Abstract

Cosheaves are a dual notion of sheaves. In this paper, we prove existence of a dual of sheafifications, called cosheafifications, in the $\infty$-category theory. We also prove that the $\infty$-category of $\infty$-cosheaves is presentable and equivalent to an $\infty$-category of left adjoint functors.

Introduction

Let $\mathcal{X}$ be a small category equipped with a pretopology $J$ and $\mathcal{V}$ be a category. Recall that a functor $\mathcal{G} : \mathcal{X} \to \mathcal{V}$ is called a precosheaf on $\mathcal{X}$. It is called a cosheaf if the diagram

\[
\prod_{j,k \in I} \mathcal{G}(u_j \times_U u_j) \rightrightarrows \prod_{i \in I} \mathcal{G}(u_i) \to \mathcal{G}(U) \tag{0.1}
\]

is a coequalizer in $\mathcal{V}$ for all $\{U_i \to U\}_i \in J$. In other words, a cosheaf is a $\mathcal{V}^{\text{op}}$-valued sheaf. This is a dual notion of sheaves. For example, the zeroth homology functor on the category of open sets is a cosheaf of abelian groups. We write $\text{PCSh}(\mathcal{X}, \mathcal{V})$ for the category of $\mathcal{V}$-valued precosheaves on $\mathcal{X}$ and $\text{CSh}(\mathcal{X}, \mathcal{V})$ for its full subcategory consisting of cosheaves. In [Pra16], Prasolov proved that the inclusion $\text{CSh}(\mathcal{X}, \mathcal{V}) \hookrightarrow \text{PCSh}(\mathcal{X}, \mathcal{V})$ admits a right adjoint, called the cosheafification functor. This is a dual notion of sheafification, which is a left adjoint functor of the natural inclusion.

In this paper, we develop a basic theory of cosheaves in $\infty$-category, and prove the existence of cosheafifications in the $\infty$-categorical situation. Let $\mathcal{X}$ be a small $\infty$-site and $\mathcal{V}$ be a symmetric closed monoidal $\infty$-category which is presentable. We define $\mathcal{V}$-valued cosheaves on $\mathcal{X}$ by using Čech nerves instead of coequalizer diagrams (Definition 3.1). The reason that we use Čech nerves is to define cosheaves as a dual of sheaves in the sense of Lurie [Lur09]. It turns
out that this definition is indeed a generalization of cosheaves in 1-category (see Remark 3.6).

Our main result is summarized as follows.

**Theorem 1** (see Theorem 3.4). Assume that \( \mathcal{X} \) and \( \mathcal{V} \) satisfy the conditions (2.2) and (3.1). Then

1. \( \text{CSh}(\mathcal{X}, \mathcal{V}) \hookrightarrow \text{PCSh}(\mathcal{X}, \mathcal{V}) \) admits a right adjoint,
2. \( \text{CSh}(\mathcal{X}, \mathcal{V}) \) is presentable, and
3. \( \text{CSh}(\mathcal{X}, \mathcal{V}) \) is equivalent to the \( \infty \)-category of left adjoint functors from \( \text{Sh}(\mathcal{X}, \mathcal{V}) \) to \( \mathcal{V} \).

The right adjoint functor constructed in (1) gives the \( \infty \)-cosheafification. The conditions (2.2) and (3.1) are satisfied in various cases including the following (cf. Examples 2.11 and 3.5): with \( \mathcal{X} \) arbitrary,

- \( \mathcal{V} \) the \( \infty \)-category \( S \) of \( \infty \)-groupoids,
- \( \mathcal{V} \) the full subcategory of \( S \) consisting of \( n \)-truncated objects for \( n \geq 0 \),
- \( \mathcal{V} \) the connective part of the \( \infty \)-derived category of a commutative ring,
- \( \mathcal{V} \) the \( \infty \)-category of connective spectra.

In [BF06], Bunge-Funk proved that for a site \( \mathcal{X} \) and an elementary topos \( \mathcal{V} \), \( \text{CSh}(\mathcal{X}, \mathcal{V}) \) is equivalent to the category of cocontinuous functors from \( \text{Sh}(\mathcal{X}, \mathcal{V}) \) to \( \mathcal{V} \). Theorem 1 (3) implies Bunge-Funk’s result in the case \( \mathcal{V} = \text{Set} \).

This paper is organized as follows. In §1, we give technical preliminaries from the \( \infty \)-category theory. In §2, we prove that the \( \infty \)-category \( \text{PCSh}(\mathcal{X}, \mathcal{V}) \) is equivalent to the \( \infty \)-category of left adjoint functors from \( \text{PSh}(\mathcal{X}, \mathcal{V}) \) to \( \mathcal{V} \) under the condition (2.2). In §3, we prove the main result.

**Notation.** In this paper, we use quasi-categories as a model of \( \infty \)-categories. We write \( S \) for the \( \infty \)-category of small \( \infty \)-groupoids. For an \( \infty \)-category \( C \), the simplicially enriched hom-set from \( X \in C \) to \( Y \in C \) is denoted by \( \text{Hom}(X, Y) \). When \( C = S \), we write \( [X, Y] = \text{Hom}_S(X, Y) \) and \( P(X) = [X, S] \).

**Acknowledgments.** I would like to thank my adviser Shohei Ma for many useful advices. This work was supported by JSPS KAKENHI 19J21433.

1 **Right non-degenerate \( \infty \)-bifunctors**

In this section, we give technical preliminaries from the theory of \( \infty \)-categories. For \( \infty \)-categories \( C_1 \) and \( C_2 \), we write \( [C_1, C_2]^L \) (resp. \( [C_1, C_2]^R \)) for the full subcategory of \([C_1, C_2]\) consisting of left (resp. right) adjoint functors. We refer to [Lur09] for the basic theory of \( \infty \)-categories.

**Definition 1.1.** Let \( C \), \( C' \) and \( \mathcal{V} \) be \( \infty \)-categories. A bifunctor \( C \times C' \rightarrow \mathcal{V} \) is called right non-degenerate, if the induced functor \( C' \rightarrow [C, \mathcal{V}] \) is fully faithful and its essential image coincides with \([C, \mathcal{V}]^L\).
Throughout this section, we fix a right non-degenerate bifunctor $\langle , \rangle : \mathcal{C} \times \mathcal{C}^\vee \to \mathcal{V}$.

**Lemma 1.2.** If $\mathcal{C}$ and $\mathcal{V}$ are presentable, then so is $\mathcal{C}^\vee$.

**Proof.** This follows from the equivalence of $\infty$-categories $\mathcal{C}^\vee \cong [\mathcal{C}, \mathcal{V}]^L$ and [Lur09, Prop. 5.5.3.8].

We write $G^\dagger$ for the functor $\langle - , G \rangle : \mathcal{C} \to \mathcal{V}$. Recall that there exists a canonical equivalence of $\infty$-categories
$$\varepsilon : [\mathcal{C}, \mathcal{V}]^L \congto (|\mathcal{V}, \mathcal{C}|^R)^{\text{op}}$$
such that $\varepsilon F$ is right adjoint to $F$ for all $F \in [\mathcal{C}, \mathcal{V}]^L$ (see [Lur09, Prop. 5.2.6.2] and its proof). Then we write $G^\dagger = \varepsilon (G^\dagger)$ for $G \in \mathcal{C}^\vee$.

**Definition 1.3.** Let $\mathcal{D}$ be a full subcategory of $\mathcal{C}$. We write $\mathcal{D}^\vee$ for the full subcategory of $\mathcal{C}^\vee$ consisting of $G \in \mathcal{C}^\vee$ such that $G^\dagger v \in \mathcal{D}$ for all $v \in \mathcal{V}$. We call $\mathcal{D}^\vee$ the dual of $\mathcal{D}$ with respect to $\langle , \rangle$.

The following property says that localizations of $\mathcal{C}$ induce a right-nondegenerate bifunctor.

**Proposition 1.4.** Let $\mathcal{D}$ be a reflective full subcategory of $\mathcal{C}$.

1. The restriction $\mathcal{D} \times \mathcal{D}^\vee \to \mathcal{V}$ of $\langle , \rangle$ is right non-degenerate.

2. If $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{V}$ are presentable, then $\mathcal{D}^\vee$ is coreflective in $\mathcal{C}^\vee$.

**Proof.** (1) For $G \in \mathcal{D}^\vee$, the functor $G_\mathcal{D} : \mathcal{V} \to \mathcal{C}$ induces $\varepsilon_\mathcal{D} : \mathcal{V} \to \mathcal{D}$. Since $\varepsilon_\mathcal{D}$ is right adjoint to $G^\dagger|_\mathcal{D}$, we obtain $(-)_\mathcal{D} : \mathcal{D}^\vee \to (|\mathcal{V}, \mathcal{C}|^R)^{\text{op}}$. By the $\infty$-commutative diagram
$$\begin{array}{ccc}
\mathcal{D} & \to & (|\mathcal{V}, \mathcal{D}|^R)^{\text{op}} \\
\downarrow & & \downarrow \\
\mathcal{C} & \to & (|\mathcal{V}, \mathcal{C}|^R)^{\text{op}},
\end{array}$$
the functor $(-)_\mathcal{D}$ is fully faithful. The composition
$$G' : \mathcal{V} \xrightarrow{G_\mathcal{D}} \mathcal{D} \to \mathcal{C}$$
is contained in $|\mathcal{V}, \mathcal{C}|^R$ for all $G \in |\mathcal{V}, \mathcal{D}|^R$. Thus there exists $G \in \mathcal{C}^\vee$ such that $G_\mathcal{D}$ is equivalent to $G'$ in $(|\mathcal{V}, \mathcal{C}|^R)^{\text{op}}$. Then we have $G_\mathcal{D} \cong G$. This means that the functor $(-)_\mathcal{D}$ is essentially surjective and thus equivalence of $\infty$-categories. Therefore, $\mathcal{D} \times \mathcal{D}^\vee \to \mathcal{V}$ induces an equivalence of $\infty$-categories
$$\mathcal{D}^\vee \congto (|\mathcal{V}, \mathcal{D}|^R)^{\text{op}} \congto [\mathcal{D}, \mathcal{V}]^L.$$

(2) By Lemma 1.2, $\mathcal{C}^\vee$ and $\mathcal{D}^\vee$ are presentable. Thus by [Lur09, Cor. 5.5.2.9], it suffices to show that $\mathcal{D}^\vee$ is closed under small colimits in $\mathcal{C}^\vee$. Since limits of right adjoint functors are computed objectwise and $\mathcal{D}$ is closed under small limits in $\mathcal{C}$, the functor $[\mathcal{V}, \mathcal{D}]^R \to [\mathcal{V}, \mathcal{C}]^R$ preserves all small limits. Therefore, the assertion follows from the equivalences $|\mathcal{C}|^{\text{op}} \congto [\mathcal{V}, \mathcal{C}]^R$ and $|\mathcal{D}|^{\text{op}} \congto [\mathcal{V}, \mathcal{D}]^R$. 

\[\square\]
Next, we consider the dual of the localization of \( C \) by a set of morphisms \( S \). Let \( C^{S,\text{loc}} \) be the full subcategory of \( C \) consisting of \( S \)-local objects (see [Lur09, Def. 5.5.4.1]). We write \( C_{S,\text{col}}^{\vee} \) for the full subcategory of \( C^{\vee} \) consisting of objects \( G \) such that for every \( f : \mathcal{F} \to \mathcal{F}' \) in \( S \), the induced morphism

\[
\langle f, G \rangle : \langle \mathcal{F}, G \rangle \to \langle \mathcal{F}', G \rangle
\]

is an equivalence in \( V \).

**Proposition 1.5.** Let \( S \) be a set of morphisms in \( C \). Then \( (C^{S,\text{loc}})^{\vee} = C_{S,\text{col}}^{\vee} \).

**Proof.** Let \( f : \mathcal{F} \to \mathcal{F}' \) be a morphism in \( C \) and let \( G \in C^{\vee} \). It suffices to show that the following two conditions are equivalent.

1. The morphism \( \mathcal{G}^! f : \mathcal{G}^! \mathcal{F} \to \mathcal{G}^! \mathcal{F}' \) is an equivalence in \( V \).
2. The morphism \( \text{Hom}_C(f, \mathcal{G}^! v) : \text{Hom}_C(\mathcal{F}', \mathcal{G}^! v) \to \text{Hom}_C(\mathcal{F}, \mathcal{G}^! v) \) is an equivalence in \( S \) for all \( v \in V \).

By the \( \infty \)-Yoneda lemma, the condition (1) is equivalent to the following.

1’ The morphism \( \text{Hom}_C(\mathcal{G}^! f, v) : \text{Hom}_C(\mathcal{F}', \mathcal{G}^! v) \to \text{Hom}_C(\mathcal{F}, \mathcal{G}^! v) \) is an equivalence in \( S \) for all \( v \in V \).

Since \( \mathcal{G}^! \) is left adjoint to \( \mathcal{G}_! \), the equivalence \( \text{Hom}_C(\mathcal{G}^! f, v) \cong \text{Hom}_C(f, \mathcal{G}^! v) \) shows that (1’) \( \Leftrightarrow \) (2).

**Remark 1.6.** When \( C \) is presentable and \( S \) is small, \( C^{S,\text{loc}} \) is reflective and can be regarded as the localization \( S^{-1}C \) of \( C \) by \( S \) (see [Lur09, Prop. 5.5.4.2 and 5.5.4.15]). Therefore, we obtain a right non-degenerate bifunctor \( S^{-1}C \times C_{S,\text{col}}^{\vee} \to V \) by Proposition 1.4.

## 2 Pairing between presheaves and precosheaves

Throughout this section, we fix a small simplicial set \( X \) and (the underlying \( \infty \)-category of) a symmetric closed monoidal \( \infty \)-category \( V \) which is complete and cocomplete. We write \( \text{PSh}(X, V) = [X^{\text{op}}, V] \) and \( \text{PCSh}(X, V) = [X, V] \). Our aim of this section is to give a canonical right non-degenerate bifunctor \( \text{PSh}(X, V) \times \text{PCSh}(X, V) \to V \). Let \( \otimes, \mathcal{H}om \) and \( \mathbb{1} \) denote the monoidal product, the internal hom and the monoidal unit of \( V \), respectively. Let \( r_* \) be the covariant functor \( V \to S \) represented by \( \mathbb{1} \) and \( r^* \) be the functor \( S \to V \) defined by \( S_* \mapsto \text{colim}(S_* \to \{ \mathbb{1} \} \hookrightarrow V) \). The pair \( (r^*, r_*) \) is a variation of free-forgetful adjunctions.

**Lemma 2.1.** The functor \( r_* \) is right adjoint to \( r^* \).

**Proof.** For each \( S_* \in S, v \in V \) and \( n \geq 0 \), there exist equivalences

\[
\text{Hom}_V(r^* S_n, v) \cong \text{Hom}_V \left( \prod_{s \in S_n} \mathbb{1}, v \right) \cong \prod_{s \in S_n} \text{Hom}_V(\mathbb{1}, v) \cong \text{Hom}_S(S_n, r_* v).
\]
Since $S_\bullet$ is equivalent to the colimit of the functor $\Delta \to \mathcal{S}$, $[n] \mapsto S_n$, and $r^*$ preserves colimits, we obtain an equivalence

$$\text{Hom}_V(r^*S_\bullet, v) \cong \text{Hom}_S(S_\bullet, r_*v)$$

which is functorial in both $S_\bullet$ and $v$. \hfill \Box

Let $Y^\infty$ be the $\infty$-Yoneda embedding $\mathcal{X} \hookrightarrow \mathcal{P}(\mathcal{X})$. By Lemma 2.1, $r^*$ and $r_*$ induce an adjunction

$$\tau : \mathcal{P}(\mathcal{X}) \rightleftarrows \text{PSh}(\mathcal{X}, V) : \mathcal{L}.$$ 

We define $Y^V$ as the composition $\tau \circ Y^\infty$. We give examples of $r^*$ and $r_*$ for some $V$.

**Example 2.2.**

1. Assume that $V = \mathcal{S}$ equipped with the cartesian monoidal structure. Then $r^*$ and $r_*$ are equivalent to the identity. Thus we obtain that $Y^V \cong Y^\infty$.

2. Assume that $V$ is the full subcategory category $S_{\leq n}$ of $\mathcal{S}$ consisting of $n$-truncated objects for $n \geq 0$. Then $r_*$ is the inclusion $S_{\leq n} \hookrightarrow \mathcal{S}$ and $r^*$ is the truncation functor.

3. Let $\Lambda$ be a commutative unital ring. Assume that $V$ is the derived $\infty$-category $D(\Lambda)$ of $\Lambda$ (see [Lur12]). Recall that $D(\Lambda)$ has a monoidal structure induced by derived tensor products. Under this monoidal structure, we obtain that $r_*(C_\bullet) \cong \text{DK}(\tau_{\geq 0}C_\bullet)$ for each $C_\bullet \in D(\Lambda)$, where $\text{DK}$ means the Dold-Kan correspondence and $\tau_{\geq 0}$ means the truncation. Thus $r^*$ coincides with the functor of singular chain complexes.

4. Assume that $V$ is the stable $\infty$-category of spectra $\text{Sp}$. Then $V$ has a symmetric monoidal structure given by smash products (see [Lur12, §4.8.2]). Since $1_\mathcal{L}$ is the sphere spectrum, $r^*$ is the suspension $\Sigma^\infty : \mathcal{S} \to \text{Sp}$ and $r_*$ is the looping $\Omega^\infty : \text{Sp} \to \mathcal{S}$.

**Lemma 2.3.** There exists an equivalence $\text{Hom}_V \cong r_* \circ \text{Hom}$ in $[V^{\text{op}} \times V, \mathcal{S}]$.

**Proof.** For each $v, w \in \mathcal{V}$, there exist equivalences

$$r_*\text{Hom}(v, w) = \text{Hom}_V(1_\mathcal{V}, \text{Hom}(v, w)) \cong \text{Hom}_V(v \otimes 1_\mathcal{V}, w) \cong \text{Hom}_V(v, w)$$

which are functorial in both $v$ and $w$. \hfill \Box

For a bifunctor of $\infty$-categories $F : C^{\text{op}} \times C \to D$, we write $\int_{c \in C} F(c, c)$ (resp. $\int^c F(c, c)$) for the $\infty$-categorical end (resp. coend) of $F$ introduced by Glasman [Gla16, Def. 2.2]. Since ends (resp. coends) are defined as a limit (resp. colimit), these commute with right (resp. left) adjoint functors. We define a bifunctor $\mathcal{H}om' : \text{PSh}(\mathcal{X}, V)^{\text{op}} \times \text{PSh}(\mathcal{X}, V) \to \mathcal{V}$ by

$$\mathcal{H}om'(\mathcal{F}, \mathcal{F}') = \int_{U \in \mathcal{X}} \text{Hom}(\mathcal{F}(U), \mathcal{F}'(U))$$

for $\mathcal{F}, \mathcal{F}' \in \text{PSh}(\mathcal{X}, V)$. The next lemma is a variation of Yoneda’s lemma for $\mathcal{H}om'$. 5
Lemma 2.4. In $\mathcal{X}^{\text{op}} \times \text{PSh}(\mathcal{X}, \mathcal{V})$, the functor $(U, \mathcal{F}) \mapsto \mathcal{F}(U)$ is equivalent to $(U, \mathcal{F}) \mapsto \mathcal{H}om(Y^V(U), \mathcal{F})$.

Proof. Since ends are defined as a limit, these commute with hom-functors. Thus we obtain an equivalence

$$\mathcal{H}om_Y(v, \mathcal{H}om(Y^V(U), \mathcal{F}))) \cong \int_{V \in \mathcal{X}} \mathcal{H}om_Y(v, \mathcal{H}om(Y^V(U)(V), \mathcal{F}(V)))$$

for each $U \in \mathcal{X}$, $\mathcal{F} \in \text{PSh}(\mathcal{X}, \mathcal{V})$ and $v \in \mathcal{V}$. We have

$$\mathcal{H}om_Y(Y^V(U)(V), \mathcal{H}om(v, \mathcal{F}(V))) \cong \mathcal{H}om_Y(v \otimes Y^V(U)(V), \mathcal{F}(V)) \cong \mathcal{H}om_Y(Y^V(U)(V), \mathcal{H}om(v, \mathcal{F}(V)))$$

By Lemmas 2.1 and 2.3, we also have

$$\mathcal{H}om_Y(Y^\infty(U)(V), \mathcal{H}om(v, \mathcal{F}(V))) \cong \mathcal{H}om_Y(Y^\infty(U)(V), \mathcal{H}om(v, \mathcal{F}(V)))$$

Hence, we obtain

$$\mathcal{H}om_Y(v, \mathcal{H}om(Y^V(U), \mathcal{F}))) \cong \int_{V \in \mathcal{X}} \mathcal{H}om_Y(Y^\infty(U)(V), \mathcal{H}om_Y(v, \mathcal{F}(V)))$$

The right-hand side is equivalent to $\mathcal{H}om_{\mathcal{P}(\mathcal{X})}(Y^\infty(U), \mathcal{H}om_Y(v, \mathcal{F}(V)))$ by [Gla16, Prop. 2.3]. Therefore, the $\infty$-Yoneda lemma in $\mathcal{X}$ gives an equivalence

$$\int_{V \in \mathcal{X}} \mathcal{H}om_Y(Y^\infty(U)(V), \mathcal{H}om_Y(v, \mathcal{F}(V))) \cong \mathcal{H}om_Y(v, \mathcal{F}(U))$$

Applying the $\infty$-Yoneda lemma in $\mathcal{V}$ to the equivalence

$$\mathcal{H}om_Y(v, \mathcal{H}om(Y^V(U), \mathcal{F}))) \cong \mathcal{H}om_Y(v, \mathcal{F}(U))$$

we obtain the assertion.

Definition 2.5. We define a bifunctor $\langle \cdot, \cdot \rangle : \text{PSh}(\mathcal{X}, \mathcal{V}) \times \text{PSh}(\mathcal{X}, \mathcal{V}) \to \mathcal{V}$ by

$$\langle \mathcal{F}, \mathcal{G} \rangle = \int_{U \in \mathcal{X}} \mathcal{F}(U) \otimes \mathcal{G}(U)$$

for $\mathcal{F} \in \text{PSh}(\mathcal{X}, \mathcal{V})$ and $\mathcal{G} \in \text{PSh}(\mathcal{X}, \mathcal{V})$. We write $\mathcal{F}^!$ for the functor $\langle - , \mathcal{G} \rangle : \text{PSh}(\mathcal{X}, \mathcal{V}) \to \mathcal{V}$, and $\mathcal{G}^!$ for the functor $\mathcal{V} \to \text{PSh}(\mathcal{X}, \mathcal{V})$ defined by $v \mapsto \mathcal{H}om(\mathcal{G}(-), v)$.

We are interested in the non-degeneracy of the paring $\langle \cdot, \cdot \rangle$, for which our main result is Proposition 2.10. We first prepare some lemmas.
Lemma 2.6. In $[\text{PSh}(\mathcal{X}, \mathcal{V})^{\text{op}} \times \text{PCSh}(\mathcal{X}, \mathcal{V})^{\text{op}} \times \mathcal{V}, \mathcal{V}]$, the functor

$$(\mathcal{F}, \mathcal{G}, v) \mapsto \mathcal{H}om(\langle \mathcal{F}, \mathcal{G} \rangle, v)$$

is equivalent to

$$(\mathcal{F}, \mathcal{G}, v) \mapsto \mathcal{H}om'(\mathcal{F}, \mathcal{G}_v).$$

Proof. For each $\mathcal{F} \in \text{PSh}(\mathcal{X}, \mathcal{V})$, $\mathcal{G} \in \text{PCSh}(\mathcal{X}, \mathcal{V})$ and $v \in \mathcal{V}$, there exist equivalences

$$\mathcal{H}om(\langle \mathcal{F}, \mathcal{G} \rangle, v) \cong \int_{U \in \mathcal{X}} \mathcal{H}om(\mathcal{F}(U) \otimes \mathcal{G}(U), v)$$

$$\cong \int_{U \in \mathcal{X}} \mathcal{H}om(\mathcal{F}(U), \mathcal{H}om(\mathcal{G}(U), v))$$

$$= \int_{U \in \mathcal{X}} \mathcal{H}om(\mathcal{F}(U), (\mathcal{G}_v)(U))$$

which are functorial in $\mathcal{F}, \mathcal{G}$ and $v$. □

By Lemmas 2.3 and 2.6, we see that $\mathcal{G}^!$ is left adjoint to $\mathcal{G}_!$. This means that the pairing (2.1) induces a functor $(-)^! : \text{PCSh}(\mathcal{X}, \mathcal{V}) \rightarrow [\text{PSh}(\mathcal{X}, \mathcal{V}), \mathcal{V}]^L$. Let Res be the functor $[\text{PSh}(\mathcal{X}, \mathcal{V}), \mathcal{V}] \rightarrow \text{PCSh}(\mathcal{X}, \mathcal{V})$ defined by $F \mapsto F \circ Y^V$. We prove that $(-)^!$ is an essential section of Res.

Lemma 2.7. The composition

$$\text{PCSh}(\mathcal{X}, \mathcal{V}) \xrightarrow{(-)^!} [\text{PSh}(\mathcal{X}, \mathcal{V}), \mathcal{V}] \xrightarrow{\text{Res}} \text{PCSh}(\mathcal{X}, \mathcal{V})$$

is equivalent to the identity of $\text{PCSh}(\mathcal{X}, \mathcal{V})$. In other words, we have

$$\langle Y^V(U), \mathcal{G} \rangle \cong \mathcal{G}_!(U)$$

which is functorial in $U \in \mathcal{X}$ and $\mathcal{G} \in \text{PCSh}(\mathcal{X}, \mathcal{V})$.

Proof. For each $\mathcal{G} \in \text{PCSh}(\mathcal{X}, \mathcal{V})$ and $v \in \mathcal{V}$, there exist equivalences

$$\mathcal{H}om(\mathcal{G}^! \circ Y^V(-), v) \cong \mathcal{H}om'(Y^V(-), \mathcal{G}_v) \cong (\mathcal{G}_v)(-) = \mathcal{H}om(\mathcal{G}(-), v)$$

by Lemmas 2.6 and 2.4. Thus Lemma 2.3 gives

$$\mathcal{H}om_v(\mathcal{G}^! \circ Y^V(-), v) \cong \mathcal{H}om_v(\mathcal{G}(-), v).$$

Therefore, the $\infty$-Yoneda lemma in $\mathcal{V}$ leads to a functorial equivalence $\text{Res}(\mathcal{G}^! \cong \mathcal{G}_!$. □

We prove a variation of the co-Yoneda lemma.
Lemma 2.8. In \([\mathcal{X}^{\text{op}} \times \text{PSh}(\mathcal{X}, \mathcal{V})], \mathcal{V}\), the functor \((V, \mathcal{F}) \mapsto \mathcal{F}(V)\) is equivalent to

\[
V \mapsto \int^{U \in \mathcal{X}} \mathcal{F}(U) \otimes \mathcal{Y}(U)(V)
\]

Proof. Since \(\text{PSh}(\mathcal{X}, \mathcal{V}) = \text{PCSh}(\mathcal{X}^{\text{op}}, \mathcal{V})\) and \(\text{PCSh}(\mathcal{X}, \mathcal{V}) = \text{PSh}(\mathcal{X}^{\text{op}}, \mathcal{V})\), Lemma 2.7 for \(\mathcal{X}^{\text{op}}\) gives a desired equivalence.

We write \([\text{PSh}(\mathcal{X}, \mathcal{V})], \mathcal{V}\) \(\star\) for the full subcategory of \([\text{PSh}(\mathcal{X}, \mathcal{V})], \mathcal{V}\) consisting of functors \(F\) such that the composition

\[
\text{PSh}(\mathcal{X}, \mathcal{V}) \times \mathcal{V}(\mathcal{F}, \text{id}) \longrightarrow \mathcal{V} \times \mathcal{V}
\]

is equivalent to

\[
\text{PSh}(\mathcal{X}, \mathcal{V}) \times \mathcal{V}(\text{id}, \mathcal{G}) \longrightarrow \text{PSh}(\mathcal{X}, \mathcal{V}) \times \text{PSh}(\mathcal{X}, \mathcal{V})
\]

in \([\text{PSh}(\mathcal{X}, \mathcal{V}) \times \text{PSh}(\mathcal{X}, \mathcal{V})], \mathcal{V}\). Lemma 2.3 shows that \([\text{PSh}(\mathcal{X}, \mathcal{V})], \mathcal{V}\) \(\star\) \(\subset\) \([\text{PSh}(\mathcal{X}, \mathcal{V})], \mathcal{V}\) \(\square\).

Lemma 2.9. A functor \(F : \text{PSh}(\mathcal{X}, \mathcal{V}) \to \mathcal{V}\) belongs to \([\text{PSh}(\mathcal{X}, \mathcal{V})], \mathcal{V}\) \(\star\) if and only if the following diagram is \(\infty\)-commutative:

Thus the assertion follows from the \(\infty\)-Yoneda lemma in \(\mathcal{V}\).

Now we can prove our non-degeneracy result on the pairing \(\langle \cdot, \cdot \rangle\).

Proposition 2.10. The functor \((-)^{\dagger} : \text{PCSh}(\mathcal{X}, \mathcal{V}) \to [\text{PSh}(\mathcal{X}, \mathcal{V}), \mathcal{V}] \star\) is an equivalence of \(\infty\)-categories and \(\text{Res}\) gives its inverse. In particular, when

\[
[\text{PSh}(\mathcal{X}, \mathcal{V}), \mathcal{V}] \star = [\text{PSh}(\mathcal{X}, \mathcal{V}), \mathcal{V}] \square
\]

holds, the pairing \(\langle \cdot, \cdot \rangle\) is right non-degenerate.

Proof. By Lemma 2.7, it suffices to show that the composition

\[
[\text{PSh}(\mathcal{X}, \mathcal{V}), \mathcal{V}] \square \xrightarrow{\text{Res}} \text{PCSh}(\mathcal{X}, \mathcal{V}) \xrightarrow{(-)^{\dagger}} [\text{PSh}(\mathcal{X}, \mathcal{V}), \mathcal{V}] \square
\]

holds, the pairing \(\langle \cdot, \cdot \rangle\) is right non-degenerate.
is equivalent to the identity of $[\text{PSh}(\mathcal{X}, \mathcal{V}), \mathcal{V}]^\bullet$. For $F \in [\text{PSh}(\mathcal{X}, \mathcal{V}), \mathcal{V}]^\bullet$ and $\mathcal{F} \in \text{PSh}(\mathcal{X}, \mathcal{V})$, we have

$$F(\mathcal{F}) \cong F \left( \int_{U \in \mathcal{X}} \mathcal{F}(U) \otimes \mathcal{V}(U)(-) \right) \cong \int_{U \in \mathcal{X}} F(\mathcal{F}(U) \otimes \mathcal{V}(U)(-))$$

by Lemma 2.8. We also have

$$\text{Res}(F)^\dagger(\mathcal{F}) = \int_{U \in \mathcal{X}} \mathcal{F}(U) \otimes F(\mathcal{V}(U))$$

by the definition of $\text{Res}(F)$. Thus the assertion follows from Lemma 2.9.

Finally, we give some examples of $\mathcal{V}$ satisfying the condition (2.2).

**Example 2.11.** (1) Clearly, $S$ and $S_{\leq n}$ for $n \geq 0$ satisfy (2.2).

(2) For a commutative unital ring $\Lambda$, let $D_{\geq 0}(\Lambda)$ be the full subcategory of $D(\Lambda)$ consisting of connective complexes. Then the Dold-Kan correspondence says that $D_{\geq 0}(\Lambda)$ satisfies (2.2). Moreover, for all $C_\bullet \in D(\Lambda)$ and $\mathcal{F} \in \text{PSh}(\mathcal{X}, D(\Lambda))$, we have natural equivalences

$$F(C_\bullet \times \mathcal{F}) \cong F((\text{colim}_{n \in \mathbb{Z}} (\tau_{\geq n} C_\bullet)[n]) \otimes (\text{colim}_{m \in \mathbb{Z}} (\tau_{\geq m} \mathcal{F})[m]))$$

$$\cong \text{colim}_{n, m \in \mathbb{Z}} F((\tau_{\geq n} C_\bullet)[n] \otimes (\tau_{\geq m} \mathcal{F})[m])$$

$$\cong \text{colim}_{n, m \in \mathbb{Z}} ((\tau_{\geq n} C_\bullet)[n] \otimes F(\mathcal{F}[m]))$$

$$\cong (\text{colim}_{n \in \mathbb{Z}} (\tau_{\geq n} C_\bullet)[n]) \otimes F(\text{colim}_{m \in \mathbb{Z}} (\tau_{\geq m} \mathcal{F})[m]))$$

$$\cong C_\bullet \otimes F(\mathcal{F})$$

by Lemma 2.9 in $D_{\geq 0}(\Lambda)$. Thus $D(\Lambda)$ also satisfies (2.2).

(3) The $\infty$-category $\text{Sp}$ satisfies (2.2). Indeed, for $E_\bullet \in \text{Sp}$, $\mathcal{F} \in \text{PSh}(\mathcal{X}, \text{Sp})$ and $F \in [\text{PSh}(\mathcal{X}, \text{Sp}), \text{Sp}]^\bullet$, we have

$$F(E_\bullet \wedge \mathcal{F}) \cong F((\text{colim}_{n \in \mathbb{Z}} (\Sigma^\infty E_n)[-n]) \wedge \mathcal{F})$$

$$\cong \text{colim}_{n \in \mathbb{Z}} F(\Sigma^\infty E_n \wedge \mathcal{F}[n])([-n])$$

$$\cong \text{colim}_{n \in \mathbb{Z}} F((\text{colim}_{E_n} 1 \wedge \mathcal{F}[n])([-n])$$

$$\cong \text{colim}_{n \in \mathbb{Z}} (\text{colim}_{E_n} F(\mathcal{F}[n]))([-n])$$

$$\cong \text{colim}_{n \in \mathbb{Z}} (\text{colim}_{E_n} F(\mathcal{F}[n]))([-n])$$

$$\cong \text{colim}_{n \in \mathbb{Z}} (\text{colim}_{E_n} F(\mathcal{F}[n]))([-n])$$

$$\cong \text{colim}_{n \in \mathbb{Z}} (\text{colim}_{E_n} F(\mathcal{F}[n]))([-n])$$

$$\cong (\text{colim}_{n \in \mathbb{Z}} (\Sigma^\infty E_n)[-n]) \wedge F(\mathcal{F})$$

$$\cong E_\bullet \wedge F(\mathcal{F})$$
\text{where } \text{colim}_{E_n} \text{ means the colimit of the constant diagram on } E_n. \text{ Similarly, the full subcategory } \text{Sp}_{\geq 0} \text{ of } \text{Sp} \text{ consisting of connective spectra also satisfies (2.2).} 

3 Proof of the main result

Let \( \mathcal{X} \) be a small \( \infty \)-category equipped with a Grothendieck topology \( J \) in the sense of [Lur09, Def. 6.2.2.1] and \( \mathcal{V} \) be a presentable symmetric closed monoidal \( \infty \)-category. In this section we prove the main result stated in the introduction. We first give a definition of \( \mathcal{V} \)-valued (co)sheaves on \( \mathcal{X} \). Recall that a family of morphisms \( \{U_i \to X\} \) is called a covering family of \( X \in \mathcal{X} \) if the smallest sieve of \( \mathcal{X}/X \) containing \( \{U_i \to X\} \) is a covering sieve. For each covering family \( U_0 \), we write \( \check{C}(U_0) : \Delta^{\text{op}} \to \text{PSh}(\mathcal{X}) \) for its \( \check{C} \)ech nerve and \( \check{C}^V(U_0) = r \circ \check{C}(U_0) \).

We define sheaves and cosheaves by using \( \check{C} \)ech nerves, instead of more familiar definition via (co)equalizer diagrams. It seems that this is more suitable for the theory of \( \infty \)-categories as developed in [Lur09]. The two definitions agree (Remark 3.6).

\textbf{Definition 3.1.} (1) A presheaf \( \mathcal{F} \in \text{PSh}(\mathcal{X}, \mathcal{V}) \) is called a sheaf if for each covering family \( U_0 \) of \( \mathcal{X} \), the composition
\[
\Delta^{\text{op}} \check{C}^V(U_0) \to \text{PSh}(\mathcal{X}, \mathcal{V})^{\text{op}} \overset{\mathcal{H}om'(-, \mathcal{F})}{\to} \mathcal{V}
\]
is a limit diagram in \( \mathcal{V} \). We write \( \text{Sh}(\mathcal{X}, \mathcal{V}) \) for the full subcategory of \( \text{PSh}(\mathcal{X}, \mathcal{V}) \) consisting of sheaves.

(2) A precosheaf \( \mathcal{G} \in \text{PCSh}(\mathcal{X}, \mathcal{V}) \) is called a cosheaf if for each covering family \( U_0 \) of \( \mathcal{X} \), the composition
\[
\Delta^{\text{op}} \check{C}^V(U_0) \to \text{PSh}(\mathcal{X}, \mathcal{V}) \overset{\mathcal{G}^!}{\to} \mathcal{V}
\]
is a colimit diagram in \( \mathcal{V} \). We write \( \text{CSh}(\mathcal{X}, \mathcal{V}) \) for the full subcategory of \( \text{PCSh}(\mathcal{X}, \mathcal{V}) \) consisting of cosheaves.

\textbf{Remark 3.2.} Assume that \( \mathcal{X} \) has all pullbacks. Then a \( \check{C} \)ech nerve of a covering family of \( \mathcal{X} \) can be regarded as a simplicial object in \( \mathcal{X} \). Thus by Lemma 2.4, \( \mathcal{F} \in \text{PSh}(\mathcal{X}, \mathcal{V}) \) is a sheaf if and only if \( \mathcal{F} \circ \check{C}(U_0) \) is a limit diagram for every \( U_0 \) by Lemma 2.7.

For a monomorphism \( i_U : j(U) \to Y^\infty \) associated with a covering sieve \( U \) (see [Lur09, Prop. 6.2.2.5]), we write \( i_U^Y = i_U^V \) and \( Y^V(U) = \pi(j(U)) \). We write \( S_\mathcal{V} = \{i_U^V \in \text{Mor}(\text{PSh}(\mathcal{X}, \mathcal{V})) \mid U \in J\} \). The following lemma assures that our definition of sheaves coincides with that of [Lur09, Def. 6.2.2.6] when \( \mathcal{V} = S_\mathcal{V} \).

\textbf{Lemma 3.3.} A presheaf \( \mathcal{F} \in \text{PSh}(\mathcal{X}, \mathcal{V}) \) is a sheaf if and only if the morphism \( \mathcal{H}om'(i_U^V, \mathcal{F}) \) is an equivalence in \( \mathcal{V} \) for all \( i_U^V \in S_\mathcal{V} \). Moreover, when \( r_* \) is conservative, i.e., it reflects equivalences, then \( \mathcal{F} \) is a sheaf if and only if it is \( S_\mathcal{V} \)-local.
Proof. Let $\Delta^{\geq 1}$ for the full subcategory of $\Delta$ consisting of $[n]$ for $n \geq 1$. Since $\tau$ preserves colimits, we have
\[
\text{colim} \, \check{C}^V(U_0)_{|_{\Delta^{\geq 1}}} \cong \tau(\text{colim} \, \check{C}^V(U_0)_{|_{\Delta^{\geq 1}}}) \cong \tau(j(U)) = Y^V(U).
\]
Thus the first half of the assertion follows from Lemma 2.4 and the second half follows from Lemma 2.3.

Now we prove the main result. This can be applied to the $\infty$-categories in the following Example 3.5.

**Theorem 3.4.** Assume that $X$ and $V$ satisfy (2.2) and also the following condition:
\[
\text{Sh}(X, V) = \text{Sh}(X, V)^{\text{Sy-loc}}.
\]
Then the following hold.

1. Under the pairing (2.1), we have $\text{CSh}(X, V) = \text{Sh}(X, V)^{\text{Sy-loc}}$.
2. The $\infty$-category $\text{Sh}(X, V)$ is presentable.
3. The $\infty$-category $\text{CSh}(X, V)$ is presentable.
4. The inclusion $\text{Sh}(X, V) \hookrightarrow \text{PSh}(X, V)$ admits a left adjoint.
5. The inclusion $\text{CSh}(X, V) \hookrightarrow \text{PCSh}(X, V)$ admits a right adjoint.
6. There exists an equivalence of $\infty$-categories $\text{CSh}(X, V) \cong [\text{Sh}(X, V), V]^L$.

**Proof.** The assertion (2) follows from Lemma 3.3 and [Lur09, Prop. 5.5.4.15]. Moreover, (1) and (2) imply (3)-(6). Indeed, since $\text{Sh}(X, V)$ is closed under limits in $\text{PSh}(X, V)$, (4) follows from (2) and [Lur09, Cor. 5.5.2.9]. Therefore, (3), (5) and (6) follow from Lemma 1.2 and Proposition 1.4. Thus it suffices to prove (1). Since $G^\dagger$ preserves colimits, a precosheaf $G \in \text{PCSh}(X, V)$ is a cosheaf if and only if the canonical morphism
\[
\langle Y^V(U), G \rangle \rightarrow \langle Y^V(U), G \rangle
\]
is an equivalence in $V$ by Lemma 2.7. Thus we obtain (1) by Proposition 1.5.

Most $\infty$-categories from Example 2.11 satisfy the condition (3.1).

**Example 3.5.** (1) Clearly, $S$ and $S_{\leq n}$ for $n \geq 0$ satisfy (3.1).

(2) The $\infty$-category $D_{\geq 0}(\Lambda)$ in Example 2.11 (2) satisfies (3.1) by Lemma 3.3. Indeed, $r_*$ is conservative by the Dold-Kan correspondence.

(3) The $\infty$-category $S_{\geq 0}$ in Example 2.11 (3) satisfies (3.1) by Lemma 3.3. Indeed, $r_*$ is conservative by [Lur12, Cor. 7.1.4.13].

**Remark 3.6.** Recall that if $X$ is a 1-category and $V = \text{Set}$, cosheaves are usually defined via coequalizer diagrams (cf. (0.1)). Theorem 3.4 (1) says that our Definition 3.1 (2) is equivalent to this more familiar definition for such $(X, V)$. Indeed, we can prove that the category of cosheaves defined by coequalizer diagrams is the dual of the category of sheaves by a similar proof. Similarly, our definition is also a generalization of ordinary cosheaves of modules.
Remark 3.7. In the context of $\infty$-topos theory, sheaves on a simplicial set $\mathcal{X}$ are sometime defined as an object of an $\infty$-topos $\mathcal{T}$ on $\mathcal{X}$. Note that there exists an $\infty$-topos which is not the $\infty$-category of sheaves on a Grothendieck topology on $\mathcal{X}$, unlike the ordinary topos theory. Then Theorem 3.4 suggests a definition of cosheaves in such a situation. Namely, we can define a cosheaf on $\mathcal{T}$ as an object of the dual $\mathcal{T}^\vee$, or equivalently, a left adjoint functor $\mathcal{T} \to \mathcal{S}$.

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