Incomplete $q$-Chebyshev Polynomials

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Abstract

In this paper, we get the generating functions of $q$-Chebyshev polynomials using $\eta_z$ operator, which is $\eta_z(f(z)) = f(qz)$ for any given function $f(z)$. Also considering explicit formulas of $q$-Chebyshev polynomials, we give new generalizations of $q$-Chebyshev polynomials called incomplete $q$-Chebyshev polynomials of the first and second kind. We obtain recurrence relations and several properties of these polynomials. We show that there are connections between incomplete $q$-Chebyshev polynomials and some well-known polynomials.

Keywords: $q$-Chebyshev polynomials, $q$-Fibonacci polynomials, Incomplete polynomials, Fibonacci number

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1 Introduction

Chebyshev polynomials are of great importance in many areas of mathematics, particularly approximation theory. The Chebyshev polynomials of the second kind can be expressed by the formula

$$U_n(x) = 2xU_{n-1}(x) - U_{n-2}(x) \quad n \geq 2,$$

with initial conditions $U_0 = 1$, $U_1(x) = 2x$ and the Chebyshev polynomials of the first kind can be defined as

$$T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x) \quad n \geq 2,$$

with initial conditions $T_0(x) = 1$, $T_1(x) = x$ in [10].

The well-known Fibonacci and Lucas sequences are defined by the recurrence relations

$$F_{n+1} = F_n + F_{n-1} \quad n \geq 1$$

$$L_{n+1} = L_n + L_{n-1} \quad n \geq 1$$

with initial conditions $F_0 = 0$, $F_1 = 1$ and $L_0 = 2$, $L_1 = 1$, respectively. In [9], Filipponi introduced a generalization of the Fibonacci numbers. Accordingly, the incomplete Fibonacci and Lucas numbers are determined by:

$$F_n(k) = \sum_{j=0}^{k} \binom{n-1-j}{j}, \quad 0 \leq k \leq \left\lfloor \frac{n-1}{2} \right\rfloor$$

and

$$L_n(k) = \sum_{j=0}^{k} \frac{n}{n-j} \binom{n-j}{j}, \quad 0 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor$$

where $n \in \mathbb{N}$. Note that $F_n\left(\left\lfloor \frac{n-1}{2} \right\rfloor \right) = F_n$ and $L_n\left(\left\lfloor \frac{n}{2} \right\rfloor \right) = L_n$. In [11], the generating functions of incomplete Fibonacci and Lucas polynomials were given by Pintér and Srivastava. For more results on the incomplete Fibonacci numbers, the readers may refer to [7,8,12,14].
We need \( q \)-integer and \( q \)-binomial coefficient. There are several equivalent definition and notation for the \( q \)-binomial coefficients \([1][6][15] \). Let \( q \in \mathbb{C} \) with \( 0 < |q| < 1 \) as an indeterminate and nonnegative integer \( n \). The \( q \)-integer of the number \( n \) is defined by

\[
\langle n \rangle_q := \frac{1 - q^n}{1 - q},
\]

with \( \langle 0 \rangle_q = 0 \). The \( q \)-factorial is defined by

\[
\langle n \rangle_q := \begin{cases} 
\langle n \rangle_q \langle n - 1 \rangle_q \ldots \langle 1 \rangle_q & \text{if } n = 1, 2, \ldots \\
1 & \text{if } n = 0.
\end{cases}
\]

The Gaussian or \( q \)-binomial coefficients are defined by

\[
\begin{align*}
\begin{bmatrix} n \end{bmatrix}_q & := \frac{\langle n \rangle_q}{\langle n-k \rangle_q \langle k \rangle_q}, & 0 \leq k \leq n \\
\end{bmatrix}_q & := \frac{\langle q \rangle_q \langle n-k \rangle_q \langle q \rangle_k}{\langle q \rangle_q} & 0 \leq k \leq n
\end{align*}
\]

or

\[
\begin{align*}
\begin{bmatrix} n \end{bmatrix}_q & := \frac{\langle q \rangle_q \langle n-k \rangle_q \langle q \rangle_k}{\langle q \rangle_q} & 0 \leq k \leq n
\end{align*}
\]

with \( \langle n \rangle_q \) for \( n < k \), where \( (x; q)_n \) is the \( q \)-shifted factorial, that is, \( (x; q)_0 = 1 \),

\[
(x; q)_n = \prod_{i=0}^{n-1} (1 - q^i x).
\]

The \( q \)-binomial coefficient satisfies the recurrence relations and properties:

\[
\begin{align*}
\begin{bmatrix} n + 1 \end{bmatrix}_q &= q^k \begin{bmatrix} n \end{bmatrix}_q + \begin{bmatrix} n \end{bmatrix}_q \begin{bmatrix} k \end{bmatrix}_q & \text{(1.3)} \\
\begin{bmatrix} n + 1 \end{bmatrix}_q &= \begin{bmatrix} n \end{bmatrix}_q + q^{n-k+1} \begin{bmatrix} n \end{bmatrix}_q & \text{(1.4)} \\
\begin{bmatrix} n \end{bmatrix}_q \begin{bmatrix} n - k \end{bmatrix}_q &= q^k \begin{bmatrix} n - k \end{bmatrix}_q + \begin{bmatrix} n - k - 1 \end{bmatrix}_q & \text{(1.5)} \\
\begin{bmatrix} n \end{bmatrix}_q \begin{bmatrix} n - k \end{bmatrix}_q &= q^k \begin{bmatrix} n - k \end{bmatrix}_q + q^n \begin{bmatrix} n - k - 1 \end{bmatrix}_q & \text{(1.6)}
\end{align*}
\]

The \( q \)-analogues of Fibonacci polynomials are studied by Carlitz in \([2]\). Also, a new \( q \)-analogue of the Fibonacci polynomials is defined by Cigler and obtain some of its properties in \([5]\). In \([6]\), Pan study some arithmetic properties of the \( q \)-Fibonacci numbers and the \( q \)-Pell numbers. Cigler defined \( q \)-analogues of the Chebyshev polynomials and some properties of these polynomials in \([3][4]\).

In this paper, we derive generating functions of \( q \)-Chebyshev polynomials of first and second kind. More generally, we define incomplete \( q \)-Chebyshev polynomials of first and second kind. We get recurrence relations and several properties of these polynomials. We show that there are the relationships between \( q \)-Chebyshev polynomials and incomplete \( q \)-Chebyshev polynomials.

### 2 \( q \)-Chebyshev Polynomials

**Definition 1.** The \( q \)-Chebyshev polynomials of the second kind are defined by

\[
\mathcal{U}_n(x, s, q) = (1 + q^n)x \mathcal{U}_{n-1}(x, s, q) + q^{n-1}s \mathcal{U}_{n-2}(x, s, q) \quad n \geq 2,
\]

with initial conditions \( \mathcal{U}_0(x, s, q) = 1 \) and \( \mathcal{U}_1(x, s, q) = (1 + q)x \) in \([3]\).

**Definition 2.** The \( q \)-Chebyshev polynomials of the first kind are defined by

\[
\mathcal{T}_n(x, s, q) = (1 + q^{n-1})x \mathcal{T}_{n-1}(x, s, q) + q^{n-1}s \mathcal{T}_{n-2}(x, s, q) \quad n \geq 2,
\]

with initial conditions \( \mathcal{T}_0(x, s, q) = 1 \) and \( \mathcal{T}_1(x, s, q) = x \) in \([3]\).
It is clear that $U_n(x, -1, 1) = U_n(x)$ and $T_n(x, -1, 1) = T_n(x)$. The $q$-Chebyshev polynomials of the second kind is determined as the combinatorial sum

\[
U_n(x, s, q) = \sum_{j=0}^{\lfloor n/2 \rfloor} q^j \binom{n-j}{j}_q \frac{(-q; q)_{n-j}}{(-q; q)_j} s^j x^{n-2j}, \quad n \geq 0
\]  

(2.3)

and the $q$-Chebyshev polynomials of the first kind is determined as

\[
T_n(x, s, q) = \sum_{j=0}^{\lfloor n/2 \rfloor} q^j \binom{n-j}{j}_q \frac{(-q; q)_{n-j-1}}{(-q; q)_j} s^j x^{n-2j}, \quad n > 0
\]  

(2.4)

with $T_0(x, s, q) = 1$ in [3].

### Table 1: Some special cases of the $q$-Chebyshev polynomials of the second kind

| $x$ | $s$ | $q$ | $U_n(x, s, q)$ | $q$-Chebyshev polynomials of the second kind |
|-----|-----|-----|----------------|--------------------------------------------|
| $x$ | $-1$ | 1   | $U_n(x)$      | Chebyshev polynomials of the second kind    |
| $x$ | $\frac{1}{2}$ | 1   | $F_{n+1}(x)$  | Fibonacci polynomials                      |
| $x$ | $\frac{1}{2}$ | 1   | $F_{n+1}$     | Fibonacci numbers                          |
| $x$ | 1    | 1   | $P_{n+1}(x)$  | Pell polynomials                           |
| $x$ | 1    | 1   | $P_{n+1}$     | Pell numbers                               |
| $\frac{1}{2}$ | 2 | $y$ | $J_{n+1}(y)$  | Jacobsthal polynomials                     |
| $\frac{1}{2}$ | 2 | 1   | $J_{n+1}$     | Jacobsthal numbers                         |

### Table 2: Some special cases of the $q$-Chebyshev polynomials of the first kind

| $x$ | $s$ | $q$ | $T_n(x, s, q)$ | $q$-Chebyshev polynomials of the first kind |
|-----|-----|-----|----------------|--------------------------------------------|
| $x$ | $-1$ | 1   | $T_n(x)$      | Chebyshev polynomials of the first kind    |
| $\frac{1}{2}$ | 1 | 1   | $L_n(x)$      | Lucas polynomials                          |
| $\frac{1}{2}$ | 1 | 1   | $L_n$         | Lucas numbers                              |
| $x$ | 1    | 1   | $Q_n(x)$      | Pell-Lucas polynomials                     |
| $x$ | 1    | 1   | $Q_n$         | Pell-Lucas numbers                         |
| $\frac{1}{2}$ | 2 | $y$ | $J_n(y)$      | Jacobsthal-Lucas polynomials               |
| $\frac{1}{2}$ | 2 | 1   | $J_n$         | Jacobsthal-Lucas numbers                   |

### 2.1 Generating Functions of $q$-Chebyshev Polynomials

Andrews [15] obtain the generating function for Schur’s polynomials, which is defined by $S_n(q) = S_{n-1}(q) - q^{n-2}S_{n-2}(q)$ for $n > 1$ with intial conditions $S_0(q) = 0$ and $S_1(q) = 1$. The generating function of $S_n(q)$ is

\[
\sum_{n=0}^{\infty} S_n(q) x^n = \frac{x}{1 - x - x^2 \eta_z}
\]  

(2.5)

where is $\eta_z$ is an operator on functions of $z$ defined by $\eta_z(f(z)) = f(qz)$ in [15]. We give the following theorems for generating functions of $q$-Chebyshev polynomials of the second and first kind with an operator $\eta_z$.

**Theorem 1.** The generating function of $q$-Chebyshev polynomials of the second kind is

\[
G(z) = \frac{1}{1 - z x - (xqz + sqz^2) \eta_z}.
\]  

(2.6)
Proof. Let
\[ G(z) = \sum_{n=0}^{\infty} U_n z^n. \]

Now we show that
\[(1 - xz - (xz + sqz^2) \eta_z) G(z) = 1.\]

Thus we write
\[(1 - xz - (xz + sqz^2) \eta_z) G(z) = \sum_{n=0}^{\infty} U_n z^n - x \sum_{n=0}^{\infty} T_{n-1} z^n - x \sum_{n=0}^{\infty} T_n q^n z^n - s \sum_{n=0}^{\infty} U_n q^{n+1} z^{n+2} = \sum_{n=0}^{\infty} U_n z^n - x \sum_{n=1}^{\infty} (1 + q^n) T_{n-1} z^n - s \sum_{n=2}^{\infty} U_n q^{n-1} z^n = U_0 + U_1 z - x(1 + q) U_0 z + \sum_{n=2}^{\infty} (U_n - x(1 + q^n) U_{n-1} - s U_{n-2} q^{n-1}) z^n. \]

Therefore we have from Eq. (2.1)
\[(1 - xz - (xz + sqz^2) \eta_z) G(z) = U_0 + U_1 z - x(1 + q) U_0 z.\]

From \(U_0 = 1\) ve \(U_1 = (1 + q)x\), we get
\[(1 - xz - (xz + sqz^2) \eta_z) G(z) = 1.\]

\[ \square \]

Theorem 2. The generating function of \(q\)-Chebyshev polynomials of the first kind is
\[ S(z) = \frac{1 - xz}{1 - xz - (xz - sqz^2) \eta_z}. \quad (2.7) \]

Proof. Let \(S(z) = \sum_{n=0}^{\infty} T_n z^n\). Then
\[(1 - xz - (xz - sqz^2) \eta_z) S(z) = \sum_{n=0}^{\infty} T_n z^n - x \sum_{n=1}^{\infty} T_{n-1} z^n - x \sum_{n=0}^{\infty} T_n q^n z^n - s \sum_{n=2}^{\infty} T_n q^{n-1} z^n = \sum_{n=0}^{\infty} T_n z^n - x \sum_{n=1}^{\infty} (1 + q^n) T_{n-1} z^n - s \sum_{n=2}^{\infty} T_n q^{n-1} z^n = T_0 + T_1 z - 2x T_0 z + \sum_{n=2}^{\infty} (T_n - x(1 + q^n) T_{n-1} - s q^{n-1} T_{n-2}) z^n \]
and we get using Eq. (2.2)
\[(1 - xz - (xz - sqz^2) \eta_z) S(z) = T_0 + T_1 z - 2T_0 xz.\]

From \(T_0 = 1\) ve \(T_1 = x\), we conclude that
\[ S(z) - xz S(z) - xz \eta_z S(z) - sqz^2 \eta_z S(z) = 1 - xz, \]
finally we obtain
\[ S(z) = \frac{1 - xz}{1 - xz - (xz - sqz^2) \eta_z}, \quad (2.8) \]

\[ \square \]
3 Incomplete $q$-Chebyshev Polynomials

In this section, we define incomplete $q$-Chebyshev polynomials of the first and second kind. We give several properties for these polynomials.

**Definition 3.** For $n$ is a nonnegative integer, the incomplete $q$-Chebyshev polynomials of the second kind are defined as

$$U^k_n(x, s, q) = \sum_{j=0}^{k} q^{j} \left[ \frac{n-j}{j} \right] \frac{(-q;q)_{n-j}}{(-q;q)_j} s^j x^{n-2j} \quad 0 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor. \quad (3.1)$$

When $k = \left\lfloor \frac{n}{2} \right\rfloor$ in (3.1), $U^k_n(x, s, q) = U_n(x, s, q)$, we get the $q$-Chebyshev polynomials of the second kind in [34]. Some special cases of the incomplete $q$-Chebyshev polynomials of the second kind are provided in Table 1.

**Definition 4.** For $n$ is a nonnegative integer, the incomplete $q$-Chebyshev polynomials of the first kind are defined by

$$T^n_k(x, s, q) = \sum_{j=0}^{k} q^{j} \left[ \frac{n-j}{n-j} \right] \frac{(-q;q)_{n-j-1}}{(-q;q)_j} s^j x^{n-2j} \quad 0 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor. \quad (3.2)$$

Some special cases of the incomplete $q$-Chebyshev polynomials of the first kind are provided in Table 2.

**Theorem 3.** The incomplete $q$-Chebyshev Polynomials of the second kind satisfy

$$U^k_{n+2} = (1 + q^{n+2}) U^k_{n+1} + q^{n+1} s U^k_{n} \quad (3.3)$$

for $0 \leq k \leq \frac{n-1}{2}$.

**Proof.** From Eq. (3.1), we can write

$$(1 + q^{n+2}) U^k_{n+1} + q^{n+1} s U^k_{n} = (1 + q^{n+2}) x \sum_{j=0}^{k} q^{j} \left[ \frac{n-j+1}{j} \right] \frac{(-q;q)_{n-j+1}}{(-q;q)_j} s^j x^{n-2j}$$

$$+ q^{n+1} s \sum_{j=0}^{k} q^{j} \left[ \frac{n-j}{j} \right] \frac{(-q;q)_{n-j-1}}{(-q;q)_j} s^j x^{n-2j}$$

$$= \sum_{j=0}^{k+1} q^{j} \left\{ (1 + q^{n+2}) \left[ \frac{n-j+1}{j} \right] + q^{n+1} s \left[ 1 + q^{j} \left[ \frac{n-j+1}{j} \right] \right] \right\}$$

$$\times \frac{(-q;q)_{n-j+1}}{(-q;q)_j} s^j x^{n-2j+2}$$

$$= \sum_{j=0}^{k+1} q^{j} \left\{ \left[ \frac{n-j+1}{j} \right] + q^{n-2j+2} \left[ \frac{n-j+1}{j} \right] \right\}$$

$$+ q^{n+j+2} \left[ \frac{n-j+1}{j} \right] \left[ \frac{n-j+1}{j} \right] \frac{(-q;q)_{n-j+1}}{(-q;q)_j} s^j x^{n-2j+2}.$$ 

Thus using Eq. (1.3) and Eq. (1.4), we get

$$(1 + q^{n+2}) x U^k_{n+1} + q^{n+1} s U^k_{n} = \sum_{j=0}^{k+1} q^{j} \left[ \frac{n-j+2}{j} \right] \frac{(-q;q)_{n-j+2}}{(-q;q)_j} s^j x^{n-2j+2}$$

$$= \sum_{j=0}^{k+1} q^{j} \left[ \frac{n-j+2}{j} \right] \frac{(-q;q)_{n-j+2}}{(-q;q)_j} s^j x^{n-2j+2}$$

$$= U^k_{n+2}. \quad \square$$
Proof. Using Eq. (3.1), we obtain polynomials of the first and second kind.

By using Eq. (3.3) and Eq. (3.5), we get

\[ \mathcal{T}^{k+2}_{n+2} = x \mathcal{U}^k_{n+1} + q^{n+1} s \mathcal{U}^{k-1}_n. \] (3.5)

Proof. Using Eq. (3.1), we obtain

\[ \mathcal{U}^k_{n+1} + q^{n+1} s \mathcal{U}^{k-1}_n = x \sum_{j=0}^{k} q^j \left[ \frac{n-j+1}{j} \right] \frac{(-q; q)_{n-j+1}}{(-q; q)_j} s^j x^{n+1-2j} + q^{n+1} s \sum_{j=0}^{k-1} q^j \left[ \frac{n-j}{j} \right] \frac{(-q; q)_{n-j}}{(-q; q)_j} s^j x^{n-2j} \]

\[ = \sum_{j=0}^{k} q^j \left\{ \left[ \frac{n-j+1}{j} \right] + q^{n+1-j+1}(1+q^j) \left[ \frac{n-j+1}{j} \right] \right\} \frac{(-q; q)_{n-j+1}}{(-q; q)_j} s^j x^{n-2j+2} \]

From Eq. (1.4) and Eq. (1.9), we get

\[ x \mathcal{U}^k_{n+1} + q^{n+1} s \mathcal{U}^{k-1}_n = \sum_{j=0}^{k} q^j \left[ \frac{n+2}{n-j+2} \right] \frac{n-j+2}{n-j} \left[ \frac{n-j+1}{j} \right] \frac{(-q; q)_{n-j+1}}{(-q; q)_j} s^j x^{n-2j+2} \]

\[ = \mathcal{T}^k_{n+2}. \]

Theorem 5. The incomplete q-Chebyshev polynomials of the first kind satisfy

\[ \mathcal{T}^{k+1}_{n+2} = (1+q^{n+1})x \mathcal{T}^{k+1}_{n+1} + q^{n+1} s \mathcal{T}^k_n \] (3.6)

for \( 0 \leq k \leq \frac{n-1}{2} \).

Proof. By using Eq. (3.3) and Eq. (3.5), we get

\[ \mathcal{T}^{k+1}_{n+2} = x \mathcal{U}^{k+1}_{n+1} + q^{n+1} s \mathcal{U}^{k-1}_n \]

\[ = (1+q^{n+1})x^2 \mathcal{U}^{k+1}_{n+1} + q^n x \mathcal{U}^k_{n+1} + q^{n+1} s (1+q^n) x \mathcal{U}^k_{n+1} + q^{2n} s^2 \mathcal{U}^{k-1}_{n+2} \]

\[ = (1+q^{n+1})x \left\{ x \mathcal{U}^k_{n+1} + q^n s \mathcal{U}^k_{n+1} \right\} + q^{n+1} s \left\{ x \mathcal{U}^k_{n+1} + q^n s \mathcal{U}^{k-1}_{n+2} \right\} \]

\[ = (1+q^{n+1})x \mathcal{T}^{k+1}_{n+1} + q^{n+1} s \mathcal{T}^k_n. \]

Corollary 2. Incomplete q-Chebyshev polynomials of the first kind satisfy the non-homogeneous recurrence relation

\[ \mathcal{T}^k_{n+2} = (1+q^{n+1}) \mathcal{T}^k_{n+1} + q^{n+1} s \mathcal{T}^k_n - q^{n+1+k^2} \left[ \frac{n}{n-k} \right] \frac{n-k}{k} \left[ \frac{n-k+1}{k} \right] \frac{(-q; q)_{n-k-1}}{(-q; q)_k} s^{k+1} x^{n-2k}. \] (3.7)

Theorem 6. For \( 0 \leq k \leq \frac{n+1}{2} \), then

\[ \mathcal{T}^k_{n+2} = x \mathcal{U}^k_{n+1}(x, q^2, s, q) + q s \mathcal{U}^{k-1}_n(x, q^2, s, q) \] (3.8)

holds.
Proof. We obtain from Eq. (3.1)

\[ x \mathcal{U}_{n+1}^k(x, q^2 s, q) + q s \mathcal{U}_{n}^{k-1}(x, q^2 s, q) = x \sum_{j=0}^{k} q^j \left\{ \binom{n-j+1}{j} \frac{(-q;q)_{n+1-j}}{(-q;q)_j} (q^2 s)^j x^{n+1-2j} \right\} \]

\[ + q s \sum_{j=0}^{k-1} q^j \left\{ \binom{n-j}{j} \frac{(-q;q)_{n-j}}{(-q;q)_j} (q^2 s)^j x^{n-2j} \right\} \]

\[ = \sum_{j=0}^{k} q^j \left\{ \binom{n-j+1}{j} + (1+q^j) \binom{n-j+1}{j-1} \right\} \frac{(-q;q)_{n+1-j}}{(-q;q)_j} s^j x^{n+2-j} \]

By using Eq. (1.3), we have

\[ x \mathcal{U}_{n+1}^k(x, q^2 s, q) + q s \mathcal{U}_{n}^{k-1}(x, q^2 s, q) = \sum_{j=0}^{k} q^j \left\{ \binom{n-j+1}{j} + (1+q^j) \binom{n-j+1}{j-1} \right\} \frac{(-q;q)_{n+1-j}}{(-q;q)_j} s^j x^{n+2-j} \]

\[ = \mathcal{T}_{n+2}^k. \]

Theorem 7. We have

\[ (1+q^{n+2}) \mathcal{T}_{n+2}^k = \mathcal{U}_{n+2}^k + q^{2n+3} s \mathcal{U}_{n}^{k-1}, \quad 0 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor. \]

Proof. From Eq. (3.3) and Eq. (3.1), we get

\[ \mathcal{U}_{n+2}^k + q^{2n+3} s \mathcal{U}_{n}^{k-1} = \left\{ (1+q^{n+2})x \mathcal{U}_{n+1}^k + q^{n+1} s \mathcal{U}_{n}^{k-1} \right\} + q^{2n+3} s \mathcal{U}_{n}^{k-1} \]

\[ = \sum_{j=0}^{k} q^j \left\{ \binom{n-j+1}{j} + q^{n+1-2j+1} (1+q^j) \binom{n-j+1}{j-1} \right\} \frac{(-q;q)_{n+1-j}}{(-q;q)_j} s^j x^{n+2-j} \]

\[ + q^{n+2} \sum_{j=0}^{k} q^j \left\{ \binom{n-j+1}{j} + q^{n+1-2j+1} (1+q^j) \binom{n-j+1}{j-1} \right\} \frac{(-q;q)_{n+1-j}}{(-q;q)_j} s^j x^{n+2-j} \]

We get the following result from Eq. (1.4) and Eq. (1.6)

\[ \mathcal{U}_{n+2}^k + q^{2n+3} s \mathcal{U}_{n}^{k-1} = \sum_{j=0}^{k} q^j \left\{ \binom{n+2}{j} \frac{(-q;q)_{n+1-j}}{(-q;q)_j} s^j x^{n+2-j} \right\} \]

\[ + q^{n+2} \sum_{j=0}^{k} q^j \left\{ \binom{n+2}{j} \frac{(-q;q)_{n+1-j}}{(-q;q)_j} s^j x^{n+2-j} \right\} \]

\[ = \mathcal{T}_{n+2}^k + q^{n+2} \mathcal{T}_{n+2}^k. \]

Lemma 1. We have

\[ \frac{d \mathcal{U}_n}{dx} = n x^{-1} \mathcal{U}_n - 2 x^{-1} \sum_{j=0}^{\left\lfloor \frac{n}{2} \right\rfloor} j q^j \left\{ \binom{n-j}{j} \frac{(-q;q)_{n-j}}{(-q;q)_j} s^j x^{n-2j} \right\} \]

(3.10)
and
\[
\frac{d T_n}{dx} = n x^{-1} T_n - 2 x^{-1} \sum_{j=0}^{[\frac{n}{2}]} j q^j \sum_{j=0}^{[\frac{n}{2}]} q^n \left[ \frac{n}{j} \right] \left( -q; q \right)_{n-j} \left( -q; q \right)_{j} s^j x^{n-2j}.
\] (3.11)

Proof. By using Eq. (2.33), we have
\[
\frac{d U_n}{dx} = \frac{d}{dx} \left( \sum_{j=0}^{[\frac{n}{2}]} q^j \sum_{j=0}^{[\frac{n}{2}]} q^n \left[ \frac{n}{j} \right] \left( -q; q \right)_{n-j} \left( -q; q \right)_{j} s^j x^{n-2j} \right)
= \sum_{j=0}^{[\frac{n}{2}]} q^j (n - 2j) \left[ \frac{n}{j} \right] \left( -q; q \right)_{n-j} \left( -q; q \right)_{j} s^j x^{n-2j-1}
= n x^{-1} U_n - 2 \sum_{j=0}^{[\frac{n}{2}]} j q^j \sum_{j=0}^{[\frac{n}{2}]} q^n \left[ \frac{n}{j} \right] \left( -q; q \right)_{n-j} \left( -q; q \right)_{j} s^j x^{n-2j-1}.
\]

Similarly, from Eq. (2.4), we get Eq. (3.11). \hfill \square

Using Lemma 1, we can prove the following theorem.

**Theorem 8.** We have
\[
\sum_{k=0}^{[\frac{n}{2}]} U_k = \left( \frac{n}{2} - \frac{n}{2} + 1 \right) U_n + \frac{x d U_n}{2 dx}.
\] (3.12)

Proof. From Eq. (3.1), we have
\[
\sum_{k=0}^{[\frac{n}{2}]} U_k = U_0 + U_1 + \ldots + U_{[\frac{n}{2}]}
= \left( q^0 \left[ \frac{n}{0} \right] \left( -q; q \right)_n x^n \right) + \left( q^0 \left[ \frac{n}{0} \right] \left( -q; q \right)_n x^n + q \left[ \frac{n}{1} \right] \left( -q; q \right)_{n-1} x^{n-2} \right) + \ldots
+ \left( q^0 \left[ \frac{n}{0} \right] \left( -q; q \right)_n x^n + q \left[ \frac{n}{1} \right] \left( -q; q \right)_{n-1} x^{n-2} + \ldots + q^{\left[ \frac{n}{2} \right]} \left[ \frac{n}{\frac{n}{2}} \right] \left( -q; q \right)_{n-\frac{n}{2}} x^{n-2\left( \frac{n}{2} \right)} \right)
= \left( \frac{n}{2} + 1 \right) \left( q^0 \left[ \frac{n}{0} \right] \left( -q; q \right)_n x^n \right) + \left( \frac{n}{2} + 1 + \ldots + q^{\left[ \frac{n}{2} \right]} \left[ \frac{n}{\frac{n}{2}} \right] \left( -q; q \right)_{n-\frac{n}{2}} x^{n-2\left( \frac{n}{2} \right)} \right)
= \left( \frac{n}{2} + 1 \right) U_n - \sum_{j=0}^{[\frac{n}{2}]} q^{\left[ \frac{n}{2} \right]} \sum_{j=0}^{[\frac{n}{2}]} q^n \left[ \frac{n}{j} \right] \left( -q; q \right)_{n-j} \left( -q; q \right)_{j} s^j x^{n-2j}
= \left( \frac{n}{2} + 1 \right) U_n - \sum_{j=0}^{[\frac{n}{2}]} q^{\left[ \frac{n}{2} \right]} \sum_{j=0}^{[\frac{n}{2}]} q^n \left[ \frac{n}{j} \right] \left( -q; q \right)_{n-j} \left( -q; q \right)_{j} s^j x^{n-2j}.
\]

Then by using Lemma 1 we get
\[
\sum_{k=0}^{[\frac{n}{2}]} U_k = \left( \frac{n}{2} - \frac{n}{2} + 1 \right) U_n + \frac{x d U_n}{2 dx}.
\] \hfill \square
Theorem 9. We have
\[ \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} T_n^k = \left( \left\lfloor \frac{n}{2} \right\rfloor - \frac{n}{2} + 1 \right) T_n + \frac{x}{2} \frac{d T_n}{dx}. \] (3.13)

Proof. We have from Eq. (3.2)

\[
\begin{align*}
\sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} T_n^k &= T_n^0 + T_n^1 + \ldots + T_n^{\left\lfloor \frac{n}{2} \right\rfloor} \\
&= (q^0 \left( \begin{array}{c} n \\ 0 \end{array} \right) \frac{(-q; q)_{n-1}}{(q; q)_0} x^n) + (q^0 \left( \begin{array}{c} n \\ 0 \end{array} \right) \frac{(-q; q)_n}{(q; q)_0} x^n + q^n \left( \begin{array}{c} n \\
-1 \end{array} \right) \frac{(-q; q)_{n-1}}{(q; q)_1} x^n + \ldots \\
&\quad + \left( q^0 \left( \begin{array}{c} n \\ 0 \end{array} \right) \frac{(-q; q)_{n-1}}{(q; q)_0} x^n + q^n \left( \begin{array}{c} n \\
-1 \end{array} \right) \frac{(-q; q)_{n-1}}{(q; q)_1} x^n + \ldots \\
&\quad + q^{\left\lfloor \frac{n}{2} \right\rfloor} \left( \begin{array}{c} n \\
-\left\lfloor \frac{n}{2} \right\rfloor \end{array} \right) \frac{(-q; q)_{n-\left\lfloor \frac{n}{2} \right\rfloor}}{(q; q)_{\frac{n}{2}}} x^n \right) + \ldots \\
&= \sum_{j=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \left( \left\lfloor \frac{n}{2} \right\rfloor + 1-j \right) q^{j^2} \frac{\left( \begin{array}{c} n-j \\
\left\lfloor \frac{n}{2} \right\rfloor \end{array} \right) \frac{(-q; q)_{n-j-1}}{(q; q)_j} x^{n-2j} \\
&= \left( \left\lfloor \frac{n}{2} \right\rfloor + 1 \right) T_n - \sum_{j=0}^{\left\lfloor \frac{n}{2} \right\rfloor} j q^{j^2} \frac{\left( \begin{array}{c} n-j \\
\left\lfloor \frac{n}{2} \right\rfloor \end{array} \right) \frac{(-q; q)_{n-j-1}}{(q; q)_j} x^{n-2j}. \\
\end{align*}
\]

Lemma 1 implies that
\[ \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} T_n^k = \left( \left\lfloor \frac{n}{2} \right\rfloor - \frac{n}{2} + 1 \right) T_n + \frac{x}{2} \frac{d T_n}{dx}. \]

4 Tables and Graphs of Incomplete $q$-Chebyshev polynomials

In this section, we display the graphs of the $q$-Chebyshev polynomials and incomplete $q$-Chebyshev polynomials. Also we give the tables of some special cases of incomplete $q$-Chebyshev polynomials and numbers.

Table 3: Some special cases of the incomplete $q$-Chebyshev polynomials of the second kind

| $x$ | $s$ | $q$ | $U_n^{s}(x, s, q)$ | Incomplete $q$-Chebyshev polynomials of the second kind |
|-----|-----|-----|---------------------|------------------------------------------------------|
| $x$ | $y$ | 1   | $F_{x+1}(x, y)$     | Incomplete bivariate Fibonacci polynomials           |
| $x$ | $y$ | 1   | $F_{x+1}(x, y)$     | Incomplete bivariate Fibonacci polynomials           |
| $x$ | 1   | 1   | $F_{x+1}(x)$        | Incomplete Fibonacci polynomials                    |
| $x$ | 1   | 1   | $F_{x+1}(x)$        | Incomplete Fibonacci polynomials                    |
| $x$ | 1   | 1   | $F_{x+1}(x)$        | Incomplete Pell polynomials                         |
| $x$ | 1   | 1   | $F_{x+1}(x)$        | Incomplete Pell polynomials                         |
| $x$ | 2   | 1   | $J_{x+1}(x)$        | Incomplete Jacobsthal polynomials                   |
| $x$ | 2   | 1   | $J_{x+1}(x)$        | Incomplete Jacobsthal polynomials                   |

The numerical results for the incomplete Chebyshev numbers, first and second kind, incomplete Fibonacci, incomplete Pell, and incomplete Jacobsthal numbers are displayed in Table 4. The numerical results for the incomplete Lucas, incomplete Pell-Lucas, and incomplete Jacobsthal-Lucas numbers are displayed in Table 6.
Table 4: Some special cases of the incomplete $q$-Chebyshev polynomials of the first kind

| $x$ | $s$ | $q$ | $T_n^s(x, s, q)$ | Incomplete $q$-Chebyshev polynomials of the first kind |
|-----|-----|-----|------------------|---------------------------------------------------------|
| $\frac{-1}{2}$ | $y$ | 1 | $\frac{1}{2}L_n^s(x, y)$ | Incomplete Bivariate Lucas polynomials |
| $\frac{1}{2}$ | 1 | 1 | $\frac{1}{2}L_n^s(x)$ | Incomplete Lucas polynomials |
| $x$ | 1 | 1 | $\frac{1}{2}Q_n^s(x)$ | Incomplete Pell-Lucas polynomials |
| $1$ | 1 | 1 | $\frac{1}{2}Q_n^s$ | Incomplete Pell-Lucas numbers |
| $\frac{1}{2}$ | $2y$ | 1 | $\frac{1}{2}J_n^s(y)$ | Incomplete Jacobsthal-Lucas polynomials |
| $\frac{1}{2}$ | 2 | 1 | $\frac{1}{2}J_n^s$ | Incomplete Jacobsthal-Lucas numbers |

Table 5: Incomplete Chebyshev numbers of the first and second kind

$$T_n^s(1, -1, 1)$$  $$U_n^s(1, -1, 1)$$

| $n/k$ | 0 | 1 | 2 | 3 | 4 | 0 | 1 | 2 | 3 | 4 |
|-------|---|---|---|---|---|---|---|---|---|---|
| 1     | 1 |   |   |   |   | 2 |   |   |   |   |
| 2     | 2 | 1 |   |   |   | 4 | 3 |   |   |   |
| 3     | 4 | 1 |   |   |   | 8 | 4 |   |   |   |
| 4     | 8 | 0 | 1 |   |   | 16| 4 | 5 |   |   |
| 5     | 16| -4| 1 |   |   | 32| 0 | 6 |   |   |
| 6     | 32| -16| 2 | 1 |   | 64| -16| 8 | 7 |   |
| 7     | 64| -48| 8 | 1 |   | 128| -64| 16| 8 |   |
| 8     | 128| -128| 32| 0 | 1 | 256| -192| 48| 8 | 9 |
| 9     | 256| -320| 112| -8| 1 | 512| -512| 160| 0 | 10|

Table 6: Incomplete Fibonacci numbers, incomplete Pell numbers and incomplete Jacobsthal numbers

$$U_n^s(\frac{1}{2}, 1, 1) = F_{n+1}^s$$  $$U_n^s(1, 1, 1) = F_{n+1}^s$$  $$U_n^s(\frac{1}{2}, 2, 1) = J_{n+1}^s$$

| $n/k$ | 0 | 1 | 2 | 3 | 4 | 0 | 1 | 2 | 3 | 4 | 0 | 1 | 2 | 3 | 4 |
|-------|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1     | 1 |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |
| 2     | 1 | 2 |   |   |   | 4 | 5 |   |   |   |   |   |   |   |   |   |
| 3     | 1 | 3 |   |   |   | 8 | 12|   |   |   |   | 1 | 5 |   |   |   |
| 4     | 1 | 4 | 5 |   |   | 16| 28 | 29|   |   |   | 1 | 7 | 11 |   |   |
| 5     | 1 | 5 | 8 |   |   | 32| 64 | 70|   |   |   | 1 | 9 | 21 |   |   |
| 6     | 1 | 6 | 12| 13|   | 64| 144 | 168| 169|   |   | 1 | 114| 35 | 43 |   |
| 7     | 1 | 7 | 17| 21|   | 128| 320 | 400| 408|   |   | 1 | 13 | 53 | 85 |   |
| 8     | 1 | 8 | 23| 33| 34| 256| 704 | 944| 984| 985|   | 1 | 15 | 75 | 155| 171|
| 9     | 1 | 9 | 30| 50| 55| 512| 1336 | 2208| 2368| 2378|   | 1 | 17 | 101| 261| 341|

In Figures 1, 2 the graphs of the $q$-Chebyshev polynomials of first and second kind for $s = -1$, $q = -0.5$, $0.5$, $0.9$, $0.9999$, $n = 0, 1, 2, 3, 4, 5$ and $-1 \leq x \leq 1$ are shown.
Table 7: Incomplete Lucas numbers, incomplete Pell-Lucas numbers and incomplete Jacobsthal-Lucas numbers

| $n/k$ | $2T_n^k(\frac{1}{2}, 1, 1) = L_n^{(k)}$ | $2T_n^k(1, 1, 1) = Q_n^{(k)}$ | $2T_n^k(\frac{1}{2}, 2, 1) = j_n^{(k)}$ |
|------|-----------------|-----------------|-----------------|
|      | 0 1 2 3 4       | 0 1 2 3 4       | 0 1 2 3 4       |
| 1    | 1 2             | 1 5             |
| 2    | 1 3 4           | 1 7             |
| 3    | 1 5 7           | 1 9 17          |
| 4    | 1 6 11          | 1 11 31         |
| 5    | 1 7 16 18       | 1 13 49 65      |
| 6    | 1 8 22 29       | 1 15 71 127     |
| 7    | 1 9 29 45 47    | 1 17 97 225 257 |
| 8    | 1 10 37 67 76   | 1 19 127 367 511|
| 9    |                 |                 |

$q = -0.5$

$q = 0.5$

$q = 0.9$

$q = 0.9999$

Figure 1: Graphs of $T_n(x, s, q)$ for $s = -1$, $q = -0.5, 0.5, 0.9, 0.9999$, $n = 0, 1, 2, 3, 4, 5$
In Figure 2: Graphs of $U_n(x, s, q)$ for $s = -1$, $q = -0.5, 0.5, 0.9, 0.9999$, $n = 0, 1, 2, 3, 4, 5$.

In Figure 3 the graphs of the incomplete $q$-Chebyshev polynomials of second kind $U_0^k(x, s, q)$ for $s = -1$, $q = -0.9, -0.5, 0.5, 0.9$, $k = 0, 1, 2, 3, 4$ are shown.

In Figure 4 the graphs of the incomplete Lucas polynomials $T_0^k(x, s, q)$ for $s = 1$, $q = -0.9, -0.5, 0.5, 0.9$, $k = 0, 1, 2, 3, 4$ are shown.

In Figure 5 the graphs of the incomplete Jacobsthal polynomials $U_0^k(x, s, q)$ for $s = 2$, $q = -0.9, -0.5, 0.5, 0.9$, $k = 0, 1, 2, 3, 4$ are shown.

In Figure 6, the graphs of incomplete Fibonacci numbers $U_0^k(x, 1, 1)$ and incomplete Lucas numbers $T_0^k(x, 1, 1)$ for $1 \leq n \leq 9$ $0 \leq k \leq k$ are shown.
Figure 3: Graphs of $U^k(x, s, q)$ for $s = -1, q = -0.9, -0.5, 0.5, 0.9, k = 0, 1, 2, 3, 4$
Figure 4: Graphs of $T_5^k(x, s, q)$ for $s = 1, q = -0.9, -0.5, 0.5, 0.9, k = 0, 1, 2$
Figure 5: Graphs of $\mathcal{U}_8^k(x, s, q)$ $s = 2, q = -0.9, -0.5, 0.5, 0.9, k = 0, 1, 2, 3, 4$

Figure 6: Graphs of incomplete Fibonacci and Lucas numbers for $1 \leq n \leq 9$, $0 \leq k \leq 4$. 
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