GENERAL REARRANGEMENT LEMMA FOR HEAT TRACE ASYMPTOTIC ON NONCOMMUTATIVE TORI

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ABSTRACT. We study a technical problem arising from the spectral geometry of noncommutative tori: the small time heat trace asymptotic associated to a general second order elliptic operator. We extend the rearrangement operators in the conformal case to the general setting using hypergeometric integrals over Grassmannians. The main result is the explicit formula of the second heat coefficient in terms of the coefficients. When specializing to examples in conformal case, we not only recover results in previous works but also obtain some extra functional relations whose validation provides experimental support to the main results. At last, we verify the relations based on combinatorial properties derived from the hypergeometric features.

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1. Introduction

We provide in this paper an upgrade to the rearrangement lemma in the computation of the small time heat trace expansion on noncommutative tori via pseudo-differential calculus. It allows us to incorporate general second order elliptic operators $P$ of the form in Eq. (2.6) and in principle, to obtain closed formulas of the functional densities of the heat coefficients (see Eq. (2.9)) written in terms of the derivatives of the coefficients of $P$. The primary application of the technical question is the spectral geometry on noncommutative tori and toric noncommutative manifolds (cf. [CL01], [BLvS13]), in which basic notions in Riemannian geometry, such as metric and curvature, are investigated in a purely operator-theoretic framework. Following Connes’s spectral paradigm, the metrics are implemented as geometric operators $P$, playing the role of the Laplacian or the squared Dirac operators. The coefficients of heat trace expansion of $P$ that we would like to understand, are the associated local invariants. In particular, by analogy with results on Riemannian manifolds, the second heat coefficient appeared in the main results encodes the full information of the scalar curvature. The conformal aspects of program has been carried out in great detail on noncommutative two tori [LM16, CM14, FK13] and toric noncommutative manifolds [Liu18c, Liu17]. More references can be found in recent surveys [LM19, FK19].

In the commutative world, curvature at the infinitesimal level appears as commutators of covariant derivatives of the metric connection. Such noncommutativity globally influences the shape of the underlying manifold. The new notion of curvature for noncommutative spaces contains an additional noncommutativity arising from the metric itself: the metric coordinates do not commute with their derivatives. The so-called rearrangement lemma defines the building blocks of such contribution. In more detail, on noncommutative tori $C^\infty(T^n_\theta)$, metric tensor $g = (g_{ij})$ becomes the coefficient matrices $A = (k_{ij}) \in GL_m(C^\infty(T^n_\theta))$ in Definition 2.1 appeared in the leading term of the underlying geometric operator $P$. Even though we can assume that the entries $k_{ij}$ mutually commute, there is no control on the commutativity when their derivatives are involved. As a result, except the Leibniz property, many other basic formulas in calculus require upgrades. Let us look at a toy example: for a derivation $\nabla$ and $k \in C^\infty(T^n_\theta)$ invertible, we can use the Leibniz property to expand:

$$\begin{align*}
\nabla^2 k^3 &= k^2(\nabla^2 k) + (\nabla k)k^2 + k(\nabla^2 k)k + 2[ k(\nabla k)(\nabla k) + (\nabla k)k(\nabla k) + (\nabla k)(\nabla k)k ] \\
&= k^2(1 + y + y^2)(\nabla^2 k) + 2k(1 + y^{(1)} + y^{(1)}y^{(2)})(\nabla k \otimes \nabla k).
\end{align*}
$$

\[(1.1)\]
Compared to $\nabla^2 k^3 = 2k^2(\nabla k) + 6k(\nabla k)(\nabla k)$ for $k$ and $\nabla k$ commute, we see that general local differential expressions $L(k, \nabla k, \nabla^2 k, \ldots)$ derived from the operator valued coordinate $k$ involve new coefficients: rearrangement operators (see (3.1) for definitions), such as $1 + y + y^2$ and $1 + y^{(1)} + y^{(1)} y^{(2)}$ appeared in Eq. (1.1) above, where $y = k^{-1}(e)$. A more interesting example is the Duhamel's formula for the derivative of the exponential of a self-adjoint $h \in C^\infty(T^\oplus_{\theta})$:

$$\nabla(e^h) = \int_0^1 e^{(1-s)h}(\nabla h)e^{sh}ds = e^h \exp(-\text{ad}_h)^{-1}(\nabla(h)), \quad \text{ad}_h = [h, \cdot].$$

What is universal behind the rearrangement operators is the spectral functions like $1 + y + y^2$, $1 + y_1 + y_1 y_2$ and $(e^x - 1)/x$. It has been observed by Connes and Moscovici that the one variable functions appeared in their work [CM14] show great resemblance to those in topology generating characteristic classes. In the conclusion of [CF19], it is pointed out that the functions obtained in the paper seems familiar in transcendence theory. The author added hypergeometric features [Liu18a, Liu18b] into the rearrangement lemma that concerns what kind of functions shall arise in the pseudo-differential approach to the heat coefficients. It turns out that the arguments in [Liu18a, Liu18b] can be adapted to handle the general situation involving $m \times m$ operator-valued coordinates where $m$ denotes the dimension. The key feature is that the spectral functions are given by hypergeometric integrals. For non-experts of special functions, “hypergeometric” refers to the property which sounds interesting from algebraic geometry point of view: namely, the integrals (Eq. (3.18)) are constructed out of the combinatorial data (cf. Eqs. (3.16) and (3.17)) of standard simplexes $\Delta^n$. In fact, for the diagonal case $F_{a}(z), \ldots, z_N$ (Eq. (3.26)) and the conformal case $H_{a}(z; m, j)$ (Eq. (3.29)), the functions belong to a general class of hypergeometric functions over Grassmannians [AK11, §3].

Back to the technical question raised at the beginning, we briefly outline the algorithm for computing heat coefficients in §4.1—§4.3. The remaining sections §4.4—§4.5 are devoted to the main results: the functional density of the $V_2$-term (as in Eq. (2.7)). Several versions of the explicit local expressions of $\nu_2(P)$ are recorded. The first one (Theorem 4.5) is presented in a compact form which has the merit for communicating the formulas. To get a precise and detailed understanding of the notations, one should look at Theorems 4.7 and 4.8 in which rearrangement operators $F_{\alpha}(A)$ are fully expanded into components. Despite the complexity of the formulas, the result can be directly applied for all potential applications. At last, we examine, in §4.5 a situation in which the coefficient matrix $A$ is diagonal. By restricting to “eigenvalues” of the general form, the simplified formulas make the underlying geometric features more transparent, which is the next step of our exploration of curvature beyond conformal geometry.

In the last two sections, we focus on the Laplacian $\Delta_{\varphi}$ representing the simplified model of conformal geometry studied in [Liu18a]. We are able to derive some functional relations (cf. Corollary 5.6) by computing the associated $V_2$-term in two ways based on the general result in §4.5. The functions on two sides of the equations are quite different at the first glance. The complete cancellation with each other yields strong support to the validation of our calculation in §4. A notable feature of our computation in §6 is that, instead of invoking the lengthy algebraic expressions, we present and manipulate the spectral functions through a basis of functions consisting of the hypergeometric family and additional $G^{(1)}_{\text{pow}}$ and $G^{(1,1)}_{\text{pow}}$ (see Lemma 5.3). The cancellation can be seen through the two features among the basis discussed
in [Liu18b]: differential and recursive relations and the action of the variational operators (especially the cyclic permutations Eq. (6.1)).

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2. Preliminaries

2.1. Noncommutative m-tori $C^\infty(\mathbb{T}_\theta^m)$. Let $\theta = (\theta_{ij}) \in M_{n \times n}(\mathbb{R})$ be a skew-symmetric matrix. The smooth noncommutative $m$-torus $C^\infty(\mathbb{T}_\theta^m) = (C^\infty(\mathbb{T}^m), \times_{\theta})$ (viewed as smooth coordinate functions on $\mathbb{T}_\theta^m$) is identical to $C^\infty(\mathbb{T}^m)$ as a topological vector space with a deformed multiplication. Similar to the existence of Fourier expansion for functions on tori, the generators of $C^\infty(\mathbb{T}_\theta^m)$ consists of $m$ unitary elements: $U_s U_s^* = U_s^* U_s = 1$, $s = 1, \ldots, m$ and for $\bar{l} = (l_1, \ldots, l_m) \in \mathbb{Z}^m$, put

\[
    a_{\bar{l}} U^\bar{l} := a_{l_1} U_1^{l_1} \cdots U_m^{l_m}, \quad a_{\bar{l}} \in \mathbb{C},
\]

then

\[
    C^\infty(\mathbb{T}_\theta^m) = \left\{ \sum_{\bar{l} \in \mathbb{Z}^m} a_{\bar{l}} U^\bar{l} \mid a_{\bar{l}} \in S(\mathbb{Z}) \text{ is a Schwartz function in } \bar{l} \right\}.
\]

The deformed multiplication is given in terms of the generators:

\[
    U_s U_l = e^{2\pi i \bar{\theta}_{sl}}, \quad U_s U_{s'} = 1, \quad 1 \leq s, l \leq m.
\]

There exist a canonical trace $\varphi_0 : C^\infty(\mathbb{T}_\theta^m) \to \mathbb{C}$ taking the constant term of an element:

\[
    \varphi_0 \left( \sum_{\bar{l}} a_{\bar{l}} U^\bar{l} \right) = a_0.
\]

We denote by $\mathcal{H}$ the corresponding GNS representation obtained by completing $C^\infty(\mathbb{T}_\theta^m)$ with respect to the inner product

\[
    \langle f, \tilde{f} \rangle := \varphi_0(\tilde{f}^* f), \quad \forall f, \tilde{f} \in C^\infty(\mathbb{T}_\theta^m).
\]

The noncommutative $m$-torus $C(\mathbb{T}_\theta^m)$ (playing the role of all continuous function on $\mathbb{T}_\theta^m$) is the $C^*$-algebra $C(\mathbb{T}_\theta^m) = C^\infty(\mathbb{T}_\theta^m)$ in which the completion is taken with respect to the operator norm from the following representation: $\rho : C^\infty(\mathbb{T}_\theta^m) \to B(L^2(\mathbb{T}_\theta^m))$, with $\mathbb{T}_\theta^m \cong \mathbb{R}^m / (2\pi \mathbb{Z})^m$.

\[
    (\rho(U_s) f)(x) := e^{i x_1 \tilde{\theta}_s}, \quad \forall f \in L^2(\mathbb{T}_\theta^m),
\]

where $\tilde{\theta}_s$ is the $s$-column of $\theta$. 
The differential calculus (moreover pseudo-differential calculus) is built upon a $C^*$-dynamical system $(C(T^m_{\theta}), \mathbb{R}^m, \sigma)$. The action $\sigma$ is periodic, namely, a lift of a $T^m$ action attached to the $\mathbb{Z}^m$ grading. In other words, $U^i$ in Eq. (2.1) are the eigenvectors:

$$\sigma_r(U^i) = e^{ir\theta^i} U^i, \; r \in \mathbb{R}^m, \; i \in \mathbb{Z}^m.$$  

(2.4)

The representation $\rho$ in Eq. (2.3) and the translation action of $\mathbb{R}^m$

$$V_r(f)(x) := f(x + r), \; r, x \in \mathbb{R}^m.$$  

form a covariant representation of the $C^*$-dynamical system: $\rho(\sigma_r(T)) = V_r \rho(T) V_r$. The smooth noncommutative torus $C^\infty(T^m_{\theta})$ consists of exactly those smooth elements in the $C^*$-dynamical system, that is, all $a \in C(T^m_{\theta})$ such that $r \rightarrow \sigma_r(a)$ is a smooth function in $r \in \mathbb{R}^m$.

If we identify $\mathbb{R}^m \cong \text{Lie}(\mathbb{R}^m)$, the Lie derivatives of the standard basis give rise to the basic derivations $\{\delta_i\}_{l=1}^m$ acting as $-i\partial_i$ on $C^\infty(T^m)$ regarding to the coordinate $(x_1, \ldots, x_m)$. More precisely, we have on the generators:

$$\delta_i(U_j) = 1_{ij} U_j, \; 1 \leq i, j \leq m.$$  

(2.5)

The basic derivations are the generators of the algebra of differential operators. In particular, a second order differential operator is always of the form:

$$P = \sum_{1 \leq s, l \leq m} k_{si} \delta_s \delta_l + \sum_{s=1}^m r_s \delta_s + p_0,$$  

(2.6)

where the coefficients $k_{si}, r_s, p_0 \in C^\infty(T^m_{\theta})$. As in the commutative case, the ellipticity concerns only the coefficients of the leading term:

**Definition 2.1.** The differential operator $P : C^\infty(T^m_{\theta}) \rightarrow C^\infty(T^m_{\theta})$ given in Eq. (2.6) is called elliptic if the coefficient matrix $A = (k_{ij})_{1 \leq i, j \leq m}$ admit a self-adjoint log. Precisely, we require that $A = \exp(\log A)$ where the matrix log $A = (h_{ij})$ is symmetric with self-adjoint and mutually commute entries:

$$h_{ij} = h_{ji} = h^*_{ij} = h^*_{ji}, \; [h_{ij}, h_{st}] = 0, \; 1 \leq i, j, s, t \leq m.$$  

As a consequence, $A$ is positive invertible with mutually commute entries.

2.2. **Spectral geometry.** If $P$ fulfills ellipticity above, Connes’s pseudo-differential calculus implies that there exists a order one pseudo-differential operator $Q$ such that $P - Q^*Q$ is of order one. Arguments in Gilkey’s book [GT85] can be applied to show that $P$ has discrete spectrum contained in a conic region of $\mathbb{C}$. In our examples, the spectrum of $P$ is contained in $[c, \infty)$ for some $c \in \mathbb{R}$. The heat operator can be defined via holomorphic functional calculus:

$$e^{-tP} = \frac{1}{2\pi i} \int_C e^{-t\lambda}(P - \lambda)^{-1} d\lambda,$$  

(2.7)

where $C$ is a suitable contour winding around the spectrum of $P$.

Our primary interest is the spectral geometry of $C^\infty(T^m_{\theta})$ in which $P$ plays the role of a geometric differential operator. In the conformal case on $C^\infty(T^m_{\theta})$ [LM16, CM14], $P$ is modeled on the Dolbeault Laplacian while on toric noncommutative manifolds $C^\infty(M)_{\theta}$ [Liu17].
comes from the squared Dirac operator. On $\mathbb{T}_\theta^m$, Even for noncommutative tori, the notion of general metrics and the associated construction of $P$ is still widely open. A recent proposal of constructing Laplace-Beltrami operators can be found in [HP19]. Among the mentioned examples, a common feature inherited from Riemann geometry is the property that the metric tensor $g = (g_{ij})$ is implemented as the coefficient matrix $A$ of the leading term while for lower order terms, $r_s$ and $p_0$ consist of the first and second derivatives of entries of $A$ respectively.

Once the metric $P$ is chosen, the corresponding local invariants can be extracted from the small time asymptotic of the heat trace functional $a \to \text{Tr}(ae^{-tP})$:

\[
\text{Tr}(ae^{-tP}) \sim t \sum_{j=0}^{\infty} t^{\frac{j-m}{2}} V_j(a, P), \ a \in C^\infty(\mathbb{T}_\theta^m),
\]

where $\text{Tr}$ is the operator trace with respect to the Hilbert space $\mathcal{H}$ defined in Eq. (2.2). Each coefficient is absolutely continuous with respect to the canonical trace $\varphi_0$ with Radon-Nikodym derivative $\psi_j(P) \in C^\infty(\mathbb{T}_\theta^m)$ (also referred as functional densities in the paper):

\[
V_j(a, P) = \varphi_0(a \psi_j(P)), \ \forall a \in C^\infty(\mathbb{T}_\theta^m).
\]

If $P$ is the scalar Laplacian $\Delta$ on a closed Riemann manifold $(M, g)$, the invariants $\psi_j(\Delta) \in C^\infty(M)$ are known as Minakshisundaram-Pleijel coefficients which can written, in principle, as polynomial functions in the derivatives of metric tensor. In particular, the first non-trivial one is $\psi_2(\Delta) = S_{g}/6$ proportional to the scalar curvature function. It gives a geometric interpretation to our main results in §4.4 as a model of the scalar curvature from the spectral geometry perspective. The spectral paradigm has also been implemented in great detail, known as spectral action principle, in the noncommutative geometry approach to standard model, see [CM08, §11] for further references.

The pseudo-differential calculus is able to, not only establish the existence of the expansion, but also provide a efficient algorithm for the computation of $\psi_j(P)$. We shall see in later sections that they can be written as finite sums of differential expressions in the coefficients of $P$: $k_{j1}, r_s$ and $p_0$. Nevertheless, the length of of $\psi_j(P)$ grows substantially as $j$ going up, cf. [CF19] for an impression of the complexity of $V_4$-term even in the conformal case. Sum up, a challenging task in the spectral geometry is to explore universal structures behind the intimidating differential expressions, so that further applications, such as related variational problems, can be carried out.

3. Hypergeometric Functions in the Rearrangement Lemma

3.1. Continuous functional calculus. Let $A$ be a unital commutative $C^*$-algebra and $\mathcal{M}_A$ be the spectrum (the space of maximal ideals). The Gelfand-Naimark theorem asserts that $A$ is isomorphic to the $C^*$-algebras of continuous functions on $\mathcal{M}_A$. We denote the $*$-isomorphism (the inverse of the Gelfand map), by:

\[
\Psi : C(\mathcal{M}_A) \to A.
\]

Let $\bar{a} = (a_1, \ldots, a_j)$ be a tuple of mutually commutative self-adjoint elements in $A$. They generate a commutative unital $C^*$-algebra $C(1, \bar{a}) \subset A$ whose spectrum $\mathcal{M}$ can be identified
with a compact subset in $\mathbb{R}^j$:

$$\mathcal{M} \subset \prod_{j=1}^{n} X_{a_j} \subset \mathbb{R}^J, \quad m \in \mathcal{M} \rightarrow (\Psi^{-1}_{\bar{a}}(a_1)(m), \ldots, \Psi^{-1}_{\bar{a}}(a_j)(m)),$$

where $\Psi^{-1}_{\bar{a}} : C(1, \bar{a}) \rightarrow C(M)$ is the Gelfand map and the evaluation $\Psi^{-1}_{\bar{a}}(a_j)(m)$ belongs to the spectrum $X_{a_j}$ of $a_j$, $1 \leq j \leq J$. For any $f(\bar{x}) \in C(M)$, we have several notations for the functional calculus:

$$f(a_1, \ldots, a_J) := \int_{\mathcal{M}} f(\bar{x}) dE^\bar{a}(\bar{x}) := \Psi_{\bar{a}}(f) \in C(1, \bar{a}),$$

where $dE^\bar{a}$ and the map $\Psi_{\bar{a}}$ are both referred to as the spectral measure when no confusion arises. In the case of $J = 1$ with a normal element, that is, $\bar{a} = (a_1)$ with $[a_1, a_1^*] = 0$, the $C^*$-algebra $C(1, a_1)$ is well-defined and the space of maximal ideals is identical to the spectrum: $\mathcal{M} \equiv X_{a_1} \subset \mathbb{C}.

3.2. Smooth functional calculus. Let $\mathcal{A}$ be a unital $C^*$-algebra as before. We shall briefly review the construction in [Les17, §3]. Consider the (algebraic) contraction map $\cdot : A^{\otimes n+1} \times A^{\otimes n} \rightarrow \mathcal{A}$, on elementary tensors, it reads:

$$a_0 \otimes \cdots \otimes a_n \cdot (\rho_1 \otimes \cdots \otimes \rho_n) = a_0 \rho_1 a_1 \cdots \rho_n a_n.$$

It makes elements of $A^{\otimes n+1}$ into linear operators from $A^{\otimes n}$ to $\mathcal{A}$. We denote the induced map by

$$\iota : A^{\otimes n+1} \rightarrow L(A^{\otimes n}, A).$$

For any $a \in A$, for $0 \leq j \leq n$, depending on the context, we denote by $a^{(j)}$ either the elementary tensor

$$a^{(j)} := 1 \otimes \cdots \otimes a \otimes \cdots \otimes 1, \quad a \text{ occurs at the } j \text{-the factor},$$

or the operator

$$a^{(j)} := \iota(1 \otimes \cdots \otimes a \otimes \cdots \otimes 1) \in L(A^{\otimes n}, A).$$

The left and right multiplications correspond to $a^{(0)}, a^{(1)} \in L(A, A)$. Generally, the superscript $(j)$ simply indicates that, the multiplication occurs at the $j$-slot of elementary tensors in $A^{\otimes n}$. For self-adjoint $a = a^* \in A$ and $k = e^a$, we put $x_a = -\text{ad}_a = [-, a]$ and $y_a = \text{Ad}_{k^{-1}} = k^{-1}(-)k$, then the associated lifted operators in $L(A^{\otimes n}, A)$ are given by:

$$x_a^{(j)} = -a^{(j-1)} + a^{(j)}, \quad y_a^{(j)} = k^{(j-1)}k^{(j)} = e^{x_a^{(j)}}, \quad 1 \leq j \leq n,$$

in which the superscript $(j)$ indicates the commutator or conjugation operator acts only on the $j$-th factor on elementary tensors. We shall make use of the inverse of the relations in Eq. (3.7) in later sections:

$$a^{(j)} = -a^{(0)} + x_a^{(1)} + \cdots + x_a^{(j)}, \quad k^{(j)} = (k^{(0)})^{-1}y_a^{(1)} \cdots y_a^{(j)}, \quad 1 \leq j \leq n.$$
The functional calculus requires different completions of the algebraic tensor products of $A$. The projective tensor product $A^{\otimes_{p} n}$ is the norm completion with respect to

$$\|a\|_\gamma := \inf \sum_s \|a(s)_0 \cdots \|a(s)_n\|,$$

where the infimum runs over all possible decomposition of $a$ as elementary tensors $a = \sum_s a(s)_0 \otimes \cdots \otimes a(s)_n$. The signature property of the projective tensor product is that the multiplication map:

$$m : A^{\otimes_{p} n} \to A$$

extends by continuity to an isomorphism of $A^{\otimes_{s} n}$. The signature property of the projective tensor product is that the multiplication map:

$$m : A^{\otimes_{p} n} \to A$$

induces a continuous map

$$\iota : A^{\otimes_{p} n+1} \to L_{\text{cont}}(A^{\otimes_{p} n}, A).$$

Now let us consider a tuple of mutually commuting self-adjoint elements $\tilde{a} = (a_1, \ldots, a_J)$ with spectra $X_{a_j}$, $1 \leq j \leq J$. As in Eq. (3.2), the space of maximal ideals $M_{\tilde{a}}$ of $C(1, \tilde{a})$ is a subset of $\prod_{j=1}^{J} X_{a_j} \subset \mathbb{R}^J$. Now let $U \subset \mathbb{R}^J$ be an open subset containing $M_{\tilde{a}}$. Followed by the restriction map $C^{\infty}(U) \to C(M_{\tilde{a}})$, the continuous functional calculus $\Psi_{\tilde{a}}$ in Eq. (3.1) leads to a map,

$$\Psi_U : C^{\infty}(U) \to C(1, \tilde{a}), \ f \in C^{\infty}(U) \to f(a_1, \ldots, a_J) := \Psi_{\tilde{a}}(f|_{M_{\tilde{a}}}).$$

The smooth functional calculus relies on the nuclearity of the Fréchet topology of $C^{\infty}(U)$, which states that the projective $\otimes_{p}$ and injective $\otimes_{s}$ tensor product agree and they are both isomorphic to the smooth functions on the Cartesian product:

$$C^{\infty}(U)^{\otimes_{p}, n+1} \cong C^{\infty}(U^{n+1}) \cong C^{\infty}(U)^{\otimes_{s}, n+1}.$$  

The injective side allows us to approximate multivariable functions $f(x_0, \ldots, x_n)$ by those of the form of separating variables. More precisely, the algebraic map $C^{\infty}(U)^{\otimes_{s}, n+1} \to C^{\infty}(U^{n+1})$:

$$f_0 \otimes \cdots \otimes f_n \to f, \text{ with } f(x_0, \ldots, x_n) := f_0(x_0) \cdots f_n(x_n)$$

extends by continuity to an isomorphism of $C^{\infty}(U)^{\otimes_{s}, n+1} \to C^{\infty}(U^{n+1})$. The projective feature implies that, after the projective completion, the algebraic map

$$\Psi^{\otimes_{s} n+1} : C^{\infty}(U)^{\otimes_{s}, n+1} \to A^{\otimes_{p} n+1} : f_0 \otimes \cdots \otimes f_n \to f_0(\tilde{a}) \otimes \cdots \otimes f_n(\tilde{a})$$

induces a continuous map

$$\Psi_{\gamma} : C^{\infty}(U^{n+1}) \to A^{\otimes_{s} n+1}. \tag{3.10}$$

As in Eq. (3.6), we denote $\tilde{a}^{(j)} = (a_1^{(j)}, \ldots, a_J^{(j)})$ and $a = (a_1^{(j)})_{j \times n}$, with $1 \leq i \leq J$ and $0 \leq j \leq n$. The operators $a_i^{(j)} \in L(A^{\otimes_{s} n}, A)$ defined in Eq. (3.6) are the images of the coordinate functions under the functional calculus:

$$\{\iota \circ \Psi_{\gamma}\}(u_i^{(j)}) = a_i^{(j)}. \tag{3.11}$$

where $\tilde{u} = (\tilde{u}^{(1)}, \ldots, \tilde{u}^{(n)}) \in U^{n+1} \subset \mathbb{R}^{J \times n}$, with $\tilde{u}^{(j)} = (u_1^{(j)}, \ldots, u_J^{(j)})$.

**Definition 3.1 (Smooth functional calculus).** Keep the notations as above. For any $f \in C^{\infty}(U^{n+1})$, we define the smooth functional calculus in the following way:

$$f(a) = f(\tilde{a}^{(0)}, \ldots, \tilde{a}^{(n)}) := (\iota \circ \Psi_{\gamma})(f) \in L_{\text{cont}}(A^{\otimes_{s} n}, A),$$
where \( \iota \) and \( \Psi_\gamma \) are defined in Eqs. 3.9 and 3.10 respectively.

**Proposition 3.1.** Consider functions given via integral representations:

\[
f(\bar{u}) = \int_{\mathcal{B}} F(p, \bar{u}) d\mu_p
\]

where \((\mathcal{B}, \mu)\) is a Borel space and \(F(p, u) : \mathcal{B} \times U^{n+1} \rightarrow C\) is continuous in \(p\) and smooth in \(\bar{u}\).

With the integrability condition: for any compact set \(K \subset U^{n+1}\) and multiindex \(\alpha\),

\[
\int_{\mathcal{B}} \sup_{\bar{u} \in K} |\partial_\alpha F(p, \bar{u})| d\mu_p < \infty,
\]

we have the Fubini type result

\[
f(\hat{\alpha}^{(0)}, \ldots, \hat{\alpha}^{(n)}) := \Psi_\gamma \left( \int_{\mathcal{B}} F(p, \cdot) d\mu_p \right) = \int_{\mathcal{B}} \Psi_\gamma (F(p, \cdot)) d\mu_p,
\]

where the last integral is a Bochner one taking values in \(A^{\otimes n+1}\).

**Proof.** The integrability for all derivatives shown in Eq. 3.12 implies the converges of the integral \(\int_{\mathcal{B}} F(p, \cdot) d\mu_p\), with respect to the Fréchet topology of \(C^\infty(U^{n+1})\), so that \(f(\bar{u})\) is smooth in \(\bar{u}\). We refer to [Les17, Theorem 3.4] for more details. \(\square\)

The integral form mentioned in the proposition above leads to a more elementary construction of the functional calculus making use of Fourier transform. For any \(f \in C^\infty(U^{n+1})\), taking any extension \(\hat{f} \in \mathcal{S}(\mathbb{R}^{n+1})\) to a Schwartz function so that it can be written as a Fourier transform, with \(\bar{u} = (u^{(j)}_i)\) and \(\xi = (\xi^{(j)}_i)\), \(1 \leq l \leq J\) and \(0 \leq j \leq n\):

\[
\hat{f}(\bar{u}) = \int_{\mathbb{R}^{n+1}} (\hat{f})(\xi) \exp \left( \sum_{l, j} i\xi^{(j)}_i u^{(j)}_i \right) d\xi,
\]

where \((\hat{f})\) denotes the normalized Fourier transform. Now we are ready to define the Schwartz functional calculus

\[
\Psi_\gamma : \mathcal{S}(\mathbb{R}^n) \rightarrow L(A^{\otimes n}, A) : f \rightarrow \Psi_\gamma(f) := f_{\mathcal{S}}(\hat{\alpha}^{(0)}, \ldots, \hat{\alpha}^{(n)}).
\]

by substituting \(u^{(j)}_i \rightarrow \bar{u}^{(j)}_i\) into the integral form above. More precisely, we have, on elementary tensors:

\[
f_{\mathcal{S}}(\hat{\alpha}) = \int_{\mathbb{R}^{n+1}} (\hat{f})(\xi) \exp \left( \sum_{l, j} i\xi^{(j)}_i u^{(j)}_i \right) (\rho_1 \otimes \cdots \otimes \rho_n) d\xi,
\]

Finally, following the substitution in Eq. 3.7, we obtain the corresponding functional calculus for the commutator and conjugation operators given in Eq. 3.7: \(\{x_{ai}, y_{ai}\} \subset L(A^{\otimes n}, A)\).
with $1 \leq i \leq J$ and $1 \leq j' \leq n$. In more detail, let $	ilde{u} = (u^{(j)}_i)_{J \times n}$ be the coordinate function in Eq. (3.11), we denote matrices, viewed as maps:
$$\tilde{x} := \tilde{x}(\tilde{u}) = (x^{(j)}_i)_{J \times n}, \quad \tilde{y} := \tilde{y}(\tilde{u}) = (y^{j'}_i)_{J \times n} : U^{n+1} \subset \mathbb{R}^{J \times (n+1)} \to \mathbb{R}^{J \times n},$$
in which the entries come from the change of variables in Eq. (3.7):
$$x^{(j')}_i := x^{(j')}_i(\tilde{u}) = -u^{(j'-1)}_i + u^{(j')}_i, \quad y^{j'}_i := y^{j'}_i(\tilde{u}) = e^{-u^{j'-1}_i} e^{u^{(j')}_i}, \quad 1 \leq i \leq J, \quad 1 \leq j' \leq n.$$
Let $\tilde{x} = (x^{(j')}_{a_l})$ and $\tilde{y} = (y^{(j')}_{a_l})$, we define
$$f_{x'}(\tilde{x}) := (f \circ \tilde{x})_{x'}, \quad f_{y'}(\tilde{y}) := (f \circ \tilde{y})_{y'},$$
whenever the right hand sides make sense as Schwartz functional calculus (cf. Eq. (3.14)) in $a = (\tilde{a}^{(0)}, \ldots, \tilde{a}^{(n)})$.

### 3.3. Hypergeometric integrals for the rearrangement lemma.

We now describe the spectral functions required in the rearrangement lemma (Proposition 4.4). For a multiindex $\alpha = (a_0, a_1, \ldots, a_n) \in \mathbb{Z}_{>0}^{n+1}$ and a point $u = (u_1, \ldots, u_n) \in \Delta^n$ in the standard $n$-simplex:
$$\Delta^n = \left\{ \sum_{i=1}^n u_j \leq 1, \; u_1 \geq 0, \ldots, u_n \geq 0 \right\} \subset \mathbb{R}^n,$$
we set:

$$\omega_{\alpha}(u) = \left( \prod_{i=1}^n \Gamma(a_i) \right)^{-1} \left( 1 - \sum_{i=1}^n u_i \right)^{a_0-1} \left( \prod_{i=1}^n u_i^{a_i-1} \right)$$

where $\Gamma(z)$ is the standard Gamma function. Observe that $\omega_{\alpha}(u)$ couples all the boundary hyperplanes of the standard $n$-simplex $\Delta^n$ with the index $\alpha$ in a multiplicative fashion. Similarly, for a tuple $\tilde{A} = (A_0, \ldots, A_n)$ of positive invertible $m \times m$ matrices, we denote
$$B_n(\tilde{A}, u) = A_0(1 - \sum_{i=1}^n u_i) + \sum_{i=1}^n A_1 u_i = A_0(1 - \sum_{i=1}^n Z_i u_i) \in M_{m \times m}(\mathbb{R}),$$
where, in the last equal sign, we have substituted, $A_l = A_0 Y_l \cdots Y_1$ with $Y_l = A_{l-1}^{-1} A_l$, so that
$$Z_l = 1 - A_0^{-1} A_l = 1 - Y_l \cdots Y_1, \quad 1 \leq l \leq n.$$
By assembling the notations together with $\xi = (\xi_1, \ldots, \xi_m) \in \mathbb{R}^m$, we introduce a family of hypergeometric integrals as below:

$$F_\theta(\tilde{A}, \xi) = \int_{\Delta^n} \omega_{\alpha}(u) \frac{(2\pi)^{m/2}}{\sqrt{\det B_n(\tilde{A}, u)}} \exp\left( \frac{1}{4} \sum_{1 \leq i, j \leq n} (B_n(\tilde{A}, u))_{ij} \xi_i \xi_j \right) \, du.$$

Let us apply the functional calculus in §3.2 to the hypergeometric family given above. The $J$-tuple (with $J = m^2$) of commuting elements comes from the entries of the coefficient matrix $A = (k_{ij}) \in GL_m(C^\infty(T^n_\theta))$ of the leading term of the differential operator $P$ in Eq. (2.6). It gives
rise to \( \tilde{A} = (A^{(1)}, \ldots, A^{(n)}) \) with \( A^{(l)} = (k_{ij}^{(l)}), \ l = 0, \ldots, n, \) where \( k_{ij}^{(l)} \in L(C^{\infty}(T^m_{\theta} \otimes n, C^{\infty}(T^m_{\theta}))) \) as in Eq. (3.6). Furthermore, matrix in Eq. (3.17) becomes

\[
B_n(u) := B_n(u, \tilde{A}) = A^{(0)}(1 - \sum_{l=1}^{n} u_l) + \sum_{l=1}^{n} A^{(l)}(1 - \sum_{l=1}^{n} Z_{l} u_l)
\]

with \( A^{(l)} = A^{(0)} Y^{(1)} \ldots Y^{(l)} \) and \( Y^{(l)} = (y^{(l)}_i) = (A^{(l-1)})^{-1} A^{(l)} \) so that:

\[
Z^{(l)}(l) = (z^{(l)}_i) = 1 - (A^{(0)})^{-1} A^{(l)} = 1 - Y^{(l)} \ldots Y^{(l)}, \ l = 1, \ldots, n.
\]

Finally, we are ready to describe the generalization of the rearrangement operators appeared in the conformal setting. They are formal differential operators acting on the algebra of polynomial symbols \( C^{\infty}(T^m_{\theta} | \xi) \). By setting \( \tilde{A} \to \hat{A} \) and \( \xi \to \hat{\xi} \) in Eq. (3.19), we obtain

\[
F_{\hat{A}}(A) := F_{\hat{A}}(\hat{A}, \hat{\xi}) : (C^{\infty}(T^m_{\theta} | \xi))^n \to C^{\infty}(T^m_{\theta} | \xi).
\]

In fact, we need \( F_{\hat{A}}(A)|_{\xi=0} : (C^{\infty}(T^m_{\theta} | \xi))^n \to C^{\infty}(T^m_{\theta}) \) with the evaluation map \( |_{\xi=0} : C^{\infty}(T^m_{\theta} | \xi) \to C^{\infty}(T^m_{\theta}) \):

\[
F_{\hat{A}}(A) = \int_{\Delta^n} \omega_{\hat{A}}(u) \frac{(2\pi)^{m/2}}{\sqrt{\det B_n(u)}} \exp \left( \frac{1}{4} \sum_{1 \leq i, j \leq n} (B_n^{-1}(u))_{ij} \hat{\xi}_i \hat{\xi}_j \right) \ d u.
\]

Since the domain involves only polynomial symbols (in \( \xi \)), the exponential of differential operators make sense as the power series:

\[
\exp \left( \sum_{i,j} B^{-1}_{ij} \hat{\xi}_i \hat{\xi}_j \right) = \sum_{N=1}^{\infty} \frac{1}{N!} \left( \sum_{i,j} B^{-1}_{ij} \hat{\xi}_i \hat{\xi}_j \right)^N = \sum_{N=1}^{\infty} \frac{1}{N!} \sum_{l_1, \ldots, l_{2N}} \left( B^{-1}_{l_1 l_2} \hat{\xi}_{l_1} \hat{\xi}_{l_2} \right) \cdots \left( B^{-1}_{l_{2N-1} l_{2N}} \hat{\xi}_{l_{2N-1}} \hat{\xi}_{l_{2N}} \right),
\]

where \( i, j \) and \( l \)'s are summed over \( 1 \) to \( m \). In practice, we often have to expand \( F_{\hat{A}}(A) \) into components:

\[
F_{\hat{A}}(A) = \sum_{N=1}^{\infty} \sum_{l_1, \ldots, l_{2N}} F_{\hat{A}}(A)_{(l_1, l_2) \rightarrow (l_{2N-1}, l_{2N})} \hat{\xi}_{l_1} \cdots \hat{\xi}_{l_{2N}}
\]

with \( F_{\hat{A}}(A)_{(l_1, l_2) \rightarrow (l_{2N-1}, l_{2N})} : C^{\infty}(T^m_{\theta} \otimes n) \to C^{\infty}(T^m_{\theta}) \):

\[
F_{\hat{A}}(A)_{(l_1, l_2) \rightarrow (l_{2N-1}, l_{2N})} = \frac{1}{4N!} \int_{\Delta^n} \omega_{\hat{A}}(u) \frac{(2\pi)^{m/2}}{\sqrt{\det B_n(u)}} B_n^{-1}(u)_{l_1 l_2} \cdots B_n^{-1}(u)_{l_{2N-1} l_{2N}} \ d u,
\]

\[
F_{\hat{A}}(A)_0 = \frac{1}{4N!} \int_{\Delta^n} \omega_{\hat{A}}(u) \frac{(2\pi)^{m/2}}{\sqrt{\det B_n(u)}} \ d u, \ \text{for } N = 0.
\]
Since $\mathbf{B} := B_n(u)$ is symmetric and has mutually commuting entries, the inverse can be computed from the adjugate matrix $\text{adj} \mathbf{B}$ of $\mathbf{B}$, which is also symmetric:

$$(\mathbf{B})^{-1} = (\det \mathbf{B})^{-1} \mathbf{B}. \quad (\mathbf{B})_{ij} = (\text{adj} \mathbf{B})_{ij} = \partial_{B_{ij}} (\det \mathbf{B}).$$

As a result, we can replace the inverse in Eq. (3.23) by derivatives of the determinant:

$$F_{\alpha}(\mathbf{A})_{(l_1l_2)\cdots(l_{2N-1}l_{2N})} = \left( \frac{(2\pi)^{m/2}}{4^N N!} \int_{\Delta^n} \omega_\alpha(u) \left[ (\det \mathbf{B})^{-N-1/2} \left( \partial_{B_{l_1l_2}} (\det \mathbf{B}) \cdots \partial_{B_{l_{2N-1}l_{2N}}} (\det \mathbf{B}) \right) \right] \right)_{(u,n)} du.$$

We remind the reader three parameters $m$, $n$ and $N$ which have been frequently used in the construction of $F_{\alpha}(\mathbf{A})$ above:

1. $m$ denotes the dimension of the noncommutative tori. It determines the length of $\xi \in \mathbb{R}^m$ and the dimensions of the matrices such as: $\mathbf{A}^{(j)}$, $\mathbf{Y}^{(j)}$, $\mathbf{Z}^{(j)}$ and $\mathbf{B}(u, n)$.
2. $n$ comes from the length of $\alpha$: for $\alpha \in \mathbb{Z}_{>1}^n$, $F_{\alpha}(\mathbf{A})$ acts on $A^{\otimes n}$ and the integration is taken over the standard $n$-simplex: $u \in \Delta^n$.
3. $N$ appears when expanding the formal differential operator $F_{\alpha}(\mathbf{A})$ (in $\xi$) in Eq. (3.22). In applications, $N$ is subject to a relation Eq. (3.28).

3.4. The diagonal case. We now assume that $\mathbf{A} = \text{diag}(k_{11}, \ldots, k_{nn})$ is diagonal, that is the leading symbol reads $p_{\text{lead}}(\xi) = \sum_{s=1}^{m} k_s \xi_s^2$ with abbreviation $k_s := k_{ss}$. All the matrices defined in the previous section are diagonal: $\{\mathbf{A}^{(l)}\}_{l=0}^{n}$, $\{\mathbf{Y}^{(l)}\}_{l=1}^{n}$, $\{\mathbf{Z}^{(l)}\}_{l=1}^{n}$, and $\mathbf{B}_n(u)$. For $s = 1, \ldots, m$:

$$A^{(l)}_{ss} = k_s^{(l)} , \quad (\mathbf{Y}^{(l)})_{ss} = y_s^{(l)} (k_s^{(l-1)})^{-1} k_s^{(l)} , \quad (\mathbf{Z}^{(l)})_{ss} = z_s^{(l)} = 1 - y_s^{(l)} \cdots y_s^{(l)},$$

and for $u = (u_1, \ldots, u_n)$

$$(\mathbf{B}_n)_{ss}(u) = k_s^{(0)} (1 - \sum_{l=1}^{N} z_s^{(l)} u_l).$$

Therefore the components of $F_{\alpha}(\mathbf{A})$ are “diagonal” in the sense that

$$F_{\alpha}(\mathbf{A})_{(l_1l_2)\cdots(l_{2N-1}l_{2N})} = (\mathbf{1}_{l_1l_2} \cdots \mathbf{1}_{l_{2N-1}l_{2N}}) F_{\alpha}(\mathbf{A})_{(l_1l_2)\cdots(l_{2N-1}l_{2N})},$$

where $\mathbf{1}_{ij}$ is the $(i, j)$-entry of the identity matrix. The components of $F_{\alpha}(\mathbf{A})$ can be written as a product in which the first factor collects the contribution from the $1$-left multiplications (entries of $\mathbf{A} := \mathbf{A}^{(0)}$), while the second factor consists of the action of the conjugation operators generated by $\mathbf{Z}^{(l)}$ or $\mathbf{Y}^{(l)}$:

$$F_{\alpha}(\mathbf{A})_{(l_1l_2)\cdots(s_Ns_N)} = \frac{1}{4^N N!} \det(\mathbf{A})^{-1/2} \left( \prod_{l=1}^{N} k_{ll} \right)^{-1} F_{\alpha}(\mathbf{z})_{s_1, \ldots, s_N}^{\cdots}.$$

---

1 The transpose of the cofactor matrix.
where $z$ denotes the matrix $z = (z^{(l)}_j)$, $1 \leq j \leq m$ and $1 \leq l \leq n$. The spectral functions of $F_a(z_{s_1, \ldots, s_N})$ are of $m \times n$ variables: $z := z(m, n) = (z^{(l)}_j)$, with $1 \leq j \leq m$ and $1 \leq l \leq n$:

\[(3.26) \quad F_a(z_{s_1, \ldots, s_N}) = (2\pi)^m \int_{\Delta_n} \omega_{\alpha}(u)(\det Z_n(z, u))^{-\frac{1}{2}} \left( \prod_{l=1}^{N} Z_n(z, u)_{s_l s_l} \right)^{-1} du,
\]

where $Z_n(z, u)_{s_l s_j}$ is the $s_l$-th diagonal entry of the $m \times m$ matrix:

$$Z_n(z, u) = \text{diag}(1 - \sum_{l=1}^{n} z^{(l)}_1 u_1, \ldots, 1 - \sum_{l=1}^{n} z^{(l)}_m u_1).$$

When $N = 0$,

$$F_a(z)_0 = (2\pi)^m \int_{\Delta_n} \omega_{\alpha}(u)(\det Z_n(z, u))^{-\frac{1}{2}} du.$$

3.5. **The conformal case.** We now further assume that $k = k_1 = \cdots = k_m$, that is, there is only one noncommutative coordinate $k \in C^\infty(\mathbb{T}_\theta^m)$ positive and invertible and $A = kI$ is a scalar matrix. The $z$-variables in Eq. (3.24) becomes: $Z^{(l)} = Z^{(l)}_I$ with $y = k^{-1}\cdot k$ and

$$Z^{(1)} = 1 - y, \quad Z^{(2)} = 1 - y^{(1)}y^{(2)}, \quad \ldots, \quad Z^{(n)} = 1 - y^{(1)}\cdots y^{(n)}.$$

With $z^{(1)}_1 = \cdots = z^{(l)}_m = z^{(l)}$, the integrand of in Eq. (3.26) is reduced to:

\[
(\det Z_n(z, u))^{-\frac{1}{2}} \left( \prod_{l=1}^{N} Z_n(z, u)_{s_l s_l} \right)^{-1} = \left( 1 - \sum_{l=1}^{n} z^{(l)}_l u_l \right)^{-\frac{m}{2} - N}.
\]

The components $F_a(z)_{s_1, \ldots, s_N}$ are all identical, in other words, only the length $N$ matters. In later computation, $F_a(A)$ turns up when integrating the resolvent approximations $b_j(\xi, \lambda)$ of odd degree in $\xi$ and consists of summands of the form

\[(3.27) \quad b_0 a_0 \rho_0 b_1 a_1 \cdots \rho_n b_n^{a_n} = (b_0^{0})^{a_0} \cdots (b_0^{n})^{a_n} \cdot (\rho_1 \otimes \cdots \otimes \rho_n)
\]

which contributes to a term $F_a(A)(\rho_1 \otimes \cdots \otimes \rho_n)$ in the final result of the heat coefficient. In Eq. (3.27), $b_0 = (p_2(\xi) - \lambda)^{-1}$ is of degree $-2$ in $\xi$, while $\rho_1(\xi) \otimes \cdots \otimes \rho_n(\xi)$ is of degree $2N^2$. Hence $N$, $\alpha$ and $j$ are subjected to the condition:

\[(3.28) \quad -2(\sum_{s=0}^{n} \alpha_s) + 2N = -j - 2, \quad \text{or} \quad N = \sum_{s=0}^{n} \alpha_s - j/2 - 1.
\]

Let $\tilde{z} = (z^{(1)}_1, \ldots, z^{(n)}_m), \alpha = (a_1, \ldots, a_n)$, all the functions $F_a((z^{(l)}_j)_{s_1, \ldots, s_N})$ in Eq. (3.26) are equal to

\[(3.29) \quad H_a(\tilde{z}; m; j) = (2\pi)^{\frac{m}{2}} \int_{\Delta_n} \omega_{\alpha}(u) \left( 1 - \sum_{l=1}^{n} z^{(l)}_l u_l \right)^{-\frac{m}{2} - \sum_{s=0}^{n} \alpha_s + j/2 + 1} du.
\]

\footnote{If $\rho_1(\xi) \otimes \cdots \otimes \rho_n(\xi)$ is of odd degree in $\xi$, it automatically killed by $F_a(A)|_{\xi=0}$ according to Eq. (3.22). This observations also explains the vanishing of all odd heat coefficients.
We set a default value for $j$: $H_\alpha(\bar{z}; m) := H_\alpha(\bar{z}; m; 2)$ when dealing with the second heat coefficient. Compared to the hypergeometric family $H_\alpha(\bar{z}; m)$ used in [Liu18a, Liu18b], we have

$$H_\alpha(\bar{z}; m) = \frac{(2\pi)^{m/2}}{\Gamma(d(\alpha, m))} H_\alpha(\bar{z}; m) = C_m \frac{\Gamma(m/2)}{\Gamma(d(\alpha, m))} H_\alpha(\bar{z}; m),$$

where $\alpha = (\alpha_0, \ldots, \alpha_n)$, and $d(\alpha, m) := d(\alpha, 2, m)$ with $d(\alpha, j, m) = \sum_0^n \alpha_i + m/2 - j$. The constant $C_m$ is the overall factor used in [Liu18a, Liu18b]:

$$C_m = 2^{m/2} \frac{\pi^{m/2}}{\Gamma(m/2)} = 2^{m/2 - 1} \text{vol}(S^{m-1}).$$

4. Heat Coefficients via Pseudo-differential Calculus

We assume, in the section, the reader’s acquaintance with Connes’s pseudo-differential calculus attached to a $C^*$-dynamical system see [Con80] and [Baa88a, Baa88b]. In recent papers, [HLPT19a, HLPT19b] give detailed discussions on on pseudo-differential operators on arbitrary noncommutative tori and [LM16, LM19] deal with pseudo-differential calculus acting on Heisenberg modules. The author

We only consider (pseudo-differential) operators acting on functions. Without extra indication, $P : \mathcal{C}^\infty(T^m_\theta) \to \mathcal{C}^\infty(T^m_\theta)$ will always denote an elliptic self-adjoint second order differential operator of the form in Eq. (2.6) whose coefficient matrix $\Lambda$ of the lead term fulfills the conditions in Definition 2.1. We would like to outline the computation of the heat coefficients in the small time asymptotic of $\text{Tr}(e^{-tP})$ in Eq. (2.6) with special focus on the $V_2$-term.

4.1. Symbol calculus. The space of parametric symbols is contained in $p(\xi, \lambda) \in \mathcal{C}^\infty(\mathbb{R}^m \times \Lambda, \mathcal{C}^\infty(T^m_\theta))$, the analog of functions on the cotangent bundle of $T^m_\theta$, where the domain of the resolvent parameter $\Lambda \subset \mathbb{C}$ is a conic subset. We will encounter only homogeneous symbols in the paper, on which the filtration (or the graded structure) is reflected on the homogeneity condition: a degree $d$ symbol satisfies, with $d \in \mathbb{R},$

$$p(c \xi, \sqrt{c} \lambda) = c^d p(\xi, \lambda), \ \forall c > 0.$$

Notice that $\lambda$ is treated as a degree two symbol since the elliptic operator in question is of second order. The key ingredient of the symbolic calculus is the formal star product $\ast$ represents the symbol of the composition of two pseudo-differential operators $P$ and $Q$:

$$\sigma(PQ) = p \ast q \sim \sum_{j=0}^\infty a_j(p, q), \ p = \sigma(P), q = \sigma(Q),$$

where $a_j(\cdot, \cdot)$ are bi-differential operators lowering the total degree by $j$.

To incorporate the notations in [Liu18c, Liu17], we put

$$\nabla_j = -i \delta_j : \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{C}^\infty(T^m_\theta)) \to \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{C}^\infty(T^m_\theta))$$

which are the horizontal covariant differentials if we think $\mathcal{C}^\infty(\mathbb{R}^m, \mathcal{C}^\infty(T^m_\theta))$ as the smooth functions on the cotangent space of $T^m_\theta$. The vertical differentials $D$ is simply the derivatives in
\[ \xi \in \mathbb{R}^m: D_s = \partial_{\xi_s}. \] Notations, for higher derivatives, such as:

\[
(D^2 p)_{st} = D_{st}^2 p := \partial_{\xi_s} \partial_{\xi_t} p, \quad (\nabla^2 p)_{st} = \nabla^2_{\xi_s} \nabla_{\xi_t} (p) = -\delta_s \delta_t (p)
\]
are freely used in later calculations. In particular, the bi-differential operators in the \(*\)-product (Eq. (4.2)) are given by:

\[
a_j(p, q) = \frac{(-i)^j}{j!} (D^j p) (\nabla^j q) := \frac{(-i)^j}{j!} \sum_{1 \leq i_1, \ldots, i_j \leq m} (D^j p)_{i_1, \ldots, i_j} (\nabla^j p)_{i_{j+1}, \ldots, i_j},
\]

where \((D^j p) (\nabla^j q)\) is the contraction of two rank \(j\) tensor: contravariant \(D^j p\) and covariant \(\nabla^j q\) respectively. Of course, the multiplication among the summands is the one induced from \(C^\infty(T^m_{\theta}).\)

For differential operators, computation of symbols is similar to the classical counterpart. In detail, they are polynomials in \(\xi\) constructed on generators in the following way: for the basic derivations \(\{\delta_j\}_{j=1}^m\) and coordinate functions \(f \in C^\infty(T^m_{\theta})\) (via left-multiplication), we have:

\[
\sigma(\nabla_j) = \sigma(-i \delta_j) = -i \xi_j, \quad \sigma(f) = f.
\]

The rest is determined via the \(*\)-product Eq. (4.2), which is a finite sum when \(p, q\) are polynomials in \(\xi\).

### 4.2. Resolvent approximation.

Let \(P\) be such an elliptic operator in of the form in Eq. (2.6) with the heat operator given in Eq. (2.7) via holomorphic functional calculus which suggests that one shall start with the resolvent \((P - \lambda)^{-1}\). We write the symbol \(\sigma(P - \lambda) = p_2(\xi, \lambda) + p_1(\xi) + p_0\), so that \(p_l\) is homogeneous of degree (cf. Eq. (4.1)) \(l, l = 0, 1, 2\). Note that the resolvent parameter \(\lambda\) is grouped with the leading terms \(p_2(\xi, \lambda) = p_2(\xi) - \lambda\) as they are both of degree 2. We assume formally \(\sigma((P - \lambda)^{-1}) \sim \sum_{j=0}^{\infty} b_j(\xi, \lambda)\) with \(b_j\) of degree \(-2-j\). The inverse is taken with respect to the \(*\)-product:

\[
(b_0 + b_1 + \cdots) \star \sigma(P) = \sum_{0 \leq j, r \leq \infty} \sum_{l=0}^{2} a_j(b_r, p_l) \sim 1.
\]

We compare two sides according to the homogeneity. Since summand \(a_j(b_r, p_l)\) is of degree \(l - 2 - r - j\), we get, by collecting terms of degree \(i = 0, -1, -2, \ldots:\)

\[
a_0(b_0, p_2) = 1,
\]

\[
\sum_{l=0}^{2} \sum_{r=0}^{N} a_{l-2-r+N}(b_r, p_l) = 0, \quad N = 1, 2, \ldots.
\]

Since \(a_0(p, q) = pq\), the first approximation \(b_0\) is simply the resolvent of the leading symbol \(p_2:\)

\[
b_0 = (p_2(\xi, \lambda))^{-1} = (p_2(\xi) - \lambda)^{-1},
\]

where the inverse is taken in \(C^\infty(\mathbb{R}^m, C^\infty(T^m_{\theta}))\), \(\forall \xi \neq 0\) whose existence is provided by the ellipticity of \(P\). By solving equation in Eq. (4.5) one by one (for \(N = 1, 2, \ldots\)), we obtain the recursive formulas of \(b_N:\)

\[
b_N = \left( \sum_{l=0}^{2} \sum_{r=0}^{N-1} a_{l-2-r+N}(b_r, p_l) \right)(-b_0).
\]
For example, the first two terms are given by:

\begin{align}
(4.7) & \quad b_1 = [a_0 (b_0, p_1) + a_1 (b_0, p_2)](-b_0) \\
(4.8) & \quad b_2 = [a_0 (b_0, p_0) + a_0 (b_1, p_1) + a_1 (b_0, p_1) + a_1 (b_1, p_2) + a_2 (b_0, p_2)](-b_0),
\end{align}

with

\begin{equation}
(4.9) \quad a_0(p, q) = pq, \quad a_1(p, q) = -i(D p)(\nabla q), \quad a_2(p, q) = -\frac{1}{2} (D^2 p)(\nabla^2 q).
\end{equation}

By carefully expand the right hand sides of Eqs. (4.7) and (4.8) according to Eq. (4.9), we get

\begin{equation}
b_2 = (b_2)_I + (b_2)_II,
\end{equation}

where the terms are grouped apropos to the number of $b_0$-factors. Here is part I:

\begin{equation}
(4.10) \quad (b_2)_I = -\frac{1}{2} b_0^2(D_s^2 p_2) \cdot (\nabla^2 s p_2) b_0 + b_0^4(D_s p_2) \cdot (D_s^2 p_2) \cdot (\nabla^2 s p_2) b_0 \\
- i b_0^2(D_s p_2)(\nabla s p_1) b_0 - b_0(p_0)b_0,
\end{equation}

and then part II:

\begin{equation}
(4.11) \quad (b_2)_II = b_0^2(D_s p_2)(D_s(\nabla s p_2)) b_0(\nabla s p_2) b_0 - b_0^2(D_s p_2)(\nabla s p_2) b_0(\nabla s p_2) b_0 \\
+ b_0^2(D_s^2 p_2)(\nabla s p_2) b_0(\nabla s p_2) b_0 - 2b_0^2(D_s^2 p_2)(D_s p_2)(\nabla s p_2) b_0(\nabla s p_2) b_0 \\
+ i b_0^2(D_s p_2)(\nabla s p_2) b_0(p_1)b_0 - i b_0(D_s p_1) b_0(\nabla s p_2) b_0 + i b_0(p_1) b_0^2(D_s p_2)(\nabla s p_2) b_0 \\
+ b_0(p_1)b_0(p_1)b_0,
\end{equation}

where summations are taken over repeated indices $s, t$ from 1 to $m$.

4.3. **Rearrangement lemma.** The resolvent approximation $\{b_j\}_{j=0}^\infty$ determines the heat coefficients in the following way.

**Proposition 4.1.** *In the light of Eq. (2.7), the heat coefficient $v_j(P)$, $j = 0, 1, 2, \ldots$, is completely determined by $b_j$ in the following way:*

\begin{equation}
(4.12) \quad v_j(P) = \frac{1}{2\pi i} \int_{\mathbb{R}^m} \int_{\mathbb{C}} e^{-\lambda} b_j(\xi, \lambda) d\lambda d\xi.
\end{equation}

**Proof.** This is a standard result in pseudo-differential calculus. One first establishes a trace formula linking the operator trace of a pseudo-differential operator with its symbol (cf. [Wid78, Theorem 5.7] for instance) and then follows the argument in [GT95, §1.8] to reach the heat coefficients. \hfill \square

Similar to the $b_2$-term given in Eqs. (4.10) and (4.11), we have, in general, that $b_j$ consists of finite sums of the form:

\begin{equation}
(4.13) \quad b_j = \sum b_0^{\alpha_0} \rho_1 b_0^{\alpha_1} \cdots \rho_n b_0^{\alpha_n} := \sum (b_0^{(0)})^{\alpha_0} \cdots (b_0^{(n)})^{\alpha_n} \cdot (\rho_1 \otimes \cdots \otimes \rho_n),
\end{equation}
where the $\rho$'s are the derivatives of $p_j$, $j = 2, 1, 0$, which are polynomial in $\xi$ and have no dependence on $\lambda$. Therefore we factor out the $\rho$'s by making used of the notations in Eq. (3.6) which leads to the rearrangement operator from $C^\infty(T^m_\theta)[\xi]^{\otimes n}$ to $C^\infty(T^m_\theta)$:

$$\rho_1 \otimes \cdots \otimes \rho_n \rightarrow \frac{1}{2\pi i} \int_{\mathbb{C}} \int_{\mathbb{C}} e^{-\lambda((b_0^{(0)})^{\alpha_0} \cdots (b_0^{(n)})^{\alpha_n})} d\lambda (\rho_1 \otimes \cdots \otimes \rho_n) d\xi.$$ 

The rearrangement lemma (Proposition 4.4) asserts that it is exactly the operator $F_{\alpha}(A)$ given by Eq. (3.21), where $A = (k_{ij}) \in GL_m(C^\infty(T^m_\theta))$ is the coefficient matrix of $p_2(\xi)$.

We first deal with the contour integral in Eq. (4.12), for which the origin is no longer a singularity. We fix $C$ to be the imaginary axis $\lambda = i x$, with $x \in \mathbb{R}$ oriented from $-\infty$ to $\infty$. By replacing $b_j$ with the summands shown in Eq. (4.13), we have arrived at the integral in Eq. (4.14), which turns out to be an hypergeometric integral (Eq. (4.15)).

**Lemma 4.2.** Let $\overline{A} = (A_0, \ldots, A_n) \in \mathbb{R}^{n+1}$, $l_0, \ldots, l_n \in \mathbb{N}_+$, denote

$$G_{l_0, \ldots, l_n}(\overline{A}) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} e^{-i x (A_0 - i x)^{-l_0} \cdots (A_n - i x)^{-l_n}} d x$$

Then $G_{l_0, \ldots, l_n}(\overline{A})$ is equal to the following confluent type hypergeometric integral:

$$G_{l_0, \ldots, l_n}(\overline{A}) = \left( \prod_0^n \Gamma(l_j) \right)^{-1} \int_{\Delta^n} (1 - \sum_1^n u_j)^{l_0-1} \left( \prod_0^n u_j^{l_j-1} \right) e^{-(A_0(1-\sum_1^n u_j)+\sum_1^n A_j u_j)} d u_1 \cdots d u_n$$

$$= \int_{\Delta^n} \omega_1(u) e^{-(A_0(1-\sum_1^n u_j)+\sum_1^n A_j u_j)} d u$$

where $\Delta^n = \{ \sum_1^n u_j \leq 1, u_1 \geq 0, \ldots, u_n \geq 0 \} \subset \mathbb{R}^n$ is the standard simplex and $\omega_1(u)$ is defined in Eq. (3.16).

**Proof.** We begin with rewriting each $(A_j - i x)^{-l_j}$, $j = 0, \ldots, n$ as a Mellin transform:

$$(A_j - i x)^{-l_j} = \frac{1}{\Gamma(l_j)} \int_0^\infty s^{l_j-1} e^{-(A_j-i x)s_j} d s_j$$

3 compared to the one in Eq. (2.7), in which the elliptic operator $P$ might have non-trivial kernel.
so that \( G_{l_0,\ldots,l_n}(\overline{A}) \) becomes:

\[
\left( \prod_{j=0}^{n} \Gamma(l_j) \right) G_{l_0,\ldots,l_n}(\overline{A})
= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{0,\infty}^{\infty} e^{(\sum_{j=0}^{n} s_{j-1})ix} \prod_{j=0}^{n} \left( s_{j-1}^{-l_j} e^{-A_j s_j} \right) ds_0 \cdots ds_n \, dx
= \int_{\mathbb{R}^n} e^{u_0 ix} (u_0 + 1 - \sum_{j=1}^{n} u_j)^{-l_0-1} e^{-A_0(u_0+\sum_{j=1}^{n} u_j)} \prod_{j=0}^{n} \left( u_j^{-l_j} e^{-A_j u_j} \right) d u_0 d x \left( d u_1 \cdots d u_n, \right)
\]

where the last line is obtained by the substitution:

\[
u_0 = \sum_{j=0}^{n} s_j - 1, \quad u_1 = s_1, \ldots, u_n = s_n.
\]

We denote

\[
f(u_0, \ldots, u_n) = (u_0 + 1 - \sum_{j=1}^{n} u_j)^{-l_0-1} e^{-A_0(u_0+\sum_{j=1}^{n} u_j)} \prod_{j=0}^{n} \left( u_j^{-l_j} e^{-A_j u_j} \right)
\]

and view it as a function in \( u_0 \). Set \( f(u_0) = 1_{\left\{ u_0 \geq \sum_{j=1}^{n} u_j - 1 \right\}}(u_0) f(u_0, \ldots, u_n) \), where \( 1_{\left\{ u_0 \geq \sum_{j=1}^{n} u_j - 1 \right\}}(u_0) \) is the characteristic function in \( u_0 \) of the set \( \left\{ u_0 \geq \sum_{j=1}^{n} u_j - 1 \right\} \subset \mathbb{R} \). The Fourier inversion theorem with respect to \( d u_0 d x \) gives:

\[
\int_{\mathbb{R}^n} \sum_{j=0}^{n} u_j^{-l_j} e^{-A_j(u_0+\sum_{j=1}^{n} u_j)} \prod_{j=0}^{n} \left( u_j^{-l_j} e^{-A_j u_j} \right) d u_0 d x
= \int_{\mathbb{R}^n} \sum_{j=0}^{n} u_j^{-l_j} f(u_0, \ldots, u_n) d u_0 d x = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{u_0 ix} f(u_0, \ldots, u_n) d u_0 d x
= f(0, u_1, \ldots, u_n) = f(0, u_1, \ldots, u_n) 1_{\left\{ \sum_{j=1}^{n} u_j \leq 1 \right\}}(u_1, \ldots, u_n).
\]

We conclude the proof by observing that \( f(0, u_1, \ldots, u_n) \) is exactly the integral (including the factor \( e^{-A_0} \)) appeared on the right hand side of (4.14) and \( \Delta^n = \{ 0, \infty \}^n \cap \{ \sum_{j=1}^{n} u_j \leq 1 \} \). \( \square \)

The next step is the integration in \( \xi \) in Eq. (4.12), which will be handled by the well-known Gaussian integral for polynomials.

**Lemma 4.3 (Gaussian Integral).** Let \( f(\xi) \) be a polynomial in \( \xi = (\xi_1, \ldots, \xi_m) \) and \( B = (B_{ij}) \) be a symmetric positive-definite \( m \times m \) matrix with the inverse denoted by \( (B^{-1})_{ij} \), then

\[
\int_{\mathbb{R}^m} e^{-\sum_{i \leq j \leq m} B_{ij} \xi_i \xi_j} f(\xi) d \xi = \frac{(2\pi)^{m/2}}{\sqrt{\text{det } B}} \exp \left( \frac{1}{4} \sum_{1 \leq i, f \leq m} (B^{-1})_{ij} \partial_{\xi_i} \partial_{\xi_j} \right) f(\xi) \bigg|_{\xi=0}
\]

where the exponential over differential operators is interpreted as power series.
Consider the elliptic operator $P$ given in Eq. (2.6) with leading symbol $p_2(\xi) = \sum_{s,t=1}^m k_{st} \xi_s \xi_t$. Denote by $A = (k_{ij})_{m \times m}$ the coefficient matrix and put $A^{(l)} = (k_{ij}^{(l)})$, $l = 1, \ldots, n$. For each $u = (u_0, \ldots, u_n) \in \Delta^n \subset \mathbb{R}^n$ in the standard simplex, let $B_n(u)$ be the coupled matrix defined in Eq. (3.19):

$$B_n(u) = A^{(0)}(1 - \sum_{l=1}^n u_l) + \sum_{l=1}^n A^{(l)} u_l = A^{(0)}(1 - \sum_{l=1}^n Z^{(l)} u_l).$$

**Proposition 4.4.** Let $\alpha = (\alpha_0, \ldots, \alpha_n) \in \mathbb{R}_{n+1}^n$, and $\rho_1 := \rho_1(\xi), \ldots, \rho_n := \rho_n(\xi) \in C^\infty(T^n_B)[\xi]$ are polynomial symbols. Put $\hat{\rho}(\xi) = \rho_1(\xi) \otimes \cdots \otimes \rho_n(\xi)$. If we apply the integral onto a typical summand appeared in the resolvent approximation, the result is $\text{The operator } F_0(A)_{|\xi=0}$ defined in (3.21) computes the integral in Eq. (4.12) applied onto summands of the resolvent approximation (see Eq. (4.13)):

$$\frac{1}{2\pi i} \int_{\mathbb{R}^m} \int_C e^{-\lambda} b_0^a \rho_1 b_0^a \rho_2 \cdots b_0^{a_n} \rho_n b_0^a d\lambda d\xi$$

$$= \int_{\Delta^n} \omega_n(u) \frac{(2\pi)^{m/2}}{\sqrt{\det B_n(u)}} \exp \left( \frac{1}{4} \sum_{1 \leq i \leq j \leq n} (B^{-1}_{n}(u))_{ij} \partial_{\xi_i} \partial_{\xi_j} \right) (\hat{\rho}(\xi)) |_{\xi=0} \quad d\xi$$

$$= F_0(A)_{|\xi=0}(\hat{\rho}(\xi))$$

**Proof.** We start with Lemma 4.2 with operator-valued arguments $A_j = p_2^{(j)}(\xi) = \sum_{s,t=1}^m k_{st}^{(j)} \xi_s \xi_t$, $j = 0, 1, \ldots, n$:

$$\frac{1}{2\pi i} \int_C e^{-\lambda} b_0^a \rho_1 b_0^a \rho_2 \cdots b_0^{a_n} \rho_n b_0^a d\lambda$$

$$= \frac{1}{2\pi i} \int_C e^{-\lambda} (p_2^{(0)} - \lambda)^{a_0} \cdots (p_2^{(n)} - \lambda)^{a_n} d\lambda (\rho_1 \otimes \cdots \otimes \rho_n)$$

$$= G_a(p_2^{(0)}, \ldots, p_2^{(n)}) (\rho_1 \otimes \cdots \otimes \rho_n)$$

$$= \int_{\Delta^n} \omega_n(u) \exp \left( - \left( p_2^{(0)}(1 - \sum_{l=1}^n u_l) + \sum_{l=1}^n p_2^{(l)} u_l \right) \right) (\rho_1 \otimes \cdots \otimes \rho_n).$$

Notice that we have silently quoted Eq. (3.13) (more than once) in which the Borel spaces are $C$ and $\Delta^n$, the verification of the integrability condition Eq. (3.12) is straightforward and left to the reader.

The coupled matrix $B_n(u)$ is the coefficient matrix of the sum:

$$p_2^{(0)}(\xi)(1 - \sum_{l=1}^n u_l) + \sum_{l=1}^n p_2^{(l)}(\xi) u_l = \sum_{0 \leq j, l \leq m} B_n(u)_{ji} \xi_j \xi_l.$$
The result follows immediately from the Gaussian integral lemma [4.3]:

$$\int_{\mathbb{R}^m} \exp \left( - \left( p_2^{(0)}(1 - \sum_{i=1}^{n} u_i) + \sum_{i=1}^{n} p_2^{(l)} u_i \right) \right) (\rho(\xi)) d\xi$$

$$= \frac{(2\pi)^{m/2}}{\sqrt{\det B_n(u)}} \exp \left( \frac{1}{4} \sum_{1 \leq i, j \leq n} (B^{-1}(u, n))_{ij} \partial_{\xi_i} \partial_{\xi_j} \right) (\rho(\xi)) \bigg|_{\xi=0}. \square$$

4.4. **General form of the second heat coefficients.** Now let us work out the rest of the computation for the $v_2$-term starting with Prop. [4.1]

$$v_2(P) = \frac{1}{2\pi i} \int_{\mathbb{R}^m} \int_{\mathcal{C}} e^{-\lambda (b_2(\xi, \lambda) \mathbb{I}_m + b_2(\xi, \lambda) \mathbb{H})} d\lambda \ d\xi,$$

where $b_2(\xi, \lambda) \mathbb{I}_m$ and $b_2(\xi, \lambda) \mathbb{H}$ are recorded in Eqs. [4.10] and [4.11]. We have just shown that the result of the integration is given by the rearrangement operators $F_0(A)|_{\xi=0} : C^\infty(\mathbb{T}_m^m) \otimes [\xi] \to C^\infty(\mathbb{T}_m^m)$. More precisely, we just have to carry out substitutions like:

- $b_2(\xi, \lambda) \mathbb{I}_m$:
  $$b_2^0(D_{st}^2 p_2) \cdot (\nabla^2 s_{st} p_2) b_0 \to F_{2,1}(A)(D_{st}^2 p_2) \cdot (\nabla^2 s_{st} p_2)$$
  $$b_2^0(D_{st}^2 p_2 \otimes \nabla s_p) b_0(\nabla t_p) b_0 \to F_{2,1,1}(A)(D_{st}^2 p_2 \otimes (\nabla t_p))$$

on individual terms in Eqs. [4.10] and [4.11]. After that, we have obtained a relatively compact version of the second heat coefficient.

**Theorem 4.5.** We shall group the summands of $v_2(P)$ in terms of the parameter $\alpha$ in $F_0(A)$:

$$v_2(P) = v_2(P)_{\mathbb{I}} + v_2(P)_{\mathbb{H}}.$$

The first part $v_2(P)_{\mathbb{I}} = v_2(P)_{\mathbb{I},2,1} + v_2(P)_{\mathbb{I},3,1} + v_2(P)_{\mathbb{I},1,1}$:

$$v_2(P)_{\mathbb{I},2,1} = \sum_{s,t} - F_{2,1}(A) \left( \frac{1}{2} D_{st}^2 p_2 (\nabla^2 s_{st} p_2) + i (D_s p_2)(\nabla s p_1) \right)$$

$$v_2(P)_{\mathbb{I},3,1} = F_{3,1}(A) \left( (D_s p_2)(D_t p_2)(\nabla^2 s_{st} p_2) \right)$$

$$v_2(P)_{\mathbb{I},1,1} = - F_{1,1}(A) \left( p_0 \right)$$

(4.16)

The second part consists of:

$$v_2(P)_{\mathbb{H}} = v_2(P)_{\mathbb{H},2,1} + v_2(P)_{\mathbb{H},3,1} + v_2(P)_{\mathbb{H},2,2,1} + v_2(P)_{\mathbb{H},1,2,1} + v_2(P)_{\mathbb{H},1,1,1},$$

in which:

$$v_2(P)_{\mathbb{H},2,1} = \sum_{s,t} F_{2,1,1}(A) \left( (D_s p_2)(D_t p_2)(\nabla^2 s_{st} p_2) \otimes (\nabla s p_2) + (D_{st}^2 p_2)(\nabla s p_2) \otimes (\nabla t p_2) \right) + i (D_s p_2)(\nabla s p_2) \otimes p_1 + i (D_s p_2) p_1 \otimes (\nabla s p_2).$$

(4.17)
and
\begin{equation}
(4.18) 
\nu_2(P)_{1,1,2,1} = i \sum F_{1,2,1}(A) \left( p_1 \otimes (D_s p_2)(\nabla_s p_2) \right),
\end{equation}
and
\begin{equation}
(4.19) 
\nu_2(P)_{1,3,1,1} = -2 \sum F_{3,1,1}(A) \left( (D_s p_2)(D_t p_2)(\nabla_s p_2) \otimes (\nabla_t p_2) \right)
\end{equation}
\begin{equation}
(4.20) 
\nu_2(P)_{2,2,1,1} = -F_{2,2,1}(A) \left( (D_s p_2)(\nabla_s p_2) \otimes (D_t p_2)(\nabla_t p_2) \right),
\end{equation}
and
\begin{equation}
(4.21) 
\nu_2(P)_{1,1,1,1} = F_{1,1,1,1}(A) \left( p_1 \otimes p_1 \right) + \sum F_{1,1,1,1}(A) \left( -l(D_s p_1) \otimes (\nabla_s p_2) \right).
\end{equation}

For practical purposes, we need the fully expanded version for all the terms above. For the differential operator $P$ given in (2.6), the symbol reads explicitly: $p_0 = p_0$ and
\begin{equation}
(4.22) 
p_2(\xi) = \sum_{i,j} k_{ij} \xi_i \xi_j, \quad p_1(\xi) = \sum_s r_s \xi_s,
\end{equation}
with derivatives (will be needed later):
\begin{equation}
(4.23) 
D_s p_2 = \sum_l k_{sl} \xi_l, \quad D_s^2 p_2 = 2k_{st}, \quad D_s p_1 = r_s,
\end{equation}
\begin{equation}
\nabla^2_{jl}(D_s^2 p_2) = 2(\nabla^2_{jl} k_{st}) \nabla_j(D_s p_1) = \nabla_j(r_s).
\end{equation}
To expand $F_0(A)$ using Eq. (3.22), we recall a general result to compute the partial derivative $\partial_{\xi_1} \cdots \partial_{\xi_{2N}}$.

Lemma 4.6. Let $\{\rho_{1}(\xi)\}_{i=1}^{n}$ be homogeneous polynomial in $\xi$ with degree $\beta_{i} = \deg \rho_{i} \geq 1$. Then the partial derivative of the product $\rho_{1} \cdots \rho_{n}$
$$
\partial_{\xi_{i_1} \cdots \xi_{i_{2N}}}^{2N} \left( \rho_{1}(\xi) \cdots \rho_{n}(\xi) \right) \big|_{\xi=0}
$$
is non-zero only when $\beta = (\beta_{1}, \ldots, \beta_{n})$ is partition of $2N$, that is $2N = \sum_{i=1}^{n} \beta_{i}$ and
$$
\partial_{\xi_{i_1} \cdots \xi_{i_{2N}}}^{2N} \left( \rho_{1}(\xi) \cdots \rho_{n}(\xi) \right) \big|_{\xi=0} = \partial_{\beta_{1}}^{(1)}(\rho_{1}(\xi)) \cdots \partial_{\beta_{n}}^{(n)}(\rho_{n}(\xi))
$$
where we split the partial derivatives according to the partition $\beta$:
$$
\partial_{\beta_{1}}^{(1)} = \frac{1}{\beta_{1}!} \prod_{i=1}^{\beta_{1}} \partial_{\xi_{i_{1}}}, \quad \partial_{\beta_{2}}^{(2)} = \frac{\beta_{1} + \beta_{2}}{\beta_{2}!} \prod_{i=\beta_{1}+1}^{\beta_{1}+\beta_{2}} \partial_{\xi_{i_{1}}}, \ldots, \partial_{\beta_{n}}^{(n)} = \frac{1}{\beta_{n}!} \prod_{i=\sum_{i=1}^{n-1} \beta_{i}+1}^{\sum_{i=1}^{n} \beta_{i}} \partial_{\xi_{i_{1}}},
$$
Last but not least, those barred $l$‘s indicate the following symmetrization (without dividing by the factorial factor) occurs:
\begin{equation}
(4.24) 
(\text{**})^{l_1, \ldots, l_{2N}} = \sum_{\tau \in S_{2N}} (\text{**})^{(l_{1}), \ldots, (l_{2N})}. \nonumber
\end{equation}
Proof. Despite the lengthy notations, operations behind is quite simple. We distribute \(2N\) partial derivatives to the factors \(\rho_1, \ldots, \rho_n\) following the general Leibniz property and then collect the non-zero terms after evaluating at \(\xi = 0\). Notice that non-trivial contribution only occurs in the situation in which \(\rho_l\) is differentiated exactly \(\beta_l\) times, \(l = 1, \ldots, n\).

Let us take Eq. (4.19) for example, \((D_s p_1)(D_t p_2)(\nabla_s p_2) \otimes (\nabla_t p_2)\) is a polynomial of degree 6 with partition \(\beta = (1, 1, 2, 2)\), thus

\[
\begin{align*}
F_{3,1,1}(A)((D_s p_1)(D_t p_2)(\nabla_s p_2) \otimes (\nabla_t p_2)) & = F_{3,1,1}(A)_{(l_1 l_2)(l_3 l_4)(l_5 l_6)}(A) \partial_{\xi_1} \cdots \partial_{\xi_6} \left[ (D_s p_1)(D_t p_2)(\nabla_s p_2) \otimes (\nabla_t p_2) \right]_{\xi = 0} \\
& = F_{3,1,1}(A)_{(l_1 l_2)(l_3 l_4)(l_5 l_6)}(A) \left( \frac{1}{4} (D_s^2 p_1)(D_t^2 p_2)(\nabla_s D_t^2 p_2) \otimes (\nabla_t D_t^2 p_2) \right). 
\end{align*}
\]

After repeating similar computation above to all summands of \(v_2(P)_l\) and \(v_2(P)_h\) in Theorem 4.5 we have arrived at fully expanded version of the \(V_2\)-term which will be recorded separately into two theorems.

Theorem 4.7. With the components of \(F_2(A)\) defined in (3.23):

\[
F_2(A)_{(l_1 l_2)(l_3 l_4)(l_5 l_6)} : C^\infty(T^m_0) \otimes \otimes \rightarrow C^\infty(T^m_0),
\]

we can further expand the summands of \(v_2(P)_l\) listed in Theorem 4.5:

\[
v_2(P)_{l, 2, 1} = \sum_{s, t, l_1 l_2} -F_{2,1}(A)_{l_1 l_2} \left( \frac{1}{2} (D_s^2 p_2)(\nabla_s^2 (D_t^2 p_2)) + i (D_s^2 p_2)(\nabla_s (D_t^2 p_1)) \right),
\]

\[
= -2 \sum_{s, t, l_1 l_2} F_{2,1}(A)_{l_1 l_2} \left( k_{st}(\nabla_s k_{l_1 l_2}) + i k_{st} (\nabla_s r_{l_2}) \right),
\]

and

\[
v_2(P)_{l, 3, 1} = \sum_{s, t, l_1, \ldots, l_4} F_{3,1}(A)_{(l_1 l_2)(l_3 l_4)} \left( \frac{1}{2} (D_s^2 p_2)(D_t^2 p_2)(\nabla_{st}^2 (D_t^2 p_2)) \right)
\]

\[
= 4 \sum_{s, t, l_1, \ldots, l_4} F_{3,1}(A)_{(l_1 l_2)(l_3 l_4)} \left( k_{st} k_{t l_2} \nabla_{st}^2 (k_{l_1 l_4}) \right),
\]

and

\[
v_2(P)_{l, 1, 1} = -F_{1,1}(A)_{\bar{b}}(\bar{p}_0).
\]

Summation in \(s, t, l_1, \ldots, l_4\) runs from 1 to \(m\). The barred letters \(\bar{l}_1, \ldots, \bar{l}_4\) indicate symmetrization (without dividing by the factorial factor, cf. Eq. (4.24)) is applied.
Theorem 4.8. Keep notations, we have the full expansion of $v_2(P)_{11}$:

\[
v_2(P)_{11,2,1,1} = \sum_{s,t,l_1,\ldots,l_4} F_{2,1,1}(A)_{l_1,l_2,l_3,l_4} \left[ \frac{1}{2} (D^2_{s_1} p_2) (\nabla_s (D^2_{l_1} p_2)) \otimes (\nabla_t (D_{l_2} p_2)) + \frac{i}{2} (D^2_{s_1} p_2) (\nabla_s (D^2_{l_1} p_2)) + \frac{i}{2} (D^2_{s_1} p_2) (\nabla_t (D_{l_2} p_2)) \right]
\]

\[= \sum_{s,t,l_1,\ldots,l_4} F_{2,1,1}(A)_{l_1,l_2,l_3,l_4} \left[ \frac{1}{2} (D^2_{s_1} p_2) (\nabla_s (D^2_{l_1} p_2)) \otimes (\nabla_t (D_{l_2} p_2)) + \frac{i}{2} (D^2_{s_1} p_2) (\nabla_s (D^2_{l_1} p_2)) + \frac{i}{2} (D^2_{s_1} p_2) (\nabla_t (D_{l_2} p_2)) \right]
\]

and

\[v_2(P)_{11,1,2,1} = \sum_{s,t,l_1,\ldots,l_4} i F_{2,1,1}(A)_{l_1,l_2,l_3,l_4} \left( \frac{1}{2} D_{l_1} p_1 \otimes (D^2_{s_1} p_2) (\nabla_s (D^2_{l_1} p_2)) \right)
\]

and

\[v_2(P)_{11,3,1,1} = -2 \sum_{s,t,l_1,\ldots,l_6} F_{3,1,1}(A)_{l_1,l_2,l_3,l_4,l_5,l_6} \left( \frac{1}{4} (D^2_{s_1} p_2) (D^2_{l_1} p_2) (\nabla_s (D^2_{l_1} p_2)) \otimes (\nabla_t (D_{l_2} p_2)) \right)
\]

and

\[v_2(P)_{11,2,2,1} = -\sum_{s,t,l_1,\ldots,l_6} F_{2,2,1}(A)_{l_1,l_2,l_3,l_4,l_5,l_6} \left( \frac{1}{4} (D^2_{s_1} p_2) (D^2_{l_1} p_2) (\nabla_s (D^2_{l_1} p_2)) \otimes (\nabla_t (D_{l_2} p_2)) \right),
\]

and

\[v_2(P)_{11,1,1,1} = -i \sum_{s,t,l_1,l_2} F_{1,1,1}(A)_{l_1,l_2} \left( (D_2 p_1) \otimes (D^2_{l_1} p_2) \right) + \sum_{l_1,l_2} F_{1,1,1}(A)_{l_1,l_2} \left( 2 (D_1 p_1) \otimes (D_{l_2} p_1) \right)
\]

\[= -i \sum_{s,t,l_1,l_2} F_{1,1,1}(A)_{l_1,l_2} \left( 2 r_1 \otimes (D^2_{l_1} p_2) \right) + \sum_{l_1,l_2} F_{1,1,1}(A)_{l_1,l_2} \left( 2 r_1 \otimes (D_{l_2} p_1) \right).
\]
4.5. **Diagonal Case.** In this section, we shall look at a special situation in which the coefficient matrix \( A = \text{diag}(k_1, \ldots, k_m) \) is diagonal, in other words, the leading symbol of \( P \) (defined in Eq. (2.6)) is of the form \( p_2(\xi) = \sum_1^m k_i \xi_i^2 \). We have seen the simplification of the rearrangement operators \( F_6(A) \) and their components in §3.4 in which the notations will be freely used. The remaining work is to simply the differential expressions on which the rearrangement operators act.

Let us start with those terms in which no symmetrization occurs. We only have to invoke Eq. (3.25) to replace \( F_d(A) \) by \( F_d(A) \):

\[
\begin{align*}
\nu_2(P)_{1,1,1} &= -F_{1,1}(A)_{0,0} = -(\det A)^{-\frac{1}{2}} F_{1,1}(A)_{0,0}, \\
\nu_2(P)_{1,2,1} &= -2F_{2,1}(A)_{11}(k_s(\nabla^2_{ss} k_l)) - 4 \sum_s F_{2,1}(A)_{ss} (ik_s(\nabla_s r_s))
\end{align*}
\]

\[
= (\det A)^{-\frac{1}{2}} \left( -\frac{1}{2} \sum_{s,l} F_{2,1}(z)_{l} \left( k_s k_l \nabla^2_{ss} k_l \right) - \sum_s F_{2,1}(z)_s (i \nabla_s r_s) \right).
\]

The computation of the symmetrization (appeared in Eqs. (4.25) to (4.29)) is straightforward, we only state the results in Lemma 4.9 and 4.10.

**Lemma 4.9.** The symmetrization over \( \{l_1, \ldots, l_4 \} \) yields \( 4! = 24 \) terms which are reduced to two cases divided as \( 8 + 16 \). Here are the summands:

\[
\begin{align*}
\sum_{s,t,l_1, \ldots, l_4} F_{3,1}(A)_{l_1,l_2,l_3,l_4} (k_{s,l_1} k_{t,l_2} \nabla^2_{ss} k_{l_3,l_4})
\end{align*}
\]

\[
= \sum_{s,t} 8F_{3,1}(A)_{ss,tt} (k_{ss}^2 \nabla^2_{ss} k_{tt}) + 16 \sum_s F_{3,1}(A)_{ss,tt} (k_{ss}^2 \nabla^2_{ss} k_{ss}),
\]

and

\[
\begin{align*}
\sum_{s,t,l_1, \ldots, l_4} F_{2,1,1}(A)_{l_1,l_2,l_3,l_4} (k_{s,l_1} (\nabla_s k_{l_2}) \otimes (\nabla_t k_{l_3,l_4}))
\end{align*}
\]

\[
= 8 \sum_{s,l} F_{2,1,1}(A)_{ss,tt} (k_{s} (\nabla_s k_{tt}) \otimes (\nabla_s k_{st})) + 16 \sum_s F_{2,1,1}(A)_{ss,tt} (k_s (\nabla_s k_{ss}) \otimes (\nabla_s k_{ss})),
\]

and

\[
\begin{align*}
\sum_{s,t,l_1, \ldots, l_4} F_{2,1,1}(A)_{l_1,l_2,l_3,l_4} (k_{s,l_1} (\nabla_s k_{l_2}) \otimes (\nabla_s k_{l_3,l_4}))
\end{align*}
\]

\[
= 8 \sum_{s,l,t} F_{2,1,1}(A)_{st,tt} (k_{s} (\nabla_s k_{tt}) \otimes (\nabla_s k_{tt})) + 16 \sum_s F_{2,1,1}(A)_{tt,tt} (k_s (\nabla_s k_{tt}) \otimes (\nabla_s k_{tt})).
\]
and
\[
\sum_{s,t,l_1,\ldots,l_4} F_{2,1,1}(A)_{(l_1l_2)(l_3l_4)} \left( k_{s,l_1} (\nabla_{s} k_{l_2l_4}) \otimes r_{l_3} \right)
\]

\begin{align*}
&= 8 F_{2,1,1}(A)_{(ss)(tt)} (k_{ss} (\nabla_{s} k_{tt}) \otimes r_{s}) + 16 F_{2,1,1}(A)_{(ss)(tt)} (k_{ss} \nabla_{s} k_{ss} \otimes r_{s}) \\
&= 8 F_{2,1,1}(A)_{(ss)(tt)} (k_{ss} r_{s} \otimes (\nabla_{s} k_{tt})) + 16 F_{2,1,1}(A)_{(ss)(tt)} (k_{ss} r_{s} \otimes \nabla_{s} k_{ss}),
\end{align*}

(4.35)

and
\[
\sum_{s,t,l_1,\ldots,l_4} F_{1,2,1}(A)_{(l_1l_2)(l_3l_4)} \left( r_{l_1} \otimes k_{l_2} (\nabla_{s} k_{l_3l_4}) \right)
\]

\begin{align*}
&= 8 \sum_{s,t} F_{1,2,1}(A)_{(ss)(tt)} (r_{s} \otimes k_{ss} (\nabla_{s} k_{tt})) + 16 \sum_{s,t} F_{1,2,1}(A)_{(ss)(ss)} (r_{s} \otimes k_{ss} (\nabla_{s} k_{ss})).
\end{align*}

(4.36)

**Lemma 4.10.** Similar to the previous lemma, we collect terms involving symmetrization over \( \{l_1, \ldots, l_6\} \) which leads to \( 6! = 720 = 384 + 96 \times 3 + 48 \) terms as shown below:

\[
\sum_{s,t,l_1,\ldots,l_6} F_{3,1,1}(A)_{(l_1l_2l_3l_4l_5l_6)} k_{s,l_1} k_{t,l_2} (\nabla_{s} k_{l_3l_4l_5l_6}) \otimes (\nabla_{t} k_{l_1l_2l_3l_4l_5l_6})
\]

\begin{align*}
&= 384 \sum_{s} F_{3,1,1}(A)_{(ss)(ss)(ss)} \left( k_{ss}^{2} (\nabla_{s} k_{ss}) \otimes (\nabla_{s} k_{ss}) \right) \\
&+ 96 \sum_{s,t} F_{3,1,1}(A)_{(ss)(tt)} (k_{ss}^{2} (\nabla_{s} k_{tt}) \otimes (\nabla_{s} k_{tt})) \\
&+ 96 \sum_{s,t} F_{3,1,1}(A)_{(ss)(ss)} \left( k_{ss}^{2} (\nabla_{s} k_{ss}) \otimes (\nabla_{s} k_{ss}) + k_{ss}^{2} (\nabla_{s} k_{tt}) \otimes (\nabla_{s} k_{ss}) \right) \\
&+ 48 \sum_{s,t,l} F_{3,1,1}(A)_{(ss)(tt)(tt)} \left( k_{ss}^{2} (\nabla_{s} k_{tt}) \otimes (\nabla_{s} k_{tt}) \right),
\end{align*}

(4.37)

and
\[
\sum_{s,t,l_1,\ldots,l_6} F_{2,2,1}(A)_{(l_1l_2l_3l_4l_5l_6)} k_{s,l_1} (\nabla_{s} k_{l_2l_3l_4l_5l_6}) \otimes k_{t,l_2} (\nabla_{t} k_{l_1l_3l_4l_5l_6})
\]

\begin{align*}
&= 384 \sum_{s} F_{2,2,1}(A)_{(ss)(ss)(ss)} (k_{ss} (\nabla_{s} k_{ss}) \otimes k_{ss} (\nabla_{s} k_{ss})) \\
&+ 96 \sum_{s,t} F_{2,2,1}(A)_{(ss)(tt)} (k_{ss} (\nabla_{s} k_{tt}) \otimes k_{ss} (\nabla_{s} k_{tt})) \\
&+ 96 \sum_{s,t} F_{2,2,1}(A)_{(ss)(ss)} \left( k_{ss} (\nabla_{s} k_{ss}) \otimes k_{ss} (\nabla_{s} k_{ss}) + k_{ss} (\nabla_{s} k_{tt}) \otimes k_{ss} (\nabla_{s} k_{ss}) \right) \\
&+ 48 \sum_{s,t,l} F_{2,2,1}(A)_{(ss)(tt)(tt)} (k_{ss} (\nabla_{s} k_{tt}) \otimes k_{ss} (\nabla_{s} k_{tt})).
\end{align*}

(4.38)
From Eq. (4.32), we see that

$$v_2(P)_{1,3,1} = \sum_{s,l} 32 F_{3,1}(A)_{(s,s)(l,l)} \left( k_{s,s}^2 (\nabla_{s,s}^2 k_{l,l}) \right) + 64 \sum_s F_{3,1}(A)_{(s,s)(s,s)} \left( k_{s,s}^2 (\nabla_{s,s}^2 k_{s,s}) \right)$$

$$= (\det A)^{-\frac{1}{2}} \left( \sum_{s,l} F_{3,1}(\mathbf{z})_{s,l} \left( \frac{k_s}{k_l} (\nabla_{s,s}^2 k_l) \right) + 2 \sum_s F_{3,1}(\mathbf{z})_s \left( (\nabla_{s,s}^2 k_s) \right) \right).$$

By adding up Eqs. (4.39), (4.31) and (4.30), we have finished the computation of $v_2(P)_1$.

**Theorem 4.11.** We group the summands of $v_2(P)_1$ according to the index

$$\alpha \in \{(1,1),(2,1),(3,1)\}$$

in $F_{\alpha}(A)$ as below:

$$(\det A)^{\frac{1}{2}} (v_2(P)_{1,1,1}) = -F_{1,1}(A)_0 (p_0),$$

$$(\det A)^{\frac{1}{2}} (v_2(P)_{1,2,1}) = -\frac{1}{2} \sum_{s,l} F_{2,1}(\mathbf{z})_{s,l} \left( \frac{k_s}{k_l} (\nabla_{s,s}^2 k_l) \right) - \sum_s F_{2,1}(\mathbf{z})_s \left( i \nabla_s r_s \right),$$

$$(\det A)^{\frac{1}{2}} (v_2(P)_{1,3,1}) = \sum_{s,l} F_{3,1}(\mathbf{z})_{s,l} \left( \frac{k_s}{k_l} (\nabla_{s,s}^2 k_l) \right) + 2 \sum_s F_{3,1}(\mathbf{z})_s \left( (\nabla_{s,s}^2 k_s) \right),$$

where summation $s, l$ run from 1 to $m$.

We can reorganize the sum in terms of the differential expressions on which $F_{\alpha}(A)$ acts.

**Theorem 4.12.** In the diagonal case, $v_2(P)_1$ is given by:

$$(\det A)^{\frac{1}{2}} v_2(P)_1 = \sum_{s,l} \left( F_{3,1}(\mathbf{z})_{s,l} - \frac{1}{2} F_{2,1}(\mathbf{z})_{s,l} \left( \frac{k_s}{k_l} (\nabla_{s,s}^2 k_l) \right) \right)$$

$$+ \sum_s 2 F_{3,1}(\mathbf{z})_s \left( (\nabla_{s,s}^2 k_s) \right) - F_{2,1}(\mathbf{z})_s \left( i \nabla_s r_s \right) - F_{1,1}(\mathbf{z})_0 (p_0),$$

where summation $s, l$ run from 1 to $m$.

**Theorem 4.13.** We group the summands of $v_2(P)_{11}$ according to the index

$$\alpha \in \{(1,1,1),(2,1,1),(1,2,1),(3,1,1),(2,2,1)\}$$

in $F_{\alpha}(A)$ as below:

$$(\det A)^{\frac{1}{2}} v_2(P)_{11,1,1,1} = -\frac{i}{2} \sum_{s,l} F_{1,1,1}(\mathbf{z})_{s,l} \left( k_{l,l}^{-1} r_s \otimes (\nabla_s k_l) \right) + \frac{1}{2} \sum_s F_{1,1,1}(\mathbf{z})_{s,s} \left( k_{s,s}^{-1} r_s \otimes r_s \right).$$
and
(4.41) \[ (\det \mathbf{A})^{\frac{1}{2}} (v_2(P)|_{s,1,1}) \]
\[ = \sum_{s,t} F_{2,1,1}(\mathbf{z}), k_s^{-1}(\nabla_s k_s) \otimes (\nabla_s k_t) + 2 \sum_{s} F_{2,1,1}(\mathbf{z})_s, k_s^{-1}(\nabla_s k_s) \otimes (\nabla_s k_s) \]
\[ + \sum_{s,t} F_{2,1,1}(\mathbf{z})_{s,t}, \left( \frac{k_s}{2k_s k_t} (\nabla_s k_s) \otimes (\nabla_s k_t) \right) + \sum_{s} F_{2,1,1}(\mathbf{z})_s, \left( \frac{k_s}{k_t^2} (\nabla_s k_s) \otimes (\nabla_s k_s) \right) \]
\[ + i \sum_{s,t} F_{2,1,1}(\mathbf{z})_{s,t}, \left( \frac{1}{2} k_s^{-1} r_s \otimes (\nabla_s k_t) \right) + i \sum_{s} F_{2,1,1}(\mathbf{z})_s, \left( k_s^{-1} r_s \otimes (\nabla_s k_s) \right) \]
\[ + i \sum_{s,t} F_{2,1,1}(\mathbf{z})_{s,t}, \left( \frac{1}{2} k_s^{-1} r_s \otimes (\nabla_s k_t) \right) + i \sum_{s} F_{2,1,1}(\mathbf{z})_s, \left( k_s^{-1} r_s \otimes (\nabla_s k_s) \right) , \]

and
(4.42) \[ (\det \mathbf{A})^{\frac{1}{2}} (v_2(P)|_{s,1,2}) \]
\[ = i F_{1,2,1}(\mathbf{z}), \left( \frac{k_s^{(1)}}{2k_s k_t} r_s \otimes (\nabla_s k_t) \right) + i F_{1,2,1}(\mathbf{z})_s, \left( \frac{k_s^{(1)}}{k_t^2} r_s \otimes (\nabla_s k_s) \right) , \]

and
(4.43) \[ - (\det \mathbf{A})^{\frac{1}{2}} v_2(P)|_{s,1,1} \]
\[ = 8 \sum_{s} F_{3,1,1}(\mathbf{z})_{s,s,s}, \left( k_s^{-1}(\nabla_s k_s) \otimes (\nabla_s k_s) \right) + 2 \sum_{s,t} F_{3,1,1}(\mathbf{z})_{s,t,t}, \left( \frac{k_s}{k_t^2} (\nabla_s k_t) \otimes (\nabla_s k_t) \right) \]
\[ + 2 \sum_{s,t} F_{3,1,1}(\mathbf{z})_{s,s,t}, \left( k_t^{-1}(\nabla_s k_s) \otimes (\nabla_s k_t) + (\nabla_s k_t) \otimes (\nabla_s k_s) \right) \]
\[ + \sum_{s,t} F_{3,1,1}(\mathbf{z})_{s,t,t}, \left( \frac{k_s}{k_t k_t} (\nabla_s k_s) \otimes (\nabla_s k_t) \right) \]

and
(4.44) \[ - (\det \mathbf{A})^{\frac{1}{2}} v_2(P)|_{s,2,1} \]
\[ = 4 \sum_{s} F_{2,2,1}(\mathbf{z})_{s,s,s}, \left( \frac{k_s^{(1)}}{k_t^2} (\nabla_s k_s) \otimes k_t(\nabla_s k_s) \right) + \sum_{s,t} F_{2,2,1}(\mathbf{z})_{s,t,t}, \left( \frac{k_s^{(1)}}{k_t^2} (\nabla_s k_t) \otimes k_s(\nabla_s k_t) \right) \]
\[ + \sum_{s,t} F_{2,2,1}(\mathbf{z})_{s,s,t}, \left( \frac{k_s^{(1)}}{k_t} (\nabla_s k_s) \otimes k_s(\nabla_s k_t) + k_s(\nabla_s k_t) \otimes k_s(\nabla_s k_s) \right) \]
\[ + \frac{1}{2} \sum_{s,t} F_{2,2,1}(\mathbf{z})_{s,t,t}, \left( \frac{k_s^{(1)}}{k_t k_t} (\nabla_s k_s) \otimes k_s(\nabla_s k_t) \right) . \]

**Proof.** By collecting terms in Eqs. (4.33), (4.34), (4.35), we obtain $v_2(P)|_{s,1,1}$. For $v_2(P)|_{s,1,2,1}$, we need (4.36). The remaining terms $v_2(P)|_{s,3,1,1}$ and $v_2(P)|_{s,2,2,1}$ are computed in Lemma 4.10. \[\square\]
Again, let us group the terms apropos of the differential expressions that are indexed in the following way:

\[
L_s^{(1)} = (\nabla_s k_s) \otimes (\nabla_t k_s), \quad L_{s,t}^{(2)} = (\nabla_s k_t) \otimes (\nabla_s k_t), \quad L_{s,t}^{(3)} = (\nabla_s k_t) \otimes (\nabla_t k_t) \\
L_{s,t}^{(4)} = (\nabla_s k_t) \otimes (\nabla_s k_s), \quad L_{s,t,l}^{(5)} = (\nabla_s k_t) \otimes (\nabla_t k_l)
\]

and

\[
L_s^{(6)} = i(\nabla_s k_s) \otimes r_s, \quad L_{s,t}^{(7)} = i(\nabla_s k_t) \otimes r_s \\
L_{s,t}^{(8)} = i r_s \otimes (\nabla_s k_s), \quad L_{s,t,t}^{(9)} = i r_s \otimes (\nabla_s k_t), \quad L_{s,t,l}^{(10)} = r_s \otimes r_s.
\]

We also make use of the substitution in Eq. (3.24) \(k_s^{(1)} = k_s y_s^{(1)} = k_s (1 - z_s^{(1)})\) to move all \(k\)-factors to the very left.

**Theorem 4.14.** In the diagonal case, \(v_2(P)_{\|L}\) consists of two parts:

\[
\begin{align*}
(\text{det} A)^{\frac{1}{2}} v_2(P)_{\|L} &= \sum_s k_s^{-1} (-8F_{3,1,1}(z)_{s,s,s} - 4(1 - z_s^{(1)})F_{2,2,1}(z)_{s,s,s} + 2F_{2,1,1}(z)_{s,s}) (L_s^{(1)}) \\
&\quad + \sum_{s,t} k_s^{-1} k_t^{-1} (-2F_{3,1,1}(z)_{s,t,s} - (1 - z_s^{(1)})F_{2,2,1}(z)_{s,t,s} + F_{2,1,1}(z)_{s,t,s} + F_{2,1,1}(z)_{s,t}) (L_{s,t}^{(2)}) \\
&\quad + \sum_{s,t} k_s^{-1} k_t^{-1} (-2F_{3,1,1}(z)_{s,t,s} - (1 - z_s^{(1)})F_{2,2,1}(z)_{s,t,s} + F_{2,1,1}(z)_{s,t}) (L_{s,t}^{(3)}) \\
&\quad + \sum_{s,t} k_s^{-1} k_t^{-1} (-2F_{3,1,1}(z)_{s,t,s} - (1 - z_s^{(1)})F_{2,2,1}(z)_{s,t,s} + F_{2,1,1}(z)_{s,t}) (L_{s,t}^{(4)}) \\
&\quad + \sum_{s,t,l} k_s^{-1} k_t^{-1} k_l^{-1} (-F_{3,1,1}(z)_{s,t,l} - 1/2 (1 - z_s^{(1)})F_{2,2,1}(z)_{s,t,l} + 1/2 F_{2,1,1}(z)_{s,t,l}) (L_{s,t,l}^{(5)})
\end{align*}
\]

and contribution from \(p_1(\xi)\) (the symbol of first order term of \(P\) in Eq. (2.6)) is grouped as below:

\[
\begin{align*}
(\text{det} A)^{\frac{1}{2}} v_2(P)_{\|L} &= \sum_s k_s^{-1} F_{2,1,1}(z)_{s,s,s} (L_s^{(6)}) + 1/2 \sum_{s,t} k_t^{-1} F_{2,1,1}(z)_{s,s,t} (L_{s,t}^{(7)}) + 1/2 \sum_s k_s^{-1} F_{1,1,1}(z)_{s,s} (L_{s}^{(10)}) \\
&\quad + \sum_s k_s^{-1} F_{2,1,1}(z)_{s,s,s} + (1 - z_s^{(1)})F_{1,2,1}(z)_{s,s,s} (L_s^{(6)}) \\
&\quad + \sum_{s,t} 1/2 k_t^{-1} F_{2,1,1}(z)_{s,t,s} + (1 - z_s^{(1)})F_{1,2,1}(z)_{s,t,s} - F_{1,1,1}(z)_{s,s} (L_{s,t}^{(9)}) \\
&\quad + \sum_{s,t,l} 1/2 k_l^{-1} F_{2,1,1}(z)_{s,t,l} + (1 - z_s^{(1)})F_{1,2,1}(z)_{s,t,l} - F_{1,1,1}(z)_{s,t,l} (L_{s,t,l}^{(8)})
\end{align*}
\]
As before, we put $x$.

Their heat coefficients are related in a similar way.

**Lemma 5.1.** The two sets of heat coefficients agree up to a conjugation of the Weyl factor:

$$v_j(\Delta_\varphi) = y^{\frac{1}{2}} (v_j(\Delta_k)), \quad j = 0, 1, 2, \ldots$$

In this case, the metrics are parametrized by only one noncommutative coordinate $k = e^h$. As before, we put $x := x_h$ (resp. $y := y_h$):

$$x_h = [\bullet, h], \quad y_h = e^{x_h} = k^{-1}(\bullet)k : C^\infty(\mathbb{T}_\theta^n) \to C^\infty(\mathbb{T}_\theta^n)$$

and the partial versions $x^{(l)}, y^{(l)} \in L(C^\infty(\mathbb{T}_\theta^n) \otimes^n, C^\infty(\mathbb{T}_\theta^n))$, $l = 1, \ldots, n$. The reduction process for the rearrangement operators

$$F(A)(t_1, t_2, \ldots, t_{2N}, t'_1, \ldots, t'_{2N}) \xrightarrow{A = \text{diag}(k_1, \ldots, k_m)} F(A)_{s_1, \ldots, s_N} \xrightarrow{k_1 = \cdots = k_m = k} H_A(z^{(1)}, \ldots, z^{(n)})$$

has been explained in §3.5 where $z = (z^{(1)}, \ldots, z^{(n)})$ is a special case of the change of variable in Eq. (3.24):

$$z^{(l)} = 1 - y^{(1)} \cdots y^{(l)}.$$

The spectral functions $H_a(z_1, \ldots, z_n)$ differs from $H_a(z_1, \ldots, z_n)$ derived in [Liu18a, Liu18b] by a Gamma-factor (see Eq. (3.30)). For the differential expressions listed in Eqs. (4.45) and (4.46), all of them are reduced to one of the two types below (after summing over $s$ or $s, t$):

$$\text{Tr} (\nabla^2 k) := \sum_{s=1}^{m} \nabla_s^2 k = -\Delta k, \quad \text{Tr} ((\nabla k)(\nabla k)) := \sum_{s=1}^{m} (\nabla_s k) \otimes (\nabla_s k).$$

In [Liu18c, Liu17] and [Liu18a, Liu18b], a conjugation trick was applied which yields substantial simplification to the computation. Namely, one starts with another Laplacian

$$\Delta_k := k \Delta = k^{\frac{1}{2}} \Delta_\varphi k^{\frac{-1}{2}} = y^{-\frac{1}{2}} (\Delta_\varphi).$$

Their heat coefficients are related in a similar way.

**Lemma 5.1.** The two sets of heat coefficients agree up to a conjugation of the Weyl factor:

$$v_j(\Delta_k) = y^{\frac{1}{2}} (v_j(\Delta_\varphi)), \quad j = 0, 1, 2, \ldots.$$
Proof. It follows from the fact that the modular operator $\Delta$ commutes with the exponential. More precisely, we have $\forall f \in C^\infty(T^m_\theta)$,
\[
\text{Tr}(f e^{-t\Delta}) = \text{Tr}(f y^{1/2} e^{-t\Delta y}) = \text{Tr}(y^{-1/2}(f) e^{-t\Delta y}).
\]
We see, by comparing the asymptotic expansion of two sides, that for $j = 0, 1, 2, \ldots$,
\[
\phi_0\left(f v_j(\Delta \varphi)\right) = \phi_0\left(y^{-1/2}(f) v_j(\Delta k)\right) = \phi_0\left(f y^{1/2}(v_j(\Delta k))\right).
\]

The Laplacian $\Delta_k$ is much easier to handle since the symbol contains only the leading term. As we can see later, §5.3 consists of the exact extra work if one attacks the heat asymptotic of $\Delta_\varphi$ directly.

5.2. Example I: $\Delta_k = k\Delta$. For $\Delta_k = k\Delta$, the symbol $\sigma(\Delta_k) = p_2$ contains only the leading term:
\[
p_2^{\Delta_k}(\xi) = k |\xi|^2, \quad p_1^{\Delta_k}(\xi) = p_0^{\Delta_k}(\xi) = 0.
\]

Proposition 5.2. The second heat coefficient consists of two parts $v_2(\Delta_k) = v_2(\Delta_k)_I + v_2(\Delta_k)_II$:
\[
v_2(\Delta_k)_I = k^{-\frac{m}{2}} G_{\Delta_k, I}(z)(\text{Tr}(\nabla^2 k)),
\]
\[
v_2(\Delta_k)_II = v_2(\Delta_k)_II = k^{-\frac{m}{2} - 1} G_{\Delta_k, II}(z^{(1)}, z^{(2)})(\text{Tr}(\nabla k(\nabla k))),
\]
where operators $z$ and $z^{(1)}, z^{(2)}$ are defined in Eq. (5.2) and the spectral functions are given in terms of hypergeometric functions as below:

\[
(5.5) \quad G_{\Delta_k, I}(z) = (m + 2) H_{3,1}(z; m) - m \frac{2}{2} H_{2,1}(z; m),
\]
and with $\bar{z} = (z_1, z_2)$,

\[
(5.6) \quad G_{\Delta_k, II}(\bar{z}) = -(m^2 + 6m + 8)\left[H_{3,1,1}(\bar{z}; m) + \frac{1}{2}(1 - z_1)H_{2,2,1}(\bar{z}; m)\right]
\]
\[
+ \frac{(m^2 + 4m + 4)}{2} H_{2,1,1}(\bar{z}; m).
\]

Proof. We shall apply reduction rules like:
\[
k_s \to k, \quad F_\alpha(z)_{s,t,l} \to H_\alpha(z),
\]
to all terms appeared in Theorems 4.12 and 4.14. For instance, a summand in Eq. 4.43 works out as below:
\[
\sum_{s,t,l} \frac{k_s}{k_t k_l} F_{3,1,1}(z)_{s,t,l} ((\nabla_s k_t) \otimes (\nabla_s k_l)) \to m^2 k^{-1} H_{3,1,1}(\bar{z}; m)(\text{Tr}(\nabla k(\nabla k))),
\]
where $\bar{z} = (z^{(1)}, z^{(2)})$. Notice that summing over $s$ yields $\text{Tr}(\nabla k(\nabla k))$ while summing over $t, l$ produces $m^2$ copies of the same term. By repeating the process, we obtain the reduction of
Eqs. (4.41), (4.43) and (4.44):

\[ v_2(\Delta_k)_{\Pi,3,1,1} \rightarrow - \left( m^2 + 6m + 8 \right) H_{3,1,1}(\vec{z}; m) (\text{Tr}(\nabla k(\nabla k))) \]

\[ v_2(\Delta_k)_{\Pi,2,2,1} \rightarrow - \frac{1}{2} \left( m^2 + 6m + 8 \right) (1 - z^{(1)}) H_{2,2,1}(\vec{z}; m) (\text{Tr}(\nabla k(\nabla k))) \]

\[ v_2(\Delta_k)_{\Pi,2,1,1} \rightarrow \frac{1}{2} \left( m^2 + 4m + 4 \right) H_{2,1,1}(\vec{z}; m) (\text{Tr}(\nabla k(\nabla k))). \]

They constitute all the non-zero contributes to \( v_2(\Delta_k)_{\Pi} \), hence we have proved Eq. (5.6). For Eq. (5.5), calculation is similar, terms in Theorem 4.11 are turned into:

\[ v_2(\Delta_k)_{\Pi,2,1} \rightarrow \frac{1}{2} m H_{2,1}(\vec{z}; m) (\text{Tr}(\nabla^2 k)), \]

\[ v_2(\Delta_k)_{\Pi,3,1} \rightarrow (m + 2) H_{3,1}(\vec{z}; m) (\text{Tr}(\nabla^2 k)). \]

\[ \boxed{} \]

In order to recover the exact formulas in \([\text{Liu18a, Liu18b}]\), we recall constants from Eqs. (3.30) and (3.31):

\[ \frac{H_{3,1,1}(\vec{z}; m)}{H_{3,1,1}(\vec{z}; m)} = \frac{H_{2,2,1}(\vec{z}; m)}{H_{2,2,1}(\vec{z}; m)} = C_m \left( \frac{m}{m^2 + 2} \right) = C_m \left( \frac{8}{m + 4}(m + 2) \right). \]

and

\[ \frac{H_{2,1,1}(\vec{z}; m)}{H_{2,1,1}(\vec{z}; m)} = C_m \left( \frac{m}{m^2 + 1} \right) = C_m \left( \frac{4}{m + 2} \right). \]

Therefore:

\[ C_m^{-1} G_{\Delta_k,\Pi}(z) = - \frac{8}{m} \left( H_{3,1,1}(\vec{z}; m) + \frac{1}{2} (1 - z^{(1)}) H_{2,2,1}(\vec{z}; m) \right) + \left( \frac{4}{m + 2} \right) H_{2,1,1}(\vec{z}; m). \]

The right hand side is exactly \( H_{\Delta_k}(\vec{z}; m) \) given in \([\text{Liu18b, Prop. 4.1}]\). Similarly, we have

\[ \frac{H_{3,1}(\vec{z}; m)}{H_{3,1}(\vec{z}; m)} = C_m \frac{4}{(m + 2)m}, \quad \frac{H_{2,1}(\vec{z}; m)}{H_{2,1}(\vec{z}; m)} = C_m \frac{2}{m} \]

thus

\[ C_m^{-1} G_{\Delta_k,1}(z) = - H_{2,1}(z; m) + \frac{4}{m} H_{3,1}(z; m), \]

which equals \( K_{\Delta_k}(z; m) \) in \([\text{Liu18b, Prop. 4.1}]\).
5.3. **Example II**: $\Delta \phi = k^{1/2} \Delta k^{1/2}$. We start with Lemma 5.3, the “chain rule” with respect to the change of variable $k^{1/2} \rightarrow k$ and then compute the symbol of $\Delta \phi$ in Lemma 5.4.

**Lemma 5.3.** Recall

\[
\delta_s(k^{1/2}) = k^{-1/2} G_{\text{pow}}(y; 1/2)(\delta_s(k)),
\]

\[
\delta_z(k^{1/2}) = k^{-1/2} G_{\text{pow}}(y; 1/2)(\delta_z^2(k)) + 2k^{-3/2} G_{\text{pow}}^{(1,1)}(y^{(1)}, y^{(2)}; 1/2)(\delta_z(s) \otimes \delta_z(k)),
\]

where $G_{\text{pow}}$ is obtained by applying divided differences to the power functions $u^j$ with $j \in \mathbb{R}$:

\[
G_{\text{pow}}^{(1)}(y; j) = u^j[1, y] = \frac{y^j - 1}{y - 1},
\]

\[
G_{\text{pow}}^{(1,1)}(y_1, y_2; j) = u^j[1, y_1, y_2] = G_{\text{pow}}^{(1)}(u; j)[y_1, y_2] = \frac{G_{\text{pow}}^{(1)}(y_1, y_2; j) - G_{\text{pow}}^{(1)}(y_1; j)}{y_2 - y_1}.
\]

We will fix $j = 1/2$ from now on and freely use the abbreviations $G_{\text{pow}}^{(1)}(y) := G_{\text{pow}}^{(1)}(y; 1/2)$ and $G_{\text{pow}}^{(1,1)}(y_1, y_2) = G_{\text{pow}}^{(1,1)}(y_1, y_2; 1/2)$ in the rest of the computation.

**Proof.** See [Liu18b Lemma 2.13].

**Lemma 5.4.** The symbol of $\Delta \phi = k^{1/2} \Delta k^{1/2}$ is given by

\[
\sigma(\Delta \phi) = k |\xi|^2 + 2k^{1/2} \delta_s(k^{1/2}) \xi_s + k^{1/2} \delta_z^2(k^{1/2}).
\]

Following the notations in Eq. (2.4), we have: the leading term agrees with the previous one, $p_2^\Delta = p_2^\Delta$, and the linear term

\[
p_1^\Delta(\xi) = \sum_{s=1}^{m} 2k^{1/2} \delta_s(k^{1/2}) \xi_s,
\]

with $r_s^\Delta(\xi) = 2k^{1/2} \delta_z(k^{1/2})$.

and the constant (in $\xi$) term $p_0^\Delta = \sum_{s=1}^{m} k^{1/2} \delta_z^2(k^{1/2})$.

**Remark.** By taking Lemma 5.3 into account and $\delta_s = -i \nabla_s$, also Eq. (5.3), we can rewrite every thing in terms of $\nabla k$ and $\nabla^2 k$:

\[
(5.7) \quad r_s^\Delta(\xi) = -2i k^{1/2} \nabla_s(k^{1/2}) = -2i G_{\text{pow}}^{(1)}(y)(\nabla_s k), \quad s = 1, \ldots, m.
\]

and

\[
(5.8) \quad p_0^\Delta = -\sum_{s=1}^{m} k^{1/2} \nabla_s^2(k^{1/2}) = G_{\text{pow}}^{(1)}(y) [\text{Tr}(\nabla^2 k)] + k^{-1} 2G_{\text{pow}}^{(1,1)}(y_1, y_2)(\text{Tr}(\nabla k)(\nabla^2 k)).
\]
Proof. Notice that \( \forall a \in C^\infty(\mathbb{T}^m_{\theta}) \), we have \([\delta_s, a] = \delta_s(a), \) thus:
\[
\left[ \delta_s^2, k^2 \right] = \delta_s[\delta_s, k^2] + [\delta_s, k^2] \delta_s = \left[ \delta_s, [\delta_s, k^2] \right] + 2[\delta_s, k^2] \delta_s = \delta_s^2(k^2) + 2\delta_s(k^2) \delta_s.
\]
We are ready to put \( \Delta_{\varphi} \) into the form of Eq. (2.6) with the help of commutators:
\[
\Delta_{\varphi} = k \Delta + k^\frac{1}{2} [\Delta, k^\frac{1}{2}] = k \Delta + k^\frac{1}{2} \sum_{s=1}^{m} \left[ \delta_s^2, k^\frac{1}{2} \right]
\]
\[
= k \sum_{s=1}^{m} \delta_s^2 + \sum_{s=1}^{m} \left( 2\delta_s(k^\frac{1}{2}) \delta_s + \delta_s^2(k^\frac{1}{2}) \right).
\]
The symbol follows immediately by replacing the differential operators \( \delta_s \rightarrow \xi_s \). Another method is to make use of the \( \ast \)-product. Notice that \( D^j |\xi|^2 = 0 \) for all \( j > 2 \). There are only three non-zero terms for \( |\xi|^2 \ast k^\frac{1}{2} \) which are given in Eq. (4.9):
\[
\sigma(\Delta k^\frac{1}{2}) = \sum_{j=0}^{2} a_j(|\xi|^2, k^\frac{1}{2}) = k^\frac{1}{2} |\xi|^2 - 2I \sum_{s=1}^{m} \nabla_s(k^\frac{1}{2}) \xi_s - \sum_{s=1}^{m} \nabla_s^2(k^\frac{1}{2}).
\]
□

Here comes the key result.

**Proposition 5.5.** The second heat coefficient of \( \Delta_{\varphi} \) is of the form \( \nu_2(\Delta_{\varphi}) = \nu_2(\Delta_{\varphi})_I + \nu_2(\Delta_{\varphi})_II \) with
\[
\nu_2(\Delta_{\varphi})_I = G_{\Delta_{\varphi}, I}(z)(\text{Tr}(\nabla^2 k)), \quad \nu_2(\Delta_{\varphi})_II = G_{\Delta_{\varphi}, II}(z^{(1)}, z^{(2)})(\text{Tr}((\nabla k)(\nabla k))),
\]
where the two traces are defined in Eq. (5.3) and the spectral functions are given by:
\begin{align*}
\text{(5.9)} & \quad G_{\Delta_{\varphi}, I}(z) = G_{\Delta_{\varphi}, I}(z) + j_{L_1, p_1, p_0}(z), \\
\text{(5.10)} & \quad G_{\Delta_{\varphi}, II}(z_1, z_2) = G_{\Delta_{\varphi}, II}(z_1, z_2) + j_{L_1, p_1, p_0}(z_1, z_2) + j_{II, L^{(0)}, \ldots, L^{(10)}}(z_1, z_2),
\end{align*}
where \( G_{\Delta_{\varphi}, I} \) and \( G_{\Delta_{\varphi}, II} \) is defined in Proposition 5.2. The remaining terms are also spanned by the hypergeometric family \( H_{\alpha}(z; m) \) in (3.29) and \( G_{\text{pow}}^{(1)} \) and \( G_{\text{pow}}^{(2)} \) in Lemma 5.3
\begin{align*}
\text{(5.11)} & \quad j_{L_1, p_1, p_0}^{(2)}(z) = -G_{\text{pow}}^{(2)}(y) \left( 2H_{2,1}(z; m) - H_{1,1}(z; m) \right), \\
\text{(5.12)} & \quad j_{L_1, p_1, p_0}^{(1,1)}(z_1, z_2) = -2 \left( y_1^{-1} G_{\text{pow}}^{(1)}(y_1) G_{\text{pow}}^{(1)}(y_2) + 2G_{\text{pow}}^{(1)}(y_1, y_2) \right) H_{2,1}(z_2; m) \\
& \quad + 2G_{\text{pow}}^{(1)}(y_1, y_2) H_{1,1}(z_2; m),
\end{align*}
and
\begin{align*}
\text{(5.13)} & \quad j_{II, L^{(0)}, \ldots, L^{(10)}}(z_1, z_2) \\
& = \left[ (2 + m) \left( G_{\text{pow}}^{(1)}(y_1) + G_{\text{pow}}^{(1)}(y_2) \right) H_{2,1,1}(z_1, z_2; m) + (2 + m) G_{\text{pow}}^{(1)}(y_1)(1 - z_1) H_{1,2,1}(z_1, z_2; m) \\
& \quad - \left( m G_{\text{pow}}^{(1)}(y_1) + 2G_{\text{pow}}^{(1)}(y_1) G_{\text{pow}}^{(1)}(y_2) \right) H_{1,1,1}(z_1, z_2; m). \right].
\end{align*}
Proof. Since $p_2^\Delta_y = p_2^\Delta_i$, the contribution from the leading term is identical to those (i.e., $G_{\Delta, \mathbf{I}}$ and $G_{\Delta, \mathbf{II}}$) obtained in Proposition 5.2. It remains to count the terms involving $p_1^\Delta_y$ and $p_0^\Delta_y$ in Theorems 4.12 and 4.14 which are $-\sum_s F_{2,1}(\mathbf{z}_{(s)}) (i \nabla_s r_s) - F_{1,1}(\mathbf{z}_0)(p_0)$ and $\nu_2(\Delta_y I_{\mathbf{I}, I_{\mathbf{II}}})$. We packed the tedious computation into the proof of Lemma 5.7 and 5.8 at the end of this section.

We see that the contribution from $p_1^\Delta_y$ and $p_0^\Delta_y$ is proportional to that of the leading term.

**Corollary 5.6.** The following relations hold:

\begin{align}
(y^\frac{1}{2} - 1)G_{\Delta, \mathbf{I}}(z) &= J^{(2)}_{l_1, p_1}(z), \\
((y_1 y_2)^{\frac{1}{2}} - 1)G_{\Delta, \mathbf{II}}(z_1, z_2) &= J^{(1,1)}_{l_1, p_1, p_2}(z_1, z_2) + J^{(2,2)}_{l_1, p_1, p_2}(z_1, z_2)
\end{align}

where $z = 1 - y$ and $z_1 = 1 - y_1, z_2 = 1 - y_1 y_2$ as in Eq. (3.24).

**Proof.** In the previous proposition, we obtain the first version of $G_{\Delta, \mathbf{I}}$ and $G_{\Delta, \mathbf{II}}$ (cf. Eqs 5.9 and 5.10) making use of the general results (Theorems 4.12 and 4.14). On the other hand, Lemma 5.1 states that $\nu_2(\Delta_y) = y^\frac{3}{2}(\nu_2(\Delta_i))$, which implies that their spectral functions agree up to a factor of $y^\frac{3}{2} = (1 - z)^\frac{3}{2}$ (or $(y_1 y_2)^\frac{3}{2} = (1 - z_2)^\frac{3}{2}$):

$$G_{\Delta, \mathbf{I}}(z) = (1 - z)^\frac{3}{2}G_{\Delta, \mathbf{I}}(z), \quad G_{\Delta, \mathbf{II}}(z_1, z_2) = (1 - z_2)^\frac{3}{2}G_{\Delta, \mathbf{II}}(z_1, z_2).$$

The relations follow immediately from comparison. □

**Lemma 5.7.** With $r_s^\Delta_y = -2 i k \frac{1}{2} \nabla_s(k \frac{1}{2})$ and $p_0^\Delta_y = -k \frac{1}{2} \sum_{l=1}^{m} \nabla^2_l(k \frac{1}{2})$, the following sum in Theorem 4.11 becomes:

\[
- \sum_s F_{2,1}(\mathbf{z}_{(s)}) (i \nabla_s r_s) - F_{1,1}(\mathbf{z}_0)(p_0) = J^{(2)}_{l_1, p_1, p_2}(z) [\text{Tr}(\nabla^2 k)] + k^{-1} J^{(1,1)}_{l_1, p_1, p_2}(z_{1}, z_{2}) [\text{Tr}(\nabla k)(\nabla k)]
\]

where

\[
J^{(2)}_{l_1, p_1, p_2}(z) = -G^{(1)}_{\text{pow}}(y) (2 H_{2,1}(z; m) - H_{1,1}(z; m))
\]

and

\[
J^{(1,1)}_{l_1, p_1, p_2}(z_1, z_2) = -2 \left( y_1^{-\frac{1}{2}} G^{(1)}_{\text{pow}}(y_1) G^{(1)}_{\text{pow}}(y_2) + 2 G^{(1,1)}_{\text{pow}}(y_1, y_2) \right) H_{2,1}(z_2; m) + 2 G^{(1,1)}_{\text{pow}}(y_1, y_2) H_{1,1}(z_2; m).
\]

**Proof.** By replacing $F_{a, b}(\mathbf{z}_{(s)})$ with $H_{a, b}(\mathbf{z})$ and plugging in $r_1^\Delta_y$ and $p_0^\Delta_y$, we see that

\[
\sum_s F_{2,1}(\mathbf{z}_{(s)}) (i \nabla_s r_s) + F_{1,1}(\mathbf{z}_0)(p_0) = (2 H_{2,1}(z; m) - H_{1,1}(z; m)) \left( \sum_s k^\frac{1}{2} \nabla^2 s k^\frac{1}{2} \right) + 2 H_{2,1}(z; m) (\nabla s k^\frac{1}{2})^2.
\]
The results follow from the changing the derivatives (according to Lemma 5.3) \( \nabla_s k^\frac{1}{2} \) and \( \nabla_s^2 k^\frac{1}{2} \) to \( \nabla_s k \) and \( \nabla_s^2 k \):

\[
(\nabla_s k^\frac{1}{2})^2 = \left( k^{-\frac{1}{2}} G_{\text{pow}}^{(1)}(y)(\nabla_s k) \right)^2 = k^{-1}(y^{(1)})^{-\frac{1}{2}} G_{\text{pow}}^{(1)}(y^{(1)}) G_{\text{pow}}^{(1)}(y^{(2)})(\nabla_s k \otimes \nabla_s k),
\]

\[
k^\frac{1}{2} \nabla_s^2 k^\frac{1}{2} = k^{-1} \left[ G_{\text{pow}}^{(1)}(y) \left( \nabla_s^2 k + 2 G_{\text{pow}}^{(1)}(y^{(1)}, y^{(2)}) \right) \right] (\nabla_s k \otimes (\nabla_s k)).
\]

\( \square \)

\textbf{Lemma 5.8}. In Theorem 4.14, terms involving \( p_1^{\Delta^j}(\xi) \) give rise to the following contribution in the conformal case:

\[
\nu_2(\Delta^j \varphi, \mathbf{h}, L^{(6)}, L^{(10)}) = J_{\mathbf{h}, L^{(6)}, L^{(10)}(z_1, z_2)}(\text{Tr}(\nabla k(\nabla k)))
\]

where the two variable function \( J_{\mathbf{h}, L^{(6)}, L^{(10)}(z_1, z_2)} \) is given by:

\[
J_{\mathbf{h}, L^{(6)}, L^{(10)}(z_1, z_2)} = \left[ (2 + m)(G_{\text{pow}}^{(1)}(y_1) + G_{\text{pow}}^{(1)}(y_2)) \right] H_{2,1,1}(z_1, z_2; m) + (2 + m)G_{\text{pow}}^{(1)}(y_1)(1 - z_1)H_{1,2,1}(z_1, z_2; m).
\]

Proof. With \( r_s^{\Delta^j} = -2i G_{\text{pow}}^{(1)}(y)(\nabla_s k) \), we first turn \( L_s^{(6)} \) to \( L_s^{(10)} \) in Eq. (4.46) into \( \text{Tr}(\nabla k(\nabla k)) \):

\[
\sum_{s=1}^{m} L_{s}^{(10)} = -4 G_{\text{pow}}^{(1)}(y^{(1)}) G_{\text{pow}}^{(1)}(y^{(2)}) \text{Tr}(\nabla k(\nabla k))
\]

\[
\sum_{s=1}^{m} L_{s}^{(6)} = 2 G_{\text{pow}}^{(1)}(y^{(1)}) \text{Tr}(\nabla k(\nabla k)), \quad \sum_{s,l=1}^{m} L_{s}^{(7)} = 2m G_{\text{pow}}^{(1)}(y^{(1)}) \text{Tr}(\nabla k(\nabla k)),
\]

\[
\sum_{s=1}^{m} L_{s}^{(8)} = 2 G_{\text{pow}}^{(1)}(y^{(2)}) \text{Tr}(\nabla k(\nabla k)), \quad \sum_{s,l=1}^{m} L_{s}^{(9)} = 2m G_{\text{pow}}^{(1)}(y^{(2)}) \text{Tr}(\nabla k(\nabla k)).
\]

Then we repeat reduction the process used in the proof of Proposition 5.2 to complete the calculation. \( \square \)

6. Verification of the Functional Relations

Last but not least, we provide some conceptual validation for the our lengthy computation by carefully examining the relations in Eqs. 5.14 and 5.15 bases on their explicit expressions given in terms of \( G_{\text{pow}}^{(1)} \), \( G_{\text{pow}}^{(1,1)} \) and the hypergeometric family \( H_{\alpha}(\bar{z}; m) \) (cf. Eqs. 5.5, 5.6 and 5.11 to 5.13).  

6.1. Preparations. First of all, let us switch to the hypergeometric family \( H_{\alpha}(\bar{z}; m) \) used in [Liu18b §5]:

\[
H_{\alpha}(\bar{z}; m) = \frac{1}{\Gamma(\sum_{j=0}^{n} \alpha_j + \frac{m}{2} - 2)} H_{\alpha}(\bar{z}; m), \quad \alpha = (\alpha_0, \ldots, \alpha_n),
\]
so that we can freely quote formulas listed there without modifications. The variables \(\{z, z_1, z_2\}\) always denote the following change of variable of \(\{y, y_1, y_2\}\):
\[
z = 1 - y, \quad z_1 = 1 - y_1, \quad z_2 = 1 - y_1 y_2.
\]
For example,
\[
H_{a,b}(z; m) = H_{a,b}(1 - y; m), \quad H_{a,b,c}(z_1, z_2; m) = H_{a,b,c}(1 - y_1, 1 - y_1 y_2; m).
\]
We often drop the arguments and write \(H_{a,b}\) or \(H_{a,b,c}\) when no confusion arises.

The two cyclic transformations are crucial:
\[
\tau_1 : C(\mathbb{R}_+) \to C(\mathbb{R}_+), \quad \tau_1(f)(y) = f(y^{-1})
\]
\[
\tau_2 : C(\mathbb{R}_+) \to C(\mathbb{R}_+), \quad \tau_2(f)(y_1, y_2) = \tilde{f}((y_1 y_2)^{-1}, y_1)
\]
defined on functions in \(y\) and in \((y_1, y_2)\) respectively. We see immediately that \(\tau_1^2 = 1\). For \(\tau_2\), we have \(\tau_2^3 = 1\). Notice that, \(\tau_1\) (resp. \(\tau_2\)) becomes linear fractional transformation with respect to \(z\) (resp. \(z_1, z_2\)). The key feature of the cyclic permutations is the fact that (cf. [Liu18b] Prop. 5.1) they permute the components of \(a\) in \(H_{a}(z; m)\):
\[
\tau_1(H_{a,b}) = (1 - z)^{a+b+m/2-2} H_{b,a},
\]
\[
\tau_2(H_{a,b,c}) = (1 - z_2)^{a+b+c+m/2-2} H_{b,c,a}.
\]
We will also need the formulas of \(\tau_2^2\):
\[
\tau_2^2(H_{a,b,c})(z_1, z_2; m) = (1 - z_1)^{a+b+c+m/2-2} H_{c,a,b}(z_1, z_2; m),
\]
\[
\tau_2^2(H_{a,b}(z_2; m)) = (1 - z_1)^{a+b+m/2-2} H_{b,a}(z_1; m).
\]
Note that Eq. (6.6) follows from Eq. (6.3) with the substitutions in Eq. (6.2):
\[
\tau_2^2(H_{a,b}(z_2; m)) = \tau_2^2(H_{a,b}(1 - y_1 y_2; m)) = H_{a,b}(1 - y_1^{-1}; m)
\]
\[= \tau_1(H_{a,b}(z_1; m)) = (1 - z_1)^{a+b+m/2-1} H_{b,a}(z_1; m).
\]

We now turn to recurrence relations. The goal is to express \(H_{a,b,c}\) and \(H_{a,b}(z_1; m)\) in terms of \(\{H_{1,1,2}, H_{1,2,1}, H_{1,2,2}\}\). For two variable functions, we need:
\[
H_{1,3,1} = \frac{((m+6)(1-y_1)-6)H_{1,2,1} + (m+2)H_{1,1,1} + 2(1-y_1 y_2)(H_{1,1,2} - y_1 H_{1,2,2})}{4(1-y_1)y_1},
\]
\[
H_{1,2,2} = \frac{H_{1,2,1} - H_{1,1,2}}{z_1 - z_2}.
\]

\footnote{In the setting of [Liu18b] §2, the cyclic transformations arise from integration by parts with respect to the modular operator.}
Therefore, \( \forall \eta_{1,1,3}, \eta_{1,2,2} \in \mathbb{C} \):

\[
\eta_{1,1,3} H_{1,1,3} + \eta_{1,2,2} H_{1,2,2} = \left[ \left( \frac{m}{4 y_1} + \frac{1}{y_1 - 1} - \frac{1}{2 y_1(y_2 - 1)} \right) \eta_{1,1,3} + \frac{\eta_{1,2,2}}{y_1(y_2 - 1)} \right] H_{1,2,1} + \left[ \frac{-1}{y_1(y_2 - 1)} \eta_{1,2,2} + \frac{y_2(y_1 y_2 - 1)}{2 y_1(y_1 - 1)(y_2 - 1)} \eta_{1,1,3} \right] H_{1,1,2} + \left[ \frac{-(2 + m)}{4 y_1(y_1 - 1)} \eta_{1,1,3} \right] H_{1,1,1}.
\]

The connection between one and two variable families is given by the divided difference operation:

\[
H_{a,1,1}(z_1, z_2; m) = (z H_{a+1,1}(z; m))[z_1, z_2]_z = \frac{z_1 H_{a+1,1}(z_1; m) - z_2 H_{a+1,1}(z_2; m)}{z_1 - z_2}.
\]

Set \( a = 1 \) and apply \( \partial_{z_1} \) or \( \partial_{z_2} \) on both sides, we see that

\[
H_{1,2,1} = \partial_{z_1} H_{1,1,1} = \frac{H_{1,2}(z_1; m) - H_{1,1,1}}{z_1 - z_2}, \quad H_{1,1,2} = \partial_{z_2} H_{1,1,1} = \frac{H_{1,2}(z_2; m) - H_{1,1,1}}{z_2 - z_1}.
\]

As a consequence, we obtain

\[
H_{1,2}(z_1; m) = (z_1 - z_2)H_{1,2,1} + H_{1,1,1}, \quad H_{1,2}(z_2; m) = (z_2 - z_1)H_{1,1,2} + H_{1,1,1}.
\]

To get \( H_{2,1}(z_1; m) \), notice that

\[
\tau^2_{(2)}(H_{1,2}(z_2; m)) = \tau^2_{(2)}(H_{1,2}(1 - y_1 y_2; m)) = H_{1,2}(1 - y_1^{-1}; m)
\]

\[
= \tau_{(1)}(H_{1,2})(z_1) = y_1^{1 - \frac{m}{2}} H_{2,1}(z_1; m),
\]

thus

\[
H_{2,1}(z_1; m) = y_1^{1 - \frac{m}{2}} \tau^2_{(2)}(H_{1,2}(z_2; m)) = y_1^{1 - \frac{m}{2}} \tau^2_{(2)}((z_2 - z_1)H_{1,1,2} + H_{1,1,1})
\]

\[
= (y_1 y_2 - 1) y_1 H_{2,1,1} + H_{1,1,1},
\]

where we have used Eqs. (6.2) and (6.4) to reach the last equal sign. Next, we replace \( H_{2,1,1} \) according to

\[
H_{2,1,1} = \frac{m + 2}{2} H_{1,1,1} - y_1 y_2 H_{1,1,2} - y_1 H_{1,2,1}.
\]

The final result reads:

\[
H_{2,1}(z_1; m) = \left[ 1 - \frac{m + 2}{2} z_2 \right] H_{1,1,1} + z_2(1 - z_2)H_{1,1,2} + z_2(1 - z_1)H_{1,2,1}.
\]

Sum up,

\[
(\eta_{2,1} H_{2,1} + \eta_{1,2} H_{1,2})(z_1; m) = y_1 \left[ (y_2 - 1) \eta_{1,2} - (y_1 y_2 - 1) \eta_{2,1} \right] H_{1,2,1} + \left[ y_1 y_2(1 - y_1 y_2) \eta_{2,1} \right] H_{1,1,2} + \left[ \eta_{1,2} + \frac{1}{2} (m(y_1 y_2 - 1) + 2 y_1 y_2) \eta_{2,1} \right] H_{1,1,1}.
\]
6.2. Verification I. Let us look at the following function

\[ V_I(z) = -z \left( \frac{4}{m} H_{3,1}(z; m) - H_{2,1} \right) + \frac{2}{m} H_{2,1}(z; m) - H_{1,1}(z; m), \]

which is obtained by taking the difference of the two sides of Eq. (5.14) with the substitution

\[ C \rightarrow H_{a, b} / \Gamma(a + b + m/2 - 2). \]

A common factor \( \Gamma(m/2) \) is dropped since we only care about \( V_I = 0 \) or not. It is easier to show that \( \tau_{(1)}(V_I) = 0 \) where the cyclic transformation \( \tau_{(1)} \) is given in Eq. (6.1). Indeed, according to Eqs. (6.1) and (6.3), we have

\[ (1 - z)^{-m/2} \tau_{(1)}(V_I)(z) = \frac{4z}{m} (1 - z) H_{1,3}(z; m) + \left[ -z + \frac{4}{m} (1 - z) \right] H_{1,2}(z; m) - H_{1,1}(z; m), \]

where the right hand side vanishes because it is the hypergeometric ODE of \( H_{1,1}(z; m) \) (cf. [Liu18b]).

6.3. Verification II. The verification of Eq. (5.15) is much more involved compared to the one-variable case in previous section. As before, we begin with change of notations \( H_a \rightarrow H_a \), the functions in Eqs. (5.6), (5.12) and (5.13) are turned into

\[ J_I = \left( C_{\text{pow}}^{(1)}(y_1) + C_{\text{pow}}^{(1)}(y_2) \right) H_{2,1,1} + y_1 C_{\text{pow}}^{(1)}(y_1) H_{1,2,1} - \frac{1}{2} C_{\text{pow}}^{(1)}(y_1) \left( m + 2 C_{\text{pow}}^{(1)}(y_2) \right) H_{1,1,1}, \]

(6.14)

\[ G_I = \left[ (y_1 y_2) \frac{1}{2} - 1 \right] \left( \frac{2 + m}{2} H_{2,1,1} - 2 H_{3,1,1} - y_1 H_{2,2,1} \right), \]

(6.15)

\[ J_I = - \left( 2 C_{\text{pow}}^{(1)}(y_1, y_2) + y_1 \frac{1}{2} C_{\text{pow}}^{(1)}(y_1) C_{\text{pow}}^{(1)}(y_2) \right) H_{2,1}(z_2; m) + \frac{m}{2} C_{\text{pow}}^{(1)}(y_1, y_2) H_{1,1}(z_2; m), \]

(6.16)

where we have suppressed the arguments of the functions \( J_I, J_I, G_I \) and \( H_{a,b,c} \), for example \( J_I := J_I(z_1, z_2; m) \). A common factor has been factored out, that is

\[ J_I = C_m I_{\Pi,(1), \ldots, L_{10}}, G_I = C_m G_{\Delta, \Pi} \rightarrow H_{1,1}, J_I = C_m I_{1,1, \ldots, L_{10}}, \]

where \( C_m = \Gamma(m/2 + 1)/2 \). We apply the cyclic transformation \( \tau_{(2)}^2 \) (cf. Eq. (6.2)) on both sides of Eq. (5.15) so that all the hypergeometric pieces \( H_{a,b,c} \), \( H_{a,b} \) appeared in Eqs. (6.14) to (6.16) start with 1 in the subscripts, that is, of the form \( H_{1,a,b} \) or \( H_{1,a} \).

Proposition 6.1. Keep the notations as above, we have

\[ \tau_{(2)}^2 G_I - J_I = \tau_{(2)}^2 (J_I). \]

In particular, Eq. (5.15) holds as well.

Proof. The verification is arranged as follows. First, we express the two sides of Eq. (6.17) as combinations of \( \{ H_{1,2,1}, H_{1,1,2}, H_{1,1,1} \} \). The calculation and the results are is postponed to Lemmas 6.2 and 6.3 respectively. The rest is to show that the corresponding coefficients are
Indeed equal. We will take advantage of the fact that $G^{(1)}_{\text{pow}}$ and $G^{(1,1)}_{\text{pow}}$ are divided difference of the power function $u^j$ (here $j = 1/2$, cf. Lemma 5.3). Recall that

1. A divided difference $f[x_0, \ldots, x_n]$ is symmetric in its $n + 1$ arguments.
2. If the function $f$ is homogeneous of degree $j \in R$, namely $f(c y) = c^j f(y)$, $\forall c > 0$,
   then the $n$-th divided difference is of homogeneity $j - n$, that is $f[c x_0, \ldots, c x_n] = c^{j-n} f[x_0, \ldots, x_n]$.

Therefore, we have the following identities that are repeatedly applied in the proof:

$$G^{(1)}_{\text{pow}}(y^{-1}) = u^{\frac{1}{2}}[1, y^{-1}]_u = y^{\frac{1}{2}} u^{\frac{1}{2}}[y, 1]_u = y^{\frac{1}{2}} G^{(1)}_{\text{pow}}(y)$$  \hspace{1cm} (6.18)

and

$$G^{(1,1)}_{\text{pow}}(y_2, (y_1 y_2)^{-1}) = u^{\frac{1}{2}}[1, y_2, y_2(y_1 y_2)^{-1}]_u = y_1^{\frac{1}{2}+1} u^{\frac{1}{2}}[y_1, y_2 y_1, 1]_u$$
\[= y_1^{\frac{1}{2}} u^{\frac{1}{2}}[y_1 y_2, 1]_u - u^{\frac{1}{2}}[1, y_1]_u = y_1^{\frac{1}{2}} u^{\frac{1}{2}}[y_1 y_2, 1]_u - u^{\frac{1}{2}}[1, y_1]_u,\]
\[\text{in which}\]
$$u^{\frac{1}{2}}[1, y_1]_u = G^{(1)}_{\text{pow}}(y_1), \quad u^{\frac{1}{2}}[1, y_1 y_2]_u = G^{(1)}_{\text{pow}}(y_1 y_2),$$
$$u^{\frac{1}{2}}[y_1 y_2, y_1]_u = y_1^{\frac{1}{2}} u^{\frac{1}{2}}[y_2, 1]_u = y_1^{\frac{1}{2}} G^{(1)}_{\text{pow}}(y_2).$$  \hspace{1cm} (6.19)

Now we are ready to simplify the functions $	ilde{c}_{1,1,1}$, $	ilde{c}_{1,1,2}$ and $	ilde{c}_{1,2,1}$ defined in Eqs. 6.25 to 6.27 to the corresponding coefficients given in Lemma 6.2 starting with $	ilde{c}_{1,1,2}$:

$$
\tilde{c}_{1,1,2} = y_1 y_2 (1 - y_1 y_2) G^{(1,1)}_{\text{pow}}(y_2, (y_1 y_2)^{-1}) = y_1^{\frac{1}{2}} y_2 \left( y_1^{\frac{1}{2}} G^{(1)}_{\text{pow}}(y_1) - G^{(1)}_{\text{pow}}(y_2) \right),
$$

which is equal to the coefficient of $H_{1,1,2}$ in Lemma 6.2. Notice that Eq. 6.19 yields different denominators for $G^{(1,1)}_{\text{pow}}(y_2, (y_1 y_2)^{-1})$ to achieving cancellation.

Up to a minus sign, $\tilde{c}_{1,2,1}$ in Eq. 6.25 consists of the following terms:

$$
\left[ y_1 (y_1 y_2 - 1) + y_2^2 (y_2 - 1) \right] G^{(1,1)}_{\text{pow}}(y_2, (y_1 y_2)^{-1}) = y_1^{\frac{1}{2}} \left[ G^{(1)}_{\text{pow}}(y_2) - 2 y_1^{\frac{1}{2}} G^{(1)}_{\text{pow}}(y_1) + y_1^{\frac{1}{2}} G^{(1)}_{\text{pow}}(y_1 y_2) \right],
$$
\[\text{so that the sum reads:}\]
\[-\tilde{c}_{1,2,1} = y_1^{\frac{1}{2}} \left[ -2 G^{(1)}_{\text{pow}}(y_1^{-1}) + G^{(1)}_{\text{pow}}(y_2) + (y_1 y_2)^{\frac{1}{2}} G^{(1)}_{\text{pow}}(y_1 y_2) \right].\]  \hspace{1cm} (6.20)

In the calculation of Eq. 6.21, we have used $(y_2 - 1) G^{(1)}_{\text{pow}}(y_2) = y_2^{\frac{1}{2}} - 1$ and the relation $(y_1 y_2)^{\frac{1}{2}} G^{(1)}_{\text{pow}}(y_1 y_2) = G^{(1)}_{\text{pow}}((y_1 y_2)^{-1})$. The relation also implies that $\tilde{c}_{1,2,1}$ is indeed equal to the corresponding coefficients in Lemma 6.2.
By adding up the three terms in Eq. (6.27):
\[ y_1(y_2 - 1)G_{\text{pow}}^{(1)}(y_2, (y_1 y_2)^{-1}) = y_1 \left( y_1 \frac{1}{2} G_{\text{pow}}^{(1)}(y_1 y_2) - y_1 \frac{3}{2} G_{\text{pow}}^{(1)}(y_1) \right), \]
\[ \frac{m}{2} (y_2 y_1 - 1)G_{\text{pow}}^{(1)}(y_2, (y_1 y_2)^{-1}) = \frac{m}{2} y_1 \left( G_{\text{pow}}^{(1)}(y_2) - y_1 \frac{3}{2} G_{\text{pow}}^{(1)}(y_1) \right), \]
\[ - y_1 y_2 \frac{1}{2} G_{\text{pow}}^{(1)}(y_2)G_{\text{pow}}^{(1)}((y_1 y_2)^{-1}) = - y_1 \frac{3}{2} G_{\text{pow}}^{(1)}(y_2)G_{\text{pow}}^{(1)}(y_1 y_2) \]
we have for the last one:
\[ \hat{c}_{1,1,1} = y_1 \left[ \frac{m}{2} G_{\text{pow}}^{(1)}(y_2) - \left( 1 + \frac{m}{2} \right) y_1 \frac{1}{2} G_{\text{pow}}^{(1)}(y_1) + y_1 \frac{3}{2} G_{\text{pow}}^{(1)}(y_1 y_2) \left( 1 - G_{\text{pow}}^{(1)}(y_2) \right) \right], \]
in which the third term can also be written as
\[ y_1 \frac{3}{2} G_{\text{pow}}^{(1)}(y_1 y_2) \left( 1 - G_{\text{pow}}^{(1)}(y_2) \right) = (y_1 y_2)^{\frac{1}{2}} G_{\text{pow}}^{(1)}(y_1 y_2)G_{\text{pow}}^{(1)}(y_2) = G_{\text{pow}}^{(1)}((y_1 y_2)^{-1}) G_{\text{pow}}^{(1)}(y_2). \]
Therefore \( \hat{c}_{1,1,1} \) agrees with the coefficient of \( H_{1,1,1} \) in Lemma 6.2.

**Lemma 6.2.** The function \( \tau_{(2)}^2 \left( G_{II} - J_{II} \right) \) belongs to the span of \( \{ H_{1,2,1}, H_{1,1,2}, H_{1,1,1} \} \) with coefficients given by:
\[
(6.22) \quad y_1^{-\frac{m}{2}} \tau_{(2)}^2 (G_{II} - J_{II})(z_1, z_2; m) = y_1^2 \left[ 2y_1^2 G_{\text{pow}}^{(1)}(y_1) - G_{\text{pow}}^{(1)}(y_2) - G_{\text{pow}}^{(1)}((y_1 y_2)^{-1}) \right] H_{1,2,1} + y_1^2 y_2 \left[ y_1 \frac{3}{2} G_{\text{pow}}^{(1)}(y_1) - G_{\text{pow}}^{(1)}(y_2) \right] H_{1,1,2} + y_1 \left[ -\left( 1 + \frac{m}{2} \right) y_1 \frac{3}{2} G_{\text{pow}}^{(1)}(y_1) + \frac{1}{2} G_{\text{pow}}^{(1)}(y_2) \left( m + 2 G_{\text{pow}}^{(1)}((y_1 y_2)^{-1}) \right) \right] H_{1,1,1}.
\]

**Proof.** We first follow Eqs. (6.2) and (6.4) to carry out the transformation \( \tau_{(2)}^2 \):
\[ y_1^{-\frac{m}{2}} \tau_{(2)}^2 (G_{II})(z_1, z_2; m) = \tau_{(2)}^2 ((y_1 y_2)^{\frac{1}{2}} - 1) \left( \frac{m + 2}{2} y_1^2 H_{1,2,1} - y_2 y_1^2 H_{1,2,2} - 2y_1^3 H_{1,3,1} \right), \]
where
\[ \tau_{(2)}^2 ((y_1 y_2)^{\frac{1}{2}} - 1) = \tau_{(2)}^2 \left( G_{\text{pow}}^{(1)}(y_1 y_2)(y_1 y_2 - 1) \right) = G_{\text{pow}}^{(1)}(y_1^{-1}) y_1^{-1}(1 - y_1) \]
and
\[ (6.23) \quad y_1^{-\frac{m}{2}} \tau_{(2)}^2 (J_{II})(z_1, z_2; m) = \left[ G_{\text{pow}}^{(1)}(y_2) + G_{\text{pow}}^{(1)}((y_1 y_2)^{-1}) \right] y_1^2 H_{1,2,1} + y_2 y_1^2 G_{\text{pow}}^{(1)}(y_2) H_{1,1,2} + \frac{1}{2} y_1 \left[ G_{\text{pow}}^{(1)}(y_2) \left( m + 2 G_{\text{pow}}^{(1)}((y_1 y_2)^{-1}) \right) \right] H_{1,1,1}. \]
Notice that \( \tau_{(2)}^2 (J_{II}) \) is already written as a span of \( \{ H_{1,1,1}, H_{1,2,1}, H_{1,1,2} \} \). For \( \tau_{(2)}^2 (G_{II}) \), we need to replace \( H_{1,2,2} \) and \( H_{1,3,1} \) whose general form has been computed in Eq. (6.9) with coefficients
\[ \eta_{1,3,1} = 2y_1^2 (y_1 - 1) G_{\text{pow}}^{(1)}(y_1^{-1}), \quad \eta_{1,2,2} = \frac{y_2}{2} \eta_{1,3,1}. \]
Therefore we have \( y_1^{-\frac{m}{2}} \tau_{(2)}^2(G_{11}) = c_{1,2,1} H_{1.2.1} + c_{1,1,2} H_{1.1,2} + c_{1,1,1} H_{1.1,1} \) where the coefficients are computed according to Eq. \((6.9)\) as below. First

\[
c_{1,2,1} = -\left(\frac{m}{2} + 1\right) y_1 (y_1 - 1) G_{\text{pow}}^{(1)}(y_1^{-1}) + \left[ \frac{m}{4} y_1^2 + \frac{1}{y_1 - 1} \left( \frac{1}{2} \frac{1}{y_1(y_2 - 1)} \right) \eta_{1,1,3} + \frac{\eta_{1,2,2}}{y_1(y_2 - 1)} \right] \eta_{1,1,3} + \frac{\eta_{1,3,1}}{y_1 - 1}
\]

\[
= -\left(\frac{m}{2} + 1\right) y_1 (y_1 - 1) G_{\text{pow}}^{(1)}(y_1^{-1}) + \left( \frac{m}{4} + 1 \right) y_1^{-1} \eta_{1,3,1} + \frac{\eta_{1,3,1}}{y_1 - 1}
\]

\[
= 2 y_1^2 G_{\text{pow}}^{(1)}(y_1^{-1})
\]

where the first two terms in the middle line cancel out. The next one:

\[
c_{1,1,2} = \frac{-1}{y_1(y_2 - 1)} \eta_{1,2,2} + \frac{y_2(y_2 - 1)}{2 y_1(y_1 - 1)(y_2 - 1)} \eta_{1,1,3}
\]

\[
= \left( \frac{y_1 y_2 - 1}{y_1 - 1} \right) \frac{y_2 \eta_{1,2,1}}{2 y_1(y_2 - 1)} = y_2 y_1^2 G_{\text{pow}}^{(1)}(y_1^{-1})
\]

The last one

\[
c_{1,1,1} = - \frac{(2 + m)}{4 y_1(y_1 - 1)} \eta_{1,1,3} = - \frac{2 + m}{2} y_1 G_{\text{pow}}^{(1)}(y_1^{-1}).
\]

With \( G_{\text{pow}}^{(1)}(y_1^{-1}) = y_1^1 G_{\text{pow}}^{(1)}(y_1) \), the total reads

\[
y_1^{-\frac{m}{2}} \tau_{(2)}^2(G_{11})(z_1, z_2; m) = y_1^1 G_{\text{pow}}^{(1)}(y_1) \left( 2y_1^2 H_{1.2.1} + y_2 y_1^2 H_{1.1,2} - \frac{2 + m}{2} y_1 H_{1.1,1} \right).
\]

We finish the computation by taking the difference of Eqs. \((6.23)\) and \((6.24)\). □

**Lemma 6.3.** The function \( \tau_{(2)}(J_1) \) can also be written as a combination of \( \{H_{1.2.1}, H_{1.1,2}, H_{1.1,1}\} \):

\[
y_1^{-\frac{m}{2}} \tau_{(2)}^2(J_1) = \tilde{c}_{1,2,1} H_{1.2.1} + \tilde{c}_{1,1,2} H_{1.1,2} + \tilde{c}_{1,1,1} H_{1.1,1},
\]

where the coefficients \( c_{1,2,1}, c_{1,1,2} \) and \( c_{1,1,1} \) are given in Eqs. \((6.25)\) to \((6.27)\) respectively.

**Proof.** Notice that \( J_1 \) in Eq. \((6.16)\) is of the form \( J_1 = a_1 H_{2.1} + \frac{m}{4} a_2 (H_{1.1} - \frac{4}{m} H_{2.1}) \) where \( H_{a,b} = H_{a,b}(z_2; m) \) and the coefficients are given by:

\[
a_1 = -y_1^{-\frac{1}{2}} G_{\text{pow}}^{(1)}(y_1) G_{\text{pow}}^{(1)}(y_2), \quad a_2 = 2 G_{\text{pow}}^{(1,1)}(y_1, y_2).
\]

For the part containing \( a_2 \), we have:

\[
y_1^{-\frac{m}{2}} \tau_{(2)}^2 \left( - \frac{4}{m} H_{2.1}(z_2; m) + H_{1.1}(z_2; m) \right) = -\frac{4}{m} y_1 H_{1.2}(z_1; m) + H_{1.1}(z_1; m)
\]

\[
= -\frac{2}{m} y_1 H_{1.2}(z_1; m) + \frac{2}{m} H_{2.1}(z_1; m),
\]
where the relation $H_{1,1} = \frac{z}{m}((1 - z)H_{1,2} + H_{2,1})$ from [Liu18b] is applied. Now we have shown that $y_1^{\frac{m}{n}} \tau^2_{(2)}(J_1) = \eta_{1,2}H_{1,2} + \eta_{2,1}H_{2,1}$ with $H_{a,b} := H_{a,b}(z_1; m)$ and

$$\eta_{1,2} = y_1 \left[ \frac{-1}{2} \tau_{(2)}^2(a_1) - \frac{1}{2} \tau_{(2)}^2(a_2) \right] = y_1 \left[ -y_2^{-\frac{1}{2}}C_{\mathrm{pow}}^{(1)}(y_2)G_{\mathrm{pow}}^{(1)}((y_1 y_2)^{-1}) - C_{\mathrm{pow}}^{(1)}(y_2, (y_1 y_2)^{-1}) \right],$$

$$\eta_{2,1} = \frac{1}{2} \tau_{(2)}^2(a_2) = G_{\mathrm{pow}}^{(1,1)}(y_2, (y_1 y_2)^{-1}).$$

Accordingly, $y_1^{\frac{m}{n}} \tau^2_{(2)}(J_1) = \tilde{c}_{1,2,1}H_{1,2,1} + \tilde{c}_{1,1,2}H_{1,1,2} + \tilde{c}_{1,1,1}H_{1,1,1}$ where the coefficients are determined by Eq. (6.12):

(6.25)

$$\tilde{c}_{1,2,1} = y_1 [(y_2 - 1)\eta_{1,2} - (y_1 y_2 - 1)\eta_{2,1}]$$

$$= y_1^2(y_2 - 1)\tau_{(2)}^2(a_1) - \frac{1}{2} \left[ y_1(y_1 y_2 - 1) + y_1^2(y_2 - 1) \right] \tau_{(2)}^2(a_2)$$

$$= -y_1^2(y_2 - 1)y_2^{-\frac{1}{2}}C_{\mathrm{pow}}^{(1)}(y_2)G_{\mathrm{pow}}^{(1)}((y_1 y_2)^{-1}) - \left[ y_1(y_1 y_2 - 1) + y_1^2(y_2 - 1) \right] G_{\mathrm{pow}}^{(1,1)}(y_2, (y_1 y_2)^{-1})$$

and

(6.26)

$$\tilde{c}_{1,1,2} = y_1 y_2 (1 - y_1 y_2)\eta_{2,1} = \frac{1}{2} y_1 y_2 (1 - y_1 y_2) \tau_{(2)}^2(a_2)$$

$$= y_1 y_2 (1 - y_1 y_2) G_{\mathrm{pow}}^{(1,1)}(y_2, (y_1 y_2)^{-1}),$$

and

(6.27)

$$\tilde{c}_{1,1,1} = \eta_{1,2} + \frac{1}{2} \left( m(y_1 y_2 - 1) + 2y_1 y_2 \right) \eta_{2,1}$$

$$= y_1 \tau_{(2)}^2(a_1) + \frac{1}{2} y_1(y_2 - 1) \tau_{(2)}^2(a_2) + \frac{m}{4} (y_1 y_2 - 1) \tau_{(2)}^2(a_2)$$

$$= y_1 y_2^{-\frac{1}{2}}G_{\mathrm{pow}}^{(1)}(y_2)G_{\mathrm{pow}}^{(1)}((y_1 y_2)^{-1}) + \left[ y_1(y_2 - 1) + \frac{m}{2} (y_1 y_2 - 1) \right] G_{\mathrm{pow}}^{(1,1)}(y_2, (y_1 y_2)^{-1}).$$

□

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