Uniform Generalization Bound on Time and Inverse Temperature for Gradient Descent Algorithm and its Application to Analysis of Simulated Annealing

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Abstract
In this paper, we propose a novel uniform generalization bound on the time and inverse temperature for stochastic gradient Langevin dynamics (SGLD) in a non-convex setting. While previous works derive their generalization bounds by uniform stability, we use Rademacher complexity to make our generalization bound independent of the time and inverse temperature. Using Rademacher complexity, we can reduce the problem to derive a generalization bound on the whole space to that on a bounded region and therefore can remove the effect of the time and inverse temperature from our generalization bound. As an application of our generalization bound, an evaluation on the effectiveness of the simulated annealing in a non-convex setting is also described. For the sample size, we derive evaluations with orders $\sqrt{n^{-1}\log(n+1)}$ and $|(\log)^4(s)|^{-1}$, respectively. Here, $(\log)^4$ denotes the 4 times composition of the logarithmic function.

Keywords: Generalization Bound, Stochastic Differential Equation, Gradient Descent, Simulated Annealing, Non-convex Optimization

1. Introduction

Numerical calculation methods have become practical due to the development of computers, and therefore it has become more and more important to guarantee their performance theoretically. In fact, almost all algorithms that achieve numerical solutions include hyperparameters, which are arbitrarily set by the user. In general, the setting of hyperparameters greatly affects the performance of the algorithm. In particular, it is important to derive an explicit evaluation of the relationship between hyperparameter settings and algorithm performance to determine the optimal hyperparameter settings.

For stochastic gradient Langevin dynamics (SGLD), which is one of the typical optimization algorithms, \cite{2,3,5,6,11,14,17,20,21,22,23,24,26,27,28,29,30}, have derived evaluations on the effectiveness of SGLD from which we can choose appropriate hyperparameter settings. Let $\mathcal{Z}$ be the set of all data points and $\ell(w; z) : \mathbb{R}^d \times \mathcal{Z} \to [0, \infty)$ denote the loss on $z \in \mathcal{Z}$ for a parameter $w \in \mathbb{R}^d$. $z_1, \ldots, z_n$ are independent and identically distributed (IID) samples generated from the distribution $\mathcal{D}$ on $\mathcal{Z}$, and we define the empirical loss by $L_n(w) = \frac{1}{n} \sum_{i=1}^{n} \ell(w; z_i)$. Then, for the step size $\eta > 0$ and the inverse temperature $\beta > 0$, SGLD is defined as follows.

$$X_{k+1}^{(n, \eta)} = X_k^{(n, \eta)} - \eta \nabla L_n(X_k^{(n, \eta)}) + \sqrt{2\eta/\beta} \epsilon_k, \quad k \geq 0. \quad (1.1)$$

Here, $\epsilon_k$ are IID, each of which obeys the $d$-dimensional standard normal distribution. Assuming the dissipativity of the loss $\ell(w; z)$ and Lipschitz continuity of its gradient, \cite{26} showed the following evaluation to (1.1), where $L(w) = E_{z \sim \mathcal{D}}[\ell(w; z)]$ denotes the expected loss for the parameter $w$ and $C_i > 0$ are constants independent of $\eta$, $n$, $\beta$, and $k$.

$$E[L(X_k^{(n, \eta)})] - \min_{w \in \mathbb{R}^d} L(w) \leq C_1 \left( \frac{e^{C_2 \beta}}{n} + \sqrt{\eta} e^{C_3 \beta} + \exp \left\{ C_2 \beta - \frac{C_4 k n}{e^{C_5 \beta}} \right\} + \log(\beta + 1) \right). \quad (1.2)$$

According to (1.2), by determining hyperparameters in the order of $\beta$, $\eta$, $k$, and $n$, SGLD (1.1) can minimize the expected loss with arbitrary accuracy.

However, SGLD (1.1) always contains a constant error with order $\beta^{-1}\log(\beta + 1)$ since it uses the fixed inverse temperature $\beta$. The algorithm that increases the inverse temperature and decreases the step size with time evolution
to remove this error term is called as simulated annealing (SA). For SA, as \([1,2]\) indicates, by setting increase and decrease rates of the inverse temperature and step size properly, the error term caused by the setting of hyperparameters except for \(n\) vanish with time evolution. Whereas, as in the first term in the R.H.S of \([1,2]\), previous works on SGLD \([22,24,26]\) only have derived generalization bounds, which are bounds for \(n\), that explode as \(\beta\) tends to infinity. Hence, it seems that the sample size \(n\) should increase with time evolution to control the effect of \(\beta\) when SA is applied. However, in general, the sample size \(n\) has its upper bound and we cannot take \(n\) arbitrarily large enough to control the effect of \(\beta\). In fact, previous works on SA \([1,2,4,6,11,12,16,17]\) have not derived generalization bounds, which indicates it is difficult to derive practical generalization bounds to the SA algorithm.

The first main result in this paper, Theorem 2.2, is a refined generalization bound to SGLD \([1,1]\). While previous works \([24,26]\) use uniform stability \([12]\), we use Rademacher complexity to make our generalization bound independent of the time and inverse temperature.

The second main result, Theorem 2.3, is the evaluation of the same form as \([1,2]\) on the effectiveness of the SA algorithm. Using the generalization bound capable of increasing the inverse temperature, we can derive a practical generalization bound to the SA algorithm.

This paper is organized as follows. In Section 2 we give accurate statements of our main results of uniform generalization bound on the time and inverse temperature for SGLD algorithm and its application to the evaluation on the SA algorithm. Sections 3 and 4 are devoted to the proof of the first and second main results, respectively. Finally, the results used in Sections 3 and 4 are stated and proved in Appendix.

2. Main Result

To formulate our first main result, we introduce the following notations. \(S = (z_1, \ldots, z_n) \in \mathcal{Z}^n\) are IID generated from the distribution \(\mathcal{D}\) on \(\mathcal{Z}\). Let \(\ell(w; z)\) be a loss function and we define the expected loss and the empirical loss by \(L(w) = E_{z \sim \mathcal{D}}[\ell(w; z)]\) and \(L_n(w) = \frac{1}{n} \sum_{i=1}^{n} \ell(w; z_i)\), respectively. For a \(d\)-dimensional Brownian motion \(W\), the initial value \(x_0 \in \mathbb{R}^d\), and the inverse temperature \(\beta > 0\), we consider

\[
dX^{(n)}_t = -\nabla L_n(X^{(n)}_t)dt + \sqrt{2/\beta}dW_t, \quad X^{(n)}_0 = x_0. \tag{2.1}
\]

We impose the following assumption on the loss function \(\ell(w; z)\), and therefore the drift coefficient \(-\nabla L_n\) of \([2.1]\).

**Assumption 2.1.** The loss \(\ell(w; z)\) is nonnegative and satisfies \(\sup_{z \in \mathcal{Z}} |\ell(0; z)| \leq B\) and \(\sup_{z \in \mathcal{Z}} \|\nabla \ell(0; z)\|_{\mathbb{R}^d} \leq A\) for some \(A, B > 0\). Thus, the expected loss \(L(w) = E_{z \sim \mathcal{D}}[\ell(w; z)]\) is well-defined. In addition, \(\ell(\cdot; z) \in C^1(\mathbb{R}^d; \mathbb{R})\) satisfies the following two conditions for all \(z \in \mathcal{Z}\).

1. (1) \((m, b)\)-dissipative for some \(m, b > 0\). Here, \(H \in C^1(\mathbb{R}^d; \mathbb{R})\) is said to be \((m, b)\)-dissipative when the following inequality holds.

\[
(\nabla H(x), x)_{\mathbb{R}^d} \geq m\|x\|_{\mathbb{R}^d}^2 - b, \quad x \in \mathbb{R}^d. \tag{2.2}
\]

2. (2) \(M\)-smooth for some \(M > 0\). Here, \(H \in C^1(\mathbb{R}^d; \mathbb{R})\) is said to be \(M\)-smooth when the following inequality holds.

\[
\|\nabla H(x) - \nabla H(y)\|_{\mathbb{R}^d} \leq M\|x - y\|_{\mathbb{R}^d}, \quad x, y \in \mathbb{R}^d. \tag{2.3}
\]

Under the notation of \([1,1]\) and \([2,1]\), previous works on SGLD \([22,24,26]\) use the following decomposition.

\[
E[L(X^{(n)}_{k,n})] - \min_{w \in \mathbb{R}^d} L(w) = \left\{ E[L(X^{(n)}_{k,n})] - E[L(X^{(n)}_{k,n})] \right\} + \left\{ E[L(X^{(n)}_{k,n})] - E[L_n(X^{(n)}_{k,n})] \right\} + \left\{ E[L_n(X^{(n)}_{k,n})] - \min_{w \in \mathbb{R}^d} L(w) \right\}. \tag{2.4}
\]

Then, the generalization bound for SGLD corresponds to the bound to the second term in the R.H.S of \([2,3]\).

The first main result in this paper is a uniform evaluation on the time and inverse temperature to the second term in the R.H.S of \([2,3]\). In the following, we denote \(f = O_{\alpha}(g)\) if there exists a constant \(C_{\alpha} > 0\) that depends only on \(\alpha\) such that \(f \leq C_{\alpha}g\) holds. Similarly, we denote \(f = \Theta_{\alpha}(g)\) when \(f \geq C_{\alpha}g\) holds.

**Theorem 2.2.** Under Assumption 2.1 for sufficiently large \(\beta > 0\) and \(\alpha_0 = (m, b, M, A, B, d)\), the following inequality holds.

\[
|E[L(X^{(n)}_{1})] - E[L_n(X^{(n)}_{1})]| \leq O_{\alpha_0} \left( \frac{\log(n + 1)}{n} + (1 + \|x_0\|_{\mathbb{R}^d}^3) \exp \left\{ - \frac{t}{e^{1/\alpha_0(\beta)}} + O_{\alpha_0}(\beta) \right\} + (1 + \|x_0\|_{\mathbb{R}^d}^2) \sqrt{\frac{\log(\beta + 1)}{\beta}} \right). \tag{2.5}
\]
Next, to formulate our second main result, we introduce the following notations. \( \gamma : [0, \infty) \to (0, \infty) \) is a strictly increasing function and for a monotone decreasing sequence \( \eta = \{ \eta_k \}_{k=1}^\infty \), we set \( T_k := \sum_{j=1}^k \eta_j \). Denoting \( \phi^{(n)}(t) = \sum_{k=1}^\infty T_k \chi(T_k, T_{k+1})(t) \), we define the SA \( Z^{(n)} \) and its discretization \( Z^{(n,\eta)} \) with initial values \( x_0 \in \mathbb{R}^d \) as follows. Here, \( \chi_{\Gamma} \) denote the indicator function of \( \Gamma \).

\[
\begin{align*}
    dZ_t^{(n)} &= -\nabla L_n(Z_t^{(n)}) dt + \sqrt{2/\gamma(s)} dW_s, \quad Z_0^{(n)} = x_0, \quad (2.5) \\
    dZ_t^{(n,\eta)} &= -\nabla L_n(Z_t^{(n,\eta)}) dt + \sqrt{2/\gamma(s)} dW_s, \quad Z_0^{(n,\eta)} = x_0. \quad (2.6)
\end{align*}
\]

Finally, for each \( s \geq 0 \), we define the function \( \alpha(s, \cdot) \) by

\[
\int_s^\infty \frac{\gamma(s)}{\gamma(u)} du = t. \quad (2.7)
\]

The properties of \( \alpha(s, \cdot) \) is described in Lemma A.8 For the function \( \gamma \) and the sequence \( \eta \), we impose the following assumption.

**Assumption 2.3.** For sufficiently large \( t > 0 \), \( \gamma(t) = (\log)^3(t) \) holds. Here, \( (\log)^k \) denotes the k times composition of the logarithmic function. Furthermore, \( \eta_k = 1/k \) for all \( k \in \mathbb{N} \). Thus, \( \lim_{k \to \infty} T_k = \infty \) holds.

The second main result in this paper is an evaluation on the effectiveness of the SA.

**Theorem 2.4.** Under Assumptions 2.1 and 2.3 for sufficiently large \( s > 0 \) and \( \alpha_1 = (m, b, M, A, B, \gamma(0), d) \), the following inequalities hold.

\[
\begin{align*}
    &E \left[ L_n(Z_{\alpha_1(s, x_0^{(n,s/\gamma)})}) \right] - \min_{w \in \mathbb{R}^d} L(w) \leq O_{\alpha_1} \left( \frac{1 + \|x_0\|_{\gamma}^s}{(\log)^{4}(s)} \right), \quad (2.8) \\
    &E \left[ L(Z_{\alpha_1(s, x_0^{(n,s/\gamma)})}) \right] - E \left[ L_n(Z_{\alpha_1(s, x_0^{(n,s/\gamma)})}) \right] \leq O_{\alpha_1} \left( \sqrt{\frac{\log(n+1)}{n}} + \frac{1 + \|x_0\|_{\gamma}^s}{(\log)^{4}(s)} \right), \quad (2.9) \\
    &E \left[ L(Z_{\alpha_1(s, x_0^{(n,s/\gamma)})}) \right] - E \left[ L(Z_{\alpha_1(n, x_0^{(n,s/\gamma)})}) \right] \leq O_{\alpha_1} \left( (1 + \|x_0\|_{\gamma}^s) \exp\{-\Omega_{\alpha_1}(s^{1/2})\} \right). \quad (2.10)
\end{align*}
\]

In particular, we have

\[
E \left[ L(Z_{\alpha_1(s, x_0^{(n,s/\gamma)})}) \right] - \min_{w \in \mathbb{R}^d} L(w) \leq O_{\alpha_1} \left( \sqrt{\frac{\log(n+1)}{n}} + \frac{1 + \|x_0\|_{\gamma}^s}{(\log)^{4}(s)} \right). \quad (2.11)
\]

Almost all of the previous works on SA \( [1, 2, 4, 9, 11, 13, 16, 19] \) consider the optimization problem on the discrete or bounded space. In addition, these works only show that the SA algorithm approaches any fixed neighborhood of minimizers of objective functions. Therefore, Theorem 2.4 is novel in that it considers the optimization problem on \( \mathbb{R}^d \) and derives the evaluation equivalent to (1.2).

**Remark 2.5.** The sequence \( \{Z_{\alpha_1(n)}^{(k)}\}_{k=0}^\infty \), which is constructed by extracting values from \( Z^{(n,\eta)} \) at each \( T_k \), has the same law as the sequence

\[
\tilde{Z}_{k+1}^{(n,\eta)} = \tilde{Z}_k^{(n,\eta)} - \eta_k \nabla L_n(\tilde{Z}_k^{(n,\eta)}) + \sqrt{\eta_k} \epsilon_k, \quad \tilde{Z}_0^{(n,\eta)} = x_0. \quad (2.11)
\]

Here, \( \eta_k = \int_{T_k}^{T_{k+1}} 2\gamma(t)^{-1} dt \).

3. Proof of Theorem 2.2

In this section, we prove our first main result, Theorem 2.2. Generalization bounds given in [24, 26] are based on uniform stability [12]. In the following, by Rademacher complexity, we derive a generalization bound capable of increasing the inverse temperature. While our bound is capable of increasing the inverse temperature, the order of it degrades from \( n^{-1} \) to \( \sqrt{n^{-1} \log(n+1)} \) compared with the results by previous works.

First, we show Lemma 3.1 below, which is based on existing results Theorems A.12 and A.13 on Rademacher complexity.
Lemma 3.1. For any \( R > 0 \), we have

\[
E \left[ \sup_{\|w\|_2 \leq R} |L(w) - L_n(w)| \right] \leq O_{M,A,B,d,R} \left( \frac{\log(n + 1)}{n} \right),
\]

Proof. Let \( \mathcal{F} = \{ \ell(w; \cdot) \mid \|w\|_{\mathbb{R}^d} \leq R \} \). If \( \|w\|_{\mathbb{R}^d} \leq R \), then \( \sup_{z \in \mathbb{R}^d} \|\nabla \ell(w; z)\|_{\mathbb{R}^d} \leq A + MR \) holds. Thus, for any \( \|w\|_{\mathbb{R}^d}, \|v\|_{\mathbb{R}^d} \leq R \), we have \( |\ell(w; z) - \ell(v; z)| \leq (A + MR)\|w - v\|_{\mathbb{R}^d} \). Furthermore, for any \( \delta > 0 \), \( \{w \in \mathbb{R}^d \mid \|w\|_{\mathbb{R}^d} \leq R \} \) can be covered by \( (\delta^{-1} R \sqrt{d} + 1)^d \) closed balls with radius \( \delta \). Therefore, with the notation in Theorem A.12, we obtain

\[
C(\mathcal{F}, n^{-1}, \|\cdot\|_{1,S}) \leq (nR(A + MR)\sqrt{d} + 1)^d.
\]

Similarly, if \( \|w\|_{\mathbb{R}^d} \leq R \), then \( |\ell(w; z)| \leq B + (A + MR)R \) holds, and therefore

\[
\sup_{f \in \mathcal{F}} \left( \frac{1}{n} \sum_{i=1}^{n} f(z_i)^2 \right)^{1/2} \leq B + (A + MR)R.
\]

As a result, applying Theorem A.12 to \( \varepsilon = n^{-1} \), we obtain

\[
\hat{R}_n(\mathcal{F}, S) \leq \frac{1}{n} + \{B + (A + MR)R\} \sqrt{\frac{2d\log(nR(A + MR)\sqrt{d} + 1)}{n}}.
\]

Theorem A.13 completes the proof.

Lemma 3.1 proves Theorem 2.2 as follows. Let \( R = \sqrt{2m^{-1}(2 + b\log 2)} \). If \( \|x\|_{\mathbb{R}^d} > R \), then we have \( L_n(x) - \min_{w \in \mathbb{R}^d} L_n(w) \geq 1 \) by Lemma A.1. Thus, for the Gibbs measure \( \pi_{\beta,L_n}(dw) \propto e^{-\beta L_n(w)}dw \),

\[
P(\|X_t^{(n)}\|_{\mathbb{R}^d} > R) \leq E[L_n(X_t^{(n)}) - \pi_{\beta,L_n}(L_n)] + E \left[ \pi_{\beta,L_n}(L_n) - \min_{w \in \mathbb{R}^d} L_n(w) \right]
\]

holds. Therefore, Lemma A.15, Proposition 4.1 in 20, and Proposition 11 in 24 yield

\[
P(\|X_t^{(n)}\|_{\mathbb{R}^d} > R) \leq O_{\alpha_0} \left( 1 + \|x_0\|_{\mathbb{R}^d}^2 \right) \exp \left\{ -\frac{t}{e^{4\alpha_0(\beta)}} + O_{\alpha_0}(\beta) \right\} + \frac{\log(\beta + 1)}{\beta}. \quad (3.1)
\]

Since Lemmas A.1 and A.10 yield \( E[|L_n(X_t^{(n)}) - L_n(X_t^{(n)})|^2]^{1/2} \leq O_{\alpha_0}(1 + \|x_0\|_{\mathbb{R}^d}^2) \),

\[
E[L(X_t^{(n)})] - E[L_n(X_t^{(n)})] \leq E \left[ \sup_{\|w\|_2 \leq R} |L(w) - L_n(w)| \right] + O_{\alpha_0} \left( 1 + \|x_0\|_{\mathbb{R}^d}^2 \right) P(\|X_t^{(n)}\|_{\mathbb{R}^d} > R)^{1/2}
\]

holds. The desired result follows from Lemma 3.1 and 3.1.

4. Proof of Theorem 2.3

In this section, we prove Theorem 2.3. As for 2.8 and 2.9, we follow the same scheme as in 19. That is, for a sufficiently large time \( s \), we approximate the SA 2.5 by SGLD

\[
dY^{(n)}(s,t) = -\nabla L_n(Y^{(n)}(s,t))dt + \sqrt{2/\gamma(s)}dW_t
\]

and reduce the problem of deriving bounds for 2.5 to that for 19. As an approximation of 2.6 by 19, we use Lemma A.9 which is a refinement of Lemma 2 in 19. On the other hand, we prove 2.10 directly using approximate reflection coupling (ARC) proposed in 26.
4.1. Proof of (2.8)

For sufficiently large $s > 0$, we consider the decomposition

$$E[L_n(Z_{\alpha(s,s^{2/3})})] - \min_{w \in \mathbb{R}^d} L_n(w) = E\left[ L_n(Z_{\alpha(s,s^{2/3})}) - \min_{w \in \mathbb{R}^d} L_n(w); \{L_n(Z_s^{(n)}) \leq (\log)^4(s)\} \right]$$

$$+ E\left[ L_n(Z_{\alpha(s,s^{2/3})}) - \min_{w \in \mathbb{R}^d} L_n(w); \{L_n(Z_s^{(n)}) > (\log)^4(s)\} \right].$$

(4.2)

By Chebyshev’s inequality and Lemmas A.1 and A.10, the second term in the R.H.S. of (4.2) has the desired bound. To derive the bound for the first term, fix arbitrarily $x \in \mathbb{R}^d$ so that $L_n(x) \leq (\log)^4(s)$ holds. In addition, suppose that $Y^{(n)}(s, \cdot)$ defined by (4.1) has an initial value $x$. Then, Lemma A.9 yields

$$\left| E[L_n(Z_{\alpha(s,s^{2/3})}) | S, Z_s^{(n)} = x] - E\left[ L_n(Y^{(n)}(s, s^{2/3})) | S\right] \right| \leq O_{\alpha_1} \left( \frac{1 + \|x\|_{\mathbb{R}^d}^2}{\sqrt{(\log)^2(s)}} \right).$$

Furthermore, applying Lemma A.15, Proposition 4.1 in [26] and Proposition 11 in [24] to the R.H.S of

$$E \left[ L_n(Y^{(n)}(s, s^{2/3})) | S\right] - \min_{w \in \mathbb{R}^d} L_n(w) \leq \left\{ E \left[ L_n(Y^{(n)}(s, s^{2/3})) | S\right] - \pi_{\gamma(s)}, L_n(L_n) \right\} + \left\{ \pi_{\gamma(s)}, L_n(L_n) - \min_{w \in \mathbb{R}^d} L_n(w) \right\},$$

we obtain

$$E \left[ L_n(Y^{(n)}(s, s^{2/3})) | S\right] - \min_{w \in \mathbb{R}^d} L_n(w) \leq O_{\alpha_1} \left( 1 + \|x\|_{\mathbb{R}^d}^2 \right) \exp \left\{ - \frac{s^{2/3}}{e^{\Omega(x)}(\gamma(s))} + O_{\alpha_1}(\gamma(s)) \right\} + \frac{\log(\gamma(s) + 1)}{\gamma(s)}.$$

In particular, since $\gamma(s) = (\log)^3(s)$ for sufficiently large $s > 0$,

$$E[L_n(Z_{\alpha(s,s^{2/3})}) | S, Z_s^{(n)} = x] - \min_{w \in \mathbb{R}^d} L_n(w) \leq O_{\alpha_1} \left( \frac{1 + \|x\|_{\mathbb{R}^d}^2}{(\log)^4(s)} \right)$$

holds. In addition, $E[\min_{w \in \mathbb{R}^d} L_n(w)] \leq \min_{w \in \mathbb{R}^d} L_n(w)$ holds. Therefore, integrating both sides with respect to $P(Z_s^{(n)} \in dx)$ on $\{ x \in \mathbb{R}^d \mid L_n(x) \leq (\log)^4(s) \}$ and taking expectation on $S$, Lemma A.10 proves the desired result.

4.2. Proof of (2.9)

As in the proof of (2.8), for sufficiently large $s > 0$, we consider the decomposition

$$E \left[ L(Z_{\alpha(s,s^{2/3})}) - E \left[ L_n(Z_{\alpha(s,s^{2/3})}) \right] \right] = E \left[ L(Z_{\alpha(s,s^{2/3})}) - L_n(Z_{\alpha(s,s^{2/3})}); \{L_n(Z_s^{(n)}) \leq (\log)^4(s)\} \right]$$

$$+ E \left[ L(Z_{\alpha(s,s^{2/3})}) - L_n(Z_{\alpha(s,s^{2/3})}); \{L_n(Z_s^{(n)}) > (\log)^4(s)\} \right].$$

(4.3)

Then, the second term in the R.H.S of (4.3) has the desired bound. To derive the bound for the first term, fix arbitrarily $x \in \mathbb{R}^d$ so that $L_n(x) \leq (\log)^4(s)$ and suppose that $Y^{(n)}(s, \cdot)$ defined by (4.1) has an initial value $x$. Then, we have by Theorem 2.2

$$\left| E \left[ L(Y^{(n)}(s, s^{2/3})) | S\right] - E \left[ L_n(Y^{(n)}(s, s^{2/3})) | S\right] \right| \leq O_{\alpha_1} \left( \sqrt{\frac{\log(n + 1)}{n}} + (1 + \|x\|_{\mathbb{R}^d}^3) \frac{\log(\gamma(s) + 1)}{\gamma(s)} \right).$$

Therefore, we can prove (2.9) in a similar manner to the proof of (2.8).

4.3. Proof of (2.10)

(2.10) can be proved in the same way as Theorem 2.4 (2) in [26]. To explain this, we introduce the following notations. For $p > 0$, define $V_p : \mathbb{R}^d \rightarrow \mathbb{R}$ by $V_p(x) = \|x\|_{\mathbb{R}^d}^p$ and let $V_p(x) = 1 + V_p(x)$. For constants $C(p)$ and $\lambda(p)$ defined by

$$\lambda(p) = \frac{mp}{2}, \quad C(p) = \lambda(p) \left\{ \frac{2}{m} \left( \frac{d + p - 2}{\gamma(0)} + b \right) \right\}^{p/2},$$

(4.4)
let
\[ C = C(2) + \lambda(2), \quad \lambda = \lambda(2). \]

For sets
\[
S_1 := \{(x, y) \in \mathbb{R}^d \times \mathbb{R}^d \mid \bar{V}_2(x) + \bar{V}_2(y) \leq 2\lambda^{-1}C\}, \\
S_2 := \{(x, y) \in \mathbb{R}^d \times \mathbb{R}^d \mid \bar{V}_2(x) + \bar{V}_2(y) \leq 4C(1 + \lambda^{-1})\},
\]
let \( R_1 \) and \( R_2 \) be the diameters of \( S_1 \) and \( S_2 \), respectively, where the diameter of a set \( \Gamma \subset \mathbb{R}^d \) is defined by \( \sup_{x,y \in \Gamma} \|x - y\|_{\mathbb{R}^d} \).

Fix \( t > 0 \) and we define \( \kappa_t \) and \( Q(\kappa_t) \) by
\[
\kappa_t := \min \left\{ \frac{1}{2} \frac{2}{C\gamma(t)(e^{2R_1} - 1 - 2R_1)} \exp \left\{ -\frac{M\gamma(t)}{8} R_1^2 \right\} \right\} \in (0, 1) \tag{4.5}
\]
and
\[
Q(\kappa) := \sup_{x \in \mathbb{R}^d} \frac{\|\nabla \bar{V}_2(x)\|_{\mathbb{R}^d}}{\max \{V_2(x), \kappa \}} = \sup_{x \in \mathbb{R}^d} \frac{2\|x\|_{\mathbb{R}^d}}{\max \{1 + \|x\|_{\mathbb{R}^d}^2, \kappa \}} = 2\sqrt{\kappa_t - \kappa_t^2} \in (0, 1),
\]
respectively. In addition, we define functions \( \varphi_t, \Phi_t : [0, \infty) \to [0, \infty) \) by
\[
\varphi_t(r) := \exp \left( -\frac{M\gamma(t)}{8} r^2 - 2Q(\kappa_t)r \right), \quad \Phi_t(r) = \int_0^r \varphi_t(s)ds.
\]
For constants \( \zeta_t, \xi_t \) and \( c_t \) defined by
\[
\frac{1}{\zeta_t} := \int_0^{R_2} \Phi_t(s)\varphi_t(s)^{-1}ds, \quad \frac{1}{\xi_t} := \int_0^{R_1} \Phi_t(s)\varphi_t(s)^{-1}ds, \quad c_t := \min \left\{ \frac{\zeta_t}{\gamma(t)}, \frac{\lambda}{2}C\lambda\kappa_t \right\},
\]
let
\[
g_t(r) := 1 - \frac{\zeta_t}{4} \int_0^{\min\{r, R_2\}} \Phi_t(s)\varphi_t(s)^{-1}ds - \frac{\xi_t}{4} \int_0^{\min\{r, R_1\}} \Phi_t(s)\varphi_t(s)^{-1}ds.
\]
Furthermore, for
\[
f_t(r) := \begin{cases} \int_0^{\min\{r, R_2\}} \varphi_t(s)g_t(s)ds, & r \geq 0 \\ \xi_t, & r < 0 \end{cases},
\]
and \( U_t(x, y) := 1 + \kappa_t \bar{V}_2(x) + \kappa_t \bar{V}_2(y) \), let
\[
\rho_{2,t}(x, y) = f_t(||x - y||_{\mathbb{R}^d})U_t(x, y), \quad x, y \in \mathbb{R}^d.
\]
Finally, for probability measures \( \mu \) and \( \nu \) on \( \mathbb{R}^d \), denoting the set of all coupling between them by \( \Pi(\mu, \nu) \), let
\[
\mathcal{W}_{\rho_{2,t}}(\mu, \nu) := \inf_{\gamma \in \Pi(\mu, \nu)} \int_{\mathbb{R}^d \times \mathbb{R}^d} \rho_{2,t}(x, y)\gamma(dx, dy). \tag{4.7}
\]
Here, for random variables \( Z_1 \) and \( Z_2 \), \( \mathcal{W}_{\rho_{2}}(\mathcal{L}(X_1), \mathcal{L}(X_2)) \) may be abbreviated as \( \mathcal{W}_{\rho_{2}}(X_1, X_2) \).

With aforementioned notations, noting the monotonicity of the function \( \gamma \), we can prove the following in the same way as Proposition 5.1 in [20].

**Proposition 4.1.** For any \( 0 \leq s \leq t \), the following inequality holds.

\[
\mathcal{W}_{\rho_{2,t}}(Z^{(n)}_s, Z^{(n)}_{\eta,s}) \leq O_{\alpha_1} \left( 1 + \|x_0\|_{\mathbb{R}^d}^3 \right) e^{-cs} \int_0^s e^{cs}u \sum_{k=0}^{\infty} \sqrt{k+1} \chi(T_k, T_{k+1})(u)du. \tag{4.8}
\]
Proposition 4.1 proves (2.10) as follows. Combining Lemma A.15 in [26] and Proposition 4.1, we obtain

\[
|E[L(Z_{t}^{(n)})] - E[L(Z_{t}^{(q,n)})]| \leq O_{\alpha_{1}} \left(1 + \|x_{0}\|^{3}_{\mathbb{R}^{d}}\right)e^{O_{\alpha_{1}}(\gamma(t))}e^{-c_{t}t} \int_{0}^{t} e^{c_{t}u} \sum_{k=0}^{\infty} \sqrt{\eta_{k+1}} \chi_{(T_{k}, T_{k+1})}(u) du.
\]  \hspace{1cm} (4.9)

Let \( k \) be the natural number such that \( T_{k} < \sqrt{t} \leq T_{k+1} \). Then \( e^{\sqrt{t} - 1} \leq k + 1 \) since \( T_{k} = \sum_{j=1}^{k} j^{-1} \leq 1 + \log k \). Therefore,

\[
\int_{0}^{t} e^{c_{t}u} \sum_{k=0}^{\infty} \sqrt{\eta_{k+1}} \chi_{(T_{k}, T_{k+1})}(u) du = \int_{0}^{\sqrt{t}} e^{c_{t}u} \sum_{k=0}^{\infty} \sqrt{\eta_{k+1}} \chi_{(T_{k}, T_{k+1})}(u) du + \int_{\sqrt{t}}^{t} e^{c_{t}u} \sum_{k=0}^{\infty} \sqrt{\eta_{k+1}} \chi_{(T_{k}, T_{k+1})}(u) du
\]

\[
\leq c_{t}^{-1}(e^{c_{t}\sqrt{t} - 1} + c_{t}^{-1}e^{-\frac{\sqrt{t}}{\sqrt{2}}} (e^{c_{t}t} - e^{c_{t}\sqrt{t}}))
\]

holds. In addition, by (4.6) and Assumption 2.3

\[
c_{t} = \Omega_{\alpha_{1}} \left(\exp\left\{-\frac{|(\log)^{2}(t)|^{O_{\alpha_{1}}(1)}}{\gamma(t)}\right\}\right)
\]

holds for sufficiently large \( t > 0 \). Hence,

\[
|E[L(Z_{t}^{(n)})] - E[L(Z_{t}^{(q,n)})]| \leq (1 + \|x_{0}\|^{3}_{\mathbb{R}^{d}})O_{\alpha_{1}} \left(\exp\left\{-\frac{\Omega_{\alpha_{1}}(t - \sqrt{t})}{|(\log)^{2}(t)|^{O_{\alpha_{1}}(1)}} + \frac{|(\log)^{2}(t)|^{O_{\alpha_{1}}(1)}}{O_{\alpha_{1}}}\right\} + \exp\left\{\frac{1}{2} + \frac{|(\log)^{2}(t)|^{O_{\alpha_{1}}(1)}}{\gamma(t)} - \frac{\sqrt{t}}{2}\right\}\right).
\]

Taking \( t = \alpha(s, s^{2/3}) \), (2.10) follows from Lemma A.8.

\[\square\]

\section{Appendix}

\subsection{Difference between SGLD and SA}

In this subsection, we prepare a result on the approximation of SA by SGLD (Lemma A.9), which is a refinement of Lemma 2 in [13].

Let \( F \in C^{1}(\mathbb{R}^{d}; \mathbb{R}) \) be \((m, b)\)-dissipative and \( M \)-smooth, and fix \( s > 0 \). We define the SA and SGLD along with the gradient \( \nabla F \) of \( F \) by

\[
dZ_{t} = -\nabla F(Z_{t}) dt + \sqrt{2/\gamma(t)} dW_{t},
\]

\[
dY(s, t) = -\nabla F(Y(s, t)) dt + \sqrt{2/\gamma(s)} dW_{t},
\]

respectively. Furthermore, for given \( 0 < r_{0} < r_{1} < r_{2} \), let

\[
\Omega_{i, F} = \{x \in \mathbb{R}^{d} \mid F(x) \leq r_{i}\}, \quad \partial\Omega_{i, F} = \{x \in \mathbb{R}^{d} \mid F(x) = r_{i}\}
\]

\hspace{1cm} \hspace{1cm} (A.3)

and for two sets \( \Gamma_{1}, \Gamma_{2} \subset \mathbb{R}^{d} \), we denote the distance between them by

\[
\text{dist}(\Gamma_{1}, \Gamma_{2}) = \inf\{\|x - y\|_{\mathbb{R}^{d}} \mid x \in \Gamma_{1}, y \in \Gamma_{2}\}.
\]

Finally, for any \( s \geq 0 \), the solutions of (A.2) and

\[
dX_{t} = -\nabla F(X_{t}) dt,
\]

(A.5)

with the same initial values \( x \in \mathbb{R}^{d} \) are denoted by \( Y^{x}(s, \cdot) \) and \( X^{x} \), and the path of \( X^{x} \) until \( t \) is denoted by \( \Gamma_{X}^{x}(t) = \{X_{t}^{x} \mid 0 \leq s \leq t\} \).

\textbf{Lemma A.1.} (Lemma 2 in [24]) For any \( c \in (0, 1) \) and \( x \in \mathbb{R}^{d} \),

\[
F(cx) + \frac{1}{2}(1 - c^{2})m\|x\|^{2}_{\mathbb{R}^{d}} + b \log c \leq F(x) \leq F(0) + \frac{1}{2}\|\nabla F(0)\|^{2}_{\mathbb{R}^{d}} + \frac{M + 1}{2}\|x\|^{2}_{\mathbb{R}^{d}}
\]
holds. In particular, for $r > 0$, $F(x) \geq r$ and $F(x) \leq r$ indicate

$$
\|x\|_{\mathbb{R}^d}^2 \geq \frac{2}{M+1} \left( r - F(0) - \frac{1}{2}\|\nabla F(0)\|_{\mathbb{R}^d}^2 \right)
$$

(A.6)

and

$$
\|x\|_{\mathbb{R}^d}^2 \leq \frac{4}{m} \left( r + \frac{1}{2} b \log 2 - \inf_{w \in \mathbb{R}^d} F(w) \right),
$$

(A.7)

respectively.

Proof. By Taylor’s theorem

$$
F(x) - F(0) = \int_0^1 \langle x, \nabla F(tx) \rangle_{\mathbb{R}^d} dt \
\geq \int_0^1 \frac{1}{t} \langle tx, \nabla F(tx) \rangle_{\mathbb{R}^d} dt \geq \int_0^1 \frac{1}{t} \left\{ m \|x\|_{\mathbb{R}^d}^2 - b \right\} dt = \frac{1}{2}(1-c^2) m \|x\|_{\mathbb{R}^d}^2 + b \log c,
$$

$$
F(x) - F(0) = \int_0^1 \langle x, \nabla F(tx) \rangle_{\mathbb{R}^d} dt \leq \|x\|_{\mathbb{R}^d} \int_0^1 (\|\nabla F(0)\|_{\mathbb{R}^d} + M \|x\|_{\mathbb{R}^d}) dt = \|\nabla F(0)\|_{\mathbb{R}^d} \|x\|_{\mathbb{R}^d} + \frac{M}{2} \|x\|_{\mathbb{R}^d}^2
$$

hold. Taking $c = 1/\sqrt{2}$, the rest of the statement follows.

Lemma A.2. For any $\delta > 0$, let

$$
\tilde{r}_0(\delta) = \frac{M+1}{2} \delta + F(0) + \frac{1}{2} \|\nabla F(0)\|_{\mathbb{R}^d}^2.
$$

(A.8)

Then, if $x \in \mathbb{R}^d$ satisfies $F(x) \geq \tilde{r}_0(2b/m)$,

$$
\|\nabla F(x)\|_{\mathbb{R}^d}^2 - \frac{2}{\beta} \Delta F(x) \geq 0
$$

(A.9)

holds for any $\beta \geq 4Md/mb$.

Proof. According to (A.6), we have

$$
\|x\|_{\mathbb{R}^d}^2 \geq \frac{2}{M+1} \left( \tilde{r}_0(2b/m) - F(0) - \frac{1}{2}\|\nabla F(0)\|_{\mathbb{R}^d}^2 \right) = \frac{2b}{m}.
$$

Thus, by the $(m, b)$-dissipativity of $F$,

$$
\|\nabla F(x)\|_{\mathbb{R}^d} \geq \frac{1}{\|x\|_{\mathbb{R}^d}} (m \|x\|_{\mathbb{R}^d}^2 - b) = m \|x\|_{\mathbb{R}^d} - \frac{b}{\|x\|_{\mathbb{R}^d}} \geq \sqrt{\frac{mb}{2}}
$$

holds. On the other hand, $M$-smoothness of $F$ indicates $\Delta F(x) \leq Md$. Therefore, if $\beta \geq 4Md/mb$, then (A.9) holds.

Lemma A.3. For any $\delta > 0$, let

$$
\tilde{r}_1(r_0, \delta) = F(0) + \frac{1}{2} \|\nabla F(0)\|_{\mathbb{R}^d}^2 + \frac{4(M+1)}{m} \left( r_0 + \frac{m \delta^2}{4} + \frac{1}{2} b \log 2 - \inf_{w \in \mathbb{R}^d} F(w) \right).
$$

(A.10)

Then, if $r_1 \geq \tilde{r}_1(r_0, \delta)$, we have $\text{dist}(0, \Omega_{0,F} \cup \partial \Omega_{1,F}) \geq \delta$.

Proof. Fix arbitrarily $x \in \Omega_{0,F}$ and $v \in \mathbb{R}^d$ such that $\|v\|_{\mathbb{R}^d} < \delta$. Then, Lemma A.1 yields

$$
F(x + v) \leq F(0) + \frac{1}{2} \|\nabla F(0)\|_{\mathbb{R}^d}^2 + \frac{M+1}{2} \|x + v\|_{\mathbb{R}^d}^2
\leq \left( F(0) + \frac{1}{2} \|\nabla F(0)\|_{\mathbb{R}^d}^2 + (M+1)\|x\|_{\mathbb{R}^d}^2 + \delta^2 \right)
\leq \left( F(0) + \frac{1}{2} \|\nabla F(0)\|_{\mathbb{R}^d}^2 + \frac{4(M+1)}{m} \left( r_0 + \frac{m \delta^2}{4} + \frac{1}{2} b \log 2 - \inf_{w \in \mathbb{R}^d} F(w) \right) \right),
$$

and therefore $x + v \in \Omega_{1,F}$ cannot hold.

□
Lemma A.4. With the notation of (A.10), let \( r_1 \geq \hat{r}_1(r_0, 1) \). If we define
\[
\varepsilon = \frac{1}{2\sqrt{2}} \left\{ \| \nabla F(0) \|_{\mathbb{R}^d}^2 + \frac{4M^2}{m} \left( r_1 + \frac{1}{2} b \log 2 - \inf_{u \in \mathbb{R}^d} F(u) \right) \right\}^{-1/2}
\]  
and
\[
\delta_0 = \text{dist} \left( \Omega_{0,F}, \bigcup_{x \in \partial \Omega_{1,F}} \Gamma_{F}^x(\varepsilon) \right),
\]  
then, \( \delta_0 \geq 1/2 \) holds.
Proof. For \( x \in \partial \Omega_{1,F} \),
\[
F(X^x_t) = r_1 - \int_{0}^{t} \| \nabla F(X^x_s) \|_{\mathbb{R}^d}^2 \, ds, \quad t \geq 0
\]  
holds. In particular, by \( F(X^x_t) \leq r_1 \) and (A.7), we have
\[
\| \nabla F(X^x_t) \|_{\mathbb{R}^d}^2 \leq 2 \| \nabla F(0) \|_{\mathbb{R}^d}^2 + 2M^2 \| X^x_t \|_{\mathbb{R}^d}^2 \leq 2 \| \nabla F(0) \|_{\mathbb{R}^d}^2 + \frac{8M^2}{m} \left( r_1 + \frac{1}{2} b \log 2 - \inf_{u \in \mathbb{R}^d} F(u) \right).
\]  
Thus, for any \( 0 \leq s \leq \varepsilon \),
\[
\| X^x_s - x \|_{\mathbb{R}^d} \leq \varepsilon \sup_{u \geq 0} \| \nabla F(X^x_u) \|_{\mathbb{R}^d} \leq \frac{1}{2}
\]  
holds. In particular, \( x \in \partial \Omega_{1,F} \) indicates \( \text{dist} \left( \partial \Omega_{1,F}, \bigcup_{x \in \partial \Omega_{1,F}} \Gamma_{F}^x(\varepsilon) \right) \leq 1/2 \). Therefore,
\[
\text{dist} \left( \Omega_{0,F}, \bigcup_{x \in \partial \Omega_{1,F}} \Gamma_{F}^x(\varepsilon) \right) \geq \text{dist}(\Omega_{0,F}, \partial \Omega_{1,F}) - \text{dist} \left( \partial \Omega_{1,F}, \bigcup_{x \in \partial \Omega_{1,F}} \Gamma_{F}^x(\varepsilon) \right)
\]  
and Lemma A.3 complete the proof.

Lemma A.5. Let \( r_0 = r_0(s) = (\log)^4(s) \) and let \( r_1 = r_1(s) = \hat{r}_1(r_0(s), 1) \). Then, for sufficiently large \( s > 0 \), \( \varepsilon \) defined by (A.11) satisfies
\[
\frac{1}{(\log)^2(s)} \leq \varepsilon \leq e^{-2M} \leq 1.
\]  
Proof. According to (A.10), for sufficiently large \( s > 0 \), we have \( 3m^{-1}(M + 1)r_0(s) \leq r_1(s) \leq 5m^{-1}(M + 1)r_0(s) \). Therefore, \( \varepsilon \geq ((\log)^2(s))^{-1} \) holds by (A.11).

Lemma A.6. For any \( \delta > 0 \), let \( \xi(\delta) = \inf \{ t \geq 0 \mid \| X^x_t - Y^x(s,t) \|_{\mathbb{R}^d} \geq \delta \} \). Then, we have
\[
P(\xi(\delta) < t) \leq \frac{4e^{Mt} \delta^2}{\pi \gamma(s)} \exp \left\{ -\frac{e^{-2M\delta^2\gamma(s)}}{4td^2} \right\}, \quad t > 0.
\]  
Proof. By the definitions of \( X^x \) and \( Y^x(s,t) \),
\[
\| X^x_t - Y^x(s,t) \|_{\mathbb{R}^d} \leq M \int_{0}^{t} \| X^x_u - Y^x(s,u) \|_{\mathbb{R}^d} du + \sqrt{\frac{2}{\gamma(s)}} \| W_t \|_{\mathbb{R}^d}
\]  
holds, and therefore we obtain
\[
\| X^x_t - Y^x(s,t) \|_{\mathbb{R}^d} \leq \sqrt{\frac{2}{\gamma(s)}} e^{Mt} \max_{0 \leq s \leq t} \| W_u \|_{\mathbb{R}^d}
\]  
by Gronwall’s lemma. Thus, \( P(\xi(\delta) < t) \leq P(\max_{0 \leq u \leq t} \| X^x_u - Y^x(s,u) \|_{\mathbb{R}^d} \geq \delta) \leq P(\max_{0 \leq u \leq t} \| W_u \|_{\mathbb{R}^d} \geq e^{-M\delta^2\gamma(s)/2}) \). Applying Problem 2.8.3 in [15] to the R.H.S. of
\[
P \left( \max_{0 \leq u \leq t} \| W_u \|_{\mathbb{R}^d} \geq e^{-M\delta^2\gamma(s)/2} \right) \leq \sum_{i=1}^{d} P \left( \max_{0 \leq u \leq t} \| W_{i,u} \| \geq e^{-M\delta^2\gamma(s)/2d^2} \right),
\]  
we obtain the desired result.
Lemma A.7. Let \( r_0 = r_0(s) \) and \( r_1 = r_1(s) \) be the same as in Lemma A.4 and let \( r_2 = r_2(s) = r_1(s) + 6 \). In addition, for any continuous process \( V \), let

\[
\tau(V) = \inf\{t \geq 0 \mid V_t \notin \Omega_{2, r}\}. \tag{A.16}
\]

Then, for sufficiently large \( s > 0 \), the following inequality holds.

\[
P(\tau(Y^x(s, \cdot)) < (\log)^2(s)) \leq \frac{2}{(\log)^2(s)} \left( 1 + \frac{8eM\delta^2}{\sqrt{\pi c(s)}} \right), \quad x \in \Omega_{0, r}. \tag{A.17}
\]

**Proof.** In this proof, we denote the underlying filtration as \( \{\mathcal{F}_t\} \). First, we only have to show \((A.17)\) for \( x \in \partial\Omega_{1, r} \).

In fact, denoting \( \theta(V) = \inf\{t \geq 0 \mid V_t \in \partial\Omega_{1, r}\} \), \( \theta(V) \leq \tau(V) \) holds for \( x \in \Omega_{0, r} \). Thus, if \((A.17)\) is true for all \( x \in \partial\Omega_{1, r} \), then

\[
P(\tau(Y^x(s, \cdot)) < (\log)^2(s)) = E\left[ P(\tau(Y^x(s, \cdot)) < (\log)^2(s) \mid \mathcal{F}_\theta) \mid \{\theta < (\log)^2(s)\} \right],
\]

and therefore the strong Markov property of \( Y^x(s, \cdot) \) yields the desired result.

To show \((A.17)\) for \( x \in \partial\Omega_{1, r} \), we define the sequence of stopping times as \( \sigma_0(V) = 0, \theta_0(V) = 0 \) and

\[
\sigma_{i+1}(V) = \inf\{t > \theta_i(V) \mid V_t \in \Omega_{0, r}\}, \quad \theta_i(V) = \inf\{t > \sigma_i(V) \mid V_t \notin \Omega_{1, r}\}, \quad i \geq 1.
\]

Let

\[
Q_F(t, V) = \exp\left\{ \frac{\gamma(s)}{2} F(V_t) - \frac{\gamma(s)}{2} F(V_0) - \frac{1}{2} \int_0^t \Delta F(V_s) ds + \frac{\gamma(s)}{4} \int_0^t \|\nabla F(V_s)\|_{\mathbb{R}^d}^2 ds \right\}.
\]

Then, by Itô’s formula, we have

\[
Q_F(t, Y^x(s, \cdot)) = \exp\left\{ \sqrt{\frac{\gamma(s)}{2}} \int_0^t (\nabla F(Y^x(s, u)), dW_u)_{\mathbb{R}^d} - \frac{\gamma(s)}{4} \int_0^t \|\nabla F(Y^x(s, u))\|_{\mathbb{R}^d}^2 du \right\}.
\]

Therefore, By Girsanov’s theorem, \( Y^x(s, \cdot) \) on \([0, \tau(Y^x(s, \cdot))]) \) under the \( Q_F(\tau(Y^x(s, \cdot))), Y^x(s, \cdot) dP \) has the same distribution as \( x + \sqrt{2/\gamma(s)} W \).

On the other hand, by \( x \in \partial\Omega_{1, r} \), if a continuous process \( V \) satisfies \( V_0 = x \), then \( F(V_u) \geq r_0(s) \) holds for any \( u \leq \sigma_1(V) \). Thus, for sufficiently large \( s > 0 \), Lemma A.4 yields

\[
\|\nabla F(V_u)\|_{\mathbb{R}^d}^2 \geq \frac{2}{\gamma(s)} \Delta F(V_u) \geq 0.
\]

Therefore, by \( \gamma(s) = (\log)^3(s) \) and \( r_2(s) - r_1(s) = 6 \), the following inequality holds on \( \{\tau(V) < \sigma_1(V)\} \).

\[
Q_F(\tau(V), V)^{-1} = \exp\left\{ \frac{\gamma(s)}{2} F(x) - \frac{\gamma(s)}{2} F(V(\tau(V))) + \frac{1}{2} \int_0^{\tau(V)} \Delta F(V_u) du - \frac{\gamma(s)}{4} \int_0^{\tau(V)} \|\nabla F(V_u)\|_{\mathbb{R}^d}^2 du \right\}
\]

\[
\leq \frac{1}{(\log^2(s))^3}.
\]

In particular, since \( \{\tau(Y^x(s, \cdot)) < \sigma_1(Y^x(s, \cdot))\} \in \mathcal{F}_{\tau(Y^x(s, \cdot))} \), denoting \( \tilde{V} = x + \sqrt{2/\gamma(s)} W \), we have

\[
P(\tau(Y^x(s, \cdot)) < \sigma_1(Y^x(s, \cdot))) = E\left[ Q_{F, \sigma_1}(\tau(V), \tilde{V})^{-1} \mid \{\tau(V) < \sigma_1(V)\} \right] \leq \frac{1}{(\log^2(s))^3}.
\]

Combining this inequality and the strong Markov property of \( Y^x(s, \cdot) \), for all \( k \in \mathbb{N} \), we obtain

\[
P(\tau(Y^x(s, \cdot)) < \sigma_k(Y^x(s, \cdot))) = \sum_{i=1}^{k} P(\sigma_{i-1}(Y^x(s, \cdot)) \leq \tau(Y^x(s, \cdot)) < \sigma_i(Y^x(s, \cdot)))
\]

\[
= \sum_{i=1}^{k} E\left[ P(\tau(Y^x(s, \cdot)) < \sigma_i(Y^x(s, \cdot)) \mid \mathcal{F}_{\theta_{i-1}(Y^x(s, \cdot))}; \{\sigma_{i-1}(Y^x(s, \cdot)) \leq \tau(Y^x(s, \cdot))\} \right] \leq \frac{k}{(\log^2(s))^3}. \tag{A.18}
\]
If we define ε and δ₀ as (A.11) and (A.12), respectively, then ζ(δ₀) defined in Lemma A.6 satisfies $P(σ₁(Y^x(s,·)) < ε) ≤ P(ζ(δ₀) < ε)$. In fact, since $Y^x(s, t₀) ∈ Ω₀,F$ for $t₀ = σ₁(Y^x(s,·))$, if $t₀ < ε$, then by the definition of δ₀

$$||Y^x(s, t₀) - X^x_{t₀}||_{R^d} ≥ \text{dist} \left(Ω₀,F, \bigcup_{y ∈ Ω₀,F} Γ^y_{t₀}(t₀) \right) ≥ δ₀$$

holds. Therefore, ζ(δ₀) ≤ t₀ < ε by the definition of ζ(δ₀). On the other hand, for sufficiently large $s > 0$, ε satisfies (A.15). Thus, Lemmas A.4 and A.6 yield

$$P(σ₁(Y^x(s,·)) < ε) ≤ \frac{8e^M d²}{(\log)^2(s)^3 \sqrt{πγ(s)}} \exp \left\{ -\frac{2Mγ(s)}{16εd²} \right\} ≤ \frac{8e^M d²}{(\log)^2(s)^3 \sqrt{πγ(s)}}$$

Whereas, for any $k ∈ N$, we have

$$σ_k(Y^x(s,·)) = σ₁(Y^x(s,·)) + \sum_{i=1}^{k-1} (σ_{i+1}(Y^x(s,·)) - σ_i(Y^x(s,·))) ≥ σ₁(Y^x(s,·)) + \sum_{i=1}^{k-1} (σ_{i+1}(Y^x(s,·)) - θ_i(Y^x(s,·))).$$

Thus, on the event \{σ_k(Y^x(s,·)) ≤ kε\}, there exists at least one $0 ≤ i ≤ k - 1$ such that $σ_{i+1}(Y^x(s,·)) - θ_i(Y^x(s,·)) < ε$. Therefore, since $P(σ_{i+1}(Y^x(s,·)) - θ_i(Y^x(s,·)) < ε) ≤ \sup_{y ∈ Ω₀,F} P(σ₁(Y^x(s,·)) < ε)$ by the strong Markov property of $Y^x(s,·)$,

$$P(σ_k(Y^x(s,·)) < kε) ≤ \frac{8ke^M d²}{(\log)^2(s)^3 \sqrt{πγ(s)}} \quad (A.19)$$

holds. Combining (A.8) and (A.10), we obtain

$$P(τ(Y^x(s,·)) < kε) ≤ P(τ(Y^x(s,·)) < σ_k(Y^x(s,·))) + P(σ_k(Y^x(s,·)) < kε) ≤ \frac{k}{(\log)^2(s)^3} \left(1 + \frac{8e^M d²}{\sqrt{πγ(s)}}\right).$$

As a result, taking $k ∈ N$ so that $(\log)^2(s)^2 ≤ k < (\log)^2(s)^2 + 1$, since $(\log)^2(s) ≤ kε$ holds by (A.15), we obtain

$$P(τ(Y^x(s,·)) < (\log)^2(s)) ≤ P(τ(Y^x(s,·)) < kε) ≤ \frac{(\log)^2(s)^2 + 1}{(\log)^2(s)^3} \left(1 + \frac{8e^M d²}{\sqrt{πγ(s)}}\right) ≤ \frac{2}{(\log)^2(s)} \left(1 + \frac{8e^M d²}{\sqrt{πγ(s)}}\right),$$

as desired. \square

**Lemma A.8.** The function $α(s,·)$ defined by (2.7) satisfies $α(s, t) ≥ s + t$ for any $t ≤ s$.

**Proof.** For each fixed $s ≥ 0$, the map $r → ∫_s^r \frac{γ(u)}{γ(u)} du$ tends to infinity as $r → ∞$. Thus, $α(s, t)$ is well-defined as the inverse of strictly increasing continuous function. By the monotonicity of $γ(t)$,

$$t = ∫_s^{α(s,t)} \frac{γ(u)}{γ(u)} du ≤ α(s, t) - s,$$

holds, and therefore $s + t ≤ α(s, t)$.

For sufficiently large $s > 0$, we have $γ(s) = (\log)^3(s)$ and $2(\log)^3(s) ≥ (\log)^3(3s)$. Thus, for $t ≤ s$,

$$∫_s^{s+2t} \frac{(\log)^3(s)}{(\log)^3(u)} du ≥ \frac{2t(\log)^3(s)}{(\log)^3(s + 2t)} ≥ t$$

holds. Therefore, $α(s, t) ≤ s + 2t$ follows from the definition of $α(s, t)$. \square

**Lemma A.9.** Let $H ∈ C^1(ℝ^d; ℝ)$ be $M$-smooth. If $r₀(s) = (\log)^4(s)$ and $h(s) ≤ s^{2/3}$, then for any $x ∈ Ω₀,F$,

$$|E_{s,x}[H(Z_{α(s,h(s))})] - E[H(Y^x(s, h(s)))]| ≤ O_{m,b,M,γ(0),H(0),∥∇H(0)∥_{R^d},F(0),∥∇F(0)∥_{R^d},d} \left(1 + \frac{∥x∥_{R^d}}{(\log)^2(s)}\right)$$

holds. Here, $E_{s,x}[·] = E[· | Z_s = x]$. 11
Proof. According to Lévy’s theorem,
\[ \tilde{W}_t := \sqrt{\frac{\gamma(s)}{2}} \int_s^t \sqrt{\frac{2}{\gamma(u)}} dW_u \]
is a new Brownian motion with respect to the time changed filtration. Setting \( u = \alpha(s, v) \), we have
\[ \int_s^t \nabla F(Z_u) du = \int_0^t \frac{\alpha(s, u)}{\gamma(s)} \nabla F(Z(s, u)) du, \quad \int_s^t \sqrt{\frac{2}{\gamma(u)}} dW_u = \sqrt{\frac{2}{\gamma(s)}} \tilde{W}_t. \]
Thus, when \( Z_s = x \), \( \tilde{Z}(s, t) = Z_{\alpha(s, t)} \) satisfies
\[ \tilde{Z}(s, t) = x - \int_0^t \frac{\alpha(s, u)}{\gamma(s)} \nabla F(Z(s, u)) du + \sqrt{\frac{2}{\gamma(s)}} \tilde{W}_t. \tag{A.20} \]
To apply the result of Section 7.6.4 in [18] to (A.20) and
\[ Y^x(s, t) = x - \int_0^t \nabla F(Y^x((s, u)) du + \sqrt{\frac{2}{\gamma(s)}} \tilde{W}_t, \]
let
\[ S_1(t) = -\sqrt{\frac{\gamma(s)}{2}} \int_0^t \left( \frac{\alpha(s, u)}{\gamma(s)} - 1 \right) \| \nabla F(Y^x(s, u)) \|^2 \|e^e \| du, \]
\[ S_2(t) = \frac{\gamma(s)}{2} \int_0^t \left( \frac{\alpha(s, u)}{\gamma(s)} - 1 \right)^2 \| \nabla F(Y^x(s, u)) \|^2 \|e^e \| du, \]
and let \( Q(t) = \exp \{ S_1(t) - \frac{1}{2} S_2(t) \} \). Then for \( \tau(Y^x(s, \cdot)) \) defined by (A.10),
\[ E_{s,x}[H(\tilde{Z}(s, h(s)))] - E[H(Y^x(s, h(s)))] = \left\{ E_{s,x}[H(\tilde{Z}(s, h(s))): \{ \tau(\tilde{Z}) \geq h(s) \}] - E[H(Y^x(s, h(s))): \{ \tau(Y^x(s, \cdot)) \geq h(s) \}] \right\} + \left\{ E[H(\tilde{Z}(s, h(s))): \{ \tau(\tilde{Z}) < h(s) \}] - E[H(Y^x(s, h(s))): \{ \tau(Y^x(s, \cdot)) < h(s) \}] \right\} = I_1 + I_2 \]
holds. In the rest of proof, we bound each of \( I_1 \) and \( I_2 \).

First, we bound \( I_1 \). Since \( Q(t) \) is a martingale on \( [0, \tau(Y^x(s, \cdot))] \), by Lemma A.10, we have
\[ |I_1| \leq \sqrt{E[H(Y^x(s, h(s)))]} \sqrt{E[|Q(h(s) \wedge \tau(Y^x(s, \cdot)) - 1|^2]} \leq O_m, b, M, \gamma(0), H(0), ||\nabla H(0)||_{l^d}, ||\nabla F(0)||_{l^d}, (1 + ||x||_{l^d}) \sqrt{E[|Q(h(s) \wedge \tau(Y^x(s, \cdot)) - 1|^2]}. \]
When \( h(s) \leq \tau(Y^x(s, \cdot)) \), setting \( v = \alpha(s, u) \), we obtain
\[ S_2(h(s) \wedge \tau(Y^x(s, \cdot))) \leq \frac{\gamma(s)}{2} \left( ||\nabla F(0)||_{l^d} + M_{r_2}(s) \right)^2 \int_s^{h(s) \wedge \tau(Y^x(s, \cdot))} \left( \frac{\alpha(s, u)}{\gamma(s)} - 1 \right)^2 du \]
\[ = \frac{\gamma(s)}{2} \left( ||\nabla F(0)||_{l^d} + M_{r_2}(s) \right)^2 \int_s^{\alpha(s, h(s))} \left( \frac{\alpha(u)}{\gamma(s)} - 1 \right)^2 \frac{\gamma(s)}{\gamma(u)} du \]
\[ \leq \frac{1}{2 \gamma(s)} \left( ||\nabla F(0)||_{l^d} + M_{r_2}(s) \right)^2 \int_s^{\alpha(s, t)} (\alpha(u) - \gamma(s))^2 du. \]
Furthermore, since we have \( 0 \leq \log(1 + r) \leq r \) for all \( r \geq 0 \),
\[ |(\log)^k(u) - (\log)^k(s)| = \left| \log \left( 1 + \frac{(\log)^{k-1}(u)}{(\log)^{k-1}(s)} - 1 \right) \right| \leq \frac{1}{(\log)^{k-1}(s)} |(\log)^{k-1}(u) - (\log)^{k-1}(s)| \]
12
holds for any \( k \). Therefore, since \( \gamma(s) = (\log)^3(s) \) and \( r_1(s) \leq O_{m,b,M,F(0),\|\nabla F(0)\|_{\text{ad}}} ((\log)^4(s)) \) hold for sufficiently large \( s > 0 \), Lemma \( A.8 \) yields

\[
S_2(h(s) \wedge \tau(Y^x(s, \cdot))) \leq O_{m,b,M,F(0),\|\nabla F(0)\|_{\text{ad}}} \left( \frac{|(\log)^4(s)|}{(\log)^3(s)} \int_s^{\alpha(s,h(s))} (u - (\log)^3(s))^2 \, du \right) \\
\leq O_{m,b,M,F(0),\|\nabla F(0)\|_{\text{ad}}} \left( \frac{|(\log)^4(s)|}{s^2 \log s^2 (\log)^2(s)^2(\log)^3(s)} \right) \left( \int_s^{\alpha(s,h(s))} (u - s)^2 \, du \right) \\
\leq O_{m,b,M,F(0),\|\nabla F(0)\|_{\text{ad}}} \left( \frac{|(\log)^4(s)|}{\log s^2 (\log)^2(s)^2(\log)^3(s)} \right).
\]

Whereas, by the martingale property of \( \tilde{Q}(t) \) = \( \exp \{ 2S_1(t) - 2S_2(t) \} \) on \([0, \tau(Y^x(s, \cdot))]\), we obtain

\[
E[|Q(h(s) \wedge \tau(Y^x(s, \cdot))) - 1|^2] = E[\tilde{Q}(h(s) \wedge \tau(Y^x(s, \cdot))) (\exp \{ S_2(h(s) \wedge \tau(Y^x(s, \cdot))) \} - 1)].
\]

As a result, since we have by \( e^r - 1 \leq re^r \)

\[
E[|Q(h(s) \wedge \tau(Y^x(s, \cdot))) - 1|^2] = E[|Q(h(s) \wedge \tau(Y^x(s, \cdot))) - 1|^2] \\
\leq O_{m,b,M,F(0),\|\nabla F(0)\|_{\text{ad}}} \left( \frac{|(\log)^4(s)|}{\log s^2 (\log)^2(s)^2(\log)^3(s)} \right)
\]

for sufficiently large \( s > 0 \),

\[
|I_1| \leq O_{m,b,M,\gamma(0),H(0),\|\nabla H(0)\|_{\text{ad}},F(0),\|\nabla F(0)\|_{\text{ad}},d} \left( \frac{1 + \|x\|_{\text{ad}}}{(\log)^2(s)} \right) \quad (A.21)
\]

holds as desired.

Finally, we bound \( I_2 \). Applying \( A.21 \) to \( H = 1 \), we obtain

\[
|P(\tau(\tilde{Z}) < h(s)) - P(\tau(Y^x(s, \cdot)) < h(s))| = |P(\tau(\tilde{Z}) \geq h(s)) - P(\tau(Y^x(s, \cdot)) < h(s))| \\
\leq O_{m,b,M,\gamma(0),F(0),\|\nabla F(0)\|_{\text{ad}},d} \left( \frac{1 + \|x\|_{\text{ad}}}{(\log)^2(s)} \right).
\]

Therefore, by Lemmas \( A.7 \) and \( A.10 \)

\[
|I_2| \leq \sqrt{E[H(\tilde{Z}(s,h(s)))^2]} \sqrt{P(\tau(\tilde{Z}) < h(s))} + \sqrt{E[H(Y^x(s,h(s)))^2]} \sqrt{P(\tau(Y^x(s, \cdot)) < h(s))} \\
\leq O_{m,b,M,\gamma(0),H(0),\|\nabla H(0)\|_{\text{ad}},F(0),\|\nabla F(0)\|_{\text{ad}},d} \left( \frac{1 + \|x\|_{\text{ad}}}{\sqrt{(\log)^2(s)}} \right)
\]

holds, and therefore the proof is completed. \( \square \)

### A.2. Moment bound

The following two lemmas can be proved in the similar manners to \( A.4 \) noting that \( \gamma \) and \( \eta \) are monotonic.

**Lemma A.10.** (Lemma A.4 in \( A.4 \)) Let \( p \geq 2 \) and let \( F \in C^1(\mathbb{R}^d; \mathbb{R}) \) be \((m,b)\)-dissipative and \( M\)-smooth. Suppose that \( Z \) is the solution of

\[
dZ_t = -\nabla F(X_t)dt + \sqrt{2/\gamma(t)}dW_t
\]

with initial value \( Z_0 \in L^p(\Omega; \mathbb{R}^d) \). Then, for any \( t \geq 0 \),

\[
E[\|Z_t\|_{\text{ad}}^p] \leq e^{-\lambda(p)t} E[\|Z_0\|_{\text{ad}}^p] + \frac{C(p)}{\lambda(p)} (1 - e^{-\lambda(p)t}) \quad (A.22)
\]

holds. Here, \( C(p) \) and \( \lambda(p) \) are constants defined by \( A.4 \).
Lemma A.11. (Lemma A.5 in [26]) Assume that $F_k \in C^1(\mathbb{R}^d; \mathbb{R})$ is $(m, b)$-dissipative and $M$-smooth for each $k$ and satisfies $\sup_{k \in \mathbb{N}} \|\nabla F_k(0)\|_{\mathbb{R}^d} \leq A$. For a sequence $\eta = \{\eta_k\}_{k=1}^{\infty}$ that decreases to 0, let $Z^{(n)}$ be the process defined by

$$dZ_t^{(n)} = -\nabla F_k(Z_t^{(n)}(\eta)) + \sqrt{2/\gamma(t)}dW_t.$$ 

Then, for all $\ell \in \mathbb{N}$,

$$\sup_{t \geq 0} E[\|Z_t^{(n)}\|^2_{\mathbb{R}^d}] \leq O_{m,b,M,\gamma, A,d,\ell, \eta}(1 + E[\|Z_0\|^2_{\mathbb{R}^d}]).$$

holds.

A.3. Results on generalization bound

Theorem A.12. (Theorem 4.5 in [25]) Let $F$ be a family of functions from $Z$ to $\mathbb{R}$. Denoting

$$\|f - \tilde{f}\|_{1,S} := \frac{1}{n} \sum_{i=1}^{n} |f(z_i) - \tilde{f}(z_i)|, \quad f, \tilde{f} \in F,$$

let, $C(F, \varepsilon, \| \cdot \|_{1,S})$ be the size of minimal $\varepsilon$-cover of $F$ with respect to $\| \cdot \|_{1,S}$. Then, if

$$\sup_{f \in F} \left( \frac{1}{n} \sum_{i=1}^{n} f(z_i)^2 \right)^{1/2} \leq c$$

holds, we have

$$\hat{R}_n(F, S) \leq \inf_{\varepsilon > 0} \left( \varepsilon + \frac{c \sqrt{2}}{\sqrt{n}} \sqrt{\log C(F, \varepsilon, \| \cdot \|_{1,S})} \right),$$

where for IID $\sigma_1, \ldots, \sigma_n$ satisfying $P(\sigma_i = 1) = P(\sigma_i = -1) = 1/2$, the empirical Rademacher complexity $\hat{R}_n(F, S)$ is defined by

$$\hat{R}_n(F, S) = \frac{1}{n} E \left[ \sup_{f \in F} \sum_{i=1}^{n} \sigma_i f(z_i) \right]. \quad (A.23)$$

Theorem A.13. (Theorem 4.1 in [25]) Let $R_n(F) = E[\hat{R}_n(F, S)]$. Then we have

$$E \left[ \sup_{f \in F} \left| E[f(z_1)] - \frac{1}{n} \sum_{i=1}^{n} f(z_i) \right| \right] \leq 4R_n(F).$$

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