The electric conductivity of a pion gas

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Abstract. The determination of transport coefficients plays a central role in characterizing hot and dense nuclear matter. In the present work we calculate the electric conductivity of hot hadronic matter by extracting it from the $\rho$ meson spectral function, as its zero-energy limit at vanishing momentum. Using hadronic many-body theory, we calculate the $\rho$ meson self-energy in a pion gas. This requires the dressing of the pion propagators in the $\rho$ self-energy with $\pi$-$\rho$ loops, and the inclusion of vertex corrections to maintain gauge invariance. The resulting spectral function is used to calculate the electric conductivity of hot hadronic matter. In particular, we analyze the transport peak of the spectral function and extract its behavior with temperature and coupling strength. Our results suggest that, while obeying lower bounds proposed by conformal field theories in the strong-coupling limit, hot pion matter is a strongly-coupled medium.

1. Introduction
Transport coefficients are an important tool for characterizing hot nuclear matter. Recent calculations of the electric conductivity in hot hadronic matter have been performed, however, their results have varied considerably [1,2,3,4,5]. Furthermore, several of the calculations yield a result below a quantum lower bound proposed in Ref. [1]. Here, we perform a quantum many-body calculation, while attempting to maintain unitarity and gauge invariance.

Our calculation is rooted in dilepton emission calculations reviewed in Ref. [6]. The dilepton emission rate is proportional to the electromagnetic current correlator ($\Pi_{EM}$),

$$\frac{dR_{l+l-}}{d^4q} = -\frac{\alpha_{EM}^2}{\pi^3 M^2} f_B(q_0, T) Im[\Pi_{EM}(M, q, T, \mu_B)].$$

(1)

Where $f_B$ denotes the Bose distribution and $\alpha_{EM} = \frac{1}{137}$. One can readily extract the electric conductivity from $\Pi_{EM}$ in the zero-momentum, low-energy limit using

$$\sigma(T) = -e^2 \lim_{q_0 \to 0} \frac{Im[\Pi_{EM}(q_0, q, 0, T)]}{q_0}.$$  

(2)

Using weak-coupling techniques in a three-flavor quark-gluon plasma (QGP), it was found that $Im[\Pi_{EM}(q_0, q, 0, T)]$ displays a Lorentzian-like transport peak centered at zero energy [7]. For zero coupling the transport peak becomes a Dirac $\delta$-function at zero energy, resulting in an infinite conductivity. As the coupling is increased the peak spreads out, allowing the extraction of a finite conductivity.
Within the vector meson dominance model, electromagnetic interactions in hadronic matter are mediated by the $\rho$, $\omega$, and $\phi$ mesons, with the primary contribution coming from the $\rho$ meson. Thus, one can relate $\Pi_{EM}$ to the $\rho$ propagator ($D_\rho$) via:

$$\Pi_{EM}(q, T) = \frac{m_\rho^4}{g_\rho^2} D_\rho(q, T) = \frac{m_\rho^4}{g_\rho^2} \frac{1}{q^2 - m_\rho^2 - \Sigma_\rho(q, T)}.$$  \hspace{1cm} (3)

The $\rho$ self-energy ($\Sigma_\rho$) characterizes the $\rho$'s interaction with the medium. The contributions of thermal baryons to $\Sigma_\rho$ were calculated for example, in Ref. [8]. However, in the low-$T$ limit thermal baryon-antibaryon excitations are suppressed and the effects of pions will become dominant. Here, we calculate the effects of in-medium pions on $\Sigma_\rho$, requiring the determination of the relevant Feynman diagrams between thermal pions and the $\rho$ meson.

In section 2 we calculate the contribution of in-medium pions to $\Sigma_\rho$. We calculate the vertex corrections required to maintain gauge invariance in section 3. In section 4 we discuss our preliminary results. We summarize and discuss future work in section 5.

2. Rho self-energy from in-medium pion cloud

In Ref. [9] it was found that there are two $\rho$ self-energy diagrams required to maintain gauge invariance in vacuum, see the left and middle panels of Fig. 1. The pertinent self-energies are given by:

$$\Sigma_\rho^{\mu\nu}(q) = ig_\rho^2 \int \frac{d^4k}{(2\pi)^4} \frac{(2k + q)^\mu(2k + q)^\nu}{(k^2 - m_\rho^2 + i\epsilon)((k + q)^2 - m_\rho^2 + i\epsilon)} - 2ig_\rho^2 g^{\mu\nu} \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 - m_\rho^2 + i\epsilon},$$ \hspace{1cm} (4)

where $g_\rho$ is the $\rho\pi\pi$ coupling. We adopt the value of $g_\rho = 5.9$, as used in Ref. [8].

We now consider the effects of thermal pions on $\Sigma_\rho$. Resonant scattering of a cloud-pion with a thermal pion via an intermediate $\rho$ modifies $\Sigma_\rho$ (see the right panel of Fig. 1), requiring the dressing of the cloud pions’ propagators with a self-energy given by:

$$\Sigma_\pi(k) = 2ig_\rho^2 \int \frac{d^4p}{(2\pi)^4} \frac{(k + p)_\mu(k + p)_\nu(-g^{\mu\nu} + \frac{(p - k)_\mu(p - k)_\nu}{m_\rho^2})}{(p^2 - m_\rho^2 + i\epsilon)((p - k)^2 - m_\rho^2 + i\epsilon)}.$$ \hspace{1cm} (5)

![Figure 1](image)

**Figure 1.** Left panel: The $\pi\pi$ loop given by the first integral in Eq. (4). Middle panel: The tadpole loop given by the second integral in Eq. (4). Right panel: The pion self-energy resulting from resonant scattering of a cloud-pion with a thermal pion through an intermediate $\rho$ meson.

We resum the pion propagators in the $\rho$ self-energy with $\Sigma_\pi$ giving:

$$\Sigma_\rho^{\mu\nu}(q, T) = ig_\rho^2 \int \frac{d^4k}{(2\pi)^4} \frac{(2k + q)^\mu(2k + q)^\nu}{(k^2 - m_\rho^2 - \Sigma_\pi(k, T))((k + q)^2 - m_\rho^2 - \Sigma_\pi(k + q, T))}$$

$$- 2ig_\rho^2 g^{\mu\nu} \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 - m_\rho^2 - \Sigma_\pi(k, T)}.$$ \hspace{1cm} (6)

If the propagators are left undressed, an infinite conductivity will result from the calculation, due to the presence of non-interacting particles.
3. Vertex corrections
Dressing the pion propagators with $\Sigma_\pi$ violates gauge invariance. This needs to be corrected by modifying the $\rho\pi\pi$ and $\rho\rho\pi\pi$ vertices, which amounts to considering the interactions of thermal pions with the vertices. We follow the lead of Refs. [8] and [10] to determine the required corrections. The calculation will be gauge invariant if the vertex corrections satisfy the following Ward-Takahashi identities:

$$q^\mu \Gamma^{(3)}_{\mu ab} = g_\rho \epsilon_{3ab} (-\Sigma_\pi (k + q) + \Sigma_\pi (k))$$

(7)

$$q^\mu \Gamma^{(4)}_{\mu\nu ab} = ig_\rho (\epsilon_{3ca} \Gamma^{(3)}_{\nu bc}(k, -q) - \epsilon_{3be} \Gamma^{(3)}_{\nu ca}(k + q, -q)),$$

(8)

where $\Gamma^{(3)}_{\mu ab}$ and $\Gamma^{(4)}_{\mu\nu ab}$ are the vertex corrections to the $\rho\pi\pi$ and $\rho\rho\pi\pi$ vertices, respectively.

Figure 2. Left: Corrections to the $\rho\pi\pi$ vertex. Right: Corrections to the $\rho\rho\pi\pi$ vertex.

One must calculate all diagrams resulting from replacing a vacuum vertex with a corresponding vertex correction, see Fig. 2. The vertex corrections contain intermediate pion and $\rho$ propagators, which must be dressed to extract a finite conductivity. We dress these propagators with widths, which are constant in energy and momentum. While introducing these widths also entails a violation of gauge invariance, it is controlled through a suppression by a thermal $\rho$ mass.

4. Results
We have not completed the calculation of all the vertex corrections at this point. However, the majority of the corrections have been calculated, and we do not expect our final results to largely change upon completion.

Figure 3. Left panel: The energy-scaled EM spectral function, $\sigma/T$ can be extracted from the zero crossing. Values for several $g_\rho$’s are displayed. Right panel: $\sigma/T$ vs $1/g_\rho$, allowing one to examine the strong-coupling limit of the calculation. The solid line shows the proposed lower bound from Ref. [1].
Our results show that by \( T = 150 \text{ MeV} \) the Lorentzian peak in \( \text{Im}[\Pi_{EM}(q_0,\vec{q}=0,T)] \) has flattened out, see the left panel of Fig. 3. As we decrease \( g_\rho \) we see a re-emergence of the Lorentzian peak, suggestive of a weakly-coupled system with a well-defined conductivity peak, not unlike the one in the perturbative QCD calculation of Ref. [7]. Our findings thus suggest that the pion gas is a strongly coupled medium. The right panel of Fig. 3 shows that as the coupling increases our calculation may approach a lower limit compatible with the quantum bound.

Figure 4. Results for the conductivity of a pion gas as a function of \( T \). The solid line shows the proposed lower bound from Ref. [1].

Figure 4 shows our results for \( \sigma/T \) as a function of \( T \) for the nominal value of \( g_\rho = 5.9 \). Our results are significantly larger than previous calculations for a pion gas compiled in Ref. [11]. However, they respect the quantum bound proposed in Ref. [1].

5. Summary and future work
Our results suggest that a.) pion matter is a strongly coupled medium b.) the conductivity obeys a proposed quantum lower bound. Once we have completed the calculation of the spectral function at zero momentum, we will implement the baryonic effects calculated in Ref. [8] into our calculation. We will then extend the calculation to finite momentum and use the results to calculate corrections to dilepton emission rates of hot hadronic matter.

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