Random Walk and Broad Distributions on Fractal Curves

Seema E Satin        A.D.Gangal

January 18, 2013

Abstract

In this paper we analyse random walk on a fractal structure, specifically fractal curves, using the recently developed calculus for fractal curves. We consider only unbiased random walk on the fractal structure and find out the corresponding probability distribution which is Gaussian like in nature, but shows deviation from the standard behaviour. Moments are calculated in terms of Euclidean distance for a von Koch curve. We also analyse Levy distribution on the same fractal structure, where the dimension of the fractal curve shows significant contribution to the distribution law by modifying the nature of moments. The appendix gives a short note on Fourier transform on fractal curves.

1 Introduction

Random walks in one dimension is a well established topic [1,2,3,4], with several applications in physics including transport phenomena and diffusive behaviour. The discrete simple random walk in 1-dimension is described by a particle taking steps, each towards the right or left with a specified probability. For unbiased walks the probability is same for right or left step [3]. In the continuum limit the number of steps $N$ goes to $\infty$, the step size $\delta$ goes to zero, such that $N\delta$ remains fixed. The random variable is rescaled to give the probability distribution [2,3], showing the diffusive behaviour.

While studying transport properties on disordered systems, one encounters subdiffusive behaviour [5]. Some physical examples are, NMR diffusometry on percolation structures [14], motion of a bead in polymer network [15] etc. In general the diffusion process can be expected to depend on both the geometry of the medium and the process. While various aspects of dependence on processes (e.g. Gaussian, Levy etc) and ordinary geometries are well studied, the dependence on fractal geometries of media is relatively unexplored.

It is expected that on fractals, subdiffusion can be due to the geometry of the underlying fractal structure on which the process takes place. In this chapter, various well established approaches in random walk problems are extended to fractal geometry. We study such a behaviour by considering a random walk on a
fractal curve. In particular, we present the analytical and numerical results for walks on a von Koch curve in $\mathbb{R}^2$. Random walk on such a curve is performed by taking a step either forward or backward along the curve itself with equal probability if the walk is unbiased. Here we consider unbiased walk only.

Such a random walk in the continuum limit leads to study of processes like diffusion on the Fractal curve $F$. Hence one can study the diffusive behaviour of particles, performing random motion on the fractal curve, by using this formulation. Appropriate and exact expressions can be obtained, for a class of walks, to describe physical processes by using this model. The simplest case is that of obtaining a probability distribution, which is same as the solution obtained by solving a diffusion equation on a fractal curve as in [17]. The study of a discrete random walk on a fractal curve $F$ is another approach towards studying such transport properties on these kind of structures.

![Figure 1: Example of a fractal curve. $\theta$ and $\theta'$ denote any two points on the curve, and $l$ gives the Euclidean distance between these two points](image)

There are several approaches to study anomalous diffusion. Some commonly used frameworks are fractional Brownian motion [6], the continuous time random walk [7], fractional diffusion equations [8,9], generalized Langevin equation and Fokker-Planck equations [10,11,12,13] etc. In most of these approaches the assumptions for validity of simplest form of Central limit theorem are not satisfied [5].

In the present case, the Central limit theorem is not violated. The departure from Gaussian distribution or, the normal diffusive behaviour, is explicitly due
to the underlying fractal geometry of space and not due to correlations or long tailed statistics.

Here, we consider a random walk on a fractal curve (von-Koch like), in $\mathbb{R}^n$. The calculus on such a curve is developed in [17], where the curve $F$ has fractal dimension $\alpha$. $\theta$ denotes a point on the curve and $J(\theta) \equiv S^\alpha_u$ denoted the mass function, which is the mass of the fractal covered up to point $\theta$, $u$ is the parameter of the curve. A short review of the Calculus is given in Appendix 2.

Random walk on such a curve is performed by taking a step either forward or backward along the curve itself with equal probability if the walk is unbiased. In this paper we consider unbiased walk only.

In various sections of this paper we find analogues of simplest aspects of random walk problems in 1-D extended to fractal curves.

2 Discrete Random Walk on a fractal curve

We consider the special class of random walks on a fractal curve $F$, where the walker covers a fixed mass $\Delta$ in each step. If the fractal curve, $F$ reduces to a real line, then the random walk on it reduces to an ordinary simple random walk as analysed in [18].

Let $C(N, \theta, \theta')$ be the number of walks that start at a point $\theta$ on the fractal curve $F$ with $\gamma$-dimension, $\alpha$ (note that $\alpha \geq 1$ in general, particularly here $1 < \alpha < 2$), and end at another point $\theta'$ on the same fractal curve, at the $N^{th}$ step. Then

$$C(N; \theta, \theta') = C(N - 1; \theta, J^{-1}(J(\theta') - \Delta)) + C(N - 1; \theta, J^{-1}(J(\theta') + \Delta)) \quad (1)$$

Equation (1) is the recursion relation for a random walk on $F$.

When $N$ i.e. number of steps is large, and $\Delta$ small, equation (1) can be approximated by a differential equation involving the $F^\alpha$-derivative. This is carried out in the following.

$$C(N + 1; \theta, \theta') = \{C(N; \theta, J^{-1}(J(\theta') - \Delta)) + C(N; \theta, J^{-1}(J(\theta') + \Delta)) - 2C(N; \theta, \theta')\}$$

$$+ 2C(N, \theta, \theta') \quad (2)$$

Taylor expanding (as given in appendix) first two terms around $J(\theta')$ in powers of $\Delta$, we get

$$C(N + 1; \theta, \theta') = (\Delta^2(D^\alpha_{F \theta'})^2C(N; \theta, \theta') + \frac{1}{12} \Delta^4(D^\alpha_{F \theta'})^4C(N; \theta, \theta') + \ldots) + 2C(N; \theta, \theta')$$

$$\approx \Delta^2(D^\alpha_{F \theta'})^2C(N; \theta, \theta') + O(\Delta^4) + 2C(N; \theta, \theta') \quad (3)$$

$$C(N + 1; \theta, \theta') - 2C(N; \theta, \theta') \approx \Delta^2(D^\alpha_{F \theta'})^2C(N; \theta, \theta') \quad (5)$$
Each step has equal probability in forward or reverse direction on the curve. The total number of steps being \( N \), there are \( 2^N \) ways of performing an \( N \) step random walk. Thus, \( P(N, \theta, \theta') \), the probability of a such an \( N \)-step random walk, starting at \( \theta \) and ending at \( \theta' \) is given by

\[
P(N, \theta, \theta') = \frac{1}{2^N} C(N, \theta, \theta')
\]

Hence we can replace \( C(N, \theta, \theta') \) by \( 2^N P(N, \theta, \theta') \) and \( C(N+1, \theta, \theta') \) by \( 2^{N+1} P(N+1, \theta, \theta') \).

Thus, equation (5) leads to

\[
P(N + 1, \theta, \theta') - P(N; \theta, \theta') \approx \Delta^2 \frac{1}{2} (D_{\theta'}^2)^2 P(N; \theta, \theta')
\]

We now assume \( N \) to be large and \( P(N, \theta, \theta') \) to be slowly varying function of \( N \). Thus we can write the above equation in the form

\[
\frac{\partial}{\partial N} P(N; \theta, \theta') = \frac{\Delta^2}{2} (D_{\theta'}^2)^2 P(N; \theta, \theta') \tag{6}
\]

The solution of the above equation can easily be obtained by using conjugacy between the \( F^\alpha \)-derivative and ordinary derivative as in [16]. This gives,

\[
P(N, \theta, \theta') = \frac{1}{\sqrt{2\pi N}} \exp\left(-\frac{(J(\theta) - J(\theta'))^2}{2\Delta^2 N}\right) \tag{7}
\]

Now, one can go to the continuum case, where one can set \( t = N\tau \), \( N \) being the number of steps in the discrete walk and \( \tau \) being the duration between the consecutive steps. Thus the expression for probability distribution in the continuum case, which is of the form given by equation (11) in the later section can be obtained and used readily.

### 3 Moments

We now consider the Euclidean distance \( L(\theta) \equiv L(w(u)) = |w(u)| \) upt a point \( \theta = w(u) \) from the origin on the fractal curve \( F \).

Consider the probability distribution which behaves as

\[
P(\theta) \equiv P(w(u) = \theta) \sim \exp(-S^2_F(u)^2) \text{ where } S^2_F(u) = J(\theta) \tag{8}
\]

with appropriate normalization. Using this form for the probability distribution we calculate the first two absolute moments on the fractal curve as follows:

\[
\langle L \rangle = \int_{C(-\infty, \infty)} L(\theta) P(\theta) d\theta \tag{9}
\]

and

\[
\langle L^2 \rangle = \int_{C(-\infty, \infty)} L^2(\theta) P(\theta) d\theta \tag{10}
\]
We evaluate the above two moments in terms of Euclidean distance, using the explicit form
\[ P(w(u) = \theta) = \frac{1}{\sqrt{2\pi At}} \exp\left(-\frac{(J(\theta) - S_p^*(u))^2}{2At}\right) \]  
(11)
as obtained in chapter [17], which is the continuum case of equation [7] as \( N \to \infty \) and \( \Delta \to 0 \) (\( N\Delta \) being held fixed).

**Heuristic Calculation of Absolute Moments**

Now, for the given gaussian like probability, equation (11), we calculate the moments in terms of Euclidean distance, for random motion on a von Koch curve. From the graph of Rise function and Euclidean distance, for a von Koch curve as given in [16], it is clear, that 
\[ c_1[L(w(u))]^\alpha < S_p^*(u) < c_2[L(w(u))]^\alpha. \]
Thus we can approximate \( S_p^*(u) \) by \( L^\alpha \), i.e
\[ S_p^*(u) \sim [L(w(u))]^\alpha \]
or in short hand notation
\[ S_p^* \sim L^\alpha \]

Thus, the first moment can be calculated analytically by using an intuitive replacement for the \( F^\alpha \)-integrals in equations (9) and (10). \( S_p^* \) is thus replaced by \( L^\alpha \) and \( dF^\theta \) is replaced by \( dL^\alpha \), keeping in view the way \( F^\alpha \)-integrals are defined in [16].

Thus,
\[ <L> = \frac{1}{\sqrt{2\pi At}} \int_{C(0,\infty)} L \exp\left(-\frac{(L^\alpha)^2}{2At}\right)dL^\alpha \]  
(12)
On substituting \( L^\alpha/\sqrt{2At} = z \) we reduce the integral to the following
\[ <L> = \text{Const.}t^{1/2\alpha} \int_0^\infty z^{1/\alpha} \exp(-z^2)dz \]  
(13)
which implies
\[ <L> \sim t^{1/2\alpha} \]  
(14)
similarly we can obtain the second moment in terms of the Euclidean distance
\[ <L^2> = \frac{1}{\sqrt{2\pi At}} \int_{C(0,\infty)} L^2 \exp\left(-\frac{(L^\alpha)^2}{2At}\right)dL^\alpha \]  
(15)
leading to the behaviour
\[ <L^2> \sim t^{1/\alpha} \]  
(16)
In the case of von Koch curve \( 1/\alpha = 0.792 \) and \( 1/2\alpha = 0.396 \). These expressions are in accordance with the numerical results presented in fig(2) and...
Figure 2: Plot of $\log < L >$ (Y-axis) vs. $\log(t)$ (X-axis). The cross denotes the value (calculated by performing $F^\alpha$ integration numerically), to which the straight line is fitted with slope 0.403, to extract the value of exponent in equation (14), for a von Koch curve.

In these figures a plot between moments and time is shown. These plots have been obtained by performing the $F^\alpha$ integrals in equations (9) and (10) numerically, for the von-Koch curve. From the plots one obtains

$$< L > \sim t^{0.403}$$

and

$$< L^2 > \sim t^{0.802}$$

Here we see a beautiful match between the expected values and numerical calculations within numerical accuracy. The plots also confirm the expected behaviour of $S_F^\alpha(u)$.

4 Broad Distributions

We have derived in section (2), the expression for probability distribution for a discrete, simple random walk on fractal curve. Now we examine random walks with some different (non Gaussian) probability distributions for the individual steps.
Figure 3: Plot of $\log \langle L^2 \rangle$ (Y-axis) vs. $\log(t)$ (X-axis). The cross denotes value (calculated by performing $F^\alpha$-integration numerically) to which straight line is fitted with slope 0.802 to extract the value of the exponent in equation (16), for a von Koch curve.

Review of Broad Distributions

We begin by summarizing some relevant results for Broad distributions from [5].

For a 1-D random walk in ordinary space, let $X_N$ denote the sum of independent random variables $l_n$, i.e

$$X_N = \sum_{n=1}^{N} l_n$$

where $l_n$ is the $n^{th}$ step size. Let us denote $p(l)$, the probability distribution for $l$(individual steps in the random walk) which is broad, that is ,it decreases for large $l$ as $l^{-1+\mu}$ with $\mu > 0$. We consider two cases,

- For $0 < \mu < 1$, $X_N$, the first moment $\langle l \rangle$ is infinite.
- For $1 < \mu < 2$, the first moment $\langle l \rangle$ is finite and the second moment $\langle l^2 \rangle$ is still infinite.

It is well known that limit distributions of the sum $X_N$ are defined by their characteristic functions which are given by $\hat{L}_{\beta,\mu}(k)$ in Fourier space, i.e $k$-space,
when \( \mu \) and \( \beta \) are parameters, \( \beta \) being the degree of asymmetry. We consider the case \( \beta = 0 \) for the stable distribution \( L_{0,\mu} \). In this case large positive and negative values of \( l_n \) occur with equal frequency. Then

\[
\tilde{L}_{0,\mu}(k) = e^{-|k|^\mu}
\]  

(18)

Its Fourier transform being

\[
L_{0,\mu}(Z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk e^{ikZ-|k|^\mu}
\]  

(19)

We set \( Z = Z_N \) where, \( Z_N = X_N/N^{1/\mu} \) or \( (X_N - \langle l \rangle_N)/N^{1/\mu} \) as \( N \to \infty \), for \( 0 < \mu < 1 \) or \( 1 < \mu < 2 \) respectively. All the important results for such a case are given in [5].

**Broad Distributions on fractal curves**

Now we consider an analogous approach for the random walk on fractal curve. In particular we take parametrizable fractal curve, the von-Koch curve with \( \gamma \)-dimension equal to \( \log 4/\log 3 \) as described in [16].

We consider the following probability distribution in the limiting case, when \( N \to \infty \). We now abuse the above notation for obvious reasons to denote \( L_{0,\mu}(\psi) \) as the distribution in Fourier space of fractal curve \( F \), where \( \psi = w(k) \) and \( S_F^\alpha(k) = J(\psi) \) as described in appendix. Then

\[
L_{0,\mu}(\psi) = \exp(-|J(\psi)|^\mu)
\]  

(20)

In what follows we will use \( \tilde{L} \) to refer to quantity on the fractal curve (space) and \( \bar{L} \) to refer to the quantity in the real line.

The inverse Fourier transform (as defined in the appendix), for the above distribution is given by

\[
\tilde{L}_{0,\mu}(\theta) = \frac{1}{2\pi} \int_{C(-\infty,\infty)} \exp(iJ(\psi)J(\theta) - |J(\psi)|^\mu) dF^\alpha \psi
\]  

(21)

The above integral in view of conjugacy between \( F^\alpha \) integral and Riemann integral becomes:

\[
\tilde{L}_{0,\mu}(y = J(\theta)) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(i\tilde{k}y - |\tilde{k}|^\mu) d\tilde{k}
\]  

(22)

For \( \mu = 2 \) we obtain the gaussian distribution case:

\[
\tilde{L}_{0,\mu}(y = J(\theta)) = \frac{1}{\sqrt{2\pi}} \exp(-\frac{y^2}{2})
\]  

(23)

Applying the fractalizing operator to the above equation we get back the expression for distribution on fractal curve.

\[
\tilde{L}_{0,\mu}(\theta) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{\{J(\theta) = S_F^\alpha(u)\}^2}{2}\right)
\]  

(24)
Similarly for the Cauchy case when $\mu = 1$ we get

$$\tilde{L}_{0,\mu} = \frac{1}{\pi(1 + \{J(\theta) = S_F^\mu(u)\}^2)}$$

Now, one can then discretize the integral in equation (22) by replacing $d\tilde{k}$ in terms of equal steps of size $\Delta \tilde{k}$, as follows:

$$\tilde{L}_{0,\mu}(y = J(\theta)) = \sum_{m=-\infty}^{\infty} \exp[i\tilde{k}_m y - |\tilde{k}_m|^\mu] \Delta \tilde{k}$$

where $\tilde{k}_m = m \Delta \tilde{k}$.

For values of $\mu$ other than 1 and 2, the expansion for large arguments is given by [5]

$$\tilde{L}_{\mu,0}(y = J(\theta)) = (\pi)^{-1} \sum_{m=1}^{\infty} (-)^{m+1} \frac{y^{-(\mu+1)}}{m!} \Gamma(1 + m\mu) \sin(\pi \mu/2)$$

The leading term of which is

$$\tilde{L}_{\mu,0}(y = J(\theta)) = (\pi)^{-1} y^{-(\mu+1)} \Gamma(1 + \mu) \sin(\pi \mu/2)$$

Applying the fractalizing transformation to the above we get the result:

$$\tilde{L}_{\mu,0}(\theta) = (\pi)^{-1} S_F^\mu(u)^{-(\mu+1)} \Gamma(1 + \mu) \sin(\pi \mu/2)$$

(25)

$$\sim L^{-\alpha(\mu+1)}$$

(26)

for large values of $J(\theta) = S_F^\mu(u)$ or $L$ where $L$ denotes the Euclidean distance.

One can see that $<L>$ is infinite for $\mu \leq 1/\alpha$ and is finite for $\mu > 1/\alpha$ for these large values of $L$. Also the second moment $<L^2>$ diverges for $\mu \leq 2/\alpha$ and is finite for $\mu > 2/\alpha$ for large values of $L$. Hence we see a scaling down of the absolute moments for Levy distributions due to the fractal structure of the underlying space.

### 5 First Passage Time

The first passage time is a standard problem in Statistical Physics [19, 3]. Here we extend it to first passage time on fractal curves.

Random walk on a fractal curve can be described by forward and backward steps along the curve that cover equal mass on the fractal curve as in section [2], so that there is one-to-one correspondence between random walk on a straight line (the parameter $u$) and that on the curve $F$.

The first passage time, or the time required to reach a certain point $\theta = w(u)$ on the curve $F$, for the first time can be defined as follows:
Definition 1 The first passage time $T(\theta)$ to a point $\theta \in F$ is given by
\[ T(\theta) = \inf \{ t : \Theta(t) = \theta \} \] (27)

where $\Theta(t)$ is the position of a random walker on $F$ at time $t$, which started at a point $\theta_0$ at time $t = 0$.

Now we explore the relation between this first passage time and the mass covered by the random walker on the fractal curve in this time.

When the walker performs forward and backward steps along the curve, let the mass covered in the $n^{th}$ step be given by $\gamma^n_F(F, u_n, u_{n-1})$, corresponding to parameter value lying between $u_{n-1}$ and $u_n$. The resultant displacement (considered on the $S^\alpha_F$-axis), of the particle performing a random walk of $M$ steps, some of which may be in forward direction and rest in the backward direction on the curve, so as to reach an arbitrary point $w(u_k)$ on $F$, is given by sum of $\gamma^n_F$ over $M$ steps. For $k < M$ such that $M = k + r$, and step size $\Delta$, for every sequence $\{u_1, u_2, \ldots, u_M\}$, such that,
\[ S^\alpha_F(u_M) = k\Delta \] (28)

Let the time duration between consecutive steps on the fractal curve $F$, be denoted by $\tau$. We calculate the maximum mass that a random walker covers in a given time $t$, such that $t = M\tau$, where $M$ is the total number of steps taken in time $t$.

Since $k = M - r$ we can write equation (28) in the following form:
\[ S^\alpha_F(u_M) = \left( \frac{t}{\tau} - r \right)\Delta \] (29)

It can be clearly seen that $S^\alpha_F(u_M)$ is maximum when $r = 0$ on the rhs of the above equation. Hence
\[ S^\alpha_F(u_M)|_{\text{max}} = \frac{t}{\tau}\Delta \] (30)

Next we calculate the minimal time required by a random walker to cover certain mass on the curve.

Conversely, let $t = M\tau$ be the time taken for $M$ steps on the curve $F$, also let $M = k + r$ as given above, then $k = \frac{t - r\tau}{\tau}$, and we can write equation (28) as
\[ S^\alpha_F(u_M) = \frac{t - r\tau}{\tau}\Delta \] (31)

Thus,
\[ t = r\tau + \frac{\tau S^\alpha_F(u_M)}{\Delta} \]

Hence we see that $t$ is minimum when $r = 0$, Hence
\[ t_{\text{min}} = \frac{\tau}{\Delta} S^\alpha_F(u_M) \] (32)
An interesting quantity, which can be analysed is the maximum Euclidean distance on a fractal curve, covered by the random walker in a minimum time $t$. In general these will depend on the geometry of the particular fractal curve of interest. The relation between Euclidean distance on a fractal curve and the staircase function [16] can only be obtained numerically, even for a simple fractal curve like the von-Koch curve. A plot between $t_{min}$ minimum time required to reach a point and $L_{max}$ (maximum Euclidean distance) calculated numerically for a von Koch curve, is shown in fig. (4). One may note a step-like behaviour in this figure. We comment on this in the results.

6 Results

It is clear from the figures (2) and (3) that a subdiffusive behaviour is addressed by the gaussian-like random walks on the von Koch curves.

Eq. (25) shows the behaviour of the stable law $L_{\mu,0}$ for values of $\theta$ for which $S_{\nu}^u(u)$ is large. This behaviour differs from the ordinary law for large deviation, when the underlying space is not a fractal and when the stable law decreases as $y^{-(1+\mu)}$. In the case of an underlying fractal space, the exponent gets multiplied by a factor of $\alpha$ the dimension of the fractal curve itself. Thus we see that from a Euclidean perspective, the behaviour gets a direct contribution
from the exponent of the distribution and the dimension of the curves. This is a striking difference. When taking straight line approximations on fractal curves, the dimension of the fractal curve contributes only indirectly. While using the above method, the fractal dimension plays a direct role in changing the nature of heavy tails of the distribution.

In fig. 4 we see a step-like behaviour, this explicitly shows that the Euclidean distance on the curve remains constant for a short while and then rises. There is a largest period of constant Euclidean distance in the figure, this depends on the geometry and the initial position of the random walker on the von Koch curve. Here we calculate the Euclidean distance from the origin or initial point (0,0) on the curve in $\mathbb{R}^2$.

**Appendix 1**

**The Fourier Transform**

From the definition of conjugacy [16]

$$\phi[f](S_F^x(u)) = f(w(u)) \tag{33}$$

The Fourier Transform on the real line, for a function $g(v)$, is defined by

$$g(v) = \int_{-\infty}^{\infty} \hat{g}(y) \exp(-ivy) dy \tag{34}$$

and the inverse Fourier Transform is

$$\hat{g}(y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} g(v) \exp(iv y) dy \tag{35}$$

In the case of a parametrizable fractal curve $F$, which is obtained by a fractalizing transformation on an interval of the real line, we propose that the Fourier space can also be obtained by the same fractalizing transformation on the interval of a real line. The interval may be $(-\infty, \infty)$.

Let $\hat{g} = \phi[f]$ and $g = \phi[f]$ also $v = S_F^x(k)$ and $y = S_F^x(u)$.

We use the notation $J(\theta) = S_F^x(u)$ and $J(\psi) = S_F^x(k)$, where $\theta = w(u)$ and $\psi = w(k)$.

Then taking Fourier transform of the LHS of equation (33) one can write

$$\phi[f](v = J(\psi)) = \int_{-\infty}^{\infty} \phi[f](y = J(\theta)) \exp(-iyv) dy$$

$$= \int_{C(-\infty, \infty)} f(\theta) \exp(-iJ(\theta)v) d\theta \tag{36}$$

where

$$C(-\infty, \infty) = \lim_{a \rightarrow -\infty, b \rightarrow \infty} C(a, b)$$
and
\[ \phi[\hat{f}](v = J(\psi)) = \phi[\int_{-\infty}^{a} f(y = J(\theta)) \exp(-iyv)dy] = \int_{C(-\infty,\infty)} f(\theta) \exp(-iJ(\theta)v)d\theta \] (37)

Comparing equations (36) and (37), we can write
\[ \phi[\hat{f}] = \phi[\hat{f}] \]

Also one can define the action of \( \phi \) in Fourier space as
\[ \hat{\phi}[f](v = J(\psi)) = \hat{f}(\psi) \] (38)

Now using conjugacy one can rewrite equation (36) as
\[ \hat{f}(\psi) = \int_{C(-\infty,\infty)} f(\theta) \exp(-iJ(\theta)J(\psi))d\theta \] (39)

Similarly, inverse transform of the above can be obtained from equation (35), which can be written as
\[ f(\theta) = \frac{1}{2\pi} \int_{C(-\infty,\infty)} \hat{f}(\psi) \exp(iJ(\theta)J(\psi))d\theta \] (40)

**Appendix 2**

**Review of Calculus on Fractal Curves**

For a set \( F \) and a subdivision \( P_{[a,b]} \), \( a < b \), \( [a,b] \subset [a_0,b_0] \) let \( w : [a,b] \to F \), then we define the mass function as follows:

\[ \gamma^{\alpha}(F,a,b) = \lim_{\delta \to 0} \inf_{P_{[a,b]}:|P| \leq \delta} \sum_{i=0}^{n-1} \frac{|w(t_{i+1}) - w(t_i)|^\alpha}{\Gamma(\alpha + 1)} \] (41)

where \( |\cdot| \) denotes the euclidean norm on \( \mathbb{R}^n \), \( 1 \leq \alpha \leq n \) and \( P_{[a,b]} = \{a = t_0, \ldots, t_n = b\} \).

The staircase function, which gives the mass of the curve upto a certain point on the fractal curve \( F \) is defined as
\[ S^\alpha_p(u) = \begin{cases} \gamma^\alpha(F,p_0,u) & u \geq p_0 \\ -\gamma^\alpha(F,u,p_0) & u < p_0 \end{cases} \] (42)

where \( u \in [a_0,b_0] \).

A point on the curve \( w(u) \equiv \theta \) and \( S^\alpha_p(u) \equiv J(\theta) \). The \( F^{\alpha} \) derivative is defined as:
\[ (D^\alpha_F f)(\theta) = F^{-1} \lim_{\theta' \to \theta} \frac{f(\theta') - f(\theta)}{J(\theta') - J(\theta)} \] (43)
The $F^\alpha$-integral is also defined and is denoted by

$$\int_{C(a,b)} f(\theta)d\alpha^\alpha \theta$$

where $C(a, b)$ is the section of the curve lying between points $w(a)$ and $w(b)$ on the fractal curve $F$.

References

[1] S.Chandrashekhar. *Stochastic problems in physics and astronomy*. Reviews of Modern Physics, 15(1), Jan. 1943.

[2] N.G. van Kampen. *Stochastic Processes in Physics and Chemistry*. Elsevier Science Publishers, 1992.

[3] G.R.Grimmet and D.R.Stirzaker 1992 Probability and random Processes (Oxford Science Publications)

[4] Federick Reif 1985 *Fundamentals of Statistical and Thermal Physics* (McGRAW-HILL International Editions)

[5] Jean-Philippe BOUCHAUD and Antoine GEORGES 1990 *Anomalous Diffusion in Disordered Media:Statistical Mechanisms, Model and Physical Applications*. Physics Reports(Review Section of Physics Letters)195, Nos. 4&5(1990)127 – 293.

[6] B.B.Mandelbrot and J.W.van Ness, SIAM (Soc. Ind.Appl Math.) Rev 10, 422 (1968).

[7] S.Havlin and D.Ben-Avraham. *Diffusion in disordered media* Ann. Rev Phys.Chem, (39) 269-290, 1988.

[8] W.R.Schneider and W.Wyss, J.Math Phys. 30, 134 (1989).

[9] R.Metzler, W.G.Glockle, and T.F.Nonnenmacher, Physica A 211, 13 (1994) ; R.Hilfer, Fractals 3, 211 (1995).

[10] H.C.Fogedby, Phys.Rev.E 58, 1690 (1998).

[11] H.C.Fogedby, Phys.Rev.Lett. 73, 2517 (1994); Phys. Rev. E 50 1657 (1994).

[12] G.M.Zaslavsky, Chaos 4, 24 (1994).

[13] K.M.Kolwankar and A. D.Gangal, Phys.Rev.Lett 80, 214 (1998).

[14] H.P.Müller, R.Kimmich, and J.Weis, Phys.Rev.E 54, 5278 (1996).

[15] F.Amblard, A.C. Maggs, B.Yurke, A.N.Paragellis, and S.Leibler, Phys.Rev.Lett. 77, 4470 (1996).
[16] Abhay Parvate, Seema Satin and A.D.Gangal. *Calculus on Fractal Curves in* \( \mathbb{R}^n \) (in print: Fractals), [arXiv:0906.0676v2][math-ph], 2009.

[17] Seema Satin, Abhay Parvate and A.D.Gangal. *Fokker Planck Equation on Fractal Curves*, [arXiv:1004.4422v2][math-ph]

[18] Joseph Rudnick and George Gaspari. *Elements of Random Walk*. Cambridge University Press.

[19] William Feller. *An Introduction to Probability Theory and Applications, Volume-I*. John Wiley and Sons, Inc., 1968.