2d gravity and matrix models

I. 2d gravity

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ABSTRACT

Some approaches to 2d gravity developed for the last years are reviewed. They are
physical (Liouville) gravity, topological theories and matrix models. A special at-
tention is paid to matrix models and their interrelations with different approaches.
Almost all technical details are omitted, but examples are presented.

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## Contents

1 Introduction .............................................. 2

2 Physical gravity ........................................... 2

2.1 Liouville theory ......................................... 2

2.1.1 Classical Liouville theory ......................... 2

2.1.2 Quantum Liouville theory ......................... 4

2.1.3 Sin-Gordon viewpoint ....................... 5

2.2 Liouville gravity ....................................... 6

2.2.1 Polyakov-KPZ-DDK approach .................. 6

2.2.2 Dressing of operators ......................... 7

2.2.3 String susceptibility ...................... 9

2.2.4 Weights of operators ...................... 10

3 Topological theories .................................... 11

4 Discrete matrix models - the first step .......... 12

4.1 A role of matrix models ....................... 13

4.2 Integrability in matrix models ................. 14

4.2.1 Toda chain equation ..................... 14

4.2.2 Determinant representation ............... 16

4.2.3 Multi-matrix models ......................... 17

4.3 Virasoro constraints .............................. 19

4.3.1 Ward identities in matrix models .......... 19

4.3.2 Conformal matrix models ................. 20

4.3.3 Ward identities in multi-matrix models .... 22

4.4 Continuum limit ..................................... 25

4.4.1 Continuum limit of Toda chain equations .... 25

4.4.2 The continuum limit of the Virasoro algebra .. 27

4.4.3 Invariant formulation of the limiting procedure .... 29

4.5 Fermionic representations and forced hierarchies . 30

4.5.1 Determinant representations of \( \tau \)-functions .... 30

4.5.2 Determinant representations for matrix model hierarchies .... 33

4.5.3 Integrable structure of CMMM - multi-component hierarchy .... 35

5 GKM approach to matrix models ............... 37
| Section | Title                                      | Page |
|---------|--------------------------------------------|------|
| 5.1     | What are the matrix models in the double scaling limit | 37   |
| 5.2     | Generalized Kontsevich Model (GKM)         | 39   |
| 5.3     | Main integrable properties of GKM           | 40   |
| 5.3.1   | Integrable structure                       | 40   |
| 5.3.2   | Reductions                                 | 41   |
| 5.4     | String equation                            | 41   |
| 5.4.1   | $L_{-1}$-constraint                        | 41   |
| 5.4.2   | Universal string equation                  | 43   |
| 5.4.3   | $W$-constraints                            | 43   |
| 6       | Conclusion                                 | 44   |
1 Introduction

After Polyakov proposed in 1981 the way to deal with non-critical strings [1], there were many attempts to get some concrete results. However, it still remains one of the most important and challenging problems in modern string theory. Indeed, instead of ordinary (string) matter theory Polyakov considered the theory of matter fields coupled to two-dimensional gravity. This is why the investigation of such theories is of great importance.

Actually, the revival of the interest to these theories occurred after 1988, mainly due to the calculation of the critical indices proposed by some groups both in light cone [2] and in conformal [3] gauges. These groups have given strong evidence in favor of the existence of self-consistent non-critical strings out of ”magic” interval of the central charges of the matter fields of the theory $1 \leq c \leq 25$.

The next serious success was achieved in the fall of 1989 when there were invented some new ”non-field” viewpoints to the theory of 2d gravity which have given efficient tools to deal with the matter theories with the central charge $c \leq 1$. Below I try to review this new development as it is observed from nowdays.

Indeed, in section 2 I just sketchily describe the ”field theory” approach of [2, 3], i.e. Liouville field theory. But this approach, which I refer to ”physical gravity” is still poorly developed, so the next sections are devoted to some new frameworks like topological theories of different types (section 3) and matrix models (sections 4 and 5). Some concluding remarks can be found in section 6. Let us emphasize that we intently present (in section 5) only sketchy review of matrix models in external fields. This very important approach deserves a separate paper. We hope to present it in the second part of the review, which is to be devoted to matrix models in applications to wider set of physical problems (like gauge theories).

In fact, the main part of the review is devoted to matrix model approach, the first, as it is the main point of authors’s interest and, the second, as just within this approach the most striking, powerful and general results were obtained. This review is goaled to non-specialists, therefore, almost all tedious technical details are omitted, but the text is provided by a number of examples and simple calculations. To get more detailed discussion, one can use references to original papers presented in all necessary places. I would like to stress that throughout the paper I mainly follow the approach developed by our group. Therefore, the references to different approaches are poorly presented. They can be found in two very detailed recent reviews [4, 5].

2 Physical gravity

2.1 Liouville theory

2.1.1 Classical Liouville theory

In this section we will briefly consider the Liouville theory having in minds that it describes, being properly interpreted, the theory of physical 2d gravity. To begin with, let us consider the theory with the canonical Liouville action:
$$S_L = \frac{1}{4\pi} \int d^2 z \left[ \frac{1}{2} \partial \phi \bar{\partial} \phi - \frac{1}{\beta^2} e^{\sqrt{2} \beta \phi} \right], \quad (2.1.1)$$

where the field depends on variables $z, \bar{z}$, standard complex variables in the Euclidean space-time and $\partial \equiv \frac{\partial}{\partial z}, \bar{\partial} \equiv \frac{\partial}{\partial \bar{z}}$. This theory has two main features - it possesses conformal invariance at the classical level and it has no constant-valued ground state solution, which results in disastrous infra-red singularities.

As for the first property, naively the theory is not conformal, since the trace of the stress tensor is equal to $T_{zz} = \frac{1}{\beta^2} e^{\sqrt{2} \beta \phi}$. The chiral component of the stress tensor for the action (2.1.1) is

$$T_{zz} = -\frac{1}{2} (\partial \phi)^2 \quad (2.1.2)$$

and so for $T_{\bar{z}z}$. However, it is not holomorphic, even on the equations of motion, since

$$\bar{\partial} T_{zz} = \frac{\sqrt{2}}{\beta} e^{\sqrt{2} \beta \phi} \partial \phi. \quad (2.1.3)$$

This naive stress tensor can be improved by adding the term which does not effect to the conservation law:

$$T_{\mu \nu} \rightarrow T_{\mu \nu} - (\partial^2 \delta_{\mu \nu} - \partial_{\mu} \partial_{\nu}) \Sigma, \quad T = T_{zz} + \partial^2 \Sigma. \quad (2.1.4)$$

Now, taking $\Sigma = \frac{1}{\sqrt{2} \beta} \phi$, we get

$$T = -\frac{1}{2} (\partial \phi)^2 + \frac{1}{\sqrt{2} \beta} \partial^2 \phi \quad (2.1.5)$$

and

$$\bar{\partial} T = \frac{\sqrt{2}}{\beta} e^{\sqrt{2} \beta \phi} \partial \phi + \frac{1}{\sqrt{2} \beta} \partial^2 \bar{\partial} \phi = 0, \quad (2.1.6)$$

with using the equations of motion.

In fact, $\Sigma$ adds to the action just full derivative, proportional to $\bar{\partial} \partial \phi$, which might be essential only when considering non-trivial boundary conditions like the presence of vacuum charge [6, 7, 8]. This term in action can be invariantly written as

$$\frac{\sqrt{2}}{\beta} R \phi, \quad (2.1.7)$$

where $R = \bar{\partial} \partial \log \rho, \rho = g_{zz}$ are the two-dimensional curvature and metric respectively. The Poisson brackets of the stress tensor (2.1.5) produce the Virasoro algebra\footnote{It is evident from vanishing of trace of the improved stress tensor on the equations of motion.} with conformal anomaly presented already at the classical level:

$$c_{\text{cl}} = \frac{6}{\beta^2}. \quad (2.1.8)$$
2.1.2 Quantum Liouville theory

Now let us treat the quantum corrections to the Liouville theory. Indeed, the theory is ill-defined due to infra-red singularities. Nevertheless, one can try to calculate the quantities not sensitive to the infra-red divergences. Say, one can consider ultra-violet renormalizations of the action \( (2.1.1) \) with the curvature term \( (2.1.7) \).

Indeed, it is not difficult to take into account all ultra-violet divergent diagramms, as in two dimensions these are only tadpoles. The calculation is rather trivial, and results in multiplicative renormalization of the coupling constant \( \frac{1}{\beta} \) of the exponential in the action \( (2.1.1) \) by the factor \( \Lambda^{-2\beta^2} \). It is the same as shift of the vacuum expectation value of field \( \phi \) by the quantity \( \delta = -\sqrt{2}\beta \log \Lambda \). Let us note that this shift depends on ultra-violet cut-off parameter \( \Lambda \) which should be made dimensionless by some infra-red cut-off parameter. This parameter can be naturally chosen to be the order of magnitude of the metric \( \rho \), as it is metric that cuts-off the theory at large distances in curved space\(^2\). Thus, we can put \( \phi \rightarrow \phi - \sqrt{2}\beta \log \Lambda \rho \). It just results in the renormalization of the coefficient in the curvature term \( (2.1.7) \):

\[
\frac{1}{2} \partial \phi \bar{\partial} \phi \rightarrow -\frac{1}{2} (\phi + \delta) \bar{\partial} \phi - \phi \bar{\partial} \delta = \frac{1}{2} \partial \phi \bar{\partial} \phi + \sqrt{2}\beta R \phi. \tag{2.1.9}
\]

It results in the central charge equal to

\[
c = 1 + 6(\beta + \frac{1}{\beta})^2. \tag{2.1.10}
\]

Now let us justify (or illuminate) this result. Let us suggest that one should still preserve the conformal invariance at the quantum level. It means that the exponential term in the action should be of the dimension \((1,1)\), i.e. should have canonical dimension. From the other hand, the conformal theory with the action \( (2.1.1) \) and additional \( \alpha R \phi \) term has the central charge \( c = 1 + 3\alpha^2 \), and the ”naive” dimension \( \Delta_\gamma = \frac{\gamma^2}{2} \) of the operator \( \exp\{\gamma \phi\} \), which it has in the gaussian model, changes to \( \Delta_\gamma = \frac{1}{2} \gamma (\alpha - \gamma) \). As this dimension is to be unit, one gets the expression for the coefficient \( \alpha \):

\[
\alpha = \sqrt{2}(\beta + \frac{1}{\beta}), \quad \beta = \frac{\alpha}{\sqrt{8}} \pm \sqrt{\frac{\alpha^2}{8} - 1}, \tag{2.1.11}
\]

which coincides with that in the formula \( (2.1.9) \). It leads to the same expression for the central charge \( (2.1.10) \).

Certainly, this consideration was rather a motivation than the rigid result. Indeed, the last reasoning is not too correct, as we treated only chiral objects throughout the discussion, but one can separate chiral and anti-chiral parts only on the equations of motion. This approach can be done completely correct only in the case of \( N = 2 \) Liouville theory \( \footnote{\text{It again can be turned to the matter of boundary conditions.}} \).

\footnote{\text{Throughout the review we omit the sign of normal ordering where this can not be misleading.}}
2.1.3 Sin-Gordon viewpoint

Now we will say some words on how to deal with the Liouville theory in a more correct way. Let us note that the exact computation of all ultra-violet contribution would lead to the complete final answer in the absence of infra-red divergences. Indeed, there are no dimensional parameters, excluding UV cut-off parameter, and, therefore, any non UV contributions to the effective action are forbidden by the dimension argument. Unfortunately, this is not the case in the Liouville theory as it has terrible infra-red divergences, and, therefore, essentially depends on the IR cut-off parameter. These IR divergences are due to the absence of finite-density constant classical solution of the theory, i.e. due to the absence of classical vacuum in the theory. In principle, this ill-defined behaviour should be expected from the very beginning as the theory is the conformal one, and, therefore, has no good particle content, finite $S$-matrix and etc. It implies that, to deal with the Liouville theory, one should consider this as the theory given in a fixed point of some larger theory. The natural choices of this larger theory are the sin- or sinh-Gordon theories, depending on whether $\beta$ is real or pure imaginary\footnote{The consideration above was equally correct for any complex $\beta$.}. It was investigated in the paper $[10]$, where the authors worked with the sin-Gordon theory (i.e. $\beta \equiv i\tilde{\beta}$ is pure imaginary) with the action\footnote{It is essential for calculations to have two different constants $\gamma_{\pm}$ in action, though one of them can be removed by a shift of the field. This shift is often infinite in the cases under consideration.}

\[ S_{SG} = \frac{1}{4\pi} \int d^2z \left[ \partial \phi \bar{\partial} \phi + \gamma_+ e^{i\sqrt{2}\tilde{\beta}\phi} + \gamma_- e^{-i\sqrt{2}\tilde{\beta}\phi} + i\tilde{\alpha} R\phi \right]. \]  (2.1.12)

For the sake of convenience, we also replaced $\alpha \to \tilde{\alpha} \equiv -i\alpha$ in the action, i.e. $c = 1 - 3\tilde{\alpha}^2$. The authors of $[10]$ generalized field theory calculation above to this more complicated case and have calculated $\beta$-functions perturbatively in $\gamma_\pm$ and $\tilde{\beta}^2 - 1$. They have observed that $\tilde{\alpha}/\tilde{\beta}$ is a RG invariant and calculated $\beta$-functions. These were:

\[ \frac{d\lambda}{dt} \equiv \beta_\beta = 2\tilde{\beta}^4\gamma_+\gamma_-, \]
\[ \frac{dq^2}{dt} \equiv \beta_\alpha = \frac{1}{16}\tilde{\alpha}^2\tilde{\beta}^2\gamma_+\gamma_-, \]
\[ \frac{d\gamma_\pm}{dt} \equiv \beta_\pm = \gamma_\pm(\tilde{\beta}^2 - 1 \mp \frac{1}{\sqrt{2}}\tilde{\alpha}\tilde{\beta}). \]  (2.1.13)

The theory appears to have both an infrared and an ultraviolet fixed points in the region of approximation. There are two essentially different regimes which were realized. One of them corresponds to RG flow from the free field theory with a background charge $\tilde{\alpha}_+$ to the free field theory with a different (and larger, in complete agreement with Zamolodchikov’s $c$-theorem $[11]$) background charge $\tilde{\alpha}_-$. In the other regime, one flows from an UV fixed point with $\gamma_+ = \gamma_- = 0$ to an IR fixed point, where $\gamma_- = 0$, but $\gamma_+ \neq 0$, i.e. one of the exponentials (marginal operator) survives in the IR limit, leading to the Liouville action.

From the equation (2.1.13) one can determine the values of parameters corresponding to this fixed point:
\[ \tilde{\beta} = \frac{\tilde{\alpha}}{\sqrt{8}} \pm \sqrt{\frac{\tilde{\alpha}^2}{8} + 1}. \]  

(2.1.14)

This is consistent with the connection (2.1.11), thus justifying our results.

The manifest calculation of [10] also reproduced the values of dimensions which were obtained above, and the generalized central charge, which on the RG trajectory is given by some monotonic decreasing function (in accordance with Zamolodchikov’s c-theorem [11]) at the fixed point describing Liouville theory reaches the value (2.1.10).

2.2 Liouville gravity

2.2.1 Polyakov-KPZ-DDK approach

In the previous section we considered Liouville theory with no reference to the main purpose of the review, that is, the description of 2d physical gravity. In the section we are going to discuss 2d physical gravity itself.

One should start from the original covariant Polyakov action for the string in flat D-dimensional euclidean target space:

\[ S = \frac{1}{4\pi} \int d^2 \xi \sqrt{g} g^{\mu\nu} \partial_{\mu} X^i \partial_{\nu} X^i, \quad i = 1, \ldots, D. \]  

(2.2.1)

The main purpose is to calculate the correlation functions through the following path integral:

\[ \langle V_1 V_2 \ldots V_n \rangle = \int D X D g^{\mu\nu} e^{-S} \]  

(2.2.2)

with the action (2.2.1). We will proceed quantizing this system in conformal gauge, though there is an equivalent light-cone gauge approach [2]. The conformal gauge approach was proposed in the original paper by Polyakov [1], and was developed in [3], where the central charge and weights of operators were obtained.

Thus, we fix conformal gauge \( g_{\mu\nu} = e^{\phi} \hat{g}_{\mu\nu} \), where \( \hat{g}_{\mu\nu} \) is a background metric. The gauge fixing procedure also introduces reparameterization ghosts \( b, c \), which are almost inessential in our consideration. Therefore, the complete action consists of three pieces: the ghost action, the Liouville action (2.1.1) with the proper curvature term and the matter field action. As the complete action should be conformally invariant and should not depend on the background metric \( g_{\mu\nu} \), the complete central charge must vanish.\(^6\)

In fact, the ghost stress tensor contributes -26 into the full central charge, and the matter fields do \( D \). Certainly, one can consider, instead of (2.2.1), an arbitrary conformal matter theory, with an arbitrary central charge \( c_m \) instead \( D \). We will do so to keep the most general expressions admitting non-integer values of the central charge.

Now we can calculate the restriction due to zero full central charge, using also the manifest expression for the Liouville central charge (2.1.10):

\(^6\)This means that the theory ought to be topological one, see the next section.
\[ c_m + c_{gh} + c_L = 0 \rightarrow c_m + (-26) + (1 + 3\alpha^2) = 0, \]

i.e.

\[ \alpha = \sqrt{\frac{25 - c_m}{3}}, \]

(2.2.4)

and \( \alpha \) was determined in the previous subsection.

It is evident from the expression (2.1.11) that the condition \( \alpha^2 < 8 \) leads to the complex values of \( \beta \) and the Liouville central charge \( c_L \) (2.1.10), i.e. to non-unitary theory. It reproduces notorious limitation on the central charge of matter (i.e. on the dimension of the target space):

\[ c_m \leq 1. \]

(2.2.5)

In fact, complex \( \beta \) just means that the chosen vacuum of the theory is unstable, and this vacuum is to pass to a different ground state. It can be understood also from the tachyon mass square which is equal to \( \frac{1 - c_m}{12} \). This means that the tachyon is absent and the vacuum state is stable only provided by the same condition \( c_m \leq 1 \).

There is another limit when \( c_L \) is also real. It corresponds to \( \alpha^2 < 0 \), i.e. \( c \geq 25 \). This implies that \( \beta \) is pure imaginary, or, what is the same after redefinition \( \phi \rightarrow i\phi \), the wrong sign of the kinetic Liouville term. It seems to spoil the unitarity of the theory too.

But there is a special case of \( c_m = 25 \), when one just reproduces the critical string with 25 space coordinates (matter fields) and one time coordinate with reversed sign in the kinetic term (Liouville field). Therefore, in such a way, we recover the mechanism to get Minkowski space from Euclidean theory, which is applicable for the critical string! Simultaneously it means that the ”physical” dimension of non-critical string is \( c_m + 1 \).

### 2.2.2 Dressing of operators

Now we are going to discuss operator content of 2d gravity. As the first step, let us note that the functional integral in gravity theory should be independent both of background metric \( \hat{g}_{\mu\nu} \) and of physical metric \( g_{\mu\nu} = \hat{g}_{\mu\nu} e^{\sqrt{2}\beta \phi} \).

The first property implies zero full central charge of the theory and was already used to connect Liouville exponential with the matter central charge \( c_m \). The second requirement was used in the previous subsection to calculate central charge of the Liouville theory (2.1.10).

Now we will consider matter field in the same context. These fields should be also renormalized (dressed) by the Lioville field due to the interaction with gravity. To calculate this dressing, we require for correlation functions of the operators to be independent of metric, instead of doing very tedious perturbative calculations.

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7This renormalization of the exponential factor - or the Liouville field - is a direct consequence of our requirement for this exponential to have the dimension (1,1). The same phenomenon can be described as well by saying that the scalar Liouville field changes the transformation law and is not scalar after taking into account the conformal anomaly (see more detailed discussion in [4]).
Thus, let us consider a matter operator $V$ which has a dimension $\Delta$. It is used to suggest that, as a result of gravity interaction, it is dressed by exponential of the Liouville field:

$$V_{dr} = e^{\gamma \phi} V. \quad (2.2.6)$$

Actually, we would have to consider more general operators which include Liouville secondaries (and maybe even ghost contributions). Fortunately, it is not necessary to do in the 2$d$ gravity case. This is due to huge invariance of the theory. The drastic reduction of the space of vertex operators is a general phenomenon occurring whenever a string model is build from CFT. Let us remind first the situation with the critical string. In the simplest treatment, one neglects the Liouville field at all, but this is not the only thing one does. There are three additional requirements to physical vertex operators in critical string models:

a) They do not contain ghost fields (in certain and natural picture)\(^8\).

b) They are Virasoro primaries (with respect to “full” Virasoro). (All the descendants, present in CFT are “gauged out”). The reason is that coupling to 2$d$ gravity implies gauging the Virasoro algebra and thus all the descendants are eliminated as gauge-non-invariant operators. They really decouple from the correlators\(^9\).

c) They are integrals (in the picture consistent with a)) of operators of conformal dimension one\(^10\).

Naive generalization of these three principles to the case of non-critical string has been suggested in \(^3\), with the only modification made at the point b), where one is supposed to take primaries of the full (matter + Liouville) Virasoro algebra, in the form of (matter primaries)$\times$(Liouville primaries) and Liouville primaries being pure exponentials. It is, however, not quite true that such a simple generalization is valid. A problem arises at the very beginning – at point a). In order to derive honestly this requirement one should prove that in every BRST cohomology class (i.e. for every physical state) there is a ghost free representative. According to \([12, 13]\) this is not true in non-critical case: there are two additional representatives of the cohomology classes, which unavoidably (and non-trivially) contain ghost fields. In the particular case of $c = 1$ model, the first ones were interpreted in \([2]\) as a ground ring, another one (the “ghost number two”) still has no nice interpretation. It seems, however, that the subsector, described by principles a),b),c) is closed by itself under OPE (modulo descendants, i.e. fields vanishing in the correlation functions). Another delicate point is that the requirement c), i.e. that the full dimension is one, permits two different choices of Liouville primaries associated with a given matter operator. In what follows we consider a subsector with one specific choice of these two. This subsector also seems to be closed under OPE in the above sense.

Thus, we restrict ourselves to this subsector and consider only the operators of the form (2.2.6), more general discussion can be found in \([14]\). Let us calculate the value of the exponent $\gamma$ in (2.2.6) requiring (1,1) dimension of the dressed operator (it was already explained in different words that it just expresses the possibility to integrate it over 2$d$ surface with no breaking conformal invariance). Then we obtain:

\(^8\)Except for one delicate point, concerning dilaton operator.
\(^9\)It is just no-ghost theorem.
\(^10\)Let us note that other (equivalent) pictures may be obtained by multiplication of dimension one integrands (constrained by a) and b)) with ghost field instead of integration.
\[ \Delta + \frac{1}{2} \gamma [\alpha - \gamma] = 1, \quad (2.2.7) \]
i.e.
\[ \gamma = \frac{1}{2} (\alpha \pm \sqrt{\alpha^2 - 8 + 8\Delta}). \quad (2.2.8) \]

Now let us say some words on the choice of sign in this expression (and that in the Liouville exponent, see \((2.1.11)\)). Hereafter, our prescription will be to choose "+", because of arguments proposed in \([13]\). There was argued that this choice selects out the local operators corresponding to the states with positive energy. We will comment more this point a bit later.

In the next subsection we use this expression to determine scaling properties of dressed operators.

### 2.2.3 String susceptibility

As correlation functions of physical operators in 2d gravity do not depend on metric, as well as on the positions of operators (since the integration over locations of all operators is to be presented), all correlation functions are just numbers. We will return to the question in the next section and now let us only note that we still need some characteristic like dimension, which differs among operators (and correlation functions). The simplest way to introduce such a characteristic is to fix the area of the surface \(A\) and investigate \(A\)-dependence of correlation functions. It is clear that this characteristic is of great importance, as it is that which governs the grow (or shrinking) of the surfaces and controls their stability. It is the general feature of all string theories that their partition functions exponentially decrease with area. The index which really differs among string theories and governs the behaviour of their ground states is pre-exponential factor.

We will now calculate this factor. Let us consider the path integral for 2d gravity (=non-critical string) theory with fixed area \(A\):

\[ Z(A) \equiv \int \mathcal{D}\phi\mathcal{D}\text{(matter)}\mathcal{D}(\text{ghosts}) e^{-S_L} \delta \left( \int d^2 \xi \sqrt{\hat{g}} e^{\sqrt{2} \beta \phi} - A \right), \quad (2.2.9) \]

where we included matter and ghost action terms into the measures and took into account that the area of the surface is nothing but the integral of the physical metric \(g_{\mu \nu} = \hat{g}_{\mu \nu} e^{\sqrt{2} \beta \phi}\) over the surface.

Though the integral \((2.2.9)\) is very complicated, its \(A\)-dependence can be easily determined. Indeed, let us do the substitution \(\phi \to \phi + \frac{1}{\sqrt{2} \beta} \log A\). Then, the exponential in the Liouville action turns into the exponential term \(e^{-\text{const} \times A}\) in the functional integral (due to the \(\delta\)-function in \((2.2.9)\)). Therefore, the only contribution of the action to the pre-factor is from the curvature term:

\[ \frac{1}{4\pi} \int d^2 \xi \sqrt{\hat{g}} \alpha R \phi \to \frac{1}{4\pi} \int d^2 \xi \sqrt{\hat{g}} \alpha R \phi + \frac{\alpha}{\sqrt{2} \beta} \log A \left( \frac{1}{4\pi} \int d^2 \xi \sqrt{\hat{g}} R \right). \quad (2.2.10) \]
Measure in the path integral (2.2.9) is evidently invariant with respect to constant shifts. Then, using the identity \( \delta(Ax) = A^{-1}\delta(x) \) and the Gauss-Bonnet theorem \( \frac{1}{4\pi} \int d^2\xi \sqrt{g}R = 1 - g \), where \( g \) is the genus of the surface, one finally obtains

\[
Z(A) \sim A^{\frac{-(1-g)-1}{2}}} e^{-\text{const}\times A} \equiv A^{(\gamma_{str}-2)(1-g)-1} e^{-\text{const}\times A},
\] (2.2.11)

i.e.

\[
\gamma_{str} = 2 - \frac{\alpha}{\sqrt{2\beta}} = \frac{1}{12} \left( c_m - 1 - \sqrt{(c_m - 25)(c_m - 1)} \right).
\] (2.2.12)

This index \( \gamma_{str} \) is called string susceptibility \(^{12}\).

### 2.2.4 Weights of operators

In the previous subsection we have calculated the index describing the dependence of the partition function on area \( A \). Analogous dependence of correlation functions can be also investigated, and defines the index which is a proper characteristic of operator in 2\(d\) gravity. More concretely, the index, which is called the weight of operator, can be determined from the normalized one-point correlation function:

\[
<V> \equiv \frac{1}{Z(A)} \int \mathcal{D}\phi \mathcal{D}(\text{matter})\mathcal{D}(\text{ghosts}) \times e^{-S_L} \delta \left( \int d^2\xi \sqrt{g} e^{\frac{1}{\sqrt{2\beta}} - A} \right) \int d^2\xi \sqrt{g} V_{dr} \sim A^{1-h}.\] (2.2.13)

Physically, this index controls the behaviour (growth) of string surface when the operator is inserted. Therefore, the standard notions of relevant, marginal or irrelevant operators still make sense. In particular, relevant operators correspond to \( h < 1 \) and dominate in the IR (large \( A \)) limit.

To calculate \( h \), we do the same steps as in course of calculation of the string susceptibility and use the manifest expression for the dressing exponent (2.2.8):

\[
<V> \sim \frac{A^{\frac{(1-g)}{2\beta} - 1 + \frac{\gamma}{\sqrt{2\beta}}}}{A^{\frac{(1-g)}{2\beta} - 1}} = A^{\gamma/\sqrt{2\beta}},
\] (2.2.14)

i.e.

\[
h = 1 - \frac{\gamma}{\sqrt{2\beta}} = \frac{\sqrt{1 - c_m + 24\Delta V} - \sqrt{1 - c_m}}{\sqrt{25 - c_m} - \sqrt{1 - c_m}}.
\] (2.2.15)

Thus, the classification of operators into relevant, marginal and irrelevant ones does not change for dressed operators. Indeed, the conditions \( \Delta = 1, \Delta > 1 \) and \( \Delta < 1 \) immediately imply the same conditions for the dressed weights \( h \) in (2.2.15).

---

\(^{11}\)We do not concretize the constant in the exponential because of its non-universal nature, in contrast to the index \( \gamma_{str} \).

\(^{12}\)Let us remark that \( \gamma_{str} \) has a complex value when \( 1 < c_m < 25 \), pointing once again to "the magic interval" of values of \( c_m \).
Let us note that the expression (2.2.15) is real and positive for \( c_m \leq 1 \) provided the choice of sign is "-" in the expressions (2.2.8) and (2.1.11). This is one more argument in favor of that choice.

Now, to have an explicit example, let us write down the values of \( \gamma_{str} \) and \( h \) for the operators in minimal models \([16, 7, 8]\). The central charge of \((p,q)\)-minimal model is equal

\[
c = 1 - \frac{6(p-q)^2}{pq},
\]

i.e.

\[
\gamma_{str} = \frac{-2|p-q|}{p+q-|p-q|}.
\]

The (Kac) spectrum of operator dimensions is given by

\[
\Delta_{m,n} = \frac{(pm - qn)^2 - (p-q)^2}{4pq}, \quad 1 \leq m \leq q-1, \quad 1 \leq n \leq p-1,
\]

i.e.

\[
h_{m,n} = \frac{|pm - qn| - |p-q|}{p+q-|p-q|}.
\]

This expression should be compared with the indices calculated in matrix models (the below).

### 3 Topological theories

We could conclude from the discussion of the previous section that correlation functions in the theory of 2d gravity have a very simple structure. Indeed, if we do not fix the area of the surface, then all correlation functions are just numbers, provided by vanishing of the full central charge of the theory. This means that it is a topological theory.

Indeed, there no variables which correlation functions could depend on, since integration over metrics is done. Certainly, it implies that the theory is in unbroken gravity phase.

Let us look at the same argument from a different viewpoint. The routine way to do (generally, non-critical) string integral was usually to fix the conformal factor and then to remove two remaining components of metric tensor by making use of the general covariance of the string action (i.e. to integrate out the diffeomorphism group). After doing this one remains with the integral over moduli of the conformal (=complex) structures and with the integral over conformal (Liouville) factor (in the case of non-critical strings). It is approximately the procedure which was implied in the previous section.

Now let us change the order of the integrations. Namely, one can integrate over metrics from the very beginning. Then, we have an integral (correlation function) which can depend only on the locations of the points (as any metric dependence is already integrated out) and still possesses the general covariance (which is again correct only in unbroken phase). It can be evidently nothing but a number.
Thus, we are led to consider the theory of 2d gravity as a topological theory. Certainly, there is plenty of different topological theories, but it is naturally to consider only 2d theories. Indeed, there are only two successful attempts to formulate a topological theory appropriate for the description of 2d gravity.

The first attempt was due to Witten [17], who proposed the theory which preserves the memory of the moduli space treating as correlation functions some integrals over moduli space. More concretely, let us consider the holomorphic linear bundle $\mathcal{F}_i$ with the fiber being the cotangent space (complex one dimensional vector space) to the Riemann surface $\Sigma$ at the marked point $x_i$. Then we can take as correlation functions the intersection numbers of these bundles. It means that we should integrate the first Chern classes of the exterior products of these bundles over (Deligne-Mumford) compactification $\mathcal{M}_{g,n}$ of the moduli space of the curve $\Sigma$ of the genus $g$ (see also [18]):

$$<\sigma_1...\sigma_n> \equiv \int_{\mathcal{M}_{g,n}} \prod_i c_1(\mathcal{F}_i)^{d_i}. \quad (3.1)$$

These correlation functions are indeed numbers (positive rational) which vanish unless $6g - 6 + 2n = 2\sum d_i$. They are related by a number of recursion relations which are equivalent to the Virasoro constraints of the Hermitean one-matrix model in the continuum limit [19]. This allows one to identify the two theories. Unfortunately, it is not so immediate to generalize the construction to multi-matrix model case, and the work in this direction is not finished yet (see, however, [20, 21]).

Another successful construction of the appropriate topological theory was based on the following general idea. Let us consider the theory whose stress tensor (or action) is anticommutator of something with a BRST operator $Q$: $T = \{*, Q\}$, $Q^2 = 0$. Then, this stress tensor describes a topological theory on the subspace of the physical states (i.e. those defined by the condition $Q|phys> = 0$):

$$T|phys > = 0. \quad (3.2)$$

In fact, since coordinate variations are generated by the stress tensor, any variation of a correlator is proportional to the the same correlator with insertion of $T$. Therefore, correlator of the physical fields, due to the condition (3.2), does not depend on the locations of the points and is a number.

A concrete realization of this general idea has been proposed in [22] (see also [23]), where twisted $N = 2$ superconformal theory was investigated. It was formulated in abstract terms of OPE, with no any Lagrangians. The Lagrangian formulation, namely, $N = 2$ supersymmetric Landau-Ginzburg theory has been proposed in [24].

The advantage of the theories of described type is that they can be investigated in some details, and they are applicable to the description of plenty of matrix models, in contrast to the theories of Witten’s type. Because of the lack of space we can not develop here all these ideas, and refer to the reviews and papers containing all necessary references [25, 26, 27].

4 Discrete matrix models - the first step
4.1 A role of matrix models

Now let us say that the results mentioned in the section 2 are the only ones obtained in the framework of physical gravity. But in the Fall of 1989 a drastic progress in 2d gravity was achieved in some other, rather uncommon for the field theorists, frameworks. In particular, an old idea proposed in [28] to replace the path integral of 2d gravity over metrics (2.2.2) by the sum over all possible triangulations of surfaces (i.e. reformulate it in target space terms) have led, after all, to what is called nowadays matrix models. Really it consists of three different components - double scaling limit [29, 30, 31], constraint algebras satisfied by the partition functions of matrix models [19, 32] and integrability properties [33].

Up to now, the approach relies only on the original "physical" arguments of triangulation with successive taking the (double scaling) continuum limit and on the comparison of the very poor predictions obtained from the physical gravity with those within matrix model treatment. These predictions include weights of operators in the theory and string susceptibility, as well as the calculations of 3-point functions (and some n-point functions in very simple channels). In fact, it means that the comparison is at the level of the spectra of the theories. It implies, more or less, that the matter of quantum measure is still out of the scope of the present investigations in physical gravity. In fact, though there are some distinguished measures in the gravity path integral (induced by the free field representation, or Polyakov’s one, or ...), it is still necessary to fix somehow these as well as the normalization factor depending on the genus of the (Riemann surface) string world sheet (i.e. on the order of the perturbation theory).

At this point, one can reverse the line of reasoning and consider matrix models as a definition of these entries. To be a good definition, such a theory should be defined naturally over all genera simultaneously. Indeed, I will try to demonstrate below that matrix models have to do with the infinite Grassmannian, which, in its turn, naturally describes the Universal Moduli Space (UMS, the space of all Riemann surfaces of all genera). Therefore, matrix models do really give the example of the theory with required properties.

Indeed, we will see that the partition function of a matrix model is a \( \tau \)-function and corresponds to a concrete point of the Grassmannian (corresponding to a Riemann surface of infinite genus) which implies that after doing the path integral (2.2.2) over UMS, one reproduces the answer corresponding to a point of the same UMS (see also [34, 35, 36]). It resembles the phenomenon realized some years ago [37] in 2d quantum integrable systems, when the exact quantum correlators satisfy classical integrable equations.

Thus, it is expected that matrix models give a correct description of the non-critical strings with \( c < 1 \), therefore, better understanding of their properties is necessary. I will try to describe briefly these properties in the next two sections. Indeed, now we can do even more - we establish the connection of matrix models with some other recent approaches to 2d gravity which can be also solved up to the very end (in contrast to physical gravity). It is possible to reach the proof of complete identity between all these theories. This should not be very surprising as all these theories calculate, in essence, the same cohomologies and reflect...
different general properties of the same final answer.

Now we just start from the matrix integral, the standard arguments in favor of it can be found in plenty of reviews \cite{4, 5, 39, 40}. Let us note that all the specific properties of matrix models will be demonstrated in this section for the example of Hermitian one matrix model. More complicated examples (in this and the next sections) will be labelled by asterisque and can be omitted without disturbing the main line of reasoning. I omit many subtle points and all evident (but maybe technically interesting) generalizations of the considered examples, referring to the original papers. Many of these generalizations can be also found in two recent reviews \cite{39, 40}.

4.2 Integrability in matrix models

4.2.1 Toda chain equation

Let us consider the integral over Hermitian $n \times n$ matrix with arbitrary potential:

$$Z_n(t) \equiv \left\{ Vol_{U(n)} n! \right\}^{-1} \int [dH] \exp\{ -\text{Tr} V(H) \}, \quad V(H) \equiv -\sum_{i=0} t_i H^i,$$

(4.2.1)

where $[dH]$ is the proper Haar measure. The standard procedure of doing this integral is, indeed, to fix some simple potential (polynomial of a finite degree), to write down the equations satisfied by the partition function (usually there are few equations as the integral (4.2.1) depends on some coefficients in the polynomial potential) and to take the double scaling limit in these equations. Instead, we consider the partition function (4.2.1) as the function of all coefficients $t_i$’s and demonstrate that all the crucial properties of matrix models (integrability and constraint algebra) can be exposed before taking the continuum limit. Moreover, in a sense, these properties are easier observed in the case.

Thus, to begin with, let us demonstrate that the partition function (4.2.1) is nothing but the $\tau$-function of the Toda chain integrable hierarchy \cite{41, 42}. We can rewrite (4.2.1) as the integral only over eigenvalues with angular variables integrated over:

$$Z_n(t) = (n!)^{-1} \int \prod_i dh_i \Delta^2(h) \exp\left\{ -\sum_{i,k} t_i h^k_i \right\},$$

(4.2.2)

Now one can apply the standard machinery of the orthogonal polynomials \cite{43}. To do this, let us define the polynomials by the orthogonality condition

$$\langle i|j \rangle \equiv \langle P_i,h \rangle = \int P_i(h) P_j(h) e^{-V(h)} dh = \delta_{i,j} e^{\phi_i(t)},$$

(4.2.3)

where $e^{\phi_i(t)}$ are the norms to be determined, the polynomial normalization being fixed by the unit coefficient in the leading term:

$$P_i(h) = \sum_{j \leq i} a_{ij} h^j, \quad a_{ii} = 1.$$

(4.2.4)

Using these polynomials, one can rewrite (4.2.2) as
\[ \mathcal{Z}_n = (n!)^{-1} \int \prod_i dh_i \det P_{k-1}(h_j) \det P_{l-1}(h_m) \exp \left\{ - \sum_{i,k} t_k h_i^k \right\} = \prod_{i=0}^{n-1} e^{\varphi_n(t)}. \] (4.2.5)

Now we are going to prove that this partition function is nothing but a \( \tau \)-function of Toda chain hierarchy. In fact, we derive only the first equation, refering to [11] for the general proof and any details.

To begin with, let us consider the scalar product \( \langle n | h | m \rangle \) for different \( m \). It is evidently zero whenever \( m - n \neq 0, \pm 1 \) (e.g., \( \langle n | n - 2 \rangle = \langle n | n - 1 \rangle + \sum_{k>1} e_k < n | n - k \rangle = 0 \) by the conditions (4.2.3) and (4.2.4)). It means that

\[ hP_n(h) = P_{n+1}(h) - p_n P_n(h) + R_n P_{n-1}(h), \] (4.2.6)

where the coefficients \( p_n \) and \( R_n \) should be determined. The latter can be trivially calculated using (4.2.6) twice:

\[ \langle n - 1 | h | n \rangle = \langle hP_{n-1}, P_n \rangle = \langle n | n \rangle + \sum_{k>0} < n | n - k \rangle = < P_{n-1}, hP_n \rangle = R_n < n - 1 | n - 1 \rangle, \] i.e.

\[ R_n(t) = e^{\varphi_n(t) - \varphi_{n-1}(t)}. \] (4.2.7)

Indeed, the recurrent relation (4.2.6) already hints at the Toda chain, as, introducing difference \( L \)-operator with the matrix elements

\[ (L)_{mn} = \delta_{m,n-1} - p_n \delta_{m,n} + R_{n+1} \delta_{m,n+1} \] (4.2.8)

with eigenfunction (Baker-Akhiezer function) \( \Psi_n(h) = \exp \{- \frac{1}{2} V(h)\} P_N(h) \), one can trivially see in (4.2.6) the action of Lax operator of the Toda chain [11], where \( h \) is, as usual, the spectral parameter:

\[ L \Psi_n(h) = h \Psi_n(h). \] (4.2.9)

Now to get the equations of Toda chain it is sufficient to obtain the correct action of the operator of the derivative with respect to the first time [11]. But more illuminating way is to derive the Toda chain equations [11]. As an example, we demonstrate here the derivation only of the first equation in the hierarchy. To do this, let us differentiate \( \langle n | n \rangle \) with respect to the first time using (4.2.3), (4.2.4) and (4.2.6):

\[ \dot{\varphi}_n e^{\varphi_n} = \frac{\partial}{\partial t_1} < n | n > = 2 \left\langle \frac{\partial P_n}{\partial t_1}, P_n \right\rangle - < n | h | n > = -p_n e^{\varphi_n}, \]

i.e. \( p_n = -\dot{\varphi}_n \), (4.2.10)

where dot means the derivative with respect to the first time. Now we differentiate the same quantity once more and, using (4.2.3), (4.2.4), (4.2.6), (4.2.7) and (4.2.10) and integrating by parts, we obtain finally the well-known Toda chain equation:

\[ \ddot{\varphi}_n e^{\varphi_n} + (\dot{\varphi}_n)^2 e^{\varphi_n} = \frac{\partial^2}{\partial t_1^2} < n | n > - < n | h^2 | n > - 2 \left\langle \frac{\partial P_n}{\partial t_1}, hP_n \right\rangle = \]

\[ = \left( p_n^2 + R_{n+1} + R_n \right) e^{\varphi_n} - 2R_n e^{\varphi_n}, \] (4.2.11)
i.e.

\[ \phi_n = R_{n+1} - R_n = e^{\phi_{n+1} - \phi_n} - e^{\phi_n - \phi_{n-1}}. \]

(4.2.12)

This equation can be rewritten in the Hirota bilinear form leading to the notion of \( \tau \)-function, key object in the theory of integrable hierarchies. In fact, let us introduce this by the formula

\[ e^{\phi(t)} = \frac{\tau_{n+1}(t)}{\tau_n(t)}. \]

(4.2.13)

Then, one can rewrite (4.2.12) in the Hirota bilinear form as

\[ \tau_n(t) \frac{\partial^2}{\partial t_1^2} \tau_n(t) - \left( \frac{\partial \tau_n(t)}{\partial t_1} \right)^2 = \tau_{n+1}(t)\tau_{n-1}(t). \]

(4.2.14)

Indeed, the main sense of the notion of \( \tau \)-function is the statement of existance the unique function depending on infinite set of times and satisfying the infinite hierarchy of equations. What we demonstrated was the only first equation of whole hierarchy, see more exhausting consideration in [44, 41].

Hereafter, we will not address to any equations and discuss only \( \tau \)-functions. What we are going to show for the next two sections is that any matrix model partition function is a \( \tau \)-function. Indeed, using (4.2.13), (4.2.5) and putting \( \tau_0 = 1 \),

\[ \tau_0 = 1, \]

(4.2.15)

one can obtain for one matrix model:

\[ \mathcal{Z}_n = \frac{\tau_n}{\tau_0} = \tau_n. \]

(4.2.16)

### 4.2.2 Determinant representation

Certainly, to demonstrate that the partition function is a \( \tau \)-function one should not check all infinite hierarchy of equations. During our consideration in the next sections, we obtain some effective and convenient representations for \( \tau \)-functions. Now let me just prove that the \( \tau \)-function of the Hermitian one matrix model can be presented in the specific determinant form which defines, actually, Toda chain \( \tau \)-function.

Let us rewrite the orthogonality condition (4.2.3) in the ”matrix form”. Namely, we introduce matrix \( A \) with matrix elements \( a_{mn} \) determined in (4.2.4), so-called moment matrix \( C \) whose matrix elements are determined by

\[ C_{ij} = \int dh h^{i+j-2} e^{-V(h)} \]

(4.2.17)

and the diagonal matrix \( J \) whose diagonal entries are just \( e^{\phi_n} \). Then one can rewrite (4.2.3) as matrix relation

\[ ACA^T = J \]

(4.2.18)

as matrix relation (4.2.3) means transposed matrix), and taking the determinant of both sides of the relation, one obtain

\[ \text{This equation is nothing but Riemann-Hilbert (factorization) problem - see [44, 41] for details.} \]
\[ Z_n = \det C_{ij}, \quad (4.2.18) \]

the conditions
\[ \partial C_*(t) \equiv \partial_k C_*(t) = \partial^k C_*(t) \equiv \partial^k C_*(t), \quad (4.2.19) \]

\[ C_{ij} = C_{i+j} \quad (4.2.20) \]

and
\[ C_n = \partial^{n-2} C_{11} \equiv \partial^{n-2} C \quad (4.2.21) \]

being satisfied (as it is evident from the manifest form of the moment matrix (4.2.17)). Thus, one can write finally
\[ Z_n = \det \partial^{i+j} C. \quad (4.2.22) \]

Note that the conditions (4.2.19) and (4.2.21) is to be satisfied for any (Toda) KP hierarchy, while (4.2.20) determines Toda chain hierarchy \[45\]. We will return to the question later.

### 4.2.3 Multi-matrix models

In this section we discuss the integrability in multi-matrix models considering at the beginning the standard multi-matrix model considered in \[46\]. Namely, its partition function is given by the matrix integral\[16\]

\[ Z^{(p)}_n \sim \int \prod_{i=1}^p [dH_i] \exp \left\{ -\text{Tr} \left[ \sum_{i=1}^p V_i(H_i) + \sum_{i=1}^{p-1} H_i H_{i+1} \right] \right\}. \quad (4.2.23) \]

This integral again can be integrated over angular variables, this time using Itzykson-Zuber formula \[17\]

\[ \int [dU]_{n \times n} e^{\text{tr}XULU^\dagger} \sim \frac{\det_{ij} e^{h_i h_j}}{\Delta(L) \Delta(X)} \quad (4.2.24) \]

and one obtains

\[ Z^{(p)}_n \sim \int \prod_{i=1}^n d h_i^{(1)} d h_i^{(p)} \Delta(h_i^{(1)}) \Delta(h_i^{(p)}) A(h_i^{(1)}, h_i^{(p)}) \equiv \]

\[ \equiv \int \prod_{i=1}^p \prod_{l=1}^n d h_i^{(l)} \Delta(h_i^{(1)}) \Delta(h_i^{(p)}) \exp \left\{ -\sum_{i,l} V_l(h_i^{(l)}) - \sum_{l=1}^{p-1} \sum_{i} h_i^{(l)} h_i^{(l+1)} \right\}. \quad (4.2.25) \]

\[16\text{For the sake of brevity, we omit all proportionality factors, which can be found, say, in } \[39\].
To apply the orthogonal polynomials in this case, one needs two sets of polynomials $Q_n(x)$ and $T_m(y)$, which satisfy the orthogonality relation:

$$\int dx dy P_n(x)Q_m(y)A(x,y) = \delta_{m,n} h_m(t_k, \bar{t}_k|c_k), \quad (4.2.26)$$

where times $t_k$ and $\bar{t}_k$ parameterize potentials $V_1$ and $V_p$ accordingly, while coefficients $c_k$ parameterize other potentials. Then, in complete analogy with one matrix model case, one can obtain the determinant representation:

$$Z_n = \det_{n \times n} C^{(p)}_{ij} \quad (4.2.27)$$

where

$$C^{(p)}_{ij} = \int dx dy x^{i-1} y^{j-1} A(x,y), \quad C^{(p)}_{11} \equiv C^{(p)} \quad (4.2.28)$$

and

$$Z_n = \det_{ij} \partial^{i-1} \bar{\partial}^{j-1} C^{(p)} \quad (4.2.29)$$

as the entries in the determinant satisfy the conditions analogous to (4.2.19) and (4.2.21), which imply that the partition function (4.2.27) is the $\tau$-function of general Toda lattice hierarchy [41, 42, 45]. This property is understood already from the fact that the partition function (4.2.29) depends essentially on two different sets of times (which correspond to positive and negative times in Toda lattice hierarchy), with the other times (potentials) in (4.2.23) just parameterizing the $\tau$-function, i.e. the point of the Grassmannian. Certainly, it would be more natural to have all times at the equal footing. In fact, we considered up to now only one possible generalization to multi-matrix models. But there is one more, from some viewpoints more attractive, extension. This is so-called "Conformal Multi-Matrix Models" (CMMM) [48, 49], which will be discussed in the next subsection. These CMMM can be described by multi-component hierarchies of $SL(p+1)$ AKNS type and depend on many sets of times.

For the sake of simplicity, we write down here only two matrix model case. Then, the model is given by the partition function (in eigenvalue variables)

$$Z_{n_1, n_2}^{\text{CMMM,2}} \sim \int \left[ \prod_{l=1}^{2} \prod_{i=1}^{n_l} dh^{(l)}_{i_l} \right]^{2} \prod_{i=1}^{2} \Delta^2(h^{(l)}_i) \prod_{i,j} (h^{(1)}_i - h^{(2)}_j) \exp \left\{ - \sum_{i,l} V_i(h^{(l)}_i) \right\}. \quad (4.2.30)$$

We will discuss later why this quite specific partition function is especially interesting and discuss its integrable structure in more details. Now it is just an example of another multi-matrix model which has a rather simple determinant representation.
where

\[ C(t) = \int dz \exp[-V_1(z)], \quad \tilde{C}(\tilde{t}) = \int dz \exp[-V_2(z)]. \]  

(4.2.32)

This representation gives rise to the \( \tau \)-function of (generalized) \( SL(3) \) AKNS system \[50, 48\]. From this viewpoint, one matrix model corresponds to usual \( SL(2) \) AKNS hierarchy, which is equivalent to Toda chain \[50\]. Certainly, analogous equivalence is absent for higher \( SL(N) \) hierarchies.

4.3 Virasoro constraints

4.3.1 Ward identities in matrix models

In the previous subsection we briefly described the integrable properties of the discrete matrix models. In this subsection we pay attention to equally important and general property, constraint algebra which is imposed on matrix models. The role of these constraint algebra is to fix the concrete \( \tau \)-function. In fact, the integrable hierarchy does not fix the \( \tau \)-function uniquely. Moreover, there is a huge space of the solution to the equations of hierarchy, which are parameterized by (subspaces) of infinite dimensional Grassmannian. It turns out that constraint algebra imposed on \( \tau \)-function fixes it usually uniquely. It means that in order to describe matrix model partition function in invariant way, one should determine what kind of \( \tau \)-function is the corresponding partition function (what hierarchy it is described) and what constraint algebra it satisfies. Again, let us start with the simplest example of Hermitian one matrix model.

We demonstrate now that the partition function (4.2.1) satisfies Virasoro algebra constraints (more precisely, its Borel subalgebra). Indeed, these are nothing but Ward identities of the model (4.2.1) \[51, 52, 53, 54\]. To see this, let us shift the variable \( H \) by small quantity \( \epsilon H \). Then, as it is nothing but the change of variables, the integral (4.2.1) will be unchanged. From the other hand, this shift results into changing of potential \( \sum t_k H^k \rightarrow \sum (H + \epsilon H^{q+1})^k = \sum_{k=1}^{q} t_k H^k + \sum_{k>q} (t_k + (k-q)t_{k-q}) H^k \). This means that these shifts are given by the action of operators \( L_q^c \equiv \sum k t_k \frac{\partial}{\partial t_k} \) producing Virasoro algebra.

Now we should also take into account the changing of the measure under the transformation above. The Jacobian of variation of Haar measure on Hermitian matrices is equal to

\[ \det \frac{\partial (H + \epsilon H^{q+1})}{\partial t_k} \sim 1 + \epsilon \text{Tr} \frac{\partial H^{q+1}}{\partial t_k} = 1 + \sum_{k=0}^{q} \text{Tr} H^k \text{Tr} H^{q-k}. \]

As any degree \( k \) of matrix \( H \) can be obtained by differentiating of the exponential of potential with respect to \( t_k \), one can
reproduce the changing of measure by the operators $\sum_{k=0}^{q} \frac{\partial^2}{\partial t_k \partial t_{q-k}}$. Note that according to the definition [4.2.1]

$$\frac{\partial}{\partial t_0} Z_n = n Z_n.$$ 

Therefore, the final result of the shift is:

$$Z_n(t) = Z_n(t) \big|_{\text{with shifted } H} = (1 + L_q) Z_n(t),$$  \hspace{1cm} (4.3.1)

i.e.

$$L_q Z_n = 0, \quad q \geq -1,$$  \hspace{1cm} (4.3.2)

where operators

$$L_q = \sum_{k=0}^{\infty} k t_k \frac{\partial}{\partial t_{k+q}} + \sum_{k=0}^{q} \frac{\partial^2}{\partial t_k \partial t_{q-k}}$$  \hspace{1cm} (4.3.3)

satisfy the commutation relations of Virasoro algebra

$$[L_n, L_m] = (n - m) L_{n+m}.$$  \hspace{1cm} (4.3.4)

As we consider only Borel subalgebra of whole Virasoro algebra, the question of the central charge is out of our consideration.

Thus, we proved that the partition function of Hermitian one matrix model satisfies Virasoro algebra constraints. Unfortunately, it is not so simple to find out an analogous symmetry in multi-matrix models [4.2.23]. So, we are going to suggest some more general way to construct matrix models possessing given symmetry, which allows one to write down multi-matrix models with $W$-symmetry, and, as a by-product, explains why conformal algebra arises in matrix models.

### 4.3.2 Conformal matrix models

In fact, now we just demonstrate that the partition function of (4.2.1) can be rewritten in terms of correlators in gaussian $c = 1$ conformal field theory and the Virasoro generators (4.3.3) actually have the well-known form of the Virasoro operators in the theory of one free scalar field. Indeed, let us consider holomorphic components of the scalar field

$$\phi(z) = \hat{Q} + \hat{P} \log z + \sum_{k \neq 0} \frac{J_{-k}}{k} z^{-k}$$

$$[J_n, J_m] = n \delta_{n+m,0}, \quad [\hat{Q}, \hat{P}] = 1$$  \hspace{1cm} (4.3.5)

and define the vacuum states

$$J_k |0\rangle = 0, \quad \langle n|J_{-k} = 0, \quad k > 0$$

$$\hat{P}|0\rangle = 0, \quad \langle N|\hat{P} = n \langle n|.$$  \hspace{1cm} (4.3.6)
"Half" of the stress-tensor components

\[ T(z) = \frac{1}{2} [\partial \phi(z)]^2 = \sum T_q z^{-q-2}, \quad T_q = \frac{1}{2} \sum_{k>0} J_{-k} J_{k+q} + \frac{1}{2} \sum_{a+b=q, a,b \geq 0} J_a J_b, \quad (4.3.7) \]

obviously vanishes the \( SL(2) \)-invariant vacuum

\[ T_q |0\rangle = 0, \quad q \geq -1. \quad (4.3.8) \]

Then, we define the Hamiltonian by

\[ H(t) = \frac{1}{\sqrt{2}} \sum_{k>0} t_k J_k = \oint_{C_0} V(z) j(z) \]

\[ V(z) = \sum_{k>0} t_k z^k, \quad j(z) = \frac{1}{\sqrt{2}} \partial \phi(z). \quad (4.3.9) \]

Now let us solve the Virasoro constraints (4.3.2) in the general form. Put differently, we derive the general partition function possessing the Virasoro symmetry (4.3.2). One can easily construct a "conformal field theory" solution of (4.3.2) in two steps. The basic "transformation"

\[ L_q \langle n | e^{H(t)} ... = (n) | e^{H(t)} T_q ... \quad (4.3.10) \]

can be checked explicitly and show how the Virasoro generators (4.3.3) transforms to those in gaussian model. As an immediate consequence, any correlator of the form

\[ \langle n | e^{H(t)} G|0\rangle \quad (4.3.11) \]

\( (n \) counts the number of zero modes, "included" in \( G \) – that is the role of the size of matrix in (4.2.1)) gives a solution to (4.3.2) provided by

\[ [T_q, G] = 0, \quad q \geq -1. \quad (4.3.12) \]

The conformal solution to (4.3.12) (and therefore to (4.3.1)) immediately comes from the basic properties of 2d conformal algebra. Indeed, any solution to

\[ [T(z), G] = 0 \quad (4.3.13) \]

is a solution to (4.3.12), and it is well-known that the solution to (4.3.13) is (by definition of the chiral algebra) a function of screening charges in the free scalar field theory given by

\[ Q_\pm = \oint J_\pm = \oint e^{\pm \sqrt{2} \phi}. \quad (4.3.14) \]

With a selection rule on zero mode it gives

\[ G = \exp Q_+ \rightarrow \frac{1}{n!} Q_+^n \quad (4.3.15) \]

\[ 17 \text{For the sake of brevity, we omit the sign of normal ordering in the evident places, say, in the expression for } T \text{ and } W \text{ in terms of free fields.} \]

\[ 18 \text{Of course, the general case might be } G \sim Q_+^{n+m} Q_-^m \text{ but the special prescription for integration contours, proposed in } [55], \text{ implies that the dependence of } m \text{ can be irrelevant and one can just put } m = 0. \text{ In fact, the problem with the choice of integration contour is a rather subtle point and discussed in } [55]. \]
In this case the solution
\[
\mathcal{Z}_n(t) = \langle n| e^{H(t)} \exp Q_+|0 \rangle
\]  
(4.3.16)
after computation of the free theory correlator, analytic continuation of the integration contour gives the result
\[
\mathcal{Z}_n = (n!)^{-1} \int \prod_{i=1}^{n} dz_i \exp \left( - \sum t_k z_i^k \right) \Delta_n^2(z) =
\]  
(4.3.17)
in the form of the matrix integral (4.2.1). This is the simplest example of CMMM mentioned above which already explains the name.

Thus, we derive Virasoro invariant matrix model of the general type. Certainly, this procedure is immediately generalized to other symmetries.

### 4.3.3 * Ward identities in multi-matrix models*

Now we are going to derive multi-matrix models invariant with respect to $W$-algebra and discuss what is the symmetry of the standard multi-matrix models (4.2.23).

Let us consider the immediate generalization of proposed procedure. Now we can use powerful tools of 2d conformal field theories, where it is well known how to generalize almost all the steps of above construction: first, instead of looking for a solution to Virasoro constraints one can impose extended Virasoro or $W$-constraints on the partition function. In such case one would get Hamiltonians in terms of multi-scalar field theory, and the second step is generalized directly using screening charges for $W$-algebras. The general scheme of solving discrete $W$-constraints looks as follows:

(i) Consider Hamiltonian as a linear combination of the Cartan currents of a level one Kac-Moody algebra $\mathcal{G}$
\[
H(t^{(1)}, \ldots, t^{(\text{rank} \mathcal{G})}) = \sum_{\lambda, k > 0} t_k^{(\lambda)} \mu_\lambda J_k,
\]  
(4.3.18)
where \( \{\mu_i\} \) are basis vectors in Cartan hyperplane, which for $SL(p+1)$ case are chosen to satisfy
\[
\mu_i \cdot \mu_j = \delta_{ij} - \frac{1}{p+1}; \quad \sum_{j=1}^{p+1} \mu_j = 0.
\]

(ii) The action of differential operators $W_i^{(a)}$ with respect to times \( \{t_k^{(\lambda)}\} \) can be now defined from the relation
\[
W_i^{(a)} \langle n| e^{H(t)} \rangle \ldots = \langle n| e^{H(t)} W_i^{(a)} \rangle \ldots , \quad a = 2, \ldots, p+1; \quad i \geq 1 - a,
\]  
(4.3.19)
where
\[
W_i^{(a)} = \oint z^{a+i-1} W^{(a)}(z)
\]
\[
W^{(a)}(z) = \sum_{\lambda} [\mu_\lambda \partial \phi(z)]^a + \ldots
\]  
(4.3.20)
are spin-\(a\) W-generators of \(W_{p+1}\)-algebra written in terms of rank \(G\)-component scalar fields [30].

(iii) The conformal solution to the discrete \(W\)-constraints arises in the form

\[
Z_n^{\{\text{CMMM},p+1\}}(t) = \langle n| e^{H(t)} G\{Q^{(\alpha)}\}|0\rangle
\]

(4.3.21)

where \(G\) is again an exponential function of screenings of level one Kac-Moody algebra (see 17 and references therein)

\[
Q^{(\alpha)} = \oint J^{(\alpha)} = \oint e^{\alpha \phi}
\]

(4.3.22)

\(\{\alpha\}\) being roots of finite-dimensional simply laced Lie algebra \(G\). The correlator (4.3.21) is still a free-field correlator and the computation gives it again in a multiple integral form

\[
Z_n^{\{\text{CMMM},p+1\}}(t) \sim \int \prod_{\alpha} \prod_{i=1}^{n_\alpha} dz_i^{(\alpha)} \exp \left(- \sum_{\lambda,k>0} \gamma_k^{(\lambda)} (\mu_\lambda \alpha)(z_i^{(\alpha)})^k \right)
\]

\[
\times \prod_{(\alpha,\beta)} \prod_{i=1}^{n_\alpha} \prod_{j=1}^{n_\beta} (z_i^{(\alpha)} - z_j^{(\beta)})^{\alpha \beta}
\]

(4.3.23)

The only difference with the one-matrix case (4.3.17)) is that the expressions (4.3.23) have rather complicated representation in terms of multi-matrix integrals, the following objects will necessarily appear

\[
\prod_{i=1}^{n_\alpha} \prod_{j=1}^{n_\beta} (z_i^{(\alpha)} - z_j^{(\beta)})^{\alpha \beta} \longrightarrow \left[\det\{H^{(\alpha)} \otimes I - I \otimes H^{(\beta)}\}\right]^{\alpha \beta},
\]

(4.3.24)

However, this is still a model with a chain of matrices and with closest neighbour interactions only (in the case of \(SL(p+1)\)).

Actually, it can be shown that CMMM, defined by (4.3.21) as a solution to the \(W\)-constraints has a very rich integrable structure and possesses a natural continuum limit [48, 49]. To pay for these advantages one should accept a slightly less elegant matrix integral with the entries like (4.3.24).

The first non-trivial example is the \(p = 3\) solution to \(W_3\)-algebra: an alternative to the conventional 2-matrix model. In this case one has six screening charges \(Q^{(\pm \alpha_i)}\) \((i = 1, 2, 3)\) which commute with

\[
W^{(2)}(z) = T(z) = \frac{1}{2} [\partial \phi(z)]^2
\]

(4.3.25)

and

\[
W^{(3)}(z) = \sum_{\lambda=1}^{3} (\mu_\lambda \partial \phi(z))^3,
\]

(4.3.26)

where \(\mu_\lambda\) are vectors of one of the fundamental representations (3 or \(\bar{3}\)) of \(SL(3)\).

The particular form of integral representation (4.3.23) depends on particular screening insertions to the correlator (4.3.21). We will concentrate on the solutions which have no denominators. One of the reasons of such choice is that these solutions possess the most simple integrable structure, though the other ones can still be analyzed in the same manner.
The simplest solutions which have no denominators correspond to specific correlators
\[ Z_n^{(CMMM,p+1)}(t) = \langle n|e^{H(t)} \prod_i \exp Q_{\alpha_i}|0 \rangle \] (4.3.27)
when we take \( \alpha_i \) to be “neighbour” (not simple!) roots: \( (\alpha_i, \alpha_j) = 1 \). In the case of \( SL(3) \) this corresponds, say, to insertions of only \( Q_{\alpha_1} \) and \( Q_{\alpha_2} \) and gives formula (4.2.30). Thus, we establish that the partition function (4.2.30) satisfies \( W^{(3)} \)-constraint algebra, the generators of which can be read off from the formula (4.3.19):

\[ W^{(3)}_q = 3 \sum_{k,l>0} kt_k t_l \frac{\partial}{\partial t_{q+k+l}} + 3 \sum_{k>0} kt_k \sum_{a+b=k+q} \frac{\partial^2}{\partial t_a \partial t_b} + \sum_{a+b+c=q} \frac{\partial^3}{\partial t_a \partial t_b \partial t_c}. \] (4.3.28)

There are many possible generalizations of the above scheme, some of them, including "supersymmetric matrix models", can be found in [48].

Thus, we derived matrix models possessing \( W \)-symmetry. The question we are going to discuss to complete this subsection is what symmetry does possess the standard multi-matrix model (4.2.23). Let us consider the simplest case of two-matrix model (4.2.23), one of the potentials, say, \( V_2 \) being truncated to be just \( cH_1^3 \). Now we repeat the procedure of the derivation of the Virasoro constraints (4.3.1). Let us do the following infinitesimal change of integration variables

\[ \delta H_2 = H_1^q, \quad q \geq 0 \]
\[ \delta H_1 = 3cH_1^q (V_1'(H_1) - H_2) + "quantum \ corrections". \] (4.3.29)

This variation of variables induces the variation of potential \( V_1 \):

\[ \delta V_1 = -3c(V')^2H_1^q + H_1^{q+1} + "quantum \ corrections". \] (4.3.30)

The first term in this expression gives rise to the "classical" part of the symmetry generators (analogous to \( L_q^3 \) above) and the second one produces the derivative \( \partial/\partial t_{q+1} \) in the Ward identities.

While the \( H_2 \)-component of the variation (4.3.29) does not change the integration measure \( dH_1 dH_2 \), this is not true for \( H_1 \)-component. The corresponding Jacobian is responsible for the “quantum” contributions to (4.3.23) and (4.3.30).

Now we just write down the result for the constraints imposed on the partition function (4.2.30):

\[ \left( -\frac{\partial}{\partial t_{q+1}} + 3c\tilde{W}^{(3)}_{q-2}\{t\} \right) \left. Z_n^{(2)}(t) \right|_{V_2=cH_1^3} = 0. \] (4.3.31)

where generators

\[ \tilde{W}^{(3)}_{q-2} = \sum_{k,l>0} kt_k t_l \frac{\partial}{\partial t_{q+k+l-2}} + \sum_{k>0} kt_k \sum_{a+b=k+q-2} \frac{\partial^2}{\partial t_a \partial t_b} + \sum_{a+b+c=q-2} \frac{\partial^3}{\partial t_a \partial t_b \partial t_c} + \sum_{a+b+c=q-2} \frac{\partial^3}{\partial t_a \partial t_b \partial t_c} + \frac{q(q-1)}{2} \frac{\partial}{\partial t_q}. \] (4.3.32)
produces some new constraint algebra.

This $\tilde{W}^{(3)}$ algebra arises in many different contexts and can be trivially generalized to any spin. The generators of this algebra have the same "classical part" as the standard $W^{(3)}$ algebra (4.3.28), but "quantum parts" are different. The set of $\tilde{W}$-constraints is closed (as well Ward identities (4.3.31)) and is an algebra quadratic in generators, similar to the standard $W$-algebra. Let us stress that considering the polynomial potential $V_2$ of a degree $K$, one obtains the Ward identity, analogous (4.3.31), but for $\tilde{W}^{(K)}$-algebra. In particular, the generic potential leads to linear $\tilde{W}^{(\infty)}$-algebra. The detailed description of the properties of $\tilde{W}$-algebra and constraints in two-matrix model (4.2.23) can be found in [58, 39].

Thus, we realized that the simple multi-matrix models (4.2.23) satisfy unusual $\tilde{W}$-constraints, while more complicated models (4.2.30) satisfy the standard $W$-constraints. In the continuum limit this difference disappears, and both these types of models have the same (double scaling) continuum limit. Unfortunately, we do not know the efficient way of taking the continuum limit in the multi-matrix models (4.2.23) (certainly, I mean by this that we have no complete understanding of the renormalization of all operators and constraint algebra, i.e. of simultaneously all correlation functions). But CMMM turns out to be more suitable for the continuum limit procedure. This is one of their advantages. We are going to discuss these questions in the next subsection.

4.4 Continuum limit

4.4.1 Continuum limit of Toda chain equations

In this subsection we discuss the bridge between above considered discrete matrix models and these are in the double scaling limit. Let us stress that, due to singularity of this procedure, it is very difficult to take this continuum limit directly in the partition function (in particular, there should be some (infinite) renormalizations), but easier to do in some equations, where the procedure is less singular (to be honest, there is a very simple way of doing the double scaling limit immediately in the partition function, but it requires some additional information and will be discussed in the next section). Indeed, we already mentioned that the crucial properties of integrability and imposed constraint algebra are preserved in the continuum limit. It implies that the most natural choice of equations to take the continuum limit would be either constraint algebra, or integrable equations. Now we work out both these ways for the simplest example of the Hermitian one matrix model (4.2.1).

Let us start with integrable equations of Toda chain hierarchy. Our further discussion is sufficiently general and does not refer to the concrete matrix model $\tau$-function, i.e. the continuum limit conserves all the space of solutions to integrable equations, and the structure of integrability. From the other hand, we demonstrate below that it also conserves constraint algebra. It is specific of the double scaling limit, which can be invariantly determined just by its properties of not disturbing all crucial matrix model properties.

Thus, now we build the map from Toda hierarchy to KdV hierarchy by limiting procedure, and then we do analogous map from discrete Virasoro algebra (4.3.3) to that in the double scaling limit. Certainly, we consider here only the first equation of the hierarchy (4.2.12). For the sake of simplicity, we consider here only the case of even potential in (4.2.1). It implies that all odd times $t_{2k+1} = 0$ and all $\varphi_n$ are supposed to be independent of them. Thus,
\[ p_n = \frac{\partial^2 \nu}{\partial t^2} = 0 \] and the first non-trivial equation (which is really the second equation of Toda chain hierarchy) has a form \((t \equiv t_2)\):

\[ \frac{\partial R_n}{\partial t} = -R_n(R_{n+1} - R_{n-1}). \quad (4.4.1) \]

This hierarchy is the reduction of Toda chain hierarchy and called Volterra hierarchy.

Indeed, we could guess that the continuum limit of Toda chain hierarchy should be KdV hierarchy by looking at the Lax operator \([4.2.8]\). Indeed, the main idea of the continuum limit in matrix models is to change the discrete variable \(n\) by the continuous time variable (which will be the first time in the continuum hierarchy). Therefore, choosing \(n = x \epsilon\) we can get in the continuum limit:

\[ L^{cont} = 2 - \epsilon^2[\partial_x^2 + (\partial \varphi(x))]. \quad (4.4.2) \]

One can recognize in this operator just Lax operator of KdV hierarchy, the eigenvalues of Toda Lax operator being distributed around zero in the continuum limit (see \([41]\)). Still for the equations the continuum limit procedure is not so simple. Indeed, doing naively, we get from \((4.4.1)\):

\[ \frac{\partial R(x)}{\partial t} = -R(x)(R(x + \epsilon) - R(x - \epsilon)) \underset{\epsilon \to 0}{\rightarrow} -R(x')R'(x'). \quad (4.4.3) \]

provided by the replace \(x = \epsilon x'\). This formula is called dispersionless KdV equation (or Bateman, or Hopf, or Kholohv-Zabolotskaya equation). This equation is too simple and corresponds to just naive "large \(n\) limit" of matrix models \([59]\). To obtain KdV equation, one should take care of non-linear term. The procedure allowing one to preserve such terms is called "double scaling limit" and is as follows.

Imagine, that in continuum limit \(R_n\) tends to a constant \(R_0\), and the function \(r(x)\) arises only as scaling approximation to this constant: \(R(x) = R_0(1 + \epsilon^s r(x))\). Then the leading term at the r.h.s. of \((4.4.3)\) is \(\epsilon RR'(x) = -2\epsilon^sr(x)(1 + \mathcal{O}(\epsilon^2, \epsilon^s))\), and instead of \((4.4.3)\) we would get:

\[ \frac{\partial r}{\partial t} = -2\epsilon^s R_0r'(x)(1 + \mathcal{O}(\epsilon^2, \epsilon^s)). \quad (4.4.4) \]

Then, by a simple change of variables\(^{19}\)

\[ T_1 = x - 2\epsilon R_0 t, \quad T_3 = \epsilon^3 R_0 t \quad (4.4.6) \]

it can be transformed into

\[ \frac{\partial r}{\partial T_3} = \epsilon^{-2}\mathcal{O}(\epsilon^2, \epsilon^s), \quad (4.4.7) \]

\(^{19}\) This change of variables is implied by the relation:

\[ \frac{\partial}{\partial t} + 2\epsilon^s R_0 \frac{\partial}{\partial x} = \left( \frac{\partial}{\partial t} + 2\epsilon^s R_0 \frac{\partial}{\partial x} \right) \frac{\partial}{\partial t} + \left( \frac{\partial}{\partial x} + 2\epsilon^s R_0 \frac{\partial}{\partial x} \right) \frac{\partial}{\partial x} = \frac{\partial}{\partial t}. \quad (4.4.5) \]
and terms at the r.h.s. also should be taken into account. Then we get:

\[ \frac{\partial r(x)}{\partial t} = -2\epsilon R_0 \left( 1 + \epsilon^s r(x) \right) \left( r'(x) + \frac{1}{6} \epsilon^2 r'''(x) + \mathcal{O}(\epsilon^4) \right) = \]

\[ = -2\epsilon R_0 \left( r'(x) + \frac{1}{6} \epsilon^2 r'''(x) + \epsilon^s r'(x) + \epsilon^2 \mathcal{O}(\epsilon^2, \epsilon^s) \right) \]

and, after the change of variables (4.4.6),

\[ \frac{\partial r(T_1)}{\partial T_3} = -\frac{1}{3} r'''(T_1) - 2\epsilon^{-2} rr'(T_1) + \mathcal{O}(\epsilon^2, \epsilon^s). \]  

(4.4.9)

One can see that the choice \( s = 2 \) is distinguished (a critical point) and at this point we get nothing but KdV equation:

\[ \frac{\partial r}{\partial T_3} = -\frac{1}{3} \frac{\partial^3 r}{\partial T_3^3} - 2r \frac{\partial r}{\partial T_1}, \]  

(4.4.10)

Thus, we obtained KdV equation. Certainly, one can get all KdV hierarchy in the same way. Moreover, one can investigate the continuum limit of the Toda (not Volterra) hierarchy, producing two equivalent KdV hierarchies (which correspond to flows in odd and even Toda times) - see [60, 61]. It reflects in some problems in the continuum limit of Virasoro algebra (see below).

Thus, the lesson, which we can extract from this continuum limit procedure is that, to get the proper limit, one should do linear transformation of times and renormalization of the partition function (or \( R_n \)). Now we use these lessons in order to study the continuum limit of Virasoro algebra.

### 4.4.2 * The continuum limit of the Virasoro algebra*

The procedure of taking the continuum limit of Virasoro algebra we demonstrate now is a bit tedious and too technical, as, in contrast to our previous consideration allows one to work with whole hierarchy simultaneously. Let me just describe a set of rules which leads to the correct answer, all details and hints can be found in [62].

It has been suggested in [32] that the square root of the partition function of the continuum limit of one-matrix model is subjected to the Virasoro constraints

\[ \mathcal{L}_q^\text{cont} \sqrt{Z}^{ds} = 0, \quad q \geq -1, \]  

(4.4.11)

where

\[ \mathcal{L}_q^\text{cont} = \sum_{k=0} \left( k + \frac{1}{2} \right) T_{2k+1} \frac{\partial}{\partial T_{2(k+q)+1}} + G \sum_{0 \leq k \leq q-1} \frac{\partial^2}{\partial T_{2k+1} \partial T_{2(q-k-1)+1}} + \]

\[ + \frac{\delta_{0,q}}{16} + \frac{\delta_{-1,q} T_1^2}{(16G)} \]  

(4.4.12)
are modes of the stress tensor

\[ T(z) = \frac{1}{2} \partial \Phi^2(z) - \frac{1}{16z^2} = \sum_{q} \frac{L_q}{z^{q+2}}. \]  

\[(4.4.13)\]

Indeed, we demonstrate that these equations which reflect the \( W^{(2)} \)-invariance of the partition function of the continuum model can be deduced from analogous constraints in Hermitian one-matrix model by taking the double-scaling continuum limit. The procedure is as follows.

In order to obtain the relation between \( W \)-invariance of the discrete and continuum models one has to consider a reduction of model \((4.2.1)\) to the pure even potential \( t_{2k+1} = 0 \). Let us denote by the \( \tau_{n}^{\text{red}} \) the partition function of the reduced matrix model

\[ \tau_{n}^{\text{red}}(t_{2k}) = \int [dH] \exp \text{Tr} \sum_{k=0} t_{2k} H^{2k} \]  

\[(4.4.14)\]

and consider the following change of the time variables

\[ g_{m} = \sum_{k \geq m} (-)^{k-m} \frac{\Gamma(k+\frac{3}{2}) a^{-k-\frac{1}{2}}}{(k-m)! \Gamma(m+\frac{3}{2})} T_{2k+1}^{*}, \]  

\[(4.4.15)\]

where \( g_{m} \equiv m t_{2m} \) and this expression can be used also for the zero discrete time \( g_{0} \equiv n \) that plays the role of the dimension of matrices in the one-matrix model. Derivatives with respect to \( t_{2k} \) transform as

\[ \frac{\partial}{\partial t_{2k}} = \sum_{m=0}^{k-1} \frac{\Gamma(k+\frac{1}{2}) a^{m+\frac{1}{2}}}{(k-m-1)! \Gamma(m+\frac{3}{2})} \frac{\partial}{\partial T_{2m+1}}, \]  

\[(4.4.16)\]

where the auxiliary continuum times \( T_{2m+1}^{*} \) are connected with “true” Kazakov continuum times \( T_{2m+1} \) via

\[ T_{2k+1} = T_{2k+1}^{*} + a \frac{k}{k+1/2} T_{2(k-1)+1}, \]  

\[(4.4.17)\]

and coincide with \( T_{2m+1} \) in the double-scaling limit when \( a \to 0 \).

Let us rescale the partition function of the reduced one-matrix model by exponent of quadratic form of the auxiliary times \( T_{2m+1}^{*} \)

\[ \tilde{\tau} = \exp \left( -\frac{1}{2} \sum_{m,k \geq 0} A_{mk} \tilde{T}_{2m+1} \tilde{T}_{2k+1} \right) \tau_{n}^{\text{red}} \]  

\[(4.4.18)\]

with

\[ A_{km} = \frac{\Gamma(k+\frac{3}{2}) \Gamma(m+\frac{3}{2})}{2 \Gamma^{2} \left( \frac{1}{2} \right)} \frac{(-)^{k+m} a^{-k-m-1}}{k! m! (k+m+1)(k+m+2)}. \]  

\[(4.4.19)\]

Then a direct though tedious calculation \([62]\) demonstrates that the relation
\[
\frac{\tilde{L}_q \tilde{\tau}}{\tau} = a^{-q} \sum_{p=0}^{q+1} C_{q+1}^p (-1)^{q+1-p} \frac{L_{2p}^{\text{red} \tau \text{red}}}{\tau^{\text{red}}},
\]

is valid, where

\[
L_{2q}^{\text{red}} \equiv \sum_{k=0} \frac{k t_{2k}}{2} \frac{\partial}{\partial t_{2(k+q)}} + \sum_{0 \leq k \leq q} \frac{\partial^2}{\partial t_{2k} \partial t_{2(q-k)}}
\]

and

\[
\tilde{\mathcal{L}}_{-1} = \sum_{k \geq 1} \left( k + \frac{1}{2} \right) T_{2k+1} \frac{\partial}{\partial T_{2(k-1)+1}} + \frac{T^2}{16G},
\]

\[
\tilde{\mathcal{L}}_0 = \sum_{k \geq 0} \left( k + \frac{1}{2} \right) T_{2k+1} \frac{\partial}{\partial T_{2k+1}},
\]

\[
\tilde{\mathcal{L}}_q = \sum_{k \geq 0} \left( k + \frac{1}{2} \right) T_{2k+1} \frac{\partial}{\partial T_{2(k+q)+1}}
\]

\[+ \sum_{0 \leq k \leq q-1} \frac{\partial}{\partial T_{2k+1}} \frac{\partial}{\partial T_{2(q-k-1)+1}} - \frac{(-)^q}{16a^q}, \quad n \geq 1.
\]

Here \(C_p^q = \frac{q!}{p!(q-p)!}\) are binomial coefficients.

These Virasoro generators differ from the Virasoro generators \(\text{(4.4.12)}\) \cite{32,19} by terms which are singular in the limit \(a \to 0\). At the same time \(L_{2p}^{\text{red} \tau \text{red}}\) at the r.h.s. of \(\text{(4.4.20)}\) do not need to vanish, since

\[
0 = L_{2p} \tau \bigg|_{t_{2k+1}=0} = L_{2p}^{\text{red} \tau \text{red}} \bigg|_{t_{2k+1}=0} + \frac{\partial^2 \tau}{\partial t_{2i+1} \partial t_{2(p-i-1)+1}} \bigg|_{t_{2k+1}=0}.
\]

It was shown in \cite{62} that these two origins of difference between \(\text{(4.4.12)}\) and \(\text{(4.4.22)}\) actually cancel each other, provided eq.\(\text{(4.4.20)}\) is rewritten in terms of the square root \(\sqrt{\tilde{\tau}}\) rather than \(\tilde{\tau}\) itself:

\[
\frac{L_q^{\text{cont} \sqrt{\tilde{\tau}}}}{\sqrt{\tilde{\tau}}} = a^{-q} \sum_{p=0}^{q+1} C_{q+1}^p (-1)^{q+1-p} \frac{L_{2p}^{\tau \text{red}}}{\tau} \left|_{t_{2k+1}=0} \right. (1 + O(a)).
\]

The proof of this cancelation, as given in \cite{62}, is not too much simple and makes use of integrable equations for \(\tau\).

### 4.4.3 * Invariant formulation of the limiting procedure

After we demonstrated the continuum limit procedure for the Hermitian one matrix case, the next question to be addressed to is how to generalize this limiting procedure to multi-matrix case. Indeed, in the case of conventional multi-matrix models this procedure is still unknown. But for CMMM it can be trivially done, what is one of the advantages of CMMM we mentioned above. To do this, let us demonstrate a more economical way to define the change
of the time-variables $t \rightarrow T$ (also discussed in [62]), implied by the scalar field formalism. The Kazakov change of the time variables (4.4.15, 4.4.16) can be deduced from the following prescription. Let us consider the free scalar field with periodic boundary conditions

$$\partial \varphi(u) = \sum_{k \geq 0} g_k u^{2k-1} + \sum_{k \geq 1} \frac{\partial}{\partial t_{2k}} u^{-2k-1},$$

(4.4.25)

and analogous scalar field with antiperiodic boundary conditions:

$$\partial \Phi(z) = \sum_{k \geq 0} \left( \left( k + \frac{1}{2} \right) T_{2k+1} z^{k-\frac{3}{2}} + \frac{\partial}{\partial \tilde{T}_{2k+1}} z^{-k-\frac{3}{2}} \right).$$

(4.4.26)

Then the equation

$$\frac{1}{\tau} \partial \Phi(z) \tilde{\tau} = a \frac{1}{\tau_{\text{red}}} \partial \varphi(u) \tau_{\text{red}}, \quad u^2 = 1 + az \quad (4.4.27)$$

generates the correct transformation rules (4.4.15), (4.4.16) and gives rise to the expression for $A_{km}$ (4.4.19). Taking the square of the both sides of the identity (4.4.27),

$$\frac{1}{\tilde{\tau}} T(z) \tilde{\tau} = \frac{1}{\tau_{\text{red}}} T(u) \tau_{\text{red}},$$

(4.4.28)

one can obtain after simple calculations that the same relation (4.4.20) is valid.

Now it is evident how to generalize this procedure to general multi-matrix models using the formalism of scalar fields with $\mathbb{Z}_p$-twisted boundary conditions. It can be found in details in the references [48, 49].

4.5 Fermionic representations and forced hierarchies

4.5.1 Determinant representations of $\tau$-functions

Beginning with this subsection we are going to spend more time discussing integrability in matrix models, which was practically out of our consideration before. In particular, now we are going to discuss the notion of $\tau$-function, main definitions and the determinant representations, which we were talking about in the second subsection. We mainly follow the references [63, 44, 45].

We restrict ourselves to consideration of ordinary KP and Toda-lattice $\tau$-functions, which can be constructed as correlators in the theory of free fermions $\psi, \tilde{\psi}$ ($b, c$-system of spin 1/2). The basic quantity is the ratio of fermionic correlators,

$$\tau_N(t, \bar{t} \mid G) \equiv \frac{\langle N \mid e^H G e^H \mid N \rangle}{\langle N \mid G \mid N \rangle}, \quad (4.5.1)$$

in the theory of free 2-dimensional fermionic fields $\psi(z), \psi^*(z)$ with the action $\int \psi^* \tilde{\partial} \psi$ (see the Appendix A1 for the definitions and notations). In the definition (4.5.1)
\[ H = \sum_{k>0} t_k J_k; \quad \bar{H} = \sum_{k>0} \bar{t}_k J_{-k}, \quad (4.5.2) \]

and the currents are defined to be

\[ J(z) = \psi^*(z)\psi(z); \quad (4.5.3) \]

where

\[
\psi(z) = \sum_z \psi_n z^n \ dz^{1/2}, \\
\psi^*(z) = \sum_z \psi^*_n z^{-n-1} \ dz^{1/2}. \quad (4.5.4)
\]

This expression for $\tau$-function is rather famous. Let us point out that this $\tau$-function is the $\tau$-function of the most general one-component hierarchy which is Toda lattice hierarchy. Putting all negative times to be zero as well as discrete index $n$, one obtains usual KP hierarchy. An element

\[ G = : \exp\{\sum_{m,n} \mathcal{G}_{mn} \psi^*_m \psi_n\} : \quad (4.5.5) \]

is an element of the group $GL(\infty)$ realized in the infinite dimensional Grassmannian. The normal ordering should be understood here with respect to any $\langle k \rangle$ vacuum,\textsuperscript{20} where vacuum states are defined by conditions

\[ \psi_m |k\rangle = 0 \quad m < k \quad , \quad \psi^*_m |k\rangle = 0 \quad m \geq k. \quad (4.5.6) \]

Any particular solution of the hierarchy depends only on the choice of the element $G$ (or, equivalently it can be uniquely described by the matrix $\mathcal{G}_{km}$). Thus, we can reformulate the sense of constraint algebra in matrix models: it just fixes the element of the Grassmannian, i.e. one can reformulate constraint algebra in terms of equations on the element of the Grassmannian. This approach was advocated in \[64\].

From commutation relations for the fermionic modes, one can conclude that any element in the form (4.5.5) rotates the fermionic modes as follows

\[ G \psi_k G^{-1} = \psi_j R_{jk} \quad , \quad G \psi^*_k G^{-1} = \psi^*_j R^{-1}_{kj}, \quad (4.5.7) \]

where the matrix $R_{jk}$ can be expressed through $\mathcal{G}_{jk}$ (see \[65\]). We will see below that the general $\tau$-function (4.5.1) can be expressed in the determinant form with explicit dependence of $R_{jk}$. In order to calculate $\tau$-function we need some more notations. Using commutation relations for the fermionic modes, one can obtain the evolution of $\psi(z)$ and $\psi^*(z)$ in times \{\(t_k\), \{\(\bar{t}_k\}\} in the form

\[
\psi(z, t) \equiv e^{H(t)} \psi(z) e^{-H(t)} = e^{\xi(t,z)} \psi(z), \quad (4.5.8) \\
\psi^*(z, t) \equiv e^{H(t)} \psi^*(z) e^{-H(t)} = e^{-\xi(t,z)} \psi^*(z); \quad (4.5.9)
\]

\textsuperscript{20}Indeed, it is rather crucial point and we will return to possible choices of the normal orderings.
\[ \psi(z, \bar{t}) \equiv e^{H(t)} \psi(z) e^{-H(t)} = e^{\xi(t,z^{-1})} \psi(z), \quad (4.5.10) \]
\[ \psi^*(z, \bar{t}) \equiv e^{H(t)} \psi^*(z) e^{-H(t)} = e^{-\xi(t,z^{-1})} \psi(z), \quad (4.5.11) \]

where

\[ \xi(t,z) = \sum_{k=1}^{\infty} t_k z^k. \quad (4.5.12) \]

Now let us define Shur polynomials by the formula:

\[ \exp\{ \sum_{k>0} t_k x^k \} \equiv \sum_{k>0} P_k(t_k) x^k. \quad (4.5.13) \]

Using this definition, from eqs. (4.5.8)-(4.5.11) one can easily obtain the evolution of the fermionic modes:

\[ \psi_k(t) \equiv e^{H(t)} \psi_k e^{-H(t)} = \sum_{m=0}^{\infty} \psi_{k,m} P_m(t), \quad (4.5.14) \]
\[ \psi_k^*(t) \equiv e^{H(t)} \psi_k^* e^{H(t)} = \sum_{m=0}^{\infty} \psi_{k,m}^* P_m(-t); \quad (4.5.15) \]
\[ \psi_k(\bar{t}) \equiv e^{\bar{H}(\bar{t})} \psi_k e^{-\bar{H}(\bar{t})} = \sum_{m=0}^{\infty} \psi_{k,m} P_m(\bar{t}); \quad (4.5.16) \]
\[ \psi_k^*(\bar{t}) \equiv e^{\bar{H}(\bar{t})} \psi_k^* e^{-\bar{H}(\bar{t})} = \sum_{m=0}^{\infty} \psi_{k,m}^* P_m(-\bar{t}). \quad (4.5.17) \]

It is useful to introduce the totally occupied state \( | - \infty \rangle \) which satisfies the requirements

\[ \psi_i^* | - \infty \rangle = 0, \quad i \in \mathbb{Z}. \quad (4.5.18) \]

Then any shifted vacuum can be generated from this state as follows:

\[ | n \rangle = \psi_{n-1} \psi_{n-2} \ldots | - \infty \rangle. \quad (4.5.19) \]

Note that the action of an any element \( G \) of the Clifford group (and, as consequence, the action of \( e^{-H(\bar{t})} \) on \( | - \infty \rangle \) is very simple: \( G | - \infty \rangle \sim | - \infty \rangle \), so using (4.5.13) and (4.5.16) one can obtain from eq.(4.5.1):

\[ \tau_n(t, \bar{t}) = \langle \langle -\infty | \psi_{n-2}^*(-t) \psi_{n-1}(-t) \psi_1(-\bar{t}) G \psi_{n-1}(-\bar{t}) \psi_{n-2}(-\bar{t}) \ldots | - \infty \rangle \sim \]
\[ \sim \det[\langle -\infty | \psi_i^*(-t) G \psi_j(-\bar{t}) G^{-1} | - \infty \rangle] |_{i,j \leq n-1}. \quad (4.5.20) \]

Using (4.5.17) it is easy to see that

\[ G \psi_j(-\bar{t}) G^{-1} = \sum_{m,k} P_m(-\bar{t}) \psi_k R_{k,j+m}. \quad (4.5.21) \]

and the “explicit” solution of the two-dimensional Toda lattice has the determinant representation:
where

\[ \tau_n(t, \bar{t}) \sim \det H_{i+n,j+n}(t, \bar{t})|_{i,j<0} , \]  

(4.5.22)

and

\[ H_{ij}(t, \bar{t}) = \sum_{k,m} R_{km} P_{k-i}(t) P_{m-j}(\bar{t}) . \]  

(4.5.23)

The ordinary solutions to KP hierarchy \[63\] correspond to the case when the whole evolution depends only of positive times \( \{t_k\} \); negative times \( \{\bar{t}_k\} \) serve as parameters which parameterize the family of points in Grassmannian and can be absorbed by re-definition of the matrix \( R_{km} \). Then \( \tau \)-function of (modified) KP hierarchy has the form

\[ \tau_n(t) = \langle n| e^{H(t)} G(\bar{t}) |n\rangle \sim \det \left[ \sum_k R_{k,j+n}(\bar{t}) P_{k-i-n}(t) \right] |_{i,j<0} , \]  

(4.5.24)

where \( G(\bar{t}) \equiv Ge^{-\bar{H}(\bar{t})} \) and

\[ R_{kj}(y) \equiv \sum_m R_{km} P_{m-j}(-y) . \]  

(4.5.25)

4.5.2 Determinant representations for matrix model hierarchies

Now let us briefly discuss the formulas of the second subsection within the developed framework. Indeed, we already have the general determinant representation (4.5.22) and would have to reduce it to concrete subhierarchies. The life is, however, not so simple as all the determinants we know from matrix models are \textit{finite} determinants, in contrast to (4.5.22). This is an essential problem. Indeed, this is due to the specific boundary condition like (4.2.15) imposed on matrix model \( \tau \)-functions. The hierarchies with this boundary condition are called \textit{forced hierarchy} \[66\] and studied in details in \[42, 45\]. The main result of this study is that these hierarchies are corresponded by the singular elements of the Grassmannian, which are induced by the fermion operators (4.5.4), but with only positive (or only negative) modes taken into account. It implies the consideration in (4.5.3) of \textit{semi}-infinite (instead of infinite) matrix of the group \( GL(\infty) \). In its turn, it leads to the \textit{finite} (instead of semi-infinite) determinant (4.5.22).

Indeed, there is the problem of continuing the boundary condition (4.2.14) to negative \( n \). Say, the Toda chain integrable equations can not fix it (it can be trivially observed from Hirota equation - see (4.2.14)). However, switching on the negative (Toda) times allows one to fix this continuation. And if one wishes to have the Toda chain as a proper reduction from Toda lattice hierarchy, it is necessary to put

\[ \tau_{-n} = 0 \]  

(4.5.26)

(the case of \( \tau_{-n} = \tau_n \), for example, corresponds to CKP system - see [12]). To do this, one should multiply the element of the Grassmannian by the projector onto positive \( n \):

\[ P_+ = \exp[\sum_{i<0} \psi_i \psi_i^*] ; \]  

(4.5.27)
with the properties:

\[ P_+ |n \rangle = \theta(n)|n \rangle, \]
\[ P_+ \psi^*_{-k} = \psi_{-k} P_+ = 0 \ , \ k > 0 \ ; \]
\[ [P_+, \psi_k] = [P_+, \psi^*_k] = 0 \ , \ k \geq 0 \ . \]

The point, however, is that after this is done, the defining properties of the original (non-forced) system are the same as those of the forced one under the condition (4.5.26) (i.e. after throwing away the half of fermionic modes and inserting the projector). Therefore, we discuss now the Toda lattice hierarchy and its Toda chain reduction with no restricting ourselves to the forced case.

Let us note that the entries (4.5.23) of the determinant (4.5.22) are not arbitrary ones, but are conditioned to satisfy the properties:

\[ \partial H_{ij}/\partial t_p = H_{i,j-p}, \ j > p > 0, \]  
\[ \partial H_{ij}/\partial \bar{t}_p = -H_{i-p,j}, \ i > p > 0, \]

which are the automatic consequences of the corresponding property of Shur polynomials:

\[ \partial P_k/\partial t_p = P_{k-p} \]

following immediately from their definition. From the properties (4.5.29) and (4.5.30) one obtains directly our conditions (4.2.19) and (4.2.21) and their analogs for the multi-matrix case (4.2.27) (in the latter case, one should change the sign in the last potential in (4.2.23), which is fixed by the sign in the Hamiltonians in \( \bar{t}_k \)). Now let us discuss the condition (4.2.20), i.e. the reduction of general Toda lattice hierarchy to Toda chain. It can be easily formulated as condition to the element \( G \) of the Grassmannian:

\[ [J_k + \bar{J}_k, G] = 0, \]  
\[ [\Lambda + \Lambda^{-1}, R] = 0 . \]

which is equivalent to constraint

\[ [\Lambda + \Lambda^{-1}, R] = 0 . \]

where \( \Lambda \) is shift matrix \( \Lambda_{ij} \equiv \delta_{i,j-1} \). In this case,

\[ Ge^{-\bar{t}_k \bar{J}_k} = e^{-\bar{t}_k \bar{J}_k} e^{-\bar{t}_k \bar{J}_k} G \ e^{\bar{t}_k \bar{J}_k} \]

and \( \tau \)-function depends (up to the trivial factor) only on times \( \{ t_k - \bar{t}_k \} \):

\[ \tau_n(t, \bar{t}) = e^{\sum k \bar{t}_k t_k} \langle n|e^{H(t-\bar{t})}G|n \rangle . \]

The reduction (4.5.33) has an important solution\(^{21}\)

\(^{21}\)Generally the solutions \( R_{nk} = R_{n+k} \) and \( R_{nk} = R_{n-k} \) are different, but for forced hierarchy, when \( \tau \)-function is the determinant of finite matrix, these two solutions are equivalent due to possibility to reflect matrix with respect to vertical axis without changing the determinant.
In this case the matrix $\hat{H}_{ij}$ defined by eq. (4.5.23) evidently satisfies the relations $H_{ij} = H_{i+j}$ and

$$(\partial_k + \partial_{\bar{k}})H_{i+j} = 0 \text{ for any } k < n-i, k < n-j$$  (4.5.37)

due to the properties (4.5.29) and (4.5.30). The property (4.5.37) certainly does not imply that the corresponding $\tau$-function depends only on difference of times because of restriction of values of $k$, but it restores correct dependence of times with taking into account of exponential in (4.5.35).

Thus, we have proved that, for the Toda chain hierarchy, the matrix $H_{ij}$ satisfies the condition

$$[H, \Lambda + \Lambda^{-1}] = 0,$$  (4.5.38)

which leads to $\tau$-function of Toda chain hierarchy (properly rescaled by exponential of bilinear form of times) which depends only on the difference of positive and negative times $t_p - \bar{t}_p$, but not on their sum (one can consider this as defining property of Toda chain hierarchy). Let us remark that only one possible solution to constraint (4.5.38) is matrix $H_{ij} = H_{i+j}$, but it should not be generally independent of the difference of times $t_p - \bar{t}_p$. Certainly, one can through out negative times as the final answer for complete objects like $\tau$-function should be really independent of this difference.

Anyway, we obtained for the $\tau$-function of Toda chain hierarchy the determinant representation with necessary conditions (4.2.19)-(4.2.21), i.e. the matrix $H_{ij}$ just corresponds to moment matrix $C_{ij}$ and we finally reproduce the result (4.2.18).

### 4.5.3 *Integrable structure of CMMM - multi-component hierarchy*

Now we are going to say some words on integrable structure which arises in CMMM. In fact, we already wrote down the determinant representation (4.2.31) for CMMM. Now there are two things which can be established. The first one is to obtain the integrable equations. It can be done immediately from the equation (4.2.31) and leads to clear generalization of the Hirota equation (1.2.14):

$$\frac{1}{\tau_{N,M}} \frac{\partial^2}{\partial t_1^2} \tau_{N,M} - \left( \frac{\partial \tau_{N,M}}{\partial t_1} \right) \left( \frac{\partial \tau_{N,M}}{\partial t_1} \right) = \frac{\tau_{N+1,M-1} \tau_{N-1,M+1}}{\tau_{N,M}^2}. $$  (4.5.39)

The second point is just the properties of the $\tau$-function corresponding to CMMM within the framework described above. The statement is that the partition function of CMMM is a $\tau$-function of a multi-component ($(p+1)$-component) Kadomtsev-Petviashvili hierarchy which obeys the constraint

$$\sum_{k=1}^{p+1} \partial/\partial t_n^{(k)} \tau^{(p+1)}(\{t\}) = 0, \quad n = 1, 2, \ldots,$$  (4.5.40)
where \( p \) is the number of matrices in CMMM. Let us demonstrate how it works in our usual simplest example of \( p = 1 \). It should be 2-component hierarchy.

The \( \tau \)-function of 2-component KP hierarchy is by definition the correlator

\[
\tau_{N,M}^{(2)}(t, \bar{t}) = \langle N, M| e^{H(t, \bar{t})} G|N + M, 0\rangle \tag{4.5.41}
\]

where

\[
H(t, \bar{t}) = \sum_{k>0} (t_k J_k^{(1)} + \bar{t}_k J_k^{(2)}) \tag{4.5.42}
\]

\[
J^{(i)}(z) = \sum J_k^{(i)} z^{-k-1} = :\psi^{(i)}(z)\psi^{(i)*}(z): \tag{4.5.43}
\]

\[
\psi^{(i)}(z)\psi^{(j)*}(z') = \frac{\delta_{ij}}{z - z'} + \ldots . \tag{4.5.44}
\]

All these formulas are trivially generalized to \((p + 1)\)-component case by introducing \( p + 1 \) fermions.

Now we are going to demonstrate that (4.3.16) is equivalent to (4.5.41) for certain \( G \) for which (4.5.41) depends only on the differences \( t_k - \bar{t}_k \). To do this we have to make use of the free-fermion representation of \( SL(2)_{k=1} \) Kac-Moody algebra:

\[
J_0 = \frac{1}{2}(\psi^{(1)}\psi^{(1)*} - \psi^{(2)}\psi^{(2)*}) = \frac{1}{2}(J^{(1)} - J^{(2)})
\]

\[
J_+ = \psi^{(2)}\psi^{(1)*} \quad J_- = \psi^{(1)}\psi^{(2)*} \tag{4.5.45}
\]

Now let us take \( G \) to be the following exponent of a quadratic form (indeed, we know that this is the correct fermionic form for exponential of screening operators)

\[
G \equiv : \exp \left( \int \psi^{(2)}\psi^{(1)*} : \right) : \tag{4.5.46}
\]

Now we bosonize the fermions

\[
\psi^{(i)*} = e^{\phi_i}, \quad \psi^{(i)} = e^{-\phi_i}
\]

\[
J^{(1)} = \partial\phi_1, \quad J^{(2)} = \partial\phi_2 \tag{4.5.47}
\]

and compute the correlator

\[
\tau_N^{(2)}(t, \bar{t}) \equiv \tau_{N,-N}^{(2)}(t, \bar{t}) = \frac{1}{N!} \langle N, -N| \exp \left( \sum_{k>0} (t_k J_k^{(1)} + \bar{t}_k J_k^{(2)}) \right) \left( \int :\psi^{(2)}\psi^{(1)*}: \right)^N |0\rangle = \frac{1}{N!} \langle N, -N| \exp \left( \int [V_1(z)J^{(1)}(z) + V_2(z)J^{(2)}(z)] \right) \left( \int :\exp(\phi_1 - \phi_2) : \right)^N |0\rangle,
\]

where \( V_{(1,2)} \equiv \sum_k (t_k, \bar{t}_k) z^k \). Introducing the linear combinations \( \sqrt{2}\phi = \phi_1 - \phi_2, \sqrt{2}\tilde{\phi} = \phi_1 + \phi_2 \) we finally get
\[ \tau_N^{(2)}(t, \bar{t}) = \frac{1}{N!} \langle \exp \left( \frac{1}{\sqrt{2}} \oint [V_1(z) + V_2(z)]\partial \bar{\phi}(z) \right) \rangle \times \langle N| \exp \left( \frac{1}{\sqrt{2}} \oint [V_1(z) - V_2(z)]\partial \phi(z) \right) \left( \int : \exp \sqrt{2}\phi : \right)^N |0\rangle = \tau_N^{(2)}(t - \bar{t}) \]  

(4.5.48)

since the first correlator is in fact independent of \( t \) and \( \bar{t} \). Thus, we proved that the \( \tau \)-function indeed depends only on the difference of two sets of times \( \{t_k - \bar{t}_k\} \). So, we obtained here a particular case of the 2-component KP hierarchy and

(i) requiring the elements of Grassmannian to be of the form (4.5.46) we actually performed a reduction to the 1-component case\(^{22}\).

(ii) we proved in (4.5.48) that this is an AKNS-type reduction for the \( \tau \)-function (4.5.47)\(^{[44, 50]}\).

The above simple example already contains all the basic features of at least all the \( A_p \) cases. Indeed, the reduction (4.5.48) is nothing but \( SL(2) \)-reduction of a generic \( GL(2) \) situation. In other words, the diagonal \( U(1) \) \( GL(2) \)-current \( \bar{J} = \frac{1}{2}(J^{(1)} + J^{(2)}) = \frac{1}{\sqrt{2}} \partial \bar{\phi} \) decouples. This is an invariant statement which can be easily generalized to higher \( p \) cases. Indeed, the screening operators for the \( p \)-matrix case are the integrals of \( SL(p + 1)_1 \) Kac-Moody currents (not \( GL(p + 1) \) ones) and, thus, the \( \tau \)-function (4.3.21)-(4.3.22) does not depend on \( \{\Sigma_{i=1}^{p+1} x^{(i)}_k\} \), i.e. we obtain the constraint (4.5.40).

As to the Toda-like representation of CMM, in the simplest \( SL(2) \)-case the result should be equivalent to the Toda chain hierarchy. In the fermionic language this connection is established by the following substitution in the element of the Grassmannian

\[ \psi^{(1)}(z) \rightarrow \psi(z), \quad \psi^{(2)}(z) \rightarrow \psi\left(\frac{1}{z}\right) \]  

(4.5.49)

and the same for \( \psi^* \)'s. This is a reflection of the fact that Toda system is described by the two marked points (say, 0 and \( \infty \)) and corresponds to two glued discs, so it can be also described by two different fermions. This might lead to a general phenomenon, when any multi-component solution to CMM is actually related to (some reduction) of a multi-component Toda lattice.

5 GKM approach to matrix models

5.1 What are the matrix models in the double scaling limit

In this section we are going to discuss the models which arise \textit{after} the continuum (double scaling) limit is taken. Indeed, now we are merely formulate a set of conditions, which uniquely defines the corresponding partition function.

\(^{22}\) Note that the idea to preserve both indices in (4.5.41) leads immediately to additional insertions either of \( \psi^{(1)*} \) or \( \psi^{(2)} \) to the right vacuum \( |0\rangle \), so that it is no longer annihilated at least by the \( T^{-1} \) Virasoro generator, or in other words this ruins the string equation. Thus only the particular reduction (4.5.46) seems to be consistent with string equation. This choice of indices just corresponds to that considered originally in \[37\].
These conditions are of two kinds. The first one describes the integrable properties. That is, the square root of the partition function of the Hermitian $K$-matrix model in the continuum limit is a $\tau$-function of the $K$-reduced KP hierarchy. Say, one-matrix model corresponds to KdV hierarchy, two-matrix model – to Boussinesq hierarchy and etc.

The second condition imposed on the partition function is so-called string equation. The role of this equation is to fix the $\tau$-function which corresponds to the matrix model. Put it differently, this equation gives the point of the Grassmannian which corresponds to the matrix model. The equation is

$$\frac{\partial}{\partial x} \mathcal{L}_{-1}^{(K)} \tau = 0,$$

where

$$\mathcal{L}_{-1}^{(K)} = \frac{1}{K} \sum_{n>1} n T_n \frac{\partial}{\partial T_{n-K}} + \frac{1}{2K} \sum_{a+b=K \atop a,b>0} a T_a b T_b + \frac{\partial}{\partial T_{1}}.$$  \hspace{1cm} (5.1.2)

Indeed, hereafter we call string equation a bit more general equation

$$\mathcal{L}_{-1}^{(K)} \tau = 0,$$

which is also correct.

It can be demonstrated [67] that above mentioned conditions imply that the $\tau$-function satisfies a whole set of constraints. More concretely, $K$-reduced $\tau$-function satisfies the set of $W$-algebra constraints:

$$W_n^{(i)} \tau^{(K)} = 0, \quad i = 2, \ldots, K; \quad n \geq -i + 1.$$  \hspace{1cm} (5.1.4)

$W$-operators here are the standard generators of Fateev-Lukyanov-Zamolodchikov $W$-algebra [56] expressed in the manifest terms of times, i.e. with creation and annihilation operators being $T_n$ and $\partial/\partial T_n$ respectively and acting on the space of function of times [32].

There is also inverse conjecture [32] which asserts that the constraints (5.1.3) have a unique solution, i.e. they select out the $\tau$-function of $K$-reduced KP hierarchy, which satisfies the string equation (5.1.3). This conjecture is still not proved, though there are strong arguments in favor of its correctness [68].

Let us note now that it is not so simple to prove the properties described above, and, in part, they are merely an invariant definition of the double scaling continuum limit. Some general argument have been used to derive these properties [19, 32], but there are still only two rigid calculations.

The first one describes the transition from discrete to continuum $W$- and Virasoro constraints and has been done for $K = 2$ case of the standard matrix models [32] and for the general case of the conformal multi-matrix models [48, 49]. Another calculation makes use of the manifest matrix integral representation of the partition function [43], which will be described in the next subsections.
5.2 Generalized Kontsevich Model (GKM)

Thus far we have described the set of constraints which uniquely defines the partition function of matrix models in the double scaling limit. Now we are going to present the manifest solution of these constraints. Indeed, for the simplest case of one-matrix model it was derived from Witten topological theory (see section 3) by Kontsevich [69]. We propose proper generalization of his result, demonstrating that our solution satisfies all necessary constraints (in contrast to the approach [69], where there has been manifestly calculated the integral (3.1)).

For the lack of space, we will briefly describe only the structures and appealing properties of the partition function, which we call Generalized Kontsevich’s Model (GKM) [70]. Unfortunately, we can not explain here the deep connections of GKM with the theory of integrable systems and develop possible generalizations, valuable in different applications. In particular, the connection with topological $N = 2$ Landau-Ginzburg theories is out of the scope of the present review. We refer to the conclusion for a short review of the existing literature.

The partition function of the GKM is defined by the following integral over $N \times N$ Hermitean matrix:

$$Z_N^{\{V\}}[M] \equiv \frac{\int e^{U(M,Y)}dY}{\int e^{-U_2(M,Y)}dY},$$

(5.2.1)

where

$$U(M,Y) = Tr[V(M + Y) - V(M) - V'(M)Y]$$

(5.2.2)

and

$$U_2(M,Y) = \lim_{\epsilon \to 0} \frac{1}{\epsilon^2} U(M, \epsilon Y),$$

(5.2.3)

is an $Y^2$-term in $U$. $M$ is also a Hermitean $N \times N$ matrix with eigenvalues $\{\mu_i\}$, $V(\mu)$ is arbitrary analytic function.

All crucial properties of GKM follows from its definition. That is, its partition function may be considered as a functional of two different variables: potential $V(\mu)$ and the infinite-dimensional Hermitean matrix $M$ with eigenvalues $\{\mu_i\}$. Partition function $Z_N^{\{V\}}$ is an $N$-independent KP $\tau$-function, considered as a function of time-variables $T_n = \frac{1}{n} Tr M^{-n}$ and the point of Grassmannian is specified by the choice of potential. The $N$-dependence enters only through the argument $M$ : we return to finite-dimensional matrices if only $N$ eigenvalues of $M$ are finite. In this sense the “continuum” limit of $N \to \infty$ is smooth.

The GKM is associated with a subset of Grassmannian, specified by additional $L_{-1}$-constraint. For particularly adjusted potentials $V(\mu) = const \cdot \mu^{K+1}$, the corresponding points in Grassmannian lies in the subvarieties, associated with $K$-reductions of KP-hierarchy, $Z^{\{V\}}$ becomes independent of all the time-variables $T_k$, and the $L_{-1}$-constraint implies the whole tower of $W_K$-algebra constraints on the reduced $\tau$-function. These properties are exactly the same as suggested for double scaling limit of the $K - 1$-matrix model, and in fact there is an identification

$$Z^{\{K\}}_{\infty} = \sqrt{\Gamma^{\{K-1\}}_{ds}}.$$  (5.2.4)
All this means that GKM provides an interpolation between double-scaling continuum limits of all multimatrix models and thus between all string models with $c \leq 1$. Moreover, this is a reasonable interpolation, because both integrable and “string-equation” structures are preserved. This is why we advertise GKM as a plausible (on-shell) prototype of a unified theory of 2d gravity.

In the next subsections we briefly comment all properties described here.

## 5.3 Main integrable properties of GKM

### 5.3.1 Integrable structure

After the shift of variables $X = Y + M$ and integration over angular components of $X$, $Z_N^{(V)}[M]$ acquires the form of

$$Z_N^{(V)}[M] = \frac{[\det \tilde{\Phi}_i(\mu_j)]}{\Delta(M)}, \quad (5.3.1)$$

where $\Delta(M) = \prod_{i<j} (\mu_i - \mu_j)$ is the Van-der-Monde determinant, and functions

$$\tilde{\Phi}_i(\mu) = [V''(\mu)]^{1/2} e^{V(\mu) - \mu V'(\mu)} \int e^{-V(x) + x V'(\mu)} x^i dx \quad (5.3.2)$$

The only assumption necessary for the derivation of (4) from (1) is the possibility to represent the potential $V(\mu)$ as a formal series in positive integer powers of $\mu$.

Formula (5.3.1) with arbitrary entries $\phi_i(\mu)$ having the asymptotics

$$\phi_i(\mu) \sim \mu^{i-1}(1 + O(\mu)) \quad (5.3.3)$$

is characteristic for generic KP $\tau$-function $\tau^G(T_n)$ in Miwa’s coordinates

$$T_n = \frac{1}{n} Tr M^{-n}, \quad n \geq 1 \quad (5.3.4)$$

and the point $G$ of Grassmannian is defined by potential $V$ through the set of basis vectors $\{\phi_i(\mu)\}$. The most immediately this set is connected with the Baker-Akhiezer (BA) function

$$\Psi_\pm(z|T_k) = e^{\sum t_k z^k / 2} e^{\pm \frac{z-n}{n}} \tau(T_n) \quad (5.3.5)$$

through the relation:

$$\Psi_+(\mu|T_k) \big|_{t_k=0, \ k \neq 1} = \sum_{l=0}^{\infty} x^l \frac{\phi_l(\mu)}{l!}. \quad (5.3.6)$$

Thus, we obtain

$$Z^{(V)}[M] = \tau^{(V)}(T_n). \quad (5.3.7)$$
The case of finite \( N \) in this formalism is distinguished by the condition that only \( N \) of the parameters \( \{ \mu_i \} \) are finite. In order to take the limit \( N \to \infty \) in the GKM (5.2.1) it is enough to bring all the \( \mu_i \)'s from infinity. In this sense this a smooth limit, in contrast to the singular conventional double-scaling limit, which one needs to take in ordinary (multi)matrix models.

Let us check that this limit is really smooth. To do this, one should change the number of Miwa variables \( N + 1 \to N \) bringing \( \mu_{N+1} \) to \( \infty \) in the determinant (5.3.1). Using the asymptotics (5.3.3), it is trivially to prove that it results in \( N \times N \) determinant of the same type. This means that \( N \) enters only as the number of Miwa variables, but not as an explicit parameter.

5.3.2 Reductions

The integral \( F^{\{V\}}[\Lambda] \), \( \Lambda \equiv \mathcal{V}'(M) \), in the numerator of (5.2.1) satisfies the Ward identity

\[
Tr \left\{ \epsilon(\Lambda) \left[ \mathcal{V}' \left( \frac{\partial}{\partial \Lambda'} \right) - \Lambda \right] \right\} F^{\{V\}}_N = 0 \tag{5.3.8}
\]

(as result of invariance under any shift of integration variable \( X \to X + \epsilon(M) \)). Let us denote the integral in (5.3.2) through \( F_i(\mathcal{V}'(\mu)) \). If potential \( \mathcal{V}(\mu) \) is restricted to be a polynomial of degree \( K + 1 \), this identity implies that the functions \( F_i(\lambda) \) obey additional relations:

\[
F_{m+Kn}(\lambda) = \lambda^n \cdot F_m(\lambda) + \sum_{i=1}^{m+Kn-1} s_i F_i(\lambda). \tag{5.3.9}
\]

Since the sum at the r.h.s. does not contribute to determinant (5.3.1), we can say that all the functions \( F_n \) are expressed through the first \( K \) functions \( F_1...F_K \) by multiplication by powers of \( \lambda = \mathcal{V}'(\mu) \). Such situation (when the basis vectors \( \phi_i \), defining the point of Grassmannian are proportional to the first \( K \) ones) corresponds to reduction of KP-hierarchy. This reduction depends on the form of \( \mathcal{V}'(\mu) \) and in the case of \( \mathcal{V}(\mu) = \mathcal{V}_K(\mu) = const \cdot \mu^{K+1} \) coincides with the well-known \( K \)-reduction of the KP-hierarchy (KdV as \( K = 2 \), Boussinesq as \( K = 3 \) etc.). Thus in such cases partition function of GKM becomes \( \tau^{\{K\}} \)-function of the corresponding hierarchy. Generic \( \tau^{\{K\}} \) possesses an important property: it is almost independent of all time-variables \( T_{nK} \). To be exact,

\[
\partial \log \tau^{\{K\}} / \partial T_{nK} = a_n = const. \tag{5.3.10}
\]

In variance with generic \( \tau^{\{K\}} \), the partition function \( Z^{\{K\}} \) of GKM obeys this condition with all \( a_n = 0 \) (see proof in [70]).

5.4 String equation

5.4.1 \( L_{-1} \)-constraint

The set of function \( \{ \tilde{\Phi}_i(\mu) \} \) in (4) is, however, not arbitrary. They are all expressed through a single function — potential \( \mathcal{V}(\mu) \), — and are in fact recurrently related:
\[ F_i(\lambda) = (\partial/\partial \lambda)^{i-1} F_1(\lambda). \] (5.4.11)

This relation is enough to prove that
\[
\frac{\partial}{\partial T_1} \log Z_{N}^{\{V\}} = -Tr M + Tr \frac{\partial}{\partial \Lambda tr} \log \det F_i(\lambda_j) \] (5.4.12)

whenever potential \( V(\mu) \) grows faster than \( \mu \) as \( \mu \to \infty \).

Thus, \( Z^{\{V\}} \) satisfies a simple identity:
\[
\frac{1}{Z^{\{V\}}} L_{-1}^{\{V\}} Z_N^{\{V\}} = \frac{\partial}{\partial T_1} \log Z_N^{\{V\}} + Tr M - Tr \frac{\partial}{\partial \Lambda tr} \log \det F_i(\lambda_j) = 0 \] (5.4.13)

where operator \( L_{-1}^{\{V\}} \) is defined to be
\[
L_{-1}^{\{V\}} = \sum_{n \geq 1} Tr [ \frac{1}{V''(M)} M^{n+1} ] \frac{\partial}{\partial T_n} + \frac{1}{2} \sum_{i,j} \frac{V''(\mu_i) - V''(\mu_j)}{\mu_i - \mu_j} + \frac{\partial}{\partial T_1} \] (5.4.14)

(the items with \( i = j \) are included into the sum). From eqs.\((5.4.13),(5.4.14)\) it follows that partition function of GKM satisfies the constraint
\[
L_{-1}^{\{V\}} \tau^{\{V\}} = 0. \] (5.4.15)

The reason why the operator \((5.4.14)\) is denoted by \( L_{-1}^{\{V\}} \) is clear from the following argument. If \( V = V_K \), the generic expression \((5.4.14)\) for the \( L_{-1} \)-operator turns into
\[
L_{-1}^{\{K\}} = \frac{1}{K} \sum_{n \geq K} nT_n \frac{\partial}{\partial T_n} - \frac{1}{2K} \sum_{a+b=K \atop a,b>0} aT_a bT_b \] (5.4.16)

The last item at the r.h.s. may be eliminated by the shift of time-variables:
\[
T_n \to \hat{T}_n^{\{K\}} = T_n + \frac{K}{n} \delta_{n,K+1}. \] (5.4.17)

Only expressed in terms of these \( \hat{T} \)'s the constraint \((5.4.14)\) acquires the form of
\[
L_{-1}^{\{K\}} \tau^{\{K\}} = \left\{ \frac{1}{K} \sum_{n \geq K \atop n \neq 0 \mod K} n\hat{T}_n \frac{\partial}{\partial \hat{T}_n} - \frac{1}{2K} \sum_{a+b=K \atop a,b>0} a\hat{T}_a b\hat{T}_b \right\} \tau^{\{K\}} = 0. \] (5.4.18)

coinciding with expression \((5.1.3)\).
5.4.2 Universal string equation

Generalization of the string equation to the case of arbitrary potential

\[ \frac{\partial}{\partial T_1} \sum_{\nu} \mathcal{L}^{(\nu)}_{-1} \tau^{(\nu)} \frac{1}{\tau^{(\nu)}} = 0 . \]  

(5.4.19)

may be transformed to the following form

\[ \sum_{n \geq -1} T_n \frac{\partial^2 \log \tau}{\partial T_1 \partial T_n} = u , \]  

(5.4.20)

where

\[ T_n \equiv Tr \left( M_n \mathcal{L}^{\nu}_{-1} \mathcal{L}^{\nu} \right) , \]  

(5.4.21)

\[ u \equiv \frac{\partial^2 \log \tau}{(\partial T_1)^2} , \quad \frac{\partial \log \tau}{\partial T_0} \equiv 0 , \quad \frac{\partial \log \tau}{\partial T_{-1}} \equiv T_1 . \]  

Using BA function (5.3.5), one can rewrite string equation (5.4.15) in the form of bilinear relation

\[ \sum_i \Psi_+ (\mu_i) \Psi_- (\mu_i) = u . \]  

(5.4.22)

5.4.3 \( \mathcal{W} \)-constraints

According to subsection 5.1 the constraint

\[ \mathcal{L}^{(K)}_{-1} \tau^{(K)} = 0 \]  

(5.4.23)

implies the entire tower of \( \mathcal{W} \)-constraints

\[ \mathcal{W}^{(K)}_{K_n} Z^{(K)} = 0, \quad k = 2, 3, ..., K; \quad n \geq 1 - k \]  

(5.4.24)

imposed on \( \tau^{(K)} \). Here \( \mathcal{W}^{(p)}_{K_n} \) is the \( n \)-th harmonics of the \( p \)-th generator of Zamolodchikov’s \( \mathcal{W}_K \)-algebra (the proper notation would be \( \mathcal{W}^{(p)}_n \), but it is a bit too complicated). There is a Virasoro Lie sub-algebra, generated by \( \mathcal{W}^{(2)}_{K_n} = \mathcal{L}^{(K)}_n \), and the particular \( \mathcal{L}^{(K)}_{-1} \) is just the operator (5.4.19). This is the true origin of our notation \( \mathcal{L}^{(\nu)}_{-1} \) in the generic situation (where the entire Virasoro subalgebra of \( W_\infty \) was not explicitly specified).

Besides being a corollary of (5.4.23), the constraints (5.4.24) can be directly deduced from the Ward identity (5.3.8). For the case of \( K = 2 \) (which is original Kontsevich’s model [89]) this derivation was given in [71] (see also [72, 73, 74] for alternative proofs). Unfortunately, for \( K \geq 3 \) the direct corollary of (5.3.8) is not just (5.4.24), but peculiar linear combinations of these constraints, e.g. for \( K = 3 \) they look like
\[ \mathcal{W}^{(3)}_{3n} Z^{(3)}_\infty = 0, \quad n \geq -2; \]
\[
\left\{ \sum_{k \geq 1} (3k - 1) \hat{T}_{3k-1} \mathcal{W}^{(2)}_{3k+3n} + \sum_{a+b=3n} \frac{\partial}{\partial T_{3a+2}} \mathcal{W}^{(2)}_{3b-3} \right\} Z^{(3)}_\infty = 0,
\]
\[
a, b \geq 0, \quad n \geq -2; \quad (5.4.25)
\]
\[
\left\{ \sum_{k \geq 1+\delta_{n+4,0}} (3k - 2) \hat{T}_{3k-2} \mathcal{W}^{(2)}_{3k+3n} + \sum_{a+b=3n} \frac{\partial}{\partial T_{3a+1}} \mathcal{W}^{(2)}_{3b-3} \right\} Z^{(3)}_\infty = 0,
\]
\[
a, b \geq 0, \quad n \geq -3.
\]

For identification of (5.4.25) with (5.4.24) one can argue that both sets of constraints possess unique, and thus coinciding, solutions. Really, only the case of \( K = 3 \) has been investigated in this case in the paper [75].

6 Conclusion

In the present review we have discussed in some details only Liouville (physical) gravity and discrete matrix models. We almost ignored topological theories, and described GKM approach very briefly, though they deserve separate reviews.

Indeed, this is the GKM approach which allows one to connect different branches and viewpoints. On one hand, for the particular case of the cubic potential (\( K = 2 \)) GKM integral can be immediately derived from the Witten topological theory [18]. On the other hand, it is possible to establish connection between GKM and \( N = 2 \) Landau-Ginzburg topological gravity [6, 27]. As topological theories of both these types are appropriate models of 2d gravity, the connections of these with GKM are of great importance.

Let us note that there are many generalizations of the original GKM integral (5.2.1). Say, one can include "negative" and "zero" time variables to obtain Toda lattice \( \tau \)-functions [45]. It allows to incorporate in GKM treatment Hermitian one-matrix model and to take the double scaling limit immediately in the partition function [45, 77].

Another possible generalization is to consider polynomials of \( \frac{1}{X} \) as the GKM potential. This case has much to do with unitary matrix integrals [68].

An absolutely different issue is the existence of different phases in GKM. The new one can be defined by another Miwa transformation of times:
\[
\hat{T}_k \equiv \frac{1}{k} \text{Tr} M^k.
\]
\[
(6.0.1)
\]
In fact, it rather determines a different asymptotics of GKM integral than a new phase. It acquires the sense of phase only when GKM integral is applied to the description of QCD [78].

A separate issue is the connection between GKM integral and character expansion of \( \tau \)-functions [29]. This point is rather important in applications both to unitary matrix models
Another limitation of this review is that we restricted ourselves to Hermitian only matrix models and only to one-cut solution in the continuum limit. In fact, the properties, say, of unitary matrix model \[80, 78, 81, 41\] is understood well too. Its integrable properties are described at discrete level \[82\] as well as in the continuum limit \[83, 84, 79\]. Proper Virasoro algebra was also investigated, again both at the discrete \[85, 51\] and continuum \[72\] levels. Different reductions were discussed \[85\] and GKM framework proposed \[72, 68\].

Nevertheless, these models have no clear physical implications for 2d gravity, but rather in 2d QCD. This model is also not included into physically essential general framework, which would be multi-component hierarchies. These have (presumably) to include at equal footing DKP hierarchy and some other reductions having to do with different 2d gravity theories, including those with \(c = 1\). These models should be associated with non-trivial critical points like multi-cut solutions \[86\].

Thus, unitary model, on one hand, requires a separate review and, on the other hand, is out of our main line. This is why we avoided to discuss them here.

This discussion might explain that it is impossible to put all the material on the small room of this review. The above-mentioned gaps will be filled in the second part of the review, devoted exclusively to matrix models in external fields, which have much to do both with 2d gravity (see section 5 above) and 2d gauge theories.

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