THE FREE WREATH PRODUCT OF A DISCRETE GROUP BY A QUANTUM
AUTOMORPHISM GROUP

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Abstract. Let $G$ be the quantum automorphism group of a finite dimensional C*-algebra $(B, \psi)$ and $\Gamma$ a
discrete group. In the paper we study the fusion rules of $\hat{\Gamma} \wr \ast G$. First we revise the representation theory of
$G$ and in particular we describe the spaces of intertwiners using non-crossing partitions. This allows to find
the fusion rules of the free wreath product in the general case of a state $\psi$. We will also prove, when $\psi$ is
a trace, the simplicity of the reduced C*-algebra and, when $\Gamma$ is moreover finite, the Haagerup property of
$L^\infty(\hat{\Gamma} \wr \ast G)$.

INTRODUCTION

The idea of the quantum automorphism group is to give a quantum version of the standard notion of
automorphism group. From the classical group theory we know that $S_N$ can be interpreted as the universal
group acting on a set of $N$ points. Wang in [Wan98] proved that, given a finite dimensional C*-algebra $B$
edowed with a state $\psi$, the category of compact quantum groups coacting on $B$ and $\psi$-invariant admits
an universal object, called quantum automorphism group and denoted $A^{\text{aut}}(B, \psi)$. In the particular case of
$B = \mathbb{C}^N$ endowed with the uniform probability measure $\psi$ it is easy to prove that $A^{\text{aut}}(\mathbb{C}^N, \psi)$ is actually
$C(S_N^+)$, the quantum (non commutative) version of $C(S_N)$.
The representation theory of $A^{\text{aut}}(B, \psi)$ was first studied by Banica in [Ban99, Ban02] who showed that, if
$\psi$ is a $\delta$-form and dim($B$) $\geq 4$, the irreducible representations and the fusion rules are the same of
$SO(3)$. In particular he studied the spaces of intertwiners between tensor products of the fundamental corepresentation
showing that there is a bijection between a linear basis of these spaces and the Temperley-Lieb diagrams.
Later in [BS09] Banica and Speicher proved that, in the case of $C(S_N^+)$ it was possible to describe the inter-
twiners in a simpler way using non-crossing partitions. Always in [BS09] this combinatorial interpretation in
terms of non-crossing partitions was extended to $O_N^+, B_N^+$ and $H_N$ (respectively the orthogonal, bistochastic
and hyperoctahedral groups) and the notion of easy quantum group was introduced to describe this kind of
quantum groups.
The definition of free wreath product by a quantum permutation group was first given by Bichon in [Bic04].
Bichon in fact proved that the free wreath product of a compact quantum group by $C(S_N^+)$ can be endowed
with a compact quantum group structure. It was later shown that the quantum reflection group $H_N^+$ is
actually isomorphic to $\widehat{\mathbb{Z}} \wr \ast S_N^+$ and its fusion rules in the case $N \geq 4$ were found in [BV09] by Banica and
Vergnioux. Lemeux in [Lem13b] generalized this result to the free wreath product $\hat{\Gamma} \wr \ast S_N^+$, where $\Gamma$ is a
discrete group and $N \geq 4$.
Using the fusion rules it has been possible in the last years to study important properties as simplicity, fullness,
factoriality, Haagerup property and weak amenability of the main matrix quantum groups, in particular Brannan in [Bra13]
proved the simplicity and the uniqueness of the trace for the reduced C*-algebra $C_r(A^{\text{aut}}(B, \psi))$ (here $\psi$ is a $\delta$-trace and dim($B$) $\geq 8$) and the Haagerup property for the von Neumann
algebra $L^\infty(A^{\text{aut}}(B, \psi))$. Lemeux in [Lem13a] showed that the quantum reflection groups $H_N^+$, $N \geq 4$ have
the Haagerup property and later in [Lem13b] extended this result to $\hat{\Gamma} \wr \ast S_N^+$ for $\Gamma$ finite. He proved also the
simplicity if $\Gamma$ is discrete.
The goal of this work is to give a further generalization of these results, considering the free wreath product
of a discrete group by a general quantum automorphism group.
The first step is to revise the representation theory of \( A^{\text{aut}}(B, \psi) \), \( \dim(B) \geq 4 \) showing that, for a general \( \delta \)-form \( \psi \) it is possible to describe the intertwiners using non-crossing partitions. This is actually reasonable because the fusion rules do not change with respect to the standard case of \( C(S^+_N) \), but it is technically more complicated to compute the coefficients of the intertwiners. Thanks to this graphical interpretation we can find the fusion rules and the irreducible representations of \( H^+_1(B, \psi)(\Gamma) := \hat{\Gamma} \cdot A^{\text{aut}}(B, \psi) \), \( \dim(B) \geq 4 \). In this case the intertwiners are interpreted using non-crossing partitions decorated with the elements of the discrete group \( \Gamma \). In the last section we present some properties of the reduced algebras in the case of a \( \delta \)-trace \( \psi \). In particular it is first proved that \( C_r(H^+_1(B, \psi)) \), when \( \dim(B) \geq 8 \), is simple and has a unique trace. If the group \( \Gamma \) is finite, we finally show that the von Neumann algebra \( L^\infty(H^+_1(B, \psi)) \), \( \dim(B) \geq 4 \) has the Haagerup property. All the results are also presented in a more general form dropping the \( \delta \) condition for \( \psi \) and proving that in this case the free wreath product can be decomposed as a free product of quantum groups where the condition still holds.

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1. The quantum automorphism group \( A^{\text{aut}}(B, \psi) \)

In this section we recall some known results concerning the quantum automorphism group and the category of non-crossing partitions in order to give a new description of its intertwining spaces using non-crossing partitions instead of Temperley-Lieb diagrams.

We start with some preliminary definitions and notations. A Woronowicz compact quantum group (see [Wor87, Wor91, Wor98]) is a couple \((A, \Delta)\) where \( A \) is a unital \( C^* \)-algebra and \( \Delta : A \rightarrow A \otimes A \) a coassociative comultiplication such that \( \Delta(A)(A \otimes 1) \) and \( \Delta(A)(1 \otimes A) \) are dense in \( A \otimes A \). Woronowicz showed that this is equivalent to define a compact quantum group as a couple \((A, \Delta)\) where \( A \) is a unital \( C^* \)-algebra and \( \Delta : A \rightarrow A \otimes A \) a \( C^* \)-homomorphism together with a family of unitary matrices \((u^\alpha)_{\alpha \in I} \), \( u^\alpha \in M_d(\alpha) \) such that:

- the \( * \)-subalgebra generated by the entries \( u^\alpha_{ij} \) of the matrices \( u^\alpha \) is dense in \( A \)
- for all \( \alpha \in I \) and \( 1 \leq i, j \leq d_\alpha \), we have \( \Delta(u^\alpha_{ij}) = \sum_{k=1}^{d_\alpha} u^\alpha_{ik} \otimes u^\alpha_{kj} \)
- for all \( \alpha \in I \), the transposed matrix \((u^\alpha)^t\) is invertible

In the following we will use in particular this second definition. A corepresentation of the compact quantum group \((A, \Delta)\) is a matrix \( v \in M_n(A) \) such that \( \Delta(v_{ij}) = \sum_{k=1}^n v_{ik} \otimes v_{kj} \). Given two corepresentations \( v \in M_n(A) \) and \( w \in M_m(A) \) the space of the intertwiners between them is \( \text{Hom}(v, w) = \{ T \in M_{m,n}(\mathbb{C}) \mid (T \otimes \text{id}_A)v = w(T \otimes \text{id}_A) \} \).

We recall that every finite dimensional \( C^* \)-algebra \( B \) is isomorphic to a multimatrix \( C^* \)-algebra, so in the following we will consider the decomposition \( B = \bigoplus_{\alpha=1}^c M_n(\alpha)(\mathbb{C}) \). Let \( \mathcal{B} = \{(e^\alpha_{ij})_{i,j=1,...,n_\alpha}, \alpha = 1,...,c\} \) be the canonical basis of matrix units and suppose \( B \) endowed with the usual multiplication \( m : B \otimes B \rightarrow B, m(e^\alpha_{ij} \otimes e^\beta_{kl}) = \delta_{jk} \delta_{\alpha \beta} e^\alpha_{il} \) and unity \( \eta : C \rightarrow B, \eta(1) = \sum_{\alpha=1}^c \sum_{i=1}^{n_\alpha} e^\alpha_{ii} \). Every finite dimensional \( C^* \)-algebra \( B \) can be endowed with a Hilbert space structure considering the scalar product \( \langle x, y \rangle = \psi(y^*x) \) induced by a faithful state \( \psi \). Normalizing the basis \( \mathcal{B} \) with respect to the norm associated to this scalar product we get the orthonormal basis \( \mathcal{B}' = \{ b^\alpha_{ij} | b^\alpha_{ij} = \psi(e^\alpha_{ij})^{-1/2} e^\alpha_{ij}, i, j = 1,...,n_\alpha, \alpha = 1,...,c \} \) which will be used in the following.

We are now ready to give one of the main definition of the paper. The quantum automorphism group of a finite dimensional \( C^* \)-algebra \((B, \psi)\) is the universal object in the category of the compact quantum groups
coacting on $B$ and leaving the state $\psi$ invariant (see [Ban99] [Wan98]). More precisely it can be defined as follows:

**Definition 1.1.** Let $B$ be a $n$-dimensional C*-algebra with multiplication $m : B \otimes B \rightarrow B$ and unity $\eta : \mathbb{C} \rightarrow B$. Let $\psi$ be a state on $B$. Consider the following universal unital C*-algebra:

$$A^{\text{aut}}(B, \psi) = \{ (u_{ij})_{i,j=1, \ldots, n} | u = (u_{i,j}) \text{ is unitary, } m \in \text{Hom}(u \otimes^2 u), \eta \in \text{Hom}(1, u) \}$$

Then $A^{\text{aut}}(B, \psi)$ endowed with the usual comultiplication $\Delta(u_{ij}) = \sum_{k=1}^{n} u_{ik} \otimes u_{kj}$ is the compact quantum automorphism group of the C*-algebra $(B, \psi)$.

**Remark 1.2.** Choosing an orthonormal basis for $B$ it is possible to transform the three defining conditions of a quantum automorphism group in a set of relations between the coefficients of $u$. The relations depend on the basis, but the quantum automorphism group generated is of course independent from this choice (see [Ban99]).

**Definition 1.3.** Let $B$ be a $n$-dimensional C*-algebra as in definition 1.1 and $\delta > 0$. A faithful state $\psi : B \rightarrow \mathbb{C}$ is a $\delta$-form if the multiplication map of $B$ and its adjoint with respect to the inner product induced by $\psi$ satisfy $mm^* = \delta \cdot \text{id}_B$.

If such a $\psi$ is also a trace then it is called a tracial $\delta$-form or a $\delta$-trace.

**Remark 1.4.** For every faithful state $\psi : M_n(\mathbb{C}) \rightarrow \mathbb{C}$ there exists $Q \in M_n(\mathbb{C})$, $Q > 0$, $\text{Tr}(Q) = 1$ such that $\psi = \text{Tr}(Q \cdot )$. In particular we notice that every such $\psi$ is a $\delta$-form, with $\delta = \text{Tr}(Q^{-1})$.

More generally, if $B = \bigoplus_{\alpha=1}^{c} M_{n_{\alpha}}(\mathbb{C})$ then every faithful state $\psi$ over $B$ is of the form $\psi = \bigoplus_{\alpha=1}^{c} \text{Tr}(Q_{\alpha} \cdot)$ for a suitable family $Q_{\alpha} \in M_{n_{\alpha}}(\mathbb{C})$, $Q_{\alpha} > 0$, $\sum_{\alpha} \text{Tr}(Q_{\alpha}) = 1$. In particular if $\text{Tr}(Q_{\alpha}^{-1}) = \delta$ for all $\alpha$ then $\psi$ is a $\delta$-form.

It is well known that every positive complex matrix is diagonalizable, so the matrices $Q_{\alpha}$ are always similar to diagonal matrices with positive real eigenvalues. The following proposition shows that we can actually study only this case because the quantum automorphism groups of a C*-algebra endowed with states associated to similar matrices are isomorphic.

**Proposition 1.5.** Let $B = \bigoplus_{\alpha=1}^{c} M_{n_{\alpha}}(\mathbb{C})$ be a finite dimensional C*-algebra and $\phi = \bigoplus_{\alpha=1}^{c} \text{Tr}(Q_{\alpha} \cdot)$, $\psi = \bigoplus_{\alpha=1}^{c} \text{Tr}(P_{\alpha} \cdot)$ two states on $B$. Suppose that the families of matrices $Q_{\alpha}$ and $P_{\alpha}$ are similar, i.e. there exists a family of unitary matrices $U_{\alpha}$ such that $P_{\alpha} = U_{\alpha}^{*} Q_{\alpha} U_{\alpha}$. Then the quantum automorphism groups $A^{\text{aut}}(B, \phi)$ and $A^{\text{aut}}(B, \psi)$ are isomorphic.

**Proof.** The proof consists in showing that, for a good choice of the basis of $B$, the two groups are generated by the same relations. Let $U = \bigoplus_{\alpha=1}^{c} U_{\alpha}$ and consider the $*$-homomorphism $\pi : (B, \psi) \rightarrow (B, \phi)$ given by $b \mapsto U b U^{*}$. We observe first that $\phi = \psi \circ \pi$; it follows that, for all $b, c \in B$, $\langle b, c \rangle_{\psi} = \psi(c^{*} b) = \phi(\pi(c^{*} b)) = \phi(\pi(c)^{*} \pi(b)) = \langle \pi(b), \pi(c) \rangle_{\phi}$ where the scalar products are induced by the two states on $B$. In particular any orthonormal basis of $B$ with respect to $\psi$ is sent by $\pi$ to an orthonormal basis with respect to $\phi$. Let $(b_{i})_{i}$ be an orthonormal basis of $(B, \psi)$ and consider the quantum group $A^{\text{aut}}(B, \psi)$ as introduced in Definition 1.1. It is generated by the coefficients of the matrix $u^{\psi}$ subject to some relations. The central points of this proof is that the quantum group $A^{\text{aut}}(B, \phi)$, considering $\pi(b_{i})$ as the basis of $B$, is generated by a matrix $u^{\phi}$ with the same relations as $A^{\text{aut}}(B, \psi)$, so the isomorphism between $A^{\text{aut}}(B, \psi)$ and $A^{\text{aut}}(B, \phi)$, using these well chosen presentations is simply given by $u^{\psi}_{ij} \mapsto u^{\phi}_{ij}$. \[\Box\]

Thanks to Proposition 1.5 in the following of this section we can always consider $B$ as a multimatirx C*-algebra endowed with a $\delta$-form $\psi$ such that the associated matrices $Q_{\alpha}$ are diagonal. We denote $Q_{i,\alpha}$ the eigenvalue in position $(i, i)$ of $Q_{\alpha}$ and observe that $\psi(e^{\psi}_{i}) = \text{Tr}(Q_{\alpha} e^{\psi}_{i}) = \delta_{i,1} Q_{i,\alpha}$.

The representation theory of $A^{\text{aut}}(B, \psi)$ is well known from [Ban02], but in order to generalize it to the wreath product with a discrete group we need a description of their intertwining spaces in terms of non-crossing partition, as it was done for $A^{\text{aut}}(C_{n}, \text{tr})$ (see [BS09]). To this purpose we recall some standard definition about non-crossing partitions.
Definition 1.6. Let $k, l \in \mathbb{N}$. We use the notation $NC(k, l)$ to denote the set of non-crossing partitions between $k$ upper points and $l$ lower points, i.e. all the possible connections between these points using strings that do not cross. A block in a non-crossing partition is a set of points connected each other, a sort of "connected component" of the diagram. A point without connections can be considered as a trivial block. The total number of blocks of $p \in NC(k, l)$ is denoted $b(p)$.

Definition 1.7. Let $p \in NC(k, l), q \in NC(v, w)$. We define the following diagram operations:

1. the tensor product $p \otimes q$ is the diagram in $NC(k + v, l + w)$ obtained by horizontal concatenation of the diagrams $p$ and $q$

2. if $l = v$ it is possible to define the composition $qp$ as the diagram in $NC(k, w)$ obtained identifying the lower points of $p$ with the upper points of $q$ and removing all the blocks eventually appeared and which does not contain neither one of the upper points of $p$ nor one of the lower points of $q$; this operation, when it is defined, is associative

3. the adjoint $p^*$ is the diagram in $NC(l, k)$ obtained turning upside-down the diagram $p$

Notation 1.8. Multiplying two non-crossing partitions $p \in NC(k, l), q \in NC(l, w)$ we get a unique partition $qp \in NC(k, w)$ but, as observed, there can be some blocks composed only of lower points of $p$/upper points of $q$ which are removed. We refer to them as central blocks and their number is denoted $cb(p, q)$.

Furthermore the vertical concatenation can produce some (closed) cycles which will not appear in the final non-crossing partition too. Intuitively they are the rectangles which are obtained when two or more central points are connected both in the upper and in the lower non-crossing partition (see example below). More formally the number of cycles is denoted $cy(p, q)$ and defined as

$$cy(p, q) = l + b(qp) + cb(p, q) - b(p) - b(q)$$

Example 1.9. To make more clear the multiplication operation and the concepts of block, central block and cycle we make an example with $p \in NC(4, 17)$ and $q \in NC(17, 5)$:

Immediately we have $b(p) = 6$, $b(q) = 7$, $b(qp) = 3$ and $cb(p, q) = 1$.

The number of cycles is then $cy(p, q) = 17 + 3 + 1 - 6 - 7 = 8$.

In the following proposition we introduce a simple but useful relation concerning the number of cycles obtained multiplying three non-crossing partitions.

Proposition 1.10. Let $p \in NC(k, l), r \in NC(l, m)$ and $s \in NC(m, v)$. Then the following relation holds:

$$cy(p, sr) = cy(p, r) + cy(rp, s) - cy(r, s)$$

Proof. Using the definition of a cycle just introduced and the associativity of the composition, relation (1.1) reduces to $cb(p, sr) = cb(p, r) + cb(rp, s) - cb(r, s)$. To finish the proof it is then enough to observe that $cb(p, sr) = cb(p, r)$ because the number of central blocks obtained concatenating $p$ and $sr$ does not depend from the non-crossing partition $s$; similarly $cb(rp, s) = cb(r, s)$.
The next goal is to assign a linear map to every non-crossing partition. To define this map we will use the following notation which generalizes the classical one.

**Notation 1.11.** Consider a diagram $p \in NC(k, l)$ and associate to every point a matrix unit of the canonical basis of $B$. Let $(b_{i_1,j_1}^{a_1}, \ldots, b_{i_k,j_k}^{a_k})$ be the ordered set of elements associated to the upper points and $(b_{r_1,s_1}^{b_1}, \ldots, b_{r_l,s_l}^{b_l})$ the elements associated to the lower points. Let $ij$ and $\alpha$ be the multi-index notation for the indices of these matrix, in particular $ij = ((i_1, j_1), \ldots, (i_k, j_k))$ and $\alpha = (\alpha_1, \ldots, \alpha_k)$: in an analogous way we define $rs$ and $\beta$. Denote by $b_v$, $v = 1, \ldots, m$ the different blocks of $p$, and let $b_v^\dagger$ $(b_v^\ddagger)$ be the upper (lower) points of the block $b_v$. This product is conventionally the identity matrix if there are no upper (lower) points in the block. Define

\[
\delta_p^{\alpha, \beta}(ij, rs) := \prod_{v=1}^{m} \psi((b_v^\dagger)^* b_v^\ddagger)
\]

**Example 1.12.** Consider the following non-crossing partition $p$ where we associated an element of the basis to every point.

\[
\begin{align*}
& b_{i_1,j_1}^{a_1} \quad b_{i_2,j_2}^{a_2} \quad b_{i_3,j_3}^{a_3} \\
& b_{r_1,s_1}^{b_1} \quad b_{r_2,s_2}^{b_2} \quad b_{r_3,s_3}^{b_3} \quad b_{r_4,s_4}^{b_4}
\end{align*}
\]

In this case the coefficient just introduced is $\delta_p^{\alpha, \beta}(ij, rs) = \psi((b_{i_1,j_1}^{a_1} b_{i_2,j_2}^{a_2} b_{i_3,j_3}^{a_3})^* b_{r_1,s_1}^{b_1} b_{r_2,s_2}^{b_2} b_{r_3,s_3}^{b_3})^* b_{r_4,s_4}^{b_4})$.

**Remark 1.13.** It is possible to give a more concrete interpretation of the coefficient $\delta_p^{\alpha, \beta}(ij, rs)$. First we notice that it can be non zero only if the indices $\alpha_k, \beta_k$ are equal in the points of a same block. In this case it will be effectively non zero if the following condition is satisfied for each block. Let $((i_{c_1}, j_{c_1}), \ldots, (i_{c_l}, j_{c_l}))$ and $((r_{d_1}, s_{d_1}), \ldots, (r_{d_w}, s_{d_w}))$ be the couples of indices of the matrix units associated, for a fixed block, to the upper points and to the lower points respectively. Then the second index of each couple must be equal to the first of the following one, assuming that:

- the first index of the first of the upper points is equal to the first index of the first of the lower points $(i_{c_1} = r_{d_1})$
- the second index of the last of the upper points is equal to the second index of the last of the lower points $(j_{c_l} = s_{d_w})$

**Definition 1.14.** We associate to every element $p \in NC(k, l)$ the linear map $T_p : B^\otimes k \rightarrow B^\otimes l$ defined by:

\[T_p(b_{i_1,j_1}^{a_1} \otimes \ldots \otimes b_{i_k,j_k}^{a_k}) = \sum_{r,s, \alpha, \beta} \delta_p^{\alpha, \beta}(ij, rs) b_{r_1,s_1}^{b_1} \otimes \ldots \otimes b_{r_l,s_l}^{b_l}\]

**Example 1.15.** The diagram $p$ associated to the multiplication map $m$ (writing explicitly $b_{i_1,j_1}^{a_1}, b_{i_2,j_2}^{a_2}$ on the upper points and $b_{r_1,s_1}^{b_1}$ on the lower point) is:

\[
\begin{align*}
& b_{i_1,j_1}^{a_1} \quad b_{i_2,j_2}^{a_2} \\
& b_{r_1,s_1}^{b_1}
\end{align*}
\]

Here, applying the definition

\[
\delta_p^{\alpha, \beta}((i_1, j_1, i_2, j_2), (r_1, s_1)) = \psi((\psi(e_{i_1,j_1}^{a_1}) e_{i_1,j_1}^{a_1})^{*} e_{i_2,j_2}^{a_2} e_{i_2,j_2}^{a_2})^{*} e_{r_1,s_1}^{b_1} e_{r_1,s_1}^{b_1}\]

\[
= \psi(e_{i_1,j_1}^{a_1})^{*} e_{i_2,j_2}^{a_2} e_{i_2,j_2}^{a_2} e_{r_1,s_1}^{b_1} e_{r_1,s_1}^{b_1}\]

\[
= \delta^{a_1, a_2} \delta^{a_1, b_1} \delta^{a_2, b_1} \psi_{i_1,j_1} e_{i_1,j_1}^{a_1} e_{i_2,j_2}^{a_2} e_{r_1,s_1}^{b_1} e_{r_1,s_1}^{b_1}\]

\[
= \delta^{a_1, b_1} \delta^{a_2, b_1} \psi_{i_1,j_1} e_{i_1,j_1}^{a_1} e_{i_2,j_2}^{a_2} e_{r_1,s_1}^{b_1} e_{r_1,s_1}^{b_1}\]

\[
= \delta^{a_1, b_1} \delta^{a_2, b_1} \psi_{i_1,j_1} e_{i_1,j_1}^{a_1} e_{i_2,j_2}^{a_2} e_{r_1,s_1}^{b_1} e_{r_1,s_1}^{b_1}\]
so the associated map $T_p : B^\otimes 2 \rightarrow B$ is given by

$$T_p(b_{i_1,j_1}^{\alpha_1} \otimes b_{i_2,j_2}^{\alpha_2}) = \delta_{\alpha_1,\alpha_2} \delta_{j_1,i_2} \psi(e_{j_1,i_1}^{\alpha_1})^{-\frac{1}{2}} b_{i_1,j_2}^{\alpha_2}$$

which is actually the multiplication $m$.

The diagram $q$ associated to the unity map $\eta$ is:

\[
\begin{array}{c}
\bullet \\
b_{1,1}^{\alpha_1} \\
\end{array}
\]

In fact $\delta_p^{\beta_1}((1,(r_1,s_1)) = \psi((e_{s_1,s_1}^{\beta_1})^{-\frac{1}{2}} e_{r_1,s_1}^{\beta_1}) = \delta_{r_1,s_1} \psi(e_{s_1,s_1}^{\beta_1})\frac{1}{2}$ so $T_q : \mathbb{C} \rightarrow B$ is given by $T_p(1) = \sum_{r_1} \psi(e_{r_1,s_1}^{\beta_1})\frac{1}{2} b_{1,1}^{\beta_1}$.

As usual the diagram of the identity map (with two basis elements) is:

\[
\begin{array}{c}
\bullet \\
b_{1,1}^{\alpha_1} \\
\bullet \\
b_{1,1}^{\beta_1} \\
\end{array}
\]

We can now state an important result of compatibility between the standard operations of tensor product, composition and adjoint of linear maps and the same operations between diagrams introduced in Definition 1.17 (see [BS09] Proposition 1.9) for the case of $\mathbb{C}^n$ with the canonical trace).

**Proposition 1.16.** Let $p \in NC(l,k), q \in NC(v,w)$. We have:

1. $T_{p \otimes q} = T_p \otimes T_q$
2. $T_{p^*} = T_p$
3. if $k = v$ then $T_{qp} = \delta^{-c(p,q)} T_q T_p$

**Proof.** The proof is essentially the same of [BS09], but it is necessary to be more careful during the computations in order to prove that this is the correct way to associate a morphism to a non-crossing partition, in particular it is crucial that $\psi$ is a $\delta$-form.

The relation 1 is clear because $\delta_p^{e_1,\beta}(ij,rs) \delta_q^{e_1',\beta'}(IJ,RS) = \delta_{p \otimes q}^{e_1,e_1'}(ij IJ, rs RS)$.

The relation 2 follows from $\delta_p^{e_1,\beta}(ij,rs) = \delta_{p^*}^{e_1,\beta}(rs,ij)$ which is true because $\psi((b_{ij}^{\alpha_1})^*) = \psi(b_{ij}^{\alpha_1})$ (here it is crucial that the eigenvalues of the matrix associated to $\psi$ are reals).

The relation 3 is less obvious and it is not possible to prove it just looking at the definition of the coefficients, because its validity strongly depends also on the geometric structure of the non-crossing partitions. In this case the relation to prove is

$$\sum_{\beta} \sum_{r,s=1}^{n_\beta} \delta_{\alpha_1,\alpha_2}^{\beta_1}(ij,rs) \delta_q^{\beta_2}(rs,RS) = \delta_{ \sum_{\beta} \sum_{r,s=1}^{n_\beta} \delta_{\alpha_1,\alpha_2}^{\beta_1}(ij,rs) \delta_q^{\beta_2}(rs,RS) = \delta_{\alpha_1,\alpha_2}^{\beta_1}(ij,RS)$$

We recall that every non-crossing partition is obtained using compositions, tensor products and adjoints of the basic morphisms $m, \eta$ and $id_B$. In order to prove the composition formula between the maps associated to two non-crossing partitions $p$ and $q$ we can think to decompose $q$ in the composition of a sequence of non-crossing partitions corresponding to elementary maps of type $id_B^{u,v} \otimes f \otimes id_B^{w,v}$ where $u, v \in \mathbb{N}$ and $f = id_B, m, m^*, \eta, \eta^*$. Supposing then that relation 3 holds for the composition of a general non-crossing partition $p$ with this kind
of maps and letting \( q = q_s \cdots q_2 q_1 \) be such a decomposition of \( q \) we have

\[
T_{qp} = T_{q_s \cdots q_2 q_1 p} = \delta^{-cy(p,q_2 \cdots q_1 p, q_s \cdots q_2 q_1)} T_{q_s \cdots q_2 q_1 p} \]

where the penultimate equality follows applying relation (1.3) is satisfied. For all the other possible situations to consider. If \( \eta \) is normalized) and \( \delta^{-cy(p,q_2 \cdots q_1 p, q_s \cdots q_2 q_1)} \) is the identity maps was replaced by \( \eta^* \) (and with a little abuse of notation we removed \( b_{R_1, S_1}^{\alpha_1} \) but did not reassign the index \( z \)). Relation (1.3) is verified because

It remains now only to show relation (1.3) when composing the map associated to \( s \) times Proposition 1.10.

Furthermore we can consider that the non-crossing partition \( p \) has only one block (or eventually two when \( f = m \); it is in fact possible to recover the multi-block case using a tensor product argument or generalizing in an obvious way the following proof. We are now ready to start the computations in the different cases. Let \( p \in NC(l, k) \) be a one block non-crossing partition. The first case that we study is the composition of \( T_p \) with \( T_q = \text{id}_B^\otimes k \). The corresponding diagram is

\[ \begin{array}{c}
\bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet \\
\end{array} \]

In this case relation (1.3) is satisfied, in fact

\[
\sum_{\beta} \sum_{r,s=1}^{\alpha} \delta_{\beta}^{-cy(i, j)} \delta_{\gamma}^{(rs)} (rs, RS) = \sum_{\beta} \sum_{r,s=1}^{\alpha} \psi((b_{R_1, S_1}^{\beta_1} \cdots b_{R_k, S_k}^{\beta_k})^* (b_{i_1, j_1}^{\beta_1} \cdots b_{i_l, j_l}^{\beta_l})) \prod_{t=1}^{k} \delta_{R_t, R_t}^{\delta} S_{R_t}
\]

with respect to the previous case in the lower non-crossing partition \( q \) one of the identity maps was replaced by \( \eta^* \) (and with a little abuse of notation we removed \( b_{R_1, S_1}^{\eta^*} \) but did not reassign the index \( z \)). Relation (1.3) is verified because
\[
\sum_{\beta} \sum_{r,s=1}^{n_{\beta}} \delta_{\beta}^{\alpha,\beta}(i,j,rs) \delta_{\beta}^{\beta,\gamma}(rs,RS) =
\sum_{\beta} \sum_{r,s=1}^{n_{\beta}} \psi((b_{r_1,s_1}^{b_{r_2,s_2}} \cdots b_{r_{k},s_{k}}^{b_{r_{k+1},s_{k+1}}})) \psi((b_{R_1,S_1}^{R_{r_1,s_1}} \cdots b_{R_{r_{k},s_{k}}^{R_{r_{k+1},s_{k+1}}}))) \prod_{t=1,t\neq z}^{k} \psi((b_{R_t,S_t}^{R_{r_t,s_t}})^* b_{r_t,s_t}) =
\sum_{\beta} \sum_{r,s=1}^{n_{\beta}} \psi((b_{r_1,s_1}^{b_{r_2,s_2}} \cdots b_{r_{k},s_{k}}^{b_{r_{k+1},s_{k+1}}}))(\delta_{r,s}) \sum_{\beta} Q_{s_1}^{\frac{1}{2}} \prod_{t=1,t\neq z}^{k} \delta_{\beta_{r_t},\delta_{R_{r_t},\delta_{S_{r_t}}}} =
\sum_{\beta} \delta_{\beta}^{\alpha,\beta}(i,j,RS) \psi((b_{R_1,S_1}^{R_{r_1,s_1}} \cdots b_{R_{r_{k},s_{k}}^{R_{r_{k+1},s_{k+1}}}})^* (b_{r_1,j_1}^{b_{r_{k+1},j_{k+1}}})) =
\delta_{\beta}^{\alpha,\beta}(i,j,RS)
\]

The third case which we analyse is the composition of \( T_p \) with \( T_q = \text{id}_B^\otimes \otimes m \otimes \text{id}_B^\otimes \). As before there are two different situations to consider: if the two upper points of \( m \) are connected just to one block or to two different blocks of the non-crossing partition \( p \). We observe that in the first sub-case a cycle appears, in fact the diagram is

As before one of the points of lower line was removed and its index (in this case \( z + 1 \)) was not reassigned to have a more clear notation. Relation (1.3) is still verified and the factor \( \delta \) due to the cycle appear. In fact we have

\[
\sum_{\beta} \sum_{r,s=1}^{n_{\beta}} \delta_{\beta}^{\alpha,\beta}(i,j,rs) \delta_{\beta}^{\beta,\gamma}(rs,RS) =
\sum_{\beta} \sum_{r,s=1}^{n_{\beta}} \psi((b_{r_1,s_1}^{b_{r_2,s_2}} \cdots b_{r_{k},s_{k}}^{b_{r_{k+1},s_{k+1}}})) \psi((b_{R_1,S_1}^{R_{r_1,s_1}} \cdots b_{R_{r_{k},s_{k}}^{R_{r_{k+1},s_{k+1}}}))) \prod_{t=1,t\neq z}^{k} \psi((b_{R_t,S_t}^{R_{r_t,s_t}})^* b_{r_t,s_t}) =
\sum_{\beta} \sum_{r,s=1}^{n_{\beta}} \psi((b_{r_1,s_1}^{b_{r_2,s_2}} \cdots b_{r_{k},s_{k}}^{b_{r_{k+1},s_{k+1}}}))(\delta_{r,s}) \sum_{\beta} Q_{s_1}^{\frac{1}{2}} \prod_{t=1,t\neq z}^{k} \delta_{\beta_{r_t},\delta_{R_{r_t},\delta_{S_{r_t}}}} =
\sum_{\beta} \delta_{\beta}^{\alpha,\beta}(i,j,RS) \psi((b_{R_1,S_1}^{R_{r_1,s_1}} \cdots b_{R_{r_{k},s_{k}}^{R_{r_{k+1},s_{k+1}}}})^* (b_{r_1,j_1}^{b_{r_{k+1},j_{k+1}}})) =
\delta \cdot \delta_{\beta}^{\alpha,\beta}(i,j,RS)
\]

The diagram corresponding to the sub-case with two blocks (with \( p \in \text{NC}(l + l', k + k') \)) is

Relation (1.3) is still verified, in fact

\[
\sum_{\beta,\beta'} \sum_{r',s'=1}^{n_{\beta'}} \delta_{\beta}^{\alpha,\beta'}(i,i',j,j',rr's's') \delta_{\beta}^{\beta',\gamma'}(rr's's',RR'S') =
\]
\[
\sum_{\beta, \beta'} \sum_{r, s=1}^{n_{\beta}} \sum_{r', s'=1}^{n_{\beta'}} \psi((b_{i_1}^{(r)}, s_{i_1}^{(r)}) \cdots b_{i_k}^{(r), s_{i_k}^{(r)})}) \psi((b_{i_1}^{(r')}, s_{i_1}^{(r')}) \cdots b_{i_k}^{(r'), s_{i_k}^{(r')}}) (b_{i_1}^{(r)}, s_{i_1}^{(r)}) \cdots (b_{i_k}^{(r'), s_{i_k}^{(r')}})
\]

where the penultimate equality is obtained simplifying \( Q_{j, \gamma}^{-1} \) with the value given by the first \( \psi \) and inserting all the \( \delta \) conditions and coefficients of its argument in the argument of the second \( \psi \). This is essentially due to the fact that the index \( j_i \) is in both arguments.

With similar computations it is finally possible to prove that the formula still holds in the remaining cases of \( \eta \) (trivial case) and \( m^* \).

\[ \square \]

**Remark 1.17.** It is interesting to observe that, with respect to the classical composition formula of the maps associated to two non-crossing partitions, in this more general case the eventual correction factor depends on the number of cycles which appear instead of the number of central blocks (see Proposition 1.16(3) and compare with Proposition 1.9(2) in [BS09]). This is due to a different choice made during the study of the two cases. In the classical case of the quantum permutation group \( S_n^+ = A^{\text{aut}}(C_n, tr) \) the \( \delta \)-form considered is the usual trace \( tr \) which is not normalized but is a 1-form, while in our general case of \( A^{\text{aut}}(B, \psi) \) the \( \delta \)-form \( \psi \) is always normalized (by definition of a state). Now the correction factor \( n^{-cb(p, q)} \) present in the classical formula can actually be seen as \( tr(1)^{-cb(p, q)} \) so it disappears considering a normalized \( \delta \)-form while on the other side it is obvious that the dependency from the cycles can be ignored if \( \delta = 1 \). Of course, for a given \( \delta \)-form, it is not possible, in general, to find a normalization constant such that, in addition, \( \delta = 1 \). In our more general context we decided to follow the definition of a state and to consider only normalized \( \delta \)-forms.

**Remark 1.18.** Proposition 1.16 allows us to define the \( C^* \)-tensor category of non-crossing partition. It will be denoted \( \mathcal{N}^C \) and:

- \( \text{Obj}(\mathcal{N}^C) = \mathbb{N} \)
- \( \text{Mor}(k, l) = \text{Span}\{p | p \in NC(k, l)\} \)

In particular in this category every object is equal to its adjoint, because for every \( k \in \mathbb{N} \) the map associated to the following diagram of \( NC(0, 2k) \) satisfies the conjugate condition.

\[ \emptyset \]

We now recall a result from [Ban99, Ban02, BS09] about the description of the intertwining spaces of the category of corepresentations of \( A^{\text{aut}}(B, \psi) \). The main difference is that the morphisms are associated to non-crossing partitions instead of Temperley-Lieb diagrams, as in [Ban02], where the \( \delta \)-form case is studied.

**Theorem 1.19.** Let \( B \) be a \( n \)-dimensional \( C^* \)-algebra, \( n \geq 4 \) and consider the compact quantum automorphism group \( A^{\text{aut}}(B, \psi) \) with fundamental representation \( u \). Then for all \( k, l \in \mathbb{N} \)

\[ \text{Hom}(u^{\otimes k}, u^{\otimes l}) = \text{span}\{T_p | p \in NC(k, l)\} \]

Furthermore the maps associated to distinct non-crossing partitions in \( NC(k, l) \) are linearly independent.
Proof. The first inclusion ($\subseteq$) comes from the fact that the morphisms in $\text{Hom}(u^\otimes k, u^\otimes l)$ are generated using linear combinations of compositions and tensor products of the maps $m, m^*, \eta, \eta^*, \text{id}$. All this "basic" maps correspond to non-crossing partitions (see Example 1.13), so the maps generated too.

The other inclusion ($\supseteq$) follows remarking that all non-crossing partitions can be obtained from the basic ones (diagrams of multiplication, unity and identity) using the operations of Definition 1.7 (and this is clear because this relation is true in the case of $(B, \psi) = (\mathbb{C}^n, \text{tr})$). The independence of the maps follows from a dimension count, as observed in [BS09], because the dimensions of the intertwining spaces computed in [Ban99] are still true in this case (see also [Ban02]). □

We recall now a proposition from [DCFY13] in order to generalize the representation theory to the case of a state $\psi$:

**Proposition 1.20.** (Brannan). Let $(B, \psi)$ be a finite-dimensional $C^*$-algebra equipped with a state. Let $B = \bigoplus_{i=1}^k B_i$ be the coarsest direct sum decomposition into $C^*$-algebras such that, for each $i$, the normalization $\psi_i$ of $\psi_{|B_i}$ is a $\delta_i$-form for a suitable $\delta_i$. Then, $A^{\text{aut}}(B, \psi)$ is isomorphic to the free product $\ast_{i=1}^k A^{\text{aut}}(B_i, \psi_i)$.

**Remark 1.21.** A suitable decomposition of $B$ always exists. If $B = \bigoplus_{\alpha=1}^\ell M_{n_{\alpha}}$ is the standard multimatrix decomposition, we first observe that the restriction of $\psi$ to every summand (after normalization) is a $\delta$-form for a suitable $\delta$. Then the summands $B_i$ of the decomposition in Proposition 1.20 are given by the direct sum of the $M_{n_{\alpha}}$ with a common $\delta$. The isomorphism is proved using universal properties.

The representation theory of a free product was studied by Wang in [Wan95] and is completely determined by the representation theory of the factors. In particular the non trivial irreducible representations are the alternating tensor product of the non trivial irreducible representations of the factors.

2. **Intertwiners and fusion rules of $\hat{\Gamma} \wr_\Lambda A^{\text{aut}}(B, \psi)$**

In this section we introduce the free wreath product of the quantum permutation group by a discrete group and study its representation theory.

**Definition 2.1.** Let $\Gamma$ be a discrete group and consider the quantum automorphism group $A^{\text{aut}}(B, \psi)$ where $\psi$ is a state on $B$. The free wreath product $\hat{\Gamma} \wr_\Lambda A^{\text{aut}}(B, \psi)$ is the universal unital $C^*$-algebra with generators $a(g) \in \mathcal{L}(B) \otimes \hat{\Gamma} \Lambda$, $A^{\text{aut}}(B, \psi)$, $g \in \Gamma$ and relations such that:

- $a(g)$ is unitary for every $g \in \Gamma$
- $m \in \text{Hom}(a(g) \otimes a(h), a(gh))$ for every $g, h \in \Gamma$
- $\eta \in \text{Hom}(1, a(\text{id}))$

This universal $C^*$-algebra can actually be endowed with a compact quantum group structure, but for this construction we need to study in more detail the generators $a(g)$.

**Notation 2.2.** Consider the matrices $a = (a_{ij,\alpha}^{kl,\beta})$ and $b = (b_{ij,\alpha}^{kl,\beta})$ with coefficients in a $C^*$-algebra where $1 \leq \alpha, \beta \leq c$, $1 \leq i, j \leq n_{\alpha}$, $1 \leq k, l \leq n_{\beta}$. Their multiplication is defined by:

$$(ab)_{ij,\alpha}^{kl,\beta} = \sum_{\gamma=1}^c \sum_{r,s=1}^n a_{rs,\gamma}^{kl,\beta} b_{ij,\alpha}^{rs,\gamma}$$

The adjoint matrix is:

$$a^* = ((a_{ij,\alpha}^{kl,\beta}^*)_{ij,\alpha}^{kl,\beta})$$

**Remark 2.3.** The generators of $\hat{\Gamma} \Lambda A^{\text{aut}}(B, \psi)$ can be seen as matrices of type $a(g) = (a_{ij,\alpha}^{kl,\beta}(g))$, $1 \leq \alpha, \beta \leq c$, $1 \leq i, j \leq n_{\alpha}$, $1 \leq k, l \leq n_{\beta}$, $g \in \Gamma$. Using the conventions introduced in Notation 2.2 the three conditions of
We observe that the matrices $a_{ij,\alpha}^r$ at the beginning of the paper we gave two equivalent definitions, for this proof we will consider the second one.

**Proof.** This proof consists in verifying that the defining properties of a compact quantum group are satisfied.

Consider the universal C*-algebra $\widehat{\Gamma}_\Delta \ast B,\psi$ and define the comultiplication map

$$\Delta : \Gamma_\Delta \ast B,\psi \longrightarrow \Gamma_\Delta \ast B,\psi \otimes \Gamma_\Delta \ast B,\psi$$

Then $(\widehat{\Gamma}_\Delta \ast B,\psi,\Delta)$ is a compact quantum group which will be denoted $H^+_\ast(B,\psi)(\Gamma)$.

**Proof.** This proof consists in verifying that the defining properties of a compact quantum group are satisfied. At the beginning of the paper we gave two equivalent definitions, for this proof we will consider the second one. We observe that the matrices $a(g)$ are unitary and, by construction, their entries generate a dense $*$-subalgebra of $\Gamma_\Delta \ast B,\psi$. The map $\Delta$ is the right one by definition. It remains only to prove that the transpose matrices $(a(g))^t$ are invertible. For every $(a(g))^t$ the inverse is given by $b(g) = \left(\frac{Q_{i,j,\alpha}^r}{Q_{j,i,\alpha}^s}\right)^{-\frac{1}{2}} \frac{a_{j,i,\alpha}^s(a^{-1})_{i,j,\alpha}^{rs}}{Q_{j,i,\alpha}^s}$, in fact we have that

$$(b(g)(a(g))^t)^{t^t} = \sum_{\alpha=1}^{c} \sum_{i,j=1}^{n_a} \left(\frac{Q_{i,j,\alpha}^r}{Q_{j,i,\alpha}^s}\right)^{-\frac{1}{2}} a_{j,i,\alpha}^s(a^{-1})_{i,j,\alpha}^{rs} = \delta_{\beta\gamma}\delta_{kr} \sum_{\alpha=1}^{c} \sum_{i,j=1}^{n_a} \left(\frac{Q_{i,j,\alpha}^r}{Q_{j,i,\alpha}^s}\right)^{-\frac{1}{2}} a_{j,i,\alpha}^s(e) = \delta_{\beta\gamma}\delta_{kr}\delta_{is}$$

so $b(g)(a(g))^t = \text{Id}$. In the same way it is possible to prove that $(a(g))^t b(g) = \text{Id}$, so $a(g)$ is invertible and $H^+_\ast(B,\psi)(\Gamma)$ is a compact quantum group.

**Remark 2.5.** The result given in Proposition 2.4 is clearly still valid for the free wreath product $H^+_\ast(B,\psi)(\Gamma)$ so the matrices $Q_{\alpha}$ associated to the state $\psi$ on $B$ are always considered to be diagonal.

The first step now is to study the representation theory of $H^+_\ast(B,\psi)(\Gamma)$ in the case of a $\delta$-form $\psi$. This result will then be extended to the case of a state $\psi$ with a result analogous to Proposition 1.20.

**Notation 2.6.** We denote by $NC_{\Gamma}(g_1,...,g_k;h_1,...,h_l)$ the set of diagrams in $NC(k,l)$ where the $k$ upper points are decorated by some $g_i \in \Gamma$ and the $l$ lower points by elements $h_j \in \Gamma$ such that in every block the product of the upper elements is equal to the product of the lower elements (with the convention that, if the block connects only upper or only lower points, the product must be the unit of $\Gamma$). For example

$$\begin{array}{cccc}
g_1 & g_2 & g_3 & g_4 \\
\bullet & \bullet & \bullet & \bullet \\
& & h_1 &
\end{array}$$

is in $NC_{\Gamma}(g_1,g_2,g_3,g_4;h_1)$ if $g_1 = e$, $g_2g_3g_4 = h_1$.

The operations between non-crossing partitions introduced in Definition 1.17 as well as the compatibility results of Proposition 1.18 naturally extend to decorated diagrams.
Definition 2.11. Let $\Gamma$ be a discrete group and $g, h, v \in \text{NC}_\Gamma(g_1, ..., g_k; h_1, ..., h_l)$, $q \in \text{NC}_\Gamma(g'_1, ..., g'_l; h'_1, ..., h'_m)$. We have:

1. $T_{p \otimes q} = T_p \otimes T_q$ where $p \otimes q \in \text{NC}_\Gamma(g_1, ..., g_k, g'_1, ..., g'_l; h_1, h'_1, ..., h_l, h'_m)$ is obtained by horizontal concatenation.
2. $T_p^* = T_{p^*}$ where $p^* \in \text{NC}_\Gamma(h_1, ..., h_l; g_1, ..., g_k)$ is obtained turning $p$ upside-down.
3. If $(h_1, ..., h_l) = (g'_1, ..., g'_l)$ then $T_{qp} = \delta^{-cy}(p,q)T_qT_p$ where $qp \in \text{NC}_\Gamma(g_1, ..., g_k; h'_1, ..., h'_m)$ is obtained by vertical concatenation.

Proof. The proof is essentially the same of Proposition 1.16. We have just to observe that the operations of tensor product, adjoint and composition always produce diagrams with an admissible decoration.

Example 2.8. The fundamental maps $m, \eta$ and $\text{id}_B$ can be represented using decorated non-crossing partitions. Their diagrams are the same diagrams introduced in Example 1.15 with all the admissible decorations. In particular for all $g, h \in \Gamma$ the multiplication $m$ corresponds to the following non-crossing partition of $\text{NC}_\Gamma(g, h; gh)$:

```
g h
\downarrow \downarrow
gh
```

while the unity $\eta$ and the identity $\text{id}_B$ correspond respectively to the following decorated diagrams in $\text{NC}_\Gamma(\emptyset; e)$ and in $\text{NC}_\Gamma(g; g)$:

```
\emptyset
\downarrow
\emptyset
```
```
g
\downarrow
\emptyset
```
```
\emptyset
\downarrow
g
```
```
\emptyset
\downarrow
\emptyset
```

Theorem 2.9. Let $\Gamma$ be a discrete group and $(B, \psi)$ be a finite dimensional C*-algebra with a $\delta$-form $\psi$ and $\dim(B) \geq 4$. The spaces of intertwiners of $\hat{\Gamma} \circ \text{out}(B, \psi)$ is spanned by the linear maps associated to some decorated non-crossing partitions. In particular for $g_1, h_j \in \Gamma$ we have

$$\text{Hom}(a(g_1) \otimes ... \otimes a(g_k), a(h_1) \otimes ... \otimes a(h_l)) = \text{span}\{T_p|p \in \text{NC}_\Gamma(g_1, ..., g_k; h_1, ..., h_l)\}$$

Proof. The first inclusion (\subseteq) is clear because the maps in the generator set of the morphisms (multiplication, unity, identity and their adjoints) are associated to the usual non-crossing partitions with all the possible decorations, so are in the span. For the other inclusion we need first to recall that all non-crossing partitions can be built using tensor product and composition of multiplication, unity, identity and their adjoints. It’s now enough to observe that it is possible to give every admissible decoration because the "product rule" (introduced in Notation 2.20) remains valid throughout all the tensoring and composition operations.

As in [Lem13b Cor 2.21] we can immediately deduce a result about basic corepresentations. The proof is identical.

Proposition 2.10. The basic corepresentations $a(g), g \in \Gamma$ of $H_\Gamma^+(B, \psi)(\Gamma)$ are irreducible and pairwise non-equivalent if $g \neq e$; the remaining corepresentation is $a(e) = 1 \oplus \omega(e)$, where $\omega(e)$ is irreducible and non-equivalent to any $a(g), g \neq e$.

Definition 2.11. Let $\Gamma$ be a discrete group and $M = < \Gamma >$ the monoid of the words over $\Gamma$. It is endowed with the following operations:

- involution: $(g_1, ..., g_k)^{-1} = (g_1^{-1}, ..., g_k^{-1})$
- concatenation: $(g_1, ..., g_k), (h_1, ..., h_l) = (g_1, ..., g_k, h_1, ..., h_l)$
- fusion: \((g_1, \ldots, g_k, h_1, \ldots, h_l) = (g_1, \ldots, g_k h_1, \ldots, h_l)\)

We can now state the main theorem, generalizing Theorem 2.25 in [Lem13b] to the case of \(H^+_{(B,\psi)}(\Gamma)\). The proof is the same of [Lem13b], because it relies only on the description of the intertwining spaces using non-crossing partitions which we proved to be possible in this more general context too.

**Theorem 2.12.** The irreducible corepresentations of \(H^+_{(B,\psi)}(\Gamma)\) are indexed by the words of \(M\) and denoted \(\omega(x), x \in M\) with involution \(\bar{\omega}(x) = \omega(\bar{x})\). In particular for \(g \in \Gamma\) we have \(\omega(g) = a(g) \oplus \delta_{g, e} 1\).

The fusion rules are:

\[
\omega(x) \otimes \omega(y) = \sum_{x=u,t \atop y=t,v} \omega(u,v) \oplus \sum_{x=u,t \atop y=t,v \atop u \neq \emptyset, v \neq \emptyset} \omega(u,v)
\]

As in [Lem13b] the same corepresentations can be indexed in a different way.

**Proposition 2.13.** Let \(L := \langle a, z_g | g \in \Gamma \rangle\) be a monoid and consider the submonoid \(S := \langle az_g a | g \in \Gamma \rangle\). There is a bijection between \(S\) and the irreducible corepresentations of \(H^+_{(B,\psi)}(\Gamma)\).

To extend these results to the case of a state \(\psi\) we use this proposition.

**Proposition 2.14.** Let \(B = \bigoplus_{\alpha=1}^c M_{n_{\alpha}}(\mathbb{C})\) be a finite-dimensional C*-algebra with a state \(\psi = \bigoplus_{\alpha=1}^c Tr(Q_{\alpha})\) on it. The state \(\psi\) restricted to every summand \(M_{n_{\alpha}}(\mathbb{C})\) (and normalized) is a \(\delta\)-form with \(\delta = Tr(Q_{\alpha}^{-1})\).

Consider the decomposition \(B = \bigoplus_{i=1}^d B_i\) where every \(B_i\) is the direct sum of all the \(M_{n_{\alpha}}(\mathbb{C})\) such that \(Tr(Q_{\alpha}^{-1})\) is a constant value denoted \(\delta_i\). Let \(\psi_i\) be the state on \(B_i\) obtained by normalizing \(\psi_i|_{B_i}\). Then

\[
\hat{\Gamma} \ast \ast \bigotimes_{i=1}^k A_{\ast \ast}^u(B_i) \cong \bigoplus_{i=1}^k \bigotimes_{i=1}^k A_{\ast \ast}^u(B_i, \psi_i)
\]

is a \(*\)-isomorphism which intertwines comultiplications.

**Proof.** To prove this result it is necessary to use the relations introduced in Remark 2.3 between the generators of the free wreath product \(\hat{\Gamma} \ast \ast \bigotimes_{i=1}^k A_{\ast \ast}^u(B, \psi)\). First we notice that the map \(mm^* \in \Hom(a(g), a(g))\) is such that \(mm^*|_{B_i} = \delta_i \text{id}_{B_i}\). The existence of this particular morphism implies that \(\delta_{ij}^{k,l} \delta_{ij}^{\alpha}(g) = 0\) for all \(i, j, k, l\) whenever \(M_{n_{\alpha}}(\mathbb{C})\) and \(M_{n_j}(\mathbb{C})\) are not in the same summand \(B_i\). This allows to simplify the defining relations above and in particular we deduce that, for every \(i\), there is a natural morphism \(\hat{\Gamma} \ast \ast \bigotimes_{i=1}^k A_{\ast \ast}^u(B_i, \psi_i) \rightarrow \hat{\Gamma} \ast \ast \bigotimes_{i=1}^k A_{\ast \ast}^u(B, \psi)\).

It follows that, by the universality of the free product construction, there is a morphism intertwining the comultiplications

\[
*_{i=1}^k(\hat{\Gamma} \ast \ast \bigotimes_{i=1}^k A_{\ast \ast}^u(B_i, \psi_i)) \rightarrow \hat{\Gamma} \ast \ast \bigotimes_{i=1}^k A_{\ast \ast}^u(B, \psi)
\]

Conversely, joining the generators and the relations of the different factors \(\hat{\Gamma} \ast \ast \bigotimes_{i=1}^k A_{\ast \ast}^u(B_i, \psi_i)\) we get exactly the generators and the relations of \(H^+_{(B,\psi)}(\Gamma)\) (in the simplified version) so by the universality of the free wreath product construction we get the inverse morphism intertwining the comultiplications

\[
\hat{\Gamma} \ast \ast \bigotimes_{i=1}^k A_{\ast \ast}^u(B, \psi) \rightarrow *_{i=1}^k(\hat{\Gamma} \ast \ast \bigotimes_{i=1}^k A_{\ast \ast}^u(B_i, \psi_i))
\]

and the quantum group isomorphism is proved. \(\Box\)

The non trivial irreducible representations of \(\hat{\Gamma} \ast \ast \bigotimes_{i=1}^k A_{\ast \ast}^u(B, \psi)\) are then given by an alternating tensor product of non trivial irreducible representations of the factors \(\hat{\Gamma} \ast \ast \bigotimes_{i=1}^k A_{\ast \ast}^u(B_i, \psi_i)\) (see [Wan95]).

We end this section with a remark about the spectral measure on a subalgebra of \(H^+_{(B,\psi)}(\Gamma)\) (\(\psi\) \(\delta\)-trace) which will be useful in the following section.
Generalisation of the proof presented in [Lem13b, Theorem 3.5] and relies on the simplicity of the generator $a$ to satisfy the Avitzour’s condition. But this follows from Remark 2.15 where we observed that, considering $a \delta$ when $\psi$ is a $\delta$-form. Let $\chi(a(e)) := (Tr \otimes id)(a(e))$ be the character of the corepresentation $a(e)$. It follows immediately from relation (2.1) that $\chi(a(e))$ is selfadjoint. To find the spectral measure of $\chi(a(e))$ it is then enough to compute the moments $h(\chi(a(e))^k)$. In particular, denoting $p_k$ the orthogonal projection onto the fixed points space $Hom(1, a(e) \otimes k)$ and using some classic results of Woronowicz (see [Wor88]) we have $h(\chi(a(e))^k) = h((Tr \otimes id)(a(e))^k) = Tr((id \otimes h)(a(e) \otimes k)) = Tr(p_k) = \dim(Hom(1, a(e) \otimes k)) = \#NC(0, k) = C_k$ where $C_k$ are the Catalan numbers. They are exactly the moments of the free Poisson law of parameter $1$, so this is the spectral measure of $\chi(a(e))$.

3. Properties of $H^+_{(B, \psi)}(\Gamma)$

3.1. Simplicity and uniqueness of the trace for the reduced algebra. In this section we prove that, under certain conditions, the reduced C*-algebra $C_r(H^+_{(B, \psi)}(\Gamma))$ is simple and has a unique trace.

Remark 3.1. The free product decomposition given in Proposition 2.14 implies, using a well known result of Wang (see [Wan95]), that the Haar measure of $\hat{\Gamma}_\ast A_{aut}(B, \psi)$ is the free product of the Haar measures of its factors. It follows that the decomposition is still true at the level of the reduced C* and von Neumann algebras, so the following isomorphisms holds:

$$(C_r(\hat{\Gamma}_{\ast} A_{aut}(B, \psi)), h) \cong \bigoplus_{i = 1}^{k} (C_r(\hat{\Gamma}_{\ast} A_{aut}(B_i, \psi_i)), h_i)$$

$$(L^\infty(\hat{\Gamma}_{\ast} A_{aut}(B, \psi)), h) \cong \bigoplus_{i = 1}^{k} (L^\infty(\hat{\Gamma}_{\ast} A_{aut}(B_i, \psi_i)), h_i)$$

where $h$ and $h_i$ are the Haar states on the respective C*-algebras.

Proposition 3.2. Let $(B, \psi)$ be a finite dimensional C*-algebra endowed with a trace $\psi$. Let $\Gamma$ be a discrete group, $|\Gamma| \geq 4$. Consider the free product decomposition of the reduced C*-algebra $C_r(H^+_{(B, \psi)}(\Gamma))$ given in Remark 3.1. If there is just one factor (i.e. $\psi$ is a $\delta$-trace) and $\text{dim}(B) \geq 2$ or there are two or more factors with $\text{dim}(B_i) \geq 4$ for all $i$, then $C_r(H^+_{(B, \psi)}(\Gamma))$ is simple and has a unique trace given by the free product of the Haar measures.

Proof. If there are two or more factors the result follows from a proposition of Avitzour (see [Avi82] Section 3). It states that, given two C*-algebras $A$ and $B$ endowed with tracial Haar states $h_A$ and $h_B$, the reduced free product C*-algebra $A \ast \text{red}_B$ is simple with unique trace if there exist two unitary elements of $Ker(h_A)$ orthogonal with respect to the scalar product induced by $h_A$ and a unitary element in $Ker(h_B)$. To show that in our case these elements exist we use a result from [DHR97] Proposition 4.1 (i) according to which, if a C*-algebra $A$ endowed with a normalized trace $\tau$ admits an abelian sub-C*-algebra $B$ so that the spectral measure corresponding to $\tau_B$ is diffuse, then there is a unitary element $u \in A$ such that $\tau(u^n) = 0$ for each $n \in Z, n \neq 0$. Our purpose is then to apply this proposition to every factor of the decomposition in order to satisfy the Avitzour’s condition. But this follows from Remark 2.13 where we observed that, considering the generator $a(e)$ of an indecomposable free wreath product $\hat{\Gamma} \ast A_{aut}(B, \psi)$, $\psi \delta$-trace, the spectral measure associated to its character $\chi(a(e))$ is the free Poisson law of parameter $1$, which is diffuse. The simplicity and uniqueness of the trace in the multifactor case is then proved.

In the second case, when $\psi$ is a $\delta$-trace and there is not a free product decomposition, the proof is actually a generalisation of the proof presented in [Lem13b, Theorem 3.5] and relies on the simplicity of $A_{aut}(B, \psi)$ for a $\delta$-trace proved in [Bra13], so we will give just a sketch of the arguments.

Let $E$ be the subset of the monoid $M$ containing only words composed using the neutral element of $\Gamma$, denoted $e$. Let $A$ be the subspace of $C_r(H^+_{(B, \psi)}(\Gamma))$ generated by the coefficients of the irreducible representations
associated to the words of $E$ and $A'$ its closure in $C_r(\mathcal{H}^+_{(B,\psi)}(\Gamma))$. We have that $A' \cong C_r(\text{Aut}(B, \psi))$. Furthermore there exists a unique conditional expectation $P: C_r(\mathcal{H}^+_{(B,\psi)}(\Gamma)) \to A'$ such that $h = h_{A'} \circ P$, where $h$ is the Haar state of $C_r(\mathcal{H}^+_{(B,\psi)}(\Gamma))$ and $h_{A'} := h_{|A'}$. Now, let $I \subseteq C_r(\mathcal{H}^+_{(B,\psi)}(\Gamma))$ be an ideal; we want to prove that it is trivial or equal to $C_r(\mathcal{H}^+_{(B,\psi)}(\Gamma))$. Consider the ideal $P(I) \subseteq A'$; the simplicity of $C_r(\text{Aut}(B, \psi))$ implies that $P(I)$ is either trivial or $A'$. If $P(I) = \{0\}$ then $I \subseteq \ker(P) \subseteq \ker(h)$. Suppose $x \in I$, then $x^*x \in I$, so $h(x^*x) = 0$. $h$ being faithful we get immediately $x = 0$ so in this case $I = \{0\}$. If $P(I) = A'$ the proof is more complicated, but the idea is that if $x \in I$ it is possible to built a sequence $(b_i)_i$ in $C_r(\text{Aut}(B, \psi))$ such that $\|1 - \sum b_i b_i^*\| < 1$ so the element $\sum b_i b_i^* \in I$ and it is invertible.

To prove the uniqueness of the trace we introduce the space $D$ generated by the coefficients of the irreducible representations associated to words in $E^\ast$, i.e. which contain at least a letter different from $e$. Let $D'$ be its closure and notice that $C_r(\mathcal{H}^+_{(B,\psi)}(\Gamma)) = A' \ast D'$. The first step of the proof consists in showing that any faithful normal trace on $C_r(\mathcal{H}^+_{(B,\psi)}(\Gamma))$ coincide with the Haar state when restricted to $D'$. The uniqueness of the trace on $C_r(\text{Aut}(B, \psi)) \cong A'$ implies the equality also when restricting to $A'$ so the proof is finished. 

3.2. Haagerup property. The aim of this subsection is to prove that, under some hypothesis, the von Neumann algebra $L^\infty(\mathcal{H}^+_{(B,\psi)}(\Gamma))$ has the Haagerup property. First we recall the basic definition.

**Definition 3.3.** Let $G$ be a compact quantum group with Haar state $h$. We say that $L^\infty(G)$ has the Haagerup property if there exists a net $(\{\varphi_x\})_{x \in A}$ of normal unital completely positive $h$-preserving maps on $L^\infty(G)$ such that the extension to $L^2(G)$ is a compact operator and converges pointwise to the identity in $L^2$-norm.

In particular we want to prove the following result.

**Proposition 3.4.** Let $\Gamma$ be a finite group and $(B, \psi)$ a finite dimensional $C^*$-algebra $(\dim(B) \geq 4)$ with a $\delta$-trace. Then $L^\infty(\mathcal{H}^+_{(B,\psi)}(\Gamma))$ has the Haagerup approximation property.

This proposition extends Theorem 3.12 in [Lem13b] and the proof uses the same arguments.

In the following $\pi: C(\mathcal{H}^+_{(B,\psi)}(\Gamma)) \to C(\text{Aut}(B, \psi))$ will be the canonical map given by $(\text{id} \otimes \pi)(a(g)) = u$ for all $g \in \Gamma$ where $u$ is the fundamental representation of $\text{Aut}(B, \psi)$. Consider the Gelfand isomorphism

$$\omega: C^*(\chi_\alpha, \alpha \in \text{Irr}(\text{Aut}(B, \psi))) \to C(\text{Spec}(\chi))$$

where $\chi$ is the character of the fundamental representation of $\text{Aut}(B, \psi)$ (i.e. $\chi = (\text{Tr} \otimes \text{id})(u)$), $\chi_\alpha$ is the character of the irreducible representation indexed by $\alpha$ and $[0, \dim(B)] \subseteq \text{Spec}(\chi) \subset \mathbb{R}$.

We recall that the irreducible representations of a compact quantum group allow to decompose the Hilbert space associated with the GNS construction, in particular, in our case, $L^2(\mathcal{H}^+_{(B,\psi)}(\Gamma)) = \bigoplus_{\alpha \in S} L^2_\alpha(\mathcal{H}^+_{(B,\psi)}(\Gamma))$ where $L^2_\alpha(\mathcal{H}^+_{(B,\psi)}(\Gamma))$ is the space generated by the coefficients of the irreducible representation indexed by $\alpha$. Let $P_\alpha: L^2(\mathcal{H}^+_{(B,\psi)}(\Gamma)) \to L^2_\alpha(\mathcal{H}^+_{(B,\psi)}(\Gamma))$ be the orthogonal projection on the summand. We finally have this crucial lemma from [Bra13].

**Lemma 3.5.** Let $I = [0, 4]$ if $\dim(B) = 4$ and $I = [4, \dim(B)]$ if $\dim(B) \geq 5$. The net $(T_{\varphi_x})_{x \in I}$ is made up of normal, unital, completely positive $h$-preserving maps on $L^\infty(\mathcal{H}^+_{(B,\psi)}(\Gamma))$ defined by:

$$T_{\varphi_x} = \sum_{\alpha \in S} \frac{\varphi_x(\chi_\alpha)}{d_\alpha} P_\alpha$$

where $\varphi_x = ev_x \circ \omega \circ \pi$ is a state on $C^*(\chi_\alpha, \alpha \in \text{Irr}(\mathcal{H}^+_{(B,\psi)}(\Gamma)))$, $ev_x$ is the evaluation function in the point $x$ and $d_\alpha$ is the dimension of the representation associated to $\alpha$. 


Proof. (Proposition 3.4) To prove that $L^\infty(H^+_\infty(B,\psi)(\Gamma))$ has the Haagerup approximation property it is necessary to show that the $L^2$ extension of the net $(T_{\varphi_x})_{x \in I}$ is a compact operator and it converges pointwise to the identity.

The proof of the compactness, being each $T_{\varphi_x}$ the direct sum of projections over finite dimensional spaces, reduces to show that the net of the coefficients vanishes at infinity, i.e. that $\frac{1}{\varphi_x(\alpha)} \to 0$ as $\alpha \to \infty$. This can be proved exactly as in [Lem13a] Propositions 3.3 and 3.4 in the case of $\hat{Z}_s \wr S_N$ because the proof relies only on the fusion rules which are essentially the same.

Following [Lem13a] Proposition 3.5] we have the pointwise convergence too.

Since the Haagerup property is stable under tracial free products (see [Boc93, Proposition 3.9]) and using Remark 3.1 we have the generalisation to the case of a trace.

Proposition 3.6. Let $(B,\psi)$ be a finite dimensional C*-algebra endowed with a trace $\psi$ and $\Gamma$ be a finite group. Consider the free product decomposition of the reduced von Neumann algebra $L^\infty(\hat{\Gamma})$, $A^{\text{aut}}(B,\psi)$ given in Remark 3.1. If for each $i$, $\dim(B_i) \geq 4$ then $L^\infty(H^+_\infty(B,\psi)(\Gamma))$ has the Haagerup property.

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