Power-law Genesis: strong coupling and galileon-like vector fields.

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Abstract

A simple way to construct models with early cosmological Genesis epoch is to employ bosonic fields whose Lagrangians transform homogeneously under scaling transformation. We show that in these theories, for a range of parameters defining the Lagrangian, there exists a homogeneous power-law solution in flat space-time, whose energy density vanishes, while pressure is negative (power-law Genesis). We find the condition for the legitimacy of the classical field theory description of such a situation. We note that this condition does not hold for our earlier Genesis model with vector field. We construct another model with vector field and power-law background solution in flat space-time, which is legitimately treated within classical field theory, violates the NEC and is stable. Upon turning on gravity, this model describes the early Genesis stage.

1 Introduction and summary.

Genesis [1] is a cosmological scenario without initial singularity. In this scenario, the Universe starts its expansion from flat space-time and zero energy density at large negative times. As the Universe evolves, the energy density and the Hubble rate grow, and eventually reach large values. If gravity is described by General Relativity, then this regime requires the domination of exotic matter which violates the Null Energy Condition, NEC (for a review see [2]). Later on, the energy density of exotic matter has to be converted into the energy density of usual matter, and the conventional cosmological evolution starts. As shown in [3], the violation of the NEC in a healthy way is possible in the context of the scalar Galileon.
theories [4]. By now, numerous ways to implement the Genesis idea have been proposed, mostly in the context of theories involving scalars (see Ref. [1, 5] for an incomplete list and Ref. [6] for topical review), but also in vector field models [7].

A straightforward way to construct a model of the early Genesis epoch is to make use of a Lagrangian which, in the absence of gravity, transforms homogeneously under scaling transformation: \( \mathcal{L} \Rightarrow \lambda^{N} \mathcal{L} \) when \( \pi_\alpha(x^\nu) \Rightarrow \lambda^{s} \pi_\alpha(\lambda x^\nu) \), where \( \pi_\alpha \) denote the non-gravitational fields in the model, and \( N \) and \( s \) are constant parameters. Then, quite generally, the model, still in the absence of gravity, has a spatially homogeneous solution \( \pi \propto |t|^{-s} \) as \( t \to -\infty \) (power-law Genesis, see Sec. 2), for which the energy density vanishes while pressure is negative. This precisely means the violation of the NEC. When gravity (described by GR) is turned on, energy density no longer stays equal to zero; instead, it increases as required for the early Genesis stage. This mechanism has been invented in Ref. [1] (with \( N = 4 \) and \( s = 1 \)) and then utilized in other contexts (see Ref. [2] for a review), including models with vector fields [7].

However, within this class of models, the coefficients in the quadratic Lagrangian for perturbations about the classical solution often tend to zero as \( t \to -\infty \), which implies that the strong coupling energy scale also tends to zero. In such a situation, the classical treatment may become problematic, cf. Refs. [1, 8]. To figure out whether or not this is the case, one should study both quadratic and interaction terms in the Lagrangian for perturbations and find the behavior of the strong coupling scale \( \Lambda \) as \( t \to -\infty \):

\[ \Lambda(t) \propto |t|^{-\sigma}. \]

This scale should be compared with the classical energy scale \( E_{cl} \), which is merely the evolution rate, and in the power-law Genesis case is given by

\[ E_{cl}(t) \propto |t|^{-1}. \]

The classical treatment is legitimate provided that \( E_{cl} \ll \Lambda \), which means

\[ \sigma \leq 1 \quad (1.1) \]

(the case \( \sigma = 1 \) is subtle: the relation \( E_{cl} \ll \Lambda \) may be valid in a restricted region of parameter space).

In this note we address this strong coupling issue in the context of the power-like models described above. This is done in Sec. 2, where we show that the requirement (1.1) is equivalent to

\[ N \leq 4. \]

We note that this property does not hold for Genesis with vector field proposed in Ref. [7]. Therefore, in Sec. 3 we construct another model with vector field and power-law background
solution that obeys (1.1); we determine the range of parameters in which the background is stable and violates the NEC in Minkowski space. For completeness, we also turn on gravity (in the form of GR) and describe the evolution of the scale factor at the early Genesis stage.

2 Strong coupling scale.

As outlined in Introduction, we consider the Lagrangian for $M$ bosonic fields $\pi_\alpha$, $\alpha = 1, 2, ..., M$, in 4d Minkowski space. Index $\alpha$ may either enumerate the fields (say, if $\pi_\alpha$ are scalars) or be Lorentz index, or both. The Lagrangian is assumed to transform homogeneously under scaling transformation

$$x^\nu \Rightarrow \lambda x^\nu, \quad \pi_\alpha(x^\nu) \Rightarrow \lambda^s \pi_\alpha(\lambda x^\nu), \quad s \neq 0.$$  

Namely,

$$\mathcal{L} \Rightarrow \lambda^N \mathcal{L}. \quad (2.1)$$

Importantly, we assume that equations of motion are second order in derivatives, even though the Lagrangian may involve second derivatives of the fields. This is the case in generalized Galileon theories [2, 4, 9] as well as in theories with Galileon-like vector fields [7].

We consider for definiteness the Lagrangians which are linear combinations of the monomials involving $n$ fields without derivatives, $m$ first derivatives and $l$ second derivatives of the fields (the argument goes through if one allows also for inverse powers of the fields):

$$(\pi_{\alpha_1}...\pi_{\alpha_n}) \cdot (\partial \pi_{\gamma_1}...\partial \pi_{\gamma_m}) \cdot (\partial^2 \pi_{\omega_1}...\partial^2 \pi_{\omega_l}) \sim [\pi]^n \cdot [\partial \pi]^m \cdot [\partial^2 \pi]^l. \quad (2.2)$$

Here

$$ns + m(s + 1) + l(s + 2) = N,$$

so that the transformaton law (2.1) holds. For a range of parameters defining the Lagrangian, there exists a homogeneous power-law solution

$$\pi_\alpha^{(0)} = \beta_\alpha |t|^{-s}, \quad (2.3)$$

with constant $\beta_\alpha$. Indeed, the term (2.2) gives a contribution to equation of motion with total number of fields equal to $(n + m + l - 1)$ and total number of derivatives $(m + 2l)$. Therefore, with the Ansatz (2.3), each of the $M$ equations of motion is proportional to $|t|^{-N+s}$ with the proportionality coefficient being a polynomial in $\beta_\alpha$. In other words, equations of motion make a system of $M$ algebraic equations for $M$ coefficients $\beta_\alpha$, which has a solution for a range of parameters entering the Lagrangian.\(^1\)

\(^1\)Unless there is some symmetry that relates coefficients of different monomials (2.2) in such a way that this algebraic system does not have a real solution.
Let us now consider perturbations about the background (2.3), \( \pi_\alpha = \pi^{(0)}_\alpha + \delta \pi \). Our purpose is to determine the time-dependence of the lowest strong coupling scale in the limit \( t \to -\infty \). We begin with quadratic Lagrangian for perturbations. Since we assume that there are no third and higher derivatives of \( \delta \pi \) in the equations of motion, there are no terms with second and higher derivatives in the quadratic Lagrangian. So, the relevant terms are, schematically, \( (\partial \delta \pi)^2 \). The monomial (2.2) in the original Lagrangian contributes to the terms \( (\partial \delta \pi)^2 \) in the quadratic Lagrangian with coefficients involving \( (n + m + l - 2) \) background fields \( \pi^{(0)} \) and \( (m + 2l - 2) \) derivatives acting on them. Hence, the structure of the quadratic Lagrangian is
\[
L^{(2)} \supset |t|^{-N+2s+2}(\partial \delta \pi)^2.
\]
This implies that canonically normalized fields are
\[
\xi_\alpha \propto |t|^{-N/2+s+1} \delta \pi_\alpha.
\] (2.4)
Their mass dimension, by definition, equals 1.

We now turn to the interactions between perturbations \( \delta \pi \). The term (2.2) induces interactions of the following form:
\[
[\pi^{(0)}]^{n-a} \cdot [\partial \pi^{(0)}]^{m-b} \cdot [\partial^2 \pi^{(0)}]^{l-c} \times [\delta \pi]^a \cdot [\partial \delta \pi]^b \cdot [\partial^2 \delta \pi]^c,
\]
where
\[
a + b + c \geq 3.
\] (2.5)
We make use of (2.3) and (2.4) and find that in terms of canonically normalized field, this interaction Lagrangian is proportional to
\[
|t|^{-N(a+b+c-2)+c-a} \times [\xi]^a \cdot [\partial \xi]^b \cdot [\partial^2 \xi]^c.
\]
On dimensional grounds, the coefficient of \([\xi]^a \cdot [\partial \xi]^b \cdot [\partial^2 \xi]^c\) in the Lagrangian is \( E_s^{-(a+2b+3c-4)} \), where \( E_s \) is the (naive) strong interaction scale (we consider the case \( a + 2b + 3c - 4 > 0 \), otherwise no constraint is obtained). Thus,
\[
E_s \propto |t|^{-N(a+b+c-2)+c-a \over a+2b+3c-4}.
\]
We require that this scale is higher than the classical energy scale \( t^{-1} \) for \( |t| \to \infty \) and get
\[
N \over 2 (a + b + c - 2) + c - a \over a + 2b + 3c - 4 < 1,
\]
or
\[
(N - 4)(a + b + c - 2) < 0.
\]
We recall (2.5) and obtain finally

\[ N \leq 4, \]

where we include the case \( N = 4 \) in which both classical and quantum strong coupling scales behave as \( |t|^{-1} \), and the quantum scale may be higher due to specific relationships between the parameters in the Lagrangian, see Ref. [1] for an example.

3 Vector field model with stable NEC-violating solution.

3.1 Early-time evolution: Minkowski space

We now construct a simple vector field model which is covariant under scaling transformation \( A_\mu(x^\nu) \to \lambda^8 A_\mu(\lambda x^\nu) \), so that the Lagrangian transforms as given by (2.1), and, furthermore, \( N \leq 4 \) to avoid strong coupling. By trial and error we arrive at the following Lagrangian with \( N = \frac{12}{5} \) and \( s = -\frac{1}{5} \):

\[ \mathcal{L} = q(D^2A^\rho \square A_\rho + kB^2 + lC^2 + u(\mathcal{F}_{\mu\nu}\mathcal{F}_\rho^\nu A^\mu A^\rho + 2A^\rho A_\rho A_{\mu} A_{\mu})), \]  

where \( q, k, l \) and \( u \) are free parameters, and

\[ \mathcal{F}_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \]
\[ D = A_{\mu\nu} A^\mu A^\nu, \]
\[ B = A_\mu A^\nu A^{\nu\lambda} A_\mu A_\lambda, \]
\[ C = A^{\mu\tau} A_\tau A^\rho A_{\mu\tau}. \]

In accordance with Sec. 2, equations of motion have a solution

\[ A_\mu^{bg} = (\beta |t|^{\frac{1}{5}}, 0, 0, 0) \]  

with constant \( \beta \). This classical evolution occurs in a weak coupling regime at early times, \( t \to -\infty \).

We now wish to figure out whether there exists a set of parameters \( q, k, l, u \) in the Lagrangian (3.1), such that the solution (3.2) is stable and violates the NEC. By solving the field equation, we find

\[ \beta^5 = \frac{20u}{3m - 5}, \]

where

\[ m = l + k + u. \]
To see the NEC-violation, we need the expression for the energy-momentum tensor of this solution:

\[ T_{\mu\nu} = \frac{2\delta(\sqrt{-g}L)}{\sqrt{-g}\delta g^{\mu\nu}} \bigg|_{g_{\rho\sigma} = \eta_{\rho\sigma}}. \]

To this end, we consider minimal coupling to the metric, i.e., set \( A_{\mu\nu} = \nabla_\nu A_\mu, \Box A_\mu = \nabla^\mu \nabla_\mu A_\rho, D = A_{\mu\nu} A_\tau A_\lambda g^{\mu\tau} g^{\nu\lambda}, \) etc., in curved space-time. The Lagrangian (3.1) can be written in the following form:

\[ \mathcal{L} = \frac{1}{2} f(D) \Box F - f(D) A_{\tau,\sigma} A^{\tau,\sigma} + L(A_\mu, A_{\lambda,\nu}) \]

where

\[ F = A_\mu A^\mu, \]
\[ f(D) = qD^2, \]
\[ L = q[kB^2 + lC^2 + u(\mathcal{F}_{\mu\nu} \mathcal{F}_\rho A^\rho A^{\mu\nu} + 2A^{\mu\nu} A_{\rho,\nu} A_{\mu,\rho})]. \]

We find

\[ T_{00} = 0, \]
\[ T_{ij} = p\delta_{ij}, \quad i, j = 1, 2, 3, \]
\[ p = \left( -\frac{1}{2} \partial_\tau f \partial^\tau F + L - f A_{\tau,\sigma} A^{\tau,\sigma} \right) \bigg|_{g_{\mu\nu} = \eta_{\mu\nu}; \ A_\mu = A_{\mu}^{bg}}. \] (3.3)

This gives

\[ p = \frac{qu^{\frac{m}{2}} 2^{\frac{12}{2}} 5^{-\frac{12}{2}} (11 - m)}{(3m - 5)^{\frac{m}{2}}} (-t)^{-\frac{m}{2}}, \quad t < 0. \] (3.4)

Thus, the background \( A_{\mu}^{bg} \) violates the NEC provided that

\[ q(11 - m) < 0. \] (3.5)

Let us consider the stability of the solution \( A_{\mu}^{bg} \). Having in mind Ref. [10], we also require subluminality of the perturbations about it. Stability conditions and conditions for the absence of superluminal perturbations for Galileon-like vector models were derived in Ref. [7]. Making use of the results of Ref. [7], it is straightforward to find that there are two ranges of parameters such that all these conditions together with (3.5) are satisfied for \( t < 0: \)

1. \( q > 0, \quad u \neq 0, \)
2. \( \frac{25}{2} < k \leq \frac{39}{2}, \quad 11 - k < l < \frac{k + 1}{9}, \)

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and

\[(II) \quad q > 0,\]
\[u \neq 0,\]
\[k > \frac{39}{2},\]
\[\frac{9 - 7k}{15} < l < -\frac{k + 1}{9},\]

Thus, our example shows that there are stable homogeneous solutions in vector theories that violate the NEC and avoid strong coupling regime at early times.

### 3.2 Turning on gravity.

Here we construct an initial stage of the cosmological Genesis scenario, similar to Ref. [1]. To this end, we turn on gravity and assume that it is described by conventional General Relativity, while the vector field is minimally coupled to metric, as described above. Importantly, all equations of motion, for both vector field and metric, remain second order in derivatives [7], just like in Horndeski theories.

In the asymptotic past, space-time is assumed to be Minkowskian, and in accordance with (3.3), (3.4), energy-momentum tensor vanishes as \(t \to -\infty\). At large but finite \(|t|\), gravitational effects on the vector field evolution are negligible, so, to the leading order in \(M_{Pl}\), the energy density and pressure are given by (3.3), (3.4). Then the Hubble parameter is obtained from

\[\dot{H} = -4\pi G(\rho + p).\]

We find

\[H = \frac{40\pi Gqu^\frac{8}{3}5^\frac{6}{5}7^\frac{5}{8}(m - 11)}{7(3m - 5)^\frac{8}{7}}(-t)^{-\frac{7}{5}}, \quad t \to -\infty.\]

Thus, the Universe undergoes accelerated expansion characteristic of the early Genesis epoch. At this stage, perturbations about the background are stable and subluminal.

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