EVERY GROUP IS THE GROUP OF SELF-HOMOTOPY EQUIVALENCES OF FINITE DIMENSIONAL CW-COMPLEX

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Abstract. We prove that any group $G$ occurs as $E(X)$, where $X$ is CW-complex of finite dimension and $E(X)$ denotes its group of self-homotopy equivalence. Thus, we generalize a well-know theorem due to Costoya and Viruel [10] asserting that any finite group occurs as $E(X)$, where $X$ is rational elliptic space.

1. Introduction

For a simply connected CW-complex $X$, we are interested in the group of self-homotopy equivalences $E(X)$, and the so-called Kahn’s realisability problem of groups [13]. Namely, if a given group $G$ can occur as $E(X)$ for some space $X$.

For finite groups, in [10] Costoya and Viruel solved completely this problem by constructing a rational elliptic space $X$ having formal dimension $n = 208 + 80|G|$, where $G$ is a certain finite graph associated with $G$ and $|G|$ denotes its order. The space $X$ satisfies $\pi_k(X) = 0$ for all $k \geq 120$. Later on in [5], it is proven that we can realise the finite group $G$ by a non-elliptic space having formal dimension $n = 120$, independently of the order of $G$.

It is worth mentioning that Kahn’s realisability problem has been solved for generic spaces in [8] but is still open in the realm of CW-complexes for infinite groups. Thus, inspired by the ideas developed in [4, 7, 9], this paper aims to solve the quoted problem for arbitrary groups and in the context of CW-complexes.

Theorem 1. For any group $G$ and any prime $p > 1114$, there exists a CW-complex $X$ such that $G \cong E(X_{(p)})$, where $X_{(p)}$ is the $p$-localization of $X$. More precisely, we have

1. $X$ is an 116-connected, 2341-dimensional, and of finite type if $G$ is finite.
2. $H_i(X, \mathbb{Z}(p))$ is a free $\mathbb{Z}(p)$-modules over a basis which in bijection with $G$ for $i = 691, q, 2314 \leq q \leq 2341$.
3. $H_i(X, \mathbb{Z}(p)) \cong \mathbb{Z}(p)$, for $i \in \{116, 152, 202, 304, 404, 2314\}$.

2. Anick’s $Z(p)$-local homotopy theory

We prove our main theorem using standard tools of Anick’s differential graded Lie algebra framework for $\mathbb{Z}(p)$-local homotopy theory which we refer to [1, 2, 3] for a general introduction to these techniques. We recall some of the notations here. Let $CW^k_m$ denote the category of $m$-connected, finite CW-complexes of dimension no greater than $k + 1$ with $m$-skeleton reduced to a point, $CW^k_{m+1}(\mathbb{Z}(p))$ denote the category obtained by $\mathbb{Z}(p)$-localizing the CW-complexes in $CW^k_m$ and $DGL^k_m(\mathbb{Z}(p))$ denote the category

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of the free differential graded Lie algebras (DGL for short) $(\mathbb{L}(W), \partial)$ in which $W$ is a free graded $\mathbb{Z}(p)$-module satisfying $W_n = 0$ for $n < m$ and $n > k$.

2.1. Homotopy in $\textbf{DGL}^k_m(\mathbb{Z}(p))$ (see [1, pp. 425-6]). If $(\mathbb{L}(W), \delta)$ is an object of $\textbf{DGL}^k_m(\mathbb{Z}(p))$, then we define the DGL $\mathbb{L}(W, sW, W'), D)$ as follows

$$W \cong W', \quad (sW)_i = W_{i-1}.$$ 

The differential $D$ is given by

$$D(w) = \delta(w), \quad D(sw) = w', \quad D(w') = 0.$$ 

Here $w'$ is the image of $w$ under the isomorphism $W \cong W'$. Define $S$ as the derivation of degree +1 on $\mathbb{L}(W, sW, W')$ given by

$$S(w) = sw, \quad S(sw) = S(w') = 0.$$ 

A homotopy between two DGL-maps $\alpha, \alpha' : (\mathbb{L}(W), \delta) \to (\mathbb{L}(W), \delta)$ is a DGL-map $F : (\mathbb{L}(W, sW, W'), D) \to (\mathbb{L}(W), \delta)$, such that $F(w) = \alpha(w)$ and $F \circ e^\theta(w) = \alpha'(w)$, where

$$e^\theta(w) = w + w' + \sum_{n \geq 1} \frac{1}{n!}(S \circ D)^n(w), \quad \theta = D \circ S + S \circ D.$$ 

Subsequently, we will need the following lemma.

**Lemma 2.1.** Let $\alpha, \alpha' : (\mathbb{L}(W_{\leq n}), \delta) \to (\mathbb{L}(W_{\leq n}), \delta)$ be two DGL-maps such that $\alpha'(w) = \alpha(w) + y$ on $W_n$ and $\alpha' = \alpha$ on $W_{\leq n-1}$. Assume that $y = \partial(z)$, where $z \in \mathbb{L}(W_{\leq n})$. Then $\alpha$ and $\alpha'$ are homotopic.

**Proof.** Define $F$ by setting

$$F(w) = \alpha'(w), \quad F(w') = -y \quad \text{and} \quad F(sw) = -z \quad \text{for} \ w \in W_n,$$

$$F(w) = \alpha(w), \quad F(w') = 0 \quad \text{and} \quad F(sw) = 0 \quad \text{for} \ w \in W_{\leq n-1}.$$ 

Then $F$ is the needed homotopy. □

Anick’s work [3] asserts that if $k < \min(m+2p-3, mp-1)$, then the homotopy category of $\textbf{CW}^{k+1}_m(\mathbb{Z}(p))$ is equivalent to the homotopy category of $\textbf{DGL}^k_m(\mathbb{Z}(p))$. Thus, given a space $X$ in $\textbf{CW}^{k+1}_m(\mathbb{Z}(p))$, Anick’s model recovers homotopy data via

$$\pi_*(X) \cong H_{* - 1}(\mathbb{L}(W), \partial) \quad \text{and} \quad H_*(X, \mathbb{Z}(p)) \cong H_{* - 1}(W, d),$$

where $d$ is the linear part of $\partial$ (see [1, Theorem 8.5]). Moreover, Anick’s theory directly implies an identification of the form

$$\mathcal{E}(X) \cong \mathcal{E}(\mathbb{L}(W)), \quad (1)$$

where $\mathcal{E}(\mathbb{L}(W))$ is the group of differential graded Lie self-homotopy equivalences of $(\mathbb{L}(W), \partial)$ modulo the relation of homotopy in $\textbf{DGL}^k_m(\mathbb{Z}(p))$ (see [3] for more details).
2.2. Strongly connected digraphs and theorem of J. de Groot. A digraph (i.e. a directed graph) \( G = (V(G), E(G)) \), where \( V(G) \) denotes the set of the vertices of \( G \) and \( E(G) \) the set of its edges, is strongly connected if for every \( v, u \in V(G) \), there exists an integer \( m \in \mathbb{N} \) and vertices \( v = v_0, v_1, \ldots, v_m = u \) such that \( (v_i, v_{i+1}) \in E(G) \) for all \( i = 0, 1, \ldots, m \).

The following theorem, due to J. de Groot ([11], pp. 96), plays a central role in this paper.

**Theorem 2.2.** Any group \( G \) is isomorphic to the automorphism group of a strongly connected digraph \( G \).

**Remark 2.3.** It should be noted that, in [11], de Groot is not considering strongly connected digraphs, but just graphs or digraphs. But any simple graph can be seen as a symmetric digraph, ([12], §1.1), if it is connected, then the associated digraph is strongly connected.

Now, let us give an outlook of the steps developed in the next sections in order to prove the main result.

- To a given group \( G \) corresponds a strongly connected digraph \( G \) such that \( G \cong \text{aut}(G) \) according to Theorem 2.2.
- To the digraph \( G \), given above, we define a differential graded Lie \( \mathbb{Z}(p) \)-algebra \( \mathcal{L}(G, 0) \). We study the group \( \mathcal{E}(\mathcal{L}(G, 0)) \) and we show that the sub-module of the cycles of degree 2313 is not trivial.
- Next, we define a new DGL \( \mathcal{L}(G, 1) \) by adding generators in degree 2314 to \( \mathcal{L}(G, 0) \) killing all the cycles of dimension 2313 in \( \mathcal{L}(G, 0) \). Likewise, we study the group \( \mathcal{E}(\mathcal{L}(G, 1)) \) and we show that the sub-module of the cycles of degree 2314 is not trivial.
- Similarly, new DGLs \( \mathcal{L}(G, s) \), \( 2 \leq s \leq 24 \), are constructed by adding generators in degree 2314 + \( s \) to \( \mathcal{L}(G, s - 1) \) killing all the cycles of dimension 2313 + \( s \) in \( \mathcal{L}(G, s - 1) \). Again, we study the group \( \mathcal{E}(\mathcal{L}(G, 25)) \) and we show that they are cycles of degree 2339 in \( \mathcal{L}(G, 25) \).
- Finally, we construct the DGL \( \mathcal{L}(G, 26) \) by adding generators in degree 2340 to \( \mathcal{L}(G, 25) \) killing all the cycles of dimension 2339 in \( \mathcal{L}(G, 25) \) and, crucially, we prove that the sub-module of the cycles of degree 2340 is trivial. This allows us to define an isomorphism of group from \( \mathcal{E}(\mathcal{L}(G, 26)) \) to \( \text{aut}(G) \).
- Consequently, using the Anick’s \( \mathbb{Z}(p) \)-local homotopy theory framework, where \( p > 1114 \) is a prime, to \( \mathcal{L}(G, 26) \) corresponds an object \( X \) in the category \( \text{DGL}_{2341}(\mathbb{Z}(p)) \) satisfying

\[
\mathcal{E}(X) \cong \mathcal{E}(\mathcal{L}(G, 26)) \cong \text{aut}(G) \cong G.
\]

3. Graded Lie \( \mathbb{Z}(p) \)-algebras associated with strongly connected digraphs

**Definition 3.1.** Let \( p > 1114 \) be a prime. For a given strongly connected digraph \( G \), with more than one vertex, we define the following DGL

\[
\mathcal{L}(G, 0) = \left\{ \sum_{1 \leq i < j \leq 6} w_{ij} x_{v, u} + \sum_{v \in V(G)} z_{(v, u)} | v \in V(G), (v, u) \in E(G), \partial \right\}.
\]

The degrees of the generators are as follows

\[
|w_1| = 115, \quad |w_2| = 151, \quad |w_3| = 201, \quad |w_4| = 303, \quad |w_5| = 403, \quad |w_6| = 2313
\]

\[
|x_v| = 690, \quad \forall v \in V(G), \quad |z_{(v, u)}| = 2337, \quad \forall (v, u) \in E(G).
\]
The differential is given by
\[
\begin{align*}
\partial(w_1) &= \partial(w_2) = \partial(w_3) = 0, \quad \partial(x_v) = 0, \quad v \in V(\mathcal{G}), \\
\partial(w_4) &= [w_2, w_2], \quad \partial(w_5) = [w_3, w_3], \\
\partial(w_6) &= \left[[w_2, w_3], (\text{ad} w_3)^3(w_2)\right] + (\text{ad} w_1)\left((\text{ad}[w_4, w_2])^4([w_1, w_2])\right), \\
\partial(z_{(v,u)}) &= (\text{ad} x_v)^3([w_1, w_2]) + (\text{ad} w_1)^7([w_2, [x_v, x_u]]) + (\text{ad} w_1)^{19}(w_2) + Y + Z,
\end{align*}
\]
where
\[
Z = \left[(\text{ad} w_1)^5(w_3), (\text{ad} w_2)^3(w_3), (\text{ad} w_2)^2([w_3, w_5])\right],
\]
\[
Y = (\text{ad} w_1)^{12}([[w_2, w_3], [w_3, w_5]]).
\]

Recall that the iterated Lie bracket of length \(k + 1\) is defined by
\[
(\text{ad} x)^k(y) = [x, [x, \ldots, [x, y]\ldots]],
\]
where \(x\) is involved \(k\) times. For the sake of simplicity, let's denote
\[
\mathcal{L} = (L(w_1, w_2, w_3, w_4, w_5, x_v \mid v \in V(\mathcal{G})), \partial).
\]

**Remark 3.2.** The prime number \(p\) and the degrees of the elements in the DGL \(\mathbb{Z}_{(p)}\)-algebra \(\mathcal{L}(\mathcal{G}, 0)\) have been meticulously selected to align with Anick’s framework. This choice is made with consideration of the constraint imposed by the dimension and connectivity of the DGL \(\mathcal{L}(\mathcal{G}, 0)\), namely the relation \(k < \min(m + 2p - 3, mp - 1)\), where \(k\) is the dimension and \(m\) is the connectivity of \(\mathcal{L}(\mathcal{G}, 0)\), mentioned in section 2. Note that as by [2], we have \(m = 115\) and \(k = 2337\), it follows that the prime \(p\) can be chosen as \(p > 1114\).

Later, we will make use of the following lemmas.

**Lemma 3.3.** For every \(v \in V(\mathcal{G})\), the following cycles are not boundaries in \(\mathcal{L}\),
\[
[x_v, [x_u, [x_t, [w_2, w_1]]]]], \quad (\text{ad} x_v)^3([w_1, w_2]), \quad (\text{ad} w_1)^7([w_2, [x_v, x_u]]), \quad (\text{ad} w_1)^{19}(w_2).
\]

**Proof.** Since the only non zero differentials are \(\partial(w_3) = [w_2, w_2], \partial(w_5) = [w_3, w_3]\) and none of the given cycles contain \(w_2\) or \(w_3\) twice, it follows that the given brackets cannot be boundaries. \(\square\)

**Lemma 3.4.** For every \(s, s' \in V(\mathcal{G})\) such that \(s \neq s'\), the following brackets
\[
[x_s, [x_s, [x_{s'}, [w_1, w_2]]]], \quad [x_s, [x_{s'}, [x_s, [w_1, w_2]]]], \quad [x_{s'}, [x_s, [x_{s'}, [w_1, w_2]]]],
\]
are linearly independent. Moreover, any linear combination of those brackets is not a boundary in \(\mathcal{L}\).

**Proof.** First, the condition \(p > 1114\) ensures that \(\mathcal{L}(\mathcal{G}, 0)\) is an object of the category \(\text{DGL}_p^{2334}\) allowing us to use Anick’s framework in [2]. Next, let \(\mathcal{T} = T(w_1, w_2, w_3, w_4, w_5, x_v \in V(\mathcal{G}))\) denotes the universal algebra of \(\mathcal{L}\). Recall that \(\mathcal{T}\) can be considered as a graded Lie algebra by defining
\[
[B, C] = BC - (-1)^{|B||C|}CB, \quad B, C \in \mathcal{T}.
\]

Note that the differential of \(\mathcal{T}\) satisfies
\[
\partial(w_4) = 2w_2^2, \quad \partial(w_3) = 2w_3^2.
\]
Next, set \(A = [w_1, w_2]\), if
\[
\mu_1[x_s, [x_s, [x_{s'}, A]]] + \mu_2[x_s, [x_{s'}, [x_s, A]]] + \mu_3[x_{s'}, [x_s, [x_s, A]]] = 0, \quad \mu_1, \mu_2, \mu_3 \in \mathbb{Z}_{(p)},
\]
Therefore, 

\[ \mu \]

Consequently, 

Likewise, as the expression 

\[ [x_s, [x_{s'}, [x_s, A]]] \]

and by using the relation (2) we obtain 

\[ [x_s, [x_{s'}, [x_s, A]]] = x_s x_{s'} x_s A - x_s x_{s'} A x_s - x_s^2 A x_{s'} + x_s A x_s x_{s'} - x_s x_{s'} x_s A + x_s x_{s'} A x_s + x_s A x_{s'} x_s - A x_s x_{s'} x_s, \]  

\[ [x_{s'}, [x_s, [x_s, A]]] = x_{s'} x_s^2 A - 2 x_{s'} x_s A x_s - x_{s'} A x_s^2 - x_{s'}^2 A x_s - 2 x_s A x_{s'} x_{s'} - A x_{s'} x_{s'} x_{s'}. \]  

Therefore \( \mu_1 x_{s'}^2 x_{s'} A \) does not appear in (7) and in (8). As a result \( \mu_1 = 0 \). Likewise, as the expression \( \mu_2 x_s x_{s'} x_s w_1 w_2 \) does not appear in (8), it follows that \( \mu_2 = 0 \). Consequently \( \mu_3 = 0 \).

Finally, since the monomials \( \mu_1 x_{s'}^2 x_{s'} w_1 w_2, \mu_2 x_s x_{s'} x_s w_1 w_2 \) and \( \mu_3 x_{s'} x_s x_s w_1 w_2 \) cannot be reached by the differential \( \partial \) according to (\[ \square \]), it follows that any linear combination of the given brackets is not a boundary in \( \mathcal{L} \).

**Lemma 3.5.** For every \( v \in V(\mathcal{G}) \), the brackets \( Y \) and \( Z \), given in (\[ \text{3.4} \]), are linearly independent. Moreover, any linear combination of \( Y \) and \( Z \) is not a boundary in \( \mathcal{L} \).

**Proof.** First, recall that 

\[ Y = (\text{ad } w_1)^2([w_2, w_3], [w_3, w_5]), \quad Z = [(\text{ad } w_1)^3(w_3), ((\text{ad } w_2)^3(w_3), (\text{ad } w_2)^2([w_3, w_5])]. \]

As \( Y \) contains exactly 2 generators \( w_3, Z \) has 3, we derive that \( Y \) and \( Z \) are linearly independent.

Next, we claim that any linear combination \( \gamma_1 Y + \gamma_2 Z \), where \( \gamma_1, \gamma_2 \in \mathbb{Z}_p \), cannot be a boundary. Indeed, using the same argument given in the previous Lemmas, expanding the three brackets in the universal algebra \( \mathcal{T} \), we get the following monomials

\[ \gamma_1 w_1^2 w_2 w_3 w_5, \quad \gamma_2 w_1^3 w_3 w_2^2 w_3 w_5, \]

and none of them cannot be reached by the differential \( \partial \) according to (\[ \text{3.5} \]).

**Remark 3.6.** Using the same arguments as in the proof of Lemmas (\[ \text{3.4} \]) and (\[ \text{3.5} \]), we can easily prove that any linear combination of the brackets

\[ [w_2, w_3], (\text{ad } w_3)^3(w_2), \quad (\text{ad } w_1)^2((\text{ad } w_4, w_2)^3([w_1, w_2])), \]

given in the relation (\[ \text{3.5} \]), is not boundary in \( \mathcal{L} \).

4. **Main result**

4.1. **Studying the group** \( \mathcal{E}(\mathcal{L}(\mathcal{G}, 0)) \). First, let \( \mathcal{L}_k(\mathcal{G}, 0) \) denote the sub-module consisting of elements of degree \( k \). Next, we initiate the process with the following Lemmas.

**Lemma 4.1.** The set of the decomposable elements in the \( \mathcal{L}_{690}(\mathcal{G}, 0) \) is empty.

**Proof.** Assume \( \Theta \in \mathcal{L}_{690}(\mathcal{G}, 0) \) is a decomposable element. then we can write

\[ |\Theta| = a_1 |w_1| + a_2 |w_2| + a_3 |w_3| + a_4 |w_4| + a_5 |w_5| = 690, \]

and by using the relation (\[ \text{2} \]) we obtain

\[ 115a_1 + 151a_2 + 201a_3 + 303a_4 + 403a_5 = 690. \]

The equation (\[ \text{3.1} \]), where \( a_1, a_2, a_3, a_4, a_5 \in \{0, 1, 2, \ldots\} \) are unknown, is called a Frobenius equation and can be solved by WOLFRAM\textsuperscript{\[ \text{1} \]} software using the following code

\[ \text{https://reference.wolfram.com/language/tutorial/Frobenius.html} \]

\[ \text{\[ \text{1} \]} \text{See the link below for more details} \]
"FrobeniusSolve\{\{115, 151, 201, 303, 403\}, 690\}".

The only solution to \([\alpha] \in E(\mathcal{L}(G,0))\), we can write
\[
\alpha(w_1) = \beta w_1, \quad \alpha(w_2) = \lambda w_2, \quad \alpha(w_3) = \gamma w_3, \quad \alpha(w_4) = qw_4, \quad \alpha(w_5) = rw_5,
\]
\[
\alpha(x_v) = \sum_{s \in V(G)} \rho(v,s) x_s,
\]
\[
\alpha(w_b) = \mu w_b + F,
\]
\[
\alpha(z(v,u)) = \sum_{(s,r) \in E(G)} \rho(v,u),(r,s) z(r,s) + B(v,u),
\]
where all the coefficients belong to \(\mathbb{Z}_p\) and where \(B(v,u)\) and \(F\) are decomposable elements in \(\mathcal{L}_{2337}(G,0)\) and \(\mathcal{L}_{2313}(G,0)\) respectively.

Remark 4.2. Almost every coefficients \(\rho(v,u),(r,s)\) and \(a(v,s)\) is zero. Moreover, as \(\alpha\) is a homotopy equivalence, then its induces an isomorphism on the indecomposables, therefore \(\beta, \lambda, q, \gamma, r \neq 0\) and at least one of the coefficients \(a(v,s)\) is not zero as well as, at least one of the coefficients \(\rho(v,u),(r,s)\) is not zero.

Lemma 4.3. If \([\alpha] \in E(\mathcal{L}(G,0))\), then \(q = \lambda^2\) and \(r = \gamma^2\).

Proof. Since \(\partial(\alpha(w_3)) = \alpha(\partial(w_3))\) and \(\partial(\alpha(w_3)) = \alpha(\partial(w_3))\), it follows that
\[
\partial(\alpha(w_3)) = q[w_2, w_2], \quad \alpha(\partial(w_3)) = [\alpha(w_3), \alpha(w_3)] = \gamma^2[w_3, w_3],
\]
\[
\partial(\alpha(w_3)) = q[w_2, w_2], \quad \alpha(\partial(w_3)) = [\alpha(w_2), \alpha(w_2)] = \lambda^2[w_2, w_2].
\]
Consequently, \(q = \lambda^2\) and \(r = \gamma^2\).

Proposition 4.4. Let \([\alpha] \in \mathcal{L}(G,0)\). There exists unique \(\phi \in \text{aut}(G)\) such that
\[
\alpha(z(v,u)) = z(\phi(v), \phi(u)) + B(v,u), \quad \forall (v,u) \in E(G),
\]
\[
\alpha(w_b) = w_b + F,
\]
\[
\alpha(x_v) = x_{\phi(v)}, \quad \forall v \in V(G),
\]
\[
\alpha(w_i) = w_i, \quad i = 1, 2, 3, 4, 5.
\]
Moreover, \(B(v,u)\) and \(F\) are cycles in \(\mathcal{L}_{2337}(G,0)\) and \(\mathcal{L}_{2313}(G,0)\) respectively.

Proof. Notice that the strong connectivity of the graph implies that for every \(v \in V(G)\), \(v\) is the starting vertex of an edge \((v, w) \in E(G)\). Therefore the coefficients in \([\alpha]\) can be entirely determined by the relation \(\alpha \circ \partial = \partial \circ \alpha\). Indeed, first we have
\[
\alpha(\partial(w_b)) = [\alpha(w_2) \alpha(, w_3)], (ad \alpha(w_3)) \gamma (ad \alpha(w_2)) + (ad \alpha(w_1)) \gamma \gamma (ad \alpha(w_1), \alpha(w_2)) \gamma (ad \alpha(w_1), \alpha(w_2))
\]
\[
= \lambda^2 \gamma^{10} \gamma^{10} (w_2, w_3), (ad \alpha(w_3)) \gamma (w_2) + \beta^3 \lambda^{13} (ad \alpha(w_1)) \gamma \gamma (ad \alpha(w_1), \alpha(w_2))
\]
\[
\partial(\alpha(w_b)) = \mu \partial(w_b) + \partial(F) = \mu (w_2, w_3), (ad \alpha(w_3)) \gamma (w_2) + \mu (ad \alpha(w_1)) \gamma (ad \alpha(w_1), \alpha(w_2)) + \partial(F).
\]
Since \(\alpha(\partial(w_b)) = \partial(\alpha(w_b))\) and \(F\) is decomposable element in \(\mathcal{L}_{2337}(G,0)\), we deduce that
\[
\mu = \lambda^2 \gamma^{10} = \beta^3 \lambda^{13}, \quad \partial(F) = 0
\]
Next, we have

\[
\alpha(\partial(z_{v,u})) = (\text{ad}(\alpha(x_v)))^3([\alpha(w_1), \alpha(w_2)]) + (\text{ad}(\alpha(w_1)))^3([\alpha(w_2), \alpha(x_v), \alpha(x_u)]) + (\text{ad}(\alpha(w_1)))^{19}(\alpha(w_2)) + \alpha(Y) + \alpha(Z),
\]

\[
\partial(\alpha(z_{v,u})) = \sum_{(r,s) \in \mathcal{E}(\mathcal{G})} \rho_{(v,u),(r,s)} \partial(z_{r,s}) + \partial(B_{(v,u)}),
\]

(13)

where \(Y, Z\) are given in (14). Next, recall that from the relations (10), we have

\[
\alpha(w_1) = \beta w_1, \quad \alpha(w_2) = \lambda \omega_2, \quad \alpha(x_v) = \sum_{s \in V(\mathcal{G})} a_{(v,s)} x_s,
\]

(14)

where almost every coefficients \(a_{(v,s)}\) is zero and at least one of them is not zero. Expanding the expression

\[
(\text{ad}\alpha(x_v))^3(\alpha(w_1), \alpha(w_2)),
\]

using (14), we get to the following brackets

\[
\beta \lambda a^2_{(v,s)} a_{(v,s')} [x_s, [x_{s'}, [x_{w_1}, w_2]]], \quad v, s, s' \in V(\mathcal{G}),
\]

(15)

\[
\beta \lambda a^2_{(v,s)} a_{(v,s')} [x_{s'}, [x_s, [x_{w_1}, w_2]]], \quad v, s, s' \in V(\mathcal{G}),
\]

\[
\beta \lambda a^2_{(v,s')} a_{(v,s')} [x_{s'}, [x_s, [x_{w_1}, w_2]]], \quad v, s, s' \in V(\mathcal{G}).
\]

These brackets are proven to be linearly independent according to Lemma 3.3. However, none of the brackets in the expression (13), giving \(\partial(\alpha(z_{v,u}))\), is formed using three generators \(x_s, x_{s'}, x_{w_2}\) with \(s \neq s'\). Moreover, by Lemma 3.4, the expressions (14) as well as the following expression

\[
\beta \lambda a^2_{(v,s)} a_{(v,s')} ([x_s, [x_{s'}, [x_{w_1}, w_2]]] + [x_{s'}, [x_s, [x_{w_1}, w_2]]] + [x_s, [x_{s'}, x_{w_1}, w_2]]),
\]

are neither trivial nor a boundaries. We now, by the formula \(\alpha(\partial(z_{v,u}))) \neq 0\), deduce that all of the coefficients \(\beta \lambda a^2_{(v,s)} a_{(v,s')}\) are nil. Since \(\beta, \lambda \neq 0\), it follows that only one coefficient among \(a_{(v,s)}\), where \(s \in V(\mathcal{G})\), is not zero. Let us denote it by \(a_{(v,t,e)}\). As a result, the formula (14) becomes \(\alpha(x_v) = a_{(v,t,e)} x_t\). Thus, there is a unique vertex \(t \in V(\mathcal{G})\) such that \(\alpha(x_v) = a_{(v,t,e)} x_t\).

Consequently, on the one hand and going back to (11) and (10), we deduce that

\[
\alpha(Y) = \beta^{12} \lambda^4 Y, \quad \alpha(Z) = \beta^5 \lambda^5 \gamma^5 Z,
\]

On the other hand, the formulas (14) become

\[
\alpha(\partial(z_{v,u})) = \beta \lambda a^3_{(v,t,e)} (\text{ad}(\alpha(x_v)))^3([w_1, w_2]) + \beta^7 \lambda a_{(v,t,e)} a_{(u,t,e)} (\text{ad}(w_1))^{13}(\alpha(w_2), [x_t, x_{s'}]) + \beta^{19}(\text{ad}(w_1))^{19}(w_2) + \beta^{12} \lambda^4 Y + \beta^5 \lambda^5 \gamma^5 Z,
\]

(16)

\[
\partial(\alpha(z_{v,u})) = \sum_{(r,s) \in \mathcal{E}(\mathcal{G})} \rho_{(v,u),(r,s)} ((\text{ad}(x_v))^3([w_1, w_2]) + (\text{ad}(w_1))^{15}([w_2, [x_t, x_{s'}]]) + (\text{ad}(w_1))^{19}(w_2) + Y + Z) + \partial(B_{(v,u)}).
Likewise, due to Lemmas 3.3 and 5.5 all the brackets in $\partial(\alpha(z_{(v,u)})) - \partial(B_{(v,u)})$ and $\alpha(\partial(z_{(v,u)}))$ are not boundaries, and by comparing the coefficients in the formula

$$\partial(\alpha(z_{(v,u)})) - \alpha(\partial(z_{(v,u)})) = 0,$$  

(17)

we deduce that all the coefficients $\rho(v,u),(r,s) = 0$ zero except $\rho(v,u),(t_v, t_u) \neq 0$ which satisfies the following equations

$$\rho(v,u),(t_v, t_u) = \beta \lambda a_{(v,t_v)}^3 = \beta^7 \lambda a_{(v,t_v)} a_{(u,t_u)} = \beta^{19} \lambda = \beta^{12} \lambda \gamma^4 = \beta^5 \lambda^5 \gamma^5.$$  

(18)

From $\beta^{19} \lambda = \beta^{12} \lambda \gamma^4 = \beta^5 \lambda^5 \gamma^5$, we deduce that

$$\beta^7 = \gamma^4, \quad \beta^{12} = \lambda^4 \gamma^5, \quad \beta^7 = \lambda^4 \gamma \implies \gamma^3 = \lambda^4,$$

therefore,

$$\beta^7^{12} = \gamma^{48} = (\beta^{12})^7 = \lambda^{28} \gamma^{35} \implies \gamma^{13} = \lambda^{28} = (\gamma^3)^7 = \gamma^{21}.$$

It follows that $\gamma^8 = 1$. As $\gamma^3 = \lambda^4$ and $\beta^7 = \gamma^4$, we deduce that $\beta = \gamma = 1$. As a result, the relation $12$ becomes $\lambda^2 = \lambda^{13}$ implying that $\lambda = 1$.

Now, the relations $18$ become

$$\rho(v,u),(t_v, t_u) = a_{(v,t_v)}^3 = a_{(v,t_v)} a_{(u,t_u)} = 1.$$

Consequently, we get

$$\rho(v,u),(t_v, t_u) = \beta = \lambda = a_{(v,t_v)} = a_{(u,t_u)} = \gamma = 1,$$

and from Lemma 4.3 it follows that $q = r = 1$. Thus, the formulas $16$ become

$$\alpha(\partial(z_{(v,u)})) = (\text{ad}(x_v))^3([w_1, w_2]) + (\text{ad}(w_1))^7([w_2, [x_v, x_v]]) + (\text{ad}(w_1))^{19}(w_2) + Y + Z,$$

$$\partial(\alpha(z_{(v,u)})) = \text{ad}(x_v)^3([w_1, w_2]) + (\text{ad}(w_1))^7([w_2, [x_v, x_v]]) + (\text{ad}(w_1))^{19}(w_2) + Y + Z + \text{ad}(w_1)^{19}(w_2) + Y + Z + \partial(B_{(v,u)}).$$

and from $17$, it follows that $\partial(B_{(v,u)}) = 0$.

Thus, going back the formulas $14$, we have proved that for every $v \in V(\mathcal{G})$, there is a unique vertex $t_v \in V(\mathcal{G})$ and for every $(v, u) \in E(\mathcal{G})$, there is a unique edge $(t_v, t_u) \in E(\mathcal{G})$ such that

$$\alpha(z_{(v,u),v}) = z_{(t_v, t_u)} + B_{(v,u)} \quad \alpha(x_v) = x_{t_v}, \quad \alpha(w_6) = w_6 + F, \quad \alpha(w_i) = w_i, \quad 1, 2, 3, 4, 5.$$

with both $B_{(v,u)}$ and $F$ being cycle as desired.

Thus, define $\phi : \mathcal{G} \to \mathcal{G}$, by $\phi(v) = t_v, \phi((v, u)) = (t_v, t_u)$, we obtain $11$. $\square$

Remark 4.5. Let $\mathcal{L}(\mathcal{G}, 27) = \mathcal{L}(\mathcal{G}, 0) \oplus \left(\mathcal{L}(h, h \in \mathcal{I}, \theta)\right)$ be the DGL obtained from $\mathcal{L}(\mathcal{G}, 0)$ by adding generators $h \in \mathcal{I}$, where $2314 \leq |h| \leq 2340$. It is easy to see that $\mathcal{L}(\mathcal{G}, 27)$ does not contain any bracket of degree $d$, where $2313 \leq d \leq 2340$, formed by using three generators from the set $\{x_v\}_{v \in V(\mathcal{G})}$. Indeed; if a bracket contains three generators from $\{x_v\}_{v \in V(\mathcal{G})}$, it follows that the sum of the degrees of the other generators forming this bracket is $d - 3 \times 690$. But we have $243 \leq d - 3 \times 690 \leq 270$ and we know that no element in $\mathcal{L}(\mathcal{G}, 27)$ has degree between 243 and 270. Consequently, for $0 \leq n \leq 27$, if the sub-module $Z_n(\mathcal{L}(\mathcal{G}, 27))$ of the cycles of degree $2313 + n$ in $\mathcal{L}(\mathcal{G}, 27)$ is not trivial, then we can choose a Hall basis for $Z_n(\mathcal{L}(\mathcal{G}, 27))$ the following set

$$\mathcal{B}_n = \left\{y_{n,1}, \ldots, y_{n,m_n}, y_{s,v}, y_{n,\{v', u\}}; v, v', u \in V(\mathcal{G})\right\},$$  

(19)
Remark 4.5, we can write $F$ is a cycle of degree 2313.

Lemma 4.8. Let $G$ be a graph and $\phi \in \text{aut}(G)$. Assume $(\mathbb{L}(W), \partial)$ is a DGL such that the sub-module $Z_{2313}(\mathbb{L}(G), 0)$ of the cycles of degree 2313 is not trivial.

Proof. First, Indeed, for instance, it is easy to check that the following bracket

$$[(\text{ad } w_3)^0(w_2), (\text{ad } w_2)^0(w_3)]$$

is a cycle of degree 2313. \hfill \square

Remark 4.7. Due to Proposition 4.4 and taking into account that $y_{s,v}$ and $y_{s,(v',u)}$ contain elements from $\{x_v\}_{v \in V(G)}$, if $[\alpha] \in E(\mathbb{L}(G), 27)$, then there is a unique $\phi \in \text{aut}(G)$ such that for every $v, v', u \in V(G)$ and $0 \leq n \leq 27$, we have

$$\alpha(y_{n,v}) = y_{n,\phi(v)}, \quad \alpha(y_{n,(v',u)}) = y_{n,(\phi(v'),\phi(u))}$$

Lemma 4.8. The sub-module $Z_{2313}(\mathbb{L}(G), 0)$ of the cycles of degree 2313 is not trivial.

Proof. First, Indeed, for instance, it is easy to check that the following bracket

$$[(\text{ad } w_3)^0(w_2), (\text{ad } w_2)^0(w_3)]$$

is a cycle of degree 2313. \hfill \square

Remark 4.7. Due to Proposition 4.4 we know that $F$, given in (11), is a cycle. Hence, by Remark 4.5 we can write $F$ as a linear combination of the elements of $B_0$.

The forthcoming lemma, presented in a broad context, is poised to play a pivotal role in the forthcoming developments.

Lemma 4.8. Let $G$ be a graph and $\phi \in \text{aut}(G)$. Assume $(\mathbb{L}(W), \partial)$ is a DGL such that the sub-module $Z_{p}(\mathbb{L}(W))$ has a Hall basis the set $\{y_1, \ldots, y_m, y_{(v',u)}\}_{v',u \in V(G)}$. Let $\mathbb{L}(U) = \mathbb{L}(W) \oplus \left\{(\mathbb{L}(t_1, \ldots, t_m, v, v', u) | v, v', u \in V(G), ; \partial)\right\}$ be the DGL obtained from $\mathbb{L}(W)$ by adding generators $t_1, \ldots, t_m, v, v', u \in V(G)$ in degree $p + 1$. The differential is given by

$$\partial(t_1) = y_1, \ldots, \partial(t_m) = y_m, \quad \partial(t_v) = y_v, \quad \partial(t_{(v',u)}) = y_{(v',u)}. \quad (20)$$

If $[\alpha] \in E(\mathbb{L}(U))$ is such that

$$\alpha(y_{i}) = y_i, \quad \alpha(y_v) = y_{\phi(v)}, \quad \alpha(y_{(v',u)}) = y_{(\phi(v'),\phi(u))}, \quad (21)$$

then we have

$$\alpha(t_{(v',u)}) = t_{(\phi(v'),\phi(u))} + C_{(v',u)}, \quad \forall v', u \in V(G)$$

$$\alpha(t_v) = t_{\phi(v)} + C_v, \quad \forall v \in V(G)$$

$$\alpha(t_k) = t_k + C_k, \quad \forall k = 1, \ldots, m.$$
where all the coefficients belong to $\mathbb{Z}_{(p)}$ and where $C_{\{v',u\}}, C_v$ and $C_k$ are decomposable elements in $\mathbb{L}_{p+1}(W)$. Therefore, using (24) we get

$$
\partial(\alpha(t_{\{v',u\}})) = \sum_{i=1}^m \sigma_i \partial(t_i) + \sum_{r \in V(\mathcal{G})} \sigma_{\{v',u\},r} \partial(t_r) + \sum_{s,t \in V(\mathcal{G})} \sigma_{\{v',u\},\{s,t\}} \partial(t_{\{s,t\}}) + \partial(C_{\{v',u\}}),
$$

$$
\partial(\alpha(t_v)) = \sum_{i=1}^m \tau_i \partial(t_i) + \sum_{r \in V(\mathcal{G})} \tau_{v,r} \partial(t_r) + \sum_{s,t \in V(\mathcal{G})} \tau_{v,\{s,t\}} t_{\{s,t\}} + \partial(C_v),
$$

$$
\partial(\alpha(t_k)) = \sum_{i=1}^m \nu_i \partial(t_i) + \sum_{r \in V(\mathcal{G})} \nu_{k,r} \partial(t_r) + \sum_{s,t \in V(\mathcal{G})} \nu_{k,\{s,t\}} t_{\{s,t\}} + \partial(C_k).
$$

(23)

Next, using (21) we get

$$
\alpha(\partial(t_{\{v',u\}})) = \alpha(y_{\{v',u\}}) = y_{\{\phi(v'),\phi(u)\}}, \quad \alpha(\partial(t_v)) = \alpha(y_v) = y_{\phi(v)}, \quad \alpha(\partial(t_k)) = \alpha(y_k) = y_k
$$

As $\partial \circ \alpha = \alpha \circ \partial$, it follows that $\partial(C_{\{v',u\}}) = \partial(C_v) = \partial(C_k) = 0$ and all the coefficients in the relations (24) are zero except $\sigma_{\{v',u\},\{\phi(v'),\phi(u)\}} = \tau_{v,v} = \nu_k = 1$.

4.2. Construction of the DGL $\mathcal{L}(\mathcal{G},1)$. We extend $\mathcal{L}(\mathcal{G},0)$ by adding generators to obtain the following DGL

$$
\mathcal{L}(\mathcal{G},1) = \mathcal{L}(\mathcal{G},0) \oplus \left( \mathbb{L}(t_{1,1}, \ldots, t_{1,m_1}, t_{1,v}, t_{1,\{v',u\}}; \mid v, v', u \in V(\mathcal{G}), \partial) \right).
$$

The degrees of the generators are as follows

$$
|t_{1,1}| = \cdots = |t_{1,m_1}| = |t_{1,v}| = |t_{1,\{v',u\}}| = 2314, \quad \forall v, v', u \in V(\mathcal{G}).
$$

The differential is given by

$$
\partial(t_{1,1}) = y_{0,1}, \ldots, \partial(t_{1,m_1}) = y_{0,m_0}, \quad \partial(t_{1,v}) = y_{0,v}, \quad \partial(t_{1,\{v',u\}}) = y_{0,\{v',u\}}.
$$

(24)

where $\mathcal{B}_0 = \{y_{0,1}, \ldots, y_{0,m_0}, y_{0,v}, y_{0,\{v',u\}}; v, v', u \in V(\mathcal{G})\}$ as in (19).

**Lemma 4.9.** If $[\alpha] \in \mathcal{E}(\mathcal{L}(\mathcal{G},1))$, then there exists a unique $\phi \in \text{aut}(\mathcal{G})$ such that

$$
\alpha(t_{1,\{v',u\}}) = t_{1,\{\phi(v'),\phi(u)\}} + C_{1,\{v',u\}}, \quad \forall v', u \in V(\mathcal{G})
$$

$$
\alpha(t_{1,v}) = t_{1,v} + C_{1,v}, \quad \forall v \in V(\mathcal{G})
$$

$$
\alpha(t_{1,k}) = t_{1,k} + C_{1,k}, \quad \forall k = 1, \ldots, m_1.
$$

$$
\alpha(z_{v,u}) = z_{\{\phi(v),\phi(u)\}} + B_{v,u}, \quad \forall (v,u) \in E(\mathcal{G}),
$$

$$
\alpha(x_i) = x_{\phi(v)}, \quad \forall v \in V(\mathcal{G}),
$$

$$
\alpha(w_i) = w_i, \quad i = 1, 2, 3, 4, 5, 6.
$$

where $C_{1,\{v',u\}}, C_{1,v}, C_{1,k}$ are cycles of degree 2314.

**Proof.** First, upon invoking Lemma 4.8 and considering Remark 4.5, we get

$$
\alpha(t_{1,\{v',u\}}) = t_{1,\{\phi(v'),\phi(u)\}} + C_{1,\{v',u\}}, \quad \forall v', u \in V(\mathcal{G})
$$

$$
\alpha(t_{1,v}) = t_{1,v} + C_{1,v}, \quad \forall v \in V(\mathcal{G})
$$

$$
\alpha(t_{1,k}) = t_{1,k} + C_{1,k}, \quad \forall k = 1, \ldots, m_1.
$$

Next, in one hand, by Proposition 4.4, we know that $\alpha(w_0) = w_0 + F$, where the cycle $F$ is a linear combination of the elements of the base $\mathcal{B}_0$. On the other hand, by (21), each element of $\mathcal{B}_0$ is a boundary. Thus, by Lemma 2.11 and the relation (24), the DGL-map $\alpha$ can be chosen, up to homotopy, such that $\alpha(w_0) = w_0$.

**Lemma 4.10.** The sub-module $Z_{2314}(\mathcal{L}(\mathcal{G},1))$ is not trivial.

Proof. It is easy to check that the two following brackets
\[
[w_3, w_3, (w_2, (\text{ad } w_1)^{14}(w_2))], \quad [x_v, [w_3, [w_3, (\text{ad } w_1)^8(w_2)]]]
\]
are cycles of degree 2314.

\[\square\]

Remark 4.11. Since \(C_{1,v}, C_{1,u}\) and \(C_{1,k}\) are cycles of degree 2314, by Remark \(4.5\) we can write each of them as a linear combination of the elements of \(B_1\).

Using the preceding process, for every \(2 \leq s \leq 24\), we construct a DGL denoted by
\[
\mathcal{L}(G,s) = \mathcal{L}(G,s-1) \oplus \left( \mathcal{L}(t_{s,1}, \ldots , t_{s,m_s-1}, t_{s,v}, t_{s,\{v',u\}}; \ | \ v, v', u \in V(G), \partial ) \right).
\]
that fulfill the following properties for all \(v, v', u \in V(G)\).

\[\begin{align*}
(1) & \quad |t_{s,1}| = \cdots = |t_{s,m_s}| = |t_{s,v}| = |t_{s,\{v',u\}}| = 2313 + s, \\
(2) & \quad \partial(t_{s,1}) = y_{s-1,1}, \ldots , \partial(t_{s,m_s-1}) = y_{s-1,m_s-1}, \quad \partial(t_{s,v}) = y_{s-1,v}, \quad \partial(t_{s,\{v',u\}}) = y_{s-1,\{v',u\}}. \\
(3) & \quad \text{All the cycles } y_{s-1,1}, \ldots , y_{s-1,m_s}; y_{s-1,v}; y_{s-1,\{v',u\}} \text{ form the base } B_s \text{ of the sub-module } Z_{2313+s}(\mathcal{L}(G,s)) \text{ as it is mentioned in Remark } 4.5. \\
(4) & \quad \text{If } [\alpha] \in \mathcal{L}(\mathcal{L}(G,24)), \text{ then there exists a unique } \phi \in \text{aut}(G) \text{ such that }
\begin{align*}
\alpha(t_{s,\{v',u\}}) &= t_{s,\{\phi(v'),\phi(u)\}}, & \forall v', u \in V(G) \\
\alpha(t_{s,v}) &= t_{s,\phi(v)}, & \forall v \in V(G) \\
\alpha(t_{s,k}) &= t_{s,k}, & \forall k = 1, \ldots , m_s. \\
\alpha(z_{v,u}) &= z_{\phi(v),\phi(u)} + B_{(v,u)}, & \forall (v,u) \in E(G), \\
\alpha(x_v) &= x_{\phi(v)}, & \forall v \in V(G), \\
\alpha(w_i) &= w_i, & i = 1, 2, 3, 4, 5, 6.
\end{align*}
\]

Lemma 4.12. The sub-module \(Z_{2337}(\mathcal{L}(G,24))\) is not trivial.

Proof. Indeed, for instance, it is easy to check that the following brackets
\[
[[x_v, x_u], [[x_2, x_4], x_2, [x_2, x_3]]], \quad [[[w_2, w_4], (\text{ad } w_1)^6((\text{ad } w_2)^3(x_v))], \\
[[w_3, w_5], [w_2, w_4], [w_3, (\text{ad } w_1)^5(w_3)]]],
\]
are cycles of degree 2337.

\[\square\]

Corollary 4.13. As \(B_{(v,u)}\) in \(\mathcal{L}(G,24)\) is a cycle, according to Proposition \(4.3\) it can be written as a linear combination of elements in \(B_{24}\).

4.3. Construction of the DGL \(\mathcal{L}(G,25)\). We extend \(\mathcal{L}(G,24)\) by adding generators to define the following DGL
\[
\mathcal{L}(G,25) = \mathcal{L}(G,24) \oplus \left( \mathcal{L}(t_{25,1}, \ldots , t_{25,m_{24}}, t_{25,v}, t_{25,\{v',u\}}; \ | \ v, v', u \in V(G), \partial ) \right).
\]
The degrees of the generators are as follows
\[
|t_{25,1}| = \cdots = |t_{25,m_{24}}| = |t_{25,v}| = |t_{25,\{v',u\}}| = 2338, \quad \forall v, v', u \in V(G).
\]
The differential is given by
\[
\partial(t_{24,1}) = y_{23,1}, \ldots , \partial(t_{24,m_{23}}) = y_{23,m_{23}}
\]
\[
\partial(t_{25,v}) = y_{24,v}, \quad \partial(t_{25,\{v',u\}}) = y_{24,\{v',u\}},
\]
where the cycles \(y_{24,1}, \ldots , y_{24,m_{24}}, y_{24,v}, y_{24,\{v',u\}}\) form a basis of \(B_{23}\) as in \(\text{[13]}\).

Lemma 4.14. The sub-module \(Z_{2338}(\mathcal{L}(G,25))\) is not trivial.
\begin{proof}
Obviously, $\left[ w_3, (\text{ad } w_2)^3(w_4) \right], \left[ w_2, w_4 \right], \left[ w_3, (\text{ad } w_1)^5(w_2) \right]$ is a 2338-cycle. \qed
\end{proof}

**Lemma 4.15.** If $[\alpha] \in \mathcal{E}(\mathcal{L}(G, 25))$, then there is a unique $\phi \in \text{aut}(G)$ such that
\[
\alpha(t_{25}, \{v^\prime, u\}) = t_{25}, \{\phi(v'), \phi(u)\} + C_{25}, \{v^\prime, u\}, \quad \forall v', u \in V(G)
\]
\[
\alpha(t_{25}, v) = t_{25}, \phi(v) + C_{25}, v, \quad \forall v \in V(G)
\]
\[
\alpha(t_{25}, k) = t_{25}, k + C_{25}, k, \quad \forall k = 1, \ldots, m_{25}.
\]
\[
\alpha(z_{(v, u)}) = z_{(\phi(v), \phi(u))}, \quad \forall (v, u) \in E(G),
\]
\[
\alpha(x_v) = x_{\phi(v)}, \quad \forall v \in V(G),
\]
\[
\alpha(w_i) = w_i, \quad i = 1, 2, 3, 4, 5, 6.
\]
where $C_{25}, \{v^\prime, u\}, C_{25}, v, C_{25}, k$ are 2338-cycles.

\begin{proof}
First, if we apply Lemma 4.8 then we get
\[
\alpha(t_{24}, \{v^\prime, u\}) = t_{24}, \{\phi(v'), \phi(u)\} + C_{24}, \{v^\prime, u\}, \quad \forall v', u \in V(G)
\]
\[
\alpha(t_{24}, v) = t_{24}, \phi(v) + C_{24}, v, \quad \forall v \in V(G)
\]
\[
\alpha(t_{24}, k) = t_{24}, k + C_{24}, k, \quad \forall k = 1, \ldots, m_{24}.
\]
Next, by Corollary 4.13 we know that $B_{(v, u)}$ is the linear combination of the elements of $B_{24}$. But, by 20, each element of $B_{24}$ is a boundary. Thus, by Lemma 2.1 and (25), the DGL-map $\alpha$ can be chosen, up to homotopy, such that $\alpha(z_{(v, u)}) = z_{(\phi(v), \phi(u))}$. \qed
\end{proof}

**Remark 4.16.** Since $C_{24}, \{v^\prime, u\}, C_{24}, v$ and $C_{24}, k$ are 2338-cycles, by Remark 4.5 we can write each of them as a linear combination of the elements of $B_{25}$.

### 4.4. Construction of the DGL $\mathcal{L}(G, 26)$

We extend $\mathcal{L}(G, 25)$ by adding generators to define the following DGL
\[
\mathcal{L}(G, 26) = \mathcal{L}(G, 25) \oplus \left( \mathbb{L}(t_{26,1}, \ldots, t_{26,m_{25}}, t_{26,v}, t_{26,\{v^\prime, u\}} \mid v, v', u \in V(G), \partial \right).
\]
The degrees of the generators are as follows
\[
|t_{26,1}| = \cdots = |t_{26,m_{25}}| = |t_{26,v}| = |t_{26,\{v^\prime, u\}}| = 2339, \quad \forall v, v', u \in V(G).
\]
The differential is given by
\[
\partial(t_{26,1}) = y_{25,1}, \ldots, \partial(t_{26,m_{25}}) = y_{25,m_{25}}
\]
\[
\partial(t_{26,v}) = y_{25,v}, \quad \partial(t_{26,\{v^\prime, u\}}) = y_{25,\{v^\prime, u\}}.
\]
where the cycles $y_{25,1}, \ldots, y_{25,m_{25}}, y_{25,v}, y_{25,\{v^\prime, u\}}$ form a basis of $B_{25}$ as in 19.

**Lemma 4.17.** The sub-module $Z_{2339}(\mathcal{L}(G, 26))$ is not trivial.

\begin{proof}
Obviously, $\left[ [w_3, [w_2, w_4]], [w_2, w_4], [w_3, (\text{ad } w_1)^5(w_2)] \right]$ is a 2339-cycle. \qed
\end{proof}

**Lemma 4.18.** Let $[\alpha] \in \mathcal{E}(\mathcal{L}(G, 26))$. There is a unique $\phi \in \text{aut}(G)$ such that
\[
\alpha(t_{26}, \{v^\prime, u\}) = t_{26}, \{\phi(v'), \phi(u)\} + C_{26}, \{v^\prime, u\}, \quad \forall v', u \in V(G)
\]
\[
\alpha(t_{26}, v) = t_{26}, \phi(v) + C_{26}, v, \quad \forall v \in V(G)
\]
\[
\alpha(t_{26}, l) = t_{26}, l + C_{26}, l, \quad \forall l = 1, \ldots, m_{25}.
\]
\[
\alpha(t_{25}, \{v^\prime, u\}) = t_{25}, \{\phi(v'), \phi(u)\}, \quad \forall v', u \in V(G)
\]
\[
\alpha(t_{25}, v) = t_{25}, \phi(v), \quad \forall v \in V(G)
\]
\[
\alpha(t_{25}, k) = t_{25}, k, \quad \forall k = 1, \ldots, m_{24}.
\]
where $C_{26}, \{v^\prime, u\}, C_{26}, v, C_{26}, l$ are 2339-cycles.
Proof. First, if we apply Lemma \textbf{4.8} then we get

\[
\alpha(t_{26,(v',u)}) = t_{26,(\phi(v'),\phi(u))} + C_{26,(v',u)}, \quad \forall v', u \in V(G)
\]
\[
\alpha(t_{26,v}) = t_{26,\phi(v)} + C_{26,v}, \quad \forall v \in V(G)
\]
\[
\alpha(t_{26,l}) = t_{26,l} + C_{26,l}, \quad \forall l = 1, \ldots, m_{25}
\]

Next, from Lemma \textbf{4.18} we know that \(C_{25,(v,u)}\), \(C_{25,v}\) and \(C_{25,k}\) are cycles which can be written as linear combinations of the elements in \(B_{25}\) due to Remark \textbf{4.5}. But, by \textbf{27}, each element of \(B_{25}\) is a boundary. Thus, by Lemma \textbf{2.1} the DGL-map \(\alpha\) can be chosen, up to homotopy, such that for every \(v, v', u \in V(G)\) and \(1 \leq k \leq m_{24}\) we have

\[
\alpha(t_{24,(v',u)}) = t_{24,(\phi(v'),\phi(u))}, \quad \alpha(t_{24,v}) = t_{24,\phi(v)}, \quad \alpha(t_{24,l}) = t_{24,l}.
\]

as desired. \(\square\)

4.5. Construction of \(\mathcal{L}(G, 27)\). Define

\[
\mathcal{L}(G, 27) = \mathcal{L}(G, 26) \oplus \left( \mathbb{L}(t_{27,1}, \ldots, t_{27,m_{26}}, t_{26,v}, t_{27,(v,u)} \mid v, v', u \in V(G)), \partial \right).
\]

The degrees of the generators are as follows

\[
|t_{27,1}| = \cdots = |t_{27,m_{26}}| = |t_{27,v}| = |t_{27,(v,u)}| = 2340, \quad \forall v, v', u \in V(G).
\]

The differential is given by

\[
\partial(t_{27,1}) = y_{26,1}, \ldots, \partial(t_{27,m_{26}}) = y_{27,m_{26}}
\]

\[
\partial(t_{27,v}) = y_{26,v}, \quad \partial(t_{27,(v,u)}) = y_{26,(v',u)}.
\]

where the cycles \(y_{26,1}, \ldots, y_{26,m_{26}}, y_{26,v}, y_{26,(v,u)}\) form a basis of \(B_{26}\) as in \textbf{19}.

Lemma 4.19. Let \([\alpha] \in \mathcal{E}(\mathcal{L}(G, 27))\). There is a unique \(\phi \in \text{aut}(G)\) such that

\[
\alpha(t_{27,(v',u)}) = t_{27,(\phi(v'),\phi(u))} + C_{27,(v',u)}, \quad \forall v', u \in V(G)
\]
\[
\alpha(t_{27,v}) = t_{27,\phi(v)} + C_{27,v}, \quad \forall v \in V(G)
\]
\[
\alpha(t_{27,l}) = t_{27,l} + C_{27,l}, \quad \forall l = 1, \ldots, m_{26}
\]

\[
\alpha(t_{26,(v',u)}) = t_{26,(\phi(v'),\phi(u))}, \quad \forall v', u \in V(G)
\]
\[
\alpha(t_{26,v}) = t_{26,\phi(v)}, \quad \forall v \in V(G)
\]
\[
\alpha(t_{26,l}) = t_{26,l}, \quad \forall k = 1, \ldots, m_{25}
\]

where \(C_{27,(v',u)}\), \(C_{27,v}\) and \(C_{27,k}\) are 2340-cycles.

Proof. First, if we apply Lemma \textbf{4.8} then we get

\[
\alpha(t_{27,(v',u)}) = t_{27,(\phi(v'),\phi(u))} + C_{27,(v',u)}, \quad \forall v', u \in V(G)
\]
\[
\alpha(t_{27,v}) = t_{27,\phi(v)} + C_{27,v}, \quad \forall v \in V(G)
\]
\[
\alpha(t_{27,l}) = t_{27,l} + C_{27,l}, \quad \forall l = 1, \ldots, m_{26}
\]

Next, from Lemma \textbf{4.18} we know that \(C_{26,(v,u)}\), \(C_{26,v}\) and \(C_{26,k}\) are cycles which can be written as linear combinations of the elements in \(B_{26}\) due to Remark \textbf{4.5}. But, by \textbf{28}, each element of \(B_{26}\) is a boundary. Thus, by Lemma \textbf{2.1} the DGL-map \(\alpha\) can be chosen, up to homotopy, such that for every \(v, v', u \in V(G)\) and \(1 \leq k \leq m_{25}\) we have

\[
\alpha(t_{25,(v',u)}) = t_{25,(\phi(v'),\phi(u))}, \quad \alpha(t_{25,v}) = t_{25,\phi(v)}, \quad \alpha(t_{25,k}) = t_{25,k}.
\]

as wanted. \(\square\)
The goal of this paragraph is to show that $Z_{2340}(\mathcal{L}(\mathcal{G}, 26))$ is trivial. Indeed, let
\[ B(w_1^{(k_1)}, w_2^{(k_2)}, w_3^{(k_3)}, w_4^{(k_4)}, w_5^{(k_5)}, x_v^{(k_v)}, x_u^{(k_u)}) \],
\[ v \neq u, \]
denote the set of all the brackets of the Hall basis of the DGL $\mathcal{L}(\mathcal{G}, 1)$ formed exactly using $k_1$ generators $w_1$, $k_2$ generators $w_2$, $k_3$ generators $w_3$, $k_4$ generators $w_4$, $k_5$ generators $w_5$, $k_u$ generators $x_v$ and $k_u$ generators $x_u$. For instance, the bracket
\[ [[x_v, x_u], [[x_v, x_u], [x_v, x_u]]] \in B(w_1^{(0)}, w_2^{(0)}, w_3^{(1)}, w_4^{(1)}, w_5^{(0)}, x_v^{(1)}, x_u^{(1)}). \]

**Lemma 4.20.** We have the following two statements.

1. **Any non-zero linear combination of elements of the set**
\[ B(w_1^{(k_1)}, w_2^{(1)}, w_3^{(k_3)}, w_4^{(2)}, w_5^{(0)}, x_v^{(k_v)}, x_u^{(k_u)}), \]
**cannot be a cycle.**

2. **Any non-zero linear combination of elements of the set**
\[ B(w_1^{(5)}, w_2^{(3)}, w_3^{(0)}, w_4^{(3)}, w_5^{(1)}, x_v^{(0)}, x_u^{(0)}), \]
**cannot be a cycle.**

**Proof.** First for the assertion (1), by expanding an arbitrary non-zero element
\[ l \in B(w_1^{(k_1)}, w_2^{(1)}, w_3^{(k_3)}, w_4^{(2)}, w_5^{(0)}, x_v^{(k_v)}, x_u^{(k_u)}), \]
in $T(w_1, w_2, w_3, w_4, \{x_v\}_{v \in V(\mathcal{G})})$, we can write $l$ as the following linear combination
\[ l = c_1 w_1^2 A_1 + c_2 w_1 w_4 A_2 + c_3 A_3 w_4 A_4 + w_1 c_4 A_5 w_4 A_6 + c_5 A_8 w_4^2 + c_6 w_3 w_4 A_10, \]
where $c_1, c_2, c_3, c_4, c_5, c_6 \in Z_{(p)}$ and where $A_1, \ldots, A_{10}$ are (non constant) expressions belonging to $T(w_1, w_2, w_3, \{x_v\}_{v \in V(\mathcal{G})})$. In other words, $A_1, \ldots, A_{10}$ are expressions of monomials composed of the generators $w_1, w_2, w_3$ and $x_v$, where $v \in V(\mathcal{G})$. Therefore, using the relations (30) we get
\[ \partial(c_1 w_1^2 A_1) = 2c_1 w_1 w_4 A_1 + 2c_1 w_4 w_1^2 A_1, \]
\[ \partial(c_2 w_1 w_4 A_2) = 2c_2 w_1 w_4 A_2 + 2c_2 w_4 w_1^2 A_2, \]
\[ \partial(c_3 A_3 w_4 A_4) = 2c_3 A_3 w_4 A_4 + 2c_3 A_4 w_4 w_4^2, \]
\[ \partial(c_4 A_5 w_4 A_6) = 2c_4 A_5 w_4 A_6 + 2c_4 A_6 w_4 w_4^2, \]
\[ \partial(c_5 A_8 w_4^2) = 2c_5 A_8 w_4^2 + 2c_5 A_8 w_4^2, \]
\[ \partial(c_6 w_3 w_4 A_10) = 2c_6 w_3 w_4 A_10 + 2c_6 w_3 w_4 w_4^2. \]

Clearly, as the generator $w_4$ is not involved in the expressions of $A_1, \ldots, A_{10}$, it follows that $2c_1 w_4 w_4^2 A_1$ does not appear in other expressions in (29). This implies that $c_1 = 0$. Likewise, the expression $c_2 w_1 w_4^2$ does not appear in the other expressions in (29). So $c_2 = 0$. Repeating the same argument, we conclude that if $\partial(l) = 0$, then $c_1 = c_2 = c_3 = c_4 = c_5 = c_6 = 0$.

Next, for the assertion (2), by using the same argument as in the first assertion and expanding an arbitrary non-zero element
\[ l' \in B(w_1^{(5)}, w_2^{(3)}, w_3^{(0)}, w_4^{(3)}, w_5^{(1)}, x_v^{(0)}, x_u^{(0)}), \]
in $T(w_1, w_2, w_3, w_4, w_5, \{x_v\}_{v \in V(\mathcal{G})})$ in this case, we can write $l'$ as a linear combination
\[ l' = b_1 w_1^2 B_1 + b_2 w_1 w_4 B_2 + b_3 B_3 w_4 B_4 + b_4 B_5 w_4 B_6 w_4 B_7 + b_5 B_8 w_4^2 + b_6 w_4 B_9 w_4 B_{10}, \]
where $b_1, b_2, b_3, b_4, b_5, b_6 \in \mathbb{Q}$ and where $B_1, \ldots, B_{10}$ are (non constant) expressions of monomials composed of the generators $w_1, w_2, w_3, w_4$ and $x_v$, where $v \in V(G)$. Therefore, using (6) we obtain
\begin{equation}
\begin{aligned}
\partial(b_1w_4^2B_1) &= 2b_1w_4^2w_1B_1 \pm 2b_1w_4w_2B_1 \pm b_1w_4w_2^2\partial(B_1), \\
\partial(b_2w_4B_2w_4) &= 2b_2w_4^2B_2w_4 \pm b_2w_4\partial(B_2)w_4 \pm 2b_2w_4B_2w_4^2, \\
\partial(b_3B_3w_4B_4w_4) &= b_3\partial(B_3)w_4B_4w_4 \pm 2b_3B_3w_2B_4w_4 \pm b_3B_3w_4\partial(B_4)w_4 \pm 2b_3B_3w_4B_4w_4^2, \\
\partial(b_4B_5w_4B_6w_4B_7) &= b_4\partial(B_5)w_4B_6w_4B_7 \pm 2b_4B_5w_4^2B_6w_4B_7 \pm b_4B_5w_4\partial(B_6)w_4B_7 \pm 2b_4B_5w_4B_6w_4B_7 \pm b_4B_5w_4B_6w_4\partial(B_7), \\
\partial(b_5B_9w_4) &= b_5\partial(B_9)w_4 \pm 2b_5B_9w_2^2w_4 \pm b_5B_9w_4w_2^2, \\
\partial(b_6B_9w_4B_{10}) &= 2b_6w_4B_9w_4B_{10} \pm b_6\partial(B_9)w_4B_{10} \pm 2b_6w_4B_9w_2B_{10} \pm b_6w_4B_9w_4\partial(B_{10}).
\end{aligned}
\end{equation}

Likewise, as the generator $w_4$ is not involved in the expressions of $B_1, \ldots, B_{10}$ and $\partial(B_1), \ldots, \partial(B_{10})$, it follows that $2b_1w_4^2B_1$ does not appear in the other expressions in (6). This implies that $b_1 = 0$. Next, the expression $2b_2w_4B_2w_4^2$ does not appear in the other expressions in (30). So $b_2 = 0$. Repeating the same argument, we conclude that if $\partial(l') = 0$, then $b_3 = b_4 = b_5 = b_6 = 0$.

\textbf{Lemma 4.21.} The module $\mathcal{L}_{2340}(G, 27)$ is spanned by brackets belonging to the following two sets
\begin{equation}
\begin{aligned}
\mathcal{B}(w_1^{(5)}, w_2^{(1)}, w_3^{(2)}, w_4^{(4)}, w_5^{(0)}, x_v^{(0)}, x_u^{(0)}) \quad \mathcal{B}(w_1^{(5)}, w_2^{(3)}, w_3^{(0)}, w_4^{(3)}, w_5^{(1)}, x_v^{(0)}, x_u^{(0)}).
\end{aligned}
\end{equation}

\textbf{Proof.} First, recall that, by construction, the DGL $\mathcal{L}(G, 26)$ is formed by elements of degrees
\begin{equation}
\begin{aligned}
115, 151, 201, 303, 403, 690, q, \quad 2313 \leq q \leq 2340.
\end{aligned}
\end{equation}
Therefore, for degree reasons, there is no bracket in $\mathcal{L}_{2340}(G, 26)$ formed with a generator of degree $q$. Next, if $\Theta \in \mathcal{L}_{2340}(G, 3)$, then we get
\begin{equation}
|\Theta| = 115a_1 + 151a_2 + 201a_3 + 303a_4 + 403a_5 + 690a_6, \quad \text{where} \quad a_1, \ldots, a_6 \in \{0, 1, 2, \ldots\} \quad \text{yielding the following Frobenius equation}
\end{equation}
\begin{equation}
115a_1 + 151a_2 + 201a_3 + 303a_4 + 403a_5 + 690a_6 = 2340.
\end{equation}
Using WOLFRAM software, we get only two solutions which are providing only two solutions which are
\begin{equation}
\begin{aligned}
a_1 &= 5, \quad a_2 = 1, \quad a_3 = 2, \quad a_4 = 4, \quad a_5 = 0, \quad a_6 = 0, \\
a_1 &= 5, \quad a_2 = 3, \quad a_3 = 0, \quad a_4 = 3, \quad a_5 = 1, \quad a_6 = 0
\end{aligned}
\end{equation}
as desired.

\textbf{Lemma 4.22.} The submodule $Z_{2340}(\mathcal{L}(G, 27))$ is trivial.

\textbf{Proof.} First, according to Lemma 4.21, $\mathcal{L}_{2340}(G, 26)$ is spanned by brackets belonging to the sets (6). Next, by Lemma 4.20 any linear combination of elements of the set $\mathcal{B}(w_1^{(5)}, w_2^{(1)}, w_3^{(2)}, w_4^{(4)}, w_5^{(0)}, x_v^{(0)}, x_u^{(0)})$ or $\mathcal{B}(w_1^{(5)}, w_2^{(3)}, w_3^{(0)}, w_4^{(3)}, w_5^{(1)}, x_v^{(0)}, x_u^{(0)})$ cannot be a cycle. Finally, let
\begin{equation}
\theta = (\gamma_1l_1 + \cdots + \gamma_kl_k) + (\mu_1h_1 + \cdots + \mu_kh_k), \quad \gamma_1, \ldots, \gamma_k, \mu_1, \ldots, \mu_k \in \mathbb{Z}[\mu],
\end{equation}
Theorem 4.24. The map $\gamma$ is a linear combination such that
\[
l_1, \ldots, l_{k_1} \in B(w^{(5)}_1, w^{(1)}_2, w^{(2)}_3, w^{(4)}_4, w^{(5)}_5, x^{(0)}_v, x^{(0)}_u),
\]
\[
h_1, \ldots, h_{k_2} \in B(w^{(5)}_1, w^{(2)}_2, w^{(3)}_3, w^{(3)}_4, w^{(1)}_5, x^{(0)}_v, x^{(0)}_u).
\]
Applying the differential $\partial$ we get
\[
\partial(\theta) = \gamma_1 \partial(l_1) + \cdots + \gamma_{k_1} \partial(l_{k_1}) + \mu_1 \partial(h_1) + \cdots + \mu_{k_2} \partial(h_{k_2}).
\]
Let's write $\partial(h_j) = h_{j,1} + h_{j,2}$, where
\[
\partial(l_1), \ldots, \partial(l_{k_1}) \in B(w^{(5)}_1, w^{(2)}_2, w^{(3)}_3, w^{(3)}_4, w^{(1)}_5, x^{(0)}_v, x^{(0)}_u),
\]
\[
h_{1,1}, \ldots, h_{k_2,1} \in B(w^{(5)}_1, w^{(2)}_2, w^{(3)}_3, w^{(3)}_4, w^{(1)}_5, x^{(0)}_v, x^{(0)}_u),
\]
\[
h_{1,2}, \ldots, h_{k_2,2} \in B(w^{(5)}_1, w^{(2)}_2, w^{(3)}_3, w^{(3)}_4, w^{(1)}_5, x^{(0)}_v, x^{(0)}_u).
\]
All the brackets $h_{1,1}, \ldots, h_{k_2,1}$ are formed using only one generator $w_5$, while in the other elements $w_5$ is not involved, so if $\theta$ is a cycle, then $\mu_1 = \cdots = \mu_{k_2} = 0$ implying also that $\gamma_1 = \cdots = \gamma_{k_1} = 0$. Hence, the only 2340-cycle in $L(G, 27)$ is zero.

Corollary 4.23. The cycles $C_{27,1}, C_{27,2}, C_{27,3}$, given in Lemma 4.19 are trivial.

Proof. It suffices to apply Lemma 4.22.

As a consequence of Remark 4.11, Lemmas 4.15, 4.18, 4.19 and Corollary 4.23, we can define a group homomorphism $\Psi(L(G, 27)) \to \text{aut}(G)$ by setting $\Psi([\alpha]) = \phi$.

Theorem 4.24. The map $\Psi$ is an isomorphism of groups.

Proof. For every $\sigma \in \text{aut}(G)$, we define
\[
\alpha_\sigma(t_{s, k}) = t_{s, k}, \quad \forall k = 1, \ldots, m_{s-1}, \quad \forall s = 1, \ldots, 27,
\]
\[
\alpha_\sigma(t_{s, \{v, u\}}) = t_{s, \{\sigma(v), \sigma(u)\}}, \quad \forall v, u \in V(G), \quad \forall s = 1, \ldots, 27,
\]
\[
\alpha_\sigma(t_{s, v}) = t_{s, \sigma(v)}, \quad \forall v \in V(G), \quad \forall s = 1, \ldots, 27,
\]
\[
\alpha_\sigma(z_{(v, u)}) = z_{(\sigma(v), \sigma(u))}, \quad \forall (v, u) \in E(G),
\]
\[
\alpha_\sigma(x_v) = x_{\sigma(v)}, \quad \forall v \in V(G),
\]
\[
\alpha_\sigma(w_i) = w_i, \quad i = 1, 2, 3, 4, 5, 6.
\]
Clearly, we have $\partial \circ \alpha_\sigma = \alpha_\sigma \circ \partial$ implying that $[\alpha_\sigma] \in E(L(G, 27))$. Hence, we get a map
\[
\Phi : \text{aut}(G) \to E(L(G, 27)), \quad \Phi(\sigma) = [\alpha_\sigma],
\]
and it is easy to check that it is the inverse of $\Psi$. Finally, $\Phi$ is a homomorphism of groups because we have $\Phi(\sigma_1 \circ \sigma_2) = [\alpha_{\sigma_1 \circ \sigma_2}] = [\alpha_{\sigma_1} \circ [\alpha_{\sigma_2}], \Phi(\sigma_1) \circ \Phi(\sigma_2)$, for all $\sigma_1, \sigma_2 \in \text{aut}(G)$.

As a consequence of Theorem 4.24 we derive the following result.

Theorem 4.25. For any group $G$ and any prime $p > 1114$, there exists a CW-complex $X$ such that $G \cong E(X(p))$, where $X(p)$ is the $p$-localization of $X$. More precisely:

1. $X$ is an 116-connected, 2341-dimensional, and of finite type if $G$ is finite.
2. $H_i(X, Z(p))$ is a free $Z(p)$-modules over a basis which in bijection with $G$ for $i = 691, q, \quad 2314 \leq q \leq 2341$.
3. $H_i(X, Z(p)) \cong Z(p)$, for $i \in \{116, 152, 202, 304, 404, 2314\}$. 
Proof. First, by Theorem 2.2, to the group $G$ corresponds a strongly connected digraph $\mathcal{G}$ such that $\text{aut}(\mathcal{G}) \cong G$. Next, to the graph $\mathcal{G}$, we can assign the DGL $L(\mathcal{G}, 27)$. Then, by Theorem 4.24, we get $E(L(\mathcal{G}, 27)) \cong \text{aut}(\mathcal{G})$. Finally, using the Anick’s $\mathbb{Z}(p)$-local homotopy theory framework, to $L(\mathcal{G}, 27)$ corresponds an object $X$ in the category $\text{DGL}_{115}(\mathbb{Z}(p))$ satisfying $E(X) \cong E(L(\mathcal{G}, 27))$ according to the identifications (1). Consequently,
$$E(X) \cong E(L(\mathcal{G}, 27)) \cong \text{aut}(\mathcal{G}) \cong G,$$
as desired. \hfill \Box

Remark 4.26. By virtue of Anick’s theory, the generators of the DGL $L(\mathcal{G}, 27)$ are in correspondence with the $\mathbb{Z}(p)$-localised cells of the CW-complex $X$ constructed in Theorem 4.25. Thus, $X$ is finite if and only if the group $G$ is finite.

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