A SIMPLE PROPERTY OF THE WEYL TENSOR
FOR A SHEAR, VORTICITY AND ACCELERATION-FREE
VELOCITY FIELD

LUCA GUIDO MOLINARI AND CARLO ALBERTO MANTICA

Abstract. We prove that, in a space-time of dimension \( n > 3 \) with a velocity field that is shear-free, vorticity-free and acceleration-free, the covariant divergence of the Weyl tensor is zero if and only if the contraction of the Weyl tensor with the velocity is zero. This extends a property found in Generalised Robertson-Walker spacetimes, where the velocity is also eigenvector of the Ricci tensor. Despite the simplicity of the statement, the proof is involved. As a product of the same calculation, we introduce a curvature tensor with an interesting recurrence property.

1. Introduction

A shear-free, vorticity-free and acceleration-free velocity field \( u_{k} \), has covariant derivative

\[ \nabla_{i}u_{j} = \varphi (g_{ij} + u_{i}u_{j}) \]  

where \( \varphi \) is a scalar field, and \( u_{k}u^{k} = -1 \). For such a vector field we prove the following results for the Weyl tensor, in space-time dimension \( n > 3 \):

Theorem 1.1. \( \nabla_{m}C_{jkl}^{m} = 0 \iff u_{m}C_{jkl}^{m} = 0 \)

Next, we introduce the following tensor, where \( E_{kl} = u^{j}u^{m}C_{jklm} \) is the electric part of the Weyl tensor:

\[ \Gamma_{ijklm} = C_{ijklm} - \frac{n - 2}{n - 3}(u_{i}u_{m}E_{kl} - u_{k}u_{m}E_{il} - u_{i}u_{l}E_{km} + u_{k}u_{l}E_{im}) \]

\[ - \frac{1}{n - 3}(g_{im}E_{kl} - g_{km}E_{il} - g_{il}E_{km} + g_{kl}E_{im}) \]

Theorem 1.2. \( \Gamma_{ijklm} \) is a generalised curvature tensor, it is totally trace-less and:

\[ u^{m}\Gamma_{jklm} = 0 \]

\[ u^{p}\nabla_{p}\Gamma_{jklm} = -2\varphi \Gamma_{jklm} \]

The tensor is zero in \( n = 4 \).

The proofs make use of various properties of “twisted” space-times, that were introduced by B. Y. Chen [3] as a generalisation of warped space-times:

\[ ds^{2} = -dt^{2} + f^{2}(\vec{x}, t)g_{\mu\nu}(\vec{x})dx^{\mu}dx^{\nu} \]

---

2010 Mathematics Subject Classification. Primary 53B30, Secondary 83C20.

Key words and phrases. Weyl tensor, twisted space-time, Generalized Robertson-Walker space-time, torse-forming vector, generalized curvature tensor.
\( f > 0 \) is the scale factor and \( g_{\mu \nu} \) is the metric tensor of a Riemannian sub-manifold of dimension \( n - 1 \). If \( f \) only depends on time, the metric is warped and the space-time is a Generalized Robertson-Walker (GRW) space-time \([2, 4, 10]\). Chen \([5]\) and the authors \([11]\) gave covariant characterisations of twisted space-times; the latter reads: a space-time is twisted if and only if there exists a time-like unit vector field \( u^i \) with the property \((1)\).

The space-time is GRW if \( u^i \) is also eigenvector of the Ricci tensor \([10]\); it is RW with the further condition that the Weyl tensor is zero, \( C_{ijkl} = 0 \).

The next two short sections collect useful results on twisted space-times, and about the Weyl tensor in \( n = 4 \).

2. Twisted space-times

We summarise some results on twisted space-times, taken from ref. \([11]\):

i) the vector field \( u_j \) is unique (up to reflection).

ii) the vector field \( u_j \) is Weyl compatible (see \([8]\) for a general presentation):

\[
(7) \quad (u_i C_{jklm} + u_j C_{kilm} + u_k C_{ijlm}) u^m = 0.
\]

This classifies the Weyl tensor as purely electric with respect to \( u_j \) \([6]\).

A contraction gives the useful property:

\[
(8) \quad C_{jklm} u^m = u_k E_{ji} - u_j E_{kl}
\]

where \( E_{jk} = C_{ijkl} u^i u^l \). It follows that \( C_{jklm} u^m = 0 \) if and only if \( E_{ij} = 0 \).

iii) the Ricci tensor has the general form

\[
(9) \quad R_{jk} = \frac{R - n \xi}{n - 1} u_j u_k + \frac{R - \xi}{n - 1} g_{jk} + (n - 2)(u_j v_k + u_k v_j - E_{jk})
\]

where \( R = R^k_k, \xi = (n - 1)(u^p \nabla_p \varphi + \varphi^2) \), and \( v^k = (g^{km} + u^k u^m) \nabla_m \varphi \) is a space-like vector.

iv) A twisted space-time is a GRW space-time if and only if \( v_j = 0 \).

3. The Weyl tensor in four-dimensional space-times

The following algebraic identity by Lovelock holds in \( n = 4 \) \([7]\), ex. 4.9):

\[
(10) \quad 0 = g_{ar} C_{bcrs} + g_{br} C_{casr} + g_{cr} C_{abrs} + g_{at} C_{bcrs} + g_{bt} C_{casr} + g_{ct} C_{abrs} + g_{as} C_{bctr} + g_{bs} C_{catr} + g_{cs} C_{abtr}
\]

It implies that \( C_{abcd} C^{abcd} = \frac{1}{4} \delta_r^s C^2 \), where \( C^2 = C_{abcd} C^{abcd} \).

The contraction of \((10)\) with \( u^r u^s \), where \( u^i \) is any time-like unit vector, gives the Weyl tensor in terms of its contractions \( u^d C_{abcd} \) and \( E_{ad} = u^b u^c C_{abcd} \):

\[
(11) \quad C_{abcd} = -u^m (u_a C_{mbcd} + u_b C_{amcd} + u_c C_{abmd} + u_d C_{abcm}) + g_{ad} E_{bc} - g_{bd} E_{ac} - g_{ac} E_{bd} + g_{bc} E_{ad}
\]

**Proposition 3.1.** If \( u^m \) is Weyl compatible, \((7)\), in \( n = 4 \) the Weyl tensor is wholly given by its electric component:

\[
(12) \quad C_{abcd} = 2(u_a u_d E_{bc} - u_a u_c E_{bd} + u_b u_c E_{ad} - u_b u_d E_{ac}) + g_{ad} E_{bc} - g_{ac} E_{bd} + g_{bc} E_{ad} - g_{bd} E_{ac}
\]

and \( C^2 = 8 E^2 \), where \( E^2 = E_{ab} E^{ab} \).
**Proof**. The property (8) is used to simplify (11). Contraction with \( u^i u^j \) of the identity \( \frac{1}{4} C^2 g_{ij} = C_{[iabc} C_{j]abc} \) and (8) give:

\[
-\frac{1}{4} C^2 = (u^i C_{[iabc})(u_j C_{j]abc}) = (u_i E_{ca} - u_c E_{ba})(u^b E^{ca} - u^c E^{ba}).
\]

Since \( E_{ca} u^c = 0 \), the result is

\[
-E_{ca} E^{ca} - E_{ba} E^{ba} = -2E^2.
\]

\[
\square
\]

**Corollary 3.2.** In a twisted space-time in \( n = 4 \), \( C_{abcd} = 0 \) if and only if \( E_{ab} = 0 \).

---

### 4. THE MAIN RESULTS

In \( n > 3 \) the second Bianchi identity for the Riemann tensor translates to an identity for the Weyl tensor [1]:

\[
\nabla_i C_{jklm} + \nabla_j C_{iklm} + \nabla_k C_{ijlm} = \frac{1}{n-3} \nabla_p (g_{jm} C_{kilm} + g_{km} C_{ijlp})
\]

(13) 

\[ + g_{im} C_{jklp} + g_{ik} C_{jlim} + g_{il} C_{kmjp} + g_{lm} C_{ikjp}).\]

As a consequence of (13), as shown in the Appendix, we obtain the intermediate result:

**Proposition 4.1.** In a twisted space-time the divergence of the Weyl tensor is:

\[
\nabla_p C_{ikm} = (n-3)(\nabla_i E_{km} - \nabla_k E_{im})
\]

(14) 

\[ + (n-2)[u_p u^p (u_i E_{km} - u_k E_{im})] + 2\varphi(u_i E_{km} - u_k E_{im})
\]

\[ + (2u_k u_m + g_{km}) \nabla_p E_{ip} - (2u_k u_m + g_{km}) \nabla_p E_{ip}.\]

**Corollary 4.2.** In a twisted space-time, if \( \nabla^p C_{jklp} = 0 \) then

\[
\nabla_p E_{pk} = 0 \quad \text{and} \quad u^p \nabla_p E_{km} = -\varphi(n-1) E_{km}
\]

(15)

**Proof.** Note the identity:

\[
u^m \nabla_p C_{kjm} = \nabla_p (u^m C_{kjm}) = \nabla_p (u^i E_{km} - u_k E_{im}) = u_i \nabla_p E_{km} - u_k \nabla_p E_{im}.\]

Another identity is:

\[
u^i \nabla_p C_{kjm} = \nabla_p (u^i C_{kjm}) - \varphi E_{km} = \nabla_p (u^m E_{pk} - u^p E_{mk}) - \varphi E_{km} = u_m \nabla_p E_{pk} - \varphi(n-1) E_{km} - u^p \nabla_p E_{km}.
\]

Together, the two identities imply the statements. \[
\square
\]

Now, we are able to extend to twisted space-times a property of GRW space-times (Theorem 3.4, [9]):

**Theorem 1.1**: In a twisted space-time of dimension \( n > 3 \):

\[
\nabla_m C_{jkl}^m = 0 \iff u^m C_{jkl}^m = 0
\]

**Proof.** If \( u^m C_{jkl} = 0 \) then \( E_{kl} = 0 \) and \( \nabla_m C_{jkl} = 0 \) follows from (14). If \( \nabla_m C_{jkl} = 0 \), the identities (15) simplify eq.(14) as follows:

\[
0 = (n-3)[(\nabla_i E_{km} - \nabla_k E_{im}) - (n-2)\varphi(u_i E_{km} - u_k E_{im})]
\]

If \( n > 3 \), a contraction with \( u^i \) gives:

\[
0 = u^i \nabla_i E_{km} + \varphi E_{km}.\]

This and the second implication in (15) mean that \( E_{kl} = 0 \) i.e. \( u^m C_{jkl} = 0 \) by (8). \[
\square
\]

The final result (20) in the Appendix, suggests the introduction of the new tensor (3), that combines the Weyl tensor with the generalized curvature tensors obtained as Kulkarni-Nomizu products of \( E_{ij} \) with \( u_i u_j \) or \( g_{ij} \).

It has the symmetries of the Weyl tensor for exchange and contraction of indices, as well as the first Bianchi identity (it is a generalized curvature tensor). Moreover
it is traceless, $\Gamma_{mn}m = 0$, and any contraction with $u$ is zero. The associated scalar $\Gamma^2 = \Gamma_{abcd}\Gamma^{abcd}$ is evaluated:

$$\Gamma^2 = C^2 - 4\frac{n-2}{n-3}E^2 \tag{17}$$

By Prop. 3.1 this tensor is identically zero in $n = 4$.

In dimension $n > 4$, Theorem 1.2 is basically the result (20) of the long calculation in the Appendix.

**Remark 4.3.** The property $\Gamma_{abcd}u^d = 0$ means that in the frame (6), where $u^0 = 1$ and space components $u^\mu$ vanish, the components $\Gamma_{abcd}$ where at least one index is time, are zero. Therefore, $\Gamma^2 > 0$ in $n > 4$ and, for the same reason, $E^2 \geq 0$. We conclude that the Weyl scalar is positive:

$$C^2 = 4\frac{n-2}{n-3}E^2 + \Gamma^2 \geq 0 \tag{18}$$

**APPENDIX**

**Proposition 4.4.** In a twisted space the following identities hold among the Weyl tensor and the contracted Weyl tensor:

$$(n-3)(u^p\nabla_p C_{ikm} + 2\varphi C_{ikm}) = (n-2)(u^p\nabla_p (u_iE_{km} - u_kE_{im}) + 2\varphi(u_iE_{km} - u_kE_{im})) \
+ (2u_ku_m + g_{km})\nabla_p E^p \tag{19}$$

$$(n-3)(u^p\nabla_p C_{iklm} + 2\varphi C_{iklm}) = (n-2)(u^p\nabla_p (u_iu_mE_{kl} - u_ku_mE_{il} - u_iu_mE_{km} + u_ku_mE_{im}) \
+ 2\varphi(u_iu_mE_{kl} - u_ku_mE_{il} - u_iu_mE_{km} + u_ku_mE_{im})) \
+ [u^p\nabla_p (g_{im}E_{kl} - g_{kl}E_{im} - g_{il}E_{km} + g_{kl}E_{im})] \
+ 2\varphi(g_{im}E_{kl} - g_{kl}E_{im} - g_{il}E_{km} + g_{kl}E_{im}) \tag{20}$$

**Proof.** Contraction of (13) with $u^j$ is:

$$u^j\nabla_j C_{ijkl} + u^j\nabla_j C_{jklm} + u^j\nabla_j C_{ijlm} = \frac{1}{n-3}(u_m\nabla_p C_{km} + u_l\nabla_p C_{ikm}) \
+ \frac{1}{n-3}\nabla_p [u^j(g_{km}C_{jil} + g_{il}C_{jkm} + g_{kl}C_{ijm} + g_{il}C_{kjm})] \
- \frac{1}{n-3}\varphi u^j u^l (g_{km}C_{jil} + g_{il}C_{jkm} + g_{kl}C_{ijm} + g_{il}C_{kjm})$$

Where possible, the vector $u^k$ is taken inside covariant derivatives to take advantage of property (8)

$$\nabla_j(u^j C_{ijkl}) - \varphi h^j_i C_{ijkl} + u^j\nabla_j C_{ijkl} + \nabla_k(u^j C_{ijlm}) - \varphi h^k_i C_{ijlm} \
= \frac{1}{n-3}(u_m\nabla_p C_{km} + u_l\nabla_p C_{ikm}) + \frac{1}{n-3}\nabla_p [g_{km}(u_pE_{li} - u_lE_{pi}) \
+ g_{im}(u_pE_{lk} - u_lE_{ik}) + g_{kl}(u_mE_{pi} - u_pE_{mi}) + g_{il}(u_pC_{mk} - u_mE_{pk}) \
+ \frac{1}{n-3}\varphi[g_{km}E_{il} - g_{im}E_{kl} - g_{kl}E_{im} + g_{il}E_{km}]$$
\[ \nabla_i (u_i E_{mk} - u_m E_{ik}) - \varphi C_{iklm} - \varphi u_i (u_i E_{mk} - u_m E_{ik}) + u^j \nabla_j C_{kilm} \]
\[ + \nabla_k (u_m E_{li} - u_l C_{mi}) - \varphi C_{iklm} - \varphi u_k (u_m E_{li} - u_l C_{mi}) \]
\[ = \frac{1}{n-3} (u_m \nabla_p C_{iklm} + u_i \nabla_p C_{iklm}) \]
\[ + \frac{1}{n-3} u^p \nabla_p [g_{km} E_{li} - g_{im} E_{lk} - g_{il} E_{km} + g_{il} C_{km}] \]
\[ + \frac{1}{n-3} \nabla^p [g_{km} u_l E_{pi} + g_{im} u_l E_{pk} + g_{ki} u_m E_{pi} - g_{il} u_m E_{pk}] \]
\[ + \frac{1}{n-3} \varphi (g_{km} E_{il} - g_{im} E_{kl} - g_{il} E_{km} + g_{il} E_{km}) \]

\( (n-3) [u_i (\nabla_i E_{mk} - \nabla_k E_{mi}) - u_m (\nabla_i E_{lk} - \nabla_k E_{li}) - 2 \varphi C_{iklm} + u^j \nabla_j C_{kilm}] \]
\[ = (u_m \nabla_p C_{iklm} + u_l \nabla_p C_{iklm}) + u^p \nabla_p [g_{km} E_{li} - g_{im} E_{lk} - g_{il} E_{km} + g_{il} C_{km}] \]
\[ - g_{km} u_l \nabla^p E_{pi} + g_{im} u_l \nabla^p E_{pk} + g_{ki} u_m \nabla^p E_{pi} - g_{il} u_m \nabla^p E_{pk} \]
\[ + 2 \varphi (g_{km} E_{il} - g_{im} E_{kl} - g_{il} E_{km} + g_{il} E_{km}) \]

Contraction with \( u^i \) yields the first result, (19):
\[ \nabla_p C_{iklm} = (n-3) (\nabla_i E_{km} - \nabla_k E_{im}) \]
\[ + (n-2) [u^p \nabla_p (u_i E_{km} - u_k E_{im}) + 2 \varphi (u_i E_{km} - u_k E_{im})] \]
\[ + (2 u_k u_m + g_{km}) \nabla_p E_{il} - (2 u_i u_m + g_{im}) \nabla_p E_{ik} \]

which is used to replace the covariant divergences \( \nabla_p C_{iklm} \) in the previous expression
\[ (n-3) [u_i (\nabla_i E_{mk} - \nabla_k E_{mi}) - u_m (\nabla_i E_{lk} - \nabla_k E_{li}) - 2 \varphi C_{iklm} + u^j \nabla_j C_{kilm}] \]
\[ = -u_m \{ (n-3) (\nabla_i E_{kl} - \nabla_k E_{li}) + (n-2) [u^p \nabla_p (u_i E_{kl} - u_k E_{li}) + 2 \varphi (u_i E_{kl} - u_k E_{li})] \]
\[ + (2 u_k u_l + g_{kl}) \nabla_p E_{il} - (2 u_i u_l + g_{il}) \nabla_p E_{ik} \}
\[ + u \{ (n-3) (\nabla_i E_{km} - \nabla_k E_{im}) + (n-2) [u^p \nabla_p (u_i E_{km} - u_k E_{im}) + 2 \varphi (u_i E_{km} - u_k E_{im})] \]
\[ + (2 u_k u_m + g_{km}) \nabla_p E_{il} - (2 u_i u_m + g_{im}) \nabla_p E_{ik} \}
\[ + u^p \nabla_p [g_{km} E_{li} - g_{im} E_{lk} - g_{il} E_{km} + g_{il} C_{km}] \]
\[ - g_{km} u_l \nabla^p E_{pi} + g_{im} u_l \nabla^p E_{pk} + g_{ki} u_m \nabla^p E_{pi} - g_{il} u_m \nabla^p E_{pk} \]
\[ + 2 \varphi (g_{km} E_{il} - g_{im} E_{kl} - g_{il} E_{km} + g_{il} E_{km}) \]

Some derivatives cancel, and we are left with
\[ (n-3) [-2 \varphi C_{iklm} - u^p \nabla_p C_{iklm}] \]
\[ = -u_m \{ (n-2) [u^p \nabla_p (u_i E_{kl} - u_k E_{li}) + 2 \varphi (u_i E_{kl} - u_k E_{li})] \}
\[ + u \{ (n-2) [u^p \nabla_p (u_i E_{km} - u_k E_{im}) + 2 \varphi (u_i E_{km} - u_k E_{im})] \}
\[ + u^p \nabla_p [g_{km} E_{li} - g_{im} E_{lk} - g_{il} E_{km} + g_{il} C_{km}] \]
\[ + 2 \varphi (g_{km} E_{il} - g_{im} E_{kl} - g_{il} E_{km} + g_{il} E_{km}) \]
[4] B-Y Chen, *A simple characterization of generalized Robertson-Walker space-times*, Gen. Relativ. Gravit. **46** (2014), 1833, 5 pp.

[5] B-Y Chen, *Rectifying submanifolds of Riemannian manifolds and torqued vector fields*, Kragujevac Journal of Mathematics **41** n.1 (2017), 93–103.

[6] S. Hervik, M. Ortaggio and L. Wylleman, *Minimal tensors and purely electric or magnetic space-times of arbitrary dimension*, Class. Quantum Grav. **30** n.16 (2013).

[7] D. Lovelock and H. Rund, *Tensors, differential forms, and variational principles* (1975, Dover reprint, 1989).

[8] C. A. Mantica and L. G. Molinari, *Weyl compatible tensors*, Int. J. Geom. Meth. Mod. Phys. **11** n.8 (2014), 1450070, 15 pp.

[9] C. A. Mantica and L. G. Molinari, *On the Weyl and Ricci tensors of Generalized Robertson-Walker space-times*, J. Math. Phys. **57** n.10 (2016) 102502, 6pp.

[10] C. A. Mantica and L. G. Molinari, *Generalized Robertson-Walker space-times, a survey*, Int. J. Geom. Meth. Mod. Phys. **14** n.3 (2017) 1730001, 27 pp.

[11] C. A. Mantica and L. G. Molinari, *Twisted Lorentzian manifolds: a characterization with torse-forming time-like unit vectors*, Gen. Relativ. Gravit. **49** (2017) 51.

L. G. Molinari (corresponding author): Physics Department, Università degli Studi di Milano and I.N.F.N. sez. Milano, Via Celoria 16, 20133 Milano, Italy – C. A. Mantica: I.I.S. Lagrange, Via L. Modignani 65, 20161, Milano, Italy, and I.N.F.N. sez. Milano.

E-mail address: Luca.Molinari@unimi.it, Carlo.Mantica@mi.infn.it