NORMAL PRESENTATION ON ELLIPTIC RULED SURFACES

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INTRODUCTION

This article deals with the normal presentation of line bundles over an elliptic ruled surface. Let $X$ be an irreducible projective variety and $L$ a very ample line bundle on $X$, whose complete linear series defines

$$
\phi_L : X \rightarrow \mathbb{P}(H^0(L))
$$

Let $S = \bigoplus_{n=0}^{\infty} S^nH^0(X, L)$ and Let $R(L) = \bigoplus_{n=0}^{\infty} H^0(X, L^\otimes n)$ be the homogenous coordinate ring associated to $L$. Then $R$ is a finitely generated graded module over $S$, so it has a minimal graded free resolution. We say that the line bundle $L$ is normally generated if the natural maps

$$
S^mH^0(X, L) \rightarrow H^0(X, L^\otimes m)
$$

are surjective for all $m \geq 2$. If $L$ is normally generated, then we say that $L$ satisfies property $N_p$, if the matrices in the free resolution of $R$ over $S$ have linear entries until the $p$th stage. In particular, property $N_1$ says that the homogeneous ideal $I$ of $X$ in $\mathbb{P}(H^0(L))$ is generated by quadrics. A line bundle satisfying property $N_1$ is also called normally presented.

Let $R = k \oplus R_1 \oplus R_2 \oplus \ldots$ be a graded algebra over a field $k$. The algebra $R$ is a Koszul ring iff $\text{Tor}_i^R(k, k)$ has pure degree $i$ for all $i$.

We would like to thank our advisor David Eisenbud for his encouragement and helpful advice. We are also glad to thank Aaron Bertram, Raquel Mallavibarrena and Giuseppe Pareschi for helpful discussions.
In this article we determine exactly (Theorem 4.1) which line bundles on elliptic ruled surface $X$ are normally presented (Yuko Homma has classified in [Ho1] and [Ho2] all line bundles which are normally generated on an elliptic ruled surface). In particular we see that numerical classes of normally presented divisors form a convex set. (See Figure 1 for the case $e(X) = -1$; recall that Num($X$) is generated by the class of a minimal section $C_0$ and by the class of a fiber $f$ and that $C_0$ is ample.) As a corollary of the above result we show that Mukai’s conjecture is true for the normal presentation of the adjoint linear series for an elliptic ruled surface.

In section 5 of this article, we show that if $L$ is normally presented on $X$ then the homogeneous coordinate ring associated to $L$ is Koszul. We also give a new proof of the following result due to Butler: if deg($L$) $\geq$ 2$g$ + 2 on a curve $X$ of genus $g$, then $L$ embeds $X$ with Koszul homogeneous coordinate ring.

To put things in perspective, we would like to recall what is known regarding these questions in the case of curves. A classical result of Castelnuovo (c.f. [C]) says that if deg($L$) $\geq$ 2$g$ + 1, $L$ is normally generated. St. Donat and Fujita ([F] and [S-D]) proved that if deg($L$) $\geq$ 2$g$ + 2, then $L$ is normally presented. These theorems have been recently generalized to higher syzygies by Green (see [G]), who proved that if deg($L$) $\geq$ 2$g$ + $p$ + 1, then $L$ satisfies the property $N_p$. One way of generalizing the above results to higher dimensions is to interpret them in terms of adjoint linear series: let $\omega_X$ be the canonical bundle of a curve $X$, and let $A$ be an ample line bundle (since $X$ is a curve, $A$ is ample iff deg($A$) > 0). If $L = \omega_X \otimes A^{\otimes 3}$ (respectively $L = \omega_X \otimes A^{\otimes p+3}$), then Castelnuovo’s Theorem (respectively Green’s Theorem) says that $L$ is normally generated (respectively satisfies property $N_p$).

Unlike the case of curves, the landscape of surfaces (not to speak of higher dimensions) is relatively uncharted. Recently Reider proved (c.f. [R]) that if $X$ is a surface over the complex numbers, then $\omega_X \otimes A^{\otimes 4}$ is very ample. Mukai has conjectured that $\omega_X \otimes A^{\otimes p+4}$ satisfies $N_p$. Some work in this direction has been done by David Butler in [B], where he studies the syzygies of adjoint linear series on ruled varieties. He proves that if the dimension of $X$ is $n$, then $\omega_X \otimes A^{\otimes 2n+1}$ is normally generated and $\omega_X \otimes A^{\otimes 2n+2np}$ is normally presented; specializing to the case of ruled surfaces, his result says that $\omega_X \otimes A^{\otimes 5}$ is normally generated and that $\omega_X \otimes A^{\otimes 8}$ is normally presented. In this article we consider not just the adjunction bundle, but any very ample line bundle on an elliptic ruled surface. In particular, we prove that $\omega_X \otimes A^{\otimes 5}$ is normally presented thereby proving Mukai’s conjecture for $p = 1$ in the case of elliptic ruled surface.

In a sequel to this article we generalize our results on normal presentation to higher syzygies. We there show the following: let $L = B_1 \otimes \ldots \otimes B_{p+1}$ be a line bundle on $X$, where each $B_i$ is base point free and ample, then $L$ satisfies property $N_p$. As a corollary we show that $\omega_X \otimes A^{\otimes 2p+3}$ satisfies property $N_p$.

1. Background material

Convention. Throughout this paper we work over an algebraic closed field $k$.

We state in this section some results we will use later. The first one is this beautiful cohomological characterization by Green of the property $N_p$. Let $L$ be a globally generated line bundle. We define the vector bundle $M_L$ as follows:

$$0 \rightarrow M_L \rightarrow H^0(L) \otimes \mathcal{O}(p) \rightarrow L \rightarrow 0 \quad (1.1)$$
In fact, the exact sequence (1.1) makes sense for any variety \( X \) and any vector bundle \( L \) as long as \( L \) is globally generated.

**Lemma 1.2.** Let \( L \) be a normally generated line bundle on a variety \( X \) such that \( H^i(L^\otimes^{-i}) = 0 \) for all \( i \geq 1 \). Then, \( L \) satisfies the property \( N_p \) iff \( H^1(L^\otimes^{p+1} L) \) vanishes for all \( 1 \leq p' \leq p \).

**Proof.** The lemma is a corollary of [GL], Lemma 1.10. \( \square \)

(1.2.1) If the \( \text{char}(k) \neq 2 \), we can obtain the vanishing of \( H^1(L^\otimes M L) \) by showing the vanishings of \( H^1(L^\otimes M L) \), because \( L^\otimes M L \) is in this case a direct summand of \( M^2 L \).

The other main tool we will use is a generalization by Mumford of a lemma of Castelnuovo:

**Theorem 1.3.** Let \( L \) be a base-point-free line bundle on a variety \( X \) and let \( F \) be a coherent sheaf on \( X \). If \( H^i(F \otimes L^{-i}) = 0 \) for all \( i \geq 1 \), then the multiplication map

\[
H^0(F \otimes L^\otimes i) \otimes H^0(L) \to H^0(F \otimes L^\otimes i+1)
\]

is surjective for all \( i \geq 0 \).

**Proof.** [Mu], p. 41, Theorem 2. Note that the assumption made there of \( L \) being ample is unnecessary. \( \square \)

It will be useful to have the following characterization of projective normality:

**Lemma 1.4.** Let \( X \) be a surface with geometric genus \( h^2(O_X) = 0 \) and let \( L \) be an ample, base-point-free line bundle. If \( H^1(L) = 0 \), then \( L \) is normally generated iff \( H^1(M L \otimes L) = 0 \).

**Proof.** The line bundle \( L \) is normally generated iff the map

\[
S^m H^0(X, L) \xrightarrow{\alpha} H^0(X, L^\otimes m)
\]

is surjective. The map \( \alpha \) fits in the following commutative diagram:

\[
\begin{array}{ccc}
H^0(L)^\otimes m & \xrightarrow{\beta} & S^m H^0(L) \\
\downarrow{\gamma_1} & & \\
H^0(L^\otimes 2) \otimes H^0(L)^\otimes m-2 & \xrightarrow{\alpha} & H^0(L^\otimes m) \\
\downarrow{\gamma_2} & & \\
\vdots & & \\
\downarrow{\gamma_{m-2}} & & \\
H^0(L^\otimes m-1) \otimes H^0(L) & \xrightarrow{\gamma_{m-1}} & H^0(L^\otimes m)
\end{array}
\]

The map \( \beta \) is surjective. From this fact it follows that the surjectivity of \( \alpha \) is equivalent to the surjectivity of \( \gamma_{m-1} \circ \cdots \circ \gamma_1 \). Theorem 1.3 implies the surjectivity of \( \gamma_{m-1}, \ldots, \gamma_{m-1} \). Hence the surjectivity of \( \gamma_1 \) implies the surjectivity of \( \alpha \). On the other hand, if \( m = 2 \) the surjectivity of \( \alpha \) implies the surjectivity of \( \gamma_1 \). Finally from (1.1) we obtain

\[
H^0(L) \otimes H^0(L) \to H^0(L^\otimes 2) \to H^1(M L \otimes L) \to H^0(L) \otimes H^1(L).
\]

Therefore the vanishing of \( H^1(L) \) implies that the surjectivity of \( \gamma_1 \) is equivalent to the vanishing of \( H^1(M L \otimes L) \). \( \square \)
2. General results on normal presentation

As mentioned in the introduction, according to our philosophy, the tensor product of two base-point-free line bundles $B_1$ and $B_2$ (provided it is ample and that certain higher cohomology groups vanish) should be normally presented. This philosophy is made concrete in the following

**Proposition 2.1.** Let $X$ be a surface with geometric genus 0 and $B_1$ and $B_2$ base-point-free line bundles such that $H^1(B_1) = H^1(B_2) = H^2(B_2 \otimes B_1^*) = H^2(B_1 \otimes B_2^*) = 0$ and let $L = B_1 \otimes B_2$. Then $H^1(M_L^{\otimes p+1} \otimes L) = 0$ for $p = 0, 1$. In particular, if $L$ is ample and $\text{char}(k) \neq 2$, then $L$ is normally presented.

We will prove a more general version of Proposition 2.1 in section 5.

From this proposition we will obtain corollaries for Enriques surfaces (Corollary 2.8) and for elliptic ruled surfaces (Theorem 4.1). To prove Proposition 2.1 we will need several lemmas and observations:

**Observation 2.2.** Let $X$ be a surface with geometric genus 0, let $P$ be an effective line bundle, and let $B$ be a line bundle such that, for some $p \in |P|$, $B \otimes \mathcal{O}_P$ is trivial or has a global section vanishing at finite subscheme of $P$ (e.g., let $B$ be base-point-free). If $H^1(P) = H^1(B) = 0$, then $H^1(B \otimes P) = 0$.

**Observation 2.3.** Let $X$ be a surface, let $P$ be an effective line bundle and $L$ a coherent sheaf. If $H^2(L) = 0$, then $H^2(L \otimes P) = 0$

**Lemma 2.4.** Let $X$ be a surface, let $B$ be a globally generated line bundle such that $H^1(B) = 0$ and let $Y$ be a curve in $X$ such that $B \otimes \mathcal{O}_Y(Y)$ is globally generated. Then $B \otimes \mathcal{O}_X(Y)$ is also globally generated.

**Proof.** The result is the surjectivity of the middle vertical arrow in the following commutative diagram:

$$
\begin{array}{ccc}
H^0(B) \otimes \mathcal{O}_X & \hookrightarrow & H^0(B \otimes \mathcal{O}_X(Y)) \otimes \mathcal{O}_X \\
\downarrow & & \downarrow \\
B & \hookrightarrow & B \otimes \mathcal{O}_X(Y) \\
\end{array}
$$

The hypothesis is that the vertical left hand side arrow and the vertical right hand side arrow are surjective. □

**Lemma 2.5.** Let $X$ be a surface with geometric genus 0, let $B_1$ and $B_2$ be two base-point-free line bundles and let $L = B_1 \otimes B_2$. If $H^1(B_1) = H^1(B_2) = 0$ and $H^2(B_2 \otimes B_1^*) = 0$, then $H^2(M_L \otimes B_1^{\otimes n}) = 0$ for all $n \geq 1$.

**Proof.** If we tensor exact sequence (1.1) with $B_1^{\otimes n}$ and take global sections, we obtain

$$
H^0(L) \otimes H^0(B_1^{\otimes n}) \xrightarrow{\alpha} H^0(L \otimes B_1^{\otimes n}) \rightarrow H^1(M_L \otimes B_1^{\otimes n}) \rightarrow H^0(L) \otimes H^1(B_1^{\otimes n}).
$$

From Observation 2.2 it follows that the vanishing of $H^1(M_L \otimes B_1^{\otimes n})$ is equivalent to the surjectivity of the multiplication map $\alpha$. In the case $n = 1$ the surjectivity of $\alpha$ follows trivially from our hypothesis and Theorem 1.3. The proof of the
surjectivity of $\alpha$ goes by induction. We show here only the case $n = 2$. We consider the commutative diagram

$$
\begin{array}{ccc}
H^0(B_1) \otimes H^0(B_1) \otimes H^0(L) & \rightarrow & H^0(B_1^{\otimes 2}) \otimes H^0(L) \\
\downarrow \gamma & & \downarrow \alpha \\
H^0(B_1) \otimes H^0(B_1 \otimes L) & \rightarrow & H^0(B_1^{\otimes 2} \otimes L)
\end{array}
$$

where the maps are the obvious ones coming from multiplication. To prove the surjectivity of $\alpha$ it suffices to prove that $\gamma$ and $\delta$ are surjective. The surjectivity of $\gamma$ follows from the surjectivity of $\alpha$ when $n = 1$. To prove the surjectivity of $\delta$, again by Theorem 1.3, it is enough to check that $H^1(L) = H^2(B_2) = 0$. This follows from the hypothesis and from the Observations 2.2 and 2.3. \qed

**Lemma 2.6.** Let $X$ be a surface with geometric genus 0, let $B_1$ and $B_2$ be two base-point-free line bundles and let $L = B_1 \otimes B_2$ be nonspecial. Let $B_1$ and $B_2$ satisfy the conditions $H^1(B_1^{\otimes 2}) = H^1(B_2) = 0$ and $H^2(B_2 \otimes B_1^* ) = H^2(B_1^{\otimes 2} \otimes B_2^* ) = 0$. If $P$ is any effective line bundle on $X$ such that either $H^1(P) = 0$ or $P \cong \mathcal{O}$, then $H^1(M_L \otimes L \otimes P) = 0$. In particular, if $L$ is ample, $L$ is normally generated.

**Proof.** If we tensor (1.1) with $L \otimes P$ and take global sections, we obtain

$$
H^0(L) \otimes H^0(L \otimes P) \xrightarrow{\alpha} H^0(L^{\otimes 2} \otimes P) \rightarrow H^1(M_L \otimes L \otimes P) \rightarrow H^0(L) \otimes H^1(L \otimes P).
$$

From Observation 2.2 it follows that $H^1(L \otimes P)$ vanishes. Therefore the vanishing of $H^1(M_L \otimes L \otimes P)$ is equivalent to the surjectivity of the multiplication map $\alpha$. To prove the surjectivity of $\alpha$ we use the same trick as in the proof of the previous lemma. We write this commutative diagram:

$$
\begin{array}{ccc}
H^0(B_2) \otimes H^0(B_1) \otimes H^0(L \otimes P) & \rightarrow & H^0(L) \otimes H^0(L \otimes P) \\
\downarrow \gamma & & \downarrow \alpha \\
H^0(B_2) \otimes H^0(L \otimes B_1 \otimes P) & \rightarrow & H^0(L^{\otimes 2} \otimes P).
\end{array}
$$

It suffices then to prove that $\gamma$ and $\delta$ are surjective and by Theorem 1.3 it is enough to check that $H^1(B_2 \otimes P) = H^2(B_2 \otimes B_1^* \otimes P) = H^1(B_1^{\otimes 2} \otimes P) = H^2(B_1^{\otimes 2} \otimes B_2^* \otimes P) = 0$. These vanishings follow trivially from the hypothesis of the lemma and from Observations 2.2 and 2.3. \qed

(2.7) **Proof of Proposition 2.1.** Observation 2.2 implies that $H^1(L)$ vanishes. Thus from Lemma 2.6 it follows that $H^1(M_L \otimes L) = 0$. This implies that the vanishing of $H^1(M_L^{\otimes 2} \otimes L)$ is equivalent to the surjectivity of the multiplication map

$$
H^0(M_L \otimes L) \otimes H^0(L) \xrightarrow{\alpha} H^0(M_L \otimes L^{\otimes 2}).
$$

(2.7.1)

To prove the surjectivity of $\alpha$ we write this commutative diagram:

$$
\begin{array}{ccc}
H^0(B_2) \otimes H^0(B_1) \otimes H^0(M_L \otimes L) & \rightarrow & H^0(L) \otimes H^0(M_L \otimes L) \\
\downarrow & & \downarrow \alpha \\
H^0(B_2) \otimes H^0(M_L \otimes L \otimes B_1) & \rightarrow & H^0(M_L \otimes L^{\otimes 2}).
\end{array}
$$
By Theorem 1.3 it is enough to check that

\[ H^1(M_L \otimes B_2) = H^1(M_L \otimes B_1^{\otimes 2}) = 0 \]
\[ H^2(M_L \otimes B_2 \otimes B_1^*) = H^2(M_L \otimes B_1^{\otimes 2} \otimes B_2^*) = 0. \]

The first two vanishings follow from Lemma 2.5. The other two follow from sequence (1.1) and from Observations 2.2 and 2.3.

If \( L \) is ample, it follows from Lemma 1.2, (1.2.1) and Lemma 1.4 that \( L \) is normally presented.

The conditions on the vanishing of cohomology required in the statement of Proposition 2.1 are not so restrictive. For instance if we take \( B_1 \) and \( B_2 \) equal and ample, the conditions on the vanishing of \( H^2 \) are automatically satisfied for surfaces with geometric genus 0. If the surface we are considering is Enriques or elliptic ruled the vanishing of \( H^1 \) also occurs. The next corollary is an outcome of these observations.

**Corollary 2.8.** Let \( X \) be an Enriques surface, let \( \text{char}(k) = 0 \) and let \( B \) be an ample line bundle on \( X \) without base points. Then \( B^{\otimes 2} \) is normally presented.

**Proof.** Since \( K_X \equiv 0 \) and \( B \) is ample, \( \omega_X \otimes B \) is also ample and by Kodaira vanishing, \( H^1(B) = 0 \). Thus we can apply Proposition 2.1.

### 3. Ampleness, base-point-freeness and cohomology of line bundles on elliptic ruled surfaces

We have shown in Corollary 2.8 that \( B^{\otimes 2} \) is normally presented if \( B \) is an ample, base-point-free line bundle over an Enriques surface. The same result is true in the case of elliptic ruled surfaces. However in this case we can do much better. In fact we will be able to characterize (c.f. Theorem 4.2) those line bundles which are normally presented. From the statement of Lemma 2.6 it is clear that the knowledge of the vanishing of higher cohomology of line bundles on elliptic ruled surfaces will be crucial for this purpose. On the other hand once we know that the tensor product of two base-point-free line bundles is normally presented, knowing in addition which line bundles on an elliptic ruled surface are base-point-free will allow us to characterize those line bundles that are normally presented. In this light we will devote this section to recalling the vanishing of cohomology of line bundles and the characterization of base-point-free line bundles on elliptic ruled surfaces.

We introduce now some notation and recall some elementary facts about elliptic ruled surfaces. Proofs for the statements of this paragraph can be found in [H], §V.2. In this and the next section \( X \) will denote a smooth elliptic ruled surface, i.e. \( X = \mathbb{P}(\mathcal{E}) \), where \( \mathcal{E} \) is a vector bundle of rank 2 over a smooth elliptic curve \( C \). We will assume \( \mathcal{E} \) to be normalized, i.e., \( \mathcal{E} \) has global sections but twists of it by line bundles of negative degree do not. Let \( \pi \) denote the projection from \( X \) to \( C \). We set \( \mathcal{O}(\mathcal{E}) = \bigwedge^2 \mathcal{E} \) and \( \epsilon = -\text{deg} \mathcal{E} \geq -1 \). We fix a minimal section \( C_0 \) such that \( \mathcal{O}(C_0) = \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1) \). The group \( \text{Num}(X) \) is generated by \( C_0 \) and by the class of a fiber, which we will denote by \( f \). If \( a \) is a divisor on \( C \), \( a f \) will denote the pullback of \( a \) to \( X \) by the projection from \( X \) to \( C \). Sometimes, when \( \text{deg} a = 1 \), we will write, by an abuse of notation, \( f \) instead of \( a f \). The canonical divisor \( K_X \) is linearly equivalent to \(-2C_0 + af\), and hence numerically equivalent to \(-2C_0 - af\).
Proposition 3.1.
Let $L$ be a line bundle on $X$, numerically equivalent to $aC_0 + bf$.
If $e = -1$:

| $a$  | $b$       | $h^0(L)$ | $h^1(L)$ | $h^2(L)$ |
|------|-----------|-----------|-----------|-----------|
| $a \geq 0$ | $b > -a/2$  | $> 0$     | $0$       | $0$       |
|       | $b = -a/2$  | $?         | $?         | $0$       |
|       | $b < -a/2$  | $0$       | $> 0$     | $0$       |
| $a = -1$ | any $b$    | $0$       | $0$       | $0$       |
| $a \leq -2$ | $b > -a/2$  | $0$       | $> 0$     | $0$       |
|       | $b = -a/2$  | $0$       | $?         | $?         |
|       | $b < -a/2$  | $0$       | $0$       | $> 0$     |

If $e \geq 0$:

| $a$  | $b$       | $h^0(L)$ | $h^2(L)$ |
|------|-----------|-----------|-----------|
| $a \geq 0$ | $b > 0$   | $> 0$     | $0$       |
|       | $b = 0$   | $?         | $0$       |
|       | $b < 0$   | $0$       | $0$       |
| $a = -1$ | any $b$   | $0$       | $0$       |
| $a \leq -2$ | $b > -e$  | $0$       | $0$       |
|       | $b = -e$  | $0$       | $?         |
|       | $b < -e$  | $0$       | $> 0$     |

Proof. If $a < 0$ it is obvious that $h^0(L) = 0$. If $a \geq 0$, one obtains the statements for $h^0(L)$ and $h^1(L)$ by pushing down $L$ to $C$ and computing the cohomology there. In the case of $e \leq 0$, we use the fact that the symmetric powers of $E$ are semistable bundles ([Mi], Corollary 3.7 and §5). Then we use the fact that, if $F$ is a semistable bundle over an elliptic curve and $\deg(F) > 0$, then $h^0(F) > 0$ and $h^1(F) = 0$ and the fact that if $\deg(F) < 0$, then $h^0(F) = 0$ and $h^1(F) > 0$. In the case $e > 0$ the computation of cohomology on $C$ is elementary, since $E$ is decomposable. If $a = -1$, $\pi^*L = P^1 \times F = L = 0$, hence $H^1(L) = 0$. 

| $a$  | $b$       | $h^1(L)$ |
|------|-----------|-----------|
| $a \geq 0$ | $b > ae$  | $0$       |
|       | $b = ae$  | $?         |
|       | $b < ae$  | $> 0$     |
| $a = -1$ | any $b$   | $0$       |
| $a \leq -2$ | $b > e(a + 1)$  | $0$       |
|       | $b = e(a + 1)$  | $?         |
|       | $b < e(a + 1)$  | $> 0$     |
The other statements in the proposition follow by duality. □

The last proposition means that the vanishing of cohomology of line bundles on $X$ is an almost numerical condition, in the sense that in most cases we can decide whether or not a particular cohomology group vanishes by simply looking at the numerical class to which the line bundle belongs. As a matter of fact, in those numerical classes in which we cannot decide, there exist line bundles for which certain cohomology group vanishes and line bundles for which it does not. We will study in more detail this situation in the case $e = -1$, because we will need for the sequel to know exactly for which line bundles the cohomology vanishes. Concretely, this knowledge will allow us to use Proposition 2.1 and Proposition 5.4 in the proofs of Theorem 4.2 and Theorem 5.7 respectively. It will be used as well in [GP]. Also we will show the existence of a smooth elliptic curve numerically equivalent to $2C_0 - f$.

**Proposition 3.2.** Let $X$ be a ruled surface with invariant $e = -1$. Then

3.2.1. There exist only three effective line bundles in the numerical class of $2C_0 - f$. They are $O(2C_0 - (e + \eta_i)f)$, where the $\eta_i$s are the nontrivial degree 0 divisors corresponding to the three nonzero torsion points in $\text{Pic}^0(C)$. The unique element in $|2C_0 - (e + \eta_i)f|$ is a smooth elliptic curve $E_i$.

3.2.2. For each $n > 1$, there are only four effective line bundles numerically equivalent to $n(2C_0 - f)$. They are $O(2nC_0 - n(e + \eta_i)f)$ and $O(2nC_0 - n\epsilon f)$. The only smooth (elliptic) curves (and indeed the only irreducible curves) in these numerical classes are general members in $|4C_0 - 2\epsilon f|$.

The number of linearly independent global sections of these line bundles is summarized in the following table:

| $n \geq 0$ | 0 | 1 | 2 | 3 | $n$ |
|------------|---|---|---|---|-----|
| $h^0(O(2nC_0 - n\epsilon f))$ | 1 | 0 | 2 | 1 | $3\left\lfloor \frac{n}{2} \right\rfloor - n + 1$ |
| $h^0(O(2nC_0 - n(e + \eta_i)f))$ | 0 | 1 | 1 | 2 | $n - \left\lfloor \frac{n}{2} \right\rfloor$ |

**Proof.** For any $p \in C$ we consider the following exact sequence:

$$0 \to H^0(O(2C_0 - pf)) \to H^0(O(2C_0)) \overset{\varphi_{p}}{\to} H^0(O_{pf}(2C_0)) \to H^1(O(2C_0 - pf)) \to 0.$$ 

Pushing forward the morphism

$$H^0(O(2C_0)) \otimes O \to O(2C_0)$$

to $C$ we obtain

$$H^0(S^2(\mathcal{E})) \otimes O_C \overset{\varphi}{\to} S^2(\mathcal{E}) \to Q \to 0.$$ 

Note that the restriction of $\varphi$ to the fiber of $H^0(S^2(\mathcal{E}) \otimes O_C$ over $p$ is precisely $\varphi_p$. Thus the points $p$ for which $H^0(O(2C_0 - pf)) \neq 0$ are exactly the ones where the rank of $\varphi$ drops. Note that $h^0(O(2C_0)) = 3$ (push down the bundle to $C$, use the fact that $C$ is a smooth curve of genus 0).
same semistability considerations as in the sketch of the proof of Proposition 3.1 to obtain the vanishing of $H^1$ and then, use Riemann-Roch.) The rank of $S^2(E)$ is also 3, so if the rank of $\varphi$ never dropped, $\varphi$ would be an isomorphism, which is not true, because, since $e = -1$, the degree of $S^2(E)$ is 3. Therefore there exists $p \in C$ such that $H^0(O(2C_0 - pf)) \neq 0$. We fix such a point $p$ and some effective divisor $E$ inside $|2C_0 - pf|$. Since $2C_0 - f$ cannot be written as sum of two nonzero numerical classes both containing effective divisors, $E$ is irreducible and reduced. By adjunction, $p_a(E) = 1$ and since $E$ dominates $C$, $E$ is indeed a smooth elliptic curve.

We prove now by induction on $n$ the following statement: for each $n \geq 0$, there are finitely many effective line bundles numerically equivalent to $2nC_0 - nf$. The result is obviously true for $n = 0$. Take now $n > 0$. We fix a divisor $\mathcal{d}'$ of degree $n - 1$. What we want to prove is that the number of points $z \in C$ such that $H^0(O(2nC_0 - \mathcal{d}f)) \neq 0$ is finite, where $\mathcal{d} = \mathcal{d}' + z$. We may assume that $H^0(O((2n - 2)C_0 - (\mathcal{d}' - p + z)f) = 0$, since, by induction hypothesis, there are only finitely many points $z$ for which this does not happen. We tensor the sequence

$$0 \to \mathcal{O}(-2C_0 + pf) \to \mathcal{O} \to \mathcal{O}_E \to 0 \quad (3.2.3)$$

by $\mathcal{O}(2nC_0 - \mathcal{d}f)$ and take global sections. Since $H^0(O((2n - 2)C_0 - (\mathcal{d} - p)f)$ is 0 and the degree of the push forward of $O((2n - 2)C_0 - (\mathcal{d} - p)f)$ to $C$ is 0, it follows that $H^1(O((2n - 2)C_0 - (\mathcal{d}' - p + z)f) = 0$. Hence we obtain that $H^0(O(2nC_0 - \mathcal{d}f) = H^0(O_E(2nC_0 - \mathcal{d}f))$. The degree of $O_E(2nC_0 - \mathcal{d}f)$ is zero, therefore $z$ is such that $H^0(O(2nC_0 - \mathcal{d}f) \neq 0$ iff $O_E(2nC_0 - \mathcal{d}f) \cong O_E$, i.e., iff $O_E(2nC_0 - \mathcal{d}'f) \cong O_E(zf)$. There are only finitely many such points $z$, since otherwise, we will have that all the fibers of the $2 : 1$ morphism from $E$ onto $C$ induced by the degree 2 divisor which is obtained as the restriction of $2nC_0 - \mathcal{d}'f$ to $E$ are members of the same $g^1_2$.

The last statement implies that the length of $Q$ is finite, and equal to $\deg S^2(E) = 3$. We claim that $Q$ is in fact supported in three distinct points $p_1, p_2, p_3$. If not, there would exist a global section of $S^2(E)$ vanishing at some point $q$ to order greater or equal than 2. In particular, $O(2C_0 - 2qf)$ would be effective, which contradicts Proposition 3.1. Our aim now is to identify $p_1, p_2, p_3$. Let $E_i$ be the unique element of $|2C_0 - pf|$. We saw before that $E_i$ is a smooth elliptic curve. Pushing down to $C$ the exact sequence (3.2.3) we obtain

$$0 \to \mathcal{O}_C \to \pi_* \mathcal{O}_{E_i} \to R^1 \pi_* \mathcal{O}(-2C_0 + pf) \to 0.$$

Since $\pi|_{E_i}$ is unramified and $E_i$ is connected, it follows that $\pi_* \mathcal{O}_{E_i} = \mathcal{O}_C \oplus \mathcal{L}$ for some line bundle $\mathcal{L}$ such that $\mathcal{L}^{\otimes 2} = \mathcal{O}_C$, but $\mathcal{L} \neq \mathcal{O}_C$. Using relative duality and projection formula one obtains that $2(p_i - e) \sim 0$ but $p_i - e \not\sim 0$. This proves the first part of the proposition.

For the second part, remember that we have already proven the existence of only finitely many effective line bundles. Therefore, for any $p \in C$, we have the exact sequence

$$0 \to H^0(S^{2n} \mathcal{E} \otimes \mathcal{O}_C((-n + 1)p)) \otimes \mathcal{O}_C \to S^{2n} \mathcal{E} \otimes \mathcal{O}_C((-n + 1)p) \to Q' \to 0.$$

The length of $Q'$ is equal to the degree of $S^{2n} \mathcal{E} \otimes \mathcal{O}_C((-n + 1)p)$, which is $4m + 1$ if $n = 2m$ and $4m + 3$ if $n = 2m + 1$, but it is also equal to the sum of the lengths of $Q$ at the points $p_1, p_2, p_3$. If the $Q$'s were all of the same length, then $Q'$ would have the same length, which is absurd. Therefore $Q'$ is not effective, and we are done.
dimensions of the linear spaces of global sections of line bundles in the numerical class of $2nC_0 - nf$. Then, the rest of the statement in 3.2.2 and the numbers in the table follow from comparing the length of $Q'$ with the sum of the dimensions of the linear spaces generated by sections corresponding to reducible divisors numerically equivalent to $2nC_0 - nf$. □

(3.2.4) We will fix once and for all a smooth elliptic curve $E$ in the numerical class of $2C_0 - f$.

(3.2.5) For a different proof of the existence of a smooth elliptic curve in the numerical class of $2C_0 - f$ see [Ho2], corollary 2.2.

In the case $e \geq 0$ we are interested in finding sections of $\pi$ whose self-intersection is near to that of $C_0$ (they will play in the sequel a role similar to that of $E$):

**Proposition 3.3.** Let $X$ be an elliptic ruled surface with invariant $e \geq 0$. The general member of $|C_0 - \mathcal{E}|$ is a smooth elliptic curve and those are the only smooth curves in the numerical class of $C_0 + ef$.

**Proof.** If $\text{det}(\mathcal{E}) \neq \mathcal{O}$ the dimension of $|C_0 - \mathcal{E}|$ is $h^0(\mathcal{O} \oplus \mathcal{O}(-\mathcal{E})) - 1 = e$. Since the dimension of $| - \mathcal{E}'|s$ is $e - 1$ for any nontrivial divisor $-\mathcal{E}'$ of degree $e$ on $C$ it is clear that not all the elements in $|C_0 - \mathcal{E}|$ are unions of $C_0$ and $e$ fibers. On the other hand this is the only way in which an element of $|C_0 - \mathcal{E}|$ can be reducible (this is because for any divisor $d$ of degree $d < e$, the dimension of $|C_0 + d|$ is $d - 1$, which implies that any element of $|C_0 + d| \neq (\mathcal{O} \oplus \mathcal{O}(\mathcal{E}'')) \oplus \mathcal{O}(\mathcal{E}'')) = 1 = e - 1$

which means that all members of $|C_0 - \mathcal{E}'|s$ are reducible.

If $\text{det}(\mathcal{E}) = \mathcal{O}$, $\mathcal{E}$ is an extension of $\mathcal{O}$ by $\mathcal{O}$. Thus the member or members of $|C_0 - \mathcal{E}| = |C_0|$ are smooth elliptic curves and $|C_0 - \mathcal{E}| = \emptyset$ for any divisor $\mathcal{E}'$ on $C$ of degree 0 different from $\mathcal{E}$. □

(3.3.1) We will fix once and for all a smooth elliptic curve $E'$ in the numerical class of $C_0 + ef$.

**Proposition 3.4 ([H], V.2.20.b and V.2.21.b.).** Let $L$ be line bundle on $X$ in the numerical class of $aC_0 + bf$.

If $e = -1$, $L$ is ample iff $a > 0$ and $a > -\frac{1}{2}b$.

If $e \geq 0$, $L$ is ample iff $a > 0$ and $b - ae > 0$.

We state now a proposition describing numerical conditions which imply base-point-freeness. The proof follows basically the one given in [Ho1] and [Ho2]. In characteristic 0 the proposition can also be proven using Reider’s theorem ([R]).

**Proposition 3.5.** Let $L$ be a line bundle on $X$ in the numerical class of $aC_0 + bf$.

If $e = -1$, $a \geq 0$, $a + b \geq 2$ and $a + 2b \geq 2$, then $L$ is base-point-free.

If $e \geq 0$, $a \geq 0$ and $b - ae \geq 2$, then $L$ is base-point-free.

**Proof.** First we consider the case $e = -1$. In the first place we prove the proposition when $a = 0$ and $b \geq 2$, when $a = b = 1$, and when $a = 2$ and $b = 0$. We state now a proposition describing numerical conditions which imply base-point-freeness. The proof follows basically the one given in [Ho1] and [Ho2]. In characteristic 0 the proposition can also be proven using Reider’s theorem ([R]).
The first case follows easily from the fact that line bundles on elliptic curves whose degrees are greater or equal than 2 are base-point-free. For the other two cases we use the fact that there are only one or two minimal sections through a given point of $X$ (c.f. [Ho2]). On the other hand, for a given point $p \in C$, there are infinitely many effective reducible divisors in $|C_0 + pf|$, namely, those consisting of the union of a divisor linearly equivalent to $C_0 + \tau f$ and a divisor linearly equivalent to $(p - \tau)f$, where $\tau$ is a degree 0 divisor on $C$. Hence, the intersection of all those reducible divisors is empty. Analogously, for a given divisor $\nu$ of degree 0 there are infinitely many effective reducible divisors in $|2C_0 + \nu f|$, namely, those consisting of the union of a divisor linearly equivalent to $C_0 + (\nu + \tau)f$ and a divisor linearly equivalent to $C_0 + (\nu - \tau)f$, and the same argument goes through.

Now we use lemma 2.4. The base-point-free line bundle $B$ will be numerically equivalent to $bf$ ($b \geq 2$), $C_0 + f$ or $2C_0$ and $Y$ will be $E$ (defined in (3.2.4)) or $C_0$ (note that since deg($L \otimes O_Y(Y)$) $\geq 2$, it follows that $L \otimes O_Y(Y)$ is base-point-free). Iterating this process we obtain the result. The only place where we have to be careful in the application of lemma 2.4 iteratively is in making sure that the base-point-free line bundles we keep obtaining are nonspecial. This problem is taken care of by proposition 3.1.

The case $e \geq 0$ is easier. The line bundle $L$ is base-point-free if it is in the numerical class of $bf$, when $b \geq 2$. Then we get the result for any other bundle satisfying the conditions in the proposition by using the lemma 2.4. The curve $Y$ in lemma 2.4 will be $E'$ (defined in (3.3.1)). Again proposition 3.1 assures us that the line bundles we obtain are nonspecial. \(\Box\)

**Remark 3.5.1.** The numerical condition of Proposition 3.5 characterizes those equivalence classes consisting entirely of base-point-free line bundles.

**Proof.** A line bundle that satisfies the above numerical conditions is base-point-free by virtue of Proposition 3.5. To prove the other implication, consider a base-point-free line bundle $L$ in the numerical class of $aC_0 + bf$, which does not satisfy the above conditions. If $e(X) = -1$, the restriction of $L$ to the elliptic curve $E$ is a base-point-free line bundle. Hence, since its degree is equal to $a + 2b < 2$, it must be the trivial line bundle, which implies that $a + 2b = 0$. Then if follows from Proposition 3.2 that the general member of the numerical class is not base-point-free (in fact it is not even effective!). If $e(X) \geq 0$, for the same reason as above, the restriction of $L$ to $C_0$ is trivial. This is only possible if $L = O(n(C_0 - ef)) \Box$

() The proof of the previous remark suggests that there exist nontrivial base-point-free line bundles with self-intersection 0. That is indeed the case. For example, if $e = -1$, the divisors $2nC_0 + nef$ for any even number $n$ greater than 0 are base-point-free; if $X = C \times P^1$, the divisors $nC_0$ and if $e \geq 1$, the divisors $n(C_0 - ef)$ are base-point-free. Hence base-point-freeness cannot be characterized numerically. However, if we assume $L$ to be ample, then the numerical conditions in Proposition 3.5 do give a characterization of base-point-freeness:

**Remark 3.5.3.** Let $L$ be a line bundle on $X$ in the numerical class of $aC_0 + bf$.

- If $e = -1$, the line bundle $L$ is ample and base-point-free iff $a \geq 1$, $a + b \geq 2$ and $a + 2b \geq 2$.
- If $e \geq 0$, the line bundle $L$ is ample and base-point-free iff $a \geq 1$ and $b - ae \geq 2$.

**Proof.** If $L$ satisfies the numerical conditions in the statement of the remark, then by propositions 3.4 and 3.5 it is ample and base point free. Now assume that
L is ample and base-point-free. If \( e = -1 \), from proposition 3.4, it follows that \( a \geq 1 \). On the other hand, since \( L \) is base-point-free, its restriction to any curve in \( X \) is also base-point-free. Consider the curves \( C_0 \) and a smooth curve \( E \) in the numerical class of \( 2C_0 - f \). The restriction of \( L \) to each of them has degree \( a + b \) and \( a + 2b \) respectively. The fact that the restriction of \( L \) to \( C_0 \) is base-point-free implies that either \( a + b \geq 2 \) or the restriction of \( L \) to \( C_0 \) is trivial. The latter is impossible since \( L \) is ample. Analogously the fact that \( L \) is ample and that the restriction of \( L \) to \( E \) is base-point-free implies that \( a + 2b \geq 2 \). If \( e \geq 0 \), by proposition 3.4, \( a \geq 1 \). Since \( L \) is as well base-point-free, by restricting \( L \) to \( C_0 \) we obtain that \( b - ae = \deg (L \otimes \mathcal{O}_{C_0}) \geq 2 \).

4. Normal presentation on elliptic ruled surfaces

We recall that in this section \( X \) denotes a ruled surface over an elliptic curve and we continue to use the notation introduced at the beginning of section 3. We have just seen which line bundles on \( X \) are ample and which are base-point-free. The next question to ask would be: “which line bundles are very ample and which are normally generated?” This problem was solved by Y. Homma in [Ho1] and [Ho2], who proved that a line bundle \( L \) on \( X \) is normally generated iff it is very ample. She also characterizes those line bundles (see Figures 1 for the case \( e = -1 \)). Homma proves as well that in the case of a normally generated line bundle \( L \), the ideal corresponding to the embedding induced by \( L \) is generated by quadratic and cubic forms. Thus the next question is to identify those line bundles which are normally presented.

(4.1) Throughout the remaining part of this section we assume that \( \text{char}(k) \neq 2 \).

We will use Proposition 2.1 and the results from Section 3 to characterize the line bundles on \( X \) which are normally presented.

**Theorem 4.2.** The condition of normal presentation depends only on numerical equivalence. More precisely, let \( L \) be a line bundle on \( X \) numerically equivalent to \( aC_0 + bf \). If \( e = -1 \), \( L \) is normally presented iff \( a \geq 1 \), \( a + b \geq 4 \) and \( a + 2b \geq 4 \). If \( e \geq 0 \), \( L \) is normally presented iff \( a \geq 1 \) and \( b - ae \geq 4 \).

**Proof.** First we prove that if a line bundle \( L \) satisfies the numerical conditions in the statement, it is normally presented. To this end we will use Proposition 2.1. The idea is to write \( L \) as tensor product of two line bundles \( B_1 \) and \( B_2 \) satisfying the numerical conditions in Proposition 3.5, and such that \( H^2(B_1 \otimes B_2^*) = H^2(B_2 \otimes B_1^*) = 0 \). Let us exhibit the line bundles \( B_1 \) and \( B_2 \) in the different cases:

If \( e = -1 \), \( L \) can be written as tensor product of \( B_1 \) and \( B_2 \), where the couple \( B_1 \) and \( B_2 \) satisfies one of the following numerical conditions:

(4.2.1) \( B_1 \equiv C_0 + n f \) and \( B_2 \equiv 2 f \) or \( C_0 + f \), for some \( n \geq 1 \); (in this case, \( 1 \leq a \leq 2 \) and \( a + b \geq 4 \)).

(4.2.2) \( B_1 \equiv 2C_0 \) and \( B_2 \equiv 2C_0 + lf \) or \( C_0 + nf \), for some \( l \geq 0 \), and some \( n \geq 1 \); (in this case \( 3 \leq a \leq 4 \) and \( a + b \geq 4 \)).

(4.2.3) \( B_1 \equiv 2C_0 + m(2C_0 - f) \) and \( B_2 \equiv 2C_0 + lf \) or \( C_0 + nf \), for some \( m \geq 1 \), some \( l \geq 1 \) and some \( n \geq 1 \); (in this case, \( a \geq 5 \) and \( a + 2b \geq 4 \)).

(4.2.4) \( B_1 \equiv 2C_0 + m(2C_0 - f) \) and \( B_2 \equiv 2C_0 \), for some \( m \geq 1 \); (in this case, \( a \geq 5 \) and \( a + 2b = 4 \)).
If \( B_1 \) and \( B_2 \) satisfy (4.2.1), (4.2.2) or (4.2.3), Proposition 3.1 implies that \( H^2(B_1 \otimes B_2) = H^2(B_2 \otimes B_1^*) = 0 \). If \( B_1 \) and \( B_2 \) satisfy (4.2.4), \( H^2(B_1 \otimes B_2^*) \) vanishes but \( H^2(B_2 \otimes B_1^*) \) might not be zero. However from Proposition 3.2 it follows that, given \( L \equiv 4C_0 + m(2C_0 - f) \), \( m \geq 1 \), we can choose \( B_1 \equiv 2C_0 + m(2C_0 - f) \) and \( B_2 \equiv 2C_0 \) such that \( L = B_1 \otimes B_2 \) and \( H^2(2B_2 \otimes B_1^*) = 0 \), hence we are done in this case.

If \( e \geq 0 \), \( L \) can be written as tensor product of \( B_1 \) and \( B_2 \), where,

\[
\text{if } a \text{ is even: } B_1 \equiv (a/2)C_0 + \lfloor b/2 \rfloor f \text{ and } B_2 \equiv (a/2)C_0 + \lceil b/2 \rceil f \quad \text{and} \quad \text{if } a \text{ is odd: } B_1 \equiv \lfloor a/2 \rfloor C_0 + \lfloor (b-e)/2 \rfloor f \text{ and } B_2 \equiv \lceil a/2 \rceil C_0 + \lceil (b+e)/2 \rceil f.
\]

Proposition 3.4 implies that \( L \) is ample, Proposition 3.5 implies that \( B_1 \) and \( B_2 \) are base-point-free and Propositions 3.1 and 3.2 imply that \( H^1(B_1) = H^1(B_2) = H^2(B_1 \otimes B_2) = H^2(B_2 \otimes B_1^*) = 0 \), so from Proposition 2.1 it follows that \( L \) is normally presented.

Now we will suppose that \( L \) is normally presented but does not satisfy the numerical conditions in the statement and we will derive a contradiction. We can assume that \( a \geq 1 \). Otherwise \( L \) would not be ample. If \( e = -1 \), we can also assume that \( a + b = 3 \) and \( a + 2b = 3 \). If not the restriction of \( L \) to either the minimal section \( C_0 \) or the curve \( E \) defined in (3.2.4) would not be very ample. Analogously, if \( e \geq 0 \), we can assume that \( b - ae = 3 \). Otherwise, the restriction of \( L \) to \( C_0 \) would not be very ample.

To obtain the contradiction we follow the same strategy: we will see that the assumption of \( L \) being normally presented forces its restriction to both \( C_0 \) and \( E \) to be also normally presented, and we will derive from that the contradiction. If \( e \geq 0 \) let \( P \) denote the line bundle \( O(C_0) \). If \( e = -1 \) and \( L \equiv C_0 + 2f \) or \( 2C_0 + f \), let \( P \) denote \( O(C_0) \). If \( e = -1 \) and \( L \equiv 3C_0 + m(2C_0 - f) \), \( m \geq 0 \), let \( P \) denote \( O(E) \). Let \( p \) denote a smooth elliptic curve in \( |P| \). We claim that

\[
H^2(\bigwedge^2 M_L \otimes L \otimes P^*) = 0 . \tag{4.2.5}
\]

To prove (4.2.5) we will prove instead the fact that \( H^2(M_L \otimes L \otimes P^*) = 0 \). Consider the following exact sequence, which arises from exact sequence (1.1),

\[
H^1(M_L \otimes L \otimes P^*) \rightarrow H^2(M_L \otimes L \otimes P^*) \\
\rightarrow H^0(L) \otimes H^2(M_L \otimes L \otimes P^*) . \tag{4.2.6}
\]

Since \( H^1(L) = 0 \), the vanishing of \( H^1(M_L \otimes L \otimes P^*) \) is equivalent to the surjectivity of the map

\[
H^0(L) \otimes H^0(L \otimes P^*) \rightarrow H^0(L \otimes P^*) .
\]

The line bundle \( L \) is base-point-free by Proposition 3.5. Therefore, by Theorem 1.3, it is enough to check that \( H^1(L \otimes P^*) = H^2(P^*) = 0 \). The vanishings follow from our choice of \( P \) and from Proposition 3.1, except the vanishing of \( H^2(P^*) \) when \( P \simeq O(E) \), which follows from Proposition 3.2 and duality. Using (1.1) we obtain that \( H^2(M_L \otimes L \otimes P^*) \) will vanish if \( H^1(L \otimes P^*) \) and \( H^2(L \otimes P^*) \) vanish. These two vanishings follow from Proposition 3.1. Therefore by (4.2.6), \( H^2(M_L \otimes L \otimes P^*) = 0 \) and \( H^2(M_L \otimes L \otimes P^*) = 0 \).
Now since $L$ is assumed to be normally presented and $H^1(L) = 0$ we have, by Lemma 1.2, that $H^1(\bigwedge^2 M_L \otimes L) = 0$. Thus from (4.2.5) it follows that

$$H^1(\bigwedge^2 (M_L \otimes \mathcal{O}_p) \otimes L) = H^1(\bigwedge^2 M_L \otimes \mathcal{O}_p) = 0 .$$

(4.2.7)

Consider the following commutative diagram, (which holds for any base-point-free line bundle $L$ and for any nontrivial line bundle $P$ such that $H^1(L \otimes P^*) = 0$):

\[
\begin{array}{ccc}
& 0 & 0 \\
0 & \downarrow & \downarrow \\
0 \rightarrow H^0(L \otimes P^*) \otimes \mathcal{O}_p & \rightarrow & H^0(L \otimes P^*) \otimes \mathcal{O}_p \rightarrow & 0 \\
& \downarrow & \downarrow \\
0 \rightarrow M_L \otimes \mathcal{O}_p & \rightarrow & H^0(L) \otimes \mathcal{O}_p \rightarrow & L \otimes \mathcal{O}_p \rightarrow 0 \\
& \downarrow & \downarrow \\
0 \rightarrow M_L \otimes \mathcal{O}_p & \rightarrow & H^0(L \otimes \mathcal{O}_p) \otimes \mathcal{O}_p \rightarrow & L \otimes \mathcal{O}_p \rightarrow 0 \\
& \downarrow & \downarrow \\
& 0 & 0 & 0 \\
\end{array}
\]

From the left hand side vertical sequence we obtain the surjection

$$\bigwedge^2 (M_L \otimes \mathcal{O}_p) \otimes L \rightarrow \bigwedge^2 M_L \otimes \mathcal{O}_p \otimes L.$$ 

Since $p$ is a curve, it follows that

$$H^1(\bigwedge^2 M_L \otimes \mathcal{O}_p \otimes L) = 0 .$$

(4.2.8)

The line bundle $L \otimes \mathcal{O}_p$ on $p$ is normally generated because $p$ is an elliptic curve and $\deg (L \otimes \mathcal{O}_p) = 3$. Thus Lemma 1.2 and (4.2.8) imply that $L \otimes \mathcal{O}_p$ is normally presented, which is impossible since the complete linear series of $L \otimes \mathcal{O}_p$ embeds $p$ as a plane cubic! \ensuremath{\Box}

(4.2.9) It follows in particular from Proposition 4.2 that the normally generated line bundles on $X$, or more precisely, their numerical classes form a convex set in $\text{Num}(X)$, as shown in Figure 1, in which we describe $\text{Num}(X)$ when $e = -1$ (we do not draw the similar picture for $e \geq 0$).

Now we reformulate Theorem 4.2 and state some corollaries, which will help us to put our result in perspective:

**Theorem 4.3.** Let $X$ be an elliptic ruled surface. A line bundle $L$ on $X$ is normally presented iff it is ample and can be written as the tensor product of two line bundles $B_1$ and $B_2$ such that every line bundle numerically equivalent to any of them is base-point-free.

**Proof.** If $L$ is normally presented, it is obviously ample and satisfies the numerical conditions of Proposition 4.2. In the proof of that proposition we showed that a line bundle satisfying the mentioned numerical conditions can be written as the tensor product of two base-point free line bundles $B_1$ and $B_2$ satisfying the conditions of Proposition 4.2.

\[
0 \rightarrow \bigwedge^2 (M_L \otimes \mathcal{O}_p) \otimes L \rightarrow \bigwedge^2 M_L \otimes \mathcal{O}_p \otimes L.
\]

(4.2.7)
Remark 3.5.1. Hence these $B_1$ and $B_2$ are such that all the line bundles in their numerical classes are base-point-free.

On the other hand, assume that $L$ is ample and isomorphic to $B_1 \otimes B_2$ and that any line bundle numerically equivalent to either $B_1$ or $B_2$ is base-point-free. Let $B_i$ be in the numerical class of $a_iC_0 + b_if$ and $L$ in the numerical class of $aC_0 + bf$. If $e = -1$, by Remark 3.5.1, $a_i + b_i \geq 2$ and $a + 2b \geq 2$. Thus we obtain that $a + b \geq 4$ and $a + 2b \geq 4$. Since $L$ is ample, $a \geq 1$. Hence, from Proposition 4.2 it follows that $L$ is normally presented. If $e \geq 0$ one argues in a similar fashion. □

**Corollary 4.4.** Let $X$ be as above. Let $B_i$ be an ample and base point free line bundle on $X$ for all $1 \leq i \leq q$. If $q \geq 2$, then $B_1 \otimes \cdots \otimes B_q$ is normally presented and if $q < 2$, in general $B_1 \otimes \cdots \otimes B_q$ is not normally presented.

**Proof.** It follows from Remarks 3.5.1 and 3.5.3 that a line bundle numerically equivalent to any of the $B_i$ is base-point-free. From Remark 3.5.3 and Proposition 3.4 it follows that $L$ is ample. Thus, by Theorem 4.3, $L$ is normally presented. □

**FIGURE 1**

In the above figure cross means that all the members in the numerical class are base point free, dashed (or lined) square means that the corresponding coordinate ring is presented by quadratic forms, dashed (or lined) disc means normally presented, annulus means normally generated, blank disc means ample, gray or hashed disc means ample and base point free.

**Corollary 4.5.** Let $X$ be as above. Let $A_i$ be an ample line bundle on $X$ for all $1 \leq i \leq q$.
$1 \leq i \leq q$. If $q \geq 4$, then $A_1 \otimes \cdots \otimes A_q$ is normally presented and if $q < 4$, in general $A_1 \otimes \cdots \otimes A_q$ is not normally presented.

Proof. From Proposition 3.4 and Remark 3.5.3 it follows that the tensor product of two ample line bundles is ample and base-point-free. Hence the corollary follows from Corollary 4.4. \(\Box\)

**Corollary 4.6.** Let $X$ be an elliptic ruled surface. Let $A_i$ be an ample line bundle on $X$ for all $1 \leq i \leq q$.

If $e = -1$ and $q \geq 5$, then $\omega_X \otimes A_1 \otimes \cdots \otimes A_q$ is normally presented. If $e = -1$ and $q < 5$, in general $\omega_X \otimes A_1 \otimes \cdots \otimes A_q$ is not normally presented.

If $e = 0$ and $q \geq 4$, then $\omega_X \otimes A_1 \otimes \cdots \otimes A_q$ is normally presented. If $e = 0$ and $q < 4$, in general $\omega_X \otimes A_1 \otimes \cdots \otimes A_q$ is not normally presented.

If $e \geq 1$ and $q \geq 3$, then $\omega_X \otimes A_1 \otimes \cdots \otimes A_q$ is normally presented. If $e \geq 1$ and $q < 3$, in general $\omega_X \otimes A_1 \otimes \cdots \otimes A_q$ is not normally presented.

Proof. Let $A_i$ be in the numerical class of $a_i C_0 + b_i f$ and $\omega_X \otimes A_1 \otimes \cdots \otimes A_q$ in the numerical class of $a C_0 + bf$. If $e = -1$, $A_i$ is ample iff $a_i \geq 1$ and $a_i + 2b_i \geq 1$ (c.f. Proposition 3.4). In particular we also have that if $A_i$ is ample, then $a_i + b_i \geq 1$. Since $\omega_X$ is numerically equivalent to $-2C_0 + f$ it follows that

$$a \geq q - 2 \geq 3 > 1,$$

$$a + b \geq q - 1 \geq 4$$

and

$$a + 2b \geq q \geq 4.$$ 

Hence by Theorem 4.2, $\omega_X \otimes A_1 \otimes \cdots \otimes A_q$ is normally presented.

If $e = 0$, $A_i$ is ample iff $a_i \geq 1$ and $b_i \geq 1$ (c.f. Proposition 3.4). Since $\omega_X$ is numerically equivalent to $-2C_0$ it follows that $a \geq q - 2 > 1$ and $b - ae \geq q \geq 4$. Hence by Theorem 4.2, $\omega_X \otimes A_1 \otimes \cdots \otimes A_q$ is normally presented.

If $e \geq 0$, $A_i$ is ample iff $a \geq 1$ and $b_i - a_i e \geq 1$ (c.f. Proposition 3.4). Since $\omega_X$ is numerically equivalent to $-2C_0 - ef$ it follows that $a \geq q - 2 \geq 1$ and $b - ae \geq q + e \geq 4$. Then by Theorem 4.2, $\omega_X \otimes A_1 \otimes \cdots \otimes A_q$ is normally presented.

The line bundles $\omega_X \otimes \mathcal{O}(4C_0)$, if $e(X) = -1$; $\omega_X \otimes \mathcal{O}(3C_0 + 3f)$, if $e(X) = 0$; and $\omega_X \otimes \mathcal{O}(2C_0 + 2(e + 1)f)$, if $e(X) \geq 1$, are not normally presented (c.f. Theorem 4.2). Thus our bound is sharp. \(\Box\)

We want to compare our results to the results known for curves. Fujita and St. Donat (c.f. [F] and [S-D]) proved that, on a curve, any line bundle of degree bigger or equal than $2g + 2$ is normally presented. The results in this section as well as Theorem 2.1 are analogous in different ways to the result by Fujita and St. Donat. The approach taken up to now to generalize this result has been to look at adjoint linear series. In this line it was conjecture by Mukai that on any surface $X$, $\omega_X \otimes A_1 \otimes \cdots \otimes A_q$ should be normally presented for all $q \geq 5$ and $A_i$ ample line bundle. Corollary 4.6 shows that this conjecture holds if $X$ is an elliptic ruled surface and that it is sharp if the invariant $e(X) = -1$. One disadvantage of this generalization is that it only gives information about a small class of line bundles.

The possible ways of generalization indicated by Theorem 4.4:

(4.7) Let $X$ be a surface. If $L$ is the product of two ample line bundles $B_1$ and $B_2$, such that every line bundle $B$ numerically equivalent to either $B_1$ or $B_2$ is base-point-free, then $L$ is normally presented.
or by Proposition 2.1:

(4.8) Let \( X \) be a surface. If \( L \) is ample and the product of two base-point-free and nonspecial line bundles, then \( L \) is normally presented;

or maybe by some combination of the two, take in account a larger class of line bundles in general. In subsequent articles we prove that both (4.7) and (4.8) hold for K3 surfaces.

We remark that Theorem 4.3, which is stronger than Corollary 4.4, can also be seen as an analogue of Fujita and St. Donat’s theorem, since the latter can be rephrased as follows:

Let \( L \) be a line bundle on a curve. Every line bundle numerically equivalent to \( L \) is normally presented iff \( L \) is ample and the tensor product of two line bundles \( B_1 \) and \( B_2 \) such that every line bundle \( B \) numerically equivalent to \( B_1 \) or \( B_2 \) is base-point-free.

However, the veracity of Theorem 4.3 seems to depend on the particular properties of elliptic ruled surfaces and the corresponding statement is false on K3 surfaces.

We generalize Corollaries 4.4 and 4.5 to higher syzygies in a forthcoming article (c.f. [GP]), by proving that the product of \( p+1 \) or more ample and base-point-free line bundles satisfies the property \( N_p \).

5. Koszul algebras

In the previous section we determined which line bundles on an elliptic ruled surface are normally presented. A question to ask is whether the coordinate ring of the embedding induced by those line bundles is a Koszul ring, since it is well known that a variety with a Koszul homogeneous coordinate ring is projectively normal and defined by quadrics. The answer to this question is affirmative not only in the case of elliptic ruled surfaces, but in all other cases with which we have dealt throughout this work, since we are able to prove that the corresponding coordinate ring to a line bundle satisfying the conditions of Proposition 2.1 is Koszul.

We introduce now some notation and some basic definitions: given a line bundle \( L \) on a variety \( X \), we recall that \( R(L) = \bigoplus_{n=0}^{\infty} H^0(X, L^{\otimes n}) \).

**Definition 5.1.** Let \( R = k \oplus R_1 \oplus R_2 \oplus \ldots \) be a graded ring and \( k \) a field. \( R \) is a Koszul ring iff \( \text{Tor}_i^R(k, k) \) has pure degree \( i \) for all \( i \).

Now we will give a cohomological interpretation, due to Lazarsfeld, of the Koszul property for a coordinate ring of type \( R(L) \). Let \( L \) be a globally generated line bundle on a variety \( X \). We will denote \( M^{(0)},L := L \) and \( M^{(1)},L := M_L \otimes L = M_{M^{(0)},L} \otimes L \). If \( M^{(1)},L \) is globally generated, we denote \( M^{(2)},L := M_{M^{(1)},L} \otimes L \). We repeat the process and define inductively \( M^{(h)},L := M_{M^{(h-1)},L} \otimes L \), if \( M^{(h-1)},L \) is globally generated. Now we are ready to state the following

**Lemma 5.2 ([P], Lemma 1).** Let \( X \) be a projective variety over an algebraic closed field \( k \). Assume that \( L \) is a base-point-free line bundle on \( X \) such that the vector bundles \( M^{(h)},L \) are globally generated for every \( h \geq 0 \). If \( H^1(M^{(h)},L \otimes L^{\otimes s}) = 0 \) for every \( h \geq 0 \) and every \( s \geq 0 \) then \( R(L) \) is a Koszul \( k \)-algebra. Moreover, if \( H^1(L^{\otimes s}) = 0 \) for every \( s \geq 1 \) the converse is also true.

Now we will prove a general result analogous to Proposition 2.1 but before that, we state the following well known
Observation 5.3. Let $\mathcal{F}$ be a locally free sheaf over a scheme $X$ and $A$ an ample line bundle. If the multiplication map $H^0(\mathcal{F} \otimes A^\otimes n) \otimes H^0(A) \to H^0(\mathcal{F} \otimes A^\otimes n+1)$ surjects for all $n \geq 0$, then $\mathcal{F}$ is globally generated.

Theorem 5.4. Let $X$ be a surface with $p_g = 0$, let $B_1$ and $B_2$ be two base-point-free line bundles and let $L = B_1 \otimes B_2$ be ample. If

$$H^1(B_1) = H^1(B_2) = H^2(B_1 \otimes B_2^*) = H^2(B_2 \otimes B_1^*) = 0,$$

then the following properties are satisfied for all $h \geq 0$:

1. $M^{(h),L}$ is globally generated.
2. $H^1(M^{(h),L} \otimes B_1^{\otimes b_1} \otimes B_2^{\otimes b_2}) = 0$ for all $b_1, b_2 \geq 0$.
3. $H^0(M^{(h),L} \otimes B_j^*) = 0$ where $j = 1, 2$.
4. $H^1(M^{(h),L} \otimes B_i \otimes B_j^*) = 0$ where $i = 1, 2$ and $j = 2, 1$.
5. $H^0(M^{(h),L} \otimes B_i^{\otimes 2} \otimes B_j^*) = 0$ where $i = 1, 2$ and $j = 2, 1$.

In particular $H^2(M^{(h),L} \otimes L^\otimes s) = 0$ for all $h, s \geq 0$, and $R(L)$ is a Koszul $k$-algebra.

Proof. We prove the lemma by induction on $h$.

If $h = 0$, property 5.4.1 means that $L$ is globally generated, which is true by hypothesis. Properties 5.4.2 to 5.4.5 mean that $H^1(B_1^{\otimes (b_1+1)} \otimes B_2^{\otimes (b_2+1)}) = H^1(B_1^{\otimes \beta_1}) = 0$ where $b_1, b_2 \geq 0$, $\beta_i = 1, 2, 3$ and $i = 1, 2$. These vanishings occur by hypothesis and Observation 2.2.

Now consider $h > 0$ and assume that the result is true for all $0 \leq h' \leq h - 1$. Let $L'$ denote $B_1^{\otimes b_1} \otimes B_2^{\otimes b_2}$, $B_j^*$, $B_i \otimes B_j^*$ or $B_i^{\otimes 2} \otimes B_j^*$ accordingly. If we tensor

$$0 \to M^{(h-1),L} \to H^0(M^{(h-1),L}) \otimes \mathcal{O} \to M^{(h-1),L} \to 0,$$  

(c.f. (1.1))

by $L \otimes L'$ and take global sections, we obtain

$$H^0(M^{(h-1),L} \otimes L \otimes L') \cong H^0(M^{(h-1),L} \otimes L \otimes L') \to$$

$$H^1(M^{(h),L} \otimes L') \to H^0(M^{(h-1),L}) \otimes H^1(L \otimes L').$$  

(5.4.6)

Since, by Observation 2.2, $H^1(L \otimes L') = 0$, properties 5.4.2 to 5.4.5 are equivalent to the surjectivity of the multiplication map $\alpha$ in the different cases. First we prove property 5.4.5. Consider the following commutative diagram:

$$\begin{array}{ccc}
H^0(M^{(h-1),L} \otimes H^0(B_i) \otimes H^0(B_i)) & \xrightarrow{\varphi_1} & H^0(M^{(h-1),L} \otimes H^0(B_i) \otimes H^0(B_i)) \\
\downarrow & & \downarrow \\
H^0(M^{(h-1),L} \otimes B_i^{\otimes 2}) \otimes H^0(B_i) & \xrightarrow{\varphi_2} & H^0(M^{(h-1),L} \otimes B_i^{\otimes 3}) \\
\downarrow & & \downarrow \\
H^0(M^{(h-1),L} \otimes B_i^{\otimes 2}) \otimes H^0(B_i) & \xrightarrow{\varphi_3} & H^0(M^{(h-1),L} \otimes B_i^{\otimes 3}).
\end{array}$$

To show the surjectivity of $\alpha$ it suffices then to show the surjectivity of $\varphi_1$, $\varphi_2$ and $\varphi_3$. To prove that these three map are surjective we use Theorem 1.3. For example, to see that $\varphi_2$ is surjective is enough by Theorem 1.3 to show that
\(H^1(M^{(h-1), L} \otimes B_i^*) = 0\) and \(H^2(M^{(h-1), L} \otimes B_i^{-2}) = 0\). We argue analogously for the other two maps and thus conclude that in order to show the surjectivity of \(\alpha\) it is enough to check that

\[
\begin{align*}
(5.4.7) \quad H^1(M^{(h-1), L} \otimes B_i^*) &= H^1(M^{(h-1), L}) = 0 \\
H^1(M^{(h-1), L} \otimes B_i) &= 0 \quad \text{and} \\
(5.4.8) \quad H^2(M^{(h-1), L} \otimes B_i^{-2}) &= H^2(M^{(h-1), L} \otimes B_i^*) = 0 \\
H^2(M^{(h-1), L}) &= 0.
\end{align*}
\]

The vanishings in (5.4.7) follow from the assumption that properties 5.4.2 and 5.4.3 hold for \(h - 1\). Proving (5.4.8) is not hard. For instance, to obtain

\[
(5.4.8.1) \quad H^2(M^{(h-1), L} \otimes B_i^{-2}) = 0
\]

we consider the following sequence that we obtain from (1.1):

\[
\begin{align*}
H^1(M^{(h-2), L} \otimes B_j \otimes B_i^*) &\to H^2(M^{(h-1), L} \otimes B_i^{-2}) \to H^2(B_j \otimes B_i^*).
\end{align*}
\]

Hence it is clear that in order to show (5.4.8.1) it is enough to check that \(H^1(M^{(h-2), L} \otimes B_j \otimes B_i^*) = 0\) and that \(H^2(B_j \otimes B_i^*) = 0\). Arguing in a similar way for the remaining vanishings in (5.4.8), we conclude that in order to prove (5.4.8) it is enough to check

\[
(5.4.9) \quad H^1(M^{(h-2), L} \otimes B_j \otimes B_i^*) = H^1(M^{(h-2), L} \otimes B_j) = 0 \\
H^1(M^{(h-2), L} \otimes B_1 \otimes B_2) = 0 \quad \text{and} \\
(5.4.10) \quad H^2(B_j \otimes B_i^*) = H^2(B_j) = H^2(B_1 \otimes B_2) = 0.
\]

The vanishings in (5.4.9) follow from the assumption that properties 5.4.2 and 5.4.4 hold for \(h - 2\). Statement (5.4.10) follows by hypothesis and Observation 2.3.

The proof of properties 5.4.3 and 5.4.4 is analogous. In fact, notice that we have implicitly proven both when we showed the surjectivity of \(\varphi_1\) and \(\varphi_2\).

Now we prove property 5.4.2. The argument is similar to the one we have use to prove 5.4.5 and we will only sketch it here in little detail. To show the surjectivity of the map \(\alpha\)

\[
\begin{align*}
H^0(M^{(h-1), L}) \otimes H^0(B_1^{b_1+1} \otimes B_2^{b_2+1}) &\to H^0(M^{(h-1), L} \otimes B_1^{b_1+1} \otimes B_2^{b_2+2})
\end{align*}
\]

(c.f. (5.4.6)), one can write a similar diagram to the one in the proof of 5.4.5. Then it is enough to prove the surjectivity of the following map, which is a composition of multiplication maps (we assume \(b_2 \geq b_1\)):

\[
\begin{align*}
H^0(M^{(h-1), L}) \otimes [H^0(B_1) \otimes H^0(B_2)]^{b_1+1} &\otimes [H^0(B_2)]^{b_2-b_1} \\
\varphi &\to H^0(M^{(h-1), L} \otimes B_1^{b_1+1} \otimes B_2^{b_2+2})
\end{align*}
\]
We show the surjectivity of the composite map $\varphi$ by showing the surjectivity of each of its components. The first component is

$$
\begin{align*}
\text{H}^0(M^{(h-1),L}) \otimes \text{H}^0(B_1) \otimes \text{H}^0(B_2) \otimes [\text{H}^0(B_1) \otimes \text{H}^0(B_2)]^{\otimes b_1} \otimes [\text{H}^0(B_2)]^{\otimes b_2-b_1} \\
\downarrow \varphi_1 \\
\text{H}^0(M^{(h-1),L} \otimes B_1) \otimes \text{H}^0(B_2) \otimes [\text{H}^0(B_1) \otimes \text{H}^0(B_2)]^{\otimes b_1} \otimes [\text{H}^0(B_2)]^{\otimes b_2-b_1}
\end{align*}
$$

Hence by Theorem 1.3 it is enough to check the vanishings of the cohomology groups $\text{H}^1(M^{(h-1),L} \otimes B_1)$ and $\text{H}^2(M^{(h-1),L} \otimes B_1^{-2})$. For the surjectivity of the second component

$$
\begin{align*}
\text{H}^0(M^{(h-1),L} \otimes B_1) \otimes \text{H}^0(B_2) \otimes [\text{H}^0(B_1) \otimes \text{H}^0(B_2)]^{\otimes b_1} \otimes [\text{H}^0(B_2)]^{\otimes b_2-b_1} \\
\downarrow \varphi_2 \\
\text{H}^0(M^{(h-1),L} \otimes B_1 \otimes B_2) \otimes [\text{H}^0(B_1) \otimes \text{H}^0(B_2)]^{\otimes b_1} \otimes [\text{H}^0(B_2)]^{\otimes b_2-b_1}
\end{align*}
$$

again by Theorem 1.3 it is enough to check the vanishings of the groups $\text{H}^1(M^{(h-1),L} \otimes B_1 \otimes B_2)$ and $\text{H}^2(M^{(h-1),L} \otimes B_1 \otimes B_2^{-2})$. We use the same argument for the remaining components of $\varphi$ and conclude that in order to prove the surjectivity of $\varphi$, it suffices to check

(5.4.11) $\text{H}^1(M^{(h-1),L} \otimes B_1^{\otimes \beta_1} \otimes B_2^{\otimes \beta_2}) = 0$, for all $\beta_1$ and $\beta_2$ satisfying one of the following conditions:

- $1 \leq \beta_1 \leq b_1 - 1$ and $\beta_2 = \beta_1 + 1$
- $1 \leq \beta_1 \leq b_1 - 1$ and $\beta_2 = \beta_1 - 2$
- $\beta_1 = b_1 + 1$ and $b_1 \leq \beta_2 \leq b_2 - 1$

and

(5.4.12) $\text{H}^2(M^{(h-1),L} \otimes B_1^{\otimes \gamma_1} \otimes B_2^{\otimes \gamma_2}) = 0$, for all $\gamma_1$ and $\gamma_2$ satisfying one of the following conditions:

- $-2 \leq \gamma_1 \leq b_1 - 2$ and $\gamma_2 = \gamma_1 + 2$
- $1 \leq \gamma_1 \leq b_1 + 1$ and $\gamma_2 = \gamma_1 - 3$
- $\gamma_1 = b_1 + 1$ and $b_1 - 1 \leq \gamma_2 \leq b_2 - 2$.

The vanishings in (5.4.11), except the vanishings of $\text{H}^1(M^{(h-1),L} \otimes B_1^*)$ and $\text{H}^1(M^{(h-1),L} \otimes B_1 \otimes B_2^*)$, follow from the assumption that property 5.4.2 holds for $h - 1$. The vanishing of $\text{H}^1(M^{(h-1),L} \otimes B_1^*)$ follows from the assumption that property 5.4.3 holds for $h - 1$. The vanishing of $\text{H}^1(M^{(h-1),L} \otimes B_1 \otimes B_2^*)$ follows from the assumption that property 5.4.4 holds for $h - 1$. To prove the vanishings in (5.4.12) we consider the following sequence that we obtain from (1.1):

$$
\begin{align*}
\text{H}^1(M^{(h-2),L} \otimes B_1^{\otimes (\gamma_1+1)} \otimes B_2^{\otimes (\gamma_2+1)}) & \rightarrow \\
\text{H}^2(M^{(h-1),L} \otimes B_1^{\otimes \gamma_1} \otimes B_2^{\otimes \gamma_2}) & \rightarrow \text{H}^2(B_1^{\otimes (\gamma_1+1)} \otimes B_2^{\otimes (\gamma_2+1)}).
\end{align*}
$$

Hence it is enough to show that these cohomology groups vanish:

(5.4.13) $\text{H}^1(M^{(h-2),L} \otimes B_1^{\otimes (\gamma_1+1)} \otimes B_2^{\otimes (\gamma_2+1)}) = 0$ and

(5.4.14) $\text{H}^2(B_1^{\otimes (\gamma_1+1)} \otimes B_2^{\otimes (\gamma_2+1)}) = 0$. 
for all $\gamma_1$ and $\gamma_2$ satisfying one of the conditions from (5.4.7.1) to (5.4.7.3).

Statement (5.4.13), except for the vanishings of $H^1(M^{(h-2)}; L \otimes B_2 \otimes B_1^{-1})$ and $H^1(M^{(h-2)}; L \otimes B_1^{\otimes 2} \otimes B_2^{-1})$, follow from the assumption that property 5.4.2 holds for $h - 2$. The vanishing of $H^1(M^{(h-2)}; L \otimes B_2 \otimes B_1^{-1})$ follows from the assumption that property 5.4.4 holds for $h - 2$. The vanishing of $H^1(M^{(h-2)}; L \otimes B_1^{\otimes 2} \otimes B_2^{-1})$ follows from the assumption that property 5.4.5 holds for $h - 2$. All the vanishings in (5.4.14) follow by hypothesis and Observation 2.3.

Finally we prove property 5.4.1. By Observation 5.3, it is enough to show that
\[
H^0(M^{(h)}; L \otimes L^{\otimes n}) \otimes H^0(L) \rightarrow H^0(M^{(h)}; L \otimes L^{\otimes n + 1})
\]
surjects for all $n \geq 0$. For that it suffices to prove the surjectivity of the map
\[
H^0(M^{(h)}; L \otimes L^{\otimes n}) \otimes H^0(B_1) \otimes H^0(B_2) \rightarrow H^0(M^{(h)}; L \otimes L^{\otimes n + 1})
\]
for all $n \geq 0$. Using Theorem 1.3, it is enough to check
\[
H^1(M^{(h)}; L \otimes B_1^{\otimes n - 1} \otimes B_2^{\otimes n}) = 0
\]
(5.4.16) 
and
\[
H^1(M^{(h)}; L \otimes B_1^{\otimes n + 1} \otimes B_2^{\otimes n - 1}) = 0
\]
(5.4.17) 

The vanishings in (5.4.16) follow from the fact, which we have just proved, that properties 5.4.2 to 5.4.4 hold for $h$. To prove (5.4.17), again by (1.1), it is enough to show that
\[
H^1(M^{(h-1)}; L \otimes B_1^{\otimes n - 1} \otimes B_2^{\otimes n + 1}) = 0
\]
(5.4.18) 
and
\[
H^1(M^{(h-1)}; L \otimes B_1^{\otimes n + 2} \otimes B_2^{\otimes n - 1}) = 0
\]
(5.4.19) 

The vanishings in (5.4.18) follow from the assumption that properties 5.4.2, 5.4.4 and 5.4.5 hold for $h - 1$ and (5.4.19) follows by hypothesis and Observation 2.3.

In particular, it follows from property 5.4.2 that $H^1(M^{(h)}; L \otimes L^{\otimes s}) = 0$ for all $h, s \geq 0$. Thus, as consequence of Lemma 5.2, the coordinate ring $R(L)$ is a Koszul $k$-algebra. $\square$

(5.5) Note that if $h = 1$ and $n = 1$, the multiplication map (5.4.15) is actually the same as (2.7.1). Moreover, the fact that $H^1(M_L \otimes L) = 0$ is a special case of 5.4.2. Hence on our way to prove Theorem 5.4, we have reproved Proposition 2.1 and therefore Theorem 5.4 may be seen as a generalization of the cited proposition.

Even though the above theorem is stated for surfaces with $p_g = 0$, the same proof (or indeed a simpler one) works for curves. Thus we obtain the following

**Theorem 5.5.** Let $C$ be a curve, let $B_1$ and let $B_2$ be two nontrivial base-point-free line bundles on $C$. If $H^1(B_1) = H^1(B_2) = 0$, then $R(L)$ is a Koszul.

**Proof.** The only properties of surfaces with $p_g = 0$ that we use in the proof of Theorem 5.4 are the fact that $H^2(C, \mathcal{L}) = 0$, and Observations 2.2 and 2.3.
Observation 2.2 is obviously still true if $X$ is a curve. Observation 2.3 and the fact that $H^2(O_X) = H^2(B_1^* \otimes B_2) = H^2(B_2 \otimes B_1) = 0$ and are trivially true for curves, hence the theorem follows from the proof of Theorem 5.4. □

Theorem 5.5 yields as a corollary the following result by David Butler (see also [Po]):

**Corollary 5.6 ([B], Theorem 3).** Let $C$ be a curve and let $L$ be a line bundle on $C$. If $\text{deg}(L) \geq 2g + 2$, then $R(L)$ is Koszul.

*Proof.* If $\text{deg}(L) \geq 2g + 2$, then $L$ can be written as tensor product of two general line bundles of degree $g + 1$. Such line bundles are base-point-free and nonspecial. □

Theorem 5.4 yields these three results:

**Corollary 5.7.** Let $X$ be an Enriques surface over an algebraic closed field of characteristic 0 and let $B$ be an ample line bundle on $X$ without base points. Then $R(B^\otimes 2)$ is Koszul.

*Proof.* The proof is analogous to the proof of Corollary 2.8. □

**Theorem 5.8.** Let $X$ be an elliptic ruled surface. Let $L$ be a line bundle on $X$ numerically equivalent to $aC_0 + bf$. If $e = -1$ and $a \geq 1$, $a + b \geq 4$ and $a + 2b \geq 4$, then $R(L)$ is Koszul. If $e \geq 0$ and $a \geq 1$ and $b - ae \geq 4$, then $R(L)$ is Koszul.

*Proof.* The proof is analogous to the proof of the first part of Theorem 4.2. □

(5.9) It is well known that for an ample line bundle $L$, the fact of $R(L)$ being Koszul implies formally the property of being normally presented (c.f. [BF], 1.16). Therefore Theorem 5.5 gives a different proof of Fujita and St. Donat’s theorem, Corollary 5.7 provides another proof of Corollary 2.8 and Theorem 5.8 provides another proof of the first part of Theorem 4.2. These proofs are less elementary, but in the case of Theorem 4.2, we have the advantage of working also in characteristic 2.

If we assume that $\text{char}(k) \neq 2$, it follows from Proposition 4.2 that the property of $R(L)$ being a Koszul algebra is characterized by the numerical conditions in the statement of Theorem 5.8. We can restate this as we did in the case of Theorem 4.2:

**Theorem 5.9.** Let $X$ be as above and let $L$ be a line bundle on $X$. Assume that $\text{char}(k) \neq 2$. Then $R(L)$ is a Koszul algebra iff it is ample and can be written as the tensor product of two line bundles $B_1$ and $B_2$ such that every line bundle numerically equivalent to any of them is base-point-free.

(5.9.1) Having in account that, on elliptic ruled surfaces, normal presentation only depends on numerical equivalence (c.f. Theorem 4.2), Theorem 5.9 can be considered as analogous to Theorem 5.5. Indeed Theorem 5.5 can be rephrased as follows:

If a line bundle $L$ on $C$ is normally presented and so is every line bundle numerically equivalent to $L$, then $R(L)$ is Koszul.
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