Notes on pulse-width modulation appropriate for a sophomore-level electronics or programming class

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Pulse-width modulation is a nice way to create pseudo analog outputs from a microcontroller (eg, an Arduino), which is limited to digital output signals. The quality of the analog output signal depends on the filter one uses to average out the digital outputs, and this filter can be analyzed with sophomore-level knowledge of resistor-capacitor circuits in charging and discharging state. These notes were used as a text supplement in a sophomore/junior level Microcontroller programming class for second and third-year Physics majors. I don’t know where this material is normally treated in the Physics curriculum and figured others might find the resource useful. Comments welcome!

I. BACKGROUND

Most of the world runs on analog measurement levels. The amount of coffee in your cup doesn’t resolve to an integer number of milliliters, and the sine wave in a residential electric line certainly doesn’t have an integer-defined voltage vs time graph.

What does “analog” mean? One way of thinking about the term is to divide the quantity you’re measuring, eg, the volume of coffee in your cup, $V_c$, by some smallest measuring unit, eg $dV = 0.5 mL$ in a graduated cylinder. If the division always resolves to an integer, the measurement is integer, and if there’s often (always) a remainder, the measurement is analog.

FIG. 1. A simple and energy-inefficient way to dim an LED is to use a potentiometer in series with the LED. Roughly half of the power is thrown away as waste heat.

The microcontroller we’ve been using in class, an Arduino Uno, only has digital outputs. Pin 13 can be on, 5v, or off, 0v, and the intermediate state, 2.5v, is impossible. If you want to dim an LED’s brightness, one way to do it is to wire the LED in series with a potentiometer, which mechanically maps electrical resistance to the rotational position of a control. As you use this “dial a resistor,” you mechanically change the (analog) resistance that’s in series with the LED. A higher resistance means less current flows through the series circuit and thus fewer electrons flow through the PN junction in the LED, which means fewer photons are emitted from the LED, which means fewer photons hit your retina, which your brain interprets as “dimmer.”

There are drawbacks to this approach: the potentiometer requires human input, and the series resistor also wastes energy as heat. One workaround to this LED dimming problem we’ve already used in lab is to toggle the LED anode from on to off at a frequency that’s faster than a human eye-brain interface can keep up with. In practice, if the LED pin is run in an on-off-on-off cycle of about 1ms per on or off period, most people see a 50% dimmed LED rather than a blinking LED. Safety note: recall that varying the on/off period from 1ms to about 20ms can induce seizures in some people - be careful!

If we want the LED to be a bit brighter, the on state can be held for 3ms while the off state remains at 1ms. Or, if we need the LED to be very dim, we can hold the on state for 500us and then the off state for 3500us. Formally, we can implement this as a “Pulse Width Modulation” routine, where the voltage the microcontroller sends is on/high/5v for a time $t_1$ and then off/low/0v for a time $t_2$. The overall period of the cycle is $T = t_1 + t_2$, graphically shown in figure 2, or mathematically, one cycle is given,

$$v(t) = 5v \text{ for } 0 < t < t_1$$
$$= 0v \text{ for } t_1 < t < t_1 + t_2. \quad (1)$$

Note, equation 1 is usually periodic, as $T = t_1 + t_2$ is usually quite short compared to the relaxation time $T_R$ of the phenomena we’re trying to drive. What is a relaxation time? When power is removed, a computer cooling fan will take a few seconds to spin down to rest. The length of time probably depends on the fan’s age, accumulated dust, and the quality of the bearings in the fan. Thus for the fan, $T_R \approx 5s$. In your eye, $T_R$ is probably on the order of a few milliseconds, because if a phenomena is faster than $\approx 5ms$, the motion appears smooth, while motions slower than this time are seen discretely.
Mathematically it can be thought of as a sum of $\Theta(t)$ functions. The microcontroller’s “rail” voltage, $V_{cc}$ is assumed to be the output level, i.e. $V_{cc} = 5v$.

Technically, a better definition of $T_R$ is to look for an exponential response, $\exp[-t/T_R]$, in the behavior of an object when the driving force is removed. To be “smoothly” controlled by PWM, the PWM period, $T = t_1 + t_2$ should be significantly less than exponential relaxation/decay time, i.e. $T \ll T_R$.

In an automobile’s internal-combustion engine, one of the pistons fires with each engine revolution. If the car’s motor is running at 2500 RPM, a piston will fire and push on the crankshaft at a rate of $\approx 42$ combustions each second. So, for an engine, $T \approx \frac{1}{42 \text{ push/rev}} \approx 24 \text{ms}$. Automotive engines are built with large, heavy, steel flywheels attached to the crankshaft. The flywheel’s angular momentum gives the system a control momentum that is to look for an exponential response, $\exp[-t/T_R]$, in the behavior of an object when the driving force is removed. To be “smoothly” controlled by PWM, the PWM period, $T = t_1 + t_2$ should be significantly less than exponential relaxation/decay time, i.e. $T \ll T_R$.

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The electrical PWM function, equation 1, has an average value, $\langle V \rangle$, that is straight-forward to compute. Skipping the integral, we have:

$$\langle V \rangle = \frac{1}{t_1 + t_2} (V_{cc} \cdot t_1 + 0 \cdot t_2) = V_{cc} \cdot \frac{t_1}{t_1 + t_2} \quad (2)$$

The amount of time the wave is held high corresponds to a “duty-cycle” level, $d = \frac{t_1}{t_1 + t_2}$, which simplifies the expression for average output, $\langle V \rangle = d \cdot V_{cc}$.

**II. A BASIC PWM IMPLEMENTATION IN C**

Implementation of equation 1 in Arduino C can take the form of a for loop with repeated delays. The loop starts with the output pin high, and then after delays corresponding to $t_1$ have elapsed, the output pin is pulled low for the remainder of the PWM cycle.

Listing 1. Simplest PWM routine

```c
int output_pin = 13;

void setup() {
    pinMode(output_pin, OUTPUT);
    digitalWrite(13, LOW);
}

void loop() {
    int j;
    int num_pwm_bins = 32;
    int brightness = 5;
    digitalWrite(output_pin, HIGH);
    for (j = 0; j < num_pwm_bins; j++) {
        if (j == brightness) {
            digitalWrite(output_pin, LOW);
        }
        delayMicroseconds(20);
    }
}
```

The rough function of this code follows. Pin 13 is used to output the PWM signal. In the Arduino environment the loop() function, line 7, runs in an infinite loop. Each iteration of loop() generates a single PWM wave, equation 1, and is broken into 32 sections via a for loop, line 14. Each section consists of a 20us delay and a check to see if the duty-cycle for the pwm wave has been met. The function generates a duty cycle of $d \approx \frac{\text{brightness}}{\text{num_pwm_bins}}$, because after “brightness” number of sections have elapsed, line 15, the output pin is pulled low.

In this implementation, $t_1 \approx 20us \cdot 5 = 100us$, $t_2 \approx 20us \cdot 27 = 540us$, and $d \approx \frac{5}{32} \cdot 5v \approx 0.78v$. Note that this implementation’s output has a granularity of $\frac{1}{42} \cdot 5v \approx 0.15v$, i.e. we can’t specify an output level with precision finer than 0.15v. If we wanted to specify an average output voltage at the 0.001v level, the “num_pwm_bins” variable, line 9, would need to satisfy the description, 0.001v = $\frac{1}{\text{num_pwm_bins}} \cdot V_{cc}$. At $V_{cc} = 5v$, this would mean $\text{num_pwm_bins} \approx 5000$.

Implicitly then, there’s a trade-off between the granularity or precision of a PWM signal and time it will take to implement. A signal with 0.001v precision takes 5000 steps · 20us/step ≈ 0.1s to implement. 0.1 seconds is an eternity in the time-frame of a microcontroller running at 20MHz!

**III. AVERAGING PWM OUTPUT WITH A LOW-PASS FILTER**

When used electrically, PWM signals are often averaged by a resistor-capacitor filter circuit, shown in figure 3. In steady-state conditions, the capacitor averages out the pulsed input by storing and releasing electric charge in the normal “RC” fashion.

Students are normally introduced to capacitor charging and discharging in their second semester of University Physics. See for example, OpenStax University Physics, vol 2, section 10.5 - RC Circuits.

The Kirchhoff voltage loop for the filter can be written
FIG. 3. A PWM averaging circuit, using a resistor $R$ and capacitor $C$. In some contexts this arrangement is described as a low-pass filter. The circuit is redrawn under charging (PWM HIGH) and discharging (PWM LOW) conditions.

FIG. 4. The typical waveform that is seen at the $V_{avg}$ low-pass filter output from figure 3. This "shark-fin" wave comes from a periodic charge-discharge cycle that’s applied by the PWM output to the RC series circuit.

in the following way,

$$V_{cc} - iR - Q/C = 0, \text{ for } 0 < t < t_1$$

$$-iR - Q/C = 0, \text{ for } t_1 < t < t_1 + t_2. \quad (3)$$

The solutions to this set of equations, $Q_1$ during $t_1$ and $Q_2$ during $t_2$, are

$$Q_1(t) = A \exp[-t/\tau] + B, \text{ and}$$

$$Q_2(t) = D \exp[-t/\tau]. \quad (4)$$

The current in the resistor is the time-derivative of the capacitor charge, $i = \frac{dQ}{dt}$, and plugging the solutions, equation 3, into the voltage equations, 4, give the relations,

$$\tau = RC$$

$$B = C V_{cc}. \quad (5)$$

At this point, amplitudes $A$ and $D$ remain undefined. The amount of charge on the capacitor must be continuous, so two additional equations of constraint are:

$$Q_1(0) = Q_2(t_1 + t_2), \text{ and}$$

$$Q_1(t_1) = Q_2(t_1). \quad (6)$$

When these boundary conditions are applied to the equation 4, you can determine $A$ and $D$ to be:

$$A = V_{cc} \cdot C \left( \frac{\exp[-t_2/\tau] - 1}{1 - \exp[-(t_1 + t_2)/\tau]} \right), \text{ and}$$

$$D = V_{cc} \cdot C \left( \frac{\exp[t_1/\tau] - 1}{1 - \exp[-(t_1 + t_2)/\tau]} \right) \quad (7)$$

IV. CHECK YOUR UNDERSTANDING QUESTIONS

1. For each of the following systems identify and estimate the (PWM) push and the averaging system. Additionally estimate PWM period, $T$, and system relaxation time, $T_R$.

(a) The two cycle engine in a chainsaw or outboard motor.
(b) A muscle fiber in your bicep.
(c) The human digestion system.
(d) A variable speed motor in a hot air furnace blower.
(e) A dimmable 120VAC light bulb.
(f) Constituents calling a congressperson’s office.
(g) A bicycle.
(h) Human locomotion, ie swimming or walking.

2. If the voltage across the LED in figure 1 is 2.2v, what fraction of the system’s power actually makes it to the LED to be used to generate light?\(^{1}\)

V. LAB 1

These problems are written in an ascending order of difficulty. Not all need to be done. This took about 4 hours of in-class lab work.

1. How closely does the the PWM listing given, 1, match the timing specifications in 1? Use an oscilloscope to inspect the output signal. Deviations from the timing model probably come from latency in the for loop.

2. How can you rewrite code listing 1 to better match the PWM timing specifications in equation 1? Hint, call delay only twice per PWM cycle. Solution:

Listing 2. PWM routine with better timing

```c
int output_pin = 13;

void setup() {
  pinMode(output_pin, OUTPUT);
  digitalWrite(13, LOW);
}

void loop() {
  int num_pwm_bins = 32;
  int brightness = 10;
  int cycle_length = 20;
  int t1 = cycle_length * brightness;
  int t2 = cycle_length * (num_pwm_bins - brightness);

  // Your code here...
```

1. Update 2023-03-15
while (1) {
    digitalWrite(output_pin, HIGH);
    delayMicroseconds(t1);
    digitalWrite(output_pin, LOW);
    delayMicroseconds(t2);
}

It is informative to compare these two PWM functions in an oscilloscope. The latency implicit in the for loop and repeated delay calls in listing 1 becomes quite obvious.

3. Write a program that generates a PWM signal that has a minimum delay of 10\(\mu\)s (0.1 MHz), and a period of 10 ms. Implicitly, this is a duty level that can be specified to 1 part in 1000. Document the accuracy of your routine with a multimeter and an oscilloscope. Specifically, what is the finest resolution of average voltage you can specify between adjacent duty levels? Also, what are the on-off times in the oscilloscope output? How closely do they match the millis/micros delay times in your code?

4. Fold your PWM routine into a function. It should take some standard outputs that you define, eg, carrier period, duty level, duration of pulse, etc. Make sure the function works - it is a good way to clarify your thinking and code!

5. Use a capacitor and a load resistor, per figure 5, connected between the PWM lead and ground, to average out the PWM signal. Inspect the averaged and unaveraged signal on the oscilloscope. Is the capacitors effect uniform across duty levels? This works best if the capacitor is large.

6. Use your PWM routine to create a sawtooth function, with \(T = 2\) sec. You’ll have to call your PWM function repeatedly with an intentionally chosen increase each time the PWM function is run. Inspect the function with an oscilloscope with and without a capacitor to average out the signal. How accurately have you created a “perfect sawtooth”? Note, in this problem you are basically creating an arbitrary function generator.

7. Use your PWM routine and a 3904 transistor (or similar) to drive an external load with your sawtooth. (A DC computer cpu fan is easy, but there are other options, eg a speaker at 20kHz . . .). Note, if you want to drive a fan, the Blum Arduino book has a reasonable recipe on pp 65-69.\(^2\). One way to check your work is to listen to the fan as the sawtooth winds up and down.

VI. LAB 2

This took about 4 more hours of in-class lab time (and the students worked on this outside of class as well!)

1. Generate a PWM wave with average value of \(3.27 \pm 0.01\) volts, measured with a multimeter. Reminder, the PWM function we’ve been working with is HIGH (5.0 v) for \(t_1\) and LOW (0 v) for \(t_2\). The cycle is periodic at a time \(t = t_1 + t_2\). How many intervals do you need to break the PWM cycle into to achieve \(\pm 0.01\) volt resolution?

2. Generate an periodic oscillating function that is \(1.22v\) for \(100\)ms and \(3.77v\) for \(200\)ms (both \(\pm 0.01v\)). Note, for this you’ll want to use a resistor-capacitor filter like that shown in figure 5.

3. Starting with Kirchhoff loops for the filter when the system in the OK and OFF states, work out the mathematical details of III. You’ll need to set up two differential equations, propose and try out two solutions, and apply boundary conditions at \(t = 0, t = t_1\), and \(t = t_1 + t_2\) to solve for all undetermined variables in the solution.

4. Build the circuit in figure 5 with known \(R, C, t_1, t_2\) values and capture the filtered output via an oscilloscope for a duration of \(\approx 2 (t_1 + t_2)\). Compare the oscilloscope output to a plot (in Mathematica or similar) for \(Q_1\) and \(Q_2\) above (recall, \(Q = CV\) for capacitors). There should be good agreement!

5. The “Ripple Voltage,” \(V_R\), in this PWM scheme is the total variation in output voltage in the averaged
PWM signal. $V_R$ is shown graphically in figure 4. Using $Q_1$ and $Q_2$, equation 4 above, create an algebraic expression for this ripple voltage, $V_R$. Then, using several different $t_1$, $t_2$, $R$, and $C$ values, test the accuracy of your prediction for $V_R$.

6. Application Question: if you hold $R$ and $C$ constant, are there $t_1$ or $t_2$ values that make the ripple voltage larger? (Application, are the PWM levels at which this averaging routing makes a noisy, unreliable signal? Or, if $t_1 + t_2$ is fixed, are there $t_1$ or $t_2$ values that make the averaging element produce a clean, smooth signal?)

7. Application Question: If you want the ripple voltage to be no more than 10% of the ideal output, $V_{CC \frac{t_1}{t_1 + t_2}}$, what constraint should be imposed on $R$ and $C$? Back up your answer with data from lab.

8. Use the averaged PWM output to create a function that will play a (musical) note of frequency $f$. A musical note can be a sine wave, $V(t) = Sin [2 \pi ft]$. The pitch (high-ness) of the note is dictated by its frequency, $f$, where $f = 1/T$, and $T$ is the period, or “length in time of one cycle of the sine wave. Like the sawtooth before, you’ll need to make repeated calls to the PWM function to approximate the shape of a sine curve. Note as well that your sinewave will be generated with a DC off-set, which you could remove with an opamp. A SEEED Grove buzzer, is an easy way to play this note. Here’s a list of Piano key frequencies, https://en.wikipedia.org/wiki/Piano_key_frequencies

9. Extra credit: Use your musical note function to play a song on a speaker. We have low power buzzers in the SEEED Grove kits, or you can use a speaker, but you might need to use an op-amp (LF411) to scale the output. Happy Birthday or Edelweiss are nice choices.

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1 The Arduino Uno board is a hardware wrapper for the Atmel ATmega328 8-bit microcontroller. The Arduino is described online at https://www.arduino.cc/.

2 Jeremy Blum, Exploring Arduino: Tools and Techniques for Engineering Wizardry, (Wiley, 1st Edition, 2013), pp. 65-69.

3 See for example, http://wiki.seeedstudio.com/Grove-Buzzer/

4 This problem is inspired by a related discussion in the “Methods of Experimental Physics” course manual at the University of Minnesota. See the link, https://sites.google.com/a/umn.edu/mxp/