RADIATIVE CORRECTIONS TO THE STEFAN-BOLTZMANN LAW

FINN RAVNDAL
Institute of Physics, University of Oslo, N-0316 Oslo, Norway

Abstract

Photons in blackbody radiation have non-zero interactions due to their couplings to virtual electron-positron pairs in the vacuum. For temperatures much less than the electron mass \( m \) these effects can be described by an effective theory incorporating the Uehling and Euler-Heisenberg interactions as dominant terms. By a redefinition of the electromagnetic field, the Uehling term is shown not to contribute. The Stefan-Boltzmann energy is then modified by a term proportional with \( T^8/m^4 \) in agreement with the semi-classical result of Barton. The same effects give a speed of light smaller than one at non-zero temperature as has also recently been derived using full QED.

When one calculates the free energy of the quark-gluon plasma in QCD, one encounters infrared divergences in higher order perturbation theory. These can be cured by resummation as shown by Kastening and Zhai. A more direct derivation of the same results was subsequently given by Braaten and Nieto using a dimensionally reduced effective theory in three dimensions. At low temperatures QCD is a strongly interacting theory and the free energy cannot be calculated in perturbation theory. But at very low energies the relevant hadronic degrees of freedom are pions described by a non-linear \( \sigma \)-model. Even though the theory is formally non-renormalizable, radiative corrections can be systematically calculated as long as one works at sufficiently low energies. Using this effective field theory Gerber and Leutwyler could then calculate perturbative corrections to the free energy of interacting pions.

Using the effective field theory approach developed by Braaten and Nieto for QCD, Andersen has shown that the thermodynamics of the electron-photon plasma of QED at high temperatures can also be derived in a more compact way than using the alternative method of resummation. But in contrast with QCD, one can use QED perturbatively also at temperatures less than the electron mass. Since only the photonic degrees are then excited, one can instead of full QED use the effective Euler-Heisenberg theory to derive the relevant thermodynamics. This approach goes back at least to a contribution by Tarrach. Since then it has been used by Barton using a semi-classical

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method to derive the first radiative correction to the energy density of black-body radiation. Tarrach wanted to calculate the effects of quantum fluctuations on the propagation of light at non-zero temperatures. An improved result by LaTorre, Pascual and Tarrach using full QED has since then been derived. We will here consider the effective theory of low-energy photons and use it to derive the above physical consequences. It is shown to represent a large reduction of calculational effort.

Integrating out the electron field in the QED partition function, we obtain the effective Lagrangian

$$L_{\text{eff}}(A) = -\frac{1}{4} F_{\mu\nu}^2 - i \text{Tr} \log [\gamma^\mu (i\partial_\mu - eA_\mu) - m]$$

At low energies it can be expanded in powers of momenta and gives to leading order

$$L_{\text{eff}} = -\frac{1}{4} F_{\mu\nu}^2 + L_U + L_{\text{EH}} + \ldots$$

where the first term

$$L_U = \frac{\alpha}{60\pi m^2} F_{\mu\nu} \Box F^{\mu\nu}$$

is the Uehling interaction due to the lowest-order vacuum polarization loop where $\alpha = e^2 / 4\pi$ is the fine structure constant and $\Box \equiv \partial_\mu \partial^\mu$. The next term

$$L_{\text{EH}} = \frac{\alpha^2}{90m^4} \left[ (F_{\mu\nu} F^{\mu\nu})^2 + \frac{7}{4} (F_{\mu\nu} \tilde{F}^{\mu\nu})^2 \right]$$

is the Euler-Heisenberg interaction where $\tilde{F}_{\mu\nu} = \frac{i}{2} \epsilon_{\mu\nu\rho\sigma} F^{\rho\sigma}$ is the dual field strength.

The equation of motion for the free field is $\Box F_{\mu\nu} = 0$ and thus we see that the Uehling term can be effectively set equal to zero as long as there is no matter present. Since the equation of motion is only satisfied by on-shell photons, one may then question the validity of this simplification where the interactions are used to generate loop diagrams with virtual photons. But since one is in general allowed to shift integration variables in the corresponding functional integrals, we see that under the transformation

$$A_\mu \to A_\mu + \frac{\alpha}{30\pi m^2} \Box A_\mu$$

the Uehling interaction is again removed. We are thus left with the effective Lagrangian

$$L_{\text{eff}}(A) = -\frac{1}{4} F_{\mu\nu}^2 + \frac{\alpha^2}{90m^4} \left[ (F_{\mu\nu} F^{\mu\nu})^2 + \frac{7}{4} (F_{\mu\nu} \tilde{F}^{\mu\nu})^2 \right]$$
describing interacting photons at low energies in the absence of matter.

The energy density of free photons at non-zero temperature $T$ is given by the Stefan-Boltzmann law $E = \frac{\pi^2}{30} T^4$. From the above effective theory we can now calculate the first quantum correction to this classical result. For dimensional reasons it must thus vary with the temperature like $T^8/m^4$. Its magnitude is most directly calculated using the Matsubara formalism where the photon field is periodic in imaginary time. In lowest order perturbation theory the correction to the free energy density is then obtained directly from the partition function as

$$\Delta F = -\frac{\alpha^2}{90m^4} \langle 7 F_{\mu\nu} F_{\alpha\beta} F^{\mu\alpha} F^{\nu\beta} - \frac{5}{2} F_{\mu\nu} F_{\alpha\beta} F^{\mu\nu} F^{\alpha\beta} \rangle$$

where we have rewritten the term involving the dual field strength tensor $\tilde{F}_{\mu\nu}$ by products of $F_{\mu\nu}$ alone. In order to evaluate this expectation value we only need the correlator

$$\langle F_{\mu\nu}(K) F_{\alpha\beta}(-K) \rangle = \frac{1}{K^2} [K_{\mu}K_{\alpha} \delta_{\mu\beta} - K_{\nu}K_{\beta} \delta_{\mu\alpha} + K_{\mu}K_{\beta} \delta_{\nu\alpha} - K_{\mu}K_{\alpha} \delta_{\nu\beta}]$$

Using Wick’s theorem we see that the result is given by the 2-loop Feynman diagram in Figure 1 which is really a product of two 1-loop diagrams. It is therefore almost trivial to evaluate compared with the corresponding 3-loop diagram in full QED. One then obtains

![Figure 1: Lowest order diagram for the photon free energy in the effective theory.](image)

$$\Delta F = -\frac{22\alpha^2}{45m^4} \int_P \int_Q \frac{(PQ)^2}{P^2Q^2}$$

where the basic sum-integral is

$$\int_P \int_Q \frac{Q_{\mu}Q_{\nu}}{Q^2} = \frac{\pi^2T^4}{90} (\delta_{\mu\nu} - 4\delta_{\mu4} \delta_{\nu4})$$
using dimensional regularization for the space integrals and zeta-function regularization for the Matsubara summations. We thus obtain

$$\Delta F = -\frac{22\pi^4\alpha^2}{125 \cdot 243} \frac{T^8}{m^4}$$

This result has previously been derived by Barton using a semi-classical method and treating the interacting photon gas as a material medium. It gives directly the pressure in the gas as $P = -F$. The entropy is given by the derivative with respect to temperature and thus the energy density follows from $\mathcal{E} = (1 - T \partial/\partial T)F$ as

$$\mathcal{E} = \frac{\pi^2}{15} T^4 + \frac{154\pi^4\alpha^2}{125 \cdot 243} \frac{T^8}{m^4}$$

Obviously the corrections due to the Euler-Heisenberg interaction are negligible for ordinary temperatures. Barton has also investigated the implications of the photon interactions for the Planck distribution of blackbody radiation.

It is now straightforward to calculate also the finite-temperature photon self-energy $\Pi_{\mu\nu}(K)$ which will follow from the Feynman diagram in Figure 2 using the above effective Euler-Heisenberg theory. At zero temperature it has previously been considered by Halter who showed that the photon remains massless as expected. Projecting out the longitudinal and transverse components in the formalism of Weldon, we then obtain

$$\Pi_L(K) = \frac{44\pi^2\alpha^2}{2025} \frac{T^4}{m^4}(K^2 + \mathbf{K}^2)$$

and

$$\Pi_T(K) = \frac{44\pi^2\alpha^2}{2025} \frac{T^4}{m^4}(-K^2 + \mathbf{K}^2)$$

Transforming these results back to Minkowski space, we can then obtain the permittivity $\epsilon$ and susceptibility $\mu$ of the photon vacuum at finite temperatures.

\[\text{Figure 2: Lowest order diagram for the photon self energy in the effective theory.}\]
In the static limit then follows

$$
\epsilon = \mu = 1 + \frac{44\pi^2\alpha^2}{2025} \frac{T^4}{m^4}
$$

The speed of light at non-zero temperatures is therefore

$$
c = 1 - \frac{44\pi^2\alpha^2}{2025} \frac{T^4}{m^4}
$$

It is reduced below its ordinary vacuum value due to the interaction of the photons with virtual electron-positron pairs as first calculated by Tarrach\(^6\). A corrected result has since then been derived in full QED from 2-loop diagrams\(^8\) and agrees with the above result obtained more directly in the effective theory.

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