Towards a Theory of The QCD String

or The Physics of a 750 MeV Boson

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w Victor Gorbenko, 1511.01908

earlier work:
w Raphael Flauger, Victor Gorbenko, 1203.1054, 1205.6805, 1301.2325, 1404.0037
w Patrick Cooper, Victor Gorbenko, Ali Mohsen, Stefano Storace 1411.0703
The major underlying question:

What is $SU(\infty)$ Yang-Mills?

Old, fascinating and famously hard. Will try to convince you that it may be the right time to attack it.
QCD is a theory of strings

Large N QCD is a theory of free strings

Can we solve this free string theory?

Bissey et al, hep-lat/0606016
✓ Confining gauge theory with a gap $\Lambda$
✓ Unbroken center symmetry

SETUP

4D theory

2D theory

Energy
✓ Confining gauge theory with a gap
✓ Unbroken center symmetry
✓ Large $N$

Energy

$\Lambda_N \sim N_c / \ell_s$
\[ \mathcal{O} = P \exp \{ i \int A \} \]

all the data from the papers by Athenodorou, Bringoltz and Teper

Looks hopeless to solve without experimental data

\[
\int \mathcal{D}A e^{-S_{YM}} \mathcal{O}(0) \mathcal{O}^\dagger(t) \rightarrow e^{-E_\mathcal{O} t} + \ldots
\]
Theories\' coordinates of the embedding of the string worldsheet into the target space. Hence their transformation rules under the full Poincaré group $\text{ISO}(1, D - 1)$ are simply those of the space-time coordinates. These are analogues of the sigma model $U$ field in the chiral pion Lagrangian. The Lorentz invariant Lagrangian is then simply a sum of local geometric invariants constructed with the help of the embedding $X^\mu$.

\[
\text{string} = \int d^2 \sigma \sqrt{\gamma} \left( \cdots + K_i^\gamma + \cdots \right)
\]

where $\gamma$ is the induced metric on the world-sheet, $\gamma^\mu_\nu = \partial^\mu X^\nu \partial_\nu X^\mu$, $K^\gamma_i$ is the second fundamental form (the extrinsic curvature) of the world-sheet. The first term in (4) is the Nambu–Goto (NG) action, the second one is the rigidity term introduced by Polyakov [8] and Kleinert [9], and dots stand for higher derivative geometric invariants.

The tension of the string $\sigma$, the rigidity parameter $\gamma^0$, and the coefficients in front of all other higher-derivative operators are free parameters of the low energy effective theory to be determined either from experiment (or from the lattice data for the QCD string), or from matching the effective theory to the microscopic theory in the UV (which can be done, for example, for cosmic strings in weakly coupled models).

Much of our discussion will deal with infinitely long strings because we are concerned with the form of the bulk action. IR effects such as finite size effects and boundary terms can be included at a later stage.

As expected, the action (4) is invariant under the linearly realized $\text{ISO}(1, 1) \times \text{SO}(D - 2)$ symmetry, which is the unbroken subgroup of $\text{ISO}(1, D - 1)$ in the presence of a long straight string. The $\text{ISO}(1, 1)$ factor acts as a worldsheet Poincaré group, and $\text{SO}(D - 2)$ acts as in (2). The remaining spatial translations act as in (1), and the action of the remaining broken boosts and rotations $J^\gamma_i$ following from the linear transformation law for $X^\mu$ is

\[
\delta^\alpha_i X^j = -\epsilon (\delta^i_j \sigma^\alpha + X^i \partial^\alpha X^j)
\]
Embedding Coordinates

\[ x^\mu = (\sigma^\alpha, \ell_s X^i(\sigma)) \]

Induced Metric

\[ h_{\alpha \beta} = \partial_\alpha x^\mu \partial_\beta x_\mu \]

\[ S_{string} = -\ell_s^{-2} \int d^2\sigma \sqrt{h} + \text{Nambu-Goto} \]

\[ \int d^2\sigma \sqrt{h} \left( \frac{1}{\alpha_0} (K^i_{\alpha \beta})^2 + \chi R \right) + \text{rigidity Euler characteristic} \]

(vanishes on-shell)
Embedding Coordinates

\[ x^\mu = (\sigma^\alpha, \ell_s X^i(\sigma)) \]

Induced Metric

\[ h_{\alpha\beta} = \partial_\alpha x^\mu \partial_\beta x_\mu \]

\[ S_{\text{string}} = -\ell_s^{-2} \int d^2 \sigma \sqrt{h} + \]

\[ \int d^2 \sigma \sqrt{h} \left( \frac{1}{\alpha_0} (K^i_{\alpha\beta})^2 + \chi R \right) + \]

\[ C_1 \ell_s^2 \int d^2 \sigma \sqrt{h} (K^i_{\alpha\beta})^4 + \cdots = C_1 \ell_s^6 \int d^2 \sigma (\partial_\alpha \partial_\beta X^i)^4 + \cdots \]

higher order non-universal terms start at \( \ell_s^6 \)

Nambu-Goto

rigidity

(Euler characteristic (vanishes on-shell))
Explains the ground state data

\[ E_0(R) = \frac{R}{\ell_s^2} - \frac{(D - 2)\pi}{6R} - \frac{(D - 2)^2 \pi^2 \ell_s^2}{72R^3} - \frac{(D - 2)^3 \pi^3 \ell_s^4}{432R^5} + \text{non-universal terms} \]

classical

Lüscher term

Figure 4: Sample connected diagrams contributing to the \( R^5 \) order corrections to the energy levels.

With Weyl symmetric ordering one has to include the contribution from the non-covariant counterterm (14) with the value of \( c_4 \) as given by (16).

An attempt to perform this calculation was made recently in [10], using the Weyl symmetric ordering. It was assumed that all the diagrams without the \( c_4 \) term add up into the light cone spectrum (17) expanded up to this order. So rather than including all the diagrams above, only the contribution of the \( c_4 \) term was calculated. It turns out that the \( c_4 \) term contributes only to the tree level shift of two-particle states. It was conjectured that the correct value of \( c_4 \) is \( \frac{(D - 2)^2}{192} \pi \), and this tree-level result was suggested as the leading correction to the light cone spectrum.

As we saw the correct value of \( c_4 \) is given by (16), and we see no reason for the diagrams with \( c_4 = 0 \) to reproduce the light cone spectrum, so that this calculation is incomplete.

However, here comes the puzzle. Recently the calculation at the \( 1/R^5 \) order was performed in the PS gauge [27] and yielded the same result as the one obtained in [10]. We should note in passing that the practical advantage of the PS gauge is that at this order one has to work with a free theory with a single interaction term. The price to pay is that one has to impose the BRST constraints to restrict to the correct physical states. The puzzle is why the calculation in the PS gauge agrees with the incomplete one using the wrong value of \( c_4 \).

The explanation is as follows. Rather than doing the full brute force calculation in static gauge one can make use of the known light cone spectrum. This is the exact spectrum of some relativistic integrable two-dimensional theory. At this order in the derivative expansion its Lagrangian takes the form

\[ L_{LC} = L_{NG} + \frac{(D - 2)^2}{192} \pi \partial \partial X_i \partial \partial X_j . \]

Here \( L_{NG} \) is the full renormalized NG action at this order in derivative expansion. For example, in dimensional regularization \( L_{NG} \) includes the evanescent term (15), with Weyl symmetric ordering \( L_{NG} \) includes the non-covariant \( c_4 \)-term with the correct value (16) of \( c_4 \).

The additional term in (50) cancels the PS annihilation amplitude and breaks non-linearly.

In fact, as will become clear momentarily they do not reproduce the light cone spectrum.
Explains the ground state data

\[ E_0(R) = \frac{R}{\ell_s^2} - \frac{(D - 2)\pi}{6R} - \frac{(D - 2)^2\pi^2\ell_s^2}{72R^3} - \frac{(D - 2)^3\pi^3\ell_s^4}{432R^5} + \text{non-universal terms} \]

with classical contributions and Lüscher term diagrams.

This looks like bad news: very hard to extract non-trivial information.
Excited states are more promising

Left-movers only:

Solid --- universal terms in $\ell_s/R$ expansion
Dashed --- light cone quantized bosonic string
Excited states are more promising

Colliding left- and right-movers:

- Spin 2
- Scalar
- Pseudoscalar

Solid --- universal terms in $\ell_s/R$ expansion
Dashed --- light cone quantized bosonic string
However, the emphasis is usually made on non-critical theories with non-zero string coupling constant. To the best of our knowledge the exact $S$-matrix (1) of a "free" critical string has not been discussed before, and we feel that the viewpoint advocated here may be useful. We present further speculations and future directions in the concluding section 7.

2. Exact $S$-Matrix of the Critical Nambu–Goto

For our purposes it will be instructive to consider the worldsheet theory from an effective field theory point of view. A detailed introduction to this approach can be found in the accompanying paper [18]. From this point of view the world-sheet theory of an infinitely long string in a $D$-dimensional Minkowski space is a theory of Goldstone bosons corresponding to the coset $ISO(D-1,1)/ISO(1,1) \times SO(D)$. Here $ISO(D-1,1)$ is the non-linearly realized target space Poincaré symmetry. Its linearly realized subgroup is a direct product of the worldsheet Poincaré symmetry $ISO(1,1)$ and of the $SO(D)$ group of transverse rotations. This is a consistent effective field theory in any number of dimensions with a cut-off scale set by the string length $\ell_s$, which physically corresponds to the string width. The effective action starts with the Nambu–Goto term and in principle has an infinite number of higher derivative corrections, corresponding to higher order geometric invariants.

Somewhat miraculously, the Nambu–Goto theory is expected, at least in a certain sense, to be renormalizable in the critical number of dimensions $D=26$ [19]. An effective field theorist would discover this by calculating loops and finding that divergences, which were expected on the basis of the naive power counting, cancel. We will discuss some aspects of these expected cancellations in section 4. We will argue that the story is somewhat subtle. In particular, the cancellations occur only for on-shell quantities. This makes it challenging to see the cancellations by a direct calculation because at low orders in perturbation theory on-shell divergences cancel because of symmetry. To see non-trivial cancellations one thus has to go rather far in the loop expansion.

For now we take a shortcut, and do not check the cancellations by brute force calculation. Instead, we deduce the properties of the resulting finite on-shell amplitudes from the known spectrum of the theory at finite volume. This is known for instance from the quantization in light-cone gauge (which is consistent with the non-linearly realized $ISO(D-1,1)$ symmetry at $D=26$). After compactification on a circle (see, e.g., [20]),

$$E_{LC}(N, \tilde{N}) = \sqrt{\frac{4\pi^2(N - \tilde{N})^2}{R^2} + \frac{R^2}{\ell_s^4} + \frac{4\pi}{\ell_s^2} \left( N + \tilde{N} - \frac{D - 2}{12} \right)}$$

Here $\ell_s$ is the string length, $N$ and $\tilde{N}$ are levels of an excited string state, so that $2\pi (N - \tilde{N})/R$ is the total Kaluza–Klein momentum of the state.

To avoid confusion, let us clarify the meaning of the subscript $LC$. It indicates that we use light-cone quantization to define the theory at the quantum level. However, equation (4) corresponds to target space energies obtained in light-cone quantization and should not be confused with the spectrum of the light-cone Hamiltonian. Classically, the target space expansion breaks down for excited states because $2\pi \ell_s/R$ is a large number!
Light Cone (GGRT) spectrum:

\[ E_{LC}(N, \tilde{N}) = \sqrt{\frac{4\pi^2 (N - \tilde{N})^2}{R^2}} + \frac{R^2}{\ell_s^4} + \frac{4\pi}{\ell_s^2} \left( N + \tilde{N} - \frac{D-2}{12} \right) \]

\( \ell_s / R \) expansion breaks down for excited states because \( 2\pi \) is a large number!

for excited states:

\[ E = \ell_s^{-1} E(p, \ell_s, \ell_s / R) \]

Let’s try to disentangle these two expansions
Finite volume spectrum in two steps:

1) Find infinite volume S-matrix
2) Extract finite volume spectrum from the S-matrix

1) is a standard perturbative expansion in $p\ell_s$
2) perturbatively in massive theories (Lüscher)
   exactly in integrable 2d theories through TBA

Relativistic string is neither massive nor integrable
But approaches integrable GGRT theory at low energies.
Thermodynamic Bethe Ansatz

Dorey, Tateo ’96

\[ \hat{p}^{(i)}_{kL} R + \sum_{j,m} 2\delta(\hat{p}^{(i)}_{kL}, \hat{p}^{(j)}_{mR}) N^{(j)}_{mR} - i \sum_{j=1}^{D-2} \int_{0}^{\infty} \frac{dp'}{2\pi} \frac{d2\delta(i\hat{p}^{(i)}_{kL}, p')}{dp'} \ln \left( 1 - e^{-R\epsilon^{j}_{R}(p')} \right) = 2\pi n^{(i)}_{kL} \]

\[ \epsilon^{i}_{L}(p) = p + \frac{i}{R} \sum_{j,k} 2\delta(p, -i\hat{p}^{(j)}_{kR}) N^{(j)}_{kR} + \frac{1}{2\pi R} \sum_{j=1}^{D-2} \int_{0}^{\infty} dp' \frac{d2\delta(p, p')}{dp'} \ln \left( 1 - e^{-R\epsilon^{j}_{R}(p')} \right) \]

\[ E(R) = R + \sum_{j,k} p^{(j)}_{kL} + \sum_{j=1}^{D-2} \int_{0}^{\infty} \frac{dp'}{2\pi} \ln \left( 1 - e^{-R\epsilon^{j}_{L}(p')} \right) \]

+right-movers

The leading order phase shift reproduces all of the GGRT spectrum
Thermodynamic Bethe Ansatz

Asymptotic Bethe Ansatz
(~Lüscher's formula)

\[ \hat{p}_{kL}^i R + \sum_{j,m} 2\delta(\hat{p}_{kL}^i, \hat{p}_{mR}^j) N_{mR}^{(j)} - \frac{i}{2\pi} \sum_{j=1}^{D-2} \int_0^\infty \frac{dp'}{2\pi} \frac{d2\delta(i\hat{p}_{kL}^i, p')}{dp'} \ln \left( 1 - e^{-Re_j^R(p')} \right) = 2\pi n_{kL}^{(i)} \]

\[ \hat{\epsilon}_L^i(p) = p + \frac{i}{R} \sum_{j,k} 2\delta(p, -i\hat{\epsilon}_{kR}^j) N_{kR}^{(j)} + \frac{1}{2\pi R} \sum_{j=1}^{D-2} \int_0^\infty \frac{dp'}{2\pi} \frac{d2\delta(p, p')}{dp'} \ln \left( 1 - e^{-Re_j^R(p')} \right) \]

\[ E(R) = R + \sum_{j,k} \hat{p}_{kL}^j + \sum_{j=1}^{D-2} \int_0^\infty \frac{dp'}{2\pi} \ln \left( 1 - e^{-Re_j^R(p')} \right) \]

+right-movers

The leading order phase shift reproduces all of the GGRT spectrum

Zamolodchikov’91
Dorey, Tateo ’96
Improve your appearance with TBA:
Colliding left- and right-movers

\[
\begin{align*}
\Delta E\ell_s & \quad \text{before} \\
\frac{R}{\ell_s} & \quad \text{spin 2} \\
& \quad \text{scalar} \\
& \quad \text{pseudoscalar}
\end{align*}
\]
Colliding left- and right-movers

Red points:
A new massive state appearing as a resonance in the antisymmetric channel!

spin 2
scalar
pseudoscalar

after
\[ S = \int d^2 \sigma \frac{1}{2} \partial_\alpha \phi \partial^\alpha \phi - \frac{1}{2} m^2 \phi^2 + Q \phi \epsilon^{\alpha\beta} \epsilon_{ij} K^i_{\alpha\gamma} K^j_{\beta\gamma} \]

\[ m \ell_s \approx 1.85^{+0.02}_{-0.03} \quad Q \approx 0.382 \pm 0.004 \]

SD, Flauger, Gorbenko, 1301.2325
Alternative and Equivalent View on TBA:

Use finite volume spectrum to reconstruct $S$-matrix
Alternative and Equivalent View on TBA:

Use finite volume spectrum to reconstruct S-matrix

GGRT spectrum

\[ E_{LC}(N, \tilde{N}) = \sqrt{\frac{4\pi^2(N - \tilde{N})^2}{R^2}} + \frac{R^2}{\ell_s^4} + \frac{4\pi}{\ell_s^2} \left( N + \tilde{N} - \frac{D - 2}{12} \right) \]

corresponds to an integrable theory with

\[ e^{2i\delta(s)} = e^{is\ell^2/4} \]

★ Time delay proportional to the collision energy
★ *Scale survives all the way to the UV!*
Could QCD string be integrable for pure glue?

No

Integrable at tree level

Universal one-loop particle production if $D \neq 26, 3$

All these one-loop amplitudes can be explicitly calculated
A simple option to restore integrability:

\[ S_{\text{string}} = -\ell_s^{-2} \int d^2 \sigma \sqrt{-\det(\eta_{\alpha\beta} + \partial_\alpha X^i \partial_\beta X^i + \partial_\alpha \phi \partial_\beta \phi)} + Q \int d^2 \sigma \phi R[X] + \ldots \]

\[ Q = \sqrt{\frac{25 - D}{48\pi}} \]

\[ e^{2i\delta(s)} = e^{is\ell^2/4} \]

This is also known as a linear dilaton background.
Another simple option to restore integrability:

\[ S_{\text{string}} = -\ell_s^{-2} \int d^2\sigma \sqrt{-\det(\eta_{\alpha\beta} + \partial_\alpha X^i \partial_\beta X^i + \partial_\alpha \phi \partial_\beta \phi)} + Q \int d^2\sigma \phi \tilde{K} \tilde{K} + \ldots \]

\[ Q = \sqrt{\frac{25 - D}{48\pi}} = \sqrt{\frac{7}{16\pi}} \approx 0.373176 \ldots \]

\[ e^{2i\delta(s)} = e^{i s \ell^2 / 4} \]

Compare to

\[ Q_{\text{lattice}} \approx 0.382 \pm 0.004 \]
What this could mean?

- **Numerology**
- In the planar limit axion becomes massless and the planar QCD string is integrable
- This is the UV asymptototics of the planar QCD string
the second option is excluded
To conclude:

(Strawman) proposal for the structure of the QCD string in D=3,4:

✴ Matter content:
Goldstones+massive antisymmetric O(D-2) tensor

✴ Integrable UV asymptotics with
\[ e^{2i\delta(s)} = e^{is\ell^2 / 4} \]

✴ Future checks: confront with lattice data for winding strings and glueball spectra