On the accuracy of the approximation of the complex exponent by the first terms of its Taylor expansion with applications

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Abstract

A new bound for the remainder term in the Taylor expansion of the complex exponent $e^{ix}$, $x \in \mathbb{R}$, is proved yielding precise moment-type estimates of the accuracy of the approximation of the characteristic function (the Fourier–Stieltjes transform) of a probability distribution by the first terms of its Taylor expansion. Namely, for an arbitrary random variable $X$ with the characteristic function $f(t) = \mathbb{E}e^{itX}$, $t \in \mathbb{R}$, and $\mathbb{E}X = 0$, $\mathbb{E}X^2 = 1$, $\mathbb{E}|X|^3 = b \geq 1$, the symbol $\mathbb{E}$ standing for the mathematical expectation, the precise bounds

\[ |\mathbb{E}X^3| \leq c(b)|X|^3, \]

\[ |f(t) - 1 + t^2/2| \leq \inf_{\lambda > 0} (\lambda|\mathbb{E}X^3| + q_3(\lambda)|X|^3) |t|^3/6 \leq b\gamma_3(b)|t|^3/6, \]

\[ |f'(t) + t| \leq \inf_{\lambda > 0} (\lambda|\mathbb{E}X^3| + q_2(\lambda)|X|^3) t^2/2 \leq b\gamma_2(b)t^2/2, \]

\[ |f''(t) + 1| \leq \inf_{\lambda > 0} (\lambda|\mathbb{E}X^3| + q_1(\lambda)|X|^3) |t| \leq b\gamma_1(b)|t| \]

are proved for all $t \in \mathbb{R}$ and $b \geq 1$, where the function $c(b) = \sqrt{0.5\sqrt{1 + 8b^{-2}} + 0.5 - 2b^{-2}}$ increases strictly monotonically varying within the limits $0 = c(1) \leq c(b) < \lim_{b \to \infty} c(b) = 1$,

\[ q_n(\lambda) = \sup_{x > 0} \frac{n!}{x^n} \left| e^{ix} - \sum_{k=0}^{n-1} \frac{(ix)^k}{k!} - \lambda \frac{(ix)^n}{n!} \right|, \quad \gamma_n(b) = \inf_{\lambda > 0} (\lambda c(b) + q_n(\lambda)). \]

Moreover, the functions $\gamma_n(b)$ increase strictly monotonically varying within the limits $\gamma_n(1) \leq \gamma_n(b) < \lim_{b \to \infty} \gamma_n(b) = 1$, $n = 1, 2, 3$, with $\gamma_3(1) < 0.5950$, $\gamma_2(1) = 2/\pi < 0.6367$, $\gamma_1(1) < 0.7247$.

Key words and phrases: probability transformation, zero bias transformation, shape bias transformation, characteristic function, $L_1$-metric, moment inequality, McLaurin series, Taylor series

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1 Introduction and motivation

As is well known, the remainder term
\[ r_n(x) = e^{ix} - \sum_{k=0}^{n-1} \frac{(ix)^k}{k!}, \quad x \in \mathbb{R}, \ n \in \mathbb{N}, \]
in the Taylor expansion of the complex exponent satisfies the precise inequality
\[ |r_n(x)| \leq |x|^n/n!, \quad x \in \mathbb{R}, \ n \in \mathbb{N}, \]  
with equality attained as \( x \to 0 \), i.e. the factor \( 1/n! \) on the r.-h. side of (1) cannot be made less. Nevertheless, this does not mean that inequality (1) is unimprovable. Indeed, in 1991 H. Prawitz [16] suggested to rearrange a part of the remainder (however, always a smaller part) to the main term and proved that:
\[ \left| r_n(x) - \frac{n}{2(n+1)} \cdot \frac{(ix)^n}{n!} \right| \leq \frac{n+2}{2(n+1)} \cdot \frac{|x|^n}{n!}, \quad x \in \mathbb{R}, \ n \in \mathbb{N}, \]  
with equality still attained as \( x \to 0 \), whence (1) immediately follows.

The advantage of bound (2) as compared with (1) becomes especially noticeable, if \( x \) is an integration variable. For example, in probability theory \( x \) may stand for the product \( tX \) of an arbitrary random variable (r.v.) \( X \) defined on some probability space \( (\Omega, \mathcal{A}, P) \) and an argument \( t \in \mathbb{R} \) of its characteristic function (ch.f.)
\[ f(t) = \mathbb{E}e^{itX} = \int_{-\infty}^{+\infty} e^{itx} dF(x), \quad F(x) = P(X < x), \ x \in \mathbb{R}, \]
which is the Fourier–Stieltjes transform of the function of bounded variation \( F(x) \) (the distribution function of the r.v. \( X \)). Namely, suppose that for some \( n \in \mathbb{N} \)
\[ \mathbb{E}|X|^n \equiv \int_{-\infty}^{+\infty} |x|^n dF(x) < \infty \]
and denote
\[ \alpha_k = \mathbb{E}X^k, \quad \beta_k = \mathbb{E}|X|^k, \quad k = 1, 2, \ldots, n, \]
\[ R_n(t) = \mathbb{E}r_n(tX) = \int_{-\infty}^{\infty} r_n(tx) dF(x) = f(t) - \sum_{k=0}^{n-1} \frac{\alpha_k(it)^k}{k!}. \]
Then, by virtue of the Jensen inequality, \( |\alpha_k| \leq \beta_k, \ k = 1, \ldots, n \). Moreover, \( \alpha_n \) may vanish for odd \( n \), for example, for any symmetric distribution (i.e., if the r.v.’s \( X \) and \( -X \) have identical distributions), whereas \( \beta_n \) may be infinitely large.

As it follows from (1),
\[ |R_n(t)| \leq \frac{\beta_n|t|^n}{n!}, \quad t \in \mathbb{R}, \]  
with equality attained at any degenerate distribution as \( t \to 0 \), i.e. the factor \( 1/n! \) on the r.-h. side of (3) cannot be made less. However, by use of inequality (2), Prawitz managed to replace the absolute moment \( \beta_n \) by the linear combination of \( |\alpha_n| \) and \( \beta_n \) with coefficients still summing up to one:
\[ |R_n(t)| \leq \frac{n|\alpha_n| + (n+2)\beta_n}{2(n+1)} \cdot \frac{|t|^n}{n!}, \quad t \in \mathbb{R}, \]  
with equality attained as \( t \to 0 \), i.e. the factor \( 1/n! \) on the r.-h. side of (4) cannot be made less.
whence (3) immediately follows by virtue of Jensen’s inequality. Prawitz also paid a special
attention to the case $n = 3$, which is very important in the problem of estimation of the accuracy
of the normal approximation to normalized sums of independent random variables with finite
third moments, and in the same paper [16] noted that the coefficient
\[ \frac{n + 2}{2(n + 1)!} = \frac{3}{48} = 0.1041 \ldots \]
at $\beta_3$ on the r.-h. side of (1) cannot be less than
\[ x_3 \equiv \sup_{x > 0} (\cos x - 1 + x^2/2)/x^3 = 0.0991 \ldots . \]
Inequality (2) stipulates natural questions: if a larger part of the remainder is rearranged
to the main term, will the factor $(n + 2)/(2(n + 1)!)$ on the r.-h. side of (2) become less or
not? If yes, then what is its least possible value and will the sum of the coefficients at the
Corresponding main term and remainder still be equal to one or will it increase? To answer
these questions, we propose to consider the functions
\[ q_n(\lambda) = \sup_{x > 0} \frac{n!}{x^n} |e^{ix} - \sum_{k=0}^{n-1} \frac{(ix)^k}{k!} - \lambda \frac{(ix)^n}{n!}|, \quad \lambda \geq 0, \quad n = 1, 2, \ldots \]
(although here the supremum over $x > 0$ can be replaced by the supremum over all $x \in \mathbb{R}$,
$x \neq 0$, we will use a less cumbersome variant), which guarantee the validity of the inequality
\[ |e^{ix} - \sum_{k=0}^{n-1} \frac{(ix)^k}{k!} - \lambda \cdot \frac{(ix)^n}{n!}| \leq q_n(\lambda) \cdot \frac{|x|^n}{n!}, \quad x \in \mathbb{R}, \ n \in \mathbb{N}, \ \lambda \geq 0. \]
Eliminating the real or the imaginary part in the definition of $q_n(\lambda)$ we observe that
\[ \inf_{\lambda \geq 0} q_n(\lambda) \geq \begin{cases} \sup_{x > 0} \frac{n!}{x^n} \Re \left( e^{ix} - \sum_{k=0}^{n-1} \frac{(ix)^k}{k!} \right) = \sup_{x > 0} \frac{n!}{x^n} \cos x - \sum_{k=0}^{(n-1)/2} \frac{(-1)^k x^{2k}}{(2k)!}, \ n \text{ is odd,} \\ \sup_{x > 0} \frac{n!}{x^n} \Im \left( e^{ix} - \sum_{k=0}^{n-1} \frac{(ix)^k}{k!} \right) = \sup_{x > 0} \frac{n!}{x^n} \sin x - \sum_{k=1}^{n/2} \frac{(-1)^{k-1} x^{2k-1}}{(2k-1)!}, \ n \text{ is even.} \end{cases} \tag{5} \]
Replacing the supremum $\sup_{x > 0}$ in the definition of $q_n(\lambda)$ by the limit $\lim_{x \to 0^+}$ we also notice that
\[ q_n(\lambda) \geq \lim_{x \to 0^+} \frac{n!}{x^n} |e^{ix} - \sum_{k=0}^{n-1} \frac{(ix)^k}{k!} - \lambda \frac{(ix)^n}{n!}| \vee \lim_{x \to \infty} \frac{n!}{x^n} |e^{ix} - \sum_{k=0}^{n-1} \frac{(ix)^k}{k!} - \lambda \frac{(ix)^n}{n!}| = |1 - \lambda| \vee \lambda, \]
for all $\lambda > 0$, hence we will consider only the interval $0 \leq \lambda < 1/2$. Inequality (2) implies that
\[ q_n(\lambda) = 1 - \lambda, \quad 0 \leq \lambda \leq \frac{n}{2(n + 1)}. \]
Define
\[ \lambda_* = \lambda_*(n) = \sup \{ \lambda \geq 0 : \sup_{0 \leq s \leq \lambda} (s + q_n(s)) = 1 \}, \]
\[
\lambda^* = \lambda^*(n) = \inf\{\lambda \geq 0 : q_n(\lambda) = \inf_{s \geq 0} q_n(s)\}, \quad n \in \mathbb{N},
\]

i.e. \(\lambda_n\) is the greatest value of \(\lambda\) that minimizes the sum \(\lambda + q_n(\lambda)\), and \(\lambda^* \geq \lambda_n\) is the least value of \(\lambda\) that minimizes \(q_n(\lambda)\). Then, actually, only \(\lambda \in [\lambda_n, \lambda^*]\) are of interest. Inequality (2) also implies that

\[
\lambda_n \geq \frac{n}{2(n + 1)},
\]

and the posed questions can be re-formulated as follows:

\[
\lambda_n(n) \geq \frac{n}{2(n + 1)}, \quad \lambda^*(n) \geq \lambda_n(n), \quad q_n(\lambda^*(n)) = ?
\]

Moreover, using the introduced functions it is easy to obtain the following estimates for \(R_n(t)\) and its derivatives, which improve (4).

**Theorem 1.** For any r.v. \(X\) with the characteristic function \(f(t)\) and \(\mathbb{E}|X|^n < \infty\) for some \(n \in \mathbb{N}\), for all \(t \in \mathbb{R}\) and \(\lambda \geq 0\) the following estimates hold:

\[
\left| f(t) - \sum_{k=0}^{n-1} \alpha_k(\lambda t)^k - \lambda \frac{\alpha_n(\lambda t)^n}{n!} \right| \leq q_n(\lambda) \frac{\beta_n|t|^n}{n!}, \quad (6)
\]

\[
\left| \frac{d^\ell f(t)}{dt^\ell} - \sum_{k=0}^{n-\ell} \frac{t^{\ell+k}\alpha_k t^k}{(n-k)!} - \lambda \frac{t^n \alpha_n t^n}{(n-k)!} \right| \leq q_n(\lambda) \frac{\beta_n|t|^{n-\ell}}{(n-\ell)!}, \quad \ell = 1, (n-1). \quad (7)
\]

**Remark 1.** If \(\lambda_n, \lambda^*\) are known, then, actually, in (6), (7) the greatest lower bounds over \(\lambda \geq 0\) can be replaced by those over smaller sets \(\lambda_n \leq \lambda \leq \lambda^*\), and it suffices to study the properties of the functions \(q_n(\lambda)\) only within the intervals \(\lambda \in [\lambda_n, \lambda^*]\).

In lemmas [2] 3 5 below, it will be demonstrated that for \(n = 1, 2, 3\)

\[
\lambda_n(1) = \frac{1}{4} = 0.25, \quad \lambda_n(2) = \frac{1}{3} = 0.3333\ldots, \quad \lambda_n(3) = \frac{3}{8} = 0.375,
\]

\[
\lambda^*(1) = \frac{\sin \theta_1^*}{\theta_1^*} = 0.3108\ldots, \quad \lambda^*(2) = 4\pi^{-2} = 0.4052\ldots, \quad \lambda^*(3) = 6 \frac{\theta_3^* - \sin \theta_3^*}{(\theta_3^*)^3} = 0.4466\ldots,
\]

where \(\theta_1^* = 2.3311\ldots, \quad \theta_3^* = 3.9958\ldots\) are, respectively, the unique roots of the equations

\[
\theta_1 \sin \theta_1 + \cos \theta_1 - 1 = 0, \quad \theta_1 \in (0, \pi),
\]

\[
\theta_3^2 + 2\theta_3 \sin \theta_3 + 6(\cos \theta_3 - 1) = 0, \quad \theta_3 \in (0, 2\pi),
\]

i.e. the functions \(\lambda + q_n(\lambda)\) are constant (equal to one) within the intervals \(0 \leq \lambda \leq n/(2(n + 1)) = \lambda_n(n)\), increase strictly monotonically for \(\lambda_n(n) \leq \lambda \leq \lambda^*(n)\), and the functions \(q_n(\lambda)\) decrease strictly monotonically for \(0 \leq \lambda \leq \lambda^*(n)\) and attain their minimum values at \(\lambda = \lambda^*(n), n = 1, 2, 3\). In addition, in lemmas [2] 3 5 below it will be proved that

\[
\inf_{\lambda \geq 0} q_1(\lambda) = q_1(\lambda^*(1)) = \sup_{x > 0} \frac{1 - \cos x}{x} = 1 - \frac{\cos \theta_1^*}{\theta_1^*} = 0.7246\ldots,
\]

\[
\inf_{\lambda \geq 0} q_2(\lambda) = q_2(4\pi^{-2}) = 2 \sup_{x > 0} \frac{x - \sin x}{x^2} = 2 \frac{x - \sin x}{x^2} \bigg|_{x=\pi} = \frac{2}{\pi} = 0.6366\ldots,
\]

\[
\inf_{\lambda \geq 0} q_3(\lambda) = q_3(\lambda^*(3)) = 6 \sup_{x > 0} \frac{\cos x - 1 + x^2/2}{x^3} = 6 \frac{\cos \theta_3^* - 1 + (\theta_3^*)^2/2}{(\theta_3^*)^3} = 0.5949\ldots (= 6\varepsilon_3),
\]
i.e., actually, for \( n = 1, 2, 3 \) inequalities hold with the equality sign. In other words, say, for \( n = 3 \) one can eliminate the imaginary part of \( (e^{ix} - 1 - ix - (ix)^2)/2 - \lambda(ix)^3/6 \) when searching the supremum in the definition of \( q_3(\lambda) \) by choosing a special value of \( \lambda = \lambda^* (3) \):

\[
\inf_{\lambda > 0} \sup_{x > 0} \frac{1}{x^3} \left( \cos x - 1 + \frac{x^2}{2} \right)^2 + \left( \sin x - x + \frac{\lambda x^3}{6} \right)^2 = \sup_{x > 0} \frac{\cos x - 1 + x^2/2}{x^3}.
\]

Note that the estimates for \( R_n(t) \) and its derivatives

\[
|f(t) - 1 - i\alpha_1 t + \alpha_2 t^2/2| \leq \min_{3/5 < \lambda < \lambda^*(3)} (\lambda |\alpha_3| + q_3(\lambda) \beta_3)|t|^3/6, \tag{8}
\]

\[
|f'(t) - i\alpha_1 + \alpha_2 t| \leq \min_{1/3 < \lambda < 4\pi/2} (\lambda |\alpha_3| + \alpha_2(t) \beta_3)|t|^2/2, \tag{9}
\]

\[
|f''(t) + \alpha_2| \leq \min_{1/4 < \lambda < \lambda^*(1)} (\lambda |\alpha_3| + q_1(\lambda) \beta_3)|t|, \tag{10}
\]

\[
|f(t) - 1| \leq \min_{1/4 < \lambda < \lambda^*(1)} (\lambda |\alpha_1| + q_1(\lambda) \beta_1)|t|, \tag{11}
\]

implied by theorem \( \square \) for \( n = 1, 2, 3 \) are precise in the sense that equalities in (8)-(11) are attained for each \( |t| \leq \theta \) at the symmetric three-point distributions of the form \( P(|X| = \theta/|t|) = t^2/\theta^2 = 1 - P(X = 0) \) (for which \( f(t) = 1 - t^2 \theta^{-2}(1 - \cos \theta) \), \( \alpha_1 = \alpha_3 = 0 \), \( \alpha_2 = 1 \), \( \beta_1 = |t|/\theta \), \( \beta_3 = \theta/|t| \)) with \( \theta = \theta_3^* \) in (8), \( \theta = \pi \) in (9), and \( \theta = \theta_1^* \) in (10), (11).

The following theorem allows to get rid of the third moment \( \alpha_3 \) in (8)-(10) with \( \mathbb{E} X \), \( \mathbb{E} (X - \mathbb{E} X)^2 \), \( \mathbb{E} |X - \mathbb{E} X|^3 \) being fixed.

**THEOREM 2.** For all \( b \geq 1 \) and any r.v. \( X \) with \( \mathbb{E} X = 0 \), \( \mathbb{E} X^2 = 1 \), \( \mathbb{E} |X|^3 = b \) \( |\mathbb{E} X^3| \leq A(b)\mathbb{E} |X|^3 \),

where

\[
A(b) = \sqrt{\frac{1}{2} \left( 1 + 8b^{-2} \right) + \frac{1}{2} - 2b^{-2}} < 1,
\]

with equality attained for each \( b \geq 1 \) at the two-point distribution

\[
P\left( X = \left( \frac{1 \pm u}{1 \pm u} \right)^{1/2} \right) = \frac{1 \pm u}{2}, \quad u = \sqrt{b \sqrt{b^2 + 8/2 - b^2/2} - 1}.
\]

Moreover, the function \( A(b) \) is concave and increases strictly monotonically varying within the limits \( 0 = A(1) \leq A(b) < \lim_{b \to \infty} A(b) = 1, b \geq 1 \). The function \( bA(b), b \geq 1 \), is concave as well.

Theorem \( \square \) improves Jensen’s inequality, which states that \( |\alpha_3|/\beta_3 \leq 1 \): actually, this ratio is strictly less than one for all distributions with zero mean and only tends to one as the normalized third moment \( \beta_3/\beta_3^2 \) goes to infinity.

Theorems \( \square \) and \( \square \) imply

**COROLLARY 1.** For all \( b \geq 1 \) and any r.v. \( X \) with \( \mathbb{E} X = 0 \), \( \mathbb{E} X^2 = 1 \), \( \mathbb{E} |X|^3 = b \) the following inequalities hold for all \( t \in \mathbb{R} \):

\[
|f(t) - 1 + t^2/2| \leq b\gamma_3(b)|t|^3/6,
\]

\[
|f'(t) + t| \leq b\gamma_2(b)t^2/2,
\]

\[
|f''(t) + 1| \leq b\gamma_1(b)|t|,
\]

\[
|f(t) - 1| \leq (\theta_1^*)^{-1}(1 - \cos \theta_1^*)\mathbb{E}|tX| \leq 0.7247 \cdot |t|\mathbb{E}|X|,
\]
where

\[ \gamma_n(b) = \min_{\lambda \in \lambda^*(n)} (\lambda A(b) + q_n(\lambda)), \quad n = 1, 2, 3, \]

moreover, the functions \( b \gamma_n(b), \gamma_n(b), n = 1, 2, 3, \) are concave and increase strictly monotonically in \( b \geq 1, \gamma_n(b) \) varying within the limits

\[ q_n(\lambda^*(n)) = \gamma_n(1) \leq \gamma_n(b) < \lim_{b \to \infty} \gamma_n(b) = 1, \quad b \geq 1, \quad n = 1, 2, 3. \]

The values of the functions \( \gamma_n(b), n = 1, 2, 3, \) for some \( b \geq 1 \) are presented in columns 2, 5, 8 of Table I. In columns 3, 6, 9, the values of \( \lambda_n = \lambda_n(b) \) are specified, that deliver minimum in the definition of \( \gamma_n(b) \), and in columns 4, 7, 10 the values of \( q_n = q_n(\lambda_n(b)) \) are presented as well.

| \( b = \lambda_n \) | \( b = \lambda_n \) | \( b = \lambda_n \) | \( b = \lambda_n \) | \( b = \lambda_n \) | \( b = \lambda_n \) | \( b = \lambda_n \) | \( b = \lambda_n \) | \( b = \lambda_n \) | \( b = \lambda_n \) | \( b = \lambda_n \) |
|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|
| 1.0001 | 0.729674 | 0.3091 | 0.7247 | 0.636620 | 0.4052 | 0.6367 | 0.594972 | 0.4466 | 0.5950 |
| 1 | 0.729674 | 0.3091 | 0.7247 | 0.636620 | 0.4052 | 0.6367 | 0.594972 | 0.4466 | 0.5950 |
| 1.001 | 0.740517 | 0.3057 | 0.7248 | 0.657374 | 0.3992 | 0.6368 | 0.617864 | 0.4407 | 0.5952 |
| 1.005 | 0.759711 | 0.2999 | 0.7253 | 0.682462 | 0.3924 | 0.6374 | 0.645582 | 0.4340 | 0.5957 |
| 1.01 | 0.773696 | 0.2960 | 0.7258 | 0.700771 | 0.3877 | 0.6380 | 0.665840 | 0.4293 | 0.5964 |
| 1.05 | 0.828077 | 0.2821 | 0.7293 | 0.772182 | 0.3714 | 0.6422 | 0.745088 | 0.4130 | 0.6005 |
| 1.10 | 0.863075 | 0.2743 | 0.7325 | 0.818315 | 0.3621 | 0.6460 | 0.796466 | 0.4038 | 0.6043 |
| 1.20 | 0.903490 | 0.2662 | 0.7370 | 0.871750 | 0.3526 | 0.6512 | 0.856138 | 0.3943 | 0.6095 |
| 1.30 | 0.927590 | 0.2618 | 0.7399 | 0.903693 | 0.3473 | 0.6547 | 0.891887 | 0.3890 | 0.6130 |
| 1.40 | 0.943762 | 0.2590 | 0.7421 | 0.925160 | 0.3440 | 0.6573 | 0.915944 | 0.3857 | 0.6156 |
| 1.50 | 0.955288 | 0.2570 | 0.7436 | 0.940474 | 0.3417 | 0.6591 | 0.933121 | 0.3834 | 0.6174 |
| 1.60 | 0.963824 | 0.2556 | 0.7448 | 0.951825 | 0.3400 | 0.6605 | 0.945860 | 0.3817 | 0.6188 |
| 1.70 | 0.970322 | 0.2546 | 0.7457 | 0.960468 | 0.3387 | 0.6616 | 0.955565 | 0.3804 | 0.6199 |
| 1.79 | 0.975371 | 0.2538 | 0.7464 | 0.967189 | 0.3378 | 0.6624 | 0.963114 | 0.3795 | 0.6208 |
| 1.90 | 0.979362 | 0.2531 | 0.7470 | 0.972502 | 0.3370 | 0.6631 | 0.969083 | 0.3787 | 0.6214 |
| 2.00 | 0.982560 | 0.2526 | 0.7475 | 0.976760 | 0.3364 | 0.6637 | 0.973868 | 0.3781 | 0.6220 |
| 3.00 | 0.995576 | 0.2506 | 0.7494 | 0.994102 | 0.3341 | 0.6659 | 0.993365 | 0.3757 | 0.6243 |
| 4.00 | 0.998416 | 0.2502 | 0.7498 | 0.997888 | 0.3336 | 0.6664 | 0.997624 | 0.3752 | 0.6248 |
| 5.00 | 0.999306 | 0.2501 | 0.7499 | 0.999075 | 0.3334 | 0.6666 | 0.999859 | 0.3751 | 0.6249 |
| \( \infty \) | 1 | 1/4 | 3/4 | 1 | 1/3 | 2/3 | 1 | 3/8 | 5/8 |

Table 1: The values of the functions \( \gamma_n(b) \) rounded up for some \( b \geq 1 \) (columns 2, 5, 8), the corresponding values of \( \lambda_n = \lambda_n(b) \) rounded down, that deliver minimum in the definition of \( \gamma_n(b) \) (columns 3, 6, 9), and the values of \( q_n = q_n(\lambda_n(b)) \) rounded up (columns 4, 7, 10) for \( n = 1, 2, 3 \).

The problem of estimation of the accuracy of the approximation of characteristic functions by polynomials was also considered in [17].

Note that the estimates given in corollary [I] are rather rough either for large \( t \), or for large \( b \). However, this defect can be corrected if the characteristic function \( f(t) \) is approximated by its derivatives (and the derivatives — by the characteristic function). Namely, the following estimates can be derived from corollary [I] and the results of [2] [19] which are obtained with the
application of the zero biased and shape biased transformations:

$$|f(t) + f'(t)| \leq 2 \sin \left(\frac{bt}{4} \wedge \frac{\pi}{2}\right) \wedge \left(\gamma_2(b) \cdot \frac{bt}{2} + \frac{t^2}{2}\right),$$

$$|f(t) + f''(t)| \leq 2 \sin \left(\frac{|bt|}{2} \wedge \frac{\pi}{2}\right) \wedge \left(\gamma_1(b) \cdot |bt| + \frac{t^2}{2}\right)$$

for all $t \in \mathbb{R}$ and any r.v. $X$ with $\mathbb{E}X = 0$, $\mathbb{E}X^2 = 1$, $\mathbb{E}|X|^3 = b \geq 1$. Note that the r.h.-sides of the last inequalities remain bounded for large $t$ as well as for large $b$.

The presented estimates for characteristic functions allow to sharpen substantially the Berry–Esseen inequality and its structural improvements (see, e.g., the recent works \[20\,24\,26\,3\,5\,4\,23\,25\,2\,21\,6\,27\,18\] and references in \[14\,12\]), non-uniform estimates of the accuracy of the normal approximation to distributions of sums of independent r.v.’s (see \[13\,8\,22\,14\,12\,11\] and references therein), as well as uniform and non-uniform moment-type estimates of the rate of convergence in limit theorems for compound and mixed compound Poisson distributions (see \[9\,7\,5\,2\,11\]).

2 Proofs

The following lemmas establish the properties of the functions $q_n(\lambda)$ and give the exact values of the quantities $\lambda_n(n)$, $\lambda^*(n)$, for $n = 3$ (lemma 2), $n = 2$ (lemma 3), and $n = 1$ (lemma 5).

**LEMMA 1** (see \[15\]). Let $\theta_3^* = 3.9958 \ldots$ be the unique root of the equation

$$x^2 + 2x \sin x + 6(\cos x - 1) = 0, \quad x \in (0, 2\pi).$$

Then

$$x_3 = \sup_{x > 0} \frac{\cos x - 1 + x^2/2}{x^3} = \frac{\cos \theta_3^* - 1 + (\theta_3^*)^2/2}{(\theta_3^*)^3} = \frac{\theta_3^* - \sin \theta_3^*}{3(\theta_3^*)^2} = 0.099161 \ldots$$

**LEMMA 2.** Let

$$\lambda^* = 6 \frac{\theta_3^* - \sin \theta_3^*}{(\theta_3^*)^3} = 0.4466 \ldots.$$

For $3/8 < \lambda \leq \lambda^*$ by $\theta_3(\lambda) \in (0, 2\pi)$ denote the unique root of the equation

$$2 \cos x(\lambda x^4 - 18x^2 + 36) - 6x \sin x(x^2(\lambda + 1) - 12) - (3 - 4\lambda)x^4 - 72 = 0, \quad x \in (0, 2\pi),$$

and $\theta_3(\lambda) = 0$ for $0 \leq \lambda \leq 3/8$. Then $\theta_3(\lambda^*) = \theta_3^*$, $q_3(\lambda) = 1 - \lambda$ for $0 \leq \lambda \leq 3/8$ and

$$q_3(\lambda) = \frac{6}{x^3} \left[ \cos x - 1 + \frac{x^2}{2} \right]^{1/2} + \left( \sin x - x + \frac{\lambda x^3}{6} \right) \left. \right|_{x = \theta_3(\lambda)}, \quad 3/8 < \lambda \leq \lambda^*.$$

Moreover, the function $\lambda + q_3(\lambda)$ is strictly increasing for $3/8 < \lambda \leq \lambda^*$, the function $q_3(\lambda)$ is strictly decreasing for $0 \leq \lambda \leq \lambda^*$. In particular,

$$q_3(3/8) = 5/8, \quad \inf_{\lambda \geq 0} q_3(\lambda) = q_3(\lambda^*) = 6 \cdot x_3 = 0.594971 \ldots.$$
Proof. Denote
\[ h(x) = h(x, \lambda) = \left( \cos x - 1 + \frac{x^2}{2} \right)^2 + \left( \sin x - x + \frac{\lambda x^3}{6} \right)^2, \quad x > 0, \]
\[ f(x) = f(x, \lambda) = 6x^{-3} \sqrt{h(x, \lambda)}, \quad x > 0. \]
Then \( h(x, \lambda) > 0, \) \( f(x, \lambda) > 0 \) for all \( x > 0, \) and
\[ q_3(\lambda) = \sup_{x > 0} f(x, \lambda), \quad \lambda \geq 0. \]
From a result of [16] it follows that \( q_3(3/8) \leq 5/8, \) hence, for all \( 0 \leq \lambda \leq 3/8 \)
\[ q_3(\lambda) \leq q_3(3/8) + 3/8 - \lambda \leq 5/8 + 3/8 - \lambda = 1 - \lambda. \]
Since \( q_3(\lambda) \geq \lim_{x \to 0^+} f(x, \lambda), \lambda \geq 0, \) we conclude that \( q_3(\lambda) = 1 - \lambda, \theta_3(\lambda) = 0 \) for all \( 0 \leq \lambda \leq 3/8. \)

Now consider \( 3/8 < \lambda < 1/2. \) For all \( x > 0 \) we have
\[ f'(x) = \frac{3xh'(x) - 6h(x)}{x^4 \sqrt{h(x)}}, \]
\[ h'(x) = 2(\cos x - 1 + x^2/2)(x - \sin x) + 2(\sin x - x + \lambda x^3/6)(\cos x - 1 + kx^2/2), \]
\[ f_1(x) = f'(x) \cdot 2x^4 \sqrt{h(x)} = 6xh'(x) - 6h(x) = 2 \cos x(\lambda x^4 - 18x^2 + 36) - 6x \sin x(x^2(\lambda + 1) - 12) + 4\lambda x^4 - 3x^4 - 72, \]
\[ f_2(x) = \frac{f'_1(x)}{2x^2} = (\lambda - 3)x \cos x - \sin x(\lambda x^2 + 9\lambda - 9) + 8\lambda x - 6x, \]
\[ f'_2(x) = 3(1 - \lambda)x \sin x + (6 - 8\lambda - \lambda x^2) \cos x + 8\lambda - 6, \]
\[ f''_2(x) = \sin x(\lambda x^2 + 5\lambda - 3) + (3 - 5\lambda)x \cos x, \]
\[ f'''_2(x) = x(7\lambda \sin x - 3 \sin x + \lambda x \cos x) = x \cos x \cdot g(x), \quad g(x) = \lambda x + (7\lambda - 3) \tan x. \]
Evidently, \( f_1(\theta_3(\lambda)) \equiv 0 \) by the definition of \( \theta_3(\lambda). \) Split the domain \( x > 0 \) into the non-overlapping intervals \( x \in (2\pi m, 2\pi (m + 1)] \equiv (0, 2\pi] + 2\pi m, m = 0, 1, 2, \ldots, \) where
\[ A + c = \{ x \in \mathbb{R}: x = a + c, \ a \in A \}, \quad A \subset \mathbb{R}, \ c \in \mathbb{R}, \]
and consider the function \( f(x) \) and its derivatives on each of these intervals. The function \( \cos x \) has the zeros \( \pi/2 + 2\pi m \) and \( 3\pi/2 + 2\pi m \) in the interval \( x \in (0, 2\pi] + 2\pi m, \) which might be the zeros of the function \( f'''_2(x) \) as well. However,
\[ f'''_2(\pi/2 + 2\pi m) = (\pi/2 + 2\pi m) \cdot (7\lambda - 3) \neq 0, \quad f'''_2(3\pi/2 + 2\pi m) = (3\pi/2 + 2\pi m) \cdot (3 - 7\lambda) \neq 0 \]
for all \( \lambda \neq 3/7, \) thus all the roots of the equation \( f'''_2(x) = 0 \) coincide with those of the function \( g(x), \) if \( \lambda \neq 3/7. \) Now consider three cases for possible values of \( \lambda \in (3/8, 1/2): \)

1. if \( 3/8 < \lambda < 3/7, \) then the function \( g(x) \) vanishes in the points \( x_1 \in (0, \pi/2) + 2\pi m \) and \( x_2 \in (\pi, 3\pi/2) + 2\pi m \) changing its sign from + to −. Since \( \cos x_1 > 0, \cos x_2 < 0, \) the function \( f'''_2(x) \) changes its sign on each of the intervals \( x \in (0, 2\pi] + 2\pi m, m \geq 0, \) only in the two points \( x_1 \in (0, \pi/2) \) (from + to −) and \( x_2 \in (\pi, 3\pi/2) \) (from − to +).
2. if $\lambda = 3/7$, then $f''_2(x) = \lambda x^2 \cos x$ changes its sign in the two points: $x_1 = \pi/2 + 2\pi m$ (from $+\to -$) and $x_2 = 3\pi/2 + 2\pi m$ (from $-\to +$).

3. if $3/7 < \lambda < 1/2$, then the function $g(x)$ vanishes in the points $x_1 \in (\pi/2, \pi) + 2\pi m$ and $x_2 \in (3\pi/2, 2\pi) + 2\pi m$ changing its sign from $-\to +$. Since $\cos x_1 < 0$, $\cos x_2 > 0$, the function $f''_2(x)$ changes its sign on each of the intervals $x \in (0, 2\pi] + 2\pi m$, $m \geq 0$, only in the two points $x_1 \in (\pi/2, \pi) + 2\pi m$ (from $+\to -$) and $x_2 \in (3\pi/2, 2\pi) + 2\pi m$ (from $-\to +$).

Summarizing what was said above we conclude that on each of the intervals $x \in (0, 2\pi] + 2\pi m$, $m \geq 0$, the function $f''_2(x)$ changes its sign exactly in two points $x_1 \in (0, \pi) + 2\pi m$ (from $+\to -$) and $x_2 \in (\pi, 2\pi) + 2\pi m$ (from $-\to +$). Thus, $x_1$ is the point of minimum and $x_2$ is the point of maximum of the function

$$f''_2(x) = \sin (\lambda x^2 + 5\lambda - 3) + (3 - 5\lambda) x \cos x.$$ 

We have

$$f''_2(0) = 0, \quad f''_2(2\pi m) = 2\pi m (3 - 5\lambda) > 0, \quad m \geq 1,$$

hence, $f''_2(x_1) > 0, f''_2(x_2) < 0$, and $f''_2(x)$ changes its sign exactly in two points $x_3 \in (x_1, \pi) + 2\pi m$ and $x_4 \in (\pi, x_2) + 2\pi m$ (from $+\to -$) and $x_4 \in (\pi, 2\pi) + 2\pi m$ (from $-\to +$).

Thus, the function

$$f''_2(x) = 3(1 - \lambda) x \sin x + (6 - 8\lambda - \lambda x^2) \cos x + 8\lambda - 6$$

has exactly two stationary points on each of the intervals $(0, 2\pi] + 2\pi m$, $m \geq 0$: $x_3 \in (0, \pi) + 2\pi m$ which is the point of maximum and $x_4 \in (\pi, 2\pi) + 2\pi m$ which is the point of minimum. For $m = 0$ we have $f''_2(0) = 0, f''_2(2\pi) = -\lambda(2\pi)^2 < 0$, consequently, $f''_2(x_3) > 0, f''_2(x_4) < 0$, and the function $f''_2(x)$ changes its sign in a unique point $x_6 \in (0, 2\pi)$ (from $+\to -$). For $m \geq 1$ we have

$$f''_2(2\pi m) = -\lambda(2\pi m)^2 < 0,$$

consequently, $f''_2(x_3) > 0, f''_2(x_4) < 0$, and $f''_2(x)$ changes its sign exactly in two points $y_5 \in (0, \pi) + 2\pi m$ (from $-\to +$) and $y_6 \in (\pi, 2\pi) + 2\pi m$ (from $+\to -$).

Thus, the function

$$f(x) \equiv \frac{f_1(x)}{2x^2} = (\lambda - 3) x \cos x - \sin (\lambda x^2 + 9\lambda - 9) + 8\lambda x - 6x,$$

has a unique stationary point $x_6$ on the interval $(0, 2\pi]$ which is the point of maximum and exactly two stationary points on each of the intervals $(0, 2\pi] + 2\pi m$ with $m \geq 1$: $y_5 \in (0, \pi) + 2\pi m$ which is the point of minimum and $y_6 \in (\pi, 2\pi) + 2\pi m$ which is the point of maximum. Since

$$f_2(0) = 0, \quad f_2(2\pi m) = 18\pi m (\lambda - 1) < 0, \quad m \geq 1,$$

we conclude that $f_2(x_6) > 0$ and the function $f'_1(x) = 2x^2 f_2(x)$ changes its sign on the interval $(0, 2\pi]$ in a unique point $x_6 \in (x_6, 2\pi) \in (0, 2\pi)$ (from $+\to -$). With $m \geq 1$ we have

$$f_2(3\pi/2 + 2\pi m) = \lambda (x^2 + 8x + 9) - 6x - 9|_{x = 3\pi/2 + 2\pi m} \geq 3/8 \cdot (x^2 + 8x + 9) - 6x - 9|_{x = 3\pi/2 + 2\pi m} = 3/8 \cdot ((3\pi/2 + 2\pi m - 4)^2 - 31) \geq 3/8 \cdot ((7\pi/2 - 4)^2 - 31) > 0,$$
and hence, the function \( f'(x) = 2x^2f_2(x) \) changes its sign exactly in two points on each of the intervals \((0, 2\pi] + 2\pi m, m \geq 1\): \( y_7 \in (y_5, 3\pi/2) + 2\pi m \in (0, 3\pi/2) + 2\pi m \) (from \(-\) to \(+\)) and \( y_8 \in (3\pi/2, 2\pi) + 2\pi m \) (from \(+\) to \(-\)).

Thus, the function

\[
f_1(x) = f'(x) \cdot 2x^4\sqrt{h(x)} = 2 \cos x(\lambda x^4 - 18x^2 + 36) - 6x \sin x(x^2(\lambda + 1) - 12) + (4\lambda - 3)x^4 - 72,
\]

where \( h(x) > 0 \), has a unique stationary point \( x_8 \) on the interval \((0, 2\pi]\) (the point of maximum), and exactly two stationary points on each of the intervals \((0, 2\pi] + 2\pi m\) with \( m \geq 1\): \( y_7 \in (0, 3\pi/2) + 2\pi m \) (the point of minimum) and \( y_8 \in (3\pi/2, 2\pi) + 2\pi m \) (the point of maximum). Since

\[
f_1(0) = 0, \quad f_1(2\pi m) = 3(2\pi m)^2((2\lambda - 1)(2\pi m)^2 - 12) < 0, \quad m \geq 1,
\]

the function \( f'(x) \) has a unique zero within the interval \((0, 2\pi)\), which is the point of maximum of the function \( f(x) \) and coincides with \( \theta_3(\lambda) \).

As regards the domain \( x > 2\pi \), we are going to prove that \( f_1(x) < 0 \) for all \( x > 2\pi \) and \( 3/8 < \lambda \leq \lambda^* \), implying that the function \( f(x) \) has no maxima for \( x > 2\pi \) and completing the proof of the relation \( q_3(\lambda) = 6f(\theta_3(\lambda), \lambda) \). Since the function \( f_1(x) \) has a unique point of maximum \( y_8 \in (3\pi/2, 2\pi) + 2\pi m \) on each of the intervals \((0, 2\pi] + 2\pi m, m \geq 1\), it suffices to prove that \( f_1(x) < 0 \) for all \( x \in (3\pi/2, 2\pi) + 2\pi m \).

Note that \( \cos x > 0, \sin x < 0 \) for \( \lambda \in (0, \pi/2] + 2\pi m = (0, \pi/2] + 3\pi/2 + 2\pi m, \) and hence for \( \lambda \leq \lambda^* < 0.4467 \)

\[
f_1(x) \leq 2 \cos x(\lambda^* x^4 - 18x^2 + 36) - 6x \sin x(x^2(\lambda^* + 1) - 12) + (4\lambda^* - 3)x^4 - 72,
\]

moreover, as it can be easily seen, \( \lambda^* x^4 - 18x^2 + 36 > 0, x^2(\lambda^* + 1) - 12 > 0 \) for all

\[
x \geq 3\pi/2 + 2\pi m \geq 7\pi/2 = 10.99 \ldots.
\]

Split the domain \( x \in (0, \pi/2] + 3\pi/2 + 2\pi m \) into two intervals: \( x \in (0, \pi/4] + 3\pi/2 + 2\pi m \) and \( x \in [\pi/4, \pi/2] + 3\pi/2 + 2\pi m \) and examine the function \( f_1(x) \) on each of them. For \( x \in (0, \pi/4] + 3\pi/2 + 2\pi m \) we have \( \cos x \leq \sqrt{2}/2, \sin x \geq -1 \) and thus

\[
f_1(x) \leq \sqrt{2}(\lambda^* x^4 - 18x^2 + 36) + 6x(x^2(\lambda^* + 1) - 12) + (4\lambda^* - 3)x^4 - 72 =
\]

\[
= (\lambda^*(\sqrt{2} + 4) - 3)x^4 + 6(\lambda^* + 1)x^3 - 18\sqrt{2}x^2 - 72x + 36(\sqrt{2} - 2) <
\]

\[
< -x^2((3 - \lambda^*(\sqrt{2} + 4))x^2 - 6(\lambda^* + 1)x + 18\sqrt{2}).
\]

Since \( 3 - \lambda^*(\sqrt{2} + 4) > 1 - 1/\sqrt{2} > 0 \), now it can be easily seen that \( f_1(x) < 0 \) for all

\[
x > \frac{3(\lambda^* + 1) + \sqrt{9(\lambda^* + 1)^2 - 18\sqrt{2}(3 - \lambda^*(\sqrt{2} + 4))}}{3 - \lambda^*(\sqrt{2} + 4)} = 10.91 \ldots,
\]

in particular, for \( x \geq 3\pi/2 + 2\pi m \geq 7\pi/2 = 10.99 \ldots \).

For \( x \in [\pi/4, \pi/2] + 3\pi/2 + 2\pi m \) we have \( \cos x \leq 1, \sin x \geq -\sqrt{2}/2 \) and thus

\[
f_1(x) \leq 2(\lambda^* x^4 - 18x^2 + 36) + 3\sqrt{2}x(x^2(\lambda^* + 1) - 12) + (4\lambda^* - 3)x^4 - 72 =
\]

\[
= 3(2\lambda^* - 1)x^4 + 3\sqrt{2}(\lambda^* + 1)x^3 - 36x^2 - 36\sqrt{2}x <
\]

\[
< -3x^2((1 - 2\lambda^*)x^2 - \sqrt{2}(\lambda^* + 1)x + 12) < 0
\]
for all \( x \in \mathbb{R} \), since the discriminant \( 2(\lambda^* + 1)^2 - 4 \cdot 12(1 - 2\lambda^*) < -0.93 \) is negative.

Thus, we have proved that the function \( f(x) = f(x, \lambda) \) attains its maximal value for \( x > 0 \) at the unique point \( x = \theta_3(\lambda) \in (0, 2\pi) \) for \( 3/8 < \lambda \leq \lambda^* \) and at the point \( x \to 0+ \) for \( 0 \leq \lambda \leq 3/8 \).

Now prove that \( q_3(\lambda^*) = 6\chi_3 \). With
\[
\lambda = \lambda^* = 6 \frac{\theta_3^* - \sin \theta_3^*}{(\theta_3^*)^3} = 18 \frac{\cos \theta_3^* - 1 + (\theta_3^*)^2/2}{(\theta_3^*)^4}
\]
(two last relations following from lemma [1], we have
\[
h(\theta_3^*, \lambda^*) = \chi_3^2(\theta_3^*)^6, \\
h'_x(x, \lambda^*)_{x=\theta_3^*} = 2(\cos \theta_3^* - 1 + (\theta_3^*)^2/2)(\theta_3^* - \sin \theta_3^*) + \]
\[
+2(\sin \theta_3^* - \theta_3^* + \lambda^*(\theta_3^*)^3/6)(\cos \theta_3^* - 1 + \lambda^*(\theta_3^*)^2/2) = \\
= 2\chi_3(\theta_3^*)^3 \cdot 3\chi_3(\theta_3^*)^2 = 6\chi_3^2(\theta_3^*)^5, \\
f_1(\theta_3^*)/6 = xh'_x(x, \lambda)|_{x=\theta_3^*} - 6h(\theta_3^*, \lambda) = 0.
\]
By virtue of the uniqueness of the root \( \theta_3(\lambda) \) of the equation \( f_1(x) = 0 \), which is equivalent to \( f'_x(x, \lambda) = 0 \) within the interval \((0, 2\pi)\), we conclude that \( \theta_3(\lambda^*) = \theta_3^* \) so that
\[
q_3(\lambda^*) = f(\theta_3(\lambda^*), \lambda^*) = f(\theta_3^*, \lambda^*) = 6 \frac{\cos \theta_3^* - 1 + (\theta_3^*)^2/2}{(\theta_3^*)^3} = 6\chi_3.
\]
Now prove the declared properties of the functions \( q_3(\lambda), \lambda + q_3(\lambda) \). Since
\[
f''_{\lambda\lambda}(x, \lambda) = \frac{x^3(\cos x - 1 + x^2/2)^2}{6((\cos x - 1 + x^2/2)^2 + (\sin x - x + \lambda x^3/6)^2)^{3/2}} > 0, \quad 0 < x < \infty,
\]
the function \( f(x, \lambda) \) is strictly convex in \( \lambda \) for all \( x \in (0, \infty) \). As it follows from what was proved, the least upper bound in the definition of \( q_3(\lambda) \) is attained for all \( 3/8 < \lambda \leq \lambda^* \) at a finite point \( x = \theta_3(\lambda) \) separated from zero:
\[
q_3(\lambda) = \sup_{x \geq 0} f(x, \lambda) = \max_{\theta_3(\lambda) \in x \leq 2\pi} f(x, \lambda), \quad \theta_3(\lambda) > 0, \quad 3/8 < \lambda \leq \lambda^*,
\]
hence for all \( \lambda_1, \lambda_2 \in (3/8, \lambda^*] \) and \( 0 \leq \alpha \leq 1 \) we have
\[
q_3(\alpha \lambda_1 + (1 - \alpha) \lambda_2) = \max_{\theta_3(\lambda) \in x \leq 2\pi} f(x, \alpha \lambda_1 + (1 - \alpha) \lambda_2) < \\
\leq \max_{\theta_3(\lambda) \in x \leq 2\pi} \left( \alpha f(x, \lambda_1) + (1 - \alpha) f(x, \lambda_2) \right) \\
\leq \alpha \sup_{x \geq 0} f(x, \lambda_1) + (1 - \alpha) \sup_{x \geq 0} f(x, \lambda_2) = \alpha q_3(\lambda_1) + (1 - \alpha) q_3(\lambda_2),
\]
i.e. the function \( q_3(\lambda) \) is strictly convex for \( 3/8 < \lambda \leq \lambda^* \) as well. Since
\[
q_3(\lambda) \geq 6 \sup_{x \geq 0} (\cos x - 1 + x^2/2)/x^3 = 6\chi_3 = q_3(\lambda^*),
\]
\( \lambda = \lambda^* \) being the unique point of minimum of the function \( q_3(\lambda) \) on the interval \( 3/8 < \lambda \leq \lambda^* \), the function \( q_3(\lambda) \) should decrease strictly monotonically for \( 3/8 < \lambda \leq \lambda^* \). For \( 0 \leq \lambda \leq 3/8 \), obviously, the function \( q_3(\lambda) = 1 - \lambda \) is strictly decreasing.

The function \( \lambda + q_3(\lambda) \) is strictly convex for \( 3/8 < \lambda \leq \lambda^* \) as a sum of a convex and a strictly convex functions, hence, it cannot be constant on any subinterval of the interval \( 3/8 < \lambda \leq \lambda^* \). On the other hand, \( \lambda + q_3(\lambda) \geq 1 \) for all \( 0 \leq \lambda \leq \lambda^* \), thus, \( \lambda + q_3(\lambda) \) should strictly increase for \( 3/8 < \lambda \leq \lambda^* \).
Lemma 3. For $1/3 < \lambda \leq 4\pi^{-2} = 0.4052\ldots$ let $\theta_2(\lambda) \in (0, \pi]$ be the unique root of the equation
\[
x(8 - \lambda x^2) \sin x + 4(\lambda x^2 + x^2 - 4) \sin^2 \frac{x}{2} - 4x^2 = 0, \quad 0 < x \leq \pi,
\]
and let \(\theta_2(\lambda) = 0\) for \(0 \leq \lambda \leq 1/3\). Then \(\theta_2(4\pi^{-2}) = \pi\), \(q_2(\lambda) = 1 - \lambda\) for \(0 \leq \lambda \leq 1/3\) and
\[
q_2(\lambda) = 2\sqrt{\left(\frac{\cos x - 1 + \lambda x^2/2}{x^2}\right)^2 + \left(\frac{x - \sin x}{x^2}\right)^2} \Bigg|_{x = \theta_2(\lambda)} , \quad \frac{1}{3} < \lambda \leq \frac{4}{\pi^2}.
\]
Moreover, the function $\lambda + q_2(\lambda)$ is strictly increasing for $1/3 < \lambda \leq 4\pi^{-2}$, the function $q_2(\lambda)$ is strictly decreasing for $0 \leq \lambda \leq 4\pi^{-2}$. In particular,
\[
q_2(1/3) = 2/3, \quad \inf_{\lambda > 0} q_2(\lambda) = q_2(4\pi^{-2}) = 2 \sup_{x > 0} \frac{x - \sin x}{x^2} = \frac{2}{\pi} = 0.636619\ldots.
\]

Proof. Denote
\[
f(x) = f(x, \lambda) = 2\sqrt{\left(\frac{\cos x - 1 + \lambda x^2/2}{x^2}\right)^2 + \left(\frac{x - \sin x}{x^2}\right)^2}, \quad x > 0.
\]
Then \(f(x, \lambda) > 0, x > 0\), and
\[
q_2(\lambda) = \sup_{x > 0} f(x, \lambda), \quad \lambda \geq 0.
\]
From the result of [16] it follows that \(q_2(1/3) \leq 2/3\), hence, for all \(0 \leq \lambda \leq 1/3\)
\[
q_2(\lambda) \leq q_2(1/3) + 1/3 - \lambda \leq 2/3 + 1/3 - \lambda = 1 - \lambda.
\]
Since \(q_2(\lambda) \geq \lim_{x \rightarrow 0^+} f(x, \lambda) = |1 - \lambda|, \lambda \geq 0\), we conclude that \(q_2(\lambda) = 1 - \lambda\) with \(\theta_2(\lambda) = 0\) for all \(0 \leq \lambda \leq 1/3\).

Now assume that \(1/3 < \lambda \leq 4\pi^{-2}\). Consider two cases of possible values of \(x\):

1. \(0 < x < 2\pi\). We have
\[
\begin{align*}
f_1(x) &\equiv f'(x) \cdot x^5 f(x)/2 = x(8 - \lambda x^2) \sin x + 4(\lambda x^2 + x^2 - 4) \sin^2(x/2) - 4x^2, \\
f_2(x) &\equiv f_1(x)/x = (4 - 4\lambda - \lambda x^2) \cos x + (2 - \lambda)x \sin x + 4(\lambda - 1), \\
f_2'(x) &\equiv (2 - 3\lambda)x \cos x + (\lambda x^2 + 3\lambda - 2) \sin x, \\
f_2''(x) &\equiv \lambda x^2 \cos x + (5\lambda - 2)x \sin x = x \cos x \cdot g(x), \quad g(x) = \lambda x + (5\lambda - 2) \tan x.
\end{align*}
\]

Obviously, \(f_1(\theta_2(\lambda)) \equiv 0\) by the definition of \(\theta_2(\lambda)\). The function \(\cos x\) has the zeros \(\pi/2\) and \(3\pi/2\) within the interval \(x \in (0, 2\pi)\), which might be the zeros of the function \(f_2''(x)\) as well. However,
\[
\begin{align*}
f_2''(\pi/2) = (5\lambda - 2) \cdot \pi/2 \neq 0, \quad f_2''(3\pi/2) = (2 - 5\lambda) \cdot 3\pi/2 \neq 0
\end{align*}
\]
for all \(\lambda \neq 2/5\), thus all the roots of the equation \(f_2''(x) = 0\) coincide with those of the function \(g(x)\), if \(\lambda \neq 2/5\). Now consider three cases of possible values of \(\lambda \in (1/3, 4\pi^{-2}]\):
(a) if $1/3 < \lambda < 2/5$, then the function $g(x)$ vanishes in some points $x_1 \in (0, \pi/2)$ and $x_2 \in (\pi, 3\pi/2)$ where it changes its sign from + to −. Since $\cos x_1 > 0$, $\cos x_2 < 0$, the function $f_2''(x)$ changes its sign only in two points $x_1 \in (0, \pi/2)$ (from + to −) and $x_2 \in (\pi, 3\pi/2)$ (from − to +).

(b) if $\lambda = 2/5$, then $f_2''(x) = \lambda x^2 \cos x$ changes its sign in two points $x_1 = \pi/2$ (from + to −) and $x_2 = 3\pi/2$ (from − to +).

(c) if $2/5 < \lambda \leq 4\pi^{-2}$, then the function $g(x)$ vanishes in some points $x_1 \in (\pi/2, \pi)$, $x_2 \in (3\pi/2, \pi, 2\pi)$ where it changes its sign from − to +. Since $\cos x_1 < 0$, $\cos x_2 > 0$, the function $f_2''(x)$ changes its sign only in two points $x_1 \in (\pi/2, \pi)$ (from + to −) and $x_2 \in (3\pi/2, 2\pi)$ (from − to +).

Summarizing what was said above we conclude that the function $f_2''(x)$ changes its sign on the interval $(0, 2\pi)$ exactly in two points $x_1 \in (0, \pi)$ (from + to −) and $x_2 \in (\pi, 2\pi)$ (from − to +). Thus, $x_1 \in (0, \pi)$ is the point of maximum and $x_2 \in (\pi, 2\pi)$ is the point of minimum of the function

$$f_2'(x) = (2 - 3\lambda)x \cos x + (\lambda x^2 + 3\lambda - 2) \sin x.$$ 

We have

$$f_2'(0) = 0, \quad f_2'(\pi) = \pi(3\lambda - 2) < 0, \quad f_2'(2\pi) = 2\pi(2 - 3\lambda) > 0,$$

hence, $f_2'(x)$ changes its sign exactly in two points $x_3 \in (0, \pi)$ (from + to −) and $x_4 \in (\pi, 2\pi)$ (from − to +).

Thus, the function

$$f_2(x) \equiv f_1(x)/x = (4 - 4\lambda - \lambda x^2) \cos x + (2 - \lambda)x \sin x + 4(\lambda - 1)$$

has exactly two stationary points $x_3 \in (0, \pi)$ which is the point of maximum and $x_4 \in (\pi, 2\pi)$ which is the point of minimum. We have $f_2(0) = 0, f_2(2\pi) = -\lambda(2\pi)^2 < 0$, hence, the function $f_2(x)$ changes its sign in a unique point $x_5 \in (0, 2\pi)$ (from + to −), which is the unique point of maximum of the function

$$f_1(x) \equiv f'(x) \cdot x^5 f(x)/2 = x(8 - \lambda x^2) \sin x + 4(\lambda x^2 + x^2 - 4) \sin^2(x/2) - 4x^2.$$ 

Since $f_1(0) = 0, f_1(\pi) = 4(\lambda\pi^2 - 4) \leq 0$ for all $\lambda \leq 4\pi^{-2}$, we conclude that the function $f_1(x)$ changes its sign in a unique point $x_6 \in (0, \pi]$ (from + to −), which is the unique point of maximum of $f(x)$ and coincides with $\theta_2(\lambda)$, since $f'(x, \lambda)|_{x=\theta_2(\lambda)} \equiv 0$.

2. $x \geq 2\pi$. For $\lambda \leq 4\pi^{-2}$ we have

$$f(x, \lambda) = 2\sqrt{\left(\frac{\cos x - 1}{x^2} + \frac{\lambda}{2}\right)^2 + \left(\frac{x - \sin x}{x^2}\right)^2} \leq 2\sqrt{\left(\frac{2}{x^2} + \frac{2}{\pi^2}\right)^2 + \left(\frac{1 + x}{x^2}\right)^2} \leq$$

$$\leq 2\sqrt{\left(\frac{1}{2\pi^2} + \frac{2}{\pi^2}\right)^2 + \left(\frac{1}{4\pi^2} + \frac{1}{2\pi}\right)^2} < 0.6268 < \frac{2}{\pi} = 0.6366 \ldots.$$
Summarizing what was said above we conclude that the function \( f(x) = f(x, \lambda) \) attains its maximum value for \( x > 0 \) at the unique point \( x = \theta_2(\lambda) \in (0, 2\pi) \), if \( 1/3 < \lambda \leq 4\pi^{-2} \), and at the point \( x = 0^+ \), if \( 0 \leq \lambda \leq 1/3 \).

Prove that \( q_2(4\pi^{-2}) = 2\pi^{-1} \). Since \( f'_x(x, 4\pi^{-2})\big|_{x = \pi} = 0 \), we conclude that \( \theta_2(4\pi^{-2}) = \pi \) by virtue of the uniqueness of the root \( \theta_2(\lambda) \) of the equation \( f'_x(x, \lambda) = 0 \). Hence,

\[
q_2(4\pi^{-2}) = f(\pi, 4\pi^{-2}) = 2\sqrt{\left(\frac{\cos x - 1 + 2\pi^{-2}x^2}{x^2}\right)^2 + \left(\frac{x - \sin x}{x^2}\right)^2}\big|_{x = \pi} = \frac{2}{\pi}.
\]

Now prove the properties of the functions \( q_2(\lambda) \), \( \lambda + q_2(\lambda) \). Since

\[
f''_{\lambda}(x, \lambda) = \frac{x^2(x - \sin x)^2}{2((\cos x - 1 + \lambda x^2/2)^2 + (x - \sin x)^2)^{3/2}} > 0, \quad 0 < x < \infty,
\]
the function \( f(x, \lambda) \) is strictly convex in \( \lambda \geq 0 \) for all \( x \in (0, \infty) \). As it follows from what has already been proved, the least upper bound in the definition of \( q_2(\lambda) \) is attained for all \( 1/3 < \lambda \leq 4\pi^{-2} \) at the finite point \( x = \theta_2(\lambda) \) separated from zero:

\[
q_2(\lambda) = \sup_{x > 0} f(x, \lambda) = \max_{\theta_2(\lambda) < x < 2\pi} f(x, \lambda), \quad \theta_2(\lambda) > 0, \quad 1/3 < \lambda \leq 4\pi^{-2},
\]

hence for all \( \lambda_1, \lambda_2 \in (1/3, 4\pi^{-2}] \) and \( 0 \leq \alpha \leq 1 \) we have

\[
q_2(\alpha \lambda_1 + (1 - \alpha) \lambda_2) = \max_{\theta_2(\lambda) < x < 2\pi} f(x, \alpha \lambda_1 + (1 - \alpha) \lambda_2) < \\
< \max_{\theta_2(\lambda) < x < 2\pi} (\alpha f(x, \lambda_1) + (1 - \alpha) f(x, \lambda_2)) \leq \\
\leq \alpha \sup_{x > 0} f(x, \lambda_1) + (1 - \alpha) \sup_{x > 0} f(x, \lambda_2) = \alpha q_2(\lambda_1) + (1 - \alpha) q_2(\lambda_2),
\]
i.e. the function \( q_2(\lambda) \) is strictly convex for \( 1/3 < \lambda \leq 4\pi^{-2} \) as well. Since

\[
q_2(\lambda) \geq 2\sup_{x > 0} (x - \sin x)/x^2 = 2\pi^{-1} = q_2(4\pi^{-2}),
\]

\( \lambda = 4\pi^{-2} \) being the unique point of minimum of the function \( q_2(\lambda) \) in the interval \( 1/3 < \lambda \leq 4\pi^{-2} \), the function \( q_2(\lambda) \) should decrease strictly monotonically for \( 1/3 < \lambda \leq 4\pi^{-2} \). For \( 0 \leq \lambda \leq 1/3 \), obviously, the function \( q_2(\lambda) = 1 - \lambda \) is strictly decreasing.

The function \( \lambda + q_2(\lambda) \) is strictly convex for \( 1/3 < \lambda \leq 4\pi^{-2} \) as a sum of a convex and a strictly convex functions, hence, it cannot be constant on any subinterval of the interval \( 1/3 < \lambda \leq 4\pi^{-2} \). On the other hand, \( \lambda + q_2(\lambda) \geq 1 \) for all \( 0 \leq \lambda \leq 4\pi^{-2} \), thus, \( \lambda + q_2(\lambda) \) should be strictly increasing for \( 1/3 < \lambda \leq 4\pi^{-2} \). \( \square \)

**Lemma 4.** Let \( \theta_1^* = 2.3311 \ldots \) be the unique root of the equation \( x \sin x + \cos x - 1 = 0 \) within the interval \((0, \pi)\). Then

\[
\kappa_1 \equiv \sup_{x > 0} \frac{1 - \cos x}{x} = \frac{1 - \cos \theta_1^*}{\theta_1^*} = \sin \theta_1^* = 0.724611 \ldots.
\]

**Proof.** Consider the function \( f(x) = (1 - \cos x)/x, \, x > 0 \). Since for \( x \geq 2\pi \) we have

\[
f(x) \leq \frac{2}{x} \leq \frac{1}{\pi} < 0.3184 < \kappa_1,
\]
it suffices only to consider \(0 < x < 2\pi\). We have
\[
f'(x) \cdot x^2 = x \sin x + \cos x - 1 = (x - \tan(x/2)) \sin x.
\]

Since \(f'(-\pi) = -2\pi^{-2} \neq 0\), all the zeros of \(f'(x)\) within the interval \((0, 2\pi)\) coincide with those of the function \(g(x) = x - \tan(x/2)\). It is easy to see that within the interval \((0, 2\pi)\) the function \(g(x)\) vanishes in a unique point \(x_1 \in (0, \pi)\) changing its sign from \(+\) to \(−\). Since \(\sin x_1 > 0\), the function \(f'(x)\) has a unique zero within the interval \((0, 2\pi)\), which coincides with \(\theta_1^*\) and delivers maximum to the function \(f(x)\).

\[\square\]

**Lemma 5.** Let
\[
\lambda^* = \frac{\sin \theta_1^*}{\theta_1^*} = 0.3108\ldots.
\]

For \(1/4 < \lambda \leq \lambda^*\) let \(\theta_1(\lambda) \in (0, \pi)\) be the unique root of the equation

\[
\cos x(2 - \lambda x^2) + (1 + \lambda)x \sin x - 2 = 0, \quad x \in (0, \pi),
\]

and \(\theta_1(0) = 0\) for \(0 \leq \lambda \leq 1/4\). Then \(\theta_1(\lambda^*) = \theta_1^*,\ \theta_1(\lambda) = 1 - \lambda\) for \(0 \leq \lambda \leq 1/4\) and

\[
q_1(\lambda) = \sqrt{\left(\frac{\cos x - 1}{x}\right)^2 + \left(\frac{\sin x - \lambda x}{x}\right)^2} \bigg|_{x=\theta_1(\lambda)}, \quad 1/4 < \lambda \leq \lambda^*.
\]

Moreover, the function \(\lambda + q_1(\lambda)\) is strictly increasing for \(1/4 < \lambda \leq \lambda^*\), the function \(q_1(\lambda)\) is strictly decreasing for \(0 \leq \lambda \leq \lambda^*\). In particular,
\[
q_1(1/4) = 3/4, \quad \inf_{\lambda > 0} q_1(\lambda) = q_1(\lambda^*) = \sup_{x > 0} \frac{1 - \cos x}{x} = \lambda_1 = 0.724611\ldots.
\]

**Proof.** Denote
\[
f(x) = f(x, \lambda) = x^{-1}\sqrt{(\cos x - 1)^2 + (\sin x - \lambda x)^2}, \quad x > 0.
\]

Then \(f(x, \lambda) > 0, \ \lambda > 0,\) and

\[
q_1(\lambda) = \sup_{x > 0} f(x, \lambda), \quad \lambda \geq 0.
\]

Notice that \(q_1(\lambda) \geq \sup_{x > 0} (\cos x - 1)/x \equiv \lambda_1 > 0.7246\) for all \(\lambda \geq 0\). From a result of [16] it follows that \(q_1(1/4) \leq 3/4\), hence, for all \(0 \leq \lambda \leq 1/4\)

\[
q_1(\lambda) \leq q_1(1/4) + 1/4 - \lambda \leq 3/4 + 1/4 - \lambda = 1 - \lambda.
\]

Since \(q_1(\lambda) \geq \lim_{x \to 0^+} f(x, \lambda) = |1 - \lambda|, \ \lambda \geq 0,\) we conclude that \(q_1(\lambda) = 1 - \lambda, \ \theta_1(\lambda) = 0\) for all \(0 \leq \lambda \leq 1/4\).

Now assume that \(1/4 < \lambda < 1/3,\) in particular, \(1/4 < \lambda \leq \lambda^*\). Consider two cases of possible values of \(x:\)

1. \(0 < x < 2\pi\). We have

\[
f_1(x) \equiv f'(x) \cdot x^2 \sqrt{(\cos x - 1)^2 + (\sin x - \lambda x)^2} = \cos x(2 - \lambda x^2) + (1 + \lambda)x \sin x - 2,
\]

\[
f_1'(x) = (1 - \lambda)x \cos x + (\lambda x^2 + \lambda - 1) \sin x,
\]

\[
f_1''(x) = \lambda x^2 \cos x + (3\lambda - 1)x \sin x = x \cos \cdot \cdot g(x), \quad g(x) = \lambda x + (3\lambda - 1) \tan x.
\]
Obviously, $f_1(\theta_1(\lambda)) \equiv 0$ by the definition of $\theta_1(\lambda)$. Within the interval $x \in (0, 2\pi)$ the function $x \cos x$ has the zeros $\pi/2$ and $3\pi/2$, which might be the zeros of the function $f''_1(x)$ as well. However, 

$$f''_1(\pi/2) = (3\lambda - 1) \cdot \pi/2 > 0, \quad f''_1(3\pi/2) = (1 - 3\lambda) \cdot 3\pi/2 < 0$$

for all $\lambda < 1/3$. Hence, all the roots of the equation $f''_1(x) = 0$ coincide with the zeros of the function $g(x)$, for all $0 \leq \lambda \leq \lambda^*$. The function $g(x)$ vanishes in the points $x_1 \in (0, \pi/2)$ and $x_2 \in (\pi, 3\pi/2)$ changing its sign from to $+$. Since $\cos x_1 > 0$, $\cos x_2 < 0$, the function $f''_1(x)$ changes its sign only in two points $x_1 \in (0, \pi/2)$ (from $+$ to $-$) and $x_2 \in (\pi, 3\pi/2)$ (from $-$ to $+$). Thus, $x_1 \in (0, \pi/2)$ is the point of maximum and $x_2 \in (\pi, 3\pi/2)$ is the point of minimum of the function $f''_1(x)$. We have 

$$f'_1(0) = 0, \quad f'_1(\pi) = \pi(\lambda - 1) < 0, \quad f'_1(2\pi) = 2\pi(1 - \lambda) > 0,$$

hence, $f'_1(x)$ changes its sign exactly in two points $x_3 \in (0, \pi)$ (from $+$ to $-$) and $x_4 \in (\pi, 2\pi)$ (from $-$ to $+$). 

Thus, the function $f_1(x)$ has exactly two stationary points $x_3 \in (0, \pi)$, the point of maximum, and $x_4 \in (\pi, 2\pi)$, the point of minimum. Since $f'_1(0) = 0$, $f'_1(2\pi) = -\lambda(2\pi)^2 < 0$, the function $f_1(x)$ changes its sign in a unique point $x_5 \in (0, 2\pi)$ (from $+$ to $-$). Moreover, $f'_1(\pi) = \lambda\pi^2 - 4 < \pi^2/3 - 4 < 0$, hence, $x_5 \in (0, \pi)$, $x_5$ delivers maximum to $f(x)$ within the interval $(0, 2\pi)$ and coincides with $\theta_1(\lambda)$.

2. $x \geq 2\pi$. For $1/4 < \lambda < 1/3$ we have

$$f(x, \lambda) = x^{-1} \sqrt{2(1 - \cos x) - 2\lambda x \sin x + \lambda^2 x^2} \leq x^{-1} \sqrt{4 + 2\lambda x + \lambda^2 x^2} \leq \sqrt{\pi^{-2} + (3\pi)^{-1} + 3^{-2}} < 0.5644 < \infty_{\lambda > 0} q_1(\lambda).$$

Summarizing what was said above we conclude that the function $f(x) = f(x, \lambda)$ attains its maximum value for $x > 0$ at the unique point $x = \theta_1(\lambda) \in (0, \pi)$, if $1/4 < \lambda < 1/3$, in particular, if $1/4 < \lambda \leq \lambda^*$, and at the point $x \to 0+$, if $0 \leq \lambda \leq 1/4$.

Prove that $q_1(\lambda^*) = \infty_{\lambda^*}$. With

$$\lambda = \lambda^* = \frac{\sin \theta_1^*}{\theta_1^*} = \frac{1 - \cos \theta_1^*}{(\theta_1^*)^2}$$

(the last relation following from the definition of $\theta_1^*$ given in lemma[4]), we have

$$f'_1(\theta_1^*) = \cos \theta_1^*(2 - \lambda^*(\theta_1^*)^2) + (1 + \lambda^*)\theta_1^* \sin \theta_1^* - 2 =$$

$$= \cos \theta_1^*(1 + \cos \theta_1^*) + \theta_1^* \sin \theta_1^* + \sin^2 \theta_1^* - 2 = \cos \theta_1^* + \theta_1^* \sin \theta_1^* - 1 = 0,$$

by the definition of $\theta_1^*$. By virtue of the uniqueness of the root $\theta_1(\lambda)$ of the equation $f_1(x) = 0$, which is equivalent to $f'_1(x, \lambda) = 0$ within the interval $(0, 2\pi)$, we conclude that $\theta_1(\lambda^*) = \theta_1^*$ and thus

$$q_1(\lambda^*) = f(\theta_1(\lambda^*), \lambda^*) = f(\theta_1^*, \lambda^*) = \frac{1 - \cos \theta_1^*}{\theta_1^*} = \infty_{\lambda^*}.$$

Now prove the properties of the functions $q_1(\lambda)$, $\lambda + q_1(\lambda)$. Since

$$f''_{\lambda\lambda}(x, \lambda) = \frac{x(\cos x - 1)^2}{((\cos x - 1)^2 + (\sin x - \lambda x)^2)^{3/2}} > 0, \quad 0 < x < \infty,$$
the function \( f(x, \lambda) \) is strictly convex in \( \lambda \) for all \( x \in (0, \infty) \). As it follows from what has been already proved, the least upper bound in the definition of \( q_1(\lambda) \) is attained for all \( 1/4 < \lambda < 1/3 \) at the finite point \( x = \theta_1(\lambda) \) separated from zero:

\[
q_1(\lambda) = \sup_{x > 0} f(x, \lambda) = \max_{\theta_1(\lambda) \leq x \leq 2\pi} f(x, \lambda), \quad \theta_1(\lambda) > 0, \quad 1/4 < \lambda < 1/3,
\]

hence for all \( \lambda_1, \lambda_2 \in (1/4, 1/3) \) and \( 0 \leq \alpha \leq 1 \) we have

\[
q_1(\alpha \lambda_1 + (1 - \alpha) \lambda_2) = \max_{\theta_1(\lambda) \leq x \leq 2\pi} f(x, \alpha \lambda_1 + (1 - \alpha) \lambda_2) < \alpha \sup_{x > 0} f(x, \lambda_1) + (1 - \alpha) \sup_{x > 0} f(x, \lambda_2) = \alpha q_1(\lambda_1) + (1 - \alpha) q_1(\lambda_2),
\]

i.e. the function \( q_1(\lambda) \) is strictly convex for \( 1/4 < \lambda < 1/3 \) as well. Since \( \lambda = \lambda^* \) being the unique point of minimum of the function \( q_1(\lambda) \) on the interval \( 1/4 < \lambda < 1/3 \), and thus, the function \( q_1(\lambda) \) should decrease strictly monotonically for \( 1/4 < \lambda \leq \lambda^* \). For \( 0 \leq \lambda \leq 1/4 \), the function \( q_1(\lambda) = 1 - \lambda \) is obviously strictly decreasing.

The function \( \lambda + q_1(\lambda) \) is strictly convex for \( 1/4 < \lambda < 1/3 \) as a sum of a convex and a strictly convex functions, hence, it cannot be constant on any subinterval of the interval \( 1/4 < \lambda < 1/3 \). On the other hand, \( \lambda + q_1(\lambda) \geq 1 \) for all \( 0 \leq \lambda < 1/3 \), thus, \( \lambda + q_1(\lambda) \) should be strictly increasing for \( 1/4 < \lambda < 1/3 \), in particular, for \( 1/4 < \lambda \leq \lambda^* \).

**Proof of theorem 2** From the results of [10] it follows that the extremal value of the linear with respect to the distribution function \( F(x) = P(X < x), x \in \mathbb{R}, \) functional

\[
E X^3 = \int_{-\infty}^{\infty} x^3 dF(x)
\]

under the three linear moment-type conditions \( E X = 0, E X^2 = 1, E |X|^3 = b \) is attained at a distribution concentrated in at most four points (i.e. the distribution function \( F(x) \) being constant almost everywhere and having at most four jumps). For each \( b \geq 1 \) there exists a unique two-point distribution which satisfies the conditions \( E X = 0, E X^2 = 1, E |X|^3 = b \). This distribution is given in the formulation of the theorem and turns the stated inequality into equality. Thus, it remains to consider three- and four-point distributions only.

Let \( X \) take exactly three different values \( x, y, z \) with the corresponding probabilities \( p, q, r > 0, p + q + r = 1 \). Without loss of generality it can be assumed that \( x > y \geq 0 > z \). From the conditions \( E X = 0, E X^2 = 1 \) we find that

\[
p = \frac{1 + yz}{(x - y)(x - z)}, \quad q = -\frac{1 + xz}{(x - y)(y - z)}, \quad r = \frac{1 + xy}{(x - z)(y - z)}, \quad xz < -1 < yz.
\]

Then

\[
E X^3 = x + y + z + xyz \equiv \alpha_3(x, y, z),
\]

\[
E |X|^3 = \frac{-z^3(1 + xy) - z^2 xy(x + y) - z(xy(1 - xy) + x^2 + y^2) + xy(x + y)}{(y - z)(x - z)} \equiv \beta_3(x, y, z).
\]

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The Lagrange function of the optimization problem $\alpha_3(x, y, z) \to \sup$ under the constraint $\beta_3(x, y, z) = b$ has the form

$$f(x, y, z, \lambda) = \alpha_3(x, y, z) + \lambda(\beta_3(x, y, z) - b).$$

In the stationary points we necessarily have

$$\frac{\partial}{\partial x}f(x, y, z, \lambda) = (1 + yz)
\left(1 + \lambda \left(1 + \frac{2z^3}{(x-z)^2(y-z)}\right)\right) = 0,$$
$$\frac{\partial}{\partial y}f(x, y, z, \lambda) = (1 + xz)
\left(1 + \lambda \left(1 + \frac{2z^3}{(x-z)(y-z)^2}\right)\right) = 0.$$

Since $xz < -1 < yz$, from these equations we find that

$$\frac{\lambda z^3(y-x)}{(x-z)^2(y-z)^2} = 0,$$

whence it follows that $\lambda = 0$ by virtue of the conditions $x > y \geq 0 > z$. If $\lambda = 0$, then the condition $f'(x, y, z, \lambda) = 0$ implies that $yz = -1$, i.e. $p = 0$, that contradicts the condition $xz < -1 < yz$ and reduces the problem to checking two-point distributions considered above.

Now let $X$ take exactly four values $t > u > v > w$ with the corresponding probabilities $p, q, r, s > 0, p + q + r + s = 1$. From the conditions $EX = 0, EX^2 = 1$ we find that

$$p = \frac{1 + uv - s(u-w)(v-w)}{(t-u)(t-v)}, \quad q = \frac{1 + tw - s(t-w)(v-w)}{(t-u)(u-v)},$$
$$r = \frac{1 + tu - s(t-w)(u-w)}{(t-v)(u-v)}.$$

Then

$$\alpha_3(s, t, u, v, w) \equiv EX^3 = t + u + v + tw-s(t-w)(u-w)(v-w).$$

Denote $\beta_3(s, t, u, v, w) = E[X]^3$. Then the Lagrange function of the optimization problem $\alpha_3(s, t, u, v, w) \to \sup$ under the constraint $\beta_3(s, t, u, v, w) = b$ has the form

$$f(s, t, u, v, w, \lambda) = \alpha_3(s, t, u, v, w) + \lambda(\beta_3(s, t, u, v, w) - b).$$

For the proof of the theorem it suffices to consider two cases:

1) $t > u > v \geq 0 > w$. In this case

$$\beta_3(s, t, u, v, w) \equiv E[X]^3 = \alpha_3(s, t, u, v, w) - 2sw^3 =$$
$$= t + u + v + tw - s(w^3 + w^2(t + u + v) - w(tu + tv + uv) + tuv).$$

In the stationary points we necessarily have

$$f'_s(s, t, u, v, w, \lambda) = (1 - \lambda)w^3 + (1 + \lambda)\left(-w^2(t + u + v) + w(tu + tv + uv) - tuv\right) = 0,$$
$$f'_t(s, t, u, v, w, \lambda) = (1 + \lambda)(1 + uv - s(u-w)(v-w)) \equiv p(t-u)(t-v)(1+\lambda) = 0.$$

Since $p > 0$ and $t > u > v$, the second equation implies that $\lambda = -1$. With this value of $\lambda$ the first equation implies that $w = 0$ contradicting the condition $w < 0$. Thus, there are no extremal distributions in this case.

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2) $t > u \geq 0 \geq v > w, u \neq v$. In this case
\[ \beta_3(s, t, u, v, w) \equiv \mathbb{E}|X|^3 = \alpha_3(s, t, u, v, w) - 2rv^3 - 2sw^3. \]

In the stationary points we necessarily have
\begin{align*}
    f_t'(s, t, u, v, w, \lambda) &= \frac{p(t-u)(2\lambda v^3 + (\lambda + 1)(t-v)^2(u-v))}{(t-v)(u-v)} = 0, \\
    f_u'(s, t, u, v, w, \lambda) &= -\frac{q(t-u)(2\lambda v^3 + (\lambda + 1)(t-v)(u-v)^2)}{(t-v)(u-v)} = 0.
\end{align*}

With the account of the conditions $p, q > 0, t > u > v$ these equations imply $\lambda = -1, v = 0$. With these values of $\lambda$ and $v$ we have
\[ f(s, t, u, 0, w, -1) = \alpha_3(s, t, u, 0, w) - \beta_3(s, t, u, 0, w) + b = 2sw^3 + b. \]

In the stationary points we necessarily have
\[ f_w'(s, t, u, v, w, \lambda) = 6sw^2 = 0, \]
whence it follows that $w = 0 = v$ contradicting the condition $w < v$. Thus, there are no extremal distributions in this case as well.

The properties of the function $A(b) = \sqrt{\frac{1}{2}\sqrt{1+8b^{-2}} + \frac{1}{2} - 2b^{-2}}$ can be established by examination the derivatives. It is easy to see that
\[ A'(b) \cdot b^{3/2} A(b) = 1 - \left(1 + 8b^{-2}\right)^{-1/2} > 0, \quad 1 \leq b < \infty, \]
i.e. $A(b)$ increases strictly monotonically for all $b \geq 1$, and
\[ A''(b) \cdot b^8 A^3(b) \left(1 + 8b^{-2}\right)^{3/2}/4 = 16\left(1 + 8b^{-2}\right)^{1/2} - b((b^2 + 8)^{1/2} + 9b) - 48 \]
decreases monotonically and, hence, attains its maximum value $(-12)$ at the point $b = 1$. Thus, $A''(b) < 0$ for all $b \geq 1$, i.e. $A(b)$ is concave. For the function $bA(b)$ we have
\[ (bA(b))'' = \left(\left(\frac{b}{2}\sqrt{b^2 + 8} + \frac{b^2}{2} - 2\right)^{1/2}\right)'' = -12\frac{b + \sqrt{b^2 + 8}}{b^2 A^3(b)(b^2 + 8)^{3/2}} < 0, \quad b > 1, \]
hence, $bA(b)$ is concave as well. \hfill \Box

References

[1] W. Hoeffding. The extrema of the expected value of a function of independent random variables. Ann. Math. Statist., 26(2):268–275, 1955.

[2] V. Korolev, I. Shevtsova. An improvement of the Berry–Esseen inequality with applications to Poisson and mixed Poisson random sums. Scand. Actuar. J., 2012(2):81–105, 2012. Available online since 04 June 2010.

[3] V. Yu. Korolev, I. G. Shevtsova. An improvement of the Berry–Esseen inequalities. Dokl. Math., 81(1):119–123, 2010.

[4] V. Yu. Korolev, I. G. Shevtsova. On the upper bound for the absolute constant in the Berry–Esseen inequality. Theory Probab. Appl., 54(4):638–658, 2010.
[5] V. Yu. Korolev, I. G. Shevtsova. Sharpened upper bounds for the absolute constant in the Berry–Esseen inequality for mixed Poisson random sums. *Dokl. Math.*, 81(2):180–182, 2010.

[6] V. Yu. Korolev, I. G. Shevtsova. A new moment-type estimate of convergence rate in the Lyapunov theorem. *Theory Probab. Appl.*, 55(3):505–509, 2011.

[7] V. Yu. Korolev, S. Ya. Shorgin. On the absolute constant in the remainder term estimate in the central limit theorem for Poisson random sums. *Probabilistic Methods in Discrete Mathematics, Proceeding of the Fourth International Petrozavodsk Conference*, 305–308. VSP, Utrecht, 1997.

[8] R. Michel. On the constant in the nonuniform version of the Berry–Esseen theorem. *Z. Wahrschein. verw. Geb.*, 55(1):109–117, 1981.

[9] R. Michel. On Berry–Esseen results for the compound Poisson distribution. *Insurance: Mathematics and Economics*, 13(1):35–37, 1993.

[10] H. P. Mulholland, C. A. Rogers. Representation theorems for distribution functions. *Proc. London Math. Soc.*, 8(2):177–223, 1958.

[11] Yu. S. Nefedova, I. G. Shevtsova. Structural improvement of nonuniform estimates for the rate of convergence in the central limit theorem with applications to Poisson random sums. *Dokl. Math.*, 84(2):675–680, 2011.

[12] Yu. S. Nefedova, I. G. Shevtsova. On non-uniform convergence rate estimates in the central limit theorem. *Theory Probab. Appl.* (in Russian), 57(1):62–97, 2012.

[13] L. Paditz. Über die Annäherung der Verteilungsfunktionen von Summen unabhängiger Zufallsgrößen gegen unbegrenzt teilbare Verteilungsfunktionen unter besonderer beachtung der Verteilungsfunktion der standardisierten Normalverteilung. Dissertation A, Technische Universität Dresden, Dresden, 1977.

[14] L. Paditz. Über eine Fehlerabschätzung im zentralen Grenzwertsatz. *Wiss. Z. Hochschule für Verkehrswe sen “Friedrich List”. Dresden.*, 33(2):399–404, 1986.

[15] H. Prawitz. Ungleichungen für den absoluten Betrag einer charakteristischen funktion. *Skand. Aktuar.-etidskr.*., (1):11–16, 1973.

[16] H. Prawitz. Noch einige Ungleichungen für charakteristische Funktionen. *Scand. Actuar. J.*, (1):49–73, 1991.

[17] L. V. Rozovskii. Accuracy of the approximation of the characteristic functions by polynomials. *J. Soviet Math.*, 36(4):532–534, 1987.

[18] I. Shevtsova. On the absolute constants in the Berry–Esseen type inequalities for identically distributed summands. [arXiv:1111.6554](http://arxiv.org/abs/1111.6554) [math.PR], 28 Nov 2011.

[19] I. Shevtsova. On the absolute constants in the Berry–Esseen type inequalities for identically distributed summands. [arXiv:1212.6775](http://arxiv.org/abs/1212.6775) [math.PR], 30 Dec 2012.

[20] I. G. Shevtsova. Sharpening of the upper bound for the absolute constant in the Berry-Esseen inequality. *Theory Probab. Appl.*, 51(3):549–553, 2007.

[21] I. G. Shevtsova. An improvement of convergence rate estimates in Lyapunov’s theorem. *Dokl. Math.*, 82(3):862–864, 2010.

[22] W. Tysiak. *Gleichmäßige und nicht-gleichmäßige Berry–Esseen Abschätzungen*. Dissertation, Gesamthochschule Wuppertal, Wuppertal, 1983.

[23] I. Tyurin. New estimates of the convergence rate in the Lyapunov theorem. [arXiv:0912.0726](http://arxiv.org/abs/0912.0726) [math.PR], 3 Dec 2009.

[24] I. S. Tyurin. On the accuracy of the Gaussian approximation. *Dokl. Math.*, 80(3):840–843, 2009.

[25] I. S. Tyurin. Refinement of the upper bounds of the constants in Lyapunov’s theorem. *Russ. Math. Surv.*, 65(3):586–588, 2010.

[26] I. S. Tyurin. On the convergence rate in Lyapunov’s theorem. *Theory Probab. Appl.*, 55(2):253–270, 2011.

[27] I. S. Tyurin. A refinement of the remainder in the Lyapunov theorem. *Theory Probab. Appl.*, 56(4):693–696, 2012.