A NOTE ON “POINTWISE INEQUALITIES FOR SOBOLEV FUNCTIONS ON OUTWARD CUSPIDAL DOMAINS”

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ABSTRACT. We establish a result which will cover the result in [3]. Let $\Omega \subset \mathbb{R}^{n-1}$ be a bounded star-shaped domain and $\Omega_\psi$ be an outward cuspidal domain with base domain $\Omega$. Then $W^{1,p}(\Omega_\psi) = M^{1,p}(\Omega_\psi)$ if and only if $W^{1,p}(\Omega) = M^{1,p}(\Omega)$.

1. INTRODUCTION

In this note, we generalize the result by Eriksson-Bique, Koskela, Malý and Zhu in [3]. Let $\Omega \subset \mathbb{R}^{n-1}$ be a bounded star-shaped domain with star center $x_0 \in \Omega$. Consider cuspidal domains of the form

$$\Omega_\psi := \{(t,x) \in (0,1) \times \mathbb{R}^{n-1}; |x-x_0| < \psi(t)\Omega\} \cup \{(t,x) \in [1,2) \times \mathbb{R}^{n-1}; |x-x_0| < \psi(1)\Omega\},$$

where $\psi: (0,1] \to (0,\infty)$ is a left continuous and increasing function. (Left continuity is required just to get $\Omega_\psi$ open. The term “increasing” is used in the non-strict sense.) The $(n-1)$-dimensional bounded star-shaped domain $\Omega$ is called the base domain of the outward cuspidal domain $\Omega_\psi$. In some sense, star-shapeness is necessary to guarantee $\Omega_\psi$ is always a domain for every left continuous

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and increasing $\psi$. One can easily see that $\Omega_\psi$ is also star-shaped, under the assumption that $\Omega$ is star-shaped. The seemingly strange cylindrical annexes are included only to exclude other singularities than the cuspidal one.

For this kind of outward cuspidal domains, we have the following result about when Sobolev spaces coincide with Hajlasz-Sobolev spaces.

**Theorem 1.2.** Let $\Omega \subset \mathbb{R}^{n-1}$ be a bounded star-shaped domain and $\psi : (0, 1] \to (0, \infty)$ be a left continuous and increasing function. Define the corresponding cuspidal domain $\Omega_\psi$ as in (1.1). Then $W^{1,p}(\Omega_\psi) = M^{1,p}(\Omega_\psi)$ if and only if $W^{1,p}(\Omega) = M^{1,p}(\Omega)$.

By the classical open mapping theorem in functional analysis, if two Banach spaces coincide as sets, we obtain the equivalence of norms immediately. In [3], authors only studied the case that the base domain is the unit ball $B^{n-1}(0, 1)$. Since unit ball is a bounded star-shaped domain for $W^{1,p}(B^{n-1}(0, 1)) = M^{1,p}(B^{n-1}(0, 1))$ for every $1 < p \leq \infty$, the result in [3] is only a special case of our result here. By the classical result due to Hajlasz [5], $W^{1,p} = M^{1,p}$ on a $W^{1,p}$-extension domain for $1 < p \leq \infty$, we have the following corollary to our theorem.

**Corollary 1.3.** Let $\Omega \subset \mathbb{R}^{n-1}$ be a bounded star-shaped $W^{1,p}$-extension domain for $1 < p \leq \infty$ and $\psi : (0, 1] \to (0, \infty)$ be a left continuous and increasing function. Define the corresponding cuspidal domain $\Omega_\psi$ as in (1.1). Then $W^{1,p}(\Omega_\psi) = M^{1,p}(\Omega_\psi)$.

2. Definitions and Preliminaries

In what follows, $U$ is always a domain and $\Omega$ is always a bounded star-shaped domain in Euclidean spaces. Typically, $c$ or $C$ will be constants that depend on various parameters and may differ even on the same line of inequalities. The Euclidean distance between points $x, y$ in the Euclidean space $\mathbb{R}^n$ is denoted by $|x - y|$. The open $m$-dimensional ball of radius $r$ centered at the point $x$ is denoted by $B^m(x, r)$.

**Definition 2.1.** A domain $\Omega \subset \mathbb{R}^n$ is called a star-shaped domain, if there exists a point $x_o \in \Omega$ such that for every $x \in \Omega$, the segment $[x, x_o]$ between $x$ and $x_o$ is contained in $\Omega$. The point $x_o$ is called the star center of $\Omega$.

Under the help of linear translations, for every bounded star-shaped domain $\Omega$, we can always assume the original point $O$ is a star center of it. Then, for every $0 < \lambda < \infty$, we define

$$\lambda \Omega := \left\{ x \in \mathbb{R}^n : \frac{x}{\lambda} \in \Omega \right\}.$$ 

We write

$$\mathbb{R}^n = \mathbb{R} \times \mathbb{R}^{n-1} := \left\{ z := (t, x) \in \mathbb{R} \times \mathbb{R}^{n-1} \right\}.$$ 

Let $\Omega \subset \mathbb{R}^{n-1}$ be a bounded star-shaped domain. We consider a left continuous increasing function $\psi : (0, 1] \to (0, \infty)$, extend the definition of $\psi$ to the interval $(0, 2)$ by setting

$$\psi(t) = \psi(1), \text{ for every } t \in (1, 2)$$

and write

$$\Omega_\psi = \{(t, x) \in (0, 2) \times \mathbb{R}^{n-1} ; |x| < \psi(t)\Omega \}.$$
The space of locally integrable functions is denoted by $L^1_{\text{loc}}(U)$. For every measurable set $Q \subset \mathbb{R}^n$ with $0 < |Q| < \infty$, and every non-negative measurable or integrable function $f$ on $Q$ we define the integral average of $f$ over $Q$ by

$$\bar{f}(Q) := \frac{1}{|Q|} \int_Q f(w) \, dw.$$ 

Let us give the definitions of Sobolev space $W^{1,p}(U)$ and Hajlasz-Sobolev space $M^{1,p}(U)$.

**Definition 2.2.** We define the first order Sobolev space $W^{1,p}(U)$, $1 \leq p \leq \infty$, as the set

$$\{ u \in L^p(U); \nabla u \in L^p(U; \mathbb{R}^n) \}.$$ 

Here $\nabla u = \left( \frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_n} \right)$ is the weak (or distributional) gradient of a locally integrable function $u$.

We equip $W^{1,p}(U)$ with the non-homogeneous norm:

$$\|u\|_{W^{1,p}(U)} = \|u\|_{L^p(U)} + \|\nabla u\|_{L^p(U)}$$

for $1 \leq p \leq \infty$, where $\|f\|_{L^p(U)}$ denotes the usual $L^p$-norm for $p \in [1, \infty]$. For $u \in L^p(\Omega)$, we denote by $\mathcal{D}_p(u)$ the class of functions $0 \leq g \in L^p(\Omega)$ for which there exists $E \subset U$ with $|E| = 0$, so that

$$|u(z_1) - u(z_2)| \leq |z_1 - z_2| (g(z_1) + g(z_2)), \quad \text{for } z_1, z_2 \in U \setminus E.$$ 

**Definition 2.3.** We define the Hajlasz-Sobolev space $M^{1,p}(U)$, $1 \leq p \leq \infty$, as the set

$$\{ u \in L^p(U), \mathcal{D}_p(u) \neq \emptyset \}.$$ 

We equip $M^{1,p}(U)$ with the non-homogeneous norm:

$$\|u\|_{M^{1,p}(U)} = \|u\|_{L^p(U)} + \inf_{g \in \mathcal{D}_p(u)} \|g\|_{L^p(U)}.$$ 

for $1 \leq p \leq \infty$. For $1 < p \leq \infty$, we write $W^{1,p}(U) =_{C} M^{1,p}(U)$ for $W^{1,p}(U) = M^{1,p}(U)$ with

$$\frac{1}{C} \|\nabla u\|_{L^p(U)} \leq \inf_{g \in \mathcal{D}_p(u)} \|g\|_{L^p(U)} \leq C \|\nabla u\|_{L^p(U)}$$

for a positive constant $C > 1$ independent of $u \in W^{1,p}(U)$. The equivalence of Sobolev space and Hajlasz-Sobolev space on bounded star-shaped domain is invariant under linear stretching.

**Lemma 2.4.** Let $\Omega \subset \mathbb{R}^n$ be a bounded star-shaped domain with $W^{1,p}(\Omega) =_{C} M^{1,p}(\Omega)$ for some $1 < p \leq \infty$. Then for every $0 < \lambda < \infty$, we have $W^{1,p}(\lambda \Omega) =_{C} M^{1,p}(\lambda \Omega)$ with a same constant $C$.

**Proof.** Fix $0 < \lambda < \infty$. Let $u \in W^{1,p}(\lambda \Omega)$ be arbitrary. Then we define a function $u_\lambda$ on $\Omega$ by setting

$$u_\lambda(z) := u(\lambda z)$$

for every $z \in \Omega$. Then, by the change of variables formula, we have $u_\lambda \in W^{1,p}(\Omega)$ with

$$\|\nabla u_\lambda\|_{L^p(\Omega)} = \lambda^{1 - \frac{2}{p}} \|\nabla u\|_{L^p(\lambda \Omega)}.$$ 

Since $W^{1,p}(\Omega) =_{C} M^{1,p}(\Omega)$, there exists a function $g_{u_\lambda} \in \mathcal{D}_p(u_\lambda)$ with

$$\frac{1}{C} \|\nabla u_\lambda\|_{L^p(\Omega)} \leq \|g_{u_\lambda}\|_{L^p(\Omega)} \leq C \|\nabla u_\lambda\|_{L^p(\Omega)}.$$
Then we define a function $g_u$ on $\lambda \Omega$ by setting

$$g_u(z) := \lambda g_{u,\lambda} \left( \frac{z}{\lambda} \right)$$

for every $z \in \lambda \Omega$. The change of variables formula implies $g_u \in D_p(u)$ with

$$\|g_u\|_{L^p(\lambda \Omega)} = \lambda^{\frac{n}{p} - 1} \|g_u\|_{L^p(\Omega)}.$$  

By combining inequalities (2.5), (2.6) and (2.7) together, we obtain the desired inequality

$$\frac{1}{C} \|\nabla u\|_{L^p(\lambda \Omega)} \leq \|g_u\|_{L^p(\lambda \Omega)} \leq C \|\nabla u\|_{L^p(\lambda \Omega)}.$$

□

3. Maximal functions

We will define a maximal function $M^T[f]$. That will vary only the first component $t$. For every $x \in \psi(1) \Omega \subset \mathbb{R}^{n-1}$ set

$$S_x := \{ t \in \mathbb{R}; (t, x) \in \Omega_\psi \}.$$  

Let $f : \Omega_\psi \to \mathbb{R}$ be measurable and let $(t, x) \in \Omega_\psi$. We define the one-dimensional maximal function in the direction of the first variable by setting

$$M^T[f](t, x) := \sup_{[a, b] \ni t} \int_{[a, b] \cap S_x} |f(s, x)| \, ds.$$  

The supremum is taken over all intervals $[a, b]$ containing $t$.

The next lemmas tell us that $M^T$ enjoys the usual $L^p$-boundedness property. See [3, Lemma 3.1] for the proof.

Lemma 3.2. Let $1 < p < \infty$. Then for every $f \in L^p(\Omega_\psi)$, $M^T[f]$ is measurable and we have

$$\int_{\Omega_\psi} |M^T[f](z)|^p \, dz \leq C \int_{\Omega_\psi} |f(z)|^p \, dz,$$

where the constant $C$ is independent of $f$.

4. Proof of the Main theorem

Let $u \in W^{1,p}(\Omega_\psi)$ be arbitrary, $1 < p < \infty$. Denote the gradient with respect to the $x$-variable by $\nabla^x$. Fix $0 < t < 2$, define the restriction of $u$ to $\{ t \} \times \psi(t) \Omega$ by setting

$$u_t(x) = u(t, x) \text{ on every } x \in \psi(1) \Omega.$$  

By Fubini’s theorem, for almost every $t \in (0, 2)$, $u_t \in W^{1,p}(\psi(t) \Omega)$. By Lemma 2.4, for every $t \in (0, 2)$ with $u_t \in W^{1,p}(\psi(t) \Omega)$, there exists a nonnegative function $g_t \in L^p(\psi(t) \Omega)$ we have

$$|u(x_1) - u(x_2)| \leq |x_1 - x_2| (g_t(x_1) + g_t(x_2))$$

for almost every $x_1, x_2 \in \psi(t) \Omega$ and

$$\frac{1}{C} \|\nabla^x u_t\|_{L^p(\psi(t) \Omega)} \leq \|g_t\|_{L^p(\psi(t) \Omega)} \leq C \|\nabla^x u_t\|_{L^p(\psi(t) \Omega)}.$$
for a constant $C$ independent of $t$. Then we define a function $g_u$ on $\Omega_\psi$ by setting
\[
g_u(t, x) := \begin{cases} g_t(x), & u_t \in W^{1,p}(\psi(t)\Omega), \\ \infty, & \text{otherwise.} \end{cases}
\]

By Fubini’s theorem and (4.3), we have $g_u \in L^p(\Omega_\psi)$ with
\[
\int_{\Omega_\psi} g_u(z)^p dz = \int_0^2 \int_{\psi(t)\Omega} g_t(x)^p dx dt \leq C \int_0^2 \int_{\psi(t)\Omega} |\nabla u_t(x)|^p dx dt < \infty.
\]

First, we introduce some results which will be used in the proof for that $W^{1,p}(\Omega) = M^{1,p}(\Omega)$ implies $W^{1,p}(\Omega_\psi) = M^{1,p}(\Omega_\psi)$. By [5], there is a bounded inclusion $\iota : M^{1,p}(\Omega_\psi) \hookrightarrow W^{1,p}(\Omega_\psi)$. To show that $\iota$ is an isomorphism, it suffices to show that its inverse $\iota^{-1}$ is both densely defined and bounded on $W^{1,p}(\Omega_\psi)$. Let $C^1(\Omega_\psi)$ be the set of continuously differentiable functions. Since $\Omega_\psi$ is bounded and star-shaped, by Theorem 1 in [11, Page 13], $C^1(\Omega_\psi) \cap W^{1,p}(\Omega_\psi)$ is dense in $W^{1,p}(\Omega_\psi)$. It suffices to show that $C^1(\Omega_\psi) \cap W^{1,p}(\Omega_\psi) \subset M^{1,p}(\Omega_\psi)$ and that for each $u \in C^1(\Omega_\psi) \cap W^{1,p}(\Omega_\psi)$ we have
\[
||u||_{M^{1,p}(\Omega_\psi)} \leq C ||u||_{W^{1,p}(\Omega_\psi)},
\]
for a positive constant independent of $u$. Next, we prove the main estimate.

**Lemma 4.4.** Let $\Omega \subset \mathbb{R}^{n-1}$ be a bounded star-shaped domain with $W^{1,p}(\Omega) = M^{1,p}(\Omega)$ for $1 < p < \infty$ and $\psi : (0, 1] \to (0, \infty)$ be a left continuous and increasing function. Define an outward cuspidal domain $\Omega_\psi$ as in (1.1). Let $z_1 = (t_1, x_1), z_2 := (t_2, x_2) \in \Omega_\psi$ be two points with $t_1 < t_2$. Suppose that $u \in W^{1,p}(\Omega_\psi) \cap C^1(\Omega_\psi)$. Then we have
\[
|u(z_1) - u(z_2)| \leq C|z_1 - z_2| \left( M^T[|\nabla u|](z_1) + M^T[|\nabla u|](z_2) + M^T[|g_u|](z_1) + M^T[|g_u|](z_2) \right).
\]

**Proof.** We will compare the change in the function via additional values $u(s, x_i)$ for some $s \in (0, 2)$. Without knowing exactly which $s$ yields an optimal estimate, we will instead average over a range of possible $s$ with the hope that, on average, the differences are better controlled. Indeed, let
\[
T_2 = \min \left\{ 2, t_2 + \frac{t_2 - t_1}{2} \right\},
T_1 = T_2 - \frac{t_2 - t_1}{2}.
\]
Notice that $t_2 \in [T_1, T_2]$ and $[T_1, T_2] \times \{ x_1, x_2 \} \subset \Omega_\psi$ by the fact that $\Omega$ is star-shaped and $\psi(t)\Omega \subset \psi(1)\Omega$ for every $0 < t < 2$. When we average over different possible $s \in [T_1, T_2]$ and use the triangle inequality we obtain that
\[
|u(z_2) - u(z_1)| \leq \frac{1}{T_2 - T_1} \int_{T_1}^{T_2} \left[ |u(t_2, x_2) - u(s, x_2)| + |u(s, x_2) - u(s, x_1)| + |u(s, x_1) - u(t_1, x_1)| \right] ds
\]
\[
(4.6)
\]
First, we estimate the terms I and III. Let \( i \in \{1, 2\} \). If \( t_i < s \), by the fundamental theorem of calculus we have

\[
|u(t_i, x_i) - u(s, x_i)| \leq \int_{t_i}^s |\nabla u(r, x_i)| \, dr \leq |t_i - s| M^r |\nabla u|(z_i) \leq 3(T_2 - T_1) M^r |\nabla u|(z_i).
\]

Similarly, (4.7) holds also if \( t_i \geq s \). Integrating with respect to \( s \) we obtain

\[
I \leq 3(T_2 - T_1) M^r |\nabla u|(z_2) \leq 2|z_2 - z_1| M^r |\nabla u|(z_2).
\]

and

\[
III \leq 3(T_2 - T_1) M^r |\nabla u|(z_1) \leq 2|z_2 - z_1| M^r |\nabla u|(z_1)
\]

Next, we apply (4.2) to the second term:

\[
II \leq \frac{C|x_1 - x_2|}{T_2 - T_1} \int_{T_1}^{T_2} (g_u(s, x_1) + g_u(s, x_2)) \, ds \leq C|x_1 - x_2| \left( \frac{1}{T_2 - T_1} \int_{T_1}^{T_2} (g_u(s, x_1) \, ds + \frac{1}{T_2 - T_1} \int_{T_1}^{T_2} (g_u(s, x_2) \, ds) \right)
\]

(4.10) \leq C|z_1 - z_2| \left( M^r |g_u|(z_1) + M^r |g_u|(z_2) \right).

Finally, by combining inequalities (4.8), (4.9), (4.10) and (4.6), we obtain the desired inequality (4.5).

Recall that a domain \( U \) is quasiconvex if there exists a constant \( C \geq 1 \) such that, for every pair of points \( x, y \in U \), there is a rectifiable curve \( \gamma \subset U \) joining \( x \) to \( y \) so that \( \text{len}(\gamma) \leq C|x - y| \).

Proof of Theorem 1.2. By [5, Theorem 7], if \( U \) is a bounded domain, \( W^{1,\infty}(U) = M^{1,\infty}(U) \) if and only if \( U \) is quasiconvex. For every left continuous and increasing function \( \psi : (0, 1] \to (0, \infty) \), \( \Omega_\psi \) is quasiconvex if and only if \( \Omega \) is quasiconvex. Hence, we have \( W^{1,\infty}(\Omega_\psi) = M^{1,\infty}(\Omega_\psi) \) if and only if \( W^{1,\infty}(\Omega) = M^{1,\infty}(\Omega) \). Thus, fix \( 1 \leq p < \infty \). By [5], we know that there is a bounded inclusion \( \iota : M^{1,p}(\Omega_\psi) \hookrightarrow W^{1,p}(\Omega_\psi) \). To show that \( \iota \) is an isomorphism it suffices to show that the dense subspace \( C^1(\Omega_\psi) \cap W^{1,p}(\Omega_\psi) \) of \( W^{1,p}(\Omega_\psi) \) is contained in \( M^{1,p}(\Omega_\psi) \), and that the restricted inverse \( \iota^{-1}|C^1(\Omega_\psi) \cap W^{1,p}(\Omega_\psi) \) is defined and bounded.

Let \( u \in C^1(\Omega_\psi) \cap W^{1,p}(\Omega_\psi) \) be arbitrary. Set

\[
\hat{g}(t, x) = M^r |\nabla u|(t, x) + g_u(t, x) + M^r |g_u|(t, x).
\]

By (4.2) and Lemma 4.4, for every \( z_1, z_2 \in \Omega_\psi \), we get the estimate

\[
|u(z_1) - u(z_2)| \leq C |z_1 - z_2| (\hat{g}(z_1) + \hat{g}(z_2)).
\]

Hence \( g := C \hat{g} \in D_p(u) \) for a suitable constant \( C > 1 \). The triangle inequality gives

\[
\int_{\Omega_\psi} |g(z)|^p \, dz \leq C \left( \int_{\Omega_\psi} M^r |\nabla u|(z)^p \, dz + \int_{\Omega_\psi} g_u(z)^p \, dz + \int_{\Omega_\psi} M^r |g_u|(z)^p \, dz \right).
\]

The inequality (4.3) leads to the estimate

\[
\int_{\Omega_\psi} g_u(z)^p \, dz \leq C \int_{\Omega_\psi} |\nabla u(z)|^p \, dz.
\]
Lemmas 3.2 leads to the estimates
\[
\int_{\Omega_\psi} |M^T[|\nabla u|(z)]^p| dz \leq C \int_{\Omega_\psi} |\nabla u(z)|^p dz
\]
and
\[
\int_{\Omega_\psi} |M^T[g_u](z)|^p dz \leq C \int_{\Omega_\psi} g_u(z)^p dz
\]
\[
\leq C \int_{\Omega_\psi} |\nabla u(z)|^p dz,
\]
which imply that \( g \in D_p(u) \) and that
\[
\|u\|_{M^{1,p}(\Omega_\psi)} \leq C \|u\|_{W^{1,p}(\Omega_\psi)}.
\]
That is, \( \iota^{-1}|C^1(\Omega_\psi)\cap W^{1,p}(\Omega_\psi) \) is both well-defined and bounded. Hence, we proved that \( W^{1,p}(\Omega) = M^{1,p}(\Omega_\psi) \) implies \( W^{1,p}(\Omega_\psi) = M^{1,p}(\Omega_\psi) \).

Next, we prove \( W^{1,p}(\Omega_\psi) = M^{1,p}(\Omega_\psi) \) will imply \( W^{1,p}(\Omega) = M^{1,p}(\Omega_\psi) \). By the same reason, it suffices to show the dense subspace \( C^1(\Omega) \cap W^{1,p}(\Omega) \) is contained in \( M^{1,p}(\Omega) \). Let \( u \in C^1(\Omega) \cap W^{1,p}(\Omega) \) be arbitrary. If \( u \equiv c \) for some constant \( c \in \mathbb{R} \), then it is easily seen that \( u \in M^{1,p}(\Omega) \) with
\[
\|u\|_{W^{1,p}(\Omega)} = \|u\|_{M^{1,p}(\Omega)}.
\]
Hence, we assume \( u \) is not a constant function, that implies \( \|\nabla u\|_{L^p(\Omega)} > 0 \).

We define a function \( \tilde{u} \) on \( \psi(1)\Omega \) by setting \( \tilde{u}(x) = u \left( \frac{x}{\psi(1)} \right) \) for every \( x \in \psi(1)\Omega \).

The change of variables formula implies
\[
\|\tilde{u}\|_{L^p(\psi(1)\Omega)} = \psi(1)^\frac{1}{p} \|u\|_{L^p(\Omega)} \quad \text{and} \quad \|\nabla \tilde{u}\|_{L^p(\psi(1)\Omega)} = \psi(1)^{1-\frac{1}{p}} \|\nabla u\|_{L^p(\Omega)}.
\]
Hence, \( \tilde{u} \in C^1(\psi(1)\Omega) \cap W^{1,p}(\psi(1)\Omega) \). By the definition of outward cuspidal domain \( \Omega_\psi \), we have
\[
\Omega_\psi := \bigcup_{x \in \psi(1)\Omega} S_x.
\]
Then, we define a function \( \hat{u} \) on \( \Omega_\psi \) by setting \( \hat{u}(t,x) := \tilde{u}(x) \) for every \( (t,x) \in \Omega_\psi \).

Since \( \psi(t)\Omega \subset \psi(1)\Omega \) for every \( t \in (0,2) \), we have \( \hat{u} \in C^1(\Omega_\psi) \) with
\[
\|\hat{u}\|_{L^p(\Omega_\psi)} \leq 2 \|\tilde{u}\|_{L^p(\psi(1)\Omega)} \quad \text{and} \quad \|\nabla \hat{u}\|_{L^p(\Omega_\psi)} \leq 2 \|\nabla \tilde{u}\|_{L^p(\psi(1)\Omega)}.
\]
Hence, \( \hat{u} \in C^1(\Omega_\psi) \cap W^{1,p}(\Omega_\psi) \). Since \( W^{1,p}(\Omega_\psi) = M^{1,p}(\Omega_\psi) \), there exists \( g \in D_p(\hat{u}) \) with
\[
\|g\|_{L^p(\Omega_\psi)} \leq C \|\nabla \hat{u}\|_{L^p(\Omega_\psi)}.
\]
By combining inequalities (4.13) and (4.14) together, we obtain
\[|\hat{u}(z_1) - \hat{u}(z_2)| \leq |z_1 - z_2|(g(z_1) + g(z_2))\]
for almost every \(z_1, z_2 \in \Omega_\psi\). Simply set \(g = \infty\) on this extra measure zero set, we can assume the last inequality holds for every \(z_1, z_2 \in \Omega_\psi\). Set \(g_t\) to be the restriction of \(g\) to \(\{t\} \times \psi(t)\Omega\) and
\[\|\nabla \chi_{\Omega_{\psi}}\|_{L^p(\psi(1)\Omega)} \leq C\|\nabla \chi\|_{L^p(\Omega)}.\]
Then, we have
\[0 < A \leq \|\nabla \hat{u}\|_{L^p(\Omega)} \leq C\|\nabla \chi\|_{L^p(\Omega)}.\]
There exists \(\hat{t} \in (1, 2)\) with
\[A \leq \|g_{\hat{t}}\|_{L^p(\psi(1)\Omega)} \leq 2A.\]
Then for every \(x_1, x_2 \in \psi(1)\Omega\), we have
\[|\hat{u}(x_1) - \hat{u}(x_2)| = |\hat{u}(\hat{t}, x_1) - \hat{u}(\hat{t}, x_2)| \leq |x_1 - x_2|(g_{\hat{t}}(x_1) + g_{\hat{t}}(x_2)).\]
Hence, we have \(g_{\hat{t}} \in D_p(\hat{u})\) with
\[\|g_{\hat{t}}\|_{L^p(\psi(1)\Omega)} \leq C\|\nabla \chi\|_{L^p(\Omega)}.\]
Define a function \(g\) on \(\Omega\) by setting
\[g(x) := \frac{1}{\psi(1)}g_{\hat{t}}(\psi(1)x) \text{ for every } x \in \Omega.\]
Then, we have
\[|u(x_1) - u(x_2)| \leq |x_1 - x_2|(g(x_1) + g(x_2))\]
for every \(x_1, x_2 \in \Omega\), and
\[\|g\|_{L^p(\Omega)} = \psi(1)^{\frac{n}{p} - 1}\|g_{\hat{t}}\|_{L^p(\psi(1)\Omega)}.\]
By combining inequalities (4.13) and (4.14) together, we obtain \(g \in D_p(u)\) with
\[\|g\|_{L^p(\Omega)} \leq C\|\nabla \chi\|_{L^p(\Omega)}\]
for a constant \(C\) independent of \(u\). Hence, we have \(C^1(\Omega) \cap W^{1,p}(\Omega) \subset M^{1,p}(\Omega)\) with
\[\|u\|_{M^{1,p}(\Omega)} \leq C\|u\|_{W^{1,p}(\Omega)}\]
as desired. In conclusion, we proved \(W^{1,p}(\Omega_\psi) = M^{1,p}(\Omega_\psi)\) implies \(W^{1,p}(\Omega) = M^{1,p}(\Omega).\)

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