We study the dynamical quantum phase transition of the critical quantum quench, in which the prequenched Hamiltonian, or the postquenched Hamiltonian, or both of them are set to be the critical points of equilibrium quantum phase transitions, we find half-quantized or unquantized dynamical topological order parameter and dynamical Chern number; these results and also the existence of dynamical quantum phase transition are all closely related to the singularity of the Bogoliubov angle at the gap-closing momentum. The effects of the singularity may also be canceled out if both the prequenched and postquenched Hamiltonians are critical, then the dynamical topological order parameter and dynamical Chern number restore to integer ones.

\[ H = \frac{1}{2} \sum_{j=1}^{N} \left[ \frac{1}{2} + \gamma \sigma_j^x \sigma_{j+1}^x + \frac{1}{2} - \gamma \sigma_j^y \sigma_{j+1}^y - g \sigma_j^z \right]. \]  

By the Jordan-Wigner transformation and Fourier transformation, the model can be transformed to

\[ H(\gamma, g) = \sum_{k>0} \eta_k^\dagger h(k) \eta_k \]  

where \( \eta_k = (c_k, e^{\dagger \gamma} k) \) and \( h(k) = \overrightarrow{d} \cdot \overrightarrow{\sigma} \), with \( \overrightarrow{\sigma} \) the Pauli matrix and \( \overrightarrow{d} = (0, \gamma \sin k, \cos k - g) \) the Bloch vector. Diagonalization of \( h(k) \) yields the dispersion relation

\[ \epsilon_k(\gamma, g) = \sqrt{(\cos k - g)^2 + 2 \gamma^2 \sin^2 k}. \]

Prepare an initial state \( |\psi_0\rangle \), which is a ground state of a prequenched Hamiltonian \( H_i = H(\gamma_i, g_i) \), and then suddenly change the system to a postquenched Hamiltonian \( H_f = H(\gamma_f, g_f) \), the system may undergo a DQPT, whose singularity is reflected in the rate function \( I(t) = -\lim_{N \to \infty} \frac{1}{N} \log |\mathcal{L}(t)|^2 \) and the dynamical free energy

\[ f(z) = -\lim_{N \to \infty} \frac{1}{N} \log Z(z) \]  

at a series of critical times. Here, the boundary function \( Z(z) = |\psi_0| e^{-i/h} |\psi_0\rangle \) is an analytic continuation of the Loschmidt amplitude

\[ \mathcal{L}(t) = |\psi_0| e^{-i/h} |\psi_0\rangle \]  

under \( z = \Re z + i \Im z \). For the XY model, \( Z(z) \) can be calculated analytically, which is

\[ Z(z) = \prod_{k>0} Z_k(z), \]  

where

\[ Z_k(z) = \cos^2 \varphi_k e^{i\epsilon_k(\gamma_f, g_f)z} + \sin^2 \varphi_k e^{-i\epsilon_k(\gamma_f, g_f)z}, \]  

where \( \varphi_k = \theta_k(\gamma_i, g_i) - \theta_k(\gamma_f, g_f) \), with \( \tan[2\theta_k(\gamma, g)] \) is defined as \( \gamma \sin k/(g - \cos k) \), \( \theta_k \) is in \( [0, \pi/2] \). The Fisher zeros of \( f(z) \) are \( z_n = 1/[2 \epsilon_k(\gamma_f, g_f)] \cdot [\ln \tan^2 \varphi_k + \pi(2n + 1)] \), with \( n = 0, 1, 2, \ldots \); when \( \varphi_k = \pm \pi/4 \), namely the initial Bloch vector \( \overrightarrow{d}_i \) and the final Bloch vector \( \overrightarrow{d}_f \) are perpendicular to each other, the real parts of \( z_n \) are zero, and we get the critical times \( t_n = t^*(n + 1/2) \), with \( t^* = \pi/\epsilon_k(\gamma_f, g_f) \), where \( k^* \) is determined by

\[ \overrightarrow{d}_i \cdot \overrightarrow{d}_f = (\cos k^* - g_i)(\cos k^* - g_f) + \gamma_i \gamma_f \sin^2 k = 0. \]
\[ \nu_D(t) = \frac{1}{2\pi} \int_0^{2\pi} \frac{\partial \phi_k(t)}{\partial k} \, dk, \]

where \( \phi_k(t) = \phi_k^\text{dyn}(t) - \phi_k^\text{eq}(t) \), with \( \phi_k^\text{eq}(t) \) the phase of the Loschmidt amplitude at momentum \( k \) and \( \phi_k^\text{dyn}(t) \) the dynamical phase

\[ L_k(t) = \mathcal{Z}_k(t) = |L_k(t)|e^{i\phi_k(t)}, \]

\[ \phi_k^\text{dyn}(t) = -\int_0^t ds \langle \psi(s) | h_f(k) | \psi(s) \rangle = c_k(\gamma_f, g_f) t \cos(2\varphi_k). \]

It is obvious that \( \nu_D \) can be understood as the winding number of the vector

\[ \vec{r}_k = (x_k, y_k) = |L_k(t)|e^{i\phi_k(t)} \]

around the origin. For the noncritical quench, the trajectory of vector \( \vec{r}_k \) is a closed loop as \( k \) varies from 0 to \( \pi \), however, for the critical quench, the trajectory in general is not closed. The reason lies in the fact that \( \vec{r}_{k\rightarrow\pi} \) can be unequal to \( \vec{r}_{k\rightarrow0} \) for the critical quantum quench; in contrary, in the noncritical quench, \( \vec{r}_{k\rightarrow0} = \vec{r}_{k\rightarrow\pi} \). Note that \( k \rightarrow 0 \) and \( k \rightarrow \pi \) are the two momenta that make \( \phi_k^\text{eq}(t) \) (always be \( n\pi \), we call the two momenta ‘fixed points’, which are also the fixed points of the evolving Bloch vector defined in Eq. (11). If \( \vec{r}_{k\rightarrow0} \neq \vec{r}_{k\rightarrow\pi} \), the two corresponding points of \( \vec{r}_k \) will be two different positions on the real axis, so the trajectory is not closed. The difference of \( \vec{r}_{k\rightarrow0} \) and \( \vec{r}_{k\rightarrow\pi} \) stems from the singularity of the Bogoliubov angle at the gap-closing momentum \( k_c \). Take the quench shown in Fig. 1(a) as an example, the gap-closing momentum of \( H_1 \) is \( k_c = 0 \), where \( \theta_{k\rightarrow0}(\gamma_0, g_0) = \pi/4 \) and \( \theta_{k\rightarrow0}(\gamma_1, g_1) = 0 \), so \( \varphi_{k\rightarrow0} = \pi/4 \), this leads to \( \vec{r}_{k\rightarrow0} = \cos[c_0(\gamma_f, g_f)] t \) according to Eqs. 3 4 8 and 8, however similar analysis for \( k \rightarrow \pi \) gives \( \vec{r}_{k\rightarrow\pi} = 1 \), so \( \vec{r}_{k\rightarrow0} \) is not equal to \( \vec{r}_{k\rightarrow\pi} \) except at some special time \( t = 2n\pi/c_0(\gamma_f, g_f) = 2n\pi \). Two examples are shown in Fig. 1(b) and (c); in Fig. 1(b), the angle swept by the vector \( \vec{r}_k \) is \( -\pi \), thus it gives a winding number of \( \nu_D = -1/2 \); however, in Fig. 1(c), the angle swept by \( \vec{r}_k \) is \( -2\pi \), so the winding number is \( \nu_D = -1 \); in both cases, the trajectories of \( \vec{r}_k \) are not closed. It is obvious that the angle swept by \( \vec{r}_k \) is related to the position of the origin, in Fig. 1(b), it is between the two points of \( \vec{r}_{k\rightarrow0} \) and \( \vec{r}_{k\rightarrow\pi} \), therefore there is a discontinuous change of the Pancharatnam geometrical phase \( \phi_k^\prime \) at the fixed points, this is the core origin of the half quantization of DTOP.

Now let us consider another example, in which the pre-quenched Hamiltonian is at the XX chain, i.e., \( \gamma_i = 0 \); in this case, there is no DQPT, and the DTOP is unquantized, as shown in Fig. 2(a). The absence of DQPT is owing to the singularity of the Bogoliubov angle at the gap-closing momentum \( k_c = \pi/3 \). Here \( k_c \) happens to be the same as the momentum \( k^* \) that satisfies Eq. (4), however, this does not mean there will be a DQPT, because at this point, the Bogoliubov angle is ill defined, so we should take the limit \( k \rightarrow k^* \), which gives \( \varphi_{k\rightarrowk^*} \approx 0.3085\pi \) and \( \varphi_{k\rightarrowk^*} \approx -0.1915\pi \), both of them are not equal to \( \pm\pi/4 \). In fact, in the whole effective Brillouin zone \([0,\pi]\), we can not find a momentum \( k^* \) that satisfies \( \varphi_{k\rightarrowk^*} = \pm\pi/4 \), so the Fisher zeros never intersect the imaginary axis, and the rate function always has no singularity. Here we can see the particularity of the critical quench, the existence of a \( k^* \) that satisfies Eq. (4) does not mean \( \varphi_{k\rightarrowk^*} = \pm\pi/4 \), i.e., the existence of a DQPT; here Eq. (4) is satisfied only because of the fact that \( d_i \) is zero at the gap-closing point. In contrary, in the noncritical quench, the condition of Eq. (4) and \( \varphi_{k^*} = \pm\pi/4 \) are the same meaning.

Also because of the singularity of the Bogoliubov angle, the trajectory of vector \( \vec{r}_k \) is closed, as shown in Fig. 2(b). \( \theta_{k\rightarrow0}(\gamma_f, g_f) = 0 \) but \( \theta_{k\rightarrow\pi}(\gamma_f, g_f) = \pi/2 \), this will lead to the discontinuous change of \( \phi_k^\prime \) and consequently the discontinuous change of \( \vec{r}_k \) at \( k = \pi/3 \), so the trajectory is not closed, i.e., \( \nu_D \) is not quantized.
\[ \rho_t = |\psi_0(k)\rangle\langle \psi_0(k)| = \frac{1}{2} \left[ 1 - \hat{d}_\theta \cdot \hat{d}_\phi \right] \] is the initial density matrix, with \(|\psi_0(k)\rangle\) the ground state of the initial Hamiltonian \(h_k(k)\), and \(\hat{d}_\theta = \hat{d}_i/|\hat{d}_i|\). The evolving Bloch vector \(\vec{d}\) can be calculated analytically [27],

\[
\vec{d} = \hat{d}_i \cos(2|\hat{d}_f|t) + 2d_f (\hat{d}_i \cdot \hat{d}_f) \sin^2(|\hat{d}_f|t) - \hat{d}_i \times \hat{d}_f \sin(2|\hat{d}_f|t). \tag{10}
\]

In Eq. (10), \(k_{\theta m}\) is the \(m\)-th fixed point of \(\vec{d}\), where the Bloch vector \(\vec{d}\) keep still during the time evolution. Generally, at such fixed point, for a noncritical quantum quench, the initial Bloch vector \(\vec{d}_i\) and \(\vec{d}_f\) are parallel (or antiparallel) to each other.

For the critical quench shown in Fig. 1, the fixed points are \(k_1 = 0\) and \(k_2 = \pi\). For the second fixed point \(k = \pi\), the initial Bloch vector \(\vec{d}_i\) and the final Bloch vector \(\vec{d}_f\) are parallel to each other, i.e., \(\varphi_{k=\pi} = 0\), this is similar to the noncritical quench. However, the first fixed point \(k = 0\) is very special, it is a gap-closing point, in such point the singularity of the two Bogoliubov angles can cancel each other.

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From the two examples, we can see that the discontinuous change of the Bogoliubov angle at the gap-closing momentum makes the mapping from the space \((k,t)\) to the surface of the Bloch sphere incomplete, so the dynamical Chern number is not integer.

Unquantization of DTOP and dynamical Chern number does not always correspond to the absence of DQPT, for example, in the quench from \((\gamma_l, g_i) = (0.5, 0.5)\) to \((\gamma_f, g_f) = (0.5, 0.8)\), although the DTOP and the dynamical Chern number are not quantized, there still exists a DQPT, and the unquantized DTOP still can be used as a detector of the DQPT, as shown in Fig. 5(a). In fact, in this example, there are two \(k^*\) satisfy Eq. (4), one is \(k^*_1 = \pi/3\), which coincides with the gap-closing momentum \(k_c\), another one is \(k^*_2 = \arccos 0.8\). From the examples shown in Fig. 2 and 4 we know that the coincidence of \(k^*_1\) and \(k_c\) can lead to the absence of singularity in the rate function \(l(t)\) and the unquantization of DTOP and dynamical Chern number, however, here \(k^*_2\) can recover the singularity in \(l(t)\); the final result is the interplay of the effects of \(k^*_1\) and \(k^*_2\), so we get the results shown in Fig. 5(a).

**Quench from noncritical point to critical point.**—If the prequenched Hamiltonian is noncritical but the postquenched Hamiltonian is critical, there is no DQPT, such as the quench from \((\gamma_l, g_i) = (1, 0.5)\) to \((\gamma_f, g_f) = (1, 1)\). In this case the momentum that satisfies Eq. (1) is \(k = 0\), which is also the gap-closing point of the postquenched Hamiltonian, i.e., \(\epsilon_0(\gamma_f, g_f) = 0\), so the critical time \(t^* = \infty\), therefore we can not find a DQPT in finite time. However in this case, we can still find a half-quantized dynamical Chern number, similar to the case shown in Fig. 4.

**Quench from critical point to critical point.**—If both the prequenched and postquenched Hamiltonian are critical, the situation becomes a little subtle. If the gap-closing momentum of the prequenched Hamiltonian is different from that of the postquenched Hamiltonian, we may get a DQPT with half-quantized DTOP and half-quantized dynamical Chern number, such as the quench from \((\gamma_l, g_i) = (1, 1)\) to \((\gamma_f, g_f) = (0, 0.5)\), which is similar to the case shown in Figs. 1 and 3. However, if the gap-closing momentum of the prequenched Hamiltonian and the postquenched Hamiltonian are the same, then the singularity of the two Bogoliubov angles can cancel out with each other, and the DTOP and dynamical Chern number.
number restore to integer ones, such as the quench from \((\gamma_i, g_i) = (2, 1)\) to \((\gamma_f, g_f) = (-2, 1)\), shown in Fig. 5(b). In this case, there are two \(k^*\) satisfy Eq. (4), which are \(k_1^* = 0\) and \(k_2^* = \arccos(-3/5)\); \(k_1^*\) coincides with the gap-closing points of both the prequenched Hamiltonian and postquenched Hamiltonian, it does not lead to any singularity in the rate function \(l(t)\), however \(k_2^*\) can lead to certain singularity in \(l(t)\), therefore there is a DQPT. For the DTOP, because both \(\epsilon_0(\gamma_i, g_i)\) and \(\epsilon_0(\gamma_f, g_f)\) are zero, we know \(\tilde{r}_{k \rightarrow 0} = 1\), which is the equal to \(\tilde{r}_{k \rightarrow \pi}\), namely the values of \(\tilde{r}_k\) are the same at the two fixed points, thus the trajectory of \(\tilde{r}_k\) is always closed, i.e., \(\nu_D\) is an integer. For the dynamical Chern number, it is easy to find that the range of the angle difference between \(\theta_k(\gamma_i, g_i)\) and \(\theta_k(\gamma_f, g_f)\) recovers to \([0, \pi/2]\), which is the same as the noncritical quench, so the evolving Bloch vector \(d^\varepsilon\) can sweep the whole Bloch sphere, thus we get \(C_{dyn} = 1\).

**Summary and discussion.**—In summary, we have studied the critical quantum quench, in which the prequenched Hamiltonian, or the postquenched Hamiltonian, or both of them are set to be the critical points of equilibrium quantum phase transitions. We demonstrate that the singularity of the Bogoliubov angle at the gap-closing momentum \(k_c\) may lead to interesting topological properties in such type of quantum quench. The singularity of this momentum can lead to discontinuous change of the Pancharatnam geometrical phase of Loschmidt amplitude, which is the origin of the unquantization of DTOP; if the momentum happens to be one of the fixed point of the Pancharatnam phase, then the DTOP can be half-quantized. For the dynamical Chern number, the trajectories of the evolving Bloch vector \(d^\varepsilon\) for \(k \rightarrow k_{c-}\) and \(k \rightarrow k_{c+}\) cut open the Bloch sphere, so the mapping from the \((k, t)\) space to the surface of the Bloch sphere is incomplete, therefore the dynamical Chern number may be half-quantized or unquantized. The coincidence of the gap-closing momentum and the fixed point is a necessary condition for half-quantized dynamical Chern number, because in the limit of this momentum, the trajectory of the evolving Bloch vector \(d^\varepsilon\) is exactly a meridian, which cut off half of the sphere.

The existence of the DQPT is also closely related to the singularity of the Bogoliubov angle, from the example shown in Fig. 2 we can see that if \(k^*\) coincides with the gap-closing momentum \(k_c\), it does not lead to any singularity in rate function, and the DQPT is absent, however, the DQPT can be recovered if there exists another \(k^*\) that does not coincide with \(k_c\). We can also see that the effects of the singularity of the Bogoliubov angles can even be canceled out if both the prequenched Hamiltonian and postquenched Hamiltonian are critical and the two gap-closing momenta are the same, in this case, the DTOP and dynamical Chern number restore to integer ones.

The conclusions in this letter are valid for the integrable system with chiral symmetry like the XY chain, it is an interesting question to generalize the research to integrable systems without chiral symmetry and non-integrable systems, and so forth. The critical quantum quench is also very possible to be realized in the experiments of ultra-cold-atomic gases[20, 21] and trapped irons[22, 24], because the noncritical quench has already been realized, what we have to do is to tune the parameters to get a critical point in the experiment.

This work is supported by the National Science Foundation of China (NSFC) under Grant Numbers 11975024, 11774002 and 11804383 and the Anhui Provincial Supporting Program for Excellent Young Talents in Colleges and Universities under Grant No. gxyqZD2019023.

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