Generalized solutions for the Euler-Bernoulli model with Zener viscoelastic foundations and distributional forces

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Abstract

We study the initial-boundary value problem for an Euler-Bernoulli beam model with discontinuous bending stiffness laying on a viscoelastic foundation and subjected to an axial force and an external load both of Dirac-type. The corresponding model equation is fourth order partial differential equation and involves discontinuous and distributional coefficients as well as a distributional right-hand side. Moreover the viscoelastic foundation is of Zener type and described by a fractional differential equation with respect to time. We show how functional analytic methods for abstract variational problems can be applied in combination with regularization techniques to prove existence and uniqueness of generalized solutions.

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1 Introduction and preliminaries

We study existence and uniqueness of a generalized solution to the initial-boundary value problem

\begin{align*}
\partial_t^2 u + Q(t, x, \partial_x)u + g &= h, \quad (1) \\
D_t^\alpha u + u &= \theta D_t^\alpha g + g, \quad (2) \\
u|_{t=0} &= f_1, \quad \partial_t u|_{t=0} = f_2, \quad (IC) \\
u|_{x=0} &= u|_{x=1} = 0, \quad \partial_x u|_{x=0} = \partial_x u|_{x=1} = 0, \quad (BC)
\end{align*}

where \( Q \) is a differential operator of the form

\[ Q u := \partial_x^2 (c(x) \partial_x^2 u) + b(x, t) \partial_x^2 u, \]

\( b, c, g, h, f_1 \) and \( f_2 \) are generalized functions, \( \theta \) a constant, \( 0 < \theta < 1 \), and \( D_t^\alpha \) denotes the left Riemann-Liouville fractional derivative of order \( \alpha \) with respect to \( t \). Problem (1)-(2) is equivalent to

\begin{equation}
\partial_t^2 u + Q(t, x, \partial_x)u + Lu = h, 
\end{equation}

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with \( L \) being the (convolution) operator given by (\( L \) denoting the Laplace transform)

\[
Lu(x,t) = \mathcal{L}^{-1}\left(\frac{1 + s^\alpha}{1 + \theta s^\alpha}\right)(t) \ast_t u(x,t),
\]

with the same initial (IC) and boundary (BC) conditions (cf. Section 3).

The precise structure of the above problem is motivated by a model from mechanics describing the displacement of a beam under axial and transversal forces connected to the viscoelastic foundation, which we briefly discuss in Subsection 1.1. We then briefly introduce the theory of Colombeau generalized functions which forms the framework for our work. Similar problems involving distributional and generalized solutions to Euler-Bernoulli beam models have been studied in [4, 12, 13, 20, 21]. The development of the theory in the paper is divided into two parts. In Section 2 we consider the initial-boundary value problem (3)-(IC)-(BC) on the abstract level. We prove, in Theorem 2.3, an existence result for the abstract variational problem corresponding to (3)-(IC)-(BC) and derive energy estimates (19) which guarantee uniqueness and serve as a key tool in the analysis of Colombeau generalized solutions. In Section 3, we first show equivalence of the system (1)-(2) with the integro-differential equation (3), and apply the results from Section 2 to the original problem in establishing weak solutions, if the coefficients are in \( L^\infty \). Afterwards we allow the coefficients to be more irregular, set up the problem and show existence and uniqueness of solutions in the space of generalized functions.

1.1 The Euler-Bernoulli beam with viscoelastic foundation

Consider the Euler-Bernoulli beam positioned on the viscoelastic foundation (cf. [2] for mechanical background). One can write the differential equation of the transversal motion

\[
\frac{\partial^2}{\partial x^2}\left(A(x) \frac{\partial^2 u}{\partial x^2}\right) + P(t) \frac{\partial^2 u}{\partial x^2} + R(x) \frac{\partial^2 u}{\partial t^2} + g(x,t) = h(x,t), \quad x \in [0,1], \ t > 0,
\]

where

- \( A \) denotes the bending stiffness and is given by \( A(x) = EI_1 + H(x-x_0)EI_2 \). Here, the constant \( E \) is the modulus of elasticity, \( I_1, I_2, I_1 \neq I_2 \), are the moments of inertia that correspond to the two parts of the beam, and \( H \) is the Heaviside jump function;
- \( R \) denotes the line density (i.e., mass per length) of the material and is of the form \( R(x) = R_0 + H(x-x_0)(R_1 - R_2) \);
- \( P(t) \) is the axial force, and is assumed to be of the form \( P(t) = P_0 + P_1 \delta(t-t_1) \), \( P_0, P_1 > 0 \);
- \( g = g(x,t) \) denotes the force terms coming from the foundation;
- \( u = u(x,t) \) denotes the displacement;
- \( h = h(x,t) \) is the prescribed external load (e.g., when describing moving load it is of the form \( h(x,t) = H_0 \delta(x-ct), H_0 \) and \( c \) are constants).

Since the beam is connected to the viscoelastic foundation there is a constitutive equation describing relation between the force of foundation and the displacement of the beam. We use the Zener generalized model given by

\[
D^\alpha_t u(x,t) + u(x,t) = \theta D^\alpha_t g(x,t) + g(x,t),
\]

where \( 0 < \theta < 1 \) and \( D^\alpha_t \) denotes the left Riemann-Liouville fractional derivative of order \( \alpha \) with respect to \( t \), defined by

\[
D^\alpha_t u(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t \frac{u(\tau)}{(t-\tau)^\alpha} d\tau.
\]
System (10)-(13) is supplied with initial conditions
\[ u(x,0) = f_1(x), \quad \partial_t u(x,0) = f_2(x), \]
where \( f_1 \) and \( f_2 \) are the initial displacement and the initial velocity. If \( f_1(x) = f_2(x) = 0 \) the only solution to (10)-(13) should be \( u \equiv g \equiv 0 \). Also, the beam is considered to be fixed at both ends, hence boundary conditions take the form
\[ u(0,t) = u(1,t) = 0, \quad \partial_x u(0,t) = \partial_x u(1,t) = 0. \]

By a change of variables \( t \mapsto \tau \) via \( t(\tau) = \sqrt{R(x)} \tau \) the problem (10)-(13) is transformed into the standard form given in (14)-(2). The function \( c \) in (11) equals \( A \) and therefore is of Heaviside type, and the function \( b \) is then given by \( b(x,t) = P(R(x)t) \) and its regularity properties depend on the assumptions on \( P \) and \( R \).

As we shall see in Section 3 standard functional analytic techniques reach as far as the following: boundedness of \( b \) together with sufficient (spatial Sobolev) regularity of the initial values \( f_1, f_2 \) ensure existence of a unique solution \( u \in L^2(0,T; H^2_0((0,1))) \)' (in fact \( u \in L^2(0,T; H^2_0((0,1))) \)) to (9) with (IC) and (BC). However, the prominent case \( b = p_0 + p_1 \delta(t-t_1) \) is clearly not covered by such a result, so in order to allow for these stronger singularities one needs to go beyond distributional solutions.

1.2 Basic spaces of generalized functions

We shall set up and solve Equation (3) subject to the initial and boundary conditions (IC) and (BC) in an appropriate space of Colombeau generalized functions on the domain \( X_T := (0,1) \times (0,T) \) (with \( T > 0 \)) as introduced in [9] and applied later on, e.g., also in [11], [13]. As a few standard references for the general background concerning Colombeau algebras on arbitrary open subsets of \( \mathbb{R}^d \) or on manifolds we may mention [9, 10, 11, 17].

We review the basic notions and facts about the kind of generalized functions we will employ below: we start with regularizing families \( (u_\varepsilon)_{\varepsilon \in (0,1)} \) of smooth functions on \( (0,1) \) (space of smooth functions on \( X_T \) all of whose derivatives belong to \( L^2 \)). We will often write \( (u_\varepsilon) \) to mean \( (u_\varepsilon)_{\varepsilon \in (0,1)} \). We consider the following subalgebras:

- **Moderate families**, denoted by \( \mathcal{E}_{M,H^{\infty}(X_T)} \), are defined by the property
  \[ \forall \alpha \in \mathbb{N}_0^n, \exists p \geq 0 : \| \partial^\alpha u_\varepsilon \|_{L^2(X_T)} = O(\varepsilon^{-p}), \quad \text{as } \varepsilon \to 0. \]

- **Null families**, denoted by \( \mathcal{N}_{H^{\infty}(X_T)} \), are the families in \( \mathcal{E}_{M,H^{\infty}(X_T)} \) satisfying
  \[ \forall q \geq 0 : \| u_\varepsilon \|_{L^2(X_T)} = O(\varepsilon^q), \quad \text{as } \varepsilon \to 0. \]

Thus moderateness requires \( L^2 \) estimates with at most polynomial divergence as \( \varepsilon \to 0 \), together with all derivatives, while null families vanish very rapidly as \( \varepsilon \to 0 \). We remark that null families in fact have all derivatives satisfy estimates of the same kind (cf. [9, Proposition 3.4(ii)])

Thus null families form a differential ideal in the collection of moderate families and we may define the Colombeau algebra as the factor algebra
\[ \mathcal{G}_{H^{\infty}(X_T)} = \mathcal{E}_{M,H^{\infty}(X_T)} / \mathcal{N}_{H^{\infty}(X_T)}. \]

A typical notation for the equivalence classes in \( \mathcal{G}_{H^{\infty}(X_T)} \) with representative \( (u_\varepsilon) \) will be \([u_\varepsilon]\). Finally, the algebra \( \mathcal{G}_{H^{\infty}((0,1))} \) of generalized functions on the interval \((0,1)\) is defined similarly and every element can be considered to be a member of \( \mathcal{G}_{H^{\infty}(X_T)} \) as well.

We briefly recall a few technical remarks from [13, Subsection 1.2]:

- If \( (u_\varepsilon) \) belongs to \( \mathcal{E}_{M,H^{\infty}(X_T)} \) we have smoothness up to the boundary for every \( u_\varepsilon \), i.e. \( u_\varepsilon \in C^{\infty}((0,1) \times [0,T]) \) (which follows from Sobolev space properties on the Lipschitz domain \( X_T \); cf. [1]) and therefore the restriction \( u|_{t=0} \) of a generalized function \( u \in \mathcal{G}_{H^{\infty}(X_T)} \) to \( t = 0 \) is well-defined by \( u_\varepsilon(\cdot,0) \in \mathcal{E}_{M,H^{\infty}((0,1))} \).
If \( v \in \mathcal{G}_{H^{\infty}((0, 1))} \) and in addition we have for some representative \( (v_\varepsilon)_\varepsilon \) of \( v \) that \( v_\varepsilon \in H^2_0((0, 1)) \), then \( v_\varepsilon(0) = v_\varepsilon(1) = 0 \) and \( \partial_x v_\varepsilon(0) = \partial_x v_\varepsilon(1) = 0 \). In particular, \( v(0) = v(1) = 0 \) and \( \partial_x v(0) = \partial_x v(1) = 0 \)

holds in the sense of generalized numbers.

Note that \( L^2 \)-estimates for parametrized families \( u_\varepsilon \in H^{\infty}(X_T) \) always yield similar \( L^\infty \)-estimates concerning \( \varepsilon \)-asymptotics (since \( H^{\infty}(X_T) \subset C^\infty(X_T) \subset W^{\infty, \infty}(X_T) \)).

The space \( H^{-\infty}(\mathbb{R}^d) \), i.e. distributions of finite order, is embedded (as a linear space) into \( \mathcal{G}_{H^{\infty}(\mathbb{R}^d)} \) by convolution regularization (cf. [3]). This embedding renders \( H^{\infty}(\mathbb{R}^d) \) a subalgebra of \( \mathcal{G}_{H^{\infty}(\mathbb{R}^d)} \).

Certain generalized functions possess distribution aspects, namely we call \( u = [(u_\varepsilon)_\varepsilon] \in \mathcal{G}_{H^{\infty}} \) associated with the distribution \( w \in \mathcal{D}' \), notation \( u \approx w \), if for some (hence any) representative \( (u_\varepsilon)_\varepsilon \) of \( u \) we have \( u_\varepsilon \to w \) in \( \mathcal{D}' \), as \( \varepsilon \to 0 \).

## 2 Preparations: An abstract evolution problem in variational form and the convolution-type operator \( L \)

In this section we study an abstract background of equation (3) subject to the initial and boundary conditions (IC) and (BC) in terms of bilinear forms on arbitrary Hilbert spaces. First we shall repeat standard results and then extend them to a wider class of problems. We shall show existence of a unique solution, derive energy estimates, and analyze the particular form of the operator \( L \) appearing in (3).

Let \( V \) and \( H \) be two separable Hilbert spaces, where \( V \) is densely embedded into \( H \). We shall denote the norms in \( V \) and \( H \) by \( \| \cdot \|_V \) and \( \| \cdot \|_H \) respectively. If \( V' \) denotes the dual of \( V \), then \( V \subset H \subset V' \) forms a Gelfand triple. In the sequel we shall also make use of the Hilbert spaces \( E_V := L^2(0, T; V) \) with the norm \( \| u \|_{E_V} := (\int_0^T \| u(t) \|_V^2 \, dt)^{1/2} \), and \( E_H := L^2(0, T; H) \) with the norm \( \| u \|_{E_H} := (\int_0^T \| u(t) \|_H^2 \, dt)^{1/2} \).

By this, \( \| \cdot \|_H \leq C \cdot \| \cdot \|_V \), \( \forall \varepsilon \in V \) (without loss of generality we may assume that \( C = 1 \)), it follows that \( \| u \|_{E_H} \leq \| u \|_{E_V}, \varepsilon \in E_V, \text{ and } E_V \subset E_H \).

The bilinear forms we shall deal with will be of the following type:

**Assumption 1.** Let \( a(t, \cdot, \cdot), a_0(t, \cdot, \cdot) \) and \( a_1(t, \cdot, \cdot), t \in [0, T], \) be (parametrized) families of continuous bilinear forms on \( V \) with

\[
a(t, u, v) = a_0(t, u, v) + a_1(t, u, v) \quad \forall u, v \in V,
\]

such that the ‘principal part’ \( a_0 \) and the remainder \( a_1 \) satisfy the following conditions:

(i) \( t \mapsto a_0(t, u, v) \) is continuously differentiable \( [0, T] \to \mathbb{R}, \) for all \( u, v \in V; \)

(ii) \( a_0 \) is symmetric, i.e., \( a_0(t, u, v) = a_0(t, v, u) \), for all \( u, v \in V; \)

(iii) there exist real constants \( \lambda, \mu > 0 \) such that

\[
a_0(t, u, u) \geq \mu \| u \|_V^2 - \lambda \| u \|_H^2, \quad \forall u \in V, \forall t \in [0, T];
\]  
\[
(7)
\]

(iv) \( t \mapsto a_1(t, u, v) \) is continuous \( [0, T] \to \mathbb{R}, \) for all \( u, v \in V; \)

(v) there exists \( C_1 \geq 0 \) such that for all \( t \in [0, T] \) and \( u, v \in V \), \( |a_1(t, u, v)| \leq C_1 \| u \|_V \| v \|_H. \)

It follows from condition (i) that there exist nonnegative constants \( C_0 \) and \( C_0' \) such that for all \( t \in [0, T] \) and \( u, v \in V \),

\[
|a_0(t, u, v)| \leq C_0 \| u \|_V \| v \|_V \quad \text{and} \quad |a_0'(t, u, v)| \leq C_0' \| u \|_V \| v \|_V,
\]

where \( a_0'(t, u, v) := \frac{d}{dt} a_0(t, u, v). \)

It is shown in [2] Ch. XVIII, p. 558, Th. 1 (see also [13] Ch. III, Sec. 8) that the above conditions guarantee unique solvability of the abstract variational problem in the following sense:
Theorem 2.1. Let \( a(t, \cdot, \cdot), t \in [0, T], \) satisfy Assumption \( \mathcal{A} \). Let \( u_0 \in \mathcal{V}, u_1 \in \mathcal{H} \) and \( f \in E_H \). Then there exists a unique \( u \in E_V \) satisfying the regularity conditions

\[
u' = \frac{du}{dt} \in E_V \quad \text{and} \quad u'' = \frac{d^2 u}{dt^2} \in L^2(0, T; V')
\] (9)

(Here time derivatives should be understood in distributional sense), and solving the abstract initial value problem

\[
\langle u''(t), v \rangle + a(t, u(t), v) = \langle f(t), v \rangle, \quad \forall v \in \mathcal{V}, \ \text{for } a.e. \ t \in (0, T) \tag{10}
\]

\[
u(0) = u_0, \quad u'(0) = u_1. \tag{11}
\]

(Note that \( \mathcal{A} \) implies that \( u \in C([0, T], \mathcal{V}) \) and \( u' \in C([0, T], V') \). Hence it makes sense to evaluate \( u(0) \in \mathcal{V} \) and \( u'(0) \in V' \) and \( \mathcal{A} \) claims that these equal \( u_0 \) and \( u_1 \), respectively.)

Remark 2.2. The precise meaning of (10) is the following: \( \forall \varphi \in \mathcal{D}(0, T) \),

\[
\langle \langle u''(t), \varphi \rangle_{(D', \mathcal{D})} + \langle a(t, u(t), \varphi) \rangle_{(D', \mathcal{D})} = \langle (f(t), \varphi) \rangle_{(D', \mathcal{D})}, \n\]
or equivalently,

\[
\int_0^T \langle u(t), \varphi''(t) \rangle \, dt + \int_0^T a(t, u(t), \varphi(t)) \, dt = \int_0^T \langle f(t), \varphi(t) \rangle \, dt.
\]

The proof of this theorem proceeds by showing that \( u \) satisfies a priori (energy) estimates which immediately imply uniqueness of the solution, and then using the Galerkin approximation method to prove existence of a solution. An explicit form of the energy estimate for the abstract variational problem (9)-(11) with precise dependence of all constants is derived in [\( \mathcal{A} \): Prop. 1.3] in the form

\[
\|u(t)\|^2_H + \|u'(t)\|^2_H \leq \left( Dr^\frac{\|u_0\|^2_H}{\min(1, \varphi)} + \|u_1\|^2_H + \int_0^t \|f(\tau)\|^2_H \, d\tau \right) \cdot e^{F_T}, \tag{12}
\]

where \( D_T := \frac{C_{u_0} + C_{u_1} + T}{\min(1, \varphi)} \) and \( F_T := \max\left\{ \frac{C_{u_0} + C_{u_1}}{\min(1, \varphi)}, \frac{C_{u_1} + T + 2}{\min(1, \varphi)} \right\} \).

2.1 Existence of a solution to the abstract variational problem

We shall now prove a similar result for a slightly modified abstract variational problem, which is to encompass our problem \( \mathcal{A} \). Here in addition to the bilinear forms we shall consider "causal" operators \( L : L^2(0, T; \mathcal{H}) \rightarrow L^2(0, T; \mathcal{H}), \forall T, < T, \) which satisfy the estimate: \( \exists C_L > 0 \) such that

\[
\|Lu\|_{L^2(0, T; \mathcal{H})} \leq C_L \|u\|_{L^2(0, T; \mathcal{H})}, \tag{13}
\]

where \( C_L \) is independent of \( T \).

Lemma 2.3. Let \( a(t, \cdot, \cdot), t \in [0, T], \) satisfy Assumption \( \mathcal{A} \). Let \( f_1 \in \mathcal{V}, f_2 \in \mathcal{H} \) and \( h \in E_H \). Let \( L : E_H \rightarrow E_H \) satisfy (13). Then there exists a \( u \in E_V \) satisfying the regularity conditions

\[
u' = \frac{du}{dt} \in E_V \quad \text{and} \quad u'' = \frac{d^2 u}{dt^2} \in L^2(0, T; V')
\]

and solving the abstract initial value problem

\[
\langle u''(t), v \rangle + a(t, u(t), v) + \langle Lu(t), v \rangle = \langle h(t), v \rangle, \quad \forall v \in \mathcal{V}, \ \text{for } a.e. \ t \in (0, T), \tag{14}
\]

\[
u(0) = f_1, \quad u'(0) = f_2. \tag{15}
\]

Moreover, we have \( u \in \mathcal{C}([0, T]; \mathcal{V}) \) and \( u' \in \mathcal{C}([0, T]; \mathcal{H}) \).
Here we give a proof based on an iterative procedure and employing Theorem 2.1 and the energy estimate (12) in each step. Notice that the precise meaning of (14) (in distributional sense) is explained in Remark 2.2.

**Proof.** Let \( u_0 \in E_H \) be arbitrarily chosen and consider the initial value problem for \( u \) in the sense of Remark 2.2.

\[
\langle u''(t), v \rangle + a(t, u(t), v) + \langle Lu_0(t), v \rangle = \langle h(t), v \rangle, \quad \forall \ v \in V, \text{ for a.e. } t \in (0, T),
\]

\[
u(0) = f_1, \quad u'(0) = f_2.
\]

By Theorem 2.1, there exists a unique \( u_1 \in E_V \) satisfying \( u_1' \in E_V, u_1'' \in L^2(0, T; V') \), and solving (16). Consider now (16) with \( Lu_1 \) instead of \( Lu_0 \). As above, by Theorem 2.1, one obtains a unique solution \( u_2 \in E_V \) with \( u_2' \in E_V \) and \( u_2'' \in L^2(0, T; V') \). Repeating this procedure we obtain a sequence of functions \( \{u_k\}_{k \in \mathbb{N}} \in E_V \), satisfying \( u_k' \in E_V, u_k'' \in L^2(0, T; V') \), and solving the following problems: for each \( k \in \mathbb{N} \),

\[
\langle u_k''(t), v \rangle + a(t, u_k(t), v) + \langle Lu_k(t), v \rangle = \langle h(t), v \rangle, \quad \forall \ v \in V, \text{ for a.e. } t \in (0, T),
\]

\[
u_k(0) = f_1, \quad u_k'(0) = f_2.
\]

Also, for all \( k \in \mathbb{N} \), \( u_k \) satisfies the energy estimate of type (12):

\[
\|u_k(t)\|_V^2 + \|u_k'(t)\|_H^2 \leq \left( D_T \|f_1\|_V^2 + \|f_2\|_H^2 + \int_0^T \|L(u_k(t)-1)(\tau)\|_H^2 d\tau \right) \cdot e^{T \cdot F_T},
\]

where the constants \( D_T \) and \( F_T \) are independent of \( k \). We claim that \( \{u_k\}_{k \in \mathbb{N}} \) converges in \( E_V \). To see this, we first note that \( u_l - u_k \) solves

\[
\langle (u_l - u_k)'', v \rangle + a(t, (u_l - u_k)(t), v) + \langle L(u_l - u_k)(t), v \rangle = 0, \quad \forall \ v \in V, \text{ for a.e. } t \in (0, T),
\]

\[
u_l - u_k(0) = 0, \quad (u_l - u_k)'(0) = 0,
\]

and \( u_l - u_k \in E_V \), with \( u_l' - u_k' \in E_V \) and \( u_l'' - u_k'' \in L^2(0, T; V') \). Moreover, the corresponding energy estimate is of the form

\[
\|u_l - u_k\|_V^2 + \|u_l - u_k\|_H^2 \leq e^{T \cdot F_T} \cdot \int_0^T \|L(u_l - u_k)(\tau)\|_H^2 d\tau.
\]

(17)

Thus,

\[
\|u_l - u_k\|_V^2 \leq e^{T \cdot F_T} \cdot \int_0^T \|L(u_l - u_k)(\tau)\|_H^2 d\tau = e^{T \cdot F_T} \cdot \|L(u_l - u_k)\|_H^2.
\]

Integrating from \( 0 \) to \( T \) and using assumption (13) on \( L \), one obtains

\[
\|u_l - u_k\|_{E_V} \leq \gamma_T \|u_l - u_k\|_{E_H},
\]

(18)

where \( \gamma_T := C_L \sqrt{T e^{T \cdot F_T}} \). Taking now \( l = k + 1 \) in (18) successively, yields

\[
\|u_{k+1} - u_k\|_{E_V} \leq \gamma_T^k \|u_1 - u_0\|_{E_H} \leq \gamma_T^k \|u_1 - u_0\|_{E_V},
\]

and hence

\[
\|u_l - u_k\|_{E_V} \leq \|u_l - u_{l-1}\|_{E_V} + \ldots + \|u_{k+1} - u_k\|_{E_V} \leq \sum_{i=0}^{k} \gamma_T^i \|u_1 - u_0\|_{E_V}.
\]

We may choose \( T_1 < T \) such that \( \gamma_T < 1 \), hence \( \sum_{i=0}^{\infty} \gamma_T^i \) converges. Note that \( t \to \gamma_t \) is increasing. By abuse of notation we denote \( L^2(0, T_1; V') \) again by \( E_V \). This further implies that \( \{u_k\}_{k \in \mathbb{N}} \) is a Cauchy sequence and hence convergent in \( E_V \), say \( u := \lim_{k \to \infty} u_k \). Similarly, one
can show convergence of $u'_k$ in $E_V$, i.e., existence of $v := \lim_{k \to \infty} u'_k \in E_V$. In the distributional setting $\lim_{k \to \infty} u'_k = u'$, and therefore $u' = v \in E_V$ (cf. [17] Ch. XVIII, p. 473, Prop. 6).

We also have to show that $u$ solves equation (14). Let $\varphi \in \mathcal{D}((0, T))$. Then

$$
\langle \langle h(t), v \rangle \rangle = \langle \langle u''(t), v \rangle \rangle + \langle \langle a(t, u(k), t), v \rangle \rangle + \langle \langle Lu(k-1)(t), v \rangle \rangle
$$

$$
= \langle \langle u_k, v \rangle \rangle' + \langle \langle a(t, u_k(t), v) \rangle \rangle + \langle \langle Lu_k(t), v \rangle \rangle
$$

$$
\to \langle \langle u, v \rangle \rangle' + \langle \langle a(t, u(t), v) \rangle \rangle + \langle \langle Lu(t), v \rangle \rangle.
$$

Here we used that $\varphi'' \in \mathcal{D}((0, T))$. Therefore $u$ solves equation (14) on the time interval $[0, T]$. The initial conditions are satisfied by construction of $u$.

It remains to extend this result on existence of a solution to the whole interval $[0, T]$.

Since $T_1$ is independent on the initial conditions, if $T > T_1$ one needs at most $\frac{T}{T_1}$ steps to reach convergence in $E_V$. In fact, one has to show regularity at the end point $T_1$ of the interval $[0, T]$ on which the solution exists, i.e.,

$$
u(T_1) \in V \quad \text{and} \quad u'(T_1) \in H.$$

To see this, it suffices to show that $u_k \to u$ in $Y_V := C([0, T_1]; V)$ and $u'_k \to u'$ in $Y_H := C([0, T_1]; H)$. From (17) and assumption (13) on $L$ we obtain

$$
\|(u_l - u_k)(t)\|_V^2 \leq e^{T_1 \cdot F_1} C_L^2 \int_0^{T_1} \|(u_l - u_k)(\tau)\|_H^2 \, d\tau \leq e^{T_1 \cdot F_1} C_L^2 \int_0^{T_1} \|(u_l - u_k)(\tau)\|_V^2 \, d\tau.
$$

Taking first the square root and then the supremum over all $t \in [0, T]$ yields

$$
\|u_l - u_k\|_{Y_V} \leq \gamma_{T_1} \|u_l - u_k\|_{Y_V}.
$$

Since $\gamma_{T_1} < 1$ this implies that $\{u_k\}_{k \in \mathbb{N}}$ is a Cauchy sequence in $Y_V$. Similarly,

$$
\|(u_l - u_k)'(t)\|_H^2 \leq e^{T_1 \cdot F_1} C_L^2 \int_0^{T_1} \|(u_l - u_k)(\tau)\|_V^2 \, d\tau,
$$

which upon taking the supremum gives

$$
\|u_l - u_k\|'_{Y_H} \leq \gamma_{T_1} \|u_l - u_k\|_{Y_V},
$$

thus $u'_k \to u'$ in $Y_H$ (due to the already established convergence of $u_k$ in $Y_V$), and $u'(T_1) \in H$. This proves the claim.

### 2.2 Energy estimates

In Section 3 we shall need a priori (energy) estimate for problem (3). In fact, for the verification of moderateness in the Colombeau setting it will be crucial to know all constants in the energy estimate precisely. Therefore, we shall now derive it.

**Proposition 2.4.** Under the assumptions of Lemma 2.3, let $u$ be a solution to the abstract variational problem (14)-(17). Then, for each $t \in [0, T]$,

$$
\|u(t)\|_V^2 + \|u'(t)\|_H^2 \leq \left( D_T \|f_1\|_V^2 + \frac{1}{\nu} \left( \|f_2\|_H^2 + \int_0^t \|h(\tau)\|_H^2 \, d\tau \right) \right) \cdot e^{F_T t},
$$

where $\nu := \min\{1, \mu\}$, $D_T := C_0 + 1 + T$, and $F_T := \max\{ \frac{C_1 + C_2 + C_4}{\nu}, \frac{C_3 + 2 + \lambda (1 + T)}{\nu} \}$. 


Proof. Setting $v := u'(t)$ in (14), we obtain (as an equality of integrable functions with respect to $t$)

$$\langle u'(t), u'(t) \rangle + a(t, u(t), u'(t)) + \langle Lu(t), u'(t) \rangle = \langle h(t), u'(t) \rangle.$$  

Since $a(t, u, v) = a_0(t, u, v) + a_1(t, u, v)$ and $\langle u'(t), u'(t) \rangle = \frac{1}{2} \frac{d}{dt} \langle u'(t), u'(t) \rangle = \frac{1}{2} \frac{d}{dt} ||u'(t)||_{L^2}^2$, we have

$$\frac{d}{dt} ||u'(t)||_{L^2}^2 = -2a_0(t, u(t), u'(t)) - 2a_1(t, u(t), u'(t)) - 2\langle Lu(t), u'(t) \rangle + 2\langle h(t), u'(t) \rangle.$$  

Integration from 0 to $t_1$, for arbitrary $0 < t_1 \leq T$, gives

$$||u'(t_1)||_{L^2}^2 - ||f_2||_{L^2}^2 = -2 \int_0^{t_1} a_0(t, u(t), u'(t)) \, dt - 2 \int_0^{t_1} a_1(t, u(t), u'(t)) \, dt - 2 \int_0^{t_1} \langle Lu(t), u'(t) \rangle \, dt + 2 \int_0^{t_1} \langle h(t), u'(t) \rangle \, dt.$$  

Note that $\frac{d}{dt} \langle u'(t), u(t) \rangle = a_0'(t, u(t), u(t)) + a_0(t, u(t), u(t)) + a_0(t, u'(t), u'(t))$ and hence, by Assumption (i), $2a_0(t, u(t), u(t)) = \frac{d}{dt} \langle u'(t), u(t) \rangle - a_0'(t, u(t), u(t))$. This yields

$$LHS := ||u'(t_1)||_{L^2}^2 + a_0(t_1, u(t_1), u'(t_1)) = ||f_2||_{L^2}^2 + a_0(0, u(0), u(0)) - \int_0^{t_1} a_0'(t, u(t), u(t)) \, dt - 2 \int_0^{t_1} a_1(t, u(t), u'(t)) \, dt - 2 \int_0^{t_1} \langle Lu(t), u'(t) \rangle \, dt + 2 \int_0^{t_1} \langle h(t), u'(t) \rangle \, dt =: RHS. \quad (20)$$  

Further, by (8), Assumption (v), the Cauchy-Schwartz inequality, the inequality $2ab \leq a^2 + b^2$, and the assumption (13) on $\lambda$ we have

$$|RHS| \leq ||f_2||_{L^2}^2 + C_0||u(0)||_{L^2}^2 + C_0' \int_0^{t_1} ||u(t)||_{L^2}^2 \, dt + 2C_1 \int_0^{t_1} ||u(t)||_{L^2}^2 \, dt + 2 \int_0^{t_1} \langle Lu(t), u'(t) \rangle \, dt + 2 \int_0^{t_1} \langle h(t), u'(t) \rangle \, dt$$

$$\leq ||f_2||_{L^2}^2 + C_0||f_1||_{L^2}^2 + (C_0' + C_1) \int_0^{t_1} ||u(t)||_{L^2}^2 \, dt + (C_1 + 2) \int_0^{t_1} ||u'(t)||_{L^2}^2 \, dt + ||Lu||_{L^2(0,t_1;L^2)} + ||h(t)||_{L^2(t_0,t_1;L^2)} \leq ||f_2||_{L^2}^2 + C_0||f_1||_{L^2}^2 + (C_0' + C_1 + C_L) \int_0^{t_1} ||u(t)||_{L^2}^2 \, dt + (C_1 + 2) \int_0^{t_1} ||u'(t)||_{L^2}^2 \, dt.$$  

Further, it follows from (7) that

$$LHS = ||u'(t_1)||_{L^2}^2 + a_0(t_1, u(t_1), u'(t_1)) \geq ||u'(t_1)||_{L^2}^2 + \mu ||u(t_1)||_{L^2}^2 - \lambda ||u(t_1)||_{L^2}^2,$$

and therefore (20) yields

$$||u'(t_1)||_{L^2}^2 + \mu ||u(t_1)||_{L^2}^2 \leq ||u'(t_1)||_{L^2}^2 + C_0||f_1||_{L^2}^2 + ||f_2||_{L^2}^2 + \int_0^{t_1} ||h(t)||_{L^2}^2 \, dt$$

$$+ (C_0' + C_1 + C_L) \int_0^{t_1} ||u(t)||_{L^2}^2 \, dt + (C_1 + 2) \int_0^{t_1} ||u'(t)||_{L^2}^2 \, dt.$$  

As shown in (13) we have that $||u(t)||_{L^2}^2 \leq (1 + t)(||f_1||_{L^2}^2 + \int_0^{t_1} ||u'(s)||_{L^2}^2 \, ds)$, hence

$$||u(t_1)||_{L^2}^2 + ||u'(t)||_{L^2}^2 \leq ||u(t)||_{L^2}^2 + \frac{1}{v} \left( ||f_2||_{L^2}^2 + \int_0^{t_1} ||h(t)||_{L^2}^2 \, dt + F_T \int_0^{t_1} (||u(t)||_{L^2}^2 + ||u'(t)||_{L^2}^2) \, dt. \right.$$  

8
where \( \nu := \min\{1, \mu\} \), \( D_T := \frac{C_0 + \lambda (1 + T)}{\nu} \) and \( F_T := \max\{ \frac{C_0}{\nu}, \frac{C_1 + \lambda (1 + T)}{\nu}\} \). The claim now follows from Gronwall’s lemma.

As a consequence of Proposition 2.4 one also has uniqueness of the solution in Lemma 2.3.

**Theorem 2.5.** Under the assumptions of Lemma 2.3 there exists a unique \( u \in E_V \) satisfying the regularity conditions \( u' \in E_V \) and \( u'' \in L^2(0, T; V') \), and solving the abstract initial value problem (17)-(18). Moreover, \( u \in C([0, T]; V) \) and \( u' \in C([0, T]; H) \).

**Proof.** Since existence of a solution is proved in Lemma 2.3, it remains to show uniqueness part of the theorem. Thus, let \( u \) and \( w \) be solutions to the abstract initial value problem (13)-(14), satisfying the regularity conditions \( u', w' \in E_V \) and \( u'', w'' \in L^2(0, T; V') \). Then \( u - w \) is a solution to the homogeneous abstract problem with vanishing initial data

\[
\langle (u - w')(t), v \rangle + a(t, (u - w)(t), v) + \langle L(u - w)(t), v \rangle = 0, \quad \forall v \in V, \text{ for a.e. } t \in (0, T),
\]

\[
(u - w)(0) = 0, \quad (u - w)'(0) = 0.
\]

Moreover, according to Proposition 2.4 \( u - w \) satisfies the energy estimates (19) with \( f_1 = f_2 = h \equiv 0 \). This implies uniqueness of the solution.

## 2.3 Basic properties of the operator \( L \)

In this subsection we analyze our particular form of the operator \( L \), relevant to the problem described in the Introduction. Therefore, we consider an operator of convolution type and seek for conditions which guarantee estimate (13).

**Lemma 2.6.** Let \( l \in L^2_{\text{loc}}(\mathbb{R}) \) with \( \text{supp} l \subset [0, \infty) \). Then for all \( T_1 \in [0, T] \), the operator \( L \) defined by \( Lu(x, t) := \int_0^T l(s)u(x, t-s)\, ds \) maps \( L^2(0, T_1; H) \) into itself, and (13) holds with \( C_L = \|l\|_{L^2(0, T)} \cdot T \).

**Remark 2.7.** We may think of \( u \) being extended by 0 outside \([0, T]\) to a function in \( L^2(\mathbb{R}; H) \), and then identify \( Lu \) with \( l \ast u \).

**Proof.** Integration of \( \|Lu(t)\|^2_H \leq \int_0^T \|l(t-s)||u(s)||H \, ds \) from 0 to \( T_1 \), \( 0 < T_1 \leq T \), yields

\[
\left( \int_0^{T_1} \|Lu(t)\|^2_H \, dt \right)^{1/2} \leq \left( \int_0^{T_1} \left( \int_0^t \|l(t-s)||u(s)||H \, ds \right)^2 \, dt \right)^{1/2} \leq \left( \int_0^{T_1} \left( \int_0^t |l(t-s)|^2 \, ds \right) \|u(s)||H \, dt \right)^{1/2} \leq \left( \int_0^{T_1} \left( \int_0^t |l(t-s)||u(s)||H \, ds \right)^2 \, dt \right)^{1/2}
\]

\[
= \int_0^{T_1} \left( \int_0^t |l(t-s)|^2 \, ds \right)^{1/2} \|u(s)||H \, ds = \|l\|_{L^2(0, T_1)} \cdot \|u\|_{L^2(0, T_1; H)} \leq \|l\|_{L^2(0, T)} \cdot T \cdot \|u\|_{L^2(0, T_1; H)},
\]

where we have used the support property of \( l \), Minkowski’s inequality for integrals (c.f [8, p. 194]), and the Cauchy-Schwarz inequality.

In the following lemma we discuss a regularization of \( L \), which will be used in Section 3.2.

**Lemma 2.8.** Let \( l \in L^1_{\text{loc}}(\mathbb{R}) \) with \( \text{supp} l \subset [0, \infty) \). Let \( \rho \in \mathcal{D}(\mathbb{R}) \) be a mollifier (\( \text{supp} \rho \subset B_1(0), \int_0^1 \rho = 1 \)). Define \( \rho_\varepsilon(t) := \gamma_\varepsilon \rho(\gamma_\varepsilon t) \), with \( \gamma_\varepsilon > 0 \) and \( \gamma_\varepsilon \to \infty \) as \( \varepsilon \to 0 \). \( l_\varepsilon := l * \rho_\varepsilon \) and \( \tilde{L}_\varepsilon u(t) := (l_\varepsilon * u)(t) \), for \( u \in E_H \). Then \( \forall \rho \in [1, \infty) \), \( l_\varepsilon \in L^p_{\text{loc}}(\mathbb{R}) \) and \( l_\varepsilon \to l \) in \( L^1_{\text{loc}}(\mathbb{R}) \).
Proof. Let $K$ be a compact subset of $\mathbb{R}$. Then

$$
\|l_{\varepsilon}\|_{L^p(K)} = \|l * \rho_{\varepsilon}\|_{L^p(K)} = \left( \int_K \int_{-\infty}^{\infty} l(\tau)\rho_{\varepsilon}(t-\tau) \, d\tau \, dt \right)^{1/p} \leq \left( \int_K \int_{K+K_1(0)} \left| l(\tau)\rho_{\varepsilon}(t-\tau) \right| \, d\tau \, dt \right)^{1/p} \leq \left( \int_K \int_{K+K_1(0)} |l(\tau)|\rho_{\varepsilon}(t-\tau) \, d\tau \, dt \right)^{1/p} \leq \int_{K+K_1(0)} \left| l(\tau)\rho_{\varepsilon}(t-\tau) \right| \, d\tau = \int_{K+K_1(0)} \|l(\tau)\|_{L^p(K+K_1(0))} \rho_{\varepsilon}(t-\tau) \, d\tau = \|l\|_{L^1(K+K_1(0))} \|\rho_{\varepsilon}\|_{L^p(K+K_1(0))}
$$

where the second inequality follows from the support properties of $l$ and $\rho$ ($t-\tau \in B_1(0), t \in K$ implies $\tau \in K+K_1(0)$), while the third inequality we used Minkowski’s inequality for integrals. Further, we shall show that $l_{\varepsilon} \to l$ in $L^1_{loc}(\mathbb{R})$. Let $K \subset \mathbb{R}$. We claim that $\int_K l_{\varepsilon} - l \to 0$, as $\varepsilon \to 0$.

By [8, Prop. 8.5], we have that $\|l(\cdot - \frac{t}{\varepsilon}) - l\|_{L^1(K)} \to 0$, as $\varepsilon \to 0$ and therefore the integrand converges to 0 pointwise almost everywhere. Since it is also bounded by $2|\rho(\tau)||l|_{L^1(K)} \in L^1(\mathbb{R})$, Lebesgue’s dominated convergence theorem implies the result. 

\[\square\]

3 Weak and generalized solutions of the model equations

We now come back to the problem (1)-(2)-(IC)-(BC) or (3)-(IC)-(BC), and hence need to provide assumptions which guarantee that it can be interpreted in the form (14), in order to the apply results obtained above. For that purpose we need to prescribe the regularity of the functions $c$ and $b$ which appear in $Q$. In Section 3.2 we shall use these results on the level of representatives to prove existence of solutions in the Colombeau generalized setting.

Thus, let $H := L^2(0,1)$ with the standard scalar product $\langle u, v \rangle = \int_0^1 u(x)v(x) \, dx$ and $L^2$-norm denoted by $\| \cdot \|_{H}$. Let $V$ be the Sobolev space $H_0^1((0,1))$, which is the completion of the space of compactly supported smooth functions $C^\infty_c((0,1))$ with respect to the norm $\|u\|_2 = (\sum_{k=0}^{2} \|u^{(k)}\|_{\infty}^2)^{1/2}$ (and inner product $\langle u, v \rangle \mapsto \sum_{k=0}^{2} \langle u^{(k)}, v^{(k)} \rangle$). Then $V' = H^{-2}((0,1))$, which consists of distributional derivatives up to second order of functions in $L^2(0,1)$, and $V \hookrightarrow H \hookrightarrow V'$ forms a Gelfand triple. With this choice of spaces $H$ and $V$ we also have that $E_V = L^2(0,T; H_0^1((0,1)))$ and $E_H = L^2((0,1) \times (0,T))$.

Let

$$
c \in L^\infty(0,1) \text{ and real}, \quad b \in C([0,T]; L^\infty(0,1)),
$$

and suppose that there exist constants $c_1 > c_0 > 0$ such that

$$
0 < c_0 \leq c(x) \leq c_1, \quad \text{for almost every } x.
$$

For $t \in [0,T]$ we define the bilinear forms $a(t, \cdot, \cdot)$, $a_0(t, \cdot, \cdot)$ and $a_1(t, \cdot, \cdot)$ on $V \times V$ by

$$
a_0(t, u, v) = \langle c(x) \partial_x^2 u, \partial_x^2 v \rangle, \quad a_1(t, u, v) = \langle b(x, t) \partial_x^2 u, v \rangle, \quad u, v \in V,
$$

where

$$
\sum_{k=0}^{2} \|a^{(k)}(t, \cdot)\|_{L^2(0,T; \mathbb{C})}^2 < \infty, \quad \text{for almost every } t, x.\quad \text{(24)}
$$

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$$

where

$$
\sum_{k=0}^{2} \|a^{(k)}(t, \cdot)\|_{L^2(0,T; \mathbb{C})}^2 < \infty, \quad \text{for almost every } t, x.\quad \text{(24)}
$$

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$$
a_0(t, u, v) = \langle c(x) \partial_x^2 u, \partial_x^2 v \rangle, \quad a_1(t, u, v) = \langle b(x, t) \partial_x^2 u, v \rangle, \quad u, v \in V,
$$

where

$$
\sum_{k=0}^{2} \|a^{(k)}(t, \cdot)\|_{L^2(0,T; \mathbb{C})}^2 < \infty, \quad \text{for almost every } t, x.\quad \text{(24)}
$$
Properties (21), (22) imply that $a_0, a_1$ defined as in (23) satisfy the conditions of Assumption 1 (cf. [13] proof of Th. 2.2). The specific form of the operator $L$ is designed to achieve equivalence of the system (11)-(2) with the equation (3), which we show in the sequel.

Let $S'_1$ denote the space of Schwartz' distributions supported in $[0, \infty)$. It is known (c.f. [15]) that for given $z \in S'_1$ there is a unique $y \in S'_1$ such that $D^t_\theta z + z = \theta D^t_\eta y + y$. Moreover, it is given by $y = \hat{L}z$, where $\hat{L}$ is linear convolution operator acting on $S'_1$ as

$$\hat{L}z := \mathcal{L}^{-1} \left( \frac{1 + s^\alpha}{1 + \theta s^\alpha} \right) *_t z, \quad z \in S'_1. \quad (25)$$

The following lemma extends the operator $\hat{L}$ to the space $E_H$.

**Lemma 3.1.** Let $\hat{L} : S'_+ \to S'_+$ be defined as in (26). Then $\hat{L}$ induces a continuous operator $L = \text{Id} + L_\alpha$ on $E_H$, where $L_\alpha$ corresponds to convolution in time variable with a function $l_\alpha \in L^1_{loc}([0, \infty))$.

**Proof.** Recall that for the Mittag-Leffler function $e_\alpha(t, \lambda)$, defined by

$$e_\alpha(t, \lambda) = \sum_{k=0}^{\infty} \frac{(-\lambda t^\alpha)^k}{\Gamma(\alpha k + 1)},$$

we have that $\mathcal{L}(e_\alpha(t, \lambda))(s) = \frac{s^{\alpha-1}}{s^{\alpha} + \lambda}$, $e_\alpha \in C^\infty((0, \infty)) \cap C([0, \infty))$ and $e'_\alpha \in C^\infty((0, \infty)) \cap L^1_{loc}([0, \infty))$ (cf. [15]). Also,

$$\mathcal{L}^{-1} \left( \frac{1 + s^\lambda}{1 + \theta s^\lambda} \right) (t) = \mathcal{L}^{-1} \left( 1 + \frac{(1 - \theta)s^\alpha}{\theta(s^\alpha + \theta)} \right) (t) = \delta(t) + \frac{1}{\theta} - 1 \right) e'_\alpha \left( t, \frac{1}{\theta} \right) =: \delta(t) + l_\alpha(t).$$

Let $u \in E_H$. Then

$$\mathcal{L}^{-1} \left( \frac{1 + s^\lambda}{1 + \theta s^\lambda} \right) (\cdot) *_t u(x, \cdot) = u(x, \cdot) + \frac{1}{\theta} - 1 \right) e'_\alpha * u(x, \cdot) \quad (26)$$

is an element in $L^2(0, T)$ for almost all $x$ (use Fubini’s theorem, $e'_\alpha \in L^1(0, T)$ and $L^1 * L^p \subset L^p$ (cf. [3])). Extend this to a measurable function on $(0, 1) \times (0, T)$, denoted by $Lu$. By Young’s inequality we have

$$\| (Lu)(x, \cdot) \|_{L^2(0, T)} \leq \| u(x, \cdot) \|_{L^2(0, T)} + \frac{1}{\theta} - 1 \| e'_\alpha * u(x, \cdot) \|_{L^2(0, T)}$$

$$\leq \| u(x, \cdot) \|_{L^2(0, T)} + \frac{1}{\theta} - 1 \| e'_\alpha \|_{L^1(0, T)} \| u(x, \cdot) \|_{L^2(0, T)},$$

hence,

$$\| Lu \|_{E_H} \leq (1 + \frac{1}{\theta} - 1 \| e'_\alpha \|_{L^1(0, T)}) \| u \|_{E_H}. \quad (27)$$

Thus, $Lu \in E_H$ and $L$ is continuous on $E_H$. \(\square\)

We may write

$$Lu := (\text{Id} + L_\alpha)u = \lambda *_t u = (\delta + l_\alpha) *_t u \quad \text{with} \quad L_\alpha u := l_\alpha *_t u, \quad l_\alpha := \frac{1}{\theta} - 1 \right) e'_\alpha(t, \frac{1}{\theta}), \quad (28)$$

and therefore the model system (11)-(2) is equivalent to Equation (3).
3.1 Weak solutions for $L^\infty$ coefficients

Now we are in a position to apply the abstract results from the previous section to the original problem.

**Theorem 3.2.** Let $b$ and $c$ be as in (31) and (32). Let the bilinear form $a(t,\cdot,\cdot)$, $t \in [0,T]$, be defined by (25) and (24), and the operator $L$ as in (23). Let $f_1 \in H^3_0((0,1))$, $f_2 \in L^2(0,1)$ and $h \in L^2((0,1) \times (0,T))$. Then there exists a unique $u \in L^2(0,T;H^3_0(0,1))$ satisfying

$$u' = \frac{du}{dt} \in L^2(0,T;H^3_0(0,1)), \quad u'' = \frac{d^2u}{dt^2} \in L^2(0,T;H^{-2}(0,1)), \quad (29)$$

and solving the initial value problem

$$\langle u''(t), v \rangle + a(t,u(t),v) + \langle Lu(t), v \rangle = 0, \quad \forall v \in H^3_0((0,1)), \quad t \in (0,T), \quad (30)$$

$$u(0) = f_1, \quad u'(0) = f_2. \quad (31)$$

(Note that, as in the abstract version, since (29) implies $u \in C([0,T],H^3_0((0,1)))$ and $u' \in C([0,T],H^{-2}((0,1)))$ it makes sense to evaluate $u(0) \in H^3_0((0,1))$ and $u'(0) \in H^{-2}((0,1))$ and (31) claims that these equal $f_1$ and $f_2$, respectively.)

**Proof.** We may apply Lemma 2.3 because the bilinear form $a$ and the operator $L$ satisfy Assumption 1 and condition (13). The latter is true according to (27) with $C_L = (1+\frac{1}{2}-1||e''_a||_{L^2(0,T)}) = 1 + ||t_o||_{L^2(0,T)}$. As noted earlier, the bilinear forms $a$, $a_0$ and $a_1$ are as in (13) (20) and (21)]. Moreover, it follows as in the proof of (13) Theorem 2.2] that $a$ satisfies Assumption 1 with

$$C_0 := ||c||_{L^\infty((0,1))}, \quad C_0' := 0, \quad C_1 := ||b||_{L^\infty((0,1) \times (0,T))}, \quad \mu := \frac{c_0}{2}, \quad \lambda := C_{1/2} \cdot c_0, \quad (32)$$

where $C_{1/2}$ is corresponding constant form Ehrling’s lemma. \[\square\]

We briefly recall two facts about the solution $u$ obtained in Theorem 3.2 (as noted similarly in (13) Section 2):

(i) Since $u(.,t) \in H^3_0((0,1))$ for all $t \in [0,T]$ and $H^3_0((0,1))$ is continuously embedded in $C^1([0,1]): v(0,t) = v(1,t) = 0, \partial_x v(0,t) = \partial_x v(1,t) = 0$ (19) Corollary 6.2) the solution $u$ automatically satisfies the boundary conditions.

(ii) The properties in (29) imply that $u$ belongs to $C^1([0,T],H^{-2}((0,1))) \cap L^2((0,T) \times (0,1))$, which is a subspace of $D'((0,1) \times (0,T))$. Thus in case of smooth coefficients $b$ and $c$ we obtain a distributional solution to the “integro-differential” equation

$$\partial_t^2 u + \partial_x^2 (c \partial_x^2 u) + b \partial_x^2 u + l \ast u = h.$$

3.2 Colombeau generalized solutions

We will prove unique solvability of Equation (3) (or equivalently, of Equations (1)-(2)) with (IC) and (BC) for $u \in \mathcal{G}_{H^\infty(X_T)}$ when $b, c, f_1, f_2, g$ and $h$ are Colombeau generalized functions, where $X_T := (0,1) \times (0,T)$.

In more detail, we find a unique solution $u \in \mathcal{G}_{H^\infty(X_T)}$ to the equation

$$\partial_t^2 u + Q(t,x,\partial_x) u + Lu = h, \quad \text{on } X_T$$

with initial conditions

$$u|_{t=0} = f_1 \in \mathcal{G}_{H^\infty((0,1))}, \quad \partial_t u|_{t=0} = f_2 \in \mathcal{G}_{H^\infty((0,1))}$$

and boundary conditions

$$u|_{x=0} = u|_{x=1} = 0, \quad \partial_x u|_{x=0} = \partial_x u|_{x=1} = 0.$$
Here $Q$ is a partial differential operator on $G_{H^\infty(X_T)}$ with generalized functions as coefficients, defined by its action on representatives in the form

$$(u_\varepsilon)_\varepsilon \mapsto (\partial^2_x(c_\varepsilon(x)\partial^2_x u_\varepsilon)) + b_\varepsilon(x,t)\partial^2_x u_\varepsilon) =: (Q_\varepsilon u_\varepsilon)_\varepsilon.$$ 

Furthermore, the operator $L$ corresponds to convolution on the level of representatives with regularizations of $l$ as given in Lemma 2.8

$$(u_\varepsilon)_\varepsilon \mapsto (l_\varepsilon * u_\varepsilon(t))_\varepsilon =: (L_\varepsilon u_\varepsilon)_\varepsilon,$$

where $l_\varepsilon = l * \rho_\varepsilon$, with $\rho_\varepsilon$ introduced in Lemma 2.8

**Lemma 3.3.** (i) If $l \in L^2_{\text{loc}}(\mathbb{R})$ and $l$ is $C^\infty$ in $(0, \infty)$ then $L$ is a continuous operator on $H^\infty(X_T)$. Thus $(u_\varepsilon)_\varepsilon \mapsto (L_\varepsilon u_\varepsilon)_\varepsilon$ defines a linear map on $G_{H^\infty(X_T)}$.

(ii) If $l \in L^1_{\text{loc}}(\mathbb{R})$ then $\forall \varepsilon \in (0,1)$ the operator $L_\varepsilon$ is continuous on $H^\infty(X_T)$ and $(u_\varepsilon)_\varepsilon \mapsto (L_\varepsilon u_\varepsilon)_\varepsilon$ defines a linear map on $G_{H^\infty(X_T)}$.

**Proof.** (i) From Lemma 2.8 with $H = L^2(0,1)$ we have that $L$ is continuous on $L^2(X_T)$ with operator norm $\|L\|_{\text{op}} \leq T \cdot \|l\|_{L^2(0,T)}$.

Let $u \in H^\infty(X_T)$ and $L(u(x,t)) = \int_0^t l(s)u(x,t-s)ds$. We have to show that all derivatives of $Lu$ with respect to both $x$ and $t$ are in $L^2(X_T)$.

- $\partial^2_x Lu(x,t) = \int_0^t l(s)\partial^2_x u(x,t-s)ds$, and hence, $\|\partial^2_x Lu\|_{L^2(X_T)} \leq T \cdot \|l\|_{L^2(0,T)}\|\partial^2_x u\|_{L^2(X_T)}$.

- $\partial^2_t \partial^2_x Lu = \partial^2_t L(\partial^2_x u)$, and since the estimates for $L(\partial^2_x u)$ are known it suffices to consider only terms $\partial^2_t Lu$. For the first order derivative we have

$$\partial_t Lu(x,t) = l(t)u(x,0) + \int_0^t l(s)\partial_t u(x,t-s)ds$$

and therefore

$$\|\partial_t Lu\|_{L^2(X_T)} \leq \|l\|_{L^2(0,T)}\|u(\cdot,0)\|_{L^2(0,1)} + T \cdot \|l\|_{L^2(0,T)}\|\partial_t u\|_{L^2(X_T)} \leq \|l\|_{L^2(0,T)}(\|u\|_{H^\infty(X_T)} + T \cdot \|u\|_{H^1(X_T)}),$$

where we have used the fact that $\text{Tr} : H^\infty(X_T) \to H^\infty((0,1))$, $u \mapsto u(\cdot,0)$ is continuous, and more precisely, $\text{Tr} : H^m(X_T) \to H^{m-1}((0,1))$ with estimates $\|\partial^2_x \partial^2_t u(\cdot,0)\|_{L^2(0,1)} \leq \|u\|_{H^m(X_T)}$, $m = m(k,l)$.

Higher order derivatives involve terms $l^{(r)}(t)\partial^r_t u(x,0), \ldots, \int_0^t l(s)\partial^r_t u(x,t-s)ds$, which can be estimated as above.

(ii) From Lemma 2.8 it follows that $l_\varepsilon \in L^2_{\text{loc}}(\mathbb{R})$, and $\|l_\varepsilon\|_{L^2(0,T)} \leq \gamma_\varepsilon^\frac{1}{2} \cdot \|l\|_{L^1(0,T)}\|\rho\|_{L^2(0,T)}$. From Lemma 2.8 we know that $L_\varepsilon$ is continuous $X_T \to X_T$, with $\|L_\varepsilon\|_{\text{op}} \leq T \cdot \|l_\varepsilon\|_{L^2(0,T)} \leq T \cdot \gamma_\varepsilon^\frac{1}{2} \cdot \|l\|_{L^1(0,T)}\|\rho\|_{L^2(0,T)}$, which is moderate. We can now proceed as in (i) to produce estimates of $\|L_\varepsilon u_\varepsilon\|_{H^r(X_T)}$, $\forall r \in \mathbb{N}$, always replacing $\|l\|_{L^2(0,T)}$ by $\gamma_\varepsilon^\frac{1}{2} \cdot \|l\|_{L^2(0,T)}$ factors. Since $\gamma_\varepsilon \leq \varepsilon^{-N}$ it follows that $(L_\varepsilon u_\varepsilon)_\varepsilon \in C_{H^\infty(X_T)}$.

**Remark 3.4.** The function $l$ as defined in (2.8) belongs to $L^2_{\text{loc}}(\mathbb{R})$, if $\alpha > 1/2$, and to $L^1_{\text{loc}}(\mathbb{R})$, if $\alpha \leq 1/2$ (which follows from the explicit form of $e_\alpha(t,\frac{1}{b})$). This means that in case $\alpha > 1/2$ we could in fact define the operator $L$ without regularization of $l$.

As in the classical case we also have to impose a condition to ensure compatibility of initial with boundary values, namely (as equation in generalized numbers)

$$f_1(0) = f_1(1) = 0.$$  

(33)
Note that if $f_1 \in \mathcal{G}_{H^\infty((0,1))}$ satisfies (33), then there is some representative $(f_{1,\varepsilon})_\varepsilon$ of $f_1$ with the property $f_{1,\varepsilon} \in H^2_0((0,1))$ for all $\varepsilon \in (0,1)$ (cf. the discussion right below Equation (28) in [13]).

Motivated by condition (22) above on the bending stiffness we assume the following about $c$: There exist real constants $c_1 > c_0 > 0$ such that $c \in \mathcal{G}_{H^\infty((0,1))}$ possesses a representative $(c_\varepsilon)_\varepsilon$ satisfying

$$0 < c_0 \leq c_\varepsilon(x) \leq c_1 \quad \forall x \in (0,1), \forall \varepsilon \in (0,1).$$

(34)

(Hence any other representative of $c$ has upper and lower bounds of the same type.)

As in many evolution-type problems with Colombeau generalized functions we also need the standard assumption that $b$ is of $L^\infty$-log-type (similar to [10]), which means that for some (hence any) representative $(b_\varepsilon)_\varepsilon$ of $b$ there exist $N \in \mathbb{N}$ and $\varepsilon_0 \in (0,1]$ such that

$$\|b_\varepsilon\|_{L^\infty(X_T)} \leq N \cdot \log(\frac{1}{\varepsilon}), \quad 0 < \varepsilon \leq \varepsilon_0.$$  

(35)

It has been noted already in [10] Proposition 1.5 that log-type regularizations of distributions are obtained in a straightforward way by convolution with logarithmically scaled mollifiers.

**Theorem 3.5.** Let $b \in \mathcal{G}_{H^\infty(X_T)}$ be of $L^\infty$-log-type and $c \in \mathcal{G}_{H^\infty((0,1))}$ satisfy (33). Let $\gamma_\varepsilon = O(\log \frac{1}{\varepsilon})$. For any $f_1 \in \mathcal{G}_{H^\infty((0,1))}$ satisfying (33), $f_2 \in \mathcal{G}_{H^\infty((0,1))}$, $h \in \mathcal{G}_{H^\infty(X_T)}$ and $l \in \mathcal{G}_{H^\infty((0,1))}$, there is a unique solution $u \in \mathcal{G}_{H^2_0(X_T)}$ to the initial-boundary value problem

$$\begin{align*}
\partial_t^2 u + Q(t,x,\partial_x u) + Lu &= h, \\
\partial_t u|_{t=0} &= f_1, \\
\partial_x u|_{x=0} &= f_2, \\
u|x=0 &= u|x=1 = 0, \\
\partial_x u|_{x=0} &= \partial_x u|_{x=1} = 0.
\end{align*}$$

(36)

Proof. Thanks to the preparations a considerable part of the proof may be adapted from the corresponding proof in [13] Theorem 3.1. Therefore we give details only for the first part and sketch the procedure from there on.

**Existence:** We choose representatives $(b_\varepsilon)_\varepsilon$ of $b$ and $(c_\varepsilon)_\varepsilon$ of $c$ satisfying (33) and (35).

Further let $(f_{1,\varepsilon})_\varepsilon$, $(f_{2,\varepsilon})_\varepsilon$, $(l_\varepsilon)_\varepsilon$, and $(h_\varepsilon)_\varepsilon$ be representatives of $f_1,f_2,l$, and $g$, respectively, where we may assume $f_{1,\varepsilon} \in H^2_0((0,1))$ for all $\varepsilon \in (0,1)$ (cf. (33)).

For every $\varepsilon \in (0,1]$ Theorem [3.2] provides us with a unique solution $u_\varepsilon \in H^1((0,T), H^2_0((0,1))) \cap H^2((0,T), H^{-2}((0,1)))$ to

$$\begin{align*}
P_\varepsilon u_\varepsilon := \partial_t^2 u_\varepsilon + Q(\varepsilon, t,x, \partial_x u_\varepsilon) + L u_\varepsilon = h_\varepsilon, \\
u_\varepsilon|_{t=0} = f_{1,\varepsilon}, \\
\partial_t u_\varepsilon|_{t=0} = f_{2,\varepsilon}, \\
u_\varepsilon|x=0 = u_\varepsilon|x=1 = 0.
\end{align*}$$

(36)

In particular, we have $u_\varepsilon \in C^1([0,T], H^{-2}((0,1))) \cap C([0,T], H^2_0((0,1)))$.

Proposition [2.4] implies the energy estimate

$$\|u_\varepsilon(t)\|_{H^2}^2 + \|\partial_t u_\varepsilon(t)\|_{L^2}^2 \leq (D_\varepsilon T) \|f_{1,\varepsilon}\|_{H^2}^2 + \|f_{2,\varepsilon}\|_{L^2}^2 + \int_0^T \|h_\varepsilon(\tau)\|_{H^2}^2 d\tau \cdot \exp(t F_\varepsilon T),$$

(37)

where with some $N$ we have as $\varepsilon \to 0$

$$(D_\varepsilon T) = (\|c_\varepsilon\|_{L^\infty} + \lambda(1+T))/\min(\mu,1) = O(\|c_\varepsilon\|_{L^\infty}) = O(1) \quad \text{(38)}$$

$$F_\varepsilon T = \max\left(\frac{\|b_\varepsilon\|_{L^\infty} + C_{L,\varepsilon}}{\|c_\varepsilon\|_{L^\infty} + 2 + \lambda(1+T)}\right) \min(\mu,1) = O(C_{L,\varepsilon} + \|b_\varepsilon\|_{L^\infty}) = O(\log(\varepsilon^{-\lambda N})), \quad \text{ (39)}$$

since $\mu$ and $\lambda$ are independent of $\varepsilon$, and $C_{L,\varepsilon} = O(\log \frac{1}{\varepsilon})$ (cf. [22]).

By moderateness of the initial data $f_{1,\varepsilon}$, $f_{2,\varepsilon}$ and of the right-hand side $h_\varepsilon$ the inequality (37) thus implies that there exists $M$ such that for small $\varepsilon > 0$ we have

$$\|u_\varepsilon\|_{L^2(X_T)}^2 + \|\partial_t u_\varepsilon\|_{L^2(X_T)}^2 + \|\partial_x u_\varepsilon\|_{L^2(X_T)}^2 + \|\partial_x u_\varepsilon\|_{L^2(X_T)}^2 = O(\varepsilon^{-M}), \quad \varepsilon \to 0.$$  

(40)

From here on the remaining chain of arguments proceeds along the lines of the proof in [13] Theorem 3.1. We only indicate a few key points requiring certain adaptions.

The goal is to prove the following properties:
1.) For every $\varepsilon \in (0, 1]$ we have $u_\varepsilon \in H^\infty(X_T) \subseteq C^\infty(X_T)$.

2.) Moderateness, i.e. for all $l, k \in \mathbb{N}$ there is some $M \in \mathbb{N}$ such that for small $\varepsilon > 0$

$$\|\partial_l^2 \partial_k^2 u_\varepsilon\|_{L^2(X_T)} = O(\varepsilon^{-M}).$$

$$(T_{l,k})$$

Note that (40) already yields $(T_{l,k})$ for $(l, k) \in \{(0, 0), (1, 0), (0, 1), (0, 2)\}$.

Proof of 1.) Differentiating (30) (considered as an equation in $\mathcal{D}'((0, 1) \times (0, T))$) with respect to $t$ we obtain

$$P_\varepsilon(\partial_t u_\varepsilon) = \partial_t h_\varepsilon - \partial_t^2 \partial_x^2 u_\varepsilon - l_\varepsilon(t) f_{1,\varepsilon} =: \tilde{h}_\varepsilon,$$

where we used $\partial_t L_\varepsilon u_\varepsilon = L_\varepsilon(\partial_t u_\varepsilon) + L_\varepsilon(t) u_\varepsilon(0)$. We have $\tilde{h}_\varepsilon \in H^1(\mathcal{D}, L^2()$ since $\partial_t h_\varepsilon \in H^\infty(\mathcal{D}, L^2(0, 1))$. $f_{1,\varepsilon} \in H^\infty(\mathcal{D}, L^2(0, 1))$, $\partial_t h_\varepsilon (x, t) \in H^\infty(X_T) \subset W^\infty,\infty(X_T)$ and $\partial_x^2 u_\varepsilon \in H^1(\mathcal{D}, L^2(0, 1))$. Furthermore, since $Q_\varepsilon$ depends smoothly on $t$ as a differential operator in $x$ and $u_\varepsilon(0) = f_{1,\varepsilon} \in H^\infty(\mathcal{D})$ we have

$$(\partial_t u_\varepsilon)(\cdot, 0) = f_{2,\varepsilon} = \tilde{f}_{1,\varepsilon} \in H^\infty(\mathcal{D}),$$

$$(\partial_t(\partial_t u_\varepsilon))(\cdot, 0) = h_\varepsilon(\cdot, 0) - Q_\varepsilon u_\varepsilon(\cdot, 0) - L_\varepsilon u_\varepsilon(\cdot, 0) = h_\varepsilon(\cdot, 0) - (Q_\varepsilon + L_\varepsilon) f_{1,\varepsilon} := \tilde{f}_{2,\varepsilon} \in H^\infty(\mathcal{D}).$$

Hence $\partial_t u_\varepsilon$ satisfies an initial value problem for the partial differential operator $P_\varepsilon$ as in (30) with initial data $\tilde{f}_{1,\varepsilon}$, $\tilde{f}_{2,\varepsilon}$ and right-hand side $\tilde{h}_\varepsilon$ instead. However, this time we have to use $V = H^2((0, 1), H^\infty((0, 1), 1))$ (replacing $H_0^2((0, 1))$ and $H = L^2((0, 1)$ in the abstract setting, which still can serve to define a Gelfand triple $V \hookrightarrow H \hookrightarrow V'$ (cf. [19] Theorem 17.4(b))) and thus allows for application of Lemma [28] and the energy estimate (39) (with precisely the same constants).

Therefore we obtain $\partial_t u_\varepsilon \in H^1(\mathcal{D}, H^2((0, 1), H^\infty((0, 1), 1)))$. $u_\varepsilon \in H^2(\mathcal{D}, H^2((0, 1), H^\infty((0, 1), 1)))$ and from the variants of (37) (with exactly the same constants $D_T^2$ and $F_T^2$) and (40) with $\partial_x u_\varepsilon$ in place of $u_\varepsilon$ that for some $M$ we have

$$\|\partial_t u_\varepsilon\|_{L^2(X_T)}^2 + \|\partial_x \partial_x u_\varepsilon\|_{L^2(X_T)}^2 + \|\partial_x^2 u_\varepsilon\|_{L^2(X_T)}^2 + \|\partial_x^2 u_\varepsilon\|_{L^2(X_T)}^2 = O(\varepsilon^{-M}) \quad (\varepsilon \rightarrow 0).$$

Thus we have proved $(T_{l,k})$ with $(l, k) = (2, 0), (1, 1), (1, 2)$ in addition to those obtained from (40) directly.

The remaining part of the proof of property 1.) requires the exact same kind of adaptions in the corresponding parts in Step 1 of the proof of [13] Th. 3.1 and we skip its details here. In particular, along the way one also obtains that

$$(T_{l,k}) \text{ holds for } l \leq 2.$$

Proof of 2.) From the estimates achieved in proving 1.) and equation (30) we deduce that

$$k_\varepsilon := \partial_x^2 (\varepsilon \partial_x^2 u_\varepsilon) = h_\varepsilon - b_\varepsilon \partial_x^2 u_\varepsilon - \partial_x^2 u_\varepsilon - L_\varepsilon u_\varepsilon$$

satisfies for all $l \in \mathbb{N}$ with some $N_l$ an estimate

$$\|\partial_l k_\varepsilon\|_{L^2(X_T)} = O(\varepsilon^{-N_l}) \quad (\varepsilon \rightarrow 0).$$

Here we are again in the same situation as in Step 2 of the proof of [13] Theorem 3.1, where now $k_\varepsilon$ plays the role of $h_\varepsilon$ there. Skipping again details of completely analogous arguments we arrive at the conclusion that the class of $(u_\varepsilon)_\varepsilon$ defines a solution to the initial value problem.

Moreover, we have by construction that $u_\varepsilon(t) \in H_0^2((0, 1))$ for all $t \in [0, T]$, hence $u(0, t) = u(1, t) = 0$ and $\partial_t u(0, t) = \partial_t u(1, t) = 0$ and thus $u$ also satisfies the boundary conditions.

Uniqueness: If $u = [(u_\varepsilon)_\varepsilon]$ satisfies initial-boundary value problem with zero initial values and right-hand side, then we have for all $q \geq 0$

$$\|f_{1,\varepsilon}\| = O(\varepsilon^q), \quad \|f_{2,\varepsilon}\| = O(\varepsilon^q), \quad \|h_\varepsilon\|_{L^2(X_T)} = O(\varepsilon^q) \quad (\varepsilon \rightarrow 0).$$

Therefore the energy estimate (37) in combination with (38) imply for all $q \geq 0$ an estimate

$$\|u_\varepsilon\|_{L^2(X_T)} = O(\varepsilon^q) \quad (\varepsilon \rightarrow 0),$$

from which we conclude that $(u_\varepsilon)_\varepsilon \in \mathcal{N}_{H^\infty(X_T)}$, i.e., $u = 0$. □
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