Mildly dissipative diffeomorphisms of the disk with zero entropy

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Abstract

We discuss the dynamics of smooth diffeomorphisms of the disc with vanishing topological entropy which satisfy the mild dissipation property introduced in [CP]. In particular it contains the Hénon maps with Jacobian up to 1/4. We prove that these systems are either (generalized) Morse Smale or infinitely renormalizable. In particular we prove for this class of diffeomorphisms a conjecture of Tresser: any diffeomorphism in the interface between the sets of systems with zero and positive entropy admits doubling cascades. This generalizes for these surface dynamics a well known consequence of Sharkovskii’s theorem for interval maps.

1 Introduction

The space of diffeomorphisms splits into two classes: those with zero entropy and those with positive entropy (by which we always mean topological entropy). The former contains Morse-Smale diffeomorphisms: their nonwandering set is formed by finitely many periodic points. The latter contains the systems exhibiting a transverse homoclinic orbit, i.e. an orbit which accumulates on the past and on the future on a same periodic orbit and which persists under small perturbations: the nonwandering set is uncountable. In particular, both classes contain $C^1$-open sets. It has been proved that Morse-Smale systems and those having a transverse homoclinic intersection define a $C^1$-dense open set [PS1][C]. However, even in the $C^1$ context, the dynamics of systems belonging to the interface of these two classes is not well understood, while in higher topologies almost nothing is known. One goal would be to characterize the systems in the boundary of the zero entropy class and in particular to try to identify, if it exists, the universal phenomenon that generates entropy.

In a more general context our central question here is the transition between simple and complicated dynamics as seen from two different angles: the fundamental angle and the applied one. The later because the transition that we consider is the trace on a Poincaré map of a transition to chaos of dissipative flows in $\mathbb{R}^3$, as it has been observed in particular in a variety of natural and engineering contexts modeled that way. This happens both in some forced damped oscillators for which one observes the formation of horseshoes for a Poincaré map and in autonomous flows where the chaos is linked to a Shil’nikov bifurcation in $C^\omega$ regularity [Shi], or even $C^{1+\text{Lip}}$ regularity [T].

We can think about two related problems when considering this central question:

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the transition to chaos (i.e., the transition from zero to positive entropy),
the transition from finitely to infinitely many periods of hyperbolic periodic orbits.

In the one-dimensional context, the natural ordering on the interval allows the development of a “combinatorial theory”, which describes properties of orbits related to this ordering. An example of results is Sharkovskii’s hierarchy of periodic orbits [Sha]; it implies in particular that any system with zero entropy only admits periodic points of period $2^n$. One paradigmatic example is the case of unimodal maps: Coullet-Tresser and independently Feigenbaum conjectured [CT, F] that the ones in the boundary of the zero entropy class are limit of a period doubling cascade with universal metric property under rather mild smoothness assumptions and are infinitely renormalizable (see also [CMMT]).

In those papers a renormalization operator was introduced and it was shown that the numerical observations could be explained if this operator, defined on an appropriate space of functions, would have a hyperbolic fixed point. The central results of the universality theory for unimodal maps have been proved by Lyubich [L] for analytic unimodal maps and extended to lower regularity in [FMP]. Partial results about multimodal maps and the associated transition to chaos have been obtained by many authors (see e.g. [MT] and references cited or citing).

- Mildly dissipative diffeomorphisms of the disc. The first step towards that universal goal in higher dimension, is to consider embeddings of the disc $\mathbb{D}$. These embeddings can be extended to as diffeomorphisms of the two-dimensional sphere by gluing a repelling disc, as detailed in [BR]. Therefore, to avoid notations we will call dissipative diffeomorphisms of the discs the $C^r$ embeddings $f: \mathbb{D} \to f(\mathbb{D}) \subset \text{Interior}(\mathbb{D})$ with $r > 1$, such that $|\det(Df(x))| < 1$ for any $x \in \mathbb{D}$. Observe that any $f$-invariant ergodic probability measure $\mu$ which is not supported on a hyperbolic sink has one negative Lyapunov exponent and another one which is non-negative. In particular for $\mu$-almost every point $x$, there exists a well-defined one-dimensional stable manifold $W^{s}(x)$. We denote $W^{s}_{\mathbb{D}}(x)$ the connected component of $W^{s}(x) \cap \mathbb{D}$ containing $x$. We strengthen the notion of dissipation:

**Definition 1.** A dissipative diffeomorphism of the disc is mildly dissipative if for any ergodic measure $\mu$ not supported on a hyperbolic sink, and for $\mu$-almost every $x$, the curve $W^{s}_{\mathbb{D}}(x)$ separates $\mathbb{D}$. That notion has been introduced for any type of surfaces in [CP], where it is shown that mild dissipation is satisfied for large classes of systems: for instance it holds for $C^2$ open sets of diffeomorphisms of the disc, and for polynomial automorphisms of $\mathbb{R}^2$ whose Jacobian is sufficiently close to 0, including the diffeomorphisms from the Hénon

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1. In dimensions 1 and 2, the topological entropy is continuous with respect to the $C^\infty$-topology by [Mi, Ka, Y]. In particular, there is no jump in entropy at the transition. For $C^1$ families on the interval, the transition to positive entropy requires infinitely many period doubling bifurcations [BH].

2. In [CT], Coullet and Tresser recognized that operator as similar to the renormalization operator introduced in Statistical Mechanics by Kennet Wilson following a prehistory in the context of high energy physics.

3. Notice that [BF] is the first paper studying cascades of period doubling in dimensions 1 and 2.

4. In [CP], these systems are called strongly dissipative diffeomorphisms since many results were only applied for systems with very small Jacobian; in the context of the disc we call them mildly dissipative, since there are classes of diffeomorphisms with not such small Jacobian, as the Hénon maps, that satisfy the main property of the definition.
family with Jacobian of modulus less than $1/4$ (up to restricting to an appropriate trapped disc). This class captures certain properties of one-dimensional maps but keeps two-dimensional features showing all the well known complexity of dissipative surface diffeomorphisms. The dynamics of the new class, in some sense, is intermediate between one-dimensional dynamics and general surface diffeomorphisms.

b – Renormalization. As it was mentioned before, the essential mechanisms for interval endomorphisms in the transition to chaos are the period doubling cascades and the main universal feature of systems in the boundary of zero entropy is that they are infinitely renormalizable. A similar result can be proved for mildly dissipative diffeomorphisms of the disc that belong to the boundary of the zero entropy class.

A diffeomorphism $f$ of the disc is renormalizable if there exist a compact set $D \subset \mathbb{D}$ homeomorphic to the unit disc and an integer $k > 1$ such that $f^i(D) \cap D = \emptyset$ for each $1 \leq i < k$ and $f^k(D) \subset D$. Moreover $f$ is infinitely renormalizable if there exists an infinite nested sequence of renormalizable attracting periodic domains with arbitrarily large periods. For instance $\text{[GvST]}$ built a $C^\infty$-diffeomorphism which has vanishing entropy and is infinitely renormalizable (see also figure 1).

**Theorem A.** For any mildly dissipative diffeomorphism $f$ of the disc whose topological entropy vanishes,

- either $f$ is renormalizable,
- or any forward orbit of $f$ converges to a fixed point.

Morse-Smale diffeomorphisms (whose non-wandering set is a finite set of hyperbolic periodic points) are certainly not infinitely renormalizable. It is natural to generalize this class of diffeomorphisms in order to allow bifurcations of periodic orbits.

**Definition 2.** A diffeomorphism is generalized Morse-Smale if:

- the $\omega$-limit set of any forward orbit is a periodic orbit,
- the $\alpha$-limit set of any backward orbit in $\mathbb{D}$ is a periodic orbit,
- the period of all the periodic orbits is bounded by some $K > 0$.

Clearly these diffeomorphisms have zero entropy. We will see in section 6 that the set of mildly dissipative generalized Morse-Smale diffeomorphisms of the disc is $C^1$ open. A stronger version of theorem A proved in section 11 (see theorem A'), states that in the renormalizable case there exist finitely many renormalizable domains such that the limit set in their complement consists of fixed points. That version implies:

**Corollary 3.** A mildly dissipative diffeomorphism of the disc with zero entropy is

- either infinitely renormalizable,
- or generalized Morse-Smale.
c – Boundary of zero entropy. The set of $C^r$ diffeomorphisms, $r > 1$, with positive entropy is $C^1$ open (see [K]). One may thus consider how positive entropy appears: a diffeomorphism belongs to the boundary of zero entropy if its topological entropy vanishes, but it is the $C^1$ limit of diffeomorphisms with positive entropy. The previous results immediately give:

**Corollary 4.** A mildly dissipative diffeomorphism of the disc in the boundary of zero entropy is infinitely renormalizable.

We may ask if the converse also holds:

**Question 5.** In the space of mildly dissipative $C^r$ diffeomorphisms of the disc, $r > 1$, can one approximate any diffeomorphism exhibiting periodic orbits of arbitrary large period by diffeomorphisms with positive entropy?

This would imply that generalized Morse-Smale diffeomorphisms are the mildly dissipative diffeomorphisms of the disc with robustly vanishing entropy. Question 5 has a positive answer if one considers $C^1$-approximations of $C^2$-diffeomorphisms (this is essentially corollary 2 in [PS2]). In a similar spirit, it is unknown (even in the $C^1$-topology) if diffeomorphisms with zero entropy are limit of generalized Morse-Smale diffeomorphisms.

**Question 6.** In the space of mildly dissipative $C^r$ diffeomorphisms of the disc, $r > 1$, can one approximate any diffeomorphism with zero entropy by a generalized Morse-Smale diffeomorphism?\(^5\)

d – Decomposition of the dynamics with zero entropy. Let us recall that Conley’s theorem (see [R2, Chapter 9.1]) decomposes the dynamics of homeomorphisms: the chain-recurrent set splits into disjoint invariant compact sets called *chain-recurrence classes*. We now describe the dynamics inside the chain-recurrence classes of mildly dissipative diffeomorphisms with zero entropy.

Let $h$ be a homeomorphism of the Cantor set $\mathcal{K}$. One considers partitions of the form $\mathcal{K} = K \cup h(K) \cup \cdots \cup h^{p-1}(K)$ into clopen sets that are cyclically permuted by the dynamics. We say that $h$ is an *odometer* if there exist such partitions into clopen sets with arbitrarily small diameters. The set of the periods $p$ is a multiplicative semi-group which uniquely determines the odometer. Each odometer is minimal and preserves a unique probability measure (this allows to talk about almost every point of the odometer $x \in \mathcal{K}$). Figure 1 represents a diffeomorphism of the disc which induces an odometer on an invariant Cantor set.

**Corollary 7.** Let $f$ be a mildly dissipative diffeomorphism of the disc with zero entropy. Then any chain-recurrence class $\mathcal{C}$ of $f$ is:

- either periodic: there exists a compact connected set $C$ and an integer $n \geq 1$ such that $\mathcal{C} = C \cup \cdots \cup f^{n-1}(C)$ and any point in $K$ is fixed under $f^n$,

- or a generalized odometer: there exists an odometer $h$ on the Cantor set $\mathcal{K}$ and a continuous subjective map $\pi : \mathcal{C} \to \mathcal{K}$ such that $\pi \circ f = h \circ \pi$ on $\mathcal{K}$. Moreover almost every point $z \in \mathcal{K}$ has at most one preimage under $\pi$.

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\(^5\)For issues related to the two last questions in the context of interval maps, see e.g. [HT] and references therein.
In addition:

- Each generalized odometer is a quasi-attractor, i.e. admits a basis of open neighborhoods $U$ satisfying $f(U) \subset U$.

- The union of the generalized odometers is an invariant compact set $\Lambda$. Outside any neighborhood of $\Lambda$ the set of periods of the periodic orbits is finite.

Corollary 7 can be compared to a recent result by Le Calvez and Tal [LT] about transitive sets of homeomorphisms of the 2-sphere with zero entropy. The methods there are quite different to ours. Note that the dissipation hypothesis is essential: for conservative systems with zero entropy, the dynamics is modeled on integrable systems, see [FH].

We do not know if there exist examples of systems exhibiting generalized odometers which are not conjugated to odometers (i.e. such that the map $\pi$ is not injective). Another problem concerns the cardinality of these classes:

**Question 8.** Does there exist a mildly dissipative diffeomorphism of the disc with zero entropy and infinitely many generalized odometers?[6]

The answer to this question is not known for general one-dimensional $C^r$-endomorphism. However for multimodal endomorphisms of the interval, the nested sequences of infinitely renormalizable domains is bounded by the number of critical points. In particular, generically the number of nested renormalizable domains is finite. This type of result is not known for surface diffeomorphisms.

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**e – Periods of renormalizable domains.** For one-dimensional multimodal maps with zero entropy, Sharkovskii’s theorem [Sha] implies that the period of the renormalizable domains are powers of 2. In the context of mildly dissipative diffeomorphisms this cannot be true, but a similar result holds when one considers renormalizable domains with “large period”:

[A degenerate $C^1$ example can be extracted from the Denjoy-like example in [BGLT].]
**Theorem B.** Let $f$ be a mildly dissipative diffeomorphism of $\mathbb{D}$ with zero topological entropy and infinitely renormalizable. There exist an open set $W$ and $m \geq 1$ such that:

- $W$ is a finite disjoint union of topological discs that are trapped by $f^m$,
- the periodic points in $\mathbb{D} \setminus W$ have period bounded by $m$,
- any renormalizable domain $D \subset W$ of $f^m$ has period of the form $2^k$; $D$ is associated to a sequence of renormalizable domains $D = D_k \subset \cdots \subset D_1 \subset W$ of $f^m$ with period $2^k$, ..., $2$. 

In other words, the period of a renormalizable domain is eventually a power of $2$, meaning that, after replacing $f$ by an iterate, the period of all the renormalizable domains are powers of $2$. As explained in the paragraph *summary of the proof* below, the proof of theorem B uses some rigidity argument.

This implies an analogue of Sharkovskii’s theorem for surface diffeomorphisms:

**Corollary 9.** Let $f$ be a mildly dissipative diffeomorphism of the disc with zero topological entropy. There exist two finite families of integers $\{n_1, \ldots, n_k\}_{k \geq 1}$ and $\{m_1, \ldots, m_\ell\}_{\ell \geq 0}$ such that the set of periods of the periodic orbits of $f$ coincides with

$$\text{Per}(f) = \{n_1, \ldots, n_k\} \cup \{m_i 2^j, 1 \leq i \leq \ell \text{ and } j \in \mathbb{N}\}.$$  \hspace{1cm} (1)

In particular, in the setting of mildly dissipative diffeomorphisms, we get an affirmative answer to the following conjecture that was formulated by one of us in 1983, and mentioned verbally since then, but appeared in a text (see [GT]) only a few years after.

**Conjecture 10 (Tresser).** In the space of $C^k$ orientation preserving embeddings of the 2-disk, with $k > 1$, which are area contracting, generically, maps which belong to the boundary of positive topological entropy have a set of periodic orbits which, except for a finite subset, is made of an infinite number of periodic orbits with periods, $m.2^k$ for a given $m$ and all $k \geq 0$.

We note that it is possible to realize any set of the form (1) as the set of periods a mildly dissipative diffeomorphism of the disc having zero entropy, whereas a diffeomorphism with positive entropy has a different set of periods (it always contains a set of the form $k.\mathbb{N}^*$). In a more general framework, theorem B is false if the dynamics is conservative (an integrable twist in the disc may admit all the periods and has vanishing entropy).

Previous works in the direction to develop a forcing theory as it follows from Sharkovskii’s theorem (see [GST]) used the ideas and language of braids. For surface diffeomorphisms, a periodic orbit defines a braid type that in turns can or not, force the positivity of topological entropy (the complement of an orbit of period three or larger in the disc can be equipped with a hyperbolic structure from where Nielsen-Thurston theory can be developed). In that sense, permutations are replaced by braids, but the discussion in braid terms cannot be reduced to a discussion in terms of periods as the conjecture formulates.

**f – Hénon family.** Given any $C^r$ endomorphisms $h$ of an interval $I \subset \mathbb{R}$ and $b_0 > 0$, there exists a disc $\mathbb{D} = I \times (-\epsilon, \epsilon)$ such that the maps defined by

$$f_0(x, y) = (h(x) + y, -bx), \quad \text{for } 0 < |b| < b_0$$  \hspace{1cm} (2)
are dissipative diffeomorphisms of $\mathbb{D}$. The (real) Hénon family is a particular case where $h$ is a quadratic polynomial. As mentioned before, the Hénon family is mildly dissipative for $0 < |b| < 1/4$ in restriction to a trapped disc. Therefore, all the theorems mentioned above can be applied to the these parameters of the Hénon family and in particular one gets the following corollary. Note that the global dynamical descriptions of the Hénon family usually suppose $|b| \ll 1$ (see [BeCa, dCLM, LM]).

**Corollary 11.** Let $f_{b,c} : (x, y) \mapsto (x^2 + c + y, -bx)$ be a Hénon map with $b \in (0, 1/4)$ and $c \in \mathbb{R}$. If the topological entropy vanishes, then:

- for any forward (resp. backward) orbit one of the following cases occurs:
  1. it escapes at infinity, i.e. it leaves any compact set,
  2. it converges to a periodic orbit,
  3. it accumulates to (a subset of) a generalized odometer;
- the set of periods has the form described in (1).

**g – Small Jacobian.** For diffeomorphisms of the disc close enough to an endomorphism of the interval and whose entropy vanishes, section [14] proves that the periods of all renormalizable domains (and so the periods of all periodic orbits) are powers of two.

More precisely, given a $C^r$ endomorphism of the interval $f_0$, there exists $b_0 > 0$ such that for any $0 < |b| < b_0$ the diffeomorphism $f_b$ is mildly dissipative. In particular all the theorems mentioned before can be applied. Assuming the Jacobian sufficiently small, a stronger property holds:

**Theorem C.** Given a family $(f_b)$ associated to a $C^2$ endomorphism of the interval as in (2), there exists $b_0 > 0$ such that, for any $b \in (0, b_0)$ and for any diffeomorphism $g$ with zero entropy in a $C^2$-neighborhood of $f_b$, there exists $n_0 \in \mathbb{N} \cup \{\infty\}$ satisfying

$$\text{Per}(g) = \{2^n, n < n_0\}. \quad (3)$$

In particular, the previous theorem can be applied to the Hénon family and one recovers one of the results in [dCLM, LM].

**h – Some differences with the one-dimensional approach.** In the context of one-dimensional dynamics of the interval, and in particular for unimodal maps, the renormalization intervals are built using the dynamics around the turning point: the boundary of the interval contains the closest iterate to the turning point of a repelling orbit (whose period is a power of two) and a preimage of that iterate. For Hénon maps with small Jacobian, although there is no notion of turning point, renormalization

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7After studying the Lorenz model for large values of the “Rayleigh number” $r$ on the advice of David Ruelle, Yves Pomeau presented this joint work at the observatory of Nice where Michel Hénon was working. He showed in particular that the time-$t$ map, for $t$ varying from 0 to 1, transforms a well chosen rectangle to an incomplete horseshoe. That night, the legend tells, Hénon extracted a model of that from his former studies of the conservative case while Pomeau and Ibanez had preferred to focus to a full double covering for which the mathematics are much simpler. “The most recognition for the least work” Hénon told to Tresser. Later, Coullet and Tresser realized that the Hénon map appears to be in the same universality class for period doubling than the one-dimensional quadratic map; this led them to conjecture in 1976 (see [CT]) that universal period doubling should be observed in fluids, since Hénon map was built to imitate a Poincaré map of the Lorenz flow in some parameters ranges, and the quadratic map is the limit as the dissipation goes to infinity, of the Hénon map.
domains are built in [dCLM, LM] by using the local stable manifold of a saddle periodic point of index 1 and its preimages (those points are the analytic continuations of the repelling points of the one-dimensional map).

Our approach can not rely neither of the notion of turning point neither on being close to well understood one-dimensional dynamics. So, the construction is different and uses the structure of the set of periodic points. Following the unstable branches, it is built a skeleton of the dynamics that allows to construct the trapping regions and the renormalization domains.

i – The renormalization operator. In [dCLM, LM] is proved that infinitely renormalizable real Hénon-like maps whose Jacobian is small enough admit an appropriately defined renormalization operator. After proper affine rescaling, the dynamics (at the period) on the renormalizable attracting domain converge to a smooth quadratic unimodal map which is nothing else that the hyperbolic fixed point of the renormalization operator for the one-dimensional dynamics.

It is not difficult to construct mildly dissipative diffeomorphisms with zero entropy which are a priori not close to a unimodal map on the interval (for instance, when the first renormalization domain has period larger than two) and in that case the renormalization scheme developed for Hénon-like maps with small Jacobian would need to be recasted. Although the present paper does not provide a well-defined renormalization operator for mildly dissipative diffeomorphism of the disk, it gives the existence of nested renormalization domains and deep renormalizations seem to drive the system towards the one-dimensional model. Indeed the renormalization domains eventually have (relative) period two; moreover the return dynamics on these domains recover certain smooth properties that are satisfied by diffeomorphism close to the one-dimensional endomorphisms (see section 3.2): [CP] associates a quotient dynamics which, on these “deep domains”, induces an endomorphism of a real tree. That raises the following question:

**Question 12.** Given a sequence of nested renormalizable domains, is it true that (after proper rescalings) the sequence of return maps generically converges to an unimodal map? One does not expect to replace “generic” by “general” because of the expected possible alternate convergence of the renormalizations to more that one fixed point: this happens in dimension 1, see e.g. [MT] and also [OET].

When \( f \) is mildly dissipative, the larger Lyapunov exponent of each generalized odometer \( \mathcal{C} \) vanishes, hence the iterates of the derivative of \( f \) on \( \mathcal{C} \) do not grow exponentially; but one can ask if a stronger property holds: given a nested sequence of renormalization domains \( (D_n) \) and their induced maps \( (f_n) \), are the derivatives \( \|Df_n|_{D_n}\| \) uniformly bounded?

j – New general tools. Few new results obtained in the present paper hold for any mildly dissipative diffeomorphism of the disk.

**Closing lemma.** One of them is a new version of the closing lemma proved in [CP] which states that for mildly dissipative diffeomorphisms of the disk, the support of any measure is contained in the closure of periodic points. Our improvement (theorem \([F']\) localizes the periodic points: given an invariant cellular connected compact set \( \Lambda \), the support of any invariant probability on that set is contained in the closure of the periodic points in \( \Lambda \). In that sense, theorem \([F']\) is a extension of a well
known result by Cartwright and Littlewood about the existence of fixed points for invariant cellular sets (see proposition 21).

No cycle. Another one is a generalization of the result proved by Pixton [P] (improving a previous work by Robinson [R1]: it states that for \( C^\infty \)-generic diffeomorphisms of the sphere, a cyclic accumulation between stable and unstable branches of periodic points can be perturbed to produce a homoclinic connection and positive entropy. Theorem [G] shows that the generic hypothesis is not needed for mildly dissipative diffeomorphisms of the disc: there is no finite sequence of fixed points such that the unstable manifold of each one accumulates on the next point and the unstable manifold of the last one accumulates on the first point (theorems [G] and [G'] in section 5). This is clear when the intersections between unstable and stable manifolds are transversal but when they just accumulate, it is more difficult. The strategy consists in building special Jordan domains (that we call Pixton discs) from the accumulation of unstable branches on stable manifolds.

**k – Summary of the proof.** In order to present the envisioned proof strategy, we first present a class of examples of infinitely renormalizable dissipative homeomorphisms of the disc (inspired by the examples in [GvST]) and we explain their main dynamical features. We use them as a prototype model for maps with zero entropy. The proofs below will show that these features (essentially) apply also for infinitely renormalizable mildly dissipative diffeomorphisms.

**Prototype models.** Let \( f_0, f_1 \) be two Morse-Smale dissipative diffeomorphisms of the disc. The limit set of \( f_0 \) is given by a fixed saddle whose unstable branches are interchanged and an attracting orbit of period two that revolves around the fixed point: the fixed point is then said to be stabilized and the attracting orbit is analogous to a period doubling sink for interval maps. The limit set of \( f_1 \) is given by a fixed attracting periodic point, a saddle of period three (also said to be stabilized) that revolves around the fixed point which anchors one of the unstable branch of the saddle periodic points, and an attracting periodic orbit (also of period three) that attracts the other unstable branch of the saddles. Both diffeomorphisms are depicted in figure 2. Observe that \( f_0 \) has an attracting disc of period 2, whose iterates belong to two different regions bounded by the
local stable manifold of the saddle; $f_1$ has an attracting disc of period three contained inside the disjoint regions bounded by the local stable manifolds of the saddle of period three (these regions, in both cases, are called decorated regions).

Given a sequence $(k_i) \in \{0, 1\}^\mathbb{N}$, one can build a sequence of dissipative diffeomorphisms $g_i = f_{k_i} \uplus f_{k_{i-1}} \uplus \cdots \uplus f_{k_0}$ with a sink of period $\tau_i := \prod_{j=1}^{\tau_i}(2 + k_j)$. The symbol $\uplus$ means that the diffeomorphism $f_{k_j}$ is pasted in the basin of the sink of $g_{j-1}$ (by writing $f_{k_j}$ as the composition of $\tau_{j-1}$ diffeomorphisms). In that way, $g_i$ has a nested sequence of attracting discs $D_0 \supset D_1 \supset \cdots \supset D_j$ of periods $\tau_0, \ldots, \tau_i$. Each diffeomorphism $g_i$ is Morse-Smale and the sequence $(g_i)$ converges to a homeomorphism whose limit set is made of periodic points and of an odometer supported on a Cantor set (the intersection of the nested sequence of attracting domain). We make some remarks: (i) The construction shows that there exist diffeomorphisms with vanishing entropy and with periodic points whose period is not $2^n$. (ii) The sequence can converge to a mildly dissipative diffeomorphism if $k_i = 0$ for $i$ large (the convergence towards a diffeomorphism is more difficult, see [GvST]). (iii) The previous construction can be performed with more pasted diffeomorphisms: the period of the saddle and the non-fixed sink may be larger; one can also consider more complicate Morse-Smale systems.

Pixton discs. The unstable branches connect the periodic points of $g_i$ and form a chain with a tree structure, see figure 3. The tree branches land at points that are:

- either attracting and may anchor unstable manifolds of points of larger period,
- or saddles whose unstable branches are exchanged at the period.

![Figure 3: Chain of periodic points associated to $f_1 \uplus f_0 \uplus f_0$; there are one saddle fixed point, a saddle of period two (at the period its unstable branches are exchanged), a sink of period four, a saddle of period twelve, and a sink of period twelve. The arrows indicate if the periodic points are saddles or sinks (on the one-dimensional structure a saddle appears as a sink).](image)

That observation will allows to reconstruct the attracting discs, see figure 4. In the first case (left of the figure), the unstable manifold of a fixed point $p$ accumulates on a fixed sink which anchors a stabilized revolving saddle with larger period: the unstable branch of $p$ has to cross the stable manifolds of the iterates of the saddle; this defines an attracting disc which contains all the periodic points attached to the sink. In the second case (right of the figure), the unstable manifold of the fixed point $p$ accumulates on a fixed saddle whose unstable branches are exchanged by the dynamics and accumulate on a sink of period 2: the unstable branch of $p$ has to cross the stable manifold of the
fixed saddle; this also defines an attracting disc which contains all the attached periodic points. We call the domains built in this way, \textit{Pixton discs}.

When all the periodic points are fixed. We now explain how to handle a general mildly dissipative diffeomorphism with zero entropy. In order to prove theorem A, one first has to show that if all the periodic points are fixed, then the limit set of the dynamics consists of only fixed points. The “no-cycle property” is crucial. Another ingredient is to prove that the $\omega$–limit set of any orbit contains a fixed point: this follows from our closing lemma (theorem F'). With these tools, one builds a filtration associated to the fixed points and conclude that the limit set of the dynamics is reduced to the set of fixed points.

Periodic structure. When there are periodic points which are not fixed, we prove that the unstable branches induce a structure as in the previous examples: they form \textit{chains} (see definition 56) that branch at points of low period to which are attached saddles of larger or equal period. A special role is played by \textit{stabilized points}: these are saddles that either are fixed and whose unstable branches are exchanged, or are not fixed but whose unstable manifold is anchored by a fixed point (see definition 50 and propositions 57 and 52). The local stable manifolds of the stabilized points bound domains called \textit{decorated regions} (see definition 53) which are two by two disjoints: indeed if two such regions intersect, the unstable manifold of a stabilized point has to cross the stable manifold of another iterate in order to accumulate on the anchoring fixed point, contradicting the fact that the entropy vanishes. The decorating regions contain all the periodic point of larger period (see definition 64, proposition 70 and 73).

Construction of trapping discs. To each unstable branch $\Gamma$, fixed by an iterate $f^n$, we build a disc that is trapped by $f^n$ and contain all the accumulation set of the branch $\Gamma$ (theorems H and H'). To each saddle accumulated by $\Gamma$ one associates a \textit{Pixton disc} which is a candidate to be trapped. These discs are bounded by arcs in $\Gamma$ and stable manifolds of saddles in the accumulation set, as in the previous examples (see lemma 81). A finite number of these Pixton discs is enough to cover the accumulation set, implying the trapping property. The closing lemma mentioned above (theorem F') is a key point for proving the finiteness.

Finiteness of the renormalization domains. A stronger version of the renormalization (theorem A') implies corollaries 4 and 7. It asserts that the number of renormalization domains required to cover the dynamics is finite. Since the renormalization discs are

Figure 4: Attracting discs obtained from an unstable branch and stable manifolds.
related to decorated regions, we have to show that the periods of the stabilized saddles is bounded (see theorem 1).

**Bound on the renormalization period.** For showing that after several renormalization steps, the renormalization periods eventually equal two (theorem B), we develop a rigidity argument: the limit attractors (the generalized odometers obtained as intersection of nested renormalizable domains of an infinitely renormalizable diffeomorphism) induce a stable lamination whose leaves vary continuously for the $C^1$-topology over sets with measure arbitrary close to one. This property follows from a $\gamma$-dissipation property (see section 3.1). In particular, for a large proportion of points, the leaves of the lamination by local stable manifolds are “parallel”. Since the renormalization domains (inside a renormalization disc obtained previously) are contained in a (relative) decorated regions, and since the measure is equidistributed between the different renormalization components, a relative renormalization period larger than two would contradict that a large proportion of local stable manifolds are parallel. A simple heuristic of that argument is the following: at small scale, the quotient by the local stable manifolds provides an interval that contains a large proportion of the points of the odometer and that is enough to recover the period doubling mantra that permeates the renormalization scheme for zero entropy maps of the interval.

l – Other attracting domains. One can wonder about the transition to chaos for dissipative diffeomorphisms on others attracting domains as it is the case of the annulus. The transition to chaos is already much more complicated on the circle than on the interval (see e.g., FT and references therein), as a result in particular of the non-triviality of the circle at the homotopy level. A prototype family that plays the role of the Hénon maps for the circle, is an annulus version of the Arnold family. Results related to the transition to chaos in that context can be found in [CKKP] and [GY].

m – Organization of the paper. The next three sections present preliminary results: section 2 describes how fixed points may be rearranged inside finitely many fixed curves, and recalls the Lefschetz formula and a fixed point criterion due to Cartwright and Littlewood; in section 3 we revisit the notion of $\gamma$-dissipation introduced in [CP] and we present a few results that allow to improve the lower bound on $\gamma$; in section 4 we state a new closing lemma.

Section 5 proves that (under the hypothesis of zero entropy and mild dissipation) there is no cycle between periodic points. This is essential to show in section 7 that periodic points are organized in chains; also in that section we introduce the notions of decoration and stabilization that provides a hierarchical organization of the chains. Section 6 discusses the notion of generalized Morse-Smale diffeomorphisms.

In section 8 we prove that the accumulation set of an unstable branch of a fixed point is contained in an arbitrarily small attracting domain and in section 9 we conclude the proof of the local renormalization (theorem A). A global version of that theorem (theorem A') is obtained in section 11; this requires to first show that the periods of the stabilized points are bounded (this is proved in section 10).

The proof of theorem B is provided in section 13; this uses the description of the chain-recurrent set (corollary 7) which is proved in section 12.

In the last two sections, we prove the results about dynamics close to interval maps and about the Hénon maps (corollary 11 and theorem C).
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2 Periodic orbits

In subsection 2.1, we analyze the different types of periodic points that could exist for a dissipative diffeomorphism of the disc. When there exist infinitely many periodic points of a given period, we rearrange them inside finitely many periodic arcs. In subsection 2.3 we recall the Lefschetz formula. In subsection 2.4 we present a kind of topological \( \lambda \)-lemma that is useful to describe the accumulation set of unstable manifolds of periodic points. In section 2.6 we recall a classical result by Cartwright and Littlewood about the existence of fixed points.

2.1 Dynamics near a periodic point

We describe the dynamics in the neighborhood of a periodic orbit. Note that up to replace \( f \) by an iterate, one may reduce to consider fixed points. When \( p \) is fixed, the eigenvalues \( \lambda_p^-, \lambda_p^+ \) of \( D_p f \) verify \( |\lambda_p^-| \leq |\lambda_p^+| \) and \( |\lambda_p^- \lambda_p^+| < 1 \).

Hyperbolic sink. When \( |\lambda_p^+| < 1 \), the point \( p \) is a hyperbolic sink. This covers in particular all the cases where \( |\lambda_p^-| = |\lambda_p^+| \). We now describe the other cases.

Stable curve. When \( |\lambda_p^-| < |\lambda_p^+| \), there exists a well defined (strong) stable manifold which is a \( C^1 \) curve. The connected component containing \( p \) is denoted by \( W_s^D(p) \). For orbits with higher period \( O \), we denote by \( W_s^D(O) \) the union of the curves \( W_s^D(p), p \in O \).

Local stable set. The local stable set of \( p \), i.e. the set of points whose forward orbit converges to \( p \) and remains in a small neighborhood of \( p \), is either a neighborhood of \( p \) (a sink), a subset of \( W_s^D(p) \), or a half neighborhood of \( p \) bounded by \( W_s^D(p) \).

Center manifold. When \( |\lambda_p^-| < |\lambda_p^+| \), the center manifold theorem asserts that there exists a \( C^1 \) curve \( \gamma \) which contains \( p \), is tangent to the eigendirection of \( D_p f \) associated to \( \lambda_p^+ \) and is locally invariant: there exists \( \varepsilon > 0 \) such that \( f(\gamma \cap B(p, \varepsilon)) \subseteq \gamma \). The two components of \( \gamma \setminus \{p\} \) are either preserved or exchanged (depending if the eigenvalue \( \lambda_p^+ \) is positive or negative). Along each component \( \Gamma \) of \( \gamma \setminus \{p\} \), the dynamics (under \( f \) or \( f^2 \)) is either attracting, repelling, or neutral (in which case \( p \) is accumulated by periodic points inside \( \Gamma \)). The type of dynamics does not depend on the choice of \( \gamma \).

Unstable branches. The unstable set \( W^u(p) \) of \( p \) is the set of points \( x \) such that the distance \( d(f^{-n}(x), f^{-n}(p)) \) decreases to 0 as \( n \to +\infty \). When it is not reduced to \( p \), it is a \( C^1 \) curve which contains \( p \). The local unstable set is defined as the set of points whose backward orbit converges to \( p \) and remains in a small neighborhood of \( p \) and observe that they are contained in the center manifold \( \gamma \). Each connected component \( \Gamma \) of \( W^u(p) \setminus \{p\} \) is called an unstable branch of \( p \).

Hyperbolic saddle. When \( |\lambda_p^-| < 1 < |\lambda_p^+| \), the point \( p \) is a hyperbolic saddle. It admits two unstable branches.
Indifferent fixed point. When $|\lambda_p^-| < 1 = |\lambda_p^+|$, the point $p$ is indifferent. We then consider the dynamics (under $f$ or $f^2$) on each side of a center manifold. When $p$ is isolated among points of period 1 and 2, it is either a sink (both components are attracting), a saddle (both components are repelling) or a saddle-node (the components are fixed, one is attracting, one is repelling): the type does not depend on the choice of the center curve $\gamma$.

Saddle with reflexion. When the unstable branches of a fixed saddle $p$ are exchanged by the dynamics, we say that $p$ is a (fixed) saddle with reflexion.

Index. For an isolated fixed point, one can define the index of that fixed point as the winding number of the vector field $f(x) - x$ around the fixed point. For dissipative diffeomorphisms the index of an isolated fixed point is:

- 1 for a sink or a saddle with reflexion,
- 0 for a saddle-node,
- $-1$ for a saddle with no reflexion.

By the classical Lefschetz formula, when the number of fixed points is finite the sum of the index of the fixed points in the disc is equal to 1.

Remark 13. A saddle-node can be considered as the degenerated case of a sink and a saddle of index $-1$ that have collided. Similarly, a fixed point with an eigenvalue less or equal to $-1$ can be considered as the collision of a fixed sink with the points of a 2-periodic orbit with positive eigenvalues. In particular, fixed saddles of index 1 may be considered as the union of a fixed sink with a saddle of period 2.

2.2 Normally hyperbolic periodic arcs

When the number of fixed points is infinite, they appear inside normally hyperbolic arcs.

Definition 14. A fixed arc is a compact $f$-invariant $C^1$ curve $I$ whose endpoints are fixed and which admits an invariant splitting $T_x\mathcal{D}|_{x \in I} = E^s \oplus F$ satisfying:

- $T_x I \subset F_x$ for each $x \in I$,
- there is $k \geq 1$ such that $|Df^k_{E^s_x}| < |Df^k_{F_x}|$ and $|Df^k_{E^s_x}| < 1$ for each $x \in I$,

It is isolated if all the fixed points in a neighborhood are contained in $I$.

A fixed point is a fixed arc: for a hyperbolic sink, the splitting is trivial $F = \{0\}$. When $I$ has two distinct endpoints $p_1, p_2$, the forward orbit of any point in the strip $W^u_\mathcal{D}(I)$ bounded by $W^u_\mathcal{D}(p_1)$ and $W^u_\mathcal{D}(p_2)$ converges to a fixed point in $I$. When $I$ is not reduced to a sink, $\mathcal{D} \setminus W^s_\mathcal{D}(I)$ has two connected components.

The unstable set of $I$ is contained in the unstable branches of the endpoints of $I$.

Definition 15. Four cases may occur for an isolated fixed arc $I$. It has the type of:

- a sink, if the orbit of any point in a neighborhood converges to a fixed point in $I$,
- a saddle with reflexion, if $I$ is a single fixed point $p$ with an eigenvalue $\lambda_p^+ \leq -1$, 


- a saddle-node, if the arc has one \( f \)-invariant unstable branches,
- a saddle with no reflexion, if the arc has two \( f \)-invariant unstable branches.

**Remark 16.** Note that if an isolated fixed arc \( I \) contains a fixed point \( p \) with an eigenvalue less or equal to \(-1\), then \( I = \{p\} \) (since the endpoints of \( I \) are fixed points). This is the only case where there may exist periodic orbits in arbitrarily small neighborhoods of \( I \). The arc \( I \) is isolated since \( p \) may be accumulated only by points with period 2.

**Proposition 17.** If \( f \) is a dissipative diffeomorphism of the disc, there exists a finite collection \( \mathcal{I} \) of disjoint isolated fixed arcs whose union contains all the fixed points of \( f \).

**Proof.** By the implicit function theorem, the set of fixed points of \( f \) is the union of a finite set of isolated fixed points and of a compact set \( K \) of fixed points \( p \) having one eigenvalue \( \lambda_p^+ \geq 1/2 \). Each isolated fixed point is an isolated fixed arc and it remains to cover \( K \) by finitely many pairwise disjoint isolated fixed arcs.

To each fixed point \( p \) having an eigenvalue \( \lambda_p^+ \geq 1/2 \), the center manifold theorem (see [BoCr]) associates a \( C^1 \) curve \( \gamma \) which contains \( p \), is tangent to the eigenspace \( F_p \) associated to the eigenvalue \( \lambda_p^+ \), and is locally invariant: \( f(\gamma) \cap \gamma \) contains a neighborhood of \( p \) in \( \gamma \); moreover, any periodic point in a neighborhood of \( p \) is contained in \( \gamma \). One can thus build an arc \( I \subset \gamma \) bounded by two fixed points, which is invariant by \( f \), normally contracted and which contains all the fixed points in a neighborhood of \( p \): it is a fixed arc, as in definition 14.

By compactness, there exists a finite family of such fixed arcs. Let us choose \( \varepsilon > 0 \) small. By decomposing the arcs, one can assume that each such arc \( I \) has diameter smaller than \( \varepsilon \), is contained in a \( C^1 \) curve \( J \) such that \( J \setminus I \) has two connected components, both of diameter larger than \( 2\varepsilon \) and such that any fixed point in the \( 2\varepsilon \)-neighborhood of \( I \) is contained in \( J \).

If there exists two arcs \( I, I' \) which intersect, one consider the larger curve associated to \( J \). We note that all the fixed points of \( I' \) are contained in \( J \). One can thus reduce \( I' \) as an arc \( \bar{I} \) such that all the fixed points of \( K \cap I' \cup I \) are contained in \( I \cup \bar{I} \) and \( I \cup \bar{I} \) is a \( C^1 \) curve. One repeat this argument for all pairs of fixed intervals. This ensures that the union of all the fixed intervals \( I \) contains \( K \) and is a union of disjoint \( C^1 \) curves. By construction, each of these curves is an isolated fixed arc.

The choice of the collection \( \mathcal{I} \) is in general not unique. Note that for any distinct \( I, I' \in \mathcal{I} \) which are not sinks, the strips \( W^s_D(I), W^s_D(I') \) are disjoint.

**Partial order:** The finite collection of fixed arcs \( \mathcal{I} \) can be partially ordered in such a way that at least one the unstable branches of the extremal points of \( I \) accumulate on \( I_{j+1} \).

### 2.3 Lefschetz formula and arcs

In the present subsection we recall the definition of index for isolated invariant arcs and “half” arcs.
Index of an arc. To any simple closed curve $\sigma \subset \mathbb{D} \setminus \text{Fix}(f)$, one associates an index $i(\sigma, f)$, which is the winding number of the vector field $f(x) - x$ along the curve. This defines for any isolated fixed arc $I$ an index $\text{index}(I, f)$: this is the index $i(\sigma, f)$ associated to any simple closed curve contained in a small neighborhood of $I$ and surrounding $I$. For arcs as described in definition 15, the index takes a value in $\{-1, 0, 1\}$, equal to 1 for a sink and a saddle with reflexion, 0 for a saddle-node, and $-1$ for a saddle with no reflexion. In particular, the index is 1 exactly when the arc has no $f$-invariant unstable branch. When $I$ is reduced to an isolated fixed point, $\text{index}(I, f)$ coincides with the usual index.

Index of a half arc. Let us consider a fixed arc $I$ which contains a fixed point $p$ having an eigenvalue $\lambda_p^+ \geq 1$ and a connected component $V$ of $\mathbb{D} \setminus W^u_0(p)$. Let us assume that $I$ is isolated in $V$, i.e. that any fixed point in a neighborhood of $I$ that belongs to $V$, also belongs to $I$. Then, one can associate an index $\text{index}(I, V, f)$: it has the value $-1/2$ if $I$ has a (local) unstable branch in $V$ that is fixed by $f$ and the value $1/2$ otherwise (in which case $I$ is semi-attracting in $V$). When $I$ is isolated, the index of $I$ is equal to the sum of the two indices associated to the two connected components of $\mathbb{D} \setminus W^u_0(p)$.

The next proposition restates the Lefschetz formula in our setting.

**Proposition 18.** Let $f$ be a dissipative diffeomorphism of the disc and $\mathcal{I}$ a set of isolated fixed arcs as in proposition 17. Then the sum of the indices $\text{index}(I, f)$ of the arcs $I \in \mathcal{I}$ is 1.

**Proof.** One can modify the dynamics inside each isolated fixed arc $I \in \mathcal{I}$ and obtain in this way a diffeomorphism $g$ satisfying:

- each $I \in \mathcal{I}$ is still a fixed arc for $g$,
- the fixed points of $g$ are all hyperbolic and contained in the union of the arcs $I$.

In particular $g$ has only finitely many fixed points. In an arc $I$, two saddles are separated by a sink. This proves that the index of any arc $I \in \mathcal{I}$ for $f$ coincides with the sum of the indices of the fixed points of $g$ that are contained in $I$.

Consequently, the sum of the indices of the arcs $I \in \mathcal{I}$ for $f$ is equal to the sum of the indices of the fixed points of $g$. From [13] proposition VII.6.6], this sums equal 1 since $g$ is a map homotopic to a constant. 

2.4 The accumulation set of unstable branches

Let $\Gamma$ be a $f$-invariant unstable branch of a fixed point $p$ and $\gamma \subset \Gamma$ be a curve which is a fundamental domain. The accumulation set of $\Gamma$ is the limit set of the iterates of $f^n(\gamma)$ as $n \to +\infty$. We say that $\Gamma$ accumulates on a set $X$ if $X$ intersects the accumulation set of $\Gamma$. These definitions naturally extend to unstable branches of periodic points. Next proposition, is a kind of topological version of the classical $\lambda$–lemma for mildly dissipative diffeomorphisms without assuming homoclinic intersections.

**Proposition 19.** Let $p, q$ be two fixed points of a mildly dissipative diffeomorphisms and let $\Gamma_p, \Gamma_q$ be two $f$-invariant unstable branches such that $\Gamma_p$ accumulates on a point of $\Gamma_q$. Then the accumulation set of $\Gamma_q$ is included in the accumulation set of $\Gamma_p$. 

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Proof. Let $U$ be a small simple connected neighborhood of $q$. There are points $y_k \in \Gamma_p$ arbitrarily close to a point $y \in \Gamma_q \cap U$ having iterates $f^{-1}(y_k), f^{-2}(y_k), \ldots, f^{-m_k}(y_k)$ in $U$ such that $f^{-m_k}(y_k)$ converge to a point $x \in U \cap W^s_0(q)$. Let $\gamma^s \subset U \cap W^s_0(q) \setminus \{q\}$ be a compact curve containing $x$ and such that both connected components of $\gamma^s \setminus \{x\}$ properly contain a fundamental domain and its iterate. Let also $\gamma^u \subset \Gamma^u(q)$ be a compact curve which properly contains a fundamental domain and its iterate. Now we take two curves $l_1^u, l_2^u$ transversal to $W^u_0(p)$ through the extremal points of $\gamma^s$ and two curves $l_1^s, l_2^s$ transversal to $\Gamma_q$ through the extremal points of $\gamma^u$. We construct the rectangles $R_n$ bounded by $l_1^u, l_2^u$ and the connected components of $f^{-n}(l_1^u), f^{-n}(l_2^u)$ inside $U \cap f^{-1}(U) \cap \cdots \cap f^{-n}(U)$ that intersect $l_1^u$ and $l_2^u$. Observe that those rectangles converges to $\gamma^s$.

Let $y_n \in W^u(p) \cap R_n$ converging to $x$. Let $l_n$ be a connected arc inside $W^u(p)$ that joins $y_n$ and $y_{n+2}$. It follows that either

1. there is a connected subsegment $l_n'$ inside $l_n \cap (R_n \cup R_{n+1} \cup R_{n+2} \cup \ldots)$ that contains $y_{n+2}$ and intersects either $l_1^u$ or $l_2^u$, or
2. there is a subsegment $l_n'$ inside $l_n$ that crosses $R_{n+1}$ and is disjoint from $l_1^u \cup l_2^u$.

In the first case, the accumulation set of $\Gamma_p$ contains a fundamental domain of a stable branch of $q$: this is a contradiction since each stable branch of $q$ contains a point in $D \setminus f(D)$. In the second case, $f^n(l_n')$ converge to $\gamma^u$ in the Hausdorff topology and from there, the proposition is concluded.

2.5 Decoration

The geometry described in the next definition is essential in this work.

Definition 20. Let $f$ be a mildly dissipative diffeomorphism of the disc. A periodic orbit $O$ which is not a sink is decorated if for each $p \in O$, one connected component of $D \setminus W^s_0(p)$ does not intersects $O$ (see figure 5).

2.6 A fixed point criterion

The following result refines Brouwer fixed point theorem inside the disc.

Figure 5: A decorated periodic orbit.
3.1 Criterion for $\gamma$-dissipation

In subsection 3.2, it is presented that for all ergodic measure as it is depicted in subsection 3.1.

The next proposition provides sufficient conditions for $\gamma$-dissipation. Observe that the hypothesis are satisfied by odometers.

**Proposition 23.** Let $f$ be a dissipative $C^r$ diffeomorphism, $r > 1$. Let $K$ be an invariant compact set which does not contain any periodic point and is uniquely ergodic and does not intersect any transitive compact set with positive entropy. Then $f$ is $\gamma$-dissipative on $K$ for all $\gamma \in (0,1)$.

**Proof.** We first claim that if a $C^r$ diffeomorphism $f$ of a surface (with $r > 1$) admits a hyperbolic ergodic measure $\mu$ with no atom, then $\text{supp}(\mu)$ is contained in a transitive set with positive topological entropy. A theorem of Katok [Ka]

A theorem of Katok [Ka] asserts the existence of periodic points $p_n$ whose orbits equidistribute towards $\mu$. Note that the points are distinct and their period goes to $+\infty$ since $\mu$ has no atom. Moreover, from the proof one can check that all the points $p_n$ belong to a compact set of points $p$ having uniform local stable and local unstable manifolds which vary continuously with $p$ (and this is not necessarily true for all the iterates of the points $p_n$). One may thus find two distinct points $p_n$ and $p_m$ close, so that their stable and unstable manifolds have transverse intersections. This implies that $f$ has a horseshoe $\Lambda$. Taking $n$ larger, one gets an increasing sequence of horseshoes whose union accumulates on $\text{supp}(\mu)$.

We now turn to the proof itself. Since $f$ is dissipative, there exists $b \in (0,1)$ such that $|\text{det} \, Df(x)| < b$ for any $x \in K$. Let us fix $\gamma \in (0,1)$ and $\varepsilon \in (0,-\frac{1}{4}(1-\gamma),\log b)$.

Let $\mu$ be the ergodic probability on $K$. Note that its upper Lyapunov exponent is non positive: otherwise, $\mu$ would be hyperbolic with no atom and the claim above would imply that $K$ intersects a transitive set with positive entropy, contradicting the assumptions of the proposition. Consequently, there exists $\ell \geq 1$ such that $1\int \log \| Df^{\ell} \| \, d\mu \leq \varepsilon/4$.

For $n \geq 1$ large enough and any $x \in K$, the distribution of the iterates $x, \ldots, f^n(x)$ is close to $\mu$, implying that for any $x, y \in K$,

$$\frac{1}{n} \log \| Df^n(x) \| \leq \varepsilon/4 + \frac{1}{n} \sum_{j=0}^{n-1} \frac{1}{\ell} \log \| Df^{\ell} (f^j(x)) \| \leq \frac{1}{2\varepsilon} + \frac{1}{\ell} \int \log \| Df^{\ell} \| \, d\mu \leq \frac{3}{4\varepsilon},$$

3 Quantitative dissipation

We recall a quantitative version of the dissipation that was introduced in [CP]:

**Definition 22.** Let $f$ be a dissipative diffeomorphism of the disc and $K$ be a $f$-invariant compact set. For $\gamma \in (0,1)$, we say that the diffeomorphism $f$ is $\gamma$-dissipative on $K$ if there is $n \geq 1$ such that for any $x, y \in K$ and any unit vector $u \in T_x \mathbb{D}$,

$$|\det Df^n(y)| < \|Df^n(x).u\|^{\gamma}.$$
For any \( x, y \in K \) and any unit vector \( u \in T_xD \), we thus have:

\[
e^{-\frac{\alpha}{2}} | \det D f^n(y) | \leq | \det D f^n(x) |.
\]

If \( u_0 \) is the unit vector in \( T_xD \) which is the most contracted under \( D f^n(x) \), we also have

\[
\| D f^n(x).u_0 \|^2 \leq | \det D f^n(x) | \leq b^n.
\]

Hence

\[
| \det D f^n(y) | \leq e^{n\frac{\alpha}{2}} \| D f^n(x).u_0 \| \leq e^{n\frac{\alpha}{2}} b^{n(1-\gamma)/2} \| D f^n(x).u_0 \|^\gamma \leq \| D f^n(x).u_0 \|^\gamma.
\]

This gives as required

\[
| \det D f^n(y) | < \| D f^n(x).u_0 \|^\gamma \leq \| D f^n(x).u \|^\gamma.
\]

So \( f^n \) is \( \gamma \)-dissipative on \( K \), provided \( n \) is chosen large enough.

### 3.2 Uniform geometry of the strong stable leaves

The next theorem has been essentially proved in \([CP]\). However, it has to be recasted to get a precise quantitative estimate.

**Theorem D.** For any \( \alpha \in (0, 1] \) and \( \varepsilon > 0 \), there exists \( \gamma \in (0, 1) \) with the following property. If \( f \) is a \( C^{1+\alpha} \) diffeomorphism which is \( \gamma \)-dissipative on an invariant compact set \( K \) which does not contain any sink, then there exists a compact set \( A \subset K \) such that:

- For any ergodic measure \( \mu \) supported on \( K \), we have \( \mu(A) > 1 - \varepsilon \).
- Each point \( x \in A \) has a stable manifold \( W^s_f(x) \) which varies continuously with \( x \) for the \( C^1 \) topology.

For the proof, we refer to \([CP]\) and the following slight changes in the results therein that are needed to be done. Let \( \tilde{\sigma}, \sigma, \tilde{\rho}, \rho \in (0, 1) \) satisfying

\[
\tilde{\sigma} > \sigma^\alpha, \quad (4)
\]

and \( A_{\tilde{\sigma}\tilde{\rho}}(f) \) be the set of points \( x \) having a direction \( E \subset T_xS \) such that for each \( n \geq 0 \)

\[
\tilde{\sigma}^n \leq \| D f^n(x)_E \| \leq \sigma^n, \quad \text{and} \quad \tilde{\rho}^n \leq \| D f^n(x)_E \|^2 \leq \rho^n. \quad (5)
\]

We recall theorem 5 and remark 2.1 in \([CP]\):

**Theorem E** (Stable manifold at non-uniformly hyperbolic points). Consider a \( C^{1+\alpha} \) diffeomorphism \( f \) with \( \alpha \in (0, 1] \). Provided \( (4) \) holds, the points in \( A_{\tilde{\sigma}\tilde{\rho}}(f) \) have a one-dimensional stable manifold which varies continuously for the \( C^1 \) topology with \( x \in S \).

To conclude theorem \([D]\) observe that it is enough to prove following proposition:
Proposition 24. Given \( \varepsilon > 0 \) and \( \alpha \in (0,1] \), there is \( \gamma > 0 \) with the following property.

Let us consider a diffeomorphism \( f \) and an invariant compact set \( K \) which does not contain any sink and where \( f \) is \( \gamma \)-dissipative. Then there exist \( \tilde{\sigma}, \sigma, \tilde{\rho}, \rho \in (0,1) \) satisfying \([4]\) such that for any ergodic measure \( \mu \) supported on \( K \), \( \mu(A_{\tilde{\sigma}\tilde{\rho}\alpha}(f)) > 1 - \varepsilon \).

Proof. This is proved in \([\text{CP}, \text{proposition 3.2}]\) in the case \( \varepsilon = 5/6, r = 2 \) and \( \gamma = 9/10 \). We explain how to adapt the proof by modifying the constants. Let us take

\[
D = \sup_{x \in K} |\text{det} Df(x)|, \quad m = \inf_{x \in K} \| Df^{-1}(x) \|^{-1},
\]

\( \tilde{\sigma} = m, \quad \tilde{\rho} = m^2 / D, \quad \sigma = D^{1-\alpha/3}, \quad \rho = D^{1-\alpha/3}. \)

Since \( f \) is \( \gamma \)-dissipative, \( D < m^\gamma \) and the condition \([4]\) is satisfied provided \((1+\alpha/9)\gamma > 1 \). Using Pliss lemma (as stated in \([\text{CP}, \text{lemma 3.1}]\)), the first condition in \([5]\), holds on a set with \( \mu \)-measure larger than \( \frac{(1-\alpha/3)\log(D)-\log(D)}{\log(\mu)} > \frac{-\alpha/3}{(1-\alpha/3-\varepsilon/2)} \). Similarly, the second condition in \([5]\) holds on a set with \( \mu \)-measure larger than \( \frac{(1-\alpha/3)\log(D)-\log(D)}{\log(\mu)+\log(D)} > \frac{-\alpha/3}{(2-\alpha/3-\varepsilon/2)} \). Hence \([5]\) holds on a set with measure larger than \( 1 - \varepsilon / 2 \) provided \( \gamma \) is chosen close to 1 so that \( \frac{-\alpha/3}{(2-\alpha/3-\varepsilon/2)} > 1 - \varepsilon / 2 \). \( \square \)

4 Closing lemmas

The following theorem is proved in \([\text{CP}]\).

Theorem F. For any mildly dissipative diffeomorphism of the disc, the support of any \( f \)-invariant probability is contained in the closure of the set of periodic points.

We state now a local version of that result. Let us recall that a compact connected set of the plane is \textit{cellular} if its complement is connected. Equivalently it is the decreasing intersection of sets homeomorphic to the unit disc.

Theorem F’ (Local version). Let \( f \) be a mildly dissipative diffeomorphism of the disc, and \( \Lambda \) be an invariant cellular connected compact set. Then the support of any \( f \)-invariant probability on \( \Lambda \) is contained in the closure of the periodic points in \( \Lambda \).

This section is devoted to the proof of theorem F’.

We may assume that \( \mu \) is ergodic and that \( \mu \) is not supported on a finite set since otherwise the conclusion of the theorem holds trivially. We have to find a periodic point in \( \Lambda \) arbitrarily close to \( x \). Note that one can replace \( f \) by \( f^2 \) and reduce to the case where \( f \) preserves the orientation. Also, by a slight modification of the boundary of the disk, it can be assumed that for almost every point the complement of the local stable manifold in the disc has two connected components.

Definition 25. For \( \mu \)-almost every point \( x \), the connected components of \( W^s_\mu(x) \setminus \{x\} \) are called \textit{stable branches} of \( x \). We say that the connected compact set \( \Lambda \) crosses a stable branch \( \sigma \) of \( x \) if there exists a connected compact set \( C \subset \Lambda \) which intersects both connected components of \( \mathbb{D} \setminus W^s_\mu(x) \) and is disjoint from \( W^s_\mu(x) \setminus \sigma \).

Remark 26. One can build connected compact sets \( C' \subset C \) satisfying the definition and contained in arbitrarily small neighborhoods of \( W^s_\mu(x) \). If this were the case there
would exist a small neighborhood $U$ of $W^s_D(x)$ such that each connected component of $C \cap U$ is disjoint from one of the connected components of $\mathbb{D} \setminus W^s_D(x)$. Hence the points in $W^s_D(x) \cap C$ would not be accumulated by points of $C$ from both components of $\mathbb{D} \setminus W^s_D(x)$: there would be a continuous partition of $C$ as points to the “left” or to the “right” of $W^s_D(x)$, contradicting the connectedness.

**Lemma 27.** Three cases occur.

- for $\mu$-almost every point $x$, the set $\Lambda$ crosses both stable branches of $x$,
- for $\mu$-almost every point $x$, the set $\Lambda$ crosses one stable branch of $x$ and is disjoint from the other one,
- for $\mu$-almost every point $x$, the set $\Lambda$ is disjoint from both stable branches of $x$.

**Proof.** We first note that the set of points such that both stable branches $W^s_D(x)$ are crossed by $\Lambda$ is forward invariant, hence is $f$-invariant on a set with full $\mu$-measure. Similarly for the set of points having only one stable branch crossed by $\Lambda$. By ergodicity, three cases occur on a set $X$ with full measure: $\Lambda$ crosses both branches of each point, or exactly one branch, or none of them.

Pesin theory gives the continuity of $W^s_D(x)$ for the $C^1$ topology on a set with positive $\mu$-measure. Up to removing from $X$ a set with zero measure, one can thus assume that each point $x \in X$, is accumulated by points $y$ of $X$ in each component of $\mathbb{D} \setminus W^s_D(x)$ such that $W^s_D(x)$ and $W^s_D(y)$ are arbitrarily close for the $C^1$ topology.

Let us consider a stable branch $\sigma$ of $x \in X$ and assume that there exists $z \in \sigma \cap \Lambda$. Since $\Lambda$ is compact and invariant, there exists $z_1, z_2 \in \sigma \setminus \Lambda$ such that $z$ belongs to the subarc $[z_1, z_2]$ of $\sigma$ connecting $z_1$ to $z_2$. Since $\Lambda$ is connected and is not contained in $W^s_D(x)$, there exists a compact connected set $C \subset \Lambda$ that intersects $[z_1, z_2]$, that is contained in a small neighborhood of $[z_1, z_2]$, and that contains a point $\zeta \in \Lambda \setminus W^s_D(x)$. Considering a point $y$ as above in the same component of $\mathbb{D} \setminus W^s_D(x)$ as $\zeta$, one deduces that the stable branch $\sigma_y$ of $y$ close to $\sigma$ is crossed by $\Lambda$. Not also that if the other stable branch $\sigma'$ of $x$ is crossed by $\Lambda$, then the stable branch $\sigma_y'$ of $y$ that is close to $\sigma_y$ is also crossed by $\Lambda$. Since $x$ and $y$ have the same number of stable branches crossed by $\Lambda$, one deduces that $\sigma$ is crossed by $\Lambda$. \hfill \Box

For proving theorem $[F]$, the three cases of lemma $[27]$ have to be addressed.

In the first case, using that both branches of local stable manifolds intersects $\Lambda$, for a point $x$ in a hyperbolic block of the measure $\mu$, we build a rectangle that contains $x$ in its interior and the boundary of the rectangle are given by two local stable manifolds of generic points of the measure and two connected arcs contained in $\Lambda$. By theorem $[F]$ there is a periodic point close to $x$ and so in the interior of the rectangle; on the other hand, by the construction of the rectangle, forward iterates of it converge to $\Lambda$; therefore, the periodic point in the interior of the rectangle, has to be in the intersection of $\Lambda$ with the rectangle.

In the second case, a similar rectangle (with boundaries given by two local stable manifolds of generic points of the measure and two connected arcs contained in $\Lambda$) can be built. However, that rectangle does not contain points of $\Lambda$ in its interior and so theorem $[F]$ does not guarantee the existence of a periodic point in $R$ (the periodic points provided by that theorem accumulate on the boundary of the rectangle). So a different strategy has to be formulated, which is described after the preparatory claim $[29]$.

For the third case, we use a slight variation of the strategy developed for the second case.
First case: $\Lambda$ crosses both stable branches of $x$. We select a neighborhood of $x$ verifying:

**Claim 28.** There is a neighborhood $R$ of $x$ whose boundary is contained in $\Lambda \cup W^s_D(x') \cup W^s_D(x'')$ where $x', x''$ are iterates of $x$.

**Proof.** From Pesin theory, there exists a set $X$ with positive measure for $\mu$ such that $W^s_D(z)$ exists and varies continuously with $z \in X$ for the $C^1$ topology. Since $\mu$ has no atom, one can furthermore require that any point $z \in X$ is accumulated in both components of $D \setminus W^s_D(z)$ by forward iterates of $z$ in $X$. Without loss of generality, one can assume that $x$ belongs to $X$ and consider two forward iterates $x', x'' \in X$ of $x$, arbitrarily close to $x$ and separated by $W^s_D(x)$. See figure 6. Since $\Lambda$ crosses both stable branches of $x$, there exists two connected compact sets $C_1, C_2 \subset \Lambda$ which intersect both curves $W^s_D(x')$, $W^s_D(x'')$ and which do not contain $x$. The connected component $R$ of $D \setminus (W^s_D(x') \cup W^s_D(x'') \cup \Lambda)$ containing $x$ has its closure contained in the interior of $D$: otherwise, there would exist an arc connecting $x$ to the boundary of $D$, contained in the strip bounded by $W^s_D(x') \cup W^s_D(x'')$, and disjoint from $\Lambda$, contradicting the connectedness of $C_1$ and $C_2$.

The volume of the iterates $f^k(R)$ and the length of the iterates $f^k(W^s_D(x'))$ and $f^k(W^s_D(x''))$ decreases to zero as $k \to +\infty$. Hence the distance between $f^k(R)$ and $\Lambda$ goes to zero when $k$ goes to $+\infty$. By applying theorem [F] there exists a periodic point $q$ in $R$. Let $\ell$ denote its period. This periodic point also belongs to $f^{k\ell}(R)$ for $k$ arbitrarily large, hence it also belongs to $\Lambda$ by our construction. The theorem follows in that case.

Second case: $\Lambda$ crosses only one stable branch of almost every point $x$. As in the proof of the previous claim, we introduce a compact Pesin block $X \subset \Lambda$ for $\mu$ with no isolated point, containing $x$ and with positive $\mu$-measure. One can replace $x$ by another point close in $X$ and require that $x$ is accumulated by $X$ in both components of $D \setminus W^s_D(x)$.

**Claim 29.** There exists $N \geq 0$ (arbitrarily large) and two points $x', x'' \in X$ such that

- $W^s_D(f^{-N}(x))$ separates $f^{-N}(x')$ and $f^{-N}(x'')$ in $D$,
- the image by $f^N$ of the strip bounded by $W^s_D(f^{-N}(x'))$ and $W^s_D(f^{-N}(x''))$ in $D$ is an arbitrarily small neighborhood $R$ of $x$,.
Figure 7: Localization when $\Lambda$ crosses one or no stable branch of $x$.

- $x''$ is a forward iterate $f^j(x')$ of $x'$,
- for any $n \geq 1$, the point $x''$ is accumulated by its forward iterates under $f^n$ in both components $D \setminus W_s(D)$.

Proof. Since the length of the iterates $f^n(W_s^a(z))$ decreases uniformly to 0 as $n$ goes to $+\infty$, the curve $f^N(W_s^a(f^{-N}(x)))$ is arbitrarily small for $N$ large enough. Note that $f^N(\partial D)$ crosses both stable branches of $x$. Considering points $x, x''$ close to $x$ in $X$, one defines a rectangle $R$ bounded by $f^N(\partial D) \cup W_s(D(x')) \cup W_s(D(x''))$.

Since $x$ has positive measure, one can choose $x', x''$ in the same orbit. Moreover, up to removing a set with zero measure, one can choose $x'$ (and $x''$) to be accumulated by its forward iterates under $f^n$ (for any $n \geq 1$) inside both components $D \setminus W_s(D)$.

In the following, one replaces $D$ by $f^N(D)$ and $f$ by $f^j$. Hence without any loss of generality one reduces to the case where:

- $W_s^a(x)$ separates $x'$ and $x''$,
- $R$ is the strip in $D$ bounded by $W_s^a(x')$ and $W_s^a(x'')$,
- $f(x') = x''$.

We now have to find a periodic point $q$ in $R \cap \Lambda$. The ergodicity of the measure will not be used anymore. We denote by $D'$ (resp. $D''$) the (open) component of $D \setminus W_s^a(x')$ (resp. of $D \setminus W_s^a(x'')$) which does not contain $W_s^a(x'')$ (resp. $W_s^a(x')$). See figure 7.

The strategy now consist in using the stable manifolds of generic points of the measure to build a forward invariant cellular set $\Delta$ that contains $\Lambda$ and such that its forward iterates converge to $\Lambda$ (see lemma 30). Then, after considering the following three sets, $\Delta' = \Delta \cap D'$, $\Delta'' = \Delta \cap D''$ and $\Delta \cap R$, we show that it is possible to build a continuous map $g$ that sends $\Delta$ into itself, coincides with an iterate of $f$ in $R$ and satisfies $g(\Delta') \cap \Delta' = \emptyset$ and $g(\Delta'') \cap \Delta'' = \emptyset$ (see lemma 32). From proposition 21 it follows that $g$ has a fixed point in $\Delta$; since that fixed point can not be neither in $\Delta'$ nor in $\Delta''$, it has to be in $R \cap \Delta$ and so it is a periodic point for $f$: since the forward iterates of $\Delta$ converges to $\Lambda$, it follows that it has to be in $\Lambda$.

The last item of the claim 29 implies that there exists a compact set $A \subset D''$ which contains arbitrarily large iterates of $x'$ and $x''$, which are contained in $f^m(X)$ for some
\( m \geq 1 \) such that \( \Lambda \) crosses a stable branch of each point \( z \in A \) (and is disjoint from the other one). The stable curves \( W^s_0(z) \) vary continuously with \( z \in A \) for the \( C^1 \) topology.

**Lemma 30.** There exists a connected compact set \( \Delta \) which has the following properties:

i. \( \Delta \) is cellular (i.e. its complement is connected).

ii. \( \Delta \) is forward invariant: \( f(\Delta) \subset \Delta \).

iii. The forward orbit of any point in \( \Delta \) accumulates on \( \Lambda \).

iv. One stable branch of \( x' \) is disjoint from \( \Delta \), the other one intersects \( \Delta \) along an arc; moreover there exists a \( \) (non-empty) arc in \( W^s_0(x') \) which contains \( x' \) in its closure and is included in the interior of \( \Delta \). The same holds for the stable branches of \( x'' \).

v. There is \( \varepsilon > 0 \) such that for any forward iterate \( z \in A \) of \( x' \), there exists a curve of size \( \varepsilon \) in \( W^s_0(z) \) containing \( z \) in its closure and included in the interior of \( \Delta \).

Let us denote \( \Delta'_0 := \Delta \cap D' \) and \( \Delta''_0 := \Delta \cap D'' \). Note that it is enough now to obtain a periodic point \( q \in R \cap \Delta \). Indeed, since the accumulation set of the forward orbit of \( q \) coincides with the orbit of \( q \), the item (ii) ensures that \( q \in \Lambda \) as required.

**Proof.** We consider for each \( z \in X \subset \Lambda \) the maximal curve \( I_z \) in \( W^s_0(z) \) bounded by points of \( \Lambda \) (possibly reduced to a point). The union \( \Delta_0 \subset \Lambda \) with all the forward iterates of the curves \( I_z \), \( z \in X \), is a forward invariant set which is compact (since the set \( X \) is compact, since the curves \( W^s_0(z) \) vary continuously with \( z \in X \) for the \( C^1 \) topology and since the length of their iterates decreases uniformly) and is connected (since \( \Lambda \) is connected). The set \( \Delta \) is obtained by filling the union \( \Delta_0 \), i.e. it coincides with the complement of the connected component of \( \mathbb{D} \setminus \Delta_0 \) which contains the boundary of \( \mathbb{D} \). Properties (i) and (ii) are satisfied.

In order to prove the property (iii), we consider a point \( y \in \Delta \). Note that if \( y \) belongs to \( \Lambda \) or to some \( W^s_0(z) \) with \( z \in X \), the conclusion of (iii) holds trivially. We thus reduce to the case where \( z \) belongs to a connected component \( C \) of \( \Delta \setminus \Delta_0 \). Note that the boundary of this component decomposes as the union of a subset of \( \Lambda \) and a set contained in the union of the \( f^{n}(W^s_0(z)) \) with \( z \in X \) and \( n \geq 0 \). Since the volume decreases under forward iterations, for \( n \) large enough the point \( f^n(z) \) gets arbitrarily close to the boundary of \( f^n(C) \). Since the length of stable manifolds \( f^n(W^s_0(z)) \) gets uniformly arbitrarily small as \( n \to +\infty \), any point in \( f^n(C) \) is arbitrarily close to \( \Lambda \) provided \( n \) is large enough, proving (iii).

By construction of \( \Delta \), for any point \( z \in X \), the intersection \( \Delta \cap W^s_0(z) \) is an arc bounded by two points of \( \Lambda \) (and not reduced to \( z \)). This is the case in particular for the intersections \( \Delta \cap W^s_0(x') \) and \( \Delta \cap W^s_0(x'') \). Since one stable branch of \( x' \) (resp. \( x'' \)) does not meet \( \Lambda \), the first half of (iv) follows.

Let \( \sigma \) be the stable branch of \( x' \) that is crossed by \( \Lambda \) and let us choose a connected compact set \( C_1 \subset \Lambda \) as in definition 25. One can choose another set \( C_2 \) which is contained in an arbitrarily small neighborhood of \( x \). Indeed, let \( f^{-k}(x') \) be a backward iterate of \( x' \) in \( X \). By choosing \( k \) large, the image \( f^{k}(W^s_0(f^{-k}(x'))) \) gets arbitrarily small. Let \( C'_2 \) be a connected set crossing a stable branch of \( f^{-k}(x') \) as in definition 25. One can choose it in a small neighborhood of \( W^s_0(f^{-k}(x')) \) (by remark 26), hence the image \( C_2 := f^{k}(C'_2) \) is contained in a small neighborhood of \( x \).

One deduces that the smallest arc \( \gamma \) connecting \( C_1 \) to \( C_2 \) inside \( W^s_0(x') \) is contained (after removing its endpoints) in the interior of \( \Delta \). Indeed, one can choose two points
Figure 8: The interior of $\Delta$ contains stable arcs.

$y_l, y_r \in X$ close to $x'$, separated by $W_s^u(x')$ and with stable curves close to $W_s^u(x')$ for the $C^1$ topology. These curves are crossed by $C_1$ and $C_2$, hence the connected components of $\mathbb{D}\setminus(C_1 \cup C_2 \cup W_s^u(y_l) \cup W_s^u(y_r))$ containing $\gamma$ is bounded away from $\partial \mathbb{D}$ and contained in $\Delta$ as claimed. See figure 8.

Since $C_2$ can be chosen in an arbitrarily small neighborhood of $x'$, one deduces that the interior of $\Delta$ contains a (non-empty) arc in $W_s^u(x')$ which contains $x'$ in its closure. The same holds for the point $x''$ and (iv) is satisfied.

As a consequence, for any $z$ in a forward iterate of $X$, the intersection $\Delta \cap W_s^u(z)$ is a finite union of arcs bounded by points of $\Lambda$.

In order to check (v), one notices that by the same argument as in the previous paragraph, for any point $z_0 \in A$, there exists a non-trivial curve $\alpha_{z_0} \subset W_s^u(z_0)$ contained in the interior of $\Delta$ and containing $z_0$ in its closure. By construction, the length of $\alpha_{z_0}$ is bounded from below for any $z \in A$ close to $z_0$. By compactness of $A$, there exists a uniform bound $\varepsilon > 0$ for all $\alpha_{z_0}$ with $z \in A$, proving (v).

Let $I' := \Delta \cap W_s^u(x')$ and $I'' := \Delta \cap W_s^u(x'')$. By (iv), these are arcs.

**Claim 31.** For $\ell \geq 1$ large, $f^\ell(I' \setminus \{x'\})$ and $f^\ell(I'' \setminus \{x''\})$ are in the interior of $\Delta$.

**Proof.** Let us consider a large forward iterate $f^{k}(x') \in A$. The image $f^{k}(I' \setminus \{x'\})$ is arbitrarily small, hence by (iv) and (v) is contained in the interior of $\Delta$. By (ii) the interior of $\Delta$ is forward invariant. This shows that for any integer $\ell$ large, the image $f^\ell(I' \setminus \{x'\})$ is contained in the interior of $\Delta$. And the same holds for $f^\ell(I'' \setminus \{x''\})$. \(\square\)

We choose $\ell \geq 1$ large such that $f^\ell(x') \in D''$. Since the stable manifolds are disjoint or coincide, this gives $f^\ell(I') \subset D''$. Note that since $\mu$ is not a periodic measure, the large forward iterate of $x'$ do not intersect $W_s^u(x')$. One can thus choose $\ell$ such that we have also $f^{\ell+1}(x') \notin D''$: this gives $f^{\ell+1}(I'') \subset \mathbb{D} \setminus D''$. We fix such an iterate $f^\ell$.

**Lemma 32.** There exists a continuous map $g$ which:

(a) maps $\Delta$ inside itself,

(b) is the restriction of an orientation-preserving homeomorphism of the plane,

(c) satisfies $g(\Delta') \cap \Delta' = \emptyset$ and $g(\Delta'') \cap \Delta'' = \emptyset$,

(d) coincides with $f^\ell$ on $R$.  

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Proof. One chooses two small neighborhoods $U', U''$ of $x'$ and $x''$. One builds a homeomorphism $\varphi$ which coincides with the identity on $R$ and near the boundary of $D$ and which sends $\Delta'$ in a small neighborhood of $I'$ and $\Delta''$ in a small neighborhood of $I''$.

More precisely, from the property (iv) of lemma 30 the curve $(U' \cap I') \setminus \{x'\}$ is contained in the interior of $\Delta$ and the stable branch of $x'$ which does not meet $I'$ is disjoint from $\Delta$. One can thus require that $\varphi(\Delta' \cap U') \subset \Delta'$ so that $f^\ell \circ \varphi(\Delta' \cap U') \subset \Delta$.

One can furthermore require that $\Delta'$ is sent in a small neighborhood of $I'$. By our choice of $\ell$, the compact arc $f^\ell(I' \setminus U')$ is contained in the interior of $\Delta$, hence one gets $f^\ell \circ \varphi(\Delta' \setminus U') \subset \Delta$. This shows that $g := f^\ell \circ \varphi$ satisfies $g(\Delta') \subset \Delta$. A similar construction in $D''$, implies that $g(\Delta'') \subset \Delta$. Since $f^\ell(\Delta) \subset \Delta$ by (ii), this implies $g(\Delta) \subset \Delta$, hence (a). The properties (b) and (d) follows from the definition of $\varphi$ and $g$.

Since $\varphi(\Delta')$ is contained in a small neighborhood of $I'$ and since $f^\ell(I') \subset D''$, one gets $g(\Delta') \subset D''$. Similarly $g(\Delta'') \subset \Delta'$. Hence property (e) holds.

From (a), the sequence $g^n(\Delta)$ is decreasing and the intersection $\widetilde{\Delta}$ is $g$-invariant. From the property (i) and as the intersection of a decreasing sequence of cellular sets, it is cellular. Together with (b), one can apply Cartwright-Littlewood’s theorem (proposition 21): the orientation preserving homeomorphism of the plane $g$ has a fixed point $q \in \Delta \subset \Delta$. From (c), the fixed point does not belong to $\Delta' \cup \Delta''$, hence it belongs to $R \cap \Delta$. From (d), it is an $\ell$-periodic point of $f$, as we wanted. The proof of the theorem follows in the second case.

**Third case:** $\Lambda$ is disjoint from the two stable branches of almost every $x$. We adapt the proof done in the second case. We can first reduce to the setting of the figure 7, where $W^s_D(x)$ separates two points $x^l$ and $x^u = f(x^l)$; $R$ is the strip bounded by $W^u_D(x^l)$ and $W^u_D(x^u)$; we have to find a periodic point $q$ in $R \cap \Lambda$.

In this case, for any iterate $f^{\ell k}(x^l)$, the set $\Lambda$ intersects $W^s_D(f^{\ell k}(x^l))$ only at $x^l$. We choose $\ell \geq 1$ large such that $f(\ell k)(x^l) \in D''$ and $f^{\ell k+1}(x^l) \in \mathbb{D} \setminus \mathbb{D}^\ell$. Note that the sets $(\Lambda \cap D') \cup \{x^h\}, (\Lambda \cap R) \cup \{x^l, x^u\}, (\Lambda \cap D'') \cup \{x^u\}$ are compact, connected, and only intersect at $x^l$ or $x^u$. The image by $f^\ell$ of the second intersects both $D''$ and $\mathbb{D} \setminus \mathbb{D}^\ell$; consequently it contains $x^u$. One deduces that the image $f^{\ell k}(\Lambda \cap D')$ does not intersect $W^s_D(x^u)$, hence is contained in $D''$. For the same reason the image $f^{\ell k}(\Lambda \cap D'')$ does not intersect $W^s_D(x^h)$, hence is contained in $\mathbb{D} \setminus \mathbb{D}^\ell$. This proves that $f^\ell$ has no fixed point in $(D' \cup D'') \cap \Lambda$. By Cartwright-Littlewood’s theorem (proposition 21), it has a fixed point in the cellular set $\Lambda$, hence in $\Lambda \cap R$ as wanted. The proof of theorem $F'$ is now complete.

5 No cycle

One says that a diffeomorphism $f$ admits a cycle of periodic orbits if there exists a sequences of periodic orbits $O_0, O_1, \ldots, O_n = O_0$ such that for each $i = 0, \ldots, n-1$, the unstable set of $O_i$ accumulates on $O_{i+1}$. The goal of this section is to prove the following:

**Theorem G.** A mildly dissipative diffeomorphisms of the disc has zero topological entropy if and only if it does not admit any cycle of periodic orbit.

This result can be localized. A set $U$ is filtrating for $f$ if it may be written as the intersection of two open sets $U = V \cap W$ such that $f(V) \subset V$ and $f^{-1}(W) \subset W$. 

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Theorem G’ (Local version). Let \( f \) be mildly dissipative diffeomorphisms of the disc and \( U \) be a filtrating set. The restriction of \( f \) to \( U \) has zero topological entropy if and only if it does not admit any cycle of periodic orbits contained in \( U \).

The non-existence of cycle of periodic orbits extend to fixed arcs. One says that a diffeomorphism \( f \) admits a cycle of fixed arcs if there is sequence of disjoint fixed arcs \( I_0, I_1, \ldots, I_n = I_0 \) such that each arc \( I_i \) admits a \( f \)-invariant unstable branch which accumulates on \( I_{i+1} \).

Corollary 33. Consider a mildly dissipative diffeomorphism \( f \) of the disc. If \( f \) has zero topological entropy, then it does not admit any cycle of fixed arcs.

The same property holds inside any filtrating set \( U \).

Proof. Let us assume that \( f \) admits such a sequence of fixed arc: for each \( i \), there exists a fixed point \( p_i \in I_i \) with a \( f \)-invariant unstable branch \( \Gamma_i \) that accumulates on \( I_{i+1} \). We may assume that the length \( n \) is minimal.

For each interval \( I_i \), we claim that one component of \( \mathbb{D} \setminus W^u_s(I_i) \) contains all the other arcs \( I_j \). Indeed, let \( \Gamma_i \) be the \( f \)-invariant unstable branch of \( I_i \) that accumulates on \( I_{i+1} \). It is the unstable branch of an endpoint \( p_i \) of \( I_i \). Since \( f \) has no cycle of fixed points, \( \Gamma_i \) is disjoint from \( W^u_s(p_i) \). As a consequence, \( \Gamma_i \) is contained in a component \( C_i \) of \( \mathbb{D} \setminus W^u_s(I_i) \), which also contains \( I_{i+1} \). If there exist arcs \( I_j, i + 1 < j \), that are not contained in \( C_i \), one considers the first one; consequently the unstable branch of \( I_{j-1} \) crosses \( W^u_s(I_j) \) and the length \( n \) of the cycle is not be minimal.

From the previous paragraph, the point \( p_{i-1} \) is contained in the component of \( \mathbb{D} \setminus W^u_s(I_i) \) which contains \( \Gamma_i \), named \( V_i \), which is is also the component bounded by \( W^u_s(p_i) \); otherwise, if \( p_{i-1} \) is not contained in \( V_i \), since \( p_0 \) is in \( V_i \) because the unstable branch of \( p_{n-1} \) that accumulates on \( I_0 \) is in \( V_0 \), it follows that for some \( i' < i \) holds that \( I_{i'} \) is in \( V_i \), but it that case, its unstable branch involved in the cycle has to cross the local stable of \( p_i \) a so the cycle would not be minimal; a contradiction.

One deduces that \( \Gamma_{i-1} \) accumulates on \( p_i \) for each \( i \). The fixed points \( p_0, p_1, \ldots, p_n \) define a cycle and by theorem G, \( f \) has positive topological entropy.

Using theorem G’ one gets the same property inside filtrating sets. \( \square \)

A simple example of a cycle is a 1–cycle associated to a fixed point: the unstable manifold of the fixed point accumulates on the stable one. In that context, in [P], it was proved that either there exists a transversal homoclinic point (and so the topological entropy is positive) or that intersection can be created by a smooth perturbation. Under the hypothesis of mild dissipation, we prove that if such a 1-cycle exists, the topological entropy is positive even if there is not transverse intersection between the invariant manifolds of the fixed point.

The end of this section is devoted to the proof of theorems G and G’.

5.1 Homoclinic orbit of a fixed point

We first consider the case of a cycle of a unique fixed point with a homoclinic orbit.

Lemma 34. Let \( f \) be mildly dissipative diffeomorphism of the disc. If there exists a point \( p \) with a fixed unstable branch \( \Gamma \) which intersects \( W^u_s(p) \), then the topological entropy of \( f \) is positive.
Proof. Let us consider $x \in W^s_D(p) \cap \Gamma$. We denote by $\gamma$ the arc of $\Gamma$ which connects $p$ to $x$. Up to replace $f$ by $f^2$, one can assume that $q$ is fixed and that the eigenvalues of $Df(p)$ are $0 < \lambda < 1 \leq \mu$. Since $f$ contract the volume, $\lambda \mu < 1$.

Let us choose some point $z^s \in W^s_D(p)$ such that $x$ belongs to the interior of the segment $[z^s, f(z^s)]$ in $W^s_D(p)$ and such that $z^s$ is not accumulated by $\Gamma$ (the point $z^s$ can be chosen as one that has a backward orbit outside the disc). One chooses two small $C^1$ arcs $\alpha, \alpha'$ transverse to $\sigma$ at $z^s$ and $f(z^s)$. Similarly, one chooses some point $z^u \in \Gamma$ such that $x$ belongs to the interior of the segment $[z^u, f(z^u)]$ in $\Gamma$ and such that the orbit of $z^u$ does not intersect the strong stable arc $W^s_D(p)$. One fixes two small arcs $\beta, \beta'$ transverse to $\gamma$ at $z^u$ and $f(z^u)$. For $n$ large, there exist four arcs $B, B' \subset f^{-n}(\beta), B' \subset f^{-n}(\beta')$ and $A \subset \alpha, A' \subset \alpha'$ which bound a rectangle $R$ whose $n$ first iterates remain close to the forward orbit of $x$ and the backward orbit of $x$. See Figure 9.

For any $\varepsilon > 0$ there is $C > 0$ such that if $n$ has been chosen large enough,

$$\min (d(B, W^s_D(p)), d(B', W^s_D(p))) \geq C^{-1} (1 + \varepsilon)^{-n} \mu^{-n},$$

$$\max (d(f^n(A), \gamma), d(f^n(A'), \gamma)) \leq C(1 + \varepsilon)^n \lambda^n.$$ 

One chooses the integer $n$ such that

$$C^{-1} (1 + \varepsilon)^{-n} \mu^{-n} > C(1 + \varepsilon)^n \lambda^n.$$ 

This is possible since the dissipation gives $\lambda \mu < 1$. In particular $f^n(R)$ “crosses” $R$.

One deduces that for any curve $\delta$ in $R$ which connects the arcs $B, B'$, the image $f^n(\delta)$ contains two curves $\delta_1, \delta_2 \subset R$ which also connect the arcs $B, B'$, and which are $\varepsilon$-separated for some $\varepsilon > 0$ independent from $\delta$. One can thus iterate $\delta$ and apply the property inductively. This implies that the topological entropy is positive.

5.2 Periods and heteroclinic orbits.

The following proposition allows to get (topological) transverse heteroclinic intersections between periodic orbits with different periods and will be used again in other sections.

**Proposition 35.** Let $f$ be a mildly dissipative diffeomorphism of the disc $f$ which preserves the orientation and with zero topological entropy. Let $p$ be a fixed point having a real eigenvalue larger or equal to 1 and $q$ be a periodic point with an unstable branch $\Gamma_q$ which is not fixed by $f$. If $\Gamma_q$ accumulates on $p$, then it intersects both components of $\mathbb{D} \setminus W^s_D(p)$.

Figure 9: Proof of lemma 34
Proof. First observe that the period of \( q \) is larger than one: if it is fixed, since the branch \( \Gamma_q \) that accumulates on the fixed points is not invariant, then both unstable branches in each component of \( \mathbb{D} \setminus W^s_q(p) \) accumulates on the same fixed point; a contradiction.

Let us assume now by contradiction that \( \Gamma_q \) intersects only one component of \( \mathbb{D} \setminus W^s_q(p) \). Since the largest eigenvalue at \( p \) is positive, the components are locally preserved by \( f \). Hence each unstable branch \( f^k(\Gamma_q) \) intersects the same components as \( \Gamma_q \). This proves that all the iterates of \( q \) are contained in a same component \( U \) of \( \mathbb{D} \setminus W^s_q(p) \).

The set \( (\partial U) \setminus \{ p \} \) is an arc that may be parametrized by \( \mathbb{R} \), hence may be endowed with an order \( < \). To each iterate \( f^j(q) \), since the other iterates \( f^j(q) \) with \( j \neq k \) are in the connected component of \( \mathbb{D} \setminus W^s_q(f^k(q)) \) that contains \( p \) (otherwise the unstable branch of some \( f^j(q) \) has to cross the stable of \( f^k(q) \)) one associates the component \( V_k \) of \( \mathbb{D} \setminus W^s_q(f^k(q)) \) which does not contain \( p \), nor the other iterates of \( q \). The components \( V_k \) are thus disjoint, hence ordered by their prints on the boundary of \( \partial U \). This induces an ordering on the iterates of \( q \).

First case. The branch \( \Gamma_q \) accumulates on a point \( x \) of \( W^s_q(p) \) which is different from \( p \). The iterates \( f^k(x) \) converge to \( p \) as \( k \rightarrow +\infty \). Since \( f \) preserves the orientation, these iterates belong to the same branch of \( W^s_q(p) \). Up to modify the parametrization of the boundary of \( U \), one can assume that the sequence \( f^k(x) \) is increasing for the order \( < \).

We choose \( y \in \Gamma_q \) close to \( x \), a small arc \( \delta \) connecting \( y \) to \( x \) and consider the arc \( \gamma \subset \Gamma_q \) connecting \( q \) to \( y \). This gives an oriented arc \( \sigma := \delta \cup \gamma \) connecting \( q \) to \( x \) in \( U \).

The set \( U \setminus (\overline{V_0} \cup \sigma) \) has two connected components that are Jordan domains. One of them (denoted by \( O \)) contains in its boundary all the forward iterates of \( x \) and the point \( p \). Up to replace \( q \) by another point in its orbit, one can assume that \( O \) contains all the iterates \( f^k(q) \neq q \), that is \( f^k(q) < q \). See figure 5.2.

Since the endpoints of \( \sigma \) (resp. of \( f(\sigma) \)) do not belong to \( f(\sigma) \) (resp. to \( \sigma \)), the algebraic intersection number between \( \sigma \) and \( f(\sigma) \) is well defined. Since \( \sigma \) is contained in the boundary of \( O \) and since the endpoints of \( f(\sigma) \) belong to \( O \), the algebraic intersection number between \( \sigma \) and \( f(\sigma) \) is zero.

This implies that for any \( k \geq 0 \), the intersection number between \( f^k(\sigma) \) and \( f^{k+1}(\sigma) \) is zero. This proves that in \( \overline{U} \setminus (f^k(\sigma) \cup W^s_q(f^k(q))) \), the points \( f^{k+1}(x) \) and \( f^{k+1}(q) \) are in the same connected component. Since \( f^k(x) < f^{k+1}(x) \), one deduces that \( f^{k+1}(q) < f^k(q) \) for any \( k \geq 0 \). This is a contradiction since when \( k+1 \) coincides with the period
of $q$, we have $f^{k+1}(q) = q > f^k(q)$.

Second case. The accumulation set of $\Gamma_q$ is disjoint from $W^s_D(p)$. We modify the previous argument. Note that in this case the stable set of $p$ contains a neighborhood of $W^s_D(p)$ in $U$. Moreover this neighborhood is foliated by strong stable curves, that we still denotes by $W^s_D(z)$.

We choose $y \in \Gamma_q$ in the stable set of $p$ and consider the oriented arc $\sigma \subset \Gamma_q$ connecting $q$ to $y$. One can choose $y$ such that the arc $\sigma$ does not intersects the component of $U \setminus W^s_D(y)$ containing $p$. Let $L$ be the half curve in $W^s_D(y)$ connecting $y$ to a point $z$ in $\partial U$. We can choose the endpoint $z$ such that $V_0 < z$. Since $V_1 < V_0$, one deduces that $f(q)$ and $f(y)$ belong to the same connected component of $U \setminus (V_0 \cup \sigma \cup L)$. See figure 5.2. In particular the algebraic intersection number between $\sigma$ and $f(\sigma)$ is zero.

For any $k \geq 0$, let $L_k$ be the half curve in $W^s_D(f^k(y))$ connecting $f^k(y)$ to a point $z_k$ in $\partial U$ such that $V_k < z_k$. Since $f^{k+1}(y)$ belongs to the strip bounded by $W^s_D(p)$ and $W^s_D(f^k(y))$, we have $z_k < z_{k+1}$. Since the algebraic intersection number between $f^k(\sigma)$ and $f^{k+1}(\sigma)$ is zero, one deduces that $V_{k+1}$ and $f^{k+1}(y)$ belong to the same component of $U \setminus (V_k \cup \sigma \cup L_k)$. In particular $V_{k+1} < V_k$ for any $k \geq 0$. As in the previous case, this is a contradiction. 

Remark 36. In the case where $f$ does not preserves the orientation, the same statement applies if one assumes that the period of $\Gamma_q$ is larger than 2 (one applies the previous proposition to $f^2$).

5.3 Cycles of fixed points

Lemma 37. If $f$ has a cycle of periodic orbits, there is an iterate $f^m$, $m \geq 1$ which has a cycle of fixed points. More precisely, there exists a fixed point $p$ for $f^m$ with a fixed unstable branch $\Gamma$ whose accumulation set contains $\Gamma$.

Proof. Let $O_0, O_1, \ldots, O_n = O_0$ be a cycle of periodic orbits. We extend periodically the sequence $(O_k)$ to any $k \in \mathbb{N}$. By invariance of the dynamics, one deduces that for each $i$ and each $p_i \in O_i$, there exists an unstable branch $\Gamma_i$ of $O_i$ which accumulates on a point of $O_{i+1}$. One deduces that for any $p_0 \in O_0$, there exists points $p_k \in O_k$, $k \geq 0$, such that an unstable branch of $p_{k-1}$ accumulates on $p_k$ for each $k \geq 1$. All the points $p_{i_n}$ belong to $O_0$, hence two of them $p_{i_1 n}, p_{i_2 n}$ should coincide.

There exists $m \geq 1$ such that all the points in $\bigcup_i O_k$ are fixed. The sequence $p_{i_1 n}, p_{i_1 n+1}, \ldots, p_{i_2 n}$ is a cycle of fixed points for $f^m$. This proves the first assertion.

Let us consider a cycle of fixed points for $g = f^m$ with minimal length. Replacing $m$ by $2m$, one can also assume that all their unstable branches are fixed. For each fixed point $p_i$ in the cycle, the other fixed points are all contained in a same component $U_i$ of $\mathbb{D} \setminus W^s_D(p_i)$: otherwise, one find a point $p_j \neq p_{j-1}$ with an unstable branch which meets both connected components and one contradicts the minimality of the cycle.

Each fixed point $p_i$ has an unstable branch $\Gamma_i$ whose accumulation set contains $p_{i+1}$. If $\Gamma_i$ intersects both components of $\mathbb{D} \setminus W^s_D(p_i)$, by proposition 19, the accumulation set contains $\Gamma_i$ and the second assertion of the lemma holds. We are thus reduced to assume that $\Gamma_i$ is contained in the closure of $U_i$. Since $p_i$ and $\Gamma_{i+1}$ are contained in $U_{i+1}$, one deduces that the accumulation set of $\Gamma_i$ contains a point of $\Gamma_{i+1}$; by proposition 19 it
contains \( \Gamma_{i+1} \). Hence the cycle is not minimal, unless \( p_i = p_{i+1} \), i.e. the cycle has only one fixed point. The second assertion holds. \( \square \)

A sequence of fixed unstable branches \( \Gamma_0, \Gamma_1, \ldots, \Gamma_n = \Gamma_0 \) associated to fixed points \( p_0, \ldots, p_n \) is a cycle of unstable branches if for each \( 0 \leq i < n \), the accumulation set of \( \Gamma_i \) contains \( \Gamma_{i+1} \). By proposition 19, this implies that for each \( i, j \), the accumulation set of \( \Gamma_i \) contains \( \Gamma_j \). We generalize lemma 34 to cycles of unstable branches.

**Lemma 38.** Let \( f \) be a mildly dissipative diffeomorphism of the disc. If there exists a sequence of fixed unstable branches \( \Gamma_0, \Gamma_1, \ldots, \Gamma_n = \Gamma_0 \) associated to fixed points \( p_0, \ldots, p_n \) such that \( \Gamma_i \) intersects \( W^s_D(p_{i+1}) \) for each \( 0 \leq i < n \), then the topological entropy of \( f \) is positive.

**Proof.** From lemma 34 for each point \( p_i \), the unstable branch \( \Gamma_i \) is contained in a connected component \( U_i \) of \( \mathbb{D} \setminus W^s_D(p_i) \). Since the accumulation set of \( \Gamma_i \) contains the other unstable branches, all the \( p_j \) and the \( \Gamma_j \) are contained in \( \overline{U_i} \). Let \( U \) be the intersection of the sets \( U_i \): it is a connected component \( U \) of \( \mathbb{D} \setminus \bigcup_i W^s_D(p_i) \) whose closure contains all the \( \Gamma_i \).

We now argue as for lemma 34. Let us assume that \( \Gamma_i \) intersects \( W^s_D(p_{i+1}) \) at some point \( x_i \). We build a rectangle \( R_i \subset U \) that stretches along a fundamental domain of \( W^s_D(p_{i+1}) \) containing \( x_i \). One chooses \( n_i \geq 1 \) large such that \( f^{n_i}(R_i) \) crosses \( R_{i+1} \). The same argument as before applies following the periodic sequence of rectangles \( R_0, R_1, \ldots, R_n = R_0 \). \( \square \)

### 5.4 Pixton discs

Let \( p \) be a fixed point with a fixed unstable branch \( \Gamma \) which is contained in its accumulation set. We introduce a notion similar to a construction in [P], which improved [R1].

A compact set \( D \subset \mathbb{D} \) is a (topological) disc if it is homeomorphic to the unit disc.

**Definition 39.** A Pixton disc associated to \( \Gamma \) is a disc \( D \) bounded by three \( C^1 \) arcs:

- an arc \( \gamma \subset \Gamma \) such that \( p \) is one endpoint,
- an arc \( \sigma \subset W^s_D(p) \) whose endpoints are \( p \) and a point \( x \neq p \) accumulated by \( \Gamma \),
- a closing arc \( \delta \) disjoint from \( f(\delta) \), joining \( \sigma \) and \( \gamma \) such that \( \delta \cap W^s_D(p) = \{x\} \).

Note that the last property implies that \( f(\delta \setminus \{x\}) \) is contained in the interior of \( D \).

**Lemma 40.** Let \( f \) be a mildly dissipative diffeomorphism of the disc with zero topological entropy and let \( p \) be a fixed point having a fixed unstable branch \( \Gamma \) which is contained in its accumulation set.

Then, there exists an aperiodic ergodic measure \( \mu \) such that

- \( \mu(D) = 0 \) for any Pixton disc \( D \) associated to \( \Gamma \),
- the closure of \( \Gamma \) contains the support of \( \mu \).

**Proof.** Note that it is enough to prove the proposition for \( f^2 \), hence one can assume that \( f \) preserves the orientation.

Let us consider a cycle of fixed unstable branches \( \Gamma_0, \Gamma_1, \ldots, \Gamma_n = \Gamma_0 \) associated to fixed points \( p_0, p_1, \ldots, p_n = p_0 \) such that \( p_0 = p, \Gamma_0 = \Gamma \), and whose cardinality \( n \) is
maximal. This exists since by proposition \[17\] all the fixed points belong to finitely many disjoint fixed arcs so that the cardinals of cycles are uniformly bounded.

By lemma \[38\], there exists \( p_i \) such that \( W_{sD}(p_i) \) is disjoint from all the unstable branches \( \Gamma_j \) for \( 1 \leq j \leq n \). By assumption there exists \( x \in W_{sD}(p_i) \) which is accumulated by all the \( \Gamma_j \). Note that the preimages of \( x \) are all well-defined in \( D \) (since \( x \) is the limit of points in \( \Gamma_i \) having infinitely many preimages in \( D \) and since \( f(D) \) is contained in the interior of \( D \)). We introduce \( K := \alpha(x) \). By construction \( K \) is contained in the accumulation set of \( \Gamma \). The goal is to show that there exists an aperiodic ergodic measure supported in \( K \) satisfying the thesis of the proposition.

**Claim 41.** Let \( D \) be a Pixton disc for an iterate \( g = f^m, m \geq 1 \):

- associated to an unstable branch \( \Gamma_D \) contained in the accumulation set of \( \Gamma_i \),
- whose closing arc \( \delta \subset \partial D \) does not meet both components of \( D \setminus W_{sD}(p_i) \).

Then the interior of \( D \) does not intersect the orbit of \( x \).

**Proof.** Let \( q \) be the periodic point associated to \( D \), which is fixed by \( g \) and let \( \gamma \cup \sigma \cup \delta \) be the boundary of \( D \). Let \( \{x_q\} = \delta \cap \sigma \).

If \( f^{-k}(x) \) belongs to the interior of \( D \), the stable manifold of \( W_{sD}(f^{-k}(x)) \) intersects the interior of \( D \) and its complement. Hence \( \delta \cup \gamma \) intersects both components of \( D \setminus W_{sD}(f^{-k}(x)) \). Note that \( \gamma \subset \Gamma_D \) does not intersect both of these components: since \( \Gamma_i \) accumulates on \( \Gamma_D \), it would imply that \( \Gamma_i \) does also and then by iteration that \( \Gamma_i \) intersects \( W_{sD}(x) \) contradicting our assumptions. As a consequence \( \delta \setminus \{x_q\} \) intersects both components of \( D \setminus W_{sD}(f^{-k}(x)) \).

Since \( g(\delta \setminus \{x_q\}) = f^m(\delta \setminus \{x_q\}) \) is contained in the interior of \( D \), one deduces that \( W_{sD}(f^{-k+m}(x)) \) intersects the interior of \( D \). By induction this implies that \( W_{sD}(f^{-k+\ell m}(x)) \) intersects the interior of \( D \) for any \( \ell \geq 0 \) and also holds that \( \delta \setminus \{x_q\} \) intersects both components of \( D \setminus W_{sD}(f^{-k+\ell m}(x)) \). But for \( \ell \) large \( W_{sD}(f^{-k+\ell m}(x)) = W_{sD}(p_i) \). One deduces that \( \delta \setminus \{x_q\} \) intersects both components of \( D \setminus W_{sD}(p_i) \), a contradiction.

**Claim 42.** Let \( D \) be a Pixton disc associated to \( \Gamma \), with a boundary \( \partial D := \gamma \cup \sigma \cup \delta \) and let \( \varepsilon > 0 \). For \( n \) large enough \( f^n(D) \) is contained in a Pixton disc \( D' \) whose closing arc \( \delta' \) has diameter smaller than \( \varepsilon \).
Proof. Replacing $D$ by $f(D)$ if necessary, one can suppose that $f^{-1}(x)$ belongs to $W_0^*(p)$. Indeed let $\delta_1$ be the connected component of $\delta \setminus \gamma$ which intersects $\sigma$ and let $\gamma_1$ be the arc in $f(\gamma)$ connecting $\gamma$ to $\delta_1$. The disc bounded by $\sigma \cup \delta_1 \cup \gamma_1$ is a Pixton disc which contains $f(D)$. One can repeat that construction and define for each $n \geq 1$ a Pixton disc $D_n$ containing $f^n(D)$ with a closing arc $\delta_n \subset \delta$. Note that $f^n(\gamma) \subset \partial f^n(D)$, has points arbitrarily close to $f^{-1}(x)$ as $n$ gets large. One can thus connect $f^n(\gamma)$ to $f^{-1}(x)$ by an arc $\delta'$ with small diameter and build a Pixton disc bounded by $\sigma' = f^{-1}(\sigma)$, $\delta'$ and an arc $\gamma' \subset f^n(\gamma)$, which by construction contains $f^n(D)$.

Let $D$ be any Pixton disc associated to $\Gamma$ and consider a Pixton disc $D' \supset f^n(D)$ given by claim 42. The assumptions of claim 41 are also satisfied by $D'$, hence it holds that $\mu(\text{Interior}(D')) = 0$. This gives $\mu(\text{Interior}(D)) = 0$. Hence either $\mu(D) = 0$, or $\mu$ is supported on the orbit of the periodic point associated to $D$. In particular if $K$ supports an aperiodic measure, the conclusion of the proposition holds. We are thus reduced to suppose that all the ergodic measures on $K$ are periodic and get a contradiction.

Claim 43. For any periodic point $q \in K$, there exists $z \in (W_0^*(q) \setminus \{q\}) \cap K$.

Proof. By definition of $K = a(x)$, if the conclusion does not hold then $x$ belongs to an unstable branch $\Gamma(q)$ of $q$.

In the case where $\Gamma(q)$ is fixed, since $x$ is accumulated by $\Gamma$, the proposition 19 proves that $\Gamma(q)$ accumulates and it is accumulated by $\Gamma$. Since the periodic cycle $\Gamma_0, \ldots, \Gamma_n$ has maximal length, $\Gamma(q)$ coincides with one of the $\Gamma_j$. This is a contradiction since $x \in \Gamma(q) \cap W_0^*(p_i)$ but $W_0^*(p_i)$ is disjoint from all the $\Gamma_j$, by our choice of $p_i$.

In the case where $q$ has larger period, the proposition 35 implies (since we have reduced to the case where $f$ preserves the orientation) that $\Gamma(q)$ intersects both components of $W_0^*(p_i)$. Since the accumulation set of $\Gamma_i$ contains $\Gamma(q)$, one deduces that $\Gamma_i$ intersects both components of $W_0^*(p_i)$, which contradicts the fact that it is disjoint from $W_0^*(p_i)$.

One deduces that any periodic point $q_0 \in K$ admits an unstable branch $\Gamma(q_0)$ (fixed by an iterate $f^n$ which may not be $f$) which is accumulated by $\Gamma$. The accumulation set of $\Gamma(q_0)$ contains a periodic point $q_1 \in K$ and then an unstable branch $\Gamma(q_1)$ accumulated by $\Gamma(q_0)$. One can build in this way infinite sequences of unstable branches $(\Gamma(q_k))_{k \in \mathbb{N}}$.

First case: there exists a periodic sequence of unstable branches. By proposition 19 there exists a branch $\Gamma(q_k)$ which is fixed by some iterate $f^n$ and whose accumulation set contains $\Gamma(q_k)$. One can thus build a Pixton disc $D$ for $f^n$ associated to this branch. One can choose the closing arc $\delta$ with arbitrarily small diameter so that it is is disjoint from one of components of $\mathbb{D} \setminus W_0^*(p_i)$.

By construction the interior of $D$ contains points of $K$, hence arbitrarily large backward iterates of $x$. This contradicts the claim 41.

Second case: there is no periodic sequence of unstable branches. Since each unstable branch $\Gamma(q_k)$ intersects $K$, it can not be contained in a normally hyperbolic arc fixed by some iterate of $f$. By proposition 17 there are at most finitely many unstable branches $\Gamma(q_k)$ for each period.

One can thus consider a sequence $(\Gamma(q_k))$ with the following property: for any $N \geq 1$, there exists $\ell \geq 1$ such that for any $k \geq \ell$, all the periodic points $q \in K$ in the accumulation set of $\Gamma(q_k)$ have period larger than $N$. 

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If $K_k$ denotes the intersection of $K$ with the orbit of the accumulation set of $\Gamma(q_k)$, we get a decreasing sequence of compact sets which do not contain periodic points of period $N$ for $k$ large enough. Let $K_\infty$ be the intersection of all the $K_k$: by construction it does not contain any periodic point; hence it supports an aperiodic measure, which contradicts our assumptions.

The proof of the proposition is now complete.

5.5 Proof of theorems G and G’

We first prove theorem G. It is well known that if $f$ has positive entropy, then it admits horseshoes \([K\alpha]\) and in particular a cycle of periodic orbits.

Conversely, let us assume that $f$ has a cycle of periodic orbits. Up to replace $f$ by an iterate, one can suppose (by lemma 37) that $f$ has a fixed unstable branch $\Gamma$ which is contained in its accumulation set. Lemma 40 then gives an aperiodic ergodic measure $\mu$ supported on the closure of $\Gamma$ such that $\mu(D) = 0$ for any Pixton disc $D$ associated to $\Gamma$. Let us denote by $\mathbb{D}_\Gamma$ the closure of the connected component of $\mathbb{D} \setminus W^n_D(p)$ which contains $\Gamma$. By our assumption, the support of $\mu$ is contained in $\mathbb{D}_\Gamma$.

**Lemma 44.** There exists a neighborhood $U$ of $p$ such that $\mu(U) = 0$. In particular, the support of $\mu$ is disjoint from $\Gamma$ and $W^n_D(p)$.

**Proof.** The measure $\mu$ is supported inside $\mathbb{D}_\Gamma$. Moreover we have $\mu(p) = 0$ since $\mu$ is not atomic. Hence if one assumes that any neighborhood of $p$ has positive $\mu$-measure, there exists some point $x \neq p$ in $W^n_D(p)$ which belongs to the support of $\mu$. One deduces that $x$ is accumulated by $\Gamma$. One can thus build a Pixton disc $D$ by closing near $f^{-1}(x)$: the disc contains a neighborhood of $x$ in $\mathbb{D}_\Gamma$, hence has positive measure. This contradicts lemma 40.

Let $W^{s,+}_D(p)$ be one of the components of $W^n_D(p) \setminus \{p\}$ which contains points accumulated by $\Gamma$. Let $\Gamma_{loc}$ be a local unstable manifold of $p$, i.e. a neighborhood of $p$ inside $\Gamma$ for the intrinsic topology. It separates small neighborhoods of $p$ in $\mathbb{D}_\Gamma$ into two components: we denote by $U^+$ the component which meets $W^{s,+}_D(p)$. See Figure 12.

![Figure 12: Quadrant separated by $\Gamma$ and $W^{s,+}(p)$](image)

Note that $\mu$-almost every point $x$ is accumulated by its orbit inside each component of $\mathbb{D} \setminus W^n_D(x)$. In particular $\Gamma$ meets these two components and intersects $W^n_D(x)$ at
some point \( z \). Iterating backward \( W^+_D(z) \), one thus gets a sequence of stable curves \( \alpha \subseteq D^+ \) such that \( f(W_n) \subseteq W_{n-1} \), \( f^n(W_n) \subseteq W^+_D(z) \), which converge to \( W^+_D(p) \) for the Hausdorff topology. We denote by \( W^+_n \) a connected component of \( W_n \setminus W^+_D(x) \) which is close to \( W^+_D(p) \) for the Hausdorff topology. By choosing \( n \) large enough, \( W^+_n \) separates \( W_n \) and \( W_{n+1} \) in \( U^+ \). See Figure 12.

Let \( x^s \in W^+_D(p) \) be a point that is not accumulated by \( \Gamma \) and let \( \beta^s \) be a small \( C^1 \) arc transverse to \( W^+_D(p) \) at \( x^s \). We also choose \( x^u \in \Gamma \) and a small \( C^1 \) arc \( \beta^u \) transverse to \( x^u \) at \( \Gamma \). For \( m \geq 1 \) large, the arcs \( f^{-m}(\beta^s) \), \( \beta^u \), \( f(\beta^u) \) and \( W^+_D(p) \) bound a rectangle \( R \). Similarly, the arcs \( f^{-1}(\beta^u) \), \( \beta^u \), \( f(m)(\beta^u) \) and \( \Gamma \) bound a rectangle \( R' \). We may choose \( W^+_n \) and \( W^+_{n+1} \) to separate \( p \) from \( \beta \) and \( m \) large enough. One thus get the following properties:

(a) \( R' \) is separated from \( R \) by \( W^+_{n-1} \) in \( U^+ \),

(b) Any point in \( R \setminus W^+_D(p) \) has a forward iterate in \( R' \).

Note that the forward iterates of \( W_n \) and \( W_{n-1} \) accumulate on the support of \( \mu \). As a consequence of lemma 44, if \( R \) has been chosen small enough, we get:

(c) The forward iterates of \( W_n \) and \( W_{n-1} \) do not meet \( R \).

Let \( D \) be a Pixton disc associated to \( \Gamma \), whose boundary is the union of three arcs: \( \sigma \in W^+_D(p), \gamma \subset \Gamma \) and a closing arc \( \delta \). We chose \( D \) so that \( \delta \) is contained in \( R \) and \( \gamma \) intersects \( R \) in only one point. Since \( f^{-1}(\gamma) \subset \gamma \), this implies that the arc \( \gamma \) is disjoint from the forward iterates of \( R \). In particular,

(d) \( R' \) is contained in \( D \).

Let us consider the two curves \( \alpha^i \subseteq W^+_n \) and \( \alpha \subseteq W^+_{n-1} \), contained in \( U^+ \cap D \) which connect \( \Gamma \) to another point of the boundary of \( D \) (and intersecting the boundary of \( D \) only at their endpoints, which by construction belong to \( \Gamma \)). Note that \( f(\alpha) \subset \alpha^i \) by definition; both are contained in \( W_{n-1} \). The curve \( \alpha \cup \gamma \) bounds a disc \( \Delta^i \) whereas the curve \( \alpha^i \cup \gamma \) bounds a disc \( \Delta^i \). Since \( W^+_n \) separates \( W^+_{n-1} \) and \( W^+_D(p) \), the discs are nested: \( \Delta^i \subseteq \Delta \). See Figure 13.

Let \( \hat{\alpha} \) and \( \hat{\alpha}^i \) denote the arcs \( \alpha, \alpha^i \) without their endpoints. Using (c), that \( f^{-1}(\gamma) \subset \gamma \) and that \( \gamma \) is disjoint from \( \hat{\alpha} \cup \hat{\alpha}^i \), one deduces that the forward iterates of \( \hat{\alpha}^i \) do not intersect \( \partial D = \sigma \cup \gamma \cup \delta \). Hence:

(e) Any forward iterate of \( \hat{\alpha} \) or \( \hat{\alpha}^i \) is either in the interior of \( D \) or disjoint from \( D \).

For \( k \) large, the images \( f^k(\alpha) \) and \( f^k(\alpha^i) \) are contained in a small neighborhood of the support of \( \mu \), hence are outside \( D \). Since \( f(\alpha) \subset \alpha^i \), one gets: \( f^{i+1}(\hat{\alpha}) \) is disjoint from \( D \) if and only if \( f(\hat{\alpha}^i) \) is disjoint from \( D \). Together with (e), one deduces that there exists \( k_0 \) such that \( f^{k_0}(\hat{\alpha}) \) is disjoint from \( D \), \( f^{k_0}(\hat{\alpha}^i) \) is in the interior of \( D \) and all the larger iterates are disjoint from \( D \). This implies the following lemma.

**Lemma 45.** There exists \( k_0 \geq 1 \) such that for any curve \( \beta \subset \Gamma \) in the interior of \( D \) and connecting \( \alpha \) to \( \alpha^i \):

- \( f^{k_0}(\beta) \) meets \( \delta \), has one endpoint in the interior of \( D \) and another one outside \( D \),
all the forward iterates of the endpoints of $f^{k_0}(\beta)$ are outside $D$.

**Proof.** Consider the iterate $k_0$ such that $f^{k_0}(\alpha)$ is inside, $f^{k_0}(\alpha')$ is outside and all larger iterates of $\alpha'$ are outside also. Note that the forward iterates of $\beta \subset \Gamma \setminus \gamma$ do not intersects $\gamma$ nor $\sigma \subset W^s_D(p)$.

From a curve $\beta$, one gets two new ones $\beta_1, \beta_2$.

**Lemma 46.** There exists $k_1$ and $\varepsilon > 0$ such that any curve $\beta$, in the interior of $D$ and connecting $\alpha$ to $\alpha'$, contains two sub-curves $\beta_1, \beta_2$ such that:

$f^{k_1}(\beta_1), f^{k_1}(\beta_2)$ are $\varepsilon$-separated, contained in $\text{Interior}(D)$ and connect $\alpha$ to $\alpha'$.

**Proof.** By Lemma 45, the curve $\beta$ contains sub curves $\bar{\beta}_1, \bar{\beta}_2$ such that:

- $\bar{\beta}_1$ (resp. $\bar{\beta}_2$) contains an endpoint $b_1$ (resp. $b_2$) of $\beta$ and a point of $f^{-k_0}(\delta)$,
- $f^{k_0}(\bar{\beta}_1)$ is disjoint from the interior of $D$,
- $f^{k_0}(\bar{\beta}_2)$ is contained in $D$.

From lemma 45 all the forward iterates of $f^{k_0+1}(b_i)$, for $i \in \{1, 2\}$ are outside $D$. From (b), there exists $k \geq k_0$ such that $f^k(\bar{\beta}_1)$ and $f^k(\bar{\beta}_2)$ have a point in the interior of $R'$, hence in the interior of $\Delta'$ (from (d)). Thus by (a) these curves contain a point of $\alpha'$ and a point of $\alpha$. Since the iterates of $\beta$ never intersects $\gamma \cup \sigma$, one deduces that for any $k' \geq k$, $f^{k'}(\bar{\beta}_1)$ and $f^{k'}(\bar{\beta}_2)$ still intersect $\alpha$ and $\alpha'$ and in particular contain two curves connecting $\alpha$ to $\alpha'$.

The integer $k$ may depend on $\beta$, but since $f^{k_0}(\beta)$ intersect $\delta \setminus W^s_D(p)$ in a compact set which does not depend on $\beta$, the integer $k$ is is uniformly bounded. One can thus find $k' = k_1$ independent from $\beta$ such that both $f^{k_1}(\bar{\beta}_1)$ and $f^{k_1}(\bar{\beta}_2)$ meet $\alpha$ and $\alpha'$.

Let us choose the minimal curve $\hat{\beta}_1 \subset \bar{\beta}_1$ which connects $b_1$ to $f^{-k_1}(\alpha')$ and the minimal curve $\hat{\beta}_2 \subset \bar{\beta}_2$ which connects $b_2$ to $f^{-k_1}(\alpha')$. In particular for any $k_0 \leq k < k_1$, the curves $f^k(\hat{\beta}_i)$ are disjoint from $\Delta'$.
One then choose $\beta_1 \subset \bar{\beta}_1$ and $\beta_2 \subset \bar{\beta}_2$ such that $f^{k_1}(\beta_1)$ and $f^{k_1}(\beta_2)$ meet $\alpha$ and $\alpha'$ at their endpoint and nowhere else.

By construction $f^{k_0}(\beta_1)$ is disjoint from $D$ and $f^{k_0}(\beta_2)$ is contained in the interior of $D$. They are contained in two different connected components of $f^{k_1-k_0}(\Delta \setminus \bar{\Delta'}) \setminus \delta$. Moreover they avoid a uniform neighborhood of $\bar{\delta} := \delta \cap f^{k_1-k_0}(\Delta \setminus \bar{\Delta'})$: indeed there exists $\ell_0$ such that any point $y$ in $\bar{\delta}$ has a forward iterate $f^\ell(y)$ in $R'$ with $\ell \leq \ell_0$. By compactness the same holds for any point in a neighborhood of $\bar{\delta}$. But by construction for any point in $f^{k_0}(\beta_1)$ and $f^{k_0}(\beta_2)$, the $k_1-k_0-1$ first iterates are disjoint from $R'$.

Hence by choosing $k_1 > k_0 + \ell$, one can ensure that $f^{k_0}(\beta_1)$ and $f^{k_0}(\beta_2)$ are disjoint from a uniform neighborhood of $\bar{\delta}$.

After having fixed $k_1$ and having chosen $\varepsilon > 0$ small enough, one deduces that the curves $f^{k_1}(\sigma_1)$, $f^{k_1}(\sigma_2)$ are $\varepsilon$-separated for some $\varepsilon > 0$ small as required.

Note that $\Gamma \setminus \gamma$ contains an arc that connect a point in $R$ with a point in $R'$: this shows that there exists a curve $\beta \subset \Gamma$ contained in the interior of $D$ which connects $\alpha$ to $\alpha'$. One then apply lemma 46 inductively: it shows that for each $\ell$, the arc $\beta$ contains $2^\ell$ orbits of length $\ell k_1$ that are $\varepsilon$-separated. One deduces that the topological entropy of $f$ is larger than $\log(2)/k_1$, hence positive.

The proof of Theorem $G$ is now complete.

The proof of Theorem $G'$ is the same, working inside the filtrating domain. □

6 Generalized Morse-Smale diffeomorphisms

We extend the definition 2 to filtrating sets:

**Definition 47.** A diffeomorphism is generalized Morse-Smale in a filtrating set $U$ if

- the $\omega$-limit set of any forward orbit in $U$ is a periodic orbit,
- the $\alpha$-limit set of any backward orbit in $U$ is a periodic orbit,
- the period of all the periodic orbits contained in $U$ is bounded by some $K > 0$.

We also say that a diffeomorphism is mildly dissipative in a filtrating set $U$ if for any ergodic measure $\mu$ for $f|_U$, which is not not supported on a hyperbolic sink, and for $\mu$-almost every $x$, $W^u_\mu(x)$ separates $U$.

**Proposition 48.** Any diffeomorphism of the disc which is mildly dissipative and generalized Morse-Smale in a filtrating set $U$ has zero topological entropy in $U$. Moreover the chain-recurrent points in $U$ are all periodic.

**Proof.** Any ergodic measure of $f|_U$ is supported on a periodic orbit, hence has zero entropy. The variational principle concludes that the topological entropy of $f|_U$ is zero.

Up to replace $f$ by an iterate, one can suppose that all the periodic points and all the unstable branches in $U$ are fixed by $f$. Let us assume by contradiction that there exists a chain-recurrent point $x$ which is not periodic. One chooses as in proposition 17 a finite collection of disjoint fixed arcs $I$ of $U$. One can require that they do not contain $x$. By our assumption, $x$ belongs to an unstable branch of an arc $I_0$, which accumulates on an arc $I_1$. Since $x$ is chain-recurrent, there exists pseudo-orbits from $I_1$ to $I_0$, hence there exists an unstable branch of $I_1$ which accumulates on another arc $I_2$ and there
exists pseudo-orbits from $I_2$ to $I_0$. Arguing inductively, one builds a sequence of fixed arcs $I_n$ in $U$ such that the unstable manifold of $I_n$ accumulates on the arc $I_{n+1}$. Since $\mathcal{I}$ is finite, this implies that there exists a cycle of arcs in $U$, contradicting corollary 33.

Proposition 49. The set of diffeomorphisms of the disc which are mildly dissipative and generalized Morse-Smale in a filtrating set $U$ is open for the $C^1$ topology.

Proof. Note that if $U$ is filtrating for $f$, it is still filtrating for diffeomorphisms close.

From proposition 17, there exists a finite collection $\mathcal{I}$ of disjoint arcs in $U$ that are fixed by an iterate $f^k$ of $f$. By normal hyperbolicity (see [BoCr]), for any $I \in \mathcal{I}$, there exists a neighborhood $V_I$ such that for any diffeomorphism $g$ that is $C^1$ close to $f$, any orbit of $g^k$ contained in $V_I$ is contained in a $g^k$-fixed arc contained in $V_I$; such an arc is still normally contracted. One deduces that any forward (resp. backward) $g^k$-orbit contained in $V_I$ accumulates on a fixed point of $g^k$.

Since $\mathcal{I}$ is finite and since the neighborhoods $V_I$ of the arcs $I$ may be chosen small, one gets a neighborhood $V = \bigcup V_I$ of the set of periodic points of $f$ with the following property: for any diffeomorphism $g$ that is $C^1$ close to $f$, the $\omega$-limit of any forward orbit of $g$ contained in $V$ is an orbit of period less or equal to $k$ and the same holds for the $\alpha$-limit of any backward orbit of $g$ contained in $V$.

The chain-recurrent set varies upper semi-continuously with the dynamics. Hence for any diffeomorphism $g$ that is close to $f$, the $\omega$-limit and the $\alpha$-limit sets of a $g$-orbit contained in $U$ is contained in $V$, and they are periodic orbits of period less or equal to $k$. This proves that $g$ is generalized Morse-Smale in $U$.

7 Stabilization, decoration, structure of periodic points

In this section $f$ is a mildly dissipative diffeomorphism of the disc with zero entropy. First we introduce and discuss two related types of configurations of saddle periodic orbits: the decoration and the stabilization (subsection 7.1). We then describe how the set of fixed points (or points of a given period) are organized through chains (see sections 7.2). Later, using the chains, we define a hierarchy between periodic points (section 7.3) and at the end, in proposition 73 we show that all periodic points are related through this hierarchy.

7.1 Stabilization and decoration

Definition 50. A periodic point $p$ is stabilized by a fixed point $q$ if one of the two following cases occurs (see figure 14):

- either $p = q$ is a fixed point, not a sink, and $D_p f$ has an eigenvalue $\lambda^+_p \leq -1$,
- or $p$ has period larger than 1 and an unstable branch $\Gamma$ which accumulates on $q$ such that either $q$ is not stabilized or $q$ is also a stabilized saddle and in that case it is required that $\Gamma$ intersects both components of $\mathbb{D} \setminus W^u_D(q)$.

Sometimes we also say that the orbit of $p$ is a stabilized periodic orbit and that $q$ is a stabilizing point. The unstable branch that accumulates on $q$ is a stabilizing branch.

Remarks 51. Let us make a few observations about stabilized and stabilizing points.
1. The first case can be considered as a degenerate case of the second: as explained in remark 13, $p$ can be considered as a 2-periodic point which has collided with the stabilizing fixed sink $q$. The stabilizing branches are hidden in $q$ in this case.

2. In the second case, $q$ could be a fixed point of any type: a sink, indifferent or a saddle (in that case, it could be either stabilized or not).

3. There may exist several stabilizing points $q$ associated to a stabilized point $p$.

We have introduced the notion of decorated periodic orbit in section 2.5.

**Proposition 52** (Stabilization implies decoration). If $f$ is a mildly dissipative diffeomorphism with zero entropy, then any periodic orbit $O$ which is stabilized by a fixed point is decorated. Each point $p \in O$ has at most one stabilizing unstable branch.

**Proof.** In the particular case where $O$ is a fixed point, the statement become trivial, we will thus assume that $O$ has period larger than 1.

Consider $p \in O$ and the connected component $C$ of $\mathbb{D} \setminus W^s_D(p)$ which does not contain the stabilizing point $q$. If one assumes that some iterate $f^j(p)$ belongs to $C$, then the unstable branch of $f^j(p)$ which accumulates on $q$ intersects both components of $\mathbb{D} \setminus W^s_D(p)$, hence intersects $W^s(p)$: this implies that $f$ has a cycle of periodic orbit, a contradiction.

We have proved that the orbit of $p$ is contained in the connecting component of $\mathbb{D} \setminus W^s_D(p)$ which contains the stabilizing branch. In particular $p$ has at most one stabilizing unstable branch.

**Definition 53.** When $p$ is stabilized by a fixed point $q$, the connected component of $\mathbb{D} \setminus W^s_D(p)$ which does not contain $q$ is called the decorated region of $p$. (In the special case where $p = q$ is a fixed point, it admits two decorated regions.)

The period of the decorated region is either the period of $p$ (when $p$ is not fixed) or 2 (when $p$ is fixed): this is the return time to the decorated region for points close to $p$.

**Proposition 54.** If $f$ is a mildly dissipative diffeomorphism with zero entropy which reverses the orientation, then each stabilized orbit has period 1 or 2.
Proof. Let us consider a stabilized periodic point \( p \) with period \( k \). By definition there exists an unstable branch \( \Gamma \) of \( p \) which accumulates on a fixed point \( q \). We denote by \( K_p \) the accumulation set of \( \Gamma \). This set is cellular and fixed by \( f^k \). Hence the set \( K := \bigcup_n f^n(K_p) \) is a cellular set fixed by \( f^k \). The complement \( \mathbb{D} \setminus K \) is an invariant annulus. Let us denote by \( B = \mathbb{R} \times (0,1) \) the universal cover with the covering automorphism \((x,t) \mapsto (x+1,t)\). The map \( f \) on the annulus lifts as a map \( h \) on \( B \) which reverses the orientation and satisfies \( h(x+1,t) = h(x,t) - (1,0) \). Let \( \gamma \) be the union of \( \Gamma \) with only one local stable branch of \( p \): it is a proper curve in the annulus which connects one end to the other one. It lifts in \( B \) as a curve \( \tilde{\gamma}_0 \) whose complement has two connected components. Repeating the construction for each iterate of \( p \) and considering the translated curves, one obtains a family of curves \((\tilde{\gamma}_n)_{n \in \mathbb{Z}}\) in \( B \) with the properties:

- \( \tilde{\gamma}_{n+k} = \tilde{\gamma}_n + (1,0) \),
- \( B \setminus \gamma_n \) has two connected components \( U_n^- \) and \( U_n^+ \) satisfying \( U_n^- \subset U_m^- \) when \( n \geq m \),
- there exists a bijection \( \tau \) of \( \mathbb{Z} \) such that \( h(\gamma_n) \subset \gamma_{\tau(n)} \).

In particular \( \tau \) is monotone. Since \( h \) reverses the orientation there exists \( a \in \mathbb{Z} \) such that \( \tau(n) = -n + a \) for each \( n \in \mathbb{Z} \). In particular either \( \tau \) has a fixed point or a point of period 2. This implies that \( p \) is either fixed or has period 2.

The previous proposition shows that when \( f \) reverses the orientation, all the decorated regions have period 2.

**Corollary 55.** If \( k \) is the period of a decorated region, then \( f^k \) preserves the orientation.

### 7.2 Structure of the set of fixed points

We introduce a notion which generalizes the fixed arcs.

**Definition 56.** Let \( p, p' \) be two fixed points. A chain for \( f \) between \( p \) and \( p' \) is a (not necessarily compact) connected set \( C \) which is the union of:

- a set of fixed points \( X \) containing \( p \) and \( p' \),
- some \( f \)-invariant unstable branches of points in \( X \).

**Proposition 57.** If \( f \) is an orientation preserving mildly dissipative diffeomorphism with zero entropy, then between any pair of fixed points \( p, p' \) there exists a chain for \( f \).

The end of this section is devoted to the proof of this proposition. Previous proposition also holds for mildly dissipative diffeomorphisms without cycles.

**Lemma 58.** If \( f \) is an orientation-preserving dissipative diffeomorphism of the disc, any \( f \)-invariant unstable branch \( \Gamma \) accumulates on a fixed point.

**Proof.** Let \( \gamma \subset \Gamma \) be a curve which is a fundamental domain. By definition the accumulation set \( \Lambda \) is an invariant compact set. Since it is arbitrarily close in the Hausdorff topology to a curve \( f^i(\gamma) \cup f^{i+1}(\gamma) \cup \cdots \cup f^j(\gamma) \), the set \( \Lambda \) is connected. Since \( f \) is dissipative, the complement \( \mathbb{D} \setminus \Lambda \) is connected. One deduces from proposition \[21\] that \( \Lambda \) contains a fixed point.

Let us consider the finite set \( \mathcal{I} \) of normally hyperbolic fixed arcs as in section \[2.2\] and recall that fixed points can be treated as arcs.
Lemma 59. For any $f$-invariant unstable branch $\Gamma$ of an arc $I \in \mathcal{I}$, the accumulation set intersects an arc $I' \in \mathcal{I}$ of index 0 or 1.

Proof. The branch $\Gamma_0 = \Gamma$ is contained in the unstable set of a fixed point $p_0$. From lemma 58 the accumulation set of $\Gamma_0$ contains a fixed point $p_1$. Let $I_1 \in \mathcal{I}$ be the fixed arc that contains $p_1$. If $I_1$ has index 0 or 1, the corollary holds. Otherwise $I_1$ has the type of a saddle with no reflexion. Since $p_0 \notin I_1$, the branch $\Gamma_0$ intersects the stable manifold of an endpoint of $I_1$. One can thus reduce to the case where $p_1$ is an endpoint of $I_1$ and where $I_1$ and $p_1$ have a common unstable branch $\Gamma_1$ which intersects the accumulation set of $\Gamma_0$. One can repeat the previous construction with the arc $I_1$ and the unstable branch $\Gamma_1$. One build in this way inductively a sequence of arcs $I_n$ with an unstable branch $\Gamma_n$ which accumulate on $I_{n+1}$. Since the number of arcs in $\mathcal{I}$ is finite, and there is no cycle (corollary 33), this sequence stops with one arc of index 0 or 1. By construction, each unstable branch $\Gamma_n$ accumulates on the unstable branch $\Gamma_{n+1}$. The proposition 19 shows that $\Gamma_0$ accumulates on $\Gamma_{\ell-1}$, hence on the last arc $I_\ell$. □

We introduce the following equivalence relation $\sim$ between fixed arcs $I, I' \in \mathcal{I}$:

$I \sim I'$: There exists a sequence of arcs $I = I_1, I_2, \ldots, I_\ell = I'$ in $\mathcal{I}$ such that for each $0 \leq i < \ell$ either $I_i$ admits a $f$-invariant unstable branch which accumulates on $I_{i+1}$ or $I_{i+1}$ admits a $f$-invariant unstable branch which accumulates on $I_i$.

Lemma 60. The relation $\sim$ has only one equivalence class.

Proof. It is enough to prove that the sum of the indices of the arcs in an equivalence class is larger or equal to 1. Then the Lefschetz formula (proposition 18) will conclude that there is at most one class.

Let $C$ be any equivalence class for $\sim$. It always contains a fixed arc of index 1:

Claim 61. The class $C$ contains a fixed arc with no $f$-invariant unstable branch.

Proof. From lemma 58 each $f$-invariant unstable branch of a fixed interval accumulates on a fixed interval. If the conclusion of the claim does not hold, one thus obtain an infinite sequence $I_n$ in $C$ such that $I_n$ admits a $f$-invariant unstable branch which accumulates on $I_{n+1}$. Since the set $\mathcal{I}$ of fixed intervals is finite, one gets a cycle between fixed interval, contradicting the conclusion of corollary 33. □

We then associate to any fixed arc of index $-1$ another arc of index 1. Let:

$\mathcal{N} := \mathbb{D} \setminus \bigcup \{W_{D}^s(I_i), I_i \in \mathcal{I} \text{ of index } -1\}$.

Claim 62. Let $U$ be a connected component of $\mathcal{N}$. Let $I \in \mathcal{I}$ be an arc of index $-1$ such that $W_{D}^s(I) \text{ bounds } U$. Then $U$ contains an arc $I' \in \mathcal{I}$ of index 1 such that $I \sim I'$.

Proof. We consider the sequences of arcs $I_1, \ldots, I_\ell$ in $\mathcal{I}$ such that $I_1 = I$, for each $k$ there exists an unstable branch of $I_k$ which accumulates on $I_{k+1}$ and each $W_{D}^s(I_k)$ either is contained in $U$ or bounds $U$. From corollary 33 such a sequence is finite. One can assume that it has maximal length.

We claim that its last element $I_\ell$ has index 1 (hence is included in $U$). If this is not the case and $I_\ell$ has index 0, it is contained in $U$ and admits an unstable branch. From lemma 59 either this unstable branch accumulated on a fixed arc contained in $U$ or it intersects one of the boundaries $W_{D}^s(I)$ of $U$. In both case, we build a new fixed arc and the sequence $I_1, \ldots, I_\ell$ is not maximal, a contradiction.

By construction the last element $I' := I_\ell$ belongs to the class $C$. □
Now we proceed to finish the proof of lemma 60. Let us choose arbitrarily a fixed arc \( I(0) \in C \) of index 1. For each arc \( I \in \mathcal{I} \) of index \(-1\), let us consider the connected component \( V \) of \( \mathbb{D} \setminus W_u^s(I) \) which does not contain \( I(0) \). Let \( U \) be the connected component of \( \mathcal{N} \) which is contained in \( V \) and whose boundary intersects \( W_u^s(I) \). The previous claim associates to it an arc \( I^I \in C \) of index 1 contained in \( U \). It is by construction different from \( I(0) \).

Note that if \( \tilde{I} \in C \) is another arc of index \(-1\), the associated arc \( \tilde{I}^I \) of index 1 is different: indeed, in each component \( U \) of \( \mathbb{D} \setminus W_u^s(I) \) which does not contain \( I(0) \), there exists a unique \( I \in \mathcal{I} \) such that \( W_u^s(I) \) bounds \( U \) and separates \( U \) from \( I(0) \).

We have shown that in \( C \) the number of arcs of index \(-1\) is smaller than the number of arcs of index 1. This concludes the proof of the lemma 60. \( \square \)

**Proof of proposition 57.** Any normally hyperbolic fixed arc is a chain for \( f \). The lemma 60 proves that the union \( C \) of arcs in \( \mathcal{I} \) with their \( f \)-invariant unstable branches is a connected set. Note that any arc is the union of a set of fixed points with \( f \)-invariant unstable branches. This shows that \( C \) of \( f \) is a chain between any pair of fixed points. \( \square \)

**Remark 63.** The proof of the proposition (and claim 62) shows the following property:

Assume that \( f \) preserves the orientation. Let \( \mathcal{I} \) be a finite collection of disjoint isolated arcs fixed by \( f \) such that for any \( I \in \mathcal{I} \) and any \( f \)-invariant unstable branch \( \Gamma \) of \( I \), any periodic point in the accumulation set of \( \Gamma \) belongs to some \( I^I \in \mathcal{I} \). Then the sum of the indices \( \text{index}(I, f) \) of all \( I \in \mathcal{I} \) is larger or equal to 1.

### 7.3 Points decreasing chain related to a stabilized point

In the present section we discuss how periodic points of larger period are related to points of lower period. Since proposition 57 holds for any (orientation preserving) iterate of \( f \), any periodic point can be related to the fixed points through a chain associated to a large iterate of \( f \). In the next definitions and propositions we show that these chains have a particular structure that link points of larger period to points of lower one.

**Definition 64.** Let \( p \) be a stabilized periodic point. A periodic point \( w \neq p \) is **decreasing chain related** to \( p \) if there exists \( k \geq 2 \) and a chain \( C \) for \( f^k \) between \( w \) and \( p \) which is contained in the closure of a decorated region of \( p \).

**Remark 65.** Note that any iterate \( f^i(C) \) of the chain is contained in the closure of a decorated region of \( f^i(p) \), hence \( f^i(w) \) is decreasing chain-related to \( p \). One deduces that the period of the decorated region of \( p \) divides the period of \( w \). We also say that the orbit of \( w \) is decreasing chain related to the orbit of \( p \).

The unstable set of a decreasing chain related point can be localized.

**Proposition 66.** If \( w \) is decreasing chain related to a stabilized periodic point \( p \), then the unstable set of \( w \) is contained in the closure of a decorated region \( V \) of \( p \).

Moreover, if the period of \( w \) is larger than the period of \( V \) and \( f \) is orientation preserving, then the closure of the unstable set of \( w \) is contained in \( V \).

**Proof.** Let us consider the two connected components of \( \mathbb{D} \setminus W_u^s(w) \) (one has to consider only periodic points which are not sinks, so as described in section 2 there is a unique stable manifold well defined). Since \( w \) belongs to a decorated region \( V \) of \( p \), one of these components \( U_1 \) is contained in \( V \). The other one is denoted by \( U_2 \). From theorem 3 (no cycle), any unstable branch \( \Gamma \) of \( w \) is contained in one of these components. If \( \Gamma \) is
included in $U_1$, then it is included in a decorated region of $p$ and the proof is concluded in that case.

We may thus assume that $\Gamma$ is included in the component $U_2$ of $D \setminus W^s_D(w)$ which contains $p$ and let us prove first that its accumulation set is contained in the closure of $V$. See figure 15. Since $w$ is decreasing chain related to $p$ then there exists a chain $C$ for an iterate $f^k$ which contains $w, p$ and which is included in the closure of the decorated region $V$ (recall definition 64). If $\Gamma$ is part of the chain $C$, by definition is included in the closure of $V$. So, let us consider the case where $\Gamma$ is not part of the chain; then there exist points of $C$ in $U_2$ which accumulate on $\Gamma$. Since the period of points in $C$ is uniformly bounded and since $\Gamma$ is an unstable branch in $U_2$, the point $w$ is not accumulated by periodic points of $C \cap U_2$. Consequently, there exists an unstable branch $\Gamma_C$ in $C$ which accumulates on a point of $\Gamma$. From proposition 19, one deduces that the accumulation set of $\Gamma$ is included in the accumulation set of $\Gamma_C$, which is contained in $\overline{V}$, and so the accumulation set of $\Gamma$ is also contained in $\overline{V}$.

In order to conclude, we distinguish three cases:

- The period of $p$ is larger than 1. Let $\Gamma_p$ be the unstable branch of $p$ which accumulates on a fixed point $q$ (not contained in $V$). If $\Gamma$ is not included in the closure of $V$, it crosses $W^\circ_D(p)$. As a consequence the accumulation set of $\Gamma$ contains the accumulation set of $\Gamma_p$, hence $q$. This is a contradiction since we have shown before that it is contained in $\overline{V}$.

- The point $p$ is a fixed saddle with reflexion: it admits an unstable branch $\Gamma_p$, which accumulates on a periodic point $q$ which is not in $\overline{V}$ (by lemma 58). One can conclude as in the previous case.

- The $p$ is a fixed point, admits an eigenvalue $\lambda^+_p = -1$, is not a sink, and not a saddle: it is accumulated by points $z \in D \setminus \overline{V}$ of period 2. If $\Gamma$ is not included in the closure of $V$, it crosses $W^\circ_D(p)$, hence it intersects the stable manifold of one of them. As a consequence the accumulation set of $\Gamma$ contains $z$. This is a contradiction since we have shown before that it is contained in $\overline{V}$.

In all the cases we have shown that any unstable branch of $w$ is contained in $\overline{V}$.

Let $k$ be the period of the decorated region of $p$. If the period of $w$ is larger than $k$ and if $f$ is orientation preserving, one applies proposition 33 to the diffeomorphism...
Corollary 67. A point $w$ which is stabilized can not be decreasing chain-related to a stabilized point $p$.

Proof. We argue by contradiction. Let us first assume that $p$ is not fixed. From proposition 66, the unstable set of each iterate $f^i(w)$ is contained in the decorated region of $f^i(p)$. Since the decorated region has period larger or equal to 2 and since $p$ is not fixed, the unstable set of $w$ does not accumulate on a fixed point, a contradiction.

When $p$ is fixed, since the unstable manifold of $w$ is contained in a decorated region, the point $w$ can only be stabilized by $p$. The definition of stabilized point for $p$ gives that $p$ is not a sink; the definition for $w$ implies that the unstable manifold of $w$ crosses $W^u_S(p)$, a contradiction.

Proposition 68. Let us consider two stabilized fixed points $p_1, p_2$ with decorated regions $V_1, V_2$. If there exists a point $w \in V_1$ that is decreasing chain- related to $p_2$, then $V_2 \subset V_1$.

Proof. By assumption $q \in V_1 \cap V_2$. Since the stable manifolds of $p_1$ and $p_2$ are disjoint, and the conclusion of the proposition does not hold, then $V_1 \subset V_2$.

Since $w$ is decreasing chain related to $p_2$ in $V_2$, there exists an iterate $f^j$ and a chain $C$ for $f^i$ included in $\overline{V_2}$ which contains both $w$ and $p_2$. In particular it intersects $W^u_S(p_1)$. This implies that there exists an unstable branch $\Gamma \subset C$ which meets both components of $\mathcal{D} \setminus W^u_S(p_1)$. One deduces from proposition 19 that the closure of the unstable set of $p_1$ is contained in the closure of $\Gamma$, hence in $\overline{V_2}$.

The closure of the unstable set $p_1$ contains a fixed point (since $p_1$ is stabilized): the only possible fixed point is $p_2$. By definition of stabilization, either $p_2$ is not stabilized or the unstable set $p_1$ meets both components of $\mathcal{D} \setminus W^u_S(p_2)$. In both cases we get a contradiction.

Corollary 69. A point $w$ can not be decreasing chain-related to two different stabilized point $p_1, p_2$.

Proof. Otherwise $w$ would belong to decorated regions $V_1$ and $V_2$ for $p_1$ and $p_2$ and respectively. The proposition 68 would imply that $V_1 \subset V_2$ and $V_2 \subset V_1$ simultaneously. A contradiction since $p_1 \neq p_2$.

One also describes the accumulation sets of $f$-invariant unstable branches.

Proposition 70. Let $z$ be a fixed point and $\Gamma$ be a $f$-invariant unstable branch of $z$. Let $C$ be a chain for some iterate $f^k$ between two periodic points $w, p$ such that:

- $p$ is stabilized by a fixed point $q$,
- $w$ is decreasing chain-related to $p$,
- $C$ is contained in the closure of a decorated region of $p$.

If the accumulation set of $\Gamma$ contains $w$, then it also contains $p$ (see figure 16).

Proof. By invariance, the points $f(w)$ and $f(p)$ and the chain $f(C)$ satisfy the same properties. Note that the decorated region of $p$ containing $w$ and the decorated region of $f(p)$ containing $f(w)$ are disjoint: this is a consequence of proposition 52 when $p$
has period larger than 1; when $p$ is fixed, this is a consequence of the fact that its two decorated regions are locally exchanged (since in the case that $p$ is fixed by definition the non-stable eigenvalue with modulus larger and equal to one is negative).

The $f$-invariant unstable branch $\Gamma$ accumulates in $w$ and $f(w)$ and so it has to intersect two different decorated regions, hence intersects the stable manifold of the orbit of $p$. This gives the conclusion.

\[\square\]

### 7.4 Lefschetz formula associated to a stabilized point

Using the notion of decreasing chain related periodic point, we define the notion of index of a decorated region.

**Index of a decorated region.** Given a decorated region $V$ of a stabilized periodic point $p$, and a multiple $n$ of the period $k$ of $V$, one can compute the total index of the set $C(p, V, n)$ of points $w \in V$ that are fixed by $f^n$ and decreasing chain-related to $p$. Note that by corollary 55 the map $f^n$ preserves the orientation.

From section 2.2 there exists a finite family $I$ of disjoint arcs that are fixed by $f^n$ and contained in $V$ such that the set of periodic points in $\bigcup_{I \in I} I$ is exactly $\{p\} \cup C(p, V, n)$. We denote by $I_0$ the arc of $I$ which contains $p$. Note that the other arcs $I \in I$ are isolated, hence have an index $\text{index}(I, f^n)$. The arc $I_0$ is maybe not isolated (in the case $p$ is a fixed stabilized point), but one can consider the index $\text{index}(I_0, V, f^n)$ of the half arc $I_0$ in the region $V$ for $f^n$ as defined in section 2.3.

Then, one defines the index of the decorated region $V$ for $f^n$ as

$$L(V, f^n) := \text{index}(I_0, V, f^n) + \sum_{I \in I \setminus \{I_0\}} \text{index}(I, f^n).$$

Observe that the number $L(V, f^n)$ does not depend on the choice of the family $I$:

**Proposition 71.** For any decorated region $V$ of a stabilized periodic point $p$, and for any multiple $n$ of the period $k$ of $V$, the index $L(V, f^n)$ equals $1/2$.

The proof is postponed to section 7.5. Before, we prove a weaker statement.

**Lemma 72.** For any decorated region $V$ of a stabilized periodic point $p$, and for any multiple $n$ of the period $k$ of $V$, the index $L(V, f^n)$ is larger or equal to $1/2$. 

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Proof. Since $f^n$ preserves the orientation, we follow the proof of proposition\footnote{67} and prove a version of remark \footnote{63} inside the decorated region $V$, after the following observations:

- for any $I \in \mathcal{I}$, any $f^n$-invariant unstable branch $\Gamma$ of $I$ is contained in $\overline{V}$ (unless when $I = I_0$ and $\Gamma$ is the unstable branch that stabilizes $p$),
- any point fixed by $f^n$ in the accumulation set of $\Gamma$ is contained in $\overline{V}$ (since such a point coincides with $p$ or is decreasing chain related to $p$).

Let:

$$\mathcal{N} := V \setminus \bigcup \{ W^s_{\mathcal{D}}(I_i), I_i \in \mathcal{I} \text{ of index } -1 \}.$$ 

The proof of claim \footnote{62} shows that any component $U$ of $\mathcal{N}$, either it contains an arc $I' \in \mathcal{I}$ of index 1 or it is the component bounded by $W^s_{\mathcal{D}}(I_0)$ and $I_0$ is semi-attracting in $V$.

To each arc $I \in \mathcal{I} \setminus \{ I_0 \}$ of index $-1$, one let $V_I$ be the component of $\mathbb{D} \setminus W^s_{\mathcal{D}}(I)$ which does not contain the stabilizing unstable branch of $I_0$. One associates by claim \footnote{62} an arc $I'$ of index 1 in the component of $\mathcal{N}$ bounded by $W^s_{\mathcal{D}}(I)$ which belongs to $V_I$.

When the arc $I_0$ has a $f^n$-invariant unstable branch in $V$ (and has half index
\footnote{62} $\text{index}(I_0, V, f^n) = -1/2$), one can also associate by the claim \footnote{62} an arc of index 1 which belongs to the component of $\mathcal{N}$ bounded by $W^s_{\mathcal{D}}(I_0)$.

The number of arcs of index 1 in $\mathcal{I}$ is thus larger or equal to the number of arcs of index $-1$, and it is larger or equal to the number of arcs of index $-1$ plus 1 in the case $\text{index}(I_0, V, f^n) = -1/2$. This proves that the sum of the indices $L(V, f^n)$ is always larger or equal to 1/2.

\[ \Box \]

7.5 Structure of the set of periodic points

The next proposition classifies the periodic points.

**Proposition 73.** For any periodic point $w$, one and only one of the possibilities occurs:

1. $w$ is fixed and either is a sink or $Df(w)$ has an eigenvalue $\geq 1$,
2. $w$ is stabilized,
3. $w$ is decreasing chain related to a stabilized periodic point.

**Proof.** The options (1) and (2) are incompatible by definition of the stabilization. Options (2) and (3) are incompatible by corollary \footnote{67}. Also (1) and (3) are incompatible by remark \footnote{65}. It remains to prove that any periodic point $w$ satisfies one of the cases.

Let $f^m$ be an orientation-preserving iterate that fixes $w$ and let $\mathcal{I}$ be a finite collection of isolated arcs fixed by $f^m$ which contains all the points fixed by $f^n$. Let $\mathcal{I}_0$ be the set of intervals $I \in \mathcal{I}$ containing a periodic point satisfying one of the cases (1), (2) or (3).

**Claim 74.** For $I \in \mathcal{I}_0$, any periodic point in $I$ satisfies the proposition. More precisely one and only one of the following cases occurs:

- the periodic points in $I$ are all fixed and not stabilized,
- $I$ contains either a stabilized point $p$ or a point decreasing chain related to a stabilized point $p$: all the other periodic points in $I$ are decreasing chain related to $p$. 

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Proof. We can assume that $I$ is not reduced to a single periodic point (in that case the statement holds immediately). We consider three cases:

If $I$ contains a fixed point $q$ with an eigenvalue $\lambda_q^+ \geq 1$, then any periodic point in $I$ is fixed and cannot be stabilized. The first case occurs.

If $I$ contains a fixed point $q$ with eigenvalue $\lambda_q^+ \leq -1$, the other periodic points in $I$ have period 2: if $q$ is not a sink, it is stabilized and the other periodic points in $I$ are decreasing chain-related to $q$; if $q$ is a sink, its basin in $I$ is bounded by a 2-periodic orbit $\{p, f(p)\}$, the other periodic points in $I$ are decreasing chain related to $p$ or $f(p)$.

If $I$ does not contain any fixed point, but contains a stabilized point $p$, then it is contained in the closure of the decorated region $V$ of $p$. Otherwise, by $f^n$-invariance, $I$ would contain the stabilized unstable branch of $p$ and its accumulation set: a contradiction since $I$ does not contain any fixed point. One deduces that any periodic point $I$ different from $p$ is decreasing chain related to $p$.

If $I$ does not contain any fixed point, nor any stabilized periodic point, but contains a point decreasing chain related to a stabilized point $p$, one deduces that $I$ is contained in a decorated region $V$ of $p$. Otherwise $I$ would intersect $W^+_D(p)$, and hence by $f^n$-invariance would contain $p$. Therefore any periodic point in $I$ is also decreasing chain related to $p$.

\[ \square \]

Claim 75. For any $I \in \mathcal{I} \setminus \mathcal{I}_0$ and any $f^n$-invariant unstable branch $\Gamma$, any periodic point in the accumulation set of $\Gamma$ belongs to some $I' \in \mathcal{I} \setminus \mathcal{I}_0$.

Proof. Let us consider an endpoint $z \in I$ with a $f^n$-invariant unstable branch $\Gamma$ whose accumulation set contains a $f^n$-invariant point $q$. Let us assume by contradiction that the interval $I' \in \mathcal{I}$ containing $q$ belongs to $\mathcal{I}_0$. We distinguish two cases.

i- The point $q$ is fixed: if $q$ satisfies case (1), then $z$ is stabilized, a contradiction; if $q$ satisfies case (2), since $z$ is not stabilized, the definition in Proposition 19 implies that $\Gamma$ does not intersect one of the components of $D \setminus W^+_D(q)$. Therefore, by definition 64 one deduces that $z$ is decreasing chain related to $q$: this is a contradiction since $I \notin \mathcal{I}_0$.

ii- The point $q$ is not fixed: Since $I' \in \mathcal{I}_0$, from the previous claim there exists a stabilized point $p$ such that all the periodic points in $I'$ are decreasing chain related to $p$ or coincide with $p$. Let $V$ be the decorated region associated to $p$ which contains $q$. By definition 64 there exists a chain $C \subset V$ for $f^n$ containing $q$ and $p$. Note that $\Gamma$ cannot intersect the region $D \setminus \overline{V}$: when $p$ is fixed, this would immediately imply that $z$ is stabilized, a contradiction; when $p$ is not fixed, this would imply (by Proposition 19) that the accumulation set of $\Gamma$ would contain the accumulation set of the stabilized branch of $p$, and then that $z$ is stabilized, a contradiction. One deduces that $I \cup I' \cup \Gamma \cup C$ is a chain for $f^n$ containing $z$ and $p$. It is contained in $\overline{V}$ hence $z$ is decreasing chain related to $p$. A contradiction.

We can now conclude the proof of the proposition. From claim 74 one can for each stabilized point $p$ consider the family $\mathcal{I}_p$ of arcs $I \in \mathcal{I}_0$ such that all the periodic points in $\mathcal{I}_p$ are decreasing chain related to $p$ or equal to $p$. One can also consider the family $\mathcal{I}_{\text{fix}}$ of arcs whose periodic points are fixed and not stabilized. The family $\mathcal{I}_0$ decomposes as the disjoint union of $\mathcal{I}_{\text{fix}}$ with the families $\mathcal{I}_p$, for $p$ stabilized.

Let $p$ be a stabilized fixed point, with decorated regions $V_1, V_2$. Lemma 72 implies

\[ \sum_{I \in \mathcal{I}_p} \text{index}(I, f^n) = L(V_1, f^n) + L(V_2, f^n) \geq 1. \]  

(6)
Let $p$ a stabilized point fixed by $f^n$ but not by $f$. It has one decorated region $V$. Let $I_p$ be the arc in $I_p$ which contains $p$. Since $p$ has an unstable branch in the region $\mathbb{D} \setminus V$, we get $\text{index}(I, \mathbb{D} \setminus V, f^n) = -1/2$. Consequently lemma 72 implies

$$\sum_{I \in I_p} \text{index}(I, f^n) = L(V, f^n) + \text{index}(I_p, \mathbb{D} \setminus V, f^n) \geq 0. \quad (7)$$

Note that if $I \in I$ contains a stabilized fixed point, then $\text{index}(I, f) = 1$, whereas for $I \in I_{fix}$ one has $\text{index}(I, f) = \text{index}(I, f^n)$. Therefore the Lefschetz formula (proposition 18) for $f$ gives

$$\sum_{I \in I_{fix}} \text{index}(I, f^n) = \sum_{I \in I_{fix}} \text{index}(I, f) = 1 - \mathbb{W}\{p \text{ fixed and stabilized}\}.$$ 

Combining the three previous inequalities give

$$\sum_{I \in I_0} \text{index}(I, f^n) \geq 1.$$ 

If one assumes that $I \setminus I_0$ is non-empty, the claim 75 and the remark 63 give

$$\sum_{I \in I \setminus I_0} \text{index}(I, f^n) \geq 1.$$ 

This gives $\sum_{I \in I} \text{index}(I, f^n) \geq 2$ which contradicts the Lefschetz formula (proposition 18). Consequently $I = I_0$ and any point fixed by $f^n$ satisfies one of the cases of the proposition 73. The proof is complete.

We can now complete the proof of the Lefschetz formula inside a decorated region.

**Proof of proposition 71.** We argue as in the proof of the proposition 71 for the orientation preserving iterate $f^n$. We consider a collection $I$ of disjoint isolated arcs fixed by $f^n$. For each stabilized point $p$, we consider the collection of arcs $I_p$ containing points decreasing chain related to $p$ and the point $p$ itself. We also consider the family $I_{fix}$ of arcs whose periodic points are fixed and not stabilized. The family $I$ is partitioned as the disjoint union of $I_{fix}$ with the families $I_{p}$, for $p$ stabilized.

Arguing as before, the conclusion of lemma 72 gives the inequality (6) for any $p$ stabilized and fixed and it gives the inequality (7) for $p$ stabilized and not fixed. If one of these inequalities is strict, one deduces $\sum_{I \in I} \text{index}(I, f^n) > 1$ and contradicts the Lefschetz formula (proposition 18). Consequently the inequalities (6) and (7) are equalities. This means that the inequality in lemma 72 is an equality and proposition 71 holds.

8 Trapping discs

A compact set $\Delta \subset \mathbb{D}$ is a *(topological)* disc if it is homeomorphic to the unit disc. It is trapping for $f$ if $f(\Delta) \subset \text{Interior}(\Delta)$. In this section we prove the following result.

**Theorem H.** Let $f$ be a mildly dissipative diffeomorphism of the disc with zero topological entropy and $\Gamma$ be a $f$-invariant unstable branch of a fixed point $p$. Then there exists a trapping disc $\Delta$ containing the accumulation set of $\Gamma$ and disjoint from $W^s(p)$.
It is enough to prove the theorem in the case where $f$ is orientation preserving. Let us consider the finite set $\mathcal{I}$ of isolated fixed arcs as introduced in section 2.2. Since there is no cycle of fixed arcs (corollary 33), the elements of $\mathcal{I}$ can be ordered as a sequence $I_1, \ldots, I_n$ such that there is no $f$-invariant unstable branch of $I_i$ which accumulates on $I_j$ when $j \geq i$. The proof first deals with the $f$-invariant unstable branches of the arcs $I_i$, by induction on $i$. In this case we have a more precise version.

**Theorem H’.** Let $f$ be a mildly dissipative diffeomorphism of the disc with zero topological entropy and $\mathcal{I}$ a set of isolated fixed arcs as introduced in section 2.2. For any $I_i \in \mathcal{I}$ and any $f$-invariant unstable branch $\Gamma$ of $I_i$, let $Z$ be the closure of the union of:

- the accumulation set $\Lambda$ of $\Gamma$,
- the arcs $I_j \in \mathcal{I}$ for $j < i$,
- the $f$-invariant unstable branches of the arcs $I_j$ for $j < i$.

Then, $\Lambda$ is included in an trapping disc $\Delta$ which is contained in an arbitrarily small neighborhood of $Z$.

In the following we will first prove the second theorem and then deduce the first. As an immediate consequence one gets:

**Corollary 76.** Let us consider an isolated fixed arc $I = I_i$ which is not reduced to a fixed point with eigenvalue $-1$. Let $U$ be an open set which contains the arcs $I_j \in \mathcal{I}$ for $j \leq i$ and the closure of their $f$-invariant unstable branches. Then there exists a trapping disc $\Delta \subset U$ which contains $I$.

One also deduces that periodic points are almost isolated in the recurrent set of $f$.

**Corollary 77.** Let us consider an isolated fixed arc $I$ which is not reduced to a fixed point with eigenvalue $-1$. Then, there exists a neighborhood $W$ of $I$ such that

- the $\alpha$-limit set of any point $z \in W$ is either disjoint from $W$ or a fixed point of $I$,
- the $\omega$-limit set of any point $z \in W$ is either disjoint from $W$ or a fixed point of $I$.

Note that a fixed point with eigenvalue $-1$ is contained in an isolated fixed arc $I'$ for $f^2$ to which the corollary may be applied. This gives:

**Corollary 78.** Any periodic orbit $O$, with period $N$, admits a neighborhood $W$ such that any ergodic measure $\mu$ satisfying $\mu(W) > 0$ is supported on a periodic orbit with period less or equal to $2N$.

The construction of the trapping domains in a small neighborhood that contains the accumulation set of $\Gamma$ in the proofs of theorem [H] and [H’] go along the following lines: 1- using a slight variation of definition 39 we build Pixton disc given by either i) arcs of $\Gamma$ and local stable manifold of stabilized periodic orbits (with period one or larger, lemma 81), ii) basin of attraction of (semi)attracting fixed points (lemma 82); 2- the union of these Pixton discs can be refined in a larger Pixton disc that contains all its iterates, the periodic points accumulated by $\Gamma$ and any decreasing chain-related point to them (corollary 83); 3) using the closing lemma (theorem F’) we prove that any point in the accumulation set of $\Gamma$ has it backward orbit in the interior of the Pixton disc described in previous item (showing that the forward iterate of the disc is contained in the disc) and that allows to perform the last step which consists in slight modification of the Pixton disc to guarantee that the forward iterate is contained in its interior.
8.1 Pixton discs revisited

We prepare here the proof of theorem [H]. We assume in this section that \( f \) preserves the orientation.

We consider an arc \( I_i \in \mathcal{I} \) and a \( f \)-invariant unstable branch \( \Gamma \) of an endpoint \( p \) of \( I_i \). Arguing by induction, we may assume that theorem [H] holds for the \( f \)-invariant unstable branches of any arc \( I_k \in \mathcal{I} \) with \( k < i \). Let \( Z \) be the invariant compact set introduced in the statement of the theorem. By assumption on the order inside the family \( \mathcal{I} \), the set \( Z \) disjoint from \( W_s^D(p) \). We choose a neighborhood \( U \) of \( Z \) disjoint from \( W_s^D(p) \).

We introduce the following notion, which is slightly different than the definition given before.

**Definition 79.** Given a \( f \)-invariant unstable branch \( \Gamma \), a Pixton disc associated to \( \Gamma \) is a closed topological disc \( D \) whose boundary is a Jordan curve which decomposes into

- a closed set \( \gamma^s \) satisfying \( f^n(\gamma^s) \subset \text{Interior}(D) \) for all \( n \) larger than some \( n_D \geq 1 \),
- and its complement \( \gamma^u \) (that could be empty) which is contained in \( \Gamma \).

**Remarks 80.** About a Pixton discs the next easy statements follow:

1. A trapping disc is a Pixton disc. Conversely a Pixton disc such that \( \gamma^u = \emptyset \) is a trapping disc. In particular, an attracting fixed point has associated a Pixton disc.
2. The forward iterates of a Pixton disc are Pixton discs.
3. If \( D_1, D_2 \) are two Pixton discs whose intersection is non-empty, then one obtains a new Pixton disc \( D \) by considering their “filled union”: this is the union of \( D_1 \cup D_2 \) with all the connected components of this set which do not contain the boundary of \( D \). By [Ke] (see also [LY]), the filled union is a disc. The new set \( \gamma^s \) is contained in the union of the sets \( \gamma_1^s, \gamma_2^s \) associated to \( D_1, D_2 \). The same holds for \( \gamma^u \).

Observe that previous remark provides the proof of the first step in the induction argument: the first arc \( I_1 \) in \( \mathcal{I} \) is an attracting arc.

In what follow until the end of the subsection, \( p, \Gamma, U \) are the fixed point, unstable arc and neighborhood defined at the beginning of the subsection. In order to prove theorem [H] we need to cover periodic points in the accumulation set of \( \Gamma \) by Pixton discs. This is done first for periodic points with period larger than 1, and later for fixed points.

**Lemma 81.** Consider a periodic orbit \( \mathcal{O} \) accumulated by \( \Gamma \) and stabilized by a fixed point \( q \). Then there exists a Pixton disc \( D \subset U \) which contains \( \mathcal{O} \) in its interior and whose stable boundary \( \gamma^s \) is contained in the stable manifold \( W_s^D(\mathcal{O}) \) of \( \mathcal{O} \).

**Proof.** We first assume that \( \tilde{w} \) has period \( \tau \geq 2 \). See figure [17]

Let us consider the universal cover \( \tilde{D} \) of \( D \backslash \{q\} \): it is homeomorphic to the strip \( \mathbb{R} \times [0,1) \) and the translation \( (x,y) \mapsto (x+1,y) \) can be chosen to be a covering automorphism which generates the fundamental group. Let \( \tilde{p} \) and \( \tilde{\Gamma} \) be lifts of \( p \) and of the unstable branch \( \Gamma \). We choose the lift \( \tilde{f} \) of \( f \) which preserves \( \tilde{p} \) and \( \tilde{\Gamma} \).

Consider \( w \in \mathcal{O} \), one of its stable branches \( W^s \subset W^D_s(w) \) connecting \( w \) to a point \( z \) in the boundary of \( D \), and \( W^u \) the unstable branch that accumulates on \( q \). The points \( w, z \) and the curve \( W := W^s \cup W^u \) lift as \( \tilde{w}, \tilde{z} \in \tilde{W} = \tilde{W}^s \cup \tilde{W}^u \). One may assume that
Figure 17: Construction of a Pixton disc: $w$ has period 2 (left) or 1 (right).

Figure 18: Proof of lemma 81.
\( \tilde{\gamma} = (0, 0) \). Note that \( \tilde{W} \) separates the strip: its complement contains two components bounded by \((-\infty, 0) \times \{0\}\) and \((0, +\infty) \times \{0\}\) respectively.

To any lift \( \tilde{w}' = f^k(\tilde{w}) + (\ell, 0) \) of any iterate \( f^k(w) \) of \( w \), one associates in a same way a curve \( \tilde{W}' \), disjoint from \( \tilde{W} \): it either lands on \((-\infty, 0) \times \{0\}\) (in which case one denotes \( \tilde{W}' < \tilde{W} \)) or on \((0, +\infty) \times \{0\}\). One defines in this way a totally ordered collection of \( \tilde{W}' \) separating sets \( \cdots \subset \tilde{W}_{n-1} < \tilde{W}_n < \tilde{W}_{n+1} < \cdots \) such that \( \tilde{W}_n + (1, 0) = \tilde{W}_{n+\tau} \). Since the point \( w \) is not fixed, the sets \( \tilde{W}_n = \tilde{W}_n^s \cup \tilde{W}_n^u \) are not fixed by \( \tilde{f} \): there exists \( j \neq 0 \) such that \( \tilde{f}(\tilde{W}_n) \subset \tilde{W}_{n+j} \) for any \( n \in \mathbb{Z} \). We may assume without loss of generality that \( j \geq 1 \). See figure [18].

The \( \tilde{f}' \)-invariant curve \( \tilde{\Gamma} \) accumulates on each set \( \tilde{f}^k(\tilde{W}) \subset \tilde{W}_{kj} \), \( k \geq 0 \). Since the sets are separating, it intersects all the sets \( \tilde{W}_n, n \geq 0 \). Note that the unstable branch \( \tilde{\Gamma} \) does not intersect the curve \( \tilde{W}_n^u \). It follows that it intersects all the \( \tilde{W}_n^u, n \geq 0 \).

For \( n \geq 1 \), let \( \tilde{\gamma}^u \) be a (open) curve in \( \tilde{\Gamma} \) which connects \( \tilde{W}_n^s \) to \( \tilde{W}_{n+\tau} = \tilde{W}_n^s + (1, 0) \) at two points \( a \in \tilde{W}_n^s \) and \( b \in \tilde{W}_n^s + (1, 0) \). Let \( \tilde{\gamma}^a \subset \tilde{W}_n^s \) be the (closed) curve which connects \( a \) to \( b \). The curve \( \tilde{\gamma}^u \cup \tilde{\gamma}^a \) projects on a simple closed curve \( \gamma = \gamma^a \cup \tilde{\gamma}^u \) of \( \mathbb{R} \) which bounds a disc \( D \). By construction, the large forward iterates of \( \gamma \) converge to the orbit of \( w \), hence are contained in \( D \). One deduces that \( D \) is a Pixton disc.

Note that the lift \( \tilde{\gamma} = \cup_{k \in \mathbb{Z}}(\tilde{\gamma}^u \cup \tilde{\gamma}^a + (k, 0)) \) separates the boundary \( \mathbb{R} \times \{0\} \) from the sets \( \tilde{W}_n^u \). This implies that the disc \( D \) contains all the unstable branches \( \tilde{f}^k(\tilde{W}^u) \) of the iterates of \( w \) and in particular the orbit \( \mathcal{O} \).

Up to replace \( D \) by a large iterate, one finds a Pixton disc whose unstable boundary \( \gamma^u \) is arbitrarily close to the limit set \( \Lambda \), whose stable boundary \( \gamma^s \subset W_0^s(w) \) has arbitrarily small diameter, and whose area is arbitrarily small. One deduces that the disc is in an arbitrarily small neighborhood of its unstable boundary, hence of \( \Lambda \). Consequently it is included in \( U \) as required.

In the case where \( w = q \) has period 1 but negative eigenvalue, we argue in a similar way. We denote by \( W_0^s \) and \( W_1^s \) the two stable branches of \( q \) and we lift them as an ordered collection of \( \cdots \subset \tilde{W}_n^s \subset \tilde{W}_{n+1}^s \subset \cdots \) such that the curves \( \tilde{W}_n^s \) lift \( W_0^s \) and the curves \( \tilde{W}_{2n+1}^s \) lift \( W_1^s \). Moreover \( W_{n+2}^s = W_n^s + (1, 0) \). Since \( f(W_0^s) \subset W_1^s \) and \( f(W_1^s) \subset W_0^s \), the curves \( \tilde{W}_n^s \) are not fixed by \( \tilde{f} \). The end of the proof is similar: we get a curve \( \tilde{\gamma} \) which separates the boundary \( \mathbb{R} \times \{0\} \) from a line \( \mathbb{R} \times \{1 - \delta\}, \delta > 0 \) small. It projects as a simple closed curve which bounds a Pixton disc containing \( q \) as required.

\[ \text{Lemma 82. Each fixed point} \ p' \ \text{accumulated by} \ \Gamma \ \text{and which does not have an eigenvalue less or equal to} \ -1 \ \text{is contained in a trapping disc} \ D \subset U. \]

\[ \text{Proof.} \] We use the inductive assumption stated before the section [8.1]. The fixed point \( p' \) belongs to an arc \( I' = I_j \) in \( \mathcal{I} \). From our choice of the order on \( \mathcal{I} \), we have \( j < i \). If \( I' \) has the type of a sink, it admits arbitrarily small neighborhoods that are trapping disc. Note that \( I' \) can not have the type of a point with reflexion (since \( p' \) does not have an eigenvalue less or equal to \(-1\)).

Consequently, we are reduced to consider the case where \( I' \) has a non-trivial bundle \( F \) and each endpoint is either attracting in the direction \( F \) or attached to a \( f \)-invariant unstable branch \( \Gamma' \) (it has the type of a saddle-node or of a saddle with no reflexion). The proposition holds for the branches \( \Gamma' \) (this is our inductive assumption). One deduces that there exists one or two trapping discs \( D' \) containing the accumulation sets of these branches and included in \( U \). Taking the union with a tubular neighborhood \( V \) of \( I' \) and
of the branches $\Gamma'$, one obtains a trapping disc $D \subset U$ which contains the fixed point $p'$. See figure 19.

**Corollary 83.** Under the setting of theorem H, there exists a collection $D$ of Pixton discs $D$ (disjoint from $W^s_D(p)$) such that:

(a) all the forward iterates $f^k(D)$ of discs $D \in D$ are included in $U$,

(b) any periodic orbit $O$ in the accumulation set of $\Gamma$ is contained in one $D \in D$,

(c) for any periodic orbit $\bar{O}$ in the accumulation set of $\Gamma$ and which is stabilized by a fixed point, there exists a Pixton disc $D \in D$ which contains the unstable set of $\bar{O}$, any periodic orbit $O$ decreasing chain-related to $\bar{O}$ and the unstable set of $O$.

**Proof.** For fixed points $p'$ accumulated by $\Gamma$, we either apply the remark 80 (when $p'$ is a sink), lemma 81 (when $p'$ is not a sink and has an eigenvalue smaller or equal to $-1$: it is then stabilized by the fixed point $q = p'$), or lemma 82 (in the other case).

For any periodic orbit $\bar{O}$ which is stabilized by a fixed point, the lemma 81 provides a Pixton disc $D \subset U$ which contains $\bar{O}$ in its interior and whose stable boundary $\gamma_s$ is contained in $W^s_D(\bar{O})$. By the no-cycle theorem (theorem G), the unstable manifold of $\bar{O}$ does not intersects $W^u_D(\bar{O}) \setminus \bar{O}$. This proves that the unstable set of $\bar{O}$ is included in $D$.

Let $O$ be a periodic orbit decreasing chain related to $\bar{O}$. For any point $w \in O$, there exists $\tilde{w} \in \bar{O}$ and a chain $C \subset D$ for an iterate of $f$ which contains $w$ and $\tilde{w}$. The closure of $C$ is fixed by an iterate of $f$ and is disjoint from $W^s_D(p)$ (since it is contained in $U$). As a consequence, it is disjoint from $W^u(p)$. Since $C$ is connected and contained in the closure of a decorated region of $\bar{O}$, it intersects at most one connected component of $\mathbb{D} \setminus W^s_D(\bar{O})$. Since the boundary of $D$ is contained in $W^s_D(\bar{O}) \cup W^u(p)$, the chain $C$ is contained in $D$. This proves that $O$ is included in $D$. By proposition 70, the unstable set of $w$ intersects at most one component of $\mathbb{D} \setminus W^u_D(\bar{O})$. It does not intersects $W^u(p)$ either. Consequently, $W^u(O)$ is also included in $D$. This gives the item (c).

From proposition 73, any periodic orbit $O$ in the accumulation set of $\Gamma$ which has period larger than 1 and is not stabilized by a fixed point is decreasing chain related to a periodic orbit $\bar{O}$ stabilized by a fixed point. Moreover from proposition 70, the stabilized orbit $\bar{O}$ is also accumulated by $\Gamma$. The item (c) provides a Pixton disc $D \subset U$ which contains $\bar{O}$ and $O$. This completes (b).

It remains to prove the item (a). By construction, the discs $D$ are contained in $U$. In the case $D$ is trapping, all its forward iterates are contained in $U$ also. In the other cases, $D$ is obtained with lemma 81 since $f$ is dissipative, the volume of $f^k(D)$ is arbitrarily small for $k$ large; since the stable boundary $\gamma_s$ is contained in the stable curve of a periodic orbit, its length $\int f^k(\gamma_s)$ gets arbitrarily small; since the unstable boundary
$\gamma^u$ is contained in $\Gamma$, one concludes that all the iterates $f^k(D)$, for $k$ large, are contained in $U$. Up to replace $D$ by a larger iterate, the property (a) is satisfied.

**8.2 Proof of theorem \([H']\)**

We first assume that $f$ preserves the orientation and we consider the setting of the section 8.1. The accumulation set $\Lambda$ of $\Gamma$ may be covered by Pixton discs.

**Lemma 84.** Let us consider the family of Pixton discs $D$ obtained by corollary 83. Then, any point $x$ in the accumulation set $\Lambda$ of $\Gamma$ has a backward iterate in the interior of one of the Pixton discs $D \in D$.

**Proof.** The proof of this lemma is done by contradiction. If the conclusion does not hold, the backward orbit of a point $x \in \Lambda$ accumulates on an invariant set $K \subset \Lambda$ that is disjoint from the interior of all the discs $D \in D$. Then $K$ supports an ergodic measure $\mu$. From the item (b) of corollary 83, this measure $\mu$ is non-atomic. The Pesin theory associates a compact set $B \subset \text{Support}(\mu)$ with $\mu(B) > 0$ such that all the points $z$ in $B$ have a stable manifold $W^s_D(z)$ which separates $D$ and varies continuously with $z \in B$ for the $C^1$ topology. We can thus choose $z \in B$ whose forward orbit is dense in the support of $\mu$ and two forward iterates $z', z'' \in B$ close to $z$ and separated by $W^s_D(z)$. In particular the region $R \subset D$ bounded by $W^s_D(z')$ and $W^s_D(z'')$ does not contain any fixed point. From the closing lemma (theorem $F'$), there exists a sequence $(w_k)$ of periodic points in $\Lambda$ converging to $z$. See picture 20.

We first assume that $w_k$ is stabilized by a fixed point $q$; since $w_k$ are in $\Lambda$, they are accumulated by $\Gamma$ and so by item (c) in corollary 83 there a Pixton disc $D \in D$ which contains the orbit of $w_k$ and its unstable set. Since $w_k \in R$ and $q \notin R$, the unstable set of $w_k$ intersects the boundary of $R$, hence the stable manifold of $z'$ or $z''$. Since the forward orbits of $z'$ and $z''$ equidistribute towards the measure $\mu$, this implies that $\mu$ is supported on $D$, hence on the boundary of $D$. This is a contradiction since the orbit of any point in the boundary of $D$ converges in the future or in the past towards $p$ or the orbit of $w_k$.

When $w_k$ is not stabilized by a fixed point, proposition 73 implies that $w_k$ is decreasing chain-related to a periodic point $\tilde{w}_k$ which is stabilized by a fixed point $q_k$ and proposition 70 implies that $\tilde{w}_k$ also belongs to $\Lambda$. In the case $\tilde{w}_k$ belongs to $R$, the
previous argument applies and gives a contradiction. We are thus reduced to the case where \( \tilde{w}_k \) does not belong to \( R \).

Let us consider a chain \( C \) for an iterate of \( f \) which contains \( w_k \) and \( \tilde{w}_k \). Let \( D \in \mathcal{D} \) be a Pixton disc associated to \( \tilde{w}_k \) as in corollary \( \text{SS} \) item (c): in particular it contains the chain \( C \). Since \( C \) is connected and intersects both \( R \) and its complement, there exists an unstable branch \( \Gamma \) of a periodic point \( w'_k \in C \) which intersects the stable curve of \( z' \) or \( z'' \). Since \( \Gamma \subset D \), this implies as before that \( \mu \) is supported on \( D \) and gives a contradiction.

We can now complete the proof of the theorem.

**End of the proof of theorem \( \text{H} \).** Let \( \Lambda \) be the accumulation set of \( \Gamma \): this is an invariant compact set. Let us consider the collection \( \mathcal{D} \) of Pixton discs given by corollary \( \text{SS} \). Let \( V \) be the union of all the open sets \( \text{Interior}(f^k(D)) \) over \( D \in \mathcal{D} \) and over all \( k \geq 0 \). This is an open set satisfying \( f(V) \subset V \subset U \). By lemma \( \text{S4} \) any point in the accumulation set \( \Lambda \) of \( \Gamma \) has a backward iterate in \( V \). Since \( V \) is forward invariant and by compactness of \( \Lambda \), there exists a finite number of Pixton discs \( f^k(D_n) \) such that the union of their interiors covers \( \Lambda \). The remark \( \text{S0}(3) \) allows to replace any two of these discs which intersect by a single Pixton disc. We repeat this inductively. Since \( \Lambda \) is connected, one gets a Pixton disc \( \tilde{D} \) whose interior contains \( \Lambda \). We denote by \( \tilde{\gamma}^s, \tilde{\gamma}^u \) its stable and unstable boundaries.

We modify \( \tilde{D} \) in order to obtain a Pixton disc satisfying some forward invariance. One chooses \( k \) large such that \( f^k(\tilde{\gamma}^u) \) is contained in a small neighborhood of \( \Lambda \), hence in \( \text{Interior}(\tilde{D}) \). From the definition of the Pixton disc we also have \( f^k(\tilde{\gamma}^s) \subset \text{Interior}(\tilde{D}) \).

One applies remark \( \text{S0}(3) \) again in order to build a Pixton disc \( D \) which contains \( \tilde{D} \cup f(\tilde{D}) \cup \cdots \cup f^{k-1}(\tilde{D}) \). Since \( f(\tilde{\gamma}^s) \subset \tilde{D} \), the stable boundary \( \gamma^s \) of \( D \) is included in \( \tilde{\gamma}^s \); in particular, the \( k \) first iterates of \( \gamma^s \) are contained in \( \text{Interior}(D) \). By construction, the \( k-1 \) first iterates of the boundary of \( D \) are contained in \( D \) and all the larger iterates are contained in \( \text{Interior}(\tilde{D}) \). This proves that the \( k-1 \) first iterates of \( \gamma^u \) are contained in \( D \) and the \( k \)-th iterate is included in \( \text{Interior}(D) \). One deduces that \( D \) is a Pixton disc whose interior contains \( \Lambda \) and which furthermore satisfies \( f(D) \subset D \) and \( f^k(D) \subset \text{Interior}(D) \).

One finally modifies \( D \) in order to build a disc \( \Delta \) trapped by \( f \): for each \( x \) in the boundary of \( D \), one considers the smallest integer \( i \geq 1 \) such that \( f^i(x) \in \text{Interior}(D) \); one chooses small closed discs \( D_{x,0}, D_{x,1}, \ldots, D_{x,i-1} \) centered at \( x, f(x), \ldots, f^{i-1}(x) \) respectively such that \( f(D_{x,j}) \subset \text{Interior}(D_{x,j+1}) \) when \( j < i-1 \) and \( f(D_{x,i}) \subset \text{Interior}(D) \).

By compactness, one selects finitely many points \( x_1, \ldots, x_m \) in the boundary of \( D \), such that the union of the interior of the \( D_{x,i,0} \) covers the boundary of \( D \). By construction, the union of \( D \) with all the discs \( D_{x_k, j} \) is a compact set \( \tilde{\Delta} \) whose image is contained in \( \text{Interior}(\tilde{\Delta}) \).

By \( \text{KK} \), \( \tilde{\Delta} \) is contained in a disc \( \Delta \) whose boundary is boundary of \( \tilde{\Delta} \). In particular, \( f(\Delta) \subset \text{Interior}(\Delta) \). We have thus obtained a trapping disc which contains \( \Lambda \). From the item (b) of corollary \( \text{SS} \) the trapping disc is disjoint from \( W^s(p) \) as required. The conclusion of the theorem \( \text{H} \) thus holds for \( \Gamma \).

Since all the forward iterates of the Pixton discs \( D \in \mathcal{D} \) are included in \( U \), the disc \( \Delta \) may be chosen in a small neighborhood of \( \tilde{U} \). This argument applied inductively concludes the proof of theorem \( \text{H} \) in the case \( f \) is orientation preserving.

When \( f \) is orientation reversing, one first considers \( f^2 \) and gets a disc \( \Delta_0 \) disjoint from \( W^s(p) \), which contains the accumulation set of \( \Gamma \) and is trapping for \( f^2 \). The filled
union \(\Delta_1\) of \(\Delta_0\) and \(f(\Delta_0)\) has the same property (see remark \[80\]), but also satisfies \(f(\Delta_0) \subset \Delta_0\). Arguing as above, one can then modify \(\Delta_1\) and get a disc \(\Delta \supset \Delta_1\) contained in an arbitrarily small neighborhood of \(\Delta_1\) which is trapping for \(f\).

### 8.3 Proof of theorem \([\text{H}]\) and its consequences

**Proof of theorem \([\text{H}]\)** From theorem \([\text{H}']\) the conclusion of theorem \([\text{H}]\) holds for the \(f\)-invariant unstable branches of the arcs \(I_i \in \mathcal{I}\). One can easily conclude for the other \(f\)-invariant branches, i.e. for the unstable branches \(\Gamma\) contained in \(I_i\). Indeed the accumulation set of such a branch belongs to an isolated fixed arc \(I \subset I_i\), which is disjoint from the unstable branch \(\Gamma\) and bounded by an endpoint \(p_i\) of \(I_i\). If \(I_i\) has the type of a sink, it admits arbitrarily small neighborhoods that are trapping discs and the proposition follows. Otherwise \(p_i\) has a \(f\)-invariant unstable branch and we know from theorem \([\text{H}']\) that its accumulation set is contained in a trapping disc \(\Delta_0\) disjoint from \(I_i\). One can then extend the disc \(\Delta_0\) with a tubular neighborhood of \(I\) and of the unstable branch of \(p\) one then gets a trapping disc \(\Delta\) which contains \(I\) (hence the accumulation set of \(\Gamma\)) as required.

**Proof of corollary \([77]\)** Since \(I\) is not reduced to a fixed point with eigenvalue \(-1\), three types are possible (see definition \[15\]).

If \(I\) has the type of a sink, the \(\omega\)-limit set of any point in an open neighborhood \(W\) is a fixed point of \(I\). Moreover by compactness, there exists \(k \geq 1\) such that \(f^k(\overline{W}) \subset W\) and \(\omega_{n \geq 0} f^n(W) = I\). Hence the \(\alpha\)-limit set of any point in \(W \setminus I\) is disjoint from \(W\).

If \(I\) has the type of a saddle with no reflexion, one applies theorem \([\text{H}]\) and consider two trapping discs \(V_1, V_2\) disjoint from \(I\) which contain the accumulation sets of the unstable branches of \(I\). The forward orbit of any point in a neighborhood \(W\) of \(I\) either intersect \(V_1 \cup V_2\) (in this case the \(\omega\)-limit set is contained in \(V_1 \cup V_2\) and is disjoint from \(I\)) or is contained in \(I\) (it is a fixed point). Let us define \(W' = V_1 \cup V_2 \cup (\cup_n f^n(W))\). By compactness, there exists \(k \geq 1\) such that \(f^k(\overline{W'}) \subset W'\). Hence the \(\alpha\)-limit set of any point in \(W\) is either disjoint from \(W\) or contained in \(W\). Since \(I\) is normally hyperbolic, any \(\alpha\)-limit set contained in \(W\) is a fixed point of \(I\).

If \(I\) has the type of a saddle-node, one consider a trapping disc \(V\) disjoint from \(I\) which contains the accumulation set of the (unique) unstable branch of \(I\). The forward orbit of any point in a neighborhood \(W\) of \(I\) either intersects \(V\) or is contained in \(I\). One introduces \(W' = V \cup (\cup_n f^n(W))\) and one argues as in the previous case.

### 8.4 Trapping discs and periodic measures

As a byproduct of the previous arguments we obtain the following property which will be useful later.

**Proposition 85.** Let \(f\) be a mildly dissipative diffeomorphism of the disc with zero entropy, and let \(\mu\) be an aperiodic invariant measure. Then for \(\mu\)-almost every point \(z\) there exists \(\varepsilon > 0\) with the following property: if \(\Delta\) is a disc trapped by \(f\) which contains a point \(\varepsilon\)-close to \(z\), then \(\mu\) is supported on \(\Delta\).

**Proof.** The argument appeared in the proof of lemma \([84]\). The point \(z\) is contained in a strip \(R\) bounded by two stable manifolds \(W^s_{\text{loc}}(z'), W^s_{\text{loc}}(z'')\) such that the forward orbits of \(z'\) and \(z''\) equidistribute towards \(\mu\), and such that \(R\) does not contain any fixed point.

If the disc \(\Delta\) contains a point close to \(z\), it intersects \(R\). Since \(\Delta\) is trapped, it also contain a fixed point. Consequently \(\Delta\) meets the stable manifold of \(z'\) or \(z''\), and therefore

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the forward orbit of a large iterate of this point. This forward orbit equidistributes towards \( \mu \). This shows that \( \mu \) is supported on \( \Delta \).

\[ \square \]

9 Local renormalization

In this section we prove the theorem \([A]\) about the existence of a renormalizable disc. We also explain in proposition \([86]\) how to renormalize inside a decorated region; this proposition is the main step to prove the global renormalization stated by theorem \([A]\) in section \([11]\).

9.1 Renormalizable diffeomorphisms, proof of theorem \([A]\)

Let \( f \) be a mildly dissipative diffeomorphism of the disc with zero entropy. We distinguish two cases, either all periodic points are fixed or not. In the first case, one has to prove that \( f \) is generalized Morse-Smale; in the second, that there is a renormalizable domain.

First case: any periodic point of \( f \) is fixed. For any \( x \in \mathbb{D} \), let \( \mu \) be an ergodic measure supported on \( \omega(x) \). By the closing lemma (theorem \([F]\)), \( \mu \) is supported on a fixed point \( p \) and in particular, the forward orbit of \( x \) accumulates on \( p \). If \( x \) does not belong to the stable set of \( p \) then \( p \) has an unstable branch and the forward orbit of \( x \) accumulates on that unstable branch. By theorem \([H]\) there exists a disc \( \Delta \) which is trapped (by \( f \) or by \( f^2 \)) containing the accumulation set of the unstable branch and disjoint from a neighborhood of \( p \). In particular, \( \omega(x) \subset \Delta \cup f(\Delta) \) and so \( \omega(x) \) does not contain \( p \); a contradiction. We have shown that any forward orbit converges to a fixed point, thus \( f \) is a generalized Morse-Smale.

Second case: there are periodic points with period larger than 1. By proposition \([73]\) there exists a stabilized periodic point \( p \).

If \( p \) has period \( k > 1 \), one considers the decorated region \( V_p \) associated to \( p \) and observe that one of the following cases holds.

2.a. there exists an unstable branch of \( p \) contained in \( V_p \),

2.b. \( p \) belongs to an arc which is fixed for \( f^k \), contained in \( \overline{V_p} \) and not reduced to \( p \),

2.c. \( p \) is a saddle-node of \( f^k \).

In the case 2.a, we can apply again theorem \([H]\) for \( f^k \) and the unstable branch of \( p \) that is contained in \( V_p \); this gives a disc \( D \subset V_p \) which is trapped by \( f^k \); since the decorated regions of the iterates of \( p \) are disjoint, the disc \( D \) is disjoint from its \( k - 1 \) first iterates.

In the first cases 2.b and 2.c, it follows immediately that there is a compact disc disjoint from its \( k - 1 \) first iterates and mapped into itself by \( f^k \).

If \( p \) is a stabilized fixed point, it is not a sink. Let \( V_p \) be one of its decorated regions. Only the cases 2.a and 2.b can occur. In case 2.a, \( p \) has two unstable branches that are exchanged by \( f \); hence there exists a disc \( D \subset V_p \) which is trapped by \( f^2 \). In case 2.b, \( p \) is accumulated by points of period 2: one can then find an arc \( I \subset V_p \) which is fixed by \( f^2 \) and disjoint from \( f(I) \) and then a disc \( D \subset V_p \) which is mapped into itself by \( f^2 \).
To summarize, in the second case we have found a disc $D$, disjoint from its first $k-1$ iterates and mapped into itself by $f^k$: the diffeomorphism is renormalizable. The theorem is now proved.

### 9.2 Renormalization inside decorated regions

The following proposition provides the renormalization inside each decorated region, refining the trapped domain inside a decorated region of a periodic point $p$ into finite disjoint periodic trapping domains that capture only the periodic points of larger period that are decreasing chain related to $p$.

**Proposition 86.** Let $f$ be a mildly dissipative diffeomorphism of the disc with zero entropy, $p$ be a stabilized periodic point with a decorated region $V$ and $k$ be the period of $V$. Then, there exists a finite number of disjoint topological disks $D_1, \ldots, D_m$ such that

- $D_1 \cup \cdots \cup D_m \subset V$,
- each $D_i$ is trapped by $f^k$,
- $D_1 \cup \cdots \cup D_m$ contains all the periodic points $q \in V$ that are decreasing chain related to $p$ in $V$ with period larger than $k$,
- conversely any periodic point in $D_1 \cup \cdots \cup D_m$ is decreasing chain related to $p$.

**Proof.** From corollary 55, the map $f^k$ preserves the orientation.

Let $\mathcal{P}$ be the set of $q \in V$ which are decreasing chain-related to $p$ such that

- either the period of $q$ is larger than $k$,
- or the period of $q$ equals $k$ and $Df^k(p)$ has an eigenvalue less or equal to $-1$.

For each $\tau > k$, $\mathcal{P}(\tau)$ will denote the set of $q \in \mathcal{P}$ with period less or equal to $\tau$.

**Lemma 87.** Any $q \in \mathcal{P}$ belongs to a disc $\Delta_q \subset V$ trapped by some iterate of $f^k$.

**Proof.** The case where $q$ is a sink is clear. One can thus assume that there exists $\tau > k$ such that $f^\tau(q) = q$ and $Df^\tau(q)$ has an eigenvalue larger or equal to 1.

We consider a finite collection $\mathcal{J}$ of disjoint isolated arcs fixed by $f^\tau$ which contains all the points that are fixed by $f^\tau$. Since $p$ has an unstable branch in $\mathbb{D} \setminus \overline{V}$, we also may assume that each arc is either contained in $\overline{V}$ or disjoint from it. As in the statement of theorem H, we write $J > J'$ if $J$ has an unstable manifold fixed by $f^\tau$ which accumulates on $J'$ and we consider all the sequences $J^0 > J^1 > \cdots > J^n$ in $\mathcal{J}$ such that $q \in J^0$.

**Claim 88.** The periodic points (different from $p$) in all the arcs $J^i$ are decreasing chain related to $p$.

**Proof.** The property holds for $J^0$ since $J^0$ contains $q \in \mathcal{P}$ and is included in $\overline{V}$. If the property holds for $J^0, \ldots, J^i$, then by proposition 66 the unstable branches of $J^i$ are contained in $\overline{V}$. By definition 64, their unstable sets only accumulate on periodic points that are decreasing chain related to $p$ or coincide with $p$. In particular $J^{i+1}$ is included in $\overline{V}$ and then satisfies the inductive property.

**Claim 89.** The unstable branches of $J^n$ do not accumulate on $p$. 

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Proof. The claim is proved inductively for all arcs $J^0, J^1, \ldots, J^n$. Hence one will assume that for any arc $J^0, \ldots, J^{n-1}$ the unstable branches do not accumulate on $p$ and by contradiction that one of the unstable branches of $J^n$ accumulates on $p$.

Up to removing some intervals $J^i$, $0 < i < n$, from the sequence $J^0, \ldots, J^n$, one can also assume that $J^i > J^i$ does not hold when $i + 1 < i'$. We first prove that for each $i \in \{0, \ldots, n-1\}$, the unstable branches of $J^i$ avoid one of the components of $D \setminus \bigcup W^s_{\Delta}(J^{i+1})$: otherwise one unstable branch $\Gamma$ of $J^i$ would accumulate on the unstable branches of $J^{i+1}$ and from proposition [19], the accumulation set of $\Gamma$ contains the accumulation set of the unstable branches of $J^{i+1}$. When $i < n-1$, this implies $J^i > J^{i+2}$, contradicting our choice on the sequence $J^0, \ldots, J^n$. When $i = n - 1$, our assumption on $J^n$ implies that $\Gamma$ accumulates on $p$, contradicting our assumption that the unstable branches of $J^{n-1}$ do not accumulate on $p$.

Recall that from corollary [25], the map $f^k$ preserves the orientation. The property obtained in the last paragraph together with proposition [35] imply that for each $i \in \{0, \ldots, n-1\}$, the period of the unstable branches of $J^i$ equals the period of the unstable branches of $J^{i+1}$.

By definition of $P$, either $q \in J^0$ has period larger than $k$, or has period $k$ and the unstable branches of $J^0$ are exchanged by $f^k$. In any case the unstable branches of $J^0$ have period larger than $k$. Consequently the same property holds for each arc $J^i$.

But by assumption an unstable branch of $J^n$ accumulate on $p$ and is contained in $\overline{\tau}$. Since $f^k$ is orientable, Proposition [35] implies that the unstable branches of $J^n$ have period $k$. This is a contradiction and the claim is proved.

Theorem [H] applied to $f^\tau$ provides discs that are trapped by $f^\tau$, that are contained in $V$ (thanks to claim [89]), and that contain the accumulation sets of the unstable branches of $J^0$. Consider a neighborhood of $J^0$. Iterating forward, it may be glued to the trapped discs. This defines a disc contained in $V$ that is trapped by $f^\tau$. The lemma [87] is proved.

Lemma 90. For any $\tau > k$ there exists a finite number of disjoint $f^k$--trapped discs whose union $U_\tau$ is included in $V$ and contains $P(\tau)$. Moreover $U_\tau \subset U_{\tau'}$ when $\tau \leq \tau'$.

Proof. Observe that $P(\tau)$ is compact. From lemma [87] there exist finitely many discs $\Delta_q \subset V$ that are trapped by some iterates of $f^k$ and contain all the points of $P(\tau)$. Up to modify slightly their boundaries if necessary, one can assume that they are transverse. As a consequence the union $U_\tau$ of the discs is a finite disjoint union of submanifolds with boundary which are trapped by an iterate of $f^k$. Since $f$ is dissipative, the components of $U_\tau$ are topological discs. The construction is performed inductively on $\tau$, so that $U_\tau \subset U_{\tau'}$ when $\tau \leq \tau'$.

It remains to prove that each component of $U_\tau$ is trapped by $f^k$ (instead of an iterate of $f^k$). Let us assume by contradiction that this is not the case: there exists $k < l \leq \tau$ and a disc $\Delta_q \subset U_\tau$ that only contains points decreasing chain related to $p$ with period larger or equal to $l$. As in the proof of lemma [87], one considers a finite collection of disjoint isolated arcs fixed by $f^\tau$ and compute their contribution to the indices of $f^k$ and $f^l$ in the decorated region $V$.

The arc $I_0$ which contains $p$ is fixed by $f^k$ and $\text{index}(I_0, V, f^k) = \text{index}(I_0, V, f^k)$. Similarly, the total contribution of the arcs contained in a disc trapped by $f^k$ equals 1 (both for the maps $f^k$ and $f^l$). But the total contribution of the arcs contained in a disc trapped by $f^l$ and not by $f^k$ equals 1 for the map $f^l$ and 0 for the map $f^k$. Consequently
the index of $f^l$ in $V$ is larger than the index of $f^k$ in $V$. This is a contradiction, since from proposition 71, the indices of $f^k$ and $f^l$ in the decorated region $V$ coincide (and equal 1/2).

**Lemma 91.** The set $\mathcal{P}$ is contained in one of the regions $U_\tau$.

**Proof.** If the conclusion of the lemma does not hold, one can find a sequence $\tau_k \to +\infty$ such that $U_{\tau_{k+1}} \setminus U_{\tau_k}$ contains a periodic $f^k$-orbit $\mathcal{O}_k$ supported on $\mathcal{P}$. Up to take a subsequence, $(\mathcal{O}_k)$ converges towards a $f^k$-invariant compact set $K \subset \overline{V}$.

Note that $K$ is aperiodic. Indeed any $x \in K$ is accumulated by a sequence $(q_n)$ of $\mathcal{P}$. If $x$ were periodic, then corollary 78 would imply that the periods of the points $q_n$ is bounded. Since the $q_n$ are decreasing chain-related to $p$, this would imply that $x$ has the same property and belongs to $\mathcal{P}$. This is a contradiction since $K$ is disjoint from the increasing union $\cup U_\tau$ which contains $\mathcal{P}$.

Let $\mu$ be an ergodic $f^k$-invariant measure supported on $K$. By construction, for $\mu$-almost every point $x$ there exist a component $D$ of $U_\tau$ which contains a point arbitrarily close to $x$ as $\tau \to +\infty$. Since $D$ is trapped by $f^k$, the proposition 85 implies that $\mu$ is supported on $D$. A contradiction since $K$ is disjoint from $U_\tau$. \hfill $\Box$

We have shown that $\mathcal{P}$ is contained in the union $U_\tau$ of finitely many disjoint disks (denoted by $D_1, \ldots, D_m$) in $V$ that are trapped by $f^k$. To conclude, we need to prove that for each disc $D_i$, any periodic point in $D_i$ is decreasing chain related to $p$.

Note that the iterates of $D_i$ do not meet the stable manifold of the orbit of $p$ (otherwise the trapping property would imply that $D_i$ contain $p$, a contradiction). In particular $D_i$ can not contain any fixed point. Observe also that $D_i$ does not contain a stabilized periodic point (since one of its unstable branches accumulates on a fixed point and has to be contained in $D_i$). Hence by proposition 73, each periodic point in $D_i$ is decreasing chain related to some stabilized periodic point. The next lemma asserts that they are necessarily decreasing chain related to $p$.

**Lemma 92.** Any periodic point $q \in D_i$ is decreasing chain related to $p$.

**Proof.** The proof is done by contradiction: we assume that $D_i$ contains $q$ which is decreasing chain related to a stabilized periodic point $p'$ which is different from $p$.

Since $D$ does not contain the stabilized point $p'$ and is trapped by $f^k$, it is disjoint from $W^s_{\mathcal{D}}(p')$. In particular, it is contained in a decorated region $V'$ of $p'$. Note that either $V \subset V'$ or $V' \subset V$. We will assume that the first case occurs (the proof in the second case is similar). By definition 64, there exists a chain $C$ for an iterate of $f$ that is contained in $\overline{V'}$ intersects $V$ and $p'$. Hence $C$ meets $W^s_{\mathcal{D}}(p)$: there is an unstable branch in $C$ which intersects the stable manifold of $p$ implying that $p$ is decreasing chain-related to $p'$. The corollary 60 then gives the contradiction. \hfill $\Box$

The proof of proposition 86 is now complete. \hfill $\Box$

## 10 Finiteness of the set of stabilized periodic points

In order to prove the global renormalization (theorem A in the next section), one need to show that the number of stabilized orbits is finite. This section is devoted to prove the following:
Theorem I. Let \( f \) be a mildly dissipative diffeomorphism of the disc with zero topological entropy. Then the set of its stabilized periodic orbits is finite.

Before proving the theorem we associate to any stabilized orbit a filtrating region.

Proposition 93. For any stabilized periodic orbit \( O \) there exist two topological disks \( U_O \subset \bar{U}_O \) that are trapped by \( f \) such that

- \( \bar{U}_O \setminus U_O \) contains \( O \) and any periodic orbit decreasing chain related to \( O \),
- any periodic orbit in \( \bar{U}_O \setminus U_O \) is either \( O \) or is decreasing chain related to \( O \).

Moreover if \( O \) is contained in a trapping disc \( \Delta \), then \( U_O \) can be chosen in \( \Delta \).

Proof. We first assume that the period \( k \) of \( O \) is larger than 1. Let \( p \in O \), let \( V \) be the decorated region associated to \( p \). Theorem [H] applied to the stabilized unstable branch \( \Gamma \) of \( p \) associates a trapping disc \( U_O \) which contains the accumulation set of \( \Gamma \). Note that if \( O \) is contained in a trapping disc \( \Delta \), then \( U_O \) can be chosen to be included in \( \Delta \).

Now, we deal with the decorated region. Let \( D_1, \ldots, D_n \) be the \( f^k \)-trapping discs given by proposition [6]. Let \( I_1, \ldots, I_\ell \) be isolated \( f^k \)-fixed arcs containing all the periodic points in \( V \) that are decreasing chain-related to \( p \) and not in \( \cup D_i \). Since any periodic point close to \( p \) is contained in \( V \) (by corollary [78], we can assume that the arcs \( I_j \) are contained in \( \bar{V} \). By definition [56] of the chains and the invariance of the arcs, each periodic point in \( I_j \) either coincides with \( p \) or is decreasing chain related to \( p \).

Each \( I_j \) admits a neighborhood \( O_j \) which is a topological disc such that if the \( \omega \)-limit set of a point \( x \) by \( f^k \) intersects \( \bar{O}_j \), then it is contained in \( I_j \) (by corollary [77]).

Let \( \bar{U}_O \) be the forward invariant set defined by the union of the forward iterates of \( O_j \), of \( D_i \) and of \( U_O \). It can be written as the union of finitely many connected sets:

- the disc \( U_O \),
- the trapping discs \( f^m(D_i) \) for \( 0 \leq m < k \) and \( 1 \leq i \leq n \),
- the connected unions \( f^m(T_j) := f^m(O_j) \cup f^{m+k}(O_j) \cup f^{m+2k}(O_j) \cup \ldots \) for \( 1 \leq j \leq \ell \) and \( 0 \leq m < k \).

By definition of the decreasing chain relation, for each \( 0 \leq m < k \), the union of the interior of the sets \( f^m(D_i) \) and \( f^m(T_j) \) for \( 1 \leq i \leq n \) and \( 1 \leq j \leq \ell \) is connected. One set \( O_{j_0} \) contains the point \( p \) so that each \( f^m(T_{j_0}) \) for \( 0 \leq m < k \) intersects \( U_O \). This proves that the interior of \( \bar{U}_O \) is connected.

By construction, the set \( \bar{U}_O \) is forward invariant. Since \( f \) contracts the volume, the interior of \( \bar{U}_O \) is simply connected, hence homeomorphic to the open disc.

Lemma 94. If \( O_1, \ldots, O_\ell \) are sufficiently small neighborhoods of \( I_1, \ldots, I_\ell \), the \( \omega \)-limit set of any point in \( \text{Closure}(\bar{U}_O) \) is contained in \( \text{Interior}(\bar{U}_O) \).

Proof. Since there is no cycle, one can enumerate the arcs \( I_j \) in a way that the unstable branches of \( I_j \) do not accumulate on \( I_j \) when \( j \geq j' \).

Claim 95. Let us consider an unstable branch \( \Gamma \) of \( I_j \). Then the \( \omega \)-limit set by \( f^k \) of any point of \( \Gamma \) belongs to \( D_1 \cup \cdots \cup D_n \cup I_1 \cup \cdots \cup I_{j-1} \).
Proof. Let us consider a $f^k$-invariant ergodic measure $\mu$ supported on the accumulation set $L$ of $\Gamma$. If $\mu$ is supported on a $f^k$-periodic orbit, this orbit is decreasing chain-related to $\mathcal{O}$, hence is contained in $D_1 \cup \cdots \cup D_n \cup I_1 \cup \cdots \cup I_\ell$. By our choice of the indices of the arcs $I_1, \ldots, I_\ell$, the periodic orbit is contained in $D_1 \cup \cdots \cup D_n \cup I_1 \cup \cdots \cup I_{j-1}$ and the claim follows in this case.

If $\mu$ is aperiodic, the closing lemma (theorem [F]) implies that there exist periodic points in $L$ which accumulate on the support of $\mu$. These periodic points are also decreasing chain-related to $\mathcal{O}$ and are contained in $D_1 \cup \cdots \cup D_n \cup I_1 \cup \cdots \cup I_{j-1}$. Passing to the limit one deduces that $\mu$ is also contained in $D_1 \cup \cdots \cup D_n \cup I_1 \cup \cdots \cup I_{j-1}$. □

Since the discs $U_\mathcal{O}$ and $D_i$ are trapped by some iterates of $f$, it is enough to prove that the $\omega$-limit set under $f^k$ of any point in $\text{Closure}(O_j)$ is contained in the union

$$U_\mathcal{O} \cup (D_1 \cup \ldots \cup D_n) \cup (I_1 \cup \cdots \cup I_j).$$

This is proved inductively. We assume that the property holds for any $j' < j$ and consequently we can suppose that the closure of

$$\Delta_{j-1} := U_\mathcal{O} \cup (D_1 \cup \ldots \cup D_n) \cup (T_1 \cup \cdots \cup T_{j-1})$$

is mapped into its interior by some iterate of $f^k$.

Let us consider any point $x$ in $\text{Closure}(O_j)$. If its $\omega$-limit set belongs to $I_j$, the inductive property holds trivially. Otherwise, $x$ has a forward iterate close to a neighborhood $W$ of a fundamental domain of the unstable branches of $I_j$. By choosing the neighborhood $O_j$ of $I_j$ small enough, the neighborhood $W$ can be chosen arbitrarily small and by the claim [95] any point in $W$ has a forward iterate by $f^k$ in the interior of $\Delta_{j-1}$. We have thus proved that if the $\omega$-limit set of $x$ is not contained in $I_j$, then a forward iterate of $x$ by $f^k$ belongs to $\Delta_{j-1}$ as required. The inductive property is proved, which concludes the proof of lemma [94]. □

From the previous lemma, one can thus modify $\widetilde{U}_\mathcal{O}$ near its boundary (as explained in the proof of theorem [F]) and define a topological disc $\overline{U}_\mathcal{O}$ which is trapped by $f$.

The limit orbits in $U_\mathcal{O}$ and $\overline{U}_\mathcal{O}$ are the same: for any point in $\overline{U}_\mathcal{O}$, the $\omega$-limit set is contained in one of the trapping discs $D_i$, or in $U_\mathcal{O}$, or in one of the arcs $I_j$. With proposition [86] this shows that any periodic orbit in $\overline{U}_\mathcal{O} \setminus U_\mathcal{O}$ either coincide with $\mathcal{O}$ or is decreasing chain related to $\mathcal{O}$.

When the period is 1, the proof is almost the same. The point $p$ is fixed and has two decorated regions $V, V'$, each one of period 2. Working with $V$, one builds $f^2$-trapping discs $D_1, \ldots, D_n$, isolated $f^2$-fixed arcs $I_1, \ldots, I_\ell$, and neighborhoods $O_1, \ldots, O_\ell$ as before. □

Proof of theorem [A] We distinguish two types of stabilized periodic orbits $\mathcal{O}$.

- First type: $\mathcal{O}$ admits trapping discs $U_\mathcal{O} \subset \overline{U}_\mathcal{O}$ as in proposition [93] such that the set of stabilized periodic orbits in $U_\mathcal{O}$ is finite.

- Second type: for any trapping discs $U_\mathcal{O} \subset \overline{U}_\mathcal{O}$ associated to $\mathcal{O}$ as in proposition [93] there exists infinitely many stabilized periodic orbits in $U_\mathcal{O}$.

The following shows that the set of stabilized periodic orbits of the first type is finite.
Claim 96. Let \((O_n)\) be a sequence of distinct stabilized periodic orbits and let \(U_{O_n} \subset \widetilde{U}_{O_n}\) be trapping discs associated to \(O\) as in proposition 93. Up to consider a subsequence, the following property holds: \(O_m \subset U_{O_n}\) for each \(m > n\).

\[\text{Proof.}\] Up to take a subsequence, one can assume that the sequence \((O_n)\) converges for the Hausdorff topology towards an invariant compact set \(K\). From the fact that periodic points are isolated from periodic points of large period (corollary 78) it follows that \(K\) does not contain any periodic point. Let \(\mu\) be an ergodic measure supported on \(K\). It is aperiodic and by proposition 85 for any \(n\) large the disc \(\widetilde{U}_{O_n}\) contains the support of \(\mu\). In particular the stabilized orbits \(O_m\) for \(m > n\) large intersect \(\widetilde{U}_{O_n}\). All the stabilized periodic points (different from the points of \(O_n\)) that are contained in \(\widetilde{U}_{O_n}\) are contained in \(U_{O_n}\), hence the orbits \(O_m\) meet and (by the trapping property) are contained in \(U_{O_n}\). Up to extract the sequence \((O_n)\), one can assume that for any \(n\), the set \(K\) is contained in \(U_{O_n}\).

Let us fix the integer \(n\). We have obtained that for \(m \geq n\) large \(O_m\) is contained in \(U_{O_n}\). Up to extract the subsequence \((O_m)_{m > n}\), one can assume that all the orbits \(O_m\) for \(m > n\) are contained in \(U_{O_n}\). By induction, one builds an extracted sequence which satisfies the required property for all integers \(m > n\).

To conclude, it is enough to show that there are not stabilized points of the second type. Let us assume now by contradiction that this is not the case. One builds inductively a sequence of stabilized periodic orbits of the second type \((O_n)\) and trapping discs \((U_n)\) satisfying for each \(n \geq 1\):

- \(U_n \subset U_{n-1}\),
- \(O_n \subset U_{n-1} \setminus U_n\),
- \(U_n\) contains infinitely many stabilized periodic orbits,
- the period of \(O_n\) is minimal among the periods of the stabilized periodic orbits of the second type contained in \(U_{n-1}\).

When \(O_n\) and \(U_n\) have been built, we choose \(O_{n+1}\) as a stabilized periodic orbits of the second type contained in \(U_n\) which minimizes the period. By proposition 93 there exists trapping discs \(U_{n+1} \subset \widetilde{U}_{n+1}\) associated to \(O_n\) and one can require that \(U_{n+1}\) is contained in the trapping \(U_n\). In particular \(O_{n+1} \subset U_n \setminus U_{n+1}\). Since \(O_{n+1}\) is of the second type, \(U_{n+1}\) contains infinitely many stabilized periodic orbits as required.

Once the sequences \((O_n)\) and \((U_n)\) have been built, one considers (up to extract a subsequence) the Hausdorff limit \(K\) of \((O_n)\). As in the proof of claim 96, it supports an ergodic measure \(\mu\) which is aperiodic. The intersection of the discs \(U_n\) defines an invariant cellular set \(\Lambda\) that contains \(K\). The closing lemma 84 implies that \(\Lambda\) contains periodic points \((p_k)\) with arbitrarily large period which accumulate on a point \(x\) of \(K\). We consider different cases:

- Some \(p_k\) belongs to stabilized periodic orbits of the second type. Since the minimal period of stabilized periodic orbits of the second type contained in \(U_n\) goes to \(+\infty\) as \(n \to +\infty\), this is a contradiction.

- Some \(p_k\) is decreasing chain related to a stabilized periodic orbit \(O\) of the second type. Each trapping disc \(U_n\) contains a fixed point \(q\) and there exists a decorated
region \( V \) of \( \mathcal{O} \) which does not contain \( q \). Up to replace \( p_k \) by one of its iterates, one can assume \( p_k \in V \). This shows that \( U_n \) meets \( V \) and its complement, hence intersects the stable set of \( \mathcal{O} \). Since \( U_n \) is a trapping disc, it contains \( \mathcal{O} \). But the minimal period of stabilized periodic orbits of the second type contained in \( U_n \) goes to \( +\infty \) as \( n \to +\infty \), and this is a contradiction.

- All the points \( p_k \) are decreasing chain related to stabilized periodic orbits of the first type. Since the number of this type of stabilized periodic orbits is finite, one can assume that the \( p_k \) are all decreasing chain related to the same stabilized periodic orbit \( \mathcal{O} \). Let us consider trapping discs \( U_{\mathcal{O}} \subset \overline{U}_{\mathcal{O}} \) as in proposition 93. All the \( p_k \) are contained in the filtrating region \( \overline{U}_{\mathcal{O}} \setminus U_{\mathcal{O}} \). Taking the limit, \( K \) meets that region. In particular, the orbits \( \mathcal{O}_n \) for \( n \) large also meet that region. This is a contradiction since the orbits \( \mathcal{O}_n \) are stabilized and the region \( \overline{U}_{\mathcal{O}} \setminus U_{\mathcal{O}} \) contains only one stabilized periodic orbit (the orbit \( \mathcal{O} \)).

In all the cases we found a contradiction. This ends the proof of theorem I. \( \square \)

11 Global renormalization

We now prove a strong version of theorem A and its corollary. The proof of corollary is then immediate and left to the reader.

**Theorem A’.** For any mildly dissipative diffeomorphism \( f \) of the disc with zero topological entropy, there exist \( \ell \geq 0 \), some disjoint topological discs \( D_1, \ldots, D_\ell \), some integers \( k_1, \ldots, k_\ell \geq 2 \) such that:

- each \( D_i \) is trapped by \( f^{k_i} \),
- the discs \( f^m(D_i) \) for \( 1 \leq i \leq \ell \) and \( 0 \leq m < k_i \) are pairwise disjoint,
- for each \( D_i \) there is a stabilized orbit such that each iterate of \( D_i \) is contained in a decorated region of an iterate of the stabilized orbit,
- \( f \) is generalized Morse-Smale in the complement of the union of the iterates of the disks \( (D_i) \) with periodic points of period smaller and equal to to \( \max\{1, k_1, \ldots, k_\ell\} \).

In particular the interior \( W \) of the union \( \bigcap_i \bigcap_{m \geq 0} f^m(D_i) \) is a filtrating open set.

11.1 Global renormalization: proof of theorem A’

We apply proposition 86 and associate to each stabilized periodic orbit \( \mathcal{O}_i \) some discs \( D_{i,1}, \ldots, D_{i,\ell_i} \) that are trapped by \( f^{k_i} \) where \( k_i \) is the period of the decorated regions associated to \( \mathcal{O}_i \). By construction each \( D_{i,j} \) is contained in a decorated region of \( \mathcal{O}_i \) and all the periodic points decreasing chain-related to \( \mathcal{O}_i \) and with period larger than \( k_i \) belong to the orbit of the \( D_{i,j} \).

The discs \( D_{i,j} \) and \( D_{i',j'} \) associated to different orbits \( \mathcal{O}_i, \mathcal{O}_{i'} \) are disjoint by lemma 92, proposition 73 and corollary 69. Since the number of stabilized orbits is finite (theorem I), we get the two first items.

Note that any periodic point which does not belong to the \( D_{i,j} \) is either fixed, or stabilized, or decreasing chain-related to a stabilized point with the same period. Hence its period is bounded by \( \max\{1, k_1, \ldots, k_\ell\} \).
Let \( x \) be any point whose \( \omega \)-limit set does not belong to a trapped disc. The limit set supports an ergodic measure \( \mu \). This measure cannot be aperiodic since the closing lemma would produce a periodic orbit with large period outside the discs \( D_{i,j} \). Hence the limit set contains a periodic orbit and by corollary 77 coincides with the periodic orbit. The theorem A’ is now proved.

11.2 Infinite renormalization: proof of corollary 3

By theorem A’, the dynamics of \( f \) reduces to a generalized Morse-Smale dynamics in a filtrating set \( D \setminus W \). If \( W = \emptyset \), the diffeomorphism \( f \) is generalized Morse-Smale and corollary 3 holds.

Each connected component of \( W \) is a topological disc \( \Delta \) which is trapped by an iterate \( f^k \) of \( f \); moreover the restriction of \( f^k \) to \( \Delta \) is a mildly dissipative diffeomorphism. One may thus apply theorem A’ inside each of these discs. Arguing inductively, one gets a new decomposition of the dynamics into a generalized Morse-Smale part and discs that are eventually trapped after a return time which increases at each step of the induction.

If \( f \) is not generalized Morse-Smale, the induction does not stop and \( f \) is infinitely renormalizable. Corollary 3 follows.

12 Chain-recurrent dynamics

We now describe in detail the dynamics of a mildly dissipative diffeomorphism with zero entropy and prove corollary 7.

12.1 Generalized odometers

**Proposition 97.** Let \( f \) be a mildly dissipative diffeomorphism of the disc, \( (D_i) \) be a sequence of topological discs and \( (k_i) \) be a sequence of integer such that

- \( D_i \) is trapped by \( f^{k_i} \) and disjoint from its \( k_i - 1 \) first iterates,
- \( D_i \subset D_{i+1} \) and \( k_i < k_{i+1} \) for each \( i \).

Then the intersection of the sets \( f^{k_i}(D_i) \cup f^{k_i+1}(D_i) \cup \cdots \cup f^{2k_i-1}(D_i) \) is a chain-recurrence class \( C \) which is a generalized odometer. In particular it supports a unique invariant measure \( \mu \) and for \( \mu \)-almost every point \( x \), the connected component of \( x \) in \( C \) is a singleton.

**Proof.** Let us denote by \( C \) the intersection of the sets \( f^{k_i}(D_i) \cup f^{k_i+1}(D_i) \cup \cdots \cup f^{2k_i-1}(D_i) \). It is a compact invariant set.

For each \( i \), let \( (O_i, h_i) \) be the cyclic permutation on the set with \( k_i \) elements. The inverse limit of the systems \( (O_i, h_i) \) defines an odometer \( (K, h) \) on the Cantor set. The sets \( D_i, \ldots, f^{k_i-1}(D_i) \) define a partition of \( C \) and induce a factor map on \( (O_i, h_i) \), hence a semi-conjugacy \( p: (C, f) \sim (K, h) \). Since the connected components of \( C \) coincide with the decreasing intersections of sequences of the form \( f^{m_i+k_i}(D_i) \), the preimages \( p^{-1}(x) \) coincide with the connected components of \( C \).

Let \( \nu \) be the unique invariant probability measure on \( (K, h) \) and let \( \mu \) be an ergodic probability on \( (C, f) \) such that \( p_*(\mu) = \nu \). The following claim shows that \( C \) is a generalized odometer.

**Claim 98.** For \( \nu \)-almost every point \( z \in K \), the preimage \( p^{-1}(z) \) is a singleton.
Proof. Let us consider a set $X \subset \mathcal{C}$ with positive $\mu$-measure which is a hyperbolic block, such that $W^s_D(x)$ varies continuously with $x \in X$. One can also find a disc $D \subset \mathbb{D}$ which contains $\mathcal{C}$ such that $f(D) \subset \text{Interior}(D)$ and whose boundary is transverse to the manifolds $W^s_D(x)$, for $x \in X$. Let $B \subset X$ be a subset with positive measure of points having arbitrarily large backward iterates $f^{-n}(x) \in X$ that are accumulated by points of $X$ on both components of $\mathbb{D} \setminus W^s_D(f^{-n}(x))$.

Let us choose $\varepsilon > 0$. For each $x \in B$, there exist backward iterates $f^{-n}(x) \in X$ such that $f^n(W^s_D(f^{-n}(x)))$ has diameter smaller than $\varepsilon/2$. As in the proof of theorem 97, one can thus find a rectangle $R$ with diameter smaller than $\varepsilon$, which contains $x$ and whose boundary is contained in $\partial f^n(D) \cup W^s_D(x') \cup W^s_D(x'')$ for two forward iterates $x', x''$ of $x$. For $i$ large enough, the disc $f^{m_i+k_i}(D_i)$ which contains $x$ is contained in $f^n(D)$, and does not meet the iterates $x', x''$, nor their stable manifolds. Consequently, $f^{m_i+k_i}(D_i)$ has diameter smaller than $\varepsilon$. Since $\varepsilon > 0$ has been chosen arbitrarily, the connected component of $\mathcal{C}$ containing $x \in B$ is reduced to $x$.

Since $x$ is arbitrary in $B$ which has positive measure and since $\mu$ is ergodic, one deduces that for $\mu$-almost every $x$, the connected component of $x$ in $\mathcal{C}$ (which coincides with $p^{-1}(p(x))$) is a singleton. Since $p_*(\mu) = \nu$, the claim follows. \qed

The claim and the characterization of the connected components of $\mathcal{C}$ prove that for $\mu$-almost every point $x$, the connected component of $x$ in $\mathcal{C}$ is a singleton.

Since the discs $D_i$ are trapped by $f^{k_i}$, any chain-recurrence class which meets $\mathcal{C}$ is contained in $\mathcal{C}$. For any $\varepsilon$, let us consider $i$ and an iterate $f^{m_i+k_i}(D_i)$ with diameter smaller than $\varepsilon$. Any forward and backward orbit in $\mathcal{C}$ intersects $f^{m_i+k_i}(D_i)$, showing that $\mathcal{C}$ is chain-transitive. This implies that $\mathcal{C}$ is a chain-recurrence class. \qed

### 12.2 Dynamics on chain-recurrence classes: proof of corollary 7

Let us apply inductively theorem 10 as in the proof of corollary 9. We obtain a decreasing sequence $(W_n)$ of trapped open sets such that the dynamics in each $\mathbb{D} \setminus \overline{W_n}$ is generalized Morse-Smale. By proposition 48, the chain-recurrent set in that region is the set of periodic points. Since their period is bounded, the chain-recurrence classes $\mathcal{C}$ of $f$ in the complement of any $W_k$ can be written as a disjoint union $\mathcal{C} = C \cup \cdots \cup f^{m-1}(C)$, where $C$ is a connected component of the set of periodic points and $f^m(C) = C$. Corollary 7 is proved when $f$ is generalized Morse-Smale.

It remains to describe the dynamics when $f$ is infinitely renormalizable, that is when the sequence $(W_n)$ is infinite. By construction the infimum of the periods of the periodic points in $W_n$ gets arbitrarily large as $n$ goes to infinity. Up to replace $(W_n)$ by the sequence $(f^m(W_n))$, one can assume that the intersection $\cap W_n$ is an invariant compact set $\Lambda$.

By construction, each connected component of $W_n$ is a topological disc $D$ which is trapped by an iterate $f^m$ and disjoint from its $m-1$ first iterates. Moreover $m$ goes uniformly to infinity as $n \to +\infty$ (since the periods in $D$ get large when $n$ increases). One deduces from proposition 97 that $\Lambda$ is a union of generalized odometers that are chain-recurrences classes of $f$. This ends the proof of corollary 7. \qed

### 13 Set of periods

In the present section we provide the proof of theorem 11 and corollary 9.
13.1 Proof of theorem $\text{B}$

We consider an infinitely renormalizable $C^r$ diffeomorphism, since for generalized Morse-Smale systems the conclusion of theorem $\text{B}$ holds immediately by taking $W = \emptyset$. From theorem $\text{A}$, if one assumes by contradiction that theorem $\text{B}$ is not satisfied, there exist a sequence of topological discs $(D_i)$, integers $k_i, \tau_i \to +\infty$, periodic orbits $(O_i)$ such that

- $D_i$ is trapped by $f^{k_i}$ and disjoint from its $k_i - 1$ first iterates,
- $O_i$ is contained in $D_i \cup f(D_i) \cup \cdots \cup f^{k_i-1}(D_i)$ and has period $\tau_i$,
- $O_i \cap f^m(D_i)$ is a stabilized orbit of $f^{k_i}$ for $0 \leq m < k_i$ and has period $\tau_i/k_i \geq 3$.

**Claim 99.** For each $i, m$, $O_i \cap f^m(D_i)$ is a decorated orbit of $f^{k_i}$ in $\mathbb{D}$.

**Proof.** The intersection $O_i \cap f^m(D_i)$ is a decorated orbit in $D = f^m(D_i)$: this means that for any points $x, y$ in the orbit, there exists a path in $D$ which connect them and is disjoint from the local stable manifolds $W^s_D(z)$ in $D$ of the other points of the orbit. In particular there exists a path in $\mathbb{D}$ which connect them and is disjoint from the local manifolds $W^s_D(z)$ in $\mathbb{D}$, proving that $O_{i+1} \cap f^m(D_i)$ is decorated in $\mathbb{D}$.

Let us choose $\alpha \in (0, \min(1, r - 1))$ and $\varepsilon \in (0, 1/4)$. Theorem $\text{D}$ associates $\gamma \in (0, 1)$ by theorem $\text{A}$ there exists a nested sequence of topological discs $\widehat{D}_i$ that are periodic and trapped with periods $\widehat{k}_i \to +\infty$ such that $D_i \subset \widehat{D}_i$. By proposition 97 the intersection of the sets $\widehat{D}_i \cup f(\widehat{D}_i) \cup \cdots \cup f^{\widehat{k}_i-1}(\widehat{D}_i)$ is a chain-recurrence class $C$ which is a generalized odometer. In particular it does not contain any periodic points and it supports a unique invariant probability $\mu$. Proposition 23 implies that $f$ is $\gamma$-dissipative on $C$, hence on the domains $\widehat{D}_i \cup f(\widehat{D}_i) \cup \cdots \cup f^{\widehat{k}_i-1}(\widehat{D}_i)$ for $i$ large enough. Theorem $\text{D}$ provides a compact set $A$ such that $W^s_D(x)$ exists and varies continuously with $x \in A$ in the $C^1$ topology and $\nu(A) > 3/4$ for any invariant probability measure supported on a neighborhood of $C$. In particular the orbits $O_i$ have at least $3\tau_i/4$ itetates in $A$.

By proposition 97 for $\mu$-almost every point $x$, the connected component of $x$ in $C$ is reduced to $\{x\}$. This implies that for any $\delta > 0$ and for $i$ large enough, at least $3\widehat{k}_i/4$ discs in the family $\widehat{D}_i \cup f(\widehat{D}_i) \cup \cdots \cup f^{\widehat{k}_i-1}(\widehat{D}_i)$ have diameter smaller than $\delta$.

The number of discs $f^m(\widehat{D}_i)$ ($0 \leq m < \widehat{k}_i$) which contain at most 2 points in $O_{i+1} \cap A$ is smaller than $(\tau_i/\widehat{k}_i - 2)^{-1} \text{Card}(O_i \setminus A)$, hence than $\tau_i/4$. Consequently there exists a disc $f^{m+\delta_k}(\widehat{D}_i)$ with diameter smaller than $\delta$ which contains at least 3 points $x, y, z$ of $A \cap O_i$. Since the three points are close, the local stable manifolds $W^s_D(x)$, $W^s_D(y)$, $W^s_D(z)$ are close for the $C^1$-topology. In particular there are coordinates in the disc such that the three curves are graphs over one of the coordinate axis. This implies that one of the stable manifolds separates the two other ones in $\mathbb{D}$. This is a contradiction since the orbit $O_i \cap f^m(D_i)$ of $f^{k_i}$ is decorated. Theorem $\text{B}$ is proved.

13.2 Proof of corollary 9

Theorem $\text{B}$ implies that there exists $m \geq 1$, a finite number of topological discs $D_1, \ldots, D_\ell$ and integers $m_1, \ldots, m_\ell$ such that

- the discs $f^i(D_i)$ with $1 \leq i \leq \ell$ and $0 \leq k < m_i$ are pairwise disjoint,
- each disc $D_i$ is trapped by $f^{m_i}$. 

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- the set $F$ of periodic points in the complement of $\cup_{i,k} f^k(D_i)$ is finite,

- each $f^k|_{D_i}$ is infinitely renormalizable and each renormalization disc $\Delta \subset D_i$ is contained in a sequence of renormalization discs $\Delta_0 = \Delta \subset \Delta_1 \subset \cdots \subset \Delta_s = D_i$ such that the period of $\Delta_j$ is the double of the period of $\Delta_{j+1}$.

Theorem A’ shows that the set of periods of each diffeomorphism $f^k|_{D_i}$ coincides with $\{2^n, n \geq 0\}$. This shows that the set of periods of $f$ coincides with $F \cup \{m_i 2^n, 1 \leq i \leq \ell \text{ and } n \in \mathbb{N}\}$.

Corollary follows. \qed

14 Dynamics close to one-dimensional endomorphisms

In this section, we prove theorem C.

14.1 Extension of one-dimensional endomorphisms

From now on, and to keep the approach described in [CP], we consider extensions of a one-dimensional endomorphisms which slightly differ from (2), but which work both for the interval and the circle: given a one-dimensional manifold $I$ (the circle $S^1$ or the interval $(0,1)$), a $C^2$ map $h : I \to I$ isotopic to the identity (such that $h(\partial I) \subset \text{Interior}(I)$ in the case of the interval), $\epsilon > 0$ small and $b \in (-1,1)$ even smaller, we get a map $f_b$ on $D := I \times (-\epsilon, \epsilon)$ defined by

$$f_b : (x, y) \to (h(x) + y, b(h(x) - x + y)).$$

Indeed for any $y \in \mathbb{R}$ close to 0 and any $x \in h(I)$, the sum $x + y$ is well defined and, since $h$ is isotopic to the identity, the difference $h(x) - x$ belongs to $\mathbb{R}$. Note that the Jacobian is constant and equal to $b$. When $|b| > 0$, the map $f_b$ is a diffeomorphism onto its image. When $b = 0$ the image $f_0(D)$ is contained in $I \times \{0\}$ and the restriction of $f_0$ coincides with $h \times \{0\}$.

Theorems 1 and 2 in [CP] assert that for $|b| > 0$ small enough, the map $f_b$ is mildly dissipative and that the same property holds for any diffeomorphism. The diffeomorphism (2) that is presented in the introduction can be handled in the same way. Indeed for $b = 0$, the map $f_0$ is an endomorphism which contracts the curves $h(x) + y = \text{cte}$ to a point: these curves are analogous to strong stable manifolds. One can check moreover that, for any ergodic measure which is not supported on a sink, the points in a set with uniform measure are far from the critical set, implying that these curves cross the domain $I \times (-\epsilon, \epsilon)$. For $|b| > 0$, the control of the uniformity of the stable manifolds ensures that for points in a set with uniform measure has local stable manifolds close to the curves $h(x) + y = \text{cte}$.

14.2 Parallel laminations

The proof of theorem C follows from the property that for points on a large set, the stable manifolds are “parallel”, i.e. do not contain decorated configurations.

**Definition 100.** A family of $C^1$-curves $\gamma : [0,1] \to \mathbb{D}$ is parallel if:
Given a diffeomorphism \( g \) any periodic orbit in the complement has period 1 or 2.

Let us choose a diffeomorphism \( g \) such that any periodic orbit in the complement is fixed by \( g \) and for any ergodic measure \( \mu \) which is not supported on a sink, \( \mu(S) > 1 - \delta \).

**Proposition 101.** Given \( \delta > 0 \) and a \( C^2 \)-endomorphism \( h \) of the interval, there exists \( b_0 > 0 \) such that for any \( b \) with \( 0 < |b| < b_0 \), and for any diffeomorphism \( g \) in a \( C^2 \)-neighborhood of \( f_0 \), there exists a compact set \( S \) such that:

- each \( x \in S \) has a stable manifold and the family \( \{W^s_b(x), x \in S\} \) is parallel;
- for any ergodic measure \( \mu \) which is not supported on a sink, \( \mu(S) > 1 - \delta \).

**Proof.** We follow and adapt the proof of theorem 2 in [CP]. Let \( K > \| Dh \| \) and fix \( L \gg K \). We introduce four numbers, depending on \( b \):

\[
\sigma(b) := L|b|, \quad \tilde{\sigma}(b) = |b|/5K, \quad \tilde{\rho}(b) := |b|/25K^2, \quad \rho = L^2 |b|.
\]

Consider the set \( A(f_b) \) of points \( x \) having a direction \( E \subset T_x D \) satisfying

\[
\forall n \geq 0, \quad \tilde{\sigma}^n \leq \| Df^n(x)|E \| \leq \sigma^n, \quad \text{and} \quad \tilde{\rho}^n \leq \frac{\| Df^n(x)|E \|}{\det Df^n(x)} \leq \rho^n.
\]

The proof of lemma 4.4 in [CP] shows that by taking \( L \) large enough, then \( \mu(A(f_b)) > 1 - \delta/2 \) for any invariant ergodic probability \( \mu \) which is not supported on a sink. Let us choose a small neighborhood \( U \) of the critical set \( \{x, Dh(x) = 0\} \). Then the measure \( \mu(U \times (-\epsilon, \epsilon)) \) is smaller than \( \delta/2 \) and on its complement, the angle between the stable manifolds \( W^s(x) \) for \( x \in A(f_b) \backslash U \times (-\epsilon, \epsilon) \) is bounded away from zero.

Having chosen \( |b| \) small enough, the leaves \( W^s_b(x) \) for \( f_b \) are \( C^1 \)-close to affine segments (theorem 1 in [CP]) and are uniformly transverse to the horizontal for points \( x \) in the set \( S := A(f_b) \backslash U \times (-\epsilon, \epsilon) \), defining a parallel lamination for \( f_b \).

When \( |b| \) is small enough, the mild dissipation is robust (see theorem 1 of [CP]) and the property extends to diffeomorphisms \( g \) that are \( C^2 \)-close to \( f_b \).

\[\square\]

### 14.3 Proof of theorem [C]

Let us choose a diffeomorphism \( g \) as in the statement of theorem [C]. Having chosen \( g \) in a small neighborhood of a diffeomorphism \( f_b \), with \( |b| \) small, ensures that proposition [101] holds for some \( \delta \in (0, 1/3) \).

In particular at least 2/3 of the iterates of any stabilized periodic orbit meets the set \( S \): the parallel property then implies that the period of any stabilized periodic orbit is 1 or 2. The theorem [A] gives disjoint renormalization discs with period 2, such that any periodic orbit in the complement has period 1 or 2.

Let us consider any one of the obtained renormalization domains \( D \) and the induced diffeomorphism \( g^2|_D \). Note that if \( O \subset D \) is a stabilized orbit of \( g^2 \), then both \( O \) and \( g(O) \) are stabilized periodic orbits of \( g^2 \) in \( D \). By proposition [101] at least 2/3 of the iterates of \( O \) or \( g(O) \) belong to \( S \). The parallel property then implies that the period of \( O \) under \( g^2 \) is 1 or 2. The theorem [A] gives smaller disjoint renormalization discs with period 4, such that any periodic orbit in the complement is fixed by \( g^4 \). Arguing inductively, one deducts that there exists renormalization discs of period \( 2^n \) such that the periodic orbits in the complement are fixed by \( g^{2^n} \). Consequently, any periodic orbit is fixed by some iterate \( g^{2^n} \), hence has a period which is a power of 2.

\[\square\]
15 Dynamics of the Hénon map

In this section we prove corollary 11.

15.1 Reduction to a dissipative diffeomorphism of the disc

The dynamics of a dissipative Hénon map is the same as the dynamics of a dissipative diffeomorphism of the disc.

Proposition 102. For any Hénon map \( f_{b,c} \) with \( |b| \in (0,1) \), there exists:

- a smooth dissipative diffeomorphism of the disc \( g: \mathbb{D} \to f(\mathbb{D}) \),
- a topological disc \( \Delta \subset \mathbb{R}^2 \),
- a homeomorphism \( h: \Delta \to \mathbb{D} \),
- a decomposition \( \mathbb{D} = \mathbb{D}_1 \cup \mathbb{D}_2 \) into half discs separated by a diameter,
- a decomposition \( \Delta = \Delta_1 \cup \Delta_2 \) with \( \Delta_i = h(D_i) \),

such that:

1. \( g(D_2) \subset \text{interior}(D_2) \) and any forward orbit of \( g|_{D_2} \) converges to a fixed point \( p_0 \);
2. \( f_{b,c}(\Delta_1) \subset \text{interior}(\Delta) \) and \( f_{b,c} = h \circ g \circ h^{-1} \) on \( \Delta_1 \);
3. the forward orbit of any \( x \in \Delta_2 \) escapes to infinity: \( \|f_{b,c}^n(x)\| \to \infty \) as \( n \to \infty \);
4. the backward orbit of any \( x \in \Delta \setminus f_{b,c}(\Delta) \) escapes to infinity: \( \|f_{b,c}^{-n}(x)\| \to \infty \) as \( n \to -\infty \);
5. any \( f_{b,c} \)-orbit which does not meet \( \Delta \) escapes to infinity in the past and future.

Remark 103. When \( |b| < 1/4 \), the diffeomorphism \( g \) is mildly dissipative. Indeed let us consider an ergodic measure \( \mu \) of \( g \) which is not supported on a sink. From item (1), it is supported on \( \mathbb{D}_1 \). From items (2), \( \nu := h^{-1}_*(\mu) \) is an ergodic measure for \( f_{b,c} \) which is not supported on a sink. From Wiman theorem (see [CP] theorem 2 and section 4.2), for \( \nu \)-almost every point \( x \), each stable curve of \( x \) is unbounded in \( \mathbb{R}^2 \), hence intersects the boundary of \( \Delta_1 \). From item (2) again, one deduces that for \( \mu \)-almost every point \( y \), each stable curve intersects the boundary of \( \mathbb{D}_1 \) and cannot meet \( \mathbb{D}_2 \) since its forward orbit is not attracted by \( p_0 \). One deduces that each stable curve of \( y \) meets the boundary of \( \mathbb{D} \), proving that \( g \) is mildly dissipative.

Proof of proposition 102. Let us fix \( R > 0 \) and define \( D = D_1 \cup D_2 \) where

\[ D_1 = [-R,R] \times [-\sqrt{|b|}R,\sqrt{|b|}R], \quad D_2 = [R,2R^2] \times [-\sqrt{|b|}R,\sqrt{|b|}R]. \]

One checks easily that if \( R \) is large enough,

\( f_{b,c}(D_1) \subset \text{interior}(D) \quad \text{and} \quad f_{b,c}(D_1 \cap D_2) \subset \text{interior}(D). \)

One then defines an embedding \( \tilde{f}: D \to \text{interior}(D) \) such that

- \( \tilde{f}|_{D_1} = f_{b,c}|_{D_1} \),
One approximates $D$ by a disc $\Delta \subset D$ with a smooth boundary: $D \setminus \Delta$ is contained in a small neighborhood of $\partial \Delta$ and such that $(R) \times \mathbb{R}$ decomposes $\Delta$ in two half discs $\Delta_1, \Delta_2$. One then chooses a diffeomorphism $h: \Delta \to \mathbb{D}$ and set $g = h \circ \tilde{f} \circ h^{-1}$. The items (1) and (2) of the proposition are then satisfied.

Note that the domain $U^+ = \{(x, y), \ |x| \geq R \text{ and } |y| \leq \sqrt{b}|x|\}$ is mapped into itself and that if $R$ has been chosen large enough then the image $(x', y')$ of $(x, y) \in U^+$ satisfies $x' > 2|x|$. Consequently the forward orbit of any point in $U^+$ escapes to infinity.

The inverse map is $f_{b,c}^{-1}(x, y) \mapsto (-y/b, x - y^2/b^2 - c)$. As before, the domain $U^- = \{(x, y), \ |x| \geq R \text{ and } |y| \geq \sqrt{b}|x|\}$ is mapped into itself by $f_{b,c}^{-1}$ and that if $R$ has been chosen large enough then the preimage $(x', y')$ of $(x, y) \in U^-$ satisfies $|y'| > 2|y|$. Consequently the backward orbit by $f_{b,c}$ of any point in $U^-$ escapes to infinity. This concludes the proof of items (3), (4), (5). $\square$

### 15.2 Proof of corollary 11

Let $g$ be the diffeomorphism given by proposition 102. Since the topological entropy of $f_{b,c}$ vanishes, the same holds for $g$. Moreover by remark 103, $g$ is mildly dissipative.

From the items (3) and (5) of proposition 102, any forward orbit by $f_{b,c}$ which does not escape to infinity accumulates in a subset $K$ of $\Delta_1$. The image $h(K)$ by the conjugacy is the limit set of a forward orbit of $g$. It is contained in a chain-recurrence class of $g$. With corollary 7 one deduces that the forward orbit of $f_{b,c}$ converges to a periodic orbit or to a subset of an odometer. From items (4) and (5), a similar conclusion holds for backward orbits.

The periodic set of $f_{b,c}$ is included in $\Delta_1$ and is conjugated by $h$ to the periodic set of $g$, once the fixed point $p_0$ has been excluded. Hence the set of periods of $f_{b,c}$ can be described from the set of periods of $g$. By corollary 9 it has the structure [1]. $\square$

### 15.3 Final remark: trapping discs for the Hénon map

We propose an alternative proof to corollary 11 in the case where the Hénon map $f_{b,c}$ is orientation-preserving (i.e. $b \in (0, 1)$). Indeed the following proposition holds. One can then find a trapping disc and directly apply corollaries 7 and 9 to $f_{b,c}$.

**Proposition 104.** Any Hénon map $f = f_{b,c}: (x, y) \mapsto (x^2 + c + y, -bx)$ with jacobian $b \in (0, 1)$ satisfies one of the following properties:

a) There is no fixed point. All the orbits escape to infinity in the past and the future.

b) The fixed points belong to a simple curve $\gamma: [0, +\infty) \to \mathbb{R}^2$ whose image is invariant and which satisfies $\gamma(t) \overset{t \to +\infty}{\longrightarrow} \infty$. Any forward (resp. backward) orbit either converges to a fixed point or escape to infinity.

c) There exist a topological disc $D \subset \mathbb{R}^2$ trapped by $f$ and a simple curve $\gamma: \mathbb{R} \to \mathbb{R}^2$ whose image is invariant and which satisfies $\gamma(t) \overset{t \to +\infty}{\longrightarrow} \infty$ and $\gamma((-\infty, 0]) \subset D$ such that all the fixed points are contained in $D \cup \gamma(\mathbb{R})$. Any backward orbit either converges to a fixed point or escape to infinity. Any forward orbit either converges to a fixed point, escape to infinity, or is trapped by $D$. 71
d) There is a fixed point with a homoclinic orbit. The topological entropy is positive.

*Sketch of the proof.* If there is no fixed point, by Brouwer theorem, any orbit \( (f^n(x))_{n \in \mathbb{Z}} \) converges to infinity when \( n \to \pm \infty \) and the proposition holds. In the following we will thus assume that there exists at least one fixed point and that the topological entropy is positive.

If \((x, y)\) is fixed, then \( x^2 + c - (1 + b)x = 0 \). Hence there exists at most two fixed points. We denote by \( q = (x_q, y_q) \) the point whose first coordinate satisfies \( x_q = \frac{1 + b}{2} + \sqrt{\frac{(1+b)^2}{4} - c} \). Note that it is a saddle-node fixed point when \( 4c = (1 + b)^2 \), and a hyperbolic saddle with positive eigenvalues otherwise.

**Claim 105.** The right unstable branch of \( q \) is a graph over \((x_q, +\infty)\). It is contained in the wandering set and the forward orbit of any point in a neighborhood escapes to infinity.

*Proof.* We first consider the case \( b > 0 \). At a point \( z = (x, y) \), one considers the direction \((1, v_z) := (1, -x + \sqrt{x^2 - b})\). Let us assume \( x \geq 0 \) and that the image \( z' = (x', v') \) satisfies \( x' \geq x \). Note that \( v_z \leq v_z' \leq 0 \). If the direction \((1, v)\) at \( z \) satisfies \(-x \leq v \leq v_z\), then the image \((1, v')\) at \( z' \) satisfies

\[-x' \leq -x \leq v' \leq v_z \leq v_z'.\]

One can thus obtain the unstable manifold of \( q \) by iterating forwardly a local half graph at \( q \) whose tangent directions \((1, v)\) satisfy \(-x < v < v_z\) at any of its points \( z \) with \( x \geq x_q \). The iterates still satisfy these inequalities. The sequence of iterated graphs converges towards the right unstable branch of \( q \), which is a graph. Since there is no fixed point satisfying \( x > x_q \), it is a graph over \((x_q, +\infty)\).

Any point \( x \) in a neighborhood of the unstable branch of \( q \) has a forward iterate in the domain \( U^+ \) introduced in the proof of proposition 102 hence escape to infinity in the future. In particular the first coordinate is strictly increasing and \( x \) admits a wandering neighborhood.

**Claim 106.** On the domain where it is a graph, the local stable manifold of \( q \) is convex. On the domain where it is a graph, the local unstable manifold of \( q \) is concave.

In the following, we denote by \((x_s, x_q)\) and \((x_u, x_q)\) the maximal open domains where the left stable and left unstable branches of \( q \) are graphs.

*Proof.* Let us consider a graph \( \Gamma = \{(t, \varphi(t))\}_{t \in \mathbb{I}} \) whose image is a graph \( \Gamma' = \{(s, \varphi'(s))\} \) such that \( f|_\Gamma \) preserves the orientation on the first projection. The slope \( v(t) = D\varphi(t) \) has an image with slope \( v'(t) = -b/(2t + v(t)) \). The derivative of the slope of the image is \( Dv'(t) = \frac{b(2 + Dv(t))}{(2t + v(t))^2} \). If \( Dv(t) \) is positive, the same holds for \( Dv'(t) \). If \( Dv'(t) \) is negative, the same holds for \( 2 + Dv(t) \), hence for \( Dv(t) \).

One deduces that: if \( \Gamma \) is concave, then \( \Gamma' \) is also concave; if \( \Gamma' \) is a convex, then \( \Gamma \) is also convex.

*First case: the left unstable or the left stable branch of \( q \) is a graph.* If the left unstable branch of \( q \) is a graph, it is bounded by the second fixed point \( p \). Hence \( W^u(q) \cup \{p\} \) is an invariant closed half line containing the two fixed points. The domain \( U = \mathbb{R}^2 \setminus (W^u(q) \cup \{p\}) \) is homeomorphic to a plane. By Brouwer theorem, any orbit which does not belong to that line escape to infinity in the domain \( U \) when \( n \to \pm \infty \). Together
with the claim \[105\], this implies that the forward (resp. backward) orbit either belongs to the stable (resp. unstable) manifold of a fixed point, or converges to infinity in \(\mathbb{R}^2\).

If the left stable branch of \(q\) is a graph, the union of the left stable branch and of the right stable branch is an invariant closed half line and a similar argument holds.

**Second case: the unstable manifold of \(q\) is not a graph and \(x_s \leq x_u\).** The left unstable branch is not a graph: there exists a point \(z_u \in W^u(q)\) with a vertical tangent space and (by lemma \[106\]) the unstable arc connecting the points \(z_u\) and \(q\) is a concave graph \(\gamma^u\) over an interval \((x_u, x_q)\). The (local) left stable branch of \(q\) is a graph over a maximal interval \((x_s, x_q)\). The tangent map at \(q\) is \(Df(q) = \begin{pmatrix} 2x_q & 1 \\ -b & 0 \end{pmatrix}\), hence the stable graph is above the unstable graph. Moreover the two graphs are disjoint: if they intersect, by concavity (lemma \[106\]) they have a transverse intersection point, and the entropy is positive, contradicting our assumption.

One can build a Jordan domain \(\Delta\) by considering the union of the unstable arc \(\gamma^u\), a vertical segment \(\gamma^v\) and a stable arc \(\gamma^s\) above \((x_u, x_q)\), see figure 21.

We claim that \(f(\Delta) \subset \Delta\). Indeed \(f(\gamma^u)\) does not crosses \(\gamma^s\), as explained above. It does not crosses \(\gamma^v\) either, since \(f^{-1}(\gamma^v)\) is a subset of a convex graph \(\{(t, cte - t^2), t \in \mathbb{R}\}\) which is tangent to the concave graph \(\gamma^u\) at the point \(f^{-1}(z_u)\). Similarly the horizontal segment \(f(\gamma^v)\) does not cross the convex graph \(\gamma^s\) since one of its endpoints is below the convex graph \(\gamma^s\) and the other one is on the graph.

One considers a domain \(D\) which is bounded by curves close to (but disjoint from) \(\gamma^u, \gamma^v, \gamma^s\). The inclination lemma implies that \(D\) is a trapped disc, see figure 21. By construction it contains a fundamental domain of the left unstable branch of \(q\). One deduces that \(\mathbb{R}^2 \setminus (W^u(q) \cup (\cap_n f^n(D)))\) is homeomorphic to the plane and does not contain any fixed point. Brouwer theorem implies that any forward (resp. backward) orbit either belongs to the stable (resp. unstable) manifold of \(q\), or intersects \(D\) (resp. belongs to \(\cap_n f^n(D)\)), or escapes to infinity.

**Third case: the left stable branch is not a graph and \(x_s > x_u\).** We perform a similar construction. The local stable graph is bounded by a point \(z_s\) with a vertical tangent space. As before, the two local graphs are disjoint and one builds a Jordan domain \(\Delta\) by considering the union of a stable arc \(\gamma^s\), a vertical segment \(\gamma^v\) and an unstable arc \(\gamma^u\) above \((x_s, x_q)\). For the same reasons as before, the boundary of \(\Delta\) does not cross its image. In this case \(f(\gamma^v)\) is an horizontal graph tangent to the convex graph \(\gamma^s\).
and hence above it. This implies $f(\Delta) \supset \Delta$, which contradicts the volume contraction of $f$. 

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