A new mechanism for saturating unstable r-modes in neutron stars

B. Haskell\textsuperscript{1,2}, K. Glampedakis\textsuperscript{3,4} & N. Andersson\textsuperscript{5}

\textsuperscript{1} Max-Planck-Institut f"ur Gravitationsphysik, Albert-Einstein-Institute, Am M"uhlenberg 1, Potsdam, D-14776, Germany
\textsuperscript{2} School of Physics, The University of Melbourne, Melbourne, Victoria 3010, Australia
\textsuperscript{3} Departamento de F"ısica, Universidad de Murcia, Murcia, E-30100, Spain
\textsuperscript{4} Theoretical Astrophysics, University of T"ubingen, Auf der Morgenstelle 10, T"ubingen, D-72076, Germany
\textsuperscript{5} Mathematical Sciences and STAG Research Centre, University of Southampton, Southampton SO17 1BJ, UK

7 May 2014

ABSTRACT
We consider a new mechanism for damping the oscillations of a mature neutron star. The new dissipation channel arises if superfluid vortices are forced to cut through superconducting fluxtubes. This mechanism is interesting because the oscillation modes need to exceed a critical amplitude in order for it to operate. Once it acts the effect is very strong (and nonlinear) leading to efficient damping. The upshot of this is that modes are unlikely to ever evolve far beyond the critical amplitude. We consider the effect of this new dissipation channel on the r-modes, that may be driven unstable by the emission of gravitational waves. Our estimates show that the fluxtube cutting leads to a saturation threshold for the instability that can be smaller than that of other proposed mechanisms. This suggests that the idea may be of direct astrophysical relevance.

1 CONTEXT

Neutron stars represent a hands-off laboratory for physics under extreme conditions, and may ultimately provide a complement to information gleaned from particle colliders like the LHC. While such terrestrial experiments probe hot plasmas at relatively low densities, the core of a neutron star requires an understanding of the cold dense part of the QCD phase diagram (Alford et al. 2008). To gain access to this information we need to accurately model how a realistic neutron star interior connects to its exterior and affects observable features.

A commonly considered example involves the cooling of the star. Which processes lead to the star cooling down and how does heat flow from the interior to the surface? By matching models of possible scenarios to X-ray data for isolated neutron stars, we may be able to constrain the theory. An excellent recent example of this is provided by the observed real-time cooling of the remnant in Cassiopeia A, which has provided the first true constraint on the superfluid transition temperature for the star’s core (Shternin et al. 2011; Page et al. 2011).

Another aspect of the problem relates to the dynamics of the star’s complex core. A neutron star undergoes a number of changes as it evolves and provided that these are dramatic enough, various stellar oscillation modes may be excited. These could, in turn, affect the emission pattern of the star (either in X-ray or radio) provided that the interior fluid motion has a significant effect on the star’s magnetosphere. This has led to the development of neutron star astero-seismology, where the aim is to use future observations to probe the star’s interior in the same way that helio-seismologists have successfully constrained the interior physics on the Sun.

A breakthrough in this area came with the observations of quasi periodic oscillations in the X-ray tails of large magnetar flares (Strohmayer & Watts 2003). Early, relatively naive, models suggested that the observed oscillations could be identified with various elastic oscillation modes of the star’s crust (Piro 2005; Samuelsson & Andersson 2007). More recent work has attempted, not yet completely successfully, to account for the anticipated strong magnetic field effects (Colaiuda & Kokkotas 2012; Gabler et al. 2013). This is a very difficult problem, but there has been clear progress in the last few years.

Since neutron stars are distant, one would expect their oscillations to be excited to detectable amplitudes only under exceptional circumstances. Such events would be rare, like the magnetar flares. However, there is an exception to this rule. Modes of oscillation may become unstable at various instances during the star’s life. Provided an unstable mode is allowed to grow large enough, such instabilities may lead to a detectable signal and may also have an indirect effect on the star’s evolution (say of the spin). A number of possible instabilities have been discussed in the literature. As far as mature neutron stars are concerned, the most promising ideas involve the Coriolis driven r-modes, which somewhat counter-intuitively may become unstable due to the gravitational waves they emit (Andersson 1998; Friedman & Morsink 1998). This has stimulated a large
body of work on the nature of the r-modes, the gravitational wave signal they would be associated with and the physics that may affect the growth of the instability. A number of possible damping and saturation mechanisms have been suggested over the last 15 years or so (Arras et al. 2003; Nayyar & Owen 2006; Bondarescu et al. 2007; 2009; Haskell et al. 2009; 2010; Andersson et al. 2011; Rezzolla et al. 2000; Bildsten & Ushomirsky 2004; Glampedakis & Andersson 2006; Gusakov et al. 2013). Nevertheless, the conclusions from state-of-the-art modelling remain relatively unaffected. The r-mode instability is likely to set a spin-threshold for neutron stars. This is an important observation since the fastest observed radio pulsars and accreting neutron stars spin well below the theoretical break-up limit (Chakrabarty et al. 2003; Patruno 2010). A mechanism is required to explain this, and the r-mode instability appears to fit the bill. Furthermore, a recent analysis of the problem has shown that the theoretical predictions for the r-mode instability window for a ‘minimal’ neutron star model, which does not include superfluidity or the appearance of exotic particles (such as hyperons or deconfined quarks) in the core, is not consistent with current X-ray observations of Low Mass X-ray Binaries (LMXBs) (Ho et al. 2011; Haskell et al. 2012). There is, therefore, a clear need to include additional effects in our modelling, such as superfluidity and superconductivity in the core of the star.

This paper introduces a new mechanism to the r-mode scenario. The argument hinges on the star’s core and builds on the fact that there is likely to be a region where superfluid neutron vortices co-exist with superconducting protons. As has been argued in different contexts, such a region may have decisive impact on the star’s dynamics. Due to the interaction between superfluid vortices and magnetic fluxtubes, any changes in the star’s vorticity (the bulk rotation or the fluid motion associated with an oscillation mode) may be coupled to the magnetic field. This suggests two scenarios. In the first, the vortices become pinned to the, more plentiful, fluxtubes. In the second scenario, the vortices can cut through the flux tubes, but at a cost. This latter process is expected to be highly dissipative. It is this possibility that we explore in this paper.

2 BRIEF SUMMARY

The fact that superfluid dynamics is damped by a mutual friction arising from the interaction between quantised vortices and other components in the mixture (typically, phonons in laboratory studies of He3 and electrons in a neutron star core) is well established. The main idea dates back to work by Hall and Vinen in 1955 (Hall & Vinen 1956). They introduced a linear friction between superfluid (Helium) vortices and the “normal” component (represented by phonons). Balancing this force to the Magnus force that would drive the vortices to move along with the superfluid condensate in the absence of friction, they deduced the functional form for the force, \( F_{mf} \), in the following, that couples the two “fluid” components in the system; the superfluid condensate and the normal component.

In the standard picture, the vortex friction, \( f_\mu \), is taken to be linear in the relative velocity \( \mathbf{u} \) between the vortices and the normal fluid;

\[
\mathbf{f}_\mu = \rho_\mu \kappa \mathbf{R} \mathbf{u}
\]  

where \( \rho_\mu \) and \( \kappa \) are, respectively, the density of the superfluid and the quantum of circulation associated with each vortex. The dimensionless friction coefficient, \( \mathbf{R} \), is assumed to be velocity-independent. The force balance that controls the motion of individual vortices leads to a linear algebraic relation \( \mathbf{u} = \mathbf{U}(\mathbf{w}) \), where \( \mathbf{w} \) is the relative velocity between the condensate and the normal fluid. Inverting this relation one finds that the relative fluid flow is damped according to

\[
\partial_t \mathbf{w} + \{\} = -\frac{1}{x_p \rho_n} \mathbf{F}_{mf}
\]

where \( \mathbf{F}_{mf} \) is obtained from \( \mathbf{f}_\mu \) by using the inferred \( \mathbf{U}(\mathbf{w}) \) relation and combining the effect for an array of vortices. The brackets in \( \{\} \) represent fluid terms that are not relevant to this discussion and \( x_p = \rho_p / \rho \) is the normal fluid fraction (\( \rho = \rho_\mu + \rho_p = \) the total density).

In the case of superfluid neutron star dynamics, one can show that a similar relation applies provided that \( \rho_\mu \) and \( \rho_p \) are taken to be the neutron and proton densities. Hence, a relation like \( \{\} \) will affect the relative motion associated with any global oscillation mode. This means that we can extract a characteristic mutual friction dissipation timescale in terms of the mode energy \( E_{mode} \) (obtained as a volume integral of the inviscid velocity field) and the rate of work \( E_{int} \) done by \( \mathbf{F}_{mf} \). Provided the damping rate is slow compared to the dynamics of the mode, the timescale is well approximated by:

\[
\tau_{mf} \approx \frac{2E_{mode}}{E_{mf}}
\]  

This timescale can be very short if the mode under consideration has a significant counter-moving component. Detailed work has shown that this is the case for the fundamental \( f \)-mode, and as a result the gravitational-wave driven instability of those modes is severely suppressed in a superfluid star (Lindblom & Mendell 1995; Andersson et al. 2009). The conclusions for the Coriolis restored r-mode is different. These modes are affected by mutual friction to a much lesser extent, essentially because they are mainly horizontal (Lee & Yoshida 2003; Haskell et al. 2009; Passamonti et al. 2009).

In the following section we outline the derivation of the new friction force. The steps involved essentially repeat the analysis of Link (2003). Having done this, we will discuss the implications for the r-mode instability. Readers that are mainly interested in the astrophysical results (or may already be familiar with the fluxtube cutting mechanism) can proceed straight to section 4.
3 THE NEW FRICTION MECHANISM

3.1 Vortex-fluxtube pinning

The interaction between superfluid neutron vortices and superconducting proton fluxtubes in the outer core of a neutron star is thought to be key to the evolution of the system, possibly linking changes in spin to the evolution of the magnetic field. An important ingredient in this problem is the energy cost associated with superfluid vortices, which are expected to be magnetised due to the entrainment effect (Alpar, Langer & Sauls 1988), cutting through superconducting fluxtubes. As a rough estimate one may consider the energy associated with superposition of a neutron vortex and a proton fluxtube. This leads to what we will refer to as the pinning energy, \( f_{\text{pin}} \), acting on each moving vortex. Ignoring geometrical factors related to direction dependence, the force per intersection is of the order (Ruderman, Zhu & Chen 1998)

\[
F_{\text{int}} \approx \frac{E_{\text{int}}}{\Lambda_s^2} = \Lambda_s^2 B_n B_p
\]  

where the London penetration length \( \Lambda_s \) (which is of the order of few tens of \( fm \)) represents the typical size of the overlap region, while \( B_n \) and \( B_p \) are the magnetic fields carried by individual vortices and fluxtubes, respectively. The force per unit length of a given vortex is then

\[
f_{\text{pin}} \approx \frac{F_{\text{int}}}{d_p}
\]

where

\[
d_p \approx \left( \frac{B}{\phi_0} \right)^{1/2} \approx 3 \times 10^3 B_{12}^{-1/2} \text{ fm}
\]

Here \( B \) (and \( B_{12} = B/10^{12} \text{G} \)) is the macroscopic core magnetic field, \( \phi_0 \) is the quantum of magnetic flux and \( d_p \) is the typical distance separating the fluxtubes.

This estimate allows us to quantify how easy it is for a vortex to cut through the array of fluxtubes in a neutron star core. A necessary condition is that vortices do not pin onto the fluxtubes, which means that the Magnus force must exceed the pinning force. To make this quantitative, let us represent the vortex and fluxtube velocities by \( u_n \) and \( u_p \), respectively. Meanwhile, the macroscopic flows (that enter the averaged two-fluid hydrodynamics) are given by \( v_n \) (for the superfluid neutrons) and \( v_p \) (for the proton condensate). If a vortex is pinned to the fluxtubes, then we expect to have \( u_n = u_p \approx v_p \). Basically, it is natural to assume that the fluxtubes move with the proton condensate. This means that the velocity difference that enters into the Magnus force is approximated by \( u_n - v_n \approx v_p - v_n \equiv w \). Given this, we can obtain a minimum velocity lag, \( w_{\text{pin}} \), between the neutron and proton fluids below which vortex pinning is likely to take place (Link 2003)

\[
w_{\text{pin}} \approx \frac{f_{\text{pin}}}{\rho \kappa} \approx 1.5 \times 10^8 B_{12}^{1/2} \text{ cm/s}
\]  

We note here that in this expression (and the ones hereafter) only the dependance with respect to the magnetic field is shown while the fluid density has been set to \( \rho = 10^{14} \text{ g/cm}^3 \), a value representative of a neutron star outer core.

The estimate \( \Delta w \) will be of central importance later. The key point is that, as long as the relative velocity \( w \) between the two fluids is below \( w_{\text{pin}} \), the vortices will not be able to move relative to the fluxtubes. Hence, the damping mechanism that we will now discuss will not act.

3.2 Kelvin-wave damping

Once the pinning can no longer balance the Magnus force and the vortices start moving, they must cut through the fluxtube array to keep going. This may be a highly dissipative process due to the excitation of Kelvin waves along the vortex. This point was first argued by Epstein & Baym (1992) for vortices moving through the lattice of nuclei in the star’s crust, and later adapted by Link (2003) to the conditions in the core that we discuss here. In an effective theory, the waves on the vortex can be treated as particles, “kelvons”, with effective mass \( \mu \) and energy \( E_k = \hbar^2 k^2/2\mu \), where \( k \) is the associated wavenumber. If we let

\[
u = u_n - u_p
\]

be the relative vortex-fluxtube velocity, then the interaction at each intersection lasts a time interval \( t_{\text{int}} \sim \Lambda_s/u \). The kelvon energy can be estimated by using this characteristic timescale in the standard formula for an oscillator; \( E_k \approx \hbar/\int_{\text{int}} \) (ultimately originating from the uncertainty principle). This then leads to the characteristic wavenumber

\[
k \approx \left( \frac{2\mu}{\hbar \Lambda_s} \right)^{1/2} \equiv \frac{1}{\Lambda_s} \left( \frac{u}{\nu_A} \right)^{1/2}
\]

Given that the characteristic velocity

\[
u_A = \hbar/2\mu \Lambda_s \approx 10^9 \text{ cm/s}
\]

we should typically have \( k \Lambda_s \ll 1 \) in the case of neutron star dynamics. A more sophisticated analysis, leading to the same final estimate, can be found in Link (2003).

In order to calculate a dissipation rate we need the kelvons produced at different intersections of the same vortex to add incoherently. This requires \( k \Delta p \gg 1 \), which in turn leads to a lower limit for the relative vortex-fluxtube velocity:

\[
u_{\text{low}} \approx 6.5 \times 10^5 B_{12} \text{ cm/s}
\]

In order for the mechanism we discuss to operate efficiently we need \( u \gg u_{\text{low}} \). Note that \( u_{\text{low}} \gg w_{\text{pin}} \) when \( B \gtrsim 10^8 \text{ G} \) or so. The estimates we present are thus still consistent for the case of LMXBs, as long as the internal magnetic field is not much stronger than the inferred exterior dipolar magnetic field strength (which is inferred to be \( \approx 10^8 \text{ G} \)). If we want to consider significantly stronger magnetic fields we would need to first understand the behaviour at velocities in the range between \( w_{\text{pin}} \) and \( u_{\text{low}} \) better.

The energy released at each vortex/fluxtube intersection was determined by Link (2003). The result is

\[
\Delta E = \frac{2 \rho \kappa}{\pi \rho \kappa} (\nu_A u)^{-1/2}
\]

This suggests that the energy loss rate (per unit volume) is

\[
\dot{E}_{\text{int}} = \frac{N_n u}{d_p^2} \Delta E
\]

where \( N_n \) is the number of vortices per unit area. Alternatively, we can use the fact that (ignoring entrainment,
which only affects the estimate by a factor of order unity (Andersson, Sidery & Comer 2006) 2Ω ref ≈ N+nκ, to get
\[ \dot{E}_{\text{cut}} = \frac{4\Omega_0}{\pi n_o \kappa^2} f^2 \left( \frac{u}{v_n} \right)^{1/2} \] (14)

We can relate the rate (12) to the work done by a drag force (exerted on a unit length vortex segment) of the general form
\[ f_D = \rho u \kappa R u \] (15)
with a velocity-dependent coefficient \( R = R(u) \). Then from
\[ \dot{E}_{\text{cut}} = f_D \cdot u \] (16)
we infer that
\[ R = R_0 \left( \frac{v_n}{u} \right)^{3/2} \] (17)
with
\[ R_0 = \frac{2}{\pi} \left( \frac{f_{\text{pin}}}{\rho n_o \kappa^2} \right)^2 \rightarrow R_0 \approx 1.4 \times 10^{-10} B_{12} \] (18)

Finally, we can construct a hydrodynamical mutual friction force density exerted on the neutron fluid by averaging the force (15) over the vortex array. This leads to
\[ f_{\text{inf}} = N_a f_D \rightarrow f_{\text{inf}} = 2 \Omega_0 n_o R_0 \left( \frac{\Omega_0}{u} \right)^{3/2} u \] (19)

The key observation here is that, as soon as the vortices start to move relative to the fluxtubes they are likely to be prevented by a very strong friction. This is obvious since \( u_{\text{pin}} \) and \( u_{\text{low}} \) are both going to be much smaller than \( v_\Lambda \). The lower the relative velocity, the stronger this damping is. In practice, this means that the vortices are unlikely to be able to keep moving and the system will be driven back towards pinning.

4 R-MODE DAMPING AND SATURATION

In the previous section we outlined the argument that leads to vortices cutting though fluxtubes being a highly dissipative process. This argument is not original, but we believe this is the first time that the discussion has been framed in the context of a mutual friction force. The final result (19) allows us to consider the mechanism in a range of relevant contexts. For example, once the dissipation due to vortex-fluxtube cutting is expressed as a mutual friction force we can adapt it for the two-fluid hydrodynamics model used to model neutron star oscillations and instabilities. As an illustration of this analysis, let us try to estimate what the effect on the gravitational-wave driven r-mode instability may be.

In order to make use of the deduced mutual friction force in a problem involving the standard two-fluid model, we first of all need to replace the dependence on the relative velocity, \( u \), between vortices and fluxtubes with the relative velocity, \( w \), between the two macroscopic fluid components. The standard approach to this, pioneered by Hall and Vinen more than half a century ago (Hall & Vinen 1953), is to first balance the vortex force (19) by the Magnus force that acts on the vortices and the invert the relation to get an expression for \( u = u(w) \). The steps involved are straightforward in the case where the friction coefficient \( R \) is constant. When \( R \) is velocity dependent the analysis becomes slightly more involved and one should in principle consider the full problem, including relative flows in the background. However, in the present case we can bypass this problem by making a couple of (potentially debatable) assumptions.

First of all, on dynamical timescales the fluxtubes can be assumed to move with the protons, which means that \( u_p \approx v_p \). It is not quite so easy to justify a similar relation between the neutron fluid and vortex velocities. To make progress, we nevertheless assume that \( u_n \approx v_n \). This would be true for free vortices and it might be a reasonable approximation in the case of vortices moving at high speed through the fluxtubes. This is, in fact, the approximation underpinning the model in Section 3 so it make sense to make this approximation here as well. With these assumptions we simply have \( u = w \).

Now, from detailed two-fluid calculations (Haskell et al. 2000) we know that unstable r-modes have a particular relative velocity contribution. In general this contribution is position dependent, due to the density dependence of the superfluid pairing gaps. In order to keep things simple, we will nevertheless assume that this contribution is proportional to the average velocity perturbation, \( v \). This leads to
\[ w = \lambda v \rightarrow w \approx \lambda \Omega R \] (20)
where \( \alpha \) is the usual (dimensionless) r-mode amplitude (e.g. Owen et al. 1998). We know from actual mode calculations that the counter-moving contribution enters at higher order in the slow-rotation expansion such that
\[ \lambda = \lambda_0 \left( \frac{\Omega}{\Omega_K} \right)^2 \] (21)
where \( \Omega_K \) is the break-up frequency and \( \lambda_0 \) is taken to be a spin-independent factor. We have ignored the radial dependence of the r-mode’s velocity field, \( v \sim (r/R)^2 \), which should be reasonable as long as the main damping effect originates in the outer core of the star where this factor is of order unity. This is, of course, a simplification but in this first proof-of-principle discussion we prefer to proceed analytically rather than turn to numerical solutions. Given this attitude, we feel that this is a natural simplification to make.

If the mode has large enough amplitude to force vortices through fluxtubes, then \( w \lesssim u_{\text{pin}} \) which means that
\[ R \lesssim 2.5 \times 10^{-3} B_{12}^{1/4} \] (22)
It is worth noting that the deduced upper limit is about a factor \( \sim 10 – 100 \) larger than the (velocity-independent) drag coefficient associated with the standard mutual friction mechanism; scattering of electrons by vortices (Alpar, Langer & Sauls 1988; Andersson, Sidery & Comer 2006).

The corresponding damping timescale can be estimated in the usual way (see e.g. Andersson & Kokkotas 2001) by making use of (3). This argument involves the r-mode energy
\[ E_{\text{mode}} \approx \frac{1}{2} \alpha^2 \Omega^2 M R^2 J \] (23)
which leads to \( \dot{J} = 0.016 \) for an \( n = 1 \) polytropic (Owen et al. 1998). The mutual friction damping rate is given by
\[ \dot{E}_{\text{mf}} = \int \dot{E}_{\text{cut}} dV \] (24)
and our estimates lead to
\[ \dot{E}_{\text{mut}} \approx \frac{4}{\pi R_s^2} \int f_{\text{pin}}^2 \left( \frac{w}{v_s} \right)^{1/2} \, dV \] (25)

That is,
\[ \dot{E}_{\text{mut}} \approx \frac{8}{\pi f_k} \int f_{\text{pin}}^2 \left( \frac{\alpha \lambda_0}{R \kappa} \right)^{1/2} \int_{R_{\text{in}}}^{R} \frac{r^3}{\rho} dr \] (26)

We have assumed that fluxtube cutting takes place in the outer part of the stellar core, in the region \( R_{\text{in}} < r \lesssim R \) (where the coexistence of a neutron superfluid and a proton superconductor is likely) and that \( \lambda_0 \) and \( \rho \) are approximately uniform.

Through these arguments we obtain an order of magnitude estimate for the mutual friction damping timescale
\[ \tau_{\text{mut}} \approx 6 \times 10^{10} \lambda_0^{-1/2} \alpha^{3/2} \nu_{500}^{-1/2} B_8^{-1} s \] (27)

where \( \nu_{500} = \nu/500 \) Hz is the scaled spin-frequency of the star (\( \nu = \Omega/2\pi \)).

If we want to consider the relevance of the proposed mechanism for various astrophysical scenarios, then we need to provide an estimate for \( \lambda_0 \). This will require a more detailed numerical calculation for realistic superfluid parameters etcetera. However, we can use previous mode-calculations to get an idea of the likely range of values for this parameter. Extracting an averaged value from the r-mode study by Haskell et al. [2000] (assuming their pinning limit) we find that \( \lambda_0 \) ought to lie in the range
\[ (\lambda_0) \approx 0.1 - 1 \] (28)

both for strong and weak superfluidity models. This result is obviously not very precise, but it will allow us to assess whether the new damping mechanism is strong enough to warrant a more detailed investigation.

In considering possible astrophysical scenarios it is important to appreciate that the features of the new mechanism are rather different from the standard mutual friction. Most importantly, the dissipation due to fluxtube cutting is a non-linear process that saturates rather than the standard mutual friction mechanism. Moreover, the dissipation due to fluxtube cutting is non-linear (where the coexistence of a neutron superfluid and a proton superconductor is likely) and that \( \lambda_0 \) and \( \rho \) are approximately uniform.

5 ASTROPHYSICS: APPLICATION TO ACCRETING SYSTEMS

An obvious astrophysical setting where the fluxtube cutting scenario may apply is in fast spinning accreting neutron stars in Low-Mass X-ray Binaries. From previous considerations of the r-mode instability in this context [Brown & Ushomirsky 2000], we know that the mode amplitude required to achieve torque balance is
\[ \alpha_{\text{acc}} \approx 1.3 \times 10^{-7} \left( \frac{L_{\text{acc}}}{10^{35} \text{erg/s}} \right)^{1/2} \nu_{500}^{-7/2} \] (31)

Balancing the two mechanisms, as would be appropriate if the fluxtube cutting allows the r-mode to grow to the precise amplitude required to prevent further spin-up in an accreting system, we have
\[ \frac{\alpha_{\text{pin}}}{\alpha_{\text{acc}}} \approx 8 \left( \frac{\lambda_0}{0.1} \right)^{-1} B_8^{1/2} \left( \frac{L_{\text{acc}}}{10^{35} \text{erg/s}} \right)^{-1/2} \nu_{500}^{1/2} \] (32)

In order for the new mechanism to play a role in explaining the observed population, one would expect to have \( \alpha_{\text{pin}}/\alpha_{\text{acc}} \approx 1 \) for the fastest spinning systems.

As an example, let us consider 4U1608-522 which spins at 620 Hz and for which the averaged accretion luminosity is \( 5 \times 10^{36} \) erg/s. If the range we have suggested for \( \lambda_0 \) is reliable, then we find that the proposed scenario would work provided the interior magnetic field in this system is:
\[ B \approx (0.9 - 3) \times 10^8 \text{ G} \] (33)

This is in the range of the expected surface fields for these mature systems. Moreover, it is natural to assume that the interior field (which may initially be much stronger than the externally visible field) of an old neutron star would be of the same order of magnitude as that in the exterior. The main point here is that our rough estimates lead to a results that appears consistent with both observations and our understanding of these systems. This makes it plausible that the new mechanism does, indeed, have a role to play in the r-mode scenario. At the very least, it warrants a more detailed investigation.

It is also worth noting an alternative strategy. We could take \( \lambda_0 \) as a “free parameter”, which would make sense given our general ignorance of the conditions in the outer core of a neutron star. This parameter could then be constrained by observations relating to the magnetic field of fast spinning accreting neutron stars. As an example of this strategy, let us consider the data for IGR J00291+5931 (taking the observational constraints from Patruno [2010]). In this case, we have a spin frequency of 600 Hz, a luminosity of \( 6 \times 10^{36} \) erg/s and a suggested external field of \( B \approx 2 \times 10^8 \) G from the spin down rate in quiescence. From (32) we find that the accretion torque could be balanced by the fluxtube cutting mechanism as long as
\[ \lambda_0 \approx 0.16 \] (34)

comfortably inside the range suggested by the mode calculations. Again, this example suggests that the new mechanism should be relevant.

Finally, it is interesting to compare the mode amplitude \( \alpha_{\text{pin}} \) (essentially the saturation amplitude associated with fluxtube cutting) against previous results on r-mode saturation due to non-linear couplings with other inertial modes.
In general, different saturation mechanisms would be competing with each other, and the one leading to the smallest mode amplitude would be physically the most relevant.

The first incarnation of r-mode saturation by non-linear mode-couplings is the model of Arras et al. (2003); that work predicts a maximum r-mode amplitude

$$\alpha_{\text{A}} \approx 1.4 \times 10^{-3} \nu_{500}^{5/2}$$

(35)

This result has been refined by the more recent calculations of Bondarescu et al. (2007, 2009), resulting in a saturation amplitude $\alpha_{\text{sat}} \approx 0.1 \nu_8$ (for simplicity we retain the spin dependence of eqn. (35) but we note that the behaviour of the Bondarescu et al. (2007, 2009) saturation amplitude shows a rather complicated behaviour as a function of time).

Comparing this more recent mode-coupling saturation amplitude with our $\alpha_{\text{pin}}$,

$$\frac{\alpha_{\text{pin}}}{\alpha_{\text{sat}}} \approx 10^{-2} \left( \frac{\nu}{0.1} \right)^{-1} \nu_{8}^{1/2} \nu_{500}^{-11/2}$$

(36)

This suggests that the fluxtube cutting mechanism is competitive as an r-mode saturation mechanism, being at least as efficient as mode-coupling. This result supports our earlier claim that a more detailed investigation of the physics of fluxtube-vortex interaction as a source of friction for the r-mode instability is needed.

6 CONCLUDING DISCUSSION

To summarize our results, we have formulated a new type of vortex mutual friction force, based on the dissipative cutting of fluxtubes by fast moving vortices, and have studied its impact on the r-mode instability in superfluid neutron stars. The non-linear dependence of this force with respect to the relative vortex-fluxtube velocity leads to a rapid damping of the r-mode above a threshold amplitude at which the vortex array is forced to unpin from the fluxtubes. Effectively, this fluxtube-cutting friction provides a natural saturation mechanism for the r-mode instability.

We have highlighted the fact that our results may have important implications for the physics of accreting neutron stars in LMXBs. We have shown that the saturation amplitude due to fluxtube cutting (represented by $\alpha_{\text{pin}}$, see eqn. 29) could be smaller than the maximum amplitude set by non-linear couplings between the r-mode and other inertial modes. Remarkably, this same amplitude could also be comparable to that required for balancing the accretion spin-up torque. In practice this means that the saturation amplitude we calculate is such that it may allow for gravitational wave emission to be setting the spin equilibrium period for some of the hotter, faster, systems (which are in fact the best candidates for gravitational wave detection). However, for slower neutron stars in LMXBs, our amplitude may be small enough to never allow the mode to grow to the point where gravitational wave emission would influence the spin evolution (or, indeed, thermal evolution) of the system. This would allow a system to ‘live’ inside the standard r-mode instability window, without the need of additional damping mechanisms to explain the observations of Haskell et al. (2012) and Mahmoodifar & Strohmayer (2013).

A more sophisticated treatment of the problem is of course necessary to accurately predict the relative strength of gravitational wave, accretion and electromagnetic spin down torques. This is of key importance for gravitational wave detection, given that recent analysis have shown that the dynamics of many sources is probably dictated by electromagnetic and accretion torques, with only a few systems likely to be interesting targets for next generation gravitational wave detectors Haskell & Patruno (2011), Patruno et al. (2012), Mahmoodifar & Strohmayer (2013).

There are aspects of the fluxtube-cutting friction that have not been discussed in any detail here. For instance, an important issue is the fact that, as any other frictional force, the mechanism discussed here should provide an additional source of heating in the stellar interior. However, calculating the rate of heating is a difficult task because the quasi-stationary state of the system is likely to be that of pinning. This ‘pinning regime’ may not actually translate to physically immobilised vortices. The system’s finite temperature may drive vortex creep with $v \sim w_{\text{pin}}$. Unfortunately, in this velocity regime our analysis breaks down, making it impossible to make any prediction about dissipation and heating. We can, nevertheless, obtain an upper limit for the heating rate by using the mode damping rate of the cutting regime. By then balancing the energy dissipation rate in (14) with the energy carried away by neutrino neutrino emission due to Cooper paring, $E_{\nu} = 1.5 \times 10^{37} T_{8}^{8}$ erg/s, with $T_{8}$ the temperature in units of $10^{8}$ K, we obtain core temperatures of order $10^{8}$ K. This temperature is consistent with the observed surface temperatures of LMXBs, especially for the faster systems which are also likely to be the most interesting for gravitational wave emission Haskell et al. (2012), Mahmoodifar & Strohmayer (2013).

A more detailed understanding of vortex-fluxtube interactions over the entire range of the expected velocities would represent a key advance in this area, relevant for many aspects of neutron star dynamics. The mechanism we have discussed may not only be crucial for our understanding of the nonlinear development of the r-mode instability; it could also impact on models of pulsar glitches Lim et al. (2012), Haskell et al. (2013) and the combined magneto-rotational evolution of neutron stars Reiterman, Zhu & Chen (1998), Glampedakis & Andersson (2011), Glampedakis et al. (2012). To make further progress we need to sharpen our computational tools and develop models that account for the mesoscopic vortex-fluxtube interactions while, at the same time, track the macroscopic fluid dynamics. This is a challenging problem but the estimates we have presented provide clear motivation for future efforts.

ACKNOWLEDGMENTS

KG is supported by the Ramón y Cajal Programme of the Spanish Ministerio de Ciencia e Innovación and by the German Science Foundation (DFG) via SFB/TR7. BH is supported by the Australian Research Council (ARC) via a Discovery Early Career Award (DECRA). NA is supported by STFC in the UK.
REFERENCES

Arras P., Flanagan E.E., Morsink S.M., Schenk A.K., Teukolsky S.A., Wasserman I., 2003, ApJ, 591, 1129
Alford M. A., Schmitt A., Rajagopal K., Schäfer T., 2008, Rev.Mod.Phys., 80, 1455
Alpar M. A., Langer S. A., Sauls J. A., 1988, ApJ, 282, 533
Andersson N., 1998, ApJ, 502, 714
Andersson N., Kokkotas K., 2001, Int. J. Mod. Phys. D, 10, 381
Andersson N., Sidery T., Comer G.L., 2006, MNRAS, 368, 162
Andersson N., Glampedakis K., Haskell B., 2009, Phys.Rev.D, 79, 103009
Andersson N., Haskell B., Comer G. L., 2010, Phys.Rev.D, 82, 023007
Baym G., Pethick C., Pines D., 1969, Nature, 224, 673
Bildsten L., Ushomirsky G., 2000, ApJ 529, L33
Bondarescu R., Teukolsky S.A., Wasserman I., 2007, Phys. Rev. D, 76, 064019
Bondarescu R., Teukolsky S.A., Wasserman I., 2009, Phys. Rev. D, 79, 104003
Brown E. F., Ushomirsky G., 2000, ApJ 536, 915
Chakrabarty D., Morgan E. H., Muno M. P., et al, 2003, Nature, 424, 42
Colaiuda A., Kokkotas K.D., 2012, MNRAS 423, 811
Epstein R. I. Baym G., 1992, ApJ, 387, 276
Friedman J. L., Morsink S. M., 1998, ApJ, 502, 714
Gabler M., Cerda-Duran P., Font J. A., Müller E., Stergioulas N., 2013, MNRAS, 430, 1811
Glampedakis K., Andersson, N., Samuelsson L., 2011, MNRAS, 410, 805
Glampedakis K., Andersson, N., 2011, ApJ Lett., 740, L35
Glampedakis K., Andersson, N., 2006, MNRAS 371, 1311
Gusakov M. E., Chugunov A. I., Kantor E. M., 2013, eprint arXiv:1305.3825
Hall H.E., Vinen W.F., 1956, Proc. R. Soc. London A, 238, 215
Haskell, B., Andersson, N., Passamonti A., 2009, MNRAS 397, 1464
Haskell, B., Andersson, N., 2010, MNRAS 408, 1897
Haskell B., Patruno A., 2011, ApJ, 738, L14
Haskell, B., Degenaar N., Ho W. C. G., 2012, MNRAS 424, 93
Haskell, B., Pizzochero P. M., Seveso S., ApJ Lett., 764, L25
Ho W. C. G., Andersson N., Haskell B., 2011, Phys.Rev.Lett., 107, 101101
Lee U., Yoshiida S., 2003, ApJ, 586, 403
Lindblom L., Mendell G., 1995, ApJ, 444, 804
Link B., 2003, Phys. Rev. Lett., 91, 101101
Link B., 2012, eprint arXiv:1211.2200
Mahmoodifar S., Strohmayer T., eprint arXiv:1302.1204
Nayyar M., Owen B. J., 2007, Phys.Rev.D 73, 084001
Owen B. J., Lindblom L., Cutler C., Schutz B. F., Vecchio A., Andersson N., 1998, Phys.Rev.D, 58, 084020
Passamonti A., Haskell, B., Andersson, N., 2009, MNRAS 396, 951
Patruno A., 2010, ApJ, 722, 909
Patruno A., Haskell B., D’Angelco C., 2012, ApJ, 746, 9
Page D., Prakash M., Lattimer J.M., Steiner A.W., 2011, Phys. Rev. Lett.,106, 081101
Piro A., 2005, ApJ 634, L153
Rezzolla L., Lamb F. K., Shapiro S. L., 2000, ApJ, 531, L139
Ruderman, M., Zhu T., Chen K., 1998, ApJ, 492, 267
Samuelsson L., Andersson N., 2007, MNRAS 374, 256
Shternin P.S., Yakovlev D.G., Heinke C.O., Ho W.C.G., Patnaude D.J., 2011, MNRAS, 412, L108
Strohmayer, T. E. and Watts, A. L., 2005, ApJ, 632, L111