Dynamics of the mean-field interacting quantum kicked rotor

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We study the dynamics of the many-body atomic kicked rotor with interactions at the mean-field level, governed by the Gross-Pitaevskii equation. We show that dynamical localization is destroyed by the interaction, and replaced by a subdiffusive behavior. In contrast to results previously obtained from a simplified version of the Gross-Pitaevskii equation, the subdiffusive exponent does not appear to be universal. By studying the phase of the mean-field wave function, we propose a new approximation that describes correctly the dynamics at experimentally relevant times close to the start of subdiffusion, while preserving the reduced computational cost of the former approximation.

I. INTRODUCTION

Ultracold quantum gases are controllable and flexible tools to simulate a great variety of condensed-matter systems [1]. In many cases, the use of equivalent models – i.e. alternative models that can be mapped onto a desired system – constitutes a promising route, at lower experimental or computational expense, to a better understanding of the underlying physics. A well-known illustration is Floquet engineering, which relies on the careful design of time-periodic systems whose stroboscopic evolution is governed by an effective static Hamiltonian featuring the desired properties [2, 3]. Floquet engineering allows mimicking to a high degree of details the physics of quantum disordered systems. In this context, the atom-optics realization [4] of the quantum kicked rotor (QKR) [5, 6] has proven to be an almost ideal quantum simulator [7]. While the associated dynamics is known to display dynamical localization, which is the analogue of Anderson localization in momentum space [8], the QKR can, in fact, be rigorously mapped onto a 1D Anderson model [9]. Moreover, by adding to the kick amplitude a temporal dependence made of $d − 1$ frequencies incommensurate with the kick frequency $T_k$, one obtains the quasiperiodic kicked rotor which maps to an Anderson model in $d$ dimensions [10–12]. Thanks to its experimental and conceptual simplicity, the QKR has been widely used to investigate Anderson-like physics experimentally: observation of Anderson localization [11], characterization of critical properties [12–15], critical dimension localization [16] or other universality classes [17].

A challenging question concerns many-body effects on Anderson localization [18, 19]. This question arises in a wide class of condensed-matter systems, from disordered superconductors [20, 21] to superfluid $^4$He in porous media [22]. In addition, the interplay between disorder and interactions is known to give rise to non-trivial collective behavior, possibly underlying complicated many-body phase transitions [23–30]. In the more restricted frame of bosonic systems in the mean-field interacting regime, where interactions are simply taken into account by a quadratic nonlinearity in the Gross-Pitaevskii Equation (GPE) [31], previous studies of the mean-field Anderson model have predicted that the localized regime should be replaced by a subdiffusive phase [32–35]. In 3D however, the full numerical simulation of the mean-field Anderson model implies a very heavy computational cost. Hence, the QKR could constitute a promising equivalent model to circumvent this issue; yet, it is still unclear whether its equivalence with the corresponding Anderson model holds in the presence of interactions, since the latter are local in position space whereas localization occurs in momentum space.

The mean-field interacting bosonic QKR is modeled by a GPE [36], see Eq. (1) below. While its dynamics at short time has been studied in various regimes [36–38], the long term effect of the interactions on dynamical localization in the exact GPE has not been studied in detail, due to strong numerical instabilities and computer resource cost. For these reasons, the long-time dynamics of the interacting QKR has instead mostly been studied with an uncontrolled approximation [that we dub the local momentum approximation (LMA), see below] [33, 35, 36, 39]. The latter predicts a subdiffusion of the kinetic energy $\langle p^2(t) \rangle \propto t^\alpha$ ($\alpha \sim 0.3 − 0.4$), thus implying a destruction of dynamical localization, which corresponds to $\alpha = 0$. Furthermore, this exponent $\alpha$ is similar to that found in the 1D interacting Anderson model at mean-field level (the equivalence between QKR and Anderson model being preserved in the frame of the LMA), and is expected to be universal [32, 33].

There is thus a need to study the exact GPE of the interacting QKR, to ensure that this subdiffusion is indeed present and to test the universality of the subdiffusion exponent. In this work, we study the asymptotic behavior of the GPE at very long times (up to $10^5$ kicks), and show that, while our data is compatible with subdiffusion, they are not described by a universal exponent. We offer an improved approximate dynamics which is numerically shown to be a better approximation of the full
GPE approximation for experimentally accessible times (about $10^3$ kicks).

II. THE NONLINEAR QUANTUM KICKED ROTOR

We consider a degenerate boson gas in a ring of circumference $L_\parallel$, tightly confined in the transverse direction with a characteristic energy scale $\hbar \omega_\perp$ and transverse dimension $L_\perp \ll L_\parallel$, such that this energy is much larger than any other energy scale in the system. In such conditions the dynamics of the transverse degrees of freedom is effectively frozen and the longitudinal dynamics is essentially one dimensional. The mean-field bosonic QKR wave function $\psi(x, t)$ is governed by the GPE

$$i\hbar \partial_t \psi = -\frac{\hbar^2 \partial_x^2 \psi}{2} + g|\psi|^2 \psi + K \cos(x) \sum_n \delta(t - n) \psi, \quad (1)$$

where $K$ is the kick amplitude, proportional to the optical potential created by a pulsed standing wave of wavenumber $k_L$, time is expressed in units of the interval $T_k$ between two kicks, lengths are in units of $(2k_L)^{-1}$ and the effective Planck constant is $\hbar = 4 \hbar k_L^2 T_k / M$ with $M$ the atom mass [12]. The dimensionless 1D nonlinear coupling constant in such units is given by

$$g = \pi k^2 k_L a \frac{\hbar}{\omega_R ML_\perp^2} N = \frac{k^2}{2} k_L a \frac{\omega_R}{\omega_{\perp}} N, \quad (2)$$

where the last equality is valid for a harmonic transverse confinement, see for instance [40], $a$ is the (3D) s-wave scattering length of the contact interaction, $N$ the number of atoms, and $\omega_R = \hbar k_L^2 / (2M)$ the recoil frequency. With typical values for Potassium atoms, one obtains $g = 1$ for $L_\perp = 5 \mu$m (or, equivalently, $\omega_{\perp} / 2\pi = 62$ Hz) and $N \sim 1600$.

We use periodic boundary conditions $\psi(L_\parallel, t) = \psi(0, t)$ where $L_\parallel = 2\pi$ is the system size and normalization

$$\int_{L_\parallel} |\psi(x, t)|^2 dz = 1.$$  

Because of this boundary condition, the spectrum of the momentum operator $p = ik \partial_x$ is discrete, $nk$ with $n$ an integer, so that the momentum space wave function $\hat{\psi}(p)$ is given by the Fourier series:

$$\psi(x) = \frac{1}{\sqrt{L_\parallel}} \sum_n \hat{\psi}(nk) e^{inx}, \quad (3)$$

with normalization $\sum_n |\hat{\psi}(nk)|^2 = 1$. We solve Eq. (1) by a real-time propagation using a finite-difference scheme.

The initial state is chosen to be delta-peaked in momentum space, $\hat{\psi}(p, t = 0) = \delta_{p, p_0}$ and results are averaged over the initial momentum $p_0$. Two classes of numerical methods can be used for this type of problems: a Crank-Nicolson scheme, which is conditionally stable and energy-conserving, but implicit in time, thus quite expensive; a split-step method, which is explicit, thus numerically less expensive, but does not conserve energy. Although preferred in the absence of kicks (due to its energy-conserving character), the Crank-Nicolson method turns out to be highly unstable when kicks are applied, while the split-step methods are more robust. However, the split-step method also presents numerical instabilities, and long-time calculations demand higher-order split-step methods to be used. For all the results presented here, we used a second-order split-step method, with a time step $\Delta t = 10^{-4}$, which prevents numerical instability in the time range $[0, 10^6]$ (verified by studying the convergence as a function of the time step). The resulting evolution of the kinetic energy is displayed in Fig. 1, for two values of $g$ and of $K$ (with $k = 2.89$). The curves are averaged over 10 different choices of $p_0 \in [k, 10k]$. While the short-time dynamics is interaction-independent, and well described by the non-interacting QKR, the long-time one features subdiffusive behavior with a non-universal subdiffusive exponent – the lines $t^{0.4}$ and $t^{0.6}$ are guides for the eye.

![FIG. 1. Evolution of the kinetic energy in the GPE QKR, for two values of $g$ and of $K$, and with $k = 2.89$. The long-time dynamics reveals a universal subdiffusion with a non-universal subdiffusive exponent – the lines $t^{0.4}$ and $t^{0.6}$ are guides for the eye.](image)
FIG. 2. Statistics of the subdiffusive exponents found for the three models studied in this work (top panel: exact GPE, middle: PAA, bottom: LMA, see Sec. III). Each histogram is obtained by varying the parameters $g$ and $K$ in $[0, 20]$, and extracting for each parameter set a subdiffusive exponent by a linear regression on the kinetic energy curves in the asymptotic regime. The uncertainty on the exponent is about 0.1 for each model. The dispersion in each set of results suggests a non-universality of the subdiffusive exponents The disagreement between the LMA and PPA with the exact GPE shows their inability to capture correctly the asymptotic subdiffusive regime.

III. ALTERNATIVE MODELS WITH APPROXIMATED DYNAMICS

In momentum space, the interaction term is nonlocal and given by the Fourier transform of $|\psi|^2\psi$, namely

$$F(p, \hat{\psi}) \equiv \frac{1}{2\pi} \sum_{p_1, p_2} \hat{\psi}^*(p_1)\hat{\psi}(p_2)\hat{\psi}(p + p_1 - p_2).$$

(4)

In Ref. [36], a simplification was introduced neglecting all “off-diagonal” contributions, thus making “by hand” the interaction local in momentum space, which amounts to replacing it by

$$F_{\text{LMA}}(p, \hat{\psi}) = \frac{\gamma}{2\pi} |\hat{\psi}(p)|^2\hat{\psi}(p).$$

(5)

where $\gamma$ could be a function of $p$ that cannot be easily determined, and that has been arbitrarily set to 1 in Ref. [36], a choice that we kept in this work. Although not rigorously justified, this “local momentum approximation” (LMA) significantly reduces the cost of the numerical integration, since the resulting GPE can be integrated in momentum space between consecutive kicks. Moreover, the simplified equation evokes a mean-field interacting Anderson model translated in the momentum space. A rationale for the LMA is to suppose that the evolution makes the phases of the different $\hat{\psi}(p, t)$ uncorrelated for large enough momentum differences, so that the integrals in Eq. (4) are dominated by the contributions of $p_1 \simeq p_2 \simeq p$. Finding a rigorous derivation of the LMA is not obvious, but this heuristic argument can be used to construct an improved approximation, which we now describe.

Writing the wave function in the amplitude-phase representation $\hat{\psi}(p) = A(p) e^{i\phi(p)}$, our starting point is the numerical observation that in the presence of kicks, $\phi(p)$ obtained from the full GPE solution appears to be a uniformly randomly distributed function of $p$. In particular, we observe numerically that replacing the phases $\phi(p)$ by independent random phases distributed uniformly in $[0, 2\pi]$, and thus $\hat{\psi}(p) \rightarrow \hat{\psi}_{\text{rand}}(p)$, does not change the (modulus of the) interaction functional $|F(p, \hat{\psi})| \simeq |F(p, \hat{\psi}_{\text{rand}})|$. We can thus treat the phase as a uniformly distributed random function of $p$.

Denoting the average over random $\phi(p)$ realizations by an overline, we have $F(p, \bar{\psi}) = 0$, because it only involves three phases. For the correlations $F(p, \hat{\psi})F^*(p', \hat{\psi})$ we have

FIG. 3. Log of $|F(p, \hat{\psi})|$ computed from a solution $\hat{\psi}$ of the GPE QKR after 200 kicks with parameters $K = 12$, $g = 12$, and $k = 2.89$, and its approximation $|F_{\text{PAA}}(p, \hat{\psi})|$. The agreement is very good, validating the PAA, see Eq. (8).
\[
\frac{F(p, \psi)F^*(p', \psi)}{(2\pi)^2} = \sum_{p_1, p_2, p_1', p_2'} A(p_1)A(p_2)A(p + p_1 - p_2)A(p_1')A(p_2')A(p' + p_1 - p_2') \times \exp \left[ -i\left(\phi(p_1) - \phi(p_2) - \phi(p + p_1 - p_2) - \phi(p_1') + \phi(p_2') + \phi(p' + p_1' - p_2')\right) \right].
\]

Using the independence of the phases, a straightforward calculation gives

\[
\frac{F(p, \psi)F^*(p', \psi)}{(2\pi)^2} = \frac{\delta_{p,p'}}{(2\pi)^2} \left( 4A(p)^2 + 2 \sum_{p_1, p_2} A(p_1)^2A(p_2)^2A(p + p_1 - p_2)^2 \right). 
\]

Furthermore, a numerical study of the phase of \(F(p, \hat{\psi})\) shows that it also appears to be uniformly distributed, and can thus be replaced by \(\phi(p)\). This suggests the following “phase-averaging approximation” (PAA) of \(F(p, \psi)\),

\[
F_{\text{PAA}}(p, \hat{\psi}) = \frac{e^{i\phi(p)}}{2\pi} \sqrt{4A(p)^2 + 2 \sum_{p_1, p_2} A(p_1)^2A(p_2)^2A(p + p_1 - p_2)^2}.
\]

Fig. 4 shows the dynamics as obtained from the GPE, LMA, and PAA, as well as the non-interacting case. While the short-time dynamics is interaction-independent (and thus approximation independent), the LMA fails to capture the regime where the interactions start to be relevant, while this is well achieved by the PAA. This improved approximation is a faithful description of the exact GPE dynamics for the typical time range \([0, 1000]\) accessible to state-of-the-art experiments. In particular, it captures well the timescale at which the interactions start to matter, contrary to the LMA. At longer, experimentally inaccessible, times, however, both the LMA and the PAA disagree with the exact integration of the GPE, probably due to the nonlinear correlations between the phases which start to build up. In particular, none of these approximations is able to successfully capture the subdiffusive exponents, as visible on Fig. 2.

IV. CONCLUSION

The study of the exact GPE dynamics has confirmed the breakdown of dynamical localization observed within the LMA but invalidated the universality of the subdiffusive exponent \(\alpha\). Moreover, we have introduced a new approximation of the GPE, the PAA, which is better justified than the LMA, yet computationally as advantageous, and allows for a good description of the dynamics up to relatively long times where interactions do play a non-negligible role. It, however, does not give the correct range of the subdiffusive exponent. The understanding of the breakdown of the PAA at longer times is left for future work. Explaining the variations of the non-universal exponent \(\alpha\) in the exact GPE is also a very interesting line of research.
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