Around a conjecture of Erdős on graph Ramsey numbers

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Abstract

For given graphs $G_1$ and $G_2$ the Ramsey number $R(G_1,G_2)$, is the smallest positive integer $n$ such that each blue-red edge coloring of the complete graph $K_n$ contains a blue copy of $G_1$ or a red copy of $G_2$. In 1983, Erdős conjectured that there is an absolute constant $c$ such that $R(G_1,G_2) \leq 2c\sqrt{m}$ for any graph $G$ with $m$ edges and no isolated vertices. Recently this conjecture was proved by B. Sudakov. In this note, using the Sudakov’s ideas we give an extension of his result and some interesting corollaries.

Keywords: Ramsey number, Erdős conjecture, Complete graph.

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1 Introduction

For given graphs $G_1$ and $G_2$ the Ramsey number $R(G_1,G_2)$, is the smallest positive integer $n$ such that each blue-red edge coloring of the complete graph $K_n$ contains a blue subgraph isomorphic to $G_1$ or a red subgraph isomorphic to $G_2$. We denote $R(G,G)$ by $R(G)$. The existence of such a positive integer is guaranteed by Ramsey’s classical result \cite{7}. Since 1970’s, Ramsey theory has grown into one of the most active areas of research within combinatorics, overlapping variously with graph theory, number theory, geometry and logic. Probably the most complicated question in this field is the estimating the Ramsey number of complete graphs. A basic result of Erdős and Szekeres \cite{6} implies the following theorem.

\textbf{Theorem 1.1} \textsuperscript{[6]} For every positive integer $n$, $R(K_n) \leq 2^{2n}$.

Using probabilistic methods, Erdős \textsuperscript{3} obtained a lower bound for $R(K_n)$.

\textbf{Theorem 1.2} \textsuperscript{[3]} For every positive integer $n > 2$, $R(K_n) \geq 2^{\frac{n}{2}}$.

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Over the last sixty years, there have been several improvements on these bounds. The best upper bound is obtained by Conlon [2]. A problem on Ramsey numbers of general graphs was posed by Erdős and Graham [5], who conjectured that among all graphs with \( m = \binom{n}{2} \) edges and no isolated vertices, the complete graph on \( n \) vertices has the largest Ramsey number. Since the number of vertices in a complete graph with \( m \) edges is a constant multiple of \( \sqrt{m} \), the best upper bound is obtained by Conlon [2]. A problem on Ramsey numbers of general graphs was posed by Erdős and Graham [5], who conjectured that among all graphs with \( m = \binom{n}{2} \) edges and no isolated vertices, the complete graph on \( n \) vertices has the largest Ramsey number. Since the number of vertices in a complete graph with \( m \) edges is a constant multiple of \( \sqrt{m} \) and Graham [5], who conjectured that among all graphs with \( m \) edges and no isolated vertices, the complete graph on \( n \) vertices has the largest Ramsey number. Since the number of vertices in a complete graph with \( m \) edges is a constant multiple of \( \sqrt{m} \), the best upper bound is obtained by Conlon [2]. A problem on Ramsey numbers of general graphs was posed by Erdős and Graham [5], who conjectured that among all graphs with \( m = \binom{n}{2} \) edges and no isolated vertices, the complete graph on \( n \) vertices has the largest Ramsey number. Since the number of vertices in a complete graph with \( m \) edges is a constant multiple of \( \sqrt{m} \), the best upper bound is obtained by Conlon [2]. A problem on Ramsey numbers of general graphs was posed by Erdős and Graham [5], who conjectured that among all graphs with \( m = \binom{n}{2} \) edges and no isolated vertices, the complete graph on \( n \) vertices has the largest Ramsey number. Since the number of vertices in a complete graph with \( m \) edges is a constant multiple of \( \sqrt{m} \), the best upper bound is obtained by Conlon [2]. A problem on Ramsey numbers of general graphs was posed by Erdős and Graham [5], who conjectured that among all graphs with \( m = \binom{n}{2} \) edges and no isolated vertices, the complete graph on \( n \) vertices has the largest Ramsey number. Since the number of vertices in a complete graph with \( m \) edges is a constant multiple of \( \sqrt{m} \), the best upper bound is obtained by Conlon [2]. A problem on Ramsey numbers of general graphs was posed by Erdős and Graham [5], who conjectured that among all graphs with \( m = \binom{n}{2} \) edges and no isolated vertices, the complete graph on \( n \) vertices has the largest Ramsey number. Since the number of vertices in a complete graph with \( m \) edges is a constant multiple of \( \sqrt{m} \), the best upper bound is obtained by Conlon [2]. A problem on Ramsey numbers of general graphs was posed by Erdős and Graham [5], who conjectured that among all graphs with \( m = \binom{n}{2} \) edges and no isolated vertices, the complete graph on \( n \) vertices has the largest Ramsey number. Since the number of vertices in a complete graph with \( m \) edges is a constant multiple of \( \sqrt{m} \), the best upper bound is obtained by Conlon [2].

**Theorem 1.5** If \( G \) is a graph with \( m \) edges and with no isolated vertices, \( R(G) \leq 2^{250\sqrt{m}} \).

Our main results in this note are the following two theorems. The first theorem is an extension of Theorem 1.3 and will be proved by the same arguments in [8] and the second one can be obtained by the first main theorem and a result in [1] on the Ramsey number of a bounded maximum degree graph and a complete graph. We give their proofs in the last section. Through this note, the notations \( \ln x \) and \( \log x \) are the logarithms in the natural base \( e \) and \( 2 \), respectively.

**Theorem 1.4** Let \( G_i, i = 1, 2, \) be a graph of order \( n_i \) with no isolated vertices and \( m \) be a positive integer with \( 2^{\frac{\log m}{\log 2}} \geq n - 27\sqrt{m} \), where \( n = \max\{n_1, n_2\} \). Also suppose that for each \( 27 \leq \alpha \leq \frac{\log^2 m}{8} \) we have the following properties:

I. There exists \( U_i \subseteq V(G_i), i = 1, 2, \) so that \( |U_i| \leq \alpha \sqrt{m} \) and \( \Delta(G_i - U_i) \leq \frac{2\sqrt{m}}{\alpha} \).

II. There exists \( V_i \subseteq V(G_i), i = 1, 2, \) so that \( |V_i| \leq m^\frac{3}{2} \) and \( \max\{R(G_1 - V_1, G_2), R(G_1, G_2 - V_2)\} \leq 2^{36\sqrt{m}} \).

Then \( R(G_1, G_2) \leq 2^{250\sqrt{m}} \).

The join of two graphs \( G \) and \( H \), denoted by \( G + H \), is a graph with vertex set \( V(G \cup H) \) and edge set \( E(G \cup H) \cup \{uv | u \in G, v \in H\} \).

**Theorem 1.5** Let \( G_1 = K_p, m \geq 27, p \leq 27\sqrt{m} + \frac{16\sqrt{m}}{\log m} \) and \( G_2 = K_l + H \) where \( l \leq 27\sqrt{m}, \Delta(H) \leq \frac{16\sqrt{m}}{\log m} \), \( n(H) \leq 2^{\frac{106\log m}{\log 2}} \) and there is \( S \subseteq V(H) \) with \( |S| \leq \sqrt{m^3} - 27\sqrt{m} \) such that \( \Delta(H - S) < \frac{\log m}{4} \).

Then \( R(G_1, G_2) \leq 2^{250\sqrt{m}} \).

Immediately we can obtained some corollaries using Theorems 1.4 and 1.5.

**Corollary 1.6** Suppose that \( G_i, i = 1, 2, \) is a graph with \( m_i \) edges and with no isolated vertices. If \( m = \max\{m_1, m_2\} \), then \( R(G_1, G_2) \leq 2^{250\sqrt{m}} \).

**Proof.** Since \( G_i, i = 1, 2, \) has at most \( 2m \) vertices, then by the Theorem 1.4 \( R(G_1, G_2) \leq R(K_{2m}, K_{2m}) \leq 2^{4m} \). If \( m \leq 60^2 \), then \( 4m \leq 250\sqrt{m} \) and so \( R(G_1, G_2) \leq 2^{250\sqrt{m}} \). So we may assume that \( m \geq 60^2 \). Obviously \( 2^{\frac{106\log m}{\log 2}} \geq 2m - 27\sqrt{m} \geq n - 27\sqrt{m} \), where \( n = \max\{n_1, n_2\} \) and \( n_i, i = 1, 2, \) is the number of vertices of \( G_i \). Again clearly for \( 27 \leq \alpha \leq \frac{\log^2 m}{8} \), the maximum degree of a graph obtained by deleting \( \alpha \sqrt{m} \) vertices of large degree of \( G_i \) is at most \( \frac{2m}{\alpha \sqrt{m}} = \frac{2\sqrt{m}}{\alpha} \). On the other hand, \( m^\frac{3}{2} \geq n \). Hence the assertion holds by Theorem 1.4.

**Corollary 1.7** Suppose \( G_i, i = 1, 2, \) is a graph of order \( n_i \) and \( n = \max\{n_1, n_2\} \). If there exists a subset \( U_i \subseteq V(G_i), i = 1, 2, \) with \( |U_i| \leq 27\sqrt{m} \) such that \( \Delta(G_i - U_i) \leq 54\frac{\sqrt{m}}{\log m} \), then \( R(G_1, G_2) \leq 2^{250\sqrt{m}} \).
**Proof.** The assertion holds by putting $m = n^2$ in Theorem 1.4. □

Using Theorem 1.5, we get the following corollaries.

**Corollary 1.8** Let $G_1 = K_p$, $m \geq 27$, $p \leq 27\sqrt{m} + \frac{18\sqrt{m}}{\log m}$ and $G_2 = K_l + qK_1$, where $l \leq 27\sqrt{m}$ and $q \leq 2^{106\log m}$. Then $R(G_1, G_2) \leq 2^{50\sqrt{m}}$.

**Corollary 1.9** For positive integers $p$ and $q$ with $q \leq 2^{27\log \frac{53}{q}}$, we have $R(K_p, K_p, q) \leq R(K_p, K_p + qK_1) \leq 2^{\frac{27}{2}p}$.

# 2 Preliminaries

In this section, we present some lemmas which will be used in the proof of Theorem 1.4. Let $G = (V, E)$ be a graph and $U \subseteq V$. The induced subgraph of $G$ on $U$ is denoted by $G[U]$. The edge density of $G[U]$ is defined by $d(U) = \frac{e(U)}{(V)^2}$, where $e(U)$ is the number of edges of $G[U]$. In an edge-coloring of $K_n$, we call an ordered pair $(X, Y)$ of disjoint subsets of vertices monochromatic if all edges in $X \cup Y$ incident to each vertex in $X$ have the same color.

**Lemma 2.1** (8) For all $k$ and $l$, every blue-red edge coloring of $K_N$, contains a monochromatic pair $(X, Y)$ with

$$|Y| \geq \left(\frac{k + l}{k}\right)^{-1}N - k - l,$$

which is red and $|X| = k$ or is blue and $|X| = l$.

**Lemma 2.2** (8) Let $0 < \epsilon \leq \frac{1}{8}$ and let $t$ and $N$ be positive integers satisfying $t \geq \epsilon^{-1}$ and $N \geq t\epsilon^{-14\epsilon t}$. Then every blue-red edge coloring of $G = K_N$ in which edge density of the induced subgraph on the blue edges is at most $\epsilon$ contains a monochromatic pair $(X, Y)$ with $|X| \geq t$ and $|Y| \geq \epsilon t N$.

**Lemma 2.3** (8) Let $G$ be a graph with $n$ vertices, maximum degree $\Delta$ and let $\epsilon \leq \frac{1}{8}$. If $H$ has $N \geq \epsilon^{4\Delta \log^* n}$ vertices and does not contain a copy of $G$, then it has a subset $S$ with $|S| \geq \epsilon^{-4\Delta \log^* n}$ and edge density $d(S) \leq \epsilon$.

**Lemma 2.4** Let $G_1$ and $G_2$ be graphs with no isolated vertices of orders $n_1$ and $n_2$, respectively, and $m$ be a positive integer such that $2^{\frac{\log m}{8}} \geq n - 27\sqrt{m}$, where $n = \max\{n_1, n_2\}$. Also suppose that for $27 \leq \alpha \leq \frac{\log m}{8}$ there exists $U_i \subseteq V(G_i)$ with $|U_i| \leq \alpha\sqrt{m}$ such that $\Delta(G_i') \leq \frac{2\sqrt{m}}{\alpha^{1/2}}$, where $G_i' = G_i - U_i$. If a blue-red edge coloring of $K_N$ has no blue copy of $G_1$ and no red copy of $G_2$ and it contains a monochromatic pair $(X, Y)$ with $|X| \geq \alpha\sqrt{m}$ and $|Y| \geq \frac{2^{12\alpha t}}{\alpha^{1/2}}\sqrt{m}$, then it also contains a monochromatic pair $(X', Y')$ with $|X'| \geq 2^{2\alpha t}\sqrt{m}$ and $|Y'| \geq 2^{-12\alpha t}\sqrt{m}|Y|$.

**Proof.** Consider a blue-red edge coloring of $K_N$ with the properties in Lemma 2.4. Assume that the color of the monochromatic pair $(X, Y)$ is blue. By the hypothesis, the induced subgraph on $Y$ contains no blue $G_1'$. Otherwise, together with $X$, we get a blue $G_1$. Let $\epsilon = 2^{-3\alpha t}$ and $t = 2^{2\alpha t}\sqrt{m}$. Since $\alpha \leq \frac{\log m}{8}$ we have $2^{\alpha t} \leq \sqrt{m}$ and so $t \geq \epsilon^{-1}$. Also note that, since $27 \leq \alpha \leq \frac{\log m}{8}$, we have $42\alpha \frac{2^{-\alpha t}}{\alpha^{1/2}} \leq 48\alpha \frac{2^{-\alpha t}}{\alpha^{1/2}}$.
and \(2^{5a \frac{i}{3}} \sqrt{m} \geq 2^{10 \frac{m}{2m}} \geq m^{\frac{3}{2}} \geq 2^{2a \frac{i}{3}} \sqrt{m} = t \). Applying Lemma 2.3 to the blue graph restricted to \(Y\), we find a subset \(S \subset Y\) with

\[
|S| \geq \epsilon^{-4\Delta(G_i)} \log |Y| \geq (2^{-3a \frac{i}{3}} - 4(2^{a - 1} \sqrt{m}) \frac{3a \frac{i}{3}}{2m}) |Y| = 2^{-72a \frac{i}{3}} \sqrt{m} |Y| \geq 2^{5a \frac{i}{3}} \sqrt{m} \geq 2^{10 \frac{m}{2m}} \geq n - 27 \sqrt{m},
\]

such that the density of the induced subgraph on the blue edges in \(S\) is at most \(\epsilon\). Then the size of \(S\) satisfies

\[
|S| \geq 2^{53a \frac{i}{3}} \sqrt{m} \geq 2^{5a \frac{i}{3}} \sqrt{m} 2^{48a \frac{i}{3}} \sqrt{m} \geq 2^{42a \frac{i}{3}} 2^{-\frac{1}{3}} \sqrt{m} = t \epsilon^{-14\epsilon t},
\]

and we can apply Lemma 2.2 to \(S\). So \(S\) contains a monochromatic pair \((X',Y')\) with \(|X'| \geq t\) and \(|Y'| \geq \epsilon^{14\epsilon t} |S|\). To complete the proof, recall that \(|S| \geq 2^{-72a \frac{i}{3}} \sqrt{m} |Y|\) and therefore

\[
|Y'| \geq \epsilon^{14\epsilon t} |S| \geq 2^{-48a \frac{i}{3}} \sqrt{m} |S| \geq 2^{-120a \frac{i}{3}} \sqrt{m} |Y|.
\]

A similar argument, which we omit, can be used to finish the proof in the case when the color of the monochromatic pair \((X,Y)\) is red.

\[\square\]

3 The Proofs

In this section, we give the proofs for the main theorems.

The proof of Theorem 1.4. Assume that \(N = 250\sqrt{m}\) and suppose for contradiction that there is a blue-red edge coloring of \(K_N\) with no blue copy of \(G_1\) and no red copy of \(G_2\). Since \(G_1\) and \(G_2\) have at most \(n\) vertices by Theorem 1.1, we have \(R(G_1,G_2) \leq R(K_n) \leq 2^{2n}\). So we may assume that \(n \geq 125\sqrt{m}\). Applying Lemma 2.1 with \(k = l = 27\sqrt{m}\), we have a monochromatic pair \((X_1,Y_1)\) with \(|X_1| \geq 27\sqrt{m}\) and

\[
|Y_1| \geq \left(\frac{k + l}{k}\right)^{-1} N - k - l \geq 4^{-27\sqrt{m}} N = 2^{196\sqrt{m}}.
\]

Define \(\alpha_1 = 27\) and \(\alpha_{i+1} = 2^{2\alpha_i + 1}\). It is easy to see \(\alpha_{i+1} \geq (\frac{4}{3})^3 \alpha_i\) for all \(i\) and therefore \(\alpha_i^{\frac{i}{3}} \leq \frac{1}{3} (\frac{4}{3})^{i-1}\). This implies that

\[
\Sigma_{j=1}^{i} \alpha_j^{\frac{i}{3}} \leq \frac{1}{3} \Sigma_{j=1}^{i} \left(\frac{3}{4}\right)^j \leq \frac{1}{3} \Sigma_{j=0}^{\infty} \left(\frac{3}{4}\right)^j = \frac{4}{3}.
\]

Since the blue-red edge coloring of \(K_N\) has no blue copy of \(G_1\) and no red copy of \(G_2\), we can repeatedly apply Lemma 2.3. After \(i\) iterations, we have a monochromatic pair \((X_{i+1},Y_{i+1})\) with \(|X_{i+1}| \geq \alpha_{i+1} \sqrt{m}\) and

\[
|Y_{i+1}| \geq 2^{-120a \frac{i}{3}} \sqrt{m} |Y_i| \geq 2^{-120a \frac{i+1}{3}} \sqrt{m} \alpha_{i+1}^{\frac{i}{3}} |Y_i| \geq 2^{-120a \frac{i+1}{3}} \sqrt{m} 2^{196\sqrt{m}} = 2^{36\sqrt{m}}.
\]

We continue iterating until the first index \(i\) such that \(\alpha_i \geq \frac{\log 3 \sqrt{m}}{8}\). Then for \(\alpha = \frac{\log 3 \sqrt{m}}{8}\) we have a monochromatic pair \((X,Y)\), \(|X| \geq \alpha \sqrt{m}\) and \(|Y| \geq 2^{36\sqrt{m}} \geq 2^{125a \frac{i}{3}} \sqrt{m}\). Then applying Lemma 2.3 one more time we obtain a monochromatic pair \((X',Y')\) with \(|X'| \geq 2^{2a \frac{i}{3}} \sqrt{m} = m^{\frac{3}{2}}\) and \(|Y'| \geq 2^{36\sqrt{m}}\). First let \((X',Y')\) be blue. By the second property in the hypothesis, since the induced subgraph on \(Y'\) contains no a red \(G_2\), it contains a blue copy of \(G_1 - V_1\) and clearly \(X' \cup (G_1 - V_1)\) contains a blue copy of \(G_1\), a contradiction. So we may assume that the monochromatic pair \((X',Y')\) is red. Clearly the subgraph on \(Y'\) does not contain a blue \(G_1\) and so it contains a red \(G_2 - V_2\) and so \(X' \cup (G_2 - V_2)\) contains a red copy of \(G_2\), a contradiction.

\[\square\]

To prove Theorem 1.5 we need the following result in [1].
Theorem 3.1 \(\square\) Let \(H\) be a graph with \(h\) vertices and chromatic number \(\chi(H) = k \geq 2\). Suppose that there is a proper \(k\)-coloring of \(H\) in which the degrees of all vertices, besides possibly those in the first color class, are at most \(r\), where \(1 \leq r(< h)\). Define \(\alpha(k, r)\) to be 1 if \(k > r\), and 0 otherwise. Then, for every integer \(m > 1\),

\[
R(H, K_m) \leq \left(\frac{100m}{\ln m}\right)^{(2r-k+2)(k-1)}(\ln m)^{\alpha(k, r)}h^r.
\]

The proof of Theorem 1.5. Let \(q = n(H - S)\), \(k = \chi(H - S)\) and \(r = \Delta(H - S)\). Since \(m \geq 27\), then \(p \leq 40\sqrt{m}\). If \(r = 0\), we add an edge, say \(e\), to \(H - S\). By Theorem 3.1 we have

\[
R(H - S, G_1) = R(H - S + e, G_1)
\]

\[
< 100pq \leq 100(27\sqrt{m} + \frac{16\sqrt{m}}{\log^2 m})2^{106 \sqrt{m}}
\]

\[
< 4000\sqrt{m}2^{106 \sqrt{m}} = 2^{\log 4000\sqrt{m} + 106 \sqrt{m}}
\]

\[
\leq 2^{13 + \frac{1}{2} \log m + 106 \sqrt{m}} < 2^{36\sqrt{m}}.
\]

Then we can use Theorem 1.4. Now suppose that \(r \geq 1\). Using Theorem 3.1, we get

\[
R(H - S, G_1) = R(H - S, K_p)
\]

\[
\leq \left(\frac{100p}{\ln p}\right)^{(2r-k+2)(k-1)}(\ln p)(q)^r
\]

\[
\leq \left(\frac{100p}{\ln p}\right)^{\frac{2r}{2}+1}(\ln p)(27\sqrt{m} + 2^{106 \sqrt{m}} - \sqrt{m}^3)^r
\]

\[
\leq \left(\frac{100p}{\ln p}\right)^{\frac{2r}{2}+1}2^{106 \sqrt{m}}r
\]

\[
(\ln p)^{\frac{2r}{2}+1}
\]

\[
\leq (4000\sqrt{m})^{\frac{2r}{2}+1}2^{106 \sqrt{m}}r y
\]

\[
= 2^{\frac{2r}{2}+1 \log 4000\sqrt{m} + 106 \sqrt{m}r} y
\]

\[
< 2^{36\sqrt{m}}.
\]

The last inequality holds, since

\[
y = \frac{1}{(\ln p)^{\frac{2r}{2}+1}} \leq 1,
\]

and

\[
\frac{r^2 + r}{2} \log 4000\sqrt{m} + 106 \sqrt{m}r = \frac{r^2 + r}{2}(\log 4000 + \log \sqrt{m}) + 106 \sqrt{m}r
\]

\[
< 6(r^2 + r) + \frac{r^2 + r}{2} \log \sqrt{m} + 106 \sqrt{m}r
\]

\[
\leq 12(\log^2 \sqrt{m}) + \log^2 \sqrt{m} \log \sqrt{m} + 26.5 \sqrt{m}
\]

\[
= \frac{12}{16} \log^2 m + \frac{1}{32} \log^3 m + 26.5 \sqrt{m}
\]

\[
< 9.5 \sqrt{m} + 26.5 \sqrt{m} = 36 \sqrt{m}.
\]

Now we use Theorem 1.4 to finish the proof. \(\square\)
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