A new approach for solving fractional RL circuit model through quadratic Legendre multi-wavelets

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Abstract: The aim of present work is to obtain the approximate solution of fractional model for the electrical RL circuit by using quadratic Legendre multiwavelet method (QLMWM). The beauty of the paper is convergence theorem and mean square error analysis, which shows that our approximate solution converges very rapidly to the exact solution and the numerical solution is compared with the classical solution and Legendre wavelets method (LWM) solution, which is much closer to the exact solution. The fractional integration is described in the Riemann-Liouville sense. The results are shows that the method is very effective and simple. In addition, using plotting tools, we compare approximate solutions of each equation with its classical solution and LWM.

Keywords: Electrical circuit; Fractional differential equation; Quadratic Legendre multiwavelet; Operational matrix of fractional integration; Block pulse function

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Introduction

In recent years, wavelets have found their place in a wide range of engineering disciplines. Wavelets are used in numerical analysis, signal analysis, system analysis, optimal control and solution of many differential and integral equations. The main characteristic of wavelets is its ability to convert the given differential equations, fractional order differential equations and integral equations to a system of nonlinear and linear algebraic equations, which are then solved by existing numerical methods. Many researchers started using various wavelets\textsuperscript{[16]} for analyzing problems of greater computational complexity and proved wavelets to be powerful tools to explore a new direction in solving fractional differential equations (FDEs) and differential equations.

Fractional calculus has many applications in numerous seemingly diverse fields of physics and engineering processes. It involving derivative and integrals of non-integer order, is the natural generalization of the classical calculus\textsuperscript{[7-9]}. Therefore many authors have been interested in studying the fractional calculus and finding accurate and efficient method for solving fractional differential equations (FDEs), since FDEs show many advantages over the simulation of natural physical processes and dynamic systems\textsuperscript{[10-12]}. The applications of fractional calculus have been demonstrated by many authors. For examples, fractional calculus in applied to model the nonlinear oscillation of earthquake\textsuperscript{[13]}, continuum and statistical mechanics\textsuperscript{[14]}, fluid-dynamic traffic\textsuperscript{[15]}, control theory\textsuperscript{[16]}, signal process\textsuperscript{[17]} and dynamics of interfaces between nano particles and subtracts\textsuperscript{[18]}

Now, we discuss the basic equations of electric circuits involving resistor with a resistance $R$ measured on ohms, an inductor with inductance $L$ measured in henries, and a capacitor with capacitance $C$ measured in farads. We investigate the following equation for RL circuit.

$$L\frac{d}{dt}J(t) + R J(t) = V$$

where $J(t)$ and $V(t)$ are the current and electric charge in the shell of capacitor with respect to time $t$ respectively.
Many authors are proposed numerical solution of FDEs of electrical circuits. Francisco Gomez et al. [19] and A. Atangana et al. [20] solving fractional order RLC circuit and Ahmed Alsaedi et al. [21] used fractional calculus for numerical solution of RC, LC and RL circuits.

The outline of this paper is as follows: In section second, we discuss some notations, definitions and preliminary facts of the fractional calculus theory. In section third, we present wavelet and Legendre wavelet and their properties. In section fourth, we shows that the function approximation and collocation points. Section fifth is devoted to the operational matrix of integration. In section sixth, we give convergence and mean square error theorems for our method. Finally, we solve fractional electric circuit equation given in section seventh, to illustrate the performance of our method.

Basic definitions of Fractional Calculus
In this section, we present some notations, definitions and preliminary facts of the fractional calculus theory which are used further in the present paper.

**Definition 2.1** The Riemann-Liouville fractional integral operator $I^\alpha$ of order $\alpha$ on the usual Lebesgue space $L[a,b]$ is given [4] by

$$I^\alpha f(t) = \begin{cases} \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)ds, & \alpha > 0 \\ \frac{a}{b}, & \alpha = 0 \end{cases}$$

It has the following properties:
(i) $I^\alpha I^\beta = I^{\alpha+\beta}$,
(ii) $I^\alpha I^\beta = I^\beta I^\alpha$,
(iii) $(I^\alpha I^\beta) f(t) = (I^\beta I^\alpha) f(t)$,
(iv) $I^\alpha (t-a)^v = \frac{\Gamma(v+1)}{\Gamma(\alpha+v+1)}(t-a)^{\alpha+v}$.

where $f \in L[a,b]$, $\alpha,\beta \geq 0$ and $v > -1$.

**Definition 2.2** For a function $f$ given on interval $[a,b]$, the Caputo definition of fractional order derivative of order $\alpha$ ($n-1 < \alpha \leq n$) of $f$ is defined [21] by

$$D^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \left( \frac{d}{dt} \right)^n \int_a^t (t-s)^{\alpha-1} f(s)ds,$$

where $t > 0$, $n$ is a integer. It has the following two basic properties for $n-1 < \alpha \leq n$ and $f \in L[a,b]$.

$$D^\alpha (I^n f(t)) = f(t)$$

and

$$I^n D^\alpha (f(t)) = f(t) - \sum_{k=0}^{n-1} f^{(k)}(0+) \frac{(t-a)^k}{k!}, \text{ } t > 0.$$  

For more details on the mathematical properties of fractional derivatives and integrals see [7-9] and [22, 23].

Legendre wavelet (LW) and quadratic Legendre multiwavelet (QLMW)

Wavelets constitute a family of functions constructed by performing translation and dilation on a single function $\psi$, where $\psi$ is a mother wavelet. We define family of continuous wavelets [24] by

$$\psi_{a,b}(t) = |a|^{-1/2} \psi(\frac{t-b}{a}), \text{ } a,b \in \mathbb{R}; \text{ } a \neq 0,$$

where $a$ is called scaling parameter and $b$ is translation parameter. If we restricted the parameters $a$ and $b$ to discrete values as $a = a_0k, b = nb_0a_0k$, where $a_0 > 1, b_0 > 0$, and $k, n$ are positive integers. We have the following family of discrete wavelets:

$$\psi_{a,b}(t) = |a|^{-1/2} \psi(a_0k t - nb_0), \text{ } k, n \in \mathbb{Z},$$

where the function $\psi_{a,b}(t)$ form a wavelet basis for $L^2(\mathbb{R})$. In particular, when $a_0 = 2$ and $b_0 = 1$, the function $\psi_{a,b}(t)$ form an orthonormal basis.
For constructing compactly supported multiwavelets [24] we used multi-resolution analysis (MRA) [25] that involves several scaling functions and associated mother multiwavelets vector [26, 31].

For constructing the quadratic Legendre multiwavelet (QLMW), the single scaling function $\phi(t)$ is replaced with vector scaling function yield:

$$\Phi(t) = [\phi_0(t), \phi_1(t), \phi_2(t)]^T,$$

where

$$\phi_0(t) = 1, \quad \phi_1(t) = \sqrt{5}(2t - 1) \quad \text{and} \quad \phi_2(t) = \sqrt{5}(6t^2 - 6t + 1), \quad 0 \leq t \leq 1.$$  \hspace{1cm} (6)

Let $\Psi(t) = [\psi_0(t), \psi_1(t), \psi_2(t)]^T$, be the corresponding mother wavelet functions, then as in [27, 28] by MRA and the condition of orthonormality, we obtain the QLMW as:

$$\begin{align*}
\psi_0(t) &= \begin{cases}
-\frac{1}{3}(120t^2 - 72t + 7), & 0 \leq t \leq \frac{1}{2} \\
\frac{1}{3}(120t^2 - 168t + 55), & \frac{1}{2} \leq t \leq 1,
\end{cases} \\
\psi_1(t) &= \begin{cases}
\sqrt{5}(30t^2 - 14t + 1), & 0 \leq t \leq \frac{1}{2} \\
\sqrt{5}(30t^2 - 46t + 17), & \frac{1}{2} \leq t \leq 1,
\end{cases} \\
\psi_2(t) &= \begin{cases}
-\frac{1}{3}(48t^2 - 18t + 1), & 0 \leq t \leq \frac{1}{2} \\
\frac{1}{3}(48t^2 - 78t + 31), & \frac{1}{2} \leq t \leq 1.
\end{cases}
\end{align*}$$  \hspace{1cm} (7)

By translating and dilating the quadratic Legendre wavelet $\Psi(t)$, we have

$$\{\psi_{j,n}(t)\} = \{2^{j/2}\psi^j(2^j t - n)\}, \quad k, n, j \in \mathbb{Z}$$

The family $\{\psi_{j,n}(t)\}$ forms an orthonormal basis for $L_2(\mathbb{R})$ and subfamily is orthonormal in $L_2[0,1]$ for $n = 0, 1, 2, ..., 2^j - 1, k = 0, 1, 2, ...$ and $j = 0, 1, 2$.

Function approximation

A function $f(t)$ defined over $[0, 1]$ and approximated as

$$f(t) = \sum_{i=0}^{\tilde{m}} g_i \phi(t) + \sum_{k=0}^{\tilde{M}} \sum_{j=0}^{2^k-1} \sum_{n=0}^{2^k-1} g_{k,n} \psi_{j,n}(t),$$  \hspace{1cm} (8)

where $g_i = \{f(t), \phi(t)\}$ and $g_{k,n} = \{f(t), \psi_{j,n}(t)\}$. If the infinite series in equation (8) is truncated then the equation (8) can be written as

$$f(t) = \sum_{i=0}^{\tilde{m}} g_i \phi(t) + \sum_{k=0}^{\tilde{M}} \sum_{j=0}^{2^k-1} \sum_{n=0}^{2^k-1} g_{k,n} \psi_{j,n}(t) = G^T \Theta(t),$$  \hspace{1cm} (9)

where $T$ indicates transposition and, $G$ and $\Theta(t)$ are $\tilde{m} \times (3 \cdot 2^{\tilde{M}+1})$ column vectors given by

$$g = [g_0, g_1, g_2, g_3, ..., g_{M(2^{\tilde{M}+1})}, g_{M(2^{\tilde{M}+1})}, g_{M(2^{\tilde{M}+1})}, ..., g_{M(2^{\tilde{M}+1})}]^T;$$

$$\Theta(x) = [\phi_0, \phi_1, \phi_2, \psi_0, \psi_1, \psi_2, ..., \psi_{M(2^{\tilde{M}+1})}, \psi_{M(2^{\tilde{M}+1})}, \psi_{M(2^{\tilde{M}+1})}, ..., \psi_{M(2^{\tilde{M}+1})}]^T.$$  \hspace{1cm} (10)

Taking the collocation points as following:

$$t_i = \frac{s-1}{3 \cdot 2^{\tilde{M}+1}}, \quad s = 1, 2, ..., 3 \cdot 2^{\tilde{M}+1}.$$  \hspace{1cm} (11)

Now, taking the quadratic Legendre multiwavelet matrix $\mathcal{A}_{\text{q}}$ [6, 29] as:
\[
\mathcal{J}_{\text{m}} = \left[ \begin{bmatrix} 0 \frac{1}{3,2^m+1} \frac{5}{3,2^m+1} \end{bmatrix} \right]
\]

(12)

For example, when \( M = 1 \) the quadratic Legendre multiwavelet is expressed as

\[
\mathcal{J}_{\text{m}} = \left[ \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \right]
\]

\[
\begin{bmatrix}
\sqrt{3} & -\sqrt{3} & 0 & 1 & \sqrt{3} & 2 & \sqrt{3} & 2 & \sqrt{3} \\
2\sqrt{3} & -2 & \sqrt{3} & 0 & 1 & \sqrt{3} & 2 & \sqrt{3} & 2 \sqrt{3} \\
\sqrt{3} & -\sqrt{3} & 0 & 1 & \sqrt{3} \sqrt{3} & 2 \sqrt{3} & 2 \sqrt{3} & 2 \sqrt{3} & 2 \sqrt{3} \\
3 & -3 & 2 \sqrt{3} & 0 & 1 & \sqrt{3} \sqrt{3} & 2 \sqrt{3} & 2 \sqrt{3} & 2 \sqrt{3} \\
\sqrt{3} & -\sqrt{3} & 0 & 1 & \sqrt{3} \sqrt{3} & 2 \sqrt{3} & 2 \sqrt{3} & 2 \sqrt{3} & 2 \sqrt{3} \\
\sqrt{3} & -\sqrt{3} & 0 & 1 & \sqrt{3} \sqrt{3} & 2 \sqrt{3} & 2 \sqrt{3} & 2 \sqrt{3} & 2 \sqrt{3} \\
3 & -3 & 2 \sqrt{3} & 0 & 1 & \sqrt{3} \sqrt{3} & 2 \sqrt{3} & 2 \sqrt{3} & 2 \sqrt{3} \\
7 & 11 & 5 & 7 & 11 & 13 & 11 & 7 & 5 & 11 \\
3 & 18 & 9 & 6 & 9 & 18 & 3 & 18 & 9 & 18 \\
\sqrt{3} & -\sqrt{3} & 0 & 1 & \sqrt{3} \sqrt{3} & 2 \sqrt{3} & 2 \sqrt{3} & 2 \sqrt{3} & 2 \sqrt{3} \\
8 \sqrt{3} & -8 \sqrt{3} & 2 & 8 \sqrt{3} & 0 & 1 & \sqrt{3} \sqrt{3} & 2 \sqrt{3} & 2 \sqrt{3} \\
3 & -3 & 2 \sqrt{3} & 0 & 1 & \sqrt{3} \sqrt{3} & 2 \sqrt{3} & 2 \sqrt{3} & 2 \sqrt{3} \\
7 \sqrt{2} & 5 \sqrt{2} & 11 \sqrt{2} & 7 \sqrt{2} & 11 \sqrt{2} & 9 & 5 \sqrt{2} & 0 & 0 & 0 & 0 \\
3 & 9 & 9 & 3 & 3 & 9 & 9 & 0 & 0 & 0 & 0 \\
\sqrt{6} & -\sqrt{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & -3 & 2 \sqrt{3} & 0 & 1 & \sqrt{3} \sqrt{3} & 2 \sqrt{3} & 2 \sqrt{3} & 2 \sqrt{3} \\
3 & -3 & 2 \sqrt{3} & 0 & 1 & \sqrt{3} \sqrt{3} & 2 \sqrt{3} & 2 \sqrt{3} & 2 \sqrt{3} \\
3 & -3 & 2 \sqrt{3} & 0 & 1 & \sqrt{3} \sqrt{3} & 2 \sqrt{3} & 2 \sqrt{3} & 2 \sqrt{3} \\
7 \sqrt{2} & 5 \sqrt{2} & 11 \sqrt{2} & 7 \sqrt{2} & 11 \sqrt{2} & 9 & 5 \sqrt{2} & 0 & 0 & 0 & 0 \\
3 & 9 & 9 & 3 & 3 & 9 & 9 & 0 & 0 & 0 & 0 \\
\sqrt{10} & 2 \sqrt{10} & 4 \sqrt{10} & 2 \sqrt{10} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & 9 & 9 & 3 & 3 & 9 & 9 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

The coefficient vector \( G^T \) is obtained by

\[
G^T = \mathcal{J}_{\text{m}}^{-1}.
\]

(13)

Quadratic Legendre multiwavelet operational matrix of fractional integral

The fractional integral of order \( \alpha \) in the Riemann-Liouville sense of the vector \( \Theta(t) \), defined in equation (2), can be approximated by quadratic Legendre multiwavelet series with quadratic Legendre multiwavelet coefficient matrix \( P^\alpha \).

\[
(I^\alpha \Theta)(t) = P^\alpha \Theta(t),
\]

(14)

where the \( P^\alpha \) is called the \( m \times m \) quadratic Legendre multiwavelet operational matrix of integral of order \( \alpha \). It shows that the operational matrix \( P^\alpha \) can be approximate [29] as:

\[
P^\alpha = \mathcal{J}_{\text{m}} F^\alpha \mathcal{J}_{\text{m}}^{-1},
\]

(15)

where \( F^\alpha \) is the operational matrix of fractional integration of order of the block-pulse function (BPFs), which given in [6]

\[
F^\alpha = \begin{bmatrix}
1 & \xi_1 & \xi_2 & \xi_3 & \cdots & \xi_{m-1} \\
0 & 1 & \xi_1 & \xi_2 & \cdots & \xi_{m-2} \\
0 & 0 & 1 & \xi_1 & \cdots & \xi_{m-3} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & 1 & \xi_{m-4} \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix},
\]
where $\xi = (\ell + 1)^{\ell+1} - 2\ell^{\ell+1} + (\ell - 1)^{\ell+1}$ [29].

Existence, Uniqueness and Convergence

In this section we discuss convergence and mean square error of Legendre wavelet method.

Theorem 6.1 [27] Let the function $f:[0,1] \to \mathbb{R}$ and $f \in C^6[0,1]$, then $G^T \Theta$ approximates $f$ with mean error bound as follows:

$$\|f - G^T \Theta\| \leq \frac{1}{312234} \sup_{x \in [0,1]} |f'(x)|,$$

where $\| \cdot \|$ denotes the norm in $L_2(R)$ space.

Proof. The proof can be seen in [30]. We see that, the factor $1/31234$ shows best approximation of function because if $k$ increases then error decrease rapidly.

Theorem 6.2 (Convergence theorem) Let the function $D^\alpha f(x) \in C^6[0,1]$ and $D^\alpha f(x)$ exists bounded second derivative, can be expressed as in equation (8) and the truncated series given in equation (9) converges towards the exact solution.

Proof. The proof can be seen in [28].

Theorem 6.3 (Mean square error) Let the function $f:[0,1] \to \mathbb{R}$ and $f \in C^6[0,1]$, then we have the following accuracy estimation:

$$E_s = \sum_{p=0}^{n} \sum_{q=0}^{\frac{3M-1}{2}} \frac{2\sqrt{3}}{2q+1(2q+5)\sqrt{2(q-3)+2p-1}} \sup_{x \in [0,1]} |f'(x)|,$$

where

$$E_s = \int_0^1 \left[ \sum_{p=0}^{n} g_p \phi_p + \sum_{p=0}^{n} \sum_{q=0}^{\frac{3M-1}{2}} g_{pq} \psi_{pq}(x) \right] dx.$$

Proof. Let us consider the quantity

$$E_s^2 = \int_0^1 \left[ \sum_{p=0}^{n} g_p \phi_p + \sum_{p=0}^{n} \sum_{q=0}^{\frac{3M-1}{2}} g_{pq} \psi_{pq}(x) \right]^2 dx,$$

which call the mean square error in approximating $f(x)$ by

Calculating the deviation of equation (17), we get

$$E_s^2 = \sum_{p=0}^{n} \sum_{q=0}^{\frac{3M-1}{2}} \int_0^1 (g_{pq})^2 \psi_{pq}^2(x) dx.$$

From theorem 2.2, we have

$$\left| g_{pq} \right| \leq \frac{2\sqrt{3}}{2q+1(2q+5)\sqrt{2(q-3)+2p-1}} \sup_{x \in [0,1]} |f'(x)|,$$

From equation (18) and equation (19), we have

$$E_s \leq \sum_{p=0}^{n} \sum_{q=0}^{\frac{3M-1}{2}} \frac{2\sqrt{3}}{2q+1(2q+5)\sqrt{2(q-3)+2p-1}} \sup_{x \in [0,1]} |f'(x)|.$$
We consider RL circuit differential equation given in equation (1). RL circuit consists only resistor, inductor and a non-variant voltage source are present in the circuit and its differential equation is given as follows

\[ LJ(t) + RJ(t) = V. \]  

(20)

with \( J(0) = J_0 \) and \( V \) is the constant voltage source.

The classical solution of equation (20) is

\[ J(t) = \left[ I_0 - \frac{VL}{R} \right] e^{-\frac{t}{L}} + \frac{VL}{R}. \]

(21)

Now, we analyse equation (20) using fractional calculus, we replace \( J(t) \) by \( D^\nu J(t) \), where \( \nu \in (0,1) \). In the sense of Riemann-Liouville derivative, we get the fractional order RL circuit and its differential equation as

\[ \left( D^\nu J \right)(t) + \frac{R}{L} J(t) = \frac{V}{L}. \]

(22)

Let \( \frac{R}{L} = \sigma^2 \) and \( \frac{V}{L} = \rho^2 \), then equation (22) become

\[ \left( D^\nu J \right)(t) + \sigma^2 J(t) = \rho^2. \]

(23)

We use equation (9) to approximate \( D^\nu J(t) \) as

\[ D^\nu J(t) = \sum_{\nu=0}^{2} \sum_{\nu=0}^{m} \sum_{\nu=0}^{n} z_{\nu} \phi(t) + \sum_{\nu=0}^{m} \sum_{\nu=0}^{n} \sum_{\nu=0}^{T} z_{\nu} \phi(t) = Z^T \Theta(t). \]

(24)

Integrating equation (24) with respect to \( t \), over \([0,t]\), we get

\[ J(t) = J(0) + Z^T P_{\nu} \Theta(t), \]

(25)

We can approximate \( J_0 \) and \( \rho^2 \) as

\[ J(0) = J_0 = \left[ J_0, J_0, \ldots, J_0 \right] \gamma \phi_{\nu} \Theta(t) \quad \text{and} \quad \rho^2 = \left[ \rho^2, \rho^2, \ldots, \rho^2 \right] \gamma \phi_{\nu} \Theta(t). \]

(26)

Substituting equations (24-26) in equation (23), we obtain

\[ Z^T \Theta(t) + \sigma^2 \left[ \left[ J_0, J_0, \ldots, J_0 \right] \gamma \phi_{\nu} \Theta(t) + Z^T P_{\nu} \Theta(t) \right] = \left[ \rho^2, \rho^2, \ldots, \rho^2 \right] \gamma \phi_{\nu} \Theta(t), \]

\[ Z^T \Theta(t) + \sigma^2 \left[ J_0, J_0, \ldots, J_0 \right] \gamma \phi_{\nu} \Theta(t) + \sigma^2 Z^T P_{\nu} \Theta(t) = \left[ \rho^2, \rho^2, \ldots, \rho^2 \right] \gamma \phi_{\nu} \Theta(t), \]

\[ Z^T + \sigma^2 \left[ J_0, J_0, \ldots, J_0 \right] \gamma \phi_{\nu} + \sigma^2 Z^T P_{\nu} = \left[ \rho^2, \rho^2, \ldots, \rho^2 \right] \gamma \phi_{\nu}, \]

\[ Z^T \left( I + \sigma^2 P_{\nu} \right) = \left( \left[ \rho^2, \rho^2, \ldots, \rho^2 \right] \gamma \phi_{\nu} - \sigma^2 \left[ J_0, J_0, \ldots, J_0 \right] \gamma \phi_{\nu} \right), \]

\[ Z^T = \left( \left[ \rho^2, \rho^2, \ldots, \rho^2 \right] \gamma \phi_{\nu} - \sigma^2 \left[ J_0, J_0, \ldots, J_0 \right] \gamma \phi_{\nu} \right) \gamma \phi_{\nu} \times \left( I + \sigma^2 P_{\nu} \right)^{-1}. \]

(27)

Hence required

\[ J(t) = \left[ J_0, J_0, \ldots, J_0 \right] \gamma \phi_{\nu} \Theta(t) + \left( \left[ \rho^2, \rho^2, \ldots, \rho^2 \right] \gamma \phi_{\nu} - \sigma^2 \left[ J_0, J_0, \ldots, J_0 \right] \gamma \phi_{\nu} \right) \gamma \phi_{\nu} \times \left( I + \sigma^2 P_{\nu} \right)^{-1} \gamma \phi_{\nu} \Theta(t). \]

(28)

By manipulate the above system (27) of linear equations we obtain the unknown vector \( Z \). Substituting these value of vector \( Z \) in equation (28), we obtain numerical results of RL circuit for different value of \( k, j, M \) and \( \alpha \). When \( \alpha = 0.999 \) and \( k = 0, 1; j = 0, 1, 2; M = 3 \) the fractional RL circuit graph behave similar to the classical solution graph for \( \nu = 1 \) shown in the Figure 1 for QLMWM and at \( k = 1; \) for LWM in Figure 2. LWM is weak at \( t = 1 \) show in the Table 1. It is show that the proposed QLMWM approach is more accurate to the LWM. Table 1 describes the effectiveness of the proposed method by comparing with the classical solution at \( \nu = 1 \).
Figure 1. Current versus time graph at 
and $M = 3$

\[ R = 10, L = 1, V = 10, V_0 = 0.01, \alpha = 0.5, 0.75 \text{ and } 0.999 \text{ with QLMW (} k = 0, 1, 2 \text{ and } M = 3 \text{.)} \]

| $t$ | $\alpha = 0.50$ | $\alpha = 0.75$ | $\alpha = 0.999$ | $\alpha = 1$ |
|-----|----------------|----------------|----------------|-----------|
| 0.1 | QLMW | LW | QLMW | LW | QLMW | LW | CS |
| 0.1 | $8.8085 \times 10^{-1}$ | $7.7867 \times 10^{-1}$ | $8.8627 \times 10^{-1}$ | $6.6899 \times 10^{-1}$ | $7.6608 \times 10^{-1}$ | $5.2978 \times 10^{-1}$ | $6.3579 \times 10^{-1}$ |
| 0.2 | $8.4025 \times 10^{-1}$ | $8.8787 \times 10^{-1}$ | $8.2145 \times 10^{-1}$ | $8.8797 \times 10^{-1}$ | $8.8126 \times 10^{-1}$ | $8.5076 \times 10^{-1}$ | $8.6602 \times 10^{-1}$ |
| 0.3 | $9.0615 \times 10^{-1}$ | $9.2552 \times 10^{-1}$ | $9.2401 \times 10^{-1}$ | $9.7257 \times 10^{-1}$ | $9.7020 \times 10^{-1}$ | $1.0111 \times 10^{-1}$ | $9.5071 \times 10^{-1}$ |
| 0.4 | $9.1714 \times 10^{-1}$ | $9.1906 \times 10^{-1}$ | $9.3957 \times 10^{-1}$ | $9.4748 \times 10^{-1}$ | $9.9041 \times 10^{-1}$ | $1.0108 \times 10^{-1}$ | $9.8186 \times 10^{-1}$ |
| 0.5 | $9.2493 \times 10^{-1}$ | $9.2404 \times 10^{-1}$ | $9.4922 \times 10^{-1}$ | $9.4989 \times 10^{-1}$ | $9.9632 \times 10^{-1}$ | $9.9895 \times 10^{-1}$ | $9.9332 \times 10^{-1}$ |
| 0.6 | $9.3091 \times 10^{-1}$ | $9.2929 \times 10^{-1}$ | $9.5608 \times 10^{-1}$ | $9.5497 \times 10^{-1}$ | $9.9862 \times 10^{-1}$ | $9.9938 \times 10^{-1}$ | $9.9755 \times 10^{-1}$ |
| 0.7 | $9.3588 \times 10^{-1}$ | $9.3393 \times 10^{-1}$ | $9.6104 \times 10^{-1}$ | $9.5953 \times 10^{-1}$ | $9.9923 \times 10^{-1}$ | $9.9967 \times 10^{-1}$ | $9.9990 \times 10^{-1}$ |
| 0.8 | $9.3952 \times 10^{-1}$ | $9.3808 \times 10^{-1}$ | $9.6504 \times 10^{-1}$ | $9.6359 \times 10^{-1}$ | $9.9969 \times 10^{-1}$ | $9.9983 \times 10^{-1}$ | $9.9966 \times 10^{-1}$ |
| 0.9 | $9.4278 \times 10^{-1}$ | $9.4162 \times 10^{-1}$ | $9.6817 \times 10^{-1}$ | $9.6713 \times 10^{-1}$ | $9.9981 \times 10^{-1}$ | $9.9986 \times 10^{-1}$ | $9.9987 \times 10^{-1}$ |
| 1   | $9.3248 \times 10^{-1}$ | $0.0000 \times 10^{-1}$ | $0.0000 \times 10^{-1}$ | $0.0000 \times 10^{-1}$ | $9.1958 \times 10^{-1}$ | $0.0000 \times 10^{-1}$ | $9.9995 \times 10^{-1}$ |

Table 1. Numerical results for RL circuit \((R = 10, L = 1, V = 10, V_0 = 0.01, \alpha = 0.5, 0.75 \text{ and } 0.999)\)

**Conclusion**

In this paper, we propose a numerical scheme based on the QLMW and operational matrix of the fractional integral to obtain numerical solutions of FDEs of electric circuits. Such analysis can be further applied to other physical models to develop a better understanding of use of wavelets in real-life problems. The solution obtained using the suggested method shows that this approach can solve the problem effectively and is very simple and easy in implementation. Moreover, the convergence and mean square error of the function approximation with QLMW basis was discussed. The
numerical solution of circuit’s FDEs demonstrates the validity and applicability of proposed method. The solutions of the electrical circuit equations are presented graphically and in tabular form with a comparison with their classical solutions and Legendre wavelets method (LWM).

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