Linear Programming Hierarchies in Coding Theory: Dual Solutions
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Abstract
The rate vs. distance problem is a long-standing open problem in coding theory. Recent papers have suggested a new way to tackle this problem by appealing to a new hierarchy of linear programs. If one can find good dual solutions to these LP’s, this would result in improved upper bounds for the rate vs. distance problem of linear codes. In this work we develop the first dual feasible solutions to the LP’s in this hierarchy. These match the best known bound for a wide range of parameters. Our hope is that this is a first step towards better solutions, and improved upper bounds for the rate vs. distance problem of linear codes.

1 Introduction
The rate vs. distance problem is a major open problem in coding theory. It seeks the largest cardinality $A(n, d)$ of a binary code of length $n$ with minimal distance $d$. Here we are interested in the range $d = \Theta(n)$ and $n \to \infty$. In this case, $A(n, d)$ is known to grow exponentially in $n$, and we consider the asymptotic maximal rate,

$$R(\delta) := \limsup_{n \to \infty} \frac{1}{n} \log_2 (A(n, \lfloor \delta n \rfloor))$$

where $0 < \delta < 1/2$ is the relative distance of the code.

The best known lower bound $R(\delta) \geq 1 - H(\delta)$ was given by Gilbert [1] for general codes and by Varshamov [2] for linear codes, where $H$ is the binary entropy function.

The best known upper bounds are the first and second linear programming (LP) bounds [3], both of which are based on Delsarte’s linear program [4]. The first LP bound

$$R(\delta) \leq H(1/2 - \sqrt{\delta(1 - \delta)})$$

is the best known upper bound for $0.273 < \delta < 1/2$. Much of what we do here revolves around this bound. The exact value of Delsarte’s LP remains unknown. However, there is strong numerical evidence [5] that the MRRW [3] bound has fully exhausted its potential to upper bound $R(\delta)$.

A code $C \subset \mathbb{F}_2^n$ is linear if it is a linear subspace. This is, of course a very strong restriction, so it stands to reason that one should be able to derive stricter upper bounds that are specific to linear codes. We denote by $A_{\text{Lin}}(n, d)$ and $R_{\text{Lin}}(\delta)$ the analogues of $A(n, d)$ and $R(\delta)$ when restricted to linear codes. Recent works [6, 7] are opening the way to linear programs stronger than Delsarte’s that hopefully improve the upper bound for linear codes.

Coregliano et. al. [6] developed a new hierarchy of linear programs whose $\ell$-th member upper-bounds $A_{\text{Lin}}(n, d)^\ell$, and converges to this quantity when $\ell = \Omega(n^2)$. The novel idea behind the new hierarchy is to consider the Cartesian product of $\ell$ copies of a code. This way, the linearity property of the code can be utilized in addition to Delsarte’s constraints.

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In our previous work [7] we employ related ideas to develop a hierarchy which is stricter than that of [6]. We also suggest a different objective function that bounds $A_{\text{Lin}}(n, d)$ instead of $A_{\text{Lin}}(n, d)^\ell$. In the present paper we consider both objective functions.

The LP hierarchies are extremely interesting as they may lead to progress in the longstanding problem of bounding $R_{\text{Lin}}(\delta)$. One natural course of action in this direction is to find good dual feasible solutions, the same way that the LP bounds were proven based on Delarte’s LP. It is challenging to find dual feasible solutions for the LP hierarchies. Due to higher dimensionality, and increased complexity of the LP’s. It is even far from trivial to find dual feasible solutions for the hierarchy which attains the first LP bound. In this work we make a first step in this direction.

1.1 Our Contribution

1. We construct a family of dual feasible solutions for the LP hierarchy, which attain the first LP bound up to $\ell \leq \log n - \log \log n$, where $\ell$ is the level in the hierarchy. These solutions apply for both linear and non-linear codes.

It is natural to ask how to apply this method to linear codes, and we provide a partial answer to this question.

2. We consider the alternative objective function, which bounds $A_{\text{Lin}}(n, d)$ instead of $A_{\text{Lin}}(n, d)^\ell$, and construct a family of feasible solutions.

In contrast with the solution alluded to in point 1 these solutions apply to all values of $\ell$. Also, while both approaches rely on solutions to Delsarte’s LP, this one treats these solutions as black boxes.

1.2 Outline of the Paper

In section 2 we provide preliminary material, including the relevant LP hierarchies (2.2), and a dual feasible solution to the LP hierarchy for linear codes, with an objective function that is linear in $A_{\text{Lin}}(n, d)$. We close with some concluding remarks in section 5.

Proofs are deferred to the end of the paper, in appendix A.

2 Preliminaries

A binary code of length $n$ is a subset $C \subset \mathbb{F}_2^n$. Throughout we only discuss binary codes. We denote by $|x| := |\{1 \leq i \leq n : x_i \neq 0\}|$ the Hamming weight of $x \in \mathbb{F}_2^n$. The Hamming distance between $x, y \in \mathbb{F}_2^n$ is $|x + y|$. The code’s distance is $\text{dist}(C) := \min\{|x + y| : x, y \in C, x \neq y\}$. The largest possible size of a binary code of length $n$ and distance $d$ is denoted

$$A(n, d) := \max \{|C| : C \subset \mathbb{F}_2^n, \text{dist}(C) \geq d\}.$$  

A code is linear if it is a linear subspace. For linear codes, this size is denoted $A_{\text{Lin}}(n, d)$.

The rate of a code is $R(C) := \frac{1}{n} \log_2 |C|$. The rate vs. distance problem is to find

$$R(\delta) := \limsup_{n \to \infty} \frac{1}{n} \log_2 (A(n, \lfloor \delta n \rfloor))$$

for every $\delta \in (0, 1/2)$.

Let $f, g : \mathbb{F}_2^n \to \mathbb{R}$. We define inner product w.r.t. the uniform measure, $\langle f, g \rangle := 2^{-n} \sum_{x \in \mathbb{F}_2^n} f(x)g(x)$. The convolution between $f, g$ is denoted $f * g$ and defined by $(f * g)(x) := 2^{-n} \sum_y f(y)g(x+y)$.

The Fourier transform of $f$ is denoted either $\mathcal{F}(f)$ or $\hat{f}$ and defined by $\hat{f}(x) := \langle f, \chi_x \rangle$, where $\chi_x(y) = (-1)^{\langle x, y \rangle}$. Fourier transform is its own inverse, up to normalization: $2^n \mathcal{F}(\mathcal{F}(f)) = f$. 

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In Fourier domain, inner product and convolution are without normalization, namely \( \langle \hat{f}, \hat{g} \rangle_F = \sum_x \hat{f}(x) \hat{g}(x) \) and \( (\hat{f} \ast_{F} \hat{g})(x) = \sum_y \hat{f}(y) \hat{g}(x+y) \). In favor of readability we omit the subscript \( F \) when possible.

By the Convolution Theorem, \( \hat{f} \ast_{F} \hat{g} = \hat{f} \cdot \hat{g} \). Similarly, \( \hat{f} \cdot \hat{g} = \hat{f} \ast_{F} \hat{g} \).

Let \( \ell \in \mathbb{N} \). We identify the space \( \mathbb{F}_2^\ell \) with the spaces \( \mathbb{F}_2^\ell \times \mathbb{F}_2^\ell \) and \( (\mathbb{F}_2^n)^\ell \). Given \( X \in \mathbb{F}_2^\ell \times \mathbb{F}_2^n \) we denote its rows by \( x_1, \ldots, x_\ell \).

2.1 Krawtchouk Polynomials

The (univariate) Krawtchouk polynomials \( \{K_0 \equiv 1, K_1, \ldots, K_n\} \) are a family of orthogonal polynomials w.r.t. binomial measure,

\[
\sum_{i=0}^{n} 2^{-n} \binom{n}{i} K_j(i) K_k(i) = \binom{n}{j} \delta_{j,k}
\]

The Krawtchouks are uniquely determined up to normalization. Here we assume the normalization \( K_i(0) = \binom{n}{i} \), for \( i = 0, \ldots, n \). The Krawtchouks are defined over \( \mathbb{R} \) but we extend their definition to the cube, writing \( K_i(x) := K_i(|x|) \), for \( x \in \mathbb{F}_2^n \).

The Fourier transform of the \( i \)-th Krawtchouk polynomial is the \( i \)-th level-set indicator \( L_i \),

\[
L_i(x) := 1_{|x|=i}, \quad \hat{K}_i(x) = L_i(x)
\]

2.2 Overview of LP Hierarchies

We describe the LP hierarchies related to the current work, without proofs. More details can be found in [6, 7].

All of the hierarchies are parameterized by three positive integers: \( n \) - the code’s length; \( d \) - the code’s distance; and \( \ell \) - the level in the hierarchy. Every LP in the hierarchy can be symmetrized and converted to an equivalent LP with multivariate Krawtchouk polynomials. For convenience we use Fourier-theoretic terminology.

2.2.1 LP Hierarchy for General Codes

Define the set of forbidden configurations as in [6],

\[
\text{ForbConfig}(n, d, \ell) := \left\{ (x_1, \ldots, x_\ell) \in (\mathbb{F}_2^n)^\ell : 1 \leq |x_i| \leq d - 1 \right\}
\]

Denote the following LP by Delsarte\((n, d, \ell)\). Its optimum is an upper bound on \( A(n, d)^\ell \).

\[
\text{maximize} \quad \sum_{X \in \mathbb{F}_2^\ell \times \mathbb{F}_2^n} f(X)
\]

subject to

\[
\begin{align*}
&f : \mathbb{F}_2^\ell \times \mathbb{F}_2^n \to \mathbb{R} \\
&f(0) = 1 \\
&f \geq 0 \\
&\hat{f} \geq 0 \\
&f(X) = 0 \quad \text{if } X \in \text{ForbConfig}(n, d, \ell)
\end{align*}
\]

We note that this hierarchy is degenerate, namely Delsarte\((n, d, \ell) = Delsarte(n, d, \ell + 1)\) for every \( \ell \). The equality is between the optimal values.
2.2.2 LP Hierarchies for Linear Codes

Define the set of forbidden configurations for linear codes,

\[ \text{ForbConfig}_{\text{Lin}}(n, d, \ell) := \{ X \in \mathbb{F}_2^{\ell \times n} : 1 \leq |x| \leq d - 1 \text{ for some } x \in \text{rowspan}(X) \} \]

Denote the following LP by Delsarte_{Lin}(n, d, \ell). Its optimum is an upper bound on \( A_{\text{Lin}}(n, d)^\ell \).

\[
\begin{align*}
\text{maximize} & \quad \sum_{X \in \mathbb{F}_2^{\ell \times n}} f(X) \\
\text{subject to} & \quad f : \mathbb{F}_2^{\ell \times n} \rightarrow \mathbb{R} \\
& \quad f(0) = 1 \\
& \quad f \geq 0 \\
& \quad \hat{f} \geq 0 \\
& \quad f(X) = 0 \quad \text{if } X \in \text{ForbConfig}_{\text{Lin}}(n, d, \ell)
\end{align*}
\]

Note that Delsarte(n, d, \ell) and Delsarte_{Lin}(n, d, \ell) differ only in their sets of forbidden configurations, (which is larger in the linear case).

The final LP hierarchy that we consider has the same set of constraints as Delsarte_{Lin}(n, d, \ell), but a different objective function. Its optimum is an upper bound to \( A_{\text{Lin}}(n, d) \), rather than \( A_{\text{Lin}}(n, d)^\ell \). Namely, it is related linearly to the code’s size.

\[
\begin{align*}
\text{maximize} & \quad \frac{1}{2^{\ell} - 1} \sum_{u \neq 0} \sum_{x \in \mathbb{F}_2^n} f(ux^\top) \\
\text{subject to} & \quad f \in \text{Delsarte}_{\text{Lin}}(n, d, \ell)
\end{align*}
\]

Here \( f \in \text{Delsarte}_{\text{Lin}}(n, d, \ell) \) means that \( f \) is feasible for Delsarte_{Lin}(n, d, \ell).

2.3 The First LP Bound

The first LP bound is obtained by constructing a dual feasible solution to Delsarte’s LP. In this section we present such a construction, which will be used in the subsequent section.

The dual of Delsarte’s LP, for binary codes of length \( n \) and distance \( d \), can be presented as follows.

**Proposition 1.** \( A(n, d) \) is upper bounded by

\[
\begin{align*}
\text{minimize} & \quad g(0)/\hat{g}(0) \\
\text{subject to} & \quad g : \mathbb{F}_2^n \rightarrow \mathbb{R} \\
& \quad \hat{g} \geq 0 \\
& \quad \hat{g}(0) > 0 \\
& \quad g(x) \leq 0 \quad \text{if } |x| \geq d
\end{align*}
\]

To turn it into an LP, we can further posit that \( \hat{g}(0) = 1 \).

All of the solutions to this dual LP, given in [3, 8, 9, 10, 11, 12], have the form

\[ g(x) = (t - |x|) \cdot \Lambda^2(x) \]

where \( t \leq d \), and \( \Lambda \) is chosen appropriately. This guarantees that constraint (5) is satisfied, and it only remains to find \( \Lambda \) that satisfies the Fourier constraints, (3) and (4). The linearity of the function \( x \mapsto (t - |x|) \) simplifies this task.

The above-mentioned solutions also share the same \( \Lambda \), with slight differences. But the different methods used to construct this \( \Lambda \) shed new light over the approach given in [6] which originated in [3]. As we explain shortly, the function \( x \mapsto |x| \) is related to the adjacency matrix of the Hamming
cube. Also, a good choice for \( \Lambda \) is the first eigenfunction of the smallest Hamming ball which satisfies a certain constraint.

To see this connection, note that \( 2(t - |x|) = K_1(x) - K_1(t) \), where \( K_1(t) = n - 2t \) is the first Krawtchouk polynomial. The Fourier transform of \( K_1 \) is \( L_1 \), the indicator function of the set \( \{ x \in \mathbb{F}_2^n : |x| = 1 \} \). Consider the operator of convolution with \( L_1 \). The matrix of this operator is the \( 2^n \times 2^n \) matrix \( A \), the adjacency matrix of Hamming cube. Namely, for any \( x, y \in \mathbb{F}_2^n 

\[
A_{x,y} = \begin{cases} 
1 & |x + y| = 1 \\
0 & \text{otherwise}
\end{cases}
\]

We include the simple proof: let \( f : \mathbb{F}_2^n \to \mathbb{R} \),

\[
2^n(L_1 \ast f)(x) = \sum_{y \in \mathbb{F}_2^n} L_1(y)f(x + y) = \sum_{i=1}^n f(x + e_i) = \sum_{y:|y + x| = 1} f(y) = (Af)(x)
\]

All papers [3, 8, 9, 10, 11, 12] find an appropriate \( \Lambda \) to establish the first LP bound. Of all these papers our approach is closest to that of [8].

Proposition 2. Let \( \varepsilon > 0 \). Let \( \Lambda : \mathbb{F}_2^n \to \mathbb{R} \) such that

(a) \( \hat{\Lambda}(0) = 1 \); (b) \( \hat{\Lambda} \geq 0 \); (c) \( A \cdot \hat{\Lambda} \geq (n - 2d + 2\varepsilon)\hat{\Lambda} \);

Then, \( g(x) := 2(d - |x|)\Lambda^2(x) \) is a feasible solution to Delsarte’s dual LP, and

\[
\frac{g(0)}{g(0)} \leq \frac{d}{\varepsilon} |\text{supp}(\hat{\Lambda})|
\]

Proposition 3. There exists a function \( \Lambda = \Lambda_{d,\varepsilon} \) which satisfies proposition [8] and its Fourier transform, \( \hat{\Lambda} \), is supported on the Hamming ball of radius \( r = n/2 - \sqrt{d(n-d)} + o(n) \).

Corollary 1 (The First LP Bound).

\[
\mathcal{R}(\delta) \leq H(1/2 + \sqrt{\delta(1-\delta)})
\]

Let us describe a function \( \Lambda \) for proposition [8]. Let \( A^{\leq r} \) be a submatrix of \( A \) corresponding to all vertices \( x \in \mathbb{F}_2^n \) of Hamming weight \( \leq r \). Namely, the adjacency matrix of the Hamming ball of radius \( r \). We choose \( \Lambda \) such that \( \hat{\Lambda} \) is the Perron eigenfunction of \( A^{\leq r} \), and pick the smallest \( r \) for which \( A^{\leq r} \) has spectral radius at least \( n - 2(d - \varepsilon) \). For more details, see the proof of proposition [8] and the remark that follows, in appendix [A].

3 Dual Solutions to the LP Hierarchies

In this section we construct a family of dual feasible solutions for the LP hierarchy Delsarte\((n,d,\ell)\). We also consider how to apply the same ideas to Delsarte Lin\((n,d,\ell)\), and the resulting complications.

As in Delsarte’s dual LP, also the duals of the hierarchies, which we define below, consist of two types of constraints: Fourier constraints, and a non-positivity constraint. Thus, we may again try to decompose \( g \) into a function which guarantees non-positivity \( (d - |x|) \), and a function geared at yielding the Fourier constraints. However, while for \( \ell = 1 \) a linear function is all you need for the non-positivity constraint, this is no longer possible when \( \ell \) grows.

Instead of a linear function, we construct a polynomial \( \Phi_{n,d,\ell} \) which is non-positive in the desired regions, and seek a function \( g_{n,d,\ell} : \mathbb{F}_2^{\ell \times n} \to \mathbb{R} \) of the form

\[
g_{n,d,\ell} = \Phi_{n,d,\ell} \cdot \Gamma_{n,d,\ell}^2
\]

as a dual feasible solution to Delsarte\((n,d,\ell)\). It turns out that for our choice of \( \Phi \), the function \( \Gamma := \Lambda^{\otimes \ell} = \Lambda \otimes \cdots \otimes \Lambda \) works, where \( \Lambda \) is from proposition [8]. In other words, the solution is obtained by a reduction from the \( \ell \)-th level to Delsarte.
The main shortcoming of our solution is its fast growth in $n$:

$$\text{value}(g_{n,d,\ell})^{1/\ell} \leq \left( en^{1/d} \right)^{\ell \log \delta} 2^{nH\left(1/2+\sqrt{d(1-\delta)}\right)+o(n)}$$

where $\delta = d/n$. When $\ell$ is too large, the first term becomes dominant and the solution’s value exponentially exceeds the first LP bound. Moreover, the hierarchy for general codes is known to be degenerate, namely, comparing optimal values,

$$(\text{Delsarte}(n,d,\ell))^{1/\ell} = \text{Delsarte}(n,d,1)$$

which means, in particular, that there exists a solution to the $\ell$-th level that has the exact same value of the solution from the previous section.

So, do the methods that we use for general codes apply to linear codes as well? In this case, we are able to construct an analogue of $\Phi$ that is suitable for linear codes, but a solution based on $\Lambda \otimes \ell$ no longer works. Instead, we suggest a reduction to a problem of the same spirit of proposition 2.

Throughout this section, we fix the parameters $n, d, \ell$, and omit their subscripts, e.g. we write $g$ instead of $g_{n,d,\ell}$. Also, we denote $\delta = d/n$.

### 3.1 General Codes - Delsarte $(n, d, \ell)$

Let us first define the dual of Delsarte $(n, d, \ell)$.

**Proposition 4.** $A(n,d)^\ell$ is upper bounded by

minimize $\ g(0)/\tilde{g}(0) \quad (9)$

subject to

\[ g : \mathbb{F}_2^{\ell \times n} \to \mathbb{R} \quad (10) \]

\[ \tilde{g} \geq 0 \quad (11) \]

\[ \tilde{g}(0) > 0 \quad (12) \]

\[ g(X) \leq 0 \quad X \in \text{AllowedConfig}(n,d,\ell) \land X \neq 0 \quad (13) \]

where $\text{AllowedConfig}(n,d,\ell)$ is the complement of the set of forbidden configurations,

\[ \text{AllowedConfig}(n,d,\ell) := \left\{ (x_1,\ldots,x_\ell) \in (\mathbb{F}_2^n)^\ell : x_i = 0 \lor |x_i| \geq d \text{ for all } i = 1,\ldots, \ell \right\} \]

We proceed to construct a feasible solution in two steps:

(1) Define a function $\Phi : \mathbb{F}_2^{\ell \times n} \to \mathbb{R}$ such that $\Phi(0) > 0$ and $\Phi$ satisfies constraint (13).

(2) Find a function $\Gamma : \mathbb{F}_2^{\ell \times n} \to \mathbb{R}$ such that $g := \Phi \cdot \Gamma^2$ is feasible, namely it satisfies constraints (11) and (12) (the remaining constraint is satisfied by construction).

Of course we want to carry out step (1) with a function $\Phi$, that makes step (2) possible.

Here is the idea behind our construction of $\Phi$ (see fig. 1): Consider a set of $2^\ell - 1$ balls, each in one subcube $\{\mathbb{F}_2^{\ell \times n}\}_{U \in \ell}$ of $\mathbb{F}_2^{\ell \times n}$. Pick the centers, the radii and the $\ell_p$-norm of the balls so that if $X \in \text{AllowedConfig}(n,d,\ell)$, it is contained in an even number of balls. For each ball define a function which is negative inside the ball, and positive outside of it. Finally, $\Phi$ is the product of these functions. If $X \in \text{AllowedConfig}(n,d,\ell)$, then $\Phi(X) \leq 0$, since it is the product of and odd number of non-positive functions, and an even number of non-negative functions.

Let us carry out steps (1) and (2).

**Step (1)** Let $m \in \mathbb{N}$ be even such that $\ell \leq \left( \frac{1+\delta}{1-\delta} \right)^m$. For every $\emptyset \neq U \subset [\ell]$, let

\[ \phi_U(x_1,\ldots,x_\ell) = \sum_{U \subset U} (n + d - 2|x_i|)^m - (n - d)^m \]
Figure 1: Illustration of AllowedConfig$(n, d, 2)$ and $\Phi_{n,d,2} \leq 0$, for $\ell = 2$. The axes are the Hamming weights of $(x_1, x_2) \in (\mathbb{F}_2^n)^2$.

Define $\Phi = \Phi_{n,d,\ell} : \mathbb{F}_2^\ell \times \mathbb{F}_2^n \rightarrow \mathbb{R}$:

$$
\Phi = \prod_{\emptyset \neq U \subset [\ell]} \phi_U
$$

**Step (II)** Let $\varepsilon > 0$ such that $(n - d + 2\varepsilon)^m - (n - d)^m = 1$, i.e.

$$
2\varepsilon = ((n - d)^m + 1)^{1/m} - (n - d)
$$

Let $\Lambda = \Lambda_{d,\varepsilon}$ from proposition 3. Define $\Gamma := \Lambda^\otimes \ell$, the tensor product of $\ell$ copies of $\Lambda$.

The following propositions establish the main result of this section, corollary 2.

**Proposition 5.**

1. $\phi_U(x_1, \ldots, x_\ell) \geq 0$ if $x_i = 0$ for some $i \in U$.
2. $\phi_U(x_1, \ldots, x_\ell) \leq 0$ if $|x_i| \geq d$ for all $i \in U$.
3. $\Phi(0) > 0$.
4. $\Phi(X) \leq 0$ if $X \in \text{AllowedConfig}(n, d, \ell)$ and $X \neq 0$.

**Proposition 6.** Let $g_{n,d,\ell} := \Phi_{n,d,\ell} \cdot (\Lambda^\otimes \ell)^2$.

1. $g_{n,d,\ell}$ is a feasible dual solution to Delsarte$(n, d, \ell)$.
2. The value of $g_{n,d,\ell}$ is

$$
\text{value}(g_{n,d,\ell}) = \frac{g_{n,d,\ell}(0)}{g_{n,d,\ell}(0)} \leq \left(\frac{e}{n^{1/d}}\right)^{2\ell \log \ell} |\text{supp}(\Lambda)|^\ell
$$

**Corollary 2.** $\text{value}(g_{n,d,\ell})^{1/\ell}$ coincides with the first LP bound for $\ell \leq \log n - \log \log n$.
3.2 Linear Codes - DelsarteLin\((n, d, \ell)\)

Let us define the dual of DelsarteLin\((n, d, \ell)\).

**Proposition 7.** \(A_{\text{Lin}}(n, d)\) is upper bounded by

\[
\text{minimize} \quad g(0)/\hat{g}(0)
\]

subject to

\[
g : \mathbb{F}_2^{\ell \times n} \to \mathbb{R}
\]

\[
\hat{g} \geq 0
\]

\[
\hat{g}(0) > 0
\]

\[
g(X) \leq 0 \quad X \in \text{AllowedConfig}_{\text{Lin}}(n, d, \ell) \land X \neq 0 \quad (13b)
\]

where \(\text{AllowedConfig}_{\text{Lin}}(n, d, \ell)\) is the complement of the set of forbidden configurations for linear codes,

\[
\text{AllowedConfig}(n, d, \ell) := \{(x_1, \ldots, x_\ell) \in (\mathbb{F}_2^n)^\ell : y = 0 \lor |y| \geq d \text{ for all } y \in \text{span}(x_1, \ldots, x_\ell)\}
\]

Note that the only difference between the dual of Delsarte\((n, d, \ell)\) and that of DelsarteLin\((n, d, \ell)\) is that constraint (13) is replaced by (13b).

Based on the function \(\Phi\) from the previous section we create a function \(\Phi_{\text{Lin}}\) which is non-positive on \(\text{AllowedConfig}_{\text{Lin}}(n, d, \ell)\). The basic building blocks of \(\Phi(X)\) were the functions \(\{n-2|x_i|\}\), where \(x_1, \ldots, x_\ell\) are the rows of \(X\). Namely, \(\Phi(X)\) acts separately and symmetrically on each row of \(X\). Therefore a solution for \(\ell = 1\) can be transformed to a solution for larger \(\ell\), as done in the previous section.

For linear codes, however, we need to consider linear combinations of \(X\)'s rows. Thus, \(\Phi_{\text{Lin}}\) is built from the functions \(\{n-2|u^\top X|\}\) for \(0 \neq u \in \mathbb{F}_2^{\ell}\). This is the set of linear multivariate Krawtchouk polynomials, a family of multivariate orthogonal polynomials. The classical Krawtchouk polynomials play a key role in earlier studies of the rate vs. distance problem. The multivariate Krawtchouk polynomials occupy an analogous position in the present theory. For more on these polynomials and their relation to the LP hierarchies, see [6, 7].

The coefficient matrix of \(n-2|u^\top X|\) in Fourier basis is the \(2^{\ell n} \times 2^{\ell n}\) matrix which we denote by \(A_u\), for any non-zero \(u \in \mathbb{F}_2^{\ell}\). This matrix is defined, for every \(X, Y \in \mathbb{F}_2^{\ell \times n}\), by

\[
A_{u, X,Y} = \begin{cases} 1 & \text{if } X + Y = u e_j \text{ for some } j = 1, \ldots, n \\ 0 & \text{otherwise} \end{cases} \quad (14)
\]

where \(e_j\) is the \(j\)-th standard basis vector in \(\mathbb{F}_2^n\). The proof is a one-liner similar to [3]. Notice that when \(\ell = 1\) this is the adjacency matrix of the Hamming cube \(\mathbb{F}_2^n\).

We turn to define \(\Phi_{\text{Lin}}\).

Let \(m \in \mathbb{N}\) be even such that \(2^{\ell-1} \leq \left(\frac{1+\delta}{1-\delta}\right)^m\).

Define \(\phi_{v, \text{Lin}} : \mathbb{F}_2^{\ell \times n} \to \mathbb{R}\)

\[
\phi_{v, \text{Lin}}(X) = \sum_{u : (u,v) \in \mathbb{F}_2^\ell} [(n+d-2|u^\top X|)^m - (n-d)^m]
\]

for \(0 \neq v \in \mathbb{F}_2^\ell\). Define \(\Phi_{\text{Lin}} = \Phi_{n,d,\ell}\) by

\[
\Phi_{\text{Lin}}(X) = \prod_{0 \neq v \in \mathbb{F}_2^\ell} \phi_{v, \text{Lin}}(X)
\]

Here is the analogue of proposition [5] for \(\Phi_{\text{Lin}}\).
Proposition 8.

1. $\phi_v^{\text{Lin}}(X) \geq 0$ if $|u^\top X| = 0$ for some $u$ for which $(v, u)_{\mathbb{F}_2} = 1$.
2. $\phi_v^{\text{Lin}}(X) \leq 0$ if $|u^\top X| \geq d$ for all $u$ for which $(v, u)_{\mathbb{F}_2} = 1$.
3. $\Phi^{\text{Lin}}(0) > 0$.
4. $\Phi^{\text{Lin}}(X) \leq 0$ if $X \in \text{AllowedConfig}_{\text{Lin}}(n, d, \ell)$ and $X \neq 0$.

One way to proceed to a feasible solution is by solving the following problem, which is based on the ideas from proposition 2.

**Problem 1.**

\[
\begin{align*}
\text{minimize} & \quad \left| \text{supp}(\hat{\Gamma}) \right| \\
\text{subject to} & \quad \hat{\Gamma}(0) = 1; \quad \hat{\Gamma} \geq 0; \quad \hat{\Phi}^{\text{Lin}} \ast \hat{\Gamma} \geq 2^{(\ell-1)(2^\ell-1)}
\end{align*}
\]

Solving problem 1 would yield the following bound.

**Proposition 9.** Let $\Gamma$ be a solution to problem 1. Then

\[
A^{\text{Lin}}(n, d) \leq \left(\epsilon n^{1/\delta}\right)^{2^\ell} \left| \text{supp}(\hat{\Gamma}) \right|^{1/\ell}
\]

Let us comment on the tensor product $\Lambda^\otimes \ell$ from the previous section, and why it is not a viable choice here. In the proof of proposition 3, we rely on the fact that

\[
\mathcal{F}[K_1(n - |x_i|)] \ast \hat{\Lambda}^\otimes \ell = A^u \hat{\Lambda}^\otimes \ell \geq (n - 2(d - \epsilon))\hat{\Lambda}^\otimes \ell
\]

where $e_i$ is the $i$-th standard basis vector in $\mathbb{F}_2^n$. An analogous proof that $\hat{\Lambda}^\otimes \ell$ is feasible for problem 1 requires (15) to apply to all $0 \neq u \in \mathbb{F}_2^d$, namely

\[
A^u \hat{\Lambda}^\otimes \ell \geq (n - 2(d - \epsilon))\hat{\Lambda}^\otimes \ell
\]

But this is not the case. Indeed, the definition of $\Lambda$ implies $\hat{\Lambda}(x) \leq (n - 2d)^{-1}$ for every $|x| = 1$. Let $u \in \mathbb{F}_2^d$ with $|u| \geq 2$, then

\[
(A^u \hat{\Lambda}^\otimes \ell)(0) = \hat{\Lambda}^{\ell - |u|}(0) \sum_{i=1}^n \hat{\Lambda}^{|u|}(e_i) \leq \frac{n}{(n - 2d)^{|u|}} < (n - 2(d - \epsilon))\hat{\Lambda}^\otimes \ell(0)
\]

### 4 Dual Feasible Solution to the Linear-Valued Objective

Changing the objective function of Delsarte_{Lin}(n, d, \ell) yields a very different dual problem. We recall the new objective, which bounds $A^{\text{Lin}}(n, d)$ instead of $A^{\text{Lin}}(n, d)^\ell$:

\[
\begin{align*}
\text{maximize} & \quad \frac{1}{2^\ell - 1} \sum_{0 \neq u \in \mathbb{F}_2^d} \sum_{x \in \mathbb{F}_2^n} f(u x^\top)
\end{align*}
\]

where $f : \mathbb{F}_2^{\ell \times n} \to \mathbb{R}$ is feasible for Delsarte_{Lin}(n, d, \ell). A particular advantage of this objective function is that now the LP is well-defined when $\ell \to \infty$. We believe that there is much to be gained from this fact. Another advantage is this: Whereas our construction from the previous step becomes too weak when $\ell$ is too large, the solutions that we provide here are good for any $\ell$.

Recall the completeness theorem of [6], which states, informally, that Delsarte_{Lin}(n, d, \ell) converges to the true value of $A^{\text{Lin}}(n, d)$ when $\ell = \Omega(n^2)$. The proof of this theorem does not apply when the objective function is (16), however numerical results from [7] show that, at least for $\ell = 2$, the objective function (16) is on par with the objective function of Delsarte_{Lin}(n, d, \ell).

Let us define the dual problem.
Proposition 10. $A_{\text{Lin}}(n, d)$ is upper bounded by

\begin{align*}
\text{minimize} & \quad g(0) \\
\text{subject to} & \quad g : \mathbb{F}_2^{d \times n} \to \mathbb{R} \\
& \quad \hat{g} \geq 0 \\
& \quad \hat{g}(0) = 1 \\
& \quad g(X) \leq 1 \quad X \in \text{AllowedConfig}_{\text{Lin}}(n, d, \ell) \\
& \quad g(ux^\top) \leq 1 - \frac{1}{2^{\ell - 1}} \quad |x| \geq d \land u \neq 0
\end{align*}

The last constraint states, in other words, that if $X \in \text{AllowedConfig}_{\text{Lin}}(n, d, \ell)$ and its rank is 1, then $g(X) \leq 1 - 1/(2^{\ell - 1})$.

Let us proceed in finding a feasible solution.

Proposition 11. Let $g_1$ be any dual feasible solution to Delsarte’s LP. Namely,

\[ \hat{g}_1 \geq 0, \quad \hat{g}_1(0) = 1, \quad g_1(x) \leq 0 \text{ if } |x| \geq d \]

Let

\[ g(X) = \begin{cases} 1 + \frac{1}{2^{\ell - 1}}(g_1(x) - 1) & X = ux^\top, \ 0 \neq u \in \mathbb{F}_2^d, \ 0 \neq x \in \mathbb{F}_2^n \\ g_1(0) & X = 0 \\ 1 & \text{otherwise} \end{cases} \]

Then, $g$ is feasible for the LP defined in proposition 10 and its value is $g_1(0)$.

The value of $g$ is equal to that of $g_1$ by construction.

Constraint (21) is satisfied because $g(X) = 1$ for every $X$ of rank $\geq 2$, regardless of the weights of its span. Constraint (22) is satisfied because $g_1(x) \leq 0$ when $|x| \geq d$.

For the remaining constraints we need the following proposition.

Proposition 12.

\[ \hat{g}(X) = \delta_0(X) + \frac{2^{-(\ell - 1)n}}{2^{\ell - 1}} \sum_{0 \neq u \in \mathbb{F}_2^d} \hat{g}_1(u^\top X) - \delta_0(u^\top X) \]

Constraints (19) and (20) follow from the proposition and the facts that $\hat{g}_1(0) = 1$ and $\hat{g}_1 \geq 0$.

5 Discussion

The new LP hierarchies [6, 7] open a new way to engage with the rate vs. distance problem for linear codes. In this work, we leverage proofs of the first LP bound to develop the first family of feasible solutions for these LPs, which attain the bound.

For the Delsarte($n, d, \ell$) hierarchy, our solutions recover the first LP bound in the range $\ell \leq \log n - \log \log n$. It is known that good solutions exist for all $\ell$, and we intend to return in future work to the search of such solutions. The holy grail of this research is proofs of tighter upper bounds on $R_{\text{Lin}}(\delta)$. A possible approach starts from the observation that a solution for Delsarte($n, d, \ell$) is also feasible for linear codes. To this end we will seek modifications of such solutions, as indicated above.

For Delsarte$_{\text{Lin}}(n, d, \ell)$, the hierarchy for linear codes, we introduced problem 1. It is based on the same methods we used for general codes. Although we still do not know whether good solutions for this problem will improve the bound, we believe that a better understanding of this problem, and in particular of the operators $\{A^u\}_{0 \neq u \in \mathbb{F}_2^d}$ (see (13)), will resolve many of the remaining mysteries.

We also considered another objective function for Delsarte$_{\text{Lin}}(n, d, \ell)$, that bounds $A_{\text{Lin}}(n, d)$ rather than $A_{\text{Lin}}(n, d)^\ell$. This hierarchy has the advantage that is it well defined when $\ell \to \infty$. The solutions we construct for this problem match the first LP bound for every $\ell$. 

10
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A Proofs

*Proof of proposition 1.* This is a particular case of proposition 4 with $\ell = 1$.

*Proof of proposition 2.* By construction, $g$ satisfies constraint 5. For constraint 4, by the convolution theorem,

$$\hat{g} = \mathcal{F}[2(d - |x|)] * \hat{\Lambda} * \hat{\Lambda} = \left((A - (n - 2d)I)\hat{\Lambda}\right) * \hat{\Lambda} \geq 2\varepsilon \hat{\Lambda} * \hat{\Lambda} \geq 0$$

where $A$ is the adjacency matrix of the Hamming cube, defined in (7). By the preceding equation,

$$\hat{g}(0) \geq 2\varepsilon(\hat{\Lambda} * \hat{\Lambda})(0) = 2\varepsilon \|\hat{\Lambda}\|_2^2 > 2\varepsilon \hat{\Lambda}^2(0) > 0$$

hence $g$ satisfies (3). Finally, let us bound the value of $g$:

$$\frac{g(0)}{\hat{g}(0)} \leq \frac{2d\Lambda^2(0)}{2\varepsilon(\Lambda * \Lambda)(0)} = \frac{d\|\hat{\Lambda}\|_2^2}{\varepsilon \|\Lambda\|_2^2}$$

Note that $\|\hat{\Lambda}\|_1 = (\hat{\Lambda}, 1_{\text{supp}(\hat{\Lambda})})$, and apply Cauchy-Schwartz inequality to complete the proof.
Proof of proposition [3]

We will use the following facts. References can be found, e.g., in [3].

Fact 1. The roots of the Krawtchouks all lie in \((0, n)\). Denote by \(z_{j,i}\) the \(j\)-th root of \(K_i\). The roots of \(K_i\) and \(K_{i+1}\) interlace: \(z_{j,i} < z_{j+1,i} \leq \sqrt{k(n-k)+o(n)}\)

Fact 2. For \(n\) large enough,
\[
z_{1,k} = n/2 - \sqrt{k(n-k)} + o(n)
\]

Fact 3 (Christoffel-Darboux formula). Let \(0 \leq j \leq n\) and define
\[
\Lambda_j(t, s) := \sum_{i=0}^{j} \binom{n}{i}^{-1} K_i(t)K_i(s)
\]
for every \(t, s \in \mathbb{R}\). Then,
\[
[K_1(s) - K_1(t)] \Lambda_j(t, s) = \frac{j+1}{j(j+1)} [K_{j+1}(s)K_j(t) - K_j(s)K_{j+1}(t)]
\]

Let us define \(\Lambda\).

Let \(r \in \mathbb{N}\) be smallest such that \(z_{1,r} \geq d - \varepsilon\), where \(z_{1,r}\) is the first root of the \(r\)-th Krawtchouk, \(K_r\). This implies \(d - \varepsilon \in [z_{1,r+1}, z_{1,r}]\).

Define
\[
\Lambda(x) = \Lambda_{d,\varepsilon}(x) := \Lambda_r(x,d - \varepsilon) = \sum_{i=0}^{r} \binom{n}{i}^{-1} K_i(d - \varepsilon)K_i(x)
\]
where \(\Lambda_r(x, d - \varepsilon)\) was defined in \((24)\).

The Fourier transform of \(\Lambda\) is
\[
\hat{\Lambda}(x) = \sum_{i=0}^{r} \binom{n}{i}^{-1} K_i(d - \varepsilon)L_i(x)
\]

because \(L_i = L_i\), which is the indicator of the set \(\{x \in \mathbb{F}_2^n : |x| = i\}\).

Let us show that \(\Lambda\) satisfies proposition [2]

(a) By \((27)\), \(\hat{\Lambda}(0) = 1\).

(b) Recall that \(K_1\) is positive in the segment \([0, z_{1,1})\); that \(z_{1,i} > z_{1,r}\) for all \(i < r\); and we chose \(r\) so that \(z_{1,r} \geq d - \varepsilon\), whence \(K_i(d - \varepsilon) \geq 0\) for every \(0 \leq i \leq r\).

Therefore, \(\hat{\Lambda} \geq 0\).

(c) The degree-1 Krawtchouk is \(K_1(t) = n - 2t\). We rearrange \(2(d - |x|)\) by adding and subtracting \(2\varepsilon\) and writing it using \(K_1\).
\[
2(d - |x|) = 2\varepsilon + K_1(x) - K_1(d - \varepsilon)
\]

Then apply the Christoffel-Darboux formula \((25)\):
\[
\mathcal{F}(2(d - |x|) \cdot \Lambda)(x) = 2\varepsilon \hat{\Lambda}(x) + \frac{r+1}{(\frac{r}{2})} [K_r(d - \varepsilon)L_{r+1}(x) - K_{r+1}(d - \varepsilon)L_r(x)]
\]

The first term is non-negative by the previous item, and the rest is also non-negative by our choice of \(r\). Therefore, \(2(d - |x|) \ast \hat{\Lambda} = A\hat{\Lambda} - (n - 2d)\hat{\Lambda} \geq 2\varepsilon \hat{\Lambda}\)
Finally, note that $\hat{\Lambda}$ is supported on the Hamming ball of radius $r$.

**Remark 1.** Our proof here is based on Krawtchouk theory and is close to [3]. It works just as well with $\Lambda_r(x, z_{1,r+1}) = \sum_{i=0}^{r} \binom{n}{i}^{-1} K_i(z_{1,r+1})K_i(x)$ which is the $\Lambda$ we described at the end of section 2.3 The first zero of $K_{r+1}$ is the spectral radius of $A^{rz}$, the adjacency matrix of the Hamming ball of radius $r$.

**Proof of corollary 1.** The cardinality of the Hamming ball of radius $r$ is $2^{nH(r/n) + o(1)}$. Choose $\varepsilon$ not too small in proposition 2, e.g. $\varepsilon = 1$. By propositions 1, 2 and 3, $A(n, d) \leq 2^{nH(1/2 - \sqrt{d/n} (1 - d/n)) + o(n)}$.

**Proof of proposition 4.** Let $g$ be a feasible solution to the LP in the proposition. Let $f$ be a feasible solution to Delsarte$(n, d, \ell)$. Then $\hat{g}(0) \sum_{X \in \mathbb{F}_2^{\times n}} f(X) = 2^{fn} \hat{g}(0) \hat{f}(0) \leq 2^{fn} \sum_{X} \hat{g}(X) \hat{f}(X) = 2^{fn} \langle \hat{g}, \hat{f} \rangle_X = \sum_{X} g(X) f(X) \leq g(0)$.

The first transition if by definition. The second is because $\hat{f} \geq 0$ and $\hat{g} \geq 0$. The fourth is by Parseval’s identity. The last transition is because, for each $0 \neq X \in \mathbb{F}_2^{\times n}$, if $X \in \text{ForbConfig}(n, d, \ell)$ then $f(X) = 0$, otherwise $g(X) \leq 0$ and $f(X) \geq 0$.

Finally, we use the fact that $A(n, d)^\ell \leq \sum_X f(X)$.

**Proof of proposition 5.**

1. $(n + d - 2|x_i|)^m \geq 0$ for every $i$ because $m$ is even. If $x_i = 0$ for some $i \in U$ then

$$\sum_{i \in U} (n + d + |x_i|)^m \geq (n + d)^m > \ell(n - d)^m \geq |U|(n - d)^m$$

The second inequality follows from the constraint on $m$.

2. Always $|x| \leq n$, so if $|x| \geq d$ then $n + d - 2|x| \leq n - d$ since $m$ is even, $(n + d - 2|x|)^m \leq (n - d)^m$. Assuming $|x_i| \geq d$ for all $i \in U$, then $\phi_U(x_1, \ldots, x_r) \leq |U|(n - d)^m - (n - d)^m = 0$

3. Obvious.
4. Let $0 \neq X \notin \text{AllowedConfig}(n, d, \ell)$. Let $V = \{1 \leq i \leq \ell : |x_i| \geq d\}$. Then $x_i = 0$ if $i \notin V$.

By item 1, $\phi_U(X) > 0$ for every $U \not\subset V$. By item 2, $\phi_U(X) \leq 0$ for every $U \subset V$. There are $2^{|V|} - 1$ non-empty subsets of $V$. $\Phi(X)$ is a product of an odd number of non-positive functions, and some positive functions. Hence $\Phi(X) \leq 0$.

Proof of proposition [6]

1. By proposition [6], $g$ satisfies [13].

It remains to show that $\hat{g} \geq 0$ and $\hat{g}(0) > 0$.

In the previous section we saw that

$$(A - (n - 2d)I)\hat{\Lambda} \geq 2\varepsilon \hat{\Lambda}$$

which implies

$$(A + dI)\hat{\Lambda} \geq (n - d + 2\varepsilon)\hat{\Lambda}$$

Repeated application of the operator $A + dI$ results in

$$(A + dI)^m \hat{\Lambda} \geq (n - d + 2\varepsilon)^m \hat{\Lambda}$$

Let $i \in [\ell]$. The function $(n + d - 2|x_i|)^m$ can be expressed as

$$(K_i(x_i) + d)^m \prod_{j \in [\ell], j \neq i} K_0(x_j)$$

because $K_0 \equiv 1$. The Fourier transform of $K_0$ is $L_0$, and convolution with $L_0$ corresponds to the identity matrix $I$. Thus, convolution with $F[(n + d - 2|x_i|)^m]$ corresponds to the matrix

$$I \otimes \cdots \otimes I \otimes (A + d)^m \otimes I \otimes \cdots \otimes I$$

Namely, the convolution operator of $F[(n + d - 2|x_i|)^m]$ interacts only with the $i$-th coordinate in $(\mathbb{F}_2^n)^{\ell}$, hence

$$F[(n + d - 2|x_i|)^m] \ast (\hat{\Lambda}^\otimes \ell) \geq (n - d + 2\varepsilon)^m(\hat{\Lambda}^\otimes \ell)$$

By linearity of the convolution operation, and by our choice of $\varepsilon$,

$$\hat{\phi}_U \ast (\hat{\Lambda}^\otimes \ell) \geq |U|((n - d + 2\varepsilon)^m - (n - d)^m)(\hat{\Lambda}^\otimes \ell)$$

for every $\emptyset \neq U \subset [\ell]$. Thus,

$$\hat{\Phi} \ast (\hat{\Lambda}^\otimes \ell) \geq \left(\prod_{j=1}^{\ell} j^{|j^\ell|}\right) \hat{\Lambda}^\otimes \ell$$

This implies that $\hat{g} \geq 0$ and $\hat{g}(0) > 0$, namely $g$ is feasible.

2. Let us compute the value of $g$. Using similar reasoning as in [28],

$$\frac{g(0)}{g(0)} \leq \frac{\Phi(0)}{\prod_{j=1}^{\ell} j^{|j^\ell|}} \left|\supp(\hat{\Lambda}^\otimes \ell)\right| = \frac{\Phi(0)}{\prod_{j=1}^{\ell} j^{|j^\ell|}} \left|\supp(\hat{\Lambda})\right|^{\ell}$$

We can pick $m \geq \frac{1}{\delta} \log \ell$. Then,

$$\Phi(0) \leq \prod_{j=1}^{\ell} j^{m((1 + \delta)^m - (1 - \delta)^m)(\binom{\ell}{j})}$$

$$\leq \left(\prod_{j=1}^{\ell} j^{\binom{\ell}{j}}\right) (n^m e^{\delta m})^{\sum_{j=1}^{\ell} \binom{\ell}{j}}$$

$$\leq \left(\prod_{j=1}^{\ell} j^{\binom{\ell}{j}}\right) n^{\frac{2^\ell\log \ell}{\gamma}} \ell^{2\ell - 1}$$
\[
\frac{g(0)}{\tilde{g}(0)} \leq \left( en^{1/\delta} \right)^{2^\ell \log \ell} \left| \text{supp}(\Lambda) \right|^{\ell}
\]

Proof of corollary \[\text{ii}\]. By propositions \[\text{iv}\] and \[\text{vi}\]
\[
A(n, d) \leq \left( en^{1/\delta} \right)^{2^\ell \log \ell} \left| \text{supp}(\Lambda) \right|
\]

The value of \( |\text{supp}(\Lambda)| \) is equivalent to the first LP bound, by corollary \[\text{i}\]. Therefore, the bound we obtained is as long as if
\[
\frac{1}{n} \log_2 \left( en^{1/\delta} \right)^{2^\ell \log \ell} = o_n(1)
\]
which is true when \( \ell \leq \log n - \log \log n \).

Proof of proposition \[\text{vii}\]. The proof is similar to that of proposition \[\text{iv}\].

Proof of proposition \[\text{viii}\].
1. From the first item of proposition \[\text{v}\] and by the choice of \( m \),
\[
\phi_v(X) \geq (n + d)^m - 2^{\ell-1}(n - d)^m > 0
\]
2. From the second item of proposition \[\text{v}\] and since for every \( v \) the number of \( u \) for which \( \chi_v(u) = -1 \) is \( 2^{\ell-1} \),
\[
\phi_v(X) \leq 2^{\ell-1}(n - d)^m - 2^{\ell-1}(n - d)^m = 0
\]
3. Obvious.
4. Let \( X \in \text{AllowedConfig}_{\text{Lin}}(n, d, \ell) \), \( X \neq 0 \). Let \( V = \{ u : |u^T X| = 0 \} \). Observe that \( V \) is a linear subspace. Let \( v \in V^\perp \setminus \{0\} \). By item 2, \( \phi_v(X) \leq 0 \). On the other hand, if \( v \in F_2^\perp \setminus V^\perp \), by item 1 \( \phi_v(X) > 0 \). So \( \Phi_{\text{Lin}}(X) \) is a product of \( 2^{\dim V} - 1 \) non-positive functions, and \( 2^\ell - 2^{\dim V} \) positive functions, hence \( \Phi_{\text{Lin}}(X) \leq 0 \).

Proof of proposition \[\text{ix}\]. Using a similar reasoning to the proof of propositions \[\text{ii}\] and \[\text{vi}\] it is not hard to see that \( g := \Phi_{\text{Lin}}^2 \) is a feasible solution to Delsarte_{\text{Lin}}(n, d, \ell), with value
\[
\frac{g(0)}{\tilde{g}(0)} \leq \frac{\Phi_{\text{Lin}}(0)}{2^{(\ell-1)(2^\ell - 1)}} \left| \text{supp}(\Lambda) \right|
\]
Also, choosing \( m \geq \ell/\delta \),
\[
\Phi_{\text{Lin}}(0) \leq \left[ 2^{\ell-1}((n + d)^m - (n - d)^m) \right]^{2^\ell - 1}
\]
\[
\leq 2^{(\ell-1)2^\ell - 1} (\epsilon \delta n)^{m2^\ell}
\]
\[
\leq 2^{(\ell-1)(2^\ell - 1)} \left( en^{1/\delta} \right)^{\ell 2^\ell}
\]
Finally, recall that \( A_{\text{Lin}}(n, d) \leq (g(0)/\tilde{g}(0))^{1/\ell} \).
Proof of proposition 10. Let $f : \mathbb{F}_2^{d \times n} \to \mathbb{R}$ be a feasible solution to DelsarteLin$(n, d, \ell)$. Let $g : \mathbb{F}_2^{d \times n} \to \mathbb{R}$ be a feasible solution to the program in the proposition.

$$A(n, d) \leq \frac{1}{2^{d-1}} \sum_{0 \neq u \in \mathbb{F}_2^d} \sum_{x \in \mathbb{F}_2^n} f(ux^\top)$$

(1) $f(0) = 1$.

(2) For $u, x \neq 0$, if $|x| \leq d - 1$ then $f(ux^\top) = 0$, otherwise $\frac{1}{2^{d-1}} \leq 1 - g(ux^\top)$.

(3) For $X \neq 0$ with rank $\geq 2$, if $X \in \text{ForbConfig}_{\text{Lin}}(n, d, \ell)$ then $f(X) = 0$, otherwise $f(X) \geq 0$ and $1 - g(X) \geq 0$.

(4) Parseval's identity.

(5) $\hat{g}(0) = 1, \hat{f} \geq 0, \hat{g} - \delta_0 \geq 0$.

Proof of proposition 12. Rewrite $g$ in a more convenient way:

$$g(X) = 1 + (g_l(0) - 1)\delta_0(X) + \frac{1}{2^{d-1}}(g_l(x) - 1)\mathbb{I}_{|X = ux^\top|}$$

where $\mathbb{I}_{|X = ux^\top|}$ is the indicator function of the set

$$\{X \in \mathbb{F}_2^{d \times n} : X = ux^\top \text{ for some } 0 \neq u \in \mathbb{F}_2^d, 0 \neq x \in \mathbb{F}_2^n\}$$

The constant function 1 is the Fourier character that corresponds to the zero vector, $\chi_0$. Its Fourier transform is Kronecker's delta function at 0, $\delta_0(X)$.

$$\hat{g}(X) = \delta_0(X) + 2^{-fn} \left[(g_l(0) - 1)\delta_0(X) + \frac{1}{2^{d-1}} \sum_{u \neq 0} \sum_{y \neq 0} \chi_X(uy^\top) \ [g_l(y) - 1]\right]$$

$$= \delta_0(X) + \frac{2^{-fn}}{2^{d-1}} \sum_{u \neq 0} \sum_{y \neq 0} \chi_X(uy^\top) \ [g_l(y) - 1]$$

It is not hard to verify that $\langle X, uy^\top \rangle = \langle u^\top X, y \rangle$, hence $\chi_X(uy^\top) = \chi_u^\top X(y)$. Thus, the inner sum over $y$ is the projection of the functions $g_l$ and 1 over the Fourier character $\chi_u^\top X$, up to normalization by $2^{-n}$.