Subfactor realisation
of modular invariants

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13 February, 2003 and revised 2 April, 2003

Dedicated to Rudolf Haag on the occasion of his eightieth birthday

Abstract

We study the problem of realising modular invariants by braided subfactors and
the related problem of classifying nimreps. We develop the fusion rule structure of
these modular invariants. This structure is useful tool in the analysis of modular data
from quantum double subfactors, particularly those of the double of cyclic groups, the
symmetric group on 3 letters and the double of the subfactors with principal graph the
extended Dynkin diagram $D_5^{(1)}$. In particular for the double of $S_3$, 14 of the 48 modular
modular invariants are nimless, and only 28 of the remaining 34 nimble invariants can
be realised by subfactors.

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1 Introduction

A prominent problem in rational conformal field theory (RCFT) is the classification of modular invariant partition functions \(Z(\tau) = \sum Z_{\lambda,\mu} \chi_\lambda(\tau) \chi_\mu^*(\tau)\) where \(\chi_\lambda(\tau) = \text{Tr}_\lambda(e^{2\pi i \tau(L_0 - c/24)})\) is the trace in the irreducible representation \(\lambda\) of the chiral algebra, with conformal Hamiltonian \(L_0\), \(\text{Im}(\tau) > 0\) and \(c\) is the central charge. This problem has been solved only for a few models, although its mathematical formulation is very simple in terms of the following modular data. For a given finite-dimensional representation of the modular group \(\text{SL}(2, \mathbb{Z})\), let \(S = [S_{\lambda,\mu}]\) and \(T = [T_{\lambda,\mu}]\) denote the matrices representing the (images) of the generators \((0 -1)\) and \((1 1)\), respectively. We further suppose that \(T\) is diagonal, \(S\) is symmetric, \(S^2\) a permutation matrix and \(S_{\lambda,0} \geq S_{0,0} > 0\) where “0” is a distinguished primary field. Then a coupling matrix \(Z\) that commutes with \(S\) and \(T\) subject to the constraints

\[Z_{\lambda,\mu} = 0, 1, 2, 3, \ldots\] and \(Z_{00} = 1\)

is called a modular invariant. These constraints reflect the physical background of the problem. The condition \(Z_{00} = 1\) reflects the uniqueness of the vacuum and will be relaxed in this work. Related to this problem is the classification of non-negative integer matrix representations (nimrep [8, Page 735]) of the original fusion rules. It arose in two a priori unrelated contexts: by Cardy [11] on boundary RCFT and di Francesco-Zuber [24] in an attempt to generalize the ADE Coxeter graphs pattern. Lately, it has been further revived by Ocneanu [45], Feng Xu [61], Böckenhauer and Evans [1, 2, 3, 4, 5, 6], Böckenhauer, Evans and Kawahigashi [7, 8, 9] and Evans [17, 18]. The set of modular invariants for a given modular data is finite [26, 5] and has also been of intense study by Cappelli et al. [10] and Gannon [27] for WZW models. The modular invariants are the torus partition functions (one loop partition function of a closed string) and the cylinder partition functions (one loop partition function of an open string) are the nimreps of the underlying RCFT. These two partition functions should be compatible [11]. We can find modular data in a wide variety of contexts. In the sequel, we will mainly be concerned with the ones arising from quantum doubles of finite groups \(G\) or those from the Wess-Zumino-Witten (WZW) models for compact Lie groups \(G\) at arbitrary levels [59]. For a fixed finite group \(G\), the primary fields are given by pairs \((a, \chi)\) where \(a\) are representatives of conjugacy classes of \(G\) and \(\chi\) are the characters of the irreducible representations of the centraliser \(C_G(a)\) of \(a \in G\).
Then the matrices \( \text{(1)} \)

\[
S_{(a,\chi),(a',\chi')} = \frac{1}{|C_G(a)||C_G(a')|} \sum_{g \in G(a,a')} \chi'(g^{-1}ag) \chi(ga'g^{-1}),
T_{(a,\chi),(a',\chi')} = \delta_{a,a'}\delta_{\chi,\chi'}\chi(a)/\chi(e)
\]

where \( G(a,a') = \{ g \in G | gag^{-1} = ga'g^{-1}a \} \) and \( e \) denotes the identity of \( G \) form the modular data of the quantum double (subfactor). This modular data can be twisted for every \( |k| \in H^3(G, S^1) \) \[4], and the subfactor interpretation of this data is in \[31\]. The parameter \( k \) is regarded as the level in this setting \[4\]. Then the quantum double of \( G \) at level 0 is precisely the model in Eq. \( (1) \). In the WZW setting (or loop group setting), the matrices \( S \) and \( T \) are the Kac-Peterson matrices constructed from the representation theory of unitary integrable highest weights modules over affine Lie algebras or in an exponentiated form from the positive energy representations of loop groups \[59\]. It has been observed \[14\] that the number of modular invariants of both the twisted group and the WZW data are small, but usually dwarfed by the swarm of modular invariants of the (untwisted) group models (cf. the 48 modular invariants for the \( G = S_3 \) with the 9 for the twisted ones).

In the framework of braided subfactors we consider a finite system of endomorphisms \( \{X_N\} \) of a type III factor \( N \) with a (non-degenerate) braiding on it, which leads to a Verlinde fusion rule algebra and produce the modular data \( S \) and \( T \) \[7\ \[51\]. Sources of modular data from braided subfactors arise from the WZW models and Ocneanu asymptotic subfactor which is regarded as the subfactor analogue of Drinfel’d quantum double construction. This quantum double subfactor is basically the same as the Longo-Rehren inclusion \[31\] and is a way of yielding braided systems from not necessarily commutative systems. The \( S \)- and \( T \)-matrices for the WZW subfactor models have been constructed by A. Wassermann \[59\] proving that they indeed coincide with the Kac-Peterson \( S \) and \( T \) matrices. The modular data from a quantum double subfactor was first established by Ocneanu \[19\ , Section 12.6\] using topological insight, later by Izumi \[31\] with an algebraic flavour (see also \[75\ \[83\]).

Fixing a braided system of endomorphisms on a type III factor \( N \), we look for inclusions \( \iota : N \leftrightarrow M \) such that its dual canonical endomorphism \( \theta = \bar{\iota} \) decomposes as a sum of endomorphisms from \( \{X_N\} \). To produce a modular invariant from such an inclusion, we first employ Longo-Rehren \( \alpha^{\pm} \)-induction method \[41\] of extending endomorphisms of \( N \) to those in \( M \) and then compute the dimensions of the intertwiner spaces \( Z_{\lambda,\mu} := \langle \alpha^+_{\lambda}, \alpha^-_{\mu} \rangle \). The matrix \( Z_{N\subset M} = [Z_{\lambda,\mu}] \) thus constructed from a braided inclusion \( N \subset M \) is a modular invariant \[7\ \[15\ , see also Sect. \[3\ or Theorem \[8\\ A modular invariant \( Z_{N\subset M} \) encodes the rich structure of the inclusion \( N \subset M \). The induced systems, which may well be non-commutative and hence not braided, encode the quantum symmetries of the physical situation we started with. Given the list of all modular invariants of a \( S \)- and \( T \)-model it is an interesting problem to determine which can be produced by braided inclusions through \( \alpha \)-induction. This is in general a difficult task and in turn related to the problem of classification of subfactors. The present work features the several known constraints a modular invariant has to fulfill in order to be produced by an inclusion as well as new ones arising from the fusion rule structure of those realised by subfactors.

In the course of previous work \[5\ \[17\ \[20\ , it was observed that modular invariants satisfy remarkable fusion rules. If the sufferable modular invariants \( \{Z_a; a\} \) which can be realised by subfactors span the commutant \( \{S, T\} \) of the modular representation, then are certainly decompositions \( Z_aZ_b = \Sigma_c m_{ab}^cZ_c \) where the \( m_{ab}^c \) are some complex coefficients. Certainly
the product $Z_aZ_b$ commutes with $S$ and $T$ and so is a modular invariant with non-negative integer entries but may not be normalised. It is therefore intriguing to ask whether any non-normalised modular invariant can be decomposed in this way with non-negative coefficients, in particular such products $Z_aZ_b$. It is in fact more natural to consider

$$Z_aZ_b^t = \Sigma_{c}n_{ab}^{c}Z_{c}$$  \hspace{1cm} (2)$$

where $t$ is the transpose. Such decompositions were noted by one of us in the context of $SU(2)$ modular invariants [5] particularly at level 16. When $N \subset M$ is a braided subfactor realising a modular invariant $Z$, then the induced $M-M$ system has $\Sigma_{\lambda,\mu}Z_{\lambda,\mu}^2$ irreducible sectors, whilst the $N-M$ system has $\Sigma_{\lambda}Z_{\lambda,\lambda}$ irreducible sectors yielding the Cappelli-Itzykson-Zuber (CIZ) graph and corresponding nimrep in the case of $SU(2)$ [8]. In the case of $Z = Z_{E_7}$, the modular invariant associated to the Dynkin diagram $E_7$, the $M-M$ system thus has 17 sectors which naturally decomposes into two induced $N-N$ orbits, on $D_{10}$ and $E_7$ graphs, $17 = 10 + 7$. This was the original motivation in [5] to write the numerical count $\Sigma_{\lambda,\mu}Z_{\lambda,\mu}^2$ as $\text{Tr}ZZ^t$, the trace of a modular invariant, since the full system $M\mathcal{X}_M$ for $Z = Z_{E_7}$ now breaks up as $\text{Tr}Z_{E_7}^2 = \Sigma_{\lambda,\mu}Z_{\lambda,\mu}^2 = 17 = 10 + 7 = \text{Tr}Z_{D_{10}} + \text{Tr}Z_{E_7}$. Remarkably not only is this numerical trace computation consistent but the matrices themselves satisfy $Z_{E_7}^2 = Z_{D_{10}} + Z_{E_7}$. This lead us to consider the role of these modular invariant fusion rules and the role of the matrix $ZZ^t$ which is nonnormalised modular invariant and to understand the full graph of $Z$ as the CIZ graph for $ZZ^t$. These ideas were used as a powerful tool in understanding the full $M-M$ graph in terms of the $N-N$ orbits in many examples - see e.g. [5] Section 6]. In [17], this analysis was taken further and in particular the programme to analyse and understand these fusion rules of modular invariants and the $M_a-M_b$ system in terms of the decomposition of the modular invariant $Z_aZ_b^t$ into normalised modular invariants was pushed further. Here we complete this programme. One of the tools we use is adopted from the work of [17] [21] in the setting of Frobenius algebras which in turn had adapted the work on $\alpha$-induction [1] [2] [3] [4] [5] [7] and used it as a tool in a braided tensor category framework. The braided Frobenius algebra product of [17] in the factor inclusion context allows us to construct inclusions whose canonical endomorphisms are the products of individual canonical endomorphisms. Then the central decomposition of the associated von Neumann algebra inclusion into subfactors yields the fusion rule algebra structure Theorem 3.6.

We note that it is known for sometime that some reasonable looking partition functions are nevertheless unphysical because they cannot appear in any consistent RCFT. With the braided inclusion approach we have an efficient machine to see if a given partition function is in fact physical. There is considerable evidence that the sufferable modular invariants are precisely those of physical interest. In the setting of algebraic quantum field theory [4], if the given factor $N$ extends to a quantum field theoretical net of factors $\{N(I)\}$ indexed by proper intervals $I \subset \mathbb{R}$ and the system $N\mathcal{X}_N$ is obtained as restricting DHR-morphisms, then a braided subfactor $N \subset M$ will provide two local nets of subfactors $\{N(I) \subset M_{\pm}(I)\}$ which provide a rigorous formulation of the left and right maximal extensions of the chiral algebra. Indeed Rehren has shown [33] with chiral observables as light-cone nets built in an observable net over 2D Minkowski space that any braided extension $N \subset M$ determines an entire local 2D conformal field theory over Minkowski space. The vacuum Hilbert space of the 2D net decomposes upon restriction to the tensor product of the left and right chiral observables precisely according to the matrix $Z$ arising from $N \subset M$ through $\alpha$-induction.
In the algebraic quantum field theory framework, $\alpha$-induction plays a critical role in taking localised DHR endorphisms to solitonic ones but the the neutral system $M_A^0$ however corresponds to proper DHR endomorphisms. Braided subfactors provide consistent unitary $6j$-symbols or Frobenius *-algebras in braided tensor categories which provide the data for computation of correlation functions [50, Section 5.3], [21]. The $6j$-symbol or connection approach and bimodules is more amenable to the type II setting and statistical mechanical framework whilst the Frobenius *-algebras or Q-systems and sectors are more amenable to the type III setting and conformal field theory framework as explained in [20]. The $N-M$ sectors (which is the CIZ graph in the $SU(2)$ setting) describes the boundary conditions, whilst the full system of $M-M$ sectors describe the defect lines.

In this work we start the analysis of modular invariants from the modular data arising from quantum double subfactors. For that we study further the structure of fusion rules of modular invariants [6, 17, 21, 20]. The first obvious case is the quantum double of the finite group subfactor $M_0 \subset M_0 \rtimes G$, where the finite group $G$ acts outerly on a type III factor $M_0$ identified by Ocneanu and later by Izumi (see [19]) to be the group-subgroup subfactor $N = M_0 \rtimes \Delta(G) \subset M_0 \rtimes (G \times G) = M$ where $\Delta(G) = \{(g, g) : g \in G\}$ denotes the diagonal subgroup of $G \times G$. The case $G = Z_2$ was studied by Böckenhauer-Evans [4] (in the context of the $SO(16n)$ WZW level 1 models). Here we consider the next models $G = Z_3$ and $S_3$ the cyclic group of order 3 and the symmetric group on 3 elements. Some obvious analogues in [17] from the previously studied models (most notably the $SU(2)$ WZW level $k$ model by Böckenhauer, Evans and Kawahigashi [8] and Ocneanu [45] where all the normalised modular invariants are indeed produced by subfactors) are no longer true (Remark 6.18). It has been announced by Ocneanu [46] that all $SU(3)$ WZW modular invariants are produced by subfactors. An outline of the history of WZW modular invariants is in [18]. The quantum $S_3$ double model has 48 modular invariants. The list of 32 modular invariants announced in [14] is not complete, we found the complete list with 16 other invariants. This extended list has been confirmed to us by T. Gannon [28]. The quantum $S_3$ double model shows a rich and complex structure. In order to have a thorough study of this model we were led to consider products of modular invariants [6, 17, 21] and in fact showing that the product of dual canonical endomorphisms are still canonical endomorphisms in the setting of braided systems whose modular invariant is the matricial product (Sect. 3). Using module category theory, Ostrik [18] has computed the possible $\text{Tr}(Z)$ and $\text{Tr}(ZZ^t)$ for the list of modular invariants $Z$ from the quantum $S_3$ double model which arise from module categories of the modular category, all labelled $(K, \psi)$ for $K$ subgroups of $S_3 \times S_3$ and $\psi$ are elements of the 2-cohomology group $H^2(K, S^1)$. With our approach we found module categories (of the $M_A^0$ system) giving rise to $\text{Tr}(Z) = 4$ which were missing in [18] since the modular invariants realised by subfactors must form a fusion rule algebra (see Theorem 3.6). Indeed there are two subgroups (conjugacy classes) of $S_3 \times S_3$ producing trace 4 matrices.

This paper is organized as follows. In Sect. 2 we review some mathematical tools, see e.g. [7], necessary in the sequel. It contains some elements of the nimrep theory following [27] and on the subfactor side the paper [8] and its generalizations. In this section we also propose some terminology for modular invariants as in Sect. 2.2. In Sect. 3 we provide the fusion rule decomposition of modular invariants using the central decomposition of braided inclusions. In Sect. 4 the relation between the subgroups of a finite group $G$ and the intermediate subfactors $N = M_0 \rtimes \Delta(G) \subset M_0 \rtimes (G \times G) = M$ is clarified. In Sect. 5 we prove that all the 8 quantum $Z_3$ double modular invariants are realised by subfactors.
Sect. 6 is devoted to the quantum $S_3$ double model. With a numerical search and employing the estimate of [6] Eq. (1.6), we found all the 48 quantum $S_3$ double modular invariants and then the Verlinde fusion matrices. We classify the modular invariants that have matching nimreps and decide which modular invariants are indeed produced by braided subfactors (we list the nimble and the sufferable ones in Corollary 6.17). In Sect. 6.6 the full systems arising from the sufferable modular invariants are displayed.

2 Preliminaries

In this section we recall the general framework of [7, 8, 14].

2.1 Braided inclusion theory

We shall consider type III von Neumann algebras with finite dimensional centres. A morphism between such algebras $A$ and $B$ shall be a faithful unital $\ast$-homomorphism $\rho : A \to B$, called a $B$-$A$ morphism, and we write $\rho \in \text{Mor}(A,B)$. We will only consider those of finite statistical dimension or inclusions of finite index. Then $d_\rho = [B : \rho(A)]^{1/2}$ is called the statistical (or quantum) dimension of $\rho$. Here $[B : \rho(A)]$ is the Jones index of the inclusion $\rho(A) \subset B$. If $\rho$ and $\sigma$ are $B$-$A$ morphisms with finite statistical dimensions, then the vector space of intertwiners $\text{Hom}(\rho,\sigma) = \{ t \in B : t\rho(a) = \sigma(a)t, a \in A \}$ is finite-dimensional, and we denote its dimension by $\langle \rho,\sigma \rangle$. A morphism conjugate to $\rho$ will be denoted $\bar{\rho} : B \to A$.

Consider a type III inclusion $N \subset M$ with $N$ a factor, and denote by $\iota$ the inclusion map. Then $\gamma = \iota \bar{\iota}$ and $\theta = \iota \iota$ are the canonical and dual canonical endomorphism of $N \subset M$ respectively. In the group case, $M^G \subset M$ where we have a outer action $\alpha$ of the finite group $G$ on say a factor $M$, then $\gamma$ decomposes as a sector into the group elements, $[\gamma] = \bigoplus_{g \in G} [\alpha_g]$, the decomposition of $\theta$ as a sector is according to irreducible representations of $G$, $[\theta] = \bigoplus_{\pi \in \hat{G}} d_\pi [\pi]$ with $d_\pi$ being the dimension of the irreducible representation of $\pi$.

Let $N\mathcal{X}_N$ denote a finite system of irreducible endomorphisms of a factor $N$ in the sense that different elements of $N\mathcal{X}_N$ are inequivalent, for any $\lambda \in N\mathcal{X}_N$ there is a representative $\bar{\lambda} \in N\mathcal{X}_N$ of the conjugate sector $[\bar{\lambda}]$ and $N\mathcal{X}_N$ is closed under composition and subsequent irreducible decomposition. We denote by $\Sigma(N\mathcal{X}_N)$ a set of representative endomorphisms of integral sums of sectors from $N\mathcal{X}_N$ [7]. The quantity $\omega = \sum d_\lambda^2$ is the global index of the system.

If $N\mathcal{X}_N = \{\rho_\lambda\}$ is a finite system and $j : N \to N^{opp}$ is the natural anti-linear isomorphism, $\rho_\lambda^{opp} = j \cdot \rho_\lambda \cdot j$, we set $B = N \otimes N^{opp}$. Longo and Rehren [11] have shown that there exists a subfactor $A \subset B$ such that $\gamma = \bigoplus_\lambda \rho_\lambda \otimes \rho_\lambda^{opp}$ is its canonical endomorphism (now referred to as the Longo-Rehren inclusion). This notion was further translated into the type II$_1$ setting by Masuda [12] who proved that Longo-Rehren inclusion and Ocneanu asymptotic inclusion are essentially the same object. Ocneanu [19] Chapter 12] has constructed a non-degenerate braiding on the $A$-$A$ system from the above quantum double inclusion $A \subset B$ and later in a more algebraic way by Izumi [31] [30]. For a subsystem $\Pi$ of $N\mathcal{X}_N$ Izumi’s Galois correspondence [31] Proposition 2.4] asserts then that there exists an intermediate subfactor $A \subset B_\Pi \subset B$ such that $\gamma_\Pi = \bigoplus_{\lambda \in \Pi} \rho_\lambda \otimes \rho_\lambda^{opp}$ is a canonical endomorphism of the inclusion $B_\Pi \subset B$.

Whenever we have a non-degenerate braiding on a system, from the Hopf link and twist we can define $S$- and $T$-matrices of type $N\mathcal{X}_N \times N\mathcal{X}_N$ [31] [56], which satisfy the Verlinde
The fusion matrices $N_{\lambda} = [N^\nu_{\lambda,\mu}, \nu]$, where $N^\nu_{\lambda,\mu} = \langle \lambda \mu, \nu \rangle$, recover the original fusion rules, i.e.

$$N_{\lambda} \cdot N_{\mu} = \sum_{\nu \in N_{\lambda,\mu}} N^\nu_{\lambda,\mu} N^\nu_{\lambda,\mu}.$$  

In our setting of a braided inclusion $\iota : N \hookrightarrow M$ of type III we are interested to extend a morphism $\lambda \in \text{End}(N)$ to another morphism in $\text{End}(M)$. We assume now that we have a (type III) inclusion $\iota : N \hookrightarrow M$ together with a finite system $N\mathcal{X}_N \subset \text{End}(N)$ which is non-degenerately braided and such that the dual canonical endomorphism $\theta = \iota \iota' \in \Sigma(N\mathcal{X}_N)$ for the inclusion $M-N$ morphism $\iota : N \hookrightarrow M$. In the above setting, we say that $N \subset M$ is a braided subfactor. One can define the $\alpha$-induced morphisms $\alpha^\pm_{\lambda} \in \text{End}(M)$ by the Longo-Rehren formula [11]:

$$\alpha^\pm_{\lambda} = \overline{c}^{-1} \circ \text{Ad}(c^{\pm}_{(\lambda, \theta)}) \circ \lambda \circ \overline{c},$$

where $c$ is the braiding, so that $\alpha^\pm_{\lambda}$ extends $\lambda$ in $N\mathcal{X}_N$, $\alpha^\pm_{\lambda} \iota = \iota \lambda$. We can define the positive integral matrix $Z_{\lambda,\mu} = \langle \alpha^+,_{\lambda}, \alpha^-_{\mu} \rangle$, normalised at the vacuum $Z_{0,0} = 1$ if $M$ is a factor, sometimes denoted by $Z_{N \subset M}$ when we want to emphasize the inclusion from which it was constructed. By [4, 18], the matrix $Z_{N \subset M}$ commutes with the modular $S$- and $T$-matrices for subfactors which also holds for inclusions under the decomposition into normalised ones (cf. Theorem 3.6). Therefore $Z$ is a modular invariant. Now we use $\alpha$-induction and the inclusion map $\iota$ to construct finite systems whose general theory has been developed in [7, 18]. Let us choose representative endomorphisms of each irreducible subsector of sectors of the form $[\iota \lambda \iota']$, $\lambda \in N\mathcal{X}_N$. Any subsector of $[\alpha^+_{\lambda}, \alpha^-_{\mu}]$ is automatically a subsector of $[\nu \nu']$ for some $\nu$ in $N\mathcal{X}_N$ and since we assume the non-degeneracy of the braiding the converse also holds true [7]. This set of sectors yields a system $M\mathcal{X}_M$ of sectors in general non-commutative (the original sectors from the system $N\mathcal{X}_N$ is commutative since it is braided). We define in a similar fashion the chiral systems $M\mathcal{X}^\pm_M$ to be the subsystems of $\beta \in M\mathcal{X}_M$ such that $[\beta]$ is an irreducible subsector of $[\alpha^+_{\lambda}]$. The neutral or ambichiral system $M\mathcal{X}^0_M$ is defined as the intersection $M\mathcal{X}^-_M \cap M\mathcal{X}^+_M$, so that we obtain $M\mathcal{X}^0_M \subset M\mathcal{X}^+_M \subset M\mathcal{X}_M$ (see e.g. [4]).

In the braided subfactor case, their global indices (the sum of the squares of the quantum dimensions) are denoted by $\omega_0, \omega_\pm$ and $\omega$, and are completely encoded in the modular invariant $Z$, namely [8, 11],

$$\omega_\pm = \sum_{\lambda} d_{\lambda} Z_{\lambda,0}^\omega = \sum_{\lambda} d_{\lambda} Z_{0,\lambda}^\omega, \quad \omega_0 = \omega_\pm^2 / \omega.$$  

To help find the irreducible sectors in each of the above system one has the relation [8, Eq. (33)]

$$\langle \alpha^+_{\lambda}, \alpha^-_{\mu} \rangle \leq \langle \theta \lambda, \mu \rangle.$$  

Finally, the neutral system $M\mathcal{X}^0_M$ inherits a non-degenerate braiding (therefore their fusion rules are commutative) by [11] whose modular matrices are denoted by $S^{\text{ext}}$ and $T^{\text{ext}}$. We can recover the matrix $Z$ from the branching coefficients $(b^+_{\tau,\lambda} = \langle \tau, \alpha^+_{\lambda} \rangle)$ as follows:

$$Z_{\lambda,\mu} = \sum_{\tau \in M\mathcal{X}^0_M} b^+_{\tau,\lambda} b^-_{\tau,\mu}.$$  

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Moreover we have the following intertwining properties of the branching coefficient matrices between original and extended $S$- and $T$-matrices [11 Theorem 6.5]:

\[ S^\text{ext} b^\pm = b^\pm S, \quad T^\text{ext} b^\pm = b^\pm T. \tag{7} \]

The modular invariant $Z$ is a permutation matrix if and only if $M \lambda_M = M \chi_M$ [3,8]. The modular invariant $Z$ produced by a subfactor $N \subset M$ is said to be of type I if $Z_{0,\lambda} = \langle \theta, \lambda \rangle$ for all $\lambda \in \mathcal{X}_N$ [3,11] (in particular $Z$ is symmetric). In this situation we say that the chiral locality holds for the subfactor $N \subset M$. Note however that there may be other braided subfactors producing the same modular matrix $Z$. In other words, the chiral locality holds if the dual canonical endomorphism is visible in the vacuum row (and hence column), so $[\theta] = \oplus Z_{0,\lambda} [\lambda]$. In the presence of chiral locality, we can usually recover the canonical endomorphism from the full $M$-$M$ system [3] by computing the dimensions $\langle \alpha^+_\lambda, \alpha^+_\mu, \gamma \rangle = \langle \alpha^+_{\lambda}, \alpha^+_{\mu} \rangle$.

If a braided subfactor $N \subset M$ produces a modular invariant $Z$, then there are intermediate subfactors $N \subset M_{\pm} \subset M$, such that $N \subset M_{\pm}$ produce symmetric modular invariants $Z_{\pm}$ (type I parents) [3]. Moreover we have $Z_{\lambda,0} = Z_{\lambda,0}^+$ and $Z_{0,\lambda} = Z_{0,\lambda}^{-}$ and the canonical endomorphisms of $N \subset M_{\pm}$ are visible from the vacuum row and column of $Z$: $[\theta_+] = \oplus Z_{0,\lambda} [\lambda]$ and $[\theta_-] = \oplus Z_{0,\lambda} [\lambda]$, which incidentally implies Eq. (5).

**Definition 2.1** A non-negative integer representation (nimrep, for short [5]) of a given modular data is an assignment of a matrix $G_{[\lambda]}$ (or $G_{\lambda}$ for short) to each primary field $\lambda$, with non-negative integer entries, preserving the fusion rules, i.e.

\[ G_{\lambda} \cdot G_{\mu} = \sum_\eta N_{\lambda,\mu}^{\eta} G_{\eta}, \quad G_0 = \text{id}, \quad \text{and } G_{\lambda} = G_{\lambda}^t \text{ for all } \lambda, \mu \]

where $N_{\lambda,\mu}^{\eta}$ are the Verlinde fusion integers from the original modular data.

By a standard argument (see e.g. [58] or [19, Page 425]), we can simultaneously diagonalize all $G_{\lambda}$’s, and moreover the eigenvalues of $G_{\lambda}$ are $S_{\mu\lambda}/S_{\mu0}$ for $\mu$ running in some multi-set (possibly with multiplicities). The cardinality of this set is the dimension of the nimrep. Two nimreps $G'$ and $G''$ are equivalent if there is a permutation matrix $P$ such that $P G_{\lambda}^t P^{-1} = G''_{\lambda}$ for all $\lambda$. There is a natural notion of direct sum of nimreps [27]. The regular nimrep is obtained, by setting $G_{\lambda} := N_{\lambda}$ where $N_{\lambda}$ is the Verlinde fusion matrix associated to the primary field $\lambda \in \mathcal{X}_N$. By the exponents of a modular invariant $Z$, we mean the multi-set $\text{Exp}$ consisting of $Z_{\lambda\lambda}$ copies of $\lambda \in \mathcal{X}_N$.

RCFT is thought to require that each physical modular invariant has at least one nimrep such that their exponents coincide. A modular invariant is nimble if it has a matching nimrep. A nimrep hereafter is meant to be compatible with some modular invariant (i.e., its spectrum and dimension are dictated by the diagonal part of the modular invariant). Hence we discard (although some of the results below remain valid) nimreps with no consistent modular invariant as being spurious nimreps. A nimless modular invariant is a modular invariant without a matching nimrep. We regard the category of nimless modular invariants the least interesting of all.

Following closely Gannon’s paper [27], let us fix a nimrep $G$. Then by Perron-Frobenius theory $S_{\lambda0}/S_{00}$ is the norm of the matrix $G_{\lambda}$ for every $\lambda \in \mathcal{X}_N$. Clearly, all $G_{\lambda}$’s are symmetric if all the exponents $\mu$ are self conjugate $\mu = \bar{\mu}$. More generally, $G_{\lambda} = G_{\eta}$ if and only if $S_{\lambda\mu} = S_{\mu\mu}$ for all $\mu \in \text{Exp}$. We conveniently enumerate some properties [27] of nimreps that we use in the sequel.
1. For a given modular data, there are only finitely many indecomposable inequivalent nimreps.

2. For every primary field $\lambda$, the norm of every connected component of $G_\lambda$ is $S_{\lambda 0}/S_{0 0}$. Moreover, $[G_\lambda]_{i,j} \leq S_{\lambda 0}/S_{0 0}$.

3. The spectrum of every matrix $G_\lambda$ is $\{S_{\lambda \mu}/S_{0 \mu} : \mu \in \text{Exp}\}$.

4. The number of indecomposable components of $G_\lambda$ is the number of exponents $\mu$ such that $S_{\lambda \mu}/S_{0 \mu} = S_{\lambda 0}/S_{0 0}$.

5. No column or row of each $G_\lambda$ can be identically equal to zero.

We finish this section with a well posed problem.

**Problem.** Classify the nimreps of a given modular data.

### 2.2 Induced nimreps for sufferable modular invariants

Let $N \subset M_a$ and $N \subset M_b$ be two braided subfactors, with $\iota_a$ and $\iota_b$ denoting the inclusion maps respectively, whose modular invariants of the given modular data we denote by $Z_a$ and $Z_b$ respectively. Let $M_a X, M_b X$ denote the irreducible representative endomorphisms from the subsectors of $[\iota_a \lambda \bar{\iota}_b]$ with $\lambda$ running over $N X N$. Then by extending ideas from [7] we can show [17] that $M_a X, M_b X$ under the left action $M_a X, M_a$ and right action of $M_b X, M_b$ is isomorphic to

$$\bigoplus_{\lambda, \mu \in N X N} H^c_{\lambda, \mu} \otimes \mathcal{T}^c_{\lambda, \mu},$$

where $H^c_{\lambda, \mu} \subseteq \bigoplus_{x \in N X M_c} \text{Hom}(\lambda \bar{\mu}, x \bar{x})$ is a certain Hilbert space of dimension $[Z_c]_{\lambda, \mu}, c = a, b$. In particular $\# M_a X M_b = \text{Tr}(Z_a(Z_b)^t)$. Moreover, we have an induced nimrep in a stronger sense as follows (see [17]). The matrices

$$\Gamma^\beta_{\lambda, \mu; \beta'} = \langle \beta \alpha^+_{\lambda, \mu}, \beta' \rangle, \quad \beta, \beta' \in M_a X M_b$$

form a nimrep (we can use either inductions $\alpha^{(a)}$ or $\alpha^{(b)}$ from the braided subfactors $N \subset M_a$ or $N \subset M_b$ respectively). Such a $Z_a Z_b$ nimrep provides us with further constraints for a modular invariant to be produced by a braided subfactor. The eigenvalues of $\Gamma^\beta_{\lambda, \mu}$ arise from $\{\chi_\zeta(\lambda) \chi_\eta(\mu)\}$, where $\chi_\lambda(\nu) = S_{\lambda \nu}/S_{\lambda 0}$, with multiplicities

$$\text{mult}(\chi_\zeta(\lambda) \chi_\eta(\mu)) = [Z_a]_{\zeta, \eta}[Z_b]_{\zeta, \eta}$$

This is a generalization of the classification result [8, Theorem 4.16]. We display in Fig. 1 the number of irreducible sectors; each vertex is labelled with an algebra $N, M_a$ or $M_b$ and a line between two algebras represents the number of irreducible sectors.

Hence if $M_a = M_b$, then $\# M X M = \text{Tr}(Z Z^t)$ and moreover we can see in particular that $M X M$ is commutative if and only if $Z_{\lambda, \mu} \leq 1$ for all $\lambda, \mu$ (so recovering [7, Theorem 6.8]). Also $b^\tau_{\tau, \lambda} \leq 1$ for all $\tau, \lambda$ if and only if $M X M_c$ is commutative (as part of [8, Eq. (4.11)]). Furthermore, if $M_b = N$ we get $\text{Tr}(Z_{N \subset M})$ irreducible $M-N$ sectors by decomposing $[\iota \lambda], \ldots$
Figure 1: Irreducible $M_a$-$M_b$ sectors from the subfactors $N \subset M_a$ and $N \subset M_b$ with modular invariants $Z_a$ and $Z_b$ for $\lambda$ running in the braided system $NAX_N$, into irreducible sectors and we recover the nimrep constructed in [8, Page 768] or [6, Page 10]:

$$G^b_{\lambda,a} := \langle \alpha^\pm a, b \rangle = \langle a\lambda, b \rangle, \quad a, b \in MAX_N.$$ 

As an application, these matrices $G_{\lambda}$ are used together with the fact that $Z := Z_{N\subset M}$ commutes with $S$ to get the following curious identity [17, Proposition 3.3]:

$$\bigoplus_{a \in MAX_N} [a\bar{a}] = \bigoplus_{\lambda,\mu \in NAX_N} Z_{\lambda,\mu}[\lambda\bar{\mu}]. \quad (11)$$

The matrices $G_{\lambda}$ do not depend on the $\pm$ induction choice. We can consider $G_{\lambda}$ as the adjacency matrix of the fusion graph of $[\alpha^\pm]$ on the $MAX_N$ sectors via left multiplication. Then the set of matrices $\{G_{\lambda}\}$ yield a nimrep of the underlying modular data [8] whose exponents match with the exponents of the modular invariant $Z_{N\subset M}$.

**Definition 2.2** A modular invariant $Z$ is said to be sufferable if it arises from a braided inclusion $N \subset M$ through the process of $\alpha$-induction $[Z]_{\lambda,\mu} = \langle \alpha^\pm_\lambda, \alpha^\pm_\mu \rangle$. It is called insufferable otherwise.

Therefore every sufferable modular invariant has a compatible nimrep. We are also interested in tackling the following well posed problem.

**Problem.** For a given modular data, classify their sufferable modular invariants. We assemble the 3 categories (from the ones that encode a very deep and rich structure to poorer ones) of modular invariants that are interesting to classify for a fixed RCFT or modular data: sufferable modular invariants, nimble but insufferable modular invariants and nimless modular invariants.

### 3 Q-systems for inclusions

Let $N$ be a factor and $\iota : N \to M$ be an inclusion in a von Neumann algebra $M$, with $\bar{\iota} : M \to N$ a conjugate endomorphism of $\iota$. Then since $\bar{\iota}\iota, \iota\bar{\iota}$ both contain the identity $id_M$, $id_N$ respectively, there are intertwining isometries, $v$ and $w_1$, in $\text{Hom}(id_N, \bar{\iota})$, $\text{Hom}(id_M, \bar{\iota})$.
respectively \[39\] \[30\]. Then \( w = \bar{\iota}(w_1) \) is an isometry in \( \text{Hom}(\theta, \theta^2) \) where \( \theta = \bar{\iota} \) is the dual canonical endomorphism which satisfy \[40\]

\[
\begin{align*}
 w^*\theta(w) &= w w^*, \\
 w^2 &= \theta(w)w, \\
 v^*w &= w^*\theta(v) = 1/d
\end{align*}
\]

with \( d = d(\iota) \). Hence \( d^2 = [M : N] \). It is convenient to represent intertwiners graphically

![Intertwiners](image)

Figure 2: Intertwiners \( t \in \text{Hom}(\rho, \sigma) \) and \( t^* \in \text{Hom}(\sigma, \rho) \)

Our convention is that an element \( t \) in the intertwiner space \( \text{Hom}(\rho, \sigma) \) is written as in Fig. 2. With this convention, we write the isometries \( w \) and \( v \) and as in Fig. 3 and Fig. 4 respectively. Then the relations of Eq. \[12\] can be displayed graphically as in Figs. 5, 6 and 7

![Diagram](image)

Figure 3: Diagrammatic representation of \( \sqrt{d(\iota)}w \) and \( \sqrt{d(\iota)}w^* \), respectively

![Diagram](image)

Figure 4: Diagrammatic representation of \( \sqrt{d(\iota)}v \) and \( \sqrt{d(\iota)}v^* \), respectively

The system \( \Theta = (\theta, v, w) \) is called a Q-system by Longo \[40\] which characterises precisely which endomorphisms can arise as dual canonical endomorphisms for \( N \subset M \). More precisely, Longo worked with equivalent notion of characterising canonical endomorphisms. It is well known that the dual sector \( [\theta] \) does not determine \( M \) uniquely up to inner conjugacy. This is an \( H^2 \) cohomological obstruction that has been studied in \[31\] \[33\] where the
following definition is proposed. Two Q-systems \((\theta_1, v_1, w_1), (\theta_2, v_2, w_2)\) are equivalent if there exists a unitary \(u \in \text{Hom}(\theta_1, \theta_2)\) such that \(v_2 = uv_1, w_2 = u\theta_1(u)w_1^*\). In particular we obtain \(\theta_2(\cdot) = u\theta_1(\cdot)u^*\) thus \([\theta_1] = [\theta_2]\). The above relations Eq. (12) mean that a Q-system is a Frobenius algebra \(A = (\theta, m, e, \Delta, \epsilon)\) where \(e \in \text{Hom}(1, \theta), m \in \text{Hom}(\theta^2, \theta), \epsilon \in \text{Hom}(\theta, 1), \Delta \in \text{Hom}(\theta, \theta^2)\) such that \((\theta, m, e)\) is an algebra, \((\theta, \Delta, \epsilon)\) is a co-algebra with the algebraic and co-algebraic structure related by

\[
(id_{\Theta} \otimes m) \circ (\Delta \otimes id_{\Theta}) = \Delta \circ m = (m \otimes id_{\Theta}) \circ (id_{\Theta} \otimes \Delta).
\]

This is in as Fig. 5 where \(m\) and \(\Delta\) are identified with \(w^*\) and \(w\) respectively. A Frobenius algebra \(A\) is said to be special [21, Definition 3.4] if there are non-zero constants \(\beta_1\) and \(\beta_A\) such that \(\varepsilon \circ e = \beta_1 id_1, m \circ \Delta = \beta_A id_A\). For a Q-system \(\Theta = (\theta, v, w), v^*v = 1\) and \(w^*w = 1\), implying that \(\Theta\) is a special \(*\)-Frobenius algebra (with \(\beta_1 = \beta_\Theta = d(\iota)\)), see Figs. 8 and 11. A Q-system \(\Theta\) is automatically a symmetric Frobenius algebra [21, Definition 3.4], as in Fig. 8 since \(id = d^2v^*w^*\theta(w)\theta(v) = d^2\theta(v^*)\theta(w^*)wv\).

For a Q-system \(\Theta = (\theta, v, w)\), a (left) \(\Theta\)-module [47] (see Fig. 10) is a pair \((U, \rho)\), where \(U \in \Sigma(N\chi_N)\) and \(\rho \in \text{Hom}(\theta \otimes U, U)\) such that

\[
\rho \circ (m \otimes \text{id}) = \rho \circ (\text{id} \otimes \rho), \quad \rho \circ (e \otimes \text{id}) = \text{id}.
\]
In particular, we can define the induced $\Theta$-modules $\text{Ind}_\Theta(U) = \theta \otimes U$ for any $U \in \Sigma(\mathcal{X}_N)$, and it is the case that any simple $\Theta$-module is a submodule of $\text{Ind}_\Theta(\lambda)$, for some $\lambda \in \mathcal{X}_N$ (see Fig. 11). A $\Theta_a$-$\Theta_b$ bimodule is a triple $(U, \rho_a, \rho_b)$ with $U \in \Sigma(\mathcal{X}_N)$, $\rho_a \in \text{Hom}(\theta_a \otimes U, U)$ and $\rho_b \in \text{Hom}(U \otimes \theta_b, U)$ such that $(U, \rho_a)$ is a left $\Theta_a$-module, $(U, \rho_b)$ a right $\Theta_b$-module so that

$$
\rho_a \circ (\text{id}_{\Theta_a} \otimes \rho_b) = \rho_b \circ (\rho_a \otimes \text{id}_{\Theta_b})
$$

as LHS of Fig. 9. A map $t : U \to V$ is an $\Theta_a$-$\Theta_b$ intertwiner if the condition on RHS of Fig. 9 holds.

Given $\mathcal{Q}$-systems $\Theta_a$ and $\Theta_b$, the simple $\Theta_a$-$\Theta_b$ bimodules can be identified with the simple $\Theta_a \otimes \Theta_b^{opp}$-modules as in Fig. 10 where $\Theta_b^{opp}$ is the opposite algebra [Remark 12]. Recall that if $\Theta = (\theta, v, w)$ is a $\mathcal{Q}$-system on a braided factor $N$, then we can define an associated opposite $\mathcal{Q}$-system $(\theta, v, \epsilon_{(\theta, \theta)} w)$ denoted by $\Theta^{opp}$ as in Fig. 11. The graphical representation of $\epsilon_{(\lambda, \mu)} \equiv \epsilon_{(\lambda, \mu)}^+$ is in Fig. 12. We remark that if $\Theta$ and $\Theta^{opp}$ are equivalent then the modular invariant $Z$ is symmetric $Z = Z'$, but by [4], there are $\mathcal{Q}$-
systems producing non-symmetric modular invariants, thus we may have $\Theta \not\cong \Theta^{\text{opp}}$. The $\Phi$ product $\Theta_a \otimes \Theta_b = \Theta_{ab}$ of two $Q$-systems $\Theta_a = (\theta_a, v_a, w_a)$ and $\Theta_b = (\theta_b, v_b, w_b)$ is $(\theta_a \theta_b, \theta_a(v_b)v_a, \theta_a(\epsilon(\theta_a, \theta_b))\theta_b^2(w_b)w_a)$ as in Fig. 12. For completion, we define here the direct sum $\Theta_a \oplus \Theta_b$ of $Q$-systems, whose associated braided inclusion is $N \subset M_a \oplus M_b$, if the inclusions $N \subset M_a, N \subset M_b$ represent $\Theta_a$, and $\Theta_b$ respectively. Let $s_a, s_b \in N$ be Cuntz generators, i.e. $s_i^*s_j = \delta_{i,j}1$ and $s_a s_a^* + s_b s_b^* = 1$. Define now

$$\begin{align*}
\theta(n) &= s_a \theta_a(n) s_a^* + s_b \theta_b(n) s_b^*, \quad n \in N \\
v &= (\sqrt{d_a s_a v_a} + \sqrt{d_b s_b v_b})/\sqrt{d(\theta)}, \\
w &= \theta(s_a)s_a w_a s_a^* + \theta(s_b) s_b w_b s_b^*.
\end{align*}$$

where $d(\theta) = d(\theta_a) + d(\theta_b)$, and $d_a = d(\theta_a), d_b = d(\theta_b)$.

**Lemma 3.1** Let $\Theta_a$ and $\Theta_b$ be $Q$-systems associated with the braided subfactors $N \subset M_a$ and $N \subset M_b$, respectively. Then we can identify the category of $\Theta_a$-$\Theta_b$ bimodules with the category of $M_a$-$M_b$ sectors $M_a \mathcal{X} M_b$. 

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Proof. Denote by \( \iota_a \) and \( \iota_b \) the inclusion maps of the subfactors \( N \subset M_a \) and \( N \subset M_b \) respectively. Every irreducible \( \beta \) in \( \Sigma(M_a \mathcal{X}_M) \) arises from the decomposition \( \iota_a \lambda \iota_b \) with \( \lambda \) in \( N \mathcal{X} N \). Define now \( \Phi : \Sigma(M_a \mathcal{X}_M) \to \Theta_a - \Theta_b \)-bimodules by \( \Phi(\beta) = \iota_a \lambda t_{\iota b} \) for \( \beta \in M_a \mathcal{X}_M \) (see RHS of Fig. 15). In particular, \( \Phi(\iota_a \lambda \iota_b) = \theta_a \lambda \theta_b \). If \( \beta, \beta' \in \Sigma(M_a \mathcal{X}_M) \), then we map an intertwiner \( t \in \text{Hom}(\beta, \beta') \) to \( \Phi(t) = \iota_a \lambda t_{\iota b} \in \text{Hom}(\iota_a \lambda t_{\iota b}, \iota_a \lambda t'_{\iota b}) = \text{Hom}(\Phi(\beta), \Phi(\beta')) \), see RHS of Fig. 16. Then \( \Phi \) is an \( \Theta_a - \Theta_b \)-morphism as in Fig. 17. Let us finally prove that \( \Phi \) is injective. Suppose that \( \iota_a \beta t_{\iota b} \simeq \iota_a \beta' t_{\iota b} \) as \( \Theta_a - \Theta_b \) bimodules. Then we prove in Fig. 18 that \( t = 1_{\iota a} \otimes t'' \otimes 1_{\iota b} \) with \( t'' \in \text{Hom}(\beta, \beta') \). If \( t \) is an isomorphism so is \( t'' \), therefore \( \beta \simeq \beta' \). We have \( \#_{M_a \mathcal{X}_M} = \text{Tr}(Z_a Z_b^1) \) by \( [17]^1 \), so since \( \Phi \) is injective

1This was shown as in Sect. 2.2 for subfactors and normalised invariants, but the results extend to inclusions using central decompositions as in Sect. 3.1 particularly as in Lemma 3.5. Similarly for the
Figure 17: Φ is an Θ\textsubscript{a}-Θ\textsubscript{b} morphism

Figure 18: Proving that \( t = 1_{\text{ia}} \otimes t'' \otimes 1_{\text{ib}} \)

\[
\#_{M_a}\mathcal{X}_{M_b} \leq \#(\text{irreducible } \Theta\textsubscript{a}-\Theta\textsubscript{b} \text{ bimodules}). \text{ On the other hand, by [21, Proposition 5.16]}^2, \\
\text{Tr}(Z_a Z_b) \geq \#(\text{irreducible } \Theta\textsubscript{a} \otimes \Theta\textsuperscript{opp}\textsubscript{b} \text{ modules}) = \#(\text{irreducible } \Theta\textsubscript{a}-\Theta\textsubscript{b} \text{ bimodules}). \text{ So} \\
\#_{M_a}\mathcal{X}_{M_b} = \#(\text{irreducible } \Theta\textsubscript{a}-\Theta\textsubscript{b} \text{ bimodules}). \text{ Therefore } \Phi \text{ is surjective.} \qed
\]

There are particular cases from Lemma \ref{lemma:specific_cases} namely, when \( N = M_a \) and \( M_a = M_b \). In the first case the lemma implies that for the Q-system \( \Theta_b =: \Theta \) with \( N \subset M \) the associated braided subfactor, we can identify the category of \( \Theta \)-modules with the category \( \Sigma(M \mathcal{X}_N) \) compatible with the left \( \mathcal{X}_N \) and right \( M \mathcal{X}_N \) actions. The irreducible \( N\cdot M \) sectors \( \beta \) in \( M \mathcal{X}_N \) arise from the decompositon of \( i\lambda \), for \( \lambda \in N \mathcal{X}_N \). (The map \( \Phi : \Sigma(M \mathcal{X}_N) \to \Theta \text{-modules} \) is given by \( \Phi(\beta) = i\beta \) for \( \beta \in M \mathcal{X}_N \). In particular \( i\lambda \mapsto i\lambda = \text{Ind}_\Theta(\lambda) \).) Combining with previous remarks, this means that we can identify the \( N\cdot M_{\text{abopp}} \) system \( N \mathcal{X}_{M_{\text{abopp}}} \) with the \( M_a\cdot M_b \) system \( M_a \mathcal{X}_{M_b} \).

The second interesting case \( \Theta\textsubscript{a}=\Theta\textsubscript{b} \) is a Q-system from a braided subfactor \( N \subset M \). Let us consider how the product \( \Theta \otimes \Theta\textsuperscript{opp} \) is related or not to the Jones basic construction generating property for \( M\cdot M \) sectors by \( \alpha \)-induced ones.

---

1. Theorem 5.18
2. Theorem 5.18
If $\Theta = (\theta, v, w)$ is a Q-system for a braided subfactor $N \subset M$ then we can understand a Q-system for the Jones basic construction $N \subset M \subset M_1$. If $M \subset M_1$ is the Jones basic construction for $N \subset M$ and $i_1 : M \subset M_1$ is the canonical inclusion map, we can naturally identify the dual canonical endomorphism $\tilde{i}_1 i_1$ of $M \subset M_1$ with the canonical endomorphism $i_1$ of $N \subset M$. Thus the dual canonical endomorphism for $N \subset M_1$ is $\tilde{i}_1 i_1 = \tilde{i}_1 i_1 = \theta^2$. The corresponding isometries are $V = wv$ and $W = \theta(wv)$, i.e. we have a new Q-system $\Theta_J$ associated with the Jones basic construction $\Theta_J = (\theta^2, wv, \theta(wv))$, see Fig. 19. On the other hand we also have another Q-system $\Theta_P$ associated with $\theta^2$ from the $\Phi$ product: namely, $\Theta_P = (\theta^2, \theta(v)w, \theta(\epsilon(\theta, \theta))^2 w(v))$ where $\epsilon$ is the braiding operator, see Fig. 20. Denote by $N \subset M_J$ and $N \subset M_P$ the braided inclusions from $\Theta_J$ and $\Theta_P$, respectively.

For each Q-system on $\theta^2$, we consider $\theta^4 \in \Sigma(X_N)$ with two different products arising from the Jones or braided products. If we label the strings $i$ and $\tilde{i}$ by $\theta$ in Figs. 19 and 20 the two Q-systems on $\theta^4$ are equivalent using the unitary $u = \epsilon(\theta, \theta^2)$ arising from the braiding, see Fig. 21 and then we pull the strings to obtain the corresponding Jones co-product, see also Fig. 22. We actually have four Q-systems on $\theta^4$. The first two arise from the Q-system $\Theta_J$ by performing the Jones and braided products, thus obtaining $\Theta_{JJ}$ and $\Theta_{JP}$, respectively. The other two $\Theta_{PJ}$ and $\Theta_{PP}$ arise similarly from the Q-system $\Theta_P$. Namely,

$\Theta_{JJ} = \left( \theta^4, \theta(wv)w, \theta^2(\theta(wv)w) \right)$,  
$\Theta_{JP} = \left( \theta^4, \theta^2(wv), \theta^2(\epsilon(\theta, \theta)^2 \theta(\theta^2(w(v)))) \right)$,  
$\Theta_{PJ} = \left( \theta^4, \theta(\epsilon(\theta, \theta))\theta^2(w)w\theta(v)v, \theta^2(\theta(\epsilon(\theta, \theta))\theta^2(w)w\theta(v)v) \right)$,  
$\Theta_{PP} = \left( \theta^4, \theta^2(\theta(v)v)(\theta(v)v), \theta^2(\epsilon(\theta, \theta)^2 \theta(\theta^2(w)v)) \right)$.
From the above Q-systems $\Theta_{JJ}$, $\Theta_{J\Phi}$, $\Theta_{\Phi J}$ we denote by $N \subset M_{JJ}$, $N \subset M_{J\Phi}$, $N \subset M_{\Phi J}$ and $N \subset M_{\Phi \Phi}$, respectively, the associated braided inclusions.

**Proposition 3.2** Let $\theta = \bar{\iota} \iota$ be the dual canonical endomorphism of a braided subfactor $\iota : N \subset M$. Then $\Theta_{JJ}$, $\Theta_{\Phi J}$ and $\Theta_{J\Phi}$ are equivalent.

**Proof.** That $\Theta_{JJ}$ and $\Theta_{J\Phi}$ are equivalent Q-systems has been proven above (cf. Figs. 21 and 22). We can also prove that $\Theta_{\Phi J}$ and $\Theta_{JJ}$ are equivalent using the unitary $u = \epsilon(\theta, \theta)$, see Fig. 23. 

In the Q-system $\Theta_{\Phi}$, we can use the relative braiding [7] and, e.g., use the unitary $u = \epsilon(\iota, \theta)$ to get the LHS of Fig. 21, but then we see that the labelling of this picture is $\bar{\iota} \iota \bar{\iota} \cdots$ whereas the one for the Jones product is $\iota \iota \cdots$ see Fig. 19, and they do not match.

![Figure 21: Equivalent Q-systems](image1)

**Remark 3.3** Let $\Theta = (\theta, v, w)$ be a Q-system on a braided system $N, \mathcal{X}_N$ such that $\theta = \lambda_0 \oplus \lambda_\sigma$ and $\lambda_\sigma^2 = \lambda_0$. Thus we can endow $\theta^2$ with two Q-systems $\Theta_J$ and $\Theta_\Phi$. We see that $\dim Z(\Theta_J) = 1$ whereas $\dim Z(\Theta_\Phi) = 1$ if $\epsilon(\sigma, \sigma) = \kappa_\sigma \text{id}$ with $\kappa_\sigma = e^{2\pi i \sigma} \neq 1$, and $\dim Z(\Theta_\Phi) = 2$ if $\epsilon(\sigma, \sigma) = \text{id}$. In the $SU(2)$ modular invariants $Z_{D_n}$ associated to the Dynkin diagrams $D_n$ [10], the endomorphism $\theta = \lambda_0 \oplus \lambda_\sigma$ is a dual canonical endomorphism [3] producing $Z_{D_n}$. Moreover $\kappa_\sigma = 1$ for $D_{\text{odd}}$ and $\kappa_\sigma \neq 1$ for $D_{\text{even}}$. Therefore $\Theta_J$ and $\Theta_\Phi$ are inequivalent in the $D_{\text{even}}$ case. In fact $M_\Phi$ is not a factor, $\dim(M_\Phi' \cap M_\Phi) = 2$, whereas $M_J$ is (always) a factor, cf. Lemma 3.4. Note that $Z_{N \subset M_\Phi} = Z_{D_{\text{even}}}$. Moreover, $\Theta_{\Phi \Phi}$ and $\Theta_{JJ}$ are also not equivalent and so $N \subset M_{\Phi \Phi}$ and $N \subset M_{JJ}$ are different inclusions.
3.1 Fusion of sufferable modular invariants

Let Θ = (θ, v, w) be a Q-system on a factor N, and N ⊂ M the associated inclusion. We will denote Hom(id, θ) by H(Θ) and let Z(Θ) := {b ∈ H(Θ) : w∗b = w∗θ(b)}, see RHS of Fig. 23. If the multiplicity of id in θ is one, the inclusion N ⊂ M is irreducible and N′ ∩ M ≃ C therefore N ⊂ M is a subfactor. In case the multiplicity of id in θ is not one, then by [10] there is still an inclusion N ⊂ M with M a von Neumann algebra with finite dimensional centre. In the following we get a condition in terms of data on N, for which M becomes a factor, thereby yielding a subfactor N ⊂ M.

\[ L_\bar{\iota} : \text{Hom}(id, \theta) \to \text{Hom}(\iota, \iota) \]

Figure 24: The algebra Z(Θ)

Lemma 3.4 Let Θ = (θ, v, w) be a Q-system with N ⊂ M the associated inclusion. Then H(Θ) ≃ N′ ∩ M and moreover Z(Θ) ≃ M′ ∩ M.

Proof. As in e.g. [7] the left Frobenius reciprocity map L_\bar{\iota} : Hom(id, \theta) → Hom(\iota, \iota) is defined as L_\bar{\iota}(b) = d_\iota(id)(b^*)w_1 with w_1 ∈ Hom(id, \iota), see Fig. 25. Note that Hom(\iota, \iota) = N′ ∩ M. Since M = Nw_1 pointwise [11] and L_\bar{\iota}(b) commutes with N, we need to prove that b ∈ Z(Θ) if and only if L_\bar{\iota}(b)w_1 = w_1L_\bar{\iota}(b). The later equation is displayed in Fig. 27. The algebraic verification of Fig. 26 is as follows. Let b ∈ H(Θ) such that L_\bar{\iota}(b)w_1 = w_1L_\bar{\iota}(b), i.e.
Let $Θ= (θ, v, w)$ be a Q-system, $N ⊂ M$ the associated braided inclusion and $p$ a projection in the finite dimensional algebra $M' ∩ M ⊂ \text{Hom}(t, i)$. Then we consider the cut-down Q-system $Θ_p$ where $θ_p = i p t_p$ with $t_p(x) = p x$. Note that $θ_p = i p t_p ∈ \Sigma(M, X_N)$. Denote by $Z_p$ the modular invariant produced by $Θ_p$. Then if we take a (finite) partition of unity $\sum_p p = 1$ of the algebra $M' ∩ M$ by minimal projections $\{p\}$, $Θ = \bigoplus_p Θ_p$.

**Lemma 3.5** Let $N ⊂ M$ be a braided inclusion and $\{p\}$ be a partition of unity by minimal projections of $M' ∩ M$. Then $Z_{N ⊂ M}$ is a sum of normalised sufferable modular invariants $Z_{N ⊂ M} = \sum_p Z_p$, so that $Z_{0,0}$ is the dimension of $M' ∩ M$.

**Proof.** Since every projection $p$ is minimal, $p(M' ∩ M)p = \mathbb{C} p$ so $M_p$ is a factor and then $[Z_p]_{00} = 1$. Observe by [17] Sect. 5) that $α$-induction from $N$ to $M$ is trivial on the relative commutant $N' ∩ M$, thus trivial on $M' ∩ M$, so that $α$-induction is the sum of the cut-down $α$- inductions. Consequently, $Z$ is additive on Q-systems. For a formal proof see the description in the remark before Theorem 5.7 in [4] of $Z$ in terms of $N-N$ and $N-M$ data, from which additivity is clear, or use [21] Proposition 5.3].
Theorem 3.6  Let $N \subset M_a$ and $N \subset M_b$ be braided inclusions, whose modular invariants we denote by $Z_a$ and $Z_b$ respectively. Then the product $Z_a Z_b^\tau$ is sufferable and decomposes as an integral sum of normalised sufferable modular invariants.

Proof. Suppose $\Theta_a$ and $\Theta_b$ are Q-systems for the inclusions $N \subset M_a$ and $N \subset M_b$ respectively. Let $\Theta_{ab} \otimes \Theta_{b}^{opp}$ denote the Q-system $\Theta_a \otimes \Theta_{b}^{opp}$ with $N \subset M_{ab}^{opp}$ the associated inclusion. Then by Lemma 3.1 and [21, Proposition 5.3], $Z_{N \subset M_{ab}^{opp}} = Z_a Z_b^\tau$. On the other hand, let us take a partition of unity of minimal projections $\{p\}$ of $M_{ab}^{opp} \cap M_{a}^{opp}$. Then $Z_{N \subset M_{ab}^{opp}} = \sum Z_p$ by Lemma 3.5 and the result is proven. 

A special case of this result when $a = b$, and $(Z_a Z_b^\tau)_{00} = 2$ was derived in [17].

Corollary 3.7  The set of sufferable modular invariants yields a fusion rule algebra.

3.2 Q-systems from 6j-symbols

Here we present in co-ordinate form the Q-system relations on the isometries $v$ and $w$. We consider only the case where the Verlinde fusion rules are multiplicity free and where $\theta$ is also multiplicity free i.e. $\langle \theta, \lambda \rangle = 0, 1$.

Let $s_i \in \text{Hom}(\lambda_i, \theta)$ be Cuntz generators, i.e. $s_i^* s_j = \delta_{i,j} 1$, $\sum_i s_i s_i^* = 1$, such that $\theta(a) = \sum_i \lambda_i(a) s_i^*$. Similarly fix an orthonormal basis $\{v_{ij}^k\}$ of $\text{Hom}(\lambda_k, \lambda_i \lambda_j)$. Then any $w \in \text{Hom}(\theta, \theta^2)$ can be written as $w = \sum w_{ij}^k s_i \lambda_i(s_j) v_{ij}^k s_k^*$ with $w_{ij}^k$ complex numbers. Since we assume $\text{dim Hom}(id, \theta) = 1$, we can take $v = s_0$. We note by [19], that the Q-relations $w^* \theta(w) = w w^*$ and $w^2 = \theta(w) w$ are equivalent as long as we assume the other relation $dv^* w = dw^* \theta(v) = id$. So $w$ has to be an isometry $w^* w = id$ and $dv^* w = w^* \theta(v) = id$ and $w^2 = \theta(w) w$. These constraints are respectively

$$\sum_{i,j} |w_{ij}^k|^2 = 1, \quad w_{0k}^k = w_{k0}^k = 1/d, \quad \text{and} \quad w_{ij}^l w_{lk}^p = \sum_{e < \theta} w_{ijk}^e w_{elp} \cdot F_{ij}^{ek}(ijk)p$$

for all $i, j, l, k, p < \theta$ and where the numbers $F_{ij}^{ek}(ijk)p \in \mathbb{C}$ are the associated 6j-symbols [21], see Fig. 28. We calculate

$$w^* w = \sum (w_{ij}^p s_r \lambda_r(s_p) v_{ij}^p s_l^*)^* (w_{ij}^k s_i \lambda_i(s_j) v_{ij}^k s_k^*) = \sum w_{ij}^l w_{ij}^t v_{ij}^k v_{ij}^* s_t^* s_i \lambda_i(s_j) v_{ij}^k s_k^* = \sum |w_{ij}^t|^2 s_t^* s_i,$$

hence $\sum_{i,j} |w_{ij}^t|^2 = 1$ for all $t$. Also

$$dv^* w = ds_0^* \sum w_{ij}^k s_i \lambda_i(s_j) v_{ij}^k s_k^* = d \sum w_{0j}^k s_j v_{0j}^k s_k^* = \sum w_{0k}^k s_k s_k^*,$$

hence $w_{0k}^k = 1/d$ for all $k$ and similarly from $dv^* \theta(v) = id$ we obtain $w_{k0}^k = 1/d$ for all $k$.

We now have

$$ww = \sum w_{ij}^k w_{pq}^r s_i \lambda_i(s_j) v_{ij}^k s_k^* \cdot s_p \lambda_p(s_q) v_{pq}^r s_r^*$$

$$= \sum w_{ij}^k w_{kq}^r s_i \lambda_i(s_j) (v_{ij}^k \lambda_k(s_q)) v_{kq}^r s_r^*$$

$$= \sum w_{ij}^k w_{kq}^r s_i \lambda_i(s_j) \lambda_j(s_q) v_{ij}^k v_{kq}^r s_r^*.$$
whereas

\[ \theta(w)w = \sum w_{ij}^k w_{lp}^q \theta(s_i \lambda_i(s_j) v_{ij}^k s_k^* \cdot s_p \lambda_p(s_q) v_{pq}^r \cdot s_r^* ) \]

\[ = \sum w_{ij}^k w_{lp}^q s_p \lambda_p(s_i) \lambda_p(s_j) v_{ij}^k \lambda_p(s_k^*) v_{pq}^r s_r^* \]

\[ = \sum w_{ij}^k w_{pk}^l \omega \sum \lambda(p(s_i) \lambda_p(s_j) v_{ij}^k v_{pk}^l s_r^* \]

\[ = \sum w_{ij}^k w_{pk}^l \omega \sum \lambda(p(s_i) \lambda_p(s_j) v_{ij}^k v_{pk}^l s_r^* \]

yielding the final relation of Eqs. (14).

In the sequel we will use the following normalisations \([21, \text{Eq. (2.38)}]\)

\[ F_{pq}^{(0jk)} = F_{ki}^{(ijk)} = F_{jp}^{(ij0)} = 1 \]  \hfill (15)

whenever they are allowed to be non-zero by the fusion rules. It is usually tremendous work to prove that a given \( \theta \in \Sigma(N \times N) \) is a dual canonical endomorphism (requiring the solution of non-linear equations). Here we construct Q-systems that will be of use in the sequel.

The Frobenius-Schur indicator \( FS_\lambda = \omega^{-1} \sum_{\mu, \nu} N_{\lambda, \mu} d_{\mu} \omega_{\mu}^2 / \omega_{\nu}^2 \) vanishes unless \( \lambda \) is self-conjugate and is +1 or −1 depending on whether \( \lambda \) is real or pseudo-real, respectively (see e.g. \([21]\)). For a simple current \( \sigma \) (i.e. \( d_\sigma = 1 \)) one has \( FS_\sigma = \omega_\sigma^2 \). There is a well known obstruction for finding a periodic inner perturbation of a given automorphism \( \sigma \) which is a priori only periodic modulo the inner automorphisms \( [\sigma^n] = [id] \), so that \( (\text{Ad}(v)\sigma)^n = id \) \([13]\). We consider this obstruction in the context of Q-systems.

**Lemma 3.8** Let \( N \times N \) be a nondegenerate braided system and suppose that there is a simple current \( \sigma \in N \times N \) such that \( [\sigma^2] = [id] \). Then \( 0 = id \oplus \sigma \) is a dual canonical endomorphism of a braided subfactor if and only if the Frobenius-Schur indicator \( FS_\sigma = 1 \).

**Proof.** The last Eq. of \([13]\) reduces to

\[ w_{ij}^l w_{lk}^p = w_{ik}^0 w_{il}^p F_{ij(k)}^{(ijk)p} + w_{jk}^1 w_{il}^p F_{ij(k)}^{(ijk)p} \]

where we put 1 = \( 0 = id \) and \( i, j, k, l = 0, 1 \). We know the 6j-symbols by e.g. \([21]\) Eqs. (2.38) and (2.47):

\[ F_{pq}^{(0jk)} = F_{ki}^{(ijk)} = F_{jp}^{(ij0)} = 1, \quad FS_\sigma = F_{00}^{(111)} = 1 \]
When we insert these $6j$-symbols into the above equation we see that we get consistent constants $w_{ij}^k$ if and only if $\text{FS}_\sigma = 1$. We then get the following solution:

$$w_{00}^0 = w_{11}^0 = w_{01}^1 = w_{10}^1 = 1/\sqrt{2} \quad \text{and} \quad w_{11}^1 = w_{00}^0 = w_{01}^0 = w_{10}^0 = 0,$$

which yield a solution for all of Eqs. (14). □

We remark that $\text{FS}_\sigma = 1$ is Rehren’s condition \cite{22} $\omega_\sigma^2 = 1$ in the net setting. Let $p$ be a prime number and \( N \mathcal{X}_N = \{\lambda_{(m,n)}\} \) be the system of endomorphisms of \( N = M_0 \times \Delta(\mathbb{Z}_p) \) obtained from the quantum double of \( M_0 \subset M_0 \times \mathbb{Z}_p \) whose sectors obey the $\mathbb{Z}_p \times \mathbb{Z}_p$ fusion rules and where $M_0$ is a type III factor. We also fix a non-degenerate braiding on $\mathcal{X}_N$. Now we look for $\theta \in \Sigma(\mathcal{X}_N)$ such that $\theta$ is a dual canonical endomorphism.

**Lemma 3.9** (i) Let $p \neq 2$ be a prime number and $H$ a subgroup of $\mathbb{Z}_p \times \mathbb{Z}_p$ isomorphic to $\mathbb{Z}_p$ and consider $\theta_H = \bigoplus_{(m,n) \in H} \lambda_{(m,n)}$. Then $\theta_H$ is a dual canonical endomorphism of some braided subfactor $N \subset M$.

(ii) The endomorphism $\theta_{\mathbb{Z}_p \times \mathbb{Z}_p}$ is a dual canonical endomorphism.

**Proof.** The $6j$-symbols are, see e.g. \cite{25} Eq. (7.340)],

\[
F_{i+j+k,i+j}^{(ijk)i+j+k} = 1, \quad i,j,k \in \mathbb{Z}_p \times \mathbb{Z}_p
\]

and the others vanish. Note also this normalization is consistent with the one used in Eq. (15). Next we consider the subfactor $N = M_0 \times \Delta(\mathbb{Z}_p) \subset M_0 \times (\mathbb{Z}_p \times \mathbb{Z}_p)$. Then the $\theta_{\Delta(\mathbb{Z}_p)} = (0,0) \oplus (1,1) \oplus \cdots \oplus (p,p)$ is its dual canonical endomorphism by \cite{37} and therefore $\theta_{\Delta(\mathbb{Z}_p)}$ is a dual canonical endomorphism. The $6j$-symbols for $\Delta(\mathbb{Z}_p)$ are obtained by restriction of those for $\mathbb{Z}_p \times \mathbb{Z}_p$, i.e.

\[
F_{i+j+k,i+j}^{(ijk)i+j+k} = 1, \quad i,j,k \in \Delta(\mathbb{Z}_p)
\]

and $F_{m,n}^{(ijk)i+j+k} = 0$ if $h \neq i + j + k$ or $m \neq j + k$ or $n \neq i + j$ with $i,j,k,h,m,n$ in $\Delta(\mathbb{Z}_p)$. Hence if we take another copy $H$ of $\mathbb{Z}_p$ in $\mathbb{Z}_p \times \mathbb{Z}_p$ we get exactly the same formula for its $6j$-symbols, i.e.

\[
F_{i+j+k,i+j}^{(ijk)i+j+k} = 1, \quad i,j,k \in H
\]

and $F_{m,n}^{(ijk)i+j+k} = 0$ if $h \neq i + j + k$ or $m \neq j + k$ or $n \neq i + j$ with $i,j,k,h,m,n$ in $H$ (because of the fusion rules). Thus solving Eq. (14) for $\theta_H$ is equivalent to solving the same equation for $\theta_{\Delta(\mathbb{Z}_p)}$. Since we already know that such a solution exists for $\Delta(\mathbb{Z}_p)$ we conclude that $\theta_H$ is also a dual canonical endomorphism of some braided subfactor $N \subset M$.

Let us now prove (ii). From part (i), $\theta_{\mathbb{Z}_p \times 0}$ and $\theta_{0 \times \mathbb{Z}_p}$ are dual canonical endomorphisms, so is their product $\theta_{\mathbb{Z}_p \times \mathbb{Z}_p}$. □

### 4 The subfactor $M_0 \rtimes \Delta(G) \subset M_0 \rtimes (G \times G)$

We identify the intermediate subfactors of the asymptotic subfactor of a finite group $G$ that arise from subgroups of $G$.

**Proposition 4.1** There is a bijection between normal subgroups $N \triangleleft G$ and subgroups $H$ of $G \times G$ containing $\Delta(G)$.

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Proof. Let $H$ be a subgroup of $G \times G$ such that $\Delta \subset H$, where $\Delta$ denotes the diagonal subgroup of $G \times G$. If $(h_1, h_2) \in H$, $(h_{1}^{-1}, h_{1}^{-1}) \in \Delta \subset H$ implies $(e, h_{2}h_{1}^{-1}) \in H$. Then $H' := \{ h \in G : (e, h) \in H \}$ is a subgroup of $G$. On the other hand, for $s \in G$, $h \in H'$, $(e, shs^{-1}) = (s, s)(e, h)(s^{-1}, s^{-1}) \in H$, therefore $shs^{-1} \in H'$ for all $s$ and so $H'$ is a normal subgroup of $G$ (and $\{e\} \times H' \subset H$). Conversely, take a normal subgroup $H'$ of $G$, and define $H := \Delta \cdot H' \times H' = \{(gH', gH') : g \in G\}$. Then $\Delta \subset H \subset G \times G$ and clearly the above $H \leftrightarrow H'$ construction is a bijection.

Then from each normal $N$ subgroup of $G$ we get an intermediate subfactor $M_0 \rtimes \Delta(G) \subset M_0 \rtimes N \subset M_0 \rtimes (G \times G)$. Clearly $\Delta(G)$ is normal in $G \times G$ if and only if $G$ is an abelian group.

In the more general case where we fix a subgroup $H$ of $G$, the principal and dual graphs of $N = M_0 \rtimes H \subset M_0 \rtimes G = M$ are computed in [37] through the Mackey induction/restriction machinery. The dual principal graph is as in Fig. 29 with $\text{res} : \pi \to \pi|_H$, $\text{ind} : \sigma \to \text{ind}^G_H\sigma$ being restriction and induction of group representations respectively.

![Figure 29: Dual principal graph of $M_0 \rtimes H \subset M_0 \rtimes G$](image)

Strictly speaking, the dual principal graph is the connected component of the above restriction/induction procedure containing the trivial representation of $G$. The principal graph is more subtle. First we take representatives of the double cosets $H \setminus G/H$. Then for each double coset $HgH$ we fix a representative $g$ and consider the stabilizer subgroup $H^g := H \cap gHg^{-1}$. Then the set of vertices of the principal graph of $N \subset M$ is labelled by pairs $(g, \sigma)$ where $g$ runs over a set of representatives of the double cosets and $\sigma$ are irreducible representations of $H^g$ [38]. Finally the principal graph is the connected component containing the trivial representation of $H$ of the graph, see Fig. 30. By [37] Proposition 31], the vertex set of the principal graph of $N \subset M$ is given by $G/\overline{N} \sqcup \overline{H}/\overline{N}$ where $\overline{N}$ is the largest normal subgroup of $G$ containing $H$. Furthermore, $G/\overline{N}$ labels the $N$-$N$

![Figure 30: Principal graph of $M_0 \rtimes H \subset M_0 \rtimes G$](image)

irreducible sectors whereas $\overline{H}/\overline{N}$ labels the $N$-$M$ ones. We note that the principal graph of $M_0 \rtimes H \subset M_0 \rtimes G$ is the dual principal graph of the inclusion $M_0^G \subset M_0^H$ (see e.g. [31].
or $[19]$. The more difficult fusion of the $N$-$N$ sectors is also carried out by Kosaki et al. $[37, 38]$.

When we are in the situation $\Delta(G) \subset G \times G$, the center $Z_G$ of $G$ is trivial if and only if $N$ is trivial. Therefore, the above Mackey induction/restriction graphs are connected if and only if $Z_G = \{e\}$. This happens, for instance, when $G$ is $S_3$ the symmetric group on 3 letters. Moreover, since

$$\Delta(G)(g,h)\Delta(G) = \Delta(G)(e,g^{-1}h)\Delta(G)$$

for any $g,h \in G$, we easily see that every double coset $\Delta(G)(g,h)\Delta(G)$ gives rise to a conjugacy class $C_{g^{-1}h}$ of $\Delta(G)$. Since the stabilizer of $(g,h)$ equals the centraliser of $g^{-1}h$, i.e.

$$(g,h)\Delta(G)(g^{-1},h^{-1}) \cap \Delta(G) = \{ x \in G : xg^{-1}h = g^{-1}hx \},$$

we have the group theoretical relation between the framework of Coste et al. $[14]$ and the one in $[37]$ which we will use in the sequel.

We now illustrate the above algorithm in some concrete examples. In the first example we display the principal graph of $N = M_0 \rtimes \Delta(S_3) \subset M_0 \rtimes (S_3 \times S_3) = M$ in Fig. 31. Here we have three $S_3 \equiv \Delta(S_3)$ double cosets:

- $S_3(e,e)S_3$ with 6 elements,
- $S_3((132),e)S_3$ with 12 elements and
- $S_3((132),(12))S_3$ with 18 elements.

We further have

$$S_3 \cap (e,e)S_3(e,e)^{-1} \simeq S_3, \quad S_3 \cap ((132),e)S_3(132,e)^{-1} \simeq Z_3$$

$$S_3 \cap ((132),(12))S_3((132),(12))^{-1} \simeq Z_2.$$

The three first vertices $[\lambda_0],[\lambda_1],[\lambda_2]$ of Fig. 31 label the trivial 1, sign $\epsilon$ and 2 dimensional $\sigma$ irreducible representations of the group $S_3$, respectively. The vertices $[\lambda_3],[\lambda_4],[\lambda_5]$ label the trivial and the other two irreducible representations of $Z_3$, respectively. Finally, the last two vertices $[\lambda_6],[\lambda_7]$ label the two irreducible inequivalent representations of $Z_2$. In the sequel, it will become clear why we choose this notation for the vertices of the principal graph of the above subfactor $N \subset M$.

The dual principal graph of the subfactor $M_0 \rtimes S_3 \subset M_0 \rtimes S_3 \times S_3$ is derived from the irreducible representations of $S_3$ whose (commutative) fusion is as follows: $\epsilon \epsilon = 1 = \epsilon 1, \quad \sigma \sigma = 1 + \epsilon + \sigma$. In this way the Mackey induction/restriction between $\widehat{S_3} \times \widehat{S_3}$ and $\widehat{S_3}$ gives rise to the graph displayed in Fig. 32 which is in turn the dual principal graph of the above subfactor by $[37]$.

The only non-trivial intermediate subfactor $M_0 \rtimes S_3 \subset P \subset M_0 \rtimes (S_3 \times S_3)$ arises from the normal subgroup $Z_3 \equiv \{ e,(123),(132) \}$ of $S_3$ by Proposition $[11]$ i.e., $P = M_0 \rtimes K$ where $K := S_3 \cdot (Z_3 \times Z_3) \simeq (Z_3 \times Z_3) \rtimes Z_2$. The group $H$ is a non-commutative subgroup of $S_3 \times S_3$ of order 18 (equal to pairs in $S_3 \times S_3$ with the same parity). The subfactor $M_0 \rtimes S_3 \subset M_0 \rtimes K$ thus has Jones index equals 3. Next we compute its principal graph. We obtain two double cosets for $S_3 \subset S_3 \cdot (Z_3 \times Z_3)$; namely $S_3(e,e)S_3 \simeq S_3$ and $S_3((12),(23))S_3$ with 12 elements. We then obtain

$$S_3 \cap (e,e)S_3(e,e)^{-1} \simeq S_3, \quad S_3 \cap ((12),(23))S_3((12),(23))^{-1} \simeq Z_3.$$
In this way, we display the induction/restriction graph in Fig. 33 where the bottom vertices and the first 3 vertices on the top label the irreducible representations of $S_3$ and the other 3 vertices label the irreducible representations of $\mathbb{Z}_3$. We explain the notation of the labels. The $N-N$ labels of the connected component containing the vertex $[\iota, P]$ in Fig. 33 is a subset of those $N-N$ vertices in Fig. 31, the others (i.e. $x, y, z$) are unrelated to the latter ones (cf. the proof of Proposition 6.12).

Note that the principal graph of $N = M_0 \rtimes S_3 \subset M_0 \rtimes (S_3 \cdot (\mathbb{Z}_3 \times \mathbb{Z}_3)) = P$ is the connected component containing the trivial representation of $S_3$. Therefore it is the Dynkin diagram $A_5$. This can alternatively be seen by noting that $S_3$ is non-normal (index 3) subgroup of $S_3 \cdot (\mathbb{Z}_3 \times \mathbb{Z}_3)$.

5 Modular data and modular invariants for cyclic groups

The modular data for finite abelian groups have been carried out explicitly in [14, Section 3.1]. In particular for cyclic groups $\mathbb{Z}_d$, the primary fields are parametrized by pairs $(m, n)$ for $m, n \in \mathbb{Z}_d$ whose conjugate is $(-m, -n)$, and the $S$ and $T$ matrices:
5.1 Subfactors for the quantum $\mathbb{Z}_3$ modular invariants

The $\mathbb{Z}_2$ case has been carried out in [3] Page 25], using a different notation. After relabelling the primary fields they match in the following manner: $Z_2 = W$, $Z_4 = X_c$, $Z_5 = X_s$, $Z_7 = Q^t$ and $Z_8 = Q$. In Table 4 we provide all products of modular invariants of the quantum $\mathbb{Z}_2$ double model. The heterotic modular invariants $Z_7$ and $Z_8$ arise from the subfactor $N \subset N \rtimes (\mathbb{Z}_2 \times \mathbb{Z}_2)$ producing another two modular invariants (namely $Z_4$ and $Z_5$) through intermediate subfactors by the application of the type I parent theorem. The last non-trivial modular invariant arise from an inclusion $N \subset N \rtimes \Delta(\mathbb{Z}_2)$. We note that here $N = \pi_0(SO(16l))^n$ with $\pi_0$ the level 1 vacuum representation [59].
because of the inequality $\langle Z \rangle$.

Proof. The dual canonical sector of $N$ is obtained in the following way, we obtain $\langle N \rangle$ has no further irreducible reducible sectors. Let $Z$ be the system arising from these endomorphisms which as sectors obey the quantum double of $Z$. As in [8, Eq. (37)]. Now let us appeal to [17, Proposition 3.3]. On one hand, $\langle Z \rangle$ gives rise to a dual canonical endomorphism $\theta_H = \oplus_{(m,n) \in H} \lambda_{(m,n)}$.

Proposition 5.1 In the case $p = 3$, the subfactors $N \subset N \rtimes H_1$, $N \subset N \rtimes H_2$ and $N \subset N \rtimes H_3$ produce the permutation modular invariants $Z_1$, $Z_2$ and $Z_3$, respectively.

Proof. The dual canonical sector of $N \subset N \rtimes H_2$ is $\langle \theta_2 \rangle = [\lambda_{00}] \oplus [\lambda_{12}] \oplus [\lambda_{21}]$. In this way, we obtain $\langle \theta_2 \lambda_{mn}, \lambda_{mn} \rangle = 1$ for all primary fields of the model. On the other hand, $\langle \theta_2 \lambda_{0i}, \lambda_{0j} \rangle = \delta_{i,j}$ where $\delta$ denotes the Kronecker function. So we have so far three irreducible $N-M$ sectors $[i \lambda_{00}], [i \lambda_{01}], [i \lambda_{02}]$. However, since $\langle \theta_2 \lambda_{0i}, \lambda_{kj} \rangle = \delta_{i-k,j}$ we have no further irreducible $N-M$ sectors. Then $\text{Tr}(Z_{N\subset M}) = 3$ where we conclude that $Z_{N\subset M} = Z_4, Z_5, Z_2$ or $Z_3$. As $\langle \theta_2 \lambda_{0i}, \lambda_{0i} \rangle = 0$ for $i = 1, 2$, $Z_{N\subset M}$ is neither $Z_4$ nor $Z_5$ because of the inequality $\langle \alpha^+_\lambda, \alpha^+_\mu \rangle \leq \langle \theta \lambda, \mu \rangle$ as in [8, Eq. (37)]. Now let us appeal to [17, Proposition 3.3]. On one hand,

$$\bigoplus_{i,j,k,r} Z_{3(i,j,kr)} [\lambda_{ij} \lambda_{kr}] = 3[\lambda_{00}] \oplus 3[\lambda_{11}] \oplus 3[\lambda_{22}]$$

and on the other,

$$\bigoplus_{a \in M \times N} [a \bar{a}] = [\bar{\lambda}_{00} \bar{i} \lambda_{00}] \oplus [\bar{\lambda}_{12} \bar{i} \lambda_{12}] \oplus [\bar{\lambda}_{21} \bar{i} \lambda_{21}] = 3[\lambda_{00}] \oplus 3[\lambda_{12}] \oplus 3[\lambda_{21}]$$

which is not the same sector. So $Z_{N\subset M}$ cannot be $Z_3$ by [17, Proposition 3.3], and must be $Z_2$.

The dual canonical sector of $N \subset N \rtimes H_3$ is $\langle \theta_3 \rangle = [\lambda_{00}] \oplus [\lambda_{11}] \oplus [\lambda_{22}]$. Since for $\theta_3$ we have $\langle \theta_3 \lambda_{0i}, \lambda_{kj} \rangle = \delta_{i+k,j}$ we get exactly 3 irreducible $N-M$ sectors $[i \lambda_{00}], [i \lambda_{01}], [i \lambda_{02}]$, so

Table 1: Fusion $Z_3 Z_3^\dagger$ of $Z_2$ modular invariants

|   | W | X | Q | Q |
|---|---|---|---|---|
| 1 | 1 | W | X | Q | Q |
| W | W | 1 | Q^i | Q | X | Q | Q |
| X | X | Q | 2X | Q | 2Q | X | Q |
| X | X | Q | 2Q | Q | 2X | X | Q |
| Q | Q | X | X | 2X | Q | Q | Q | Q |


\( \text{Tr}(Z_{N \subset M}) = 3 \) implying that \( Z_{N \subset M} = Z_4, Z_5, Z_2 \) or \( Z_3 \). Elimination of both \( Z_4 \) and \( Z_5 \) is similar to the above case.

We also have

\[
\bigoplus_{a \in M \mathcal{X}_N} [a \tilde{a}] = [\bar{\lambda}_{00} \bar{\imath} \lambda_{00}] \oplus [\bar{\lambda}_{11} \bar{\imath} \lambda_{11}] \oplus [\bar{\lambda}_{22} \bar{\imath} \lambda_{22}] = 3[\lambda_{00}] \oplus 3[\lambda_{11}] \oplus 3[\lambda_{22}]
\]

and on the other hand,

\[
\bigoplus_{i,j,k,r} Z_{2(i,j,k,r)}[\lambda_{ij} \bar{\lambda}_{kr}] = 3[\lambda_{00}] \oplus 3[\lambda_{12}] \oplus 3[\lambda_{21}].
\]

Therefore again by [17, Proposition 3.3], \( Z_{N \subset N \times H_3} = Z_3 \).

\( \square \)

**Proposition 5.2** The quantum double modular invariants \( Z_4 \) and \( Z_5 \) are realised by the subfactors \( N \subset N \times (\mathbb{Z}_3 \times \{0\}) \) and \( N \subset N \times (\{0\} \times \mathbb{Z}_3) \).

**Proof.** For the subfactor \( N \subset N \times (\{0\} \times \mathbb{Z}_3) \), \( \theta_5 = [\lambda_{00}] \oplus [\lambda_{01}] \oplus [\lambda_{02}] \) is its dual canonical sector. Computing we find that \( \langle \theta_5 \lambda_{ij}, \lambda_{ij} \rangle = 1 \), for all \( i, j \). \( \langle \theta_5 \lambda_{00}, \lambda_{01} \rangle = \langle \theta_5 \lambda_{00}, \lambda_{02} \rangle = \langle \theta_5 \lambda_{10}, \lambda_{11} \rangle = \langle \theta_5 \lambda_{10}, \lambda_{12} \rangle = \langle \theta_5 \lambda_{20}, \lambda_{21} \rangle = \langle \theta_5 \lambda_{20}, \lambda_{22} \rangle = 1 \). The other intertwiner spaces are zero. Hence \( [\lambda_{00}] = [\lambda_{01}], [\lambda_{02}], [\lambda_{10}]=[\lambda_{11}], [\lambda_{12}],[\lambda_{20}]=[\lambda_{21}]=[\lambda_{22}] \). Then \( \text{Tr}(Z_{N \subset M}) = 3 \), implying that \( Z_{N \subset M} = Z_4, Z_5, Z_2 \) or \( Z_3 \). Since \( \langle \theta_5 \lambda_{00}, \lambda_{10} \rangle = 0 \), \( Z_{N \subset M} \) cannot be \( Z_4 \). Now we employ [17, Proposition 3.3]. First, we use \( \theta_5 \) and obtain:

\[
[\epsilon] := \bigoplus_{a \in M \mathcal{X}_N} [a \tilde{a}] = [\bar{\lambda}_{00} \bar{\imath} \lambda_{00}] \oplus [\bar{\lambda}_{10} \bar{\imath} \lambda_{10}] \oplus [\bar{\lambda}_{20} \bar{\imath} \lambda_{20}] = 3[\lambda_{00}] \oplus 3[\lambda_{01}] \oplus 3[\lambda_{02}].
\]

For \( Z = Z_2 \) or \( Z_3 \),

\[
\bigoplus_{i,j,k,r} Z_{2(i,j,k,r)}[\lambda_{ij} \bar{\lambda}_{kr}]
\]

has been computed in the proof of Proposition 5.1. It is in either case different from the above computation of \([\epsilon]\). So \( Z_{N \subset M} \) cannot be the permutation modular invariants \( Z_2 \) and \( Z_3 \). Therefore \( Z_{N \subset N \times (\{0\} \times \mathbb{Z}_3)} = Z_5 \). In a similar way, we prove that \( N \subset N \times (\mathbb{Z}_3 \times \{0\}) \) produces \( Z_4 \).

\( \square \)

**Proposition 5.3** The modular invariants \( Z_6, Z_7, Z_8 \) are sufferable.

**Proof.** We already know that \( Z_2 \) and \( Z_3 \) are sufferable, then so is \( Z_3 Z_2 = Z_6 \) by Section 8. The same with \( Z_7 = Z_2 Z_5 \).

\( \square \)

**Corollary 5.4** For \( i = 1, 2, 3, 4, 5 \), the symmetric quantum \( \mathbb{Z}_3 \) double modular invariant \( Z_i \) is produced by the subfactor \( N \subset N \times H_i \). Moreover \( H_6 \) produces \( Z_6, Z_7 \) and \( Z_8 \).

The global index of \( N \mathcal{X}_N \) is \( \omega = 9 \). In the \( H_1, H_2, H_3, H_6 \) cases, we have \( \mathcal{X}_M^0 = \mathcal{X}_M^1 = \mathcal{X}_M \) and as sectors they are \( \mathbb{Z}_3 \times \mathbb{Z}_3 \). In the cases \( H_4, H_5, H_7, H_6, \mathcal{X}_M^0 = \{(0,0)\} \), the sectors of \( \mathcal{X}_M^1 \) are isomorphic to \( \mathbb{Z}_3 \) and those from \( \mathcal{X}_M \) isomorphic to \( \mathbb{Z}_3 \times \mathbb{Z}_3 \). The indecomposable nirmreps can be derived by [27, Proposition 4]. We also have Table 2 for the (unique) fusion among the (sufferable) quantum \( \mathbb{Z}_3 \) modular invariants. Note that \( H^2(\mathbb{Z}_3 \times \mathbb{Z}_3, S^1) \) is responsible for \( H_6 \) producing three different modular invariants.
We have that $H^3(\mathbb{Z}_3, S^1) = \mathbb{Z}_3 = \{0, 1, 2\}$ by e.g. [14]. For $k = 1$ (the level 1 case), all the primary fields are simple currents again and the $S$ and $T$ matrices are as follows:

$$S = \frac{1}{3} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & z & z & z & z & z & z \\ 1 & z & z & z & z & z & z \\ 1 & z & z & w & v & u & w \\ 1 & z & z & u & v & w & u \\ 1 & z & z & v & u & w & v \\ 1 & z & z & w & v & u & w \\ 1 & z & \bar{z} & w & v & u & w \end{pmatrix}, \quad T = \text{diag}(1, 1, 1, u, v, w, u, \bar{w}),$$

where $u = \exp[2\pi i/9]$, $v = \exp[8\pi i/9]$, $w = \exp[4\pi i/9]$, $z = \exp[-2\pi i/3]$.

These modular data arise from the affine $SU(9)_1$. The modular invariants are:

$Z_1 = \sum_{i,j} \chi_{ij} \chi_{ij}^*$, $Z_2 = \chi_{00} \chi_{00}^* + (\chi_{01} \chi_{02}^* + \chi_{10} \chi_{21}^* + \chi_{11} \chi_{20}^* + \chi_{12} \chi_{22}^* + \text{c.c.})$ and $Z_3 = \sum_{i,j} \chi_{i0} \chi_{j0}^*$.

Since $T_{10,10}$ is a 9th root of unity, by [52] Lemma 4.4 we can choose representatives in each simple current sector such that $\{\lambda_{ij}\} \simeq \mathbb{Z}_9$. Moreover the sector $[\theta] = \bigoplus_{ij} [\lambda_{ij}]$ is a dual canonical sector of the subfactor $N \subset N \rtimes \mathbb{Z}_9$. This produces in turn the above $Z_2$ permutation invariant. Also as $\{\lambda_{00}, \lambda_{01}, \lambda_{02}\}$ is a subgroup of $\mathbb{Z}_9$, $[\theta] = [\lambda_{00}] \oplus [\lambda_{01}] \oplus [\lambda_{02}]$ is a dual canonical endomorphism of $N \subset N \rtimes (\{0\} \times \mathbb{Z}_3)$ which produces the above $Z_3$ modular invariant. In this way, we conclude that all the quantum $\mathbb{Z}_3$ double level 1 modular invariants are sufferable and their fusion rules are as in Table 3. The level $k = 2$ modular data is obtained by the complex conjugation of those for the level $k = 1$ case.

Table 2: Fusion $Z_aZ_b^t$ of $\mathbb{Z}_3$ level 0 modular invariants

| $Z_1$ | $Z_2$ | $Z_3$ | $Z_4$ | $Z_5$ | $Z_6$ | $Z_7$ | $Z_8$ |
|-------|-------|-------|-------|-------|-------|-------|-------|
| $Z_1$ | $Z_2$ | $Z_3$ | $Z_4$ | $Z_5$ | $Z_6$ | $Z_7$ | $Z_8$ |
| $Z_2$ | $Z_3$ | $Z_6$ | $Z_7$ | $Z_3$ | $Z_4$ | $Z_5$ | $Z_8$ |
| $Z_3$ | $Z_6$ | $Z_2$ | $Z_4$ | $Z_7$ | $Z_5$ | $Z_8$ | $Z_3$ |
| $Z_4$ | $Z_7$ | $Z_3$ | $Z_2$ | $Z_5$ | $Z_8$ | $Z_3$ | $Z_6$ |
| $Z_5$ | $Z_8$ | $Z_4$ | $Z_7$ | $Z_2$ | $Z_5$ | $Z_3$ | $Z_6$ |
| $Z_6$ | $Z_3$ | $Z_2$ | $Z_7$ | $Z_4$ | $Z_5$ | $Z_8$ | $Z_3$ |
| $Z_7$ | $Z_4$ | $Z_1$ | $Z_3$ | $Z_2$ | $Z_5$ | $Z_6$ | $Z_8$ |
| $Z_8$ | $Z_5$ | $Z_6$ | $Z_7$ | $Z_1$ | $Z_8$ | $Z_3$ | $Z_2$ |

Table 3: Fusion $Z_aZ_b^t$ of $\mathbb{Z}_3$ levels 1 or 2 modular invariants

| $Z_1$ | $Z_2$ | $Z_3$ |
|-------|-------|-------|
| $Z_1$ | $Z_1$ | $Z_2$ | $Z_3$ |
| $Z_2$ | $Z_2$ | $Z_1$ | $Z_3$ |
| $Z_3$ | $Z_3$ | $Z_3$ | $Z_3$ |

5.2 Quantum $\mathbb{Z}_3$ double level 1 and 2 modular invariants

We have that $H^3(\mathbb{Z}_3, S^1) = \mathbb{Z}_3 = \{0, 1, 2\}$ by e.g. [14].
6 Subfactors and nimreps for the quantum $S_3$ double modular invariants

6.1 The modular data and the Verlinde fusion matrices

The symmetric group $S_3 = Z_3 \times Z_2$ case has eight primary fields: $(e, \psi_i)$ for $i = 0, 1, 2$ where $\psi_0, \psi_1, \psi_2$ are the characters associated to the trivial, sign and two dimensional representation (resp.) of $S_3$: $((123), \varphi_k)$, $k = 0, 1, 2$ for the 3 characters of $Z_3$; and $((12), \varphi'_k)$, for the 2 characters of $Z_2$. We label these primary fields $\Phi$ by 0, 1, \ldots, 7 as in [14]. The modular data are:

$$S = \frac{1}{6} \begin{pmatrix}
1 & 1 & 2 & 2 & 2 & 3 & 3 \\
1 & 1 & 2 & 2 & 2 & -3 & -3 \\
2 & 2 & 4 & -2 & -2 & 0 & 0 \\
2 & 2 & -2 & 4 & -2 & 0 & 0 \\
2 & 2 & -2 & -2 & 4 & 0 & 0 \\
2 & 2 & -2 & -2 & 4 & 0 & 0 \\
3 & -3 & 0 & 0 & 0 & 0 & 3 & -3 \\
3 & -3 & 0 & 0 & 0 & 0 & -3 & 3
\end{pmatrix}$$

$$T = \begin{pmatrix}
\exp[2\pi\sqrt{-1}/3], & \exp[4\pi\sqrt{-1}/3], & 1, & 1, & 1, & \exp[2\pi\sqrt{-1}/3], & \exp[4\pi\sqrt{-1}/3], & 1, & 1, & 1
\end{pmatrix}.$$ 

The primary fields 0 and 1 are the only simple currents of our model. Next we determine explicitly all Verlinde fusion matrices $[53]$, $N_i, i = 0, 1, \ldots, 7$ since they will be employed in the calculation of the dimensions of intertwiner spaces later. They are computed from the Verlinde formula [53] and can also be derived from [53] Page 279, which as quadratic forms are:

$$N_0 = |x_0|^2 + |x_1|^2 + |x_2|^2 + |x_3|^2 + |x_4|^2 + |x_5|^2 + |x_6|^2 + |x_7|^2,$n

$$N_1 = x_0x_1^* + x_1x_0^* + |x_2|^2 + |x_3|^2 + |x_4|^2 + |x_5|^2 + x_6x_7^* + x_7x_6^*,$n

$$N_2 = x_0x_2^* + x_2x_0^* + x_1x_2^* + x_2x_1^* + |x_4|^2 + x_3(x_4 + x_5^*) + (x_4 + x_5)x_3^* + x_4x_5^* + x_5x_4^* + |x_6 + x_7|^2,$n

$$N_3 = x_0x_3^* + x_3x_0^* + x_1x_3^* + x_3x_1^* + |x_3|^2 + x_2(x_4 + x_5^*) + (x_4 + x_5)x_2^* + x_4x_5^* + x_5x_4^* + |x_6 + x_7|^2,$n

$$N_4 = x_0x_4^* + x_4x_0^* + x_1x_4^* + x_4x_1^* + |x_4|^2 + x_5(x_2 + x_3)^* + (x_2 + x_3)x_5^* + x_3x_2^* + x_2x_3^* + |x_6 + x_7|^2,$n

$$N_5 = x_0x_5^* + x_5x_0^* + x_1x_5^* + x_5x_1^* + |x_5|^2 + x_4(x_2 + x_3^*) + (x_2 + x_3)x_4^* + x_2x_3^* + x_3x_2^* + |x_6 + x_7|^2,$n

$$N_6 = x_0x_6^* + x_6x_0^* + x_1x_6^* + x_7x_1^* + (x_2 + x_3 + x_4 + x_5)(x_6^* + x_7^*) + (x_6 + x_7)(x_2 + x_3 + x_4 + x_5)^*,$n

$$N_7 = x_0x_7^* + x_7x_0^* + x_1x_7^* + x_6x_1^* + (x_2 + x_3 + x_4 + x_5)(x_6^* + x_7^*) + (x_6 + x_7)(x_2 + x_3 + x_4 + x_5)^*.$n

In particular all the primary fields are self-conjugate and moreover the Frobenius-Schur indicator $\text{FS}_\lambda = 1$ for all $\lambda \in \mathcal{N} \mathcal{X}$. 

6.2 The modular invariants of the quantum $S_3$ double

We present here the quantum $S_3$ double modular invariants. There are precisely 48 modular invariants. The identity matrix and the permutation matrix corresponding to the
interchange $2 \leftrightarrow 3$ are the permutation invariants $Z_1$ and $Z_2$ respectively. Then we get three new modular symmetric invariants:

$$|\chi_0 + \chi_1|^2 + 2|\chi_2|^2 + 2|\chi_3|^2 + 2|\chi_4|^2 + 2|\chi_5|^2 + k(\chi_2 \chi_3^* + \chi_3 \chi_2^* - |\chi_2|^2 - |\chi_3|^2)$$

where $k = 0, 1, 2$, producing the matrices $Z_5, Z_6$ and $Z_7$ respectively. We have two more symmetric modular invariants ($k = \pm 1$), whose matrices we denote by $Z_3, Z_4$, respectively:

$$|\chi_0 + \chi_1 + \chi_2 + \chi_3|^2 + k(\chi_2 \chi_3^* + \chi_3 \chi_2^* - |\chi_2|^2 - |\chi_3|^2).$$

Set $s_1 = \chi_0 + \chi_1 + \chi_2 + \chi_3$, $s_2 = \chi_0 + \chi_1 + 2\chi_2$, $s_3 = \chi_0 + \chi_1 + 2\chi_3$, $s_4 = \chi_0 + \chi_2 + \chi_6$ and $s_5 = \chi_0 + \chi_3 + \chi_6$. Then for every pair $(i, j)$, $i, j = 1, 2, 3, 4, 5$, $Z_{ij} := s_is_j^*$ is a modular invariant partition function. These 32 modular invariants were found in [14] but the list is not complete. The extra sixteen are listed below:

$$Z_{(1)} = |\chi_0 + \chi_1 + \chi_3|^2 + 2|\chi_3|^2 + |\chi_4|^2 + |\chi_5|^2,$$

$$Z_{(2)} = |\chi_0 + \chi_1 + \chi_3|^2 + 2|\chi_3|^2 + |\chi_4|^2 + |\chi_5|^2,$$

$$Z_{(3)} = |\chi_0 + \chi_1 + \chi_2|^2 + |\chi_2|^2 + 2|\chi_3|^2 + |\chi_4|^2 + |\chi_5|^2,$$

$$Z_{(4)} = |\chi_0 + \chi_1 + \chi_2|^2 + 2|\chi_2|^2 + |\chi_3|^2 + |\chi_4|^2 + |\chi_5|^2,$$

$$Z_{(22)} = |\chi_0 + \chi_2|^2 + |\chi_1 + \chi_2|^2 + |\chi_6|^2 + |\chi_7|^2,$$

$$Z_{(33)} = |\chi_0 + \chi_3|^2 + |\chi_1 + \chi_3|^2 + |\chi_6|^2 + |\chi_7|^2,$$

$$Z_{(44)} = |\chi_0 + \chi_4|^2 + \chi_1 \chi_6^* + \chi_2 \chi_6^* + \chi_6 \chi_4^* + \chi_6 \chi_5^* + |\chi_7|^2,$$

$$Z_{(55)} = |\chi_0 + \chi_3|^2 + \chi_1 \chi_6^* + \chi_3 \chi_6^* + \chi_6 \chi_4^* + \chi_6 \chi_5^* + |\chi_7|^2,$$

$$Z_{(32)} = (\chi_0 + \chi_3)(\chi_0 + \chi_2)^* + (\chi_1 + \chi_3)(\chi_1 + \chi_2)^* + |\chi_6|^2 + |\chi_7|^2$$

$$Z_{(23)} = Z_{(32)}^*, Z_{(45)} = (\chi_0 + \chi_3)(\chi_0 + \chi_2)^* + \chi_1 \chi_6^* + \chi_3 \chi_6^* + \chi_6 \chi_4^* + \chi_6 \chi_5^* + |\chi_7|^2,$$

$$Z_{(54)} = Z_{(45)}^*,$$

$$Z_{(21)} = (\chi_0 + \chi_1 + \chi_3)(\chi_0 + \chi_1 + \chi_2)^* + 2\chi_3 \chi_2^* + \chi_2 \chi_3^* + |\chi_4|^2 + |\chi_5|^2,$$

$$Z_{(12)} = Z_{(21)}^*,$$

$$Z_{(61)} = (\chi_0 + \chi_1 + \chi_3)(\chi_0 + \chi_1 + \chi_2)^* + \chi_3 \chi_2^* + \chi_2 \chi_3^* + |\chi_3|^2 + |\chi_4|^2 + |\chi_5|^2$$

$$Z_{(16)} = Z_{(61)}^*. $$

After a basis change as in [13] Page 698], the $S$ and $T$ matrices become permutation matrices, corresponding to the permutations $(06)(23)(45)$ and $(345)(67)$ of $S_8$. Then one easily sees that the dimension of the commutant $\{S, T\}'$ is 11.

### 6.3 Nimless modular invariants

In the sequel we find nimless modular invariants as an application of the nimrep theory developed in [7] and [27]. We use the graphs of norm 2 whose number of vertices is less or equal to six (see [29] or [27] Page 58] together with their Perron-Frobenius eigenvalues as displayed in Fig. 4 [which can be derived from Table 4]. The notation $A^0_n, D^0_n$ refers to tadpole graphs as illustrated in Fig. 4. In any possible nimrep, the matrices $G_{[\lambda]}$ are symmetric because the primary fields are self-conjugate.

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| Graph | Eigenvalues                  | Range       |
|-------|-----------------------------|-------------|
| $A_{n}^{(1)}$, $n \geq 1$ | $2 \cos(2\pi k/(n + 1))$ | $0 \leq k \leq n$ |
| $D_{n}^{(1)}$, $n \geq 4$ | $0, 0, 2 \cos(\pi k/(n - 2))$ | $0 \leq k \leq n - 2$ |
| $E_{6}^{(3)}$ | $\pm 2, \pm 1, \pm 1, 0$ |             |
| $E_{8}^{(3)}$ | $\pm 2, \pm \sqrt{2}, \pm 1, 0, 0$ |             |
| $E_{8}^{(1)}$ | $\pm 2, \pm 2 \cos(\pi/5), \pm 1, \pm 2 \cos(2\pi/5), 0$ |             |
| $A_{n}^{d}$, $n \geq 1$ | $2 \cos(k\pi/n)$ | $0 \leq k < n$ |
| $D_{n}^{6}$, $n \geq 3$ | $0, 2 \cos(2\pi k/(2n - 3))$ | $0 \leq k \leq n - 2$ |

Table 4: Connected graphs with norm two and their extended ADE graphs

**Proposition 6.1** The modular invariants $Z_6$, $Z_{11}$, $Z_{14}$, $Z_{15}$ and the transposes $Z_{41}$, $Z_{51}$ are nimless.

**Proof.** Let us assume that there is a nimrep for the modular invariant $Z_6$, then its exponents are $\{0, 1, 2, 3, 4^2, 5^2\}$. So the spectrum of the matrix $G[5]$ is $\{-1^4, 2^4\}$. In the sequel if $a$ is primary field or eigenvalue with multiplicity $n$, we abbreviate this by $a^n$. Hence $G[5]$ has four indecomposable components each of them with norm two. By Fig. 34 (or Table 4) we conclude that there is no $(n + 1)$ by $(n + 1)$ matrix with spectrum is $\{-1^n, 2\}$ for $n = 1, 2, 3, 4, 5$. Therefore $Z_6$ is nimless.

If $Z_{11}$ is nimble then as its exponents are the primary fields $\{1, 2, 3\}$, we find that the spectrum of $G[2]$ is $\{-1, 2^3\}$. Hence, $G[2]$ has 3 irreducible components, one of them with spectrum being $\{-1, 2\}$. There is no 2 by 2 matrix whose spectrum is given by this set as seen in Fig. 34. Let us suppose that there is a nimrep for $Z_{15}$, so Exp= $\{0, 3\}$. Then we are looking for a 2 by 2 symmetric non-negative matrix $G[3]$ with eigenvalues $S_{20}/S_{00} = 2$ and $S_{23}/S_{03} = -1$. Again such a matrix cannot exist. The same can be carried out (it is actually in [27] Page 38) for $Z_{14}$ whose exponents are $\{0, 2\}$. □

**Proposition 6.2** The modular invariants $Z_{(21)}$, $Z_{(61)}$, $Z_{(3)}$, $Z_{(4)}$, $Z_{(1)}$, $Z_{(12)}$, $Z_{(2)}$ and $Z_{(16)}$ are nimless.

**Proof.** The exponents of the trace 6 modular invariants $Z_{(61)}$, $Z_{(16)} = Z_{(61)}^\tau$ are $\{0, 1, 2, 3, 4, 5\}$. If a nimrep exists then the spectrum of $G[2]$ is $\{-1^3, 2^3\}$ so $G[2]$ has three connected components. However there is no norm two matrix with spectrum $\{-1, 2\}$ or $\{-1^3, 2\}$ (see Fig. 34). Therefore $Z_{(61)}$ and $Z_{(16)}$ are nimless. The exponents of the trace 6 matrix $Z_{(3)}$ are $\{0, 1, 2^2, 4, 5\}$. The spectrum of $G[7]$ in a nimrep (if it exists) is also $\{-1^3, 2^3\}$, therefore $Z_{(3)}$ is nimless by the above argument. The exponents of the trace 8 matrix $Z_{(4)}$ are $\{0, 1, 2^2, 3, 4, 5\}$. If a nimrep exists, then the spectrum of $G[2]$ is $\{-1^3, 2^5\}$. Hence $G[2]$ has five connected components. By the above arguments such a matrix $G[2]$ cannot exist. So $Z_{(4)}$ is nimless. The interchange of the primary fields 2 and 3 gives us the exponents of $Z_{(2)}$. Therefore $Z_{(2)}$ is also nimless. The exponents of $Z_{(21)}$ and $Z_{(12)}$ are $\{0, 1, 2^2, 6, 7\}$. If they are nimble, then the spectrum of $G[5]$ is $\{-1, 2^3\}$. There is however no norm 2 graph whose spectrum is $\{-1, 2\}$. Hence these modular invariants are nimless. Finally, from $Z_{(3)}$ we obtain the exponents of $Z_{(1)}$ by the interchange of the primary fields 2 and 3. So $Z_{(1)}$ is nimless. □
Figure 34: Connected graphs with norm two and \( \# \text{vertices} \leq 6 \) together with their eigenvalues (cf. Table 4)

### 6.4 Nimble modular invariants

We produce here by hand the list of quantum \( S_3 \) double modular invariants that have a compatible nimrep.

**Proposition 6.3** The trace one modular invariants \( Z_{25}, Z_{52}, Z_{34} \) and \( Z_{43} \) have unique nimreps.

**Proof.** The primary field 0 is the only exponent. We have trivially the (unique) nimrep:

\[
G_0 = G_1 = 1, \quad G_2 = G_3 = G_4 = G_5 = 2, \quad G_6 = G_7 = 3. \]

**Proposition 6.4** The trace two modular invariants \( Z_3, Z_{23}, Z_{32}, Z_{45}, Z_{54}, Z_{(45)} \) and \( Z_{(54)} \) have unique nimreps.

**Proof.** The exponents of \( Z_3, Z_{23} \) and \( Z_{32} \) are \( \{0, 1\} \), and we obtain (unique) nimrep:

\[
G_0 = G_1 = \text{id}, \quad G_2 = G_3 = G_4 = G_5 = 2\text{id}, \quad G_6 = G_7 = \begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix}. \]

We likewise construct the unique nimrep for both \( Z_{45} \) and \( Z_{54} \). Their exponents are \( \{0, 6\} \). The spectrum of every \( G_\lambda \) is provided in Table 5. We therefore find that \( G_0 = \text{id}, \)
Table 5: Eigenvalues of $G_\lambda$ for $Z_{45}$, $Z_{54}$ and $Z_{(45)}$, $Z_{(54)}$

| $Z_{45}, Z_{54}$ | 0 | 6 |
|------------------|---|---|
| $G_{[0]}$       | 1 | 1 |
| $G_{[1]}$       | 1 | -1 |
| $G_{[2]}$       | 2 | 0 |
| $G_{[3]}$       | 2 | 0 |
| $G_{[4]}$       | 2 | 0 |
| $G_{[5]}$       | 2 | 0 |
| $G_{[6]}$       | 3 | 1 |
| $G_{[7]}$       | 3 | -1 |

| $Z_{(45)}, Z_{(54)}$ | 0 | 7 |
|----------------------|---|---|
| $G_{[0]}$            | 1 | 1 |
| $G_{[1]}$            | 1 | -1 |
| $G_{[2]}$            | 2 | 0 |
| $G_{[3]}$            | 2 | 0 |
| $G_{[4]}$            | 2 | 0 |
| $G_{[5]}$            | 2 | 0 |
| $G_{[6]}$            | 3 | -1 |
| $G_{[7]}$            | 3 | 1 |

$G_{[1]} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $G_{[2]} = G_{[3]} = G_{[4]} = G_{[5]} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, $G_{[6]} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$, and $G_{[7]} = G_{[1]}G_{[6]}$.

Finally, the exponents of the trace 2 matrices $Z_{(45)}$ and $Z_{(54)}$ are \{0, 7\}. The list of the eigenvalues of every matrix $G_\lambda$ is as in Table 5. We find the following (unique) nimrep for these modular invariants:

$$G_{[0]} = \text{id}, \quad G_{[1]} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad G_{[2]} = G_{[3]} = G_{[4]} = G_{[5]} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

$$G_{[6]} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \quad G_{[7]} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}.$$

Proposition 6.5 The trace three modular invariants $Z_{24}$, $Z_{42}$, $Z_{35}$, $Z_{53}$, $Z_{44}$, $Z_{(55)}$ and $Z_{(44)}$ have unique nimreps.

Proof. The exponents of $Z_{44}$ are \{0, 2, 6\}. So the spectrum of each non-negative matrix $G_\lambda$ is provided in Table 6. Using Fig. 34, we can fix (i.e. up to permutations) $G_{[2]}$ as

$$G_{[2]} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

Then by the fusion rules, $G_{[2]} \cdot G_{[3]} = G_{[4]} + G_{[5]} = 2G_{[3]}$. This equation holds true only when $G_{[3]} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ using Fig. 34. Moreover, we obtain

$$\begin{pmatrix} 4 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 2 & 2 \end{pmatrix} = G_{[2]} \cdot G_{[2]} = G_{[0]} + G_{[1]} + G_{[2]} = G_{[1]} + \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$$

which implies that $G_{[1]} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$.
Now we look for a 3 by 3 symmetric matrix $G[7]$ with entries from \{0,1,2,3\} whose trace equals $3 - 1 = 2$. From the fusion rules

$$G[7] \cdot G[7] = G[0] + G[2] + G[3] + G[4] + G[5] = G[0] + G[2] + 3G[3] = \begin{pmatrix} 6 & 3 & 3 \\ 3 & 2 & 1 \\ 3 & 1 & 2 \end{pmatrix}$$

so in particular the entries of $G[7]$ are from \{0,1,2\}. We can now easily check that the trace 2 matrix $G[7]$ has to be $G[7] = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$. Finally, let us look for a 3 by 3 matrix

$$G[6] = \begin{pmatrix} a & b & c \\ b & y & f \\ c & f & d \end{pmatrix}$$

whose entries belong to \{0,1,2,3\}. In this way, by the fusion rules,

$$\begin{pmatrix} 2a & b + c & b + c \\ 2b & y + f & y + f \\ 2c & d + f & d + f \end{pmatrix} = G[6] \cdot G[2] = G[6] + G[7] = \begin{pmatrix} 2a & b + 1 & c + 1 \\ b + 1 & y & f + 1 \\ c + 1 & f + 1 & d \end{pmatrix}.$$ 

But we also know that the fusion rules are commutative, so

$$\begin{pmatrix} 2a & b + c & b + c \\ 2b & y + f & y + f \\ 2c & d + f & d + f \end{pmatrix} = \begin{pmatrix} 2a & 2b & 2c \\ 2a & b + c & y + f & f + d \\ 2b & c + 1 & f + 1 & d \end{pmatrix}$$

leading to the conclusion: $a = 2, b = c = d = y = 1, f = 0.$

![Table 6: Eigenvalues of $G[\lambda]$ for $Z_{44}$, $Z_{35}$ and $Z_{53}$](image)

Next we consider $Z_{35}$ (and $Z_{53}$), with its exponents \{0,3^2\}. The spectra of the matrices $G[\lambda]$ are in Table 6. So using Fig. 33 $G[0] = G[1] = \text{id}, \quad G[3] = 2\text{id},$

$$G[2] = G[4] = G[5] = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$ 

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Now we look for fusion matrices $G_{[6]} = G_{[7]} = \begin{pmatrix} a & b & c \\ b & d & e \\ c & e & f \end{pmatrix}$ with $a, b, c, d, e, f = 0, 1, 2, 3$.

By the fusion rules,

$$G_{[6]} \cdot G_{[6]} = G_{[0]} + G_{[2]} + G_{[3]} + G_{[4]} + G_{[5]} = G_{[0]} + 3G_{[2]} + G_{[3]} = \begin{pmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{pmatrix}.$$ 

Now $G_{[6]}$ is symmetric with positive spectrum and its square is the matrix with 3 everywhere. So by spectral theory, $G_{[6]}$ is unique and has to be the matrix with 1 everywhere. Now it is easy to check that this $G_{[6]}$ is consistent with the other fusion rules. Therefore the modular invariants $Z_{35}$ and $Z_{53}$ are nimble. The exponents of $Z_{24}$ and $Z_{42}$ are $\{0, 2^2\}$. The spectra of the 3 by 3 matrices $G_{[\lambda]}$ are in Table 8. We then compute as in the $Z_{44}$ case and get the following nimrep: $G_{[0]} = G_{[1]} = \text{id}$, $G_{[2]} = 2\text{id}$,

$$G_{[3]} = G_{[4]} = G_{[5]} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \quad G_{[5]} = G_{[6]} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$ 

| $Z_{(55)}$ | 0 | 3 | 7 |
|----------|---|---|---|
| $G_{[0]}$ | 1 | 1 | 1 |
| $G_{[1]}$ | 1 | 1 | -1 |
| $G_{[2]}$ | 2 | -1 | 0 |
| $G_{[3]}$ | 2 | 2 | 0 |
| $G_{[4]}$ | 2 | -1 | 0 |
| $G_{[5]}$ | 2 | -1 | 0 |
| $G_{[6]}$ | 3 | 0 | -1 |
| $G_{[7]}$ | 3 | 0 | 1 |

| $Z_{(44)}$ | 0 | 2 | 7 |
|----------|---|---|---|
| $G_{[0]}$ | 1 | 1 | 1 |
| $G_{[1]}$ | 1 | 1 | -1 |
| $G_{[2]}$ | 2 | 2 | 0 |
| $G_{[3]}$ | 2 | -1 | 0 |
| $G_{[4]}$ | 2 | -1 | 0 |
| $G_{[5]}$ | 2 | -1 | 0 |
| $G_{[6]}$ | 3 | 0 | -1 |
| $G_{[7]}$ | 3 | 0 | 1 |

Table 7: Eigenvalues of $G_{[\lambda]}$ for $Z_{(44)}$, $Z_{(55)}$, $Z_{24}$ and $Z_{42}$

The exponents of the trace 3 matrix $Z_{(55)}$ are $\{0, 3, 7\}$. The list of the eigenvalues of every matrix $G_{[\lambda]}$ is as in Table 7. Since this table is the same (up to a permutation of the primary fields) as that for $Z_{44}$, we conclude that $Z_{(55)}$ is nimble. We have Exp=$\{0, 2, 7\}$ for the trace 3 matrix $Z_{(44)}$. The eigenvalues of $G_{\lambda}$ are provided in Table 7. Thus, a permutation of matrices of $Z_{(55)}$ give rise to a nimrep of $Z_{(44)}$.

In the following we use $A \oplus B$ for the block matrix $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ and $A \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ for block matrix $\begin{pmatrix} 0 & A \\ A & 0 \end{pmatrix}$.

**Proposition 6.6** The trace four modular invariants $Z_{13}$, $Z_{31}$, $Z_{12}$, $Z_{21}$, $Z_{(32)}$ and $Z_{(23)}$ have unique nimreps.

**Proof.** The primary fields $\{0, 1, 2^2\}$ are the exponents of both $Z_{12}$ and $Z_{21}$. The eigenvalues are in Table 8. We have following nimrep for $Z_{21}$:

$$G_{[0]} = G_{[1]} = \text{id}, \quad G_{[2]} = 2\text{id},$$

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\[ G[3] = G[4] = G[5] = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}, \quad G[6] = G[7] = \begin{pmatrix} 0 & 2 & 2 & 2 \\ 2 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \end{pmatrix}. \]

A nimrep for \( Z_{12} \) is obtained from that of \( Z_{13} \) by interchanging the primary fields 2 and 3.

The exponents of the trace 4 matrices \( Z_{(32)} \) and \( Z_{(23)} \) are \{0, 1, 6, 7\}. The spectra of the matrices \( G[\lambda] \) are in Table 8. We find the following nimrep for these modular invariants:

\[
G[0] = \text{id}, \quad G[1] = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad G[2] = G[3] = G[4] = G[5] = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \oplus \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \\
G[6] = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad G[7] = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
\]

\[ \square \]

| \( Z_{12}, Z_{21} \) | 0 | 1 | 2 | 3 |
|---------------------|---|---|---|---|
| \( G[0] \)        | 1 | 1 | 1 | 1 |
| \( G[1] \)        | 1 | 1 | 1 | 1 |
| \( G[2] \)        | 2 | -2 | 2 | 2 |
| \( G[3] \)        | 2 | 2 | -1 | -1 |
| \( G[4] \)        | 2 | 2 | -1 | -1 |
| \( G[5] \)        | 2 | 2 | -1 | -1 |
| \( G[6] \)        | 3 | -3 | 0 | 0 |
| \( G[7] \)        | 3 | -3 | 0 | 0 |

Table 8: Eigenvalues of \( G[\lambda] \) for \( Z_{12}, Z_{21}, Z_{(32)} \) and \( Z_{(23)} \)

**Proposition 6.7** The trace six modular invariants \( Z_{(22)}, Z_{2}, Z_{4}, Z_{7}, Z_{22}, Z_{33} \) and \( Z_{(33)} \) have unique nimreps.

**Proof.** The exponents of the trace 6 matrix \( Z_{(22)} \) are \{0, 1, 2^2, 6, 7\}. The spectra of the matrices \( G[\lambda] \) are as in Table 8. Then, \( G[0] = \text{id} \) and we can fix

\[
G[3] = G[4] = G[5] = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.
\]

By the fusion rules \( G[3]^2 - G[3] - G[0] = G[1] \) we see that

\[
G[1] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.
\]

Next let us use the fusion rules \( G[2]G[1] = G[2] \) and \( G[2] \cdot G[3] = G[4] + G[5] = 2G[3] \). We then obtain the (unique) matrix

\[
G[2] = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \oplus \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}.
\]
Table 10: We can compute and find that the following set of matrices form a nimrep for $G$

Next we use the commutation of $G_{[6]}$ with $G_{[1]}$, $G_{[2]}$ and $G_{[3]}$, together with the fusion rule $G_{[6]}^2 = G_{[0]} + G_{[2]} + 3G_{[3]}$ to get the matrix

$$G_{[6]} = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Finally, the fusion rule $G_{[6]}G_{[1]} = G_{[7]}$ alone determines $G_{[7]}$, namely:

$$G_{[7]} = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Therefore $Z_{(22)}$ is nimble. Since $Z_{(33)}$ is obtained from $Z_{(22)}$ by permuting the primary fields 2 and 3, we conclude that $Z_{(33)}$ is also nimble. In a similar manner we obtain the following nimrep for the permutation invariant $Z_2$: $G_{[0]} = \text{id}$,

$$G_{[1]} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \end{pmatrix}, G_{[3]} = G_{[2]} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix},$$

$$G_{[4]} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 1 \end{pmatrix}, G_{[5]} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix},$$

$$G_{[6]} = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \end{pmatrix}, G_{[7]} = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \end{pmatrix}.$$

The exponents of $Z_4$ are $\{0, 1, 2^2, 3^2\}$. The spectra of the fusion matrices $G_{[\lambda]}$ are in Table 10. We can compute and find that the following set of matrices form a nimrep for $Z_4$, $G_{[0]} = G_{[1]} = \text{id}$,

$$G_{[2]} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}, G_{[3]} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \otimes \begin{pmatrix} 1 & 1 \end{pmatrix},$$

$$G_{[4]} = G_{[5]} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \end{pmatrix}, G_{[6]} = G_{[7]} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \end{pmatrix}.$$

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We fix the system $N \mathcal{X}^1_N$ arising from the quantum $S_3$ double, regarding the primary field $i$ for $i = 0, \ldots, 7$ as a morphism $\lambda_i \in N \mathcal{X}^1_N$ (see [3] and Fig. 31). Since a modular invariant that can be produced by a subfactor is nimble we immediately get that those from Proposition 6.1 and 6.2 are insufferable because they are nimless.

We now find a $\theta \in \Sigma(N \mathcal{X}^1_N)$ that cannot be a dual canonical endomorphism of any braided subfactor.

**Lemma 6.8** The sector $[\lambda_0] \oplus [\lambda_1] \oplus [\lambda_2] \oplus [\lambda_3]$ cannot be that of the dual canonical endomorphism of a subfactor

**Proof.** Let us suppose that $[\theta] = [\lambda_0] \oplus [\lambda_1] \oplus [\lambda_2] \oplus [\lambda_3]$ is a dual canonical sector of some type III subfactor with inclusion map $\iota$. Using Verlinde fusion matrices $\langle \theta \lambda_0, \lambda_0 \rangle = 1$ so $[\iota \lambda_0] = [\iota]$ is irreducible, $\langle \theta \lambda_0, \lambda_2 \rangle = \langle \theta \lambda_0, \lambda_3 \rangle = 1$, thus $[\iota \lambda_0]$ is in the irreducible decomposition of $[\iota \lambda_2]$ and $[\iota \lambda_3]$. Moreover $\langle \theta \lambda_2, \lambda_2 \rangle = \langle \theta \lambda_2, \lambda_3 \rangle = 3$, $\langle \theta \lambda_2, \lambda_3 \rangle = 0$ which is incompatible with the fact that $[\iota \lambda_0]$ appears in the decomposition of both $[\iota \lambda_2]$ and $[\iota \lambda_3]$. Therefore, $\theta$ cannot be a dual canonical endomorphism of any subfactor. \hfill \Box

**Proposition 6.9** The nimble modular invariants $Z_3$, $Z_4$, $Z_{12}$, $Z_{13}$, $Z_{21}$ and $Z_{31}$ cannot be realised by subfactors.

**Proof.** Assume any one of the listed invariants is realised by a subfactor $N \subset M$. Then $s_1$ is visible in either the first row or first column. The type I parent theorem [11] Theorem 4.7,
yields an intermediate subfactor $N \subset P \subset M$ such that $N \subset P$ has dual canonical sector described by $s_1$, i.e. $[\lambda_0] \oplus [\lambda_1] \oplus [\lambda_2] \oplus [\lambda_3]$. This is however impossible by Lemma 6.8. □

### 6.5 Subfactors producing quantum $S_3$ double modular invariants

We propose in this Section to study the remaining modular invariants of $S_3$. As above we fix the system $N\Lambda_N$ arising from the group $S_3$ and look for subfactors $N \subset M$ of type III with inclusion map $\iota$ whose dual canonical endomorphism $\theta = \iota \in \Sigma(N\Lambda_N)$ such that the associated modular invariant $Z_{N \subset M} = \langle \alpha_\lambda^+, \alpha_\mu^- \rangle$ produces an invariant from the list of $S_3$. The fundamental inclusion (the quantum double of the subfactor $M_0 \subset M_0 \rtimes S_3$) is the subfactor $N := M_0 \times \Delta(S_3) \subset M_0 \times (S_3 \times S_3) =: M$ where $M_0$ is a type III factor [12]. In Table 11 we provide the list of 28 quantum $S_3$ double modular invariants that are nimble and possibly sufferable that will be studied further here.

**Proposition 6.10** The above fundamental subfactor realises the symmetric modular invariant $Z_{55}$. The dual canonical endomorphism $\theta$ is visible in the vacuum block of $Z_{55}$ (so that chiral locality holds).

**Proof.** The irreducible $N$-$N$ sectors of $N \subset M$ are labelled by $\hat{S}_3 \sqcup \hat{S}_2$ whereas the $N$-$M$ ones are labelled by $S_3$ and finally the $M$-$M$ irreducible sectors by $\hat{S}_3 \times \hat{S}_3 \times \hat{S}_3$. The $N$-$M$ star vertex (i.e. the trivial representation of $S_3$) is connected to the $N$-$N$ star vertex, i.e. $[\lambda_0]$, the trivial representation of $Z_3$, i.e. $[\lambda_3]$, and trivial representation of $Z_2$, i.e. $[\lambda_6]$. Hence the dual canonical sector of $N \subset M$ is $[\theta] = [\lambda_0] \oplus [\lambda_3] \oplus [\lambda_6]$. Let us compute the modular invariant $Z_{N \subset M}$ for this braided subfactor. From Verlinde fusion matrices, $N_0$, $N_3$ and $N_6$, we obtain $\langle \theta\lambda_0, \lambda_0 \rangle = \langle \theta\lambda_1, \lambda_1 \rangle = \langle \theta\lambda_2, \lambda_2 \rangle = \langle \theta\lambda_4, \lambda_4 \rangle = \langle \theta\lambda_5, \lambda_5 \rangle = 1$, $\langle \theta\lambda_3, \lambda_3 \rangle = \langle \theta\lambda_6, \lambda_6 \rangle = \langle \theta\lambda_7, \lambda_7 \rangle = 2$, $\langle \theta\lambda_0, \lambda_1 \rangle = \langle \theta\lambda_0, \lambda_2 \rangle = \langle \theta\lambda_0, \lambda_4 \rangle = \langle \theta\lambda_0, \lambda_5 \rangle = 0$, $\langle \theta\lambda_0, \lambda_3 \rangle = \langle \theta\lambda_0, \lambda_6 \rangle = 1$, $\langle \theta\lambda_1, \lambda_7 \rangle = \langle \theta\lambda_1, \lambda_2 \rangle = \langle \theta\lambda_1, \lambda_4 \rangle = 0$, $\langle \theta\lambda_1, \lambda_5 \rangle = \langle \theta\lambda_1, \lambda_6 \rangle = 0$, $\langle \theta\lambda_1, \lambda_3 \rangle = \langle \theta\lambda_1, \lambda_7 \rangle = 1$. So $[\lambda_0], [\lambda_1], [\lambda_2]$ are irreducible $N$-$M$ sectors and $[\lambda_3] = [i\lambda_0] \oplus [i\lambda_1]$. Computing further, $\langle \theta\lambda_2, \lambda_3 \rangle = 0$, $\langle \theta\lambda_2, \lambda_4 \rangle = \langle \theta\lambda_2, \lambda_5 \rangle = \langle \theta\lambda_2, \lambda_6 \rangle = \langle \theta\lambda_2, \lambda_7 \rangle = 1$, hence, $[\lambda_4] = [i\lambda_5] = [i\lambda_2], [\lambda_6] = [i\lambda_0] \oplus [i\lambda_2]$ and $[\lambda_7] = [i\lambda_1] \oplus [i\lambda_2]$. Therefore $\text{Tr}(Z_{N \subset M}) = 3$, which implies that $Z_{N \subset M} = Z_{44}, Z_{55}, Z_{44}, Z_{55}, Z_{24}, Z_{35}, Z_{12}$ or $Z_{53}$. Next we use the inequality $\langle \alpha_\lambda^+, \alpha_\mu^- \rangle \leq \langle \theta\lambda, \mu \rangle$ valid by [8] Eq. (37)) to eliminate more matrices from the above list: since $\langle \theta\lambda_0, \lambda_2 \rangle = 0$ but $Z_{44,0,2} = Z_{24,0,2} = 1$, $Z_{N \subset M}$ cannot be either $Z_{44}$ nor $Z_{24}$; also, $\langle \theta\lambda_0, \lambda_1 \rangle = 0$ and $Z_{42,0,1} = Z_{53,0,1} = 1$, so $Z_{N \subset M}$ cannot be $Z_{42}$ or $Z_{53}$; finally, as $\langle \theta\lambda_1, \lambda_0 \rangle = 0$ and $Z_{35,1,0} = 1$, $Z_{N \subset M}$ cannot be $Z_{35}$. Similarly we use the 02-entry of

| $\text{Tr}(Z)$ | $\text{Tr}(Z^{Z'})$ |
|---------------|------------------|
| 1 | 8, 9, 12 |
| 2 | $Z_{45}, Z_{54}, Z_{45}, Z_{54}$ |
| 3 | $Z_{44}, Z_{55}, Z_{44}, Z_{55}$ |
| 4 | $Z_{33}, Z_{23}$ |
| 5 | $Z_2$ |
| 6 | $Z(22), Z(33)$ |
| 7 | $Z_7, Z_{22}, Z_{33}$ |
| 10 | $Z_{55}$ |

Table 11: The $(\text{Tr}(Z), \text{Tr}(Z^{Z'}))$ of nimble $S_3$ modular invariants

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$Z_{(44)}$ to rule it out. Finally, using the curious identity [17, Proposition 3.3] we conclude that indeed $Z_{N \subseteq M} = Z_{55}$. □

**Proposition 6.11** The modular invariants $Z_1$ and $Z_5$ are realised by subfactors.

**Proof.** The identity modular invariant $Z_1$ can always be obtained by the trivial subfactor $N = 1$. The simple currents are $\lambda_0$ and $\lambda_1$, which form the commutative group $Z_2$. By Lemma 3.3, $[\theta] = [\lambda_0] \oplus [\lambda_1]$ is dual sector since $w_{\lambda_1}^2 = 1$. Now we use Verlinde fusion matrices $N_0$ and $N_1$ and obtain: $\langle \theta \lambda_0, \lambda_0 \rangle = \langle \theta \lambda_1, \lambda_1 \rangle = \langle \theta \lambda_6, \lambda_6 \rangle = \langle \theta \lambda_7, \lambda_7 \rangle = \langle \theta \lambda_0, \lambda_1 \rangle = \langle \theta \lambda_5, \lambda_5 \rangle = 1$. Therefore, $[\lambda_0] = [\lambda_1]$ and $[\lambda_6] = [\lambda_7]$ provide two irreducible $N$-$M$ sectors where $\iota$ denotes the inclusion map $N \hookrightarrow M$. On the other hand, $\langle \theta \lambda_2, \lambda_2 \rangle = \langle \theta \lambda_3, \lambda_3 \rangle = \langle \theta \lambda_4, \lambda_4 \rangle = \langle \theta \lambda_5, \lambda_5 \rangle = 2$, so the sectors $[\lambda_2], [\lambda_3], [\lambda_4], [\lambda_5]$ decompose into two irreducible $N$-$M$ sectors (which at this stage might be equal). However since $\langle \theta \lambda_0, \lambda_2 \rangle = \langle \theta \lambda_0, \lambda_3 \rangle = \langle \theta \lambda_0, \lambda_4 \rangle = \langle \theta \lambda_5, \lambda_5 \rangle = \langle \theta \lambda_6, \lambda_6 \rangle = \langle \theta \lambda_6, \lambda_7 \rangle = \langle \theta \lambda_7, \lambda_7 \rangle = 2$, we get a further $2+2+2+2=8$ new irreducible $N$-$M$ sectors, making a total of 10. Hence $Z_{N \subseteq M} = Z_{55}$ since it is the only trace 10 quantum $S_3$ double modular invariant. □

**Proposition 6.12** The modular invariant $Z_{(33)}$ is produced by the subfactor $N = M_0 \rtimes S_3 \subseteq M_0 \rtimes (S_3 \rtimes S_3)$.

**Proof.** We consider the following intermediate subfactor

$$N = M_0 \rtimes \Delta(S_3) \subseteq_{\iota_1} M_0 \rtimes (S_3 \rtimes (S_3 \times S_3)) \subseteq_{\iota_2} M_0 \rtimes (S_3 \times S_3) = M.$$ 

Then $[\iota_1 \iota_2]$ is a subsector of $[\theta_{N \subseteq M}] = [\iota_1 \iota_2 \iota_3 \iota_4]$ because $[\lambda_0]$ is a subsector of $[\iota_2 \iota_2]$. Since $d_{\iota_1 \iota_2} = 3$ we have that the dual canonical sector of $M_0 \rtimes S_3 \subseteq M_0 \rtimes (S_3 \rtimes (S_3 \times S_3))$ is $[\theta] = [\lambda_0] \oplus [\lambda_3]$. So now we use again the Verlinde fusion matrices $N_0$ and $N_3$ and compute the dimensions of the intertwiner spaces in order to get the $N$-$M$ induced system:

$$\langle \theta \lambda_0, \lambda_0 \rangle = \langle \theta \lambda_1, \lambda_1 \rangle = \langle \theta \lambda_2, \lambda_2 \rangle = \langle \theta \lambda_4, \lambda_4 \rangle = \langle \theta \lambda_5, \lambda_5 \rangle = 1, \langle \theta \lambda_3, \lambda_3 \rangle = \langle \theta \lambda_6, \lambda_6 \rangle = \langle \theta \lambda_7, \lambda_7 \rangle = 2, \langle \theta \lambda_0, \lambda_1 \rangle = \langle \theta \lambda_0, \lambda_2 \rangle = \langle \theta \lambda_0, \lambda_4 \rangle = \langle \theta \lambda_0, \lambda_5 \rangle = \langle \theta \lambda_0, \lambda_6 \rangle = \langle \theta \lambda_0, \lambda_7 \rangle = 0, \langle \theta \lambda_0, \lambda_3 \rangle = \langle \theta \lambda_1, \lambda_3 \rangle = 1, \langle \theta \lambda_0, \lambda_5 \rangle = \langle \theta \lambda_0, \lambda_6 \rangle = \langle \theta \lambda_0, \lambda_7 \rangle = 0, \langle \theta \lambda_0, \lambda_3 \rangle = \langle \theta \lambda_2, \lambda_3 \rangle = \langle \theta \lambda_2, \lambda_5 \rangle = \langle \theta \lambda_2, \lambda_6 \rangle = \langle \theta \lambda_2, \lambda_7 \rangle = 0, \langle \theta \lambda_2, \lambda_4 \rangle = \langle \theta \lambda_2, \lambda_5 \rangle = \langle \theta \lambda_2, \lambda_6 \rangle = \langle \theta \lambda_2, \lambda_7 \rangle = 1. \tag{32}$$

Then $[\iota_0], [\iota_1], [\iota_2]$ are distinct irreducible sectors and $[\iota_3] = [\iota_0] \oplus [\iota_1], [\iota_5] = [\iota_4] = [\iota_2], [\iota_6] = [a] \oplus [b]$ and $[\iota_7] = [a] \oplus [c]$ with $[a]$, $[b]$ and $[c]$ three new irreducible sectors. Therefore, $\text{Tr}(Z_{M_0 \rtimes S_3 \subseteq M_0 \rtimes (S_3 \rtimes (S_3 \times S_3))) = Z_{(33)}, Z_2, Z_7, Z_{22} \text{ or } Z_{33}$. Recall that in general $\langle \alpha^+, \alpha^- \rangle \leq \langle \theta, \mu \rangle$ (see [5, Eq. (37)]). So since for our concrete $\theta$, $\langle \theta, \lambda_1 \rangle = 0$ then $Z_{M_0 \rtimes S_3 \subseteq M_0 \rtimes (S_3 \rtimes (S_3 \times S_3))}$ cannot be $Z_2, Z_{22}, Z_{33}$ because the 01-entry of those matrices is 1. The $Z_{(33)}$ matrix is ruled out using its 02-entry. Hence $Z = Z_2, Z_{(33)}$. We will use the curious identity [17, Proposition 3.3] (see Sect. 2.2) to eliminate one of them. The RHS of that identity using $Z_2$ is

$$\bigoplus_{\lambda, \mu} (Z_2)_{\lambda, \mu} \langle \lambda | \mu \rangle = 6 | \lambda_0 \rangle + 2 | \lambda_1 \rangle + 2 | \lambda_2 \rangle + 2 | \lambda_3 \rangle + 5 | \lambda_4 \rangle + 5 | \lambda_5 \rangle.$$
The LHS is (with $\theta = [\lambda_0] + [\lambda_3]$):
\[
\bigoplus_{x \in M \times X_N} [x \bar{x}] = [\lambda_0 \theta \lambda_0] + [\lambda_1 \theta \lambda_1] + [\lambda_2 \theta \lambda_2] + [a \bar{a}] + [\bar{b}] + [\bar{c}].
\]

Since $\lambda_3$ appears at least four times on the LHS, and only twice on the RHS using the modular invariant $Z_2$, we conclude using the curious identity [17 Proposition 3.3] that indeed $Z = Z_{(33)}$.

**Proposition 6.13** *The permutation modular invariant $Z_2$ is sufferable.*

**Proof.** Let us consider the endomorphism $\theta = \lambda_0 \oplus \lambda_4 \in \Sigma(X \times N)$. We prove now that $\theta$ is indeed a dual canonical endomorphism using the results of Sect. 3. First we equip $\theta$ with a Q-system structure. The final Eq. in (14) reduces to
\[
w^{0}_{ij} w^{0}_{lj} = w^{0}_{jk} w^{0}_{kl} F^{(ijk)l}_{ijkl} + w^{0}_{jkl} F^{(ijkl)}_{ijkl} = 0 \hspace{1cm} (17)
\]
with $i, j, k, l, p = 0, 4$ and the $F$’s are the $6j$-symbols arising from the fusion rules of the the quantum $S_3$ double model. In the sequel we will use the normalisations from [21 Eq. (3.11)] (cf. [29 Eq. (3.4)]: $F^{(jkl)}_{ijkl} = \delta_{ijl}$, $F^{(ijkl)}_{ijkl} = \delta_{ijk}$. As in [21 Eq. (3.11)], $F^{(ij)k}_{(i)jk} = \sqrt{\frac{d_k}{d_i d_j}}$ whenever $N_{ij}^k \neq 0$ by considering the diagonal case $G_{ij} = N_{ij}$. Then we also have the identity $F^{(iii)l}_{(i)l} F^{(iii)}_{(i)l} = 1/d_i$ which holds true by the pentagon equation
\[
F^{(iii)l}_{(i)l} F^{(iii)}_{(i)l} = \sum_s F^{(iii)l}_{(i)ls} F^{(iii)}_{s l} F^{(iii)}_{(i)l} = F^{(iii)l}_{(i)l} F^{(iii)}_{(i)l} = F^{(iii)}_{(i)l}.
\]

Now we find the unitary matrix $F^{(444)4}$. Note first that since $\lambda_0, \lambda_1$ and $\lambda_4$ are self conjugate, the entries of $F^{(444)4}$ are real numbers by [21 Eq. (3.13)]. So far we have $F^{(444)4}_{00} = 1/\sqrt{2} = F^{(444)4}_{04}$, $F^{(444)4}_{40} = 1/d_4 = 1/2$, $F^{(444)4}_{40} = 1/2$. Note that
\[
\sum_{s=0,1,4} F^{(444)4}_{1s} F^{(444)4}_{s4} = 0
\]
so $F^{(444)4}_{14} \neq 0$ and we have the following four equations by the orthogonality relations
\[
F^{(444)4}_{00} F^{(444)4}_{04} + F^{(444)4}_{04} F^{(444)4}_{14} + F^{(444)4}_{04} F^{(444)4}_{40} = \sqrt{2}/4 + F^{(444)4}_{14}/2 + \sqrt{2}F^{(444)4}_{44}/2 = 0 \hspace{1cm} (18)
\]
\[
F^{(444)4}_{04} F^{(444)4}_{04} + F^{(444)4}_{41} F^{(444)4}_{44} + F^{(444)4}_{44} F^{(444)4}_{44} = 1/2 + F^{(444)4}_{14} = F^{(444)4}_{41} + F^{(444)4}_{44} F^{(444)4}_{44} = 1, \hspace{1cm} (19)
\]
\[
F^{(444)4}_{10} F^{(444)4}_{04} + F^{(444)4}_{11} F^{(444)4}_{14} + F^{(444)4}_{14} F^{(444)4}_{41} = 1, \hspace{1cm} (20)
\]
\[
F^{(444)4}_{10} F^{(444)4}_{04} + F^{(444)4}_{11} F^{(444)4}_{14} + F^{(444)4}_{14} F^{(444)4}_{41} = 0. \hspace{1cm} (21)
\]
Now we can insert the value of $F_{14}^{(444)4}$ from Eq. (18) in Eq. (19) to obtain $2F_{14}^{(444)4} + 3(F_{44}^{(444)4})^2 = 0$. Then $F_{44}^{(444)4} = 0$ or $F_{44}^{(444)4} = -2/3$. Let us suppose that $F_{44}^{(444)4} = -2/3$. Then replacing this value in Eq. (18) we find $F_{14}^{(444)4} = \sqrt{2}/3$. But then solving Eq. (20) and Eq. (21) we get different values for $F_{11}^{(444)4}$. Hence $F_{44}^{(444)4} = 0$ and thus $F_{14}^{(444)4} = -\sqrt{2}/2$ and $F_{11}^{(444)4} = 1/2$. So the matrix $F^{(444)4}$

\[
F^{(444)4} = \begin{pmatrix}
F_{00}^{(444)4} & F_{01}^{(444)4} & F_{04}^{(444)4} \\
F_{10}^{(444)4} & F_{11}^{(444)4} & F_{14}^{(444)4} \\
F_{40}^{(444)4} & F_{41}^{(444)4} & F_{44}^{(444)4}
\end{pmatrix} = \begin{pmatrix}
1/2 & 1/2 & \sqrt{2}/2 \\
1/2 & 1/2 & -\sqrt{2}/2 \\
\sqrt{2}/2 & -\sqrt{2}/2 & 0
\end{pmatrix}.
\]

Since we get exactly the same (above) equations when we replace the $\lambda_3$ by $\lambda_2$, we conclude that $F^{(444)4} = F^{(333)3}$, hence since $[\lambda_0] \oplus [\lambda_3]$ is a dual canonical sector by Proposition 6.12 we can conclude that $[\theta] = [\lambda_0] \oplus [\lambda_3]$ is a dual canonical sector of a braided subfactor $N \subset M$ by Sect. 6. We now find the modular invariant attached to $\theta$. We compute and see that $\langle \theta_0, \lambda_0 \rangle = \langle \theta_1, \lambda_1 \rangle = \langle \theta_2, \lambda_2 \rangle = \langle \theta_3, \lambda_3 \rangle = 1$, $\langle \theta_4, \lambda_4 \rangle = \langle \theta_5, \lambda_5 \rangle = \langle \theta_6, \lambda_6 \rangle = \langle \theta_7, \lambda_7 \rangle = 2$, $\langle \theta_0, \lambda_1 \rangle = \langle \theta_0, \lambda_2 \rangle = \langle \theta_0, \lambda_3 \rangle = 1$, $\langle \theta_0, \lambda_4 \rangle = \langle \theta_0, \lambda_5 \rangle = \langle \theta_0, \lambda_6 \rangle = \langle \theta_1, \lambda_1 \rangle = \langle \theta_1, \lambda_2 \rangle = \langle \theta_1, \lambda_3 \rangle = 1$, $\langle \theta_2, \lambda_2 \rangle = \langle \theta_2, \lambda_3 \rangle = \langle \theta_2, \lambda_4 \rangle = \langle \theta_2, \lambda_5 \rangle = \langle \theta_2, \lambda_6 \rangle = 1$, $\langle \theta_3, \lambda_3 \rangle = \langle \theta_3, \lambda_4 \rangle = \langle \theta_3, \lambda_5 \rangle = \langle \theta_3, \lambda_6 \rangle = 2$, $\langle \theta_4, \lambda_4 \rangle = \langle \theta_4, \lambda_5 \rangle = \langle \theta_4, \lambda_6 \rangle = 1$, $\langle \theta_5, \lambda_5 \rangle = \langle \theta_5, \lambda_6 \rangle = 3$, $\langle \theta_6, \lambda_6 \rangle = 4$, $\langle \theta_7, \lambda_7 \rangle = 5$. Hence $F^{(444)4} = 0$ and thus $F_{14}^{(444)4} = -\sqrt{2}/2$ and $F_{11}^{(444)4} = 1/2$. So the matrix $F^{(444)4}$

\[
F^{(444)4} = \begin{pmatrix}
F_{00}^{(444)4} & F_{01}^{(444)4} & F_{04}^{(444)4} \\
F_{10}^{(444)4} & F_{11}^{(444)4} & F_{14}^{(444)4} \\
F_{40}^{(444)4} & F_{41}^{(444)4} & F_{44}^{(444)4}
\end{pmatrix} = \begin{pmatrix}
1/2 & 1/2 & \sqrt{2}/2 \\
1/2 & 1/2 & -\sqrt{2}/2 \\
\sqrt{2}/2 & -\sqrt{2}/2 & 0
\end{pmatrix}.
\]

Remark 6.14 We can likewise prove that $\theta = \lambda_0 \oplus \lambda_5 \in \Sigma(\mathcal{X}_N)$ is a dual canonical endomorphism. It also produces the permutation modular invariant $Z_2$.

Proposition 6.15 The modular invariants $Z_{(22)}$ and $Z_{(44)}$ are sufficientable.

Proof. The modular invariant $Z_{(22)}$ is sufficientable because it is a product of sufficientable modular invariants $Z_{(22)} = Z_2 Z_{(33)} Z_2$ (see previous Propositions 6.12 and 6.13). We now prove that $Z_{(44)}$ is sufficientable. We already know that $Z_{(22)}$ is sufficientable and moreover it is a type I modular invariant with $[\theta] = [\lambda_0] \oplus [\lambda_2]$. The global indices of $\mathcal{M} \mathcal{X}_M$, $\mathcal{M}^+ \mathcal{X}_M$ and $\mathcal{M} \mathcal{X}_M^+$ are $\omega = 36, \omega_+ = 36/3 = 12, \omega_0 = 12^2/36 = 4$ respectively. We compute the $\mathcal{X}_+^-$ chiral systems and conclude that the irreducible decompositions are as follows: $[\alpha_0], [\alpha_1^+]$, $[\alpha_2^+] = [\alpha_0] \oplus [\alpha_1^+]$, $[\alpha_3^+] = [\alpha_0^+] \oplus [\alpha_1^+]$, $[\alpha_4^+] = [\alpha_0^+] \oplus [\alpha_1^+] \oplus [\alpha_2^+]$, $[\alpha_5^+] = [\alpha_0^+] \oplus [\alpha_1^+] \oplus [\alpha_2^+] \oplus [\alpha_3^+]$, $[\alpha_6^+] = [\alpha_0^+] \oplus [\alpha_1^+] \oplus [\alpha_2^+] \oplus [\alpha_3^+] \oplus [\alpha_4^+]$, $[\alpha_7^+] = [\alpha_0^+] \oplus [\alpha_1^+] \oplus [\alpha_2^+] \oplus [\alpha_3^+] \oplus [\alpha_4^+] \oplus [\alpha_5^+]$. From the entries of the modular invariant $Z_{(22)}$, we see that $[\alpha_1^+] = [\alpha_1^-]$ denoted henceforth by $[\alpha_1]$, $[\alpha_6^+(-2)] = [\alpha_6^-]$ denoted from now by $[\alpha_6^-]$ and similarly $[\alpha_7^+]$. Thus $\mathcal{M} \mathcal{X}_M^+ = \{\alpha_0, \alpha_1, \alpha_6^+, \alpha_7^+\}$. Also the $\mathcal{X}_-^+\text{-chiral}$ systems are $\mathcal{M} \mathcal{X}_M^+ = \{\alpha_0, \alpha_1, \alpha_6^-, \alpha_7^+\}$.

In particular, the branching coefficient matrix $b = [b_{\tau, \lambda}]$ with $b_{\tau, \lambda} = \langle \tau, \lambda \rangle$, $\tau \in \mathcal{M} \mathcal{X}_M^+$:

\[
b = \begin{pmatrix}
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}.
\]
is the dual canonical sector of $S$ and $\sigma$ by permuting the second $v$.

By [6], there is a non-degenerate braiding on the neutral system $\Lambda^0_M \cdot Z^t_M$ of $Z_{(22)}$. Let $S^{\text{ext}}$ and $T^{\text{ext}}$ be the modular matrices of the extended system $\Lambda^0_M \cdot Z^t_M$. So by [4, Theorem 6.5], $S^{\text{ext}} b = b S, T^{\text{ext}} b = b T$. Since we are in the fortunate situation that

$$bb^t = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

is invertible, we can solve the above equations and get $S^{\text{ext}}$ and $T^{\text{ext}}$ uniquely. They are the modular data of the quantum double of $\mathbb{Z}_2$ presented in [5] or [4, Page 285] or Eq. 10 by permuting the second $v$ and fourth $c$ labels. Since

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

is sufferable by [5], so is $b^t M b = Z_{(41)}$. If $N \subset M$ produces $Z_{(22)}$, then $N \subset M \times_v \mathbb{Z}_2$ realises $Z_{(44)}$ whose dual canonical endomorphism is $\theta = \sigma b^{\text{ext}}$ with $[\theta^{\text{ext}}] = [\lambda_0] \oplus [\alpha_2^{(2)}]$ and $\sigma$ is the $\sigma$-restriction [5] defined by $\sigma_\beta = i \beta$. We compute and find that $[\theta] = [\lambda_0] \oplus [\lambda_2] \oplus [\lambda_7]$ is the dual canonical sector of $N \subset M \times \mathbb{Z}_2$.

---

Table 12: Fusion $Z_a Z^t_b$ of $S_3$ modular invariants from Table 11

By [5], there is a non-degenerate braiding on the neutral system $\Lambda^0_M \cdot Z^t_M$ of $Z_{(22)}$. Let $S^{\text{ext}}$ and $T^{\text{ext}}$ be the modular matrices of the extended system $\Lambda^0_M \cdot Z^t_M$. So by [4, Theorem 6.5], $S^{\text{ext}} b = b S, T^{\text{ext}} b = b T$. Since we are in the fortunate situation that

$$bb^t = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

is invertible, we can solve the above equations and get $S^{\text{ext}}$ and $T^{\text{ext}}$ uniquely. They are the modular data of the quantum double of $\mathbb{Z}_2$ presented in [5] or [4, Page 285] or Eq. 10 by permuting the second $v$ and fourth $c$ labels. Since

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

is sufferable by [5], so is $b^t M b = Z_{(41)}$. If $N \subset M$ produces $Z_{(22)}$, then $N \subset M \times_v \mathbb{Z}_2$ realises $Z_{(44)}$ whose dual canonical endomorphism is $\theta = \sigma b^{\text{ext}}$ with $[\theta^{\text{ext}}] = [\lambda_0] \oplus [\alpha_2^{(2)}]$ and $\sigma$ is the $\sigma$-restriction [5] defined by $\sigma_\beta = i \beta$. We compute and find that $[\theta] = [\lambda_0] \oplus [\lambda_2] \oplus [\lambda_7]$ is the dual canonical sector of $N \subset M \times \mathbb{Z}_2$. □

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In Tables 12 and 13 we present products \(Z_a Z_b^t\) of the 28 matrices of Table 11 together with their (unique) decomposition into normalised modular invariants. Since by Sect. 8 products of sufferable modular invariants are sufferable, we conclude that all the 28 nimble modular invariants of the Table 11 are sufferable as follows. We already know that \(Z_1, Z_2, Z_5, Z_{55}, Z_{(44)}, Z_{(22)}, Z_{(33)}\) are sufferable. The modular invariant \(Z_7\) is sufferable because \(Z_7 = Z_2 Z_5\). Then \(Z_{(23)} = Z_{(22)} Z_{(33)}\) is sufferable and thus so is its conjugate \(Z_{(32)} = Z_{(23)}^t\) as \(Z_7\) is sufferable because \(Z_7 = Z_2 Z_5\). Then \(Z_{(23)} = Z_{(22)} Z_{(33)}\) is sufferable and thus so is its conjugate \(Z_{(32)} = Z_{(23)}^t\). As \(Z_7 Z_{55} = Z_{25}\) we also conclude that \(Z_{25}\) and its conjugate \(Z_{52}\) are sufferable; so is \(Z_{22}\) by the type I parent theorem 4. As \(Z_5 Z_{52} = Z_{32}, Z_{32}\) and its conjugate \(Z_{23}\) are sufferable; so is \(Z_{33}\) by the type I theorem. As \(Z_{55} Z_{5} = Z_{35}, Z_{35}\) and its conjugate \(Z_{53}\) are also sufferable. Then \(Z_5 Z_{(44)} = Z_{24}, Z_{(44)} Z_{33} = Z_{43}\) and \(Z_{55} Z_{24} = Z_{54}\), so \(Z_{24}, Z_{42}, Z_{34}, Z_{43}, Z_{54}, Z_{45}\) are sufferable. Again by the type I parent theorem 4, \(Z_{(44)} Z_{(33)} = Z_{(54)}\) we conclude that \(Z_{(54)}\) and its conjugate \(Z_{(45)}\) are sufferable. The same with \(Z_{(55)}\) since \(Z_{(55)} = Z_{(33)} Z_{(54)}\).

**Remark 6.16** In the framework of [48], every (conjugacy class) subgroup of \(S_3 \times S_3\) gives rise to a sufferable modular invariant (although different subgroups may well give the same modular invariant). The trace of \(Z\) attached to \(H\) is easily computed: we choose representatives \(\{g\}\) of the double cosets \(\Delta(S_3) \setminus S_3 \times S_3 / H\) then

\[
\text{Tr}(Z) = \sum_g \# \text{irred}(\Delta(S_3) \cap g H g^{-1}).
\]

With our approach we managed to show that the trace 4 modular invariants \(Z_{(32)}\) and \(Z_{(23)}\) are sufferable which do not appear in Ostrik’s Table 4.4 in [48] where there are 20 subgroups of \(S_3 \times S_3\). However, there are two more (conjugacy classes) subgroups \(H_{21} = (Z_3 \times 0) \cdot (\Delta(Z_2))\) and \(H_{22} = (\Delta(Z_2)) \cdot (0 \times Z_3)\), both isomorphic to \(S_3\) which are missing in [48] Table 4.1, which produce \(\text{Tr}(Z) = 4\) modular invariants. Therefore \(Z = Z_{(23)}, Z_{(32)}\).

For the sake of completeness, we summarize the 48 modular invariants in each category: nimless, spurious nimble and sufferable. The span of the sufferable modular invariants is \(\{S,T\}'\).

**Corollary 6.17**

1. The nimless \(S_3\) modular invariants are the 14 matrices

\[
\begin{array}{cccccccc}
Z_6 & Z_{11} & Z_{14} & Z_{15} & Z_{21} & Z_{22} & Z_{24} & Z_{25} \\
Z_{(3)} & Z_{(4)} & Z_{(1)} & Z_{(12)} & Z_{(2)} & Z_{(16)} & Z_{(61)}
\end{array}
\]

2. The nimble but insufferable modular invariants are the 6 matrices

\[
\begin{array}{cccc}
Z_6 & Z_7 & Z_8 & Z_{21} & Z_{31} \\
Z_{35} & Z_{42} & Z_{43} & Z_{44} & Z_{45}
\end{array}
\]

3. The sufferable modular invariants are the 28 matrices

\[
\begin{array}{cccccccccccc}
Z_1 & Z_2 & Z_3 & Z_5 & Z_7 & Z_{22} & Z_{23} & Z_{25} & Z_{33} & Z_{(22)} & Z_{(33)} & Z_{55} & Z_{24} & Z_{25} & Z_{34} \\
Z_{35} & Z_{42} & Z_{43} & Z_{44} & Z_{45} & Z_{52} & Z_{53} & Z_{54} & Z_{(42)} & Z_{(45)} & Z_{(55)} & Z_{(54)} & Z_{(23)}
\end{array}
\]

6.6 The full systems for the sufferable modular invariants

The global index \(\omega = \sum \eta S_{00}^2 / S_{00}^2\) of our system \(\mathcal{N}_N\) is 36.
\[
\begin{array}{cccccccccc}
\times & Z_{23} & Z_{43} & Z_{33} & Z_{24} & Z_{34} & Z_{34} & Z_{25} & Z_{35} & Z_{45} & Z_{(23)} \\
Z_{23} & Z_{23} & Z_{25} & Z_{25} & Z_{33} & Z_{43} & Z_{45} & Z_{45} & Z_{45} & Z_{45} & Z_{(23)} \\
Z_{24} & Z_{24} & Z_{24} & Z_{24} & Z_{24} & Z_{24} & Z_{24} & Z_{24} & Z_{24} & Z_{24} & Z_{24} \\
Z_{25} & Z_{25} & Z_{25} & Z_{25} & Z_{25} & Z_{25} & Z_{25} & Z_{25} & Z_{25} & Z_{25} & Z_{25} \\
Z_{33} & Z_{33} & Z_{33} & Z_{33} & Z_{33} & Z_{33} & Z_{33} & Z_{33} & Z_{33} & Z_{33} & Z_{33} \\
Z_{34} & Z_{34} & Z_{34} & Z_{34} & Z_{34} & Z_{34} & Z_{34} & Z_{34} & Z_{34} & Z_{34} & Z_{34} \\
Z_{35} & Z_{35} & Z_{35} & Z_{35} & Z_{35} & Z_{35} & Z_{35} & Z_{35} & Z_{35} & Z_{35} & Z_{35} \\
Z_{45} & Z_{45} & Z_{45} & Z_{45} & Z_{45} & Z_{45} & Z_{45} & Z_{45} & Z_{45} & Z_{45} & Z_{45} \\
\end{array}
\]

Table 13: Fusion $Z_aZ_b^c$ of $S_3$ modular invariants from Table 11 (cont.)

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Case $Z_1$

The structure of the induced full $M$-$M$ system for the trivial modular invariant $Z_1$ is given by the original Verlinde algebra of modular data. Since $\langle \alpha_\lambda^+, \alpha_\mu^- \rangle = \delta_{\lambda \mu}$, $\alpha_\lambda^+ = \alpha_\lambda^-$, and all disjoint. Then $\lambda \mapsto \alpha_\lambda$ is an isomorphism. Hence $M^X M \simeq_M X^\pm M \simeq_M X^0 = N\lambda^N$ for any subfactor $N \subset M$ realising the trivial modular invariant. We display the fusion graph of $[\lambda_6]$ in Fig. 35. The full systems for the permutation invariant $Z_2$ are similar, obtained by permuting $\lambda_2$ and $\lambda_3$ in those for $Z_1$.

![Figure 35: $Z_1$, fusion graph of $[\alpha_{\lambda_6}]$](image)

Case $Z_5$

The simple current invariant $Z_5$ yields 20 irreducible $M$-$M$ sectors, whose fusion is non-commutative by [2] Corollary 6.9. The global indices of $\omega_\pm = 18, \omega_0 = 9$. We compute now the chiral systems $M^X_M^\pm$ and then the branching coefficients. By the fusion rules, Frobenius reciprocity and homomorphism property of the $\omega$-induction, we have

$$\langle \alpha_\lambda^+, \alpha_\mu^- \rangle = \langle \alpha_\lambda^+, \alpha_\mu^+, \alpha_0 \rangle = \sum_\eta N_{\lambda\mu}^\eta Z_\eta = N_{\lambda\mu}^0 + N_{\lambda\mu}^1 = \langle \alpha^- \lambda, \alpha^- \mu \rangle.$$  

We therefore find, using the original Verlinde fusion matrices, that $[\alpha_0^\pm] = [\alpha_1^\pm], [\alpha_2^\pm] = [\alpha_2^{(1)}] \oplus [\alpha_2^{(2)}], [\alpha_2^\pm] = [\alpha_2^{(1)}] \oplus [\alpha_2^{(2)}], [\alpha_4^\pm] = [\alpha_4^{(1)}] \oplus [\alpha_4^{(2)}], [\alpha_5^\pm] = [\alpha_5^{(1)}] \oplus [\alpha_5^{(2)}], [\alpha_6^\pm] = [\alpha_6^\pm], \forall [\alpha_0^\pm], [\alpha_2^i], [\alpha_4^i] \text{ and } [\alpha_6^\pm]$ irreducible sectors ($i = 1, 2; j = 2, 3, 4, 5$). Since the dimension function is additive, $d_{\alpha_\lambda^{i(\pm)}} = 1$ for all $i = 1, 2; j = 2, 3, 4, 5$. Of course $d_{\alpha_6^{i(\pm)}} = 3$. We conclude that the commutative neutral system as sectors is formed with 9 automorphisms $[\alpha_0^\pm], [\alpha_6^{i(\pm)}], \forall i = 1, 2; j = 2, 3, 4, 5$ which is either $Z_3 \times Z_3$ or $Z_9$. However, irred($[\alpha_0^\pm]_n^i \in \{[\alpha_0^\pm], [\alpha_6^{i(\pm)}]\}, i = 1, 2; j = 2, 3, 4, 5$. This implies that it cannot be $Z_9$ therefore is has to be $Z_3 \times Z_3$. Hence the system $M^X_M^\pm = M^X_M^0 \cup \{\alpha_6^\pm\}$, with the other fusion rules given by $[\alpha_6^{i(\pm)}][\alpha_6^{i(\pm)}] = [\alpha_6^{i(\pm)}][\alpha_6^{i(\pm)}], \forall i = 1, 2; j = 2, 3, 4, 5$.

Next we compute the other 9 irreducible induced $M$-$M$ sectors. Using Frobenius reci-
procity and the homomorphism property of the $\alpha$-induction

$$\langle \alpha_6^+\alpha_6^-, \alpha_6^+\alpha_6^- \rangle = \langle \alpha_6^+\alpha_6^-, \alpha_6^+\alpha_6^- \rangle = \sum_{\eta,\mu} N_6^\eta N_6^\mu \langle \alpha_6^+\alpha_6^-, \alpha_6^+\alpha_6^- \rangle = \sum_{\eta,\mu} N_6^\eta N_6^\mu Z_{\eta\mu} = 9$$

we get another 9 automorphisms $[\rho_k]$, with $k = 1, \ldots, 9$. Furthermore, as above

$$\langle \alpha_6^+\alpha_6^-, \alpha_6^+\alpha_6^- \rangle = \sum_{\eta,\mu} N_6^\eta N_6^\mu = 9$$

so the fusion graphs of $[\alpha_6^+]$ are displayed in Fig. 36.

In this figure, we use straight lines for the fusion graph with $[\alpha_6^+]$ whereas dashed lines for that with $[\alpha_6^-]$. Vertices from the neutral system $M\mathcal{X}_M^0$ are marked with larger dots. The decomposition $Z_5^2 = 2Z_5$ is reflected by the layering of the full system as two orbits $M\mathcal{X}_M^+$ and $M\mathcal{X}_M^+\alpha_6^-$. The full system has a C*-algebra structure (which is a weak C*-Hopf algebra of dimension 20, see [49]) as follows

$$\text{Fusion}(M\mathcal{X}_M) \simeq \mathbb{C}^4 \oplus \text{Mat}_2 \oplus \text{Mat}_2 \oplus \text{Mat}_2 \oplus \text{Mat}_2.$$ 

![Figure 36: $Z_5$, fusion graphs of $[\alpha_6^+]$, where $Z_5^2 = 2Z_5$](image-url)
The branching coefficient matrix is

\[
\mathbf{b} = \begin{pmatrix}
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}.
\]

For the local realism when the dual canonical endomorphism \( \theta = \lambda_0 \oplus \lambda_1 \) we can compute the canonical endomorphism \( \gamma \) using the machinery developed in \([33] \) Corollary 3.19. It is, as a sector, \( [\gamma] = [\text{id}] \oplus [\alpha_2] \) since \( \langle \alpha_1^+, \gamma \rangle = (\theta \lambda_1, \lambda_0) = \delta_{1,0} + \delta_{1,3} \).

The full system of \( Z_7 \) is the same as that for \( Z_5 \) (up to permutation of sectors).

**Case \( Z_{(22)} \)**

As in the proof and notation of \([6, 13] \) the neutral system \( _M \mathcal{X}_M^0 = \{ \alpha_0, \alpha_1, \alpha_6^2, \alpha_7^2 \} \) is isomorphic to the quantum double of \( Z_2 \). The \( \mathcal{X}^\pm \)-chiral systems are \( _M \mathcal{X}_M^\pm = \{ \alpha_0, \alpha_1, \alpha_6^2, \alpha_7^2, \alpha_3^\pm, \alpha_6^\pm(1) \} \), which are actually isomorphic to \( \widehat{S}_3 \times Z_2 \) as sectors. \( \)From both chiral systems we have so far eight \( M-M \) irreducible sectors.

Now \( \langle \alpha_3^+, \alpha_3^- \rangle = N_{3,3}^5 \) and moreover the dimension (of the intertwining spaces) of \( \alpha_3^+ \alpha_3^- \) with any sector of both chiral systems vanishes. Hence the irreducible decomposition of \( \alpha_3^+ \alpha_3^- \) provides us with two new \( M-M \) irreducible sectors. Now \( \langle \alpha_3^+ \alpha_6^-, \alpha_3^+ \alpha_6^- \rangle = 3 \) and \( \langle \alpha_3^+ \alpha_6^-, \alpha_3^+ \rangle = \langle \alpha_3^+ \alpha_6^-, \alpha_3^- \rangle = 1 \) and the intertwining spaces of \( \alpha_3^+ \alpha_6^- \) with any other irreducible \( M-M \) sector vanishes. Therefore we have two more \( M-M \) irreducible sectors \([\alpha_3^+ \alpha_6^-, \alpha_3^+ \alpha_6^-]\). So \( [\alpha_3^+ \alpha_6^-] = [(\alpha_3^+ \alpha_6^-)] = [(\alpha_3^+ \alpha_6^-)] = [(\alpha_3^+ \alpha_6^-)] \). The full system \( _M \mathcal{X}_M \) decomposes into two sheets according to \( Z_{(22)} = Z_{(22)} + Z_{(22)} \), with six elements each.

The full systems of \( Z = Z_{(32)}, Z_{(33)}, Z_{(23)} \) are obtained by permutations of those for \( Z_{(22)} \).

**Case \( Z_{55} \)**

For the sufferable modular invariant \( Z_{55} \), the cardinality of the full \( M-M \) system is \( \text{Tr}(Z_{55}Z_{55}^*) = 9 \) whose fusion rules are in turn commutative \([7] \) Corollary 6.9], and with global indices \( \omega_\pm = 6, \omega_0 = 1 \). For the \( \mathcal{X}^\pm \)-chiral systems, we use

\[
\langle \alpha_\lambda^+, \alpha_\mu^- \rangle = \langle \alpha_\lambda^+, \alpha_\mu^- \rangle = \sum_\eta N_{\lambda \mu}^\eta Z_{\eta \eta} = N_{\lambda \mu}^0 + N_{\lambda \mu}^3 + N_{\lambda \mu}^6 = \langle \alpha_\lambda^-, \alpha_\mu^- \rangle.
\]

So computing we get that \( _M \mathcal{X}_M^\pm = \{ \alpha_0, \alpha_1^+, \alpha_2^\pm \} \) with \( [\alpha_3^\pm] = [\alpha_0] \oplus [\alpha_1^\pm] \). \( [\alpha_5^\pm] = [\alpha_5^\pm] \). Since \( \omega_\pm = 6 \), and \( d_{\alpha_2^\pm} = 2 \) we easily conclude that the sectors of \( _M \mathcal{X}_M^\pm \) are \( \widehat{S}_3 \). Hence the sectors from \( _M \mathcal{X}_M \) are \( \widehat{S}_3 \times \{1\} \) and those from \( _M \mathcal{X}_M^- \) with \( \{1\} \times \widehat{S}_3 \).
We display the fusion the all \(M-M\) system together with the fusion graphs of both \([\alpha_2^+\]) on the LHS and \([\alpha_1^+\]) on the RHS in Fig. 37. In this figure, we use straight lines for the fusion graphs of \([\alpha_2^+\]) and \([\alpha_1^+\]) whereas dashed lines for those of \([\alpha_2^-\]) and \([\alpha_1^-\]). We also encircled the \(\chi^+\)-chiral sectors with small circles and with larger circles those for the \(\chi^-\)-chiral system. The full system decomposes into three sheets \(M\chi^+/M\chi^+M\alpha_1^-\cdot M\chi^+/M\alpha_2^-\) according to \(Z_{55}^2 = 3Z_{55}\). The canonical sector of the fundamental inclusion \(N \subset M\) is \([\gamma] = [\text{id}] \oplus [\alpha_1^+\alpha_1^-] \oplus [\alpha_2^+\alpha_2^-]\) by [3] Corollary 3.19 because \([\alpha_1^+\alpha_1^-\gamma] = (\alpha_2^+\alpha_2^-\gamma) = 1\).

We also conclude that the full system of \(Z_{44}, Z_{45}, Z_{54} M\chi^-\) as sectors are \(S_3 \times \hat{S}_3\).

Figure 37: \(Z_{55}\), fusion graphs of \([\alpha_2^+\]) and \([\alpha_1^+\]) where \(Z_{55}^2 = 3Z_{55}\)

**Case \(Z_{22}\)**

The global indices of \(M\chi^+\) and \(M\chi^0\) are, respectively, \(\omega_+ = 6\) and \(\omega_0 = 1\). The dimension of the intertwiner spaces in the \(\chi^+\)-chiral system is obtained by \(\langle \alpha_1^+, \alpha_1^- \rangle = \sum N_{\chi^+} = N_{\chi^0} = N_{\chi^+} + N_{\chi^-} + N_{\chi^+} - N_{\chi^-}\). We then find that \(M\chi^+/M\chi^+ = \{\alpha_0, \alpha_3^{(1)}+, \alpha_3^{(2)}+, \alpha_6^{(3)}\}\) with \([\alpha_0] = [\alpha_1^+], [\alpha_2] = 2[\alpha_0], [\alpha_3^+] = [\alpha_4^+] = [\alpha_5^+] = [\alpha_3^{(1)}] \oplus [\alpha_3^{(2)}], [\alpha_7^+] = [\alpha_6^+] \oplus [\alpha_6^{(1)}] \oplus [\alpha_6^{(2)}] \oplus [\alpha_6^{(3)}]\). Since \(Z_{22}\) is symmetric we have \(M\chi^+/M\chi^+ = \{\alpha_0, \alpha_3^{(1)}-, \alpha_3^{(2)}-, \alpha_6^{(1)}-, \alpha_6^{(2)}-, \alpha_6^{(3)}\}\). We can conclude that \(M\chi^+/M\chi^+\) as sectors are \(S_3\). Hence by the global indices we have \(M\chi^+/M\chi^+ = S_3 \times S_3\) which decomposes into six sheets of \(S_3\) according to \(Z_{22}^2 = 6Z_{22}\). We also have for \(Z = Z_{33}, Z_{34}, Z_{32}\) that \(M\chi^+/M\chi^+ = S_3 \times S_3\).

**Case \(Z_{25}\)**

We easily find that \(M\chi^+M\chi^+\) as sectors \(S_3 \times \hat{S}_3\), \(M\chi^0\) as sectors \(S_3\), \(M\chi^-\) as sectors \(\hat{S}_3\) and \(M\chi^0 = \{\alpha_0\}\). For \(Z = Z_{24}, Z_{35}, Z_{34}\) we get isomorphic full systems as for \(Z_{25}\). Note that \(M\chi^+\) and \(M\chi^-\) are not isomorphic – indeed their cardinalities are different and moreover one system is commutative whilst the other is not. Thus the decompositions of the full system according to \(Z_{25}Z_{25}^\dagger = 3Z_{22}\) and \(Z_{25}^\dagger Z_{25} = 6Z_{55}\) are different.
Case $Z_{(44)}$

The global indices of $\mathcal{X}_M^\pm$ and $\mathcal{X}_M^0$ are $\omega_\pm = 12$ and $\omega_0 = 4$ respectively. We compute the $\mathcal{X}^\pm$-chiral systems and conclude that the irreducible decompositions are as follows: $[\alpha_0]$, $[\alpha_1^\pm]$ and $[\alpha_2^\pm]$ are the chiral parts of $S_3$ graphs of $[\alpha_3^\pm]$ drawn on the RHS of Fig. 38). The decomposition $Z_{(44)} = Z_{(22)} + Z_{44}$ means that the full system can be decomposed into a six element sheet $\mathcal{X}_M^+ = \mathcal{X}_M^0 \cup \{\alpha_3^+, \alpha_6^{(1)}\}$ from the chiral part of $Z_{(22)}$ and a further three element sheet $\{\alpha_3^-, \alpha_6^{(1)}\}$ from the chiral part of $Z_{44}$.

![Diagram](image)

Figure 38: $Z_{(44)}$: fusion graph of $[\alpha_3^+]$ and $[\alpha_5^+]$ where $Z_{(44)}^2 = Z_{(22)} + Z_{44}$

The branching coefficient matrix is as in Eq. (22). Therefore by the proof of Proposition 6.15 the extended modular matrices $S^{\text{ext}}$ and $T^{\text{ext}}$ are the quantum $Z_2$ double model.

The full system for $Z = Z_{(45)}, Z_{(55)}, Z_{(54)}$ are similar to that for $Z_{(44)}$.

Remark 6.18 It was the case with $SU(2)$ modular invariants (which are all sufferable) that:

$$ [\theta]_Z = \bigoplus_{\{\lambda: \text{FS}_\lambda = 1\}} Z_{\lambda,\lambda}[\lambda] $$

is always a dual canonical sector producing the same $Z$, see [17]. So since in the quantum $S_3$ double case, where the Frobenius-Schur indicator $\text{FS}_\lambda = 1$ for all $\lambda \in N\mathcal{X}_N$, we would
naturally ask whether $\theta_Z$ is a dual endomorphism reproducing the sufferable modular invariant $Z$. However in general this no longer holds true in our current quantum $S_3$ double model as follows. Consider for example the following sectors:

\[ [\theta_{Z_1}] = [\lambda_0] \oplus [\lambda_1] \oplus [\lambda_2] \oplus [\lambda_3] \oplus [\lambda_4] \oplus [\lambda_5] \oplus [\lambda_6] \oplus [\lambda_7] \]
\[ [\theta_{Z_2}] = [\lambda_0] \oplus [\lambda_1] \oplus [\lambda_2] \oplus [\lambda_3] \oplus [\lambda_4] \oplus [\lambda_5] \oplus [\lambda_6] \oplus [\lambda_7] \]
\[ [\theta_{Z_{55}}] = [\lambda_0] \oplus [\lambda_3] \oplus [\lambda_6] \]

Amongst these, only $\theta_{Z_{55}}$ is a dual canonical endomorphism producing $Z_{55}$ by Proposition 6.10. The endomorphisms $\theta_{Z_2}$ gives rise to inconsistent dimensions of intertwiner spaces (since $\langle \theta_{\lambda_4}, \lambda_5 \rangle = 0$ and $\langle \theta_{\lambda_0}, \lambda_4 \rangle = \langle \theta_{\lambda_0}, \lambda_5 \rangle = 1$), whereas $\theta_{Z_1}$ would give a trace 6 or trace 9 matrix instead of 8 if it reproduced $Z_1$.

7 The three quantum doubles of $D^{(1)}_5$

There are precisely three subfactors whose principal graph is the extended Dynkin diagram $D^{(1)}_5$. Izumi [30, Page 622] has written down the modular data from the Longo-Rehren inclusion of these subfactors. One coincides with that from the quantum $S_3$ double and the other two lead to the following modular data:

\[ S = \frac{1}{6} \begin{pmatrix}
1 & 1 & 2 & 2 & 2 & 2 & 3 & 3 \\
1 & 1 & 2 & 2 & 2 & 2 & -3 & -3 \\
2 & 2 & 4 \cos[4\pi/9] & 4 \cos[8\pi/9] & 4 \cos[2\pi/9] & -2 & 0 & 0 \\
2 & 2 & 4 \cos[8\pi/9] & 4 \cos[2\pi/9] & 4 \cos[4\pi/9] & -2 & 0 & 0 \\
2 & 2 & 4 \cos[2\pi/9] & 4 \cos[8\pi/9] & 4 \cos[4\pi/9] & -2 & 0 & 0 \\
2 & 2 & -2 & -2 & -2 & 4 & 0 & 0 \\
3 & -3 & 0 & 0 & 0 & 0 & 3 & -3 \\
3 & -3 & 0 & 0 & 0 & 0 & -3 & 3
\end{pmatrix}, \]

\[ T = \text{diag}(1, 1, \varphi, \varphi^4, \varphi^7, 1, 1, -1) \]

where $\varphi = \exp[\pm 2\pi i/9]$. All the $N$-$N$ sectors $[\lambda_i]$, $i = 0, \ldots, 7$, are self-conjugate and moreover the Frobenius-Schur indicator $FS_\lambda = 1$ for all $\lambda \in N\mathcal{X}_N$.

7.1 The modular invariants

All the $N$-$N$ sectors $[\lambda_i]$, $i = 0, \ldots, 7$, are self-conjugate and moreover the Frobenius-Schur indicator $FS_\lambda = 1$ for all $\lambda \in N\mathcal{X}_N$.

The dimension of the commutant \{S, T\} is 6. With a numerical search and employing the estimate of [5, Eq. (1.3)] we find the list of modular invariants for the above model:

\[ Z_1 = |\chi_0|^2 + |\chi_1|^2 + |\chi_2|^2 + |\chi_3|^2 + |\chi_4|^2 + |\chi_5|^2 + |\chi_6|^2 + |\chi_7|^2, \]
\[ Z_2 = |\chi_0 + \chi_1|^2 + 2|\chi_2|^2 + 2|\chi_3|^2 + 2|\chi_4|^2 + 2|\chi_5|^2, \]
\[ Z_3 = |\chi_0 + \chi_5|^2 + (\chi_1 + \chi_5)\chi_6^* + \chi_6(\chi_1 + \chi_5)^* + |\chi_7|^2, \]
\[ Z_4 = |\chi_0 + \chi_5 + \chi_6|^2, \]
\[ Z_5 = |\chi_0 + \chi_5|^2 + |\chi_1 + \chi_5|^2 + |\chi_6|^2 + |\chi_7|^2, \]
\[ Z_6 = (\chi_0 + \chi_5 + \chi_6)(\chi_0 + \chi_1 + 2\chi_5)^*, \]
\[ Z_7 = |\chi_0 + \chi_1 + \chi_5|^2 + |\chi_2|^2 + |\chi_3|^2 + |\chi_4|^2 + 2|\chi_5|^2, \]
\[ Z_8 = (\chi_0 + \chi_1 + 2\chi_5)(\chi_0 + \chi_5 + \chi_6)^*, \]
\[ Z_9 = |\chi_0 + \chi_1 + 2\chi_5|^2. \]
The fundamental dual canonical sector is \( [\theta] = [\lambda_0] \oplus [\lambda_5] \oplus [\lambda_6] \) producing the trace 3 matrix \( Z_4 \). Considering the two (even) endpoints of the graph \( D_5^{(1)} \) which both have quantum dimension equals 1 and applying Izumi’s Galois correspondence \[34\] there exists an intermediate subfactor \( N \subset P \subset M \) with \( N \subset M \) being the above subfactor that produces \( Z_4 \). Since the Jones index of \( N \subset P \) is 3 and its dual canonical sector, a subsector of \( [\theta] \) must be \( [\theta_{N \subset P}] = [\lambda_0] \oplus [\lambda_5] \). This endomorphism produces a type I modular invariant, of trace 6, \( Z_5 \). The simple currents sector \( [\theta] = [\lambda_0] \oplus [\lambda_1] \) is a dual canonical sector. It produces the trace 10 matrix \( Z_2 \). Also \( Z_8 = Z_2 Z_4 \) is sufferable by Theorem \[36\] and so is its conjugate \( Z_6 \). Then by the type I parent theorem \[34\], \( Z_9 \) is sufferable as well. By the same type of computation used in Proposition \[16.15\] we can conclude that \( Z_3 \) is also sufferable. The spectrum of \( G_5 \) is \( \{-1^3, 2^3\} \) if a nimrep exists for \( Z_7 \), but there is no graph with this spectrum (as was observed by Gannon \[27\]). Hence we have the following classification for the (normalized) modular invariants of \( LR(D_5^{(1)}, w = \exp(-2\pi i/3)) \). The modular invariant \( Z_7 \) is nimless, but the modular invariants \( Z_1, Z_2, Z_3, Z_4, Z_5, Z_6, Z_8 \) and \( Z_9 \) are all sufferable (hence nimble). We also have the fusion of sufferable modular invariants in Table \[14\]. The decompositions in this table are unique.

|          | \( Z_1 \) | \( Z_2 \) | \( Z_3 \) | \( Z_4 \) | \( Z_5 \) | \( Z_6 \) | \( Z_8 \) | \( Z_9 \) |
|----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|
| \( Z_1 \) | \( Z_1 \) | \( Z_2 \) | \( Z_3 \) | \( Z_4 \) | \( Z_5 \) | \( Z_8 \) | \( Z_6 \) | \( Z_9 \) |
| \( Z_2 \) | \( 2Z_2 \) | \( Z_2 \) | \( Z_8 \) | \( Z_9 \) | \( 2Z_8 \) | \( Z_9 \) | \( 2Z_9 \) |
| \( Z_3 \) | \( Z_3 \) | \( Z_6 \) | \( Z_4 + Z_5 \) | \( Z_4 + Z_8 \) | \( Z_5 + Z_6 \) | \( 3Z_4 \) | \( Z_6 + Z_9 \) | \( 3Z_8 \) |
| \( Z_4 \) | \( Z_4 \) | \( Z_6 \) | \( Z_4 + Z_6 \) | \( 3Z_4 \) | \( Z_4 + Z_6 \) | \( Z_6 \) | \( 3Z_6 \) | \( 3Z_6 \) |
| \( Z_5 \) | \( Z_5 \) | \( Z_9 \) | \( Z_4 + Z_8 \) | \( Z_5 + Z_9 \) | \( 3Z_4 \) | \( Z_6 + Z_9 \) | \( 3Z_8 \) | \( 3Z_9 \) |
| \( Z_6 \) | \( Z_6 \) | \( 2Z_6 \) | \( 3Z_4 \) | \( 3Z_4 \) | \( 3Z_6 \) | \( 6Z_4 \) | \( 3Z_6 \) | \( 6Z_6 \) |
| \( Z_8 \) | \( Z_8 \) | \( Z_9 \) | \( Z_6 + Z_8 \) | \( 3Z_4 \) | \( 3Z_8 + Z_9 \) | \( 3Z_8 \) | \( 3Z_9 \) | \( 3Z_9 \) |
| \( Z_9 \) | \( Z_9 \) | \( 2Z_9 \) | \( 3Z_8 \) | \( 3Z_8 \) | \( 3Z_8 \) | \( Z_6 + Z_9 \) | \( 3Z_9 \) | \( 6Z_9 \) |

Table 14: Fusion \( Z_aZ_b^* \) of sufferable \( D_5^{(1)} \) modular invariants

Acknowledgement. The first named author is supported by the EU QSNG network in Quantum Spaces - Noncommutative Geometry, and the EPSRC network ABC–KLM, whilst the second named author is supported by FCT (Portugal) under grant BD/9704/96 and CAMGDS-IST. We are extremely grateful for their financial assistance.

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