BOSONIC FORMULAS FOR $\widehat{sl}_2$ COINVARIANTS

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ABSTRACT. We derive bosonic-type formulas for the characters of $\widehat{sl}_2$ coinvariants.

1. Introduction

Let us fix a natural number $k$ which we call level. Let $L_l$ be the integrable $\widehat{sl}_2$ module of level $k$ and of weight $l$, generated by a highest weight vector $v_l$ such that

$$h_0v_l = lv_l, \quad f_0^{l+1}v_l = e_1^{k-l+1}v_l = f_iv_l = e_{i+1}v_l = h_iv_l = 0, \quad i \in \mathbb{Z}_{<0},$$

where $e_i, f_i, h_i$ are the standard generators of $\widehat{sl}_2$. Then the space of coinvariants $L_{e,l}^k$ is the quotient space

$$L_{e,l}^k = L_l/(e_iL_l, \quad i \in \mathbb{Z}_{\geq 0}).$$

This quotient is naturally doubly graded by actions of $h_0$ and the degree operator $d$.

In this paper we present bosonic-type formulas for the corresponding character

$$\chi_{e,l}^k(q, z) = \text{Tr}_{L_{e,l}^k} q^d z^{h_0}.$$

The spaces of coinvariants $L_{e,l}^k$ were studied in the series of papers [2], [3], [4] (the space $L_{e,l}^k$ is denoted there by $L_{0,\infty}^{(k,l)}$), where several other spaces with the same character $\chi_l$ are given. One such space, a space of rigged configurations, produced a fermionic-type formula for $\chi_{e,l}^k(q, z)$, see Theorems 3.5.2, 3.5.3 in [4]. Another description via “combinatorial paths” led to a difference equation for $\chi_{e,l}^k(q, z)$. This difference equation was the main theme in all proofs of the papers [2], [3], [4]. It also plays the central role in this paper. Let us recall the construction.

For $0 \leq i \leq l \leq k$, denote by $C_{i}[i]$ the set of pairs $(a; b) = (a_0, a_1 \ldots; b_0, b_1, \ldots)$, where $a_i, b_i$ are non-negative integers such that only finitely many are different from zero, and

$$a_0 = i, \quad a_r + b_{r+1} + a_{r+1} \leq k, \quad b_r + a_r + b_{r+1} \leq k, \quad \sum_{s=m}^{n} b_s \leq k + \sum_{s=m+1}^{n-2} a_s, \quad (1.1)$$

where $r \geq 0, -1 \leq m < n \leq \infty$. Here we set $b_{\infty} = k$, $b_{-1} = l$, $b_0 = a_{-1} = a_\infty = 0$. Then by Corollary 5.4.10 of [3],

$$\chi_i(q, z) = \sum_{i=0}^{l} \chi_{i,l}(q, z, z^{-1}), \quad \chi_{i,l}(q, z_1, z_2) := \sum_{(a,b) \in C_{i}[i]} q^{\sum j(a_j+b_j)} z_1^{\sum b_j} z_2^{\sum a_j+b_j}.$$
By Proposition 3.3.1 in [3], we have a difference equation of the form

\[
\chi_{i,l}(q, z_1, z_2) = \sum_{i',l'} M_{i,l}^{i',l'} (q, z_1, z_2) \chi_{i',l'}(q, z_1, qz_2), \quad \chi_{i,l}(q, z_1, 0) = \delta_{i,0} \delta_{l,0},
\]

where the matrix \(M\) is given by (2.1) below.

The set \(C_l[i]\) can be thought of as the set of integer points in a polytope in an infinite-dimensional space cut out by the hyperplanes (1.1). Then one may expect an existence of a bosonic-type formula for \(\chi_{i,l}\) in the spirit of [6], written as a sum over vertices of the characters of the corresponding cones. Such a bosonic formula was studied in a simpler situation in [5]. However, our polytope is rather complicated and a direct approach does not look promising.

Instead we use the difference equation to reduce our infinite-dimensional problem to a problem in two dimensions. The non-zero entries of the matrix \(M\) are written as

\[
M_{i,l}^{i',l'} = m_1^i m_2^l m_3^{i'} m_4^{l'},
\]

where the \(m_j\) are monomials of the form

\[
q^{\alpha_j} z_1^{\beta_j} z_2^{\gamma_j}, \quad \alpha_j, \beta_j, \gamma_j \in \mathbb{Z}.
\]

Moreover, for a generic \((i, l)\), the set of \((i', l')\) such that \(M_{i,l}^{i',l'} \neq 0\) is a pentagon (see Figure 1). Given a monomial \(n_1, n_2\) of the form (1.2), these properties allow us to rewrite \(\sum_{i',l'} M_{i,l}^{i',l'} n_1^{i'} n_2^{l'}\) as a sum of 5 rational functions corresponding to the vertices of the pentagon. We denote these five terms by \(A(n_1^{i_1} n_2^{l_1}), \ldots, E(n_1^{i_5} n_2^{l_5})\). Then we can formally write

\[
\chi_{i,l} = \lim_{N \to \infty} (A + B + C + D + E)^N (\delta_{i,0} \delta_{l,0}).
\]

Expanding the RHS we obtain an expression for the character as a sum over all possible non-commutative monomials in the operators \(A, \ldots, E\).

This naive expansion suffers from difficulties for two reasons. First, the number of terms grows exponentially as \(N \to \infty\). Second, there are terms which contain zero denominators. The aim of the present paper is to find a formula for \(\chi_{i,l}\) as a sum over a subset of monomials, such that the number of terms grows polynomially and that all terms are well defined. Informally the existence of such a formula means that a huge cancellation takes place. The result is given in Theorem 6.1. In the preceding sections we explain the origin of this formula. For that purpose, an appropriate language is provided by what we call summation graph; see Figure 3. Monomials in the operators \(A, B, C, D, E\) are identified with directed paths in this graph, and the cancellation pattern can be visualized. Terms corresponding to the same path (of infinite length) can be summed up with the use of Jackson’s \(6\Phi_5\) formula (see (7.1) below). Then, all terms which cancel do so in pairs. Therefore, we need to perform no additional summation to observe the desired cancellation. We emphasize that the actual proof of Theorem 6.1 is done by a direct computation and is logically independent of these considerations.

Our final answer for \(\chi_{i,l}\) given in Section 7 is a sum of 18 families of meromorphic functions. The functions in each family are parametrized by 3 non-negative integers.
Each of them has a factorized form
\[ m_0 m_1 m_2 m_3 \prod_r (1 - f_r) \prod_s (1 - g_s), \]
where \( m_j, f_r, g_s \) are monomials as in (1.2). Note that the structure of this formula does not depend on \( i, l \) in contrast to the fermionic formulas.

We expect that one can write in a similar fashion bosonic formulas for solutions of certain class of difference equations. Another example of such equations and the corresponding formulas are given in [4].

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2. The Difference Equation

Fix a natural number \( k \in \mathbb{Z}_{>0} \). Define a matrix \( M(q, z_1, z_2) = (M_{i,l}^{r,s}) \) where \( 0 \leq i \leq l \leq k, 0 \leq i' \leq l' \leq k \), by the formulas
\[
M_{i,l}^{r,s} = \begin{cases} 
(qz_1 z_2)^{l-i} z_2^i & \text{if } l-i \leq i' \leq k-i; \\
(qz_1 z_2)^{l-i} z_2^i & \text{if } i' < l-i \leq k-i; \\
0 & \text{otherwise.}
\end{cases}
\] (2.1)

It is a square matrix of size \((k+1)(k+2)/2\). We denote \( M(q, z_1, 0) \) simply by \( M(0) \).

All entries of the matrix \( M(0) \) are equal to 0 or 1.

For example, for \( k = 1 \), we have
\[
M = \begin{pmatrix}
1 & q z_1 z_2 & 1 \\
0 & 1 & 1 \\
0 & 0 & 0
\end{pmatrix}, \quad M(0) = \begin{pmatrix}
1 & 0 & 1 \\
0 & 1 & 1 \\
0 & 0 & 0
\end{pmatrix},
\]
where the components are arranged in the order \((i, l) = (0, 0), (0, 1), (1, 1)\).

Consider the system of difference equations
\[
\chi(q, z_1, z_2) = M(q, z_1, z_2) \chi(q, z_1, q z_2)
\] (2.2)
for a vector \( \chi(q, z_1, z_2) = (\chi_{i,l}) \) where \( 0 \leq i \leq l \leq k \), and \( \chi_{i,l} \) is a formal power series in the variables \( q, z_2 \), whose coefficients are formal Laurent series in \( z_1^{-1} \). We impose the following initial condition
\[
\chi_{i,l}(q, z_1, 0) = \delta_{i,0} \delta_{l,0}.
\] (2.3)

Lemma 2.1. The system (2.2), (2.3) has a unique solution in \( \mathbb{C}[q, z_2][(z_1^{-1})^m] \) where \( m = (k+1)(k+2)/2 \).

Proof. Let us write \( \chi(q, z_1, z_2) = \sum_{j=0}^{\infty} f_j(q, z_1) z_2^j \), where \( f_j(q, z_1) \in \mathbb{C}[q][[(z_1^{-1})^m] \). By (2.3), \( f_0(q, z_1)_{i,l} = (\delta_{i,0} \delta_{l,0}) \). Comparing the coefficients of \( z_1^n \) in (2.2), we obtain
\[
f_n(q, z_1) = q^n M(0) f_n(q, z_1) + \ldots,
\]
where the dots denote terms which depend on \( f_j(q, z_1) \) with \( j < n \). There exists the inverse matrix \((\text{Id} - q^n M(0))^{-1}\) whose coefficients are power series in \( q \) alone, and therefore the vector \( f_n(q, z_1) \) is uniquely determined via \( f_j(q, z_1) \) with \( j < n \).
For \( N \in \mathbb{Z}_{\geq 0} \), define vector valued functions \( \chi^{(N)}(q, z_1, z_2) = (\chi_{i,l}^{(N)}) \) \((0 \leq i \leq l \leq k)\) recursively by

\[
\chi^{(N+1)}(q, z_1, z_2) = M(q, z_1, z_2) \chi^{(N)}(q, z_1, qz_2), \tag{2.4}
\]

\[
\chi_{i,l}^{(0)}(q, z_1, z_2) = \delta_{i,0}\delta_{l,0}.
\]

The function \( \chi_{i,l}^{(N)} \) is a polynomial in \( q, z_1, z_2 \) with non-negative integer coefficients.

**Lemma 2.2.** As \( N \) tends to infinity, the limit of \( \chi_{i,l}^{(N)}(q, z_1, z_2) \) exists in \( \mathbb{C}[[q]][z_1, z_2] \) and is equal to \( \chi_{i,l}(q, z_1, z_2) \).

**Proof.** Note that

\[
M(0)\chi^{(0)}(q, z_1, z_2) = \chi^{(0)}(q, z_1, z_2),
\]

and that for \( m \geq n, M(q, z_1, q^m z_2) \equiv M(0) \mod q^n \). Therefore, for \( N \geq n \) we have,

\[
\chi^{(N)}(q, z_1, z_2) = M(q, z_1, z_2)M(q, z_1, qz_2) \ldots M(q, z_1, q^{N-1}z_2)\chi^{(0)}(q, z_1, q^N z_2)
\]

\[
\equiv M(q, z_1, z_2)M(q, z_1, qz_2) \ldots M(q, z_1, q^{n-1}z_2)M(0)^{N-n}\chi^{(0)}(q, z_1, z_2)
\]

\[
\equiv M(q, z_1, z_2)M(q, z_1, qz_2) \ldots M(q, z_1, q^{n-1}z_2)\chi^{(0)}(q, z_1, z_2)
\]

\[
\mod q^n.
\]

Therefore the limit of \( \chi_{i,l}^{(N)}(q, z_1, z_2) \) exists as \( N \) tends to infinity. Then it is equal to \( \chi_{i,l}(q, z_1, z_2) \) by the uniqueness part of Lemma 2.1.

**Corollary 2.3.** The components of \( \chi(q, z_1, z_2) \) are power series in \( q \) with coefficients in \( \mathbb{Z}_{\geq 0}[z_1, z_2] \).

The main purpose of this paper is to derive a bosonic-type formula for \( \chi(q, z_1, z_2) \).

### 3. Bosonic Formula for a Triangle and a Rectangle

The sum in the difference equations (2.2), (2.4) is taken over the union of a triangle \( B_1D_1E \) and a rectangle \( AB_2D_2C \) shown in Figure 1. We divide the pentagon into two regions (the triangle \( B_1D_1E \) and the rectangle \( AB_2D_2C \)) since the formulas we apply will be different for these two.

The following two lemmas give bosonic type expressions for a sum over integer points inside a triangle and a rectangle. These lemmas are easily proved by a direct computation.

**Lemma 3.1.** Let \( a, b \) be integers such that \( a \leq b \). Then

\[
\sum_{a \leq m \leq b} x^m y^n = \frac{(xy)^a}{(1-x)(1-xy)} + \frac{x^b y^n}{(1-x^{-1})(1-y)} + \frac{(xy)^b}{(1-y^{-1})(1-(xy)^{-1})}. \tag{3.1}
\]
Lemma 3.2. Let $a, b, c, d$ be integers such that $a \leq b + 1$ and $c \leq d + 1$. Then

\[
\sum_{a \leq m \leq b \atop c \leq n \leq d} x^m y^n = \frac{x^a y^c}{(1 - x)(1 - y)} + \frac{x^a y^d}{(1 - x)(1 - y^{-1})} + \frac{x^b y^c}{(1 - x^{-1})(1 - y)} + \frac{x^b y^d}{(1 - x^{-1})(1 - y^{-1})}. \tag{3.2}
\]

Note that Lemmas 3.1, 3.2 state an equality of rational functions. Therefore, the equality still holds if we choose to expand the RHS into power series in any consistent way. There are four choices for the expansion of (3.2), depending on whether $x, y$ are taken in the neighborhood of 0 or $\infty$. For (3.1) there are 6 ways to expand since there is an additional choice of $xy$ being in the neighborhood of 0 or $\infty$.

The lemmas 3.1, 3.2 are simple examples of writing a character of a convex polytope as a sum over vertices of characters of the corresponding cones. The second lemma in fact follows from an even more simple example, namely from the expression for a 1-dimensional segment:

\[
\sum_{a \leq m \leq b} x^m = \frac{x^a}{1 - x} + \frac{x^b}{1 - x^{-1}}.
\]

We remark that the Weyl formula for the characters of irreducible finite dimensional modules over semisimple Lie algebras have a similar structure.

4. The extremal operators

Let $P, Q, R$ be monomials of the form $q^\alpha x_1^\beta x_2^\gamma$ with $\alpha, \beta, \gamma \in \mathbb{Z}_{\geq 0}$. We denote by $[P, Q, R]$ the $(k + 1)(k + 2)/2$-dimensional vector whose $(i, l)$-th component is given by

\[
[P, Q, R]_{i,l} = [P(1 : \frac{Q}{P} : \frac{R}{P})]_{i,l} = P^{k-l}Q^{l-i}R^i.
\]
In this notation we have

\[ \chi_{i,l}^{(0)}(q, z_1, z_2) = \delta_{i,0}\delta_{l,0} = [1, 0, 0]_{i,l}, \]

\[ \chi_{i,l}^{(1)}(q, z_1, z_2) = z_2^i\delta_{l,i} = [1, 0, z_2]_{i,l}, \]

\[ \chi_{i,l}^{(2)}(q, z_1, z_2) = \sum_{j=0}^{k-l} q^{l-i+j}z_2^{l+j} = \frac{[1, qz_2, z_2]_{i,l}}{1 - qz_2} + \frac{[qz_2, z_2, z_2]_{i,l}}{1 - (qz_2)^{-1}}. \] (4.1)

Consider the linear span \( V \) of vectors

\[ f[P, Q, R], \quad f \in \text{C}[q, z_2]((z_1^{-1})) \]. (4.2)

Normally we will write an element of \( V \) simply as \( v \), without exhibiting the dependence on \( q, z_1, z_2 \) explicitly. We say \( v \) is simple if it has the form (4.2). We call \( f \) and \( [P, Q, R] \) the scalar part and the vector part, respectively. Since \( f[aP, aQ, aR] = fa^k[P, Q, R] \) holds for a monomial \( a \), the scalar part and the vector part are determined up to this freedom.

For a vector-valued function \( g(q, z_1, z_2) \), let

\[ (Sg)(q, z_1, z_2) = g(q, z_1, qz_2) \] (4.3)

denote the \( q \)-shift operator in \( z_2 \). We introduce the extremal operators \( A, B, C, D, E \) by the formulas

\[ A(f[P, Q, R]) = S \left( \frac{f[P, Q, q^{-1}z_2P]}{(1 - z_1z_2Q/P)(1 - R/Q)} \right), \]

\[ B(f[P, Q, R]) = S \left( \frac{f(1 - Q/P)[P, R, q^{-1}z_2P]}{(1 - z_1z_2Q/P)(1 - Q/R)(1 - R/P)} \right), \]

\[ C(f[P, Q, R]) = S \left( \frac{f[z_1z_2Q, Q, q^{-1}z_2P]}{(1 - (z_1z_2)^{-1}P/Q)(1 - R/Q)} \right), \]

\[ D(f[P, Q, R]) = S \left( \frac{(1 - (z_1z_2)^{-1})f[z_1z_2Q, R, q^{-1}z_2P]}{(1 - (z_1z_2)^{-1}P/Q)(1 - (z_1z_2)^{-1}R/Q)(1 - Q/R)} \right), \]

\[ E(f[P, Q, R]) = S \left( \frac{f[R, R, q^{-1}z_2P]}{(1 - z_1z_2Q/R)(1 - P/R)} \right), \]

extended by linearity. The meaning of the RHS is as follows. For a monomial \( X = q^\alpha z_1^\beta z_2^\gamma \) \((\alpha, \gamma \in \mathbb{Z}_{\geq 0}, \beta \in \mathbb{Z})\) such that \( \alpha + \gamma > 0 \) or \( \alpha = \gamma = 0, \beta < 0, 1/(1 - X^\pm) \) means its expansion in non-negative powers of \( X \). We define an operator \( G(= A, B, C, D, E) \) on \( f[P, Q, R] \) when the denominators appearing in the RHS are all of this form. Otherwise we do not define \( G(f[P, Q, R]) \).

The main point of introducing the operators \( A, B, C, D, E \) is:

**Lemma 4.1.** Suppose operators \( A, B, C, D, E \) are defined on a simple vector \( v \), and let \( M(q, z_1, z_2) \) be the matrix (2.4). Then we have

\[ M(q, z_1, z_2)v(q, z_1, qz_2) = ((A + B + C + D + E)v)(q, z_1, z_2). \]
Proof. It suffices to consider the case $v = [P, Q, R]$. Written out explicitly, the $(i, l)$-component of the LHS reads
\[
\sum_{i - l \leq i' \leq l - i} (qz_1z_2)^{i' - l} z_2^i S(P)^{k - i'} S(Q)^{i' - l} S(R)^{i'}
+ \sum_{i' < l - i \leq i' \leq k - i} (qz_1z_2)^{i' - l + i} z_2^i S(P)^{k - i'} S(Q)^{i' - l} S(R)^{i'}.
\]
We apply Lemmas 3.1 and 3.2 to obtain seven terms corresponding to vertices in Figure 1. Namely
\[
M(q, z_1, z_2)v(q, z_1, qz_2) = ((B_1 + E + D_1 + A + B_2 + D_2 + C) v)(q, z_1, z_2), \quad (4.4)
\]
where
\[
B_1([P, Q, R]) = S \left( \frac{[P, R, q^{-1}z_2P]}{(1 - z_1z_2Q/P)(1 - R/P)} \right),
\]
\[
B_2([P, Q, R]) = S \left( \frac{[P, R, q^{-1}z_2P]}{(1 - z_1z_2Q/Q)(1 - R/Q)} \right),
\]
\[
D_1([P, Q, R]) = S \left( \frac{[z_1z_2Q, R, q^{-1}z_2P]}{(1 - (z_1z_2)^{-1}P/Q)(1 - (z_1z_2)^{-1}R/Q)} \right),
\]
\[
D_2([P, Q, R]) = S \left( \frac{[z_1z_2Q, R, q^{-1}z_2P]}{(1 - (z_1z_2)^{-1}P/Q)(1 - R/Q)} \right).
\]
The lemma follows from the identities $B = B_1 + B_2$, $D = D_1 + D_2$. \hfill \Box

If the operators $A, B, C, D, E$ were always defined, the lemma could be applied repeatedly to write the character as
\[
\chi^{(N)}(q, z_1, z_2) = (A + B + C + D + E)^{N-1}[1, 0, z_2],
\]
\[
\chi(q, z_1, z_2) = (A + B + C + D + E)^\infty[1, 0, z_2]. \quad (4.5)
\]
Unfortunately it is not the case. For example,
\[
C^3B_2C^3A^3E[P, Q, R] = f(P, Q, R)[q^3 z_1^2 \tilde{R}, q^2 z_2 \tilde{R}, q^3 z_1 z_2^2 \tilde{R}], \quad \tilde{R} = S^5(R).
\]
Therefore the operators $BC^3B^3AE$ and $ECB^3C^3E$ are not defined because $0$ is produced in the denominator. (Note however that the sum $(B+E)C^3B^3AE$ is well defined.) This example shows that there is no subspace $W$ which contains $[1, 0, 0]$ and is stable under $A, B, C, D, E$. Such a difficulty does not arise in the case treated in [1].

Nevertheless it is possible to express $\chi(q, z_1, z_2)$ as a sum over a subset of non-commutative monomials in $A, B, C, D, E$, with all terms being defined in the sense above. In Theorem 6.1 below we write down the formula and show directly that it is the unique solution of the difference equation (2.2). Informally speaking, it means that the rest of the terms in (4.5) cancel with each other, including the non-defined ones. In the next two sections we analyze the mechanism of the cancellation and explain how the formula was found. This part is meant to motivate Theorem 6.1, although it is logically unnecessary.
5. Summation Graph

In this section we prepare some language for Section 6.

A monomial $M$ is an ordered composition of operators $G_1G_2\ldots G_n$, where $G_i \in \{A,B,C,D,E\}$. We call $n$ the degree of $M$.

Now, our goal is to find different monomials which, when acted upon a simple vector $v \in V$, give rise to the same vector part. For example, the vector parts of $BC([P,Q,R])$ and $BD([P,Q,R])$ are both equal to $S^2([z_1z_2Q,q^{-1}z_2P,z_1z_2^2q^{-2}Q])$. Therefore the sum $B(C+D)$ applied to a simple vector is again a simple vector. Note that even though some monomials may not be defined, the action of a monomial on the vector part is always defined.

Introduce the maps $\sigma_G : \{1,2,3\} \to \{1,2,3\}$, where $G \in \{A,B,C,D,E\}$, by the formulas

\[
\sigma_A = (1,2,1), \quad \sigma_B = (1,3,1), \quad \sigma_C = (2,1,2), \quad \sigma_D = (2,3,1), \quad \sigma_E = (3,3,1),
\]

where $\sigma_G = (\sigma_G(1), \sigma_G(2), \sigma_G(3))$.

We define the map $\sigma_M$ for all monomials $M$ by the product rule

$$\sigma_{M_1M_2} = \sigma_{M_2}\sigma_{M_1}.$$ 

In other words $\sigma$ is an anti-homomorphism of the semi-group of non-commutative monomials in $A,B,C,D,E$ to the semi-group of endomorphisms of $\{1,2,3\}$.

The maps $\sigma_M$ describe how $M$ permutes $P,Q,R$ in the vector part. Namely,

$$M([P_1,P_2,P_3]) = S^{\deg M}\langle f[u_1\sigma_P(1), u_2\sigma_P(2), u_3\sigma_P(3)] \rangle,$$

where $u_1,u_2,u_3$ denote factors independent of $P_1,P_2,P_3$.

Introduce a directed graph $\Gamma$, which we call summation graph (see Figure 2). The vertices of $\Gamma$ are non-empty subsets $I$ of $\{1,2,3\}$. Each vertex emits 5 arrows, labeled $A, B, C, D, E$. The arrow with label $G$ which starts from a vertex $I$ ends at the vertex $\sigma_G(I)$.

Note that the summation graph $\Gamma$ has three floors consisting of subsets $I$ of given cardinality. We picture vertices corresponding to subsets of larger cardinality above those corresponding to subsets of smaller cardinality. Then no arrow goes “up”.

In our picture, arrows with the same beginning and end are represented by one arrow with several labels. For arrows $G_1, G_2$ with a common source $I \subset \{1,2,3\}$, we write $G_1 + G_2$ if and only if the $i$-th component of the vector parts of $G_1([P,Q,R])$ and $G_2([P,Q,R])$ coincide for all $i \in I$. For example, the vector parts of $A([P,Q,R])$ and $C([P,Q,R])$ coincide in the second and the third components. Therefore we write $A+C$ above the arrows coming from $(2)$ and $(2,3)$, but do not do so for $(1,2,3)$.

We identify monomials in $A,B,C,D,E$ with (oriented) paths in $\Gamma$ starting at $(1,2,3)$, by reading a monomial $M$ from left to right and choosing the corresponding arrows accordingly. For example, the monomial $ACB$ corresponds to the path

$$(1,2,3) \xrightarrow{A} (1,2) \xrightarrow{C} (2) \xrightarrow{B} (3).$$

The vector part of $M([P_1,P_2,P_3])$ corresponding to a path ending at a vertex $I$ depends only on $P_i$ with $i \in I$. Therefore we have the following obvious lemma.
Figure 2. Summation graph (here $\sum = A + B + C + D + E$).

**Lemma 5.1.** Let $M$ be a monomial. Suppose that the corresponding path in the summation graph ends at a source of an arrow $G_1 + \cdots + G_s$, where $G_i$ are distinct elements of $\{A, B, C, D, E\}$. Then the operator $M(G_1 + \cdots + G_s)$ maps simple vectors $v \in V$ to simple vectors if it is defined.

We will use such summations in Section 6 to combine several terms in (1.3) in one. It turns out that the sum of the corresponding scalar parts in our computations is completely factorized to "linear" factors. We warn the reader that this is not necessarily the case in general (e.g. in the bottom floor for finite $N$).

6. Cancellation in the case $N = \infty$

From now on we concentrate on the case $N = \infty$. The purpose of this section is to describe the structure of the resulting formula and the way how it was obtained. In the argument below, we assume that all compositions of operators are defined. After finding the final formula we turn to proving it by direct means.

In formula (1.3), the RHS is a sum of monomials in $A, B, C, D, E$ of infinite degree applied to $[1, 0, z_2]$. Our first step is to choose another vector in place of $[1, 0, z_2]$, so that it suffices to deal only with monomials of finite degree.

Let $M$ be an arbitrary monomial of infinite degree and $N$ be a finite degree of $MGN([1, 0, z_2]) = 0$ holds for $G = C, D, E$.

The reason is as follows. We claim, that if $G((1, 0, z_2]) \neq 0$, then all components of its vector part contain a factor $z_2$. Indeed, if $N = 1$ then $C([1, 0, z_2]) = D([1, 0, z_2]) = 0$, and the vector part of $E([1, 0, z_2])$ is $[qz_2, qz_2, z_2]$. If $N \neq 1$, then the only component which does not depend on $z_2$ in $N([1, 0, z_2])$ may be the first one, and our claim follows.
Then \([z_2 P, z_2 Q, z_2 R] = z_2^k [P, Q, R]\) has an overall factor \(z_2^k\). It vanishes when \(M\) is applied, because of an infinite shift \(S^\infty\).

The formula (6.5) would then take the form

\[
\chi(q, z_1, z_2) = \sum_{\deg M < \infty} M(A + B)^\infty([1, 0, z_2]). \tag{6.1}
\]

In the language of paths in the summation graph \(\Gamma\), the formula (6.1) means that we sum over all paths of finite length which start at \((1, 2, 3)\) and end at \((1)\).

We compute \((A + B)^\infty([1, 0, z_2])\) using the identity

\[
(A + B)^n = A^n + \sum_{j=0}^{n-1} A^j B(A + B)^{n-j-1}.
\]

We have explicitly

\[
A^n B(A + B)^m([1, 0, z_2]) = \frac{[1, q^{n+1} z_2, z_2]}{(1 - q^{n+1} z_2) \prod_{j=0}^{m-1} (1 - q^j) \prod_{j=n+2}^{2n+m+1} (1 - q^j z_1 z_2)}.
\]

and \(A([1, 0, z_2]) = 0\). Taking the limit \(n \to \infty\) we obtain the following definition.

Introduce the vector \(v_\infty \in V\) by the formulas

\[
v_\infty = \sum_{n=1}^{\infty} f_n v_n, \tag{6.2}
\]

where

\[
v_n = [1, q^n z_2, z_2], \quad f_n = \frac{(-1)^{n-1} q^{n(n-1)/2} (1 - q^{2n} z_1 z_2^2)}{(q)_\infty (q^{n+1} z_1 z_2^2)\infty (q)_{n-1} (1 - q^n z_2)} \tag{6.3}.
\]

We have an identity \((A + B)v_\infty = v_\infty\) (see Lemma 7.1 below).

Now, we describe the mechanism of the cancellation and the structure of the final formula.

First of all we infer, on the basis of examples, that paths which pass through vertices (2) or (3) cancel out. In other words the surviving monomials correspond to paths in the summation graph which never go to the bottom level until the tail \((A + B)^\infty\) is reached. We have no direct proof of this statement.

After that reduction, we still have several cycles which paths can wrap around in the middle floor. Let us set

\[
L = (C + D)D(B + D + E), \quad \bar{L} = D(B + D + E)(C + D).
\]

We have several operator identities which explain a part of the cancellation. For example, we have

\[
BE = 0,
\]

which implies in particular that if \(E\) cycle appears after \(L\) then the corresponding term is zero. We have also the identities valid for any \(m \in \mathbb{Z}_{\geq 0}\),

\[
X \bar{L}^m D(A + C) = 0, \quad X \bar{L}^m B = 0 \quad (X = A, C), \quad
Y L^m (C + D)D(A + C) = 0, \quad Y L^m (C + D)B = 0 \quad (Y = B, E),
\]
which imply that the cycle \( D(A + C), (C + D)B \) give zero contributions. (We stress
that we do not use any of these operator identities for our proofs.) So, we are left with
the cycles \( A \) (at the vertex \((1, 2)\)), \( E \) (at the vertex \((1, 3)\)) and \( L \) (starting at the vertex
\((1, 3)\)).

Finally, there is an ordering of the cycles in the following sense. We call a path in
the summation graph \( \Gamma \) *good* if the following 3 conditions are fulfilled:

- it ends at the vertex \((1, 3)\),
- it does not contain both \( A \) and \( E \) cycles,
- it enters neither \( A \) nor \( E \) cycles after passing through \( L \) cycle.

A path which is not good is called bad. We infer that in the formula (6.1) bad paths
cancel out and only good paths survive.

These considerations lead us to the following theorem.

**Theorem 6.1.** We have the following identity

\[
\chi(q, z_1, z_2) = \sum_{n,m,s=0}^{\infty} \left( D^n A^m BL^s v_\infty + D^n C A^m BL^s v_\infty + D^n E^{m+1} L^s v_\infty +
\right.
\]

\[
+ D^n C A^m D(B + D + E)L^s v_\infty + D^n A^{m+1} D(B + D + E)L^s v_\infty \right), \tag{6.4}
\]

where \( v_\infty \) is defined in (6.2) and (6.3). All terms in the RHS are defined.

**Proof.** For the proof we use the explicit formulas for each term in the RHS given in
Section 7. By a direct computation we verify that they are all defined, and that each
operator \( A, B, C, D, E \) are also defined on them.

Substitution \( z_2 = 0 \) to the RHS clearly gives 0 (because of the nontrivial vector part)
unless \( l = l = 0 \), when we get 1 (coming from the term \( Bv_\infty \)). By the uniqueness part
of Lemma 2.1, it is enough to show that multiplication by \( A + B + C + D + E \) does
not change the RHS.

There are two things to show. First, we have to show that multiplication by \( A + B + C + D + E \) reproduces all the terms, and second, that the extra terms appearing
cancel out.

The first statement, informally speaking, can be reformulated as follows. If a mono-
mial \( GM \) with \( G \in \{A, B, C, D, E\} \) is good, then monomial \( M \) itself is also good. We
verify it by case checking. The nontrivial cases are \( BL^s v_\infty, EL^s v_\infty, CD^2 L^s v_\infty \) and
\( AD^2 L^s v_\infty \) with \( s \in \mathbb{Z}_{>0} \). We have to show that \( L^s v_\infty \) and \( D^2 L^s v_\infty \) occur in the RHS.
Let us consider the case of \( L^s v_\infty \), the other case is similar.

Expanding \( L \) into a sum of monomials, we see that all monomials in \( L^s v_\infty \) are good,
except for the monomial \( D^3 A^s B v_\infty \), which we replace by the sum of good monomials
\( \sum_{n=0}^{\infty} D^{3s} A^n B v_\infty \), using the identity (see Lemma 7.1)

\[
(1 - A)^{-1} B v_\infty = \sum_{n=0}^{\infty} A^n B v_\infty = v_\infty.
\]

For the second statement, we claim that cancellations always happen in pairs. We list
the necessary equalities which are checked using the explicit formulas given in Section
7.
The cancellations of bad monomials of type $E$, given by replacing first factor $q$ on the RHS is a power series in $z$. Then each function with "linear" factors of the form $(1-q^s)^{\pm 1}, (1-q^s z_1)^{\pm 1}, (1-q^s z_2)^{\pm 1}, (1-q^s z_1 z_2)^{\pm 1}$ with $s \geq 0$.

The cancellations of bad monomials of type $EM$, where $M$ is a good monomial are given by replacing first factor $B$ in the above formulas by $E$.

Note that if $M$ is good then $DM$ is also good.

Note that each term in the RHS of Theorem 6.1 is a product of a monomial in $q, z_1, z_2$ with "linear" factors of the form $(1-q^{s+1})^{\pm 1}, (1-q^{s-1})^{\pm 1}, (1-q^s z_1)^{\pm 1}, (1-q^s z_2)^{\pm 1}, (1-q^s z_1 z_2)^{\pm 1}$ with $s \geq 0$.

Also note that in order to prove the cancellation (or in order to write closed formulas) one has to break 5 families in the RHS of (6.4) into 18 families. Then each function on the RHS is a power series in $q$ whose coefficients are formal Laurent series in $z_1^{-1}$.
over the ring of polynomials $\mathbb{Z}[z_2]$. We know by corollary 2.3 that terms containing $z_1^{-1}$ disappear after the summation.

Finally, we remark that some cancellations of the above happen between monomials of different degrees, for example $CEv_\infty + CE^2v_\infty = 0$. However, we could rewrite this equality as a sum of monomials of the same degree: $CE(A + B)v_\infty + CE^2v_\infty = 0$.

7. The Explicit Formula

We start with an identity which plays a crucial role in our computations:

$$\frac{(q)_\infty}{(qx)_\infty(qy)_\infty} = \sum_{n=1}^{\infty} \frac{(-1)^n(q^n(n-1)/2)(1 - q^{2n}xy)}{(q)_{n-1}(q^{n+1}xy)_\infty(1 - q^n x)(1 - q^n y)}.$$  \hfill (7.1)

In fact this is a special case of Jackson’s $6 \Phi_5$ formula which states (see for example [7] p.102, formula (3.4.2.3)):

$$\sum_{n=0}^{\infty} \frac{(1 - q^{2n}a)(a)_n(b)_n(c)_n(d)_n(qa/\text{bcd})^n}{(1 - a)(q)_n(qa/b)_n(qa/c)_n(qa/d)_n} = \frac{(qa)_\infty(qa/cd)_\infty(qa/\text{bd})_\infty(qa/bc)_\infty}{(qa/b)_\infty(qa/c)_\infty(qa/d)_\infty(qa/\text{bcd})_\infty}.$$  \hfill (7.1)

Note that $v_\infty$ is a sum of infinitely many simple vectors with vector parts $[1, q^n z_2, z_2]$ and the first component of these vector parts is always 1. Therefore, if $M$ is a monomial corresponding to a path which ends at vertex (1) in the summation graph, then $M[1, q^n z_2, z_2]$ does not depend on $n$, and $Mv_\infty$ is a simple vector. In particular, all formulas below are simple vectors. The summation of scalar parts can be always explicitly performed using (7.1).

Now we finish with explicit formulas for the terms in the RHS of the formula in Theorem 6.1. These formulas are obtained by an explicit computation, induction on $m, n, s$ and using formula (7.1).

We define quadratic forms $\alpha, \beta, \gamma$ and $\delta$ by

$$\alpha_{n,m,s} = 3(n + s)^2 + 2ms, \quad \beta_{n,m,s} = 3(n + s)^2 + m^2 + 4ms + 3mn, \quad \gamma_{n,m,s} = \frac{21}{2}n^2 + \frac{1}{2}m^2 + \frac{13}{2}s^2 + 3nm + 15ns + 5ms, \quad \delta_{n,m,s} = \frac{11}{2}n^2 + \frac{3}{2}m^2 + 5s^2 + 5nm + 10ns + 6ms.$$

Then we have the following formulas.

Lemma 7.1. We have $(A + B)v_\infty = v_\infty$. Moreover, $A^n f_1 v_1 = f_{n+1} v_{n+1}$ and $Bv_\infty = f_1 v_1$.

Proof. The identity $A^n f_1 v_1 = f_{n+1} v_{n+1}$ is obvious. The identity $Bv_\infty = f_1 v_1$ is equivalent to (7.1) with $x = q^2 z_1 z_2^2$ and $y = 1$. \hfill $\square$
\[ D^{3n-2} A^m B L^s v_\infty = (-1)^{n+m+s}(z_1 z_2^2)^{n+s-z_2^2} q^{\gamma_{n,m,s}-1/2} n - 1/2 m - 1/2 s + 1 \times \]
\[ \frac{(q z_1 z_2)_{2n-1}(q^{3n+m+s} z_1 z_2^2)_s(1 - q^{6n+2m+2s-2} z_1 z_2^2)}{(q)_\infty(q)_{2n-2}(q)_{2n+m+2s-1}(q^{3n+m+s-1} z_2^2)_n(s) (q^{2n+m+2s} z_1 z_2)_n(q^{4n+m+2s-1} z_1 z_2^2)_\infty} \times [q^{\alpha_{n,m,s}+m}(z_1 z_2^2)^{n+s}(1: q^{-2n-m-2s}(z_1 z_2)^{-1} : q^{1-4n-m-2s}(z_1 z_2)^{-1})], \quad n \geq 1, m, s \geq 0, \]

\[ D^{3n-1} A^m B L^s v_\infty = (-1)^{n+m+s}(z_1 z_2^2)^{n+s-z_2^2} q^{\gamma_{n,m,s}-1/2} n - 1/2 m - 1/2 s \times \]
\[ \frac{(q z_1 z_2)_{2n-1}(q^{3n+m+s+1} z_1 z_2^2)_s(1 - q^{6n+2m+2s} z_1 z_2^2)}{(q)_\infty(q)_{2n-1}(q)_{2n+m+2s}(q^{3n+m+s} z_2^2)_n(s) (q^{2n+m+2s+1} z_1 z_2)_n(q^{4n+m+2s+1} z_1 z_2^2)_\infty} \times [q^{\alpha_{n,m,s}+(z_1 z_2^2)^{n+s}}(1: q^{-2n}(z_1 z_2)^{-1} : q^{2n+m+2s} z_2)], \quad n \geq 1, m, s \geq 0, \]

\[ D^{3n-2} C A^m B L^s v_\infty = (-1)^{n+m+s}(z_1 z_2^2)^{n+s} z_2^2 q^{\gamma_{n,m,s}-1/2} n - 1/2 m - 1/2 s + 1 \times \]
\[ \frac{(q z_1 z_2)_{2n-1}(q^{3n+m+s} z_1 z_2^2)_s(1 - q^{6n+2m+2s-2} z_1 z_2^2)}{(q)_\infty(q)_{2n-1}(q)_{2n+m+2s-1}(q^{3n+m+s-1} z_2^2)_n(s) (q^{2n+m+2s} z_1 z_2)_n(q^{4n+m+2s+1} z_1 z_2^2)_\infty} \times [q^{\alpha_{n,m,s}+m}(z_1 z_2^2)^{n+s}(1: q^{-2n-m-2s}(z_1 z_2)^{-1} : q^{2n+m+2s} z_2)], \quad n \geq 1, m, s \geq 0, \]

\[ D^{3n-1} C A^m B L^s v_\infty = (-1)^{n+m+s}(z_1 z_2^2)^{n+s} z_2^2 q^{\gamma_{n,m,s}+1/2} n + 1/2 m + 1/2 s \times \]
\[ \frac{(q z_1 z_2)_{2n-1}(q^{3n+m+s+1} z_1 z_2^2)_s(1 - q^{6n+2m+2s} z_1 z_2^2)}{(q)_\infty(q)_{2n-1}(q)_{2n+m+2s}(q^{3n+m+s} z_2^2)_n(s) (q^{2n+m+2s+1} z_1 z_2)_n(q^{4n+m+2s+1} z_1 z_2^2)_\infty} \times [q^{\alpha_{n,m,s}+(z_1 z_2^2)^{n+s}}(1: q^{4n+m+2s} z_2 : q^{2n+m+2s} z_2)], \quad n \geq 1, m \geq 1, s \geq 0, \]

\[ D^{3n-3} C A^m B L^s v_\infty = (-1)^{n+m+s+1}(z_1 z_2^2)^{n+s} z_2^2 q^{\gamma_{n,m,s}-1/2} n - 1/2 m - 1/2 s + 2 \times \]
\[ \frac{(q z_1 z_2)_{2n-2}(q^{3n+m+s+1} z_1 z_2^2)_s(1 - q^{6n+2m+2s-2} z_1 z_2^2)}{(q)_{2n-2}(q)_{2n+m+2s-1}(q^{3n+m+s-1} z_2^2)_n(s) (q^{2n+m+2s+1} z_1 z_2)_n^{-1}(q^{4n+m+2s-1} z_1 z_2^2)_\infty} \times [q^{\alpha_{n,m,s}+m}(z_1 z_2^2)^{n+s}(1: q^{-2n+1}(z_1 z_2)^{-1} : q^{2-4n-m-2s}(z_1 z_2)^{-1})], \quad n \geq 1, m \geq 0, s \geq 0, \]
\[ D^{3n-1} A^{m+1} D(B + D + E) L^{-1} v_\infty = (-1)^{n+m+s} (z_1 z_2)^{n+s} q^{\gamma_{m,s} - \frac{11}{2} n - \frac{3}{2} m - \frac{5}{2} s} \times \]
\[
\times (q z_1 z_2)^{2n-1} (q^{3n+m+s+1} z_1 z_2)_{s-1} (1 - q^{6n+2m+2s} z_1 z_2^2) \times \]
\[
\times (q)^{\gamma_{m,s}} (z_1 z_2)^{2n-1} (q^{3n+m+s+1} z_1 z_2)_{s-1} (1 - q^{6n+2m+2s} z_1 z_2^2) \times \]
\[
\times [q^{\gamma_{m,s}} (z_1 z_2)^{n+s} (1 : q^{-2n} (z_1 z_2)^{-1} : q^{-4n-m-2s} (z_1 z_2)^{-1})], \quad n \geq 1, m \geq 0, s \geq 1, \]

\[ D^{3n-2} A^m D(B + D + E) L^s v_\infty = (-1)^{n+m+s+1} (z_1 z_2)^{n+s} q^{\gamma_{m,s} - \frac{1}{2} n - \frac{1}{2} m - \frac{1}{2} s + 1} \times \]
\[
\times (q z_1 z_2)^{2n-1} (q^{3n+m+s} z_1 z_2)_{n+s} (1 - q^{6n+2m+2s-2} z_1 z_2^2) \times \]
\[
\times (q)^{\gamma_{m,s}+m} (z_1 z_2)^{2n+2} (q^{2n+m+2s} z_1 z_2)^{n} (1 : q^{-2n-m-2s} (z_1 z_2)^{-1} : q^{2n+2m+2s} z_1 z_2), \quad n \geq 0, m \geq 1, s \geq 1, \]

\[ D^{3n} A^m D(B + D + E) L^{s-1} v_\infty = (-1)^{n+m+s+1} (z_1 z_2)^{n+s} q^{\gamma_{m,s} - \frac{1}{2} n - \frac{1}{2} m - \frac{1}{2} s} \times \]
\[
\times (q z_1 z_2)^{2n-1} (q^{3n+m+s+1} z_1 z_2)_{s-1} (1 - q^{6n+2m+2s-2} z_1 z_2^2) \times \]
\[
\times (q)^{\gamma_{m,s}} (z_1 z_2)^{2n-1} (q^{3n+m+s+1} z_1 z_2)_{s-1} (1 - q^{6n+2m+2s} z_1 z_2^2) \times \]
\[
\times [q^{\gamma_{m,s}} (z_1 z_2)^{n+s} (1 : q^{-2n-m-2s} (z_1 z_2)^{-1} : q^{2n+2m+2s} z_1 z_2)], \quad n \geq 1, m \geq 1, s \geq 1, \]

\[ D^{3n-2} C A^{m-1} D(B + D + E) L^s v_\infty = (-1)^{n+m+s} (z_1 z_2)^{n+s} q^{\gamma_{m,s} - \frac{11}{2} n - \frac{3}{2} m - \frac{5}{2} s} \times \]
\[
\times (q z_1 z_2)^{2n-1} (q^{3n+m+s+1} z_1 z_2)_{s-1} (1 - q^{6n+2m+2s} z_1 z_2^2) \times \]
\[
\times (q)^{\gamma_{m,s}+m} (z_1 z_2)^{2n+2} (q^{2n+m+2s} z_1 z_2)^{n} (1 : q^{-2n-m-2s} (z_1 z_2)^{-1} : q^{2n+2m+2s} z_1 z_2), \quad n \geq 1, m \geq 1, s \geq 1, \]

\[ D^{3n-1} C A^m D(B + D + E) L^{s-1} v_\infty = (-1)^{n+m+s+1} (z_1 z_2)^{n+s} q^{\gamma_{m,s} - \frac{11}{2} n - \frac{3}{2} m - \frac{5}{2} s} \times \]
\[
\times (q z_1 z_2)^{2n-1} (q^{3n+m+s+1} z_1 z_2)_{s-1} (1 - q^{6n+2m+2s} z_1 z_2^2) \times \]
\[
\times (q)^{\gamma_{m,s}} (z_1 z_2)^{2n-1} (q^{3n+m+s+1} z_1 z_2)_{s-1} (1 - q^{6n+2m+2s} z_1 z_2^2) \times \]
\[
\times [q^{\gamma_{m,s}} (z_1 z_2)^{n+s} (1 : q^{-2n-m-2s} (z_1 z_2)^{-1} : q^{2n+2m+2s} z_1 z_2)], \quad n \geq 1, m \geq 0, s \geq 1, \]

\[ D^{3n-3} C A^m D(B + D + E) L^s v_\infty = (-1)^{n+m+s+1} (z_1 z_2)^{n+s} q^{\gamma_{m,s} - \frac{11}{2} n - \frac{3}{2} m - \frac{5}{2} s + 2} \times \]
\[
\times (q z_1 z_2)^{2n-2} (q^{3n+m+s+1} z_1 z_2)_{s-1} (1 - q^{6n+2m+2s} z_1 z_2^2) \times \]
\[
\times (q)^{\gamma_{m,s}} (q)^{2n-2} (q^{3n+m+s+1} z_1 z_2)_{s-1} (1 - q^{6n+2m+2s} z_1 z_2^2) \times \]
\[
\times [q^{\gamma_{m,s}} (z_1 z_2)^{n+s} (1 : q^{-2n+1} (z_1 z_2)^{-1} : q^{2n+2m+2s} z_1 z_2)], \quad n \geq 1, m \geq 0, s \geq 0, \]
\[ D^{3n-2}E^{2m+1}L^sv_\infty = (-1)^{n+m+1}(z_1z_2)^{n+s}z_2^m q^{\delta_{n,m,s}} - \frac{3}{2} n - \frac{1}{2} m (q)^{-1} \times \]
\[ (q^{-2s}z_1)_s(qz_1z_2)_{n-1} \times [q^{\beta_{n,m,s}}(z_1z_2)^{n+s}z_2^m (1 : q^{-2n-m-2s}(z_1z_2)^{-1} : q^{-n_{z_1}^{-1}})], \quad n \geq 1, m \geq 0, s \geq 0, \]

\[ D^{3n-1}E^{2m+1}L^sv_\infty = (-1)^{n+m}(z_1z_2)^{n+s}z_2^m q^{\delta_{n,m,s}} - \frac{3}{2} n - \frac{1}{2} m \times \]
\[ (q^{-2s}z_1)_s(qz_1z_2)_n \times [q^{\beta_{n,m,s}}(z_1z_2)^{n+s}z_2^m (1 : q^n : q^{-n-m-2s})], \quad n \geq 0, m \geq 1, s \geq 0, \]

\[ D^{3n-2}E^{2m+2}L^sv_\infty = (-1)^{n+m}(z_1z_2)^{n+s}z_2^m q^{\delta_{n,m,s}} - \frac{3}{2} n - \frac{1}{2} m \times \]
\[ (q^{-2s}z_1)_s(qz_1z_2)_{n-1} \times [q^{\beta_{n,m,s}}(z_1z_2)^{n+s}z_2^m (1 : q^n : q^{-n-m-2s})], \quad n \geq 1, m \geq 0, s \geq 0, \]

\[ D^{3n-1}E^{2m}L^sv_\infty = (-1)^{n+m}(z_1z_2)^{n+s}z_2^m q^{\delta_{n,m,s}} - \frac{3}{2} n - \frac{1}{2} m \times \]
\[ (q^{-2s}z_1)_s(qz_1z_2)_n \times [q^{\beta_{n,m,s}}(z_1z_2)^{n+s}z_2^m (1 : q^{-2n-m-2s}(z_1z_2)^{-1} : q^{-n_{z_1}^{-1}})], \quad n \geq 1, m \geq 1, s \geq 0, \]

\[ D^{3n}E^{2m}L^sv_\infty = (-1)^{n+m}(z_1z_2)^{n+s}z_2^m q^{\delta_{n,m,s}} + \frac{1}{2} n + \frac{1}{2} m + 2s \times \]
\[ (q^{-2s}z_1)_s(qz_1z_2)_n \times [q^{\beta_{n,m,s}}(z_1z_2)^{n+s}z_2^m (1 : q^n : q^{-n+m+2s})], \quad n \geq 0, m \geq 1, s \geq 0. \]

Our convention for expansions of R.H.S. of these formulas into power series can be now written in the form:

\[ |q| \ll 1, \quad |z_1^{-1}| \ll 1, \quad |z_2| \ll 1, \quad |z_1z_2| \sim 1. \]
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