ON THE TOP-DIMENSIONAL COHOMOLOGY OF ARITHMETIC CHEVALLEY GROUPS

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Abstract. Let \( K \) be a number field with ring of integers \( \mathcal{O} \) and let \( G \) be a Chevalley group scheme not of type \( E_6, F_4 \) or \( G_2 \). We use the theory of Tits buildings and a result of Tóth on Steinberg modules to prove that \( H^{\text{vcd}}(G(\mathcal{O}); \mathbb{Q}) = 0 \) if \( \mathcal{O} \) is Euclidean.

1. Introduction

In this article, we obtain the following result about the cohomology of arithmetic groups:

**Theorem 1.1.** Let \( K \) be a number field, \( \mathcal{O} \) the ring of integers in \( K \) and \( G \) a Chevalley–Demazure group scheme of type \( A_n, B_n, C_n, D_n, E_6 \) or \( E_7 \). If \( \mathcal{O} \) is Euclidean, then the rational cohomology of \( G(\mathcal{O}) \) vanishes in its virtual cohomological dimension \( \text{vcd} = \text{vcd}(G(\mathcal{O})) \),

\[ H^{\text{vcd}}(G(\mathcal{O}); \mathbb{Q}) = 0. \]

As \( \text{SL}_n \) and \( \text{Sp}_{2n} \) are the simply-connected Chevalley–Demazure schemes of types \( A_{n-1} \) and \( C_n \), respectively, Theorem 1.1 is a common generalisation of results of Lee–Szczarba [LS76] (for \( \text{SL}_n(\mathcal{O}) \)) and Brück–Patzt–Sroka [Sro21, Chapter 5] (for \( \text{Sp}_{2n}(\mathbb{Z}) \), building on work of Gunnells [Gun00]).

There are two main ingredients in the proof of Theorem 1.1. The first is work of Borel–Serre [BS73] who proved that the groups in question are virtual Bieri–Eckmann duality groups. This allows one to study their high-dimensional rational cohomology by analysing their low-dimensional homology with coefficients in the so called Steinberg module. The second ingredient is a result of Tóth [Tót05] that gives a generating set of this module. He shows that in the cases covered in Theorem 1.1, the Steinberg module is cyclic as a \( G(\mathcal{O}) \)-module. This generalises results by Ash–Rudolph [AR79] and Gunnells [Gun00] in the cases of \( \text{SL}_n \) and \( \text{Sp}_{2n} \), respectively.

The Steinberg module can be described as the top-dimensional homology group of an associated Tits building. Previous vanishing results in the settings of \( \text{SL}_n \) and \( \text{Sp}_{2n} \) used explicit descriptions of the buildings that were specific for the corresponding types; see [CFP19, Sections 1.1 and 4] for type \( A \) and [Sro21, Chapter 5 and Definition 60] for type \( C \). We prove a building-theoretic generalisation of the key step in [CFP19, Section 4] for all types; see Proposition 3.3. This enables us to show that cohomology vanishing in the virtual cohomological dimension always...
follows if one can show that the Steinberg module is generated by “integral apartment classes”; see Theorem 3.4. The generation by integral apartments classes, in turn, is the content of Tóth’s result.

There are two assumptions in Theorem 1.1. The first is that $\mathcal{G}$ be not of type $E_8$, $F_4$ or $G_2$. This is due to the same hypothesis in Tóth’s work and comes from the fact that, in these cases, there is no maximal parabolic subgroup whose unipotent radical is abelian [Tót05, Section 5]. This makes certain computations harder in these cases [Tót05, second paragraph after Theorem 2]. The second assumption is that $\mathcal{O}$ be Euclidean, which is also a restriction in Tóth’s work. However, Euclideanity seems to be a natural assumption for a general statement in the style of Theorem 1.1. This is among other things indicated by work of Miller–Patzt–Wilson–Yasaki [MPWY20] who obtain non-vanishing results for $\mathcal{G} = SL_n$ and certain non-Euclidean PIDs $\mathcal{O}$.

The condition that $\mathcal{O}$ should at least be a PID is necessary in a strong sense, at least for $\mathcal{G} = SL_n$ [CFP19, Theorem D] and $\mathcal{G} = Sp_{2n}$ [BH23, Theorem 1.1].

In type $A$, for the group $SL_n(\mathcal{O})$, even stronger vanishing results are already known: Church–Putman [CP17] showed that the rational cohomology of this group vanishes also one degree below its virtual cohomological dimension if $\mathcal{O} = \mathbb{Z}$, and Kupers–Miller–Patzt–Wilson [KMPW22] proved the same result for $\mathcal{O}$ the Gaussian or Eisenstein integers. Brück–Miller–Patzt–Sroka–Wilson [BMP+22] extended this to vanishing of the rational cohomology two degrees below the virtual cohomological dimension for $\mathcal{O} = \mathbb{Z}$. These results confirm parts of a conjecture by Church–Farb–Putman [CFP14] who asked whether it was generally true that

$$H^{\text{vcd}(SL_n(\mathbb{Z})) - 1}(SL_n(\mathbb{Z}); \mathbb{Q}) = 0 \text{ if } i < n - 1 = \text{rk}(SL_n)).$$

In light of Theorem 1.1, one is tempted to ask whether vanishing behaviour similar to Eq. (1.1) might also occur for other arithmetic Chevalley groups.

**Question 1.2.** Let $\mathbb{K}$ be a number field, $\mathcal{O}$ the ring of integers in $\mathbb{K}$ and $\mathcal{G}$ a Chevalley–Demazure group scheme. If $\mathcal{O}$ is Euclidean, is it true that

$$H^{\text{vcd}(\mathcal{G}(\mathcal{O})) - i}(\mathcal{G}(\mathcal{O}); \mathbb{Q}) = 0 \text{ for all } i < \text{rk}(\mathcal{G})?$$

Currently, evidence for such a vanishing pattern is given by Theorem 1.1, the above mentioned results in type $A$ and work of Brück–Patzt–Sroka [BPS23] in type $C$ that shows that $H^{\text{vcd}(Sp_{2n}(\mathbb{Z})) - 1}(Sp_{2n}(\mathbb{Z}); \mathbb{Q})$ is trivial for $n \geq 2$.

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## 2. Background

### 2.1. Coxeter groups and Coxeter complexes.

Given a finite set $S$, consider a symmetric matrix $M = (m_{s,t})_{s,t \in S}$ whose diagonal entries equal one and all other entries are $\infty$ or integers greater than one. A group $W$ with presentation

$$W = \langle s \in S | (st)^{m_{s,t}} = 1 \text{ for all } s, t \text{ with } m_{s,t} < \infty \rangle$$

is called a *Coxeter group*. The pair $(W, S)$ is the corresponding *Coxeter system*, and $S$ is the *Coxeter generating set*. The *rank* of the system $(W, S)$ is the cardinality $|S|$ of the given generating set. We write $\ell(w)$ for the word length of $w \in W$ with respect to the generating set $S$. The system $(W, S)$ is called *spherical* if the underlying Coxeter group $W$ is finite. The reader is referred to standard textbooks, such as [Hum72, GP00, AB08], for further background on Coxeter groups.
Definition 2.1 ([AB08, Chapter 3]). Let $(W, S)$ be a Coxeter system. Given $J \subseteq S$, the subgroup $W_J := \langle J \rangle \leq W$ is called a (proper) standard parabolic subgroup of $W$. The Coxeter complex of $(W, S)$, denoted by $\Sigma(W, S)$, is the simplicial complex whose vertices are the cosets (in $W$) of its maximal standard parabolic subgroups, and where $g_0 W_{S \setminus \{s_0\}}, \ldots, g_k W_{S \setminus \{s_k\}}$ span a simplex if and only if $g_0 W_{S \setminus \{s_0\}} \cap \cdots \cap g_k W_{S \setminus \{s_k\}} \neq \emptyset$.

Left multiplication on cosets induces an action of $W$ on $\Sigma(W, S)$.

Lemma 2.2 ([AB08, Section 2.5 and Proposition 1.108]). Suppose $(W, S)$ is spherical. Then, $\Sigma(W, S)$ is $W$-equivariantly homeomorphic to the unit $|S| - 1$-sphere in Euclidean space $\mathbb{R}^{|S|}$ where any (conjugate of a) Coxeter generator $s \in S$ of $W$ acts on $\mathbb{R}^{|S|}$ as a linear reflection.

The Coxeter complex $\Sigma(W, S)$ has a distinguished simplex $C$ of maximal dimension $|S| - 1$, which is called the fundamental chamber of $\Sigma(W, S)$ and corresponds to the intersection of all maximal standard parabolic subgroups, 

$$C = \{ W_{S \setminus \{s\}} \mid s \in S \}.$$ 

Lemma 2.2 justifies the following definition.

Definition 2.3. Let $(W, S)$ be a spherical Coxeter system. The standard apartment class of $\Sigma(W, S)$ is the generator 

$$[\Sigma(W, S)] \in \tilde{H}_{|S| - 1}(\Sigma(W, S); \mathbb{Z}) \cong \mathbb{Z}$$

with underlying chain 

$$\sum_{w \in W} (-1)^{\ell(w)} w \cdot C \in \tilde{C}_{|S| - 1}(\Sigma(W, S); \mathbb{Z}).$$

Corollary 2.4. In the setting of Lemma 2.2, any Coxeter generator $s \in S$ of $W$ acts by $(-1)$ on the standard apartment class of $\Sigma(W, S)$,

$$s \cdot [\Sigma(W, S)] = -[\Sigma(W, S)].$$

2.2. Chevalley groups and their Weyl groups. We briefly introduce the group schemes that will come up in our work. All of the material presented in this section is standard and can be found in multiple seminal works on the topic, such as [Che55, Ree64, Kos66, Ste16]. We shall mostly follow Steinberg’s notation [Ste16].

Let $\Phi$ be a (reduced, crystallographic) irreducible root system — such root systems have been classified and form the seven classical types $A_n$, $B_n$, $C_n$, $D_n$, $E_6$, $E_7$, $E_8$, $F_4$ and $G_2$; cf. [Hum72]. Let $\mathfrak{g}_\Phi$ be the corresponding complex simple Lie algebra and $\Lambda$ the lattice of weights [Hum72] of some complex representation $\mathfrak{g}_\Phi \to gl(V)$. By the work of Chevalley [Che55] (independently by Ree [Ree64]) and Demazure [DG70], one can construct from such data $(\Phi, \Lambda)$ a unique representative functor [Kos66, Wat79] sending commutative unital rings to groups. We denote this functor by $G_\Phi^\Lambda$ and call it a Chevalley–Demazure group scheme of type $\Phi$. Given a (commutative) ring $R$ (with unity), we call the group of $R$-points $G_\Phi^\Lambda(R)$ a Chevalley–Demazure group, or more briefly a Chevalley group (over $R$).

Denoting by $\Lambda_{\text{sc}}$ (resp. by $\Lambda_{\text{ad}}$) the full lattice of weights (resp. the root lattice spanned by $\Phi$) of $\mathfrak{g}_\Phi$, we have $\Lambda_{\text{sc}} \subseteq \Lambda \subseteq \Lambda_{\text{ad}}$. The containment is reflected on groups: there exist central isogenies $G_\Phi^\Lambda_{\text{sc}} \hookrightarrow G_\Phi^\Lambda \twoheadrightarrow G_\Phi^\Lambda_{\text{ad}}$ (The ‘largest’ scheme on the left is universal or simply-connected, and the ‘smallest’ on the right is adjoint.) The following are typical examples of (infinite) families of Chevalley–Demazure groups,

$$G_\Phi^\Lambda_{\text{sc}} = \text{SL}_{n+1}, \quad G_\Phi^\Lambda_{\text{ad}} = \text{PGL}_{n+1}, \quad G_\Phi^\Lambda_{\text{sc}} = \text{Spin}_{2n+1}, \quad G_\Phi^\Lambda_{\text{ad}} = \text{Sp}_{2n}, \quad G_\Phi^\Lambda = \text{SO}_{2n}.$$
For the remainder of the paper we usually omit the root system and the lattice of weights and write $G := G^\Lambda_s$ to simplify notation.

Fixing an arbitrary total ordering on $\Phi$ gives rise to a subset of simple roots $\Delta$. That is, every $\alpha \in \Phi$ is a unique $\mathbb{Z}$-linear combination of elements of $\Delta$, in a way that the coefficients are all either positive or negative. In particular, $\Delta$ allows us to define the set $\Phi^+$ of positive roots — i.e., those roots whose coefficients with respect to $\Delta$ are all positive. Similarly, $\Phi^- = -\Phi^+$. The rank of $\Phi$, denoted $rk(\Phi)$, is the cardinality of $\Delta$ — it does not depend on the choice of a subset of simple roots. The rank of $G = G^\Lambda_s$ is defined as $rk(\Phi)$.

A choice of subset of simple roots $\Delta \subseteq \Phi$ gives rise to a $\mathbb{Z}$-subscheme in $\mathcal{G} := G^\Lambda_s$ that is isomorphic to $G^\Lambda_m$ of the group scheme $\mathcal{G}$ a and call it the $H$ by $G$ that is isomorphic to $\mathcal{G}$. The subschemes $X_{\alpha} \subseteq \mathcal{G}$ and the lattice of weights and write $\mathbb{Z}$-group scheme $G_m(R) \cong (R^\times, \cdot)$, cf. [DG70, Exposé XXII]. We denote such a subgroup by $\mathcal{H} \leq \mathcal{G}$ and call it the standard ($\mathbb{Z}$-split) maximal torus of $\mathcal{G}$.

The structure of Chevalley–Demazure groups is very much constrained by certain subgroups determined by roots. Given $\alpha \in \Phi^+$, one constructs an embedding over $\mathbb{Z}$ of the group scheme $(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix})$ into $\mathcal{G} = G^\Lambda_s$ that is normalised by the torus $\mathcal{H} \leq \mathcal{G}$. The image of this subscheme is denoted by $X_{\alpha} \subseteq \mathcal{G}$. Similarly, the opposite root $-\alpha$ gives a $\mathbb{Z}$-embedding $(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}) \hookrightarrow \mathcal{G}$, whose image is denoted by $X_{-\alpha}$ and is also normalised by $\mathcal{H}$. The subgroups $X_{\alpha}$ and $X_{-\alpha}$ are called unipotent root subgroups of $\mathcal{G}$. We remark that both are isomorphic to the additive $\mathbb{Z}$-group scheme $G_m(R) \cong (R, +)$. Turning to the group of $R$-points, a unipotent root element attached to $\alpha \in \Phi^+$ is the image $x_{\alpha}(r) \in X_{\alpha}(R)$ of the unipotent matrix $(\begin{smallmatrix} 1 & r \\ 0 & 1 \end{smallmatrix})$ under the above mentioned embedding, where $r \in R$. (Similarly for $-\alpha$ with $(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix})$.)

The Weyl group $N(\mathcal{H})/\mathcal{H}$ of a Chevalley–Demazure group scheme $\mathcal{G}$ is the quotient of $N(\mathcal{H}) \leq \mathcal{G}$ — i.e., the normaliser of $\mathcal{H}$ in $\mathcal{G}$ — by the torus $\mathcal{H} \leq \mathcal{G}$. As the notation suggests, this construction is functorial; cf. [DG70, Exposé XXII.3]. On the other hand, the root system $\Phi$ also gives rise to a Coxeter group $W_\Phi$ [Hum72], which has the same classical type as $\Phi$. This group acts by reflections on $\mathbb{R}^{rk(\Phi)} = \mathbb{R}^{|\Delta|}$ preserving the set of roots $\Phi \subseteq \mathbb{R}^{rk(\Phi)}$. (Here we interpret the set of simple roots $\Delta$ as forming a basis for $\mathbb{R}^{rk(\Phi)}$.) Denote by $S = \{s_\alpha \mid \alpha \in \Delta\}$ the Coxeter generating set of $W_\Phi$. The relationship between these two groups is described in the sequel. Given an arbitrary base ring $R$, define for each $\alpha \in \Phi$

$$w_{\alpha} := x_{\alpha}(1)x_{-\alpha}(1)^{-1}x_{\alpha}(1) \in N(\mathcal{H})(R) \leq \mathcal{G}(R),$$

which is the image of $(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix})$ under the embedding $\langle (\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}), (\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}) \rangle \hookrightarrow \langle X_{\alpha}, X_{-\alpha} \rangle \leq \mathcal{G}$.

**Theorem 2.5** (Chevalley [Che55, §III], Demazure [DG70, Exposé XXII.3]). Let $R$ be an arbitrary commutative unital ring. There is an isomorphism

$$f : N(\mathcal{H})(R)/\mathcal{H}(R) \rightarrow W_\Phi$$

that maps $w_{\alpha}^{-1}\mathcal{H}(R)$ to $s_\alpha$ for all $\alpha \in \Delta$.

We write $\tilde{S} = \{w_{\alpha} \in \mathcal{G}(R) = G^\Lambda_s(R) \mid \alpha \in \Delta\}$ and denote by $\tilde{W} = \langle \tilde{S} \rangle \leq \mathcal{G}(R)$ the subgroup of $\mathcal{G}(R)$ generated by $\tilde{S}$. The group $\tilde{W}$ is called the extended Weyl group of $\Phi$.

**2.3. The spherical Building of a Chevalley group.** We now specialise to the case where $R = K$ is a field. Following Tits [Tit74], every Chevalley group $\mathcal{G}(K)$ gives rise to a highly-symmetrical simplicial complex on which $\mathcal{G}(K)$ acts with strong transitivity properties. As before, we let $\mathcal{H}$ be an arbitrary, but fixed, maximal split torus of $\mathcal{G} = G^\Lambda_s$.

**Definition 2.6.** The standard Borel subgroup of $\mathcal{G}(K) = G^\Lambda_s(K)$ is the subgroup

$$B(K) = \langle \mathcal{H}(K), X_{\alpha}(K) \mid \alpha \in \Phi^+ \rangle \leq \mathcal{G}(K).$$
The (proper) standard parabolic subgroups of \( G(\mathbb{K}) \) are the proper subgroups that contain \( B(\mathbb{K}) \). The spherical (or Tits) building of \( G(\mathbb{K}) \), denoted by \( \Delta(\mathbb{K}) \), is the simplicial complex whose vertices are the cosets of maximal standard parabolic subgroups of \( G(\mathbb{K}) \), and \( g_0 P_0(\mathbb{K}), \ldots, g_k P_k(\mathbb{K}) \) span a simplex if and only if 
\[
g_0 P_0(\mathbb{K}) \cap \cdots \cap g_k P_k(\mathbb{K}) \neq \emptyset.
\]
Left multiplication on cosets induces an action of \( G(\mathbb{K}) \) on \( \Delta(\mathbb{K}) \).

It should be stressed that, due to the central isogenies, the spherical building \( \Delta(\mathbb{K}) \) depends only on the ground field \( \mathbb{K} \) and on the root system \( \Phi \); but not on the lattice of weights \( \Lambda \); cf. [Tit74, Proposition 5.4]. That is, \( \Delta(\mathbb{K}) \cong \Delta(\mathbb{K}) \cong \Delta(\mathbb{K}) \). Moreover, as the centre of \( G(\mathbb{K}) \) acts trivially on \( \Delta(\mathbb{K}) \) by definition, the action of an element \( g \in G(\mathbb{K}) \) on \( \Delta(\mathbb{K}) \) coincides with that of its image \( g(\mathbb{K}) \in G(\mathbb{K}) \).

It is immediate from the above definition that \( \Delta(\mathbb{K}) \) contains a canonical maximal simplex \( C \), called the fundamental chamber, corresponding to the intersection of all maximal standard parabolic subgroups. Let \( \tilde{W} \) denote the extended Weyl group defined after Theorem 2.5. We call the subcomplex \( \Sigma \subset \Delta(\mathbb{K}) \) spanned by \( \{ \tilde{w} C \mid \tilde{w} \in \tilde{W} \} \) the standard apartment in the Tits building \( \Delta(\mathbb{K}) \). Note that the action of \( G(\mathbb{K}) \) on \( \Delta(\mathbb{K}) \) restricts to an action of \( \tilde{W} \) on \( \Sigma \).

**Theorem 2.7** (Tits [Tit74, Theorem 5.2]). Let \( f : N(\mathcal{H})(\mathbb{K})/\mathcal{H}(\mathbb{K}) \to W_\Phi \) be the isomorphism of Theorem 2.5. There is an isomorphism of simplicial complexes
\[ F : \Sigma \to \Sigma(W_\Phi, S) \]
between the standard apartment \( \Sigma \) and the Coxeter complex \( \Sigma(W_\Phi, S) \) such that for each \( \tilde{w} \in \tilde{W} \), the action of \( \tilde{w} \) on \( \Sigma \) agrees with the action of \( f(\tilde{w} \mathcal{H}(\mathbb{K})) \) on \( \Sigma(W_\Phi, S) \).

Recall that the Coxeter complex \( \Sigma(W_\Phi, S) \) is a simplicial \((|S| - 1)\)-sphere (see Lemma 2.2). Theorem 2.7 therefore allows us to associate a unique homology class to the standard apartment \( \Sigma \) of \( \Delta(\mathbb{K}) \) by pulling back the standard apartment class \( [\Sigma(W_\Phi, S)] \in \tilde{H}_{rk(\Phi) - 1}(\Sigma(W_\Phi, S); \mathbb{Z}) \) of the Coxeter complex (see Definition 2.3). Namely, fixing for each \( w \in W_\Phi \) a representative \( \tilde{w} \) for \( w \in \tilde{W} \leq G(\mathbb{K}) \), the standard apartment class \( [\Sigma] \) in the Tits building is
\[ [\Sigma] = F_{\ast}^{-1}[\Sigma(W_\Phi, S)] = \sum_{w \in W_\Phi} (-1)^{\ell(w)} \tilde{w} \cdot C \in \tilde{H}_{rk(\Phi) - 1}(\Delta(\mathbb{K}); \mathbb{Z}). \]

The Solomon–Tits Theorem says that the \( G(\mathbb{K}) \)-translates of this class generate the entire homology of the building:

**Theorem 2.8** (Solomon–Tits [Sol69]). For all \( i \neq rk(\Phi) - 2 \), the reduced homology \( \tilde{H}_i(\Delta(\mathbb{K}); \mathbb{Z}) \) is trivial. The \( \mathbb{Z}[G(\mathbb{K})] \)-module \( \mathcal{S} \) := \( \tilde{H}_{rk(\Phi) - 1}(\Delta(\mathbb{K}); \mathbb{Z}) \) is generated (as an abelian group) by \( G(\mathbb{K}) \)-translates of \( [\Sigma] \).

**Definition 2.9.** The \( \mathbb{Z}[G(\mathbb{K})] \)-module \( \mathcal{S} \) from Theorem 2.8 is called the Steinberg module for \( G(\mathbb{K}) \). Its generators \( g \cdot [\Sigma], g \in G(\mathbb{K}) \), are called apartment classes.

2.4. Actions of arithmetic subgroups on spherical buildings. If \( \mathbb{K} \) is a number field with ring of integers \( \mathcal{O} \) and \( \mathcal{G} = G(\mathcal{O}) \), the subgroup \( \mathcal{G}(\mathcal{O}) \leq G(\mathbb{K}) \) is an example of an arithmetic subgroup of \( G(\mathbb{K}) \); cf. [Mar91, Chapter I.3]. Of course, \( \mathcal{G}(\mathcal{O}) \) inherits from \( G(\mathbb{K}) \) a natural action on the spherical building \( \Delta(\mathbb{K}) \). Borel–Serre showed that this action reveals a lot of cohomological information about \( G(\mathcal{O}) \). In particular, they proved the following.


Theorem 2.10 (Borel–Serre [BS73]). Let $G$ be a Chevalley–Demazure group scheme, and $\mathcal{O}$ the ring of integers of an algebraic number field $\mathbb{K}$. Write $r$ for the (real) dimension of the symmetric space associated to $G(\mathbb{O})_{\mathbb{Z}}$. Then the virtual cohomological dimension of $G(\mathcal{O})$ is $\text{vcd}(G(\mathcal{O})) = r - \text{rk}(\Phi)$ and, for every $i$, there is an isomorphism

$$H^{\text{vcd}(G(\mathcal{O}))-i}(G(\mathcal{O}); \mathbb{Q}) \cong H_i(G(\mathcal{O}); \mathbb{Q} \otimes \mathbb{Z} \, \mathcal{St}).$$

The theorem of Borel–Serre above follows from their construction of the bordification for the symmetric space attached to $G(\mathcal{O})_{\mathbb{Z}}$, and the fact that extension (resp. Weil restriction) of scalars preserves parabolics; see [BS73, Theorem 8.41 and Section 9] and [Mar91, Section 1.7].

3. Vanishing of cohomology

In this section we prove our main theorem. Recall that the action of $G(\mathcal{O})$ on $\Delta(G(\mathbb{K}))$ is induced by left multiplication on cosets of parabolics $gP(\mathbb{K}) \subset G(\mathbb{K})$.

Definition 3.1. An apartment class $[\mathcal{A}] \in \mathcal{St}$ is called integral if it is a $G(\mathcal{O})$-translate of the standard apartment class $[\Sigma]$. That is, $[\mathcal{A}] = \gamma \cdot [\Sigma]$ for some $\gamma \in G(\mathcal{O})$.

Generalising works of Ash–Rudolph [AR79] and Gunnells [Gun00], Tóth established the following.

Theorem 3.2 (Tóth [Tót05]). Suppose the Chevalley–Demazure group scheme $G$ is not of type $E_6$, $F_4$ or $G_2$. If the ring of integers $\mathcal{O}$ is Euclidean, then the Steinberg module $\mathcal{St}$ for $G(\mathbb{K})$ is generated (as an abelian group) by integral apartment classes.

Although Tóth uses the terminology ‘simple Chevalley groups’ (which might be mistaken as adjoint), we remark that the schemes he considers are in fact of the form $G^\Lambda_{\mathbb{A}}$ for any $\Lambda$; cf. [Tót05, second paragraph of Section 2]. Moreover, it is straightforward to verify that his results do hold for all $G^\Lambda_{\mathbb{A}}$ (except the excluded types $E_6$, $F_4$, $G_2$) since Chevalley’s commutator formulae [Che55] are valid for all $G^\Lambda_{\mathbb{A}}$ regardless of the lattice $\Lambda$.

The following is a key ingredient for our proof of Theorem 1.1. It can be seen as a building-theoretic generalisation of an argument used by Church–Farb–Putman in the setting of $\text{SL}_n$ [CFP19, Proof of Theorem C, last paragraph].

Proposition 3.3. If the apartment class $[\mathcal{A}] \in \mathcal{St}$ is a translation of the standard apartment class by an element of the normaliser of $G(\mathcal{O})$ in $G(\mathbb{K})$, then there exists $\gamma \in G(\mathcal{O})$ such that $\gamma \cdot [\mathcal{A}] = -[\mathcal{A}]$.

Proof. Let $[\mathcal{A}] = \gamma_1 \cdot [\Sigma]$ with $\gamma_1 \in N_G(\mathbb{K})(G(\mathcal{O}))$ and $[\Sigma]$ the standard apartment class be given. Let $W_\Phi$ be the Coxeter group associated to $\Phi$, and choose a Coxeter generator $s_\alpha \in W_\Phi$ (hence a reflection of the underlying Euclidean space $\mathbb{R}^{\text{rk}(\Phi)}$). By Theorem 2.5, we can find a lift of $s_\alpha$ in $G(\mathcal{O})$, i.e., there exists $\gamma_2 \in G(\mathcal{O})$ such that $\gamma_2 H(\mathcal{O})$ maps to $s_\alpha$ under the isomorphism

$$f : N(H)(\mathcal{O})/H(\mathcal{O}) \to W_\Phi.$$

By Theorem 2.7, we have

$$\gamma_2 \cdot [\Sigma] = \gamma_2 \cdot F_\alpha^{-1}([\Sigma(W_\Phi, S)]) = F_\alpha^{-1}(s_\alpha \cdot [\Sigma(W_\Phi, S)]).$$

As $s_\alpha$ reverses orientation by Corollary 2.4, we have $s_\alpha \cdot [\Sigma(W_\Phi, S)] = -[\Sigma(W_\Phi, S)]$.

Now define $\gamma := \gamma_1 \gamma_2^{-1}$, which lies in $G(\mathcal{O})$ because $\gamma_1$ normalises it. Putting the above together, it follows that

$$\gamma \cdot [\mathcal{A}] = (\gamma_1 \gamma_2^{-1}) \cdot (\gamma_1 \cdot [\Sigma]) = (\gamma_1 \gamma_2) \cdot [\Sigma] = \gamma_1 \cdot (\gamma_2 \cdot [\Sigma]) = \gamma_1 \cdot (-[\Sigma]) = -[\mathcal{A}],$$

as desired. \qed
In what follows we always consider $\mathbb{Q}$ to be a trivial $G$-module, where $G$ is either $G(\mathcal{O})$ or $G(K)$.

**Theorem 3.4.** Let $K$ be a number field, $\mathcal{O}$ the ring of integers of $K$ and $G$ a Chevalley–Demazure group scheme. If the Steinberg module $St$ for $G(K)$ is generated by integral apartment classes, then

$$H^\text{red}(G(\mathcal{O}))(\mathcal{O}); \mathbb{Q}) = 0.$$ 

**Proof.** By Theorem 2.10, there is an isomorphism

$$H^\text{red}(G(\mathcal{O}))(\mathcal{O}); \mathbb{Q}) \cong H_0(G(\mathcal{O}); \mathbb{Q} \otimes \mathbb{Z} St).$$

The right hand side, in turn, is isomorphic to the module of co-invariants of the $\mathbb{Q}[G(\mathcal{O})]$-module $\mathbb{Q} \otimes \mathbb{Z} St$,

$$H_0(G(\mathcal{O}); \mathbb{Q} \otimes \mathbb{Z} St) \cong (\mathbb{Q} \otimes \mathbb{Z} St)G(\mathcal{O}) \cong \mathbb{Q} \otimes (\mathbb{Z}[G(\mathcal{O})]) St;$$

see, for instance, [Bro94, Chapter III.1]. Since $St$ is assumed to be generated by integral apartment classes, it therefore suffices to prove that for every integral apartment class $[A] \in St$ and every $q \in \mathbb{Q}$, we have

$$q \otimes \mathbb{Z}[G(\mathcal{O})] [A] = 0 \in \mathbb{Q} \otimes \mathbb{Z}[G(\mathcal{O})] St.$$

Let $q$ and $[A]$ be given, with $[A]$ integral. By Proposition 3.3, we can find $\gamma \in G(\mathcal{O})$ such that $\gamma[A] = -[A]$. As $G(\mathcal{O})$ acts trivially on $\mathbb{Q}$, it follows that

$$q \otimes \mathbb{Z}[G(\mathcal{O})] [A] = q \cdot \gamma \otimes \mathbb{Z}[G(\mathcal{O})] [A] = q \otimes \mathbb{Z}[G(\mathcal{O})] \cdot [A] = -(q \otimes \mathbb{Z}[G(\mathcal{O})] [A]).$$

Hence $q \otimes \mathbb{Z}[G(\mathcal{O})] [A] = 0$ since $\text{char}(\mathbb{Q}) = 0$, which concludes the proof. \qed

**Remark 3.5.** The proof of Theorem 3.4 actually shows that if $St$ is generated by integral apartment classes, then $H_0(G(\mathcal{O}); R \otimes \mathbb{Z} St) = 0$ for all rings $R$ in which $2$ is invertible.

**Proof of Theorem 1.1.** By Theorem 3.2, the Steinberg modules for the Chevalley groups $G(K)$ of the given types are generated by integral apartment classes. The claim thus follows from Theorem 3.4. \qed

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