Fault Tolerant Quantum Filtering and Fault Detection for Quantum Systems

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Abstract

This paper aims to determine the fault tolerant quantum filter and fault detection equation for a class of open quantum systems coupled to laser fields and subject to stochastic faults. In order to analyze open quantum systems where the system dynamics involve both classical and quantum random variables, a quantum-classical probability space model is developed. Using a reference probability approach, a fault tolerant quantum filter and a fault detection equation are simultaneously derived for this class of open quantum systems. An example of two-level open quantum systems subject to Poisson-type faults is presented to illustrate the proposed method. These results have the potential to lead to a new fault tolerant control theory for quantum systems.

Key words: Open quantum systems; quantum-classical probability model; fault tolerant quantum filtering; fault detection.

1 Introduction

The theory of filtering, which in a broad sense is a scheme considering the estimation of the system states from noisy signals and/or partial observations, plays a significant role in modern engineering science. A filter propagates our knowledge about the system states given all observations up to the current time and provides optimal estimates of the system states. From the fundamental postulates of quantum mechanics, one is not allowed to make noncommutative observations of quantum systems in a single realization or experiment. Any quantum measurement yields in principle only partial information about the system. This fact makes the theory of quantum filtering extremely useful in measurement based feedback control of quantum systems, especially in the field of quantum optics ([Rouchon & Ralph (2015]), [Wiseman & Milburn (2010)]). A system-probe interaction setup in quantum optics is used as the typical physical scenario concerning the extraction of information about the quantum system from continuous measurements ([Belavkin (1992)], [Gardiner & Zoller (2000)]). The quantum system under consideration, e.g., a cloud of atoms trapped inside a vacuum chamber, is interrogated by probing it with a laser beam. After interaction with the electromagnetic radiation (laser), the free electrons of the atoms are accelerated and can absorb energy. This energy is then emitted into the electromagnetic field as photons which can be continuously detected through a homodyne detector ([Wiseman & Milburn (2010)]). Using the continuous integrated photocurrent generated by the homodyne detector one can conveniently estimate the internal states of the atoms. To find the optimal estimates is then precisely the goal of quantum filtering theory. A very early approach to quantum filtering was presented in a series of papers by Belavkin dating back to the early 1980s ([Belavkin (1980]), [Belavkin (1992)]), which was developed in the framework of continuous nondemolition quantum measurement using the operational formalism from Davies’s precursor work ([Davies (1969)]). In the physics community, the theory of quantum filtering was also independently developed in the early 1990s ([Carmichael (1993)]), named “quantum trajectory theory” in the context of quantum optics.

Particular emphasis is given to the work by Bouten et al. ([Bouten et al. (2007)]) where quantum probability theory was used in a rigorous way and a quantum filter for a laser-atom interaction setup in quantum optics was derived using a quantum reference probability method. A basic idea in quantum probability theory is an isomorphic equivalence between a commutative subalgebra of quantum operators on a Hilbert space and a classical (Kolmogorov) probability space through the spectral theorem, from which any probabilistic quantum operation within the commutative subalge-
bra can be associated with a classical random variable. The complete quantum probability model is treated as the non-commutative counterpart of Kolmogorov’s axiomatic characterisation of classical probability. Similar to the classical case ([Bertsekas & Tsitsiklis (2002)]), the optimal estimate of any observable is given by its quantum expectation conditioned on the history of continuous nondemolition quantum measurements of the electromagnetic field. The quantum filter was derived in terms of Itô stochastic differential equations using a reference probability method.

In practice, classical randomness may be introduced directly into the system dynamics of quantum systems ([Ruschhaupt et al. (2012)]). For example, the system Hamiltonian of a superconducting quantum system may contain classical randomness due to the existence of stochastic fluctuations in magnetic flux or gate voltages ([Dong et al. (2015)]). A spin system may be subject to stochastically fluctuating fields that will introduce classical randomness into the system dynamics ([Dong & Petersen (2012)]). For an atom system subject to a laser beam, the occurrence of stochastic faults in the laser device may cause the introduction of classical randomness into the dynamics of the atom system ([Viola & Knill (2003)], [Khodjasteh & Lidar (2005)]). For an open quantum system, the system may evolve randomly and the system dynamics may involve two kinds of randomness, i.e., quantum randomness due to intrinsic quantum indeterminacy and classical randomness arising from the imprecise behaviour of macroscopic devices. These two kinds of randomnesses are independent because they have completely different origins. Because of changes in the system evolution, the quantum filter in ([Bouten et al. (2007)]) cannot directly provide optimal statistical inference of the internal states of the atoms and must be redesigned. However, many probabilistic operations for random quantum observables and classical random variables are not easy to handle in the framework of quantum probability theory built on a deterministic type commutative-algebra-normal-state structure. It is desirable to reformulate the quantum probability model such that both classical and quantum randomness can be easily analyzed. In this paper, we introduce a quantum-classical probability model that is built on a random commutative algebra equipped with a more general normal state. The quantum-classical probability space is described by a quadruple, and the quantum probability space and the classical probability space can be considered as its special cases. When we concentrate on a class of open quantum systems subject to stochastic faults, this quantum-classical probability model provides a convenient tool to simultaneously derive the fault tolerant quantum filter and fault detection equations for this class of systems.

This paper is organized as follows. Section 2 describes the class of open quantum systems under consideration in this paper. A quantum-classical probability space model is presented in Section 3. In Section 4, the fault tolerant quantum filter and fault detection equations are simultaneously derived for open quantum systems using the quantum-classical probability space model. An example of two-level quantum systems with Poisson-type faults is illustrated. Section 5 concludes this paper.

2 Heisenberg Dynamics of Open Quantum Systems

In this work, we concentrate on an open quantum system that has been widely investigated in quantum optics ([Wiseman & Milburn (2010)], [Qi et al. (2013)], [van Handel et al. (2005)]). The quantum system under consideration is a cloud of atoms in weak interaction with an external laser probe field which is continuously monitored by a homodyne detector ([Bouten et al. (2007)], [Mirrahimi & van Handel (2007)]). Such a quantum system can be described by quantum stochastic differential equations driven by quantum noises $B(t)$ and $B^\dagger(t)$ ([Wiseman & Milburn (2010)]). The dynamics of the quantum system are described by the following quantum stochastic differential equation:

$$
\begin{align*}
\dot{U}(t) &= \left\{ \left( -iH(t) - \frac{1}{2} L^\dagger L \right) \right\} U(t),
\end{align*}
$$

with initial condition $U(0) = I$ and $i = \sqrt{-1}$. Here $U(t)$ describes the Heisenberg-picture evolution of the system operators and $H(t)$ is the system Hamiltonian. In terms of the system states, if $\rho_0$ is a given system state, we write $\rho_0 = \pi_0 \otimes |\psi\rangle \langle \psi|$, where $|\psi\rangle$ represents the vacuum state. The system operator $L$, together with the field operator $b(t) = B(t)$ models the interaction between the system and the field. From quantum Itô rule, one has ([Gardiner & Zoller (2000)])

$\begin{align*}
\dot{b}(t)dB^\dagger(t) &= dt, \\
\dot{b}^\dagger(t)dB(t) &= dB(t)b(t) = dB^\dagger(t)dB(t) = 0.
\end{align*}$

The atom system and the laser field form a composite system and the Hilbert space for the composite system is given by $\mathcal{H}_p \otimes \mathcal{E} = \mathcal{H}_p \otimes \mathcal{E} \otimes \mathcal{E}_i$, where we have exhibited the continuous temporal tensor product decomposition of the Fock space $\mathcal{E} = \mathcal{E}_j \otimes \mathcal{E}_i$ into the past and future components ([Belavkin (1992)], [Holevo (1991)]). It is assumed that $\dim(\mathcal{H}_p) = n < \infty$. The atomic observables are described by self-adjoint operators on $\mathcal{H}_p$. Any system observable $X$ at time $t$ is given by $X(t) = j(t)(X) = U^\dagger(t)(X \otimes I)U(t)$. It is noted that (1) is written in Itô form, as all stochastic differential equations in this paper.

In practice, the system Hamiltonian may change randomly because of, e.g., faulty control Hamiltonians that appear in the system dynamics at random times ([Viola & Knill (2003)], [Khodjasteh & Lidar (2005)]) or random fluctuations of the external electromagnetic

\footnote{We have assumed $h=1$ by using atomic units in this paper.}
from time \( \text{dB} \)

Similarly, the time evolution operator \( B \) (Sect. 2) can be described by a Hermitian operator \( A \). We have that \( A \) is a random unitary operator and \( X(t) = j_t(X) \) is a random observable, both depending on the stochastic process \( F(t) \).

In this paper, for simplicity we still write \( U(t) \) instead of the functional form \( U(F,t) \). One can conclude that the commutativity of observables is preserved, that is, \( [j_t(A), j_t(B)] = 0 \) if \( [A, B] = 0 \) where \( A, B \) are two system observables in \( \mathcal{H}_\mathcal{F} \). Here the commutator is defined by \( [A, B] = AB - BA \). In addition, from (1) one can see that \( U(t) \) depends on \( B(t') \) and \( B^\dagger(t') \), \( 0 \leq t' < t \), since the increments \( dB(t) \) and \( dB^\dagger(t) \) point to the future evolution. Consequently,

\[
[U(t), dB(t)] = [U(t), dB^\dagger(t)] = 0. \tag{2}
\]

Similarly, the time evolution operator \( U(t,s) = U(t)U^\dagger(s) \) from time \( s \) to time \( t \) depends only on the field operators \( dB(s') \) and \( dB^\dagger(s') \) with \( s < s' \leq t \). Thus,

\[
[U(t,s), B(\tau)] = [U(t,s), B^\dagger(\tau)] = 0, \quad \tau \leq s. \tag{3}
\]

In quantum experiments, generally measurement is performed on the field. Using homodyne detectors, the observation process is given by \( Y(t) = j_t(Q(t)) = \sum_{j=1}^n j_t(Q_j(t))U(t) \) where \( Q(t) = B(t) + B^\dagger(t) \) is the real quadrature of the input field. The operator \( Q(t) \) commutes with itself at different times, i.e., \( [Q(t), Q(s)] = 0 \). When the field is initialized in the vacuum state, \( Q(t) \) is isomorphically equivalent to a real Wiener process (\([\text{Gardiner & Zoller} (2000)]\)). Combining (2) and (3) with the fact that \( [\mathbb{I} \otimes Q(t), X \otimes \mathbb{I}] = 0 \), it is easy to show that: (i) \( Y(t), Y(s) = 0 \) at all times \( s,t \) and (ii) \( Y(s), X(t) = 0, \forall s \leq t \). These two properties guarantee that (i) \( Y(t) \) can be continuously monitored, and (ii) it is possible to obtain the conditional statistics of an observable \( X(t) \) based on the history of \( Y(t) \). In addition, by using the quantum \( It\hat{\sigma} \) rule, one has

\[
dY(t) = U(t)(L + L^\dagger)U(t)dt + dQ(t), \tag{4}
\]

from which \( Y(t) \) looks like \( j_t(L + L^\dagger) = U(t)(L + L^\dagger)U(t) \) with a noise \( Q(t) \).

### 3 Quantum-Classical Probability Space

Let \( (\Omega, \mathcal{F}, \mathcal{P}) \) be a complete classical probability space on which we have a right continuous and complete filtration \( \{\mathcal{F}_t\}_{t \geq 0} \) of sub-\( \sigma \)-fields of \( \mathcal{F} \). In the sequel, \( \mathbb{E}\{\cdot\} \) denotes the mathematical expectation operator with respect to the given probability measure \( \mathcal{P} \).

When classical random variables are introduced into the Hamiltonian of an open quantum system, the standard quantum filter will fail to produce (least mean square) optimal estimates of the system states. The quantum probability theory (\([\text{Bouten et al.} (2007)]\), \([\text{Hudson & Parthasarathy} (1984)]\)), which is built on a deterministic type commutative-algebraic-normal-state structure, cannot provide a simple formulation for applications with many probabilistic operations on random observables and classical random variables. This brings difficulties in applying the quantum probability theory in analyzing quantum systems that evolve randomly, like the case we considered in Sect. 2.

In this section, we introduce a new quantum-classical probability space which can easily deal with both classical and quantum randomnesses. To begin with, we briefly introduce the quantum probability theory. Let \( \mathcal{H} \) be a complex Hilbert space and \( \mathcal{H}(\mathcal{H}) \) be the set of all bounded operators on \( \mathcal{H} \). We first discuss the case that \( \dim(\mathcal{H}) = n < \infty \). It is known that the foundations of quantum mechanics can be also formulated in a similar language to the classical Kolmogorov’s probability theory (\([\text{Gardiner & Zoller} (2000)]\)). The basic ideas are as follows. Based on the spectral theorem (\([\text{Akhiiezer & Glazman} (1981)]\)), any self-adjoint operator \( A \) on \( \mathcal{H} \) admits a spectral decomposition \( A = \sum_{j=1}^n a_jP_j \), where \( \{a_j\} \subset \mathbb{R} \) are the eigenvalues of \( A \) and \( \{P_j\} \) are the corresponding orthogonal projection operators which form a resolution of the identity, i.e., \( P_jP_k = \delta_{jk}P_k \) and \( \sum_{j=1}^n P_j = I \). For any continuous function \( f : \mathbb{R} \to \mathbb{C} \), one has \( f(A) = \sum_{j=1}^n f(a_j)P_j \). Thus the set \( \mathcal{A} = \{X : X = f(A), \forall f : \mathbb{R} \to \mathbb{C}\} \) forms a commutative \( * \)-algebra generated by \( A \). That is, arbitrary linear combinations, products and adjoints of operators in \( \mathcal{A} \) are still in \( \mathcal{A} \), \( I \in \mathcal{A} \) and all elements of \( \mathcal{A} \) commute. A mapping \( \mathbb{P} : \mathcal{A} \to \mathbb{C} \) is called a normal state on \( \mathcal{A} \) if it is positive and normalized, i.e., \( \mathbb{P}(X) \geq 0 \) if \( X \geq 0 \) and \( \mathbb{P}(I) = 1 \). From Theorem 7.1.12 in (\([\text{Kadison & Ringrose} (1983)]\)), there is always a density operator \( \rho \) such that \( \mathbb{P}(X) = \text{Tr}(\rho X) \), where \( \rho = \rho^\dagger, \text{Tr}(\rho) = 1 \) and \( \rho \geq 0 \). Note that \( P_j \in \mathcal{A} \) are exactly the events one can distinguish by measuring \( A \) and their probabilities are given by \( \mathbb{P}(A_j) \) if the system has a density operator \( \rho \). We have the following lemma.

**Lemma 3.1** (\([\text{Bouten et al.} (2007)]\)) (Equivalence theorem, finite-dimensional case). Let \( \mathcal{A} \) be a commutative \( * \)-algebra of operators on a finite-dimensional Hilbert space \( \mathcal{H} \), and let \( \mathbb{P} \) be a normal state on \( \mathcal{A} \). There is a classical probability space \( (\Omega', \mathcal{F}', \mathcal{P}') \) and a \( * \)-isomorphism \( t \) from \( \mathcal{A} \) onto the set of measurable functions on \( \Omega' \), and moreover \( \mathbb{P}(X) = \mathbb{E}_{\mathcal{P}'}(t(X)), \forall X \in \mathcal{A} \).

Thus a commutative \( * \)-algebra structure is completely equivalent to a classical probability space. The pair

\footnote{A \( * \)-isomorphism \( t \) is a linear bijection with \( t(XY) = t(X) t(Y) \) and \( t(X^\dagger) = t(X)^\dagger \). Here \( t \) depends only on a unitary operator \( U \) by which all elements of the algebra \( \mathcal{A} \) can be diagonalized. One can always find such an operator \( U \) since all elements of \( \mathcal{A} \) commute.}
Quantum probability theory ([Gardiner & Zoller (2000)], [Breuer & Petruccione (2002)], [Bouten et al. (2007)]) is built on the $*$-algebra structure of deterministic operators on a Hilbert space. In many physical situations, however, quantum systems may evolve randomly because of the introduction of classical random processes, as discussed in Section 2. In this case, the system evolution is described by a random unitary operator $U_R(t)$ that depends on some classical random variable (vector) $R$, and any system observable $A$ at time $t$ will be in the form of a random observable $A(R,t) = U_R(t)A U_R^*(t)$. In this paper, we assume that $R$ is defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and takes values in a finite set $\{R_1, \ldots, R_m\}$. In each single quantum measurement of the system observable, we actually go through two realizations: (i) the choice of a sample point $\omega \in \Omega$, and (ii) the quantum measurement performed on a deterministic observable $A(R(\omega),t)$. As a result, given a system state $\rho$, the average observed value of $A(R,t)$ is given by $\hat{\mathbb{P}}(A(R,t))$, where $\hat{\mathbb{P}}$ is a linear map $\hat{\mathbb{P}}(\cdot) = \mathbb{E}\{\text{Tr}(\rho(\cdot))\}: \mathcal{B}(\mathcal{H}) \rightarrow \mathbb{R}$.

The measurement results of $A(R,t)$ contain information of the random variable $R$, which makes it natural to ask whether the joint statistics of $A(R,t)$ and some classical random variables depending on $R$ can be well defined. This is indeed possible because we can equivalently treat any classical random variable $v(R)$ as a random observable $v(R)$ under $\hat{\mathbb{P}}$ on the Hilbert space, since $\mathbb{E}_{\rho} (e^{i v(R)}) = \hat{\mathbb{P}} (e^{i v(R)})$ for any density operator $\rho$. In other words, $v(R)$ and $v(R)$ share the same characteristic function. Without loss of generality, here we suppose $v$ is a scalar function satisfying $v(R_j) \neq v(R_k)$, $j \neq k$. These results can be directly extended to a multi-dimensional case. It is clear that $v(R)$ commutes with all quantum operators in $\mathcal{H}$ (this is exactly a property of classical random variables). Then we have the following result.

**Lemma 3.2.** For any given self-adjoint operator $A$, random unitary operator $U_R(t)$ and scalar function $v(R)$, the set of random self-adjoint operators $\mathcal{A} = \{X : X = f(v(R)A(R,t)), \forall f : \mathbb{R} \rightarrow \mathbb{C}\}$ forms a commutative $*$-algebra on $\mathcal{H}$ and has a normal state $\hat{\mathbb{P}}$. In addition, any normal state on $\mathcal{A}$ can be written in the form of $\mathbb{E}\{\text{Tr}(\rho(\cdot))\}$ for some density operator $\rho$.

**Proof.** The normality of $\hat{\mathbb{P}}$ can be verified from the fact that $\hat{\mathbb{P}}(X) \geq 0$ if $X \geq 0$ and $\hat{\mathbb{P}}(I) = 1$. The first part of Lemma 3.2 can be obtained if $\mathcal{A}$ has a basis consisting of projection operators. From the spectral theorem, one has $A(R,t) = \sum_{j=1}^{n} \lambda_j P_j(R,t)$ and $\nu(R) = \sum_{k=1}^{n} \nu_k I_{R}=R_k$, where $P_j(R,t) = U_R^*(t)P_j U_R(t)$ and $I_{R}=R_k$ is the indicator function of the event “$R = R_k$”. Then any element in $\mathcal{A}$ can be written as a linear combination of the projection operators $\{I_{R}=R_j P_j(R,t)\}, k \in \{1, \ldots, n\}, j \in \{1, \ldots, n\}$. The conclusion of the second part follows directly from Theorem 7.1.12 in ([Kadison & Ringrose (1983)]).

**Remark 3.1.** One can observe that both $\mathcal{A}$ and $U_R^*(t)\mathcal{A} U_R(t)$ are commutative subalgebras of $\mathcal{A}$. In addition, $\mathcal{A}$ contains the $\sigma-$algebra generated by the classical random variable $v(R)$ as a special case.

**Remark 3.2.** Any classical variable $v(R) \in \mathcal{A}$.

From Lemmas 3.1 and 3.2, we have the following result.

**Theorem 3.1.** (General equivalence theorem, finite-dimensional case). Let $\mathcal{A}$ be a random commutative $*$-algebra of operators on the Hilbert space $\mathcal{H}$, equipped with a normal state $\hat{\mathbb{P}}$. There exist a probability space $(\Omega', \mathcal{F}', P')$ and a $*$-isomorphism $t$ from $\mathcal{A}$ onto the set of measurable functions on $\Omega'$, and moreover $\hat{\mathbb{P}}(X) = \mathbb{E}_{\rho}(t(X)), \forall X \in \mathcal{A}$.

Thus a random commutative $*$-algebra $\mathcal{A}$ with a normal state $\hat{\mathbb{P}}$ is completely equivalent to a classical probability space. In other words, when the discussion is restricted to a random commutative $*$-algebra, any probabilistic operation can be defined directly in terms of the associated classical probability space. In particular, we consider the conditional expectation which will be useful in subsequent analysis.

Let $Y_i \in \mathcal{A}$ be a self-adjoint operator. Then $Y_i$ and $\mathcal{A}$ generate a larger random commutative $*$-algebra, which is isomorphic to a classical probability space through a linear mapping $t$, based on Theorem 3.1. By using classical probability theory, the conditional expectation can be directly defined as $\hat{\mathbb{P}}(Y_{i} | \mathcal{A}) = t^{-1}(E_{\rho}(t(Y_{i})) \sigma \{t(\mathcal{A})\})$. This discussion can be extended to any operator $Y \in \mathcal{A}$. To be specific, $Y$ can be written as $Y = Y_1 + iY_2$, where $Y_1 = \frac{Y + Y^*}{2}$ and $Y_2 = \frac{Y - Y^*}{2i}$. Thus, $\hat{\mathbb{P}}(Y | \mathcal{A}) = \hat{\mathbb{P}}(Y_1 | \mathcal{A}) + i\hat{\mathbb{P}}(Y_2 | \mathcal{A})$, with $\hat{\mathbb{P}}(Y_1 | \mathcal{A})$ and $\hat{\mathbb{P}}(Y_2 | \mathcal{A})$ well defined.

Following the same idea in classical probability theory, the following definition of conditional expectation is given.

**Definition 3.1.** (Quantum-classical conditional expectation, finite-dimensional case) Consider a random commutative
A *-algebra \(\mathcal{A}\) equipped with a normal state \(\hat{\mathbb{P}}\). The map \(\hat{\mathbb{P}}(\cdot|\mathcal{A}) : \mathcal{A} \rightarrow \mathcal{A}\) is called (a version of) the conditional expectation from \(\mathcal{A}\) onto \(\mathcal{A}\) if \(\hat{\mathbb{P}}(\hat{\mathbb{P}}(X)|\mathcal{A})Y = \hat{\mathbb{P}}(XY)\) for all \(X \in \mathcal{A}, Y \in \mathcal{A}\).

When the Hilbert space is of finite dimension, an explicit expression for conditional expectation can be obtained. Let \(\{\hat{P}_j\}\) be the set of basis projection operators of \(\mathcal{A}\) and \(X \in \mathcal{A}\). Then a version of the conditional expectation is given by

\[
\hat{\mathbb{P}}(X|\mathcal{A}) = \sum_{\hat{P}(j) \neq 0} \frac{\hat{P}(\hat{P}_jX)}{\hat{P}(\hat{P}_j)} \hat{P}_j.
\]

Here we discuss two special cases of the expression (5).

Case 1. Suppose \(\mathcal{A}\) is a deterministic commutative *-algebra, e.g., \(\mathcal{A} = \mathbb{C}\), and \(X \in \mathcal{A}\) is a deterministic operator. Then the basis \(\{\hat{P}_j\}\) is also deterministic and one has

\[
\hat{\mathbb{P}}(X|\mathcal{A}) = \sum_{\hat{P}(j) \neq 0} \frac{\hat{P}(\hat{P}_jX)}{\hat{P}(\hat{P}_j)} \hat{P}_j,
\]

where \(\hat{\mathbb{P}}(\cdot) = \text{Tr}(\rho(\cdot))\). In this case, (5) reduces to the expression of quantum conditional expectation in Equation (2.10) of ([Bouten et al. (2007)]).

Case 2. Set \(A = I\), then \(\mathcal{A}\) in Lemma 3.2 reduces to a \(\sigma\)-field generated by a classical random variable \(v(R)\). Let \(X = xI\) with \(x\) being a random variable on \((\Omega, \mathcal{F}, \mathbb{P})\). Then one has

\[
\hat{\mathbb{P}}(X|\mathcal{A}) = \sum_{E(1_{v(R)=v}) \neq 0} \frac{\mathbb{E}(x1_{v(R)=v})}{\mathbb{E}(1_{v(R)=v})} 1_{v(R)=v}
= \mathbb{E}(x|v(R)),
\]

which is exactly the expression of classical conditional expectation ([Bertsekas & Tsitsiklis (2002)]).

Thus the defined conditional expectation contains both classical and quantum conditional expectations as special cases. Note that Definition 3.1 also allows us to define the expectation of classical random variables conditioned on random observables, and vice versa. The above analysis can be directly extended to infinite-dimensional Hilbert spaces and we go directly to the following fundamental theorems and definitions without mentioning the details.

**Theorem 3.2.** (Equivalence theorem). Let \(\mathcal{C}\) be a random commutative von Neumann algebra equipped with a normal state \(\hat{\mathbb{P}}\). Then there exist a probability space \((\Omega', \mathcal{F}', \mathbb{P}')\) and a *-isomorphism \(t\) from \(\mathcal{C}\) onto the algebra of bounded measurable complex functions on \(\Omega'\), such that \(\hat{\mathbb{P}}(X) = \mathbb{E}_{\mathbb{P}'}(t'(X)), X \in \mathcal{C}\).

**Proof.** Theorem 3.2 follows from Theorem 3.3 in ([Bouten et al. (2007)]).

**Definition 3.2.** (Quantum-classical probability space) A quantum-classical probability space is a quadruple \((\Omega, \mathcal{F}, \mathcal{N}, \hat{\mathbb{P}})\), where \(\mathcal{N}\) is a von Neumann algebra.

The structure of quantum-classical probability space embodies the basic idea of how we treat quantum and classical randomnesses: they should be treated differently but not necessarily separately. The noncommutative quantum-classical probability space \((\Omega, \mathcal{F}, \mathcal{N}, \hat{\mathbb{P}})\) describes the statistics of all possible observations and has an (infinite) number of commutative quantum-classical probability subspaces \((\Omega, \mathcal{F}, \mathcal{C}, \hat{\mathbb{P}})\), each of which is isomorphically equivalent to a classical probability space according to Theorem 3.2. The particular commutative quantum-classical probability subspace under consideration is determined by the specific experiment we choose to perform. It must be noted that each commutative quantum-classical probability subspace still contains the classical probability space \((\Omega, \mathcal{F}, \mathbb{P})\). That is, one is free to measure any classical random variable.

**Definition 3.3.** (Quantum-classical conditional expectation) Let \(\mathcal{C}\) be a commutative von Neumann algebra equipped with a normal state \(\hat{\mathbb{P}}\). Then the map \(\hat{\mathbb{P}}(\cdot|\mathcal{C})\) is called (a version of) the quantum-classical conditional expectation from \(\mathcal{C}\) onto \(\mathcal{C}\), if \(\hat{\mathbb{P}}(\hat{\mathbb{P}}(X|\mathcal{C})Y) = \hat{\mathbb{P}}(XY)\) for all \(X \in \mathcal{C}\) and \(Y \in \mathcal{C}\).

**Theorem 3.3.** The conditional expectation of Definition 3.3 exists and is unique with probability one (any two versions \(P\) and \(Q\) of \(\hat{\mathbb{P}}(\cdot|\mathcal{C})\) satisfies \(\|P - Q\|_{\hat{\mathbb{P}}} = 0\), where \(\|X\|_{\hat{\mathbb{P}}} = \hat{\mathbb{P}}(X^*X)\)). Moreover, \(\hat{\mathbb{P}}(X|\mathcal{C})\) is the least mean square estimate of \(X\) given \(\mathcal{C}\) in the sense that \(\|X - \hat{\mathbb{P}}(X|\mathcal{C})\| \leq \|X - Y\|\) for all \(Y \in \mathcal{C}\).

**Proof.** Theorem 3.3 follows from Theorem 3.16 in ([Bouten et al. (2007)]).

**Remark 3.3.** The elementary properties of classical conditional expectation, for example, linearity, positivity, the tower property and “taking out what is known” ([Bertsekas & Tsitsiklis (2002)]), still hold for the above defined conditional expectation. Proofs follow directly from classical cases and are omitted here.

We end this section with a quantum-classical Bayes formula.

**Theorem 3.4.** (Quantum-classical Bayes formula) Consider a quantum-classical probability space \((\Omega, \mathcal{F}, \mathcal{N}, \hat{\mathbb{P}})\) and let \(\mathcal{C} \subset \mathcal{N}\) be a commutative von Neumann algebra. Suppose a new probability measure \(\hat{\mathbb{P}}\) is defined by \(d\hat{\mathbb{P}} = Ad\mathbb{P}\), where the \(\mathcal{F}\)-measurable random variable \(A\) is the Radon-Nikson derivative. Choose \(V \in \mathcal{C}\) such that \(V^*V > 0\) and \(\hat{\mathbb{P}}(AV^*V) = 1\). Then we can define on \(\mathcal{C}\) a new normal state \(\hat{\mathbb{Q}}\) by \(\hat{\mathbb{Q}}(X) = \hat{\mathbb{P}}(AV^*XV)\) and

\[
\hat{\mathbb{Q}}(X|\mathcal{C}) = \frac{\hat{\mathbb{P}}(AV^*XV/\mathcal{C})}{\hat{\mathbb{P}}(AV^*V/\mathcal{C})}, \quad \forall X \in \mathcal{C}.
\]
Proof. Let $Y$ be any element of $\mathcal{F}$. Then we have

\[
\tilde{P}(\tilde{P}(\Lambda^t V|\mathcal{F}) Y) = \tilde{P}(\Lambda^t V Y)
\]
\[
= \tilde{Q}(XY) = \tilde{Q}(\tilde{Q}(XY)|\mathcal{F})
\]
\[
= \tilde{P}(\Lambda^t \tilde{Q}(X|\mathcal{F}) Y) = \tilde{P}(\Lambda^t \tilde{V} \tilde{Q}(X|\mathcal{F}) Y)
\]
\[
= \tilde{P}(\tilde{P}(\Lambda^t V|\mathcal{F}) \tilde{Q}(X|\mathcal{F}) Y).
\]  

(9)

Let $Y = (\tilde{P}(\Lambda^t V|\mathcal{F}) - \tilde{P}(\Lambda^t V|\mathcal{F}) \tilde{Q}(X|\mathcal{F}))$, then from (9) we have $\|Y\|_p = 0$. In other words, $\tilde{P}(\Lambda^t V|\mathcal{F}) = \tilde{P}(\Lambda^t V|\mathcal{F}) \tilde{Q}(X|\mathcal{F}) \tilde{P}$ almost surely. □

Remark 3.4. Theorem 3.4 contains both quantum Bayes formula ([Bouten et al. (2007)]) and classical Bayes formula ([Bertsekas & Tsitsiklis (2002)]) as special cases.

4 Fault Tolerant Quantum Filtering and Fault Detection

4.1 Fault tolerant quantum filter and fault detection equation

In classical (non-quantum) engineering, apparatus may suffer from malfunctions or degradation events (faults), especially after a long running time or when working in difficult environments. The occurrence of faults can often make the system evolve far from its desired or normal operating conditions and can lead to a drastic change in the system behaviour. Thus this is a phenomenon that needs to be seriously considered. Recall the quantum systems described in Section 2. In the laser-atom interaction realization, the laser field is often treated in a classical way and it generates an electromagnetic field at the position of the atom. Then the laser-atom interaction can be described by a dipole interaction Hamiltonian which depends on the intensity of the classical electromagnetic field ([Ruschhaupt et al. (2012)]). Therefore, if the macroscopic laser device suffers from a fault, e.g., it produces a faulty electromagnetic field, an unexpected additional Hamiltonian will be introduced into the quantum system. In this case, the system Hamiltonian in (1) will be given by $H(F(t))$ where $F(t)$ is the fault process.

In practice, the system may transit between a finite number of different faulty modes at random times. This makes it desirable to model the fault process on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ by a continuous-time Markov chain $\{F(t)\}_{t>0}$ adapted to $\{\mathcal{F}_t\}_{t>0}$ ([Davis (1975]), [Hibey & Charalambous (1999)], [Elliott et al. (1995)]). The state space of $F(t)$ is often chosen to be the finite set $\mathbb{S} = \{e_1, e_2, ..., e_N\}$ (for some positive integer $N$) of canonical unit vectors in $\mathbb{R}^N$. Let $p_t = (p_{t1}, p_{t2}, ..., p_{tN})^T$ be the probability distribution of $F(t)$, i.e., $p_{tk} = \mathbb{P}(F(t) = e_k), k = 1, 2, ..., N$ and suppose the Markov process $F(t)$ has a so-called Q matrix or transition rate matrix $\Pi = (a_{jk}) \in \mathbb{R}^{N \times N}$. Then $p_t$ satisfies the forward Kolmogorov equation $\frac{dp_t}{dt} = \Pi p_t$.

$\Pi$ is a Q matrix, we have $a_{jk} = -\sum_{j \neq k} a_{jk}$, and $a_{jk} \geq 0, j \neq k$. Then $F(t)$ is a corol process ([Elliott et al. (1995)]) that satisfies the following stochastic differential equation:

\[
dF(t) = \Pi F(t)dt + dM(t),
\]  

(10)

where $M(t) = F(t) - F(0) - \int_0^t \Pi F(t^-)d\tau$ is an $\{\mathcal{F}_t\}$ martingale ([Elliott et al. (1995)]) and satisfies

\[
\sup_{0 \leq t \leq T} \mathbb{E}(|M(t)|^2) < \infty.
\]

One goal of this paper is to derive the equations of the fault tolerant quantum filter and fault detection for this class of open quantum systems. To be specific, we use a reference probability approach to find the least-mean-square estimates of a system observable $X \in \mathcal{B}(\mathcal{H})$ at time $t$ and the fault process $F(t)$ for the quantum system under consideration, given the observation process $Y(s), 0 \leq s \leq t$. This can be accomplished if we can obtain the following estimates:

\[
\sigma^2_t(X) = \tilde{P}((F(t), e_j) U^t X U(t)|\mathcal{F}),
\]  

(11)

where $\mathcal{F}$ is the commutative von Neumann algebra generated by $Y(s)$ up to time $t$, and $<, >$ is the inner product in $\mathbb{R}^N$. From the previous analysis, one has $< F(t), e_j > U^t(t)XU(t) \in \mathcal{F}$, which guarantees that the conditional expectation (11) is well defined.

It follows from (3) that for $0 \leq t$,

\[
U^t(t)Q(s)U(t) = U^t(s)U^t(t, s)Q(s)U(t, s)U(s)
\]
\[
= U^t(s)Q(s)U(s) = Y(s),
\]  

(12)

which implies that $\mathcal{F}$ can be rewritten as $\mathcal{F} = U^t(t)\mathcal{L}U(t)$ where $\mathcal{L}$ is the commutative von Neumann algebra generated by $Q(s)$ up to time $t$. From quantum probability theory, we know that $Q(t)$ under the vacuum state is equivalent to a classical Wiener process ([Gardiner & Zoller (2000)]). This fact makes it simpler to design a quantum filter in terms of $Q(t)$ because it is convenient to manipulate $Q(t)$ using the quantum Itô formula ([Hudson & Parthasarathy (1984)]). Next, we will use a quantum analog of the classical change-of-measure technique to obtain an explicit expression for $\sigma^2_t(X)$.

Define an operator $V(t)$ that satisfies the quantum stochastic differential equation

\[
dV(t) = \left\{ -iH(F(t)) - \frac{1}{2} L^*L \right\} dt + LdQ(t) \right\} V(t),
\]  

(13)

with $V(0) = I$. Then $V(t) \in \mathcal{L}'_t$ and we have the following lemma.
Lemma 4.1. For any system observable \( X \in \mathcal{B}(\mathcal{H}) \), the conditional expectation in (11) can be rewritten as

\[
\sigma^f_i(X) = U^\dagger(t) \frac{\hat{P}(\langle F(t), e_j \rangle V^\dagger(t)XV(t)|\mathcal{E}_t)}{\hat{P}(V^\dagger(t)V(t)|\mathcal{E}_t)} U(t).
\] (14)

Proof. See the Appendix. \( \square \)

Write

\[
\pi^f_i(X) = U^\dagger(t)\hat{P}(\langle F(t), e_j \rangle V^\dagger(t)XV(t)|\mathcal{E}_t)U(t),
\] (15)

which is the unnormalized conditional expectation. Since \( \sum_{j=1}^N \langle F(t), e_j \rangle = 1 \), we have

\[
\sigma^f_i(X) = \frac{\pi^f_i(X)}{\sum_{j=1}^N \pi^f_i(I)}.
\] (16)

An explicit expression for \( \pi^f_i(X) \) can now be obtained.

Theorem 4.1. (Unnormalized fault tolerant quantum filtering equation) The unnormalized conditional expectation \( \pi^f_i(X) \) satisfies the following quantum stochastic differential equation:

\[
d\pi^f_i(X) = \left( \sum_{k=1}^N a_{jk} \pi^f_i(X) + \pi^f_i(\mathcal{L}_{L,H}(e_j)(X)) \right) dt
+ \pi^f_i( XL + L^tX)dY(t),
\] (17)

where the so-called Lindblad generator is given by

\[
\mathcal{L}_{L,H}(X) = i[H,X] + L^tXL - \frac{1}{2}(L^tLX + XL^tL).
\]

Proof. Using the \( It\dot{o} \) product rule, and from (10) and (13), we obtain

\[
\langle F(t), e_j \rangle V^\dagger(t)XV(t)
= \langle F(0), e_j \rangle X + \int_0^t \langle \Pi F(s), e_j \rangle V^\dagger(s)XV(s)ds
+ \int_0^t V^\dagger(s)XV(s) dM(s), e_j
+ \int_0^t \langle F(s), e_j \rangle d(V^\dagger(s)XV(s)).
\] (18)

Taking conditional expectation with respect to \( \mathcal{E}_t \) on both sides of (18) while using the mutual independence of \( \{Q(t), M(t), F(0)\} \), we obtain

\[
\hat{P}(\langle F(t), e_j \rangle V^\dagger(t)XV(t)|\mathcal{E}_t)
= \hat{P}(\langle F(0), e_j \rangle X)
+ \int_0^t \hat{P}(\langle \Pi F(s), e_j \rangle V^\dagger(s)XV(s)ds|\mathcal{E}_t)
+ \int_0^t \hat{P}(\langle F(s), e_j \rangle V^\dagger(s)(\mathcal{Q}(F(s))(X)
+ XL + L^tX)V(s)ds|\mathcal{E}_t)
= \hat{P}(\langle F(0), e_j \rangle X)
+ \int_0^t \hat{P}(\langle \Pi F(s), e_j \rangle V^\dagger(s)XV(s)|\mathcal{E}_t)ds
+ \int_0^t \hat{P}(\langle F(s), e_j \rangle V^\dagger(s)(\mathcal{Q}(F(s))(X)
+ XL + L^tX)V(s)|\mathcal{E}_t)ds.
\] (19)

In addition,

\[
\langle \Pi F(s), e_j \rangle = \langle F(s), \Pi^T e_j \rangle = \left( F(s), \sum_{k=1}^N a_{jk} e_k \right)
= \sum_{k=1}^N a_{jk} \langle F(s), e_k \rangle.
\] (20)

Let \( h^f_i(X) = \hat{P}(\langle F(t), e_j \rangle V^\dagger(t)XV(t)|\mathcal{E}_t) \). Then we have \( \pi^f_i(X) = U^\dagger(t)h^f_i(X)U(t) \). From (19) and (20), \( h^f_i(X) \) satisfies the following stochastic differential equation:

\[
dh^f_i(X) = \left( \sum_{k=1}^N a_{jk} h^f_i(X) + h^f_i(\mathcal{L}_{L,H}(e_j)(X)) \right) dt
+ h^f_i( XL + L^tX)dQ(t).
\] (21)

From Definition 3.3, we know \( h^f_i(X) \in \mathcal{E}_t \). Using the \( It\dot{o} \) formula, we have

\[
d\pi^f_i(X) = (U(t) + dU(t))^\dagger dh^f_i(X)(U(t) + dU(t))
= \left( \sum_{k=1}^N a_{jk} \pi^f_i(X) + \pi^f_i(\mathcal{L}_{L,H}(e_j)(X)) \right) dt
+ \pi^f_i( XL + L^tX)dQ(t)
+ \pi^f_i( XL + L^tX)dM(s), e_j
+ \pi^f_i( XL + L^tX)dY(t),
\] (22)

which is exactly (17). \( \square \)

Theorem 4.2. (Normalized fault tolerant quantum filtering equation) The normalized conditional expectation \( \sigma^f_i(X) \)
satisfies the following quantum stochastic differential equation:

\[ d\sigma_t^j(X) = \left( \sum_{k=1}^N a_{jk} \pi_t^k(X) + \sigma_t^j \left( Z_{L_H(e_j)}(X) \right) \right) dt + \left( \sigma_t^j(XL + L^tX) - \sigma_t^j(X) \sum_{k=1}^N \sigma_t^k(L + L^t) \right) dW(t), \tag{23} \]

where \( W(t) = Y(t) - \int_0^t \sum_{k=1}^N \sigma_t^k(L + L^t) ds \) is called innovation process and is a Wiener process under \( \tilde{P} \).

**Proof.** From Theorem 4.1, we have

\[ d\pi_t^j(I) = \sum_{k=1}^N a_{jk} \pi_t^k(I) dt + \pi_t^j(L + L^t) dY(t), \tag{24} \]

since \( Z_{L_H(e_j)}(I) = 0 \).

In addition, it follows from the properties of the Q matrix that

\[ d \sum_{k=1}^N \pi_t^k(I) = \sum_{j=1}^N \sum_{k=1}^N a_{jk} \pi_t^k(I) dt + \sum_{k=1}^N \pi_t^k(L + L^t) dY(t) = \sum_{k=1}^N \pi_t^k(L + L^t) dY(t). \tag{25} \]

Equation (16) can be rewritten as

\[ \sum_{k=1}^N \pi_t^k(I) \sigma_t^j(X) = \pi_t^j(X). \tag{26} \]

Differentiating both sides of (26) based on the quantum Itô rule yields

\[ d \sum_{k=1}^N \pi_t^k(I) \sigma_t^j(X) + d \sigma_t^j(X) + \sum_{k=1}^N \pi_t^k(I) d\sigma_t^j(X) = d\pi_t^j(X). \]

It is noted that \( [\sigma_t^j(X), dY(t)] = 0 \) because \( \sigma_t^j(X) \in \mathcal{Y} \). From (24)-(27), one has

\[ \left( \sum_{k=1}^N \pi_t^k(I) + \sum_{k=1}^N \pi_t^k(L + L^t) dY(t) \right) d\sigma_t^j(X) = d\pi_t^j(X) - \sum_{k=1}^N \pi_t^k(L + L^t) \sigma_t^j(X) dY(t). \tag{27} \]

From (17) and (26), one has

\[ \left( \sum_{k=1}^N \pi_t^k(I) \right)^{-1} d\pi_t^j(X) = \left( \sum_{k=1}^N a_{jk} \pi_t^k(X) + \sigma_t^j \left( Z_{L_H(e_j)}(X) \right) \right) dt + \sigma_t^j(XL + L^tX) dY(t). \tag{28} \]

Then dividing both sides of (27) by \( \sum_{k=1}^N \pi_t^k(I) \) yields

\[ \left( I + \sum_{k=1}^N \sigma_t^k(L + L^t) dY(t) \right) d\sigma_t^j(X) = \left( \sum_{k=1}^N a_{jk} \pi_t^k(X) + \sigma_t^j \left( Z_{L_H(e_j)}(X) \right) \right) dt + \sigma_t^j(XL + L^tX) dY(t). \tag{29} \]

By multiplying both sides of (29) with \( I - \sum_{k=1}^N \sigma_t^k(L + L^t) dY(t), (23) \) can be obtained using the fact \( dY(t) dY(t) = dt \).

Next, note \( \sum_{k=1}^N \sigma_t^k(L + L^t) = \tilde{P}(U^t(t)(L + L^t) U(t)|\mathcal{F}_t) \in \mathcal{F}_t \). Thus one can prove that \( W(t) \) is a commutative process which is equivalent to a classical stochastic process under \( \tilde{P} \) according to Theorem 3.2.

In addition, let \( K \in \mathcal{F}_s, s \leq t \), then

\[ \tilde{P}(W(s)|\mathcal{F}_s) = \tilde{P}(W(s)) \]

\[ = \tilde{P} \left( Y(t) K - \int_0^t \tilde{P}(U^t(\tau)(L + L^t) U(\tau)|\mathcal{F}_\tau) K d\tau \right) \]

\[ = \tilde{P} \left( Y(t) K - \int_0^t \tilde{P}(U^t(\tau)(L + L^t) U(\tau)|\mathcal{F}_\tau) K d\tau \right. \]

\[ - \int_s^t U^t(\tau)(L + L^t) U(\tau) d\tau \]

\[ = \tilde{P}(W(s) K) \tilde{P}((Q(t) - Q(s)) K) = \tilde{P}(W(s) K). \tag{30} \]

Therefore, \( \tilde{P}(W(t)|\mathcal{F}_t) = W(s), s \leq t \), which means \( W(t) \) is a \( \mathcal{F}_t \)–martingale. Finally, \( dW(t)dW(t) = dY(t) dY(t) = dt \). Then \( W(t) \) is a Wiener process using Levy’s Theorem ([Karatsas & Shreve (1991)]).

**Remark 4.1.** Since our discussion is under the Heisenberg picture, \( \tilde{P} \) is fixed. Based on Theorem 3.2, (23) is a classical recursive stochastic differential equation driven by the classical Wiener process \( W(t) \), and \( Y(t) \) can be replaced by its classical observation process counterpart. As a result, (23) can be directly implemented on a classical signal processor.

**Remark 4.2.** The coupled system of stochastic differential equations (23) is the normalized conditional expectation
of \( \langle F(t), e_j \rangle U^\dagger(t)XU(t) \), given \( \mathcal{H}_f \). When \( \pi_{jk} = 0, \forall j \neq k \), this system is decoupled and reduces to the well-known quantum filtering equation of \( U^\dagger(t)XU(t) \) given \( \mathcal{H}_f \) ([Belavkin (1992)], [Bouten et al. (2007)]).

Normally, the open quantum system is defined on a finite dimensional Hilbert space \( \mathcal{H}_f \). Noting that \( \sigma_j^f \) is linear, identity preserving and positive, it is a normal state on \( \mathcal{B}(\mathcal{H}_f) \).

From Corollary 4.1, Optionally, the open quantum system is defined on a finite dimensional Hilbert space \( \mathcal{H}_f \). Noting that \( \sigma_j^f \) is linear, identity preserving and positive, it is a normal state on \( \mathcal{B}(\mathcal{H}_f) \). Based on Lemma 3.2, there exists a density operator \( \rho^f \) such that \( \sigma_j^f(X) = \mathbb{E}\{\text{Tr} \{ \rho^f \{ \langle F(t), e_j \rangle X \} \} \} = \text{Tr} \{ \rho^f \} \) with \( \rho^f = \mathbb{E}\{ \langle F(t), e_j \rangle \rho^f \} \). The following is a corollary of Theorem 4.2.

Corollary 4.1. Let \( \rho^f \) be the random operator that satisfies \( \sigma_j^f(X) = \text{Tr} \{ \rho^f \} \) for all system observables \( X \in \mathcal{B}(\mathcal{H}_f) \).

Then \( \rho^f \) satisfies the following stochastic differential equation

\[
d\rho^f = \sum_{j=1}^{N} a_{jk} \rho^f_{k} + \mathcal{L}_{L,H(e_j)}^-(\rho^f) dt
+ \left( L\rho^f_{L} + \rho^f_{L} L^\dagger - \rho^f_{L} \sum_{k=1}^{N} \text{Tr} (\rho^f_{k} (L + L^\dagger)) \right) dW(t),
\]

with \( \rho^f_0 = \mathbb{E}\{ \langle F(0), e_j \rangle \} \pi_0 \). Here \( \mathcal{L}_{L,H(e_j)}^- \) is the adjoint Lindblad generator:

\[
\mathcal{L}_{L,H(e_j)}^-(X) = -i[H,X] + LX L^\dagger - \frac{1}{2} (L^\dagger LX + XL^\dagger L).
\]

Note \( \rho^f \) is not a density matrix because it is not defined in terms of the conditional expectation of real system observables. In fact, we have

\[
\mathbb{P}(U^\dagger(t)XU(t) | \mathcal{H}_f) = \sum_{k=1}^{N} \sigma_k^f(X).
\]

Let \( \rho_t \) be the random density matrix that satisfies \( \mathbb{P}(U^\dagger(t)XU(t) | \mathcal{H}_f) = \text{Tr} \{ \rho_t X \} \). We have

\[
\rho_t = \sum_{k=1}^{N} \rho^k_t, \text{ with } \text{Tr} \{ \rho_t \} = 1 \text{ and } \rho_0 = \pi_0.
\]

From Corollary 4.1, \( \rho_t \) satisfies

\[
d\rho_t = \left( -\sum_{k=1}^{N} i[H(e_k), \rho^k_t] + L \rho_t L^\dagger - \frac{1}{2} L^\dagger L \rho_t - \frac{1}{2} \rho_t L^\dagger L \right) dt
+ (L \rho_t + \rho_t L^\dagger - \rho_t \text{Tr} (L + L^\dagger) \rho_t) dW(t).
\]

Equation (34) is the fault tolerant quantum stochastic master equation.

In addition, the conditional probability densities of the fault process are given by

\[
\hat{\rho}_t = \mathcal{P}(F(t) = e_j | \mathcal{H}_f) = \hat{\mathbb{P}}(\langle F(t), e_j \rangle | \mathcal{H}_f) = \sigma_j^f(t),
\]

which satisfy the following coupled equations using Theorem 4.2:

\[
d\hat{\rho}_t = \sum_{k=1}^{N} a_{jk} \hat{\rho}_k dt
+ \left( \sigma_j^f(L + L^\dagger) - \hat{\rho}_j \sum_{k=1}^{N} \sigma_k^f(L + L^\dagger) \right) dW(t).
\]

Let \( \hat{\rho}_t = [\hat{\rho}_1^f, ..., \hat{\rho}_N^f] \). Then (36) can be rewritten in a vector form as

\[
d\hat{\rho}_t = \Pi \hat{\rho}_t dt + G(t) dW(t),
\]

where \( G(t) = \sum_{k=1}^{N} e_k \sigma_k^f(L + L^\dagger) - \hat{\rho}_i \sum_{k=1}^{N} \sigma_k^f(L + L^\dagger) \).

Equation (37) is the corresponding fault detection equation.

The system of coupled equations (36) or the vector form (37) represents the conditional probability distribution that the system is under any faulty mode. It can be used to determine whether a particular type of fault has happened within the system at time \( t \). A possible criteria for fault detection is

\[
\text{The } j\text{th fault happens, if } \hat{\rho}_j^f \geq p_0,
\]

where \( 1 \geq p_0 > 0 \) is a threshold value chosen by the users.

4.2 Application to Two-level Quantum Systems

Two-level quantum systems (qubits) play a significant role in quantum information processing. For a two-level system, the filter equations reduce to a finite set of stochastic differential equations. In this case, \( \mathcal{H}_f = \mathbb{C}^2 \). We select the coupling strength operator \( L = \sigma_- \) and the free Hamiltonian \( H_0 = \sigma_+ \).

Assume that a fault occurs at time \( T \), at which time a new Hamiltonian \( H_f = \sigma_y \) is introduced into the system. Following ([Davis (1975)]), we assume that \( f(t) \) is a Poisson process with rate \( \lambda \), stopped at its first jump time \( T \). That is,

\[
f(t) = \begin{cases} 0, & \text{if } t < T \\ 1, & \text{if } t \geq T \end{cases}
\]

and \( T \) is an exponential random variable with probability distribution

\[
P(T \leq t) = 1 - e^{-\lambda t}.
\]

From ([Elliott et al. (1995)]), the process \( M(t) = f(t) - \lambda \min(t,T) \) is a martingale and the process \( f(t) \) satisfies

\[
df(t) = \lambda (1 - f(t)) dt + dM(t).
\]
Also, we consider \( f(0) = 0 \) only (because \( f(t) \) stops at its first jump). Let \( F(t) = [1 - f(t), f(t)]' \). Then \( F(t) \) takes values in \( \{e_1, e_2\} \) and satisfies

\[
dF(t) = \begin{bmatrix} -\lambda & 0 \\ \lambda & 0 \end{bmatrix} F(t) + \begin{bmatrix} -1 \\ 1 \end{bmatrix} dM(t). \tag{42}
\]

Hence, the coupled quantum filtering equations are given by

\[
\begin{align*}
dp_1^t &= \left( -p_1^t + \mathcal{L}_{L,H_0}^*(\rho_1^t) \right) dt + (Lp_1^t + p_1^t L^t - p_1^t \sum_{k=1}^2 \text{Tr}(\rho_k^t (L + L^t))) dW(t), \\
dp_2^t &= \left( p_2^t + \mathcal{L}_{L,H_0}^*(\rho_2^t) \right) dt + (Lp_2^t + p_2^t L^t - p_2^t \sum_{k=1}^2 \text{Tr}(\rho_k^t (L + L^t))) dW(t).
\end{align*}
\]

Write

\[
\begin{aligned}
\rho_1^t &= \frac{1}{2} (\alpha(t) I + x_1(t) \sigma_x + y_1(t) \sigma_y + z_1(t) \sigma_z), \\
\rho_2^t &= \frac{1}{2} (1 - \alpha(t)) I + x_2(t) \sigma_x + y_2(t) \sigma_y + z_2(t) \sigma_z,
\end{aligned}
\]

where \( \sigma_j, j \in \{x, y, z\} \) are the Pauli matrices as follows:

\[
\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \text{and} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

Then we obtain seven coupled equations for the seven coefficients related to the fault tolerant quantum stochastic master equation:

\[
\begin{align*}
d\alpha(t) &= -\alpha(t) dt + (x_1(t) - \alpha(t) (x_1(t) + x_2(t))) dW(t) \\
dx_1(t) &= -\frac{1}{2} x_1(t) + 2 y_1(t) dt + (\alpha(t) + z_1(t) - x_1(t) (x_1(t) + x_2(t))) dW(t) \\
dy_1(t) &= (2 x_1(t) - \frac{1}{2} x_2(t) - y_1(t) (x_1(t) + x_2(t))) dW(t) \\
dz_1(t) &= -\alpha(t) + 2 z_1(t) dt - (x_1(t) + x_2(t) z_1(t)) dW(t) \\
dx_2(t) &= (x_1(t) - \frac{1}{2} x_2(t) - 2 y_2(t) + 2 z_2(t)) dt + (1 - \alpha(t) - x_2(t) (x_1(t) + x_2(t)) + z_2(t)) dW(t) \\
dy_2(t) &= (y_1(t) + 2 x_2(t) - \frac{1}{2} y_2(t)) dt - (x_1(t) + x_2(t)) y_2(t) dW(t) \\
dz_2(t) &= (-1 + \alpha(t) + z_1(t) - 2 x_2(t) - z_2(t)) dt - (x_2(t) + (x_1(t) + x_2(t)) z_2(t)) dW(t)
\end{align*}
\]

The fault detection equation is given by

\[
d\hat{p}_t = \Pi \hat{p}_t dt + G(t) dW(t). \tag{43}
\]

where \( G(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} - \hat{p}_t (x_1(t) + x_2(t)) \). The innovation process \( W(t) \) is given by \( W(t) = y(t) - \int_0^t x_1(s) + x_2(s) ds \).

5 Conclusions

In this paper, an approach to solving the problem of fault tolerant quantum filtering and fault detection for a class of laser-atom open quantum systems has been developed. A new quantum-classical probability space model is proposed to enable us to deal with both classical and quantum randomness in a simple way. By describing the stochastic fault process as a finite-state jump Markov chain and using a reference probability approach, a set of coupled stochastic differential equations satisfied by the conditional system states and fault process estimates are derived. An application to two-level quantum systems under Poisson type faults is also presented. The proposed approaches provide a new avenue to deal with various cases where classical randomness appear in quantum systems and have the potential for wide applications in analyzing and synthesizing open quantum systems.

Appendix

Proof of Lemma 4.1. Let \( \mathcal{Q} = \mathcal{Q}_{t}(X) = \mathbb{P}(U^\dagger(t)XU(t)) \).

Let \( K(t) \) be any element of \( \mathcal{B} \), then \( K(t) = U^\dagger(t)K_o U(t) \) for some \( K_o \in \mathcal{B} \). Note the scalar valued function \( \langle F(t), e_j \rangle \in \mathcal{L}' \) and \( X \in \mathcal{L}' \). We have

\[
\mathbb{P}(\langle F(t), e_j \rangle U^\dagger(t)XU(t)|\mathcal{B}) K = \mathbb{P}(\langle F(t), e_j \rangle U^\dagger(t)XU(t)|\mathcal{B}) K(t) = \mathbb{P}(\langle F(t), e_j \rangle U^\dagger(t)XK_o U(t)) = \mathcal{Q}(\langle F(t), e_j \rangle X|\mathcal{B}) K_o(t) = \mathbb{P}(U^\dagger(t)\mathcal{Q}(\langle F(t), e_j \rangle X|\mathcal{B})K_o U(t)) = \mathbb{P}(U^\dagger(t)\mathcal{Q}(\langle F(t), e_j \rangle X|\mathcal{B}) U(t) K(t)). \tag{44}
\]

Letting \( K(t) = (\mathbb{P}(\langle F(t), e_j \rangle U^\dagger(t)XU(t)|\mathcal{B})) - U^\dagger(t)\mathcal{Q}(\langle F(t), e_j \rangle X|\mathcal{B}) U(t) \) yields

\[
\mathbb{P}(\langle F(t), e_j \rangle U^\dagger(t)XU(t)|\mathcal{B}) = U^\dagger(t)\mathcal{Q}(\langle F(t), e_j \rangle X|\mathcal{B}) U(t) \tag{45}
\]

almost surely under \( \mathbb{P} \).

In addition, suppose the system is initialized at \( \pi_0 = \sum_k p_k |\alpha_k \rangle \langle \alpha_k| \) and we define a curve \( |\psi_k(t)\rangle = U(t)|\alpha_k \rangle \otimes \)
Using the fact that $dB(t)\mid \psi(t) = 0$, one obtains (see Equation (6.13) in [Holevo (1991)])

$$d\mid \psi_k(t)\rangle = \{(\{iH(\tau) - \frac{1}{2}L^2L\}dt + LdQ(t)\}\mid \psi_k(t)\rangle.$$

(46)

In other words, $U(t)\mid (\alpha_k \otimes \psi(t))\rangle = V(t)\mid (\alpha_k \otimes \psi(t))\rangle$ since $U(0) = V(0) = I$. After some mathematical manipulation, one obtains $\text{Tr}(p_\theta U^\dagger(t)XU(t)) = \text{Tr}(p_\theta V^\dagger(t)XV(t))$ which leads to

$$\tilde{\mathbb{P}}(\langle F(t), e_j \rangle U^\dagger(t)XU(t)) = \tilde{\mathbb{P}}(\langle F(t), e_j \rangle V^\dagger(t)XV(t)).$$

(47)

Then we can apply Theorem 3.4 with $\Lambda = 1$, $A = \langle F(t), e_j \rangle X \in \mathcal{L}_f'$, $V = V(t)$ and $\mathcal{C} = \mathcal{D}_f$ and obtain

$$\tilde{\mathbb{Q}}'\langle F(t), e_j \rangle X \mid \mathcal{D}_f = \frac{\tilde{\mathbb{P}}(\langle F(t), e_j \rangle V^\dagger(t)XV(t) \mid \mathcal{D}_f)}{\tilde{\mathbb{P}}(V^\dagger(t)XV(t) \mid \mathcal{D}_f)}.$$

(48)

Lemma 4.1 can be concluded combing (45) and (48). □

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