CHARACTERIZATION OF GENERALIZED 
YOUNG MEASURES GENERATED BY 
A-FREE MEASURES 

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Abstract. We organize a robust analytical framework in terms of the space 
$\text{BV}^A(\mathbb{R}^d)$ of functions with bounded $A$-variation, where $A$ is a partial differential operator satisfying Murat’s constant rank property. This perspective enable us to carry constructions available for gradients into the $A$-free framework (introduced by Dacorogna and Fonseca & Müller). In particular, this allows the gluing and localization of $A$-free measures without modifying the underlying $A$-free constraint. We combine these advances with delicate geometric constructions to give a full characterization of the class of generalized Young measures generated by sequences of $A$-free measures (where $A$ is an operator of arbitrary order). The main characterization result is stated in terms of a well-known separation property involving the class of $A$-quasiconvex integrands. We give a second characterization in terms of the tangent Young measures being $A$-free Young measures. Lastly, we show that the inclusion 
$L^1(\Omega) \cap \ker A \hookrightarrow M(\Omega) \cap \ker A$ 
is dense with respect to the area-strict convergence of measures.

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1. INTRODUCTION

The last decades have witnessed an extensive development of the study of non-convex variational energies related to equilibrium configurations of materials in a wide range of physical models (such as the study of crystalline solids and thermelastic materials, linear elasticity, perfect plasticity, micromagnetics, and ferromagnetics, among others [10, 14, 19, 27]). Often, these models consist in a minimization principle for integrals of the form

\[ u \mapsto I_f(u) = \int_\Omega f(x, u(x)) \, dx, \]

where \( f : \Omega \times \mathbb{R}^N \to \mathbb{R} \) satisfies a uniform \( p \)-growth condition

\[ |f(z)| \lesssim 1 + |z|^p, \]

and the candidate configurations \( u : \Omega \to \mathbb{R}^N \) obey a set of physical laws determined by a system of linear PDE’s

\[ A u = 0 \text{ in the sense of distributions on } \Omega. \]

In these circumstances, designs with near to minimal energy exhibit compatible equilibrium behavior at microscopical scales, while, at larger scales, configurations adapt by gluing together the low energy patterns allowed by the governing equations in (2). This interplay conveys the formation of finer and finer oscillations, often resulting in some form of \( L^p \)-weak convergence \( u_j \rightharpoonup u \) when \( p > 1 \), or weak-* convergence (in the sense of measures) when \( p = 1 \) [3, 4, 8, 12, 24, 25, 32, 33, 44, 45]. In general, such weak forms of convergence are incompatible with the lower semicontinuity of the energy, which is usually the starting point of a minimization principle. Additionally, the case \( p = 1 \) is often ill-posed in the sense that, independently of the PDE-constraint, a solution to the minimization problem may fail to exist. Indeed, since \( L^1 \) is not a reflexive space, it naturally lacks compactness properties, and therefore one is left to extend the variational setting (1)-(2) to the minimization of the extended energy functional

\[ \mu \mapsto I_f(\mu) := \int_\Omega f(x, \mu^{ac}(x)) \, dx + \int_{\mathbb{R}^N} f^\infty \left( \frac{d\mu}{d|x|^*} \right) |x|^*, \]

defined for measure-valued configurations \( \mu \in \mathcal{M}(\Omega; \mathbb{R}^N) \cap \ker A \). Here, \( f^\infty \) is the strong recession function of \( f \) defined (provided that it exists) as

\[ f^\infty(x, z) := \lim_{x', t \to \infty} \frac{f(x', tz')}{t} \quad \text{for all } x \in \overline{\Omega} \text{ and } z \in \mathbb{R}^N. \]

Here we focus in the case \( p = 1 \), which requires a careful study of the oscillation and concentration phenomena occurring along weak-* convergent sequences of measures satisfying (2). In this regard, an equivalent approach towards the understanding of (1)-(2) is to fully characterize all the parametrized measures, in this case generalized Young measures (see [2, 21]), generated by uniformly bounded sequences \( (\mu_j) \subset \mathcal{M}(\Omega; \mathbb{R}^N) \cap \ker A \). Let us recall that, formally, a sequence \( (\mu_j) \subset \mathcal{M}(\Omega; \mathbb{R}^N) \) is said to generate the generalized Young measure \( (\nu_x, \lambda, \nu^\infty_x) \) \( x \in \overline{\Omega} \) if and only if

\[ T_f(\mu_j) \to \int_{\mathbb{R}^N} \left( \int_{\mathbb{R}^N} f(x, z) \, d\nu_x(z) \right) \, dx \]
\[ + \int_{\mathbb{R}^{N-1}} \left( \int_{\mathbb{R}^{N-1}} f^\infty(x, z) \, d\nu^\infty_x(z) \right) \, d\lambda(x), \]
for all sufficiently regular integrands \( f : \Omega \times \mathbb{R}^N \to \mathbb{R} \) with linear growth at infinity as above. Here, \((\nu_x, \lambda, \nu_x^\infty)_{x \in \mathbb{R}^d}\) is a triple consisting of a non-negative measure \( \lambda \in \mathcal{M}(\Omega) \) and families of probability measures \( \{\nu_x\}, \{\nu_x^\infty\} \subset \text{Prob}(\mathbb{R}^N) \).

The work of Young [50–52] and the use of parametrized measures plays a fundamental role in representing generalized solutions of optimal control problems. In the calculus of variations, it also serves the purpose of addressing the relaxation of integrals. However, the study of (classical) Young measures from the point of view of partial differential equations started with the work of Tartar & Murat, who, motivated by problems in continuum mechanics and electromagnetism, introduced the theory of compensated compactness [37,39,47,48]. With this sophisticated theory, they were able to explain the interaction between the PDE-constraint and the quadratic constitutive relations for linearized models \((p = 2)\).

The first full characterization of Young measures in the PDE-constrained context was achieved for nonlinear variational models where the configuration is the gradient \( \nabla w \) of a Sobolev function \( w \in W^{1,p}(\Omega; \mathbb{R}^m) \); this corresponds to the PDE-constraint \( A = \text{curl} \) and the space \( \mathbb{R}^N = \mathbb{R}^m \otimes \mathbb{R}^d \) in the context of \((1)-(2)\).

In this particular case, it is well-known [1,11,34] that a sufficient and necessary condition for the lower semicontinuity of \((1)\) is given by Morrey’s quasiconvexity condition. The characterization of \(L^p\)-gradient Young measures, which is due to Kinderlehrer & Pedregal [28,29], accounts for the validity of Jensen’s inequality between gradient Young measures and all quasiconvex integrands. More precisely, they showed that a weak-* measurable map \( \nu : \Omega \to \text{Prob}(\mathbb{R}^m \otimes \mathbb{R}^d) \) —corresponding to the family \( \{\nu_x\} \) in the generalized setting described above—is a (classical) Young measure generated by a \(p\)-equi-integrable sequence of gradients \((\nabla w_j)\) if and only if \( \nabla w_j \rightharpoonup \nabla w \) in \(L^p\) and

\[
(4) \quad h(\nabla w(x)) \leq \int_{\mathbb{R}^m \otimes \mathbb{R}^d} h(z) \, d\nu_x(z) \quad \text{at } \mathcal{L}^d\text{-a.e. } x \in \Omega,
\]

for all quasiconvex integrands \( h : \mathbb{R}^m \otimes \mathbb{R}^d \to \mathbb{R} \) with \(p\)-growth at infinity; this characterization result holds for \(p = 1\), however it requires the generating sequences to be equi-integrable. The extension of this result to generalized Young measures generated by gradients (which is instead associated to the space \(BV(\Omega; \mathbb{R}^m)\) of functions of bounded variation) is due to Kristensen & Rindler [32]. There, the authors show that a generalized Young measure \((\nu, \lambda, \nu^\infty)\) is generated by a sequence of gradient measures if and only if the same condition \((4)\) holds for \(\nu\). Surprisingly, this conveys that the singular part of \((\nu, \lambda, \nu^\infty)\) remains somewhat unconstrained; this owes to Alberti’s Rank One Theorem and the recent rigidity result for rank-one convex and positively homogeneous functions contained in [30].

The efforts to establish a formal Young measure \(A\)-free variational theory initiated with the work of Dacorogna [16], who studied \(A\)-free maps \(u\) which are represented by potentials \(u = Bv\) where \(B\) is a suitable first-order operator. However, it was the seminal work of Fonseca & Müller which laid the foundations for an \(A\)-free setting under a more general assumption. More precisely, they studied the case when \(A\) is a first-order homogeneous partial differential as in \((5)\), and \(A\) satisfies the so-called constant rank property (see \((6)\) below).\(^1\) The authors generalized Morrey’s notion of quasiconvexity to the \(A\)-free setting and showed that the necessary and sufficient condition for the lower semicontinuity of \((1)\), under \(p\)-growth and \(p\)-equi-integrability assumptions, was precisely the \(A\)-quasiconvexity of the integrand. Let us recall that a Borel integrand \(h : \mathcal{W} \to \mathbb{R}\) is called \(A\)-quasiconvex

\(^1\) A recent result of Raita [43] establishes that, in fact, Dacorogna’s assumption and the constant rank assumption are equivalent under the \(p\)-equi-integrability assumption; our results extend this to the case \(p = 1\) when no equi-integrability is assumed.
if
\[ h(z) \leq \int_{[0,1]^d} h(z + w(y)) \, dy \quad \text{for all } z \in \mathbb{W}, \]
and all periodic \( w \in C^\infty_{\text{per}}([0,1]^d; \mathbb{W}) \) satisfying
\[ \int_{[0,1]^d} w = 0 \quad \text{and} \quad \mathcal{A} w = 0 \quad \text{on } \mathbb{R}^d. \]

Moreover, they also extended the characterization theorem of Kinderlehrer & Pedregal to the \( \mathcal{A} \)-free setting by showing that a weak-\( \star \) measurable map \( \nu : \Omega \rightarrow \text{Prob}(\mathbb{W}) \) is a Young measure generated by a \( p \)-equi-integrable sequence of \( \mathcal{A} \)-free maps \( (u_j) \) if and only if the following three conditions hold:

(i) The \( p \)th moment of \( \nu \) is bounded, that is,
\[ \int_{\Omega} \int_{\mathbb{W}} |z|^p \, d\nu_x(z) \, dx < \infty; \]

(ii) there exists \( u \in L^p(\Omega; \mathbb{W}) \) such that \( \mathcal{A} u = 0 \) and
\[ u(x) \equiv \int_{\mathbb{W}} z \, d\nu_x(z) \quad \text{as functions in } L^p(\Omega; \mathbb{W}), \]

(iii) and, at \( \mathbb{L}^d \)-almost every \( x \in \Omega \), the Jensen-type inequality
\[ h(u(x)) \leq \int_{\mathbb{W}} h(z) \, d\nu_x(z) \, dx, \]
holds and all \( \mathcal{A} \)-quasiconvex integrands \( h : \mathbb{W} \rightarrow \mathbb{R} \) with \( p \)-growth at infinity.

It is worthwhile to remark that the main challenge of extending these results from the space of gradients to the \( \mathcal{A} \)-free setting, at that time, was the lack of a potential structure of \( \mathcal{A} \)-free fields. In this regard, the presentation of Kinderlehrer & Pedregal’s for gradients departs significantly from the \( \mathcal{A} \)-free setting; gradients allow for localizations of the form \( w \mapsto \nabla(\varphi w) \), which do not break the \( \mathcal{A} \)-free constraint (here \( \mathcal{A} = \text{curl} \)). Instead, manipulations had to be carried out at the level of the \( \mathcal{A} \)-free field \( u \), requiring a projection on \( \ker \mathcal{A} \) (this is the only point where the constant rank condition is crucially used).

In the generalized Young measure framework, full characterization results are restricted to gradients (as discussed previously) and symmetrized gradients [18]. These results, however, rely on the strong rigidity of gradient and symmetric gradients (see [44, 45]). In general, the only available result is a partial characterization due to Bafá, Matias & Santos. There, the authors characterize all (generalized) Young measures generated by \( \mathcal{A} \)-free measures, under the following somewhat restrictive assumptions:

1. The operator \( \mathcal{A} \) is defined on its essential domain. This means that its associated principal symbol \( \mathcal{A} \) satisfies
\[ \mathcal{A}(\xi) \in \text{Lin}(\mathbb{W}, \mathbb{X}), \quad \xi \in \mathbb{R}^d \setminus \{0\}, \]
for some finite dimensional spaces \( \mathbb{W}, \mathbb{X} \) with
\[ \mathbb{W} = \text{span}(\Lambda_{\mathcal{A}}), \]
where \( \Lambda_{\mathcal{A}} \) is the wave cone associated to \( \mathcal{A} \); see the section below for the correspondent definitions.

2. The operator \( \mathcal{A} \) is assumed to be of first-order. This implies that its associated principal symbol map \( \xi \mapsto \mathcal{A}(\xi) \) is a linear map. In turn, this allows for homogenization-type arguments which unfortunately fail for higher order operators.
3. The characterization is restricted to Young measures generated measures $\mu_j \rightharpoonup \mu$, where the limiting measure $\mu$ satisfies the following Morrey semi-norm bound

$$[\mu]_{1,1+\alpha} := \sup_{r>0} \frac{|\mu|(B_r(x))}{r^{1+\alpha}} < \infty \quad \text{for some } \alpha > 0.$$  

This Morrey-type bound on $\mu$ is in general too restrictive for applications; it rules out measures with Hausdorff dimension equal to one. For instance, every closed smooth curve $\Gamma : [0,1] \to \Omega \subset \mathbb{R}^d$ defines a solenoidal measure (divergence-free measure) by setting

$$\gamma = \tau_\Gamma \mathcal{H}^1 \setminus \Gamma.$$  

Here, $\tau_\Gamma$ is a normal tangent vector-field of $\Gamma$. For such $\gamma$, the Morrey semi-norm bound above fails for all $\alpha > 0$, that is, $[\gamma]_{1,1+\alpha} = \infty$. In particular, Assumption $(\dagger)$ prevents the characterization of Young measures generated by normal 1-currents.

1.1. **Set-up and main results.** The purpose of this work is to give a full characterization of all generalized Young measures generated by $A$-free measures, therefore extending the (classical) Young measure characterization of Fonseca & Müller [23] into a more general setting which allows the appearance of mass concentrations along the sequence in the case $p = 1$. In addition, the main contribution of this work is the closing of the gap between Dacorogna’s $A$-free potential framework and the constant rank $A$-free framework when no equi-integrability is assumed. This however does not immediately follow from Raita’s algebraic characterization of the symbol of constant rank operators [43, Theorem 1]; see also [41]. Our strategy departs from previous ones (even in the case of gradients) in the sense that we do not work with averaged Young measure approximations. Instead, we work with Lebesgue-point properties and the gluing of local generating sequences in the level of a suitable potential.

Here and in what follows, $\Omega \subset \mathbb{R}^d$ is an open and bounded set with $\mathcal{L}^d(\partial \Omega) = 0$ (where $\mathcal{L}^d$ denotes the $d$-dimensional Lebesgue measure). We work with homogeneous partial differential operators $A$ of the form

$$A = \sum_{|\alpha|=k} A_\alpha \partial^\alpha = 0, \quad A_\alpha \in \operatorname{Lin}(\mathcal{W}; \mathbb{X}),$$  

where $\mathbb{X}, \mathcal{W}$ are finite dimensional inner product euclidean spaces. Here $\alpha \in \mathbb{N}_0^d$ is a multi-index with modulus $|\alpha| = \alpha_1 + \cdots + \alpha_d$ and $\partial^\alpha$ represents the distributional derivative $\partial^{\alpha_1}\cdots \partial^{\alpha_d}$. Our main assumption on $A$ is that it satisfies the following constant rank condition introduced by Murat [38] (see also [23]): there exists a positive integer $r$ such that

$$\text{rank } A(\xi) = r \quad \text{for all non-zero directions } \xi \in \mathbb{R}^d,$$

where

$$A(\xi) := \sum_{|\alpha|=k} A_\alpha \xi^\alpha \in \operatorname{Lin}(\mathcal{W}; \mathbb{X}), \quad \xi \in \mathbb{R}^d,$$

is the principal symbol associated to the operator $A$. We also recall the notion of wave cone associated to $A$, which plays a fundamental role in the study of $A$-free fields and first originated in the work of Murat & Tartar. The wave cone associated to the principal symbol $A$ is the cone

$$\Lambda_A = \bigcup_{|\xi|=1} \ker A(\xi) \subset \mathcal{W}.$$  

Notice that the wave cone contains those Fourier amplitudes along which it is possible to construct highly oscillating $A$-free fields. More precisely, $P \in \Lambda_A$ if and only if there exists $\xi \neq 0$ such that $A(Ph(x \cdot \xi)) = 0$ for all $h \in \mathbb{C}^b(\mathbb{R})$. 


Before stating our main result, let us give the rigorous definitions of generalized Young measures (as introduced in [21], and later extended in [2]) and of generalized $A$-free Young measures (which are those Young measures generated by $A$-free measures).

**Definition 1.1** (generalized Young measure). A triple $(\nu, \lambda, \nu^\infty)$ is called a generalized Young in $\Omega$ with values in $W$ provided that

(i) $\nu : \Omega \to \text{Prob}(W)$ is a weak-$\star$ measurable map,

(ii) $\lambda \in M^+(\Omega)$ is a non-negative Radon measure on $\Omega$, and

(iii) $\nu^\infty : \Omega \to \text{Prob}(S_W)$ is a weak-$\star$ $\lambda$-measurable map, where $S_W$ is the unit sphere in $W$.

The set of such Young measures will be denoted by $Y(\Omega; W)$.

**Definition 1.2** ($A$-free Young measure). The Young measure $(\nu, \lambda, \nu^\infty)$ is said to be an $A$-free Young measure if there exists a uniformly bounded sequence $(\mu_j) \subset M(\Omega; W)$ such that

$$\| A \mu_j \|_{W^{1,q}(\Omega)} \to 0$$

for some $1 < q < \frac{1}{d-1}$, and it satisfies

$$T_f(\mu_j) \to \int \langle f, \nu \rangle \, d\mathcal{L}^d + \int \langle f^\infty, \nu^\infty \rangle \, d\lambda$$

for all integrands $f \in E(\Omega, W)$; see Section 2.2 for the precise definition.

With these considerations in mind, we are now in position to state our main characterization results. The first result extends the Hahn–Banach-type characterization from Theorem 4.1 in [23] in terms of the validity of Jensen’s inequality for all $A$-quasiconvex integrands with linear growth at infinity. The second one characterizes $A$-free Young measures in terms of their tangent Young measure approximations in the spirit of [46]; definitions of tangent Young measures are postponed to Section 2.2.

**Theorem 1.1** (dual characterization). Let $\Omega \subset \mathbb{R}^d$ be an open and bounded Lipschitz domain and let $(\nu, \lambda, \nu^\infty) \in Y(\Omega; W)$ be a generalized Young measure with $\lambda(\partial \Omega) = 0$. Then, the triple $(\nu, \lambda, \nu^\infty)$ is an $A$-free Young measure if and only if

(i) the first moment of the Young measure is bounded, that is,

$$\int_\Omega \int_W |z| \, d\nu_x(z) \, dx + \lambda(\Omega) < \infty;$$

(ii) there exists $\mu \in M(\Omega; W)$ such that $A \mu = 0$ and

$$\mu = \langle \text{id}, \nu \rangle \mathcal{L}^d + \langle \text{id}, \nu^\infty \rangle \lambda;$$

(iii) at $\mathcal{L}^d$-almost every $x \in \Omega$, the Jensen inequality

$$h(\mu^\text{ac}(x)) \leq \langle h, \nu_x \rangle + \langle h^\#, \nu^\infty_x \rangle \lambda^\text{ac}(x)$$

holds for all $A$-quasiconvex integrands $h : W \to \mathbb{R}$ with linear growth at infinity;

(iv) and, at $\lambda^s$-almost every $x \in \Omega$, the singular part verifies the constraint

$$\text{supp}(\nu^\infty_x) \subset \text{span}\{A_k\}.$$
Here, $f^\#$ is the upper recession function of $f$ defined as

$$f^\#(x, A) := \limsup_{x' \to x} \limsup_{z' \to A} \frac{f(x', tz')}{t},$$

for all $x \in \Omega$ and $z \in W$

which (differently from $f^\infty$) always exists; this recession function can be seen to be upper-semicontinuous and positively 1-homogeneous.

**Remark 1.1** (unconstrained singular part). If $A$ is defined in its essential domain

$$\mathbb{W}_k := \text{span}\{A_k\} = \mathbb{W},$$

then $\nu^\infty$ is fully unconstrained since (iv) is equivalent to the trivial set inclusion

$$(iv') \quad \text{supp}(\nu^\infty_x) \subset W.$$

We list a few relevant $A$-free structures where this condition is satisfied, which means that the singular part in the characterization given above is unconstrained:

- gradient Young measures,
- symmetric gradient Young measures,
- solenoidal Young measures (generated by divergence-free fields),
- Young measures generated by normal $m$-currents without boundary.

**Remark 1.2.** Condition (iv) at singular points is equivalent to requiring that, at $\lambda^s$-almost every $x \in \Omega$, the following Jensen-type inequality

$$(iv') \quad h^\# \left( \frac{d\mu}{d|\mu|^s} \right) \leq \int_{S_k} h^\#(z) \, d\nu^\infty(z),$$

holds for every $A$-quasiconvex integrand $h : W \to \mathbb{R}$ with linear growth at infinity; this follows directly from the Structure Theorem for $A$-free measures [17] and the rigidity property of $A$-quasiconvex and positively 1-homogeneous functions established in [30].

**Theorem 1.2** (local characterization). Let $\Omega \subset \mathbb{R}^d$ be an open and bounded Lipschitz domain and let $(\nu, \lambda, \nu^\infty) \in \mathbf{Y}(\Omega; W)$ be a generalized Young measure with $\lambda(\partial \Omega) = 0$. Then, the triple $(\nu, \lambda, \nu^\infty)$ is an $A$-free measure if and only if

(i) there exists $\mu \in \mathcal{M}(\Omega; W)$ such that $A\mu = 0$ and

$$\mu = \langle \text{id}, \nu \rangle \mathcal{L}^d + \langle \text{id}, \nu^\infty \rangle \lambda;$$

(ii) at $(\mathcal{L}^d + \lambda^s)$-almost every $x \in \Omega$, there exists a tangent Young measure

$$\sigma \in \text{Tan}(\nu, x) \in \mathbf{Y}(\mathbb{R}^d; W)$$

which is an $A$-free Young measure on $\mathbb{R}^d$.

Lastly, we show that an $A$-free measure defined on an open domain $\Omega \subset \mathbb{R}^d$ can be approximated in the area-strictly sense of measures, by a sequence of $A$-free functions of $C^\infty$-class on $\Omega$. This approximation result is of relevance to certain minimization principles involving the relaxation of functionals of the form

$$u \mapsto \int_{\Omega} f(x, u(x)), \quad u \in L^1(\Omega; W), \quad A\mu = 0,$$

Frequently, it has been accepted to impose a geometric assumption on $\Omega$ which guarantees the approximation of $A$-free measures by $L^1$-integrable $A$-free fields in the strict sense of measures (see for instance [4, 5, 36]). More precisely, it has been frequently assumed that $\Omega$ is a strictly star-shaped domain, i.e., there exists $x \in \Omega$ such that

$$(\Omega - x) \subset \rho(\Omega - x) \quad \text{for all } \rho > 1.$$
The approximation result contained in Theorem 1.3 below allows, in particular, to dispense with this assumption on the geometry of $\Omega$ in such variational models. In order to state this result we need to introduce the following basic concept. The area functional of a measure is defined as

\[
\text{Area}(\mu, \Omega) := \int_{\Omega} \sqrt{1 + |\mu^\text{loc}(x)|^2} \, dx + |\mu^\ast|(\Omega), \quad \mu \in \mathcal{M}(\Omega; \mathbb{R}^N).
\]

In addition to the well-known weak-$*$ convergence of measures, we say that a sequence $(\mu_j)$ converges area-strictly to $\mu$ in $\mathcal{M}(\Omega; \mathbb{W})$ if

\[
\mu_j \rightharpoonup^* \mu \quad \text{in} \quad \mathcal{M}(\Omega; \mathbb{W}) \quad \text{and} \quad \text{Area}(\mu_j, \Omega) \to \text{Area}(\mu, \Omega).
\]

This notion of convergence turns out to be stronger than the conventional strict convergence of measures, which means that

\[
\mu_j \rightharpoonup^* \mu \quad \text{in} \quad \mathcal{M}(\Omega; \mathbb{W}) \quad \text{and} \quad |\mu_j|(\Omega) \to |\mu|(\Omega).
\]

The approximation result is contained in the following theorem:

**Theorem 1.3.** Let $\Omega \subset \mathbb{R}^d$ be an open and bounded Lipschitz domain and let $\mu \in \mathcal{M}(\Omega; \mathbb{W})$ be an $\mathcal{A}$-free measure. Then, there exists a sequence of fields $(u_j) \subset \text{L}^1(\Omega; \mathbb{W})$ satisfying

\[
u_j \rightharpoonup^* \mu \quad \text{as measures in} \quad \mathcal{M}(\Omega; \mathbb{W}),
\]

\[
\text{Area}(\nu_j, \mathbb{L}^d, \Omega) \to \text{Area}(\mu, \Omega),
\]

\[
\mathcal{A} \nu_j = 0 \quad \text{on} \quad \Omega.
\]

**Remark 1.3.** The regularity of the recovery sequence $(u_j)$ can be lifted to be of class $\text{C}^k(\Omega; \mathbb{W})$ for any $k \in \mathbb{N}$.

We close the characterization of $\mathcal{A}$-free measures with an application of Theorem 1.3 which allows us to re-define $\mathcal{A}$-free measures in terms of a pure $\mathcal{A}$-free constraint:

**Corollary 1.1** (pure-constraint). Let $\Omega \subset \mathbb{R}^d$ be an open Lipschitz set and let $(\nu, \lambda, \nu^\infty) \in \mathcal{Y}(\Omega; \mathbb{W})$ be a generalized Young measure with $\lambda(\partial \Omega) = 0$.

The following are equivalent:

1. $(\nu, \lambda, \nu^\infty)$ is an $\mathcal{A}$-free Young measure,
2. $(\nu, \lambda, \nu^\infty)$ is generated by $\mathcal{A}$-free measures.

### 1.2. Young measures generated by boundaries of normal currents.

The following framework has recently received more attention in light of the new ideas proposed to study certain dislocation models which are related to functionals defined on normal 1-currents without boundary (boundaries of 2-currents) [15, 26].

Let $1 \leq m \leq d$ be an integer. The space of $m$-dimensional currents consists of all distributions $T \in \mathcal{D}'(\Omega; \Lambda_m \mathbb{R}^d)$. Dualizing the action of the exterior derivative on the cochain complex of smooth differential forms, one defines the boundary of the $m$-dimensional current $T$ as the $(m-1)$-current acting on $\mathcal{C}^0_c(\Omega; \Lambda^{m-1} \mathbb{R}^d)$ as $\partial T[\omega] = T(d\omega)$. The space $\mathcal{N}_m(\Omega)$ of $m$-dimensional normal currents is defined as the space of $m$-currents $T$, such that both $T$ and $\partial T$ can be represented by a measure, that is,

\[
\mathcal{N}_m(\Omega) \cong \left\{ T \in \mathcal{M}(\Omega; \Lambda_m \mathbb{R}^d) : \partial T \in \mathcal{M}(\Omega; \Lambda_{m-1} \mathbb{R}^d) \right\}.
\]

The boundary operator on $\mathcal{N}_m(\Omega)$ defines a first-order operator $d^\ast$ of the form (5), with a principal symbol $d^\ast(\xi) : \Lambda_m \mathbb{R}^d \to \Lambda_{m-1} \mathbb{R}^d$ acting on $m$-vectors as the interior multiplication

\[
d^\ast(\xi)[v] = v_\ast \xi^\ast \quad \text{where} \quad \langle v_\ast \xi^\ast, z^\ast \rangle = \langle v, \xi^\ast \wedge z^\ast \rangle.
\]
Definition 1.3 (current Young measures without boundary). A Young measure \( \nu \in Y(\Omega; \wedge_m \mathbb{R}^d) \) is called an \( m \)-current Young measure without boundary if there exists a sequence of uniformly bounded \( m \)-currents \((T_j) \subset N_m(\Omega)\) satisfying
\[
\partial T_j = 0 \quad \text{on } \Omega
\]
and
\[
T_j \rightharpoonup \nu \quad \text{on } \Omega.
\]

Since the exterior derivative satisfies \( d^2 = 0 \), a simple duality argument shows that the boundary operator also satisfies \( \partial^2 = 0 \). In particular, every Young measure generated by a sequence of boundaries of \((m + 1)\)-currents is an \( m \)-current Young measure without boundary.

In [9] the authors give, as a byproduct of their results, a partial characterization of all \( m \)-dimensional current Young measures without boundary (in fact, a full characterization when \( m > 1 \); compare Assumption (†) with the dimensional estimates \( T \ll \mathcal{I}^m \) for currents established by Federer in [22], see also [6, 7]). Hence, our Theorem 1.1 extends the aforementioned partial result to a full characterization of \( m \)-current Young measures without boundary. In order to state this formally, let us first define the notion of \( \partial_m \)-quasiconvexity. We say that a Borel map \( h : \wedge_m \mathbb{R}^d \to \mathbb{R} \) is called \( \partial_m \)-quasiconvex if
\[
h(v) \leq \int_{[0,1]^d} h(v + \omega(y)) \, dy \quad \text{for all } v \in \wedge_m \mathbb{R}^d,
\]
and all smooth fields \( \omega \in C^\infty_{\text{per}}([0,1]^d; \wedge_m \mathbb{R}^d) \) satisfying
\[
\partial \omega = 0 \quad \text{and} \quad \int_{[0,1]^d} \omega = 0.
\]

Corollary 1.2. Let \( \nu = (\nu, \lambda, \nu^\infty) \in Y(\Omega; \wedge_m \mathbb{R}^d) \) be a Young measure with bounded first moment and \( \lambda(\partial \Omega) = 0 \). Then, \( \nu \) is an \( m \)-current Young measure without boundary if and only if there exists an \( m \)-current \( T \in N_m(\Omega) \) such that
\[
\partial T = 0 \quad \text{and} \quad T = \langle \text{id}, \nu \rangle \mathcal{L}^d + \langle \text{id}, \nu^\infty \rangle \lambda;
\]
and, for \( \mathcal{L}^d \)-almost every \( x \in \Omega \), it holds that
\[
h(T^{ac}(x)) \leq \langle h, \nu_x \rangle + \langle h^\#, \nu^\infty_x \rangle \lambda^{ac}(x),
\]
for all Borel \( \partial_m \)-quasiconvex integrands \( h : \wedge_m \mathbb{R}^d \to \mathbb{R} \) with linear growth at infinity.

Somewhat related, for systems of divergence-free fields we obtain the following characterization (recall that divergence-free fields are essentially 1-currents without boundary):

Corollary 1.3 (divergence-free Young measures). Let \( \nu = (\nu, \lambda, \nu^\infty) \in Y(\Omega; \mathbb{R}^d \otimes \mathbb{R}^d) \) be a Young measure satisfying \( \lambda(\partial \Omega) = 0 \). Then, there exists a uniformly bounded sequence of matrix-valued measures \((\mu_j) \subset \mathcal{M}(\Omega; \mathbb{R}^d \otimes \mathbb{R}^d)\) satisfying
\[
\text{div } \mu_j \coloneqq \sum_{j=1}^d \partial_j \mu_{ij} = 0 \quad \text{in the sense of distributions on } \Omega
\]
and
\[
\mu_j \rightharpoonup \nu \quad \text{on } \Omega,
\]
if and only if
(i) the first moment of \( \nu \) is bounded, i.e.,
\[
\int \int_{\mathbb{R}^d \otimes \mathbb{R}^d} |z| \, dv_x(z) \, dx + \lambda(\Omega) < \infty;
\]
(ii) the barycenter $\mu$ of $\nu$ satisfies
\[ \text{div} \mu = 0 \quad \text{in the sense of distributions on } \Omega; \]
(iii) and, at $\mathcal{L}^d$-almost every $x \in \Omega$, it holds
\[ h(\mu^{ac}(x)) \leq \int_{\mathbb{R}^d \otimes \mathbb{R}^d} h(z) \, d\nu(z) + \lambda^{ac}(x) \int_{\mathbb{R}^d \otimes \mathbb{R}^d} h^\#(z) \, d\nu^\infty(z), \]
for all div-quasiconvex Borel integrands $h : \mathbb{R}^d \otimes \mathbb{R}^d \to \mathbb{R}$ with linear growth at infinity.

Here, a Borel integrand $h : \mathbb{R}^d \otimes \mathbb{R}^d \to \mathbb{R}$ is called div-quasiconvex provided that
\[ h(M) \leq \int_{[0,1]^d} h(M + w(y)) \, dy \quad \text{for all } M \in \mathbb{R}^d \otimes \mathbb{R}^d, \]
and all divergence-free matrix fields $w \in C^\infty_{\text{per}}([0,1]^d; \mathbb{R}^d \otimes \mathbb{R}^d)$ with zero mean value.

2. Preliminaries

The $d$-dimensional torus is denoted by $\mathbb{T}^d$, and by $Q$ we denote the closed $d$-dimensional unit cube $[-1/2,1/2]^d$. We denote by $Q_c(x)$ the open cube centered at $x \in \mathbb{R}^d$.

2.1. Geometric measure theory. Let $X$ be a locally convex space. We denote by $C_c(X)$ the space of compactly supported and continuous functions on $X$, and by $C_0(X)$ we denote its completion with respect to the $\| \cdot \|_\infty$ norm. The space $C_c(X)$ is not a complete normed space in the usual sense, however, it is a complete metric space as the inductive union of Banach spaces $C_0(K_m)$ where $K_m \subset X$ are compact and $K_m \nearrow X$. By the Riesz representation theorem, the space $M_0(X)$ of bounded signed Radon measures on $X$ is the dual of $C_0(X)$; a local argument of the same theorem states that the space $M(X)$ of signed Radon measures on $X$ is the dual of $C_c(X)$. We notate by $M^+(X)$ the subset of non-negative measures. Since $C_0(X)$ is a Banach space, the Banach–Alaoglu theorem and its characterizations hold and in particular bounded sets of $M_0(X)$ are weak-$*$ metrizable. On the other hand, the local compactness of $C_c(X)$ permits the existence of a complete and separable metric $d_*$ on $M(X)$ with the property that convergence with respect to that metric is equivalent to the weak-$*$ convergence in $M_{\text{loc}}(X)$ (see Remark 14.15 in [35]):
\[ d_*(\mu_j, \mu) \to 0 \quad \Leftrightarrow \quad \mu_j \rightharpoonup \mu \quad \text{in } M(X). \]

In a similar manner, for a finite dimensional euclidean space $\mathbb{W}$, $M(X; \mathbb{W})$ and $M(X; \mathbb{W})$ will denote the spaces of $\mathbb{W}$-valued bounded Radon measures and $\mathbb{W}$-valued Radon measures respectively. The space $M(X)$ is a normed space endowed with the total variation norm
\[ |\mu|(X; \mathbb{W}) := \sup \left\{ \int_X \varphi \, d\mu : \varphi \in C_0(X; E), \|\varphi\|_\infty \leq 1 \right\}. \]

The set of all positive Radon measures on $X$ with total variation equal to one is denoted by
\[ \text{Prob}(X) := \left\{ \nu \in M^+(X) : \nu(X) = 1 \right\}; \]
the set of probability measures on $X$.

The Riesz' Representation Theorem states that every vector-valued measure $\mu \in M(\Omega; \mathbb{W})$ can be written as
\[ \mu = g_\mu |\mu| \quad \text{for some } g_\mu \in L^\infty(\Omega; |\mu|; \mathbb{W}), \]
This decomposition is commonly referred as the *polar decomposition* of $\mu$. The set $L_\mu$ of points $x_0 \in \Omega$ where

$$
\lim_{r \downarrow 0} \int_{Q_r(x_0)} |g_\mu(x) - g_\mu(x_0)| \, d|\mu|(x) = 0
$$

is satisfied, is called the set of $\mu$-Lebesgue points. This set conforms a full $|\mu|$-measure set of $\Omega$, i.e., $|\mu|(\Omega \setminus L_\mu) = 0$. In what follows, we shall always work with good representatives of $\mu$-integrable maps. If $\mu \in L^1_{L^1}(\Omega, \mu; \mathbb{W})$, then $\mu$ satisfies

$$
g(x) = \lim_{r \downarrow 0} \int_{Q_r(x)} g(y) \, d\mu(y) \quad \text{for all } x \in L_\mu \subset \Omega.
$$

If $\mu, \lambda$ are Radon measures over $\Omega$, and $\lambda \geq 0$, then the Besicovitch Differentiation Theorem states that there exists a set $E \subset \Omega$ of zero $\lambda$-measure such that

$$
\lim_{r \downarrow 0} \frac{\mu(Q_r(x))}{\lambda(Q_r(x))} = \frac{d\mu}{d\lambda}(x) \quad \text{for any } x \in \text{supp}(\lambda) \setminus E,
$$

where $\frac{d\mu}{d\lambda} \in L^1_{\text{loc}}(\Omega, \lambda; \mathbb{W})$ is the Radon–Nykodým derivative of $\mu$ with respect to $\lambda$.

Another resourceful representation of a measure is given by the *Radon–Nykodým–Lebesgue decomposition* which we shall frequently denote as

$$
\mu = \mu^{ac} \mathcal{L}^d + \mu^s,
$$

where as usual $\mu^{ac} := \frac{d\mu}{d\mathcal{L}^d} \in L^1_{\text{loc}}(\Omega; \mathbb{W})$, $|\mu^s| \perp \mathcal{L}^d$.

2.1.1. **Push-forward.** If $T : \Omega \to \Omega'$ is Borel measurable, the *image* or *push-forward* of $\mu$ under $T$ is defined by the formula

$$
T[\mu](E) = \mu(T^{-1}(E)) \quad \text{for every Borel set } E \subset \Omega'.
$$

Whenever $g : \Omega' \to [-\infty, \infty]$ is a Borel map, then

$$
\int_E g(y) \, dT[\mu](y) = \int_{T^{-1}(E)} g(T(x)) \, d\mu(x).
$$

2.1.2. **Tangent measures.** In this section we recall the notion of tangent measure as introduced by Preiss [40]. Let $\mu \in \mathcal{M}(\Omega; \mathbb{W})$ and consider the map $T_{x,r}(y) := (y - x)/r$, which blows up $B_r(x_0)$, the open ball around $x_0 \in \Omega$ with radius $r > 0$, into the open unit ball $B_1$. The push-forward of $\mu$ under $T_{x,r}$ is given by the measure

$$
T_{x,r}[\mu](E) := \mu(x_0 + rE), \quad \text{for all Borel } E \subset r^{-1}(\Omega - x).
$$

A non-zero measure $\tau \in \mathcal{M}(\mathbb{R}^d; \mathbb{W})$ is said to be a *tangent measure* of $\mu$ at $x_0 \in \mathbb{R}^d$, if there exist sequences $r_m \downarrow 0$ and $c_m > 0$ such that

$$
c_m T_{x,r_m}[\mu] \rightharpoonup \tau \quad \text{in } \mathcal{M}(\mathbb{R}^d; \mathbb{W});
$$

in this case the sequence $c_m T_{x,r_m}[\mu]$ is called a *blow-up sequence*. We write $\text{Tan}(\mu, x_0)$ to denote the set of all such tangent measures.

Using the canonical zero extension that maps the space $\mathcal{M}(\Omega; \mathbb{W})$ into the space $\mathcal{M}(\mathbb{R}^d; \mathbb{W})$ we may use most of the results contained in the general theory for tangent measures when dealing with tangent measures defined on smaller domains. The following theorem, due to Preiss, states that one may always find tangent measures.

**Theorem 2.1** (Theorem 2.5 in [40]). If $\mu$ is a Radon measure over $\mathbb{R}^d$, then $\text{Tan}(\mu, x) \neq \emptyset$ for $\mu$-almost every $x \in \mathbb{R}^d$. 

This property of Radon measures measures will play a silent, but fundamental role, in our results. We shall use it to “amend” the current lack of a Poincaré inequality for general domains; this, because (8) acts as an artificial extension operator for tangent measures restricted to the unit cube \( Q \subset \mathbb{R}^d \). Returning to the properties of tangent measures, one can show (see Remark 14.4 in [35]) that, for a tangent measure \( \tau \in \text{Tan}(\mu, x_0) \), it is always possible to choose the scaling constants \( c_m > 0 \) in the blow-up sequence to be

\[
c_m := c \mu(x_0 + r_n U)^{-1},
\]

for any open and bounded set \( U \subset \mathbb{R}^d \) containing the origin and with the property that \( \sigma(U) > 0 \), for some positive constant \( c = c(U) \); this process may involve passing to a subsequence. Then, from [40, Thm 2.6(1)] it follows that at \( \mu \)-almost every \( x \in \Omega \) we can find \( \tau \in \text{Tan}(\mu, x) \) as the weak-* limit a blow-up sequence of the form

\[
\frac{1}{|\mu|(Q_{r_n}(x))} T_{x, r_n}[\mu] \stackrel{\ast}{\rightharpoonup} \tau \quad \text{in } \mathcal{M}(\mathbb{R}^d ; \mathbb{W}), \quad |\tau|(Q) = |\tau|\left(\overline{Q}\right) = 1.
\]

Yet another special property of tangent measures is that at, \( |\mu| \)-almost every \( x \in \mathbb{R}^d \), it holds that

\[
\tau \in \text{Tan}(\mu, x) \quad \text{if and only if } \quad |\tau| \in \text{Tan}(|\mu|, x),
\]

\[
\tau = g_\mu(x)|\tau|;
\]

which in particular conveys that tangent measures are generated by strictly-converging blow-up sequences. If \( \mu, \lambda \in \mathcal{M}_{\text{loc}}^+(\mathbb{R}^d) \) are two Radon measures with the property that \( \mu \ll \lambda \), i.e., that \( \mu \) is absolutely continuous with respect to \( \lambda \), then (see Lemma 14.6 of [35])

\[
(9) \quad \text{Tan}(\mu, x) = \text{Tan}(\lambda, x) \quad \text{for } \mu \text{-almost every } x \in \mathbb{R}^d.
\]

Then, a consequence of (9) and Lebesgue’s differentiation theorem is that

\[
(10) \quad \text{Tan}(\mu, x) = \left\{ \alpha \mu^{\alpha}(x) \mathcal{L}^d : \alpha \in (0, \infty) \right\}, \quad \text{at } \mathcal{L}^d \text{-a.e. } x \in \mathbb{R}^d.
\]

In fact, if \( f \in L^1(\Omega, \mathbb{W}) \), then it an easy consequence from the Lebesgue Differentiation Theorem that

\[
(11) \quad \frac{1}{r^d} \cdot T_{x_0, r}[f \mathcal{L}^d] \to f(x_0) \mathcal{L}^d \quad \text{strongly in } L^1_{\text{loc}}(\mathbb{R}^d, \mathbb{W}).
\]

### 2.2. Integrands and Young measures.

Generalized Young measures conform a set of dual objects to the integrands in \( \mathbf{E}(\Omega; \mathbb{W}) \). We recall briefly some aspects of this theory, which was introduced by DiPERNA and MAJDA in [21] and later extended in [2,32]. For \( f \in C(\Omega \times \mathbb{W}) \) we define the transformation

\[
(Sf)(x, \hat{z}) := (1 - |\hat{z}|) f \left( x, \frac{\hat{z}}{1 - |\hat{z}|} \right), \quad (x, \hat{z}) \in \Omega \times \mathbb{B}_\mathbb{W},
\]

where \( \mathbb{B}_\mathbb{W} \) denotes the open unit ball in \( \mathbb{W} \). Then, \( Sf \in C(\Omega \times \mathbb{B}_\mathbb{W}) \). We set

\[
\mathbf{E}(\Omega; \mathbb{W}) := \{ f \in C(\Omega \times \mathbb{W}) : Sf \text{ extends to } C(\Omega \times \mathbb{B}_\mathbb{W}) \}.
\]

In particular, all \( f \in \mathbf{E}(\Omega; \mathbb{W}) \) have linear growth at infinity, i.e., there exists a positive constant \( M \) such that \( |f(x, z)| \leq M(1 + |z|) \) for all \( x \in \Omega \) and all \( z \in \mathbb{W} \). With the norm

\[
\|f\|_{\mathbf{E}(\Omega; \mathbb{W})} := \|Sf\|_\infty, \quad f \in \mathbf{E}(\Omega; \mathbb{W}),
\]
the space $E(\Omega; W)$ turns out to be a Banach space. Also, by definition, for each $f \in E(\Omega; W)$ the limit
\[
f^\infty(x, z) := \lim_{x' \to x \atop z' \to z \atop t \to \infty} \frac{f(x', t z')}{t}, \quad (x, z) \in \overline{\Omega} \times W,
\]
exists and defines a positively 1-homogeneous function called the strong recession function of $f$. Moreover every $f \in E(\Omega; W)$ satisfies
\[
f(x, z) = (1 + |z|)Sf\left(x, \frac{z}{1 + |z|}\right) \text{ for all } x \in \Omega \text{ and } z \in W.
\]
For an integrand $f \in E(\Omega; W)$ and a Young measure $\nu \in Y(\Omega; W)$, we define a duality pairing between $f$ and $\nu$ as
\[
\langle f, \nu \rangle := \int_\Omega \langle f, \nu \rangle_x \, dx + \int_{\overline{\Omega}} \langle f^\infty, \nu^\infty \rangle_x \, d\lambda(x)
\]
\[
= \int_\Omega \left( \int_W f(x, z) \, d\nu_x(z) \right) \, dx
\]
\[
+ \int_{\overline{\Omega}} \left( \int_{\partial W} f^\infty(x, \hat{z}) \, d\nu^\infty(\hat{z}) \right) \, d\lambda(x).
\]
The barycenter of a Young measure $\nu \in Y(\Omega; \mathbb{R}^N)$ is defined as the measure
\[
\|\nu\| := \langle \text{id}, \nu \rangle \mathcal{L}^d \Omega + \langle \text{id}, \nu^\infty \rangle \lambda \in \mathcal{M}(\overline{\Omega}; \mathbb{R}).
\]
In many cases it will be sufficient to work with functions $f \in E(\Omega; W)$ that are Lipschitz continuous. The following density lemma can be found in [32, Lemma 3].

**Lemma 2.1.** There exist countable families of non-negative functions $\{\varphi_p\} \subset C(\overline{\Omega})$ and Lipschitz integrands $\{h_q\} \subset E(\Omega; W)$ such that, for any given two Young measures $\nu_1, \nu_2 \in Y(\Omega; W)$,
\[
\|\nu_1 \otimes h_q, \nu_2\| = \|\varphi_p \otimes h_q, \nu_2\| \quad \forall p, q \in \mathbb{N} \implies \nu_1 = \nu_2.
\]
Since $Y(\Omega; W)$ is contained in the dual space of $E(\Omega; W)$ via the duality pairing $\langle \cdot, \cdot \rangle$, we say that a sequence of Young measures $(\nu_j) \subset Y(\Omega; W)$ weak-$*$ converges to $\nu \in Y(\Omega; W)$, in symbols $\nu_j \rightharpoonup^* \nu$, if
\[
\|f, \nu_j\| \to \|f, \nu\| \quad \text{for all } f \in E(\Omega; W).
\]
Fundamental for all Young measure theory is the following compactness result, see [32, Section 3.1] for a proof.

**Lemma 2.2** (compactness). Let $(\nu_j) \subset Y(\Omega; W)$ be a sequence of Young measures satisfying
\[
\begin{align*}
\text{(i)} \quad & \text{the family } \left\{ x \mapsto \langle |\cdot|, \nu_j \rangle : j \in \mathbb{N} \right\} \text{ is uniformly bounded in } L^1(\Omega), \\
\text{(ii)} \quad & \sup_j \lambda_j(\overline{\Omega}) < \infty.
\end{align*}
\]
Then, there exists a subsequence (not relabeled) and $\nu \in Y(\Omega; W)$ such that $\nu_j \rightharpoonup^* \nu$ in $Y(\Omega; W)$.

The Radon–Nykodym–Lebesgue decomposition induces a natural embedding
\[
\mathcal{M}(\Omega; W) \hookrightarrow Y(\Omega; W)
\]
via the identification $\mu \mapsto \delta_\mu = (\delta_{\mu^a}, |\mu^s|, \delta_{\mu^c})$. 
Definition 2.1. We say that a sequence of measures \((\mu_j) \subset \mathcal{M}(\Omega; W)\) generates the Young measure \(\nu\) in \(\Omega\) if and only if
\[
\delta_{\mu_j} \rightharpoonup \nu \quad \text{in} \quad Y(\Omega; W) \quad \text{in} \quad E(\Omega, W)^*.
\]
We shall represent this convergence symbolically as
\[
\mu_j \rightharpoonup \nu \quad \text{on} \quad \Omega.
\]

Notice that the generation of a Young measure is a local property in the sense that
\[
\mu_j \rightharpoonup \nu \quad \text{on} \quad \Omega \quad \iff \quad \mu_{L_x(r)} \rightharpoonup \nu_{L_x(r)} \quad \text{for all open} \quad \omega \subset \Omega \quad \text{with} \quad \lambda_\nu(\partial \omega).
\]

The proof of the following result follows the same principles used in the proof of [4, Lem. 2.15] with \(A \equiv 0\).

**Proposition 2.1.** Let \(\nu \in Y(\Omega; W)\) be a Young measure generated by a sequence of the form \((u_j \mathcal{L}^d)\). If there exists another sequence \((v_j) \subset L^1(\Omega; W)\) that satisfies
\[
\lim_{j \to \infty} \|u_j - v_j\|_{L^1(\Omega)} = 0,
\]
then
\[
v_j \mathcal{L}^d \rightharpoonup \nu \quad \text{on} \quad \Omega.
\]

The following notion of \(L^1\)-shift of a Young measure was introduced by KRISTENSEN & RINDLER in [32] in the context of gradient Young measures. In this work, these will only be used to deal with the fact that \(W\) might be in fact larger than \(W_\lambda\) in the proof of Theorem 1.1. To our knowledge, it has not been impossible to give a proof of Theorem 1.1 by means of an \(L^1\)-approximation (see for instance the discussion in Section 5 of [18]).

**Definition 2.2** (\(L^1\)-shifts). If \(\nu = (\nu, \lambda, \nu^\infty) \in Y(\Omega; W)\) is a generalized Young measure and \(v \in L^1_{loc}(\Omega, W)\), we define the \(v\)-shift of \(\nu\) to be the Young measure
\[
\Gamma_v[\nu] := (\nu \ast \delta_{-v}, \lambda, \nu^\infty) \in Y(\Omega, W).
\]

If \(f \in E(\Omega; W)\), then
\[
\langle f, \Gamma_v[\nu] \rangle = \int_{\Omega} \langle f, \nu \rangle_{x+v(x)} \, d\mathcal{L}^d(x) + \int_{\Omega} \langle f^\infty, \nu^\infty \rangle_{x} \, d\lambda(x).
\]
For a subset \(X \subset L^1(\Omega, W)\) we write
\[
\text{Shift}_X[\nu] := \{ \Gamma_v[\nu] : v \in X \}.
\]

2.2.1. **Tangent Young measures.** Similarly to the case of measures, we can define the push-forward of Young measures. If \(T : \Omega \to \Omega'\) is Borel, the push-forward of \(\nu = (\nu, \lambda, \nu^\infty) \in Y(\Omega, W)\) under \(T\) is the Young measure acting on \(f \in E(\Omega', W)\) as
\[
\langle f, T[\nu] \rangle = \left\langle f \circ (T, \text{id}_W), \nu \right\rangle = \int_{\Omega'} \langle f, \nu \rangle \, dT[d\mathcal{L}^d] + \int_{\Omega'} \langle f^\infty, \nu^\infty \rangle \, dT[\lambda].
\]
Suppose that \(x \in \Omega\). A non-zero locally finite Young measure \(\sigma \in Y(\mathbb{R}^d; W)\) is said to be a tangent Young measure of \(\nu\) at \(x\) if there exist sequences \(r_m \searrow 0\) and \(c_m > 0\) such that
\[
c_m \cdot T_{x,r_m}[\nu] \rightharpoonup \sigma \quad \text{in} \quad E(\mathbb{R}^d; W)^*.
\]
The set of tangent Young measures of \(\nu\) at \(x \in \Omega\) will be denoted as \(Tan(\nu, x)\). Since Young measures can be seen, via disintegration, as Radon measures over \(\Omega \times W\), the
property of tangent measures contained in Theorem 2.1 lifts to a similar principle for tangent Young measures:

**Proposition 2.2.** If \( \nu = (\nu; \lambda, \nu^\infty) \in Y(\Omega; \mathbb{R}^d) \) is a Young measure, then
\[
\text{Tan}(\nu, x) \neq \emptyset \text{ for } (\mathcal{L}^d + \lambda^*)\text{-almost every } x \in \Omega.
\]

Young measures also enjoy a Lebesgue-point property in the sense that a tangent Young measure \( \sigma \in \text{Tan}(\nu, x) \) truly represents the values of \( \nu \) around \( x \). More precisely, we have the following localization principle for \((\mathcal{L}^d + \lambda^*)\)-almost every \( x_0 \in \Omega \): every tangent measure \( \sigma \in \text{Tan}(\nu, x_0) \) is a homogeneous Young measure of the form
\[
(15) \quad \sigma = (\nu_{x_0}, \tau, \nu^\infty_{x_0}), \quad \text{where } \tau \in \text{Tan}(\lambda, x).
\]
This property tells us that certain aspects of the weak-* measurable maps \( \nu \) and \( \nu^\infty \) belonging to \( \nu \) can be effectively studied by looking at tangent measures of \( \nu \) itself. In a similar fashion to (8), at every \( x_0 \) where Proposition 2.3 holds, we may find a tangent Young measure \( \sigma \in \text{Tan}(\nu, x_0) \) as in (15) with
\[
(16) \quad \tau(\partial Q) = 0,
\]
and \( \sigma \) is generated by a blow-up sequence as in (14) where
\[
(17) \quad c_m := \left\{ \begin{array}{ll}
\mathcal{L}^d(Q_r(x))^{-1} & \text{if } x \text{ is a regular point of } \lambda, \\
\lambda^*(Q_r(x))^{-1} & \text{if } x \text{ is a singular point of } \lambda.
\end{array} \right.
\]
in any case \( c_m \) can be taken to be \((\| \cdot \|, \nu \mathcal{L} Q_r(x))^{-1}\). At singular points we may assume without loss of generality that
\[
(18) \quad \frac{1}{\lambda^*(Q_r(x))} T_{x, r_n}[\nu^{\text{ac}}] \mathcal{L}^d \to 0 \text{ strongly in } L^1_{\text{loc}}(\mathbb{R}^d; \mathbb{W}).
\]

We state two general localization principles for Young measures, one at regular points and another one at singular points. These are well-established results, for a proof we refer the reader to [44, 45]; see also the Appendix in [4].

**Proposition 2.3.** Let \( \nu = (\nu; \lambda, \nu^\infty) \in Y(\Omega; \mathbb{W}) \) be a generalized Young measure. Then for \( \mathcal{L}^d \)-a.e. \( x_0 \in \Omega \) there exists a regular tangent Young measure \( \sigma = (\gamma, \gamma, \sigma^\infty) \in \text{Tan}(\nu, x_0) \), that is,
\[
\{ \sigma \} \in \text{Tan}(\nu, x_0), \quad \sigma_y = \nu_{x_0} \text{ a.e.,}
\]
\[
\lambda = \frac{d\nu}{d\mathcal{L}^d}(x_0) \mathcal{L}^d \in \text{Tan}(\lambda, x_0), \quad \sigma^\infty = \nu^\infty_{x_0} \lambda_{\gamma} \text{-a.e.}
\]

**Proposition 2.4.** Let \( \nu = (\nu; \lambda, \nu^\infty) \in Y(\Omega; \mathbb{W}) \) be a generalized Young measure. Then there exists a set \( S \subset \Omega \) with \( \lambda^*(\Omega \setminus S) = 0 \) such that for all \( x_0 \in S \) there exists a singular tangent Young measure \( \sigma = (\gamma, \gamma, \sigma^\infty) \in \text{Tan}(\nu, x_0) \), that is,
\[
\{ \sigma \} \in \text{Tan}(\nu, x_0), \quad \sigma_y = \delta_{x_0} \text{ a.e.,}
\]
\[
\gamma \in \text{Tan}(\lambda^*_x, x_0), \quad \gamma(Q) = 1, \quad \gamma(\partial Q) = 0, \quad \sigma^\infty = \nu^\infty_{x_0} \gamma \text{-a.e.}
\]

### 2.3. \( \mathcal{A} \)-quasiconvexity

We state a well-known sectional convexity result for \( \mathcal{A} \)-quasiconvex integrands.

#### 2.4. \( \mathcal{D} \)-convexity

Let \( \mathcal{D} \) be a balanced cone of directions in \( \mathbb{W} \), that is, we assume that \( tA \in \mathcal{D} \) for all \( A \in \mathcal{D} \) and every \( t \in \mathbb{R} \). A real-valued function \( h: \mathbb{W} \to \mathbb{R} \) is said to be \( \mathcal{D} \)-convex provided its restrictions to all line segments in \( \mathbb{W} \) with directions in \( \mathcal{D} \) are convex.

We recall the following \( \Lambda_{\mathcal{A}} \)-convexity property of \( \mathcal{A} \)-quasiconvex functions contained in lemma from [23, Proposition 3.4] for first-order operators and in [4, Lemma 2.19] for the general case:
Lemma 2.3. If \( h : \mathcal{W} \to \mathbb{R} \) is an integrand with linear growth at infinity and \( h \) is \( \mathcal{A} \)-quasiconvex, then \( h \) is \( \Lambda_h \)-convex.

Corollary 2.1. Let \( f \in C(\mathcal{W}) \) be an integrand and let \( z_0 \in \mathcal{W} \). Then,
\[
Q_{\mathcal{A}}f(z_0 + z) = Q_{\mathcal{A}}h(z) \quad \text{for all } z \in \mathcal{W}_h,
\]
where \( h = f(z_0 + p \cdot) \) and \( p : \mathcal{W} \to \mathcal{W}_h \) is the canonical projection.

Moreover, \( Q_{\mathcal{A}}h > -\infty \) if and only if \( Q_{\mathcal{A}}f(z_0) > -\infty \).

Proof. Since every every \( \mathcal{A} \)-free periodic function \( w \in C_c^\infty(\mathbb{T}^d; \mathcal{W}) \) (where the subscript stands for average zero) in fact satisfies \( w(y) \in \mathcal{W}_h \) for all \( y \in \mathbb{T}^d \) (see [4, Section 2.5]), it follows that
\[
\int_{\mathbb{T}^d} f(z_0 + z + w(y)) \, dy = \int_{\mathbb{T}^d} h(z + w(y)) \quad \text{for all } \mathcal{A} \text{-free } w \in C_c^\infty(\mathbb{T}^d; \mathcal{W}).
\]

We deduce from this equivalence that
\[
Q_{\mathcal{A}}f(z_0 + z) = Q_{\mathcal{A}}h(z).
\]

The \( \Lambda_h \) convexity of \( \mathcal{A} \)-quasiconvex functions and the fact that \( \Lambda_h \) precisely spans \( \mathcal{W}_h \) imply that \( Q_{\mathcal{A}}f(z_0) > -\infty \) if and only if \( Q_{\mathcal{A}}f(z_0 + z) > -\infty \) for all \( z \in \mathcal{W}_h \).

This and the previous identity prove the second assertion. \( \square \)

3. Analysis of constant rank operators

Since the results contained in this section will be used for both \( \mathcal{A} \) and its associated potential \( \mathcal{B}[D] \) (see the discussion below), we have decided to work with the more canonical notation \( \mathcal{A}[D] \) for operators as in (5) which links the operator directly to its principal symbol. In order to maintain an economical notation, we what will often write \( \Lambda u \) to denote \( \mathcal{A}[D]u \). We will write \( k_\mathcal{A} \) to denote the order of \( \mathcal{A} \), and \( k_\mathcal{B} \) to denote the order of \( \mathcal{B} \). To denote that an \( L^p \)-integrable function or measure \( \mu \) has mean zero on \( \Omega \) we will write \( \mu \in L^p_\mu(\Omega; \mathcal{W}) \) and \( \mu \in M(\Omega; \mathcal{W}) \) respectively.

The aim of this section is to give a simple extension of the well-known \( L^p \)-multiplier projections for constant rank operators established by Fonseca & Müller in [23]. This will allow us to construct \( \mathcal{BV}^\mathcal{A}(\mathbb{R}^d) \) and \( \mathcal{BV}^\mathcal{A}(\mathbb{T}^d) \), the spaces of functions of bounded \( \mathcal{A} \)-variation in \( \mathbb{R}^d \) and in \( \mathbb{T}^d \) the \( d \)-dimensional flat torus. Most importantly, we will show that an embedding of the form
\[
\mathcal{BV}^\mathcal{A} \hookrightarrow \mathcal{W}^{k_\mathcal{A},-1,1}
\]
holds. This compactness result will be one of the angular stones behind the gluing of \( \mathcal{A} \)-free Young measures, and subsequently one of the angular stone supporting proofs of Theorems 1.1 and 1.2.

We denote by \( \mathbb{T}^d \cong \mathbb{R}^d/\mathbb{Z}^d \) the \( d \)-dimensional flat torus. The Fourier transform acts on periodic measures by the formula
\[
\hat{\mu}(\xi) = \mathfrak{F}u(\xi) := \int_{\mathbb{T}^d} e^{-2\pi i x \cdot \xi} \, d\mu(x), \quad \mu \in M(\mathbb{T}^d; \mathcal{W}).
\]

Smooth periodic functions are represented by \( \mathfrak{F}^{-1} \) through the trigonometric sum
\[
u(x) = \sum_{\xi \in \mathbb{Z}^d} \hat{u}(\xi) e^{2\pi i x \cdot \xi}.
\]

The choice to primarily work with Fourier coefficients lies in the following characterization for constant rank operators due to Raita [43, Thm. 1]. Let \( \mathcal{A}[D] \) be an operator from \( \mathcal{W} \) to \( \mathcal{X} \) as in (5). Then \( \mathcal{A}[D] \) satisfies the constant rank condition if
and only if then there exists an operator $\mathcal{B}[D]$ from $\mathcal{V}$ to $\mathcal{W}$ as in (5) —where $\mathcal{V}$ is a finite dimensional inner product space depending on $\mathcal{A}[D]$— and satisfying

$$\text{Im } \mathcal{B} (\xi) = \ker \mathcal{A} (\xi) \quad \text{for all } \xi \in \mathbb{R}^d \setminus \{0\}. \quad (19)$$

Here and in what follows $\mathcal{A}[D]$ and $\mathcal{B}[D]$ shall always be assumed to satisfy the relation (19), we call $\mathcal{B}[D]$ the associated potential to $\mathcal{A}[D]$. \footnote{The class of operators $\mathcal{B}[D]$ satisfying (19) may have more than element.}

In particular, it is shown there that every $\mathcal{A}[D]$-free potential field is the $\mathcal{B}[D]$-gradient of a suitable potential. The precise statement is the following:

**Lemma 3.1** (Lemma 5 in [43]). Let $w \in C^\infty(\mathbb{T}^d; \mathcal{W})$. Then $w$ satisfies

$$\mathcal{A}[D]w = 0 \quad \text{and} \quad \int_{\mathbb{T}^d} w = 0,$$

if and only if there exists a potential $u \in C^\infty(\mathbb{T}^d; \mathcal{V})$ such that

$$\mathcal{B}[D]u = w \quad \text{and} \quad \int_{\mathbb{T}^d} u = 0.$$

The main idea behind the proof of (19) rests on an old theorem of DECCELL [20, Theorem 3] which serves to show that $\mathcal{A}^\dagger (\xi)$, the Moore–Penrose inverse of $\mathcal{A} (\xi)$, is a rational tensor of degree $-(k_A).$ \footnote{The Moore–Penrose inverse $M^\dagger$ of a tensor $M \in \mathcal{X} \otimes \mathcal{W}^*$ satisfies $MM^\dagger = \text{proj}_{\text{Im } M}$ and $M^\dagger M = \text{proj}_{\ker M^\dagger}$.} The family $\{\mathcal{B}^\dagger (\xi)\}_{\xi \in \mathbb{Z}^d}$ defines a multiplier

$$u(x) = \sum_{\xi \in \mathbb{Z}^d \setminus \{0\}} \mathcal{B}^\dagger (\xi) \hat{w} (\xi) e^{2\pi i x \cdot \xi}, \quad \text{for } w \in C^\infty(\mathbb{T}^d; \mathcal{W}) \text{ with } \hat{w}(0) = 0,$$

thus reflecting the necessity of the average zero assumption. \footnote{Notice that, in full space, the kernel $K = \mathcal{F}^{-1}\mathcal{B}^\dagger$ may not necessarily be a singular integral.} Moreover, following [43] one readily checks with (19), if $\mathcal{A}[D]w = 0$, then

$$(\mathcal{B}[D]u)(x) = \sum_{\xi \in \mathbb{Z}^d \setminus \{0\}} \mathcal{B}(\xi)\mathcal{B}^\dagger (\xi) \hat{w} (\xi) e^{2\pi i x \cdot \xi} = \sum_{\xi \in \mathbb{Z}^d \setminus \{0\}} \text{proj}_{\ker \mathcal{A}(\xi)} \hat{w} (\xi) e^{2\pi i x \cdot \xi} = w(x), \quad \text{for } w \in C^\infty(\mathbb{T}^d; \mathcal{W}).$$

On the other hand, going back to the celebrated work of MÜLLER & FONSECA [23], it is well-known that (see [4, Lemma 2.13]), for exponents $1 < p < \infty$, there exists a $(p, p)$-multiplier $\{u \mapsto \pi_A u\}$ acting on Fourier coefficients as the linear projection $\text{proj}_{\ker \mathcal{A}(\xi)}$ and satisfying $\int_{\mathbb{T}^d} \pi_A u = 0$,

$$\mathcal{A}[D](\pi_A u) = 0, \quad \text{and} \quad \|u - \pi_A u\|_{L^p(\mathbb{T}^d)} \leq C_p\|\mathcal{A} w\|_{W^{-1,p}(\mathbb{T}^d)}, \quad (20)$$

for all $u \in C^\infty(\mathbb{T}^d; \mathcal{W})$ with $\int_{\mathbb{T}^d} u = 0$. (The Mikhlin Multiplier yields a similar $(p, p)$-multiplier on the space $C^\infty_c(\mathbb{R}^d; \mathcal{W})$.) Using the identity of the Moore–Penrose pseudo-inverse $M^\dagger M = \text{id} - \text{proj}_{\ker M}$ we get

$$u - \pi_A u = \tilde{\mathcal{F}}^{-1}((\text{id}_\mathcal{W} - P_{\ker A})\mathcal{F} u) = \tilde{\mathcal{F}}^{-1}(\mathcal{A}^\dagger \mathcal{F}(\mathcal{A}[D]u)) =: u_A.$$

Hence, following the ideas in the proof of [49, Proposition 8.22] we may define a kernel $G : \mathbb{R}^d \setminus \{0\} \to \mathcal{W} \otimes \mathcal{W}^*$ such that $\hat{G}(\xi) = |\xi|^{k_A} \mathcal{A}^\dagger (\xi)$, which is $-(k_A - s)$-homogeneous; so that $G$ is homogeneous of order $-(d - (k_A - s))$.

$$|\xi|^{k_A} \mathcal{F} u_A (\xi) = \hat{G}(\xi) \mathcal{F}(\mathcal{A}[D]w)(\xi) = \frac{1}{|\xi|^{k_A - s}} \mathcal{A}^\dagger \left( \frac{\xi}{|\xi|} \right) \mathcal{F}(\mathcal{A}[D]u)(\xi).$$
Now, let us recall that \( \xi \mapsto A^\dagger(\xi/|\xi|) \) is a 0-homogeneous map which is smooth when restricted to \( \mathbb{S}^{d-1} \). In particular, the family \( \{A^\dagger(\xi/|\xi|)\}_{\xi \in \mathbb{Z}^d} \) defines a \((p,p)\)-multiplier (see Proposition 2.13 in [23] and references therein). Then the identity above yields the estimate

\[
\|u_h\|_{W^{s,p}(\mathbb{T}^d)} \leq c_p \left( \|Au\|_{W^{-((k_h-s),p)}(\mathbb{T}^d)} \right) \quad \text{for all } u \in C^\infty_0(\mathbb{T}^d; \mathbb{W}),
\]

and all \( s \in (k_h - d, k_h] \). (Here \( W^{0,p} := L^p \).

Precisely, this hints there is scale of fractional spaces where the Fonseca–Müller representative map \( \{u \mapsto u_h\} \) can be extended continuously, and more importantly, where the projection possesses strong compactness properties. Let us introduce the fractional spaces associated to \( A[D] \):

**Definition 3.1.** Let \( s \in (k_h - d, k_h] \) and let \( 1 \leq p < \infty \). We define the scale of fractional sub-spaces of measures

\[
W^{s,p}_A(\mathbb{T}^d) := \left\{ u \in \mathcal{M}_h(\mathbb{T}^d; \mathbb{W}) : A[D]u \in W^{-(k_h-s),p} \right\}.
\]

Endowed with the norm

\[
\|u\|_{W^{s,p}_A(\mathbb{T}^d)} := \|u\|_{(\mathbb{T}^d)} + \|Au\|_{W^{-(k_h-s),p}(\mathbb{T}^d)},
\]

these are Banach spaces (this follows from the continuity of the operator \( A[D] \) with respect to distributional convergence). Since it will be of use, we also define

\[
\dot{W}^{s,p}_A(\mathbb{T}^d) := \left\{ u \in \mathcal{M}_2(\mathbb{T}^d; \mathbb{W}) : A[D]u \in W^{-(k_h-s),p} \right\},
\]

and the homogeneous norm

\[
\|u\|_{\dot{W}^{s,p}_A(\mathbb{T}^d)} := \|Au\|_{W^{-(k_h-s),p}(\mathbb{T}^d)}.
\]

We state the following extension of the Fonseca–Müller \( A \)-representative map (see also the associated critical-regime embeddings in [42] for the \( \mathbb{B} \)-potential setting)

**Lemma 3.2.** Let \( s \in (k_h - d, k_h] \) and let \( 1 < p < \infty \). There exists a continuous linear map \( T_h : W^{s,p}_A(\mathbb{T}^d) \to \dot{W}^{s,p}(\mathbb{T}^d) \) satisfying

1. \( T_h u = u_h \) for all \( u \in C^\infty_0(\mathbb{T}^d; \mathbb{W}) \),
2. \( A(T_h u) = Au \),
3. there exists \( c_p > 0 \) (depending solely on \( p \)) such that

\[
\|u_h\|_{W^{s,p}(\mathbb{T}^d)} \leq c_p \|Au\|_{W^{-(k_h-s),p}(\mathbb{T}^d)} \quad \text{for all } u \in \dot{W}^{s,p}_A(\mathbb{T}^d).
\]

**Proof.** This follows from (21) and an approximation argument.

**Remark 3.1.** Notice that by choosing \( s = 0 \) in the previous lemma we recover the classical Fonseca–Müller estimate (20).

A direct consequence of the previous Lemma and the embeddings for fractional Sobolev–Slobodetskii spaces is the following compactness property:

**Corollary 3.1.** Let \( s' \in (0,1] \) and let \( 1 < q < \frac{d}{d-s'} \). Then, there exist positive constants \( c, c_q \) such that

\[
\|D^{k_h-1} T_h u\|_{W^{1-s',q}(\mathbb{T}^d)} \leq c_q \|Au\|_{W^{-s',q}(\mathbb{T}^d)} \leq c_q \cdot c \cdot d^*_s(Au,0)
\]

for all measures \( u \in \dot{W}^{k_h-s',q}_A(\mathbb{T}^d) \).
Proof. The first inequality follows from the previous lemma by taking $s' = k_A - s$ with $s \in (k_A - 1, k_A]$, and the embedding for fractional Sobolev spaces discussed below; in fact, the estimate holds for all $1 < q < \infty$. However, we will need to constrain the threshold of $q$’s in order to obtain the second inequality. The compact embedding for fractional Sobolev–Slobodetskii spaces with $s', q > d$ read $W^{s', q}(\mathbb{T}^d) \hookrightarrow C(\mathbb{T}^d)$. If $q'$ is the dual Sobolev exponent of $q$, then a duality argument gives

$$
\left( M_2(\mathbb{T}^d), d_* \right) \hookrightarrow W^{-s', q}(\mathbb{T}^d) \quad \text{for all } 1 < q < \frac{d}{d - s'}.
$$

Recall that the space of measures on the left hand side is a separable and complete metric space when equipped with the metric $d_*$. Moreover, this metric has the following property:

$$
\lim_{j \to \infty} d_*(\mu_j, \mu) = 0 \text{ if and only if } \mu_j \rightharpoonup \mu \text{ in } M(\mathbb{T}^d);
$$

for more details about the metric $d_*$ see for instance Remark 14.15 in [35]. □

Definition 3.2 ($[A[D]$-representatives). Let $G = \mathbb{R}^d$, $\mathbb{T}^d$. From this point on we will denote indistinctly by $\pi_A$ the projections from Lemma 3.2 and Corollary 3.1. If $\mu \in M(G)$ satisfies the total variation bound $|A u|_s(G) < \infty$, then we will write

$$
u_A = T_A \mu_A = \begin{cases} \mathfrak{F}^{-1}([A^\dagger A]F u) & \text{if } G = \mathbb{T}^d \\ \mathfrak{F}^{-1}([A^\dagger A]F u) & \text{if } G = \mathbb{R}^d \end{cases},
$$

which we term as the $[A[D]$-representative of $\mu$.

Remark 3.2. Note that if $\mu$ is compactly supported, then, up to a re-scaling argument, we can always define $\mu_A$ (both with respect to the projection in global space $\mathbb{R}^d$ and the one on $\mathbb{T}^d$). However, we should remain aware of the implications of uncertainty principle, which implies that

- $\mathfrak{F}^{-1}([A^\dagger A]F u)$ is integrable, but it may not be compactly supported,
- $\mathfrak{F}^{-1}([A^\dagger A]F u)$ is periodic, but it is not integrable on $\mathbb{R}^d$.

Both projections and estimates contained in Lemma 3.2 and Corollary 3.1 can be stated for measures in the whole space $M(\mathbb{R}^d; \mathbb{W})$ via analogous methods through the use of the classical Fourier transform. The transfer of the Foneca–Müller multiplier is a direct consequence of the Mikhlin Multiplier Theorem applied for smooth 0-order homogeneous multipliers.

In light of this discussion we summarize the estimates obtained in this section with the following compact embeddings for constant rank operators:

Lemma 3.3. Let $A[D]$ be a partial differential operator as in (5), and assume that $A[D]$ satisfies the constant rank condition (6).

Then, for $G = \mathbb{T}^d$ or $G = \mathbb{R}^d$ it holds that

$$
T_A \left[ W^{s', p}_A(G; \mathbb{W}) \right] \hookrightarrow \mathfrak{W}^{[s]},^p(G; \mathbb{W}),
$$

where $1 \leq p < \infty$ and $s \in (0, k_A]$; the embedding is compact provided that $s > [s]$.  


3.1. Localization estimates. We end this section with a useful observation for estimates concerning the localization by cut-off functions. Let \( \varphi \in \mathcal{C}_{c}^{\infty} (\mathbb{R}^{d}; \mathcal{W}) \), the commutator of \( A \) on \( \varphi \) is the partial differential operator
\[
[A, \varphi] := A \circ \varphi - \varphi \circ A,
\]
where \( \varphi \) is regarded as a zero order operator. More precisely, acting on distributions \( \eta \in \mathcal{D}'(\mathbb{R}^{d}; \mathcal{W}) \) as \( A(\varphi \cdot \eta) - \varphi \cdot A\eta \). Notice that \( [A, \varphi] \) is a partial differential operator of order \((k_{A} - 1)\) with smooth coefficients depending solely on the symbol \( A \) and the first \( k_{A} \) derivatives of \( \varphi \).

In particular, if \( \mu \in \mathcal{M}_{\text{loc}}(\mathbb{R}^{d}; \mathcal{W}) \) satisfies \( A\mu \in W^{-k_{A}, p}_{\text{loc}}(\mathbb{R}^{d}) \), then by virtue of the embedding (22) with \( s' = 1 \) we get
\[
A(\varphi \mu) = \varphi \cdot A\mu + [A, \varphi] \mu \in W^{-k_{A}, p}_{\text{loc}}(\mathbb{R}^{d}),
\]
and
\[
\| \varphi \mu \|_{W^{-k_{A}, p}_{\text{loc}}(\mathbb{R}^{d})} \lesssim |\mu|(K) \quad \text{for all supp}(\varphi) \subset K \subset \mathbb{R}^{d}.
\]

4. Functions of bounded \( A \)-variation

We are now in position to introduce the spaces \( BV^{A}(\mathbb{R}^{d}) \) and \( BV^{A}(\mathbb{T}^{d}) \) of bounded \( A \)-variation when \( A[D] \) has the form (5) and satisfies Murat’s constant rank property (6). Similar spaces, such as the \( A \)-Sobolev spaces \( W^{k_{A}, p}(\Omega) \), have been introduced in [5] (mostly for notation purposes); see also [13], where the authors adapt this notion to define the space of functions of bounded \( A \)-variation when \( A[D] \) is a first-order complex \( C \)-elliptic operator.

Let us begin by analyzing the definition of \( BV^{A} \)-spaces when \( A[D] \) is assumed to be an elliptic operator of order \( k \) from \( \mathcal{W} \) to \( X \). The regularizing effects of \( A[D] \), for solutions to the equation
\[
A[D]u = f \in \mathcal{M}_{\text{loc}}(\mathbb{R}^{d}; X), \quad u \in \mathcal{M}_{\text{loc}}(\mathbb{R}^{d}; \mathcal{W}),
\]
restrict the class of all possible solutions to \( u \in L^{1}(\mathbb{R}^{d}; \mathcal{W}) \). There is therefore no loss of generality in defining
\[
BV^{A}(\mathbb{R}^{d}) := \{ u \in L^{1}(\mathbb{R}^{d}; \mathcal{W}) : A[D]u \in \mathcal{M}_{\text{loc}}(\mathbb{R}^{d}; X) \},
\]
and to endow this space with the norm \( \| u \|_{BV^{A}(\mathbb{R}^{d})} := \| u \|_{L^{1}(\mathbb{R}^{d})} + |A\mu|(\mathbb{R}^{d}) \). Defined in this way, this space satisfies the embedding \( BV^{A} \hookrightarrow W^{k - 1, 1} \), which follows from classical Fourier transform arguments. When \( A[D] \) is not an elliptic operator, the issue here is that there are in general too many possible solutions of (25), hindering the set contention \( BV^{A}(\mathbb{R}^{d}) \subset L^{1}(\mathbb{R}^{d}; \mathcal{W}) \). We propose the following natural definition which endows \( BV^{A} \) with a much richer structure
\[
BV^{A}(\mathbb{R}^{d}) := \{ u_{A} : |u|(\mathbb{R}^{d}) + |A\mu|(\mathbb{R}^{d}) < \infty \}.
\]
And, since it will be of more use for our applications, we also define
\[
BV^{A}(\mathbb{T}^{d}) := \{ u_{A} : u \in \mathcal{M}_{1}(\mathbb{T}^{d}; \mathcal{W}), |A\mu|(\mathbb{T}^{d}) < \infty \}.
\]

For the sake of completeness we also define a fractional scale of \( A \)-Sobolev Spaces in the range \( s \in (-d + k_{A}, k] \) as follows
\[
W^{s, p}_{A}(\mathbb{R}^{d}) := \{ u_{A} : \| u \|_{L^{p}(\mathbb{R}^{d})} + \| A\mu \|_{W^{-(k_{A} - s), p}(\mathbb{R}^{d})} < \infty \}, \quad 1 \leq p < \infty,
\]
and
\[
W^{s, p}(\mathbb{T}^{d}) := \{ u_{A} : u \in \mathcal{M}_{1}(\mathbb{T}^{d}; \mathcal{W}), \| A\mu \|_{W^{-(k_{A} - s), p}(\mathbb{T}^{d})} < \infty \}, \quad 1 \leq p < \infty.
\]

The main results contained in this work hinge on the following Rellich–Kondrachov-type embeddings which follow directly from the proof of Corollary 3.1.
Lemma 4.1. Let $A[D]$ be a partial differential operator as in (5), and assume that $A[D]$ satisfies the constant rank condition (6). Let $G$ be either $T^d$ or $\mathbb{R}^d$, then
$$\text{BV}^A(G) \subset \text{W}^k\text{A}^{-1,1}(G; \mathbb{W}).$$

It will also be convenient to count with a version of Lemma 3.1 for measures. This is contained in the following result which follows from a mollification argument and Lemma 3.1 itself.

Proposition 4.1. Let $A[D]$ be an homogeneous linear partial differential operator as in (5) and satisfying the constant rank condition (6). Let $B[D]$ be its associated potential, then
$$\left\{ B[D]v : v \in \text{BV}^B(T^d) \right\} = \left\{ u \in M_d(T^d; \mathbb{W}) : A[D]u = 0 \right\}.$$ 

If in place of working on $T^d$ we work on $\mathbb{R}^d$, we can, in general, only say that gradients of (compactly supported) functions of bounded $B$-variation are $A[D]$-free fields. In fact, the slightly more general property holds:

Lemma 4.2. Let $A[D]$ be an homogeneous linear partial differential operator from $\mathbb{W}$ to $\mathbb{X}$ as in (5) and let $B[D]$ be its associated potential. Let $u \in S'(\mathbb{R}^d; \mathbb{V})$ be a tempered distribution. Then
$$A[D] \circ B[D]u = 0 \quad \text{in the sense of distributions on } \mathbb{R}^d.$$ 

Proof. We denote by $\mathcal{F}$ the Fourier transform for tempered distributions defined on $\mathbb{R}^d$. Since $u$ is a tempered distribution, so is $A[D](B[D]u)$. Therefore, by a density argument, it suffices to test with all $\eta \in S(\mathbb{R}^d; \mathbb{X})$. In what follows $A^*(\xi)$ denotes the $L^2$-adjoint of $A(\xi)$ and $A^*[D]$ is the partial differential operator associated to it; and similarly for $B[D]$. A simple calculation with the Fourier transform yields (recall that $A(\xi) \circ B(\xi) \equiv 0$)
$$\langle B[D]u, A[D]^* \eta \rangle = \langle \mathcal{F}u, (A \circ B)^* \mathcal{F}^{-1} \eta \rangle = 0$$
for all $\eta \in S(\mathbb{R}^d; \mathbb{X})$. Since $\mathcal{F}$ induces an isometry of $S(\mathbb{R}^d; \mathbb{X})$ onto itself, the sought assertion follows from the inclusion $C_c^\infty(\mathbb{R}^d; \mathbb{X}) \hookrightarrow S(\mathbb{R}^d; \mathbb{X})$.  

5. Helmholtz-type decompositions

As we have seen in the previous section, the multiplier $\pi_A$ is the $L^2$-projection onto the kernel of $A$ in frequency space. In particular this means that it is self-adjoint when $p = 2$. Indeed, if $u, v \in L^2(\mathbb{R}^d; \mathbb{W})$, then Plancherel’s identity gives
$$\langle \pi_A u, v \rangle = \langle \text{proj}_{A(\xi)} \mathcal{F}u, \mathcal{F}v \rangle = \langle \mathcal{F}u, \text{proj}_{A(\xi)} \mathcal{F}v \rangle = \langle u, \pi_A v \rangle.$$ 

In the case that $A v = 0$, we furthermore deduce $\langle u_A, v \rangle = 0$. Classical duality identities for unbounded linear operators and a version of Lemma 4.2 yield $u_A \in \text{ker} A[D] \subset A^*[D]$. This shows that

$$B^* u_A = 0 \quad \text{for all } u \in L^2(\mathbb{R}^d; \mathbb{W}).$$ 

Yet, another way of seeing this is to consider the minimization of the Dirichlet energy
$$v \mapsto \int_{\mathbb{R}^d} |v|^2, \quad \text{where } A w = A u.$$ 

Once again, by Plancherel’s identity, one verifies that the minimizer $v_0$ satisfies
$$\tilde{v}_0(\xi) = \text{proj}_{\text{ker} A(\xi)} \hat{u}(\xi) \quad \text{for all } \xi \in \mathbb{R}^d.$$
and hence \( u_\lambda \) is the unique minimizer of (27). The fact that \( \text{Im} \mathbb{B}[D] \subset \ker A[D] \) (see Lemma 4.2), allows for a comparison principle with variations of the form \( \{ t \mapsto u_\lambda + t \mathbb{B} \varphi \} \) in (27) whose corresponding variational inequality is equivalent to the identity
\[
\int_{\mathbb{R}^d} u_\lambda \cdot \mathbb{B} \varphi = 0 \quad \text{for all } \varphi \in C_0^\infty(\mathbb{R}^d; X),
\]
which is precisely the weak formulation of (26). In particular, every \( u \in L^2(\mathbb{R}^d; \mathbb{W}) \) has an \( L^2 \)-decomposition of the form
\[
u = \mathbb{B}^{\ast} v + z, \quad \text{where } \mathbb{B}^{\ast} v = 0 \text{ and } z \in \ker A.
\]
Observe that when \( A = \text{curl} \) then \( \mathbb{B}^{\ast} = \text{div} \), and therefore this decomposition is the well-known Helmholtz decomposition
\[
u = \mathbb{B}^{\ast} v + \nabla w, \quad \nabla \cdot v = 0.
\]

Let us now work the general case for compactly supported measures. In what follows \( u \in \mathcal{M}(\mathbb{R}^d; \mathbb{W}) \) is a compactly supported measure and \( u_\delta := u * \rho_\delta \), where \( \rho_\delta \in C^\infty(\mathbb{R}^d) \) is a standard mollifier at scale \( \delta > 0 \). Motivated by the fact that multipliers commute with translations we verify that
\[
(u_\delta)_{\lambda} = \mathcal{F}^{-1}(\hat{[A^\top A]} \hat{u} * \hat{\rho}_\delta) = \mathcal{F}^{-1}(\hat{A^\top A} \hat{u}) * \delta = u_\lambda * \rho_\delta.
\]

We deduce from (26) and the fact that \( u_\delta \in L^2(\mathbb{R}^d; \mathbb{W}) \) that \( u_\lambda \) is the weak-\( \ast \) limit of \( \mathbb{B}^\ast[D] \)-free measures. The same results can be carried in the \( d \)-dimensional torus. Even more, if indeed we work with Fourier coefficients, the discussion above and Lemma 4.1 give the following decomposition:

**Theorem 5.1** (Helmholtz decomposition). Let \( \mu \in \mathcal{M}(\mathbb{R}^d; \mathbb{W}) \) be a compactly supported measure. Then
\[
\mathbb{B}^{\ast} \mu_\lambda = 0 \quad \text{in the sense of distributions on } \mathbb{R}^d.
\]

In particular \( \mu \) can be decomposed into a \( \mathbb{B}^{\ast} \)-free field and \( \mathbb{B} \)-potential as
\[
\mu = \mu_\lambda + \mathbb{B} u, \quad \text{where } \mathbb{B} u = \mu - \mu_\lambda.
\]

**Theorem 5.2** (Helmholtz decomposition II). Let \( \Omega \subset \mathbb{R}^d \) be an open and bounded domain and let \( \mu \in L^p(\Omega; \mathbb{W}) \) for some \( 1 < p < \infty \). Then, \( \mu \) can be decomposed as
\[
\mu = v + \mathbb{B} u,
\]
where
\[
\mathbb{B}^{\ast} v = 0 \quad \text{in the sense of distributions on } \Omega,
\]
\[
\|v\|_{L^p(\Omega)} \leq c_{\Lambda, \mathbb{B}, p} \|\mu\|_{L^p(\Omega)},
\]
and
\[
\|D^{\mathbb{B}^{\ast}} v\|_{L^p(\Omega)} \leq c_{\Lambda, \mathbb{B}, p} \|\mu\|_{L^p(\Omega)}.
\]

**Proof.** Let us first assume that \( \Omega \subset Q \). We may consider
\[
\tilde{\mu} := 1_{\Omega} \mu - \langle \mu \rangle_{\Omega} \in L^p_\mathcal{F}(\mathbb{T}^d; \mathbb{W}).
\]

Accordingly, we deduce \( \tilde{\mu} \in W^{-k_\Lambda, p}(\mathbb{T}^d) \) and \( \|A \tilde{\mu}\|_{W^{-k_\Lambda, p}(\mathbb{T}^d)} \leq 2\|\mu\|_{L^p(\Omega)} \). Notice this latter estimate may indeed be a very raw estimate when \( A \tilde{\mu} \) is small or zero (cf. Remark 5.1). Using the projection \( \pi_{\Lambda} \) we may split \( \tilde{\mu} \) as
\[
\tilde{\mu} = \tilde{\mu}_\lambda + \pi_{\Lambda} \tilde{\mu}, \quad \mathbb{B}^{\ast} \tilde{\mu}_\lambda = 0 \text{ on } \mathbb{T}^d,
\]
and
\[
\|\tilde{\mu}_\lambda\|_{L^p(\mathbb{T}^d)}, \|\pi_{\Lambda} \tilde{\mu}\|_{L^p(\mathbb{T}^d)} \lesssim c_{p, \Lambda} \|\mu\|_{L^p(\Omega)}.
\]
Since $\pi_A \tilde{\mu} \in L^p(T^d; W)$ is an $A$-free map, with the aid of Lemma 4.2 (with the operator $A$) and Lemma 3.2 (with the operator $B$) we may find a potential $\tilde{u} \in BV^B(T^d; W)$ satisfying

$$\tilde{u} = \tilde{u}_B, \quad B\tilde{u} = \pi_A \tilde{\mu}$$

and

$$\|\tilde{u}\|_{W^{k,p}(T^d; W)} \leq c_{p,B} \|B\tilde{u}\|_{L^p(T^d)} = c_{p,B} \|\pi_A \tilde{\mu}\|_{L^p(T^d)} \lesssim c_{p,A} \cdot c_{p,B} \|\mu\|_{L^p(T^d)}.$$

The sought assertion follows by taking $v = [\tilde{\mu}_A + (\mu)\Omega] \mid \Omega$ and $u = \tilde{u} \mid \Omega$.

The general case goes as follows. Let $r := \text{diam}(\Omega)/2$ so that $T_{x,r}(\Omega) \subseteq Q$ for some $x \in \Omega$ and apply the previous argument to $\mu_r := \mu(x + r\cdot)$ to get

$$\mu_r = v_r + Bu_r, \quad B^*v = 0 \text{ on } T_{x,r}(\Omega),$$

where

$$\|v_r\|_{L^p(T_{x,r}(\Omega))} \lesssim c_{p,A} \|\mu_r\|_{L^p(T_{x,r}(\Omega))},$$

$$\|D^{ks}u_r\|_{L^p(T_{x,r}(\Omega))} \lesssim c_{p,A} \cdot c_{p,B} \|\mu_r\|_{L^p(T_{x,r}(\Omega))}.$$ 

By re-scaling back with $v := v_r \circ T_{x,r}$ and $u := r^{ks} \cdot u_r \circ T_{x,r}$ we conclude that

$$\mu = v + Bu, \quad B^*v = 0,$$

with

$$\|v\|_{L^p(\Omega)} \lesssim c_{p,A} \|\mu\|_{L^p(\Omega)},$$

$$\|D^{ks}u\|_{L^p(\Omega)} \lesssim c_{p,A} \cdot c_{p,B} \|\mu\|_{L^p(\Omega)}.$$ 

This finishes the proof.

\[\square\]

Remark 5.1 (assumptions on the domain). The assertions of Theorem 5.2 with $A = \text{curl}$ recover a classical Helmholtz-type decomposition

$$\mu = v + \nabla u, \quad \nabla \cdot v = 0 \text{ on } \Omega.$$

If moreover, $\Omega$ is assumed to be a simply connected and Lipschitz domain, and $\text{curl}\mu = 0$, then the argument in the proof of Theorem 5.2 is in line with De Rham’s Theorem in the sense that

$$\mu = \nabla u \text{ for some } u \in W^{1,2}(\Omega).$$

Indeed, in this case the extension $\tilde{\mu}$ is still a curl-free field on $Q$ and $\tilde{\mu}_{\text{curl}} = 0$, whence (up to a constant) we get $v = 0$. Notice however that if $\Omega$ is not assumed to be a simply connected and Lipschitz domain, then even when $\text{curl}\mu = 0$, it might occur that

$$\text{curl} \tilde{\mu} \neq 0.$$

This, in turn, conveys the appearance of a non-trivial divergence-free part in the decomposition of $\mu$. For general operators $A$, this phenomenon tells us that for general domains $\Omega$, one cannot expect to define

$$\mu_h = \tilde{\mu}_A \text{ when } \mu \in L^p(\Omega; W).$$
6. Proof of the Approximation Theorem 1.3

Proof. Step 1. An area-strictly converging recovery sequence. Let \( \{\varphi_i\}_{i \in \mathbb{N}} \subset C^\infty_c(\Omega) \) be a locally finite partition of unity of \( \Omega \). Set
\[
\mu_{(i)} := \mu \varphi_i \in \mathcal{M}(\Omega; \mathbb{R}^N).
\]
Following Step 2 in the proof of [4, Thm 1.7], we recall there exists a sequence \((\tilde{u}_j) \subset C^\infty(\Omega; W)\) defined as (there, denoted by \( u_j \))
\[
\tilde{u}_j := \sum_{i=1}^{\infty} \mu_{i,j},
\]
where each \( \mu_{i,j} := \mu_{(i)} * \rho_{\delta_{(j)}} \) is compactly supported in \( \Omega \) (and \( \delta_{(j)} \downarrow 0 \)). Here \( \rho \) is a standard normalized mollifier at scale \( \delta > 0 \). Moreover, it is shown that \( \tilde{u}_j \xrightarrow{L^d} \mu \) area strictly-converges to \( \mu \) in the sense that
\[
\text{Area}(\tilde{u}_j, \Omega) \to \text{Area}(\mu, \Omega).
\]

Step 2. Construction of the A-free sequence. Let us begin with a few observations about the measures \( \mu_{i,j} \) defined above:
(a) Every \( \mu_{i,j} \) is a compactly supported on \( \Omega \). Therefore we may naturally consider each \( \mu_{i,j} \) as an element of \( C^\infty_c(\mathbb{R}^d; W) \); and every \( \mu_{(i)} \) as an element of \( \mathcal{M}_b(\mathbb{R}^d; W) \).
(b) The operator \( A \) acts on \( \mu_{i,j} \) as
\[
A \mu_{i,j} = A \mu_{(i)} * \rho_{\delta_{(i)}} =: f_i * \rho_{\delta_{(i)}} =: f_{i,j}, \quad \int_{\mathbb{T}^d} f_i = 0.
\]
(c) As \( \{\varphi_i\}_{i \in \mathbb{N}} \) is a locally finite partition, we can take linear operators inside and outside arbitrary sums subjected to it. In particular,
\[
\sum_{i=1}^{\infty} f_i = A \left( \sum_{i=1}^{\infty} \mu_{(i)} \right) = A \mu = 0 \text{ locally on } \Omega.
\]
By (23)-(24) we get \( f_{i,j} \to f_i \) strongly in \( W^{-k_i,q}(\mathbb{R}^d) \), for all \( 1 < q < \frac{d}{d-1} \).

Let \( i \in \mathbb{N} \) be an arbitrary but fixed index. Resting on observation (a), let us denote
\[
v_{i,j} := (\mu_{i,j})_h \quad \text{and} \quad v_i := (\mu_{(i)})_h,
\]
the respective \( A \)-representatives from by Lemma 3.2 (in its version for \( \mathbb{R}^d \) instead of \( \mathbb{T}^d \)). It follows from the estimate in the same lemma that
\[
\|v_{i,j} - v_i\|_{L^q(\mathbb{R}^d)} \leq c_q \|f_{i,j} - f_i\|_{W^{-k_i,q}(\mathbb{R}^d)} \xrightarrow{(c)} 0.
\]
Taking a further subsequence if necessary, we may assume that
\[
\|v_i - v_{i,j}\|_{L^q(\mathbb{R}^d)} \leq \frac{1}{2^j \times j}.
\]
Now, let us look at the translations \( w_{i,j} := \mu_{i,j} - v_{i,j} \in \mathcal{M}_b(\mathbb{R}^d; W) \). Note that these are \( A \)-free on \( \mathbb{R}^d \), which leads us to define our candidate for an \( A \)-free recovery sequence. Consider the sequence consisting of the elements
\[
u_j := \sum_{i=1}^{\infty} (w_{i,j} + v_i) \xrightarrow{L^d} \mu, \quad j \in \mathbb{N}.
\]

Claim 1. Each \( u_j \xrightarrow{L^d} \mu \) is \( A \)-free.
Arguing as in (d) — and by this observation itself, we get
\[ \mathcal{A} u_j = \sum_{i=1}^{\infty} \mathcal{A} v_i = \sum_{i=1}^{\infty} f_i = 0 \] locally on \( \Omega \).

**Claim 2.** The sequence \((u_j)\) area-strictly converges to \( \mu \).
Since \((\tilde{u}_j, \mathcal{L}^d)\) already area-converges to \( \mu \), it suffices to show that \( \tilde{u}_j \) and \( u_j \) are asymptotically \( L^1 \)-close to each other (which is sufficient to ensure the asymptotic closeness of the area functional which has a uniformly Lipschitz integrand). This is easily verified since
\[
\| u_j - \tilde{u}_j \|_{L^1(\Omega)} \leq \sum_{i=1}^{\infty} \| v_i - v_{i,j} \|_{L^2(\mathbb{R}^d)} \leq \frac{30}{j}.
\]

This proves the second claim, which finishes the proof. \(\square\)

7. Decomposition of generating sequences

**Lemma 7.1** (decomposition of generating sequences). Let \( \nu = (\nu, \lambda, \nu^\infty) \in Y(\Omega, \mathcal{W}) \) be an \( \mathcal{A} \)-free young measure and let \( \Omega' \subsetneq \Omega \) be a Lipschitz open subset with \( \lambda(\partial\Omega') = 0 \). Then, the barycenter measure \( [\nu] \) can be decomposed into the \( \mathcal{B} \)-gradient of a potential \( u \in BV^\mathcal{B}(\Omega') \) and an \( \mathcal{A} \)-free field \( v \in L^1(\Omega') \), that is,
\[
[\nu] |_{\mathcal{B}} = Bu + v, \quad \lambda v = 0 \text{ on } \Omega'.
\]
Moreover, there exists a sequence \((u_h) \subset BV^\mathcal{B}(\Omega')\) satisfying
- \( u_h \equiv u \) on a neighborhood of \( \partial\Omega' \),
- \( u_h \to u \) in \( W^{k\lambda-1,1}(\Omega') \), and
- \( Bu_h + v \mathcal{L}^d \mathcal{Y}_{\Omega'} \nu \) in \( Y(\Omega'; \mathcal{W}) \).

**Proof.** We may (without loss of generality) assume that \( \Omega' \subset \Omega \subset Q \). Let \((\mu_j)\) be a sequence of \( \mathcal{A} \)-free functions generating \( \nu \). Let \( \varphi \in C_c^\infty(\Omega) \) be a cut-off function of \( \Omega' \) subjected to \( \Omega_1 \). Lastly, let us write \( \mu = [\nu] \) to denote the barycenter measure of \( \nu \). We define a sequence of compactly supported measures on \( \Omega \) (and \( Q \)) by setting \( \sigma_j := \varphi \mu_j \) and \( \sigma_0 := \varphi \mu \). Using the trivial extension by zero, we may regard each \( \sigma_j \) as an element of \( \mathcal{M}_d(\mathbb{T}^d, \mathcal{W}) \); here \( \mathbb{T} = \int \mu \). Notice that thanks to Lemma 3.2 and to the embedding (22) we obtain
\[
\mathbb{A} w_j = [\mathbb{A}, \varphi]\mu_j \implies f_j := \mathbb{A} w_j \in W^{-k\lambda, q}(\mathbb{T}^d)
\]
\[
\implies w_j \in W^{0, q}(\mathbb{T}^d)
\]
\[
\implies (w_j)_k \in L^q(\mathbb{T}^d),
\]
for all \( 1 < q < \frac{6}{5}k \). Here, in the first implication, we have used that commutator \([\mathbb{A}, \varphi] := \varphi \mathbb{A} - \mathbb{A} \varphi \) is a partial differential operator of order at most \((k\lambda - 1)\) with smooth coefficients depending only on \( \|\varphi\|_{k, \infty} \) and the coefficients of \( \mathbb{A} \). Moreover, since \( d_\lambda(\mu_j, \mu) \to 0 \), then \( f_j \to f_0 := \mathbb{A} \sigma_0 \in W^{-k\lambda, q}(\mathbb{T}^d) \). Hence, it follows from Lemma 3.2 that
\[
\| (w_j)_k - (w_0)_k \|_{L^q(\mathbb{T}^d)} \leq c_q \| f_j - f_0 \|_{W^{-k\lambda, q}(\mathbb{T}^d)} \to 0,
\]
whence
\[
(w_j)_k \to (w_0)_k \text{ strongly in } L^q(\mathbb{T}^d), \quad \mathbb{A}(w_j)_k = f_j.
\]

Firstly, for each \( i \in \mathbb{N}_0 \), this allows us to write \( \sigma_j = z_j + [(w_j)_k + \mathcal{F}_j] \mathcal{L}^d \mathcal{W} \) where \( z_j \in \mathcal{M}_d(\mathbb{T}; \mathcal{W}) \) is an \( \mathcal{A} \)-free measure (indeed \( \mathbb{A}(w_j)_k = \mathbb{A} \sigma_j \)). We may apply the
exactness property contained in Proposition 4.1 on the sequence \((z_j)\), to obtain yet another sequence \((u_j) \subset BV^{\mathcal{B}}(\mathbb{T}^d)\) satisfying \(\mathbb{B}(u_j) = \mathbb{B}u_j = z_j\). Notice that by construction, the sequence \(\{(u_j)_{\mathcal{B}}\}\) is uniformly bounded in \(BV^{\mathcal{B}}(\mathbb{T}^d)\). In fact, Corollary 3.1 applied to the operator \(\mathbb{B}[D]\) ensures that

\[
\|(u_j)_{\mathcal{B}} - (u_0)_{\mathcal{B}}\|_{W^{k_3-1,1}(\mathbb{T}^d)} \leq c_q \cdot c_d\{z_j, z_0\} \to 0.
\]

We readily check that

\[
\mathbb{B}(u_0) + (w_0)_{\mathcal{B}} \mathcal{L}^d + \sigma_j \mathcal{L}^d \equiv \sigma_0 \equiv \mu \quad \text{as measures in } \mathcal{M}(\Omega'; \mathbb{W}).
\]

Since \(\text{supp}(f_j) \subset \Omega \setminus \overline{\Omega'}\) for all \(j \in \mathbb{N}_0\), this implies \((w_j)_{\mathcal{B}}\) is an \(\mathcal{A}\)-free measure on \(\Omega\). This proves the first assertion on the decomposition of the barycenter \(\mu\) on \(\Omega\).

Moreover, since \(\lambda(\partial\Omega') = 0\), we obtain

\[
\mathbb{B}(u_j)_{\mathcal{B}} + (w_j)_{\mathcal{B}} \mathcal{L}^d + \sigma_j \mathcal{L}^d \equiv \mu_j \overset{Y}{\rightharpoonup} \nu \quad \text{on } \Omega'.
\]

Furthermore, since \((w_j)_{\mathcal{B}} \rightarrow (w_j)_{\mathcal{A}}\) strongly in \(L^q\), it follows from Proposition 2.1 that

\[
\mathbb{B}(u_j)_{\mathcal{B}} + (w_0)_{\mathcal{B}} \mathcal{L}^d + \sigma_j \mathcal{L}^d \overset{Y}{\rightharpoonup} \nu \mathcal{L}_\Omega \Omega'.
\]

Next, we address the boundary adjustment of the generating sequence above. For a positive real \(t > 0\) we define \(\Omega'_t = \{ x \in \Omega' : \text{dist}(x, \partial\Omega') > t \}\). Fix \(\varphi_t \in C_c^\infty(\Omega')\), a cut-off of \(\Omega'_t\) with \(\varphi \equiv 0\) on \(\Omega'_t\), and such that \(\|\varphi_t\|_{k_3, \infty} \lesssim t^{-k_3}\). Let \(\delta_n \searrow 0\) be an infinitesimal sequence of positive reals. We define a sequence with \((u_0)_{\mathcal{B}}\)-boundary values by setting

\[
u_{h,j} := \varphi_{\delta_n}(u_j)_{\mathcal{B}} - (u_0)_{\mathcal{B}} + (u_0)_{\mathcal{B}}, \quad u_{h,j} \equiv (u_0)_{\mathcal{B}} \text{ on } \Omega' \setminus \Omega'_0.
\]

For fixed \(h\), it follows from (31) that there exists \(j = j(h)\) such that

\[
\| (u_j)_{\mathcal{B}} - (u_0)_{\mathcal{B}} \|_{W^{k_3-1,1}(\Omega')} \leq \frac{\delta_n}{h} \quad \text{for all } n \geq j(h).
\]

In particular, setting \(u_h := u_{h,j(h)}\), we can estimate the total variation of \(\mathbb{B}u_h\) as

\[
|\mathbb{B}u_h|_{\Omega'} \lesssim \|\varphi_{\delta_n}\|_{k_3, \infty} \cdot \|(u_j)_{\mathcal{B}} - (u_0)_{\mathcal{B}}\|_{W^{k_3-1,1}(\Omega')} + |\mathbb{B}u_0|_{\Omega'} \leq \frac{1}{h} + |\mathbb{B}u_0|_{\Omega'}.
\]

Notice that not only this implies that \((\mathbb{B}u_h)\) is uniformly bounded, but also that the sequence does not charge the boundary \(\partial\Omega\). Therefore, up to extracting a subsequence (which we will not relabel), the sequence generates a Young measure on \(\Omega'\) which does not carry mass into the boundary, i.e.,

\[
\mathbb{B}u_h + v \mathcal{L}^d \overset{Y}{\rightharpoonup} \sigma \quad \text{in } \Omega', \quad \lambda_{\sigma}(\partial\Omega') = 0,
\]

where \(v : = [(w_0)_{\mathcal{B}} + \sigma_j \mathcal{L}^d\). On the other hand, our construction gives the equivalence of measures \(\mathbb{B}u_h = \mathbb{B}(u_{j(h)})_{\mathcal{B}}\) when these are restricted to the set \(\Omega'_{\delta_n}\). Since \(\delta_n \searrow 0\), we deduce from (32)-(33) that \(\sigma \equiv \nu\) on \(\Omega'\), and therefore

\[
\mathbb{B}u_h + v \mathcal{L}^d \overset{Y}{\rightharpoonup} \nu \quad \text{in } \Omega',
\]

with \(u_h \equiv (u_0)_{\mathcal{B}}\) on a neighborhood of \(\partial\Omega'\). This finishes the proof. \(\square\)

The previous lemma establishes that concentrations generated along an \(\mathcal{A}\)-free sequences are, in fact, only carried by \(\mathbb{B}\)-gradients. More generally, we have the following result.
Corollary 7.1 (decomposition of blow-up sequences). Let $\nu = (\nu, \lambda, \nu^\infty) \in Y(\Omega; W)$ be an $A$-free measure and let $\sigma \in \text{Tan}(\nu, x)$ be a tangent Young measure. Then, for every Lipschitz domain $\omega \subset \mathbb{R}^d$ with $\lambda(\partial \omega) = 0$, there exist a potential $u \in BV^d(\omega)$ and a vector $z \in W$ such that

$$\mathbb{B} u + z \partial_x \mathbb{L} \omega = [\sigma] \mathbb{L} \omega.$$ 

Moreover, there exists a sequence $(u_j) \subset BV^d(\omega)$ satisfying $u_j \to u$ in $W^{1,1}(\omega)$ and such that

$$\mathbb{B} u_j + z \partial_x \mathbb{L} \omega \to [\sigma] \mathbb{L} \omega \quad \text{in} \ Y(\omega; W).$$

Furthermore, if $x \in \Omega$ is a singular point of $\nu$, then $z = 0 \in W$.

Proof. The locality property (13) of Young measures and the local decomposition of generating sequences given in Lemma 7.1 imply that it is enough to show the assertion when

$$\mu_j = \mathbb{B} u_j + v \partial_x \mathbb{L} \omega \to [\nu] \mathbb{L} \omega \quad \text{in} \ Y(\omega; W), \quad v \in L^1(\Omega; W), \quad A v = 0.$$ 

We consider two cases: when $x \in \Omega$ is a regular or a singular point of $\lambda$.

**Regular points:** Every tangent Young measure $\sigma \in \text{Tan}(\nu, x)$ is generated by a sequence of the form

$$\frac{c}{r^j} \cdot T_{x,r} [\mathbb{B} u_j + v \partial_x \mathbb{L}] \to [\sigma] \mathbb{L} \omega \quad \text{in} \ Y(\mathbb{R}^d, W).$$

Recall however from (11) that

$$\frac{c}{r^j} \cdot T_{x,r} [v \partial_x \mathbb{L}] \to [v(x) \partial_x \mathbb{L}] \quad \text{strongly in} \ L^1_{\text{loc}}(\mathbb{R}^d, W).$$

Hence, from the linearity of the push-forward and the compactness of Young measures, it follows that (here we use that $\lambda(\partial \omega) = 0$).

$$\frac{c}{r^j} \cdot T_{x,r} [\mathbb{B} u_j + v(x) \partial_x \mathbb{L}] \to [\sigma] \mathbb{L} \omega \quad \text{in} \ Y(\omega, W).$$

The assertion follows by taking $z = v(x)$.

**Singular points:** This proof is easier since instead of (34)-(35) we have

$$\frac{c}{\lambda^*(Q_j(x))} \cdot T_{x,r} [\mathbb{B} u_j + v \partial_x \mathbb{L}] \to [\sigma] \mathbb{L} \omega \quad \text{in} \ Y(\mathbb{R}^d, W).$$

Recall however from (18) that

$$\frac{c}{\lambda^*(Q_j(x))} \cdot T_{x,r} [v \partial_x \mathbb{L}] \to [\theta] \quad \text{strongly in} \ L^1(\mathbb{R}^d, W).$$

Therefore, using the exact same arguments as before (with different normalization constants) yields $z = 0$, and $\sigma = \theta$ with $\theta$ a tangent $B$-Young measure.

This completes the proof.

□

Corollary 7.2. If $\nu \in Y(\Omega, W)$ is an $A$-free Young measure, then

$$\text{Tan}(\nu, x) \subset \text{Shift}_{\text{BY}} \left\{ \mathbb{B} \text{YM}(\mathbb{R}^d) \right\} \quad \text{for} \ \mathbb{L}^d \text{ almost every} \ x \in \Omega,$$

and

$$\text{Tan}(\nu, x) \subset \mathbb{B} \text{YM}(\mathbb{R}^d) \quad \text{for} \ \lambda^* \text{ almost every} \ x \in \Omega.$$

Proof. **Regular points:** Keeping the same notation and following (34)-(35) we may assume (this may involve passing to a further subsequence if necessary) that

$$\mathbb{B} u_j \to \theta \quad \text{and} \quad \frac{c}{r^j} \cdot T_{x,r} [\mathbb{B} u_j + z \partial_x \mathbb{L}] \to [\theta] \quad \text{in} \ Y(\theta, \lambda_0, \theta^\infty);$$

and therefore $\theta = (\theta_x, \lambda_0, \theta^\infty) \in \text{Tan}(\theta, x)$ is an homogeneous Young measure.
Let \( j_0 \) be sufficiently large so that \( \mathcal{W} \subset T_{x,r_{j_0}}(\Omega) \) and re-adjust the index of the sequence by the translation \( j \leftrightarrow j + j_0 \). In this way \( u_j \in BV^\infty(\omega) \) for all \( j \) and hence \( \theta \in \mathcal{YM}(\omega) \). Now, fix \( f \in \mathbf{E}(\omega, \mathcal{W}) \) and consider the shifted integrand \( g = f(\cdot + z, \cdot) \) which also belongs to \( \mathbf{E}(Q, \mathcal{W}) \) and satisfies \( g^\infty = f^\infty \). Then, by definition,

\[
\left\langle f, \nu \right\rangle = \lim_{j \to \infty} \left\langle f, \mu_j \right\rangle = \lim_{j \to \infty} \left\langle g, \theta \right\rangle
\]

\[
= \int f_{y+z} \, d\theta_y \, d\mathcal{L}^d(y) + \int f_y \, d\theta_y \, d\mathcal{L}^d(y) = \int f \, d(\theta * \delta_z) \, d\mathcal{L}^d + \int f^\infty \, d\theta^\infty \, d\lambda_y.
\]

This shows that

\[ \nu = (\theta_x * \delta_{-z}, \lambda_y, \theta^\infty_x). \]

We conclude that \( \Gamma_{v(x)}[\nu] = \theta \), whence we show \( \nu \) is a \( \mathcal{W} \)-shift of a \( \mathcal{B} \)-Young measure on \( \omega \).

**Singular points:** The proof at singular points follows the exact same reasoning except that in this case \((36)-(37)\) lead to \( z = v(x) = 0 \in \mathcal{W} \), and therefore there is no effective shift. \( \square \)

### 8. Proof of the local characterization Theorem 1.2

**Proof. Sufficiency.** This is straightforward from the definition of \( \mathcal{A} \)-free Young measure, a blow-up, and a diagonalization argument.

**Necessity.** Let \( \{\varphi_p \otimes h_q\}_{p,q \in \mathbb{N}} \subset \mathbf{E}(\Omega; \mathcal{W}) \) be the countable family from Lemma 2.1 which separates \( \mathbf{Y}(\Omega; \mathcal{W}) \). Let \( \nu = (\nu, \lambda, \nu^\infty) \in \mathbf{Y}(\Omega; \mathcal{W}) \) as in the assumptions of Theorem 1.2 and let us write \( \mu = |\nu| \) to denote the barycenter of \( \nu \). Consider also the positive measure

\[ \Lambda := \mathcal{L}^d + \lambda^s \in \mathcal{M}^+(\Omega). \]

It follows from the main assumption, that there exists a \( \Lambda \)-measure set \( B \subset \Omega \) with the following property: at every \( x \in B \) there exists a tangent Young measure \( \sigma = (\nu_x, \kappa, \nu^\infty_x) \in \text{Tan}(\nu, x) \) satisfying \((16)\) and (without carrying the \( x \)-dependence on several of the following elements)

\[
(38) \quad \left\langle \varphi \otimes h, \nu^{(r_j)} \right\rangle = c_j \left\langle (\varphi \circ T_{x,r_j}) \otimes h, \nu \right\rangle \rightarrow \left\langle \varphi \otimes h, \sigma \right\rangle \quad r_j \searrow 0,
\]

\[
(39) \quad c_j^{-1}(x) = \|\cdot\|_{\nu \mathcal{L}Q_{r_j}(x)}, \quad \lambda(Q_{r_j}(x)) = \lambda(Q_{r_j}(x)) > 0.
\]

In what follows we shall simply write \( c_{r_j} = c_{r_j}(x) \) when no possible confusion arises. Particular consequences the convergence above are the following: at every \( x \in B \) we can find a blow-up sequence

\[
(40) \quad c_j \cdot T_{x,r_j}[\lambda] \rightharpoonup \kappa, \quad \kappa(\partial Q) = 0,
\]

and (composing with the identity map \( \text{id}_{\mathcal{W}} \)) also

\[
(41) \quad \gamma_j := c_j \cdot T_{x,r_j}[\mu] \rightharpoonup [\sigma], \quad ||[\sigma]||(Q) \leq 1.
\]
Moreover, a version of Lemma 7.1 (applied to the sets $\Omega = \mathbb{R}^d$ and $\Omega' = Q$) and Corollary 7.2 allows us to re-write $[\sigma]$ and the blow-up sequence $\gamma_j$ as:

\begin{align}
[\sigma] \mathcal{L} Q &= z \mathcal{L}^d + \mathbb{B} u, \quad z := v(x) \in W, \\
\gamma_j &= v_j \mathcal{L}^d + \mathbb{B} u(i), \\
u^{(r_j)} &\to u \text{ in } W^{k-1,1}(Q). \\
v^{(r_j)} &\to z \text{ in } L^1(Q).
\end{align}

Step 1. Construction of a disjoint cover of $B$. Fix $m \in \mathbb{N}$. At every $x \in \Omega$ we define $\rho_m(x)$ as the supremum over all radii $0 < r_j(x) \leq \frac{1}{m}$ (where $r_j(x) \searrow 0$ is the sequence from the previous step at a given $x \in B$) such that

\begin{align}
\|\langle \varphi \otimes h_q, u^{(r_j)} \rangle \rangle - \langle \varphi \otimes h_q, [\sigma] \rangle \| &\leq \frac{1}{m} \quad \forall \, q \leq m, \\
|\gamma_m|(Q) &\leq 2, \\
\|u^{(r_j)} - u\|_{W^{k-1,1}(Q)} &\leq \frac{1}{m^{k+1}}, \\
\|v^{(r_j)} - u\|_{L^p(Q)} &\leq \frac{1}{m}.
\end{align}

Next, define the cover (of open cubes) with centers in $B$ given by

$$
Q_m := \{ Q_{r_j}(x) \subset \Omega : x \in B, r_j(x) \leq \rho_m(x) \}.
$$

Notice that, since $\rho_m(x) > 0$ exists for all $x \in B$, then $Q_m$ is a fine cover of $B$ and hence we may apply Besicovitch’s Covering Theorem (with the measure $\Lambda$) to find a disjoint sub-cover $O_m = \{ Q_{x,m} \}$, where each $Q_{x,m}$ is of the form $Q_{r_m(x)}$ for some $0 < r_m(x) \leq \rho_m(x) \leq \frac{1}{m}$, and

$$
\Lambda(\Omega \setminus O_m) = 0, \quad O_m := \bigcup_{Q_x \in O_m} Q_x.
$$

Step 3. An adjusted generating sequence of $\sigma$. Let $x \in B$ be fixed and let $\sigma = \sigma(x)$ be the $A$-free Young measure from the beginning of the proof. Yet another application of Lemma 7.1 yields a sequence $(w_h) \subset BV^B(Q)$ satisfying

\begin{align}
w_h &\to u \text{ in } W^{k-1,1}(Q), \\
\tilde{\gamma}_m := z \mathcal{L}^d \mathcal{L} Q + \mathbb{B} w_h &\to \sigma \text{ in } Y(Q; W).
\end{align}

Let $h = H(m) \in \mathbb{N}$ be sufficiently large so that

\begin{align}
\|\mathbb{B} u_h\|\leq 2, & \quad \text{for all } h \geq H(m) \\
\|u_h - u\|_{W^{k-1,1}(Q)} &\leq \frac{1}{m^{k+1}}, \quad \text{for all } h \geq H(m).
\end{align}

Next, let $\eta_m, \varphi_m \in C^\infty_{\text{cut}}([0, 1])$ be two (disjoint) cut-off functions which satisfy the following properties:

\begin{align}
\|\eta_m\|_{k, \infty} &\leq \|\varphi_m\|_{k, \infty} \leq \eta_m \|\varphi_m\|_{k, \infty} \leq m^{k+1} \\
\|Q_{1, -\frac{1}{m}} \leq \|\varphi_m\|_{k, \infty} \leq \|Q_{1, -\frac{1}{m}} \leq \|\varphi_m\|_{k, \infty} \leq m^{k+1}
\end{align}

Step 4. Boundary adjustment for generating sequences of $\sigma$. The next step is to define an $A$-free sequence generating $\sigma$ on $Q$ which also has a blow-up of $\mu$ as boundary values. This should allow us to freely glue each of this approximations together while keeping the $A$-free constraint
Fix \( m \in \mathbb{N} \) and let \( Q_{x,m} \in \mathcal{O}_m \). We begin by constructing a sequence on \( Q \), which we shall later translate to each \( Q_{x,m} \in \mathcal{O}_m \). Bearing in mind all the \( x \)-dependencies that we have omitted in the previous steps, define the \( \mathcal{A} \)-free sequence

\[
q_{h,m} := B(\varphi_m(w_h - u)) + Bu + z \mathcal{L}^d
generating \text{sequence}
\]

\[
+ \mathcal{B}(\eta_m(u^{(r_m)} - u)) + \eta_m(v^{(r_m)} - z) \mathcal{L}^d
\]

\[
= [B, \varphi_m](w_h - u) + z \mathcal{L}^d + \varphi_m \cdot Bw_h
\]

\[
+ (1 - \varphi_m - \eta_m)Bu + [B, \eta_m](u^{(r_m)} - u)
\]

\[
+ \eta_m \cdot Bu^{(r_m)} + \eta_m(v^{(r_m)} - z) \mathcal{L}^d.
\]

Here, let us recall that the commutator \([B, \chi] := B \circ \chi - \chi \circ B\) is a differential operator of order at most \( k_B - 1 \) (with coefficients involving the coefficients of \( B \) and the \( k_B \)th order derivatives of \( \chi \)). By this token, if \( h \geq H(m) \), we may estimate the total variation of \( q_{h,m} \) as

\[
\left| q_{h,m}(Q) \right| \lesssim \| \varphi_m \|_{k,\infty} \cdot \| w_h - u \|_{W^{k_B - 1}(Q)} + \| Bw_h \|_{W^k(Q)} + \| B \|_{W^{\infty,k_B - 1}(Q)} \cdot \| u \|_{W^{k_B - 1}(Q)} + \| B \|_{L^\infty(Q)} \left( \| u \|_{L^p(Q)} + \| v^{(r_m)} - z \|_{L^p(Q)} \right)
\]

whence it is established that \( q_{h,m}(h) \in \mathcal{M}(Q; \mathcal{Y}) \). In fact, we get that \( \limsup_{m \to \infty} |q_m| (Q \setminus Q_{1 - \frac{1}{m}}) = 0 \); this follows from the property \( \kappa(\partial Q) = 0 \). Therefore, passing to further subsequence of the \( h \)'s if necessary (not relabeled), we may assume that

\[
q_{h,m} \overset{Y}{\rightarrow} \tilde{\sigma} in Y(Q; \mathcal{Y}), \quad \lambda_{\tilde{\sigma}}(\partial Q) = 0.
\]

On the other hand, observe that \( q_{h,m} L(Q_{1 - \frac{1}{m}}, \mathcal{Y}) \equiv z \mathcal{L}^d + Bw_{h(m)} \) and hence, by the locality of Young measures, it must hold \( \tilde{\sigma} \equiv \sigma \) in \( Y(Q_{1 - \frac{1}{m}}, \mathcal{Y}) \). Since this holds for all \( h \in \mathbb{N} \) and neither \( \tilde{\sigma} \) or \( \sigma \) charge the boundary \( \partial Q \), it follows that

\[
q_{h,m} \overset{Y}{\rightarrow} \sigma \text{ in } Y(Q_{1 - \frac{1}{m}}, \mathcal{Y}) \text{ as } h \to \infty,
\]

\[
q_{h,m} \equiv v^{(r_m)} \mathcal{L}^d + Bu^{(r_m)} \equiv \gamma_m \text{ as measures on } Q \setminus Q_{1 - \frac{1}{m}}.
\]

In particular, the uniform bound above and \((40)\) ensure that we may find another subsequence \( h(m) \geq H(m) \) satisfying

\[
\left( \| I_Q \otimes h_q, \delta_{q_{h(m),m}} \| - \| I_Q \otimes h_q, \sigma \| \right) \leq \frac{1}{m} \quad \text{for all } q \leq m.
\]

**Step 5. Gluing together and generating \( \nu \).** So far, we have constructed generating sequences for specific tangent Young measures of \( \nu \) on every \( x \) where there is a cube \( Q_{x,m} \subset \mathcal{O}_m \). The rest of the proof can be summarized in the following two steps: First, we construct an \( \mathcal{A} \)-free sequences by gluing the pull-backs of the \( q_{h(m),m}'s \) to each \( Q_{x,m} \); and second, we show the new global sequence is uniformly bounded.

**Step 5a. Gluing the generating sequences.** For fixed \( x \in \mathbb{R}^d \) and \( r > 0 \) we define the map \( G_{x,r(y)} = (T_{x,r})^{-1} = x + ry \), which is defined for all \( y \in \mathbb{R}^d \). Fix a cube \( Q_{x,m} \) in \( \mathcal{O}_m \) and define an \( \mathcal{A} \)-free measure there by setting

\[
U_m := c_m^{-1} \cdot G_{x,r_m} q_{h(m),m} \in \mathcal{M}(Q_{x,m}; \mathcal{Y}).
\]
Notice that

\[ U_m \equiv c_m^{-1} \cdot G_{x,r_m} [\gamma_m] \]
\[ \equiv c_m^{-1} \cdot c_m (T_{x,r_m} \circ G_{x,r_m}) [\mu] \]
\[ \equiv \mu \quad \text{as measures on } Q \setminus Q_{1-\frac{1}{m}}. \]

Therefore, the measure defined as

\[ \tau_m(\text{d}y) := \begin{cases} U_m(\text{d}y) & \text{if } y \in Q_{x,m} \\ \mu(\text{d}y) & \text{if } y \in \Omega \setminus O_m \end{cases} \]

is well-defined in \( \Omega \), and (cf. (56)) is also \( \mathcal{A} \)-free on \( \Omega \). Moreover, its total variation in \( \Omega \) can be controlled as follows (recall that the push-forward is a mass preserving operation)

\[ |\tau_m|(\Omega) \leq \sum_{Q_{x,m} \in Q_m} |U_m|(Q_{x,m}) + |\mu|(\Omega \setminus O_m) \]
\[ \leq \sum_{Q_{x,m} \in Q_m} c_m^{-1} |h_{h(m),m}|(Q) \]
\[ \leq \sum_{Q_{x,m} \in Q_m} c_m^{-1} \cdot \left( \frac{3}{m} + 5|\gamma_m|(Q) \right) \]
\[ \leq 10 \cdot \sum_{Q_{x,m} \in Q_m} \| \nu \mathbb{L} Q_{x,m} \| \]
\[ \leq \left( 1 + |\cdot, \nu \right) < \infty. \]

**Step 5b.** The new \( \mathcal{A} \)-free sequence generates \( \nu \). This is the last step. To check that \( \nu \) is indeed an \( \mathcal{A} \)-free Young measure in \( \Omega \), it suffices to check that \( \tau_m \) generates \( \nu \) in \( \Omega \). First, we estimate how close \( U_m \) is from generating \( \nu \) on \( Q_{x,m} \). Fix \( \varphi \in C(\Omega) \). Every cube \( Q_{x,m} \in Q_m \) has diameter at most \( \sqrt{dm}^{-1} \) and therefore there exists a modulus of continuity (depending solely on \( \varphi \)) such that \( \| \varphi(x) - \varphi \|_{\infty}(Q_{x,m}) \leq \omega(m^{-1}) \) for all \( Q_{x,m} \in Q_m \); the same bound holds for any dilation of \( \varphi \) on the corresponding dilation of \( Q_{x,m} \).

Let \( q \in \mathbb{N} \) and let \( M_q \) to be the linear growth constant of \( h_q \). We define

\[ \delta(m) := \omega(m^{-1}) \left( 12M_q \| 1 + |\cdot, \nu \| + 2\| \varphi \|_{\infty} m^{-1} \right) \searrow 0 \quad \text{as } m \to \infty. \]
Let $m \geq q$. Regarding $U_m$ as an element of $\mathcal{M}(\Omega; \mathcal{W})$ through the trivial extension by zero, we obtain the estimate
\[
\|\langle \varphi \otimes h_q, \nu \rangle, Q_{x,m} \rangle - \langle \varphi \otimes h_q, \nu \rangle, Q_{x,m} \rangle \|
\leq c_m \|\langle \varphi \otimes G_{x',r_m} \otimes h_q, \delta_{qh(m),m} \rangle - \langle \varphi \otimes h_q, \nu \rangle, Q_{x,m} \rangle \|
+ c_m \omega(m^{-1}) \cdot M_q[\mathcal{L}^d(Q) + |qh(m),m|(Q)]
\leq c_m \|\langle \varphi \otimes G_{x',r_m} \otimes h_q, \nu \rangle, Q_{x,m} \rangle - \langle \varphi \otimes h_q, \nu \rangle, Q_{x,m} \rangle \|
+ 2\|\varphi\|_\infty m^{-1} + M_q[\mathcal{L}^d(Q) + |qh(m),m|(Q)]
\leq c_m \|\langle \varphi \otimes G_{x',r_m} \otimes h_q, \nu \rangle, Q_{x,m} \rangle - \langle \varphi \otimes h_q, \nu \rangle, Q_{x,m} \rangle \|
+ 2\|\varphi\|_\infty m^{-1} + \omega(m^{-1}) \cdot M_q[\mathcal{L}^d(Q) + |qh(m),m|(Q)] + c_m \|\varphi\|_\infty m^{-1}
\leq \omega(m^{-1}) \left[ 12 M_q[\|1 \cdot \nu \|, Q_{x,m}] + 2\|\varphi\|_\infty m^{-1} \right].
\]

Therefore, adding up these estimates for each cube $Q_{x,m}$ on $Q_m$ yields
\[
\|\langle \varphi \otimes h_q, \delta_{x,m} \rangle - \langle \varphi \otimes h_q, \nu \rangle \|
\leq \sum_{Q_{x,m} \in Q_m} \|\langle \varphi \otimes h_q, \delta_{U_m} \rangle - \langle \varphi \otimes h_q, \nu \rangle, Q_{x,m} \rangle \| + |\mu|(O_m)
\leq \delta(m).
\]

This shows that $\langle \varphi \otimes h_q, \delta_{x,m} \rangle \to \langle \varphi \otimes h_q, \nu \rangle$ as $m \to \infty$, and, in particular, this holds for $\varphi = \varphi_p$ for any $p \in \mathbb{N}$.

Conclusion. Since the family $\{\varphi_p \otimes h_q\}_{p,q\in \mathbb{N}}$ separates $E(\Omega; \mathcal{W})$, we conclude that

$\tau_m \xrightarrow{\text{weak-}*} \nu$ on $\Omega$.

Since $\nu \in \mathcal{A}Y_{\mu}(\Omega) := \{ \tau \in \mathcal{A}Y(\Omega) : |\tau| = \mu, \tau$ is an $\mathcal{A}$-free Young measure $\}$ is weak-* closed in $(E(\Omega; \mathcal{W}))^*$ (see for instance Lemma 5.3 in [9]; see also the discussion at the beginning of the next section), this proves that $\nu$ is an $\mathcal{A}$-free measure as desired.

\[\square\]

9. Proof of the Dual Characterization Theorem 1.1

Let $\mu \in \mathcal{M}(\Omega; \mathcal{W})$ be an $\mathcal{A}$-free measure. We define the set

$\mathcal{A}Y_{\mu,0}(\Omega) := \{ \nu = (\nu, \lambda, \nu^\infty) \in \mathcal{A}Y(\Omega; \mathcal{W}) : \lambda(\partial \Omega) = 0, |\nu| = \mu \}.$

The proof of the following proposition is contained in Lemma 5.3 of [9]. There the authors state their main results under additional assumptions (see for instance (1)). However, the proof of this specific proposition makes not use of such assumptions and can be worked out by verbatim in our context.

Proposition 9.1. The set $\mathcal{A}Y_{\mu,0}(\Omega)$ is weak-* closed in $(E(\Omega; \mathcal{W}))^*$.

The main step towards the proof of the characterization result Theorem 1.1 rests in showing the following convexity property. Once this is established the
proof of Theorem 1.1 follows by relaxation argument and the geometric version of Hahn–Banach’s Theorem argument.

**Theorem 9.1.** The set $\mathcal{A}Y_{\mu,0}(\Omega)$ is a convex set.

**Proof.** Fix $0 < \theta < 1$ and let

$$
\nu_1 = (\nu_1, \lambda_1, \nu_1^\infty), \nu_2 = (\nu_2, \lambda_2, \nu_2^\infty) \in \mathcal{A}Y_{\mu,0}(\Omega).
$$

We also write $\nu_\theta := \theta \nu_1 + (1 - \theta)\nu_2 \in (E(\Omega; \mathcal{W}))^\ast$. Our goal is to show that $\nu_\theta$ is an $\mathcal{A}$-free Young measure on $\Omega$. To show this we will exhibit a sequence of $\mathcal{A}$-free measures on $\Omega$ which generate the functional $\nu_\theta$.

Since this will be a fairly long and technical proof we will begin by describing a brief program of the proof. The foundation of our proof lies in a careful inspection of the infinitesimal qualitative behavior of points $x \in \Omega$ with respect to our Young measures $\nu_1, \nu_2$. The qualitative understanding of the set of tangent Young measures of $\nu_i$ ($i = 1, 2$) at a given $x \in \Omega$ will be decisive in the choice of construction of an $\mathcal{A}$-free recovery sequence for $\nu_\theta$ about that point. Once every point and their local constructions are established, the rough idea is to use Besicovitch’s Covering Theorem to build a partition of $\Omega$ into disjoint tiles, each of which retrieves the infinitesimal properties of $\nu_i$ and hence the recovery sequences of $\nu_\theta$ about their center points. The one but last step is to glue the aforementioned $\mathcal{A}$-free recovery sequences from each tile into a globally $\mathcal{A}$-free sequence which generates an arbitrarily close a piece-wise constant approximation of $\nu_\theta$. The conclusion of the argument then follows from a diagonalization argument between the larger scale of piece-wise constant approximations of $\nu_\theta$ where we glue the recovery sequences, and the smaller scale where the corresponding recovery sequences are effectively constructed.

**Step 1. Qualitative analysis of points.**

Since we are trying to capture the fine properties of $\nu_1$ and $\nu_2$ simultaneously, it will be convenient to define the measure $\Lambda := \lambda_1^\ast + \lambda_2^\ast$, which is a suitable substitute candidate to keep track of the interactions between singular points of $\lambda_1$ and $\lambda_2$. We start by distinguishing regular points and singular points. It follows from the Radon–Nykodym theorem that at $(\mathcal{L}^d + \Lambda)$-almost everywhere $x \in \Omega$ one of the following properties hold: either

\begin{equation}
(58) \quad x \in \text{reg}(\Omega) := \left\{ x \in \Omega : \frac{d\Lambda}{d\mathcal{L}^d}(x) = \lim_{r \downarrow 0} \frac{\Lambda(Q_r(x))}{(2r)^d} = 0 \right\}
\end{equation}

is a regular point, or

\begin{equation}
(59) \quad x \in \text{sing}(\Omega) := \left\{ x \in \Omega : \frac{d\mathcal{L}^d}{d\Lambda}(x) = \lim_{r \downarrow 0} \frac{(2r)^d}{\Lambda(Q_r(x))} = 0 \right\}
\end{equation}

is a singular point. Throughout this proof we shall call points with the first property (which holds $\mathcal{L}^d$-almost everywhere) regular points, and points satisfying the second property (which holds $\Lambda$-almost everywhere) will be called singular points; we shall only consider points $x \in \Omega$ that are either regular or singular points. In addition, we may assume without any loss of generality that the limits

$$
\frac{d\lambda_i^\ast}{d\Lambda}(x) = \lim_{r \downarrow 0} \frac{\lambda_i^\ast(Q_r(x))}{\Lambda(Q_r(x))}, \quad i = \{1, 2\},
$$

exist at every singular point $x \in \Omega$. Next, we further partition $\text{sing}(\Omega)$ into sets which render precise information about the size relation between $\lambda_1$ and $\lambda_2$. More
and the fact that
\[ \frac{d\lambda_2^r}{d\lambda_1^r}(x) = 0, \]

2.1 which separates points in
\[ N, x \]
behavior of \( \text{Tan}(\Lambda) \)
regular and singular points, and the latter by their weights with respect to
\[ N \]
then, up to modifying \( N \), we may assume that \( g_1, g_2 \) are \( \Lambda \)-measurably continuous and

\[
\begin{align*}
    &x \in G_0 \implies g_1(x) = \lim_{r \downarrow 0} \frac{\lambda_1^r(Q_r(x))}{\lambda_2^r(Q_r(x))} = 0, \\
    &x \in G_1 \implies g_1(x) = \lim_{r \downarrow 0} \frac{\lambda_2^r(Q_r(x))}{\lambda_1^r(Q_r(x))} \in (0, \infty), \\
    &x \in G_\infty \implies g_2(x) = \lim_{r \downarrow 0} \frac{\lambda_2^r(Q_r(x))}{\lambda_1^r(Q_r(x))} = 0.
\end{align*}
\]

Step 1a. Tangential properties of singular points. So far we have separated regular and singular points, and the latter by their weights with respect to \( \lambda_1 \) and \( \lambda_2 \). The next step is to separate points in \( \text{sing}(\Omega) \) with respect to the qualitative behavior of \( \text{Tan}(\Lambda, x) \).

1. If there exists a tangent measure \( \tau \in \text{Tan}(\Lambda, x) \) which does not charge points, i.e.,
\[ \tau(\{y\}) = 0 \text{ for all } y \in \mathbb{R}^d, \]
then we write \( x \in \mathcal{R} \). Every \( x \in \mathcal{R} \) has the following property (see Corollary A.2): if \( \Theta \in (0, 1) \), \( g \) is a \( \Lambda \)-measurable map, and \( x \) is a \( \Lambda \)-Lebesgue point of \( g \), then there exist (a) a sequence of infinitesimal radii \( r_h \downarrow 0 \) and (b) a sequence of open Lipschitz sets \( D_h \subset Q_{r_h} \) satisfying
\[
\Lambda(x + \partial D_h) = 0, \quad \lim_{h \to \infty} \frac{\Lambda(x + D_h)}{\Lambda(Q_{r_h}(x))} = \Theta,
\]
and
\[
\lim_{h \to \infty} \int_{x + D_h} |g - g(x)| \, d\Lambda = 0.
\]
In particular, if \( x \in G_1 \), then
\[
\lim_{h \to \infty} \frac{\Lambda(x + D_h)}{\Lambda(Q_{r_h}(x))} = \lim_{h \to \infty} \frac{\lambda_1^r(x + D_h)}{\lambda_1^r(Q_{r_h}(x))} = \lim_{h \to \infty} \frac{\lambda_2^r(x + D_h)}{\lambda_2^r(Q_{r_h}(x))} = \Theta.
\]

2. If otherwise (1) does not hold for any tangent measure of \( \Lambda \) at \( x \), we write \( x \in \mathcal{S} \). It follows from Lemma A.3 and the fact that blow-ups of blow-ups are blow-ups (see Theorem 2.12 in [40]) that
\[ x \in \mathcal{S} \implies \delta_0 \in \text{Tan}(\Lambda, x). \]

Step 1b. Selection of points with Lebesgue-type properties. We now turn to the selection of points which later shall be the centers of the tile partitions. As usual let \( \{f_{p,q}\}_{p,q \in \mathbb{N}} \subset E(\Omega; \mathcal{W}) \) be the family from Lemma 2.1 which separates points in \( (E(\Omega; \mathcal{W}))^* \).
Up to removing a set of \(\mathcal{L}^d\)-measure zero, we may assume that every \(x \in \text{reg}(\Omega)\) is a Lebesgue point of the maps
\[
\left\{ x \mapsto \langle f_{p,q} \nu_x \rangle_x + \langle f_{p,q} \nu_x^\infty \rangle_x \lambda_i(x) \right\} \quad i = 1, 2; \quad p, q \in \mathbb{N}.
\]

About singular points \(x \in \text{sing}(\Omega)\) we will be more careful and set \(B_i^\infty \subset \text{sing}(\Omega)\) to be the set of \(\lambda_i\)-Lebesgue points of the family of maps
\[
\left\{ x \mapsto \langle f_{p,q} \nu_x \rangle_x \right\} \quad i = 1, 2; \quad p, q \in \mathbb{N}.
\]

Each \(B_i^\infty\) has full \(\lambda_i\)-full measure on \(\Omega\) and hence \(B_1^\infty \cup B_2^\infty\) has full \(\Lambda\)-measure on \(\Omega\). Therefore, in what follows there will be no loss of generality in assuming that \(\text{sing}(\Omega) = B_1^\infty \cup B_2^\infty\); this union may not be disjoint.

**Step 2. Building a partition of cubes with good fine properties.** Let \(m \in \mathbb{N}\), in this step we will address the construction of a full \(\Lambda\)-measure partition of \(\Omega\) with \(O(m^{-1})\)-asymptotic approximation Lebesgue-type properties. To begin, let us define a fine cover of \(L := \text{reg}(\Omega) \cup \text{sing}(\Omega)\). At every \(x \in L\) we define
\[
\rho_m(x) := \sup \left\{ 0 \leq r \leq \frac{1}{m} : r \text{ satisfies the } (P_m(x)) \text{ property} \right\}.
\]

A radius \(r\) is said to satisfy \((P_m(x))\) provided the following continuity properties hold for \(i = 1, 2\) and all indexes \(p, q \leq m\):

If \(x \in \text{reg}(\Omega)\), then
\[
\frac{\Lambda(Q_r(x))}{(2r)^d} \leq \frac{1}{m},
\]
\[
\int_{Q_r(x)} |\langle f_{p,q} \nu_x \rangle_y - \langle f_{p,q} \nu_x \rangle_x| \, dy \leq \frac{1}{m},
\]
\[
\int_{Q_r(x)} |\langle f_{p,q} \nu_x^\infty \rangle_y \cdot \lambda_i^\infty(y) - \langle f_{p,q} \nu_x^\infty \rangle_x \cdot \lambda_i^\infty(x)| \, dy \leq \frac{1}{m}.
\]

If \(x \in \text{sing}(\Omega)\), then
\[
\int_{Q_r(x)} (1 + |\cdot|, \nu) \, dy + \int_{Q_r(x)} \lambda_i^\infty(y) \, dy \leq \frac{1}{m} \cdot \Lambda(Q_r(x)),
\]
\[
\int_{Q_r(x)} |\langle f_{p,q} \nu_x^\infty \rangle_y - \langle f_{p,q} \nu_x^\infty \rangle_x| \, d\lambda_i^\infty(y) \leq \frac{1}{m} \quad \forall x \in B_i^\infty,
\]
\[
\frac{\lambda_i^\infty(Q_r(x))}{\Lambda(Q_r(x))} \leq \frac{1}{m} \quad \text{if } x \in G_0,
\]
\[
\frac{\lambda_i^\infty(Q_r(x))}{\Lambda(Q_r(x))} \leq \frac{1}{m} \quad \text{if } x \in G_\infty.
\]

If \(x \in \mathcal{R} \cap G_1\), then we require
\[
\left| \frac{\lambda_i(x + D_r)}{\lambda_i(Q_r(x))} - \Theta \right| \leq \frac{1}{m}, \quad \Lambda(x + \partial D_r) = 0.
\]

If \(x \in G_0\) or \(x \in G_\infty\), then we can only find \(D_r\) satisfying (67) for \(\lambda_2\) and \(\lambda_1\) respectively.

Lastly, if \(x \in \mathcal{S} \cap G_1\), then
\[
\frac{\lambda_i(A_r)}{\lambda_i(Q_r(x))} \leq \frac{1}{m},
\]
where
\[
A_r := Q_r(x) \setminus Q_{s_r}(x), \quad \Lambda(\partial A_r) = 0, \quad \frac{s_r}{r} \leq \frac{1}{m}.
\]
Moreover, $s_r$ can be chosen sufficiently small so that
\begin{equation}
\| \Delta_{s_r} u - u \|_{W^{s,1}(Q_x)} \leq \frac{\Lambda(Q_x(x))}{m},
\end{equation}
where $\mu L Q_r(x) = Bu + v L^d$ is the decomposition provided by Lemma 7.1 for $\mu$ on $Q_r(x)$. Here we have used the short notation
\[
\Delta_{s_r} w := w(\cdot \pm s_r e_1), \quad s_r := s_r(x),
\]
for the translations of a function $w$.

Now, this is indeed a large amount of smallness conditions to keep track, but they are all fundamental if one wishes to avoid (trivial) partitions which do not reflect the behavior of $\nu_1, \nu_2$ appropriately.

**Claim 1.** $\rho_m(x) > 0$ for all $x \in L$.

**Proof of Claim 1.** Most of the properties are easy to check: Properties (60)-(62) and (63)-(66) follow directly from the construction and the Lebesgue properties discussed in Step 1b. Property (67) is a consequence of Step 1a.(1). We focus in showing (68)-(69) which will follow from the fact that $\delta_0 \in \text{Tan}(\Lambda, x)$. Indeed, in this case we may a sequence of infinitesimal radii $r_j \downarrow 0$ such that
\[
\gamma_j := \frac{1}{\Lambda(Q_{x,j}(x))} \cdot T_{x,r_j}[\Lambda] \xrightarrow{\delta_0} \delta_0 \text{ locally in } M(\mathbb{R}^d).
\]
Then, by the strict convergence of the blow-up sequence we deduce that
\[
\lim_{j \to \infty} \Lambda(Q_{sr_j}) = \lim_{j \to \infty} \gamma_j(Q_{sr_j}(x)) = \lim_{j \to \infty} \frac{\Lambda(Q_{sr_j}(x))}{\Lambda(Q_{r_j}(x))} = 1 \quad \forall s \in (0,1).
\]
In particular, since $x \in G_1$, we conclude that
\[
\lim_{j \to \infty} \frac{\Lambda(Q_{sr_j}(x))}{\Lambda(Q_{r_j}(x))} = \lim_{j \to \infty} \frac{\lambda_i(Q_{sr_j}(x))}{\lambda_i(Q_{r_j}(x))} = 1, \quad i = 1, 2.
\]
Choosing $s \leq \frac{1}{r_j}$ in a way that $s_{r_j} := s r_j$ satisfies the required properties for $\Lambda_{r_j}$ and $Q_{r_j}(x)$ (this can be done by slightly modifying each $r_j$ in the blow-up sequence), we exhibit an infinitesimal sequence $r_j$ (and their associated $s_{r_j}$) satisfying (68)-(69).

This proves the claim.

In particular, the cover
\[
Q_m := \{ Q_r(x) : x \in L, 0 < r \leq \rho_m(x) \text{ with } \Lambda(\partial Q_r(x)) = 0 \}
\]
conforms a fine cover of $L$ to which we may apply Besicovitch’s Covering Theorem: There exists a sub-cover $\Omega_m \subset Q_m$ of disjoint cubes satisfying
\begin{equation}
\Lambda(\Omega \setminus \Omega_m) = 0 \quad \text{and} \quad \Lambda(\partial Q_2) = 0 \quad \text{for all } Q_x \in \Omega_m.
\end{equation}
Here, we have set $\Omega_m := \cup_{Q_x \in \Omega_m} Q_x$.

**Step 3. Piece-wise homogeneous approximations of $\nu_i$.** The idea behind defining $O_m$ is to construct a piece-wise homogeneous approximation of $\nu_1, \nu_2$ of order $\frac{1}{m}$ as follows: Fix $i \in \{1, 2\}$ and define, through duality, a sequence of functionals in $(E(\Omega, W))^*$ acting as
\[
\left( \langle f, \nu_1^{(m)} \rangle \right) := \sum_{x \in \text{reg}(\Omega), Q_x \in \Omega_m} \left( \int_{Q_x} \langle f, \nu_1 \rangle_x dy + \int_{Q_x} \langle f^\infty, \nu_i^\infty \rangle_x \lambda_i^{ac}(x) dy \right) + \sum_{x \in \Omega_m} \int_{Q_x} \langle f^\infty, \nu_i^\infty \rangle_x d\lambda_i^*(y).
\]
The fact that these functionals are in fact Young measures follows directly from (70), the weak-\(*\) measurability properties of \(\nu_1\) and \(\nu_2\), and the fact that simple Borel maps are measurable with respect to any Radon measure.

**Claim 2.** As \(m \to \infty\) it holds that

\[
\nu^{(m)}_i \overset{\ast}{\rightharpoonup} \nu_i \quad \text{in } (E(\Omega, W))^*, \quad i = 1, 2.
\]

**Proof of Claim 2.** Let \(p, q \in \mathbb{N}\) (we shall simply write \(f = f_{p,q}\)). First, we show that

\[
\lim_{m \to \infty} \left| \int_{\Omega} \langle f, \nu_i \rangle \, d\mathcal{L}^d + \int_{\Omega} \langle f^\infty, \nu_i^\infty \rangle \, d(\lambda_i^{ac} \mathcal{L}^d) \right.
\]

\[
- \sum_{Q_s \in \mathcal{C}_m} \int_{Q_s} \langle f, \nu_i \rangle_x \, dy + \int_{Q_s(x)} \lambda_i^{ac}(x) \langle f^\infty, \nu_i^\infty \rangle_x \, dy \right| = 0. \tag{71}
\]

We consider \(p, q \leq m \in \mathbb{N}\). We may estimate (cf. (70)) the difference of the integrals above by the sum of the two non-negative quantities

\[
I_m := \left| \sum_{Q_s \in \mathcal{C}_m} \int_{Q_s} \langle f, \nu_i \rangle \, d\mathcal{L}^d + \int_{\Omega} \langle f^\infty, \nu_i^\infty \rangle \, d(\lambda_i^{ac} \mathcal{L}^d) \right|
\]

and

\[
II_m := \sum_{Q_s \in \mathcal{C}_m} \left| \int_{Q_s} \langle f, \nu_i \rangle_y - \langle f, \nu_i \rangle_x \, dy \right.
\]

\[
+ \int_{Q_s(x)} \langle f^\infty, \nu_i^\infty \rangle_y \lambda_i^{ac}(y) - \langle f^\infty, \nu_i^\infty \rangle_x \lambda_i^{ac}(x) \, dy \right|.
\]

Using (63) and the linear growth of \(|f| \leq M_f(1 + |\cdot|)\) we obtain

\[
I_m \leq [1 + M_f] \sum_{Q_s \in \mathcal{C}_m} \int_{Q_s} \langle 1 + |\cdot|, \nu_i \rangle \, d\mathcal{L}^d(y) + \int_{Q_s(x)} \lambda_i^{ac} \, d\mathcal{L}^{d}(y)
\]

\[
\leq \frac{1}{m} \sum_{Q_s \in \mathcal{C}_m} \mathcal{A}(Q_s) \leq \frac{1}{m} \mathcal{A}(\Omega).
\]

It follows that \(\lim_{m \to \infty} I_m = 0\).

On the other hand, we use (61)-(62) to bound \(II_m\) as

\[
II_m \leq \sum_{Q_s \in \mathcal{C}_m} \left( \int_{Q_s} |\langle f, \nu_i \rangle_y - \langle f, \nu_i \rangle_x | \, dy \right.
\]

\[
+ \int_{Q_s} |\langle f^\infty, \nu_i^\infty \rangle_y \lambda_i^{ac}(y) - \langle f^\infty, \nu_i^\infty \rangle_x \lambda_i^{ac}(x) | \, dy \right)
\]

\[
\leq \frac{2}{m} \sum_{Q_s \in \mathcal{C}_m} \mathcal{L}^d(Q_s) \leq \frac{2}{m} \mathcal{L}^d(\Omega).
\]

This shows that \(\lim_{m \to \infty} II_m = 0\), whence (71) follows.

To prove the claim we are left to show that

\[
\lim_{m \to \infty} \left| \sum_{Q_s \in \mathcal{C}_m} \int_{Q_s} \langle f^\infty, \nu_i^\infty \rangle_y - \langle f^\infty, \nu_i^\infty \rangle_x \, d\lambda_i \right| = 0.
\]
We may estimate the integrand above, for fixed $m \in \mathbb{N}$, by

$$
\sum_{Q_x \in \mathcal{O}_m} \int_{Q_x} \left| \langle f^\infty, \nu^\infty \rangle_x - \langle f^\infty, \nu^\infty \rangle_x \right| \, d\lambda^x
$$

\[ \leq \frac{1}{m} \sum_{Q_x \in \mathcal{O}_m} \Lambda(Q_x) \leq \frac{1}{m} \Lambda(\Omega). \]

Since $\{f_{p,q}\}$ separates $(\mathcal{E}(\Omega; \mathcal{W}))^*$, this proves Claim 2.

\[ \square \]

**Step 4. Construction of a global $\mathcal{A}$-free recovery sequence.** Let us fix $m \in \mathbb{N}$. Next, we define candidate recovery sequences for $\nu_\theta$ on $Q_x \in \mathcal{O}_m$. This will be done depending on whether $x$ belongs to $\mathcal{R}$ or $\mathcal{S}$ where these sets are the ones defined in Step 1a.

**Step 4a. Cubes $Q_x \in \mathcal{O}_m$ centered at $x \in \mathcal{R} \cup \text{reg}(\Omega)$.** We recall from step 1a and (67) that, if $x \in \mathcal{R}$, then there are open Lipschitz sets $D_x \subset Q_x \subset \Omega$ satisfying

\[ \Lambda(\partial D_x) = 0, \]

\[ \left| \frac{\lambda_i(D_x)}{\lambda_i(Q_x(x))} - \theta \right| \leq \frac{1}{m} \quad \text{whenever } x \in G_1, \]

\[ \text{and} \]

\[ \left| \frac{\lambda_2(D_x)}{\lambda_2(Q_x(x))} - \theta \right| \leq \frac{1}{m} \quad \text{whenever } x \in G_0, \]

and

\[ \left| \frac{\lambda_1(D_x)}{\lambda_1(Q_x(x))} - \theta \right| \leq \frac{1}{m} \quad \text{whenever } x \in G_\infty. \]

On the other hand, since $\nu_1, \nu_2$ are $\mathcal{A}$-free Young measures on $\Omega$, we may apply Lemma 7.1 to find sequences (to avoid adding unnecessary notation, we will omit the $x$-dependence of these sequences) of $\mathcal{A}$-free measures $(u_j) \subset \mathcal{M}(D_x; \mathcal{W})$ and $(v_j) \subset \mathcal{M}(Q_x \setminus \overline{D}_x; \mathcal{W})$ satisfying

\[ u_j \equiv \mu \text{ on a neighborhood of } \partial D_x \quad \text{and} \quad u_j \overset{Y}{\rightharpoonup} \nu_1 \text{ on } D_x, \]

\[ v_j \equiv \mu \text{ on a neighborhood of } \partial(Q_x \setminus \overline{D}_x) \quad \text{and} \quad v_j \overset{Y}{\rightharpoonup} \nu_2 \text{ on } (Q_x \setminus \overline{D}_x). \]

The same construction applies with for $x \in \text{reg}(\Omega)$ with the exception that we require $D_x \subset Q_x$ to satisfy

\[ \frac{\mathcal{L}^d(D_x)}{\mathcal{L}^d(Q_x(x))} = \theta \quad \text{and} \quad \mathcal{L}^d(\partial D_x) = 0. \]

It follows from the uniformity of the Lebesgue measure that this can always be achieved for some open Lipschitz $D_x \subset Q_x$; in this case the set $D_x$ can be chosen to be a strip of width $\theta$ or an open concentric cube of $Q_x$ of side $\theta^\star$.

In what follows we shall write

\[ Q_x^1 = D_x \quad \text{and} \quad Q_x^2 = Q_x \setminus \overline{D}_x. \]

Notice that by construction the measures

\[ w_j = w_j^x \equiv \mathbb{1}_{Q_x^2} u_j + \mathbb{1}_{Q_x^1} v_j \]

are $\mathcal{A}$-free on $Q_x$ for all $j \in \mathbb{N}$. Moreover, the $w_j$'s can be extended by $\mu$ outside $Q_x$ and particular this extension preserves the $\mathcal{A}$-free constraint. Moreover, in virtue
of (76)-(77) and the locality of the weak-$*$ convergence of Young measures it holds that
\[
\text{(79)} \quad w_j \rightharpoonup^{\ast} \nu_1 \mathbb{L} Q_x^1 + \nu_2 \mathbb{L} Q_x^2 \quad \text{and} \quad \nu_1 \mathbb{L} Q_x^1 + \nu_2 \mathbb{L} Q_x^2 \quad \text{in} \quad A\mathcal{Y}(Q_x).
\]
Therefore, upon re-adjusting the sequence \((w_j)\) we may assume that
\[
\text{(80)} \quad \sup_{j \in \mathbb{N}} |w_j|(Q_x) \leq 2\left( \left\| \cdot \right\|_{\nu_1, \mathbb{L} Q_x} + \left\| \cdot \right\|_{\nu_2, \mathbb{L} Q_x} \right)\).
\]

\[\text{Step 4b. Cubes } Q_x \in C_m \text{ centered at } x \in \mathcal{S}.\]

The constructions in these cubes will be completely different and it will consist of separating the generating sequences of \(\nu_1, \nu_2\) locally. Once again, by Lemma 7.1, we may find sequences of potentials \((u_j), (w_j) \subset BV^2(Q_x)\) such that
\[
u_j, w_j \equiv u \text{ on a neighborhood of } \partial Q_x, \quad u_j, w_j \to u \quad \text{in} \quad W^{k,-1,1}(Q_x),
\]
where \(\mu \mathbb{L} Q_x = \mathbb{B}u + v \mathcal{L}^d\) for some \(v \in L^1(Q_x; \mathbb{W})\). Moreover,
\[
\mathbb{B}u_j + v \mathcal{L}^d \rightharpoonup^{\ast} \nu_1 \quad \text{in} \quad \mathcal{Y}(Q_x; \mathbb{W}),
\]
and
\[
\mathbb{B}w_j + v \mathcal{L}^d \rightharpoonup^{\ast} \nu_2 \quad \text{in} \quad \mathcal{Y}(Q_x; \mathbb{W}).
\]

Now, let \(\varphi\) be a cut-off function satisfying (here \(Q_x = Q_{r}(x)\))
\[
\mathbb{I}_{Q_{r/2}}(x) \leq \varphi \leq \mathbb{I}_{Q_{r/4}}(x), \quad \left\| \varphi \right\|_{k, \infty} \lesssim r^{-dk}.
\]

Due to the \(L^p\)-continuity of the translations, we may choose \(n_1 = n_1(m) \in \mathbb{N}\) to be sufficiently large so that
\[
\text{(81)} \quad \left\| u_j - u \right\|_{W^{k-1,1}(Q_x)} + \left\| w_j - u \right\|_{W^{k-1,1}(Q_x)} \leq \frac{r^{dk}}{m} \cdot \frac{\Lambda(Q_x)}{m}
\]
for all \(j \geq n_1\).

We are now in position to define our recovery sequence candidate for \(\nu_0\) on \(Q_x\) by setting
\[
q_j = q_j^0 := \mathbb{B}(\varphi[\theta_1 \Delta_{-\nu_j} u_j + \theta_2 \Delta_{-\nu_j} w_j] + \mathbb{B}u + v), \quad j \in \mathbb{N},
\]

The purpose of this sequence is to shift \(u_j\) and \(v_j\) apart from each other, while preserving the \(\mu\)-boundary conditions near \(\partial Q_x\) (see Figure 2 below). Clearly, \((q_j)\) is a sequence of \(\mathcal{A}\)-free measures on \(Q_x\) with \(q_j \equiv \mu\) on a neighborhood of \(\partial Q_x\) and \(q_j \approx \theta_1 u_j + \theta_2 v_j\) on \(Q_{r/2}(x)\). Notice that this construction differs from the previous one (when \(x \in \mathcal{R}\)) in the sense that the \(\theta_i\)-weights are incorporated by simple multiplication. In general, this construction is too naive to work. However, in this case, it works because we have \(\Lambda \approx \delta_0\) in \(Q_x\).
Let us fix $j \in \mathbb{N}$. Writing $u = \theta_1 u + \theta_2 u$ and adding a zero, we may express $q_j$ as

$$q_j = \theta_1 \varphi \cdot \Delta_{-s_x} \mathcal{B}(u_j - u) + \theta_2 \varphi \cdot \Delta_{s_x} \mathcal{B}(w_j - u) + \theta_1 \Delta_{-s_x} [\mathcal{B}, \varphi](u_j - u) + \theta_2 \Delta_{s_x} [\mathcal{B}, \varphi](w_j - u)$$

$$+ \theta_1 [\mathcal{B}, \varphi](\Delta_{s_x} u - u) + \theta_2 [\mathcal{B}, \varphi](\Delta_{s_x} u - u) + \varphi \cdot \mathcal{B}(\theta_1 \Delta_{-s_x} u + \theta_2 \Delta_{s_x} u - u) + \mu,$$

where as usual the commutator $[\mathcal{B}, \varphi] = \mathcal{B}(D) \cdot \chi - \chi \cdot \mathcal{B}(D)$ is a partial differential operator of order at most $(k_B - 1)$ and whose coefficients depend solely on $\|\varphi\|_{k, \infty}$ and the principal symbol $\mathcal{B}$. In particular, we obtain the following estimate for the total variation of $q_j$:

$$|q_j|(Q_x) \lesssim_{\mathcal{B}} |\mathcal{B}w_j|(Q_x) + |\mathcal{B}w_j|(Q_x) + |\mu|(Q_x) + \|\varphi\|_{k, \infty} \left( \|u_j - u\|_{W^{k-1,1}(Q_x)} + \|w_j - u\|_{W^{k-1,1}(Q_x)} + \|\Delta_{-s_x} u - u\|_{W^{k-1,1}(Q_x)} + \|\Delta_{s_x} u - u\|_{W^{k-1,1}(Q_x)} \right)$$

$$\lesssim_{\mathcal{B}} |\mathcal{B}u_j|(Q_x) + |\mathcal{B}w_j|(Q_x) + |\mu|(Q_x) + \frac{\Lambda(Q_x)}{m} \quad \forall j \geq n.$$

In particular, upon re-adjusting the sequence $(q_j)$ we may assume that

$$\sup_{j \in \mathbb{N}} |q_j|(Q_x) \lesssim_{\mathcal{B}} |\mu|(Q_x) + \frac{\Lambda(Q_x)}{m},$$

and $q_j \overset{Y}{\underset{\sigma^*}{\to}} \mathcal{A}Y(Q_x)$.

Observe that if $f = f_{\rho, q}$ with $p, q \leq m$, then

$$\left| \theta_1 \cdot \left\langle f, \nu_\ell \right\rangle - \theta_1 \cdot \left\langle f, \nu_\ell \mathbf{L} Q_{s_x}(x) \right\rangle \right| \leq M_f \left\langle 1 + |\cdot|, \nu_\ell \mathbf{L} A_x \right\rangle \lesssim \frac{2}{m} \Lambda(Q_x).$$

Hence, there exists $n_1 \leq n_2 = n_2(m) \in \mathbb{N}$ such that (with $\gamma_j^1 = v \mathcal{L}^d + \mathcal{B}u_j$)

$$\left\langle f, \delta_{\theta_1 \gamma_j^1} \mathbf{L} Q_x \right\rangle = \int_{Q_x} \langle f, \theta_1 (\gamma_j^1)^{ac} \rangle \mathcal{L}^d + \int_{Q_x} \theta_1 \langle f^{\infty}, (\gamma_j^1)^{s} \rangle \, d\lambda$$

$$\overset{(63)}{=} \int_{Q_{s_x}(x)} \theta_1 \langle f^{\infty}, (\gamma_j^1)^{s} \rangle \, d\lambda + M_f \cdot O(m^{-1}) \cdot \Lambda(Q_x)$$

$$\overset{(63)}{=} \theta_1 \cdot \left\langle f, \delta_{\theta_1 \gamma_j^1} \mathbf{L} Q_{s_x}(x) \right\rangle + 2M_f \cdot O(m^{-1}) \cdot \Lambda(Q_x)$$
for all \( j \geq n_1 \). An analogous estimate holds for \( \nu_2, w_j \), and \( \theta_2 \). Let us set \( R_x := Q_x \setminus ((Q_x \cap (x - s_x \varepsilon_1))^\prime \cup (Q_x \cap (x + s_x \varepsilon_1)^\prime)) \). Then, by the definition of \( q_j \), a similar argument combined with the right translations \( \pm s_x \varepsilon_1 \) yield (for \( j \geq n_1 \))

\[
\left\langle f, \delta_{q_j} \right| L \left( R_x \right) \right\rangle = O(m^{-1})[M_f \cdot \Lambda(Q_x)],
\]

\[
\left\langle f, \delta_{q_j} \right| L \left( Q_x \setminus \overline{R_x} \right) \right\rangle = \theta_1 \cdot \left\langle f, \delta_{q_j} \right| L \left( Q_x \right) \right\rangle + \theta_2 \cdot \left\langle f, \delta_{q_j} \right| L \left( Q_x \right) \right\rangle + O(m^{-1})[M_f \cdot \Lambda(Q_x)].
\]

Combining these estimates we obtain upon re-adjusting the sequence of \( j \)'s (recall that we had written \( f = f_{p,q} \))

\[
\left\langle f_{p,q}, \sigma^2 \right| L \left( Q_x \right) \right\rangle = \lim_{j \to \infty} \left\langle f_{p,q}, \delta_{q_j} \right| L \left( Q_x \right) \right\rangle \\
= \left\langle f_{p,q}, \nu_0 \right| L \left( Q_x \right) \right\rangle + O(m^{-1})[M_f \cdot \Lambda(Q_x)]
\]

whenever \( p, q \leq m \).

**Step 4c. Gluing the local recovery sequences.**

Every cube \( Q_x \in \mathcal{O}_m \) is centered at some \( x \in L \) and since

\[
\text{reg}(\Omega) \cup \mathcal{R} \cup \mathcal{S} = \text{reg}(\Omega) \cup \text{sing}(\Omega) = L,
\]

the constructions in Steps 4a and 4b indeed cover all possible scenarios which can present. The next task is to glue the recovery sequences together to obtain an \( \mathcal{A} \)-free global recovery sequence of the \( O(m^{-1}) \)-approximation of \( \nu_0 \). For each \( m \in \mathbb{N} \), let us define the sequence

\[
w_j^{(m)}(dy) := \begin{cases} 
    w_j^x(dy) & \text{if } x \in \mathcal{R} \cup \text{reg}(\Omega) \\
    q_j^x(dy) & \text{if } x \in \mathcal{S} \\
    \mu(dy) & \text{elsewhere}
\end{cases}, \quad y \in \Omega.
\]

Notice that by construction each \( w_j^{(m)} \) is \( \mathcal{A} \)-free since each \( w_j^x \) and \( q_j^x \) is \( \mathcal{A} \)-free on \( Q_x \) and has \( \mu \)-boundary values in an open neighborhood of \( \partial Q_x \).

**Step 4d. Generation of the \( (m^{-1}) \)-approximations of \( \nu_0 \).**

Appealing to the locality of weak-* convergence of Young measures, we show next that if \( p, q \leq m \), then (as \( j \to \infty \))

\[
\lim_{j \to \infty} \left\langle f_{p,q}, \delta_{w_j^{(m)}} \right| L \left( \mathbb{R}_x \right) \right\rangle = \left\langle f_{p,q}, \nu_0^{(m)} \right| L \left( \mathbb{R}_x \right) \right\rangle \\
+ O(m^{-1}) \cdot M_{f_{p,q}} \cdot \Lambda(\Omega),
\]

where \( \nu_0^{(m)} \) is the Young measure which acts on \( f \in \mathcal{E}(\Omega, \mathbb{W}) \) by the representation formula

\[
\left\langle f, \nu_0^{(m)} \right| L \left( \mathbb{R}_x \right) \right\rangle = \sum_{i=1,2} \sum_{Q_x \in \mathcal{O}_m} \left\langle f, \nu_i \right| L \left( Q_x \right) \right\rangle \\
= \sum_{Q_x \in \mathcal{O}_m} \theta_i \cdot \left\langle f, \nu_i \right| L \left( Q_x \right) \right\rangle + \left\langle f, \nu_\mu \right| L \left( \mathbb{R}_x \right) \right\rangle.
\]

Later, in the next step, we will show these Young measures are indeed \( O(m^{-1}) \)-approximations of \( \nu_0 \). This, together with a diagonalization argument with (84) will imply that \( \nu_0 \in \mathcal{A}Y_{\mu,0}(\Omega) \).
First, we show that the sequence \((w^{(m)}_{j,m})_{j,m} \subset M(\Omega, W)\) has uniformly bounded total variation on \(\Omega\). There is no loss of generality in assuming that \(f_{1,1} = \mathbb{1}_\Omega \otimes \cdot\), and therefore

\[
|w^{(m)}_{j,m}|(\Omega) \leq 2 \left( \left\langle |\cdot|, \nu_1 \mathbf{L} Q_x \right\rangle + \left\langle |\cdot|, \nu_2 \mathbf{L} Q_x \right\rangle \right) + \sum_{Q_x \in Q_m} \left( |\mu|(Q_x) + \frac{\Lambda(Q_x)}{m} \right)
\]

\[
\leq 3 \left( \left\langle \mathbb{1}_\Omega \otimes \cdot, \nu_1 \right\rangle + \left\langle \mathbb{1}_\Omega \otimes \cdot, \nu_2 \right\rangle \right) + \frac{1}{m} \Lambda(\Omega) + |\mu|(\Omega).
\]

This shows

\[
\sup_{m \in \mathbb{N}} \left( \sup_{j \in \mathbb{N}} |w^{(m)}_{j,m}|(\Omega) \right) \leq \sup_{m \in \mathbb{N}} C(m) < \infty,
\]

as desired.

Since \((w^{(m)}_{j,m})\) is uniformly bounded on \(\Omega\), the desired limit approximation in \((84)\) follows from 1) the locality of weak-\(*\) convergence of Young measures, 2) the generation properties \((76)-(77)\) for points in \(\text{reg}(\Omega) \cup \mathcal{R}\), 3) the generation property at singular points in \(\mathcal{S}\) the \((83)\), and 4) the fact that \(\mathcal{O}_m\) is a full \((\mathcal{L}^d + \Lambda)\)-partition of \(\Omega\).

**Step 5.** The sequence \(\nu_{\theta}^{(m)}\) approximates \(\nu_\theta\). Next we show that

\[
\lim_{m \to \infty} \left\langle f_{p,q}, \nu_{\theta}^{(m)} \right\rangle = \left\langle f_{p,q}, \nu_\theta \right\rangle \text{ for all } p, q \in \mathbb{N}.
\]

Accordingly, fix \(p, q \in \mathbb{N}\) and choose \(m \geq p, q\). Let us, for the sake of simplicity, write \(f = f_{p,q}\) and \(f^\infty = f^\infty_{p,q}\). Due to the high amount of terms and estimates involving this argument, let us write

\[
\left\langle f, \nu_{\theta}^{(m)} \right\rangle = I_1 + I_2 + II + III_1 + III_2 + IV + V_1 + V_2,
\]

where each term contains partial sums subjected to a decomposition of the mesh \(\mathcal{O}_m\) in the following way:
(a) the cubes $Q_x$ around regular points $x \in \text{reg}(\Omega)$. For $i \in \{1, 2\}$, the corresponding partial sum is given by

$$
I_i := \sum_{Q_x \in \mathcal{O}_m \atop x \in \text{reg}(\Omega)} \left( \int_{Q_x} \left( \langle f, \nu_i \rangle + \langle f^\infty, \nu_i \rangle \cdot \lambda_i^\text{ac} \right) \, dL^d + \int_{Q_x} \langle f^\infty, \nu_i^\infty \rangle \, d\lambda_i^\text{ac} \right)
$$

$$
\text{(60)-(62)} \quad \sum_{x \in \text{reg}(\Omega) \atop Q_x \in \mathcal{O}_m} \left( \langle f, \nu_i \rangle_x + \langle f^\infty, \nu_i^\infty \rangle \lambda_i^\text{ac}(x) \right) \cdot L^d(Q_x)
$$

$$
+ \sum_{x \in \text{reg}(\Omega) \atop Q_x \in \mathcal{O}_m} O(m^{-1}) \cdot (\text{Lip}(f) + 1) \cdot L^d(Q_x)
$$

$$
\text{(78)} \quad \theta_1 \cdot \sum_{x \in \text{reg}(\Omega) \atop Q_x \in \mathcal{O}_m} \int_{Q_x} \left( \langle f, \nu_i \rangle_x + \langle f^\infty, \nu_i^\infty \rangle \lambda_i^\text{ac}(x) \right) \, dL^d
$$

$$
+ O(m^{-1}) \cdot (\text{Lip}(f) + 1) \cdot L^d(\Omega)
$$

$$
\text{[61)-(62]} \quad \theta_i \sum_{Q_x \in \mathcal{O}_m \atop x \in \text{reg}(\Omega)} \|f, \nu_i, L \|_{Q_x}
$$

$$
+ O(m^{-1}) \cdot (\text{Lip}(f) + 2) \cdot L^d(\Omega)
$$

(b) and now we cover the singular set $\text{sing}(\Omega)$, starting with the cubes around singular points $x \in R \cap G_0$

$$
II := \sum_{Q_x \in \mathcal{O}_m \atop x \in R \cap G_0} \int_{Q_x} \langle f, \nu_1 \rangle \, dL^d + \int_{Q_x} \langle f^\infty, \nu_1^\infty \rangle \, d\lambda_1
$$

$$
+ \sum_{Q_x \in \mathcal{O}_m \atop x \in R \cap G_0} \int_{Q_x} \langle f, \nu_2 \rangle \, dL^d + \int_{Q_x} \langle f^\infty, \nu_2^\infty \rangle \, d\lambda_2
$$

$$
\text{[63]-[65]} \quad \sum_{Q_x \in \mathcal{O}_m \atop x \in R \cap G_0 \cap B_2^\infty} \left( \langle f^\infty, \nu_1^\infty \rangle_x \lambda_1^\text{ac}(Q_1^x) + M_f \cdot O(m^{-1}) \cdot (\Lambda(Q_1^x)) \right)
$$

$$
+ \sum_{Q_x \in \mathcal{O}_m \atop x \in R \cap G_0 \cap B_2^\infty} \left( \langle f^\infty, \nu_2^\infty \rangle_x \lambda_2^\text{ac}(Q_2^x) + M_f \cdot O(m^{-1}) \cdot (\Lambda(Q_2^x)) \right)
$$

$$
\text{[66]-[74]} \quad \theta_2 \cdot \sum_{Q_x \in \mathcal{O}_m \atop x \in R \cap G_0 \cap B_2^\infty} \int_{Q_x} \langle f^\infty, \nu_2^\infty \rangle_x \, d\lambda_2^\text{ac}
$$

$$
+ O(m^{-1}) \cdot (M_f + \text{Lip}(f)) \cdot \Lambda(\Omega).
$$

In the first equality we have strongly used the $\Lambda$-Lebesgue property for the sets $Q_x$ which is justified in Step 1a(1); the precise statement is contained in Corollary A.2. The same argument will be used in the estimates (c) and (d) below;
(c) passing to points $x \in \mathcal{R} \cap G_1$ (in this case $x \in B_1^\infty \cap B_2^\infty$). For $i = 1, 2$ the partial sum reads

$$III_i := \sum_{Q_x \in O_m} \int_{Q_x} \langle f, \nu_i \rangle \, d\mathcal{L}^d + \int_{Q_x} \langle f^\infty, \nu_i^\infty \rangle \, d\lambda_i^s$$

$$= \sum_{Q_x \in O_m} \left( \langle f^\infty, \nu_i^\infty \rangle_x \cdot \lambda_i^s(Q_x^i) + M_f \cdot O(m^{-1}) \cdot \Lambda(Q_x) \right)$$

$$+ \sum_{Q_x \in O_m} O(m^{-1}) \cdot \lambda_i(Q_x)$$

$$= \theta_i \cdot \sum_{Q_x \in O_m} \int_{Q_x} \langle f^\infty, \nu_i^\infty \rangle_x \, d\lambda_i^s$$

$$+ O(m^{-1}) \cdot (M_f + \text{Lip}(f) + 1) \Lambda(Q_x);$$

(d) and to finally cover $\mathcal{R}$, the singular points $x \in \mathcal{R} \cap G_\infty$: an analogous estimate to the one derived in (b) gives

$$IV := \sum_{Q_x \in O_m} \int_{Q_x} \langle f, \nu_1 \rangle \, d\mathcal{L}^d + \int_{Q_x} \langle f^\infty, \nu_1^\infty \rangle \, d\lambda_1$$

$$+ \sum_{Q_x \in O_m} \int_{Q_x} \langle f, \nu_2 \rangle \, d\mathcal{L}^d + \int_{Q_x} \langle f^\infty, \nu_2^\infty \rangle \, d\lambda_2$$

$$= \theta_1 \cdot \sum_{Q_x \in O_m} \int_{Q_x} \langle f^\infty, \nu_1^\infty \rangle_x \, d\lambda_1^s$$

$$+ O(m^{-1}) \cdot (M_f + \text{Lip}(f)) \cdot \Lambda(\Omega).$$

(e) Lastly, the cubes with centers $x \in S$ which by definition are simply given by

$$V_i := \sum_{Q_x \in O_m} \langle f, \nu_0 \mathbf{1}_{Q_x} \rangle.$$

Since the singular set can be split into the disjoint union

$$\text{sing}(\Omega) = \mathcal{R} \cup S$$

$$= (\mathcal{R} \cap G_0 \cap B_2^\infty) \cup (\mathcal{R} \cap G_1 \cap B_1^\infty \cap B_2^\infty) \cup (\mathcal{R} \cap G_\infty \cap B_1^\infty) \cup S,$$
and since every possible cube \( Q_x \in \mathcal{O}_m \) is centered at one (and only one) of the previous four sets, we deduce from inspecting the terms \( I, II, III, IV, V \) that
\[
\left\langle f, \nu_0^{(m)} \right\rangle = \sum_{i=1,2} \theta_i \left( \sum_{Q_x \in \mathcal{O}_m} \left( \int_{Q_x} f(x, \nu_i) \, d\mathcal{L}^d(y) + \int_{Q_x} \left\langle f^\infty, \nu_i^{\infty} \right\rangle \, d\lambda_i^s(x) \, d\mathcal{L}^d(y) \right) \\
+ \sum_{x \in \{(\text{sing}(\Omega) \cap B_m^\infty) \setminus S, x \in \mathcal{O}_m} \int_{Q_x} \left\langle f^\infty, \nu_i^{\infty} \right\rangle \, d\lambda_i^s(y) \right) + \sum_{Q_x \in \mathcal{O}_m} \left\langle f, \nu_0 LQ_x \right\rangle \\
+ O(m^{-1}) \cdot \left( M_f + \text{Lip}(f) + 2 \right) \cdot \left( \Lambda(\Omega) + \mathcal{L}^d(\Omega) \right)
\]
\[
= \theta_1 \cdot \left\langle f, \nu_1^{(m)} \right\rangle + \theta_2 \cdot \left\langle f, \nu_2^{(m)} \right\rangle \\
+ O(m^{-1}) \cdot \left( M_f + \text{Lip}(f) + 2 \right) \cdot \left( \Lambda(\Omega) + \mathcal{L}^d(\Omega) \right).
\]

**Conclusion.** Let us recall that
\[
\sup_{m \in \mathbb{N}} \left( \sup_{j \in \mathbb{N}} |w_j^{(m)}|(\Omega) \right) \leq \sup_{m \in \mathbb{N}} C(m) < \infty,
\]
where \( C(m) \) is the constant from (85). Returning to the estimate (84), we may then, by a diagonalization argument, define a sequence of \( \mathcal{A} \)-free measures
\[
w^{(m)} := w_j^{(m)} \in \mathcal{M}(\Omega; \mathbb{W}),
\]
satisfying (cf. Claim 2 and Step 5)
\[
\lim_{m \to \infty} \left\langle f, \delta_{w^{(m)}} \right\rangle = \lim_{m \to \infty} \left\langle f, \nu_0^{(m)} \right\rangle = \theta_1 \lim_{m \to \infty} \left\langle f, \nu_1^{(m)} \right\rangle + \theta_2 \lim_{m \to \infty} \left\langle f, \nu_2^{(m)} \right\rangle \\
= \theta_1 \left\langle f, \nu_1 \right\rangle + \theta_2 \left\langle f, \nu_2 \right\rangle \quad \text{for all } f \in \{f_{p,q}\}_{p,q \in \mathbb{N}}.
\]
Moreover, it follows from the compactness of Young measures and the separation Lemma 2.1 that the convergence above implies (this may involve passing to a subsequence)
\[
w^{(m)} \rightharpoonup \nu_0 \quad \text{on } \Omega.
\]
This finishes the proof. \( \square \)

9.1. **Proof of Theorem 1.1. Necessity.** Conditions (i)-(ii) are obvious since, by definition, an \( \mathcal{A} \)-free measure is generated by a uniformly bounded sequence of (asymptotically) \( \mathcal{A} \)-free measures. On the other hand, conditions (iii)-(iv) were established in [4]: condition (iii) is contained in Proposition 3.1, and condition (iv) in Lemma 3.2; condition (iv') is contained in Proposition 3.3.

**Sufficiency.** Let us first recall the main relaxation result given in [4], this will be used later in the proof.

**Theorem 9.2.** Let \( f : \Omega \times \mathbb{W} \to [0, \infty) \) be a continuous integrand that is Lipschitz in its second argument, uniformly over the \( x \)-variable. Assume also that \( f \) has linear growth at infinity and is such that there exists a modulus of continuity \( \omega \) satisfying
\[
|f(x, z) - f(y, z)| \leq \omega(|x - y|)(1 + |z|) \quad \text{for all } x, y \in \Omega, \ A \in \mathbb{W}.
\]
Further suppose that the strong recession function \( f^\infty \) exists. Then, for the functional
\[
\mathcal{G}[u] := \int_{\Omega} f(x, u(x)) \, dx, \quad u \in L^1(\Omega; \mathbb{W}),
\]
the weak-$\star$ (sequential) lower semicontinuous envelope defined by
$$\overline{\mathcal{T}}[\mu] := \inf \left\{ \liminf_{j \to \infty} G[u_j] : (u_j) \subset L^1(\Omega; \mathbb{W}), u_j \rightharpoonup^d \mu, ||A u_j||_{W^{1,q}} \to 0 \right\},$$
where $\mu \in \mathcal{M}(\Omega; \mathbb{W})$ is an $\mathcal{A}$-measure and $1 < q < \frac{d}{d-1}$, is given by
$$\overline{\mathcal{T}}[\mu] = \int_{\Omega} f(x, \mu^w(x)) \, dx + \int_{\Omega} f^\infty(x, \mu^s).$$

**Remark 9.1** (on the assumptions of Theorem 9.2). Firstly, the requirement that $f$ is a non-negative integrand can be dropped provided that
(a) $Q_A f(x, \cdot) > -\infty$ on a dense subset of $\Omega$, and
(b) the relaxation $\overline{\mathcal{T}}$ is restricted to weak-$\star$ limit measures satisfying $|\mu|(\partial \Omega)$.

The relaxation result cited above is first proved for $L^1$-fields (here one uses the finiteness of $Q_A f$; see also [9, Theorem 1.2] for a proof of this when $f(x, z) = f(z)$), the general case follows by strict-convergence approximation and Reshetnyak’s Continuity Theorem. The condition $|\mu|(\partial \Omega)$ prevents the recovery sequence to carry mass towards the boundary, where we do not have $\mathcal{A}$-free Jensen-type inequalities (essential for the lower bound). For the same reason, the recovery sequences for such measures only generate Young measures whose singular part does not charge the boundary.

The regularity assumptions from the Theorem above are all covered by integrands in the class $E(\Omega; \mathbb{W})$. Indeed, since $Sf \in C(\Omega \times \mathbb{W})$, it follows that there exists a modulus of continuity $\omega$ satisfying
$$|Sf(x, z) - Sf(y, z)| \leq \omega(|x - y|).$$
It follows from the representation (12) that
$$|f(x, z) - f(x, y)| \leq \omega(|x - y|) \cdot (1 + |z|).$$
And, clearly $f^\infty$ exists and is in fact continuous. Therefore, we may apply the results of Theorem 9.2 for (possibly non-negative) integrands $E(\Omega; \mathbb{W})$ whenever (a) and (b) are satisfied.

We are now ready to begin with the proof of Theorem 1.1.

We show that at $(\mathcal{L}^d + \lambda^s)$-almost every $x_0 \in \Omega$ there exists an $\mathcal{A}$-free tangent Young measure $\nu_0 \in \text{Tan}(\nu, x_0)$. Since the proof at regular points and the proof at singular are completely analogous (with the exception that one uses conditions (iii) and (iv) on each of them respectively), we shall only work out the proof at singular points in detail:

Let $\nu \in Y(\Omega; \mathbb{W})$ be a Young measure satisfying the necessity assumptions of Theorem 1.1 on a set $D \subset \Omega$ of full $\lambda^s$-measure with the following property: if $x_0 \in D$, then
- $\text{Tan}(\lambda^s, x_0) \neq \emptyset$,
- condition (iv) holds,
- the Structure Theorem [17] ensures that
$$P_0 := \frac{d\nu}{d\lambda^s}(x_0) = (\text{id}, \nu^\infty_{x_0}) \in \Lambda_h.$$

Accordingly, the set $\text{Tan}(\nu, x_0)$ is non-empty and there exists a tangent Young measure $\nu_0$ there satisfying (see Proposition 2.4)
$$\nu_0 = (\delta_0, \lambda_0, \nu^\infty_{x_0}).$$

Our goal is to show that $\nu_0$ is locally an $\mathcal{A}$-free measure and then conclude by applying the local characterization contained in Theorem 1.2. Since the proof is only local, to argue this it suffices to show $\nu_0$ is an $\mathcal{A}$-free measure on the unit cube.
\(Q\). For a similar reason, there is no loss of generality in assuming that \(\gamma(\partial Q) = 0\). Observe that, condition (iv) or condition (iv') on \(\nu\) implies that
\[
\langle g^\infty, \nu_0^\infty \rangle \geq g^\infty(\langle \text{id}, \nu_0^\infty \rangle)
\]
for all continuous \(\mathcal{A}\)-quasiconvex integrands \(g : \mathcal{W} \rightarrow \mathbb{R}\) with linear growth at infinity.

Let \([\nu_0] = \tau\) be the barycenter of \(\sigma\). A version for homogeneous Young measures of Theorem 9.1 ensures that the set
\[
\mathcal{A}\mathcal{Y}_{0,\tau}^{\text{sing}}(Q) := \left\{ \sigma = (\sigma_y, \lambda_\sigma, \sigma_\infty) \in \mathcal{Y}(Q; \mathcal{W}) : \lambda_\sigma(Q) = \lambda_\sigma(Q) = 1, |\sigma| = \tau, \text{ and } \sigma \text{ is an } \mathcal{A}\text{-free homogeneous Young measure} \right\},
\]
is a weak-\(\ast\) and convex subset of the space of \((\mathcal{E}(Q; \mathcal{W}))^\ast\). Notice that here our construction departs from the similar approach presented in [17] for BD-Young measures where the authors are able to work truly with tangent singular Young measures in the sense that \(\sigma_y = \delta_0\) for \(\mathcal{L}^d\)-a.e. \(y \in Q\). This subtle difference will play a role in Step 2. It follows from the geometric version of the Hahn–Banach theorem that \(\mathcal{A}\mathcal{Y}_{0,\tau}^{\text{sing}}(Q)\) coincides with the set
\[
\mathcal{Y} := \bigcap \left\{ H : \mathcal{A}\mathcal{Y}_{0,\tau}^{\text{sing}}(Q) \subset H, H \in \mathcal{E}(Q; \mathcal{W})^\ast \text{ weak-}\ast\text{ closed affine half-space} \right\}.
\]
For every such \(H\), there exists \(f_H \in \mathcal{E}(Q; \mathcal{W})\) such that
\[
H = \{ \ell \in \mathcal{E}(Q; \mathcal{W})^\ast : \ell(f_H) \geq s_H > -\infty \}.
\]
Let \(H\) be an arbitrary but fixed weak-\(\ast\) closed half-plane as in the definition of \(\mathcal{Y}\). Accordingly,
\[
\left\langle f_H, \sigma \right\rangle \geq s_H \quad \text{for all } \sigma \in \mathcal{A}\mathcal{Y}_{0,\tau}^{\text{sing}}(Q).
\]

**Step 1. Reduction to integrands defined on the essential domain.** First, we show there exists an integrand \(g_H \in \mathcal{E}(Q; \mathcal{W}_\Lambda)\) satisfying
\[
\left\langle \check{g}_H \otimes 1_{\mathcal{W}_\Lambda}, \sigma \right\rangle = \left\langle f_H, \sigma \right\rangle \quad \text{for all } \sigma \in \mathcal{A}\mathcal{Y}_{0,\tau}^{\text{sing}}(Q) \cup \{\nu_0\}.
\]
We define the integrand
\[
g_H := f_H \circ (\text{id}_\mathcal{W} \times 1), \quad \text{where } \text{id}_\mathcal{W} : \mathcal{W}_\Lambda \rightarrow \mathcal{W} \text{ is the canonical linear inclusion}.
\]
Notice that \(g_H \in \mathcal{E}(Q; \mathcal{W}_\Lambda)\) and \(\check{g}_H := g_H \otimes 1_{\mathcal{W}_\Lambda} \in \mathcal{E}(Q; \mathcal{W})\). Moreover, by Corollary 2.1 we get
\[
Q_{A\mathcal{Y}} \otimes 1_{\mathcal{W}_\Lambda} = Q_{A\check{g}_H}.
\]
Let \(\sigma \in \mathcal{A}\mathcal{Y}_{0,\tau}^{\text{sing}}(Q)\). By virtue of Lemma 3.2 in [4], it follows that \(\text{supp}(\sigma_\infty^\infty) \subset \mathcal{W}_\Lambda\) for \(\lambda_\sigma\)-almost all \(y \in Q\) (here we use that \(\sigma\) is a homogeneous non-trivial Young measure on \(Q\) and that \(P_0 \in \Lambda_\Lambda\)). In particular,
\[
\left\langle \check{g}_H, \sigma \right\rangle = \left\langle f_H, \sigma \right\rangle \geq s_H \quad \text{for all } \sigma \in \mathcal{A}\mathcal{Y}_{0,\tau}^{\text{sing}}(Q);
\]
the same argument, this time using the the assumption (iv) gives \(\text{supp}(\nu_0^\infty) \subset \mathcal{W}_\Lambda\) for \(\lambda_0\)-almost every \(y \in Q\), works to show that
\[
\left\langle \check{g}_H, \nu_0 \right\rangle = \left\langle f_H, \nu_0 \right\rangle.
\]

**Step 2. Boundedness of \(Q_{A\mathcal{Y}}\).** Next, we use strongly the fact that \(g_H\) is defined on the essential domain of \(\mathcal{A}\) to apply Lemma 5.5 in [9], which states that there exists a dense subset \(D \subset Q\) such that \(Q_{A\mathcal{Y}}(y, \cdot)\) has linear growth for all \(y \in D\), or equivalently, that (see Lemma 2.5 in [31])
\[
Q_{A\mathcal{Y}}(y, 0) > -\infty \quad \text{for all } y \in D.
\]
Let us remark that, even though the cited work poses stronger assumptions on $A$, the proof of this result only requires the integrand to be defined in the essential domain of $A$. The way the proof works is to show first that $Q_{AH}(\cdot, \tau^{ac}(\cdot)) > -\infty$ on $D$ a dense subset of $Q$. This is achieved by using the finiteness $\|g_H, \delta_x\|$ and the separation property $\|g_H, \sigma\| \geq s_H$. It is precisely here where it is used that Young measures $\sigma \in AY_{\tau,0}$ are allowed to have a non-trivial absolutely continuous part, since then the separation $\|g_H, \sigma\| \geq s_H$ is sufficiently rich to satisfy the assumptions of Lemma 5.5 in [9].

Step 3. $\nu_0 \in H$. The previous step and (87) show that (a) in in Remark 9.1 is satisfied for $f = \tilde{g}_H$ and $\Omega = Q$. Let us recall that $|\tau(\partial Q)| = 0$ so that (b) in Remark 9.1 is satisfied. Hence we may apply Theorem 9.2 with $\tau, g_H$ and $Q$ to find a recovery sequence $(u_j) \subset L^1(Q; W)$ such that

$$\lim_{j \to \infty} \int_Q \hat{g}_H(u_j) \, dy \leq \int_Q Q_{AH}(\tau^{ac}) \, dy + \int_Q Q_{AH}(\sigma).$$

Passing to a further subsequence if necessary, we may assume $u_j, L^d \rightharpoonup Y, \sigma$ for some $A$-free Young measure $\sigma \in Y(Q; W)$. Again, condition (b) on $\tau$ and the discussion in Remark 9.1 ensure that $\sigma \in AY_{\tau,0}$. This gives $\langle \tilde{g}_H, \sigma \rangle = \langle f_H, \sigma \rangle \geq s_H$. On the other hand, since $\tilde{g}_H \geq Q_{AH}$, it follows from (iv) and Remark 1.2 that (here we use that $Q_{AH}$ has linear growth at infinity and that $\lambda_0 \in \text{Prob}(Q)$)

$$\langle f_H, \nu_0 \rangle \geq \langle Q_{AH}, \nu_0 \rangle \geq \int_Q [(Q_{AH}(y, \cdot), \delta_0) + (Q_{AH}(y, \cdot)^\infty, \nu_{x_0})] \lambda_0^\infty(y) \, dy + \int_Q Q_{AH}(\lambda_0^\infty \cdot P_0) \, L^d + \langle Q_{AH}^{\infty}(P_0 \lambda_0^\infty) \rangle$$

where in the one but last inequality we have dealt with the absolutely continuous part with the aid of the following property: every $\Lambda_\chi$-convex function $g$ with linear growth at infinity satisfies (see for instance Lemma 2.5 in [30])

$$g(z) + g^{\infty}(P) \geq g(z + P) \quad \text{for all } z \in W \text{ and } P \in \Lambda_\chi.$$

Hence, by combining (89)-(90) we obtain

$$\langle f_H, \nu_0 \rangle \geq \lim_{j \to \infty} \int_Q \hat{g}_H(u_j) \, dy = \langle \tilde{g}_H, \sigma \rangle = \|f_H, \sigma\| \geq s_H.$$

This shows that $\nu_0 \in H$.

Conclusion. Since $H$ was arbitrarily chosen, the last conclusion shows that $\nu_0 \in Y$, which by construction is precisely $AY(Q)^{\text{sing}}_{\tau,0}$. This shows that, at $\lambda^\text{sing}$-almost every $x_0 \in Q$, every tangent Young measure is locally an $A$-free measure. Since the same holds at $L^d$-almost everywhere (see discussion below), then by virtue of Theorem 1.2 we conclude that

$\nu$ is an $A$-free measure.
**Comments on the case of regular points.** The proof is very similar to the one of singular points with the following considerations

Step 1. We work with the set of Young measures

\[ \mathcal{A} \mathcal{Y}^{\text{reg}}_{0,\tau}(Q) := \{ \sigma = (\sigma_y, \gamma_\sigma, \sigma_y^\infty) \in \mathcal{Y}(Q; \mathcal{W}) : \gamma_\sigma = \lambda^{ac}(x) \mathcal{L}^d, [\sigma] = \tau, \text{ and } \sigma \text{ is an } \mathcal{A}\text{-free homogeneous Young measure} \}, \]

which is weak-* closed and convex with respect in \((\mathcal{E}^{\text{res}}(Q; \mathcal{W}))^*\), where (see Section 2.5 in [17])

\[ \mathcal{E}^{\text{res}}(Q; \mathcal{W}) := \{ 1 \otimes h : 1 \otimes h \in \mathcal{E}(Q; \mathcal{W}) \}. \]

The reason why we can work with this simpler class lies in the fact that \(\lambda_\sigma\) is a uniform measure for all \(\sigma \in \mathcal{A} \mathcal{Y}^{\text{reg}}_{0,\tau}(Q)\). Hence, \(\mathcal{E}^{\text{res}}(Q; \mathcal{W})\) indeed separates the homogeneous Young measures on \(Q\), this is the property on which the weak-* closedness and the convexity hinge on.

Step 2. Notice that in this case we do not have the control of the Structure Theorem \(P_1 \in \Lambda_\mathcal{A}\), therefore we must define

\[ g_{\theta}(z) := f_{\theta}(\tau^{ac}(x_0) + i[z]) \text{ for all } z \in \mathbb{W}_\lambda. \]

Note that by Corollary 7.2, it holds that \(\Gamma_{\theta}[\sigma]\) is an homogeneous \(\mathcal{A}\)-free Young measure time with zero barycenter if and only if \(\sigma\) is an homogeneous \(\mathcal{A}\)-free Young measure time with barycenter \(\tau^{ac}(x_0) \mathcal{L}^d \mathcal{L}^d Q\). And therefore,

\[ \text{Shifts}_{\tau^{ac}(x_0)} \{ \mathcal{A} \mathcal{Y}^{\text{reg}}_{0,\tau^{ac}(x_0)}\mathcal{Y}^{ac}(Q) \} = \mathcal{A} \mathcal{Y}^{\text{reg}}_{0,0}(Q). \]

On the other hand, the Young measures \(\tilde{\sigma}\) in the latter set satisfy the crucial property \(\text{supp}(\tilde{\sigma}_y) \subset \mathbb{W}_\lambda\) for \(\mathcal{L}^d\)-a.e. \(y \in Q\) (this follows from the analysis in Section 2.5 in [4]). In particular, we obtain

\[ \left\langle f_{\theta}, \sigma \right\rangle = \left\langle g_{\theta}, \Gamma_{\tau^{ac}(x_0)}[\sigma] \right\rangle, \]

and from assumption (iii) we get

\[ \left\langle f_{\theta}, \nu_0 \right\rangle = \left\langle g_{\theta}, \Gamma_{\tau^{ac}(x_0)}[\nu_0] \right\rangle. \]

Here, \(\Gamma_{\tau^{ac}(x_0)}[\sigma]\) is the shifted Young measure \((\sigma \ast \delta_{\tau^{ac}(x_0)}, \lambda^{ac}(x) \mathcal{L}^d, \sigma^{\infty})\).

Step 3. The proof carries out similarly with the aid of (iii) in place of (iv).

This finishes the proof. \(\Box\)

**Appendix A. Some technical lemmas for Radon measures**

This section is devoted to address some technical results which are essential to the proof of Theorem 9.1.

**Lemma A.1.** Let \(\lambda\) be a probability measure on the unit open cube \(Q \subset \mathbb{R}^d\) and assume that \(\lambda\) does not charge points in \(Q\), that is, \(\lambda(\{x\}) = 0\) for all \(x \in Q\). Let \(\theta \in (0,1)\), then there exists a convex Borel set \(D \subset Q\) satisfying

\[ Q_r \subset D \subset \overline{Q}_r \text{ for some } r \in (0,1), \]

and

\[ \lambda(C) = \theta. \]

**Proof.** In the case when \(d = 1\) we define the monotone non-decreasing map

\[ r \mapsto \varphi(r) := \int_{-r}^{r} \, d\lambda, \quad r \in (0,1), \]

Figure 3. Generic shape of the approximation set $D$ from Lemma A.1 ($d = 3$); composed by the first open cube approximation $S_1 \subset Q$, the second-step 2-dimensional relatively open caps $S_2 \subset \partial S_1$, and the last-step 1-dimensional relatively open caps $S_3 \subset \partial S_2$.

which in particular is a function of bounded variation. Notice that, in this case, the one dimensional BV-theory and the assumption on $\lambda$ give

$$\varphi'({\{r\}}) = \varphi(r^+) - \varphi(r^-) = \lambda({\{r\}}) = 0.$$  

This proves $\varphi$ is in fact continuous in the interval $(0, 1)$. Therefore, since $\varphi(0) = 0$ (again by assumption) and $\varphi(1) = 1$, then the Mean Value Theorem ensures there exists $\tau \in (0, 1)$ with $\lambda(Q_r) = \varphi(r) = \theta$.

The case $d > 1$ requires a co-area-type argument. The first step is to define the map

$$\varphi(r) := \int_{Q_r} d\lambda, \quad r \in (0, 1),$$

is monotone non-decreasing and satisfies $\lim_{r \searrow 0} \varphi(r) = 0$ and $\lim_{r \nearrow 1} \varphi(r) = 1$. In particular,

$$|\varphi'({\{r\}})| = \varphi(r^+) - \varphi(r^-) = \lambda(\partial Q_r) \quad \text{for all } r \in (0, 1).$$

Set $r := \sup \{ s \in (0, 1) : \varphi(s) \leq \theta \}$. Clearly, if $\varphi$ is lower semicontinuous at $r$, then we can set $C_1 = Q_r$ which automatically satisfies the conclusions of the Lemma (in this case is not necessary to use the assumption that $\lambda$ does not charge points). However, in general one cannot expect this as for instance there might be mass sitting on $\lambda(\partial Q_r)$. We may then assume that $\theta_1 := \lambda(Q_r) = \varphi(r^-) \leq \theta$ and

$$0 \leq \theta - \theta_1 \leq \varphi(r^+) - \varphi(r^-).$$

(91)

This will be our first approximation. The second step to carry the same approximation now on $\{q_1, \ldots, q_{2d}\}$, the $(2d)$ open $(d-1)$-dimensional faces of $\partial Q_r$, which are axis-directional translated $(d-1)$-dimensional cubes (centered at the origin) in $\mathbb{R}^{d-1}$; the number $(2d)$ of faces will not be relevant for our construction. Define the maps

$$\varphi_{i,2}(s) := \int_{q_i} d\lambda, \quad s \in (0, 1), \ i \in \{1, \ldots, 2d\}.$$  

Repeating the same procedure as in the first step on each of the faces simultaneously, we update the error of our estimate to

$$0 \leq \theta - \theta_2 := \theta - \theta_1 - \sum_{i=1}^{2d} \varphi_{i,2}(r_i^{-}),$$

where $r_i^{-}$ are the corresponding points.
where \( r_1 := \sup \{ s \in (0, r) : \sum_{i=1}^{2d} \varphi_{i,2}(s) \leq \theta - \varphi(r^-) \} \). Notice that
\[
\theta_2 = \lambda(C_2),
\]
where
\[
C_2 := Q_r \cup q_{i_1}^1 \cup \cdots \cup q_{i_1}^{2d}.
\]
Moreover, by construction (since each one of the added faces are concentric to each face of \( Q_r \)), \( C_2 \) is a (semi-open/semi-closed) convex set satisfying \( Q_r \subset C_2 \subset \overline{Q_r} \).

There are two cases, either \( \theta = \theta_2 \) and then \( \lambda(C_1) = \theta \), or \( \theta - \theta_2 > 0 \) and we must keep adding bits of \( \partial Q_r \), which may require to perform a similar argument on the \((d-2)\)-dimensional concentric faces of \( \partial Q_r \) and the \((d-2)\)-dimensional faces of each \( q_i^k, 1 \leq i \leq 2d \). In general, the \((j+1)\)th step is to iterate this argument (when \( \theta - \theta_j > 0 \)) on all the possible \((d-j)\)-dimensional faces of \( \partial Q_r \) and all the other \((d-j)\)-dimensional faces resulting of adding the previous \((d-j+1)\)-dimensional concentric cubical caps. The key part of the construction is that, at the end of the \((j+1)\)th step, one obtains a convex set \( C_j \subset C_{j+1} \) satisfying \( Q_r \subset C_{j+1} \subset Q_r \) and
\[
0 \leq \theta - \lambda(C_{j+1}) = \theta - \theta_{j+1} \quad \text{for some} \quad \theta_{j+1} \geq \theta_j.
\]

We now argue why there exists \( 1 \leq j \leq d \) such that \( \theta_j = \theta \). If \( d = 2 \), then by the same argument that in the one-dimensional case we get that the maps \( \varphi_{i,2} \) are continuous, and hence it must be that \( \theta_2 \) reaches \( \theta \). If \( d = 3 \), then the maps \( \varphi_{i,3} \) of the third step are continuous and hence at most \( \theta_3 \) reaches \( \theta \). In general, the description of this procedure is tedious, but it is inductively natural and always reaches and endpoint (at most after \( d \)-steps) where we find a convex set \( C \) containing the origin and satisfying \( Q_r \subset C \subset Q_r \) and
\[
\lambda(C) = \theta.
\]

This finishes the proof.

\[ \square \]

**Corollary A.1.** Let \( \lambda \) be a probability measure on the unit open cube \( Q \subset \mathbb{R}^d \) and assume that \( \lambda \) does not charge points in \( Q \). Let \( \theta \in (0,1) \) and let \( \varepsilon > 0 \). Then, there exists an open Lipschitz set \( D \subset Q \) satisfying
\[
\lambda(\partial D) = 0, \quad \text{and} \quad |\lambda(D) - \theta| \leq \varepsilon.
\]

**Proof.** From the previous lemma we may find a set \( D \) satisfying \( \lambda(D) = \theta \). Now, from the inner and outer regularity of Radon measures we may find a compact set \( K \) and an open set \( O \) such that \( K \subset D \subset O \) satisfying
\[
0 \leq \lambda(D) - \lambda(K) < \frac{\varepsilon}{2} \quad \text{and} \quad 0 \leq \lambda(O) - \lambda(D) < \frac{\varepsilon}{2}.
\]

Moreover, since \( \text{dist}(K,O^\partial) > 0 \), there exists a Lipschitz open set \( K \subset C \subset O \) with \( \rho := \text{dist}(C,K \cup O^\partial) > 0 \). On the other hand, since \( C \) is a Lipschitz compact set, the family of sets
\[
T_\delta = \{ x \in Q : \text{dist}(x,C) < \delta \}, \quad \delta > 0,
\]
is a a family of open Lipschitz sets satisfying \( K \subset T_\delta \subset O \) for some \( \delta \in [0, \delta_1] \), where \( 0 < \delta_1 \ll \rho \). Furthermore, if \( 0 < s < t < \delta_1 \), then \( \partial T_s \cap \partial T_t = \emptyset \). In particular, since \( \lambda \) is a Radon measure, there exists a full \( \mathcal{L}^1 \)-measure subset of \( I \subset [0, \delta_1] \) such that
\[
\lambda(\partial T_\delta) = 0 \quad \text{for all} \quad \delta \in I.
\]

Let us choose \( \delta_0 \in I \) and recall from our construction that \( K \subset T_{\delta_0} \subset O \). Hence,
\[
0 \leq |\lambda(T_{\delta_0}) - \lambda(D)| \leq \lambda(O) - \lambda(K) < \varepsilon.
\]

This finishes the proof.

\[ \square \]
Lemma A.2 (shrinking sequence). Let $\lambda$ be a positive Radon measure on $\Omega$ and assume there exists a tangent measure $\tau \in \text{Tan}(\lambda, x)$ which does not charge points on $\mathbb{R}^d$. Then, for every $\theta \in (0, 1)$, there exists an infinitesimal sequence $r_m \downarrow 0$ and a sequence of open Lipschitz sets $R_m \subset Q_{r_m}(x)$ satisfying
$$\lambda(\partial R_m) = 0 \quad \text{for all } m \in \mathbb{N},$$
and
$$\lim_{m \to \infty} \frac{\lambda(R_m)}{\lambda(Q_{r_m}(x))} = \theta.$$

Proof. Since $T_{0,r}[\tau]$ belongs to $\text{Tan}(\lambda, x)$ for all $r > 0$, we may assume without any loss of generality that $\tau$ is a good blow-up as in (8). That is, there exists an infinitesimal sequence $r_j \downarrow 0$ such that
$$\frac{1}{\lambda(Q_{r_j}(x))} T_{x,r_j}[\lambda] \rightharpoonup \tau \quad \text{in } \mathcal{M}(\mathbb{R}^d, \mathbb{W}), \quad |\tau|(Q) = |\tau|(\overline{Q}) = 1.$$

By assumption and Corollary A.1 we may find a sequence of Lipschitz open sets $(D_m)_{m \in \mathbb{N}}$ satisfying $\subset D_m \subset Q$ for all $m \in \mathbb{N}$. Moreover,
$$\tau(\partial D_m) = 0 \quad \text{and} \quad |\tau(D_m) - \theta| \leq \frac{1}{m}.$$

Hence, for fixed $m$, we deduce from the strict-convergence of the blow-up sequence that
$$\lim_{j \to \infty} \frac{\lambda(x + r_j D_m)}{\lambda(Q_{r_j}(x))} = \tau(D_m) = \theta + O(m^{-1}).$$

Moreover, up to a small re-scaling at each $m$ we may assume without loss of generality that $\lambda(x + r_j \partial D_m) = 0$. A standard diagonalization argument yields a subsequence $r_m := r_{(j(m))} \downarrow 0$ such that
$$\lim_{m \to \infty} \frac{\lambda(R_m)}{\lambda(Q_{r_m}(x))} = \theta, \quad R_m := x + r_m D_m \subset Q_{r_m}(x).$$

By construction, the sequence of sets $(R_m)_{m \in \mathbb{N}}$ has the desired properties. \qed

Corollary A.2. Let $\lambda$ be a positive Radon measure on $\Omega$ and assume there exists a tangent measure $\tau \in \text{Tan}(\lambda, x)$ which does not charge points. Let $f$ be a $\lambda$-measurable map and assume furthermore that $x$ is a $\lambda$-Lebesgue point of $f$. Then, for every $\theta \in (0, 1)$, there exist a sequence $r_m \downarrow 0$ and a sequence of Lipschitz open sets $D_m \subset Q_{r_m}$ satisfying
$$\lambda(\partial D_m) = 0, \quad \lim_{m \to \infty} \frac{\lambda(x + D_m)}{\lambda(x + Q_{r_m})} = \theta,$$
and
$$\lim_{m \to \infty} \int_{x + D_m} |f(y) - f(x)| \, d\lambda(y) = 0.$$

Proof. The existence of the sequence of open Lipschitz sets $(D_m)$ satisfying the first two properties follows directly from the previous corollary. The third property follows from the estimate
$$\int_{x + D_m} |f(y) - f(x)| \, d\lambda(y) \leq \frac{1}{\theta + O(m^{-1})} \int_{Q_{r_m}(x)} |f(y) - f(x)| \, d\lambda(y),$$
and the fact that $x$ is a $\lambda$-Lebesgue point of $f$. \qed

Lemma A.3. Let $\lambda$ be a positive Radon measure on $\Omega$ and assume there exists $x \in \Omega$ such that $\lambda(\{x\}) > 0$. Then $\text{Tan}(\lambda, x) = \{ c e_0 : c > 0 \}$. 

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Proof. Let $\tau \in \tan(\lambda, x)$ and set $0 < \alpha = \lambda(\{x\})$. Since $\tan(\lambda, x)$ is a $d$-cone, it is enough to show that $\tau = \delta_0$ when $\tau$ is a probability measure on $Q$. Moreover, we may also assume the blow-up sequence converging to $\tau$ has the form
\[
\gamma_j = \frac{1}{\lambda(Q_{x_j}(x))} T_{x_j, r_j} [\lambda] \rightharpoonup \tau \text{ in } \mathcal{M}(\mathbb{R}^d; \mathbb{W}).
\]
It follows from the strict-convergence $\gamma \to \tau$ on $Q$ that
\[
\tau(Q_s) = \lim_{j \to \infty} \gamma_j(Q_s) = \frac{1}{\alpha_s} \lim_{j \to \infty} \lambda(Q_{x_j}(x)) = 1 \text{ for all } s > 0.
\]
Since $\tau$ is a probability measure on $Q$, this shows that $\tau \ll Q = \delta_0$ as desired. $\square$

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