Zero-Point Momentum in Complex Media

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Abstract. In this work we apply field regularization techniques to formulate a number of new phenomena related to momentum induced by electromagnetic zero-point fluctuations. We discuss the zero-point momentum associated with magneto-electric media, with moving media, and with magneto-chiral media.

PACS. 42.50.Lc Quantum fluctuations, quantum noise, and quantum jumps – 12.20.Ds Specific calculations – 3.70.+k Theory of quantized fields – 11.10.Gh Renormalization

1 Introduction

It is well-known and widely accepted that zero-point fluctuations affect the physics on both microscopic and macroscopic scales, upon creating forces between materials. The Casimir effect [1] - the attractive force between two macroscopic metallic plates - and physically equivalent to the Lifshitz effect when it comes to dielectric media - is undoubtedly the most famous effect. Also the 1/r^6 Van der Waals force and its retarded 1/r^7 equivalent, the Casimir-Polder force [2], between microscopic polarizable atoms can be understood as direct manifestations of zero-point energy [3].

Electromagnetic zero-point energy has been the subject of many fundamental work, sometimes heavily debated in literature, since it often lacks experimental verification. The most fundamental aspect of vacuum fluctuations is their Lorentz invariance. It can be shown that only isotropic radiation with power spectrum ω^3, obviously the one associated with modes whose density is proportional to ω^2/c_0^3 and whose zero-point energy is 1/2hω [4] is Lorentz-invariant. The Unruh effect [6] is another fundamental result and concerns observers with constant acceleration a. They see the zero-point energy emerge as a Planck law with temperature kT = ha/2πc_0 [5,4]. This effect has never been observed. On the basis of the relativistic equivalence principle, gravity should create the same effect, known as Hawking radiation [7].

The most controversial aspect of vacuum energy is the UV catastrophe [8]. The vacuum energy density is arbitrarily large at large frequencies. No rigorous mathematical tool seems to exist so far to deal with this problem. Fortunately, the divergence does not affect the Casimir force, since it formally drops out, as can be seen for instance using the Euler-Maclaurin summation formula [9]. Yet, the energy itself may still be an observable. In 1993, Schwinger [9] attempted to explain the electromagnetic energy observed from strongly oscillating water bubbles - an acousto-optical effect called sonoluminescence - in terms of the zero-point energy released by the contracting bubble. Upon disregarding vacuum fluctuations with frequencies beyond the UV, Schwinger concluded that the excess Casimir energy of a sphere with radius a and dielectric constant ε be equal to E_c = hω^2/εc^2(1 - 1/√ε) which has the right scaling - it decreases with decreasing volume - and about the right order of magnitude if the cut-off is properly chosen. Another longstanding issue, first raised by Dirac in 1934 [10], is the observation that zero-point energy should be gravitationally active and should thus appear as a contribution to the cosmological constant in the Einstein equations. If this is true, the divergence of zero-point energy comes in rudely. The Planck length √hG/c_0^3 = 10^{-33}m is the only available cut-off for vacuum modes hand but this would still lead to impossible cosmological scenarios. Dirac concluded that the large zero-point energy is absent for a still mysterious reason.

So good motivations exist to search for regularization techniques that deal with the divergence in a different way. Milton et al [11] proposed a more sophisticated regularization scheme, where only the finite part of the zero-point energy is considered, due to the finite geometry [13]. For a spherical bubble they find E_c ~ +h(ε - 1)^2/a^4, which has clearly the wrong scaling to explain sonoluminescence, and which is orders of magnitude smaller than the estimate by Schwinger. The same regularizations have been tested for a variety of space-time structures to solve the cosmological constant problem (see [14] for a review). Here the sign is also an issue since the cosmological constant is believed to be positive, leading to a negative pressure and an expansion that accelerates. The sign of the Casimir force itself had already been an issue in the sixties. In 1956 Casimir himself proposed [15] that the vacuum force exerted on a spherical metallic shell with surface A = 4πa^2 might have the similar attractive form
\[
F = -\alpha \hbar c_0 a^2 / 4\pi a^4 = -\alpha \hbar c_0 / a^2 \]
as was found for the plates, though with a different unknown constant \(\alpha > 0\). He speculated that this force might stabilize the Coulomb repulsion \(F = +e^2 / a^2\) of the electron. This would provide a first calculation for the fine structure constant, since stability would impose \(e^2 / \alpha \hbar c_0 = \alpha\). Unfortunately, the regularized Casimir force on a metallic surface was shown by Boyer to be repulsive \([5]\).

Only a few years ago, a new controversial effect triggered by zero-point fluctuations was put forward, this time addressing their momentum in magneto-electric (ME) media. In any medium the dielectric constant can be affected by external electric and magnetic fields according to \(\Delta \varepsilon = \chi \vec{k} \cdot (\vec{E}_0 \times \vec{B}_0)\), with \(\vec{k}\) the unit wave vector. This leads to different optical properties for photons propagating along or opposite to the vector \(\vec{E}_0 \times \vec{B}_0 = \vec{S}_0\) though, unlike the Faraday effect, independent on circular polarization. Ref. \([15]\) considered the radiative momentum of photons in ME media. His predictions are made in a context that is already controversial in itself, since the momentum of photons in matter is still heavily debated \([17]\).

The momentum density of zero-point fluctuations in a ME medium was found to be,

\[
p = \frac{1}{32\pi^3} \left( \mu^{-1} + \varepsilon \right) \frac{\hbar \omega^2}{c^4} \chi S_0 \quad (1)
\]

Like Schwinger in his attempt to explain sonoluminescence, Feigel regularized by adopting an UV cut-off for the zero-point spectrum, arguing that at very high frequencies the ME optical response should vanish. His choice of a lower cut-off wavelength of 0.1 nm is based on the fact that optical ME has been observed in the X-ray. He assumed a typical ME effect \(\chi S_0 \approx 10^{-11}\), and with mass densities typically equal to 1 g/cm\(^3\) this would lead to typical speeds of \(v \approx 30\) nm/s, likely to be too small to be measurable. However, our literature study revealed that ME materials exist such as FeGaO\(_3\) for which \(\chi S_0 \approx 10^{-4}\) is observed down to wavelengths of order 2 Å in the X-ray \([18]\). The prediction in Eq. (1) would lead to much larger speeds, up to centimeters per second, that should be observable in experiments.

The calculation of zero-point momentum in ME media puts forward a revolutionary prediction (by APS Focus \([19]\) referred to as “momentum from nothing”), with a clear order of magnitude estimate. This work is a new occasion to question cut-off procedures for zero-point modes. They break the Lorentz-invariance of the quantum vacuum provocatively and indeed the end-result (1) is so much Lorentz-variant, that it is not even likely to be repairable. But most of all, like in the Schwinger theory of sonoluminescence and in the cosmological constant debate, the cut-off procedures give “inelegant” and “unreasonable” results. As for zero-point momentum, the real QED vacuum is known to have a frequency-independent ME response \(\chi \sim \hbar e^4 / m_e^2 c_0^2\) \([20]\), so that Formula 1 predicts a finite zero-point momentum density of “empty” vacuum up to \(10^{30}\) times larger than the momentum density \(E_0 \times B_0 / 4\pi c_0\) associated with the applied fields. In matter, the cut-off procedure is often justified as a crude way of dealing with dispersion, but the results above suggest that this may not be the whole story, and that in reality the UV catastrophe is nonexistent for a yet unknown reason.

In this work we investigate how the zero-point momentum emerges if one applies the field regularization techniques that have been proposed in literature. This technique would eliminate the Schwinger theory as an explanation for sonoluminescence \([11, 12]\). It is not our intention of this work to advocate regularization techniques. We wish here to come to a quantitative prediction by assuming the validity of these techniques. They are well defined mathematically and straightforward to implement numerically, even in symbolic software. However, to our knowledge nobody has ever been able to assign the removed, diverging terms to the values of observable constants, as it should be in a good renormalizable theory. Also experimental tests are rare. Brevik et al. \([12]\) and Barton \([13]\) regularize the zero-point energy of a dielectric sphere with volume \(V = 4\pi a^3 / 3\) and dielectric constant \(\varepsilon\) and show that this method is equivalent to a dimensional regularization of the Van der Waals energy between the atoms constituting the sphere,

\[
\int_V d^3r \int_V d^3r' \left( -\frac{23a^2}{4\pi |r - r'|^2} \right) \rightarrow \frac{23(\varepsilon - 1)^2}{1536\pi a} \quad (2)
\]

In section 2 we show that this regularization appears again in the expression for zero-point momentum. In a previous Letter we have already applied the field regularization methods proposed by Kong and Ravndal \([21]\) for the ME zero-point momentum in the Casimir geometry, and concluded that the effect survives the regularization, but that its value is reduced by some 20 orders of magnitude. In the present work we address a genuine finite object: a ME sphere. This makes the regularized expression above subject to experimental tests, since the momentum of a finite object is a measurable quantity, much more than energy. This would be an indirect test for the regularization of zero-point motion in general, and such knowledge could be of vital importance to proceed for instance in the cosmological constant debate. We will also consider two other situations where zero-point momentum might show up: a moving sphere and a magneto-chiral sphere. Both cases reveal a surprise. For the moving sphere zero-point fluctuations seem to achieve a (regularized) momentum proportional to the velocity of the sphere. This would thus contribute to the mass of the sphere! We will show that the problem of a moving dipole is actually not UV divergent and that a precise prediction is obtained for the contribution of zero-point modes to the mass of a polarizable atom. Finally, for a magneto-chiral sphere, we will present a microscopic argument why the contribution of zero-point motion to momentum should vanish. We hope that this gives deeper insight into the microscopic nature of chirality.
2 Bi-anisotropic sphere

Starting point of our theoretical study is the set of macroscopic Maxwell equations - expressed in Gaussian units - applied to bi-anisotropic matter [22]. Such media are described by a general linear "constitutive" relation between the macroscopic electromagnetic fields \( \mathbf{D}, \mathbf{H} \), and the microscopic fields \( \mathbf{E}, \mathbf{B} \),

\[
\mathbf{D} = \varepsilon \mathbf{E} + \chi \cdot \mathbf{B} \\
\mathbf{H} = -\chi^T \cdot \mathbf{E} + \mu^{-1} \cdot \mathbf{B}
\]

The constitutive tensors \( \varepsilon \) and \( \mu \) are assumed real-valued symmetric, the constitutive, bi-anisotropic tensor \( \chi \) is assumed real-valued. In this first work we wish to exclude the presence of optical dispersion and absorption. In inhomogeneous media all tensors depend on the position vector \( \mathbf{r} \). Time-dependence can be allowed as well provided the variation is much slower than the typical cycle oscillation of the electromagnetic fields, so that we can still work at constant frequency. The best-known case of optical bi-anisotropy is undoubtedly rotatory power, which can be described by the symmetric tensor \( \chi_{ij} = g d_{ij} \), with \( g \) a pseudo scalar, induced by some microscopic chirality.

ME media can be modelled by the anti-symmetric choice \( \chi_{ij} = \chi ( E_i^0 B_j^0 - B_i^0 E_j^0 ) \), with \( \chi \) a scalar. These relations are combined with two Maxwell’s equations applied to homogeneous media all tensors depend on the position vector \( \mathbf{r} \).

This leads to the following wave equation,

\[
\frac{\partial^2}{\partial t^2} \varepsilon (\mathbf{r}) - i \frac{\omega}{c_0} \phi_p \cdot \chi^T (\mathbf{r}) + i \frac{\omega}{c_0} \chi (\mathbf{r}) \cdot \phi_p \\
- \phi_p \cdot \mu (\mathbf{r})^{-1} \cdot \phi_p \] \( \mathbf{E} = -\frac{4 \pi i \omega}{c_0^2} \mathbf{J}_q \) (7)

in terms of the hermitian tensor operator \( \phi_{num,p} \equiv i \epsilon_{numlp} \). From this equation we can identify the interaction between matter and radiation,

\[
\mathbf{V}(\mathbf{r}, \mathbf{p}) \equiv \frac{\omega^2}{c_0^2} [1 - \varepsilon(\mathbf{r})] + i \frac{\omega}{c_0} \phi_p \cdot \chi^T (\mathbf{r}) - i \frac{\omega}{c_0} \chi (\mathbf{r}) \cdot \phi_p \\
- \phi_p \cdot [1 - \mu (\mathbf{r})^{-1}] \cdot \phi_p
\]

Upon combining the macroscopic Maxwell-equations, the constitutive equations and the Lorentz-force \( \mathbf{E} + \mathbf{v} / c_0 \times \mathbf{B} \), we can arrive at the following momentum conservation law,

\[
\partial_t \left( \frac{1}{4 \pi c_0} \mathbf{E} \times \mathbf{B} + \rho \mathbf{v} \right) = \nabla \cdot ( - \rho \mathbf{v} \mathbf{v} + \mathbf{T}_0 ) \tag{9}
\]

with the symmetric vacuum stress tensor \( \mathbf{T}_0 = ( E_i E_j^* + B_i B_j^* ) / 4 \pi - \delta_{ij} \mathbf{E} \) with \( \mathbf{E} = ( \mathbf{E} \cdot \mathbf{E} + \mathbf{B} \cdot \mathbf{B} ) / 8 \pi \) the electromagnetic energy density. Upon integrating Eq. (9) far beyond the physical size of the object we get,

\[
\frac{d}{dt} \int d^3 \mathbf{r} \left( \frac{1}{4 \pi c_0} \mathbf{E} \times \mathbf{B}^* + \rho \mathbf{v} \right) = \lim_{r \to \infty} \int d \mathbf{S} \cdot \mathbf{T}_0 \tag{10}
\]

To work out the flow of momentum to infinity, expressed by the surface integral on the right, we consider a radiation field \( I(\omega, \mathbf{k}) \) incident on a conservative, confined bi-anisotropic obstacle. In the far field the electromagnetic polarization is orthogonal to the direction of propagation \( d \mathbf{S} \). As a result the terms \( E_i E_j^* \) and \( B_i B_j^* \) of \( \mathbf{T}_0 \) can be seen not to contribute to momentum. After some algebra we find that,

\[
\lim_{r \to \infty} \int d \mathbf{S} \cdot \mathbf{T}_0 \sim \int_0^\infty d \omega \int d \mathbf{k}_{in} \int d \mathbf{k}_{out} I(\omega, \mathbf{k}_{in}) \\
\frac{d \sigma(\mathbf{k}_{in}, \mathbf{k}_{out})}{d \Omega} (\mathbf{k}_{in} - \mathbf{k}_{out}) \tag{11}
\]

Simultaneous P-T symmetry guarantees that \( d \sigma(\mathbf{k}_{in}, \mathbf{k}_{out}, \chi) = d \sigma(\mathbf{k}_{out}, \mathbf{k}_{in}, \chi) \) [24]. Hence, if the radiation field \( I(\omega, \mathbf{k}) \) is isotropic, and this is true when the obstacle is subject to zero-point radiation, the momentum flow to infinity vanishes. In particular, no inelastic effects occur due to recoil effects. As a result, the total momentum

\[
m \mathbf{v} + \int d^3 \mathbf{r} \frac{1}{4 \pi c_0} \mathbf{E} \times \mathbf{B}^*
\]

is a conserved quantity. We will refer to the first term as the kinematic momentum and to the second term as the radiative momentum. The above formula for total momentum agrees with the more sophisticated theory by Nelson [25] and we refer to this work for a more detailed discussion in relation to the Abraham-Minkowski controversy [17,26] about which term is the real “radiation momentum” and which part constitutes the genuine momentum of “matter”. In particular, the contribution of radiation to momentum found here is equal to neither the “Abraham value” \( -\frac{1}{4 \pi c_0} \mathbf{E} \times \mathbf{H} \), nor the “Minkowski value” \( -\frac{1}{4 \pi c_0} \mathbf{D} \times \mathbf{B} \). We emphasize that for an isotropic monochromatic wave field scattered from a finite object, the space integral is perfectly finite since the integrand is confined in and around the object. The problems will appear when integrating over a power spectrum that diverges itself as \( \omega^3 \).

Another pertinent remark is that the momentum conservation expressed by Eq. (10) continues to be valid if the constitutive tensors, the tensor \( \chi(\mathbf{r}) \) in particular, are time-dependent. Constitutive equations with time-dependent coefficients can be justified when the variation of the coefficients is slow compared to the variation of the fields themselves. This becomes a delicate issue for vacuum fluctuations that comprise all frequencies and that can thus be arbitrarily slow. The regularization techniques show that
for an object of size \( a \), the typical frequency that contributes to the momentum equals \( c_0/a \). In cut-off procedures even higher frequencies dominate. This leaves enough room to turn on the external fields adiabatically. Since a perfect symmetry exists between the wave vectors \( \mathbf{p} \) and \(-\mathbf{p}\) if \( \chi = 0 \), we anticipate \( \mathbf{E} \times \mathbf{B} \) to vanish before turning on the external fields. The conservation law \([10]\) thus leads us to the conclusion that after having turned on the fields, the object achieves a velocity given by,

\[
m \mathbf{v} = -\frac{1}{4\pi c_0} \int d^3r \langle 0 | \mathbf{E} \times \mathbf{B} | 0 \rangle
\]

(12)

where \( \langle 0 | \cdots | 0 \rangle \) stands for vacuum expectation. For the product of two electric fields this expectation can be obtained from the fluctuation-dissipation theorem, which at zero temperature takes the form,

\[
\langle 0 | E_i(\omega, \mathbf{r}) \mathbf{E}^\dagger_j(\omega', \mathbf{r}') | 0 \rangle = -4\hbar \omega^2 \epsilon_0 \text{Im} G_{ij}(\omega, \mathbf{r}, \mathbf{r}') \times 2\pi \delta(\omega - \omega')
\]

(13)

with \( G \) the (classical) Green’s tensor associated with the wave equation \([7]\). It can be straightforwardly verified that the momentum of zero-point fluctuations is expressed as,

\[
P_{\text{rad}, i} = \frac{1}{4\pi c_0} \int d^3r \langle 0 | \mathbf{E} \times \mathbf{B} | 0 \rangle_i = -\frac{\hbar}{\pi c_0} \times \text{Im} \int_0^\infty d\omega \int \frac{d^3k}{(2\pi)^3} \omega (k_i G_{jj}(\omega, \mathbf{k}, \mathbf{k}) - k_j G_{ij}(\omega, \mathbf{k}, \mathbf{k}))
\]

(14)

For a genuine empty vacuum this reduces to the familiar expression \( 2 \times \int d^3k/(2\pi)^3 \frac{1}{2}\hbar \mathbf{k} \), which is zero in view of the perfect symmetry in \( \mathbf{k} \).

In the following we shall consider a sphere with a dielectric constant \( \epsilon \) slightly different from one and a weak bi-anisotropic tensor \( \chi \), both confined and constant in the sphere. We shall expand the Green’s function in the potential interaction \([\mathbb{S}]\). Only contributions linear in \( \chi \) can survive the symmetry between \( \mathbf{k} \) and \(-\mathbf{k} \). We leave technical details to the Appendices. The first order Born approximation to \( G \) involves one scattering from the sphere expressed by \( G^{(1)}(\mathbf{k}, \mathbf{k}) = G_0(\omega, \mathbf{k}) \cdot \mathbf{V}(\omega, \mathbf{k}) \cdot G_0(\omega, \mathbf{k}) \) in terms of the free-space propagator \( G_0(\omega, \mathbf{k}) \). The frequency integral can easily be performed and we find,

\[
m \mathbf{v}_i \sim \epsilon_{ijk} \chi_{jk} a^3 \int d^3k \hbar \mathbf{k}
\]

(15)

Dimensional regularization puts the integral to zero, and no zero-point momentum - proportional to the volume of the object - is found in this order. Note that this contribution is considered by Feigel, and handled using a cut-off.

We proceed with the second order Born approximation, and collect the terms proportional to \( (\epsilon - 1)\chi \). This involves one normal and one bi-anisotropic scattering. We write \( P_{\text{rad}} = \hbar (I_{ij} - I_{jj}) \) with

\[
I_{ij} = -\frac{1}{\pi c_0^3} \text{Im} \int_0^\infty d\omega \int \frac{d^3k}{(2\pi)^3} \frac{d^3k'}{(2\pi)^3} k_i \left[ G_0(\mathbf{k}) \cdot \mathbf{V}(\omega, \mathbf{k}, \mathbf{k}') \cdot G_0(\mathbf{k}') \cdot \mathbf{V}(\omega, \mathbf{k}', \mathbf{k}) \cdot G_0(\mathbf{k}) \right]_{ij}
\]

(16)

We leave the technical details to Appendix A. Important is that all terms are proportional to the stocked Casimir-Polder energy, that we will regularize as has been proposed in Eq. \([2]\). The final result is,

\[
m \mathbf{v}_i = -P^i_{\text{rad}} = \eta \frac{\hbar}{a}(\epsilon - 1) \epsilon_{imn} \chi_{mn}
\]

(17)

with (see Appendix) \( \eta = (I_0 - I_1 + C/3 - A/3 + D/3 - E/2)/192\pi^2 = 0.007909 \).

### 2.1 Magneto-electric sphere

We can insert the choice for a ME medium \([22][23]\): \( \chi_{mn} = g_{\text{EM}}(E_n B_m - E_m B_n) \) to find that

\[
m \mathbf{v} = 2\eta \times \frac{\hbar}{a}(\epsilon - 1) g_{\text{EM}} \mathbf{E} \times \mathbf{B}
\]

(18)

We recall that this relation applies after having turned on the fields \( \mathbf{E}, \mathbf{B} \) adiabatically. For a piece of FeGaO\(_3\) (mass density 4.5 g/cm\(^3\), \( g_{\text{EM}} E B \approx 10^{-4} \)) of size \( a = 1\mu m \) we find the unmeasurable speed \( v = 10^{-20} \) m/s, some 12 orders of magnitude smaller than the value predicted by Eq. \([1]\).

### 2.2 Moving sphere

It is well known that a sphere with a dielectric constant \( \epsilon \), moving with a speed \( \mathbf{v} \) much smaller than the speed of light, possesses a bi-anisotropic tensor \( \chi_{ij} = (1 - \epsilon)\epsilon_{ijk} c_k/c_0 \) \([4]\). In this work we systematically neglect dispersion of the dielectric constant. We note however that the familiar dispersion for the dielectric constant beyond the plasma frequency, \( \epsilon = 1 - \omega_p^2/\omega^2 \) \([27]\) will not be able to render the first Born approximation \([15]\) finite. This term would still diverge like \( \int dk k \). This suggests that the neglect of dispersion is not at the origin of the UV catastrophe, and that the problem is more fundamental.

Equation \([17]\) thus applies and we can write,

\[
P_{\text{rad}} = -2\eta \times \frac{\hbar}{a c_0}(\epsilon - 1)^2 \mathbf{v}
\]

(19)

The moving sphere thus drags along with him a radiative zero-point momentum with opposite sign. This is a rather revolutionary prediction, since it implies that the mass of the object is reduced by its finite size and its polarizability! We emphasize that this result follows from a dimensional regularization that is still arguably controversial. It
is difficult to say whether dispersion would eliminate the divergence. A scaling argument suggests that a dispersion as least as fast as \( \varepsilon^{-1} \sim 1/\omega^4 \) is required to give a finite result for the second order in the Born expansion.

Note that in this particular case it is possible to book-keep the diverging momentum of the zero-point fluctuations into the bulk mass of sphere, i.e. the one measured for large \( \alpha \gg 1 \). For a dielectric sphere of size 1 \( \mu \)m the change is mass is of order \( 10^{-9}m_e \) and thus completely negligible. One could speculate about this effect on atomic scale \( (a = a_0, \) a polarizability density of order \( a_0^3 \) so that \( \varepsilon - 1 \approx 1 \)). This would yield a mass reduction of order \( 10^{-4}m_e \) (or 50 eV), still small enough to be unobserved. It is even more speculative to apply Eq. (19) on the electron scale \((10^{-15}m)\). Here the mass associated with zero-point momentum becomes of the same order of magnitude as the rest mass itself, that is of order \( m_e = 0.5 \) MeV. Inspired by the original argument by Casimir to explain electron stability \( \alpha \) one could even propose that the momentum of the electron is purely due to zero-point motion. If we propose \( \mathbf{P}_{\text{rad}} = \alpha(\hbar/\pi r_0c_0)\mathbf{v} := m_0\mathbf{v} \) we find that \( \alpha \) should be equal to the fine structure constant \( e^2/\hbar c_0 \). Unfortunately, for the dielectric sphere we find the opposite sign, just like the Casimir force was also seen to be repulsive \[5\], but it is fascinating that this argument gives the right order of magnitude for \( \alpha \). We remark that the value found above for \( \eta \) is close to the fine-structure constant.

### 2.3 Moving dipole

In the following we consider a moving electric dipole and calculate semi-classically the radiative momentum associated with the zero-point fluctuations, modified by the presence of the dipole. The polarization is assumed to be point like, and if the dipole is moving with speed \( \mathbf{v} \) the light-matter interaction becomes

\[
\mathbf{V} = (1 - \varepsilon) \frac{\omega^2}{c_0^2} U|0\rangle \langle 0| - i \frac{\omega}{c_0} \phi_\mathbf{p} \cdot \chi^T U|0\rangle \langle 0| + i \frac{\omega}{c_0} U|0\rangle \langle 0| \chi \phi_\mathbf{p}
\]

(20)

with \( U \) interpreted as a small physical volume associated with the dipole, \( \alpha = (\varepsilon - 1) \) its polarizability density, and the bi-anisotropic tensor \( \chi = (1 - \varepsilon)(\varepsilon \cdot \mathbf{v}/c_0) \). The advantage of this interaction is that the full Born series can be summed, although momentum integrals have to be regularized \[28\]. For \( \mathbf{v} = 0 \), the \( t \)-matrix is found from

\[
t_0 = -\frac{1}{(\alpha \omega^2)^{-1} + G_0(r = 0)} = \frac{-4\pi \Gamma \omega^2/c_0^2}{\omega_0^2 - \omega^2 - \frac{2\imath}{3} \Gamma \omega_0^2 \omega/c_0^2}
\]

(21)

The second familiar formula is obtained when a momentum regularization is adopted for the diverging \( k \)-integral of the Green’s tensor \[28\]. The divergence of the longitudinal part \( A_L/\omega^2 > 0 \) can be absorbed into the polarizability by defining

\[
\frac{1}{\alpha(0)} := \frac{1}{\alpha} + A_L
\]

A similar “satisfactory regularization” procedure is not possible for the transverse Green’s tensor. Dimensional regularization introduces \( \Gamma = 1/4\pi \) and the resonant frequency \( \omega_0 = c_0(\alpha(0)\Gamma)^{-1/2} \).

The regularization for the isotropic dipole produces an “acceptable” result for its scattering amplitude. In Appendix B it is established that the inclusion of bi-anisotropic effects does not lead to new singularities. We can then use Eq. (14) to calculate the vacuum expectation value for \( \mathbf{E} \times \mathbf{B} \). We find

\[
\mathbf{P}_{\text{rad}}(\omega) = \frac{2\hbar}{\pi} \text{Im} \frac{t_0}{\alpha \omega^2} \mathbf{v}
\]

(22)

It can easily be checked that the frequency integral of \( \text{Im} t_0/\omega^2 \) converges and equals \(-\pi(2)\alpha(0)\omega_0/c_0^2 \). Hence we arrive at the final result,

\[
\mathbf{P}_{\text{rad}} = -\frac{\alpha(0)}{\alpha c_0^2} \frac{\hbar \omega_0}{c_0^2} \mathbf{v}
\]

(23)

We conclude that the moving dipole drags a momentum associated with zero point opposite to its kinematic momentum. This can be interpreted as a reduction of the kinematic mass. The minus sign was also found earlier for the moving sphere.

It is surprising to see that the ratio of real to bare polarizability density comes in. For a small dielectric sphere difference between \( \alpha \) and \( \alpha(0) \) can be attributed to depolarization induced by surface charges, but if we want to apply this the model to an atom, \( \alpha \) is usually supposed to be unmeasurable. We can now imagine two scenarios. If this depolarization is negligible, typically true when \( \alpha \approx \alpha(0) \approx U \), the front factor in \( \mathbf{P}_{\text{rad}} \) equals one, and the mass would be reduced by an amount \( \hbar \omega_0/c_0^2 \).

For a typical resonant transition at a few eV this would modify the Hydrogen mass by roughly one part in \( 10^9 \). This is roughly the same value estimated in the previous section on the basis of a dielectric atom. If however \( \alpha \gg \alpha(0) \approx U \), the finite polarizability density is fully governed by surface depolarization, described here by the regularization scalar \( A_L \), then \( \mathbf{P}_{\text{rad}} = 0 \). Unfortunately, the present semi-classical approach is not able to predict the value of \( \alpha(0)/\alpha \). A quantum theory is needed.

We expect in general that zero-point motion does not generate energy flow, not even in bi-anisotropic. This means that the quantum expectation value of the Poynting vector should vanish. In a bi-anisotropic the latter is not necessarily proportional to the momentum. For the moving dipole it can be checked explicitly that the quantum expectation value of the Poynting vector \( \mathbf{S} = c_0 \mathbf{E} \times \mathbf{H}/4\pi \), indeed vanishes.
3 Magneto-chiral object

The optical properties of a homogeneous magneto-chiral (MC) material can be characterized by a contribution to the index of refraction that is independent on polarization, and linear in the magnetic field [29]. Symmetry arguments impose that the sign of this contribution is different for opposite enantiomers, and opposite for counter propagating beams. Thus typically ∆n ∼ gk · B₀ with g a material pseudo scalar related to microscopic chirality. This phenomenon can seen as a collective effect of rotatory power, with optical bi-anisotropy χij = gδij, and the Faraday effect that contributes iVεijkBk to the dielectric constant, with V the Verdet constant.

If we accept this macroscopic description of optical MC, the “radiative” momentum of a MC sphere created by zero-point motion can be calculated in just the same way as was done earlier. The second order Born approximation generates optical MC by means of products of chiral and Faraday-type terms, that have to be regularized when we integrate over all frequencies of the vacuum. In a attempt to be more realistic one could accept that the Verdet constant behaves like V ∼ χ². This phenomenon can seen as a collective effect of rotatory power, with optical bi-anisotropy χij = gδij, and the Faraday effect that contributes iVεijkBk to the dielectric constant, with V the Verdet constant.

A second approach consists of accepting the unavoidable heterogeneous structure of space that underlies spatial chirality. One can propose a simple optical model to describe a chiral “molecule” in terms of a chiral distribution of N > 4 classical dipoles. If the dipoles are subject to the Zeeman effect, this molecule exhibits MC properties in the optical scattering, that are particularly revealed when we average over orientations to restore spherical symmetry [30].

The scattering amplitude of the MC molecule - linearized in the external magnetic field - was obtained in Ref. [30]. It can be inserted into expression (14) to find the radiative momentum. The end result can be expressed in terms of a trace of a complex 3N × 3N matrix involving two Lévi-Civita tensor densities, and the scattering amplitude τ₀ of the dipoles found earlier. A straightforward analysis leads us - quite surprisingly - to exactly the same expression as for the “diffuse supercurrent” that was considered by us in Ref. [30]. The possibility of such a current, directed along the magnetic field and not involving the familiar gradient of energy density (familiar from Fick’s law), was investigated for random media with chiral scatterers, but with negative result. In the present context we thus conclude that the chiral object does not carry any vacuum momentum, not at any frequency, when the magnetic field is turned on.

One can try to analyse this conclusion. In the microscopic picture, Poynting vector and radiative momentum are proportional at any point, since locally μ = 1 and χ = 0. Since we do not expect any macroscopic energy current - quantified by the average Poynting vector over some large volume - to occur in vacuum (yet this statement is hard to prove in heterogeneous, complex media), we might anticipate that also the macroscopic radiative momentum must vanish,

\[ \mathbf{P}_{\text{rad}} = \int \frac{1}{4\pi c_0} \mathbf{E} \times \mathbf{B} |_{\text{0}} = \frac{1}{c_0^2} \int d^3 \mathbf{r} \langle 0 | \mathbf{S} | 0 \rangle = 0 \]  (25)

This second equality does not hold in the macroscopic description of MC, and the two deviate at any point. In the microscopic picture, dispersion and spatial structure have been taken into account much more realistically than in the macroscopic constitutive description. This example thus shows that one has to be careful in applying macroscopic Maxwell equations to fundamental issues whose origin is truly microscopic. The macroscopic, regularized outcome [24] is thus probably wrong.

4 Conclusions

The purpose of this work was to come to concrete expressions for the momentum of zero-point motion in complex media. This constitutes a new and unique occasion to “test” regularization methods for vacuum properties in experiments, since momentum is directly observable, much more than energy. Three cases have been discussed for which the momentum of zero-point motion does not seem to vanish “trivially”. For a non-absorbing sphere subject to both an external electric and magnetic field we find a radiative momentum inversely proportional to its radius. To this end regularization techniques had to be adopted to render the outcome finite. A confrontation of this prediction to future experiments may thus shed new light on the validity of regularization methods in general. The same procedure leads to a radiative momentum of zero-point fluctuations of a moving sphere. This effect in principle lowers the kinetic mass of the sphere. The same conclusion is reached for a moving dipole, thus reassuring that - at least in this case - the prediction is not an artifact of the macroscopic model. At last, regularization techniques have been applied to a sphere exhibiting both rotatory power and the Faraday effect. Here it is possible to come up with a more microscopic description, using Faraday-active dipoles in a chiral geometry. In this case the zero-point momentum is rigorously equal to zero, although the macroscopic models yields a finite result.

In the future we hope to develop fully quantum-mechanical descriptions of magneto-electric objects and moving dipoles. The calculations have also been done for idealized media,
free from dispersion and absorption. Clearly, this has to be improved in the future. The Lorentz-invariance of zero-
point motion is also an important aspect that must be
given attention.

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A Calculation of $I_{ijl}$

In this Appendix we calculate the $I_{ijl}$ defined by Eq. (16).
For simplicity we put $c_0 = 1$. Since the constitutive pa-
rameters are assumed not to vary with frequency, the
frequency integral of this object can be performed using
Cauchy contour integration. Since $G_0 = (1 - k k / \omega^2)(\omega^2 - k^2 + i \epsilon)^{-1}$ has a longitudi-
nal part and a part proportional to the identity, we can essen-
tially discriminate three different contributions to Eq. (16), with either 0, 1, or 2 longitudinal propagators in the above expression.

In the absence of any longitudinal part, the frequency
integral of any contribution proportional to $(\epsilon - 1)\chi$
will be of the form

$$
\int_0^\infty \frac{d \omega}{\omega^2} \frac{1}{(\omega^2 - k^2 + i \epsilon)^2} \frac{1}{\omega^2 - k^2 + i \epsilon} = \frac{\pi i}{4} \frac{k + 2k'}{k + k'}^2
$$

(26)

We can write $V(\omega, k', k) = \tilde{V}(k, k') \theta_{kk'}$ with

$$
\theta_{kk'} = \int_B d^3x \exp[i(\mathbf{k} - \mathbf{k'}) \cdot \mathbf{x}]
$$

(27)

where in our case the integral is to be carried out over a
sphere. This transforms Eq. (16) into,

$$
I_{ijl}^{(0)} = -\frac{1}{4} (\epsilon - 1)(\epsilon_{jmn} \chi_{lm} + \chi_{jmn} \epsilon_{lmi}) \int_B d^3x \int_B d^3y \int_B d^3k
$$

(28)

$$
\int_B d^3k \int_B d^3k' \left(2\pi\right)^3 \exp[i \mathbf{k} \cdot (\mathbf{x} - \mathbf{y})] \exp[-i \mathbf{k'} \cdot (\mathbf{x} - \mathbf{y})]
$$

$$
\frac{k + 2k'}{(k + k')^2} \chi(k_n + k'_n)
$$

The four integrals restore complete spherical symmetry, so
that the tensor $k_n(k_n + k'_n)$ must lead to a factor propor-
tional to $\delta_{n/3}$. The factor of proportionality then im-
dediately follows by contraction. If we re-scale $k = |\mathbf{x} - \mathbf{y}|$ and $k' = |\mathbf{x} - \mathbf{y}| y$ we arrive at the following expression,

$$
I_{ijl} = \frac{-K(B)}{48\pi^3 (\epsilon - 1)}(I_0 - I_1)(\epsilon_{jmn} \chi_{lm} + \chi_{jmn} \epsilon_{lmi})(28)
$$

where we have introduced the volume integral

$$
K(B) = \int_B d^3x \int_B d^3y \frac{1}{|\mathbf{x} - \mathbf{y}|^7}
$$

and the two scalars

$$
I_0 = \int_0^\infty dp \int_0^\infty dq q^3 p + 2q \frac{p + q}{(p + q)^2} j_0(p)j_0(q)
$$

(29)

$$
I_1 = \int_0^\infty dp \int_0^\infty dq q^3 p + 2q \frac{p + q}{(p + q)^2} j_1(p)j_1(q)
$$

(30)

A factor $e^{-(p+q)}$ can be added to ensure convergence.

We can repeat this calculation in the presence of one
longitudinal propagator $(-k k / \omega^2)(\omega^2 - k^2 + i \epsilon)^{-1}$. The
frequency integral now becomes,

$$
\int_0^\infty \frac{d \omega}{\omega^2} \frac{1}{(\omega^2 - k^2 + i \epsilon)^2} \frac{1}{\omega^2 - k^2 + i \epsilon} = \frac{\pi i}{4} \frac{1}{k + k'}^2
$$

(31)
We give the end-result of a long calculation that involves angular averaging of four-rank tensors, that generate Bessel functions of higher order, and that make the calculation of scalars more involved. We find,

\[ I_{ij}^{(1)} = \frac{K(B)}{16\pi^4} (\varepsilon - 1) \times \]
\[ \left[ (D_1 + \frac{C - A}{15}) \epsilon_{lmn} (\delta_{ij} \epsilon_{nlm} + \delta_{il} \epsilon_{njm}) \right. \]
\[ + \left. (D_3 + \frac{C - A}{15}) (\epsilon_{mli} \chi_{jm} + \epsilon_{mji} \chi_{lm}) \right] \tag{32} \]

with \( 6D_1 + 9D_3 = D \) and \( 12D_1 + 3D_3 = E = E_1 + E_2 + E_3 \) in terms of

\[ A = \int_0^\infty dp \int_0^\infty dq \frac{p^4 q^3}{(p + q)^2} j_1(p) j_1(q) \]
\[ C = \int_0^\infty dp \int_0^\infty dq \frac{p^4 q^3}{(p + q)^2} j_0(p) j_0(q) \]
\[ D = \int_0^\infty dp \int_0^\infty dq \frac{p^4 q^3}{(p + q)^2} j_0(p) j_0(q) \]
\[ E = \frac{1}{(4\pi)^2} \int d^3p \int d^3q \frac{(p \cdot q)^2}{p + q} \exp[i(p \cdot q) \cdot \hat{r}] \tag{33} \]

The final contribution involves two longitudinal propagators. Since the angular average over a tensor of order 6 appears it is convenient to perform the contractions first. We find that,

\[ I_{ij}^{(2)} - f_{ij}^{(2)} = -\frac{K(B)E}{32\pi^4} (\varepsilon - 1) \epsilon_{lmn} \chi_{nm} \tag{34} \]

We conclude that all contributions generate the stock Casimir-Polder energy \( K(B) \) of Eq. (2) as the only diverging element. If we adopt dimensional regularization for this object,

\[ K \rightarrow -\frac{\pi^2}{12a} \tag{35} \]

To calculate the double momentum integrals above we can move the \( p \)-integral to the imaginary axis and formulate the resulting integral in the full complex plane using polar coordinates \((r, \phi)\). We find

\[ I_0 = -12 \int_0^{\pi/2} d\phi \cos 5\phi \cos \phi \sin^4 \phi = 0.589 \cdots \]
\[ I_1 = -6 \int_0^{\pi/2} d\phi \sin^2 \phi \cos \phi (3 \sin 2\phi \sin 5\phi + 2 \cos 3\phi) \]
\[ = 4.123 \cdots \]
\[ A = 4 \int_0^{\pi/2} d\phi \sin^2 \phi \cos \phi \cos 2\phi (2 \cos 3\phi + 3 \sin 2\phi \sin 5\phi) \]
\[ = 1.374 \cdots \]
\[ C = 24 \int_0^{\pi/2} d\phi \sin^4 \phi \cos \phi \cos 5\phi \cos 2\phi = -1.767 \cdots \]
\[ E_1 = -24 \int_0^{\pi/2} d\phi \sin^2 \phi \cos \phi (1.5 \sin 2\phi \sin 5\phi + \cos 3\phi) \]
\[ = 8.246 \cdots \]
\[ E_2 = -4 \int_0^{\pi/2} d\phi \cos^2 \phi (3 + 4 \sin 2\phi \sin 4\phi + 3 \sin \phi \sin 3\phi + \cos \phi \cos 3\phi) \]
\[ = -13.744 \cdots \]
\[ E_3 = 6 \int_0^{\pi/2} d\phi \cos \phi (6 \cos \phi - 2 \cos 2 \phi \cos 3\phi \]
\[ - \sin^2 2\phi \cos 5\phi) = 24.74 \cdots \]

B Radiative momentum of moving dipole

The scattering matrix of the moving dipole can be calculated from the Born series \((e_0 = 1)\)

\[ T_{kk'} = V_{kk'} + \int \frac{d^3k''}{(2\pi)^3} V_{kk''} \cdot G_0(k'') \cdot V_{k''k'} + \cdots \]

which for the bi-anisotropic point dipole can be fully summed up to

\[ T_{kk'} = t_0 + \frac{t_0}{(1 - e_0)^2} \left[ -i \phi k \cdot \chi^T + i \chi \phi k' \right] \tag{36} \]

Here, \( t_0 \) is the \( t \)-matrix of the isotropic dipole, given in the text. We can develop Eq. (14) to

\[ \begin{align*}
\mathbf{q} \cdot \mathbf{P}_{\text{rad}}(\omega) &= -\frac{\hbar}{\pi} \text{Im} \ Tr \left[ \int \frac{d^3k}{(2\pi)^3} (\mathbf{k} \cdot \mathbf{q} - \mathbf{kq}) \cdot G_0(k) \cdot \\
& \quad \cdot \left[ (\mathbf{\epsilon} \cdot \mathbf{k}) \cdot (\mathbf{\epsilon} \cdot \mathbf{v}) + (\mathbf{\epsilon} \cdot \mathbf{v}) \cdot (\mathbf{\epsilon} \cdot \mathbf{k}) \right] \cdot G_0(k) \right]
\end{align*} \]

We have \((\mathbf{\epsilon} \cdot \mathbf{k}) \cdot (\mathbf{\epsilon} \cdot \mathbf{v}) + (\mathbf{\epsilon} \cdot \mathbf{v}) \cdot (\mathbf{\epsilon} \cdot \mathbf{k}) = \mathbf{k} \mathbf{v} + \mathbf{v} \mathbf{k} - 2 \mathbf{k} \cdot \mathbf{v}\). It is convenient to use the identity, valid to order \( \mathbf{v} \),

\[ G_0(k)[-\mathbf{k} \mathbf{v} - \mathbf{v} \mathbf{k}] G_0(k) = G_0(k + \frac{\mathbf{v}}{2}) - G_0(k - \frac{\mathbf{v}}{2}) \]

This brings us to

\[ \mathbf{q} \cdot \mathbf{P}_{\text{rad}}(\omega) = -\frac{\hbar}{\pi} \text{Im} \ Tr \int \frac{d^3k}{(2\pi)^3} (\mathbf{q} \cdot \mathbf{v} - \mathbf{qv}) \cdot G_0(k) \]

Since \( \int d^3k (2\pi)^3 G_0(k) = -1/\alpha^2 - 1/t_0 \) this reduces to

\[ \mathbf{q} \cdot \mathbf{P}_{\text{rad}}(\omega) = \left( \mathbf{q} \cdot \mathbf{v} \right) \times \frac{2\hbar}{\pi} \text{Im} \ \frac{t_0(\omega)}{\alpha^2} \tag{37} \]