Existence, uniqueness and optimal decay rates for the 3D compressible Hall-magnetohydrodynamic system

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\textbf{Abstract} We are concerned with the study of the Cauchy problem to the 3D compressible Hall-magnetohydrodynamic system. We first establish the unique global solvability of strong solutions to the system when the initial data are close to a stable equilibrium state in critical Besov spaces. Furthermore, under a suitable additional condition involving only the low frequencies of the data and in $L^2$-critical regularity framework, we exhibit the optimal time decay rates for the constructed global solutions. The proof relies on an application of Fourier analysis to a mixed parabolic-hyperbolic system, and on a refined time-weighted energy functional.

\textbf{Key words:} well-posedness; optimal decay rates; compressible Hall-magnetohydrodynamic system; Besov spaces.

\section{Introduction and main results}

In this paper, we consider the following 3D compressible Hall-magnetohydrodynamic equations [1]:

\begin{equation}
\begin{aligned}
\partial_t \rho + \text{div}(\rho u) &= 0, \\
\partial_t (\rho u) + \text{div}(\rho u \otimes u) - \mu \Delta u - (\lambda + \mu) \nabla \text{div} u + \nabla p &= (\nabla \times H) \times H, \\
\partial_t H - \nabla \times (u \times H) + \nu \nabla \times (\nabla \times H) + \nabla \times \left( \frac{(\nabla \times H) \times H}{\rho} \right) &= 0, \\
\text{div} H &= 0,
\end{aligned}
\end{equation}

where $\rho(t, x), u(t, x), H(t, x)$ denote, respectively, the density, velocity, and magnetic field. $p = p(\rho)$ is pressure satisfying $p'(\rho) > 0$ and for all $\rho > 0$. The Lamé coefficients $\mu$ and $\lambda$ satisfy the physical conditions

\begin{equation}
\mu > 0, \quad 2\mu + 3\lambda > 0,
\end{equation}

\textsuperscript{*}Research supported by the National Natural Science Foundation of China (11501332,11171034,11371221), the Natural Science Foundation of Shandong Province (ZR2015AL007), and Young Scholars Research Fund of Shandong University of Technology.

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which ensures that the operator $-\mu\Delta - (\lambda + \mu)\nabla \text{div}$ is a strongly elliptic operator and $\nu > 0$ is the magnetic diffusivity acting as a magnetic diffusion coefficient of the magnetic field. Here, we simply set $\nu = 1$ since its size does not play any role in our analysis. In this paper, we are concerned with the Cauchy problem of the system (1.1) in $\mathbb{R}_+ \times \mathbb{R}^3$ subject to the initial data

$$(\rho, u, H)|_{t=0} = (\rho_0, u_0, H_0).$$

(1.3)

In many current physics problems, Hall-MHD is required. For example, when magnetic shear is large, which precisely occurs during reconnection events, the influence of the Hall term becomes dominant. However, this term is usually small and can be neglected in laminar situations, which is why conventional MHD models ignore it. When $\rho = \text{const}$, system (1.1) becomes incompressible Hall-MHD system, which has received many studies. The first systematic study of Hall-MHD is due to Lighthill [24] followed by Campos [4]. The Hall-MHD is indeed needed for such problems as magnetic reconnection in space plasmas [22], star formation [3], and neutron stars [27]. A physical review on these questions can be found in [26]. Mathematical derivations of Hall-MHD equations from either two-fluids or kinetic models can be found in [1] and in this paper, the first existence result of global weak solutions is given. In recent years, a number of works have been dedicated to this model, we refer readers to Refs [1, 5, 6, 7, 16, 17, 18, 30] for more discussions. When $\nabla \cdot u = 0$, system (1.1) becomes the density-dependent Hall-MHD system, which has been investigated by many authors, and for more details, see [15, 20].

When the Hall effect term $\nabla \times \left( \frac{1}{\rho}(\nabla \times H) \times H \right)$ is neglected, system (1.1) reduces to well-known compressible MHD system. There are many results regarding the global existence of the solutions and the decay of the smooth solutions to the compressible MHD equations, see [23, 28, 29] and references therein. However, to the best of our knowledge, very few results have been established on the dynamics of the global solutions to the 3D compressible Hall-MHD system, especially on the temporal decay of the solutions. Recently, Fan et al. [16] first obtained the global existence and the optimal decay rates of smooth solutions to the 3D compressible Hall-MHD equations (1.1) where the initial data are close to an equilibrium state in $H^3(\mathbb{R}^3_x)$ and belong to $L^1(\mathbb{R}^3_x)$. Later, Gao and Yao [21] improved the result from [16]. They proved the global existence of strong solutions by the standard energy method under the condition that the initial data are close to the constant equilibrium state in the lower regular spaces $H^2(\mathbb{R}^3_x)$ and obtained optimal decay rates for the constructed global strong solutions in $L^2$-norm if the initial data belong to $L^1$ additionally. More recently, Xu et al. [31] proved the global existence and temporal decay rates of the solutions to the system (1.1) when the initial data are close to a stable equilibrium state in $H^3(\mathbb{R}^3_x) \cap \dot{H}^{-s}(\mathbb{R}^3_x)$ for some $s \in [0, 3/2)$ by using a pure energy method. Obviously, all these results for compressible Hall-MHD equations (1.1) need higher smoothness for the initial data. The price to pay however is that assuming higher smoothness precludes from using a critical function spaces framework. To
our knowledge, few results have been established for the 3D compressible Hall-MHD equations in critical spaces. In the present article, we first study the global well-posedness in critical regularity framework. In order to tackle the problem, we need make scaling analysis of the system (1.1). Different from compressible MHD equations, Hall-term breaks the natural scaling of the Navier-Stokes equations. In fact, if we set the density $\rho = \text{const}$ and the fluid velocity $u \equiv 0$, then (1.1) is reduced to
\[
\partial_t H - \nabla \times \left( (\nabla \times H) \times H \right) = 0, \quad \text{in} \quad \mathbb{R}_+ \times \mathbb{R}^3.
\] (1.4)
Assume that magnetic field $H$ satisfy (1.4), then the function $H_l(x, t) = H(lx, l^2t)$ for $l > 0$ form a solution to (1.4) again. So, $\nabla H$ of (1.4) has the same scaling with the fluid velocity $u$ to the usual Navier-Stokes equations. Motivated by the scaling of 3D compressible Navier-Stokes equations and (1.4), we find that the function space for the velocity $u$ is similar to $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ and the regularity of the density $c = \rho - \bar{\rho}$ and the magnetic field $H$ is one order higher than that of the velocity $u$, and hence a function space for $c$ and $H$ which is similar to $\dot{H}^{\frac{3}{2}}$ would be a candidate. Unfortunately, the function space $\dot{H}^{\frac{3}{2}}(\mathbb{R}^3)$ does not turn to be a good candidate for $c$ and $H$ since $\dot{H}^{\frac{3}{2}}(\mathbb{R}^3)$ is not included in $L^\infty$. Thus, it seems more natural to choose homogeneous Besov space $\dot{B}^{\frac{3}{2}}_{2,1}$ for the density $c$ and the magnetic field $H$ since $\dot{B}^{\frac{3}{2}}_{2,1}$ is continuously embedded into $L^\infty$. As in [10, 11, 12, 13], the different dissipative mechanisms of low frequencies and high frequencies inspire us to deal with $c$ and $H$ in $\dot{B}^{\frac{3}{2}}_{2,1}$. However, we can not obtain the desired bounds directly in critical regularity framework. In particular, there is a difficulty coming from the convection term $u \cdot \nabla c$ in transport equations in high frequencies, as one derivative loss about the function $c$ will appear no matter how smooth is $u$ if it is viewed as a perturbation term. To overcome the difficulty, we need establish a uniform estimate for a mixed hyperbolic-parabolic linear system with convection terms (See Proposition 3.1). Combining with some nonlinear estimates and the standard continuity arguments, we obtain global existence and uniqueness of strong solutions to the system (1.1) in critical Besov spaces if the initial density is close to a positive constant. Next, one may wonder how global strong solutions constructed above look like for large time. Under a suitable additional condition involving only the low frequencies of the data and in $L^2$-critical regularity framework, we exhibit the optimal time decay rates for the constructed global strong solutions. In this part, our main ideas are based on an application of Fourier analysis to a linearized parabolic-hyperbolic system and on a refined time-weighted energy functional. In low frequencies, making good use of the decay estimates of Green’s function for the linearized system and Duhamel’s principle, one can obtain the desired estimates. In high frequencies, we can deal with the estimates of the nonlinear terms in the system employing the Fourier localization method, the energy method and commutator estimates. Finally, in order to close the energy estimates, we exploit some decay estimates with gain of regularity for the high.
frequencies of $\nabla u, \nabla^2 H$.

Now we state our main results as follows:

**Theorem 1.1** Assume that $(\rho_0 - 1, u_0, H_0) \in \tilde{B}_{2,1}^{3/2} \times \tilde{B}_{2,1}^{1} \times \tilde{B}_{2,1}^{1/2}$ and with no loss of generality that $\rho = 1$ and $p'(1) = 1$. Then there exists a constant $\eta > 0$ such that if

$$
\|\rho_0 - 1\|_{\tilde{B}_{2,1}^{3/2}} + \|u_0\|_{\tilde{B}_{2,1}^{1}} + \|H_0\|_{\tilde{B}_{2,1}^{1/2}} \leq \eta,
$$

then the Cauchy problem (1.1)-(1.3) admits a unique global solution $(\rho - 1, u, H)$ satisfying that for all $t \geq 0$,

$$
X(t) \overset{\text{def}}{=} \| (\rho - 1, u, H) \|_{L_1^\infty(\tilde{B}_{2,1}^{3/2})} + \| (\rho - 1, u, H) \|_{L_1^\infty(\tilde{B}_{2,1}^{3/2})} + \| u \|_{L_1^\infty(\tilde{B}_{2,1}^{3/2})} + \| (\rho - 1, H) \|_{L_1^\infty(\tilde{B}_{2,1}^{3/2})} \leq \| \rho_0 - 1 \|_{\tilde{B}_{2,1}^{3/2}} + \| u_0 \|_{\tilde{B}_{2,1}^{1}} + \| H_0 \|_{\tilde{B}_{2,1}^{1/2}}. 
$$

**Theorem 1.2** Let the data $(\rho_0 - 1, u_0, H_0)$ satisfy the assumptions of Theorem 1.1. Denote $\langle \tau \rangle \overset{\text{def}}{=} \sqrt{1 + \tau^2}$ and $\alpha \overset{\text{def}}{=} \frac{5}{2} - \varepsilon$ with $\varepsilon > 0$ arbitrarily small. There exists a positive constant $c$ so that if in addition

$$
D_0 \overset{\text{def}}{=} \| (\rho_0 - 1, u_0, H_0) \|_{B_{2,\infty}}^{\frac{3}{2}} \leq c,
$$

then the global solution $(\rho - 1, u, H)$ given by Theorem 1.1 satisfies for all $t \geq 0$,

$$
D(t) \leq C \left( D_0 + \| (\nabla \rho_0, u_0, \nabla H_0) \|_{B_{2,1}^{\frac{3}{2}}} \right) 
$$

with

$$
D(t) \overset{\text{def}}{=} \sup_{s \in (-\frac{3}{2}, 0]} \left( \langle \tau \rangle^{\frac{3}{2} + \frac{1}{4} s} (\rho - 1, u, H) \right)_{L_1^\infty(\tilde{B}_{2,1}^{3/2})} + \| \langle \tau \rangle^{\alpha} (\nabla \rho, u, \nabla H) \|_{L_1^\infty(\tilde{B}_{2,1}^{3/2})} \leq \| \rho_0 - 1 \|_{\tilde{B}_{2,1}^{3/2}} + \| u_0 \|_{\tilde{B}_{2,1}^{1}} + \| H_0 \|_{\tilde{B}_{2,1}^{1/2}}. 
$$

**Remark 1.3** In Theorem 1.1, we extend the existence result of 3D incompressible Hall-MHD system in critical Besov spaces in [7] to 3D compressible case.

**Remark 1.4** In Theorem 1.2, we obtain the optimal decay rates for the 3D compressible Hall-MHD equations (1.1) in critical regularity framework. Additionally, the regularity index $s$ can take both negative and nonnegative values, rather than only nonnegative integers, which improves the classical decay results in high Sobolev regularity, such as [16, 21, 31]. In fact, for the solution $(\rho - 1, u, H)$ constructed in Theorem 1.2, employing homogeneous Littlewood-Paley decomposition for $\rho - 1$, we have

$$
\Lambda^s (\rho - 1) = \sum_{q \in \mathbb{Z}} \hat{\Lambda}_q \Lambda^s (\rho - 1).
$$
Thus
\[ \| \Lambda^s (\rho - 1) \|_{L^2} \lesssim \sum_{q \in \mathbb{Z}} \| \Delta_q \Lambda^s (\rho - 1) \|_{L^2} = \| \Lambda^s (\rho - 1) \|_{\dot{B}^0_{2,1}}. \]

Based on the Bernstein inequalities and the low-high frequencies decomposition, we may write
\[ \sup_{t \in [0,T]} \langle t \rangle^{\frac{3}{2}+\frac{s}{2}} \| \Lambda^s (\rho - 1) \|_{\dot{B}^0_{2,1}} \lesssim \| \langle t \rangle^{\frac{3}{2}+\frac{s}{2}} (\rho - 1) \|_{L^\infty_t (\dot{B}^0_{2,1})} + \| \langle t \rangle^{\frac{3}{2}+\frac{s}{2}} (\rho - 1) \|_{L^2_t (\dot{B}^0_{2,1})}. \]

If follows from Inequality (1.8) and definitions of \( D(t) \) and \( \alpha \) that
\[ \| \langle t \rangle^{\frac{3}{2}+\frac{s}{2}} (\rho - 1) \|_{L^\infty_t (\dot{B}^0_{2,1})} \lesssim D_0 + \| (\nabla \rho_0, u_0, \nabla H_0) \|_{L^\infty_t (\dot{B}^0_{2,1})} \quad \text{if} \quad -\frac{3}{2} < s \leq 2 \]
and that, because we have \( \alpha \geq \frac{3}{2}+\frac{s}{2} \) for all \( s \leq 3/2 \),
\[ \| \langle t \rangle^{\frac{3}{2}+\frac{s}{2}} (\rho - 1) \|_{L^2_t (\dot{B}^0_{2,1})} \lesssim D_0 + \| (\nabla \rho_0, u_0, \nabla H_0) \|_{L^2_t (\dot{B}^0_{2,1})} \quad \text{if} \quad s \leq 3/2. \]

This yields the following desired result for \( \rho - 1 \)
\[ \| \Lambda^s (\rho - 1) \|_{L^2} \leq C (D_0 + \| (\nabla \rho_0, u_0, \nabla H_0) \|_{L^2_t (\dot{B}^0_{2,1})} \langle t \rangle^{-\frac{3}{2}+\frac{s}{2}} ) \quad \text{if} \quad -\frac{3}{2} < s \leq \frac{3}{2}, \]
where the fractional derivative operator \( \Lambda^s \) is defined by \( \Lambda^s f = \mathcal{F}^{-1}(|\cdot|^s \mathcal{F} f) \). Similarly, we have
\[ \| \Lambda^s u \|_{L^2} \leq C (D_0 + \| (\nabla \rho_0, u_0, \nabla H_0) \|_{L^2_t (\dot{B}^0_{2,1})} \langle t \rangle^{-\frac{3}{2}+\frac{s}{2}} ) \quad \text{if} \quad -\frac{3}{2} < s \leq \frac{1}{2}, \]
\[ \| \Lambda^s H \|_{L^2} \leq C (D_0 + \| (\nabla \rho_0, u_0, \nabla H_0) \|_{L^2_t (\dot{B}^0_{2,1})} \langle t \rangle^{-\frac{3}{2}+\frac{s}{2}} ) \quad \text{if} \quad -\frac{3}{2} < s \leq \frac{3}{2}. \]

In particular, taking \( s = 0 \) leads back to the standard optimal \( L^1 - L^2 \) decay rate of \( (\rho - 1, u, H) \) as in [16, 21].

**Remark 1.5** Due to the embedding \( L^1(\mathbb{R}^3) \hookrightarrow \dot{B}^{-\frac{3}{2}}_{2,\infty}(\mathbb{R}^3) \), our results in Theorem 1.2 extend the known conclusions in [16, 21]. In particular, our condition involves only the low frequencies of the data and is based on \( L^2(\mathbb{R}^3) \)-norm framework.

**Notations.** We assume \( C \) be a positive generic constant throughout this paper that may vary at different places and denote \( A \leq CB \) by \( A \lesssim B \).

## 2 Littlewood-Paley theory and some useful lemmas

Let us introduce the Littlewood-Paley decomposition. Choose a radial function \( \varphi \in \mathcal{S}(\mathbb{R}^N) \) supported in \( C = \{ \xi \in \mathbb{R}^N, \frac{3}{4} \leq |\xi| \leq \frac{5}{4} \} \) such that
\[ \sum_{q \in \mathbb{Z}} \varphi(2^{-q} \xi) = 1 \quad \text{for all} \quad \xi \neq 0. \]
The homogeneous frequency localization operators \( \hat{\Delta}_q \) and \( \hat{S}_q \) are defined by

\[
\hat{\Delta}_q f = \varphi(2^{-q}D)f, \quad \hat{S}_q f = \sum_{k \leq q-1} \hat{\Delta}_k f \quad \text{for} \quad q \in \mathbb{Z}.
\]

With our choice of \( \varphi \), one can easily verify that

\[
\hat{\Delta}_q \hat{\Delta}_k f = 0 \quad \text{if} \quad |q-k| \geq 2 \quad \text{and} \quad \hat{\Delta}_q(\hat{S}_{k-1} f \hat{\Delta}_k f) = 0 \quad \text{if} \quad |q-k| \geq 5.
\]

We denote the space \( Z'(\mathbb{R}^N) \) by the dual space of \( Z(\mathbb{R}^N) = \{ f \in S(\mathbb{R}^N); D^\alpha \hat{f}(0) = 0; \forall \alpha \in \mathbb{N}^N \text{ multi-index} \} \), it also can be identified by the quotient space of \( S'(\mathbb{R}^N)/P \) with the polynomials space \( P \). The formal equality

\[
f = \sum_{q \in \mathbb{Z}} \hat{\Delta}_q f
\]

holds true for \( f \in Z'(\mathbb{R}^N) \) and is called the homogeneous Littlewood-Paley decomposition.

The following Bernstein’s inequalities will be frequently used.

**Lemma 2.1** [8] Let \( 1 \leq p_1 \leq p_2 \leq +\infty \). Assume that \( f \in L^{p_1}(\mathbb{R}^N) \), then for any \( \gamma \in (\mathbb{N} \cup \{0\})^N \), there exist constants \( C_1, C_2 \) independent of \( f \), \( q \) such that

\[
\text{supp} \hat{f} \subseteq \{ ||\xi|| \leq A_0 2^q \} \Rightarrow \| D^\gamma f \|_{p_2} \leq C_1 2^{q(\frac{1}{p_1} - \frac{1}{p_2})} \| f \|_{p_1},
\]

\[
\text{supp} \hat{f} \subseteq \{ A_1 2^q \leq ||\xi|| \leq A_2 2^q \} \Rightarrow \| f \|_{p_1} \leq C_2 2^{-q||\gamma||} \sup_{|\beta|=|\gamma|} \| D^\beta f \|_{p_1}.
\]

**Definition 2.2** Let \( s \in \mathbb{R}, 1 \leq p, r \leq +\infty \). The homogeneous Besov space \( \dot{B}^s_{p,r} \) is defined by

\[
\dot{B}^s_{p,r} = \{ f \in Z'(\mathbb{R}^N): \| f \|_{\dot{B}^s_{p,r}} < +\infty \},
\]

where

\[
\| f \|_{\dot{B}^s_{p,r}} \overset{\text{def}}{=} \left\| 2^{qs} \| \hat{\Delta}_q f(t) \|_p \right\|_r.
\]

**Remark 2.3** Some properties about the Besov spaces are as follows

- **Derivation:**

\[
\| f \|_{\dot{B}^s_{2,1}} \approx \| D f \|_{\dot{B}^{s-1}_{2,1}};
\]

- **Algebraic properties:** for \( s > 0 \), \( \dot{B}^s_{2,1} \cap L^\infty \) is an algebra;

- **Interpolation:** for \( s_1, s_2 \in \mathbb{R} \) and \( \theta \in [0,1] \), we have

\[
\| f \|_{\dot{B}^{s_1}_2 \cap (1-\theta) s_2} \leq \| f \|_{\dot{B}^{s_1}_2}^{\theta} \| f \|_{\dot{B}^{s_2}_2}^{(1-\theta)}.
\]
Definition 2.4 Let $s \in \mathbb{R}$, $1 \leq p, \rho, r \leq +\infty$. The homogeneous space-time Besov space $L^p_T(\dot{B}^s_{p,r})$ is defined by

$$L^p_T(\dot{B}^s_{p,r}) = \left\{ f \in \mathbb{R} \times \mathcal{Z}'(\mathbb{R}^N) : \|f\|_{L^p_T(\dot{B}^s_{p,r})} < +\infty \right\},$$

where

$$\|f\|_{L^p_T(\dot{B}^s_{p,r})} \overset{\text{def}}{=} \left\| \|2^q s \| \dot{\Delta}^q f \|_{L^p_T} \right\|_{L^p_T}.$$  

We next introduce the Besov-Chemin-Lerner space $\tilde{L}^q_T(\dot{B}^s_{p,r})$ which is initiated in [9].

Definition 2.5 Let $s \in \mathbb{R}$, $1 \leq p, q, r \leq +\infty$, $0 < T \leq +\infty$. The space $\tilde{L}^q_T(\dot{B}^s_{p,r})$ is defined by

$$\tilde{L}^q_T(\dot{B}^s_{p,r}) = \left\{ f \in \mathbb{R} \times \mathcal{Z}'(\mathbb{R}^N) : \|f\|_{\tilde{L}^q_T(\dot{B}^s_{p,r})} < +\infty \right\},$$

where

$$\|f\|_{\tilde{L}^q_T(\dot{B}^s_{p,r})} \overset{\text{def}}{=} \left\| \|2^q s \| \dot{\Delta}^q f(t) \|_{L^q(0,T;L^p)} \right\|_{L^p_T}.$$  

Obviously, $\tilde{L}^1_T(\dot{B}^s_{p,1}) = L^T_T(\dot{B}^s_{p,1})$. By a direct application of Minkowski’s inequality, we have the following relations between these spaces

$$L^p_T(\dot{B}^s_{p,r}) \hookrightarrow \tilde{L}^p_T(\dot{B}^s_{p,r}), \text{ if } r \geq p,$$

$$\tilde{L}^p_T(\dot{B}^s_{p,r}) \hookrightarrow L^p_T(\dot{B}^s_{p,r}), \text{ if } p \geq r.$$  

To deal with functions with different regularities for high frequencies and low frequencies, motivated by [11, 12], it is more effective to work in hybrid Besov spaces. We remark that using hybrid Besov spaces has been crucial for proving global well-posedness for compressible systems in critical spaces (see [10, 11, 12]).

Definition 2.6 Let $s, t \in \mathbb{R}$. We set

$$\|f\|_{\dot{B}^s_{2,1}} = \sum_{q \leq 0} 2^{qs} \|\dot{\Delta}^q f\|_{L^2} + \sum_{q > 0} 2^{qs} \|\dot{\Delta}^q f\|_{L^2}.$$  

For $m = -\left[\frac{N}{2} + 1 - s\right]$, we define

$$\dot{B}^{s,t}_{2,1} = \left\{ f \in \mathcal{S}'(\mathbb{R}^N) : \|f\|_{\dot{B}^{s,t}_{2,1}} < \infty \right\}, \text{ if } m < 0,$$

$$\dot{B}^{s,t}_{2,1} = \left\{ f \in \mathcal{S}'(\mathbb{R}^N)/\mathcal{P}_m : \|f\|_{\dot{B}^{s,t}_{2,1}} < \infty \right\}, \text{ if } m \geq 0.$$  

Remark 2.7 Some properties about the hybrid Besov spaces are as follows

- $\dot{B}^{s,s}_{2,1} = \dot{B}^s_{2,1};$

- If $s \leq t$, then $\dot{B}^{s,t}_{2,1} = \dot{B}^s_{2,1} \cap \dot{B}^t_{2,1}$. Otherwise, $\dot{B}^{s,t}_{2,1} = \dot{B}^s_{2,1} + \dot{B}^t_{2,1}$. In particular, $\dot{B}^{s,s}_{2,1} \hookrightarrow L^\infty$ as $s \leq \frac{N}{2};$
• **Interpolation:** for \( s_1, s_2, t_1, t_2 \in \mathbb{R} \) and \( \theta \in [0,1] \), we have

\[
\| f \|_{B^{\theta s_1 + (1-\theta)s_2, \theta t_1 + (1-\theta)t_2}_{2,1}} \leq \| f \|_{B^{\theta}_{2,1}}^{\theta} \| f \|_{B^{s_2, t_2}_{2,1}}^{1-\theta},
\]

- If \( s_1 \leq s_2 \) and \( t_1 \geq t_2 \), then \( \dot{B}^{s_1, t_1}_{2,1} \hookrightarrow \dot{B}^{s_2, t_2}_{2,1} \).

We have the following properties for the product in Besov spaces and hybrid Besov spaces.

**Proposition 2.8** [14] For all \( 1 \leq r, p, p_1, p_2 \leq +\infty \), there exists a positive universal constant such that

\[
\| fg \|_{B^{s_1+s_2}_{p,r}} \lesssim \| f \|_{L^\infty} \| g \|_{B^{s_1}_{p,r}} + \| g \|_{L^\infty} \| f \|_{B^{s_2}_{p,r}}, \quad \text{if } s > 0;
\]

\[
\| fg \|_{B^{s_1+s_2}_{p,r}} \lesssim \| f \|_{B^{s_1}_{p,r}} \| g \|_{B^{s_2}_{p,r}}, \quad \text{if } s_1, s_2 < \frac{N}{p}, \quad \text{and } s_1 + s_2 > 0;
\]

\[
\| fg \|_{B^{s_1}_{p,r}} \lesssim \| f \|_{B^{s_1}_{p,r}} \| g \|_{B^{s_2}_{p,r}}, \quad \text{if } |s| < \frac{N}{p};
\]

\[
\| fg \|_{B^{s_1}_{2,1}} \lesssim \| f \|_{B^{s_1}_{2,1}} \| g \|_{B^{s_2}_{2,1}}, \quad \text{if } s \in (-d/2, d/2].
\]

**Proposition 2.9** [12] For all \( s_1, s_2 > 0 \), there exists a positive universal constant such that

\[
\| fg \|_{\dot{B}^{s_1+s_2}_{2,1}} \lesssim \| f \|_{L^\infty} \| g \|_{\dot{B}^{s_1}_{2,1}} + \| g \|_{L^\infty} \| f \|_{\dot{B}^{s_2}_{2,1}}.
\]

For all \( s_1, s_2 \leq \frac{N}{2} \) such that \( \min\{s_1 + t_1, s_2 + t_2\} > 0 \), there exists a positive universal constant such that

\[
\| fg \|_{\dot{B}^{s_1+1+t_1, \frac{N}{2}+s_2+t_2-\frac{N}{2}}_{2,1}} \lesssim \| f \|_{\dot{B}^{s_1+1+t_1}_{2,1}} \| g \|_{\dot{B}^{s_2+t_2}_{2,1}}.
\]

For the composition of functions, we have the following estimates.

**Proposition 2.10** [12] Let \( s > 0 \) and \( u \in \dot{B}^{s}_{2,1} \cap L^\infty \).

(i) If \( F \in W^{[s]+2,\infty}_{loc}(\mathbb{R}^N) \) with \( F(0) = 0 \), then \( F(u) \in \dot{B}^{s}_{2,1} \). Moreover, there exists a function of one variable \( C_0 \) depending only on \( s \) and \( F \), and such that

\[
\| F(u) \|_{\dot{B}^{s}_{2,1}} \leq C_0(\| u \|_{L^\infty}) \| u \|_{\dot{B}^{s}_{2,1}}.
\]

(ii) If \( u, v \in \dot{B}^{N}_{2,1} \), \( (v-u) \in \dot{B}^{s}_{2,1} \) for \( s \in (-\frac{N}{2}, \frac{N}{2}] \) and \( G \in W^{[s]+3,\infty}_{loc}(\mathbb{R}^N) \) satisfies \( G'(0) = 0 \), then \( G(v) - G(u) \in \dot{B}^{s}_{2,1} \) and there exists a function of two variables \( C \) depending only on \( s \), \( N \) and \( G \), and such that

\[
\| G(v) - G(u) \|_{\dot{B}^{s}_{2,1}} \leq C(\| u \|_{L^\infty}, \| v \|_{L^\infty})(\| u \|_{\dot{B}^{s}_{2,1}} + \| v \|_{\dot{B}^{s}_{2,1}}) \| v-u \|_{\dot{B}^{s}_{2,1}}.
\]

Throughout this paper, the following estimates for the convection terms arising in the linearized systems will be used frequently.
Proposition 2.11 [12] Let $F$ be an homogeneous smooth function of degree $m$. Suppose that $-N/2 < s_1, t_1, s_2, t_2 \leq 1 + N/2$. The following two estimates hold

\[ |(F(D) \Delta_q (v \cdot \nabla a), F(D) \Delta_q a)| \]
\[ \leq C \gamma q 2^{-q\phi^{1+2(q)-m}} \|v\|_{B^{m+1}_{2,1}} \|a\|_{B^{m+2}_{2,1}} \|F(D) \Delta_q a\|_2, \]

\[ |(F(D) \Delta_q (v \cdot \nabla a), \Delta_q b)| + (\Delta_q (v \cdot \nabla b), F(D) \Delta_q a)| \]
\[ \leq C \gamma q \|v\|_{B^{m+1}_{2,1}} \|F(D) \Delta_q a\|_2 \|b\|_{B^{m+2}_{2,1}} \]
\[ + 2^{-q\phi^{1+2(q)-m}} \|a\|_{B^{m+2}_{2,1}} \|\Delta_q b\|_2, \]

where $(\cdot, \cdot)$ denotes the $2$-inner product, $\sum_{q \in \mathbb{Z}} \gamma_q \leq 1$ and the operator $F(D)$ is defined by $F(D)f := F^{-1} F(\xi) Ff$, $\phi^{\alpha, \beta}(q)$ is the following characteristic function on $\mathbb{Z}$

\[ \phi^{\alpha, \beta}(q) = \begin{cases} \alpha, & \text{if } q \leq 0, \\ \beta, & \text{if } q \geq 1. \end{cases} \]

Proposition 2.12 [11] Let $1 \leq p, p_1 \leq \infty$, $1 \leq r \leq \infty$ and $\sigma \in \mathbb{R}$. There exists a constant $C > 0$ depending only on $\sigma$ such that for all $q \in \mathbb{Z}$, we have

\[ \|v \cdot \nabla, \partial_t \Delta_q a\|_{L^p} \leq C c_q 2^{-q(\sigma-1)} \|\nabla v\|_{B^{p_1}_{\sigma+1}}, \quad \text{for } -\min\left(\frac{d}{p_1}, \frac{d}{p}\right) < \sigma \leq 1 + \min\left(\frac{d}{p_1}, \frac{d}{p}\right), \]
\[ \|v \cdot \nabla, \Delta_q a\|_{L^p} \leq C \gamma q 2^{-q\sigma} \|\nabla v\|_{B^{p_1}_{\sigma+1}} \|a\|_{B^{p_1}_{\sigma+1}}, \quad \text{for } -\min\left(\frac{d}{p_1}, \frac{d}{p}\right) < \sigma < 1 + \frac{d}{p_1}, \]

where the commutator $[\cdot, \cdot]$ is defined by $[f, g] = fg - gf$ and $(c_j)_{j \in \mathbb{Z}}$ denotes a sequence such that $\sum_{q \in \mathbb{Z}} c_q \leq 1$.

Proposition 2.13 [2] Let $\sigma \in \mathbb{R}$, $(p, r) \in [1, \infty]^2$ and $1 \leq p_2 \leq p_1 \leq \infty$. Let $u$ satisfy

\[
\begin{cases}
\partial_t u - \mu \Delta u = f, \\
|u|_{t=0} = u_0.
\end{cases}
\]  

Then for all $T > 0$ the following a priori estimates is fulfilled

\[ \frac{1}{\mu^{r_1}} \|u\|_{L^p_{t}\left(B^{r_1}_{p, r_1}\right)} \lesssim \|u_0\|_{B^{r_1}_{p_2, r_1}} + \frac{1}{\mu^{r_2-1}} \|f\|_{L^p_{t}\left(B^{r_2-1}_{p_2, r_2}\right)}. \]

Remark 2.14 The solutions to the following Lamé system

\[
\begin{cases}
\partial_t u - \mathcal{A} u = f, \\
|u|_{t=0} = u_0,
\end{cases}
\]

are also fulfill (2.2).

We finish this subsection by listing an elementary but useful inequality.
Lemma 2.15 \cite{25} (a) Let \( r_1, r_2 > 0 \) satisfy \( \max\{r_1, r_2\} > 1 \). Then
\[
\int_0^t (1 + t - \tau)^{-r_1}(1 + \tau)^{-r_2} d\tau \leq C(r_1, r_2)(1 + t)^{-\min\{r_1, r_2\}}.
\]
(b) Let \( r_1, r_2 > 0 \) and \( f \in L^1(0, +\infty) \). Then
\[
\int_0^t (1 + t - \tau)^{-r_1}(1 + \tau)^{-r_2} f(\tau) d\tau \leq C(r_1, r_2)(1 + t)^{-\min\{r_1, r_2\}} \int_0^t |f| d\tau.
\]

3 Reformulation of the Original System (1.1) and A priori estimates for linearized system with convection terms

3.1 Reformulation of the Original System (1.1)

We first reformulate the original system (1.1) into a different form. For the magnetic field \( H \), we have the following identities:
\[
\nabla (|H|^2) = 2(H \cdot \nabla)H + 2(\nabla \times H) \times H,
\]
\[
\nabla \times (\nabla \times H) = \nabla \text{div}H - \Delta H,
\]
and
\[
\nabla \times (u \times H) = u(\text{div}H) - H(\text{div}u) + H \cdot \nabla u - u \cdot \nabla H = -H(\text{div}u) + H \cdot \nabla u - u \cdot \nabla H
\]
with \( \text{div}H = 0 \).

We will denote that \( c = \rho - 1 \). Then, in term of the new variables \( (c, u, H) \), system (1.1)-(1.3) becomes
\[
\begin{aligned}
\partial_t c + u \cdot \nabla c + \text{div}u &= f, \\
\partial_t u + u \cdot \nabla u - Au + \nabla c &= g, \\
\partial_t H + u \cdot \nabla H - \Delta H &= h, \\
\text{div}H &= 0, \\
(c, u, H)|_{t=0} &= (c_0, u_0, H_0),
\end{aligned}
\]
where
\[
f = -c \text{div}u,
\]
\[
g = -L_1(c)Au + L_2(c)\nabla c - L_3(c) \left( \frac{1}{2} |H|^2 - H \cdot \nabla H \right),
\]
\[
h = -H(\text{div}u) + H \cdot \nabla u - \nabla \times \left( L_3(c)(\nabla \times H) \times H \right),
\]
with
\[
Au = \mu \Delta u + (\lambda + \mu) \nabla \text{div}u, \quad L_1(c) = \frac{c}{1 + c}, \quad L_2(c) = \frac{p'(1 + c)}{1 + c} - 1, \quad L_3(c) = \frac{1}{c + 1}.
\]
3.2 A priori estimates for linearized system with convection terms

Next, we investigate some a priori estimates for the following linearized system with convection terms

\[
\begin{align*}
\partial_t c + v \cdot \nabla c + \nabla u &= f, \\
\partial_t u + v \cdot \nabla u - \mu \Delta u + (\lambda + \mu) \nabla \text{div} u + \nabla c &= g, \\
\partial_t H + v \cdot \nabla H - \Delta H &= h, \\
(c, u, H)|_{t=0} &= (c_0, u_0, H_0).
\end{align*}
\]

(3.2)

We prove the following proposition and show a uniform estimate for a mixed hyperbolic-parabolic linear system with convection terms. What is crucial in this work is the smoothing effect on the velocity \(u\) and magnetic fields \(H\) and a \(L^1\) decay on \(\rho - \bar{\rho}\) (this plays a key role to control the pressure term).

**Proposition 3.1** Denote

\[
V(t) := \int_0^t \|v(\tau)\|_{\dot{B}^{\frac{5}{2}}_{2,1}} d\tau.
\]

Let \((c, u, H)\) be a solution of the system (3.2) on \([0, t)\). Then the following estimates hold for \(t \in [0, T)\)

\[
\|c\|_{\dot{L}^{\infty}([0, t]; \dot{B}^{\frac{1}{2}}_{2,1})} + \|u\|_{\dot{L}^{\infty}([0, t]; \dot{B}^{\frac{1}{2}}_{2,1})} + \|H\|_{\dot{L}^{\infty}([0, t]; \dot{B}^{\frac{1}{2}}_{2,1})} \\
+ \int_0^t \|c(\tau)\|_{\dot{B}^{\frac{1}{2}}_{2,1}} d\tau + \int_0^t \|u(\tau)\|_{\dot{B}^{\frac{1}{2}}_{2,1}} d\tau + \int_0^t \|H(\tau)\|_{\dot{B}^{\frac{1}{2}}_{2,1}} d\tau \\
\lesssim e^{CV(t)} \left(\|c_0\|_{\dot{B}^{\frac{1}{2}}_{2,1}} + \|u_0\|_{\dot{B}^{\frac{1}{2}}_{2,1}} + \|H_0\|_{\dot{B}^{\frac{1}{2}}_{2,1}} + \int_0^t \|f(\tau)\|_{\dot{B}^{\frac{1}{2}}_{2,1}} d\tau \\
+ \int_0^t \|g(\tau)\|_{\dot{B}^{\frac{1}{2}}_{2,1}} d\tau + \int_0^t \|h(\tau)\|_{\dot{B}^{\frac{1}{2}}_{2,1}} d\tau\right).
\]

(3.3)

**Proof.** To prove the proposition, we first localize the system (3.2) according to the homogeneous Littlewood-Paley decomposition. Then each dyadic block can be estimated by using energy method. Applying the operator \(\dot{\Delta}_q\) to the system (3.2), we deduce that \((\dot{\Delta}_q c, \dot{\Delta}_q u, \dot{\Delta}_q H)\) satisfies

\[
\begin{align*}
\partial_t \dot{\Delta}_q c + \dot{\Delta}_q (v \cdot \nabla c) + \text{div} \dot{\Delta}_q u &= \dot{\Delta}_q f, \\
\partial_t \dot{\Delta}_q u + \dot{\Delta}_q (v \cdot \nabla u) - \mu \dot{\Delta}_q u + (\lambda + \mu) \text{div} \dot{\Delta}_q u + \nabla \dot{\Delta}_q c &= \dot{\Delta}_q g, \\
\partial_t \dot{\Delta}_q H + \dot{\Delta}_q (v \cdot \nabla H) - \Delta \dot{\Delta}_q H &= \dot{\Delta}_q h.
\end{align*}
\]

(3.4)

Taking the \(L^2\)-scalar product of the first equation of (3.2) with \(\dot{\Delta}_q c\) and \(-\Delta \dot{\Delta}_q c\), the second equation with \(\dot{\Delta}_q u\), the third equation with \(\dot{\Delta}_q H\) and \(\Delta \dot{\Delta}_q H\) respectively, we obtain the following five identities

\[
\frac{1}{2} \frac{d}{dt} \|\dot{\Delta}_q c\|^2_{L^2} + (\dot{\Delta}_q (v \cdot \nabla c)|\dot{\Delta}_q c\) + (\text{div} \dot{\Delta}_q u|\dot{\Delta}_q c\) = (\dot{\Delta}_q f|\dot{\Delta}_q c),
\]

(3.5)

\[
\frac{1}{2} \frac{d}{dt} \|
abla \dot{\Delta}_q c\|^2_{L^2} + (\dot{\Delta}_q (v \cdot \nabla c) - \Delta \dot{\Delta}_q c\) + (\text{div} \dot{\Delta}_q u - \Delta \dot{\Delta}_q c\) = (\dot{\Delta}_q f - \Delta \dot{\Delta}_q c),
\]

(3.6)
\[
\frac{1}{2} \frac{d}{dt} \| \Delta_q u \|^2_{L^2} + (\Delta_q (v \cdot \nabla u) | \Delta_q u) + \mu \| \nabla \Delta_q u \|^2_{L^2} + (\lambda + \mu) \| \text{div} \Delta_q u \|^2_{L^2} + (\nabla \Delta_q c | \Delta_q u) = (\Delta_q f | \Delta_q u),
\]
(3.7)

\[
\frac{1}{2} \frac{d}{dt} \| \Delta_q H \|^2_{L^2} + (\Delta_q (v \cdot \nabla H) | \Delta_q H) + \| \Delta \Delta_q H \|^2_{L^2} = (\Delta_q g | \Delta_q H),
\]
(3.8)

and

\[
\frac{1}{2} \frac{d}{dt} \| \nabla \Delta_q H \|^2_{L^2} + (\Delta_q (v \cdot \nabla H) | \Delta_q H) + \| \Delta \Delta_q H \|^2_{L^2} = (\Delta_q g | - \Delta \Delta_q H).
\]
(3.9)

Next, we derive an identity involving \((\nabla \Delta_q c | \Delta_q u)\). For this purpose, we apply the operator \(\nabla\) to the first equation in (3.2) and take the \(L^2\) scalar product with \(\Delta_q u\), then calculate the scalar product of the second equation in (3.2) with \(\nabla \Delta_q c\), and then by summing up the results, we get

\[
\frac{d}{dt} (\nabla \Delta_q c | \Delta_q u) - \| \text{div} \Delta_q u \|^2_{L^2} + (\lambda + 2\mu) (\text{div} \Delta_q c | \Delta_q c) + \| \nabla \Delta_q c \|^2_{L^2}
\]

\[
+ (\nabla \Delta_q (v \cdot \nabla c) | \Delta_q u) + (\Delta_q (v \cdot \nabla u) | \nabla \Delta_q c)
\]

\[
= (\nabla \Delta_q f | \Delta_q u) + (\Delta_q g | \nabla \Delta_q c).
\]
(3.10)

We now define

\[
\alpha_q^2 = \| \Delta_q c \|^2_{L^2} + (\lambda + 2\mu) A \| \nabla \Delta_q c \|^2_{L^2} + \| \Delta_q u \|^2_{L^2} + \| \Delta_q H \|^2_{L^2} + \| \nabla \Delta_q H \|^2_{L^2}
\]

\[
+ 2A (\nabla \Delta_q c | \Delta_q u),
\]

where \(A = \frac{\lambda + \mu}{\mu} > 0\). Then, there exist two positive constants \(c_1\) and \(c_2\) such that

\[
c_1 \alpha_q^2 \leq \| \Delta_q c \|^2_{L^2} + \| \nabla \Delta_q c \|^2_{L^2} + \| \Delta_q u \|^2_{L^2} + \| \Delta_q H \|^2_{L^2} + \| \nabla \Delta_q H \|^2_{L^2} \leq c_2 \alpha_q^2,
\]

for \(M \in (1/(2\mu + \lambda), 2/(\mu + \lambda))\), we have

\[
|2(\nabla \Delta_q c | \Delta_q u)| \leq M \| \Delta_q u \|^2_{L^2} + \frac{1}{M} \| \nabla \Delta_q c \|^2_{L^2}.
\]

Thus,

\[
\alpha_q \approx \begin{cases} \| (\Delta_q c, \Delta_q u, \Delta_q H) \|_{L^2}, & \text{for } q \leq q_0, \\ \| (\nabla \Delta_q c, \Delta_q u, \nabla \Delta_q H) \|_{L^2}, & \text{for } q > q_0. \end{cases}
\]

Combining with (3.5)-(3.10), it yields, with the help of Proposition 2.11, that

\[
\frac{1}{2} \frac{d}{dt} \alpha_q^2 + (\mu + \lambda - A) \| \text{div} \Delta_q u \|^2_{L^2} + \mu \| \nabla \Delta_q u \|^2_{L^2} + A \| \nabla \Delta_q c \|^2_{L^2}
\]

\[
= - (\Delta_q (v \cdot \nabla c | \Delta_q c) + (2\mu + \lambda) A (\Delta_q (v \cdot \nabla c | \Delta_q c) - (\Delta_q (v \cdot \nabla u) | \Delta_q u)
\]

\[
- (\Delta_q (v \cdot \nabla H) | \Delta_q H) + (\Delta_q (v \cdot \nabla H) | \Delta \Delta_q H) - A (\nabla \Delta_q (v \cdot \nabla c) | \Delta_q u)
\]

\[
- A (\Delta_q (v \cdot \nabla u) | \nabla \Delta_q c) + (\Delta_q f | \Delta_q c) - (2\mu + \lambda) A (\Delta_q f - \Delta \Delta_q c) + (\Delta_q f | \Delta_q u)
\]

\[
\lesssim \alpha_q \| \Delta_q f \|_{L^2} + \| \nabla \Delta_q f \|_{L^2} + \| \Delta_q g \|_{L^2} + \| \Delta_q h \|_{L^2} + \| \nabla \Delta_q h \|_{L^2}
\]

\[
+ 2^{-\frac{8}{3}} \gamma V' \| (c, u, H) \|_{B_{2,1}^{\frac{1}{2}} \times B_{2,1}^{\frac{1}{2}} \times B_{2,1}^{\frac{1}{2}}}.\]
Thus, it follows

\[
\begin{align*}
\frac{1}{2} \frac{d}{dt} |c|^2 + c_0 \min(2^{2q}, 1)|c|^2 \\
\leq \gamma_q 2^{-\frac{q}{2}} \left[ \left\| f \right\|_{B^{\frac{1}{2}, \frac{3}{4}}_{2, 1}} + \left\| g \right\|_{B^{\frac{1}{2}, \frac{3}{4}}_{2, 1}} + \left\| h \right\|_{B^{\frac{1}{2}, \frac{3}{4}}_{2, 1}} + V'(c, u, H) \right\|_{B^{\frac{1}{2}, \frac{3}{4}} \times B^{\frac{1}{2}, \frac{3}{4}} \times B^{\frac{1}{2}, \frac{3}{4}}}
\right] a_q,
\end{align*}
\]

which implies that

\[
2^{\frac{q}{2}} a_q + c_0 \int_0^t \min(2^{2q}, 1)2^{\frac{q}{2}} a_q(\tau)d\tau
\]

\[
\leq 2^{\frac{q}{2}} a_q(0) + C \gamma_q \int_0^t \left[ \left\| f \right\|_{B^{\frac{1}{2}, \frac{3}{4}}_{2, 1}} + \left\| g \right\|_{B^{\frac{1}{2}, \frac{3}{4}}_{2, 1}} + \left\| h \right\|_{B^{\frac{1}{2}, \frac{3}{4}}_{2, 1}} + V' \sum_q 2^{q(s-1)} a_q \right]d\tau.
\]

Thus, by Gronwall’s inequality, we have

\[
\begin{align*}
\left\| c \right\|_{L^\infty([0, t]; B^{\frac{1}{2}, \frac{3}{4}}_{2, 1})} + \left\| u \right\|_{L^\infty([0, t]; B^{\frac{1}{2}, \frac{3}{4}}_{2, 1})} + \left\| H \right\|_{L^\infty([0, t]; B^{\frac{1}{2}, \frac{3}{4}}_{2, 1})} \\
+ \int_0^t \left\| c(\tau) \right\|_{B^{\frac{1}{2}, \frac{3}{4}}_{2, 1}}d\tau + \int_0^t \left\| u(\tau) \right\|_{B^{\frac{1}{2}, \frac{3}{4}}_{2, 1}}d\tau + \int_0^t \left\| H(\tau) \right\|_{B^{\frac{1}{2}, \frac{3}{4}}_{2, 1}}d\tau
\end{align*}
\]

\[
\leq c^{CV(t)} \left( \left\| c_0 \right\|_{B^{\frac{1}{2}, \frac{3}{4}}_{2, 1}} + \left\| u_0 \right\|_{B^{\frac{1}{2}, \frac{3}{4}}_{2, 1}} + \left\| H_0 \right\|_{B^{\frac{1}{2}, \frac{3}{4}}_{2, 1}} + \int_0^t \left\| f(\tau) \right\|_{B^{\frac{1}{2}, \frac{3}{4}}_{2, 1}}d\tau
\right)
\]

\[
+ \int_0^t \left\| g(\tau) \right\|_{B^{\frac{1}{2}, \frac{3}{4}}_{2, 1}}d\tau + \int_0^t \left\| h(\tau) \right\|_{B^{\frac{1}{2}, \frac{3}{4}}_{2, 1}}d\tau \right).
\]

(3.12)

Based on the damping effect for \(c\), we get the smoothing effect of \(u\) by considering (3.2) with \(\nabla c\) being seen as a source term. Furthermore, we also exploit the smoothing effect of \(H\) by heat equation of (3.2),. Thanks to (3.12), it suffices to state the proof for the high frequencies only for \(u\) and \(H\). From (3.7) and Proposition 2.11, we have

\[
\frac{1}{2} \frac{d}{dt} \left\| \hat{\Delta}_q u \right\|_{L^2}^2 + C2^{2q} \left\| \hat{\Delta}_q u \right\|_{L^2}^2
\]

\[
= -(\nabla \hat{\Delta}_q c \cdot \hat{\Delta}_q u) - (\hat{\Delta}_q (v \cdot \nabla u) \cdot \hat{\Delta}_q u) + (\hat{\Delta}_q f \cdot \hat{\Delta}_q u)
\]

\[
\leq \left\| \hat{\Delta}_q u \right\|_{L^2} \left( 2^q \left\| \hat{\Delta}_q c \right\|_{L^2} + \left\| \hat{\Delta}_q g \right\|_{L^2} + V'(t) 2^{-\frac{q}{2}} \left\| u \right\|_{B^{\frac{1}{2}, \frac{3}{4}}_{2, 1}} \right).
\]

It follows that

\[
\frac{d}{dt} \sum_{q \geq q_0} 2^{\frac{q}{2}} \left| \hat{\Delta}_q u \right|_{L^2}^2 + C \sum_{q \geq q_0} 2^{\frac{q}{2}} 2^{2q} \left| \hat{\Delta}_q u \right|_{L^2}^2
\]

\[
\leq \sum_{q \geq q_0} 2^{\frac{q}{2}} \left( 2^q \left| \hat{\Delta}_q c \right|_{L^2} + \left| \hat{\Delta}_q g \right|_{L^2} + V'(t) 2^{-\frac{q}{2}} \left| u \right|_{B^{\frac{1}{2}, \frac{3}{4}}_{2, 1}} \right)
\]

\[
\leq \sum_{q \geq q_0} 2^{\frac{q}{2}} 2^{2q} \left| \hat{\Delta}_q c \right|_{L^2} + \left| \hat{\Delta}_q g \right|_{B^{\frac{1}{2}, \frac{3}{4}}_{2, 1}} + V'(t) \left| u \right|_{B^{\frac{1}{2}, \frac{3}{4}}_{2, 1}},
\]
which implies, with the help of (3.12), that
\[
\int_0^t \sum_{q \geq q_0} 2^{\frac{3}{2}} 2^{2q} \| \Delta_q u(\tau) \|_{L^2} d\tau \\
\lesssim \| u_0 \|_{B^{\frac{1}{2}}_{2,1}} + \int_0^t \sum_{q \geq q_0} 2^{\frac{3}{2}} 2^{2q} \| \Delta_q c(\tau) \|_{L^2} d\tau + \int_0^t \| g(\tau) \|_{B^{\frac{1}{2}}_{2,1}} d\tau + V(t) \sup_{\tau \in [0,t]} \| u \|_{B^{\frac{1}{2}}_{2,1}}.
\]
(3.13)

\[
\lesssim e^{CV(t)} (\| c_0 \|_{B^{\frac{1}{2}}_{2,1}} + \| u_0 \|_{B^{\frac{1}{2}}_{2,1}} + \| H_0 \|_{B^{\frac{1}{2}}_{2,1}} + \int_0^t \| f(\tau) \|_{B^{\frac{1}{2}}_{2,1}} d\tau + \int_0^t \| h(\tau) \|_{B^{\frac{1}{2}}_{2,1}} d\tau).
\]
Similarly, from (3.8), Proposition 2.11 and Remark 2.7, we have
\[
\int_0^t \sum_{q \geq q_0} 2^{\frac{3}{2}} 2^{2q} \| \Delta_q H(\tau) \|_{L^2} d\tau \\
\lesssim \| H_0 \|_{B^{\frac{1}{2}}_{2,1}} + \int_0^t \| g(\tau) \|_{B^{\frac{1}{2}}_{2,1}} d\tau + V(t) \sup_{\tau \in [0,t]} \| H \|_{B^{\frac{1}{2}}_{2,1}}.
\]
(3.14)

\[
\lesssim e^{CV(t)} (\| c_0 \|_{B^{\frac{1}{2}}_{2,1}} + \| u_0 \|_{B^{\frac{1}{2}}_{2,1}} + \| H_0 \|_{B^{\frac{1}{2}}_{2,1}} + \int_0^t \| f(\tau) \|_{B^{\frac{1}{2}}_{2,1}} d\tau + \int_0^t \| h(\tau) \|_{B^{\frac{1}{2}}_{2,1}} d\tau).
\]

Combining with (3.13)-(3.14) and (3.12), we finally conclude that (3.3). Thus, we complete the proof of Proposition 3.1.

4 Global existence for initial data near equilibrium

In this section, we are going to show that if the initial data satisfy
\[
\| c_0 \|_{B^{\frac{1}{2}}_{2,1}} + \| u_0 \|_{B^{\frac{1}{2}}_{2,1}} + \| H_0 \|_{B^{\frac{1}{2}}_{2,1}} \leq \eta
\]
for some sufficiently small \( \eta \), then there exists a positive constant \( M \) such that
\[
\| (c, u, H) \|_{E^{\frac{3}{2}}} \leq M \eta,
\]
where
\[
E^{\frac{3}{2}} \overset{\text{def}}{=} (L^1(\mathbb{R}^+; B^{\frac{3}{2}}_{2,1}) \cap C(\mathbb{R}^+; B^{\frac{1}{2}}_{2,1})) \times (L^1(\mathbb{R}^+; B^{\frac{3}{2}}_{2,1}) \cap C(\mathbb{R}^+; B^{\frac{1}{2}}_{2,1}))
\times (L^1(\mathbb{R}^+; B^{\frac{3}{2}}_{2,1}) \cap C(\mathbb{R}^+; B^{\frac{3}{2}}_{2,1})).
\]
This uniform estimate will enable us to extend the local solution \((c, u, H)\) obtained within an iterative scheme as in [12] to a global one. To this end, we use a contradiction argument. Define
\[
T_0 = \sup \{ T \in [0, \infty) : \| (c, u, H, \theta) \|_{E^{\frac{3}{2}}} \leq M \eta \}.
\]
with $M$ to be determined later. Suppose that $T_0 < \infty$. We apply linear estimates in Proposition 3.1 to the solutions of the reformulated system (3.1) such that for all $t \in [0, T_0]$, the following estimates hold

\[
\begin{aligned}
\|c\|_{L^\infty([0,t];B^\frac{1}{2};_{2,1})} + \|u\|_{L^\infty([0,t];B^\frac{1}{4};_{2,1})} + \|H\|_{L^\infty([0,t];B^\frac{1}{4};_{2,1})} \\
+ \int_0^t \|c(\tau)\|_{B^\frac{1}{2};_{2,1}} d\tau + \int_0^t \|u(\tau)\|_{B^\frac{1}{4};_{2,1}} d\tau + \int_0^t \|H(\tau)\|_{B^\frac{1}{4};_{2,1}} d\tau \\
\lesssim c^{CV(t)} \left(\|c_0\|_{B^\frac{1}{2};_{2,1}} + \|u_0\|_{B^\frac{1}{4};_{2,1}} + \|H_0\|_{B^\frac{1}{4};_{2,1}} + \int_0^t \|f(\tau)\|_{B^\frac{1}{4};_{2,1}} d\tau \right) \\
+ \int_0^t \|g(\tau)\|_{B^\frac{1}{2};_{2,1}} d\tau + \int_0^t \|h(\tau)\|_{B^\frac{1}{4};_{2,1}} d\tau \right).
\end{aligned}
\]  

(4.1)

where

\[ V(T_0) = \int_0^{T_0} \|u(\tau)\|_{B^\frac{5}{2};_{2,1}} d\tau. \]

In what follows, we derive some estimates for the nonlinear terms $f$, $g$ and $h$. First, by Proposition 2.9, we have

\[
\|f\|_{L^1_0(B^\frac{1}{2};_{2,1})} \lesssim \|c\|_{L^\infty_0(B^\frac{1}{4};_{2,1})} \|\text{div}\|_{L^1_0(B^\frac{3}{2};_{2,1})} \lesssim M^2 \eta^2.
\]

(4.2)

Next, we bound the term $g$. By the embedding $B^\frac{1}{2};_{2,1} \hookrightarrow B^\frac{3}{2};_{2,1} \hookrightarrow L^\infty$ and Proposition 2.10, we get

\[
\|L_1(c)\|_{B^\frac{2}{2};_{2,1}} \leq C_0(\|c\|_{L^\infty}) \|c\|_{B^\frac{3}{2};_{2,1}} \lesssim \|c\|_{B^\frac{3}{2};_{2,1}},
\]

thus

\[
\|L_1(c)Au\|_{L^1_0(B^\frac{3}{2};_{2,1})} \lesssim \|L_1(c)\|_{L^\infty_0(B^\frac{3}{2};_{2,1})} \|Au\|_{L^1_0(B^\frac{3}{2};_{2,1})} \lesssim \|c\|_{L^\infty_0(B^\frac{3}{2};_{2,1})} \|Au\|_{L^1_0(B^\frac{3}{2};_{2,1})} \lesssim \|c\|_{L^\infty_0(B^\frac{3}{2};_{2,1})} \|u\|_{L^1_0(B^\frac{5}{2};_{2,1})} \lesssim M^2 \eta^2.
\]

Similarly,

\[
\left\| L_3(c) \left( \frac{1}{2} \nabla |H|^2 - H \cdot \nabla H \right) \right\|_{L^1_0(B^\frac{3}{2};_{2,1})} \lesssim \left\{ 1 + \|L_1(c)\|_{L^\infty_0(B^\frac{3}{2};_{2,1})} \right\} \|H \cdot \nabla H\|_{L^1_0(B^\frac{1}{2};_{2,1})} \lesssim \left\{ 1 + \|L_1(c)\|_{L^\infty_0(B^\frac{3}{2};_{2,1})} \right\} \|H\|_{L^\infty_0(B^\frac{1}{2};_{2,1})} \|\nabla H\|_{L^1_0(B^\frac{1}{2};_{2,1})} \lesssim \left\{ 1 + \|L_1(c)\|_{L^\infty_0(B^\frac{3}{2};_{2,1})} \right\} \|H\|_{L^\infty_0(B^\frac{1}{2};_{2,1})} \|H\|_{L^1_0(B^\frac{5}{2};_{2,1})} \lesssim C(1 + \|L_1(c)\|_{L^\infty_0(B^\frac{3}{2};_{2,1})}) \|H\|_{L^\infty_0(B^\frac{1}{2};_{2,1})} \|H\|_{L^1_0(B^\frac{5}{2};_{2,1})} \lesssim (1 + M \eta) M^2 \eta^2.
\]
Therefore, we omit it. This completes the proof of the existence in Theorem 1.1. We employ a classical Friedrich’s approximation and compactness method (cf. [11, 12, 13]), we can establish the global existence of strong solutions of the system (3.1) as follows. Employing Proposition 2.8, we infer that

\[ \| L_2(c) \nabla c \|_{L^1_t(B_2^{1,1})} \lesssim \| L_2(c) \|_{L^2_t(B_2^{3,1})} \| c \|_{L^2_t(B_2^{3,1})} \]

\[ \lesssim \| c \|^2_{L^2_t(B_2^{3,1})} \]

\[ \lesssim \| c \|_{L^\infty_t(B_2^{1,2})} \| c \|_{L^1_t(B_2^{5,2})} \]

\[ \lesssim M^2 \eta^2. \]

Therefore,

\[ \| g \|_{L^1_t(B_2^{\frac{1}{2},1})} \leq C(1 + M \eta) M^2 \eta^2. \]  \hfill (4.3)

Finally, we bound the term \( h \) as follows. Employing Proposition 2.8, we infer that

\[ \| H \cdot \nabla u - \text{div} H \|_{L^1_t(B_2^{\frac{1}{2},1})} \lesssim \int_0^{T_0} \| H \|_{B_2^{\frac{1}{2},1}} \| \nabla u \|_{B_2^{\frac{3}{2},1}} \, d\tau \]

\[ \lesssim \| H \|_{L^\infty_t(B_2^{1,2})} \| u \|_{L^1_t(B_2^{5,2})} \]

\[ \lesssim M^2 \eta^2, \]

\[ \| \nabla \times (L_3(c)(\nabla \times H) \times H) \|_{L^1_t(B_2^{\frac{1}{2},1})} \]

\[ \lesssim \left( 1 + \| L_1(c) \|_{L^\infty_t(B_2^{3,1})} \right) \| (\nabla \times H) \times H \|_{L^1_t(B_2^{5,2})} \]

\[ \lesssim \left( 1 + \| L_1(c) \|_{L^\infty_t(B_2^{3,1})} \right) \| H \|_{L^\infty_t(B_2^{3,1})} \| \nabla \times H \|_{L^1_t(B_2^{5,2})} \]

\[ \lesssim \left( 1 + \| L_1(c) \|_{L^\infty_t(B_2^{3,1})} \right) \| H \|_{L^\infty_t(B_2^{3,1})} \| H \|_{L^1_t(B_2^{5,2})} \]

\[ \lesssim (1 + M \eta) M^2 \eta^2. \]

Hence, we gather that

\[ \| h \|_{L^1_t(B_2^{\frac{1}{2},1})} \lesssim (1 + M \eta) M^2 \eta^2. \]  \hfill (4.4)

Substituting (4.2)-(4.4) into (4.1), we obtain that

\[ \| (c, u, H) \|_{L^\infty_t(B_2^{\frac{1}{2},1})} \leq C_1 e^{C_1 M \eta} (1 + M \eta) M^2 \eta^2. \]  \hfill (4.5)

Choose \( M = 8C_1 \), for \( \eta \) small enough such that

\[ e^{C_1 M \eta} \leq 2, \quad (1 + M \eta) M^2 \eta \leq 2, \]  \hfill (4.6)

which implies that

\[ \| (c, u, H) \|_{L^\infty_t(B_2^{\frac{1}{2},1})} \leq \frac{1}{2} M \eta. \]

This is a contradiction with the definition of \( T_0 \). As a consequence, we conclude that \( T_0 = \infty \). Based on the above uniform estimates, employing a classical Friedrich’s approximation and compactness method (cf. [11, 12, 13]), we can establish the global existence of strong solutions of the system (3.1). Here, we omit it. This completes the proof of the existence in Theorem 1.1.
5 Uniqueness

In this section, we will address uniqueness of strong solutions to the system (3.1). For this purpose, suppose that \((c_i, u_i, H_i)_{i=1,2}\) in \(\mathcal{E}_{T}^\frac{1}{r}\) solve (3.1) with the same initial data. Define

\[
\delta c = c_2 - c_1, \quad \delta u = u_2 - u_1, \quad \delta H = H_2 - H_1.
\]

Then, \((\delta c, \delta u, \delta H)\) satisfies the following system

\[
\begin{cases}
\partial_t \delta c + u_2 \cdot \nabla \delta c + \text{div}\delta u = \delta f, \\
\partial_t \delta u + u_2 \cdot \nabla \delta u - A\delta u + \nabla \delta c = \delta g, \\
\partial_t \delta H + u_2 \cdot \nabla \delta H - \Delta \delta H = \delta h,
\end{cases}
\]

where

\[
\delta f = -\delta u \cdot \nabla c_1 - \delta c \text{div} u_2 - c_1 \text{div}\delta u, \quad \delta g = \sum_{i=1}^{8} \delta g_i, \quad \delta h = \sum_{i=1}^{8} \delta h_i,
\]

with

\[
\begin{align*}
\delta g_1 &= -u_2 \cdot \nabla \delta u, & \delta h_1 &= \delta u \cdot \nabla H_1, \\
\delta g_2 &= -\delta u \cdot \nabla u_1, & \delta h_2 &= H_2 \nabla \delta u, \\
\delta g_3 &= -(L_1(c_2) - L_1(c_1)) A u_2, & \delta h_3 &= \delta H \cdot \nabla u_1, \\
\delta g_4 &= -L_1(c_1) A \delta u, & \delta h_4 &= -H_2 \text{div}\delta u, \\
\delta g_5 &= -\nabla (K_0(c_2) - K_0(c_1)), & \delta h_5 &= -\delta H \text{div} u_1, \\
\delta g_6 &= -(L_3(c_2) - L_3(c_1)) \nabla H_2 \cdot H_2, & \delta h_6 &= -\nabla \times \left[ (L_3(c_2) - L_3(c_1)) (\nabla \times H_2) \times H_2 \right], \\
\delta g_7 &= -L_3(c_1) \nabla \delta H \cdot H_2, & \delta h_7 &= \nabla \times \left[ L_3(c_1) (\nabla \times \delta H) \times H_2 \right], \\
\delta g_8 &= -L_3(c_1) \nabla H_1 \cdot \delta H, & \delta h_8 &= \nabla \times \left[ L_3(c_1) (\nabla \times H_1) \times \delta H \right],
\end{align*}
\]

and

\[
K_0(z) = \int_{0}^{z} L_2(y) dy.
\]

Applying Proposition 3.1 to the system (5.1), we get

\[
\| (\delta c, \delta u, \delta H) \|_{\mathcal{E}_{T}^\frac{1}{r}} \lesssim e^{\int_{0}^{T} \| u_2(t) \|_{B_{2,1}^\frac{3}{2}} dt} \left( \| \delta f \|_{L_1^1(B_{2,1}^\frac{3}{2})} + \| \delta g \|_{L_1^1(B_{2,1}^\frac{3}{2})} + \| \delta h \|_{L_1^1(B_{2,1}^\frac{3}{2})} \right).
\]

(5.2)

Let us observe that \(\partial_t c_i \in L_{loc}^1(B_{2,1}^\frac{3}{2})\), and hence \(c_i \in C(B_{2,1}^\frac{3}{2}) \cap L^\infty(B_{2,1}^\frac{3}{2})(i = 1, 2)\). This entails \(c_i \in C([0, \infty) \times \mathbb{R}^3)\). On the other hand, if \(\eta\) is sufficiently small, we have

\[
|c_i(t, x)| \leq \frac{1}{4} \quad \text{for all} \quad t \geq 0 \quad \text{and} \quad x \in \mathbb{R}^3.
\]
Continuity in time for $c_2$ thus yields the existence of a time $T > 0$ such that

$$\|c_i(t)\|_{L^\infty} \leq \frac{1}{2} \quad \text{for} \quad i = 1, 2 \quad \text{and} \quad t \in [0, T].$$

Employing Propositions 2.8-2.10 and Remarks 2.3-2.7, we easily infer that

$$\|\delta f\|_{L^1_t(\dot{B}^\frac{3}{2})} \lesssim \|c_1\|_{L^\infty_t(\dot{B}^\frac{3}{2})} \|\delta u\|_{L^1_t(\dot{B}^\frac{3}{2})} + \|\text{div} u_2\|_{L^1_t(\dot{B}^\frac{3}{2})} \|\delta c\|_{L^\infty_t(\dot{B}^\frac{3}{2})},$$

$$\|\delta g_1\|_{L^1_t(\dot{B}^\frac{3}{2})} + \|\delta g_2\|_{L^1_t(\dot{B}^\frac{3}{2})} + \|\delta g_3\|_{L^1_t(\dot{B}^\frac{3}{2})} + \|\delta g_4\|_{L^1_t(\dot{B}^\frac{3}{2})} \lesssim \|u_2\|_{L^2_t(\dot{B}^\frac{3}{2})} \|\nabla \delta u\|_{L^2_t(\dot{B}^\frac{3}{2})} + \|u_1\|_{L^2_t(\dot{B}^\frac{3}{2})} \|\nabla u_2\|_{L^1_t(\dot{B}^\frac{3}{2})} + T \left(\|c_1\|_{L^\infty_t(\dot{B}^\frac{3}{2})} + \|c_2\|_{L^\infty_t(\dot{B}^\frac{3}{2})}\right) \|\delta c\|_{L^\infty_t(\dot{B}^\frac{3}{2})}$$

$$+ \|c_1\|_{L^\infty_t(\dot{B}^\frac{3}{2})} \|\nabla^2 \delta u\|_{L^1_t(\dot{B}^\frac{3}{2})},$$

$$\|\delta g_5\|_{L^1_t(\dot{B}^\frac{3}{2})} + \|\delta g_6\|_{L^1_t(\dot{B}^\frac{3}{2})} + \|\delta g_7\|_{L^1_t(\dot{B}^\frac{3}{2})} + \|\delta g_8\|_{L^1_t(\dot{B}^\frac{3}{2})} \lesssim \left(1 + \|c_1\|_{L^\infty_t(\dot{B}^\frac{3}{2})} + \|c_2\|_{L^\infty_t(\dot{B}^\frac{3}{2})}\right) \|\delta c\|_{L^\infty_t(\dot{B}^\frac{3}{2})} \|\nabla H_2 \cdot H_2\|_{L^1_t(\dot{B}^\frac{3}{2})}$$

$$+ \left(1 + \|c_1\|_{L^\infty_t(\dot{B}^\frac{3}{2})}\right) \|\nabla H_2 \cdot \delta H\|_{L^1_t(\dot{B}^\frac{3}{2})},$$

$$\|\delta g_9\|_{L^1_t(\dot{B}^\frac{3}{2})} + \|\delta g_{10}\|_{L^1_t(\dot{B}^\frac{3}{2})} \lesssim \left(1 + \|c_1\|_{L^\infty_t(\dot{B}^\frac{3}{2})} + \|c_2\|_{L^\infty_t(\dot{B}^\frac{3}{2})}\right) \|\delta c\|_{L^\infty_t(\dot{B}^\frac{3}{2})} \|H_2\|_{L^1_t(\dot{B}^\frac{3}{2})}$$

$$+ \left(1 + \|c_1\|_{L^\infty_t(\dot{B}^\frac{3}{2})}\right) \|\nabla H_2\|_{L^1_t(\dot{B}^\frac{3}{2})} \|\delta H\|_{L^1_t(\dot{B}^\frac{3}{2})}.$$
\[ \|\delta h\|_{L^4_t(B_{2,1}^{1/2})} + \|\delta h\|_{L^4_t(B_{2,1}^{1/2})} + \|\delta h\|_{L^4_t(B_{2,1}^{1/2})} + \|\delta h\|_{L^4_t(B_{2,1}^{1/2})} + \|\delta h\|_{L^4_t(B_{2,1}^{1/2})} \]

\[ \lesssim \|\delta u\|_{L^\infty_t(B_{2,1}^{1/2})} \|\nabla H\|_{L^4_t(B_{2,1}^{1/2})} + \|\nabla \delta u\|_{L^4_t(B_{2,1}^{1/2})} \|H_2\|_{L^4_t(B_{2,1}^{1/2})} + \|\delta H\|_{L^4_t(B_{2,1}^{1/2})} \|u_1\|_{L^4_t(B_{2,1}^{1/2})} \]

\[ \lesssim \|\delta u\|_{L^\infty_t(B_{2,1}^{1/2})} \|H_1\|_{L^4_t(B_{2,1}^{1/2})} + \|\delta u\|_{L^4_t(B_{2,1}^{1/2})} \|H_2\|_{L^4_t(B_{2,1}^{1/2})} + \|\delta H\|_{L^4_t(B_{2,1}^{1/2})} \|u_1\|_{L^4_t(B_{2,1}^{1/2})} , \]

and

\[ \|\delta h_6\|_{L^4_t(B_{2,1}^{1/2})} + \|\delta h_7\|_{L^4_t(B_{2,1}^{1/2})} + \|\delta h_8\|_{L^4_t(B_{2,1}^{1/2})} \]

\[ \lesssim \left( \|L_3(c_2) - L_3(c_1)(\nabla \times H_2) \times H_2\|_{L^4_t(B_{2,1}^{1/2})} + \|L_3(c_1)(\nabla \times \delta H) \times H_2\|_{L^4_t(B_{2,1}^{1/2})} + \|L_3(c_1)(\nabla \times H_1) \times \delta H\|_{L^4_t(B_{2,1}^{1/2})} \right) \]

\[ \lesssim \left( 1 + \|c_1\|_{L^\infty_t(B_{2,1}^{3/2})} + \|c_2\|_{L^\infty_t(B_{2,1}^{3/2})} \right) \|\delta c\|_{L^4_t(B_{2,1}^{1/2})} \|\nabla \times H_2\|_{L^4_t(B_{2,1}^{1/2})} + \left( 1 + \|c_1\|_{L^\infty_t(B_{2,1}^{3/2})} \right) \|\nabla \times \delta H\|_{L^4_t(B_{2,1}^{1/2})} \]

\[ \lesssim \left( 1 + \|c_1\|_{L^\infty_t(B_{2,1}^{3/2})} + \|c_2\|_{L^\infty_t(B_{2,1}^{3/2})} \right) \|\delta c\|_{L^4_t(B_{2,1}^{1/2})} \|H_2\|_{L^4_t(B_{2,1}^{1/2})} + \left( 1 + \|c_1\|_{L^\infty_t(B_{2,1}^{3/2})} \right) \|H_1\|_{L^4_t(B_{2,1}^{1/2})} \|\delta H\|_{L^4_t(B_{2,1}^{1/2})} + \|H_1\|_{L^4_t(B_{2,1}^{1/2})} \|\delta H\|_{L^4_t(B_{2,1}^{1/2})} \]

Substituting those estimates back into (5.2), we eventually get

\[ \| (\delta c, \delta u, \delta H) \|_{E_T^{1/2}} \leq Z(T) \| (\delta c, \delta u, \delta H) \|_{E_T^{1/2}} \]
with

\[
Z(T) = e^{\int_0^T \frac{d}{eta^2_1} \left[ \|c_1\|_{L^\infty_t(B^{\frac{1}{2}}_{2,1})} + \|u_2\|_{L^1_t(B^{\frac{3}{2}}_{2,1})} + \|u_2\|_{L^3_t(B^{\frac{3}{2}}_{2,1})} + \|u_1\|_{L^3_t(B^{\frac{3}{2}}_{2,1})} + \right. \\
+ \left. \left( 1 + \|c_1\|_{L^\infty_t(B^{\frac{1}{2}}_{2,1})} + \|c_2\|_{L^\infty_t(B^{\frac{3}{2}}_{2,1})} \right) \|u_2\|_{L^1_t(B^{\frac{3}{2}}_{2,1})} + T \left( \|c_1\|_{L^\infty_t(B^{\frac{3}{2}}_{2,1})} + \|c_2\|_{L^\infty_t(B^{\frac{3}{2}}_{2,1})} \right) \right) \\
+ \left( 1 + \|c_1\|_{L^\infty_t(B^{\frac{1}{2}}_{2,1})} + \|c_2\|_{L^\infty_t(B^{\frac{3}{2}}_{2,1})} \right) \|H_2\|_{L^1_t(B^{\frac{3}{2}}_{2,1})} \|H_2\|_{L^1_t(B^{\frac{3}{2}}_{2,1})} \\
+ \left( 1 + \|c_1\|_{L^\infty_t(B^{\frac{1}{2}}_{2,1})} \right) \|H_2\|_{L^1_t(B^{\frac{3}{2}}_{2,1})} + \left( 1 + \|c_1\|_{L^\infty_t(B^{\frac{1}{2}}_{2,1})} \right) \|H_2\|_{L^1_t(B^{\frac{3}{2}}_{2,1})} \\
+ \|H_1\|_{L^1_t(B^{\frac{3}{2}}_{2,1})} + \|H_2\|_{L^1_t(B^{\frac{3}{2}}_{2,1})} + \|u_1\|_{L^1_t(B^{1}_{2,1})} \\
+ \left( 1 + \|c_1\|_{L^\infty_t(B^{\frac{1}{2}}_{2,1})} + \|c_2\|_{L^\infty_t(B^{\frac{3}{2}}_{2,1})} \right) \|H_2\|_{L^1_t(B^{\frac{3}{2}}_{2,1})} \|H_2\|_{L^1_t(B^{\frac{3}{2}}_{2,1})} \\
+ \left( 1 + \|c_1\|_{L^\infty_t(B^{\frac{1}{2}}_{2,1})} \right) \|H_2\|_{L^1_t(B^{\frac{3}{2}}_{2,1})} + \left( 1 + \|c_1\|_{L^\infty_t(B^{\frac{1}{2}}_{2,1})} \right) \|H_1\|_{L^1_t(B^{\frac{3}{2}}_{2,1})} \right].
\]

We notice that \(\limsup_{T \to 0^+} Z(T) \leq C \|c_1\|_{L^\infty_t(B^{\frac{1}{2}}_{2,1})} \). This is because all other terms involve an integral in time in \(L^1\) or \(L^2\) sense so that as \(T\) goes to zero, all those integrals will converge to zero. Thus, if \(\eta > 0\) is sufficiently small, we get

\[
\|\langle \delta c, \delta u, \delta H \rangle \|_{C^{1\/2}_T} = 0,
\]

for certain \(T > 0\) small enough. Thus, we have shown uniqueness on a small time interval \([0, T]\) such that \((c_1, u_1, H_1) = (c_2, u_2, H_2)\). This completes the proof of the uniqueness in Theorem 1.1.

6 Time decay estimates

At last, we exhibit the time decay estimates of the strong solutions to the system (1.1) for initial data close to a stable equilibrium state in critical regularity framework. We divide it into several steps.

Step 1: Bounds for the low frequencies

We study the following system

\[
\begin{aligned}
\partial_t c + \text{div} u &= f_1, \\
\partial_t u - Au + \nabla c &= f_2, \\
\partial_t H - \Delta H &= f_3, \\
\text{div} H &= 0, \\
(c, u, H) \mid_{t=0} &= (c_0, u_0, H_0),
\end{aligned}
\]

where \(f_1 \equiv f - u \cdot \nabla c\), \(f_2 \equiv g - u \cdot \nabla u\), \(f_3 \equiv h - u \cdot \nabla H\).
Denoting by $A(D)$ the semi-group associated to (6.1), we have for all $q \in \mathbb{Z}$,
\[
\begin{pmatrix}
\hat{\Delta}_q c(t) \\
\hat{\Delta}_q u(t) \\
\hat{\Delta}_q H(t)
\end{pmatrix} = e^{tA(D)} \begin{pmatrix}
\hat{\Delta}_q c_0 \\
\hat{\Delta}_q u_0 \\
\hat{\Delta}_q H_0
\end{pmatrix} + \int_0^t e^{(t-\tau)A(D)} \begin{pmatrix}
\hat{\Delta}_q f_1(\tau) \\
\hat{\Delta}_q f_2(\tau) \\
\hat{\Delta}_q f_3(\tau)
\end{pmatrix} d\tau.
\tag{6.2}
\]
From an explicit computation of the action of $e^{tA(D)}$ in Fourier variables (see e.g. [23]), we discover that there exist positive constants $c_0$ and $C$ depending only on $q_0$ and such that
\[
|\mathcal{F}(e^{tA(D)}u)(\xi)| \leq Ce^{-c_0t|\xi|^2}|\mathcal{F}u(\xi)| \quad \text{for all } |\xi| \leq 2^{q_0}.
\]
Therefore, using Parseval’s equality and the definition of $\hat{\Delta}_q$, we get for all $q \leq q_0$,
\[
\|e^{tA(D)}\hat{\Delta}_q U\|_{L^2} \lesssim e^{-c_02^{2q_0}t}\|\hat{\Delta}_q U\|_{L^2}.
\]
Hence, multiplying by $t^{\frac{3}{2} + \frac{q}{2}}2^{qs}$ and summing up on $q \leq q_0$,
\[
t^{\frac{3}{2} + \frac{q}{2}} \sum_{q \leq q_0} 2^{qs}\|e^{tA(D)}\hat{\Delta}_q U\|_{L^2} \lesssim \sum_{q \leq q_0} 2^{qs} e^{-c_02^{2q_0}t}\|\hat{\Delta}_q U\|_{L^2} t^{\frac{3}{2} + \frac{q}{2}}
\]
\[
\lesssim \sum_{q \leq q_0} 2^{q(s+\frac{3}{2})} e^{-c_02^{2q_0}t}\|\hat{\Delta}_q U\|_{L^2} 2^{q(s+\frac{3}{2})} t^{\frac{3}{2} + \frac{s}{2}}
\]
\[
\lesssim \|U\|_{B_{2,\infty}^{\frac{3}{2}}} \sum_{q \leq q_0} 2^{q(s+\frac{3}{2})} e^{-c_02^{2q_0}t}\|\hat{\Delta}_q U\|_{L^2} 2^{q(s+\frac{3}{2})} t^{\frac{3}{2} + \frac{s}{2}}
\]
\[
\lesssim \|U\|_{B_{2,\infty}^{\frac{3}{2}}} \sum_{q \leq q_0} 2^{q(s+\frac{3}{2})} e^{-c_02^{2q_0}t}\|\hat{\Delta}_q U\|_{L^2} 2^{q(s+\frac{3}{2})} t^{\frac{3}{2} + \frac{s}{2}}.
\tag{6.3}
\]
As for any $\sigma > 0$ there exists a constant $C_{\sigma}$ so that
\[
\sup_{t \geq 0} \sum_{q \in \mathbb{Z}} t^{\frac{3}{2} + \frac{q}{2}} 2^{qs} e^{-c_02^{2q_0}t} \leq C_{\sigma}.
\tag{6.4}
\]
We get from (6.3) and (6.4) that for $s > -3/2$,
\[
\sup_{t \geq 0} t^{\frac{3}{2} + \frac{q}{2}} \|e^{tA(D)}U\|_{B_{2,1}^s}^{\ell} \lesssim \|U\|_{B_{2,\infty}^{\frac{3}{2}}} \|U\|_{B_{2,\infty}^{\frac{3}{2}}}.
\]
It is also obvious that for $s > -3/2$,
\[
\|e^{tA(D)}U\|_{B_{2,1}^s}^{\ell} \lesssim \|U\|_{B_{2,\infty}^{\frac{3}{2}}} \sum_{q \leq q_0} 2^{q(s+\frac{3}{2})} \lesssim \|U\|_{B_{2,\infty}^{\frac{3}{2}}}.
\]
So, setting $\langle t \rangle \overset{\text{def}}{=} \sqrt{1 + t^2}$, we arrive at
\[
\sup_{t \geq 0} \langle t \rangle^{\frac{3}{2} + \frac{q}{2}} \|e^{tA(D)}U\|_{B_{2,1}^s}^{\ell} \lesssim \|U\|_{B_{2,\infty}^{\frac{3}{2}}}^{\frac{3}{2}},
\tag{6.5}
\]
and thus, taking advantage of Duhamel’s formula,
\[
\left\| \int_0^t e^{(t-\tau)A(D)} (f_1, f_2, f_3)(\tau) d\tau \right\|_{B_{2,1}^s}^{\ell} \lesssim \int_0^t \langle t - \tau \rangle^{-(\frac{3}{2} + \frac{q}{2})} \| (f_1, f_2, f_3)(\tau) \|_{B_{2,\infty}^{\frac{3}{2}}}^{\ell} d\tau.
\tag{6.6}
\]
We claim that for all $s \in (-3/2, 2]$, then we have for all $t \geq 0$,
\[
\int_0^t \langle t - \tau \rangle^{-(\frac{3}{2} + \frac{q}{2})} \| (f_1, f_2, f_3)(\tau) \|_{B_{2,\infty}^{\frac{3}{2}}}^{\ell} d\tau \lesssim \langle t \rangle^{-(\frac{3}{2} + \frac{q}{2})} (X^2(t) + D^2(t) + D^3(t)).
\tag{6.7}
\]
Owing to the embedding $L^1 \hookrightarrow B_{2,\infty}^{-\frac{3}{2}}$, it suffices to prove (6.7) with $\| (f_1, f_2, f_3)(\tau) \|_{L^1}^{\ell}$ instead of $\| (f_1, f_2, f_3)(\tau) \|_{B_{2,\infty}^s}^{\ell}$. 

To bound the term with $f_1$, we use the following decomposition:

$$f_1 = u \cdot \nabla c + c \, \text{div} \, u^t + c \, \text{div} \, u^h.$$ 

Now, from Hölder’s inequality, the embedding $\dot{B}^0_{2,1} \hookrightarrow L^2$, the definitions of $D(t), \alpha$ and Lemma 2.15, one may write for all $s \in (-\frac{3}{2}, 2]$,

$$\int_0^t \langle t - \tau \rangle^{-\left(\frac{3}{2} + \frac{s}{2}\right)} \| (u \cdot \nabla c) (\tau) \|_{L^1} \, d\tau$$

$$\lesssim \int_0^t \langle t - \tau \rangle^{-\left(\frac{3}{2} + \frac{s}{2}\right)} \| u \|_{L^2} \| \nabla c \|_{L^2} \, d\tau$$

$$\lesssim \int_0^t \langle t - \tau \rangle^{-\left(\frac{3}{2} + \frac{s}{2}\right)} \| u \|_{\dot{B}^0_{2,1}} \| \nabla c \|_{\dot{B}^0_{2,1}} \, d\tau$$

$$\lesssim \int_0^t \langle t - \tau \rangle^{-\left(\frac{3}{2} + \frac{s}{2}\right)} \left( \| u \|_{\dot{B}^0_{2,1}} + \| u \|_{\dot{B}^0_{2,1}}^h \right) \left( \| \nabla c \|_{\dot{B}^0_{2,1}} + \| \nabla c \|_{\dot{B}^0_{2,1}}^h \right) \, d\tau$$

$$\lesssim \left( \sup_{0 \leq \tau \leq t} \langle \tau \rangle^{\frac{s}{2}} \| u (\tau) \|_{\dot{B}^0_{2,1}} \right) \left( \sup_{0 \leq \tau \leq t} \langle \tau \rangle^{\frac{s}{2}} \| \nabla c (\tau) \|^h_{\dot{B}^0_{2,1}} \right) \int_0^t \langle t - \tau \rangle^{-\left(\frac{3}{2} + \frac{s}{2}\right)} \langle \tau \rangle^{-\frac{3}{2}} \, d\tau$$

$$+ \left( \sup_{0 \leq \tau \leq t} \langle \tau \rangle^{\frac{s}{2}} \| u (\tau) \|^h_{\dot{B}^0_{2,1}} \right) \left( \sup_{0 \leq \tau \leq t} \langle \tau \rangle^{\frac{s}{2}} \| \nabla c (\tau) \|_{\dot{B}^0_{2,1}} \right) \int_0^t \langle t - \tau \rangle^{-\left(\frac{3}{2} + \frac{s}{2}\right)} \langle \tau \rangle^{-\frac{3}{2}} \, d\tau$$

$$+ \left( \sup_{0 \leq \tau \leq t} \langle \tau \rangle^{\frac{s}{2}} \| u (\tau) \|^h_{\dot{B}^0_{2,1}} \right) \left( \sup_{0 \leq \tau \leq t} \langle \tau \rangle^{\frac{s}{2}} \| \nabla c (\tau) \|^h_{\dot{B}^0_{2,1}} \right) \int_0^t \langle t - \tau \rangle^{-\left(\frac{3}{2} + \frac{s}{2}\right)} \langle \tau \rangle^{-\frac{3}{2}} \, d\tau$$

$$\lesssim D^2 (t) \int_0^t \langle t - \tau \rangle^{-\left(\frac{3}{2} + \frac{s}{2}\right)} \langle \tau \rangle^{-\min\left(2, \alpha + \frac{s}{2}, 2\alpha\right)} \, d\tau$$

$$\lesssim \langle t \rangle^{-\left(\frac{3}{2} + \frac{s}{2}\right)} D^2 (t).$$

The term $c \, \text{div} \, u^t$ may be treated along the same lines, we have

$$\int_0^t \langle t - \tau \rangle^{-\left(\frac{3}{2} + \frac{s}{2}\right)} \| \text{div} \, u^t \|_{L^1} \, d\tau$$

$$\lesssim \int_0^t \langle t - \tau \rangle^{-\left(\frac{3}{2} + \frac{s}{2}\right)} \left( \| c \|_{\dot{B}^0_{2,1}} + \| c \|_{\dot{B}^0_{2,1}}^h \right) \| \nabla u \|_{\dot{B}^0_{2,1}} \, d\tau$$

$$\lesssim \left( \sup_{0 \leq \tau \leq t} \langle \tau \rangle^{\frac{s}{2}} \| c (\tau) \|_{\dot{B}^0_{2,1}} \right) \left( \sup_{0 \leq \tau \leq t} \langle \tau \rangle^{\frac{s}{2}} \| u (\tau) \|_{\dot{B}^0_{2,1}} \right) \int_0^t \langle t - \tau \rangle^{-\left(\frac{3}{2} + \frac{s}{2}\right)} \langle \tau \rangle^{-\frac{3}{2}} \, d\tau$$

$$+ \left( \sup_{0 \leq \tau \leq t} \langle \tau \rangle^{\frac{s}{2}} \| c (\tau) \|^h_{\dot{B}^0_{2,1}} \right) \left( \sup_{0 \leq \tau \leq t} \langle \tau \rangle^{\frac{s}{2}} \| u (\tau) \|^h_{\dot{B}^0_{2,1}} \right) \int_0^t \langle t - \tau \rangle^{-\left(\frac{3}{2} + \frac{s}{2}\right)} \langle \tau \rangle^{-\frac{3}{2}} \, d\tau$$

$$\lesssim D^2 (t) \int_0^t \langle t - \tau \rangle^{-\left(\frac{3}{2} + \frac{s}{2}\right)} \langle \tau \rangle^{-\min\left(2, \alpha + \frac{s}{2}\right)} \, d\tau$$

$$\lesssim \langle t \rangle^{-\left(\frac{3}{2} + \frac{s}{2}\right)} D^2 (t).$$
Regarding the term with $c \, \text{div} \, u^h$, we use that if $t \geq 2$,

$$
\int_0^t \langle t - \tau \rangle^{-(\frac{3}{4} + \frac{1}{4})} \| (c \, \text{div} \, u^h)(\tau) \|_{L^1} \, d\tau \\
\lesssim \int_0^t \langle t - \tau \rangle^{-(\frac{3}{4} + \frac{1}{4})} \| c(\tau) \|_{L^2} \| \text{div} \, u(\tau) \|_{L^2} \, d\tau \\
\lesssim \int_0^t \langle t - \tau \rangle^{-(\frac{3}{4} + \frac{1}{4})} \| c(\tau) \|_{B^0_{2,1}} \| \text{div} \, u(\tau) \|_{B^0_{2,1}}^h \, d\tau \\
\lesssim \int_0^1 \langle t - \tau \rangle^{-(\frac{3}{4} + \frac{1}{4})} \| c(\tau) \|_{B^0_{2,1}} \| \text{div} \, u(\tau) \|_{B^0_{2,1}}^h \, d\tau \\
+ \int_1^t \langle t - \tau \rangle^{-(\frac{3}{4} + \frac{1}{4})} \| c(\tau) \|_{B^0_{2,1}} \| \text{div} \, u(\tau) \|_{B^0_{2,1}}^h \, d\tau
$$

\[ \overset{\text{def}}{=} I_1 + I_2. \]

Remembering the definitions of $X(t)$ and $D(t)$, we obtain

$$
I_1 = \int_0^1 \langle t - \tau \rangle^{-(\frac{3}{4} + \frac{1}{4})} \| c(\tau) \|_{B^0_{2,1}} \| \text{div} \, u(\tau) \|_{B^0_{2,1}}^h \, d\tau \\
\lesssim \langle t \rangle^{-(\frac{3}{4} + \frac{1}{4})} \sup_{0 \leq \tau \leq 1} \| c(\tau) \|_{B^0_{2,1}} \int_0^1 \| \text{div} \, u(\tau) \|_{B^0_{2,1}}^h \, d\tau \\
\lesssim \langle t \rangle^{-(\frac{3}{4} + \frac{1}{4})} \sup_{0 \leq \tau \leq 1} \| c(\tau) \|_{B^0_{2,1}} \int_0^1 \| u(\tau) \|_{B^0_{2,1}}^h \, d\tau \\
\lesssim \langle t \rangle^{-(\frac{3}{4} + \frac{1}{4})} D(1) X(1)
$$

and, using the fact that $\langle \tau \rangle \approx \tau$ when $\tau \geq 1$,

$$
I_2 = \int_1^t \langle t - \tau \rangle^{-(\frac{3}{4} + \frac{1}{4})} \| c(\tau) \|_{B^0_{2,1}} \| \text{div} \, u(\tau) \|_{B^0_{2,1}}^h \, d\tau \\
\lesssim \int_1^t \langle t - \tau \rangle^{-(\frac{3}{4} + \frac{1}{4})} \left( \| c(\tau) \|_{B^0_{2,1}}^{1/2} + \| c(\tau) \|_{B^0_{2,1}}^h \right) \| \text{div} \, u(\tau) \|_{B^0_{2,1}}^h \, d\tau \\
\lesssim \left( \sup_{1 \leq \tau \leq t} \langle \tau \rangle^{\frac{1}{2}} \langle \tau \rangle \right) \left( \sup_{1 \leq \tau \leq t} \| \tau \nabla u(\tau) \|_{B^0_{2,1}}^h \right) \int_1^t \langle t - \tau \rangle^{-(\frac{3}{4} + \frac{1}{4})} \langle \tau \rangle^{-\frac{3}{4}} \langle \tau \rangle^{-1} \, d\tau \\
+ \left( \sup_{1 \leq \tau \leq t} \langle \tau \rangle^{\alpha} \langle \tau \rangle \right) \left( \sup_{1 \leq \tau \leq t} \| \tau \nabla u(\tau) \|_{B^0_{2,1}}^h \right) \int_1^t \langle t - \tau \rangle^{-(\frac{3}{4} + \frac{1}{4})} \langle \tau \rangle^{-\alpha} \langle \tau \rangle^{-1} \, d\tau \\
\lesssim D^2(t) \int_1^t \langle t - \tau \rangle^{-(\frac{3}{4} + \frac{1}{4})} \langle \tau \rangle^{-\min(\alpha + 1, \frac{3}{4})} \, d\tau \\
\lesssim \langle t \rangle^{-(\frac{3}{4} + \frac{1}{4})} D^2(t).
$$

Therefore, for $t \geq 2$, we arrive at

$$
\int_0^t \langle t - \tau \rangle^{-(\frac{3}{4} + \frac{1}{4})} \| \text{cd} \, \text{iv} \, u^h(\tau) \|_{L^1} \, d\tau \\
\lesssim \langle t \rangle^{-(\frac{3}{4} + \frac{1}{4})} \left( D^2(t) + X^2(t) \right).
$$

(6.10)
The case $t \leq 2$ is obvious as $\langle t \rangle \approx 1$ and $\langle t - \tau \rangle \approx 1$ for $0 \leq \tau \leq t \leq 2$, and

$$\int_0^t \|c \text{ div } u^h\|_{L^1} d\tau \leq \|c\|_{L_t^\infty(L^2)} \|\text{div } u^h\|_{L_t^1(L^2)}$$

$$\lesssim \|c\|_{L_t^\infty(B^0_{2,1})} \|\text{div } u^h\|_{L_t^1(B^0_{2,1})} \lesssim \|c\|_{L_t^\infty(B^0_{2,1})} \|u^h\|_{L_t^1(B^2_{2,1})}$$

(6.11)

From (6.8)-(6.11), we get

$$\int_0^t (t - \tau)^{-\left(\frac{4}{7} + \frac{4}{7}\right)} \|f_1(\tau)\|_{B^2_{2,\infty}} d\tau \lesssim \langle t \rangle^{-\left(\frac{4}{7} + \frac{4}{7}\right)} (X^2(t) + D^2(t)).$$

Next, in order to bound the term of (6) corresponding to $f_2$, we use the following decomposition

$$f_2 = g - u \cdot \nabla u$$

$$= -u \cdot \nabla u^\ell - u \cdot \nabla u^h - L_1(c)Au + L_2(c)\nabla c$$

$$- L_3(c)(\frac{1}{2} \nabla |H|^2 - H \cdot \nabla H).$$

Similar to (6.9)-(6.11), we have

$$\int_0^t (t - \tau)^{-\left(\frac{4}{7} + \frac{4}{7}\right)} \|u \cdot \nabla u^\ell\|_{L^1} d\tau \lesssim \langle t \rangle^{-\left(\frac{4}{7} + \frac{4}{7}\right)} D^2(t)$$

and

$$\int_0^t (t - \tau)^{-\left(\frac{4}{7} + \frac{4}{7}\right)} \|u \cdot \nabla u^h(\tau)\|_{L^1} d\tau \lesssim \langle t \rangle^{-\left(\frac{4}{7} + \frac{4}{7}\right)} (D^2(t) + X^2(t)).$$

For $L_1(c)Au$, we write that

$$L_1(c)Au = L_1(c)Au^\ell + L_1(c)Au^h,$$

where $L_1$ stands for some smooth function vanishing at 0. Now, we have

$$\int_0^t \langle t - \tau \rangle^{-\left(\frac{4}{7} + \frac{4}{7}\right)} \|L_1(c)Au^\ell\|_{L^1} d\tau$$

$$\lesssim \int_0^t \langle t - \tau \rangle^{-\left(\frac{4}{7} + \frac{4}{7}\right)} \left(\|c\|_{B^0_{2,1}}^h + \|c\|_{B^0_{2,1}}^h\right) \|Au^\ell\|_{B^0_{2,1}} d\tau$$

$$\lesssim \left(\sup_{\tau \in [0,t]} \langle \tau \rangle^\frac{4}{7} \|c(\tau)\|_{B^0_{2,1}}^h\right) \left(\sup_{\tau \in [0,t]} \langle \tau \rangle^\frac{4}{7} \|Au^\ell\|_{B^0_{2,1}}^h\right) \int_0^t \langle t - \tau \rangle^{-\left(\frac{4}{7} + \frac{4}{7}\right)} \langle \tau \rangle^{-\frac{4}{7}} d\tau$$

$$+ \left(\sup_{1 \leq \tau \leq t} \langle \tau \rangle^\alpha \|c(\tau)\|_{B^0_{2,1}}^h\right) \left(\sup_{\tau \in [0,t]} \langle \tau \rangle^\frac{4}{7} \|Au^\ell\|_{B^0_{2,1}}^h\right) \int_0^t \langle t - \tau \rangle^{-\left(\frac{4}{7} + \frac{4}{7}\right)} \langle \tau \rangle^{-\frac{4}{7} + \alpha} d\tau$$

$$\lesssim D^2(t) \int_0^t \langle t - \tau \rangle^{-\left(\frac{4}{7} + \frac{4}{7}\right)} \langle \tau \rangle^{-\min\left(\frac{4}{7} + \frac{4}{7}\right)} d\tau$$

$$\lesssim \langle t \rangle^{-\left(\frac{4}{7} + \frac{4}{7}\right)} D^2(t).$$
To handle the term $L_1(c)Au^h$, we consider the cases $t \geq 2$ and $t \leq 2$ separately. If $t \geq 2$ then we write

$$
\int_0^t \langle t - \tau \rangle^{-(\frac{3}{4} + \frac{3}{2})} \|L_1(c)Au^h\|_{L^1} d\tau \lesssim \int_0^t \langle t - \tau \rangle^{-(\frac{3}{4} + \frac{3}{2})} \|c\|_{\dot{B}^{0}_{2,1}} \|Au^h\|_{\dot{B}^{0}_{2,1}} d\tau
$$

$$
\lesssim \int_0^1 \langle t - \tau \rangle^{-(\frac{3}{4} + \frac{3}{2})} \|c\|_{\dot{B}^{0}_{2,1}} \|Au^h\|_{\dot{B}^{0}_{2,1}} d\tau + \int_1^t \langle t - \tau \rangle^{-(\frac{3}{4} + \frac{3}{2})} \|c\|_{\dot{B}^{0}_{2,1}} \|Au^h\|_{\dot{B}^{0}_{2,1}} d\tau
$$

$$
\equiv K_1 + K_2.
$$

From the definitions of $X(t)$ and $D(t)$, we obtain

$$
K_1 = \int_0^1 \langle t - \tau \rangle^{-(\frac{3}{4} + \frac{3}{2})} \|c\|_{\dot{B}^{0}_{2,1}} \|Au^h\|_{\dot{B}^{0}_{2,1}} d\tau
$$

$$
\lesssim \langle t \rangle^{-(\frac{3}{4} + \frac{3}{2})} \left( \sup_{\tau \in [0,1]} \|c(\tau)\|_{\dot{B}^{0}_{2,1}} \right) \int_0^1 \|u\|_{\dot{B}^{0}_{2,1}} d\tau
$$

$$
\lesssim \langle t \rangle^{-(\frac{3}{4} + \frac{3}{2})} D(1) X(1),
$$

and, using the fact that $\langle \tau \rangle \approx \tau$ when $\tau \geq 1$,

$$
K_2 = \int_1^t \langle t - \tau \rangle^{-(\frac{3}{4} + \frac{3}{2})} \|c\|_{\dot{B}^{0}_{2,1}} \|Au^h\|_{\dot{B}^{0}_{2,1}} d\tau
$$

$$
\lesssim \int_1^t \langle t - \tau \rangle^{-(\frac{3}{4} + \frac{3}{2})} \left( \|c\|_{\dot{B}^{0}_{2,1}} + \|c\|_{\dot{B}^{0}_{2,1}} \right) \|Au^h\|_{\dot{B}^{0}_{2,1}} d\tau
$$

$$
\lesssim \int_1^t \langle t - \tau \rangle^{-(\frac{3}{4} + \frac{3}{2})} \|c\|_{\dot{B}^{0}_{2,1}} \|Au^h\|_{\dot{B}^{0}_{2,1}} d\tau + \int_1^t \langle t - \tau \rangle^{-(\frac{3}{4} + \frac{3}{2})} \|c\|_{\dot{B}^{0}_{2,1}} \|Au^h\|_{\dot{B}^{0}_{2,1}} d\tau
$$

$$
\lesssim \left( \sup_{0 \leq \tau \leq t} \langle \tau \rangle^{\frac{3}{2}} \|c(\tau)\|_{\dot{B}^{0}_{2,1}} \right) \left( \sup_{0 \leq \tau \leq t} \|\tau \nabla u(\tau)\|_{\dot{B}^{0}_{2,1}} \right) \int_1^t \langle t - \tau \rangle^{-(\frac{3}{4} + \frac{3}{2})} \left( \langle \tau \rangle - \frac{3}{2} \right) d\tau
$$

$$
+ \left( \sup_{0 \leq \tau \leq t} \langle \tau \rangle^{\alpha} \|\nabla c(\tau)\|_{\dot{B}^{0}_{2,1}} \right) \left( \sup_{0 \leq \tau \leq t} \|\tau \nabla u(\tau)\|_{\dot{B}^{0}_{2,1}} \right) \int_1^t \langle t - \tau \rangle^{-(\frac{3}{4} + \frac{3}{2})} \langle \tau \rangle^{-(\alpha + 1)} d\tau
$$

$$
\lesssim D^2(t) \int_1^t \langle t - \tau \rangle^{-(\frac{3}{4} + \frac{3}{2})} \langle \tau \rangle^{-\min(\alpha + 1, 7)} d\tau
$$

$$
\lesssim \langle t \rangle^{-(\frac{3}{4} + \frac{3}{2})} D^2(t).
$$

Thus, for $t \geq 2$, we arrive at

$$
\int_0^t \langle t - \tau \rangle^{-(\frac{3}{4} + \frac{3}{2})} \|L_1(c)Au^h\|_{L^1} d\tau \lesssim \langle t \rangle^{-(\frac{3}{4} + \frac{3}{2})} (X^2(t) + D^2(t)).
$$

The case $t \leq 2$ is obvious as $\langle t \rangle \approx 1$ and $\langle t - \tau \rangle \approx 1$ for $0 \leq \tau \leq t \leq 2$.

$$
\int_0^t \|L_1(c)Au^h\|_{L^1} d\tau
$$

$$
\lesssim \int_0^t \|c\|_{\dot{B}^{0}_{2,1}} \|Au^h\|_{\dot{B}^{0}_{2,1}} d\tau
$$

$$
\lesssim \left( \sup_{\tau \in [0,1]} \|c(\tau)\|_{\dot{B}^{0}_{2,1}} \right) \int_0^1 \|u\|_{\dot{B}^{0}_{2,1}} d\tau
$$

$$
\lesssim D(t) X(t).
$$
Similar to (6.8), we have
\[ \int_0^t \langle t - \tau \rangle^{-\left(\frac{2}{5} + \frac{1}{4}\right)} \| L_2(c) \nabla c \|_{L^1} \, d\tau \lesssim \langle t \rangle^{-\left(\frac{2}{3} + \frac{1}{4}\right)} D^2(t). \]
To bound the term \( L_3(c) \left( \frac{1}{2} \nabla |H|^2 - H \cdot \nabla H \right) \), it suffices to consider \( L_3(c) H \cdot \nabla H \) as follows
\[ \int_0^t \langle t - \tau \rangle^{-\left(\frac{2}{3} + \frac{1}{4}\right)} \left\| L_3(c) \left( \frac{1}{2} \nabla |H|^2 - H \cdot \nabla H \right) \right\|_{L^1} \, d\tau \]
\[ \lesssim \int_0^t \langle t - \tau \rangle^{-\left(\frac{2}{3} + \frac{1}{4}\right)} \| L_3(c) \|_{L^2} \| H \cdot \nabla H \|_{L^2} \, d\tau \]
\[ \lesssim \int_0^t \langle t - \tau \rangle^{-\left(\frac{2}{3} + \frac{1}{4}\right)} \| H \|_{B^0_{2,1}} \| \nabla H \|_{B^0_{2,1}} \, d\tau \]
\[ + \int_0^t \langle t - \tau \rangle^{-\left(\frac{2}{3} + \frac{1}{4}\right)} \| c \|_{B^0_{2,1}} \| H \|_{B^0_{2,1}} \| \nabla H \|_{B^0_{2,1}} \, d\tau \]
\[ \overset{\text{def}}{=} L_1 + L_2. \]
We estimate the two terms \( L_1 \) and \( L_2 \) in the following,
\[ L_1 \lesssim \int_0^t \langle t - \tau \rangle^{-\left(\frac{2}{3} + \frac{1}{4}\right)} \left( \| H \|_{B^0_{2,1}} + \| H \|_{B^0_{2,1}} \right) \left( \| \nabla H \|_{B^0_{2,1}} \right) \left( \| \nabla H \|_{B^0_{2,1}} \right) \, d\tau \]
\[ \lesssim \left( \sup_{0 \leq \tau \leq t} \langle \tau \rangle^{\frac{2}{3}} \| H(\tau) \|_{B^0_{2,1}} \right) \left( \sup_{0 \leq \tau \leq t} \langle \tau \rangle^{\frac{2}{3}} \| H(\tau) \|_{B^0_{2,1}} \right) \int_0^t \langle t - \tau \rangle^{-\left(\frac{2}{3} + \frac{1}{4}\right)} \langle \tau \rangle^{-\frac{3}{4} - \frac{3}{4}} \, d\tau \]
\[ + \left( \sup_{0 \leq \tau \leq t} \langle \tau \rangle^{\frac{2}{3}} \| H(\tau) \|_{B^0_{2,1}} \right) \left( \sup_{0 \leq \tau \leq t} \langle \tau \rangle^{\frac{2}{3}} \| \nabla H(\tau) \|_{B^0_{2,1}} \right) \int_0^t \langle t - \tau \rangle^{-\left(\frac{2}{3} + \frac{1}{4}\right)} \langle \tau \rangle^{-\frac{3}{4} + \frac{2}{3}} \, d\tau \]
\[ + \left( \sup_{0 \leq \tau \leq t} \langle \tau \rangle^{\frac{2}{3}} \| H(\tau) \|_{B^0_{2,1}} \right) \left( \sup_{0 \leq \tau \leq t} \langle \tau \rangle^{\frac{2}{3}} \| \nabla H(\tau) \|_{B^0_{2,1}} \right) \int_0^t \langle t - \tau \rangle^{-\left(\frac{2}{3} + \frac{1}{4}\right)} \langle \tau \rangle^{-2\alpha} \, d\tau \]
\[ \lesssim D^2(t) \int_0^t \langle t - \tau \rangle^{-\left(\frac{2}{3} + \frac{1}{4}\right)} \langle \tau \rangle^{-\min\left(\frac{11}{4}, \frac{3}{4} + 2\alpha \right)} \, d\tau \]
\[ \lesssim \langle t \rangle^{-\left(\frac{2}{3} + \frac{1}{4}\right)} D^2(t), \]
and
\[ L_2 \lesssim \int_0^t \langle t - \tau \rangle^{-\left(\frac{2}{3} + \frac{1}{4}\right)} \left( \| c \|_{B^0_{2,1}} + \| c \|_{B^0_{2,1}} \right) \| H \|_{B^0_{2,1}} \| \nabla H \|_{B^0_{2,1}} \, d\tau \]
\[ \lesssim \left( \sup_{0 \leq \tau \leq t} \langle \tau \rangle^{\frac{2}{3}} \| c(\tau) \|_{B^0_{2,1}} \right) \int_0^t \langle t - \tau \rangle^{-\left(\frac{2}{3} + \frac{1}{4}\right)} \langle \tau \rangle^{-\frac{3}{4}} \| H \|_{B^0_{2,1}} \| \nabla H \|_{B^0_{2,1}} \, d\tau \]
\[ + \left( \sup_{0 \leq \tau \leq t} \langle \tau \rangle^{\alpha} \| c(\tau) \|_{B^0_{2,1}} \right) \int_0^t \langle t - \tau \rangle^{-\left(\frac{2}{3} + \frac{1}{4}\right)} \langle \tau \rangle^{-2\alpha} \| H \|_{B^0_{2,1}} \| \nabla H \|_{B^0_{2,1}} \, d\tau \]
\[ \lesssim D^3(t) \int_0^t \langle t - \tau \rangle^{-\left(\frac{2}{3} + \frac{1}{4}\right)} \langle \tau \rangle^{-\min\left(\frac{2}{3}, 2\alpha + \frac{3}{4} \right)} \, d\tau \]
\[ \lesssim \langle t \rangle^{-\left(\frac{2}{3} + \frac{1}{4}\right)} D^3(t). \]
Thus,
\[ \int_0^t \langle t - \tau \rangle^{-\left(\frac{2}{3} + \frac{1}{4}\right)} \| f_2(\tau) \|_{B^0_{2,1}} \, d\tau \lesssim \langle t \rangle^{-\left(\frac{2}{3} + \frac{1}{4}\right)} \left( X^2(t) + D^2(t) + D^3(t) \right). \]
Finally, we bound the term $f_3$. Similar to (6.8)-(6.11), we have

$$
\int_0^t \langle t - \tau \rangle^{-\left(\frac{\alpha}{2} + \frac{3}{2}\right)} \left( \|u \cdot \nabla H\|_{L^1} + \|H(\text{div } u)\|_{L^1} + \|H \cdot \nabla u\|_{L^1} \right) d\tau \\
\lesssim \langle t \rangle^{-\left(\frac{\alpha}{2} + \frac{3}{2}\right)} \left( X^2(t) + D^2(t) \right).
$$

To bound the term $\nabla \times (L_3(c)(\nabla \times H) \times H)$, employing Bernstein’s inequality and Proposition 2.8, we conclude that

$$
\int_0^t \langle t - \tau \rangle^{-\left(\frac{\alpha}{2} + \frac{3}{2}\right)} \left\| \nabla \times (L_3(c)(\nabla \times H) \times H) \right\|_{\dot{B}^{\frac{3}{2}, \infty}_2} \, d\tau \\
\lesssim \int_0^t \langle t - \tau \rangle^{-\left(\frac{\alpha}{2} + \frac{3}{2}\right)} \left\| L_3(c)(\nabla \times H) \times H \right\|_{\dot{B}^{\frac{3}{2}, \frac{1}{2}}_2} \, d\tau \\
\lesssim \int_0^t \langle t - \tau \rangle^{-\left(\frac{\alpha}{2} + \frac{3}{2}\right)} \left\| H \right\|_{\dot{B}^{\frac{3}{2}, 1}_2, \dot{B}^{\frac{3}{2}, \frac{1}{2}}_2} \left\| \nabla H \right\|_{\dot{B}^{\frac{3}{2}, \frac{1}{2}}_2} \, d\tau \\
+ \int_0^t \langle t - \tau \rangle^{-\left(\frac{\alpha}{2} + \frac{3}{2}\right)} \left\| c \right\|_{\dot{B}^{\frac{3}{2}, 1}_2} \left\| H \right\|_{\dot{B}^{\frac{3}{2}, 1}_2, \dot{B}^{\frac{3}{2}, \frac{1}{2}}_2} \left\| \nabla H \right\|_{\dot{B}^{\frac{3}{2}, \frac{1}{2}}_2} \, d\tau \\
\overset{\text{def}}{=} M_1 + M_2.
$$

We bound the terms $M_1$ and $M_2$ as follows

$$
M_1 \lesssim \int_0^t \langle t - \tau \rangle^{-\left(\frac{\alpha}{2} + \frac{3}{2}\right)} \left( \|H\|_{\dot{B}^{\frac{3}{2}, \frac{1}{2}}_2}^{\varepsilon} + \|H\|_{\dot{B}^{\frac{3}{2}, \frac{1}{2}}_2}^{\eta} \right) \left( \|\nabla H\|_{\dot{B}^{\frac{3}{2}, \frac{1}{2}}_2} - \|\nabla H\|_{\dot{B}^{\frac{3}{2}, \frac{1}{2}}_2} \right) \, d\tau \\
\lesssim \left( \sup_{0 \leq \tau \leq t} \langle \tau \rangle^{\frac{3}{2}} \|H(\tau)\|_{\dot{B}^{\frac{3}{2}, \frac{1}{2}}_2}^{\varepsilon} \right) \left( \sup_{0 \leq \tau \leq t} \langle \tau \rangle^{\frac{3}{2}} \|H(\tau)\|_{\dot{B}^{\frac{3}{2}, \frac{1}{2}}_2}^{\eta} \right) \int_0^t \langle t - \tau \rangle^{-\left(\frac{\alpha}{2} + \frac{3}{2}\right)} \langle \tau \rangle^{-\frac{3}{2}} \, d\tau \\
+ \left( \sup_{0 \leq \tau \leq t} \langle \tau \rangle^{\frac{3}{2}} \|H(\tau)\|_{\dot{B}^{\frac{3}{2}, \frac{1}{2}}_2}^{\varepsilon} \right) \left( \sup_{0 \leq \tau \leq t} \langle \tau \rangle^{\frac{3}{2}} \|\nabla H(\tau)\|_{\dot{B}^{\frac{3}{2}, \frac{1}{2}}_2}^{\eta} \right) \int_0^t \langle t - \tau \rangle^{-\left(\frac{\alpha}{2} + \frac{3}{2}\right)} \langle \tau \rangle^{-\frac{3}{2}} \, d\tau \\
+ \left( \sup_{0 \leq \tau \leq t} \langle \tau \rangle^{\frac{3}{2}} \|H(\tau)\|_{\dot{B}^{\frac{3}{2}, \frac{1}{2}}_2}^{\varepsilon} \right) \left( \sup_{0 \leq \tau \leq t} \langle \tau \rangle^{\frac{3}{2}} \|\nabla H(\tau)\|_{\dot{B}^{\frac{3}{2}, \frac{1}{2}}_2}^{\eta} \right) \int_0^t \langle t - \tau \rangle^{-\left(\frac{\alpha}{2} + \frac{3}{2}\right)} \langle \tau \rangle^{-\frac{3}{2}} \, d\tau \\
\lesssim D^2(t) \int_0^t \langle t - \tau \rangle^{-\left(\frac{\alpha}{2} + \frac{3}{2}\right)} \langle \tau \rangle^{-\frac{3}{2}} \, d\tau \\
\lesssim \langle t \rangle^{-\left(\frac{\alpha}{2} + \frac{3}{2}\right)} D^2(t),
$$

and

$$
M_2 \lesssim \int_0^t \langle t - \tau \rangle^{-\left(\frac{\alpha}{2} + \frac{3}{2}\right)} \left( \|c\|_{\dot{B}^{\frac{3}{2}, \frac{1}{2}}_2}^{\varepsilon} + \|c\|_{\dot{B}^{\frac{3}{2}, \frac{1}{2}}_2}^{\eta} \right) \left( \nabla H\right)_{\dot{B}^{\frac{3}{2}, \frac{1}{2}}_2}^{-\frac{1}{2}} \, d\tau \\
\lesssim \left( \sup_{0 \leq \tau \leq t} \langle \tau \rangle^{\frac{3}{2}} \|c(\tau)\|_{\dot{B}^{\frac{3}{2}, \frac{1}{2}}_2}^{\varepsilon} \right) \int_0^t \langle t - \tau \rangle^{-\left(\frac{\alpha}{2} + \frac{3}{2}\right)} \langle \tau \rangle^{-\frac{3}{2}} \left( \|H\|_{\dot{B}^{\frac{3}{2}, \frac{1}{2}}_2} \left\| \nabla H \right\|_{\dot{B}^{\frac{3}{2}, \frac{1}{2}}_2}^{-\frac{1}{2}} \right) \, d\tau \\
+ \left( \sup_{0 \leq \tau \leq t} \langle \tau \rangle^{\frac{3}{2}} \|c(\tau)\|_{\dot{B}^{\frac{3}{2}, \frac{1}{2}}_2}^{\varepsilon} \right) \int_0^t \langle t - \tau \rangle^{-\left(\frac{\alpha}{2} + \frac{3}{2}\right)} \langle \tau \rangle^{-\frac{3}{2}} \left( \|H\|_{\dot{B}^{\frac{3}{2}, \frac{1}{2}}_2} \left\| \nabla H \right\|_{\dot{B}^{\frac{3}{2}, \frac{1}{2}}_2}^{-\frac{1}{2}} \right) \, d\tau \\
\lesssim D^3(t) \int_0^t \langle t - \tau \rangle^{-\left(\frac{\alpha}{2} + \frac{3}{2}\right)} \langle \tau \rangle^{-\frac{3}{2}} \, d\tau \\
\lesssim \langle t \rangle^{-\left(\frac{\alpha}{2} + \frac{3}{2}\right)} D^3(t).
$$
Hence
\[ \int_0^t \langle t - \tau \rangle^{-\left(\frac{3}{2} + \frac{3}{2}\right)} \| f_\delta(\tau) \|_{L_{t}^{2}, B_{x}^{0, \infty}}^{} d\tau \lesssim \langle t \rangle^{-\left(\frac{3}{2} + \frac{3}{2}\right)} \left( X^2(t) + D^2(t) + D^3(t) \right). \]
Thus, we complete the proof of (6.7). Combining with (6.5) and (6.7), we conclude that for all \( t \geq 0 \) and \( s \in (-\frac{3}{2}, 2) \),
\[ \langle t \rangle^{\frac{3}{2} + \frac{3}{2}} \| (c, u, H) \|_{L_{t}^{1}}^{} \lesssim D_0 + X^2(t) + D^2(t) + D^3(t). \tag{6.12} \]

**Step 2: Decay estimates for the high frequencies of \((\nabla c, u, \nabla H)\)**

Now, the starting point is Inequality (3.11) which implies that for \( q \geq q_0 \) and for some \( c_0 = c(q_0) > 0 \), we have
\[ \frac{1}{2} \frac{d}{dt} \alpha_q^2 + c_0 \alpha_q^2 \leq \left( \| (f_q, g_q, h_q, \nabla f_q, \nabla h_q) \|_{L^2} + \| R_q(u, c) \|_{L^2} + \| R_q(u, u) \|_{L^2} \right. \]
\[ + \| R_q(u, H) \|_{L^2} + \| \tilde{R}_k(u, c) \|_{L^2} + \| \tilde{R}_k(u, H) \|_{L^2} + \| \nabla u \|_{L^\infty} \alpha_q \bigg) \alpha_q, \]
where
\[ f_q = \hat{\Delta} q f, \quad g_q = \hat{\Delta} q g, \quad h_q = \hat{\Delta} q h, \]
in which
\[ R_q(u, b) \overset{\text{def}}{=} [u \cdot \nabla, \hat{\Delta} q] b = u \cdot \nabla \hat{\Delta} q b - \hat{\Delta} q (u \cdot \nabla b) \quad \text{for} \quad b \in \{ c, u, H \}, \]
\[ \tilde{R}_q(u, b) \overset{\text{def}}{=} [u \cdot \nabla, \partial_i \hat{\Delta} q] b = u \cdot \nabla \partial_i \hat{\Delta} q b - \partial_i \hat{\Delta} q (u \cdot \nabla b) \quad \text{for} \quad b \in \{ c, H \}. \]

After time integration, we discover that
\[ e^{\alpha_q t} \alpha_q(t) \leq \alpha_q(0) + \int_0^t \left( \| (f_q, g_q, h_q, \nabla f_q, \nabla h_q) \|_{L^2} + \| R_q(u, c) \|_{L^2} + \| R_q(u, u) \|_{L^2} \right. \]
\[ \left. + \| R_q(u, H) \|_{L^2} + \| \tilde{R}_k(u, c) \|_{L^2} + \| \tilde{R}_k(u, H) \|_{L^2} + \| \nabla u \|_{L^\infty} \right) d\tau. \]

For \( q \geq q_0 \), we have \( \alpha_q \approx \| (\nabla \Delta_q c, \Delta_q u, \nabla \Delta_q H) \|_{L^2} \). Then,
\[ \langle t \rangle^\alpha \| (\nabla \Delta_q c, \Delta_q u, \nabla \Delta_q H)(t) \|_{L^2} \lesssim \langle t \rangle^\alpha e^{-\alpha q t} \| (\nabla \Delta_q c, \Delta_q u, \nabla \Delta_q H)(0) \|_{L^2} \]
\[ + \langle t \rangle^\alpha \int_0^t e^{\alpha q (t - \tau)} \left( \| (f_q, g_q, h_q, \nabla f_q, \nabla h_q) \|_{L^2} + \| R_q(u, c) \|_{L^2} + \| R_q(u, u) \|_{L^2} \right. \]
\[ + \| R_q(u, H) \|_{L^2} + \| \tilde{R}_k(u, c) \|_{L^2} + \| \tilde{R}_k(u, H) \|_{L^2} + \| \nabla u \|_{L^\infty} \alpha_q \bigg) d\tau \]
and thus, multiplying both sides by \( 2^\frac{x}{2} \), taking the supremum on \([0, T]\), and summing up over \( q \geq q_0 \),
\[ \| \langle t \rangle^\alpha (\nabla c, u, \nabla H) \|_{L^1_T(B_x^{0, \infty})}^h \lesssim \| (\nabla c_0, u_0, \nabla H_0) \|_{B_x^{0, \infty}}^h + \sum_{q \geq q_0} \sup_{0 \leq t \leq T} \langle t \rangle^\alpha \int_0^t e^{\alpha q (t - \tau)} 2^\frac{x}{2} S_q \right) \right) \]
with \( S_q \overset{\text{def}}{=} \sum_{i=1}^{7} S_q^i \) and

\[
S_q^1 = \|(f, g, h, q, \nabla f, \nabla h, q)\|_{L^2}, \quad S_q^2 = \|R_q(u, c)\|_{L^2}, \quad S_q^3 = \|R_q(u, c)\|_{L^2}, \quad S_q^4 = \|R_q(u, H)\|_{L^2}, \quad S_q^5 = \|\tilde{R}_q(u, c)\|_{L^2}, \quad S_q^6 = \|\tilde{R}_q(u, H)\|_{L^2}, \\
S_q^7 = \|\nabla u\|_{L^\infty} \|(\Delta_q \nabla c, \Delta_q u, \Delta_q \nabla H)\|_{L^2}.
\]

Bounding the sum, for \( 0 \leq t \leq 2 \), and taking advantage of Proposition 2.12, we end up with

\[
\sum_{q \geq q_0} \sup_{0 \leq t \leq 2} (t)^{\frac{3}{2}} \int_0^t e^{c_1(t-\tau)} 2^{\frac{q}{2}} S_q(\tau) \, d\tau \lesssim \int_0^2 (\|(f, g, h, \nabla f, \nabla h)\|_{B_{2,1}^{\frac{3}{2}}}^h + \|\nabla u\|_{B_{2,1}^{\frac{3}{2}}}^\| c, u, H, \nabla c, \nabla H \|_{B_{2,1}^{\frac{3}{2}}}^\| \) \, d\tau \tag{6.14}
\]

\[
\lesssim \int_0^2 (\|(\nabla f, g, \nabla h)\|_{B_{2,1}^{\frac{1}{2}}}^h + \|\nabla u\|_{B_{2,1}^{\frac{3}{2}}}^\| c, u, H, \nabla c, \nabla H \|_{B_{2,1}^{\frac{3}{2}}}^\| \) \, d\tau \\
\overset{\text{def}}{=} Q_1 + Q_2.
\]

From Propositions 2.8-2.10, we estimate the terms \( Q_1 \) and \( Q_2 \) as follows

\[
\int_0^2 \|\nabla f\|_{B_{2,1}^{\frac{1}{2}}}^h \, d\tau \lesssim \int_0^2 \|f\|_{B_{2,1}^{\frac{3}{2}}}^h \, d\tau \\
\lesssim \int_0^2 \|c\|_{\dot{B}_{2,1}^{\frac{3}{2}}}^h \|\nabla u\|_{\dot{B}_{2,1}^{\frac{3}{2}}}^\| \, d\tau \\
\lesssim \int_0^2 \|c\|_{\dot{B}_{2,1}^{\frac{3}{2}}}^h \|\nabla u\|_{\dot{B}_{2,1}^{\frac{3}{2}}}^\| \, d\tau \\
\lesssim \|c\|_{L^\infty(\dot{B}_{2,1}^\frac{3}{2})} \|u\|_{L^1(\dot{B}_{2,1}^\frac{3}{2})} \lesssim X^2(2),
\]

\[
\int_0^2 \|g\|_{B_{2,1}^{\frac{1}{2}}}^h \, d\tau \lesssim \int_0^2 \|L_1(c)Au\|_{B_{2,1}^{\frac{3}{2}}}^h \, d\tau + \int_0^2 \|L_2(c)\nabla c\|_{B_{2,1}^{\frac{1}{2}}}^h \, d\tau \\
+ \int_0^2 \|L_3(c)(\frac{1}{2} \nabla |H|^2 - H \cdot \nabla H)\|_{B_{2,1}^{\frac{1}{2}}}^h \, d\tau \\
\lesssim \int_0^2 \|c\|_{\dot{B}_{2,1}^{\frac{3}{2}}}^h \|Au\|_{\dot{B}_{2,1}^{\frac{3}{2}}}^\| \, d\tau + \int_0^2 \|c\|_{\dot{B}_{2,1}^{\frac{3}{2}}}^h \|\nabla c\|_{\dot{B}_{2,1}^{\frac{1}{2}}}^\| \, d\tau \\
+ \int_0^2 (1 + \|L_1(c)\|_{\dot{B}_{2,1}^{\frac{3}{2}}}^\| ) \|H \cdot \nabla H\|_{\dot{B}_{2,1}^{\frac{1}{2}}}^\| \, d\tau \\
\lesssim \|c\|_{L^\infty(\dot{B}_{2,1}^{\frac{1}{2}})} \|u\|_{L_1^\infty(\dot{B}_{2,1}^{\frac{3}{2}})} + \|c\|_{L_1^\infty(\dot{B}_{2,1}^{\frac{3}{2}})}^\| \|H\|_{L_1^\infty(\dot{B}_{2,1}^{\frac{3}{2}})}^2 \\
+ (1 + \|c\|_{L^\infty(\dot{B}_{2,1}^{\frac{1}{2}})}) \|H\|_{L_1^\infty(\dot{B}_{2,1}^{\frac{3}{2}})} \|H\|_{L_1^\infty(\dot{B}_{2,1}^{\frac{3}{2}})}^\| \\
\lesssim X^2(2) + X^3(2),
\]
\[ \int_0^2 \| \nabla h \|_{B^{2}_{2,1}}^h \, d\tau \lesssim \int_0^2 \| h \|_{B^{2}_{2,1}}^h \, d\tau \]
\[ \lesssim \int_0^2 \| H(\div u) \|_{B^{2}_{2,1}}^h \, d\tau + \int_0^2 \| H \cdot \nabla u \|_{B^{2}_{2,1}}^h \, d\tau \]
\[ + \int_0^2 \| \nabla \times (L_3(c)(\nabla \times H) \times H) \|_{B^{2}_{2,1}}^h \, d\tau \]
\[ \lesssim \int_0^2 \| H \cdot \nabla u \|_{B^{2}_{2,1}}^h \, d\tau + \int_0^2 \| \nabla \times (L_3(c)(\nabla \times H) \times H) \|_{B^{2}_{2,1}}^h \, d\tau \]
\[ \lesssim \| H \|_{L^{\infty}(B^{\frac{1}{2}}_{2,\tau})} \| u \|_{L^1(B^{\frac{3}{2}}_{2,1})} + (1 + \| c \|_{L^\infty(B^{\frac{1}{2}}_{2,\tau})}) \| \nabla H \|_{L^1(\dot{B}^{\frac{3}{2}}_{2,1})} \| H \|_{L^{\infty}(B^{\frac{3}{2}}_{2,\tau})} \]
\[ \lesssim X^2(2) + X^3(2), \]

and
\[ \int_0^2 \| \nabla u \|_{B^{2}_{2,1}}^h \| (c, u, H, \nabla c, \nabla H) \|_{B^{2}_{2,1}}^h \, d\tau \]
\[ \lesssim \| \nabla u \|_{L^1(B^{\frac{3}{2}}_{2,1})} \| (c, u, H, \nabla c, \nabla H) \|_{L^{\infty}(B^{\frac{1}{2}}_{2,1})} \]
\[ \lesssim \| u \|_{L^1(B^{\frac{3}{2}}_{2,1})} \| (c, H) \|_{L^{\infty}(B^{\frac{3}{2}}_{2,1})} + \| u \|_{L^{\infty}(B^{\frac{3}{2}}_{2,1})} \]
\[ \lesssim X^2(2). \]

Therefore, for the case \( t \leq 2 \),
\[ \sum \sup_{q \geq q_0} \left( (t)^{\alpha} \int_0^t e^{\gamma(t-s)} \frac{1}{2} S_q \, d\tau \right) \lesssim X^2(2) + X^3(2). \tag{6.15} \]

To bound the supremum on \([2, T]\), we split the integral on \([0, t]\) into integrals on \([0, 1]\) and \([1, t]\), respectively. The \([0, 1]\) part of the integral is easy to handle, we have
\[ \sum \sup_{q \geq q_0} \left( (t)^{\alpha} \int_0^1 e^{\gamma(t-s)} \frac{1}{2} S_q \, d\tau \right) \leq \sum \sup_{2 \leq t \leq T} \left( (t)^{\alpha} \int_0^1 e^{\gamma(t-s)} \frac{1}{2} S_q \, d\tau \right) \]
\[ \lesssim \int_0^1 \sum_{q \geq q_0} e^{\gamma(s)} S_q \, d\tau. \]

Hence
\[ \sum \sup_{q \geq q_0} \left( (t)^{\alpha} \int_0^1 e^{\gamma(t-s)} \frac{1}{2} S_q \, d\tau \right) \lesssim X^2(1) + X^3(1). \tag{6.16} \]

Let us finally consider the \([1, t]\) part of the integral for \( 2 \leq t \leq T \). We shall use repeatedly the following inequalities
\[ \| \nabla u \|_{L^{\infty}(B^{\frac{3}{2}}_{2,1})} \lesssim D(t), \tag{6.17} \]
and
\[ \| \nabla H \|_{L^{\infty}(B^{\frac{3}{2}}_{2,1})} \lesssim D(t), \tag{6.18} \]
which are straightforward as regards the high frequencies of \( u \) and \( H \) and stem from
\[ \| \nabla u \|_{L^{\infty}(B^{\frac{3}{2}}_{2,1})} \lesssim \| (\tau)^{\frac{1}{2}} \nabla u \|_{L^{\infty}(B^{\frac{3}{2}}_{2,1})} \lesssim \| (\tau)^{\frac{1}{2}} u \|_{L^{\infty}(B^{\frac{3}{2}}_{2,1})} \lesssim D(t), \]
\[ \| \nabla H \|_{L^{\infty}(B^{\frac{3}{2}}_{2,1})} \lesssim D(t), \]
as well as
\[ \| \tau \nabla H \|_{L^p_t(B_{2,1}^\frac{3}{2})} \lesssim \| \langle \tau \rangle^{\frac{3}{2}} \nabla H \|_{L^p_t(B_{2,1}^\frac{3}{2})} \lesssim \| \langle \tau \rangle^{\frac{3}{2}} \tau H \|_{L^p_t(B_{2,1}^\frac{3}{2})} \lesssim D(t), \]
for the low frequencies of \( u \) and \( H \).

Regarding the contribution of \( S^1_q \), by Lemma 2.15 we first notice that
\[
\sum_{q \geq q_0} \sup_{2 \leq t \leq T} \left( \langle t \rangle^\alpha \int_1^t e^{c_0(t-t')} 2^{\frac{5}{2}} S^1_q(t') d\tau \right)
= \sum_{q \geq q_0} \sup_{2 \leq t \leq T} \left( \langle t \rangle^\alpha \int_1^t e^{c_0(t-t')} 2^{\frac{5}{2}} \| (f_q, g_q, h_q, \nabla f_q, \nabla h_q) \|_{L^2} d\tau \right)
\lesssim \| \tau^\alpha (f, g, h, \nabla f, \nabla h) \|_{L^2_t(B_{2,1}^\frac{1}{2})}^h
\lesssim \| \tau^\alpha (\nabla f, g, \nabla h) \|_{L^2_t(B_{2,1}^\frac{1}{2})}^h.
\]
Now, product laws in tilde spaces ensures that
\[
\| \tau^\alpha \nabla f \|_{L^2_t(B_{2,1}^\frac{1}{2})}^h \lesssim \| \tau^{\alpha-1} c \|_{L^2_t(B_{2,1}^\frac{3}{2})}^h \| \tau \|_{L^2_t(B_{2,1}^\frac{3}{2})}. \tag{6.19}
\]
The high frequencies of the first term of the r.h.s is obviously bounded by \( D(T) \). That is,
\[
\| \tau^{\alpha-1} c \|_{L^2_t(B_{2,1}^\frac{3}{2})}^h \lesssim \| \tau^\alpha c \|_{L^2_t(B_{2,1}^\frac{3}{2})}^h \lesssim D(T). \tag{6.19}
\]
As for the low frequencies of the first term of the r.h.s, we notice that for all small enough \( \epsilon > 0 \),
\[
\| \tau^{\alpha-1} c \|_{L^2_t(B_{2,1}^\frac{3}{2})}^h \lesssim \| \tau^{\alpha-1} c \|_{L^2_t(B_{2,1}^\frac{3}{2}-2\epsilon)}^h
\lesssim \| \tau^{\alpha-\frac{3}{2}+\epsilon} \tau^{-\epsilon} c \|_{L^2_t(B_{2,1}^\frac{3}{2}-2\epsilon)}^h
\lesssim \| \tau^{\frac{3}{2}-\epsilon} c \|_{L^2_t(B_{2,1}^\frac{3}{2}-2\epsilon)}^h
\lesssim D(T). \tag{6.20}
\]
Combining with (6.19) and (6.20), we obtain
\[
\| \tau^{\alpha-1} c \|_{L^2_t(B_{2,1}^\frac{3}{2})}^h \lesssim D(T). \tag{6.21}
\]
Similarly,
\[
\| \tau^{\alpha-1} H \|_{L^2_t(B_{2,1}^\frac{3}{2})}^h \lesssim D(T). \tag{6.22}
\]
Therefore, using (6.17) and (6.21) we get
\[
\| \tau^\alpha \nabla f \|_{L^2_t(B_{2,1}^\frac{3}{2})}^h \lesssim D^2(T).
\]
Noticing that \( g = L_2(c) \nabla c + L_1(c) Au - L_3(c) (\frac{1}{2} \nabla |H|^2 - H \cdot \nabla H) \), we shall use repeatedly the following inequality
\[
\| c \|_{L^2_t(B_{2,1}^\frac{3}{2})} \lesssim \| c \|_{L^2_t(B_{2,1}^\frac{3}{2})} \lesssim X(T). \tag{6.23}
\]
Employing (6.19) and (6.23), we obtain
\[
\|\tau^\alpha L_2(c) \nabla c^\ell\|_{L^\infty_T(B^1_{2,1})} \lesssim \|\tau c\|_{L^\infty_T(B^1_{2,1})} \|\tau^{\alpha-1} \nabla c\|_{L^\infty_T(B^1_{2,1})}^\ell \\
\lesssim \|\tau c\|_{L^\infty_T(B^1_{2,1})} D(T) \\
\lesssim \left(\|\tau c\|_{L^\infty_T(B^1_{2,1})}^\ell + \|\tau c\|_{L^\infty_T(B^1_{2,1})}^h\right) D(T) \\
\lesssim \left(\|\tau^{\frac{3}{2}-\varepsilon} c\|_{L^\infty_T(B^1_{2,1})}^\ell + \|\langle\tau\rangle^{\alpha} c\|_{L^\infty_T(B^1_{2,1})}^h\right) D(T) \\
\lesssim D^2(T).
\]

Thus,
\[
\|\tau^\alpha L_2(c) \nabla c\|_{L^\infty_T(B^1_{2,1})}^h \lesssim X^2(T) + D^2(T).
\]

From (6.17) and (6.21), we also see that
\[
\|\tau^\alpha L_1(c)Au\|_{L^\infty_T(B^1_{2,1})}^h \lesssim \|\tau \nabla^2 u\|_{L^\infty_T(B^1_{2,1})} \|\tau^{\alpha-1} c\|_{L^\infty_T(B^1_{2,1})} \\
\lesssim D^2(T).
\]

Employing Propositions 2.8-2.10 and (6.22), for the term \(L_3(c)\left(\frac{1}{2} \nabla |H|^2 - H \cdot \nabla H\right)\), we have
\[
\left\|\tau^\alpha L_3(c)\left(\frac{1}{2} \nabla |H|^2 - H \cdot \nabla H\right)\right\|_{L^\infty_T(B^1_{2,1})}^h \\
\lesssim \|L_3(c)\|_{L^\infty_T(B^1_{2,1})} \|\tau^{\alpha} H \cdot \nabla H\|_{L^\infty_T(B^1_{2,1})} \\
\lesssim \left(1 + \|L_1(c)\|_{L^\infty_T(B^1_{2,1})}^\alpha\right) \left\|\tau^{\alpha-1} H\right\|_{L^\infty_T(B^1_{2,1})} \|\nabla H\|_{L^\infty_T(B^1_{2,1})} \|\nabla H\|_{L^\infty_T(B^1_{2,1})} \\
\lesssim \left(1 + \|L_1(c)\|_{L^\infty_T(B^1_{2,1})}^\alpha\right) \left\|\tau^{\alpha-1} H\right\|_{L^\infty_T(B^1_{2,1})}^2 \\
\lesssim X^2(T) + D^2(T) + D^4(T).
\]
For the second term on the right-side of (6.25), by virtue of (6.18), (6.22) and (6.23) we obtain
\[
\left\| \left\| \tau^\alpha \nabla \times \left( L_3(c)(\nabla \times H) \times H \right) \right\|_{L_t^\infty(B_{2,1}^{\frac{3}{2}})} \right\|
\leq \left\| \tau^\alpha L_3(c)(\nabla \times H) \times H \right\|_{L_t^\infty(B_{2,1}^{\frac{3}{2}})}
\leq \left\| L_3(c) \right\|_{L_t^\infty(B_{2,1}^{\frac{3}{2}})} \left\| \tau^\alpha H \cdot \nabla H \right\|_{L_t^\infty(B_{2,1}^{\frac{3}{2}})}
\leq \left( 1 + \left\| L_1(c) \right\|_{L_t^\infty(B_{2,1}^{\frac{3}{2}})} \right) \left\| \tau^{-\alpha-1} H \right\|_{L_t^\infty(B_{2,1}^{\frac{3}{2}})} \left\| \tau \nabla H \right\|_{L_t^\infty(B_{2,1}^{\frac{3}{2}})}
\leq X^2(T) + D^2(T) + D^4(T).
\]
We end up with
\[
\sum_{q \geq q_0} \sup_{2 \leq t \leq T} \left( \tau^\alpha \int_1^t e^{c_0(\tau-t)} 2^{\frac{3}{2}} S_q^4(\tau) d\tau \right) \leq X^2(T) + D^2(T) + D^4(T).
\]
(6.26)
To bound the term with $S_q^2$, we use the fact that
\[
\int_1^t e^{c_0(\tau-t)} \left\| R_q(u, c) \right\|_{L^2} d\tau \leq \left\| R_q(\tau u, \tau^{-\alpha-1} a) \right\|_{L_t^\infty(L^2)} \int_1^t e^{c_0(\tau-t)} \tau^{-\alpha} d\tau.
\]
Hence, thanks to Lemma 2.15 and Proposition 2.12,
\[
\sum_{q \geq q_0} \sup_{2 \leq t \leq T} \left( \tau^\alpha \int_1^t e^{c_0(\tau-t)} 2^{\frac{3}{2}} S_q^2(\tau) d\tau \right)
\leq \sum_{q \geq q_0} \sup_{2 \leq t \leq T} \left( \tau^\alpha \int_1^t e^{c_0(\tau-t)} 2^{\frac{3}{2}} \left\| R_q(u, c) \right\|_{L^2} d\tau \right)
\leq \left\| \tau \nabla u \right\|_{L_t^\infty(B_{2,1}^{\frac{3}{2}})} \left\| \tau^{-\alpha-1} c \right\|_{L_t^\infty(B_{2,1}^{\frac{3}{2}})}.
\]
The first term on the right-side of the above inequality may be bounded thanks to (6.17), and the high frequencies of the last one on the right-side are obviously bounded by $D(T)$. To bound the term $\left\| \tau^{-\alpha-1} c \right\|_{L_t^\infty(B_{2,1}^{\frac{3}{2}})}$, we have the following inequality
\[
\left\| \tau^{-\alpha-1} c \right\|_{L_t^\infty(B_{2,1}^{\frac{3}{2}})} \leq \left\| \tau^{-\alpha-1} c \right\|_{L_t^\infty(B_{2,1}^{\frac{3}{2}}-2\epsilon)}
\leq \left\| \tau^{\alpha-2+\epsilon} \tau^{1-\epsilon} c \right\|_{L_t^\infty(B_{2,1}^{\frac{3}{2}}-2\epsilon)}
\leq D(T).
\]
We eventually get
\[
\sum_{q \geq q_0} \sup_{2 \leq t \leq T} \left( \tau^\alpha \int_1^t e^{c_0(\tau-t)} 2^{\frac{3}{2}} S_q^2(\tau) d\tau \right) \leq D^2(T).
\]
(6.27)
Similarly, we have
\[
\sum_{q \geq q_0} \sup_{2 \leq t \leq T} \left( \tau^\alpha \int_1^t e^{c_0(\tau-t)} 2^{\frac{2}{3}} (S_q^3(\tau) + S_q^4(\tau)) d\tau \right) \leq D^2(T).
\]
(6.28)
Finally, using product laws, (6.17), (6.21), (6.22) and Lemma 2.15, we obtain

\[
\sum_{q \geq q_0} \sup_{2 \leq t \leq T} \left( t^\alpha \int_1^t e^{c_0(t-\tau)} 2^{\frac{2}{q}} S_q(\tau) \, d\tau \right) \lesssim \sum_{q \geq q_0} \sup_{2 \leq t \leq T} \left( t^\alpha \int_1^t e^{c_0(t-\tau)} 2^{\frac{2}{q}} \| \tilde{R}_q(u, c) \|_{L^2} \, d\tau \right)
\]

(6.29)

\[
\lesssim \| \tau \nabla u \|_{L^\infty_T(\hat{B}_{Z_1}^{\frac{1}{2}})} \| \tau^{\alpha-1} \nabla c \|_{L^\infty_T(\hat{B}_{Z_1}^{\frac{1}{2}})} \sup_{2 \leq t \leq T} \left( t^\alpha \int_1^t e^{c_0(t-\tau)} \tau^{-\alpha} \, d\tau \right)
\]

\[
\lesssim D^2(T),
\]

(6.30)

and

\[
\sum_{q \geq q_0} \sup_{2 \leq t \leq T} \left( t^\alpha \int_1^t e^{c_0(t-\tau)} 2^{\frac{2}{q}} S_q(\tau) \, d\tau \right) \lesssim \sum_{q \geq q_0} \sup_{2 \leq t \leq T} \left( t^\alpha \int_1^t e^{c_0(t-\tau)} 2^{\frac{2}{q}} \| \tilde{R}_q(u, H) \|_{L^2} \, d\tau \right)
\]

(6.31)

\[
\lesssim \| \tau \nabla u \|_{L^\infty_T(\hat{B}_{Z_1}^{\frac{1}{2}})} \| \tau^{\alpha-1} \nabla H \|_{L^\infty_T(\hat{B}_{Z_1}^{\frac{1}{2}})} \sup_{2 \leq t \leq T} \left( t^\alpha \int_1^t e^{c_0(t-\tau)} \tau^{-\alpha} \, d\tau \right)
\]

\[
\lesssim D^2(T).
\]

Putting all the above inequalities (6.26)-(6.31) together, we conclude that

\[
\sum_{q \geq q_0} \sup_{2 \leq t \leq T} \left( t^\alpha \int_1^t e^{c_0(t-\tau)} 2^{\frac{2}{q}} S_q(\tau) \, d\tau \right) \lesssim X^2(T) + X^3(T) + D^2(T) + D^4(T).
\]

(6.32)

Then plugging (6.15), (6.16) and (6.32) into (6.13) yields

\[
\| (\tau)^{\alpha} (\nabla c, u, \nabla H) \|_{L^\infty_T(\hat{B}_{Z_1}^{\frac{1}{2}})} \lesssim \| (\nabla c_0, u_0, \nabla H_0) \|_{B_{Z_1}^{\frac{1}{2}}} + X^2(T) + X^3(T) + D^2(T) + D^4(T).
\]

(6.33)

**Step 3: Decay estimates with gain of regularity for the high frequencies of** \( \nabla u, \Delta H \).

This step is devoted to bounding the last two terms of \( D(t) \). We first deal with the term \( \| \tau \nabla u \|_{L^\infty_T(\hat{B}_{Z_1}^{\frac{1}{2}})} \) and shall use the fact that the velocity \( u \) satisfies

\[
\partial_t u - Au = F := -\nabla c - u \cdot \nabla c - L_1(c)Au + L_2(c)\nabla c - L_3(c)(\frac{1}{2}\nabla |H|^2 - H \cdot \nabla H).
\]

(6.34)

So,

\[
\partial_t(t\tilde{A}u) - A(t\tilde{A}u) = \tilde{A}u + tAF.
\]
We deduce from Remark 2.14 that
\[
\| \tau A u \|^h_{L_t^\infty (B_T^+) } \leq \| A u \|^h_{L_t^1 (B_T^+ ) } + \| \tau F \|^h_{L_t^\infty (B_T^{1/3} ) },
\]
whence, using the bounds given by Theorem 1.1,
\[
\| \tau \nabla u \|^h_{L_t^\infty (B_T^+ ) } \leq X(0) + \| F \|^h_{L_t^\infty (B_T^{1/3} ) } , \tag{6.35}
\]
In order to bound the first term of \( F \), we notice that, because \( \alpha \geq 1 \) and according to (6.33), we have
\[
\| \tau \nabla c \|^h_{L_t^\infty (B_T^+ ) } \leq \| \tau c \|^h_{L_t^\infty (B_T^+ ) } \leq X(0) + X^2(t) + X^3(t) + D^2(t) + D^4(t).
\]
Furthermore, from (6.17) and the definition of \( X(t) \), we have
\[
\| \tau u \cdot \nabla u \|^h_{L_t^\infty (B_T^+ ) } \leq \| u \|^h_{L_t^\infty (B_T^+ ) } \| \tau \nabla u \|_{L_t^\infty (B_T^+ ) } \leq X(t) D(t)
\]
and
\[
\| \tau L_1(c) A u \|^h_{L_t^\infty (B_T^+ ) } \leq \| c \|^h_{L_t^\infty (B_T^+ ) } \| \tau \nabla u \|_{L_t^\infty (B_T^+ ) } \leq X(t) D(t).
\]
Next, product and composition estimates adapted to tilde spaces give
\[
\| \tau L_2(c) \nabla c \|^h_{L_t^\infty (B_T^+ ) } \leq \| \tau \nabla c \|^h_{L_t^\infty (B_T^+ ) } \| \nabla c \|^h_{L_t^\infty (B_T^+ ) } \leq D^2(t).
\]
Employing (6.22) and the definition of \( X(t) \), we get
\[
\| \tau L_3(c) \left( \frac{1}{2} \nabla |H|^2 - H \cdot \nabla H \right) \|^h_{L_t^\infty (B_T^+ ) } \leq \| L_3(c) \|^h_{L_t^\infty (B_T^+ ) } \| \tau H \cdot \nabla H \|^h_{L_t^\infty (B_T^+ ) } \leq (1 + \| L_1(c) \|^h_{L_t^\infty (B_T^+ ) } ) \| H \|^h_{L_t^\infty (B_T^+ ) } \| \tau \nabla H \|^h_{L_t^\infty (B_T^+ ) } \leq X^2(t) + D^2(t) + X^4(t).
\]
Therefore,
\[
\| \tau \nabla u \|^h_{L_t^\infty (B_T^+ ) } \leq X(0) + X^2(t) + X^3(t) + X^4(t) + D^2(t) + D^4(t) . \tag{6.36}
\]
Finally, in order to bound the term of \( \| \tau \nabla H \|^h_{L_t^\infty (B_T^+ ) } \), we shall use the fact that the magnetic \( H \) satisfies
\[
\partial_t H - \Delta H = G := \mp u \cdot \nabla H - H(\text{div} u) + H \cdot \nabla u - \nabla \times (L_3(c)(\nabla \times H) \times H). \tag{6.37}
\]
Furthermore,
\[ \partial_t (t \nabla^2 H) - \Delta (t \nabla^2 H) = \nabla^2 H + t \nabla^2 G. \]

From Proposition 2.13, we have
\[ \| \tau \nabla^2 H \|^h_{L^\infty_t(B_z)} \lesssim \| \nabla^2 H \|^h_{L^1_t(B_z)} + \| \tau \nabla^2 G \|^h_{L^\infty_t(B_z)}. \]

Using the bounds given by Theorem 1.1, we get
\[ \| \tau \nabla^2 H \|^h_{L^\infty_t(B_z)} \lesssim X(0) + \| \tau G \|^h_{L^\infty_t(B_z)}. \] (6.38)

Bounding the first two terms of \( G \), we notice that
\[
\| \tau u \cdot \nabla H \|^h_{L^\infty_t(B_z)} \lesssim \| u \|^h_{L^\infty_t(B_z)} \| \tau \nabla H \|^h_{L^\infty_t(B_z)} \lesssim \| u \|^h_{L^\infty_t(B_z)} \| \tau \nabla H \|^h_{L^\infty_t(B_z)} \lesssim X(t) \left( \| \tau \nabla H \|^\ell_{L^\infty_t(B_z)} + \| \tau \nabla H \|^h_{L^\infty_t(B_z)} \right) \] (6.39)
\[
\lesssim X(t) \left( \| \tau^{2-\varepsilon} H \|^\ell_{L^\infty_t(B_z)} + \| \tau \nabla H \|^h_{L^\infty_t(B_z)} \right) \lesssim X(t) D(t) \]

and
\[
\| \tau H \cdot \nabla u \|^h_{L^\infty_t(B_z)} \lesssim \| H \|^h_{L^\infty_t(B_z)} \| \tau \nabla u \|^h_{L^\infty_t(B_z)} \lesssim \| H \|^h_{L^\infty_t(B_z)} \| \tau \nabla u \|^h_{L^\infty_t(B_z)} \lesssim X(t) D(t). \] (6.40)

The third term of \( G \) is similar to the second one, using (6.22), we obtain
\[
\| \tau \nabla \times (L_3(c)(\nabla \times H) \times H) \|^h_{L^\infty_t(B_z)} \lesssim \| \tau \nabla \times (L_3(c)(\nabla \times H) \times H) \|^h_{L^\infty_t(B_z)} \lesssim \| \tau \nabla \times (L_3(c)(\nabla \times H) \times H) \|^h_{L^\infty_t(B_z)} \lesssim \| L_3(c) \|^h_{L^\infty_t(B_z)} \| \tau H \cdot \nabla H \|^h_{L^\infty_t(B_z)} \lesssim \| L_3(c) \|^h_{L^\infty_t(B_z)} \| \tau H \cdot \nabla H \|^h_{L^\infty_t(B_z)} \lesssim (1 + \| c \|^h_{L^\infty_t(B_z)}) \| H \|^h_{L^\infty_t(B_z)} \| \tau \nabla H \|^h_{L^\infty_t(B_z)} \| \tau \nabla H \|^h_{L^\infty_t(B_z)} \lesssim X^2(t) + D^2(t) + X^4(t). \]

Hence, reverting to (6.38), we get
\[ \| \tau \nabla^2 H \|^h_{L^\infty_t(B_z)} \lesssim X(0) + X^2(t) + D^2(t) + X^4(t). \] (6.41)
Finally, adding up the obtained inequality to (6.12), (6.33) and (6.36) yields for all \( t \geq 0 \),

\[
D(t) \lesssim X(0) + D_0 + \| (\nabla c_0, u_0, \nabla H_0) \|^h_{B_{2,1}^{\frac{1}{2}}} + X^2(t) + X^3(t) + X^4(t) + D^2(t) + D^3(t) + D^4(t)
\]

\[
\lesssim D_0 + \| (\nabla c_0, u_0, \nabla H_0) \|^h_{B_{2,1}^{\frac{1}{2}}} + X^2(t) + X^3(t) + X^4(t) + D^2(t) + D^3(t) + D^4(t),
\]

where we have used \( X(0)^{\mathbb{L}} = \| (c_0, u_0, H_0) \|^{\mathbb{L}}_{B_{2,1}^{\frac{1}{2}}} \lesssim \| (c_0, u_0, H_0) \|^{\mathbb{L}}_{B_{2,1}^{-\frac{3}{2},\infty}} \). As Theorem 1.1 ensures that \( X(t) \) is small, one can conclude that (1.5) is fulfilled for all time if \( D_0 \) and \( \| (\nabla c_0, u_0, \nabla H_0) \|^h_{B_{2,1}^{\frac{1}{2}}} \) are small enough. This completes the proof of Theorem 1.2.

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