A Design Method of Chaotic Synchronous Multi-stable Manifold

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Abstract. Line spectra in the low frequency band of the underwater radiated noise are the main factors affecting the acoustic stealth performance of submarines. The chaotification technique for line spectra reduction can realize spectra reconstruction and reduce line spectra intensity by converting line spectra in the low frequency band into chaotic spectra in the broadband frequency. Previous studies have shown that the method of the generalized synchronization of chaos can generate continuous chaos of the system under variable operating conditions, and can reduce the amplitude significantly in some cases, which provides a new idea for the theory of the chaotification technique for line spectra reduction. However, this method can only select the appropriate synchronization control strategy of chaos through "trial and error" method to achieve chaos with small amplitude. Therefore, a design method of multiple stable synchronous manifolds including stable chaotic attractors with small amplitude is proposed. This method is verified and illustrated by the coupled Duffing system, and applied to the two degree-of-freedom (DOF) nonlinear vibration isolation system (VIS).

1. Introduction

With its superior concealment and assault, the submarine has become one of the most important maritime military forces since its naissance. In physical fields such as sound, infrared, electromagnetic, and wake, only sound waves can travel long distances in seawater, which becomes the main signal for exposing the submarine's whereabouts. After World War II, through the unremitting efforts of experts and scholars in the field of vibration and noise control of submarine, spectra intensity in the high frequency band of the underwater radiated noise has been greatly reduced, and the line spectra in low-frequency bands has increasingly become a major factor affecting the safety of submarines [1].

Therefore, scholars from various countries have begun to focus on how to reduce the spectra intensity and hide the information about the submarine carried by the line spectrum. Among them, Zhu [2-7] innovatively proposed the chaotification technique for line spectra reduction, which can transform the line spectra into a broadband chaotic spectra through the nonlinear VIS in chaotic state, thereby weaken and reconstruct the spectra signal in the radiated water sound of submarine. In order to realize the continuous chaos of the VIS, a chaotification technique for line spectra reduction based on generalized synchronization of chaos is proposed by Yu [3]. Lou [8] realized the continuous chaos of the hard Duffing VIS by using the generalized synchronization of chaos based on a parameter-driven method. Sang [9] accomplished the generalized synchronization of chaos of the heterogeneous two-dimensional delay systems by using the stability theory of functional differential equations, and simplified the method of the generalized synchronization of chaos. Yang [10] actualized continuous chaos with small amplitude of 2 DOF high-static-low-dynamic-stiffness VIS by using the generalized synchronization of chaos base on a open-loop plus nonlinear closed-loop coupling method. However, at present, in order to generate continuous chaos with small amplitude by utilizing the method of the generalized synchronization of chaos, the “trial and error” is the only way to select the appropriate chaotic synchronization control strategy. Consequently, there is still no systematic method.

In this paper, a design method of multiple stable synchronous manifolds is proposed, which ensures that there are chaotic attractors with small amplitude in the phase space, thus achieving the continuous chaotic motion of the small amplitude of the VIS.


2. Study on the design method

2.1 Presentation of the method

Taking the one-way coupling system as the research object:

\[ \dot{x} = F(x) \]  
\[ \dot{y} = G(y) + kP(x) \]

(1a)
(1b)

Where \( x \) and \( y \) represent the state vector of the driving system \( F(x) \) and the response system \( G(y) \), respectively. \( P(x) \) denotes the control function and \( k \) is a scalar representing the coupling coefficient.

Under the premise that the driving system \( F(x) \), the response system \( G(y) \) and the control function \( P(x) \) are known, Yu [11] obtained the functional relationship between the state vector \( y \) of the response system and the state vector \( x \) of the driving system by the approximate central manifold method [12], that is, the synchronous manifold \( y = H(x) \). In this paper, a reverse thinking is adopted. Under the premise that the driving system \( F(x) \) and the response system \( G(y) \) are known, and the control function \( P(x) \) is unknown, a design method of multiple stable synchronous manifold is proposed. The detailed idea is as follows: Firstly, try to construct a synchronous manifold \( y = H(x) \) according to the need, and design the control function \( P(x) \) by the approximate central manifold method, and solve the simultaneous equations combined with \( y = H(x) \), the driving system and the response system including the unknown \( P(x) \) to obtain the specific expression of \( P(x) \). Then, the obtained \( P(x) \) is substituted into the coupled system. According to the stability theorem of synchronous manifold proposed by Yu [13], it is judged whether \( y = H(x) \) is stable in the coupled system containing the specific \( P(x) \). If \( y = H(x) \) is stable, the stable synchronous manifold \( y = H(x) \) is obtained. If \( y = H(x) \) is unstable, then \( y = H(x) \) needs to be perturbed and the above process is repeated until the resulting synchronous manifold is stable. Finally, the approximate central manifold method is used to find other synchronous manifolds of the coupled system under the control of \( P(x) \) and determine its stability.

2.2 Verification and illustration

This design method is verified and illustrated by the coupled Duffing system as below. The single DOF oscillator under harmonic excitation is selected as the driving system:

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{x}_3 \\
\dot{x}_4
\end{bmatrix} =
\begin{bmatrix}
x_2 \\
-wx_1 - x_1^2 - x_3 + f \cos(\omega t) \\
-\omega x_4 \\
\omega x_3
\end{bmatrix}
\]

(2)

Where the introduction of \( x_3 \) and \( x_4 \) transforms the driving system into an autonomous system, and \( x_3 = \cos(\omega t) \), \( x_4 = \sin(\omega t) \) respectively. Set the parameters of the driving system as below: \( \omega = 3.9311, w = 4, v = 3, u = 0.15, f = 9 \).

![Figure 1. The phase diagram(a) and the poincare map(b) of the driving system](image-url)
When the initial condition is \((0, 0)\), the phase diagram and the Poincaré map are shown in figure 1, and the power spectra are shown in figure 2. It can be seen from figure 1 and figure 2 that the movement trajectory of the driving system is entangled and spread over a certain phase space, and the power spectra exhibit the characteristics of continuity. It can be concluded that the driving system is in a chaotic state of motion under the condition of these parameters.

\[ y = H(x) = A_1 + B_1 x \]

Where \( A_1 = [0, 0, 0, 0]^T \),

\[ B_1 = \begin{bmatrix} 0 & 0.033 & -0.625 & -0.056 \\ -0.133 & -0.005 & 0.080 & 2.457 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \]

It is worth noting that multiple attempts are required to construct this synchronous manifold. Firstly, considering the case where the matrix \( B_1 \) in the synchronous manifold retains one bit after the decimal point, the parameters are adjusted according to the obtained control function \( P(x) \). Then, the above steps are repeated by sequentially considering the case where the matrix \( B_1 \) in the synchronous manifold retains two and three bit after the decimal point. The form of the resulting control function \( P(x) \) should be quite simple, making it easy to implement control strategy when applied to engineering practice.

Since \( y = H(x) \) is differentiable, then there is

\[ \dot{y} = D_y \cdot H(x) \dot{x} \]

Substitute equation (1a) and equation (1b) into equation (5) and get
The control function $P(x)$ is designed by using the approximate central manifold method, expand $P(x)$ into a Taylor series and keep the first two terms: $P(x) = Mx + N$. Substitute $y = H_1(x)$ into the equation (6), and get

$$P(x) = \begin{bmatrix} 0.01 & 0.0005 & -0.031416 & -0.000625 \\ 0.0005 & -1 & 0.007127 & -0.01938 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0$$

(7)

Ignore the calculation error within the allowable range and round up $P(x)$ to

$$P(x) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

(8)

According to the stability theorem of synchronous manifold proposed by Yu [13], the following inference can be drawn: If the maximum condition Lyapunov exponent (CLE) of one-way coupled system (1) is less than 0, then the synchronous manifold is asymptotically stable. If the maximum CLE of the one-way coupled system (1) is bigger than 0, the synchronous manifold is unstable. Substitute the obtained $P(x)$ into the coupled system, and the maximum CLE of the system under the action of mapping $H_1(x)$ is $-0.7449$, indicating that the corresponding synchronous manifold is stable.

Another synchronous manifold of the coupled system under the control of $P(x)$ is obtained by using the approximate central manifold method:

$$y = H_2(x) = A_2 + B_2x$$

(9)

Where $A_2 = [-1, 0, 0, 0]^T$, $B_2 = \begin{bmatrix} -0.100 & 0.025 & -0.584 & -0.035 \\ 0 & 0.004 & 0.088 & 2.295 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

In this case, the values of $B_2$ are exactly three decimal places. Under the action of mapping $y = H_2(x)$, the maximum CLE of the system is $-0.7502$. Similarly, it can be concluded that the corresponding synchronization manifold is stable.

![Figure 3](image-url)  
Figure 3. The phase diagram(a) and the attraction basins(b) of the coexistent attractors of the coupled Duffing system when $k=0.1$
The validity of this design method will be proved by numerical simulation. Due to different initial conditions, the system operates in two different chaotic attractors, and the phase diagram is shown in figure 3(a). When the initial condition is (0,0,0,0), the system runs on a stable attractor A1 with smaller amplitude. When the initial condition is (0,0,4,0), the system runs on a stable attractor A2 with larger amplitude. The coexistent chaotic attractors all have their attraction basins, as shown in figure 3(b). The black region is the attraction basin of the attractor with smaller amplitude, and the gray region is the attraction basin of the attractor with larger amplitude. It can be seen that there are two stable synchronous manifolds with different amplitudes in the coupled Duffing system, which proves the validity of the proposed method.

### 3. Further application of the method

In order to further extend the application range of this design method, the application of the method in the 2 DOF nonlinear VIS is considered. The chaotic single DOF oscillator represented by equation (1) is still chosen as the driving system. A 2 DOF nonlinear VIS with control function $P(x)$ is selected as the response system:

$$\begin{bmatrix}
\dot{y}_1 \\
\dot{y}_2 \\
\dot{y}_3 \\
\dot{y}_4 \\
\dot{y}_5 \\
\dot{y}_6
\end{bmatrix} =
\begin{bmatrix}
y_2 \\
-\delta_1(y_2 - y_4) - (y_1 - y_5) + \xi_1(y_1 - y_5)^3 -(y_1 - y_5)^3 + d \cos(\omega t)
y_4 \\
-\mu\delta_2 y_1 - \mu p_2 y_3 + \mu \delta_3(y_2 - y_4) + \mu (y_1 - y_5) - \mu \xi_2(y_1 - y_5)^3 + \mu (y_1 - y_5)^3 + kP(x)
y_5 \\
-\omega y_6
\end{bmatrix}$$

The response system is still under the chaotification control of the driving system in a manner of state-driven generalized synchronization, where $k=0.1$, the introduction of $y_3$ and $y_6$ transforms the response system into an autonomous system, and $y_5=x_3=\cos(\omega t)$, $y_6=x_4=\sin(\omega t)$ respectively. Set the parameters of the response system as below: $\delta_1=0.8$, $\xi_1=2$, $d=20$, $\mu=0.2$, $p_2=2$.

Try to construct a synchronous manifold:

$$y=H_s(x)=A_3+B_3x$$

where $A_3=[0, 0, 0, 0, 0]^T$, 

$$B_3 =
\begin{bmatrix}
0.110 & -0.050 & -1.234 & 0.399 \\
0.200 & 0.118 & 1.118 & 4.850 \\
-0.027 & 0.054 & -0.022 & -0.186 \\
-0.218 & -0.035 & -0.239 & 0.086 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}$$

The control function $P(x)$ is designed by using the approximate central manifold method, expand $P(x)$ into a power series and keep the first two terms: $P(x)=Mx+N$. Here, $P(x)$ can be expressed as follows:

$$P(x)=
\begin{bmatrix}
m_{11} & m_{12} & m_{13} & m_{14} & 0 & 0 \\
m_{21} & m_{22} & m_{23} & m_{24} & 0 & 0 \\
m_{31} & m_{32} & m_{33} & m_{34} & 0 & 0 \\
m_{41} & m_{42} & m_{43} & m_{44} & 0 & 0 \\
m_{51} & m_{52} & m_{53} & m_{54} & 0 & 0 \\
m_{61} & m_{62} & m_{63} & m_{64} & 0 & 0
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5 \\
x_6
\end{bmatrix} +
\begin{bmatrix}
n_1 \\
n_2 \\
n_3 \\
n_4 \\
n_5 \\
n_6
\end{bmatrix}$$

Substitute $y=H_s(x)$, the equation of the driving system and the response system with the control function $P(x)$ into the equation (6) and get

\begin{align}
\begin{bmatrix}
0.110 & -0.050 & -1.234 & 0.399 \\
0.200 & 0.118 & 1.118 & 4.850 \\
-0.027 & 0.054 & -0.022 & -0.186 \\
-0.218 & -0.035 & -0.239 & 0.086 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5 \\
x_6
\end{bmatrix} +
\begin{bmatrix}
n_1 \\
n_2 \\
n_3 \\
n_4 \\
n_5 \\
n_6
\end{bmatrix} =
\begin{bmatrix}
m_{11} & m_{12} & m_{13} & m_{14} & 0 & 0 \\
m_{21} & m_{22} & m_{23} & m_{24} & 0 & 0 \\
m_{31} & m_{32} & m_{33} & m_{34} & 0 & 0 \\
m_{41} & m_{42} & m_{43} & m_{44} & 0 & 0 \\
m_{51} & m_{52} & m_{53} & m_{54} & 0 & 0 \\
m_{61} & m_{62} & m_{63} & m_{64} & 0 & 0
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5 \\
x_6
\end{bmatrix} +
\begin{bmatrix}
n_1 \\
n_2 \\
n_3 \\
n_4 \\
n_5 \\
n_6
\end{bmatrix}
\end{align}
\[
P(x) = \begin{bmatrix} 0 & -0.005 & 0.005089 & -0.009774 & 0 & 0 \\ -0.006 & 2.007 & 0.01435 & 0.012302 & 0 & 0 \\ 0.02 & -0.001 & -0.061846 & 0.00442 & 0 & 0 \\ 0.0004 & 2.0043 & 0.013146 & -0.003471 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}
\]

(12)

Ignore the calculation error within the allowable range and round up \( P(x) \) to

\[
P(x) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}
\]

(13)

Substitute the obtained \( P(x) \) into the coupled system, and the maximum CLE of the system under the action of mapping \( y = H_3(x) \) is -0.0594, indicating that the corresponding synchronous manifold \( y = H_3(x) \) is stable.

Another synchronous manifold of the coupled system under the control of \( P(x) \) is obtained by using the approximate central manifold method:

\[
y = H_4(x) = A_4 + B_4 x
\]

(14)

where \( A_4 = [1, 0, 0, 0, 0]^T \),

\[
B_4 = \begin{bmatrix} 0.063 & -0.034 & -1.196 & 0.316 \\ 0.137 & 0.068 & 0.934 & 4.700 \\ -0.016 & 0.051 & -0.028 & -0.168 \\ -0.206 & -0.023 & -0.199 & 0.111 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\]

In this case, the values of \( B_4 \) are exactly three decimal places. Under the action of mapping \( y = H_4(x) \), the maximum CLE of the system is -0.2033. Similarly, it can be concluded that the corresponding synchronization manifold \( y = H_4(x) \) is stable.

![Figure 4. The phase diagram (a) and the attraction basins (b) of the coexistent attractors of the 2 DOF nonlinear VIS when k=0.1](image-url)
Due to different initial conditions, the system operates in two different chaotic attractors, and the phase diagram is shown in figure 4(a). When the initial condition is (0,0,0,-5,2,0), the system runs on a stable attractor A1 with smaller amplitude. When the initial condition is (0,0,0,-5,-4,6), the system runs on a stable attractor A2 with larger amplitude. The coexistent chaotic attractors all have their attraction basins, as shown in figure 4(b). The black region is the attraction basin of the attractor A1 with smaller amplitude, and the gray region is the attraction basin of the attractor A2 with larger amplitude. It can be seen that there are two stable synchronous manifolds with different amplitudes in the 2 DOF nonlinear VIS, which proves the validity of the proposed method.

4. Conclusions
Line spectra in the low frequency band of the underwater radiated noise are the main factors affecting the acoustic stealth performance of submarines. The chaotification technique for line spectra reduction can realize spectra reconstruction and reduce line spectra intensity. The method of the generalized synchronization of chaos can generate continuous chaos of the system under variable operating conditions, and can reduce the amplitude significantly in some cases, which provides a new idea for the theory of the chaotification technique for line spectra reduction. However, at present, in order to generate continuous chaos with small amplitude by utilizing the method of the generalized synchronization of chaos, the "trial and error" method is the only way to select the appropriate synchronization control strategy of chaos. Consequently, there is still no systematic method. In this paper, therefore, a design method of multiple stable synchronous manifold is proposed to make the system have the chaotic attractor with small amplitude in phase space. This method is verified and illustrated by the coupled Duffing system, and applied to the 2 DOF nonlinear VIS.

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References
[1] Liu P and Liu G L 2013 Acoust. Elec. Eng. 112(4) 26-27(in Chinese)
[2] Zhu S J, Jiang R J and He L 2003 J. Nav. Uni. Eng. 15(1) 19-22(in Chinese)
[3] Lou J J 2006 Nav. Uni. Eng. PhD Thesis (in Chinese)
[4] Yu X 2008 Nav. Uni. Eng. PhD Thesis (in Chinese)
[5] Zhang Z H, Zhu S J and Lou J J 2011 J. Vib. Sho. 30(7) 40-44(in Chinese)
[6] Zhang J, Xu D L, Li Y L and Zhou J X 2014 J. Phys. 63(18) 120-30(in Chinese)
[7] Zhang Z H, Zhu S J and He Q W 2012 J. Vib. Eng. 25(01) 30-37(in Chinese)
[8] Lou J J, Zhang H, Yu X and Zhu S J 2015 J. Vib. Sho. 34(14) 106-09(in Chinese)
[9] Sang J Y, Wang J and Yue L J 2010 J. Phys. 59(11) 7618-22(in Chinese)
[10] Yang C, Chai K, Lou J J and Zhu S J 2014 J. Vib. Eng. 2018(04) 620-28 (in Chinese)
[11] Yu X, Zhu S J and Liu S Y 2008 J. Phys. 57(05) 2761-69(in Chinese)
[12] Brown R 1998 Phys. Rev. Lett. 81(22) 4835-4838
[13] Yu X, Zhu S J and Liu S Y 2007 J. S. V. 306 835-848.