MAXIMAL EIGENVALUES OF A CASIMIR OPERATOR
AND MULTIPLICITY-FREE MODULES

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Abstract. Let \( g \) be a finite-dimensional complex semisimple Lie algebra and \( b \) a Borel subalgebra. Then \( g \) acts on its exterior algebra \( \wedge g \) naturally. We prove that the maximal eigenvalue of the Casimir operator on \( \wedge g \) is one third of the dimension of \( g \), that the maximal eigenvalue \( m_i \) of the Casimir operator on \( \wedge^i g \) is increasing for \( 0 \leq i \leq r \), where \( r \) is the number of positive roots, and that the corresponding eigenspace \( M_i \) is a multiplicity-free \( g \)-module whose highest weight vectors correspond to certain ad-nilpotent ideals of \( b \). We also obtain a result describing the set of weights of the irreducible representation of \( g \) with highest weight a multiple of \( \rho \), where \( \rho \) is one half the sum of positive roots.

1. Introduction

Let \( g \) be a finite-dimensional complex semisimple Lie algebra and \( U(g) \) its universal enveloping algebra. The study of the \( g \)-module structure of its exterior algebra \( \wedge g \) has a long history. Although this module structure is still not fully understood, Kostant has done a lot of important work on it; see for example [2] and [3].

Let \( \text{Cas} \in U(g) \) be the Casimir element with respect to the Killing form. Let \( m_i \) be the maximal eigenvalue of \( \text{Cas} \) on \( \wedge^i g \) and \( M_i \) be the corresponding eigenspace. Let \( p \) be the maximal dimension of commutative subalgebras of \( g \). In [2] it is proved that \( m_i \leq i \) for any \( i \) and \( m_i = i \) for \( 0 \leq i \leq p \), and if \( m_i = i \), then \( M_i \) is a multiplicity-free \( g \)-module whose highest weight vectors correspond to \( i \)-dimensional abelian ideals of \( b \). The integer \( p \) for all the simple Lie algebras was determined by Malcev, and Suter gave a uniform formula for \( p \) in [6].

Fix a Cartan subalgebra \( \mathfrak{h} \) of \( g \) and a set \( \Delta^+ \) of positive roots. Let \( \rho \in \mathfrak{h}^* \) be one half the sum of all the positive roots. For any \( \lambda \in \mathfrak{h}^* \), let \( V_\lambda \) denote the irreducible representation of \( g \) with highest weight \( \lambda \).

In this paper we will prove the following result, which extends some theorems of Kostant. Let \( n = \dim g \), \( l = \text{rank} g \) and \( r \) be the number of positive roots.

Theorem 1.1 (Theorem 3.2). (1) One has \( m_i \leq n/3 \) for \( i = 0, 1, \cdots, n \), and \( m_i = n/3 \) if and only if \( i = r, r+1, \cdots, r+l \). For \( s = 0, 1, \cdots, l \), \( M_{r+s} = \binom{l}{s} V_{2\rho} \).
(2) For \(0 \leq i < r\) one has \(m_i < m_{i+1}\). For \(1 \leq i \leq r\), \(M_i\) is a multiplicity-free \(\mathfrak{g}\)-module, whose highest weight vectors correspond to certain ad-nilpotent ideals of \(\mathfrak{b}\). In fact \(\bigoplus_{i=0}^{r} M_i\) is also a multiplicity-free \(\mathfrak{g}\)-module.

Note that Kostant already proved in Remark 1.2 of [4] that \(m_i \leq n/3\) for all \(i\). Thus (1) of this theorem provides further information about the corresponding eigenspace of the maximal eigenvalue \(n/3\).

This result relates \(M_i\) to ad-nilpotent ideals of \(\mathfrak{b}\), which are classified in [5]. But it will be complicated to determine those ad-nilpotent ideals of \(\mathfrak{b}\) corresponding to the highest weight vectors of \(M_i\).

To prove this theorem, we need the following interesting result.

**Proposition 1.2** (Proposition 2.1). Let \(k \in \mathbb{Z}^+\). The set of weights of \(V_{k\rho}\) (whose dimension is \((k+1)^r\)) is

\[
\{ \sum_{i=1}^{r} c_i \alpha_i | \alpha_i \in \Delta^+, c_i = -k/2, -k/2 + 1, \cdots, k/2 - 1, k/2 \}.
\]

2. Weights of a representation with highest weight a multiple of \(\rho\)

Let \(\mathfrak{g}\) be a finite-dimensional complex semisimple Lie algebra. Fix a Cartan subalgebra \(\mathfrak{h}\) of \(\mathfrak{g}\) and a Borel subalgebra \(\mathfrak{b}\) of \(\mathfrak{g}\) containing \(\mathfrak{h}\). Let \(\Delta\) be the set of roots of \(\mathfrak{g}\) with respect to \(\mathfrak{h}\) and \(\Delta^+\) be the set of positive roots whose corresponding root spaces lie in \(\mathfrak{b}\). Let \(\Lambda = h^*\) be the lattice of \(\mathfrak{g}\)-integral linear forms on \(\mathfrak{h}\) and \(\Delta^+\) be the subset of dominant integral linear forms. Let \((, )\) be the bilinear form on \(\mathfrak{h}^*\) induced by the Killing form. Let \(l = \dim \mathfrak{h}, r = |\Delta^+|\) and \(n = l + 2r = \dim \mathfrak{g}\).

Assume \(\Delta^+ = \{\alpha_1, \alpha_2, \cdots, \alpha_r\}\).

For any \(\lambda \in \Lambda\), let \(\pi_{\lambda} : \mathfrak{g} \to \text{End}(V_{\lambda})\) be the irreducible representation of \(\mathfrak{g}\) with highest weight \(\lambda\), and \(\Gamma(V_{\lambda})\) be the set of weights, with multiplicities. Any \(\gamma \in \Gamma(V_{\lambda})\) will appear \(k\) times if the dimension of the \(\gamma\)-weight space is \(k\). For example \(\Gamma(\mathfrak{g}) = \Delta \cup \{0, \cdots, 0\}\) \((l\text{ times})\). If \(U \subset V_{\lambda}\) is an \(\mathfrak{h}\)-invariant subspace, then we will also use \(\Gamma(U)\) to denote the the set of weights of \(U\) with multiplicities and define

\[
\langle U \rangle = \sum_{\gamma \in \Gamma(U)} \gamma.
\]

For any \(S \subset \Gamma(V_{\lambda})\), we also define \(\langle S \rangle = \sum_{\gamma \in S} \gamma\).

Let \(\rho \in \mathfrak{h}^*\) be one half the sum of all the positive roots. For any \(k \in \mathbb{Z}^+\), the representation \(V_{k\rho}\) of \(\mathfrak{g}\) has dimension \((k+1)^r\) by Weyl’s dimension formula. The following result describes the set of weights of \(V_{k\rho}\), which is well-known if \(k = 1\) (see e.g. [7]).

**Proposition 2.1.** The set of weights of \(V_{k\rho}\) is

\[
\Gamma(V_{k\rho}) = \{ \sum_{i=1}^{r} c_i \alpha_i | \alpha_i \in \Delta^+, c_i = -k/2, -k/2 + 1, \cdots, k/2 - 1, k/2 \},
\]

\[
\{ \gamma \in \mathfrak{h}^* | \gamma = c_1 \alpha_1 + \cdots + c_r \alpha_r \text{ and } c_i \geq 0 \text{ for all } i \}.
\]
or equivalently,
\[ \Gamma(V_{k\rho}) = \{ k\rho - \sum_{i=1}^{r} c_i \alpha_i | \alpha_i \in \Delta^+, c_i = 0, 1, \cdots, k. \} . \]

Proof. By Weyl’s denominator formula,
\[ \prod_{i=1}^{r} \left( e^{k+\frac{1}{2} \alpha_i} - e^{-k-\frac{1}{2} \alpha_i} \right) = \sum_{w \in W} \operatorname{sgn}(w) e^{w((k+1)\rho)}. \]

Then for \( c_i = -k/2, -k/2 + 1, \cdots, k/2 - 1, k/2 \) with \( i = 1, \cdots, r \),
\[
\sum_{c_1, \cdots, c_r} e^{\sum_{i=1}^{r} c_i \alpha_i} = \prod_{i=1}^{r} \left( e^{(-\frac{k}{2}) \alpha_i} + e^{(-\frac{k}{2}+1) \alpha_i} + \cdots + e^{(\frac{k}{2}-1) \alpha_i} + e^{(\frac{k}{2}) \alpha_i} \right) \\
= \prod_{i=1}^{r} \frac{e^{\frac{k+1}{2} \alpha_i} - e^{-\frac{k+1}{2} \alpha_i}}{e^{\frac{1}{2} \alpha_i} - e^{-\frac{1}{2} \alpha_i}} \\
= \frac{\sum_{w \in W} \operatorname{sgn}(w) e^{w((k+1)\rho)}}{\prod_{i=1}^{r} \left( e^{\frac{1}{2} \alpha_i} - e^{-\frac{1}{2} \alpha_i} \right)} = \operatorname{char}(V_{k\rho}). \]

Let \( \operatorname{Cas} \in U(\mathfrak{g}) \) be the Casimir element corresponding to the Killing form. For any \( \lambda \in \Gamma \), define \( \operatorname{Cas}(\lambda) = (\lambda + \rho, \lambda + \rho) - (\rho, \rho) \).

The following result is well-known.

**Lemma 2.2.** If \( \lambda \in \Lambda \), then \( \operatorname{Cas}(\lambda) \) is the scalar value taken by \( \operatorname{Cas} \) on \( V_\lambda \). For any \( \mu \in \Gamma(V_{\lambda}) \) one has \( \operatorname{Cas}(\mu) \leq \operatorname{Cas}(\lambda) \) and \( \operatorname{Cas}(\mu) < \operatorname{Cas}(\lambda) \) if \( \mu \neq \lambda \).

### 3. Maximal eigenvalues of a Casimir operator and the corresponding eigenspaces

Let \( \wedge \mathfrak{g} \) be the exterior algebra of \( \mathfrak{g} \). Then \( \mathfrak{g} \) acts on \( \wedge \mathfrak{g} \) naturally. Let \( m_i \) be the maximal eigenvalue of \( \operatorname{Cas} \) on \( \wedge^i \mathfrak{g} \) and \( M_i \) be the corresponding eigenspace.

One knows that \( \wedge^i \mathfrak{g} \) is isomorphic to \( \wedge^{n-i} \mathfrak{g} \) as \( \mathfrak{g} \)-modules for each \( i \), so one has
\[ m_i = m_{n-i} \]

and
\[ M_i \cong M_{n-i} \]

Let \( p \) be the maximal dimension of commutative subalgebras of \( \mathfrak{g} \). Kostant showed that \( m_i \leq i \) and \( m_i = i \) for \( 0 \leq i \leq p \), and if \( m_i = i \), then \( M_i \) is spanned by \( \wedge^k \mathfrak{a} \), where \( \mathfrak{a} \) runs through \( k \)-dimensional commutative subalgebras of \( \mathfrak{g} \).

A nonzero vector \( w \in \wedge \mathfrak{g} \) is called decomposable if \( w = z_1 \wedge z_2 \wedge \cdots \wedge z_k \) for some positive integer \( k \), where \( z_i \in \mathfrak{g} \). In this case let \( \mathfrak{a}(w) \) be the corresponding \( k \)-dimensional subspace spanned by \( z_1, z_2, \cdots, z_k \).
Theorem 3.1 (Proposition 6 and Theorem 7 of [2]). (1) Let

\[ w = z_1 \wedge z_2 \wedge \cdots \wedge z_k \in \wedge^k g \]

be a decomposable vector. Then \( w \) is a highest weight vector if and only if \( a(w) \) is \( b \)-normal, i.e., \([b, a(w)] \subset a(w)\). In this case the highest weight of the simple \( g \)-module generated by \( w \) is \( \langle a(w) \rangle \).

Thus there is a one-to-one correspondence between all the decomposably-generated simple \( g \)-submodules of \( \wedge^k g \) and all the \( k \)-dimensional \( b \)-normal subspaces of \( g \).

(2) Let \( a_1, a_2 \) be any two ideals of \( b \) lying in \( n \). Then \( \langle a_1 \rangle = \langle a_2 \rangle \) if and only if \( a_1 = a_2 \). Thus, if \( V_1 \subset \wedge^3 g, V_2 \subset \wedge^3 g \) are two decomposably-generated simple \( g \)-submodules which correspond to ideals of \( b \) lying in \( n \), then \( V_1 \) is equivalent to \( V_2 \) if and only if \( V_1 = V_2 \).

Theorem 3.2. (1) One has

\[ m_i = \max \{ ||\rho + \gamma_1 + \cdots + \gamma_i||^2 - ||\rho||^2 \mid \{\gamma_t \mid t = 1, \cdots, i\} \subset \Gamma(g) \} \]

for any \( i \).

(2) One has \( m_i \leq n/3 \) for \( i = 0, 1, \cdots, n \), and \( m_i = n/3 \) if and only if \( i = r, r + 1, \cdots, r + l \). For \( s = 0, 1, \cdots, l \), \( M_{r+s} = \left( \frac{1}{s} \right) V_{2s} \).

(3) For \( 0 \leq k < r \) one has \( m_k < m_{k+1} \). For \( 1 \leq k \leq r \), \( M_k \) is a multiplicity-free \( g \)-module, whose highest weight vectors correspond to those \( k \)-dimensional ad-nilpotent ideals \( a \) of \( b \) such that \( \text{Cas}(\langle a \rangle) = m_k \). In fact \( \bigoplus_{k=0}^{r} M_k \) is also a multiplicity-free \( g \)-module.

Proof. (1) For \( j = 1, \cdots, r \), let \( x_j \) (resp. \( y_j \)) be a weight vector corresponding to \( \alpha_j \) (resp. \( -\alpha_j \)). Let \( \{h_1, \cdots, h_t\} \) be a basis of \( \mathfrak{h} \). Then

\[ A = \{x_1, \cdots, x_r, y_1, \cdots, y_r, h_1, \cdots, h_t\} \]

is a basis of \( g \) consisting of weight vectors. Then

\[ B_i = \{a_1 \wedge a_2 \wedge \cdots \wedge a_i \mid a_j \in A\} \]

is a basis of \( \wedge^i g \) consisting of weight vectors. Let

\[ C_i = \{v \in B_i \mid \text{Cas}(\langle a(v) \rangle) = m_i\} \]

Then by Corollary 2.1 of [2], \( M_i \) is the direct sum of simple \( g \)-modules with highest weight vectors \( v \in C_i \). It is clear that

\[ \text{Cas}(\langle a(v) \rangle) = ||\rho + \gamma_1 + \cdots + \gamma_i||^2 - ||\rho||^2 \]

if the weight of \( a_j \) is \( \gamma_j \); thus (1) follows.

(2) For any \( S = \{\gamma_j \mid j = 1, \cdots, i\} \subset \Gamma(g) \), \( \langle S \rangle \) is a weight of \( \pi_{2\rho} \) by Proposition 2.1. Thus by Lemma 2.2 \( \text{Cas}(\langle S \rangle) \leq \text{Cas}(2\rho) = 8||\rho||^2 = n/3 \), as \( ||\rho||^2 = n/24 \) by Freudenthal’s strange formula. So \( m_i = n/3 \) if and only if there exists \( S \subset \Gamma(g) \) such that \( |S| = i \) and \( \langle S \rangle = 2\rho \). Then \( S \) must be of the form \( \{x_1, \cdots, x_r, h_{j_1}, \cdots, h_{j_s}\} \) and thus \( r \leq i \leq r + l \). For \( 0 \leq s \leq l \), it is clear that

\[ C_{r+s} = \{x_1 \wedge \cdots \wedge x_r \wedge h_{j_1} \wedge \cdots \wedge h_{j_s} \mid 1 \leq j_1 < j_2 < \cdots < j_s \leq l\} \]

thus \( M_{r+s} = \left( \frac{1}{s} \right) V_{2s} \).
(3) We first show that \( m_{k+1} > m_k \) for \( 0 \leq k < r \), which clearly holds in the case \( k = 0 \). Assume \( 1 \leq k < r \). Let \( v = a_1 \wedge \cdots \wedge a_k \in C_k \). Then \( v \) is a highest weight vector of \( M_k \), whose weight is \( \langle S \rangle \) with \( S = \Gamma(a(v)) \). Then \( \text{Cas}(\langle S \rangle) = m_k \), and \([b, a(v)] \subset a(v)\) by Theorem 3.1 (1). Recall that for \( \gamma = \sum_{i=1}^l k_i \gamma_i \in \Delta^+ \), where \( \{ \gamma_i | i = 1, \ldots, l \} \) is the set of simple roots, its height is defined as \( \sum_{i=1}^l k_i \). Choose a positive root \( \alpha \) in \( \Delta^+ \setminus S \) (which is nonempty as \( k < r \)) with largest height. Set \( T = S \cup \{ \alpha \} \). Let \( a \in A \) be the \( \alpha \)-weight vector and let \( u = v \wedge a \in B_{k+1} \). By the choice of \( \alpha \) it is clear that \([b, a(u)] \subset a(u)\); thus \( u \) is also a highest weight vector, whose weight is \( \langle T \rangle = \langle S \rangle + \alpha \). As

\[
\langle \langle T \rangle, \alpha \rangle = \langle \langle S \rangle, \alpha \rangle + \langle \alpha, \alpha \rangle > 0,
\]

\( \langle S \rangle \in \Gamma(V_\lambda) \) with \( \lambda = \langle T \rangle \). Then

\[
m_{k+1} \geq \text{Cas}(\langle T \rangle) > \text{Cas}(\langle S \rangle) = m_k.
\]

Now assume \( 1 \leq k \leq r \). Let \( v = a_1 \wedge \cdots \wedge a_r \in C_k \), and let \( S = \Gamma(a(v)) \). We will show that \( S \subset \Delta^+ \). If not, let \( S' = S \setminus (S \cap (-S)) \). Then \( \langle S' \rangle = \langle S \rangle \) and \( |S'| = t < k \). Thus \( m_k = \text{Cas}(S) = \text{Cas}(S') \leq m_t \), which contradicts the previous result. Thus for \( 1 \leq k \leq r \) one always has \( S \subset \Delta^+ \).

Any \( v \in C_k \) is a highest weight vector, so \([b, a(v)] \subset a(v)\). If \( 1 \leq k \leq r \), we have just showed \( \Gamma(a(v)) \subset \Delta^+ \). Thus \( a(v) \) is an ad-nilpotent ideal of \( b \). Let \( \lambda(v) = (a(v)) \). Then

\[
M_k = \bigoplus_{v \in C_k} V_{\lambda(v)}.
\]

By Theorem 3.1 (2), if \( v_1, v_2 \in C_k \) with \( v_1 \neq v_2 \), then \( a(v_1) \neq a(v_2) \) and \( \lambda(v_1) \neq \lambda(v_2) \). Thus \( M_k \) is a multiplicity-free \( g \)-module, whose highest weight vectors corresponding to the ad-nilpotent ideals \( a \) of \( b \) such that \( \text{Cas}(\langle a \rangle) = m_k \). By Theorem 3.1 (2) one can further get that \( \bigoplus_{k=0}^r M_k \) is also a multiplicity-free \( g \)-module. \( \square \)

Note that Kostant already showed in Remark 1.2 of \([4]\) that \( m_i \leq n/3 \) for all \( i \) using different arguments.

Remark 3.3. Considering the isomorphism of \( g \)-modules \( \wedge^k g \) and \( \wedge^{n-k} g \), \( \wedge^k g \) is multiplicity-free for \( 0 \leq k \leq r \) and \( n-r \leq k \leq n \). For \( r \leq k \leq r+l \) \( (r+l = n-r) \), we have showed that \( M_k \) is primary of type \( \pi_{2\rho} \). As \( g \)-modules one has \( \wedge^l g = 2^l V_\rho \otimes V_\rho \) (see \([3]\) ), so \( \wedge g \) contains exactly \( 2^l \) copies of \( V_\rho \), which is just \( \bigoplus_{s=0}^l M_{r+s} \).

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