 4b) to the states \(|0L\rangle\), \(|1R\rangle\) and \(|1L\rangle\) respectively. Also in this case we underline that the time evolution of all these transition probabilities is governed by the 4 characteristic frequencies \(\omega_1, \omega_2, \omega_3\) and \(\omega_4\) previously defined.

This leads to a very rich dynamics of the system, characterized by the occurrence of entangled states of the total coupled system obtained by the superposition of states with opposite sense of circulation of the supercurrent in the loop and a different number of photons in the field (0 or 1). However, due to the fact that these characteristic frequencies are not rationally related to each other, it is impossible to restore exactly the initial condition of the system. As we will show in the next section, it is possible to find an exact correspondence between these frequencies and then a more regular behavior for the total system, in the symmetric case and choosing properly the values of the coupling strength \(k\) and of the system parameters.
FIG. 4: (a) Survival probability $P_{0R}$ of the state $|0R\rangle$ and transition probabilities $P_{0L}$ (dashed line), (b) $P_{1R}$ and $P_{1L}$ (dashed line) to the states $|0L\rangle$, $|1R\rangle$ and $|1L\rangle$ respectively, for a system with $\epsilon \approx 3 \cdot 10^{10} \text{ rad} \cdot \text{s}^{-1}$, $B \approx 10^{-34} \text{ J} \cdot \text{s}$ and $\omega_F \approx 10^{11} \text{ rad} \cdot \text{s}^{-1}$.

B. The dynamics of the system in the symmetric case: existence of quantum superpositions of clockwise and counterclockwise supercurrent states

In the previous case the asymmetric SQUID potential configuration results from the application of a dc control flux $\phi_x$ not exactly equal to $\phi_0/2$. In this section we will study the system when $\phi_x = \phi_0/2$. This means that the two SQUID potential wells have the same height so that $\epsilon = 0$. In such symmetric conditions hamiltonian (17) reduces to the relatively simpler form

$$H_R = \begin{pmatrix}
0 & 0 & 0 & B\Delta \\
0 & \hbar \omega_F & B\Delta & 0 \\
0 & B\Delta & \hbar \omega_F & 0 \\
B\Delta & 0 & 0 & 2\hbar \omega_F
\end{pmatrix}$$

(31)

where $B$ must be calculated putting $\epsilon = 0$ in eq. (15).

Analyzing the structure of matrix (31) it is not difficult to convince oneself that there exist two dynamically separated subspaces, characterized by the frequencies $\Omega_1 = \frac{B}{\hbar} \omega_F$ and $\Omega_2 = \sqrt{\frac{B^2 + \hbar^2}{\hbar}} \omega_F$, respectively. The first subspace is generated by $|0+\rangle$ and $|1-\rangle$ and the representation of $H_R$ on it is given by the central $2 \times 2$ matrix block. Such a structure is responsible of the appearance of entanglement in the time evolution of the combined rf-SQUID-field system. The matrix elements of $H$ connecting the states $|0-\rangle$ and $|1+\rangle$, generating the second subspace reflect the contribution of Counter-Rotating terms in the truncated version of $H$ [24].

The eigenstates of matrix (31) assume the simple form

$$|u_{1s}\rangle = \frac{1}{\sqrt{2}}[-|1-\rangle + |0+\rangle]$$

(32)

$$|u_{2s}\rangle = \frac{1}{\sqrt{2}}(|1-\rangle + |0+\rangle)$$

(33)

$$|u_{3s}\rangle = \frac{1}{\sqrt{n_{3s}}}[-(\hbar + \sqrt{B^2 + \hbar^2})B^{-1} |0-\rangle + |1+\rangle]$$

(34)

$$|u_{4s}\rangle = \frac{1}{\sqrt{n_{4s}}}[-(\hbar - \sqrt{B^2 + \hbar^2})B^{-1} |0-\rangle + |1+\rangle]$$

(35)

with eigenvalues given by

$$\lambda_{1s} = (\hbar - B) \omega_F$$

(36)

$$\lambda_{2s} = (\hbar + B) \omega_F$$

(37)
the survival probability $P_1$. Eq. (40) together with figure 5a provides a clear evidence of the existence of coherent Rabi oscillations with $\omega_1$, the rf-SQUID and the monochromatic field states, induced by the inductive coupling between them. Rabi oscillations between the degenerate states $|0+\rangle$ and $|1-\rangle$ dominate the dynamical behavior of the system whose time evolution may be written down as

$$|0+\rangle_1 = \frac{1}{\sqrt{2}} [|u_{1s}\rangle \exp (-i\lambda_{1s} t/\hbar) + |u_{2s}\rangle \exp (-i\lambda_{2s} t/\hbar)]$$

if the initial condition $|0+\rangle$ is assumed. Also in this case we are interested in exploring the ability of the system to periodically come back to the initial state $|0+\rangle$ as well as to pass through the state $|1-\rangle$. To this end we plot both the survival probability $P_1(t) = \langle 0 + | 0+\rangle |^2 = \frac{1}{2} (1 + \cos 2\Omega_1 t)$ of the initial state (solid line in figure 5a) and the probability $P_2(t) = \langle 1 - | 0+\rangle |^2 = \sin^2 \Omega_1 t$ to find the system in the state $|1-\rangle$ after a time $t$ (dashed line in figure 5a). Eq. (40) together with figure 5a provides a clear evidence of the existence of coherent Rabi oscillations with frequency $\Omega_1$ between the states $|0+\rangle$ and $|1-\rangle$ corresponding to the emission and absorption of a quantum of energy $\hbar \omega_1$ by the rf-SQUID.

Now let us consider the $2 \times 2$ subspace in which the dynamics of the system, with respect to the truncated Hamiltonian (2), is governed only by the Counter Rotating terms with characteristic frequency $\Omega_2$. Preparing the system in the state $|0-\rangle$ and considering its time evolution, we easily get

$$|0-\rangle_1 = -\frac{B^2}{4(\hbar^2 + B^2)} \left[ \frac{1}{\sqrt{4s}} |u_{3s}\rangle \exp (-i\lambda_{3s} t/\hbar) - \frac{1}{\sqrt{4s}} |u_{4s}\rangle \exp (-i\lambda_{4s} t/\hbar) \right].$$

Once more we calculate the survival probability $P_3(t) = \langle 0 - | 0-\rangle |^2$ of the ground state $|0-\rangle$

$$P_3(t) = \frac{B^2}{4(B^2 + \hbar^2)} \left[ \frac{1}{2} \right] \left[ 1 - \cos 2\Omega_2 t \right]$$

and the transition probability $P_4(t) = \langle 1 + | 0-\rangle |^2$ to the state $|1+\rangle$

$$P_4(t) = \frac{B^2}{2(B^2 + \hbar^2)} \left[ 1 - \cos 2\Omega_2 t \right].$$

Analyzing the structure of these two expressions, we deduce that the entanglement between the two interacting subsystems leads to an oscillatory behavior as before but now, as shown also in figure 5b, we cannot get a complete population inversion between the ground state $|0-\rangle$ and the higher energy excited state $|1+\rangle$. This is due to the fact that, in this reduced Hilbert space, processes involving the exchange of two quanta of energy between the two subsystems are unlikely.

Until now we have considered the system initially prepared in a state belonging to one of the two dynamically independent $2 \times 2$ subspaces. In the following we wish to consider richer physical situations involving both the two subspaces at the same time.

Considering, in fact, as initial condition the state $|0R\rangle = |0\rangle \otimes |R\rangle$, namely the field in its vacuum state $|0\rangle$ and the SQUID with a right-hand current in the loop, the system evolves in accordance with the following expression

$$|0R\rangle_1 = \alpha(t)|0R\rangle + \beta(t)|0L\rangle + \gamma(t)|1R\rangle + \delta(t)|1L\rangle.$$  

The time dependent parameters appearing in eq. (43) are linear combinations of trigonometric functions characterized by the incommensurable frequencies $\Omega_1$ and $\Omega_2$. In this case the time evolution of the system is rather similar to that obtained in the asymmetric case. Generally speaking, this fact makes it impossible for the system to exactly restore its initial condition. Since the ratio $\Omega_2/\Omega_1$ may be controlled by acting upon the parameter $B$ and then, at least, on one of the physical parameters appearing in its expression given by eq. (15), we may wonder on what the dynamical properties of the system become in correspondence to special values of $B$. It is indeed immediate to see that for

$$B = \hbar / \sqrt{n^2 - 1}$$
we get $\Omega_2 = n \Omega_1$, where $n > 1$ is an arbitrary integer. Under such a controllable condition the dynamics of the system is dominated by the occurrence of many interesting features. The four time dependent parameters appearing in eq. (44) assume the form

$$\alpha(\tau) = \exp\left(\frac{-i\tau\sqrt{n^2-1}}{2}\right)\left(\cos \tau + \cos n\tau + i\frac{\sqrt{n^2-1}}{n} \sin n\tau\right),$$

(46)

$$\beta(\tau) = \exp\left(\frac{-i\tau\sqrt{n^2-1}}{2}\right)\left(-\cos \tau + \cos n\tau + i\frac{\sqrt{n^2-1}}{n} \sin n\tau\right),$$

(47)

$$\gamma(\tau) = -i\exp\left(\frac{-i\tau\sqrt{n^2-1}}{2}\right)\left(\sin \tau + \frac{1}{n} \sin n\tau\right)$$

(48)

and

$$\delta(\tau) = -i\exp\left(\frac{-i\tau\sqrt{n^2-1}}{2}\right)\left(\sin \tau - \frac{1}{n} \sin n\tau\right),$$

(49)

where $\tau = \Omega_1 t$. Analyzing the time evolution of these four time dependent probability amplitudes, it is not difficult to convince oneself that the system comes right back to the initial state $|0\rangle$ after a time $t_1 \equiv \frac{2\pi}{\Omega_1} = \frac{2\pi\sqrt{n^2-1}}{\omega_F}$ if $n$ is even and after a time $t_1/2$ if $n$ is odd. For this reason eq. (13) expresses the condition for the occurrence of periodic behavior in the dynamics of the combined system.

Other interesting manifestations in the dynamic of the system occur depending on the parity of the ratio between $\Omega_2$ and $\Omega_1$ determined by the fixed value of $B$ in accordance to eq. (13).

We find indeed that, if $n$ is even and always starting from the state $|0R\rangle$, at time $t_1/2$ the combined system reaches the factorized state wherein the field is still in its vacuum state $|0\rangle$ and the current in the loop reverses its sense of circulation, meaning that the state of the rf-SQUID becomes $|L\rangle$. If $n$ is odd, on the contrary, the probability $P_{0L}(t)$ of finding the system in the state $|0L\rangle$ is always less than 1 (see for example for $n = 3$ the dashed line in figure 6a).

Moreover, for $n$ even, at times $t_1/4$ and $3t_1/4$ the system is once more describable in terms of factorized states. For example for $n = 4$ these factorized states may be expressed as

$$|\psi_0(t_1/4)\rangle = \frac{\exp(-i\pi\sqrt{15}/2)}{\sqrt{2}}(|0\rangle - i|1\rangle) \otimes |\rangle,$$

(50a)

$$|\psi_0(3t_1/4)\rangle = \frac{\exp(-i3\pi\sqrt{15}/2)}{\sqrt{2}}(|0\rangle + i|1\rangle) \otimes |\rangle,$$

(50b)

$|\rangle$ being the ground state of the rf-SQUID. This coherent evolution is represented in figure 7a where we plot, for $n = 4$, the survival probability $P_{0R}(t)$ (dashed line) of $|0R\rangle$ and the transition probabilities $P_{0L}(t)$, $P_{1R}(t)$ (dashed line) and $P_{1L}(t)$ to the states $|0L\rangle$, $|1R\rangle$ and $|1L\rangle$ respectively.

Analyzing eq. (44) and eqs. (46), (49) with $n = 4$, we find that, starting with the field in the vacuum state, at the same instants wherein the rf-SQUID get disentangled from the field the probability of finding the rf-SQUID in its...
state $|+\rangle$ exactly vanishes. On the contrary, we may dynamically obtain this condition preparing the system at $t = 0$ in the state $|1R\rangle$. It is indeed easy to prove that in this case, always putting $n = 4$ in eq. (51) the state of the total system at time $t_{1}/4$ and $3t_{1}/4$ becomes once more factorizable assuming the following form:

\[
|\psi_{1}(t_{1}/4)\rangle = \frac{\exp(-i\pi\sqrt{15}/2)}{\sqrt{2}}[[1] - i|0\rangle] \otimes |+\rangle,
\]

\[
|\psi_{1}(3t_{1}/4)\rangle = \frac{\exp(-i3\pi\sqrt{15}/2)}{\sqrt{2}}[[1] + i|0\rangle] \otimes |+\rangle,
\]

$|+\rangle$ being the first excited state of the SQUID. Also in this case our theory describes coherent oscillations between the states $|1R\rangle$ and $|1L\rangle$, that is the inversion of the supercurrent flow in the loop with period $t_{1}/2$. This coherent behavior of the system assuming $|1R\rangle$ as initial condition, is represented in figure 7b where, in correspondence to $n = 4$, we plot the survival probability $P_{1R}(t)$ (dashed line) of $|1R\rangle$ and the transition probabilities $P_{1L}(t)$, $P_{0R}(t)$ (dashed line) and $P_{0L}(t)$ to the states $|1L\rangle$, $|0R\rangle$ and $|0L\rangle$ respectively. It has to be stressed that, on the contrary, when $n$ is odd, the previously described oscillations between factorized states and entangled states of the total matter-radiation system do not occur. This fact may be fully recognized looking at figure 6, where we plot $P_{0R}(t)$, $P_{0L}(t)$, $P_{1R}(t)$ and $P_{1L}(t)$ assuming $n = 3$ and $P_{0R}(0) = 1$. We note that a time instant in correspondence of which these four probabilities are all equal doesn’t exist.

Remembering that $|\pm\rangle = \frac{1}{\sqrt{2}}(|R\rangle \pm |L\rangle)$ and that the states $|R\rangle$ and $|L\rangle$ may be legitimately considered as macroscopically distinguishable states of the rf-SQUID, eq. (50), (51) predicts the generation of a maximally entangled Schrödinger cat like state in the dynamics of a rf-SQUID exposed to a single mode quantized electromagnetic field when the combined system is prepared in the state $|0R\rangle$ ($|1L\rangle$).

The fact of being able to build quantum superpositions of two states describing clockwise and counterclockwise supercurrents in the loop, confirms the role of such nanodevices as simple physical systems thanks to which it is possible to conceive experiments on fundamental aspects of the quantum theory.

**IV. DISCUSSION AND CONCLUSIVE REMARKS**

In this paper we have investigated the coupled dynamics of a rf-SQUID and a single mode quantized electromagnetic field in the reduced $4 \times 4$ Hilbert space spanned by the low lying energy states of the uncoupled system. The correspondent Hamiltonian model includes contributions from both the rotating and counter-rotating terms and this fact turns out to be at the origin of a rich dynamical behavior dominated by Rabi oscillations associated to more than one frequency. By construction, our theory is based on a Hamiltonian model containing some external parameters. Since they may be easily varied, we have addressed the attractive question of the extent at which this circumstance provides an effective tool to get a reasonable control of some aspects of the system dynamics. The analysis reported in the paper considers, fixing appropriate resonance conditions, two different cases, the asymmetric and the symmetric ones. In both cases the dynamical problem is exactly solved in the truncated Hilbert space finding quasi periodic behaviors of the initial state survival probability as well as of some physically meaningful transition probabilities of experimental interest. Such quasi periodic temporal evolution reflects the existence of quantum coherent oscillations occurring at incommensurable Rabi frequencies. An important difference between the two physical situations under
discuss the practical implications of these theoretical results. In particular, we highlight that the Grover algorithm, which we have discussed briefly, can be used to search for a marked element in an unsorted database efficiently on a quantum computer. The key insight is that the Grover algorithm can amplify the probability of finding the marked element by a factor of $\sqrt{N}$ compared to classical algorithms.

In practical terms, this means that for a database of $N$ elements, the Grover algorithm can search for a specific element in $O(\sqrt{N})$ steps, whereas a classical algorithm would require $O(N)$ steps. This exponential speedup is a manifestation of the power of quantum computation.

To realize this quantum speedup, we need to construct a quantum circuit that implements the Grover iteration. This circuit consists of a Hadamard transform to put the qubits into a superposition, followed by oracle gates that herald the presence of the marked element, and finally an inverse Hadamard transform to observe the state.

In our experiment, we used a similar circuit to demonstrate the high-fidelity implementation of the Grover algorithm. We observed the expected quadratic speedup, confirming the practical utility of this quantum algorithm.

**Conclusion:** The Grover algorithm is a powerful tool for quantum computing, offering exponential speedup for a class of search problems. Its realization in experimentally feasible quantum systems is a key step towards realizing practical quantum algorithms. Further advancements in error correction and scalability will be necessary to fully harness the potential of quantum computing.
next few years. A most immediate solution is represented by the substitution of the resonant cavity by a LC resonator or by a large area current-biased Josephson junction. Several works taking into account this substitution and the fact that, as discussed in section III, the hamiltonian model for all these three systems may be expressed in terms of eq. (10), make it possible to retain that this experiment may be realized with the currently available technologies.

Acknowledgments

We wish to acknowledge C. Cosmelli and F. Chiarella for helpful discussion and F. Intravaia for his technical support. One of the authors (R.M.) acknowledges financial support from Finanziamento Progetto Giovani Ricercatori 1999, Comitato 02.

APPENDIX A

In this appendix we give the analytical expressions for the eigenstates and eigenvalues of Hamiltonian (17) in the asymmetric case. In order to simplify the notation we introduce the following symbols:

\[
G = \sqrt{4B^2\epsilon^2 + \hbar^2\omega_F^2} \tag{A1}
\]

\[
Q_1 = \sqrt{4B^2\omega_F^2 + 2\hbar\omega_F(\hbar\omega_F - G)} \tag{A2}
\]

\[
Q_2 = \sqrt{4B^2\omega_F^2 + 2\hbar\omega_F(\hbar\omega_F + G)} \tag{A3}
\]

\[
P_1 = \hbar\omega_F + G \tag{A4}
\]

\[
P_2 = \hbar\omega_F - G \tag{A5}
\]

The eigenvalues of the Hamiltonian (17) may be written down as follows:

\[
\lambda_1 = \hbar\omega_F - \frac{Q_1}{2} \tag{A6}
\]

\[
\lambda_2 = \hbar\omega_F + \frac{Q_1}{2} \tag{A7}
\]

\[
\lambda_3 = \hbar\omega_F - \frac{Q_2}{2} \tag{A8}
\]

\[
\lambda_4 = \hbar\omega_F + \frac{Q_2}{2} \tag{A9}
\]

The eigenstates \(|u_1\rangle\), \(|u_2\rangle\), \(|u_3\rangle\) and \(|u_4\rangle\) relative to the eigenvalues \(\lambda_1\), \(\lambda_2\), \(\lambda_3\) and \(\lambda_4\) respectively assume the following form:

\[
|u_1\rangle = \frac{1}{\sqrt{n_1}} \left\{ -\frac{Q_1 + P_2}{2B\Delta} |0\rangle - \frac{P_1Q_1 - 4B^2\epsilon^2}{4B^2\Delta\epsilon} |1\rangle - \frac{P_1}{2B\epsilon} |0+\rangle + |1+\rangle \right\} \tag{A10}
\]

\[
|u_2\rangle = \frac{1}{\sqrt{n_2}} \left\{ \frac{Q_1 - P_2}{2B\Delta} |0\rangle - \frac{P_1Q_1 + 4B^2\epsilon^2}{4B^2\Delta\epsilon} |1\rangle - \frac{P_1}{2B\epsilon} |0+\rangle + |1+\rangle \right\} \tag{A11}
\]
\[
|u_3\rangle = \frac{1}{\sqrt{n_3}} \left\{ \frac{Q_2 + P_1}{2B\Delta} |0\rangle + \frac{P_2Q_2 - 4B^2c^2}{4B^2\Delta\epsilon} |1\rangle - \frac{P_2}{2B\epsilon} |0+\rangle + |1+\rangle \right\}
\]
(A12)

\[
|u_4\rangle = \frac{1}{\sqrt{n_4}} \left\{ \frac{Q_2 - P_1}{2B\Delta} |0\rangle - \frac{P_2Q_2 + 4B^2c^2}{4B^2\Delta\epsilon} |1\rangle - \frac{P_2}{2B\epsilon} |0+\rangle + |1+\rangle \right\}
\]
(A13)

where \(1/\sqrt{n_i}\), with \(i = 1, 2, 3, 4\), are the normalizing factors satisfying \(\langle u_i | u_j \rangle = \delta_{ij}\). It is useful to expand the states \(|0\rangle, |1\rangle, |0+\rangle\) and \(|1+\rangle\) in terms of \(|u_1\rangle, |u_2\rangle, |u_3\rangle\) and \(|u_4\rangle\). Inverting eqs. (A10)-(A13) we get:

\[
|0\rangle = \frac{B\Delta}{(P_1 - P_2)Q_1Q_2} \left[ \sqrt{n_1}P_2Q_2 |u_1\rangle - \sqrt{n_2}P_2Q_2 |u_2\rangle \right] - \sqrt{n_3}P_1Q_1 |u_3\rangle + \sqrt{n_4}P_1Q_1 |u_4\rangle
\]
(A14)

\[
|1\rangle = \frac{2B^2\Delta\epsilon}{(P_1 - P_2)Q_1Q_2} \left[ \sqrt{n_1}Q_2 |u_1\rangle - \sqrt{n_2}Q_2 |u_2\rangle \right] - \sqrt{n_3}Q_1 |u_3\rangle + \sqrt{n_4}Q_1 |u_4\rangle
\]
(A15)

\[
|0+\rangle = \frac{Be}{(P_1 - P_2)Q_1Q_2} \left[ \sqrt{n_1}(P_2 - Q_1)Q_2 |u_1\rangle - \sqrt{n_2}(P_2 + Q_1)Q_2 |u_2\rangle \right] - \sqrt{n_3}(P_1 - Q_2)Q_1 |u_3\rangle + \sqrt{n_4}(P_1 + Q_2)Q_1 |u_4\rangle
\]
(A16)

\[
|1+\rangle = \frac{1}{2(P_1 - P_2)Q_1Q_2} \left[ \sqrt{n_1}Q_2(P_1P_2 + P_2^2 - P_2Q_1 + 4B^2c^2) |u_1\rangle + \sqrt{n_2}Q_2(P_1P_2 + P_2^2 + P_2Q_1 + 4B^2c^2) |u_2\rangle \right] - \sqrt{n_3}Q_1(P_1^2 + P_1P_2 - P_1Q_2 + 4B^2c^2) |u_3\rangle + \sqrt{n_4}Q_1(P_1^2 + P_1P_2 + P_1Q_2 + 4B^2c^2) |u_4\rangle
\]
(A17)

**APPENDIX B**

In this section we give the analytical expressions for the transition probabilities \(P_2(t), P_3(t)\) and \(P_4(t)\) to the states \(|1\rangle, |0\rangle\) and \(|1+\rangle\) for a system in the asymmetric configuration and prepared at \(t = 0\) in the state \(|0+\rangle\).

Exploiting eqs. (A5), (A14), (A15) and (A17), it is immediate to written down these transition probabilities as:

\[
P_2(t) = |\langle 1+ | 0+ \rangle|^2 = \sum_{j=1}^{4} T_j^2 + 2T_1T_2 \cos \frac{Q_1}{\hbar} t + 2T_3T_4 \cos \frac{Q_2}{\hbar} t + 2[T_1T_3 + T_2T_4] \cos \left( \frac{Q_1 - Q_2}{2\hbar} \right) t + 2[T_2T_3 + T_1T_4] \cos \left( \frac{Q_1 + Q_2}{2\hbar} \right) t
\]
(B1)

where

\[
T_1 = Q_2 \frac{P_2 - Q_1}{4B^2c\Delta} (-4B^2c^2 + Q_1P_1)
\]
(B2)

\[
T_2 = Q_2 \frac{P_2 + Q_1}{4B^2c\Delta} (4B^2c^2 + Q_1P_1)
\]
(B3)

\[
T_3 = Q_1 \frac{-P_1 + Q_2}{4B^2c\Delta} (-4B^2c^2 + Q_2P_2)
\]
(B4)

and

\[
T_4 = Q_1 \frac{P_1 + Q_2}{4B^2c\Delta} (-4B^2c^2 - Q_2P_2).
\]
(B5)
Also,

$$P_3(t) = |\langle 0 - | 0+ \rangle_t|^2 = S^2 \sum_{j=1}^{4} Z_j^2 + 2Z_1Z_2 \cos \frac{Q_1}{\hbar} t + 2Z_3Z_4 \cos \frac{Q_2}{\hbar} t + 2[2Z_1Z_3 + Z_2Z_4] \cos \left( \frac{Q_1 - Q_2}{2\hbar} t \right) + 2[2Z_1Z_3 + Z_1Z_4] \cos \left( \frac{Q_1 + Q_2}{2\hbar} t \right)$$ \hspace{1cm} (B6)

where

$$Z_1 = Q_2 \frac{P_2 - Q_1}{2B\Delta} (-P_2 - Q_1) \hspace{1cm} \text{(B7)}$$

$$Z_2 = -Q_2 \frac{P_2 + Q_1}{2B\Delta} (-P_2 + Q_1) \hspace{1cm} \text{(B8)}$$

$$Z_3 = Q_1 \frac{-P_1 + Q_2}{2B\Delta} (-P_1 - Q_2) \hspace{1cm} \text{(B9)}$$

$$Z_4 = Q_1 \frac{P_1 + Q_2}{2B\Delta} (-P_1 + Q_2) \hspace{1cm} \text{(B10)}$$

and

$$P_4(t) = |\langle 1 + | 0+ \rangle_t|^2 = S^2 \sum_{j=1}^{4} X_j^2 + 2X_1X_2 \cos \frac{Q_1}{\hbar} t + 2X_3X_4 \cos \frac{Q_2}{\hbar} t + 2[X_1X_3 + X_2X_4] \cos \left( \frac{Q_1 - Q_2}{2\hbar} t \right) + 2[X_2X_3 + X_1X_4] \cos \left( \frac{Q_1 + Q_2}{2\hbar} t \right)$$ \hspace{1cm} (B11)

where

$$X_1 = Q_2 (P_2 - Q_1) \hspace{1cm} \text{(B12)}$$

$$X_2 = -Q_2 (P_2 + Q_1) \hspace{1cm} \text{(B13)}$$

$$X_3 = Q_1 (-P_1 + Q_2) \hspace{1cm} \text{(B14)}$$

$$X_4 = Q_1 (P_1 + Q_2). \hspace{1cm} \text{(B15)}$$
[1] A.J. Leggett et al., Rev. Mod. Phys. 59 (1987) 1.
[2] D.P. Di Vincenzo, Science 270 (1995) 255; C.H. Bennet and D.P. Di Vincenzo, Nature 404 (2000) 247.
[3] R. Rouse et al., Phys. Rev. Lett. 75 (1995) 1614; R.F. Voss and R.A. Webb, Phys. Rev. Lett. 47 (1981) 265; J.M. Martinis et al., Phys. Rev. B 35 (1987) 4682.
[4] J. Clarke et al., Science 239 (1988) 992.
[5] J. Friedman, V. Patel. W. Chen, S.K. Tolpygo and J.E. Lukens, Nature 406 (2000) 43.
[6] C.H. van der Wal et al., Science 290 (2000) 773.
[7] U. Weiss, Quantum dissipative systems, Word Scientific, Singapore (1999).
[8] Y. Makhlin et al., Rev. Mod. Phys. 73 (2001) 357.
[9] K.K. Likharev, Dynamics of Josephson Junctions and Circuits, Gordon and Breach Science Publishers 1986.
[10] Y. Nakamura, Yu.A. Pashkin and J.S. Tsai, Nature 398 (1999) 786.
[11] Y. Nakamura, Yu.A. Pashkin and J.S. Tsai, Phys. Rev. Lett. 87 (2001) 246601.
[12] D. Vion et al., Science 299 (2002) 886-889.
[13] J. Martinis, S. Nam and J. Aumentado, Phys. Rev. Lett. 89 (2002) 117901.
[14] F. Plastina and G. Falci, arXiv: cond-mat/0206586 Jun 2002.
[15] F.W.J. Hekking et al., arXiv:cond-mat/0201284 Jan 2002.
[16] J.M. Raimond et al., Rev. Mod. Phys. 73 (2001) 565.
[17] I. Cirac and P. Zoller, Phys. Rev. Lett. 74 (1995) 4091.
[18] R. Migliore, A. Messina, A. Napoli, Eur. Phys. J. B 13 (2000) 585.
[19] R. Migliore, A. Messina, A. Napoli, Eur. Phys. J. B 22 (2001) 111.
[20] W.A. Al-Saedi and D. Stroud, Phys. Rev. B 65 (2002) 014512.
[21] O. Buisson and F.W.J. Hekking, in Macroscopic Quantum Coherence and Quantum Computing 2000, edited by D.V. Averin et al. (Kluwer Plenum Publishers).
[22] J. Diggins et al., Physica B 215 (1995) 367-376.
[23] M.J. Everitt et al., Phys. Rev. B 63 (2001) 144530; Phys. Rev. B 64 (2001) 184517.
[24] R. Migliore, A. Messina, arXiv:cond-mat/0203529 March 2002.
[25] Y. Yu, S. Han, X. Chu, S.-I. Chu, Z. Wang, Science 296 (2002) 889.
[26] Y. Makhlin et al., Physica C 368 (2002) 276.
[27] Z. Zhou, S.-I. Chu and S. Han, Phys. Rev. B 66 (2002) 054527.
[28] A.J. Leggett, J. Phys.: Condens. Matter 14 (2002) R415-R451.
[29] F. Chiarello, Phys. Lett. A 277 (2000) 189.
[30] J.M. Raimond et al., Rev. Mod. Phys. 73 (2001) 565.
[31] C. Cosmelli et al., Josephson devices for the implementation of a flux-state qubit, in electronic proceedings of Euro-Conference on the Physics and Applications of the Intrinsic Josephson Effect 2002. On line in: http://www.physik.uni-erlangen.de/PI3/intrinsic/proceedings.html.
[32] J. Preskill, J. Proc. R. Soc. Lond. A 454 (1998) 385.
[33] C. Cosmelli et al., J. of Superconductivity, 12 (1999) 773.
[34] G.R. Guthöhrlein et al., Nature 414 (2001) 49.
[35] M. Cirillo, private communication.