Spaces with contravariant and covariant affine connections and metrics

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Abstract

The theory of spaces with different (not only by sign) contravariant and covariant affine connections and metrics \((L_n, g)\)-spaces is worked out within the framework of the tensor analysis over differentiable manifolds and in a volume necessary for the further considerations of the kinematics of vector fields and the Lagrangian theory of tensor fields over \((L_n, g)\)-spaces. The possibility of introducing different (not only by sign) affine connections for contravariant and covariant tensor fields over differentiable manifolds with finite dimensions is discussed. The action of the deviation operator, having an important role for deviation equations in gravitational physics, is considered for the case of contravariant and covariant vector fields over differentiable manifolds with different affine connections (called \(L_n\)-spaces). A deviation identity for contravariant vector fields is obtained. The notions covariant, contravariant, covariant projective and contravariant projective metric are introduced in \((L_n, g)\)-spaces. The action of the covariant and the Lie differential operator on the different type of metrics is found. The notions of symmetric covariant and contravariant (Riemannian) connection are determined and presented by means of the covariant and contravariant metric and the corresponding torsion tensors. The different types of relative tensor fields (tensor densities) as well as the invariant differential operators acting on them are considered. The invariant volume element and its properties under the action of different differential operators are investigated.

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1 Introduction

In the present review, the differentiable manifolds with different (not only by sign) contravariant and covariant affine connections and metrics [spaces with contravariant and covariant affine connections and metrics, \((\mathcal{L}_n, g)\)-spaces] are considered as models of the space-time. On the grounds of the differential-geometric structures of the \((\mathcal{L}_n, g)\)-spaces the kinematics of vector fields and the dynamics of tensor fields has been worked out as useful tools in mathematical models for description of physical interactions and especially the gravitational interaction in the modern gravitational physics. The general results found for differentiable manifolds with different (not only by sign) contravariant and covariant affine connections and metrics can be specialized for spaces with one affine connection and a metric [the s. c. \((\mathcal{L}_n, g)\)-spaces] as well as for (pseudo) Riemannian spaces with or without torsion [the s.c. \(U_n\)- and \(V_n\)-spaces]. The most results are given either in index-free form or in a co-ordinate, or in a non-co-ordinate basis. The main objects taken in such type of investigations can be given in the following scheme

**Spaces with contravariant and covariant affine connections and metrics**

**differential operators**

- covariant differential operator,
- Lie differential operator,
- operator of curvature,
- deviation operator,
- extension operator,
- affine connections, metrics,
- special tensor fields, tensor densities, invariant volume element

**Kinematic characteristics**

of contravariant vector fields
relative velocity (shear, rotation and expansion velocities),
relative acceleration (shear, rotation and expansion accelerations),
development equations,
geodesic and auto-paralel equations, 
Fermi-Walker transports,
conformal transports

| Lagrangian theory of tensor fields
| Lagrangian density,
| variational principles,
| Euler-Lagrange’s equations,
| energy-momentum tensors

In this review we consider only the elements of the first part of the above scheme related to spaces with contravariant and covariant affine connections and metrics. The review appears as an introduction to the theory of the \( (\mathcal{L}_n, g) \)-spaces. It contains formulas necessary for the development of the mechanics of tensor fields and for constructing mathematical models of different dynamical systems described by the use of the main objects under consideration. The general results found for differentiable manifolds with different (not only by sign) contravariant and covariant affine connections and metrics can be specialized for spaces with one affine connection and a metric [the s. c. \((L_n, g)\)-spaces] as well as for (pseudo) Riemannian spaces with or without torsion [the s.c. \(U_n\)- and \(V_n\)-spaces]. The most results are given either in index-free form or in a co-ordinate, or in a non-co-ordinate basis. This has been done to facilitate the reader in choosing the right form of the results for his own further considerations. The main conclusions are summarized in the last section.

The \((\mathcal{L}_n, g)\)-spaces have interesting properties which could be of use in the theoretical physics and especially in the theoretical gravitational physics. In these type of spaces the introduction of a contravariant non-symmetric affine connection for contravariant tensor fields and the introduction of a symmetric (Riemannian, Levi-Civita connection) for covariant tensor fields is possible. On this grounds we can consider flat spaces \([\mathcal{M}_n, g)\)-spaces] with predetermined torsion for the contravariant vector fields and with torsion-free connection for the covariant vector fields. In analogous way these type of structures could be induced in (pseudo) Riemannian spaces \([\mathcal{V}_n, g)\)-spaces].

1.1 Space-time geometry and differential geometry

In the last few years new attempts \[1\] - \[3\] have been made to revive the ideas of Weyl \[4\], \[5\] for using manifolds with independent affine connection and metric (spaces with affine connection and metric) as a model of space-time in the
theory of gravitation. In such spaces the connection for co-tangent vector fields (as dual to the tangent vector fields) differs from the connection for the tangent vector fields only by sign. The last fact is due to the definition of dual vector bases in dual vector spaces over points of a manifold, which is a trivial generalization of the definition of dual bases of algebraic dual vector spaces from the multi-linear algebra. On the one hand, the hole modern differential geometry is built as a rigorous logical structure having as one of its main assumption the canonical definition for dual bases of algebraic dual vector spaces (with equal dimensions). On the other hand, the possibility of introducing a non-canonical definition for dual bases of algebraic dual vector spaces (with equal dimensions) has been pointed out by many mathematicians who have not exploited this possibility for further evolution of the differential-geometric structures and its applications. The canonical definition of dual bases of dual spaces is so naturally embedded in the ground of the differential geometry that no need has occurred for changing it. But the last time evolution of the mathematical models for describing the gravitational interaction on a classical level shows a tendency to generalizations, using spaces with affine connection and metric, which can also be generalized using the freedom of the differential-geometric preconditions. It has been proved that an affine connection, which in a point or over a curve in Riemannian spaces can vanish (a fact leading to the principle of equivalence in ETG), can also vanish under a special choice of the basic system in a space with affine connection and metric. The last fact shows that the equivalence principle in the ETG could be considered only as a physical interpretation of a corollary of the mathematical apparatus used in this theory. Therefore, every differentiable manifold with affine connection and metric can be used as a model for space-time in which the equivalence principle holds. But if the manifold has two different (not only by sign) connections for tangent and cotangent vector fields, the situation changes and is worth being investigated.

The basic notions in the differential geometry related to the notions considered in this review are defined for the most part in textbook and monographs on differential geometry [see for example].

2 Algebraic dual vector spaces. Contraction operator

The notion of algebraic dual vector space can be introduced in a way in which the two vector spaces (the considered and its dual vector space) are two independent (finite) vector spaces with equal dimensions.

Let $X$ and $X^*$ be two vector spaces with equal dimensions $\dim X = \dim X^* = n$. Let $S$ be an operator (mapping) such that to every pair of elements $u \in X$ and $p \in X^*$ sets an element of the field $K$ ($R$ or $C$), i.e.

$$S : (u, p) \rightarrow z \in K , u \in X , p \in X^* .$$

(1)
Definition 1  The operator (mapping) $S$ is called contraction operator $S$ if it is a bilinear symmetric mapping, i.e. if it fulfils the following conditions:

(a) $S(u, p_1 + p_2) = S(u, p_1) + S(u, p_2)$, $\forall u \in X, \forall p_i \in X^*$, $i = 1, 2$.
(b) $S(u_1 + u_2, p) = S(u_1, p) + S(u_2, p)$, $\forall u_i \in X$, $i = 1, 2$, $\forall p \in X^*$.
(c) $S(\alpha u, p) = S(u, \alpha p) = \alpha S(u, p)$, $\alpha \in K$.
(d) Nondegeneracy: if $u_1, \ldots, u_n$ are linear independent in $X$ and $S(u_i, p) = 0$, $i = 1, \ldots, n$, then $p$ is the null element in $X^*$. In an analogous way, if $p_1, \ldots, p_n$ are linear independent in $X^*$ and $S(u, p_i) = 0$, $i = 1, \ldots, n$, then $u$ is the null element in $X$.
(e) Symmetry: $S(u, p) = S(p, u)$, $\forall u \in X$, $\forall p \in X^*$.

Let $e_1, \ldots, e_n$ be an arbitrary basis in $X$, and let $e_1', \ldots, e_n'$ be an arbitrary basis in $X^*$. Let $u = u^i e_i \in X$ and $p = p^k e^k \in X^*$. From the properties (a) - (c) it follows that

$$S(u, p) = f^k_i u^i p^k, \quad (2)$$

where

$$f^k_i = S(e_i, e^k) = S(e^k, e_i) \in K. \quad (3)$$

In this way the result of the action of the contraction operator $S$ is expressed in terms of a bilinear form. The property non-degeneracy (d) means the non-degeneracy of the bilinear form. The result $S(u, p)$ can be defined in different ways by giving arbitrary numbers $f^k_i \in K$ for which the condition $\det(f^k_i) \neq 0$ and, at the same time, the conditions (a) - (d) are fulfilled.

Definition 2  (Mutually) algebraic dual vector spaces. The spaces $X$ and $X^*$ are called (mutually) dual spaces if a contraction operator acting on them is given and they are considered together with this operator $\langle i.e. \langle X, X^*, S \rangle \rangle$ with $\dim X = n = \dim X^*$ defines the two (mutually) dual spaces $X$ and $X^*$.

The definition for (mutually) algebraic dual spaces allows for a given vector space $X$ an infinite number of vector spaces $X^*$ (in different ways dual to $X$) to be constructed. In order to avoid this non-uniqueness Efimov and Rosendorn introduced the notion equivalence between dual vector spaces [which is an additional condition to the definition of (mutually) dual spaces].

Definition 3  Equivalent dual to $X$ vector spaces. Let $X_1^*$ and $X_2^*$ be two $n$-dimensional vector spaces, dual to $X$. If a linear isomorphism exists between them, such that

$$S(u, p) = S(u, p'), \ \forall u \in X, \forall p \in X_1^*, p' \in X_2^*, \quad (4)$$

where $p'$ is the element of $X_2^*$, corresponding to $p$ of $X_1^*$ by means of the linear isomorphism, then $X_1^*$ and $X_2^*$ are called equivalent dual to $X$ vector spaces.

Proposition 4  All linear (vector) spaces, dual to a given vector space $X$, are equivalent to each other.
For proving this proposition, it is enough to be shown that if for \( X \) and \( X^\ast \) is given an arbitrary \( S \), then for an arbitrary basis \( e_1, ..., e_n \in X \) one can find an unique dual to it basis \( e^1, ..., e^n \) in the space \( X^\ast \), i.e. \( e^1, ..., e^n \in X^\ast \) can be found in an unique way so that \( S(e_i, e^k) = f^k_i \), where \( f^k_i \in K \) are preliminary given numbers [20]. The proof is analogous to the proof given by Efimov and Rosendorn [7] for the case \( S = C : C(e_k, e^l) = g^l_k, \ g^l_k = 1 \) for \( k = i, \ g^l_k = 0 \) for \( k \neq i \). \( C(e_k, e^l) = g^l_k \) means that the dual to \( \{ e_k \} \) basic vector field \( e^l \) is orthogonal to all basic vectors \( e_k \) for which \( k \neq i \). The contraction operator \( C \) is the corresponding to the canonical approach mapping

\[
C(u, p) = C(p, u) = p(u) = p_i . u^i . \tag{5}
\]

The new definition of algebraic dual spaces is as a matter in fact corresponding to that in the common approach. Only the dual basic vector \( e^i \) is not orthogonal to the basic vectors \( e_k : S(e_k, e^i) = f^k_i \neq g^l_k \). It is enough to be noticed that for an arbitrary element \( p \in X^\ast \) the corresponding linear form

\[
S(u, p) = p_i \cdot u^i = p_i \cdot f^i_k \cdot u^k = p_i \cdot u^i \tag{6}
\]

is given, where \( p_1, ..., p_n \) are the constant components of a given vector \( p \in X^\ast \). The last equality can be written also in the form

\[
S(u, p) = S(p, u) = p(u) = p_i . u^i . \tag{7}
\]

Remark 1 The generalization of the notion of algebraic dual spaces for the case of vector fields over a differentiable manifold is a trivial one. The vector fields ars considered as sections of vector bundles over a manifold. The vector bases become dependent on the points of the manifold and the numbers \( f^i_j \) are considered as functions over the manifold.

Remark 2 If the basic vectors in the tangential space \( T_x(M) \) at a point \( x \) of a manifold \( M \) (\( \dim M = n \)) are the co-ordinate vector fields \( \partial_i \) and in the dual vector space (the co-tangent space) \( T^*_x(M) \) the basis \( \{ dx^k \} \) is defined as a dual to the basis \( \{ \partial_i \} \), where \( dx^k \) are the differentials of the co-ordinates \( x^k \) of the point \( x \) in a given chart, then \( S(\partial_i, dx^k) = f^k_i [f^k_i \in C^\infty(M)] \). After multiplication of the last equality with \( f^l_k \) and taking into account the relation \( f^k_j \cdot f^l_k = g^l_k \) the condition follows \( S(\partial_i, f^l_k \cdot dx^k) = g^l_k \), which is equivalent to the result of the action of the contraction operator \( C \) over the vectors \( \partial_i \) and \( e^l \), where \( e^l = f^l_k \cdot dx^k \). The new vectors \( e^l \) are not in the general case co-ordinate differentials of the co-ordinates \( x^l \) at \( x \in M \). They would be differentials of new co-ordinates \( x^l = x^l'(x^k) \) if the relation \( dx^l = A_k l \cdot dx^k \) is connected with the condition \( e^l = dx^l \) and \( x^l = \int_0^l dx^l \). In analogous way, for the case, when \( S(f^l_i \cdot \partial_i, dx^k) = g^l_k \) the new vectors \( e_i = f^l_i \cdot \partial_i \) in the general case are not again co-ordinate vector fields \( \partial_i \); \( e_i \) would be again co-ordinate vector fields if by changing the charts (the co-ordinates) at a point \( x \in M \) the condition \( f^l_i = \frac{\partial x^l}{\partial x^i} \) is fulfilled.
Thus, the definition of algebraic dual vector fields over manifolds by means of the contraction operator $S$ as a generalization of the contraction operator $C$ allows considerations including functions $f^i \, j(x^k)$ instead of the Kronecker symbol $\delta^i_j$.

The contraction operator $S$ can be easily generalized to a multilinear contraction operator $\mathcal{S}$.

3 Contravariant and covariant affine connections.

Covariant differential operator

3.1 Affine connection. Covariant differential operator

The notion affine connection can be defined in different ways but in all definitions a linear mapping is given, which to a given vector of a vector space over a point $x$ of a manifold $M$ juxtaposes a corresponding vector from the same vector space at this point. The corresponding vector is identified as vector of the vector space over another point of the manifold $M$. The way of identification is called transport from one point to another point of the manifold.

Vector and tensor fields over a differentiable manifold are provided with the structure of a linear (vector) space by defining the corresponding operations at every point of the manifold.

Definition 5 Affine connection over a differentiable manifold $M$. Let $V(M)$ ($\dim M = n$) be the set of all (smooth) vector fields over the manifold $M$. The mapping $\nabla : V(M) \times V(M) \to V(M)$, by means of $\nabla(u, w) \to \nabla_u w$, $u, w \in V(M)$, with $\nabla_u$ as a covariant differential operator along the vector field $u$ (s. the definition below), is called affine connection over the manifold $M$.

Definition 6 A covariant differential operator (along the vector field $u$). The linear differential operator (mapping) $\nabla_u$ with the following properties

(a) $\nabla_u (v + w) = \nabla_u v + \nabla_u w$, $u, v, w \in V(M)$,
(b) $\nabla_u (f \cdot v) = (uf) \cdot v + f \cdot \nabla_u v$, $f \in C^r(M)$, $r \geq 1$,
(c) $\nabla_u (v + w) = \nabla_u v + \nabla_u w$,
(d) $\nabla_u (v \cdot f) = f \cdot \nabla_u v$,
(e) $\nabla_u (v \otimes w)$ = $\nabla_u v \otimes w + v \otimes \nabla_u w$ (Leibniz rule), $\otimes$ is the sign for the tensor product,

is called covariant differential operator along the vector field $u$.

The result of the action of the covariant differential operator $\nabla_u v$ is often called covariant derivative of the vector field $v$ along the vector field $u$.

In a given chart (coordinate system), the determination of $\nabla e_\alpha e_\beta$ in the basis $\{e_\alpha\}$ defines the components $\nabla^\gamma_{\alpha \beta}$ of the affine connection $\nabla$.

$$\nabla e_\alpha e_\beta = \nabla^\gamma_{\alpha \beta} e_\gamma, \quad \alpha, \beta, \gamma = 1, ..., n.$$ (8)
\{\nabla_{\alpha\beta}^\gamma\}\) have the transformation properties of a linear differential geometric object \[\[\ref{10},\ 27]\].

**Definition 7** Space with affine connection. Differentiable manifold \(M\), provided with affine connection \(\nabla\), i.e. the pair \((M, \nabla)\), is called space with affine connection.

### 3.2 Contravariant and covariant affine connections

The action of the covariant differential operator on a contravariant (tangential) co-ordinate basic vector field \(\partial_i\) over \(M\) along another contravariant co-ordinate basic vector field \(\partial_j\) is determined by the affine connection \(\nabla = \Gamma\) with components \(\Gamma_{ij}^k\) in a given chart (co-ordinate system) defined through

\[
\nabla_{\partial_j} \partial_i = \Gamma_{ij}^k \partial_k .
\]

For a non-coordinate contravariant basis \(e_\alpha \in T(M)\), \(T(M) = \cup_{x \in M} T_x(M)\),

\[
\nabla_{e_\beta} e_\alpha = \Gamma_{\alpha\beta}^\gamma e_\gamma .
\]

**Definition 8** Contravariant affine connection. The affine connection \(\nabla = \Gamma\) induced by the action of the covariant differential operator on contravariant vector fields is called contravariant affine connection.

The action of the covariant differential operator on a covariant (dual to contravariant) basic vector field \(\partial^\alpha\) \([\partial^\alpha \in T^*(M)\), \(T^*(M) = \cup_{x \in M} T^*_x(M)\)] along a contravariant basic (non-co-ordinate) vector field \(e_\beta\) is determined by the affine connection \(\nabla = P\) with components \(P_{\beta\gamma}^\alpha\) defined through

\[
\nabla_{e_\beta} \partial^\alpha = P_{\beta\gamma}^\alpha \partial^\gamma .
\]

For a co-ordinate covariant basis \(dx^i\)

\[
\nabla_{\partial_j} dx^i = P_{kj}^i dx^k .
\]

**Definition 9** Covariant affine connection. The affine connection \(\nabla = P\) induced by the action of the covariant differential operator on covariant vector fields is called covariant affine connection.

**Definition 10** Space with contravariant and covariant affine connections (\(\mathcal{L}_n\)-space). The differentiable manifold provided with contravariant affine connection \(\Gamma\) and covariant affine connection \(P\) is called space with contravariant and covariant affine connections.
The connection between the two connections $\Gamma$ and $P$ is based on the connection between the two dual spaces $T(M)$ and $T^*(M)$, which on its side is based on the existence of the contraction operator $S$. Usually commutation relations are required between the contraction operator and the covariant differential operator in the form

$$S \circ \nabla_u = \nabla_u \circ S.$$  \hfill (13)

If the last operator equality in the form $\nabla_{\partial_h} \circ S = S \circ \nabla_{\partial_h}$ is used for acting on the tensor product $dx^i \otimes \partial_j$ of two basic vector fields $dx^i \in T^*(M)$ and $\partial_j \in T(M)$, then the relation follows

$$f^i_{\ j, k} = \Gamma^i_{\ jk} f^i_{\ t} + P^i_{\ tk} f^t_{\ j} \quad \text{and} \quad f^i_{\ j, k} := \partial_k f^i_{\ j} \text{ (in a co-ordinate basis).}$$ \hfill (14)

The last equality can be considered from two different points of view:

1. If $P^i_{\ jk}(x^l)$ and $\Gamma^i_{\ jk}(x^l)$ are given as functions of co-ordinates in $M$, then the equality appears as a system of equations for the unknown functions $f^i_{\ j}(x^l)$. The solutions of these equations determine the action of the contraction operator $S$ on the basic vector fields for given components of both connections. The integrability conditions for the equations can be written in the form

$$R^m_{\ jkl} f^i_{\ m} + P^i_{\ mkl} f^m_{\ j} = 0 ,$$ \hfill (15)

where $R^m_{\ jkl}$ are the components of the contravariant curvature tensor, constructed by means of the contravariant affine connection $\Gamma$, and $P^i_{\ mkl}$ are the components of the covariant curvature tensor, constructed by means of the covariant affine connection $P$, where $[R(\partial_i, \partial_j)]dx^k = P^k_{\ l ij} dx^l$, $[R(\partial_i, \partial_j)]\partial_k = R^l_{\ k ij} \partial_l$, $R(\partial_i, \partial_j) = \nabla_{\partial_i} \nabla_{\partial_j} - \nabla_{\partial_j} \nabla_{\partial_i}$.

2. If $f^i_{\ j}(x^l)$ are given as functions of the co-ordinates in $M$, then the conditions for $f^i_{\ j}$ determine the connection between the components of the contravariant affine connection $\Gamma$ and the components of the covariant affine connection $P$ on the grounds of the predetermined action of the contraction operator $S$ on basic vector fields.

If $S = C$, i.e. $f^i_{\ j} = g^i_{\ j}$, then the conditions for $f^i_{\ j}$ are fulfilled for every $P = -\Gamma$, i.e.

$$P^i_{\ jk} = -\Gamma^i_{\ jk} .$$ \hfill (16)

This fact can be formulated as the following proposition

**Proposition 11** $S = C$ is a sufficient condition for $P = -\Gamma$ ($P^i_{\ jk} = -\Gamma^i_{\ jk}$).

**Corollary 12** If $P \neq -\Gamma$, then $S \neq C$, i.e. if the covariant affine connection $P$ has to be different from the contravariant affine connection $\Gamma$ not only by sign, then the contraction operator $S$ has to be different from the canonical contraction operator $C$ (if $S$ commutes with the covariant differential operator).

The corollary allows the introduction of different (not only by sign) contravariant and covariant connections by using contraction operator $S$, different from the canonical contraction operator $C$. 

10
Example 13 If $f^i_{\ j} = e^{\varphi} g^i_j$, where $\varphi \in C^r(M)$, $\varphi \neq 0$, then $P^i_{jk} = -\Gamma^i_{jk} + \varphi_{,k} g^i_j$.

3.3 Covariant derivatives of contravariant tensor fields

The action of a covariant differential operator along a contravariant vector field $u$ is called transport along a contravariant vector field $u$ (or transport along $u$).

The result of the action of the covariant differential operator on a tensor field is called covariant derivative of this tensor field.

The result $\nabla_u V$ of the action of $\nabla_u$ on a contravariant tensor field $V$ is called covariant derivative of a contravariant tensor field $V$ along a contravariant vector field $u$ (or covariant derivative of $V$ along $u$).

The action of the covariant differential operator on contravariant tensor fields with rank $> 1$ can be determined in a trivial manner on the grounds of the Leibniz rule which the operator obeys. Then the action of the operator $\nabla_{\partial_i}$ on a tensor basis $\partial_A = \partial_{j_1} \otimes \ldots \otimes \partial_{j_l}$ can be written in the form

$$\nabla_{\partial_i} \partial_A = \nabla_{\partial_i} [\partial_{j_1} \otimes \ldots \otimes \partial_{j_l}] = (\nabla_{\partial_i} \partial_{j_1} \otimes \partial_{j_2} \otimes \ldots \otimes \partial_{j_l}) + \ldots + (\partial_{j_1} \otimes \nabla_{\partial_i} \partial_{j_2} \otimes \ldots \otimes \partial_{j_l}) + \ldots + (\partial_{j_1} \otimes \ldots \otimes \nabla_{\partial_i} \partial_{j_l}) =$$

$$= \Gamma_{j_1,j_2}^{i_1,j_3} \partial_{i_1} \otimes \partial_{j_2} \otimes \ldots \otimes \partial_{j_l} + \ldots + \Gamma_{j_1,j_2}^{i_1,j_3} \partial_{i_1} \otimes \ldots \otimes \partial_{i_l} =$$

$$= (\sum_{k=1}^{l} g_{j_k}^l g_{j_1}^{i_k}, g_{j_2}^{i_2}, \ldots, g_{j_{k-1}}^{i_{k-1}}, g_{j_{k+1}}^{i_{k+1}}, \ldots, g_{j_l}^{i_l}) \Gamma_{ij}^j (\partial_{i_1} \otimes \ldots \otimes \partial_{i_l}) .$$

If we introduce the abbreviations

$$S_{Am}^{Bi} = - \sum_{k=1}^{l} g_{j_k}^i g_{j_1}^{i_k}, g_{j_2}^{i_2}, \ldots, g_{j_{k-1}}^{i_{k-1}}, g_{j_{k+1}}^{i_{k+1}}, \ldots, g_{j_l}^{i_l} ,$$

$$\Gamma_{Aj}^B = - S_{Am}^{Bi} \Gamma_{ij}^j , \quad A = j_1 \ldots j_l , \quad B = i_1 \ldots i_l ,$$

then $\nabla_{\partial_i} \partial_A$ can be written in the form

$$\nabla_{\partial_i} \partial_A = \Gamma_{Aj}^B \partial_B = - S_{Am}^{Bi} \Gamma_{ij}^j \partial_B .$$

The quantities $S_{Am}^{Bi}$ obey the following relations

(a) $S_{Bi}^{Aj} S_{Ak}^{Cl} = - g_{k}^{l} S_{Bk}^{Cl} , \quad \text{dim} \ M = n, \ l = 1, \ldots, N ,$

(b) $S_{Bi}^{Bj} = - N g_{Bi}^{C} g_{Cj}^{l} ,$

(c) $S_{Bi}^{Ai} = - N g_{Bi}^{A} ,$

where

$$g_{Bi}^{A} = g_{i_1}^{j_1}, g_{i_2}^{j_2}, \ldots, g_{i_{m-1}}^{j_{m-1}}, g_{i_{m}}^{j_{m}}, g_{i_{m+1}}^{j_{m+1}}, \ldots, g_{i_{l}}^{j_{l}}$$

is defined as multi-Kronecker symbol of rank $l$

$$g_{Bi}^{A} = 1 \quad \text{if} \ i_k = j_k \quad \text{for all} \ k \ \text{simultaneously},$$

$$= 0 \quad \text{if} \ i_k \neq j_k , \quad k = 1, \ldots, l .$$

The covariant derivative along a contravariant vector field $u$ of a contravariant tensor field $V = V^A \partial_A$ can be written in a co-ordinate basis as

$$\nabla_u V = (V^A, i_{\ i} + \Gamma_{Ai}^B V^B), u_\partial A = V^A, i_{\ i} u_\partial A ,$$

$$= V^A, i_{\ i} u_\partial A ,$$
where
\[ V^A_{;i} = V^A_{,i} + \Gamma^A_{Bi}V^B \]

is called first covariant derivative of the components \( V^A \) of the contravariant tensor field \( V \) along a contravariant co-ordinate basic vector field \( \partial_i \)
\[ \nabla_{\partial_i} V = V^A_{;i} \partial_A . \]

(23)

In an analogous way we find for the second covariant derivative \( \nabla_\xi \nabla_u V \)
\[ \nabla_\xi \nabla_u V = (V^A_{;ij}.u^j + V^A_{;j}.u^j_{;i}).\xi^i \partial_A = (V^A_{;j}.u^j_{;i}).\xi^i \partial_A , \]

where
\[ V^A_{;ij} = (V^A_{;ij}).u^j + \Gamma^A_{Bi}.V^B_{;ij} - \Gamma^k_{ji}.V^A_{;k} \]

is the second covariant derivative of the components \( V^A \) of the contravariant vector field \( V \). Here
\[ \nabla_\xi \nabla_u V - \nabla_u \nabla_\xi V = [(V^A_{;j} - V^A_{;j;i}).u^j_{;i} + V^A_{;j}(u^j_{;i} - \xi^j_{;i}u^i)].\partial_A . \]

(24)

3.4 Covariant derivatives of covariant tensor fields

In analogous way the covariant derivative of a covariant vector field can be written in the form
\[ \nabla_u p = (p_{i,j} + P^k_{ij}.p_k).u^j_{;i} = p_{i;j}.u^j_{;i} , \quad p \in T^*(M) , \]

(in a co-ordinate basis).

(25)

The action of the covariant differential operator on covariant tensor fields with rank > 1 is generalized in a trivial manner on the grounds of the Leibniz rule, which holds for this operator. Then the action of the operator \( \nabla_{\partial_j} \) on the basis \( dx^A = dx^j \otimes \ldots \otimes dx^k \) can be written in the form
\[ \nabla_{\partial_j} dx^B = P^B_{Aj}.dx^A = -S^B_{Am} P^m_{ij} dx^A , \]

(26)

where \( P^B_{Aj} = -S^B_{Am} P^m_{ij} \).

The covariant derivative of a covariant tensor field \( W = W_A dx^A = W_B.e^B \) can be written in the form
\[ \nabla_u W = (W_{A,j} + P^R_{Aj}.W_B).u^j_{;i} dx^A = W_{A;j}.u^j_{;i} dx^A ; \]

(in a co-ordinate basis).

(27)

The form of the covariant derivative of a mixed tensor field follows from the form of the derivative of contravariant and covariant basic tensor fields, and the Leibniz rule
\[ \nabla_u K = \nabla_u (K^A_{ Bj}.\partial_A \otimes dx^B) = K^A_{ Bj;u^j_{;i}}.\partial_A \otimes dx^B = (K^A_{ Bj} + \Gamma^C_{ Bj}.K^C_{ B;i} + P^D_{ Bj} K^D_{ A;i}).u^j_{;i} \partial_A \otimes dx^B , \]

(in a co-ordinate basis).

(28)
If the Kronecker tensor is defined in the form

\[ K r = g^i_j \partial_i \otimes dx^j = g^\alpha_\beta e_\alpha \otimes e^\beta, \]

then the components of the contravariant and the covariant affine connection differ from each other by the components of the covariant derivative of the Kronecker tensor, i.e.

\[ \Gamma^i_{jk} + P^i_{jk} = g^i_{;j}, \quad \Gamma^\alpha_{\beta\gamma} + P^\alpha_{\beta\gamma} = g^\alpha_{\beta/\gamma} . \]

**Remark 3** In the special case, when \( S = C \) and in the canonical approach \( g^i_{;j} = 0 (g^\alpha_{\beta/\gamma} = 0) \).

### 4 Lie differential operator

The Lie differential operator \( \mathcal{L}_\xi \) along the contravariant vector field \( \xi \) appears as an other operator, which can be constructed by means of a contravariant vector field. Its definition can be considered as a generalization of the notion Lie derivative of tensor fields \[ [28], [19], [11], [29] \].

**Definition 14** \( \mathcal{L}_\xi := \text{Lie differential operator along the contravariant vector field } \xi \text{ with the following properties:} \)

(a) \( \mathcal{L}_\xi : V \rightarrow \bar{V} = \mathcal{L}_\xi V , V, \bar{V} \in \otimes^i(M) . \)
(b) \( \mathcal{L}_\xi : W \rightarrow \bar{W} = \mathcal{L}_\xi W , W, \bar{W} \in \otimes^k(M) . \)
(c) \( \mathcal{L}_\xi : K \rightarrow \bar{K} = \mathcal{L}_\xi K , K, \bar{K} \in \otimes^i_k(M) . \)
(d) Linear operator with respect to tensor fields,

\( \mathcal{L}_\xi(\alpha.V_1 + \beta.V_2) = \alpha.\mathcal{L}_\xi V_1 + \beta.\mathcal{L}_\xi V_2 , \alpha, \beta \in F(R \text{ or } C) , V_i \in \otimes^i(M) , i = 1, 2 , \)
\( \mathcal{L}_\xi(\alpha.W_1 + \beta.W_2) = \alpha.\mathcal{L}_\xi W_1 + \beta.\mathcal{L}_\xi W_2 , W_i \in \otimes^k(M) , i = 1, 2 , \)
\( \mathcal{L}_\xi(\alpha.K_1 + \beta.K_2) = \alpha.\mathcal{L}_\xi K_1 + \beta.\mathcal{L}_\xi K_2 , K_i \in \otimes^i_k(M) , i = 1, 2 . \)
(e) Linear operator with respect to the contravariant field \( \xi \),

\( \mathcal{L}_\xi(\alpha, \beta, u) = \alpha.\mathcal{L}_\xi + \beta.\mathcal{L}_u , \alpha, \beta \in F(R \text{ or } C) , \xi, u \in T(M) . \)
(f) Differential operator, obeying the Leibniz rule,

\( \mathcal{L}_\xi(S \otimes U) = \mathcal{L}_\xi S \otimes U + S \otimes \mathcal{L}_\xi U , S \in \otimes^m(M) , U \in \otimes^k_i(M) . \)
(g) Action on function \( f \in C^r(M) , r \geq 1 , \)

\( \mathcal{L}_\xi f = \xi f , \xi \in T(M) . \)
(h) Action on contravariant vector field,

\( \mathcal{L}_\xi u = [\xi, u] , \xi, u \in T(M) , [\xi, u] = \xi \circ u - u \circ \xi , \)
\( \mathcal{L}_\xi e_\alpha = [\xi, e_\alpha] = -e_\alpha \xi^\beta - \xi^\gamma C^C\gamma_{\alpha \beta}.e_\beta , \)
\( \mathcal{L}_\xi e_\beta = [e_\alpha, e_\beta] = C^\alpha_{\beta \gamma}.e_\gamma , C^\alpha_{\beta \gamma} \in \otimes^1(M) , \)
\( \mathcal{L}_\xi \partial_i = -\xi^j_i.\partial_j , \quad \mathcal{L}_\xi \partial_j = [\partial_i, \partial_j] = 0 . \)
(i) Action on covariant basic vector field,

\( \mathcal{L}_\xi e^\alpha = k^\alpha_{\beta}(\xi).e^\beta , \quad \mathcal{L}_\xi e^\alpha = k^\alpha_{\beta/\gamma}.e^\beta , \)
\( \mathcal{L}_\xi dx^i = k^i_j(\xi).dx^j , \quad \mathcal{L}_\xi dx^i = k^i_{jk}.dx^j . \)
The action of the Lie differential operator on a covariant basic vector field is determined by its action on a contravariant basic vector field and the commutation relations between the Lie differential operator and the contraction operator $S$.

### 4.1 Lie derivatives of contravariant tensor fields

The Lie differential operator $\mathcal{L}_\xi$ along a contravariant vector field $\xi$ appears as another operator which can be constructed by means of a contravariant vector field. It is an operator mapping a contravariant tensor field $V$ in a contravariant tensor field $\tilde{V} = \mathcal{L}_\xi V$.

The action of the Lie differential operator along a contravariant vector field $\xi$ is called **dragging-along the contravariant vector field $\xi$** (or **dragging-along $\xi$**).

The result of the action $(\mathcal{L}_\xi V)$ of the Lie differential operator $\mathcal{L}_\xi$ on $V$ is called **Lie derivative of the contravariant tensor field $V$ along a contravariant vector field $\xi$** (or **Lie derivative of $V$ along $\xi$**).

The commutator of two Lie differential operators

$$[\mathcal{L}_\xi, \mathcal{L}_u] = \mathcal{L}_\xi \circ \mathcal{L}_u - \mathcal{L}_u \circ \mathcal{L}_\xi \quad (33)$$

has the properties:

(a) Action on a function

$$[\mathcal{L}_\xi, \mathcal{L}_u]f = (\mathcal{L}_\xi \circ \mathcal{L}_u - \mathcal{L}_u \circ \mathcal{L}_\xi)f = [\xi, u]f = (\nabla_\xi, \nabla_u)f \quad , \quad f \in C^r(M) \quad , \quad r \geq 2 \quad .$$

(b) Action on a contravariant vector field

$$[\mathcal{L}_\xi, \mathcal{L}_u]v = (\mathcal{L}_\xi \circ \mathcal{L}_u - \mathcal{L}_u \circ \mathcal{L}_\xi)v = \mathcal{L}_\xi \mathcal{L}_uv - \mathcal{L}_u \mathcal{L}_\xi v = \mathcal{L}_v \mathcal{L}_u \xi = -\mathcal{L}_u \mathcal{L}_v \xi = \mathcal{L}_\xi \mathcal{L}_uv \quad .$$

(c) The Jacobi identity

$$< [\mathcal{L}_\xi, [\mathcal{L}_u, \mathcal{L}_v]] > \equiv [\mathcal{L}_\xi, [\mathcal{L}_u, \mathcal{L}_v]] + [\mathcal{L}_v, [\mathcal{L}_\xi, \mathcal{L}_u]] + [\mathcal{L}_u, [\mathcal{L}_v, \mathcal{L}_\xi]] \equiv 0 \quad . \quad (34)$$

The Lie derivative of a contravariant vector field

$$\mathcal{L}_\xi u = [\xi, u] = (\mathcal{L}_\xi u^i) \partial_i = (\xi^k \cdot u^i \xi_{,k} - u^k \cdot \xi_{i,k}) \partial_i \quad , \quad (35)$$

where

$$\mathcal{L}_\xi u^i = \xi^k \cdot u^i \xi_{,k} - u^k \cdot \xi_{i,k} \quad (36)$$

is called **Lie derivative of the components $u^i$ of a vector field $u$ along a contravariant vector field $\xi$** (or **Lie derivative of the components $u^i$ along $\xi$** in a co-ordinate basis).

In a non-co-ordinate basis the Lie derivative can be written in an analogous way as in a co-ordinate basis

$$\mathcal{L}_\xi u = [\xi, u] = (\mathcal{L}_\xi u^\alpha) e_\alpha = (\xi^\beta \cdot e_{\beta} u^\alpha - u^\beta \cdot e_\beta \xi_{,\alpha} + C_{\beta \gamma} \cdot \xi^\beta \cdot e_\gamma) e_\alpha \quad , \quad (37)$$
where

$$\mathcal{L}_\xi u^\alpha = \xi^\beta . e_\beta u^\alpha - u^\beta . e_\beta \xi^\alpha + C_{\beta\gamma}^\alpha . \xi^\beta . u^\gamma$$  \hspace{1cm} (38)$$
is called Lie derivative of the components $u^\alpha$ of the contravariant vector field $u$ along a contravariant vector field $\xi$ in a non-co-ordinate basis (or Lie derivative of the components $u^\alpha$ along $\xi$). $\mathcal{L}_\xi u$ can be written in the form

$$\mathcal{L}_\xi u = (e_\beta u^\alpha - C_{\gamma\beta}^\alpha . u^\gamma).e_\alpha = u^\alpha /_{\beta} . e_\alpha = -\mathcal{L}_u e_\beta = (\mathcal{L}_e u^\alpha).e_\alpha ,$$  \hspace{1cm} (39)$$

where

$$\mathcal{L}_e u^\alpha = u^\alpha /_{\beta} = e_\beta u^\alpha - C_{\gamma\beta}^\alpha . u^\gamma .$$  \hspace{1cm} (40)$$

The second Lie derivative $\mathcal{L}_\xi \mathcal{L}_u v$ will have in a non-co-ordinate basis the form

$$\mathcal{L}_\xi \mathcal{L}_u v = [\xi^\beta . e_\beta (\mathcal{L}_u v^\alpha) - (\mathcal{L}_u v^\beta).e_\beta \xi^\alpha - C_{\gamma\beta}^\alpha . (\mathcal{L}_u v^\gamma).\xi^\beta].e_\alpha = (\mathcal{L}_\xi \mathcal{L}_u v^\alpha).e_\alpha ,$$  \hspace{1cm} (41)$$

where $\mathcal{L}_\xi \mathcal{L}_u v^\alpha$ is $\xi^\beta . e_\beta (\mathcal{L}_u v^\alpha) - (\mathcal{L}_u v^\beta).e_\beta \xi^\alpha - C_{\gamma\beta}^\alpha . (\mathcal{L}_u v^\gamma).\xi^\beta$ is called second Lie derivative of the components $v^\alpha$ along $u$ and $\xi$ in a non-co-ordinate basis.

The action of the Lie differential operator on a contravariant tensor field with rank $k > 1$ can be generalized on the basis of the validity of the Leibniz rule under the action of this operator on the bases of the tensor fields.

The result of the action of the operator $\mathcal{L}_\xi$ on a basis $\partial_A = \partial_{j_1} \otimes \ldots \otimes \partial_{j_k}$ can be found by the use of the already known relation $\mathcal{L}_\xi \partial_{j_k} = -\mathcal{L}_\partial_{j_k} \xi = -\xi^m . \partial_{j_k} . \partial_m$. Then $\mathcal{L}_\xi \partial_A = S_A m . \xi^m . \partial_B$, and

$$\mathcal{L}_\xi V = \mathcal{L}_\xi (V^A . \partial_A) = (\mathcal{L}_\xi V^A) . \partial_A = (\xi^k V^A . k + S_{Bk} A^l V^B . \xi^k l). \partial_A ,$$  \hspace{1cm} (42)$$

where $\mathcal{L}_\xi V^A = \xi^k V^A . k + S_{Bk} A^l V^B . \xi^k l$ is the Lie derivative of the components $V^A$ of a contravariant tensor field $V$ along a contravariant vector field $\xi$ in a co-ordinate basis (or Lie derivative of the components $V^A$ along $\xi$ in a co-ordinate basis).

For $\mathcal{L}_\xi \mathcal{L}_u V$ we obtain

$$\mathcal{L}_\xi \mathcal{L}_u V = (\mathcal{L}_\xi \mathcal{L}_u V^A) . \partial_A = [\xi^k (\mathcal{L}_u V^A) . k + S_{Bk} A^l (\mathcal{L}_u V^B) . \xi^k l]. \partial_A ,$$  \hspace{1cm} (43)$$

where $\mathcal{L}_\xi \mathcal{L}_u V^A = \xi^k (\mathcal{L}_u V^A) . k + S_{Bk} A^l (\mathcal{L}_u V^B) . \xi^k l$ is called second Lie derivative of the components $V^A$ along $u$ and $\xi$ in a co-ordinate basis.

The result of the action of the Lie differential operator $\mathcal{L}_\xi$ on a non-co-ordinate basis $e_A$ can be found in an analogous way as that for a co-ordinate basis. Since

$$\mathcal{L}_\xi e_\beta = -\xi^\alpha /_{\beta} . e_\alpha , \hspace{0.5cm} \xi^\alpha /_{\beta} = e_\beta \xi^\alpha - C_{\gamma\beta}^\alpha . \xi^\gamma ,$$  \hspace{1cm} (44)$$

$$\mathcal{L}_\xi e_A = \mathcal{L}_\xi [e_{a_1} \otimes \ldots \otimes e_{a_l}] = (\mathcal{L}_\xi e_{a_1} \otimes e_{a_2} \ldots \otimes e_{a_l}) + (e_{a_1} \otimes \mathcal{L}_\xi e_{a_2} \otimes \ldots \otimes e_{a_l}) + \ldots + (e_{a_1} \otimes \ldots \otimes \mathcal{L}_\xi e_{a_l}) = S_{Aa} B^j . \xi^a /_{\beta} . e_B ,$$  \hspace{1cm} (45)$$

\begin{align*}
A &= a_1 \ldots a_l ,
B &= \beta_1 \ldots \beta_l ,
\end{align*}
then
\[ \mathcal{L}_\xi e_A = S_{\alpha A} B^\beta \xi^\alpha / / \beta e_B \text{ and } \mathcal{L}_\xi V = \mathcal{L}_\xi (V^A e_A) = (\xi^\alpha e_A V^A + S_{\alpha B} A^\beta V^B \xi^\alpha / / \beta) e_A . \] (46)

The explicit form of the expression \( S_{\alpha B} A^\beta \xi^\alpha / / \beta \) can be given as
\[ S_{\alpha B} A^\beta \xi^\alpha / / \beta = S_{\alpha B} A^\beta e_\beta \xi^\alpha - S_{\alpha B} A^\beta C_{\gamma \beta} \xi^\gamma , \] (47)

and if we introduce the abbreviations
\[ C_{\gamma \beta} A = S_{\alpha B} A^\beta C_{\gamma \beta} \alpha = -S_{\alpha B} A^\beta C_{\gamma \beta} \alpha , \] (48)
\[ S_{\alpha B} A^\beta \xi^\alpha / / \beta = S_{\alpha B} A^\beta e_\beta \xi^\alpha - C_{\alpha B} A^\beta \xi^\alpha , \] (49)

then \( \mathcal{L}_\xi V^A \) can be written in the forms
\[ \mathcal{L}_\xi V^A = \xi^\alpha e_\alpha V^A + S_{\alpha B} A^\beta V^B \xi^\alpha / / \beta = \xi^\alpha e_\alpha V^A + S_{\alpha B} A^\beta V^B (e_\beta \xi^\alpha - C_{\gamma \beta} \alpha \xi^\gamma) = \xi^\alpha V^A / / \alpha + S_{\alpha B} A^\beta V^B e_\beta \xi^\alpha . \] (50)

\( \mathcal{L}_\xi V^A \) is called Lie derivative of the components \( V^A \) of a contravariant tensor field \( V \) along \( \xi \) in a non-co-ordinate basis. Here
\[ V^A / / \alpha = e_\alpha V^A - S_{B \beta} A^\gamma V^B C_{\alpha \beta} \gamma = e_\alpha V^A - C_{A \alpha} B^\beta V^B . \] (51)

In a non-co-ordinate basis the relations are valid
\[ \mathcal{L}_{e_\alpha} V = V^A / / \alpha e_A , \quad \mathcal{L}_{e_\alpha} e_A = -C_{A \alpha} B^\beta e_B . \] (52)

The quantity
\[ S_{\alpha B} A^\beta = -\sum_{k=1}^{l} g_{i_1}^{i_1} \cdots g_{i_{k-1}}^{j_{k-1}} g_{i_k}^{i_k} \cdots \cdots g_{i_{k+1}}^{j_{k+1}} \cdots g_{i_l}^{j_l} , \] (53)

where \( l = 1, \ldots, N, B = j_1 \ldots j_l, A = i_1 \ldots i_l, \) is the multi-contraction symbol with rank \( N \).

### 4.2 Connections between the covariant and the Lie differentiations

The action of the covariant differential operator and the action of the Lie differential operator on functions are identified with the action of the contravariant vector field in the construction of both operators. The contravariant vector field acts as a differential operator on functions over a differentiable manifold \( M \)
\[ \nabla_\xi f = \xi f = \mathcal{L}_\xi f = \xi^\alpha e_\alpha f , \quad f \in C^r (M) , \quad \xi \in T(M) . \]
If we compare the Lie derivative with the covariant derivative of a contravariant vector field in a non-co-ordinate (or co-ordinate) basis

\[
\mathcal{L}_\xi u = (\mathcal{L}_\xi u^\alpha).e_\alpha = (\xi^\beta e_\beta u^\alpha - u^\beta e_\beta \xi^\alpha + C_{\beta\gamma}^\alpha \xi^\beta u^\gamma).e_\alpha ,
\]

\[
\nabla_\xi u = (u^\alpha /\beta \xi^\beta).e_\alpha = (\xi^\beta e_\beta u^\alpha + \Gamma^\alpha_{\gamma\beta} u^\gamma \xi^\beta).e_\alpha ,
\]

we will see that both expressions have a common term of the type \( \xi u^\alpha = \xi^\beta e_\beta u^\alpha \) allowing a relation between the two derivatives.

After substituting \( e_\beta u^\alpha = u^\alpha /\beta - \Gamma^\alpha_{\gamma\beta} u^\gamma \) and \( e_\beta \xi^\alpha = \xi^\alpha /\beta - \Gamma^\alpha_{\gamma\beta} \xi^\gamma \) in the expression for \( \mathcal{L}_\xi u \) we obtain

\[
\mathcal{L}_\xi u^\alpha = u^\alpha /\beta \xi^\beta - \xi^\alpha /\beta \xi^\beta u^\alpha - T_{\beta\gamma}^\alpha \xi^\beta u^\gamma = u^\alpha /\beta \xi^\beta - (\xi^\alpha /\beta - T_{\beta\gamma}^\alpha \xi^\gamma).u^\beta ,
\]

where

\[
T_{\beta\gamma}^\alpha = \Gamma^\alpha_{\beta\gamma} - C_{\beta\gamma}^\alpha = - T_{\gamma\beta}^\alpha ,
\]

\[
\mathcal{L}_\xi u = (\mathcal{L}_\xi u^\alpha).e_\alpha = (u^\alpha /\beta \xi^\beta - \xi^\alpha /\beta \xi^\beta u^\alpha - T_{\beta\gamma}^\alpha \xi^\beta u^\gamma).e_\alpha = \nabla_\xi u - \nabla u \xi = T(\xi, u) ,
\]

with

\[
T(\xi, u) = T_{\beta\gamma}^\alpha \xi^\beta u^\gamma , e_\alpha = - T(u, \xi) , \quad T(e_\beta, e_\gamma) = T_{\beta\gamma}^\alpha .e_\alpha .
\]

The contravariant vector field \( T(\xi, u) \) is called (contravariant) torsion vector field (or torsion vector field).

If we use the equality following from the expression for \( \mathcal{L}_u v^\beta \)

\[
v^\beta /\gamma, u^\gamma - u^\beta /\gamma, v^\gamma = \mathcal{L}_u v^\beta + T_{\alpha\gamma}^\beta u^\alpha .v^\gamma
\]

in the expression

\[
\nabla_\nu \nabla_v \xi - \nabla_v \nabla_\nu \xi = [(\xi^\alpha /\beta /\gamma - \xi^\alpha /\gamma /\beta).v^\beta u^\gamma + \xi^\alpha /\beta (v^\beta /\gamma, u^\gamma - u^\beta /\gamma, v^\gamma)].e_\alpha ,
\]

then

\[
\nabla_\nu \nabla_v \xi - \nabla_v \nabla_\nu \xi = \left[(\xi^\alpha /\beta /\gamma - \xi^\alpha /\gamma /\beta).v^\beta u^\gamma + \xi^\alpha /\beta (\mathcal{L}_u v^\beta + T_{\beta\gamma}^\alpha v^\gamma).e_\alpha ,\right.
\]

\[
\left. = (\xi^\alpha /\beta /\gamma - \xi^\alpha /\gamma /\beta).v^\beta u^\gamma + \xi^\alpha /\beta (\mathcal{L}_u v^\beta + T_{\beta\gamma}^\alpha v^\gamma).e_\alpha ,\right.
\]

\[
T_{\beta\gamma}(u, v) = T_{\beta\gamma}^\alpha .u^\beta .v^\gamma , \quad \nabla T(u, v) \xi = \xi^\alpha /\beta T_{\beta\gamma}^\beta .u^\gamma .v^\beta .e_\alpha ,
\]

\[
\nabla_\nu \nabla_v \xi - \nabla_v \nabla_\nu \xi - \nabla T(u, v) \xi = (\xi^\alpha /\beta /\gamma - \xi^\alpha /\gamma /\beta).v^\beta u^\gamma .e_\alpha + \nabla T(u, v) \xi ,
\]

or

\[
\nabla_\nu \nabla_v \xi - \nabla_v \nabla_\nu \xi - \nabla T(u, v) \xi = (\xi^\alpha /\beta /\gamma - \xi^\alpha /\gamma /\beta).v^\beta u^\gamma .e_\alpha ,
\]

\[
\nabla_\nu \nabla_v \xi - \nabla_v \nabla_\nu \xi - \nabla T(u, v) \xi = (\xi^\alpha /\beta /\gamma - \xi^\alpha /\gamma /\beta).v^\beta u^\gamma .e_\alpha .
\]

In a co-ordinate basis the contravariant torsion vector will have the form

\[
T(\xi, u) = T_{kl}^i .\xi^k .u^l .\partial_i = (\Gamma^i_{lk} - \Gamma^i_{kl}).\xi^k .u^l .\partial_i ,
\]
\[ T_{kl}^i = \Gamma^i_{lk} - \Gamma^i_{kl}, \quad T(\partial_k, \partial_l) = T_{kl}^i \partial_i. \] (63)

The Lie derivative \( L_\xi u \) can be now written as
\[ \begin{aligned}
L_\xi u &= (L_\xi u^i) \partial_i = (u^i \cdot_\xi \xi^k - u^k \xi^i \cdot_\xi \xi^l - T_{kl}^i \cdot_\xi \xi^l \cdot_\partial \partial_i, \partial_i, \quad (64)
\end{aligned} \]

The connection between the covariant derivative and the Lie derivative of a contravariant tensor field can be found in an analogous way as in the case of a contravariant vector field.

4.3 Lie derivative of covariant basic vector fields

4.3.1 Lie derivative of covariant co-ordinate basic vector fields

The commutation relations between the Lie differential operator \( L_\xi \) and the contraction operator \( S \) in the case of basic co-ordinate vector fields can be written in the form
\[ \begin{aligned}
L_\xi \circ S(dx^i \otimes \partial_j) &= S \circ L_\xi(dx^i \otimes \partial_j), \\
L_\xi \circ S(e^a \otimes e_\beta) &= S \circ L_\xi(e^a \otimes e_\beta), \quad (65)
\end{aligned} \]

where
\[ \begin{aligned}
L_\xi \circ S(dx^i \otimes \partial_j) &= \xi f^i_j, \\
S \circ L_\xi(dx^i \otimes \partial_j) &= S(L_\xi dx^i \otimes \partial_j) + S(dx^i \otimes L_\xi \partial_j). \quad (66)
\end{aligned} \]

By means of the non-degenerate inverse matrix \((f^i_j)^{-1} = (f^j_i)\) and the connections \( f^i_{jk} f^j_l = g^i_l, f^k_{ij} f^j_l = g^k_l \), after multiplication of the equality for \( k^i_j(\xi) \) with \( f^m_j \) and summation over \( j \), the explicit form for \( k^i_j(\xi) \) is obtained in the form
\[ k^i_j(\xi) = f^i_{lj} \xi^k \cdot_\xi \partial_k + f^i_{lk} \cdot_\xi \partial_i f^j_l. \] (67)

For \( L_\partial dx^i = k^i_j(\partial_k) dx^j = k^i_{jk} dx^j \) it follows the corresponding form
\[ \begin{aligned}
L_\partial dx^i &= k^i_{jk} dx^j = f^i_{lj} \cdot_\partial_i f^j_l, \\
k^i_{jk} &= f^i_{lj} \cdot_\partial_i f^j_l. \quad (68)
\end{aligned} \]

On the other hand, from the commutation relations between \( S \) and the covariant differential operator \( \nabla_\xi \), the connection between the partial derivatives of \( f^i_j \) and the components of the contravariant and covariant connections \( \Gamma \) and \( P \) follows in the form
\[ f^i_{lj} = P^i_{mk} f^m_l + \Gamma^i_{lk} f^i_m. \] (69)

After substituting the last expression in the expressions for \( k^i_j(\xi) \) and for \( k^i_{jk} \), the corresponding quantities are obtained in the forms
\[ k^i_{jk}(\xi) = f^i_{lj} \xi^k \cdot_\xi \partial_k + (P_{jk}^i f_{lk} f^i_m + \Gamma^i_{lk} f^i_m) \xi^k, \] (70)
\[ k^i_\ j \partial_k = k^i_\ jk = P^i_\ jk + f^i_\ j \Gamma^m_\ ik \ f^i_\ m , \]

\[ \mathcal{L}_\xi dx^i = [f^i_\ j \xi^k_\ j \ f^i_\ k + (P^i_\ jk + f^i_\ j \Gamma^m_\ ik \ f^i_\ m) \xi^k] dx^i , \quad (71) \]

\[ \mathcal{L}_\partial dx^i = k^i_\ jk \ f^j_\ k = \left( P^i_\ jk + f^i_\ j \Gamma^m_\ ik \ f^i_\ m \right) dx^j . \quad (72) \]

If we introduce the abbreviations

\[ \xi^i_\ j = f^i_\ k \xi^k_\ j \ f^i_\ l , \quad \Gamma^i_\ jk = f^i_\ j \Gamma^m_\ ik \ f^i_\ m , \quad (73) \]

then the Lie derivatives of covariant co-ordinate basic vector fields \( dx^i \) along the contravariant vector fields \( \xi \) and \( \partial_k \) can be written in the forms

\[ \mathcal{L}_\xi dx^i = [\xi^i_\ j + (P^i_\ jk + \Gamma^i_\ jk) \xi^k] dx^j , \quad \mathcal{L}_\partial dx^i = (P^i_\ jk + \Gamma^i_\ jk) dx^j . \quad (74) \]

### 4.3.2 Lie derivative of covariant non-co-ordinate basic vector fields

Analogous to the case of covariant co-ordinate basic vector fields the Lie derivatives of covariant non-co-ordinate basic vector fields can be obtained in the form

\[ \mathcal{L}_\xi e^\alpha = \left[ \xi^\alpha_\ /\beta + (P^\alpha_\ /\gamma + \Gamma^\alpha_\ /\gamma) \xi^\gamma \right] e^\beta = \]

\[ = e^\beta \xi^\alpha + (P^\alpha_\ /\gamma + \Gamma^\alpha_\ /\gamma + C^\beta_\ /\gamma) \xi^\gamma \] \[ \mathcal{L}_\partial e^\alpha = (P^\alpha_\ /\gamma + \Gamma^\alpha_\ /\gamma) e^\beta , \quad (75) \]

where

\[ \xi^\alpha_\ /\beta = f^\alpha_\ \gamma \xi^\gamma_\ /\beta \ f^\delta_\ \gamma \ f^\beta_\ \delta \ f^\gamma_\ \sigma \ f^\sigma_\ \beta = \\
\quad = e^\beta \xi^\alpha + (P^\beta_\ /\gamma + \Gamma^\beta_\ /\gamma) \xi^\gamma \] \[ e^\beta \xi^\alpha = f^\alpha_\ \gamma \xi^\gamma_\ /\beta \ f^\beta_\ \gamma \ f^\gamma_\ \sigma \ f^\beta_\ /\gamma = f^\alpha_\ \gamma \xi^\gamma_\ /\beta \ f^\beta_\ /\gamma \ f^\gamma_\ \sigma \ f^\beta_\ /\gamma + C^\beta_\ /\gamma \ f^\gamma_\ \sigma = \\
\quad = \Gamma^\beta_\ /\gamma \ f^\gamma_\ \sigma \ f^\beta_\ /\gamma . \quad (77) \]

### 4.4 Lie derivatives of covariant tensor fields

The action of the Lie differential operator on covariant vector and tensor fields is determined by its action on covariant basic vector fields and on the functions over \( M \).

In a co-ordinate basis the Lie derivative of a covariant vector field \( p \) along a contravariant vector field \( \xi \) can be written in the forms

\[ \mathcal{L}_\xi p = \mathcal{L}_\xi [p_i dx^i] = (\mathcal{L}_\xi p_i) dx^i = \]

\[ = [p_i, k \xi^k + p_j \xi^j_\ k + p_j (P^j_\ ik + \Gamma^j_\ ik) \xi^k] dx^i = \quad (78) \]

\[ \mathcal{L}_\partial p = [p_i, k \xi^k + \xi^k_\ /l p_k + T^j_\ ik \ f^j_\ l \ f^k_\ m] dx^i , \]

where

\[ \xi^i_\ j = f^i_\ k \xi^k_\ j \ f^i_\ l , \quad \Gamma^i_\ jk = f^i_\ j \Gamma^m_\ ik \ f^i_\ m , \quad (\text{in a co-ordinate basis}) . \quad (79) \]
In a non-co-ordinate basis the Lie derivative $\mathcal{L}_\xi p$ has the forms

$$\mathcal{L}_\xi p = \mathcal{L}_\xi(p_\alpha \cdot e^\alpha) = (\mathcal{L}_\xi p_\alpha) \cdot e^\alpha =$$

$$\{ (e_\gamma p_\alpha + P_\alpha^\beta p_\beta) \xi^\gamma + p_\beta \cdot [\xi^\alpha, e_\beta^\gamma + (\Gamma^\gamma_{\alpha \beta} + \tau^\gamma_{\alpha \beta}) \xi^\gamma] \} \cdot e^\alpha =$$

$$= (p_\alpha^\beta \cdot \xi^\beta + \xi^\alpha /\alpha \cdot p_\beta + T_{\gamma \alpha \beta} \cdot p_\beta \cdot \xi^\gamma) \cdot e^\alpha ,$$

where

$$\xi^\beta /\beta = \frac{f^\beta}{\beta} \delta \xi^\gamma /\gamma \cdot f_\alpha \gamma , \quad \tau^\beta = \frac{f_\alpha}{\alpha} \delta \tau^\beta /\tau \cdot f^\beta /\beta ,$$

$$T_{\beta \gamma}^\alpha = \Gamma_{\gamma \beta}^\alpha - \Gamma_{\beta \gamma}^\alpha - C_{\beta \gamma}^\alpha , \quad \text{(in a non-co-ordinate basis). (81)}$$

The action of the Lie differential operator on covariant tensor fields is determined by its action on basic tensor fields.

In a co-ordinate basis

$$\mathcal{L}_\xi W = \mathcal{L}_\xi(W_A \cdot dx^A) = (\xi^\alpha (W_A)) \cdot dx^A + W_A \cdot \mathcal{L}_\xi dx^A =$$

$$= (\mathcal{L}_\xi W_A) \cdot dx^A , \quad W \in \otimes (M),$$

$$\mathcal{L}_\xi dx^B = -k^m_n (\xi) S_{Am} B_n \cdot dx^A , \quad \mathcal{L}_\xi dx^m = k^m_n (\xi) \cdot dx^n ,$$

$$\mathcal{L}_\xi dx^B = [-\xi^F B S_{Ak} B_l - S_{Am} B_n (P_{nl}^m + \Gamma^m_{nl}) \xi^l] \cdot dx^A .$$

After introducing the abbreviations

$$\Gamma_{Ak}^B = -S_{Aji} B_j \Gamma_{jk}^i , \quad P_{Ak}^B = -S_{Aji} B_j P_{jk}^i , \quad (83)$$

$\mathcal{L}_\xi W$ can be written in the form

$$\mathcal{L}_\xi W = (\mathcal{L}_\xi W_A) \cdot dx^A =$$

$$= [\xi^k W_A]_k - \xi^k \cdot S_{Ak} B_l W_B + (P_{Al}^B + \Gamma_{Al}^B) W_B \cdot \xi^l] \cdot dx^A ,$$

where

$$\mathcal{L}_\xi W_A = \xi^k W_A]_k - \xi^k \cdot S_{Ak} B_l W_B + (P_{Al}^B + \Gamma_{Al}^B) W_B \cdot \xi^l =$$

$$= \xi^k W_A]_k - S_{Ak} \Gamma_{ij} W_B (\xi^j \cdot B_l - T_{lj} \xi^l) =$$

$$= \xi^k W_A]_k - S_{Ak} B_l W_B \cdot (\xi^j \cdot B_l - T_{lj} \xi^l) ,$$

$$\mathcal{L}_\partial W_A = W_A \partial + (P_{A_j}^B + \Gamma_{A_j}^B) W_B ,$$

$$\mathcal{L}_\partial dx^B = -S_{Aji} B_l (\xi^j \cdot + \Gamma_{lj}^i) \cdot dx^A = (P_{A_j}^B + \Gamma_{A_j}^B) dx^A .$$

The second Lie derivative of the components $W_A$ of the covariant tensor field $W$ can be written in the form

$$\mathcal{L}_\xi \mathcal{L}_\xi W_A = \xi^k \cdot (\xi^l W_A) \cdot \xi^j - \xi^k \cdot S_{Ak} B_l \cdot \mathcal{L}_\xi W_B + (P_{Al}^B + \Gamma_{Al}^B) \xi^l \cdot \mathcal{L}_\xi W_B .$$

In a non-co-ordinate basis $\mathcal{L}_\xi W$ has the form

$$\mathcal{L}_\xi W = (\xi^k W_A) \cdot e^A + W_B \cdot (\mathcal{L}_\xi e^B) = (\xi^k W_A) \cdot e^A ,$$

(88)
where

\[ \mathcal{L}_\xi W_A = \xi^\beta . e_\beta W_A - S_A^\alpha B^\beta . W_B . e_\beta \xi^\alpha + (P_B^\beta + \Gamma_B^\beta A^\gamma + C_A^\gamma B) . W_B . \xi^\gamma, \]

\[ \Gamma_B^\beta A^\gamma = -S_A^\alpha B^\beta . \Omega^\alpha_\beta^\gamma, \quad -C_A^\gamma B = -S_A^\alpha B^\beta . \Omega^\beta_\gamma, \]

\[ \mathcal{L}_e^\beta e^B = (P_B^\beta + \Gamma_B^\beta A^\gamma + C_A^\gamma B) . e^A, \]

\[ \mathcal{L}_e^\beta W_A = e^\beta W_A + (P_B^\beta + \Gamma_B^\beta A^\gamma + C_A^\gamma B) . W_B. \]

The second Lie derivative of \( W_A \) in a non-co–ordinate basis has the form

\[ \mathcal{L}_\xi \mathcal{L}_u W_A = \xi^\beta . e_\beta (\mathcal{L}_u W_A) - S_A^\alpha B^\beta (\mathcal{L}_u W_B) . e_\beta \xi^\alpha + (P_B^\beta + \Gamma_B^\beta A^\gamma + C_A^\gamma B) . \xi^\gamma. \mathcal{L}_u W_B. \] (89)

The Lie derivatives of covariant basic tensor fields can be given in terms of the covariant derivatives of the components of the contravariant vector field \( \xi \) and the torsion tensor

\[ \mathcal{L}_\xi e^B = \left[ -S_A^\alpha B^\beta . \xi^\alpha /\beta + P_B^\beta . \xi^\gamma + \Gamma_B^\beta A^\gamma . \xi^\gamma \right] . e^A, \] (91)

where \( \Gamma_B^\beta A^\gamma = S_A^\alpha B^\beta . T^\alpha \beta^\gamma \). \( \mathcal{L}_\xi W_A \) will then have the form

\[ \mathcal{L}_\xi W_A = \xi^\beta . W_A /\beta - S_A^\alpha B^\beta . W_B . (\xi^\alpha /\beta - T^\lambda_\alpha^\gamma \xi^\gamma) = \xi^\beta . W_A /\beta - S_A^\alpha B^\beta . W_B . (\xi^\alpha /\beta - T^\lambda_\alpha^\gamma \xi^\gamma). \] (92)

The generalization of the Lie derivatives for mixed tensor fields is analogous to that for covariant derivatives of mixed tensor fields.

4.5 Classification of linear transports with respect to the connections between contravariant and covariant affine connections

By means of the Lie derivatives of covariant basis vector fields, a classification can be proposed for the connections between the components \( \Gamma^\gamma_{jk} \) \( (\Gamma^\gamma_{jk}) \) of the contravariant affine connection \( \Gamma \) and the components \( P^i_{jk} \) \( (P^i_{jk}) \) of the covariant affine connection \( P \). On this basis, linear transports (induced by the covariant differential operator or by connections) and draggings-along (induced by the Lie differential operator) can be considered as connected with each other through commutation relations of both operators with the contraction operator.
Transplant condition

\[ P_\beta \gamma + \Gamma^\alpha \gamma \beta + C^\alpha \gamma \beta = F^\alpha \beta \gamma \, , \]
\[ P^i_\beta \gamma + \Gamma^i_\beta \gamma = F^i \beta \gamma \, . \]

Type of dragging—along and transports

\[ \mathcal{L}_{\xi^e} e^\alpha = F^\alpha \beta \gamma e^\beta \, , \]
\[ \mathcal{L}_{\partial_\kappa} dx^i = F^{i \beta \gamma} \pi_k dx^i \, . \]

Transport with arbitrary dragging-along

\[ \mathcal{L}_{\xi^e} e^\alpha = \mathcal{A}_e, \mathcal{A}_e + C^\alpha \beta \gamma e^\beta \, , \]
\[ \mathcal{L}_{\partial_\kappa} dx^i = \mathcal{A}_e, \pi_k dx^i \, . \]

Transport with co-linear dragging-along

\[ \mathcal{L}_{\xi^e} e^\alpha = C^\beta \gamma e^\beta \, , \]
\[ \mathcal{L}_{\partial_\kappa} dx^i = 0 \, . \]

Transport with invariant dragging-along

\[ \mathcal{L}_{\partial_\kappa} dx^i = 0 \, . \]

Table 1. Relations between transport conditions and types of dragging-along

The classification of the relations between the affine connections is analogous to the classification proposed by Schouten [27] and considered by Schmutzer [30].

5 Curvature operator. Bianchi identities

5.1 Curvature operator

One of the well-known operators constructed by means of the covariant and the Lie differential operators which has been used in the differential geometry of differentiable manifolds is the curvature operator.

Definition 15 Curvature operator. The operator

\[ R(\xi, u) = \nabla_\xi \nabla_u - \nabla_u \nabla_\xi - \nabla_{\xi u} = [\nabla_\xi, \nabla_u] - \nabla_{[\xi, u]} \, , \quad \xi, u \in T(M) \, , \quad (93) \]

is called curvature operator (or operator of the curvature).

1. Action of the curvature operator on a function of a class \( C^r(M), r \geq 2 \), over a manifold \( M \)

\[ [R(\xi, u)]^f = 0 \, , \quad f \in C^r(M), \quad r \geq 2 \, . \]

2. \( [R(\xi, u)]^f v = f [R(\xi, u)]^v \), \( f \in C^r(M), \quad r \geq 2 \), \( v \in T(M) \).

3. Action of the curvature operator on a contravariant vector field

\[ [R(\xi, u)]^v = \nabla_\xi \nabla_u v - \nabla_u \nabla_\xi v - \nabla_{\xi \nabla_u} v = \]
\[ = [(v^i, f j, \beta_\gamma) \cdot u^i, \xi_j + v^i, f j, \beta_\gamma \cdot T^j_\beta_\gamma, \pi^\delta \cdot u^\gamma] \cdot e_\delta = \]
\[ = [(v^i, f j, k - v^i, f j, k) \cdot u^i, \xi_j + v^i, f j, k \cdot T^j_\beta_\gamma, \pi^\delta \cdot u^\gamma] \cdot \partial_\kappa \, . \quad (94) \]

In such a way, we can find for \( \forall \xi \in T(M) \) and \( \forall u \in T(M) \) the relation in a co-ordinate basis

\[ v^i, f j, k = v^i, f j, k \cdot T^j_\beta_\gamma, \pi^\delta \cdot u^\gamma \, , \quad (95) \]
where
\[ R^i_{\ jkl} = \Gamma^i_{jl,k} - \Gamma^i_{jk,l} + \Gamma^i_{jl}.\Gamma^i_{mk} - \Gamma^m_{jk}.\Gamma^i_{ml} \] (96)
are called components of the (contravariant) curvature tensor (Riemannian tensor) in a co-ordinate basis.

4. Action of the curvature operator on contravariant tensor fields.
For \( V = V^A.e_A = V^B.\partial_B, V \in \otimes^1(M) \) and the bases \( e_A \) and \( \partial_B \) the following relations can be proved using the properties of \( S_{Ak} B^l \) and \( \Gamma^B_{Ai} = \)
\[
[R(\xi, u)](f.V) = f.[R(\xi, u)V],
\]
(97)
\[ [R(\xi, u)]V = V^A.[R(\xi, u)e_A = V^B.[R(\xi, u)]\partial_B, \]
(98)
\[ [R(\partial_j, \partial_i)]\partial_A = R^B_{Aji}.\partial_B = -S_{Ak} B^l.R^k_{lji}.\partial_B, \]
(99)
where
\[ R^B_{Aji} = -S_{Ak} B^l.R^k_{lji}, \quad S_{Ak} B^l/i = 0, \] (100)
\[ R^B_{Aji} = \Gamma^B_{Ai,j} - \Gamma^B_{Aj,i} + \Gamma^C_{Ai}.\Gamma^B_{Cj} - \Gamma^C_{Aj}.\Gamma^B_{Ci}, \] (101)
\[ [R(\xi, u)]V = -S_{Bk} A^l.V^B.R^k_{iij}.\xi^i.u^j.\partial_A. \] (102)

On the other side, it follows from the explicit construction of \( [R(\xi, u)]V \)
\[
[R(\xi, u)]V = (V^A :i;j - V^A :j;i + V^A :k.T^j_{ki}).u^i.\xi^j.\partial_A, \]
\[ V^A :i;j - V^A :j;i = -S_{Bk} A^l.V^B.R^k_{iij}.T^j_{ki}.V^A :k, \]
\[ [R(\partial_j, \partial_i)] - \nabla_T(\partial_j, \partial_i)]V = (V^A :i;j - V^A :i;j).\partial_A. \] (103)

5. The action of the curvature operator on covariant vector fields is determined by its structure and by the action of the covariant differential operator on covariant tensor field.

In a co-ordinate basis
\[
[R(\xi, u)]p = (\nabla_\xi \nabla_u - \nabla_u \nabla_\xi - \nabla_\xi u)p = p_i.P^i_{vkj}.\xi^k.u^v.dx^i =
\]
\[ = (p_{i;j;k} - p_{k;ij} + T^j_{k;i}.p_{ij}).u^i.\xi^k.dx^i, \] (104)
\[ [R(\partial_k, \partial_l)]dx^i = P^i_{jkl}.dx^j, \] (105)
\[
Q^i_{jkl} = P^i_{jl,k} - P^i_{jk,l} + P^m_{jk}.P^i_{ml} - P^m_{jl}.P^i_{mk} = -P^i_{jkl} \]
(106)
are called components of the covariant curvature tensor in a co-ordinate basis.

Special case: \( S = C : f^i_j = g^i_j : P^i_{jk} + \Gamma^i_{jk} = 0 \).
\[ P^i_{jkl} = -R^i_{jkl}. \] (107)

In a non-co-ordinate basis:
\[
[R(\xi, u)]p = p_\alpha .P^\alpha_{\beta\gamma}\xi^\beta.u^\gamma.e^\alpha =
\]
\[ = (p_{\alpha/\beta/\gamma} - p_{\alpha/\beta/\gamma} + T^\gamma_{\beta\gamma}.p_{\alpha/\beta}).\xi^\beta.u^\gamma.e^\alpha, \] (108)
\[ P^\alpha_{\delta\beta\gamma} = e_\beta P^\alpha_{\delta\beta} - e_\gamma P^\alpha_{\delta\beta} + P^\alpha_{\delta\beta}.P^\alpha_{\gamma\delta} - P^\alpha_{\delta\gamma}.P^\alpha_{\beta\delta} - C_{\beta\gamma}^\sigma .P^\alpha_{\delta\sigma}. \]  
\( (109) \)

\[ P^\alpha_{\delta\beta\gamma} = -P^\alpha_{\delta\gamma\beta} \] are called components of the covariant curvature tensor in a non-co-ordinate basis.

For a covariant tensor field \( W = W_A dx^A = W_C.e^C \in \otimes_k(M) \) we have the relation in a co-ordinate basis

\[ W_A;i;\beta - W_A;j;i = S_A m.B_B P_m ni j + W_A;l.T_{ij,l}, \]  
\( (110) \)

and in a non-co-ordinate basis

\[ W_A/\beta;/\gamma - W_A/\gamma;/\beta = S_A .B_B P^\alpha m P^\delta/\alpha \delta/\beta + W_A/\delta;T_{\beta\gamma}, \]  
\( (111) \)

### 5.2 Bianchi identities

If we write down the cycle of the action of the curvature operator on contravariant vector fields, i.e., if we write

\[ < [R(\xi, u)]v > = [R(\xi, u)]v + [R(v, \xi)]u + [R(u, v)]\xi, \]  
\( (112) \)

and put the explicit form of every term in the cycle, then by the use of the covariant and the Lie differential operator, after some (not so difficult) calculations, we can find identities of the type written in the form

\[ [R(\xi, u)]v + [R(v, \xi)]u + [R(u, v)]\xi \equiv T(T(T(\xi, u), v), u) + T(T(\xi, u), v, \xi) + \]  
\[ \frac{\partial}{\partial \xi}(T(u, v)) + \frac{\partial}{\partial v}(T(\xi, u)) + \frac{\partial}{\partial u}(T(\xi, v)), \]  
\( (113) \)

or in the form

\[ < [R(\xi, u)]v > \equiv T(T(\xi, u), v) > + < \frac{\partial}{\partial \xi}(T(u, v)) > . \]  
\( (114) \)

The identities are called Bianchi identities of first type (or of the type 1), where

\[ < T(T(\xi, u), v) > \equiv T(T(T(\xi, u), v), u) + T(T(\xi, u), v, \xi) + \]  
\[ \frac{\partial}{\partial \xi}(T(u, v)) + \frac{\partial}{\partial v}(T(\xi, u)) + \frac{\partial}{\partial u}(T(\xi, v)) . \]

By the use of the curvature operator and the covariant differential operator a new operator \((\nabla_w R)(\xi, u)\) can be constructed in the form

\[ (\nabla_w R)(\xi, u) = [\nabla_w, R(\xi, u)] - R(\nabla_w \xi, u) - R(\xi, \nabla_w u) , \]  
\( (115) \)

where

\[ [\nabla_w, R(\xi, u)] = \nabla_w \circ R(\xi, u) - R(\xi, u) \circ \nabla_w , \]  
\( w, \xi, u \in T(M) . \)

\((\nabla_w R)(\xi, u)\) has the structure

\[ (\nabla_w R)(\xi, u) = \nabla_w \nabla_w \xi \nabla_w - \nabla_w \nabla_u \xi + \nabla_u \nabla_w \xi - \nabla_w \nabla_u \xi \]  
\[ + \nabla_u \nabla_w \xi - \nabla_u \nabla_u \xi + \nabla_w \nabla_u \xi - \nabla_u \nabla_u \xi + \]  
\[ \nabla_u \nabla_u \xi , \]  
\( (116) \)

\[ \nabla_u \nabla_u \xi . \]
This operator obeys the s. c. Bianchi identity of second type (or of the type 2.)

\[ < (\nabla_w R)(\xi, u) > \equiv R(w, T(\xi, u)) > , \] (117)

where

\[ < (\nabla_w R)(\xi, u) > \equiv (\nabla_w R)(\xi, u) + (\nabla_u R)(w, \xi) + (\nabla_\xi R)(u, w) , \]
\[ < R(w, T(\xi, u)) > \equiv R(w, T(\xi, u)) + R(u, T(w, \xi)) + R(\xi, T(u, w)) . \]

The Bianchi identity of type 2. can be written in a co-ordinate or in a non-co-ordinate basis as an identity of the components of the contravariant curvature tensor

\[ R^i_{\ j<kl:m>} \equiv R^i_{\ j<kn}.T_{lm>}^n \equiv -R^i_{\ jn<k}.T_{lm>}^n , \] (118)

where

\[ R^i_{\ j<kl:m>} \equiv R^i_{\ jkl;m} + R^i_{\ jmkl} + R^i_{\ jlmk} , \]
\[ R^i_{\ j<kn}.T_{lm>}^n \equiv R^i_{\ jkn}.T_{lm}^n + R^i_{\ jmn}.T_{kl}^n + R^i_{\ jir}.T_{mk}^r . \] (119)

For the commutator

\[ [\nabla_w, R(\xi, u)] = \nabla_w \circ R(\xi, u) - R(\xi, u) \circ \nabla_w \]

the following commutation identity is valid:

\[ < [\nabla_w, R(\xi, u)] > \equiv - < R(w, \mathcal{L}_\xi u) > , \] (120)

where

\[ < [\nabla_w, R(\xi, u)] > \equiv [\nabla_w, R(\xi, u)] + [\nabla_u, R(w, \xi)] + [\nabla_\xi, R(u, w)] , \]
\[ < R(w, \mathcal{L}_\xi u) > \equiv R(w, \mathcal{L}_\xi u) + R(u, \mathcal{L}_w \xi) + R(\xi, \mathcal{L}_u w) . \] (121)

The curvature operator and the Bianchi identities have been applied in differentiable manifolds with one affine connection. They can also find applications in considerations concerning the characteristics of differentiable manifolds with affine connections and metrics. The structure of the curvature operator induces a construction of an other operator called deviation operator.

6 Deviation operator

By means of the structure of the curvature operator,

\[ R(\xi, u) = \nabla_\xi \nabla_u - \nabla_u \nabla_\xi - \nabla_{\mathcal{L}_{\xi} u} = [\nabla_\xi, \nabla_u] - \nabla_{[\xi, u]} , \] (122)

the commutator \([\nabla_w, R(\xi, u)] \ [w, \xi, u \in T(M)]\) can be presented in the form

\[ [\nabla_w, R(\xi, u)] = [\nabla_w, \mathcal{L}_\Gamma(\xi, u)] + [\nabla_u, [\nabla_\xi, \nabla_u]] - [\nabla_w, [\mathcal{L}_\xi, \nabla_u]] , \] (123)

where

\[ \mathcal{L}_\Gamma(\xi, u) = \mathcal{L}_\xi \nabla_u - \nabla_u \mathcal{L}_\xi - \nabla_{\mathcal{L}_{\xi} u} = [\mathcal{L}_\xi, \nabla_u] - \nabla_{[\xi, u]} . \] (124)

The operator \(\mathcal{L}_\Gamma(\xi, u)\) appears as a new operator, constructed by means of the Lie differential operator and the covariant differential operator. [31] - [34].
Definition 16 The operator $\mathcal{L}\Gamma(\xi, u)$ is called deviation operator. Its properties contain the relations:

1. Action of the deviation operator on a function $f : [\mathcal{L}\Gamma(\xi, u)]f = 0$, $f \in C^r(M), r \geq 2$.
2. Action of the deviation operator on a contravariant vector field:

$$[\mathcal{L}\Gamma(\xi, u)]v = f[\mathcal{L}\Gamma(\xi, u)]v , \xi, u, v \in TM, \quad [\mathcal{L}\Gamma(\xi, u)]v = v^\alpha[v\mathcal{L}\Gamma(\xi, u)]\partial_\alpha = u^\gamma v^\beta[v\mathcal{L}\Gamma(\xi, \partial_\gamma)]\partial_\beta = v^\beta v^\gamma[v\mathcal{L}\Gamma(\xi, \partial_\gamma)]\partial_\beta.$$ 

The connections between the action of the deviation operator and that of the curvature operator on a contravariant vector field can be given in the form

$$[\mathcal{L}\Gamma(\xi, u)]v = [R(\xi, u)]v + [\nabla_u \nabla_v - \nabla_v \nabla_u]\xi - T(\xi, \nabla_u v) + \nabla_v T(\xi, v).$$  \hspace{1cm} (125)

In a co-ordinate basis $[\mathcal{L}\Gamma(\xi, \partial_i)]\partial_k$ has the form

$$[\mathcal{L}\Gamma(\xi, \partial_i)]\partial_k = \{\xi^i, \partial_i - R_{\ ijk}^l \xi^j + (T_{jk}^i \xi^j)\partial_i\} \partial_k = (L_{\xi} \Gamma_k^i)\partial_i,$$ \hspace{1cm} (126)

where

$$\nabla_{\partial_j}(T(\xi, \partial_i)) = T(\xi, \nabla_{\partial_j} \partial_i) = (T_{ij}^k \xi^i)\partial_j \partial_k.$$ 

$L_{\xi} \Gamma_k^i$ is called Lie derivative of contravariant affine connection along the contravariant vector field $\xi$. It can be written also in the form

$$L_{\xi} \Gamma_k^i = \xi^i, \partial_i + \xi^j \Gamma_k^j - \xi^j, \partial_j \Gamma_k^j + \xi^j, \partial_i \Gamma_j^i + \xi^j, \partial_j \Gamma_i^j.$$ \hspace{1cm} (127)

By means of $L_{\xi} \Gamma_k^i$ the expression for $[\mathcal{L}\Gamma(\xi, u)]v$ can be presented in the form

$$[\mathcal{L}\Gamma(\xi, u)]v = v^k, u^l, (L_{\xi} \Gamma_k^i)\partial_i =$$

$$= [\xi^j, \partial_j - R_{\ ijk}^l \xi^j + (T_{jk}^i \xi^j)\partial_i]v^k, u^l \xi^j.$$ \hspace{1cm} (128)

In this way, the second covariant derivative $\nabla_u \nabla_v \xi$ of the contravariant vector field $\xi$ can be presented by means of the deviation operator in the form

$$\nabla_u \nabla_v \xi = ([R(u, \xi)]v) + \nabla_{\xi} \nabla_u v - L_{\xi}(\nabla_u v) - \nabla_u [T(\xi, v)] + [\mathcal{L}\Gamma(\xi, u)]v =$$

$$= ([R(u, \xi)]v) + \nabla_{\xi} \nabla_u v - \nabla_u L_{\xi} v - \nabla_{L_{\xi} v} v - \nabla_u [T(\xi, v)].$$ \hspace{1cm} (129)

For $v = u$ the last identity is called generalized deviation identity \[31\]. It is used for analysis of deviation equations in spaces with affine connection and metric ($L_n$-spaces, $U_n$-spaces and $V_n$-spaces), where deviation equations are considered with respect to their structure and solutions \[32\] - \[33\], and as a theoretical ground for gravitational wave detectors in (pseudo) Riemannian spaces without torsion ($V_n$-spaces) \[10\] - \[12\]. Deviation equations of Synge and Schild and its generalization for $(L_n, g)$-spaces are considered in \[29\].

2. Action of the deviation operator on contravariant tensor field

$$[\mathcal{L}\Gamma(\xi, u)]V = u^\gamma V^A, (L_{\xi} \Gamma_k^i)\partial_i = u^\gamma V^B, (L_{\xi} \Gamma_k^i)\partial_i =$$

$$= -(S_{BA} A^\beta V^B, L_{\xi} \Gamma_k^i)\partial_i, u^\gamma e_A = V \in \otimes^k(M),$$ \hspace{1cm} (130)

26
where
\[ L_\xi \Gamma^A_{\beta\gamma} = -S_{B\alpha} A^B \xi \Gamma^A_{\beta\gamma}, \]
\[ \langle [\mathcal{L}\Gamma(\xi, u)]e_B \rangle = \langle (L_{\Gamma}^A_{\beta\gamma})_{e_A} \rangle , \]
\[ = -S_{B\alpha} A^B, [\xi^\alpha / \beta / \gamma - R^\beta_{\beta\delta} \xi^\delta + (T_{\delta\beta\gamma}^\alpha / \gamma)]_{e_A} . \] (131)

3. The deviation operator obeys identity analogous to the 1. type Bianchi identity for the curvature operator
\[ \langle [\mathcal{L}\Gamma(\xi, u)]v \rangle \equiv \langle (\nabla_\xi \nabla_v - \nabla_{\nabla_\xi u})v \rangle + \langle T(T(\xi, u), v) \rangle \]
\[ = -\langle T(u, \nabla_\xi v) \rangle, \xi, u, v \in T(M) , \] (132)
where
\[ \langle [\mathcal{L}\Gamma(\xi, u)]v \rangle = [\mathcal{L}\Gamma(\xi, u)]v + [\mathcal{L}\Gamma(v, \xi)]u + [\mathcal{L}\Gamma(u, v)]\xi , \]
\[ \langle (\nabla_\xi \nabla_v - \nabla_{\nabla_\xi u})v \rangle = (\nabla_\xi \nabla_v - \nabla_{\nabla_\xi u})v + (\nabla_v \nabla_\xi u - \nabla_{\nabla_\xi v})u + \]
\[ + (\nabla_\xi \nabla_v - \nabla_{\nabla_\xi u})\xi , \]
\[ \langle T(u, \nabla_\xi v) \rangle = T(u, \nabla_\xi v) + T(v, \nabla_\xi u) + \nabla_\xi (\nabla_v u) \]; (133)

In a non-co-ordinate basis this identity obtains the form
\[ (L_{\xi} \Gamma_{\alpha\beta}^\gamma).v^\alpha.w^\beta + (L_{\xi} \Gamma_{\alpha\beta}^\gamma)_{e_A} \xi^\alpha.w^\beta + (L_{\xi} \Gamma_{\alpha\beta}^\gamma)_{e_A} \xi^\beta \equiv \]
\[ \equiv \xi^\gamma / / \alpha / \beta : v^\alpha.w^\beta + u^\gamma / / \alpha / \beta \xi^\alpha.w^\beta + u^\gamma / / \alpha / \beta \xi^\beta + \]
\[ + T_{\alpha\beta} T_{\rho\delta} \xi^\gamma.w^\beta - \]
\[ - T_{\alpha\beta} \xi^\gamma.w^\beta / / \rho / \delta .w^\delta + \xi^\alpha.w^\beta / / \rho / \delta .w^\beta .w^\delta .w^\gamma \] (134)

The commutator of the covariant differential operator and the deviation operator obeys the following identity
\[ \langle [\nabla_w, L_\xi \Gamma(\xi, u)] \rangle \equiv \langle [\nabla_w, [L_\xi, \nabla_u]] \rangle - \langle R(w, L_\xi u) \rangle , \] (135)
where
\[ \langle [\nabla_w, L_\xi \Gamma(\xi, u)] \rangle = \langle [\nabla_w, \nabla_w \Gamma(\xi, u)] \rangle + \langle [\nabla_u, \nabla_w \Gamma(v, \xi)] \rangle + \langle [\nabla_v, \nabla_u \Gamma(\xi, v)] \rangle , \]
\[ \langle [\nabla_w, [L_\xi, \nabla_u]] \rangle = \langle [\nabla_w, [L_\xi, \nabla_u]] \rangle + \langle [\nabla_u, [L_\xi, \nabla_v]] \rangle + \langle [\nabla_v, [L_\xi, \nabla_w]] \rangle , \]
\[ \langle R(w, L_\xi u) \rangle = R(w, L_\xi u) + R(u, L_\xi w) + R(\xi, L_w w) \]
\[ , \xi, u, w \in T(M) . \]

4. The action of the deviation operator on covariant vector fields is determined by its structure and especially by the Lie differential operator.

In a non-co-ordinate basis
\[ [L_\xi(\xi, v)] e^\alpha = L_\xi \nabla_\xi e^\alpha - \nabla_{\nabla_\xi e^\alpha} e_\xi - \nabla_{\xi e^\alpha} e^\alpha = (L_\xi P_{\beta\gamma}^\delta).e^\beta , \] (136)
where
\[ L_\xi P_{\beta\gamma}^\delta = \xi^\delta.e_{\beta\gamma} P_{\beta\gamma}^\delta + P_{\beta\gamma}^\delta.e_{\beta\gamma} \xi^\delta + P_{\beta\gamma}^\delta.(P_{\beta\gamma}^\rho + \Gamma_{\beta\gamma}^\rho + C_{\beta\gamma}^\rho).\xi^\rho - \]
\[ - e_{\beta\gamma}(e_{\rho\gamma} - e_{\gamma\rho}).e_{\beta\gamma} P_{\beta\gamma}^\rho + \Gamma_{\beta\gamma}^\rho + C_{\beta\gamma}^\rho).\xi^\rho - \]
\[ - P_{\beta\gamma}^\delta[e_{\gamma \xi} - e_{\xi \gamma}].e_{\beta\gamma} P_{\beta\gamma}^\rho + \Gamma_{\beta\gamma}^\rho + C_{\beta\gamma}^\rho).\xi^\rho + \]
\[ + P_{\beta\gamma}^\delta.(e_{\gamma \xi} - C_{\gamma \xi} - \xi^\rho) \] (137)
\[ e_\beta \xi^\gamma = f_\beta^\sigma \epsilon \sigma \epsilon \xi^\gamma = f_\kappa^\beta \xi^\kappa, \quad \Gamma_{\beta \rho}^\gamma = f_\kappa^\gamma \epsilon \beta \epsilon \kappa \epsilon \rho. \tag{138} \]

The expression for \( \mathcal{L}_\xi P_{\beta \gamma}^a \) can also be written in the form
\[ \mathcal{L}_\xi P_{\beta \gamma}^a = - P_{\beta \gamma \delta}^a \xi^\delta - \xi_{\beta ; \gamma} + T_{\beta \delta} \xi^\gamma + T_{\beta \delta} \xi^\delta. \tag{139} \]

\( \mathcal{L}_\xi P_{\beta \gamma}^a \) is called Lie derivative of the components \( P_{\beta \gamma}^a \) of the covariant affine connection \( P \) in a non-co-ordinate basis.

Special case: \( S = \epsilon^\sigma C : f^\alpha_\beta = \epsilon^\sigma g_{\beta}^\alpha, \quad f^i_j = \epsilon^\sigma g^i_j, \quad f_\beta^\gamma = \epsilon^\sigma g^\beta_\gamma. \)
\[ \mathcal{L}_\xi P_{\beta \gamma}^a = - P_{\beta \gamma \delta}^a \xi^\delta - \xi_{\beta ; \gamma}^i + T_{\beta \delta}^i \xi^\gamma + T_{\beta \gamma}^i \xi^\delta. \tag{140} \]

The Lie derivative of the components of the covariant affine connection \( P \) could be used in considerations related to deviation equations for covariant vector fields.

### 7 Extended covariant differential operator. Extended derivative

If \( \Gamma_{\beta \gamma} \) are components of a contravariant affine connection \( \Gamma \) and \( P_{\beta \gamma} \) are components of a covariant affine connection \( P \) in a given (here co-ordinate) basis in a \((L_n, g)\)-space, then \( \Gamma_{\beta \gamma} \) and \( P_{\beta \gamma} \) correspond to a new \([\text{extended with respect to } \nabla_u, \ u \in T(M)]\) covariant differential operator \( \epsilon \nabla_u \)
\[ \epsilon \nabla_u \xi^i = \Gamma_{\beta \gamma}^i \xi^\beta - A_{\beta \gamma}^{i j k} \xi^j \xi^k, \quad \epsilon \nabla_u \xi^i = P_{\beta \gamma}^i \xi^\beta - B_{\beta \gamma}^{i j k} \xi^j \xi^k, \quad A, B \in \otimes^2(M), \]
are components (in the same basis) of a new contravariant affine connection \( \Gamma \) and a new covariant affine connection \( P \) respectively.

\( \Gamma \) and \( P \) correspond to a new \([\text{extended with respect to } \nabla_u, \ u \in T(M)]\) covariant differential operator \( \epsilon \nabla_u \)
\[ \epsilon \nabla_u \xi^i = \Gamma_{\beta \gamma}^i \xi^\beta - A_{\beta \gamma}^{i j k} \xi^j \xi^k, \quad \epsilon \nabla_u \xi^i = P_{\beta \gamma}^i \xi^\beta - B_{\beta \gamma}^{i j k} \xi^j \xi^k, \quad A, B \in \otimes^2(M), \]
with the same properties as the covariant differential operator \( \nabla_u \).

If we choose the tensors \( \Gamma \) and \( P \) with certain predefined properties, then we can find \( \Gamma \) and \( P \) with predetermined characteristics. For instance, we can find \( \nabla_u \) for which \( \epsilon \nabla_u g = 0, \ \forall u \in T(M) \), \( g \in \otimes^2(M) \), although \( \nabla_u g \neq 0 \) for the covariant affine connection \( P \). On the other side, \( \Gamma_{\beta \gamma} \) and \( P_{\beta \gamma} \) are related to each other on the basis of the commutation relations of \( \epsilon \nabla_u \) with the contraction operator \( S \). From
\[ \nabla_u \circ S = S \circ \nabla_u, \quad (S \circ \epsilon \nabla_u)(\partial_j \otimes dx^i) = \Gamma_{\beta \gamma}^{i j} f^j_i - A_{\beta \gamma}^{i j k} f^j_i f^k_i + P_{\beta \gamma}^{i j} f^j_i f^k_i, \]
we have
\[ (S \circ \epsilon \nabla_u)(\partial_j \otimes dx^i) = \Gamma_{\beta \gamma}^{i j} f^j_i - A_{\beta \gamma}^{i j k} f^j_i f^k_i + P_{\beta \gamma}^{i j} f^j_i f^k_i, \]
\(( \nabla \partial_s \circ S)(\partial_j \otimes dx^j) = \nabla \partial_h (S(\partial_j \otimes dx^j)) = \nabla \partial_h (f^i_j) = \partial_k (f^i_j) = f^i_{j,k} \).

Therefore,

\[ f^i_{j,k} = \Gamma^i_{jk} f^i_l + \tilde{A}^i_{jk} f^i_l + P^i_{lk} f^j_j - \tilde{B}^i_{lk} f^l_j . \]

Since \((\nabla \partial_h \circ S)(\partial_j \otimes dx^j) = (S \circ \nabla \partial_h)(\partial_j \otimes dx^j)\) leads to the relation \(f^i_{j,k} = \Gamma^i_{jk} f^i_l + P^i_{lk} f^j_j\), we obtain the connection between \(\tilde{A}^i_{jk}\) and \(\tilde{B}^i_{jk}\) in the form \(\tilde{A}^i_{jk} f^i_l + \tilde{B}^i_{lk} f^l_j = 0\).

Therefore, \(\tilde{B}^i_{jk} = - \tilde{A}^i_{mk} f^i_l f^j_m = - \tilde{A}^i_{jk} + \tilde{A}^i_{jl} f^j_m = - \tilde{B}^i_{jl} f^j_m\).

We can write \(\nabla \partial_h\) as a covariant differential operator because \(f \mathbf{dx}^j\) appears as a mixed tensor field of second rank but acting on tensor fields as a covariant differential operator because \(\mathbf{dx}^j\) is defined as covariant differential operator on tensor field in \(\mathbb{R}^n\).

**Definition 17** Extended to \(\nabla_u\) covariant differential operator. The linear differential operator \(\mathbf{dx}^j\) : \(v \to \nabla_u v = \tilde{v}, v, \tilde{v} \in \otimes^k T(M)\), with the properties of \(\nabla_u\).

From the properties of \(\mathbf{dx}^j\) and \(\nabla_u\) the properties of the operator \(\tilde{A}_u\) follow

\[ \tilde{A}_u : v \to \mathbf{dx}^j u , \quad u \in T(M) , \quad v, \mathbf{dx}^j u \in \otimes^k T(M) . \]

(a) \(\tilde{A}_u (v + w) = \tilde{A}_u v + \tilde{A}_u w , \quad v, w \in \otimes^k T(M) . \)

(b) \(\tilde{A}_u (f v) = f \tilde{A}_u v , \quad f \in C^r(M) . \)

(c) \(\tilde{A}_u w = \tilde{A}_u w + \tilde{A}_u w . \)

(d) \(\tilde{A}_uf v = f \tilde{A}_u v . \)

(e) \(\tilde{A}_u f = 0 . \)

(f) \(\tilde{A}_u (v \otimes w) = \tilde{A}_u v \otimes w + v \otimes \tilde{A}_u w , \quad v \in \otimes^k T(M) , w \in \otimes^m T(M) . \)

(g) \(\tilde{A}_u \circ S = S \circ \tilde{A}_u\) (commutation relation with \(\mathbf{dx}^j\)).

All properties of \(\tilde{A}_u\) correspond to the properties of \(\mathbf{dx}^j\) and \(\nabla_u\) as well defined covariant differential operators. In fact, \(\tilde{A}_u\) can be defined as \(\tilde{A}_u = \nabla_u - \mathbf{dx}^j\). If \(\tilde{A}_u\) is a given mixed tensor field, then \(\mathbf{dx}^j\) can be constructed in an unique way.

On the grounds of the above considerations we can formulate the following proposition:

**Proposition 18** To every covariant differential operator \(\nabla_u\) and a given tensor field \(\tilde{A}_u \in \otimes^k T(M)\) acting as a covariant differential operator on tensor field in a \((\mathbb{R}^n, g)\)-space corresponds an extended covariant differential operator \(\mathbf{dx}^j = \nabla_u - \tilde{A}_u\).
In accordance to its property (c): \( A_{u+v} = A_u + A_v \), \( A_u \) has to be linear to \( u \). On the other side, \( \overline{A}_u \) as a mixed tensor field of second rank can be represented by the use of the existing in \((\mathcal{L}_n, g)\)-space contravariant and covariant metrics \( \overline{\mathcal{F}} \) and \( g \) respectively in the form \( \overline{A}_u = \overline{\mathcal{F}}(A_u) \), where \( A_u \) is a covariant tensor field of second rank constructed by the use of a tensor field \( C \) and a contravariant vector field \( u \) in such a way that \( A_u \) is linear to \( u \). There are at least three possibilities for construction of a covariant tensor field of second rank \( A_u \) in such a way that \( A_u \) is linear to \( u \), i.e.

1. \( A_u = C(u) = A_{ij}(u)^{i} dx^i \otimes dx^j \), \( A = A_{ijk} dx^i \otimes dx^j \otimes dx^k \in \otimes_3(M) \), \( u \in T(M) \).
2. \( A_u = C(u) = \nabla u B = B_{ij,k} u^k dx^i \otimes dx^j \), \( B = B_{ij} dx^i \otimes dx^j \in \otimes_2(M) \), \( u \in T(M) \).
3. \( A_u = C(u) = A(u) + \nabla u B = A_{ijk} u^k dx^i \otimes dx^j \), \( A = A_{ijk} dx^i \otimes dx^j \otimes dx^k \in \otimes_3(M) \), \( u \in T(M) \); \( \nabla u B = B_{ij,k} u^k dx^i \otimes dx^j \), \( B = B_{ij} dx^i \otimes dx^j \in \otimes_2(M) \), \( u \in T(M) \).

An extended covariant differential operator \( \nabla u = \nabla u - \overline{A}_u \) can obey additional conditions determining the structure of the mixed tensor field \( \overline{A}_u \) (acting on tensor fields as a covariant differential operator). One can impose given conditions on \( \nabla u \) leading to determined properties of \( \overline{A}_u \) and vice versa: one can impose conditions on the tensor field \( \overline{A}_u \) leading to determined properties of \( \nabla u \).

Every extended covariant differential operator as well as every covariant differential operator have their special type of transports of covariant vector fields.

## 8 Metrics

The notion contraction operator has been introduced, acting on two vectors belonging to two different vector spaces with equal dimensions in a point of a differentiable manifold \( M \) and juxtaposing to them a function over \( M \). If a contraction operator is acting on two vectors belonging to one and the same vector space, then this operator is connected with the notion metric.

**Definition 19** Metric. Contraction operator \( S \) acting on two vectors of one and the same vector space and mapping them to an element of the field \( F \) (\( \mathbb{R} \) or \( \mathbb{C} \)).

**Definition 20** Metric over a differentiable manifold \( M \). Contraction operator \( S \), acting on two vector fields which vectors in every given point \( x \in M \) belong to one and the same vector space, i.e. \( S : (u, v) \to S(u, v) \in C^\prime(M) \), \( u_x, v_x \in N_x(M) \).

## 8.1 Covariant metric
**Definition 21** Covariant metric. Contraction operator $S$, acting on two contravariant vector fields over a manifold $M$, which action is identified with the action of a covariant symmetric tensor field of rank two on the two vector fields, i.e.

$$S(u, v) = S(g, q) = S(g, u \otimes v) = S(g \otimes (u \otimes v)), \quad q = u \otimes v. \ (141)$$

The tensor $g = g_{\alpha\beta}.e^\alpha.e^\beta = g_{ij}.dx^i.dx^j$ is called covariant metric tensor field (covariant metric) and $g(x) = g_x \in \otimes_{2f_x}(M)$ is called covariant metric tensor (covariant metric) at a point $x \in M$.

(a) Action of the covariant metric on two contravariant vector fields in a co-ordinate basis

$$g(u, v) = g_{ij}.u^i.v^j = g_{ij}.u^i.v^j = g_{ij}.u^i.v^j = u^i.v^j = u^i.v^j, \quad u^i = g_{ij}.u^j, \quad u_j = g_{ij}.u^i. \ (142)$$

**Remark 4** $g(u, v)$ is also called scalar product of the contravariant vector fields $u$ and $v$ over the manifold $M$. When $v = u$, then

$$g(u, u) = g_{ij}.u^i.u^j = g_{ij}.u^i.u^j = u^i.u^j = u^i.u^j = := u^2 = \pm |u|^2 = \pm l^2_u, \quad (144)$$

and $g(u, u) = u^2 = \pm l^2_u$ is called the square of the length of the contravariant vector field $u$.

(b) The action of the covariant metric on a contravariant vector field $u$ can be introduced by means of the contraction operator $S$ in a co-ordinate basis as

$$g(u) := S^j.k(g, u) = S^j.k(g_{ij}.dx^i.dx^j, u^k.dk) = g_{ij}.u^k.S^j.k(dx^i.dx^j, dk) = g_{ij}.u^k.f^j.k.dx^i = g_{ij}.u^k.dx^i = g_{ij}.u^k.dx^i = u(g), \quad g_{ij} = g_{ij}.f^j.k. \ (145)$$

**Remark 5** The abbreviation $u(g)$ is equivalent to the abbreviation $(u)(g) := S(u, g)$. It should not be considered as the result of the action of the contravariant vector field $u$ on $g$. Such action of $u$ on $g$ is (until now) not defined.

The action of the covariant metric $g$ on a contravariant vector field $u$ considered in index form (in a given basis) is called *lowering indices* by means of $g$. The result of the action of $g$ on $u \in T(M)$ is a covariant vector field $g(u) \in T^*(M)$. On this ground, $g$ can be defined as linear mapping (operator) which maps every element of $T(M)$ in a corresponding element of $T^*(M)$, i.e. $g : u \rightarrow g(u) \in T^*(M), \ u \in T(M)$. 

31
8.1.1 Covariant symmetric affine connection

In a non-co-ordinate basis the covariant affine connection \( P \) will have the form

\[
P^\gamma_{\alpha\beta} = \overline{P}^\gamma_{\alpha\beta} + \frac{1}{2} U^\gamma_{\alpha\beta},
\]

(146)

where

\[
\overline{P}^\gamma_{\alpha\beta} = \frac{1}{2} (P^\gamma_{\alpha\beta} + P^\gamma_{\beta\alpha} + C^\gamma_{\alpha\beta}),
\]

(147)

\[
U^\gamma_{\alpha\beta} = P^\gamma_{\alpha\beta} - P^\gamma_{\beta\alpha} - C^\gamma_{\alpha\beta} = -U^\gamma_{\beta\alpha}.
\]

The components of the covariant derivative of covariant metric tensor field \( g \) can be presented by means of the covariant symmetric affine connection. If

\[
g_{\alpha\beta;\gamma} = e^\gamma_{\alpha\beta} + \overline{P}^\gamma_{\alpha\gamma} g_{\delta\beta} + \overline{P}^\gamma_{\beta\gamma} g_{\delta\alpha}
\]

(148)

is the covariant derivative of the components \( g_{\alpha\beta} \) of the covariant metric tensor \( g \) with respect to the covariant symmetric affine connection \( \overline{P} \) in a non-co-ordinate basis, then

\[
g_{\alpha\beta;\gamma} = g_{\alpha\beta;\gamma} + \frac{1}{2} (U^\gamma_{\alpha\beta} g_{\delta\gamma} + U^\gamma_{\beta\gamma} g_{\delta\alpha}).
\]

(149)

On the other side, the components of the covariant symmetric affine connection \( \overline{P}^\gamma_{\alpha\beta} \) can be written in the form

\[
g_{\delta\gamma} \overline{P}^\delta_{\alpha\beta} = -\{\alpha\beta, \gamma\} + K_{\alpha\beta\gamma} + C_{\alpha\beta\gamma} + \frac{1}{2} (g_{\delta\alpha} U^\delta_{\beta\gamma} + g_{\delta\beta} U^\delta_{\alpha\gamma}) =
\]

(150)

where

\[
\{\alpha\beta, \gamma\} = \frac{1}{2} (e_{\alpha} g_{\beta\gamma} + e_{\beta} g_{\alpha\gamma} - e_{\gamma} g_{\alpha\beta}),
\]

(151)

\[
K_{\alpha\beta\gamma} = \frac{1}{2} (g_{\alpha\gamma}/\beta + g_{\beta\gamma}/\alpha - g_{\alpha\beta}/\gamma),
\]

\[
C_{\alpha\beta\gamma} = \frac{1}{2} (g_{\delta\alpha} C_{\beta\gamma}^\delta + g_{\delta\beta} C_{\alpha\gamma}^\delta + g_{\delta\gamma} C_{\alpha\beta}^\delta).
\]

By means of the last expressions \( P^\gamma_{\alpha\beta} \) can be represented in the form

\[
g_{\delta\gamma} \overline{P}^\delta_{\alpha\beta} = -\{\alpha\beta, \gamma\} + K_{\alpha\beta\gamma} + U_{\alpha\beta\gamma} + C_{\alpha\beta\gamma},
\]

(152)

where

\[
U_{\alpha\beta\gamma} = \frac{1}{2} (g_{\delta\alpha} U^\delta_{\beta\gamma} + g_{\delta\beta} U^\delta_{\alpha\gamma} + g_{\delta\gamma} U^\delta_{\alpha\beta}).
\]

(153)

In the special case, when the condition \( g_{\alpha\beta;\gamma} = 0 \) is required, then the following proposition can be proved:

**Proposition 22** The necessary and sufficient condition for \( g_{\alpha\beta;\gamma} = 0 \) is the condition

\[
g_{\delta\gamma} \overline{P}^\delta_{\alpha\beta} = -\{\alpha\beta, \gamma\} + C_{\alpha\beta\gamma}.
\]

(154)
The proof follows immediately from (150).

In a co-ordinate basis the covariant derivative of the components $g_{ij}$ of $g$ can in analogous way be presented by means of the components $P^l_{jk}$ of the covariant symmetric affine connection

$$g_{ij; k} = g_{ij, k} + \mathcal{P}^l_{ik}g_{lj} + \mathcal{P}^l_{jk}g_{il} + \frac{1}{2}(U^l_{ik}g_{lj} + U^l_{jk}g_{il}) = g_{ij/k} + \frac{1}{2}(U^l_{ik}g_{lj} + U^l_{jk}g_{il}) , \quad (155)$$

where

$$g_{ij/k} = g_{ij, k} + \mathcal{P}^l_{ik}g_{lj} + \mathcal{P}^l_{jk}g_{il} . \quad (156)$$

8.1.2 Action of the Lie differential operator on the covariant metric

In a co-ordinate basis $\mathcal{L}_\xi g$ will take the form

$$\mathcal{L}_\xi g = (\mathcal{L}_\xi g_{ij})dx^idx^j = [g_{ij,k}\xi^k + g_{kj}\xi^k + g_{ik}\xi^k + (g_{kj}, T^k_{ij} + g_{ik}, T^k_{ij})].\xi^l].dx^i.dx^j . \quad (157)$$

The following relations are also fulfilled:

$$\mathcal{L}_\xi[g(u,v)] = \xi[g(u,v)] = (\mathcal{L}_\xi g)(u,v) + g(\mathcal{L}_\xi u,v) + g(u, \mathcal{L}_\xi v) , \quad \mathcal{L}_\xi g(u) = (\mathcal{L}_\xi g)(u) + g(\mathcal{L}_\xi u) , \quad \xi, u, v \in T(M) . \quad (158)$$

The action of the Lie differential operator is called dragging-along a contravariant vector field. On the basis of draggings-along the metric tensor field $g$ notions as arbitrary (non-metric) draggings-along, quasi-projective draggings-along, conformal motions and motions can be defined and considered in analogous way as in $(L_n, g)$-spaces. Here we will only define different types of draggings-along:

1. Arbitrary (non-metric) draggings-along

$$\mathcal{L}_\xi g = q_\xi , \quad \forall \xi \in T(M) , \quad q_\xi \in \otimes_{\text{sym}}^2(M) ,$$

2. Quasi-projective draggings-along

$$\mathcal{L}_\xi g = \frac{1}{2}[p \otimes g(\xi) + g(\xi) \otimes p] , \quad \xi \in T(M) , \quad p \in T^*(M) .$$

3. Conform-invariant draggings-along (conformal motions)

$$\mathcal{L}_\xi g = \lambda g , \quad \lambda \in C^r(M) , \quad \xi \in T(M) .$$

4. Isometric draggings-along (motions)

$$\mathcal{L}_\xi g = 0 , \quad \xi \in T(M) .$$

For all types of draggings-along changes of the scalar product of two contravariant vector fields and the changes of the length of these fields can be found and used in the analogous way as in $(L_n, g)$-spaces.
8.2 Covariant projective metric

If a covariant metric field \( g \) is given and there exists a contravariant vector field \( u \) which square of the length \( g(u, u) = e \neq 0 \), then a new covariant tensor field can be constructed orthogonal to the vector field \( u \). It possesses properties analogous to these of the covariant tensor field \( g \) acting on contravariant vector fields in every orthogonal to \( u \):

\[
(T^\perp_x u(M) = \{ \xi_x \} : g_x(\xi_x, u_x) = 0), \quad g_x \in \otimes_{\text{sym}}^2 x(M).
\]

**Definition 23** Covariant projective metric. Covariant metric, orthogonal to a given non-isotropic (non-null) vector field \( u \) \([e = g(u, u) \neq 0] \), i.e. covariant metric \( h_u \) satisfying the condition \( h_u(u) = u(h_u) = 0 \) and constructed by means of the covariant metric \( g \) and \( u \) in the form

\[
h_u = g - \frac{1}{g(u, u)} g(u) \otimes g(u) = g - \frac{1}{e} g(u) \otimes g(u) . \quad (159)
\]

The properties of the covariant projective metric follow from its construction and from the properties of the covariant metric \( g \):

(a) \( h_u(u) = u(h_u) = 0 \), \( g(u)(u) = g(u, u) = e \).

(b) \( h_u(u, u) = 0 \).

(c) \( h_u(u, v) = h_u(v, u) = 0 \), \( \forall v \in T(M) \).

8.3 Contravariant metric

**Definition 24** Contravariant metric. Contraction operator \( S \), acting on two covariant vector fields over a manifold \( M \) which action is identified with the action of a contravariant symmetric tensor field of rank two on the two vector fields, i.e.

\[
S(p, q) = \mathcal{F}(p, q) := S(\mathcal{F}, w) := S(\mathcal{F}, p \otimes q) = S(\mathcal{F} \otimes (p \otimes q)) , \quad w = p \otimes q ,
\]

The tensor field \( \mathcal{F} = g^{\alpha\beta} e_\alpha e_\beta = g^{ij} \partial_i \partial_j \) is called contravariant metric tensor field (contravariant metric). \( \mathcal{F}(x) = \mathcal{F}_x \in \otimes^2 x(M) \) is called contravariant metric tensor in \( x \in M \).

The properties of the contravariant metric are determined by the properties of the contraction operator and its identification with the contravariant symmetric tensor field of rank 2. On this basis the following properties can be proved:

(a) Action of the contravariant metric on two covariant vector fields in a co-ordinate basis

\[
\mathcal{F}(p, q) = g^{kl} f^i_k f^j_l \cdot p_i q_j = g^{ij} p_i q_j = g^{kl} p_k^\alpha q^\alpha = p_j^\alpha q^\alpha ,
\]

\[
p^\alpha_k = f^i_k \cdot p_i , \quad q^\alpha_i = f^j_i \cdot q_j , \quad p^\alpha_j = g^{ij} p_i .
\]

[When \( q = p \), then \( \mathcal{F}(p, p) = p^2 = \pm | p |^2 \) is called square of the length of the covariant vector field \( p \).]
(b) Action of the contravariant metric $\overline{g}$ on covariant vector field

$$\overline{g}(p) = p(\overline{f}) = g^{ij}p^j \partial_i = g^{ij}p^j \partial_i = g^{ij}p^j \partial_i = g^{ij}p^j \partial_i,$$

$$g^{ij} = g^{il}f^k_p = g^{ik}p^k \partial_i = g^{ik}p^k \partial_i = p^i \partial_i,$$

The action of the contravariant metric $\overline{g}$ on covariant vector field $p$ in a given basis is called raising of indices by means of the contravariant metric. The result of this action is a contravariant vector field $\overline{g}(p)$. On this basis $\overline{g}$ can be defined as a linear mapping (operator) which maps an element of $T^*(M)$ in an element of $T(M)$:

$$\overline{g}: p \rightarrow \overline{g}(p) \in T(M) \ , \ p \in T^*(M) .$$

The connection between the contravariant and covariant metric can be determined by the conditions

$$\overline{g}[g(u)] = u , \quad u \in T(M) , \quad g[\overline{g}(p)] = p , \quad p \in T^*(M) . \quad \text{(160)}$$

In a co-ordinate basis these conditions take the forms:

$$g^{ij}\overline{g}_{jk} = g^i_k , \quad g_{ij}\overline{g}^{jk} = g^k_i . \quad \text{(161)}$$

From the last expressions the relation follows

$$g[\overline{g}] = g_{ij}\overline{g}^{ij} = n , \quad \overline{g}[g] = g^{ij}\overline{g}_{ij} = n , \quad \dim M = n . \quad \text{(162)}$$

8.3.1 Contravariant symmetric affine connection

From the transformation properties of the components of the contravariant affine connection, it follows that the quantity

$$\frac{1}{2}(\Gamma^\gamma_{\alpha\beta} + \Gamma^\gamma_{\beta\alpha} - C_{\alpha\beta\gamma}) \text{ or } \frac{1}{2}(\Gamma^k_{ij} + \Gamma^k_{ji})$$

has the same transformation properties as the contravariant affine connection itself. This fact can be used as usual for representing the contravariant affine connection by means of its symmetric and anti-symmetric part in the form

$$\Gamma^k_{ij} = \Gamma^k_{ij} - \frac{1}{2}T^k_{ij} , \quad \Gamma^k_{ij} = \frac{1}{2}(\Gamma^k_{ij} + \Gamma^k_{ji}) , \quad T^k_{ij} = \Gamma^k_{ji} - \Gamma^k_{ij} ,$$

$$\Gamma^\gamma_{\alpha\beta} = \Gamma^\gamma_{\alpha\beta} - \frac{1}{2}T^\gamma_{\alpha\beta} , \quad \Gamma^\gamma_{\alpha\beta} = \frac{1}{2}(\Gamma^\gamma_{\alpha\beta} + \Gamma^\gamma_{\beta\alpha} - C_{\alpha\beta\gamma}) , \quad \frac{1}{2}(\Gamma^\gamma_{\alpha\beta} + \Gamma^\gamma_{\beta\alpha} - C_{\alpha\beta\gamma}) , \quad \text{(163)}$$

$$T^\gamma_{\alpha\beta} = \Gamma^\gamma_{\beta\alpha} - \Gamma^\gamma_{\alpha\beta} - C_{\alpha\beta\gamma} , \quad \text{in a co-ordinate basis ,}$$

$$\Gamma^\gamma_{\alpha\beta} = \Gamma^\gamma_{\alpha\beta} - \frac{1}{2}T^\gamma_{\alpha\beta} \quad \text{in a co-ordinate basis .}$$

$\Gamma^k_{ij}$ ($\Gamma^\gamma_{\alpha\beta}$) are called components of the contravariant symmetric affine connection in a co-ordinate (respectively in a non-co-ordinate) basis.
The components of the covariant derivative of the contravariant metric tensor field $\mathbf{g}$ can be represented by means of the contravariant symmetric affine connection. If we introduce the abbreviations

$$g^{\alpha\beta} \gamma = e_{\gamma} g^{\alpha\beta} + \Gamma_{\delta\gamma}^{\beta} g^{\delta\alpha} + \Gamma_{\delta\gamma}^{\delta} g^{\alpha\beta} \quad \text{(in a non-co-ordinate basis)}$$

$$g^{ij} /k = g^{ij} ,k + \Gamma_{ik}^{j} g^{ij} + \Gamma_{ik}^{j} g^{ij} \quad \text{(in a co-ordinate basis)}$$

where $g^{\alpha\beta} \gamma$ is the covariant derivative of the components of the contravariant metric tensor $\mathbf{g}$ with respect to the contravariant symmetric affine connection $\Gamma$ in a non-co-ordinate basis, then

$$g^{\alpha\beta} \gamma = g^{\alpha\beta} \gamma - \frac{1}{2} (T_{\delta\gamma}^{\alpha} g^{\delta\beta} + T_{\delta\gamma}^{\beta} g^{\alpha\delta}) . \quad (165)$$

By means of the explicit expression for $g^{\alpha\beta} /\kappa . g^{\kappa\gamma}$ and the usual method for expressing the components of the symmetric affine connection the components of the contravariant symmetric affine connection can be represented in the form

$$g^{\alpha\delta} g^{\beta\kappa} \Gamma_{\delta\kappa}^{\gamma} = \frac{1}{2} \left( g^{\alpha\gamma} . e_{\kappa} g^{\beta\delta} + g^{\beta\gamma} . e_{\kappa} g^{\alpha\delta} + g^{\alpha\delta} . e_{\kappa} g^{\beta\gamma} \right) - \frac{1}{2} g^{\gamma\delta} (g^{\beta\kappa} T_{\delta\kappa}^{\alpha} + g^{\alpha\kappa} T_{\delta\kappa}^{\beta}) . \quad (166)$$

where

{\{\alpha\beta,\gamma\}} = \frac{1}{2} \left( g^{\alpha\kappa} . e_{\kappa} g^{\beta\gamma} + g^{\beta\kappa} . e_{\kappa} g^{\alpha\gamma} - g^{\gamma\kappa} . e_{\kappa} g^{\alpha\beta} \right),

$$\overline{a}^{\beta\gamma} = \frac{1}{2} \left( g^{\beta\gamma} /\kappa . g^{\alpha\kappa} + g^{\alpha\gamma} /\kappa . g^{\beta\kappa} - g^{\alpha\beta} /\kappa . g^{\gamma\kappa} \right), \quad (167)$$

{\{\alpha\beta,\gamma\}} are called Christoffel symbols of the first kind for the contravariant symmetric affine connection in a non-co-ordinate basis.

The components $\Gamma_{\alpha\beta}^{\gamma}$ of the contravariant affine connection $\Gamma$ can be written by means of the last abbreviations in the form

$$g^{\alpha\delta} g^{\beta\kappa} \Gamma_{\delta\kappa}^{\gamma} = -\{\alpha\beta,\gamma\} + \frac{1}{2} \left( g^{\gamma\delta} (g^{\beta\kappa} T_{\delta\kappa}^{\alpha} + g^{\alpha\kappa} T_{\delta\kappa}^{\beta}) \right) - \frac{1}{2} g^{\gamma\delta} (g^{\beta\kappa} T_{\delta\kappa}^{\alpha} + g^{\alpha\kappa} T_{\delta\kappa}^{\beta}) , \quad (171)$$

By using the connections between the components of the covariant metric, the components of the contravariant metric and their derivatives

$$g_{\alpha\kappa} g^{\beta\gamma} = g_{\alpha\kappa} , g^{\beta\gamma} , e_{\kappa} g^{\delta\gamma} = -g^{\delta\gamma} , e_{\kappa} (g_{\alpha\delta}) , \quad (170)$$

the components of the contravariant affine connection $\Gamma$ can be represented in a non-co-ordinate basis in the form

$$\Gamma_{\alpha\beta}^{\gamma} = \{\alpha\beta\} - \frac{1}{2} \left( \overline{a}_{\alpha\beta}^{\gamma} - \overline{a}_{\alpha\beta}^{\gamma} \right) , \quad (171)$$
where

$$\{\gamma_{\alpha\beta}\} = \frac{1}{2} g^{\gamma\delta}[e_\beta(g_{\alpha\delta}) + e_\alpha(g_{\beta\delta}) - e_\delta(g_{\alpha\beta})] = -g_{\gamma\rho} g_{\alpha\sigma} \{\rho\sigma, \gamma\}, \quad \Gamma_{\alpha\beta}^\gamma = -g_{\gamma\rho} g_{\alpha\sigma} \Gamma_{\rho\sigma\gamma}, \quad C_{\alpha\beta}^\gamma = -g_{\gamma\rho} g_{\alpha\sigma} C_{\rho\sigma\gamma}. \quad (172)$$

$$\{\gamma_{\alpha\beta}\}$$ are called generalized Christoffel symbols of the second kind for the contravariant symmetric affine connection in a non-co-ordinate basis.

In analogous way, when a contravariant and covariant metric fields are given, the covariant affine connection can be presented by means of the both types of tensor metric fields in the form

$$P_{\alpha\beta}^\gamma = -\{\gamma_{\alpha\beta}\} + K_{\alpha\beta}^\gamma + U_{\alpha\beta}^\gamma + C_{\alpha\beta}^\gamma, \quad (173)$$

where

$$\{\gamma_{\alpha\beta}\} = g^{\gamma\sigma}\{\alpha\beta, \gamma\}, \quad K_{\alpha\beta}^\gamma = g^{\gamma\sigma} K_{\alpha\beta\sigma}, \quad U_{\alpha\beta}^\gamma = g^{\gamma\sigma} U_{\alpha\beta\sigma}, \quad C_{\alpha\beta}^\gamma = g^{\gamma\sigma} C_{\alpha\beta\sigma}. \quad (174)$$

$$\{\gamma_{\alpha\beta}\}$$ are called generalized Christoffel symbols of the second kind for the covariant symmetric affine connection in a non-co-ordinate basis.

The same expressions can be obtained also in a co-ordinate basis.

For the special case, when the condition of vanishing of the covariant derivatives of the contravariant metric with respect to the contravariant symmetric affine connection is required, i.e. $g^{\alpha\beta} ; \gamma = 0$, then the components of the contravariant symmetric affine connection can be written in the form

$$\Gamma_{\alpha\beta}^\gamma = \{\gamma_{\alpha\beta}\} - C_{\alpha\beta}^\gamma. \quad (175)$$

The last expression is the necessary and sufficient condition for $g^{\alpha\beta} ; \gamma = 0$.

In a co-ordinate basis the necessary and sufficient condition for $g^{ij} / k = 0$ takes the form

$$\Gamma_{ij}^k = \{\gamma_{ij}\}. \quad (176)$$

On the basis of the connection between the covariant derivative of the contravariant tensor metric field and the covariant derivative of the covariant tensor metric field

$$\nabla_\xi g = -g(\nabla_\xi \overline{g}) g, \quad (\nabla_\xi \overline{g})[g(u)] = -\overline{g}(\nabla_\xi g)(u), \forall \xi, \forall u \in T(M), \quad \nabla_\xi \overline{g} = -\overline{g}(\nabla_\xi g) \overline{g}, \quad (\nabla_\xi \overline{g})(\overline{g}(p)) = -g[(\nabla_\xi \overline{g})(p)], \quad \forall \xi \in T(M), \quad \forall p \in T^*(M),$$

one can prove that there is one-to-one correspondence between the transports of $g$ and $\overline{g}$. Every transport of the covariant tensor metric field $g$ induces a corresponding transport of the contravariant tensor metric field $\overline{g}$ and vice versa.

### 8.4 Contravariant projective metric

The notion of contravariant projective metric with respect to a non-isotropic (non-null) contravariant vector field $u$ can be introduced in two different ways:
(a) by definition
\[ h^u = \bar{g} - \frac{1}{g(u,u)} u \otimes u = \bar{g} - \frac{1}{e} u \otimes u, \quad e = g(u,u) \neq 0, \quad (175) \]

(b) by inducing from the covariant projective metric using the relations between the covariant and contravariant metric
\[ h^u = \bar{g}(h_u)\bar{g} = \bar{g} - \frac{1}{e} u \otimes u, \quad \bar{g}(g(u) \otimes g(u))\bar{g} = u \otimes u. \quad (176) \]

\( h^u \) is called contravariant projective metric with respect to the non-isotropic contravariant vector field \( u \).

The properties of the contravariant projective metric are determined by its structure.

9 Bianchi identities for the covariant curvature tensor

9.1 Bianchi identity of first type for the covariant curvature tensor

The existence of contravariant and covariant metrics allow us to consider the action of the curvature operator on a covariant vector field \( g(v) = g_{\alpha\beta} v^\beta e^\alpha = g_{ij} v^j dx^i \), constructed by the use of the covariant metric \( g \) and a contravariant vector field \( v \).

The identity
\[ < \bar{g} \{ [R(\xi, u)]g(v) \} > \equiv < \bar{g}([R(\xi, u)]g(v)) > - < [R(\xi, u)]v > \equiv \]
\[ < \bar{g}([R(\xi, u)]g(v)) > - < T(T(\xi, u), v) > - < (\nabla_\xi T)(u, v) > \quad (177) \]
is called Bianchi identity of first type (of the type 1.) for the covariant curvature tensor.

In a coordinate basis the Bianchi identity of first type will have the forms
\[ P^{i}_{<ijk>} \equiv - g^{m\bar{\tau}} R^\tau_{m<ij} g_{k>n} , \quad (178) \]
\[ R^{i}_{<ijk>} \equiv - g^{m\bar{\tau}} g_{mn} P^{n}_{<ijk>} \equiv T^{l}_{<ij} T^{k}_{<ij} + T^{m}_{<ij} T_{mk} > l. \quad (179) \]

It is obvious that the form of the Bianchi identity of first type for the components of the covariant curvature tensor is not so simple as the form of the Bianchi identity for the components of the contravariant curvature tensor.

9.2 Bianchi identity of second type for the covariant curvature tensor

The action of the operator \((\nabla_\nu R)(\xi, u)\) can be extended to an action on covariant vector and tensor fields in an analogous way as in the case of contravariant
we can find the identity
\[
\nabla_w \{ R(\xi, u) \} p = [(\nabla_w R)(\xi, u)] p + [R(\nabla_w \xi, u)] p + [R(\xi, \nabla_w u)] p + [R(\xi, u)](\nabla_w p), \quad w, \xi, u \in T(M), \quad p \in T^*(M) ,
\]
we can find the identity
\[
< (\nabla_w R)(\xi, u) > p \equiv < R(w, T(\xi, u)) > p ,
\]
where
\[
< (\nabla_w R)(\xi, u) > p = [\{ \nabla_w R(\xi, u) \} p + [\{ \nabla_u R \}(w, \xi)] p + [\{ \nabla_{\xi} R \}(u, w)] p ,
\]
\[
< R(w, T(\xi, u)) > p = [R(w, T(\xi, u))] p + [R(u, T(w, \xi))] p + [R(\xi, u)](\nabla_w p).
\]
The identity (181) is called Bianchi identity of second type (of type 2.) for the covariant curvature tensor.

The Bianchi identity of second type will have the form in a coordinate basis
\[
P^{i}_{j<kl;m>} = P^{i}_{jkl;m} + P^{i}_{jmk;l} + P^{i}_{jlm;k} \equiv
P^{i}_{j<kn} . T_{lm} n + P^{i}_{jkn} . T_{lm} n + P^{i}_{j<kl} . T_{nm} dV_{n}, (183)
\]

10 Invariant volume element

10.1 Definition and properties

The notion of volume element of a manifold \( M \) can be generalized to the notion of invariant volume element [14].

**Definition 25** The volume element of a manifold \( M \) (dim \( M = n \))

\[
d^{(n)x} = d^{(n)x} = dx^1 \wedge \ldots \wedge dx^n \quad \text{(in a co-ordinate basis),}
\]
\[
dV_{n} = e^1 \wedge \ldots \wedge e^n \quad \text{(in a non-co-ordinate basis).}
\]

The properties of the volume element could be represented as follows:

\[
d^{(n)x} = \frac{1}{\sqrt{\det(A_{\alpha'}^\alpha)}} \varepsilon_A \omega^A = \frac{1}{\sqrt{\det(A_{\alpha'}^\alpha)}} \varepsilon_A . d\tilde{x}^A , \quad d^{(n')x'} = J^{-1} . d^{(n)x} , \quad dV'_{n} = J^{-1} . dV_{n} ,
\]
where \( J = \det(A_{\alpha'}^\alpha) = \det(\partial x^j / \partial x'^i) \), \( dV'_{n} = e'^1 \wedge \ldots \wedge e'^n \), \( \varepsilon_A = \varepsilon_{i_1} \ldots \varepsilon_{i_n} \), \( \omega^A = dx^{i_1} \wedge \ldots \wedge dx^{i_n} \), \( \varepsilon_A \) is the Levi-Civita symbol [14],

\[
\varepsilon_A . \omega^{A'} = J^{-1} . \varepsilon_A . \omega^A \quad \varepsilon_A . \omega^A = J . \varepsilon_A . \omega^{A'} ,
\]
\[
\varepsilon_{A'} . d\tilde{x}^{A'} = J^{-1} . \varepsilon_A . d\tilde{x}^A \quad \varepsilon_A . d\tilde{x}^A = J . \varepsilon_{A'} . d\tilde{x}^{A'},
\]
\[
d^{(n)x} = \frac{1}{\sqrt{\det(A_{\alpha'}^\alpha)}} \varepsilon_A \omega^A = \frac{1}{\sqrt{\det(A_{\alpha'}^\alpha)}} \varepsilon_A . d\tilde{x}^A , \quad d^{(n')x'} = J^{-1} . d^{(n)x} .
\]

The transformation properties of the volume element are corresponding to these of a tensor density of the weight \( \omega = -\frac{1}{2} \). Therefore, for the construction
of an invariant volume element (keeping its form and independent of the choice
of a full anti-symmetric tensor basis) it is necessary the volume element to be
multiplied with a tensor density with the weight \( \omega = \frac{1}{2} \) and rank 0. Since the
covariant metric tensor field is connected with the basic characteristics of con-
travariant (and covariant) vector fields and determines along with them notions
(such as length of a contravariant vector, cosine of the angle between two con-
travariant vectors) which in the Euclidean geometry are related to the notion
volume element, the covariant metric tensor density \( \tilde{Q}_g \) with a weight \( \omega = \frac{1}{2} \)
and rank 0 \((\tilde{Q}_g = |d_g|^{\frac{1}{2}})\) appears as a suitable multiplier to a volume element.

**Definition 26** The invariant volume element \( d\omega \) of a manifold \( M \) (dim \( M = n \)).

\[
d\omega = \sqrt{-d_g} d^{(n)} x := \frac{1}{n!} \varepsilon_A \omega^A, \quad \omega^A = \sqrt{-d_g} \omega^A, \quad d_g < 0,
\]

(invariant volume element in a co-ordinate basis),

\[
d\omega = \sqrt{-d_g} dV_n, \quad d_g < 0,
\]

(invariant volume element in a non-co-ordinate basis).

From the transformation properties of \( \sqrt{-d_g} \) : \( \sqrt{-d'_g} = \pm J \sqrt{-d_g} \) the in-
variance of the invariant volume element follows: \( d\omega' = \pm d\omega \), where

\[
d\omega' = \sqrt{-d'_g} d^{(n)} x' \quad \text{(in a co-ordinate basis)},
\]

\[
d\omega' = \sqrt{-d'_g} dV'_n \quad \text{(in a non-co-ordinate basis)}.
\]

**Remark 6** The sign \((-)\) in \( \pm d\omega \) can be omitted because of the identical config-
uration (order, orientation) of the basic vector fields in the old and in the new
tensor basis.

From the definition of the invariant volume element the relations connected
with its structure follow:

\[
d\omega' = \frac{1}{n!} \sqrt{-d'_g} \varepsilon_A \omega^A \omega^A = \frac{1}{n!} \sqrt{-d_g} \varepsilon_A d\omega^A = d\omega.
\]

**10.2 Action of the covariant differential operator on an
invariant volume element**

The action of the covariant differential operator on an invariant volume element
is determined by its action on the elements of the construction of the invariant
volume element (the Levi-Civita symbols, the full anti-symmetric tensor basis,
the metric tensor density). From \( d\omega = \frac{1}{n!} \varepsilon_A \omega^A \) and \( \nabla_{\xi}(d\omega) \), it follows

\[
\nabla_{\xi}(d\omega) = \nabla_{\xi}[\frac{1}{n!} \varepsilon_A \omega^A] = \frac{1}{n!}[(\xi \varepsilon_A) \omega^A + \varepsilon_A \nabla_{\xi} \omega^A].
\]
\[ \nabla_\xi(d\omega) \text{ can be written in the form} \]
\[ \nabla_\xi(d\omega) = \frac{1}{2} g[\nabla_\xi g], \frac{1}{n!} \varepsilon_A \omega^A = \frac{1}{2} g[\nabla_\xi g], d\omega . \quad (188) \]

\( \nabla_\xi(d\omega) \) is called \textit{covariant derivative of the invariant volume element} \( d\omega \) along the contravariant vector field \( \xi \).

### 10.3 Action of the Lie differential operator on an invariant volume element

The action of the Lie differential operator on an invariant volume element is determined in analogous way as the action of the covariant differential operator

\[ \mathcal{L}_\xi(d\omega) = \frac{1}{n!} \varepsilon_A \mathcal{L}_\xi \omega^A = \frac{1}{n!} \left[ (\xi \varepsilon_A) \omega^A + \varepsilon_A \mathcal{L}_\xi \omega^A \right] = \frac{1}{n!} \varepsilon_A \mathcal{L}_\xi \omega^A , \quad (189) \]

After some computation, it follows for \( \mathcal{L}_\xi(d\omega) \)

\[ \mathcal{L}_\xi(d\omega) = \frac{1}{n!} \varepsilon_A \frac{1}{2} g[\mathcal{L}_\xi g], \omega^A = \frac{1}{2} g[\mathcal{L}_\xi g], \frac{1}{n!} \varepsilon_A \omega^A , \]
\[ \mathcal{L}_\xi(d\omega) = \frac{1}{2} g[\mathcal{L}_\xi g], d\omega . \quad (190) \]

\( \mathcal{L}_\xi(d\omega) \) is called \textit{Lie derivative of the invariant volume element} \( d\omega \) along the contravariant vector field \( \xi \).

\textit{Special case:} Metric transports \((\nabla_\xi g = 0) : \nabla_\xi(d\omega) = 0 \).

\textit{Special case:} Isometric draggings along \( \text{(motions)} \) \((\mathcal{L}_\xi g = 0) : \mathcal{L}_\xi(d\omega) = 0 \).

\[ \cdots \]

In some cases, when the conservation of the volume is required as an additional condition, one can introduce a new covariant differential operator or a new Lie differential operator which do not change the invariant volume element, i. e. they act on \( d\omega \) in an analogous way as \( \nabla_\xi \) and \( \mathcal{L}_\xi \) act on constant functions.

### 10.4 Covariant differential operator preserving the invariant volume element

The variation of the invariant volume element \( d\omega \) under the action of the covariant differential operator \( \nabla_\xi \)

\[ \nabla_\xi(d\omega) = \frac{1}{2} g[\nabla_\xi g], d\omega \]

allows the introduction of a new covariant differential operator \( \omega \nabla_\xi \) preserving by its action the invariant volume element.
**Definition 27** \( \omega \nabla \xi \) is a covariant differential operator preserving the invariant volume element \( d\omega \) along a contravariant vector field \( \xi \)

\[
\omega \nabla \xi = \nabla \xi - \frac{1}{2} g^{\gamma \delta} g_{\gamma \delta} \xi .
\]

The properties of \( \omega \nabla \xi \) are determined by the properties of the covariant differential operator and the existence of a covariant metric tensor field \( g \) connected with its contravariant metric tensor field \( g \):

(a) Action on an invariant volume element \( d\omega \):

\[
\omega \nabla \xi (d\omega) = 0 , \quad (191)
\]

It follows from the definition of \( \omega \nabla \xi \) and (188).

(b) Action on a contravariant basic vector field:

\[
\omega \nabla \partial_j \partial_i = (\Gamma^k_{ij} - \frac{1}{2} g^m_{ij} \cdot g_{mjk} \cdot g^k_l) \partial_k . \quad (192)
\]

(c) Action on a covariant basic vector field:

\[
\omega \nabla \partial_j dx^i = (P^i_{kj} - \frac{1}{2} g^m_{kj} \cdot g_{mik} \cdot g^k_l) dx_k . \quad (193)
\]

(d) Action on a function \( f \) over \( M \)

\[
\omega \nabla \xi f = \xi f - \frac{1}{2} g^{\gamma \delta} g_{\gamma \delta} \xi f , \quad f \in C^r(M) , \quad r \geq 1 . \quad (194)
\]

If we introduce the abbreviations

\[
Q_\beta = g^{\gamma \delta} g_{\gamma \delta} \xi , \quad Q_j = g^{\gamma \delta} g_{\gamma \delta} \xi_j , \quad (195)
\]

\[
Q = Q_\beta \xi^\beta = Q_j . dx^j , \quad (196)
\]

\[
\omega \Gamma^\gamma_{\alpha \beta} = \Gamma^\gamma_{\alpha \beta} - \frac{1}{2} g^m_{ij} Q_{mij} , \quad \omega P^\alpha_{\gamma \beta} = P^\alpha_{\gamma \beta} - \frac{1}{2} g^m_{ij} Q_{mij} , \quad (197)
\]

\[
Q_\xi = g^{\gamma \delta} \nabla_\xi g = Q_\beta \xi^\beta = Q_j . \xi^j = 2 . \theta_\xi , \quad (198)
\]

then \( \omega \nabla \xi \), (192), and (193) can be written in the form

\[
\omega \nabla \xi = \nabla \xi - \frac{1}{2} Q_\xi , \quad (199)
\]

\[
\omega \nabla e_\alpha = \omega \Gamma^\gamma_{\alpha \beta} e_\gamma , \quad \omega \nabla \partial_j \partial_i = \omega \Gamma^k_{ij} \partial_k , \quad (200)
\]

\[
\omega \nabla e_\alpha = \omega P^\alpha_{\gamma \beta} e_\gamma , \quad \omega \nabla \partial_j dx^i = \omega P^i_{kj} dx^k . \quad (201)
\]

\( \omega \Gamma^\gamma_{\alpha \beta} \) are called components of the *contravariant affine connection* \( \omega \Gamma \) preserving the invariant volume element \( d\omega \) in a non-co-ordinate basis, \( \omega P^\alpha_{\gamma \beta} \) are called components of the *covariant affine connection* \( \omega P \) preserving the invariant volume element \( d\omega \) in a non-co-ordinate basis.
Since $\omega^{\Gamma\gamma}_{\alpha\beta}$ and $\omega^{P\gamma}_{\alpha\beta}$ differ from $\Gamma^{\gamma}_{\alpha\beta}$ and $P^{\gamma}_{\alpha\beta}$ respectively with the components of a mixed tensor field $\frac{1}{2}g^{\gamma}_{\alpha\beta}Q_{\beta}$ of rank 3, $\omega^{\Gamma}$ and $\omega^{P}$ will have the same transformation properties as the affine connections $\Gamma$ and $P$ respectively.

The action of $\omega^{\nabla_{\xi}}$ on a contravariant vector field $u$ can be written in the form

$$\omega^{\nabla_{\xi}}u = \nabla_{\xi}u - \frac{1}{2}Q_{\xi}.u.$$  \hspace{1cm} (202)

If $u$ is considered as a tangential vector field to a curve $x^i(\tau)$, i.e.

$$u = \frac{d}{d\tau} = u^\alpha.e_\alpha = u^i.\partial_i , \quad u^i = \frac{dx^i}{d\tau} ,$$  \hspace{1cm} (203)

$$u^\alpha = A_i^\alpha.u^i = A_i^\alpha.\frac{dx^i}{d\tau} , \quad e_\alpha = A_\alpha^k.\partial_k , \quad A_i^\alpha.A_\alpha^k = g^k_i ,$$  \hspace{1cm} (204)

and the parameter $\tau$ is considered as a function of another parameter $\lambda$ [with one to one (injective) mapping between $\tau$ and $\lambda$], i.e.

$$\tau = \tau(\lambda) , \quad \lambda = \lambda(\tau) ,$$  \hspace{1cm} (205)

$$u = \frac{d}{d\tau} = \frac{d\lambda}{d\tau}.\frac{d}{d\lambda} = \frac{d\lambda}{d\tau}.v , \quad v = \frac{d}{d\lambda} ,$$  \hspace{1cm} (206)

then $\omega^{\nabla_{\xi}}u$ can be represented by means of the vector field $v$ and $\nabla_{\xi}v$ in the form

$$\omega^{\nabla_{\xi}}u = \frac{d\lambda}{d\tau}.\nabla_{\xi}v + [\xi(\frac{d\lambda}{d\tau}) - \frac{1}{2}Q_{\xi}.\frac{d\lambda}{d\tau}]v .$$  \hspace{1cm} (207)

If an additional condition for a relation between $\lambda$ and $\tau$ is given in the form

$$\xi(\frac{d\lambda}{d\tau}) - \frac{1}{2}Q_{\xi}.\frac{d\lambda}{d\tau} = 0 ,$$  \hspace{1cm} (208)

then for an arbitrary vector field $\xi$ a solution for $\lambda = \lambda(\tau)$ exists in the form

$$\lambda = \lambda_0 + \lambda_1.\int[\exp(\frac{1}{2}\int Q_i.dx^i)]d\tau , \quad Q_i = Q_i(x^k) , \quad \lambda_0, \lambda_1 = \text{const.}.$$  \hspace{1cm} (209)

and the connection between $\omega^{\nabla_{\xi}}u$ and $\nabla_{\xi}v$ is obtained in the form

$$\omega^{\nabla_{\xi}}u = \frac{d\lambda}{d\tau}.\nabla_{\xi}v = [\lambda_1.\exp(\frac{1}{2}\int Q_i.dx^i)].\nabla_{\xi}v , \quad \lambda_1 = \text{const.}.$$  \hspace{1cm} (210)

It follows from the last expression that there is a possibility the action of $\omega^{\nabla_{\xi}}$ on a contravariant vector field $u$ (as a tangential vector field to a given curve) to be juxtaposed to the action of $\nabla_{\xi}$ on the corresponding to vector field $u$ vector field $v$ (obtained after changing the parameter of the curve). If the vector field $v$ fulfils the condition for an auto-parallel transport along $\xi$, induced by the covariant differential operator $\nabla_{\xi}$ ($\nabla_{\xi}v = 0$), then the vector field $u$ will also fulfil the auto-parallel condition along $\xi$ induced by the covariant differential operator $\omega^{\nabla_{\xi}}$ ($\omega^{\nabla_{\xi}}u = 0$).
The action of $\omega \nabla \xi$ on a metric tensor field $g$ can be presented in the form

$$\omega \nabla \xi g = \nabla \xi g - \frac{1}{2} Q \xi \cdot g .$$  \hspace{1cm} (211)

After contraction of both components of $\omega \nabla \xi g$ with $g$, i.e., for $\overline{g}[\omega \nabla \xi g] = g^{\overline{\alpha \beta}} (\omega \nabla \xi g)_{\alpha \beta}$, the equality

$$\overline{g}[\omega \nabla \xi g] = (1 - \frac{n}{2}) Q \xi$$  \hspace{1cm} (212)

follows.

The trace free part of $\omega \nabla \xi g$. 

$$\omega \nabla \xi g = \omega \nabla \xi g - \frac{1}{n} \overline{g}[\omega \nabla \xi g] \cdot g ,$$  \hspace{1cm} (213)

by means of (212) can be written in the form

$$\omega \nabla \xi g = \omega \nabla \xi g + \frac{n - 2}{2n} Q \xi \cdot g .$$  \hspace{1cm} (214)

Using this form, $\omega \nabla \xi g$ can be presented by means of its trace free part and its trace part in the form

$$\omega \nabla \xi g = \omega \nabla \xi g - \frac{n - 2}{2n} Q \xi \cdot g ,$$  \hspace{1cm} (215)

where $\overline{g}[\omega \nabla \xi g] = 0$.

Special case: $\dim M = n = 2$: $\omega \nabla \xi g = \omega \nabla \xi g$, $\overline{g}[\omega \nabla \xi g] = 0$.

Special case: $\dim M = n = 4$: $\omega \nabla \xi g = \omega \nabla \xi g - \frac{1}{4} Q \xi \cdot g$.

The covariant differential operator preserving the invariant volume element

does not obey the Leibniz rule when acting on a tensor product $Q \otimes S$ of two tensor fields $Q$ and $S$

$$\omega \nabla \xi (Q \otimes S) = \omega \nabla \xi Q \otimes S + Q \otimes \omega \nabla \xi S + \frac{1}{4} Q \xi \cdot Q \otimes S ,$$  \hspace{1cm} (216)

where $Q \in \otimes^k (M)$, $S \in \otimes^m (M)$.

10.5 Trace free covariant differential operator. Weyl’s transport. Weyl’s space

The description of the gravitational interaction and its unification with the other types of interactions over differentiable manifolds with affine connections and metric $([L_n, g]$)-spaces) induces the introduction of an affine connection with a corresponding covariant differential operator $^s \nabla \xi$ constructed by means of $\nabla \xi$ and $Q \xi$ in the form

$$^s \nabla \xi = \nabla \xi - \frac{1}{n} Q \xi , \hspace{1cm} \dim M = n .$$  \hspace{1cm} (217)
The action of $^\ast \nabla_\xi$ on a covariant metric tensor field $g$ is determined as
\begin{equation}
^\ast \nabla_\xi g = \nabla_\xi g - \frac{1}{n} Q_\xi g , \tag{218}
\end{equation}

obeying the condition
\begin{equation}
\mathcal{F}[^\ast \nabla_\xi g] = 0 . \tag{219}
\end{equation}

On the basis of this relation the covariant differential operator $^\ast \nabla_\xi$ is called a \textit{trace free covariant differential operator}.

If the transport of $g$ by the trace free covariant differential operator $^\ast \nabla_\xi$ obeys the condition
\begin{equation}
^\ast \nabla_\xi g = 0 , \tag{220}
\end{equation}

equivalent to the condition for $\nabla_\xi g$
\begin{equation}
\nabla_\xi g = \frac{1}{n} Q_\xi g . \tag{221}
\end{equation}

then the transport is called \textit{Weyl's transport}.

The covariant vector field
\begin{equation}
\overline{Q} = \frac{1}{n} Q \tag{222}
\end{equation}

is called \textit{Weyl’s covector field}.

A differentiable manifold $M$ (dim $M = n$) with affine connection and metric, over which for every contravariant vector field $\xi \in T(M)$ the transport of $g$ is a Weyl transport, is called \textit{Weyl's space with torsion} (\textit{Weyl-Cartan space}) $Y_n$.

The trace free covariant differential operator $^\ast \nabla_\xi$ is connected with the covariant differential operator $^\omega \nabla_\xi$ preserving the invariant volume element $d\omega$ through the relation
\begin{equation}
^\omega \nabla_\xi = ^\ast \nabla_\xi - \frac{n-2}{2n} Q_\xi = \nabla_\xi - \frac{1}{2} Q_\xi . \tag{223}
\end{equation}

The action of the two operators $^\omega \nabla_\xi$ and $^\ast \nabla_\xi$ would be identical, if dim $M = n = 2$ ($Q_\xi \neq 0$) or if $Q_\xi = 0$.

The components $\Gamma^\gamma_\alpha_\beta$, of the affine connection $\Gamma$ can be represented by means of the components of the affine connections corresponding to the operators $^\omega \nabla_\xi$ and $^\ast \nabla_\xi$.

$\nabla_{e_\beta} e_\alpha$ can be written in the form
\begin{equation}
\nabla_{e_\beta} e_\alpha = \frac{1}{2} (\nabla_{e_\alpha} e_\beta + \nabla_{e_\beta} e_\alpha - [e_\alpha, e_\beta]) = \frac{1}{2} T(e_\alpha, e_\beta) , \tag{224}
\end{equation}

corresponding to the representation of $\Gamma^\gamma_\alpha_\beta$ in the form
\begin{equation}
\Gamma^\gamma_\alpha_\beta = \frac{1}{2} (\Gamma^\gamma_\alpha_\beta + \Gamma^\gamma_\beta_\alpha - C_{\alpha_\beta} \gamma) - \frac{1}{2} T^\gamma_\alpha_\beta = \Gamma^\gamma_\alpha_\beta - \frac{1}{2} T^\gamma_\alpha_\beta . \tag{225}
\end{equation}
If we introduce the abbreviations
\[ g\nabla e_\beta e_\alpha = \frac{1}{2}(\nabla e_\alpha e_\beta + \nabla e_\beta e_\alpha - [e_\alpha, e_\beta]) = \Gamma^\gamma_{\alpha\beta} e_\gamma, \quad (226) \]
\[ ^s\nabla e_\beta e_\alpha = Q_{\alpha\beta} \gamma e_\gamma = (\Gamma^\gamma_{\alpha\beta} - \frac{1}{n}g^\gamma_{\alpha\beta}Q_\beta)e_\gamma, \quad (227) \]
then
\[ \nabla e_\beta e_\alpha = ^s\nabla e_\beta e_\alpha + \frac{1}{n}Q_\beta e_\alpha, \quad (228) \]
\[ \nabla e_\beta e_\alpha = g\nabla e_\beta e_\alpha + \frac{1}{2}T(e_\beta, e_\alpha). \quad (229) \]
From (224), (226), and (228), it follows that
\[ \nabla e_\beta e_\alpha = \frac{1}{2}[g\nabla e_\beta e_\alpha + \frac{1}{2}T(e_\beta, e_\alpha) + \frac{1}{n}Q_\beta e_\alpha + ^s\nabla e_\beta e_\alpha]. \quad (230) \]
The last equality corresponds to the representation of \( \Gamma^\gamma_{\alpha\beta} \) in the form
\[ \Gamma^\gamma_{\alpha\beta} = \frac{1}{2}(\Gamma^\gamma_{\alpha\beta} - \frac{1}{2}T_{\alpha\beta} \gamma + \frac{1}{n}g^\gamma_{\alpha\beta}Q_\beta + Q_{\alpha\beta} \gamma). \quad (231) \]
In analogous way, using the relations
\[ \omega\nabla e_\beta e_\alpha = \omega\Gamma^\gamma_{\alpha\beta} e_\gamma = (\Gamma^\gamma_{\alpha\beta} - \frac{1}{2}g^\gamma_{\alpha\beta}Q_\beta)e_\gamma = \nabla e_\beta e_\alpha - \frac{1}{2}Q_\beta e_\alpha, \quad (232) \]
\[ \nabla e_\beta e_\alpha = \omega\nabla e_\beta e_\alpha + \frac{1}{2}Q_\beta e_\alpha, \quad (233) \]
\[ \nabla e_\beta e_\alpha = g\nabla e_\beta e_\alpha + \frac{1}{2}T(e_\beta, e_\alpha). \quad (234) \]
From (228) and (229), the connection between \( g\nabla e_\beta e_\alpha \) and \( ^s\nabla e_\beta e_\alpha \) follows in the form
\[ g\nabla e_\beta e_\alpha = ^s\nabla e_\beta e_\alpha + \frac{1}{2}T(e_\beta, e_\alpha) + \frac{1}{n}Q_\beta e_\alpha, \quad (235) \]
equivalent to the connection between \( \Gamma^\gamma_{\alpha\beta} \) and \( Q_{\alpha\beta} \gamma \)
\[ \Gamma^\gamma_{\alpha\beta} = Q_{\alpha\beta} \gamma + \frac{1}{2}T_{\alpha\beta} \gamma + \frac{1}{n}g^\gamma_{\alpha\beta}Q_\beta. \quad (236) \]
On the other side, there is a connection between \( g\nabla e_\beta e_\alpha \) and \( \omega\nabla e_\beta e_\alpha \)
\[ g\nabla e_\beta e_\alpha = \omega\nabla e_\beta e_\alpha - \frac{1}{2}T(e_\beta, e_\alpha) + \frac{1}{2}Q_\beta e_\alpha, \quad (237) \]
corresponding to the connection between \( \Gamma^\gamma_{\alpha\beta} \) and \( Q_{\alpha\beta} \gamma \)
\[ \Gamma^\gamma_{\alpha\beta} = \omega\Gamma^\gamma_{\alpha\beta} + \frac{1}{2}T_{\alpha\beta} \gamma + \frac{1}{2}g^\gamma_{\alpha\beta}Q_\beta. \quad (238) \]
10.6 Lie differential operator preserving the invariant volume element

The action of the Lie differential operator $\mathcal{L}_\xi$ on the invariant volume element $d\omega$

$$\mathcal{L}_\xi(d\omega) = \frac{1}{2} \mathfrak{g}[\mathcal{L}_\xi g], d\omega,$$

allows the construction of a new Lie differential operator preserving the invariant volume element $d\omega$.

**Definition 28** $\omega \mathcal{L}_\xi := \text{Lie differential operator preserving the invariant volume element } d\omega \text{ along a contravariant vector field } \xi$

$$\omega \mathcal{L}_\xi = \mathcal{L}_\xi - \frac{1}{2} \mathfrak{g}[\mathcal{L}_\xi g].$$

The properties of $\omega \mathcal{L}_\xi$ are determined by the properties of the Lie differential operator and the existence of a covariant metric tensor field $g$ connected with a contravariant metric tensor field $\mathfrak{g}$.

(a) Action on the invariant volume element $d\omega$:

$$\omega \mathcal{L}_\xi(d\omega) = 0.$$ (239)

It follows from the definition of $\omega \mathcal{L}_\xi$ and (190).

(b) Action on a contravariant basis vector field:

$$\omega \mathcal{L}_{e_\alpha} e_\beta = \mathcal{L}_{e_\alpha} e_\beta - \frac{1}{2} \mathfrak{g}[\mathcal{L}_{e_\alpha} g].e_\beta = (C_{\alpha\beta}^\gamma - \frac{1}{2} g^{\rho\sigma}.\mathcal{L}_{e_\alpha} g_{\rho\sigma} g_{\beta}^\gamma), e_\gamma,$$ (240)

$$\omega \mathcal{L}_{\partial_i} \partial_j = -\frac{1}{2} g^{\gamma\beta}.\mathcal{L}_{\partial_i} g_{\beta\gamma} \partial_j.$$ (241)

(c) Action on a covariant basis vector field:

$$\omega \mathcal{L}_{e_\alpha} e^\beta = \mathcal{L}_{e_\alpha} e^\beta - \frac{1}{2} \mathfrak{g}[\mathcal{L}_{e_\alpha} g].e^\beta = k_{\gamma\alpha}^\beta, e^\gamma - \frac{1}{2} \mathfrak{g}[\mathcal{L}_{e_\alpha} g].e^\beta,$$ (242)

$$\omega \mathcal{L}_{\partial_i} dx^j = k_{mi}^j .dx^m - \frac{1}{2} \mathfrak{g}[\mathcal{L}_{\partial_i} g], dx^j.$$ (243)

(d) Action on a function $f$:

$$\omega \mathcal{L}_\xi f = \xi f - \frac{1}{2} \mathfrak{g}[\mathcal{L}_\xi g].f, \quad f \in C^r(M), \quad r \geq 1.$$ (244)

If we introduce the abbreviations

$$P_\beta = \mathfrak{g}[\mathcal{L}_{e_\beta} g] = g^{\rho\sigma}.\mathcal{L}_{e_\beta} g_{\rho\sigma},$$ (245)

$$P_j = \mathfrak{g}[\mathcal{L}_{\partial_j} g] = g^{\rho\sigma}.\mathcal{L}_{\partial_j} g_{\rho\sigma},$$ (246)
\[ P = P_\beta . e^\beta = P_\beta . d x^\beta, \]  
\[ P_\xi = g[\xi g] = 2, \theta_\xi, \]  
\[ \tilde{C}_{\alpha \beta}^\gamma = C_{\alpha \beta}^\gamma - \frac{1}{2} P_\alpha . g_\beta^\gamma, \tilde{C}_{\alpha \beta}^\gamma \neq -\tilde{C}_{\beta \alpha}^\gamma, \]  
then \( \omega \mathcal{L}_\xi \) and \ref{240} \div \ref{244} can be written in the form
\[ \omega \mathcal{L}_\xi = \mathcal{L}_\xi - \frac{1}{2} P_\xi, \]  
\[ \omega \mathcal{L}_e \xi e^\alpha e^\beta = \mathcal{L}_e \xi e^\alpha e^\beta - \frac{1}{2} P_\alpha . e^\beta = \]  
\[ = (C_{\alpha \beta} \gamma - \frac{1}{2} g_\beta^\gamma P_\alpha) . e_\gamma = \tilde{C}_{\alpha \beta} \gamma . e_\gamma, \]  
\[ \omega \mathcal{L}_\partial_\alpha \partial_\beta = - \frac{1}{2} P_\partial_\alpha \partial_\beta, \]  
\[ \omega \mathcal{L}_e \xi e^\alpha e^\beta = \mathcal{L}_e \xi e^\alpha e^\beta - \frac{1}{2} P_\alpha . e^\beta = \]  
\[ = k_{\gamma \alpha}^\beta . e_\gamma - \frac{1}{2} P_\alpha . e^\beta, \]  
\[ \omega \mathcal{L}_\partial_\alpha . d x^i = k_{mi}^j . d x^m - \frac{1}{2} P_i . d x^i, \]  
\[ \omega \mathcal{L}_\xi f = \xi f - \frac{1}{2} P_\xi . f, \quad f \in C^r(M), \quad r \geq 1. \]  

The commutator of two Lie differential operators preserving \( d\omega \) has the following properties:

(a) Action on a function \( f \):
\[ [\omega \mathcal{L}_\xi, \omega \mathcal{L}_u] f = (\mathcal{L}_\xi u) f + \frac{1}{2} (u P_\xi - \xi P_u) f = \]  
\[ = (\mathcal{L}_\xi u + \frac{1}{2} (u P_\xi - \xi P_u)) f, \quad f \in C^r(M), \quad r \geq 2. \]  

(b) Action on a contravariant vector field:
\[ [\omega \mathcal{L}_\xi, \omega \mathcal{L}_u] v = [\mathcal{L}_\xi, \mathcal{L}_u] v + \frac{1}{2} (u P_\xi - \xi P_u) v = \]  
\[ = \{\mathcal{L}_\xi, \mathcal{L}_u\} + \frac{1}{2} (u P_\xi - \xi P_u) v, \quad \xi, u, v \in T(M). \]  

(c) Satisfies the Jacobi identity
\[ < [\omega \mathcal{L}_\xi, \omega \mathcal{L}_u], \omega \mathcal{L}_v > = [\omega \mathcal{L}_\xi, \omega \mathcal{L}_u] \omega \mathcal{L}_v > + [\omega \mathcal{L}_v, \omega \mathcal{L}_\xi] \omega \mathcal{L}_u > + [\omega \mathcal{L}_u, \omega \mathcal{L}_v] \omega \mathcal{L}_\xi > \equiv 0. \]  

The different types of differential operators acting on the invariant volume element can be used for description of different physical systems and interactions over a differentiable manifold with affine connections and metric interpreted as a model of the space-time.
11 Conclusions

The main conclusions following from the obtained results could be grouped together in the following statements:

1. A contraction operator $S$, commuting with the covariant differential operator and with the Lie differential operator, for which the affine connection $P$ determined by the covariant differential operator for covariant tensor fields is different (not only by sign) from the affine connection $\Gamma$ determined by the covariant differential operator for contravariant tensor fields can be introduced over every differentiable manifold. The components (in a co-ordinate or in a non-co-ordinate basis) of both affine connections $P^i_{jk}$ and $\Gamma^i_{jk}$ differ to each other by the components of the Kronecker tensor. At least three cases could be distinguished:

- **(a)** $\tilde{g}^i_{jk} := 0 : P^i_{jk} + \Gamma^i_{jk} = 0$, $[P^i_{jk} \text{ differs only by sign from } \Gamma^i_{jk} \text{ (canonical case: } S := C)]$,

- **(b)** $\hat{g}^i_{jk} := \varphi_{,k} \cdot g^i_{jk} : P^i_{jk} + \Gamma^i_{jk} = \varphi_{,k} \cdot g^i_{jk}$, $\varphi \in C^r(M)$, $[P^i_{jk} \text{ differs from } \Gamma^i_{jk} \text{ by the derivative of an invariant function } \varphi \in C^r(M), r \geq 2, \text{ along a basis vector field } (\partial_k \text{ or } e_k), \text{ and the components of the Kronecker tensor in the given basis}]$,

- **(c)** $\tilde{g}^i_{jk} = q^i_{jk} : P^i_{jk} + \Gamma^i_{jk} = q^i_{jk}$, $q \in \otimes^1_2(M)$, $[P^i_{jk} \text{ differs from } \Gamma^i_{jk} \text{ by the covariant derivative } q^i_{jk} \text{ of the Kronecker tensor along a basis vector field } \partial_k \text{ (or } e_k)]$.

In the cases (b) and (c) the Lie derivatives of covariant tensor fields depend also on structures determined by the affine connections in contrast to the case (a), where the covariant derivative and the Lie derivative of covariant tensor fields are independent of each other structures (although the fact that the Lie derivatives can be expressed by means of the covariant derivatives).

On the grounds of the obtained results the kinematics of vector fields has been worked out \[26\], \[47\] - \[50\]. The Lagrangian theory for tensor fields has been considered \[51\] and applied to the Einstein theory of gravitation as a special case of a Lagrangian theory of tensor fields \[\mathbb{R}\] over $V_n$-spaces ($n = 4$).

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