Foundation of Symbol Theory
for Analytic Pseudodifferential Operators. I

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Abstract. A new symbol theory for pseudodifferential operators in the complex analytic category is given. This theory provides a cohomological foundation of symbolic calculus.

Introduction

The aim of this series of papers is to establish a complete symbol theory for the sheaf $E^\mathbb{R}_X$ of pseudodifferential operators in the complex analytic category. Here we distinguish a little difference between the usage of hyphenation in the words “pseudodifferential” and “pseudo-differential.” The latter might be more familiar than the former for most of readers. To clarify this distinction, we have to mention some of history.

The notion of the pseudo-differential operators in the analytic category was introduced by Boutet de Monvel and Kreé [11] and by Boutet de Monvel [10] for the real domain and by Sato, Kawai and Kashiwara [29] for the complex domain about forty years ago. Note that [11] introduced the notion in the category of ultradifferentiable functions of Gevrey class which contains the analytic category for a special case and treated operators of finite order. On the other hand, [10] and [29] considered operators of infinite order and these operators play an essential role in [29] and in Kashiwara and Kawai [18]. The notion is effectively used not only in the analysis of differential equations in analytic category but in various fields of mathematics and now it becomes one of the most basic tools in analysis as well as pseudo-differential operators in $C^\infty$ category. There are a number of references which use pseudo-differential operators and we can not cite all of them. We only cite Björk [9], Hörmander [12], Kashiwara and Schapira [22], Kumano-go [25], Liess [27], Schapira [30] and Trèves [31], and refer the readers to the references cited in those books or manuscripts.

The definition of the pseudo-differential operators given in [10] used oscillatory integrals and analytic symbols, while [29] employed the cohomology theory. One of the advantages of the latter theory is invariance which comes from the cohomology theory. But most of analysts are familiar with the former theory because it is comprehensible as an analogy of pseudo-differential operators in $C^\infty$ category through symbol theory and it does not require heavy algebraic tools such as derived categories and exact sequences. Symbol theory for pseudo-differential operators was also developed in [29], where the sheaf of them was denoted by $P_X$. This sheaf is recently denoted by $E^\mathbb{R}_X$ after the work of Kashiwara and

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Schapira [19] and called the sheaf of microdifferential operators (of finite or infinite order). Note that it is a subsheaf of $\mathcal{E}^\mathbb{R}_X$. The notion of symbols defined in [29] is different from that of [10]. A symbol of a microdifferential operator (or a pseudo-differential operator in the sense of [29]) is a sequence with an index in $\mathbb{Z}$ of holomorphic functions defined on the cotangent bundle $T^*X$ with some homogeneity and growth conditions. On the other hand, a pseudo-differential operator in the sense of [10] is defined by a total symbol $p(x, \xi)$ which is real analytic in $(x, \xi)$ satisfying a growth condition in $\xi$ variables. The relation between those two theories was clarified by Kataoka [23]. He defined symbols of operators in $\mathcal{E}^\mathbb{R}_X$ by using the Radon transform and through his theory, we knew that pseudo-differential operators of [10] is obtained by restriction of $\mathcal{E}^\mathbb{R}_X$ to the real domain.

The essential idea of the definition of $\mathcal{E}^\mathbb{R}_X$ was given in [29] but the definition itself was not given there explicitly. The definition first appeared in the work of Kashiwara and Kawai [17], where the notation $\mathcal{P}^\mathbb{R}_X$ was used, although the name of the sections of the sheaf was not given. As well as the case of $\mathcal{E}^\infty_X$, we use the notation $\mathcal{E}^\mathbb{R}_X$ instead of $\mathcal{P}^\mathbb{R}_X$ after [19] and we call the sections of $\mathcal{E}^\mathbb{R}_X$ pseudodifferential operators after [2].

Since the symbol theory developed in [23] was not published, some parts of it were supplied by the first author [2] and the theory played a role in the analysis of operators of infinite order (cf. Aoki [1], [3], Aoki, Kawai, Koike and Takei [7], Aoki, Kawai and Takei [8], Kajitani and Wakabayashi [14], Kataoka [24], Uchikoshi [32]). Some systematic description of the theory has been included in the book of Aoki, Kataoka and Yamazaki [6]. The foundation of the symbol theory of $\mathcal{E}^\mathbb{R}_X$ at the present stage is, however, quite unsatisfactory. There are two issues: first one is that, as Kamimoto and Kataoka have pointed out in their work [16, Example 1.1], the space of the kernel functions which comes from standard Čech representation of cohomology groups is not closed under composition of kernel functions defined by naive integration employed in [2], [6]. Regarding this issue, [16] gives a possible solution by introducing the notion of formal kernels. Second issue is that the relation between the action of operators by integration of kernel functions and canonical action through cohomological definition was not clarified. We note that the notion of formal kernels given in [16] has not yet given a solution to this issue directly. Thus we think we have to provide a complete symbol theory of $\mathcal{E}^\mathbb{R}_X$ which solves these issues.

We mention that some modifications of the symbol theory are given by Uchikoshi [33] and by Ishimura [13] for microlocal operators and non-local operators in the analytic category, respectively. But there are analogous issues in these theories.

In this series of papers, we establish a new symbol theory of $\mathcal{E}^\mathbb{R}_X$ which completely fits in the cohomological definition of the sheaf. In the first part, we present a foundation of symbol theory for $\mathcal{E}^\mathbb{R}_X$. Our main idea is to introduce a redundant parameter, which we call an apparent parameter, in the definitions of (real) holomorphic microfunctions and symbols. By introducing this parameter, cohomological definition of operation such as composition, formal adjoint, coordinate transformation, etc. are directly related to those of symbols (see Kashiwara-Schapira [20], [21]). To clarify the relation between Čech cohomology classes and symbols, we fully use the theory of the action of microdifferential
operators on microfunctions established by Kashiwara and Kawai ([17], [18]). We also develop a theory of formal symbols which was firstly introduced for operators of infinite order by [10] and generalized by [1], [2] and by Laurent [26]. The formal symbol theory established in this article is exactly based on the cohomological definition of $\mathcal{E}_X^\infty$. To develop this theory, we employ an idea introduced by [26]. Our forthcoming second paper will be devoted to the symbol theory for operators with Gevrey growth and the cohomology theory for Whitney holomorphic functions. It will be useful for applications.

The plan of this paper, the first part, is as follows. In Section 1, we prepare a proposition of the local cohomology group theory on a vector space which we shall use in this article. Section 2 gives a new formulation of the sheaf of (real) holomorphic microfunctions utilizing an apparent parameter. Applying this formulation, we give a cohomological representation of pseudodifferential operators in Section 3. In Section 4, we define symbol spaces with an apparent parameter. The relation between symbols and cohomological representation of pseudodifferential operators is clarified in Section 5. Sections 6 and 7 are devoted to establish a theory of formal symbols with an apparent parameter for pseudodifferential operators. Logically we can skip these two sections since our theory developed up to Section 5 provides the equivalence of our present theory and the symbol theory given in [2], [6] without cohomology theoretical foundation. But we think we need these two sections since the relation between formal symbols and kernel functions can be understood directly from the viewpoint of these sections. Of course, they provides a unified treatment of symbols of pseudodifferential operators with previous sections. Basic algebraic operations in terms of symbols are given with cohomology theoretical foundation in these sections.

In Appendix A, we confirm the compatibility of actions of pseudodifferential operators on the sheaf of holomorphic microfunctions. Appendix B gives a general construction of the sheaf of microfunctions which can manage the symbol mapping on the space of kernel functions with respect to arbitrary coverings.

This work, especially, the idea of introducing redundant parameter, is inspired by [16]. The authors would like to express their sincere thanks to Professor K. Kataoka and Dr. S. Kamimoto. They also thank Professor T. Kawai and Professor Y. Okada for encouragement to them.

§ 1. Local Cohomology Groups on a Vector Space

We denote by $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ the sets of integers, of real numbers and of complex numbers respectively. Further, set $\mathbb{N} := \{m \in \mathbb{Z}; m > 0\}$, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and $\mathbb{C}^\times := \{c \in \mathbb{C}; c \neq 0\}$.

Let $X$ be a finite dimensional $\mathbb{R}$-vector space, and define an open proper sector $S \subset \mathbb{C}$ by

$$S := \{\eta \in \mathbb{C}; a < \arg \eta < b, 0 < |\eta| < r\}$$

for some $0 < b - a < \pi$ and $r > 0$. We set $\hat{X} := X \times \mathbb{C}_\eta$ with coordinates $(x, \eta)$, and let $\pi_\eta: \hat{X} \ni (x, \eta) \mapsto x \in X$ be the canonical projection. Let $G \subset X$ be a closed subset (not necessarily convex) and $U \subset X$ an open neighborhood of the origin. In this section we give another representation of local cohomology groups $H^k_{Gru}(U; \mathcal{F})$ for a sheaf $\mathcal{F}$.
on $X$. For this purpose, we need some preparations. Let $Z$ be a closed subset in $X$ and \( \varphi : X \times [0, 1] \to X \) a continuous deformation mapping which satisfies the following conditions:

(i) \( \varphi(x, 1) = x \) for any $x \in X$ and \( \varphi(z, s) = z \) for any $z \in Z$.

(ii) \( \varphi(\varphi(x, s), 0) = \varphi(x, 0) \) for any $s \in [0, 1]$ and $x \in X$.

(iii) We set \( \rho_{\varphi}(x, s) := |\varphi(x, s) - \varphi(x, 0)|. \)

Then $\rho_{\varphi}(x, s)$ is a strictly increasing function of $s$ outside $Z$, i.e. if $s_1 < s_2$, we have $\rho_{\varphi}(x, s_1) < \rho_{\varphi}(x, s_2)$ for any $x \in X \setminus Z$.

Let us set, for short

\[
\rho_{\varphi}(x) := \rho_{\varphi}(x, 1) = |\varphi(x, 1) - \varphi(x, 0)| = |x - \varphi(x, 0)|. 
\]

Here we remark

\[
\rho_{\varphi}(\varphi(x, s)) = |\varphi(x, s) - \varphi(\varphi(x, s), 0)| = |\varphi(x, s) - \varphi(x, 0)| = \rho_{\varphi}(x, s). 
\]

Let us see a typical example of such a deformation mapping.

1.1. **Example.** Let $\zeta$ be a unit vector in $X := \mathbb{C}^n$ and $Z = \{x \in X; \langle x, \zeta \rangle = 0\}$ with $\langle x, \zeta \rangle := \sum_{i=1}^{n} x_i \zeta_i$. Define the deformation mapping $\varphi : X \times [0, 1] \to X$ by

\[
\varphi(x, s) := x + (s - 1)\langle x, \zeta \rangle \overline{\zeta}. 
\]

Here $\overline{\zeta}$ denotes the complex conjugate of $\zeta$. Note that $\varphi(x, 1) = x$ and $\varphi(x, 0)$ gives the orthogonal projection to the complex hyperplane $Z$ with respect to the standard Hermitian metric $|x| = \langle x, \overline{x} \rangle^{1/2}$.

Let $\varrho > 0$ a positive constant. We define the subsets in $\widehat{X}$ by

\[
\begin{align*}
\widehat{G} &:= \{(\varphi(x, s), \eta) \in \widehat{X}; \rho_{\varphi}(x) \leq \varrho|\eta|, 0 \leq s \leq 1, x \in G\}, \\
\widehat{U} &:= \{(x, \eta) \in U \times S; \rho_{\varphi}(x) < \varrho|\eta|\}.
\end{align*}
\]

Note that $\widehat{G} \cap \widehat{U}$ is a closed subset in $\widehat{U}$. Then we have the following proposition.

1.2. **Proposition.** Let $\mathcal{F}$ be a complex of Abelian sheaves on $X$. Assume that $U$ satisfies

\[
\sup_{x \in \widehat{U}} \rho_{\varphi}(x) < \varrho r.
\]

Then there exists the following isomorphism:

\[
R\Gamma_{\widehat{G}\cap\widehat{U}}(U; \mathcal{F}) \cong R\Gamma_{\widehat{G}\cap\widehat{U}}(\widehat{U}; \pi_{\eta}^{-1}\mathcal{F}).
\]

**Proof.** Since $\pi_{\eta}^{-1}(G) \cap \widehat{U}$ is closed in $\widehat{G} \cap \widehat{U}$ and $\widehat{U}$ is open in $\pi_{\eta}^{-1}(U)$, we obtain

\[
\mathbb{Z}_{\pi_{\eta}^{-1}(G) \cap \widehat{U}} \to \mathbb{Z}_{\pi_{\eta}^{-1}(G)} \to \mathbb{Z}_{\pi_{\eta}^{-1}(G) \cap \pi_{\eta}^{-1}(U)} = \pi_{\eta}^{-1}\mathbb{Z}_{\pi_{\eta}^{-1}(U)} = \pi_{\eta}^{-1}\mathbb{Z}_{\pi_{\eta}^{-1}(U)}[-2],
\]

and this induces the canonical morphism

\[
R\pi_{\eta}^{-1}\mathbb{Z}_{\pi_{\eta}^{-1}(U)} \to \mathbb{Z}_{\pi_{\eta}^{-1}(U)}[-2].
\]
Hence, we have
\[
R\Gamma_{G\cap U}(U; \mathcal{F}) \simeq R\text{Hom}_{Z_X}(\mathbb{Z}_{G\cap U}, \mathcal{F}) \to R\text{Hom}_{Z_X}(R\pi_{\eta}^*\mathbb{Z}_{G\cap U}, \mathcal{F})[-2] \\
\simeq R\text{Hom}_{Z_X}(\mathbb{Z}_{G\cap \hat{U}}, \pi_{\eta}^1\mathcal{F})[-2] \simeq R\Gamma_{G\cap \hat{U}}(\hat{U}; \pi_{\eta}^{-1}\mathcal{F}),
\]
and to show Proposition 1.2, it suffices to prove that (1.3) is an isomorphism. We first give some properties of \(\varphi\) and \(\rho_\varphi\).

(1) If \(\varphi(x, s) = \varphi(x', s')\) holds, we have \(\varphi(x, 0) = \varphi(\varphi(x, s), 0) = \varphi(\varphi(x', s'), 0) = \varphi(x', 0)\).

In particular, \(\rho_\varphi(x, s) = \rho_\varphi(x', s')\).

(2) For any \(x^* \in X\) we set
\[
G(x^*) := \{(g, t) \in G \times [0, 1]; \varphi(g, t) = x^*\}.
\]

If \(G(x^*) \neq \emptyset\), there exists \((x, s) \in G(x^*)\) such that \(\rho_\varphi(x)\) attains the value
\[
a(x^*) := \inf\{\rho_\varphi(g); (g, t) \in G(x^*)\}.
\]

Let us compute \(R\pi_{\eta}^*\mathbb{Z}_{G\cap \hat{U}}\). If \(x^* \notin U\), clearly we have \((R\pi_{\eta}^*\mathbb{Z}_{G\cap \hat{U}})_x^* = 0\). Hence in what follows, we assume \(x^* \in U\), in particular, \(\pi_{\eta}^{-1}(x^*) \cap \hat{U} \neq \emptyset\) holds thanks to the assumption. Then it follows from the definition and the properties above that if \(\pi_{\eta}^{-1}(x^*) \cap \hat{G} \neq \emptyset\),
\[
\pi_{\eta}^{-1}(x^*) \cap \hat{G} \simeq \{\eta \in \mathbb{C}; a(x^*) \leq 0|\eta|\}.
\]

We also have
\[
\pi_{\eta}^{-1}(x^*) \cap \hat{U} \simeq \{\eta \in S; \rho_\varphi(x^*) < 0|\eta|\}.
\]

By taking these observations into account, we can calculate \((R\pi_{\eta}^*\mathbb{Z}_{G\cap \hat{U}})_x^*\), for \(x^* \in U\) as follows. If \(x^* \in G\), we see that the subset \(\pi_{\eta}^{-1}(x^*) \cap \hat{G} \neq \emptyset\) and we get
\[
\pi_{\eta}^{-1}(x^*) \cap \hat{G} \simeq \{\eta \in \mathbb{C}; \rho_\varphi(x^*) \leq 0|\eta|\}
\]
because of \((x^*, 1) \in G(x^*)\) and
\[
\rho_\varphi(x^*) = \rho_\varphi(x^*, 1) = \rho_\varphi(x, s) \leq \rho_\varphi(x, 1) = \rho_\varphi(x)
\]
for any \((x, s) \in G(x^*)\). Hence we have
\[
\pi_{\eta}^{-1}(x^*) \cap \hat{G} \cap \hat{U} \simeq \{\eta \in S; \rho_\varphi(x^*) < 0|\eta|\},
\]
which implies
\[
(R\pi_{\eta}^*\mathbb{Z}_{G\cap \hat{U}})_x^* = R\Gamma_c(\pi_{\eta}^{-1}(x^*) \cap \hat{G} \cap \hat{U}; \mathbb{Z}_{\hat{X}}) = \mathbb{Z}[-2].
\]

On the other hand, if \(x^* \notin G\), we obtain
\[
(R\pi_{\eta}^*\mathbb{Z}_{G\cap \hat{U}})_x^* = R\Gamma_c(\pi_{\eta}^{-1}(x^*) \cap \hat{G} \cap \hat{U}; \mathbb{Z}_{\hat{X}}) = 0.
\]

As a matter of fact, if \(\pi_{\eta}^{-1}(x^*) \cap \hat{G} = \emptyset\), the claim clearly holds. Otherwise, we have \(\rho_\varphi(x^*) < a(x^*)\) which can be shown by the following argument. Let \((x, s)\) be a point in \(G(x^*)\) with \(\rho_\varphi(x) = a(x^*)\). Since \(x^* \notin G\), \(x \in G\) and \(x^* = \varphi(x, s)\), we have \(x \notin Z\) and \(s < 1\). From these facts,
\[
\rho_\varphi(x^*) = \rho_\varphi(x^*, 1) = \rho_\varphi(x, s) < \rho_\varphi(x, 1) = \rho_\varphi(x) = a(x^*)
\]
follows. Hence we have
\[ \pi^{-1}_\eta(x^*) \cap \bar{G} \cap \tilde{U} \simeq \{ \eta \in S; a(x^*) \leq \varrho|\eta| \}, \]
which implies the claim.

Summing up, we have obtained
\[ (R\pi_\eta \mathcal{Z}_{G, \tilde{U}})_{x^*} = \begin{cases} \mathbb{Z}[-2] & (x^* \in G \cap U), \\ 0 & \text{(otherwise)}, \end{cases} \]

hence (1.3) is an isomorphism. This completes the proof.

\[ \square \]

1.3. Remark. Without (1.2), we have the following claim by the same argument as that in the proof above: Set \( U' := \{ x \in U; \rho_x(x) < \varrho r \} \). Then there exists the canonical isomorphism
\[ R\Gamma_{G, \tilde{U}}(U'; \mathcal{F}) \simeq R\Gamma_{G, \tilde{U}}(\tilde{U}; \pi^{-1}_\eta \mathcal{F}). \]

§2. Holomorphic Microfunctions with an Apparent Parameter

Let \( X \) be an \( n \)-dimensional \( \mathbb{C} \)-vector space with the coordinates \( z = (z_1, \ldots, z_n) \), and \( Y \) the closed complex submanifold of \( X \) defined by \( \{ z' = 0 \} \) where \( z = (z', z'') \) with \( z' := (z_1, \ldots, z_d) \) for some \( 1 \leq d \leq n \). Set \( \tilde{X} := X \times \mathbb{C} \), and let \( \pi_\eta: \tilde{X} \ni (z, \eta) \mapsto z \in X \) be the canonical projection as in Section 1. In what follows, we denote an object defined on the space \( \tilde{X} \) by a symbol with \( \tilde{\circ} \) like \( \tilde{U}_\kappa \) etc. For any \( z \in \mathbb{C}^n \), we set \( ||z|| := \max_{1 \leq i \leq n} |z_i| \). Let \( \mathcal{O}_X \) be the sheaf of holomorphic functions on \( X \), and \( \mathcal{E}^\mathbb{R}_{Y|X} \) the sheaf of real holomorphic microfunctions along \( Y \) on the conormal bundle \( T^*_YX \) to \( Y \). Let \( z_0 = (0, z''_0) \in Y \) and \( z_0^* = (z''_0, \zeta_0') \in T^*_YX \) with \( |\zeta_0'| = 1 \). Set
\[ f_1(z) := \langle z', \zeta_0' \rangle, \quad f'(z) := z' - \langle z', \zeta_0' \rangle \zeta_0'. \]

2.1. Remark. The subsequent arguments can be applied to a general family of a function \( f_1(z) \) and a mapping \( f'(z) \), that enables us to develop the theory not only on a vector space but also on a complex manifold. It is, however, rather technical. Hence we put such a generalization in Appendix B.

By the definition of \( \mathcal{E}^\mathbb{R}_{Y|X} \), we have
\[ \mathcal{E}^\mathbb{R}_{Y|X, z_0} = \lim_{\varrho \to 0} H^d_{G_{\varrho, L} \cap U}(U; \mathcal{O}_X). \]

Here \( U \subset X \) ranges through open neighborhoods of \( z_0 \), and \( G_{\varrho, L} \) denotes the closed set
\[ G_{\varrho, L} := \{ z \in X; \varrho^2 |f'(z)| \leq |f_1(z)|, f_1(z) \in L \}, \]
where \( L \subset \mathbb{C} \) ranges through closed convex cones with \( L \subset \{ \tau \in \mathbb{C}; \text{Re} \tau > 0 \} \cup \{0\} \). Now we apply the result in the previous section to the case above. We take the open sector \( S_{r, \theta} \) defined by
\[ S_{r, \theta} := \{ \eta \in \mathbb{C}; |\arg \eta| < \theta, 0 < |\eta| < r \} \]
for \( 0 < \theta < \pi/2 \) and \( r > 0 \) as an \( S \) in the previous section. The vector \( \zeta \) is taken to be the image of \( \zeta_0' \) by the canonical mapping \( (T^*_YX)_{z_0} \to (T^*_X)_{z_0} = \mathbb{C}^n \). We adopt the deformation mapping given in Example 1.1 and assume that \( U \) is sufficiently small so
that the assumption of Proposition 1.2 is satisfied. Therefore there exists the canonical isomorphism

\[ R^\Gamma_{G_{e,L}\cap U}(U; \mathcal{O}_X) \cong R^\Gamma_{G_{e,L}\cap \widehat{U}_{e,r,\theta}}(\widehat{U}_{e,r,\theta}; \pi_\eta^{-1}\mathcal{O}_X), \]

where \( \widehat{G}_{e,L} \) and \( \widehat{U}_{e,r,\theta} \) are defined by (1.1) with respect to \( G = G_{e,L} \) and \( U \). By easy computations, these sets are given by

\[ \hat{G}_{e,L} = \{(z, \eta) \in \hat{X}; g|f'(z)| \leq |\eta|, f_1(z) \in L\}, \]
\[ \hat{U}_{e,r,\theta} = \{(z, \eta) \in U \times S_{r,\theta}; |f_1(z)| < g|\eta|\}, \]

respectively. Thus, from the exact sequence

\[ 0 \to \pi_\eta^{-1}\mathcal{O}_X \to \mathcal{O}_{\hat{X}} \xrightarrow{\partial \eta} \mathcal{O}_{\hat{X}} \to 0, \]

we obtain the following distinguished triangle:

\[ R^\Gamma_{G_{e,L}\cap U}(U; \mathcal{O}_X) \to R^\Gamma_{G_{e,L}\cap \widehat{U}_{e,r,\theta}}(\widehat{U}_{e,r,\theta}; \mathcal{O}_{\hat{X}}) \xrightarrow{\partial \eta} R^\Gamma_{G_{e,L}\cap \widehat{U}_{e,r,\theta}}(\widehat{U}_{e,r,\theta}; \mathcal{O}_{\hat{X}}) \xrightarrow{+1}. \]

We will see later the fact

\[ \lim_{\theta, r, \theta, L, U} H^k_{G_{e,L}\cap \widehat{U}_{e,r,\theta}}(\widehat{U}_{e,r,\theta}; \mathcal{O}_{\hat{X}}) = 0 \quad (k \neq d). \]

Hence we have reached the following definition and theorem.

2.2. Definition. We define

\[ \hat{C}^R_{Y|X,z_0^*} := \lim_{e,r,\theta, L, U} H^d_{G_{e,L}\cap \widehat{U}_{e,r,\theta}}(\widehat{U}_{e,r,\theta}; \mathcal{O}_{\hat{X}}), \]

where \( U \subset X \) and \( L \subset \mathbb{C} \) range through open neighborhoods of \( z_0 \) and closed convex cones in \( \mathbb{C} \) with \( L \subset \{ \tau \in \mathbb{C}; \text{Re} \tau > 0 \} \cup \{0\} \) respectively, and the subsets \( \widehat{U}_{e,r,\theta} \) and \( \hat{G}_{e,L} \) are given in (2.1). Further we define

\[ C^R_{Y|X,z_0^*} := \text{Ker}(\partial \eta; \hat{C}^R_{Y|X,z_0^*} \to \hat{C}^R_{Y|X,z_0^*}). \]

Therefore, we obtain:

2.3. Theorem. There exists the following canonical isomorphism

\[ C^R_{Y|X,z_0^*} \cong C^R_{Y|X,z_0^*}. \]

Let us show (2.2). We may assume \( z_0^* = (z_0'' = (z_0', \zeta_0') = (0; 1, 0, \ldots, 0) \). Let \( \kappa := (r, r', \varrho, \theta) \in \mathbb{R}^4 \) be a 4-tuple of positive constants with

\[ 0 < \theta < \frac{\pi}{2}, \quad 0 < \varrho < 1, \quad 0 < r < \varrho r'. \]

Then we set

\[ S_\kappa := \{ \eta \in \mathbb{C}; 0 < |\eta| < r, |\arg \eta| < \frac{\theta}{4} \} \]

and define the open subset

\[ \hat{U}_\kappa := \bigcap_{i=2}^n \{ (z, \eta) \in X \times S_\kappa; |z_1| < \varrho|\eta|, |z_i| < r' \}. \]
We also define the closed cone
\[ \tilde{G}_\kappa := \bigcap_{i=2}^{d} \{(z, \eta) \in \tilde{X}; |\arg z_1| \leq \frac{\pi}{2} - \theta, \varrho|z_i| \leq |\eta|\}. \]

By using these subsets, we introduce objects corresponding to \( \tilde{C}_{Y|X,z_0^*}^R \) and \( C_{Y|X,z_0^*}^R \) at \( z_0^* \), which are easily manipulated by Čech cohomology groups.

2.4. Definition. We define
\[ \tilde{C}_{Y|X}^R(\kappa) := H^d_{\tilde{G}_\kappa \cap \tilde{U}_\kappa}(\tilde{U}_\kappa; \mathcal{O}_{\tilde{X}}), \]
\[ C_{Y|X}^R(\kappa) := \ker(\partial_\eta; \tilde{C}_{Y|X}(\kappa) \to \tilde{C}_{Y|X}(\kappa)). \]

Clearly we have
\[ \tilde{C}_{Y|X,z_0}^R = \lim_{\kappa} \tilde{C}_{Y|X}(\kappa) \text{ and } C_{Y|X,z_0}^R = \lim_{\kappa} C_{Y|X}(\kappa), \]
since families of closed cones and open subsets appearing in inductive limits of the both sides are equivalent with respect to inclusion of sets.

2.5. Proposition. If \( k \neq d \), then
\[ H^k_{\tilde{G}_\kappa \cap \tilde{U}_\kappa}(\tilde{U}_\kappa; \mathcal{O}_{\tilde{X}}) = 0. \]

In particular, (2.2) holds.

Proof. We set
\[ \tilde{V}_\kappa^{(i)} := \{(z, \eta) \in \tilde{U}_\kappa; \frac{\pi}{2} - \theta < \arg z_1 < \frac{3\pi}{2} + \theta\}, \]
\[ \tilde{V}_\kappa^{(i)} := \{(z, \eta) \in \tilde{U}_\kappa; \varrho|z_i| > |\eta|\} \quad (2 \leq i \leq d). \]

Since each \( \tilde{V}_\kappa^{(i)} \) is pseudoconvex and \( \tilde{U}_\kappa \setminus \tilde{G}_\kappa = \bigcup_{i=1}^{d} \tilde{V}_\kappa^{(i)} \), we have \( H^k_{\tilde{G}_\kappa \cap \tilde{U}_\kappa}(\tilde{U}_\kappa; \mathcal{O}_{\tilde{X}}) = 0 \) for \( k > d \). Let us show the assertion for \( k < d \). As \( \varrho < 1 \) and \( r < \varrho r' \) hold, we have
\[ R\Gamma_{\tilde{G}_\kappa \cap \tilde{U}_\kappa}(\tilde{U}_\kappa; \mathcal{O}_{\tilde{X}}) \simeq R\Gamma_{\tilde{G}_\kappa \cap \tilde{K}}(\tilde{K}; \mathcal{O}_{\tilde{X}}), \]
where
\[ \tilde{K} := \bigcap_{i=d+1}^{n} \{(z, \eta) \in X \times S_\kappa; |z_1| < \varrho|\eta|, |z_i| < r'\}. \]

Let us consider the holomorphic mapping on \( \tilde{X} \) defined by
\[ \varphi(z, \eta) := (z_1, \eta z_2, \ldots, \eta z_d, z'' \wedge, \eta). \]

Since \( \varphi \) is bi-holomorphic on \( X \times \mathbb{C}^\times \), we have
\[ R\Gamma_{\tilde{G}_\kappa \cap \tilde{K}}(\tilde{K}; \mathcal{O}_{\tilde{X}}) \simeq R\Gamma_{\tilde{K} \cap \tilde{K}}(\tilde{K} \setminus \tilde{K}; \mathcal{O}_{\tilde{X}}). \]

Here we set \( \tilde{K} := \tilde{K}_1 \cap \tilde{K}_2 \) with
\[ \tilde{K}_1 := \{(z, \eta) \in \tilde{X}; |\arg z_1| \leq \frac{\pi}{2} - \theta\}, \quad \tilde{K}_2 := \bigcap_{i=2}^{d} \{(z, \eta) \in \tilde{X}; \varrho|z_i| \leq 1\}. \]

Then we have the distinguished triangle
\[ R\Gamma_{\tilde{K} \cap \tilde{K}}(\tilde{K}; \mathcal{O}_{\tilde{X}}) \to R\Gamma_{\tilde{K} \cap \tilde{K}}(\tilde{K} \setminus \tilde{K}_1; \mathcal{O}_{\tilde{X}}) \to R\Gamma_{\tilde{K} \cap \tilde{K}_1}(\tilde{K} \setminus \tilde{K}_1; \mathcal{O}_{\tilde{X}}) \to R\Gamma_{\tilde{K} \cap \tilde{K}_1}(\tilde{K} \setminus \tilde{K}_1; \mathcal{O}_{\tilde{X}}) \to R\Gamma_{\tilde{K} \cap \tilde{K}_2}(\tilde{K} \setminus \tilde{K}_2; \mathcal{O}_{\tilde{X}}) \to R\Gamma_{\tilde{K} \cap \tilde{K}_2}(\tilde{K} \setminus \tilde{K}_2; \mathcal{O}_{\tilde{X}}) \to R\Gamma_{\tilde{K} \cap \tilde{K}_2}(\tilde{K} \setminus \tilde{K}_2; \mathcal{O}_{\tilde{X}}) \to R\Gamma_{\tilde{K} \cap \tilde{K}_2}(\tilde{K} \setminus \tilde{K}_2; \mathcal{O}_{\tilde{X}}). \]

Hence the claim of the proposition follows from the following well-known lemma. ∎
2.6. Lemma. Let $D$ be a closed disk with positive radius in $C$ and $U$ a pseudoconvex open subset in $C^n$. Then

$$H^\nu_{D^k \times U}(C^k \times U; \mathcal{O}_{C^{k+m}}) = 0 \quad (\nu \neq k).$$

Furthermore, for any pseudoconvex open subsets $U_1 \subset U_2$ in $C^n$ which are non-empty and connected, the following canonical morphism is injective:

$$H^k_{D^k \times U_2}(C^k \times U_2; \mathcal{O}_{C^{k+m}}) \rightarrow H^k_{D^k \times U_1}(C^k \times U_1; \mathcal{O}_{C^{k+m}}).$$

Next, we set

$$U_k := \bigcap_{i=2}^n \{ z \in X; |z_1| < r^i, |z_i| < r^i \},$$

$$G_k := \bigcap_{i=2}^d \{ z \in X; |\arg z_1| \leq \frac{\pi}{2} - \theta, \theta^2|z_i| \leq |z_1| \}.$$ 

2.7. Corollary. If $k \neq d$, then

$$H^k_{G_k \cap U_k}(U_k; \mathcal{O}_X) = 0,$$

and there exists the following exact sequence:

$$0 \rightarrow H^d_{G_k \cap U_k}(U_k; \mathcal{O}_X) \rightarrow \tilde{G}_Y^R[X](\kappa) \xrightarrow{\partial_n} \tilde{C}_Y^R[X](\kappa) \rightarrow 0.$$ 

Proof. We set

$$V_{(1)}^\kappa := \{ z \in U_k; \frac{\pi}{2} - \theta < \arg z_1 < \frac{3\pi}{2} + \theta \},$$

$$V_{(i)}^\kappa := \{ z \in U_k; \theta^2|z_i| > |z_1| \} \quad (2 \leq i \leq d).$$

Since each $V_{(i)}^\kappa$ is pseudoconvex and $U_k \setminus G_k = \bigcup_{i=1}^d V_{(i)}^\kappa$, we have $H^k_{G_k \cap U_k}(U_k; \mathcal{O}_X) = 0$ for $k > d$. By Proposition 1.2 and Remark 1.3, we have the following distinguished triangle

$$R\Gamma_{G_k \cap U_k}(U_k; \mathcal{O}_X) \rightarrow R\Gamma_{\widehat{G}_k \cap \widehat{U}_k}(\widehat{U}_k; \mathcal{O}_X) \xrightarrow{\partial_n} R\Gamma_{G_k \cap U_k}(\widehat{U}_k; \mathcal{O}_X) \xrightarrow{\Gamma} .$$

By Definition 2.4 and Proposition 2.5, we have (2.6) and $H^k_{G_k \cap U_k}(U_k; \mathcal{O}_X) = 0$ for $k < d$. 

Note that, since

$$\widehat{U}_k \subset \pi^{-1}_\eta(U_k), \quad \pi^{-1}_\eta(G_k) \cap \widehat{U}_k \subset \widehat{G}_k \cap \widehat{U}_k,$$

the morphism $H^d_{G_k \cap U_k}(U_k; \mathcal{O}_X) \rightarrow \tilde{C}_Y^R[X](\kappa)$ is defined by a natural way associated with inclusion of sets. By Proposition 2.5 and (2.6), we obtain the following corollary.

2.8. Corollary. Let $z_0^\kappa = (0; 1, 0, \ldots, 0)$. Then there exist isomorphisms

$$H^d_{G_k \cap U_k}(U_k; \mathcal{O}_X) \xrightarrow{\sim} \tilde{C}_Y^R[X](\kappa) \downarrow \tilde{C}_Y^R[X,z_0^\kappa] \xrightarrow{\sim} \lim_{\kappa} \tilde{C}_Y^R[X](\kappa).$$
We now consider a Čech representation of $C^\infty_{Y|X}(\kappa)$. Recall $\hat{V}^{(i)}_\kappa \subset \hat{X}$ of (2.5) and $V^{(i)}_\kappa \subset X$ of (2.7) for $1 \leq i \leq d$. Let $\mathcal{P}_d$ be the set of all the subsets of $\{1, \ldots, d\}$ and $\mathcal{P}_d^\vee \subset \mathcal{P}_d$ consisting of $\alpha \in \mathcal{P}_d$ with $\#\alpha = d - 1$ ($\#\alpha$ denotes the number of elements in $\alpha$). For $\alpha \in \mathcal{P}_d$, we define

$$
\hat{V}^{(\alpha)}_\kappa := \bigcap_{i \in \alpha} \hat{V}^{(i)}_\kappa, \quad V^{(\alpha)}_\kappa := \bigcap_{i \in \alpha} V^{(i)}_\kappa.
$$

In what follows, the symbol $\ast$ denotes the set $\{1, \ldots, d\}$ by convention, for example,

$$
\hat{V}^{(\ast)}_\kappa := \hat{V}^{(1, \ldots, d)}_\kappa = \bigcap_{i=1}^d \hat{V}^{(i)}_\kappa.
$$

As each $\hat{V}^{(\alpha)}_\kappa$ (resp. $V^{(\alpha)}_\kappa$) is pseudoconvex, we have

$$
\hat{\mathcal{C}}^\infty_{Y|X}(\kappa) = \Gamma(\hat{V}^{(\ast)}_\kappa; \mathcal{O}_{\hat{X}}) / \sum_{\alpha \in \mathcal{P}_d^\vee} \Gamma(\hat{V}^{(\alpha)}_\kappa; \mathcal{O}_{\hat{X}}),
$$

$$
\mathcal{C}^\infty_{Y|X}(\kappa) = \{ u \in \hat{\mathcal{C}}^\infty_{Y|X}(\kappa); \partial_1 u = 0 \},
$$

$$
H^d_{G_\kappa \cap U_{\kappa}}(U_{\kappa}; \mathcal{O}_X) = \Gamma(V^{(\ast)}_\kappa; \mathcal{O}_X) / \sum_{\alpha \in \mathcal{P}_d^\vee} \Gamma(V^{(\alpha)}_\kappa; \mathcal{O}_X).
$$

Since $\hat{V}^{(\alpha)}_\kappa \subset \pi^{-1}(V^{(\alpha)}_\kappa)$ holds, we can regard a holomorphic function $\varphi$ on $V^{(\alpha)}_\kappa$ as that on $\hat{V}^{(\alpha)}_\kappa$, and thus, we have the natural morphism $\Gamma(V^{(\alpha)}_\kappa; \mathcal{O}_X) \to \Gamma(\hat{V}^{(\alpha)}_\kappa; \mathcal{O}_{\hat{X}})$. This induces the canonical morphism between the Čech cohomology groups

$$
H^d_{G_\kappa \cap U_{\kappa}}(U_{\kappa}; \mathcal{O}_X) = \frac{\Gamma(V^{(\ast)}_\kappa; \mathcal{O}_X)}{\sum_{\alpha \in \mathcal{P}_d^\vee} \Gamma(V^{(\alpha)}_\kappa; \mathcal{O}_X)} \to \{ u \in \frac{\Gamma(\hat{V}^{(\ast)}_\kappa; \mathcal{O}_{\hat{X}})}{\sum_{\alpha \in \mathcal{P}_d^\vee} \Gamma(\hat{V}^{(\alpha)}_\kappa; \mathcal{O}_{\hat{X}})}; \partial_1 u = 0 \} = C^\infty_{Y|X}(\kappa).
$$

Clearly this morphism coincides with (2.8), hence it gives an isomorphism by Corollary 2.8.

§ 3. COHOMOLOGICAL REPRESENTATION OF $\mathcal{E}^\infty_{X, \kappa}$ WITH AN APPARENT PARAMETER

We inherit the same notation from the previous section. Set $X^2 := X \times X$ with the coordinates $(z, w)$, and let $(z, w, \eta)$ be coordinates of $\tilde{X}^2 := X^2 \times \mathbb{C}$. Let $\Delta \subset X^2$ be the diagonal set. We identify $X$ with $\Delta$, and

$$
T^*X = \{(z; \zeta) \} \simeq \{(z, z; \zeta, -\zeta)\} = T^*_\Delta X^2.
$$

Let $\mathcal{E}^\infty_{X}$ denote the sheaf of pseudodifferential operators on the cotangent bundle $T^*X$ of $X$, and $z^*_0 = (z_0; \zeta_0) \in T^*X$ with $|\zeta_0| = 1$. Set

$$
f_{\Delta,1}(z, w) := \langle z - w, \zeta_0 \rangle, \quad f^*_\Delta(z, w) := z - w - \langle z - w, \zeta_0 \rangle \overline{\zeta_0}.
$$

See also Appendix B for a generalization of the mappings above and the following arguments on a complex manifold. For a closed convex cone $L \subset \mathbb{C}$, set

$$
G_{\Delta, \varphi, L} := \{(z, w) \in X^2; \varphi^2 |f^*_\Delta(z, w)| \leq |f_{\Delta,1}(z, w)|, f_{\Delta,1}(z, w) \in L \}.
$$

Then it follows from the definition of $\mathcal{E}^\infty_{X, \kappa}$ that we have

$$
\mathcal{E}^\infty_{X, z^*_0} = \lim_{\varphi, L, U} H^n_{G_{\Delta, \varphi, L} \cap U}(U; \mathcal{O}^{(0, n)}_{X^2}).
$$
Here $\mathcal{O}^{(0,n)}_{X^2}$ is the sheaf of holomorphic $n$-forms with respect to $dw_1, \ldots, dw_n$, $U \subset X^2$ and $L \subset \mathbb{C}$ range through open neighborhoods of $(z_0, \zeta_0)$ and closed convex cones in $\mathbb{C}$ with $L \subset \{ \tau \in \mathbb{C}; \Re \tau > 0 \} \cup \{0\}$ respectively.

Now we introduce the corresponding cohomology group with an apparent parameter. Set, for an open subset $U \subset X^2$ and a closed convex cone $L \subset \mathbb{C}$,

$$\hat{U}_{\Delta, \varrho, \rho, \theta} := \{(z, w, \eta) \in U \times S_{\tau, \varrho}; |f_{\Delta, 1}(z, w)| < \varrho|\eta|\},$$

$$\hat{G}_{\Delta, \varrho, \rho, \theta} := \{(z, w, \eta) \in \hat{X}^2; \varrho |f'_{\Delta, 1}(z, w)| \leq |\eta|, f_{\Delta, 1}(z, w) \in L\}.$$

3.1. Definition. We set

$$\hat{E}^{\mathbb{R}}_{\hat{X}, z_0^*} := \lim_{\varrho, \rho, \theta, \Delta, L \to \Delta, \varrho, \rho, \theta} H^n_{\hat{G}_{\Delta, \varrho, \rho, \theta} \cap \hat{U}_{\Delta, \varrho, \rho, \theta}}(\hat{U}_{\Delta, \varrho, \rho, \theta}; \mathcal{O}^{(0,n,0)}_{\hat{X}^2}).$$

Here $\mathcal{O}^{(0,n,0)}_{\hat{X}^2}$ is the sheaf of holomorphic $n$-forms with respect to $dw_1, \ldots, dw_n$, $U \subset X^2$ and $L \subset \mathbb{C}$ range through open neighborhoods of $(z_0, \zeta_0)$ and closed convex cones in $\mathbb{C}$ with $L \subset \{ \tau \in \mathbb{C}; \Re \tau > 0 \} \cup \{0\}$ respectively. Further we define

$$E^{\mathbb{R}}_{\hat{X}, z_0^*} := \text{Ker}(\partial_{\eta}; \hat{E}^{\mathbb{R}}_{\hat{X}, z_0^*} \to \hat{E}^{\mathbb{R}}_{\hat{X}, z_0^*}).$$

From the consequence of the previous section, the following theorem immediately follows.

3.2. Theorem. There exists the canonical isomorphism

$$\mathcal{O}^{\mathbb{R}}_{\hat{X}, z_0^*} \simeq E^{\mathbb{R}}_{\hat{X}, z_0^*}.$$

We assume $z_0^* = (z_0; \zeta_0) = (0; 1, 0, \ldots, 0)$ in what follows and consider a Čech representation of $E^{\mathbb{R}}_{\hat{X}, z_0^*}$. Let $\kappa = (r, r', \varrho, \theta) \in \mathbb{R}^4$ be parameters satisfying the conditions (2.3). Then we define

$$\hat{U}_{\Delta, \kappa} := \bigcap_{i=2}^n \{(z, w, \eta) \in \hat{X}^2; \|z\| < r', \eta \in S_\kappa; |z_1 - w_1| < \varrho|\eta|, |z_i - w_i| < r'\},$$

$$\hat{G}_{\Delta, \kappa} := \bigcap_{i=2}^n \{(z, w, \eta) \in \hat{X}^2; |\arg(z_1 - w_1)| \leq \frac{\pi}{2} - \theta, \varrho |z_i - w_i| \leq |\eta|\}.$$

We also set

$$U_{\Delta, \kappa} := \bigcap_{i=2}^n \{(z, w) \in X^2; \|z\| < r', |z_1 - w_1| < \varrho r, |z_i - w_i| < r'\},$$

$$G_{\Delta, \kappa} := \bigcap_{i=2}^n \{(z, w) \in X^2; |\arg(z_1 - w_1)| \leq \frac{\pi}{2} - \theta, \varrho^2 |z_i - w_i| \leq |z_1 - w_1|\}.$$

3.3. Definition. We define

$$\hat{E}^{\mathbb{R}}_{X}(\kappa) := H^n_{\hat{G}_{\Delta, \kappa} \cap \hat{U}_{\Delta, \kappa}}(\hat{U}_{\Delta, \kappa}; \mathcal{O}^{(0,n,0)}_{\hat{X}^2}),$$

$$E^{\mathbb{R}}_{X}(\kappa) := \text{Ker}(\partial_{\eta}; \hat{E}^{\mathbb{R}}_{X}(\kappa) \to \hat{E}^{\mathbb{R}}_{X}(\kappa)).$$

Note that

$$\hat{E}^{\mathbb{R}}_{X, z_0^*} = \lim_{\kappa \to \hat{X}} \hat{E}^{\mathbb{R}}_{X}(\kappa) \quad \text{and} \quad E^{\mathbb{R}}_{X, z_0^*} = \lim_{\kappa \to X} E^{\mathbb{R}}_{X}(\kappa).$$
hold. Then by employing the coordinates transformation \((z, w) \mapsto (z, z - w)\), it follows from Proposition 2.5, Corollaries 2.7 and 2.8 that the both complexes

\[
RG_{G,\Delta,\kappa} \cap U_{\Delta,\kappa} (\hat{U}_{\Delta,\kappa}; \mathcal{O}_{X^2}^{(0,n,0)}),
\]

\[
RG_{G,\Delta,\kappa} \cap U_{\Delta,\kappa} (U_{\Delta,\kappa}; \mathcal{O}_{X^2}^{(0,n)}) \simeq RG_{G,\Delta,\kappa} \cap U_{\Delta,\kappa} (\hat{U}_{\Delta,\kappa}; R\mathcal{H}om_{\mathcal{O}_{X^2}}(\mathcal{O}_{X^2}/\mathcal{O}_{X^2}\partial_{\eta}, \mathcal{O}_{X^2}^{(0,n,0)}))
\]

are concentrated in degree \(n\), and we have the canonical isomorphism

\[
H^n_{G,\Delta,\kappa} \cap U_{\Delta,\kappa} (U_{\Delta,\kappa}; \mathcal{O}_{X^2}^{(0,n)}) \simeq E^n_{X}(\kappa).
\]

Furthermore we have

\[
\hat{\mathcal{O}}_{X^{\ast}} = \lim_{\kappa} H^n_{G,\Delta,\kappa} \cap U_{\Delta,\kappa} (U_{\Delta,\kappa}; \mathcal{O}_{X^2}^{(0,n)}),
\]

By these facts, we get

\[
\hat{\mathcal{O}}_{X^{\ast}} = \lim_{\kappa} E^n_{X}(\kappa).
\]

Now we give the Čech representations of these cohomology groups. Recall that the open subset \(\hat{U}_{\Delta,\kappa} \subset \hat{X}^2\) is defined by

\[
\bigcap_{i=2}^{n} \{(z, w, \eta) \in X^2 \times S_\kappa; \|z\| < r', |z_1 - w_1| < \eta, |z_i - w_i| < r'\}.
\]

Here the open sector \(S_\kappa\) was given by (2.4). Set

\[
\hat{V}_{\Delta,\kappa}^{(i)} := \{(z, w, \eta) \in \hat{U}_{\Delta,\kappa}; \frac{\pi}{2} - \theta < \text{arg}(z_1 - w_1) < \frac{3\pi}{2} + \theta\},
\]

\[
\hat{V}_{\Delta,\kappa}^{(i)} := \{(z, w, \eta) \in \hat{U}_{\Delta,\kappa}; \eta |z_i - w_i| > |\eta|\} \quad (2 \leq i \leq n).
\]

We also set

\[
V_{\Delta,\kappa}^{(i)} := \{(z, w) \in U_{\Delta,\kappa}; \frac{\pi}{2} - \theta < \text{arg}(z_1 - w_1) < \frac{3\pi}{2} + \theta\},
\]

\[
V_{\Delta,\kappa}^{(i)} := \{(z, w) \in U_{\Delta,\kappa}; \eta^2 |z_i - w_i| > |z_1 - w_1|\} \quad (2 \leq i \leq n).
\]

For any \(\alpha \in \mathcal{P}_n\), the subset \(\hat{V}_{\Delta,\kappa}^{(\alpha)}, V_{\Delta,\kappa}^{(\alpha)}\) etc. are defined in the same way as those in (2.9).

Then, using these coverings, we have

\[
\hat{E}_{X}(\kappa) = \Gamma(\hat{V}_{\Delta,\kappa}^{(\ast)}; \mathcal{O}_{X^2}^{(0,n,0)})/ \sum_{\alpha \in \mathcal{P}_n} \Gamma(\hat{V}_{\Delta,\kappa}^{(\alpha)}; \mathcal{O}_{X^2}^{(0,n,0)}),
\]

\[
E_{X}(\kappa) = \{K \in \hat{E}_{X}(\kappa); \partial_{\eta} K = 0\},
\]

\[
H^n_{G,\Delta,\kappa} \cap U_{\Delta,\kappa} (U_{\Delta,\kappa}; \mathcal{O}_{X^2}^{(0,n)}) = \Gamma(V_{\Delta,\kappa}^{(\ast)}; \mathcal{O}_{X^2}^{(0,n)})/ \sum_{\alpha \in \mathcal{P}_n} \Gamma(V_{\Delta,\kappa}^{(\alpha)}; \mathcal{O}_{X^2}^{(0,n)}).
\]

Let us take any \(K(z, w) \, dw = [\psi(z, w, \eta) \, dw] \in E_{X}(\kappa)\) and \(f(z) = [u(z, \eta)] \in C_{Y \mid X}(\kappa)\) with representatives \(\psi(z, w, \eta) \, dw \in \Gamma(\hat{V}_{\Delta,\kappa}^{(\ast)}; \mathcal{O}_{X^2}^{(0,n,0)})\) and \(u(z, \eta) \in \Gamma(\hat{V}_{\Delta,\kappa}^{(\ast)}; \mathcal{O}_{X^2})\) respectively, which were introduced in the previous section. We will define the action \(\mu_{K}\) on \(C_{Y \mid X}(\kappa)\) associated with the kernel \(K(z, w) \, dw\). For that purpose, we first introduce the paths of the integration related to \(\mu_{K}\). Let \((z, \eta) \in \hat{X}\). Set \(\beta_0 := \frac{\theta}{2} e^{-\sqrt{\pi}(\pi + \theta)/2}\) and...
For any \( z, \eta; \varrho, \theta \) we set
\[
\gamma(z, \eta; \varrho, \theta) := \{ w_i = z_i + \left( \frac{|\eta|}{\varrho} + \varepsilon \right) e^{2\pi i t} : 0 \leq t \leq 1 \}.
\]

Define the real \( n \)-dimensional chain in \( X \) made from these paths by
\[
\gamma(z, \eta; \varrho, \theta) := \gamma_1(z, \eta; \varrho, \theta) \times \gamma_2(z, \eta; \varrho) \times \cdots \times \gamma_n(z, \eta; \varrho) \subset X,
\]
\[
\overline{\gamma}(z, \eta; \varrho, \theta) := \overline{\gamma}_1(z, \eta; \varrho, \theta) \times \overline{\gamma}_2(z, \eta; \varrho) \times \cdots \times \overline{\gamma}_n(z, \eta; \varrho) \subset X.
\]

Let \( \pi_2 : \hat{X}^2 \ni (z, w, \eta) \mapsto (w, \eta) \in \hat{X} \) be the canonical projection. For \( \alpha \in \mathcal{P}_n \) and \( \beta \in \mathcal{P}_d \), we set
\[
\overline{W}^{(\alpha, \beta)}_\kappa := \overline{W}^{(\alpha)}_\kappa \cap \pi_2^{-1}(\overline{W}^{(\beta)}_\kappa),
\]
\[
\overline{W}^{(\ast, \ast)}_\kappa := \overline{W}^{(\ast)}_\kappa \cap \pi_2^{-1}(\overline{W}^{(\ast)}_\kappa).
\]
We also set \( \overline{W}^{(\alpha, \ast)}_\kappa := \overline{W}^{(\alpha, (1, \ldots, d))}_\kappa \) and \( \overline{W}^{(\ast, \beta)}_\kappa := \overline{W}^{((1, \ldots, n), \beta)}_\kappa \). Then the following lemma is easily obtained by elementary computations.

3.4. Lemma. Let \( \kappa = (\hat{\varrho}, \hat{\varrho}', \hat{\rho}, \hat{\theta}) \in \mathbb{R}^4 \) satisfying
\[
0 < \hat{\varrho} = \hat{r}, \quad 0 < \hat{\varrho}' < \frac{\hat{r}'}{2}, \quad 0 < \hat{\theta} < \frac{\theta}{4}, \quad 0 < \hat{\rho} < \frac{\rho}{2} \sin \frac{\theta}{4},
\]
and the corresponding conditions to (2.3). Then the following hold for sufficiently small \( \varepsilon > 0 \):

1. For any \( (z, \eta) \in \hat{W}^{(\ast)}_\kappa \), in \( \hat{X}^2 \)
\[
\{ z \} \times \gamma(z, \eta; \varrho, \theta) \times \{ \eta \} \subset \hat{W}^{(\ast, \ast)}_\kappa.
\]
Here \( \{ z \} \times \gamma(z, \eta; \varrho, \theta) \times \{ \eta \} \) denotes the product of these three subsets in \( \hat{X}^2 = X \times X \times \mathbb{C} \).

2. For any \( (z, \eta) \in \hat{W}^{(\beta)}_\kappa \) with \( \beta \in \mathcal{P}_d \),
\[
\{ z \} \times \gamma(z, \eta; \varrho, \theta) \times \{ \eta \} \subset \hat{W}^{(\ast, \beta)}_\kappa.
\]

3. For any \( (z, \eta) \in \hat{W}^{((1, \ldots, d))}_\kappa \),
\[
\{ z \} \times \overline{\gamma}(z, \eta; \varrho, \theta) \times \{ \eta \} \subset \hat{W}^{((1, \ldots, n), \ast)}_\kappa,
\]
Furthermore
\[
\{ z \} \times \partial \gamma(z, \eta; \varrho, \theta) \times \{ \eta \} \subset \hat{W}^{(\ast, \ast)}_\kappa,
\]
where \( \partial \gamma(z, \eta; \varrho, \theta) \) denotes the boundary of \( \gamma(z, \eta; \varrho, \theta) \) as a real \( n \)-dimensional chain.
Now we are ready to define the action $\mu_K$ of $K(z,w)dw \in E^\mathbb{R}_X(\kappa)$ on $C^\mathbb{R}_{Y|X}(\kappa)$.

3.5. **Theorem.** The bi-linear morphism

$$
\mu: E^\mathbb{R}_X(\kappa) \otimes C^\mathbb{R}_{Y|X}(\kappa) \to C^\mathbb{R}_{Y|X}(\tilde{\kappa})
$$

defined by

$$
K(z,w)dw \otimes f(z) = [\psi(z,w,\eta)dw] \otimes [u(z,\eta)]
\implies \mu(Kdw \otimes f) := \int_{\gamma(z,\eta;\theta)} \psi(z,w,\eta)u(w,\eta)dw
$$

is well defined. Here $\tilde{\kappa}$ is a 4-tuple of positive constants satisfying the conditions given in Lemma 3.4. In particular, there exists the following linear morphism:

$$
\mu_K: C^\mathbb{R}_{Y|X}(\kappa) \ni f(z) \mapsto \mu(Kdw \otimes f) \in C^\mathbb{R}_{Y|X}(\tilde{\kappa}).
$$

3.6. **Remark.** The same result holds for $\psi(z,w,\tau,\eta)dw$ and $u(w,\tau,\eta)$ with additional holomorphic parameters $\tau$.

**Proof of Theorem 3.5.** For any $\varphi(z,w,\eta) \in \Gamma(\hat{\mathcal{W}}^{(*,*)}_{\kappa}; \hat{\mathcal{O}}_{\hat{X}})$, set

$$
\mu(\varphi)(z,\eta) := \int \varphi(z,w,\eta)dw.
$$

Note that, by Lemma 3.4 (1) we have $\mu(\varphi)(z,\eta) \in \Gamma(\hat{\mathcal{V}}^{(\alpha,\beta)}_{\kappa}; \hat{\mathcal{O}}_{\hat{X}})$.

3.7. **Lemma.** Assume that $\varphi(z,w,\eta) \in \Gamma(\hat{\mathcal{W}}^{(\alpha,\beta)}_{\kappa}; \hat{\mathcal{O}}_{\hat{X}})$ with $\alpha \in \mathcal{P}^\vee_n$ and $\beta = *$ or with $\alpha = *$ and $\beta \in \mathcal{P}^\vee_d$. Then $\mu(\varphi)(z,\eta) \in \Gamma(\hat{\mathcal{V}}^{(\beta)}_{\kappa}; \hat{\mathcal{O}}_{\hat{X}})$ for some $\beta \in \mathcal{P}^\vee_d$.

**Proof.** If $\varphi(z,w,\eta) \in \Gamma(\hat{\mathcal{W}}^{(\alpha,*)}_{\kappa}; \hat{\mathcal{O}}_{\hat{X}})$ for some $\alpha \in \mathcal{P}^\vee_n$, we have

$$
\begin{align*}
\mu(\varphi)(z,\eta) &\in \Gamma(\hat{\mathcal{V}}^{(\alpha)}_{\kappa}; \hat{\mathcal{O}}_{\hat{X}}) & (\alpha = \{2, \ldots, n\}), \\
\mu(\varphi)(z,\eta) &= 0 & \text{(otherwise)}. 
\end{align*}
$$

Here we remark that the first fact comes from Lemma 3.4 (3) by deforming the path of integration to $\gamma(z,\eta;\theta,\theta)$. In the same way, by Lemma 3.4 (2), it follows that if $\varphi(z,w,\eta) \in \Gamma(\hat{\mathcal{W}}^{(*)\beta}_{\kappa}; \hat{\mathcal{O}}_{\hat{X}})$ for some $\beta \in \mathcal{P}^\vee_d$,

$$
\mu(\varphi)(z,\eta) \in \Gamma(\hat{\mathcal{V}}^{(*)\beta}_{\kappa}; \hat{\mathcal{O}}_{\hat{X}}).
$$

It follows from Lemma 3.7 that $\mu$ induces the canonical morphism

$$
\mu: \frac{\Gamma(\hat{\mathcal{W}}^{(*)\alpha}_{\kappa}; \hat{\mathcal{O}}_{\hat{X}})}{\sum_{(\alpha,\beta) \in A} \Gamma(\hat{\mathcal{W}}^{(*)\alpha\beta}_{\kappa}; \hat{\mathcal{O}}_{\hat{X}})} \to \frac{\Gamma(\hat{\mathcal{V}}^{(*)\beta}_{\kappa}; \hat{\mathcal{O}}_{\hat{X}})}{\sum_{\beta \in \mathcal{P}^\vee_d} \Gamma(\hat{\mathcal{V}}^{(*)\beta}_{\kappa}; \hat{\mathcal{O}}_{\hat{X}})} = \hat{\mathcal{C}}^\mathbb{R}_{Y|X}(\tilde{\kappa})
$$

where $A := \{(\alpha, *); \alpha \in \mathcal{P}^\vee_n\} \sqcup \{(*, \beta); \beta \in \mathcal{P}^\vee_d\}$. Furthermore, we have the canonical morphism

$$
E^\mathbb{R}_X(\kappa) \otimes C^\mathbb{R}_{Y|X}(\kappa) \to \frac{\Gamma(\hat{\mathcal{W}}^{(*)\alpha}_{\kappa}; \hat{\mathcal{O}}_{\hat{X}})}{\sum_{(\alpha,\beta) \in A} \Gamma(\hat{\mathcal{W}}^{(*)\alpha\beta}_{\kappa}; \hat{\mathcal{O}}_{\hat{X}})}
$$
by $[\psi(z, w, \eta) \, dw] \otimes [u(z, \eta)] \mapsto [\psi(z, w, \eta) \, u(w, \eta) \, dw]$. Hence we have obtained the morphism

$$
\mu: E_X^\mathbb{R}(\mathcal{K}) \otimes C_Y^\mathbb{R}(\mathcal{K}) \ni [\psi(z, w, \eta) \, dw] \otimes [u(z, \eta)] \mapsto \left[ \int_{\gamma(z, \eta \otimes \theta)} \psi(z, w, \eta) \, u(w, \eta) \, dw \right] \in C_Y^\mathbb{R}(\tilde{\mathcal{K}}).
$$

Thus to complete the proof, it suffices to show the image of $\mu$ is contained in $C_Y^\mathbb{R}(\tilde{\mathcal{K}})$. We have

$$
\begin{align*}
\partial_\eta \int_{\gamma(z, \eta \otimes \theta)} \psi(z, w, \eta) \, u(w, \eta) \, dw &= \int_{\gamma(z, \eta \otimes \theta)} \left[ \tau \psi(z, z_1 + \tau \eta, w', \eta) \, u(z_1 + \tau \eta, w', \eta) \right]_{\tau = \theta_0}^{\beta_1} \, dw_2 \cdots \, dw_n \\
&+ \int \partial_\eta \psi(z, w, \eta) \, u(w, \eta) \, dw + \int \psi(z, w, \eta) \, \partial_\eta u(w, \eta) \, dw.
\end{align*}
$$

By Lemma 3.4 (3), the first term belongs to $\Gamma(V^{(\lambda_1, \ldots, \lambda_n)}_\mathcal{K}; \mathcal{O}_{\tilde{X}})$. For the second and third terms, as each integrand belongs to $\sum_{(\alpha, \beta) \in A} \Gamma(V^{(\alpha, \beta)}_\mathcal{K}; \mathcal{O}_{\tilde{X}})$, the corresponding integral also belongs to $\sum_{\beta \in \beta_d} \Gamma(V^{(\beta)}_\mathcal{K}; \mathcal{O}_{\tilde{X}})$. Hence we have obtained $\partial_\eta \mu([\psi \, dw] \otimes [u]) = 0 \in C_Y^\mathbb{R}(\tilde{\mathcal{K}})$, which implies $\mu([\psi \, dw] \otimes [u]) \in C_Y^\mathbb{R}(\tilde{\mathcal{K}})$. The proof is complete.

As a corollary of the theorem, we have the result on the composition on $E_X^\mathbb{R}(\mathcal{K})$.

**3.8. Corollary.** Let $\tilde{\mathcal{K}} = (\tilde{r}, \tilde{r}', \tilde{\theta}, \tilde{\theta}) \in \mathbb{R}^4$ satisfying

$$
0 < \tilde{r} < r, \quad 0 < \tilde{r}' < \frac{r'}{8}, \quad 0 < \tilde{\theta} < \frac{\theta}{4}, \quad 0 < \tilde{\theta} < \frac{\theta}{2} \sin \frac{\theta}{4},
$$

and the corresponding conditions to (2.3). Then there exists the bi-linear morphism

$$
\mu: E_X^\mathbb{R}(\mathcal{K}) \otimes E_X^\mathbb{R}(\mathcal{K}) \rightarrow E_X^\mathbb{R}(\tilde{\mathcal{K}})
$$

defined by

$$
[\psi_1(z, w, \eta) \, dw] \otimes [\psi_2(z, w, \eta) \, dw] \mapsto \left[ \int_{\gamma(z, \eta \otimes \theta)} \psi_1(z, \tilde{w}, \eta) \, \psi_2(\tilde{w}, w, \eta) \, d\tilde{w} \right] \, dw.
$$

**Proof.** By employing the coordinates transformation $z = z' + w$, $\tilde{w} = \tilde{w}' + w$ and $w = u$, the integration above becomes

$$
\int_{\gamma(z', \eta \otimes \theta)} \psi_1(z' + w, \tilde{w}', w, \eta) \, \psi_2(\tilde{w}' + w, w, \eta) \, d\tilde{w}'.
$$

Then, under the new coordinates $(z', \tilde{w}', w)$, the $\psi_2$ (resp. the result of the integration) can be regarded a holomorphic microfunction along $\{\tilde{w}' = 0\}$ (resp. $\{z' = 0\}$). Hence, by noticing the simple fact that $|w_1| < \frac{r'}{2}$ and $|\tilde{w}_1'| < \frac{r'}{2}$ imply $|\tilde{w}_1'| < r'$, we can easily obtain the result by the theorem.

The following theorem can be shown by the same arguments as in Kashiwara-Kawai [18]. We give the detailed proof in Appendix A for the reader’s convenience. See also Theorem B.8 for the corresponding claim at an arbitrary point in $T^*X$. 
3.9. Theorem. The action
\[ \mathcal{E}_{X,z_0^*}^R \otimes \mathcal{E}_{Y|X,z_0^*}^R = \lim_{\kappa} (E_X^R(\kappa) \otimes C_{Y|X}(\kappa)) \xrightarrow{\mu} \lim_{\kappa} C_{Y|X}(\kappa) = \mathcal{E}_{Y|X,z_0^*}^R \]
coincides with the cohomological action of \( \mathcal{E}_{X,z_0^*}^R \) on \( \mathcal{E}_{Y|X,z_0^*}^R \).

As an immediate corollary, we have:

3.10. Corollary. The multiplication of the ring \( \mathcal{E}_{X,z_0^*}^R \) coincides with the composition defined by
\[ \mathcal{E}_{X,z_0^*}^R \otimes \mathcal{E}_{X,z_0^*}^R = \lim_{\kappa} (E_X^R(\kappa) \otimes E_X^R(\kappa)) \xrightarrow{\mu} \lim_{\kappa} E_X^R(\kappa) = \mathcal{E}_{X,z_0^*}^R. \]

§ 4. Symbols with an Apparent Parameter

Let \( X := \mathbb{C}^n \) and consider \( T^*X \simeq X \times \mathbb{C}^n = \{(z;\zeta)\} \). Let \( \pi: T^*X \rightarrow X \) be the canonical projection. If \( V \subset T^*X \) is a conic set and \( d > 0 \), we set
\[ V[d] := \{(z;\zeta) \in V; \|\zeta\| \geq d\}. \]

For any open conic subset \( \Omega \subset T^*X \) and \( \rho \geq 0 \), we set
\[ \Omega_\rho := \mathrm{Cl}\left[\bigcup_{(z,\zeta) \in \Omega} \{(z + z';\zeta + \zeta') \in \mathbb{C}^{2n}; \|z'\| \leq \rho, \|\zeta'\| \leq \rho\|\zeta\|\}\right]. \]

Here \( \mathrm{Cl} \) means the closure. In particular, \( \Omega_0 = \mathrm{Cl}\Omega \). For any \( d > 0 \) and \( \rho \in [0,1[\), we set for short:
\[ d_\rho := d(1-\rho). \]

Let \( U, V \) be conic subsets of \( T^*X \). Then we write \( V \sqsubset U \) if \( V \) is generated by a compact subset of \( U \). We recall the definition of symbols of \( \mathcal{E}_{X}^R \):

4.1. Definition (see [2], [6]). Let \( \Omega \subset T^*X \) be an open conic subset.

(1) We call \( P(z,\zeta) \) a symbol on \( \Omega \) if there exist \( d > 0 \) and \( \rho \in ]0,1[\) such that \( P(z,\zeta) \in \Gamma(\Omega_\rho[d_\rho]; \mathcal{E}_{T^*X}) \), and for any \( h > 0 \) there exists \( C_h > 0 \) such that
\[ |P(z,\zeta)| \leq C_h e^{h\|\zeta\|} \quad ((z;\zeta) \in \Omega_\rho[d_\rho]). \]

We denote by \( \mathcal{S}(\Omega) \) the set of symbols on \( \Omega \).

(2) We call \( P(z,\zeta) \) a null-symbol on \( \Omega \) if there exist \( d > 0 \) and \( \rho \in ]0,1[\) such that \( P(z,\zeta) \in \Gamma(\Omega_\rho[d_\rho]; \mathcal{E}_{T^*X}) \), and there exist \( C, \delta > 0 \) such that
\[ |P(z,\zeta)| \leq Ce^{-\delta\|\zeta\|} \quad ((z;\zeta) \in \Omega_\rho[d_\rho]). \]

We denote by \( \mathcal{N}(\Omega) \) the set of null-symbols on \( \Omega \).

(3) For any \( z_0^* \in T^*X \), we set
\[ \mathcal{S}_{z_0^*} := \lim_{\Omega \ni z_0^*} \mathcal{S}(\Omega) \supset \mathcal{N}_{z_0^*} := \lim_{\Omega \ni z_0^*} \mathcal{N}(\Omega) \]
where \( \Omega \subset T^*X \) ranges through open conic neighborhoods of \( z_0^* \).
Next, set for short

$$S := S_{\kappa}$$

for some $r, \theta \in ]0, 1/2[$ (recall (2.4)). In particular we always assume that $|\eta| < 1/2$ for any $\eta \in S$. For $Z \subseteq S$, we set $m_Z := \min_{\eta \in Z} |\eta| > 0$.

4.2. Definition. We define a set $\mathcal{H}(\Omega; S)$ as follows: $P(z, \zeta, \eta) \in \mathcal{H}(\Omega; S)$ if

(i) $P(z, \zeta, \eta) \in \Gamma(\Omega_{\rho}[d_{\rho}] \times Z; \sigma_{T^*X \times \mathbb{C}})$ for some $d > 0$ and $\rho \in ]0, 1[$,

(ii) there exists $\delta > 0$ so that for any $Z \subseteq S$, there exists a constant $C_Z > 0$ satisfying

$$|P(z, \zeta, \eta)| \leq C_Z e^{-\delta \|\eta\|} \quad ((z; \zeta, \eta) \in \Omega_{\rho}[d_{\rho}] \times Z).$$

4.3. Lemma. If $P(z, \zeta, \eta) \in \mathcal{H}(\Omega; S)$, it follows that $\partial_\eta P(z, \zeta, \eta) \in \mathcal{H}(\Omega; S)$.

Proof. We assume that $P(z, \zeta, \eta) \in \Gamma(\Omega_{\rho}[d_{\rho}] \times Z; \sigma_{T^*X \times \mathbb{C}})$. For any $Z \subseteq S$, we take $\delta' \in ]0, \delta[\] as

$$Z' := \bigcup_{\eta \in Z} \{\eta' \in \mathbb{C} : |\eta - \eta'| \leq \delta' |\eta|\} \subseteq S.$$ Then by the Cauchy inequality, there exists a constant $C_{Z'} > 0$ such that

$$|\partial_\eta P(z, \zeta, \eta)| \leq \frac{1}{\delta' |\eta|} \sup_{|\eta - \eta'| = \delta' |\eta|} |P(z, \zeta, \eta')| \leq \frac{C_{Z'} e^{-\delta' \|\eta\|/2}}{\delta' m_Z} \quad ((z; \zeta, \eta) \in \Omega_{\rho}[d_{\rho}] \times Z). \quad \square$$

4.4. Proposition. Let $P(z, \zeta, \eta) \in \Gamma(\Omega_{\rho}[d_{\rho}] \times Z; \sigma_{T^*X \times \mathbb{C}})$. Assume that $\partial_\eta P(z, \zeta, \eta) \in \mathcal{H}(\Omega; S)$.

(1) The following conditions are equivalent:

(i) there exists a constant $\nu > 0$ satisfying the following: for any $Z \subseteq S$ there exists a constant $C_Z > 0$ such that

$$|P(z, \zeta, \eta)| \leq C_Z e^{\nu \|\eta\|} \quad ((z; \zeta, \eta) \in \Omega_{\rho}[d_{\rho}] \times Z).$$

(ii) for any $h > 0$ and $Z \subseteq S$ there exists constant $C_{h,Z} > 0$ such that

$$|P(z, \zeta, \eta)| \leq C_{h,Z} e^{h \|\eta\|} \quad ((z; \zeta, \eta) \in \Omega_{\rho}[d_{\rho}] \times Z).$$

(2) Assume that $P(z, \zeta, \eta)$ satisfies the equivalent conditions of (1) (resp. $P(z, \zeta, \eta) \in \mathcal{H}(\Omega; S)$). Then for any $\eta_0 \in S$, it follows that $P(z, \zeta, \eta_0) \in \mathcal{H}(\Omega)$ (resp. $P(z, \zeta, \eta_0) \in \mathcal{N}(\Omega)$) and further $P(z, \zeta, \eta) - P(z, \zeta, \eta_0) \in \mathcal{H}(\Omega; S)$.

Proof. (1) (i) $\implies$ (ii). For any $h > 0$, we choose $\eta_0 \in S \cap \mathbb{R}$ as $\nu \eta_0 < h$. Then there exists a constant $C_{\eta_0} > 0$ such that

$$|P(z, \zeta, \eta_0)| \leq C_{\eta_0} e^{\nu \eta_0 \|\eta\|} \leq C_{\eta_0} e^{h \|\eta\|} \quad ((z; \zeta, \eta) \in \Omega_{\rho}[d_{\rho}]).$$

For any $Z \subseteq S$, let $Z' \subseteq S$ be the convex hull of $Z \cup \{\eta_0\}$. Since $\partial_\eta P(z, \zeta, \eta) \in \mathcal{H}(\Omega; S)$, we can find $\delta > 0$ and a constant $C_{Z'} > 0$ such that for any $(z; \zeta, \eta) \in \Omega_{\rho}[d_{\rho}] \times Z \subset \Omega_{\rho}[d_{\rho}] \times Z'$ the following holds:

$$|P(z, \zeta, \eta)| = |P(z, \zeta, \eta_0) + \int_{\eta_0}^{\eta} \partial_\eta P(z, \zeta, \tau) d\tau| \leq C_{\eta_0} e^{h \|\eta\|} + |\eta - \eta_0| C_{Z'} e^{-\delta m_{Z'}} \|\eta\|$$

$$\leq (C_{\eta_0} + rC_{Z'}) e^{h \|\eta\|}.$$
(ii) $\implies$ (i). For any $Z \in S$, we take $0 < h \leq m_Z$. Then there exists $C_{h,Z} > 0$ such that

$$|P(z, \zeta, \eta)| \leq C_{h,Z}e^{h\|\eta\|} \leq C_{h,Z}e^{h\|\eta\|} \quad ((z; \zeta, \eta) \in \Omega_P[d_p] \times Z).$$

(2) Taking $Z = \{\eta_0\}$, we see that $P(z, \zeta, \eta_0)$ is a $\mathcal{F}(\Omega)$ by (1). Set $\delta_0 := \delta[\eta_0]$. As in the proof of (i) $\implies$ (ii) in (1), we see that for any $(z; \zeta, \eta) \in \Omega_P[d_p] \times Z \subset \Omega_P[d_p] \times Z'$ the following holds: if $|\eta| \geq |\eta_0|

$$|P(z, \zeta, \eta) - P(z, \zeta, \eta_0)| = \left| \int_{\eta_0}^{\eta} \partial_{\eta} P_{\nu}(z, \zeta, \tau) d\tau \right| \leq |\eta - \eta_0| C_{h,Z}e^{-\delta_0|\eta_0\|} \leq r C_{h,Z}e^{-\delta_0|\eta_0\|},$$

and if $|\eta| \leq |\eta_0|

$$|P(z, \zeta, \eta) - P(z, \zeta, \eta_0)| \leq r C_{h,Z}e^{-\delta_0|\eta_0\|} \leq r C_{h,Z}e^{-\delta_0|\eta_0\|}.$$  

Hence $P(z, \zeta, \eta) - P(z, \zeta, \eta_0) \in \mathcal{N}(\Omega; S)$. If $P(z, \zeta, \eta) \in \mathcal{N}(\Omega; S)$, the proof is same. \hfill \square

4.5. Definition. (1) We define a set $\mathcal{S}(\Omega; S)$ as follows: $P(z, \zeta, \eta) \in \mathcal{S}(\Omega; S)$ if

(i) $P(z, \zeta, \eta) \in \Gamma(\Omega_P[d_p] \times S; \mathcal{O}_{T^*X \times \mathbb{C}})$ for some $d > 0$ and $\rho \in \]0, 1[,$

(ii) $\partial_{\eta} P(z, \zeta, \eta) \in \mathcal{N}(\Omega; S)$,

(iii) $P(z, \zeta, \eta)$ satisfies the equivalent conditions of Proposition 4.4.

Note that $\mathcal{N}(\Omega; S) \subset \mathcal{S}(\Omega; S)$ holds by Lemma 4.3.

(2) For $z^*_0 \in T^*_X$, we set

$$\mathcal{S}_{z^*_0} := \lim_{\Omega \to S} \mathcal{S}(\Omega; S) \supset \mathcal{M}_{z^*_0} := \lim_{\Omega \to S} \mathcal{M}(\Omega; S).$$

Here $\Omega \in T^*_X$ ranges through open conic neighborhoods of $z^*_0$, and the inductive limits with respect to $S$ are taken by $r, \theta \to 0$ in (4.1).

We call each element of $\mathcal{S}(\Omega; S)$ (resp. $\mathcal{N}(\Omega; S)$) a symbol (resp. null-symbol) on $\Omega$ with an apparent parameter in $S$. It is easy to see that $\mathcal{S}(\Omega; S)$ is a $\mathbb{C}$-algebra under the ordinary operations of functions, and $\mathcal{N}(\Omega; S)$ is a subalgebra. By definition, we can regard that

$$\mathcal{F}(\Omega) = \{P(z, \zeta, \eta) \in \mathcal{S}(\Omega; S); \partial_{\eta} P(z, \zeta, \eta) = 0\} \subset \mathcal{S}(\Omega; S),$$

$$\mathcal{N}(\Omega) = \mathcal{F}(\Omega) \cap \mathcal{N}(\Omega; S) \subset \mathcal{N}(\Omega; S).$$

Hence we have an injective mapping $\mathcal{F}(\Omega)/\mathcal{N}(\Omega) \hookrightarrow \mathcal{S}(\Omega; S)/\mathcal{N}(\Omega; S)$. Moreover

4.6. Proposition. There exists the following isomorphism:

$$\mathcal{F}(\Omega)/\mathcal{N}(\Omega) \cong \mathcal{S}(\Omega; S)/\mathcal{N}(\Omega; S).$$

Proof. Let $P(z, \zeta, \eta) \in \mathcal{S}(\Omega; S)$. We fix $\eta_0 \in S$. Then by Proposition 4.4, we have $P(z, \zeta, \eta_0) \in \mathcal{F}(\Omega)$ and $[P(z, \zeta, \eta)] = [P(z, \zeta, \eta_0)] \in \mathcal{S}(\Omega; S)/\mathcal{N}(\Omega; S)$. \hfill \square

4.7. Definition. We set

$$:P(z, \zeta, \eta): := P(z, \zeta, \eta) \text{ mod } \mathcal{N}(\Omega; S) \in \mathcal{S}(\Omega; S)/\mathcal{N}(\Omega; S)$$

which is called the normal product or the Wick product of $P(z, \zeta, \eta)$. 

§ 5. Kernel Functions and Symbols

In this section, we shall establish the correspondence of kernel functions and symbols. For this purpose, first we define two mappings that give the correspondence above. Set \( z_0^* = (z_0; \zeta_0) := (0; 1, 0, \ldots, 0) \). Take any element \( K(z, w) dw = \int [\psi(z, w, \eta)] dw \in \lim_{\kappa \to 0} E^R_X(\kappa) \). Then a representative \( \psi(z, z + w, \eta) \) of \( K(z, z + w) \) is holomorphic on \( \bigcap_{i=1}^n \{ (z, w, \eta) \in \mathbb{C}^{2n} \times S; \|z\| < r', \frac{1}{\rho} |\eta| < |w_i| < r', |w_1| < \rho |\eta|, |\arg w_1| < \frac{\pi}{2} + \theta \} \).

5.1. Definition. We set
\[
\sigma(\psi)(z, \zeta, \eta) := \int_{\gamma(0, \eta, \epsilon, \theta)} \psi(z, z + w, \eta) e^{\langle w, \zeta \rangle} dw.
\]

In Proposition 5.4 below, we show that \( \sigma \) induces a mapping \( \mathcal{E}^R_X(z_0^*) \to \mathcal{G}_{z_0^*}/\mathcal{N}_{z_0^*} \).

In order to construct the inverse of \( \sigma \), we make full use of the following family of functions (see Laurent [26, p.39]):

5.2. Definition. We set
\[
\Gamma_\nu(t, \eta) := \begin{cases} 1 & (\nu = 0), \\ \\
\frac{1}{(\nu - 1)!} \int_0^\eta e^{-s\zeta} s^{\nu - 1} ds & (\nu \in \mathbb{N}). \end{cases}
\]

Let \( z_0^* = (0; 1, 0, \ldots, 0) \in \hat{T}^* X \), and \( P(z, \zeta, \eta) \in \mathcal{G}_{z_0^*} \). By Proposition 4.4, for any sufficiently small \( \eta_0 > 0 \) we have \( P(z, \zeta, \eta) - P(z, \zeta, \eta_0) \in \mathcal{N}_{z_0^*} \). We may assume that \( \|\zeta\| = |\zeta_1| \) on a neighborhood of \( z_0^* \). We develop \( P(z, \zeta, \eta_0) \) into the Taylor series with respect to \( \zeta/\zeta_1 = (\zeta_2/\zeta_1, \ldots, \zeta_n/\zeta_1) \):

\[
P(z, \zeta, \eta_0) = \sum_{\alpha \in \mathbb{N}^{n-1}} P_\alpha(z, \zeta_1, \eta_0) \left( \frac{\zeta}{\zeta_1} \right)^\alpha.
\]

Then we set \( P_\alpha^B(z, \zeta_1, \eta) := P_\alpha(z, \zeta_1, \eta_0) \zeta_1^{1/\alpha} \Gamma_\alpha(\zeta_1, \eta) \) and
\[
P_\alpha^B(z, \zeta, \eta) := \sum_{\alpha \in \mathbb{N}^{n-1}} P_\alpha^B(z, \zeta_1, \eta) \left( \frac{\zeta}{\zeta_1} \right)^\alpha.
\]

5.3. Definition. Under the preceding notation, we set
\[
\varpi_\alpha(P)(z, w_1, \eta) := \int_d^\infty P_\alpha^B(z, \zeta_1, \eta) \frac{e^{-w_1 \zeta_1}}{\zeta_1^{1/\alpha}} d\zeta_1
\]
\[
= \int_d^\infty P_\alpha(z, \zeta_1, \eta_0) \Gamma_\alpha(\zeta_1, \eta) e^{-w_1 \zeta_1} d\zeta_1,
\]
and further define
\[
\varpi(P)(z, z + w, \eta) := \sum_{\alpha \in \mathbb{N}^{n-1}} \frac{\alpha! \varpi_\alpha(P)(z, w_1, \eta)}{(2\pi \sqrt{-1})^n (w')^{\alpha+1} n^{\alpha+1}}.
\]

Here we set \( w' := (w_2, \ldots, w_n) \) and \( 1_{n-1} := (1, \ldots, 1) \). In Proposition 5.5 below, we show that \( \varpi \) induces a mapping \( \mathcal{G}_{z_0^*}/\mathcal{N}_{z_0^*} \to \mathcal{E}^R_X(z_0^*) \).
5.4. Proposition. The $\sigma$ in Definition 5.1 induces the linear mapping

$$\sigma : \mathcal{E}^R_{X,z^*} \xrightarrow{\lim \limits_{\kappa \rightarrow \psi}} \mathcal{E}^R_{X,T}$$

$$K(z, w) dw = [\psi(z, w, \eta) dw] \xrightarrow{\sigma} \sigma(K)(z, \zeta) = [\sigma(\psi)(z, \zeta, \eta)].$$

The $\sigma$ does not depend on the choice of the path of the integration.

We call $\sigma$ the symbol mapping, and $\sigma(K)$ the symbol of $K(z, w) dw \in \mathcal{E}^R_{X,z^*}$.

Proof. We expand

$$\psi(z, z + w, \eta) = \sum_{\alpha \in \mathbb{Z}^{n-1}} \frac{\psi_{\alpha}(z, w_1, \eta)}{(2\pi \sqrt{-1})^{n-1}(w')^{\alpha+1}_{n-1}}.$$

If $\alpha_i + 1 \leq 0$ for some $2 \leq i \leq n$, this term is zero in $\lim \limits_{\kappa \rightarrow \psi} \mathcal{E}^R_{X}(\kappa)$, hence we may assume from the beginning that

$$\psi(z, z + w, \eta) = \sum_{\alpha \in \mathbb{N}^{n-1}} \frac{\psi_{\alpha}(z, w_1, \eta)}{(2\pi \sqrt{-1})^{n-1}(w')^{\alpha+1}_{n-1}}.$$

Here

$$\psi_{\alpha}(z, w_1, \eta) := \int_{|w_2| = c|\eta|, \ldots, |w_n| = c|\eta|} \psi(z, z_1 + w_1, z' + \tilde{w}', \eta)(\tilde{w}')^{\alpha} d\tilde{w}'$$

for $c > \frac{1}{\theta}$. Hence we may assume that $\psi(z, z + w, \eta)$ is holomorphic on

$$\bigcap_{i=2}^{n}(z, w, \eta) \in \mathbb{C}^{2n} \times S; \|z\| < r', |w_1| < \rho|\eta|, |\arg w_1| < \frac{\pi}{2} + \theta, \frac{|\eta|}{|w_1|} < \rho \big\}.$$

Take $c' > 0$, an open conic neighborhood $\Omega = \lim \limits_{\text{conic}} T^*X$ and $0 < \rho < 1$ as

$$\Omega_{\rho} \in \lim_{\text{conic}} \{ (z, \zeta) \in \mathbb{C}^{2n}; \|z\| \leq r', \|\zeta\| \leq c'|\zeta_1|, |\arg \zeta_1| \leq r''\}.$$

Taking $c'$ small enough, we can assume that $\|\zeta\| = |\zeta_1|$ on $\Omega_{\rho}$. We have

$$\sigma(\psi)(z, \zeta, \eta) = \int_{\gamma_1(0, \eta; \theta)} dw_1 \int_{\gamma_2(0, \eta; \theta)} \cdots \int_{\gamma_n(0, \eta; \theta)} \sum_{\alpha \in \mathbb{N}_{n-1}} \frac{\psi_{\alpha}(z, w_1, \eta)}{(2\pi \sqrt{-1})^{n-1}(w')^{\alpha+1}_{n-1}} e^{w\zeta_1} dw'$$

$$= \sum_{\alpha \in \mathbb{N}_{n-1}} \frac{(c')^{\alpha}}{\alpha!} \int_{\gamma_1(0, \eta; \theta)} \psi_{\alpha}(z, w_1, \eta) e^{w_1\zeta_1} dw_1.$$

We can change $\gamma_1(0, \eta; \theta) = \{ w_1 = |\eta| s e^{2\pi \sqrt{-1} t}; 0 \leq t \leq 1 \}$ with $0 < \rho^{-1} < s'$. Deforming $\gamma_1(0, \eta; \theta)$, we can see that for any $h > 0$ we have $e^{h|\eta_1|}$ holds if $|\arg \zeta_1| < r''$ and $w_1 \in \gamma_1(0, \eta; \rho, \theta)$. Thus we have

$$|e^{w_1\zeta_1}| = e^{Re(w_1, \zeta_1)} \leq \exp \left( Re(w_1, \zeta_1) + \sum_{i=2}^{n} |w_i| \zeta_i \right)$$

$$\leq e^{h|\eta_1| + (n-1)c's'|\zeta_1|} = e^{(h+(n-1)c's')|\eta_\zeta||\.}$$
Fix $d > 0$. Take $Z \in S$. Then there exists a constant $C > 0$ such that

$$
\left| \int_{\gamma(0,\eta;\varphi,\theta)} \psi(z, z + w, \eta) e^{(w, \zeta)} dw \right| \leq C e^{(h + (n-1)\epsilon')\|\eta\|} ((z; \zeta, \eta) \in \Omega_\rho[d_\rho] \times Z),
$$

that is, we can see that $\sigma(\psi)(z, \zeta, \eta) \in \Gamma(\Omega_\rho[d_\rho] \times S; \mathcal{E}_{T^*X \times \mathbb{C}})$ and satisfies (4.4). If $|\psi(z, w, \eta) dw| = 0 \in \lim_{\kappa} E_X^R$, we may assume that there is $\delta' > 0$ such that $|e^{(w, \zeta_1)}| \leq e^{-\delta|\eta\zeta_1|}$ holds if $|\arg \zeta_1| < r''$ and $w_1 \in \mathcal{T}_1(0; \eta; \varphi, \theta)$. We choose $\epsilon'$ so small that $\delta := \delta' - (n-1)\epsilon' > 0$. Then there exists a constant $C > 0$ such that

$$
\left| \int_{\gamma(0,\eta;\varphi,\theta)} \psi(z, z + w, \eta) e^{(w, \zeta)} dw \right| \leq C e^{((n-1)\epsilon' - \delta')|\eta\zeta_1|} \leq C e^{-\delta\|\eta\|} ((z; \zeta, \eta) \in \Omega_\rho[d_\rho] \times Z),
$$

that is, $\sigma(\psi)(z, \zeta, \eta) \in \mathfrak{N}_{z_0}$. Further we can prove that

$$
\int_{\gamma(0,\eta;\varphi,\theta)} \psi(z, z + w, \eta) e^{(w, \zeta)} dw - \int_{\gamma(0,\eta;\varphi,\theta)} \psi(z, z + w, \eta) e^{(w, \zeta)} dw \in \mathfrak{N}_{z_0}.
$$

Note that

$$
\partial_\eta \sigma(\psi)(z, \zeta, \eta) = \int_{\gamma_2(0,\eta;\varphi) \times \cdots \gamma_n(0,\eta;\varphi)} \left[ \tau \psi(z, z_1 + \tau \zeta, z_1 + \tau \zeta, \eta \tau + w, \eta) e^{\tau \eta \zeta_1 + (w', \zeta')} \right]_{\tau = \beta_0}^{\beta_1} dw_2 \cdots dw_n
$$

$$
+ \int_{\gamma(0,\eta;\varphi,\theta)} \partial_\eta \psi(z, z + w, \eta) e^{(w, \zeta)} dw.
$$

Since $\partial_\eta \psi(z, w, \eta)$ is a zero class, $\partial_\eta \sigma(\psi)(z, \zeta, \eta) \in \mathfrak{N}_{z_0}$. Thus we see that $\sigma(\psi)(z, \zeta, \eta) \in \mathfrak{S}_{z_0}$ and $\sigma$ is well defined.

5.5. Proposition. The $\varpi$ in Definition 5.3 induces the linear mapping

$$
\varpi: \mathfrak{S}_{z_0}/\mathfrak{N}_{z_0} \ni :P(z, \zeta, \eta) : \mapsto \varpi(\{P\}) := \left[ \varpi(P)(z, w, \eta) dw \right] \in \mathcal{E}_{X, z_0}^R.
$$

This mapping is independent of the choice of either $\eta_0$ or the path of the integration.

We call $\varpi(\{P\})$ the kernel of $\{P\} \in \mathfrak{S}_{z_0}/\mathfrak{N}_{z_0}$.

Proof. We need the following estimate to prove that $\varpi$ is well defined:

5.6. Lemma. Assume that $\text{Re}(\eta \tau) \geq 2\delta_0|\eta \tau| > 0$ for some $\delta_0 \in ]0, \frac{1}{2}[$. Then for any $\nu \in \mathbb{N}$,

$$
|\Gamma_\nu(\tau, \eta)| \leq \frac{|\eta|^{\nu}}{\nu!}, \quad |\tau^{\nu} \Gamma_\nu(\tau, \eta)| \leq \frac{e^{-\delta_0|\eta \tau|}}{\delta_0^{-1}}.
$$

Proof. We have (5.8) as follows:

$$
|\Gamma_\nu(\tau, \eta)| \leq \frac{1}{(\nu - 1)!} \int_0^{\eta|} e^{-s\tau s^{\nu-1}} ds \leq \frac{1}{(\nu - 1)!} \int_0^{\eta|} s^{\nu-1} ds = \frac{|\eta|^{\nu}}{\nu!}.
$$

By the definition of $\Gamma$-function and induction on $\nu$, we have

$$
1 - \tau^{\nu} \Gamma_\nu(\tau, \eta) = \frac{\tau^{\nu}}{(\nu - 1)!} \int_0^{\infty} e^{-s\tau s^{\nu-1}} ds = \sum_{k=0}^{\nu-1} \frac{(\eta \tau)^k}{k!} e^{-\eta \tau}.
$$
Therefore, we have
\[
|1 - \tau^n \Gamma_n(\tau, \eta)| = \left| \sum_{k=0}^{n-1} \frac{(\delta_0 \eta \tau)^k}{k! \delta_0^k} e^{-\eta \tau} \right| \leq \frac{e^{-\delta_0 |\eta \tau|}}{\delta_0 \gamma^{-1}} = e^{-\delta_0 |\eta \tau|}.
\]
\[
\leq e^{\delta_0 |n \eta|} e^{-\delta_0 |\eta \tau|} = e^{-\delta_0 |\eta \tau|}.
\]
\[\square\]

Recall (5.1) and (5.2). There exist sufficiently small \(r_0, \theta' > 0\) and sufficiently large \(d > 0\) such that \(P_\alpha(z, \zeta_1, \eta_0)\) is holomorphic on a common neighborhood of \(D\) for each \(\alpha \in \mathbb{N}_0^{n-1}\), where
\[
D := \{(z, \zeta_1) \in \mathbb{C}^{n+1}; \|z\| \leq r_0, |\arg \zeta_1| \leq \theta', |\zeta_1| \geq d\}.
\]
It follows from the Cauchy inequality that we can take a constant \(K > 0\) so that for each \(h > 0\) there exists \(C_h > 0\) such that for every \(\alpha \in \mathbb{N}_0^{n-1}\),
\[
|P_\alpha(z, \zeta_1, \eta_0)| \leq C_h K^{|\alpha|} e^{h|\zeta_1|} ((z, \zeta_1) \in D).
\]
We take \(\delta_0 \in \left]\frac{1}{2}, \frac{1}{2}\right[\) as \(\text{Re}(\eta \zeta_1) \geq 2\delta_0 |\eta \zeta_1| > 0\) if \(\eta \in S\) and \(|\arg \zeta_1| \leq \theta'\). Take \(\varepsilon > 0\) as
\[
0 < \frac{K\varepsilon}{\delta_0} < \frac{1}{2}.
\]
For any \(Z \in S\), we chose \(h = \frac{\delta_0 m Z}{2}\). Then by (5.9), for \((z, \zeta_1) \in D \times Z\) and \(|\zeta_1| \leq \varepsilon|\zeta_1| (2 \leq i \leq n)\) we have
\[
|P(z, \zeta, \eta) - P^B(z, \zeta, \eta)| = \left| \sum_{|\alpha|=1} P_\alpha(z, \zeta_1, \eta_0) (1 - \zeta_1^{|\alpha|} \Gamma_0(\zeta_1, \eta)) \right| \leq \delta_0 C_h e^{-\delta_0 |\eta \zeta_1|/2} \sum_{|\alpha|=1} \frac{(K\varepsilon / \delta_0)^{|\alpha|}}{2^n} \leq 2^{n-1} \delta_0 C_h e^{-\delta_0 |\eta \zeta_1|/2},
\]
where we remark that \(\#\{|\alpha| \in \mathbb{N}_0^{n-1}; |\alpha| = i\} = \left(\begin{array}{c} n + i - 2 \\ i \end{array}\right) \leq 2^{n+i-2}\). Therefore we see
\[
P(z, \zeta, \eta) - P^B(z, \zeta, \eta) = P(z, \zeta, \eta) - P(z, \zeta, \eta_0) + P(z, \zeta, \eta_0) - P^B(z, \zeta, \eta) \in \mathbb{H}_{z_0}.
\]
Further by (5.8) and (5.10), there exists a constant \(K > 0\) so that for each \(h > 0\) there exists \(C_h > 0\) such that for every \(\alpha \in \mathbb{N}_0^{n-1}\) and \((z, \zeta_1, \eta) \in D \times S\), we have
\[
\frac{|P^B_\alpha(z, \zeta_1, \eta)|}{|\zeta_1|^{\alpha|}} \leq \frac{C_h (K|\eta|)^{|\alpha|} e^{h|\zeta_1|}}{|\alpha|!}.
\]
We can take a sufficiently small \(\delta_1, \delta' > 0\) such that
\[
\{w_1 \in \mathbb{C}; |\arg w_1| < \delta' + \frac{\pi}{2}\} \subset \bigcup \{w_1 \in \mathbb{C}; \Re(w_1 \zeta_1) \geq \delta_1 |w_1 \zeta_1|\},
\]
and we set
\[
L := \{(z, w_1) \in \mathbb{C}^{n+1}; \|z\| < r_0, |\arg w_1| < \delta' + \frac{\pi}{2}\}.
\]
By (5.12), for any \(k \in \mathbb{N}\) there exists \(C_k > 0\) such that for every \(\alpha \in \mathbb{N}_0^{n-1}\),
\[
\frac{|P^B_\alpha(z, \zeta_1, \eta)|}{|\zeta_1|^{\alpha|}} \leq \frac{C_k (K|\eta|)^{|\alpha|} e^{\delta_1 |\zeta_1|/k}}{|\alpha|!} ((z, \zeta_1, \eta) \in D \times S).
\]
By changing the direction of the integration in the complex $\tau$-plane, $\varpi_\alpha(P)(z, w_1, \eta)$ extends analytically to the domain $L \times S$. Set

$$L_k := \{(z, w_1) \in \mathbb{C}^{n+1}; \|z\| < r_o, |\arg w_1| < \delta' + \frac{\pi}{2}, \frac{2}{k} < |w_1|\}. \tag{5.15}$$

Then by (5.14) and (5.8) for any $\eta \in S$ we have

$$\sup\{|\varpi_\alpha(P)(z, w_1, \eta)|; (z, w_1) \in L_k\} \leq \frac{2kC_k}{\delta ! |\alpha|!} (K|\eta|)^{|\alpha|}. \tag{5.16}$$

Therefore the right-hand side of (5.4) converges locally uniformly in

$$V_k := \bigcap_{2 \leq i \leq n}(z, w, \eta) \in \mathbb{C}^{2n} \times S; (z, w_1) \in L_k, K|\eta| < |w_i\}. \tag{5.17}$$

Hence $\varpi(P)(z, z + w, \eta)$ is a holomorphic function defined on the set

$$V := \bigcup_{k=1}^{\infty}V_k = \bigcap_{2 \leq i \leq n}(z, w, \eta) \in \mathbb{C}^{2n} \times S; \|z\| < r_o, |\arg w_1| < \delta' + \frac{\pi}{2}, K|\eta| < |w_i|\}. \tag{5.18}$$

Next, we have

$$\partial_\eta \varpi_\alpha(P)(z, w_1, \eta) = \frac{1}{(|\alpha| - 1)!} \int_\delta^\infty \tilde{P}_\alpha(z, \zeta_1, \eta_0) e^{-(\eta + w_1)\zeta_1} |\alpha|! - 1 d\zeta_1. \tag{5.19}$$

Let $Z \in S$. Then choosing $h = \delta mZ$ in (5.10), for $\|z\| < r_o, \eta \in Z$ and $|w_1| < \frac{\delta |\eta|}{2}$, we have

$$|\partial_\eta \varpi_\alpha(P)(z, w_1, \eta)| \leq \frac{C_{\delta mZ}(K|\eta|)^{|\alpha|}}{(|\alpha| - 1)!} \int_\delta^\infty e^{\delta |\eta| - |w_1| |\zeta_1|} |\alpha|! - 1 d\zeta_1$$

$$\leq \frac{C_{\delta mZ}(K|\eta|)^{|\alpha|}}{(|\alpha| - 1)!} \int_\delta^\infty e^{-(\delta |\eta| - |w_1| |\zeta_1|)} |\alpha|! - 1 d\zeta_1 \leq 2C_{\delta mZ}(K|\eta|)^{|\alpha|}. \tag{5.20}$$

Hence $\varpi(P)(z, z + w, \eta)$ is holomorphic on

$$\bigcup_{z \in S} \bigcap_{2 \leq i \leq n}(z, w, \eta) \in \mathbb{C}^{2n} \times Z; \|z\| < r_o, |w_1| < \frac{\delta |\eta|}{2}, K|\eta| < |w_i|\}$$

$$= \bigcap_{2 \leq i \leq n}(z, w, \eta) \in \mathbb{C}^{2n} \times S; \|z\| < r_o, |w_1| < \frac{\delta |\eta|}{2}, K|\eta| < |w_i|\}.$$ This entails that $|\partial_\eta \varpi(P)(z, w, \eta) dw| = 0 \in \mathcal{E}_{\mathcal{M}, x_0}^\mathcal{M}$. If $P(z, \zeta, \eta) \in \mathcal{M}_{x_0}$, there exists a constant $\delta, C, K > 0$ so that for every $\alpha \in N_0^{n-1}$,

$$|P_\alpha(z, \zeta_1, \eta_0)| \leq CK^{|\alpha|} e^{-\delta |\zeta_1|} \quad ((z, \zeta_1) \in D). \tag{5.21}$$

Thus if $|w_1| < \frac{\delta}{2}$, we have

$$|\varpi_\alpha(P)(z, w_1, \eta)| \leq \frac{C (K|\eta|)^{|\alpha|}}{|\alpha|!} \int_\delta^\infty e^{-(\delta |w_1| |\zeta_1|)} |\alpha|! - 1 d\zeta_1 \leq \frac{2C (K|\eta|)^{|\alpha|}}{\delta |\alpha|!}. \tag{5.22}$$

Thus $\varpi(P)(z, z + w, \eta)$ is holomorphic on

$$\bigcap_{2 \leq i \leq n}(z, w, \eta) \in \mathbb{C}^{2n} \times S; \|z\| < r_o, |w_1| < \frac{\delta}{2}, K|\eta| < |w_i|\},$$
hence \([\varpi(P)(z, w, \eta) \, dw] = 0\). If we change \(\eta_0\) or \(d\) in (5.3), for the same reasoning as above, we see that \([\varpi(P)(z, w, \eta) \, dw] = 0\). Therefore we obtain a well-defined linear mapping (5.7).

Now we shall prove our fundamental theorem for the symbol theory:

5.7. Theorem. The mappings \(\sigma\) and \(\varpi\) are inverse to each other. In particular

\[
\sigma: \mathcal{E}^\mathbb{R}_{X, z_0^*} \rightarrow \mathcal{G}_{z_0^*}/\mathcal{M}_{z_0^*}.
\]

Proof. We may assume that \(z_0^* = (0; 1, 0, \ldots, 0)\), and we may also assume \(\|\zeta\| = |\zeta_1|\) on a neighborhood of \(z_0^*\) in the course of proof.

**Step 1.** We shall show \(\sigma \cdot \varpi = 1: \mathcal{G}_{z_0^*}/\mathcal{M}_{z_0^*} \rightarrow \mathcal{G}_{z_0^*}/\mathcal{M}_{z_0^*}\). Let \(P(z, \zeta, \eta) \in \mathcal{G}_{z_0^*}\). Assume that \(P^B(z, \zeta, \eta) \in \mathcal{G}_{z_0^*}\) is holomorphic on a neighborhood of

\[
\tilde{V} := \bigcap_{2 \leq i \leq n} \{(z, \zeta, \eta) \in \mathbb{C}^{2n} \times S; \|z\| < r_0, |\zeta_1| \geq d, d|\arg \zeta_1| \leq 1, d|\zeta_i| \leq |\zeta_1|\}.
\]

By the definition of \(\varpi\), we have

\[
\sigma \cdot \varpi(P)(z, \zeta, \eta) = \int_{\gamma(0, \eta, \vartheta, \theta)} \varpi(P)(z, z + w, \eta) e^{(w, \zeta)} \, dw
\]

\[
= \int \, dw \sum_{\gamma(0, \eta, \vartheta, \theta)} \alpha! e^{(w, \zeta)} (2\pi \sqrt{-1})^n (w')^{\alpha+1} \int_d \hat{P}^B_{\alpha}(z, \zeta_1, \eta) \frac{e^{-w_1 \xi_1}}{\xi_1^{\alpha|\theta|}} \, d\xi_1
\]

\[
= \sum_{\alpha \in \mathbb{N}_{0}^{n-1}} (\zeta')^{\alpha} \int \, dw_1 \frac{e^{w_1 \xi_1}}{2\pi \sqrt{-1}} \int_d \hat{P}^B_{\alpha}(z, \zeta_1, \eta) \frac{e^{-w_1 \xi_1}}{\xi_1^{\alpha|\theta|}} \, d\xi_1.
\]

We set

\[
\tilde{V}_\varepsilon := \bigcap_{i=2}^{n} \{(z, \zeta) \in \mathbb{C}^{2n}; \|z\| < r_0, |\zeta_1| > \frac{d}{\varepsilon}, |\arg \zeta_1| \leq \varepsilon, |\zeta_i| \leq \varepsilon|\zeta_1|\}.
\]

We deform the path of integration \(\int_d \, d\xi_1\) in two ways as follows: Let \(\delta > 0\) be a sufficiently small constant and \(d^\pm\) intersection points of the circle \(|\tau| = d\) and \(\{\xi_1 \in \mathbb{C}; \pm \text{Im} \xi_1 = \delta \text{Re} \xi_1 > 0\}\). Let \(\Sigma_{\pm}\) be paths starting from \(d\), first going to \(d^\pm\) along the circle and
next going to the infinity along the half lines \( \{ \xi_1 \in \mathbb{C}; \pm \text{Im} \xi_1 = \delta \text{Re} \xi_1 > 0 \} \) respectively (see Figure 1). According to these deformations, we divide the path \( \gamma_1(0, \eta; \varrho, \theta) \) into two parts:

\[
\gamma_1^\pm(0, \eta; \varrho, \theta) := \gamma_1(0, \eta; \varrho, \theta) \cap \{ w_1 \in \mathbb{C}; \pm \text{Im} w_1 > 0 \}.
\]

We take \( a \in \gamma_1(0, \eta; \varrho, \theta) \cap \mathbb{R} \). Now we can change the order of integration in \( I \) (cf. Figure 1) and obtain:

\[
\sigma \cdot \varpi(P)(z, \zeta, \eta) = \sum_{\alpha \in \mathbb{N}_0^{n-1}} (\zeta')^\alpha \left( \int_{\Sigma_+} \frac{P_{\alpha}^B(z, \zeta_1, \eta)}{2\pi \sqrt{-1} \xi_1^{[\alpha]}} \int_{\gamma_1^+(0, \eta; \varrho, \theta)} e^{w_1(\zeta_1 - \xi_1)} \, dw_1 \right.
\]

\[
+ \int_{\Sigma_-} \frac{P_{\alpha}^B(z, \zeta_1, \eta)}{2\pi \sqrt{-1} \xi_1^{[\alpha]}} \int_{\gamma_1^-(0, \eta; \varrho, \theta)} e^{w_1(\zeta_1 - \xi_1)} \, dw_1 \bigg) \sum_{\alpha \in \mathbb{N}_0^{n-1}} (\zeta')^\alpha \int_{\Sigma_-} \frac{P_{\alpha}^B(z, \zeta_1, \eta)}{2\pi \sqrt{-1} \xi_1^{[\alpha]}(\zeta_1 - \xi_1)} \, d\xi_1
\]

\[
+ \sum_{\alpha \in \mathbb{N}_0^{n-1}} (\zeta')^\alpha \int_{\Sigma_+} \frac{P_{\alpha}^B(z, \zeta_1, \eta)}{2\pi \sqrt{-1} \xi_1^{[\alpha]}(\zeta_1 - \xi_1)} \, d\xi_1.
\]

Here we remark that \( a > 0 \) can be taken as sufficiently small. Further we set

\[
I := \sum_{\alpha \in \mathbb{N}_0^{n-1}} (\zeta')^\alpha \int_{\Sigma_-} \frac{P_{\alpha}^B(z, \zeta_1, \eta)}{2\pi \sqrt{-1} \xi_1^{[\alpha]}(\zeta_1 - \xi_1)} \, d\xi_1,
\]

\[
I^- := -\sum_{\alpha \in \mathbb{N}_0^{n-1}} (\zeta')^\alpha \int_{\Sigma_-} \frac{P_{\alpha}^B(z, \zeta_1, \eta)}{2\pi \sqrt{-1} \xi_1^{[\alpha]}(\zeta_1 - \xi_1)} \, d\xi_1,
\]

\[
I^+ := \sum_{\alpha \in \mathbb{N}_0^{n-1}} (\zeta')^\alpha \int_{\Sigma_+} \frac{P_{\alpha}^B(z, \zeta_1, \eta)}{2\pi \sqrt{-1} \xi_1^{[\alpha]}(\zeta_1 - \xi_1)} \, d\xi_1.
\]

Then \( \sigma \cdot \varpi(P)(z, \zeta, \eta) = I + I^- + I^+ \). Let us recall that we have discussed in \( |\arg \zeta_1| \leq \varepsilon; |\zeta_1| \geq \frac{d}{\varepsilon} > d \) and \( |\zeta_1| \leq \varepsilon |\zeta_1| \) \((2 \leq i \leq n)\). We can find \( \varepsilon_0, c > 0 \) such that \( |\zeta_1 - \xi_1| \geq c |\zeta_1| \geq \frac{cd}{\varepsilon} \) and \( \text{Re}(\zeta_1 \beta_1 \eta) \leq -2c |\eta \zeta_1| \) hold for any \( \varepsilon \in [0, \varepsilon_0] \) and \( (\zeta_1, \eta) \in \Sigma_- \times S \). Further there exists a constant \( h_0 > 0 \) such that \( \text{Re}(\beta_1 \eta \zeta_1) \geq 2h_0 |\eta \zeta_1| \) holds for any \( \xi_1 \in \Sigma_+ \setminus \{|\xi_1| = d\} \) and \( \eta \in S \). For any \( Z \subseteq S \), choose \( h = h_Z > 0 \) as \( h_Z < h_0 m_Z \) in (5.12). Hence replacing \( \varepsilon > 0 \) as \( 2K \varepsilon \leq c \) on \( V_{\varepsilon} \times Z \) we have

\[
|I^-| \leq \sum_{|\alpha|=0}^\infty \frac{C_{h_Z}(K|\eta \zeta_1|)^{|\alpha|} e^{-2c|\eta \zeta_1|}}{2\pi c d |\alpha|!} \times \left( e^{(h_Z + |\beta_1| \eta)|d} \int_{|\xi_1| = d} |d\xi_1| + \int_d^{\infty} e^{-(2h_0 |\eta| - h_Z)|\xi_1|} d|\xi_1| \right)
\]

\[
\leq \frac{2^{n-2} C_{h_Z} e^{-c|\eta \zeta_1|}}{c} \left( e^{(h_Z + |\beta_1| \eta)|d} + \frac{e^{-h_0 dr}}{2\pi h_0 dm_Z} \right).
\]

Hence we see that \( I^- \in M_{z_0} \). Similarly, we have

\[
|I^+| \leq \frac{2^{n-2} C_{h_Z} e^{-c|\eta \zeta_1|}}{c} \left( e^{(h_Z + |\beta_1| \eta)|d} + \frac{e^{-h_0 dr}}{2\pi h_0 dm_Z} \right),
\]
hence \( I^+ \in \mathfrak{N}_{\zeta_0} \). Now we consider \( I \). For any \( K \subseteq \{ \zeta_1 \in \mathbb{C}; |\arg \zeta_1| \leq \varepsilon \} \), we see that the integral operator
\[
\int_{\Sigma_\pm} \frac{e^{a(\zeta_i - \zeta_1)}}{2\pi \sqrt{-1}(\zeta_1 - \zeta_i)} d\zeta_i d\zeta_1
\]
has the Cauchy kernel with a damping factor since \(- \text{Re}(a \zeta_i) < 0\). Hence,
\[
I = \sum_{\alpha \in \mathbb{N}_0^{\alpha-1}} P_\alpha^R(z, \zeta_1, \eta) \left( \frac{\zeta'}{\zeta_1} \right) = P_\alpha(z, \zeta, \eta)
\]
holds if \( \zeta_1 \) is located in the domain surrounded by \( \Sigma_- - \Sigma_+ \). Thus we have
\[
\sigma \cdot \varpi(P)(z, \zeta, \eta) - P(z, \zeta, \eta) = \sigma \cdot \varpi(P)(z, \zeta, \eta) - P^R(z, \zeta, \eta) + P^R(z, \zeta, \eta) - P(z, \zeta, \eta) \in \mathfrak{N}_{\zeta_0}^+,
\]
that is, \( \sigma \cdot \varpi = 1: \mathcal{G}_{\zeta_0}^+ / \mathfrak{N}_{\zeta_0}^+ \simeq \mathcal{G}_{\zeta_0}^+ / \mathfrak{N}_{\zeta_0}^+ \).

**Step 2.** Let \( P = [\psi(z, w, \eta) dw] \in \mathcal{F}_{\mathcal{K}^+}^+ \). Then we can assume that a representative \( \psi(z, z + w, \eta) \) has the form as in (5.5). By Proposition 5.4, each coefficient \( P_\alpha(z, \zeta_1, \eta) \) in (5.1) is written as
\[
P_\alpha(z, \zeta_1, \eta) = \frac{\zeta_1^{[\alpha]}}{\alpha!} \int_{\gamma_1(0, \eta_0, \varrho, \theta)} \psi_\alpha(z, p, \eta) e^{p \xi_1} dp.
\]
We assume that each \( \psi_\alpha(z, p, \eta) \) is holomorphic on
\[
\{(z, p, \eta) \in \mathbb{C}^{n+1} \times S; \|z\| < 2r_0, \|w_1\| < \varrho|\eta|, |\arg p| < \frac{\pi}{2} + \theta \}.
\]
Fix \( \eta_0 \in S \cap \mathbb{R} \), and we take \( \varrho' > \varrho \) as \( \varrho'|\eta| < |\eta_0| \) for any \( \eta \in S \). By (5.6), there exists \( c > 0 \) and for any \( Z \subseteq S \) there exists \( C_Z > 0 \) such that for any \( \eta \in Z \),
\[
\sup \{|\psi_\alpha(z, p, \eta)|; \|z\| \leq r_0, p \in \gamma_1(0, \eta_0, \varrho, \theta)\} \leq C_Z(c|\eta|)^{|\alpha|+n-1}.
\]
By the definition, we have
\[
\varpi \cdot \sigma(\psi)(z, z + w, \eta) = \frac{1}{\sum_{|\alpha|=0}^{\infty} 2\pi \sqrt{-1}(w')^{\alpha+1} I_{n-1}} \int_{d}^{\infty} d\xi \zeta^{[\alpha]} \int_{\gamma_1(0, \eta_0, \varrho, \theta)} \Gamma_{[\alpha]}(\zeta_1, \eta) e^{-w_1 \xi_1} \int_{\gamma_1(0, \eta_0, \varrho, \theta)} \psi_\alpha(z, p, \eta) e^{p \xi_1} dp.
\]
We set
\[
\varpi' \cdot \sigma(\psi)(z, z + w, \eta) := \sum_{|\alpha|=0}^{\infty} \frac{1}{2\pi \sqrt{-1}(w')^{\alpha+1} I_{n-1}} \int_{d}^{\infty} d\xi \zeta^{[\alpha]} \int_{\gamma_1(0, \eta_0, \varrho, \theta)} \psi_\alpha(z, p, \eta) e^{p \xi_1} dp.
\]
We assume that \( \text{Re}(\eta \zeta_1) \geq 2\delta_0|\eta| \zeta_1 > 0 \) for some \( \delta_0 \in \left]0, \frac{1}{2}\right[ \). We deform the path \( \gamma_1(0, \eta_0, \varrho, \theta) \) as \( |e^{p \xi_1}| \leq e^{-|\delta_0|\eta| \zeta_1|/2} \) for \( |\arg \zeta_1| \leq \varepsilon \). Then by (5.9) and (5.21), for any \( Z \subseteq S \), there exists \( C_Z > 0 \) such that if \( 2|w_1| < \delta_0|\eta| \) and \( c|\eta| < \delta_0|w_1| \) (\( 2 \leq i \leq n \)), we have
\[
\sum_{|\alpha|=0}^{\infty} \frac{1}{2\pi \sqrt{-1}(w')^{\alpha+1} I_{n-1}} \int_{d}^{\infty} d\xi \zeta^{[\alpha]} \int_{\gamma_1(0, \eta_0, \varrho, \theta)} \psi_\alpha(z, p, \eta) e^{p \xi_1} dp \leq \sum_{|\alpha|=1}^{\infty} C_Z|\gamma_1(0, \eta_0, \varrho, \theta)|^{(c|\eta|)^{|\alpha|+n-1}} \int_{d}^{\infty} \frac{e^{-(\delta_0|\eta|/2-|w_1|)\zeta_1}}{\delta_0^{(|\alpha|-1)}} d\zeta_1.
\]
We may assume that \(|\gamma_1(0, \eta_0; \varrho, \theta)|\) denotes the length of \(\gamma_1(0, \eta_0; \varrho, \theta)\). Next we consider

\begin{equation}
\int_{\gamma_1(0, \eta_0; \varrho, \theta)} \psi_a(z, p, \eta) e^{i\xi_1} dp - \int_{\gamma_1(0, \eta_0; \varrho, \theta)} \psi_a(z, p, \eta_0) e^{i\xi_1} dp = \int dp e^{i\xi_1} \int_{\eta_0}^{\eta} \partial_\eta \psi_a(z, p, \tau) d\tau.
\end{equation}

Since \(\partial_\eta \psi(z, w, \eta)\) is holomorphic on \(|w_1| < \varrho|\eta|\), as in (5.21) there exists \(c\) and for any \(S \subset Z\) there exists \(C_Z > 0\) such that for any \(\eta \in Z\),

\[
\sup\{|\psi_a(z, p, \eta) - \psi_a(z, p, \eta_0)|; \|z\| \leq r_0, |p| \leq \varrho|\eta|\} \leq C_Z (c|\eta|)^{|\alpha|+n-1}.
\]

Thus we can change the path of the integration in (5.22) as \(e^{i\xi_1} \leq e^{-c|\eta|\xi_1}\) for \(|\arg \xi_1| \leq \varepsilon\). Hence if \(\eta \in Z\), \(|w_1| < c'|\eta|\) and \(c|\eta| < \delta_0|w_i|\) \((2 \leq i \leq n)\), we have

\[
\left| \sum_{|\alpha| = 0}^{\infty} \frac{C_Z |\gamma_1(0, \eta_0; \varrho, \theta)||c|\eta||^{|\alpha|+n-1}}{2\pi (w')^{\alpha+1} n!} \int_\gamma \frac{d\zeta}{e^{i\zeta} \zeta} \right| \leq \sum_{|\alpha| = 0}^{\infty} \frac{C_Z |\gamma_1(0, \eta_0; \varrho, \theta)||c|\eta||^{|\alpha|+n-1}}{2\pi (c'|\eta| - |w_1|)^{\alpha+1} n!} \int_\gamma \frac{d\zeta}{e^{i\zeta} \zeta} < \infty.
\]

Summing up we can prove that \([\varpi' \cdot \sigma(\psi)(z, w, \eta) \, dw] = [\varpi \cdot \sigma(\psi)(z, w, \eta) \, dw] \in \lim_{\kappa \to X} \hat{E}'_{\kappa}(\kappa)\).

We may assume that \(\varpi' \cdot \sigma(\psi)(z, z + w, \eta)\) is holomorphic on

\[
\bigcap_{2 \leq i < n} \{ (z, w, \eta) \in \mathbb{C}^{2n} \times S; \|z\| < r_0, |\arg w_1| < \delta' + \frac{\pi}{2}, c|\eta| < |w_i| \}
\]

with some constants \(r_0, \delta', c > 0\).

\[
\bigcap_{2 \leq i < n} \{ (z, w, \eta) \in \mathbb{C}^{2n} \times S; \|z\| < r_0, 0 < |w_1|, |\arg w_1| < \delta' + \frac{\pi}{2}, c|\eta| < |w_i| \}
\]

with some constants \(r_0, \delta', c > 0\). Let \(\gamma_1'\) be a path starting from \(\beta_0 \eta_0\), ending at \(\beta_1 \eta_0\) and detouring \(w_1\) clockwise as in Figure 2. If \(\text{Re}(p - w_1) < 0\), we have

\[
\int_{\gamma_1} e^{(p-w_1) \xi_1} d\xi_1 = - \frac{e^{(p-w_1) d}}{p - w_1}.
\]
and the right-hand side extends analytically. Thus on the common domain of definition we have

\[
\varpi' \cdot \sigma(\psi)(z, z + w, \eta) = \sum_{\alpha \in \mathbb{N}_0^n} \frac{-1}{(w')^{\alpha+1-n}} \int_{\gamma_1(0, \eta_0; \theta)} \frac{\psi_{\alpha}(z, p, \eta) e^{(p-w_1)d}}{2\pi \sqrt{-1} (p-w_1)} \, dp \\
= \sum_{\alpha \in \mathbb{N}_0^n} \frac{-1}{(w')^{\alpha+1-n}} \int_{\gamma_1(0, \eta_0; \theta) \backslash \gamma_1} \frac{\psi_{\alpha}(z, p, \eta) e^{(p-w_1)d}}{2\pi \sqrt{-1} (p-w_1)} \, dp \\
+ \sum_{\alpha \in \mathbb{N}_0^n} \frac{1}{(w')^{\alpha+1-n}} \int_{\gamma_1} \frac{\psi_{\alpha}(z, p, \eta) e^{(p-w_1)d}}{2\pi \sqrt{-1} (p-w_1)} \, dp \\
= \psi(z, z + w, \eta) + \Pi,
\]

where

\[
\Pi := \sum_{\alpha \in \mathbb{N}_0^n} \frac{1}{(w')^{\alpha+1-n}} \int_{\gamma_1} \frac{\psi_{\alpha}(z, p, \eta) e^{(p-w_1)d}}{2\pi \sqrt{-1} (p-w_1)} \, dp.
\]

As in (5.21), there exist \( c, c_1 > 0 \) and for any \( Z \in S \), there exists \( C_Z > 0 \) such that

\[
\left| \frac{e^{(p-w_1)d}}{p-w_1} \right| \leq C_Z, \quad |\psi_{\alpha}(z, p, \eta)| \leq C_Z(c|\eta|)^{\alpha+n-1},
\]

hold on \( \{||z|| < c_1, |w_1| < c_1|\eta|, p \in \gamma'_1, \eta \in Z\} \). Thus, on \( \{||z|| < c_1, |w_1| < c_1|\eta|, \eta \in Z\} \) we have

\[
|\Pi| \leq \sum_{\alpha \in \mathbb{N}_0^n} \frac{C_Z^2|\gamma'_1|}{2\pi} \left| \left( \frac{c|\eta|}{w'} \right)^{\alpha+1-n} \right|.
\]

By taking \( \delta > 0 \) as \( c\delta < 1 \) and \( \delta < c_1 \), we see that \( \Pi \) is holomorphic on

\[
\bigcup_{Z \in S} \bigcap_{i=2}^n \{ (z, w, \eta) \in \mathbb{C}^{2n} \times Z; ||z|| < \delta, |w_1| < c_1|\eta| < \delta^2|w_1| \} \\
= \bigcap_{i=2}^n \{ (z, w, \eta) \in \mathbb{C}^{2n} \times S; ||z|| < \delta, |w_1| < c_1|\eta| < \delta^2|w_1| \}.
\]

Thus \( \varpi \cdot \sigma = 1 \): \( \delta_{X,z_0}^R \simeq \delta_{X,z_0}^* \).

By Steps 1 and 2, we see that \( \sigma^{-1} = \varpi \), hence \( \sigma: \delta_{X,z_0}^R \simeq \mathcal{G}_{z_0}/\mathcal{N}_{z_0} \).

Let \( P \in \mathcal{G}_{z_0} \), and consider \( [\varpi(P)(z, w, \eta) \, dw] \in \lim_{\kappa} E_X^R(\kappa) \). Here we can assume that \( \varpi(P)(z, z+w, \eta) \) is holomorphic on \( V \) in (5.18). Take \( c_0 > 1 \) such that \( c_0 \Re \zeta_1 \geq |\zeta_1| \) for \( |\arg \zeta_1| \leq \theta' \). In (5.10), we take \( \{\varepsilon_{\nu}\}^{\infty}_{\nu=1} \subset \mathbb{R}_{>0} \) and \( C > 0 \) as

\[
1 \gg \varepsilon_1 > \varepsilon_2 > \cdots > \varepsilon_\nu \rightarrow 0, \quad \frac{2C\varepsilon_{\nu}/2}{\varepsilon_\nu} \leq 2^\nu C,
\]

Set \( \varepsilon_{\alpha} := \varepsilon_{|\alpha|} \) for short, and we define

\[
\varpi_{0,\alpha}(P)(z, w_1) := \int_{d} P_{\alpha}(z, \zeta_1, \eta_0) \Gamma_{|\alpha|}(\zeta_1, c_0 \varepsilon_{\alpha} - w_1) e^{-w_1 \zeta_1} \, d\zeta_1,
\]

\[
\varpi_0(P)(z, z+w) := \sum_{\alpha \in \mathbb{N}_0^n} \frac{\alpha! \varpi_{0,\alpha}(P)(z, w_1)}{(2\pi \sqrt{-1})^n (w')^{\alpha+1-n}}.
\]
5.8. **Theorem.** (1) The $\varpi_0$ induces the mapping $\varpi_0 : \mathfrak{S}_{z_0} / \mathcal{N}_{z_0} \to \mathcal{E}_{X,z_0}$.  
(2) It follows that $[\varpi(P)(z, w, \eta) \, dw] = [\varpi_0(P)(z, w) \, dw] \in \lim_{\kappa} E_{X}^R(\kappa)$, and the following diagram is commutative:

$$
\begin{array}{c}
\mathfrak{S}_{z_0} / \mathcal{N}_{z_0} \\
\approx
\end{array} \xrightarrow{\varpi_0} \begin{array}{c} \mathfrak{S}_{z_0} / \mathcal{N}_{z_0} \\
\approx
\end{array} \begin{array}{c} \mathfrak{S}_{z_0} / \mathcal{N}_{z_0} \\
\approx
\end{array} \lim_{\kappa} E_{X}^R(\kappa).
$$

(5.23)

Here, the isomorphism $\mathcal{E}_{X,z_0} \approx \mathfrak{S}_{z_0} / \mathcal{N}_{z_0}$ is induced by

$$
\psi(z, w) \, dw \mapsto \sigma(\psi)(z, \zeta, \eta_0) = \int_{\gamma(0, \eta_0; \theta)} \psi(z, z + w) e^{(w, \zeta)} \, dw
$$

for any fixed $\eta_0 \in S$.

5.9. **Remark.** (1) The isomorphism $\mathcal{E}_{X,z_0} \approx \mathfrak{S}_{z_0} / \mathcal{N}_{z_0}$ is established in [2], [5] and [6].

(2) From the diagram (5.23), we obtain an explicit description of the isomorphism $\mathcal{E}_{X,z_0} \approx \lim_{\kappa} E_{X}^R(\kappa)$.

**Proof.** If $\varepsilon_\nu < \delta_1 |w_1|$, $\Re(w_1 \zeta_1) \geq \delta_1 |w_1|$, and $0 \leq t \leq 1$, we have

$$
|e^{-t(c_0 \varepsilon_\nu - w_1) + w_1} \zeta_1| = e^{-t c_0 \varepsilon_\nu \Re \zeta_1 - (1-t) \Re (w_1 \zeta_1)} \leq e^{-t \varepsilon_\nu |\zeta_1| - (1-t) \varepsilon_\nu |\zeta_1|} = e^{-\varepsilon_\nu |\zeta_1|}.
$$

Thus

$$
|\Gamma_\nu(\zeta_1, c_0 \varepsilon_\nu - w_1) e^{-w_1} \zeta_1| = 
$$

$$
\left| \frac{1}{(\nu - 1)!} \int_0^{c_0 \varepsilon_\nu - w_1} e^{-s w_1 + \zeta_1 s^\nu - 1} ds \right| = 
$$

$$
\left| \frac{1}{(\nu - 1)!} \int_0^{c_0 \varepsilon_\nu - w_1} e^{-t(c_0 \varepsilon_\nu - w_1) + w_1} \zeta_1 t^\nu dt \right| 
$$

$$
\leq \frac{(c_0 \varepsilon_\nu + |w_1|)^\nu e^{-|\zeta_1|}}{(\nu - 1)!} \int_0^\nu \nu^{-1} dt = \frac{(c_0 \varepsilon_\nu + |w_1|)^\nu e^{-|\zeta_1|}}{\nu!}.
$$

Set

$$
L'_\alpha := \{(z, w_1) \in \mathbb{C}^{n+1} ; \|z\| < r_0, |\arg w_1| < \delta' + \frac{\pi}{2}, \varepsilon_\alpha < \delta_1 |w_1|\}.
$$

Then taking $h = \frac{\varepsilon_\alpha}{2}$ in (5.10) we have

$$
\sup_{L'_\alpha} |\varpi_0,\alpha(P)(z, w_1)| \leq C_{\varepsilon_\alpha/2} (K(c_0 \varepsilon_\alpha + |w_1|))^{\alpha} \frac{\int_0^\infty e^{-|\zeta_1|/2} d|\zeta_1|}{|\alpha|!} 
$$

$$
\leq \frac{2C_{\varepsilon_\alpha/2} (K(c_0 \varepsilon_\alpha + |w_1|))^{\alpha}}{\varepsilon_\alpha |\alpha|!} \leq C (2K(c_0 \varepsilon_\alpha + |w_1|))^{\alpha}.
$$

If $(z, w_1) \in L'_\nu$ and $2K(c_0 \varepsilon_\nu + |w_1|) < |w_1|$, we have

$$
|\varpi_0(P)(z, z + w)| \leq \sum_{\alpha \in \mathbb{N}_{n-1}} \frac{C}{(2\pi)^{n} |(w')^{1_{n-1}}| \left(\frac{2K(c_0 \varepsilon_\alpha + |w_1|)}{w'}\right)^{\alpha}} < \infty,
$$

hence $\varpi_0(P)(z, z + w)$ is holomorphic
Therefore \( \varpi_0(P)(z, w) \, dw \) defines a germ of \( \mathcal{S}^k_{X, t_0} \). There exists \( \delta > 0 \) such that \( \Re(\eta \zeta_1) \geq 2\delta|\eta \zeta_1| \) for any \( \eta \in S \) and \( |\arg \zeta_1| \leq \theta' \). Suppose that \( |w_1| < \delta|\eta| \). For any \( Z \in S \), there exists \( N_z \in \mathbb{N} \) such that \( \delta m_2 \geq \varepsilon \nu \) for any \( \nu \geq N_z \). Thus if \( 0 \leq t \leq 1 \), we have

\[
|e^{-(1-t)c_0\varepsilon_\nu + t(\eta + w_1)\zeta_1}| = e^{-(1-t)c_0\varepsilon_\nu} \Re \zeta_1 - t(\eta + w_1) \zeta_1) \leq e^{-(1-t)c_0\varepsilon_\nu-|w_1|} |\zeta_1| \\
\leq e^{-(1-t)c_0\varepsilon_\nu-|t\delta m_2\zeta_1|} \leq e^{-\varepsilon_\nu|\zeta_1|}.
\]

Thus we have

\[
|\langle \mathcal{F}_\eta \zeta_1, \eta \rangle \rangle \rangle - \mathcal{F}_\eta \zeta_1, c_0\varepsilon_\nu - w_1 \rangle \rangle \rangle \rangle |e^{-w_1\zeta_1}| = \left| \frac{1}{(\nu - 1)!} \int_0^{\eta} e^{-(s+w_1)\zeta_1} s^{\nu-1} ds \right| \\
\leq \left| \frac{(|\eta| + \delta|\eta| + c_0\delta m_2)\varepsilon e^{-\varepsilon_\nu|\zeta_1|}}{(\nu - 1)!} \right| \int_0^{1} t^{\nu-1} dt \leq \frac{(1 + \delta + c_0\delta|\eta|)\varepsilon e^{-\varepsilon_\nu|\zeta_1|}}{\nu!}. 
\]

Set \( K_1 := K(1 + \delta + c_0\delta) \). Choosing \( h = \frac{\varepsilon_\alpha}{2} \) in (5.10) (|\alpha| \geq N_z) we have

\[
|\varpi_\alpha(P)(z, w_1, \eta) - \varpi_0(P)(z, w_1)| \leq \frac{C\varepsilon_{2\alpha}(K_1|\eta|)\varepsilon e^{-\varepsilon_\alpha|\zeta_1|}}{|\alpha|!} \int_0^{1} e^{-\varepsilon_\alpha|\zeta_1|/2} d|\zeta_1| \leq \frac{2C\varepsilon_{2\alpha}(K_1|\eta|)\varepsilon}{|\alpha|!} \leq \frac{C(2K_1|\eta|)^{|\alpha|}}{|\alpha|!}.
\]

Thus if \( 2K_1|\eta| < |w_1| \), we have

\[
|\varpi(P)(z, z + w, \eta) - \varpi_0(P)(z, z + w)| \leq \left| \frac{\alpha!(\varpi_\alpha(P)(z, w_1, \eta) - \varpi_\alpha(P)(z, w_1))}{(2\pi\sqrt{-1})^n (w')^{\alpha+1}|\gamma_{n-1}} \right| + \sum_{|\alpha| \geq N_z} \frac{C}{(2\pi)^{|\alpha|}(w')^{1|\alpha_{n-1}}} \left( \frac{2K_1|\eta|}{w'} \right)^{|\alpha|} \leq \frac{C(2K_1|\eta|)^{|\alpha|}}{|\alpha|!}.
\]

hence \( \varpi(P)(z, z + w, \eta) - \varpi_0(P)(z, z + w) \) is holomorphic on

\[
\bigcup_{Z \in S} \bigcap_{2 \leq \zeta \leq n} \{ (z, w, \eta) \in \mathbb{C}^2n \times Z; \|z\| < r_0, |w_1| < \delta|\eta|, 2K_1|\eta| < |w_1| \} \\
= \bigcap_{2 \leq \zeta \leq n} \{ (z, w, \eta) \in \mathbb{C}^2n \times Z; \|z\| < r_0, |w_1| < \delta|\eta|, 2K_1|\eta| < |w_1| \}.
\]

Therefore we have \( |\varpi(P)(z, w, \eta) \, dw| = |\varpi_0(P)(z, w) \, dw| \in \lim_{\kappa} \mathbb{E}^k_X(\kappa) \).
for any $h > 0$ there exists a constant $C_h > 0$ such that
\[
|P_\nu(z, \xi)| \leq \frac{C_h \nu! A^\nu e^{h||\xi||}}{||\xi||^\nu} (\nu \in \mathbb{N}_0, (z; \xi) \in \Omega_\rho[d_\rho]).
\]

(2) $P(t; z, \xi, \eta) = \sum_{\nu=0}^\infty t^\nu P_\nu(z, \xi, \eta) \in \mathcal{N}_c(\Omega)$ is an element of \( \mathcal{N}_c(\Omega) \) if there exists a constant $A > 0$ satisfying the following: for any $h > 0$ there exists a constant $C_h > 0$ such that
\[
\left|\sum_{\nu=0}^{m-1} P_\nu(z, \xi)\right| \leq \frac{C_h m! A^m e^{h||\xi||}}{||\xi||^m} (m \in \mathbb{N}, (z; \xi) \in \Omega_\rho[d_\rho]).
\]

(3) We set
\[
\mathcal{N}_{c, z_0} := \lim_{\Omega \to \Omega} \mathcal{N}_c(\Omega) \supset \mathcal{N}_{c, z_0}^* := \lim_{\Omega \to \Omega} \mathcal{N}_c(\Omega).
\]

We call each element of $\mathcal{N}_c(\Omega)$ (resp. $\mathcal{N}_c(\Omega)$) a classical formal symbol (resp. classical formal null-symbol) on $\Omega$.

6.2. Definition. Let $t$ be an indeterminate. Then we define a set $\mathcal{N}_c(\Omega; S)$ as follows: $P(t; z, \xi, \eta) = \sum_{\nu=0}^\infty t^\nu P_\nu(z, \xi, \eta) \in \mathcal{N}_c(\Omega; S)$ if

(i) $P(t; z, \xi, \eta) \in \Gamma(\Omega_\rho[d_\rho] \times S; \mathcal{O}_T) \llbracket t \rrbracket$ for some $d > 0$ and $\rho \in ]0, 1[$,

(ii) there exists a constant $A > 0$, and for any $Z \in S$, $h > 0$, there exists $C_{h,Z} > 0$ such that
\[
\left|\sum_{\nu=0}^{m-1} P_\nu(z, \xi, \eta)\right| \leq \frac{C_{h,Z} m! A^m e^{h||\xi||}}{||\xi||^m} (m \in \mathbb{N}, (z; \xi, \eta) \in \Omega_\rho[d_\rho] \times Z).
\]

6.3. Definition. We define a set $\mathcal{N}_c(\Omega; S)$ as follows: $P(t; z, \xi, \eta) = \sum_{\nu=0}^\infty t^\nu P_\nu(z, \xi, \eta) \in \mathcal{N}_c(\Omega; S)$ if

(i) $P(t; z, \xi, \eta) \in \Gamma(\Omega_\rho[d_\rho] \times S; \mathcal{O}_T) \llbracket t \rrbracket$ for some $d > 0$ and $\rho \in ]0, 1[$,

(ii) there exists a constant $A > 0$, and for any $Z \in S$, $h > 0$ there exists $C_{h,Z} > 0$ such that
\[
|P_\nu(z, \xi, \eta)| \leq \frac{C_{h,Z} \nu! A^\nu e^{h||\xi||}}{||\xi||^\nu} (\nu \in \mathbb{N}_0, (z; \xi, \eta) \in \Omega_\rho[d_\rho] \times Z).
\]

(iii) $\partial_\eta P(t; z, \xi, \eta) \in \mathcal{N}_c(\Omega; S)$.

We call each element of $\mathcal{N}_c(\Omega; S)$ (resp. $\mathcal{N}_c(\Omega; S)$) a classical formal symbol (resp. classical formal null-symbol) on $\Omega$ with an apparent parameter in $S$.

6.4. Lemma. $\mathcal{N}_c(\Omega; S) \subset \mathcal{N}_c(\Omega; S)$.

Proof. We assume (6.1). Take $C', B > 0$ as $2(\nu + 1)A^\nu+1 \leq d_\rho C'B^\nu$ and $2A^\nu \leq C'B^\nu$ for any $\nu \in \mathbb{N}_0$. Then for any $\nu \in \mathbb{N}$ and $(z; \xi, \eta) \in \Omega_\rho[d_\rho] \times Z$ we have
\[
|P_\nu(z, \xi, \eta)| = \left|\sum_{i=0}^\nu P_i(z, \xi, \eta) - \sum_{i=0}^{\nu-1} P_i(z, \xi, \eta)\right| \leq \left|\sum_{i=0}^\nu P_i(z, \xi, \eta)\right| + \left|\sum_{i=0}^{\nu-1} P_i(z, \xi, \eta)\right|
\leq \frac{C_{h,Z} (\nu + 1)! A^\nu+1 e^{h||\xi||}}{||\xi||^\nu+1} + \frac{C_{h,Z} \nu! A^\nu e^{h||\xi||}}{||\xi||^\nu} \leq \frac{C'C_{h,Z} \nu! B^\nu e^{h||\xi||}}{||\eta||^\nu||\eta||^\nu}.\]
Next, for any \( Z \in S \), take \( \delta' \) and \( Z' \) as in (4.3). Then by the Cauchy inequality, for any \( h > 0 \) there exists a constant \( C_{h,z'} > 0 \) such that for any \( m \in \mathbb{N} \) and \( (z; \zeta, \eta) \in \Omega_{\rho}[d_\rho] \times Z \),

\[
\left| \sum_{\nu=0}^{m-1} \partial_\nu P_\nu(z, \zeta, \eta) \right| \leq \frac{1}{\delta'|\eta|} \sup_{|\eta'-\eta|=\delta'|\eta|} \left| \sum_{\nu=0}^{m-1} P_\nu(z, \zeta, \eta') \right| \leq \frac{C_{h,z'} m! (2A)^m e^{h|\zeta|}}{\delta'|\eta|}.
\]

\[\square\]

For any \( z_0^* \in T^* X \), we set

\[
\hat{\mathcal{G}}_{cl,z_0} := \lim_{\alpha \to S} \hat{\mathcal{G}}_{cl}(\Omega; S) \supset \hat{\mathcal{N}}_{cl,z_0} := \lim_{\alpha \to S} \hat{\mathcal{N}}_{cl}(\Omega; S).
\]

6.5. **Proposition.** Let \( P(t; z, \zeta, \eta) \in \hat{\mathcal{G}}_{cl}(\Omega; S) \). Then for any \( \eta_0 \in S \), it follows that \( P(t; z, \zeta, \eta_0) \in \hat{\mathcal{G}}_{cl}(\Omega) \) and \( P(t; z, \zeta, \eta) - P(t; z, \zeta, \eta_0) \in \hat{\mathcal{N}}_{cl}(\Omega; S) \).

**Proof.** Set \( A_0 := A/|\eta_0| > A \). For any \( h > 0 \), there exists a constant \( C_{h,\eta_0} > 0 \) such that for any \( (z; \zeta) \in \Omega_{\rho}[d_\rho] \) the following holds:

\[
|P_\nu(z, \zeta, \eta_0)| \leq \frac{C_{h,\eta_0} \nu! A_0^\nu e^{h|\zeta|}}{|\zeta|^\nu};
\]

that is, \( P(t; z, \zeta, \eta_0) \in \hat{\mathcal{G}}_{cl}(\Omega) \). For any \( Z \subseteq S \), let \( Z' \subseteq S \) be the convex hull of \( Z \cup \{\eta_0\} \).

Since

\[
P_\nu(z, \zeta, \eta) = P_\nu(z, \zeta, \eta_0) + \int_{\eta_0}^{\eta} \partial_\eta P_\nu(z, \zeta, \tau) d\tau,
\]

and \( \partial_\eta P(t; z, \zeta, \eta) \in \hat{\mathcal{N}}_{cl}(\Omega; S) \), there exists \( A > 0 \) so that for any \( h > 0 \) we can find a constant \( C_{h,z'} > 0 \) such that for any \( m \in \mathbb{N} \) and \( (z; \zeta, \eta) \in \Omega_{\rho}[d_\rho] \times Z \subseteq \Omega_{\rho}[d_\rho] \times Z' \) the following holds: if \( |\eta| \geq |\eta_0| \)

\[
\left| \sum_{\nu=0}^{m-1} \left( P_\nu(z, \zeta, \eta) - P_\nu(z, \zeta, \eta_0) \right) \right| = \left| \sum_{\nu=0}^{m-1} \int_{\eta_0}^{\eta} \partial_\eta P_\nu(z, \zeta, \tau) d\tau \right| \leq \left| \eta - \eta_0 \right| \frac{C_{h,z'} m! A_0^m e^{h|\zeta|}}{|\eta_0| |\zeta|^m} \leq \frac{rC_{h,z'} m! A_0^m e^{h|\zeta|}}{|\eta_0| |\zeta|^m},
\]

and if \( |\eta| \leq |\eta_0| \)

\[
\left| \sum_{\nu=0}^{m-1} \left( P_\nu(z, \zeta, \eta) - P_\nu(z, \zeta, \eta_0) \right) \right| \leq \frac{rC_{h,z'} m! A_0^m e^{h|\zeta|}}{|\eta_0| |\zeta|^m} \leq \frac{rC_{h,z'} m! A_0^m e^{h|\zeta|}}{|\eta_0| |\zeta|^m}.
\]

Hence \( P(t; z, \zeta, \eta) - P(t; z, \zeta, \eta_0) \in \hat{\mathcal{N}}_{cl}(\Omega; S) \).

Since \( |\eta_0| < |\zeta| \) for any \( \eta \in S \), we can regard that

\[
\hat{\mathcal{G}}_{cl}(\Omega) = \{ P(t; z, \zeta, \eta) \in \hat{\mathcal{G}}_{cl}(\Omega; S) ; \partial_\eta P(t; z, \zeta, \eta) = 0 \} \subset \hat{\mathcal{G}}_{cl}(\Omega; S),
\]

\[
\hat{\mathcal{N}}_{cl}(\Omega) = \hat{\mathcal{G}}_{cl}(\Omega) \cap \hat{\mathcal{N}}_{cl}(\Omega; S) \subset \hat{\mathcal{N}}_{cl}(\Omega; S).
\]

Hence we have an injective mapping \( \hat{\mathcal{G}}_{cl}(\Omega) / \hat{\mathcal{N}}_{cl}(\Omega) \to \hat{\mathcal{G}}_{cl}(\Omega; S) / \hat{\mathcal{N}}_{cl}(\Omega; S) \). Moreover

6.6. **Proposition.** \( \hat{\mathcal{G}}_{cl}(\Omega) / \hat{\mathcal{N}}_{cl}(\Omega) \simeq \hat{\mathcal{G}}_{cl}(\Omega; S) / \hat{\mathcal{N}}_{cl}(\Omega; S) \).

**Proof.** Let us take any \( P(t; z, \zeta, \eta) \in \hat{\mathcal{G}}_{cl}(\Omega; S) \). We fix \( \eta_0 \in S \). Then by Proposition 6.5, we have \( P(t; z, \zeta, \eta_0) \in \hat{\mathcal{G}}_{cl}(\Omega) \) and \( [P(t; z, \zeta, \eta)] = [P(t; z, \zeta, \eta_0)] \in \hat{\mathcal{G}}_{cl}(\Omega; S) / \hat{\mathcal{N}}_{cl}(\Omega; S). \) \[\square\]
6.7. Proposition. \( \mathfrak{H}_{cl}(\Omega; S) \cap \Gamma(\Omega_{\rho}[d_{\rho}] \times S; \Theta_{T^*X \times \mathbb{C}}) = \mathfrak{H}(\Omega; S) \).

Proof. Let \( P(z, \zeta, \eta) \in \mathfrak{H}_{cl}(\Omega; S) \cap \Gamma(\Omega_{\rho}[d_{\rho}] \times S; \Theta_{T^*X \times \mathbb{C}}) \). Then there exists \( A > 0 \) and for any \( Z \in S, h > 0 \) there exists \( C_{h,Z} > 0 \) such that

\[
|P(z, \zeta, \eta)| \leq \frac{C_{h,Z} \nu! A^{\nu} e^{h\|\zeta\|}}{\|\eta\zeta\|^\nu} \quad (\nu \in \mathbb{N}_0, (z; \zeta, \eta) \in \Omega_{\rho}[d_{\rho}] \times Z).
\]

Then for any \( \zeta \) with \( \|\zeta\| \geq d_{\rho} \), taking \( \nu \) as the integer part of \( \frac{\|\eta\zeta\|}{A} \), we have

\[
|P(z, \zeta, \eta)| \leq C_{h,Z} \left( \frac{2\pi\|\eta\zeta\|}{A} \right)^{1/2} e^{(h-\|\eta\|/A)\|\zeta\|^{-1}}.
\]

We choose \( h = 2hA = m_z \). Hence \( e^{(h-\|\eta\|/A)\|\zeta\|} \leq e^{-\|\eta\zeta\|/(2A)} \). Then we can find \( \delta, C_Z' > 0 \) such that

\[
C_{h,Z} \left( \frac{2\pi\|\eta\zeta\|}{A} \right)^{1/2} e^{-\|\eta\zeta\|/(2A)} \leq C_Z'e^{-\delta\|\eta\zeta\|}.
\]

Here \( \delta \) is not depend on \( Z \). Thus (4.2) holds. Conversely, assume (4.2). Set \( A := \frac{1}{\delta} \).

Then for any \( m \in \mathbb{N}_0 \) and \( h > 0 \), we have

\[
|P(z, \zeta, \eta)| \leq C_Z e^{-\delta\|\eta\zeta\|} \leq C_Z m! A^m \frac{\|\eta\zeta\|^m}{\|\eta\zeta\|^m} \leq \frac{C_Z m! A^m e^{h\|\zeta\|}}{\|\eta\zeta\|^m}.
\]

Hence \( P(z, \zeta, \eta) \in \mathfrak{H}_{cl}(\Omega; S) \cap \Gamma(\Omega_{\rho}[d_{\rho}] \times S; \Theta_{T^*X \times \mathbb{C}}) \). \( \square \)

By Proposition 6.7 we can also regard \( \mathfrak{S}(\Omega; S) = \widehat{\mathfrak{S}}_{cl}(\Omega; S) \cap \Gamma(\Omega_{\rho}[d_{\rho}] \times S; \Theta_{T^*X \times \mathbb{C}}) \).

Moreover

6.8. Theorem. Let \( \Omega \subset T^*_X \) be any sufficiently small neighborhood of \( z_0^* = (z_0; \zeta_0) \in \mathfrak{T}_X \). Then for any \( P(t; z, \zeta, \eta) = \sum_{\nu=0}^{\infty} t^\nu P_{\nu}(z, \zeta, \eta) \in \widehat{\mathfrak{S}}_{cl}(\Omega; S) \), there exists \( P(z, \zeta) \in \mathcal{S}(\Omega) \) such that

\[
P(t; z, \zeta, \eta) - P(z, \zeta) \in \mathfrak{H}_{cl}(\Omega; S).
\]

Proof. We may assume that \( \zeta_0 = (1, 0, \ldots, 0), \; \Omega \subset \mathfrak{S}_{conic}\{ (z; \zeta); \Re \zeta_1 \geq 2\delta_0|\zeta_1| \} \) for some \( 0 < \delta_0 < \frac{1}{2} \), \( \|\zeta\| = |\zeta_1| \) on \( \Omega_{\rho}[d_{\rho}] \), and \( P_\nu(z, \zeta, \eta) \in \Gamma(\Omega_{\rho}[d_{\rho}] \times S; \Theta_{T^*X \times \mathbb{C}}) \). Fix \( \eta_0 \in S \).

Thus \( P(t; z, \zeta, \eta_0) \in \mathfrak{S}_{cl}(\Omega) \) by Proposition 6.5. Set \( A_0 := A/|\eta_0| \). We take a constant \( a \) as

\[
0 < a < \min\{1, \frac{1}{2A_0}\}
\]

and set \( B := \max\{\frac{1}{\delta_0 A}, \frac{A_0}{2\delta_0}\} \). Using the function \( \Gamma_\nu(\tau, a) \) in Definition 5.2, we set

\[
P(z, \zeta) := \sum_{\nu=0}^{\infty} P_\nu(z, \zeta, \eta_0) \zeta_1^\nu \Gamma_\nu(\zeta_1, a).
\]

By (6.3) and (5.8) for any \( h > 0 \) on \( \Omega_{\rho}[d_{\rho}] \) we have

\[
|P(z, \zeta)| \leq \sum_{\nu=0}^{\infty} \frac{C_{h,\nu}! A_{\nu}^\nu e^{h\|\zeta\|}}{\|\zeta\|^\nu \nu!} \leq 2C_{h} e^{h\|\zeta\|}.\]
Therefore $P(z, \zeta) \in \mathcal{S}(\Omega)$. On the other hand, for any $m \in \mathbb{N}$, on $\Omega_{\rho}[d_{\rho}]$ we have

$$|P(z, \zeta) - \sum_{\nu=0}^{m-1} P_{\nu}(z, \zeta, \eta_0)| \lesssim \sum_{\nu=1}^{m-1} |P_{\nu}(z, \zeta, \eta_0)| \left|1 - \zeta^{\nu}_{1} \Gamma_{\nu}(\zeta_{1}, a)\right| + \sum_{\nu=m}^{\infty} |P_{\nu}(z, \zeta, \eta_0)| \zeta^{\nu}_{1} \Gamma_{\nu}(\zeta_{1}, a)|.$$

For any $1 \leq \nu \leq m - 1$, we have $e^{-a \delta_{0}||\zeta||} = e^{-a \delta_{0}||\zeta||} \leq \frac{(m-\nu)!}{(a \delta_{0}||\zeta||)^{m-\nu}}$. Thus by (5.9) and (6.3) we have

$$\sum_{\nu=1}^{m-1} |P_{\nu}(z, \zeta, \eta_0)| \left|1 - \zeta^{\nu}_{1} \Gamma_{\nu}(\zeta_{1}, a)\right| \leq \sum_{\nu=1}^{m-1} \frac{\delta_{0} C_{h \nu} ! A_{0}^{\nu} e^{h ||\zeta||} e^{-a \delta_{0}||\zeta||}}{\left(\delta_{0} ||\zeta||\right)^{\nu}} \leq \frac{\delta_{0} C_{h m} ! B^{m} e^{h ||\zeta||}}{\left(\delta_{0} ||\zeta||\right)^{m}} \sum_{\nu=1}^{m} (a A_{0})^{\nu} \leq \frac{\delta_{0} C_{h m} ! B^{m} e^{h ||\zeta||}}{||\zeta||^{m}}.$$

Next, since

$$\int_{0}^{a} e^{-2 \delta_{0} s ||\zeta||} s^{k+m-1} ds < a^{k} \int_{0}^{\infty} e^{-2 \delta_{0} s ||\zeta||} s^{m-1} ds = \frac{a^{k}(m-1)!}{(2 \delta_{0} ||\zeta||)^{m}},$$

we have

$$\sum_{\nu=m}^{\infty} |P_{\nu}(z, \zeta, \eta_0)| \zeta^{\nu}_{1} \Gamma_{\nu}(\zeta_{1}, a) \leq C_{h} e^{h ||\zeta||} \sum_{\nu=m}^{\infty} \frac{\nu A_{0}^{\nu} |\zeta_{1}|^{\nu}}{||\zeta||^{\nu}} \int_{0}^{a} e^{-2 \delta_{0} s ||\zeta||} s^{\nu-1} ds \leq C_{h} e^{h ||\zeta||} \sum_{k=0}^{\infty} (k+m) A_{0}^{k+m} \int_{0}^{a} e^{-2 \delta_{0} s ||\zeta||} s^{k+m-1} ds \leq C_{h} (m-1)! A_{0}^{m} e^{h ||\zeta||} \sum_{k=0}^{\infty} (k+m)(a A_{0})^{k} \leq C_{h} (m-1)! B^{m} e^{h ||\zeta||} \frac{2(m+1)}{||\zeta||^{m}} \leq \frac{4 C_{h} m! B^{m} e^{h ||\zeta||}}{||\zeta||^{m}}.$$

Thus by Proposition 6.5, we have

$$P(t; z, \zeta, \eta) - P(z, \zeta) = (P(t; z, \zeta, \eta) - P(t; z, \zeta, \eta_0)) + (P(t; z, \zeta, \eta_0) - P(z, \zeta))$$

$$\in \hat{\mathfrak{N}}_{cl}(\Omega; S) + \mathfrak{A}_{cl}(\Omega) \subset \hat{\mathfrak{N}}_{cl}(\Omega; S). \quad \square$$

6.9. Corollary. Let $\Omega \subset T^{*} X$ be any sufficiently small neighborhood of $z_{0}^{*} = (z_{0}; \zeta_{0}) \in \check{T}^{*} X$. Then $\mathfrak{S}(\Omega; S)/\mathfrak{M}(\Omega; S) \simeq \hat{\mathfrak{S}}_{cl}(\Omega; S)/\hat{\mathfrak{N}}_{cl}(\Omega; S)$.

6.10. Definition. As in the case of $\mathfrak{S}(\Omega; S)$, for any $P(t; z, \zeta, \eta) \in \hat{\mathfrak{S}}_{cl}(\Omega; S)$ we set

$$:P(t; z, \zeta, \eta): := P(t; z, \zeta, \eta) \mod \hat{\mathfrak{N}}_{cl}(\Omega; S) \in \hat{\mathfrak{S}}_{cl}(\Omega; S)/\hat{\mathfrak{N}}_{cl}(\Omega; S)$$

which is also called the normal product or the Wick product of $P(t; z, \zeta, \eta)$.

Take $\Omega \subset T^{*} \mathbb{C}^{n}$. Let $z = (z_{1}, \ldots, z_{n})$ and $w = (w_{1}, \ldots, w_{n})$ be local coordinates on a neighborhood of $\text{Cl} \pi(\Omega) \subset X$, and $(z; \zeta_{1}, (w; \lambda))$ corresponding local coordinates on a neighborhood of $\text{Cl} \Omega$. Let $z = \Phi(w)$ be the coordinate transformation. We define $J_{\Phi}^{*}(z')$ by the relation $\Phi^{-1}(z') = J_{\Phi}^{*}(z')$. Then $J_{\Phi}^{*}(z, z)\lambda = \frac{\partial w}{\partial z}(z)\lambda = \zeta.$
Moreover if \( P = \epsilon < \Omega \) we have
\[
\Phi^* P(t; w, \lambda, \eta) := e^{(\partial_{z', \partial z})} P(t; \Phi(w), \zeta' + \bar{J}_\Phi^*(\Phi(w) + z', \Phi(w))\lambda, \eta) \bigg|_{z' = 0},
\]
i.e.
\[
(\Phi^* P)_v(w, \lambda, \eta) = \sum_{k + |\alpha| = \nu} \frac{1}{\alpha!} \partial_{\zeta}^\alpha \partial_{z'}^\alpha P_k(\Phi(w), \zeta' + \bar{J}_\Phi^*(\Phi(w) + z', \Phi(w))\lambda, \eta) \bigg|_{z' = 0}.
\]

6.11. **Theorem.** (1) \( \Phi^* P(t; w, \lambda, \eta) \) defines an element of \( \widehat{\mathcal{S}}_c(\Omega; S) \) with respect to \( (w; \lambda) \). Moreover if \( P(t; z, \zeta, \eta) \in \widehat{\mathcal{N}}_c(\Omega; S) \), it follows that \( \Phi^* P(t; w, \lambda, \eta) \in \widehat{\mathcal{N}}_c(\Omega; S) \).

(2) \( 1^* \) is the identity, and if \( z = \Phi(w) \) and \( w = \Psi(v) \) are complex coordinate transformations, \( \Psi^* \Phi^* = (\Phi \Psi)^* \) holds.

**Proof.** (1) Assume that \( P_\nu(z, \zeta, \eta) \in \Gamma(\Omega_\rho[d_\rho] \times S; \mathcal{E}_{T^*X \times \mathbb{C}}) \). Note that \( \partial_\eta \Phi^* P(t; w, \lambda, \eta) = \Phi^* (\partial_\eta P)(t; w, \lambda, \eta) \). We may assume that on a neighborhood of \( \Omega_\rho \), there exist \( c > 1 \) and \( 0 < c' < 1 \) such that
\[
c' \|\lambda\| < \|\zeta\| = \left\| t \left[ \frac{\partial w}{\partial z}(z) \right] \lambda \right\| < c \|\lambda\|
\]
We can choose \( 0 < \epsilon < 1 \) such that \( c' - c \epsilon > 0 \), and that there exists \( \delta > 0 \) such that if \( \|z'\| \leq \delta \), it follows that
\[
\left\{ \begin{array}{l}
\| \bar{J}_\Phi(z + z', z)\lambda - \zeta\| \leq \epsilon \|\zeta\|, \\
\|z'\| \leq \epsilon \|\zeta'\| \leq \|\zeta\|.
\end{array} \right.
\]
Moreover take \( \rho' \in [0, \rho] \) and replacing \( \epsilon, \delta > 0 \) if necessary, setting
\[
\Omega_\rho' := \bigcup_{\Omega_\rho}[d_\rho \times Z, (z; \zeta' + \bar{J}_\Phi^*(z + z', z)\lambda); \|z'\| \leq \delta, \|\zeta'\| \leq \epsilon \|\zeta\|]
\]
we have
\[
\Omega_\rho' \in \mathcal{N}_c[\mathcal{E}_{T^*X \times \mathbb{C}}].
\]
Thus for any \( h > 0, Z \in S \) and \( (z; \zeta' + \bar{J}_\Phi^*(z + z', z)\lambda, \eta) \in \Omega_\rho'[d_\rho] \times Z \), we have
\[
|P_k(z, z' + \bar{J}_\Phi^*(z + z', z)\lambda, \eta) - C_h x k! A^k \|\zeta' + \bar{J}_\Phi^*(z + z', z)\lambda\| \|\eta(\zeta' + \bar{J}_\Phi^*(z + z', z)\lambda)\|^k\|.
\]
Set \( \Phi(w) := z \). If \( (z; \zeta, \eta) \in \Omega_\rho'[d_\rho] \times Z \), we have
\[
\left( \frac{1}{\alpha!} \right) \left| \partial_{\zeta}^\alpha \partial_{z'}^\alpha P_k(z, \zeta' + \bar{J}_\Phi^*(z + z', z)\lambda, \eta) \right|_{z' = 0} \leq \sup_{|\alpha| \leq 0} \|P_k(z, \zeta' + \bar{J}_\Phi^*(z + z', z)\lambda, \eta)\|
\]
\[
\leq \frac{\alpha!}{(\epsilon \delta \|\zeta\|)^{|\alpha|}} \sup_{|\alpha| \leq 0} \|P_k(z, \zeta' + \bar{J}_\Phi^*(z + z', z)\lambda, \eta)\|
\]
for any fixed \( z \), to show that
\[
\text{Set } B := \frac{2}{\varepsilon \delta c'} \text{ and replacing } \varepsilon, \delta > 0 \text{ as } C := \frac{\varepsilon \delta c' A}{2(c' - c \varepsilon)} < 1. \text{ Then if } \| \zeta \| \geq c' \| \lambda \| \geq (\nu + 1)d_{\rho'}, \text{ we have}
\]
\[
\left\| \langle \Phi^* P \rangle_{\nu} (w, \lambda, \eta) \right\| \leq C_{h, Z} e^{2hc|\lambda|} \sum_{k=0}^{\nu} \frac{k! A^k}{(\varepsilon \delta c' \| \lambda \|)^{\nu-k}} \sum_{|\alpha| = \nu-k} \frac{\alpha!}{(\varepsilon \delta c' \| \lambda \|)^{\nu-k}} \leq C_{h, Z} e^{2hc|\lambda|} \sum_{k=0}^{\nu} \frac{k! A^k}{(\varepsilon \delta c' \| \lambda \|)^{\nu-k}} \leq 2^{\nu-1} C_{h, Z} m! e^{2hc|\lambda|} \leq \frac{2^{\nu-1} C_{h, Z} m! e^{2hc|\lambda|}}{(1-C) \| \eta \| \nu}. \]

Next, if \( P(t; z, \zeta, \eta) \in \mathfrak{H}_{cl}(\Omega; S) \), for any \( m \in \mathbb{N} \)
\[
\left\| \sum_{\nu=0}^{m-1} \frac{1}{\alpha!} \partial_{\zeta}^{\alpha} \partial_{z}^{\nu} P_k(z, \zeta', \eta) \right\|_{\eta'=0} \leq \frac{C_{h, Z} m! A^m e^{2hc|\lambda|}}{(\varepsilon \delta c' \| \lambda \|)^{m}}. \]

Hence setting \( \Phi(w) = \), we have
\[
\left\| \sum_{\nu=0}^{m-1} \frac{1}{\alpha!} \partial_{\zeta}^{\alpha} \partial_{z}^{\nu} P_k(z, \zeta', \eta) \right\|_{\eta'=0} \leq \frac{C_{h, Z} m! A^m e^{2hc|\lambda|}}{(\varepsilon \delta c' \| \lambda \|)^{m}}. \]

(2) It is trivial that \( 1^* \) is the identity. In order to prove that \( \Phi^* \Phi = (\Phi \Phi)^* \), it is enough to show that \( \Phi^* \Phi^* P(t; v, \xi, \eta) = (\Phi \Phi)^* P(t; v, \xi, \eta) \) for any \( P(t; z, \zeta, \eta) \in \mathcal{E}_{T^* X \times S(z_0, \zeta_0, \eta_0)} [t] \) for any fixed \( (z_0; \zeta_0) \in \text{Cl} \Omega \). Note that \( \left[ \frac{\partial_v}{\partial_\zeta} \right] \lambda = \zeta \) and \( \left[ \frac{\partial_v}{\partial_\eta} \right] (w) \xi = \lambda \).

6.12. Lemma (see [4], [28]). For any \( n \)-tuple \( A(t; z, \zeta) = (A_1(z, \zeta), \ldots, A_n(z, \zeta)) \) of holomorphic functions, and holomorphic function \( Q(z, \zeta) \), the following holds:
\[
e^{\langle \partial_{\zeta}, \partial_z \rangle} Q(z, \zeta) e^{z(A(z, \zeta))} \bigg|_{z=0} = e^{\langle \partial_{\zeta}, \partial_z \rangle} \sum_{\alpha \in \mathbb{N}_0^n} \frac{1}{\alpha!} \partial_{\zeta}^{\alpha} \left( Q(z, \zeta) A(z, \zeta)^\alpha \right) \bigg|_{z=0}. \]

Proof. We have:
\[
e^{\langle \partial_{\zeta}, \partial_z \rangle} Q(z, \zeta) e^{z(A(z, \zeta))} \bigg|_{z=0} = \sum_{\alpha, \beta \in \mathbb{N}_0^n} \frac{1}{\alpha! \beta!} \partial_{\zeta}^{\beta} \partial_{z}^{\beta} Q(z, \zeta) z^{\alpha} A(z, \zeta)^\alpha \bigg|_{z=0}
= \sum_{\alpha, \beta} \sum_{\gamma < \beta} \left( \beta \right) \frac{1}{\alpha! \beta!} \partial_{\zeta}^{\beta} \left( \frac{\partial^\gamma z^{\alpha}}{\partial z^{\gamma}} \partial_{z}^{\beta-\gamma} Q(z, \zeta) A(z, \zeta)^\alpha \right) \bigg|_{z=0}.\]
\[
\begin{align*}
&= \sum_{\alpha \leq \beta} \frac{1}{\alpha!(\beta - \alpha)!} \partial_\zeta^\beta \partial_{z-Q}^{\beta-\alpha}(Q(z, \zeta) A(z, \zeta)^\alpha)|_{z=0} \\
&= e^{(\partial_\zeta, \partial_z)} \sum_{\alpha \in \mathbb{N}_0^n} \frac{1}{\alpha!} \partial_\zeta^\alpha (Q(z, \zeta) A(z, \zeta)^\alpha)|_{z=0}.
\end{align*}
\]

**6.13. Remark.** Take \(\tau \in \mathbb{C}\) and \(z' \in \mathbb{C}^n\) such that \(|\tau z'|\) is sufficiently small, hence \(J^*_\Phi(z + \tau z', z)\) is holomorphic. Then by Lemma 6.12 for any holomorphic function \(Q(\zeta)\), we have
\[
e^{(\partial_\zeta, \partial_z)} Q(\zeta') e^{(z', J^*_\Phi(z + \tau z', z) \lambda - \zeta_0)|_{\zeta'=0}} = \sum_{\alpha} \frac{1}{\alpha!} \partial_\zeta^\alpha \left( Q(\zeta')(J^*_\Phi(z + \tau z', z) \lambda - \zeta_0) \right)|_{\zeta'=0}
\]

Hence as a formal power series with respect to \(t\), we have
\[
e^{(\partial_\zeta, \partial_z)} Q(\zeta') e^{(z', J^*_\Phi(z + \tau z', z) \lambda - \zeta_0)|_{\zeta'=0}} = \sum_{\alpha} \frac{1}{\alpha!} \partial_\zeta^\alpha \partial_{z'}^\alpha Q(\zeta' + J^*_\Phi(z + z', z) \lambda - \zeta_0)|_{\zeta'=0}
\]

In what follows, we use this type of arguments.

By Lemma 6.12 and Remark 6.13, setting \(\Phi(w) = z\) we have
\[
e^{(\partial_{z'}, \partial_{z})} P(t; z, \zeta_0 + \zeta', \eta) e^{(J^*_\Phi(z + \tau z', z) \lambda - \zeta_0)|_{\zeta'=0}} = \sum_{\nu=0}^{\infty} \frac{1}{\nu!} \partial_{z'}^\nu \partial_{\zeta'}^\nu Q(\zeta' + J^*_\Phi(z + z', z) \lambda - \zeta_0)|_{\zeta'=0}
\]

On the other hand, we have
\[
w + J^*_\Phi(z + z', z) z' = \Phi^{-1}(z) + \Phi^{-1}(z + z') - \Phi^{-1}(z) = \Phi^{-1}(z + z').
\]

Hence we have
\[
J^*_\Phi(w + J^*_\Phi(z + z', z) z', w) J^*_\Phi(z + z', z) z' = J^*_\Phi(\Phi^{-1}(z + z'), \Phi^{-1}(z)) J^*_\Phi(z + z', z)(z + z' - z)
\]

\[
= J^*_\Phi(\Phi^{-1}(z + z'), \Phi^{-1}(z)) (\Phi^{-1}(z + z') - \Phi^{-1}(z))
\]

\[
= \Phi^{-1}(z + z') - \Phi^{-1}(z) = J^*_\Phi(z + z', z) z'.
\]

Thus as a formal power series with respect to \(t\), we have
\[
J^*_\Phi(w + J^*_\Phi(z + t z', z) t z', w) J^*_\Phi(z + t z', z) z' = J^*_\Phi(w + t z', z) z'.
\]
Set $z_0 := \Phi(w_0)$ and \( \frac{\partial \Phi}{\partial z}(z_0) \lambda_0 = \zeta_0 \). Then on a neighborhood of \((w_0; \eta_0)\), setting \( z = \Phi(w) = \Phi \Psi(v) \) we have

\[
\Psi^* \Phi^* P(t; v, \xi, \eta) = e^{(\partial_x, \partial_x')} \Phi^* P(t; w, \lambda_0 + \lambda', \eta) e^{(J^*_\theta (w + tw', w') \xi)} e^{-\langle \eta', \lambda_0 \rangle} \bigg|_{\lambda' = 0}^{w' = 0} \\
= e^{(\partial_x, \partial_x')} \left( \left( \left( J^*_\theta (z + tz', z') \lambda_0 + \lambda', \eta \right) \right) e^{(J^*_\theta (w + tw', w') \xi)} e^{(\langle \eta, \lambda_0 \rangle)} \right) \bigg|_{\lambda' = 0}^{w' = 0}.
\]

By Lemma 6.12 and Remark 6.13 we have

\[
e^{(\partial_x, \partial_x')} e^{(J^*_\theta (z + tz', z') \lambda_0 + \lambda', \eta)} e^{(J^*_\theta (w + tw', w') \xi)} e^{-\langle \eta', \lambda_0 \rangle} \bigg|_{\lambda' = 0}^{w' = 0} = \left( \Phi \Psi^* \right) P(t; v, \xi, \eta).
\]

6.14. **Definition.** Under the notation above, we define a coordinate transformation \( \Phi^* \) associated with \( \Phi \) by

\[
\Phi^* (P^*): (t; w, \lambda, \eta) := \Phi^* P(t; w, \lambda, \eta).
\]

6.15. **Lemma.** Let \( \Omega \subset T^* X \) be a conic open subset, \( d > 0 \), \( \rho \in ]0, 1[ \). Assume that \( P(z, \zeta, \eta) \in \Gamma(\Omega \rho [d \rho] \times S; \Theta_{T^* X \times \mathbb{C}}) \) and \( n, m, N \in \mathbb{N}_0 \) satisfy the following:

- for any \( h > 0 \) and \( Z \in S \), there exists a constant \( C_{h,Z} > 0 \) such that for any \( \rho' \in ]0, \rho[ \), on \( \Omega \rho [d \rho] \times Z \)

\[
|P(z, \zeta, \eta)| \leq \frac{C_{h,Z} e^{h \| \zeta \|}}{(\rho - \rho')^{N \| \eta \| \zeta \|^m}}.
\]

Then for any \( \alpha, \beta \in \mathbb{N}_0^n \) \((|\alpha| + |\beta| > 0)\), \( \rho' \in ]0, \rho[ \), and \((z; \zeta, \eta) \in \Omega \rho [d \rho] \times Z \), the following hold:

1. If \( n \in \mathbb{N} \), set \( C_{\nu} := \frac{\nu + 1}{\nu} \). Then

\[
|\partial_z^\alpha \partial_\zeta^\beta P(z, \zeta, \eta)| \leq \frac{C_{h,Z} C_{\nu} e^{N (\nu + 1) |\alpha + \beta|} |\beta|! e^{2h \| \zeta \|}}{(\rho - \rho')^{(\alpha + \beta) |\alpha + \beta| + N \| \eta \| \zeta \|^m + |\beta|}} \quad (\beta \neq 0),
\]

\[
|\partial_z^\alpha P(z, \zeta, \eta)| \leq \frac{C_{h,Z} e^{N (\nu + 1) |\alpha|} |\alpha|! e^{h \| \zeta \|}}{(\rho - \rho')^{(\alpha + \beta) |\alpha| + N \| \eta \| \zeta \|^m}} \quad (\beta = 0).
\]
If \( \nu = 0 \), then
\[
|\partial_z^\alpha \partial_\zeta^\beta P(z, \zeta, \eta)| \leq \frac{C_{h,Z} \alpha! \beta! e^{2h||\zeta||}}{(\rho - \rho')^{\alpha+\beta}(1 - \rho + \rho')^m ||\eta\zeta||^{m+|\beta|}} \quad (\beta \neq 0),
\]
\[
|\partial_z^\alpha P(z, \zeta, \eta)| \leq \frac{C_{h,Z} \alpha! e^{h||\zeta||}}{(\rho - \rho')^{|\alpha|}||\eta\zeta||^m} \quad (\beta = 0).
\]

Proof. (1) Set \( \rho'' := \rho' + \frac{\rho - \rho'}{\nu + 1} \). Note that for any \((z, \zeta) \in \Omega_{\rho'}[d_{\rho'}]\) and \((z', \zeta')\) with \(||z'|| \leq \frac{\rho - \rho'}{\nu + 1} \) and \(||\zeta'|| \leq \frac{(\rho - \rho')||\eta\zeta||}{\nu + 1} < ||\zeta||\), we have \((z + z'; \zeta + \zeta') \in \Omega_{\rho''}[d_{\rho''}]\). Indeed, by the definition there exists \((z_0, \zeta_0) \in \Omega\) such that \(||z - z_0|| \leq \rho'\) and \(||\zeta - \zeta_0|| \leq \rho'||\zeta_0||\), hence we have \(||\zeta|| \leq (\nu + 1)||\zeta_0|| \leq 2||\zeta_0||\). Recall that we assumed that \(S \subset \{ \eta \in \mathbb{C}; |\eta| < \frac{1}{2} \}\). Therefore
\[
||\zeta + \zeta' - \zeta_0|| \leq \rho'||\zeta_0|| + \frac{\rho - \rho'}{\nu + 1}||\eta\zeta|| \leq \rho''||\zeta_0|| = \rho''||\zeta_0||.
\]
Further we have
\[
2||\zeta|| \geq ||\zeta + \zeta'|| \geq \left(1 - \frac{|\eta(\rho - \rho')|}{\nu + 1} \right)||\zeta|| \geq \left(1 - \frac{\rho - \rho'}{\nu + 1} \right)||\zeta|| \geq d(1 - \rho''),
\]
hence \((z + z'; \zeta + \zeta') \in \Omega_{\rho''}[\nu + 1]d_{\rho''}\). Thus replacing \(\rho'\) with \(\rho''\) in (6.8), for any \(h > 0\) and \((z; \zeta, \eta) \in \Omega_{\rho'}[d_{\rho'}] \times Z\) we have
\[
\sup_{|\zeta'| - (\rho - \rho', \nu + 1)}|P(z + z', \zeta + \zeta', \eta)| \leq \frac{C_{h,Z} e^{2h||\zeta||}}{(1 - \frac{1}{\nu + 1})^{N\nu} (1 - \frac{\rho - \rho'}{\nu + 1})^{m}(\nu + 1)^{m}} \leq \frac{C_{h,Z} e^{2h||\zeta||}}{(\rho - \rho')^{N\nu} ||\eta\zeta||^m}.
\]

Therefore if \(\beta \neq 0\), we have
\[
|\partial_z^\alpha \partial_\zeta^\beta P(z, \zeta, \eta)| \leq \frac{(\nu + 1)^{|\alpha+\beta|} \alpha! \beta!}{(\rho - \rho')^{\alpha+\beta} ||\eta\zeta||^{|\beta|}} \sup_{|\zeta'| - (\rho - \rho', \nu + 1)}|P(z + z', \zeta + \zeta', \eta)| \leq \frac{C_{h,Z} e^{N(\nu + 1)^{|\alpha+\beta|} \alpha! \beta!} e^{2h||\zeta||}}{(\rho - \rho')^{\alpha+\beta} ||\eta\zeta||^m + N\nu}.
\]

If \(\beta = 0\), we have
\[
|\partial_z^\alpha P(z, \zeta, \eta)| \leq \frac{(\nu + 1)^{|\alpha|} \alpha!}{(\rho - \rho')^{\alpha}} \sup_{|\zeta'| - (\rho - \rho', \nu + 1)}|P(z + z', \zeta, \eta)| \leq \frac{C_{h,Z} e^{N(\nu + 1)^{|\alpha|} \alpha!} e^{h||\zeta||}}{(\rho - \rho')^{\alpha} ||\eta\zeta||^m + N\nu}.
\]

(2) We may choose \(|z'| = \rho - \rho'\) and \(|\zeta'| = (\rho - \rho')||\eta\zeta||\). \(\square\)
6.16. **Theorem.** For any $P(t; z, \zeta), Q(t; z, \zeta) \in \mathcal{S}_{cl}(\Omega; S)$, set

$$Q \circ P(t; z, \zeta, \eta) := e^{t(\partial_{\zeta}, \partial_{\zeta}, \partial_{\zeta})}Q(t; z, \zeta', \eta) P(t; z', \zeta, \eta) \bigg|_{z' = z, \zeta' = \zeta}$$

$$= e^{t(\partial_{\zeta}, \partial_{\zeta})}Q(t; z, \zeta + \zeta', \eta) P(t; z + z', \zeta, \eta) \bigg|_{z' = 0}.$$ 

1. $Q \circ P(t; z, \zeta, \eta) \in \mathcal{S}_{cl}(\Omega; S)$. Moreover if either $P(t; z, \zeta, \eta)$ or $Q(t; z, \zeta, \eta)$ is an element of $\mathcal{S}_{cl}(\Omega; S)$, it follows that $Q \circ P(t; z, \zeta, \eta) \in \mathcal{S}_{cl}(\Omega; S)$.

2. $R \circ (Q \circ P) = (R \circ Q) \circ P$ holds.

3. Let $\Phi(w) = z$ be a holomorphic coordinate transformation. Then

$$\Phi^* Q \circ \Phi^* P(t; w, \lambda, \eta) = \Phi^*(Q \circ P)(t; w, \lambda, \eta).$$

**Proof.** (1) We assume $P(t; z, \zeta, \eta) = \sum_{\nu = 0}^{\infty} t^\nu P_\nu(z, \zeta, \eta), Q(t; z, \zeta, \eta) = \sum_{\nu = 0}^{\infty} t^\nu Q_\nu(z, \zeta, \eta) \in \Gamma(\Omega_\rho[d_\rho] \times S; \mathcal{O}_{T^*X \times C})[t]$. If we set $Q \circ P(t; z, \zeta, \eta) = \sum_{\nu = 0}^{\infty} t^\nu R_\nu(z, \zeta, \eta)$, we have

$$R_\nu(z, \zeta, \eta) = \sum_{|\alpha| + k + l = \nu} \frac{1}{\alpha!} \partial_\zeta^\alpha Q_\nu(z, \zeta, \eta) \cdot \partial_z^k P_k(z, \zeta, \eta).$$

Therefore $R(t; z, \zeta, \eta) \in \Gamma(\Omega_\rho[d_\rho] \times S; \mathcal{O}_{T^*X \times C})[t]$. For any $\rho' \in [0, \rho[, k \in \mathbb{N}_0$ and $Z \subseteq S$ on $\Omega_{\rho'}[d_{\rho'}] \times Z \subseteq \Omega_{\rho}[d_{\rho}] \times Z$, we have $|P_k(z, \zeta, \eta)|, |Q_k(z, \zeta, \eta)| \leq \frac{C_{h, Z}k! A^k e^{h\|z\|}}{\|\eta\zeta\|^{k}}$. Hence by Lemma 6.15, we have

$$|\partial^\alpha Q_\nu| \leq \frac{C_{h, Z}! A^k e^{2h\|z\|}}{(\rho - \rho')^{|\alpha|}(1 - \rho + \rho')^l \|\eta\zeta\|^{l+|\alpha|}},$$

$$|\partial^\alpha P_k| \leq \frac{C_{h, Z}! A^k e^{h\|z\|}}{(\rho - \rho')^{|\alpha|} \|\eta\zeta\|^{l}}.$$ (6.10)

We choose $\rho' \in [0, \rho[$ as $C := \frac{(\rho - \rho')^2 A}{2(1 - \rho + \rho')} < 1$, and set $B := \frac{1}{(\rho - \rho')^2}$. Then on $\Omega_{\rho'}[d_{\rho'}] \times Z$ we have

$$|R_\nu| \leq \sum_{k + l = \nu} \frac{C_{h, Z}^2 k! l! A^{k+l} e^{3h\|z\|}}{(1 - \rho + \rho')^l \|\eta\zeta\|^{l+|\alpha|}} \sum_{|\alpha| = \nu - k - l} \frac{1}{\alpha!} \frac{\rho - \rho')^{|\alpha|} \|\eta\zeta\|^{k+l+|\alpha|}}$$

$$\leq \frac{2^n C_{h, Z}^{2} B^l e^{3|z\|}}{\|\eta\zeta\|^l} \sum_{k + l = \nu} C_{h, Z}^2 k! l! e^{3h\|z\|}$$

Assume that $P(t; z, \zeta, \eta) \in \mathcal{S}_{cl}(\Omega; S)$. Then by (6.10) for any $m \in \mathbb{N}$ we have

$$\left| \sum_{k = 0}^{m-1} \partial_z^k P_k(z, \zeta, \eta) \right| \leq \frac{C_{h, Z}! A^m e^{2\|z\|}}{(\rho - \rho')^{|\alpha|} \|\eta\zeta\|^m}.$$

Then on $\Omega_{\rho'}[d_{\rho'}] \times Z$ we have

$$\left| \sum_{k = 0}^{m-1} \partial_z^k P_k(z, \zeta, \eta) \right| \leq \frac{C_{h, Z}^2 A^m e^{3h\|z\|}}{(\rho - \rho')^{|\alpha|} \|\eta\zeta\|^m(1 - \rho + \rho')^l (\rho - \rho')^2 |\alpha|}.$$
Therefore \( Q \circ P(t; z, \zeta, \eta) \in \mathcal{H}_c(\Omega; S) \). For the same reasoning, we can show that if \( Q(t; z, \zeta, \eta) \in \mathcal{H}_c(\Omega; S) \), we have \( Q \circ P(t; z, \zeta, \eta) \in \mathcal{H}_c(\Omega; S) \). In particular, since

\[
\partial_\eta (Q \circ P)(t; z, \zeta, \eta) = \partial_\eta Q \circ P(t; z, \zeta, \eta) + Q \circ \partial_\eta P(t; z, \zeta, \eta),
\]

we see that if \( P(t; z, \zeta, \eta), Q(t; z, \zeta, \eta) \in \mathcal{E}_c(\Omega; S) \), we have \( Q \circ P(t; z, \zeta, \eta) \in \mathcal{E}_c(\Omega; S) \).

The proof of (2) is easy.

(3) We may show the equality around each point \((\Phi^{-1}(z_0), [\frac{\partial z}{\partial w}(w_0)]_0) = (w_0, \lambda_0)\) as a formal series. Set \( z := \Phi(w) \). We remark that

\[
\partial^\alpha \beta \alpha e^{(J_\Phi(z+z''; z)z'', \lambda+\lambda')}(\Phi^{-1}(z+z'') - w)^\alpha,
\]

and hence as a formal power series with respect to \( t \), we have

\[
\frac{\partial^\alpha}{\partial^\alpha} e^{(J_\Phi(z+z''; z)z'', \lambda+\lambda')}(\Phi^{-1}(z+z'') - w)^\alpha.
\]

Further since

\[
J_\Phi(z+z'', z) + J_\Phi(z+z''+z', z'') = \Phi^{-1}(z+z''+z') - \Phi^{-1}(z) = J_\Phi(z+z''+z', z)(z''+z'),
\]

as a formal power series with respect to \( t \), we have

\[
J_\Phi(z + tz'', z) + J_\Phi(z + tz'' + tz', z + z'')z' = J_\Phi(z + tz'' + tz', z)(z'' + z').
\]

Therefore we have

\[
\Phi^* Q \circ \Phi^* P(t; w, \lambda, \eta) = e^{(\partial_{\lambda} \Phi_{\beta}(\partial_\lambda \Phi_{\gamma} \partial_\gamma P)(t; z, \zeta, \eta)} e^{(J_\Phi(z+z''; z)z'', \lambda+\lambda')e^{-(z'', \zeta_0)} \cdot e^{(\partial_{\lambda} \Phi_{\beta}(\partial_\lambda \Phi_{\gamma} \partial_\gamma P)(t; w, \lambda, \eta)} e^{(J_\Phi(z+z''; z)z'', \lambda+\lambda')e^{-(z'', \zeta_0)}}}
\]

\[
= e^{(\partial_{\lambda} \Phi_{\beta}(\partial_\lambda \Phi_{\gamma} \partial_\gamma P)(t; z, \zeta_0, \zeta)} e^{(J_\Phi(z+z''; z)z'', \lambda+\lambda')e^{-(z'', \zeta_0)}}
\]

\[
= e^{(\partial_{\lambda} \Phi_{\beta}(\partial_\lambda \Phi_{\gamma} \partial_\gamma P)(t; z, \zeta_0, \zeta, \eta)} e^{(J_\Phi(z+z''; z)z'', \lambda+\lambda')e^{-(z'', \zeta_0)}}
\]

and

\[
\frac{\partial^\alpha}{\partial^\alpha} e^{(J_\Phi(z+z''; z)z'', \lambda+\lambda')}(\Phi^{-1}(z+z'') - w)^\alpha.
\]
Moreover if (1) \( F \) or (2) \( W \) e have (6.11) \( S \), we set

\[
S = \Phi^*(P(t; z, \zeta, \eta)) \otimes dz = \Phi^*(P(t; z, \zeta, \eta)) \otimes dz^*.
\]

Here \( \Phi^* \) also stands for the pull-back of differential forms, and (6.11) means that

\[
\det \frac{\partial z}{\partial w} \Phi^*(P^*) = (\det \frac{\partial z}{\partial w} \Phi^* P)^* = (\Phi^* P)^* \circ \det \frac{\partial z}{\partial w}.
\]

Proof. (1) For \( P(t; z, \zeta, \eta) = \sum_{i=0}^{\infty} t^i P_i(z, \zeta, \eta) \), we set \( P^*(t; z, \zeta, \eta) = \sum_{i=0}^{\infty} t^i P_i^*(z, \zeta, \eta) \). Then

\[
P^*(z, \zeta, \eta) = \sum_{|\alpha|+k=1} \frac{1}{\alpha!} \partial_\zeta^\alpha \partial_z^\alpha P_k(t; z, -\zeta, \eta).
\]

As in the proof of Theorem 6.16, we can show \( P^*(t; z, \zeta, \eta) \in \tilde{\mathcal{S}}_c(O^a; S) \), and that if \( P(t; z, \zeta) \in \tilde{\mathcal{N}}_c(O; S) \), we have \( P^*(t; z, \zeta) \in \tilde{\mathcal{N}}_c(O^a; S) \). Moreover

\[
P^{**}(t; z, \zeta, \eta) = e^{\ell(\partial_\zeta \partial_z)} e^{-\ell(\partial_\zeta \partial_z)} \Phi^*(P(t; z, -\zeta, \eta)) = P(t; z, \zeta, \eta).
\]

(2) We have

\[
(Q \circ P)^*(t; z, \zeta, \eta) = e^{\ell(\partial_\zeta \partial_z)} (Q \circ P(t; z, -\zeta, \eta))
\]

\[
= e^{\ell(\partial_\zeta \partial_z)} (e^{-\ell(\partial_\zeta \partial_z)} Q(t; z, -\zeta - \zeta', \eta) P(t; z + z', -\zeta, \eta) \bigg|_{z'=0})
\]

\[
= e^{\ell(\partial_\zeta \partial_z)} e^{-\ell(\partial_\zeta \partial_z)} (Q(t; z + z', -\zeta - \zeta', \eta) P(t; z + z', -\zeta - \zeta', \eta) \bigg|_{z'=0})
\]

\[
= e^{\ell(\partial_\zeta \partial_z)} e^{\ell(\partial_\zeta \partial_z)} (P(t; z + z', -\zeta - \zeta', \eta)) Q^*(t; z + z', \zeta + \zeta', \eta) \bigg|_{z'=0} = P^* \circ Q^*(t; z, \zeta, \eta).
\]

(3) (i) For any holomorphic function \( a(z, \eta) \), we have

\[
\Phi^*(a(z, \eta)^* \otimes dz) = \Phi^*(a(z, \eta) \otimes dz) = a(\Phi(z, \eta)) \otimes \det \frac{\partial z}{\partial w} dw = a(\Phi(z, \eta)) \det \frac{\partial z}{\partial w} \otimes dw,
\]

\[
\Phi^*(a(z, \eta) \otimes dz)^* = (a(\Phi(z, \eta)) \det \frac{\partial z}{\partial w})^* \otimes dw = a(\Phi(z, \eta)) \det \frac{\partial z}{\partial w} \otimes dw.
\]
Set $J := \frac{\partial z}{\partial w}$ for short. Since $\Phi^*(\zeta) = \sum_{k=1}^{n} \frac{\partial w_k}{\partial z_i} \lambda_k$, we have

$$\Phi^*((\zeta_0) \otimes dz) = -\Phi^*(\zeta_0 \otimes dz) = -\sum_{k=1}^{n} \frac{\partial w_k}{\partial z_i} \lambda_k \odet \frac{\partial z}{\partial w} \odw = -J \sum_{k=1}^{n} \frac{\partial w_k}{\partial z_i} \lambda_k \otimes dw,$$

$$\Phi^*(\zeta_0 \otimes dz)^* = (\det J \sum_{k=1}^{n} \frac{\partial w_k}{\partial z_i} \lambda_k)^* \otimes dw$$

$$= -\sum_{k=1}^{n} \left( \det J \frac{\partial w_k}{\partial z_i} \eta_k + \frac{\partial \det J}{\partial w_k} \frac{\partial w_k}{\partial z_i} + \det J \frac{\lambda_k}{\partial z_i} \right) \otimes dw.$$

Here we remark

$$\frac{\partial \det J}{\partial w_k} = \det J \Tr \left( J^{-1} \frac{\partial J}{\partial w_k} \right), \quad \frac{\partial J^{-1}}{\partial w_k} = -J^{-1} \frac{\partial J}{\partial w_k} J^{-1}.$$

Hence we have

$$\sum_{k=1}^{n} \left( \frac{\partial \det J}{\partial w_k} \frac{\partial w_k}{\partial z_i} + \det J \frac{\partial w_k}{\partial w_k} \right)$$

$$= \det J \sum_{k=1}^{n} \left( \Tr \left( J^{-1} \frac{\partial J}{\partial w_k} \right) \frac{\partial w_k}{\partial z_i} - \frac{\partial \det J}{\partial w_k} \frac{\partial w_k}{\partial z_i} \right)$$

$$= \det J \sum_{k=1}^{n} \left( \frac{\partial w_k}{\partial z_i} \frac{\partial^2 w_k}{\partial w_k \partial w_k} - \frac{\partial w_k}{\partial z_i} \frac{\partial^2 w_k}{\partial w_k \partial w_k} \right) = 0.$$

Therefore $\Phi^*((\zeta_0) \otimes dz) = \Phi^*(\zeta_0 \otimes dz)^*$.

(ii) If $P$ and $Q$ satisfy (6.11), then

$$\Phi^*((Q \circ P) \otimes dz) = \Phi^*(P^* \circ Q^* \otimes dz) = \Phi^*(P^*) \circ \Phi^*(Q^*) \otimes \odet \frac{\partial z}{\partial w} \odw$$

$$= \det \frac{\partial z}{\partial w} \Phi^*(P^*) \circ \Phi^*(Q^*) \otimes \odw = (\Phi^*P)^* \circ \odet \frac{\partial z}{\partial w} \Phi^*(Q^*) \otimes \odw$$

$$= (\Phi^*P)^* \circ \odet \frac{\partial z}{\partial w} \odw = \Phi^*(Q \circ P)^* \circ \odet \frac{\partial z}{\partial w} \odw$$

$$= (\odet \frac{\partial z}{\partial w} \Phi^*(Q \circ P))^* \otimes \odw = (\odet \frac{\partial z}{\partial w} \Phi^*(Q \circ P) \otimes \odw)^* = \Phi^*(Q \circ P \otimes dz)^*$$

(iii) Take any point $(z_0; \zeta_0) \in \Omega_\rho$ and consider the Taylor expansion

$$P(t; z, \zeta, \eta) = \sum_{\alpha} p_\alpha(t; z, \zeta, \eta) (\zeta - \zeta_0)^\alpha = \sum_{\alpha} p_\alpha(t; z, \zeta, \eta) \circ (\zeta - \zeta_0)^\alpha.$$ 

We may prove (6.11) in a formal sense. Then by induction and (i), (ii), we obtain

$$\Phi^*(P^* \otimes dz) = \sum_{\alpha} \Phi^*((p_\alpha(t; z, \zeta, \eta) \circ (\zeta - \zeta_0)^\alpha) \otimes dz)$$

$$= \sum_{\alpha} \Phi^*(p_\alpha(t; z, \zeta, \eta) \circ (\zeta - \zeta_0)^\alpha \otimes dz)^* = \Phi^*(P \otimes dz)^*.$$ 

\[ \Box \]

6.19. Definition. For any $\hat{\mathcal{H}}(\Omega; S) / \hat{\mathcal{H}}_c(\Omega; S)$, we define the formal adjoint by

$$(\hat{\mathcal{H}}(\Omega; S) / \hat{\mathcal{H}}_c(\Omega; S))^* := \hat{\mathcal{H}}(\Omega^\alpha; S) / \hat{\mathcal{H}}_c(\Omega^\alpha; S).$$
6.20. **Remark.** We identify $X$ with the diagonal set of $X \times X$. Then the sheaf $\mathcal{D}_X^\infty$ of holomorphic differential operators of infinite order on $X$ is defined by

$$\mathcal{D}_X^\infty := H^n_X(\Omega_{X \times X}^{(0,n)}) .$$

Recall that for any open subset $U \subset X$, a section $P(z, \partial_z) = \sum_{\alpha} a_{\alpha}(z) \partial_z^\alpha \in \Gamma(U; \mathcal{D}_X^\infty)$ is given by the following equivalent conditions:

1. For any $W \in U$ and $h > 0$ there exists $C > 0$ such that

$$\sup_{z \in W} |a_{\alpha}(z)| \leq \frac{Ch^{|\alpha|}}{\alpha!} .$$

2. Set $P(z, \zeta) := \sum_{\alpha} a_{\alpha}(z) \zeta^\alpha$. For any $W \in U$ and $h > 0$ there exists $C > 0$ such that

$$\sup_{z \in W} |P(z, \zeta)| \leq Ce^{h\|\zeta\|} .$$

$\mathcal{D}_X^\infty$ is a sheaf of rings on $X$, and:

1. the coordinate transform is given by

$$\Phi^*P(w, \lambda) = e^{(\partial_{\psi}, \partial_{\zeta})} P(\Phi(w), \lambda') + iL_\Phi^*P(\Phi(w) + \lambda', \Phi(w)) \lambda)\big|_{\zeta' = 0} ,$$

2. the product is given by

$$Q \circ P(z, \zeta) = e^{(\partial_{\psi}, \partial_{\zeta})} Q(z, \zeta + \zeta') P(z + \zeta', \zeta')\big|_{\zeta' = 0} ,$$

3. the formal adjoint is given by

$$P^*(z, \zeta) = e^{(\partial_{\psi}, \partial_{\zeta})} P(z, -\zeta) .$$

From our point of view, the sheaf $\mathcal{E}_X^\mathbb{R}$ on $T^*X$ can be defined $\mathcal{E}_X^\mathbb{R} := \mathcal{D}_X^\infty$, and on $\check{T}^*X$, associated conic sheaf with

$$\text{Op}(\check{T}^*X) \ni V \mapsto \lim_{S'} \mathcal{S}(\mathbb{R}_{>0}V; S)/\mathcal{M}(\mathbb{R}_{>0}V; S) = \mathcal{J}(\mathbb{R}_{>0}V)/\mathcal{N}(\mathbb{R}_{>0}V) ,$$

or equivalently

$$\text{Op}(\check{T}^*X) \ni V \mapsto \lim_{S'} \mathcal{S}_\check{\mathbb{C}}(\mathbb{R}_{>0}V; S)/\mathcal{M}_\check{\mathbb{C}}(\mathbb{R}_{>0}V; S) = \mathcal{J}_\check{\mathbb{C}}(\mathbb{R}_{>0}V)/\mathcal{N}_\check{\mathbb{C}}(\mathbb{R}_{>0}V) .$$

6.21. **Theorem.** Let $[\psi(z, w, \eta) \, dw]$, $[\varphi(z, w, \eta) \, dw] \in \mathcal{E}_{X, z_0}^\mathbb{R}$. Set for short

$$\sigma(\psi) \odot \sigma(\varphi)(z, \zeta, \eta) := \sum_{\alpha} \frac{1}{\alpha!} \partial_{\zeta}^\alpha \sigma(\psi)(z, \zeta, \eta) \partial_{\zeta}^\alpha \sigma(\varphi)(z, \zeta, \eta) .$$

Then the following hold:

1. $\sigma(\psi) \odot \sigma(\varphi)(z, \zeta, \eta) \in \mathcal{E}_{z_0}^\mathbb{R}$.
2. $\sigma(\psi) \odot \sigma(\varphi)(z, \zeta, \eta) - \sigma(\psi) \odot \sigma(\varphi)(t; z, \zeta, \eta) \in \mathcal{H}_{z_0}^\mathbb{R}$.
3. $\sigma(\mu(\psi \otimes \varphi))(z, \zeta, \eta) - \sigma(\psi) \odot \sigma(\varphi)(z, \zeta, \eta) \in \mathcal{H}_{z_0}^\mathbb{R}$.

In other words, the product in $\mathcal{E}_{X, z_0}^\mathbb{R}$ coincides with that of the classical symbols given in Theorem 6.16 through the symbol mapping $\sigma$. 
Proof. We assume that \( \sigma(\psi)(z, \zeta, \eta), \sigma(\varphi)(z, \zeta, \eta) \in \Gamma(\Omega_{\rho}[d_{\rho}] \times S; \mathcal{O}_{T^{*}X \times \mathbb{C}}). \)

(1) Take \( r' \in \left]0, \rho[. \right. \) Fix any \( Z \subseteq S \) and \( h > 0. \) In \( \gamma(0, \eta; \varrho, \theta), \) we can change \( \gamma_{i}(0, \eta; \varrho) \) as \( \{w_{i} = |\eta|s' e^{2\pi \sqrt{-1} t}; 0 \leq t \leq 1\} \) with \( 0 < \varrho^{-1} < s' \) \((2 \leq i \leq n), \) and \( \gamma_{1}(0, \eta; \varrho, \theta) \subseteq \{|w_{1}| < |\eta|s'\}. \) Therefore, we have

\[
|\partial_{\zeta}^{\alpha} \sigma(\psi)(z, \zeta, \eta)| = \left| \partial_{\zeta}^{\alpha} \int_{\gamma(0, \eta; \varrho, \theta)} \psi(z, z + \bar{w}, \eta) e^{(\bar{w}, \zeta)} d\bar{w} \right| = \left| \int_{\gamma(0, \eta; \varrho, \theta)} \bar{w}^{\alpha} \psi(z, z + \bar{w}, \eta) e^{(\bar{w}, \zeta)} d\bar{w} \right|
\leq (|\eta|s')^{\alpha} C_{h, z} e^{h||\zeta||}.
\]

For the same reason, we have

\[
|\partial_{\eta} \partial_{\zeta}^{\alpha} \sigma(\psi)(z, \zeta, \eta)| = |\partial_{\zeta}^{\alpha} \partial_{\eta} \sigma(\psi)(z, \zeta, \eta)| \leq (|\eta|s')^{\alpha} C e^{-\delta||\eta||}.
\]

In the same way taking \( \|z\| \leq r'_{0} < r', \) for some \( R > 0 \) we have

\[
|\partial_{\eta} \partial_{\zeta}^{\alpha} \sigma(\varphi)(z, \zeta, \eta)| \leq \frac{C_{h, z} \alpha! e^{h||\zeta||}}{R^{\alpha}},
\]

\[
|\partial_{\eta} \partial_{\zeta}^{\alpha} \sigma(\varphi)(z, \zeta, \eta)| \leq \frac{C_{z} \alpha! e^{-\delta||\eta||}}{R^{\alpha}}.
\]

Hence taking \( r \) small enough as \( rs' < R, \) we have

\[
|\sigma(\psi) \circ \sigma(\varphi)(z, \zeta, \eta)| \leq \frac{C_{h, z}^{2} e^{2h||\zeta||}}{(1 - |\eta|s'/R)^{\alpha}}.
\]

For any \( (z, \zeta, \eta) \in \Omega_{\rho}[d_{\rho}] \times Z, \) choosing \( h = \delta m_{z}/2, \) we have

\[
|\partial_{\eta}(\sigma(\psi) \circ \sigma(\varphi))(z, \zeta, \eta)|
= \sum_{\alpha} \frac{1}{\alpha!} \left( \partial_{\eta} \partial_{\zeta}^{\alpha} \sigma(\psi)(z, \zeta, \eta) \partial_{\zeta}^{\alpha} \sigma(\varphi)(z, \zeta, \eta) + \partial_{\zeta}^{\alpha} \sigma(\psi)(z, \zeta, \eta) \partial_{\eta} \partial_{\zeta}^{\alpha} \sigma(\varphi)(z, \zeta, \eta) \right)
\leq \frac{4C_{h, z} C e^{-\delta||\eta||/2}}{(1 - |\eta|s'/R)^{\alpha}}.
\]

(2) Fix \( m \in \mathbb{N}. \) Then by Lemma 6.15, under the same notation of proof of (1), for any \( \beta, \gamma \in \mathbb{N}_{0}^{n} \) with \( |\beta| = m, \) on \( \Omega_{\rho}[d_{\rho}] \times Z \) we have

\[
|\partial_{\zeta}^{\beta+\gamma} \sigma(\psi)(z, \zeta, \eta)| = \left| \partial_{\zeta}^{\beta+\gamma} \int_{\gamma(0, \eta; \varrho, \theta)} \psi(z, z + \bar{w}, \eta) e^{(\bar{w}, \zeta)} d\bar{w} \right|
= \left| \partial_{\zeta}^{\beta} \int_{\gamma(0, \eta; \varrho, \theta)} \bar{w}^{\gamma} \psi(z, z + \bar{w}, \eta) e^{(\bar{w}, \zeta)} d\bar{w} \right|
\leq \frac{(|\eta|s')^{\gamma} C_{h, z} m! A^{m} e^{2h||\zeta||}}{(\rho - \rho')^{m} ||\eta||^{m}}.
\]
Therefore, setting $B := \frac{2A}{(\rho - \rho') R}$ we obtain

$$\left| \sigma(\psi) \otimes \sigma(\varphi)(z, \zeta, \eta) - \sum_{|\alpha| = 0}^{m-1} \frac{1}{\alpha!} \partial^\alpha \sigma(\psi)(z, \zeta, \eta) \partial^\alpha \sigma(\varphi)(z, \zeta, \eta) \right|$$

$$= \left| \sum_{|\alpha| \geq m} \frac{1}{\alpha!} \partial^\alpha \sigma(\psi)(z, \zeta, \eta) \partial^\alpha \sigma(\varphi)(z, \zeta, \eta) \right|$$

$$\leq \sum_{|\beta| = m} \sum_{|\gamma| = 0} \frac{1}{(\beta + \gamma)!} \partial^{\beta + \gamma} \sigma(\psi)(z, \zeta, \eta) \partial^{\beta + \gamma} \sigma(\varphi)(z, \zeta, \eta)$$

$$\leq \frac{C^2 m! A^m e^{b|\zeta|}|\eta|^m}{(\rho - \rho')^m |\eta|^m} \sum_{|\beta| = m} \sum_{|\gamma| = 0} \frac{(|\eta| s')^{|\gamma|}}{R^{\beta + |\gamma|}} \leq \frac{2^{n-1} C^2 m! B^m e^{b|\zeta|}|\eta|^m}{(1 - |\eta| s'/R)^n |\eta|^m}.$$ 

(3) Take two paths $\gamma(0, \eta; \tilde{\theta}, \tilde{\theta})$ and $\gamma(0, \eta; \theta', \theta')$. Here we take $\tilde{\theta}$ is sufficiently smaller than $\theta$, and next we take $\theta'$ is sufficiently smaller than $\tilde{\theta}$. Hence we may assume

$$\mu(\psi \otimes \varphi)(z, z + w, \eta) = \int_{\gamma(0, \eta; \tilde{\theta}, \tilde{\theta})} \psi(z, z + \bar{w}, \eta) \varphi(z + \bar{w}, z + w, \eta) d\bar{w},$$

$$\sigma(\mu(\psi \otimes \varphi))(z, \zeta, \eta) = \int_{\gamma(0, \eta; \theta', \theta')} \mu(\psi \otimes \varphi)(z, z + w, \eta) e^{(w, \zeta)} dw$$

Then we find

$$\sigma(\psi) \otimes \sigma(\varphi)(z, \zeta, \eta) = \sum_{\alpha} \frac{1}{\alpha!} \int_{\gamma(0, \eta; \theta, \theta)} \bar{w}^\alpha \psi(\bar{w}, z + \bar{w}, \eta) e^{(w, \zeta)} d\bar{w} \int_{\gamma(0, \eta; \theta, \theta)} \partial^\alpha \varphi(\bar{z}, \bar{w}, \eta) e^{(\bar{w}, \zeta)} d\bar{w}$$

$$= \frac{1}{\alpha!} \int_{\gamma(0, \eta; \theta, \theta)} \bar{w}^\alpha \psi(\bar{w}, z + \bar{w}, \eta) e^{(w, \zeta)} d\bar{w} \int_{\gamma(0, \eta; \theta, \theta)} \partial^\alpha \varphi(\bar{z}, \bar{w}, \eta) e^{(\bar{w}, \zeta)} d\bar{w}$$

$$+ \sum_{\alpha} \frac{1}{\alpha!} \int_{\gamma(0, \eta; \theta, \theta)} \bar{w}^\alpha \psi(\bar{w}, z + \bar{w}, \eta) e^{(w, \zeta)} d\bar{w} \int_{\gamma(0, \eta; \theta, \theta)} \partial^\alpha \varphi(\bar{z}, \bar{w}, \eta) e^{(\bar{w}, \zeta)} d\bar{w}$$

By the Cauchy integration theorem, $\gamma_1(0, \eta; \theta, \theta) \cap (-\gamma_1(0, \eta; \tilde{\theta}, \tilde{\theta}))$ can be changed to the following two segment paths:

$$\left[ \frac{\partial \eta}{2} e^{-\sqrt{-1}(\pi + \tilde{\theta})/2}, \frac{\partial \eta}{2} e^{-\sqrt{-1}(\pi + \tilde{\theta})/2} \right], \left[ \frac{\partial \eta}{2} e^{\sqrt{-1}(\pi + \tilde{\theta})/2}, \frac{\partial \eta}{2} e^{\sqrt{-1}(\pi + \tilde{\theta})/2} \right].$$

Then we can find $\delta > 0$ such that on the two paths above $\Re(w_1, \zeta_1) \leq -\delta |\eta \zeta_1|$ holds. Thus as in (1) we can see

$$\sum \frac{1}{\alpha!} \int_{\gamma(0, \eta; \theta, \theta)} \bar{w}^\alpha \psi(\bar{w}, z + \bar{w}, \eta) e^{(w, \zeta)} d\bar{w} \int_{\gamma(0, \eta; \theta, \theta)} \partial^\alpha \varphi(\bar{z}, \bar{w}, \eta) e^{(\bar{w}, \zeta)} d\bar{w}.$$
We have
\[
\psi(z, z + \bar{\omega}, \eta) \left( \sum_{\alpha} \frac{\bar{\omega}^\alpha}{\alpha!} \partial_z^\alpha \varphi(z, z + w, \eta) \right) e^{(\bar{\omega} + w, \zeta)} d\bar{\omega} dw
\]
\[
= \int \psi(z, z + \bar{\omega}, \eta) \varphi(z + \bar{\omega}, z + \bar{\omega} + w, \eta) e^{(\bar{\omega}, \zeta)} d\bar{\omega} dw
\]
\[
= \int d\bar{\omega} \int \psi(z, z + \bar{\omega}, \eta) \varphi(z + \bar{\omega}, z + w, \eta) e^{(w, \zeta)} d\bar{\omega}
\]
\[
+ \int d\bar{\omega} \int \psi(z, z + \bar{\omega}, \eta) \varphi(z + \bar{\omega}, z + w, \eta) e^{(w, \zeta)} d\bar{\omega}.
\]

We consider
\[
\sigma(\mu(\psi \otimes \varphi))(z, \zeta, \eta) = \int d\bar{\omega} \int \psi(z, z + \bar{\omega}, \eta) \varphi(z + \bar{\omega}, z + w, \eta) e^{(w, \zeta)} d\bar{\omega}
\]
\[
= \int d\bar{\omega} \int \psi(z, z + \bar{\omega}, \eta) \varphi(z + \bar{\omega}, z + w, \eta) e^{(w, \zeta)} d\bar{\omega}.
\]

By the Cauchy integration theorem, \(\gamma_1(0, \eta; \theta') \cap (-\gamma_1(0, \eta; \theta))\) can be changed to the following two segment paths:
\[
\left[ \frac{\bar{\omega}}{2} e^{-\sqrt{-1} (\pi + \theta)/2}, \frac{\bar{\omega}'}{2} e^{-\sqrt{-1} (\pi + \theta')/2} \right], \quad \left[ \frac{\bar{\omega}}{2} e^{\sqrt{-1} (\pi + \theta)/2}, \frac{\bar{\omega}'}{2} e^{\sqrt{-1} (\pi + \theta)/2} \right].
\]

Then we can find \(\delta > 0\) such that on the two paths above Re\(\langle w_1, \zeta_1 \rangle \leq -\delta |\eta\zeta_1|\) holds. Thus we can see
\[
\sigma(\mu(\psi \otimes \varphi))(z, \zeta, \eta) - \int d\bar{\omega} \int \psi(z, z + \bar{\omega}, \eta) \varphi(z + \bar{\omega}, z + w, \eta) e^{(w, \zeta)} d\bar{\omega} \in \mathcal{N}_0.
\]

Next we consider two segment paths
\[
\left[ \frac{\bar{\omega}}{2} e^{-\sqrt{-1} (\pi + \theta)/2}, \frac{\bar{\omega}}{2} e^{-\sqrt{-1} (\pi + \theta)/2} + \bar{\omega}_1 \right], \quad \left[ \frac{\bar{\omega}}{2} e^{\sqrt{-1} (\pi + \theta)/2} + \bar{\omega}_1, \frac{\bar{\omega}}{2} e^{\sqrt{-1} (\pi + \theta)/2} \right].
\]

Since \(\tilde{\varrho}\) is sufficiently smaller than \(\varrho\) and \(|\tilde{\omega}_1| \leq \frac{\tilde{\varrho} |\eta|}{2}\), we can find \(\delta > 0\) such that on the two paths above Re\(\langle w_1, \zeta_1 \rangle \leq -\delta |\eta\zeta_1|\) holds. Therefore we can conclude that
\[
\int d\bar{\omega} \int \psi(z, z + \bar{\omega}, \eta) \varphi(z + \bar{\omega}, z + w, \eta) e^{(w, \zeta)} d\bar{\omega} \in \mathcal{N}_0.
\]

The proof is complete. \(\square\)

6.22. **Remark.** Let \([\psi(z, w, \eta) dw] \in \mathcal{E}_{X, z_0}^R\). Then we can also prove the following:

1. We have
\[
: P^*(t; z, \zeta, \eta) : = \int_{\gamma(0, \eta, \theta)} \psi(z - w, z, \eta) e^{-(w, \zeta)} dw.
\]
(2) Let \( z = \Phi(w) \) be a complex coordinate transformation. Then (see (B.4))
\[
\Phi^* P(t; w, \lambda, \eta) = \int \psi(z, z', \eta) e^{(\Phi^{-1}(z') - \Phi^{-1}(z), \lambda)} dz'.
\]

§ 7. **Formal Symbols with an Apparent Parameter**

7.1. **Definition** (see [2], [6]). Let \( t \) be an indeterminate.

1. \( P(t; z, \zeta) = \sum_{\nu=0}^{\infty} t^\nu P_\nu(z, \zeta) \) is an element of \( \hat{\mathcal{T}}(\Omega) \) if \( P_\nu(z, \zeta) \in \Gamma(\Omega_\rho[(\nu + 1)d_\rho]; \mathcal{O}_{T^* X}) \)

for some \( d > 0 \) and \( \rho \in ]0, 1[, \) and there exists a constant \( A \in ]0, 1[ \) satisfying the following: for any \( h > 0 \) there exists a constant \( C_h > 0 \) such that
\[
|P_\nu(z, \zeta)| \leq C_h A^\nu h^{|\zeta|} \quad (\nu \in \mathbb{N}_0, (z; \zeta) \in \Omega_\rho[(\nu + 1)d_\rho]).
\]

2. Let \( P(t; z, \zeta) = \sum_{\nu=0}^{\infty} t^\nu P_\nu(z, \zeta) \in \hat{\mathcal{T}}(\Omega) \). Then \( P(t; z, \zeta) \) is an element of \( \hat{\mathcal{N}}(\Omega) \) if there exists a constant \( A \in ]0, 1[ \) satisfying the following: for any \( h > 0 \) there exists a constant \( C_h > 0 \) such that
\[
\left| \sum_{\nu=0}^{m-1} P_\nu(z, \zeta) \right| \leq C_h A^m h^{|\zeta|} \quad (m \in \mathbb{N}, (z; \zeta) \in \Omega_\rho[md_\rho]).
\]

3. For \( z_0^* \in T^* X \), we set
\[
\hat{\mathcal{T}}(z_0^*) := \lim_{\Omega \to} \hat{\mathcal{T}}(\Omega) \supset \hat{\mathcal{N}}(z_0^*) := \lim_{\Omega \to} \hat{\mathcal{N}}(\Omega).
\]

We call each element of \( \hat{\mathcal{T}}(\Omega) \) (resp. \( \hat{\mathcal{N}}(\Omega) \)) a formal symbol (resp. formal null-symbol) on \( \Omega \).

For \( U \subset S \) and \( m \in \mathbb{N} \), we set
\[
(\Omega_\rho * U)[md_\rho] := \{(z; \zeta, \eta) \in \Omega_\rho \times U; \|\eta\zeta\| \geq md_\rho\} \subset \Omega_\rho[md_\rho] \times U.
\]

7.2. **Definition.** Let \( t \) be an indeterminate. We say that \( P(t; z, \zeta, \eta) = \sum_{\nu=0}^{\infty} t^\nu P_\nu(z, \zeta, \eta) \) is an element of \( \hat{\mathcal{N}}(\Omega; S) \) if

(i) \( P_\nu(z, \zeta, \eta) \in \Gamma(\Omega_\rho * S)[(\nu + 1)d_\rho]; \mathcal{O}_{T^* X \times X \subset X} \) for some \( d > 0 \) and \( \rho \in ]0, 1[, \)

(ii) there exists a constant \( A \in ]0, 1[ \), and for any \( Z \subset S, h > 0 \) there exists \( C_{h, Z} > 0 \) such that
\[
\left| \sum_{\nu=0}^{m-1} P_\nu(z, \zeta, \eta) \right| \leq C_{h, Z} A^m h^{|\zeta|} \quad (m \in \mathbb{N}, (z; \zeta, \eta) \in (\Omega_\rho * Z)[md_\rho]).
\]

7.3. **Definition.** (1) We say that \( P(t; z, \zeta, \eta) = \sum_{\nu=0}^{\infty} t^\nu P_\nu(z, \zeta, \eta) \) is an element of \( \hat{\mathcal{G}}(\Omega; S) \) if

(i) \( P_\nu(z, \zeta, \eta) \in \Gamma(\Omega_\rho * S)[(\nu + 1)d_\rho]; \mathcal{O}_{T^* X \times X \subset X} \) for some \( d > 0 \) and \( \rho \in ]0, 1[, \)

(ii) there exists a constant \( A \in ]0, 1[ \), and for any \( Z \subset S, h > 0 \), there exists \( C_{h, Z} > 0 \) such that
\[
|P_\nu(z, \zeta, \eta)| \leq C_{h, Z} A^\nu h^{|\zeta|} \quad (\nu \in \mathbb{N}_0, (z; \zeta, \eta) \in (\Omega_\rho * Z)[(\nu + 1)d_\rho]).
\]

(iii) \( \partial_\eta P(t; z, \zeta, \eta) \in \hat{\mathcal{G}}(\Omega; S). \)
We call each element of \( \mathcal{S}(\Omega; S) \) (resp. \( \mathcal{N}(\Omega; S) \)) a formal symbol (resp. formal null-symbol) on \( \Omega \) with an apparent parameter in \( S \).

7.4. Lemma. \( \mathcal{N}(\Omega; S) \subset \mathcal{S}(\Omega; S) \).

Proof. We assume (7.1). For any \( \nu \in \mathbb{N} \) and \( (z; \zeta, \eta) \in (\Omega_{\rho} \ast Z)[(\nu + 1)d_{\rho}] \subset (\Omega_{\rho} \ast Z)[\nu d_{\rho}] \), we have

\[
|P_\nu(z, \zeta, \eta)| = \left| \sum_{i=0}^{\nu} P_i(z, \zeta, \eta) - \sum_{i=0}^{\nu-1} P_i(z, \zeta, \eta) \right| \leq \sum_{i=0}^{\nu} |P_i(z, \zeta, \eta)| + \sum_{i=0}^{\nu-1} |P_i(z, \zeta, \eta)|
\leq C_{h,Z} A^{\nu+1} e^{h\|\zeta\|} + C_{h,Z} A^{\nu} e^{h\|\zeta\|} \leq C_{h,Z}(A + 1) A^{\nu} e^{h\|\zeta\|}.
\]

Next, for any \( Z \subset S \), take \( \delta' \) and \( Z' \) as in (4.3). Then by the Cauchy inequality, for any \( h > 0 \) there exist constants \( C_{h,Z}, R > 0 \) such that for any \( m \in \mathbb{N} \) and \( (z; \zeta, \eta) \in (\Omega_{\rho} \ast Z)[m(2d_{\rho})] \), the following holds:

\[
\frac{1}{|\delta'|} \sup_{|\eta| = |\eta'| = \delta'|} \left| \sum_{\nu=0}^{m-1} P_\nu(z, \zeta, \eta) \right| \leq \frac{C_{h,Z} A^m e^{h\|\zeta\|}}{\delta'mZ}.
\]

We set

\[
\mathcal{E}_{x_0} := \lim_{\Omega \to S} \mathcal{S}(\Omega; S) \supset \mathcal{N}_{x_0} := \lim_{\Omega \to S} \mathcal{N}(\Omega; S).
\]

7.5. Proposition. Let \( P(t; z, \zeta, \eta) \in \mathcal{S}(\Omega; S) \). Then for any \( \eta_0 \in S \), \( P(t; z, \zeta, \eta_0) \in \mathcal{F}(\Omega) \) and \( P(t; z, \zeta, \eta) - P(t; z, \zeta, \eta_0) \in \mathcal{N}(\Omega; S) \).

Proof. Set \( d' := d/|\eta_0| > d \). Then for any \( h > 0 \), there exists a constant \( C_{h,\eta_0} > 0 \) such that

\[
|P_\nu(z, \zeta, \eta_0)| \leq C_{h,\eta_0} A^{\nu} e^{h\|\zeta\|} \quad ((z; \zeta) \in (\Omega_{\rho} \ast (\nu + 1)d')).
\]

Therefore \( P(t; z, \zeta, \eta_0) \in \mathcal{F}(\Omega) \). For any \( Z \subset S \), let \( Z' \subset S \) be the convex hull of \( Z \cup \{\eta_0\} \).

Since

\[
P_\nu(z, \zeta, \eta) = P_\nu(z, \zeta, \eta_0) + \int_{\eta_0}^\eta \partial_\eta P_\nu(z, \zeta, \tau) d\tau
\]

and \( \partial_\eta P(t; z, \zeta, \eta) \in \mathcal{N}(\Omega; S) \), there exists a constant \( A \in [0, 1] \), and for any \( h > 0 \) we can find a constant \( C_{h,Z}, R > 0 \) such that for any \( m \in \mathbb{N} \) and \( (z; \zeta, \eta) \in (\Omega_{\rho} \ast Z)[md_{\rho}] \subset (\Omega_{\rho} \ast Z')[md'] \), the following holds:

\[
\left| \sum_{\nu=0}^{m-1} (P_\nu(z, \zeta, \eta) - P_\nu(z, \zeta, \eta_0)) \right| \leq |\eta - \eta_0| C_{h,Z} A^m e^{h\|\zeta\|} \leq r C_{h,Z} A^m e^{h\|\zeta\|}.
\]

Hence \( P(t; z, \zeta, \eta) - P(t; z, \zeta, \eta_0) \in \mathcal{N}(\Omega; S) \).

7.6. Proposition. There exists the following isomorphism:

\[
\mathcal{F}(\Omega)/\mathcal{N}(\Omega) \cong \mathcal{S}(\Omega; S)/\mathcal{N}(\Omega; S).
\]
Proof. We regard that
\[ \hat{\mathcal{T}}(\Omega) = \{ P(t; z, \zeta, \eta) \in \hat{\mathcal{S}}(\Omega; S); \partial_{\eta} P(t; z, \zeta, \eta) = 0 \} \subset \hat{\mathcal{S}}(\Omega; S), \]
\[ \hat{\mathcal{N}}(\Omega) = \hat{\mathcal{T}}(\Omega) \cap \hat{\mathcal{N}}(\Omega; S) \subset \hat{\mathcal{N}}(\Omega; S). \]
Hence we have an injective mapping \( \hat{\mathcal{T}}(\Omega) / \hat{\mathcal{N}}(\Omega) \rightarrow \hat{\mathcal{S}}(\Omega; S) / \hat{\mathcal{N}}(\Omega; S) \).

Let \( P(t; z, \zeta, \eta) \in \hat{\mathcal{S}}(\Omega; S) \) and fix \( \eta_0 \in S \) arbitrary. Then by Proposition 7.5, we have \( P(t; z, \zeta, \eta_0) \in \hat{\mathcal{T}}(\Omega) \) and \( P(t; z, \zeta, \eta) - P(t; z, \zeta, \eta_0) \in \hat{\mathcal{N}}(\Omega; S) \).
\[ \square \]

7.7. Lemma. \( \hat{\mathcal{S}}_{cl}(\Omega; S) \subset \hat{\mathcal{S}}(\Omega; S) \) and \( \hat{\mathcal{N}}_{cl}(\Omega; S) \subset \hat{\mathcal{N}}(\Omega; S) \).

Proof. Let \( P(t; z, \zeta, \eta) \in \hat{\mathcal{S}}_{cl}(\Omega; S) \), and assume (6.2). We replace \( d \) as \( B := \frac{A}{d} \in [0,1[ \) if necessary. Hence on \( (\Omega_\rho * Z)(\nu + 1)d_\rho \), we have
\[ |P_\nu(z, \zeta, \eta)| \leq \frac{C_{h, z} \nu^e^{h||\eta||}}{(\nu + 1)^\nu} \left( \frac{A}{d} \right)^{\nu} \leq C_{h, z} \nu^e B^{h||\eta||}. \]
The proof of \( \hat{\mathcal{N}}_{cl}(\Omega; S) \subset \hat{\mathcal{N}}(\Omega; S) \) is similar.
\[ \square \]

7.8. Proposition. \( \hat{\mathcal{N}}(\Omega; S) \cap \Gamma(\Omega_\rho [d_\rho] \times S; \sigma_{T^*X_c}) = \mathfrak{N}(\Omega; S). \)

Proof. If \( P(z, \zeta, \eta) \in \hat{\mathcal{N}}(\Omega; S) \cap \Gamma(\Omega_\rho [d_\rho] \times S; \sigma_{T^*X_c}) \), we set \( \delta := -\frac{2\log A}{d_\rho} > 0 \), and for any \( Z \in S \), take \( h = \delta m_Z \). For each \( (z; \zeta, \eta) \in \Omega_\rho [d_\rho] \times Z \), we take \( m \) as the integral part of \( \frac{\|\eta\zeta\|}{d_\rho} \), hence \( (m + 1)d_\rho > \|\eta\zeta\| \geq md_\rho \). Thus there exists \( C_Z > 0 \) such that
\[ |P(z, \zeta, \eta)| \leq C_Z A^m e^{\delta m_Z} ||\eta\zeta|| \leq C_Z A^{\|\eta\zeta\| / d_\rho - 1} e^{\delta m_Z} ||\eta\zeta|| \leq \frac{C_Z}{A} e^{\delta \|\eta\zeta\| - 2\delta ||\eta\zeta||} \leq \frac{C_Z}{A} e^{-\delta ||\eta\zeta||}. \]
Hence we have (4.2). Conversely, by Proposition 6.7 and Lemma 7.7 we have
\[ \mathfrak{N}(\Omega; S) = \hat{\mathcal{N}}_{cl}(\Omega; S) \cap \Gamma(\Omega_\rho [d_\rho] \times S; \sigma_{T^*X_c}) \subset \hat{\mathcal{N}}(\Omega; S) \cap \Gamma(\Omega_\rho [d_\rho] \times S; \sigma_{T^*X_c}). \]
\[ \square \]

7.9. Theorem. Let \( z_0^* \in \hat{T}^*X \) and \( P(t; z, \zeta, \eta) \in \hat{\mathcal{S}}_{z_0^*} \). Then there exists \( \tilde{P}(z, \zeta, \eta) \in \mathfrak{S}_{z_0^*} \) such that \( P(t; z, \zeta, \eta) - \tilde{P}(z, \zeta, \eta) \in \hat{\mathcal{N}}_{z_0^*} \).

Proof. We may assume that \( z_0^* = (0; 1,0, \ldots, 0) \). We fix \( \eta_0 \in S \cap \mathbb{R} \). Then by Proposition 7.5 we have \( P(t; z, \zeta, \eta_0) \in \hat{\mathcal{T}}(\Omega) \) and \( P(t; z, \zeta, \eta) - P(t; z, \zeta, \eta_0) \in \hat{\mathcal{N}}(\Omega; S) \). We use the notation of the proof in Theorem 5.7. We develop \( P_\nu(z, \zeta, \eta_0) \) into the Taylor series with respect to \( (\zeta_2/\zeta_1, \ldots, \zeta_n/\zeta_1) \):
\[ P_\nu(z, \zeta, \eta_0) = \sum_{\alpha \in \mathbb{N}^{n-1}_0} P_{\nu, \alpha}(z, \zeta_1, \eta_0) \left( \frac{\zeta'}{\zeta_1} \right)^{\alpha}. \]
Then there exist sufficiently small \( r_0, \theta' > 0 \) and sufficiently large \( d > 0 \) such that \( P_{\nu, \alpha}(z, \zeta_1, \eta_0) \) is holomorphic on a common neighborhood of \( D[(\nu + 1)d] \) for each \( \alpha \in \mathbb{N}^{n-1}_0 \) where
\[ D[(\nu + 1)d] := \{ (z, \zeta_1) \in \mathbb{C}^{n+1}; \|z\| \leq r_0, \|\arg \zeta_1\| \leq \theta', \|\zeta_1\| \geq d(\nu + 1) \}. \]
It follows from the Cauchy inequality that we can take constants $K > 0$ and $A \in ]0, 1[$ so that for each $h > 0$ there exists $C_h > 0$ such that for every $\alpha \in \mathbb{N}^{n-1}$,

$$|P_{\nu, \alpha}(z, \zeta_1, \eta_0)| \leq C_h A^\nu K |\alpha| e^{|\beta|} \quad ((z, \zeta_1) \in D[(\nu + 1)d]).$$

We set $P_{\nu, \alpha}^B(z, \zeta_1, \eta)$ and $P_{\nu}^B(z, \zeta, \eta)$ as in (5.2). Then as in (5.11), for there exists $\delta_0 > 0$ and for any $Z \Subset S$, there exists $C'_Z$ such that for any $(z, \zeta_1, \eta) \in D[(\nu + 1)d] \times Z$ and $|\zeta_1| \leq \varepsilon |\zeta_1|$ we have

$$|P_{\nu}(z, \zeta, \eta_0) - P_{\nu}^B(z, \zeta, \eta)| \leq 2^n C'_Z A^\nu e^{-\delta_0 |\eta_0| / |\zeta_1|^{1/2}}.$$

Thus shrinking $\Omega$ if necessary, setting $A_1 := e^{-\delta_0 d_{\nu}/2} \in ]0, 1[$, for any $m \in \mathbb{N}$, we have on $(\Omega_\rho \ast Z)[md_\rho]$

$$\left| \sum_{\nu=0}^{m-1} (P_{\nu}(z, \zeta, \eta_0) - P_{\nu}^B(z, \zeta, \eta)) \right| \leq C Z A^m \frac{1}{1 - A}.$$

Hence

$$P(t; z, \zeta, \eta) - P^B(t; z, \zeta, \eta) = P(t; z, \zeta, \eta) - P(t; z, \zeta, \eta_0) + P(t; z, \zeta, \eta_0) - P^B(t; z, \zeta, \eta) \in \mathcal{H}(\Omega; S).$$

We set

$$\varpi_{\alpha}(P_{\nu})(x, w_1, 1) := \int_{(\nu + 1)d} P_{\nu, \alpha}^B(z, \zeta_1, \eta) \frac{e^{-w_1 \zeta_1}}{\zeta_1^\alpha} d\zeta_1.$$

Recall $L$ of (5.13) and $L_k$ of (5.15). Hence as in (5.16), $\varpi_{\alpha}(P_{\nu})(x, w_1, 1)$ extends analytically to the domain $L \times S$, and for any $\eta \in S$ we have

$$\sup \{ |\varpi_{\alpha}(P_{\nu})(x, w_1, 1)|; (x, w_1) \in L_k \} \leq \frac{2k C_{k, \nu} A^\nu}{\delta_1 |\alpha|!} (K |\eta|)^{|\alpha|}.$$

Now we define

$$\varpi(P_{\nu})(z, z + w, \eta) := \sum_{\alpha \in \mathbb{N}_0^{n-1}} \frac{\alpha! \varpi_{\alpha}(P_{\nu})(z, w_1, 1)}{(2\pi \sqrt{-1})^n (w')^{a+1} w^{-a-1}}.$$

The right-hand side converges locally uniformly $V_k$ of (5.17). Hence $\varpi(P_{\nu})(z, z + w, \eta)$ is a holomorphic function defined on $V$ of (5.18). Hence we can define

$$\overline{P}_{\nu}(z, \zeta, \eta) := \int_{(0, \eta, \theta)} \varpi(P_{\nu})(z, z + w, \eta) e^{(w, \xi)} dw$$

$$= \sum_{\alpha \in \mathbb{N}_0^{n-1}} (\zeta')^{\alpha} \int_{(0, \eta, \theta)} dw_1 \frac{e^{w_1 \zeta_1}}{2\pi \sqrt{-1}} \int_{(\nu + 1)d} P_{\nu, \alpha}^B(z, \zeta_1, \eta) \frac{e^{-w_1 \zeta_1}}{\xi_1^\alpha} d\zeta_1.$$

By virtue of (7.2), there exist conic neighborhood $\Omega$ of $z_0^\ast$, $\rho \in ]0, 1[$ and $d > 0$ such that

$$\overline{P}_{\nu}(z, \zeta, \eta) \in \Gamma(\Omega_{\rho}[d_\rho] \times S; \mathcal{E}_T^* \times \mathbb{C})$$

and for any $h > 0$ and $Z \Subset S$ there exist constants $C_{h, Z} > 0$ such that

$$|\overline{P}_{\nu}(z, \zeta, \eta)| \leq C_{h, Z} A^\nu e^{h |\xi|} \quad ((z; \zeta, \eta) \in \Omega_{\rho}[d_\rho] \times Z).$$

We set

$$\hat{V}_\varepsilon[(\nu + 1)d] := \bigcap_{i=2}^n \{(z, \zeta) \in \mathbb{C}^{2n}; \|z\| < r_0, |\zeta_1| \geq \frac{(\nu + 1)d}{\varepsilon}, |\arg \zeta_1| \leq \varepsilon, |\zeta_i| \leq \varepsilon |\zeta_1|\}. $$
Recall $\Sigma_{\pm}$ (cf. Figure 1), and we set $\Sigma_{\nu} := \{(\nu + 1)\xi_1 \in \mathbb{C}; \xi_1 \in \Sigma_{\pm}\}$. Then we have

$$\tilde{P}_\nu(z, \zeta, \eta) = I_\nu + I^-_\nu + I^+_\nu,$$

where

$$I_\nu := \sum_{\alpha \in \mathbb{N}^n_{-1}} \frac{P^B_{\nu, \alpha}(z, \zeta_1, \eta)}{\sqrt{-1} \xi_1 } e^{a(\xi_1 - \zeta_1)} d\xi_1,$$

$$I^-_\nu := -\sum_{\alpha \in \mathbb{N}^n_{-1}} \frac{P^B_{\nu, \alpha}(z, \zeta_1, \eta)}{2\sqrt{-1} \xi_1 } e^{\beta_1(\eta)(\xi_1 - \zeta_1)} d\xi_1,$$

$$I^+_\nu := \sum_{\alpha \in \mathbb{N}^n_{0}} \frac{P^B_{\nu, \alpha}(z, \zeta_1, \eta)}{2\sqrt{-1} \xi_1 } e^{\beta_1(\eta)(\xi_1 - \zeta_1)} d\xi_1.$$

On $\tilde{V}_S[(\nu + 1)d] \times Z$, as in (5.19) and (5.20) we have

$$|I^-_\nu| \leq \frac{2^{n-2}C_{K_S} A^\nu e^{-c|\kappa| \zeta_1}}{c} \left( e^{(h_z + |\beta_1| \tau) d} + \frac{e^{-h_0dra}}{2\pi h_0d\rho Z} \right).$$

(7.4)

$$|I^+_\nu| \leq \frac{2^{n-2}C_{K_S} A^\nu e^{-c|\kappa| \zeta_1}}{c} \left( e^{(h_z + |\beta_1| \tau) d} + \frac{e^{-h_0dra}}{2\pi h_0d\rho Z} \right).$$

Further

$$I_\nu = \sum_{\alpha \in \mathbb{N}^n_{-1}} P^B_{\nu, \alpha}(z, \zeta_1, \eta) \left( \xi_1^\alpha \right) = P^B_{\nu}(z, \zeta, \eta)$$

holds if $\zeta_1$ is located in the domain surrounded by $\Sigma_{\nu} - \Sigma_{\nu}$. Therefore, shrinking $\Omega$ and replacing $d$ with a larger one if necessary, by (7.4), there exists $\delta > 0$ and for any $Z \in S$ there exist $C_Z > 0$ such that on $\Omega_{\rho}[(\nu + 1)d] \times Z$, the following holds:

$$|\tilde{P}_\nu(z, \zeta, \eta) - P^B_{\nu}(z, \zeta, \eta)| \leq C_Z A^\nu e^{-\delta ||\kappa||}.$$

We set $\tilde{P}(t; z, \zeta, \eta) := \sum_{\nu=0}^{\infty} t^\nu \tilde{P}_\nu(z, \zeta, \eta)$. We set $A_2 := e^{-\delta d_\nu} \in [0, 1]$. Then for any $m \in \mathbb{N}$, on $(\Omega_{\rho} * Z)[md_\rho]$ we have

$$\left| \sum_{\nu=0}^{m-1} (\tilde{P}_\nu(z, \zeta, \eta) - P^B_{\nu}(z, \zeta, \eta)) \right| \leq \frac{C_Z A_2^m}{1 - A},$$

i.e. $\tilde{P}(t; z, \zeta, \eta) - P^B(t; z, \zeta, \eta) \in \hat{\mathcal{M}}_{z_0}$. In particular by Lemma 7.4,

$$\partial_\eta \tilde{P}(t; z, \zeta, \eta) = \partial_\eta (\tilde{P}(t; z, \zeta, \eta) - P^B(t; z, \zeta, \eta)) + \partial_\eta P^B(t; z, \zeta, \eta) \in \hat{\mathcal{M}}_{z_0}.$$

Thus $\tilde{P}(t; z, \zeta, \eta) \in \mathcal{G}_{z_0}$. By (7.3), we can define

$$\tilde{P}(z, \zeta, \eta) := \sum_{\nu=0}^{\infty} \tilde{P}_\nu(z, \zeta, \eta) \in \Gamma(\Omega_{\rho}[d_\rho] \times S; \sigma_{T^*X \times \mathbb{C}}),$$

and we have

$$|\tilde{P}(z, \zeta, \eta)| \leq \frac{C_h Z e^{h||\kappa||}}{1 - A} \quad ((z; \zeta, \eta) \in \Omega_{\rho}[d_\rho] \times Z),$$

$$\left| \tilde{P}(z, \zeta, \eta) - \sum_{\nu=0}^{m-1} \tilde{P}_\nu(z, \zeta, \eta) \right| \leq \frac{C_h Z A^m e^{h||\kappa||}}{1 - A} \quad ((z; \zeta, \eta) \in \Omega_{\rho}[md_\rho] \times Z).$$

i.e. $\tilde{P}(z, \zeta, \eta) - \tilde{P}(t; z, \zeta, \eta) \in \hat{\mathcal{M}}_{z_0}$. Moreover by Lemma 7.4 and Proposition 7.8, we have

$$\partial_\eta \tilde{P}(z, \zeta, \eta) = \partial_\eta (\tilde{P}(z, \zeta, \eta) - \tilde{P}(t; z, \zeta, \eta)) + \partial_\eta \tilde{P}(t; z, \zeta, \eta).$$
By Proposition 7.8 and Theorem 7.9, we obtain an isomorphism

\[ P(t; z, \zeta, \eta) - \tilde{P}(t; z, \zeta, \eta) = P(t; z, \zeta, \eta) - P^g(t; z, \zeta, \eta) + P^g(t; z, \zeta, \eta) - \tilde{P}(t; z, \zeta, \eta) + \tilde{P}(t; z, \zeta, \eta) - \tilde{P}(t; z, \zeta, \eta) = \tilde{P}(t; z, \zeta, \eta) \in \mathfrak{H}_{z_0}^\star . \]

\[ \mathfrak{H}_{z_0}^\star \cap \Gamma(\Omega_\rho[d_\rho] \times S; \mathcal{O}_{T^*X \times \mathbb{C}}) = \mathfrak{H}_{z_0}^\star . \]

Therefore \( \bar{P}(z, \zeta, \eta) \in \mathfrak{G}_{z_0}^\star \) and

\[ P(t; z, \zeta, \eta) - \bar{P}(z, \zeta, \eta) = P(t; z, \zeta, \eta) - P^g(t; z, \zeta, \eta) + P^g(t; z, \zeta, \eta) - \bar{P}(t; z, \zeta, \eta) + \bar{P}(t; z, \zeta, \eta) - \bar{P}(t; z, \zeta, \eta) = \bar{P}(t; z, \zeta, \eta) \in \mathfrak{H}_{z_0}^\star . \]

\[ 7.10. \textbf{Theorem.} \text{ For any } z_0^* \in \mathfrak{T}^*X, \text{ the inclusions } \mathfrak{G}_{z_0}^\star \subset \mathfrak{G}_{cl,z_0}^\star \subset \mathfrak{H}_{z_0} \text{ and } \mathfrak{N}_{z_0} \subset \mathfrak{N}_{cl,z_0} \subset \mathfrak{N}_{z_0} \text{ induce} \]

\[ \mathfrak{G}_{z_0}^\star / \mathfrak{N}_{z_0} \cong \mathfrak{G}_{cl,z_0}^\star / \mathfrak{N}_{cl,z_0} \cong \mathfrak{H}_{z_0} / \mathfrak{H}_{z_0}^\star . \]

\[ \text{Proof. By Proposition 7.8 and Theorem 7.9, we obtain an isomorphism } \mathfrak{G}_{z_0}^\star / \mathfrak{N}_{z_0} \cong \mathfrak{G}_{z_0}^\star / \mathfrak{N}_{z_0}^\star, \text{ and we shall show that this isomorphism is compatible with } \mathfrak{G}_{z_0}^\star / \mathfrak{N}_{z_0} \cong \mathfrak{G}_{cl,z_0}^\star / \mathfrak{N}_{cl,z_0} \text{ in Corollary 6.9. For any } P(t; z, \zeta, \eta) \in \mathfrak{G}_{cl,z_0}^\star \subset \mathfrak{G}_{z_0}^\star, \text{ by Theorems 6.8 and 7.9, there exist } P'(z, \zeta, \eta), P''(z, \zeta, \eta) \in \mathfrak{G}_{z_0}^\star \text{ such that} \]

\[ \left\{ \begin{array}{l}
P(t; z, \zeta, \eta) - P'(z, \zeta, \eta) \in \mathfrak{N}_{cl,z_0}^\star, \\
P(t; z, \zeta, \eta) - P''(z, \zeta, \eta) \in \mathfrak{H}_{z_0}^\star.
\end{array} \right. \]

Then, by Propositions 6.7 and 7.8 we have

\[ P'(z, \zeta, \eta) - P''(z, \zeta, \eta) \in \mathfrak{G}_{z_0}^\star \cap \mathfrak{N}_{z_0} \cap \Gamma(\Omega_\rho[d_\rho] \times S; \mathcal{O}_{T^*X \times \mathbb{C}}) = \mathfrak{H}_{z_0}^\star = \mathfrak{N}_{cl,z_0}^\star \cap \Gamma(\Omega_\rho[d_\rho] \times S; \mathcal{O}_{T^*X \times \mathbb{C}}). \]

\[ \mathfrak{H}_{z_0}^\star = \mathfrak{N}_{cl,z_0}^\star \cap \Gamma(\Omega_\rho[d_\rho] \times S; \mathcal{O}_{T^*X \times \mathbb{C}}). \]

\[ 7.11. \textbf{Remark.} \text{ Summing up, we have the following commutative diagram:} \]

\[ \mathcal{E}_{X,z_0}^\mathbb{R} \xrightarrow{\mathfrak{F}_{z_0}} \mathcal{J}_{z_0}^\star / \mathcal{N}_{z_0} \xrightarrow{\tilde{\mathfrak{F}}_{cl,z_0}} \mathcal{J}_{cl,z_0}^\star / \mathcal{N}_{cl,z_0} \xrightarrow{\tilde{\mathfrak{F}}_{z_0}} \mathcal{J}_{z_0} / \mathcal{N}_{z_0} \xrightarrow{\lim_{\kappa} E_{X}(\kappa)} \mathfrak{G}_{z_0}^\star / \mathfrak{N}_{z_0} \xrightarrow{\tilde{\mathfrak{G}}_{cl,z_0}} \mathfrak{G}_{cl,z_0}^\star / \mathfrak{N}_{cl,z_0} \xrightarrow{\tilde{\mathfrak{G}}_{z_0}} \mathfrak{G}_{z_0}^\star / \mathfrak{H}_{z_0}^\star . \]

\[ 7.12. \textbf{Definition.} \text{ As in the case of } \mathfrak{G}(\Omega; S) \text{ and } \tilde{\mathfrak{G}}_{cl}(\Omega; S), \text{ for any } P(t; z, \zeta, \eta) \in \tilde{\mathfrak{G}}(\Omega; S) \text{ we set} \]

\[ \mathfrak{P}(t; z, \zeta, \eta) := P(t; z, \zeta, \eta) \bmod \tilde{\mathfrak{N}}(\Omega; S) \in \tilde{\mathfrak{G}}(\Omega; S) / \tilde{\mathfrak{N}}(\Omega; S) \]

which is also called the normal product or the Wick product of \( P(t; z, \zeta, \eta) \).

We use the notation of Theorem 6.11. For any \( P(t; z, \zeta, \eta) \in \tilde{\mathfrak{G}}(\Omega; S) \), we also set

\[ \phi^* P(t; w, \lambda, \eta) := e^{i(\partial_{\zeta} \varphi(z) + \partial_{\zeta'} \varphi(z'))} P(t; \Phi(w), \zeta + \varphi(z), \Phi(w), \zeta + \varphi(z'), \Phi(w)) \bmod \Phi^* P(t; w, \lambda, \eta) \bigg|_{z = 0}. \]

\[ 7.13. \textbf{Theorem.} \text{ (1) } \phi^* P(t; w, \lambda, \eta) \in \tilde{\mathfrak{G}}(\Omega; S) \text{ with respect to coordinate system } (w; \lambda). \]

Further if \( P(t; z, \zeta, \eta) \in \tilde{\mathfrak{N}}(\Omega; S) \), it follows that \( \phi^* P(t; w, \lambda, \eta) \in \tilde{\mathfrak{N}}(\Omega; S) \).

(2) \( \mathfrak{I}^* \) is the identity, and for complex coordinate transformations \( z = \Phi(w) \) and \( w = \Psi(v) \), it follow that \( \Psi^* \phi^* P(t; v, \xi, \eta) - (\Phi \Psi)^* P(t; v, \xi, \eta) \in \tilde{\mathfrak{N}}(v; \xi) \).
Proof. (1) Suppose that $P_k(z, \zeta, \eta) \in \Gamma((\Omega^\rho \ast S)[(k + 1)d^\rho]; \mathcal{O}_{T^* X \times \mathbb{C}})$. We also assume (6.4), (6.5) and (6.6), hence for any $h > 0$ there exists $C_{h,z} > 0$ such that for any $(z; \zeta' + \mathbb{J}_{\Phi}^\ast(z + \zeta', z)\lambda) \in (\Omega^\rho \ast Z)[(k + 1)d^\rho]$ we have

$$|P_k(z, \zeta' + \mathbb{J}_{\Phi}^\ast(z + \zeta', z)\lambda, \eta)| \leq C_{h,z} A^k e^{h\|\zeta' + \mathbb{J}_{\Phi}^\ast(z + \zeta', z)\lambda\|}.$$ 

Hence if $(z; \zeta, \eta) \in (\Omega^\rho \ast Z)[(k + 1)d^\rho]$, instead of (6.7) we have

$$\frac{1}{\alpha!} \partial_{\zeta}^\alpha \partial_{\eta}^\alpha P_k(z, \zeta' + \mathbb{J}_{\Phi}^\ast(z + \zeta', z)\lambda, \eta)|_{\zeta'=0} \leq \frac{C_{h,z} A^k e^{2hc\|\alpha\|}}{\varepsilon d^\rho \|\alpha\|\|\lambda\|\|\eta\|\|\zeta\|}.$$ 

We may assume that $\frac{1}{2} < A < 1$. Replacing $d > 0$ if necessary, we may assume $C := \frac{\varepsilon d^\rho}{2} > 4$, hence $CA > 2$. Hence if $\|\eta\| \geq c'\|\eta\| \geq (\nu + 1)d^\rho$, we have

$$\left| (\mathbb{F} \ast P)_\nu(w, \lambda, \eta) \right| \leq C_{h,z} \sum_{k=0}^{\nu} \frac{A^k e^{2hc\|\lambda\|}}{\varepsilon d^\rho \|\alpha\|\|\lambda\|\|\eta\|\|\zeta\|} \leq C_{h,z} A^\nu e^{2hc\|\lambda\|} \sum_{k=0}^{\nu} \frac{2^{n-1} C_{h,z} A^\nu e^{2hc\|\lambda\|}}{CA - 1} \leq 2^{n-1} C_{h,z} A^\nu e^{2hc\|\lambda\|}.$$ 

If $P(t; z, \zeta, \eta) \in \mathbb{H}(\Omega; S)$, for any $m \in \mathbb{N}$ and $(z, \zeta, \eta) \in (\Omega^\rho \ast Z)[md^\rho]$, we have

$$\left| \sum_{\nu=0}^{m-1} (\mathbb{F} \ast P)_\nu(w, \lambda, \eta) \right| \leq C_{h,z} A^\nu e^{2hc\|\lambda\|} \sum_{k=0}^{m-1} \frac{1}{CA} \leq C_{h,z} A^\nu e^{2hc\|\lambda\|}.$$ 

(2) Set $v^* := (v; \xi)$. By Theorem 7.9, we can find $P_0(z, \zeta, \eta) \in \mathbb{G}_{z_0} \subset \mathbb{G}_{c_{1z_0}}$ such that

$$P(t; z, \zeta, \eta) - P_0(z, \zeta, \eta) \in \mathbb{H}_{v^*}.$$ 

By Theorem 6.11, we have $\mathbb{F} \ast \mathbb{F}^\ast P_0(v, \xi, \eta) = (\mathbb{F} \mathbb{F}^\ast) P_0(v, \xi, \eta)$. Hence by (1) we obtain

$$\mathbb{F} \ast \mathbb{F}^\ast P(t; v, \xi, \eta) = (\mathbb{F} \mathbb{F}^\ast) P_0(v, \xi, \eta).$$ 

7.14. Theorem. For any $P(t; z, \zeta, \eta), Q(t; z, \zeta, \eta) \in \mathbb{H}(\Omega; S)$, set

$$Q \circ P(t; z, \zeta, \eta) := e^{t(\zeta_0 \partial_{\zeta} \partial_{\eta})} Q(t; z, \zeta', \eta) \bigg|_{\zeta'=\zeta} = e^{t(\zeta_0 \partial_{\zeta} \partial_{\eta})} Q(t; z, \zeta + \zeta', \eta) \bigg|_{\zeta'=0}.$$ 

(1) $Q \circ P(t; z, \zeta, \eta) \in \mathbb{H}(\Omega; S)$. Moreover if either $P(t; z, \zeta, \eta)$ or $Q(t; z, \zeta, \eta)$ is an element of $\mathbb{H}(\Omega; S)$, it follows that $Q \circ P(t; z, \zeta, \eta) \in \mathbb{H}(\Omega; S)$. 


(2) \( R \circ (Q \circ P) = (R \circ Q) \circ P \) holds.

(3) Let \( \Phi(w) = z \) be a holomorphic coordinate transformation. Then

\[
\Phi^*Q \circ \Phi^*P(t; w, \lambda, \eta) - \Phi^*(Q \circ P)(t; w, \lambda, \eta) \in \mathfrak{N}_{(w; \lambda)}.
\]

Proof. (1) We assume that \( P_i(t; z, \lambda, \eta) \) and \( Q_i(t; z, \lambda, \eta) \) are given in (3) and (2), respectively, for \( P(t; z, \lambda, \eta) = \sum_{i=0}^{\infty} t^i P_i(t; z, \lambda, \eta) \) and \( Q(t; z, \lambda, \eta) = \sum_{i=0}^{\infty} t^i Q_i(t; z, \lambda, \eta) \). Set \( Q \circ P(t; z, \lambda, \eta) = \sum_{i=0}^{\infty} t^i R_i(z, \lambda, \eta) \). Then

\[
R_i(z, \lambda, \eta) = \sum_{|\alpha|+k+l=\nu} \frac{1}{\alpha!} \partial^\alpha Q_i(z, \lambda, \eta) \partial^\alpha P_k(z, \lambda, \eta).
\]

Hence \( R_i(z, \lambda, \eta) \in \Gamma((\Omega_\rho \ast Z)[(\nu+1)d_\rho]; \mathcal{E}_{T^*X \times \mathbb{C}}) \). We shall prove \( R(t; z, \lambda, \eta) \in \mathfrak{F}(\Omega; S) \). Let \( Z \in S \). Note that for any \( (z, \lambda, \eta) \in (\Omega_\rho \ast Z)[(k+1)d_\rho] \) and \( (z', \lambda', \eta') \) with \( \|\eta'\| \leq \rho - \rho' \) and \( \|\zeta'\| \leq (\rho - \rho')\|\eta\| < \|\zeta\| \), we have \( (z+z', \lambda', \eta') \in \Omega_\rho[(k+1)d_\rho] \). Moreover as in (6.9) we have

\[
\|\eta(z', \zeta')\| \geq (1 - \rho + \rho')\|\eta\| \geq (k+1)d(1 - \rho').
\]

For any \( \rho' \in [0, \rho] \) and \( h > 0 \), on \( (\Omega_\rho \ast Z)[(k+1)d_\rho] \) we have

\[
|P_k(z, \lambda, \eta)|, |Q_k(z, \lambda, \eta)| \leq C_{h,Z} A^k e^{h\|\zeta\|}.
\]

Hence in the same way as in the proof of Lemma 6.15, on \( (\Omega_\rho \ast Z)[(k+1)d_\rho] \) we have

\[
|\partial^\alpha Q_k(z, \lambda, \eta)\| \leq \frac{C_{h,Z} \alpha! A^k e^{2h\|\zeta\|}}{(\rho - \rho')|\alpha|\|\eta\| |\zeta\|},
\]

(7.5)

\[
|\partial^\alpha P_k(z, \lambda, \eta)\| \leq \frac{C_{h,Z} \alpha! A^k e^{2h\|\zeta\|}}{(\rho - \rho')|\alpha|}.
\]

We replace \( d, \rho' > 0 \) as \( C := \frac{2}{Ad_\rho(\rho - \rho')^2} \leq 1 \), and chose \( C' > 0 \) and \( B \in A[1] \) as \( (\nu+1)A^\nu \leq C'B^\nu \) for any \( \nu \in \mathbb{N}_0 \). Since \#\{\{(k, l) \in \mathbb{N}_0^2; k+l = \nu - |\alpha|\} \leq \nu - |\alpha| + 1 \leq \nu + 1 \), for any \( (z; \lambda, \eta) \in (\Omega_\rho \ast Z)[(\nu+1)d_\rho] \), we have

\[
|R_i(z, \lambda, \eta)| \leq \sum_{\nu=|\alpha|+k+l} C_{h,Z}^2 A^k e^{2h\|\zeta\|} \sum_{i=0}^{\nu} 2^{i+n-1} C_{h,Z} e^{2h\|\zeta\|} \sum_{i=0}^{\nu} C_i \leq 2^{n-1} C_{h,Z}^2 B^\nu e^{2h\|\zeta\|} \sum_{i=0}^{\nu} C_i \leq \frac{2^{n-1} C' C_{h,Z}^2 B^\nu e^{2h\|\zeta\|}}{1 - C}.
\]

Next, we assume \( P(t; z, \lambda, \eta) \in \mathfrak{F}(\Omega; S) \). Then, instead of (7.5), we have that for any \( m \in \mathbb{N} \), on \( (\Omega_\rho \ast Z)[md_\rho] \) we have

\[
|\sum_{k=0}^{m-1} \partial^\alpha P_k(z, \lambda, \eta)| \leq \frac{C_{h,Z} \alpha! A^m e^{h\|\zeta\|}}{(\rho - \rho')|\alpha|},
\]

thus we have

\[
|\sum_{\nu=0}^{m-1} R_i(z, \lambda, \eta)| = \left| \sum_{i=0}^{m-1} \frac{1}{\alpha!} \partial^\alpha Q_i(z, \lambda, \eta) \sum_{k=0}^{m-i-1} \partial^\alpha P_k(z, \lambda, \eta) \right|
\]
By Theorem 6.16, we have

\[ \sum_{i+|\alpha|=0}^m C_{h,z}^2 \alpha! A^{m-|\alpha|} e^{3h\|\xi\|} \leq 2^{n-1} C_{h,z}^2 A^m e^{3h\|\xi\|} \sum_{i+\nu=0}^{m-1} C^\nu \]

The proof in the case of \( Q(t; z, \zeta, \eta) \in \hat{\mathcal{H}}(\Omega; S) \) is similar. In particular, since

\[ \partial_\eta(Q \circ P)(t; z, \zeta, \eta) = \partial_\eta Q \circ P(t; z, \zeta, \eta) + Q \circ \partial_\eta P(t; z, \zeta, \eta), \]

we see that if \( P(t; z, \zeta, \eta), Q(t; z, \zeta, \eta) \in \hat{\mathcal{G}}(\Omega; S) \), we have \( Q \circ P(t; z, \zeta, \eta) \in \hat{\mathcal{G}}(\Omega; S) \).

(2) is easily obtained.

(3) Set \( v^* := (v; \xi) \). By Theorem 7.9, we can find \( P_0(z, \zeta, \eta), Q_0(z, \zeta, \eta) \in \hat{\mathcal{G}}_{z_0^*} \subset \hat{\mathcal{G}}_{cl,z_0^*} \)

such that

\[ P(t; z, \zeta, \eta) - P_0(z, \zeta, \eta), \quad Q(t; z, \zeta, \eta) - Q_0(z, \zeta, \eta) \in \hat{\mathcal{H}}_{z_0^*}. \]

By Theorem 6.16, we have

\[ \Phi^* Q_0 \circ \Phi^* P_0(w, \lambda, \eta) = \Phi^*(Q_0 \circ P_0)(w, \lambda, \eta). \]

Hence by (1) we obtain

\[
\Phi^* P(t; w, \lambda, \eta) = \Phi^* Q_0 \circ \Phi^* P_0(t; w, \lambda, \eta) + \Phi^*(Q - Q_0) \circ \Phi^* P(t; w, \lambda, \eta)
\]

Henceby (1) we obtain

\[
= \Phi^* Q_0 \circ \Phi^* (P - P_0)(t; w, \lambda, \eta)
\]

\[
= \Phi^*(Q_0 \circ P_0)(t; w, \lambda, \eta) + \Phi^*(Q - Q_0) \circ \Phi^* P(t; w, \lambda, \eta)
\]

\[
= \Phi^*(Q_0 \circ P_0)(t; w, \lambda, \eta)
\]

\[
\equiv \Phi^*(Q_0 \circ P_0)(t; w, \lambda, \eta)
\]

\[
\equiv \Phi^*(Q \circ P)(t; w, \lambda, \eta). \]

We can also prove (cf. see the proof of Theorems 6.18 and 7.14):

7.15. Theorem. For any \( P(t; z, \zeta, \eta) \in \hat{\mathcal{G}}(\Omega; S) \) set

\[ P^*(t; z, \zeta, \eta) := e^{t(\partial_{\zeta}, \partial_\eta)} P(t; z, -\zeta, \eta). \]

(1) \( P^*(t; z, \zeta, \eta) \in \hat{\mathcal{G}}(\Omega^a; S) \) and \( P^{**} = P \). Moreover if \( P(t; z, \zeta, \eta) \in \hat{\mathcal{H}}(\Omega; S) \), it follows that \( P^*(t; z, \zeta, \eta) \in \hat{\mathcal{H}}(\Omega^a; S) \).

(2) \((Q \circ P)^*(t; z, \zeta, \eta) = P^* \circ Q^*(t; z, \zeta, \eta). \)

(3) Let \( \Phi(w) = z \) be a holomorphic coordinate transformation. Then on \( \hat{\mathcal{G}}(\Omega^a; S) \otimes \Omega_X \)

\[ \Phi^*(P(t; z, \zeta, \eta) \otimes dz) = \Phi^*(P(t; z, \zeta, \eta) \otimes dz). \]


**Appendix A. The Compatibility of Actions**

The purpose of this appendix is to show Theorem 3.9. We follow the same notation as those in Section 3. Set $\tilde{\pi}_1 : \tilde{X}^2 \ni (z, w, \eta) \mapsto (z, \eta) \in \tilde{X}$ and $\tilde{\pi}_2 : \tilde{X}^2 \ni (z, w, \eta) \mapsto (w, \eta) \in \tilde{X}$. We also define the canonical projections $\pi_1 : X^2 \rightarrow X$ and $\pi_2 : X^2 \rightarrow X$ in the same way. Note that we consider the problem at $z^*_0 = (0; 1, 0, \ldots, 0)$. Then we have the following commutative diagram:

(A.1) \[ \begin{array}{c}
\mathcal{E}^\mathbb{R}_{X,z^*_0} \otimes \mathcal{E}^\mathbb{R}_{Y|X,z^*_0} \\
\downarrow \\
H^n_{G,\Delta_K \cap U_K}(U_{\Delta_K}; \Theta^{(0,n)}_{X^2}) \otimes H^n_{G,\Delta_K \cap U_K}(U_K; \Theta_X) \rightarrow H^n_{G,\Delta_K \cap U_K}(U_{\Delta_K}; \Theta_X) \rightarrow H^n_{G,\Delta_K \cap U_K}(U_K; \Theta_X),
\end{array} \]

where the down injective morphisms are described in §2. We will explain the other morphisms appearing in the diagram above. The top horizontal arrow in (A.1) is associated with the cohomological action of $\mathcal{E}^\mathbb{R}_X$ to $\mathcal{E}^\mathbb{R}_{Y|X}$. The second horizontal arrow $\mu^c$ in (A.1) is given by the chain of morphisms

(A.2) \[ H^n_{G,\Delta_K \cap U_K}(U_{\Delta_K}; \Theta^{(0,n)}_{X^2}) \otimes H^n_{G,\Delta_K \cap U_K}(U_K; \Theta_X) \rightarrow H^{n+d}_{G,\Delta_K \cap U_K}(U_K; \Theta^{(0,n)}_{X^2}) \rightarrow H^n_{G,\Delta_K \cap U_K}(U_{\Delta_K}; \Theta_X). \]

Here we set

\[ G := G_{\Delta_K} \cap \pi_2^{-1}(G_K), \quad U := U_{\Delta_K} \cap \pi_2^{-1}(U_K). \]

The first morphism in (A.2) is the usual cohomological cup product and the second morphism in (A.2) is the cohomological residue morphism. Note that since $G \subset \pi_2^{-1}(G_K)$ and $G \cap \pi_2^{-1}(K) \Subset U$ for any compact subset $K \Subset U_K$, the second morphism in (A.2) is well defined. The third horizontal arrow $\mu^c$ in (A.1) is defined by the cohomological cup product and residue mapping in the same way as that for the second horizontal arrow. Therefore, to show the theorem, it suffices to prove that the third horizontal arrow $\mu^c$ and $\mu$ defined in Theorem 3.5 coincide. Furthermore clearly the following diagram with respect to the cup product

\[ \begin{array}{c}
\Gamma(\hat{V}_{\Delta_K}; \Theta^{(0,n)}_{X^2}) \otimes \Gamma(\tilde{V}_{\Delta_K}; \Theta_{\tilde{X}}) \\
\sum_{\alpha \in \mathcal{P}_\delta} \Gamma(\tilde{V}_{\Delta_K}; \Theta^{(0,n)}_{X^2}) \otimes \Gamma(\tilde{V}_{\Delta_K}; \Theta_{\tilde{X}}) \\
\sum_{\beta \in \mathcal{P}_\delta} \Gamma(\tilde{V}_{\Delta_K}; \Theta^{(0,n)}_{X^2}) \otimes \Gamma(\tilde{V}_{\Delta_K}; \Theta_{\tilde{X}}) \\
\sum_{(\alpha, \beta) \in \Delta} \Gamma(\tilde{V}_{\Delta_K}; \Theta^{(0,n)}_{X^2}) \otimes \Gamma(\tilde{V}_{\Delta_K}; \Theta_{\tilde{X}})
\end{array} \]

commutes. Here we set

\[ \hat{G} := G_{\Delta_K} \cap \tilde{\pi}_2^{-1}(\hat{G}_K), \quad \hat{U} := U_{\Delta_K} \cap \tilde{\pi}_2^{-1}(\hat{U}_K). \]

Hence the problem is reduced to the following proposition.
A.1. Proposition. The diagram below commutes:

\[
\begin{array}{ccc}
\Gamma(\mathcal{W}^{(s,*)}_{\kappa}; \mathcal{O}_{X^2}^{(0,n,0)}) & \xrightarrow{\psi^c} & \Gamma(\mathcal{V}^{(s)}_{\kappa}; \mathcal{O}_{X}^{(0,n,0)}) \\
\sum_{(\alpha, \beta) \in A} \Gamma(\mathcal{W}^{(\alpha,\beta)}_{\kappa}; \mathcal{O}_{X^2}^{(0,n,0)}) & \xrightarrow{\mu} & \sum_{\beta \in P_d} \Gamma(\mathcal{V}^{(\beta)}_{\kappa}; \mathcal{O}_{X}^{(0,n,0)})
\end{array}
\]

(A.3)

Here \( \mu^c \) is given by the cohomological residue morphism and \( \mu \) is given by

\[
u(z, \omega, \eta) \, dw \mapsto \int_{\gamma(z, \omega, \eta)} u(z, \omega, \eta) \, dw.
\]

Proof. We first define the closed subsets in \( T := \{(w_1, \eta) \in \mathbb{C}^2; \, |\arg \eta| < \theta \} \) by

\[
L_{\theta, \eta} := \{(w_1, \eta) \in T; \, |\arg w_1| \leq \frac{\pi}{2} - \frac{3\theta}{4} + \arg \eta\},
\]

\[
L'_{\theta, \eta} := \{(w_1, \eta) \in L_{\theta, \eta}; \, |w_1| \leq \frac{\theta}{4} |\eta|\}.
\]

Note that \( T \setminus L_{\theta, \eta} \) and \( T \setminus L'_{\theta, \eta} \) are pseudoconvex open subsets. Then the top horizontal morphism \( \mu^c \) in (A.3) can be decomposed to the chain of morphisms:

\[
H^{n+d}_{G \cap U'}(\hat{U}; \mathcal{O}_{X^2}^{(0,n,0)}) \xrightarrow{\psi^c} H^{n+d}_{G \cap U'}(\hat{U}; \mathcal{O}_{X^2}^{(0,n,0)}) \xrightarrow{\psi^c} H^{n+d}_{G \cap U'}(\hat{U}; \mathcal{O}_{X^2}^{(0,n,0)}) \xrightarrow{\psi^c} H^{n+d}_{G \cap U'}(\hat{U}; \mathcal{O}_{X^2}^{(0,n,0)})
\]

Here we will explain all the subsets appearing in the chain above. Set

\[
\hat{U}' := \hat{U} \cap \tilde{\pi}^{-1} (\hat{U}_\kappa) = \hat{U}_{\Delta, \kappa} \cap \tilde{\pi}^{-1} (\hat{U}_\kappa) \cap \tilde{\pi}^{-1} (\hat{U}_\kappa),
\]

\[
\hat{K} := \bigcap_{i=2}^{n} \{(z, w, \eta); \, (z_1 - w_1, \eta) \in L_{\theta, \eta}, \, |z_i - w_i| \leq |\eta|\};
\]

\[
\hat{K}' := \bigcap_{i=2}^{n} \{(z, w, \eta); \, (z_1 - w_1, \eta) \in L'_{\theta, \eta}, \, |z_i - w_i| \leq |\eta|\}.
\]

Note that \( \hat{G}_{\Delta, \kappa} \cap \hat{U}' \subset \hat{K} \cap \hat{U}' \) holds. Then \( \hat{G}_k \) (\( 1 \leq k \leq 4 \)) are defined by

\[
\hat{G}_1 := \hat{K} \cap \tilde{\pi}_2^{-1} (\hat{G}_\kappa), \quad \hat{G}_2 := \hat{K}' \cap \tilde{\pi}_2^{-1} (\hat{G}_\kappa),
\]

\[
\hat{G}_3 := \hat{G}_2 \cap \tilde{\pi}_1^{-1} (\hat{G}_\kappa), \quad \hat{G}_4 := \hat{K}' \cap \tilde{\pi}_1^{-1} (\hat{G}_\kappa).
\]

The morphism \( \psi^c_{\theta} \) is nothing but the residue morphism. The other morphisms are canonical ones associated with the inclusion of sets. If we take \( \hat{\omega} \) of \( \hat{K} \) sufficiently small, we have \( \hat{U}' \cap \hat{G}_1 = \hat{U}' \cap \hat{G}_2 \). Therefore the canonical morphism \( \psi^c_{\theta} \) becomes an isomorphism. Furthermore, as \( \hat{G}_2 \subset \tilde{\pi}_1^{-1} (\hat{G}_\kappa) \) holds, we get \( \hat{U}' \cap \hat{G}_2 = \hat{U}' \cap \hat{G}_3 \), thus the canonical morphism \( \psi^c_{\theta} \) is also an isomorphism. The corresponding morphisms of Čech cohomology groups are given by the following chain:

\[
\Gamma(\mathcal{W}^{(s,*)}_{\kappa}; \mathcal{O}_{X^2}^{(0,n,0)}) \xrightarrow{\psi_1} \Gamma(\mathcal{W}^{(s,*)}_{\kappa}; \mathcal{O}_{X^2}^{(0,n,0)}) \xrightarrow{\psi_2} \Gamma(\mathcal{W}^{(s,*)}_{\kappa}; \mathcal{O}_{X^2}^{(0,n,0)})
\]

(A.4)

\[
\sum_{(\alpha, \beta) \in A} \Gamma(\mathcal{W}^{(\alpha,\beta)}_{\kappa}; \mathcal{O}_{X^2}^{(0,n,0)}) \xrightarrow{\psi_2} \sum_{(\alpha, \beta) \in A} \Gamma(\mathcal{V}^{(\alpha,\beta)}_{\kappa}; \mathcal{O}_{X^2}^{(0,n,0)})
\]

\[
\sum_{(\alpha, \beta) \in A} \Gamma(\mathcal{V}^{(\alpha,\beta)}_{\kappa}; \mathcal{O}_{X^2}^{(0,n,0)}) \xrightarrow{\psi_2} \sum_{(\alpha, \beta) \in A} \Gamma(\mathcal{V}^{(\alpha,\beta)}_{\kappa}; \mathcal{O}_{X^2}^{(0,n,0)})
\]

\[
\sum_{(\alpha, \beta) \in A} \Gamma(\mathcal{V}^{(\alpha,\beta)}_{\kappa}; \mathcal{O}_{X^2}^{(0,n,0)}) \xrightarrow{\psi_2} \sum_{(\alpha, \beta) \in A} \Gamma(\mathcal{V}^{(\alpha,\beta)}_{\kappa}; \mathcal{O}_{X^2}^{(0,n,0)})
\]
Since we have similar claims as in Lemma 3.4, we have 
tative of
Furthermore, we set

\[ \psi_3 \cong \frac{Z^{n+d}(\mathfrak{M}_3; \mathcal{O}^{(0,n,0)}_X)}{B^{n+d}(\mathfrak{M}_3; \mathcal{O}^{(0,n,0)}_X)} \xrightarrow{\psi_3} \Gamma(\tilde{W}^{(s,s)}_4; \mathcal{O}^{(0,n,0)}_X) \xrightarrow{\psi_4} \sum_{(\alpha, \beta) \in \Lambda} \Gamma(W^{(\alpha, \beta)}_4; \mathcal{O}^{(0,n,0)}_X) \xrightarrow{\psi_5} \sum_{\beta \in P_d} \Gamma(\tilde{V}^{(s)}_\kappa; \mathcal{O}_X) \].

We also explain all the subsets appearing in (A.4). Set

\[ \hat{O}^{(1)} := \{(z, w, \eta) \in \tilde{U}^t; (z_1 - w_1, \eta) \notin L_{\varrho, \theta}\}, \]
\[ \hat{O}^{(i)} := \{(z, w, \eta) \in \tilde{U}^t; |z_i - w_i| > |\eta|\} \quad (i = 2, \ldots, n), \]
\[ \hat{O}^{(1)} := \{(z, w, \eta) \in \tilde{U}^t; (z_1 - w_1, \eta) \notin L_{\varrho, \theta}\}, \]
\[ \hat{O}^{(i)} := \{(z, w, \eta) \in \tilde{U}^t; |z_i - w_i| > |\eta|\} \quad (i = 2, \ldots, n). \]

Note that these open subsets are pseudoconvex. Then the coverings \( \{W_{1}^{(\alpha, \beta)}\} \) etc. appearing in (A.4) are given by

\[ \tilde{W}^{(\alpha, \beta)}_1 := \hat{O}^{(\alpha)} \cap \tilde{\pi}^{-1}_2 (\tilde{V}^{(\beta)}_\kappa), \]
\[ \tilde{W}^{(\alpha, \beta)}_2 := \hat{O}^{(\alpha)} \cap \tilde{\pi}^{-1}_2 (\tilde{V}^{(\beta)}_\kappa), \]
\[ \tilde{W}^{(\alpha, \beta, \gamma)}_3 := \hat{O}^{(\alpha)} \cap \tilde{\pi}^{-1}_2 (\tilde{V}^{(\beta)}_\kappa) \cap \tilde{\pi}^{-1}_1 (\tilde{V}^{(\gamma)}_\kappa), \]
\[ \tilde{W}^{(\alpha, \beta)}_4 := \hat{O}^{(\alpha)} \cap \tilde{\pi}^{-1}_2 (\tilde{V}^{(\beta)}_\kappa), \]

and \( Z^{n+d}(\mathfrak{M}_3; \mathcal{O}^{(0,n,0)}_X) \) (resp. \( B^{n+d}(\mathfrak{M}_3; \mathcal{O}^{(0,n,0)}_X) \)) stands for the \( n + d \) cocycle group (resp. the \( n + d \) coboundary group) of \( \check{C}ech \) complex \( \mathcal{C}^{*}(\mathfrak{M}_3; \mathcal{O}^{(0,n,0)}_X) \) with respect to the covering \( \mathfrak{M}_3 := \{\tilde{W}^{(i,j,k)}_3\}_{1 \leq i, j, k \leq n} \). Let \( Pdw \in H^{n+d}_{Gr}(\tilde{U}; \mathcal{O}^{(0,n,0)}_X) \), and \( u dw = u(z, w, \eta) \) \( dw \in \Gamma(\tilde{W}^{(s,s)}_2; \mathcal{O}^{(0,n,0)}_X) \) the corresponding representative of the \( \check{C}ech \) cohomology group. Let us trace the images of \( P \) and \( u \) by the chain of morphisms.

**Step 1.** Set \( P_1 dw := \psi_1^c(P dw) \) and \( u_1 dw := \psi_1(u dw) \). Then clearly \( u_1 dw \) is a representative of \( P_1 dw \) and we have

\[ \mu(u dw) = \mu(u_1 dw) \mod \sum_{\beta \in P_d} \Gamma(\tilde{V}^{(\beta)}_\kappa; \mathcal{O}_X), \]

where \( \mu \) was defined in the statement of the proposition.

**Step 2.** As \( \psi_2^c \) is an isomorphism, there exists \( P_2 dw \) with \( P_1 dw = \psi_2^c(P_2 dw) \). Then we can find a representative \( u_2 dw \in \Gamma(\tilde{W}^{(s,s)}_2; \mathcal{O}^{(0,n,0)}_X) \) of \( P_2 dw \) such that

\[ u_1 dw - \psi_2(u_2 dw) \in \sum_{(\alpha, \beta) \in \Lambda} \Gamma(\tilde{W}^{(\alpha, \beta)}_1; \mathcal{O}^{(0,n,0)}_X). \]

Since we have similar claims as in Lemma 3.4, we have \( \mu(u_2 dw) \in \Gamma(\tilde{V}^{(s)}_\kappa; \mathcal{O}_X) \) and

\[ \mu(u_1 dw) = \mu(u_2 dw) \mod \sum_{\beta \in P_d} \Gamma(\tilde{V}^{(\beta)}_\kappa; \mathcal{O}_X). \]

Furthermore, we set

\[ \tilde{\gamma}(z, \eta; \varrho, \theta) := \gamma(z, \eta; \varrho, \theta) \lor (-\tilde{\gamma}(z, \eta; \varrho, \theta)), \]
\[ \mu'(u_2 dw) := \int_{\tilde{\gamma}(z, w, \eta; \varrho, \theta)} u_2(z, w, \eta) \ dw. \]

Note that the real \( n \)-dimensional chain \( \tilde{\gamma}(z, \eta; \varrho, \theta) \) in \( X \) becomes a product of closed paths where each path is homotopic to the circle in \( \mathbb{C}_{w_i} \ (i = 1, \ldots, n) \), in particular, we have
\( \partial \gamma = \emptyset \). By the same claim as Lemma 3.4 (3), we get \( \mu'(u_2 \, dw) \in \sum_{\beta \in P_d} \Gamma(\mathcal{V}_{\kappa}^{(\beta)}; \mathcal{O}_\bar{X}) \). Hence we have obtained

\[
\mu(u \, dw) = \mu'(u_2 \, dw) \mod \sum_{\beta \in P_d} \Gamma(\mathcal{V}_{\kappa}^{(\beta)}; \mathcal{O}_\bar{X}).
\]

**Step 3.** As \( \psi^c_3 \) is an isomorphism, there exists \( P_3 \, dw \) with \( P_2 \, dw = \psi^c_3(P_3 \, dw) \). Then we can take a representative

\[
u_3 \, dw = \{u_3^{(\alpha, \beta', \beta')} \}_{(\alpha, \beta', \beta') \in A^{3}_{n+d}} \in \mathbb{Z}^{n+d}(\mathcal{W}_3; \mathcal{O}_{\bar{X}^2}^{(0,n,0)}) 
\]

of \( P_3 \, dw \), where we set

\[
A^{3}_{n+d} := \{(\alpha, \beta, \beta') \in P_n \times P_d \times P_d; \#\alpha + \#\beta + \#\beta' = n + d\}.
\]

Since the covering \( \{\mathcal{W}_3^{(\alpha, \beta)}\} \) is finer than \( \{\mathcal{W}_3^{(\alpha, \beta', \beta')}\} \), we get

\[
u_2 \, dw - u_3^{(s,s,0)} \, dw \in \sum_{(\alpha, \beta) \in A} \Gamma(\mathcal{W}_3^{(\alpha, \beta)}; \mathcal{O}_{\bar{X}^2}^{(0,n,0)}),
\]

and thus, we obtain

\[
\mu'(u_2 \, dw) = \mu'(u_3^{(s,s,0)} \, dw) \mod \sum_{\beta \in P_d} \Gamma(\mathcal{V}_{\kappa}^{(\beta)}; \mathcal{O}_\bar{X})
\]

for which we have:

**A.2. Lemma.** The following holds:

\[
\mu'(u_3^{(s,0,s)} \, dw) = \mu'(u_3^{(s,s,0)} \, dw) \mod \sum_{\beta \in P_d} \Gamma(\mathcal{V}_{\kappa}^{(\beta)}; \mathcal{O}_\bar{X}).
\]

**Proof.** We set \( *^{\vee k} := * \setminus \{k\} \). By the cocycle condition for \( u_4 \, dw \), we have

\[
(-1)^{n+d}(u_3^{(s,s,0)} \, dw - u_3^{(s,s^{\vee k},(d))} \, dw) + \sum_{i=1}^{n} (-1)^i u_3^{(s^{\vee i},s,(d))} \, dw + \sum_{i=1}^{n} (-1)^{n+i} u_3^{(s^{s^{\vee i}},(d))} \, dw = 0.
\]

Hence, by the same claim as Lemma 3.4 (2), we obtain

\[
\mu'(u_3^{(s,s,0)} \, dw) = \mu'(u_3^{(s,0,s^{\vee k},(d))} \, dw) \mod \sum_{\beta \in P_d} \Gamma(\mathcal{V}_{\kappa}^{(\beta)}; \mathcal{O}_\bar{X}).
\]

By repeating the same argument, we obtain the result. \( \square \)

Summing up, we have

\[
\mu(u \, dw) = \mu'(u_3^{(s,0,s)} \, dw) \mod \sum_{\beta \in P_d} \Gamma(\mathcal{V}_{\kappa}^{(\beta)}; \mathcal{O}_\bar{X}).
\]

**Step 4.** Set \( P_4 \, dw := \psi^c_4(P_3 \, dw) \). Clearly \( u_4 \, dw := \psi_4(u_3 \, dw) \) is given by \( u_3^{(s,0,s)} \, dw \) which is a representative of \( P_4 \). By the previous step, we have

\[
\mu(u \, dw) = \mu'(u_4 \, dw) \mod \sum_{\beta \in P_d} \Gamma(\mathcal{V}_{\kappa}^{(\beta)}; \mathcal{O}_\bar{X}).
\]
The Final Step. \( \psi_5^c \) is given by the residue morphism. Then the subsets \( \hat{U} ', \hat{G}_4 \) and the chain \( \hat{\gamma}(z, \eta; \varrho, \theta) \) satisfy the geometrical situation under which the following Lemma A.3 holds. Hence it follows from the lemma that the representative of \( \psi_5^c(v_4 \, dw) \) is given by \( \mu'(u_4 \, dw) \). Therefore we have the conclusion that a representative of \( \mu^c(P \, dw) \) is given by \( \mu(u \, dw) \). This completes the proof for the proposition. \( \square \)

We first clarify a geometrical situation. Let \( X := \mathbb{C}^\ell_x \) and \( Y := \mathbb{C}^n_w \). Let \( Z \) (resp. \( U \)) be a closed (resp. a Stein open) subset in \( X \), and let \( K_i \) (resp. \( W_i \)) be a closed (resp. a Stein open) subset in \( X \times \mathbb{C}^m_w \) \((i = 1, \ldots, n)\). The mappings \( \pi: X \times Y \rightarrow X \), \( \pi_i: X \times Y \rightarrow X \times \mathbb{C}^m_w \) and \( \tau_i: X \times \mathbb{C}^m_w \rightarrow X \) denote the canonical projections respectively. In this situation, the following conditions are also assumed.

(i) The subset \( U \setminus Z \subset X \) has a covering \( \{U^{(j)}\}_{j=1}^m \) of Stein open subsets for an \( m \leq \ell \).

(ii) The subset \( W_i \setminus K_i \) is Stein in \( X \times \mathbb{C}^m_w \) for \( 1 \leq i \leq n \).

(iii) The mapping \( \tau_i: K_i \cap W_i \rightarrow X \) is proper for \( 1 \leq i \leq n \).

(iv) \( U \subset \pi(n \prod_{i=1}^n \pi^{-1}(W_i)) \).

Set \( V^{(i)} := \pi_i^{-1}(W_i \setminus K_i) \) and

\[
K := \pi^{-1}(Z) \cap \prod_{i=1}^n \pi_i^{-1}(K_i), \quad W := \pi^{-1}(U) \cap \prod_{i=1}^n \pi_i^{-1}(W_i),
\]

\[
W^{(\alpha, \beta)} := \pi^{-1}(U^{(\alpha)}) \cap V^{(\beta)} \quad (\alpha \in \mathcal{P}_m, \beta \in \mathcal{P}_n).
\]

We also denote by \( \gamma_i(z) \subset \mathbb{C}^m_w \) a closed path in \( \tau_i^{-1}(z) \cap W_i \) (regarded as a subset in \( \mathbb{C}^m_w \)) turning around each component of \( \tau_i^{-1}(z) \cap K_i \) once with anti-clockwise direction.

A.3. Lemma. Under the situation described above, there exists the following commutative diagram:

\[
\begin{array}{ccc}
H^{m+n}_{K \cap W} (W; \mathcal{G}^{(0,n)}_{X \times Y}) & \overset{\mu^c}{\rightarrow} & H^{m}_{Z \cap U} (U; \mathcal{G}_X) \\
\downarrow & & \downarrow \\
\Gamma(W^{(s,*)}; \mathcal{G}^{(0,n)}_{X \times Y}) & \overset{\mu}{\rightarrow} & \Gamma(U^{(*)}; \mathcal{G}_X)
\end{array}
\]  

where \( \mu \) is defined by

\[
u(z, w) \, dw \mapsto \int_{\gamma_1(z) \times \cdots \times \gamma_n(z)} u(z, w) \, dw,
\]

and \( (\mathcal{P}_m \times \mathcal{P}_n)^\vee \) denotes \( \{ (\alpha, \beta) \in \mathcal{P}_m \times \mathcal{P}_n : \#\alpha + \#\beta = m + n - 1 \} \).

Proof. The lemma immediately follows form [18, Corollary 3.1.4]. However, for the reader’s convenience, we will give a proof in what follows. Clearly we can apply the induction with respect to \( n \), and thus, we may assume from the beginning that \( n = 1 \); i.e. \( Y = \mathbb{C} \). Then we show the claim by the induction for \( m \geq 0 \).

First we prove the lemma for \( m = 0 \). Let \( \mathcal{S}^\bullet_{X \times \mathbb{C}} \) (resp. \( \mathcal{S}^\bullet_X \)) be the \( \bar{\partial} \) complex of \( \mathcal{G}^{(0,1)}_{X \times \mathbb{C}} \) (resp. \( \mathcal{G}_X \)) with coefficients in the sheaf of distributions on \( X \times \mathbb{C} = \mathbb{R}^{2\ell} \times \mathbb{R}^2 \) (resp.
Then we have the following diagram:

\[
\begin{array}{ccc}
H^1_{K \cap W}(W; \mathcal{O}_{X \times \mathbb{C}}^{(0,1)}) & \xrightarrow{\sim} & \Gamma(W \setminus K; \mathcal{O}_{X \times \mathbb{C}}^{(0,1)}) / \Gamma(W; \mathcal{O}_{X \times \mathbb{C}}^{(0,1)}) \\
\downarrow \iota & & \downarrow \\
\lim_{\overline{K}} H^1_{\overline{K} \cap W}(W; \mathcal{S}^*_{X \times \mathbb{C}}) & \xleftarrow{\delta} & \lim_{\overline{K}} \Gamma(W \setminus \overline{K}; \mathcal{O}_{X \times \mathbb{C}}^{(0,1)}) / \Gamma(W; \mathcal{O}_{X \times \mathbb{C}}^{(0,1)}) \\
\downarrow f_\epsilon & & \downarrow \\
H^0 \Gamma(U; \mathcal{S}^*_X) & \xrightarrow{\sim} & \Gamma(U; \mathcal{O}_X).
\end{array}
\]

Here $\overline{K}$ ranges through closed subsets in $W$ such that $K \subset \text{Int} \overline{K}$ and $\pi|_W: \overline{K} \to X$ is proper. Here $\text{Int} \overline{K}$ denotes the interior of $\overline{K}$. The morphism $\delta$ is given by $u \mapsto \partial \tilde{u}$, where $\tilde{u}$ is a distribution extension of $u$ to $W$ with $u = \tilde{u}$ on $W \setminus \overline{K}$. The morphism $f_\epsilon$ is nothing but the integration along the fiber of $\pi: X \times \mathbb{C} \to X$ for distributions. Note that the element in $H^1_{K \cap W}(W; \mathcal{S}^*_{X \times \mathbb{C}})$ is a real differential 2-form as $\mathcal{S}^*_{X \times \mathbb{C}}$ is the $\overline{\partial}$ complex of $\mathcal{O}_{X \times \mathbb{C}}^{(0,1)}$. Then the commutativity of the lower square in (A.5) comes from the Stokes formula. Let $\mathcal{B}^*_{X \times \mathbb{C}}$ be a $\overline{\partial}$ complex of $\mathcal{O}_{X \times \mathbb{C}}^{(0,1)}$ with coefficients in the sheaf of hyperfunctions on $X \times \mathbb{C} = \mathbb{R}^{2\ell} \times \mathbb{R}^2$. Then we have $H^1_{K \cap W}(W; \mathcal{B}^*_{X \times \mathbb{C}}) = H^1 \Gamma_{K \cap W}(W; \mathcal{B}^*_{X \times \mathbb{C}})$. The morphism $\delta'$ is given by $u \mapsto \partial \tilde{u}$, where $\tilde{u}$ is an extension of $u$ to $W$ as an element of the flabby sheaf. The morphism $\iota$ is the composition of morphisms

\[
H^1 \Gamma_{K \cap W}(W; \mathcal{B}^*_{X \times \mathbb{C}}) \to \lim_{\overline{K}} H^1 \Gamma_{\overline{K} \cap W}(W; \mathcal{B}^*_{X \times \mathbb{C}}) \cong \lim_{\overline{K}} H^1 \Gamma_{\overline{K} \cap W}(W; \mathcal{S}^*_{X \times \mathbb{C}}).
\]

Let $\tilde{u}'$ (resp. $\tilde{u}$) be a distribution (resp. hyperfunction) extension of $u \in \Gamma(W \setminus K; \mathcal{O}_{X \times \mathbb{C}}^{(0,1)})$ to $W$ with $\tilde{u}' = u$ on $W \setminus \overline{K}$ (resp. $\tilde{u} = u$ on $W \setminus K$). Since $\text{supp}(\tilde{u} - \tilde{u}') \subset \overline{K}$, we have

\[
\partial \tilde{u} - \partial \tilde{u}' = 0 \in \lim_{\overline{K}} H^1 \Gamma_{\overline{K} \cap W}(W; \mathcal{B}^*_{X \times \mathbb{C}}),
\]

which implies the commutativity of the upper square in (A.5). Hence, as the residue morphism $\mu^\epsilon$ is the composition $f_\epsilon \circ \iota$ by definition, we have obtained the claim of the lemma for the case $m = 0$. Now suppose that the claim of the lemma is true for $0, \ldots, m - 1$. We will show the lemma for $m$. Let us consider the commutative diagram between exact sequences:

\[
\begin{array}{cccc}
H^m_{K \cap W}(W; \mathcal{O}_{X \times \mathbb{C}}^{(0,1)}) & \to & H^m_{K \cap W(U^{(m)}); \mathcal{O}_{X \times \mathbb{C}}^{(0,1)}} & \to H^{m+1}_{K \cap W}(W; \mathcal{O}_{X \times \mathbb{C}}^{(0,1)}) \\
\downarrow \mu^\epsilon & & \downarrow \mu^\epsilon & & \downarrow \\
H^{m-1}_{Z \cap U}(U; \mathcal{O}_X) & \to & H^{m-1}_{Z \cap U(U^{(m)}); \mathcal{O}_X} & \to H^m_{Z \cap U}(U; \mathcal{O}_X) \\
\end{array}
\]

where $W^{(m)} := W \cap \pi^{-1}(U^{(m)})$, $Z' := U \setminus \bigcup_{i=1}^{m-1} U^{(i)}$ and $K' := \pi^{-1}(Z') \cap \pi_{1}^{-1}(K_1)$. We also have the commutative diagram between exact sequences:

\[
\begin{array}{cccc}
\Gamma(W'^{(s,1)}; \mathcal{O}_{X \times \mathbb{C}}^{(0,1)}) & \to & \Gamma(W'^{(s,1)}; \mathcal{O}_{X \times \mathbb{C}}^{(0,1)}) & \to \Gamma(W^{(s,1)}; \mathcal{O}_{X \times \mathbb{C}}^{(0,1)}) \\
\sum_{\alpha \in \mathcal{P}_{m-1}} \mu & & \sum_{\alpha \in \mathcal{P}_{m-1}} \mu^\epsilon & & \sum_{\alpha \in \mathcal{P}_{m-1}} \\
\Gamma(U'^{(s)}; \mathcal{O}_X) & \to & \Gamma(U'^{(m)}; \mathcal{O}_X) & \to \Gamma(U^{(m)}; \mathcal{O}_X) \\
\sum_{\alpha \in \mathcal{P}_{m-1}} & & \sum_{\alpha \in \mathcal{P}_{m-1}} & & \sum_{\alpha \in \mathcal{P}_{m-1}} \\
\Gamma(U^{(s)}; \mathcal{O}_X) & \to & \Gamma(U^{(m)}; \mathcal{O}_X) & \to \Gamma(U^{(m)}; \mathcal{O}_X) \\
\end{array}
\]
Here \( \{W^n(\alpha, 1)\}, \{W^n(\alpha, 1)\}, \{U^n(\alpha)\}, \{U^n(\alpha)\} \) are the corresponding the coverings of \( W \setminus K', W^{(m)} \setminus K', U \setminus Z', U^{(m)} \setminus Z' \) respectively. By the induction hypothesis, the first and the second \( \mu^c \) and \( \mu \) in (A.6) and (A.7) coincide. Hence the third ones in the both diagrams also coincide. The proof is complete. 

**APPENDIX B. GENERAL CONSTRUCTION OF** \( C^R_{Y|X,z_0} \)

In this appendix, we will extend theories developed in Sections 2 and 3 to a general family of Čech coverings, that enables us to define the symbol mapping \( \sigma \) in a general complex manifold. We continue to use the same notation as those in Section 2 unless we specify them. Let \( X \) be an \( n \)-dimensional complex manifold with a system of local coordinates \( z = (z_1, \ldots, z_n) \), and \( Y \) a closed complex submanifold of \( X \) which is defined locally by \( \{z' = 0\} \) where \( z = (z', z'') \) with \( z' := (z_1, \ldots, z_d) \) for some \( 1 \leq d \leq n \).

Set \( \hat{X} := X \times \mathbb{C} \), and let \( \pi_\eta: \hat{X} \ni (z, \eta) \mapsto z \in X \) be the canonical projection. Let \( z_0 = (0, z_0') \in Y \) and \( z_0'' = (z_0'; \zeta_0') \in T^*_0X \) with \( \zeta_0' \neq 0 \).

Let \( \chi = \{f_1(z), \ldots, f_d(z)\} \) be a sequence of holomorphic functions in an open neighborhood of \( z_0 \) satisfying the following conditions:

1. \( df_1(z_0) \wedge df_2(z_0) \wedge \cdots \wedge df_d(z_0) \neq 0 \).
2. \( f_1, \ldots, f_d \) belong to the defining ideal \( \mathcal{I}_Y \) of \( Y \).
3. We have
   \[
   \left[ \frac{\partial f}{\partial z}(z_0) \right] e = (c_0', 0) \in (T^*X)_{z_0}.
   \]

where \( f(z) := (f_1(z), f_2(z), \ldots, f_d(z)) \) and \( e := (1, 0, \ldots, 0) \in \mathbb{C}^d \).

We denote by \( \Xi(z_0^*) \) the set of sequences satisfying the conditions above. Set

\[
G_{\varrho, L}^\chi := \{ z \in X; \varrho^2|f'(z)| \leq |f_1(z)|, f_1(z) \in L \},
\]

where \( \varrho > 0 \) and \( L \subset \mathbb{C} \) is a closed convex cone with \( L \subset \{ \tau \in \mathbb{C}; \text{Re} \tau > 0 \} \cup \{0\} \). We also set, for an open neighborhood \( U \) of \( z_0 \) in \( X \),

\[
\hat{G}_{\varrho, \varrho, \eta}^\chi := \{(z, \eta) \in \hat{X}; \varrho|f'(z)| \leq |\eta|, f_1(z) \in L \},
\]

\[
\hat{G}_{\varrho, \varrho, \eta}^\chi := \{(z, \eta) \in U \times S^1; |f_1(z)| < \varrho|\eta| \}.
\]

Now we define

\[
\hat{C}_{Y|X,z_0}^{\chi} : = \lim_{\varrho, \varrho, \varrho} \hat{G}_{\varrho, \varrho, \eta}^\chi \cap U \ni (\hat{x}, \hat{\eta}) \mapsto (\hat{x}, \hat{\eta}),
\]

\[
C_{Y|X,z_0}^{\chi} : = \text{Ker}(\partial_\eta: \hat{C}_{Y|X,z_0}^{\chi} \to \hat{G}_{Y|X,z_0}^{\chi}).
\]

Then, by the same reasoning as that in Section 2, we have the isomorphisms

\[
C_{Y|X,z_0}^{\chi} \cong \lim_{\varrho, \varrho, \varrho} \hat{G}_{\varrho, \varrho, \eta}^\chi \cap U \ni (U; \partial_\chi) \ni C_{Y|X,z_0}^{\chi},
\]

where these isomorphisms are associated with the natural inclusions of sets and the canonical morphism \( \pi_\eta^{-1} \mathcal{I}_X \to \mathcal{I}_\hat{X} \) as we have seen in Section 2. Hence, for any \( \chi_1 \) and \( \chi_2 \) in \( \Xi(z_0^*) \), two modules \( C_{Y|X,z_0}^{\chi_1} \) and \( C_{Y|X,z_0}^{\chi_2} \) are isomorphic through \( C_{Y|X,z_0}^{\chi_1} \). Using this fact,
we replace the definition of $C^\mathbb{R}_{Y|X,z_0}$ introduced in Section 2 with a slightly generalized one. From now on, we write $M^\mathbb{R}_{Y|X,z_0} := \lim_{\delta \to 0} H^{d}_{\varepsilon, L} \cap U(U; \mathcal{O}_X)$ for short.

B.1. Definition. We denote by $C^\mathbb{R}_{Y|X,z_0}$ the isomorphism class $\{C^\mathbb{R}_{Y|X,z_0}^\chi\}_{\chi \in \Xi(z^*_0)}$ of $C^\mathbb{R}_{Y|X,z_0}$ indexed by $\chi \in \Xi(z^*_0)$. In the same way, the isomorphism class $M^\mathbb{R}_{Y|X,z_0}$ is defined by $\{M^\mathbb{R}_{Y|X,z_0}^\chi\}_{\chi \in \Xi(z^*_0)}$.

By a direct consequence of the construction above, we have the morphism of $C^\mathbb{R}_{Y|X,z_0}$ associated with a coordinates transformation. Let $w = (w', w'')$ be a system of local coordinates of a copy of $X$ where $Y$ is locally defined by $w' = 0$, and $z = \Phi(w)$ a local coordinates transformation in an open neighborhood of $w_0 \in Y$ satisfying $\Phi(Y) \subset Y$ and $z_0 = \Phi(w_0)$. Set $\hat{\Phi} := \Phi \times 1_{\eta}$. Then it induces the sheaf morphism

$$\hat{\Phi}^{-1} \mathcal{O}_X \ni \varphi \mapsto \varphi \circ \hat{\Phi} \in \mathcal{O}_{\hat{X}}.$$ 

Let $w_0^* := (w_0; [\frac{\partial \Phi}{\partial w}\varphi(w_0)](\zeta_0', 0)) \in T^*_Y X$. It is easy to see

$$\chi \circ \Phi := \{f_1 \circ \Phi, \ldots, f_d \circ \Phi\} \in \Xi(w_0^*)$$

for any $\chi = \{f_1, \ldots, f_d\} \in \Xi(z^*_0)$. Hence we have the morphism $C^\mathbb{R}_{Y|X,z_0} \to C^\mathbb{R}_{Y|X,w_0}$ defined by $[u(z, \eta)] \to [u(\Phi(w), \eta)]$, which gives $\hat{\Phi}^* : C^\mathbb{R}_{Y|X,z_0} \to C^\mathbb{R}_{Y|X,w_0}$. This morphism is compatible with the morphism $\Phi^* : \mathcal{O}^\mathbb{R}_{Y|X,z_0} \to \mathcal{O}^\mathbb{R}_{Y|X,w_0}$ associated with the coordinates transformation $\Phi$ because the both morphisms are induced from the same coordinates transformation of holomorphic functions.

Next we consider a Čech representation of $C^\mathbb{R}_{Y|X,z_0}$ for $\chi = \{f_1, \ldots, f_d\} \in \Xi(z^*_0)$. Set

$$U^\chi_\kappa \ := \bigcup_{i=2}^{d}\{z = (z', z'') \in X; |f_i(z)| < \varrho r, |f_i(z)| < r', \|z'' - z_0''\| < r'\},$$

$$\hat{U}^\chi_\kappa \ := \bigcup_{i=2}^{d}\{(z, \eta) = (z', z''', \eta) \in X \times S^d_{\kappa}; |f_i(z)| < \varrho|\eta|, |f_i(z)| < r', \|z'' - z_0''\| < r'\},$$

where $\|z''\|$ denotes max$\{|z_{d+1}|, \ldots, |z_n|\}$. We also define

$$V^\chi_{\kappa(1)} := \{z \in U^\chi_\kappa; \frac{\pi}{2} - \theta < \arg f_1(z) < \frac{3\pi}{2} + \theta\},$$

$$V^\chi_{\kappa(i)} := \{z \in U^\chi_\kappa; \varrho^2|f_i(z)| > |f_1(z)|\} \quad (2 \leq i \leq d),$$

$$\hat{V}^\chi_{\kappa(1)} := \{(z, \eta) \in \hat{U}^\chi_\kappa; \frac{\pi}{2} - \theta < \arg f_1(z) < \frac{3\pi}{2} + \theta\},$$

$$\hat{V}^\chi_{\kappa(i)} := \{(z, \eta) \in \hat{U}^\chi_\kappa; \varrho|f_i(z)| > |\eta|\} \quad (2 \leq i \leq d).$$

Then it follows from the same arguments in Section 2 that we have

$$C^\mathbb{R}_{Y|X,z_0} = \lim_{\kappa \to 0} \Gamma(\hat{V}^\chi_{\kappa(1)}); \mathcal{O}_{\hat{X}})/(\sum_{\alpha \in \mathcal{P}_d} \Gamma(V^\chi_{\kappa(\alpha)}); \mathcal{O}_X),$$

$$\widehat{C}^\mathbb{R}_{Y|X,z_0} = \lim_{\kappa \to 0} \Gamma(\hat{V}^\chi_{\kappa(1)}); \mathcal{O}_{\hat{X}})/(\sum_{\alpha \in \mathcal{P}_d} \Gamma(V^\chi_{\kappa(\alpha)}); \mathcal{O}_X; \partial_y u = 0),$$

$$M^\mathbb{R}_{Y|X,z_0} = \lim_{\kappa \to 0} \Gamma(V^\chi_{\kappa(1)}); \mathcal{O}_X)/(\sum_{\alpha \in \mathcal{P}_d} \Gamma(V^\chi_{\kappa(\alpha)}); \mathcal{O}_X).$$
Let us recall the definitions of the paths $\gamma_1(z, \eta; \varrho, \theta)$ and $\gamma_i(z, \eta; \varrho)$ in $\mathbb{C}$ which were given in Section 2. In this appendix, we take slightly modified paths. Set

$$\gamma_1(\eta; \varrho, \theta) := -\gamma_1(0, \eta; \varrho, \theta), \quad \gamma_i(\eta; \varrho) := \gamma_i(0, \eta; \varrho) \quad (i > 1).$$

We define the real $d$-dimensional chain in $\mathbb{C}^d$

$$\gamma(z; \eta; \varrho, \theta) := \gamma_1(z; \eta; \varrho) \times \gamma_2(z; \eta; \varrho) \times \cdots \times \gamma_d(z; \eta; \varrho).$$

Then, for any $(0, z'') \in Y$ near $z_0$, we also define the real $d$-dimensional chain in $\mathbb{C}_d^d$ by

$$\gamma^\chi(z'', \eta; \varrho, \theta) := \{z' \in \mathbb{C}; f(z', z'') \in \gamma(\eta; \varrho, \theta)\},$$

where $f(z) := (f_1(z), \ldots, f_d(z)) : \mathbb{C}^n \to \mathbb{C}^d$ for $\chi = \{f_1, \ldots, f_d\} \in \Xi(z_0^\ast)$ and the orientation of $\gamma^\chi$ is determined by that of $\gamma$ through $f$.

Let us introduce the symbol spaces

$$\mathcal{G}_{Y|X,z_0^\ast} := \lim_{\Omega \to \zeta_0^\ast} \mathcal{G}_{Y|X}(\Omega; S) \supset \mathcal{M}_{Y|X,z_0^\ast} := \lim_{\Omega \to \zeta_0^\ast} \mathcal{M}_{Y|X}(\Omega; S),$$

$$\mathcal{J}_{Y|X,z_0^\ast} := \lim_{\Omega \to \zeta_0^\ast} \mathcal{J}_{Y|X}(\Omega) \supset \mathcal{N}_{Y|X,z_0^\ast} := \lim_{\Omega \to \zeta_0^\ast} \mathcal{N}_{Y|X}(\Omega).$$

Here $\Omega \in T_{Y}^\ast X$ ranges through open conic neighborhoods of $z_0^\ast$, and the inductive limits with respect to $S$ are taken by $r_0, \theta \to 0$. The sets $\mathcal{G}_{Y|X}(\Omega; S), \mathcal{M}_{Y|X}(\Omega; S), \mathcal{J}_{Y|X}(\Omega)$ and $\mathcal{N}_{Y|X}(\Omega)$ are defined in the same way as in Section 4. Then we can define the mapping $\hat{\sigma}^\chi : C_{Y|X,z_0^\ast} \to \mathcal{G}_{Y|X,z_0^\ast}/\mathcal{M}_{Y|X,z_0^\ast}$ by

$$\hat{\sigma}^\chi([u])(z'', \zeta', \eta) := \int_{\gamma^\chi(z'', \eta; \varrho, \theta)} u(z', z'', \eta) e^{-(z', \zeta')} dz'$$

for $u(z', z'', \eta) \in \Gamma(\tilde{V}_\kappa^{\chi(\ast); \mathcal{G}_X})$ with a suitable $\kappa$. Similarly we get the mapping $M_{Y|X,z_0^\ast} \to \mathcal{J}_{Y|X,z_0^\ast}/\mathcal{N}_{Y|X,z_0^\ast}$ by

$$\sigma^\chi([v])(z'', \zeta') := \int_{\gamma^\chi(z'', \eta; 0, \theta)} v(z', z'', \eta) e^{-(z', \zeta')} dz'$$

for $v(z', z'', \eta) \in \Gamma(V_\kappa^{\chi(\ast); \mathcal{G}_X})$ with a suitable $\kappa$ and a sufficiently small fixed $\eta_0 > 0$.

Now we have the following theorem.

**B.2. Theorem.** The morphisms $\hat{\sigma}^\chi$ and $\sigma^\chi$ induce the well-defined mappings $\hat{\sigma} : C_{Y|X,z_0^\ast} \to \mathcal{G}_{Y|X,z_0^\ast}/\mathcal{M}_{Y|X,z_0^\ast}$ and $\sigma : M_{Y|X,z_0^\ast} \to \mathcal{J}_{z_0^\ast}/\mathcal{N}_{z_0^\ast}$ respectively. To be more precise, if $\chi_1, \chi_2 \in \Xi(z_0^\ast)$ and $[u_1] \in C_{Y|X,z_0^\ast}$ and $[u_2] \in C_{Y|X,z_0^\ast}$ determining the same element in $\mathcal{G}_{Y|X,z_0^\ast}$, it follows that $\hat{\sigma}^{\chi_1}([u_1]) = \hat{\sigma}^{\chi_2}([u_2]) \in \mathcal{G}_{Y|X,z_0^\ast}/\mathcal{M}_{Y|X,z_0^\ast}$. Similarly, for $[v_1] \in M_{Y|X,z_0^\ast}$ and $[v_2] \in M_{Y|X,z_0^\ast}$ giving the same element in $\mathcal{J}_{Y|X,z_0^\ast}$, it follows that $\sigma^{\chi_1}([v_1]) = \sigma^{\chi_2}([v_2]) \in \mathcal{J}_{Y|X,z_0^\ast}/\mathcal{N}_{Y|X,z_0^\ast}$.

**Proof.** By a linear coordinate transformation, we may assume $z_0^\ast = (0; 1, 0, \ldots, 0) \in T_Y^\ast X$ and $z_0 = 0 \in X$. We need the following easy lemma.
B.3. Lemma. Let $g(z)$ be a holomorphic function in an open neighborhood of $z_0$. Assume that $g(z) \in \mathcal{I}_y$ and $\frac{\partial g}{\partial z}(z_0) = (1, 0, \ldots, 0)$. Then, for $\varphi$ and $\theta$, there exists a sufficiently small $\varepsilon > 0$ such that

$$\text{Re } g(z) \geq \varepsilon|\eta| \quad (z' \in \partial \gamma(\varphi; \theta), \ \eta \in S_{\kappa}, |z''| \leq \varepsilon, |\eta| < \varepsilon),$$

where $\partial \gamma$ denotes the boundary of $\gamma(\varphi; \theta)$.

Proof. The Taylor expansion of $g(z)$ along $Y$ is given by

$$g(z) = \psi_1(z'')z_1 + \psi_2(z'')z_2 + \cdots + \psi_d(z'')z_d + O(|z'|^2)$$

with $\psi_1(0) = 1$ and $\psi_k(0) = 0$ ($k \geq 2$). The claim immediately follows from this. \hfill \Box

Let $\chi = \{f_1, \ldots, f_d\} \in \Xi(z_0^*)$. Set $f(z) := (f_1(z), \ldots, f_d(z))$ and, for $z''$ near 0, we write by $f_{w''}^{-1}(z')$ the mapping $f(z', z'')$ regarded as a mapping of the variable $z'$ with a fixed $z''$. Then, by the coordinates transformation, we have

$$\hat{\sigma}^x([u])(z'', \zeta, \eta) = \int_{(\gamma(\eta; \varphi; \theta))} u(f_{w''}^{-1}(w'), z'', \eta) e^{-\langle f_{w''}^{-1}(w'), \zeta \rangle} \det[\partial_w f_{w''}^{-1}] dw',$$

$$\sigma^x([v])(z'', \zeta) = \int_{(\gamma(\eta; \varphi; \theta))} v(f_{w''}^{-1}(w'), z'', \eta) e^{-\langle f_{w''}^{-1}(w'), \zeta \rangle} \det[\partial_w f_{w''}^{-1}] dw'.$$

Therefore, by applying Lemma B.3 to the first coordinate function of $f_{w''}^{-1}(w')$, we have the commutative diagram below:

$$\begin{array}{ccc}
M_{Y|X,z_0^*}^{\mathbb{R}, x} & \xrightarrow{\sigma^x} & \mathcal{J}_{Y|X,z_0^*}^{\mathbb{R}, x} / \mathcal{N}_{Y|X,z_0^*} \\
\downarrow & & \downarrow \\
C_{Y|X,z_0^*}^{\mathbb{R}, x} & \xrightarrow{\hat{\sigma}^x} & \mathcal{G}_{Y|X,z_0^*} / \mathcal{N}_{Y|X,z_0^*}.
\end{array}$$

Since the first down-arrow $M_{Y|X,z_0^*}^{\mathbb{R}, x} \simeq C_{Y|X,z_0^*}^{\mathbb{R}, x}$ is isomorphic, to show the theorem, it suffices to prove the last claim in the theorem. We first consider a special case.

B.4. Lemma. Let $\chi_1 = \{f_1, \ldots, f_d\}$, $\chi_2 = \{f_1, \ldots, f_{d-1}, g\} \in \Xi(z_0^*)$. Then the last claim in Theorem B.2 holds for these $\chi_1$ and $\chi_2$.

Proof. Let $v_1 \in \Gamma(V_{\kappa}^{\chi_1}(\zeta), \mathcal{O}_X)$ and $v_2 \in \Gamma(V_{\kappa}^{\chi_2}(\zeta), \mathcal{O}_X)$ some $\kappa$ which give the same element in $\mathcal{G}_{Y|X,z_0^*}^{\mathbb{R}, x}$. Let us consider the coordinates transformation

$$w = (w', w'') = f(z) = (f_1(z), \ldots, f_d(z), z_{d+1}, \ldots, z_n),$$

and let $w_0 = f(z_0) = 0$, $w_0^* = (0, 1, 0, \ldots, 0)$. Clearly the coordinates transformation changes $\chi_1$ and $\chi_2$ to $\tilde{\chi}_1 = (w_1, \ldots, w_d)$ and $\tilde{\chi}_2 = (w_1, \ldots, w_{d-1}, h)$ with $h(w) = g \circ f^{-1}$ respectively. Further, we have

$$\sigma^{\tilde{\chi}_1}([v_1])(w'', \zeta') = \int_{(\gamma(\eta; \varphi; \theta))} v_1(f^{-1}(w)) e^{-\langle f_{w''}^{-1}(w'), \zeta \rangle} \det[\partial_w f_{w''}^{-1}] dw',$$

$$\sigma^{\tilde{\chi}_2}([v_2])(w'', \zeta') = \int_{f_{w''}(\gamma^x(\zeta''; \eta_0; \varphi; \theta))} v_2(f^{-1}(w)) e^{-\langle f_{w''}^{-1}(w'), \zeta \rangle} \det[\partial_w f_{w''}^{-1}] dw'.$$
It follows from definitions that $v_1(f^{-1}(w))$ and $v_2(f^{-1}(w))$ are holomorphic in $V^{(s)}_{\kappa} = V^{\chi_1,(s)}_{\kappa}$ and $V^{\chi_2,(s)}_{\kappa}$ respectively. We can also see

$$f_{w''}(\gamma^{\chi_2}(w'', \eta_0, \varrho, \theta)) = \gamma^{\chi_2}(w'', \eta_0, \varrho, \theta).$$

Since $\tilde{\chi}_2$ belongs to $\Xi(z_0^*)$, we have $\partial_{w_d} h(w_0) \neq 0$. This implies that, by keeping $\eta_0$ of $\gamma_d(\eta_0, \rho)$ unchanged and by taking $\eta_0$ of other $\gamma_k$ $(1 \leq k \leq d-1)$ so small if needed, the chain $\gamma(\eta_0, \rho, \theta) \times \{w''\}$ belongs the common domain of $V^{\chi_1,(s)}_{\kappa}$ and $V^{\chi_2,(s)}_{\kappa}$ for a sufficiently small $w''$. In particular, we can replace $f_{w''}(\gamma^{\chi_2}(w'', \eta_0, \varrho, \theta))$ in (B.1) with $\gamma(\eta_0; \varrho, \theta)$. Therefore the problem can be reduced to the case where $\chi_1 = \{z_1, \ldots, z_d\}$, $\chi_2 = \{z_1, \ldots, z_{d-1}, g\}$ and the morphisms $\sigma^{\chi_1}$ and $\sigma^{\chi_2}$ are replaced with the same morphism

$$\sigma(v)(z'', \zeta') := \int_{\gamma(\eta_0; \varrho, \theta)} v(z', z'') e^{-(\varphi(z), \zeta')} dz'$$

for some $\varphi(z) = (\varphi_1(z), \ldots, \varphi_d(z))$ with $\{\varphi_1(z), \ldots, \varphi_d(z)\} \in \Xi(z_0^*) (z_0^* = (0, 1, 0, \ldots, 0))$. Let us show the lemma for this case. We have the diagram

$$M^{\Xi, \chi_1}_{Y|X, z_0^*} \cong \lim_{\varrho, L, U} H^d_{\varrho, L; \Gamma_{\varrho, L} \cap U}(U; \sigma_X) \cong M^{\Xi, \chi_2}_{Y|X, z_0^*}.$$ 

The corresponding diagram of Čech representations is given by

$$\lim_{\kappa} \Gamma(V^{\chi_1,(s)}_{\kappa}; \sigma_X) \bigg/ \sum_{\alpha \in P_d^y} \Gamma(V^{\chi_1,(s)}_{\kappa}; \sigma_X) \cong \lim_{\kappa} Z^d(\Xi_{\kappa}; \sigma_X) \bigg/ \sum_{\alpha \in P_d^y} \Gamma(V^{\chi_2,(s)}_{\kappa}; \sigma_X),$$

where the covering $\Xi_{\kappa} = \{T_{\kappa}^{(i)}\}_{i=1}^{d+1}$ is given by

$$T_{\kappa}^{(i)} = V^{\chi_1,(i)}_{\kappa} \cap V^{\chi_2,(i)}_{\kappa} \quad (1 \leq i \leq d-1), \quad T_{\kappa}^{(d)} = V^{\chi_1,(d)}_{\kappa} \cap U^{\chi_2}_{\kappa}, \quad T_{\kappa}^{(d+1)} = V^{\chi_2,(d)}_{\kappa} \cap U^{\chi_1}_{\kappa}.$$ 

We also note that, for $v = \{v^{(\beta)}\} \in Z^d(\Xi_{\kappa}; \sigma_X) \subset \sum_{\beta \in A_d} \Gamma(T_{\kappa}^{(\beta)}; \sigma_X)$, we have

$$v_1 := \iota_1(v) = v^{(1, \ldots, d)}, \quad v_2 := \iota_2(v) = v^{(1, \ldots, d-1, d+1)}.$$ 

Hence, to complete the proof, it suffices to show $\sigma([v_1]) = \sigma([v_2])$. Since $v$ satisfies a cocycle condition, we have

$$v_2 - v_1 = (-1)^d \sum_{1 \leq k \leq d} (-1)^k v^{(s^w_k)}.$$ 

By modifying the path of the integration, we obtain $\sigma(v^{(2, \ldots, d+1)}) \in \mathcal{K}_{Y|X, z_0^*}$. Furthermore we get $\sigma(v^{(s^w_k)}) = 0$ if $2 \leq k < d$. Hence we have obtained $\sigma(v_2) = \sigma(v_1) \in \mathcal{K}_{Y|X, z_0^*}$. \hfill $\square$

By repeated application of Lemma B.4, we can show the last claim of the theorem for the case $\chi_1 = \{f_1, f_2, \ldots, f_d\}$, $\chi_2 = \{g, f_2, \ldots, f_d\} \in \Xi(z_0^*)$. Hence the theorem immediately follows from the lemma below. This completes the proof. \hfill $\square$

**B.5. Lemma.** Let $\chi_1 = \{f_1, f_2, \ldots, f_d\}$, $\chi_2 = \{g, f_2, \ldots, f_d\} \in \Xi(z_0^*)$. Then the last claim in Theorem B.2 holds for these $\chi_1$ and $\chi_2$. 
Proof. By the same argument in that of the proof of Lemma B.4 and by noticing Lemma B.3, the problem can be reduced to the case $\chi_1 = (z_1, z_2, \ldots, z_d)$ and $\chi_2 = (g, z_2, \ldots, z_n)$ with the morphism $\sigma$ defined by (B.2). Then we have the diagram (B.3) and the corresponding one by Čech representations is given by

$$\lim_{\kappa \to} \Gamma(V_{\kappa}^{X_1(z)}; \Omega_X) \big/ \sum_{\alpha \in \mathcal{P}_d} \Gamma(V_{\kappa}^{X_1(z)}; \Omega_X) \simeq \lim_{\kappa \to} Z^d(\mathcal{S}_\kappa; \Omega_X) \big/ B^d(\mathcal{S}_\kappa; \Omega_X),$$

where the covering $\mathcal{S}_\kappa = \{T^{(i,j)}\}_{i,j=1}^d$ is defined by

$$T^{(i,j)} := V_{\kappa}^{X_1(z)} \cap V_{\kappa}^{X_2(z)} \supset U_{\kappa}^{X_1} \cap U_{\kappa}^{X_2}.$$  

Furthermore the morphisms $\nu_1$ and $\nu_2$ are given by

$$v_1 := \nu_1(v) = (v^{(1)}, v_2 := \nu_2(v) = (v^{(2)},$$

respectively for $v = \{v^{(1)}, v^{(2)}\} \in Z^d(\mathcal{S}_\kappa; \Omega_X) \subset \sum \Gamma(T^{(i,j)}; \Omega_X)$. Then, by employing the same argument in the proof of Lemma A.2 in Appendix A, we have $\nu(v_1) = \nu(v_2) \in \mathcal{S}Y_{\nu_1(X)} \cap \mathcal{S}Y_{\nu_2(X)}$. The proof is complete. $\square$

Now we compute behavior of a symbol by a coordinates transformation. Let $z = (z', z'') = (\Phi'(w), \Phi''(w)) = \Phi(w)$ be a local coordinates transformation near $w_0 \in Y$ with $\Phi(Y) \subset Y$ and $z_0 = \Phi(w_0)$ where $Y$ is also defined by $w' = 0$ under the system of local coordinates $w = (w', w'')$. We denote by $\Phi''(w')$ the mapping $z' = \Phi'(w', w'')$ regarded as a mapping of the variable $w'$ with a fixed $w''$. Set $w_0^* := (w_0; d\Phi'(w_0)(\zeta_0, 0)) \in T_{Y}^*X_0$. Let $\chi = \{\chi_1, \ldots, \chi_2\}$ and $[u] \in C_{\mathcal{S}Y_{\nu_1(X)}}^0$ with $u(z, \eta) \in \Gamma(V_{\kappa}^{X(z)}; \Omega_X)$ for some $\kappa$. Then we get $\tilde{\sigma}([u]) = [u(\Phi'(w), \eta)] \in C_{\mathcal{S}Y_{\nu_2(X)}}^0$. Hence we have obtained

$$\tilde{\sigma}(\tilde{\sigma}([u]))(w'', \lambda', \eta) = \int \nu_2(\Phi'(w'), \eta) e^{-\langle w', \lambda' \rangle} dw' = \int \nu_2(\Phi'(w', \lambda'), \eta) e^{-\langle w', \lambda' \rangle} dw'.$$  

Let us consider the corresponding generalization of $E_{X, z_0}^n$. Hereafter we follow the same notations as those in Section 3. Let $X$ be an $n$-dimensional complex manifold. Set $X^2 := X \times X$ with a system of local coordinates $(z, w)$ and $\mathcal{X}^2 := X^2 \times \mathbb{C}$ with local coordinates $(z, w, \eta)$. Let $\Delta \subset X^2$ be the diagonal set identified with $X$ and $z_0^* = (z_0, \zeta_0) \in T^*X = T_{\Delta}^*X^2$ with $\zeta_0 \neq 0$. Let $\{f_1(z), \ldots, f_n(z)\}$ be a sequence of holomorphic functions in an open neighborhood of $z_0$ of $X$ satisfying the conditions:

(1) $df_1(z_0) \wedge \cdots \wedge df_n(z_0) \neq 0$.

(2) we have $f(z_0) = 0$ and $\frac{\partial f}{\partial z}(z_0)e = \zeta_0 \in (T^*X)_{z_0}$ where $f(z) := (f_1(z), \ldots, f_n(z))$ and $e := (1, 0, \ldots, 0) \in \mathbb{C}^n$.

We denote by $\mathcal{E}_\Delta(z_0)$ the set of such a sequence. Let $\chi = \{f_1, \ldots, f_n\} \in \mathcal{E}_\Delta(z_0)$ and set $f_{\Delta, i}(z, w) := f_i(z) - f_i(w)$,

$$f_{\Delta}(z, w) = (f_{\Delta, 1}(z, w), \ldots, f_{\Delta, n}(z, w)) := (f_{\Delta, 1}(z, w), f_{\Delta}(z, w)).$$
Define, for an open neighborhood $U \subset X^2$ of $(z, z_0)$ and a closed convex cone $L \subset \mathbb{C}$ with $L \subset \{ \tau \in \mathbb{C}; \Re \tau > 0 \} \cup \{0\}$,

$$G^X_{\Delta, e, L} := \{(z, w) \in X^2; \varrho^2|f'_{\Delta}(z, w)| \leq |f_{\Delta, 1}(z, w)|, f_{\Delta, 1}(z, w) \in L\},$$

and

$$\hat{G}^X_{\Delta, e, L} := \{(z, w, \eta) \in U \times S_{r, \theta}; |f_{\Delta, 1}(z, w)| < \varrho|\eta|\},$$

$$\hat{G}^X_{\Delta, e, L} := \{(z, w, \eta) \in \hat{X}^2; \varrho|f'(z, w)| \leq |\eta|, f_{\Delta, 1}(z, w) \in L\}.$$

We also define

$$\hat{E}^n_{X, z_0} := \lim_{\varrho, r, \theta, U} H^n_{\varrho, r, \theta, L, U}(\hat{G}^X_{\Delta, e, r, \theta}; \mathcal{O}^{(0, n, 0)}_{X^2}),$$

$$E^X_{x, z_0} := \text{Ker}(\partial_\eta : \hat{E}^n_{X, z_0} \to \hat{E}^n_{X, z_0}),$$

$$M^X_{x, z_0} := \lim_{\varrho, U} H^n_{\varrho, r, \theta, L, U}(U; \mathcal{O}^{(0, n)}_{X^2}).$$

Then we obtain isomorphisms

$$\mathcal{E}^R_{x, z_0} \cong M^X_{x, z_0} \cong E^X_{x, z_0}.$$
Let $\gamma(z, \eta; \varrho, \theta)$ be an $n$-dimension real chain in $\mathbb{C}^n$ defined in Section 3. Then we define the $n$-dimensional real chain in $\mathbb{C}^n$ by

$$\gamma^\lambda(z, \eta; \varrho, \theta) = f^{-1}_A(-\gamma(0, \eta; \varrho, \theta)) = f^{-1}(\gamma(f(z), \eta; \varrho, \theta)),$$

where $f_{A,z}(w)$ is the morphism $f_{A}(z, w) = f(z) - f(w)$ regarded as a function of $w$ for a fixed $z$ and the orientation of $\gamma^\lambda$ is induced from that of $\gamma$ by $f^{-1}$. Then we can define the mapping $\hat{\sigma}^\lambda: E^{\mathbb{R}, X}_{X, z_0^\ast} \to \mathfrak{S}_{z_0^\ast}/\mathfrak{N}_{z_0^\ast}$ by

$$\hat{\sigma}^\lambda([Kdw])(z, \zeta, \eta) := \int K(z, w, \eta) e^{(w-z, \zeta)} dw$$

for $K(z, w, \eta) dw \in \Gamma(\hat{V}_{A, \kappa}^{X(z)}; \mathfrak{O}_{X}^{(0, n, 0)})$ with a suitable $\kappa$. Similarly we have the mapping $M^{\mathbb{R}, X}_{X, z_0^\ast} \to \mathfrak{S}_{z_0^\ast}^* / \mathfrak{N}_{z_0^\ast}$ by

$$\sigma^\lambda([Kdw])(z, \zeta) := \int K(z, w) e^{(w-z, \zeta)} dw$$

for $K(z, w) dw \in \Gamma(\hat{V}_{A, \kappa}^{X(z)}; \mathfrak{O}_{X}^{(0, n)})$ with a suitable $\kappa$ and a sufficiently small fixed $\eta_0 > 0$.

As an immediate consequence of Theorem B.2, we have obtained the following corollary.

**B.7. Corollary.** There exist the well-defined symbol morphisms

$$\hat{\sigma}: E^{\mathbb{R}, X}_{X, z_0^\ast} \to \mathfrak{S}_{z_0^\ast}/\mathfrak{N}_{z_0^\ast}, \quad \sigma: M^{\mathbb{R}, X}_{X, z_0^\ast} \to \mathfrak{S}_{z_0^\ast}^* / \mathfrak{N}_{z_0^\ast},$$

induced by $\hat{\sigma}^\lambda$ and $\sigma^\lambda$ respectively.

Let us consider a coordinates transformation. Let $z = \Phi(w)$ be a local coordinates transformation of $X$ near $w_0 \in X$ with $z_0 = \Phi(w_0)$. We take $(z, z', \eta)$ and $(w, w', \eta)$ as the corresponding systems of local coordinates of $X^2$ respectively and the associated local coordinates transformation $\hat{\Phi}$ of $\hat{X}^2$ is defined by $(z, z', \eta) = (\Phi(w), \Phi(w'), \eta)$. Set $w_0^\ast := (w_0; d\Phi(w_0)(\zeta_0)) \in T^*X$. Let $\chi = \{z_1, \ldots, z_n\}$ and $[Kdz'] \in E^{\mathbb{R}, X}_{X, z_0^\ast}$ with $K(z, z', \eta) dz' \in \Gamma(\hat{V}_{A, \kappa}^{X(z)}; \mathfrak{O}_{X}^{(0, n, 0)})$ for some $\kappa$. Then, by the same argument as in $C^{\mathbb{R}, X}_{Y|X, z_0^\ast}$, we get

$$\hat{\Phi}^\ast([Kdz']) = [\hat{\Phi}^\ast(Kdz')] = [K(\Phi(w), \Phi(w'), \eta) \det[\partial_{w'} \Phi(w')] dw'] \in E^{\mathbb{R}, X}_{X, z_0^\ast}. $$

Hence we have obtained

$$\hat{\sigma}^\lambda(\hat{\Phi}^\ast([Kdz']))(w, \lambda, \eta) = \int e^{(w'-w, \lambda)} \hat{\Phi}^\ast(Kdz')$$

$$\gamma^\lambda(\eta; \varrho, \theta)$$

$$\int K(z, z', \eta) e^{(\varphi^{-1}(z') - \varphi^{-1}(z), \lambda)} dz'.$$

Finally we shall consider the action on $C^{\mathbb{R}, X}_{X, z_0^\ast}$ associated with $E^{\mathbb{R}, X}_{X, z_0^\ast}$. Let $z_0^\ast = (z_0; \zeta_0) = (0, z_0'; \zeta_0', 0) \in \hat{T}_{X}X \subset X^2X$, $x_C \in \Xi(z_0^\ast)$ and $x_E \in \Xi(\Delta(z_0^\ast))$. Assume $x_C \subset x_E$; that is, $x_C$ and $x_E$ are given by $\{f_1, \ldots, f_d\}$ and $\{f_1', \ldots, f_d', \ldots, f_n\}$ respectively. Note that, for any $x_C \in \Xi(z_0^\ast)$, we can always find a $x_E \in \Xi(\Delta(z_0^\ast))$ with $x_C \subset x_E$. Let $[u] \in C^{\mathbb{R}, X}_{Y|X, z_0^\ast}$ with
u(w, η) ∈ Γ(V_{χC(∗)}; Ω_X) and [Kdw] ∈ E^{R,χ}_X,0 with K(z, w, η)dw ∈ Γ(V_{Δ,κ}^{χ_E(∗)}; Ω_{X^2}(0,n,0)) for some κ. Then we have the morphism

\[ \mu^{χ_E} : E^{R,χ}_X,0 \otimes C^{R,χ}_{Y|X^2,0} \ni [Kdw] \otimes [u] \to \int_{γ_{χ_E}^{χ_E}(z, w, η, θ)} K(z, w, η)u(w, η)dw \in C^{R,χ}_{Y|X^2,0}, \]

which is well defined. Indeed, by the coordinates transformation \((\tilde{z}, \tilde{w}) = (f(z), f(w))\), the situation can be reduced to one studied in Section 3.

**B.8. Theorem.** The family \(\{μ^{χ_E}\}_{χ_E ∈ Ξ_X(z^0)}\) of morphisms constructed above induces the well-defined morphism \(μ : E^{R,χ}_X,0 \otimes C^{R,χ}_{Y|X^2,0} \to C^{R,χ}_{Y|X^2,0}\). Furthermore \(μ\) coincides with the action of \(E^{R,χ}_X,0\) on \(C^{R,χ}_{Y|X^2,0}\).

**Proof.** It suffices to show that, for \(χ_C \subset χ_E\), the following diagram commutes:

\[
\begin{array}{ccc}
E^{R,χ}_X,0 \otimes C^{R,χ}_{Y|X^2,0} & \xrightarrow{\mu^{χ}} & C^{R,χ}_{Y|X^2,0} \\
\downarrow & & \downarrow \\
E^{R,χ}_X,0 \otimes C^{R,χ}_{Y|X^2,0} & \xrightarrow{μ^{χ_E}} & C^{R,χ}_{Y|X^2,0} 
\end{array}
\]

We denote by \(ι\) the isomorphism \(E^{R,χ}_X,0 \cong E^{R,χ}_X,0\) and by the same symbol the one \(C^{R,χ}_{Y|X^2,0} \cong C^{R,χ}_{Y|X^2,0}\). Let \(u(w, η) ∈ Γ(V_{χC(∗)}; Ω_X)\) and \(K(z, w, η)dw ∈ Γ(V_{Δ,κ}^{χ_E(∗)}; Ω_{X^2}(0,n,0))\) for some κ. We define the coordinates transformations

\[ ϕ(z) = ϕ_1(z) = f^{-1}(z), \quad ϕ_2(z, w) = (f^{-1}(z), f^{-1}(w)), \]

\[ \hat{ϕ}_1(z, η) = (f^{-1}(z), η), \quad \hat{ϕ}_2(z, w, η) = (f^{-1}(z), f^{-1}(w), η). \]

It follows from the fact \(χ_E \circ \Phi = \{z_1, \ldots, z_n\}\) and Theorem 3.9 in Section 3 that we have

\[ μ^ϕ(ι^{-1}(\hat{ϕ}_2^Kdw) \otimes ι^{-1}(\hat{ϕ}_1^*[u])) = ι^{-1} \circ μ^{χEΦ}(\hat{ϕ}_2^Kdw) \otimes \hat{ϕ}_1^*[u]). \]

By the coordinates transformation law of the integration, we get

\[ μ^{χEΦ}(\hat{ϕ}_2^Kdw) \otimes \hat{ϕ}_1^*[u]) = \hat{ϕ}_1^* \circ μ^{χE}(Kdw) \otimes [u]. \]

Furthermore, it follows from functorial properties that \(ι^{-1} \circ \hat{ϕ}_k^* = ϕ_k^* \circ ι^{-1}\) (\(k = 1, 2\)) and \(μ^ϕ\) and \(Ψ^*\) commute. Hence we have obtained

\[ Φ^*_1 \circ μ^ϕ(ι^{-1}([Kdw]) \otimes ι^{-1}([u])) = Φ^*_1 \circ ι^{-1} \circ μ^{χE}([Kdw] \otimes [u]), \]

which implies \(μ^ϕ(ι^{-1}([Kdw]) \otimes ι^{-1}([u])) = ι^{-1} \circ μ^{χE}([Kdw] \otimes [u])\). This completes the proof. \(\Box\)

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