Two–poles $R$-matrices

Michel Talon*

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Abstract

We study integrable dynamical systems described by a Lax pair involving a spectral parameter. By solving the classical Yang–Baxter equation when the $R$-matrix has two poles we show that they can be interpreted as natural motions on a twisted loop algebra.
1 Introduction

Let us consider some dynamical system whose equations of motion have been written under a Lax form \([1]\):

\[
\frac{dL}{dt} = [L, M]
\]  

(1)

Here, \(L\) and \(M\) belong to some Lie algebra \(G\), and we assume the existence of globally defined maps (i.e., the Lax pair, \(L\) and \(M\)) from the phase space of our dynamical system to \(G\) such that the equations of motion are equivalent to (1).

Such a Lax formulation, when it exists, is generally not unique. In particular there exists a general mechanism, known as dual moment maps \([2]\), allowing to switch between two different Lax formulations of a dynamical system (frequently one with an \(N \times N\) Lax pair, and the other one a \(2 \times 2\) Lax pair). Moreover, for some systems, it is necessary in order to obtain a Lax formulation to introduce an auxiliary parameter \(\lambda\), called the spectral parameter, and to consider a Lax pair \(L(\lambda), M(\lambda)\) explicitly dependent on the spectral parameter. This means that the Lie algebra \(G\) in such cases is a loop algebra \(g \otimes \mathbb{C}[\lambda, \lambda^{-1}]\) where \(g\) is an ordinary matrix Lie algebra.

A nice example is provided by the Toda chain. Considering the open chain as a Dynkin diagram naturally leads to a Lax formulation without spectral parameter. The similar construction for the closed chain leads to a loop algebra. Moreover there exists an alternative approach due to Sklyanin \([3]\) which builds a Lax pair by considering the product of \(2 \times 2\) “local” monodromy matrices depending on a spectral parameter. Several related examples are similarly discussed in \([4, 5]\).

In spite of the arbitrariness involved in a Lax formulation, it is a good first step towards the solution of the dynamical problem, since it immediately allows to find conserved quantities. As a matter of fact, eq. (1) implies conservation of the spectrum of \(L\), as first emphasized by Lax \([4]\). In other words, the eigenvalues of the matrix \(L\) (in any representation of \(G\)) are constants of motion. When \(L\) depends on a spectral parameter \(\lambda\) the spectral curve, i.e. the curve of equation \(\det (L(\lambda) - \mu) = 0\) in the \((\lambda, \mu)\)-plane, is similarly conserved. If \(L(\lambda)\) depends algebraically on \(\lambda\), this is the algebraic equation of a compact Riemann surface. Under quite general conditions Semenov-Tian-Shanskii \([6]\) has shown that the solution of the dynamical problem can be found using abelian functions defined on this Riemann surface. In any instance equation (1) strongly suggests a geometrical formulation of the dynamical problem as the flow induced in \(G^*\) by Kirillov’s symplectic structure \([7]\).

At this point it is important to notice that such a flow is not in general integrable. In other words, the eigenvalues of \(L\), while conserved, and globally defined, do not allow for a solution of the equations of motion. It was Liouville who first pointed out \([8]\) that, in order to solve a system on a phase space of dimension \(2n\), it is necessary and sufficient to know \(n\) integrals of motion, globally defined, and in involution (i.e. the Poisson bracket of any two integrals of motion is zero). Of
course the actual Hamiltonian must belong to this set. Let us notice that, according to Darboux theorem, it is always possible to complete the Hamiltonian locally by \((n - 1)\) quantities in involution in order to get a symplectic basis, so Liouville condition is non empty only when the constants of motion are globally defined, as is the case for a Lax spectrum.

It is remarkable that this Liouville formulation leads to a very simple condition in the Lax setting, as first pointed out by Babelon and Viallet \([9]\). They have shown that the eigenvalues of \(L\) are in involution if and only if there exists an \(R\)-matrix, i.e. a function \(R\), globally defined on phase space with values in \(G \otimes G\), such that:

\[
\{L \otimes L\} = [R, L \otimes 1] - [R^{\Pi}, 1 \otimes L]
\]

In this equation, denoting \(L = \sum_i L^i e_i\) where \(L^i\) is a dynamical quantity and \((e_i)\) a basis of \(G\) then \(\{L \otimes L\} = \sum_{i,j} \{L^i, L^j\} e_i \otimes e_j\). Similarly \(R = \sum_{i,j} R^{i,j} e_i \otimes e_j\) and \(R^{\Pi}\) is the “transposed” quantity \(R^{\Pi} = \sum_{i,j} R^{j,i} e_i \otimes e_j\). Finally \([R, L \otimes 1] = \sum_{i,j,k} R^{i,j} L^k [e_i, e_k] \otimes e_j\), so that the right-hand side is expressed in terms of the Lie algebra structure.

Moreover the Jacobi conditions on the Poisson bracket lead to a constraint on the \(R\)-matrix. Under some simplifying hypothesis this constraint may be brought under a form closely related (but not identical) to the semi-classical limit of the Yang–Baxter equation. We shall call this equation the classical Yang–Baxter equation. A nice feature is that the classical Yang–Baxter equation becomes expressed entirely in terms of the Lie algebra structure of \(G\). This opens the way to the study of the solutions of this equation (in the spirit of the Belavin–Drinfel’d analysis \([10]\)) in order to partially classify integrable systems.

It appears by looking at examples that the available \(R\)-matrices either do not involve a spectral parameter, or fall into various classes with respect to it. As a matter of fact, antisymmetric \(R\)-matrices (i.e such that \(R^{\Pi} = -R\)) exactly obey Belavin–Drinfel’d’s analysis and may be classified into rational, trigonometric, and elliptic type. Let us recall that rational antisymmetric \(R\)-matrices are understood (in contrast to trigonometric, and elliptic ones). A simple result is that one–pole matrices for a simple Lie algebra \(g\) are of the form:

\[
R = \frac{\Pi}{\lambda - \mu}.
\]

Here, \(G\) is a loop algebra \(g \otimes \mathbb{C}[[\lambda, \lambda^{-1}]]\) and \(R\) a constant element of \(G \otimes G\), viewed as a function \(R(\lambda, \mu)\), and we are considering the situation in which there is only one pole for \(\mu = \lambda\). Finally \(\Pi\) is the exchange operator for the underlying Lie algebra \(g\): \(\Pi(x \otimes y) = y \otimes x\). The antisymmetry of \(R\) results from the symmetry of \(\Pi\) and the antisymmetry of \((\lambda - \mu)\).

There is no systematic study of non–antisymmetric \(R\)-matrices but examples show that two–poles non antisymmetric \(R\)-matrices are involved quite generally.
We do not know concrete examples involving rational $R$-matrices with more than two poles, although it is easy to construct such $R$-matrices by a meaning procedure [4]. It should be noted that trigonometric solutions can be obtained by an allowed deformation of such $n$-poles solutions [11].

In the following we shall recall the derivation of the classical Yang–Baxter equation, and then completely solve it for two-poles $R$-matrices, under some natural hypothesis. This is a simplification and elaboration of the argument of [12]. We shall then interpret the solution as implying that the Lax pair [1] describes an Adler-Kostant-Symes system, see [13, 14, 15], on a twisted loop algebra. This geometrical situation naturally leads to integrable systems, as emphasized notably by Reiman and Semenov-Tian-Shanski [5]. In some sense this partially answers the classification problem for integrable systems, showing that the above scheme appears necessarily in a wide variety of situations. Unfortunately the cases with $R$-matrices involving dynamical variables, or no spectral parameter (notably for the Calogero model, see [16]) are not covered by such a geometrical formulation.

2 The classical Yang–Baxter equation

The central point of this paper being the study of the classical Yang–Baxter equation, we shall recall here its derivation, which is somewhat tricky. In order to get a neat proof it is convenient to consider that the objects occurring in eq. (2) are in fact living in $T(U(G))$, where $U(G)$ is the universal algebra on the Lie algebra $G$, containing notably $1, X$ for $X \in G$, $XY$ for $X$ and $Y \in G$, such that $XY - YX = [X, Y]$ and so on. Then $T(U(G))$ is the tensorial algebra on $U(G)$ containing elements such as $1 \otimes X, 1 \otimes XY, X \otimes Y$ for $X, Y \in G$ and so on. Elements in $T^n(U(G))$ can be multiplied in the natural component-wise way, for example $(X \otimes Y)(X' \otimes Y') = (XX') \otimes (YY')$ so that in particular $[X \otimes 1, X' \otimes Z] = [X, X'] \otimes Z$. The commutators occurring in eq. (2) can be so interpreted.

It is then further convenient to introduce the following notations:

\[L_1 = L \otimes 1 \otimes 1, \quad L_2 = 1 \otimes L \otimes 1, \quad L_3 = 1 \otimes 1 \otimes L,\]

\[R_{12} = \sum_{ij} R^{ij} e_i \otimes e_j \otimes 1, \quad R_{23} = \sum_{ij} R^{ij} 1 \otimes e_i \otimes e_j, \quad R_{31} = \sum_{ij} R^{ij} e_j \otimes 1 \otimes e_i\]

and the similar objects $R_{21} = R_{12}^\Pi, \ R_{32} = R_{23}^\Pi, \ R_{13} = R_{31}^\Pi$ obtained by $R^{ij} \rightarrow R^{ji}$ which are elements of $T^3(U(G))$.

Now, under the simplifying hypothesis that $R$ is a constant matrix, independent of the dynamical variables, it is easy to convince oneself that the iteration of equation (2) leads to:

\[\{\{L \otimes L\} \otimes L\} = [R_{12}[R_{13}, L_1]] - [R_{12}[R_{31}, L_3]] + [R_{21}[R_{32}, L_3]] - [R_{21}[R_{23}, L_2]]\]
Were the $R$-matrix to contain dynamical terms, there would appear terms with derivatives of $R$ in this equation. Moreover the circular permutations involved in Jacobi identity for the Poisson bracket reduce to the sum of circular permutations on indices $(1, 2, 3)$.

In so doing the term involving $L_1$ may be written:

$$[R_{12}[R_{13}, L_1]] + [R_{13}[R_{12}, L_1]] + [R_{32}[R_{13}, L_1]] - [R_{23}[R_{12}, L_1]]$$

Noticing that Jacobi identity for the commutator in $T^3(U(G))$ may be used, the first and second term produce $[[R_{12}, R_{13}], L_1]$. Moreover $[R_{32}, L_1] = [R_{23}, L_1] = 0$ trivially so that the two other terms can also be written as similar commutators, leading to the condition:

$$[[R_{12}, R_{13}] + [R_{12}, R_{23}] + [R_{32}, R_{13}], L_1] = 0$$

Here, one introduces a second simplifying hypothesis: one assumes that $L$ is general enough so that the only natural solution to this compatibility condition is given by the classical Yang–Baxter condition:

$$[R_{12}, R_{13}] + [R_{12}, R_{23}] + [R_{32}, R_{13}] = 0.$$  \hspace{1cm} (4)

Some remarks are in order:

- Equation (4) implies the same equation with permutations of indices as may be easily derived by inserting appropriate operators $\Pi$, hence no new condition is obtained with the terms involving $L_2$ and $L_3$.

- This equation is only valid as a constraint for a constant $R$-matrix since otherwise terms with derivatives of $R$ occur (particularly $\{R, L\}$).

- Eq. (4) is a sufficient condition for eq. (2) to be consistent with the Jacobi identity, but the extent of its necessity is not clear.

- Finally this equation is closely related to the semi–classical limit of the quantum Yang-Baxter equation, but not identical.

More precisely the compatibility equation for the quantum commutation relations:

$$R \cdot T_1 \otimes T_2 = T_2 \otimes T_1 \cdot R$$

is known as the Yang–Baxter equation. It reads:

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}.$$  \hspace{1cm} (4)

Considering the $h$–expansion: $R = 1 + h r + h^2 + \ldots$ and collecting terms up to order $h^2$ in the above equation, one sees that $s$ terms cancel while $r$ terms lead to:

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0.$$
The difference with eq. (4) is the occurrence of $r_{23}$ and not $r_{32}$. As a matter of fact, if one assumes an antisymmetric $R$-matrix, one gets $R_{32} = - R_{23}$ and eq. (4) reduces exactly to the semi-classical limit of the quantum Yang–Baxter equation. However, there is no justification for such an hypothesis and concrete mechanical examples lead to non–antisymmetric $R$-matrices [4].

3 Two–poles solutions of the Yang–Baxter equation

For many interesting integrable models the Lax matrix lives in a loop algebra, i.e. may be written $L(\lambda)$ where $\lambda$ is a spectral parameter and $L(\lambda)$ may be seen as a function with values in a simple Lie algebra $\mathcal{G}$. Then eq. (2) reads:

$$\{L(\lambda) \otimes L(\mu)\} = [R(\lambda, \mu), L(\lambda) \otimes 1] - [R(\mu, \lambda), 1 \otimes L(\mu)]$$

where the function $R(\lambda, \mu)$ has values in $\mathcal{G} \otimes \mathcal{G}$.

The symplectic structure of the dynamical system is characterized by $R(\lambda, \mu)$ and notably by its pole structure. In particular the simplest example of solution of eq. (4) is given by the one–pole antisymmetric solution given by eq. (3). This solution plays a central role in the general discussion of antisymmetric solutions of eq. (4) by Belavin and Drinfel’d [10]. Nevertheless, many interesting mechanical examples require a non–antisymmetric $R$-matrix, and up to now nothing more complicated than a two–pole solution of eq. (4) had to be considered. Accordingly we shall look for solutions of eq. (4) of the form:

$$R(\lambda, \mu) = \frac{A}{\lambda - \mu} + \frac{B}{\lambda + \mu} \quad (5)$$

Since a Lax matrix $L(\lambda)$ may be multiplied by $f(\lambda)$ for any analytic $f$ and moreover $\lambda$ may be changed into $g(\lambda)$, eq. (5) represents a general two–pole $R$-matrix, with $A, B \in \mathcal{G}$. In the spirit of Belavin and Drinfel’d analysis, we shall assume that $\mathcal{G}$ is a simple Lie algebra, and obtain $A$ and $B$ such that eq. (4) is satisfied. Then as we have shown in [7] there is no allowed deformation of this solution by functions of $\lambda$ and $\mu$ up to the above mentioned freedom of redefinition.

The substitution of ansatz (5) in equation (4) leads to the following condition:

$$\begin{align*}
[A_{12}, A_{13}] &+ \frac{[B_{12}, B_{13}]}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)} + \frac{[A_{12}, B_{13}]}{(\lambda_1 - \lambda_2)(\lambda_1 + \lambda_3)} + \frac{[A_{12}, B_{13}]}{(\lambda_1 - \lambda_2)(\lambda_1 + \lambda_3)} + \\
[B_{12}, A_{13}] &+ \frac{[A_{12}, A_{23}]}{(\lambda_1 + \lambda_2)(\lambda_1 + \lambda_3)} + \frac{[B_{12}, B_{23}]}{(\lambda_1 + \lambda_2)(\lambda_1 + \lambda_3)} + \\
[A_{12}, B_{23}] &+ \frac{[B_{12}, A_{23}]}{(\lambda_1 - \lambda_2)(\lambda_2 + \lambda_3)} + \frac{[A_{32}, A_{13}]}{(\lambda_3 - \lambda_2)(\lambda_1 - \lambda_3)} + \\
[A_{12}, B_{23}] &+ \frac{[B_{12}, A_{23}]}{(\lambda_1 - \lambda_2)(\lambda_2 + \lambda_3)} + \frac{[A_{32}, A_{13}]}{(\lambda_3 - \lambda_2)(\lambda_1 - \lambda_3)} + \\
[B_{12}, A_{23}] &+ \frac{[A_{12}, B_{23}]}{(\lambda_1 + \lambda_2)(\lambda_2 + \lambda_3)} + \frac{[B_{12}, A_{23}]}{(\lambda_1 + \lambda_2)(\lambda_2 + \lambda_3)} \quad (6)
\end{align*}$$
\[
\frac{[B_{32}, B_{13}]}{(\lambda_3 + \lambda_2)(\lambda_1 + \lambda_3)} + \frac{[A_{32}, B_{13}]}{(\lambda_3 - \lambda_2)(\lambda_1 + \lambda_3)} + \frac{[B_{32}, A_{13}]}{(\lambda_3 + \lambda_2)(\lambda_1 - \lambda_3)} = 0
\]

Taking the pole in \((\lambda_1 - \lambda_2)\) one gets:
\[
\frac{[A_{12}, A_{13} + A_{23}]}{\lambda_1 - \lambda_3} + \frac{[A_{12}, B_{13} + B_{23}]}{\lambda_1 + \lambda_3} = 0
\]
which must be true for any \(\lambda_1, \lambda_3\) whence:
\[
[A_{12}, A_{13} + A_{23}] = [A_{12}, B_{13} + B_{23}] = 0.
\]

Similarly taking the pole in \((\lambda_1 + \lambda_3)\) one gets:
\[
[B_{12}, B_{13} - A_{23}] = [B_{12}, A_{13} - B_{23}] = 0.
\]

Conversely if these 4 conditions are satisfied, they are also true with permuted indices, so that the classical Yang–Baxter equation is satisfied.

In order to analyze further these conditions it is highly convenient to use a trick introduced for this purpose by Belavin and Drinfel’d, known as dualization. Since \(\mathcal{G}\) is a simple Lie algebra, it is equipped with an essentially unique invariant scalar product denoted \((,\,\,)\). Then to each element \(A = u \otimes v\) spanning \(\mathcal{G} \otimes \mathcal{G}\) we can associate the endomorphism of \(\mathcal{G}\):
\[
X \rightarrow \langle v, X \rangle u
\]
We shall denote by the same letter \(A\) this element of \(\text{End}(\mathcal{G}) \simeq \mathcal{G} \otimes \mathcal{G}^*\) since this association is an isomorphism. It enjoys the nice property that the transposed endomorphism \(T_A\) such that \((AX, Y) = (X, T_A Y)\) is associated to the operator \(v \otimes u \in \mathcal{G} \otimes \mathcal{G}\) which is the image of \(u \otimes v\) by \(\Pi\).

In order to obtain the dualized form of the above four equations, it is convenient to proceed as follows: write \(A = \sum \alpha u_\alpha \otimes v_\alpha\) with \(u_\alpha, v_\alpha \in \mathcal{G}\) and for any \(X, Y, Z \in \mathcal{G}\) consider the scalar product:
\[
(X \otimes Y \otimes Z, [A_{12}, A_{13} + A_{23}]) = \\
\sum_{\alpha, \beta} (X, [u_\alpha, u_\beta])(Y, v_\alpha)(Z, v_\beta) + (X, u_\alpha)(Y, [v_\alpha, u_\beta])(Z, v_\beta) = \\
(X, [AY, AZ]) + (Y, [TAX, AZ]) = (X, [AY, AZ] - A[Y, AZ])
\]
by using the invariance of the scalar product and the definition of the transposition. Hence, \([A_{12}, A_{13} + A_{23}] = 0\) if and only if the dualized operator obeys \(A[X, AY] = [AX, AY]\) as an element of \(\text{End}(\mathcal{G})\) for any \(X, Y \in \mathcal{G}\).

By dualizing similarly the above four equations one gets finally:
\[
A[X, AY] = [AX, AY] \quad ; \quad A[X, BY] = [AX, BY] \quad (6)
\]
\[ B[X, AY] = -[BX, BY] \] (7)
\[ B[X, BY] = -[BX, AY] \] (8)

Assuming, as in the analysis of Belavin and Drinfel’d that the dualized endomorphism \( R(\lambda, \mu) \) is invertible at least for one couple \((\lambda_0, \mu_0)\) one sees that any \( Y \in \mathcal{G} \) can be written under the form \( AY_1 + BY_2 \). Then equations (6) are equivalent to:
\[ A[X, Y] = [AX, Y] \quad \forall X, Y \in \mathcal{G}. \]

Since \( \mathcal{G} \) is simple and \( A \neq 0 \) this implies that \( A \) is invertible (since \( \text{Ker} A \) is an ideal of \( \mathcal{G} \)). Then one can normalize so that \( A = I \). Then eq. (7) with \( \sigma = -B \in \text{End} (\mathcal{G}) \) reads \( \sigma[X, Y] = [\sigma X, \sigma Y] \), hence \( \sigma \) is an automorphism of \( \mathcal{G} \). Finally equation (8) reads \( \sigma[X, \sigma Y] = [\sigma X, Y] = [\sigma X, \sigma^2 Y] \) hence \( [\sigma X, Y - \sigma^2 Y] = 0 \) \( \forall X, Y \in \mathcal{G} \). But \( \sigma X \) is arbitrary (noticing that \( \text{Ker} (\sigma) \) is an ideal of \( \mathcal{G} \), hence \( \sigma \) is bijective), so that with \( Z = (\sigma^2 - 1)Y \) one has \( \text{ad} Z = 0 \). Then for any \( T \in \mathcal{G} \), \( (Z, T) = \text{Tr} \text{ad} Z \text{ad} T = 0 \) hence \( Z = 0 \) since \( \mathcal{G} \) is simple and the Killing form is not degenerate. Finally \( \sigma^2 = 1 \) meaning that \( \sigma \) is an involutive automorphism of \( \mathcal{G} \).

Finally, \( R(\lambda, \mu) = 1/(\lambda - \mu) - \sigma/(\lambda + \mu) \) or in the usual non dualized form:
\[ R(\lambda, \mu) = \frac{\Pi}{\lambda - \mu} - \frac{\sigma \otimes 1 \Pi}{\lambda + \mu} \] (9)

This is the general two–pole solution of the classical Yang–Baxter equation under the above mentioned hypothesis. A similar \( n \)–pole solution can be constructed using an automorphism \( \sigma \) such that \( \sigma^m = 1 \). It reads:
\[ R = \sum_{n=0}^{m-1} \frac{\sigma^n \otimes 1 \Pi}{e^n \lambda - \mu} \quad \epsilon = e^{\frac{2\pi i}{m}} \]

4 Geometrical interpretation

Mechanical systems with the above \( R \)-matrices can be interpreted geometrically in the framework of the Adler-Kostant-Symes scheme [13, 14, 15]. We shall briefly sketch the relevant ideas. In the present situation, one may identify \( \mathcal{G} \) and \( \mathcal{G}^* \), and introduce the natural Kirillov’s symplectic bracket on \( \mathcal{G} \). For linear functions of the form \( f_X(Y) = (X, Y) \) \( (X, Y) \in \mathcal{G} \) the value of the Poisson bracket is \( \{f_X, f_Y\} = f_{[X, Y]} \). This extends naturally to products of linear functions, and finally to arbitrary functions on \( \mathcal{G} \). This Poisson bracket is degenerate since invariant functions on \( \mathcal{G} \) have a vanishing bracket with any other functions. One gets a non–degenerate symplectic structure by restricting oneself to \textit{orbits} of the Lie action on \( \mathcal{G} \). Now the idea is to interpret the Lax equation of motion (1) as describing the flow on such an orbit according to an appropriate Lie structure.
As a matter of fact it is impossible to get enough information by using just one Lie structure on \( G \). The clever idea of Adler, Kostant, Symes is to use the interplay between two Lie structures on \( G \). Then one chooses dynamical systems parametrizing orbits of one of the Lie structures, with the associated symplectic structure, while integrable hamiltonians are given as invariants of the other Lie structure, and may easily be shown to Poisson commute. A natural way to produce such a situation is to take for \( G \) a loop algebra \( g \otimes \mathbb{C}[\lambda, \lambda^{-1}] \) and for the second Lie algebra structure on \( G \), impose the vanishing of brackets between positive and negative powers of \( \lambda \), carefully treating the zero–power.

It has been emphasized by Semenov-Tian-Shanskii \([8]\), that the classical \( R \)-matrix appearing in eq. (2) is simply a way to define a second Lie algebra structure on \( G \), namely, for the dualized \( R \):

\[
[X, Y]_R = [X, RY] + [RX, Y]
\]

so that the Lax equation (1) is the natural equation of motion for the corresponding AKS–scheme. It should be remarked that this works for a “constant” \( R \)-matrix, and that the condition for \([ \ ]_R \) to be a Lie bracket boils down to the classical Yang-Baxter equation.

We shall now indicate the corresponding geometrical interpretation of our two–pole \( R \)-matrix. According to eq. (10) it is necessary to interpret the dualized \( R \)-matrix \( R = 1/(\lambda - \mu) - \sigma/(\lambda + \mu) \). The relation with positive and negative powers in the loop algebra is provided by writing formally:

\[
\frac{1}{\lambda - \mu} = \frac{1}{2} \sum_{n=0}^{\infty} \left( \frac{\mu^n}{\lambda^{n+1}} - \frac{\lambda^n}{\mu^{n+1}} \right) \quad \frac{1}{\lambda + \mu} = \frac{1}{2} \sum_{n=0}^{\infty} \left( (-1)^n \frac{\mu^n}{\lambda^{n+1}} + \frac{\lambda^n}{\mu^{n+1}} \right)
\]

Terms such as \( \mu^n/\lambda^{n+1} \) may be dualized by considering their action on the loop algebra:

\[
\left( \frac{\mu^n}{\lambda^{n+1}}, X(\mu) \right) = \sum_k \left( \frac{\mu^n}{\lambda^{n+1}}, X_k \right) = X_k \frac{\lambda^n}{\lambda^{n+1}}
\]

i.e. negative powers are reproduced while positive ones are killed. So doing, on gets the completely dualized form of eq. (8):

\[
R = -\frac{1 + \hat{\sigma}}{2} (P_+ - P_-)
\]

where \( P_\pm \) are projectors on positive and negative powers, while \( \hat{\sigma} \) is the extension of \( \sigma \) to the loop algebra: \((\hat{\sigma})(X(\lambda)) = \sigma \cdot X(-\lambda)\).

Here the projector \((1 + \hat{\sigma})/2\) restricts us to the well–known twisted loop algebra \([9]\) associated to an involutive automorphism, and the \( R \)-matrix (9) identifies to the structure introduced notably by Reiman and Semenov-Tian-Shanskii \([3]\), i.e an AKS scheme on a twisted algebra. We have just shown that this construction is the more general compatible with the above mentioned hypothesis.

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