$p$-adic CCR: Galois group representations, cyclic dynamics, and zeta-functions

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We consider a model of cyclic time evolution for Kochubei’s $p$-adic realization of the canonical commutation relations (CCR). Connections to Kubota-Leopoldt $p$-adic zeta-functions and to arithmetic quantum theories such as the Bost-Connes model are examined.

I. INTRODUCTION

The physical interpretation of arithmetic quantum theories has gained attention in recent years. Briefly, these quantum theories are those which exhibit Dirichlet $L$-series as thermal partition functions in their statistical formulation. Grounded on mathematical work of Julia and Bost-Connes, it was realized that there exist similarities to certain aspects of string theory at finite temperature, or to other statistical systems such as the two-dimensional Ising model. For example, the pole of the Riemann $\zeta$-function $\zeta(\beta)$ at $\beta = 1$ may be understood as an indicator of a phase transition at Hagedorn temperature $\beta^{-1} = 1$, and the functional equation for the $\zeta$-function may be conceived as a duality relation in analogy to the Kramers-Wannier relation for the two-dimensional Ising model. However, arithmetic quantum theories have not been treated as truly physical theories so far, which in other words means that until now none of them has been identified as a realistic model of a physical system.

The motivation of this note is the question whether there are $p$-adic quantum systems that share certain aspects of arithmetic quantum theories. In contrast to conventional quantum theory, $p$-adic quantum systems such as the one investigated here are realized on spaces over the field $Q_p$ of $p$-adic numbers ($p$-adic fields, which in addition are algebraically closed are not regarded here.). Our main result is that the probably most elementary $p$-adic quantum system, namely the quantum harmonic oscillator whose generators satisfy the canonical commutation relations (CCR), exhibits a structural similarity to arithmetic quantum theories. Based on Kochubei’s $p$-adic construction of the CCR, we show that the Galois group $\text{Gal}(Q_p/Q)$ of the abelian extension $Q_p$ of the rational numbers $Q$ naturally leads to a representation $\rho$ on a one-dimensional vector space containing the oscillator’s ground state. This leads us to a proposition stating that there exists a group homomorphism from the Galois group $\text{Gal}(Q_p/Q)$ to the group of automorphisms acting on the algebra of bounded linear operators over the Banach space of continuous functions on $Z_p$. From the perspective of $C^\ast$-dynamical systems our representation motivates us to to regard $\text{Gal}(Q_p/Q)$ as the one-paramater group of cyclic time evolution for the $p$-adic quantum system. Further, the one-dimensional representation itself turns out to be directly related to an Artin $p$-adic $L$-function associated to $\rho$. In our case the latter is equal to the Kubota-Leopoldt $p$-adic $\zeta$-function. We close with comments on the physical implications of our results and indicate obvious connections to the Bost-Connes arithmetic quantum system.

II. $p$-ADIC CCR

We follow the work of Kochubei. Let $p$ be a prime number, $Q_p$ the field of $p$-adic numbers, $Z_p$ the ring of $p$-adic integers. Denote $C(Z_p, Q_p)$ the Banach space of $Q_p$-valued continuous functions on $Z_p$ equipped with supremum norm. For $n \geq 1$ the functions

$$P_n(x) = \frac{x(x-1)\cdots(x-n+1)}{n!}$$

with $P_0(x) := 1$ form complete orthonormal system, i.e. the Mahler basis, of $C(Z_p, Q_p)$. Thus every function $f \in C(Z_p, Q_p)$ admits a unique expansion

$$f(x) = \sum_{n=0}^{\infty} c_n P_n(x), \quad c_n \in Q_p$$

with $|c_n|_p \to 0$, and $\|f\|_p = \sup_n |c_n|_p$, and with $|\cdot|_p$ being the absolute value of a $p$-adic number. Consider now the following operators on $C(Z_p, Q_p)$:

$$(a^+ f)(x) = x f(x-1)$$

$$(a^- f)(x) = f(x+1) - f(x).$$

This construction is well defined as it directly follows from the definition of the $\| \cdot \|_p$ norm that $a^\pm$ are bounded operators. A direct calculation of the commutator gives the CCR

$$[a^-, a^+] = 1.$$ 

Moreover, we obtain

$$a^- P_n = P_{n-1}, \quad n \geq 1, \quad a^- P_0 = 0$$

$$a^+ P_n = (n+1) P_{n+1}, \quad n \geq 0$$

thus $a^+, a^-$ are clearly analogues of the creation and annihilation operators. Setting $H = a^+ a^-$ we obtain $HP_n = nP_n$, and $[H, a^\pm] = \pm a^\pm$. The operator $H$ has a complete system of eigenvectors and its discrete spectrum coincides with the set of non-negative integers.
$Z_p$. The whole spectrum of $H$, however, equals $Z_p$, that is the $|·|_p$-closure of $Z_p$ in $Q_p$. The density of $Z_p$ in $Z_p$ implies that the kernel of $a^-$ consists of constant functions on $Z_p$. This fact may be used to show that $a^-$, $a^+$ form an irreducible representation of the CCR. A remarkable property is that every Banach space over $Q_p$ having an infinite countable orthonormal basis is isomorphic to $C(Z_p, Q_p)$.

### III. GROUND STATE STRUCTURE AND GALOIS GROUP REPRESENTATIONS

As ground state of our $p$-adic quantum mechanical system we denote any continuous function $Ω ∈ C(Z_p, Q_p)$ with $a^-Ω = 0$ and $Ω|_{Z_p} = 1$. If we enlarge the domain of ground states and consider continuous functions $Ω : X → Q_p ∈ C(X, Q_p)$, $Z_p ⊂ X$, then we may expect more than just the trivial case, i.e. a globally constant mapping, because now any locally constant function on $Z_p$ with $Ω|_{Z_p} = 1$ may be regarded as a ground state, too. It is then convenient to look at the equivalence class $Ω$ of such ground states defined upon the relation

$$Ω_1 ∼ Ω_2 ⇔ Ω_1, Ω_2 ∈ C(X, Q_p) : Ω_1|_{Z_p} = Ω_2|_{Z_p} = 1.$$  

As $Q_p$ is a field there are $p-1$ primitive roots of unity $µ^{p-1} = \{ζ_1, …, ζ_{p-1}\}, ζ_j ∈ Z_p$ for all $j ∈ [1, p-1]$, together being the only roots of $x^{p-1} - 1$ in $Q_p$. These roots are the image of the Teichmüller character $ω : Z_p^* → µ^{p-1}$, with the representation property $ω(α)ω(β) = ω(αβ)$, $α, β ∈ Z_p^*$. Since $Q_p$ is an extension field of the field of the rational numbers $Q$, one has the corresponding Galois group $Gal(Q_p/Q) ≃ Z_p^*$. Consider the one-dimensional $Q_p$-vector space $V_0 = \{cΩ : c ∈ Q_p\}$. Then, for every fixed $κ_0 ∈ \{0, …, p-2\}$ the Teichmüller character $ω$ admits a faithful one-dimensional representation $ρ_{κ_0} : Gal(Q_p/Q) → Aut(V_0)$ of the Galois group, viz.

$$ρ_{κ_0}(α)Ω = ω(α)^{κ_0}Ω, ∀α ∈ Z_p^*.$$  

(5)

We stress that $ρ_{κ_0}$ is a continuous representation, since $ω$ extends to a (uniformly) continuous function $ω ∈ C(Z_p, Q_p)$. Since $Z_p^* ≃ Z_{p-1}$, equation (5) may also be written as

$$ρ_{κ_0}(α)Ω = ω_{κ_0}^{κ_0}Ω,$$  

(6)

for a suitably chosen $t(α) ∈ [0, …, p-2]$ and $ζ_{p-1}$ is the primitive root of unity of degree $p - 1$.

Let $B(C(Z_p, Q_p)) ≃ B$ be the algebra over $Q_p$ of bounded linear operators on $C(Z_p, Q_p)$, i.e. $B = \{A ∈ B : ||A|| < ∞, ∀f ∈ C(Z_p, Q_p)\}$. Our next task is to show that the above representation $ρ$ naturally induces a representation $′ : Gal(Q_p/Q) → Aut(B)$. Every $A ∈ B$ may uniquely be represented as a set $A'$ of ordered pairs: $A' = \{(f, Af) : f ∈ C(Z_p, Q_p)\}$. Obviously, the collection of all such sets forms an algebra $B' ≃ B$. We introduce the van der Put basis $\{e_n : n ∈ N_0\}$ of $C(Z_p, Q_p)$ as follows:

e_0 = 1 and for $n > 0$, $e_n$ is the characteristic function of the disc $D_n = \{x ∈ Z_p : |x - n|_p < 1/n\}$. Then for every $f ∈ C(Z_p, Q_p)$ one obtains the uniformly convergent series:

$$f(x) = ∑_{n=0}^{∞} v_n e_n(x)$$

with $v_0 = f(0)$ and $v_n = f(n) - f(n-)$; $n_-$ is defined through the Hensel expansion given for any $n ∈ N$: $n = n_0 + n_1p + … + n_sp^{s-1}$ with $n_s ≠ 0$. Then $n_- = n_0 + n_1p + … + n_{s-1}p^{s-1}$. Thus for every $A' ∈ B'$ and every $f ∈ C(Z_p, Q_p)$ we have

$$(f, Af) = (∑_{n=0}^{∞} v_n(f)e_n(x), ∑_{n=0}^{∞} v_n(Af)e_n(x)).$$  

(7)

But every $e_n$ is locally constant on $Z_p$; this is because $N$ is dense in $Z_p$ and each disc $D_n$ exhausts all the elements of $Z_p$. Hence, we may write

$$(f, Af) = (∑_{n=0}^{∞} v_n(f)Ω, ∑_{n=0}^{∞} v_n(Af)Ω).$$  

(8)

As the expansion in the van der Put basis is uniformly convergent, we have the following mapping due to equation (8)

$$ρ'_{κ_0} : t → (∑_{n=0}^{∞} v_n(f)Ω, ∑_{n=0}^{∞} v_n(Af)ζ_{κ_0}^{κ_0}Ω).$$  

(9)

with $t ∈ Z_{p-1}$. Since $ρ_{κ_0}$ is faithful this mapping determines an automorphism on $B'$, and so it does on $B$. Thus in summary, we have the proposition

**Proposition III.1.** Let the tuple $(C(Z_p, Q_p), a^±, B)$ be the $p$-adic quantum mechanical system as previously defined. Let further $Ω ∈ V_0$ denote the ground state of this system. Then for every $κ_0 ∈ Z_{p-1}$ there is a faithful representation $ρ_{κ_0} : Gal(Q_p/Q) → Aut(B)$ uniquely determined by the one-dimensional continuous representation $ρ_{κ_0} : Gal(Q_p/Q) → Aut(V_0)$.

### IV. p-ADIC ζ-FUNCTIONS

Finite dimensional representations of Galois groups associated to field extensions exhibit a remarkable relation to certain $L$-functions. In particular, due to the pioneering work of Deligne and Ribet, one can naturally define a $p$-adic $L$-function $L_p(s, ρ)$ to any even representation $ρ$ on a vector space over a $p$-adic field such as is $Q_p$. This function is referred to as $p$-adic Artin $L$-function associated to $ρ$. Let $C_p$ be the smallest extension field of $Q_p$ that is algebraically closed and complete with respect to $|·|_p$. Then a one-dimensional representation of $Gal(C_p/Q)$ generally reads as

$$ρ(σ_n) = χ(α)$$

for $σ_n e = χ(α) e$.
where $e$ is the basis of vector space $V = \mathbb{C}_p e$. Here, $\rho$ is even when $\chi(-1) = 1$. For even representations of this kind the $p$-adic Artin $L$-function associated to $\rho$ becomes the $p$-adic Dirichlet $L$-function originally introduced by Kubota and Leopoldt. If we now restrict our view to extension field $Q_p$, then the one-dimensional representations of $\text{Gal}(Q_p/Q)$ are those given by equation \[ (r) \] and - equivalently - by equation \[ (r) \] with even characters $\chi = \omega^\kappa_0$. In this case the Kubota-Leopoldt $p$-adic Dirichlet $L$-function is defined through a $Q_p$-valued measure $\mu(x)$ and it reads (For an introduction to $p$-adic integration and $p$-adic $L$-functions, see \[ \text{[7].} \])

\[
\zeta_{p,\kappa_0}(s) = \frac{1}{\langle r \rangle^{1-s} \omega(r)^\kappa_0 - 1} \int_{Z_p} (x)^{-s} \omega(x)^{\kappa_0-1} d\mu(x)
\]  

(10)

where $r$ is any integer prime to $p$, $\langle r \rangle = r/\omega$, $\langle x \rangle = x/\omega$, and the parameter $\kappa_0 \in \{0, \ldots, p-2\}$ depicts the branches of the $p$-adic $\zeta$-function. $p$-adic Dirichlet $L$-functions have the remarkable property that they interpolate complex $L$-adic Dirichlet $\zeta$-functions at algebraic values, i.e. values for $1 - k$ with $k \in N_0$. So, for example, if we choose $p = 2$ then we have

\[
\zeta_{2,0}(1-k) = (1 - 2^{k-1}) \zeta(1-k) = (2^{k-1} - 1) \frac{B_k}{k},
\]

where $\zeta$ is the Riemann $\zeta$-function and $B_k$ is the $k$th Bernoulli number. Despite this close analogy to complex Dirichlet $L$-functions, $p$-adic Dirichlet $L$-functions are less understood than their complex counterparts. For instance, a general functional equation for $p$-adic Dirichlet $L$-functions is unknown, and their values for natural numbers larger than one are not known either.

\section{Discussion}

Proposition\[ \text{[III.1]} \] plays a similar role for our $p$-adic quantum mechanical system as does a proper group homeomorphism $\alpha : R \to \text{Aut}(A)$ play for some $C^*$-dynamical system within the framework of conventional quantum mechanics, i.e. the role of time evolution within the algebra of observables. In the latter, $A$ usually is the $C^*$-algebra of bounded linear operators on some complex Hilbert space. Despite this obvious similarity there are significant differences. Recall that $Q_p$ is not an algebraically closed field; for example, the equation $x^2 + 1 = 0$ does not have a solution in $Q_p$. Therefore, our construction shows that with $p$-adic analysis over $Q_p$ it is possible to represent a non-trivial quantum mechanical system and its time evolution on a real (in the field-theoretic sense) one-dimensional vector space—something that is totally missing for one-dimensional (Hilbert) spaces over $R$. In fact, if we try to apply our results to the case of real numbers and consequently identify $Q_\infty = R$ then the corresponding Galois group becomes $\text{Gal}(R/Q) \simeq \{\text{id}\}$, i.e. the trivial group. Now the only roots of unity in $R$ are $\{\pm 1\}$. Since there is no continuous homeomorphism $\rho : \{\pm 1\} \to \{\pm 1\}$, there is no non-trivial time evolution either. For the $p$-adic case, on the other hand, we have a cyclic time structure defined for each branch that is depicted by a value of $\kappa_0$. Surely, the physical meaning of a $p$-adic time parameter $t \in Z_{p-1}$ is far from being clear, since the usual complete ordering of real numbers is lost here. The latter property, however, has always been considered as being essential for a physical time parameter. So, $p$-adic quantum systems such as the one investigated here may indicate that $R$ does not give the only possible parametrization of time.

Finally, we want to briefly mention an obvious relation between the $p$-adic quantum system as introduced in this note and the arithmetic quantum system of Bost and Connes. In both systems the Galois group of an abelian extension of $Q$ has been represented on the corresponding algebra of observables. And in both systems a Dirichlet $L$-function naturally occurs as a consequence. In the Bost-Connes system this $L$-function is the Riemann $\zeta$-function $\zeta(\beta)$ which at the same time turns out to be the partition function of the quantum statistical system at temperature $\beta^{-1}$. Therefore, one may wonder whether it is admissible to regard $\zeta_{p,\kappa_0}(s)$ as a $p$-adic partition function, and whether $s$ may become the inverse $p$-adic temperature in any reasonable sense. Further, it is interesting to observe that our $p$-adic quantum system possesses an extra symmetry not present in the Bost-Connes system: the one-paramenter group representation of $\text{Gal}(Q_p/Q)$ appears twice as realized through the parameters $\kappa_0$ and $t$. Thus $\text{Gal}(Q_p/Q)$ plays a double-role here in the sense that it is a symmetry group of the quantum system as well as it is the system’s dynamical group representing time.

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