Entanglement dynamics for two harmonic oscillators coupled to independent environments

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Abstract
In this paper we study the entanglement evolution between two harmonic oscillators having different free frequencies each leaking into an independent bath. We use an exact solution valid in the weak coupling limit and in the short time non-Markovian regime. The reservoirs are identical and characterized by an Ohmic spectral distribution with Lorentz–Drude cut-off. This work is an extension of the case reported by Vasile et al. (2009 Phys. Rev. A 80 062324) where the oscillators have the same free frequency.

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(Some figures in this article are in colour only in the electronic version.)

1. Introduction
Entanglement is considered a fundamental feature and a resource in the context of quantum computation and information physics [2]. However, entanglement between quantum systems is easily destroyed when their unavoidable interaction with the external environment is taken into account [3]. The features and timing of the disentanglement phenomenon depend strongly on the particular physical system and environment under investigation. However, even if disentanglement cannot be avoided, quantum information and computation tasks can still be performed when the disentanglement time is much longer than the time needed to run the quantum task. From this point of view, it becomes of fundamental importance to investigate the mechanisms and features of disentanglement in those systems used in the context of quantum computation and information.

Continuous variable (CV) quantum systems are some of the possible candidates for quantum protocols [4]. We consider here the problem of two independent harmonic oscillators each one leaking in a bosonic structured reservoir [5–11]. This situation has been considered also by the present author in [1]. However here, as an extension, we study the case of oscillators with different frequencies, initially in a twin-beam entangled (TWB) state. Our aim is to investigate the entanglement evolution of two identical high-$T$ Ohmic reservoirs with a Lorentz–Drude cut-off. In this kind of system, whose dynamics are determined by a generalized Hu–Paz–Zhang master equation [12], all the initial Gaussian states remain Gaussian during the evolution. For this reason we are entitled to use the separability criterion for CV systems, introduced by Simon [13], as a marker of the entanglement between the oscillators.

The paper is organized as follows. In section 2, we introduce the physical model and its master equation in the weak coupling limit. In section 3, we review the concept of the two-mode Gaussian squeezed state, or TWB state, and provide the solution to the master equation. In section 4, we introduce the separability criteria for bipartite continuous variable systems and the Lorentz–Drude spectral function. Moreover, we study the entanglement dynamics as a function of the initial amount of entanglement and the relative values of the free oscillator frequencies and the spectral function cut-off frequency. Finally, in section 5, we give a brief summary of our work.

2. The master equation
Our system consists of two non-interacting quantum harmonic oscillators with frequencies $\omega_1$ and $\omega_2$. Each oscillator is coupled to its own bosonic reservoir. The total Hamiltonian
can be written as
\[ H = \sum_j \hbar \omega_j a_j^\dagger a_j + \sum_{j,k} \hbar \omega_{jk} b_{jk}^\dagger b_{jk} \\
+ \sum_{j,k} \gamma_{jk} (a_j + a_j^\dagger)(b_{jk} + b_{jk}^\dagger), \]
(1)
where \( j = 1, 2 \), \( \omega_{1k} \) and \( \omega_{2k} \) are the frequencies of the reservoir modes, \( a_j \) (\( a_j^\dagger \)) and \( b_{jk} \) (\( b_{jk}^\dagger \)) are the annihilation (creation) operators of the system and reservoir harmonic oscillators, respectively, and \( \gamma_{jk} \) describes the coupling between the \( j \)th oscillator and the \( k \)th mode of its environment. We assume reservoirs with the same spectral structure and equally coupled to the oscillators. The dynamics of the two oscillators can be described through the following non-Markovian local in time master equation [12]:
\[ \rho(t) = \sum_j \frac{1}{i \hbar} [H_0^j, \rho(t)] - \Delta_j(t) [X_j, [X_j, \rho(t)]] \\
+ \Pi_j(t) [X_j, \{ P_j, \rho(t) \}] \\
+ \frac{i}{2} r_j(t) [X_j, \rho(t)] - i \gamma_j(t) [X_j, \{ P_j, \rho(t) \}], \]
(2)
where \( \rho(t) \) is the reduced density matrix, \( H_0^j \) is the free Hamiltonian of the \( j \)th oscillator, and \( X_j = \frac{1}{\sqrt{2}}(a_j + a_j^\dagger) \), \( P_j = \frac{1}{\sqrt{2}}(a_j^\dagger - a_j) \) are the quadrature operators. Interaction with the reservoirs is taken into account through the time-dependent coefficients of equation (2). The quantities \( \Delta_j(t) \) and \( \Pi_j(t) \) describe diffusion processes, \( \gamma_j(t) \) is a damping term and \( r_j(t) \) renormalizes the free oscillator frequencies \( \omega_j \). For environments in thermal equilibrium and in the weak coupling limit in \( (r_j(t) \text{ being negligible)}, they read
\[ \Delta_j(t) = \alpha^2 \int_0^t ds \int_0^\infty d\omega J(\omega) [2N(\omega) + 1] \cos(\omega s) \cos(\omega_j s) \\
\Pi_j(t) = \alpha^2 \int_0^t ds \int_0^\infty d\omega J(\omega) [2N(\omega) + 1] \cos(\omega s) \sin(\omega_j s), \\
\gamma_j(t) = \alpha^2 \int_0^t ds \int_0^\infty d\omega J(\omega) \sin(\omega s) \sin(\omega_j s). \]
(3)

3. Gaussian states and separability conditions

As initial states for our system we consider the class of two-mode squeezed states (or TWBs) obtained by applying a two-mode squeezing operator to the vacuum state of the oscillators [4]. They belong to the class of Gaussian states characterized by a Gaussian characteristic function \( \chi_0(\Lambda) = \exp\{-\frac{1}{2} \Lambda^T \sigma_0 \Lambda\} \), where \( \sigma_0 \) is the covariance matrix and \( \Lambda^T = (x_1, p_1, x_2, p_2) \). For a TWB state we have
\[ \sigma_0 = \begin{pmatrix} A_0 & C_0 \\ C_0^T & A_0 \end{pmatrix}, \]
(4)
with \( A_0 = \text{Diag}(a, a) \), \( C_0 = \text{Diag}(c, -c) \), \( a = \cosh(2r)/2 \) and \( c = \sinh(2r)/2 \). The TWB state is thus determined only by the squeezing parameter \( r \), which also determines the initial amount of entanglement. The bigger the value of \( r \), the larger the initial entanglement. Using the characteristic function solution [14], it is trivial to verify that the characteristic function at time \( t \) maintains its Gaussian character \( \chi_t(\Lambda) = \exp\{-\frac{1}{2} \Lambda^T \sigma_t \Lambda\} \), where
\[ \sigma_t = \begin{pmatrix} A_t^{(1)} & C_t \\ C_t^T & A_t^{(2)} \end{pmatrix}, \]
(5)
with
\[ A_t^{(i)} = A_0 e^{-\Gamma_i t} + \left( \Delta_0^{(i)} + \Delta_0^{(i)} - \Pi_0^{(i)} - \Delta_0^{(i)} + \Pi_0^{(i)} \right), \]
(6)
\[ C_t = \begin{pmatrix} c e^{-\Gamma_1 t} \sin(\omega_1 + \omega_2 t) & c e^{-\Gamma_1 t} \sin(\omega_1 + \omega_2 t) \\ -c e^{-\Gamma_2 t} \sin(\omega_1 + \omega_2 t) & -c e^{-\Gamma_2 t} \sin(\omega_1 + \omega_2 t) \end{pmatrix}, \]
(7)
with \( \Gamma_i(t) = 2 \int_0^t \gamma(s) ds \). Moreover, because we are interested in short time non-Markovian dynamics only, we defined \( \Delta_0^{(i)} = \int_0^t \Delta(s) ds \), and the following secular coefficients: \( \Delta_\alpha^{(i)}(t) \simeq \int_0^t \Delta_\alpha(s) \cos(2\omega_\alpha(t-s)) ds \). Details of the calculations can be found in [1, 11, 14]. If a suitable environment spectrum is provided, all previous coefficients can be evaluated analytically in the high-temperature limit. This is the case for the Lorentz–Drude Ohmic distribution we introduce in the next section. In equations (6) and (7), we omitted the explicit time dependence for all the appearing coefficients.

4. Entanglement dynamics

In this section, we investigate the entanglement dynamics between the oscillators using the separability criterion of Simon [13]. This criterion is well suited in the context of two-mode Gaussian states because it represents a necessary and sufficient condition for separability and it depends only on the analytic form of the covariance matrix. In the case of the time-dependent and non-symmetric covariance matrix (5), the Simon criterion is equivalent to the following algebraical inequality [5]:
\[ S(t) = \det[A_t^{(1)} A_t^{(2)}] + \frac{1}{4} - [\det C_t] \geq 0 \]
(8)
with
\[ J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \]
(9)
\( S(t) \) is called the separability function. When \( S(t) < 0 \) the state is entangled; otherwise it is separable.

We now consider a particular situation with two reservoirs in thermal equilibrium at high temperature \( T \) \( (h\omega_0 \ll k_B T) \) characterized by a Lorentz–Drude Ohmic spectral function [3]:
\[ J(\omega) = \frac{\omega^2}{\pi} \frac{\omega}{\omega^2 + \omega_c^2}. \]
(10)
where \( \omega_0 \) is the cut-off frequency of the distribution. Hitherto we fix the temperature through the condition \( k_B T / \hbar \omega_0 = 100 \) and the coupling constant to the value \( \alpha = 0.1 \). Moreover, we use a dimensionless time variable \( \tau = \omega_0 t \) for the evolution of the separability. With these choices we have only two free parameters \( x_i = \omega_0 / \omega_i \) (\( i = 1, 2 \)), namely resonance parameters, describing the relative positions of the free oscillator frequencies \( \omega_0 \) with respect to the spectral cut-off frequency \( \omega_0 \). In [1] it has been observed the existence of two different resonance parameter regions characterized by different qualitative and quantitative behaviours of entanglement, \( x_1 = x_2 \geq 1 \) and \( x_1 = x_2 \ll 1 \). Exploiting this result we investigate three different regimes: (i) \( x_1, x_2 \geq 1 \), (ii) \( x_1, x_2 \ll 1 \) and (iii) \( x_1 \ll 1, x_2 \geq 1 \).

A second degree of freedom for our analysis is the choice of the amount of entanglement in the initial TWB state, given by the value of the squeezing parameter \( r \). In principle we do not have any limitation in the choice of the value of \( r \); however, in real situations it is not possible to realize two-mode squeezed states with \( r \geq 2 \) [4]. On the other hand, when \( r \approx 0.01 \) the amount of entanglement is so small that, at least in the high temperature limit, disentanglement is a very fast process almost independent of \( x_1 \) and \( x_2 \). For these reasons we restrict our investigation to the intermediate region of \( 0.01 \leq r \leq 1 \).

We start the analysis considering the case \( x_1, x_2 \geq 1 \). We show the entanglement evolution in figure 1 where we fixed the squeezing parameter to the value \( r = 1 \). In this regime, the dynamics is characterized by entanglement sudden death (ESD) appearing already in the short time non-Markovian region for any value of \( r \). We also fixed the value of the parameter \( x_1 = 1 \) and plotted three curves corresponding to \( x_2 = 1 \) (solid blue), \( x_2 = 10 \) (dashed red) and \( x_2 = 100 \) (dotted black). We observe that the disentanglement time \( \tau_{\text{dis}} \), defined as \( S(\tau_{\text{dis}}) = 0 \), increases only slightly as the adimensional frequency detuning \( \Delta x = |x_1 - x_2| \) increases.

The case \( x_1, x_2 \ll 1 \) is reported in figure 2. As emphasized in detail in [1], this regime is characterized by a more variegated entanglement dynamics, showing oscillations, ESD and revivals. We report the situation relative to an initially small amount of entanglement corresponding to \( r = 0.04 \) and \( x_1 = 0.1 \). The three curves correspond to \( x_2 = 0.1 \) (solid blue), \( x_2 = 0.2 \) (dashed red) and \( x_2 = 0.3 \) (dotted black). In all cases, there are entanglement oscillations and ESD, while an entanglement revival is also present when \( x_2 = 0.2 \). As in the previous regime of figure 1, we notice that disentanglement time is larger when \( \Delta x \) increases with a fixed value of \( x_1 \). However, here the differences in disentanglement time are much more evident. Also oscillations are damped out as detuning increases.

Finally, we look at the regime \( x_1 \ll 1 \) and \( x_2 \geq 1 \) whose results are shown in figure 3 with the choice \( r = 0.1 \). The blue solid line corresponds to \( x_1 = x_2 = 0.1 \), the red dashed line to \( x_1 = 0.1, x_2 = 1 \) and the black dotted line to \( x_1 = x_2 = 1 \).
From our investigation we can extract two main results. When the oscillators have different frequencies ($\omega_1 \neq \omega_2$), the entanglement dynamics show behaviour that is intermediate between the cases of frequencies both equal to $\omega_1$ and both equal to $\omega_2$. Moreover, the leading role in the disentanglement process seems to be played by the oscillator with higher resonance parameter $x$, which forces the entanglement to evolve in a way similar to the case of both parameters equal to the larger of the two, as shown in figure 3.

5. Summary

In this paper, we extended the results obtained in [1] to the case in which the two oscillators have different frequencies and therefore they interact with different parts of the reservoir spectra. We considered the particular case of an Ohmic distribution in the high temperature limit for different initial TWB states and different values of the free oscillator frequencies. Quantitative differences in the entanglement dynamics can be observed in particular when both resonance parameters are in the regime $x \ll 1$, where the evolution changes also for a small value of $\Delta x$. On the contrary, the dynamics is not strongly affected in the opposite regime even for large detuning. Moreover, when the two frequencies lie in the opposite parameter’s regime, the dynamics is similar to the case $x \gg 1$, characterized by a lack of oscillations and fast ESD.

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