On additive doubling and energy

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Abstract

We discuss some ideas related to the polynomial Freiman-Ruzsa conjecture. We show that there is a universal $\epsilon > 0$ so that any subset of an abelian group with subtractive doubling $K$ must be polynomially related to a set with additive energy at least $\frac{1}{K}$. This means that the main difficulty in proving the polynomial Freiman-Ruzsa conjecture consists in studying sets whose energy is greater than that implied by their doubling. One example is a generalized arithmetic progression of high dimension which cannot occur in finite characteristic.

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1 Introduction

We are interested in studying the structure of a subset $A$ of some additive group, which satisfies a doubling condition

$|A + A| \leq K|A|$.

One type of result along these lines is the class of so-called “Freiman theorems”.

Theorem 1.1. There is a function $f$ from the positive reals to the positive reals so that if $A \subset F_p^n$, where $p$ is a prime and $F_p$ the field of $p$ elements and

$|A + A| \leq K|A|$, 

then there exists a subspace \( H \) of \( F_p^n \) so that \( A \) is contained in some translate of \( H \) and
\[
|H| \leq f(K)|A|.
\]

Various authors have proved results along these lines, \([F73], [R99], [GT07a], [GT07b]\). There is no better possible \( f \) than \( p^K \) and this was essentially obtained by Green and Tao \([GT07b]\) in the setting where \( p = 2 \).

One may hope for a better result, namely the polynomial Freiman-Ruzsa conjecture:

**Conjecture 1.2.** There is a universal constant \( C > 0 \), so that if \( A \subset F_p^n \) with
\[
|A + A| \leq K|A|,
\]
then there exists \( z \in F_p^n \) and \( H \) a subspace so that
\[
|H| \leq K^C|A|,
\]
and
\[
|A \cap (z + H)| \geq K^{-C}|A|.
\]

The best results towards this conjecture in the case \( p = 2 \) were obtained by Green and Tao \([GT07a]\) using an energy-incrementation method. One obtained a subset of \( A \) well situated with respect to some subspace \( H \) by taking subsets of \( A \) with gradually larger additive energy. The result they obtained was a subset \( H \) with
\[
|A \cap (z + H)| \geq e^{-C\sqrt{K}}|A|,
\]
This was obtained by showing that given a set in \( F_2^n \) with additive energy \( E(A) > \frac{1}{K} \), one can find a hyperplane \( P \) so that the additive energy \( E(A \cap P) \) was at least \( \frac{1}{K} + \frac{1}{K^2} \).

This argument of Green and Tao at first seemed to us ripe for improvement since it is rather inefficient for the best examples that we knew. In characteristic 2, the best example we know for a set with additive energy \( \frac{1}{K} \) is a set \( A \) with additive doubling \( K \) obtained from a subspace \( H \) and a random set \( R \) of cardinality \( K \) by
\[
A = R + H.
\]
In this case, if we take a hyperplane \( P \) which contains \( H \) we would expect to catch only half of the random set and double the energy straight away. The difficulty which is at the heart of the polynomial Freiman Ruzsa conjecture is to decompose a set \( A \) with small doubling into its structured and random parts.

In the case that \( A = R + H \), we observed that one could identify \( H \) by fixing an element \( t \) of \( A + A \), defining \( A[t] \) as that subset of \( A \) which lies in \( t + A \) and then calculating \( A[t] + A[t] \).
We wanted to see how much of this structure remains for a general set with small doubling constant.

In the end, we failed to prove the polynomial Freiman Ruzsa conjecture, but we showed the following. We supposed that $|A + A| \leq K|A|$ and we discovered that there is a large subset of $A + A$ which has additive energy at least $K^{-\frac{36}{37}}$. This could not be used in an induction because additive energy is weaker than small sum set. However, we find it an interesting and fundamental result in its own right and it is the topic of this paper.

We remark that the failure of the induction to close is a fundamental failure of our method of proof, which works equally well independent of the characteristic. In characteristic zero, there is a basic example of a set with large additive energy whose energy is significantly greater than the reciprocal of its doubling constant, namely an arithmetic progression with large dimension. This example needs to appear in any statement of polynomial Freiman Ruzsa for characteristic zero. However, we don’t believe that anything like it can appear in the finite characteristic setting. If we could show that the extreme examples for polynomial Freiman Ruzsa in finite characteristic were not of this nature, but rather had subtractive doubling comparable to the reciprocal of the additive energy then our result would yield the conjecture.

## 2 Preliminary Lemmas

In the above discussion we restricted attention to the $p = 2$ case but our method applies equally well when $A$ is contained in an arbitrary finite abelian group, provided difference sets are used in place of sumsets. We give our argument in this general setting.

We first give two useful formulations of the Cauchy-Schwarz inequality.

**Lemma 2.1.** Let $A$ and $B$ be sets and let $f : A \to B$. Then

$$|\{(a, a') : f(a) = f(a')\}| \geq \frac{|A|^2}{|B|}.$$  

**Proof.** We calculate that

$$|\{(a, a') : f(a) = f(a')\}| = \sum_{b \in B} |f^{-1}(b)|^2,$$

and applying the Cauchy-Schwarz inequality, we find

$$\sum_{b \in B} |f^{-1}(b)|^2 \geq \frac{1}{|B|} (\sum_{b \in B} |f^{-1}(b)|)^2 = \frac{|A|^2}{|B|^2}.$$

\[\Box\]
Lemma 2.2. Let $B_1$ and $B_2$ be sets. For each $\alpha \in B_1$, let $B_\alpha \subset B_2$ with $|B_\alpha| \geq \rho |B_2|$.

Then
\[
\sum_{\alpha \in B_1} \sum_{\beta \in B_1} |B_\alpha \cap B_\beta| \geq \rho^2 |B_1|^2 |B_2|.
\]

Proof. By assumption
\[
\sum_{x \in B_2} \sum_{\alpha \in B_1} 1_{B_\alpha}(x) \geq \rho |B_1||B_2|.
\]

Applying Cauchy-Schwarz, we find
\[
\sum_{x \in B_2} \left( \sum_{\alpha \in B_1} 1_{B_\alpha}(x) \right)^2 \geq \rho^2 |B_1|^2 |B_2|.
\]

But this is precisely what we were to prove. \qed

We will make frequent use of the dyadic pigeonhole principle throughout. We give two formulations of this principle here.

Lemma 2.3. Let $f$ be a positive real valued function defined on a finite set $S$ obeying
\[
\theta \|f\|_\infty \leq f(s) \leq \|f\|_\infty
\]

for all $s \in S$. There exists $0 \leq j \leq \log_2 \theta^{-1}$ so that
\[
|\{s \in S : 2^{-j-1}\|f\|_\infty < f(s) \leq 2^{-j}\|f\|_\infty\}| \geq \frac{1}{1 - \log_2 \theta} |S|
\]

Proof. Let $k \in \mathbb{N}$ be the largest positive integer so that $\theta \leq 2^{-k+1}$. For each $0 \leq j \leq k-1$, define
\[
S_j = \{s \in S : 2^{-j-1}\|f\|_\infty < f(s) \leq 2^{-j}\|f\|_\infty\}.
\]

Since
\[
S = \bigcup_{j=0}^{k-1} S_j,
\]

there must exist some $0 \leq j \leq k - 1$ so that $|S_j| \geq \frac{1}{k} |S|$. The result follows since $k \leq 1 + \log_2 \frac{1}{\theta}$. \qed

Lemma 2.4. Let $f$ be a nonnegative valued function on a finite set $S$, not identically zero, and define $\theta \in (0, 1]$ by
\[
\frac{1}{|S|} \sum_{s \in S} f(s) = \theta \|f\|_\infty.
\]

Then there exists a nonnegative integer $0 \leq k \leq \log_2 \frac{2}{\theta}$ so that
\[
|\{s \in S : 2^{-k-1}\|f\|_\infty < f(s) \leq 2^{-k}\|f\|_\infty\}| \geq \frac{2^{k-1}\theta}{\log_2 \left(\frac{2}{\theta}\right)} |S|.
\]
Proof. Let \( S' = \{ s \in S : f(s) \geq \frac{\theta}{2} \| f \|_\infty \} \). Since

\[
\sum_{s \in S \setminus S'} f(s) < \frac{\theta}{2} \| f \|_\infty |S|,
\]
then

\[
\frac{\theta}{2} \| f \|_\infty |S| \leq \sum_{s \in S'} f(s) \leq \| f \|_\infty |S'|,
\]
and therefore \( |S'| \geq \frac{\theta}{2} |S| \).

Then

\[
\frac{\theta}{2} \| f \|_\infty |S| \leq \sum_{j=0}^{1+\log_2 \theta - 1} \sum_{s \in S_j} f(s)
\]
and so there exists \( 0 \leq k \leq \log_2 \frac{2}{\theta} \) so that

\[
\frac{\theta}{2 \log_2 \frac{2}{\theta}} |S| \| f \|_\infty \leq \sum_{s \in S_k} f(s) \leq 2^{-k} |S_k| \| f \|_\infty
\]

Let \( Z \) be an abelian group and \( A \subset Z \) a finite subset. The energy of \( A \) is defined by

\[
E(A) = |A|^{-3} \sum_{z \in Z} |(z + A) \cap A|^2 = |A|^{-3} \{(a_1, a_2, a_3, a_4) \in A^4 : a_1 - a_2 = a_3 - a_4\}.
\]

Our main lemma shows that if one set is essentially invariant under another set, then at least one of the sets has large energy.

Lemma 2.5. Let \( 0 < \rho \leq 1 \) and suppose \( B_1 \) and \( B_2 \) are two subsets of \( Z \), and suppose

\[
B_1 \subset \{ z \in Z : |(z + B_2) \cap B_2| \geq \rho |B_2| \}.
\]

Then

\[
E(B_1) \geq \frac{\rho^4}{16 \log_2 (\frac{4}{\rho^2})^2} \frac{|B_1|}{E(B_1)|B_2|}
\]

Proof. Applying Lemma 2.2 we obtain

\[
\sum_{b \in B_1} \sum_{b' \in B_1} |(b + B_2) \cap (b' + B_2) \cap B_2| \geq \rho^2 |B_1|^2 |B_2|
\]

By Lemma 2.4 we find \( 0 \leq k \leq \log_2 \frac{2}{\rho} \) and \( \Omega \subset B_1^k \) so that

\[
|\Omega| \geq \frac{2^{k-1} \rho^2}{\log_2 (\frac{4}{\rho^2})} |B_1|^2
\]
and
\[ 2^{-k-1}|B_2| \leq |(b + B_2) \cap (b' + B_2) \cap B_2| \leq 2^{-k}|B_2| \]
for all \((b, b') \in \Omega\).

Define \(S = \{z \in \mathbb{Z} : |(z + B_2) \cap B_2| \geq 2^{-k-1}|B_2|\}\). Then
\[ |B_2|^3 E(B_2) = \sum_{z \in \mathbb{Z}} |(z + B_2) \cap B_2|^2 \]
\[ \geq \sum_{z \in S} |(z + B_2) \cap B_2|^2 \geq |S| 2^{-2k-2}|B_2|^2, \]
giving the bound
\[ |S| \leq 2^{2k+2}|B_2|E(B_2). \]

On the other hand,
\[ |B_1|^3 E(B_1) = \left| \{(a_1, a_2, a_3, a_4) \in B_1^4 : a_1 - a_2 = a_3 - a_4\} \right| \]
\[ \geq \left| \{(g, g') \in \Omega^2 : -(g) = -(g')\} \right| \geq \frac{|\Omega|^2}{|-(\Omega)|} \]
\[ \geq \frac{|\Omega|^2}{|S|} \geq \frac{2^{2k-2} \rho^4}{(\log_2(\frac{1}{\rho}))^2 2^{2k+2}|B_2|E(B_2)}, \]
which gives the desired lower bound on \(E(B_1)\).

\[ \square \]

3 Main result

Given a set \(\Omega \subset \mathbb{Z}^2\), we define \(- : \Omega \to \mathbb{Z}\) by \(-(a, b) = a - b\). Given \(A \subset \mathbb{Z}\) and \(t \in A - A\), we write
\[ A[t] = (A + t) \cap A. \]
Then \(|A[t]| = |\{(a, b) \in A^2 : a - b = t\}|\).

We let \(K\) be fixed throughout. For two quantities \(A\) and \(B\), we write \(A \precsim B\) if there exists a constant \(C\), independent of \(K\), so that \(A \leq CB\). We write \(A \sim B\) if \(A \precsim B\) and \(B \precsim A\). We write \(A \precsim B\) if for each \(\delta > 0\) there exists a constant \(C_\delta > 0\), independent of \(K\), so that
\[ A \leq C_\delta K^\delta B. \]

We write \(A \approx B\) if \(A \precsim B\) and \(B \precsim A\).

**Theorem 3.1.** There exists universal constants \(\epsilon > 0\) and \(C > 0\) so that if \(A\) is a finite subset of an abelian group \(\mathbb{Z}\) satisfying \(|A - A| = K|A|\), then there exists \(A' \subset A - A\) satisfying
\[ |A'| \geq K^{-C}|A| \]

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and
\[ E(A') \gtrsim \frac{1}{K^{1-\epsilon}}. \]

Furthermore, we can take \( \epsilon = \frac{1}{37} \).

The strategy of the proof is to investigate several reasonable candidates for \( A' \). We will always assume these candidate sets do not satisfy the energy bound required in the theorem, since we would be done otherwise. But if a candidate set fails the desired energy bound we then use Lemma 2.5 to put a lower bound on the energy of a related set. Continuing in this manner, we eventually find a set \( A' \) as desired in the theorem.

We assume \( E(A - A) < \frac{1}{K^{1-\epsilon}} \) throughout, since otherwise we could take \( A' = A - A \). Likewise, we assume \( E(A) \lesssim \frac{1}{K^{1-\epsilon}} \).

For each \( t \in A - A \), we have
\[ |A| \leq |A[t] - A| \leq K|A|; \]
we therefore obtain an exponent \( 0 \leq \beta(t) \leq 1 \) so that \( |A[t] - A| = K^\beta(t)|A| \).

**Lemma 3.2.** Let \( \epsilon > 0 \) and suppose \( A \) is a finite subset of an abelian group \( \mathbb{Z} \) satisfying \( |A - A| = K|A| \) and \( E(A) \lesssim \frac{1}{K^{1-\epsilon}} \). There exists a set \( T \subset A - A \) and an exponent \( 0 \leq \beta \leq 1 \) so that
\[ |A[t]| \geq \frac{|A|}{2K} \] \hspace{1cm} (3.1)
and
\[ |A[t] - A| \sim K^\beta|A| \]
for all \( t \in T \), and in addition,
\[ |T| \gtrsim K^{-\epsilon}|A - A| \]

**Proof.** Letting \( H = \{(a, a') \in A^2 : |A[a - b]| \geq \frac{|A|}{2K}\} \), we first observe
\[ \sum_{(a,a') \in A^2 \setminus H} 1 = \sum_{t \in A - A} |A[t]| \leq \frac{|A|}{2K}|A - A| = \frac{1}{2}|A|^2, \]
and therefore \( H \geq \frac{1}{2}|A|^2 \).

On the other hand,
\[ \frac{|H|^2}{|-(H)|} \leq \sum_{t \in -(H)} |A[t]|^2 \lesssim \frac{|A|^3}{K^{1-\epsilon}}, \]
and therefore
\[ |-(H)| \gtrsim \frac{K^{1-\epsilon}|H|^2}{|A|^3} \geq K^{1-\epsilon}|A| = K^{-\epsilon}|A - A|. \]
We have (3.1) for all \( t \in -(H) \).

For each \( t \in -(H) \), \( |A[t] - A| \sim K^{\beta(t)}|A| \); by Lemma 2.3, there exists \( T \subset -(H) \) and \( 0 \leq \beta \leq 1 \) so that \( |A[t] - A| \sim K^{\beta}|A| \) for all \( t \in T \), and \(|T| \gtrsim |-(H)|\). The theorem follows since \(|-(H)| \gtrsim K^{-\epsilon}|A - A|\).

\[ \square \]

### 3.1 The Large \( \beta \) case: \( (\beta \geq \frac{1}{2} + \frac{7}{4} \epsilon) \)

**Lemma 3.3.** There exists \( 0 \leq \alpha \leq \beta \) and a set \( X \subset A - A \) so that \( |X| \gtrsim K^{-\alpha}|A - A| \) and
\[
|\{ t \in T : x \in A[t] - A \}| \gtrsim K^{\alpha+\beta-\epsilon}|A|
\]
for each \( x \in X \).

**Proof.**
\[
\sum_{t \in T} |A[t] - A| \gtrsim K^{\beta-\epsilon}|A||A - A|,
\]
but
\[
\sum_{t \in T} |A[t] - A| = \sum_{x \in A - A} |\{ t \in T : x \in A[t] - A \}|.
\]
The conclusion follows from Lemma 2.4. \( \square \)

Suppose \( x \in A[t] - A \) for some \( t \in T \subset A - A \). Then
\[
x \in t + A - A
\]
and so
\[
t \in x + A - A.
\]
Thus, \( t \in (x + A - A) \cap (A - A) \). In particular, if \( x \in X \), then
\[
|(x + A - A) \cap (A - A)| \gtrsim K^{\alpha+\beta-\epsilon}|A| = K^{\alpha+\beta-\epsilon-1}|A - A|,
\]
since \( x \in A[t] - A \) for at least \( K^{\alpha+\beta-\epsilon}|A| \) values of \( t \). Applying Lemma 2.5 with \( B_1 = X \), \( B_2 = A - A \), and \( \rho = K^{\alpha+\beta-\epsilon-1} \), we obtain
\[
E(X) \gtrsim K^{4\alpha+4\beta-4\epsilon-4}K^{1-2\epsilon}K^{\alpha} \geq K^{4\beta-6\epsilon-3}.
\]
If \( \beta \geq \frac{1}{2} + \frac{7}{4} \epsilon \) then
\[
E(X) \gtrsim \frac{1}{K^{1-\epsilon}},
\]
giving the conclusion of the Theorem with \( A' = X \). This proves the theorem provided \( \beta \geq \frac{1}{2} + \frac{7}{4} \epsilon \).
3.2 The Small $\beta$ case. ($\beta < \frac{7}{12} - \frac{4}{3} \epsilon$)

Lemma 3.4. For each $t \in T$, there exists a set $G_1(t) \subset A[t]^2$ and $0 \leq \alpha(t) \leq \beta$ so that
\[ |G_1(t)| \gtrsim K^{-\alpha(t)} A[t]^2 \]
and
\[ |(a - a' + A) \cap A| \gtrsim K^{\alpha(t) - \beta} A | \]
for all $(a, a') \in G_1(t)$.

Proof. We begin by observing
\[ \sum_{(a', a) \in A[t]^2} |(a' - A) \cap (a - A)| \geq K^{2\beta} |A[t]|^2 |A[t] - A| = K^{\beta} |A[t]|^2 |A|. \]
By Lemma 2.3, we obtain $0 \leq \alpha(t) \leq \beta$ and a set $G_1(t) \subset A[t]^2$ so that
\[ |G_1(t)| \gtrsim K^{-\alpha(t)} A[t]^2 \]
and
\[ |(a' - A) \cap (a - A)| \gtrsim K^{\alpha(t) + \beta} A | \]
for all $(a, a') \in G_1(t)$. But
\[ |(a' - A) \cap (a - A)| = |(A - a') \cap (A - a)| = |(a - a' + A) \cap A|. \]

By Lemma 2.3, we may find $0 \leq \alpha \leq \beta$ and $T' \subset T$ so that $|T'| \gtrsim |T|$ and so that $|\alpha(t) - \alpha| \leq \log_K 2$ for all $t \in T'$.

Lemma 3.5. For each $t \in T'$, there exists $\gamma(t) \geq 0$ and $G(t) \subset G_1(t)$ so that
\[ |G(t)| \gtrsim |G_1(t)|, \]
\[ |-(G(t))| \gtrsim K^{\gamma(t) - \alpha} |A[t]| \]
\[ |(A[t])[x]| \sim K^{-\gamma(t)} |A[t]| \]
for all $x \in -(G(t))$, and
\[ E(A[t]) \gtrsim K^{-\alpha + \gamma(t)} \]

Proof. Let $G'(t) = \{(a, b) \in G_1(t) : |(A[t])[a - b]| \geq \frac{|G_1(t)|}{2} \}$. Then
\[ \sum_{(a,b) \in G_1(t) \setminus G'(t)} 1 = \sum_{x \in -(G_1(t) \setminus G'(t))} |(A[t])[x]| \leq \frac{|G_1(t)|}{2}, \]
and so $|G'(t)| \geq \frac{|G_1(t)|}{2}$.

Next, we observe

\[
\frac{|G_1(t)|}{2 - |(G_1(t))|} \geq \frac{|G_1(t)|}{2|A[t] - A|} \gtrsim \frac{K^\alpha|A[t]|^2}{K^\beta|A|} = K^{-1-\alpha-\beta|A[t]|}.
\]

Coupled with $|(A[t])[x]| \leq |A[t]|$, we conclude there exists $G(t) \subset G'(t)$ and $\gamma(t) \geq 0$ so that $|G(t)| \gtrsim |G'(t)|$ and $|(A[t])[x]| \sim K^{\gamma(t)}|A[t]|$ for all $x \in -(G(t))$.

Next, we observe

\[
|G(t)| = \sum_{x \in -(G(t))} |(A[t])[x]| \approx K^{-\gamma(t)}|A[t]| - (G(t)),
\]

whereas we also see

\[
|G(t)| \gtrsim |G_1(t)| \gtrsim K^{-\alpha|A[t]|^2},
\]

giving the desired bound on $|-(G(t))|$.

Finally,

\[
|A[t]|^2 E(A[t]) = \sum_{x \in A[t] - A[t]} |(A[t])[x]|^2 \geq \sum_{x \in -(G(t))} |(A[t])[x]|^2 \gtrsim K^{\gamma(t)-\alpha|A[t]|}K^{-2\gamma(t)}|A[t]|^2,
\]

giving the desired bound on $E(A[t])$. \qed

Let $G(t)$ be the set found in the previous lemma. We may assume the stronger bound $|-(G(t))| \gtrsim K^{1-2\alpha-\epsilon|A[t]|}$. Indeed, if $\gamma(t) \leq 1 - \alpha - \epsilon$ then

\[
E(A[t]) \gtrsim \frac{1}{K^{1-\epsilon}};
\]

we thus obtain the conclusion of the theorem taking $A' = A[t]$. We may therefore assume $\gamma(t) > 1 - \alpha - \epsilon$, hence

\[
|-(G(t))| \gtrsim K^{1-2\alpha-\epsilon|A[t]|} \gtrsim K^{-1-2\alpha-\epsilon|A-A|}
\]

for all $t \in T'$.

Now define

\[
X' = \bigcup_{t \in T'} -(G(t))
\]

and

\[
g(x) = |\{t \in T' : x \in -(G(t))\}|.
\]
Lemma 3.6. There exists $X \subset X'$ and $\eta \geq 0$ so that

$$|X| \gtrapprox K^{-1-2\alpha-2\epsilon+\eta}|A-A|$$

and

$$g(x) \sim K^{-\eta}|A-A|$$

for all $x \in X$.

Proof. We first observe $|X'| \leq |A-A|$. Then

$$\frac{1}{|X'|} \sum_{x \in X'} g(x) = \frac{1}{|X'|} \sum_{t \in T'} |(G(t))| \gtrapprox \frac{|T'|}{|X'|} K^{-1-2\alpha-\epsilon}|A-A|$$

$$\gtrapprox \frac{K^{-\epsilon}|A-A|}{|A-A|} K^{-1-2\alpha-\epsilon}|A-A| \gtrapprox K^{-1-2\alpha-2\epsilon}|A-A|.$$ 

Also, $g(x) \leq |T'| \lesssim K^{-\epsilon}|A-A| \leq |A-A|$. The result now follows from Lemma 2.4.

If $x \in X$ then $g(x) \gtrapprox K^{-\eta}|A-A|$, so $x \in -(G(t))$ for at least $\sim K^{-\eta}|A-A|$ values $t \in T'$. Therefore, $x \in A[t] - A \subset t + A - A$ for at least $\sim K^{-\eta}|A-A|$ values of $t \in T'$, so $t \in x + A - A$ for each of these $t$. But each such $t$ is also in $A - A$, so

$$|(x + A - A) \cap (A - A)| \gtrapprox K^{-\eta}|A-A|$$

for each $x \in X$.

Applying Lemma 2.5 with $B_1 = X, B_2 = A - A$ and $\rho = K^{-\eta}$, we obtain

$$E(X) \gtrapprox K^{-4\eta} K^{1-\epsilon} K^{-1-2\alpha-2\epsilon+\eta} = K^{-2\alpha-3\epsilon-3\eta}.$$ 

If $E(X) \gtrapprox \frac{1}{K^{1-\epsilon}}$, we obtain the conclusion of the theorem by taking $A' = X$; we may therefore assume

$$\eta > \frac{1}{3}(1 - 2\alpha - 4\epsilon).$$

Our lower bound on $X$ is now

$$|X| \gtrapprox K^{\frac{1}{3} - \frac{8}{3}\alpha - \frac{10}{3}\epsilon}|A|.$$ 

But for each $t \in T'$ we have

$$|(a - a' + A) \cap A| \gtrapprox K^{\alpha+\beta}|A|$$

for all $(a, a') \in G(t)$. Therefore,

$$|(x + A) \cap A| \gtrapprox K^{\alpha-\beta}|A|$$

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for all $x \in X$.

Using Lemma 2.5 with $B_1 = X$, $B_2 = A$ and $\rho = K^{\alpha - \beta}$ we obtain

$$E(X) \gtrsim K^{4(\alpha - \beta)}K^{1 - \epsilon}K^{1 - \frac{s}{3} - \frac{10}{3} \epsilon} = K^{\frac{4}{3} + \frac{4}{3} \alpha - 4\beta - \frac{13}{3} \epsilon} \geq K^{\frac{4}{3} - 4\beta - \frac{13}{3} \epsilon}.$$  

If $E(X) \gtrsim \frac{1}{K^{\alpha - \beta}}$ we obtain the conclusion of the theorem upon taking $A' = X$. We may therefore assume

$$\frac{4}{3} - 4\beta - \frac{13}{3} \epsilon < -1 + \epsilon,$$

so

$$\frac{7}{12} - \frac{4}{3} \epsilon < \beta.$$  

But we already proved the theorem in the case $\beta \geq \frac{1}{2} + \frac{7}{12} \epsilon$. Taking $\epsilon = \frac{1}{37}$, the $\beta > \frac{7}{12} - \frac{4}{3} \epsilon$ range is now subsumed by the $\beta \geq \frac{1}{2} + \frac{7}{12} \epsilon$ case. This concludes the proof of our main theorem.

References

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