TWO-POINT CORRELATION FUNCTIONS AND UNIVERSALITY FOR THE ZEROS OF SYSTEMS OF $SO(n+1)$-INVARIANT GAUSSIAN RANDOM POLYNOMIALS

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ABSTRACT. We study the two-point correlation functions for the zeroes of systems of $SO(n+1)$-invariant Gaussian random polynomials on $\mathbb{R}^n$ and systems of $\text{Isom}(\mathbb{R}^n)$-invariant Gaussian analytic functions. Our result reflects the same "repelling," "neutral," and "attracting" short-distance asymptotic behavior, depending on the dimension, as was discovered in the complex case by Bleher, Shiffman, and Zelditch. For systems of the $\text{Isom}(\mathbb{R}^n)$-invariant Gaussian analytic functions we also obtain a fast decay of correlations at long distances.

We then prove that the correlation function for the $\text{Isom}(\mathbb{R}^n)$-invariant Gaussian analytic functions is "universal," describing the scaling limit of the correlation function for the restriction of systems of the $SO(k+1)$-invariant Gaussian random polynomials to any $n$-dimensional $C^2$ submanifold $M \subset \mathbb{R}^k$. This provides a real counterpart to the universality results that were proved in the complex case by Bleher, Shiffman, and Zelditch. (Our techniques also apply to the complex case, proving a special case of the universality results of Bleher, Shiffman, and Zelditch.)

1. INTRODUCTION

This paper concerns the $SO(n+1)$-invariant ensemble of Gaussian random polynomials on $\mathbb{R}^n$ and the $\text{Isom}(\mathbb{R}^n)$-invariant ensemble of Gaussian random analytic functions on $\mathbb{R}^n$. The $SO(n+1)$-invariant ensemble consists of random polynomials of the form:

$$F(X) := \sum_{|\alpha|=d} \sqrt{\frac{d!}{\alpha!}} a_\alpha X^\alpha,$$

where $X \in \mathbb{R}^{n+1}$ and the $a_\alpha$ are independent and identically distributed (iid) on the standard normal distribution, $\mathcal{N}(0,1)$. Here, we use the following multi-index notation: for any $\alpha \in (\mathbb{Z}_{\geq 0})^{n+1}$, one defines:

$$X^\alpha := \prod_{i=1}^{n+1} X_i^{\alpha_i}, \quad |\alpha| := \sum_{i=1}^{n+1} \alpha_i \quad \text{and} \quad \left(\frac{d!}{\alpha!}\right) = \frac{d!}{\prod_{j=1}^{n+1} \alpha_j!}.$$

We will study the simultaneous zeroes on the projective space $\mathbb{P}^n$ of the systems:

$$F : \mathbb{R}^{n+1} \to \mathbb{R}^n \quad \text{where} \quad F = (F_1(X), F_2(X), \ldots, F_n(X)),$$

where each $F_i$ is an independently chosen random function of the form in Equation (1). Almost surely, the common zero set of $F$ will be finitely many points. We equip $\mathbb{R}^n$ with the Riemannian metric obtained from its double cover by the unit sphere $S^n \subset \mathbb{R}^{n+1}$. The simultaneous zeroes of ensemble (4) are invariant under the isometries by elements of $SO(n+1)$; see Section 2. Because of this symmetry, authors have described this ensemble as the "most natural" ensemble of a random polynomials defined on $\mathbb{R}^n$. For this reason, it has been extensively studied by Kostlan-Edelman$[10]$, Shub-Smale$[22]$, and others.

The $\text{Isom}(\mathbb{R}^n)$-invariant ensemble of Gaussian random analytic functions is defined by the following:

$$f : \mathbb{R}^n \to \mathbb{R}^n \quad \text{where} \quad f = (f_1(x), f_2(x), \ldots, f_n(x)),$$

where $a_\alpha$ are iid on the standard normal distribution, $\mathcal{N}(0,1)$. We will show in Section 2 that the zeroes of this ensemble are invariant under all isometries of $\mathbb{R}^n$. We will see shortly that this ensemble is intimately tied to the $SO(n+1)$-invariance ensemble in the scaling limit as the degree $d \to \infty$.

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The probability density of the zeros of the system (3) is defined to be

\[ \rho(x) = \lim_{\delta \to 0} \frac{1}{\text{Vol}(N_\delta(x))} \Pr(\exists \text{ a zero of } F \text{ in } N_\delta(x)), \]

where \( N_\delta(x) := \{ y \in \mathbb{R}^n \mid \text{dist}(x, y) < \delta \} \). It follows from the invariance that this ensemble (3) has a constant density of zeroes given by

\[ \rho_d(x) = \pi^{-\frac{n+1}{2}} \Gamma \left( \frac{n+1}{2} \right) d^2 \]

see, for example, [10] Sec. 7.2. Note: the volume of the real projective space is \( \pi^{-\frac{n+1}{2}} \Gamma \left( \frac{n+1}{2} \right)^{-1} \), so the expected number of zeroes is simply \( d^2 \). The analogous definition applies to the ensemble (4) which, because of the invariance under isometries of \( \mathbb{R}^n \) has constant density

\[ \rho(x) = \pi^{-\frac{n+1}{2}} \Gamma \left( \frac{n+1}{2} \right). \]

The correlation function between the zeros of the system (3) at the two points \( x \) and \( y \) in \( \mathbb{R}^n \) is defined to be

\[ K_{n,d}(x, y) := \lim_{\delta \to 0} \frac{\Pr(\exists \text{ a zero of } F \text{ in } N_\delta(x) \text{ and } \exists \text{ a zero of } F \text{ in } N_\delta(y))}{\Pr(\exists \text{ a zero of } F \text{ in } N_\delta(x)) \Pr(\exists \text{ a zero of } F \text{ in } N_\delta(y))}. \]

It follows from the \( SO(n+1) \) invariance that \( K_{n,d}(x, y) \) depends only on the distance between \( x \) and \( y \). For this reason, we can write \( K_{n,d}(x, y) \equiv K_{n,d}(t) \), where \( t = \text{dist}_{\mathbb{R}^n}(x, y) \). Similarly, for any \( x, y \in \mathbb{R}^n \), the two point correlation function \( K_{n}(x, y) \) between zeros of (4) depends only on \( \text{dist}_{\mathbb{R}^n}(x, y) \). We have

**Theorem 1.** For any \( x \neq y \in \mathbb{R}^n \), let \( t = \text{dist}_{\mathbb{R}^n}(x, y) \). If \( d \geq 3 \), then the correlation function between zeros of the \( SO(n+1) \)-invariant ensemble satisfies

\[ K_{n,d}(x, y) \equiv K_{n,d}(t) = A_{n,d} t^{2-n} + O(t^{3-n}), \quad \text{where} \quad A_{n,d} = \left( \frac{d-1}{d^2} \right) \frac{\sqrt{\pi} \Gamma \left( \frac{n+2}{2} \right)}{2 \Gamma \left( \frac{n+1}{2} \right)}. \]

**Theorem 2.** For any \( x \neq y \in \mathbb{R}^n \), let \( t = \text{dist}_{\mathbb{R}^n}(x, y) \). The correlation function between zeros of the \( \text{Isom}(\mathbb{R}^n) \)-invariant ensemble satisfies the following short-range asymptotics

\[ K_{n}(x, y) \equiv K_{n}(t) = A_{n} t^{2-n} + O(t^{3-n}), \quad \text{where} \quad A_{n} = \frac{\sqrt{\pi} \Gamma \left( \frac{n+2}{2} \right)}{2 \Gamma \left( \frac{n+1}{2} \right)}, \]

and the following long-range asymptotics:

\[ K_{n}(t) = 1 + O \left( t e^{-\frac{t^2}{2}} \right). \]

It is clear from the proofs of Theorems 1 and 2 that there is a connection between the limit as the degree \( d \to \infty \) for the \( SO(n+1) \)-invariant ensemble and the \( \text{Isom}(\mathbb{R}^n) \)-invariant ensemble. One way to make this precise is by giving \( \mathbb{R}^n \) the metric induced from the double cover by the sphere of radius \( \sqrt{d} \). Denoting the resulting correlation function by \( \tilde{K}_{n,d}(t) \), it is straightforward to show that \( \tilde{K}_{n,d}(t) \) converges to \( K_{n}(t) \) uniformly on compact subsets of \((0, \infty)\).

However, we will consider a more general situation, showing that \( K_{n}(x, y) \) serves as the universal correlation function in the scaling limit \( d \to \infty \) for the restriction of the \( SO(k+1) \)-invariant ensemble to any \( n \)-dimensional \( C^2 \) submanifold \( M \subset \mathbb{R}^k \). To do so, it is more convenient for us to keep the metric on \( \mathbb{R}^k \) fixed and scale the points within the tangent space to \( M \).

Given a \( C^2 \) submanifold \( M \subset \mathbb{R}^k \) having dimension \( n \), the restrictions of \( n \) of the polynomials chosen iid from the \( SO(k+1) \)-invariant ensemble has a well-defined zero set which again consists a.s. of finitely many points. We give \( M \subset \mathbb{R}^k \) the metric induced by the double cover of \( \mathbb{R}^k \) by the unit sphere \( S^k \). More specifically, we obtain a Riemannian metric on \( M \) using the inclusion of tangent spaces \( T_p M \subset T_p \mathbb{R}^k \). When restricted to a sufficiently small neighborhood of the origin, the orthogonal projection \( \text{proj}_p : T_p M \to M \) provides a system of local coordinates on \( M \). We will use these systems of local coordinates to study the correlation between zeros of the restriction of the \( SO(k+1) \) invariant ensemble to \( M \).
Theorem 3. Let \( M \subset \mathbb{R}^k \) be a \( C^2 \) submanifold of dimension \( n \) and \( K_{n,d,M}(x, y) \) denote the correlation function between zeros of \( n \) polynomials chosen iid from the degree \( d \) \( SO(k+1) \) invariant ensemble restricted to \( M \). Then, for any \( p \in M \) and any \( x, y \in T_p M \) we have
\[
K_{n,d,M} \left( \text{proj}_p \left( \frac{x}{\sqrt{d}} \right), \text{proj}_p \left( \frac{y}{\sqrt{d}} \right) \right) = K_n(x, y) + O \left( \frac{1}{\sqrt{d}} \right).
\]
The constant in the estimate is uniform on compact subsets of \( T_p M \times T_p M \setminus \text{Diag} \), where \( \text{Diag} = \{(x, y) \in T_p M \times T_p M : x = y\} \).

Our techniques are largely based on those of Bleher and Di [2], who use the Kac-Rice formula (see Section 3 below) to study the \( n \)-point correlation functions for the \( SO(1,1) \) and \( SO(2) \)-invariant polynomials in one variable. Moreover, our results in the higher dimensional real case yield the exact same short-distance asymptotic behavior (with a different constant) as those of Bleher, Shiffman, and Zelditch [5,6,7] in the complex case. These asymptotic behaviors can be interpreted as “repelling” for \( n = 1 \), “neutral” for \( n = 2 \), and “attracting” for \( n \geq 3 \). See Figure 1 for numerical plots of \( K_n(t) \) for \( n = 1, 2, \) and \( n = 3 \).

We remark that calculation of the leading order asymptotics is more delicate in the real case than in the complex case because one cannot apply Wick’s Theorem to the real Kac-Rice formula.

Theorem 3 above provides a real analog of the celebrated universality results that were obtained in the complex setting by Bleher, Shiffman, and Zelditch [5,6]. Thus, the plots shown in Figure 1 depict the universal scaling limits of the correlation functions for any submanifold \( M \subset \mathbb{R}^k \) of dimension 1, 2, or 3.

The scaling limit used in Theorem 3 is needed to get a universal correlation function. This is illustrated in Section 8 where we show that when restricted to a parabola \( y = bx^2 \) the leading term from the correlation between zeros for the \( SO(3) \)-invariant polynomials of degree 3 near \( x = 0 \) depends non-trivially on \( b \). More generally, it can be interesting to ask how the geometry of \( M \) affects the correlation function \( K \) for finite degree \( d \).

The proof of Theorem 3 easily adapts to complex setting: The \( SU(k+1) \)-invariant ensemble of polynomials are obtained by interpreting the variables in (1) as complex and replacing the real Gaussians \( \alpha_x \) with complex Gaussians. The \( \text{Isom}(\mathbb{C}^n) \)-invariant ensemble of Gaussian analytic functions on \( \mathbb{C}^n \) is obtained by making the same adaptations to (4). We obtain:

Theorem 4. Let \( M \subset \mathbb{C}^k \) be an complex analytic submanifold of dimension \( n \) and \( K_{n,d,M}(x, y) \) denote the correlation function between zeros of \( n \) polynomials chosen iid from the degree \( d \) \( SO(k+1) \) invariant ensemble restricted to \( M \). Then, for any \( p \in M \) and any \( x, y \in T_p M \) we have
\[
K_{n,d,M} \left( \text{proj}_p \left( \frac{x}{\sqrt{d}} \right), \text{proj}_p \left( \frac{y}{\sqrt{d}} \right) \right) = K_n(x, y) + O \left( \frac{1}{\sqrt{d}} \right).
\]
The constant in the estimate is uniform on compact subsets of \( T_p M \times T_p M \setminus \text{Diag} \), where \( \text{Diag} = \{(x, y) \in T_p M \times T_p M : x = y\} \).

This serves as a weaker version of the results from [5,6] in that \( M \) is required to be embedded in projective space (instead of being an arbitrary Kähler manifold), the line bundle is the hyperplane bundle (corresponding to the \( SU(n+1) \)-invariant ensemble), and only two-point correlation functions are considered. On the other hand, in the work of [5,6] the manifold \( M \) is assumed to be compact. No such assumption is made in Theorems 3 and 4. For example, they can be applied at any smooth point of the correlation functions for any submanifold because one cannot apply Wick’s Theorem to the real Kac-Rice formula.

For general background on Gaussian random analytic functions and polynomials, we refer the reader to [13,14,22] and their references therein. Specifically to correlation functions, we refer the reader to the three papers listed above in the previous paragraph, as well as the works of Bogomolny, Bohigas, and Leboeuf [8], Tao and Vu [23], Bleher and Kridal [4], and Bleher and Di [3].

Our work fits in with the context of the emerging field “random real algebraic geometry.” For example, Theorem 3 applies to the restriction of the \( SO(k+1) \) ensemble to the smooth locus of a real-algebraic subset of \( \mathbb{R}^k \). We refer the reader to the works of Ibragimov-Zaporozhets [15], Burgisser [9], Nastasescu [19], Lerario-Lundberg [17], Gayet-Welschinger [12], and Fyodorov-Lerario-Lundberg [11].

The remainder of the paper will be organized as follows: In the following Section 2 we study the invariance properties of the ensembles from (3) and (4). We then use the invariance to reduce Theorems 1 and 2 to suitable versions in affine coordinates (Theorem 3). In Section 3 we recall the Kac-Rice Formulae for the density and the
Figure 1. Universal two-point limiting correlation functions $K_n(t)$ for $n = 1, 2,$ and 3, demonstrating the repelling, neutral and attracting behaviors. For $n = 1$, the graph is obtained from Formula (5.35) in [2]. For $n = 2$ and $n = 3$, the graphs were computed using Monte Carlo integration applied to formula (59) with $10^7$ and $10^6$ points, respectively, for each $t$. The data was smoothed out by replacing each value with the average of it and the 14 nearest neighboring points.
correlation functions, the main tools used in our proof. In Section 4 we compute the covariance matrices needed to prove Theorem 8 as well as their determinants, inverses, etc. Theorem 8 consists of two statements (short-distance asymptotics and long-distance asymptotics), which are proved in Sections 5 and 6 respectively. Section 7 is dedicated to proving Theorem 3 about universality of the scaling limit. Section 8 provides an example showing that for finite degree the leading asymptotics depends on the geometry of the submanifold \( M \subset \mathbb{R}^k \). In Section 9 we explain the changes that need to be made to the proof of Theorem 3 in order to prove the complex version, Theorem 11.

Appendix A contains the proof of a general estimate which is used in Sections 6 and 7. In Appendix B, we prove a result regarding the volume of random parallelotopes which is needed in Section 5.

**Notations:** Let \( \text{diag}_k(A) \) denote the block-diagonal matrix with \( k \) copies of the square matrix \( A \) along the diagonal.

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2. Invariance Properties and Reduction of Theorems 1 and 2 to Local Coordinates

**Lemma 5.** The zeroes of the system \( F \) given in (3) are invariant under the action of \( SO(n+1) \).

**Proof.** Each \( F_i(X) \) defines a Gaussian process on \( \mathbb{R}^{n+1} \), with mean 0 and covariance function

\[
E(F_i(X)F_i(Y)) = \sum_{\alpha = d}^{d} \binom{d}{\alpha} X^\alpha Y^{\alpha} = (X \cdot Y)^d.
\]

Since any Gaussian process is uniquely determined by its first and second moments [13, Theorem 2.1], this process is invariant under \( SO(n+1) \). Therefore, the zeros within \( \mathbb{R}P^n \) are also invariant under the action of \( SO(n+1) \). \( \square \)

The following lemma justifies our consideration of \( f(x) \) in (4) as actually defining a random function.

**Lemma 6.** Almost surely, the power series over \( \alpha \) in (4) converges uniformly on compact subsets of \( \mathbb{R}^n \) and moreover is real analytic on \( \mathbb{R}^n \).

**Proof.** The proof of Lemma 2.2.3 from [14] applies to show (4) almost surely converges uniformly on compact subsets of \( \mathbb{C}^n \) and hence defines a random complex analytic function on \( \mathbb{C}^n \). By restricting the resulting functions to \( \mathbb{R}^n \), we obtain the desired result. \( \square \)

**Proposition 7.** The zeroes of the system \( f = (f_1, \ldots, f_n) \) from (4) are invariant under any isometry of \( \mathbb{R}^n \). That is, for any open set \( U \subset \mathbb{R}^n \) and any isometry \( I : \mathbb{R}^n \to \mathbb{R}^n \), we have

\[
\Pr(f \text{ has a zero in } U) = \Pr(f \text{ has a zero in } I(U)).
\]

**Proof.** The zeroes of \( f \) are the same as those of

\[
g := (g_1, g_2, \ldots, g_n) \quad \text{where} \quad g_i(x) := e^{-\frac{1}{2}||x||^2} f_i(x).
\]

Each \( g_i(x) \) defines a Gaussian process on \( \mathbb{R}^n \), with mean 0 and covariance function

\[
E(g_i(x)g_i(y)) = e^{-\frac{1}{2}||x||^2 + ||y||^2} \sum_{\alpha} \frac{x^\alpha y^{\alpha}}{\alpha!} = e^{-\frac{1}{2}||x||^2 + ||y||^2} \prod_{i=1}^{n} \left( \sum_{\alpha_i = 0}^{\infty} \frac{(x_i y_i)^{\alpha_i}}{\alpha_i!} \right) = e^{-\frac{1}{2}||x-y||^2}.
\]

The result follows because (13) is clearly invariant under isometries of \( \mathbb{R}^n \). \( \square \)

We will now use these invariance properties to reduce the proofs of Theorems 1 and 2 to a particularly simple pairs of points and to local coordinates. The two points

\[
x = \left[ 1 : 0 : \cdots : \frac{t}{2} \right] \quad \text{and} \quad y = \left[ 1 : 0 : \cdots : \frac{t}{2} \right]
\]

(given here in homogeneous coordinates) have distance

\[
\text{dist}(x, y) = 2 \arctan \left( \frac{t}{2} \right) = t + O(t^3).
\]

Thus, in order to prove Theorem 1, it suffices to verify (8) for this pair of points.
Note that \((x_1, \ldots, x_n) \mapsto [1 : x_1 : \ldots : x_n]\) provides a system of local coordinates in a neighborhood of \(p\) and \(q\). In these coordinates, the \(SO(n + 1)\)-invariant ensemble becomes
\[
 f_d = (f_{d,1}(x), f_{d,2}(x), \ldots, f_{d,n}(x)),
\]
where each \(f_{d,i}\) is chosen independently of the form
\[
 f_d(x) = \sum_{|\alpha| \leq d} \sqrt{\frac{d!}{(d - |\alpha|)! \prod_{i=1}^n \alpha_i!}} a_\alpha x^\alpha
\]
where \(d \geq 1\) and the \(a_\alpha\) are iid on the standard normal distribution \(\mathcal{N}(0, 1)\).

In summary: Let \(\mathcal{K}_{n,d}(x, y)\) and \(\mathcal{K}_n(x, y)\) denote the correlation functions between zeros of the \(SO(n + 1)\)-invariant ensemble, expressed in affine coordinates \((15)\), and between zeros of the \(\text{Isom}(\mathbb{R}^n)\)-invariant ensemble \((4)\), respectively, and let
\[
 \mathcal{K}_{n,d}(t) := \mathcal{K}_{n,d}((0, \ldots, 0, -t/2), (0, \ldots, 0, t/2)) \quad \text{and} \quad \mathcal{K}_n(t) := \mathcal{K}_n((0, \ldots, 0, -t/2), (0, \ldots, 0, t/2))
\]
In order to prove Theorems \([1]\) and \([1]\), it suffices to prove:

**Theorem 8.** We have:

1. the following short-range asymptotics:
\[
 \mathcal{K}_{n,d}(t) = A_{n,d} t^{2-n} + O(t^{3-n}), \quad \text{and} \quad \mathcal{K}_n(t) = A_n t^{2-n} + O(t^{3-n}),
\]
   where \(A_{n,d}\) and \(A_n\) are given in \((8)\) and \((7)\), respectively, and

2. the following long-range asymptotics:
\[
 \mathcal{K}_n(t) = 1 + O \left( t e^{-\frac{2}{t}} \right).
\]

3. **Kac-Rice Formula**

The main technique used in this paper is the Kac-Rice Formula \([16, 20, 21]\). Suppose \(h = (h_1, h_2, \ldots, h_n)\) is a Gaussian random function on \(\mathbb{R}^n\). We begin with the Kac-Rice formula for the density of zeros. Consider the random vector
\[
 v := \begin{bmatrix} h_1(x) & \nabla h_1(x) & \ldots & h_n(x) & \nabla h_n(x) \end{bmatrix}^\top,
\]
where each gradient vector is concatenated into the vector at the indicated location. The vector \(v\) is a Gaussian column vector of dimension \(n(n + 1)\).

Let \(\xi\) be the \(n \times n\) matrix whose rows are \(\xi_1, \ldots, \xi_n\) and let \(u = [\xi_1 \ldots \xi_n]^\top\) be the vector obtained by concatenating the rows of \(\xi\).

**Proposition 9.** Suppose the covariance matrix \(C = (E v_i v_j)_{i,j=1}^{n(n+1)}\) of the vector \((\text{7})\) is positive definite. Then, the density of zeros of the system \(h\) is:
\[
 \rho_n (x) = \frac{1}{(2\pi)^{n(n+1)/2} \sqrt{\det C}} \int \frac{\det \xi e^{-\frac{1}{2} (\Omega u, u)}}{\det C_{\mathbb{R}^n}} du.
\]
where \(\Omega\) is the matrix of the elements of \(C^{-1}\) left after removing the rows and columns that correspond to the elements \(h_1(x)\) i.e., all of the rows and columns with indices congruent to 1 modulo \(n + 1\).

The Kac-Rice formula for the 2-point correlation function is a simple modification: Consider the random vector
\[
 v := \begin{bmatrix} h_1(x) & \nabla h_1(x) & h_1(y) & \nabla h_1(y) & \ldots & h_n(x) & \nabla h_n(x) & h_n(y) & \nabla h_n(y) \end{bmatrix}^\top,
\]
where each gradient vector is concatenated into the vector at the indicated location. The vector \(v\) is a Gaussian column random vector of dimension \(2n(n + 1)\).

Let \(\xi\) and \(\eta\) be the \(n \times n\) matrices whose rows are \(\xi_1, \ldots, \xi_n\) and \(\eta_1, \ldots, \eta_n\), respectively. Let \(u = [\xi_1 \eta_1 \xi_2 \eta_2 \ldots \xi_n \eta_n]^\top\), the vector formed by alternating the vectors \(\xi_i\) and \(\eta_i\).
Proposition 10. Suppose the covariance matrix $\mathbf{C} = (E(v_i v_j))_{i,j=1}^{n(n+1)}$ of the vector $(\mathbf{v}_i)$ is positive definite. Then, the two-point correlation function for the zeroes of the system $\mathbf{h}$ is:

$$K_n(x, y) = \frac{1}{(2\pi)^{n(n+1)} \rho(x)\rho(y) \sqrt{\det \mathbf{C}}} \int_{\mathbb{R}^{2n^2}} |\det \xi| |\det \eta| e^{-\frac{1}{2}((\mathbf{B}\eta, \mathbf{u}))} d\mathbf{u},$$

where $\mathbf{C}$ is the covariance matrix of the elements of $\mathbf{C}^{-1}$ left after removing the rows and columns that correspond to the elements $h_i(x)$ and $h_j(y)$, i.e., all of the rows and columns with indices congruent to 1 modulo $n+1$.

Propositions 9 and 10 are easily obtained from Theorem 2.2 of Remark 11. within the proof of Theorems 2.1 and 2.3 from Theorem 6.2 by using the suitable Gaussian density $D_k(0, \xi, \xi)$ in their formula 38 (and normalizing by the density at the two points, in the case of Proposition 10).

Remark 11. Within the proof of Theorems 2.1 and 2.3 from Theorem 6.2 it is shown that the correlation measure is absolutely continuous off of the diagonal $x = y$ (hence the name 'correlation function'). Thus, in the definition of $K(x, y)$ one need not use round balls $N_0(x)$ and $N_0(y)$. Rather, any sequence of neighborhoods of $x$ and $y$ suitable for computing a Radon-Nikodym derivative will suffice.

On certain occasions we will need the following lemma, which is proved in Appendix A, to make estimates involving the Kac-Rice formulae (18) and (20).

Lemma 12. We have:

1. For any positive definite $n^2 \times n^2$ matrix $\mathbf{A}$

$$\int_{\mathbb{R}^{2n^2}} |\det \xi| e^{-\frac{1}{2}(\mathbf{B}\eta, \mathbf{u})} - \int_{\mathbb{R}^{2n^2}} |\det \xi| e^{-\frac{1}{2}(\mathbf{A}\eta, \mathbf{u})} d\mathbf{u} = O \left( ||\mathbf{A} - \mathbf{B}||_\infty^{1/2} \right)$$

for any $2n^2 \times 2n^2$ matrix $\mathbf{B}$ sufficiently close to $\mathbf{A}$. (Here $\mathbf{u}$ is as in (18) and $|| \cdot ||_\infty$ denotes the maximum entry of the matrix.)

2. For a positive definite $2n^2 \times 2n^2$ matrix $\mathbf{A}$ there exists $D > 0$ such that

$$\int_{\mathbb{R}^{2n^2}} |\det \xi| |\det \eta| e^{-\frac{1}{2}(\mathbf{B}\eta, \mathbf{u})} - \int_{\mathbb{R}^{2n^2}} |\det \xi| |\det \eta| e^{-\frac{1}{2}(\mathbf{A}\eta, \mathbf{u})} d\mathbf{u} = O \left( ||\mathbf{A} - \mathbf{B}||_\infty^{1/2} \right)$$

for any $2n^2 \times 2n^2$ matrix $\mathbf{B}$ sufficiently close to $\mathbf{A}$. (Here $\mathbf{u}$ is as in (20).)

4. Calculation of the covariance matrices, their inverses, and $\mathbf{O}$

Let $\mathbf{C}_{n,d} = \mathbf{C}_{n,d}(t)$ and $\mathbf{C}_n = \mathbf{C}_n(t)$ be the covariance matrix for vectors $(\mathbf{19})$ applied to $f_d$ (Equation 15) and $f$ (Equation 4), respectively, at the points

$$x = \left(0, \ldots, 0, \frac{t}{2} \right) \quad \text{and} \quad y = \left(0, \ldots, 0, \frac{t}{2} \right).$$

Lemma 13. Both $\mathbf{C}_{n,d}$ and $\mathbf{C}_n$ are of the form diag$_n(\mathbf{C})$, where $\mathbf{C} = \begin{bmatrix} \mathbf{A}_+ & \mathbf{B}^T \\ \mathbf{B} & \mathbf{A}_- \end{bmatrix}$, where $\mathbf{A}$ and $\mathbf{B}$ are the following

$$\begin{pmatrix} \alpha & 0 & \ldots & 0 & \pm \delta \\ 0 & \beta & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ldots & \beta & 0 & 0 \\ \pm \delta & 0 & \ldots & 0 & \gamma \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} \mu & 0 & \ldots & 0 & \nu \\ 0 & \eta & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ldots & \eta & 0 & 0 \\ -\nu & 0 & \ldots & 0 & \tau \end{pmatrix}.$$

Proof. Since the coefficients of $f_{d,i}$ and $f_{d,j}$ (respectively $f_i$ and $f_j$) are independent when $i \neq j$, only the entries of $\mathbf{C}_{n,d}$ (respectively $\mathbf{C}_n$) with $i = j$ will have nonzero values. Thus, the covariance matrices will have the following block-diagonal structure:

$$\mathbf{C}_{n,d} = \text{diag}_n(\mathbf{C}_{n,d}) \quad \text{and} \quad \mathbf{C}_n = \text{diag}_n(\mathbf{C}_n),$$
where $\tilde{C}_{n,d}$ corresponds to the first $2n + 2$ entries of $v$ (and similarly for $\tilde{C}_{n}$). These entries correspond to $f_{d,1}$ and $f_1$, respectively. For ease of notation, we’ll drop the subscript $1$: $f_d \equiv f_{d,1}$, $f \equiv f_1$.

\begin{align}
E(f_d(x)f_d(x)) &= (1 + ||x||^2)^d = \left(1 + \frac{t^2}{4}\right)^d = E(f_d(y)f_d(y)) \\
E \left( f_d(x) \frac{\partial f_d(x)}{\partial x_i} \right) &= \frac{1}{2} \frac{\partial E(f_d(x)f_d(x))}{\partial x_i} = d \cdot x_i \cdot (1 + ||x||^2)^{d-1} \\
&= -d \frac{t}{2} \left(1 + \frac{t^2}{4}\right)^{d-1} = -E \left( f_d(y) \frac{\partial f_d(y)}{\partial y_i} \right) \\
E \left( \frac{\partial f_d(x)}{\partial x_i} \frac{\partial f_d(x)}{\partial x_i} \right) &= \frac{1}{4} \frac{\partial^2 E(f_d(x)f_d(x))}{\partial x_i^2} + \frac{1}{4} \frac{\partial E(f_d(x)f_d(x))}{\partial x_i} \\
&= d \left(1 + \frac{t^2}{4}\right)^{d-1} = E \left( \frac{\partial f_d(y) \partial f_d(y)}{\partial y_i \partial y_j} \right) \quad \text{for } i \neq n \\
E \left( \frac{\partial f_d(x)}{\partial x_i} \frac{\partial f_d(x)}{\partial x_j} \right) &= \frac{1}{4} \frac{\partial^2 E(f_d(x)f_d(x))}{\partial x_i \partial x_j} = d \cdot (d - 1) x_i x_j \cdot (1 + ||x||^2)^{d-2} \\
&= 0 = E \left( \frac{\partial f_d(y) \partial f_d(y)}{\partial y_i \partial y_j} \right) \quad \text{for } i \neq j.
\end{align}
We will do the computations of det(C), \( C^{-1} \), and \( \Omega \) (the submatrix of \( C^{-1} \) that is used in the Kac-Rice formula) in this general form and then revert back to \( C_{n,d} \) and \( C_n \), when necessary. The determinant of \( C \) is

\[
\det(C) = (\beta^2 - \eta^2)^{n(n-1)} \left( \alpha \gamma - \alpha \tau - \delta^2 - 2 \delta \nu + \gamma \mu - \mu \tau - \nu^2 \right)^n \left( \alpha \gamma + \alpha \tau - \delta^2 + 2 \delta \nu - \gamma \mu - \mu \tau - \nu^2 \right)^n
\]

We immediately find:

**Lemma 14.** For all \( t > 0 \), \( C_n \) is positive definite. For \( d \geq 3 \) and sufficiently small \( t > 0 \), \( C_{n,d} \) is positive definite.

**Proof.** It is a general fact from probability theory that the covariance matrix of a random vector is positive semi-definite. For \( C \), equation (44) becomes

\[
\det(C) = \left( e^{\frac{1}{2} t^2} - e^{-\frac{1}{2} t^2} \right)^{n(n-1)} \left( e^{\frac{1}{2} t^2} - e^{-\frac{1}{2} t^2} + t^2 \right)^n \left( e^{\frac{1}{2} t^2} - e^{-\frac{1}{2} t^2} - t^2 \right)^n,
\]

which is positive for all \( t > 0 \).

For \( C_{n,d} \) we have

\[
\det(C_{n,d}) = \frac{d^{2n^2+n(d-1)^2+n(d-2)^n}}{12^n} t^{2n^2+6n} + O(t^{2n^2+6n+1}),
\]

which is positive for \( d \geq 3 \) and \( t > 0 \) sufficiently small. \( \square \)

Applying a suitable permutation to the rows and columns of \( \tilde{C} \), one obtains a block matrix with one \( 4 \times 4 \) block and \( n-1 \) copies of the same \( 2 \times 2 \) block. Because of this, \( \tilde{C}^{-1} \) will have the same block structure and it can readily be computed to be

\[
\tilde{C}^{-1} = \begin{bmatrix}
D_+ & E_+ \\
E_- & D_-
\end{bmatrix}
\]

where \( D_\pm \) and \( E_\pm \) are the following \( (n+1) \times (n+1) \) matrices:

\[
D_\pm = 
\begin{bmatrix}
\frac{\alpha \gamma^2 - \alpha \tau^2 - \delta \gamma^2 - 2 \delta \nu \tau - \gamma \nu \mu}{\Delta} & 0 & \ldots & 0 & \frac{\alpha \gamma + \alpha \tau - \delta^2 - \delta \mu \tau + \delta \nu^2 - \gamma \mu \nu}{\Delta} \\
0 & \frac{\beta}{\beta^2 - \eta^2} & \ldots & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \ldots & \frac{\beta}{\beta^2 - \eta^2} & \frac{\alpha \gamma + \alpha \tau - \delta^2 - \delta \mu \tau + \delta \nu^2 - \gamma \mu \nu}{\Delta} & 0 \\
\frac{-\delta^2 \tau - 2 \delta \nu \gamma \mu - \mu \tau^2 - \nu^2 \tau}{\Delta} & 0 & \ldots & 0 & \frac{\alpha \gamma - \alpha \tau^2 + 2 \delta \mu \nu - \gamma \mu \nu}{\Delta} \\
0 & \frac{-\eta}{\beta^2 - \eta^2} & \ldots & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \ldots & \frac{-\eta}{\beta^2 - \eta^2} & 0 & 0 \\
\frac{\alpha \delta + \alpha \gamma + \delta \nu^3 - \delta \mu \nu - \gamma \mu \nu^3}{\Delta} & 0 & \ldots & 0 & \frac{-\alpha \delta + \alpha \gamma + \delta \nu^3 + \delta \mu \nu - \gamma \mu \nu^3}{\Delta}
\end{bmatrix}
\]

\[
E_\pm = 
\begin{bmatrix}
\alpha \delta + \alpha \gamma + \delta \nu^3 - \delta \mu \nu - \gamma \mu \nu^3 & 0 & \ldots & 0 & \frac{\alpha \delta + \alpha \gamma + \delta \nu^3 + \delta \mu \nu - \gamma \mu \nu^3}{\Delta} \\
0 & \frac{-\eta}{\beta^2 - \eta^2} & \ldots & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \ldots & \frac{-\eta}{\beta^2 - \eta^2} & 0 & 0 \\
\frac{\alpha \delta + \alpha \gamma + \delta \nu^3 - \delta \mu \nu - \gamma \mu \nu^3}{\Delta} & 0 & \ldots & 0 & \frac{-\alpha \delta + \alpha \gamma + \delta \nu^3 + \delta \mu \nu - \gamma \mu \nu^3}{\Delta}
\end{bmatrix}
\]
Where, 
\[
\Delta = \alpha^2 \gamma^2 - \alpha^2 \tau^2 - 2 \alpha \delta^2 \gamma - 4 \alpha \delta \nu \tau - 2 \alpha \gamma \nu^2 + \delta^4 + 2 \delta^2 \mu \tau - 2 \delta^2 \nu^2 + 4 \delta \gamma \mu \nu - \gamma^2 \mu^2 + \mu^2 \tau^2 + 2 \mu \nu^2 \tau + \nu^4
\]

We therefore have:

**Lemma 15.** \(\mathbf{\Omega} = \text{diag}_n \left( \tilde{\mathbf{\Omega}} \right)\) with \(\tilde{\mathbf{\Omega}} = \begin{bmatrix} \tilde{\mathbf{\Omega}}_{1,1} & \tilde{\mathbf{\Omega}}_{1,2} \\ \tilde{\mathbf{\Omega}}_{2,1} & \tilde{\mathbf{\Omega}}_{2,2} \end{bmatrix}\), where:

\[
\tilde{\mathbf{\Omega}}_{1,1} = \tilde{\mathbf{\Omega}}_{2,2} = \text{diag} \left( \frac{\beta}{\beta^2 - \eta^2}, \ldots, \frac{\beta}{\beta^2 - \eta^2} \right),
\]

\[
\tilde{\mathbf{\Omega}}_{1,2} = \tilde{\mathbf{\Omega}}_{2,1} = \text{diag} \left( \frac{\eta}{\beta^2 - \eta^2}, \ldots, \frac{\eta}{\beta^2 - \eta^2} \right).
\]

We notice that there exists a permutation matrix \(\mathbf{Q}\) such that

\[
\mathbf{M} := \mathbf{Q}^\top \mathbf{\Omega} \mathbf{Q} = \text{diag} \left( \begin{array}{cccc} \mathbf{M}_1, & \ldots, & \mathbf{M}_1, & \ldots, \mathbf{M}_1, \mathbf{M}_2 \end{array} \right),
\]

where \(\mathbf{M}_1 = \begin{bmatrix} \frac{\beta}{\beta^2 - \eta^2} & -\frac{n}{\beta^2 - \eta^2} \\ -\frac{n}{\beta^2 - \eta^2} & \frac{\beta}{\beta^2 - \eta^2} \end{bmatrix}\) and \(\mathbf{M}_2 = \begin{bmatrix} \frac{\alpha^2 \gamma - \alpha \delta \nu + 2 \delta \mu \nu - \gamma \mu^2}{\Delta} & -\frac{\alpha^2 \gamma + 2 \alpha \delta \nu - \delta^2 \mu - \mu^2 \tau - \mu \nu^2}{\Delta} \\ -\frac{\alpha^2 \gamma - \alpha \delta \nu - \delta^2 \mu - \mu^2 \tau - \mu \nu^2}{\Delta} & \frac{\alpha^2 \gamma + 2 \alpha \delta \nu - \delta^2 \mu - \mu^2 \tau - \mu \nu^2}{\Delta} \end{bmatrix}\).

**Lemma 16.** We can orthogonally diagonalize the matrix \(\mathbf{M}\) using \(\mathbf{P} = \text{diag}_n \left( \begin{bmatrix} \sqrt{\frac{\beta}{\beta^2 - \eta^2}} \\ \sqrt{\frac{\beta}{\beta^2 - \eta^2}} \end{bmatrix} \right)\), obtaining

\[
\Lambda := \mathbf{P}^\top \mathbf{M} \mathbf{P} = \text{diag} \left( \begin{array}{cccc} \lambda_1, & \ldots, & \lambda_1, & \ldots, \lambda_1, \lambda_2, \lambda_3, \lambda_4 \end{array} \right),
\]

where

\[
\lambda_1 = \frac{1}{\beta - \eta}, \quad \lambda_2 = \frac{1}{\beta + \eta}, \quad \lambda_3 = \frac{\alpha + \mu}{\alpha \gamma - \alpha \tau - \delta^2 - 2 \nu \delta + \gamma \mu - \mu \tau - \nu^2}, \quad \text{and} \quad \lambda_4 = \frac{\alpha - \mu}{\alpha \gamma + \alpha \tau - \delta^2 + 2 \nu \delta - \gamma \mu - \mu \tau - \nu^2}.
\]

We will need the following calculation in Section 5.

\[
(\lambda_1 \lambda_2)^{-\frac{n}{2}} (\lambda_3 \lambda_4)^{-\frac{n}{2}} \sqrt{\det \mathbf{C}} = (\alpha^2 - \mu^2)^{-\frac{n}{2}} \left( \frac{1 + \frac{\delta^2}{4}}{1 - \frac{\delta^2}{4}} \right)^{2d} = d^{-\frac{n}{2}} t^{-n} + O(t^{-n+2}) \quad \text{for Ensemble (15)}.
\]

\[
\left( e^{\frac{\alpha}{2}} - e^{-\frac{\alpha}{2}} \right)^{-\frac{n}{2}} = t^{-n} + O(t^{-n+2}) \quad \text{for Ensemble (4)}.
\]
Let $\Omega_{n,d}$ and $\Omega_n$ be the matrices $\Omega$ corresponding to $C_{n,d}$ and $C_n$. We will denote the eigenvalues of $\Omega_{n,d}$ by $\lambda_{d,1}, \lambda_{d,2}, \lambda_{d,3}, \lambda_{d,4}$ and the eigenvalues of $\Omega_n$ to be $\lambda_1, \lambda_2, \lambda_3, \lambda_4$. From (50) and the entries of $C_n$ computed in Lemma 13 we find:

$$
\lambda_1 = \left( e^{2\tau} - e^{-2\tau} \right)^{-1}, \quad \lambda_2 = \left( e^{2\tau} + e^{-2\tau} \right)^{-1}, \quad \lambda_3 = \frac{e^{2\tau} + e^{-2\tau}}{t^2 + e^{2\tau} - e^{-2\tau}}, \quad \text{and} \quad \lambda_4 = \frac{e^{2\tau} - e^{-2\tau}}{t^2 + e^{2\tau} - e^{-2\tau}}.
$$

In the proof of the short-range asymptotics we will need the following:

$$
\lambda_{1,1}^{-1/2} = \sqrt{\frac{d(d-1)}{2}} t + O(t^3), \quad \lambda_{2,1}^{-1/2} = \sqrt{2d} + O(t) \quad \lambda_{3,1}^{-1/2} = \sqrt{d(d-1)} t \quad \lambda_{4,1}^{-1/2} = \sqrt{\frac{d(d^2 - 3d + 2)}{12}} t^2 + O(t^3),
$$

which can be computed doing a low order expansion of the entries of $C_{n,d}$ computed in Lemma 13. Meanwhile, from (52) we have:

$$
\lambda_1^{-1/2} = \frac{t}{\sqrt{2}} + O(t^2), \quad \lambda_2^{-1/2} = \sqrt{2} + O(t^2) \quad \lambda_3^{-1/2} = t + O(t^3) \quad \lambda_4^{-1/2} = \frac{1}{\sqrt{12}} t^2 + O(t^3),
$$

**Remark 17.** These asymptotics were determined using the Maple computer algebra system [1]. However, they are simple enough that one can check them by hand.

## 5. Proof of Part (1) from Theorem 8: Short-range asymptotics

We apply the Kac-Rice formula to the covariance matrices $C_{n,d}$ and $C_n$ and the submatrices $\Omega_{n,d}$ and $\Omega_n$ of their inverses, as computed in Section 4. It applies because, by Lemma 14, $C$ is positive definite for all $t > 0$ and $C_{n,d}$ is positive definite for all $d \geq 3$ and sufficiently small $t > 0$. The proof will be the nearly same for each, so we will work with $K_{n,d}(t)$ at the very end of the section.

We apply the diagonalization of $\Lambda \equiv \Lambda_{n,d} = (QP)^T \Omega_{n,d} (QP)$ from (48) and (49) to the Kac-Rice formula (20) to obtain

$$
K_{n,d}(t) = \frac{1}{(2\pi)^{n(n+1)/2}} \rho_d(x) \rho_d(y) \sqrt{\det C_{n,d}} \int_{\mathbb{R}^{2n^2}} |\det \xi(\tau)||\det \eta(\tau)| e^{-\frac{1}{2}(\Lambda \tau, \tau)} d\tau.
$$

where $\tau := [\tau_1, \tau_2, \ldots, \tau_n]^T := P^T Q^T u$, where $\tau_i = [\tau_{i,1}, \tau_{i,2}, \ldots, \tau_{i,2n}]$ for each $1 \leq i \leq n$. In these new variables, $\xi$ and $\eta$ become the new matrices $\xi(\tau)$ and $\eta(\tau)$, whose entries are defined by

$$
\xi_{i,j}(\tau) = \frac{\sqrt{2}}{2} \left( -\tau_{i,2j-1} + \tau_{i,2j} \right) \quad \text{and} \quad \eta_{i,j}(\tau) = \frac{\sqrt{2}}{2} \left( \tau_{i,2j-1} + \tau_{i,2j} \right) \quad \text{for} \quad i, j \leq n.
$$

The reason for diagonalizing $\Omega_{n,d}$ was to change the exponent into a form conducive to forming $n$ sets of $2n$-dimensional spherical coordinates so that $(\Lambda \tau, \tau)$ becomes $\sum_{k=1}^{2n} \rho_k^2$. Let $w := [r_1, r_2, \ldots, r_n, \theta_1, \theta_2, \ldots, \theta_n]$, where $\theta_i = [\theta_{i,1}, \theta_{i,2}, \ldots, \theta_{i,2n-1}]$. Let

$$
\tau_{i,j} = \begin{cases} 
\lambda_{1,1}^{-1/2} r_1 \left( \prod_{k=1}^{\frac{j-1}{2}} \sin \theta_{i,k} \right) \left( \cos \theta_{i,\frac{j+1}{2}} \right), \quad \text{if} \ j \text{ is odd and} \ j \neq 2n - 1, \\
\lambda_{1,1}^{-1/2} r_2 \left( \prod_{k=1}^{\frac{n-1}{2}} \sin \theta_{i,k} \right) \left( \cos \theta_{i,n} \right), \quad \text{if} \ j = 2n, \\
\lambda_{2,1}^{-1/2} r_3 \left( \prod_{k=1}^{\frac{n-1}{2}} \sin \theta_{i,k} \right) \left( \cos \theta_{i,\frac{n+1}{2}} \right), \quad \text{if} \ j \text{ is even and} \ j \neq 2n, \\
\lambda_{3,1}^{-1/2} r_4 \left( \prod_{k=1}^{\frac{2n-1}{2}} \sin \theta_{i,k} \right), \quad \text{if} \ j = 2n - 1.
\end{cases}
$$

Thus, $d\tau$ becomes $(\lambda_{1,1} \lambda_{2,2} \lambda_{3,1} \lambda_{4,1})^{-\frac{1}{2}} \left( \prod_{k=1}^{n} r_k^{2n-1} d \left(S^{2n-1}\right)^n \right) d\tau$, where $S^{2n-1}$ denotes the unit sphere in $\mathbb{R}^{2n}$ and $dS^{2n-1}$ denotes the element of volume of the sphere. Let $\phi_{i,j}$ be the trigonometric product in $\tau_{i,j}$.
so that \( \tau_{i,j} = \frac{1}{2} \phi_{i,j}. \) After this variable change, we see that

\[
(58) \quad \xi_{i,j} (r, \theta) = \frac{\sqrt{\gamma}}{2} r_{i} \left( -\lambda_{m,d}^{\frac{1}{2}} \phi_{i,j} + 1 + \lambda_{m+1,d}^{\frac{1}{2}} \phi_{i,j} \right) \quad \text{and} \quad \eta_{i,j} (r, \theta) = \frac{\sqrt{\gamma}}{2} r_{j} \left( \lambda_{m,d}^{\frac{1}{2}} \phi_{i,j} - 1 + \lambda_{m+1,d}^{\frac{1}{2}} \phi_{i,j} \right)
\]

where \( m = 1 \) when \( j \neq n \) and \( m = 3 \) when \( j = n \).

Thus, in the new spherical coordinates, we have:

\[
(59) \quad K_{n,d} (t) = \frac{(\lambda_{1,d} \lambda_{2,d})}{(2\pi)^{n(n+1)}} \int_{\mathbb{R}^{n+1}_+} \frac{1}{r_{i}^{2n-1} \rho_{d}(x) \rho_{d}(y)} \left( \frac{\sqrt{\gamma}}{2} \sum_{k=1}^{n} \int_{\mathbb{S}^{n-1}} | \det \xi (r, \theta) | | \det \eta (r, \theta) | e^{-\frac{1}{t} \sum_{k=1}^{n} r_{i}^{2n-1} d (\mathbb{S}^{n-1})^{n}} \right) dr.
\]

Using the asymptotic behavior of \( \lambda_{1,d}^{-1/2}, \lambda_{2,d}^{-1/2}, \lambda_{3,d}^{-1/2}, \) and \( \lambda_{4,d}^{-1/2} \) expressed in (53), we notice that each the elements of the \( n \)th column of each determinant vanishes linearly with \( t \). Therefore, we factor out \( t \) from each column to prevent these columns from vanishing in the limit as \( t \) goes to 0. Also note that each element of row \( i \) in both \( \xi \) and \( \eta \) are linear with \( r_{i} \). Thus, we let \( \hat{\xi} (\theta, t) \) and \( \hat{\eta} (\theta, t) \) denote the resulting matrices when \( t \) is factored from the \( n \)th column and \( r_{i} \) is factored from each row of each matrix, \( \xi \) and \( \eta \), respectively. Using Fubini’s Theorem, we can now split the integral from (59) into an integral over the radii and an integral over the angles:

\[
(60) \quad K_{n,d} (t) = \frac{(\lambda_{1,d} \lambda_{2,d})}{(2\pi)^{n(n+1)}} \int_{\mathbb{R}^{n+1}_+} \frac{1}{\rho_{d}(x) \rho_{d}(y)} \left( \frac{\sqrt{\gamma}}{2} \sum_{k=1}^{n} \int_{\mathbb{S}^{n-1}} | \det \hat{\xi} (\theta, t) | | \det \hat{\eta} (\theta, t) | d (\mathbb{S}^{n-1})^{n} \right) dr.
\]

Using the definition of the Gamma function, (60) simplifies to:

\[
(62) \quad K_{n,d} (t) = \frac{2^{n^2} \Gamma (n+1)^{n}}{(2\pi)^{n(n+1)} \rho_{d}(x) \rho_{d}(y)} \left( \frac{(\lambda_{1,d} \lambda_{2,d})}{(2\pi)^{n(n+1)}} \int_{\mathbb{R}^{n+1}_+} \frac{1}{\rho_{d}(x) \rho_{d}(y)} \left( \frac{\sqrt{\gamma}}{2} \sum_{k=1}^{n} \int_{\mathbb{S}^{n-1}} | \det \hat{\xi} (\theta, t) | | \det \hat{\eta} (\theta, t) | d (\mathbb{S}^{n-1})^{n} \right) dr \right).
\]

By (53) we have that

\[
(63) \quad \frac{(\lambda_{1,d} \lambda_{2,d})}{\sqrt{\det C_{n,d}}} | \xi | = 2^{-\frac{n^2}{2} - \frac{n(n-1)}{2}} \frac{\Gamma (n+1)^{n}}{(2\pi)^{n(n+1)}} \int_{\mathbb{R}^{n+1}_+} \rho_{d}(x) \rho_{d}(y) \left( \frac{\sqrt{\gamma}}{2} \sum_{k=1}^{n} \int_{\mathbb{S}^{n-1}} | \det \hat{\xi} (\theta, t) | | \det \hat{\eta} (\theta, t) | d (\mathbb{S}^{n-1})^{n} \right) dr.
\]

Meanwhile, by (53), each entry of \( \hat{\xi} \) and \( \hat{\eta} \) is of the form: constant (potentially 0) plus \( O(t) \). Therefore,

\[
(64) \quad \int_{(\mathbb{S}^{n-1})^{n}} | \det \hat{\xi} (\theta, t) | | \det \hat{\eta} (\theta, t) | d (\mathbb{S}^{n-1})^{n} = D_{n} + O (t),
\]

where

\[
(65) \quad D_{n} = \lim_{t \to 0} \int_{(\mathbb{S}^{n-1})^{n}} | \det \hat{\xi} (\theta, t) | | \det \hat{\eta} (\theta, t) | d (\mathbb{S}^{n-1})^{n}.
\]

Finally, we also have from (5) that

\[
(66) \quad \rho_{d}(x) = \rho_{d}(y) = \pi^{-\frac{n+1}{2}} \Gamma \left( \frac{n+1}{2} \right) d^{\frac{n}{2}} + O(t^2).
\]
Therefore,

\[ K_{n,d}(t) = A_{n,d} t^{2-n} + O(t^{4-n}) \]

where

\[ A_{n,d} = \left( \frac{2n^2 \pi^{n+1}}{(2\pi)^n (n+1)!} \right) \frac{d^{n+1}}{d\theta^{n+1}} \]

We now compute the constant \( D_n \). From Equations (58), (53) and (54):

\[ \lim_{t \to 0} \xi_{i,j} (\theta, t) = \lim_{t \to 0} \eta_{i,j} (\theta, t) = \sqrt{d} \phi_{i,2j} \quad \text{for } j < n, \quad \text{and} \]

\[ \lim_{t \to 0} \hat{\xi}_{i,j} (\theta, t) = \lim_{t \to 0} \hat{\eta}_{i,j} (\theta, t) = \sqrt{\frac{d(d-1)}{2}} \phi_{i,2n-1} \quad \text{for } j = n. \]

Let \( \mu (\theta) \) be the resulting matrix when \( \sqrt{d} \) is factored out of the first through \( n-1 \)st columns and \( \sqrt{\frac{d(d-1)}{2}} \) is factored out of the \( n \)th column of \( \lim_{t \to 0} \hat{\xi} (\theta, t) \). If we do the same process with \( \lim_{t \to 0} \mu (\theta, t) \), we obtain the same result with the sign changed in the \( n \)th column. Therefore,

\[ D_n = \frac{d^n (d-1)}{2} \int_{S^{2n-1}} \det \mu (\theta)^2 d(S^{2n-1})^n. \]

From (57), we notice that each entry of row \( i \) of \( \mu (\theta) \) contains a factor of \( \prod_{j=1}^{n} \sin \theta_{i,j} \). Thus, we can take this factor out of each row of the matrix, removing any dependence of the determinant on \( \theta_{i,j} \) for \( j \leq n \). Let \( \nu (\theta) \) denote the matrix that remains after removing these factors.

We can then split the integral into two, an integral over \( B^n \), where \( B := [0, \pi]^n \), corresponding to \( \theta_{i,j} \) for \( j \leq n \), and an integral over \( (S^{n-1})^n \), corresponding to \( \theta_{i,j} \) for \( j > n \):

\[ D_n = \frac{d^n (d-1)}{2} \int_{B^n} \prod_{i,j \leq n} (\sin \theta_{i,j})^{2n+1-j} dB^n \int_{(S^{n-1})^n} \det \nu (\theta)^2 d(S^{n-1})^n. \]

Here we have used that \( dS^{2n-1} = \prod_{j=1}^{n} \sin \theta_{j}^{2n-1-j} d\theta dS^{n-1} \). The former integral in this product can be calculated recursively with integration by parts to be:

\[ \int_{\mathbb{R}^n} \prod_{i,j \leq n} (\sin \theta_{i,j})^{2n+1-j} d\theta^n = \left( \prod_{j=1}^{n} \int_{0}^{\pi} (\sin \theta)^{2n+1-j} d\theta \right)^n = \left( \pi \left( \frac{\pi}{2} \right)^n \right)^n = \left( \pi \left( \frac{\pi}{2} \right)^n \right)^n. \]

The calculation of the latter integral follows from Proposition 24 of Appendix B.

\[ \int_{(S^{n-1})^n} \det \nu^2 d(S^{n-1})^n = \left( \frac{\Gamma \left( \frac{3n}{2} \right)}{\Gamma \left( \frac{n+1}{2} \right)} \right)^{n-1} \left( \frac{\Gamma \left( \frac{n+1}{2} \right)}{\Gamma \left( \frac{n}{2} \right)} \right)^n = \left( \frac{\Gamma \left( \frac{n+1}{2} \right)}{\Gamma \left( \frac{n}{2} \right)} \right)^n. \]

Therefore,

\[ A_{n,d} = \left( \frac{2n^2 \pi^{n+1} \Gamma (n+1)}{(2\pi)^n (n+1)!} \right) \frac{d^{n+1}}{d\theta^{n+1}} \]

\[ \frac{d^n (d-1)}{2} \frac{\sqrt{\pi}}{\Gamma \left( \frac{n+2}{2} \right)} \frac{\Gamma \left( \frac{n+2}{2} \right)}{\Gamma \left( \frac{n+1}{2} \right)} = C_{n,d}. \]

Thus,

\[ K_{n,d} (t) = \left( \frac{d^n (d-1)}{2} \frac{\sqrt{\pi}}{\Gamma \left( \frac{n+2}{2} \right)} \frac{\Gamma \left( \frac{n+2}{2} \right)}{\Gamma \left( \frac{n+1}{2} \right)} \right) t^{2-n} + O(t^{4-n}), \]

as stated in Part (2) from Theorem 3.

The only differences when computing \( K_n (t) \) instead of \( K_{n,d} (t) \) are:
(1) $\lambda_1^{-1/2} = \sqrt{2} + O(t^2)$ instead of $\sqrt{2d} + O(t^2)$ and $\lambda_3^{-1/2} = t + O(t^2)$ rather than $\lambda_3^{-1/2} = \sqrt{\frac{d-1}{d}} t + O(t^2)$.

(2) the factor of $d^{-\frac{1}{2}n}$ in (51) is missing, and

(3) the factors of $d^{\frac{1}{2}n}$ are missing from the expression for the density of the zeros.

One can readily check that this results in the factor of $\frac{d^{-\frac{1}{2}n}}{\sqrt{\det g_{ij}}}$ being removed from the constant:

$$K_n(t) = \frac{\sqrt{\pi}}{2}\frac{\Gamma\left(\frac{n+2}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)} t^{2-n} + O(t^{3-n})$$

$\square$ (Part (1) of Theorem 8).

6. Proof of Part (2) from Theorem 8: Long-range asymptotics

It will be convenient to apply the Kac-Rice formulae to the ensemble $g$ given in (12), which has the same zeros as the Isom($\mathbb{R}^n$)-invariant ensemble $f$. Let $C_{n,g}$ denote the covariance matrix applied to random vector $|\Omega|^2$ for this ensemble. Recall that $x$ and $y$ are given by (23). The following covariances can be computed from those in (35,43) and the product rule:

(76) $E(g(x)g(x)) = E(g(y)g(y)) = 1$,

(77) $E\left(g(x)\frac{\partial g(x)}{\partial x_j}\right) = E\left(g(y)\frac{\partial g(y)}{\partial y_j}\right) = 0$,

(78) $E\left(\frac{\partial g(x)}{\partial x_j}\frac{\partial g(x)}{\partial x_j}\right) = E\left(\frac{\partial g(y)}{\partial y_j}\frac{\partial g(y)}{\partial y_j}\right) = 1$,

(79) $E\left(\frac{\partial g(x)}{\partial x_i}\frac{\partial g(x)}{\partial x_j}\right) = E\left(\frac{\partial g(y)}{\partial y_i}\frac{\partial g(y)}{\partial y_j}\right) = 0$ if $i \neq j$,

(80) $E(g(x)g(y)) = e^{-\frac{1}{2}||x-y||^2} = e^{-\frac{t}{2}}$,

(81) $E\left(g(x)\frac{\partial g(x)}{\partial y_j}\right) = -E\left(g(y)\frac{\partial g(x)}{\partial x_j}\right) = e^{-\frac{1}{2}||x-y||^2} (x_j - y_j) = \begin{cases} 0 & \text{if } j \neq n, \\ -te^{-\frac{t}{2}} & \text{if } j = n. \end{cases}$

(82) $E\left(\frac{\partial g(x)}{\partial x_i}\frac{\partial g(y)}{\partial y_j}\right) = e^{-\frac{1}{2}||x-y||^2} \left(1 - (x_i - y_i)^2\right) = \begin{cases} e^{-\frac{t}{2}} & \text{if } i \neq n, \\ e^{-\frac{t}{2}}(1 - t^2) & \text{if } i = n. \end{cases}$

(83) $E\left(\frac{\partial g(x)}{\partial x_j}\frac{\partial g(y)}{\partial y_k}\right) = -e^{-\frac{1}{2}||x-y||^2} (x_j - y_j) (x_k - y_k) = 0$.

Remark that $C_{n,g}$ has the structure asserted in (13) and that

$$\det(C_{n,g}) = \left(1 - e^{-t^2}\right)^n \left(1 + e^{-\frac{t}{2}t^2 - e^{-t^2}}\right)^n \left(1 - e^{-\frac{t}{2}t^2 - e^{-t^2}}\right)^n > 0$$

for all $t > 0$, so that $C_{n,g}$ is positive definite.

The proof will rely upon two facts:

(84) $$(\det C_{n,g})^{-\frac{1}{2}} = 1 + O(t^4e^{-t^2})$$ and

(85) $$||I - \Omega_{n,g}||_\infty = O(t^2e^{-t^2}),$$

where $|| \cdot ||_\infty$ denotes the maximum entry of the matrix. The former can be obtained from expression (44). The latter follows from the calculations above and Lemma 15 expressing $\Omega_{n,g}$ in terms of the entries of $C_{n,g}$.

The covariance matrix for random vector (17) is the identity, by (76,79) above. Thus, the Kac-Rice Formula for the density of zeros of $g(x)$ gives

$$1 = \frac{\rho(x)\rho(y)}{\rho(x)\rho(y)} = \frac{1}{(2\pi)^{n+1}} \int_{\mathbb{R}^{n+2}} |\det \xi| e^{-\frac{1}{2}(\xi, \xi)} d\xi \left(\frac{1}{(2\pi)^{n+1}} \int_{\mathbb{R}^{n+2}} |\det \eta| e^{-\frac{1}{2}(\eta, \eta)} d\eta\right)$$

(86) $$= \frac{1}{(2\pi)^{n(n+1)}\rho(x)\rho(y)} \int_{\mathbb{R}^{n+2}} |\det \xi| |\det \eta| e^{-\frac{1}{2}(\xi, \eta)} d\xi d\eta.$$
Since $C_{n,g}$ is positive definite, the Kac-Rice formula for two-point correlations (20) and Equation (86) give

$$|K_n(t) - 1| = \left| \frac{1}{(2\pi)^{n(n+1)}} \rho(x)\rho(y) \sqrt{\det C_{n,g}} \int_{\mathbb{R}^{2n^2}} |\det \xi| |\det \eta| e^{-\frac{x}{2}(\Omega_{n,g}u,u)} du \right| - 1$$

(87)

$$= \frac{1}{(2\pi)^{n(n+1)} \rho(x)\rho(y) \sqrt{\det C_{n,g}}} \cdot \left| \int_{\mathbb{R}^{2n^2}} |\det \xi| |\det \eta| e^{-\frac{x}{2}(\Omega_{n,g}u,u)} du - \sqrt{\det C_{n,g}} \int_{\mathbb{R}^{2n^2}} |\det \xi| |\det \eta| e^{-\frac{x}{2}(u,u)} du \right|$$

(88)

$$\leq \frac{\sqrt{\det C_{n,g}}}{(2\pi)^{n(n+1)} \rho(x)\rho(y)} \left| \int_{\mathbb{R}^{2n^2}} |\det \xi| |\det \eta| e^{-\frac{x}{2}(\Omega_{n,g}u,u)} du - \int_{\mathbb{R}^{2n^2}} |\det \xi| |\det \eta| e^{-\frac{x}{2}(u,u)} du \right|$$

(89)

$$+ \frac{1 - (\det C_{n,g})^{-\frac{1}{2}}}{(2\pi)^{n(n+1)} \rho(x)\rho(y)} \int_{\mathbb{R}^{2n^2}} |\det \xi| |\det \eta| e^{-\frac{x}{2}(u,u)} du.$$

Equation (89) is $O(t^4e^{-t^2})$ by (84).

From Lemma (12) Part 2, using $A = I$ and $B = \Omega_{n,g}$, we have

$$\left(\int_{\mathbb{R}^{2n^2}} |\det \xi| |\det \eta| e^{-\frac{x}{2}(\Omega_{n,g}u,u)} du \right) - \int_{\mathbb{R}^{2n^2}} |\det \xi| |\det \eta| e^{-\frac{x}{2}(u,u)} du = O \left(||I - \Omega_{n,g}||_{1/2}^2\right) = O \left(te^{-\frac{x^2}{2}}\right).$$

(90)

Therefore, $|K(t) - 1| = O \left(te^{-\frac{x^2}{2}}\right)$, so we obtain the desired result.

\[ \square \] (Part 2) of Theorem (3).

7. PROOF OF THEOREM (3) UNIVERSALITY

We will need the following:

**Lemma 18.** For any $x = (0, 0, \ldots, s, t)$ and $y = (0, 0, \ldots, 0, u)$ with $x \neq y$:

1. the covariance matrices $C$ corresponding to random vectors (17) and (19) for the Isom($\mathbb{R}^n$)-invariant ensemble (12) is positive definite, and
2. the submatrices $\Omega$ of $C^{-1}$ defined in the Kac-Rice formulae (18) and (20), respectively, are positive definite.

**Proof.** We give the proof when $C$ corresponds to the vector (19), from the Kac-Rice formula for the correlation function (20), leaving the necessary modifications for vector (17) to the reader.

It is a general fact from probability theory that the covariance matrix of a random vector is positive semi-definite. Thus, it will be sufficient for us to check that $\det(C) > 0$ for $x \neq y$.

We substitute $x = (0, 0, \ldots, s, t)$ and $y = (0, 0, \ldots, 0, u)$ into the covariances computed in Equations (35) and (43) from the proof of Lemma (13) obtaining that $C$ is of the form $\text{diag}_n(\tilde{C})$, where $\tilde{C} = \begin{bmatrix} A & B^T \\ B & D \end{bmatrix}$ and $A, B, \text{ and } D$ are the following $(n + 1) \times (n + 1)$ matrices:

$$A = e^{s^2 + t^2}, \quad B = e^{st}, \quad \text{and } \begin{bmatrix} 1 & 0 & \cdots & s & t \\ 0 & 1 & \ddots & 0 & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ s & \ddots & 1 + s^2 & st & \vdots \\ t & 0 & \cdots & st & 1 + t^2 \end{bmatrix}, \begin{bmatrix} 1 & 0 & \cdots & 0 & u \\ 0 & 1 & \ddots & 0 & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ s & \ddots & 1 & su & \vdots \\ t & 0 & \cdots & 0 & 1 + tu \end{bmatrix}.$$
The former has determinant 
\[ (93) \]
\[ D = e^{u^2} \begin{bmatrix} 1 & 0 & \cdots & 0 & u \\ 0 & 1 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ u & 0 & \cdots & 0 & 1 + u^2 \end{bmatrix}. \]

After applying a suitable permutation to the rows and columns, \( \tilde{C} \) becomes a block-diagonal matrix with \( n - 2 \) copies of
\[ \begin{bmatrix} e^{s^2+t^2} & e^{st} \\ e^{st} & e^{u^2} \end{bmatrix} \]
and one copy of
\[ \begin{bmatrix} e^{s^2+t^2} & se^{s^2+t^2} & te^{s^2+t^2} & e^{st} & se^{st} & te^{st} \\ se^{s^2+t^2} & (1 + s^2) e^{s^2+t^2} & ste^{s^2+t^2} & 0 & e^{st} & 0 \\ te^{s^2+t^2} & ste^{s^2+t^2} & (1 + t^2) e^{s^2+t^2} & ue^{st} & sue^{st} & (1 + tu) e^{stu} \\ e^{st} & 0 & ue^{st} & e^{u^2} & 0 & ue^{u^2} \\ se^{st} & e^{st} & sue^{st} & 0 & e^{a^2} & 0 \\ te^{st} & 0 & (1 + tu) e^{stu} & ue^{u^2} & 0 & (1 + u^2) e^{u^2} \end{bmatrix} \]
The former has determinant \( e^{6tu} (e^{s^2+(t-u)^2} - 1) \), which is positive. The latter has determinant equal to
\[ e^{6tu} \left( (e^{s^2+(t-u)^2} - e^{2s^2+2(t-u)^2}) \left( (s^2 + (t - u)^2)^2 + 3 \right) + e^{3s^2+3(t-u)^2} - 1 \right). \]
Without the exponential prefactor, this equals \( (e^r - e^{2r}) (r^2 + 3) + e^{3r} - 1 \), for \( r = s^2 + (t - u)^2 \), which one can also check is positive for all \( r > 0 \).
Since \( C \) is positive definite, so is \( C^{-1} \). After applying a suitable permutation to the rows and columns of \( C^{-1} \) the \( n^2 \times n^2 \) principal minor is \( \Omega \), which is therefore positive definite. \( \square \)

**Proof of Theorem**

After applying a suitable isometry from \( SO(k + 1) \) we can assume:

1. \( p = [0 : \cdots : 0 : 1] \) and hence \( T_p \mathbb{R}^k \) span \( (e_1, \ldots, e_k) \) \( \in \mathbb{R}^{k+1} \),
2. \( T_p M = \text{span}(e_1, \ldots, e_n) \subset T_p \mathbb{R}^k \), and
3. \( x = (0, \ldots, 0, s, t) \) and \( y = (0, \ldots, 0, u) \).

Since the ensemble on \( \mathbb{R}^k \) is invariant under elements of \( SO(k + 1) \) and since we have rotated the submanifold \( M \) under the same isometry, the correlation function remains the same.

If the homogeneous coordinates on \( \mathbb{R}^k \) are denoted \( [Z_1, \ldots, Z_{k+1}] \) we now work in the affine coordinates \( z_1 = Z_1/Z_{k+1}, \ldots, z_k = Z_k/Z_{k+1} \). We identify \( T_p M \) with the subspace \( \mathbb{R}^n \subset \mathbb{R}^k \). Because of our choice (2) above, \( M \) is locally expressed as a graph of a \( C^2 \) function \( \psi : \mathbb{R}^n \to \mathbb{R}^{k-n} \) that satisfies
\[ \psi(0) = 0 \quad \text{and} \quad D\psi(0) = 0. \]

In affine coordinates, the \( SO(k + 1) \)-invariant polynomials are
\[ f_d(z) = \sum_{|\alpha| \leq d} \sqrt{\frac{d!}{(d - |\alpha|)!}} a_\alpha z^\alpha \quad \text{where} \quad \left( d \frac{d!}{(d - |\alpha|)!} \prod_{i=1}^n \alpha_i ! \right) 
\]
and the \( a_\alpha \) are iid on the standard normal distribution \( \mathcal{N}(0, 1) \).
The correlation between zeros
\[ K_{n,d,M} \left( \text{proj}_p \left( \frac{x}{\sqrt{d}} \right), \text{proj}_p \left( \frac{y}{\sqrt{d}} \right) \right) \]
is the same as the correlation between zeros for the pull-back of this ensemble to the tangent space under $\text{proj}_p$, which is given by systems of $n$ functions chosen iid of the form

$$h_{d,\psi} \left( \frac{x}{\sqrt{d}} \right) := f_d \left( \frac{x}{\sqrt{d}}, \psi \left( \frac{x}{\sqrt{d}} \right) \right)$$

on $\mathbb{R}^n \equiv T_pM$. This follows because one need not use round balls in the definition (7) of the correlation function—any sequence of neighborhoods converging to the points $p$ and $q$ that is sufficiently nice for computing a Radon-Nikodym derivative suffices (see Remark 11). If one uses round balls $N_\delta \left( \text{proj}_p \left( \frac{x}{\sqrt{d}} \right) \right)$ and $N_\delta \left( \text{proj}_p \left( \frac{y}{\sqrt{d}} \right) \right)$ in the definition of $K_{n,d,M} \left( \text{proj}_p \left( \frac{x}{\sqrt{d}} \right), \text{proj}_p \left( \frac{y}{\sqrt{d}} \right) \right)$, then their preimages under the $C^2$ mapping $\text{proj}_p$ will be suitable neighborhoods for defining the correlation function for the pull-back (96).

Thus, it is sufficient for us to prove that

$$K_{n,d,\psi} \left( \left( \frac{x}{\sqrt{d}} \right), \left( \frac{y}{\sqrt{d}} \right) \right) = K_n(x,y) + O \left( \frac{1}{\sqrt{d}} \right),$$

where $K_{n,d,\psi} \equiv K_{d,\psi}$ denotes the correlation function for systems of $n$ functions chosen iid of the form (96).

Let $\psi(x) = (\psi_1(x), \ldots, \psi_k(x))$. If we write $\alpha = (\beta, \gamma)$ with $\beta \in \mathbb{Z}_n^+$ and $\gamma \in \mathbb{Z}_k^+$, then (96) becomes

$$H_{d,\psi} \left( \frac{x}{\sqrt{d}} \right) = \sum_{|\beta| + |\gamma| \leq d} b_{(\beta,\gamma)} \left( \frac{x}{\sqrt{d}} \right)^\beta \left( \psi \left( \frac{x}{\sqrt{d}} \right) \right)^\gamma$$

where $b_{(\beta,\gamma)} := \sqrt{\left( \frac{d}{|\beta| \gamma |\gamma|} \right)^a_{(\beta,\gamma)}}$ and $\left( \frac{d}{|\beta| \gamma} \right)^a_{(\beta,\gamma)} = \frac{d!}{(d - |\beta| - |\gamma|)!} \prod_{i=1}^{n} \beta_i! \prod_{i=1}^{k} \gamma_i!$.

As before, the coefficients $a_{(\beta,\gamma)}$ are iid on the standard normal distribution $\mathcal{N}(0,1)$.

**Proposition 19.** We have

1. Let $C_{d,\psi}$ and $\mathbf{C}$ be the covariance matrix for vector (17) applied to the systems $h_{d,\psi}(x)$ and $f(x)$, respectively, and let $\Omega_{d,\psi}$ and $\Omega$ denote the submatrices of $C^{-1}_{d,\psi}$ and $C^{-1}$ defined in the Kac-Rice formula for density (17). Then,

$$C_{d,\psi} = \mathbf{C} + O \left( \frac{1}{d} \right) \quad \text{and} \quad \Omega_{d,\psi} = \Omega + O \left( \frac{1}{d} \right),$$

where the constants implicit in the notation depends uniformly on compact subsets of $\mathbb{R}^n$.

2. Let $C_{d,\psi}$ and $\mathbf{C}$ be the covariance matrix for vector (17) applied to the systems $h_{d,\psi}(x)$ and $f(x)$, respectively, and let $\Omega_{d,\psi}$ and $\Omega$ denote the submatrices of $C^{-1}_{d,\psi}$ and $C^{-1}$ defined in the Kac-Rice formula for density (17). Then,

$$C_{d,\psi} = \mathbf{C} + O \left( \frac{1}{d} \right),$$

where the constants implicit in the $O$-notation depends uniformly on compact subsets of $\mathbb{R}^n \times \mathbb{R}^n$.

If $x = (0, \ldots, 0, s, t)$ and $y = (0, \ldots, 0, u)$, with $x \neq y$, then

$$\Omega_{d,\psi} = \Omega + O \left( \frac{1}{d} \right),$$

with the constant depending uniformly on compact subsets of $\mathbb{R}^2 \times \mathbb{R} \setminus \{(s, t) = (0, u)\}$.

**Proof.** We will prove Part 2, leaving the necessary (simple) modifications for Part 1 to the reader.

We will first show that for $\mathcal{H}$ the result depends on $\psi$ only to order $1/d$:

$$C_{d,\psi} = C_{d,0} + O \left( \frac{1}{d} \right).$$
Because $\psi$ is $C^2$, there exists a constant $A > 0$ independent of $d$ such that for any multi-indices $\beta \in \mathbb{Z}^n_+$ and $\gamma \in \mathbb{Z}^{k-n}_+$ we have:

\begin{align}
\left\| \frac{\partial}{\partial x_i} \left( \frac{x}{\sqrt{d}} \right)^\beta \right\| &= O \left( \left( \frac{A}{\sqrt{d}} \right)^{|\beta|} \right) \\
\left\| \psi \left( \frac{x}{\sqrt{d}} \right)^\gamma \right\| &= O \left( \left( \frac{A}{d} \right)^{|\gamma|} \right) \\
\left\| \frac{\partial}{\partial x_i} \psi \left( \frac{x}{\sqrt{d}} \right)^\gamma \right\| &= O \left( \left( \frac{A}{d} \right)^{|\gamma|} \right) 
\end{align}

Both the constant $A$ and the multiplicative constants (implicit in the $O$ notation) depend continuously on $x$.

We can now prove that for any $x, y \in \mathbb{R}^n$

\begin{align}
\sum_{|\beta|+|\gamma| \leq d, |\gamma| \geq 1} \left( \frac{d}{\beta \gamma} \right) \left( \frac{x}{\sqrt{d}} \right)^\beta \left( \psi \left( \frac{x}{\sqrt{d}} \right) \right)^\gamma \left( \frac{y}{\sqrt{d}} \right)^\beta \left( \psi \left( \frac{y}{\sqrt{d}} \right) \right)^\gamma &= O \left( \frac{1}{d} \right), \\
\sum_{|\beta|+|\gamma| \leq d, |\gamma| \geq 1} \left( \frac{d}{\beta \gamma} \right) \frac{\partial}{\partial x_i} \left( \frac{x}{\sqrt{d}} \right)^\beta \left( \psi \left( \frac{x}{\sqrt{d}} \right) \right)^\gamma \left( \frac{y}{\sqrt{d}} \right)^\beta \left( \psi \left( \frac{y}{\sqrt{d}} \right) \right)^\gamma &= O \left( \frac{1}{d} \right), \\
\sum_{|\beta|+|\gamma| \leq d, |\gamma| \geq 1} \left( \frac{d}{\beta \gamma} \right) \frac{\partial}{\partial x_i} \left( \frac{x}{\sqrt{d}} \right)^\beta \left( \psi \left( \frac{x}{\sqrt{d}} \right) \right)^\gamma \frac{\partial}{\partial y_j} \left( \frac{y}{\sqrt{d}} \right)^\beta \left( \psi \left( \frac{y}{\sqrt{d}} \right) \right)^\gamma &= O \left( \frac{1}{d} \right),
\end{align}

where we can have $x = y$ and/or $i = j$ and where the constant implicit in the big-O notation depends uniformly on compact subsets of $\mathbb{R}^n \times \mathbb{R}^n$.

The proofs of each of these are essentially the same, so we’ll prove (104), leaving the proofs of (102) and (103) for the reader. We have

\begin{align}
\frac{\partial}{\partial x_i} \left( \frac{x}{\sqrt{d}} \right)^\beta \psi \left( \frac{x}{\sqrt{d}} \right)^\gamma \left( \frac{y}{\sqrt{d}} \right)^\beta \psi \left( \frac{y}{\sqrt{d}} \right)^\gamma &= O \left( \frac{1}{d} \right), \\
\frac{\partial}{\partial x_i} \left( \frac{x}{\sqrt{d}} \right)^\beta \psi \left( \frac{x}{\sqrt{d}} \right)^\gamma \frac{\partial}{\partial y_j} \left( \frac{y}{\sqrt{d}} \right)^\beta \psi \left( \frac{y}{\sqrt{d}} \right)^\gamma &= O \left( \frac{1}{d} \right), \\
\frac{\partial}{\partial x_i} \left( \frac{x}{\sqrt{d}} \right)^\beta \psi \left( \frac{x}{\sqrt{d}} \right)^\gamma \frac{\partial}{\partial y_j} \left( \frac{y}{\sqrt{d}} \right)^\beta \psi \left( \frac{y}{\sqrt{d}} \right)^\gamma &= O \left( \frac{1}{d} \right),
\end{align}

by (99), (101). In particular,

\begin{align}
\sum_{|\beta|+|\gamma| \leq d, |\gamma| \geq 1} \left( \frac{d}{\beta \gamma} \right) \left( \frac{x}{\sqrt{d}} \right)^\beta \psi \left( \frac{x}{\sqrt{d}} \right)^\gamma \left( \frac{y}{\sqrt{d}} \right)^\beta \psi \left( \frac{y}{\sqrt{d}} \right)^\gamma &\leq \left( \frac{B}{\sqrt{d}} \right)^{|\beta|+|\gamma|} \\
\frac{B}{d} \sum_{|\beta|+|\gamma| \leq d, |\gamma| \geq 1} \left( \frac{B}{d} \right)^{|\beta|+|\gamma|} &\leq \frac{B}{d} \sum_{|\beta|+|\gamma| \leq d, |\gamma| \geq 1} \left( \frac{B}{d} \right)^{|\beta|+|\gamma|} \leq \frac{B}{d} \left( 1 + 2 \frac{B}{d} \right)^d \leq \frac{C}{d}.
\end{align}
The estimate (98) follows immediately. For example,
\[
E \left( \frac{\partial h_{d,\psi}(x) \partial h_{d,\psi}(y)}{\partial x_i \partial y_j} \right)
= \sum_{|\beta| \leq d} \frac{d}{\beta} \frac{\partial}{\partial x_i} \left( \frac{x}{\sqrt{d}} \right)^{\beta} \frac{\partial}{\partial y_j} \left( \frac{y}{\sqrt{d}} \right)^{\beta} + \sum_{|\beta| \geq d} \frac{d}{\beta} \frac{\partial}{\partial x_i} \left( \frac{x}{\sqrt{d}} \right)^{\beta} \frac{\partial}{\partial y_j} \left( \frac{\psi \left( \frac{x}{\sqrt{d}} \right)}{\sqrt{d}} \right)^{\beta} \frac{\partial}{\partial y_j} \left( \frac{\psi \left( \frac{y}{\sqrt{d}} \right)}{\sqrt{d}} \right)^{\beta}
= E \left( \frac{\partial h_{d,0}(x) \partial h_{d,0}(y)}{\partial x_i \partial y_j} \right) + O \left( \frac{1}{d} \right).
\]

We will now show that
\[
(109) \quad C_{d,0} = C + O \left( \frac{1}{d} \right).
\]
It follows from a calculation analogous to that from Lemma [13] but with a rescaling by $1/\sqrt{d}$ and the fact that
\[
\left( 1 + \frac{x}{d} \right)^{d} = e^{x} + O \left( \frac{1}{d} \right),
\]
with the constant depending uniformly on $x \in \mathbb{R}$. Rather than including the computation for each of the eight different types of expectations, we simply list one of the more complicated ones here:
\[
E \left( \frac{\partial h_{d,0}(x) \partial h_{d,0}(y)}{\partial x_i \partial y_j} \right) = \left( 1 + \frac{x \cdot y}{d} + \frac{x_i y_i (d-1)}{d} \right) \left( 1 + \frac{x \cdot y}{d} \right)^{d-2}
\]
\[
E \left( \frac{\partial f(x) \partial f(y)}{\partial x_i \partial y_j} \right) = (1 + x_i y_i) e^{x \cdot y}.
\]

If $x = (0, \ldots, 0, s, t)$ and $y = (0, \ldots, 0, u)$, with $x \neq y$, Lemma [13] gives that $C$ is positive definite. Then, there is a neighborhood of $C$ in the space of all $2n^2 \times 2n^2$ matrices on which taking the inverse is a differentiable map. Therefore, $C_{d,\psi}^{-1} = C^{-1} + O \left( \frac{1}{d} \right)$ and hence $\Omega_{d,\psi} = \Omega + O \left( \frac{1}{d} \right)$. \hfill \Box

Because we’ve normalized to have $x = (0, \ldots, 0, s, t)$ and $y = (0, \ldots, 0, u)$, with $x \neq y$, Lemma [13] gives that $C$ is positive definite. Therefore, $C_{d,\psi}$ will also be positive definite for $d$ sufficiently large, allowing us to apply the Kac-Rice formula (20) repeated below to (96). We will show that the result differs by $O \left( \frac{1}{d} \right)$ from the result when applying it to the Isom($\mathbb{R}^n$)-invariant ensemble (4).

\[
(20) \quad K_n(x, y) = \frac{1}{(2\pi)^{(n+1)}} \rho(x) \rho(y) \sqrt{\det C} \int_{\mathbb{R}^{2n^2}} |\det \xi| |\det \eta| e^{-\frac{1}{2}(\xi^T \Omega \eta + u^T \xi)} du.
\]

Let us first consider the prefactor from (20). To show that $\rho_d(x) = \rho(x) + O \left( \frac{1}{\sqrt{d}} \right)$, we apply Part 1 of Lemma [12] to the Kac-Rice formula for density (18). This follows because the matrix $\Omega$ is positive definite, by Lemma [13] (using that we have normalized that $x = (0, \ldots, 0, s, t)$ and $y = (0, \ldots, 0, u)$, with $x \neq y$), and because $\Omega_d = \Omega + O \left( \frac{1}{d} \right)$, by Part 1 of Proposition [19]. The same estimate holds at $y$.

Since $C_d = C + O \left( \frac{1}{d} \right)$, with $C$ positive definite, it follows immediately that
\[
\frac{1}{\sqrt{\det C_d}} = \frac{1}{\sqrt{\det C}} + O \left( \frac{1}{d} \right).
\]

We now consider the integral in (20). Lemma [18] gives that $\Omega$ is positive definite. By Part 2 of Proposition [19] we have that $\Omega_d = \Omega + O \left( \frac{1}{d} \right)$. Thus, Part 2 of Lemma [12] gives that the integral from (20) when applied to the ensemble $h_d$ (Equation 96) differs by $O \left( \frac{1}{\sqrt{d}} \right)$ from the result when applying it to the Isom($\mathbb{R}^n$)-invariant ensemble (Equation 4), with the constant depending uniformly on compact subsets of $\mathbb{R}^2 \times \mathbb{R} \setminus \{(s, t) = (0, u)\}$. \hfill \Box[Theorem 3]
8. Finite degree restrictions to submanifolds $M \subset \mathbb{RP}^k$ depend on the geometry

We present a simple example illustrating that the constant from the leading term in the correlation function can depend on the geometry of a submanifolds $M \subset \mathbb{RP}^k$ if the degree $d$ of the ensemble is finite. Thus, it is not possible to prove a universal formula for the short-range asymptotics at finite degree.

We consider the ensemble $SO(3)$-invariant polynomials of degree 3 because for degrees 1 and 2 the covariance matrix for random vector (19) is not positive definite. Restricted to a system of affine coordinates $(x, y)$, each such polynomial has the form

$$F_3(x, y) = a_{0,0} + \sqrt{3}\alpha_{1,0}x + \sqrt{3}\alpha_{0,1}y + \sqrt{6}\alpha_{1,1}xy + \sqrt{3}\alpha_{2,0}x^2 + \sqrt{3}\alpha_{0,2}y^2 + \sqrt{3}\alpha_{2,1}x^2y + \sqrt{3}\alpha_{1,2}xy^2 + a_{3,0}x^3 + a_{0,3}y^3,$$

where each of the coefficients is chosen iid with respect to the standard normal distribution, $N(0, 1)$.

Let $M \subset \mathbb{R}^2$ be given by $y = bx^2$. As in the previous section, we parameterize $M$ by the x coordinate, in this case forming the one-variable ensemble of random polynomials that depend on $b$ as a parameter:

$$H_{3,a}(x) = a_{0,0} + \sqrt{3}\alpha_{1,0}x + \sqrt{3}\alpha_{0,1}bx^2 + \sqrt{6}\alpha_{1,1}bx^3 + \sqrt{3}\alpha_{2,0}x^2 + \sqrt{3}\alpha_{0,2}bx^4 + \sqrt{3}\alpha_{2,1}bx^4 + \sqrt{3}\alpha_{1,2}x^5 + a_{3,0}x^3 + a_{0,3}bx^6,$$

In order for our results from Sections 4 and 5 to apply here, we multiply by the prefactor: $h_{3,b} := \left(1 + \frac{x^2}{2}\right)^3 H_{3,b}(x)$.

We will use the Kac-Rice formula (20) to show that the value of $b$ affects the constant term in the short-range asymptotics for the correlation function $K(x, y)$ with $x = -\frac{1}{2}$ and $y = \frac{1}{2}$.

An easy calculation using the Kac-Rice formula (18) for density of zeros gives that

$$\rho(x) = \rho(y) = \frac{\sqrt{3}}{\pi} + O(t^2),$$

where the constant term agrees with the result (3) that is stated in the introduction for the $SO(2)$-invariant polynomials of degree 3.

The covariance matrix for random vector (19) applied to the ensemble $h_{3,b}$ with $x = -\frac{1}{2}$ and $y = \frac{1}{2}$ is

$$C = \begin{bmatrix} \alpha & \delta & \mu & -\nu \\ \delta & \gamma & \nu & \tau \\ \mu & \nu & \alpha & -\delta \\ -\nu & \tau & -\delta & \gamma \end{bmatrix}$$

where

$$\alpha = \left(\frac{k^2 t^4 + 4t^2 + 16}{4906}\right)^3, \quad \delta = -\frac{3t}{1024} \left(\frac{k^2 t^4 + 2}{1024}\right) \left(\frac{k^4 t^4 + 4t^4 + 16}{256}\right)^2, \quad \mu = \left(\frac{k^2 t^4 - 4t^2 + 16}{4906}\right)^3,$$

$$\gamma = \frac{3k^2 t^4 + 12t^2 + 48}{256} \left(\frac{3k^4 t^4 + 13k^2 t^4 + 16k^2 t^2 + 12t^2 + 16}{1024}\right)^2, \quad \nu = \frac{3t}{1024} \left(\frac{k^2 t^4 - 2}{1024}\right) \left(\frac{k^4 t^4 - 4t^2 + 16}{256}\right)^2,$$

and

$$\tau = -\frac{3k^2 t^4 - 12t^2 + 48}{256} \left(\frac{3k^4 t^4 - 13k^2 t^4 + 16k^2 t^2 + 12t^2 - 16}{1024}\right).$$

This matrix has exactly the same structure as that from Sections 4 and 5 in the case that the dimension is 1. Therefore, the submatrix $\Omega$ of $C^{-1}$ from the Kac-Rice formula (20) is diagonalized in precisely the same fashion, with the eigenvalues satisfying

$$\lambda_3^{-1/2} = \sqrt{6(1 + b^2)} \ t + O(t^2) \quad \text{and} \quad \lambda_4^{-1/2} = \sqrt{\frac{1}{2} + 3b^2} \ t^2 + O(t^3).$$

(We call them $\lambda_3$ and $\lambda_4$ in order to be consistent with the previous sections.) We also have

$$\det C = \left(54b^4 + 63b^2 + 9\right) t^8 + O(t^{10}).$$

The calculation of the short-range asymptotics done in Section 5 applies here, with the minor modifications to $\lambda_3^{-1/2}$, $\lambda_4^{-1/2}$, and $\det C$ listed above. One obtains
Proposition 20. The correlation between zeros for ensemble \( \Omega \) satisfies

\[
K \left( \frac{-t}{2}, \frac{t}{2} \right) = \frac{\pi}{2\sqrt{3}} (1 + b^2) t + O(t^3).
\]

In particular, the leading term depends on the curvature of \( M \) at 0.

Question 21. In the general setting of \( M \subset \mathbb{R}P^k \) how does the constant in the leading order asymptotics near \( p \in M \) depend on the local geometry of \( M \) at \( p \)?

9. UNIVERSALITY IN THE COMPLEX SETTING

We begin by adapting the Kac-Rice formulae \((18)\) and \((20)\) to the complex setting. As the modifications are nearly identical, we will discuss the formula for correlations, leaving the formula for density to the reader.

Suppose that \( h = (h_1, h_2, \ldots, h_n) : \mathbb{C}^n \to \mathbb{C}^n \) is a Gaussian random analytic function with complex Gaussian coefficients. Let \( \xi \) and \( \eta \) be the \( n \times n \) complex matrices whose rows are \( \xi_1, \ldots, \xi_n \) and \( \eta_1, \ldots, \eta_n \), respectively. Let \( u = [\xi_1 \eta_1 \xi_2 \eta_2 \ldots \xi_n \eta_n]^T \), the vector formed by alternating the vectors \( \xi_i \) and \( \eta_i \).

Proposition 22. Suppose that the covariance matrix \( C = E(\xi_i\xi_j^*) \) of the random vector \( \xi_{ij} \) is positive definite. Then, the two-point correlation function for the zeroes of the system \( h \) is:

\[
K_n(x, y) = \frac{1}{\pi^{n(n+1)}|\rho(x)\rho(y)|} \det C \int \det(\xi^* \xi) \det(\eta^* \eta) e^{-(\Omega_{\mathbb{U}} \xi \xi^* + \Omega_{\mathbb{U}} \eta \eta^*)} du d\overline{u},
\]

where \( \Omega \) is the matrix of the elements of \( \mathbb{C}^{n+1} \) left after removing the rows and columns that correspond to the elements \( h_i(x) \) and \( h_i(y) \), i.e., all of the rows and columns with indices congruent to 1 modulo \( n + 1 \), * denotes conjugate transpose, and \( \overline{,} \) denotes the Hermitian inner product.

This follows from \((6)\) Theorem 2.1 by using the suitable Gaussian density \( D_k(0, \xi, z) \) in their formula 32 and normalizing by the density at the two points \( x \) and \( y \). (In \((6)\), the authors use a simplification of Theorem 2.1 to a somewhat different formula \((46)\). We use \((112)\) in order to make the parallel with \((20)\) more apparent.)

With these modifications to the Kac-Rice formulae, the proof of Theorem \((14)\) adapts nearly verbatim from the proof of Theorem \((5)\). We list below the simple modifications that need to be checked:

1. The proof of Lemma \((12)\) adapts easily to the integral expression in \((112)\) and to the analogous formula for the densities \( \rho(x) \) and \( \rho(y) \).

2. The proof of Lemma \((18)\) is easily adapted. More specifically, the determinant of the \( 6 \times 6 \) block analogous to \((23)\) equals:

\[
e^{-\text{Re}(\Omega)} \left( e^{(|s|)^2 + (|t-u|)^2} - e^{2(|s|)^2 + 2(|t-u|)^2} \right) \left( (s)^2 + (t-u)^2 \right)^2 + e^{3(|s|)^2 + 3(|t-u|)^2} - 1,
\]

which is positive for \( (s, t) \neq (0, u) \).

3. The proof of Proposition \((19)\) applies after verifying that, when expressed in local coordinates, the covariance matrix for the rescaled \( SU(n+1) \)-invariant polynomials differs from that of the \( \text{Isom}(\mathbb{C}^n) \)-invariant ensemble by \( O \left( \frac{1}{R} \right) \). (These covariances are listed in \((6)\) Sections 2.4 and 4.)

APPENDIX A. PROOF OF LEMMA \((12)\)

We will need the following lemma to prove Lemma \((12)\)

Lemma 23. We have

1. For any positive definite \( n^2 \times n^2 \) matrix \( A \), there exists a constants \( D > 0 \) such that, for \( R \) sufficiently large,

\[
\left| \int_{|u| > R} |\det \xi| e^{-\frac{1}{2} \langle A \xi, \xi \rangle} du \right| \leq e^{-DR}.
\]

2. For any positive definite \( n^2 \times n^2 \) matrix \( A \), there exists constants \( D > 0 \) such that, for \( R \) sufficiently large,

\[
\left| \int_{|u| > R} |\det \xi||\det \eta| e^{-\frac{1}{2} \langle A \xi, \eta \rangle} du \right| \leq e^{-DR}.
\]
Proof. We will prove Part 2 of the lemma, leaving the necessary modifications for Part 1 to the reader. First, we note that \((Au, u) \geq \lambda_{\text{min}}||u||^2\). The left side of inequality (114) is bounded by

\[
T := \int_{||u|| > R} |\det\xi| |\det\eta| e^{-\frac{x}{2}\lambda_{\text{min}}||u||^2} du \leq \text{Vol}([S^{2n-2} - 1])n!^2 \int_0^{\infty} e^{2n^2 + 2n - 1 - \frac{x}{2}\lambda_{\text{min}}} r^2 dr,
\]

using that \(|\det\xi| |\det\eta| \leq n!^2||u||^{2n^2}. Let \(a := 2n^2 + 2n - 1\) and \(b := \frac{\lambda_{\text{min}}}{2}.

\[
\int_0^{\infty} e^{a r^2} dr \leq \int_0^{\infty} e^{-br^2 + ar^2} dr = \frac{\sqrt{\pi} e^{\frac{a}{2}}}{2} \text{erfc}\left(\frac{\sqrt{b}R - \frac{a}{2\sqrt{b}}}{2}\right).
\]

The result then follows because \(\text{erfc}(x) := \frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-t^2} dt \leq e^{-x^2}. \)

Proof of Lemma 12. Like in the proof of Lemma 23 we will prove Part 2 of Lemma 12 leaving the necessary modifications for Part 1 to the reader.

We first split the left side of the inequality into integrals with \(||u|| < R\) and \(||u|| > R\) for some \(R\):

\[
\int_{\mathbb{R}^{2n^2}} |\det\xi| |\det\eta| e^{-\frac{x}{2}(Bu, u)} - \int_{\mathbb{R}^{2n^2}} |\det\xi| |\det\eta| e^{-\frac{x}{2}(Au, u)} du
\leq \int_{||u|| < R} |\det\xi| |\det\eta| e^{-\frac{x}{2}(Au, u)} \left(e^{-\frac{x}{2}((B - A)u, u)} - 1\right) du
\]

\[
+ \int_{||u|| > R} |\det\xi| |\det\eta| e^{-\frac{x}{2}(Bu, u)} du
\]

For \(B\) sufficiently close to \(A\), \(\min (Bu, u) \sqrt{2} \geq (Au, u)\). Thus, Lemma 23 gives that the two latter summands of (117) are both bound by \(e^{-DR}\).

We use Hölder’s Inequality to bound the first summand in (117) with

\[
\int_{||u|| < R} |\det\xi| |\det\eta| e^{-\frac{x}{2}(Au, u)} du \lesssim \left\| e^{-\frac{x}{2}((B - A)u, u)} - 1 \right\|_{L^\infty(||u|| < R)} = D_5 \left\| e^{-\frac{x}{2}((B - A)u, u)} - 1 \right\|_{L^\infty(||u|| < R)}.
\]

Let \(L = A - B\). Since \(e^{\frac{x}{2} - 1} < x\) as \(x\) approaches 1,

\[
\left\| e^{\frac{x}{2}(Lu, u)} - 1 \right\|_{L^\infty(||u|| < R)} \leq \left\| (Lu, u) \right\|_{L^\infty(||u|| < R)} \leq 2n^2 \left\| L \right\|_{\infty} \left\| u \right\|_{2} ||u||_{< R} \leq 2n^2 R^2 \left\| L \right\|_{\infty}.
\]

Therefore, we have

\[
\int_{\mathbb{R}^{2n^2}} |\det\xi| |\det\eta| e^{-\frac{x}{2}(Au, u)} \left(e^{-\frac{x}{2}((B - A)u, u)} - 1\right) du \leq 2n^2 D_5 R^2 \left\| L \right\|_{\infty} + 2e^{-DR}
\]

The result follows if we set \(R = \left\| A - B \right\|_{\infty}^{-1/4}. \)

Appendix B. Moments of the Volume of a Random Unit Parallelepiped

The derivation of the formula for the density (6) and the short-range asymptotics from Theorems 1 and 2 require the following formula:
Proposition 24. Consider $n$ random unit vectors in $\mathbb{R}^n$ chosen independently with respect to spherical measure. The $k$th moment of the volume $V$ of the parallelotope formed by these vectors is

$$E[V^k] = \left( \frac{\Gamma \left( \frac{n}{2} \right)}{\Gamma \left( \frac{n-k}{2} \right)} \right)^{n-1} \prod_{i=1}^{k} \frac{\Gamma \left( \frac{n-i+1}{2} \right)}{\Gamma \left( \frac{n+1}{2} \right)}.$$

This result is proved in greater generality in [13] (see Equation 23); however, we will give a simple derivation of the formula.

Proof. The method for finding this moment involves fixing each of the vectors $v_i$ that determine the parallelotope one at a time with respect to the parallelotope in $\mathbb{R}^i$ described by the previous $i-1$ vectors, weighting each newly added vector based on the probability of obtaining a given height off of the previous $(i-1)$–parallelotope.

Thus, after fixing the first vector, we see that the height of the second vector off of the first is $\cos (\theta)$ and the probability density of obtaining this height is

$$\frac{\sin^{n-2} \theta}{\int_0^{\pi} \sin^{n-2} \theta \, d\theta}.$$ 

Yet, from the next vector on, the vector can vary in two directions, along two different spheres, one of dimension $i-1$ and the other of dimension $n-i-1$, in order to maintain the same height. Thus, for the $i$th vector, we have that the probability density of obtaining each height $\cos \theta$ for an angle $\theta$ off the normal vector of the base given by the first $i-1$ vectors is

$$\frac{\sin^{n-i-1} \theta \cos^{i-1} \theta}{\int_0^{\pi} \sin^{n-i-1} \theta \cos^{i-1} \theta \, d\theta}.$$ 

To express the $k$th power of the volume, we multiply the $k$th power of each of the heights together. Thus,

$$E[V^k] = \prod_{i=1}^{n-1} \left( \frac{\int_0^{\pi/2} \sin^{n-i-1} \theta \cos^{i-1+k} \theta \, d\theta}{\int_0^{\pi/2} \sin^{n-i-1} \theta \cos^{i-1} \theta \, d\theta} \right) = \left( \frac{\Gamma \left( \frac{n}{2} \right)}{\Gamma \left( \frac{n+k}{2} \right)} \right)^{n-1} \prod_{i=1}^{k} \frac{\Gamma \left( \frac{n-i+1}{2} \right)}{\Gamma \left( \frac{n+1}{2} \right)}.$$

$$= \left( \frac{\Gamma \left( \frac{n}{2} \right)}{\Gamma \left( \frac{n+k}{2} \right)} \right)^{n-1} \prod_{i=1}^{k} \frac{\Gamma \left( \frac{n-i+1}{2} \right)}{\Gamma \left( \frac{n+1}{2} \right)}.$$

□

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