Can physics laws be derived from monogenic functions?

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This is a paper about geometry and how one can derive several fundamental laws of physics from a simple postulate of geometrical nature. The method uses monogenic functions analysed in the algebra of 5-dimensional spacetime, exploring the 4-dimensional waves that they generate. With this method one is able to arrive at equations of relativistic dynamics, quantum mechanics and electromagnetism. Fields as disparate as cosmology and particle physics will be influenced by this approach in a way that the paper only suggests. The paper provides an introduction to a formalism which shows prospects of one day leading to a theory of everything and suggests several areas of future development.

1 Introduction

The editor’s invitation to write a chapter for this book about ether and the Universe led me to think how my recent work had anything to do with ether, because the word was never used previously in my writings. It will become clear in the following sections that the concept of a privileged frame or absolute motion underlies all the argument. When one accepts the existence of a preferred frame,
the question of attaching that frame to some observable feature of the Universe is immediate. This question is addressed in Sec. 8 but we can anticipate that galaxy clusters are fixed and can be seen as the anchors for the preferred frame. This statement seems inconsistent with the observation that clusters of galaxies move relative to each other but it is resolved invoking an hyperspherical symmetry in the Universe that is revealed by the choice of appropriate coordinates.

The relationship between geometry and physics is probably stronger in the General Theory of Relativity (GTR) than in any other physics field. It is the author’s belief that a perfect theory will eventually be formulated, where geometry and physics become indistinguishable, so that the complete understanding of space properties, together with proper assignments between geometric and physical entities, will provide all necessary predictions, not only in relativistic dynamics but in physics as a whole.

We don’t have such perfect theory yet, however the author intends to show that GTR and Quantum Mechanics (QM) can be seen as originating from monogenic functions in the algebra of the 5-dimensional spacetime $G_{4,1}$. These functions can generate a null displacement condition, thus reducing the dimensionality by one to the number of dimensions we are all used to. Besides generating GTR and QM, the same space generates also 4-dimensional Euclidean space where dynamics can be formulated and is quite often equivalent to the relativistic counterpart; Euclidean relativistic dynamics resembles Fermat’s principle extended to 4 dimensions and is thus designated as 4-Dimensional Optics (4DO).

Our goal is to show how the important equations of physics, such as relativity equations and equations of quantum mechanics, can be put under the umbrella of a common mathematical approach. We use geometric algebra as the framework but introduce monogenic functions with their null derivatives in order to advance the concept. Furthermore, we clarify some previous work in this direction and identify the steps to take in order to complete this ambitious project.

Since A. Einstein formulated dynamics in 4-dimensional spacetime, this space is recognized by the vast majority of physicists as being the best for formulating the laws of physics. However, mathematical considerations lead to several alternative 4D spaces. For example, the Euclidean 4-dimensional space of 4DO is equivalent to the 4D spacetime of GTR when the metric is static, and therefore the geodesics of one space can be mapped one-to-one with those of the other. Then one can choose to work in the space that is more suitable. We build upon previous work by ourselves and by other authors about null geodesics, regarding the condition that all material particles must follow null geodesics of 5D space:

The implication of this for particles is clear: they should travel on null 5D geodesics. This idea has recently been taken up in the literature, and has a considerable future. It means that what we perceive as massive particles in 4D are akin to photons in 5D.
Accordingly, particles moving on null paths in 5D \( (dS^2 = 0) \) will appear as massive particles moving on timelike paths in 4D \( (ds^2 > 0) \) ...

We actually improve on these null displacement ideas by introducing the more fundamental monogenic condition, deriving the former from the latter and establishing a common first principle.

The only postulates in this paper are of a geometrical nature and can be summarized in the definition of the space we are going to work with; this is the 4-dimensional null subspace of the 5-dimensional space with signature \((- + + + +)\). The choice of this geometric space does not imply any assumption for physical space up to the point where geometric entities like coordinates and geodesics start being assigned to physical quantities like distances and trajectories. Some of those assignments will be made very soon in the exposition and will be kept consistently until the end in order to allow the reader some assessment of the proposed geometric model as a tool for the prediction of physical phenomena.

Mapping between geometry and physics is facilitated if one chooses to work always with non-dimensional quantities; this is done with a suitable choice for standards of the fundamental units. From this point onwards all problems of dimensional homogeneity are avoided through the use of normalizing factors listed below for all units, defined with recourse to the fundamental constants: \( \hbar \rightarrow \) Planck constant divided by \( 2\pi \), \( G \rightarrow \) gravitational constant, \( c \rightarrow \) speed of light and \( e \rightarrow \) proton charge.

| Length          | Time          | Mass          | Charge |
|-----------------|---------------|---------------|--------|
| \( \sqrt{G\hbar/c^3} \) | \( \sqrt{G\hbar/c^5} \) | \( \sqrt{\hbar c/G} \) | \( e \) |

This normalization defines a system of non-dimensional units (Planck units) with important consequences, namely: 1) All the fundamental constants, \( \hbar, G, c, e \), become unity; 2) a particle’s Compton frequency, defined by \( \nu = mc^2/\hbar \), becomes equal to the particle’s mass; 3) the frequent term \( GM/(c^2r) \) is simplified to \( M/r \).

5-dimensional space can have amazing structure, providing countless parallels to the physical world; this paper is just a limited introductory look at such structure and parallels. The exposition makes full use of an extraordinary and little known mathematical tool called geometric algebra (GA), a.k.a. Clifford algebra, which received an important thrust with the works of David Hestenes \([5]\). A good introduction to GA can be found in Gull et al. \([6]\) and the following paragraphs use basically the notation and conventions therein. A complete course on physical applications of GA can be downloaded from the internet \([7]\); the same authors published a more comprehensive version in book form \([8]\). An accessible...
presentation of mechanics in GA formalism is provided by Hestenes [9]. This is the subject of first section, where some essential GA concepts and notation are introduced.

Section two deals with monogenic function in flat 5D spacetime, deriving special relativity and the free particle Dirac equation from this simple concept. 4DO appears here as a perfect equivalent to special relativity, where trajectories can be understood as normals to 4-dimensional plane-like waves. The following section improves on this by allowing for curved space, introducing the notion of refractive index tensor. Section five examines the variational principle applied in both 4DO and GTR spaces to justify the equivalence of geodesics between the two spaces for static metrics. Refractive index is then related to its sources and the sources tensor is defined. The case of a central mass is examined and the links to Schwarzschild’s metric are thoroughly discussed. Electromagnetism and electrodynamics are formulated as particular cases of refractive index in section seven and the sources tensor is here related to a current vector. The next section introduces the hypothesis of an hyperspherical symmetry in the Universe, which would call for the use of hyperspherical coordinates; the consequences for cosmology would include a complete dismissal of dark matter for a flat rate Hubble expansion. Before the conclusion, section nine shows how the monogenic condition is effective in generating an \( SU(4) \) symmetry group and makes some advances towards a relation with the standard model of particle physics.

## 2 Introduction to geometric algebra

Geometric algebra is not usually taught in university courses and its presence in the literature is scarce; good reference works are [5, 7, 8]. We will concentrate on the algebra of 5-dimensional spacetime because this will be our main working space; this algebra incorporates as subalgebras those of the usual 3-dimensional Euclidean space, Euclidean 4-space and Minkowski spacetime. We begin with the simpler 5D flat space and progress to a 5D spacetime of general curvature.

The geometric algebra \( G_{4,1} \) of the hyperbolic 5-dimensional space we consider is generated by the coordinate frame of orthonormal basis vectors \( \sigma_\alpha \) such that

\[
\begin{align*}
(\sigma_0)^2 &= -1, \\
(\sigma_i)^2 &= 1, \\
\sigma_\alpha \cdot \sigma_\beta &= 0, \quad \alpha \neq \beta.
\end{align*}
\]

(2.1)

Note that the English characters i, j, k range from 1 to 4 while the Greek characters \( \alpha, \beta, \gamma \) range from 0 to 4. See the Appendix A for the complete notation convention used.

Any two basis vectors can be multiplied, producing the new entity called a bivector. This bivector is the geometric product or, quite simply, the product;
this product is distributive. Similarly to the product of two basis vectors, the product of three different basis vectors produces a trivector and so forth up to the fivevector, because five is the dimension of space.

We will simplify the notation for basis vector products using multiple indices, i.e. \( \sigma_\alpha \sigma_\beta \equiv \sigma_{\alpha\beta} \). The algebra is 32-dimensional and is spanned by the basis

- 1 scalar, \( 1 \),
- 5 vectors, \( \sigma_\alpha \),
- 10 bivectors (area), \( \sigma_\alpha \sigma_\beta \),
- 10 trivectors (volume), \( \sigma_\alpha \sigma_\beta \sigma_\gamma \),
- 5 tetravectors (4-volume), \( i \sigma_\alpha \),
- 1 pseudoscalar (5-volume), \( i \equiv \sigma_{01234} \).

Several elements of this basis square to unity:

\[
(\sigma_i)^2 = (\sigma_{0i})^2 = (\sigma_{0ij})^2 = (i\sigma_0)^2 = 1. \tag{2.2}
\]

It is easy to verify the equations above; suppose we want to check that \( (\sigma_{0ij})^2 = 1 \). Start by expanding the square and remove the compact notation \( (\sigma_{0ij})^2 = \sigma_0 \sigma_0 \sigma_i \sigma_j \sigma_0 \sigma_0 \sigma_i \sigma_j \), then swap the last \( \sigma_j \) twice to bring it next to its homonymous; each swap changes the sign, so an even number of swaps preserves the sign: \( (\sigma_{0ij})^2 = \sigma_0 \sigma_i (\sigma_j)^2 \sigma_0 \sigma_j \). From the third equation (2.1) we know that the squared vector is unity and we get successively \( (\sigma_{0ij})^2 = \sigma_0 \sigma_0 \sigma_i \sigma_j = -(\sigma_0)^2 (\sigma_i)^2 = -(\sigma_0)^2 \); using the first equation (2.1) we get finally \( (\sigma_{0ij})^2 = 1 \) as desired.

The remaining basis elements square to \(-1\) as can be verified in a similar manner:

\[
(\sigma_0)^2 = (\sigma_i)^2 = (\sigma_{ijk})^2 = (i\sigma_i)^2 = i^2 = -1. \tag{2.3}
\]

Note that the pseudoscalar \( i \) commutes with all the other basis elements while being a square root of \(-1\); this makes it a very special element which can play the role of the scalar imaginary in complex algebra.

We can now address the geometric product of any two vectors \( a = a^\alpha \sigma_\alpha \) and \( b = b^\beta \sigma_\beta \) making use of the distributive property

\[
ab = \left(-a^0 b^0 + \sum_i a^i b^i\right) + \sum_{\alpha \neq \beta} a^\alpha b^\beta \sigma_{\alpha\beta}; \tag{2.4}
\]

and we notice it can be decomposed into a symmetric part, a scalar called the inner or interior product, and an anti-symmetric part, a bivector called the outer or exterior product.

\[
ab = a \cdot b + a \wedge b, \quad ba = a \cdot b - a \wedge b. \tag{2.5}
\]
Reversing the definition one can write inner and outer products as

$$a \cdot b = \frac{1}{2} (ab + ba), \quad a \wedge b = \frac{1}{2} (ab - ba). \quad (2.6)$$

The inner product is the same as the usual "dot product," the only difference being in the negative sign of the $a_0b_0$ term; this is to be expected and is similar to what one finds in special relativity. The outer product represents an oriented area; in Euclidean 3-space it can be linked to the "cross product" by the relation $\text{cross}(a, b) = -\sigma_{123}a \wedge b$; here we introduced bold characters for 3-dimensional vectors and avoided defining a symbol for the cross product because we will not use it again. We also used the convention that interior and exterior products take precedence over geometric product in an expression.

When a vector is operated with a multivector the inner product reduces the grade of each element by one unit and the outer product increases the grade by one. We will generalize the definition of inner and outer products below; under this generalized definition the inner product between a vector and a scalar produces a vector. Given a multivector $a$ we refer to its grade-$r$ part by writing $<a>_r$; the scalar or grade zero part is simply designated as $<a>$. By operating a vector with itself we obtain a scalar equal to the square of the vector’s length

$$a^2 = aa = a \cdot a + a \wedge a = a \cdot a. \quad (2.7)$$

The definitions of inner and outer products can be extended to general multivectors

$$a \cdot b = \sum_{\alpha, \beta} \langle <a>_{\alpha} <b>_{\beta} \rangle |_{\alpha - \beta}, \quad (2.8)$$

$$a \wedge b = \sum_{\alpha, \beta} \langle <a>_{\alpha} <b>_{\beta} \rangle |_{\alpha + \beta}. \quad (2.9)$$

Two other useful products are the scalar product, denoted as $<ab>$ and commutator product, defined by

$$a \times b = ab - ba. \quad (2.10)$$

In mixed product expressions we will always use the convention that inner and outer products take precedence over geometric products, thus reducing the number of parenthesis.

We will encounter exponentials with multivector exponents; two particular cases of exponentiation are specially important. If $u$ is such that $u^2 = -1$ and $\theta$
is a scalar
\[
e^{u\theta} = 1 + u\theta - \frac{\theta^2}{2!} - u\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + \ldots
\]

Conversely if \( h \) is such that \( h^2 = 1 \)
\[
e^{h\theta} = 1 + h\theta + \frac{\theta^2}{2!} + h\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + \ldots
\]

The exponential of bivectors is useful for defining rotations; a rotation of vector \( a \) by angle \( \theta \) on the \( \sigma_{12} \) plane is performed by
\[
a' = e^{\sigma_{12}\theta/2}ae^{\sigma_{12}\theta/2} = \tilde{R}aR; \quad (2.13)
\]

Similarly, if we had made \( a = \sigma_2 \), the result would have been \(-\sin \theta \sigma_1 + \cos \theta \sigma_2\).

If we use \( B \) to represent a bivector whose plane is normal to \( \sigma_0 \) and define its norm by \( |B| = (BB)^{1/2} \), a general rotation in 4-space is represented by the rotor
\[
R \equiv e^{-B/2} = \cos \left( \frac{|B|}{2} \right) - \frac{B}{|B|} \sin \left( \frac{|B|}{2} \right). \quad (2.15)
\]

The rotation angle is \( |B| \) and the rotation plane is defined by \( B \). A rotor is defined as a unitary even multivector (a multivector with even grade components only) which squares to unity; we are particularly interested in rotors with bivector
components. It is more general to define a rotation by a plane (bivector) then by an axis (vector) because the latter only works in 3D while the former is applicable in any dimension. When the plane of bivector $B$ contains $\sigma_0$, a similar operation does not produce a simple rotation but produces a boost, eventually combined with a rotation. Take for instance $B = \sigma_{01} \theta/2$ and define the transformation operator $T = \exp(B)$; a transformation of the basis vector $\sigma_0$ produces

$$d' = \tilde{T} \sigma_0 T = e^{-\sigma_{01} \theta/2} \sigma_0 e^{\sigma_{01} \theta/2}$$

$$= \left( \cosh \frac{\theta}{2} - \sigma_{01} \sinh \frac{\theta}{2} \right) \sigma_0 *$$

$$* \left( \cosh \frac{\theta}{2} + \sigma_{01} \sinh \frac{\theta}{2} \right)$$

$$= \cosh \theta \sigma_0 + \sinh \theta \sigma_1. \quad (2.16)$$

In 5-dimensional spacetime of general curvature, we introduce 5 coordinate frame vectors $g_\alpha$, the indices follow the conventions set forth in Appendix A. We will also assume this spacetime to be a metric space whose metric tensor is given by

$$g_{\alpha\beta} = g_\alpha \cdot g_\beta; \quad (2.17)$$

the double index is used with $g$ to denote the inner product of frame vectors and not their geometric product. The space signature is $(-+++)$, which amounts to saying that $g_{00} < 0$ and $g_{ii} > 0$. A reciprocal frame is defined by the condition

$$g_\alpha \cdot g_\beta = \delta_\alpha^\beta. \quad (2.18)$$

Defining $g^{\alpha\beta}$ as the inverse of $g_{\alpha\beta}$, the matrix product of the two must be the identity matrix; using Einstein’s summation convention this is

$$g^{\alpha\gamma} g_{\gamma\beta} = \delta_\alpha^\beta. \quad (2.19)$$

Using the definition (2.17) we have

$$\left(g^{\alpha\gamma} g_{\gamma\beta}\right) \cdot g_\beta = \delta_\alpha^\beta; \quad (2.20)$$

comparing with Eq. (2.18) we determine $g^\alpha$ with

$$g^\alpha = g^{\alpha\gamma} g_{\gamma}. \quad (2.21)$$

If the coordinate frame vectors can be expressed as a linear combination of the orthonormed ones, we have

$$g_\alpha = n^\beta_\alpha \sigma_\beta, \quad (2.22)$$

where $n^\beta_\alpha$ is called the refractive index tensor or simply the refractive index; its 25 elements can vary from point to point as a function of the coordinates.\[2]
When the refractive index is the identity, we have $g_{\alpha} = \sigma_{\alpha}$ for the main or direct frame and $g^0 = -\sigma_0, g^i = \sigma_i$ for the reciprocal frame, so that Eq. (2.18) is verified. In this work we will not consider spaces of general curvature but only those satisfying condition (2.22).

The first use we will make of the reciprocal frame is for the definition of two derivative operators. In flat space we define the vector derivative

$$\nabla = \sigma^\alpha \partial_\alpha.$$  \hspace{1cm} (2.23)

It will be convenient, sometimes, to use vector derivatives in subspaces of 5D space; these will be denoted by an upper index before the $\nabla$ and the particular index used determines the subspace to which the derivative applies; For instance $m\nabla = \sigma^m \partial_m = \sigma^1 \partial_1 + \sigma^2 \partial_2 + \sigma^3 \partial_3$. In 5-dimensional space it will be useful to split the vector derivative into its time and 4-dimensional parts

$$\nabla = -\sigma_0 \partial_t + \sigma^i \partial_i = -\sigma_0 \partial_t + i\nabla.$$  \hspace{1cm} (2.24)

The second derivative operator is the covariant derivative, sometimes called the Dirac operator, and it is defined in the reciprocal frame $g^\alpha$

$$D = g^\alpha \partial_\alpha.$$  \hspace{1cm} (2.25)

Taking into account the definition of the reciprocal frame (2.18), we see that the covariant derivative is also a vector. In cases such as those we consider in this work, where there is a refractive index, it will be possible to define both derivatives in the same space.

We define also second order differential operators, designated Laplacian and covariant Laplacian respectively, resulting from the inner product of one derivative operator by itself. The square of a vector is always a scalar and the vector derivative is no exception, so the Laplacian is a scalar operator, which consequently acts separately in each component of a multivector. For 4 + 1 flat space it is

$$\nabla^2 = -\frac{\partial^2}{\partial t^2} + i\nabla^2.$$  \hspace{1cm} (2.26)

One sees immediately that a 4-dimensional wave equation is obtained by zeroing the Laplacian of some function

$$\nabla^2 \psi = \left( -\frac{\partial^2}{\partial t^2} + i\nabla^2 \right) \psi = 0.$$  \hspace{1cm} (2.27)

This procedure will be used in the next section for the derivation of special relativity and will be extended later to general curved spaces.
3 Monogenic functions and waves in flat space

It turns out that there is a class of functions of great importance, called monogenic functions, \( \psi \) characterized by having null vector derivative; a function \( \psi \) is monogenic if and only if

\[
\nabla \psi = 0. \tag{3.1}
\]

A monogenic function has by necessity null Laplacian, as can be seen by dotting Eq. (3.1) with \( \nabla \) on the left. We are then allowed to write

\[
\sum_i \partial_i \psi = \partial_{00} \psi. \tag{3.2}
\]

This can be recognized as a wave equation in the 4-dimensional space spanned by \( \sigma_i \) which will accept plane wave type solutions of the general form

\[
\psi = \psi_0 e^{i(p_\alpha x^\alpha + \delta)}, \tag{3.3}
\]

where \( \psi_0 \) is an amplitude whose characteristics we shall not discuss for now, \( \delta \) is a phase angle and \( p_\alpha \) are constants such that

\[
\sum_i (p_i)^2 - (p_0)^2 = 0. \tag{3.4}
\]

By setting the argument of \( \psi \) constant in Eq. (3.3) and differentiating we can get the differential equation

\[
p_\alpha dx^\alpha = 0. \tag{3.5}
\]

The first member can equivalently be written as the inner product of the two vectors \( p \cdot dx = 0 \), where \( p = \sigma^\alpha p_\alpha \). In 5D hyperbolic space the inner product of two vectors can be null when the vectors are perpendicular but also when the two vectors are null; since we have established that \( p \) is a null vector, Eq. (3.5) can be satisfied either by \( dx \) normal to \( p \) or by \( (dx)^2 = 0 \). In the former case the condition describes a 3-volume called wavefront and in the latter case it describes the wave motion. Notice that the wavefronts are not surfaces but volumes, because we are working with 4-dimensional waves.

The condition describing wave motion can be expanded as

\[
-(dx^0)^2 + \sum (dx^i)^2 = 0. \tag{3.6}
\]

This is a purely scalar equation and can be manipulated as such, which means we are allowed to rewrite it with any chosen terms in the second member; some of those manipulations are particularly significant. Suppose we decide to isolate \( (dx^4)^2 \) in the first member: \( (dx^4)^2 = (dx^0)^2 - \sum (dx^m)^2 \). We can then rename coordinate \( x^4 \) as \( \tau \), to get the interval squared of special relativity for space-like displacements

\[
d\tau^2 = (dx^0)^2 - \sum (dx^m)^2. \tag{3.7}
\]
We have thus derived the space-like part of special relativity as a consequence of monogeneity in 5D hyperbolic space and simultaneously justified the physical interpretation for coordinates $x^0$ and $x^4$ as time and proper time, respectively. A different manipulation of Eq. (3.6) has great significance because it leads to the 4DO concept. If we isolate $(dx^0)^2$ and replace $x^0$ by the letter $t$, we see that time becomes the interval in Euclidean 4D space

$$dt^2 = \sum (dx^i)^2.$$  

(3.8)

From this we conclude that the monogenic condition produces plane waves whose wavefronts are 3D volumes but can be represented by wavefront normals, just as it happens in standard optics with electromagnetic waves.

Several readers may be worried with the fact that proper time is a line integral and not a coordinate in special relativity and so $d\tau$ should not be allowed to appear on the rhs of the equation. To this we will argue that the manipulations we have done, collapsing 5D spacetime into 4 dimensions through a null displacement condition and then promoting one of the coordinates into interval, is exactly equivalent to the process of defining a light cone in Minkowski spacetime and then applying Fermat’s principle to define an Euclidean 3D metric on the light cone; we have just upgraded the procedure by including one extra dimension.

The Dirac equation can also be derived from the monogenic condition but since it appears formulated in terms of matrices in all textbooks we will have to rewrite Eq. (3.1) also in terms of matrices, so that our GA manipulations can also be understood as matrix operations. This is easily achieved if we assign our frame vectors to Dirac matrices that square to the identity matrix or minus the identity matrix as appropriate; the following list of assignments can be used but others would be equally effective.

$$
\sigma^0 \equiv \begin{pmatrix} i & 0 & 0 & 0 \\
0 & -i & 0 & 0 \\
0 & 0 & i & 0 \\
0 & 0 & 0 & -i \end{pmatrix},
\sigma^1 \equiv \begin{pmatrix} 0 & 0 & 0 & 1 \\
0 & 0 & -1 & 0 \\
0 & -1 & 0 & 0 \\
1 & 0 & 0 & 0 \end{pmatrix},
\sigma^2 \equiv \begin{pmatrix} 0 & i & 0 & 0 \\
-i & 0 & 0 & 0 \\
0 & 0 & 0 & -i \\
0 & 0 & i & 0 \end{pmatrix},
\sigma^3 \equiv \begin{pmatrix} 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \end{pmatrix},
\sigma^4 \equiv \begin{pmatrix} 0 & 0 & 0 & -i \\
0 & 0 & i & 0 \\
0 & -i & 0 & 0 \\
i & 0 & 0 & 0 \end{pmatrix}.
$$

(3.9)

$1$There are 16 possible $4 \times 4$ Dirac matrices, of which we must choose 5 such that $(\sigma_0)^2 = -I$, $(\sigma_i)^2 = I$ and $\sigma_\alpha \sigma_\beta = -\sigma_\beta \sigma_\alpha$, for $\alpha \neq \beta$; the present choice will simplify our symmetry discussions further along.
There is no need to adopt different notations to refer to the frame vectors or to their matrix counterparts because the context will usually be sufficient to determine what is meant.

We can check that matrices $\sigma^\alpha$ form an orthonormal basis of 5D space by defining the inner product of square matrices as

$$A \cdot B = \frac{AB + BA}{2}. \quad (3.10)$$

It will then be possible to verify that the inner product of any two different $\sigma$-matrices is null, $(\sigma^0)^2 = -I$ and $(\sigma^i)^2 = I$; these are the conditions defining an orthonormal basis expressed in matrix form. A more formal approach to this subject would lead us to invoke the isomorphism between the complex algebra of $4 \times 4$ matrices and Clifford algebra $G_{4,1}$, the geometric algebra of 5D spacetime.\([14]\).

It will now be convenient to expand the monogenic condition (3.1) as $(\sigma^\mu \partial_\mu + \sigma^4 \partial_4) \psi = 0$. If this is applied to the solution (3.3) and the derivative with respect to $x^4$ is evaluated we get

$$(\sigma^\mu \partial_\mu + \sigma^4 i p_4) \psi = 0. \quad (3.11)$$

Let us now multiply both sides of the equation on the left by $\sigma^4$ and note that matrix $\sigma^4 \sigma^0$ squares to the identity while the 3 matrices $\sigma^4 \sigma^m$ square to minus identity; we rename these products as $\gamma$-matrices in the form $\gamma^\mu = \sigma^4 \sigma^\mu$. Rewriting the equation in this form we get

$$(\gamma^\mu \partial_\mu + i p_4) \psi = 0. \quad (3.12)$$

The only thing this equation needs to be recognized as Dirac’s is the replacement of $p_4$ by the particle’s mass $m$; simultaneously we assign the energy $E$ to $p_0$ and 3D momentum $p$ to $\sigma^m p_m$.

We turn now our attention to the amplitude $\psi_0$ in Eq. (3.3) because we know that the Dirac equation accepts solutions which are spinors and we want to find out their equivalents in our formulation. Applying the monogenic condition to Eq. (3.3) we see that the following equation must be verified

$$\psi_0 (\sigma^\alpha p_\alpha) = 0. \quad (3.13)$$

If the $\sigma$s are interpreted as matrices, remembering that $p$ is null, the only way the equation can be verified is by $\psi_0$ being some constant multiplied by the matrix in parenthesis, which is a matrix representation of $p$. We can set the multiplying constant to unity and $\psi_0$ becomes equal to $p$; the wavefunction $\psi$ can then be interpreted as a Dirac spinor. The wave function in Eq. (3.3) can now be given a different form, taking in consideration the previous assignments

$$\psi = A (\sigma_4 m + p \mp \sigma_0 E) e^{i(\pm Et + p \cdot x + m \tau + \delta)}. \quad (3.14)$$
where $A$ is the amplitude and $x = \sigma_m x^m$ is the 3-dimensional position.

In order to separate left and right spinor components we use a technique adapted from Ref. [8]. We choose an arbitrary $4 \times 4$ matrix which squares to identity, for instance $\sigma_4$, with which we form the two idempotent matrices $(I + \sigma_4)/2$ and $(I - \sigma_4)/2$. These matrices are called idempotents because they reproduce themselves when squared. These idempotents absorb any $\sigma_4$ factor; as can be easily checked $(I + \sigma_4)\sigma_4 = (I + \sigma_4)$ and $(I - \sigma_4)\sigma_4 = -(I - \sigma_4)$.

Obviously we can decompose the wavefunction $\psi$ as

$$\psi = \psi_+ \frac{I + \sigma_4}{2} + \psi_- \frac{I - \sigma_4}{2} = \psi_+ + \psi_-.$$  (3.15)

This apparently trivial decomposition produces some surprising results due to the following relations

$$e^{i\theta} (I + \sigma_4) = (\cos \theta + i \sin \theta)(I + \sigma_4) = (I \cos \theta + i \sigma_4 \sin \theta)(I + \sigma_4) = e^{i\sigma_4 \theta} (I + \sigma_4).$$  (3.16)

and similarly

$$e^{i\theta} (I - \sigma_4) = e^{-i\sigma_4 \theta} (I - \sigma_4).$$  (3.17)

If we had chosen a different idempotent the result would have been similar; we will see how the various idempotents are arranged in a symmetry group and it has been argued that they may be related to elementary particles.[15]

4 Relativistic dynamics

When working in curved spaces the monogenic condition is naturally modified, replacing the vector derivative $\nabla$ with the covariant derivative $D$. A generalized monogenic function is then a function that verifies the equation

$$D\psi = 0.$$  (4.1)

Similarly to what happens in flat space, the covariant Laplacian is a scalar and a monogenic function must verify the second order differential equation

$$D^2 \psi = 0.$$  (4.2)

It is possible to write a general expression for the covariant Laplacian in terms of the metric tensor components (see [16, Section 2.11]) but we will consider only situations where that complete general expression is not needed.

$\sigma^4$ matrix is the same as matrix $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$. 

1
When Eq. (4.1) is multiplied on the left by $D$, we are applying second derivatives to the function, but we are simultaneously applying first order derivatives to the reciprocal frame vectors present in the definition of $D$ itself. We can simplify the calculations if the variations of the frame vectors are taken to be much slower than those of function $\psi$ so that frame vector derivatives can be neglected. With this approximation, the covariant Laplacian becomes $D^2 = g^{\alpha\beta} \partial_{\alpha\beta}$ and Eq. (4.2) can be written

$$g^{\alpha\beta} \partial_{\alpha\beta} \psi = 0. \quad (4.3)$$

This equation can have a solution of the type given by Eq. (3.3) if again the derivatives of $p_\alpha$ are neglected. This approximation is usually of the same order as the former one and should not be seen as a second restriction. Inserting Eq. (3.3) one sees that it is a solution if

$$g^{\alpha\beta} p_\alpha p_\beta = 0. \quad (4.4)$$

This equation is the curved space equivalent to Eq. (3.4) and it means that the square of vector $p = g^{\alpha} p_\alpha$ is zero, that is, $p$ is a vector of zero length; for this reason it is called a null vector or nilpotent. Vector $p$ is the momentum vector and should not be confused with 4-dimensional conjugate momentum vectors defined below.

We arrive again at Eq. (3.5) and the condition describing 4D wave motion can be expanded as

$$g_{\alpha\beta} dx^\alpha dx^\beta = 0. \quad (4.5)$$

This condition effectively reduces the spatial dimension to four but the resulting space is non-metric because all displacements have zero length. We will remove this difficulty by considering two special cases. First let us assume that vector $g_0$ is normal to the other frame vectors so that all $g_{0i}$ factors are zeroed; condition (4.5) becomes

$$g_{00}(dx^0)^2 + g_{ij} dx^i dx^j = 0. \quad (4.6)$$

All the terms in this equation are scalars and we are allowed to rewrite it with $(dx^0)^2$ in the lhs

$$(dx^0)^2 = -\frac{g_{ij}}{g_{00}} dx^i dx^j. \quad (4.7)$$

We could have arrived at the same result by defining a 4-dimensional displacement vector

$$dx^0 v = \frac{-1}{\sqrt{g_{00}}} g_i dx^i; \quad (4.8)$$

and then squaring it to evaluate its length; $v$ is a unit vector called velocity because its definition is similar to the usual definition of 3-dimensional velocity; its components are

$$v_i = \frac{dx^i}{dx^0}. \quad (4.9)$$
Being unitary, the velocity can be obtained by a rotation of the $\sigma_4$ frame vector

$$v = \hat{R}\sigma_4 R. \quad (4.10)$$

The rotation angle is a measure of the 3-dimensional velocity component. A null angle corresponds to $v$ directed along $\sigma_4$ and null 3D component, while a $\pi/2$ angle corresponds to the maximum possible 3D component. The idea that physical velocity can be seen as the 3D component of a unitary 4D vector has been explored in several papers but see [17].

Equation (4.8) projects the original 5-dimensional space into an Euclidean signature 4 dimensional space, where an elementary displacement is given by the variation of coordinate $x^0$. In the particular case where $g_0 = \sigma_0$ the displacement vector simplifies to $dx^0v = g_i dx^i$ and we can see clearly that the signature is Euclidean because the four $g_i$ have positive norm. Although it has not been mentioned, we have assumed that none of the frame vectors is a function of coordinate $x_0$.

Returning to Eq. (4.6) we can now impose the condition that $g_4$ is normal to the other frame vectors in order to isolate $(dx^4)^2$ instead of $(dx^0)^2$, as we did before;

$$(dx^4)^2 = -\frac{g_{\mu\nu}}{g_{44}} dx^\mu dx^\nu. \quad (4.11)$$

We have now projected onto 4-dimensional space with signature $(+ - - -)$, known as Minkowski signature. In order to check this consider again the special case with $g_0 = \sigma_0$ and the equation becomes

$$(dx^4)^2 = \frac{1}{g_{44}} (dx^0)^2 - \frac{g_{mn}}{g_{44}} dx^m dx^n; \quad (4.12)$$

the diagonal elements $g_{ii}$ are necessarily positive, which allows a verification of Minkowski signature. Contrary to what happened in the previous case, we cannot now obtain $(dx^4)^2$ by squaring a vector but we can do it by consideration of the bivector

$$dx^4v = \frac{1}{\sqrt{g_{44}g^4}} g_{\mu} g^4 dx^\mu. \quad (4.13)$$

All the products $g_{\mu} g^4$ are bivectors because we imposed $g_4$ to be normal to the other frame vectors. When $(dx^4)^2$ is evaluated by an inner product we notice that $g_0 g^4$ has positive square while the three $g_m g^4$ have negative square, ensuring that a Minkowski signature is obtained. Naturally we have to impose the condition that none of the frame vectors depends on $x^4$. Bivector $v$ is such that $v^2 = vv = 1$ and it can be obtained by a Lorentz transformation of bivector $\sigma_{04}$.

$$v = \hat{T}\sigma_{04}T, \quad (4.14)$$
where $T$ is of the form $T = \exp(B)$ and $B$ is a bivector whose plane is normal to $\sigma_4$. Note that $T$ is a pure rotation when the bivector plane is normal to both $\sigma_0$ and $\sigma_4$.

In special relativity it is usual to work in a space spanned by an orthonormed frame of vectors $\gamma_{\mu}$ such that $(\gamma_0)^2 = 1$ and $(\gamma_m)^2 = -1$, producing the desired Minkowski signature \[8\]. The geometric algebra of this space is isomorphic to the even sub-algebra of $G_{4,1}$ and so the area element $d\mathbf{x}^4$ can be reformulated as a vector called relativistic 4-velocity. The four $\gamma$ bivectors are defined in a similar way to the $\gamma$ matrices used in Eq. (3.12), which is to be expected from the isomorphism between geometric and matrix algebras already mentioned.

Equations (4.7) and (4.11) define two alternative 4-dimensional spaces, those of 4-dimensional optics (4DO), with metric tensor $-g_{ij}/g_{00}$ and general theory of relativity (GTR) with metric tensor $-g_{\mu\nu}/g_{44}$, respectively; in the former $x^0$ is an affine parameter while in the latter it is $x^4$ that takes such role. In fact Eq. (4.11) only covers the spacelike part of GTR space, because $(dx^4)^2$ is necessarily non-negative. Naturally there is the limitation that the frame vectors are independent of both $x^0$ and $x^4$, equivalent to imposing a static metric, and also that $g_{0i} = g_{\mu4} = 0$. Provided the metric is static, the geodesics of 4DO can be mapped one-to-one with spacelike geodesics of GTR and we can choose to work on the space that best suits us for free fall dynamics. For a physical interpretation of geometric relations it will frequently be convenient to assign new designations to the 5D coordinates that acquire the role of affine parameter in the null subspace. We recall the assignments $x^0 \equiv t$ and $x^4 \equiv \tau$; total derivatives with respect to these coordinates will receive a special notation: $df/dt = \dot{f}$ and $df/d\tau = \dot{f}$.

Unless otherwise specified, we will assume that the frame vector associated with coordinate $x^0$ is unitary and normal to all the others, that is $g_0 = \sigma_0$ and $g_{0i} = 0$. Recalling from Eq. (4.7), these conditions allow the definition of 4DO space with metric tensor $g_{ij}$. Although we could try a more general approach, we would loose the possibility of interpreting time as a line element and this, as we shall see, provides very interesting and novel interpretations of physics’ equations. In many cases it is also true that $g_4$ is normal to the other frame vectors and we have seen that in those cases we can make metric conversions between GTR and 4DO; as we shall see, electromagnetism requires a non-normal $g_4$ and so we leave this possibility open.

For the moment we will concentrate on isotropic space, characterized by orthogonal refractive index vectors $g_i$ whose norm can change with coordinates but is the same for all vectors. Normally we relax this condition by accepting that the three $g_m$ must have equal norm but $g_4$ can be different. The reason for this relaxed isotropy is found in the parallel we make with physics by assigning dimensions 1 to 3 to physical space. Isotropy in a physical sense need only be concerned with these dimensions and ignores what happens with dimension 4.
We will therefore characterize an isotropic space by the refractive index frame \( g_0 = \sigma_0, \ g_m = n_r \sigma_m, \ g_4 = n_4 \sigma_4. \) Indeed we could also accept a non-orthogonal \( g_4 \) within the relaxed isotropy concept but we will not do so for the moment.

Equation (4.7) can now be written in terms of the isotropic refractive indices as
\[
dr^2 = (n_r)^2 \sum_m (dx^m)^2 + (n_4 d\tau)^2. \tag{4.15}\]

Spherically symmetric static metrics play a special role; this means that the refractive index can be expressed as functions of \( r \) if we adopt spherical coordinates. The previous equation then becomes
\[
dr^2 = (n_r)^2 \left[ dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2) \right] + (n_4 d\tau)^2. \tag{4.16}\]

Since we have \( g_4 \) normal to the other vectors we can apply metric conversion and write the equivalent quadratic form for GTR
\[
d\tau^2 = \left( \frac{dr}{n_4} \right)^2 - \left( \frac{n_r}{n_4} \right)^2 \left[ dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2) \right]. \tag{4.17}\]

In the case of a central mass, we can examine how the Schwarzschild metric in GTR can be transposed to 4DO. The usual form of the metric is
\[
d\tau^2 = \left( 1 - \frac{2M}{\chi} \right) dr^2 - \left( 1 - \frac{2M}{\chi} \right)^{-1} d\chi^2 - \chi^2 (d\theta^2 + \sin^2 \theta d\varphi^2); \tag{4.18}\]
where \( M \) is the spherical mass and \( \chi \) is the radial coordinate, not the distance to the centre of the mass. This form is non-isotropic but a change of coordinates can be made that returns the expression to isotropic form (see D’Inverno [18, section 14.7]):
\[
\chi = \left( \chi - M \right) / 2; \tag{4.19}\]
and the new form of the metric is
\[
d\tau^2 = \left( 1 - \frac{M}{2r} \right)^2 \left( 1 + \frac{M}{2r} \right) dr^2 - \left( 1 + \frac{M}{2r} \right)^4 \left[ dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\varphi^2) \right]. \tag{4.20}\]

From this equation we immediately define two coefficients, which are called refractive index coefficients,
\[
n_4 = \frac{1 + \frac{M}{2r}}{1 - \frac{M}{2r}}, \quad n_r = \left( 1 + \frac{M}{2r} \right)^3 \frac{1 - \frac{M}{2r}}{1 - \frac{M}{2r}}. \tag{4.21}\]
These refractive indices provide a 4DO Euclidean space equivalent to Schwarzschild metric, allowing 4DO to be used as an alternative to GTR. Recalling that we derived trajectories from solutions (3.3) of a 4-dimensional wave equation (4.3), it becomes clear that orbits can also be seen as 4-dimensional guided waves by what could be described as a 4-dimensional optical fibre. Modes are to be expected in these waveguides and we shall say something about them later on.

5 Fermat’s principle in 4 dimensions

Fermat’s principle applies to optics and states that the path followed by a light ray is the one that makes the travel time an extremum; usually it is the path that minimizes the time but in some cases a ray can follow a path of maximum or stationary time. These solutions are usually unstable, so one takes the view that light must follow the quickest path. In Eq. (4.7) we have defined a time interval associated with a 4-dimensional elementary displacement, which allows us to determine, by integration, a travel time associated with displacements of any size along a given 4-dimensional path. We can then extend Fermat’s principle to 4D and impose an extremum requirement in order to select a privileged path between any two 4D points. Taking the square root to Eq. (4.7)

\[
\frac{dt}{\sqrt{-g_{ij}g^{00}dx^i dx^j}}. \tag{5.1}
\]

Integrating between two points \(P_1\) and \(P_2\)

\[
t = \int_{P_1}^{P_2} \sqrt{-\frac{g_{ij}dx^i dx^j}{g^{00}}} = \int_{P_1}^{P_2} \sqrt{-\frac{g_{ij}}{g^{00}}\dot{x}^i \dot{x}^j} dt. \tag{5.2}
\]

In order to evaluate the previous integral one must know the particular path linking the points by defining functions \(x^i(t)\), allowing the replacement \(dx^i = \dot{x}^i dt\). At this stage it is useful to define a Lagrangian

\[
L = -\frac{g_{ij}}{2g^{00}}\dot{x}^i \dot{x}^j. \tag{5.3}
\]

The time integral can then be written

\[
t = \int_{P_1}^{P_2} \sqrt{2L} dt. \tag{5.4}
\]

Time has to remain stationary against any small change of path; therefore we envisage a slightly distorted path defined by functions \(x^i(t) + \varepsilon \chi^i(t)\), where \(\varepsilon\) is arbitrarily small and \(\chi^i(t)\) are functions that specify distortion. Since the
distortion must not affect the end points, the distortion functions must vanish at those points. The time integral will now be a function of $\varepsilon$ and we require that

$$\left. \frac{dt(\varepsilon)}{d\varepsilon} \right|_{\varepsilon=0} = 0.$$  \hfill (5.5)

Now, the Lagrangian \((5.3)\) is a function of $x^i$, through $g_{\alpha\beta}$ and also an explicit function of $\dot{x}^i$. Allowing for a path change, through $\varepsilon$ makes $t$ in Eq. \((5.4)\) a function of $\varepsilon$

$$t(\varepsilon) = \int_{P_1}^{P_2} \sqrt{2L(x^i + \varepsilon \chi^i + \dot{x}^i + \varepsilon \dot{\chi}^i)} \, dt. \quad (5.6)$$

This can now be derived with respect to $\varepsilon$

$$\left. \frac{dt(\varepsilon)}{d\varepsilon} \right|_{\varepsilon=0} = \left[ \int_{P_1}^{P_2} \frac{1}{\sqrt{2L}} \left( \frac{\partial L}{\partial \dot{x}^i} \dot{\chi}^i + \frac{\partial L}{\partial \chi^i} \dot{\chi}^i \right) \, dt \right]_{\varepsilon=0}. \quad (5.7)$$

Note that the first term on the rhs can be written

$$\int_{P_1}^{P_2} \frac{1}{\sqrt{2L}} \frac{\partial L}{\partial \dot{x}^i} \dot{\chi}^i \, dt = \int_{P_1}^{P_2} \frac{\partial (\sqrt{2L})}{\partial \dot{x}^i} \dot{\chi}^i \, dt. \quad (5.8)$$

This can be integrated by parts

$$\int_{P_1}^{P_2} \frac{\partial (\sqrt{2L})}{\partial \dot{x}^i} \dot{\chi}^i \, dt = \left[ \frac{\partial (\sqrt{2L})}{\partial \dot{x}^i} \dot{\chi}^i \right]_{P_1}^{P_2} - \int_{P_1}^{P_2} \frac{d}{dt} \left( \frac{\partial (\sqrt{2L})}{\partial x^i} \right) \chi^i \, dt. \quad (5.9)$$

The first term on the second member is zero because $\chi^i$ vanishes for the end points; replacing in Eq. \((5.7)\)

$$\left. \frac{dt(\varepsilon)}{d\varepsilon} \right|_{\varepsilon=0} = \frac{1}{\sqrt{2}} \int_{P_1}^{P_2} \left[ \frac{d}{dt} \left( -\frac{1}{\sqrt{L}} \frac{\partial L}{\partial \dot{x}^i} \right) + \frac{1}{\sqrt{L}} \frac{\partial L}{\partial x^i} \right] \dot{\chi}^i \, dt. \quad (5.10)$$

The rhs must be zero for arbitrary distortion functions $\chi^i$, so we conclude that the following set of four simultaneous equations must be verified

$$\frac{d}{dt} \left( \frac{1}{\sqrt{L}} \frac{\partial L}{\partial \dot{x}^i} \right) = \frac{1}{\sqrt{L}} \frac{\partial L}{\partial x^i}; \quad (5.11)$$

these are called the Euler-Lagrange equations.

Consideration of Eqs. \((4.3)\) and \((4.11)\) allows us to conclude that the Lagrangian defined by \((5.3)\) can also be written as $L = v^2/2$ and must always equal 1/2. From the Lagrangian one defines immediately the conjugate momenta

$$v_i = \frac{\partial L}{\partial \dot{x}^i} = -g_{ij} \dot{x}^j. \quad (5.12)$$
Notice the use of the lower index \((v_i)\) to represent momenta while velocity components have an upper index \((v^i)\). The conjugate momenta are the components of the conjugate momentum vector

\[
v = \frac{g^iv_i}{\sqrt{-g_{00}}} \tag{5.13}
\]

and from Eq. (2.18)

\[
\sqrt{-g_{00}}v = g^iv_i = g^i g_{ij} \dot{x}^j = g^j \dot{x}^j. \tag{5.14}
\]

The conjugate momentum and velocity are the same but their components are referred to the reciprocal and refractive index frames, respectively.\(^1\) Notice also that by virtue of Eq. (3.4) it is also

\[
v_i = \frac{p_i}{p_0}. \tag{5.15}
\]

The Euler-Lagrange equations \((5.11)\) can now be given a simpler form

\[
\dot{v}_i = \partial_i L. \tag{5.16}
\]

This set of four equations defines trajectories of minimum time in 4DO space as long as the frame vectors \(g_\alpha\) are known everywhere, independently of the fact that they may or may not be referred to the orthonormed frame via a refractive index. By definition these trajectories are the geodesics of 4DO space, spanned by frame vectors \(g_i/\sqrt{-g_{00}}\), with metric tensor \(-g_{ij}/g_{00}\).

Following an exactly similar procedure we can find trajectories which extremize proper time, defined by taking the positive square root of Eq. (4.11). The Lagrangian is now defined by

\[
\mathcal{L} = -\frac{1}{2} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu. \tag{5.17}
\]

Consequently the conjugate momenta are

\[
v_\mu = \frac{\partial \mathcal{L}}{\partial \ddot{x}^\mu} = -\frac{g_{\mu\nu} \ddot{x}^\nu}{g_{44}}. \tag{5.18}
\]

From Eq. \((3.4)\) we have \(v_\mu = p_\mu/p_4\); the associated Euler-Lagrange equations are

\[
\dot{v}_\mu = \partial_\mu \mathcal{L}. \tag{5.19}
\]

"These are, by definition, spacelike geodesics of GTR with metric tensor \(-g_{\mu\nu}/g_{44}\) and we have thus defined a method for one-to-one geodesic mapping between 4DO and spacelike GTR. Recalling the conditions for this mapping to be valid, all the frame vectors must be independent of both \(t\) and \(\tau\) and \(g_0\) and \(g_4\) must be normal to the other 3 frame vectors. In tensor terms, all the \(g_{\alpha\beta}\) must be independent from \(t\) and \(\tau\) and \(g_{0i} = g_{\mu4} = 0.\)"

\(^1\)In most cases \(g_{00} = -1\), the velocity can be conveniently written \(v = g_i \dot{x}^i\) and conjugate momenta \(v_i = g_{ij} \dot{x}^j\).
6 The sources of refractive index

The set of 4 equations (5.16) defines the geodesics of 4DO space; particularly in cases where there is a refractive index, it defines trajectories of minimum time but does not tell us anything about what produces the refractive index in the first place. Similarly the set of equations (5.19) defines the geodesics of GTR space without telling us what shapes space. In order to analyse this question we must return to the general case of a refractive frame \( g_\alpha \) without other impositions besides the existence of a refractive index.

Considering the momentum vector

\[
p = p_\alpha s^\alpha = p_\alpha n_\beta^\alpha \sigma^\beta, \quad (6.1)
\]

with \( n_\alpha \gamma n^\beta_\gamma = \delta^\beta_\alpha \), we will now take its time derivative. Using Eq. (B.4)

\[
\dot{p} = \dot{x} \cdot (Dp) = \dot{x} \cdot G. \quad (6.2)
\]

By a suitable choice of coordinates we can always have \( g^0 = \sigma^0 \). We can then invoke the fact that for an elementary particle in flat space the momentum vector components can be associated with the concepts of energy, 3D momentum and rest mass as \( p = E\sigma^0 + \mathbf{p} + m\sigma^4 \) (see Sec. 3.) If this consequence is extended to curved space and to mass distributions, we write \( p = E\sigma^0 + \mathbf{p} + mg^4 \), where now \( E \) is energy density, \( \mathbf{p} = pm^4 \mathbf{g} \) is 3D momentum density and \( m \) is mass density. The previous equation then becomes

\[
\dot{E}\sigma^0 + \dot{\mathbf{p}} + mg^4 = \dot{x} \cdot G. \quad (6.3)
\]

When the Laplacian is applied to the momentum vector the result is still necessarily a vector

\[
D^2 p = S. \quad (6.4)
\]

Vector \( S \) is called the sources vector and can be expanded into 25 terms as

\[
S = (D^2 n^\beta_\alpha)\sigma_\beta p^\alpha = S^\beta_\alpha \sigma_\beta p^\alpha; \quad (6.5)
\]

where \( p^\alpha = g^{\alpha\beta} p_\beta \). Tensor \( S^\alpha_\beta \) contains the coefficients of the sources vector and we call it the sources tensor. The sources tensor influences the shape of geodesics as we shall see in one particularly important situation. One important consequence that we don’t pursue here is that by zeroing the sources vector one obtains the wave equation \( D^2 p = 0 \), which accepts gravitational wave solutions.

If \( \sigma^0 \) is normal to the other frame vectors we can write \( p = E(\sigma^0 + \mathbf{v}) \) in the reciprocal frame, with \( \mathbf{v} \) a unit vector or \( p = E(-\sigma^0 + \mathbf{v}) \) in the direct frame. Equation (6.2) can then be given the form

\[
\dot{E}(\sigma^0 + \mathbf{v}) + E\dot{\mathbf{v}} = \sigma_0 + \mathbf{v} \cdot G. \quad (6.6)
\]

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Since \( G \) can have scalar and bivector components, the scalar part must be responsible for the energy change, while the bivector part rotates the velocity \( v \). The bivector part of \( G \) is generated by \( D \wedge p \), which allows a simplification of the previous equation to

\[
\dot{v} = v \cdot (D \wedge v),
\]  
(6.7)

if the frame vectors are independent of \( t \). This equation is exactly equivalent to the set of Euler-Lagrange equations (5.16) but it was derived in a way which tells us when to expect geodesic movement or free fall.

We will now investigate spherically symmetric solutions in isotropic conditions defined by Eq. (4.16); this means that the refractive index can be expressed as functions of \( r \). The vector derivative in spherical coordinates is of course

\[
D = \frac{1}{n_r} \left( \sigma_r \partial_r + \frac{1}{r} \sigma_\theta \partial_\theta + \frac{1}{r \sin \theta} \sigma_\phi \partial_\phi \right) - \sigma_t \partial_t + \frac{1}{n_4} \sigma_\tau \partial_\tau.
\]  
(6.8)

The Laplacian is the inner product of \( D \) with itself but the frame vectors’ derivatives must be considered; all the derivatives with respect to \( t, r \) and \( \tau \) are zero and the non-zero ones are

\[
\begin{align*}
\partial_\theta \sigma_r &= \sigma_\theta, & \partial_\phi \sigma_r &= \sin \theta \sigma_\phi, \\
\partial_\theta \sigma_\theta &= -\sigma_r, & \partial_\phi \sigma_\theta &= \cos \theta \sigma_\phi, \\
\partial_\theta \sigma_\phi &= 0, & \partial_\phi \sigma_\phi &= -\sin \theta \sigma_r - \cos \theta \sigma_\theta.
\end{align*}
\]  
(6.9)

After evaluation the curved Laplacian becomes

\[
D^2 = \frac{1}{(n_r)^2} \left( \partial_{rr} + \frac{2}{r} \partial_r - \frac{n_r'}{n_r} \partial_r + \frac{1}{r^2} \partial_{\theta \theta} + \right.
\]

\[\left. + \frac{\cot \theta}{r^2} \partial_\theta + \frac{\csc^2 \theta}{r^2} \partial_{\phi \phi} \right) - \partial_{tt} + \frac{1}{(n_4)^2} \partial_{\tau \tau}.
\]  
(6.10)

The search for solutions of Eq. (6.4) must necessarily start with vanishing second member, a zero sources situation, which one would implicitly assign to vacuum; this is a wrong assumption as we will show. Zeroing the second member implies that the Laplacian of both \( n_r \) and \( n_4 \) must be zero; considering that they are functions of \( r \) we get the following equation for \( n_r \)

\[
n_r'' + 2n_r' \frac{r}{n_r} - \left( \frac{n_r'}{n_r} \right)^2 = 0,
\]  
(6.11)

with general solution \( n_r = b \exp(a/r) \). It is legitimate to make \( b = 1 \) because the refractive index must be unity at infinity. Using this solution in Eq. (6.10) the Laplacian becomes

\[
D^2 = e^{-a/r} \left( \partial_{rr} + \frac{2}{r} \partial_r + \frac{a}{r^2} \partial_r + \frac{1}{r^2} \partial_{\theta \theta} + \right.
\]

\[\left. + \frac{\cot \theta}{r^2} \partial_\theta + \frac{\csc^2 \theta}{r^2} \partial_{\phi \phi} \right) - \partial_{tt} + \frac{1}{(n_4)^2} \partial_{\tau \tau};
\]  
(6.12)
which produces the solution \( n_4 = n_r \). So space must be truly isotropic and not relaxed isotropic as we had allowed. The solution we have found for the refractive index components in isotropic space can correctly model Newton dynamics, which led the author to adhere to it for some time [17]. However if inserted into Eq. (4.11) this solution produces a GTR metric which is verifiably in disagreement with observations; consequently it has purely geometric significance.

The inadequacy of the isotropic solution found above for relativistic predictions deserves some thought, so that we can search for solutions guided by the results that are expected to have physical significance. In the physical world we are never in a situation of zero sources because the shape of space or the existence of a refractive index must always be tested with a test particle. A test particle is an abstraction corresponding to a point mass considered so small as to have no influence on the shape of space; in reality a point particle is a black hole in GTR, although this fact is always overlooked; one wonders how a black hole is postulated not to influence space geometry. A test particle must be seen as source of refractive index itself and its influence on the shape of space should not be neglected in any circumstances. If this is the case the solutions for vanishing sources vector may have only geometric meaning, with no connection to physical reality.

The question is then what should go into the second member of Eq. (6.4) in order to find physically meaningful solutions. If we are testing gravity we must assume some mass density to suffer gravitational influence; this is what is usually designated as non-interacting dust, meaning that some continuous distribution of non-interacting particles follows the geodesics of space. Mass density is expected to be associated with \( S^4_4 \); on the other hand we are assuming that this mass density is very small and so we use flat space Laplacian to evaluate it. We consequently make an \textit{ad hoc} proposal for the sources vector in the second member of Eq. (6.4)

\[ S = -\nabla^2 n_4 \sigma_4. \]  

Equation (6.4) becomes

\[ D^2 \dot{x} = -\nabla^2 n_4 \sigma_4; \]  

as a result the equation for \( n_r \) remains unchanged but the equation for \( n_4 \) becomes

\[ n_4'' + \frac{2n_4'}{r} - \frac{n_4' n_4'}{n_r} = -n_4'' + \frac{2n_4'}{r}. \]  

When \( n_r \) is given the exponential form found above, the solution is \( n_4 = \sqrt{n_r} \). This can now be entered into Eq. (4.11) and the coefficients can be expanded in series and compared to Schwarzschild’s for the determination of parameter \( a \). The final solution, for a stationary mass \( M \) is

\[ n_r = e^{2M/r}, \quad n_4 = e^{M/r}. \]  

(6.16)
The equivalent GTR space is characterized by the quadratic form
\[ d\tau^2 = e^{-2M/r}dt^2 - e^{2M/r} \sum_m (dx^m)^2. \] (6.17)

Expanding in series of \( M/r \) the coefficients of this metric one would find that the lower order terms are exactly the same as for Schwarzschild’s and so the predictions of the metrics are indistinguishable for small values of the expansion variable. Montanus \[19\] arrives at the same solutions with a different reasoning; Yilmaz was probably the first author to propose this metric \[20, 21, 22\].

Equation (6.14) can be interpreted in physical terms as containing the essence of gravitation. When solved for spherically symmetric solutions, as we have done, the first member provides the definition of a stationary gravitational mass as the factor \( M \) appearing in the exponent and the second member defines inertial mass as \( \nabla^2 n_4 \). Gravitational mass is defined with recourse to some particle which undergoes gravitational influence and is animated with velocity \( v \) and inertial mass cannot be defined without some field \( n_4 \) acting upon it. Complete investigation of the sources tensor elements and their relation to physical quantities is not yet done; it is believed that 16 terms of this tensor have strong links with homologous elements of stress tensor in GTR, while the others are related to electromagnetic field.

7 Electromagnetism in 5D spacetime

Maxwell’s equations can easily be written in the form of Eq. (6.4) if we don’t impose the condition that \( g_4 \) should remain normal the other frame vectors; as we have seen in section 5 this has the consequence that there will be no GTR equivalent to the equations formulated in 4DO.

We will consider the non-orthonormed reciprocal frame defined by
\[ g^\mu = \sigma^\mu, \quad g^4 = \frac{q}{m} A_\mu \sigma^\mu + \sigma^4; \] (7.1)
where \( q \) and \( m \) are charge and mass densities, respectively, and \( A = A_\mu \sigma^\mu \) is the electromagnetic vector potential, assumed to be a function of coordinates \( t \) and \( x^m \) but independent of \( \tau \). The associated direct frame has vectors
\[ g_\mu = \sigma_\mu - \frac{q}{m} A_\mu \sigma_4, \quad g_4 = \sigma_4; \] (7.2)
and one can easily verify that Eq. (2.18) is obeyed. The momentum vector in the reciprocal frame is \( p = E \sigma^0 + p_m \sigma^m + qA_\mu \sigma^\mu + m\sigma^4 \) and \( G \) in the second member of Eq. (6.2) is \( G = qDA \). We will assume \( D \cdot A \) to be zero, as one usually does in electromagnetism; also \( D \) can be replaced by \( \mu \nabla \) because the vector potential does not depend on \( \tau \). It is convenient to define the Faraday
bivector $F = \mu \nabla A$, similarly to what is done in Ref. [8]; the dynamics equation then becomes
\[
\dot{p} + q \dot{A} = q \dot{x} \cdot F; \tag{7.3}
\]
and rearranging
\[
\dot{p} = q \dot{x} \cdot F - q \dot{A}. \tag{7.4}
\]
The first term in the second member is the Lorentz force and the second term is due to the radiation of an accelerated charge.

Recalling the wave displacement vector Eq. (B.1) we have now
\[
dx = \sigma_\alpha dx^\alpha - \frac{q}{m} A_\mu \sigma_4 dx^\mu. \tag{7.5}
\]
This corresponds to a refractive index tensor whose non-zero terms are
\[
n^{\alpha \alpha} = 1, \quad n^4_\mu = -\frac{q}{m} A_\mu. \tag{7.6}
\]
According to Eq. (6.5) the sources tensor has all terms null except for the following
\[
S^4_\mu = -\frac{q}{m} D^2 A_\mu; \tag{7.7}
\]
where $D$ is the covariant derivative given by
\[
D = g^{\alpha \beta} \partial_\alpha = \sigma^\mu \partial_\mu + (\sigma^4 + \frac{q}{m} A_\mu \sigma^\mu) \partial_4. \tag{7.8}
\]
We can then define the current vector $J$ verifying
\[
\mu \nabla^2 A = \mu \nabla F = J, \tag{7.9}
\]
where
\[
J = -\frac{m}{q} S^4_\mu \sigma^\mu. \tag{7.10}
\]
Please refer to [8, Chap. 7] or to [7, Part 2] to see how these equations generate classical electromagnetism.

In free space we make $J = 0$ and Eq. (7.9) accepts plane wave solutions for $F$ which are of course electromagnetic waves. Notice that these solutions propagate in directions normal to proper time, which is perfectly consistent with the classical relativistic formulation.

The Dirac equation for a free particle has been derived from the 5-dimensional monogenic condition in Sec. 3 but we are now in position to include the effects of an EM field. Because we are working in geometric algebra, our quantum mechanics equations will inherit that character but the isomorphism between the geometric algebra of 5D spacetime, $G_{4,1}$, and complex algebra of $4 \times 4$ matrices, $M(4, \mathbb{C})$, ensures that they can be translated into the more usual Dirac matrix
formalism. Electrodynamics can now be implemented in the same way used in Sec. 7 to implement classical electromagnetism. The monogenic condition must now be established with the covariant derivative given by Eq. (7.8)

$$\sigma^\mu \partial_\mu \psi + \left( \sigma^4 + \frac{q}{m} A^\mu \sigma^\mu \right) \partial_4 \psi = 0. \quad (7.11)$$

Multiplying on the left by $\sigma^4$ and taking $\partial_4 \psi = \imath m \psi$

$$\left[ \gamma^\mu (\partial_\mu + \imath q A_\mu) + \imath m \right] \psi = 0. \quad (7.12)$$

This equation can be compared to what is found in any quantum mechanics textbook.

It is now adequate to say a few words about quantization, which is inherent to 5D monogenic functions. We have already seen that these functions are 4-dimensional waves, that is, they have 3-dimensional wavefronts normal to the direction of propagation. Whenever the refractive index distribution traps one of these waves a 4-dimensional waveguide is produced, which has its own allowed propagating modes. In the particular case of a central potential, be it an atom’s or a galaxy’s nucleus, we expect spherical harmonic modes, which produce the well known electron orbitals in the atom and have unknown manifestations in a galaxy.

### 8 Hyperspherical coordinates

Deriving physical equations and predictions from purely geometrical equations is an exercise whose success depends on the correct assignment of coordinates to physical entities; the same space will produce different predictions if different options are taken for coordinate assignment. In the previous sections we assumed that empty space could be modelled by an assignment of time, three spatial directions and proper time to five orthogonal directions in 5D spacetime. We are now going to experiment with a different assignment of flat space coordinates, which will explore the possibility that physics and the Universe have an inbuilt hyperspherical symmetry. The exercise consists on assigning coordinate $x^4 = \tau$ to the radius of an hypersphere and the three $x^m$ coordinates to distances measured on the hypersphere surface; time, $x^0$, will still be measured along a direction normal to all others. If the hypersphere radius is very large we will not be able to notice the curvature on everyday phenomena, in the same way as everyday displacements on Earth don’t seem curved to us. The Universe as a whole will manifest the consequences of its hyperspherical symmetry; using the Earth as a 3-dimensional analogue of an hyperspherical Universe, although our everyday life is greatly unaffected by Earth’s curvature the atmosphere senses this curvature and shows manifestations of it in winds and climate. What we
propose here is an exercise consisting of an arbitrary assignment between coordinates and physical entities; the validity of such exercise can only be judged by the predictions it allows and how well they conform with observations.

Hyperspherical coordinates are characterized by one distance coordinate, $\tau$ and three angles $\rho, \theta, \phi$; following the usual procedure we will associate with these coordinates the frame vectors $\{\sigma_\tau, \sigma_\rho, \sigma_\theta, \sigma_\phi\}$. The position vector for one point in 5D space is quite simply

$$x = t\sigma_t + \tau\sigma_\tau. \quad (8.1)$$

In order to write an elementary displacement $dx$ we must consider the rotation of frame vectors, but we don’t need to think hard about it because we can extend what is known from ordinary spherical coordinates.

$$dx = \sigma_0 dt + \sigma_4 d\tau + \tau\sigma_\rho d\rho + \tau\sin\rho\sigma_\theta d\theta + \tau\sin\rho\sin\theta\sigma_\phi d\phi. \quad (8.2)$$

Just as before, we consider only null displacements to obtain time intervals;

$$dt^2 = d\tau^2 + \tau^2 [d\rho^2 + \sin^2\rho (d\theta^2 + \sin^2\theta d\phi^2)]. \quad (8.3)$$

The velocity vector, $v = \dot{x} - \sigma_0$, can be immediately obtained from the displacement vector dividing by $dt$

$$v = \sigma_0 \dot{t} + \tau\sigma_\rho \dot{\rho} + \tau\sin\rho\sigma_\theta \dot{\theta} + \tau\sin\rho\sin\theta\sigma_\phi \dot{\phi}. \quad (8.4)$$

Geodesics of flat space are naturally straight lines, no matter which coordinate system we use, however it is useful to derive geodesic equations from a Lagrangian of the form (5.3); in hyperspherical coordinates the Lagrangian becomes

$$2L = v^2 = \dot{\tau}^2 + \tau^2 [\dot{\rho}^2 + \sin^2\rho (\dot{\theta}^2 + \sin^2\theta \dot{\phi}^2)]. \quad (8.5)$$

Because de Lagrangian is independent of $\phi$ we can establish a conserved quantity

$$J_\phi = \tau^2 \sin^2\rho \sin^2\theta \phi. \quad (8.6)$$

It may seem strange that any physically meaningful relation can be derived from the simple coordinate assignment that we have made, that is, proper time is associated with hypersphere radius and the three usual space coordinates are assigned to distances on the hypersphere radius. This unexpected fact results from the possibility offered by hyperspherical coordinates to explore a symmetry in the Universe that becomes hidden when we use Cartesian coordinates. In the real world we measure distances between objects, namely cosmological objects, rather than angles; we have therefore to define a distance coordinate, which is obviously $r = \tau\rho$. It does not matter where in the Universe we place the origin for $r$ and we find it convenient to place ourselves on the origin.
Radial velocities \( \dot{r} \) measure movement in a radial direction from our observation point; we are particularly interested in this type of movement in order to find a link to the Hubble relation. Applying the chain rule and then replacing \( \rho \)

\[
\dot{r} = \rho \dot{\tau} + \rho \dot{\tau} = \frac{\dot{\tau}}{\tau} r + \rho \dot{\tau}.
\]  

(8.7)

We expect objects that have not suffered any interaction to move along \( \sigma \tau \); from (8.4) we see that this implies \( \dot{\rho} = \dot{\theta} = \dot{\phi} = 0 \) and then \( \dot{\tau} \) becomes unity. Replacing in the equation above and rearranging

\[
\dot{r} = \frac{1}{r} \tau.
\]

(8.8)

What this equation tells us is exactly what is expressed by the Hubble relation. The value of \( \tau \) can be taken as constant for any given observation because the distance information is carried by photons and these preserve proper time, as we have seen in our discussion about electromagnetic waves.\(^1\) The first member of the equation is the definition of the Hubble parameter and we can then write \( H = 1/\tau \). In this way we find the physical meaning of coordinate \( \tau \) as being the Universe’s age.

Underlying the present discussion there is an assumption a preferred frame where stillness means moving along \( \sigma \tau \); there is no question of equivalent inertial frames here. This preferred frame is obviously attached to the observable still objects in the Universe which are galaxy clusters, as much as we can tell. This is far from the orthodox point of view, because galaxy clusters are seen as moving relative to each other and so cannot possible define a fixed frame. But in our formulation still objects move in straight lines along the proper time direction and keep their angular separations constant; this is naturally perceived as increasing mutual distances. If there is any relation between our formulation and an ether it must be found in the fact that movement has an absolute meaning, so it is defined relative to something that is fixed; we call the fixed reference a preferred frame while other authors call it ether.

How does the use of hyperspherical coordinates affect dynamics in our laboratory experiments? We would like to know if these coordinates need only be considered in problems of cosmological scale or, on the contrary, there are implications for everyday experiments. The answer implies rewriting (8.2) with distance rather than angle coordinates; replacing \( \rho \),

\[
dx = \sigma_0 dt + \left( \sigma_4 - \frac{r}{\tau} \sigma_\rho \right) d\tau + \sigma_\rho dr + r(\sigma_\theta d\theta + \sin \theta \sigma_\phi d\phi).
\]

(8.9)

\(^1\)In order to preserve proper time photons must travel on the hypersphere surface and thus don’t follow geodesics.
Evaluating time intervals from the null displacement condition, as before

\[
\text{dr}^2 = \left[1 + \left(\frac{\tau}{\tau}\right)^2\right] \text{d}\tau^2 - 2\frac{\tau}{\tau} \text{d}\tau \text{dr} + \text{dr}^2 + r^2 (\text{d}\theta^2 + \sin^2 \theta \text{d}\phi^2). \tag{8.10}
\]

This would be a version of (3.8) in spherical coordinates, were it not for the extra terms with powers of \(r/\tau\) in the second member. The coefficient \(r/\tau\) implies a comparison between the distance from the object to the observer and the size of the Universe; remember that \(\tau\) is both time and distance in non-dimensional units. We can say that ordinary special relativity will apply for objects which are near us, but distant objects will show in their movement an effect of the Universe’s hyperspherical nature.

With Eqs. (6.16) we have established the refractive indices \(n_r\) and \(n_4\) to account for the dynamics near a massive sphere using Cartesian coordinates; since this is frequently applied on a cosmological scale, we must find out how the dynamics is modified by the use of hyperspherical coordinates. Using the refractive indices and hyperspherical coordinates, noting that \(n_r = n_4^2\), Eq. (4.7) becomes

\[
\text{dr}^2 = n_4^2 \text{d}\tau^2 + n_4^4 \text{r}^2 \text{d}\rho^2. \tag{8.11}
\]

Dividing both members by \(\text{dr}^2\) and reversing the equation

\[
n_4^2 \text{r}^2 + n_4^4 \text{r}^2 \text{d}\rho^2 = 1; \tag{8.12}
\]

and replacing \(\tau \rho\) by \(\dot{r} - r \dot{\tau}/\tau\)

\[
n_4^2 \dot{r}^2 + n_4^4 \left[ \dot{r}^2 + \left(\frac{\tau}{\tau}\right)^2 r^2 - 2 \tau \dot{r} \frac{r}{\tau} \right] = 1. \tag{8.13}
\]

Dividing both members by \(n_4^4 r^2\) and rearranging results in the equation

\[
\left(\frac{\dot{r}}{r}\right)^2 = \left(\frac{1}{n_4^2} - \frac{\dot{\tau}^2}{n_4^2} \right) \frac{1}{r^2} - \left(\frac{\dot{\tau}}{\tau}\right)^2 + 2 \frac{\dot{\tau} \dot{r}}{\tau r}. \tag{8.14}
\]

As a further step we take the refractive index coefficients from Schwarzschild’s metric (4.21) or those of from the exponential metric (6.16) and expand the second member in series of \(M/r\) taking only the two first terms.

\[
\left(\frac{\dot{r}}{r}\right)^2 \approx 1 - \frac{\dot{\tau}^2}{r^2} + \frac{(2 \dot{\tau}^2 - 4) M}{r^3} - \left(\frac{\dot{\tau}}{\tau}\right)^2 + 2 \frac{\dot{\tau} \dot{r}}{\tau r}. \tag{8.15}
\]

The previous equation applies to bodies moving radially under the influence of mass \(M\) located at the origin which is, remember, the observer’s position. For
comparison we derive the corresponding equation in Cartesian coordinates; starting with (8.12) it is now
\[ n_4^2 \dot{t}^2 + n_4^4 \dot{r}^2 = 1; \quad (8.16) \]
dividing by \( n_4^4 r^2 \) and rearranging
\[
\left( \frac{\dot{r}}{r} \right)^2 = \left( \frac{1}{n_4^4} - \frac{\dot{\tau}^2}{n_4^4} \right) \frac{1}{r^2} \approx \frac{1 - \dot{\tau}^2}{r^2} + \frac{(2 \dot{\tau}^2 - 4)M}{r^3}. \quad (8.17)
\]

If we want to apply these equations to cosmology it is easiest to follow the approach of Newtonian cosmology, which produces basically the same results as the relativistic approach but presumes that the observer is at the centre of the Universe [18, 23]. In order to adopt a relativistic approach we need equations that replace Einstein’s in 4DO. A set of such was proposed above Eq. (6.4) but their application in cosmology has not yet been tested, so we will have to defer this more correct approach to future work. The strategy we will follow here is to consider a general object at distance \( r \) from the observer, moving away from the latter under the gravitational influence of the mass included in a sphere of radius \( r \). If we designate by \( \mu \) the average mass density in the Universe, then mass \( M \) in (8.15) is \( 4 \pi \mu r^3 / 3 \); this will have to be considered further down.

Friedman equation governs standard cosmology and can be derived both from Newtonian and relativistic dynamics, with different consequences in terms of the overall size of the Universe and the observer’s privileged position. From the cited references we write Friedman equation as
\[
\left( \frac{\dot{r}}{r} \right)^2 = \frac{8 \pi}{3} \mu + \frac{\Lambda}{3} - \frac{k}{r^2}; \quad (8.18)
\]
with \( \Lambda \) a cosmological constant and \( k \) the curvature constant; the gravitational constant was not included because it is unity in non-dimensional units and the equation is written in real, not comoving, coordinates. In order to compare (8.15) with Friedman equation there is a problem with the last term because the Hubble parameter \( \dot{r}/r \) does not appear isolated in the first member; we will find a way to circumvent the problem later on but first let us look at what (8.15) tells us when the mass density is zeroed. In this case \( n_4 = 1 \) and we find from (8.12) that \( \dot{\tau} \) is unity, unless \( \dot{\rho} \) is non-zero, for which we can find no reasonable explanation. Replacing \( n_4 \) and \( \dot{\tau} \) with unity in (8.15) we find that \( \dot{r}/r = 1/\tau \), confirming what had already been found in (8.8). Comparing with Friedman equation, this corresponds to a flat Universe with a critical mass density \( \mu = \mu_c \); it is immediately obvious that \( \mu_c = 3 / (8 \pi \tau^2) \). Let us not overlook the importance of this conclusion because it completely removes the need for a critical density if the Universe is flat; remember this is one of the main reasons to invoke dark matter in standard cosmology. Notice also that this conclusion does not depend on a
privileged observer, because it is just a consequence of space symmetry and not of dynamics.

Let us now see what happens when we consider a small mass density; here we are talking about matter that is observed or measured in some way but not postulated matter. The matter density that we will consider is of the order of 1% of the presently accepted value. It is therefore just a perturbation of the flat solution that we described above and the fact that we are presuming a privileged observer has to be taken just for this perturbation. The first thing we note when we consider matter density is that $\dot{\tau} < 1$, because there is now a component of the velocity vector along $\sigma_p$. Ideally we should solve the Euler-Lagrange equations resulting from (8.12) in order to find $\dot{\tau}$ and $\dot{\rho}$ but this is a difficult process and we shall carry on with just a qualitative discussion. Considering that we are discussing a perturbation it is legitimate to make $\dot{r}/r \approx \dot{\tau}/\tau$ and the two last terms in the second member of (8.15) can be combined into one single term $(\dot{\tau}/\tau)^2$, the same as we encountered for the flat solution, albeit with a numerator slightly smaller than unity. The first term has now become slightly positive and we can see from Friedman equation that this corresponds to a negative curvature constant, $k$, and to an open Universe. Lastly the second term includes the mass $M$ of a sphere with radius $r$ and can be simplified to $8\pi\mu(\dot{\tau}^2 - 2)/3$; this has the effect of a negative cosmological constant; the combined effect of the two terms is expected to close the Universe [23, 24]. The previous discussion was done in qualitative terms, making use of several approximations, for which reason we must question some of the findings and expect that after more detailed examination they may not be quite as anticipated; in particular there is concern about the refractive indices used, which were derived in Cartesian coordinates both by the author and those that preceded him in using an exponential metric; it may happen that the transposition to hyperspherical coordinates has not been properly made, with consequences in the perturbative analysis that was superimposed on the flat solution. The latter, however, is totally independent of such concerns and allows us to state that the assumption of hyperspherical symmetry for the Universe dispenses with dark matter in accounting for the gross of observed expansion.

Dark matter is also called in cosmology to account for the extremely high rotation velocities found in spiral galaxies [25, 26] and we will now take a brief look at how hyperspherical symmetry can help explain this phenomenon. Galaxy dynamics is an extremely complex subject, which we do not intend to explore here due to lack of space but most of all due to lack of author’s competence to approach it with any rigour; we will just have a very brief outlook at the equation for flat orbits, to notice that an effect similar to the familiar Coriollis effect on Earth can arise in an expanding hyperspherical Universe and this could explain most of the observed velocities on the periphery of galaxies. Let us recall (8.9),
divide by $dr$ and invoke null displacement to obtain the velocity

$$
v = \left( \sigma_4 - \frac{r}{\tau} \sigma_p \right) \dot{\tau} + \sigma_p \dot{r} + r (\sigma_\theta \dot{\theta} + \sin \theta \sigma_\phi \dot{\phi}).
$$

(8.19)

If orbits are flat we can make $\theta = \pi/2$ and the equation simplifies to

$$
v = \tau \sigma_4 + \left( \dot{r} - \frac{r \dot{\tau}}{\tau} \right) \sigma_p + r \phi \sigma_\phi.
$$

(8.20)

Suppose now that something in the galaxy is pushing outwards slightly, so that the parenthesis is zero; this happens if $\dot{r}/r = \dot{\tau}/\tau$ and can be caused by a pressure gradient, for instance. The result is that (8.20) now accepts solutions with constant $r \phi$, which is exactly what is observed in many cases; swirls will be maintained by a radial expansion rate which exactly matches the quotient $\dot{\tau}/\tau$. In any practical situation $\dot{\tau}$ will be very near unity and the quotient will be virtually equal to the Hubble parameter; thus the expansion rate for sustained rotation is $\dot{r}/r \approx H$. If applied to our neighbour galaxy Andromeda, with a radial extent of 30 kpc, using the Hubble parameter value of 81 km s$^{-1}$/Mpc, the expansion velocity is about 2.43 km s$^{-1}$; this is to be compared with the orbital velocity of near 300 km s$^{-1}$ and probably within the error margins. An expansion of this sort could be present in many galaxies and go undetected because it needs only be of the order of 1% the orbital velocity.

### 9 Symmetries of $G_{4,1}$ algebra

In this algebra it is possible to find a maximum of four mutually annihilating idempotents, which generate with 0 an additive group of order 16; for a demonstration see Lounesto [14, section 17.5]. Those idempotents can be generated by a choice of two commuting basis elements which square to unity; for the moment we will use $\sigma_{023}$ and $\sigma_{014}$. The set of 4 idempotents is then given by

$$
f_1 = \frac{(1 + \sigma_{023})(1 + \sigma_{014})}{4}, \quad f_2 = \frac{(1 + \sigma_{023})(1 - \sigma_{014})}{4},
$$

$$
f_3 = \frac{(1 - \sigma_{023})(1 - \sigma_{014})}{4}, \quad f_4 = \frac{(1 - \sigma_{023})(1 + \sigma_{014})}{4}.
$$

(9.1)

Using the matrices of Sec. 3 to make matrix replacements of $\sigma_{023}$ and $\sigma_{014}$ one can find matrix equivalents to these idempotents; those are matrices which have only one non-zero element, located on the diagonal and with unit value. $SU(3)$ symmetry can now be demonstrated by construction of the 8 generators...
\[ \lambda_1 = \sigma_{02}(f_1 + f_2) = \frac{\sigma_3 + \sigma_{02}}{2}, \]
\[ \lambda_2 = \sigma_{03}(f_1 + f_2) = \frac{-\sigma_2 + \sigma_{03}}{2}, \]
\[ \lambda_3 = f_1 - f_2 = \frac{\sigma_{014} - \sigma_{1234}}{2}, \]
\[ \lambda_4 = -\sigma_1(f_2 + f_3) = \frac{-\sigma_1 - \sigma_{04}}{2}, \]
\[ \lambda_5 = -\sigma_4(f_2 + f_3) = \frac{-\sigma_4 + \sigma_{01}}{2}, \]
\[ \lambda_6 = \sigma_{012}(f_1 + f_3) = \frac{\sigma_{012} + \sigma_{034}}{2}, \]
\[ \lambda_7 = -\sigma_{024}(f_1 + f_3) = \frac{\sigma_{013} - \sigma_{024}}{2}, \]
\[ \lambda_8 = \frac{f_1 + f_2 - 2f_3}{\sqrt{3}} = \frac{2\sigma_{023} + \sigma_{014} + \sigma_{1234}}{2\sqrt{3}}. \]

These have the following matrix equivalents

\[ \lambda_1 \equiv \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \lambda_2 \equiv \begin{pmatrix} 0 & -j & 0 & 0 \\ j & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \lambda_3 \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \]
\[ \lambda_4 \equiv \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \lambda_5 \equiv \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -j & 0 \\ 0 & j & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \lambda_6 \equiv \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \]
\[ \lambda_7 \equiv \begin{pmatrix} 0 & 0 & -j & 0 \\ 0 & 0 & 0 & 0 \\ 0 & j & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \lambda_8 \equiv \left( \frac{1}{\sqrt{3}} \right) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \]

which reproduce Gell-Mann matrices in the upper-left 3 \times 3 corner \[15, 27, 28\]. Since the algebra is isomorphic to complex 4 \times 4 matrix algebra, one expects to find higher order symmetries; Greiner and Müller \[27\] show how one can add 7 additional generators to those of \(SU(3)\) in order to obtain \(SU(4)\) and the same procedure can be adopted in geometric algebra. We then define the following
additional $SU(4)$ generators

\[
\begin{align*}
\lambda_9 &= \sigma_1 (f_1 + f_4) = \frac{\sigma_1 - \sigma_{04}}{2}, \\
\lambda_{10} &= \sigma_4 (f_1 + f_4) = \frac{\sigma_4 + \sigma_{01}}{2}, \\
\lambda_{11} &= -\sigma_{012} (f_2 + f_4) = -\frac{\sigma_{012} - \sigma_{034}}{2}, \\
\lambda_{12} &= \sigma_{024} (f_2 + f_4) = \frac{\sigma_{013} + \sigma_{024}}{2}, \\
\lambda_{13} &= \sigma_3 (f_3 + f_4) = \frac{\sigma_3 - \sigma_{02}}{2}, \\
\lambda_{14} &= \sigma_2 (f_3 + f_4) = \frac{\sigma_2 + \sigma_{03}}{2}, \\
\lambda_{15} &= \frac{f_1 + f_2 + f_3 - 3f_4}{\sqrt{6}} = \frac{\sigma_{023} - \sigma_{014} - \sigma_{1234}}{\sqrt{6}}.
\end{align*}
\]  

(9.4)

Once again, making the replacements with Eq. (3.9) produces the matrix equivalent generators

\[
\begin{align*}
\lambda_9 &\equiv \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, & \lambda_{10} &\equiv \begin{pmatrix} 0 & 0 & 0 & -j \\ 0 & 0 & 0 & 0 \\ j & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, & \lambda_{11} &\equiv \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \\
\lambda_{12} &\equiv \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -j \\ 0 & 0 & 0 & 0 \\ 0 & j & 0 & 0 \end{pmatrix}, & \lambda_{13} &\equiv \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, & \lambda_{14} &\equiv \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -j \\ 0 & 0 & j & 0 \end{pmatrix}, \\
\lambda_{15} &\equiv \frac{1}{\sqrt{6}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix}.
\end{align*}
\]  

(9.5)

The standard model involves the consideration of two independent $SU(3)$ groups, one for colour and the other one for isospin and strangeness; if generators $\lambda_1$ to $\lambda_8$ apply to one of the $SU(3)$ groups we can produce the generators of the second group by resorting to the basis elements $\sigma_3$ and $\sigma_{04}$. The new set of 4 idempotents is then given by

\[
\begin{align*}
f_1 &= \frac{(1 + \sigma_3) (1 + \sigma_{04})}{4}, & f_2 &= \frac{(1 + \sigma_3) (1 - \sigma_{04})}{4}, \\
f_3 &= \frac{(1 - \sigma_3) (1 - \sigma_{04})}{4}, & f_4 &= \frac{(1 - \sigma_3) (1 + \sigma_{04})}{4}.
\end{align*}
\]  

(9.6)
Again a set of $SU(3)$ generators can be constructed following a procedure similar to the previous one

\[
\begin{align*}
\alpha_1 &= \sigma_{02}(f_1 + f_2) = \frac{\sigma_{02} + \sigma_{03}}{2}, \\
\alpha_2 &= \sigma_{01}(f_1 + f_2) = \frac{\sigma_{01} + \sigma_{034}}{2}, \\
\alpha_3 &= f_1 - f_2 = \frac{\sigma_{04} - \sigma_{034}}{2}, \\
\alpha_4 &= \sigma_2(f_2 + f_3) = \frac{\sigma_2 + \sigma_{024}}{2}, \\
\alpha_5 &= -\sigma_1(f_2 + f_3) = \frac{-\sigma_1 - \sigma_{014}}{2}, \\
\alpha_6 &= \sigma_4(f_1 + f_3) = \frac{\sigma_{04} - \sigma_{034}}{2}, \\
\alpha_7 &= \sigma_{012}(f_1 + f_3) = \frac{\sigma_{012} + \sigma_{1234}}{2}, \\
\alpha_8 &= \sqrt{\frac{3}{2}}(f_1 + f_2 - 2f_3) = \frac{2\sigma_3 + \sigma_{04} + \sigma_{034}}{2\sqrt{3}}.
\end{align*}
\] (9.7)

This new $SU(3)$ group is necessarily independent from the first one because its matrix representation involves matrices with all non-zero rows/columns, while the group generated by $\lambda_1$ to $\lambda_8$ uses matrices with zero fourth row/column. In the following section we will discuss which of the two groups should be associated with colour.

At the end of Sec. 3 we used one particular idempotent to split the wavefunction into left and right spinors and here we discuss how the different idempotents are related to the symmetries discussed above, suggesting a relation between idempotents and the different elementary particles. We have already established that each set of 4 idempotents is generated by a pair of commuting unitary basis elements. Let any two such basis elements be denoted as $h_1$ and $h_2$; then the product $h_3 = h_1 h_2$ is itself a third commuting basis element. For consistence we choose, as before,

\[ h_1 \equiv \sigma_{023}, \quad h_2 \equiv \sigma_{014}; \] (9.8)

to get

\[ h_3 \equiv \sigma_{1234}, \] (9.9)

which commutes with the other two as can be easily verified. The result of this exercise is the existence of triads of commuting unitary basis elements but no tetrads of such elements. We are led to state that a general unitary element is a linear combination of unity and the three elements of one triad

\[ h = a_0 + a_1 h_1 + a_2 h_2 + a_3 h_3. \] (9.10)
Since $h$ is unitary and the three $h_m$ commute we can write

$$h^2 = [(a_0)^2 + (a_1)^2 + (a_2)^2 + (a_3)^2] + 2(a_0a_1 - a_2a_3)h_1 + 2(a_0a_2 - a_1a_3)h_2 + 2(a_0a_3 - a_1a_2)h_3 = 1 \quad (9.11)$$

The only form this equation can be verified is if the term in square brackets is unity while all the others are zero. We then get a set of four simultaneous equations with a total of sixteen solutions, as follows: 8 solutions with one of the $a_\mu$ equal to $\pm 1$ and all the others zero, 6 solutions with two of the $a_\mu$ equal to $-1/2$ and the other two equal to $1/2$ and 2 solutions with all the $a_\mu$ simultaneously $\pm 1/2$. The $a_\mu$ coefficients play the role of quantum numbers which determine the particular idempotent that goes into Eq. (3.17); these unusual quantum numbers are expressed in terms of the $SU(4)$ generators $\lambda_3$, $\lambda_8$ and $\lambda_{15}$ in Table I in order to highlight the symmetries. We don’t propose here any direct relationship between the various idempotents and the known elementary particles, although the fact that the standard model gauge symmetry group is found as direct consequence of the monogenic condition which itself generates the Dirac equation is rather intriguing.

Table 1: Coefficients for the various unitary elements.

| $a_0$ | $a_1$ | $a_2$ | $a_3$ | $\lambda_3$ | $\lambda_8$ | $\lambda_{15}$ |
|-------|-------|-------|-------|-------------|-------------|-------------|
| 1     | 0     | 0     | 0     | 0           | 0           | 0           |
| 0     | 1     | 0     | 0     | 2/$\sqrt{3}$ | $\sqrt{2}/3$ |             |
| 0     | 0     | 1     | 0     | 1/$\sqrt{3}$ | $-\sqrt{2}/3$ |             |
| 0     | 0     | 0     | 1     | $-1$       | $1/\sqrt{3}$ | $-\sqrt{2}/3$ |
| -1    | 0     | 0     | 0     | 0           | 0           | 0           |
| 0     | -1    | 0     | 0     | $-2/\sqrt{3}$ | $-\sqrt{2}/3$ |             |
| 0     | 0     | -1    | 0     | $-1$       | $-1/\sqrt{3}$ | $\sqrt{2}/3$ |
| 0     | 0     | 0     | -1    | 1           | $-1/\sqrt{3}$ | $\sqrt{2}/3$ |
| $-1/2$ | $-1/2$ | 1/2   | 1/2   | 0           | 0           | $-\sqrt{3}/2$ |
| $-1/2$ | 1/2   | $-1/2$ | 1/2   | $-1$       | $1/\sqrt{3}$ | $1/\sqrt{6}$ |
| $-1/2$ | 1/2   | 1/2   | $-1/2$ | 1           | $1/\sqrt{3}$ | $1/\sqrt{6}$ |
| 1/2   | $-1/2$ | $-1/2$ | 1/2   | $-1$       | $-1/\sqrt{3}$ | $-1/\sqrt{6}$ |
| 1/2   | $-1/2$ | 1/2   | $-1/2$ | 1           | $-1/\sqrt{3}$ | $-1/\sqrt{6}$ |
| 1/2   | 1/2   | $-1/2$ | $-1/2$ | 0           | 0           | $\sqrt{3}/2$ |
| 1/2   | 1/2   | 1/2   | 1/2   | 0           | 2/$\sqrt{3}$ | $-1/\sqrt{6}$ |
| $-1/2$ | $-1/2$ | $-1/2$ | $-1/2$ | 0           | $-2/\sqrt{3}$ | $1/\sqrt{6}$ |
10 Conclusion and future work

Monogenic functions applied in the algebra of 5-dimensional spacetime have been shown to originate laws of fundamental physics in such diverse areas as relativistic dynamics, quantum mechanics and electromagnetism, with possible, still unclear, consequences for cosmology and particle physics. To say that those functions provide us with a theory of everything is certainly unwarranted at this stage but it is clear that there is a case for much greater effort being invested in their study.

There are unanswered questions in the present work. For instance, how can we avoid an ad hoc definition of inertial mass or what is the true relation between the symmetries generated by monogenic functions and elementary particles? In spite of its various loose ends, the formalism is perfectly capable of unifying relativistic dynamics, quantum mechanics and electromagnetism, which in itself is no small achievement. Certain developments seem relatively straightforward but they must be made, even if no knew predictions are expected. Applying monogenic functions to the Hydrogen atom should not be difficult because the form of the Dirac equation we arrived at is perfectly equivalent to the standard one; one should then find the same solutions but in a GA formalism. In the same line one could try to solve the equation for a central gravitational potential, being certain to find quantum states. It is not clear how important these could be in planetary mechanics or galaxy dynamics.

Gravitational waves are predicted by the monogenic function formalism as we pointed out but did not investigate. How important are they and what chance is there of them being detected by experiment? We don’t know the answer and we don’t know what difficulties lie on the path of those who try to solve the equations; this is an open area. The sources tensor must be clearly understood and directly related to geometry; at the moment all densities, mass, electromagnetic energy, etc. must be inserted in the equations but one would expect that a perfect theory would produce such densities out of nothing. In previous papers we suggested that a recursive, non-linear, equation could be the answer to the problem but the concept has not yet been formalized and there are no clear ideas for achieving such goal.

In conclusion, the present work opens the gate of a path that will possibly lead to an entirely new formulation and understanding of physics but this path is very likely to have many hurdles to jump and several dead ends to avoid.

A Indexing conventions

In this section we establish the indexing conventions used in the paper. We deal with 5-dimensional space but we are also interested in two of its 4-dimensional
subspaces and one 3-dimensional subspace; ideally our choice of indices should clearly identify their ranges in order to avoid the need to specify the latter in every equation. The diagram in Fig. 1 shows the index naming convention used in this paper; Einstein’s summation convention will be adopted as well as the

\[ \{0, 1, 2, 3, 4\} \]

Figure 1: Indices in the range \{0, 4\} will be denoted with Greek letters \( \alpha, \beta, \gamma \). Indices in the range \{0, 3\} will also receive Greek letters but chosen from \( \mu, \nu, \xi \). For indices in the range \{1, 4\} we will use Latin letters \( i, j, k \) and finally for indices in the range \{1, 3\} we will use also Latin letters chosen from \( m, n, o \).

compact notation for partial derivatives \( \partial_{\alpha} = \partial / \partial x^\alpha \).

**B Time derivative of a 4-dimensional vector**

If there is a refractive index the wave displacement vector can be written as

\[ dx = g_\alpha dx^\alpha = n^\alpha_\beta \sigma_\beta dx^\alpha. \tag{B.1} \]

Because this vector is nilpotent, by virtue of Eq. (4.6), the five coordinates are not independent and we can divide both members by \( dx^0 = dt \) defining the nilpotent vector

\[ \dot{x} = g_0 + g_i \dot{x}^i = n^\alpha_0 \sigma_\alpha + n^\beta_i \sigma_\beta \dot{x}^i. \tag{B.2} \]

Suppose we have a 5D vector \( a = \sigma_\alpha a^\alpha \) and we want to find its time derivative along a path parameterized by \( t \), that is all the \( x^i \) are functions of \( t \). We can write

\[ \dot{a} = \partial_\beta a^\alpha x^\beta \sigma_\alpha; \tag{B.3} \]

where naturally \( \dot{x}^0 = 1 \). Remembering the definition of covariant derivative \( (2.25) \) and Eq. (B.2) we can modify this equation to

\[ \dot{a} = x^\beta g_\beta \cdot g^\beta \partial_\beta a^\alpha \sigma_\alpha = \dot{x} \cdot (Da). \tag{B.4} \]

We have expressed vector \( a \) in terms of the orthonormed frame in order to avoid vector derivatives but the result must be independent of the chosen frame.
This procedure has an obvious dual, which we arrive at by defining

\[ \dot{x} = g_\mu \dot{x}^\mu + g_4. \]  

(B.5)

The proper time derivative of vector \( a \) is then

\[ \dot{a} = \dot{x} \cdot (Da). \]  

(B.6)

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