Abstract

An evolutionarily stable strategy (ESS) was originally defined as a static concept but later given a dynamic characterization. A well known theorem in evolutionary game theory says that an ESS is an attractor of replicator dynamics but not every attractor is an ESS. We search for a dynamic characterization of ESSs in quantum games and find that in certain asymmetric bi-matrix games evolutionary stability of attractors can change as the game switches between its two forms, one classical and the other quantum.

1 Introduction

Theory of quantum games \cite{1,2} has attracted significant attention during recent years. One of the area where classical game theory \cite{3} has been quite successful is the evolutionary dynamics in a population. Evolutionary game theory \cite{4} is a growing area of research where the individuals of a population are treated as players participating in a game. The players are not rational individuals and their moves or strategies are inherited traits. In initial studies Maynard Smith and Price \cite{5} considered anonymous players who are randomly matched in pairs to play a bi-matrix game. Allowing higher payoff strategies to gradually displace strategies with lower payoffs introduces a dynamics in the population. Much of the evolutionary game theory deals with the concept of an evolutionarily stable strategy (ESS), which is characterized by the condition that if all individuals choose this strategy, then no other strategy can spread in a population.

Earlier we studied \cite{6,7,8,9,10} evolutionary stability of Nash equilibria (NE) related to the quantization of classical games. We found that in certain games evolutionary stability of NE can change as the game switches between its classical and quantum forms. We considered evolutionary stability in a scheme to play a quantum game proposed by Marinatto and Weber \cite{11}.
Maynard Smith and Price \[5\] introduced the idea of an ESS essentially as a static concept. Nothing in the definition of an ESS guarantees that the dynamics of evolution in small mutational steps will necessarily converge the process of evolution to an ESS. In fact directional evolution may also become responsible for the establishment of strategies that are not evolutionarily stable \[12\].

What are the advantages involved in the dynamic approach \[13\] towards theory of ESSs? Such an approach can be seen in the context of Liapunov’s classic definition of stability of equilibria for general dynamical systems. This definition can also be adapted for the stability of a NE. A pair of strategies \((p^*, q^*)\) is Liapunov stable when for every trajectory starting somewhere in a small neighborhood of radius \(\epsilon > 0\) around a point representing the pair \((p^*, q^*)\) another small neighborhood of radius \(\delta > 0\) can be defined such that the trajectory stays in it. When every trajectory starting in a small neighborhood of radius \(\sigma > 0\) around the point \((p^*, q^*)\) converges to \((p^*, q^*)\) the strategy pair \((p^*, q^*)\) becomes an attractor. Trajectories are defined by the dynamics underlying the game.

Taylor and Jonker \[14\] introduced a dynamics into evolutionary games with the hypothesis that the growth rate of those playing each strategy is proportional to the advantage of that strategy. This hypothesis is now understood as one of many different forms of replicator dynamics \[12, 15\]. In simple words assume that \(x_i\) is the frequency (i.e. relative proportion) of the individuals using strategy \(i\) and \(\mathbf{x}\), where \(\mathbf{x}^T = [x_1, x_2, ..., x_i, ..., x_n]\) and \(T\) is the transpose, is a vector whose components are the frequencies with \(\sum_{i=1}^{n} x_i = 1\). Let \(P_i(\mathbf{x})\) be the average payoff for using \(i\) when the population is in the state \(\mathbf{x}\). Let \(\bar{P} = \sum x_j P_j\) be the average success in the population. The replicator equation is, then, written as \[16\]

\[
\dot{x}_i = x_i(P_i(\mathbf{x}) - \bar{P})
\]  

where the dot is derivative w.r.t time. In case the payoff matrix is given as \(A = (a_{ij})\) with \(a_{ij}\) being the average payoff for strategy \(i\) when the other player uses \(j\). The average payoff for the strategy \(i\) in the population (with the assumption of random encounters of the individuals) is \((Ax)_i = a_{i1}x_1 + ... + a_{in}x_n\) and the Eq. \(1\) becomes

\[
\dot{x}_i = x_i((Ax)_i - \mathbf{x}^T A \mathbf{x})
\]  

The population state is then given as a point in \(n\) simplex \(\triangle\) \[17\]. The hypothesis of Taylor and Jonker \[14\] gives a flow on \(\triangle\) whose flow lines represent the evolution of the population. In evolutionary game theory it is agreed \[4\] that every ESS is an attractor of the flow defined on \(\triangle\) by the replicator equation \(1\), however, the converse does not hold: an attractor is not necessarily an ESS.

We now ask a question: is it possible that a non-ESS attractor of replicator dynamics in a classical game becomes an ESS for some quantum form of the same game. This possibility, besides strengthening our previous results about the relationships between parameters of initial quantum state and evolutionary
stability of a NE, gives a dynamic ground to the relevance of the theory of ESSs in quantum games.

We main result in this paper is that the above possibility exists, indeed, in certain types of games. Quantization, thus, can change non-ESS attractor of replicator dynamics into an ESS or conversely.

2 Equilibria and attractors of replicator dynamics

Early studies about the attractors of replicator dynamics by Schuster, Sigmund and Wolff [18, 19] reported the dynamics of enzymatic actions of chemicals in a mixture when their relative proportions could be changed. For example in the case of a mixture of three chemicals added in a correct order, such that corresponding initial conditions are in the basin of an interior attractor, it becomes a stable cooperative mixture of all three chemicals. But if they are added in a wrong order the initial conditions then lie in another basin and only one of the chemicals survives with others two excluded. Eigen and Schuster [18, 19, 20] also studied resulting dynamics in the evolution of macromolecules before the advent of life.

Schuster and Sigmund [21] applied the dynamics to animal behavior in Battle of Sexes game and described the evolution of strategies by treating it as a dynamical system. They wrote replicator Eqs. (2) for the following general bi-matrix

$$
\begin{bmatrix}
(a_{11}, b_{11}) \\
(a_{12}, b_{12}) \\
(a_{21}, b_{21}) \\
(a_{22}, b_{22})
\end{bmatrix}
$$

where a male can play pure strategies $X_1$, $X_2$ and a female can play pure strategies $Y_1$, $Y_2$ respectively. Let in a population engaged in this game the frequencies of $X_1$ and $X_2$ are $x_1$ and $x_2$ respectively. Similarly the frequencies of $Y_1$ and $Y_2$ are $y_1$ and $y_2$ respectively. Obviously

$$x_1 + x_2 = y_1 + y_2 = 1$$

where $x_i \geq 0$, $y_i \geq 0$, for $i = 1, 2$ (4)

the replicator equations (2) for the matrix (3) with conditions (4) are, then, written as

$$\begin{align}
\dot{x} &= x(1-x) \{ y(a_{11} - a_{12} - a_{21} + a_{22}) + (a_{12} - a_{22}) \} \\
\dot{y} &= y(1-y) \{ x(b_{11} - b_{12} - b_{21} + b_{22}) + (b_{12} - b_{22}) \}
\end{align}$$

(5)
where \( x_1 = x \) and \( y_1 = y \). These equations are of Lotka-Volterra type describing the evolution of two populations consisting of predator and prey \[22\]. Schuster and Sigmund \[21\] simplified the problem by taking

\[
\begin{align*}
    a_{11} &= b_{11} = a_{22} = b_{22} = 0 \\
    a_{12} &= a \quad a_{21} = b \quad \text{and} \\
    b_{12} &= c \quad b_{21} = d
\end{align*}
\]

which does not restrict the generality and the replicator Eqs. \(5\) remain similar.

Payoffs to the male \( P_M(x, y) \) and to the female \( P_F(x, y) \) when the male plays \( X_1 \) with probability \( x \) (i.e. he plays \( X_2 \) with the probability \( 1 - x \)) and the female plays \( Y_1 \) with the probability \( y \) (i.e. she plays \( Y_2 \) with the probability \( 1 - y \)) are written as \[23\]

\[
\begin{align*}
P_M(x, y) &= x^T My \\
P_F(x, y) &= x^T Fy
\end{align*}
\]

where \( M = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \), \( F = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \) and \( x = \begin{bmatrix} x \\ 1 - x \end{bmatrix} \), \( y = \begin{bmatrix} y \\ 1 - y \end{bmatrix} \)

and \( T \) is for transpose.

Suppose now a quantum form of the matrix game \[3\] is played using Mar- inatatto and Weber’s scheme \[11\]. The players have at their disposal following initial quantum state

\[
|\psi_{ini}\rangle = \sum_{i,j=1,2} c_{ij} |ij\rangle
\]

with normalization

\[
\sum_{i,j=1,2} |c_{ij}|^2 = 1
\]

where 1 corresponds to the pure strategies \( X_1 \) or \( Y_1 \) and 2 corresponds to the pure strategies \( X_2 \) or \( Y_2 \). The constants \( c_{ij} \) for \( i, j = 1, 2 \) are complex numbers in general. Players apply unitary operators on the quantum state with classical probabilities and payoffs to them are decided later by a measurement on final state \[11\]. Male and Female players apply the identity operator \( \hat{I} \) on the initial state \( |\psi_{ini}\rangle \) with the probabilities \( x \) and \( y \) respectively. Both the players also apply the Pauli’s spin-flip operator \( \hat{\sigma}_x \) with probabilities \( (1 - x) \) and \( (1 - y) \) respectively. The operator \( \hat{\sigma}_x \) changes the state \( |1\rangle \) to \( |2\rangle \) and \( |2\rangle \) to \( |1\rangle \). Payoffs to both players are written \[10\] in a similar form as in the Eq. \(7\)

\[
\begin{align*}
P_M(x, y) &= x^T \omega y \\
P_F(x, y) &= x^T \chi y
\end{align*}
\]
\( \omega \) and \( \chi \) are quantum forms \(^{10}\) of the payoff matrices \( M \) and \( F \) respectively i.e.

\[
\omega = \begin{bmatrix} \omega_{11} & \omega_{12} \\ \omega_{21} & \omega_{22} \end{bmatrix} \quad \text{and} \quad \chi = \begin{bmatrix} \chi_{11} & \chi_{12} \\ \chi_{21} & \chi_{22} \end{bmatrix}
\]  

(11)

where

\[
\begin{align*}
\omega_{11} &= a_{11}|c_{11}|^2 + a_{12}|c_{12}|^2 + a_{21}|c_{21}|^2 + a_{22}|c_{22}|^2 \\
\omega_{12} &= a_{11}|c_{12}|^2 + a_{12}|c_{11}|^2 + a_{21}|c_{22}|^2 + a_{22}|c_{21}|^2 \\
\omega_{21} &= a_{11}|c_{21}|^2 + a_{12}|c_{22}|^2 + a_{21}|c_{11}|^2 + a_{22}|c_{12}|^2 \\
\omega_{22} &= a_{11}|c_{22}|^2 + a_{12}|c_{21}|^2 + a_{21}|c_{12}|^2 + a_{22}|c_{11}|^2
\end{align*}
\]  

(12)

similarly

\[
\begin{align*}
\chi_{11} &= b_{11}|c_{11}|^2 + b_{12}|c_{12}|^2 + b_{21}|c_{21}|^2 + b_{22}|c_{22}|^2 \\
\chi_{12} &= b_{11}|c_{12}|^2 + b_{12}|c_{11}|^2 + b_{21}|c_{22}|^2 + b_{22}|c_{21}|^2 \\
\chi_{21} &= b_{11}|c_{21}|^2 + b_{12}|c_{22}|^2 + b_{21}|c_{11}|^2 + b_{22}|c_{12}|^2 \\
\chi_{22} &= b_{11}|c_{22}|^2 + b_{12}|c_{21}|^2 + b_{21}|c_{12}|^2 + b_{22}|c_{11}|^2
\end{align*}
\]  

(13)

For the initial product state \( |\psi_{ini}\rangle = |11\rangle \) the matrices \( \omega \) and \( \chi \) are same as \( M \) and \( F \) respectively. The classical game is, therefore, embedded in the quantum game. Simplified matrices \( \omega \) and \( \chi \) can be obtained by the assumption of Eq. \(^{b}\) i.e.

\[
\begin{align*}
\omega_{11} &= a|c_{12}|^2 + b|c_{21}|^2, \quad \omega_{12} = a|c_{11}|^2 + b|c_{22}|^2 \\
\omega_{21} &= a|c_{22}|^2 + b|c_{11}|^2, \quad \omega_{22} = a|c_{21}|^2 + b|c_{12}|^2 \\
\chi_{11} &= c|c_{12}|^2 + d|c_{21}|^2, \quad \chi_{12} = c|c_{11}|^2 + d|c_{22}|^2 \\
\chi_{21} &= c|c_{22}|^2 + d|c_{11}|^2, \quad \chi_{22} = c|c_{21}|^2 + d|c_{12}|^2
\end{align*}
\]  

(14)

The replicator Eqs. \(^{b}\) can now be written in the following ‘quantum’ form

\[
\begin{align*}
x^* &= x(1-x)[aK_1 + bK_2 - (a+b)(K_1 + K_2)y] \\
y^* &= y(1-y)[cK_1 + dK_2 - (c+d)(K_1 + K_2)x]
\end{align*}
\]  

(15)

where \( K_1 = |c_{11}|^2 - |c_{21}|^2 \) and \( K_2 = |c_{22}|^2 - |c_{12}|^2 \). These equations reduce to Eqs. \(^{b}\) for \( |\psi_{ini}\rangle = |11\rangle \) i.e. \( |c_{11}|^2 = 1 \). Similar to classical version \(^{21}\) the dynamics \(^{b}\) has five rest or equilibrium points \( x = 0, y = 0; \quad x = 0, y = 1; \quad x = 1, y = 0; \quad x = 1, y = 1; \) and an interior equilibrium point

\[
\begin{align*}
x &= \frac{cK_1 + dK_2}{(c+d)(K_1 + K_2)}, \quad y = \frac{aK_1 + bK_2}{(a+b)(K_1 + K_2)}
\end{align*}
\]  

(16)
This equilibrium point is the same as in the classical game \[21\] for \(|\psi_{ini}\rangle = |11\rangle\) i.e.

\[
x = \frac{c}{c + d}, \quad y = \frac{a}{a + b}
\]

(17)

We use the method of linear approximation \[22\] at equilibrium points to find the general character of phase diagram of the system \[15\]. Write the system \[15\] as

\[
\begin{align*}
\dot{x} &= X(x, y), \\
\dot{y} &= Y(x, y)
\end{align*}
\]

(18)

The matrix for linearization \[22\] is

\[
\begin{bmatrix}
X_x & X_y \\
Y_x & Y_y
\end{bmatrix}
\]

(19)

where, for example, \(X_x\) denotes \(\frac{\partial X}{\partial x}\). The matrix is evaluated at each equilibrium point in turn. Writing these terms as

\[
\begin{align*}
X_x &= (1 - 2x) \{aK_1 + bK_2 - (a + b)(K_1 + K_2)y\} \\
X_y &= -x(1 - x)(a + b)(K_1 + K_2) \\
Y_x &= -y(1 - y)(c + d)(K_1 + K_2) \\
Y_y &= (1 - 2y) \{cK_1 + dK_2 - (c + d)(K_1 + K_2)x\}
\end{align*}
\]

(20)

and the characteristic equation \[22\] at an equilibrium point is obtained from

\[
\left| \begin{array}{cc}
(X_x - \lambda) & X_y \\
Y_x & (Y_y - \lambda)
\end{array} \right| = 0
\]

(21)

The patterns of phase paths around equilibrium points classify the points into a few principal cases. Suppose \(\lambda_1, \lambda_2\) are roots of the characteristic Eq. \[21\]. A few cases are as follows:

(1). \(\lambda_1, \lambda_2\) real, different, non-zero, and same sign. If \(\lambda_1, \lambda_2 > 0\) then the equilibrium point is an unstable node or a repellor. If \(\lambda_1, \lambda_2 < 0\) the node is stable or an attractor.

(2). \(\lambda_1, \lambda_2\) real, different, non-zero, and opposite sign. The equilibrium point is a saddle point.

(3). \(\lambda_1 = \lambda_2 = \alpha + i\beta, \beta \neq 0\) The equilibrium is a stable spiral (attractor) if \(\alpha < 0\), an unstable spiral (repellor) if \(\alpha > 0\), a centre if \(\alpha = 0\).

Consider an equilibrium or rest point \(x = 1, y = 0\) written simply as \((1, 0)\). At this point the characteristic Eq. \[21\] has these roots

\[
\lambda_1 = -aK_1 - bK_2, \quad \lambda_2 = -cK_2 - dK_1
\]

(22)

For the classical game, i.e. \(|\psi_{ini}\rangle = |11\rangle\), these roots are \(\lambda_1 = -a, \lambda_2 = -d\). Therefore in case \(a, d > 0\) the equilibrium point \((1, 0)\) is an attractor in the classical game. Every ESS is also an attractor but the converse is not true. We
now write down the conditions that make the attractor (1, 0) also an ESS. The game of the matrix (3) with simplifications given in Eq. (6) is an asymmetric game and the equilibrium (1, 0) is an ESS if it is a strict NE. The strict NE conditions for the point (1, 0) are

\[
P_M(1, 0) - P_M(x, 0) = (1 - x)\{a(|c_{11}|^2 - |c_{21}|^2) + \\
b(|c_{22}|^2 - |c_{12}|^2)\} > 0
\]

\[
P_F(1, 0) - P_F(1, y) = y\{c(|c_{11}|^2 - |c_{12}|^2) + \\
d(|c_{22}|^2 - |c_{21}|^2)\} > 0
\]

for all \(x, y \in [0, 1]\) with \(x \neq 1\) and \(y \neq 0\). In classical game, therefore, (1, 0) is an ESS when both \(a, c > 0\). A comparison of the strict inequalities (23) with the roots (22) of the characteristic Eq. (21) show that in case \(|c_{11}|^2 = |c_{22}|^2\) the inequalities (23) guarantee that both \(\lambda_1\) and \(\lambda_2\) are negative and consequently an ESS is an attractor and an attractor is an ESS.

We study three cases:
(a) The equilibrium point (1, 0) is an attractor in classical as well as a quantum form of the game. However it is not an ESS in the classical game but is an ESS in the quantum game.
(b) Point (1, 0) is an attractor in classical as well as a quantum game. However, it is an ESS in classical game but not an ESS in the quantum game.
(c) An interior point is a saddle (center) in the classical game but it becomes a centre (saddle) in a quantum form of the game.

2.1 Case (a)

Let the constants \(a, b, c\) and \(d\) are such that \(a, d > 0\) and \(b, c < 0\). The equilibrium point (1, 0) is, then, a non-ESS attractor in classical game. Select the parameters of the initial state \(c_{ij}\) such that \(|c_{21}|^2 < |c_{22}|^2 < |c_{11}|^2 < |c_{12}|^2\) with the normalization in Eq. (9). The equilibrium (1, 0) is now an ESS in the quantum form of the game.

2.2 Case (b)

In case \(a, c, d > 0\) and \(b < 0\) the point (1, 0) is an ESS attractor of the classical game. Select now the parameters \(c_{ij}\) of the initial quantum state such that \(|c_{22}|^2 < |c_{21}|^2 < |c_{11}|^2 < |c_{12}|^2\) and \(c(|c_{11}|^2 - |c_{12}|^2) < d(|c_{21}|^2 - |c_{22}|^2)\). The equilibrium (1, 0) is a non-ESS attractor of the corresponding quantum game.

2.3 Case (c)

At the interior equilibrium point \((x, y)\) of Eq. (16) the terms of the matrix of linearization of Eq. (20) are
\[ X_x = 0, \quad Y_y = 0 \]
\[ X_y = \frac{-(cK_1 + dK_2)(cK_2 + dK_1)(a + b)}{(c + d)^2(K_1 + K_2)} \]
\[ Y_x = \frac{-(aK_1 + bK_2)(aK_2 + bK_1)(c + d)}{(a + b)^2(K_1 + K_2)} \]

(24)

the roots of the characteristic Eq. [21] are numbers \( \pm \lambda \) where

\[ \lambda = \sqrt{\frac{(aK_1 + bK_2)(aK_2 + bK_1)(cK_1 + dK_2)(cK_2 + dK_1)}{(a + b)(c + d)(K_1 + K_2)^2}} \]

(25)

the term in square root can be a positive or negative real number. Therefore, a saddle (center) in classical game can be a center (saddle) in certain quantum form of the game. A saddle or a center in a classical (quantum) game can not be, however, an attractor or a repellor in quantum (classical) form of the game.

3 Discussion

In classical evolutionary game theory attractors of a dynamics and ESSs are usually studied with reference to population models. Extending these ideas to quantum settings requires an assumption of a population of individuals having access to quantum mechanical operators and entangled states. What is the possible relevance of such an assumption in real world? Evolutionary quantum computation (EQC) [24] is such an example. In EQC an ensemble of quantum subsystems is considered changing continually such a way as to optimize some measure of emergent patterns between the system and its environment. This optimization can thought to be related to equilibria and even to some stability property of the equilibria. Nature of quantum interaction deciding stability of equilibria imply that optimization itself depends on it. Brain itself has been proposed as an evolutionary quantum computer.

Has the ESS idea a relevance only in population models? For two players case a meaning of ESS exists when the usual term ‘frequency’ is replaced with ‘fraction of the total time’. Two quantum interacting molecules modelled as players in a game will involve considerations of evolutionary stability and how it depends on the interaction pattern.

The scheme proposed by Marinatto and Weber [11] allows consideration of the relationship between quantization and evolutionary stability in matrix games. In classical ESS theory pure strategies can be combined with probabilities that sum up to one. Similar things happen in this scheme to play a quantum game. Nevertheless, the quantum aspect gives more ‘dimensions’ to a classical matrix game and stability properties of NE, and also attractors, can be studied by starting the game with different initial states. ESS idea is extended to quantum games as a static concept [6] but we showed in this paper that it can
also be dynamically characterized in such games. It then provides an alternative way for studying dynamic quantum games.

An important aspect by which evolutionary game theory is different from classical game theory is the role and need of rational decision makers [15]. Classical game theory was developed under the assumptions of rational decision makers. In evolutionary game theory, on the other hand, an individual’s ‘strategy’ is an inherited trait usually called a ‘phenotype’. A population is an abstract entity of interacting individuals with genetically determined strategies. This approach makes unnecessary the need for rational decision makers. In our effort to extend the ideas of evolutionary game theory to quantum games no rationality is associated to decision makers whose actions are quantum mechanical. Such decisions can be made, for example, in a group of interacting molecules without an assumption of consciousness associated with them.

We believe quantum game theory can provide a role for quantum mechanics in self organization of interacting molecules. Quantum mechanics is long known to play role in keeping the atoms together in molecules. We believe that quantum game theory paves the way for an equally important role for quantum mechanics in evolution and development of self organization and complexity in molecular systems. This aspect arises exciting new questions about quantum role in origin of life and also in origin of consciousness.

The ESS idea in population biology was developed in an attempt to understand complex behaviors in animal societies. The goal was to model evolutionary processes in populations of interacting individuals and to explain why certain states in the population are stable against perturbations induced by mutations. We do not see why the ESSs, and also other concepts of dynamic stability of equilibria, should be useless in the context of the rise of self organization in groups of interacting molecules. Our results show that quantum mechanics has strong and important roles in selection of stable solutions in a system of interacting ‘entities’. These entities can do quantum actions on quantum states and may simply consist of a collection of molecules. We believe that if stability of solutions or equilibria can be affected by quantum interactions then it provides a new approach towards theories of rise of complexity in groups of quantum interacting entities.

Out of two perspectives, on what should be an outcome of evolution, the matrix game theory provides one and the other is provided by optimization models [25]. In optimization models the selection is frequency-independent and evolution is imagined as a hill-climbing process. Optimal solution is obtained where fitness is maximized. Evolutionary optimization is the basis of evolutionary and genetic algorithms and forms a different approach than ESSs in matrix games. These are not, however, in direct contradiction and give different outlooks on evolutionary process. We believe evolutionary optimization is another area where a role for quantum mechanics exists and quantum game theory provides hints to find it.
4 Conclusion

In this paper, using Marinatto and Weber’s scheme [11] to play a quantum game, we explored how quantization of matrix games can give or take away evolutionary stability to attractors of replicator dynamics when it is the underlying process of the game. We considered the effects of quantization on a saddle or a center of the dynamics. We found quantization can be responsible for changing the evolutionary stability of an attractor of the dynamics. These results give a dynamic characterization to our previous results which treated the ESS idea as a static concept. We suggest these results can be of interest in evolutionary quantum computing and also in evolutionary optimization, both of which involve quantum interactions between ‘entities’ constituting a population.

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