KLR ALGEBRAS AND THE BRANCHING RULE I: THE CATEGORICAL
GELFAND-TSETLIN BASIS IN TYPE $A_n$

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ABSTRACT. We define a quotient of the category of finitely generated modules over the cyclo-
tomic Khovanov-Lauda-Rouquier algebra for type $A_n$ and show it has a module category structure
over a direct sum of certain cyclotomic Khovanov-Lauda-Rouquier algebras of type $A_{n-1}$, this
way categorifying the branching rules for $\mathfrak{sl}_{n+1} \supset \mathfrak{sl}_n$. Using this we provide an elementary
proof of Khovanov-Lauda’s cyclotomic conjecture. We show that continuing recursively gives the
Gelfand-Tsetlin basis for type $A_n$. As an application we prove a conjecture of Mackaay, Stošić
and Vaz concerning categorical Weyl modules.

1. INTRODUCTION

Let $A$ and $B$ be associative algebras, $M$ a (left) $B$-module and $f : A \to B$ a map of algebras.
Then $A$ acts on $M$ through $f$ by the formula $a.m = f(a)m$ for $a$ in $A$ and $m$ in $M$. This
procedure turns each (left) module over $B$ into a (left) module over $A$. It is well known that
this operation defines a functor between the categories of modules over the respective algebras.
Each homomorphism of algebras $f : A \to B$ gives rise to a functor of restriction between their
categories of representations

$$\text{res}_B^A : B - \text{mod} \to A - \text{mod}$$

defined by $M \mapsto fM = fB \otimes_B M$ for left $A$-modules $N$ and $B$-modules $M$. Here $fM$
means the structure of $A$-module on the $M$ as defined above: $a.m := f(a)m$.

In general an irreducible object $M$ in $B - \text{mod}$ is not sent to an irreducible over $A$ but we can
restrict to categories of modules which are totally reducible: any module is isomorphic to a direct
sum of irreducible modules, determined up to isomorphism, which is unique up to permutation
of its summands. In this case $\text{res}_B^A(M)$ decompose as a direct sum of irreducibles over $A$ and the
obtention of such a decomposition gives the branching rule of $B$ with respect to $A$.

The study of the branching rules has its roots in group theory and were first obtained in a sys-
tematic way in the study of the representations of the classical groups. They were subsequently
extended to categories of representations of other types of algebras like for example associative
algebras, Lie algebras, Hopf algebras and quantum groups. Besides being a useful tool in the
study of the representations of the objects under consideration, the branching rules have been ex-
tensively studied in theoretical physics where they have found important applications in the study
of systems through reduction of its group of symmetry to one of its subgroups (see [21, 11], the
review [20] and the references therein).

Let us consider the case of Kac-Moody algebras associated to finite quivers. For each em-
bedding $\Gamma_2 \hookrightarrow \Gamma_1$ of quivers there is an embedding of the Kac-Moody algebras $A_{\Gamma_2} \hookrightarrow A_{\Gamma_1}$.
associated to $\Gamma_1$ and $\Gamma_2$. If we restrict to the categories of integrable representations then every irreducible integral representation $V(A_{\Gamma_1})$ of $A_{\Gamma_1}$ is isomorphic as a representation of $A_{\Gamma_2}$ to a direct sum of irreducibles [8]. In some cases a general procedure exists to obtain the branching rules for this embedding, but in the general case has to work out the result case by case (see [3, 21] for a general treatment of the branching rule for classical Lie algebras). These results extend to the quantum version of Kac-Moody algebras, which is the case we are interested in. Further application of the branching rule eventually gets us to a direct sum of irreducibles over the one-dimensional Kac-Moody algebra (one-dimensional spaces therefore). Including this collection of spaces back into $V(A_{\Gamma_1})$ defines a distinguished basis which is an example of a canonical basis and is called the Gelfand-Tsetlin basis after [7] (see also [6]). Besides its interest to representation theory, Gelfand-Tsetlin have been applied to problems in mathematical physics [20].

In a remarkable series of papers [14, 15, 23] M. Khovanov, A. Lauda and independently R. Rouquier introduced a family of Hecke algebras associated to a quiver (see also [24]). These quiver Hecke algebras, which became known as KLR algebras, have shown to have a rich representation theory ([3, 9, 18, 19]) but more immediate to us in this paper is the fact that the KLR algebra associated to a quiver $\Gamma$ categorifies the lower half of the quantum version of the Kac-Moody algebra associated to $\Gamma$, which means that the latter is isomorphic to the Grothendieck ring of the former. For each dominant integral weight $\lambda$ the KLR algebra $R_{\Gamma}$ admits a quotient, denoted $R_{\Gamma,\lambda}^\lambda$, which is called a cyclotomic quotient after [3], and whose Grothendieck group is isomorphic to the integral representation $V_{\lambda}$ of $A(\Gamma)$: the category of graded modules over $R_{\Gamma,\lambda}^\lambda$, finite in each degree, admits a categorical action of $A(\Gamma)$ which descends to the Grothendieck group yielding a representation which is isomorphic to $V_{\lambda}$ [13, 25].

In this paper we concentrate on the case where $\Gamma_1$ is the Dynkin diagram of type $A_n$ and $\Gamma_2$ is the Dynkin diagram obtained from $\Gamma_1$ by removing the vertex labeled $n$ (and the corresponding edge), and investigate the consequences for the corresponding KLR algebras and its cyclotomic quotients. The inclusion of quivers $A_{n-1} \hookrightarrow A_n$ determines an inclusion $R_{n-1} \hookrightarrow R$ between the respective KLR algebras. This gives rise to functors of restriction and induction between their categories of representations which turn out to descend to the usual inclusion and projection maps between the corresponding (one-half) quantum Kac-Moody algebras. This approach needs to be modified to work with cyclotomic quotients. In this case there is a projection of the cyclotomic KLR algebra $R_{A_n}^\lambda$ to a direct sum of cyclotomic KLR algebras $\bigoplus_{\mu \in \tau(\lambda)} R_{A_{n-1}}^\mu$ with the set $\tau(\lambda)$ being determined combinatorially from $\lambda$. We obtain a functor

$$\Pi: R_{A_n}^\lambda \mod \rightarrow \bigoplus_{\mu \in \tau(\lambda)} R_{A_{n-1}}^\mu \mod$$

between their categories of graded, finite dimensional modules which is full, essentially bijective, and commutes with the categorical action of the Kac-Moody algebra given by $A_n$. Continuing recursively we end up in the category of one-dimensional modules over a collection of one-dimensional algebras $R_{A_0}^\mu$ which are labeled by certain sequences of partitions $\lambda, \lambda^{(n-1)}, \ldots, \lambda^{(1)}$, each $\lambda^{(i)}$ being a partition with exactly $i$ parts. There is a categorical action of the Kac-Moody algebra of $A_n$ on the functors $R_{A_{n+1}}^\lambda \mod \rightarrow R_{A_0}^\mu \mod \cong k \mod$ which descends to
an action on the Grothendieck group, which means that these functors categorify the elements of the Gelfand-Tsetlin basis. These functors can be interpreted as the preimages under \( \Pi \) of the one-dimensional modules over \( R_{\Lambda_0}^\mu \), giving a realization of the Gelfand-Tsetlin basis in terms of some special objects in the category of modules over \( R_{\Lambda_n}^\Lambda \).

One consequence of the categorical branching rules is that we can use it to provide an easy proof of Khovanov and Lauda’s cyclotomic conjecture from [14]. As another application of the categorical branching rules we prove a conjecture in [22] about categorical Weyl modules for the \( q \)-Schur algebra. Namely we prove that the cyclotomic KLR algebra is isomorphic to a certain endomorphism algebra constructed in [22] as part of the \( q \)-Schur categorification to give a conjectural categorification of the Weyl module \( W_\lambda \). As a consequence we obtain that the aforementioned endomorphism algebra indeed categorifies \( W_\lambda \), this way proving a second conjecture in [22].

This paper was motivated by an attempt to lift the recursive formulas for link polynomials in [12] and [27] to statements between the corresponding link homology theories (see [27] [28] for further explanations and [26] for developments). This is the first output of the program outlined in [26]. We have tried to make this paper reasonably self-contained with the exception of Section 6 where we assume familiarity with [22].

2. Quantum \( \mathfrak{sl}_{n+1} \), the Branching Rule and the Gelfand-Tsetlin Basis

In this section we review the basics about quantum \( \mathfrak{sl}_{n+1} \), its irreducible representations, the branching rule for \( \mathfrak{sl}_{n+1} \to \mathfrak{sl}_n \), and the Gelfand-Tsetlin basis. We also fix notation and recollect some results that will be used in this paper.

2.1. Quantum \( \mathfrak{sl}_{n+1} \) and its irreducible representations. We denote the weight lattice \( \Lambda^{n+1} \) and the root lattice \( X^{n+1} \). Let \( \alpha_1, \ldots, \alpha_n \) be the simple roots and \( \alpha_1^\vee, \ldots, \alpha_n^\vee \) the coroots. Any weight \( \bar{\lambda} \) can be written as \( \bar{\lambda} = (\lambda_1, \ldots, \lambda_n) \) where \( \lambda_i = \alpha_i^\vee(\bar{\lambda}) \). Denote the set of dominant integral weights by

\[
\Lambda_+^{n+1} = \{ \bar{\lambda} \in \Lambda^{n+1} | \alpha_i^\vee(\bar{\lambda}) \in \mathbb{Z}_{\geq 0} \text{ for all } i = 1, \ldots, n \}.
\]

Let also

\[
a_{ij} = \alpha_i^\vee(\alpha_j) = \begin{cases} 2 & \text{if } j = i \\ -1 & \text{if } j = i \pm 1 \\ 0 & \text{else} \end{cases}
\]

be the entries of the Cartan matrix of \( \mathfrak{sl}_{n+1} \).
The quantum special linear algebra $U_q(sl_{n+1})$ is the associative unital $\mathbb{Q}(q)$-algebra generated by the Chevalley generators $F_i, E_i$ and $K_i^{\pm 1}$, for $1, \ldots, n$, subject to the relations

$$K_i K_j = K_j K_i \quad K_i K_i^{-1} = K_i^{-1} K_i = 1$$

$$K_i F_j K_i^{-1} = q^{-\alpha_{ij}} F_j \quad K_i E_j K_i^{-1} = q^{\alpha_{ij}} E_j$$

$$E_i F_j - F_j E_i = \delta_{ij} \frac{K_i - K_i^{-1}}{q - q^{-1}}$$

$$F_i^2 F_j - (q + q^{-1}) F_i F_j F_i + F_j F_i^2 = 0 \quad \text{if} \ |i - j| = 1$$

$$E_i^2 E_j - (q + q^{-1}) E_i E_j E_i + E_j E_i^2 = 0 \quad \text{if} \ |i - j| = 1$$

$$F_i F_j = F_j F_i, \quad E_i E_j = E_j E_i \quad \text{if} \ |i - j| > 1.$$ 

For $\lambda = (i_1, \ldots, i_k)$ we define $F_\lambda = F_{i_1} \cdots F_{i_k}$ and $E_\lambda = E_{i_k} \cdots E_{i_1}$. The reason for this convention will be clear later when we introduce the diagrammatics.

The lower half $U^- (sl_{n+1}) \subset U_q (sl_{n+1})$ quantum algebra is the subalgebra generated by the $F_i$s (analogously for the upper half $U^+ (sl_{n+1})$).

Recall that a subspace $V_\mu$ of a finite dimensional $U_q(sl_{n+1})$-module $V$ is called a weight space if

$$K_i^{\pm 1} v = q^{\pm \mu_i} v$$

for all $v \in V_\mu$ and that $V$ is called a weight module if

$$V = \bigoplus_{\mu \in \Lambda^{n+1}} V_\mu.$$ 

A weight module $V$ is called a highest weight module with highest weight $\bar{\lambda}$ if there exists a nonzero weight vector vector $v_{\bar{\lambda}} \in V_{\bar{\lambda}}$ such that $E_i v_{\bar{\lambda}} = 0$ for $i = 1, \ldots, n$. For each $\bar{\lambda} \in \Lambda^{n+1}$ there exists a unique irreducible highest weight module with highest weight $\bar{\lambda}$. In the sequel we will drop the $U_q$ and write $sl_{n+1}$-module instead of $U_q(sl_{n+1})$-module.

Let $\phi$ be the anti-involution on $U_q(sl_{n+1})$ defined by

$$\phi (K_i^{\pm 1}) = K_i^{\mp 1} \quad \phi(F_i) = q^{-1} K_i E_i \quad \phi(E_i) = q^{-1} K_i^{-1} F_i.$$ 

The $q$-Shapovalov form $\langle - , - \rangle$ is the unique nondegenerate symmetric bilinear form on the highest weight module $V(\bar{\lambda})$ satisfying

$$\langle v_{\bar{\lambda}}, v_{\bar{\lambda}} \rangle = 1$$

$$\langle uv, v' \rangle = \langle v, \phi(u)v' \rangle \quad \text{for all} \ u \in U_q(sl_{n+1}) \ \text{and} \ v, v' \in V(\lambda)$$

$$f\langle v, v' \rangle = \langle f v, v' \rangle = \langle v, fv' \rangle \quad \text{for any} \ f \in \mathbb{Q}(q) \ \text{and} \ v, v' \in V(\lambda).$$
2.2. The \( q \)-Schur algebra. In this subsection we give a brief review the \( q \)-Schur algebra \( S_q(n, d) \) following the exposition in [22] (see [22] and the references therein for more details). The Schur algebra appears naturally in the context of (polynomial) representations of \( U_q(\mathfrak{g}l_n) \), which is the starting point of this subsection. The root and weight lattices are very easy to describe for quantum \( \mathfrak{g}l_n \). Let \( \epsilon_i = (0, \ldots, 1, \ldots, 0) \in \mathbb{Z}^n \), with 1 being on the \( i \)-th coordinate for \( i = 1, \ldots, n \). Let also \( \hat{\alpha}_i = \epsilon_i - \epsilon_{i+1} \in \mathbb{Z}^n \) and \( (\epsilon_i, \epsilon_j) = \delta_{i,j} \) be the Euclidean inner product on \( \mathbb{Z}^n \) (in this basis the \( \mathfrak{sl}_n \) roots can be expressed by \( \alpha_i = \hat{\alpha}_i - \hat{\alpha}_{i+1} \)).

The quantum general linear algebra \( U_q(\mathfrak{g}l_n) \) is the associative unital \( \mathbb{Q}(q) \)-algebra generated by \( K_i, K_i^{-1} \), for \( i = 1, \ldots, n \), and \( F_i, E_i \), for \( i = 1, \ldots, n-1 \), subject to the relations

\[
K_iK_j = K_jK_i \quad K_iK_i^{-1} = K_i^{-1}K_i = 1
\]

\[
E_iF_j - F_jE_i = \delta_{i,j} \frac{K_iK_i^{-1} - K_i^{-1}K_i}{q - q^{-1}}
\]

\[
K_iF_j = q^{-(\epsilon_i, \alpha_j)}F_jK_i \quad K_iE_j = q^{(\epsilon_i, \alpha_j)}E_jK_i
\]

and

\[
\begin{align*}
F_i^2F_j - (q + q^{-1})F_iF_jF_i + F_jF_i^2 &= 0 \\
E_i^2E_j - (q + q^{-1})E_iE_jE_i + E_jE_i^2 &= 0 \quad \text{if } |i - j| = 1 \\
F_iF_j - F_jF_i &= 0 \\
E_iE_j - E_jE_i &= 0 \quad \text{if } |i - j| > 1.
\end{align*}
\]

In the Beilinson-Lusztig-MacPherson [2] idempotented version of quantum groups, the Cartan subalgebras are “replaced” by algebras generated by orthogonal idempotents corresponding to the weights. To understand their definition, recall that \( K_i \) acts as \( q^{\lambda_i} \) on the \( \lambda \)-weight space of any weight representation. The idempotented version of \( U_q(\mathfrak{g}l_n) \) can be obtained from \( U_q(\mathfrak{g}l_n) \) by adjoining orthogonal idempotents \( 1_{\lambda} \), for \( \lambda \in \mathbb{Z}^n \), and adding the relations

\[
\begin{align*}
1_{\lambda}1_{\nu} &= \delta_{\lambda, \nu}1_{\mu} \\
F_i1_{\lambda} &= 1_{\lambda+\hat{\alpha}_i}F_i \\
E_i1_{\lambda} &= 1_{\lambda+\hat{\alpha}_i}E_i \\
K_i1_{\lambda} &= q^{\lambda_i}1_{\lambda}.
\end{align*}
\]

The idempotent quantum \( \mathfrak{g}l_n \) is then defined by

\[
\hat{U}(\mathfrak{g}l_n) \cong \bigoplus_{\lambda, \mu \in \mathbb{Z}^n} 1_{\lambda}U_q(\mathfrak{g}l_n)1_{\mu}.
\]

Note that \( \hat{U}(\mathfrak{g}l_n) \) is not unital anymore because \( 1 = \sum_{\lambda \in \mathbb{Z}^n} 1_{\lambda} \) would be an infinite sum. In this setting the \( q \)-Schur algebra occurs naturally as a quotient of idempotent \( U_q(\mathfrak{g}l_n) \), which happens
to be very easy to describe. Let
\[ \Lambda(n, d) = \{ \lambda \in \mathbb{N}^n : \sum_i \lambda_i = d \} \]

be a weight (sub)lattice and the highest weights be elements in
\[ \Lambda^+(n, d) = \{ \lambda \in \Lambda(n, d) : \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \}. \]

The \( q \)-Schur algebra \( S_q(n, d) \) can be defined as the quotient of idempotented quantum \( \mathfrak{gl}_n \) by the ideal generated by all idempotents \( 1_\lambda \) such that \( \lambda \in \Lambda^+(n, d) \). Thus we have a finite presentation of \( S_q(n, d) \) as the associative unital \( \mathbb{Q}(q) \)-algebra generated by \( 1_\lambda \), for \( \lambda \in \Lambda(n, d) \), and \( F_i, E_i \), for \( i = 1, \ldots, n - 1 \), subject to the relations
\[
1_\lambda 1_\mu = \delta_{\lambda, \mu} 1_\lambda \\
\sum_{\lambda \in \Lambda(n, d)} 1_\lambda = 1 \\
F_i 1_\lambda = 1_{\lambda - \alpha_i} F_i \\
E_i 1_\lambda = 1_{\lambda + \alpha_i} E_i \\
E_i E_{-j} - E_{-j} E_i = \delta_{ij} \sum_{\lambda \in \Lambda(n, d)} [\lambda_i - \lambda_i + 1] 1_\lambda
\]

We use the convention that \( 1_\mu x 1_\nu = 0 \), if \( \mu \) or \( \nu \) is not contained in \( \Lambda(n, d) \).

The irreducibles \( W_\lambda \), for \( \lambda \in \Lambda^+(n, d) \), can be constructed as subquotients of \( S_q(n, d) \), called Weyl modules. Let \( < \) denote the lexicographic order on \( \Lambda(n, d) \). For any \( \lambda \in \Lambda^+(n, d) \), we have
\[
W_\lambda \cong 1_\lambda S_q(n, d) / [\mu > \lambda].
\]

Here \([\mu > \lambda] \) is the ideal generated by all elements of the form \( 1_\mu x 1_\lambda \), for some \( x \in S_q(n, d) \) and \( \mu > \lambda \).

2.3. Branching rules. Recall that a partition with \( m \) parts is a sequence of nonnegative integers \( (\lambda_1, \ldots, \lambda_m) \) with \( \lambda_1 \geq \cdots \geq \lambda_m \). Partitions are in bijection with Young diagrams. We follow the convention where Young diagrams are left justified and lines are enumerated from top to bottom. The bijection sends \( \lambda \) to the Young diagram with \( \lambda_i \) boxes in the \( i \)th line. From now on we denote them by the same symbols.

There is a well known relation between integral dominant weights of \( \mathfrak{sl}_{n+1} \) and partitions with \( n + 1 \) parts. For each such partition \( \lambda \) there is an integral dominant weight \( \bar{\lambda} \in \Lambda_{n+1}^+ \) defined by
\[
\bar{\lambda}_i := \lambda_i - \lambda_{i+1}.
\]

If we want to use partitions to describe the finite dimensional irreducible of \( \mathfrak{sl}_{n+1} \) we can write \( V_{\lambda}^\mathfrak{sl}_{n+1} \) to denote the irreducible \( \mathfrak{sl}_{n+1} \)-module \( V_{\bar{\lambda}} \) without any ambiguity. Of course there are several partitions giving the same element of \( \Lambda_{n+1}^+ \), but there is only one if we fix the value of \( \lambda_{n+1} = 0 \).
For a partition \( \lambda \) with \( n + 1 \) parts denote by \( \tau(\lambda) \) the set of all partitions \( \mu \) with \( n \) parts satisfying
\[
\lambda_i \leq \mu_i \leq \lambda_{i+1}.
\]

We denote by \( V^\text{sl}_{n+1}_\lambda \) the irreducible finite dimensional representation of \( \text{sl}_n \) of highest weight \( \lambda \). For the embedding \( \text{sl}_n \hookrightarrow \text{sl}_{n+1} \) corresponding to adding one vertex to the Dynkin diagram of \( \text{sl}_n \) the branching rule \([11]\) says that
\[
V^\text{sl}_{n+1}_\lambda \cong \bigoplus_{\mu \in \tau(\lambda)} V^\text{sl}_n_\mu
\]
is an isomorphism of \( \text{sl}_n \)-modules. This decomposition is multiplicity free that is, each of the \( V^\text{sl}_n_\mu \) occurs at most once in the sum (2).

Define \( \tilde{\tau}_k(\lambda) \) as the set of all the Young diagrams obtained from \( \lambda \) by the removal of \( k \) boxes, no more than one from each column. The inbetweenness condition \([11]\) is the same as requiring that \( \mu \) is in exactly one of the \( \tilde{\tau}_k(\lambda) \) for some \( k \). In other words,
\[
\tau(\lambda) = \bigoplus_{k \geq 0} \tilde{\tau}_k(\lambda).
\]

Let \( \tilde{\tau}_k(\lambda) \) be the set of all the \( \bar{\mu} \in \Lambda^+ \) for \( \mu \in \tau_k(\lambda) \) and for \( 1 \leq i_1 \leq \cdots \leq i_k \leq n + 1 \) denote by \( \bar{\mu}(i_1 \cdots i_k) \in \tilde{\tau}_k(\lambda) \) the weight obtained by removal of exactly one box from each of the \( i_1, \ldots, i_k \) th lines of \( \lambda \), in the order given.

For practical purposes the set \( \tilde{\tau}_k(\lambda) \) is best described using maps \( \Lambda^{n+1} \to \Lambda^{n+1} \) and \( \Lambda^{n+1} \to \Lambda^{n} \). For \( p_\lambda : \Lambda^{n+1} \to \Lambda^{n} \) the projection
\[
p_\lambda(\bar{\mu}_1, \ldots, \bar{\mu}_n) = (\bar{\mu}_1, \ldots, \bar{\mu}_{n-1})
\]
and \( (i_1, \ldots, i_k) \) as above we define \( \xi'_i : \Lambda^{n+1} \to \Lambda^{n+1} \) and \( \xi_{i_1 \cdots i_k} : \Lambda^{n+1} \to \Lambda^{n} \) by
\[
\xi'_i(\bar{\lambda}) = (\bar{\lambda}_1, \ldots, \bar{\lambda}_{i-1} + 1, \bar{\lambda}_i - 1, \ldots, \bar{\lambda}_{n-1}, \bar{\lambda}_n)
\]
\[
\xi_{i_1 \cdots i_k}(\bar{\lambda}) = \xi'_{i_k} \cdots \xi'_{i_1}(\bar{\lambda})
\]
and \( \xi_i(\bar{\lambda}) = p_\lambda \xi'_i \). We say that \( \xi_i \) is \( \lambda \)-dominant if \( \xi_i(\bar{\lambda}) \) is in \( \Lambda^+_n \) whenever \( \bar{\lambda} \) is in \( \Lambda^{n+1}_+ \) and that \( \xi_{i_1 \cdots i_k} \) is \( \lambda \)-dominant if for each \( j \leq k \) the map \( \xi_{i_1 \cdots j} \) is \( \lambda \)-dominant.

We have
\[
\bar{\mu}(i_1 \cdots i_k) = \xi_{i_k} \cdots \xi_{i_1}(\bar{\lambda})
\]

Keeping this notation in mind we denote by \( \xi_i(\lambda) \) the partition obtained from \( \lambda \) by the removal of one box from its \( i \) th line with \( \xi_{i_1 \cdots i_k}(\lambda) \) meaning the one obtained by the removal of \( k \) boxes, one for each \( i_r \) th line. We see that \( \lambda \)-positivity of \( \xi_{i_1 \cdots i_k} \) is equivalent of the requirement that no two boxes are removed from the same column of \( \lambda \). For later use we denote by \( D^k_\lambda \) the set of all \( \lambda \)-dominant \( \xi_{i_1 \cdots i_k} \) and define \( D_\lambda = \bigcup_{k \geq 0} D^k_\lambda \). This way \( \tau(\lambda) \) can be also seen as the set of all the \( \xi_{i_1 \cdots j}(\lambda) \) with \( \xi_{i_1 \cdots j} \) in \( D_\lambda \).
2.4. The Gelfand-Tsetlin basis. We can reapply the branching rule (2) recursively until we end up with a direct sum of 1-dimensional spaces corresponding to a final decomposition of each irreducible of \( \mathfrak{sl}_2 \) into 1-dimensional \( \mathbb{Q}(q) \)-vector spaces.

We say a sequence \((\mu^{(n+1)}, \ldots, \mu^{(1)})\) of partitions, where \(\mu^{(j)}\) has \(j\) parts, is a Gelfand-Tsetlin pattern for \(\mu^{(n+1)}\) if each consecutive pair \((\mu^{(j)}, \mu^{(j-1)})\) satisfy the inbetweeness condition (1). Denote by \(S(\lambda)\) the set of all the Gelfand-Tsetlin patterns for \(\lambda\).

The Gelfand-Tsetlin patterns for \(\lambda\) are the paths followed in the sequence of weight lattices
\[
\Lambda_+^{n+1} \rightarrow \Lambda_+^n \rightarrow \cdots \rightarrow \Lambda_+^1 = \mathbb{N}_0
\]
in going from \(V_{\lambda}^{\mathfrak{sl}_{n+1}}\) to each of the 1-dimensional spaces occurring at the end. Since the decomposition (2) is multiplicity free there is a 1-1 correspondence between \(S(\lambda)\) and the set of all these 1-dimensional spaces. Let \(V_{S(\lambda)}\) be the \(\mathbb{Q}(q)\)-linear spanned by \(S(\lambda)\). We write \(|s\rangle\) for a Gelfand-Tsetlin pattern \(s\) seen as an element of \(V_{S(\lambda)}\). It turns out that it is isomorphic to \(V_{\lambda}^{\mathfrak{sl}_{n+1}}\) not only as a vector space but as \(\mathfrak{sl}_{n+1}\)-modules.

The \(\mathfrak{sl}_{n+1}\)-action on \(V_{S(\lambda)}\) can be obtained through a procedure which, in some sense, is the reverse of the direct sum decomposition (2) using the branching rule. Whilst the generators \(\{E_i, F_i\}_{i \in \{1, \ldots, n-1\}}\) preserve the weight spaces \(V_{\mu}^{\mathfrak{sl}_n}\) on the right-hand side of (2), the generators \(E_n\) and \(F_n\) move between the different \(V_{\mu}^{\mathfrak{sl}_n}\): let \(\phi\) be the isomorphism \(V_{\lambda}^{\mathfrak{sl}_{n+1}} \rightarrow \bigoplus_{\mu \in \tau(\lambda)} V_{\mu}^{\mathfrak{sl}_n}\) in (2). Then the \(\mathfrak{sl}_n\)-action on \(\bigoplus_{\mu \in \tau(\lambda)} V_{\mu}^{\mathfrak{sl}_n}\) extends to an \(\mathfrak{sl}_{n+1}\)-action if we define
\[
E_n v := \phi E_n \phi^{-1} v \quad \text{and} \quad F_n v := \phi F_n \phi^{-1} v
\]
for \(v \in \bigoplus_{\mu \in \tau(\lambda)} V_{\mu}^{\mathfrak{sl}_n}\). This is a consequence of \(\phi E_i \phi^{-1} v = E_i v\) and \(\phi F_i \phi^{-1} v = F_i v\) for all \(v \in \bigoplus_{\mu \in \tau(\lambda)} V_{\mu}^{\mathfrak{sl}_n}\). We can continue this procedure until we get the desired 1-dimensional spaces and regard the \(\mathfrak{sl}_{n+1}\)-action on them as an action on \(V_{S(\lambda)}\). The basis of \(V_{S(\lambda)}\) given by the Gelfand-Tsetlin patterns is called the Gelfand-Tsetlin basis for \(V_{\lambda}^{\mathfrak{sl}_{n+1}}\). This basis was first defined by I. M. Gel’fand and M. L. Tsetlin in [7] for the Lie algebra \(\mathfrak{gl}(n)\). The explicit form of action of the generators of the Lie algebra \(\mathfrak{gl}(n)\) on the Gelfand-Tsetlin basis can be found for example in [21][29].

3. KLR algebras and their cyclotomic quotients

In this section we describe the quiver Hecke algebras which were introduced by Khovanov and Lauda in [14] and independently by Rouquier in [23]. We concentrate on the particular case of type \(A_n\). The KLR algebra \(R_{n+1}\) associated to the quiver \(A_n\) is the algebra generated by \(\mathbb{C}\)-linear combinations of isotopy classes of braid-like planar diagrams where each strand is labeled by a simple root of \(\mathfrak{sl}_{n+1}\). Strands can intersect transversely to form crossings and they can also carry dots. Multiplication is given by concatenation of diagrams and the collection of such diagrams is subject to relations (4)-(6) below (for the sake of simplicity we write \(i\) instead \(\alpha_i\) when labeling a strand). We read diagrams from bottom to top by convention and therefore the diagram for the
product \(a.b\) is the diagram obtained by stacking the diagram for \(a\) on the top of the one for \(b\).

\[
\begin{cases}
0 & \text{if } i = j \\
\text{if } j = i \pm 1 \\
\text{else}
\end{cases}
\]

\[
\begin{cases}
\text{if } i = k = j \pm 1 \\
0 & \text{else}
\end{cases}
\]

Algebra \(R_{n+1}\) is graded with the degrees given by

\[
\deg\left(\begin{array}{c}
\alpha \\
\beta
\end{array}\right) = -a_{ij} \quad \deg\left(\begin{array}{c}
\alpha \\
\beta
\end{array}\right) = a_{ii}.
\]

The following useful relation follows from (6) and will be used in the sequel.

\[
\sum_{\ell_1 + \ell_2 = d-1} \ell_1, \ell_2 = d
\]

Let \(\beta = \sum_{j=1}^{n} \beta_j \alpha_i\) and let \(R_{n+1}(\beta)\) be the subalgebra generated by all diagrams of \(R_{n+1}\) containing exactly \(\beta_i\) strands labeled \(i\). We have

\[
R_{n+1} = \sum_{\beta \in \Lambda^+_{n+1}} R_{n+1}(\beta).
\]

We also denote by

\[
R_{n+1}(k\alpha_n) = \bigoplus_{\beta' \in \Lambda^+_{n-1}} R_{n+1}(\beta' + k\alpha_n)
\]
the subalgebra of $R_{n+1}$ containing exactly $k$ strands labeled $n$. With this notation we have

\[(9)\quad R_{n+1} = \bigoplus_{k \geq 0} R_{n+1}(k \alpha_n).\]

For a sequence $\underline{i} = (i_1, \ldots, i_k)$ with $i_j$ corresponding to the simple root $\alpha_{i_j}$ we write $1_{\underline{i}}$ for the idempotent formed by $k$ vertical strands with labels in the order given by $\underline{i}$.

\[1_{\underline{i}} = \begin{array}{cccc}
i_1 & i_2 & \cdots & i_k \\
\end{array}\]

We write $1_{\ell \underline{\ell}}$ if the sequence of labels $\underline{i} = \underline{j} \underline{j}'$ can be written as a concatenation of sequences and we are only interested in the $\underline{\ell}$ part. We also write $x_{r,\underline{i}}$ for the diagram consisting of a dot on the $r$th strand of $1_{\underline{i}}$.

\[x_{r,\underline{i}} = \begin{array}{cccc}
\cdots & \cdot & \cdots \\
i_1 & i_r & i_k \\
\end{array}\]

For $\beta$ as above we denote by $\text{Seq}(\beta)$ the set of all sequences $\underline{i}$ of simple roots in which $i_j$ appears exactly $\beta_j$ times. The identity of $R_{n+1}(\beta)$ is then given by

\[1_{R_{n+1}(\beta)} = \sum_{\underline{i} \in \text{Seq}(\beta)} 1_{\underline{i}}.\]

We have

\[R_{n+1}(\beta) = \sum_{\underline{i} \in \text{Seq}(\beta)} 1_{\underline{i}} R_{n+1} 1_{\underline{i}}.\]

If $e \in R_{n+1}$ is an idempotent then there is a (right) projective module $eP = eR_{n+1}$. For $e = 1_{\underline{i}}$ this is the projective spanned by all diagrams whose labels end up in the sequence $\underline{i}$. We can define the left projective $P_e$ in a similar way.

Denote by $R_{n+1} - \text{mod}$ and $R_{n+1} - \text{pmod}$ the categories of graded finitely generated right $R_{n+1}$-modules and of graded finitely generated projective right $R_{n+1}$-modules respectively. For idempotents $e, e'$ we have

\[\text{Hom}_{R_{n+1}-\text{mod}}(eP, e'P) = e'R_{n+1}e.\]

For a graded algebra $A$ we denote by $K'_0(A)$ the Grothendieck group of finitely generated graded projective $A$-modules and write $K_0(A)$ for $\mathbb{Q}(q) \otimes_{\mathbb{Z}[q,q^{-1}]} K'_0(A)$.

There is a pair of functors on $R_{n+1} - \text{mod}$ which descend to the Grothendieck group giving it the structure of a twisted bialgebra (see [14] for the details).
Theorem 3.1 (Khovanov-Lauda [14], Rouquier [23]). The Grothendieck group $K_0(R_{n+1})$ is isomorphic to the lower half $U^-(\mathfrak{sl}_{n+1})$ through the map that takes $[\varphi P]$ to $F_{\varphi}^*$.

This is an isomorphism of twisted bialgebras but we do not pursue this direction in this paper.

3.1. Categorical inclusion and projection for KLR algebras. Let $\Gamma_n$ and $\Gamma_{n-1}$ the Dynkin diagrams associated to $\mathfrak{sl}_{n+1}$ and $\mathfrak{sl}_n$ respectively and consider the inclusion $\Gamma_{n-1} \hookrightarrow \Gamma_n$ that adds a vertex at the end of $\Gamma_{n-1}$ and the corresponding edge:

```
1 2 \cdots n-2 n-1 \hookrightarrow 1 2 \cdots n-2 n-1 n
```

This induces an inclusion of KLR algebras

$$\iota: R_n \hookrightarrow R_{n+1} \quad x \mapsto x$$

which coincides with the obvious map coming from the decomposition

$$R_{n+1} = \bigoplus_{k \geq 0} R_{n+1}(k\alpha_n) \cong R_n + \bigoplus_{k \geq 1} R_{n+1}(k\alpha_n).$$

The functors of inclusion and restriction induced by $\iota$

$$\text{Ind}_\iota: R_n \mod \to R_{n+1} \mod \quad \text{Res}_\iota: R_{n+1} \mod \to R_n \mod$$

are biadjoint, take projectives to projectives and descend to the natural inclusion and projection maps between the Grothendieck groups.

To see this we notice that the dual construction takes the projection $\rho: R_{n+1} \to R_n$ in the decomposition (10) to form the functor of restriction of scalars and its left and right adjoints, the functors of extension of scalars and coextension of scalars by $\rho$ respectively. Recall that $\rho$ endows $R_n$ with a structure of $(R_{n+1}, R_n)$-bimodule, where the structure of left $R_{n+1}$-module is given by $r.b = \rho(r)b$ for $b \in R_n$, $r \in R_{n+1}$. The same procedure can be used to give $R_n$ a structure of $(R_n, R_{n+1})$-bimodule. We use the notation $n+1(R_n)$ and $n(R_n)$ for $R_n$ seen as a $(R_{n+1}, R_n)$-bimodule and $(R_n, R_{n+1})$-bimodule respectively. Then we have the functors

$$\text{Res}_\rho: R_n \mod \to R_{n+1} \mod \quad \text{Ext}_\rho, \text{CoExt}_\rho: R_{n+1} \mod \to R_n \mod$$

with

$$\text{Ext}_\rho(M) = M \otimes_{R_{n+1}} n+1(R_n), \quad \text{CoExt}_\rho(M) = \text{Hom}_{R_{n+1} \mod}(n(R_n)_{n+1}, M)$$

for a right $R_{n+1}$-module $M$. We have that the functors $\text{Ext}_\rho$ and $\text{CoExt}_\rho$ coincide and we also have isomorphisms of functors $\text{Ext}_\rho \cong \text{Res}_\iota$ and $\text{Res}_\rho \cong \text{Ind}_\iota$. 
3.2. Factoring idempotents. In this section we give some properties of $R_{n+1} - p \mod$ that will be used in the sequel. For $\nu = \sum_{j=1}^{n-1} \nu_j \alpha_j \in \Lambda^+_n$ and for an ordered sequence $i_1, \ldots, i_k$ we define

$$\nu(i) = \sum_{j<i} \nu_j \alpha_j + \sum_{j\geq i} (\nu_j - 1) \alpha_j,$$

and $\nu(i_1 \cdots i_k)$ as the result of iteration of (11) from $i = i_k$ to $i = i_1$.

For each $i \in \{1, \ldots, n\}$ let $p_i$ be the idempotent $1_{i,i+1,\ldots,n} \in R_{n+1}$ and for $1_\nu \in R_{n+1}(\nu(i))$ let $e'(p_i, \nu) \in R_{n+1}(\nu + \alpha_k)$ be the idempotent obtained by horizontal composition of the diagram for $p_i$ at the left of the one for $1_\nu$.

$$e'(p_i, \nu) = \begin{array}{cccc} i & 1 & \cdots & n \\ \vdots & \vdots & \cdots & \vdots \\ j_i & j_{i+1} & \cdots & j_n \\ \end{array}$$

This generalizes easily to $R_{n+1}(k\alpha_n)$. In this case we denote by $p_{i_1\cdots i_k} \in R_{n+1}(k\alpha_n)$ the idempotent $1_{i_1,i_1+1,\ldots,n} \in R_{n+1}$ and for $1_\nu \in R_{n+1}(\nu(i_1 \cdots i_k))$ is defined as the horizontal concatenation placing the diagram of $p_{i_1\cdots i_k}$ at the left of the one for $1_\nu$.

$$e'(p_{i_1\cdots i_k}, \nu) = \begin{array}{cccc} i_{i_1} & \cdots & \cdots & \cdots \\ \vdots & \vdots & \cdots & \vdots \\ j_{i_1} & j_{i_2} & \cdots & j_{i_k} \\ \end{array}$$

In the case $1 \leq i_1 \leq \cdots \leq i_k \leq n + 1$ we write $e'(p_{i_1\cdots i_k}, \nu)$ instead of $e'(p_{i_1\cdots i_k}, \nu)$. We next introduce the notion of factoring a diagram through a family of idempotents $e_i$.

**Definition 3.2.** We say that $Y \in R_{n+1}(\nu + \alpha_n)$ factors through the family of idempotents $\{e_j\}_{j \in J}$ if it can be written as a sum $\sum_{j \in J} c_j X_j$ with all $c_j \in k$ nonzero and where each $X_j$ is in $R_{n+1}(\nu + \alpha_n) e_j R_{n+1}(\nu + \alpha_n)$ with $j$ minimal.

**Proposition 3.3.** Every $1_\nu \in R_{n+1}(\nu + \alpha_n)$ factors through the family $\{e'(p_j, \nu)\}_{j \in J}$ for some indexing set $J$.

**Proof.** The idempotent $1_\nu$ consists of $|\nu|$ parallel vertical strands labeled $j_1, \ldots, j_{|\nu|}$ in that order from left to right. We give an algorithm to obtain the factorization as claimed.

**Step 1:** Take the single strand labeled $n$ and start pushing it to the left by application of the move in (4) until we find a strand labeled $n - 1$.

$$1_\nu = \begin{array}{cccc} \cdots & \cdots & \cdots & \cdots \\ j_{r+2n-1} & j_r & j_r' & n \\ \end{array} \quad \Rightarrow \quad \begin{array}{cccc} \cdots & \cdots & \cdots & \cdots \\ j_{r+2n-1} & j_r & j_r' & n \\ \end{array}$$
Step 2: Pass to the strand labeled \( n - 1 \). There are three cases to consider: we can have (i) \( j_{r-2} = n - 1 \), (ii) \( j_{r-2} = n - 2 \) or (iii) \( j_{r-2} \neq n - 2, n - 1 \).

(i) If \( j_{r-2} = n - 1 \) we use the identity

\[
\begin{array}{c|c|c|c|c}
\hline
n & n & n - 1 & n - 1 & n - 1 \\
\hline
\end{array}
\]

which follows easily from (4) and (5). We see that \( 1 \cdots n-1,n-1,n-1 \cdots \) factors through \( 1 \cdots n-1,n,n-1 \cdots \). This reduces the number of strands on the left of the strand labeled \( n \). We then apply Step 1 to the block formed by strand labeled \( n \) and the one labeled \( n - 1 \) immediately on its left.

(ii) If \( j_{r-2} = n - 2 \) we apply Step 1 to the block formed by the strands labeled \( n - 2, n - 1, n \). We can proceed until we find a strand labeled \( n - 3, n - 2 \) or \( n - 1 \). If we find a strand labeled \( n - 3 \) we repeat Step 1 to the block formed by the strands labeled \( n - 3, n - 2, n - 1 \) and \( n \). Case we find a strand labeled \( n - 2 \) we are in the situation of (i) with \( n \) replaced by \( n - 1 \). Case we find a strand labeled \( n - 1 \) we use (5) to obtain that \( 1 \cdots n-1,n-2,n-1,n-1 \cdots \) factors through \( 1 \cdots n-1,n-2,n-2,n-2 \cdots \) and through \( 1 \cdots n-2,n-1,n-1 \cdots \). In the first case we can apply (4) to obtain that \( 1 \cdots n-1,n-1,n-2,n-2 \cdots \) factors through \( 1 \cdots n-1,n-1 \cdots \) which is the case (i). In the second case we apply (i) to the strands labeled \( n - 1, n - 1, n \). Either way we reduce the number of strands on the left of the one labeled \( n \).

(iii) If \( j_{r-2} \neq n - 2, n - 1 \) we apply Step 1 to the block formed by the strands labeled \( n - 1, n \) until we find a strand labeled \( n - 2 \) or \( n - 1 \). We then proceed like in (ii). We then proceed recursively: each time we get a diagram factoring through \( 1 \cdots s,s+1,\cdots,n-1,n \cdots \) we apply Step 1 to the entire block formed by the strands labeled \( s \) to \( n \) until we find a strand labeled \( j \) for \( s - 1 \leq j \leq n - 1 \). We then apply the move in (4) to pull this strand to the left of the one labeled \( j - 1 \) obtaining the configuration below.

\[
\begin{array}{c|c|c|c|c}
\hline
\hline
y &=& \cdots & \cdots & \cdots & \cdots & \cdots \\
\hline
j & s & j - 2 & j - 1 & j & n \\
\hline
\end{array}
\]

Using (5) in the region factoring through \( 1 \cdots j,j-1,j \cdots \) one obtains that \( y \) factors through the idempotent \( 1 \cdots j,j-1,j \cdots \) and through \( 1 \cdots j-1,j,j,\cdots \). Using (4) on the first term we can slide the strand labeled \( j - 1 \) to the right of the one labeled \( n \) and then apply the procedure described above in (i) to \( 1 \cdots j,j+1 \cdots \). To the second term we apply the procedure of (i) to \( 1 \cdots j,j,j+1 \cdots \). Again, in either case we obtain a linear combination of terms each having less strands on the left of the one labeled \( n \). The procedure ends when we obtain a linear combination of diagrams, each one factoring through and idempotent of the form \( e'(p_i,\hat{1}) \) as claimed.

We now take care of the case of \( k > 1 \).

**Lemma 3.4.** Suppose \( k = 2 \). Then \( e'(p_{i_1 i_2},\hat{1}) \) factors through a family \( \{ e'(j_1,j_2,\hat{1}) \}_{j \in J} \) for some indexing set \( J \).
Proof. Suppose we have
\[ \tilde{e}'(p_{ji}, j) = \begin{array}{ccccccc}
 & & & & & & \\
 j & j+1 & n & i & j-1 & j & j+1 & n
\end{array} \]
Using (4) we slide the first strand labeled \( n \) from the left to the right until it encounters a strand labeled \( n-1 \) to obtain a factorization through \( 1_{j\cdots n-1, i, \cdots, n-2, n-1, n, n^*} \):

\[ (13) \]

We now use (5) in the \( (n) - (n-1) - (n) \) part on the right of the diagram to obtain

The first term factors through the idempotent \( 1_{j\cdots n-1, i, \cdots, n-2, n-1, n, n^*} \) and the second through \( 1_{j\cdots n-1, i, \cdots, n-2, n-1, n, n^*} \). For the first term it is easy to see that we can do the same as in (13) to slide the entire block formed by the strands labeled \( 1 \cdots j, n - 2 \) to the right of the two strands labeled \( n \) to obtain a factorization through \( 1_{j\cdots n-1, n, n, i, \cdots, n-2, n-1, n, n^*} \) which is of the form \( e'(p_{jn}, j) \) for some \( j \), as wanted. For the second term we slide the first strand labeled \( n-1 \) from the left to the right until it finds the strand labeled \( n-2 \):

Applying (5) to the part containing \( (n-1) - (n-2) - (n-1) \) we see that it factors through

\[ (14) \]

Using (4) we can put the first term in the form

and we see that it factors through \( 1_{j\cdots n-2, n-1, n-1, n, n^*} \) which in turn factors (twice) through an idempotent of the form \( e'(p_{j_{n-1}}, j') \) as wanted (this uses the identity (12)). For the second term in (14) we start by sliding the first strand labeled \( n-2 \) from the left and repeat the procedure.
After having slid all the strands at the left of the first one labeled $i$ to the right we end up with a factorization through the family $\{e'((p_{j,n-r}, I_j))\}_{r=0,\ldots,n-j}$ and through a term of the form

$$1_{i \cdots j-1,j,j+1,j+1,\ldots,n-1,n,n^*}.$$  

Notice that after the first strand labeled $j-1$ all the strands are labeled in pairs each two strands with the same label appearing consecutively. Applying the identity (12), to the part labeled $(j) - (j) - (j+1)$ we get a factorization through $1_{i \cdots j-1,j,j+1,j+1,\ldots,n-1,n,n^*}$. Doing the same to the part containing $(j+1) - (j+2) - (j+2)$ we get a factorization through $1_{i \cdots j-1,j,j+1,j+2,j+1,\ldots,n-1,n-1,n,n^*}$ which factors through

$$1_{i \cdots j-1,j,j+1,j+2,j+1,\ldots,n-1,n,n^*}$$

(this uses $\Box$) between the two consecutive strands labeled $j$ and $j+2$. It is clear that we will end up with a factorization through $e'(p_{ij}, I_j)$ for some $I_j$. This way we see that $\tilde{e}'(p_{1i2j}, I_j)$ factors through the family $\{e'((p_{j,n-s}, I_j))\}_{s=0,\ldots,n-j} \cup \{e'(p_{ij}, I_j)\}$ for some idempotents $\{1_{\underline{\lambda}}\}_{r=0,\ldots,n-j}$ and $1_{\underline{\lambda}}$.

**Proposition 3.5.** Every $1_{\underline{i}}$ in $R_{n+1}(\nu + k\alpha_n)$ factors through a family $\{e'(p_{i_1\cdots i_k}, I_{\underline{i}})\}_{n-i_k \leq I}$ for some indexing set $I$.

**Proof.** We use a combination of Proposition 3.3 and Lemma 3.4. By application of the method described in the proof of Proposition 3.3 to the leftmost strand of $1_{\underline{i}}$ labeled $n$ we factorize it through some family the idempotents $\{e'(p_{a_1}, a_1)\}$ where each $1_{\underline{a}}$ is in $R_n(\nu(a_1) + (k-1)\alpha_n)$. Repeating the procedure for the newly created $1_{\underline{a}}$ we get that $1_{\underline{a}}$ factors through $\{e'(p_{b_1b_2}, b_2)\}$ for $1_{b_1}$ in $R_n(\nu(b_1b_2) + (k-2)\alpha_n)$. But each of the $\tilde{e}'(p_{b_1b_2}, b)$ factors through a family $\{e'(p_{c_1c_2}, I_{\underline{c}})\}$ from Lemma 3.4 and therefore $1_{\underline{b}}$ factors through a family $\{e'(p_{d_1d_2}, I_{\underline{d}})\}$. Then pass to the third leftmost strand labeled $n$. From Proposition 3.3 each of the $e'(p_{d_1d_2}, I_{\underline{d}})$ factors through a family $\{e'(p_{d_1d_2f_3}, f_3)\}$ with $d_1 \leq d_2$ as before and $1_{f_3} \in R(\nu(d_1d_2f_3) + (k-3)\alpha_n)$. For each of these terms we apply Lemma 3.4 again: if $d_2 \leq f_3$ do nothing otherwise factorize $e'(p_{d_1d_2f_3}, f_3)$ through $\{e'(p_{d_1g_3}, g)\}$ where $g_3 \leq f_3$. If $d_1 \leq g_3$ do nothing otherwise apply Lemma 3.4 to the $p_{d_1}p_{g_3}$ part to obtain a factorization through $\{e'(p_{h_1h_2}, I_{\underline{h}})\}$ with $h_1 \leq h_2$. This procedure slides strands to the space between the second and third strands labeled $n$ and therefore we need to apply Proposition 3.3 to the third strand labeled $k$ again and repeat the procedure described above. Notice that applying Proposition 3.3 and Lemma 3.4 amounts of “sliding” locally some strands to the right of a strand labeled $n$. This means that each time we apply each of these procedures to a strand labeled $n$ we decrease the number of strands on its left. This means that the process terminates with the factorization of $1_{\underline{i}}$ through a family $\{e'(p_{m_1m_2m_3}, I_{\underline{m}})\}$ with each $1_{m} \in R_{n+1}(\nu(m_1m_2m_3) - (k-3)\alpha_n)$. We now repeat the whole procedure to the fourth strand labeled $n$. Finiteness of the number of strands on its left implies that application of Proposition 3.3 and Lemma 3.4 as above allows factorizing $1_{\underline{i}}$ through $\{e'(p_{n_1n_2n_3}, I_{\underline{n}})\}$. Proceeding recursively with the remaining strands labeled $n$ we get that $1_{\underline{i}}$ factors through a family $\{e'(p_{\ell_1\ell_2\ell_3\cdots\ell_k}, I_{\underline{\ell}})\}_{\ell \in L}$ with each $1_{\ell} \in R_{n+2}(\nu(\ell_1 \cdots \ell_k))$, as claimed. $\square$

3.3. **Cyclotomic KLR-algebras.** Fix a partition $\lambda$ with $n+1$ parts for once and for all and let $I^\lambda$ the two-sided KLR-algebras and the branching rule I: Gelfand-Tsetlin basis 15
Definition 3.6. The cyclotomic KLR algebra $R^\lambda_{n+1}$ is the quotient of $R_{n+1}$ by the two-sided ideal $I^\lambda$.

Differently from the standard convention in the literature we label cyclotomic KLR algebras by partitions instead of integral dominant weights. This convention will be useful later. In terms of diagrams we are taking the quotient of $R_{n+1}$ by the two-sided ideal generated by all the diagrams of the form

$$\lambda \bigg| \overset{j_1}{\ldots} \bigg| \overset{j_2}{\ldots} \bigg| \overset{j_k}{\ldots}$$

where the leftmost strand has $\bar{\lambda}_{j_1}$ dots on it. We always label the leftmost region of a diagram with a partition $\lambda$ to indicate it is in $R^\lambda_{n+1}$. The following was proved in [25].

Lemma 3.7. The cyclotomic KLR algebra $R^\lambda_{n+1}$ is Frobenius.

Projective modules over $R^\lambda_{n+1}$ are defined the same way as for $R_{n+1}$, we write $e^\lambda P$ for $e R^\lambda_{n+1}$. Denote by $R^\lambda_{n+1} - \text{mod}$ and by $R^\lambda_{n+1} - \text{pmod}$ the categories of finitely generated graded $R^\lambda_{n+1}$-modules and finitely generated graded projective $R^\lambda_{n+1}$-modules respectively. The module category structure in the $R^\lambda_{n+1} - \text{mod}$ was studied in [13, 25]. Here we describe the necessary to proceed through this paper. Let $i_i: R^\lambda_{n+1}(\nu) \rightarrow R^\lambda_{n+1}(\nu + \alpha_i)$ be the map obtained by adding a vertical strand labeled $i$ on the right of a diagram from $R^\lambda_{n+1}$. The categorical $\mathfrak{sl}_{n+1}$-action on $R^\lambda_{n+1}$ is obtained by the pair of biadjoint exact functors defined by

$$F^\lambda_i = \begin{array}{c} R^\lambda_{n+1}(\nu) - \text{mod} \\ \xrightarrow{\text{Ind}_{i_i}} \end{array} R^\lambda_{n+1}(\nu + \alpha_i) - \text{mod}$$

and

$$E^\lambda_i = \begin{array}{c} R^\lambda_{n+1}(\nu + \alpha_i) - \text{mod} \\ \xrightarrow{\text{Res}_{i_i}(\lambda - \nu - 1)} \end{array} R^\lambda_{n+1}(\nu) - \text{mod}$$

The Khovanov-Lauda cyclotomic conjecture [14] was proved by Brundan and Kleshchev [3] based on Ariki’s categorification theorem [11]:

Theorem 3.8. There is an isomorphism of $\mathfrak{sl}_{n+1}$-representations

$$K_0(R^\lambda_{n+1}) \cong \mathcal{V}^{\mathfrak{sl}_{n+1}}_{\lambda}.$$ 

In Subsection 5.2 we give an alternative, elementary proof using the categorical branching rule. Theorem 3.8 was subsequently extended to affine type $A$ by Brundan and Kleshchev [4] and to all types by Kang and Kashiwara [13] and independently by Webster [25]. Webster also proved that

$$\text{gdim Hom}_{R^\lambda_{n+1} - \text{mod}}(P, P') = \langle [P], [P'] \rangle,$$

where $\langle \ , \ \rangle$ is the $q$-Shapovalov form.

All the results in Subsection 3.2 descend to the cyclotomic setting. In particular they allow a presentation of the category $R^\lambda_{n+1}(\nu + k\alpha_n) - \text{pmod}$ in terms of the collection of projectives $\{e^\lambda(P_{i_1, \ldots, i_k})\}$ that turns out to be useful later.
4. Categorical branching rules

4.1. Categorical branching rules. We have a direct sum decomposition of algebras

\[ R_{n+1}^\lambda \cong \bigoplus_{k \geq 0} R_{n+1}^\lambda(k\alpha_n), \]

where \( R_{n+1}^\lambda(k\alpha_n) \subseteq R_{n+1}^\lambda \) is the subalgebra generated by the diagrams in \( R_{n+1}^\lambda \) containing exactly \( k \) strands labeled \( n \). We also have

\[ R_{n+1}^\lambda \mod \cong \bigoplus_{k \geq 0} R_{n+1}^\lambda(k\alpha_n) \mod. \]

Clearly \( R_{n+1}^\lambda(0) \cong R_{n}^{\rho_{\lambda}(\lambda)} \), where \( \rho_{\lambda}(\lambda) : \Lambda_{+}^{n+1} \to \Lambda_{+}^{n} \) is the projection given in (3). We want to identify each block of \( R_{n+1}^\lambda \mod \) with the categorification of the \( sl_n \)-representations in (2) in the sense that \( R_{n+1}^\lambda(k\alpha_n) \mod \) will give the \( sl_n \) irreducibles obtained by removing exactly \( k \) boxes from the Young diagram for \( V_\lambda^{\Lambda} \).

We start by defining a special class of idempotents in \( R_{n+1}^\lambda \).

**Definition 4.1.** The idempotent \( e(p_{i_1\ldots i_k}, j) \in R_{n+1}^\lambda(\nu + k\alpha_n) \) is said to be a *special idempotent*, denoted \( e(p_{i_1\ldots i_k}, j) \), if \( \xi_{i_1\ldots i_k} \) is \( \lambda \)-dominant.

The property of \( \lambda \)-dominancy of \( \xi_{i_1\ldots i_k} \) implies that \( \nu(i_1 \cdots i_k) \) is in \( \Lambda_{+}^{n} \).

In the following we give the maps between some cyclotomic KLR algebras that are necessary to obtain the categorical branching rule.

**Lemma 4.2.** For each \( k \geq 0 \) there is a surjection of algebras

\[ R_{n+1}^\lambda(k\alpha_n) \xrightarrow{\pi_k} \bigoplus_{\xi_{i_1\ldots i_k} \in \mathcal{D}_{\lambda}^\lambda} R_{n}^{\xi_{i_1\ldots i_k}(\lambda)}. \]

**Proof.** We first prove that for each \( \xi_{i_1\ldots i_k} : \Lambda_{+}^{n+1} \to \Lambda_{+}^{n} \) we have a surjection of algebras

\[ R_{n+1}^\lambda(k\alpha_n) \xrightarrow{\pi_{i_1\ldots i_k}} R_{n}^{\xi_{i_1\ldots i_k}(\lambda)}. \]

To this end it is enough to show that for each \( \xi_{i_1\ldots i_k} \) as above, the subalgebra

\[ A_{i_1\ldots i_k} := \bigoplus_{\xi \in \text{Seq}(\alpha_1, \ldots, \alpha_n)} e(p_{i_1\ldots i_k}, \xi)(R_{n+1}^{\lambda}(k\alpha_{n-1}))e(p_{i_1\ldots i_k}, \xi) \]

projects onto \( R_{n+1}^{\xi_{i_1\ldots i_k}(\lambda)} \). Let \( \tilde{A}_{i_1\ldots i_k} \subset A_{i_1\ldots i_k} \) be the subalgebra generated by all diagrams having a representative given by diagrams consisting of \( k \) blocks of vertical strands on the left, where the \( n - i_r + 1 \) strands which belong to the \( r \)th block from the left are labeled \( i_r, i_r + 1, \ldots, n \) in that order, as below

\[
\begin{array}{cccccc}
\lambda & \cdots & \cdots & \cdots & \cdots & \cdots \\
1 & n & i_2 & n & \cdots & n \\
\end{array}
\]

\[ R_{n+1}^\lambda(0, \alpha_n) \]

\[ \cdots \]
and let $\widetilde{A}_{i_1 \ldots i_k}$ be its complement vector space. Let also $\tilde{A}_{i_1 \ldots i_k}^\lambda$ be the quotient of $\widetilde{A}_{i_1 \ldots i_k}$ by the two sided ideal generated by all diagrams of the form

\[
\begin{array}{cccccccc}
\lambda & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
n & i_1 & i_2 & \ldots & n & i_k & \ldots & j_1 & \ldots & j_e \\
\end{array}
\]

where $\zeta = \xi_{i_1 \ldots i_k}(\lambda)$. The algebras $\tilde{A}_{i_1 \ldots i_k}^\lambda$ and $R_{n_{i_1 \ldots i_k}}^\lambda(\lambda)$ are isomorphic.

We start with the case $k = 1$ and use it to prove the general case by recursion. To this end we compute

\[
X_i(j, r_j) = \begin{array}{c}
\lambda \\
n & i & i+1 & \ldots & j & n \end{array}
\]

We have several cases to consider, $j < i - 1$, $j = i - 1$, $j = i$ and $i < j < n$.

For $j < i - 1$ we have

\[
X_i(j, r_j) = \begin{array}{c}
\lambda \\
n & i & i+1 & \ldots & j & n \end{array}
\]

which follow easily from relations (4) and (6).

For $j = i - 1$ we have

\[
X_i(i - 1, r_{i-1}) = \begin{array}{c}
\lambda \\
n & i & i+1 & \ldots & j & n \end{array}
\]

\[
= \begin{array}{c}
\lambda \\
n & i & i+1 & \ldots & r_{i-1} \end{array} + \begin{array}{c}
\lambda \\
n & i & i+1 & \ldots & r_{i-1} \end{array}
\]

where we used relation (4). Sliding the newly created dot close to the original $r_{i-1}$ dots on the first term and applying relation (4) to both terms we get

\[
X_i(i - 1, r_{i-1}) = \begin{array}{c}
\lambda \\
n & i & i+1 & \ldots & n \end{array} + \begin{array}{c}
\lambda \\
n & i & i+1 & \ldots & n \end{array}
\]

(17) \[
X_i(i - 1, r_{i-1}) = \begin{array}{c}
\lambda \\
n & i & i+1 & \ldots & n \end{array} + \begin{array}{c}
\lambda \\
n & i & i+1 & \ldots & n \end{array}
\]
which consists of a term in $\tilde{A}_i$ and a term in $\tilde{A}_i^\perp$.

For $j = i$ we compute

$$X_i(i, r_i) = \sum_{\ell_1 + \ell_2 = r_i - 1} \lambda_{\ell_1, \ell_2}$$

where we used relation (8) followed by (4). Using (8) to slide the $\ell_2$ dots to to upper part of the first strand from the left gives

$$X_i(i, r_i) = \sum_{\ell_1 + \ell_2 = r_i - 1} \lambda_{\ell_1, \ell_2}$$

For $\lambda_i = 1, 0$ we have $X_i(i, r_i) = 0$ for all $r_i \geq 0$ and so it is enough to consider the case $r_i > 1$ here. For $r_i > 1$ the first term is in $\tilde{A}_i^\perp$ and the second term is in $\tilde{A}_i^\perp$ unless $\ell_1 = \kappa_1 = 0$. This results in

$$X_i(i, r_i) = \sum_{\ell_1 + \ell_2 = r_i - 1} \lambda_{\ell_1, \ell_2}$$

Finally for $i < j < n$ we have

$$X_i(j, r_j) = \sum_{\ell_1 + \ell_2 = r_j - 1} \lambda_{\ell_1, \ell_2}$$
The first term is in $\tilde{A}_i \perp$. For the second we use the result of the case of $i = j$, done above, to obtain

$$X_i(j, r_j) = \lambda \underbrace{\cdots}_{i+1} r_j \underbrace{\cdots}_{n} + \text{terms in } \tilde{A}_i \perp$$

Taking $r_i = \bar{\lambda}_i$, we see that $X_i(j, \bar{\lambda}_j)$ consists of a sum of a term in $R_{n+1}^\xi(\lambda)$ with terms in $\tilde{A}_i \perp$, which shows that $R_{n+1}^\lambda(\alpha_n)$ projects onto $R_n^\xi(\lambda)$. We call this projection $\pi_i$. The kernel of $\pi_i$ is the two-sided ideal generated by the elements in $\tilde{A}_i \perp$ involved above. Proceeding recursively one gets that $\pi_{1\ldots i_k}$ is a surjection of algebras. The lemma now follows from the observation that $R_{n+1}^\lambda(\alpha_n)$ projects canonically onto $\bigoplus_{\xi_{1\ldots i_k} \in \mathcal{D}_\lambda^k} A_{1\ldots i_k}$.

□

Summing over $k$ in Lemma 4.2 we have the following.

**Corollary 4.3.** We have a surjection of algebras

$$R_{n+1}^\lambda \xrightarrow{\pi^\lambda} \bigoplus_{\xi_{1\ldots i_k} \in \mathcal{D}_\lambda^k} R_n^\xi_{1\ldots i_k}(\lambda).$$

Fix a $k \geq 1$ and let

$$\Pi^\lambda_k := \text{ext}^\lambda_k : R_{n+1}^\lambda(\alpha_n) \mod \to \bigoplus_{\xi_{1\ldots i_k} \in \mathcal{D}_\lambda^k} R_n^\xi_{1\ldots i_k}(\lambda) \mod$$

and $\text{res}^\lambda_k : \bigoplus_{\xi_{1\ldots i_k} \in \mathcal{D}_\lambda^k} R_n^\xi_{1\ldots i_k}(\lambda) \mod \to R_{n+1}^\lambda(\alpha_n) \mod$ be respectively the functors of extension of scalars and restriction of scalars by the map $\pi_k$ from Lemma 4.2.

Using the surjections $\pi_{i_1\ldots i_k} : R_{n+1}^\lambda(\alpha_n) \to R_n^\xi_{1\ldots i_k}(\lambda)$ for each $\xi_{1\ldots i_k}$ that are inherited from the map $\pi_k$ from Lemma 4.2 we call

$$\Pi_{1\ldots i_k}^\lambda : R_{n+1}^\lambda(\alpha_n) \mod \to R_n^\xi_{1\ldots i_k}(\lambda) \mod$$

and

$$\text{res}_{i_1\ldots i_k}^\lambda : R_n^\xi_{1\ldots i_k}(\lambda) \mod \to R_{n+1}^\lambda(\alpha_n) \mod$$

the components of $\Pi_k^\lambda$ and $\text{res}_k^\lambda$. Functors $\Pi_{1\ldots i_k}^\lambda$ take projectives to projectives.

**Lemma 4.4.** The functors $\Pi_k^\lambda$ and $\text{res}_k^\lambda$ are biadjoint.

**Proof.** The functor $\Pi_k^\lambda$ is left adjoint to $\text{res}_k^\lambda$ by definition. We have to show that it is also its right adjoint. In other words, we have to show that it coincides with the functor of co-extension
of scalars by $\pi_k$:

$$\text{Coext}_k^\lambda : R^\lambda_{n+1}(k\alpha_n) - \text{mod} \to \bigoplus_{\xi_{i_1\ldots i_k} \in \mathcal{D}_k^\lambda} R^\xi_{i_1\ldots i_k}(\lambda) - \text{mod}$$

$$M \mapsto \text{Hom}_{R^\lambda_{n+1}(k\alpha_n)}\left(\bigoplus_{\xi_{i_1\ldots i_k} \in \mathcal{D}_k^\lambda} R^\xi_{i_1\ldots i_k}(\lambda), M\right)$$

Since every object in these categories have a presentation by projectives it is enough to show that they coincide as functors from $R^\lambda_{n+1}(k\alpha_n) - \text{pmod}$ to $(\bigoplus_{\xi_{i_1\ldots i_k} \in \mathcal{D}_k^\lambda} R^\xi_{i_1\ldots i_k}(\lambda)) - \text{pmod}$.

For a projective $\underline{i}P$ in $R^\lambda_{n+1}(k\alpha_n) - \text{pmod}$ the projective $\Pi^\lambda_{k}(\underline{i}P)$ has a basis given by all the diagrams that start in a sequence determined by $e(p_{i_1\ldots i_k}, \underline{i}')$ and end in the sequence $\underline{i}$, where the strands corresponding to $p_{i_1\ldots i_k}$ do not carry any dots and are not allowed to cross among themselves. The algebra $\bigoplus_{\xi_{i_1\ldots i_k} \in \mathcal{D}_k^\lambda} R^\xi_{i_1\ldots i_k}(\lambda)$ acts by inclusion in $\bigoplus_{\xi_{i_1\ldots i_k} \in \mathcal{D}_k^\lambda} \mathcal{A}^\xi_{i_1\ldots i_k}(\lambda)$ followed by composition on the bottom of a diagram from $\Pi^\lambda_{k}(\underline{i}P)$. This coincides with the definition of the functor of co-extension by $\pi_k$ on objects. The same argument works in the check that both functors coincide on morphisms as well. □

**Lemma 4.5.** The functor $\Pi^\lambda_{k}$ is full and essentially surjective.

**Proof.** Fullness of $\Pi^\lambda_{k}$ is a consequence of the surjectivity of $\pi_k$ and the definition of $\Pi^\lambda_{k}$. The same argument can be used to prove that each component $\Pi^\lambda_{i_1\ldots i_k}$ is essentially surjective, since every projective $\underline{i}P$ in $R^\xi_{i_1\ldots i_k}(\lambda) - \text{pmod}$ can be obtained as $\Pi^\lambda_{i_1\ldots i_k}(e(p_{i_1\ldots i_k}, \underline{i}P))$ for $e(p_{i_1\ldots i_k}, \underline{i}P)$ in $R^\lambda_{n+1}(k\alpha_n) - \text{pmod}$. □

**Lemma 4.6.** Each functor $\Pi^\lambda_{i_1\ldots i_k}$ intertwines the categorical $\mathfrak{sl}_n$-action.

**Proof.** It is clear that the projection map $\pi_{i_1\ldots i_k} : R^\lambda_{n+1}(k\alpha_n) \to R^\xi_{i_1\ldots i_k}(\lambda)$ commutes with the map $\phi_j : R^\lambda_{n+1}(k\alpha_n) \to R^\lambda_{n+1}(k\alpha_n)$ that adds a vertical strand labeled $j \in \{1, \ldots, n-1\}$ on the right of a diagram from $R^\lambda_{n+1}(k\alpha_n)$. This induces a natural isomorphism of functors $\Pi^\lambda_{i_1\ldots i_k}E^\lambda_j \cong E^\xi_{i_1\ldots i_k}(\lambda)\Pi^\lambda_{i_1\ldots i_k}$. Now consider $E^\lambda_j$ and $E^\xi_{i_1\ldots i_k}(\lambda)$. Recall that for a projective $\underline{i}P$ in $R^\lambda_{n+1}(k\alpha_n)$ with $\underline{i} \in \text{Seq}(\beta)$ the projective $E^\lambda_j(\underline{i}P)$ has a basis given by all diagrams starting in a sequence $\underline{i}'j \in \text{Seq}(\beta)$ for fixed $j$ and ending up in the sequence $\underline{i}$ and that $\Pi^\lambda_{i_1\ldots i_k}(\underline{i}P)$ has an analogous description. The isomorphism between $\Pi^\lambda_{i_1\ldots i_k}E^\lambda_j$ and $E^\xi_{i_1\ldots i_k}(\lambda)\Pi^\lambda_{i_1\ldots i_k}$ now follows from comparison between the vector spaces $\Pi^\lambda_{i_1\ldots i_k}E^\lambda_j(\underline{i}P)$ and $E^\xi_{i_1\ldots i_k}(\lambda)\Pi^\lambda_{i_1\ldots i_k}(\underline{i}P)$. □

**Proposition 4.7.** Each functor $\Pi^\lambda_{i_1\ldots i_k}$ descends to a surjection

$$K_0(\Pi^\lambda_{i_1\ldots i_k}) : K_0(R^\lambda_{n+1}(k\alpha_n)) \to K_0(R^\xi_{i_1\ldots i_k}(\lambda))$$

of $\mathfrak{sl}_n$-representations.
Finally define the functor

\[ \Pi^\lambda := \bigoplus_{k \geq 0} \Pi_k^\lambda : R^\lambda_{n+1} \mod \rightarrow \bigoplus_{\xi_{i_1 \cdots i_k} \in \mathcal{D}_\lambda} R^\xi_{i_1 \cdots i_k} \mod. \]

Functor \( \Pi^\lambda \) is full, essentially surjective and intertwines the \( \mathfrak{s}_n \)-action by Lemmas 4.4 to 4.6.

Combining Proposition 4.7 with Theorem 3.8 we have the main result of this section, which follows easily by counting dimensions.

**Theorem 4.8.** Function \( \Pi^\lambda \) descends to an isomorphism of \( \mathfrak{s}_n \)-representations

\[ K_0(\Pi^\lambda) : V^\mathfrak{s}_{n+1} \simeq K_0(R^\lambda_{n+1}) \xrightarrow{\cong} K_0 \left( \bigoplus_{\xi_{i_1 \cdots i_k} \in \mathcal{D}_\lambda} R^\xi_{i_1 \cdots i_k} \right) \cong \bigoplus_{\mu \in \tau(\lambda)} V^{\mathfrak{s}_\mu}. \]

**Corollary 4.9.** The functor \( \Pi^\lambda \) is injective on objects.

**Proof.** From the results of Subsection 3.2 translated into the cyclotomic setting we see that it is enough to prove that \( \Pi^\lambda \) is injective on the collection of objects which can be expressed as direct sums of projectives over \( R^\lambda_{n+1} \) of the form \( e'(p_{i_1 \cdots i_k}) \). We proceed by induction on the reverse order of the lexicographic order on the \( p_{i_1 \cdots i_k} \)s (this is induced by the lexicographic order on the \( k \)-tuples \( (i_1, \ldots, i_k) \in \mathbb{Z}^k_{>0} \). The base case \( (i_1, \ldots, i_k) = (n, \ldots, n) \) being trivial we proceed to the general case. The fact that \( \Pi^\lambda e'(p_{i_1 \cdots i_k}) P = 0 \) implies that

\[ \text{Hom}_{R^\lambda_{n+1}}(e'(p_{i_1 \cdots i_k}) P, e(p_{p_1 \cdots p_k} P) = 0 \]

for all \( p_{i_1 \cdots i_k} \) greater that \( p_{i_1 \cdots i_k} \) in the lexicographic order. By the induction hypothesis this implies that

\[ \text{Hom}_{R^\lambda_{n+1}}(e'(p_{i_1 \cdots i_k}) P, e(p_{p_{m_1 \cdots m_k}} P) = 0 \]

for all \( p_{m_1 \cdots m_k} \) greater that \( p_{i_1 \cdots i_k} \) in the lexicographic order, since all but the special projectives in \( \{ e'(p_{m_1 \cdots m_k}) P \}_{p_{m_1 \cdots m_k} > p_{i_1 \cdots i_k}} \) are zero. Since every object in \( R^\lambda_{n+1} \) is isomorphic to a direct summand in \( \bigoplus_{p_{i_1 \cdots i_k} \in \mathcal{D}_\lambda} P^\lambda(s_{i_1 \cdots i_k}) \) this implies that

\[ \text{End}_{R^\lambda_{n+1}}e'(p_{i_1 \cdots i_k}) P = e'(p_{i_1 \cdots i_k}) R^\lambda_{n+1} e'(p_{i_1 \cdots i_k}) \]

is one-dimensional, for any diagram other than the one consisting only of vertical strands without dots factors through elements in a sum of spaces of the form (19), as explained in Subsection 3.2. In particular this implies that \( e'(p_{i_1 \cdots i_k}) P \) is indecomposable, which contradicts Theorem 4.8 for otherwise \( K_0(\Pi^\lambda)(e'(p_{i_1 \cdots i_k}) P) \) would be nonzero since \( K_0(e'(p_{i_1 \cdots i_k}) P) \) is nonzero and \( K_0(\Pi^\lambda) \) is an isomorphism.

Lemma 4.5 and Corollary 4.9 altogether imply that the category \( \bigoplus_{\xi_{i_1 \cdots i_k} \in \mathcal{D}_\lambda} R^\xi_{i_1 \cdots i_k} \mod \) contains all the objects of \( R^\lambda_{n+1} \mod \).
4.2. Categorical projection. Although $\Pi^\lambda_k$ has a nice behavior on projectives this is not the case for $\text{res}_k^\lambda$. Functor $\text{res}_k^\lambda$ takes a projective $\mathcal{P}$ in $R_{n+1}^\lambda(k\alpha_n)$ - mod to the $R_{n+1}^\lambda(k\alpha_n)$-module $\mathcal{P} \otimes_{\mathcal{P}_{\Pi^\lambda_k}} \mathcal{P}$, which is a quotient of the projective $\mathcal{P}$ over $R_{n+1}^\lambda(k\alpha_n)$. Module $\mathcal{P}$ has a presentation by the span of the subset of the set of all diagrams from $i_1 \cdots i_k$ whose strands can be regarded as belonging to two groups, one consisting of the usual KLR-strands from $\mathcal{P}$ satisfying the KLR relations, and other consisting of $\sum_{s=1}^k (n - i_s + 1)$ strands, with labels that at any height are ordered from left to right according to $p_i$. Strands from the second group cannot intersect among themselves nor carry dots but they can intersect strands from the first group. We can also regard a diagram in $i_1 \cdots i_k \mathcal{P}$ as the overlap of a diagram from $\mathcal{P}$ and $\sum_{s=1}^k (n - i_s + 1)$ strands that run parallel to each other, do not carry dots, have labels determined by $p_i$, and end up at the left of all strands from $\mathcal{P}$, as for example in

![Diagram]

We have that $\Pi^\lambda_{\ell_1, \ldots, \ell_r}(i_1 \cdots i_k \mathcal{P})$ is isomorphic to $\mathcal{P}$ if $(\ell_1, \ldots, \ell_r) = (i_1 \cdots i_k)$ and is zero otherwise. We also have that the functor $\bigoplus_{\xi \in \mathcal{D}_\lambda} \Pi^\lambda_{\xi_1 \cdots \xi_k} \text{res}_k^\lambda$ is isomorphic to the identity functor of $\bigoplus_{\xi \in \mathcal{D}_\lambda} R_{n+1}^\lambda(\xi) - \text{mod}$. We can think of the set of the composite functors $\{\Pi^\lambda_{\xi_1 \cdots \xi_k} \text{res}_k^\lambda\}_{\xi \in \mathcal{D}_\lambda}$ as a categorical version of a complete system of orthogonal idempotents. This is another way of seeing that $R_{n+1}^\lambda - \text{mod}$ contains $\bigoplus_{\xi \in \mathcal{D}_\lambda} R_{n+1}^\lambda(\xi) - \text{mod}$.

The collection of diagrams described above can be given the structure of associative $k$-algebra if we do not force the labels on the top to be ordered according to $e(p_{i_1 \cdots i_k}, j)$ (of course, the labels of the strands from the second group have to end up in the order determined by $p_{i_1 \cdots i_k}$) and impose the relations inherited from the KLR relations (4)-(6). Denote this algebra $\tilde{R}_{n+1}^\lambda(k\alpha_n; p_{i_1 \cdots i_k}, j)$ and define

**Definition 4.10.** $\tilde{R}_{n+1}^\lambda(k\alpha_n; p_{i_1 \cdots i_k}) = \bigoplus_{\xi} \tilde{R}_{n+1}^\lambda(k\alpha_n; p_{i_1 \cdots i_k}, j)$.

Each element of $\tilde{R}_{n+1}^\lambda(k\alpha_n; p_{i_1 \cdots i_k})$ can be thought of as an overlap of two diagrams, one from $R_{n+1}(\nu(i_1 \cdots i_k))$ for some $\nu \in \Lambda_{\Pi^\lambda_k}^{n+1}$ and another one from the 1-dimensional algebra consisting of the single element given by $\sum_{s=1}^k (n - i_s + 1)$ vertical strands without dots and labeled in the order determined by $p_i$ (see Subsection 5.2 for the definition of $\nu(i_1 \cdots i_k)$).

**Definition 4.11.** We define the algebras $\tilde{R}_{n+1}^\lambda(k\alpha_n)$ and $\tilde{R}_{n+1}^\lambda$ by

$$\tilde{R}_{n+1}^\lambda(k\alpha_n) = \bigoplus_{\xi \in \mathcal{D}_\lambda} \tilde{R}_{n+1}^\lambda(k\alpha_n; p_{i_1 \cdots i_k})$$
Lemma 4.12. We have an isomorphism of algebras $\tilde{R}^\lambda_{n+1}(k\alpha_n; p_{i_1\ldots i_k}) \cong R^\lambda_{n+1}(k\alpha_n)/J_{i_1\ldots i_k}$. Moreover, we also have

**Lemma 4.13.** Module $i_1\ldots i_k \sum\mathcal{L}$ is projective as a module over $\tilde{R}^\lambda_{n+1}(k\alpha_n)$. 

**Proof.** We have that the element $e(p_{i_1\ldots i_k}; \sum\mathcal{L})$ is an idempotent in $\tilde{R}^\lambda_{n+1}(k\alpha_n)$ and the module $i_1\ldots i_k \sum\mathcal{L} = e(p_{i_1\ldots i_k}; \sum\mathcal{L})\tilde{R}^\lambda_{n+1}(k\alpha_n)$. □

Functor $\text{res}^\lambda_k$ can be regarded as a functor from $\bigoplus_{\xi_{i_1\ldots i_k} \in \mathcal{D}_k} R^\xi_{i_1\ldots i_k}(\lambda) - \text{mod}$ to $\tilde{R}^\lambda_{n+1}(k\alpha_n) - \text{mod}$, the latter category seen as the quotient of $R^\lambda_{n+1}(k\alpha_n) - \text{mod}$ by $\sum\xi_{i_1\ldots i_k} \in \mathcal{D}_k J_{i_1\ldots i_k}$. This functor takes projectives to projectives. With this in mind we see that the quotient functor

\[(20) Q_k : R^\lambda_{n+1}(k\alpha_n) - \text{mod} \rightarrow \tilde{R}^\lambda_{n+1}(k\alpha_n) - \text{mod} \]

is isomorphic to the functor $\text{res}^\lambda_k \Pi^\lambda_k$. Moreover, the functor $\Pi^\lambda$ descends to a functor

\[\tilde{\Pi}^\lambda : \tilde{R}^\lambda_{n+1} - \text{mod} \rightarrow \bigoplus_{\xi_{i_1\ldots i_k} \in \mathcal{D}^\lambda} R^\xi_{i_1\ldots i_k}(\lambda) - \text{mod} \]

which is full and essentially surjective by Lemma 4.15. It is also faithful by the definition of the projections $\pi_k$. The categories $\tilde{R}^\lambda_{n+1} - \text{mod}$ and $\bigoplus_{\xi_{i_1\ldots i_k} \in \mathcal{D}^\lambda} R^\xi_{i_1\ldots i_k}(\lambda) - \text{mod}$ are therefore equivalent. Denote by

\[\tilde{\Pi}^\lambda_k : \tilde{R}^\lambda_{n+1}(k\alpha_n) - \text{mod} \rightarrow \bigoplus_{\xi_{i_1\ldots i_k} \in \mathcal{D}^\lambda_k} R^\xi_{i_1\ldots i_k}(\lambda) - \text{mod} \]

and

\[\tilde{\text{res}}^\lambda_k : \bigoplus_{\xi_{i_1\ldots i_k} \in \mathcal{D}_k^\lambda} R^\xi_{i_1\ldots i_k}(\lambda) - \text{mod} \rightarrow \tilde{R}^\lambda_{n+1}(k\alpha_n) - \text{mod} \]

The functors induced by $\Pi^\lambda_k$ and $\text{res}^\lambda_k$ from Subsection 4.1. It is not hard to see that the composite functors $\tilde{\text{res}}^\lambda_k \tilde{\Pi}^\lambda_k$ and $\tilde{\Pi}^\lambda_k \tilde{\text{res}}^\lambda_k$ are isomorphic to the identity functors on $\tilde{R}^\lambda_{n+1} - \text{mod}$ and $\bigoplus_{\xi_{i_1\ldots i_k} \in \mathcal{D}^\lambda_k} R^\xi_{i_1\ldots i_k}(\lambda) - \text{mod}$. 

and

\[\tilde{R}^\lambda_{n+1} = \bigoplus_{k \geq 0} \tilde{R}^\lambda_{n+1}(k\alpha_n). \]
5. CATEGORIZING THE GELFAND-TSETLIN BASIS

5.1. Recovering the categorical $\mathfrak{sl}_{n+1}$-action. A key step in constructing a categorical $\mathfrak{sl}_{n+1}$-action on the categorified Gelfand-Tsetlin basis consists in recovering the categorical $\mathfrak{sl}_{n+1}$-action on $\Pi^\lambda(R_{n+1}^{\lambda} - \text{mod})$ from its categorical action on $R_{n+1}^{\lambda} - \text{mod}$. This amounts to understand the interplay between the functors $E_n^\lambda, F_n^\lambda, \Pi^\lambda$ and $\text{res}_k^\lambda$. For a module $M$ in $R_{n+1}^{\lambda} - \text{mod}$ we want to see how the functors $E_n^\lambda, F_n^\lambda$ allow to move between the components $\Pi_{i_1 \ldots i_k}(M)$ of the different categories $R_{n}^{\xi_{i_1 \ldots i_k}^\lambda} - \text{mod}$.

Contrary to the case of $\Pi_{i_1 \ldots i_k}^\lambda$ the functors $\text{res}_{i_1 \ldots i_k}^\lambda$ do not intertwine the categorical $\mathfrak{sl}_n$-action. Nevertheless we can define functors

$$E_j^{\xi_{i_1 \ldots i_k}^\lambda}(\lambda), F_j^{\xi_{i_1 \ldots i_k}^\lambda}(\lambda): R_{n}^{\xi_{i_1 \ldots i_k}^\lambda}(\lambda) - \text{mod} \to R_{n}^{\xi_{i_1 \ldots j_r}^\lambda}(\lambda) - \text{mod}$$

for $j = 1, \ldots, n-1$, by

$$E_j^{\xi_{i_1 \ldots i_k}^\lambda}(\lambda) = \Pi_{i_1 \ldots j_r}^\lambda E_j^\lambda \text{res}_{\xi_{i_1 \ldots i_k}}^\lambda$$

and

$$F_j^{\xi_{i_1 \ldots i_k}^\lambda}(\lambda) = \Pi_{i_1 \ldots j_r}^\lambda F_j^\lambda \text{res}_{\xi_{i_1 \ldots i_k}}^\lambda$$

Lemma 5.1. For $j \in \{1, \ldots, n-1\}$ the functors $E_j^{\xi_{i_1 \ldots i_k}^\lambda}(\lambda)$ and $F_j^{\xi_{i_1 \ldots i_k}^\lambda}(\lambda)$ are zero unless $(i_1, \ldots, i_k) = (j_1, \ldots, j_r)$. In this case they coincide with the functors $E_j^{\xi_{i_1 \ldots i_k}^\lambda}(\lambda)$ and $F_j^{\xi_{i_1 \ldots i_k}^\lambda}(\lambda)$ inherit from the structure of categorical $\mathfrak{sl}_{n+1}$-module on $R_{n+1}^{\lambda} - \text{mod}$.

Proof. From Lemma 4.6 we have, for $j \in \{1, \ldots, n-1\}$,

$$E_j^{\xi_{i_1 \ldots i_k}^\lambda}(\lambda) = \Pi_{i_1 \ldots j_r}^\lambda E_j^\lambda \text{res}_{\xi_{i_1 \ldots i_k}}^\lambda \cong E_j^{\xi_{j_1 \ldots j_r}^\lambda}(\lambda) \Pi_{i_1 \ldots j_r}^\lambda \text{res}_{\xi_{i_1 \ldots i_k}}^\lambda.$$

The claim follows from the fact that the functor $\Pi_{i_1 \ldots j_r}^\lambda \text{res}_{\xi_{i_1 \ldots i_k}}^\lambda$ is the identity functor acting on $R_{n}^{\xi_{i_1 \ldots i_k}^\lambda}(\lambda)$ if the sequences $(i_1, \ldots, i_k)$ and $(j_1, \ldots, j_r)$ are equal or the zero functor if they are different. The same reasoning proves the case of $F_j^{\xi_{i_1 \ldots i_k}^\lambda}(\lambda)$.

□

Definition 5.2. For $i \in \{1, \ldots, n-1\}$ we define the functors

$$F_j^\xi := \bigoplus_{\xi_{i_1 \ldots i_k} \in D_\lambda} F_j^{\xi_{i_1 \ldots i_k}^\lambda}(\lambda) \quad \text{and} \quad E_j^\xi := \bigoplus_{\xi_{i_1 \ldots i_k} \in D_\lambda} E_j^{\xi_{i_1 \ldots i_k}^\lambda}(\lambda)$$

with the obvious source and target categories.

Let us now treat the case of the functors $F_n^\lambda$ and $E_n^\lambda$. Each object $i_1 \ldots i_k \frac{1}{\lambda} L$ in $\tilde{R}_{n+1}^{\lambda} - \text{pmod}$ is also an object in $R_{n+1}^{\lambda} - \text{mod}$ which is not projective in general. It is not hard to see that the projective cover of $i_1 \ldots i_k \frac{1}{\lambda} L$ in $R_{n+1}^{\lambda} - \text{mod}$ is $\iota_{i_1 \ldots i_k}^{\lambda} P$. Lemmas 5.7 and 4.3 together with Corollary 4.9 imply that every object in $R_{n+1}^{\lambda} - \text{pmod}$ arises this way. For an endofunctor $G$ acting on $R_{n+1}^{\lambda} - \text{pmod}$ we define a functor $\tilde{G}$ on $\tilde{R}_{n+1}^{\lambda} - \text{pmod}$ as follows. For an object $M$
in $\tilde{R}_{n+1}^\lambda - \text{pmod}$ we define $\tilde{G}(M)$ as $QG(P(M))$ where $P(M)$ is the projective cover of $M$ in $R_{n+1}^\lambda - \text{mod}$ and $Q = \oplus_{k \geq 0} Q_k$ is the quotient functor from Eq. (20). The action of $\tilde{G}$ on a morphism $f$ in $\text{Hom}_{\tilde{R}_{n+1}^\lambda - \text{pmod}}(M, M')$ is defined in an analogous way. This operations are well defined, because the composite $QG'(PQG(P(M)))$ is isomorphic to $QG'GP(M)$ for $G'$ an endofunctor on $R_{n+1}^\lambda - \text{pmod}$. For morphisms $f, f'$ we observe that $P(QG(f))$ equals $G(f'f)$ yielding $\tilde{G}(f'f) = \tilde{G}(f')\tilde{G}(f)$.

**Lemma 5.3.** The pair of (biadjoint) endofunctors $\{\tilde{F}_n^\lambda, \tilde{E}_n^\lambda\}$ take projectives to projectives and define a categorical $\mathfrak{sl}_2$-action on $\tilde{R}_{n+1}^\lambda - \text{pmod}$.

This action extends canonically to a categorical action on $\tilde{R}_{n+1}^\lambda - \text{mod}$. We now use this result to construct a $\mathfrak{sl}_2$-pair of functors $\{F_n^\xi, E_n^\xi\}$ acting on the category $\bigoplus_{\xi_1, \ldots, \xi_k \in \mathcal{D}_\lambda} \tilde{R}_{n}^{\xi_1 \cdots \xi_k}(\lambda) - \text{mod}$. We first define the functors

$$F_n^{\xi_1 \cdots \xi_k}(\lambda), E_n^{\xi_1 \cdots \xi_k}(\lambda) : \tilde{R}_{n}^{\xi_1 \cdots \xi_k}(\lambda) - \text{mod} \to \tilde{R}_{n}^{\xi_{j_1} \cdots \xi_{j_r}}(\lambda) - \text{mod}$$

by

$$F_n^{\xi_{j_1} \cdots \xi_{j_r}}(\lambda) = \tilde{F}_n^{\lambda_{j_1} \cdots \lambda_{j_r}} \tilde{F}_n^{\lambda} \tilde{\text{res}}^{\lambda}_{\xi_1 \cdots \xi_k}$$

and

$$E_n^{\xi_{j_1} \cdots \xi_{j_r}}(\lambda) = \tilde{E}_n^{\lambda_{j_1} \cdots \lambda_{j_r}} \tilde{E}_n^{\lambda} \tilde{\text{res}}^{\lambda}_{\xi_1 \cdots \xi_k}$$

Both functors are zero if the sequences $(i_1, \ldots, i_k)$ and $(j_1, \ldots, j_r)$ have the same length, since this would mean that the source and target categories would correspond to weights of the form $\nu + k\alpha_n$ with $\nu = \nu_1 \alpha_1 + \cdots + \nu_{n-1} \alpha_{n-1}$ with common $k$ (all diagrams in $\tilde{R}_{n+1}^\lambda - \text{pmod}$ and $R_{n+1}^\lambda - \text{pmod}$ would contain the same number of strands labeled $n$) and both functors $E_n^\lambda$ and $F_n^\lambda$ (and $\tilde{F}_n^\lambda$, $\tilde{E}_n^\lambda$) change the number of strands labeled $n$. Summing over all $i_1 \cdots i_k$ and all $k$ we get the functors

$$F_n^\xi, E_n^\xi : \bigoplus_{\xi_1 \cdots \xi_k \in \mathcal{D}_\lambda} \tilde{R}_{n}^{\xi_1 \cdots \xi_k}(\lambda) - \text{mod} \to \bigoplus_{\xi_1 \cdots \xi_k \in \mathcal{D}_\lambda} \tilde{R}_{n}^{\xi_{j_1} \cdots \xi_{j_r}}(\lambda) - \text{mod}$$

given by

$$F_n^\xi := \bigoplus_{\xi_1 \cdots \xi_k \in \mathcal{D}_\lambda} \bigoplus_{\xi_{j_1} \cdots \xi_{j_r} \in \mathcal{D}_\lambda} F_n^{\xi_{j_1} \cdots \xi_{j_r}}(\lambda)$$

and

$$E_n^\xi := \bigoplus_{\xi_1 \cdots \xi_k \in \mathcal{D}_\lambda} \bigoplus_{\xi_{j_1} \cdots \xi_{j_r} \in \mathcal{D}_\lambda} E_n^{\xi_{j_1} \cdots \xi_{j_r}}(\lambda).$$

**Proposition 5.4.** The functors $F_n^\xi$ and $E_n^\xi$ take projectives to projectives and define a categorical $\mathfrak{sl}_2$-action on

$$\bigoplus_{\xi_1 \cdots \xi_k \in \mathcal{D}_\lambda} \tilde{R}_{n}^{\xi_1 \cdots \xi_k}(\lambda) - \text{mod}.$$
Proof. The first claim is a consequence of the definition of functors \( \{ \tilde{F}^\lambda_n, \tilde{E}^\lambda_n \} \). Biadjointness is a consequence of biadjointness of the pair \( \{ F_n, E_n \} \) and the definition of \( \{ F^\xi_n, E^\xi_n \} \). Since \( \bigoplus_{\xi_1 \ldots \xi_k \in D_\lambda} \tilde{\text{res}}_{\xi_1 \ldots \xi_k} \tilde{\Pi}_n^\lambda \) is the identity functor on \( \tilde{R}_{n+1}^\lambda(k\alpha_n) \mod \), we have

\[
F^\xi_n E^\xi_n = \bigoplus_{\xi_1 \ldots \xi_k \in D_\lambda} \Pi_{j_1 \ldots j_r} \tilde{\Pi}_n^\lambda \tilde{\text{res}}_{\xi_1 \ldots \xi_k} \tilde{\Pi}_n^\lambda \tilde{\text{res}}_{\xi_1 \ldots \xi_k} \Pi_{\xi_1 \ldots \xi_k} \tilde{\text{res}}_{\xi_1 \ldots \xi_k} \Pi_{\xi_1 \ldots \xi_k}
\]

and

\[
E^\xi_n F^\xi_n = \bigoplus_{\xi_1 \ldots \xi_k \in D_\lambda} \tilde{\Pi}_n^\lambda \tilde{\Pi}_n^\lambda \tilde{\text{res}}_{\xi_1 \ldots \xi_k} \tilde{\Pi}_n^\lambda \tilde{\text{res}}_{\xi_1 \ldots \xi_k} \Pi_{\xi_1 \ldots \xi_k} \tilde{\text{res}}_{\xi_1 \ldots \xi_k} \Pi_{\xi_1 \ldots \xi_k}
\]

and the claim follows. \( \square \)

Corollary 5.5. The functors \( \{ F^\xi_j, E^\xi_j \}_{j \in [1, \ldots, n]} \) define a categorical \( \mathfrak{sl}_{n+1} \)-action on

\[
\bigoplus_{\xi_1 \ldots \xi_k \in D_\lambda} R^\xi_{n+1}(\lambda) \mod.
\]

Corollary 5.6. With the action \( E^\xi_n \) and \( F^\xi_n \) as above the surjection \( K_0(\Pi^\lambda) \) in Proposition 4.7 is a surjection of \( \mathfrak{sl}_{n+1} \)-representations.

Proof. It is enough to show that the functor \( \Pi^\lambda \) intertwines the categorical \( \mathfrak{sl}_2 \)-action defined by \( \{ F^\xi_n, E^\xi_n \} \). For an object \( M \) in \( R^\xi_{n+1} \mod \) we have

\[
F^\xi_n \Pi^\lambda(M) = \tilde{\Pi}^\lambda \tilde{F}_n^\lambda \tilde{\text{res}}^\lambda \Pi^\lambda(M)
\]

\[
= \tilde{\Pi}^\lambda Q F_n^\lambda P(\tilde{\text{res}}^\lambda \Pi^\lambda(M))
\]

\[
= \tilde{\Pi}^\lambda \tilde{\text{res}}^\lambda \Pi^\lambda F_n^\lambda P(\tilde{\text{res}}^\lambda \Pi^\lambda(M))
\]

\[
= \Pi^\lambda F_n^\lambda P(\tilde{\text{res}}^\lambda \Pi^\lambda(M))
\]

The claim now follows from a comparison between the vector spaces \( P(\tilde{\text{res}}^\lambda \Pi^\lambda(M)) \) and \( M \). \( \square \)

5.2. The cyclotomic quotient conjecture revisited. We can now give an elementary proof of the Khovanov-Lauda cyclotomic conjecture in Type A. Recall that from Proposition 4.7 we have a surjection of \( \mathfrak{sl}_{n+1} \)-representations

\[
K_0(\Pi^\lambda) : K_0(R^\lambda_{n+1}) \to \bigoplus_{\xi_1 \ldots \xi_k \in D_\lambda} K_0(R^\xi_{n+1}(\lambda))
\]
and so, if we know that \( K_0(R^{\mathfrak{sl}_1,\ldots,\mathfrak{sl}_n}_{\mathfrak{sl}_n}^{\lambda}) \cong V^{\mathfrak{sl}_1,\ldots,\mathfrak{sl}_n}_{\mathfrak{sl}_n}^{\lambda} \) we are done. The cyclotomic conjecture for \( \mathfrak{sl}_{n+1} \) follows from the cyclotomic conjecture for \( \mathfrak{sl}_2 \) by recursion, which in turn is a consequence of the fact that in \( R^2_2 \) we have \( 1_{\mu + 1} = 0 \), where \( 1_{\mu + 1} = 0 \) is the diagram consisting of \( \mu + 1 \) vertical parallel strands.

### 5.3. Classes of special indecomposables and the Gelfand-Tsetlin basis

Applying the procedure described in Section 4 recursively we end up with a direct sum of \( d_{\lambda} := \dim(V^{\mathfrak{sl}_n}_{\mathfrak{sl}_n}^{\lambda}) \) one-dimensional \( \k \)-vector spaces. We now reverse this procedure.

**Definition 5.7.** For each Gelfand-Tsetlin pattern \( s \in S(\lambda) \) we define a functor

\[
\text{res}^s := \text{res}^s_{\mu_0} \circ \text{res}^s_{\mu_{n-1}} \circ \cdots \circ \text{res}^s_{\mu^1} : \k \to R_{n+1}^\lambda \to \ker(\pi_s).
\]

Here each of the functors \( \text{res}^s_{\mu^j} \) corresponding to the surjection \( \pi_{\mu_{j+1},\mu_j} : R^\mu_{j+1} \to R^\mu_j \), as in Lemma 4.2, which uniquely determines a sequence \((i_1,\ldots,i_k)\). Functor \( \text{res}^s \) takes the one-dimensional \( \k \)-module \( \k \) to the module \( e(p_{i_1,\ldots,i_k}) \) which is the \( R_{n+1}^\lambda \)-module consisting of \( n \) sets of non-intersecting strands, labeled by the order given by the \( p_{i_1,\ldots,i_k} \), carrying no dots, and ending in the sequence determined by the idempotent \( e(p_{i_1,\ldots,i_k}) \). To keep the notation simple from now on we write \( \pi_s \) instead of \( \pi_{i_1,\ldots,i_k} \) and \( e(s) \) instead of \( e(p_{i_1,\ldots,i_k}) \).

**Definition 5.8.** The algebra \( \check{R}_{n+1}^\lambda \) is defined as the quotient

\[
\check{R}_{n+1}^\lambda = \bigoplus_{s \in S(\lambda)} R_{n+1}^\lambda / \ker(\pi_s).
\]

As in the case of the algebras \( \check{R}_{n+1}^\lambda \) of Subsection 4.2 each algebra \( R_{n+1}^\lambda / \ker(\pi_s) \) admit a presentation by diagrams consisting of groups of strands labeled by the entries \( s(j) \) of the string \( s \). Strands within the same group cannot cross among themselves and are labeled in the order given by the idempotent \( e(p_{i_1,\ldots,i_k}) \) determined by \( s(j) \).

**Lemma 5.9.** We have \( \text{Hom}_{R_{n+1}^\lambda / \ker(\pi_s)}(e(s)L,e(s)L) = 0 \) if \( s \neq s' \).

**Lemma 5.10.** Module \( e(s)L \) is projective indecomposable as a module over \( \check{R}_{n+1}^\lambda \).

Denote by \( \check{\psi} : R_{n+1}^\lambda \to \check{R}_{n+1}^\lambda \) the projection map. The action of functors \( \{F^\lambda_i, E^\lambda_i\}_{i = 1,\ldots,n} \) on the collection of all the modules \( e(s)L \) in \( \check{R}_{n+1}^\lambda / \ker(\pi_s) \) admits the action of the functors \( \check{\psi} : \check{R}_{n+1}^\lambda / \ker(\pi_s) \to \check{R}_{n+1}^\lambda / \ker(\pi_{s'}) \) for \( s \neq s' \). Nevertheless if we change the action of the \( F^\lambda_i \)’s and of the \( E^\lambda_i \)’s on \( \check{R}_{n+1}^\lambda / \ker(\pi_s) \) we obtain a commutative diagram

\[
\begin{array}{ccc}
R_{n+1}^\lambda / \ker(\psi) & \xrightarrow{F^\lambda_i, E^\lambda_i} & R_{n+1}^\lambda / \ker(\psi) \\
\downarrow{\check{\psi}} & & \downarrow{\check{\psi}} \\
\check{R}_{n+1}^\lambda / \ker(\pi_s) & \xrightarrow{\check{F}^\lambda_i, \check{E}^\lambda_i} & \check{R}_{n+1}^\lambda / \ker(\pi_s)
\end{array}
\]

(22)
To this end we define  
\[ \tilde{F}_j^\lambda, \tilde{E}_j^\lambda : \tilde{R}_{n+1}^\lambda \rightarrow \tilde{R}_{n+1}^\lambda \mod \] 
as the composite functors 
\[ \tilde{F}_j^\lambda = \bigoplus_{s_1, s_2, s_3 \in S(\lambda)} \tilde{\text{res}}_{s_3}^s \tilde{\Pi}_{s_1}^{s_2} F_j^\lambda \tilde{\text{res}}_{s_1}^s \Pi_{s_3}^{s_2} \tilde{E}_j^\lambda = \bigoplus_{s_1, s_2, s_3 \in S(\lambda)} \tilde{\text{res}}_{s_3}^s \tilde{\Pi}_{s_1}^{s_2} E_j^\lambda \tilde{\text{res}}_{s_1}^s \Pi_{s_3}^{s_2}. \]

**Lemma 5.11.** Functions \( \tilde{F}_j^\lambda \) and \( \tilde{E}_j^\lambda \) are biadjoint and take projectives to projectives.

**Proposition 5.12.** The collection of endofunctors \( \{ \tilde{F}_j^\lambda, \tilde{E}_j^\lambda \}_{j \in \{1, \ldots, n\}} \) defines a categorical \( \mathfrak{s}l_{n+1} \)-action on \( \tilde{R}_{n+1}^\lambda \mod \). The functor \( \tilde{\Psi} \) intertwines the categorical \( \mathfrak{s}l_{n+1} \)-action.

**Theorem 5.13.** There is an isomorphism of \( \mathfrak{s}l_{n+1} \) representations 
\[ K_0(\tilde{R}_{n+1}^\lambda) \xrightarrow{\tilde{\Psi}} \tilde{\Psi}^\mathfrak{s}l_{n+1} \]
taking the projective \( e(s)L \) to the Gelfand-Tsetlin basis element \( |s\rangle \).

**Proof.** The surjection \( \tilde{\Psi} \) of algebras induces a surjective map between the Grothendieck groups 
\[ K_0(\tilde{\Psi}) : K_0(R_{n+1}^\lambda) \xrightarrow{\sim} K_0(\tilde{R}_{n+1}^\lambda) \]
intertwining the action of \( \mathfrak{s}l_{n+1} \) which is an isomorphism if \( K_0(\tilde{R}_{n+1}^\lambda) \) is not zero, by Schur’s lemma. To prove it is not zero we use the categorical branching rule to reduce the size of the category \( \tilde{R}_{n+1}^\lambda \mod \) recursively until we get something with nonzero \( K_0 \). Chose a string \( s \in S(\lambda) \). Each surjection \( \pi_{i_1, \ldots, i_k} : R_{n+1}^\lambda(k\alpha_n) \rightarrow R_{n+1}^\lambda(k\alpha_n) \) from Lemma 4.2 induces a surjection 
\[ \tilde{\pi}_{i_1, \ldots, i_k} : \tilde{R}_{n+1}^\lambda(k\alpha_n) \rightarrow \tilde{R}_{n+1}^\lambda(k\alpha_n) \]
which in turn results in a map 
\[ K_0(\tilde{\pi}_{i_1, \ldots, i_k}) : K_0(\tilde{R}_{n+1}^\lambda(k\alpha_n)) \rightarrow K_0(\tilde{R}_{n+1}^\lambda(k\alpha_n)) \]
which is surjective. Continuing recursively we end up with a chain of surjections 
\[ K_0(\tilde{R}_{n+1}^\lambda(k\alpha_n)) \rightarrow \cdots \rightarrow K_0(\tilde{R}_{n+1}^\lambda) \]
which implies that 
\[ K_0(\tilde{R}_{n+1}^\lambda) = \bigoplus_{k \geq 0} K_0(\tilde{R}_{n+1}^\lambda(k\alpha_n)) \]
is nonzero.

The second claim follows from the fact that every indecomposable in \( R_{n+1}^\lambda \mod \) splits under \( \tilde{\Psi} \) into a direct sum of indecomposables in \( \tilde{R}_{n+1}^\lambda \mod \), each one labeled by an element of \( S(\lambda) \) together with the fact that the number of projective indecomposables is the same in both categories and the already established result that the map \( K_0(\tilde{\Psi}) \) is an isomorphism.

The results above allow us to give a presentation of the category \( R_{n+1}^\lambda \mod \) in terms of the Gelfand-Tsetlin basis using the idempotents \( e(s) \).

**Proposition 5.14.** Every object in \( R_{n+1}^\lambda \mod \) is isomorphic to a direct summand of some \( e(s)P \{ \varepsilon_s \} \), for some \( s \in S(\lambda) \) and some shift \( \varepsilon_s \).

**Proof.** Every object in \( \tilde{R}_{n+1}^\lambda \mod \) is a quotient of an object in \( R_{n+1}^\lambda \mod \). An inductive argument, starting with the modules \( i_1, \ldots, i_k \) of Subsection 4.2 shows that each object \( e(s)L \) in \( \tilde{R}_{n+1}^\lambda \mod \) has a projective cover in \( R_{n+1}^\lambda \mod \) which coincides with \( e(s)P \). The claim follows from this observation together with Lemmas 3.7, 4.5 and Corollary 4.9.
**Theorem 5.15.** The isomorphism $K_0(R_{n+1}^\lambda) \to V_{\lambda}^{\text{sl}_{n+1}}$ of Theorem 3.8 sends the projective $e(s)P$ to the Gelfand-Tsetlin basis element $\{ s \}$. 

This basis is not orthogonal with respect to the $q$-Shapovalov form, but it can be used to redefining another bilinear form $(\ , \ )$ on $V_{\lambda}^{\text{sl}_{n+1}}$ as 

$$( [P], [P'] ) := \text{gdim} \, \text{Hom}_{R_{n+1}^\lambda \mod k} (P, P')$$

for $P$, $P'$ objects in $R_{n+1}^\lambda \mod$, clearly giving $(e(s)P, e(s')P) = 0$ if $s \neq s'$. 

5.4. A functorial realisation of the Gelfand-Tsetlin basis. For each $s \in S(\lambda)$ we also have functors 

$$\Pi^s := \Pi^{\mu^1} \cdots \Pi^{\mu^n} \Pi^\lambda : R_{n+1}^\lambda \mod \to k \mod$$

$$\tilde{\Pi}^s := \tilde{\Pi}^{\mu^1} \cdots \tilde{\Pi}^{\mu^n} \tilde{\Pi}^\lambda : \tilde{R}_{n+1}^\lambda \mod \to k \mod$$

and 

$$\tilde{\res}^s := \tilde{\res}^\lambda \tilde{\res}^{\mu^n} \cdots \tilde{\res}^{\mu^1} : k \mod \to \tilde{R}_{n+1}^\lambda \mod$$

with the obvious definition of the categories $\tilde{R}_{ij}^{\mu^j} \mod$. 

**Lemma 5.16.** Functors $\Pi^s$ have orthogonal hom-spaces, in the sense that for an $R_{n+1}^\lambda$-module $M$ we have that $\text{Hom}_{R_{n+1}^\lambda \mod} (\Pi^s(M), \Pi^{s'}(M)) = 0$ if $s \neq s'$. 

Let $\mathcal{G}T(\lambda)$ denote the category of functors 

$$\text{Fun} : R_{n+1}^\lambda \mod \to k \mod$$

There are endofunctors acting on $\mathcal{G}T(\lambda)$ defined by 

$$F_i^{\mathcal{G}T} \phi(M) := \bigoplus_{r,s \in S(\mu^{i-1})} \tilde{\Pi}^r F_i^{\mu^{i-1}} \tilde{\res}^s \phi(M)$$

and 

$$E_i^{\mathcal{G}T} \phi(M) := \bigoplus_{r,s \in S(\mu^{i-1})} \tilde{\Pi}^r E_i^{\mu^{i-1}} \tilde{\res}^s \phi(M)$$

for $\phi$ a functor $R_{n+1}^\lambda \mod \to k \mod$, $M$ an $R_{n+1}^\lambda$-module and $i = 1, \ldots, n$. 

**Lemma 5.17.** Each pair of functors $F_i^{\mathcal{G}T}$, $E_i^{\mathcal{G}T}$ is biadjoint. 

**Proposition 5.18.** The collection of functors $\{ F_i^{\mathcal{G}T}, E_i^{\mathcal{G}T} \}_{i=1,\ldots,n}$ defines a categorical $\text{sl}_{n+1}$-action on $\mathcal{G}T(\lambda)$. 

**Conjecture 5.19.** We have an isomorphism of $\text{sl}_{n+1}$-modules $K_0(\mathcal{G}T(\lambda)) \cong V_{\lambda}^{\text{sl}_{n+1}}$ that takes $\Pi^s$ to the Gelfand-Tsetlin basis element $\{ s \}$. 
6. CYCLOTOMIC KLR ALGEBRAS CATEGORIFY WEYL MODULES

6.1. The $q$-Schur categorification. In [22] a diagrammatic categorification of the $q$-Schur algebra was constructed using a quotient of Khovanov and Lauda’s categorified quantum groups from [16][17]. Khovanov and Lauda’s categorified quantum $\mathfrak{sl}_n$ consists of a 2-category $\mathcal{U}(\mathfrak{sl}_n)$ defined from the following data. The objects are weights $\lambda \in \mathbb{Z}^{n-1}$. The 1-morphisms are products of symbols $\lambda F_i \lambda$ (with $\lambda_j' = \lambda_j + 1$ if $j = i \pm 1$, $\lambda_j' = \lambda_j - 2$ if $j = i$, and $\lambda_j' = \lambda_j$ otherwise) and $\lambda E_i \lambda$ (with $\lambda_j' = \lambda_j - 1$ if $j = i \pm 1$, $\lambda_j' = \lambda_j + 2$ if $j = i$, and $\lambda_j' = \lambda_j$ otherwise) with the convention that says that $\lambda F_i \mu \nu F_i \lambda$ and $\lambda E_i \mu \nu E_i \lambda$ are zero unless $\mu = \nu$. The 2-morphism of $\mathcal{U}(\mathfrak{sl}_n)$ are given by planar diagrams in a strip generated by oriented arcs that can intersect transversely and can be decorated with dots (closed oriented 1 manifolds are allowed). This graphical calculus is a generalization of the KLR algebras to a calculus where the strands can travel in all directions in the sense that it gives the KLR diagrammatics when we restrict strands to travel only downwards. The boundary of each arc is decorated with a 1-morphism. These 2-morphisms are subject to a set of relations which we do not give here (see [16, 22] for details).

In [22] Khovanov and Lauda’s categorified quantum $\mathfrak{sl}_n$ was upgraded to a categorification $\mathcal{U}(\mathfrak{gl}_n)$ of quantum $\mathfrak{gl}_n$ (taking Khovanov and Lauda’s diagrams and relations of $\mathcal{U}(\mathfrak{sl}_n)$ with $\mathfrak{gl}_n$-weights) and define the categorification of $S_q(n,d)$ as the quotient of $\mathcal{U}(\mathfrak{gl}_n)$ by 2-morphisms factoring through a weight not in $\Lambda(n,d)$.

**Definition 6.1.** The category $S(n,d)$ is the quotient of $\mathcal{U}(\mathfrak{gl}_n)$ by the ideal generated by all 2-morphisms containing a region with a label not in $\Lambda(n,d)$.

The main result of [22] is that $S(n,d)$ categorifies the $q$-Schur algebra from Subsection 2.2.

**Theorem 6.2 ([22]).** There is an isomorphism of $\mathbb{Q}(q)$-algebras

$$\gamma : \hat{S}(n,d) \cong K_0(\text{Kar}(S(n,d))).$$

6.2. Categorical Weyl modules. Recall that

$$W_\lambda \cong 1_\lambda \hat{S}(n,d)/[\mu > \lambda]$$

where “$>$” is the lexicographic order, is an irreducible for $\hat{S}(n,d)$ and that all irreducibles can be obtained this way. It was conjectured in [22] that it is easy to categorify the irreducible representations $W_\lambda$, for $\lambda \in \Lambda^+(n,d)$, using the category $S(n,d)$.

**Definition 6.3.** For any $\lambda \in \Lambda^+(n,d)$, let $1_\lambda S(n,d)$ be the category whose objects are the 1-morphisms in $S(n,d)$ of the form $1_\lambda x$ and whose morphisms are the 2-morphisms in $S(n,d)$ between such 1-morphisms. Note that $1_\lambda S(n,d)$ does not have a monoidal structure, because two 1-morphisms $1_\lambda x$ and $1_\lambda y$ cannot be composed in general. Alternatively one can see $1_\lambda S(n,d)$ as a graded ring, whose elements are the morphisms.

**Definition 6.4.** Let $\mathcal{V}_\lambda$ be the quotient of $1_\lambda S(n,d)$ by the ideal generated by all diagrams which contain a region labeled by $\mu > \lambda$.

There is a natural categorical action of $S(n,d)$, and therefore of $\mathcal{U}(\mathfrak{sl}_n)$, on $\mathcal{V}_\lambda$, defined by putting a diagram in $S(n,d)$ on the right-hand side of a diagram in $\mathcal{V}_\lambda$. This action descends to
an action of $\hat{S}(n,d) \cong K_0(\text{Kar}(S(n,d)))$ on $K_0(\text{Kar}(V_\lambda))$. The map $\gamma$ from Theorem 6.2 induces a well-defined linear map $\gamma_\lambda : W_\lambda \to K_0(\text{Kar}(V_\lambda))$, which intertwines the $\hat{S}(n,d)$-actions. It was proved in [22] that $\gamma_\lambda$ is surjective and it was conjectured that it is an isomorphism. Since $W_\lambda$ is irreducible, we have $K_0(\text{Kar}(V_\lambda)) \cong V_\lambda$ or $K_0(\text{Kar}(V_\lambda)) = 0$. So it suffices to show that $K_0(\text{Kar}(V_\lambda)) \neq 0$.

From now on we regard $R_{n+1}^\lambda$ as the category whose objects are sequences of simple roots and morphisms are KLR diagrams. Let $\mathcal{N}_\lambda$ be the two-sided ideal generated by diagrams of $R_{n+1}^\lambda$ containing a bubble of positive degree in its leftmost region.

**Definition 6.5.** The category $\tilde{\mathcal{V}}_\lambda$ is the quotient of $V_\lambda$ by $\mathcal{N}_\lambda$.

The ideal $\mathcal{N}_\lambda$ is virtually-nilpotent and therefore $\tilde{\mathcal{V}}_\lambda$ has the same Grothendieck group as $V_\lambda$ (see [22] Sec. 7) where it was also explained that this quotient satisfies the cyclotomic condition from Definition 3.6. In [22] there was defined a functor from $R_{n+1}^\lambda$ to $\tilde{\mathcal{V}}_\lambda$ which is the identity on objects and morphisms where the strands in the diagrams of $R_{n+1}^\lambda$ are seen as secretly oriented downwards. This functor is clearly full and essentially surjective and it was conjectured to be faithful. We denote this functor $\Phi_\lambda$. The main result of this section is the following.

**Theorem 6.6.** The functor $\Phi_\lambda$ is faithful and therefore an equivalence of categories.

**Proof.** We can decorate the regions of the diagrams of $R_{n+1}^\lambda$ with $\mathfrak{sl}_{n+1}$-weights, starting with a $\lambda$ in the leftmost region and subtracting $\varepsilon_j - \varepsilon_{j+1}$ any time we cross a strand labeled $j$. In other words, if the region on the left of strand labeled $j$ is decorated with the weight $\lambda'$ then the label of the region immediately at its right is $\lambda' - \varepsilon_j + \varepsilon_{j+1} = (\lambda'_1, \ldots, \lambda'_j - 1, \lambda'_{j+1} + 1, \ldots, \lambda'_{n+1})$.

We first prove that if $X \in R_{n+1}^\lambda(\beta)$ contains a region labeled by $\mu \not\in \Lambda^{\mathfrak{sl}_{n+1}}$ then $X = 0$. It is enough to assume that $\mu$ is the label of its rightmost region. Moreover we can assume that $\mu_{n+1} < 0$. For suppose $\mu_j < 0$ and $\mu_i \geq 0$ for $i > k$. Then we can use the decomposition in [13] and the fact that $\Pi^\lambda$ is injective on objects to obtain an array of diagrams, each one in a distinct $R_{n}^{\xi_{1:1:k}(\lambda)}$, but all having the weight $(\mu_1, \ldots, \mu_n)$ in its rightmost region. A recursive application of this procedure yields therefore an array of diagrams in a direct sum of cyclotomic KLR algebras $\bigoplus_i R_{j}^\beta$, all of them with the rightmost region decorated with $(\mu_1, \ldots, \mu_j)$. We can assume further that $X$ is of the form $1_\Sigma$ for some sequence $r$ of simple roots.

Assume that one of the components $\Pi^{\xi_{1:1:k}(\lambda)}1_\Sigma$ is nonzero. Then we have a nonzero diagram in $R_{m}^\lambda$ connecting the special idempotent $e(p_{1:1:k}, \Sigma')$ to $1_\Sigma$, as in

$$
\begin{array}{c}
\lambda \\
\vdots \\
n \\
\vdots \\
\tilde{\mu} \\
\vdots \\
\mu
\end{array}
$$

Recall that the strands ending in $p_{1:1:k}$ in the bottom do not cross each other nor carry any dots. All strands labelled $n$ must end at the bottom among the ones corresponding to $p_{1:1:k}$. We label
The region close to the bottom of $1_\lambda$ and immediately at the right of the last strand labelled $n$, counted from the left.

Let $|\alpha_{n-1}|_\beta$ and $|\alpha_n|_\beta$ be the number of strands labelled $n-1$ and $n$ in $R^\lambda_{n-1}(\beta)$ respectively. Since $\mu_n < 0$ we must have $|\alpha_{n-1}|_\beta < |\alpha_n|_\beta$ which means that $\mu_n \leq \mu_n < 0$. This implies that $e(\rho_{12}, \mu')$ is not a special idempotent, which is a contradiction. This forces the component $\Pi^\lambda_{\mu-\mu}(\lambda)1_\lambda$ to be the zero diagram. The reasoning above applies to all components $\Pi^\lambda_{\mu-\mu}(\lambda)1_\lambda$ and altogether, it implies that $\Pi^\lambda 1_\lambda = 0$. Since $\Pi^\lambda$ is injective we conclude that $1_\lambda = 0$ in $R^\lambda_{n+1}(\beta)$.

**Corollary 6.7.** We have an isomorphism of $\hat{S}(n, d)$ representations

$$K_0(\text{Kar}(V_\lambda)) \cong W_\lambda.$$
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