The Study of Color Octet Matrix Elements Through \( J/\psi \) and \( \Upsilon \) Production in \( e^+e^- \) Annihilation

Yi-Jie Li \(^a\), Guang-Zhi Xu \(^b\), Yu-Jie Zhang \(^{b,c}\), Kui-Yong Liu \(^a\)

\(^a\)Department of Physics, Liaoning University, Shenyang 110036, China
\(^b\)School of Physics, Beihang University, Beijing 100191, China
\(^c\)CAS Center for Excellence in Particle Physics, Beijing 100049, China

E-mail: yijiegood@gmail.com, still200@gmail.com, nophy0@gmail.com, liukuiyong@lnu.edu.cn

ABSTRACT: The next-to-leading radiative corrections and relativistic corrections to the inclusive heavy quarkonium production in \( e^+e^- \rightarrow J/\psi + LH \) and \( e^+e^- \rightarrow \Upsilon + LH \) are calculated within non-relativistic QCD frame. And the cross section with different the long distance matrix elements has been given. We find that the inclusive heavy quarkonium production process in \( e^+e^- \) annihilation can give a strong restrictions on the color octet matrix elements. And the long distance matrix elements of P-wave channel for \( J/\psi(\psi') \) would be very small. More measurement of \( J/\psi(\psi') \) and \( \Upsilon \) production associated with light hadron at BESIII, Belle, and BelleII can give more information about the color octet matrix elements.
1 Introduction

The CO (color octet) mechanism was introduced to describe the production and decay of heavy quarkonium in nonrelativistic quantum chromodynamics (NRQCD). Differing from the case in the CS (color singlet) model, in a CO production process the intermediate pair of quark and antiquark can be created at short distances with CO and then formed the non color quarkonium at long distances by emitting or absorbing soft gluons. The processes at short distances are called as short-distance coefficients, which are scaled by the strong coupling constant $\alpha_s$ and calculated perturbatively. The processes at long distances are called as LDMEs (long distance matrix elements), which are scaled by the relative velocity $v$ and cannot be calculated perturbatively. The LDMEs are universal, process-independence, and must be determined by experimental extraction, potential model, or lattice QCD calculations.

The NRQCD CO mechanism acquire some significant successes since it was proposed. The surplus production for transverse momentum of $J/\psi$ and $\psi'$ at the Tevatron seems to be a powerful evidence in favor of CO mechanism. The contribution given by the LO (leading order) calculations in CS model counts for less than 5 percent of the experimental data on the unpolarized cross section. For the CO case, it introduced three extra phenomenological parameters, the LDMEs of CO states $3S_1^{[8]}$, $1S_0^{[8]}$, $3P_0^{[8]}$, which are fitted to the experimental data. Then the result of CO make the gap between CS LO prediction and the experiment data appeared. The calculated of inclusive direct $J/\psi$ photon-production at NLO (next-to-leading order) CO cross section at HERA, and in the $\gamma + \gamma \rightarrow J/\psi + X$ production at DELPHI also suggested CO mechanism is indeed realized the production mechanism.

However, the predictions within the CO model to the polarization of $J/\psi$ hadron production are transverse polarization that much more contradicts with unpolarized result measured by the Tevatron. Many theoretical efforts were made to solve this puzzle, including...
the NLO radiative corrections for production [? ? ? ? ? ? ? ? ? ] and polarization [? ? ? ? ? ? ? ]. And the NLO relativistic corrections to $J/\psi$ hadronic production are considered too[? ? ? ]. But the three CO LDMEs fitting by the three groups were incompatible with each other’s[? ? ? ]. To determine these values, several efforts were made including a global fit with the world’s unpolarized data[? ], which are inclusive $J/\psi$ production data from various hadron-production, photon-production, two-photon scattering, and $e^+e^-$ annihilation [? ]. With the fit of the polarization data[? ], and the feed-down contributions[? ] are also considered. Unfortunately, their results were violently incompatible with others. More information about NRQCD and heavy quarkonium physics can be found in Ref.[? ] and the related papers.

The CO mechanism in heavy quarkonium production at $e^+e^-$ process annihilation are not the same as the hadron-production case. In the $e^+e^-$ annihilation at B-factories[? ], puzzles on NRQCD involving the inclusive and exclusive $J/\psi$ production[? ? ? ? ? ? ? ] had been solved by the higher-order corrections, including the radiative corrections[? ? ? ? ? ? ? ? ? ? ? ? ? ? ], the relativistic corrections [? ? ? ? ? ? ? ? ? ? ? ? ? ], and the $\mathcal{O}(\alpha_s v^2)$ corrections [? ? ? ]. But the NLO corrections to the CS piece [? ? ? ] can full the experimental measurements the magnitude of the cross sections $\sigma[J/\psi + c\bar{c} + X_{non-cc}]$ measured by Belle[? ]. It seems that the CS may give the main contribution, and leave little room for CO state in the $e^+e^-$ collider experiments. The CO LDMEs of $e^+e^-$ collider process may be smaller than that expected at the hadron colliders. So the test of the university to the high-order LDMEs in NRQCD is still interesting and helpful to comprehend the production mechanism of heavy quarkonium, especially, the CO mechanism. At the same time, the exclusive cross sections $e^+e^- \rightarrow J/\psi + \pi^+\pi^-$ are measured by Belle[? ] and BESIII[? ].

In this paper, we concentrate on the CO pieces to the inclusive production of $J/\psi$ and $\Upsilon$ production associated with LH (light hadrons) in $e^+e^-$ annihilation. For the processes $e^+e^- \rightarrow J/\psi + LH$, there are two subprocess, $e^+e^- \rightarrow q\bar{q} + J/\psi$ and $e^+e^- \rightarrow J/\psi + g$, at parton level. The former process can be ignored at the energy of the B factories[? ] and we will discuss the latter process only. The octet signature of the latter process emerges near the endpoint region of the energy of $J/\psi$ in the leading order calculation[? ]. Two CO intermediate states $^1S^0_0$ and $^3P^8_1$ appeared here.

For the process $e^+e^- \rightarrow \Upsilon + LH$, we need to consider the contribution from intermediate states $^1S^0_0$ and $^3P^{[8]}_1$, too. This process have the same property as the processes $e^+e^- \rightarrow J/\psi + LH$, but heaver mass. It is a ideal process to test LDMEs of $^1S^0_0$ and $^3P_{[8]}$. On the other hand, the $e^+e^-$ annihilation process can give little room for CO state for $J/\psi$, the $e^+e^- \rightarrow \Upsilon + LH$ process may give a strong limit to LDMEs for $\Upsilon$.

Motivated by this idea, in this paper, we compute the relativistic correction and QCD correction to the octet process $J/\psi$ and $\Upsilon$ production to next order in $v^2$ and $\alpha_s$. And give the cross section according to LDMEs which were fitted by the three groups. The paper is organized as follows. Sec.2 introduces the framework of calculation. And Sec.3 gives the numerical results. Finally, Sec.4 presents a summary.
2 The framework of calculation

This section introduces the framework of calculation for the relativistic corrections and the radiative corrections within the NRQCD factorization frame.

The amplitude for \( e^+ e^- \rightarrow J/\psi(1S_0^8/3P_j^8) + g \) is given by the following expression in NRQCD[? ? ]

\[
\mathcal{M}[J/\psi(1S_0^8/3P_j^8) + g] = \sqrt{2M_{J/\psi}} \sum_m d_m(1S_0^8/3P_j^8)\langle J/\psi|O_m(1S_0^8/3P_j^8)|0\rangle
\]

where the factor \( \sqrt{2M_{J/\psi}} \) originates from the relativistic normalization factor[? ]. The \( d_m \) are the short-distance coefficients and \( O_m \) are the NRQCD operators which are defined as

\[
O_m(1S_0^8) = \psi^\dagger T\left(-\frac{i}{2} \vec{D}\right)^{2m}\chi,
\]
\[
O_m(3P_0^8) = \frac{1}{\sqrt{3}} \psi^\dagger T\left(-\frac{i}{2} \vec{D}\right)^{2m}\left(-\frac{i}{2} \vec{D} \cdot \sigma\right)\chi,
\]
\[
O_m(3P_1^8) = \frac{1}{\sqrt{2}} \psi^\dagger T\left(-\frac{i}{2} \vec{D}\right)^{2m}\left(-\frac{i}{2} \vec{D} \times \sigma\right)\chi,
\]
\[
O_m(3P_2^8) = \psi^\dagger T\left(-\frac{i}{2} \vec{D}\right)^{2m}\left(-\frac{i}{2} \vec{D}(i\sigma^3)\right)\chi.
\]

The short-distance coefficients can be obtained by matching the \( c\bar{c} \) production process in NRQCD with that in full QCD.

\[
\mathcal{M}[c\bar{c}(1S_0^8/3P_j^8) + g]_{\text{pQCD}} = \sum_m d_m(1S_0^8/3P_j^8)\langle c\bar{c}|O_m(1S_0^8/3P_j^8)|0\rangle
\]
\[
= \begin{cases} 
2E_q \sqrt{N_c^2 - 1} \sum_m d_m(1S_0^8)q^{2m}, \\
2E_q \sqrt{N_c^2 - 1} \sum_m d_m(3P_j^8)q^{2m+1}
\end{cases} \tag{2.3}
\]

The result in the second line of above expression was obtained applying the relativistic normalization of CO \( c\bar{c} \)

\[
\langle c\bar{c}|O_0(1S_0^8)c\bar{c} \rangle = (2E_q)^2(N_c^2 - 1),
\]
\[
\langle c\bar{c}|O_0(3P_j^8)c\bar{c} \rangle = (2E_q)^2q^2(N_c^2 - 1), \tag{2.4}
\]

where \( E_q = \sqrt{m_c^2 + q^2} \) is the energy of \( c(\bar{c}) \) in the \( J/\psi \) rest frame. Then the short-distance coefficients \( d_m \) can be calculated by

\[
d_m(1S_0^8) = \frac{1}{(2m)!} \frac{\partial^{2m}}{\partial q^{2m}} \left[ \mathcal{M}[c\bar{c}(1S_0^8) + g] \right]_{q=0}^{2E_q \sqrt{N_c^2 - 1}},
\]
\[
d_m(3P_j^8) = \frac{1}{(2m + 1)!} \frac{\partial^{2m+1}}{\partial q^{2m+1}} \left[ \mathcal{M}[c\bar{c}(3P_j^8) + g] \right]_{q=0}^{2E_q \sqrt{N_c^2 - 1}}. \tag{2.5}
\]
Using Eq.\,(2.5) and Eq.\,(2.1), one will get the final expressions of the NRQCD amplitude
\[\mathcal{M}[J/\psi(1S_0^8/3P^8_j) + g] = \sqrt{2M_{J/\psi}}<J/\psi|\mathcal{O}_0(1S_0^8/3P^8_j)|0> \sum_m d_m(1S_0^8/3P^8_j)|q|^{2m}\] (2.6) where the LDMEs \(|q|^{2m}\) are defined as \(<J/\psi|\mathcal{O}_m|0>^{2m}/<J/\psi|\mathcal{O}_0|0>^{2m}\). The perturbative QCD process \(\mathcal{M}(c\bar{c}(1S_0^8/3P^8_j)+g)\) can be computed using the projection operator method
\[\mathcal{M}[c\bar{c}(1S_0^8/3P^8_j)+g] = \int \frac{d\Omega_q}{N_L} Y_{LL}(\hat{q}) Tr\{\mathcal{M}(c+\bar{c}+g)[P_{ss} \otimes \pi_8]\},\] (2.7) where the normalization factor \(N_L\) satisfy
\[N_0 = \int d\Omega_q Y_L(\hat{q}) = \sqrt{4\pi},\]
\[N_1 \epsilon^\mu(L_z) = \int d\Omega_q Y_{Lz}(\hat{q}) \epsilon^\mu /|q| = \sqrt{4\pi/3} \epsilon^\mu(L_z).\] (2.8)
Where \(\epsilon\) is the orbit polarization vector for \(P\)-wave state. The \(\pi_8\) is the octet color projector and defined by \(\pi_8 = (3i; 3j|8a) = \sqrt{2T^8}\). The spin projectors \(P_{ss}\) for spin-singlet and spin-triplet are given as the bellow expressions in terms of the momenta of the c and \(\bar{c}\) in covariant form
\[P_{00}(p_c, p_{\bar{c}}) = \frac{1}{2\sqrt{2}(E_q + m_c)}(-p_{\bar{c}} + m_c)\gamma_5 \frac{p_c + p_{\bar{c}} + 2E_q}{2E_q}(p_c + m_c),\]
\[P_{1s}(p_c, p_{\bar{c}}) = \frac{1}{2\sqrt{2}(E_q + m_c)}(-p_{\bar{c}} + m_c)\epsilon^j(s_z) \frac{p_c + p_{\bar{c}} + 2E_q}{2E_q}(p_c + m_c),\] (2.9)
We can set \(p_c = P/2 + q\) and \(p_{\bar{c}} = P/2 - q\) in the calculations, where \(P\) is the barycenter momentum of the \(c\bar{c}\) pair, etc. the \(J/\psi\)'s momentum.

### 2.1 The \(O(\alpha_s, v^2)\) correction
Now, we first consider the amplitude expanded to next leading order in \(v^2\). We expand the \(J/\psi\) mass as[? ?]
\[\frac{M_{J/\psi}}{2m_c} = 1 + \frac{1}{2m_c^2} <J/\psi|\mathcal{O}_1|0> + \mathcal{O}(v^4),\] (2.10) and substitute this expression into Eq.\,(2.6) to expand to next leading order with omitting the higher-order terms
\[\mathcal{M}[J/\psi(1S_0^8) + g] = \frac{1}{\sqrt{N_c^2 - 1}} \{ \frac{\mathcal{M}[c\bar{c}(1S_0^8) + g]}{\sqrt{M_c}} <J/\psi|\mathcal{O}_0(1S_0^8)|0> \}
+ \frac{1}{2} \frac{\partial^2}{\partial |q|^2} \frac{\mathcal{M}[c\bar{c}(1S_0^8) + g]}{\sqrt{E_q}} <J/\psi|\mathcal{O}_1(1S_0^8)|0>,\]
\[\mathcal{M}[J/\psi(3P_2^8) + g] = \frac{1}{\sqrt{N_c^2 - 1}} \{ \frac{\partial}{\partial |q|} \frac{\mathcal{M}[c\bar{c}(3P_2^8) + g]}{\sqrt{E_q}} <J/\psi|\mathcal{O}_0(3P_2^8)|0> \}
+ \frac{1}{6} \frac{\partial^3}{\partial |q|^3} \frac{\mathcal{M}[c\bar{c}(3P_2^8) + g]}{\sqrt{E_q}} <J/\psi|\mathcal{O}_1(3P_2^8)|0>.\] (2.11)
To evaluate these expressions, we expand $M[\bar{c}c + g]/\sqrt{E_q}$ to the order of $|q|^2$ for $1S^0_0$ state (and $|q|^3$ for $3P^0_3$ state). Notice that, in the amplitude expressions, $|q|$ emerges in the forms of $E_q$ and the four momentum $q$. We now expand them to the order of $|q|^3$. It’s easy to abstain

$$E_q = m_c + \frac{|q|^2}{2m_c} + \mathcal{O}(|q|^4). \quad (2.12)$$

To expand $q$, we first write down the expression of $J/\psi$ momentum $P$ and $q$ in the rest frame of $J/\psi$,

$$P^\mu_{\text{c.m.}} = (2E_q, 0), \quad q^\mu_{\text{c.m.}} = (0, q) = |q|(0, n), \quad (2.13)$$

where $n$ is the unit direction vector. When boosting to an arbitrary frame, $q$ is dependant of $E_q$, etc. $|q|$, generally, so we can expand $q$ as

$$q^\alpha = |q|n^\alpha(E_q) = \left(|q|n^\alpha + \frac{|q|^3}{2m_c} \frac{\partial n^\alpha}{\partial E_q}\right)_{E_q \to m_c} + \mathcal{O}(|q|^5). \quad (2.14)$$

Form Eq.(2.7), in the amplitude calculation, it needs to integrate over the azimuthal angle of $q$ and it is convenient to compute the tensor integral as bellow

$$\int \frac{d\Omega}{\sqrt{4\pi}} Y^*_{00}(\hat{q}) = 1,$$

$$\int \frac{d\Omega}{\sqrt{4\pi}} Y^*_{00}(\hat{q}) q^\alpha = 0,$$

$$\int \frac{d\Omega}{\sqrt{4\pi}} Y^*_{00}(\hat{q}) q^\alpha q^\beta = \frac{q^2}{3} \Pi^{\alpha\beta} |E_q \to m_c| + \mathcal{O}(q^4),$$

$$\int \frac{d\Omega}{\sqrt{4\pi/3}} Y^*_{1L_z}(\hat{q}) q^\alpha = |q|c_{L_z}^\alpha = (|q|c_{L_z}^\alpha + \frac{|q|^3}{2m_c} \frac{\partial c_{L_z}^\alpha}{\partial E_q})_{E_q \to m_c} + \mathcal{O}(|q|^5),$$

$$\int \frac{d\Omega}{\sqrt{4\pi/3}} Y^*_{1L_z}(\hat{q}) q^\alpha q^\beta = 0,$$

$$\int \frac{d\Omega}{\sqrt{4\pi/3}} Y^*_{1L_z}(\hat{q}) q^\alpha q^\beta q^\gamma = \frac{|q|^3}{5} \left(\Pi^{\alpha\beta} c_{L_z}^\gamma + \Pi^{\alpha\gamma} c_{L_z}^\beta + \Pi^{\beta\gamma} c_{L_z}^\alpha\right)_{E_q \to m_c} + \mathcal{O}(|q|^5). \quad (2.15)$$

Insert Eq.(2.15) to Eq.(2.7), the final expressions of the NLO expanding of the amplitudes can be written as
\[
M_S = M_S \big|_{q \to 0, E_q \to m_c} + \frac{q^2}{2m_c} \frac{\partial M_S}{\partial E_q} \big|_{q \to 0, E_q \to m_c} + \frac{q^2}{6} (\Pi^{\alpha \beta} \frac{\partial^2 M_S}{\partial q^\alpha \partial q^\beta}) \big|_{q \to 0, E_q \to m_c} + \mathcal{O}(q^4),
\]

\[
M_P = \left[ (|q| \varepsilon_L^a \partial_2 M_S + \frac{|q|^3}{2m_c} \partial_2 E_q + \frac{|q|^3}{2m_c} \varepsilon_L^a \partial E_q \frac{\partial M_P}{\partial q^\alpha} \big|_{q \to 0}) E_q \right]_{E_q \to m_c} + \mathcal{O}(q^5),
\]

(2.16)

where, \( \Pi^{\alpha \beta} = -g^{\alpha \beta} + \frac{P^\alpha P^\beta}{m_c^2} \) and we define \( M_S \equiv M[\bar{c}c(1S^0) + g]/\sqrt{E_q} \) and \( M_P \equiv M[\bar{c}c(3P^8) + g]/\sqrt{E_q} \). Actually, since the terms of derivative of \( \varepsilon_L^a \) and the tensor in the amplitude, it is not easy to compute the next leading order amplitude generally. In the cross section expansion, it is convenient to compute the derivative of squared amplitudes instead of that make the derivative then square the amplitudes. We now evaluate the differential cross sections taking advantage of the expansion of the amplitudes and Eq.(2.11)

\[
d\sigma_{S,P} = \frac{1}{2s} d\phi_2 \sum |M[J/\psi + g]|^2,
\]

(2.17)

where the subscripts \( S, P \) note it’s for \( 1S^0 \) or \( 3P^8 \) states, and the \( \sum \) means obtaining the sum of NRQCD amplitudes square \( M^2 \) over the final-state color and polarization and the average over the ones of the initial states. \( d\phi_2 \) is the two-body phase space and using Eq.(2.10), we can also expand it to the next leading order as

\[
d\phi_2 = d\cos \theta \frac{\lambda^{1/2}(s, M^2_{J/\psi}, 0)}{16\pi s} = d\cos \theta \frac{1}{16\pi s} \left[ (s - 4m_c^2) - 4 \frac{\langle J/\psi | \mathcal{O}_1 \rangle |0\rangle}{\langle J/\psi | \mathcal{O}_0 \rangle |0\rangle} + \mathcal{O}(v^4) \right]
= d\phi_2^{(0)} \left[ 1 - \frac{4}{s(1-r)} \frac{\langle \mathcal{O}_1 \rangle}{\langle \mathcal{O}_0 \rangle} \right] + \mathcal{O}(v^4),
\]

(2.18)

where \( \lambda(x,y,z) = x^2 + y^2 + z^2 - 2(xy + yz + xz) \). The scalar parameter \( r \) is defined as \( r = 4m_c^2/s \). \( \langle \mathcal{O}_i \rangle \) represents \( \langle 0| \mathcal{O}_i |0\rangle = \langle 0| \mathcal{O}_i^\dagger |J/\psi\rangle \langle J/\psi | \mathcal{O}_i |0\rangle \) for \( i = 0, 1 \).

The final expressions of the NLO differential cross sections for the \( 1S^0 \) and \( 3P^8 \) state are
given as:
\[
\frac{d\sigma}{d\phi} = \frac{1}{8s} d\phi^2 \left\{ M_S M_S^* \left[ \langle O_0 \rangle - \frac{4}{s(1-r)} \langle O_1 \rangle \right] \right.
\]
\[
+ \left[ \frac{1}{2m_c} \frac{\partial (M_S M_S^*)}{\partial E_{q}} + \frac{1}{3} \Pi^{\alpha\beta} \text{Re} \left( \frac{\partial^2 M_S}{\partial q^\alpha \partial q^\beta} M_S^* \right) \right] \langle O_1 \rangle \right\} _{q \to 0, E_q \to m_c} + \mathcal{O}(v^4),
\]
\[
\frac{d\sigma}{d\phi} = \frac{1}{8s} d\phi^2 \left\{ \varepsilon_L^\alpha \varepsilon_L^\beta \frac{\partial M_P}{\partial q^\alpha} \frac{\partial M_P^*}{\partial q^\beta} \left[ \langle O_0 \rangle - \frac{4}{s(1-r)} \langle O_1 \rangle \right] \right.
\]
\[
+ \left[ \frac{1}{2m_c} \frac{\partial}{\partial E_{q}} (\varepsilon_L^\alpha \varepsilon_L^\beta) \frac{\partial M_P}{\partial q^\alpha} \frac{\partial M_P^*}{\partial q^\beta} \right] \langle O_1 \rangle \right\} _{q \to 0, E_q \to m_c}
\]
\[
+ \mathcal{O}(v^4).
\]

For the $e^+ e^- \to J/\psi + LH$ process, the total differential cross section can be computed by summing the $d\sigma_S$ and $d\sigma_P$. Our LO results are consistent with Ref.[? ]. The k factors of NLO differential cross sections to the LO ones are shown as
\[
k^{(2)}(1S^0_0) = 1 - \left( \frac{5}{6} + \frac{r}{1-r} \right) \langle v^2 \rangle
\]
\[
k^{(2)}(3P^8_0) = 1 - \left[ \frac{13 - 28r + 55r^2}{10(1-r)(1-3r)} + \frac{r}{1-r} \right] \langle v^2 \rangle
\]
\[
k^{(2)}(3P^1_0) = 1 - \left[ \frac{11 - 29r - 42r^2 + (11 - 33r + 42r^2)x}{10(1-r)[1 + 2r - (1 - 2r)x]} + \frac{r}{1-r} \right] \langle v^2 \rangle
\]
\[
k^{(2)}(3P^8_1) = 1 - \left[ \frac{7 + 11r - 188r^2 - 90r^3 + (7 - 49r + 112r^2 - 90r^3)x}{10(1-r)[1 + 6r + 6r^2 + (1 - 6r + 6r^2)x]} + \frac{r}{1-r} \right] \langle v^2 \rangle
\]

(20)
where
\[
\langle v^2 \rangle = \langle O_1 \rangle \frac{m_c^2}{\langle O_0 \rangle}.
\]

(21)

And $x$ represents $\cos^2 \theta$. With the approximation $r \to 0$, the ratios, the short-distance coefficients of relativistic corrections to leading order coefficients, are $-5/6, -13/10, -11/10,$ and $-7/10$ for $1S^0_0, 3P^8_0, 3P^1_0,$ and $3P^8_1$ states, respectively. These results are consistent with Ref.[? ]. The contributions from phase space are determined by $-r/(1-r)\langle v^2 \rangle$, which are at the order of $\mathcal{O}(rv^2)$. We also calculated the NLO QCD corrections[? ]. The process $e^+ e^- \to \Upsilon + LH$ can be treated as the process $e^+ e^- \to J/\psi + LH$ directly.

3 Numerical analysis

3.1 $e^+ e^- \to J/\psi + LH$ Process

For numerical analysis, we choose the input parameters following Ref.[? ] with $m_c = 1.55 GeV$, $\alpha_s(\mu) = 0.245 \pm 0.03$. And the branch rate: $B(\psi(2S) \to J/\psi) = 59.5\%$ [? ].
The cross section in $\mathcal{O}(\alpha_s, v^2)$ would be

$$
\sigma(e^+e^- \to J/\psi + LH) = \left[ (21 - 10\langle v^2 \rangle) \frac{\langle 0|\mathcal{O}(1S_0^8)|0 \rangle}{GeV^3} + (35 - 16\langle v^2 \rangle) \frac{\langle 0|\mathcal{O}(3P^8_J)|0 \rangle}{GeV^5} \right] pb. \tag{3.1}
$$

With the selection of the matrix elements $\langle v^2 \rangle$ ranging from 0.1 to 0.3, we could estimate the contributions from the relativistic corrections reduce the cross sections with $\alpha_s$ corrections by 5% to 14%. The experimental data for the total cross section of the non-$c\bar{c}$ inclusive $J/\psi$ production measured by Belle is

$$
\sigma[e^+e^- \to J/\psi + X_{non-cc}] = 0.43 \pm 0.13 pb \tag{3.2}
$$

But the calculations of the CS piece up to order of $\alpha_s$ imply to be enough to the data and leave little room to CO channels. And the contributions of the relativistic corrections to the CS piece also can raise the predicted values of the cross sections. And it would suppress the magnitudes of the CO LDMEs. In Ref., they considered the definition of the combined CO LDMEs,

$$
M_k = \langle 0|\mathcal{O}(1S_0^8)|0 \rangle + k\langle 0|\mathcal{O}(3P^8_J)|0 \rangle/m_c^2, \tag{3.3}
$$

and got an upper limit as shown in Eq.(3.4). With the negative corrections from relativistic effects, the limit value can enhance by a factor by 8% to 20% and reaches to $2.4 \times 10^{-2}GeV^3$, i.e.

$$
M_{4.0, v^2 \sim 0.3}^{\alpha_s, v^2} < 2.4 \times 10^{-2}GeV^3. \tag{3.4}
$$

Then we compare the region the LDMEs in Eq.(3.4) with the elements fitted with the hadron production processes in the paper. Butenschon and Kniehl fitted the inclusive $J/\psi$ production from KEKB, LEP II, RHIC, HERA, Tevatron, and LHC with data selected as $p_T > 1 GeV$ for photon production and two-photon scattering and $p_T > 3 GeV$ for hadron production. Chao, Ma, Shao, Wang, and Zhang fitted the production and polarization of $J/\psi$ from the Tevatron with $p_T > 7 GeV$. Gong, Wan, Wang, and Zhang fitted the production and polarization of prompt $J/\psi$, $\psi(2S)$, and $\chi_{cJ}$ from the Tevatron and LHC. We show the value of LDMEs of $J/\psi$ and $\psi'$ in Table 1. And the LDMEs of $^3S_1^1$ are set as :

$$
\langle 0|\mathcal{O}(J/\psi(3S_1^1))|0 \rangle = 1.16 GeV^3 \tag{3.5}
$$

We can find that most of the LDMEs fitted with the hadron production processes does not in the region of Eq.(3.4). The $\sigma[J/\psi + LH]$ measured by Belle give a very strong restriction to the LDMEs.

We give the cross section as a function of energy in the following graphs. For all the figures below the solid lines present leading order cross section, the dash-dot lines are the $O(\alpha_s, v^2)$ correction cross section, and the dash lines present the $O(\alpha_s)$ correction cross section. The
blue area show the uncertainty of $\alpha_s$ and $v^2$ where $\alpha_s = 0.245 \pm 0.03$ and $v^2 = 0.2 \pm 0.1$. Firstly we give the short distance of the cross section as a function of energy without any LDMEs in Figure 1. From this figure we know the short distance of the cross section $3 P_J^8$ is 10 times or more greater than $1 S_0^8$. So the fitted LDMEs of $3 P_J^8$ must be small. The contribution from $3 S_1^1$ are small, so it is ignored here.

The cross section of the $e^+e^- \rightarrow e^+e^- \rightarrow e^+e^- \rightarrow c\bar{c}(1 S_0^8) + g$ and $e^+e^- \rightarrow c\bar{c}(3 P_J^8) + g$ and the total cross section with the LDMEs fitted by three groups are listed in Table 1. Figure 2 and 3 show the cross section as a function of the energy of center of mass system, and the value of LDMEs are corresponding to the first line and the second line in Table 1 which were fitted by Butenschoen and Kniehl[8]. In Figure 2 the LDMEs are fitted without the feed down contribution. But in Figure 3, the feed down contribution are taken into account. These two group of LDMEs have a negative value of $3 P_J^8$. For the short distance of P-wave is very large, the total cross section give negative values. Figure 4 is the cross section correspond the LDMEs fitted by Gong, et al.[8] in the last line of the Table 1. They consider the contribution from $\psi(2s)$, and the LDMEs of $3 P_J^8$ is small and negative. The total cross section have a greater uncertainty than other groups. Figure 5 and 6 are the cross section with the LDMEs fitted by Chao, et al.[9] correspond the third, the fourth, and the fifth lines in Table 1. The LDMEs in the third line of the Table 1 have a positive value of $3 P_J^8$ corresponding to Figure 5. And the other two line are two special cases, which are show in Figure 6. The total cross section have a larger value.

We give the value of the cross section in Table 2 correspond to the three groups LDMEs. From the table we know the contribution of $3 S_1^1$ is very small as in Figure 1. The contribution from CS is small and can be neglected. However the short distance of the the process $e^+e^- \rightarrow J/\psi(1 S_0^8) + g$ at BESIII in the Table 2 is about 500. When we product the LDMEs of the cross section of $e^+e^- \rightarrow J/\psi(1 S_0^8) + g$ the value is enough to satisfy the experiment for all the three groups, and they do not need to consider the contribution from $\langle 3 P_J^8 \rangle$ any more. On the other hand, the short distance of $\langle 3 P_J^8 \rangle$ channel is so large up to $10^4$, so the fitted LDMEs of $\langle 3 P_J^8 \rangle$ must be very small close to 0. We product the LDMEs of $\langle 3 P_J^8 \rangle$ which fitted by the

| $\langle 0 | O^{J/\psi(3 P_J^8)} | 0 \rangle / m_c^2$ | $\langle 0 | O^{J/\psi(3 P_J^8)} | 0 \rangle / m_c^2$ | $\langle 0 | O^{J/\psi(3 P_J^8)} | 0 \rangle / m_c^2$ |
|---------------|---------------|---------------|
| Butenschoen, et al. with feed down | 4.97 | -1.51 | -- |
| Chao, et al. set1 | 8.9 | 0.52 | -- |
| set2 | 0 | 2.25 | -- |
| set3 | 11 | 0 | -- |
| Gong, et al. | 9.7 | -0.89 | -0.012 |

The value of $\langle 0 | O^{J/\psi(3 P_J^8)} | 0 \rangle / m_c^2$ is a little difference with the References for $m_c = 1.55$ GeV here.
Figure 1. The cross section of inclusive CO $J/\psi$ production.
three groups to the short distance part, show in Figure 2, 3, 4, 5, 6 and Table 2. A lot of sets of the cross sections are negative. We can see the LDMEs of \(^3 P^3_J\) not only small but also have a great uncertainty. So we can conclude that the LDMEs of P-wave are very small.

### 3.2 \(e^+e^- \rightarrow \Upsilon + LH\) Process

For the process \(e^+e^- \rightarrow \Upsilon + LH\), we consider the feed down contribution from \(\Upsilon(2S)\) and \(\Upsilon(3S)\). And neglected the contribution from the CS fock state \(^3 S^1\) as in the \(J/\psi\) process. We choose the input parameters as \(m_b = 4.75 GeV\), \(\alpha_s(\mu) = 0.245 \pm 0.03\). And the branch rate: \(B(\Upsilon(2S) \rightarrow \Upsilon(1S)) = 26.52\%\) \(^{[?]}\), \(B(\Upsilon(3S) \rightarrow \Upsilon(1S)) = 6.57\%\) \(^{[?]}\). The value of LDMEs had been fitted by Gong, et al.\(^{[?]}\), which are given in Table 3.

**Table 2.** The cross sections if \(pb\) for \(J/\psi\) production at BESIII. The label "Butenschoen, et al.\(^{[?]}\), "Chao, et al.\(^{[?]}\), and "Gong, et al.\(^{[?]}\) present the LDMEs in the numerical cross sections calculation are chosen as which are fitted by Butenschoen, et al.\(^{[?]}\), Chao, et al.\(^{[?]}\), and Gong, et al.\(^{[?]}\), respectively.

| \(\sqrt{s}\) (GeV) | short distance coefficients | \(\langle^3 S^1_1\rangle\) GeV\(^{-3}\)pb | \(\langle^1 S^0_0\rangle\) GeV\(^{-3}\)pb | \(\langle^3 P^8_J\rangle\) GeV\(^{-5}\)pb |
|-----------------|-----------------------------|------------------|------------------|------------------|
| 4.0             | 2.32                        | 513.3 ± 112.9    | 10620.6 ± 6809.9 |
| 4.2             | 2.25                        | 520.6 ± 100.0    | 8637.9 ± 2292.0  |
| 4.4             | 2.16                        | 499.2 ± 88.1     | 6118.0 ± 1136.0  |
| 4.6             | 2.06                        | 464.3 ± 77.3     | 4368.8 ± 678.3   |
| 4.8             | 1.95                        | 424.6 ± 67.6     | 3202.0 ± 449.9   |

| \(\sqrt{s}\) (GeV) | Butenschoen, et al. | Chao, et al. | Gong, et al. |
|-----------------|---------------------|--------------|--------------|
|                 | \(\langle^1 S^0_0\rangle\) \(pb\) | \(\langle^3 P^8_J\rangle\) \(pb\) | \(\langle^1 S^0_0\rangle\) \(pb\) | \(\langle^3 P^8_J\rangle\) \(pb\) |
| 4.0             | 13.7 ± 2.9          | −63.6 ± 15.5  | 40.0 ± 8.6   | 39.2 ± 9.5   | 43.6 ± 9.4 | −54.2 ± 30.5 |
| 4.2             | 12.8 ± 2.6          | −43.5 ± 9.2   | 37.6 ± 7.5   | 26.8 ± 5.7   | 40.9 ± 8.1 | −31.6 ± 12.5 |
| 4.4             | 11.8 ± 2.2          | −30.7 ± 6.0   | 34.4 ± 6.5   | 18.9 ± 3.7   | 37.5 ± 7.1 | −22.3 ± 7.2   |
| 4.6             | 10.6 ± 1.9          | −22.4 ± 4.1   | 31.1 ± 5.6   | 13.8 ± 2.5   | 33.8 ± 6.1 | −16.6 ± 4.7   |
| 4.8             | 9.5 ± 1.7           | −16.8 ± 2.9   | 27.8 ± 4.9   | 10.3 ± 1.8   | 30.3 ± 5.3 | −12.7 ± 3.3   |

**Table 3.** The LDMEs for \(\Upsilon(1S), \Upsilon(2S), \Upsilon(3S)\) which was fitted by Gong, et al.\(^{[?]}\).

| \(H\)     | \(\langle 0|O^H(\langle^1 S^0_0\rangle)|0\rangle/10^{-2}GeV^3\) | \(\langle 0|O^H(\langle^3 P^8_J\rangle)|0\rangle/m_b^2/10^{-2}GeV^3\) |
|-----------|--------------------------------------------------|--------------------------------------------------|
| \(\Upsilon(1S)\) | 13.9                                             | 0.004                                             |
| \(\Upsilon(2S)\) | 6.0                                              | −0.003                                            |
| \(\Upsilon(3S)\) | 2.8                                              | −0.001                                            |
We give the cross section of $e^+e^- \to \Upsilon + LH$ use the LDMEs fitted by Gong, et al.[?] in Figure 7. The cross section of $e^+e^- \to \Upsilon + LH$ is so small that we have produced 100 to the cross section for convenient. We know the cross section of P wave is smaller than s-wave about an order. The cross section of P wave is a small negative value and it can give little contribution to the total cross section. Although, the value of P wave is positive, it is too small to give contribution to the total cross section. So we can get the same conclusion as $e^+e^- \to J/\psi + g$ process, that the LDMEs of P-wave channel are so small that tend to zero. We also give the value of cross section in Table 4 below, to see the conclusion clearly.

**Table 4.** The cross section for $e^+e^- \to \Upsilon + g$ P-wave and S-wave with the LDMEs fitted by Gong, et al. [?].

| $\sqrt{s}$ (GeV) | short distance | Gong, et al. |
|------------------|----------------|--------------|
|                  | $\langle S_0^8 \rangle$ | $\langle P_3^8 \rangle$ | $\langle S_0^8 \rangle$ | $\langle P_3^8 \rangle$ |
| $GeV$            | $GeV^{-3} pb$ | $GeV^{-5} pb$ | $x10^{-2} pb$ | $x10^{-2} pb$ |
| 10.0             | 0.3 ± 0.5     | -22.9 ± 4.5  | 3.3 ± 5.9     | 15.4 ± 30.8   |
| 10.2             | 0.7 ± 0.5     | -8.7 ± 3.0   | 7.3 ± 5.7     | -6.6 ± 14.1   |
| 10.4             | 0.9 ± 0.5     | 6.3 ± 11.6   | 10.4 ± 5.5    | -4.2 ± 7.8    |
| 10.6             | 1.2 ± 0.5     | 5.3 ± 9.5    | 13.0 ± 5.3    | -3.7 ± 5.9    |
| 10.8             | 1.4 ± 0.4     | 7.6 ± 5.7    | 14.9 ± 5.1    | -5.0 ± 3.6    |
| 11.0             | 1.6 ± 0.4     | 7.1 ± 4.0    | 16.3 ± 4.9    | -5.1 ± 2.5    |
| 11.2             | 1.7 ± 0.4     | 7.0 ± 2.7    | 17.2 ± 4.7    | -4.4 ± 1.8    |
| 11.4             | 1.8 ± 0.4     | 6.3 ± 2.0    | 17.9 ± 4.5    | -3.9 ± 1.3    |
| 11.6             | 1.9 ± 0.4     | 5.6 ± 1.5    | 18.2 ± 4.3    | -3.4 ± 1.0    |
| 11.8             | 1.9 ± 0.4     | 4.9 ± 1.2    | 18.4 ± 4.1    | -2.9 ± 0.8    |
| 12.0             | 1.9 ± 0.4     | 4.3 ± 0.9    | 18.3 ± 3.9    | -2.6 ± 0.6    |
| 12.2             | 1.9 ± 0.3     | 3.8 ± 0.8    | 18.2 ± 3.7    | -2.2 ± 0.5    |
| 12.4             | 1.9 ± 0.3     | 3.3 ± 0.6    | 18.0 ± 3.6    | -1.9 ± 0.4    |
| 12.6             | 1.8 ± 0.3     | 2.9 ± 0.5    | 17.6 ± 3.4    | -1.7 ± 0.3    |
| 12.8             | 1.8 ± 0.3     | 2.6 ± 0.4    | 17.3 ± 3.3    | -1.5 ± 0.3    |
| 13.0             | 1.7 ± 0.2     | 2.3 ± 0.4    | 16.9 ± 3.1    | -1.4 ± 0.2    |

4 Summary

The NLO radiative corrections and relativistic corrections to the inclusive heavy quarkonium production in $e^+e^- \to J/\psi + LH$ and $e^+e^- \to \Upsilon + LH$ are calculated within NRQCD frame. And the cross section with different the long matric elements has been given. We find that the inclusive heavy quarkonium production process in $e^+e^-$ annihilation can give a strong restrictions on the CO LDMEs. And the LDMEs of P-wave channel for $J/\psi(\psi')$ would be
very small. More measurement of $J/\psi(\psi')$ and $\Upsilon$ production associated with light hadron at BESIII, Belle, and BelleII can give more information about the CO LDMEs.
Figure 2. The cross section of inclusive CO $J/\psi$ production without the feed down contribution in LDMEs. The figures are $1S_{0}^{[8]}$ piece, $3P_{J}^{[8]}$ piece, and the total cross section, respectively. The LDMEs are chosen as the first row in Table 1, which are fitted by Butenschoen, et al. [?].
\[ \sigma(e^+e^- \rightarrow J/\psi(g)) \sqrt{s} \]

\[ \alpha_s = 0.245 \pm 0.03, \quad \langle v_2 \rangle = 0.2 \pm 0.1 \]

\( \text{LO} \)

\( \text{NLO}(\alpha_s) \)

\( \text{NLO}(\alpha_s + v_2^2) \)

**Figure 3.** The cross section of inclusive CO \( J/\psi \) production with the feed down contribution in LDMEs. The LDMEs are chosen as the second row in Table 1, which are fitted by Butenschoen, et al. [?].
Figure 4. The cross section of inclusive CO \( J/\psi \) production with the feed down contribution in LDMEs. The LDMEs are chosen as the last row in Table 1, which are fitted by Gong, et al. [?].
Figure 5. The cross section of inclusive CO $J/\psi$ production without the feed down contribution in LDMEs. The LDMEs are chosen as the "set1" row in Table 1, which are fitted by Chao, et al. [?].
Figure 6. The cross section of inclusive CO $J/\psi$ production without the feed down contribution in LDMEs. The LDMEs are chosen as the "set2" and "set3" row in Table 1 respectively, which are fitted by Chao, et al.\cite{?}. 
Figure 7. The cross section of inclusive CO Υ production. The LDMEs are chosen as which is fitted by Gong, et al. [8].
\[ \sigma(e^+ e^- \to \Upsilon(1S)) \times 100 \sqrt{s} \]

\[ \alpha_s = 0.245 \pm 0.03, \quad \langle v^2 \rangle = 0.2 \pm 0.1 \]

\[ \text{LO} \quad \text{NLO}(\alpha_s) \quad \text{NLO}(\alpha_s + v^2) \]
\( \sigma(e^+e^- \to \Upsilon(3P_J)g) \times 100 \)
$\sigma(e^+e^- \rightarrow \Upsilon g) \times 100$

$\alpha_s = 0.245 \pm 0.03$, $\langle v^2 \rangle = 0.2 \pm 0.1$

LO

NLO($\alpha_s$)

NLO($\alpha_s + v^2$)