Electromagnetic fields in Schwarzschild and Reissner-Nordström geometry.
Quantum corrections to the black hole entropy

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Abstract: Using standard coordinates, the Maxwell equations in the Reissner-Nordström geometry are written in terms of a couple of scalar fields satisfying Klein-Gordon like equations. The density of states is derived in the semi-classical approximation and the first quantum corrections to the black hole entropy is computed by using the brick-wall model.

1 Introduction

In the last decade, many efforts have been made in order to understand the deep origin of the black hole entropy [1-4] first introduced by Bekenstein [5-7] (for a recent review see Ref. [8]), but widely embraced only after the Hawking’s demonstration of black hole thermal radiation [9-11]. Some possible interpretations and methods of calculation have been proposed, but, at the moment, no one of them seems to be the true answer [2, 12-16]. Here we recall the ’t Hooft proposal [2, 17], in which the black hole entropy is identified with the entropy of the quantum fields surrounding the black hole itself. Since the density of states approaching the horizon diverges, in order to avoid divergences in the entropy, he has to introduce a cutoff parameter of the order of the Planck’s length, which is interpreted as the position of a “brick wall” (the brick wall model). He also computed the contribution to the entropy of a Schwarzschild black hole due to a scalar field using a semi-classical approximation. After this, quantum corrections to the Bekenstein-Hawking entropy, due to a scalar field, have been computed by different methods for Schwarzschild [18, 19] Reissner-Nordström [20, 21] and also for Kerr-Newman [22] black holes (for a recent review on quantum corrections to black hole entropy see for example Ref. [19]).

Also in the case of scalar fields, due to technical difficulties, one has to make some suitable approximation. Some authors directly consider the Rindler space, which can be considered as an approximation of the Schwarzschild case for very large mass. In the Rindler space, the contribution to the entropy due to scalar and also higher spin fields have been considered in
Refs. [25–27], where in particular it has been shown that, depending on the method of calculation used, the contribution of the electromagnetic field is not just twice the scalar one, but it contains some unexpected anomalous surface terms.

In the present paper we focus our attention on the electromagnetic field in Reissner-Nordström background. We solve the Maxwell equations and then compute the contribution to the black hole entropy by using the brick wall model and show that no anomalous term is present. All results are valid for the Schwarzschild black hole in the trivial limit of vanishing charge. Maxwell equations in Schwarzschild, Reissner-Nordström and also Kerr metric are usually solved by using Newman-Penrose formalism (see for example Ref. [28] and Ref. [29] for a complete treatment of solutions in such a formalism), which is not familiar to many readers. For this reason here we prefer to use a more conventional method, which consists in solving Maxwell equations for the electromagnetic potential in standard coordinates in a suitable gauge.

The paper is organized as follows. In Section 2 we consider the Maxwell equations for the electromagnetic potential in the Reissner-Nordström background and show that they reduce to the ones of a couple of independent scalar fields, which we solve in the semi-classical approximation, following the ‘t Hooft’s original work [2, 17]. In this way we easily derive the expected contribution to the Bekenstein-Hawking entropy in Section 3.

As usual we use natural units in which $G = \hbar = c = k = 1$.

## 2 Electromagnetic fields in Schwarzschild and Reissner-Nordström black holes

Here we study the electromagnetic waves in the Reissner-Nordström background (the solutions in the Schwarzschild geometry will be obtained as a limiting case for $Q \to 0$), that is the non-static solutions of the equations

$$
\nabla_i F^{ij} = 0, \quad i, j = 0, ..., 3,
$$

in the metric $g_{ij}$ given by

$$
\begin{align*}
\text{ds}^2 &= -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right) dt^2 + \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1} dr^2 + r^2 d\sigma^2, \\
F^{ij} &\text{ being the electromagnetic field strength, } \nabla_i \text{ the covariant derivative, } M \text{ and } Q \text{ the mass and the charge of the black hole respectively and finally } d\sigma^2 \text{ is the metric on the unit sphere, which is usually written in polar coordinates } \{\vartheta, \varphi\} \text{, but for our purposes the (complex) stereographic coordinates are more convenient. Then we write } \\
d\sigma^2 &= d\vartheta^2 + \sin^2 \vartheta \, d\varphi^2 = \frac{4}{(1 + \bar{z}z)^2} dz \, d\bar{z},
\end{align*}
$$

where

$$
z = \frac{\sin \vartheta e^{i\varphi}}{1 - \cos \vartheta}, \quad \bar{z} = \frac{\sin \vartheta e^{-i\varphi}}{1 - \cos \vartheta}.
$$

In such a coordinates the non-vanishing components of the metric read

$$
\begin{align*}
g_{00} &\equiv g_{tt} = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right), \\
g_{11} &\equiv g_{rr} = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1}, \\
g_{23} &\equiv g_{z\bar{z}} = \frac{2r^2}{(1 + \bar{z}z)^2} = g_{32} \equiv g_{\bar{z}z}.
\end{align*}
$$

In terms of the electromagnetic potential $A_k$, Eqs. (2.1) become

$$
\Box A_k - \nabla_j \nabla_k A^j = \Box A_k - \nabla_k \nabla_j A^j - R_{kj} A^j = 0
$$

(2.6)
and finally, after some calculations we can put them in the more useful form

\[ L A_k = A_j \partial_k \Gamma^j + 2g^{rs} \Gamma^j r_k \partial_s A_j + \partial_k \nabla_j A^j, \tag{2.7} \]

where \( \Gamma^j = g^{rs} \Gamma^j r_s \), \( \square = g^{ij} \nabla_i \nabla_j \) is the Dalem-bert's operator while \( L \) represents the Dalem-bert's operator acting on functions, that is

\[ L = \frac{1}{\sqrt{|g|}} \partial_j \sqrt{|g|} g^{ij} \partial_j. \tag{2.8} \]

From Eq. (2.7), after straightforward calculations we get

\[ L A_0 = -2 \left( \frac{M}{r^2} - \frac{Q^2}{r^3} \right) (\partial_0 A_1 - \partial_1 A_0) + \partial_0 \nabla_j A^j, \tag{2.9} \]
\[ L A_1 = -2 \left( \frac{M}{r^2} - \frac{Q^2}{r^3} \right) \left[ \partial_1 A_1 - (g^{00})^2 \partial_0 A_0 \right] + \frac{2}{r^2} \left( 1 - \frac{2M}{r} \right) A_1 + \frac{2g^{33}}{r^2} (\partial_2 A_3 + \partial_3 A_2) + \partial_1 \nabla_j A^j, \tag{2.10} \]
\[ L A_2 = -\frac{2z(1 + z\bar{z})}{r^2} \nabla_2 A_2 + \frac{2g^{31}}{r} (\partial_1 A_2 - \partial_2 A_1) + \partial_2 \nabla_j A^j, \tag{2.11} \]
\[ L A_3 = -\frac{2z(1 + z\bar{z})}{r^2} \nabla_3 A_3 + \frac{2g^{11}}{r} (\partial_1 A_3 - \partial_3 A_1) + \partial_3 \nabla_j A^j. \tag{2.12} \]

In order to select the physical degrees of freedom, now we fix the gauge \( A_0 = 0 \). In this way Eq. (2.9) gives the constraint

\[ \nabla_j A^j - 2 \left( \frac{M}{r^2} - \frac{Q^2}{r^3} \right) A_1 = \frac{g^{rr}}{r^2} \partial_r (r^2 A_r) + g^{z\bar{z}} (\partial_z A_{\bar{z}} + \partial_{\bar{z}} A_z) = 0, \tag{2.13} \]

while Eqs. (2.10)-(2.12) simplify to

\[ L A_r = -\frac{2}{r^2} \partial_r (r g^{rr} A_r), \tag{2.14} \]
\[ \left[ L + \frac{2z(1 + z\bar{z})}{r^2} \partial_{\bar{z}} \right] A_z = \frac{2g^{rr}}{r} \partial_r A_z - \frac{2}{r} \left( 1 - \frac{3M}{r} + \frac{2Q^2}{r^2} \right) \partial_{\bar{z}} A_r, \tag{2.15} \]
\[ \left[ L + \frac{2z(1 + z\bar{z})}{r^2} \partial_{\bar{z}} \right] A_{\bar{z}} = \frac{2g^{rr}}{r} \partial_r A_{\bar{z}} - \frac{2}{r} \left( 1 - \frac{3M}{r} + \frac{2Q^2}{r^2} \right) \partial_{\bar{z}} A_r. \tag{2.16} \]

It has to be noted that for any function \( \psi(t, r, z, \bar{z}) \) one has

\[ \left[ L + \frac{2z(1 + z\bar{z})}{r^2} \partial_{\bar{z}} \right] \partial_z \psi = \partial_z L \psi, \]
\[ \left[ L + \frac{2z(1 + z\bar{z})}{r^2} \partial_{\bar{z}} \right] \partial_{\bar{z}} \psi = \partial_{\bar{z}} L \psi. \]

This means that the variables can be easily separated by putting \( A_z = \partial_z \psi, A_{\bar{z}} = \pm \partial_{\bar{z}} \psi \) (note that in principle one could choose two different functions \( \psi \), but this is unnecessary, since only the sum enters the other equations). Now, one can directly verify that two classes of independent eigenfunctions \( A \equiv (A_t, A_r, A_z, A_{\bar{z}}) \) of Eqs. (2.14)-(2.16), satisfying the constraint, Eq. (2.13), can be put in the form

\[ A^{(1)} \equiv \left( 0, 0, \frac{1}{2l(l + 1)\omega} \partial_z \Phi, \frac{1}{-2l(l + 1)\omega} \partial_{\bar{z}} \Phi \right), \tag{2.17} \]
\[ A^{(2)} \equiv \left( 0, \sqrt{\frac{l(l+1)}{2\omega^3}} \Phi, \frac{g^{rr}}{\sqrt{2l(l+1)\omega^3}} \partial_z \partial_r \Phi, \frac{\partial r}{\sqrt{2l(l+1)\omega^3}} \partial_z \partial_r \Phi \right), \]  

(2.18)

where \( \Phi(t, r, z, \bar{z}) = e^{-i\omega t} f(r) Y_l^m(z, \bar{z}) \) is a scalar field satisfying the equation

\[ \Box \Phi = \frac{2g^{rr}}{r} \partial_r \Phi. \]  

(2.19)

In the tortoise coordinate

\[
    r^* = r + \frac{r^2}{r_+ - r_-} \ln(r - r_+) - \frac{r^2}{r_+ - r_-} \ln(r - r_-),
\]

\[
    r_\pm = M \pm \sqrt{M^2 - Q^2},
\]

the radial part of the field satisfies the ordinary differential equation

\[
\left[ \frac{d^2}{dr^*^2} + \omega^2 - V_l(r^*) \right] f(r^*) = 0, \quad V_l(r^*) = \frac{l(l+1)}{r^2} g^{rr},
\]  

(2.20)

with the normalization property

\[
\int |f(r^*)|^2 dr^* = 1.
\]  

(2.21)

\( r_+ = r_h \) is the radius of the horizon. The above solutions form a set of orthonormal eigenfunctions with respect to the scalar product \[30, 31\]

\[
\left( A^{(1)}, A^{(2)} \right) = i \int g^{ij} \left( A_i^{(1)} \partial_j A_j^{(2)} - \partial_i A_i^{(1)} A_j^{(2)} \right) g_{zz} g_{rr} dz d\bar{z} dr.
\]  

(2.22)

Note that the more general expression for the scalar product has been specialized to our particular case.

### 3 One-loop contribution to the entropy

The (leading) one-loop contribution to the entropy of the black hole due to the electromagnetic field can be easily computed using the brick-wall method \[2, 17\]. The computation is parallel to the original one given by ’t Hooft. In the WKB approximation, the energy spectrum is given by

\[
\int k_l(r^*) dr^* = \int k_l(r) g_{rr} dr = 2\pi \hbar n, \quad n \in \mathbb{N},
\]  

(3.1)

where

\[
k_l(r^*)^2 = \omega^2 - V_l(r^*) = E^2 - \frac{l(l+1)g^{rr}}{r^2} = k_l(r)^2.
\]  

(3.2)

Following ’t Hooft, we consider the field in the region \( r_h + \varepsilon < r < R \) and suppose it to satisfy Dirichlet boundary conditions. Then, the number of eigenstates with energy smaller than \( E \) read

\[
\nu(E) = \frac{1}{\pi} \sum_l (2l+1)n = \sum_l (2l+1) \int_{r_h+\varepsilon}^R \sqrt{E^2 - V_l(r)} g_{rr} dr
\]

\[
\sim \frac{1}{\pi} \int_0^{\frac{E^2}{\hbar^2} + 1/4} d\lambda \int_{r_h+\varepsilon}^R \sqrt{E^2 - \frac{\lambda - 4}{\lambda} g_{rr}^2} dr,
\]  

(3.3)
where we have put $\lambda = h^2 (l + 1/2)^2$ and $h^2 \sum_l (2l + 1) \to \int d\lambda$. The extreme of integration in the variable $\lambda$ is due to the fact that $k_t(r)$ has to be positive. The integration in $\lambda$ can be performed and so

$$
\nu(E) \sim \frac{2}{3\pi} \int_{r_h+\varepsilon}^{R} \left( E^2 + \frac{g_{rr}^*}{4r^2} \right)^{3/2} r^2 g_{rr}^* \, dr
$$

$$
= \frac{2}{3\pi} \int_{\varepsilon}^{r_h} \frac{r_h^4}{x^2} \left( 1 + \frac{x}{r_h} \right)^4 \left( E^2 + \frac{x}{4r_h^3(1 + x/r_h)^2} \right)^{3/2} \, dx
$$

$$
+ \frac{2}{3\pi} \int_{r_h}^{R-r_h} \frac{x^2}{x} \left( 1 + \frac{r_h}{x} \right)^4 \left( E^2 + \frac{1}{4x^2(1 + r_h/x)^3} \right)^{3/2} \, dx
$$

$$
\sim \frac{2r_h^6 E^3}{3\pi \varepsilon(r_h - r_-)^2} + \frac{VE^3}{6\pi} + ... \quad (3.4)
$$

where in the latter expression only the leading divergences have been written down ($V$ is the volume of the spherical box). The derivative of $\nu(E)$ in Eq. (3.4) represents the density of states with energy $E$.

Now for the partition function one easily gets

$$
\ln Z = -\sum_{\nu} \ln \left( 1 - e^{-\beta E_{\nu}} \right) = \beta \int_{0}^{\infty} \frac{\nu(E)}{e^{\beta E} - 1} \, dE
$$

$$
\sim \frac{\pi^2 V}{90 \beta^3} + \frac{2\pi^3 r_h^6}{45 \varepsilon \beta^3 (r_h - r_-)^2} + ... \quad (3.5)
$$

which agrees with a similar expression in Ref. [21] and reduces to the expression given in Refs. [4, 17] in the the limit $Q \to 0$, that is $r_h = 2M$, $r_- = 0$. The first term on the right hand side of the latter equation is the usual one proportional to the volume, while the second is a divergent contribution due to the presence of the horizon and is interpreted as a quantum contribution to the black hole entropy due to the matter field. Taking into account that we have two independent scalar fields both satisfying Eq. (2.19), we finally get

$$
S_{RN} = - (\beta \partial_{\beta} - 1) \ln Z = \frac{16\pi^3 r_h^6}{45 \varepsilon \beta^3 (r_h - r_-)^2}. \quad (3.6)
$$

As already anticipated in the introduction, the leading term in the one-loop contribution to the entropy due to the electromagnetic field is exactly twice the one due to the scalar field. In our derivation we do not obtain anomalous terms of the kind obtained for the Rindler case in Refs. [25–27]. In any case, as suggested in Ref. [27], such terms are non physical and have to be discharged.

All results of this section have a good limit for $Q \to 0$ and so they are valid also for the Schwarzschild black hole, with the simple substitution $r_h = 2M$, $r_- = 0$.

At the equilibrium temperature $T_H = \frac{r_h - r_-}{4\pi l^2}$ the entropy reads

$$
S_{T=T_H} = \frac{\sqrt{M^2 - Q^2}}{90 \varepsilon} \sim \frac{1}{45} \left( \frac{r_h}{l} \right)^2, \quad l^2 = \frac{4r_h^2 \varepsilon}{r_h - r_-} \quad (3.7)
$$

where the cutoff parameter $\varepsilon$ has been expressed in terms of the proper distance $l$.

## 4 Conclusion

We have written the Maxwell equations in the Reissner-Nordst"om background in terms of a couple of scalar fields satisfying a Klein-Gordon like equation. In this way we have shown that
the first quantum correction to the black hole entropy due to the electromagnetic field, which we have computed by using the semi-classical approximation, is exactly twice the one which one has for a scalar field. This means that the anomalous contributions, which one has for the Rindler case \cite{25,27}, are not present in this context.

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