Dynamics of several rigid bodies in a two-dimensional ideal fluid and convergence to vortex systems

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Abstract

We consider the motion of several solids in a bounded cavity filled with a perfect incompressible fluid, in two dimensions. The solids move according to Newton’s law, under the influence of the fluid’s pressure. On the other hand the fluid dynamics is driven by the 2D incompressible Euler equations, which are set on the time-dependent domain corresponding to the cavity deprived of the sets occupied by the solids. We assume that the fluid vorticity is initially bounded and that the circulations around the solids may be non-zero. The existence of a unique corresponding solution, à la Yudovich, to this system, up to a possible collision, follows from the arguments in [11].

The main result of this paper is to identify the limit dynamics of the system when the radius of some of the solids converge to zero, in different regimes, depending on how, for each body, the inertia is scaled with the radius. We obtain in the limit some point vortex systems for the solids converging to particles and a form of Newton’s law for the solids that have a fixed radius; for the fluid we obtain an Euler-type system. This extends the earlier works [7], which deals with the case of a single small heavy body immersed in an incompressible perfect fluid occupying the rest of the plane, [8], which deals with the case of a single small light body immersed in an incompressible perfect fluid occupying the rest of the plane, and [9] which deals with the case of a single small, heavy or light, body immersed in an irrotational incompressible perfect fluid occupying a bounded plane domain.

In particular we consider for the first time the case of several small rigid bodies, for which the strategy of the previous papers cannot be adapted straightforwardly, despite the partial results recently obtained in [10]. The main difficulty is to understand the interaction, through the fluid, between several moving solids. A crucial point of our strategy is the use of normal forms of the ODEs driving the motion of the solids in a two-steps process. First we use a normal form for the system coupling the time-evolution of all the solids to obtain a rough estimate of the acceleration of the bodies. Then we turn to some normal forms that are specific to each small solid, with an appropriate modulation related to the influence of the other solids and of the fluid vorticity. Thanks to these individual normal forms we obtain some precise uniform a priori estimates of the velocities of the bodies, and then pass to the limit. In the course of this process we make use of another new main ingredient of this paper, which is an estimate of the fluid velocity with respect to the solids, uniformly with respect to their positions and radii, and which can be seen as an refinement of the reflection method for a div/curl system with prescribed circulations.1

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Contents

1 Introduction and statement of the main result .................................................. 6
  1.1 The fluid-solid system .................................................................................. 6
  1.2 The problem of small solids ......................................................................... 7
  1.3 Main result ..................................................................................................... 9

2 Preliminaries ........................................................................................................ 11
  2.1 Solid variables and configuration spaces ..................................................... 11
  2.2 Potentials and decomposition of the fluid velocity ....................................... 12
  2.3 Brief description of the proof and organization of the paper ...................... 15

3 Estimates on the potentials .................................................................................. 17
  3.1 An auxiliary Dirichlet problem ..................................................................... 17
      3.1.1 Existence of solutions for problem (3.2) .............................................. 17
      3.1.2 A potential for a standalone solid ....................................................... 19
      3.1.3 A construction of the potential in the presence of small solids .......... 21
      3.1.4 Asymptotic behavior for problem (3.2) .............................................. 23
      3.1.5 Shape derivatives of potentials solving Dirichlet problems ............... 27
      3.1.6 Transposing to the Neumann problem .............................................. 29
  3.2 Estimates of the Kirchhoff potentials .............................................................. 30
      3.2.1 The Kirchhoff potentials ................................................................... 30
      3.2.2 Shape derivatives of the Kirchhoff potentials .................................... 32
  3.3 Estimates on the circulation stream function ................................................. 34
      3.3.1 Estimates on the reflected circulation stream function ....................... 34
      3.3.2 Shape derivatives of the reflected circulation stream function ........... 36
      3.3.3 Reflected circulation stream function of a phantom solid ................. 38
  3.4 Estimates of the Biot-Savart kernel .................................................................. 39
      3.4.1 Biot-Savart kernel .............................................................................. 39
      3.4.2 Shape derivatives of the Biot-Savart kernel ....................................... 40

4 First a priori estimates ......................................................................................... 41
  4.1 Vorticity estimates ......................................................................................... 41
  4.2 Energy estimates ............................................................................................ 42
  4.3 Rough estimate for the acceleration of the bodies ........................................ 43
      4.3.1 A decomposition of the velocity .......................................................... 43
      4.3.2 Proof of the acceleration estimates ...................................................... 44

5 Introduction of the modulations ......................................................................... 48
  5.1 Decomposition of the fluid velocity focused on a small solid ....................... 48
  5.2 Approximation of the \( \kappa \)-th exterior field ................................................. 49
  5.3 Definition of the modulations ........................................................................ 53

6 Normal forms ....................................................................................................... 53
  6.1 Statement of the normal form ....................................................................... 53
  6.2 Starting point of the proof: rewriting the solid equation with various terms .... 54
  6.3 Treatment of the simplest terms .................................................................... 56
  6.4 Exterior acceleration term ............................................................................. 56
  6.5 Main gyroscopic term .................................................................................... 60
  6.6 Added mass term ............................................................................................ 63
  6.7 Conclusion of the proof of the normal form ................................................ 65

7 Modulated energy estimates ................................................................................ 65
8 Passage to the limit

8.1 A change of variable ................................................................. 66
8.2 First step and compactness ...................................................... 67
  8.2.1 Fixing $\varepsilon_0$ and $T$ .................................................. 67
  8.2.2 Using compactness ............................................................ 68
8.3 Limit dynamics of the fluid ...................................................... 68
8.4 Limit dynamics of the solids of fixed size ................................. 70
8.5 Limit dynamics of the small solids and end of the proof of Theorem 2 71
8.6 Proof of Theorem 3 ................................................................. 73
Index

Characteristics of the solids
\( \varepsilon_\kappa \): scale factor for the \( \kappa \)-th solid, 7
\( \varepsilon \): scale factors for all solids at once, 8
\( \bar{\varepsilon} \): scale factors for all shrinking solids at once, 8
\( |\bar{\varepsilon}| \): total size of small solids, 8

Indices
\( \mathcal{P}(i) \): set of indices of large solids, 8
\( \mathcal{P}_{(ii)} \): set of indices of small massive solids, 8
\( \mathcal{P}_{(iii)} \): set of indices of small light solids, 8
\( \mathcal{P} \): set of indices of all small solids, that is \( \mathcal{P}_{(ii)} \cup \mathcal{P}_{(iii)} \), 8
\( N(i), N(ii), N(iii), N_3 \): cardinals of the sets of indices, 8

Solid position
\( h_\kappa \): position of the center of mass of the \( \kappa \)-th solid, 6
\( \vartheta_\kappa \): angle of the \( \kappa \)-th solid with respect to its initial position, 6
\( q_\kappa \): position and angle of the \( \kappa \)-th solid, 8
\( q \): positions and angles of all the solids at once, 8
\( q(i) \): positions and angles of all the final solids, 8

Solid velocity
\( p_\kappa \): linear and angular velocity of the \( \kappa \)-th solid, 11
\( p \): linear and angular velocity of all solids at once, 11
\( \bar{p}_\kappa \): linear and scaled angular velocity of the \( \kappa \)-th solid, 11
\( \bar{p} \): linear and scaled angular velocity of all solids at once, 11
\( v_{S_\kappa} \): \( \kappa \)-th solid velocity vector field, 6
\( \bar{p}_\kappa \): modulated velocity of the \( \kappa \)-th solid, 53
\( \bar{p} \): modulated velocity of all the solids at once, 53

Domains and admissible configurations
\( \Omega \): whole domain with fluid and solids, 6
\( S_\kappa \): \( \kappa \)-th solid, 7
\( \tilde{\mathcal{F}}(q) \): fluid domain, 8
\( \tilde{\mathcal{F}}(q(i)) \): final fluid domain, 8
\( Q_\delta \): bundle of shrinking bodies positions at distance \( > 2\delta \) one from another, 12
\( Q_\delta \): bundle of shrinking bodies positions and vorticity at distance \( > 2\delta \) one from another, 12
\( Q_\delta^{\text{sh}} \): \( Q_\delta \) with small solids of size less than \( \varepsilon_0 \), 12
\( Q_\delta^{\text{ss}} \): \( Q_\delta \) with small solids of size less than \( \varepsilon_0 \), 12

Potentials
\( \varphi_{\kappa,j} \): Kirchhoff potentials in \( \mathcal{F} \), 12
\( \varphi_{\kappa,j} \): standalone Kirchhoff potentials in \( \mathbb{R}^2 \backslash S_\kappa \), 13
(\( \tilde{\varphi}_{\kappa,j} \): final Kirchhoff potentials in \( \tilde{\mathcal{F}}(q(i)) \), 13
\( \psi_\kappa \): circulation potential in \( \mathcal{F}(q) \), 14
\( \psi_\kappa \): standalone circulation potential in \( \mathbb{R}^2 \backslash S_\kappa \), 14
\( \tilde{\psi}_\kappa \): final circulation potential in \( \tilde{\mathcal{F}}(q(i)) \), 14
\( K[\omega] \): Biot-Savart kernel in \( \mathcal{F}(q) \), 14
\( \tilde{K}[\omega] \): final Biot-Savart kernel in \( \tilde{\mathcal{F}}(q(i)) \), 15

Velocity fields
\( u \): fluid velocity field, 6
\( u^{\text{pot}} \): potential part of the velocity field, 15
\( u^{\text{ext}} \): exterior part of the velocity field, 43
\( u_\kappa^{\text{pot}} \), \( u_\kappa^{\text{ext}} \): decomposition of \( u^\kappa \) focused on \( S_\kappa \), 48
\( \tilde{u}_\kappa \): approximation of the \( \kappa \)-th exterior field defined in \( \tilde{F}_\kappa \), 49
\( \hat{\mu}_\kappa^{ext}, \hat{\mu}_\kappa^{pot} \): decomposition of \( u^\sigma \) focused on \( S_\kappa \) with modulated potential part, 54

Modulations
\( V_\kappa \): linear approximation of the \( \kappa \)-th exterior field \( u^\sigma_{ext} \), generating modulations, 51
\( \alpha_{\kappa,i} \): first-order modulations, 53
\( \beta_{\kappa,i} \): second-order modulations, 53

Miscellaneous
\( \mathcal{V}_\nu(A) \): \( \nu \)-neighborhood of \( A \subset \Omega \), 12
\( \xi_{\kappa,i} \): affine vector field centered on the \( \kappa \)-th solid, 12
\( K_\kappa \): vector space of affine vector fields, 49
\( \text{Kir}_\kappa \): transformation of affine vector fields in Kirchhoff vector fields, 49
\( \text{Kir}_\kappa \): transformation of affine vector fields in standalone Kirchhoff vector fields, 49
1 Introduction and statement of the main result

1.1 The fluid-solid system

The general situation that we describe is that of \( N \) solids immersed in a bounded domain of the plane. The total domain (containing the fluid and the solids) is denoted by \( \Omega \), that is a nonempty bounded open connected set in \( \mathbb{R}^2 \), with smooth boundary. In the domain \( \Omega \) are embedded \( N \) solids \( S_1, \ldots, S_N \), which are nonempty, simply connected and closed sets with smooth boundaries. To simplify, we assume that \( \Omega \) is simply connected and that the solids \( S_1, \ldots, S_N \) are not discs (though the general case could be treated similarly). We will systematically suppose them to be at positive distance one from another and from the outer boundary \( \partial \Omega \) during the whole time interval:

\[
\forall t, \ \forall \kappa \in \{1, \ldots, N\}, \ S_\kappa(t) \subset \Omega, \ \text{dist}(S_\kappa(t), \partial \Omega) > 0 \ \text{and} \ \forall \lambda \neq \kappa, \ \text{dist}(S_\kappa(t), S_\lambda(t)) > 0. \quad (1.1)
\]

Their positions depend on time, so we will denote them \( S_1(t), \ldots, S_N(t) \). Since they are rigid bodies, each solid \( S_\kappa(t) \) is obtained through a rigid movement from \( S_\kappa(0) \). The rest of the domain, occupied by the fluid, will be denoted by \( F(t) \) so that

\[
F(t) = \Omega \setminus (S_1(t) \cup \cdots \cup S_N(t)).
\]

Let us now describe the dynamics of the fluid and of the solids.

Dynamics of the fluid. The fluid is supposed to be inviscid and incompressible, and consequently driven by the incompressible Euler equation. We denote \( u = u(t, x) \) the velocity field (with values in \( \mathbb{R}^2 \)) and \( \pi = \pi(t, x) \) the (scalar) pressure field, both defined for \( t \) in some time interval \( [0, T] \) and \( x \in F(t) \). The incompressible Euler equation reads

\[
\begin{cases}
\partial_t u + (u \cdot \nabla) u + \nabla \pi = 0, \\
\text{div} \ u = 0,
\end{cases}
\text{for} \ t \in [0, T], \ x \in F(t).
\]

This equation is supplemented by boundary conditions which correspond to the non-penetration condition, precisely

\[
u \cdot n = 0 \ \text{on} \ \partial \Omega, \ \text{and} \ u \cdot n = v_{S,\kappa} \cdot n \ \text{on} \ \partial S_\kappa \ \text{for} \ \kappa \in \{1, \ldots, N\},
\]

where \( n \) denotes the unit normal on \( \partial F(t) \) directed outside \( F(t) \) and \( v_{S,\kappa} \) denotes the velocity field of the solid \( S_\kappa \).

Hence there is no difference with the classical situation, except the fact that the space-time domain is not cylindrical.

Dynamics of the solids. To describe the position of the \( \kappa \)-th solid \( S_\kappa \), we denote \( h_\kappa \) and \( \vartheta_\kappa \) the position of its center of mass and its angle with respect to its initial position. Correspondingly, the solid’s position at time \( t \) is obtained by the following rigid movement with respect to its initial position:

\[
S_\kappa(t) = h_\kappa(t) + R(\vartheta_\kappa(t)) (S_\kappa(0) - h_\kappa(0)),
\]

where \( R(\vartheta) \) is the linear rotation of angle \( \vartheta \), that is

\[
R(\vartheta) = \begin{pmatrix}
\cos(\vartheta) & -\sin(\vartheta) \\
\sin(\vartheta) & \cos(\vartheta)
\end{pmatrix}.
\]

Note also that the velocity field of the solid \( S_\kappa \) mentioned in (1.3) is given by

\[
v_{S,\kappa}(t, x) := h'_\kappa(t) + \vartheta'_\kappa(t)(x - h_\kappa(t))^\perp,
\]

where \( (x_1, x_2)^\perp := (-x_2, x_1) \). Now we denote the mass and momentum of inertia of the solid \( S_\kappa \) by \( m_\kappa \) and \( J_\kappa \) respectively. The assumption is that the solids evolve according to Newton’s law, under the influence of the fluid’s pressure on its boundary. Hence the equations of \( h_\kappa \) and \( \vartheta_\kappa \) read

\[
\begin{cases}
m_\kappa h'_\kappa(t) = \int_{\partial S_\kappa(t)} \pi(t, x) n(t, x) \, ds(x), \\
J_\kappa \vartheta'_\kappa(t) = \int_{\partial S_\kappa(t)} \pi(t, x)(x - h_\kappa(t))^\perp \cdot n(t, x) \, ds(x),
\end{cases}
\text{in} \ [0, T].
\]
Remark 1.1. It could be possible to add some external forces such as gravity in the right hand side of (1.7) with only minor modifications in the reasonings below.

Initial conditions. The system is supplemented with initial conditions:

- At initial time the solids $S_1, \ldots, S_N$ occupy the positions $S_{1,0}, \ldots, S_{N,0}$ such that

$$\forall \kappa \in \{1, \ldots, N\}, \ S_{\kappa,0} \subset \Omega, \ \text{dist}(S_{\kappa,0}, \partial \Omega) > 0 \ \text{and} \ \forall \lambda \neq \kappa, \ \text{dist}(S_{\kappa,0}, S_{\lambda,0}) > 0. \quad (1.8)$$

We introduce the initial values of the centers of masses $h_{1,0}, \ldots, h_{N,0}$, and the angles $\vartheta_{1,0} = \cdots = \vartheta_{N,0} = 0$ (by convention), which characterize these positions. We denote $F_0$ the corresponding initial fluid domain.

- The solids have initial velocities $(h'_{\kappa}, \vartheta'_{\kappa})(0) = (h'_{\kappa,0}, \vartheta'_{\kappa,0}) \in \mathbb{R}^3$ for $\kappa \in \{1, \ldots, N\}$,

- The circulations of velocity around the solids $S_1, \ldots, S_N$, gathered as $\gamma = (\gamma_1, \ldots, \gamma_N)$, are given,

- We consider an initial vorticity $\omega_0 \in L^\infty(F_0)$.

Note that this data is sufficient to reconstruct the initial velocity field $u_0 \in C^0(F_0; \mathbb{R}^3)$ in a unique way, see (2.23). In particular $\text{curl} \ u_0 = \omega_0$ and $\int_{S_{\kappa}} u_0 \cdot \tau \, ds = \gamma_\nu$ for $\nu = 1, \ldots, N$, where $\tau$ is the unit clockwise tangent vector field.

Cauchy theory à la Yudovich. The system (1.2)-(1.7) admits a suitable Cauchy theory in the spirit of Yudovich [30]. Precisely, by a straightforward adaptation of the arguments of [11], we obtain the following result where initial conditions are given, as described above.

Theorem 1. Given the initial conditions above, there is a unique maximal solution $(h_1, \vartheta_1, \ldots, h_N, \vartheta_N, u)$ in the space $C^2([0,T^*)^{3N} \times [L^\infty_{loc}(F(t); \mathbb{R}^3); \mathcal{L}(F(t); \mathbb{R}^3)] \cap C^0([0,T^*]; W^{1,q}(F(t); \mathbb{R}^3))$ (for all $q$ in $[1, +\infty)$) of System (1.2)-(1.7) for some $T^* > 0$. Moreover, as $t \rightarrow T^*$,

$$\min \left\{ \min \{ \text{dist}(S_\kappa(t), \partial \Omega), \kappa \in \{1, \ldots, N\} \}, \min \{ \text{dist}(S_\kappa(t), S_\lambda(t)), \kappa, \lambda \in \{1, \ldots, N\}, \lambda \neq \kappa \} \right\} \rightarrow 0.$$

Finally the velocity circulations around the solids $\gamma = (\gamma_1, \ldots, \gamma_N)$ are constant in time.

Above, $\mathcal{L}(F(t); \mathbb{R}^3)$ stands for the space of log-Lipschitz vector fields on $F(t)$; we recall that $\mathcal{L}(X)$ that is the space of functions $f \in L^\infty(X)$ such that

$$\|f\|_{\mathcal{L}(X)} := \|f\|_{L^\infty(X)} + \sup_{x \neq y} \frac{|f(x) - f(y)|}{(x - y)(1 + \ln|x - y|)} < +\infty.$$

Also we used the slightly abusive notation $L^\infty(0, T; \mathcal{L}(F(t); \mathbb{R}^3))$: it describes the space of functions defined for almost all $t$, with values for such $t$ in $\mathcal{L}(F(t))$, with a uniform log-Lipschitz norm. We will quite systematically use such notations from the cylindrical case to describe our situation. There should be no ambiguity coming from this abuse of notation.

Theorem 1 indicates in particular that the lifespan of the solutions is only limited by a possible collision between solids or between a solid and the boundary. Regarding the issue of collisions we refer to [14], [15] and the recent paper [3].

1.2 The problem of small solids

The main question raised by this paper is to determine a limit system when some of the solids $S_1, \ldots, S_N$ shrink to a point. To describe this problem, we will denote the scale of the $\kappa$-th solid by $\epsilon_\kappa$, and suppose that the $\kappa$-th solid $S_\kappa$ is obtained initially by applying a homothety of ratio $\epsilon_\kappa$ and center $h_{\kappa,0}$ on the solid of fixed size $S_{\kappa,0}$:

$$S_{\kappa,0} = h_{\kappa,0} + \epsilon_\kappa(S_{\kappa,0}^1 - h_{\kappa,0}). \quad (1.9)$$
The three sets of solids. Now let us be more specific about the indices \( \kappa \). The set of indices \( \{1, \ldots, N\} \) is split in three:

\[
\{1, \ldots, N\} = \mathcal{P}_1 \cup \mathcal{P}_2 \cup \mathcal{P}_3 \quad \text{with} \quad \mathcal{P}_1 := \{1, \ldots, N(i)\}, \quad \mathcal{P}_2 := \{N(i) + 1, \ldots, N(i) + N(ii)\}, \quad \mathcal{P}_3 := \{N(i) + N(ii) + 1, \ldots, N\},
\]

Corresponding respectively to the solids:

- (i) of fixed size and inertia:
  
  \[
  \text{for } \kappa \in \mathcal{P}_1, \quad \varepsilon_\kappa = 1, \quad m_\kappa^s = m_\kappa^1, \quad J_\kappa^s = J_\kappa^1. \tag{1.10}
  \]

- (ii) of size going to zero but with fixed mass:
  
  \[
  \text{for } \kappa \in \mathcal{P}_2, \quad \varepsilon_\kappa \to 0^+, \quad m_\kappa^s = m_\kappa^1, \quad J_\kappa^s = \varepsilon_\kappa^2 J_\kappa^1, \tag{1.11}
  \]

- (iii) of size and mass converging to zero:
  
  \[
  \text{for } \kappa \in \mathcal{P}_3, \quad \varepsilon_\kappa \to 0^+, \quad m_\kappa^s = \varepsilon_\kappa m_\kappa^1, \quad J_\kappa^s = \varepsilon_\kappa^{a+2} J_\kappa^1 \quad \text{for some } a_\kappa > 0. \tag{1.12}
  \]

Remark 1.2. Case (iii) encompasses the case of fixed density, for which \( a_\kappa = 2 \). This is actually the main motivation for the difference in the scaling of \( m_\kappa^s \) and \( J_\kappa^s \).

It will be useful to consider the indices corresponding to small solids (here \( s \) stands for small):

\[
\mathcal{P}_s := \mathcal{P}_2 \cup \mathcal{P}_3 = \{N(i) + 1, \ldots, N\}, \quad N_s := N(ii) + N(iii). \tag{1.13}
\]

We collect the various \( \varepsilon_\kappa \) as follows:

\[
\varepsilon = (\varepsilon_1, \ldots, \varepsilon_N), \quad \text{and} \quad \varepsilon = (\varepsilon_{N(i)} + 1, \ldots, \varepsilon_N).
\]

The total size of small solids will be denoted as follows

\[
|\varepsilon| := \sum_{\kappa \in \mathcal{P}_s} \varepsilon_\kappa. \tag{1.14}
\]

For \( \varepsilon_0 > 0 \), we will write \( \varepsilon < \varepsilon_0 \) or \( \varepsilon \leq \varepsilon_0 \) to express that the inequality is valid for each coordinate.

We assume, for any \( \kappa \) in \( \mathcal{P}_s \), that \( h_{\kappa,0} \) is in \( \Omega \) so that \( S_{\kappa,0}^\varepsilon \subset \Omega \) for \( \varepsilon_\kappa \) small enough. Up to a redefinition of \( S_{\kappa,0}^\varepsilon \) we may assume that

\[
S_{\kappa,0}^\varepsilon \subset \Omega \quad \text{for all } \varepsilon_\kappa \leq 1. \tag{1.15}
\]

Description of the position of the solids. Grouping the positions of the center of mass and angles together, we denote the position variable as follows:

\[
q_\kappa = (h_\kappa, \theta_\kappa)^T \quad \text{and} \quad q = (q_1, \ldots, q_N).
\]

It follows that the \( \kappa \)-th solid is determined by \( q_\kappa \) and \( \varepsilon_\kappa \); we will denote it by \( S_\kappa(\varepsilon_\kappa, q_\kappa) \), or in a simpler manner \( S_\kappa^\varepsilon(q_\kappa) \). Moreover when it does not play an important role in the discussion or when it is clear, we will drop the exponent \( \varepsilon \) and/or the dependence on \( q_\kappa \) to lighten the notations.

When one considers only the non-shrinking solids, it is useful to introduce

\[
q_{(i)} = (q_1, \ldots, q_{N(i)}).
\]

Fluid domains. Corresponding to the above notations, the fluid domain is

\[
\mathcal{F}^\varepsilon(q) = \Omega \setminus (S_1^\varepsilon(q_1) \cup \cdots \cup S_N^\varepsilon(q_N)).
\]

When the small solids have disappeared, it remains merely the final domain

\[
\tilde{\mathcal{F}}(q_{(i)}) = \Omega \setminus (S_1(q_1) \cup \cdots \cup S_{N(i)}(q_{N(i)})). \tag{1.16}
\]
Initial conditions. We consider the initial vorticity $\omega_0$, the circulations around the solids $\gamma = (\gamma_1, \ldots, \gamma_N)$, the initial solid positions $q_0 = (q_1, q_2, \ldots, q_N) = (h_1^0, \ldots, h_N^0)$ and the initial solid velocities $p_0 = (p_1, p_2, \ldots, p_N) = (h_1^0, \ldots, h_N^0)$ fixed independently of $\varepsilon$. Moreover we assume that $\gamma_\varepsilon \neq 0$ when $\varepsilon = 0$.

To be more precise on the vorticity, we set for $\varepsilon > 0$ the space $L^{\infty}_c(\mathcal{P})$ of essentially bounded functions $f$ satisfying that for almost all $x \in \mathcal{F}(\varepsilon)$ such that $d(x, \mathcal{S}_\varepsilon) \leq \delta$ for some $\delta \in \mathcal{P}$, one has $f(x) = 0$. Now we suppose that

$$
\omega_0 \in L^{\infty}_c(\mathcal{P}, [S_{1,0} \cup \ldots \cup S_{N(\varepsilon),0} \cup \{h_j, \ j \in \mathcal{P}\}]).
$$

Hence for some $\delta > 0$ and for suitably small $\varepsilon$, one has $\omega_0 \in L^{\infty}_{c,\delta}(\mathcal{F}_0)$.

We are now in position to state our main result.

### 1.3 Main result

We first introduce a convention. To express convergences in domains that actually depend on the solutions themselves, we will take the convention to extend the vorticity $\omega$ and the velocity $u$ (defined in $\mathcal{F}(t)$) by 0 inside $\mathcal{S}_1, \ldots, \mathcal{S}_N$. In the same way, the limit vorticity and velocity (defined in $\tilde{\mathcal{F}}(t)$) are extended by 0 inside $\mathcal{S}_1, \ldots, \mathcal{S}_{N(\varepsilon)}$ as well.

Our main result is as follows.

**Theorem 2.** Under the above assumptions there exists $\varepsilon_0 > 0$ and some $T > 0$ such that the following holds. To each family $\varepsilon$ of scale factors with $\varepsilon \leq \varepsilon_0$ we associate the corresponding maximal solution $(\varepsilon, u^\varepsilon)$ on $[0, T_\varepsilon)$ given by Theorem 1. Then the maximal existence times $T^\varepsilon$ satisfy $T^\varepsilon > T$, and, as $\varepsilon \to 0^+$, up to a subsequence, one has

$$
\lim_{\varepsilon \to 0^+} u^\varepsilon = u^* \quad \text{in} \quad C^0([0, T]; L^p(\Omega)) \quad \text{for} \quad q \in [1, 2),
$$

$$
\lim_{\varepsilon \to 0^+} \omega^\varepsilon = \omega^* \quad \text{in} \quad C^0([0, T]; L^{\infty}(\Omega) - w^*),
$$

$$
\lim_{\varepsilon \to 0^+} h^\varepsilon_{\kappa} = h^*_{\kappa} \quad \text{in} \quad \left\{ \begin{array}{ll}
W^{2,\infty}(0, T) \quad \text{weak} - * \quad \text{for} \ \kappa \in \mathcal{P}_{(i)} \cup \mathcal{P}_{(ii)}, \\
W^{1,\infty}(0, T) \quad \text{weak} - * \quad \text{for} \ \kappa \in \mathcal{P}_{(iii)},
\end{array} \right.
$$

$$
\lim_{\varepsilon \to 0^+} \vartheta^\varepsilon_{\kappa} = \vartheta^*_{\kappa} \quad \text{in} \quad W^{2,\infty}(0, T) \quad \text{weak} - * \quad \text{for} \ \kappa \in \mathcal{P}_{(i)},
$$

and at the limit the following system holds in the final domain:

$$
\text{div } u^* = 0 \quad \text{in} \quad \tilde{\mathcal{F}}(q^*_{(i)}),
$$

$$
\text{curl } u^* = \omega^* + \sum_{\kappa \in \mathcal{P}_s} \gamma_\kappa \delta h^*_\kappa \quad \text{in} \quad \tilde{\mathcal{F}}(q^*_{(i)}),
$$

$$
u u^* \cdot n = \left(h^*_{\kappa} + (\vartheta^*_{\kappa}(x - h^*_\kappa)^{-1})^2 \right) \cdot n \quad \text{on} \quad \partial \mathcal{S}_\kappa(q^*_{\kappa}) \quad \text{for} \ \kappa \in \mathcal{P}_{(i)},
$$

$$\int_{\partial \mathcal{S}_\kappa(q^*_{\kappa})} u^* \cdot \tau \, ds = \kappa_\gamma \quad \text{for} \ \kappa \in \mathcal{P}_{(i)},
$$

where $q^*_{\kappa} = (h^*_{\kappa}, \vartheta^*_{\kappa})^T$ and $q^*_{(i)} = (q^*_{1(\varepsilon)}, \ldots, q^*_{N(\varepsilon)})$.

$$
\hat{c}_t \omega^* + \text{div } (u^* \omega^*) = 0 \quad \text{in} \quad [0, T] \times \tilde{\mathcal{F}}(q^*_{(i)}(t)),
$$

for all $t \in [0, t]$, $-(\hat{c}_t u^* + (u^* \cdot \nabla) u^*)$ is a gradient in $\tilde{\mathcal{F}}(q^*_{(i)}(t)) \setminus \{h^*_\kappa(t), \kappa \in \mathcal{P}_s\}$ regular in the neighborhood of $\bigcup_{\kappa=1}^{N_{(i)}} \partial \mathcal{S}_\kappa(q^*_{\kappa})$, which we denote $\nabla \pi^*$, (1.23)

$$
\begin{cases}
m_{\kappa}(h^*_{\kappa})''(t) = \int_{\partial \mathcal{S}_\kappa(q^*_{\kappa})} \pi^*(t, x) n(t, x) \, ds(x), \\
J_{\kappa}(\vartheta^*_{\kappa})''(t) = \int_{\partial \mathcal{S}_\kappa(q^*_{\kappa})} \pi^*(t, x) (x - h^*_\kappa(t))^{-1} \cdot n(t, x) \, ds(x),
\end{cases}
$$

in $[0, T]$ for $\kappa \in \mathcal{P}_{(i)}$, (1.24)
\[ m_{\kappa}(h^*_\kappa)'' = \gamma_{\kappa} \left[ \left( h^*_\kappa \right)' - u^*_\kappa(t, h^*_\kappa) \right]_{\frac{1}{\gamma}} \text{ in } [0, T] \quad \text{for } \kappa \in \mathcal{P}_{(ii)}, \quad (1.25) \]
\[ \langle h^*_\kappa \rangle' = u^*_\kappa(t, h^*_\kappa) \text{ in } [0, T] \quad \text{for } \kappa \in \mathcal{P}_{(iii)}, \quad (1.26) \]

where \( u^*_\kappa \) is the “desingularized version” of \( u \) at \( h^*_\kappa \) defined by

\[ u^*_\kappa(t, x) = u^*(t, x) - \frac{\gamma_{\kappa} (x - h^*_\kappa(t))_{\frac{1}{\gamma}}}{2\pi |x - h^*_\kappa(t)|^{\gamma}}, \quad t \in [0, T], \quad x \in \tilde{F}(q^*_\kappa(t)). \quad (1.27) \]

On the limit system. Theorem 2 identifies the limit dynamics of a family of solutions of the system (1.2)-(1.7), when some of the solids shrink to points, as a system compound of the Euler-type system (1.21)-(1.22) for the fluid, the Newton’s laws (1.24) for the solids that have a fixed radius and the point vortex systems (1.25)-(1.26) for the limit point particles. The interest of (1.23) is to give a meaning for the trace of the limit fluid pressure \( \pi^* \) on the boundary of the solids that have a fixed radius; this gives a sense to the right hand sides in (1.24). Regarding the solids with a vanishing radius the limit equation is not the same in case (ii) and in case (iii), as we can see in (1.25)-(1.26). A common feature is that the limit equation is independent of the shape of the rigid body which has shrunk.\(^2\)

In case (ii) the rigid body reduces at the limit in a point-mass particle which satisfies the second order differential equation (1.25). This type of systems has already been discussed by Friedrichs in [4, Chapter 3], see also [12]. The force in the right hand side of (1.25) extends the classical Kutta-Joukowski force, as it is a gyroscopic force orthogonally proportional to its relative velocity and proportional to the circulation around the body. The Kutta-Joukowski-type lift force was originally studied in the case of a single body in an irrotational unbounded flow at the beginning of the 20th century in the course of the first mathematical investigations of aeronautics; see for example [19].

In case (iii) the rigid body reduces at the limit in a massless point particle which satisfies the first order differential equation (1.26), which can be seen as a classical point vortex equation, its vortex strength being given by the circulation around the rigid body. Historically the point vortex system, which dates back to Helmholtz, Kirchhoff, Kelvin and Poincaré, has been seen as a simplification of the the 2D incompressible Euler equations when the vorticity of the fluid is concentrated in a finite number of points, see for instance [25]. The key feature of the derivation of the point vortex equations from the 2D incompressible Euler equations is that the self-interaction has to be discarded. Theorem 2 proves that such equations can also be obtained as the limit of the dynamics of rigid bodies of type (iii). The desingularization of the background fluid velocity \( u^* \) mentioned in (1.27) precisely corresponds to the cancellation of the self-interaction.

On the other hand the genuine fluid vorticity \( \omega^* \) is convected by the background fluid velocity \( u^* \), according to (1.22). A precise decomposition of the velocity field \( u^* \) obtained in the limit will be given below, see (2.26). Systems mixing an evolution equation for absolutely continuous vorticity such as (1.22) and some evolution equations for point vortices such as (1.26) have been coined as vortex-wave systems by Marchioro and Pulvirenti in the early 90s, see [25].

On the lifespan, on the convergences, and on the uniqueness. Observe that the existence of a common lifetime for a subsequence \( \varepsilon \to 0^+ \) is a part of the result, as Theorem 1 does not provide any quantitative information on the existence times \( T^\varepsilon \) before collisions.

Let us also stress that the convergences in (1.19) are different depending on whether the rigid body has a positive mass in the limit or not. Indeed the weaker convergence obtained in Case (iii) is associated with the degeneracy of the solid dynamics into a first order equation. Except for some well-prepared initial data the convergence is indeed limited to the weak-* topology of \( W^{1,\infty}(0, T) \). We refer here to [1] for partial results regarding multi-scale features of the time-evolution of some toy models of the limit system above which attempts to give more insight on this issue. The issue of the uniqueness of the solution to the limit system and the associated issue of the convergence of the whole sequence, not only a subsequence, is a delicate matter. We refer to [24, 25, 17] for some positive results concerning the vortex-wave system with massless point vortices (the system occupying the whole plane). In the case\(^2\)

\(^2\)However let us recall that we assume that the solids \( S_1, \ldots, S_N \) are not discs. The case of a disk is peculiar as several degeneracies appear in this case. We refer to [9] for a complete treatment of this case for a single small body of type (ii) or (iii) immersed in a irrotational incompressible perfect fluid occupying the full plane or a bounded plane domain; in particular it is shown that the case of a homogeneous disk is rather simple whereas the case of a non-homogeneous disk requires appropriate modifications.
of several massive point vortices, we refer to the recent work [18] which gives results when the initial vorticity is bounded, compactly supported and locally constant in a neighborhood of the point vortices. A key ingredient in all these uniqueness results is that the point vortices stay away one from another and remain distant from the support of the vorticity (or at least, that the vorticity remains constant in their neighborhood.)

In the particular cases where uniqueness holds and the point vortices and the vorticity remain distant, we can improve a bit the statement of Theorem 2 into the following one.

**Theorem 3.** Suppose the assumptions of Theorem 2 to be satisfied, and suppose moreover that for this data the limit system (1.21)-(1.27) admits a unique solution in \([0, T^*]\) (of class \(W^{2,\infty}_{\text{loc}}([0, T^*])\) for the solids and the massive point vortices, \(W^{1,\infty}_{\text{loc}}([0, T^*])\) for the massless point vortices, and \(C^0([0, T^*]; L^\infty(\Omega) - w^*\) for the vorticity) for which for all \(t \in [0, T^*]\), the point vortices and the large solids do not meet one another and do not meet the support of vorticity nor the outer boundary. Then the maximal existence times \(T^*\) satisfy \(\liminf_{\epsilon \to 0} T^* \geq T^*\), and the convergences (1.17)-(1.20) hold on any time interval \([0, T] \subset [0, T^*]\) and are valid without restriction to a subsequence.

**On the relationships with earlier results.** Theorem 2 extends results obtained in the earlier works [7], which deals with the case of a single small body of type (ii) immersed in an incompressible perfect fluid occupying the rest of the plane, [8], which deals with the case of a single small body of type (iii) immersed in an incompressible perfect fluid occupying the rest of the plane, and [9], which deals with the case of a single small body of type (ii) or (iii) immersed in an irrotational incompressible perfect fluid occupying a bounded plane domain. In particular we consider for the first time the case of several small rigid bodies, for which the strategies of the previous papers cannot be adapted straightforwardly, despite the results recently obtained in [10] in the case of several rigid bodies of type (i). Indeed the main difficulty is to understand the influence of solids between themselves, and to analyze it to understand how the coupling at leading order disappears in the limit. This is made more difficult by the fact that each solid possesses its own scale.

**On the relationships with the case of the Navier-Stokes equations.** Let us mention that the Euler system is a rough modeling for a fluid in a neighborhood of rigid boundaries as even a slight amount of viscosity may drastically change the behavior of the fluid close to the boundary, due to boundary layers, and sometimes even in the bulk of the fluid when the boundary layers detach from the boundary. While the Navier-Stokes equations certainly represent a better choice in terms of modeling, it is certainly useful to first understand the case of the Euler equations. In this direction let us mention that Gallay has proven in [5] that the point vortex system can also be obtained as vanishing viscosity limits of concentrated smooth vortices driven by the incompressible Navier-Stokes equations, see also the recent extension to vortex-wave systems in [26].

## 2 Preliminaries

In this section, we introduce some notations and basic tools that are needed in the sequel. Then we describe briefly the proof and the organization of the rest of the paper.

### 2.1 Solid variables and configuration spaces

Below we introduce notations for the solid velocities and for the admissible configurations of the location of the solids and of the support of the vorticity.

**Solid velocities.** The solid velocities will be denoted as follows:

\[
p_\kappa = \left( h_\kappa, \vartheta_\kappa \right)^T, \quad \tilde{p}_\kappa = \left( h_\kappa, \xi_\kappa \vartheta_\kappa \right)^T, \quad p = (p_1, \ldots, p_N) \quad \text{and} \quad \tilde{p} = (\tilde{p}_1, \ldots, \tilde{p}_N).
\]  

For \(i \in \{1, 2, 3\}\), \(p_{\kappa,i}\) denotes the \(i\)-th coordinate of \(p_\kappa\). In terms of these coordinates, (1.6) reads as follows

\[
v_{\mathcal{S},\kappa}(t,x) = \sum_{i=1}^{3} p_{\kappa,i} \xi_{\kappa,i},
\]  

11
with \( \xi_{\kappa,i} = e_i \) for \( i = 1,2 \) and \( \xi_{\kappa,3} = (x - h_{\kappa}) \perp \) on \( \partial S_{\kappa} \) (this anticipates the notation (2.7)). Above \( e_1 \) and \( e_2 \) are the unit vectors of the canonical basis.

**Admissible configurations.** We introduce notations for the spaces of configuration of the solids which can also possibly incorporate the configuration for the vorticity. Given \( \delta > 0 \), we let

\[
\mathcal{Q}_\delta := \{(\mathbf{z}, \mathbf{q}) \in (0,1)^N \times \mathbb{R}^3 : \forall \nu, \mu \in \{1, \ldots, N\} \text{ s.t. } \nu \neq \mu, \ d(S_{\nu}^\kappa(\mathbf{q}), S_{\mu}^\kappa(\mathbf{q})) > 2\delta \text{ and } d(S_{\nu}^\kappa(\mathbf{q}), \partial \Omega) > 2\delta \}. \tag{2.3}
\]

\[
\mathcal{Q}_\delta := \{(\mathbf{z}, \mathbf{q}, \omega) \in (0,1)^N \times \mathbb{R}^3 \times L^\infty(\Omega) : (\mathbf{z}, \mathbf{q}) \in \mathcal{Q}_\delta \text{ and } \forall \mu \in \{1, \ldots, N\}, \ d(S_{\nu}^\kappa(\mathbf{q}), \text{Supp}(\omega)) > 2\delta \}. \tag{2.4}
\]

Given \( \varepsilon_0 > 0 \), we refine the above sets by limiting the size of small solids as follows

\[
\mathcal{Q}^{\varepsilon_0}_\delta := \{(\mathbf{z}, \mathbf{q}) \in \mathcal{Q}_\delta / \mathbf{z} < \varepsilon_0 \} \text{ and } \mathcal{Q}^{\varepsilon_0}_\delta := \{(\mathbf{z}, \mathbf{q}, \omega) \in \mathcal{Q}_\delta / \mathbf{z} < \varepsilon_0 \}, \tag{2.5}
\]

where as before \( \varepsilon < \varepsilon_0 \) expresses that \( \varepsilon_i < \varepsilon_0 \) for all \( i \in \mathcal{P}_s \).

**\( \nu \)-neighborhoods in \( \Omega \).** In many situations, it will be helpful to consider some neighborhoods of the solids or of their boundaries; we therefore denote for \( A \subset \Omega \) and \( \nu > 0 \):

\[
\mathcal{V}_\nu(A) := \{x \in \Omega / d(x, A) < \nu \}. \tag{2.6}
\]

For instance the above conditions for \( \mathcal{Q}_\delta \) can be rephrased in the form \( \mathcal{V}_\delta(S_{\kappa}(\mathbf{q})) \cap \mathcal{V}_\delta(S_{\xi}(\mathbf{q})) = \emptyset \) and so on.

### 2.2 Potentials and decomposition of the fluid velocity

Below we first recall the definition of the so-called Kirchhoff potentials and the associated notion of added inertia. Then we introduce the stream functions for the circulation terms, the hydrodynamic Biot-Savart operator and we finally conclude by recalling the standard decomposition of the velocity field in terms of vorticity, solid velocities and circulations.

**The Kirchhoff potentials.** First, for \( \kappa \in \{1, \ldots, N\} \) and \( j \in \{1, \ldots, 5\} \) we introduce the function \( \xi_{\kappa,j}(\mathbf{q}, \cdot) : \partial \mathcal{F}(\mathbf{q}) \rightarrow \mathbb{R}^2 \) as follows:

- on \( \partial \mathcal{F}(\mathbf{q}) \setminus \partial S_{\kappa} \), \( \xi_{\kappa,j}(\mathbf{q}, \cdot) := 0 \),
- on \( \partial S_{\kappa} \),
  \[
  \begin{cases}
  \xi_{\kappa,j}(\mathbf{q}, x) = e_j \text{ for } j = 1,2, \\
  \xi_{\kappa,3}(\mathbf{q}, x) := (x - h_{\kappa}) \parallel, \\
  \xi_{\kappa,4}(\mathbf{q}, x) := -(x_1 + h_{\kappa,1}, x_2 - h_{\kappa,2}) \text{ and } \xi_{\kappa,5}(\mathbf{q}, x) := (x_2 - h_{\kappa,2}, x_1 - h_{\kappa,1}).
  \end{cases} \tag{2.7}
  \]

We denote by

\[
K_{\kappa,j}(\mathbf{q}, \cdot) := n \cdot \xi_{\kappa,j}(\mathbf{q}, \cdot)
\]

the normal trace of \( \xi_{\kappa,j} \) on \( \partial \mathcal{F}(\mathbf{q}) \), where \( n \) denotes the unit normal vector pointing outside \( \mathcal{F}(\mathbf{q}) \). We introduce the Kirchhoff potentials \( \varphi_{\kappa,j}(\mathbf{q}, \cdot) \), as the unique (up to an additive constant) solutions in \( \mathcal{F}(\mathbf{q}) \) of the following Neumann problems:

\[
\Delta \varphi_{\kappa,j} = 0 \quad \text{in} \ \mathcal{F}(\mathbf{q}), \tag{2.8a}
\]

\[
\frac{\partial \varphi_{\kappa,j}}{\partial n}(\mathbf{q}, \cdot) = K_{\kappa,j}(\mathbf{q}, \cdot) \quad \text{on} \ \partial \mathcal{F}(\mathbf{q}). \tag{2.8b}
\]

We fix the additive constant by requiring (for instance) that

\[
\int_{\partial \mathcal{S}_{\kappa}(\mathbf{q})} \varphi_{\kappa,j} \, ds = 0.
\]
In the same spirit, we define the standalone Kirchhoff potentials as the solutions in \( \mathbb{R}^2 \setminus S_n(q) \) of the following Neumann problem:

\[
\Delta \tilde{\varphi}_{n,j} = 0 \quad \text{in} \ \mathbb{R}^2 \setminus S_n(q),
\]

\[
\frac{\partial \tilde{\varphi}_{n,j}}{\partial n}(q, \cdot) = K_{n,j}(q, \cdot) \quad \text{on} \ \partial S_n(q),
\]

\[
\nabla \tilde{\varphi}_{n,j}(x) \rightarrow 0 \quad \text{as} \ |x| \rightarrow +\infty,
\]

\[
\int_{\partial S_n} \tilde{\varphi}_{n,j}(x) \, ds(x) = 0.
\]

We underline that this potential is defined as if \( S_n \) were alone in the plane, and consequently merely depends on the position \( q_n \).

We also define the final Kirchhoff potentials corresponding to the domain \( \tilde{\mathcal{F}}(q(i)) \) where small solids have disappeared as to satisfy

\[
\Delta \tilde{\varphi}_{n,j} = 0 \quad \text{in} \ \tilde{\mathcal{F}}(q(i)),
\]

\[
\frac{\partial \tilde{\varphi}_{n,j}}{\partial n}(q, \cdot) = K_{n,j}(q, \cdot) \quad \text{on} \ \partial \tilde{\mathcal{F}}(q(i)).
\]

**Inertia matrices.** We first define the (diagonal) \( 3N \times 3N \) matrix of genuine inertia by \( \mathcal{M}_g = (\mathcal{M}_{g,k,i,k',i'})_{1 \leq k, i, k', i' \leq 3} \) with

\[
\mathcal{M}_{g,k,i,k',i'} = \delta_{k,k'}\delta_{i,i'}(\delta_{i\in[1,2]}m_k + \delta_{i,3}J_k).
\]

The \( 3N \times 3N \) matrix of added inertia is defined by \( \mathcal{M}_a = (\mathcal{M}_{a,k,i,k',i'}) \) with

\[
\mathcal{M}_{a,k,i,k',i'}(q) = \int_{\mathcal{F}(q)} \nabla \varphi_{k,i}(q, \cdot) \cdot \nabla \varphi_{k',i'}(q, \cdot) \, dx.
\]

This allows to define the total mass matrix \( \mathcal{M}(q) \) by

\[
\mathcal{M}(q) = \mathcal{M}_g + \mathcal{M}_a(q).
\]

We also define the \( \kappa \)-th added inertia matrix as the \( 3 \times 3 \) matrix defined by

\[
(\mathcal{M}_{a,k})_{i,j}(q) = \int_{\mathcal{F}(q)} \nabla \varphi_{k,i}(q, \cdot) \cdot \nabla \varphi_{k,j}(q, \cdot) \, dx,
\]

and the \( \kappa \)-th standalone added inertia matrix as the \( 3 \times 3 \) matrix defined by

\[
(\tilde{\mathcal{M}}_{a,k})_{i,j}(q) = \int_{\mathbb{R}^2 \setminus S_n(q)} \nabla \tilde{\varphi}_{k,i} \cdot \nabla \tilde{\varphi}_{k,j} \, dx.
\]

Finally, when the small solids have disappeared, we also consider the \( 3N(i) \times 3N(i) \) final added mass matrix \( \tilde{\mathcal{M}}_a(q(i)) = (\mathcal{M}_{a,k,i,k',i'}(q(i))) \) defined by

\[
\tilde{\mathcal{M}}_{a,k,i,k',i'}(q(i)) = \int_{\mathcal{F}(q(i))} \nabla \tilde{\varphi}_{k,i}(q(i), \cdot) \cdot \nabla \tilde{\varphi}_{k',i'}(q(i), \cdot) \, dx.
\]

**Remark 2.1.** All those added mass matrices are Gram matrices, and consequently symmetric and positive semi-definite. Moreover, an elementary consequence of our assumption that the solids \( S_1, \ldots, S_N \) are not balls is that they are symmetric positive definite matrices, as Gram matrices of independent families of vectors. This will be of particular interest for the standalone added mass matrices \( \mathcal{M}_{a,1}, \ldots, \mathcal{M}_{a,N} \). In the case of balls, these matrices are singular. In that case, mass-vanishing small solids require a different treatment (see [9]).
Stream functions for the circulation terms. To take into account the circulations of velocity around the solids, we introduce for each \( \kappa \in \{1, \ldots, N\} \) the stream function \( \psi_\kappa = \psi_\kappa(\mathbf{q}, \cdot) \) defined on \( \mathcal{F}(\mathbf{q}) \) of the harmonic vector field which has circulation \( \delta_{\kappa\nu} \) around \( \partial \mathcal{S}_\nu(\mathbf{q}) \) for \( \nu = 1, \ldots, N \). More precisely, for every \( \mathbf{q} \), there exist unique constants \( C_{\kappa,\nu}(\mathbf{q}) \in \mathbb{R} \) such that the unique solution \( \psi_\kappa(\mathbf{q}, \cdot) \) of the Dirichlet problem:

\[
\begin{align*}
\Delta \psi_\kappa(\mathbf{q}, \cdot) &= 0 & \text{in } & \mathcal{F}(\mathbf{q}) \\
\psi_\kappa(\mathbf{q}, \cdot) &= C_{\kappa,\nu}(\mathbf{q}) & \text{on } & \partial \mathcal{S}_\nu(\mathbf{q}), \ \nu = 1, \ldots, N, \\
\psi_\kappa(\mathbf{q}, \cdot) &= 0 & \text{on } & \partial \Omega,
\end{align*}
\]

satisfies

\[
\int_{\partial \mathcal{S}_\nu(\mathbf{q})} \frac{\partial \psi_\kappa}{\partial n}(\cdot, \cdot) \, ds = -\delta_{\nu\kappa}, \ \nu = 1, \ldots, N.
\]

These functions \( \psi_\kappa \) have their standalone counterparts, the stream functions \( \hat{\psi}_\kappa = \hat{\psi}_\kappa(\mathbf{q}, \cdot) \) defined on \( \mathbb{R}^2 \setminus \mathcal{S}_\kappa(\mathbf{q}) \) of the harmonic vector field which has circulation 1 around \( \partial \mathcal{S}_\kappa(\mathbf{q}) \). They are defined as follows: for every \( \mathbf{q} \), there exists a unique constant \( C_{\kappa}(\mathbf{q}) \in \mathbb{R} \) such that the unique solution \( \hat{\psi}_\kappa(\mathbf{q}, \cdot) \) of the Dirichlet problem:

\[
\begin{align*}
\Delta \hat{\psi}_\kappa(\mathbf{q}, \cdot) &= 0 & \text{in } & \mathbb{R}^2 \setminus \mathcal{S}_\kappa(\mathbf{q}) \\
\hat{\psi}_\kappa(\mathbf{q}, \cdot) &= C_{\kappa}(\mathbf{q}) & \text{on } & \partial \mathcal{S}_\kappa(\mathbf{q}), \\
\nabla \hat{\psi}_\kappa(\mathbf{q}, x) &\to 0 & \text{as } & |x| \to +\infty,
\end{align*}
\]

satisfies

\[
\int_{\partial \mathcal{S}_\kappa(\mathbf{q})} \frac{\partial \hat{\psi}_\kappa}{\partial n}(\cdot, \cdot) \, ds = -1.
\]

This allows to introduce the following vector depending merely on \( \mathcal{S}_\kappa \), that is on \( \varepsilon_{\kappa} \) and \( \mathbf{q}_\kappa \):

\[
\zeta^\kappa_{\varepsilon}(\mathbf{q}_\kappa) = -\int_{\partial \mathcal{S}_\kappa} (x - h_{\kappa}) \frac{\partial \hat{\psi}_\kappa}{\partial n}(\mathbf{q}_\kappa, x) \, ds(x) = R(\partial_{\kappa}) \zeta^\kappa_0(\mathbf{q}_\kappa, 0) = \varepsilon_{\kappa} R(\partial_{\kappa}) \zeta^\kappa_1(\mathbf{q}_\kappa, 0).
\]

To simplify the notations, we denote \( \zeta^\kappa_{0,0} := \zeta^\kappa_1(\mathbf{q}_\kappa, 0) \). This is referred to as the conformal center of solid.

Finally, as for the Kirchhoff potentials, we can introduce the final stream functions for the circulation \( \tilde{\psi}_\kappa(\mathbf{q}_{(i)}), \ \kappa = 1, \ldots, N_{(i)}, \) defined in \( \tilde{\mathcal{F}}(\mathbf{q}_{(i)}) \). Here \( \tilde{\psi}_\kappa(\mathbf{q}_{(i)}) \) is the stream function of the harmonic vector field which has circulation \( \delta_{\kappa\nu} \) around \( \partial \mathcal{S}_\nu(\mathbf{q}) \) for \( \nu = 1, \ldots, N_{(i)} \). It can be obtained as follows: for every \( \mathbf{q}_{(i)} \), there exist unique constants \( \tilde{C}_{\kappa,\nu}(\mathbf{q}_{(i)}) \in \mathbb{R} \) such that the unique solution \( \tilde{\psi}_\kappa(\mathbf{q}_{(i)}, \cdot) \) of the Dirichlet problem:

\[
\begin{align*}
\Delta \tilde{\psi}_\kappa(\mathbf{q}_{(i)}, \cdot) &= 0 & \text{in } & \tilde{\mathcal{F}}(\mathbf{q}_{(i)}) \\
\tilde{\psi}_\kappa(\mathbf{q}_{(i)}, \cdot) &= \tilde{C}_{\kappa,\nu}(\mathbf{q}_{(i)}) & \text{on } & \partial \mathcal{S}_\nu(\mathbf{q}_{(i)}), \ \nu = 1, \ldots, N_{(i)}, \\
\tilde{\psi}_\kappa(\mathbf{q}_{(i)}, \cdot) &= 0 & \text{on } & \partial \Omega,
\end{align*}
\]

satisfies

\[
\int_{\partial \mathcal{S}_\nu(\mathbf{q}_{(i)})} \frac{\partial \tilde{\psi}_\kappa}{\partial n}(\mathbf{q}_{(i)}, \cdot) \, ds = -\delta_{\nu\kappa}, \ \nu = 1, \ldots, N_{(i)}.
\]

Biot-Savart kernel. Following \[22, 23\] we introduce two hydrodynamic Biot-Savart operators as follows. Given \( \omega \in L^2(\mathcal{F}) \), we define the velocities \( K[\omega] \) and \( \tilde{K}[\omega] \) as the solutions of

\[
\begin{align*}
\text{div } K[\omega] &= 0 & \text{in } & \mathcal{F}(\mathbf{q}), \\
\text{curl } K[\omega] &= \omega & \text{in } & \mathcal{F}(\mathbf{q}), \\
K[\omega] \cdot n &= 0 & \text{on } & \partial \mathcal{F}(\mathbf{q}), \\
\int_{\partial \mathcal{S}_\nu} K[\omega] \cdot \tau \, ds &= 0 & \text{for } & \nu = 1, \ldots, N,
\end{align*}
\]
and
\[
\begin{aligned}
\text{div } \tilde{K} + \omega &= 0 \quad \text{in } \tilde{\mathcal{F}}(q), \\
\text{curl } \tilde{K} + \omega &= 0 \quad \text{in } \tilde{\mathcal{F}}(q), \\
\tilde{K} \cdot n &= 0 \quad \text{on } \partial \tilde{\mathcal{F}}(q), \\
\int_{\partial S_{\nu}} \tilde{K} \tau \, ds &= 0 \quad \text{for } \nu = 1, \ldots, N_{(q)}.
\end{aligned}
\] (2.22)

These are the standard and the final Biot-Savart operators, respectively.

**Standard decomposition of the velocity field.** These potentials allow to decompose the velocity field \( u \) in several terms. Since it is the unique solution to the following div / curl system:
\[
\begin{aligned}
\text{div } u &= 0 \quad \text{in } \mathcal{F}(q), \\
\text{curl } u &= \omega \quad \text{in } \mathcal{F}(q), \\
\mathcal{L} u &= \mathcal{L} \omega \quad \text{on } \partial \mathcal{F}(q), \\
\int_{\partial S_{\nu}} u \tau \, ds &= \gamma_{\nu} \quad \text{for } \nu = 1, \ldots, N,
\end{aligned}
\] (2.23)

we have the standard decomposition of the velocity field \( u \):
\[
\begin{aligned}
u \in \{1, \ldots, N\}, \quad \iota \in \{1, 2, 3\},
\end{aligned}
\] (2.24)

We introduce the following notation for the first term in the decomposition: we let \( u^\text{pot} \) be the potential part of the fluid velocity:
\[
\begin{aligned}
u \in \{1, \ldots, N\}, \quad \iota \in \{1, 2, 3\},
\end{aligned}
\] (2.25)

Note that the velocity field \( u^* \) obtained in the limit (see (1.21)) can be decomposed as in (2.24) with the “final” quantities:
\[
\begin{aligned}
u \in \{1, \ldots, N_{(q)}\}, \quad \iota \in \{1, 2, 3\},
\end{aligned}
\] (2.26)

**2.3 Brief description of the proof and organization of the paper**

Let us now give a rough idea of the proof. One of the main difficulties to pass to the limit is to obtain uniform estimates as the sizes of the small solids go to zero. A standard energy estimate proves insufficient since the energy is not bounded as the size of small solids diminish (notice that the energy of a point vortex is infinite). The hardest case is the one of small and massless solids, for which the kinetic energy gives the weakest information. We explain first the main ideas to obtain uniform estimates in the case of a single solid (\( N = 1 \)), and then we explain some additional arguments needed in the case of several solids.

**Case of a single solid.** The starting point consists in decomposing the velocity field using the potentials described above. In particular, one extracts the singularity due to the fixed velocity circulation along the solid by decomposing \( u^* \) in the form:
\[
\begin{aligned}
u \in \{1, \ldots, N_{(q)}\}, \quad \iota \in \{1, 2, 3\},
\end{aligned}
\] (2.27)
where \( u^{reg} \) is the “regular part” of the velocity. Then we inject this decomposition in (1.7), which we can rewrite
\[
\mathcal{M}_p \hat{p}_{1,i} = -\int_{\partial \mathcal{F}} (\partial_t u + (u \cdot \nabla) u) \cdot \nabla \Phi_{1,i} \, dx.
\] (2.28)

The fact that we use the standalone circulation stream function in (2.27) allows to get rid of the most singular terms arising in the right hand side of (2.28) when using the decomposition (2.27). This is due to the following properties
\[
\partial_t \nabla \hat{\psi}_1 + \nabla (\nu_{S,1} \cdot \nabla \hat{\psi}_1) = 0 \quad \text{and} \quad \int_{\partial S_1} |\nabla \hat{\psi}_1|^2 K_{1,i} \, ds = 0,
\]
which will be proved in a more general setting in (4.26), and which allow to treat the terms containing \( \partial_t \nabla \hat{\psi}_1 \) and \( |\nabla \hat{\psi}_1|^2 \). Then the most singular remaining term is linear in \( \nabla \hat{\psi}_1 \). Studying this term, we see that, in order to have a chance to perform an energy estimate in which this term does not give a too strong contribution (we will say that this term is gyroscopic or more precisely weakly gyroscopic), it is necessary to consider a modulated variable
\[
\tilde{p} = p - \text{modulation}(\varepsilon, q, p, u^c).
\]

This modulation is imposed by the system, and one must incorporate it in the other terms of the equation and show that they do not contribute too strongly to the time evolution of the modulated energy associated with \( \tilde{p} \). This will give a normal form of the equation. To obtain this normal form, it is needed to decompose \( u^{reg} \) in (2.27) in a potential part \( u^{pot} \) (only due to the movement of the solid) and an “exterior” part \( u^{ext} \), this exterior part being actually the source of the modulation. The terms that arise when taking \( u^{pot}, u^{ext} \) and the modulation into account will either be proven to contribute mildly to the modulated energy or be incorporated in the estimate as added inertia terms.

Case of several solids. When several solids are present, a new serious difficulty appears: if we write a normal form such as described above for each small solid then the equations are coupled by terms associated with other solids which may include up to second derivatives in time. Because of this difficulty the strategy used in our previous papers [7, 8, 9, 10] seems to fail. To overcome this difficulty we use again normal forms of the ODEs driving the motion of the solids but in a two-steps process. First we use a normal form for the system coupling the time-evolution of all the solids to obtain a rough estimate of the acceleration of the bodies. Then we turn to normal forms that are specific to each small solid, with an appropriate modulation related to the influence of the other solids and of the fluid vorticity. This specific normal form allows in particular to take into account the specific scaling associated with each solid. The previous rough estimate of the acceleration is used here to prove that the coupling due to the acceleration of the other solids is weaker than expected in the limit. Then, thanks to these individual normal forms, we obtain precise uniform \( a \text{ priori} \) estimates of the velocities of the bodies.

After uniform estimates are obtained, we use compactness arguments to pass to the limit. The normal forms obtained above play a central role to describe the dynamics in the limit of the small solids. For what concerns the large solids, we must study in particular the convergence of the pressure near their boundary.

Organization of the sequel of the paper. A central tool to develop the arguments above is a careful description of the potentials used in the decomposition (2.24) of the velocity field. Indeed we analyze their behavior as the size of some of the solids go to zero, and of their derivative with respect to position. We use an extension of the reflection method for a div/curl system with prescribed circulations, see Section 3. In Section 4 we prove the first \( a \text{ priori} \) estimates on the system. This encompasses in particular vorticity estimates, (not yet modulated) energy estimates and the above-mentioned rough acceleration estimates. Then in Section 5 we describe the modulations, and explain in particular how they are determined and estimated. Then in Section 6 we establish our normal forms. This allows to obtain the modulated energy estimates in Section 7. Finally in Section 8 we pass to the limit.
3 Estimates on the potentials

In this section, we show how the various potentials appearing in the decomposition (2.24) of the velocity (including the Kirchhoff potentials $\varphi_{k,i}$, the circulation stream functions $\psi_k$ and the stream function associated with the Biot-Savart kernel $K[\omega]$) can be approximated and estimated by using in particular their standalone counterparts in $\mathbb{R}^3\setminus \mathcal{S}_c$ or their final counterparts in $\mathcal{F}$.

Convention on the higher-order H"older spaces. Throughout this section, we will take the following convention for the $C^{k,\alpha}$-seminorms, $k \geq 1, \alpha \in (0,1)$, when considered on a curve. The 0-th order H"older seminorms $|\cdot|_0$ are the standard ones, and for a open set $\mathcal{O}$ in $\mathbb{R}^2$, we also consider the same seminorms $|\cdot|_{C^{k,\alpha}(\mathcal{O})}$ as usual. For a smooth curve $\gamma$ on the plane and $k \geq 1$, we set for $f \in C^{k,\alpha}(\gamma)$:

$$|f|_{C^{k,\alpha}(\gamma)} := \inf \left\{ |u|_{C^{k,\alpha}(\mathcal{O})} : u \text{ is an extension of } f \text{ to some neighborhood } \mathcal{O} \text{ of } \gamma \right\}$$

$$\|f\|_{C^{k,\alpha}(\gamma)} := \|f\|_{\operatorname{Lip}(\gamma)} + |f|_{C^{k,\alpha}(\gamma)}.$$  

For a fixed curve $\gamma$, this is equivalent to the usual norm $\|f\|_x = |\partial_x f|_0$ (due to the existence of continuous extension operators), but the constants in this equivalence of norms are not uniform as a curve shrinks (due to curvature terms in $\partial_x f$).

To study the above mentioned potentials we begin the section by considering an auxiliary general problem.

3.1 An auxiliary Dirichlet problem

In this subsection we consider a general problem of Dirichlet type that will be helpful to study all the functions used in the decomposition (2.24) and their behavior as $\varkappa$ goes to 0. The general idea is that the Dirichlet boundary conditions will be merely satisfied up to an additive constant on each component of the boundary, but in return we impose a zero-flux condition on these components.

To be more specific, we consider the general situation of a domain $\Omega$ in which are embedded $N$ solids $\mathcal{S}_1, \ldots, \mathcal{S}_N$, such as described before. The fluid domain is then $\mathcal{F} := \Omega \setminus (\mathcal{S}_1 \cup \cdots \cup \mathcal{S}_N)$. Note that the results of this subsection will be applied not only to $\mathcal{F}^e$ such as described in the introduction, but also in other domains (such as $\mathcal{F}$ or a domain in which one of the small solids has been removed).

We consider $N$ functions $\alpha_\kappa \in C^\infty(\partial \mathcal{S}_\kappa; \mathbb{R})$, $\kappa = 1, \ldots, N$, and a function $\alpha_\Omega \in C^\infty(\partial \Omega; \mathbb{R})$, and study the following problem

$$\begin{aligned}
\Delta \delta[\alpha_1, \ldots, \alpha_N; \alpha_\Omega] &= 0 \quad \text{in } \mathcal{F}, \\
\delta[\alpha_1, \ldots, \alpha_N; \alpha_\Omega] &= \alpha_\Omega \quad \text{on } \partial \Omega, \\
\delta[\alpha_1, \ldots, \alpha_N; \alpha_\Omega] &= \alpha_\kappa + c_\kappa \quad \text{on } \partial \mathcal{S}_\kappa \quad \text{for } \kappa \in \{1, \ldots, N\}, \\
\int_{\partial \mathcal{S}_\kappa} \partial_\nu \delta[\alpha_1, \ldots, \alpha_N; \alpha_\Omega](x) \, ds(x) &= 0 \quad \text{for } \kappa \in \{1, \ldots, N\}.
\end{aligned}$$

where the unknowns are the function $\delta[\alpha_1, \ldots, \alpha_N; \alpha_\Omega]$ defined in $\mathcal{F}$ and the constants $c_1, \ldots, c_N$.

3.1.1 Existence of solutions for problem (3.2)

A general existence result. The existence of solutions to problem (3.2) is granted by the following statement. For the moment, all solids are considered of fixed size.

Lemma 3.1. Given $N$ functions $\alpha_\kappa \in C^\infty(\partial \mathcal{S}_\kappa; \mathbb{R})$, $\kappa = 1, \ldots, N$, and a function $\alpha_\Omega \in C^\infty(\partial \Omega; \mathbb{R})$, there exist a unique function $\delta[\alpha_1, \ldots, \alpha_N; \alpha_\Omega]$ and unique constants $c_1, \ldots, c_N$ solution to System (3.2).

Proof of Lemma 3.1. We first introduce the solution $\delta[\alpha_1, \ldots, \alpha_N; \alpha_\Omega]$ of the standard Dirichlet problem

$$\begin{aligned}
\Delta \delta[\alpha_1, \ldots, \alpha_N; \alpha_\Omega] &= 0 \quad \text{in } \mathcal{F}, \\
\delta[\alpha_1, \ldots, \alpha_N; \alpha_\Omega] &= \alpha_\Omega \quad \text{on } \partial \Omega, \\
\delta[\alpha_1, \ldots, \alpha_N; \alpha_\Omega] &= \alpha_\kappa \quad \text{on } \partial \mathcal{S}_\kappa \quad \text{for } \kappa \in \{1, \ldots, N\}.
\end{aligned}$$
Then we correct this solution by means of the following ones: for $\kappa \in \{1, \ldots, N\}$ one defines $h_\kappa$ as the unique solution to
\[
\begin{cases}
\Delta h_\kappa = 0 & \text{in } \mathcal{F}, \\
h_\kappa = 0 & \text{on } \partial \Omega, \\
h_\kappa = 0 & \text{on } \partial \mathcal{S}_\kappa, \\
h_\kappa = 1 & \text{on } \partial \mathcal{S}_\kappa,
\end{cases}
\]
Obviously, this family is linearly independent (it is connected to the first De Rham cohomology space of $\mathcal{F}$). Then it remains to prove that the linear mapping from $\text{Span}\{h_1, \ldots, h_N\}$ to $\mathbb{R}^N$, defined by
\[
\mathfrak{R} : h \mapsto \left(\int_{\partial \mathcal{S}_1} \partial_n h(x) \, ds(x), \ldots, \int_{\partial \mathcal{S}_N} \partial_n h(x) \, ds(x)\right)
\]
is an isomorphism. This is easy, since when $h$ belongs to its kernel, one has
\[
\int_{\mathcal{F}} |\nabla h|^2 \, dx = \int_{\partial \mathcal{F}} h \partial_n h \, ds(x) = 0.
\]
Hence since $h = 0$ on $\partial \Omega$, we deduce $h = 0$ in $\mathcal{F}$. \qed

**Uniform estimates for fixed sizes.** In the sequel, a case of particular interest is the case of the “final” fluid domain where all small solids have been removed (hence the fluid domain is larger). Therefore we consider a domain $\Omega$ in which are embedded $N(i)$ solids $S_1, \ldots, S_{N(i)}$ of fixed size, each of them being obtained by a rigid movement from a fixed shape, such as described before (in particular we still use the notation $S_i(q_i)$). The fluid domain is then $\tilde{\mathcal{F}} := \Omega \setminus (S_1 \cup \cdots \cup S_{N(i)})$. We obtain a sort of maximum principle for $\mathcal{D}[\alpha_1, \ldots, \alpha_{N(i)}; \alpha_0]$ as long as the solids remain a distance at least $\delta > 0$ one from another and from the outer boundary.

**Lemma 3.2.** Let $\delta > 0$. There exists a constant $C > 0$ depending merely on $\delta$, $\Omega$, and the shapes of $S_1, \ldots, S_{N(i)}$ such that for any
\[
q_{(i)} = (q_1, \ldots, q_{N(i)}) \in Q_{(i),\delta} := \left\{(q_1, \ldots, q_{N(i)}) \in \mathbb{R}^{3N(i)} \middle| \forall i \in \{1, \ldots, N(i)\}, \text{ dist}(S_i(q_i), \partial \Omega) > 2\delta \right\},
\]
for any functions $\alpha_\lambda \in C^{\infty}(\partial \mathcal{S}_\lambda; \mathbb{R})$, $\lambda = 1, \ldots, N(i)$ and any function $\alpha_0 \in C^{\infty}(\partial \Omega; \mathbb{R})$, one has
\[
\|\mathcal{D}[\alpha_1, \ldots, \alpha_{N(i)}; \alpha_0]\|_{L^\infty(\tilde{\mathcal{F}})} \leq C\|\alpha_1, \ldots, \alpha_{N(i)}; \alpha_0\|_{L^\infty(\partial \mathcal{F})}. \tag{3.4}
\]
In particular, $\mathcal{D}[\alpha_1, \ldots, \alpha_{N(i)}; \alpha_0]$ can be defined for any functions $\alpha_\lambda \in C^0(\partial \mathcal{S}_\lambda; \mathbb{R})$, $\lambda = 1, \ldots, N(i)$ and any function $\alpha_0 \in C^0(\partial \Omega; \mathbb{R})$.

Before getting to the proof of Lemma 3.2 we state the following uniform Schauder estimates, see e.g. \cite[p. 98]{6}.

**Lemma 3.3.** Let $\delta > 0$. There exists a uniform constant $C > 0$ such that for all $q_{(i)} \in Q_{(i),\delta}$ the following Schauder estimate holds for $u \in C^{2, 1/2}(\tilde{\mathcal{F}}(q_{(i)}))$:
\[
\|u\|_{C^{2, 1/2}(\tilde{\mathcal{F}}(q_{(i)}))} \leq C \left(\|\Delta u\|_{C^{1/2}(\tilde{\mathcal{F}}(q_{(i)}))} + \|u\|_{C^{2, 1/2}(\tilde{\mathcal{F}}(q_{(i)}))} \right).
\]

**Proof of Lemma 3.3.** First one establishes the result locally by using smooth diffeomorphisms close to the identity from $\mathcal{F}(q_{(i)})$ to $\tilde{\mathcal{F}}(q_{(i)})$ when $q_{(i)}$ is close to $q_{(i)}$. Using elliptic regularity for smooth operators with coefficients close to those of the Laplacian, this yields the result in the neighborhood of $q_{(i)}$. One concludes by compactness of $Q_{(i),\delta}$. We omit the details. \qed

We now prove Lemma 3.2.
Proof of Lemma 3.2. We consider \( \alpha_\lambda \in C^\infty(\partial \Omega; \mathbb{R}) \), \( \lambda = 1, \ldots, N(\iota) \) and \( \alpha_\Omega \in C^\infty(\partial \Omega; \mathbb{R}) \) and prove (3.4); the conclusion that \( \delta \) can be extended to continuous functions follows then immediately by density. We examine the proof of Lemma 3.1: we see that \( \delta [\alpha_1, \ldots, \alpha_N; \alpha_\Omega] \) satisfies the maximum principle, and hence (3.4). It remains to prove that the correction in \( \text{Span}\{h_1, \ldots, h_N\} \) can be estimated in the same way.

It follows from Lemma 3.3 that the functions \( h_\lambda \) are uniformly bounded in \( C^{2, \frac{1}{2}}(\tilde{\mathcal{F}}) \). This involves in particular that the integrals

\[
\int_{\partial \mathcal{S}_h} \partial_n \delta [\alpha_1, \ldots, \alpha_N; \alpha_\Omega] \, dx = \int_{\partial \mathcal{F}} \delta [\alpha_1, \ldots, \alpha_N; \alpha_\Omega] \partial_n h_\lambda \, dx, \quad \lambda = 1, \ldots, N,
\]
can be bounded uniformly in terms of \( \| (\alpha_1, \ldots, \alpha_N; \alpha_\Omega) \|_{L^\infty(\tilde{\mathcal{F}})} \). It remains to prove that the isomorphism \( \mathcal{R} \) defined in (3.3) is uniformly invertible for \( q(\iota) \in \mathcal{Q}(\iota, \delta) \). Let \( h \in \text{Span}\{h_1, \ldots, h_N\} \), say \( h = \sum_{\lambda=1}^{N(\iota)} \rho_\lambda h_\lambda \). We observe that for some positive constant \( C \):

\[
\sum_{\lambda \in \mathcal{P}(\iota)} |\rho_\lambda| \leq C \| h \|_{H^{1/2}(\partial \mathcal{F})}, \tag{3.5}
\]
since the functions in \( \text{Span}\{h_1, \ldots, h_N\} \) are constant on \( \partial \mathcal{F} \). Now we have

\[
\int_{\mathcal{F}} |\nabla h|^2 \, dx = \int_{\partial \mathcal{F}} h \partial_n h \, ds(x) \leq \sum_{\lambda \in \mathcal{P}(\iota)} |\rho_\lambda| \left( \int_{\partial \mathcal{S}_h} \partial_n h \, ds(x) \right) \leq C \| h \|_{H^{1/2}(\partial \mathcal{F})} \sum_{\lambda \in \mathcal{P}(\iota)} \left( \int_{\partial \mathcal{S}_h} \partial_n h \, ds(x) \right),
\]
where we have used that \( h = \rho_\lambda \) on \( \partial \mathcal{S}_h \). Moreover, by the trace inequality (which is uniform in \( \mathcal{Q}(\iota, \delta) \) by straightforward localization arguments),

\[
\| h \|_{H^{1/2}(\partial \mathcal{F})} \leq C \| h \|_{H^1(\mathcal{F})},
\]
and, since for \( h \) in \( \text{Span}\{h_1, \ldots, h_N\} \) we have \( h = 0 \) on \( \partial \Omega \), by Poincaré’s inequality (which is also uniform in \( \mathcal{Q}(\iota) \), since it merely depends on the diameter of the domain),

\[
\| h \|^2_{H^1(\mathcal{F})} \leq C \int_{\mathcal{F}} |\nabla h|^2 \, dx.
\]

Gathering the inequalities above we deduce that

\[
\| h \|^2_{H^{1/2}(\partial \mathcal{F})} \leq C \sum_{\lambda=1}^{N(\iota)} \left( \int_{\partial \mathcal{S}_h} \partial_n h \, ds(x) \right).
\]

The conclusion follows by using again (3.5).

\[\square\]

### 3.1.2 A potential for a standalone solid

Now we consider the situation where the single solid \( S_\kappa \), rather than being embedded in \( \Omega \) together with other solids \( S_\nu, \nu \neq \kappa \), is alone in the plane. This will play a central role in the description of the asymptotic behavior of the general potentials as some solids shrink to points.

To be more specific, we consider the solid \( S_\kappa \) obtained by a rigid movement and a homothety of scale \( \varepsilon_\kappa \) with respect to its counterpart of size 1 at initial position:

\[
S_\kappa = S_\kappa(\varepsilon_\kappa, \theta_\kappa) = h_\kappa + \varepsilon_\kappa R(\theta_\kappa)(S_\kappa^0 - h_\kappa, 0),
\]
and we study the above outer Dirichlet problem on \( \mathbb{R}^2 \setminus S_\kappa^\varepsilon \). Precisely we show the following.

**Proposition 3.4.** Let \( \varepsilon_\kappa > 0 \) and let \( \alpha \in C^\infty(\partial S^\varepsilon_\kappa; \mathbb{R}) \). Then there exists a unique constant \( \hat{\varepsilon}_\kappa[\alpha] \) and a unique function \( \hat{f}_\kappa[\alpha] \in C^\infty(\mathbb{R}^2 \setminus S^\varepsilon_\kappa) \) solution to the system

\[
\begin{cases}
\Delta \hat{f}_\kappa[\alpha] = 0 & \text{in } \mathbb{R}^2 \setminus S^\varepsilon_\kappa, \\
\hat{f}_\kappa[\alpha](x) = \alpha + \hat{\varepsilon}_\kappa[\alpha] & \text{on } \partial S^\varepsilon_\kappa, \\
\hat{f}_\kappa[\alpha](x) \to 0 & \text{as } |x| \to +\infty.
\end{cases}
\tag{3.6}
\]
Moreover one has the following estimates, where the constant \( C \) merely depends on \( S^1_{k,0} \) and \( k \in \mathbb{N} \setminus \{0, 1\} \) (hence is independent of \( \varepsilon_k \)):

\[
\| \hat{h}_k[\alpha] \|_{L^\infty(\mathbb{R}^2 \setminus S^1_{\varepsilon_k,0})} \leq 2|\alpha|\|\alpha\|_{L^\infty(\mathbb{R}^2 \setminus S^1_{\varepsilon_k,0})} \quad \text{and} \quad |\hat{c}_k[\alpha]| \leq |\alpha|\|\alpha\|_{L^\infty(\mathbb{R}^2 \setminus S^1_{\varepsilon_k,0})},
\]

\[
\varepsilon_k \| \nabla \hat{h}_k[\alpha] \|_{L^\infty(\mathbb{R}^2 \setminus S^1_{\varepsilon_k,0})} + \varepsilon_k^{k+\frac{1}{2}} \hat{h}_k[\alpha] \|_{C^{k+\frac{1}{2}}(\mathbb{R}^2 \setminus S^1_{\varepsilon_k,0})} \leq C \left( |\alpha|\|\alpha\|_{L^\infty(\mathbb{R}^2 \setminus S^1_{\varepsilon_k,0})} + \varepsilon_k^{k+\frac{1}{2}}|\alpha|_{C^{k+\frac{1}{2}}(\mathbb{R}^2 \setminus S^1_{\varepsilon_k,0})} \right),
\]

and

\[
\forall x \text{ s.t. } |x - h_\alpha| \geq C \varepsilon_k, \quad |\hat{h}_k[\alpha](x)| \leq C \frac{\varepsilon_k}{|x - h_\alpha|}\|\alpha\|_{L^\infty(\mathbb{R}^2 \setminus S^1_{\varepsilon_k,0})}
\]

\[
\text{and } |\nabla \hat{h}_k[\alpha](x)| \leq C \frac{\varepsilon_k}{|x - h_\alpha|^2}\|\alpha\|_{L^\infty(\mathbb{R}^2 \setminus S^1_{\varepsilon_k,0})}.
\]

\section*{Remark 3.5.} Notice that Estimate (3.9) and the divergence theorem involve that

\[
\int_{\partial S^1_x} \hat{c}_k[\alpha] \, ds = 0.
\]

\section*{Proof of Proposition 3.4.} We proceed in two steps.

\section*{Step 1.} We first consider the case when \( \varepsilon_k = 1 \). Since the above estimates are invariant by translation and rotation, without loss of generality, we can suppose that \( \theta_\alpha = 0 \) and that 0 is in the interior of \( S^1_{1,0} \). Identifying \( \mathbb{R}^2 \) and \( \mathbb{C} \), we use the inversion \( z \mapsto 1/z \) with respect to 0. Denoting the Riemann sphere by \( \hat{\mathbb{C}} \), we set \( \Omega' := \left\{ 1/z, \, z \in \hat{\mathbb{C}} \setminus S^1_1 \right\} \) (which is a regular bounded domain since 0 is in the interior of \( S^1_{1,0} \)), and consider the Dirichlet problem:

\[
\Delta \theta = 0 \text{ in } \Omega' \text{ and } \theta(z) = \alpha(1/z) \text{ for } z \in \partial \Omega'.
\]

Notice that 0 \( \in \partial \Omega' \) because it is the image of the point at infinity by the inversion \( z \mapsto 1/z \). Then we can set for \( z \in S^1_{1,0} :\)

\[
\hat{\theta}[\alpha](z) = \theta(1/z) - \theta(0) \quad \text{and} \quad \hat{c}_k[\alpha] = -\theta(0).
\]

By conformality of the inversion \( z \mapsto 1/z \), this function satisfies (3.6). Conversely, starting from a solution to (3.6), we can invert and obtain a solution to (3.11) up to an additive constant on the boundary condition, which we remove. This proves the uniqueness of the solution to (3.6).

Now (3.7) is a direct consequence of (3.12) and of the maximum principle. Estimate (3.8) is also a consequence of (3.12): we make use of Schauder’s estimates in \( \Omega' \), then we invert using that \( d(\mathbb{R}^2 \setminus S^1_{1,0}, 0) > 0 \). Let us now focus on (3.9). The function

\[
\eta(z) := \hat{c}_z[\theta](z) = \hat{c}_x[\theta](z) - i\hat{c}_y[\theta](z)
\]

is holomorphic in \( \Omega' \). We call \( a_k(\eta), \, k \in \mathbb{N} \), the coefficients of its power series expansion at 0, so that

\[
\eta(z) = \sum_{k \geq 0} a_k(\eta) z^k.
\]

We introduce \( r > 0 \) such that the circle \( S(0, r) \) lies inside \( \Omega' \) at positive distance from \( \partial \Omega' \). Using interior elliptic estimates (see e.g. [6, Theorem 2.10, p. 23]), we see that \( \|\eta\|_{C^n(S(0, r))} \leq C|\alpha|\|\alpha\|_{L^\infty(\mathbb{R}^2 \setminus S^1_{\varepsilon_k,0})} \) for some constant \( C > 0 \) merely depending on \( S^1_{1,0} \). Then, by using the Cauchy integral formula on \( S(0, r) \), we deduce that there exists \( C_\alpha > 0 \) depending only on \( S^1_{1,0} \) such that \( |a_k(\eta)| \leq C_\alpha^k|\alpha|_{L^\infty(\mathbb{R}^2 \setminus S^1_{\varepsilon_k,0})} \) for all \( k \in \mathbb{N} \). Now, by (3.12), (3.13) and (3.14),

\[
\hat{c}_z[\alpha](z) = \frac{1}{z^2} \sum_{k \geq 0} a_k(\eta) z^k.
\]

Thus \( |\nabla \hat{\theta}[\alpha](x)| \leq C_\alpha|z|^{-2}|\alpha|_{L^\infty(\mathbb{R}^2 \setminus S^1_{\varepsilon_k,0})} \) for \( |z| \) large enough, for instance \( |z - h_\alpha| \geq 2C_\alpha \). But for \( |z - h_\alpha| \) large enough (depending on \( S^1_{1,0} \) only) we have that \( |z - h_\alpha| \leq 2|z| \). Hence we deduce the second inequality in (3.9), and then the first one by integration from infinity.
Step 2. Obtaining the estimates for arbitrary \( \varepsilon_\kappa > 0 \) is just a matter of rescaling. We call \( \hat{\mathcal{F}}_\kappa \) the potential obtained above in the exterior domain \( \mathbb{R}^2 \setminus S^\kappa \) and \( \hat{\mathcal{F}}_\kappa \) the corresponding potential in \( \mathbb{R}^2 \setminus \bar{S}^\kappa \). Given \( \alpha \in C^\infty(\bar{\mathcal{C}}_\kappa; \mathbb{R}) \) we set \( \alpha^\varepsilon(x) = \alpha(\varepsilon_\kappa x) \) defined on \( \mathcal{C}_\kappa \). Then clearly
\[
\forall x \in \mathbb{R}^2 \setminus S^\kappa, \quad \hat{\mathcal{F}}_\kappa[\alpha](x) = \hat{\mathcal{F}}_\kappa[\alpha^\varepsilon](x/\varepsilon_\kappa), \quad \nabla \hat{\mathcal{F}}_\kappa[\alpha](x) = \frac{1}{\varepsilon_\kappa} \nabla \hat{\mathcal{F}}_\kappa[\alpha^\varepsilon](x/\varepsilon_\kappa).
\]
The estimates (3.7)–(3.9) follow; Estimate (3.8) in particular is just the rescaled Schauder estimate (note that the seminorms defined in (3.1) scale in the same way as Hölder seminorms on open sets).

3.1.3 A construction of the potential in the presence of small solids

Now we consider again the situation of a domain \( \Omega \) in which are embedded \( N \) solids, among which \( N_{(i)} \) stay of fixed size and \( N_q \) are shrinking. The only constraints that we will use is dist(\( \partial S_{\kappa}, \partial S_q \)) \( \geq \delta \) for \( \kappa \neq \nu \) and dist(\( \partial S_{\kappa}, \partial \Omega \)) \( \geq \delta \) for all \( \kappa \) where \( \delta > 0 \) is fixed. The constants that follow will merely depend on \( \delta, \Omega \) and on the shape of the unscaled solids \( S^\kappa \) at size 1. In particular they are independent of \( \varepsilon_{N_{(i)}+1}, \ldots, \varepsilon_N \) (as long as they are small enough) and of the exact positions of the solids (as long as the above constraints are satisfied).

In this context we give a particular construction of \( \mathcal{F}[\alpha_1, \ldots, \alpha_N; \alpha_\Omega] \), inspired by the method of successive reflections (see e.g. [20] and references therein). The solution \( \mathcal{F}[\alpha_1, \ldots, \alpha_N; \alpha_\Omega] \) will be obtained by means of the inversion of an operator on
\[
(\eta_1, \ldots, \eta_N, \eta_\Omega) \in E_{\mathcal{F}} := C^0(\partial S_1) \times \cdots \times C^0(\partial S_N) \times C^0(\partial \Omega),
\]
which will be a perturbation of the identity by a contractive map.

Let us describe this contractive map. We first recall that \( \hat{\mathcal{F}} \) refers to the larger fluid domain where the small solids have been removed, see (1.16). Correspondingly, \( \hat{\mathcal{F}} = \partial S_1 \cup \cdots \cup \partial S_{N_{(i)}} \cup \partial \Omega \). Now given \( (\eta_1, \ldots, \eta_N, \eta_\Omega) \in E_{\mathcal{F}} \) we first introduce \( \tilde{\mathcal{G}} = \hat{\mathcal{G}}[\eta_1, \ldots, \eta_{N_{(i)}}; \eta_\Omega] \) and \( \tilde{\mathcal{C}}_\lambda = \hat{\mathcal{C}}_\lambda[\eta_1, \ldots, \eta_{N_{(i)}}; \eta_\Omega] \) as the solution in \( \hat{\mathcal{F}} \) of the Dirichlet problem
\[
\begin{aligned}
-\Delta \tilde{\mathcal{G}} &= 0 \quad \text{in } \hat{\mathcal{F}}, \\
\tilde{\mathcal{G}} &= \eta_\Omega \quad \text{on } \partial \Omega, \\
\tilde{\mathcal{C}}_\lambda &= \lambda \quad \text{on } \partial S_\kappa, \quad \forall \lambda = 1, \ldots, N_{(i)}, \\
\int_{\partial S_\kappa} \tilde{\mathcal{G}} \, dS(x) &= 0, \quad \forall \lambda = 1, \ldots, N_{(i)}.
\end{aligned}
\tag{3.15}
\]
This problem has a solution as described in Lemma 3.2. Note in particular that Lemma 3.2 brings the following estimate:
\[
\|\tilde{\mathcal{G}}\|_{L^p(\hat{\mathcal{F}})} \leq C\|(\eta_1, \ldots, \eta_{N_{(i)}}, \eta_\Omega)\|_{L^p(\partial S_1 \times \cdots \times \partial S_{N_{(i)}} \times \partial \Omega)}.
\tag{3.16}
\]

Next we introduce the function \( m = m[\eta_1, \ldots, \eta_N; \eta_\Omega] \) in \( \mathcal{F} \) by
\[
m := \tilde{\mathcal{G}} + \sum_{\lambda \in \mathcal{P}_s} \hat{\mathcal{F}}_\lambda[\eta_\lambda - \tilde{\mathcal{G}}_{\mathcal{C}_\lambda}] \quad \text{with} \quad \tilde{\mathcal{G}} = \hat{\mathcal{G}}[\eta_1, \ldots, \eta_{N_{(i)}}; \eta_\Omega],
\tag{3.17}
\]
where as in (1.13), we have denoted \( \mathcal{P}_s = \{N_{(i)} + 1, \ldots, N\} \) the set of indices for shrinking solids. Note that \( m \) is the unique solution to the following Dirichlet problem of type (3.2) (for some constants \( c_1, \ldots, c_N \)):
\[
\begin{aligned}
-\Delta m &= 0 \quad \text{in } \mathcal{F}, \\
m &= \eta_\Omega + \sum_{\lambda \in \mathcal{P}_s} \hat{\mathcal{F}}_\lambda[\eta_\lambda - \tilde{\mathcal{G}}_{\mathcal{C}_\lambda}] \quad \text{on } \partial \Omega, \\
m &= \eta_\nu + \sum_{\lambda \in \mathcal{P}_s \setminus \{\nu\}} \hat{\mathcal{F}}_\lambda[\eta_\lambda - \tilde{\mathcal{G}}_{\mathcal{C}_\lambda}] + c_\nu \quad \text{on } \partial S_\nu \quad \text{for } \nu \in \mathcal{P}_{(i)}, \\
m &= \eta_\nu + \sum_{\lambda \in \mathcal{P}_s \setminus \{\nu\}} \hat{\mathcal{F}}_\lambda[\eta_\lambda - \tilde{\mathcal{G}}_{\mathcal{C}_\lambda}] + c_\nu \quad \text{on } \partial S_\nu \quad \text{for } \nu \in \mathcal{P}, \\
\int_{\partial S_\nu} c_\nu m \, dS(x) &= 0, \quad \forall \nu = 1, \ldots, N,
\end{aligned}
\tag{3.18}
\]
where for the last equation we have used (3.6), (3.10), (3.15) and the divergence theorem.

21
Our goal is to prove that one can put the solution \( \delta[\alpha_1, \ldots, \alpha_N; \alpha_\Omega] \) of (3.2) in the form \( \mathbf{m}[\eta_1, \ldots, \eta_N; \eta_\Omega] \) with \( \eta_1, \ldots, \eta_N, \eta_\Omega \) determined from \( \alpha_1, \ldots, \alpha_N, \alpha_\Omega \). For that we define the operator \( \mathcal{T} : E_{\bar{\Omega}} \to E_{\bar{\Omega}} \) by

\[
\mathcal{T}[\eta_1, \ldots, \eta_N; \eta_\Omega] := \begin{cases} 
\sum_{\lambda \in \mathcal{P}_\mathcal{S}_1} \widehat{\lambda}[\eta_\lambda - \bar{\eta}_{\bar{\Omega} \mathcal{S}_\mathcal{S}_1}] & \text{on } \partial \bar{F} = \partial S_1 \cup \cdots \cup \partial S_{N(i)} \cup \partial \Omega, \\
\sum_{\lambda \in \mathcal{P}_\mathcal{S}_1 \setminus \{\lambda\}} \widehat{\lambda}[\eta_\lambda - \bar{\eta}_{\bar{\Omega} \mathcal{S}_\mathcal{S}_1}] & \text{on } \partial \mathcal{S}_\nu, \text{ for } \nu \in \mathcal{P}_\mathcal{S}.
\end{cases}
\]

where again \( \bar{\eta} = \bar{\eta}[\eta_1, \ldots, \eta_{N(i)}; \eta_\Omega] \). Then

\[
\mathbf{m}[\eta_1, \ldots, \eta_N; \eta_\Omega] = \begin{cases} 
\mathbf{m}[\eta_1, \ldots, \eta_N; \eta_\Omega] & \text{on } \partial \Omega, \\
\mathbf{m}[\eta_1, \ldots, \eta_N; \eta_\Omega] + c_\nu & \text{on } \partial \mathcal{S}_\nu, \nu = 1, \ldots, N.
\end{cases}
\]

Now we have the following lemma, where we recall that \( \bar{\eta} = (\varepsilon_{N(i)} + 1, \ldots, \varepsilon_N) \).

**Lemma 3.6.** There exists \( \varepsilon_0 > 0 \) depending only on \( \delta, \Omega \) and on the shape of the unscaled solids \( \mathcal{S}_1 \) such that if \( \varepsilon \leq \varepsilon_0 \), then \( \mathcal{T} \) is a \( \frac{1}{2} \)-contraction.

**Proof of Lemma 3.6.** The main argument is that the value of \( \mathcal{T}[\eta_1, \ldots, \eta_N; \eta_\Omega] \) on a connected component of the boundary, say \( \partial \mathcal{S}_\nu \), is actually given by a sum of restrictions on \( \partial \mathcal{S}_\nu \) of potentials generated on other connected components of the boundary (and the same holds for \( \partial \Omega \)). We first see that by Lemma 3.2, \( \bar{\eta} \) satisfies (3.16). Then we use (3.9): for \( \nu \neq \lambda \), this allows to estimate \( \widehat{\lambda}[\eta_\lambda - \bar{\eta}_{\bar{\Omega} \mathcal{S}_\mathcal{S}_1}] \) on the \( \delta \)-neighborhood \( \mathcal{V}_\delta(\partial \mathcal{S}_\nu) \) of \( \partial \mathcal{S}_\nu \) (see (2.6)) by

\[
\left\| \widehat{\lambda}[\eta_\lambda - \bar{\eta}_{\bar{\Omega} \mathcal{S}_1}] \right\|_{L^p(\mathcal{V}_\delta(\partial \mathcal{S}_\nu))} \leq C \varepsilon_{\lambda}( \left\| (\eta_1, \ldots, \eta_{N(i)}, \eta_\Omega) \right\|_{L^p(\delta S_1 \times \cdots \times \varepsilon S_{N(i)} \times \varepsilon \Omega)} + \left\| \eta_\lambda \right\|_{L^p(\delta \Omega)}),
\]

and the same holds for \( \mathcal{V}_\delta(\partial \Omega) \).

By the definition (3.19) of \( \mathcal{T} \), we deduce that on \( \partial \bar{F} = \partial S_1 \cup \cdots \cup \partial S_{N(i)} \cup \partial \Omega \),

\[
\left\| \mathcal{T}[\eta_1, \ldots, \eta_N; \eta_\Omega] \right\|_{L^p(\partial \bar{F})} \leq C \sum_{\lambda \in \mathcal{P}_\mathcal{S}_1} \varepsilon_{\lambda}( \left\| \eta_\lambda \right\|_{L^p(\delta \mathcal{S}_1)} + \left\| (\eta_1, \ldots, \eta_{N(i)}, \eta_\Omega) \right\|_{L^p(\delta S_1 \times \cdots \times \varepsilon S_{N(i)} \times \varepsilon \Omega)} + \left\| \eta_\lambda \right\|_{L^p(\delta \Omega)}),
\]

while on \( \partial \mathcal{S}_\nu \) for \( \nu \in \mathcal{P}_\mathcal{S} \), we get

\[
\left\| \mathcal{T}[\eta_1, \ldots, \eta_N; \eta_\Omega] \right\|_{L^p(\partial \mathcal{S}_\nu)} \leq C \sum_{\lambda \in \mathcal{P}_\mathcal{S}_1 \setminus \{\lambda\}} \varepsilon_{\lambda}( \left\| \eta_\lambda \right\|_{L^p(\delta \mathcal{S}_1)} + \left\| (\eta_1, \ldots, \eta_{N(i)}, \eta_\Omega) \right\|_{L^p(\delta S_1 \times \cdots \times \varepsilon S_{N(i)} \times \varepsilon \Omega)})
\]

Hence the operator \( \mathcal{T} \) is a \( \frac{1}{2} \)-contraction if \( \varepsilon \) is small enough.

Now we consider such an \( \varepsilon \). From Lemma 3.6 we infer that \( \text{Id} + \mathcal{T} \) is invertible. We deduce the following lemma.

**Lemma 3.7.** Given \( (\alpha_1, \ldots, \alpha_N; \alpha_\Omega) \) in \( E_{\bar{\Omega}} \) we introduce

\[
(\beta_1, \ldots, \beta_N, \beta_\Omega) := (\text{Id} + \mathcal{T})^{-1}(\alpha_1, \ldots, \alpha_N, \alpha_\Omega).
\]

Then

\[
\delta[\alpha_1, \ldots, \alpha_N; \alpha_\Omega] = \mathbf{m}[\beta_1, \ldots, \beta_N; \beta_\Omega].
\]

**Proof of Lemma 3.7.** From (3.18), (3.20) and (3.22), we see that \( \mathbf{m}[\beta_1, \ldots, \beta_N; \beta_\Omega] \) is the unique solution to (3.2) corresponding to the boundary data \( (\alpha_1, \ldots, \alpha_N; \alpha_\Omega) \).

We finish this paragraph by noticing the fact that \( \mathcal{T} \) has important regularizing properties. Recall that \( \delta \) was introduced at the beginning of Subsection 3.1.3.
Lemma 3.8. Given \( \delta > 0 \), there exists \( \varepsilon_0 > 0 \) such that for all \( \varpi \) with \( \varpi \leq \varepsilon_0 \), for all \( k \in \mathbb{N} \), there exists a positive constant \( C \) merely depending on \( k, \delta, \Omega \) and on the unscaled solids \( S^1_\lambda \) such that for any \( (\eta_1, \ldots, \eta_N; \eta_0) \in E_{\delta, \varpi} \), one has

\[
\| T(\eta_1, \ldots, \eta_N; \eta_0) \|_{C^{1, \frac{1}{2}}(\mathbb{R}^2)} \leq C \| (\eta_1, \ldots, \eta_N; \eta_0) \|_{\mathcal{X}}. 
\]

Proof of Lemma 3.8. We introduce for each \( \nu \in \{1, \ldots, N\} \) a neighborhood \( U_\nu \) of \( \partial S_\nu \) of size \( O(\delta) \), and hence independent of \( \varepsilon_\nu \). More precisely, for \( \nu \in \mathcal{P}_s \), we let \( U_\nu = V_{\delta/2}(S_\nu) \) (where we recall the notation (2.6)). For \( \nu \in \mathcal{P}_s \), we let \( U_\nu = B(h_\nu, \delta/2) \) and we notice that for suitably small \( \varpi \), one has \( S_\nu \subset B(h_\nu, \delta/8) \). We also introduce some neighborhood \( U'_\nu \) of \( S_\nu \) depending only on \( \delta \) and satisfying \( \overline{U'_\nu} \subset U_\nu \): for instance for \( \nu \in \mathcal{P}_s \), we consider \( U'_\nu = V_{\delta/4}(S_\nu) \) and for \( \nu \in \mathcal{P}_s \), we let \( U'_\nu = B(h_\nu, \delta/4) \). In the same way we introduce the \( \delta/2 \)-neighborhood (respectively \( \delta/4 \)-neighborhood) \( U_0 \) (resp. \( U'_0 \)) of \( \partial \Omega \). Then by interior elliptic regularity estimates we find a positive constant \( C = C(k, U_\nu, U'_\nu) \) such that for any harmonic function \( f \) on \( U_\nu \) one has

\[
\| f \|_{C^{1, \frac{1}{2}}(U'_\nu)} \leq C \| f \|_{L^\infty(U_\nu)}. 
\]

We apply it to \( \widehat{f}_\lambda[\eta_\nu - \eta_0] \) for \( \lambda \neq \nu \) to get a Hölder estimate on \( U'_\nu \) and restrict it to \( \partial S_\nu \) and \( \partial \Omega \) (which is trivial with the convention (3.1)). Finally we use (3.19) and (3.21). This ends the proof of Lemma 3.8.

\[\square\]

3.1.4 Asymptotic behavior for problem (3.2)

In this paragraph we study the behavior of the solutions (3.2) as some of the embedded solids shrink to points. Let \( \varpi \) satisfy the assumptions of Lemma 3.6. We consider a particular case of \( \delta[\alpha_1, \ldots, \alpha_N; \alpha] \), when all \( \alpha_k \) but one are zero and \( \alpha_0 = 0 \) as well. Let \( \kappa \in \{1, \ldots, N\} \) and \( \alpha_\kappa \in C^0(\partial S_\kappa; \mathbb{R}) \). We denote

\[
f_\kappa[\alpha_\kappa] := \delta[0, \ldots, 0, \alpha_\kappa, 0, \ldots, 0, 0, 0], \tag{3.23}
\]

where \( \alpha_\kappa \) is on the \( \kappa \)-th position. The first result of this section, concerning the case when the \( \kappa \)-th solid is small, is the following one. We recall the notation \( \mathcal{P}_s \) for the set of indices for shrinking solids, see (1.13), and the notation (2.6) for a \( \nu \)-neighborhood.

Proposition 3.9. Let \( \delta > 0 \). There exists \( \varepsilon_0 > 0 \) such that the following holds. There exists a constant \( C > 0 \) depending only on \( \delta, \Omega, k \geq 2 \) and the reference solids \( S^1_\lambda, \lambda = 1, \ldots, N \), such that for any \( \varpi \) such that \( \varpi \leq \varepsilon_0 \), for any \( \kappa \in \mathcal{P}_s \), for any \( q \in Q_\delta \), for any \( \alpha^\varpi \in C^\infty(\partial S_\kappa^\varpi; \mathbb{R}) \), one has

\[
\| \nabla f_\kappa[\alpha^\varpi] - \nabla \hat{f}_\kappa[\alpha^\varpi] \|_{L^\infty(F^\varpi)} \leq C \varepsilon_\kappa \| \alpha^\varpi \|_{L^\infty(\partial S^\varpi_\kappa)}, \tag{3.24}
\]

\[
\| f_\kappa[\alpha^\varpi] \|_{C^{1, \frac{1}{2}}(\partial S^\varpi_\kappa)} + \sum_{\lambda \in \mathcal{P}_s \setminus \{\kappa\}} \varepsilon^\lambda \| f_\kappa[\alpha^\varpi] \|_{C^{1, \frac{1}{2}}(\partial S^\varpi_\kappa)} \leq C \varepsilon_\kappa \| \alpha^\varpi \|_{L^\infty(\partial S^\varpi_\kappa)}, \tag{3.25}
\]

where \( f_\kappa[\alpha^\varpi] \in C^\infty(F^\varpi(q)) \) is the unique solution given by (3.23), \( \hat{f}_\kappa[\alpha^\varpi] \in C^\infty(\mathbb{R}^2 \setminus S^\varpi_\kappa) \) is the unique solution to (3.6).

Let us highlight that there is no Hölder norm in the right hand side of (3.25), as opposed to (3.8).

Proof of Proposition 3.9. First, we fix \( \varepsilon_0 \) so that Lemma 3.6 and Lemma 3.8 apply. We let the \( (N + 1) \)-tuple \( \mathbf{A} \) be

\[
\mathbf{A} := (0, \ldots, 0, \alpha, 0, \ldots, 0, 0),
\]

where \( \alpha \) is on the \( \kappa \)-th position and we introduce

\[
\mathbf{B} = (\beta_1, \ldots, \beta_N, \beta_0) := (I + T)^{-1}(\mathbf{A}). \tag{3.26}
\]

Then according to Lemma 3.7 we have

\[
f_\kappa[\alpha] = m[\mathbf{B}] \text{ in } \mathcal{F}. \tag{3.27}
\]
Now relying on (3.17), we arrive at the formula
\[ f_\kappa[\alpha] - \hat{f}_\kappa[\alpha] = \tilde{g}_\beta + \sum_{\lambda \in \mathcal{P}_\kappa} \hat{\lambda} \left[ \beta_\lambda \right] \text{ in } \mathcal{T}, \tag{3.28} \]
with
\[ \tilde{g}_\beta := \tilde{g}[\beta_1, \ldots, \beta_{N(i)}; \beta_\Omega] \text{ and for } \lambda \in \mathcal{P}_\kappa, \hat{\lambda} := \begin{cases} \beta_\lambda - \tilde{g}_\beta|_{\partial \mathcal{S}_\kappa} & \text{when } \lambda \neq \kappa, \\
\beta_\lambda - \tilde{g}_\beta|_{\partial \mathcal{S}_\kappa} - \alpha & \text{when } \lambda = \kappa. \end{cases} \tag{3.29} \]

Our goal is to estimate the right-hand side of (3.28). A first step is to estimate \( B - A \). To that purpose we first notice that
\[ B - A = -\mathcal{T}(B) = -\mathcal{T} \circ (I + \mathcal{T})^{-1}(A). \tag{3.30} \]
Due to Lemma 3.6, we have \( \|(I + \mathcal{T})^{-1}\|_{C^k(\partial \mathcal{F})} \leq 2 \), so in particular we deduce
\[ \|B - A\|_{L^\infty(\partial \mathcal{F})} \leq \|\mathcal{T}\|_{L^\infty(\partial \mathcal{F})}. \tag{3.31} \]
Now when computing \( \mathcal{T}(A) \) with (3.19), we see that the function \( \tilde{g} \) involved in (3.19) and the constants \( \tilde{c}_\lambda \) from (3.15) are zero because the only non-trivial boundary data \( \alpha \) is located on a small solid \( \mathcal{S}_\kappa, \kappa \in \mathcal{P}_\kappa \). Hence (3.19) gives
\[ \mathcal{T}(A) = \begin{cases} \hat{f}_\kappa[\alpha] & \text{on } \partial \Omega \text{ and on } \partial \mathcal{S}_\lambda \text{ for } \lambda \in \{1, \ldots, N\} \backslash \{\kappa\}, \\
0 & \text{on } \partial \mathcal{S}_\kappa. \end{cases} \tag{3.32} \]
We deduce from (3.31), (3.32), (3.9) and the separation between the connected components of the boundary, that
\[ B = A + \mathcal{O}(\varepsilon_\kappa \|\alpha\|_{L^\infty(\partial \mathcal{S}_\kappa)}) \text{ in } L^\infty(\partial \mathcal{F}). \tag{3.33} \]
Now we obtain higher order estimates. By (3.32), (3.9) and interior elliptic regularity estimates, \( \|\mathcal{T}(A)\|_{C^{k,\frac{1}{2}}(\partial \mathcal{F})} \leq C\varepsilon_\kappa \|\alpha\|_{L^\infty(\partial \mathcal{S}_\kappa)} \). By (3.33) and Lemma 3.8, \( \|\mathcal{T}(B - A)\|_{C^{k,\frac{1}{2}}(\partial \mathcal{F})} \leq C\varepsilon_\kappa \|\alpha\|_{L^\infty(\partial \mathcal{S}_\kappa)} \). We deduce
\[ \|\mathcal{T}(B)\|_{C^{k,\frac{1}{2}}(\partial \mathcal{F})} \leq C\varepsilon_\kappa \|\alpha\|_{L^\infty(\partial \mathcal{S}_\kappa)}, \tag{3.34} \]
which together with (3.30) gives
\[ B = A + \mathcal{O}(\varepsilon_\kappa \|\alpha\|_{L^\infty(\partial \mathcal{S}_\kappa)}) \text{ in } C^{k,\frac{1}{2}}(\partial \mathcal{F}). \tag{3.35} \]
Now the terms in the right-hand side of (3.28) can be estimated as follows. By (3.35), the fact that \( A_i = 0 \) for \( i = 1, \ldots, N(i) \), uniform Schauder estimates in \( \tilde{F} \) (Lemma 3.3) and (3.29),
\[ \|
\tilde{g}_\beta\|_{C^{k,\frac{1}{2}}(\partial \mathcal{F})} \leq C\varepsilon_\kappa \|\alpha\|_{L^\infty(\partial \mathcal{S}_\kappa)}. \tag{3.36} \]
Let us now turn to the estimate of \( \hat{\lambda}[\tilde{\beta}_\lambda], \lambda \in \mathcal{P}_\kappa \). From \( B - A = (\beta_1 - \delta_{\kappa,1}\alpha, \ldots, \beta_N - \delta_{\kappa,N}\alpha, \beta_\Omega) \), (3.35), (3.36) and (3.29), we infer that for all \( \lambda \in \mathcal{P}_\kappa \), \( \|
\tilde{\beta}_\lambda\|_{C^{k,\frac{1}{2}}(\partial \mathcal{F})} \leq C\varepsilon_\kappa \|\alpha\|_{L^\infty(\partial \mathcal{S}_\kappa)}. \)
Recalling the convention (3.1) we deduce that
\[ \left\| \bar{\beta}_\lambda - \frac{1}{|\partial \mathcal{S}_\lambda|} \int_{\partial \mathcal{S}_\lambda} \bar{\lambda} \right\|_{L^\infty(\partial \mathcal{F})} \leq C\varepsilon_\kappa \varepsilon_\lambda \|\alpha\|_{L^\infty(\partial \mathcal{S}_\kappa)}. \]
Using (3.8) on the solid \( \mathcal{S}_\lambda \) and the fact that the operators \( \hat{\lambda} \) do not see constants we deduce
\[ \forall \lambda \in \mathcal{P}_\kappa, \quad \| \nabla \hat{\lambda}[\tilde{\beta}_\lambda] \|_{L^\infty(\partial \mathcal{F} \cup \partial \mathcal{S}_\lambda)} + \varepsilon_\lambda^{\frac{1}{2}} \| \hat{\lambda}[\tilde{\beta}_\lambda] \|_{C^{k,\frac{1}{2}}(\partial \mathcal{F} \cup \partial \mathcal{S}_\lambda)} \leq C\varepsilon_\kappa \|\alpha\|_{L^\infty(\partial \mathcal{S}_\kappa)}. \tag{3.37} \]
Then interior regularity for Laplace equation involves that in the \( \delta \)-neighborhood \( \mathcal{N}_\delta(\partial \mathcal{F} \cup \partial \mathcal{S}_\lambda) \) of \( \partial \mathcal{F} \cup \partial \mathcal{S}_\lambda \)
\[ \forall \lambda \in \mathcal{P}_\kappa, \quad \| \hat{\lambda}[\tilde{\beta}_\lambda] \|_{C^{k,\frac{1}{2}}(\mathcal{N}_\delta(\partial \mathcal{F} \cup \partial \mathcal{S}_\lambda))} \leq C\varepsilon_\kappa \|\alpha\|_{L^\infty(\partial \mathcal{S}_\kappa)}. \tag{3.38} \]
Now (3.28), (3.36), (3.37) and (3.38) give (3.24) and

$$\left| f_k[\alpha^c] - \hat{f}_k[\alpha^c] \right|_{C^k, (V_k(\delta F))} + \sum_{\lambda \in P_s} \epsilon_k^k \left| f_k[\alpha^c] - \hat{f}_k[\alpha^c] \right|_{C^k, (V_k(\delta S_\alpha^{\epsilon}))} \leq C \epsilon_k \| \alpha^c \|_{L^\infty(\partial S_\delta^{\epsilon})}.$$  

Now we estimate \( \left| \hat{f}_k[\alpha^c] \right|_{C^k, (V_k(\delta F^{\epsilon}(\partial S_\delta^{\epsilon}))} \) with (3.9) and interior regularity estimate for the Laplace equation to arrive at (3.25). This ends the proof of Proposition 3.9.

There is a corresponding result in the situation where the non trivial boundary data is not given on a small solid, but rather on solids of fixed size and on the outer boundary \( \partial \Omega \).

**Proposition 3.10.** Let \( \delta > 0 \) and \( k \geq 2 \). There exist two positive constants \( C \) and \( \epsilon_0 \) depending only on \( \delta, \Omega \) and the reference solids \( S_1^{\lambda}, \lambda = 1, \ldots, N \) \( (C \) depending moreover on \( k) \), such that for any \( \Omega \) with \( \Omega \leq \epsilon_0 \), the following holds. Fix \( q \in \Omega_\delta \) and consider for each \( \kappa \in \{1, \ldots, N_i(\delta)\} \) a function \( \alpha_\kappa \in C^0(\partial S_\alpha^{\epsilon}); R \), and let \( \alpha_\Omega \in C^0(\partial \Omega; \mathbb{R}) \). Let

$$\delta_\alpha := \delta[\alpha_1, \ldots, \alpha_{N_i(\delta)}, 0, \ldots, 0; \alpha_\Omega] \in C^0(\partial F^{\epsilon}(q)),$$

and \( \tilde{\delta}_\alpha := \tilde{g}[\alpha_1, \ldots, \alpha_{N_i(\delta)}; \alpha_\Omega] \) in \( C^0(\partial F^{\epsilon}(q)) \) where \( \tilde{g} \) is given by (3.15). Then

$$\left\| \nabla \delta_\alpha - \nabla \tilde{\delta}_\alpha + \sum_{\lambda \in P_s} \nabla \hat{f}_k \left[ \tilde{g}_\alpha \mid S_\lambda^{\epsilon} \right] \right\|_{L^\infty(\partial F^{\epsilon})} \leq C(\epsilon_\alpha) \left( \| \alpha_\Omega \|_{L^\infty(\partial \Omega)} + \sum_{\kappa \in \mathcal{P}_\delta} \| \alpha_\kappa \|_{L^\infty(\partial S_\delta^{\epsilon})} \right), \tag{3.40}$$

where \( \tilde{g}^{\epsilon} \) is defined in (1.14), and

$$\| \delta_\alpha - \tilde{\delta}_\alpha \|_{C^k, (V_k(\delta F^{\epsilon}))} + \sum_{\nu \in P_s} \epsilon_k = \left| \delta_\alpha \right|_{C^k, (V_k(\delta S_\delta^{\epsilon}))} \leq C \left( \| \alpha_\Omega \|_{L^\infty(\partial \Omega)} + \sum_{\kappa \in \mathcal{P}_\delta} \| \alpha_\kappa \|_{L^\infty(\partial S_\delta^{\epsilon})} \right). \tag{3.41}$$

Moreover, uniformly for \( \alpha_1, \ldots, \alpha_{N_i(\delta)} \) and \( \alpha_\Omega \) in a bounded set of \( C^0 \) and for in \( q \in \Omega_\delta \), one has for all \( \lambda \in P_s \), as \( \epsilon_\alpha \to 0^+ \),

$$\left\| \nabla \hat{f}_k \left[ \tilde{g}_\alpha \mid S_\lambda^{\epsilon} \right] \right\|_{L^p(\mathbb{R}_x^{\lambda}; S_\lambda^{\epsilon})} \quad \text{is bounded,}$$

$$\left\| \nabla \hat{f}_k \left[ \tilde{g}_\alpha \mid S_\lambda^{\epsilon} \right] \right\|_{L^p(\mathbb{R}_x^{\lambda}; S_\lambda^{\epsilon})} \quad \to 0 \quad \text{for} \ p < +\infty$$

and

$$\left\| \nabla \hat{f}_k \left[ \tilde{g}_\alpha \mid S_\lambda^{\epsilon} \right] \right\|_{C^k(\mathbb{R}_x^{\lambda}; \mathbb{R}_x^{\lambda}; \mathbb{R}_x^{\lambda}; \mathbb{R}_x^{\lambda})} \quad \to 0 \quad \text{for any} \ c > 0 \ \text{and} \ k \in \mathbb{N}. \tag{3.42}$$

**Proof of Proposition 3.10.** We proceed as in the proof as Proposition 3.9. We introduce

$$A := (\alpha_1, \ldots, \alpha_{N_i(\delta)}, 0, \ldots, 0, \alpha_\Omega),$$

and define \( B = (\beta_1, \ldots, \beta_N; \beta_\Omega) \) again by (3.26). Then Lemma 3.7 states that \( \delta_\alpha = m[B] \) in \( \partial F \). Here, instead of (3.28), (3.17) allows to write

$$\delta_\alpha = \tilde{g}_\beta + \sum_{\lambda \in P_s} \hat{f}_k \left[ \tilde{g}_\alpha \mid S_\lambda^{\epsilon} \right] \text{ with } \tilde{g}_\beta := \tilde{g}^{\epsilon}(\beta_1, \ldots, \beta_{N_i(\delta)}; \beta_\Omega).$$

Consequently

$$\delta_\alpha = \tilde{g}_\beta + \sum_{\lambda \in P_s} \hat{f}_k \left[ \tilde{g}_\alpha \mid S_\lambda^{\epsilon} \right] = \tilde{g}_\beta + \sum_{\lambda \in P_s} \hat{f}_k \left[ \tilde{g}_\alpha \mid S_\lambda^{\epsilon} \right] = \tilde{g}_\beta + \sum_{\lambda \in P_s} \hat{f}_k \left[ \tilde{g}_\alpha \mid S_\lambda^{\epsilon} \right] \text{ with } \tilde{g}_\beta := \tilde{g}^{\epsilon}(\beta_1, \ldots, \beta_{N_i(\delta)}; \beta_\Omega). \tag{3.43}$$

To establish (3.40), we estimate the right-hand side of (3.43), starting with an estimate of \( B - A \). Instead of (3.32), we now obtain from (3.19) that

$$T(A) = - \sum_{\lambda \in P_s} \hat{f}_k \left[ \tilde{g}_\alpha \mid S_\lambda^{\epsilon} \right] \text{ on } \partial F \text{ and } T(A) = - \sum_{\lambda \in P_s \setminus \{\nu\}} \hat{f}_k \left[ \tilde{g}_\alpha \mid S_\lambda^{\epsilon} \right] \text{ on } \partial S_\nu \text{ for } \nu \in P_s.$$
Again, \( T(A) \) on \( \partial S_p \) is obtained as traces of harmonic functions generated by non-homogeneous data on boundaries of solids different from \( S_p \). Now Lemma 3.2 involves that

\[
\|\tilde{g}_\beta\|_{L^p(\partial\mathcal{F})} \leq C\|A\|_{L^p(\partial\mathcal{F})},
\]

where with a slight abuse of notation we have set \( \|A\|_{L^p(\partial\mathcal{F})} := \|\alpha\|_{L^p(\Omega)} + \sum_{\kappa \in \mathcal{P}_o} \|\alpha_\kappa\|_{L^p(\partial S_\kappa)} \). By (3.9) and interior regularity estimates,

\[
\|T(A)\|_{C^{k, \frac{1}{2}}(\partial F)} \leq C|\mathcal{F}|\|A\|_{L^p(\partial\mathcal{F})}.
\]

Using (3.31) we therefore obtain

\[
B - A = O(|\mathcal{F}|)\|A\|_{L^p(\partial\mathcal{F})} \text{ in } L^p(\partial\mathcal{F}),
\]

in place of (3.33). Using Lemma 3.8, we deduce

\[
\|T(B - A)\|_{C^{k, \frac{1}{2}}(\partial\mathcal{F})} \leq C|\mathcal{F}|\|A\|_{L^p(\partial\mathcal{F})}.
\]

We arrive at

\[
|T(B)|_{C^{k, \frac{1}{2}}(\partial\mathcal{F})} \leq C|\mathcal{F}|\|A\|_{L^p(\partial\mathcal{F})},
\]

which replaces (3.34). Since \( B = A - T(B) \),

\[
B = A + O(|\mathcal{F}|)\|A\|_{L^p(\partial\mathcal{F})} \text{ in } C^{k, \frac{1}{2}}(\partial\mathcal{F}).
\]

Then we deduce estimates on the right-hand side of (3.43). First by (3.45) and the uniform Schauder elliptic estimates in \( \mathcal{F} \) for \( \delta \)-admissible configurations (Lemma 3.3),

\[
\|\tilde{g}_\beta - \tilde{g}_\alpha\|_{C^{k, \frac{1}{2}}(\partial\mathcal{F})} \leq C|\mathcal{F}|\|A\|_{L^p(\partial\mathcal{F})}.
\]

Next, for \( \lambda \in \mathcal{P}_s \), by (3.45) and the fact that \( A_\lambda = 0 \) for \( \lambda \in \mathcal{P}_s \), \( \|\beta_\lambda\|_{C^{k, \frac{1}{2}}(\partial S_\lambda)} \leq C|\mathcal{F}|\|A\|_{L^p(\partial\mathcal{F})} \), and consequently

\[
\|\beta_\lambda - \frac{1}{|\partial S_\lambda|} \int_{\partial S_\lambda} \beta_\lambda\|_{L^p(\partial S_\lambda)} \leq C|\mathcal{F}|\|A\|_{L^p(\partial\mathcal{F})}.
\]

All the same from (3.46) we deduce

\[
\|\tilde{g}_\beta - \tilde{g}_\alpha - \frac{1}{|\partial S_\lambda|} \int_{\partial S_\lambda} (\tilde{g}_\beta - \tilde{g}_\alpha)\|_{L^p(\partial S_\lambda)} \leq C|\mathcal{F}|\|A\|_{L^p(\partial\mathcal{F})}.
\]

Hence with (3.8) and the fact that the operators \( \hat{T}_\mu \) do not see constants we deduce that for all \( \lambda \in \mathcal{P}_s \),

\[
\|\nabla^k_\lambda [\beta_\lambda]\|_{L^p(\mathbb{R}^2 \setminus \partial S_\lambda)} + \|\nabla^k_\lambda [\tilde{g}_\beta|\partial S_\lambda - \tilde{g}_\alpha|\partial S_\lambda]\|_{L^p(\mathbb{R}^2 \setminus \partial S_\lambda)} \leq C|\mathcal{F}|\|A\|_{L^p(\partial\mathcal{F})},
\]

\[
\varepsilon^{\frac{k}{2}} \left( \|\hat{T}_\lambda [\beta_\lambda]\|_{C^{k, \frac{1}{2}}(\partial S_\lambda)} + \|\hat{T}_\lambda [\tilde{g}_\beta|\partial S_\lambda - \tilde{g}_\alpha|\partial S_\lambda]\|_{C^{k, \frac{1}{2}}(\partial S_\lambda)} \right) \leq C|\mathcal{F}|\|A\|_{L^p(\partial\mathcal{F})}.
\]

Putting together (3.43), (3.46) and (3.47) we obtain (3.40).

Now to get (3.41), we estimate the right-hand side of (3.43) in \( C^{k, \frac{1}{2}}(V_\delta(\partial\mathcal{F})) \) and in \( C^{k, \frac{1}{2}}(V_\nu(\partial S_\nu)) \) for \( \nu \in \mathcal{P}_s \). For the first term in (3.43) we simply use (3.46). We now focus on the two remaining sums. First, we can estimate them in \( C^{k, \frac{1}{2}}(V_\delta(\partial\mathcal{F})) \) thanks to (3.47) and local elliptic estimates. Let us now fix in \( \nu \in \mathcal{P}_s \) and estimate these two remaining sums of (3.43) in \( C^{k, \frac{1}{2}}(V_\nu(\partial S_\nu)) \). We first use (3.47) and interior elliptic regularity to deduce that

\[
\sum_{\lambda \in \mathcal{P}_s \setminus \{\nu\}} \|\hat{T}_\lambda [\beta_\lambda]\|_{C^{k, \frac{1}{2}}(V_\delta(\partial S_\nu))} + \sum_{\lambda \in \mathcal{P}_s \setminus \{\nu\}} \|\hat{T}_\lambda [\tilde{g}_\beta|\partial S_\lambda - \tilde{g}_\alpha|\partial S_\lambda]\|_{C^{k, \frac{1}{2}}(V_\delta(\partial S_\nu))} \leq C|\mathcal{F}|\|A\|_{L^p(\partial\mathcal{F})}.
\]
For the remaining terms corresponding to $\lambda = \nu$, we use (3.48). Altogether, putting these estimates in (3.43) we obtain the uniform estimate
\[
\left| \partial_n - \tilde{g}_n + \sum_{\lambda \in P_s} \hat{f}_\lambda \left[ \tilde{B}_\lambda^i(\partial \mathcal{S}_\lambda) \right] \right|_{C^k+\frac{1}{2}(V_p(\partial \mathcal{F}))} + \sum_{\nu \in P_p} \varepsilon_{\nu}^{k-\frac{1}{2}} \left| \partial_n - \tilde{g}_n + \sum_{\lambda \in P_s} \hat{f}_\lambda \left[ \tilde{B}_\lambda^i(\partial \mathcal{S}_\lambda) \right] \right|_{C^k+\frac{1}{2}(V_p(\partial \mathcal{S}_\lambda))} \leq C \| \mathbf{A} \|_{L^\infty(\partial \mathcal{F})}.
\]

Now using (3.44) and interior regularity estimate, we have uniformly in $q \in Q_s$:
\[
\forall \lambda \in P_s, \quad \| \tilde{g}_n \|_{C^k+\frac{1}{2}(V_p(\partial \mathcal{S}_\lambda)))} \leq C \| \mathbf{A} \|_{L^\infty(\partial \mathcal{F})}.
\]
(3.49)
This implies in particular $\| \tilde{g}_n - \tilde{g}_n(h_\lambda) \|_{L^\infty(\partial \mathcal{S}_\lambda)} \leq C \| \mathbf{A} \|_{L^\infty(\partial \mathcal{F})} \varepsilon_\lambda$ for $\lambda \in P_s$. Hence using Proposition 3.4 and the invariance of $\tilde{f}_\lambda$ with respect to additive constant, we obtain a uniform estimate
\[
\forall \lambda \in P_s, \quad \varepsilon_\lambda^{k-\frac{1}{2}} \left| \tilde{f}_\lambda \left[ \tilde{B}_\lambda^i(\partial \mathcal{S}_\lambda) \right] \right|_{C^k+\frac{1}{2}(\mathbb{R}^2 \setminus \mathcal{S}_\lambda)} \leq C \| \mathbf{A} \|_{L^\infty(\partial \mathcal{F})}.
\]
Moreover by (3.9) and interior regularity estimates, one has
\[
\forall \lambda \in P_s, \quad \left| \tilde{f}_\lambda \left[ \tilde{B}_\lambda^i(\partial \mathcal{S}_\lambda) \right] \right|_{C^k+\frac{1}{2}(\mathbb{R}^2 \setminus \mathcal{S}_\lambda)} \leq C \| \mathbf{A} \|_{L^\infty(\partial \mathcal{F})}.
\]
This gives (3.41).

We now turn to (3.42). Since (3.42) corresponds to a phenomenon that we will meet at different stages of the paper, we encapsulate it in a lemma which establishes the smallness of some correctors on small solids.

**Lemma 3.11.** Let $\lambda \in P_s$. Let $\varepsilon_n \in (0,1]$ and $\varepsilon_n \to 0$. Let $(g_n)$ a sequence of functions $g_n : \partial \mathcal{S}_\lambda^\kappa \to \mathbb{R}$ such that, with our convention on the Hölder spaces, $\| g_n \|_{C^k+\frac{1}{2}(\partial \mathcal{S}_\lambda^\kappa)} \leq C$. Then, as $n \to +\infty$, $\nabla \tilde{f}_\lambda[g_n]$ is bounded in $L^\infty(\mathbb{R}^2 \setminus \mathcal{S}_\lambda^\kappa)$, $\nabla \tilde{f}_\lambda[g_n] \to 0$ for any $c > 0$ and $k \in \mathbb{N}$, and $\nabla \tilde{f}_\lambda[g_n] \to 0$ in $L^p(\Omega, \mathcal{S}_\lambda^\kappa)$, $p < +\infty$.

**Proof of Lemma 3.11.** We first observe that up to an additional constant on $\partial \mathcal{S}_\lambda^\kappa$, one has $\| g_n \|_{L^\infty(\partial \mathcal{S}_\lambda^\kappa)} \leq C \varepsilon_n$. Then the boundedness of $\nabla \tilde{f}_\lambda[g_n]$ in $L^\infty(\mathbb{R}^2 \setminus \mathcal{S}_\lambda^\kappa)$ is a consequence of (3.8). Moreover the second part of the lemma follows from (3.9). The third assertion is a consequence of the first two. \qed

Now (3.42) is a direct consequence of Lemma 3.11 and of (3.49). This ends the proof of Proposition 3.10. \qed

**Remark 3.12.** Note that, since $\varepsilon_\kappa = 1$ for $\kappa \in P_s$, Estimate (3.24) is also valid in this case. Indeed due to (3.40)-(3.42) and (3.8) we see that that $\nabla \tilde{f}_\kappa[\mathbf{a}^\kappa]$ and $\nabla \tilde{f}_\kappa[\mathbf{a}^\kappa]$ are both of size $\mathcal{O}(\| \mathbf{a}^\kappa \|)$.

### 3.1.5 Shape derivatives of potentials solving Dirichlet problems

In this paragraph, we estimate the shape derivatives of potentials solving Dirichlet problems. This will be useful to estimate the time-derivative of some velocity fields in forthcoming paragraphs. We refer to [13, 27] for general references on shape differentiation.

Let us first recall a way to write these shape derivatives. We consider a reference configuration $\overline{q}$ in $Q$. Given $\mu \in \{1, \ldots, N\}$, $\kappa \in \{1, 2, 3\}$ and $p_\kappa^\mu = (p_{\kappa, \mu}^\mu, \omega_{\kappa, \mu}^\mu) \in \mathbb{R}^3$, we define $h_\mu(t) = \overline{q}_\mu + \ell h_\mu(t)$ and consider in $\mathbb{R}^2$ a smooth time-dependent vector field such that $\xi_\mu^\kappa(t, x) = \ell h_\mu(t) + \omega_{\kappa, \mu}^\mu(x - h_\mu(t))^\perp$ in a neighborhood of $\partial \mathcal{S}_\mu(\overline{q})$ and $\xi_\mu^\kappa(0, x) = 0$ in a neighborhood of $\partial \mathcal{F}(\overline{q}) \setminus \partial \mathcal{S}_\mu(\overline{q})$. We associate then the corresponding flow $(s, x) \mapsto T^\mu_s(s, x)$ (for $s$ small and $x \in \mathcal{F}(\overline{q})$) that satisfies
\[
\ell \frac{\partial T^\mu_s(s, x)}{\partial s} = \xi_\mu^\kappa(s, T^\mu_s(s, x)), \quad T^\mu_0(0, x) = x.
\]
For small $s$, $T^*_\mu(s, \cdot)$ sends $F(\bar{q})$ into $F(\bar{q} + sp^*_\mu)$, where we denote by $p^*_\mu \in \mathbb{R}^{3N}$ the vector given by $p^*_\mu = (\delta_{\kappa_\mu}p^*_\kappa)_{\kappa=1...N}$. Then the shape derivative of a potential $\varphi = \varphi(q, x)$ (defined and regular on $\bigcup_{q \subset Q}(q) \times \partial F(q)$) with respect to $q_\mu$ is then obtained as

$$
\frac{\partial \varphi}{\partial q_\mu}(q, x) \cdot p^*_\mu = \frac{d}{ds} \varphi(q + sp^*_\mu, x) \big|_{s=0} = \frac{d}{ds} \varphi(q + sp^*_\mu, T^*_\mu(s, x)) \big|_{s=0} - \frac{\partial \varphi}{\partial x}(q, x) \cdot \xi^*_\mu(0, x).
$$

This is actually independent of the choice of the family of diffeomorphisms $T^*_\mu(s, \cdot) : F(\bar{q}) \rightarrow F(\bar{q} + sp^*_\mu)$ as long as $T^*_\mu(0, \cdot) = \text{Id}$, $\partial_s T^*_\mu(0, \cdot) = \xi^*_\mu(0, x)$ on $\partial S_\mu(q)$ and $\partial_s T^*_\mu(0, \cdot) = 0$ on $\partial F(q) \cap \partial S_\mu(q)$. We set

$$
\frac{\partial \varphi}{\partial q_{\mu, m}} := \frac{\partial \varphi}{\partial q_\mu} e_m,
$$

where $(e_1, e_2, e_3)$ is the canonical basis of $\mathbb{R}^3$.

**Lemma 3.13.** Consider a regular family of functions $(\Phi(q, \cdot))_{q \in Q}$, with $\Phi(q, \cdot) : F(q) \rightarrow \mathbb{R}$ satisfying

$$
-\Delta \hat{\Phi}(q, \cdot) = 0 \text{ in } F(q),
$$

and $\hat{\Phi}(q, \cdot) = \alpha(q, \cdot)$ on $\partial F(q)$, where $\alpha$ is a smooth function on $\bigcup_{q \subset Q}(q) \times \partial F(q)$. Then for $\mu \in \{1, \ldots, N\}$ and $m \in \{1, 2, 3\}$ the shape derivative $\frac{\partial \Phi(q, \cdot)}{\partial q_{\mu, m}}$ is the solution to the system

$$
\begin{cases}
-\Delta \hat{\Phi}(q, \cdot) = 0 \text{ in } F(q), \\
\frac{\partial \hat{\Phi}(q, \cdot)}{\partial q_{\mu, m}} = \frac{\partial \alpha(q, \cdot)}{\partial q_{\mu, m}} + \left( \frac{\partial \alpha(q, \cdot)}{\partial x} - \frac{\partial \hat{\Phi}(q, \cdot)}{\partial x} \right) \cdot nK_{\mu, m} \text{ on } \partial F(q).
\end{cases}
$$

**Remark 3.14.** Note that the material derivative $\frac{\partial \Phi(q, \cdot)}{\partial q_{\mu, m}} + \frac{\partial \Phi(q, \cdot)}{\partial x} \cdot nK_{\mu, m}$ is well-defined for functions $\alpha$ defined on the boundary $\bigcup_{q \subset Q}(q) \times \partial F(q)$ in the $(q, x)$ plane, because the vector $(\delta_{\mu, m}, (\xi_{\mu, m} \cdot n)n)$ is tangent to it, where $\delta_{\mu, m}$ denotes the vector in $\mathbb{R}^{3N}$ for which only the coordinate corresponding to $(\mu, m)$ is nonzero and is equal to 1. Alternatively, we may smoothly extend $\alpha$ in $\bigcup_{q \subset Q}(q) \times \partial F(q)$ and define the partial derivatives with respect to $q_{\mu, m}$ and $x$ independently.

**Proof of Lemma 3.13.** That $\frac{\partial \Phi(q, \cdot)}{\partial q_{\mu, m}}$ is harmonic in $F(q)$ is just a matter of commuting derivatives. For what concerns the boundary condition, we use that $\Phi(q, x) = \alpha(q, x)$ on $\partial F(q)$ to infer that for any $p^*_\mu \in \mathbb{R}^3$, $\Phi(q + sp^*_\mu, T^*_\mu(s, x)) = \alpha(q + sp^*_\mu, T^*_\mu(s, x))$ for small $s$ and $x \in \partial F(q)$, where as before $p^*_\mu = (\delta_{\kappa_\mu}p^*_\kappa)_{\kappa=1...N}$. Differentiating with respect to $s$, we deduce

$$
\frac{\partial \Phi(q, \cdot)}{\partial q_\mu} \cdot p^*_\mu + \frac{\partial \Phi(q, \cdot)}{\partial x} \cdot \xi^*_\mu = \frac{\partial \alpha}{\partial q_\mu} \cdot p^*_\mu + \frac{\partial \alpha}{\partial x} \cdot \xi^*_\mu \text{ on } \partial F(q).
$$

It follows that

$$
\frac{\partial \Phi(q, \cdot)}{\partial q_\mu} \cdot p^*_\mu = \frac{\partial \alpha}{\partial q_\mu} \cdot p^*_\mu + \frac{\partial \alpha - \Phi(q, \cdot)}{\partial x} \cdot \xi^*_\mu \text{ on } \partial F(q).
$$

It remains to notice that since $\Phi(q, x) = \alpha(q, x)$ on the boundary, the gradient of $\alpha(\cdot) - \Phi(q, \cdot)$ with respect to $x$ on the boundary is normal. With $\xi^*_\mu \cdot n = \sum_{m=1}^N p^*_{\mu, m}K_{\mu, m}$, we reach the conclusion.

The equivalent of Lemma 3.13 holds for the variant of the Dirichlet problem that we considered above.

**Corollary 3.15.** Consider a smooth function $\alpha$ on $\bigcup_{q \subset Q}(q) \times \partial F(q)$ and a regular family of functions $(\hat{\Phi}(q, \cdot))_{q \in Q}$, with $\hat{\Phi}(q, \cdot) : F(q) \rightarrow \mathbb{R}$ and a regular family of constants $(c_1(q), \ldots, c_N(q))_{q \in Q}$ which are solution to

$$
\begin{cases}
-\Delta \hat{\Phi}(q, \cdot) = 0 \text{ in } F(q), \\
\hat{\Phi}(q, \cdot) = \alpha(q, \cdot) + c_\lambda(q) \text{ on } \partial S_\lambda(q), \quad \forall \lambda \in \{1, \ldots, N\}, \\
\hat{\Phi}(q, \cdot) = \alpha(q, \cdot) \text{ on } \partial \Omega, \\
\int_{\partial S_\lambda} \hat{\Phi}(q, x) ds = 0, \quad \forall \lambda \in \{1, \ldots, N\}.
\end{cases}
$$
Then for $\mu \in \{1, \ldots, N\}$ and $m \in \{1, 2, 3\}$ the shape derivative $\frac{\partial \hat{\Phi}(q, \cdot)}{\partial q_{\mu,m}}$ is the solution to the system

$$
\begin{align*}
-\Delta \left( \frac{\partial \hat{\Phi}(q, \cdot)}{\partial q_{\mu,m}} \right) &= 0 \quad \text{in } \mathcal{F}(q), \\
\frac{\partial \hat{\Phi}(q, \cdot)}{\partial q_{\mu,m}} &= \frac{\partial \hat{\alpha}(q, \cdot)}{\partial q_{\mu,m}} + \left( \frac{\partial \hat{\alpha}(q, \cdot)}{\partial x} - \frac{\partial \hat{\Phi}(q, \cdot)}{\partial x} \right) \cdot n \ K_{\mu,m} + \hat{c}_\lambda(q) \ \text{on } \partial S_\lambda(q), \ \forall \lambda \in \{1, \ldots, N\}, \\
\int_{\partial S_\lambda} \left( \frac{\partial \hat{\Phi}(q, \cdot)}{\partial q_{\mu,m}} \right) \ ds &= 0, \ \forall \lambda \in \{1, \ldots, N\},
\end{align*}
$$

(3.52)

for some constants $c_1(q), \ldots, c_N(q)$.

**Proof of Corollary 3.15.** We check the validity of the various equations in (3.52). As for Lemma 3.13, the first equation is obtained by commuting derivatives with respect to $x$ and $q$. To obtain the second equation, we observe that the shape derivative of a constant function with respect to $x$ on $\partial S_\lambda$ (for each $q$) is a constant function on $\partial S_\lambda$. Let us highlight that the regularity with respect to $q$ is a consequence of the construction and of the regularity for the usual Dirichlet problem. The third equation is trivial.

Finally we see that the flux of $\hat{\Phi}(q, \cdot)$ across $\partial S_\lambda$ for $\lambda \neq \mu$ and across $\partial \Omega$ is zero, since these components of the boundary are fixed and the flux of $\Phi(q, \cdot)$ across them is zero for all $q$. Considering that $\frac{\partial \hat{\Phi}(q, \cdot)}{\partial q_{\mu,m}}$ is harmonic and using the divergence theorem, it follows that the flux across $\partial S_\mu$ of $\frac{\partial \hat{\Phi}(q, \cdot)}{\partial q_{\mu,m}}$ is zero as well.

\[ \square \]

**Remark 3.16.** In both (3.50) and (3.52), we may write

$$
\frac{\partial \hat{\alpha}(q, \cdot)}{\partial q_{\mu,m}} + \left( \frac{\partial \hat{\alpha}(q, \cdot)}{\partial x} - \frac{\partial \hat{\Phi}(q, \cdot)}{\partial x} \right) \cdot n \ K_{\mu,m} = \frac{\partial \alpha(q, \cdot)}{\partial q_{\mu,m}} + \left( \frac{\partial \alpha(q, \cdot)}{\partial x} - \frac{\partial \Phi(q, \cdot)}{\partial x} \right) \cdot \xi_{\mu,m}.
$$

This is just a matter of stopping the computation at (3.51), or of keeping in mind that, since $\alpha(q, \cdot) - \Phi(q, \cdot)$ is constant on the boundary, its tangential derivative is zero.

### 3.1.6 Transposing to the Neumann problem

Let us now describe how the analysis of the paragraphs above can be transposed to the Neumann problem. Given $\kappa \in \{1, \ldots, N\}$, $q \in Q$, and $\beta \in C^\infty(\partial S_\kappa; \mathbb{R})$ such that

$$
\int_{\partial S_\kappa} \beta(x) \ ds(x) = 0,
$$

(3.53)

we consider the solution $\hat{f}_\kappa^N[\beta] \in C^\infty(\mathcal{F}(q))$ (unique up to an additive constant) of the Neumann problem

$$
\begin{align*}
\Delta \hat{f}_\kappa^N[\beta] &= 0 \quad \text{in } \mathcal{F}(q), \\
\hat{c}_n \hat{f}_\kappa^N[\beta] &= 0 \quad \text{on } \partial \mathcal{F}(q) \setminus \partial S_\kappa, \\
\hat{c}_n \hat{f}_\kappa^N[\beta] &= \beta \quad \text{on } \partial S_\kappa,
\end{align*}
$$

(3.54)

and $\hat{f}_\kappa^N[\beta] \in C^\infty(\mathbb{R}^2 \setminus S_\kappa)$ be the solution (unique up to an additive constant) of the standalone Neumann problem

$$
\begin{align*}
\Delta \hat{f}_\kappa^N[\beta] &= 0 \quad \text{in } \mathbb{R}^2 \setminus S_\kappa, \\
\nabla \hat{f}_\kappa^N[\beta](x) &\to 0 \quad \text{as } |x| \to +\infty, \\
\hat{c}_n \hat{f}_\kappa^N[\beta] &= \beta \quad \text{on } \partial S_\kappa.
\end{align*}
$$

Condition (3.53) allows to write the function $\beta$ as

$$
\beta = \hat{c}_r \mathcal{B}.
$$

Then the following result is elementary to check.
Lemma 3.17. One has the correspondence $\nabla \hat{\varphi}_k^N[\beta] = \nabla^+ f_k[B]$ and $\nabla \hat{\varphi}_k^N[\beta] = \nabla^- f_k[B]$. In particular, one can apply Proposition 3.4 to $\hat{\varphi}_k^N[\beta]$ and Propositions 3.9 and 3.10 and $\hat{\varphi}_k^N[\beta]$ with $\|B\|_{L^\infty(\partial S_k^\varepsilon)} = O(\varepsilon_k \|\beta\|_{L^\infty(\partial S_k^\varepsilon)})$ in the right-hand side in place of $|a^k|_{L^\infty(\partial S_k^\varepsilon)}$.

Of course, in the same way, we can consider the Neumann counterpart of $\hat{\varphi}$ defined in (3.2), say $\hat{\varphi}^N[\beta_1,\ldots,\beta_N;\beta_0]$, and in the same way obtain the correspondence with $\hat{\varphi}[B_1,\ldots,B_N;B_0]$ where $B_1, \ldots, B_N$ and $B_0$ are primitives of $\beta_1, \ldots, \beta_N$ and $\beta_0$ on $\partial S_1, \ldots, \partial S_N$ and $\partial \Omega$, respectively.

In the sequel we will use mainly the case of the Neumann problem.

3.2 Estimates of the Kirchhoff potentials

In this paragraph we apply the above results in the case of the Kirchhoff potentials defined in (2.8) and study their shape derivatives as well.

3.2.1 The Kirchhoff potentials

We first recall several properties of the standalone Kirchhoff potentials for instance in [8].

Lemma 3.18. The standalone Kirchhoff potentials $\hat{\varphi}_{k,k}^\varepsilon$, $k \in \{1, \ldots, N\}$, $k \in \{1, \cdots, 5\}$, have the following properties:

- for fixed $q_k$, $\hat{\varphi}_{k,k}^\varepsilon(x - h_k) = \varepsilon_k^{1+\delta_{k,3}} \varphi_{k,k} \left( \frac{x - h_k}{\varepsilon_k} \right)$
  and $\nabla \hat{\varphi}_{k,k}^\varepsilon(x - h_k) = \varepsilon_k^{\delta_{k,3}} \nabla \varphi_{k,k} \left( \frac{x - h_k}{\varepsilon_k} \right)$, \hspace{1cm} (3.55)

- $\nabla \hat{\varphi}_{k,k}^\varepsilon(x) = O \left( \frac{\varepsilon_k^{2+\delta_{k,3}}}{|x - h_k|^2} \right)$ at infinity, \hspace{1cm} (3.56)

- $\varepsilon_k^{-\delta_{k,3}} \nabla \hat{\varphi}_{k,k}^\varepsilon$ is bounded in $\mathbb{R}^2 \setminus S_k^\varepsilon$ and $\hat{\varphi}_{k,k}^\varepsilon = O(\varepsilon_k^{1+\delta_{k,3}})$ on $\partial S_k^\varepsilon$. \hspace{1cm} (3.57)

Remark 3.19. It is elementary to check that given $q_k$, we recover the $k$-th standalone Kirchhoff potentials at $q_k$ at their equivalent at the basic position through

$$
\left( \begin{array}{c}
\hat{\varphi}_{k,1}(q_k, h_k + R(\theta) x) \\
\hat{\varphi}_{k,2}(q_k, h_k + R(\theta) x)
\end{array} \right) = R(-\theta) \left( \begin{array}{c}
\hat{\varphi}_{k,1}(0, x) \\
\hat{\varphi}_{k,2}(0, x)
\end{array} \right), \quad \hat{\varphi}_{k,3}(q_k, h_k + R(\theta) x) = \hat{\varphi}_{k,3}(0, x),
$$

and

$$
\left( \begin{array}{c}
\hat{\varphi}_{k,4}(q_k, h_k + R(\theta) x) \\
\hat{\varphi}_{k,5}(q_k, h_k + R(\theta) x)
\end{array} \right) = R(2\theta) \left( \begin{array}{c}
\hat{\varphi}_{k,4}(0, x) \\
\hat{\varphi}_{k,5}(0, x)
\end{array} \right).
$$

Consequently, all the estimates on the standalone Kirchhoff potentials are independent of the position $q_k$.

We have the following first statement regarding the behavior of the Kirchhoff potentials $\varphi_{k,k}$ in $\mathcal{F}^\varepsilon$ for small values of $\varepsilon_k$.

Proposition 3.20. For $\delta > 0$, there exists $\varepsilon_0 > 0$ depending only on $\delta, \Omega$ and the shape of the reference solids $S_\lambda^1$, $\lambda = 1, \ldots, N$, such that for any $\varphi$ with $\|\varphi\| \leq \varepsilon_0$, the following holds. Let $k \in \{1, \ldots, N\}$, $k \in \{1, \cdots, 5\}$ and $\ell \in \mathbb{N}\setminus\{0,1\}$. For some constant $C > 0$ independent of $\varphi$, the following holds uniformly for $q \in Q_\lambda$:

$$
\|\nabla \varphi_{k,k} - \nabla \hat{\varphi}_{k,k}^\varepsilon\|_{L^\infty(\mathcal{F}^\varepsilon(q))} \leq C \varepsilon_k^{2+\delta_{k,3}}, \hspace{1cm} (3.58)
$$

$$
\|\varphi_{k,k}\|_{C^{\ell+\delta_{k,3}}(V_\lambda(\partial \Omega))} + \sum_{\lambda \in P_\lambda(\kappa)} \varepsilon_k^{\ell+\delta_{k,3}} \|\varphi_{k,k}\|_{C^{\ell+\delta_{k,3}}(V_\lambda(\partial S_\lambda))} \leq C \varepsilon_k^{2+\delta_{k,3}}, \hspace{1cm} (3.59)
$$

$$
\|\nabla \varphi_{k,k}\|_{L^\infty(\mathcal{F}^\varepsilon(q))} \leq C \varepsilon_k^{\delta_{k,3}} \text{ and } \nabla \varphi_{k,k}(x) = O \left( \frac{\varepsilon_k^{2+\delta_{k,3}}}{|x - h_k|^2} \right) \text{ for } x \in \mathcal{F}^\varepsilon(q) \text{ s.t. } |x - h_k| \geq C \varepsilon_k. \hspace{1cm} (3.60)
$$
and one has, up to an additional constant on each connected component of the boundary,

\[
\varphi_{\kappa,k} = \begin{cases} 
O(\varepsilon_k^{2+\delta_{k\geq 3}}) & \text{on } \partial \Omega, \\
O(\varepsilon_k^{2+\delta_{k\geq 3}}\varepsilon_{\mu}) & \text{on } \partial S_\mu \text{ if } \mu \neq \kappa, \\
\tilde{\varphi}_{\kappa,k} + O(\varepsilon_k^{2+\delta_{k\geq 3}}) = O(\varepsilon_k^{1+\delta_{k\geq 3}}) & \text{on } \partial S_\kappa.
\end{cases}
\tag{3.61}
\]

**Proof of Proposition 3.20.** We use Lemma 3.17 with \( \beta = K_{\kappa,k} \), hence we may apply to it Proposition 3.9 if \( \kappa \in P_0 \) and Remark 3.12 otherwise. Since \( |K_{\kappa,k}|_{L^\infty(\partial S_\kappa)} = O(\varepsilon_k^{\delta_{k\geq 3}}) \), we obtain from (3.24) and (3.25) that (3.58) and (3.59) hold. To obtain (3.60) we use (3.58) together with (3.55) and (3.56). For what concerns (3.61), it suffices then integrate \( \nabla \varphi_{\kappa,k} - \nabla \tilde{\varphi}_{\kappa,k} \) on \( \partial S_\mu \) taking into account (3.58) and (3.56) when \( \mu \neq \kappa. \)

**Remark 3.21.** The Kirchhoff potentials \( \varphi_{\kappa,k} \) are defined up to a single additional constant (while the aforementioned additional constants in (3.61) many differ from one connected component of the boundary to the other). We can however normalize this global additional constant so that

\[
\varphi_{\kappa,k} = O(\varepsilon_k^{1+\delta_{k\geq 3}}) \text{ on } \partial \Omega \text{ and } \varphi_{\kappa,k} = O(\varepsilon_k^{2+\delta_{k\geq 3}}) \text{ on } \partial \Omega \setminus \partial \kappa.
\tag{3.62}
\]

It suffices for instance to take \( \varphi_{\kappa,k}(X) = \varphi_{\kappa,k}(X) \) for some point \( X \in \partial \Omega \) (and integrate starting from this point).

In the case of Kirchhoff potentials corresponding to a solid of fixed size, we have the following more accurate result.

**Proposition 3.22.** Let \( \delta > 0 \). There exists \( \varepsilon_0 > 0 \) such that for all \( \varpi \) with \( \varpi \leq \varepsilon_0 \) the following holds. Let \( \kappa \in P_0, k \in \{1, 2, 3\} \). Let \( \ell \in \mathbb{N} \setminus \{0, 1\} \). Then for some constant \( C > 0 \) independent of \( \varpi \), the following holds uniformly for \( q \in Q_5 \):

\[
\left\| \nabla \varphi_{\kappa,k} - \nabla \tilde{\varphi}_{\kappa,k} + \sum_{\lambda \in P_\varpi} \nabla \lambda[\tilde{\varphi}_{\kappa,k}] \right\|_{L^\infty(\Omega_\varpi)} \leq C|\varpi|,
\tag{3.63}
\]

\[
|\varphi_{\kappa,k} - \tilde{\varphi}_{\kappa,k}|_{C^{\ell-\frac{2}{3}}(\Omega_\varpi)} + \sum_{\nu \in P_\varpi} \varepsilon_\nu^{\ell-\frac{4}{3}} |\varphi_{\kappa,k}|_{C^{\ell-\frac{2}{3}}(\Omega_\varpi)} \leq C.
\tag{3.64}
\]

and the terms \( \nabla \lambda[\tilde{\varphi}_{\kappa,k}] \) are bounded in \( L^\infty(\mathbb{R}^2 \setminus S_\lambda) \), converge to 0 in \( C^{\ell}(\Omega_\varpi \setminus \partial S_\lambda) \) for all \( c > 0 \) and \( \ell \in \mathbb{N} \) and in \( L^p(\Omega_\varpi \setminus \partial S_\lambda) \), \( p < +\infty \).

**Proof of Proposition 3.22.** We let \( \varepsilon_0 \) as in Lemma 3.6 and we reason as for Proposition 3.20, using the correspondence between Dirichlet and Neumann problems (Lemma 3.17) and Proposition 3.10.

This has the following corollary on the added mass matrix. Recall that the added mass matrices where defined in (2.12)–(2.15).

**Corollary 3.23.** Let \( \delta > 0 \). There exist constants \( C > 0 \) and \( \varepsilon_0 > 0 \) such that for all \( \kappa, \kappa' \in \{1, \ldots, N\} \) and all \( i, i' \in \{1, 2, 3\} \), as long as \( (\varepsilon, q) \in Q_5^0 \),

\[
|\mathcal{M}_{a,\kappa,i,\kappa',i'} - \delta_{\kappa,k'} \mathcal{M}_{a,\kappa,i,i'}| \leq C \varepsilon_k^{2+\delta_{k\geq 3}} \varepsilon_{\kappa'}^{2+\delta_{\kappa'\geq 3}}.
\tag{3.65}
\]

Moreover one has, uniformly for \( q \in Q_5 \),

\[
\mathcal{M}_{a,\kappa,i,\kappa',i'} \longrightarrow \delta_{\kappa\kappa'} \delta_{i,i'} \mathcal{M}_{a,\kappa,i,i'} \quad \text{as} \quad \varepsilon \longrightarrow 0.
\tag{3.66}
\]

**Proof of Corollary 3.23.** We first write

\[
\mathcal{M}_{a,\kappa,i,\kappa',i'} = \int_{\partial S_{\kappa'}} \varphi_{\kappa,i} K_{\kappa',i'} ds,
\tag{3.67}
\]

and notice that this formula is insensitive to a constant added to \( \varphi_{\kappa,i} \). Estimate (3.65) is then a direct consequence of (3.61). The convergence (3.66) follows in the same way from Proposition 3.22.
Remark 3.24. Notice that both (3.65) and (3.66) prove the convergence to 0 of $M_{a,\kappa,i,i',\nu}$ when $\kappa$ or $\kappa'$ belongs to $P_s$. When both indices $\kappa$ and $\kappa'$ belong to $P_{(i)}$, (3.65) merely proves that it remains bounded. Notice also that, as a consequence of (3.55), $M_{a,\kappa,i,i',\nu}$ satisfies the scale relation $M_{a,\kappa,i,i',\nu} = \mathcal{E}_{a,\kappa,i,i',\nu} + \frac{\pi}{\kappa^2 + \delta_3 + \delta_3'} \tilde{M}_{a,\kappa,i,i',\nu}$.

3.2.2 Shape derivatives of the Kirchhoff potentials

In this paragraph, we estimate the shape derivatives of the Kirchhoff potentials. An expression of the proof for this expression by relying on the results of Section 3.1.5 (and extend it for indices 4 and 5).

Precisely we consider the shape derivative $\frac{\partial \varphi_{\lambda,\ell}}{\partial q_{\mu,m}}(q,\cdot)$ of the Kirchhoff potentials $\varphi_{\lambda,\ell}$ for $\lambda \in \{1,\ldots,N\}$ and $\ell \in \{1,\ldots,5\}$ with respect to the variable $q_{\mu,m}$ for $\mu = 1,\ldots,N$, $m = 1,2,3$.

Lemma 3.25. For $\lambda = 1,\ldots,N$, $\mu = 1,\ldots,N$, $m = 1,2,3$, the function $\frac{\partial \varphi_{\lambda,\ell}}{\partial q_{\mu,m}}(q,\cdot)$ is harmonic in $F(q)$ and satisfies:

$$\frac{\partial}{\partial n} \left( \frac{\partial \varphi_{\lambda,\ell}}{\partial q_{\mu,m}} \right)(q,\cdot) = 0 \text{ on } \partial F(q) \setminus \partial S_\mu,$$

$$\frac{\partial}{\partial n} \left( \frac{\partial \varphi_{\lambda,\ell}}{\partial q_{\mu,m}} \right)(q,\cdot) = \frac{\partial}{\partial \tau} \left[ \left( \frac{\partial \varphi_{\lambda,\ell}}{\partial \tau} - (\xi_{\mu,m} \cdot \tau) \right) \xi_{\mu,m} \right] + \delta_{\ell = 3} \delta_{m = 1,2} \frac{\partial}{\partial \tau} (\xi_{\lambda,\ell} \cdot e_m) \text{ on } \partial S_\mu. \quad (3.69)$$

We recall that the notation $\xi_{\lambda,\ell}$ is defined in (2.7).

Proof of Lemma 3.25. As previously, we translate the Neumann problem defining the Kirchhoff potential $\varphi_{\lambda,\ell}$ into a Dirichlet problem (or in other words, we consider the harmonic conjugate of $\varphi_{\lambda,\ell}$). Hence we introduce the function $\varphi_{\lambda,\ell}^*$ and the constants $c_1,\ldots,c_N$ that satisfy

$$\begin{aligned}
-\Delta \varphi_{\lambda,\ell}^* &= 0 \text{ in } F(q), \\
\varphi_{\lambda,\ell}^* &= J_{\lambda,\ell} + c_{\lambda} \text{ on } \partial S_{\lambda}(q), \\
\varphi_{\lambda,\ell}^* &= c_{\kappa} \text{ on } \partial S_{\kappa}(q), \quad \forall \kappa \neq \lambda, \\
\varphi_{\lambda,\ell}^* &= 0 \text{ on } \partial \Omega, \\
\int_{\partial S_{\lambda}} \partial_n \varphi_{\lambda,\ell}^* ds &= 0, \quad \forall \kappa \in \{1,\ldots,N\},
\end{aligned}$$

where $J_{\lambda,\ell}$ is a primitive of $K_{\lambda,\ell}$ on $S_{\lambda}$. Namely we take $J_{\lambda,\ell} = 0$ on $\partial F \setminus \partial S_{\lambda}$, and on $\partial S_{\lambda}$,

$$J_{\lambda,\ell} = -x_2 \text{ if } \ell = 1, \quad J_{\lambda,\ell} = x_1 \text{ if } \ell = 2, \quad J_{\lambda,\ell} = \frac{|x-h_{\lambda}|^2}{2} \text{ if } \ell = 3,$$

$$J_{\lambda,\ell} = (x_1-h_{\lambda,1})(x_2-h_{\lambda,2}) \text{ if } \ell = 4 \text{ and } J_{\lambda,\ell} = \frac{(x_1-h_{\lambda,1})^2 - (x_2-h_{\lambda,2})^2}{2} \text{ if } \ell = 5. \quad (3.70)$$

We extend $J_{\lambda,\ell}$ in the neighborhood of these boundaries by the same formulas. In particular, one has the relation

$$\nabla J_{\lambda,\ell} = -\xi_{\lambda,\ell}^* \text{ in the neighborhood of } \partial F. \quad (3.71)$$

Then $\nabla \varphi_{\lambda,\ell} = \nabla^\perp \varphi_{\lambda,\ell}^*$ in $F(q)$, and thus $\nabla \left( \frac{\partial \varphi_{\lambda,\ell}}{\partial q_{\mu,m}} \right) = \nabla^\perp \left( \frac{\partial \varphi_{\lambda,\ell}^*}{\partial q_{\mu,m}} \right)$ in $F(q)$. By Corollary 3.15, we find

$$\frac{\partial \varphi_{\lambda,\ell}}{\partial q_{\mu,m}} = \delta_{\lambda \mu} \delta_{\ell = 3} \frac{\partial}{\partial q_{\mu,m}} \left( \frac{\partial \varphi_{\lambda,\ell}^*}{\partial q_{\mu,m}} \right) + \left( \delta_{\lambda \mu} \nabla J_{\lambda,\ell} \cdot n - \partial_n \varphi_{\lambda,\ell}^* \right) K_{\mu,m} + c'_{\kappa} \text{ on } \partial S_{\kappa}(q), \quad \kappa \in \{1,\ldots,N\},$$

$$\frac{\partial \varphi_{\lambda,\ell}}{\partial q_{\mu,m}} = 0 \text{ on } \partial \Omega.$$
We compute $\frac{\partial J_{\lambda,\ell}}{\partial q_{\mu,m}}$ as follows:

$$
\frac{\partial J_{\lambda,\ell}}{\partial q_{\mu,m}} = \delta_{\ell,3}\delta_{m1} \nabla J_{\lambda,\ell} \cdot e_m \quad \text{on} \quad \partial \mathcal{S}_\lambda.
$$

Since $\partial \tau \varphi_{\lambda,\ell} = -\partial_n \varphi_{\lambda,\ell}^*$ and $\partial_n \left( \frac{\partial \varphi_{\lambda,\ell}}{\partial q_{\mu,m}} \right) = \partial \tau \left( \frac{\partial \varphi_{\lambda,\ell}^*}{\partial q_{\mu,m}} \right)$, using (3.71) we obtain (3.69).

This allows us to prove the following estimates on the shape derivatives of the Kirchhoff potentials.

**Proposition 3.26.** Let $\delta > 0$. There is $\varepsilon_0 > 0$ such that for all $\mathcal{X}$ such that $\mathcal{X} \subset \varepsilon_0$, for $\lambda, \mu, \kappa \in \{1, \ldots, N\}$, for $\ell \in \{1, 2, 3\}$ and $m \in \{1, 2, 3, 4, 5\}$, uniformly for $q \in \mathcal{Q}_\delta$, one has

$$
\frac{\partial \varphi_{\lambda,\ell}}{\partial q_{\mu,m}} = O(\varepsilon_\lambda^{3+2\delta_{\lambda,\mu}+\varepsilon_\delta_{\mu,m}+2\delta_{\mu,\kappa}}) \quad \text{on} \quad \partial \mathcal{S}_\kappa \quad \text{(up to an additive constant)},
$$

(3.72)

where we recall the convention (2.7) on $\partial \mathcal{S}_\kappa$.

$$
\left| \frac{\nabla \varphi_{\lambda,\ell}}{\partial q_{\mu,m}} \right|_{L^\infty(\mathcal{F})} = O(\varepsilon_\lambda^{2+\delta_{\mu,\kappa}+\varepsilon_\delta_{\mu,m}+\varepsilon_\delta_{\mu,\kappa}}),
$$

(3.73)

$$
\nabla \frac{\partial \varphi_{\lambda,\ell}}{\partial q_{\mu,m}}(x) = O(\varepsilon_\lambda^{2+\delta_{\mu,\kappa}+\varepsilon_\delta_{\mu,m}+\varepsilon_\delta_{\mu,\kappa}}) \quad \text{for} \quad x \quad \text{such that} \quad d(x, \mathcal{S}_\mu) \geq \delta.
$$

(3.74)

**Proof of Proposition 3.26.** We proceed in three steps.

**Step 1.** By Lemma 3.25,

$$
\frac{\partial \varphi_{\lambda,\ell}}{\partial q_{\mu,m}} = \tilde{p}_\mu^V[\partial \mathcal{B}], \quad \text{(3.75)}
$$

where we recall that $\tilde{p}_\mu^V$ was defined in (3.54) and where $\mathcal{B}$ is given on $\partial \mathcal{S}_\mu$ by a primitive of the data (3.69) on $\mathcal{S}_\mu$:

$$
\mathcal{B} = \frac{\left( \frac{\partial \varphi_{\lambda,\ell}}{\partial \tau} - (\xi_{\lambda,\ell} \cdot \tau) \right) (\xi_{\mu,m} \cdot n) + \delta_{\ell,3}\delta_{m1} \xi_{\lambda,\ell} \cdot e_m}{n} \quad \text{on} \quad \partial \mathcal{S}_\mu,
$$

where we recall the convention (2.7) on $\xi_{\lambda,\ell}$ (in particular, this is 0 away from $\mathcal{S}_\lambda$).

**Step 2.** Now we evaluate $\mathcal{B}$ on $\partial \mathcal{S}_\mu$. For $\lambda \neq \mu$, Proposition 3.20 gives directly

$$
\varepsilon_\lambda^{\frac{1}{2}} |\varphi_{\lambda,\ell}|_{C^{1,\frac{1}{2}}(\mathcal{V}(\partial \mathcal{S}_\mu))} \leq C \varepsilon_\lambda^{\frac{1}{2}},
$$

where $\mathcal{V}(\partial \mathcal{S}_\mu)$ is given on $\partial \mathcal{S}_\mu$.

In the case $\mu = \lambda$, by Proposition 3.20, for $j \geq 2$, one has $\varepsilon_\lambda^{\frac{1}{2}} |\varphi_{\lambda,\ell}|_{C^{1,\frac{j}{2}}(\mathcal{V}(\partial \mathcal{S}_\mu))} \leq C \varepsilon_\lambda^{\frac{1}{2}+\delta_{\lambda,3}}$. Moreover from Proposition 3.4, using the scale relation (3.55), we see that $\varepsilon_\lambda^{\frac{1}{2}} |\varphi_{\lambda,\ell}|_{C^{1,\frac{j}{2}}(\mathcal{F})} \leq C \varepsilon_\lambda^{\frac{1}{2}+\delta_{\lambda,3}}$. We deduce that

$$
\varepsilon_\lambda^{\frac{1}{2}} |\varphi_{\lambda,\ell}|_{C^{1,\frac{j}{2}}(\mathcal{V}(\partial \mathcal{S}_\mu))} \leq C \varepsilon_\lambda^{\frac{1}{2}+\delta_{\lambda,3}}.
$$

On the other hand, for all $\mu$ (including $\lambda$), the tangent $\tau$ on $\partial \mathcal{S}_\mu$ satisfies itself $\varepsilon_\mu^{\frac{j}{2}} |\tau|_{C^{1,\frac{j}{2}}(\partial \mathcal{S}_\mu)} \leq C$ (this is a scaling argument consistent with (3.1)). For what concerns the $L^\infty$ norm, it follows from Propositions 3.20 and 3.22 that $\|\nabla \varphi_{\lambda,\ell}\|_{L^\infty(\partial \mathcal{S}_\mu)} = O(\varepsilon_\lambda^{2+\delta_{\mu,\kappa}+\varepsilon_\delta_{\mu,m}})$. We deduce with the Leibniz rule that for all $\mu \in \{1, \ldots, N\}$

$$
\varepsilon_\mu^{\frac{j}{2}} |\partial \tau \varphi_{\lambda,\ell}|_{C^{1,\frac{j}{2}}(\partial \mathcal{S}_\mu)} \leq C \varepsilon_\lambda^{2+\delta_{\mu,\kappa}+\varepsilon_\delta_{\mu,m}}.
$$

It then follows that

$$
\|\mathcal{B}\|_{L^\infty(\partial \mathcal{S}_\mu)} + \varepsilon_\mu^{\frac{j}{2}} |\mathcal{B}|_{C^{1,\frac{j}{2}}(\partial \mathcal{S}_\mu)} = O(\varepsilon_\lambda^{2+\delta_{\mu,\kappa}+\varepsilon_\delta_{\mu,m}}).
$$

(3.76)

**Step 3.** Now we deduce estimates on $\tilde{p}_\mu^V[\partial \mathcal{B}]$ as follows: we apply Lemma 3.17, Proposition 3.9 and Remark 3.12 to $\tilde{p}_\mu^V[\partial \mathcal{B}]$ to obtain that for $\mu \in \{1, \ldots, N\}$,

$$
\nabla \tilde{p}_\mu^V[\partial \mathcal{B}] = \nabla \tilde{p}_\mu^V[\partial \mathcal{B}] + O(\varepsilon_\mu |\mathcal{B}|_{L^\infty(\partial \mathcal{S}_\mu)}) \quad \text{in} \quad L^\infty(\mathcal{F}(\mu)).
$$

(3.77)

To estimate $\nabla \tilde{p}_\mu^V[\partial \mathcal{B}]$, we use Proposition 3.4 and (3.76). Hence (3.74) is a consequence of (3.77) and (3.9), and (3.73) follows from (3.8). We deduce (3.72) by integrating (3.73) (if $\kappa = \mu$) and (3.74) (otherwise) over $\partial \mathcal{S}_n$. The estimate on $\partial \Omega$ is performed in the same way.

\[\Box\]
3.3 Estimates on the circulation stream function

In this section we study the circulation stream functions \( \psi_k^\varepsilon \), for \( \kappa = 1, \ldots, N \), introduced in (2.17).

We first recall several elementary properties of the standalone circulation stream functions \( \psi_k \), for \( \kappa = 1, \ldots, N \), introduced in (2.18). We refer for instance to [7] for a proof.

Lemma 3.27. For \( \varepsilon_\kappa = 1 \),

\[
\hat{\psi}_k^\varepsilon(h_\kappa, \theta_\kappa, x) = \hat{\psi}_k^\varepsilon((0,0), R(-\theta_\kappa)(x-h_\kappa)),
\]

(3.78)

for fixed \( q_\kappa \),

\[
\nabla \hat{\psi}_k^\varepsilon(x-h_\kappa) = \frac{1}{\varepsilon_\kappa} \nabla \hat{\psi}_k^\varepsilon \left( \frac{x-h_\kappa}{\varepsilon_\kappa} \right),
\]

(3.79)

the function \( \hat{c}_1\hat{\psi}_k - i\hat{c}_2\hat{\psi}_k \) admits the following Laurent series expansion for \( C \) such that \( S^1_\kappa \subset B(0, C) \),

\[
\hat{c}_1\hat{\psi}_k - i\hat{c}_2\hat{\psi}_k = \frac{1}{2i\pi z} + \sum_{k \geq 2} \frac{a_k}{z^k} \text{ for } z = x_1-h_{1,\kappa} + i(x_2-h_{2,\kappa}) \text{ and } |z| \geq C.
\]

(3.80)

Note in particular that (3.79)-(3.80) involve

\[
\nabla \hat{\psi}_k^\varepsilon(x) = \left( \begin{array}{c} (x-h_\kappa)^1 \frac{1}{2\pi |x-h_\kappa|^2} + \mathcal{O} \left( \frac{\varepsilon_\kappa}{|x-h_\kappa|^2} \right) \end{array} \right) \text{ for } |x-h_\kappa| \geq C\varepsilon_\kappa,
\]

(3.81)

and consequently

\[
(x-h_\kappa)^1 \cdot \nabla \hat{\psi}_k^\varepsilon(x) = \frac{1}{2\pi} + \mathcal{O}(\varepsilon_\kappa) \text{ for } |x-h_\kappa| \geq \mathcal{O}(1).
\]

(3.82)

The \( \mathcal{O}(\varepsilon_\kappa) \) above can be taken in any norm, because this function is harmonic, since

\[
(x-h_\kappa)^1 \cdot \nabla \hat{\psi}_k^\varepsilon(x) = \text{Re}[i(z-h_\kappa)(\hat{c}_1\hat{\psi}_k - i\hat{c}_2\hat{\psi}_k)].
\]

(3.83)

We are now in position to study \( \psi_k^\varepsilon \).

3.3.1 Estimates on the reflected circulation stream function

For \( \kappa = 1, \ldots, N \), we consider in the difference between the circulation stream function \( \psi_\kappa \) and its standalone version \( \hat{\psi}_\kappa \), that is

\[
\psi_\kappa^\varepsilon := \psi_\kappa - \hat{\psi}_\kappa.
\]

(3.84)

By (2.17) and (2.18) there are some constants \( c_\lambda \), for \( \lambda = 1, \ldots, N \), such that

\[
\begin{align*}
\Delta \psi_\kappa^\varepsilon &= 0 \text{ in } \mathcal{F}, \\
\psi_\kappa^\varepsilon &= c_\kappa \text{ on } \partial \mathcal{S}_\kappa, \\
\psi_\kappa^\varepsilon &= -\hat{\psi}_\kappa + c_\nu \text{ on } \partial \mathcal{S}_\nu, \forall \nu \neq \kappa, \\
\int_{\partial \mathcal{S}_\nu} \partial_\nu \psi_\kappa^\varepsilon &= 0, \text{ for all } \nu = 1, \ldots, N.
\end{align*}
\]

(3.85)

Thus \( \psi_k^\varepsilon \) can be considered as a “reflected” circulation stream function: one can view it as the part of \( \psi_\kappa \) due to the response of the domain to the standalone stream function \( \hat{\psi}_\kappa \). We have the following estimates on \( \psi_k^\varepsilon \).

Lemma 3.28. Let \( \delta > 0 \). There exists \( \varepsilon_0 > 0 \) such that the following holds. Let \( \kappa \in \{1, \ldots, N\} \) and \( k \in \mathbb{N} \). There exists \( C > 0 \) such that for any \( \overline{\varepsilon} \) such that \( \overline{\varepsilon} \leq \varepsilon_0 \) and any \( \underline{q} \in \mathcal{Q}_\delta \), one has

\[
\| \nabla \psi_\kappa^\varepsilon \|_{L^\infty(\mathcal{F})} \leq C,
\]

\[
\forall \lambda \in \{1, \ldots, N\}, \quad \varepsilon_\lambda^{-\frac{1}{2}} \| \psi_\kappa^\varepsilon \|_{C^{0,\frac{1}{2}}(\partial \mathcal{S}_\lambda)} \leq C.
\]

(3.86)
Proof of Lemma 3.28. We let

\[ \mathbf{A} := (\hat{\psi}_k|_{\delta S_1}, \ldots, \hat{\psi}_k|_{\delta S_N}, 0, \ldots, 0, \hat{\psi}_k|_{\partial \Omega}) \quad \text{and} \quad \hat{\mathbf{A}} := (\hat{\psi}_k|_{\delta S_1}, \ldots, \hat{\psi}_k|_{\delta S_N}, \hat{\psi}_k|_{\partial \Omega}), \]

where moreover we replace the \( \kappa \)-th element \( \hat{\psi}_k|_{\delta S_\kappa} \) with 0 whenever \( \kappa \in \mathcal{P}_1 \).  \( \tag{3.88} \)

With Propositions 3.9 and 3.10 in mind, we rewrite \( \psi^r_k \) as

\[ \psi^r_k = -\delta[\mathbf{A}] - \sum_{\kappa \in \mathcal{P}_1 \setminus \{ \kappa \}} f_\kappa[\hat{\psi}_k|_{\delta S_\kappa}]. \]  \( \tag{3.89} \)

Due to Lemma 3.27, \( \nabla \hat{\psi}_\kappa \) is bounded on \( \{ x / d(x, \partial S_\kappa) \geq \delta \} \), and hence so is \( \hat{\psi}_\kappa \). Thanks to interior elliptic estimates we may even obtain that

\[ \varepsilon_\kappa^{-1} \| \hat{\psi}_\kappa - \hat{\psi}_\kappa(h_\nu) \|_{L^\infty(\delta S_\kappa)} + |\hat{\psi}_\kappa|_{C^{1,1}(\delta S_\kappa)} \]  is bounded for \( \nu \neq \kappa \). \( \tag{3.90} \)

With uniform Schauder estimates in \( \bar{\mathcal{F}} \) (Lemma 3.3), this involves that \( \| \bar{\mathbf{A}} \|_{C^{1,1}(\bar{\mathcal{F}})} \) is bounded. With Proposition 3.10 we deduce that \( \delta[\mathbf{A}] \) gives a bounded contribution to (3.86) and (3.87).

For what concerns the second term in (3.89), we use Proposition 3.9 and (3.90). It remains then to estimate the corresponding combination of standalone potentials \( \hat{f}_\nu[\hat{\psi}_k|_{\delta S_\kappa}] \) for \( \nu \in \mathcal{P}_1 \setminus \{ \kappa \} \). The conclusion follows from Proposition 3.4.

In addition to these uniform estimates, one may describe the limit of these circulation vector fields. For that we rely on the decomposition

\[ \nabla^\perp \hat{\psi}_\kappa = \nabla^\perp \hat{\psi}_\kappa + \nabla^\perp \psi^r_k, \]  \( \tag{3.91} \)

and introduce two particular velocity vector fields that appear in the limit. For \( \kappa \in \mathcal{P}_s \), we denote

\[ H_\kappa(x) := \frac{(x - h_\kappa)^\perp}{2\pi |x - h_\kappa|^2}, \]  \( \tag{3.92} \)

and for \( \kappa \in \mathcal{P}_s \), the potential \( \hat{\psi}^r_k \) as the solution (up to an additive constant) of

\[ \begin{align*}
\Delta \hat{\psi}^r_k &= 0 \quad \text{in} \quad \bar{\mathcal{F}}(q_\kappa), \\
\nabla^\perp \hat{\psi}^r_k(x) \cdot n(x) &= -H_\kappa(x) \cdot n(x) \quad \text{on} \quad \partial \Omega \cup \bigcup_{\nu \in \mathcal{P}_1 \setminus \{ \kappa \}} \partial S_\nu, \\
\int_{\partial S_\kappa} \nabla^\perp \hat{\psi}^r_k \cdot \tau \, ds &= 0 \quad \text{for} \quad \nu \in \mathcal{P}_1. 
\end{align*} \]

(3.93)

It is straightforward to see that for any \( \kappa \in \mathcal{P}_s \),

\[ H_\kappa + \nabla^\perp \hat{\psi}^r_k = \bar{K}[\delta h_\kappa] \quad \text{in} \quad \bar{\mathcal{F}}(q_\kappa). \]

(3.94)

Then we have the following convergences, where all vector fields are put to 0 inside the solids.

**Proposition 3.29.** Let \( \delta > 0 \). Uniformly for \( q \in \mathcal{Q}_s \), one has as \( \varepsilon \to 0 \) for any \( \kappa \in \mathbb{N}, p < +\infty \) and any \( c > 0 \):

\[ \forall \kappa \in \mathcal{P}_1, \quad \nabla^\perp \hat{\psi}_\kappa \rightharpoonup H_\kappa(x) \quad \text{in} \quad L^p(\Omega) \quad \text{for} \quad p \in [1,2], \quad \text{and in} \quad C^k, \]

\( \forall \kappa \in \mathcal{P}_1, \quad \nabla^\perp \psi^r_k \rightharpoonup \nabla^\perp \hat{\psi}^r_k \quad \text{in} \quad L^p(\Omega) \) and \( \text{in} \ L^\infty, \)

\[ \forall \kappa \in \mathcal{P}_1, \quad \nabla^\perp \psi^r_k \rightharpoonup \nabla^\perp \hat{\psi}^r_k \quad \text{in} \quad L^p(\Omega) \) and \( \text{in} \ L^\infty, \]

(3.95)

**Proof of Proposition 3.29.** We begin with the proof of (3.95). Considering \( \kappa \in \mathcal{P}_1 \) and \( p \in [1,2] \), we first cut the integral in two:

\[ \int_{\Omega_{\delta S_\kappa}} \left| \nabla^\perp \hat{\psi}^r_k - H_\kappa(x) \right|^p \, dx \leq \int_{B(h_\kappa, C\varepsilon x)} \left| \nabla^\perp \hat{\psi}^r_k - H_\kappa(x) \right|^p \, dx + \int_{\Omega \setminus B(h_\kappa, C\varepsilon x)} \left| \nabla^\perp \hat{\psi}^r_k - H_\kappa(x) \right|^p \, dx, \]

35
Using again Lemma 3.11, we see that each of the terms satisfy (2.20). This gives (3.97). We recall that

Proof of Lemma 3.30. We proceed in two steps.  

converges to 0 in where was defined in (3.81). Again, due to Lemma 3.11, each of the terms converges to 0 in (x ∈ Ω/d(x, ∪ν∈Pν, Sν) ≥ c). Now by Proposition 3.10,

We now prove (3.96). Let

The proof of (3.97) is analogous. Let κ ∈ P(Ω). Here (3.84) and (3.89) give

where A was defined in (3.88). Again, due to Lemma 3.11, each of the terms above converges to 0 in (p < +∞) and in \(L^\infty((x ∈ Ω/d(x, ∪ν∈Pν, Sν) ≥ c))\). Moreover, Proposition 3.10 gives us that

Using again Lemma 3.11, we see that each of the terms above converges to 0 in (Ω/d(x, ∪ν∈Pν, Sν) ≥ c). It remains to observe that here \(\hat{\psi}_κ - \hat{g}(\hat{A}) = \psi_κ\) in \(\tilde{F}\) since both sides satisfy (2.20). This gives (3.97).

3.3.2 Shape derivatives of the reflected circulation stream function

Here we are interested in differentiating \(\psi_κ^\varepsilon\) with respect to \(q_{\mu,m}\).

Lemma 3.30. Let \(δ > 0\). There exist \(\varepsilon_0 > 0\) and \(C > 0\) such that for all \(κ, \mu \in \{1, \ldots, N\}\), \(m \in \{1, 2, 3\}\), for all \(q \in Q_δ\),

Prove of Lemma 3.30. We proceed in two steps.
Step 1. We rely on (3.85) and use Corollary 3.15 and Remark 3.16 to write
\[
\frac{\partial \psi^\varepsilon_k}{\partial \eta_{\mu,m}} = \begin{cases} 
-\delta_{\lambda\neq \kappa} \frac{\partial \hat{\psi}_k}{\partial \eta_{\mu,m}} - (\delta_{\lambda\neq \kappa} \nabla \hat{\psi}_k + \nabla \psi^\varepsilon_k) \cdot \xi_{\mu,m} + c^\lambda & \text{on } \partial \mathcal{S}_\lambda \text{ for } \lambda = 1, \ldots, N, \\
-\delta_{\mu\neq \kappa} \frac{\partial \hat{\psi}_k}{\partial \eta_{\mu,m}} & \text{on } \partial \Omega.
\end{cases}
\]  
(3.100)

We now study the various terms in the first line of (3.100). Due to Lemma 3.27 we have
\[
\frac{\partial \hat{\psi}_k}{\partial \eta_{\kappa,m}} = -\nabla \hat{\psi}_k \cdot \xi^*_\kappa_m \text{ in } \mathbb{R}^d \setminus \mathcal{S}_\kappa \text{ with } \xi^*_\kappa_j(q, x) := e_j \text{ for } j = 1, 2 \text{ and } \xi^*_\kappa,3(q, x) := (x - h_\kappa)^+ \text{ in } \mathbb{R}^d.
\]  
(3.101)

The term \(\delta_{\lambda\neq \kappa} \delta_{\mu\kappa} \frac{\partial \hat{\psi}_k}{\partial \eta_{\mu,m}}\) merely gives a contribution when \(\mu = \kappa\) on all the connected components of the boundary but \(\partial \mathcal{S}_\mu = \partial \mathcal{S}_\kappa\). Due to (3.81) and (3.101), this contribution satisfies, up to an additional constant,
\[
\left\| \delta_{\mu\neq \kappa} \delta_{\mu\kappa} \frac{\partial \hat{\psi}_k}{\partial \eta_{\mu,m}} \right\|_{L^\infty(\mathcal{S}_\lambda)} \leq C \varepsilon_{\mu,3}^{\delta_{\mu,3}}.
\]

Using inner regularity for the Laplace equation and (3.83), the same holds in \(C^{k,\frac{1}{2}}(\mathcal{F} \setminus \mathcal{V}_0(\partial \mathcal{S}_\kappa))\). Hence, up to an additive constant, we deduce
\[
\left\| \delta_{\lambda\neq \kappa} \delta_{\mu\kappa} \frac{\partial \hat{\psi}_k}{\partial \eta_{\mu,m}} \right\|_{L^\infty(\mathcal{S}_\lambda)} \leq C \varepsilon_{\lambda,3}^{\delta_{\mu,3}}.
\]

Let us now turn to the second term, which merely gives a contribution on \(\partial \mathcal{S}_\lambda\) when \(\lambda = \mu \neq \kappa\) (recall (2.7)). By Lemma 3.27 and (3.81), we see that the term \(\delta_{\lambda\neq \kappa} \nabla \hat{\psi}_k \cdot \xi_{\mu,m}\) gives a contribution of order \(\varepsilon_{\mu,3}^{\delta_{\mu,3}}\) in \(L^\infty\)-norm and in \(C^{k,\frac{1}{2}}\)-norm on \(\partial \mathcal{S}_\mu\).

Finally we consider the last term, which only again gives a contribution on \(\partial \mathcal{S}_\lambda\) when \(\lambda = \mu\). By Lemma 3.28, the term \(\nabla \hat{\psi}_k \cdot \xi_{\mu,m}\) gives a contribution of size \(\varepsilon_{\mu,3}^{k+1}\) in \(L^\infty\)-norm and at worst of order \(O(\varepsilon_{\mu,3}^{\delta_{\mu,3}} \varepsilon_{\lambda}^{\kappa-\frac{1}{2}})\) in \(C^{k,\frac{1}{2}}\)-norm on \(\partial \mathcal{S}_\mu\).

Gathering these estimates we obtain, up to an additive constant on each connected component \(\mathcal{S}_\lambda\) of the boundary, that for \(k \geq 1\),
\[
\left\| \frac{\partial \psi^\varepsilon_k}{\partial \eta_{\mu,m}} \right\|_{L^\infty(\mathcal{S}_\lambda)} \leq C \varepsilon_{\mu,3}^{\delta_{\mu,3}} \varepsilon_{\lambda}^{\kappa+\frac{1}{2}} \left\| \frac{\partial \psi^\varepsilon_k}{\partial \eta_{\mu,m}} \right\|_{C^{k,\frac{1}{2}}(\mathcal{S}_\lambda)} \leq C \varepsilon_{\mu,3}^{\delta_{\mu,3}} \varepsilon_{\lambda}^{\kappa+\frac{1}{2}},
\]
\[
\left\| \frac{\partial \psi^\varepsilon_k}{\partial \eta_{\mu,m}} \right\|_{L^\infty(\mathcal{S}_\lambda)} \leq C \varepsilon_{\mu,3}^{\delta_{\mu,3}} \text{ and } \left\| \frac{\partial \psi^\varepsilon_k}{\partial \eta_{\mu,m}} \right\|_{C^{k,\frac{1}{2}}(\mathcal{S}_\lambda)} \leq C \varepsilon_{\mu,3}^{\delta_{\mu,3}}.
\]

Step 2. As before we write
\[
\frac{\partial \psi^\varepsilon_k}{\partial \eta_{\mu,m}} = \mathcal{J} \left[ \frac{\partial \psi^\varepsilon_k}{\partial \eta_{\mu,m} \mid \mathcal{S}_1}, \ldots, \frac{\partial \psi^\varepsilon_k}{\partial \eta_{\mu,m} \mid \mathcal{S}_b \setminus \mathcal{S}_1} \right] + \sum_{\lambda \in \mathcal{P}_\lambda} \mathcal{J}_\lambda \left[ \frac{\partial \psi^\varepsilon_k}{\partial \eta_{\mu,m} \mid \partial \mathcal{S}_\lambda} \right].
\]

The \(\mathcal{J}\) term is bounded in \(W^{1,\infty}(\tilde{\mathcal{F}})\) due Proposition 3.10, the above estimates and the uniform Schauder estimates in \(\tilde{\mathcal{F}}\) (Lemma 3.3). The \(\mathcal{J}_\lambda\) terms can be replaced by their standalone counterpart \(\mathcal{J}_\lambda\) thanks to Proposition 3.9. These \(\mathcal{J}_\lambda\) are estimated by Proposition 3.4 which gives the estimates in (3.98).

Concerning (3.99), by the above considerations, we only need to discuss the contribution of the \(\mathcal{J}_\lambda\) terms. Mixing (3.8) and (3.9), and distinguishing \(x \in B(h_\lambda, C\varepsilon) \setminus \mathcal{S}_\lambda\) and \(x \in \mathcal{F} \setminus B(h_\lambda, C\varepsilon)\), we see that
\[
\forall x \in \mathcal{F}, \quad \left\| \nabla \hat{\psi}_k \frac{\partial \psi^\varepsilon_k}{\partial \eta_{\mu,m} \mid \mathcal{S}_\lambda} \right\| (x) \leq C \varepsilon_{\lambda} \left\| \frac{\partial \psi^\varepsilon_k}{\partial \eta_{\mu,m} \mid \mathcal{S}_\lambda} \right\|_{L^\infty(\mathcal{S}_\lambda)} + \varepsilon_{\mu,3}^{\kappa+\frac{1}{2}} \left\| \frac{\partial \psi^\varepsilon_k}{\partial \eta_{\mu,m} \mid \mathcal{S}_\lambda} \right\|_{C^{k,\frac{1}{2}}(\partial \mathcal{S}_\lambda)}.
\]  
(3.102)

Now we put the above inequality to the power \(p\) and integrate. We can inject \(\Omega\) in some ball \(B(h_\lambda, R)\) with \(R > 0\) fixed so that we write \(\mathcal{F} \subset B(h_\lambda, R) \setminus B(h_\lambda, C\varepsilon)\) for some positive \(C\). The result follows. \(\square\)
3.3.3 Reflected circulation stream function of a phantom solid

In this paragraph, we extend the above estimates on the reflected circulation stream function \( \psi_\kappa^r \) to a slight variant. This variant will play an important role in the definition of the modulation and in the passage to the limit, in particular for what concerns the desingularization (1.27).

For \( \kappa \in \mathcal{P}_s \) we first introduce the following "\( \kappa \)-augmented" fluid domain as follows:

\[
\tilde{\mathcal{F}}_\kappa(q) := \mathcal{F}(q) \cup S_\kappa(q).
\]

Note in particular that

\[
\partial \tilde{\mathcal{F}}_\kappa(q) = \partial \mathcal{F}(q) \setminus \partial S_\kappa(q) = \partial \Omega \cup \bigcup_{\kappa \in \{1, \ldots, N\} \setminus \{\kappa\}} \partial S_\kappa.
\]

Now we introduce \( \psi_\kappa^{r, \varphi} \) as the solution in \( \tilde{\mathcal{F}}_\kappa(q) \) (together with constants \( c_\lambda, \lambda \in \{1, \ldots, N\} \setminus \{\kappa\} \)) to the system:

\[
\begin{cases}
\Delta \psi_\kappa^{r, \varphi} = 0 & \text{in } \tilde{\mathcal{F}}_\kappa(q), \\
\psi_\kappa^{r, \varphi} = -\hat{\psi}_\kappa + c_\lambda & \text{on } \partial S_\lambda(q), \quad \lambda \in \{1, \ldots, N\} \setminus \{\kappa\}, \\
\psi_\kappa^{r, \varphi} = -\hat{\psi}_\kappa & \text{on } \partial \Omega, \\
\int_{\partial S_\kappa(q)} \partial_n \psi_\kappa^{r, \varphi} \, ds = 0 & \text{for } \nu \in \{1, \ldots, N\} \setminus \{\kappa\}.
\end{cases}
\]

(3.104)

The only difference indeed between \( \psi_\kappa^r \) and \( \psi_\kappa^{r, \varphi} \) is that the constraint \( \psi_\kappa^r = c_\kappa \) on \( \partial S_\kappa \) in (3.85) has disappeared in (3.104), and that the domain is \( \tilde{\mathcal{F}}_\kappa \) rather than \( \mathcal{F} \). Adapting the arguments above we obtain the following result.

**Lemma 3.31.** Let \( \delta > 0 \). There exists \( \varepsilon_0 > 0 \) such that the following holds. Let \( \kappa \in \mathcal{P}_s \) and \( k \in \mathbb{N} \). There exists \( C > 0 \) such that for any \( \Omega \) such that \( \varepsilon \leq \varepsilon_0 \) and any \( q \in \Omega \), one has

\[
\left\| \nabla \psi_\kappa^{r, \varphi} \right\|_{L^p(\mathcal{F})} \leq C, \quad \text{and } \forall \lambda \in \{1, \ldots, N\}, \quad \varepsilon_\lambda^{\delta_\kappa, \varphi, k} \left( \psi_\kappa^{r, \varphi} \right) \|_{C^{k-\frac{1}{2}}(\partial_\kappa S_\lambda)} \leq C, \tag{3.105}
\]

\[
\left\| \nabla \frac{\partial \psi_\kappa^{r, \varphi}}{\partial q_{\mu, m}} \right\|_{L^p(\mathcal{F} \setminus \partial_\kappa S_\mu)} \leq C \varepsilon_\mu^{\delta_\mu} \quad \text{and } \quad \left\| \nabla \frac{\partial \psi_\kappa^{r, \varphi}}{\partial q_{\mu, m}} \right\|_{L^p(\partial_\kappa S_\mu)} \leq C \varepsilon_\mu^{-1+\delta_\mu+\delta_{\varphi, \kappa}}. \tag{3.106}
\]

Moreover, uniformly for \( q \in \Omega \), one has as \( \varepsilon \to 0 \) for any \( k \in \mathbb{N} \), \( p < +\infty \) and any \( c > 0 \):

\[
\nabla^+ \psi_\kappa^{r, \varphi} \longrightarrow \nabla^+ \hat{\psi}_\kappa \quad \text{in } L^p(\Omega) \quad \text{and in } L^p \left( \{ x \in \Omega : d(x, \bigcup_{\nu \in \mathcal{P}_s \setminus \{\kappa\}} S_\nu) \geq c \} \right). \tag{3.107}
\]

**Proof of Lemma 3.31.** This is a mere adaptation of Lemmas 3.28 and 3.30 and of (3.96). Hence we only stress the variations in the proofs.

To get (3.105), the main point is that (3.89) has to be replaced by

\[
\psi_\kappa^{r, \varphi} = -\mathcal{Y}(A) - \sum_{\nu \in \mathcal{P}_s \setminus \{\kappa\}} \mathcal{F}^\nu \left[ \hat{\psi}_{\kappa, \nu} \right] \tag{3.108}
\]

where the potentials \( \mathcal{Y} \) and \( \mathcal{F} \) correspond to the domain \( \tilde{\mathcal{F}}_\kappa \) rather than \( \mathcal{F} \), and where we define the \( N \)-tuple \( A := (\hat{\psi}_{\kappa, \nu} \mathcal{F}^\nu S_\nu(q), 0, \ldots, 0, \hat{\psi}_{\kappa, \nu} \mathcal{F}^\nu \Omega) \), where \( N \) corresponds to \( N-1 \) solids plus \( \Omega \) (because there is no boundary \( \partial S_\kappa \)). Then the same argument as in Lemma 3.28 applies to obtain (3.105), using Propositions 3.9 and 3.10 in the domain with \( N-1 \) solids \( \tilde{\mathcal{F}}_\kappa \).

Concerning the estimate (3.106) of the shape derivative, when \( \mu \neq \kappa \), it suffices to make the slight correction to the boundary condition (3.100):

\[
\frac{\partial \psi_\kappa^{r, \varphi}}{\partial q_{\mu, m}} = \begin{cases}
\frac{\partial \hat{\psi}_{\kappa}}{\partial q_{\mu, m}} - \left( \nabla \psi_\kappa^{r, \varphi} + \nabla \hat{\psi}_\kappa \right) \cdot \xi_{\mu, m} + c_\lambda \text{ on } \partial S_\lambda & \text{for } \lambda \in \{1, \ldots, N\} \setminus \{\kappa\}, \\
\frac{\partial \psi_\kappa}{\partial q_{\mu, m}} & \text{on } \partial \Omega.
\end{cases}
\]
Then the same reasoning as in Lemma 3.30 applies. Importantly enough, \( \partial S_\kappa \) is now in the bulk of the domain \( \tilde{\mathcal{F}}_\kappa \) so that the standalone potentials (see (3.102)) give a bounded contribution to \( \nabla \frac{\partial \hat{\psi}_{\kappa,m}}{\partial \psi_{\kappa,m}} \) in the neighborhood of \( S_\kappa \).

When \( \mu = \kappa \), the situation is a bit different, because \( \frac{\partial}{\partial \psi_{\kappa,m}} \) is no longer a shape derivative (the domain \( \tilde{\mathcal{F}}_\kappa \) does not depend on \( \psi_{\kappa,m} \)) but a simple derivative with respect to a parameter. The boundary condition becomes \( \frac{\partial \hat{\psi}_{\kappa,m}}{\partial \psi_{\kappa,m}} = \frac{\partial \hat{\psi}_{\kappa,m}}{\partial \psi_{\kappa,m}} \) on \( \partial \tilde{\mathcal{F}}_\kappa \), and the boundedness of \( \varepsilon_k \delta_m \nabla \frac{\partial \hat{\psi}_{\kappa,m}}{\partial \psi_{\kappa,m}} \) (here in the whole \( \tilde{\mathcal{F}}_\kappa \)) follows as before.

Finally, to prove (3.107), we rely again on (3.108) and reason as for (3.97). We approximate \( \nabla \hat{\psi}^\kappa \mathbf{A} \) by \( \nabla \hat{\mathbf{g}}(\mathbf{A}) \) with the same \( \hat{\mathbf{g}} \) and the same \( \mathbf{A} := (\hat{\psi}_{\kappa|\partial S_1}, \ldots, \hat{\psi}_{\kappa|\partial S_{N(i)}}, \hat{\psi}_{\kappa|\partial \Omega}) \) as in the proof of (3.97) (since \( \kappa \in \mathcal{P}_s \)). Hence we obtain the same limit.

### 3.4 Estimates of the Biot-Savart kernel

#### 3.4.1 Biot-Savart kernel

The following will be useful for both the *a priori* estimates and the passage to the limit. We consider \( \omega \in L^2(F) \) and compare the generated velocity \( K[\omega] \) in \( F \) (in the domain with all solids) and the generated velocity \( \tilde{K}[\omega] \) in \( \tilde{F} \) (in the larger domain with only solids of family (i)) as defined in (2.21) and (2.22). In particular we prove that these velocity fields are bounded independently of \( \mathbf{z} \). Precisely we have the following result.

**Lemma 3.32.** Let \( \delta > 0 \). There exists \( \varepsilon_0 > 0 \) such that the following holds. For any \( p \in (2, +\infty] \), there exist \( C > 0 \) such that for any \( (\varepsilon, \mathbf{q}, \omega) \in \Omega^\infty_\kappa \), one has

\[
|K[\omega]|_{L^p(F)} \leq C|\omega|_{L^p(F)} \quad \text{and} \quad \varepsilon^{\frac{k-1}{2}}|K[\omega]|_{C^{k-1,\frac{1}{2}}(\partial S_\kappa)} \leq C|\omega|_{L^p(F)}, \forall \lambda = 1, \ldots, N. \tag{3.109}
\]

In the same way, there exists \( \varepsilon_0 > 0 \) and for each \( p \in (1, +\infty) \), there exist \( C > 0 \) such that for any \( (\varepsilon, \mathbf{q}) \in Q^\infty_\kappa \), any \( f \in L^p(F^\kappa(\mathbf{q}); \mathbb{R}^2) \) such that \( \text{dist}(\text{Supp}(f), \partial F^\kappa(\mathbf{q})) \geq \delta \),

\[
\|K[\text{div}(f)]\|_{L^p(F^\kappa)} \leq C\|f\|_{L^p(F^\kappa)}. \tag{3.110}
\]

Finally, uniformly for \( (\mathbf{q}, \omega) \) such that \( (\varepsilon, \mathbf{q}, \omega) \in \Omega^\infty_\kappa \) when \( \mathbf{z} \) is small and \( \omega \) is bounded in \( L^\infty \),

\[
\left| K[\omega] - \tilde{K}[\omega] \right|_{L^p(F(\mathbf{q}))} \to 0 \quad \text{for} \quad p \in (2, +\infty)
\]

and

\[
\left| K[\omega] - \tilde{K}[\omega] \right|_{L^p(\{x \in \tilde{F}: d(x, \omega \in \partial S_\kappa \geq \varepsilon\})} \to 0 \quad \text{as} \quad \varepsilon \to 0. \tag{3.111}
\]

**Remark 3.33.** Actually, our proof only needs \( \omega \) or \( f \) to be supported away from the small solids.

**Proof of Lemma 3.32.** For \( \delta > 0 \), we let \( \varepsilon_0 \) as in Lemma 3.6. Clearly, the difference \( \tilde{R}[\omega] := K[\omega] - \tilde{K}[\omega] \) satisfies

\[
\begin{aligned}
\text{div} \tilde{R}[\omega] &= \text{curl} \tilde{R}[\omega] = 0 \quad \text{in} \quad F(\mathbf{q}), \\
\tilde{R}[\omega] \cdot n &= 0 \quad \text{on} \quad \partial \tilde{F}(\mathbf{q}), \\
\tilde{R}[\omega] \cdot n &= -\tilde{K}[\omega] \cdot n \quad \text{on} \quad \partial F \setminus \partial \tilde{F}(\mathbf{q}), \\
\int_{\partial S_\kappa} \tilde{R}[\omega] \cdot \tau \, ds &= 0 \quad \text{for} \quad \nu = 1, \ldots, N.
\end{aligned}
\]

In particular one can write \( \tilde{R}[\omega] = \nabla \tilde{\varphi}[\omega] \) where

\[
\tilde{\varphi}[\omega] = - \sum_{\kappa \in \mathcal{P}_s} f_\kappa \left[ \tilde{\Psi}[\omega]|_{\partial S_\kappa} \right], \tag{3.112}
\]

and where \( \tilde{\Psi}[\omega] \) is a stream function for \( \tilde{K}[\omega] \), that is, \( \tilde{K}[\omega] = \nabla \tilde{\Psi}[\omega] \).
We first estimate $\bar{K}[\omega]$. As for Lemma 3.3, we have uniform Calderón-Zygmund estimates (see e.g. [6, Lemma 9.17]) in $\bar{F}$ as long as $q_{(i)} \in Q_{(i),\delta}$. It follows that for each $p \in (1, +\infty)$, one has a uniform constant $C > 0$ such that
\[
\|\bar{K}[\omega]\|_{W^{1,p}(\bar{F})} \leq C\|\omega\|_{L^p(\bar{F})}. \tag{3.113}
\]
Then we invoke Sobolev embedding for $p > 2$ to get the bound
\[
\|\bar{K}[\omega]\|_{L^p(\bar{F})} \leq C\|\omega\|_{\infty}. \tag{3.114}
\]
This embedding is also uniform in $\bar{F}$ as long as $q_{(i)} \in Q_{(i),\delta}$: it suffices to use an extension operator inside each solid and use the embedding in $\Omega$. We notice that since $\omega$ is distant from the solids, by inner regularity for the Laplace equation, we have
\[
\|\bar{K}[\omega]\|_{C^{\frac{1}{2}}(V_q(\bar{F}))} \leq C\|\omega\|_{\infty} \quad \text{and} \quad \|\bar{\Psi}[\omega]\|_{C^{\frac{1}{2}}(V_q(\bar{F}))} \leq C\|\omega\|_{\infty}. \tag{3.115}
\]
Now we apply Proposition 3.9 and Proposition 3.4 to each term in the right-hand-side of (3.112). This gives
\[
\|\tilde{R}[\omega]\|_{L^p(\bar{F})} \leq C\|\omega\|_{L^p(\bar{F})} \quad \text{and} \quad \varepsilon^{k-\frac{1}{2}}\|\tilde{R}[\omega]\|_{C^{k-\frac{1}{2}}(\partial\bar{S}_q)} \leq C\|\omega\|_{L^p(\bar{F})}, \forall \lambda = 1, \ldots, N.
\]
We consequently deduce (3.109) with (3.114).

The convergence (3.111) of $\tilde{R}[\omega]$ to 0 as $\varepsilon \to 0$ is proven as (3.42): it is a consequence of Lemma 3.11 and (3.115).

Finally (3.110) is proven in the same way, albeit in a weaker context. Denoting by $K_{2z}$ the Biot-Savart operator in the full plane, such that
\[
\begin{aligned}
&\text{div } K_{2z}[\omega] = 0 \quad \text{in } \mathbb{R}^2, \\
&\text{curl } K_{2z}[\omega] = \omega \quad \text{in } \mathbb{R}^2, \\
&K_{2z}[\omega](x) \to 0 \quad \text{as } x \to +\infty,
\end{aligned}
\]
we recall that $K_{2z} \circ \text{div } = \nabla^\perp \Delta_{\mathbb{R}^2}^{-1} \text{div}$ is a Calderón-Zygmund operator which sends $L^p(\mathbb{R}^2)$ into itself (for $p \in (1, +\infty)$). It remains to check that the correction to obtain $K[\text{div } f]$ is also estimated uniformly in $L^p(F_q)$. Thanks to the constraint on the support of $f$, it is again a consequence of interior elliptic estimates and of Propositions 3.9, 3.10 and 3.4. $\square$

### 3.4.2 Shape derivatives of the Biot-Savart kernel

In this paragraph, for fixed $\omega$, we estimate the shape derivative $\frac{\partial K[\omega]}{\partial \eta_{\mu,m}}$.

**Lemma 3.34.** Let $\delta > 0$, $\mu \in \{1, \ldots, N\}$, $m \in \{1, 2, 3\}$. There exists $C > 0$ and $\varepsilon_0 > 0$ such that for any $(\varepsilon, q, \omega) \in \Omega_s$,
\[
\left\|\frac{\partial K[\omega]}{\partial \eta_{\mu,m}}\right\|_{L^p(\mathbb{R}^2 \setminus \overline{\Delta_{\mathbb{R}^2}^{-1} B(q)})} \leq C\varepsilon^{\delta_3 |m|} \|\omega\|_{L^p(\mathbb{R}^2)}, \quad \left\|\frac{\partial K[\omega]}{\partial \eta_{\mu,m}}\right\|_{L^p(V_q(\partial S_q))} \leq C\varepsilon^{\delta_3 |m|} \|\omega\|_{L^p(\mathbb{R}^2)}, \quad \left\|\frac{\partial K[\omega]}{\partial \eta_{\mu,m}}\right\|_{L^p(\mathbb{R}^2 \setminus \overline{\Delta_{\mathbb{R}^2}^{-1} B(q)})} \leq C\varepsilon^{\delta_1 + \delta_3} \|\omega\|_{L^p(\mathbb{R}^2)}
\]
for $p < 2$.

**Proof of Lemma 3.34.** Here we first introduce $K_{1\Omega}[\omega]$ that satisfies
\[
\begin{aligned}
&\text{div } K_{1\Omega}[\omega] = 0 \quad \text{in } \Omega, \\
&\text{curl } K_{1\Omega}[\omega] = \omega \quad \text{in } \Omega, \\
&K_{1\Omega}[\omega] \cdot n = 0 \quad \text{on } \partial \Omega.
\end{aligned}
\tag{3.116}
\]
(Recall that we suppose $\Omega$ simply connected to simplify.) Note that $K_{1\Omega}[\omega]$ (whose shape derivative is obviously zero) can be put in the form $K_{1\Omega}[\omega] = \nabla^\perp \Psi_{1\Omega}[\omega]$ with $\Psi_{1\Omega}[\omega] = \Delta_{\Omega}^{-1} \omega$, where $\Delta_{\Omega}^{-1}$ is the usual
The inverse of the Laplacian with homogeneous Dirichlet boundary conditions on \( \partial \Omega \). Now the difference \( R[\omega] = K[\omega] - K_\Omega[\omega] \) satisfies

\[
\begin{align*}
\text{div } R[\omega] &= \text{curl } R[\omega] = 0 & \text{in } \mathcal{F}(q), \\
R[\omega] \cdot n &= -K_\Omega[\omega] \cdot n & \text{on } \partial S_\nu & \text{for } \nu = 1, \ldots, N, \\
R[\omega] \cdot n &= 0 & \text{on } \partial \Omega, \\
\int_{\partial S_\nu} R[\omega] \cdot \tau \, ds &= 0 & \text{for } \nu = 1, \ldots, N.
\end{align*}
\] (3.117)

It follows that \( R[\omega] \) can be put in the form \( R[\omega] = \nabla^2 \eta[\omega] \) with

\[
\begin{align*}
\Delta \eta[\omega] &= 0 & \text{in } \mathcal{F}(q), \\
\eta[\omega] &= -\Psi_\Omega[\omega] + c_\nu & \text{on } \partial S_\nu, & \text{for } \nu = 1, \ldots, N, \\
\eta[\omega] &= 0 & \text{on } \partial \Omega, \\
\int_{\partial S_\nu} \partial_n \eta[\omega] \, ds &= 0 & \text{for } \nu = 1, \ldots, N.
\end{align*}
\]

Consequently, using Corollary 3.15 we find that for some constants \( c'_\lambda, \lambda \in \{1, \ldots, N\} \), one has

\[
\frac{\partial \eta[\omega]}{\partial q_{\mu,m}} = 0 \text{ on } \partial \Omega \quad \text{and} \quad \frac{\partial \eta[\omega]}{\partial q_{\mu,m}} = c'_\lambda \text{ on } \partial S_\lambda \text{ for } \lambda \neq \mu,
\]

while on \( \partial S_\mu(q) \), one has

\[
\frac{\partial \eta[\omega]}{\partial q_{\mu,m}} = (-\partial_n \Psi_\Omega[\omega] + \partial_n \eta[\omega]) K_{\mu,m} + c'_\mu
\]

\[
= (K[\omega] + R[\omega]) \cdot \tau K_{\mu,m} + c'_\mu = (K[\omega] \cdot \tau) K_{\mu,m} + c'_\mu.
\]

Using Lemma 3.32, we can bound this boundary condition as in the proofs of Proposition 3.26 or Lemma 3.30, so that we obtain for some uniform constant \( C > 0 \)

\[
\left\| \frac{\partial \eta[\omega]}{\partial q_{\mu,m}} \right\|_{L^s(\partial S_\mu)} + \varepsilon \left\| \frac{\partial \eta[\omega]}{\partial q_{\mu,m}} \right\|_{L^s(\partial S_\mu)} \leq C \varepsilon^{\delta_{\mu,m}} |\omega|_{L^s(\mathcal{F}(q))}.
\]

Then we use that \( \frac{\partial \eta[\omega]}{\partial q_{\mu,m}} = 1 \mu \left( \frac{\partial \eta[\omega]}{\partial q_{\mu,m}} \right)_{\partial S_\mu} \) in \( \mathcal{F}(q) \), Propositions 3.9 and 3.10 to approximate it by the functions \( \tilde{\eta}_{\lambda} \) and \( \tilde{\gamma} \), and we estimate the latter by Proposition 3.4. The estimate in \( L^p \) norm is exactly the same as (3.99). We omit the details. \( \square \)

### 4 First a priori estimates

In this section we establish several a priori estimates on the system: on the fluid vorticity, on a renormalized energy of the system, giving a first bound of the solid velocities (which will be improved later on by modulated energy estimates), and a rough estimate of the solid accelerations. As we will see, these accelerations estimates are not straightforward and rely on a global normal form for the solids equations. They will help uncouple a bit the equations and obtain individual normal forms for the solid equations in Section 6.

#### 4.1 Vorticity estimates

**Lemma 4.1.** For a solution to System (1.2)-(1.7) and \( p \in [1, +\infty] \), \( |\omega|_p \) is conserved over time and given \( \delta > 0 \), \( |K[\omega]|_{L^p} \) is bounded independently of \( t \) and \( \varepsilon \) as long as \( (\varepsilon, q, \omega) \in \Omega_\delta \).

**Proof.** The first statement is due to

\[
\partial_t \omega^\varepsilon + (u^\varepsilon \cdot \nabla) \omega^\varepsilon = 0 \text{ in } \mathcal{F}^\varepsilon,
\]

and Liouville’s theorem. The second follows then from Lemma 3.32. \( \square \)
4.2 Energy estimates

This subsection is devoted to a sort of energy estimate, which gives a first bound of \( \hat{\rho}_\kappa \) (recall the definition in (2.1)).

**Proposition 4.2.** Let \( \delta > 0 \). There exist \( C > 0 \) and \( \varepsilon_0 > 0 \) such that as long as \( (\varepsilon, q, \omega) \in \Omega^\varepsilon_0 \), the solutions \((u^\varepsilon, h^\varepsilon, \vartheta^\varepsilon)\) of the system satisfy

\[
\forall \kappa \in \{1, \ldots, N\}, \quad |\varepsilon_{\kappa} \gamma_{\kappa} (\vartheta^\varepsilon_{\kappa}) \hat{\rho}_\kappa| \leq C. \tag{4.2}
\]

Let us mention that this estimate will be improved in the sequel, by considering modulated energy estimates.

**Proof of Proposition 4.2.** We first consider the total energy of the system:

\[
E(t) := \frac{1}{2} \sum_{\kappa \in \{1, \ldots, N\}} (m_\kappa |h_\kappa'|^2 + J_\kappa |\vartheta \kappa'|^2) + \frac{1}{2} \int_{\mathcal{F}(t)} |u|^2 \, dx. \tag{4.3}
\]

For a solution to (1.2)-(1.7), this energy \( E(t) \) is conserved over time. This is proven by multiplying (1.2) by \( u \), the equations in (1.7) by \( h_\kappa' \) and \( \vartheta_\kappa' \), respectively, summing and integrating by parts. Now the conservation of \( E(t) \) is insufficient to reach Proposition 4.2 directly because the energy is not bounded as \( \varepsilon \) goes to 0. This is due to the circulation part of the fluid velocity (see the second term in the decomposition (2.24)) corresponding to small solids. To circumvent this difficulty we will rather use the following quantity:

\[
\frac{1}{2} \sum_{\kappa \in \{1, \ldots, N\}} (m_\kappa |h_\kappa'|^2 + J_\kappa |\vartheta \kappa'|^2) + \frac{1}{2} \int_{\mathcal{F}(t)} |u^{\text{pot}}|^2 \, dx, \tag{4.4}
\]

where the potential part of the fluid velocity \( u^{\text{pot}} \) was defined in (2.25). Since, by (2.13),

\[
\frac{1}{2} \sum_{\kappa \in \{1, \ldots, N\}} (m_\kappa |h_\kappa'|^2 + J_\kappa |\vartheta \kappa'|^2) + \frac{1}{2} \int_{\mathcal{F}(t)} |u^{\text{pot}}|^2 \, dx = \frac{1}{2} \mathcal{M} \mathbf{p} \cdot \mathbf{p},
\]

in order to prove Proposition 4.2, it is sufficient to show that the quantity above is bounded independently of \( t \) and \( \varepsilon \). Indeed, once this is obtained, one uses \( \mathcal{M}_\mathbf{s} \leq \mathcal{M} \) to get a bound on \( \hat{\rho}_\kappa \) for \( \kappa \in \mathcal{P}_{(I)} \cup \mathcal{P}_{(II)} \), and one uses \( \mathcal{M}_\mathbf{a} \leq \mathcal{M} \) together with Corollary 3.23 and Remarks 2.1 and 3.24 to deduce a bound on \( \varepsilon_{\kappa} \hat{\rho}_\kappa \) when \( \kappa \in \mathcal{P}_{(III)} \).

To prove that the quantity in (4.4) is bounded we rely on the decomposition (2.24) of the fluid velocity. We call \( u^c \) the circulation part of the fluid velocity, that is second term in the right-hand side of (2.24):

\[
u^c := \sum_{\kappa \in \{1, \ldots, N\}} \gamma_\kappa \nabla^\perp \psi_\kappa(q(t), \cdot).
\]

Since \( K[\omega] \) is orthogonal to \( u^{\text{pot}} \) in \( L^2(\mathcal{F}(q)) \) (as follows from an integration by parts), we can decompose the energy (4.3) as

\[
E(t) = \frac{1}{2} \sum_{\kappa \in \{1, \ldots, N\}} (m_\kappa |h_\kappa'|^2 + J_\kappa |\vartheta \kappa'|^2) + \frac{1}{2} \int_{\mathcal{F}(t)} |u^{\text{pot}}(t, \cdot)|^2 \, dx
+ \frac{1}{2} \int_{\mathcal{F}(t)} |K[\omega]|^2 \, dx + \int_{\mathcal{F}(t)} u^c(t, \cdot) \cdot (K[\omega] + u^{\text{pot}})(t, \cdot) \, dx + \frac{1}{2} \int_{\mathcal{F}(t)} |u^c|^2 \, dx.
\]

Proposition 4.2 then follows from the assumptions on the initial data, Lemma 4.1, the conservation of \( E(t) \) and the following lemma.

**Lemma 4.3.** For \( \delta > 0 \), there exists \( \varepsilon_0 > 0 \) such that the following properties hold as long as \( (\varepsilon, q, \omega) \in \Omega^\varepsilon_0 \):

\[
\int_{\mathcal{F}(t)} |u^c(t, \cdot)|^2 \, dx - \int_{\mathcal{F}(0)} |u^c(0, \cdot)|^2 \, dx \text{ is bounded independently of } t \text{ and } \varepsilon, \tag{4.5}
\]

\[
\int_{\mathcal{F}(t)} u^c(t, \cdot) \cdot (K[\omega] + u^{\text{pot}})(t, \cdot) \, dx \text{ is bounded independently of } t \text{ and } \varepsilon. \tag{4.6}
\]
Proof of Lemma 4.3. We first notice that the vector fields $\nabla \psi_{\nu}$ are orthogonal one to another in $L^2$ as follows from an integration by parts. Hence to prove (4.5) it suffices to prove that the circulation stream functions $\psi_{\nu}$ satisfy

$$\int_{\mathcal{F}(t)} |\nabla \psi_{\nu}(q(t), \cdot)|^2 \, dx - \int_{\mathcal{F}(0)} |\nabla \psi_{\nu}(q(0), \cdot)|^2 \, dx$$

is bounded independently of $t$ and $\varepsilon$.

We use Lemma 3.28; consequently it suffices to prove that for all $\nu$, the standalone circulation stream function $\hat{\psi}_{\nu}$ satisfies

$$\int_{\mathcal{F}(t)} |\nabla \hat{\psi}_{\nu}(q(t), \cdot)|^2 \, dx - \int_{\mathcal{F}(0)} |\nabla \hat{\psi}_{\nu}(q(0), \cdot)|^2 \, dx$$

is bounded independently of $t$ and $\varepsilon$. (4.7)

Now using Lemma 3.27, we see that

$$\int_{\mathcal{F}(t)} |\nabla \hat{\psi}_{\nu}(q(t), \cdot)|^2 \, dx = \int_{\mathcal{F}(t)} |\nabla \hat{\psi}_{\nu}(q(0), \cdot)|^2 \, dx.$$

Then

$$\left| \int_{\mathcal{F}(t)} |\nabla \hat{\psi}_{\nu}(q(t), \cdot)|^2 \, dx - \int_{\mathcal{F}(0)} |\nabla \hat{\psi}_{\nu}(q(0), \cdot)|^2 \, dx \right| \leq \int_{\Delta_{\nu}} |\nabla \hat{\psi}_{\nu}(q(0), \cdot)|^2 \, dx,$$

where $\Delta_{\nu}$ is the symmetric difference $(R(-\partial_{\nu})(\mathcal{F}(t) - h_{\nu}(t)) + h_{\nu, 0}) \Delta \mathcal{F}(0)$. Since $(\varepsilon, \mathbf{q}, \omega) \in \Omega^{\delta}$ and $\mathcal{F}(t) \subset \Omega$, there is $R > 0$ independent of $\varepsilon$ such that $\Delta_{\nu} \subset B(h_{\nu, 0}, R) \setminus B(h_{\nu, 0}, \delta)$. Hence using (3.81), we arrive at (4.7) and hence at (4.5).

To get (4.6) we first integrate by parts:

$$\int_{\mathcal{F}(t)} \nabla \psi_{\nu} \cdot (K[\omega] + u_{\text{pot}}) \, dx = -\int_{\mathcal{F}(t)} \psi_{\nu} \omega \, dx + \int_{\partial \mathcal{F}(t)} \psi_{\nu}(K[\omega] + u_{\text{pot}}) \cdot \tau \, ds(x).$$

The part of the second integral on $\partial \Omega$ vanishes due to (2.17), and the parts of the second integral on each $\partial \mathcal{S}_{\lambda}$, $\lambda = 1, \ldots, N$, vanish as well because $\psi_{\nu}$ is constant on each connected component of the boundary and $K[\omega] + u_{\text{pot}}$ has zero-circulation on each $\partial \mathcal{S}_{\lambda}$. Now the first term is bounded independently of $t$ and $\varepsilon$, because $\psi_{\nu}$ is bounded on the support of $\omega$: this can be seen by integrating $\nabla \psi_{\nu}$ from some point in $\partial \Omega$ and using Lemmas 3.27 and 3.28 and to the remoteness of $\mathcal{S}_{\lambda}$ to the support of $\omega$ (due to $(\varepsilon, \mathbf{q}, \omega) \in \Omega^{\delta}$).

Hence the proof of Proposition 4.2 is complete. \qed

4.3 Rough estimate for the acceleration of the bodies

The goal of this subsection is to prove the following statement.

Proposition 4.4. Let $\delta > 0$. There exists $C > 0$ and $\varepsilon_0 > 0$ such that the solutions $(u^\varepsilon, h^\varepsilon, \partial^\varepsilon)$ of the system satisfy, as long as $(\varepsilon, \mathbf{q}, \omega) \in \Omega^{\delta}$,

$$\forall \kappa \in \{1, \ldots, N\}, \quad |e_{\kappa} \delta \omega \psi(\psi) \partial^\varepsilon| \leq C(1 + |\partial^\varepsilon|).$$

The rest of the subsection is devoted to the proof of Proposition 4.4.

4.3.1 A decomposition of the velocity

The proof of Proposition 4.4 relies on the following decomposition of the velocity.

Definition 4.5. We decompose the velocity field $u^\varepsilon$ as follows:

$$u^\varepsilon = u_{\text{pot}} + \sum_{\nu \in \{1, \ldots, N\}} \gamma_{\nu} \nabla \psi_{\nu} + u_{\text{ext}},$$

where the potential part of the velocity $u_{\text{pot}}$ was defined in (2.25). We will call $u_{\text{ext}}$ the exterior part of the velocity field.
Notice the difference between (4.9) and the standard decomposition (2.24), in that the circulation potentials considered here are standalone, following the strategy hinted in Section 2.3, and developed below, see in particular the treatment of the term $T_3$ in (4.26).

Comparing the standard decomposition (2.24) of $u^\varepsilon$ and (4.9), we see with (3.84) that

$$u^{\text{ext}} = K[\omega] + \sum_{\nu \in \{1, \ldots, N\}} \gamma_\nu \nabla^\perp \psi_\nu^\varepsilon.$$ \hspace{1cm} (4.10)

An important property of the decomposition (4.9) is given by the following lemma, concerning the field $u^{\text{ext}}$ associated with a solution to System (1.2)–(1.7).

**Lemma 4.6.** Given $\delta > 0$, there exist some constants $\varepsilon_0$ and $C > 0$ such that, for a solution to the system, as long as $(\varepsilon, q, \omega) \in \Omega_\delta^0$, one has for $u^{\text{ext}}$ considered as a function of $(t,x)$:

$$\|u^{\text{ext}}\|_{L^\infty(F(q))} \leq C,$$ \hspace{1cm} (4.11)

$$\|\partial_t u^{\text{ext}}\|_{L^\infty(V_b(\partial F) \cup \cup_{\nu \in \mathcal{P}_S} \partial \mathcal{S}_\nu)} \leq C(1 + |\slashed{\nabla}\varepsilon|),$$ \hspace{1cm} (4.12)

$$\|\partial_t u^{\text{ext}}\|_{L^\infty(V_b(\partial \mathcal{S}_\nu)))} \leq C\varepsilon_\nu^{-1}(1 + |\slashed{\nabla}\varepsilon|), \quad \forall \nu \in \mathcal{P}_S.$$ \hspace{1cm} (4.13)

**Proof of Lemma 4.6.** First, (4.11) follows from directly from (4.10) and Lemmas 3.28 and 3.32. For what concerns (4.12)-(4.13), we start with

$$\partial_t u^{\text{ext}} = K[\partial_t \omega^\varepsilon] + \sum_{\mu \in \{1, \ldots, N\}} \sum_{\nu \in \{1, \ldots, N\}} \gamma_\nu \nabla^\perp \psi_\nu^\varepsilon \cdot p_{\mu, \nu}.$$ \hspace{1cm} (4.14)

The shape derivatives of $K[\omega^\varepsilon]$ and $\nabla^\perp \psi_\nu^\varepsilon$ with respect to $q_{\mu, \nu}$ are estimated separately in $L^\infty(V_b(\partial \mathcal{S}_\mu(q)))$ and in $L^\infty(F(q) \setminus \cup_{\nu \in \mathcal{P}_S} \partial \mathcal{S}_\nu(q)))$ by using Lemma 3.34 and Lemma 3.30 respectively. Observing that $\varepsilon_\mu^{\text{adm}}|p_{\mu, \nu}| = |\slashed{\nabla}p_{\mu, \nu}|$, it follows that the second term in (4.14) gives a contribution as in (4.12)-(4.13).

It remains to study

$$K[\partial_t \omega^\varepsilon] = -K[\text{div}(u^\varepsilon \omega^\varepsilon)].$$ \hspace{1cm} (4.15)

We estimate $u^\varepsilon \omega^\varepsilon$ using the decomposition (2.24). Using that $(\varepsilon, q, \omega) \in \Omega_\delta^0$, the energy estimates and (3.60), we deduce that $\|u^{\text{pot}} \omega^\varepsilon\|_{L^\infty(F(q))} \leq C$. Using that $(\varepsilon, q, \omega) \in \Omega_\delta^0$ and Lemmas 3.27 and 3.28, we also find that $\omega^\varepsilon \sum_{\nu \in \{1, \ldots, N\}} \gamma_\nu \nabla^\perp \psi_\nu^\varepsilon$ is bounded in $L^\infty(F(q))$. With (4.11), we finally deduce that

$$\|u^{\text{ext}} \omega^\varepsilon\|_{L^\infty(F(q))} \leq C.$$ \hspace{1cm} (4.16)

With (3.110) and (4.9), this gives

$$\|K[\text{div}(u^\varepsilon \omega^\varepsilon)]\|_{L^p(F(q))} \leq C, \quad \forall p \in (1, +\infty).$$ \hspace{1cm} (4.17)

By using the support of vorticity and local elliptic estimates near the boundaries one concludes that $K[\text{div}(u^\varepsilon \omega^\varepsilon)]$ is bounded in $L^\infty(V_b(\partial F(q)))$, and (4.12)-(4.13) follow.

### 4.3.2 Proof of the acceleration estimates

We are now in position to prove Proposition 4.4.

**Proof of Proposition 4.4.** We cut the proof in several steps.

**Step 1.** By (1.2), (2.8) and an integration by parts we write the solid equation (1.7) as

$$(\mathcal{M}_g(p^\varepsilon)')_{\kappa, j} = -\int_{F(q)} (\partial_t u^\varepsilon + (u^\varepsilon \cdot \nabla)u^\varepsilon) \cdot \nabla \varphi_{\kappa, j} \, dx,$$ \hspace{1cm} (4.18)

where we recall the notation (2.11). Next we inject the decomposition (4.9) of $u^\varepsilon$. In the right-hand side, we extract from $\partial_t u^\varepsilon$ the part corresponding to

$$\partial_t u^{\text{pot}} = \sum_{\mu \in \{1, \ldots, N\}} \sum_{\nu \in \{1, \ldots, N\}} \left[ p_{\mu, \nu} \nabla \varphi_{\mu, \nu} + p_{\mu, \nu} (\nabla \varphi_{\mu, \nu})' \right].$$ \hspace{1cm} (4.19)
When injected in (4.18), the first term in (4.19) gives the added mass term \( \frac{\nabla |u^\varepsilon|^2}{2} + \omega^\varepsilon (u^\varepsilon)^\perp \).

When injected in (4.18), the first term in the right hand side of (4.20) can be integrated by parts to arrive at

\[-\frac{1}{2} \int_{\partial S_n(q)} |u^\varepsilon|^2 K_{\kappa,j} \, ds.\]

Then we develop the square

\[|u^\varepsilon|^2 = \left| u^\text{pot} + \sum_{\nu \in \{1, \ldots, N\}} \gamma_\nu \nabla^\perp \hat{\psi}_\nu + u^\text{ext} \right|^2,\]

by separating between

\[\gamma_\kappa \nabla^\perp \hat{\psi}_k \text{ and } u^\text{pot} + u^\text{ext} + \sum_{\nu \in \{1, \ldots, N\} \setminus \{\kappa\}} \gamma_\nu \nabla^\perp \hat{\psi}_\nu,\]

to arrive at

\[(M_\rho(p^\varepsilon)' + M_\sigma(p^\varepsilon)')_{\kappa,j} = T_1 + \ldots + T_7,\]

where

\[T_1 := -\sum_{\lambda, \mu \in \{1, \ldots, N\}, l, m = 1, 2, 3} \int_{F(q)} p_{\lambda, \ell} p_{\mu, m} \frac{\partial \nabla \phi_{\lambda, \ell}}{\partial q_{\mu, m}} \cdot \nabla \phi_{\kappa,j} \, dx,\]

\[T_2 := -\sum_{\nu \in \{1, \ldots, N\}} \gamma_\nu \int_{F(q)} \partial_\ell u^\text{ext} \cdot \nabla \phi_{\kappa,j} \, dx,\]

\[T_3 := \int_{F(q)} \partial_\ell u^\text{ext} \cdot \nabla \phi_{\kappa,j} \, dx,\]

\[T_4 := -\frac{1}{2} \int_{\partial S_n(q)} \left| \gamma_\kappa \nabla^\perp \hat{\psi}_k \right|^2 K_{\kappa,j} \, ds,\]

\[T_5 := -\frac{1}{2} \int_{\partial S_n(q)} \left| u^\text{pot} + u^\text{ext} + \sum_{\nu \in \{1, \ldots, N\} \setminus \{\kappa\}} \gamma_\nu \nabla^\perp \hat{\psi}_\nu \right|^2 K_{\kappa,j} \, ds,\]

\[T_6 := -\gamma_\kappa \int_{\partial S_n(q)} \left( u^\text{pot} + u^\text{ext} + \sum_{\nu \in \{1, \ldots, N\} \setminus \{\kappa\}} \gamma_\nu \nabla^\perp \hat{\psi}_\nu \right) \cdot \nabla^\perp \hat{\psi}_k \, ds,\]

\[T_7 := \int_{F(q)} \omega u^\perp \cdot \nabla \phi_{\kappa,j} \, dx.\]

**Step 2.** We now estimate these seven terms. In this proof it will be convenient to take the convention of Remark 3.21 for the Kirchhoff potentials.

**Estimate of T_1.** We first integrate by parts:

\[\int_{F(\ell)} p_{\lambda, \ell} p_{\mu, m} \frac{\partial \nabla \phi_{\lambda, \ell}}{\partial q_{\mu, m}} \cdot \nabla \phi_{\kappa,j} \, dx = p_{\lambda, \ell} p_{\mu, m} \int_{\partial S_n} \frac{\partial \phi_{\lambda, \ell}}{\partial q_{\mu, m}} K_{\kappa,j} \, ds.\]
• First case: \( \lambda = \mu \). Then either \( \kappa = \lambda = \mu \) and this integral is \( O(\varepsilon_\lambda^{1+\delta_3} \varepsilon_\mu^{\delta_2} \varepsilon_\kappa^{\delta_4}) \) (the additional power of \( \varepsilon_\lambda \) comes from the integration on \( \partial S_\kappa = \partial S_\lambda \)), or \( \kappa \neq \lambda = \mu \) and the integral is \( O(\varepsilon_\lambda^{2+\delta_3} \varepsilon_\mu^{\delta_2} \varepsilon_\kappa^{1+\delta_4}) \).

• Second case: \( \lambda \neq \mu \). Then either \( \kappa \neq \mu \) and we see the integral is \( O(\varepsilon_\lambda^{2+\delta_3} \varepsilon_\mu^{2+\delta_2} \varepsilon_\kappa^{1+\delta_4}) \), or \( \kappa = \mu \) and the integral is \( O(\varepsilon_\lambda^{2+\delta_3} \varepsilon_\mu^{\delta_2} \varepsilon_\kappa^{1+\delta_4}) \).

We recall that \( \varepsilon_\mu^{\delta_2} |p_{\mu,m}| = |\tilde{\nu}_{\mu,m}| \). Using the energy estimates provided by Proposition 4.2 (which give \( \varepsilon_\lambda^{1+\delta_3} p_{\lambda,t} \) bounded), we see that in all cases, the term in (4.22) is at least estimated by \( O(|\tilde{\nu}_{\mu,m}|^{\delta_4}) \) (the worst case being the first one where \( \kappa = \lambda = \mu \)).

**Estimate of \( T_2 \).** We first deduce from Lemma 3.27 that
\[
\partial_t \hat{\psi}_\nu + v S_{\nu,\nu} \cdot \nabla \hat{\psi}_\nu = 0 \quad \text{and} \quad \partial_t \nabla \hat{\psi}_\nu + (v S_{\nu,\nu} \cdot \nabla) \nabla \hat{\psi}_\nu = \partial_x \nabla \hat{\psi}_\nu,
\]
where we denote by \( v S_{\nu,\nu} \) the \( \nu \)-th solid vector field, see (1.6). Using the formulas
\[
\nabla (a \cdot b) = (a \cdot \nabla) b + (b \cdot \nabla) a - a \cdot \nabla \text{curl}(b) - b \cdot \nabla \text{curl}(a),
\]
curl\((x^+) = 2 \) and \( (a \cdot \nabla) x^+ = a^+ \), we find
\[
\partial_t \nabla \hat{\psi}_\nu + \nabla \left( v S_{\nu,\nu} \cdot \nabla \hat{\psi}_\nu \right) = 0. \tag{4.23}
\]
By an integration by parts it follows that
\[
\int_{\mathcal{F}(q)} \partial_t \nabla \hat{\psi}_\nu \cdot \nabla \varphi_{\kappa,j} \, dx = - \int_{\partial S_\kappa(q)} v S_{\nu,\nu} \cdot \nabla \hat{\psi}_\nu K_{\kappa,j} \, ds.
\]
Now when \( \nu = \kappa \) it is straightforward to estimate this term by \( O(\varepsilon_\kappa^{\delta_4})|\tilde{\nu}\kappa| \) since \( \nabla \hat{\psi}_\kappa = O(1/\varepsilon_\kappa) \) on \( \partial S_\kappa \). When \( \nu \neq \kappa \), one can use the divergence theorem inside \( S_\kappa \):
\[
\int_{\partial S_\kappa(q)} v S_{\nu,\nu} \cdot \nabla \hat{\psi}_\nu K_{\kappa,j} \, ds = - \int_{S_\kappa(q)} \text{div} \left( h'_\nu + \phi'_\nu (x - h_\nu)^+ \cdot \nabla \hat{\psi}_\nu \right) \varphi_{\kappa,j} \, dx. \tag{4.24}
\]
Now on the one hand using (3.81) and interior regularity estimates for the Laplace equation, we obtain
\[
\int_{S_\kappa(q)} \text{div} \left( h'_\nu \cdot \nabla \hat{\psi}_\nu \right) \varphi_{\kappa,j} \, dx = \int_{S_\kappa(q)} \varphi_{\kappa,j} \cdot \nabla (h'_\nu \cdot \nabla \hat{\psi}_\nu) \, dx = O(\varepsilon_\kappa^{2+\delta_3}) |h'_\nu|.
\]
On the other hand, we use (3.82) and see that
\[
\int_{S_\kappa(q)} \text{div} \left( (\phi'_\nu (x - h_\nu)^+ \cdot \nabla \hat{\psi}_\nu) \right) \varphi_{\kappa,j} \, ds = O(\varepsilon_\kappa^{2+\delta_3}) |\phi'_\nu|.
\]
Altogether the term \( T_2 \) can be estimated by
\[
T_2 = O(\varepsilon_\kappa^{2+\delta_3}) |\tilde{\nu}_\nu|. \tag{4.25}
\]

**Estimate of \( T_3 \).** We first integrate by parts to find
\[
\int_{\mathcal{F}(q)} \partial_t \nabla \varphi_{\kappa,j} \, dx = \int_{\mathcal{F}(q)} \partial_t \nabla \varphi_{\kappa,j} \cdot n \varphi_{\kappa,j} \, ds.
\]
By Lemma 4.6 (using (4.12) on \( \partial \Omega \) and (4.13) on the rest of the boundary), we have \( \| \partial_t \nabla \varphi_{\kappa,j} \|_{L^1(\mathcal{F}(q))} = O(1 + |\tilde{\nu}|) \). We use (3.62) to estimate the Kirchhoff potential \( \varphi_{\kappa,j} \) on the boundary and infer that
\[
T_3 = O(\varepsilon_\kappa^{1+\delta_3})(1 + |\tilde{\nu}|).
\]
Estimate of $T_4$. We have for any $j \in \{1, 2, 3\}$

$$
\int_{\partial S_\kappa} |\gamma_\kappa \nabla \hat{\psi}_\kappa|^2 K_{\kappa,j} \, ds = 0.
$$

(4.26)

This is a consequence of Blasius’ lemma, see e.g. [7, p. 511]. This also a direct consequence of Lamb’s lemma (see Lemma 6.10 below).

Estimate of $T_5$. Using Lemma 4.6, Proposition 3.20 and (3.81) we see that

$$
|u^{\text{pot}} + u^{\text{ext}} + \sum_{\nu \in \{1, \ldots, N\} \setminus \{i\}} \gamma_\nu \nabla \hat{\psi}_\nu| \leq C \left( 1 + |\hat{p}_\kappa| + \sum_{\nu \neq \kappa} \varepsilon_\kappa^3 |\hat{p}_\nu| \right) \text{ on } \partial S_\kappa.
$$

(4.27)

Considering that $K_{\kappa,j} = O(\varepsilon_\kappa^{\delta_{j3}})$ and that we integrate over $\partial S_\kappa$, using the energy estimates, we deduce that this term can be bounded by $C \varepsilon_\kappa^{\delta_{j3}} (1 + |\hat{p}_\kappa|)$.

Estimate of $T_6$. Using (4.27), the energy estimates, $\nabla \hat{\psi}_\kappa = O(1/\varepsilon_\kappa)$ on $\partial S_\kappa$ and again that $\partial S_\kappa$ is of size $O(\varepsilon_\kappa)$, we see that this term is also estimated by $C \varepsilon_\kappa^{\delta_{j3}} (1 + |\hat{p}_\kappa|)$.

Estimate of $T_7$. We use the decomposition (4.9) of $u^5$, the compactness of the support of $\omega^5$ in $\mathcal{F}(q)$ due to $(\varepsilon, q, \omega) \in \Omega_5^\infty$, the decay of the Kirchhoff potentials (3.60), the energy estimates, (3.81) and (4.11) to conclude that this term is of order $O(\varepsilon_\kappa^{1+\delta_{j3}})$.

**Step 3.** Gathering what precedes we have established, recalling (2.13),

$$
\left| (\mathcal{M} \mathbf{p})_{\kappa,j} \right| \leq C \varepsilon_\kappa^{\delta_{j3}} (1 + |\hat{p}|).
$$

(4.28)

Now define the “homogeneous” inertia matrix $\mathcal{M}^\circ$ as the total inertia matrix $\mathcal{M}$ where we divide each $(\kappa,j)$-th row and each $(\kappa,j)$-th column by $\varepsilon_\kappa^{\delta_{j3}}$. Then (4.28) translates now into

$$
\left| (\mathcal{M}^\circ (\hat{p})')_{\kappa,j} \right| \leq C (1 + |\hat{p}|).
$$

We now introduce the matrix $\mathcal{M}^\circ$ as the total homogeneous inertia matrix $\mathcal{M}^\circ$ where each $(\kappa,j)$-th column is divided by $\varepsilon_\kappa^{\epsilon_{\kappa}^{(2,\alpha_\kappa)} P_{\ell,m}}$, where we recall that $\alpha_\kappa$ was introduced in (1.12). Calling $\hat{p}$ the vector with $(\kappa,j)$-th coordinate $\varepsilon_\kappa^{\epsilon_{\kappa}^{(2,\alpha_\kappa)} P_{\ell,m}} \hat{p}_{\kappa,j}$, we hence have

$$
\mathcal{M}^\circ (\hat{p})' = \mathcal{M}^\circ \hat{p}'.
$$

Hence to end the proof of Proposition 4.4, it remains to prove that $(\mathcal{M}^\circ)^{-1}$ is bounded independently of $\varepsilon$ at least for small $\varepsilon$. Now gathering the rows and columns of $\mathcal{M}^\circ$ according to families (i), (ii) and (iii), we have a block matrix:

$$
\mathcal{M}^\circ = \begin{pmatrix}
A_{(i)(i)} & A_{(i)(ii)} & A_{(i)(iii)} \\
A_{(ii)(i)} & A_{(ii)(ii)} & A_{(ii)(iii)} \\
A_{(iii)(i)} & A_{(iii)(ii)} & A_{(iii)(iii)}
\end{pmatrix}.
$$

Using Corollary 3.23 we see that the entries of the added mass matrix $\mathcal{M}_a$ that correspond to different solids satisfy:

$$
(\mathcal{M}_a)_{\lambda, \ell, \mu, m} = O(\varepsilon_\lambda^{2+\delta_{33}+2^2+\delta_{m3}}) \quad \text{for } \lambda \neq \mu, \ \ell, m = 1, 2, 3.
$$

(4.29)

Moreover, using Corollary 3.23 and Remark 2.1, we see that for $\lambda \in P_{(iii)}$ and $\ell, m \in \{1, 2, 3\}$,

$$
\mathcal{M}_a,_{\lambda, \ell, \lambda, m} = \varepsilon_\lambda^{2+\delta_{33}+\delta_{3m}} \tilde{M}_{a,\lambda, \ell, m} + O(\varepsilon_\lambda^{4+\delta_{33}+\delta_{3m}}),
$$

where $\tilde{M}_{a,\lambda}$ is a fixed symmetric positive-definite matrix.
Relying on the genuine mass and (1.10)-(1.11) for the first two families, and either on the genuine mass (when \( \alpha_c \leq 2 \)) or the added mass (when \( \alpha_c > 2 \)) and (1.12) for the third family, we deduce that the diagonal blocks \( A_{(i)(i)} \), \( A_{(ii)(ii)} \) and \( A_{(iii)(iii)} \) are uniformly invertible. Moreover we also see that the blocks above the diagonal \( A_{(i)(ii)} \), \( A_{(i)(iii)} \) and \( A_{(ii)(iii)} \) remain bounded. Hence by Cramer’s rule the upper triangular block matrix
\[
\mathcal{M}^u := \begin{pmatrix} A_{(i)(i)} & A_{(ii)(i)} & A_{(iii)(i)} \\ 0 & A_{(ii)(ii)} & A_{(iii)(ii)} \\ 0 & 0 & A_{(iii)(iii)} \end{pmatrix},
\]
whose determinant is \( \det(A_{(i)(i)}) \det(A_{(ii)(ii)}) \det(A_{(iii)(iii)}) \), is uniformly invertible. As can be seen from Neumann’s series, when \( ||\mathcal{M}^u - \mathcal{M}^n|| \leq \frac{1}{2||\mathcal{M}^n||} \) for some matrix norm, then \( \mathcal{M}^u \) is invertible with \( ||(\mathcal{M}^u)^{-1}|| \leq 2||\mathcal{M}^n|| \). Since from (4.29) the blocks under the diagonal \( A_{(i)(i)} \), \( A_{(ii)(i)} \) and \( A_{(iii)(i)} \) converge to zero, we see that \( \mathcal{M}^u \) is uniformly invertible for suitably small \( \varepsilon \). The result follows.

\[ \square \]

5 Introduction of the modulations

In this section, we introduce the modulations that will play a central role in the normal forms of Section 6 and consequently in the modulated energy estimates of Section 7 and in the passage to the limit of Section 8.

5.1 Decomposition of the fluid velocity focused on a small solid

In this section, we merely consider \( \kappa \) in \( \mathcal{P}_s \), because only the small solids will actually be concerned with the modulations. To define the modulation, we first introduce a decomposition of the velocity field in the same spirit as (4.9), but here more focused on the \( \kappa \)-th solid.

**Definition 5.1.** For each \( \kappa \) in \( \mathcal{P}_s \), we introduce the following decomposition
\[
\mathbf{u}^* = \mathbf{u}^*_{\text{pot}} + \gamma_{\kappa} \nabla^\perp \hat{\mathbf{\psi}}_{\kappa} + \mathbf{u}^*_{\text{ext}} \quad \text{with} \quad \mathbf{u}^*_{\text{pot}} := \sum_{i=1,2,3} \mathbf{p}_{\kappa,i} \nabla \varphi_{\kappa,i}. \tag{5.1}
\]

We will refer to \( \mathbf{u}^*_{\text{pot}} \) as potential part of the decomposition (5.1), \( \gamma_{\kappa} \nabla^\perp \hat{\mathbf{\psi}}_{\kappa} \) as its circulation part, and \( \mathbf{u}^*_{\text{ext}} \) as the \( \kappa \)-th exterior field.

When comparing with the decomposition (4.9), we see that
\[
\mathbf{u}^*_{\text{ext}} = \mathbf{u}^*_{\text{ext}} + \sum_{\nu \neq \kappa} \mathbf{p}_{\kappa,i} \nabla \varphi_{\kappa,i} + \sum_{\nu \neq \kappa} \gamma_{\nu} \nabla^\perp \hat{\mathbf{\psi}}_{\nu}. \tag{5.2}
\]

The \( \kappa \)-th exterior field will play a central role in the definition of the modulation. In (5.1), the first two vector fields can be thought as “attached” to \( \mathcal{S}_{\kappa} \) (to its velocity and to the constant circulation around it), while \( \mathbf{u}^*_{\text{ext}} \) corresponds to the vector field to which \( \mathcal{S}_{\kappa} \) “is subjected” from the exterior (which includes the reflections of \( \nabla^\perp \hat{\mathbf{\psi}}_{\nu} \) on \( \partial \Omega \) and the other solids).

We first note that, due to (5.1), \( \mathbf{u}^*_{\text{ext}} \) satisfies the following div-curl system
\[
\begin{aligned}
\text{div } \mathbf{u}^*_{\text{ext}} &= 0 \quad \text{in } \mathcal{F}(\mathbf{q}), \\
\text{curl } \mathbf{u}^*_{\text{ext}} &= \mathbf{\omega}^* \quad \text{in } \mathcal{F}(\mathbf{q}), \\
\mathbf{u}^*_{\text{ext}} \cdot \mathbf{n} &= -\gamma_{\kappa} \nabla^\perp \hat{\mathbf{\psi}}_{\kappa} \cdot \mathbf{n} + \sum_{\nu \neq \kappa} \mathbf{p}_{\kappa,i} \nabla \varphi_{\kappa,i} \cdot \mathbf{n} \quad \text{on } \partial \mathcal{F}(\mathbf{q}), \\
\int_{\partial \mathcal{S}_{\kappa}} \mathbf{u}^*_{\text{ext}} \cdot \mathbf{t} ds &= \delta_{\nu \neq \kappa} \gamma_{\nu} \quad \text{for } \nu = 1, \ldots, N.
\end{aligned} \tag{5.3}
\]

Recall that \( \nabla^\perp \hat{\mathbf{\psi}}_{\kappa} \) is tangent to \( \partial \mathcal{S}_{\kappa} \); it follows in particular that \( \mathbf{u}^*_{\text{ext}} \cdot \mathbf{n} = 0 \) on \( \partial \mathcal{S}_{\kappa} \).

We have the following estimate of the \( \kappa \)-th exterior field \( \mathbf{u}^*_{\text{ext}} \).
Let us now describe a vector field $V$. Note in particular that $\nabla$ to deduce that this term is bounded. Concerning the circulation part, due (3.81) we have $\nabla \cdot \mathcal{P}_\nu = O(1)$ on $\partial \mathcal{S}_\kappa$ for $\nu \neq \kappa$, which also yields a bounded term.

**Lemma 5.2.** Let $\delta > 0$. There exists $\varepsilon_0 > 0$ and $C > 0$ such that for all $\varepsilon$ with $\varepsilon \leq \varepsilon_0$, as long as $(\varepsilon, q, \omega) \in \Omega_8^0$:

$$|v^{\text{ext}}_\kappa|_{L^\infty(\partial \mathcal{S}_\kappa)} \leq C.$$  

**Proof.** Thanks to Lemma 4.6, we only have to estimate the two sums in the right-hand side of (5.2). For that purpose, we rely on the fact that the sums are over $\nu \neq \kappa$. Concerning the Kirchhoff potential parts we can use $\nabla \phi_{\nu,i} = O(\varepsilon^{2+\delta,3})$ on $\partial \mathcal{S}_\kappa$ (Proposition 3.20) and the energy estimates (Proposition 4.2) to deduce that this term is bounded. Concerning the circulation part, due (3.81) we have $\nabla \cdot \mathcal{P}_\nu = O(1)$ on $\partial \mathcal{S}_\kappa$ for $\nu \neq \kappa$, which also yields a bounded term. \hfill $\square$

### 5.2 Approximation of the $\kappa$-th exterior field

The goal of this paragraph is to show how $u^{\text{ext}}_\kappa$ can be approximated on $\partial \mathcal{S}_\kappa$ by a linear combination of four basic vector fields. For this we introduce the following notations. Recalling (2.7), we denote for each $\kappa \in \{1, \ldots, N\}$

$$K_\kappa = K_\kappa(q) := \text{Span} \{\xi_{\kappa,1}, \xi_{\kappa,2}, \xi_{\kappa,3}, \xi_{\kappa,4}, \xi_{\kappa,5}\} \text{ and } K_{\kappa,s} = K_{\kappa,s}(q) := \text{Span} \{\xi_{\kappa,1}, \xi_{\kappa,2}, \xi_{\kappa,4}, \xi_{\kappa,5}\}.$$  

Note in particular that $\xi_{\kappa,3}$ is excluded from $K_{\kappa,s}$. Together with these spaces, we define the linear operator $\text{Kir}_\kappa$, defined on $K_\kappa$, transforming an affine vector field in the corresponding linear combination of Kirchhoff vector fields; it is defined by

$$\text{Kir}_\kappa(\xi_{\kappa,i}) = \nabla \varphi_{\kappa,i} \text{ for all } i = 1, 2, 3, 4, 5.$$  

(5.4)

This operator depends implicitly on $q$ and $\varepsilon$. (Actually one may notice that $K_\kappa$ and $K_{\kappa,s}$ do not depend on $q$ or $\varepsilon$; but the operators $\text{Kir}_\kappa$ do.) Similarly we introduce

$$\text{Kir}_\kappa(\xi_{\kappa,i}) = \nabla \varphi_{\kappa,i} \text{ for all } i = 1, 2, 3, 4, 5.$$  

(5.5)

It is an direct consequence of Proposition 3.20 that

$$|\text{Kir}_\kappa(\xi_{\kappa,i}) - \text{Kir}_\kappa(\xi_{\kappa,i})| \leq C\varepsilon^{2+\delta,3} \text{ on } \partial \mathcal{S}_\kappa.$$  

(5.6)

Let us now describe a vector field $V_\kappa \in K_{\kappa,s}$ that generates our approximation of $u^{\text{ext}}_\kappa$. Having (5.3) in mind, we first introduce the solution $\bar{u}^k = \bar{u}^k(q, p, \cdot, \cdot) \in \tilde{F}_k(q)$ (recall that this domain was introduced in (3.103)) of the following system:

$$\begin{cases}
\text{div } \bar{u}_k = 0 \text{ in } \tilde{F}_k(q), \\
\text{curl } \bar{u}_k = \omega \text{ in } \tilde{F}_k(q), \\
\bar{u}_k \cdot n = -\gamma_\nu \nabla \cdot \varphi_{\kappa} \cdot n + \sum_{\mu \neq \kappa} \sum_{i=1}^3 p_{\nu,i} \nabla \varphi_{\nu,i} \cdot n \text{ on } \partial \tilde{F}_k(q), \\
\bar{u}_k \cdot \tau ds = \gamma_\nu \text{ for } \nu \in \{1, \ldots, N\}\setminus\{\kappa\}.
\end{cases}$$  

(5.7)

We start with the following lemma which estimates $\bar{u}_k$ regardless of the fact that it comes from a solution to System (1.2)–(1.7). We recall the notation (2.6) for $V_\mu(\partial \mathcal{S}_\kappa)$.

**Lemma 5.3.** Given $\delta > 0$ there exist constants $\varepsilon_0$ and $C > 0$ such that as long as $(\varepsilon, q, \omega) \in \Omega_8^0$, for all $\kappa \in P_\kappa$, all $\mu \in \{1, \ldots, N\}$ and $m \in \{1, 2, 3\}$, one has:

$$|\bar{u}_k|_{L^\infty(V_\mu(\partial \mathcal{S}_\kappa))} \leq C \left(1 + \|\omega\|_\infty + \sum_{\nu \neq \kappa} \varepsilon_\nu^2 \| \hat{p}_\nu \|_{\mathcal{P}_\kappa} \right)$$

and

$$\left|\frac{\partial}{\partial \mu \cdot \omega} \bar{u}_k\right|_{L^\infty(V_\mu(\partial \mathcal{S}_\kappa))} \leq C \varepsilon_\mu^m \left(1 + \|\omega\|_\infty + \sum_{\nu \neq \kappa} \varepsilon_\nu |\hat{p}_\nu|\right).$$

**Proof of Lemma 5.3.** The proof is roughly the same as for Lemma 4.6 with the exception that we consider functions of $(q, x)$ rather than $(t, x)$ and that the domain is no longer $\mathcal{F}(q)$ but $\tilde{F}_k(q)$. This latter difference actually simplifies the proof because it avoids the singularity in the neighborhood of $\mathcal{S}_\kappa$. We
call $\varphi^\kappa_{\nu,m}$ the various Kirchhoff potentials in $\tilde{F}_\kappa(q)$, $\nu \in \{1, \ldots, N\} \setminus \{\kappa\}$, $i \in \{1, 2, 3\}$, $K^\kappa$ the Biot-Savart operator in $\tilde{F}_\kappa(q)$, and $\psi^\kappa_{\nu,m}$ for $\nu \in \{1, \ldots, N\} \setminus \{\kappa\}$, the various circulation stream functions in $\tilde{F}_\kappa(q)$. We recall that for $\nu = \kappa$, $\psi^\kappa_{\kappa,m}$ was defined in (3.104). Correspondingly we see from (5.7) and (3.104) that $\tilde{u}_\kappa$ can be decomposed as follows:

$$
\tilde{u}_\kappa = \sum_{\nu \neq \kappa} p_\nu \nabla \varphi^\kappa_{\nu} + \sum_{\nu \neq \kappa} \gamma_\nu \nabla \psi^\kappa_{\nu} + K^\kappa[\omega] + \gamma_\kappa \nabla \psi^\kappa_{\kappa,m} \text{ in } \tilde{F}_\kappa(q). 
$$

We observe that the statements of Section 3 that were written in a general fluid domain $\mathcal{F}$ are valid in particular in the domain $\tilde{F}_\kappa(q)$. This has the following consequences:

- The estimates of Propositions 3.20 and 3.26 are valid for the Kirchhoff potentials $\varphi^\kappa_{\nu}$,

- Decomposing the circulation stream functions $\psi^\kappa_{\nu,m}$ for $\nu \in \{1, \ldots, N\} \setminus \{\kappa\}$, as in (3.84) by introducing the potential $\psi^\kappa_{\nu,r}$ so that

$$
\psi^\kappa_{\nu} = \tilde{\psi}_{\nu} + \psi^\kappa_{\nu,r} \text{ in } \tilde{F}_\kappa(q),
$$

the function $\psi^\kappa_{\nu,r}$ satisfies the estimates of Lemmas 3.28 and 3.30,

- The estimates of Lemmas 3.32 and 3.34 are valid for the Biot-Savart operator $K^\kappa$ in $\tilde{F}_\kappa(q)$.

Finally we recall that the particular term $\nabla ^\perp \psi^\kappa_{\kappa,m}$ was studied in Lemma 3.31.

Now we proceed as in Lemma 4.6. Concerning the bound on $\|\tilde{u}_\kappa\|_{L^\infty(V_\delta(\tilde{\mathcal{S}}_\kappa))}$, we treat the various terms in the right-hand side of (5.8) as follows:

- the terms $p_\nu \nabla \varphi^\kappa_{\nu}$ are of order $\epsilon^2 q_{\nu}$ in $V_\delta(\partial \mathcal{S}_\kappa)$ by Proposition 3.20,

- the terms $\nabla ^\perp \psi^\kappa_{\nu,m}$ are bounded thanks to Lemma 3.28 and the fact that $V_\delta(\partial \mathcal{S}_\kappa)$ is a distance $O(1)$ from $\mathcal{S}_\nu$,

- the term $K^\kappa[\omega]$ is bounded thanks to Lemma 3.32,

- the term $\nabla ^\perp \psi^\kappa_{\kappa,m}$ is bounded thanks to Lemma 3.31.

Concerning the bound on the shape derivative $\partial_{q_{\kappa,m}} \tilde{u}_\kappa$, we proceed as follows, for $\mu \neq \kappa$:

- the terms $p_\nu \nabla \partial_{q_{\kappa,m}} \varphi^\kappa_{\nu}$ are estimated in $V_\delta(\partial \mathcal{S}_\kappa)$ by (3.74) in Proposition 3.26,

- for the terms $\nabla ^\perp \partial_{q_{\kappa,m}} \psi^\kappa_{\nu,m}$, $\nu \neq \kappa$, we use the decomposition (5.9). For $\partial_{q_{\kappa,m}} \nabla \psi_{\nu}$ (which vanishes unless $\mu = \nu$), we use (3.101), (3.81), (3.82) and the fact that $V_\delta(\partial \mathcal{S}_\kappa)$ is a distance $O(1)$ from $\mathcal{S}_\nu$. For $\partial_{q_{\kappa,m}} \nabla ^\perp \psi^\kappa_{\nu,m}$ we use Lemma 3.30 (that is valid in $\tilde{F}_\kappa$) and again the fact that $V_\delta(\partial \mathcal{S}_\kappa)$ is a distance $O(1)$ from $\partial \tilde{F}_\kappa$,

- the term $\partial_{q_{\kappa,m}} K^\kappa[\omega]$ is estimated thanks to Lemma 3.34, using again the fact that $V_\delta(\partial \mathcal{S}_\kappa)$ is a distance $O(1)$ from $\partial \tilde{F}_\kappa$,

- the term $\partial_{q_{\kappa,m}} \nabla ^\perp \psi^\kappa_{\kappa,m}$ is bounded by $C\epsilon_{\mu,m}$ in $V_\delta(\partial \mathcal{S}_\kappa)$ thanks to Lemma 3.31.

Finally, when $\mu = \kappa$, only the last term in (5.8) actually depends on $q_{\kappa}$. This dependence —despite the fact that $\tilde{u}_\kappa$ is defined in $\tilde{F}_\kappa$— is due to the boundary conditions in (3.104). The derivative of this term with respect to $q_{\kappa,m}$ is again estimated by $C\epsilon_{\mu,m}$ in $V_\delta(\partial \mathcal{S}_\kappa)$ thanks to Lemma 3.31.

This concludes the proof of Lemma 5.3. \[ \Box \]

We remark that outside of the support of $\omega$, $\nabla \tilde{u}_\kappa$ is a traceless $2 \times 2$ symmetric matrix; hence it is of the form

$$
\begin{pmatrix}
-a & b \\
b & a
\end{pmatrix}.
$$

When $(\varepsilon, q, \omega) \in \Omega_\delta$, $h_\kappa$ is outside of the support of $\omega$ for each $\kappa \in \mathcal{P}_\delta$; consequently we can set $(V_{\kappa,j})_{j=1,2,4,5}$ as follows

$$
\begin{pmatrix}
V_{\kappa,1} \\
V_{\kappa,2}
\end{pmatrix} := \tilde{u}_\kappa(h_\kappa) \text{ and } \begin{pmatrix}
V_{\kappa,4} \\
V_{\kappa,5} \\
V_{\kappa,4}
\end{pmatrix} := \nabla \tilde{u}_\kappa(h_\kappa),
$$

(5.10)
We proceed in four steps. We start with the estimates on $V_\kappa$ and the following estimates are satisfied for some $\kappa > 0$ independent of $\varepsilon$:

\[
V_\kappa := \sum_{i \in \{1,2,3,4,5\}} V_{\kappa,i} \xi_{\kappa,i} = V_\kappa = \tilde{u}_\kappa(q, h_\kappa) + (x - h_\kappa) \cdot \nabla_x \tilde{u}_\kappa(q, h_\kappa). \tag{5.11}
\]

We are now in position to state our approximation result.

**Proposition 5.4.** Let $\delta > 0$. There exists $\varepsilon_0 > 0$ such that for each $\kappa \in \mathcal{P}_s$ and for $\varepsilon < \varepsilon_0$, the following holds. Consider the vector field $u_{\kappa}^{ext}$ introduced in the decomposition (5.1) of the solution $u^\varepsilon$ of System (1.2)-(1.7) and $V_\kappa$ defined in (5.11). Then $V_\kappa$ belongs to $C^1([0,T]; \mathcal{K}_{\kappa,s})$ and there exists a family (parameterized by $\varepsilon$) of functions $u_{\kappa}^{ext}$ in $C^1([0,T]; \mathcal{C}(\mathcal{F}(q_{\kappa}^{ext})))$ such that, as long as $(\varepsilon, q, \nu, \omega) \in \Omega_s$,

\[
u_1 \nabla \nu_2 = \kappa \gamma \kappa_4, \quad \text{due to (3.60), it is of order } O(\varepsilon)\text{.}
\]

Concerning the time-derivative of $\tilde{u}_\kappa$, we have

\[
\partial_t \tilde{u}_\kappa(t,x) = \sum_{\nu \neq \kappa} p'_\nu \nabla \varphi'_\nu + K^\nu[\partial_t \omega] + \sum_{m \in \{1, \ldots, N\}} p_{\mu_m} \frac{\partial \tilde{u}_\kappa}{\partial \mu_m}.
\]

To estimate the first term, we use the acceleration estimates (4.8): since the contribution of $p'_\nu$ is through $p'_\nu \nabla \varphi'_\nu$, due to (3.60), it is of order $O(\varepsilon \beta^\nu)$ in $\mathcal{V}_\kappa(\partial \mathcal{S}_\kappa)$ and consequently bounded. The term $K^\nu[\partial_t \omega]$ is shown to be bounded in $L^p(\mathcal{F})$ exactly as in (4.15) and (4.17). Due to the support of $\omega$, it is hence bounded in $L^\infty(\mathcal{V}_\kappa(\partial \mathcal{S}_\kappa))$. Finally, the last term is of order $O(\beta_{\tilde{u}_\kappa})$ thanks to Lemma 5.3 and energy estimates (4.2). This proves that $\partial_t \tilde{u}_\kappa$ is bounded in $\mathcal{V}_\kappa(\partial \mathcal{S}_\kappa)$, so that by interior elliptic regularity:

\[
\|\partial_t \tilde{u}_\kappa\|_{L^\infty(\mathcal{V}_\kappa(\partial \mathcal{S}_\kappa))} \leq C(1 + \|\beta\|).
\tag{5.16}
\]

The bounds on $V_\kappa$ in (5.13)-(5.14) follow, using (5.10), (5.15) and (5.16). It remains to prove the bounds (5.13)-(5.14) on $u_{\kappa}^{ext}$.

**Step 2.** Let us now relate the function $u_{\kappa}^{ext}$ defined by (5.12) to $\tilde{u}_\kappa$. First, we use (5.3), (5.7) and the support of $\omega$ to infer that $u_{\kappa}^{ext} - \tilde{u}_\kappa$ satisfies

\[
\begin{align*}
\text{div}(u_{\kappa}^{ext} - \tilde{u}_\kappa) &= 0 \quad \text{in } \mathcal{F}(q), \\
\text{curl}(u_{\kappa}^{ext} - \tilde{u}_\kappa) &= 0 \quad \text{in } \mathcal{F}(q), \\
(u_{\kappa}^{ext} - \tilde{u}_\kappa) \cdot n &= 0 \quad \text{on } \partial \mathcal{F}(q) \setminus \partial \mathcal{S}_\kappa, \\
(u_{\kappa}^{ext} - \tilde{u}_\kappa) \cdot n &= -\tilde{u}_\kappa \cdot n \quad \text{on } \partial \mathcal{S}_\kappa, \\
\int_{\partial \mathcal{S}_\kappa} (u_{\kappa}^{ext} - \tilde{u}_\kappa) \cdot \tau \, ds &= 0 \quad \text{for } \nu = 1, \ldots, N.
\end{align*}
\]

51
Recalling the notation (3.54), this gives that
\[ u^{ext}_\kappa - \tilde{u}_\kappa = -\nabla^N \frac{\partial}{\partial t} \tilde{u}_\kappa |_{\partial S_\kappa} \cdot n, \]  
(5.17)
Then we use a Taylor expansion of \( \tilde{u}_\kappa \) in the neighborhood of \( S_\kappa \). Using local elliptic regularity estimates on \( \tilde{u}_\kappa \) (which is harmonic in the \( \delta \)-neighborhood of \( S_\kappa \)), we may estimate the second derivatives of \( \tilde{u}_\kappa \) in \( L^\infty \) in some neighborhood of \( S_\kappa \) by its \( L^\infty \) norm in a larger neighborhood and hence by \( C \| \tilde{u}_\kappa \|_\infty \). It follows (recalling (5.11)) that we may write in the \( \delta \)-2-neighborhood of \( S_\kappa \)
\[ \tilde{u}_\kappa(q, x) = V_\kappa + R_\kappa(q, x) \quad \text{with} \quad |R_\kappa(q, x)| \leq C \| \tilde{u}_\kappa \|_\infty |x - h_\kappa|^2, \]  
(5.18)
where we omit temporarily the dependence of \( \tilde{u}_\kappa \) on \( p \) and \( \omega \) to lighten the notations.

Recalling (3.54) and (5.4) we observe that
\[ \text{Kir}_{\kappa} V_\kappa = \nabla^N \frac{\partial}{\partial t} [V_\kappa \cdot n]. \]  
(5.19)
Hence by (5.17), (5.18) and (5.19) we arrive at (5.12) with
\[ u^{ext}_\kappa(t, x) := \varepsilon^{-2}_\kappa \left\{ R_\kappa(q(t), x) - \nabla^N \frac{\partial}{\partial t} \left[ R_\kappa(q(t), x) \right] |_{\partial S_\kappa} \cdot n \right\}. \]  
(5.20)

**Step 4.** We finally estimate \( \partial_t u^{ext}_\kappa \). To that purpose we introduce the stream function \( \tilde{\gamma}_\kappa \) of \( \tilde{u}_\kappa \), so that \( \tilde{u}_\kappa = \nabla^\perp \tilde{\gamma}_\kappa \) and we define
\[ \alpha^R_\kappa = \alpha^R_\kappa(q, x) := \tilde{\gamma}_\kappa - \sum_{i \in \{1, 2, 3\}} V_{\kappa, i} J_{\kappa, i}, \]
with \( J_{\kappa, i} \) defined in (3.70). By (3.71), Lemma 3.17, (5.11) and (5.18) we have
\[ \nabla^N \frac{\partial}{\partial t} \left[ R_\kappa(q(t), \cdot) \right] |_{\partial S_\kappa} \cdot n] = \nabla^\perp f_\kappa \left[ \alpha^R_\kappa \right]. \]
Hence (5.20) translates into:
\[ u^{ext}_\kappa = \varepsilon^{-2}_\kappa \left\{ R_\kappa(q(t), \cdot) - \nabla^\perp f_\kappa \left[ \alpha^R_\kappa \right] \right\}. \]
(5.24)
Thus
\[ \partial_t u^{ext}_\kappa(t, x) := \varepsilon^{-2}_\kappa \left\{ \partial_t R_\kappa(t, x) - \nabla^\perp f_\kappa \left[ \partial_t \alpha_\kappa(t, x) \right] - \sum_{\mu \in \{1, \ldots, N\}} \sum_{m \in \{1, 2, 3\}} \frac{\partial \nabla^\perp f_\kappa \left[ \alpha_\kappa(t, \cdot) \right]}{\partial q_{\mu, m}} \right\}, \]
(5.24)
where
\[ \mathfrak{R}_\kappa(t, x) := R_\kappa(q^*(t), \mathbf{p}^*(t), \omega(t), x) \quad \text{and} \quad \alpha_\kappa(t, x) := \alpha^R_\kappa(q^*(t), \mathbf{p}^*(t), \omega(t), x). \]
Relying on (5.11) and (5.18), a computation gives
\[ \partial_t \mathfrak{R}_\kappa(t, x) = \frac{\partial \tilde{u}_\kappa}{\partial t}(t, x) - \frac{\partial \tilde{u}_\kappa}{\partial t}(t, h_\kappa) - (x - h_\kappa) \cdot \nabla \frac{\partial \tilde{u}_\kappa}{\partial t}(t, h_\kappa) - \nabla^2 \tilde{u}_\kappa(t, h_\kappa) \cdot h'_\kappa \otimes (x - h_\kappa). \]
With (5.15) and (5.16) we deduce
\[
\|\partial_t R_k\|_{L^\infty(V_\epsilon(\partial\mathcal{S}_\epsilon))} \leq C\varepsilon_k(1 + |\hat{p}|).
\] (5.25)

Since \( R_k(a_k) = \nabla^2 \alpha \), it follows, using again interior elliptic regularity, that we may estimate the second term in (5.24) as follows:
\[
\|\partial_t a_k(t,\cdot) - \partial_t a_k(t, h_k)\|_{L^\infty(\mathcal{S}_\epsilon)} + \varepsilon^{k+\frac{1}{2}} |\partial_t a_k(t,\cdot)|_{C^{k+\frac{1}{2}}(\mathcal{S}_\epsilon)} \leq C\varepsilon_k^2(1 + |\hat{p}|).
\] (5.26)

With Propositions 3.9 and 3.4, this gives
\[
\|\nabla \partial_t [\hat{a}_k(t,\cdot)]\|_{L^\infty(\partial\mathcal{S}_\epsilon)} \leq C\varepsilon_k(1 + |\hat{p}|).
\]

Concerning the third term in (5.24), we use Corollary 3.15, where here the function \( a_k(t,\cdot) \) is fixed. We find
\[
\frac{\partial^2 \tilde{f}_k[a_k(t,\cdot)]}{\partial q_{\mu,m}} = (\nabla a_k^q - \nabla \tilde{f}_k[a_k^q]) \cdot n K_{\mu,m} + c'_\lambda \text{ on } \partial\mathcal{S}_\lambda \text{ and } \frac{\partial^2 \tilde{f}_k[a_k(t,\cdot)]}{\partial q_{\mu,m}} = 0 \text{ on } \partial\Omega.
\]

With (5.21)-(5.22)-(5.23) and Propositions 3.4 and 3.9, we conclude that
\[
\left\| \nabla \frac{\partial^2 \tilde{f}_k[a_k(t,\cdot)]}{\partial q_{\mu,m}} \right\|_{L^\infty(\partial\mathcal{S}_\epsilon)} \leq C\varepsilon_k^\beta m^3.
\]

Injecting in (5.24) we find the last estimate of (5.14), which concludes the proof of Proposition 5.4. \( \square \)

### 5.3 Definition of the modulations

We conclude this section by introducing the first-order modulations \( \alpha_{k,i} \) and the second-order modulations \( \beta_{k,i} \), for \( \kappa \in \mathcal{P}_\mathfrak{s} \) and \( i = 1, 2 \). We set
\[
\text{for } \kappa \in \mathcal{P}_\mathfrak{s}, \quad \alpha_{k,i} := V_{k,i} \text{ for } i = 1, 2, \quad \text{and } \quad \begin{pmatrix} \beta_{k,1} \\ \beta_{k,2} \end{pmatrix} := \begin{pmatrix} -V_{k,4} & V_{k,5} \\ V_{k,5} & V_{k,4} \end{pmatrix} \zeta^\mathfrak{s}(q_k).
\] (5.27)

We recall that \( \zeta^\mathfrak{s}(q_k) \) is defined in (2.19). We notice in passing that due to Proposition 5.4 and the scale relation in (2.19), the modulations can be estimated as follows:
\[
|\alpha_{k,i}| \leq C \quad \text{and} \quad |\beta_{k,i}| \leq C\varepsilon_k.
\] (5.28)

The first-order modulations will play a central role in the normal forms of Section 6 and hence in the modulated energy estimates of Section 7, but also in the passage to the limit in Section 8. The second-order modulations \( \beta_{k,1} \) and \( \beta_{k,2} \) disappear in the limit, but play an important role in the normal forms, in Subsection 6.5 (see Lemma 6.12).

### 6 Normal forms

In this section, we present normal forms for the dynamics of small solids. It will be useful for both the modulated energy estimates (for solids of family (iii)) and the passage to the limit (for solids of family (ii) and (iii)).

#### 6.1 Statement of the normal form

**Proposition 6.1.** Let \( \delta > 0 \). There exists \( \varepsilon_0 > 0 \) such that for \( \varepsilon \leq \varepsilon_0 \), the following holds. Consider the corresponding solutions \( (u_\epsilon^\mathfrak{s}, h^\mathfrak{s}, \vartheta^\mathfrak{s}) \) of the system, for each \( \kappa \in \mathcal{P}_\mathfrak{s} \) the exterior field \( u_\epsilon^\mathfrak{s} \) defined by (5.1), and \( V_\epsilon \) defined by (5.10) together with its coordinates \( (V_{k,i})_{i\in\{1,2,4,5\}} \) in the decomposition (5.11). Introduce the modulated variable \( \overline{p} = (\overline{p}_1, \ldots, \overline{p}_N) \) as follows: for \( i \in \{1,2,3\} \)
\[
\overline{p}_{\kappa,i} = p_{\kappa,i} \quad \text{for } \kappa \in \mathcal{P}_\mathfrak{s}, \quad \overline{p}_{\kappa,i} = p_{\kappa,i} - \delta \epsilon \alpha_{\kappa,i} + \beta_{\kappa,i} \quad \text{for } \kappa \in \mathcal{P}_\mathfrak{s},
\] (6.1)
with \( \alpha_{\kappa,i} \) and \( \beta_{\kappa,i} \) given by (5.27), and the time-dependent vector field \( B_\kappa = (B_{\kappa,j})_{j=1,2,3} \) given by

\[
B_{\kappa,j} := -\gamma_\kappa \sum_{k=1}^3 \bar{p}_{\kappa,k} \int_{t - \delta_{\kappa}}^t \partial_\alpha \hat{\psi}_\kappa \xi_{\kappa,k}^\perp \cdot \xi_{\kappa,j} \, ds. \tag{6.2}
\]

Then as long as \((\varepsilon, q, \omega) \in \Omega_\delta^0\), for each \( \kappa \in \mathcal{P}_s \), one has

\[
\mathcal{M}_{g,\kappa} \dot{p}'_\kappa + \mathcal{M}_{a,\kappa} \bar{p}'_\kappa + \frac{1}{2} \mathcal{M}_{a,n} \bar{p}'_\kappa = A_\kappa(t) + B_\kappa(t) + C_\kappa(t) + D_\kappa(t), \tag{6.3}
\]

where the term \( A_\kappa \) is weakly nonlinear in the sense that for some \( K > 0 \) independent of \( \varepsilon \), for \( j \in \{1,2,3\} \),

\[
|A_{\kappa,j}(t)| \leq K \varepsilon_\kappa^{2+\delta/3} (1 + |\hat{p}(t)|), \tag{6.4}
\]

the term \( C_\kappa \) is gyroscopic of lower order in the sense that for all times,

\[
C_\kappa(t) \cdot \bar{p}_\kappa(t) = 0, \tag{6.5}
\]

and moreover for some \( K > 0 \) independent of \( \varepsilon \), one has for \( j \in \{1,2,3\} \)

\[
|C_{\kappa,j}(t)| \leq K \varepsilon_\kappa^{1+\delta/3} (1 + |\hat{p}(t)|^2), \tag{6.6}
\]

and the term \( D_\kappa \) is weakly gyroscopic in the sense that it satisfies for some \( K > 0 \) independent of \( \varepsilon \),

\[
\left| \int_0^t D_\kappa(\tau) \cdot \bar{p}(\tau) \, d\tau \right| \leq K \varepsilon_\kappa^2 \left( 1 + t + \int_0^t |\hat{p}_\kappa(\tau)|^2 \, d\tau \right), \tag{6.7}
\]

and moreover for some \( K > 0 \) independent of \( \varepsilon \), one has for \( j \in \{1,2,3\} \)

\[
|D_{\kappa,j}(t)| \leq K \varepsilon_\kappa^{1+\delta/3}. \tag{6.8}
\]

We recall the notation (2.11) and (2.12) for the matrices \( \mathcal{M}_{g,\kappa} \) and \( \mathcal{M}_{a,\kappa} \).

Let us highlight that \( B_\kappa \) satisfies

\[
B_\kappa \cdot \bar{p}_\kappa = 0. \tag{6.9}
\]

We will refer to this term as the main gyroscopic term.

**Remark 6.2.** Note the distinction between the modulated variable \( \bar{p}_\kappa \) (for which \( \bar{p}_{\kappa,3} = \hat{\psi}_\kappa \)) on the left-hand side and the scaled variable \( \hat{p}_\kappa \) (with \( \hat{p}_{\kappa,3} = \varepsilon_\kappa \hat{\psi}_\kappa \)) on the right-hand side.

The rest of the section is devoted to the proof of Proposition 6.1.

### 6.2 Starting point of the proof: rewriting the solid equation with various terms

Given \( \delta > 0 \), we first let \( \varepsilon_0 > 0 \) small enough so that all the statements of Sections 3 to 5 apply. To prove the normal form (6.3), we will use a variant of the decomposition (5.1), which is better adapted to modulated variables.

**Definition 6.3.** For each \( \kappa \in \mathcal{P}_s \), we introduce the following decomposition

\[
u^\varepsilon = \bar{u}_{\kappa}^{pot} + \gamma_\kappa \nabla^\perp \hat{\psi}_\kappa + \bar{u}_{\kappa}^{ext} \quad \text{with} \quad \bar{u}_{\kappa}^{pot} := \sum_{j \in \{1,2,3\}} \bar{p}_{\kappa,j} \nabla \varphi_{\kappa,j}. \tag{6.10}
\]

In particular, comparing the decompositions (5.1) and (6.10) we see that

\[
\bar{u}_{\kappa}^{ext} = v_{\kappa}^{ext} + \sum_{j=1}^2 (\alpha_{\kappa,j} + \beta_{\kappa,j}) \nabla \varphi_{\kappa,j}. \tag{6.11}
\]
Proof of Proposition 6.1. We first observe that, by the first equation of (1.2) and by (4.20), the fluid pressure $\pi^\varepsilon$ satisfies:

$$\nabla \pi^\varepsilon = -\partial_t u^\varepsilon - \nabla \left( \frac{|u^\varepsilon|^2}{2} \right) - \omega^\varepsilon u^\varepsilon .$$  \hspace{1cm} (6.12)

Then by (1.7), (2.11), (2.8) and an integration by parts we obtain that, for $\kappa \in \mathcal{P}_s$ and $j \in \{1,2,3\}$,

$$(M_b p')_{\kappa,j} = -I_{\kappa,j} - J_{\kappa,j} - L_{\kappa,j},$$  \hspace{1cm} (6.13)

where

$$I_{\kappa,j} := \int_{\mathcal{F}(q)} \partial_t u^\varepsilon \cdot \nabla \varphi_{\kappa,j} \, dx, \quad J_{\kappa,j} := \int_{\mathcal{F}(q)} \nabla \left( \frac{|u^\varepsilon|^2}{2} \right) \cdot \nabla \varphi_{\kappa,j} \, dx,$$

and

$$L_{\kappa,j} := \int_{\mathcal{F}(q)} \omega^\varepsilon u^\varepsilon \cdot \nabla \varphi_{\kappa,j} \, dx. \hspace{1cm} (6.14)$$

By (6.10)

$$I_{\kappa,j} = I^1_{\kappa,j} + I^2_{\kappa,j} + I^3_{\kappa,j},$$  \hspace{1cm} (6.15)

where

$$I^1_{\kappa,j} := \gamma_{\kappa} \int_{\mathcal{F}(q)} \partial_t \hat{\psi}_k \cdot \nabla \varphi_{\kappa,j} \, dx,$$

$$I^2_{\kappa,j} := \int_{\mathcal{F}(q)} \partial_t \hat{\psi}_k \cdot \nabla \varphi_{\kappa,j} \, dx,$$

$$I^3_{\kappa,j} := \int_{\mathcal{F}(q)} \partial_t \hat{\psi}_k \cdot \nabla \varphi_{\kappa,j} \, dx. \hspace{1cm} (6.16)$$

Concerning $J_{\kappa,j}$, we integrate by parts to obtain

$$J_{\kappa,j} = \int_{\partial \mathcal{S}(q)} \left[ \frac{|u^\varepsilon|^2}{2} \right] K_{\kappa,j} \, ds. \hspace{1cm} (6.19)$$

Given two vector fields $a$ and $b$ on $\partial \mathcal{S}_\kappa$ we define

$$Q_{\kappa,j}(a,b) := \int_{\partial \mathcal{S}(q)} a \cdot b K_{\kappa,j} \, ds \quad \text{and} \quad Q_{\kappa,j}(a) := Q_{\kappa,j}(a,a). \hspace{1cm} (6.20)$$

By (6.10), we obtain, for $\kappa \in \mathcal{P}_s$ and $j \in \{1,2,3\}$,

$$J_{\kappa,j} = J^1_{\kappa,j} + J^2_{\kappa,j} + J^3_{\kappa,j} + J^4_{\kappa,j} + J^5_{\kappa,j} + J^6_{\kappa,j},$$  \hspace{1cm} (6.21)

where

$$J^1_{\kappa,j} := \frac{1}{2} Q_{\kappa,j} (\gamma_{\kappa} \nabla \hat{\psi}_k),$$  \hspace{1cm} (6.22)

$$J^2_{\kappa,j} := \gamma_{\kappa} Q_{\kappa,j} (\nabla \hat{\psi}_k, \hat{\pi}_k^\text{pot} + \hat{\pi}_k^\text{ext} - \hat{v}_{S_\kappa}),$$  \hspace{1cm} (6.23)

$$J^3_{\kappa,j} := \gamma_{\kappa} Q_{\kappa,j} (\nabla \hat{\psi}_k, \hat{v}_{S_\kappa}),$$  \hspace{1cm} (6.24)

$$J^4_{\kappa,j} := \frac{1}{2} Q_{\kappa,j} (\hat{\pi}_k^\text{pot}),$$  \hspace{1cm} (6.25)

$$J^5_{\kappa,j} := \frac{1}{2} Q_{\kappa,j} (\hat{\pi}_k^\text{ext}),$$  \hspace{1cm} (6.26)

$$J^6_{\kappa,j} := Q_{\kappa,j} (\hat{\pi}_k^\text{pot}, \hat{\pi}_k^\text{ext}).$$  \hspace{1cm} (6.27)

where we recall that $\hat{v}_{S_\kappa}$ is the $\kappa$-th solid vector field, see (1.6). In order to reach (6.3), the rest of the proof consists in combining (6.13), (6.15) and (6.21), and regrouping and treating the various terms above, for $\kappa \in \mathcal{P}_s$ and $j \in \{1,2,3\}$, in the following way:

$$-(M_b p')_{\kappa,j} = \underbrace{L_{\kappa,j}}_{\text{Lemma 6.4}} + \underbrace{J^1_{\kappa,j}}_{\text{Lemma 6.5}} + \underbrace{J^2_{\kappa,j}}_{\text{Lemma 6.6}} + \underbrace{J^4_{\kappa,j}}_{\text{Lemma 6.7}} + \underbrace{J^5_{\kappa,j}}_{\text{Lemma 6.8}} + \underbrace{J^3_{\kappa,j}}_{\text{Lemma 6.9}} + \underbrace{J^2_{\kappa,j}}_{\text{Lemma 6.10}} + \underbrace{J^4_{\kappa,j}}_{\text{Lemma 6.11}} + \underbrace{J^6_{\kappa,j}}_{\text{Lemma 6.12}}.$$  \hspace{1cm} (6.28)

For the rest of this section we fix $\kappa \in \mathcal{P}_s$ and $j \in \{1,2,3\}.$
6.3 Treatment of the simplest terms

We start with the term $L_{\kappa,j}$ defined in (6.14), recalling that a term is said weakly nonlinear when it satisfies the same inequality than $A_{\kappa}$ in (6.4).

**Lemma 6.4.** The term $L_{\kappa,j}$ is weakly nonlinear.

**Proof of Lemma 6.4.** This is an immediate consequence of (4.16) and Proposition 3.20, since, due to $(\varepsilon, \xi, \omega) \in Q_0^0$, the support of the vorticity is at distance more than $\delta$ from $\partial S_{\kappa}$.

For the term $J^1_{\kappa,j}$ defined in (6.22), (4.26) has established the following result.

**Lemma 6.5.** One has $J^1_{\kappa,j} = 0$.

Next we combine the $I^1_{\kappa,j}$ defined in (6.16) and the term $J^3_{\kappa,j}$ defined in (6.24).

**Lemma 6.6.** One has $I^1_{\kappa,j} + J^3_{\kappa,j} = 0$.

**Proof of Lemma 6.6.** We have

$$I^1_{\kappa,j} + J^3_{\kappa,j} = \gamma_{\kappa} \int_{S_{\kappa}(q)} \bar{\partial}_{\kappa} \nabla \psi_{\kappa,j} \cdot \nabla \varphi_{\kappa,j} \, dx + \gamma_{\kappa} \int_{S_{\kappa}(q)} \nu_{\kappa} \cdot \nabla \psi_{\kappa,j} \, K_{\kappa,j} \, dx$$

$$= \gamma_{\kappa} \int_{S_{\kappa}(q)} \left[ \bar{\partial}_{\kappa} \nabla \psi_{\kappa,j} + \nabla \left( \nu_{\kappa} \cdot \nabla \psi_{\kappa,j} \right) \right] \cdot \nabla \varphi_{\kappa,j} \, dx.$$

We conclude with (4.23).

For the term $J^5_{\kappa,j}$ defined in (6.26), we have the following result.

**Lemma 6.7.** The expression $J^5_{\kappa,j}$ is weakly nonlinear.

**Proof of Lemma 6.7.** By Proposition 5.4 and (6.11),

$$\pi_{\kappa}^{xt} = (\text{Id} - \text{Kir}_{\kappa}) V_{\kappa} + \varepsilon_{\kappa} u_{\kappa}^{r} + \sum_{k=1}^{2} \left( \alpha_{\kappa,k} + \beta_{\kappa,k} \right) \nabla \varphi_{\kappa,k} \text{ in } \mathcal{F}.$$

Using (5.27), we obtain

$$\pi_{\kappa}^{xt} = V_{\kappa} + \sum_{k=1}^{2} \beta_{\kappa,k} \nabla \varphi_{\kappa,k} - \sum_{k=4}^{5} V_{\kappa,k} \nabla \varphi_{\kappa,k} + \varepsilon_{\kappa} u_{\kappa}^{r} \text{ in } \mathcal{F}.$$  \hspace{1cm} (6.29)

Using (5.28), (6.20), $|\xi_{\kappa,k}|_{L^2(\partial S_{\kappa})} = O(\varepsilon_{\kappa})$ for $k = 4, 5$, $|\partial S_{\kappa}| = O(\varepsilon_{\kappa})$ and (3.60) we see that

$$J^5_{\kappa,j} = Q_{\kappa,j}(V_{\kappa}) + O(\varepsilon_{\kappa}^{2+\delta_{j}}).$$

Now integrating by parts inside $S_{\kappa}$, we obtain

$$Q_{\kappa,j}(V_{\kappa}) = \int_{S_{\kappa}} \text{div} \left( |V_{\kappa}|^2 \xi_{\kappa,j} \right) \, dx = O(\varepsilon_{\kappa}^{2+\delta_{j}}),$$

which concludes the proof of Lemma 6.7.

6.4 Exterior acceleration term

Here we deal with the exterior acceleration term $I^3_{\kappa,j}$ defined in (6.18).

**Lemma 6.8.** The term $I^3_{\kappa,j}$ is weakly nonlinear.

**Proof of Lemma 6.8.** In this proof, by convenience, we will again take the convention of Remark 3.21 for the Kirchhoff potentials. We start by integrating by parts and subdivide the boundary integral:

$$I^3_{\kappa,j} = \int_{\partial \Omega} \bar{\partial}_{\kappa} \pi_{\kappa}^{xt} \cdot n \varphi_{\kappa,j} \, ds + \int_{\partial S_{\kappa}} \bar{\partial}_{\kappa} \pi_{\kappa}^{xt} \cdot n \varphi_{\kappa,j} \, ds + \sum_{\nu \neq \kappa} \int_{\partial S_{\nu}} \bar{\partial}_{\kappa} \pi_{\kappa}^{xt} \cdot n \varphi_{\kappa,j} \, ds.$$  \hspace{1cm} (6.30)
\textbf{Step 1.} We first consider the second term in the right hand side of (6.30). From (6.29), we see that

\[
\tilde{\partial}_t \pi^{ext}_\kappa = V'_\kappa + \sum_{k=1}^2 \beta'_{\kappa,k} \nabla \varphi_{\kappa,k} - \sum_{k=4}^5 V'_{\kappa,k} \nabla \varphi_{\kappa,k} + \sum_{k=1}^2 \sum_{m=\{1,\ldots, N\}}^2 \beta_{\kappa,k} p_{\mu,m} \frac{\partial}{\partial q_{\mu,m}} \varphi_{\kappa,k} \right. \\
\left. - \sum_{k=4}^5 \sum_{m=\{1,\ldots, N\}}^2 V_{\kappa,k} p_{\mu,m} \frac{\partial}{\partial q_{\mu,m}} \varphi_{\kappa,k} + \varepsilon^2 \partial^2 \varphi_{\kappa,k} \text{ on } \partial \mathcal{S}_\kappa. \quad (6.31)
\]

From Proposition 5.4 and Proposition 3.20, we immediately see that the first and third terms in the right-hand side of (6.31) are of order \(O(1 + |\hat{\pi}|)\). Moreover, from (2.19) and (5.27), we see that

\[
\left(\frac{\beta_{\kappa,1}}{\beta_{\kappa,2}}\right)' = \left(-\frac{V'_{\kappa,4}}{V'_{\kappa,5}}, \frac{V'_{\kappa,5}}{V'_{\kappa,4}}\right) + \frac{\beta''_{\kappa}}{\beta_{\kappa,2}} \left(\frac{V_{\kappa,4}}{V_{\kappa,5}} \frac{V_{\kappa,5}}{V_{\kappa,4}}\right) (\varphi_{\kappa}(q_{\kappa}))^{\frac{\beta_{\kappa}}{\beta_{\kappa,2}}} \quad (6.32)
\]

Using Proposition 5.4 and (2.19) again, we see that this term is also of order \(O(1 + |\hat{\pi}|)\). Concerning the last two terms in (6.31), we use Proposition 3.26, (5.13) and (5.28) to deduce that they are of order \(O(1 + |\hat{\pi}|)\) as well. We conclude that

\[
\|\tilde{\partial}_t \pi^{ext}_\kappa\|_{L^\infty(\partial \mathcal{S}_\kappa)} \leq C(1 + |\hat{\pi}|).
\]

Using (3.62) and that \(|\partial \mathcal{S}_\kappa| = O(\varepsilon_\kappa)\), we deduce that

\[
\left| \int_{\partial \mathcal{S}_\kappa} \tilde{\partial}_t \pi^{ext}_\kappa \cdot n \varphi_{\kappa,j} ds \right| \leq C \varepsilon^{2+\delta_{\kappa}}(1 + |\hat{\pi}|). \quad (6.33)
\]

\textbf{Step 2.} We now consider the integral over \(\partial \Omega\) that is the first term in the right hand side of (6.30). Recalling (5.3) and (6.11) we observe that \(\pi^{ext}_\kappa \cdot n = -\gamma_\kappa \nabla \cdot \hat{\psi}_\kappa \cdot n\) on \(\partial \Omega\). Thus, on \(\partial \Omega\),

\[
\tilde{\partial}_t \pi^{ext}_\kappa \cdot n = -\gamma_\kappa (\tilde{\partial}_t \nabla \cdot \hat{\psi}_\kappa) \cdot n = \gamma_\kappa \left(v_{\kappa} \cdot \nabla \cdot \hat{\psi}_\kappa\right) \cdot n,
\]

thanks to (4.23). Therefore with (3.82), we deduce \(\tilde{\partial}_t \pi^{ext}_\kappa \cdot n = O(|\hat{\pi}|)\). On the other hand, by (3.62), \(\varphi_{\kappa,j} = O(\varepsilon^{2+\delta_{\kappa}})\) on \(\partial \Omega\) and therefore

\[
\left| \int_{\partial \Omega} \tilde{\partial}_t \pi^{ext}_\kappa \cdot n \varphi_{\kappa,j} ds \right| \leq C \varepsilon^{2+\delta_{\kappa}}(1 + |\hat{\pi}|). \quad (6.34)
\]

\textbf{Step 3.} Finally we address the integrals in the right hand side of (6.30) which are over \(\partial \mathcal{S}_\nu\) for \(\nu \neq \kappa\). By (5.2) and (6.11),

\[
\pi^{ext}_\kappa = u^{ext}_\kappa + \sum_{\lambda \neq \kappa}^{3} \sum_{i=1}^{3} p_{\lambda,i} \nabla \varphi_{\lambda,i} + \sum_{\lambda \neq \kappa}^{3} \gamma_{\lambda} \nabla \cdot \hat{\psi}_{\lambda} + \sum_{i=1}^{2} (\alpha_{\kappa,i} + \beta_{\kappa,i}) \nabla \varphi_{\kappa,i},
\]

so that

\[
\left| \int_{\partial \mathcal{S}_\nu} \tilde{\partial}_t \pi^{ext}_\kappa \cdot n \varphi_{\kappa,j} ds \right| = E^{\kappa,1}_{\nu,j} + \ldots + E^{\kappa,6}_{\nu,j}. \quad (6.35)
\]
where
\[
E^{\nu,1}_{\kappa,j} := \int_{\partial S_\nu} \hat{\varphi}_t u^x t \cdot n \varphi_{\kappa,j} \, ds,
\]
\[
E^{\nu,2}_{\kappa,j} := \sum_{\lambda \neq \kappa} \sum_{i=1}^{3} p_{\lambda,i} \int_{\partial S_\nu} \varphi_{\lambda,i} \varphi_{\kappa,j} \, ds,
\]
\[
E^{\nu,3}_{\kappa,j} := \sum_{\mu \in \{1, \ldots, N\}} \sum_{\gamma \neq \kappa} \sum_{i=1}^{3} p_{\lambda,i} p_{\mu,m} \int_{\partial S_\nu} \frac{\partial \varphi_{\lambda,i}}{\partial \mu} \cdot n \varphi_{\kappa,j} \, ds,
\]
\[
E^{\nu,4}_{\kappa,j} := \sum_{\mu \in \{1, \ldots, N\}} \gamma \int_{\partial S_\nu} \varphi_t \nabla L_{\lambda} \cdot n \varphi_{\kappa,j} \, ds,
\]
\[
E^{\nu,5}_{\kappa,j} := \sum_{\mu \in \{1, \ldots, N\}} \sum_{i=1}^{2} (\alpha_{\kappa,i} + \beta_{\kappa,i}) p_{\mu,m} \int_{\partial S_\nu} \frac{\partial \varphi_{\kappa,i}}{\partial \mu} \cdot n \varphi_{\kappa,j} \, ds,
\]
\[
E^{\nu,6}_{\kappa,j} := \sum_{i=1}^{2} (\alpha_{\kappa,i} + \beta_{\kappa,i}) \int_{\partial S_\nu} \nabla \varphi_{\kappa,i} \cdot n \varphi_{\kappa,j} \, ds.
\]

Estimate of $E^{\nu,1}_{\kappa,j}$. By (4.12) and (4.13), $||\hat{\varphi}_t u^x t||_{L^\infty(\partial S_\nu)} = O(\varepsilon_\nu^{-1}(1 + ||\hat{\varphi}||))$ and by (3.62), with $\nu \neq \kappa$, $||\varphi_{\kappa,j}||_{L^\infty(\partial S_\nu)} = O(\varepsilon_{\kappa}^{2+\delta_j})$ so that, by integration on $\partial S_\nu$,
\[
E^{\nu,1}_{\kappa,j} = O(\varepsilon_{\kappa}^{2+\delta_j}(1 + ||\hat{\varphi}||)).
\] (6.36)

Estimate of $E^{\nu,2}_{\kappa,j}$. First by definition of the Kirchhoff potentials, see (2.7) and (2.8),
\[
E^{\nu,2}_{\kappa,j} = \sum_{i=1}^{3} \int_{\partial S_\nu} \varphi_{\kappa,j} K_{\nu,i} \, ds.
\] (6.37)

By Proposition 4.4, $||\hat{\varphi}_t||_{H^\infty(\partial S_\nu)} = O(\varepsilon_{\nu}^{-2\delta_0 - \delta_1(\nu \mu)}(1 + ||\hat{\varphi}||))$, and by (3.65) and (3.67), the integral in the right hand side of (6.37) is $O(\varepsilon_{\nu}^{2+\delta_j}(1 + ||\hat{\varphi}||))$, so that, since $\varepsilon_{\nu}^{\delta_j} p_{\nu,i} = \hat{p}_{\nu,i}$,
\[
E^{\nu,2}_{\kappa,j} = O(\varepsilon_{\kappa}^{2+\delta_j}(1 + ||\hat{\varphi}||)).
\] (6.38)

Estimate of $E^{\nu,3}_{\kappa,j}$. By Lemma 3.25,
\[
E^{\nu,3}_{\kappa,j} := \sum_{m=1}^{3} \int_{\partial S_\nu} \varphi_{\kappa,j} K_{\nu,m} \, ds.
\] (6.39)

By an integration by parts
\[
\int_{\partial S_\nu} \varphi_t \left[ \left( \frac{\partial \varphi_{\lambda,j}}{\partial \tau} - (\xi_{\nu,m} \cdot n) \right) \varphi_{\kappa,j} \right] \, ds = - \int_{\partial S_\nu} \left( \frac{\partial \varphi_{\kappa,j}}{\partial \tau} - (\xi_{\nu,m} \cdot n) \right) \varphi_{\lambda,j} \, ds.
\]

By (3.60),
\[
\left| \frac{\partial \varphi_{\lambda,j}}{\partial \tau} \right|_{L^\infty(\partial S_\nu)} = O(\varepsilon_{\kappa}^{2+\delta_j})
\] and
\[
\left| \frac{\partial \varphi_{\lambda,j}}{\partial \tau} \right|_{L^\infty(\partial S_\nu)} = O(\varepsilon_{\kappa}^{2+\delta_j}).
\]

By integration on $\partial S_\nu$, using that $\varepsilon_{\nu}^{\delta_j} p_{\lambda,i} = \tilde{p}_{\lambda,i}$ and (4.2), we obtain that the first term of the right hand side of (6.39) is $O(\varepsilon_{\nu}^{2+\delta_j}||\hat{\varphi}||)$. On the other hand, by (3.65), (3.67) and Remark 3.24, the second integral in the right hand side of (6.39) is of order $O(\varepsilon_{\nu}^{2+\delta_j})$ so that by (4.2), we arrive at
\[
E^{\nu,3}_{\kappa,j} = O(\varepsilon_{\nu}^{2+\delta_j}||\hat{\varphi}||).
\] (6.40)

Estimate of $E^{\nu,4}_{\kappa,j}$. We deal with the term $E^{\nu,4}_{\kappa,j}$ by distinguishing two cases:
• First case: \( \lambda \neq \nu \). By (4.23),
\[
\int_{\partial S_\nu} \partial_t \nabla \hat{\psi}_\lambda \cdot n \varphi_{\kappa,j} \, ds = - \int_{\partial S_\nu} \nabla \left( v_{S,\lambda} \cdot \nabla \hat{\psi}_\lambda \right) \cdot n \varphi_{\kappa,j} \, ds.
\]

By (3.82) and the remark below (3.82), we find
\[
\| \nabla \left( v_{S,\lambda} \cdot \nabla \hat{\psi}_\lambda \right) \cdot n \|_{L^\infty(\partial F \cap \partial S_\nu)} = O(|\hat{\beta}_\lambda|). \tag{6.41}
\]

Hence since \( \nu \neq \lambda \) we deduce with (3.62)
\[
\int_{\partial S_\nu} \partial_t \nabla \hat{\psi}_\lambda \cdot n \varphi_{\kappa,j} \, ds = O(\varepsilon_\kappa^{2+\delta_j}|\hat{\beta}_\nu|).
\]

• Second case: \( \lambda = \nu \). Using an integration by parts and (4.23) we find
\[
\int_{\partial S_\nu} \partial_t \nabla \hat{\psi}_\lambda \cdot n \varphi_{\kappa,j} \, ds = \int_{\partial S_\nu} \nabla \left( v_{S,\nu} \cdot \nabla \hat{\psi}_\nu \right) \cdot n \varphi_{\kappa,j} \, ds
\]
\[
- \int_{\partial S_\nu} \nabla \left( v_{S,\nu} \cdot \nabla \hat{\psi}_\nu \right) \cdot n \varphi_{\kappa,j} \, ds.
\]

With another integration by parts, the first term in the right hand side of (6.42) is transformed into
\[
- \int_{\partial S_\nu} v_{S,\nu} \cdot \nabla \hat{\psi}_\nu K_{\kappa,j} \, ds.
\]

Proceeding as for (4.25), we see that this term can be estimated by \( O(\varepsilon_\kappa^{2+\delta_j}|\hat{\beta}_\nu|) \). We decompose the second term in the right hand side of (6.42) into
\[
\int_{\partial F \cap \partial S_\nu} \nabla \left( v_{S,\nu} \cdot \nabla \hat{\psi}_\nu \right) \cdot n \varphi_{\kappa,j} \, ds
\]
\[
- \int_{\partial S_\nu} \nabla \left( v_{S,\nu} \cdot \nabla \hat{\psi}_\nu \right) \cdot n \varphi_{\kappa,j} \, ds.
\]

We use (6.41) and (3.62) to deduce that the terms in the right hand side of (6.42) are of order \( O(\varepsilon_\kappa^{2+\delta_j}|\hat{\beta}_\nu|) \) (using \( |\partial S_\nu| = O(\varepsilon_\kappa) \) for the last one).

Gathering the two cases we finally arrive at
\[
E_{\kappa,j}^{\mu,4} = O(\varepsilon_\kappa^{2+\delta_j}|\hat{\beta}_\nu|). \tag{6.43}
\]

**Estimate of** \( E_{\kappa,j}^{\mu,5} \). By Lemma 3.25,
\[
E_{\kappa,j}^{\mu,5} = \frac{3}{2} \sum_{m=1}^{2} \left( \alpha_{\kappa,i} + \beta_{\kappa,i} \right) p_{\nu,m} \int_{\partial S_\nu} \hat{\partial}_n \left( \frac{\partial \varphi_{\kappa,i}}{\partial q_{\nu,m}} \right) \varphi_{\kappa,j} \, ds.
\]

For such indices, by (3.73), \( \| \nabla \frac{\partial \varphi_{\kappa,i}}{\partial q_{\nu,m}} \|_{L^\infty(\partial S_\nu)} = O(\varepsilon_\kappa^{2+\beta_{\nu,i}}) \) (recall that \( \nu \neq \kappa \)). Combining with (5.28), (3.62) and \( |\partial S_\nu| = O(\varepsilon_\nu) \), we arrive at
\[
E_{\kappa,j}^{\mu,5} = O(\varepsilon_\kappa^{4+\delta_j}|\hat{\beta}_\nu|). \tag{6.44}
\]

**Estimate of** \( E_{\kappa,j}^{\mu,6} \). Since \( \nu \neq \kappa \), by definition of the Kirchhoff potentials, see (2.7) and (2.8),
\[
E_{\kappa,j}^{\mu,6} = 0. \tag{6.45}
\]
Step 4. Gathering (6.35), (6.36), (6.38), (6.40), (6.43), (6.44) and (6.45) we deduce that for \( \nu \neq \kappa, \)
\[
\left| \int_{\partial S_\nu} \hat{e}_n \hat{\pi}_{\kappa}^{\text{ext}} \cdot n \varphi_{\kappa,j} \, ds \right| \leq C \varepsilon_{\kappa}^{2+\delta_J} (1 + |\hat{p}|). \tag{6.46}
\]
Finally combining (6.30), (6.33), (6.34) and (6.46) we conclude the proof of Lemma 6.8. \( \square \)

6.5 Main gyroscopic term

In this section we study the term \( J_{\kappa,j}^2 \) defined in (6.23). We recall that \( \kappa \in \mathcal{P}_s \).

Lemma 6.9. The term \( J_{\kappa,j}^2 \) can be put in the form
\[
J_{\kappa,j}^2 = B_\kappa + A_\kappa + D_\kappa,
\]
where \( B_\kappa = (B_{\kappa,j})_{j=1,2,3} \) is the main gyroscopic term given by (6.2), the term \( A_\kappa \) is weakly nonlinear in the sense of (6.4) and the term \( D_\kappa \) is weakly gyroscopic in the sense of (6.7)-(6.8).

Proof of Lemma 6.9. We first notice that from (5.1) and (6.10) we have the sense of (6.4) where
\[
\tilde{B}_\kappa = \kappa \otimes \kappa - \kappa \otimes \kappa - 2 \kappa \otimes \kappa
\]
Thus for \( j = 1,2,3 \) (recalling the notation (6.20)), we have

\[
J_{\kappa,j}^2 = J_{\kappa,j}^2 + \varepsilon^2 \kappa_\kappa Q_{\kappa,j}(\nabla^\perp \psi_\kappa, \tilde{\mathbf{a}}_\kappa^j), \tag{6.47}
\]
where
\[
J_{\kappa,j}^2 := \kappa_\kappa Q_{\kappa,j} \left( \nabla^\perp \psi_\kappa, (\text{Id} - \hat{\text{K}} \kappa)(V_\kappa - v_{S_{\kappa}}) \right). \tag{6.48}
\]
Using (5.6), (5.13), \(|\nabla^\perp \psi_\kappa|_{L^\infty(\partial S_\kappa)} = O(1/\varepsilon_{\kappa})\) and \(|\partial S_\kappa| = O(\varepsilon_{\kappa})\), we see that the last term in (6.47) is weakly nonlinear.

To deal with the term \( J_{\kappa,j}^2 \), we first observe that, by (2.2), (5.11), (5.27) and (6.1),
\[
V_\kappa - v_{S_{\kappa}} = - \sum_{k=1}^3 \tilde{p}_{\kappa,k} \xi_{\kappa,k} - \sum_{k=1}^5 \beta_{\kappa,k} \xi_{\kappa,k} + \sum_{k=1}^5 \frac{J_0}{5} V_{\kappa,k} \xi_{\kappa,k}. \tag{6.49}
\]
We are therefore led to estimate \( Q_{\kappa,j} \left( \nabla^\perp \psi_\kappa, (\text{Id} - \hat{\text{K}} \kappa)(\xi_{\kappa,k}) \right) \), for \( \kappa \in \mathcal{P}_s, j = 1,2,3 \) and \( k = 1,2,4,5 \).

We will rely on the following classical result.

Lemma 6.10. Let \( S_0 \) a smooth compact simply connected domain of \( \mathbb{R}^2 \). For any pair of vector fields \( u, v \) in \( C^\infty(\mathbb{R}^2 \setminus S_0; \mathbb{R}^2) \) satisfying \( \text{div} \, u = \text{div} \, v = \text{curl} \, u = \text{curl} \, v = 0 \) in \( \mathbb{R}^2 \setminus S_0 \) and \( u(x) = O(1/|x|) \) and \( v(x) = O(1/|x|) \) as \( |x| \to +\infty \), one has, for any \( j = 1,2,3, \)
\[
\int_{\partial S_0} (u \cdot v) K_0^j(0, \cdot) \, ds = \int_{\partial S_0} \xi_0(0, \cdot) \cdot \left( (u \cdot n)v + (v \cdot n)u \right) \, ds.
\]
We refer to [19, Article 134a. (3) and (7)] for a proof of Lemma 6.10; see also [9, Lemma 4.6]).

Lemma 6.10 has the following consequence.

Lemma 6.11. For all \( j = 1,2,3 \) and \( k = 1,2,3,4,5 \), we have
\[
Q_{\kappa,j}(\nabla^\perp \psi_\kappa, \xi_{\kappa,k}) \nabla^\perp \hat{\psi}_\kappa \xi_{\kappa,k} - \nabla^\perp \hat{\psi}_\kappa \xi_{\kappa,k} = \int_{\partial S_\kappa} \hat{e}_n \hat{\psi}_\kappa \xi_{\kappa,k} \cdot \xi_{\kappa,j} \, ds. \tag{6.50}
\]
Proof of Lemma 6.11. First, using that the vector field $\nabla^\perp \hat{\psi}_p$ is tangent to $\partial S_p$, we split the integral into two parts

$$Q_{\kappa,j}(\nabla^\perp \hat{\psi}_p, \xi_{\kappa,k} - \nabla \hat{\varphi}_{\kappa,k}) = \int_{\partial S_p} \nabla^\perp \hat{\psi}_p(\xi_{\kappa,k} \cdot \tau)\ K_{\kappa,j}\ ds - \int_{\partial S_p} \nabla^\perp \hat{\psi}_p \cdot \nabla \hat{\varphi}_{\kappa,k}
$$

Then thanks to Lemma 6.10, we transform the second integral as

$$-\int_{\partial S_p} \xi_{\kappa,j} \cdot \left((\nabla \hat{\varphi}_{\kappa,k} \cdot n)\nabla^\perp \hat{\psi}_p\right)\ ds.$$

Finally, since $\nabla \hat{\varphi}_{\kappa,k} \cdot n = K_{\kappa,k} = -\xi_{\kappa,k} \cdot \tau$, we observe that

$$-\int_{\partial S_p} \xi_{\kappa,j} \cdot \left((\nabla \hat{\varphi}_{\kappa,k} \cdot n)\nabla^\perp \hat{\psi}_p\right)\ ds = \int_{\partial S_p} \nabla^\perp \hat{\psi}_p(\xi_{\kappa,k} \cdot \tau)(\xi_{\kappa,j} \cdot \tau)\ ds,$$

and we arrive at (6.50).

Now with (6.49) and Lemma 6.11, we consequently transform (6.48) into

$$\vec{J}_{\kappa,j}^2 = B_{\kappa,j} + \vec{J}_{\kappa,j}^2,$$

where we recall that $B_{\kappa} = (B_{\kappa,j})_{j=1,2,3}$ is the main gyroscopic term given by (6.2) and where

$$\vec{J}_{\kappa,j}^2 := -\gamma_{\kappa} \sum_{k=1}^{2} \beta_{\kappa,k} \int_{\partial S_p} \nabla^\perp \hat{\psi}_p(\xi_{\kappa,k} \cdot \xi_{\kappa,j})\ ds + \gamma_{\kappa} \sum_{k=4}^{5} V_{\kappa,k} \int_{\partial S_p} \nabla^\perp \hat{\psi}_p(\xi_{\kappa,k} \cdot \xi_{\kappa,j})\ ds.$$

We have the following lemma, which is the main reason for the choice of $\beta_{\kappa,1}$ and $\beta_{\kappa,2}$ in (5.27).

**Lemma 6.12.** Define $\beta_{\kappa,1}$ and $\beta_{\kappa,2}$ by (5.27). Then one has the following relation for $j = 1, 2$,

$$\sum_{k=1}^{2} \beta_{\kappa,k} \int_{\partial S_p} \nabla^\perp \hat{\psi}_p(\xi_{\kappa,k} \cdot \xi_{\kappa,j})\ ds = \sum_{k=4}^{5} V_{\kappa,k} \int_{\partial S_p} \nabla^\perp \hat{\psi}_p(\xi_{\kappa,k} \cdot \xi_{\kappa,j})\ ds.
$$

**Proof of Lemma 6.12.** This is a direct consequence of (2.7), (2.18d) and (2.19): for $j = 1, 2$ and $k = 1, 2$ one finds

$$\int_{\partial S_p} \nabla^\perp \hat{\psi}_p(\xi_{\kappa,k} \cdot \xi_{\kappa,j})\ ds = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}_{j=1,2},$$

while for $j = 1, 2$ and $k = 4, 5$ one has

$$\int_{\partial S_p} \nabla^\perp \hat{\psi}_p(\xi_{\kappa,k} \cdot \xi_{\kappa,j})\ ds = \begin{pmatrix} \zeta_{\kappa,2} & \zeta_{\kappa,1} \\ -\zeta_{\kappa,2} & -\zeta_{\kappa,1} \end{pmatrix}_{j=1,2}.$$

Hence (6.51) is equivalent to $\beta_{\kappa,2} = \zeta_{\kappa,2} V_{\kappa,4} + \zeta_{\kappa,1} V_{\kappa,5}$ and $-\beta_{\kappa,1} = \zeta_{\kappa,1} V_{\kappa,4} - \zeta_{\kappa,2} V_{\kappa,5}$, that is, exactly the second relation of (5.27).

From Lemma 6.12 we readily deduce that $\vec{J}_{\kappa,1}^2 = \vec{J}_{\kappa,2}^2 = 0$. Hence it remains only to study

$$\vec{J}_{\kappa,3}^2 = -\gamma_{\kappa} \sum_{k=1}^{2} \beta_{\kappa,k} \int_{\partial S_p} \nabla^\perp \hat{\psi}_p(\xi_{\kappa,k} \cdot \xi_{\kappa,3})\ ds + \gamma_{\kappa} \sum_{k=4}^{5} V_{\kappa,k} \int_{\partial S_p} \nabla^\perp \hat{\psi}_p(\xi_{\kappa,k} \cdot \xi_{\kappa,3})\ ds =: D_{\kappa,1}^3 + D_{\kappa,2}^3.$$

Let us show that the term $\vec{J}_{\kappa,3}^2 = (0, 0, D_{\kappa,1}^3 + D_{\kappa,2}^3)^T$ is weakly gyroscopic. First, with (3.79), (5.13) and (5.28) and $\|\xi_{\kappa,k}\|_{L^\infty(S_p)} = O(\varepsilon_\kappa)$ for $k = 4, 5$, it is easy to check that it satisfies (6.8). Let us now prove (6.7) by treating the two terms $(0, 0, D_{\kappa,1}^3)^T$ and $(0, 0, D_{\kappa,2}^3)^T$ separately.

We start with the term $(0, 0, D_{\kappa,1}^3)^T$. Here (2.19) gives for $k = 1, 2$

$$\int_{\partial S_p} \nabla^\perp \hat{\psi}_p(\xi_{\kappa,k} \cdot \xi_{\kappa,3})\ ds = \zeta_{\kappa}(q_\kappa) \cdot e_k.$$
Moreover, due to (5.27) we have
\[ \sum_{k=1}^{2} \beta_{k,k} \zeta_{k}(q_{k}) \cdot e_{k} = \zeta_{k}(q_{k}) \cdot \mathcal{A}(V_{k}) \zeta_{k}(q_{k}) \text{ where } \mathcal{A}(V_{k}) := \begin{pmatrix} -V_{k,4} & V_{k,5} \\ V_{k,5} & V_{k,4} \end{pmatrix}. \]

Since the matrix \( \mathcal{A}(V_{k}) \) is a traceless symmetric 2 \times 2 matrix, we have \( R(\vartheta)^{*} \mathcal{A}(V_{k}) = \mathcal{A}(V_{k}) R(\vartheta) \) so that, using again (2.19),
\[ \sum_{k=1}^{2} \beta_{k,k} \zeta_{k}(q_{k}) \cdot e_{k} = \varepsilon_{k}^{2} \hat{\zeta}_{k,0} \cdot \mathcal{A}(V_{k}) R(2\vartheta_{k}) \hat{\zeta}_{k,0}. \]

It follows that
\[ \int_{0}^{t} \overline{P}_{k,3}(\tau) D_{3}^{1}(\tau) \, d\tau = -\gamma_{k} \varepsilon_{k}^{2} \hat{\zeta}_{k,0} \cdot \int_{0}^{t} \theta^{*}_{k}(\tau) R(2\vartheta_{k}(\tau)) \hat{\zeta}_{k,0} \, d\tau. \]

By integration by parts we infer
\[ \int_{0}^{t} \theta^{*}_{k}(\tau) R(2\vartheta_{k}(\tau)) \hat{\zeta}_{k,0} \, d\tau = -\frac{1}{2} \int_{0}^{t} \mathcal{A}(V_{k}(\tau)) R(2\vartheta_{k}(\tau)) \hat{\zeta}_{k,0} \, d\tau + \frac{1}{2} \left[ \mathcal{A}(V_{k}(\tau)) R(2\vartheta_{k}(\tau)) \hat{\zeta}_{k,0} \right]_{0}^{t}. \]

Since we can bound the right hand side by \( C(1 + \|V_{k}\|_{\infty} + t \|V_{k}'\|_{\infty}) \), the estimate (6.7) for the term \( (0,0,D_{3}^{1})^{T} \) follows from Proposition 5.4.

We now consider the term \( (0,0,D_{3}^{2})^{T} \). In that case, the integrals are given by
\[ \int_{\mathcal{S}_{n}} \hat{\vartheta}_{k} \hat{\psi}_{k} \hat{\zeta}_{k,4} \cdot \xi_{k,3} \, ds = \int_{\mathcal{S}_{n}} \hat{\vartheta}_{k} \hat{\psi}_{k} \left[ (x_{2} - h_{k,1})^{2} - (x_{1} - h_{k,1})^{2} \right] \, ds \]
and
\[ \int_{\mathcal{S}_{n}} \hat{\vartheta}_{k} \hat{\psi}_{k} \hat{\zeta}_{k,5} \cdot \xi_{k,3} \, ds = 2 \int_{\mathcal{S}_{n}} \hat{\vartheta}_{k} \hat{\psi}_{k} (x_{1} - h_{k,1})(x_{2} - h_{k,2}) \, ds. \]

We notice that
\[ (x - h_{k}) \cdot (x - h_{k}) + (x - h_{k}) \cdot (x - h_{k}) = \left( \begin{array}{cc} -2(x_{1} - h_{k,1})(x_{2} - h_{k,2}) & (x_{1} - h_{k,1})^{2} - (x_{2} - h_{k,2})^{2} \\ (x_{1} - h_{k,1})^{2} - (x_{2} - h_{k,2})^{2} & 2(x_{1} - h_{k,1})(x_{2} - h_{k,2}) \end{array} \right) \]
and consequently
\[ \sum_{k=4}^{5} V_{k,k} \int_{\mathcal{S}_{n}} \hat{\vartheta}_{k} \hat{\psi}_{k} \cdot \xi_{k,3} \, ds \]
\[ = e_{1} \cdot \left( \int_{\mathcal{S}_{n}} \hat{\vartheta}_{k} \hat{\psi}_{k} \left[ (x - h_{k}) \cdot (x - h_{k}) + (x - h_{k}) \cdot (x - h_{k}) \right] \, ds \right) \begin{pmatrix} -V_{k,5} \\ -V_{k,4} \end{pmatrix}. \]

Now the matrix between parentheses can be rewritten as
\[ \int_{\mathcal{S}_{n}} \hat{\vartheta}_{k} \hat{\psi}_{k} \left[ (x - h_{k}) \cdot (x - h_{k}) + (x - h_{k}) \cdot (x - h_{k}) \right] \, ds \]
\[ = e_{1} \cdot R(\vartheta_{k}) \left[ \int_{\mathcal{S}_{n,0}} \hat{\vartheta}_{k} \hat{\psi}_{k,0} (x \cdot x + x \cdot x) \, ds \right] R(\vartheta_{k})^{*}. \]

Call \( \mathbf{Z} \) the time-independent matrix between brackets. Since \( \mathbf{Z} \) is a traceless symmetric 2 \times 2 matrix, we have \( R(\vartheta_{k}) \mathbf{Z} R(\vartheta_{k})^{*} = \mathbf{Z} R(-2\vartheta_{k}) \), so that
\[ \sum_{k=4}^{5} V_{k,k} \int_{\mathcal{S}_{n}} \hat{\vartheta}_{k} \hat{\psi}_{k} \cdot \xi_{k,3} \, ds = -e_{1} \cdot \mathbf{Z} R(-2\vartheta_{k}) \begin{pmatrix} V_{k,5} \\ V_{k,4} \end{pmatrix}, \]

Now we deduce
\[ \int_{0}^{t} \overline{P}_{k,3}(\tau) D_{3}^{2}(\tau) \, d\tau = -\gamma_{k} e_{k}^{2} e_{1} \cdot \mathbf{Z} \int_{0}^{t} \theta^{*}_{k}(\tau) R(-2\vartheta_{k}) \begin{pmatrix} V_{k,5} \\ V_{k,4} \end{pmatrix} \, d\tau, \]
and we conclude as for the term \( (0,0,D_{3}^{1})^{T} \) by using an integration by parts in time and the estimates of Proposition 5.4.
6.6 Added mass term

In this section we combine the term \( I_{k,j}^2 \) defined in (6.17), the term \( J_{k,j}^3 \) defined in (6.25) and the term \( J_{k,j}^6 \) defined in (6.27). We recall the notation (2.12) for the added mass matrix \( \mathcal{M}_{a,k} \) which is time-dependent and that we say that a term is gyroscopic of lower order when it satisfies (6.5) and (6.6).

**Lemma 6.13.** The term \( I_{k,j}^2 + J_{k,j}^4 + J_{k,j}^6 \) can be put in the form

\[
I_{k,j}^2 + J_{k,j}^4 + J_{k,j}^6 = \mathcal{M}_{a,k}\dot{\mathcal{P}}_k + \frac{1}{2}\mathcal{M}'_{a,k}\mathcal{P}_k + A_k + C_k,
\]

where the term \( A_k \) is weakly nonlinear and where the term \( C_k \) is gyroscopic of lower order.

**Proof of Lemma 6.13.** We proceed in three steps.

**Step 1.** Using the definition of \( \mathcal{P}_k \) in (6.10), we find, for \( j \in \{1, 2, 3\},

\[
I_{k,j}^2 = (\mathcal{M}_{a,k}\mathcal{P}_k)_{k,j} + \sum_{i=1}^{3} \sum_{\nu \in \{1, \ldots, N\}} \int_{\mathcal{F}(t)} \mathcal{P}_{k,i} \frac{\partial \dot{\varphi}_{k,i}}{\partial \mathcal{Q}_{\nu,k}} \cdot \nabla \varphi_{k,j} \, dx.
\]

On the other hand, by Reynolds’ transport theorem,

\[
(\mathcal{M}'_{a,k})_{ij} = \sum_{\nu \in \{1, \ldots, N\}} \sum_{k \in \{1, 2, 3\}} \int_{\mathcal{F}(t)} \mathcal{P}_{k,j} \frac{\partial \varphi_{k,i}}{\partial \mathcal{Q}_{\nu,k}} \cdot \nabla \varphi_{k,j} \, dx + \sum_{\nu \in \{1, \ldots, N\}} \sum_{k \in \{1, 2, 3\}} \int_{\mathcal{F}(t)} \nabla \varphi_{k,i} \cdot \mathcal{P}_{k,j} \frac{\partial \varphi_{k,j}}{\partial \mathcal{Q}_{\nu,k}} \, dx
\]

so that

\[
(\mathcal{M}_{a,k}\mathcal{P}_k + \frac{1}{2}\mathcal{M}'_{a,k}\mathcal{P}_k)_{j} = I_{k,j}^2 + \frac{1}{2} \sum_{i=1}^{3} \sum_{\nu \in \{1, \ldots, N\}} \int_{\mathcal{F}(t)} \mathcal{P}_{k,i} \mathcal{P}_{k,j} \left( \nabla \varphi_{k,i} \cdot \frac{\partial \varphi_{k,j}}{\partial \mathcal{Q}_{\nu,k}} - \frac{\partial \varphi_{k,i}}{\partial \mathcal{Q}_{\nu,k}} \cdot \nabla \varphi_{k,j} \right) \, dx
\]

\[
+ \frac{1}{2} \sum_{i=1}^{3} \int_{\partial \mathcal{F}(t)} \mathcal{P}_{k,i} \cdot \nabla \varphi_{k,j} \cdot (u_{pot} \cdot n) \, ds.
\]

We focus on the last term in the right-hand side. The idea is to replace \( u_{pot} \cdot n \) with \( \mathcal{P}_{k} \cdot n \), up to an error term. Adding and subtracting (6.25) in the right-hand side, and using (6.10) we find

\[
(\mathcal{M}_{a,k}\mathcal{P}_k + \frac{1}{2}\mathcal{M}'_{a,k}\mathcal{P}_k)_{j} = I_{k,j}^2 + J_{k,j}^4
\]

\[
+ \frac{1}{2} \sum_{i=1}^{3} \sum_{\nu \in \{1, \ldots, N\}} \int_{\mathcal{F}(t)} \mathcal{P}_{k,i} \mathcal{P}_{k,j} \left( \nabla \varphi_{k,i} \cdot \frac{\partial \varphi_{k,j}}{\partial \mathcal{Q}_{\nu,k}} - \frac{\partial \varphi_{k,i}}{\partial \mathcal{Q}_{\nu,k}} \cdot \nabla \varphi_{k,j} \right) \, dx
\]

\[
+ \frac{1}{2} \int_{\partial \mathcal{F}(t)} \mathcal{P}_{k} \cdot \nabla \varphi_{k,j} \cdot (u_{pot} - \mathcal{P}_{k} \cdot n) \, ds.
\]

Call \( C_{k,j}^1 \) the expression in the second line of (6.52) and \( C_{k,j}^2 \) the expression in the third line of (6.52). It is clear that \( C_{k,j}^1 = (C_{k,1}^1, C_{k,2}^1, C_{k,3}^1)^T \) and \( C_{k,j}^2 = (C_{k,1}^2, C_{k,2}^2, C_{k,3}^2)^T \) satisfy the property \( \mathcal{P}_{k} \cdot C_{k,j}^1 = \mathcal{P}_{k} \cdot C_{k,j}^2 = 0 \).

Using (3.72) and an integration by parts we see that \( C_{k,j}^1 \) satisfies (6.6). Using (6.1), (6.10), (3.60) and (5.28) we see that independently of \( \mathcal{P} \), we have

\[
\| \mathcal{P}_{k} \|_{L^\infty(\partial \mathcal{S}_a)} \leq C (1 + |\mathcal{P}_{k}|) \quad \text{and} \quad \| \mathcal{P}_{k} \|_{L^\infty(\partial \mathcal{S}_a)} \leq C \mathcal{E}_{k} (1 + |\mathcal{P}_{k}|) \quad \text{for} \quad \nu \neq k. \quad (6.53)
\]

63
With $|\partial S_\kappa| = O(\varepsilon_\kappa)$, we deduce that $C^2_\kappa$ satisfies (6.6). Consequently the terms $C^1_\kappa$ and $C^2_\kappa$ are gyroscopic of lower order (in the sense of (6.5) and (6.6)).

**Step 2.** Hence we now focus on the last term in the right-hand side of (6.52). We first consider the integral away from $\partial S_\kappa$:

$$
\int_{\partial F(t) \setminus \partial S_\kappa} \nabla \varphi_{\kappa,j} \cdot (u^\text{pot} - \tilde{u}^\text{pot}_\kappa) \cdot n \, ds = \sum_{\nu \neq \kappa} \int_{\partial S_\nu} \nabla \varphi_{\kappa,j} \cdot u^\text{pot} \cdot n \, ds,
$$

since $\tilde{u}^\text{pot}_\kappa \cdot n = 0$ on $\partial F(t) \setminus \partial S_\kappa$ and since moreover $u^\text{pot} \cdot n$ vanishes on $\partial \Omega$. From (4.9) we have

$$
u, i \in \{1, \ldots, N\},
$$

Hence it remains to consider the term

$$
(\hat{\nu}^\text{pot}_\kappa \cdot \nabla \varphi_{\kappa,j}) \cdot n = \sum_{\ell = 1}^2 (\alpha_{\kappa,\ell} + \beta_{\kappa,\ell}) K_{\kappa,\ell} \text{ on } \partial S_\kappa.
$$

Hence with (5.28) we see that this factor is bounded. We want now to replace in this integral the factor $\tilde{u}^\text{pot}_\kappa \cdot \nabla \varphi_{\kappa,j}$ by $\hat{\nu}^\text{pot}_\kappa \cdot \nabla \hat{\varphi}_{\kappa,i}$, where we set

$$
\hat{\nu}^\text{pot}_\kappa := \sum_{i = 1}^3 p_{\kappa,i} \nabla \hat{\varphi}_{\kappa,i}.
$$

Similarly to (6.53), we have

$$
\|\hat{\nu}^\text{pot}_\kappa\|_{L^\infty(\partial S_\kappa)} \leq C(1 + |\hat{\nu}_\kappa|).
$$

Using (3.60) in Proposition 3.20, (6.53), the boundedness of (6.55) and (6.56), we find

$$
\frac{1}{2} \int_{\partial S_\kappa} \nu^\text{pot}_\kappa \cdot \nabla \varphi_{\kappa,j} ((u^\text{pot} - \nu^\text{pot}_\kappa) \cdot n) \, ds = N_{\kappa,j} + O(\varepsilon_\kappa^{2+\delta_3}(1 + |\hat{\nu}_\kappa|))
$$

where

$$
N_{\kappa,j} := \frac{1}{2} \int_{\partial S_\kappa} \hat{\nu}^\text{pot}_\kappa \cdot \nabla \hat{\varphi}_{\kappa,j} ((u^\text{pot} - \nu^\text{pot}_\kappa) \cdot n) \, ds.
$$

Of course the last term in the right-hand side of (6.57) is weakly nonlinear.

**Step 3.** Hence it remains to consider the term $N_{\kappa,j}$. Using (6.55) and applying Lemma 6.10 to $N_{\kappa,j}$ we deduce that

$$
N_{\kappa,j} = \frac{1}{2} \sum_{\ell = 1}^2 \left( \int_{\partial S_\kappa} (\alpha_{\kappa,\ell} + \beta_{\kappa,\ell}) \xi_\ell \cdot ((\hat{\nu}^\text{pot}_\kappa \cdot n) \nabla \hat{\varphi}_{\kappa,j} + (\nabla \hat{\varphi}_{\kappa,j} \cdot n) \hat{\nu}^\text{pot}_\kappa) \, ds - \hat{N}_{\kappa,j} + C^3_{\kappa,j},
$$

where

$$
\hat{N}_{\kappa,j} := \frac{2}{\ell = 1} \int_{\partial S_\kappa} (\alpha_{\kappa,\ell} + \beta_{\kappa,\ell}) \xi_\ell \cdot \hat{u}^\text{pot}_\kappa K_{\kappa,j} \, ds
$$

and

$$
C^3_{\kappa,j} := \frac{1}{2} \sum_{\ell = 1}^2 (\alpha_{\kappa,\ell} + \beta_{\kappa,\ell}) \xi_\ell \cdot (\hat{u}^\text{pot}_\kappa \cdot n) \nabla \hat{\varphi}_{\kappa,j} - (\nabla \hat{\varphi}_{\kappa,j} \cdot n) \hat{\nu}^\text{pot}_\kappa) \, ds.
$$
As before, we see that \( C_k^3 = (C_{k,1}^3, C_{k,2}^3, C_{k,3}^3)^T \) is gyroscopic of lower order, and we are left with the term \( \tilde{N}_{k,j} \). We recombine \( \tilde{N}_{k,j} \) with \( J_{k,j}^6 = Q_{k,j}(\tilde{\nu}_k^\text{pot}, \tilde{\nu}_k^{\text{ext}}) \) as follows:

\[
\tilde{N}_{k,j} - J_{k,j}^6 = \int_{\partial \mathcal{S}_k} \left( \tilde{\nu}_k^\text{pot} \cdot \left[ -\tilde{\nu}_k^{\text{ext}} + \sum_{\ell=1}^2 (\alpha_{k,\ell} + \beta_{k,\ell}) \xi_\ell \right] \right) K_{k,j} ds + Q_{k,j}(\tilde{\nu}_k^{\text{ext}}, \tilde{\nu}_k^\text{pot} - \tilde{\nu}_k^\text{pot}). \tag{6.59}
\]

By (6.11), (2.28) and Lemma 5.2, \( \|\tilde{\nu}_k^{\text{ext}}\|_\infty \leq C \). Hence as before, with (3.58) we can estimate the last term in (6.59) by \( O(\varepsilon_k^{2+\delta/3}(1 + |\tilde{p}_k|)) \). Concerning the first term in (6.59), using (6.29), (5.11) and (5.27) we find

\[
\left[ -\tilde{\nu}_k^{\text{ext}} + \sum_{\ell=1}^2 (\alpha_{k,\ell} + \beta_{k,\ell}) \xi_\ell \right] = 2 \sum_{\ell=1}^2 \beta_{k,\ell}(\xi_k - \nabla \phi_{k,\ell}) - \sum_{\ell=4}^5 V_{k,\ell}(\xi_k - \nabla \phi_{k,\ell}) - \varepsilon_k^2 u_k \quad \text{on } \partial \mathcal{S}_k.
\]

Since \( \beta_{k,\ell} = O(\varepsilon_k) \) for \( \ell = 1, 2 \) and \( \|\xi_k\|_{L^6(\partial \mathcal{S}_k)} = O(\varepsilon_k) \) for \( \ell = 4, 5 \), using (3.60) and (5.13) we see that these terms are all (at least) of order \( O(\varepsilon_k) \) in \( L^2 \) norm on \( \partial \mathcal{S}_k \). Using \( |\partial \mathcal{S}_k| = O(\varepsilon_k) \) and (5.66), this gives the estimate

\[
\tilde{N}_{k,j} - J_{k,j}^6 = O(\varepsilon_k^{2+\delta/3}(1 + |\tilde{p}_k|)).
\]

Going back to (6.54) and (6.57) and taking into account the above treatment of (6.58), we deduce that

\[
\frac{1}{2} \int_{\mathcal{F}(t)} \tilde{\nu}_k^\text{pot} \cdot \nabla \phi_{k,j}(\mu^\text{pot} - \tilde{\nu}_k^\text{pot}) \cdot n \, ds = J_{k,j}^6 + C_k^3 + O(\varepsilon_k^{2+\delta/3}(1 + |\tilde{p}_k|)). \tag{6.60}
\]

Of course the last term in (6.60) is weakly nonlinear. Then injecting (6.60) in (6.52) we obtain the desired result. \( \square \)

### 6.7 Conclusion of the proof of the normal form

Gathering (6.28), Lemmas 6.4, 6.5, 6.6, 6.7, 6.8, 6.11 and 6.13 we conclude the proof of Proposition 6.1. \( \square \)

### 7 Modulated energy estimates

This section is devoted to the following crucial \textit{a priori} estimate.

**Proposition 7.1.** Let \( \delta > 0 \). There exists \( \varepsilon_0 > 0 \) such that for all \( \kappa, \tilde{p}_k \) is bounded as long as \( (\mathcal{S}, q, \omega) \) stays in \( \mathcal{Q}^{\varepsilon_0}_\delta \).

**Proof of Proposition 7.1.** We only consider \( \kappa \in \mathcal{P}_{(i)} \), since the boundedness of \( \tilde{p}_k \) was already obtained for \( \kappa \in \mathcal{P}_{(i)} \cup \mathcal{P}_{(ii)} \), see Proposition 4.2. Now we consider (6.3) and multiply it by \( \tilde{\nu}_k \); using (6.1), we find, as long as \( (\mathcal{S}, q, \omega) \in \mathcal{Q}^{\varepsilon_0}_\delta \):

\[
\left( \mathcal{M}_\kappa \tilde{\nu}_k + \frac{1}{2} \mathcal{M}' \tilde{\nu}_k \right) \cdot \tilde{\nu}_k = A_k \cdot \tilde{\nu}_k + B_k \cdot \tilde{\nu}_k + C_k \cdot \tilde{\nu}_k + D_k \cdot \tilde{\nu}_k - \mathcal{M}_{g,k} V' \cdot \tilde{\nu}_k,
\]

where \( V_k := (\alpha_{k,1} + \beta_{k,1}, \alpha_{k,2} + \beta_{k,2}, 0)^T \). We observe that the left-hand side equals \( \frac{1}{2} (\mathcal{M}_\kappa \tilde{\nu}_k \cdot \tilde{\nu}_k)' \) and that the second and third terms in the right-hand side vanish, see (6.5) and (6.9). Concerning the last term, we use (6.1), (5.13)-(5.14) (recalling that \( \alpha_{k,1} \) and \( \beta_{k,1} \) are given by (5.27)) and (1.12); we find

\[
|M_{g,k} V' \cdot \tilde{\nu}_k| \leq C \sum_{j=1}^2 \varepsilon_k^{2+\delta/3} |\tilde{\nu}_{k,j}|(1 + |\tilde{p}_k|)
\]

Integrating over time and using (6.4) and (6.7) we deduce

\[
|M_{g,k} \tilde{\nu}_k(t) - M_{g,k} \tilde{\nu}_k \cdot \tilde{\nu}_k(0)| \leq C \int_0^t \sum_{j=1}^2 \varepsilon_k^{\min(2,\alpha_k)+\delta/3} |\tilde{\nu}_{k,j}|(1 + |\tilde{p}_k|) + K \varepsilon_k^2 \left( 1 + t + \int_0^t |\tilde{\nu}_k|^2 \right). \tag{7.1}
\]
Now we introduce the slight variant \( \tilde{p}_\kappa \) of the modulated variable:
\[
\tilde{p}_{\kappa,i} = \hat{p}_{\kappa,i} - \delta_{i(1,2)}(\alpha_{\kappa,i} + \beta_{\kappa,i}).
\]
The only difference between \( \tilde{p}_\kappa \) and \( \bar{p}_\kappa \) lies in the third coordinate \( i = 3 \): \( \tilde{p}_{\kappa,3} = \varepsilon_{\kappa} \tilde{\varphi}_3^\kappa \) while \( \bar{p}_{\kappa,3} = \tilde{\varphi}_3^\kappa \). In particular
\[
\sum_{j=1}^{3} \varepsilon_{\kappa,j}^3 |p_{\kappa,j}| = \sum_{j=1}^{3} |\bar{p}_{\kappa,j}|.
\]
Next we introduce the \( 3 \times 3 \) matrix \( M^\kappa \) whose entries are given by \( M^\kappa_{\kappa,ij} = \varepsilon_{\min(2,\alpha_{\kappa})} \delta_{ij} M_{\kappa,ij} \) for \( i, j = 1, 2, 3 \). We have
\[
M_{\kappa} \bar{p}_{\kappa} \cdot \tilde{p}_{\kappa} = \varepsilon^\min(2,\alpha_{\kappa}) M^\kappa \bar{p}_{\kappa} \cdot \tilde{p}_{\kappa}.
\]
Hence using \( \tilde{p}_\kappa \) and \( M^\kappa \), (7.1) allows to write, with \( \varepsilon^2_{\kappa} \leq \varepsilon^\min(2,\alpha_{\kappa}) \):
\[
|M^\kappa \bar{p}_{\kappa} \cdot \tilde{p}_{\kappa}(t) - M^\kappa \bar{p}_{\kappa} \cdot \tilde{p}_{\kappa}(0)| \leq C \left[ \int_0^t (1 + |\tilde{p}_{\kappa}|) |\tilde{p}_{\kappa}| + \left( 1 + t + \int_0^t |\tilde{p}_{\kappa}|^2 \right) \right].
\]
Now there are two cases:
- If \( \alpha_{\kappa} > 2 \), then relying on the added mass one has, using Corollary 3.23 and Remark 2.1, that \( |(M^\kappa)^{-1}| \leq C \) independently of \( \varepsilon \) and \( t \).
- If \( \alpha_{\kappa} \leq 2 \), then we rely on the genuine mass matrix and conclude as well that \( |(M^\kappa)^{-1}| \leq C \) independently of \( \varepsilon \) and \( t \).
Consequently in both cases we can invert by \( M^\kappa \) and reach for all \( \kappa \in \mathcal{P}_{(iii)} \):
\[
|\tilde{p}_{\kappa}|^2(t) \leq C \int_0^t (1 + |\tilde{p}_{\kappa}|) |\tilde{p}_{\kappa}| + K \left( 1 + t + \int_0^t |\tilde{p}_{\kappa}|^2 \right) + C|\tilde{p}_{\kappa}|^2(0).
\]
From (5.28), we see that \( |\tilde{p}_{\kappa}| \leq C(1 + |\bar{p}_{\kappa}|) \) and \( |\bar{p}_{\kappa}| \leq C(1 + |\tilde{p}_{\kappa}|) \). We sum over \( \kappa \in \mathcal{P}_{(iii)} \) and use that we already have a bound on \( \tilde{p}_{\kappa} \) for \( \kappa \in \mathcal{P}_{(i)} \cup \mathcal{P}_{(ii)} \). We deduce that for some constant \( K \) depending only on the geometry, \( \delta \) and the initial condition, one has:
\[
|\tilde{p}|^2(t) \leq K \left( 1 + t + \int_0^t |\tilde{p}|^2 \right).
\]
We conclude by Gronwall’s lemma (which we can apply on any time-interval for which \( (\varepsilon, q, \omega) \in \Omega^\beta \)).

8 Passage to the limit

8.1 A change of variable

A difficulty to prove the convergences is the dependence of the domain on \( \varepsilon \). This dependence is twofold: first it depends directly on \( \varepsilon \) because the small solids occupy a zone depending on this parameter; and then it depends on \( \varepsilon \) because the solution does, and all solids whether small or of fixed size are located according to the variable \( q^\varepsilon \). We can temper the difficulty associated with the second dependence by using an adequate family of diffeomorphisms which we now describe. It will not solve the first difficulty but will help with the second one; in particular it will allow to be more precise on the convergences in the neighborhoods of large solids.

First, we define the following partial set of coordinates for the solids:
\[
q^\varepsilon := (q_1, \ldots, q_{N_{(i)}}, h_{N_{(i)}+1}, \ldots, h_N).
\]  
(8.1)

This corresponds to the coordinates in which we will actually pass to the limit. Given \( \delta > 0 \), we introduce the following configuration space \( Q^\delta_\varepsilon \):
\[
Q^\delta_\varepsilon := \{ q \in \mathbb{R}^{3N_{(i)}+2N_\varepsilon} : \forall \kappa, \lambda \in \mathcal{P}_{(i)}, \kappa \neq \lambda, \forall \mu, \nu \in \mathcal{P}_{(s)}, \mu \neq \nu, \ \ d(S_\kappa(q), S_\lambda(q)) > 2\delta, |h_\mu - h_\nu| > \delta, \ d(S_\kappa(q), h_\nu) > 2\delta, \ d(S_\kappa(q), \partial \Omega) > 2\delta \text{ and } d(h_\nu, \partial \Omega) > 2\delta \}.
\]
with the obvious abuse of notation for \( \partial S_\kappa(q) \). We denote \( q_0 \) the initial value of \( q^\varepsilon \) (which does not depend on \( \varepsilon \)). We have the following statement.

**Lemma 8.1.** There exist a neighborhood \( \mathcal{U}_{q_0} \) of \( q_0 \) in \( \mathcal{Q} \) and a smooth mapping \( T : q \mapsto T_q \) from \( \mathcal{U}_{q_0} \) into the group \( \text{Diff}(\Omega) \) of the diffeomorphisms of \( \Omega \), independent of \( \varepsilon \) (provided that \( \varepsilon \) is small enough), such that \( T_q = \text{Id}_q \), such that for all \( q \in \mathcal{U}_{q_0} \), \( T_q \) is an orientation and area-preserving diffeomorphism of \( \Omega \), which sends \( S_\kappa(q_0) \) to \( S_\kappa(q) \) for \( \kappa \in \mathcal{P}_i \), \( h_{\kappa,0} \) to \( h_{\kappa} \) for \( \kappa \in \mathcal{P}_s \) and such that for all \( q \in \mathcal{U}_{q_0} \), \( T_q \) is rigid in a neighborhood of each \( S_\kappa(q_0) \) for \( \kappa \in \mathcal{P}_i \), is a translation in a neighborhood of \( h_{\kappa,0} \) for \( \kappa \in \mathcal{P}_s \) and is equal to identity in a neighborhood of \( \partial \Omega \).

**Proof of Lemma 8.1.** The construction of such a mapping is easy and classical. We first introduce \( W_\kappa \) as the \( \delta \)-neighborhood of \( S_\kappa \) for \( \kappa \in \mathcal{P}_i \) and of \( h_{\kappa} \) for \( \kappa \in \mathcal{P}_s \). Given \( q \) close to \( q_0 \), we define \( T_q \) in \( W_\kappa \) as the unique rigid movement sending \( q_0^\kappa \) to \( q_\kappa \) for \( \kappa \in \mathcal{P}_i \), as the unique translation sending \( h_{\kappa,0}^\kappa \) to \( h_{\kappa}^\kappa \) for \( \kappa \in \mathcal{P}_s \) and as the identity in a neighborhood of \( \partial \Omega \). Then we extend \( T_q \) as a global diffeomorphism on \( \Omega \): it suffices to write \( T_q \) in \( W_\kappa \) as the flow of a vector field as in Paragraph 3.1.5 and to use extensions of vector fields. To make sure to conserve the zero-divergence of these vector fields, we extend their stream functions. \( \square \)

8.2 First step and compactness

8.2.1 Fixing \( \varepsilon_0 \) and \( T \).

Given an initial data \( (\gamma, q_0, p_0, \omega_0) \) we first set (having (1.15) in mind):

\[
D := \min \{ D_\varepsilon, \varepsilon \in (0,1]^N \},
\]

where

\[
D_\varepsilon := \min \left\{ \min \{ \text{dist}(S_{\lambda,0}^\varepsilon, S_{\mu,0}^\varepsilon), \lambda \neq \mu \}, \min \{ \text{dist}(S_{\lambda,0}^\varepsilon, \partial \Omega), \lambda = 1, \ldots, N \}, \right. \\
\left. \min \{ \text{dist}(S_{\lambda,0}^\varepsilon, \text{Supp}(\omega_0)), \lambda = 1, \ldots, N \} \right\},
\]

and we observe that \( D > 0 \). Next we set

\[
\delta := \frac{D}{2},
\]

and apply Proposition 7.1 with this \( \delta \). We deduce some \( \varepsilon_0 > 0 \) and some \( C_1 > 0 \) such that, as long as \( (\varepsilon, q, \omega) \) stays in \( \Omega_{\delta_0}^\varepsilon \), one has

\[
\forall \kappa \in \{1, \ldots, N\}, \quad |p_\kappa| \leq C_1.
\]

We reduce if necessary \( \varepsilon_0 > 0 \) so that all intermediate results from Sections 3 to 7 and Subsection 8.1 hold as well.

We deduce from the existence of \( C_1 \) the existence of \( C_2 > 0 \) such that as long as \( (\varepsilon, q, \omega) \) stays in \( \Omega_{\delta_0}^\varepsilon \), one has

\[
\forall \kappa \in \{1, \ldots, N\}, \quad |v_{\varepsilon,\kappa}| \leq C_2 \quad \text{in} \quad S_\kappa,
\]

\[
|u^\varepsilon(t,x)| \leq C_2 \quad \text{on} \quad \mathcal{F}_\delta(q(t)) := \left\{ x \in \mathcal{F}(q) \cap \bigcup_{\kappa \in \mathcal{P}_s} S_\kappa \right\} > \delta \}
\]

To get (8.4), we used the decomposition (4.9) and Proposition 3.20, Lemma 3.27 and Lemma 4.6 to estimate the three terms in this decomposition. We let

\[
\mathcal{C} := \max(C_1, C_2) \quad \text{and} \quad T := \frac{D}{8C}.
\]

Then using a continuous induction argument, we see as a consequence of (4.1) and the fact that the solids move with velocity \( v_{\varepsilon,\kappa} \) that, provided that \( \varepsilon < \varepsilon_0 \), one has \( (\varepsilon, q, \omega) \) belongs to \( \Omega_{\delta_0}^\varepsilon \) for all \( t \in [0,T] \), and in particular all the above \( a \ priori \) estimates are true on \( [0,T] \).

In the sequel, reducing \( T \) if necessary, we may ask that for all \( t \in [0,T] \), \( \tilde{q}^\varepsilon(t) \in \mathcal{U}_{q_0} \), where the neighborhood \( \mathcal{U}_{q_0} \) was defined in Lemma 8.1.
8.2.2 Using compactness

As a consequence of the a priori estimates given in Lemma 4.1 and Propositions 4.2 and 7.1, we have that $p^K_{\nu,i}$ is bounded in $W^{2,\infty}(0,T)$ for $\nu \in \mathcal{P}_{(i)} \cup \mathcal{P}_{(ii)}$ and in $W^{1,\infty}(0,T)$ for $\nu \in \mathcal{P}_{(iii)}$, and that $\omega$ is bounded in $L^{\infty}((0,T) \times \Omega)$. Hence we may extract a subsequence (that we abusively still denote by an exponent $\varepsilon$) such that

\begin{align}
q^K_{\varepsilon} & \rightharpoonup q^K_{\ast} \text{ in } W^{2,\infty}(0,T) \text{ weak} - \ast \text{ for } \nu \in \mathcal{P}_{(i)}, \\
h^K_{\varepsilon} & \rightharpoonup h^K_{\ast} \text{ in } W^{2,\infty}(0,T) \text{ weak} - \ast \text{ for } \nu \in \mathcal{P}_{(ii)}, \\
h''_{\varepsilon} & \rightharpoonup h''_{\ast} \text{ in } W^{1,\infty}(0,T) \text{ weak} - \ast \text{ for } \nu \in \mathcal{P}_{(iii)}, \\
\omega^{\varepsilon} & \rightharpoonup \omega^{\ast} \text{ in } L^{\infty}((0,T) \times \Omega) \text{ weak} - \ast.
\end{align}

The fact that we can improve the convergence (8.9) to the convergence

\begin{equation}
\omega^{\varepsilon} \rightharpoonup \omega^{\ast} \text{ in } C^{0}([0,T];L^{\infty}(\Omega) - w^*),
\end{equation}

is obtained as in [21, Appendix C]: this comes from the fact that, thanks to (4.16), we have an a priori bound on $\partial_{\nu} \omega^{\varepsilon} = -\div (u^{\varepsilon} \omega^{\varepsilon})$ in $L^{\infty}(0,T;W^{-1,p}(\Omega))$. Note in particular that the convergences (1.18), (1.19) and (1.20) are contained in the above convergences. Moreover convergences (8.6) and (8.7) have naturally the following consequence:

\begin{align}
p^K_{\varepsilon} & \rightharpoonup p^K_{\ast} = (g^K_{\ast})' \text{ in } W^{1,\infty}(0,T) \text{ weak} - \ast \text{ for } \nu \in \mathcal{P}_{(i)}, \\
h_{\varepsilon}'' & \rightharpoonup h_{\ast}'' \text{ in } W^{1,\infty}(0,T) \text{ weak} - \ast \text{ for } \nu \in \mathcal{P}_{(ii)} \text{ and in } L^{2}(0,T) \text{ weak} - \ast \text{ for } \nu \in \mathcal{P}_{(iii)}.
\end{align}

8.3 Limit dynamics of the fluid

Let us now see how the convergences above involve the convergence (1.17) of the velocity field $u^{\varepsilon}$ to $u^{\ast}$ satisfying (1.21). We recall that we take the convention to extend all the vector fields by 0 inside the solids. The family of diffeomorphisms in Subsection 8.1 will be helpful here. We denote

\[ q^{\ast} := (q_{1}^{\ast}, \ldots, q_{N(i)}^{\ast}, h_{N(i)+1}^{\ast}, \ldots, h_{N(i)}^{\ast}). \]

To obtain the convergence of $u^{\varepsilon}$ we rely on the decomposition (2.24) and show that each term converges towards its final counterpart (2.26). This is done in three separate lemmas.

\textbf{Lemma 8.2.} As $\varepsilon \to 0$ for $p \in [1,2]$:

\[ K_{q^{\ast}}^{\varepsilon} [\omega^{\varepsilon}] \circ T_{q^{\ast}}^{\varepsilon} \rightharpoonup K_{q^{\ast}}^{\varepsilon} [\omega^{\ast}] \circ T_{q^{\ast}}^{\varepsilon} \text{ in } C^{0}([0,T];L^{p}(\tilde{F}(q^{\ast}))), \]

where $q^{\ast}_{(i)} := (q_{1}^{\ast}, \ldots, q_{N(i)}^{\ast})$.

\textbf{Lemma 8.3.} Let $p < +\infty$. As $\varepsilon \to 0$:

\begin{align*}
p_{\nu,i} \nabla \varphi^{\nu,i}_{\varepsilon}(q^{\ast}, T_{q^{\ast}}^{\varepsilon}(\cdot)) & \rightharpoonup p_{\nu,i} \nabla \varphi^{\nu,i}_{\varepsilon}(q^{\ast}, T_{q^{\ast}}^{\varepsilon}(\cdot)) \text{ in } L^{p}(0,T;L^{p}(\tilde{F}(q^{\ast}))) \text{ for } \nu \in \mathcal{P}_{(i)}, \\
L^{\omega}_{\nu,i} \nabla \varphi^{\nu,i}_{\varepsilon}(q^{\ast}, T_{q^{\ast}}^{\varepsilon}(\cdot)) & \rightharpoonup 0 \text{ in } L^{\omega}(0,T;L^{p}(\tilde{F}(q^{\ast}))) \text{ for } \nu \in \mathcal{P}_{(i)} \text{ and in } L^{2}(0,T;L^{p}(\tilde{F}(q^{\ast}))) \text{ for } \nu \in \mathcal{P}_{(iii)}. \end{align*}

\textbf{Lemma 8.4.} As $\varepsilon \to 0$: for $\nu \in \mathcal{P}_{(i)}$:

\[ \nabla^{\perp} \psi^{\nu}_{\varepsilon}(q^{\ast}, T_{q^{\ast}}^{\varepsilon}(\cdot)) \rightharpoonup \nabla^{\perp} \psi^{\nu}_{\varepsilon}(q^{\ast}, T_{q^{\ast}}^{\varepsilon}(\cdot)) \text{ in } L^{p}(0,T;L^{p}(\tilde{F}(q^{\ast}))) \text{ for } p < +\infty, \]

and for $\nu \in \mathcal{P}_{s}$:

\[ \nabla^{\perp} \psi^{\nu}_{\varepsilon}(q^{\ast}, T_{q^{\ast}}^{\varepsilon}(\cdot)) \rightharpoonup K_{q^{\ast}_{(i)}}[\delta \psi^{\nu}_{\varepsilon}] \circ T_{q^{\ast}}^{\varepsilon} \text{ in } L^{p}(0,T;L^{p}(\tilde{F}(q^{\ast}))) \text{ for } p < 2. \]
Proof of Lemma 8.2. For all \( t \in (0, T) \) we write, using the triangle inequality and recalling that all vector fields are filled with \( 0 \) inside the solids,

\[
|K_{q'}^\varepsilon[\omega^\varepsilon] \circ \mathcal{T}_{q'}^\varepsilon - K_{q',i}^\varepsilon[\omega^\varepsilon] \circ \mathcal{T}_{q'}^\varepsilon|_{L^p(\Omega)} \leq |K_{q'}^\varepsilon[\omega^\varepsilon] \circ \mathcal{T}_{q'}^\varepsilon - \bar{K}_{q',i}^\varepsilon[\omega^\varepsilon] \circ \mathcal{T}_{q'}^\varepsilon|_{L^p(\Omega)} + |\bar{K}_{q',i}^\varepsilon[\omega^\varepsilon - \omega^\ast] \circ \mathcal{T}_{q'}^\varepsilon|_{L^p(\Omega)},
\]

For what concerns the first term, since \( \mathcal{T}_{q'}^\varepsilon \) is measure-preserving, we have

\[
|K_{q'}^\varepsilon[\omega^\varepsilon] \circ \mathcal{T}_{q'}^\varepsilon - \bar{K}_{q',i}^\varepsilon[\omega^\varepsilon] \circ \mathcal{T}_{q'}^\varepsilon|_{L^p(\Omega)} \leq |K_{q'}^\varepsilon[\omega^\varepsilon] - \bar{K}_{q',i}^\varepsilon[\omega^\varepsilon]|_{L^p(\Omega)},
\]

which converges to zero uniformly in time thanks to Lemma 3.32. The convergence of the third term (uniformly in time) comes from (8.10): it involves the convergence of \( K_{q'}^\varepsilon[\omega^\varepsilon] \) to \( K_{q'}^\varepsilon[\omega^\varepsilon] \) (recall (3.116)) in \( C^0([0, T]; L^p(\Omega)) \) for \( p < +\infty \) due to the classical compactness of the operator \( K_{q'}^\varepsilon : L^p(\Omega) \to L^p(\Omega) \) (due to the Calderon-Zygmund estimate \( \|K_{q'}^\varepsilon[\omega]\|_{W^{1,p}(\Omega)} \leq C\|\omega\|_{L^p(\Omega)} \) and the Rellich-Kondrachov theorem.) Note that using the support of vorticity and interior regularity, this involves the convergence in \( C^0([0, T]; C^k(\bar{\Omega})) \) for each \( \lambda = 1, \ldots, N \). It remains to check that the correction \( R[\omega - \omega^\ast] \) defined in (3.117) converges to 0 in \( C^0([0, T]; L^p(\Omega)) \). This is again a consequence of Propositions 3.9 and 3.10.

Finally, concerning the second term, we consider the function

\[
[0, 1] \to L^p(\bar{\Omega}), \quad s \mapsto \bar{K}_{q',i}^\varepsilon[q', i] + s(q' - q^\varepsilon).
\]

It is well-defined for small enough \( \varepsilon \) (due to the convergences (8.6)-(8.8)), so that \( q' + s(q' - q^\varepsilon) \) belongs to the neighborhood \( \mathcal{U}_{q'} \) of Lemma 8.1, and its derivative with respect to \( s \) is bounded by

\[
C|q' - q^\varepsilon| \left( \frac{\delta \bar{K}}{\delta q} \right)_{L^p(\bar{\Omega})} + \frac{\delta \bar{K}}{\delta \varepsilon} \bigg|_{L^p(\bar{\Omega})}.
\]

(8.12)

Together with Lemma 3.34 and (3.113), this establishes Lemma 8.2.

\[ \square \]

Proof of Lemma 8.3. Here we write for \( \nu \in \mathcal{P}(i) \):

\[
|\nabla \bar{\varphi}_{\nu,i}(q'; \mathcal{T}_{q'}^\varepsilon(\cdot)) - \nabla \bar{\varphi}_{\nu,i}(q'; \mathcal{T}_{q'}^\varepsilon(\cdot))|_{L^p(\Omega)} \leq |\nabla \bar{\varphi}_{\nu,i}(q'; \mathcal{T}_{q'}^\varepsilon(\cdot)) - \nabla \bar{\varphi}_{\nu,i}(q'; \mathcal{T}_{q'}^\varepsilon(\cdot))|_{L^p(\Omega)} + |\nabla \bar{\varphi}_{\nu,i}(q'; \mathcal{T}_{q'}^\varepsilon(\cdot)) - \nabla \bar{\varphi}_{\nu,i}(q'; \mathcal{T}_{q'}^\varepsilon(\cdot))|_{L^p(\Omega)}. \tag{8.13}
\]

The first term in the right-hand side converges to zero as shown by Proposition 3.22. For the second we reason as in the proof of Lemma 8.3: we consider the function

\[
s \mapsto \nabla \bar{\varphi}_{\nu,i}(q'; s(q' - q^\varepsilon) + s(q' - q^\varepsilon) \bigg|_{\mathcal{T}_{q'}^\varepsilon(\cdot)}),
\]

where the abusive notation \( q' + s(q' - q^\varepsilon) \) means that we add \( s(q' - q^\varepsilon) \) only on the coordinate of \( q' \) corresponding to \( q_i \). Now we estimate the derivative as in (8.12). The \( x \)-derivative is bounded thanks to the uniform Schauder estimates in \( \bar{\Omega} \), the \( q \) derivative by following the proof of Proposition 3.26 by elliptic regularity in \( \bar{\Omega} \). With (8.6), this proves the convergence of the left-hand side of (8.13) to zero. The conclusion follows then from (8.11) for solids of family \( i \).

Concerning small solids, the convergence to 0 of the Kirchhoff potentials (uniform with respect to \( \mathcal{P} \)) comes from Proposition 3.20, and one concludes in the same way with (8.11).

\[ \square \]

Proof of Lemma 8.4. Here we write for \( \nu \in \mathcal{P}(i) \) and all \( t \in [0, T] \):

\[
|\nabla^\perp \bar{\psi}_\nu(q'; \mathcal{T}_{q'}^\varepsilon(\cdot)) - \nabla^\perp \bar{\psi}_\nu(q'; \mathcal{T}_{q'}^\varepsilon(\cdot))|_{L^p(\bar{\Omega})} \leq |\nabla^\perp \bar{\psi}_\nu(q'; \mathcal{T}_{q'}^\varepsilon(\cdot)) - \nabla^\perp \bar{\psi}_\nu(q'; \mathcal{T}_{q'}^\varepsilon(\cdot))|_{L^p(\bar{\Omega})} + |\nabla^\perp \bar{\psi}_\nu(q'; \mathcal{T}_{q'}^\varepsilon(\cdot)) - \nabla^\perp \bar{\psi}_\nu(q'; \mathcal{T}_{q'}^\varepsilon(\cdot))|_{L^p(\bar{\Omega})},
\]

The convergence to zero of the first term in the right-hand side, uniformly in \( \mathcal{P} \), is a consequence of Proposition 3.29. The convergence of the second term is due to (8.6) and the regularity of \( \nabla^\perp \bar{\psi}_\nu \) with respect to \( \mathcal{P} \) (using for instance Lemma 3.30 and (4.23)).

69
For \( \nu \in \mathcal{P}_s \), for \( p \in [1, 2] \) and all \( t \in [0, T] \) we have:

\[
\| \nabla \psi^\nu_L (q^\nu, \mathcal{T}_q^p (\cdot)) - \tilde{K}_{q^\nu} (h^\nu) \cdot \mathcal{T}_q^p \|_{L^p (\mathcal{F} (q_0))} \leq \| \nabla \psi^\nu_L (q^\nu, \mathcal{T}_q^p (\cdot)) - \tilde{K}_{q^\nu} (h^\nu) \cdot \mathcal{T}_q^p \|_{L^p (\mathcal{F} (q_0))} \\
+ \| \tilde{K}_{q^\nu} (h^\nu) \cdot \mathcal{T}_q^p - \tilde{K}_{q^\nu} (h^\nu) \cdot \mathcal{T}_q^p \|_{L^p (\mathcal{F} (q_0))}.
\]

The convergence to zero of the first term in the right-hand side is due to Proposition 3.29, (3.91) and (3.94). Concerning the second one, by (3.94)

\[
\| \tilde{K}_{q^\nu} (h^\nu) \cdot \mathcal{T}_q^p - \tilde{K}_{q^\nu} (h^\nu) \cdot \mathcal{T}_q^p \|_{L^p (\mathcal{F} (q_0))} \leq \| \nabla \psi^\nu_L (h^\nu) \cdot \mathcal{T}_q^p - \nabla \psi^\nu_L (h^\nu) \cdot \mathcal{T}_q^p \|_{L^p (\mathcal{F} (q_0))} \\
+ \| H (\cdot - h^\nu) \cdot \mathcal{T}_q^p - H (\cdot - h^\nu) \cdot \mathcal{T}_q^p \|_{L^p (\mathcal{F} (q_0))}.
\]

Due to the uniform convergence of \( h^\nu \) to \( h^\nu \) both terms converge to zero, the first one by regularity with respect to \( h \) of \( \nabla \psi^\nu_L \), the second one by continuity of the translations in \( L^p \).

Now the convergence (1.17) to \( u^\nu \) satisfying (1.21) is a direct consequence of Lemmas 8.2, 8.3, 8.4, and of the decompositions (2.24) and (2.26). Moreover one obtains (1.22) by passing to the limit in (4.1) thanks to (8.9) and (1.17).

### 8.4 Limit dynamics of the solids of fixed size

To pass to the limit in the equation of the solids of family \( (i) \), we must pass to the limit in the pressure. To that purpose, we observe that the convergences described in Subsection 8.3 are actually stronger when one restricts the space domain to the \( \delta \)-neighborhood of \( \partial \mathcal{S}_K \) for \( \kappa \in \mathcal{P}_i \), and, for \( \kappa \in \mathcal{P}_s \), to an annulus \( B (h_\kappa, \delta) \setminus B (h_\kappa, \delta / 2) \). This is given in the following statement.

**Lemma 8.5.** For \( \kappa \in \{1, \ldots, N\} \) we let \( U^\kappa \) the \( \delta / 2 \)-neighborhood of \( \partial \mathcal{S}_K (q_0) \) whenever \( \kappa \in \mathcal{P}_i \) and we let \( U^\kappa = B (h_\kappa, \delta) \setminus B (h_\kappa, 3 \delta / 4) \) whenever \( \kappa \in \mathcal{P}_s \). Then one has

\[
u \circ \mathcal{T}_q^p \longrightarrow u \circ \mathcal{T}_q^p \text{ in } W^{1, \infty} (0, T; C^k (U^\kappa)) \quad \text{for all } k \in \mathbb{N}.
\]

**Proof of Lemma 8.5.** This is due to the support of \( \omega \) and the remoteness of small solids from it (since \( (\mathbf{r}, \mathbf{q}, \omega) \in \Omega^{p_0} \)), which allow to improve the convergences of Lemmas 8.2, 8.3 and 8.4 to the weak-\( \ast \) one in \( W^{1, \infty} (0, T; C^k (U^\kappa)) \). Since we already have the convergence in a weaker space, it suffices to prove the boundedness of \( u \circ \mathcal{T}_q^p \) in \( W^{1, \infty} (0, T; C^k (U^\kappa)) \). That \( u \circ \mathcal{T}_q^p \) remains bounded in \( L^\infty (0, T; C^k (U^\kappa)) \) is a direct consequence of the support of \( \omega \) and interior elliptic regularity, since it is already bounded in \( L^\infty (0, T; L^p (\mathcal{F}_0)) \).

For what concerns \( \partial_t (u \circ \mathcal{T}_q^p) \) we have

\[
\partial_t (u \circ \mathcal{T}_q^p) = \begin{cases}
\partial_t u \circ \partial_t (\mathcal{T}_q^p) \quad &\text{in } U^\kappa \text{ for } \kappa \in \mathcal{P}_i, \\
\partial_t u \circ \partial_t \mathcal{T}_q^p \quad &\text{in } U^\kappa \text{ for } \kappa \in \mathcal{P}_s,
\end{cases}
\]

so that we only have to estimate \( \partial_t (u \circ \mathcal{T}_q^p) \). Again, by interior elliptic estimates, it suffices to bound it in \( L^\infty \) in a slightly larger set. We rely on decomposition (4.9):

- \( \partial_t u^{\ast \ast} \) is bounded in \( C^0 ([0, T] \times U^\kappa) \) thanks to Lemma 4.6,
- the terms \( \partial_t \nabla \psi_L \) for \( \nu \neq \kappa \) are bounded in \( C^0 ([0, T] \times U^\kappa) \) thanks to (4.23), (3.81)-(3.82) and the remoteness of \( U^\kappa \) from \( \partial \mathcal{S}_\nu \),
- all the same the term \( \partial_t \nabla \psi_L \) is bounded in \( C^0 ([0, T] \times U^\kappa) \) thanks to (4.23), (3.81)-(3.82) and to the choice of \( U^\kappa \) (that is at positive distance from \( \partial \mathcal{S}_K \) when \( \kappa \in \mathcal{P}_s \)),
- the boundedness of \( \partial_t u^{\ast \ast} \) follows from Proposition 3.26, acceleration estimates (Proposition 4.4) and Proposition 3.20 (again thanks to the choice of \( U^\kappa \)).

\[\square\]
A first consequence of Lemma 8.5 is (1.23). Indeed, due to (1.21) and (1.22), we have
\[
\text{curl}(\partial_t u^* + (u^* \cdot \nabla)u^*) = 0 \quad \text{in} \quad \tilde{F}(q_0^* (t)).
\]
For each \( \kappa \in \{1, \ldots, N\} \), we introduce a smooth simple closed loop \( \gamma_\kappa \) in \( U_\kappa^t \). Then (1.2) involve that for all \( t \in [0, T] \) and all \( \varepsilon \), one has
\[
\int_{\gamma_\kappa} (\partial_t u^* + (u^* \cdot \nabla)u^*) (t, \cdot) \cdot \tau \, ds = 0.
\]
Passing to the limit with Lemma 8.5 we infer that for all \( \kappa \in \{1, \ldots, N\} \),
\[
\int_{\gamma_\kappa} (\partial_t u^* + (u^* \cdot \nabla)u^*) \cdot \tau \, ds = 0.
\]
This establishes (1.23).

Next we deduce (1.24). It follows from Lemma 8.5 that in a vicinity of \( \partial S_\kappa \) for \( \kappa \in \mathcal{P}(1) \), the convergence of the pressure is improved: recalling that
\[
\nabla \pi^\varepsilon = -\partial_t u^\varepsilon - (u^\varepsilon \cdot \nabla)u^\varepsilon \quad \text{and} \quad \nabla \pi^* = -\partial_t u^* - (u^* \cdot \nabla)u^*,
\]
Lemma 8.5 involves that
\[
\nabla \pi^\varepsilon \circ T_q^t \longrightarrow \nabla \pi^* \circ T_q^t \quad \text{in} \quad L^\infty(0, T; C^0_b(V_{b/2}(\partial S_\kappa))) \text{weak-}^*.
\]
From (1.7) we deduce, for all \( \kappa \in \mathcal{P}(1) \):
\[
\begin{cases}
m_\kappa (h_\kappa^\varepsilon)^\nu(t) = R(\vartheta_\kappa^\varepsilon) \int_{\partial S_\kappa(\partial q_\kappa)} \pi^\varepsilon(t, T_q^t(x)) \, n(t, T_q^t(x)) \, ds(x), \\
J_\kappa (\vartheta_\kappa^\varepsilon)^\nu(t) = \int_{\partial S_\kappa(\partial q_\kappa)} \pi^\varepsilon(t, T_q^t(x)) (x - h_{\kappa,0})^\nu \cdot n(t, T_q^t(x)) \, ds(x).
\end{cases}
\]
This involves the passage to the limit in (1.7) for the first family, from which we deduce (1.24).

\section{Limit dynamics of the small solids and end of the proof of Theorem 2}

To get the convergence on small solids we go back to the normal form (6.3). Let \( \kappa \in \mathcal{P}_s \). Since we now know that \( \tilde{p}^\varepsilon \) is bounded, using (6.4), (6.6) and (6.8), we infer that the terms \( A_\kappa, C_\kappa \) and \( D_\kappa \) converge to zero strongly in \( L^\infty(0, T) \).

Now we use two lemmas, where we recall that \( \tilde{p}_\kappa \) is the modulated variable (before the passage to the limit) given by (6.1).

\begin{lemma}
When \( \kappa \in \mathcal{P}_s \), the term \( \mathcal{M}_{a,\kappa} \tilde{p}_\kappa + \frac{1}{2} \mathcal{M}'_{a,\kappa} \tilde{p}_\kappa \) converges to 0 in \( W^{-1,\infty}(0, T) \) as \( \varepsilon \) goes to 0.
\end{lemma}

\textit{Proof of Lemma 8.6.} We proceed in three steps.

\textbf{Step 1.} First \( \mathcal{M}_{a,\kappa} \) converges strongly to 0 in \( L^\infty(0, T) \) due to Corollary 3.23. Since \( \tilde{p}_\kappa \) is bounded, it follows that \( (\mathcal{M}_{a,\kappa} \tilde{p}_\kappa)' \) converges to 0 in \( W^{-1,\infty}(0, T) \).

\textbf{Step 2.} By Reynold’s transport theorem:
\[
\mathcal{M}'_{a,\kappa,i,j} = \sum_{\nu=1}^{N} \int_{\mathcal{F}(q)} \left( p_\nu \frac{\partial \varphi_{\kappa,i}}{\partial q_\nu} \right) \cdot \nabla \varphi_{\kappa,j} \, dx + \sum_{\nu=1}^{N} \int_{\mathcal{F}(q)} \nabla \varphi_{\kappa,i} \cdot \left( p_\nu \frac{\partial \varphi_{\kappa,j}}{\partial q_\nu} \right) \, dx \\
+ \int_{\partial \mathcal{F}(q)} (u^\varepsilon \cdot n) \nabla \varphi_{\kappa,i} \cdot \nabla \varphi_{\kappa,j} \, ds.
\]
By an integration by parts the first two terms are transformed into integrals over \( \partial S_\kappa \) with some integrands which are bounded according to Proposition 3.26. Therefore these two terms converge to 0.
uniformly in time. For the third one, we first notice that \( u^\varepsilon \cdot n = u^\text{pot} \cdot n \) is bounded (thanks to Propositions 3.20 and 7.1). Now using again Proposition 3.20 we see that on \( \partial F(q) \backslash \partial S_\kappa \) the integrand is of order \( O(\varepsilon^4 + h_2 + b_1) \) and that on \( \partial S_\kappa \) it is bounded. Since \( |\partial S_\kappa| = O(\varepsilon_\kappa) \), we obtain the convergence of this term to 0 as well. Thus \( \mathcal{M}_{\kappa, \kappa} \) converges to 0 in \( L^\infty(0,T) \) as \( \varepsilon \) goes to 0.

**Step 3.** Since
\[
\mathcal{M}_{\kappa, \kappa} + \frac{1}{2} \mathcal{M}'_{\kappa, \kappa} = (\mathcal{M}_{\kappa, \kappa})' - \frac{1}{2} \mathcal{M}'_{\kappa, \kappa},
\]
the result follows.

**Lemma 8.7.** When \( \kappa \in \mathcal{P}_s \), one has the uniform convergence in \([0,T]\) as \( \varepsilon \) goes to 0:
\[
\left( \begin{array}{c} B_{\kappa,1} \\ B_{\kappa,2} \end{array} \right) \to \gamma_\kappa ((h_\kappa^*)' - u_\kappa^*(h_\kappa^*))^\perp.
\]

**Proof of Lemma 8.7.** We consider the writing of \( B_\kappa \) in (6.2). Using (6.1) and (2.17d), and then (5.27), (5.28) and (5.10), we see that
\[
\left( \begin{array}{c} B_{\kappa,1} \\ B_{\kappa,2} \end{array} \right) = \gamma_\kappa \left( (h_\kappa^*)' - \left( \frac{\alpha_1 + \beta_1}{\alpha_2 + \beta_2} \right) \right)^\perp = \gamma_\kappa \left( (h_\kappa^*)' - \tilde{u}_\kappa^*(h_\kappa^*) \right)^\perp + o(1).
\]

It remains to prove that
\[
\tilde{u}_\kappa^*(h_\kappa^*) \to u_\kappa^*(h_\kappa^*) \text{ uniformly in time as } \varepsilon \to 0. \tag{8.14}
\]
To prove (8.14), we first establish the convergence for \( p \in [1,2) \)
\[
\tilde{u}_\kappa^* \circ \mathcal{T}_\kappa \to u_\kappa^* \circ \mathcal{T}_\kappa \text{ in } L^\infty(0,T;L^p(\mathcal{F}_\kappa)). \tag{8.15}
\]
This derives from (5.8) and the equivalents of Lemmas 8.2, 8.3 and 8.4 in the domain \( \mathcal{F}_\kappa \) where there is no \( S_\kappa \):}
\[
\nabla \varphi_{\nu}^{\varepsilon, \kappa} \circ \mathcal{T}_\kappa \to \nabla \varphi_{\nu} \circ \mathcal{T}_\kappa \text{ for } \nu \in \mathcal{P}_{(i)},
\]
\[
\nabla \varphi_{\nu}^{\varepsilon, \kappa} \circ \mathcal{T}_\kappa \to 0 \text{ for } \nu \in \mathcal{P}_{s \backslash \{s\}},
\]
\[
\nabla \psi_{\nu}^{\varepsilon, \kappa} \circ \mathcal{T}_\kappa \to \nabla \tilde{\psi}_{\nu} \circ \mathcal{T}_\kappa \text{ for } \nu \in \mathcal{P}_{(i)},
\]
\[
\nabla \psi_{\nu}^{\varepsilon, \kappa} \circ \mathcal{T}_\kappa \to \tilde{K}[\delta_{\kappa^2}] \circ \mathcal{T}_\kappa \text{ for } \nu \in \mathcal{P}_{s \backslash \{s\}},
\]
\[
K^{\varepsilon, \kappa}[\omega_{\kappa}] \circ \mathcal{T}_\kappa \to \tilde{K}[\omega^*] \circ \mathcal{T}_\kappa.
\]
Moreover using (3.107) and reasoning as in Lemma 8.4
\[
\nabla \psi_{\kappa}^{\varepsilon, \kappa, \nu} \circ \mathcal{T}_\kappa \to \nabla \tilde{\psi}_{\kappa} \circ \mathcal{T}_\kappa = \{ \tilde{K}[\delta_{\kappa^2}] - H_{\kappa} \} \circ \mathcal{T}_\kappa,
\]
where we recall that \( \tilde{\psi}_{\kappa} \) was defined in (3.93) and \( \tilde{\psi}_{\kappa}^{\varepsilon, \kappa} \) in (3.104). This allows to deduce (8.15) using the decomposition (5.8) of \( \tilde{u}_\kappa \). Then using inner regularity for the Laplace equation, we see that the convergence (8.15) actually holds in \( L^\infty(0,T;C^8(\mathcal{V}_3(S_\kappa))) \) since there is no vorticity near \( S_\kappa \). With the uniform convergence of \( h_\kappa^* \) toward \( h_\kappa^* \), this gives (8.14).

Hence we obtain (1.25) and (1.26) by passing to the limit in (6.3) using the assumption that \( \gamma_\kappa \neq 0 \) when \( \kappa \in \mathcal{P}_{(iii)} \) (see the last paragraph of Section 1.2) for the latter. This concludes the proof of Theorem 2. 

\[
\Box
\]
8.6 Proof of Theorem 3

In this subsection, we briefly sketch the proof of Theorem 3. Hence we consider the particular case where the data ensures the uniqueness of the solution to the limit system, together with the separation of point vortices, of solids of fixed size and of the vorticity support in the limit. Since the limit system enjoys uniqueness in this situation, the convergence without restriction to a subsequence is commonplace: let us explain why the maximal existence times $T^\varepsilon$ satisfy $\liminf_{\varepsilon \to 0} T^\varepsilon \succeq T^*$ and the convergences (1.17)-(1.20) hold on any time interval $[0, T] \subset [0, T^*)$.

Consider $T > 0$; denoting $S^\varepsilon_\kappa(t) := S_\kappa(\eta^\varepsilon_\kappa(t))$ for $\kappa \in \mathcal{P}_0$ and $S^\varepsilon_\kappa(t) := \{b^\varepsilon_\kappa(t)\}$ for $\kappa \in \mathcal{P}_s$, due to the assumption on the limit system, we can find $d_T > 0$ such that

$$\forall t \in [0, T], \quad \forall \kappa \in \{1, \ldots, N\}, \quad d(S^\varepsilon_\kappa(t), \text{Supp}(\omega^\varepsilon(t))) \geq d_T, \quad d(S^\varepsilon_\kappa(t), \partial \Omega) \geq d_T$$

and $\forall \lambda \in \{1, \ldots, N\}\setminus\{\kappa\}$, $d(S^\varepsilon_\lambda(t), S^\varepsilon_\kappa(t)) \geq d_T$.

Reducing $d_T$ if necessary, we assume that $d_T \leq D$ where $D$ was defined in (8.2). We now introduce

$$T_{\text{max}} := \sup \left\{ \tau \in [0, T] \mid \exists \varepsilon_0 > 0, \forall t \in [0, \tau], \forall \varepsilon < \varepsilon_0, \forall \kappa \in \{1, \ldots, N\}, \quad d(S^\varepsilon_\kappa(t), \text{Supp}(\omega^\varepsilon(t))) \geq d_T/2, \quad d(S^\varepsilon_\kappa(t), \partial \Omega) \geq d_T/2 \right\}$$

Due to the analysis of Subsections 8.2–8.5, we have $T_{\text{max}} \geq T$ where $T$ was defined in (8.5). Moreover, the convergence analysis of Subsections 8.2–8.5 can be carried out in any $[0, \tau] \subset [0, T_{\text{max}}]$ since we merely use a minimal distance between the solids and between the solids and the vorticity support to obtain the estimates. Hence to conclude, it suffices to prove that $T_{\text{max}} = T$.

Arguing by contradiction, we suppose that $T_{\text{max}} < T$. Using the convergences (1.19), it is easy to see that for $\tau < T_{\text{max}}$, for suitably small $\varepsilon$, we do have $d(S^\varepsilon_\kappa(t), S^\varepsilon_\tau(t)) \geq 3d_T/4$ and $d(S^\varepsilon_\kappa(t), \partial \Omega) \geq 3d_T/4$ on $[0, \tau]$ so that the limitation $T_{\text{max}} < T$ can only come from the vorticity. But using the definition of $T_{\text{max}}$, (1.19), the decomposition (2.24) and the estimates of Section 3, we see that for $\tau < T_{\text{max}}$, for suitably small $\varepsilon$, one has the uniform log-Lipschitz estimate on the support of $\omega$:

$$\|u^\varepsilon(t, \cdot)\|_{L^\infty(F \cup \bigcup_{\kappa \in \mathcal{P}_s}(V_{d_T/4}(b^\varepsilon_\kappa(t))))} \leq C \text{ uniformly for } t \in [0, \tau].$$

Moreover, reasoning as in Lemma 8.5, we see that for $p \in (1, +\infty)$,

$$\|\partial_t u^\varepsilon(t, \cdot)\|_{L^p(F \cup \bigcup_{\kappa \in \mathcal{P}_s}(V_{d_T/4}(b^\varepsilon_\kappa(t))))} \leq C \text{ uniformly for } t \in [0, \tau].$$

This implies that the convergence (1.17) can be supplemented by

$$u^\varepsilon(t, \cdot) \rightharpoonup u^*(t, \cdot) \text{ in } C^0([0, \tau] ; C^0(F \setminus \bigcup_{\kappa \in \mathcal{P}_s}(V_{d_T/4}(b^\varepsilon_\kappa(t))))).$$

This involves the convergence of the corresponding flows on $\text{Supp}(\omega_0)$. In particular, $\text{Supp}(\omega^\varepsilon(t))$ converges to $\text{Supp}(\omega^*(t))$ uniformly in time for the Hausdorff distance. Since the convergence analysis of Subsections 8.2–8.5 is valid on any $[0, \tau] \subset [0, T_{\text{max}})$, we deduce that one can find for any such $\tau$ an $\varepsilon_0 > 0$ such that for $\varepsilon < \varepsilon_0$, for all $\kappa \in \{1, \ldots, N\}$, $d(S^\varepsilon_\kappa(t), \text{Supp}(\omega^\varepsilon(t))) \geq 3d_T/4$ on $[0, \tau]$. This puts $T_{\text{max}} < T$ and the boundedness of the velocity of the vorticity support and of the solids in contradiction. This ends the proof of Theorem 3. \qed

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