An Upper Bound to Zero-Delay Rate Distortion via Kalman Filtering for Vector Gaussian Sources

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Abstract—We deal with zero-delay source coding of a vector Gaussian autoregressive (AR) source subject to an average mean squared error (MSE) fidelity criterion. Toward this end, we consider the nonanticipative rate distortion function (NRDF) which is a lower bound to the causal and zero-delay rate distortion function (RDF). We use the realization scheme with feedback proposed in [1] to model the corresponding optimal “test-channel” of the NRDF, when considering vector Gaussian AR(1) sources subject to an average MSE distortion. We give conditions on the vector Gaussian AR(1) source to ensure asymptotic stationarity of the realization scheme (bounded performance). Then, we encode the vector innovations due to Kalman filtering via lattice quantization with subtractive dither and memoryless entropy coding. This coding scheme provides a tight upper bound to the zero-delay Gaussian RDF. We extend this result to vector Gaussian AR sources of any finite order. Further, we show that for infinite dimensional vector Gaussian AR sources of any finite order, the NRDF coincides with the zero-delay RDF. Our theoretical framework is corroborated with a simulation example.

I. INTRODUCTION

Zero-delay source coding is desirable in various real-time applications, such as, in signal processing [2] and networked control systems [3–5]. Zero-delay codes form a subclass of causal source codes (see [6]), namely, codes where the reproduced source samples depends on the source samples in a causal manner. However, zero-delay source coding compared to causal source coding allow the reproduction of each source sample at the same time instant that the source sample is encoded. Unfortunately, causal source coding does not exclude the possibility of long blocks of quantized samples, which may cost arbitrary end-to-end delays.

Zero-delay codes (and causal codes) in contrast to non-causal codes cannot achieve the classical rate distortion function (RDF). Indeed, an open problem in information theory is quantifying the gap between the optimal performance theoretically attainable (OPTA) by non-causal codes, and the OPTA by causal and zero-delay codes, hereinafter denoted by \( R_{\text{ZD}}^p(D) \) and \( R_{\text{op}}^p(D) \), respectively. Notable exceptions where this gap is explicitly found are memoryless sources [6], stationary sources in high rates [7], and zero mean stationary scalar Gaussian sources with average mean squared error (MSE) distortion [8].

Throughout the years, the interest in zero-delay applications is growing, thus, initiating further research on characterizing the fundamental limitations of the OPTA by zero-delay codes. Unfortunately, it turns out that \( R_{\text{ZD}}^p(D) \) is very hard to compute and for this reason there has been a turn in studying variants of classical RDF that perform as tight as possible to \( R_{\text{ZD}}^p(D) \).

In this paper, we derive a tight upper bound to zero-delay source coding for vector Gaussian AR sources subject to an average MSE distortion. We consider nonanticipative rate distortion function (NRDF) (see, e.g., [8, eq. (1)]) which gives a tighter lower bound to \( R_{\text{op}}^p(D) \) compared to the classical RDF (see, e.g., [6, eq. (1)])). Then, we employ the feedback realization scheme proposed in [1, Fig. IV.3], that corresponds to the optimal “test-channel” of NRDF for vector Gaussian AR(1) sources and average MSE distortion. Further, we give conditions to asymptotically stabilize the performance of the specific scheme. By invoking standard techniques using entropy coded dither quantizer (ECDQ) [10, 11] on the innovations’ encoder of the feedback realization scheme, we derive the tight upper bound. In addition, we show how to generalize our scheme to vector Gaussian AR sources of any finite order. If the vector dimension of the Gauss AR source tends to infinity, we show that the \( R_{\text{op}}^p(D) \) coincides with the NRDF. We demonstrate our results with a numerical example.

Notation: We let \( \mathbb{R} = (-\infty, \infty) \), \( \mathbb{N}_0 = \{0,1,\ldots\} \). For \( t \in \mathbb{N}_0 \). We denote a sequence of RVs by \( x^n \triangleq (x_0, \ldots, x_n) \) and its realization by \( \hat{x}^n = \hat{x}^n, x_j \in \mathcal{X}, \ j = 0, \ldots, n \). The distribution of the RV \( x \) on \( \mathcal{X} \) is denoted by \( P_X(dx) \equiv P(dx) \). The conditional distribution of RV \( y \) given \( x = x \) is denoted by \( P_{Y|x}(dy|x) \equiv P(dy|x) \). The transpose of a matrix \( S \) is denoted by \( S^T \). For a square matrix \( S \in \mathbb{R}^{p \times p} \) with entries \( S_{ij} \) on the \( i^{th} \) row and \( j^{th} \) column, we denote by \( \text{diag}(S) \) the matrix having \( S_{ii}, \ i = 1, \ldots, p \), on its diagonal and zero elsewhere.

II. PROBLEM STATEMENT

In this paper we consider the zero-delay source coding setting illustrated in Fig. 1. In this setting, the \( p \)-dimensional (vector) Gaussian source is governed by the following discrete-time linear time-invariant state-space model

\[
x_{t+1} = Ax_t + Bw_t, \ t \in \mathbb{N}_0,
\]

where \( A \in \mathbb{R}^{p \times p} \), and \( B \in \mathbb{R}^{p \times q} \) are known, \( x_0 \in \mathbb{R}^p \sim \mathcal{N}(0; \Sigma_{x_0}) \) is the initial state, and the noise process \( w_t \in \mathbb{R}^q \).
is an i.i.d. Gaussian $N(0; I_{q_xq_y})$ sequence, independent of $x_0$. We allow $A$ to have eigenvalues outside the unit circle which means that $x_t$ can be unstable.

The system operates as follows. At every time step $t$, the encoder observes the source $x^t$ and produces a single binary codeword $z_t$ from a pre-defined set of codewords $Z_t$ of at most a countable number of codewords. Since the source is random, $z_t$ and its length $l_t$ are random variables. Upon receiving $z_t$, the decoder produces an estimate $\hat{y}_t$ of the source sample. We assume that both the encoder and decoder process information without delay and they are allowed to have infinite memory of the past.

Fig. 1: A zero-delay source coding scenario.

The analysis of the noiseless digital channel is restricted to the class of instantaneous variable-length binary codes $z_t$. The countable set of all codewords (codebook) $Z_t$ is time-varying to allow the binary representation $z_t$ to be an arbitrarily long sequence. The encoding and decoding policies are described by sequences of conditional probability distributions as $\{P(dz_t | z^{t-1}, x^t): t \in \mathbb{N}_0\}$ and $\{P(dy_t | y^{t-1}, z^t): t \in \mathbb{N}_0\}$, respectively. At $t = 0$, we assume $P(dz_0 | z^{-1}, x^0) = P(dz_0 | x_0)$ and $P(dy_0 | y^{0}, z^0) = P(dy_0 | z_0)$.

The design in Fig. 1 is required to yield an asymptotic average distortion $\limsup_{n \to \infty} \frac{1}{n+1} \mathbb{E}\{d(x^n, y^n)\} \leq D$, where $D > 0$ is the pre-specified distortion level, $d(x^n, y^n) \triangleq \sum_{t=0}^{n} \|x_t - y_t\|_2^2$. The objective is to minimize the expected average codeword length denoted by $\limsup_{n \to \infty} \frac{1}{n+1} \sum_{t=0}^{n} \mathbb{E}(L_t)$, over all encoding-decoding policies. These design requirements are formally cast by the following optimization problem:

$$R^{\infty}_{2D}(D) \triangleq \inf_{n \to \infty} \lim_{n \to \infty} \frac{1}{n+1} \sum_{t=0}^{n} \mathbb{E}(L_t) \leq D,$$

(2)

s. t. $\limsup_{n \to \infty} \frac{1}{n+1} \mathbb{E}\{d(x^n, y^n)\} \leq D$,

determining the OPTA by zero-delay codes.

III. PRELIMINARIES

In this section, we give the definition of NRDF of vector Gaussian AR sources subject to an average MSE distortion.

Define the source distribution by $P(dx^n) \triangleq \prod_{t=0}^{n} P(dx_t | x^{t-1})$, the reconstruction distribution by $P(dy^n | x^n) \triangleq \prod_{t=0}^{n} P(dy_t | y^{t-1}, x^t)$, and the joint distribution by $P(dx^n, dy^n) \triangleq P(dx^n) \otimes P(dy^n | x^n)$. The marginal on $y_t \in \mathcal{Y}$, $P(dy_t | y^{t-1})$, is induced by the joint distribution $P(dx^n, dy^n)$. We assume that at $t = 0$, $P(dy_0 | y^{0}, x^0) = P(dy_0 | x_0)$.

Given the previous distributions, we introduce the mutual information between $x^n$ and $y^n$ as follows:

$$I(x^n; y^n) \triangleq \sum_{t=0}^{n} \mathbb{E} \log \frac{P(y_t | y^{t-1}, x^t)}{P(y_t | y^{t-1})},$$

where $\mathbb{E}\{\cdot\}$ is the expectation with respect to the joint distribution $P(dx^n, dy^n)$.

**Definition 1.** (NRDF with average MSE distortion)

(1) The finite-time NRDF is defined by

$$R_{0,n}^{\infty}(D) \triangleq \inf_{P(dy_0 | y^{0}, x_0)} \lim_{n \to \infty} \frac{n+1}{n+1} I(x^n; y^n),$$

assuming the infimum exists.

(2) The per unit time asymptotic limit of (1) is defined by

$$R^{\infty}(D) = \lim_{n \to \infty} R_{0,n}^{\infty}(D),$$

assuming the infimum exists and the limit exists and it is finite.

If one replaces $\liminf_{n \to \infty}$ by $\inf$ in (4), then an upper bound to $R^{\infty}(D)$ is obtained, defined as follows:

$$\hat{R}^{\infty}(D) \triangleq \inf_{P(dy_0 | y^{0}, x_0)} \lim_{n \to \infty} \frac{n+1}{n+1} R_{0,n}^{\infty}(D),$$

where $P^{\infty}(dy_t | x, x_t) = P^{\infty}(dy_t | x_t)$ is the stationary or time-invariant reconstruction distribution.

It is shown in [12] Theorem 6.6] that provided the limit in (5) exists, and the source is stationary (or asymptotically stationary) then $R^{\infty}(D) = \hat{R}^{\infty}(D)$.

The optimization problem of Definition 1 in contrast to the one given in (2) is convex (see e.g., [13]). In addition, for the source model (4) and the average MSE distortion, then, by [1] Theorems 1, the optimal “test channel” corresponding to (4) is of the form

$$P^* (dy_t | y^{t-1}, x^t) = P^* (dy_t | y^{t-1}, x_t), \quad t \in \mathbb{N}_0,$$

(6)

where at $t = 0$, $P^* (dy_0 | y^{0}, x_0) = P^* (dy_0 | x_0)$, and the corresponding joint process $\{(x_t, y_t): t \in \mathbb{N}_0\}$ is jointly Gaussian.

IV. ASYMPTOTICALLY STATIONARY FEEDBACK REALIZATION SCHEME VIA KALMAN FILTERING

The authors in [1] Theorem 2] realized the optimal “test channel” of (4) with the feedback realization scheme illustrated in Fig. IV.3 that corresponds to a realization of the form:

$$y_t = E_t H_t E_t (x_t - \tilde{x}_{t|t-1}) + E_t \Phi_t \Theta_t v_t + \tilde{x}_{t|t-1},$$

(7)

where $H_t \triangleq \Phi_t \Theta_t$ is a scaling matrix; $v_t$ is an independent Gaussian noise process with $N(0; \Sigma_{v_t})$, $\Sigma_{v_t} = \text{diag}(V_t)$ independent of $x_0$; the error $x_t - \tilde{x}_{t|t-1}$ is Gaussian with $N(0; \Omega_{x_t|t-1})$, and $\tilde{x}_{t|t-1} \triangleq \mathbb{E}\{x_t | y^{t-1}\}$; the error $x_t - \tilde{x}_{t|t}$ is Gaussian with $N(0; \Omega_{x_t|t})$, and $\tilde{x}_{t|t} \triangleq \mathbb{E}\{x_t | y^{t}\}$. Moreover,
Innovations Encoder
\[ \{ \tilde{x}_{i[t-1]}, \; \Pi_{i[t-1]} \} \text{ are given by the following Kalman filter equations:} \]

**Prediction:**
\[
\hat{x}_{i[t-1]} = A\hat{x}_{i-1[t-1]}, \quad \hat{x}_{0[0]} = \mathbb{E}\{x_0\}, \tag{8a}
\]
\[
\Pi_{i[t-1]} = A\Pi_{i-1[t-1]}A^\top + BB^\top, \quad \Pi_{0[0]} = \Sigma_{x_0}, \tag{8b}
\]

**Update:**
\[
\hat{x}_{i[t]} = \hat{x}_{i[t-1]} + G_t \tilde{k}_t, \quad \hat{x}_{0[0]} = \mathbb{E}\{x_0\}, \tag{8c}
\]
\[
k_t \triangleq y_t - \tilde{x}_{i[t-1]} \text{ (innovation)}, \tag{8d}
\]
\[
\Pi_{i[t]} = \Pi_{i[t-1]} - G_t S_t G_t^\top, \quad \Pi_{0[0]} = \Sigma_{x_0}, \tag{8e}
\]
\[
G_t = \Pi_{i[t-1]}(E_t^\top H_t E_t) S_t^{-1}, \quad \text{(Kalman Gain)}
\]

where \( H_t = \text{diag}\{1 - \Delta_t^\top\}, \ \Theta_t = \sqrt{H_t \Delta_t \Sigma_v^{-1}}, \ \Phi_t = \Theta^{-1} H_t, \ \Delta_t = \text{diag}\{\delta_t\}, \ \Lambda_t = E_t \Pi_{i[t-1]} E_t^\top = \text{diag}\{\lambda_t\}, \)
\nand \( E_t \in \mathbb{R}^{p \times p} \) is an orthogonal matrix. By substituting (8a) in (7) we can also deduce that \( \mathbf{P}^* (d_{i[t]}y^{t-1}, x_t) = \mathbf{P}^* (d_{i[t]}y^{t-1}, x_t) \).

The realization scheme of (7) becomes asymptotically stationary (stable) if one of the following two conditions hold: (1) \( A \) is stable, i.e., all eigenvalues have magnitude less than one; (2) the pair \( (A, B) \) is (completely) stabilizable (see e.g., [14, p. 342]). This means that \( A \overset{\text{stab}}{\rightarrow} \lim_{t \rightarrow \infty} \Pi_{i[t-1]} < \infty, \ \Pi \triangleq \lim_{t \rightarrow \infty} \Pi_{i[t]} < \infty, \ E_t \equiv E, \ \Sigma_v \equiv \Sigma_v. \)

Next, we briefly discuss the resultingly asymptotically stationary realization scheme depicted in Fig. 2.

**Preprocessing at Encoder:** Introduce the estimation error \( \{ \tilde{k}_t : \; t \in \mathbb{N}_0 \} \), where \( \tilde{k}_t \triangleq x_t - \hat{x}_{i[t-1]}, \; t \in \mathbb{N}_0 \) with (error) covariance \( \Pi_{i[t-1]} \), \( t \in \mathbb{N}_0 \). Under conditions (1) or (2), we ensure that \( \Pi \triangleq \lim_{t \rightarrow \infty} \Pi_{i[t-1]} < \infty \) and it is unique. The error covariance matrix \( \Pi \) is diagonalized by introducing an orthogonal matrix \( E \) (invertible matrix) such that \( E \Pi E^\top = \text{diag}\{\lambda\} \triangleq \Lambda. \) To facilitate the computation, we introduce the scaling process \( \{ \gamma_t \in \mathbb{R}^p : \; t \in \mathbb{N}_0 \} \), where \( \gamma_t \triangleq E k_t, \; t \in \mathbb{N}_0 \) has independent Gaussian components.

**Preprocessing at Decoder:** Analogously, we introduce the innovations process \( \{ \tilde{k}_t : \; t \in \mathbb{N}_0 \} \) defined by (8b) and the scaling process \( \{ \tilde{\gamma}_t : \; t \in \mathbb{N}_0 \} \) defined by \( \tilde{\gamma}_t \overset{\text{def}}{=} \Theta \tilde{k}_t \), with \( \tilde{\gamma}_t \overset{\text{def}}{=} (\Phi \tilde{k}_t + \nu_t), \; \nu_t \overset{\text{dist}}{=} N(0; \bar{\Sigma}_v) \), and \( \{ \Phi, \Theta \} \) are the asymptotic limits of \( \Phi_t \) and \( \Theta_t \), respectively.

The fidelity criterion \( ||k_t - \hat{k}_t||^2 \) at each \( t \) is not affected by the above processing of \( \{ x_t, \; v_t \} : \; t \in \mathbb{N}_0 \), in the sense that the preprocessing at both the encoder and decoder do not affect the form of the squared error distortion function, that is,
\[
||x_t - y_t||^2 = ||k_t - \hat{k}_t||^2, \quad t \in \mathbb{N}_0. \tag{10}
\]

Moreover, using basic properties of conditional entropy (see, e.g., [15, Eq. (IV.35)]), it can be shown that
\[
R^{na}(D) \overset{\text{def}}{=} \lim_{n \rightarrow \infty} \frac{1}{n+1} \sum_{t=0}^{n} I(x_t; y_t|y^{t-1}) \tag{11}
\]
\[\text{s.t.} \quad \frac{1}{n+1} \mathbb{E}\{d(x^n, y^n)\} \leq D,\]

and
\[
R^{na,a}k^n, \hat{k}^n(D) \overset{\text{def}}{=} \lim_{n \rightarrow \infty} \frac{1}{n+1} \sum_{t=0}^{n} I(k_t; \hat{k}_t) \tag{12}
\]
\[\text{s.t.} \quad \frac{1}{n+1} \mathbb{E}\{d(k^n, \hat{k}^n)\} \leq D,\]

are equivalent expressions.

In addition, the steady state values of (8b) is \( \Pi = \Pi A' A + BB' \). The end-to-end MSE distortion of the scheme in Fig. 2 is
\[
\lim_{t \rightarrow \infty} E\{(x_t - y_t)^\top (x_t - y_t)\}
\]
\[= \lim_{t \rightarrow \infty} \text{trace}\{E\{(x_t - \hat{x}_{i[t]})(x_t - \hat{x}_{i[t]})^\top\}\} = \text{trace}(\Pi') \leq D.\]

Hence, following [11, Theorem 2], the per unit time asymptotic limit of Gaussian NRDF subject to the total MSE distortion can be expressed as follows.
\[
R^{na}(D) = \min_{\text{trace}(\Pi') \leq D} \frac{1}{2} \log \max \left(1, \frac{|A' A + BB'|}{|\Pi'|}\right). \tag{13}
\]

Clearly, the optimization problem in (13) is a log-determinant minimization problem and can be solved using,
for instance, Karush-Kuhn-Tucker conditions [16] Chapter 5.5.3] or semidefinite programming (SDP). A way of solving [15] is proposed in [17]. However, compared to that work, our realization scheme is implemented with feedback to take into account the effect of unstable sources in the dynamical system.

V. UPPER BOUND TO ZERO-DELAY GAUSSIAN RDF

In this section, we derive an upper bound to the zero-delay Gaussian RDF using a subactively dithered uniform scalar quantizer (SDUSQ) on the feedback realization scheme of Fig. 2. The SDUSQ scheme was introduced in [10] and since then it has been used in several papers (see, e.g., [4], [8], [18]) under various realization setups. However, it has never been documented for the realization scheme proposed in this work. Here, we consider the vector Gaussian AR(1) source of (11), and we quantize each time step \( t \) over \( p \) independently operating SDUSQ, with their outputs being jointly entropy coded conditioned to the dither. We extend our results when using vector quantization showing that at infinite dimensional vectors, the space-filling loss due to compression and the entropy coding extinguishes, i.e., \( R^{\text{ua}}(D) \) and \( R^{\text{zD}}_D(D) \) coincide.

A. Scalar quantization

Next, we use the asymptotically stationary feedback realization scheme illustrated in Fig. 2 to design an efficient \{encoder/quantizer, decoder\} pair.

We select the quantizer step size \( \Delta \) so that the covariance of the resulting quantization error meets \( \Sigma_v \). The encoder does not quantize the observed state \( x_t \) directly. Instead, it quantizes the deviation of \( x_t \) from the linear estimate \( \hat{x}_{t|t-1} \) of \( x_t \). This method is known in least squares estimation theory as innovations approach and, therefore, the encoder is named as an innovations encoder. We consider the zero-delay source coding setup illustrated in Fig. 2 with the additional change of the \( p \)-parallel additive white Gaussian noise (AWGN) channels with \( p \) independently operating SDUSQ. This is illustrated in Fig. 3. Note that, all matrices and scalings adopted in Fig. 2 still hold when the aforementioned replacement is applied.

For each time step \( t \), the input to the quantizer, is a scaled estimation error defined as follows

\[
\alpha_t = A k_t, \quad A = \Phi E.
\] (14)

Moreover, \( \alpha_t \) is an \( \mathbb{R}^p \)-valued random process. The parallel \( p \)-dimensional AWGN channel is replaced by \( p \) independently operating SDUSQ, hence we can design the covariance matrix \( \Sigma_v \) of the AWGN channels corresponding to the \( p \)-parallel AWGN channels in such a way, that for each \( t \), each diagonal entry \( V_{ii}, i = 1, \ldots, p \), i.e., \( \Sigma_v = \text{diag} \{ V \} \), to correspond to a quantization step size \( \Delta_i \), \( i = 1, \ldots, p \), such that

\[
V_{ii} = \frac{\Delta_i^2}{12}, \quad i = 1, \ldots, p.
\] (15)

This results creates a multi-input multi-output (MIMO) transmission of parallel and independent SDUSQ. We apply SDUSQ to each component of \( \alpha_t \), i.e.,

\[
\beta_{t,i} = Q_{\Delta_i}(\alpha_{t,i}), \quad i = 1, \ldots, p
\] (16)

and we let \( r_t \) be the \( \mathbb{R}^p \)-valued random process of dither signals whose individual components \( \{ r_{t,1}, \ldots, r_{t,p} \} \) are mutually independent and uniformly distributed random variables \( r_{t,i} \sim \text{Unif}(\frac{-\Delta_i}{2}, \frac{\Delta_i}{2}) \) independent of the corresponding source input components \( \alpha_{t,i}, \forall i \). The output of the quantizer is given by

\[
\hat{\beta}_{t,i} = Q_{\Delta_i}(\alpha_{t,i} + r_{t,i}), \quad i = 1, \ldots, p.
\] (17)

Note that \( \hat{\beta}_t = (\hat{\beta}_{t,1}, \ldots, \hat{\beta}_{t,p}) \) can take a countable number of possible values. In addition, by construction (see Fig. 2), the sequences \( \{ \alpha_t : t = 0, 1, \ldots \} \) and \( \{ \beta_t : t = 0, 1, \ldots \} \) are not Gaussian any more since by applying the change illustrated in Fig. 3 \( \{ \alpha_t : t = 0, 1, \ldots \} \) and \( \{ \beta_t : t = 0, 1, \ldots \} \) contain samples of the uniformly distributed process \( \{ r_t : t = 0, 1, \ldots \} \). As a result, the Kalman filter in Fig. 2 is no longer the least mean square estimator.

Entropy coding: In what follows, we apply joint entropy coding across the vector dimension \( p \) and memoryless coding across the time, that is, at each time step \( t \) the output of the quantizer \( \hat{\beta}_t \) is conditioned to the dither to generate a codeword \( z_t \). The decoder reproduces \( \beta_t \) by subtracting the dithered signal \( r_t \) from \( \hat{\beta}_t \). Specifically, at every time step \( t \), we require that a message \( \hat{\beta}_t \) is mapped into a codeword \( z_t \in \{0,1\}^l \) designed using Shannon codes [19] Chapter 5.4. For a RV \( x \), the codes constructed based on Shannon coding scheme give an instantaneous (prefix-free) code with expected code length that satisfies the bounds

\[
H(x) \leq E(l) \leq H(x) + 1.
\] (18)
If \( x \) is a \( p \)-dimensional random vector then the normalized version of (13) gives
\[
\frac{H(x)}{p} \leq \frac{E(I)}{p} \leq \frac{H(x)}{p} + \frac{1}{p}
\]
(19)
\[\alpha_t \in \mathbb{R}^p, \beta_t \in \mathbb{R}^p\]
\[\xi_t = \{\xi_{t,i} : i = 1,\ldots, p\}, \xi_{t,i} - \text{Unif}\left(-\frac{\Delta}{2}, \frac{\Delta}{2}\right)\]
\[\{\xi_{t,i} : i = 1,\ldots, p\}, \xi_{t,i} - \text{Unif}\left(-\frac{\Delta}{2}, \frac{\Delta}{2}\right)\]  
Fig. 4: An equivalent model to Fig. 3b based on scalar uniform additive noise channel.

Since the SDUSQ operates using memoryless entropy coding over time, the following theorem holds.

**Theorem 1.** (Upper bound)
Consider the realization of the zero-delay source-coding scheme illustrated in Fig. 2 with the change of AWGN channel with \( p \)-parallel independently operating SDUSQ illustrated in Fig. 3. If the vector process \( \{\beta_t : t = 0,1,\ldots\} \) of the quantized output is jointly entropy coded conditioned to the dither signal values in a memoryless fashion for each \( t \), then the operational Gaussian zero-delay rate, \( R_{\text{ZD}}^{\text{op}}(D) \), satisfies
\[
R_{\text{ZD}}^{\text{op}}(D) \leq R_{\text{M}}(D) + \frac{p}{2} \log_2 \left( \frac{\pi e}{6} \right) + 1
\]
(20)
where \( p \) is the dimension of the state-space representation given in (1), while the average MSE distortion achieves the end-to-end average distortion \( D \) of the system.

**Proof.** See Appendix A.

The previous main result combined with the lower bound on Gaussian zero-delay RDF, leads to the following corollary.

**Corollary 1.** (Bounds on zero-delay RDF)
Consider the realization of the zero-delay source-coding scheme illustrated in Fig. 2 with the change of AWGN channel with \( p \)-parallel independently operating SDUSQ as illustrated in Fig. 3. Then, for vector (stable or unstable) Gaussian AR(1) sources the following bounds hold
\[
R_{\text{M}}(D) \leq R_{\text{ZD}}^{\text{op}}(D) \leq R_{\text{M}}(D) + \frac{p}{2} \log_2 \left( \frac{\pi e}{6} \right) + 1
\]
(21)

**Proof.** This is obtained using the fact that \( R_{\text{M}}(D) \leq R_{\text{ZD}}^{\text{op}}(D) \), (13) and Theorem 1.

**Theorem 2.** (Generalization)
The bounds derived in Corollary 1 based on the realization scheme of Fig. 2 hold for vector Gaussian sources of any order.

**Proof.** See Appendix B.

In the next remark, we draw connections to existing results in the literature.

**Remark 1.** (Relations to existing results)
(1) For stationary stable scalar-valued Gaussian AR sources, our upper bound in Theorem 7 coincides with the bound obtained in [8] Theorem 7. However, the upper bound in [8] is obtained using a realization scheme with four filters instead of only one that we use in our scheme. In addition, our result takes into account unstable Gaussian sources too.

(2) Compare to [8], we use ECDQ based on a different realization setup that results into obtaining different lower and upper bounds.

Next, we employ Theorem 1 to demonstrate a simulation example.

**Example 1.** We consider a two-dimensional unstable Gaussian AR(1) source as follows:
\[
x_{t+1} = \begin{bmatrix} -1.3 & 0.4 \\ -0.3 & 0 \end{bmatrix} x_t + \begin{bmatrix} 1 \\ 0 \end{bmatrix} w_t,
\]
(22)
where \( x_t \in \mathbb{R}^2 \), the parameter matrix \( A \) is unstable because one of its eigenvalues, denoted by \( \lambda_1(A) \), has magnitude greater than one, the pair \( (A,B) \) is stabilizable and \( w_t \sim N(0; I_{2 \times 2}) \). By invoking SDPT3 [24] we plot the theoretical attainable lower and upper bounds to the zero-delay RDF. This is illustrated in Fig. 5. As expected from theory, \( R_{\text{M}}(D) \geq \sum_{\lambda_1(A) > 1} \log |\lambda_1(A)| \approx 0.263 \) bits/source sample.

![Fig. 5: Bounds on \( R_{\text{ZD}}^{\text{op}}(D) \) via the scheme of Fig. 2](image-url)
VI. CONCLUSIONS AND FUTURE DIRECTIONS

We considered zero-delay source coding of a vector Gaussian AR source under MSE distortion. Based on a feedback realization scheme that quantizes the innovations of a Kalman filter with a SDUSQ, we derived an upper bound to the zero-delay RDF. We discussed the performance of this scheme when using lattice quantization. For infinite dimensions we observed that the NRDF coincide with the zero-delay RDF. An illustrative example is presented to support our findings.

As an ongoing research, we will apply the proposed coding scheme based on SDUSQ to find the actual operational rates corresponding to the zero-delay RDF. Moreover, we will examine similar coding schemes for fixed-length coding rate.

APPENDIX A
PROOF OF THEOREM 1

In the realization scheme proposed in Fig. 2 with the change of AWGN channel with $p$-parallel independently operating SDUSQ, the operational rate for each $t$ is equal to the conditional entropy $H(\tilde{\beta}_t|t)$ where $\tilde{\beta}_t = \{\tilde{\beta}_1, \ldots, \tilde{\beta}_{t,p}\}$, $\sum_{i=1}^{p} \tilde{\beta}_i = Q_{\Delta}(\alpha_{t,i} + r_{t,i}), \ i = 1, \ldots, p$, i.e., the entropy of the quantized output $\tilde{\beta}_t$ conditioned on the $t$-value of the dither signal $r_t$. This leads to the following analysis:

$$H(\tilde{\beta}_t|t) \overset{(a)}{=} I(\alpha_t; \beta_t)$$

$$\overset{(b)}{=} I(\alpha_t; \alpha_t + \xi_t)$$

$$= h(\alpha_t + \xi_t)$$

$$\overset{(c)}{=} h(\alpha_t^G + v_t) - h(\alpha_t^G) + D(\xi_t||v_t) - D(\alpha_t + \xi_t||\alpha_t^G + v_t)$$

$$\overset{(d)}{\leq} I(\alpha_t^G; \alpha_t^G + v_t) + P \log_2 \left( \frac{\pi e}{6} \right)$$

$$= I(\alpha_t^G; \beta_t^G) + P \log_2 \left( \frac{\pi e}{6} \right),$$

(25)

where $(a)$ follows from [10] Theorem 1; $(b)$ follows from the fact that the quantization noise is $\xi_t = \beta_t - \alpha_t$ (see Fig. 3); $(c)$ follows from the fact that the relative entropy $D(x||y) = h(x) - h(x')$, see e.g., [19] Theorem 8.6.5; $(d)$ follows from the fact that $D(\alpha_t + \xi_t||\alpha_t^G + v_t) \geq 0$, with equality if and only if $\xi_t \sim \{0, 1, \ldots\}$ becomes a Gaussian distribution; $(e)$ from the fact that the differential entropy $h(v_t)$ of a Gaussian random vector with covariance $\Sigma_v = \text{diag}(V)$ is

$$h(v_t) = \frac{1}{2} \log_2 (2\pi e)^p |\Sigma_v| = \frac{1}{2} \log_2 (2\pi e) V_{ii},$$

and the entropy $h(\xi_t)$ of the uniformly distributed random vector $\xi_t = \{\xi_{t,i}; \ i = 1, 2, \ldots, p\}$, $\xi_{t,i} \sim \text{Unif} \left( -\frac{\Delta}{2}, \frac{\Delta}{2} \right)$ is

$$h(\xi_t) = \sum_{i=1}^{p} \frac{1}{2} \log_2 \Delta^2,$$

Since we have that $V_{ii} = \frac{\Delta^2}{12}, \ i = 1, \ldots, p$, the result follows.

Next, note that for $n = 0, 1, \ldots$, the following inequality holds in Fig. 2

$$I(x^n; y^n) \overset{(a)}{=} \sum_{t=0}^{n} I(x_t; y_t|y^{t-1})$$

$$\overset{(b)}{=} \sum_{t=0}^{n} I(k_t; \tilde{k}_t)$$

$$\overset{(c)}{=} \sum_{t=0}^{n} I(\alpha_t; \beta_t),$$

(26)

where $(a)$ follows from the structural properties of specific extremum problem resulting in the realization of Fig. 2 (see, e.g., the analysis in Section 11 and [15] Remark IV.5.1); $(b)$ follows from the analysis in [15] Equation (35); $(c)$ follows from the fact that $E, \Phi, \theta$ are invertible matrices and as a result the information from $k_t$ to $\tilde{k}_t$ is the same as from $\alpha_t$ to $\beta_t$ (information lossless operation).

Since we are assuming joint memoryless entropy coding of $p$ independently operating scalar uniform quantizers with subtractive dither, then by [15], for $t = 0, 1, \ldots, n$, we obtain

$$\sum_{t=0}^{n} E(I_t) \leq \sum_{t=0}^{n} (H(\tilde{\beta}_t|t) + 1)$$

$$\overset{(a)}{\leq} \sum_{t=0}^{n} \left( I(\alpha_t^G; \beta_t^G) + \frac{P}{2} \log_2 \left( \frac{\pi e}{6} \right) + 1 \right)$$

$$\overset{(b)}{\leq} I(x^n; \gamma^n) + \sum_{t=0}^{n} \frac{P}{2} \log_2 \left( \frac{\pi e}{6} \right) + (n + 1),$$

(27)

where $(a)$ follows by (25) and $(b)$ follows from (26).

Then, by first taking the per unit time limiting expression in (27) and then the infimum, we obtain

$$\inf \limsup \frac{1}{n+1} \sum_{t=0}^{n} E(I_t) \leq \inf \limsup \frac{1}{n+1} I(x^n; \gamma^n) + \frac{P}{2} \log_2 \left( \frac{\pi e}{6} \right) + 1$$

$$\overset{(a)}{=} R^{\text{opt}} \leq R^{\text{na}}(D) + \frac{P}{2} \log_2 \left( \frac{\pi e}{6} \right) + 1,$$

(28)

where $(a)$ follows by (2) and (3) respectively, and $R^{\text{na}}(D)$ is the upper bound expression of $R^{\text{na}}(D)$ for the vector Gaussian AR(1) source model given by (1).

Finally, by assumptions, (i.e., $A$ is stable or $(A, B)$ stabilizable) the innovations’ Gaussian source is asymptotically stationary. Since $R^{\text{na}}(D) = R^{\text{na}}(k^*, k^*)$, then at steady state, we have $R^{\text{na}}(D) = R^{\text{na}}(D)$ and the result follows.

APPENDIX B
PROOF OF THEOREM 2

Assume the following vector Gaussian AR($s$) process, where $s$ is a positive integer, in state space representation.

$$x_{t+1} = \sum_{j=1}^{s} A_j x_{t-j+1} + B w_t,$$

(29)
where $A_j \in \mathbb{R}^{p \times p}$, and $B \in \mathbb{R}^{p \times q}$ are deterministic matrices, $x_0 \in \mathbb{R}^p \sim N(0; \Sigma_{x_0})$ is the initial state, and $w_t \in \mathbb{R}^q \sim N(0; I_{q \times q})$ is an i.i.d. Gaussian sequence, independent of $x_0$. Clearly, for $s = 1$, (29) gives as a special case the source model described by (1). Next, we show that (29) can be expressed as an augmented vector Gaussian AR(1) process as follows

$$\tilde{x}_{t+1} = \tilde{A} \tilde{x}_t + \tilde{B} \tilde{w}_t,$$

(30)

where $\tilde{A} \in \mathbb{R}^{sp \times sp}$, and $\tilde{B} \in \mathbb{R}^{sp \times sq}$ are deterministic matrices, $\tilde{x}_0 \in \mathbb{R}^{sp} \sim N(0; \Sigma_{\tilde{x}_0})$ is the initial state with $\Sigma_{\tilde{x}_0}$ being the covariance of the initial state, and $\tilde{w}_t \in \mathbb{R}^{sq} \sim N(0; \Sigma_{\tilde{w}_t})$ is an i.i.d. Gaussian sequence, independent of $\tilde{x}_0$.

The proof employs a simple augmentation of the state process. First, note that the state space model of (29) can be modified as follows.

$$\begin{bmatrix}
    x_{t+1} \\
    x_t \\
    \vdots \\
    x_{t-s+2}
\end{bmatrix} =
\begin{bmatrix}
    A_1 & A_2 & \ldots & A_{s-1} & A_s \\
    I & 0 & \ldots & 0 & 0 \\
    0 & I & \ddots & 0 & \vdots \\
    \vdots & \vdots & \ddots & \ddots & \vdots \\
    0 & 0 & \ldots & I & 0
\end{bmatrix}
\begin{bmatrix}
    x_t \\
    x_{t-1} \\
    \vdots \\
    x_{t-s+1}
\end{bmatrix}
+ \begin{bmatrix}
    B_0 & 0 & \ldots & 0 \\
    0 & 0 & \ldots & 0 \\
    \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & \ldots & 0
\end{bmatrix}
\begin{bmatrix}
    w_t \\
    0 \\
    \vdots \\
    0
\end{bmatrix},$$

(31)

Then, (31) can be written in an augmented state space form as follows.

$$\tilde{x}_{t+1} = \tilde{A} \tilde{x}_t + \tilde{B} \tilde{w}_t,$$

(32)

where

$$\begin{array}{c}
\begin{bmatrix}
    x_{t+1} \\
    x_t \\
    \vdots \\
    x_{t-s+2}
\end{bmatrix} \in \mathbb{R}^{sp}, \\
\begin{bmatrix}
    x_t \\
    x_{t-1} \\
    \vdots \\
    x_{t-s+1}
\end{bmatrix} \in \mathbb{R}^{sp},
\end{array}
$$

$$\tilde{A} = \begin{bmatrix}
    A_1 & A_2 & \ldots & A_{s-1} & A_s \\
    I & 0 & \ldots & 0 & 0 \\
    0 & I & \ddots & 0 & \vdots \\
    \vdots & \vdots & \ddots & \ddots & \vdots \\
    0 & 0 & \ldots & I & 0
\end{bmatrix} \in \mathbb{R}^{sp \times sp},$$

$$\tilde{B} = \begin{bmatrix}
    B_0 & 0 & \ldots & 0 \\
    0 & 0 & \ldots & 0 \\
    \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & \ldots & 0
\end{bmatrix} \in \mathbb{R}^{sp \times sq}, \quad \tilde{w}_t = \begin{bmatrix}
    w_t \\
    0 \\
    \vdots \\
    0
\end{bmatrix} \in \mathbb{R}^{sq}.$$

The augmented state space representation of (32) is an augmented vector Gaussian AR(1) process which can then be applied to the feedback design of Fig. 2 obtaining the same bound as in Theorem 1. This completes the proof. □