Continuous-time autoregressive moving-average processes in Hilbert space

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Abstract We introduce the class of continuous-time autoregressive moving-average (CARMA) processes in Hilbert spaces. As driving noises of these processes we consider Lévy processes in Hilbert space. We provide the basic definitions, show relevant properties of these processes and establish the equivalents of CARMA processes on the real line. Finally, CARMA processes in Hilbert space are linked to the stochastic wave equation and functional autoregressive processes.

1 Introduction

Continuous-time autoregressive moving-average processes, or CARMA for short, play an important role in modelling the stochastic dynamics of various phenomena like wind speed, temperature variations and economic indices. For example, based on such models, in [1] the author analyses fixed-income markets while in [8] and [13] the dynamics of weather factors at various locations in Europe and Asia are modelled. Finally, in [7], [4] and [17] continuous-time autoregressive models for commodity markets like power and oil are studied.

CARMA processes constitute the continuous-time version of autoregressive moving-average time series models. In this paper we generalize these processes to a Hilbert space context. The crucial ingredient in the extension is a ”multivariate” Ornstein-Uhlenbeck process with values in a Hilbert space. There already exists an analysis of infinite dimensional Lévy-driven Ornstein-Uhlenbeck processes, and we refer the reader to the survey [3]. Moreover, matrix-valued operators and their
semigroups play an important role. In [12] a detailed semigroup theory for such operators is developed. We review some of the results from [3] and [12] in the context of Hilbert-valued CARMA processes, as well as providing some new results for these processes.

Let us recall the definition of a real-valued CARMA process. We follow [11] and first introduce the multivariate Ornstein-Uhlenbeck process \( \{Z(t)\}_{t \geq 0} \) with values in \( \mathbb{R}^p \) for \( p \in \mathbb{N} \) by

\[
dZ(t) = C_p Z(t) \, dt + e_p \, dL(t), \quad Z(0) = Z_0 \in \mathbb{R}^p.
\]

Here, \( L \) is a one-dimensional square integrable Lévy process with zero mean defined on a complete probability space \( (\Omega, \mathcal{F}, P) \) with filtration \( \mathcal{F} = \{\mathcal{F}_t\}_{t \geq 0} \), satisfying the usual hypotheses. Furthermore, \( e_i \) is the \( i \)th canonical unit vector in \( \mathbb{R}^p \), \( i = 1, \ldots, p \). The \( p \times p \) matrix \( C_p \) takes the particular form

\[
C_p = \begin{bmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \vdots & \ddots & 0 & 1 \\
-\alpha_p & -\alpha_{p-1} & \cdots & -\alpha_1 &
\end{bmatrix},
\]

for constants \( \alpha_i > 0, i = 1, \ldots, p \).

We define a continuous-time autoregressive process of order \( p \) by

\[
X(t) = e_1^\top Z(t), \quad t \geq 0,
\]

where \( e_1^\top \) means the transpose of \( e \in \mathbb{R}^p \). We say that \( \{X(t)\}_{t \geq 0} \) is a CAR(\( p \))-process. For \( q \in \mathbb{N} \) with \( p > q \), we define a CARMA(\( p, q \))-process by

\[
X(t) = b^\top Z(t), \quad t \geq 0,
\]

where \( b \in \mathbb{R}^p \) is the vector \( b = (b_0, b_1, \ldots, b_{q-1}, 1, 0, \ldots, 0)^\top \in \mathbb{R}^p \), where \( b_0 = 1 \) and \( b_i = 0, i = q + 1, \ldots, p - 1 \). Note that \( b = e_1 \) yields a CAR(\( p \))-process. Using the Euler-Maryuama approximation scheme, the CARMA(\( p, q \))-process \( \{X(t)\}_{t \geq 0} \) on a discretized time grid can be related to an autoregressive moving average time series process of order \( p, q \) (see [8] Eq. (4.17)). An explicit dynamics of the CARMA(\( p, q \))-process \( \{X(t)\}_{t \geq 0} \) is (see e.g. [9] Lemma 10.1)

\[
X(t) = b^\top \exp(tC_p)Z_0 + \int_0^t b^\top \exp((t-s)C_p)e_p \, dL(s),
\]

where \( \exp(tC_p) \) is the matrix exponential of \( tC_p \), the matrix \( C_p \) multiplied by time \( t \).

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1 The odd labelling of these constants is explained by the relationship with autoregressive time series, where \( \alpha_i \) is related (but not one-to-one) to the regression of lag \( i, i = 1, \ldots, p \).
If $C_p$ has only eigenvalues with negative real part, then the process $X$ admits a limiting distribution $\mu_X$ with characteristic exponent (see (11))

$$\hat{\mu}_X(z) := \lim_{t \to \infty} \log \mathbb{E} \left[ e^{izX(t)} \right] = \int_0^\infty \mathbb{E} \left[ (b^\top \exp(sC_p)e_p)^2 \right] ds.$$  

Here, $\Psi_b$ is the log-characteristic function of $L(1)$ (see e.g. [2]) and log the distinguished logarithm (see e.g. [19], Lemma 7.6). In particular, if $L = \sigma B$ with $\sigma > 0$ constant and $B$ a standard Brownian motion, we find

$$\hat{\mu}_X(z) = -\frac{1}{2} e^{2\sigma^2} \int_0^\infty (b^\top \exp(sC_p)e_p)^2 ds,$$

and thus $X$ has a Gaussian limiting distribution $\mu_X$ with zero mean and variance $\sigma^2 \int_0^\infty (b^\top \exp(sC_p)e_p)^2 ds$.

When $X$ admits a limiting distribution, we have a stationary representation of the process $X$ such that $X(t) \sim \mu_X$ for all $t \in \mathbb{R}$, namely,

$$X(t) = \int_{-\infty}^t b^\top \exp((t-s)C_p)e_p dL(s), \quad (6)$$

where $L$ is now a two-sided Lévy process. This links CARMA($p,q$)-processes to the more general class of Lévy semistationary (LSS) processes, defined in [4] as

$$X(t) := \int_{-\infty}^t g(t-s)\sigma(s)dL(s), \quad (7)$$

for $g: \mathbb{R}_+ \to \mathbb{R}$ being a measurable function and $\sigma$ a predictable process such that $s \mapsto g(t-s)\sigma(s)$ for $s \leq t$ is integrable with respect to $L$. Indeed, LSS processes are again a special case of so-called ambit fields, which are spatio-temporal stochastic processes originally developed in [5] for modelling turbulence. In fact, the infinite dimensional CARMA processes that we are going to define in this paper will form a subclass of ambit fields, as we will see in Section 4. We note that CARMA processes with values in $\mathbb{R}^n$ have been defined and analysed by [16], [20] and recently in [14].

2 Definition of CARMA processes in Hilbert space

Given $p \in \mathbb{N}$, and let $H_i$ for $i = 1, \ldots, p$ be separable Hilbert spaces with inner products denoted by $\langle \cdot, \cdot \rangle_i$ and associated norms $| \cdot |_i$. We define the product space $H := H_1 \times \ldots \times H_p$, which is again a separable Hilbert space equipped with the inner product $\langle x, y \rangle := \sum_{i=1}^p \langle x_i, y_i \rangle_i$ and the induced norm denoted $| \cdot | = \sum_{i=1}^p | \cdot |_i$ for $x = (x_1, \ldots, x_p), y = (y_1, \ldots, y_p) \in H$. The projection operator $\mathcal{P}_i : H \to H_i$ is defined as $\mathcal{P}_i x = x_i$ for $x \in H, i = 1, \ldots, p$. It is straightforward to see that its adjoint $\mathcal{P}_i^* : H_i \to H$ is given by $\mathcal{P}_i^* x = (0, \ldots, 0, x_i, 0, \ldots, 0)$ for $x \in H_i$, where the $x$ appears in the $i$th coordinate of the vector consisting of $p$ elements. If $U$ and $V$
are two separable Hilbert spaces, we denote $L(U, V)$ the Banach space of bounded linear operators from $U$ to $V$, equipped with the operator norm $\| \cdot \|_{\text{op}}$. The Hilbert-Schmidt norm for operators in $L(U, V)$ is denoted $\| \cdot \|_{\text{HS}}$, and $L^2(U, V)$ denotes the space of Hilbert-Schmidt operators. If $U = V$, we simply write $L(U)$ for $L(U, U)$.

Let $A_i : H_{p+1-i} \to H_p$, $i = 1, \ldots, p$ be $p$ (unbounded) densely defined linear operators, and $I_i : H_{p+2-i} \to H_{p+1-i}$, $i = 2, \ldots, p$ be another $p - 1$ (unbounded) densely defined linear operators. Define the linear operator $\mathcal{C}_p : H \to H$ represented as a $p \times p$ matrix of operators

$$
\mathcal{C}_p = \begin{bmatrix}
0 & I_p & 0 & \ldots & 0 \\
0 & 0 & I_{p-1} & \ldots & 0 \\
\vdots & \vdots & \ddots & \ldots & \ddots \\
0 & \ldots & \ldots & I_2 & \ldots & I_1 \\
A_p & A_{p-1} & \ldots & A_1 \\
\end{bmatrix}
$$

(8)

Since the $A_i$'s and $I_i$'s are densely defined, $\mathcal{C}_p$ has domain

$$
\text{Dom}(\mathcal{C}_p) = \text{Dom}(A_p) \times (\text{Dom}(A_{p-1}) \cap \text{Dom}(I_p)) \times \ldots \times (\text{Dom}(A_1) \cap \text{Dom}(I_2)),
$$

which we suppose is dense in $H$. We note in passing that typically, $H_1 = H_2 = \ldots = H_p$ and $I_i = \text{Id}$, the identity operator on $H_i$, $i = 1, \ldots, p$. Then $\text{Dom}(\mathcal{C}_p) = \text{Dom}(A_p) \times \text{Dom}(A_{p-1}) \times \ldots \times \text{Dom}(A_1)$, which is dense in $H$.

On a complete probability space $(\Omega, \mathcal{F}, P)$ with filtration $\mathcal{F} = \{\mathcal{F}_t\}_{t \geq 0}$ satisfying the usual hypotheses, denote by $L := \{L(t)\}_{t \geq 0}$ a zero-mean square-integrable $H_p$-valued Lévy process with covariance operator $Q$ (i.e., a symmetric non-negative definite trace class operator), see e.g. [18, Sect. 4.9]. Consider the following stochastic differential equation. For $t \geq 0$,

$$
dZ(t) = \mathcal{C}_p Z(t) dt + \mathcal{P}_p \mathcal{L}(t), \quad Z(0) := Z_0 \in H. \quad (9)
$$

The next proposition states an explicit expression for $Z := \{Z(t)\}_{t \geq 0}$:

**Proposition 1.** Assume that $\mathcal{C}_p$ defined in (8) is the generator of a $C_0$-semigroup \{\mathcal{S}_p(t)\}_{t \geq 0} on H. Then the $H$-valued stochastic process $Z$ given by

$$
Z(t) = \mathcal{S}_p(t) Z_0 + \int_0^t \mathcal{S}_p(t-s) \mathcal{P}_p \mathcal{L}(s) \, ds
$$

is the unique mild solution of (9).

**Proof.** We have that $\mathcal{S}_p(t-s) \mathcal{P}_p \in L(H_p, H)$, and moreover, since $\| \mathcal{P}_p \|_{\text{op}} = 1$ it follows

$$
\| \mathcal{S}_p(t-s) \mathcal{P}_p \|_{\text{op}} \leq 1, \quad \| \mathcal{S}_p(t-s) \|_{\text{op}} \| \mathcal{P}_p \|_{\text{op}} \leq K e^{c(t-s)} \| Q \|_{\text{HS}}^{1/2} \| Q \|_{\text{HS}}^{1/2}
$$

by the general exponential growth bound on the operator norm of a $C_0$-semigroup (see e.g. [12, Prop. I.5.5]). Thus, for all $t \geq 0$,
\[
\int_0^t \| \mathcal{P}(t-s) \mathcal{P}^*_p Q^{1/2} \|^2_{HS} ds \leq \frac{K}{2c} e^{2ct} \| Q^{1/2} \|^2_{HS} < \infty
\]

because \( Q \) is trace class by assumption. The stochastic integral with respect to \( L \) in the definition of \( Z \) is therefore well-defined. Hence, the result follows directly from the definition of mild solutions in [18, Def. 9.5].

From now on we restrict our attention to operators \( C_p \) in (8) which admit a \( C^0 \)-semigroup \( \{ S_p(t) \}_{t \geq 0} \). We remark in passing that in the next section we will provide a recursive definition of the semigroup \( \{ S_p(t) \}_{t \geq 0} \) in a special situation where all involved operators are bounded except \( A_1 \).

A CARMA process with values in a Hilbert space is defined next:

**Definition 1.** Let \( U \) be a separable Hilbert space. For \( L_U \in L(H, U) \), define the \( U \)-valued stochastic process \( X := \{ X(t) \}_{t \geq 0} \) by

\[
X(t) := L_U S_p(t) Z_0 + \int_0^t L_U S_p(t-s) P^*_p dL(s),
\]

for \( t \geq 0 \). Note that by linearity of the stochastic integral we can move the operator \( L_U \) inside. Furthermore, the stochastic integral is well-defined since \( L_U \in L(H, U) \) and thus has a finite operator norm.

The continuous-time autoregressive (CAR) processes constitute a particularly interesting subclass of the CARMA\((p, U, L_U)\)-processes:

**Definition 2.** The CARMA\((p, H_1, \mathcal{P}_1)\)-process \( \{ X(t) \}_{t \geq 0} \) from Definition 1 is called an \( H_1 \)-valued CAR\((p)\)-process.

The explicit dynamics of an \( H_1 \)-valued CAR\((p)\)-process becomes

\[
X(t) = \mathcal{P}_1 S_p(t) Z_0 + \int_0^t \mathcal{P}_1 S_p(t-s) \mathcal{P}^*_p dL(s),
\]

for \( t \geq 0 \). In this paper we will be particularly focused on \( H_1 \)-valued CAR\((p)\)-processes.
Remark that the process $\mathbf{L} := \mathcal{P}_p L$ defines an $H$-valued Lévy process which has mean zero and is square integrable. Its covariance operator is easily seen to be $\mathcal{P}_p^* Q \mathcal{P}_p$.

It is immediate to see that an $H_1$-valued CAR(1) process is an Ornstein-Uhlenbeck process defined on $H_1$, with

$$dX(t) = A_1X(t)dt + dL(t),$$

and thus

$$X(t) = \mathcal{J}_1(t)Z_0 + \int_0^t \mathcal{J}_1(t-s)dL(s),$$

being its mild solution.

An $H_1$-valued CAR($p$) process for $p > 1$ can be viewed as a higher-order (indeed, a $p$th order) stochastic differential equation, as we now discuss. To this end, suppose that $\text{Ran}(A_q) \subset \text{Dom}(I_2)$ and $\text{Ran}(I_2) \subset \text{Dom}(I_{q+1})$, and assume that there exist $p-1$ linear (unbounded) operators $B_1, B_2, \ldots, B_{p-1}$ such that

$$I_p \cdots I_2A_q = B_q I_p \cdots I_{q+1}. \quad (12)$$

for $q = 1, \ldots, p-1$. We note that $I_p \cdots I_2 : H_p \rightarrow H_1$ and hence $I_p \cdots I_2A_q : H_{p+1-q} \rightarrow H_1$. Moreover, $I_p \cdots I_{q+1} : H_{p+1-q} \rightarrow H_1$, and therefore $B_q : H_1 \rightarrow H_1$. We suppose that $\text{Dom}(B_q)$ is so that $\text{Dom}(B_q I_p \cdots I_{q+1}) = \text{Dom}(A_q)$, and we note that $\text{Dom}(A_q)$ is the domain of the operator $I_p \cdots I_2A_q$. For completeness, we define the operator $B_p : H_1 \rightarrow H_1$ as

$$B_p := I_p \cdots I_2A_p. \quad (13)$$

We see that this definition is consistent with the inductive relations in (12) (we suppose that $B_p$ is a linear (possibly unbounded) operator with domain $\text{Dom}(B_p) = \text{Dom}(A_p)$). With this at hand, we introduce the operator-valued $p$th-order polynomial $Q_p(\lambda)$ for $\lambda \in \mathbb{C}$,

$$Q_p(\lambda) = \lambda^p - B_1\lambda^{p-1} - B_2\lambda^{p-2} - \cdots - B_{p-1}\lambda - B_p. \quad (14)$$

By definition, $X(t) = \mathcal{P}_1 Z(t)$, which is the first coordinate in the vector $Z(t) = (Z_1(t), \ldots, Z_p(t))^\top \in H$. From (13) and the definition of the operator matrix $\mathcal{C}_p$ in (8), we find that $Z'_1(t) = I_pZ_2(t), Z'_2(t) = I_{p-1}Z_3(t), \ldots, Z'_{p-1}(t) = I_2Z_p(t)$ and finally

$$Z'_p(t) = A_pZ_1(t) + \cdots + A_1Z_1(t) + L(t).$$

Here, $L(t)$ is the formal time derivative of $L$. By iteration, we find that $Z^{(q)}_1(t) = I_pI_{p-1} \cdots I_{p-(q-1)}Z_{q+1}(t)$ for $q = 1, \ldots, p-1$. Thus,
Hilbert-valued CARMA processes

\[ Z_i^{(p)} = \frac{d}{dt} Z_i^{(p-1)} = I_p \cdots I_2 Z_p'(t) \]

\[ = I_p \cdots I_2 A_p Z_1(t) + I_p \cdots I_2 A_{p-1} Z_2(t) + \cdots + I_p \cdots I_2 A_1 Z_p(t) + I_p \cdots I_2 \dot{L}(t) \]

\[ = B_p Z_1(t) + B_{p-1} Z_2(t) + \cdots + B_1 Z_2^{(p)}(t) + \cdots + B_1 Z_1^{(p-1)}(t) + I_p \cdots I_2 \dot{L}(t). \]

In the last equality we made use of (12) and (13). We find that an \( H_1 \)-valued CAR(\( p \)) process \( X(t) = \mathcal{P}_1 Z(t) \) can be viewed as a solution to the \( p \)-th order stochastic differential equation formally expressed by

\[ Q_p \left( \frac{d}{dt} \right) X(t) = I_p \cdots I_2 \dot{L}(t). \tag{15} \]

Re-expressing Eq. (15) we find the stochastic differential equation

\[ dX^{(p-1)}(t) = \left( \sum_{q=1}^{p} B_q X^{(p-q)}(t) \right) dt + I_p \cdots I_2 dL(t). \tag{16} \]

If \( H_1 = \cdots = H_p \) and \( \mathcal{C}_p \) is a bounded operator, then \( B_q = I_q \cdots I_2 A_q \) in (12) whenever \( I_q \cdots I_2 A_q \) commutes with \( I_p \cdots I_{q+1} \). In this sense the condition (12) is a specific commutation relationship on \( A_p \) and the operators \( I_2, \ldots, I_p \). In the particular case \( I_i = \text{Id} \) for \( i = 2, \ldots, p \), then we trivially have \( A_q = B_q \) for \( q = 1, \ldots, p \).

We end this section with showing that the stochastic wave equation can be viewed as an example of a Hilbert-valued CAR(2)-process. To this end, let \( H_2 := L^2(0,1) \), the space of square integrable functions on the unit interval, and consider the stochastic partial differential equation

\[ \frac{\partial^2 Y(t,x)}{\partial t^2} = \frac{\partial^2 Y(t,x)}{\partial x^2} + \frac{\partial L(t,x)}{\partial t}, \tag{17} \]

with \( t \geq 0 \) and \( x \in (0,1) \). We can rephrase this wave equation as

\[ d \left[ \begin{array}{c} Y(t,x) \\ \frac{\partial Y(t,x)}{\partial t} \end{array} \right] = \left[ \begin{array}{ccc} 0 & \text{Id} \\ \Delta & 0 \end{array} \right] \left[ \begin{array}{c} Y(t,x) \\ \frac{\partial Y(t,x)}{\partial t} \end{array} \right] dt + \left[ \begin{array}{c} 0 \\ \text{dL}(t,x) \end{array} \right], \tag{18} \]

with \( \Delta = \partial^2 / \partial x^2 \) being the Laplace operator. The eigenvectors \( e_n(x) := \sqrt{2} \sin(\pi n x) \), \( n \in \mathbb{N} \), for \( \Delta \) form an orthonormal basis of \( L^2(0,1) \). Introduce the Hilbert space \( H_1 \) as the subspace of \( L^2(0,1) \) for which \( |f|_2^2 := \pi^2 \sum_{n=1}^{\infty} n^2 \langle f, e_n \rangle_2^2 < \infty \). Following Example B.13 in [18],

\[ \mathcal{C}_2 = \left[ \begin{array}{ccc} 0 & \text{Id} \\ \Delta & 0 \end{array} \right] \]

generates a \( C_0 \)-semigroup \( \mathcal{S}_2(t) \) on \( H := H_1 \times H_2 \). The Laplace operator \( \Delta \) is a self-adjoint negative definite operator on \( H_1 \). The semigroup \( \mathcal{S}_2(t) \) can be represented as

\[ \mathcal{S}_2(t) = \left[ \begin{array}{ccc} \cos(-\Delta)^{1/2} t & -(\Delta)^{-1/2} \sin(-\Delta)^{1/2} t \\ -(\Delta)^{1/2} \sin(-\Delta)^{1/2} t & \cos(-\Delta)^{1/2} t \end{array} \right]. \tag{19} \]
In the previous equality, we define for a real-valued function \( g \) the linear operator \( g(\Delta) \) using functional calculus, i.e., \( g(\Delta)f = \sum_{n=1}^{\infty} g(-\pi^2 n^2) \langle f, e_n \rangle e_n \) whenever this sum converges. These considerations show that the wave equation is a specific example of a CAR(2)-process.

### 3 Analysis of CARMA processes

In this section we derive some fundamental properties of CARMA processes in Hilbert spaces.

#### 3.1 Distributional properties

We state the conditional characteristic functional of a CARMA\((p, U, \mathcal{L}_U)\)-process in the next proposition.

**Proposition 2.** Assume \( X \) is a CARMA\((p, U, \mathcal{L}_U)\)-process. Then, for \( x \in U \),

\[
E \left[ e^{i\langle X(t), x \rangle_U} \mid \mathcal{F}_s \right] = \exp \left( i \langle \mathcal{L}_U \mathcal{P}_p(t) Z_0, x \rangle_U + \int_0^{t-s} \psi_L \langle \mathcal{P}_p \mathcal{P}_p^*(u) \mathcal{L}_U x \rangle_U \, du \right) \times \exp \left( i \int_0^s \mathcal{L}_U \mathcal{P}_p(t-u) \mathcal{P}_p^* dL(u), x \rangle_U \right),
\]

for \( 0 \leq s \leq t \). Here, \( \psi_L \) is the characteristic exponent of the Lévy process \( L \).

**Proof.** From (10) it holds for \( 0 \leq s \leq t \),

\[
X(t) = \mathcal{L}_U \mathcal{P}_p(t) Z_0 + \int_0^t \mathcal{L}_U \mathcal{P}_p(t-u) \mathcal{P}_p^* dL(u) + \int_s^t \mathcal{L}_U \mathcal{P}_p(t-u) \mathcal{P}_p^* dL(u).
\]

The Lévy process has independent increments, and \( \mathcal{F}_s \)-measurability of the first stochastic integral thus yields

\[
E \left[ e^{i\langle X(t), x \rangle_U} \mid \mathcal{F}_s \right] = \exp \left( i \langle \mathcal{L}_U \mathcal{P}_p(t) Z_0, x \rangle_U + i \int_0^s \mathcal{L}_U \mathcal{P}_p(t-u) \mathcal{P}_p^* dL(u), x \rangle_U \right) \times E \left[ \exp \left( i \int_s^t \mathcal{L}_U \mathcal{P}_p(t-u) \mathcal{P}_p^* dL(u), x \rangle_U \right) \right].
\]

The result follows from [18, Chapter 4].

Suppose now that \( L = W \), an \( H_p \)-valued Wiener process. Then the characteristic exponent is

\[
\psi_W(h) = -\frac{1}{2} \langle Qh, h \rangle_p,
\]
for \( h \in H_p \). Hence, from Prop. [2] it follows that,

\[
\mathbb{E} \left[ e^{i(t,x)u} \mid \mathcal{F}_s \right] = \exp \left( i \langle \mathcal{L}_U \mathcal{F}_P(t) Z_0, x \rangle_U + i \int_0^s \mathcal{L}_U \mathcal{F}_P(t-u) \mathcal{P}_P^* dW(u), x \rangle_U \right) \\
\times \exp \left( -\frac{1}{2} \int_0^{t-s} \langle \mathcal{L}_U \mathcal{F}_P(u) \mathcal{P}_P^* Q \mathcal{F}_P^*(u) \mathcal{L}_U x, x \rangle_U du \right)
\]

We find that \( X(t) \mid \mathcal{F}_s \) for \( s \leq t \) is a Gaussian process in \( H_1 \), with mean

\[
\mathbb{E}[X(t) \mid \mathcal{F}_s] = \mathcal{L}_U \mathcal{F}_P(t) Z_0 + \int_0^s \mathcal{W}_U \mathcal{F}_P(t-u) \mathcal{P}_P^* dL(u)
\]

and covariance operator

\[
\text{Var}(X(t) \mid \mathcal{F}_s) = \int_0^{t-s} \mathcal{L}_U \mathcal{F}_P(u) \mathcal{P}_P^* Q \mathcal{F}_P^*(u) L_u du,
\]

where the integral is interpreted in the Bochner sense. If the semigroup \( \mathcal{F}_P(u) \) is exponentially stable, then \( X(t) \mid \mathcal{F}_s \) admits a Gaussian limiting distribution with mean zero and covariance operator

\[
\lim_{t \to \infty} \text{Var}(X(t) \mid \mathcal{F}_s) = \int_0^\infty \mathcal{L}_U \mathcal{F}_P(u) \mathcal{P}_P^* Q \mathcal{F}_P^*(u) L_u du.
\]

This is the invariant measure of \( X \). We remark in passing that measures on \( H \) are defined on its Borel \( \sigma \)-algebra.

In [3] there is an analysis of invariant measures of Lévy-driven Ornstein-Uhlenbeck processes. We discuss this here in the context of the Ornstein-Uhlenbeck process \( \{Z(t)\}_{t \geq 0} \) defined in (9). Assume \( \mu_Z \) is the invariant measure of \( \{Z(t)\}_{t \geq 0} \), and recall the definition of its characteristic exponent \( \hat{\mu}_Z(x) \).

\[
\hat{\mu}_Z(x) = \log \mathbb{E} \left[ e^{i(x,Z(t))} \right].
\]

(20)

Here, \( x \in H \) and \( \log \) is the distinguished logarithm (see e.g. [19, Lemma 7.6]). If \( Z_0 \sim \mu_Z \), then, in distribution, \( Z_0 = Z(t) \) for all \( t \geq 0 \) and it follows that the characteristic exponent of \( \mu_Z \) satisfies,

\[
\hat{\mu}_Z(x) = \hat{\mu}_Z(\mathcal{F}_P(t)x) + \int_0^t \psi_L(\mathcal{P}_P^* u)x du
\]

(21)

for any \( x \in H \) and \( t \geq 0 \). Following [3], \( \mu_Z \) becomes an operator self-decomposable distribution since,

\[
\mu_Z = \mathcal{F}_P(t) \mu_Z * \mu_t.
\]

(22)

Here, \( \mu_t \) is the distribution of \( \int_0^t \mathcal{F}_P(u) \mathcal{P}_P^* dL(u), * \) is the convolution product of measures and \( \mathcal{F}_P(t) \mu_Z := \mu_Z \circ \mathcal{F}_P(t)^{-1} \) is a probability distribution on \( H \), given by
After appealing to the stochastic Fubini theorem, it follows from the explicit expression of \( Z(t) \) that the semigroup \( \{S_p(t)\}_{t \geq 0} \) is exponentially stable if and only if \( \lambda \in \sigma(C_p) \) where \( \sigma(C_p) \) denotes the spectrum of \( C_p \), and only if \( \Re\{\lambda\} < 0 \) for all \( \lambda \in \sigma(C_p) \). Recall from Section 1 that a real-valued CARMA process admits a limiting distribution if and only if all the eigenvalues of \( C_p \) have negative real part. In general, by Thm. V.1.11 in [12], the semigroup \( \{S_p(t)\}_{t \geq 0} \) is exponentially stable if and only if \( \{\lambda \in \mathbb{C} | \Re\{\lambda\} > 0\} \) is a subset of the resolvent set \( \rho(C_p) \) of \( C_p \) and \( \sup_{\Re\{\lambda\} > 0} ||R(\lambda, C_p)|| < \infty \). Here, \( R(\lambda, C_p) \) is the resolvent of \( C_p \) for \( \lambda \in \rho(C_p) \).

### 3.2 Semimartingale representation

Let us study a semimartingale representation of the CAR(\( p \)) process. We have the following proposition:

**Proposition 3.** For \( p \in \mathbb{N} \) with \( p > 1 \), assume that \( C_p \) defined in (8) is the generator of a \( C_0 \)-semigroup \( \{S_p(t)\}_{t \geq 0} \). Then the \( H_1 \)-valued CAR(\( p \)) process \( X \) given in Definition 2 has the representation

\[
X(t) = \mathcal{P}_1 S_p(t) Z_0 + \mathcal{P}_1 C_p \int_0^t \int_0^u S_p(u-s) \mathcal{P}_p^s dL(s) du,
\]

for all \( t \geq 0 \).

**Proof.** From [12] Ch. II, Lemma 1.3, we have that

\[
S_p(t) = \text{Id} + C_p \int_0^t S_p(s) ds.
\]

But for any \( x \in H \), it is simple to see that \( \mathcal{P}_1 \text{Id} \mathcal{P}_p^s x = 0 \) when \( p > 1 \). Therefore it holds

\[
\mathcal{P}_1 S_p(t) \mathcal{P}_p^s = \mathcal{P}_1 C_p \int_0^t S_p(s) \mathcal{P}_p^s ds.
\]

The integral on the right-hand side is in Bochner sense as an integral of operators. After appealing to the stochastic Fubini theorem, it follows from the explicit expression of \( X(t) \) in (11).
\[ X(t) = \mathcal{P}_t \mathcal{I}_p(t) \mathbf{Z}_0 + \int_0^t \mathcal{P}_t \mathcal{I}_p(t-s) \mathcal{P}_p^* dL(s) \]

\[ = \mathcal{P}_t \mathcal{I}_p(t) \mathbf{Z}_0 + \int_0^t \mathcal{P}_t \mathcal{I}_p(u) \mathcal{P}_p^* du dL(s) \]

\[ = \mathcal{P}_t \mathcal{I}_p(t) \mathbf{Z}_0 + \mathcal{P}_t \int_0^t \mathcal{P}_p(u-s) \mathcal{P}_p^* dL(s). \]

We know from [12, Ch. II, Lemma 1.3] that \( \int_0^t \mathcal{I}_p(u-s) \mathcal{P}_p^* du \in \text{Dom}(\mathcal{C}_p) \).

We demonstrate that \( \int_0^t \mathcal{I}_p(u-s) \mathcal{P}_p^* dudL(s) \in \text{Dom}(\mathcal{C}_p) \): First we recall that \( L = \mathcal{P}_p^* L \) is an \( H \)-valued square-integrable Lévy process with mean zero. From the semigroup property,

\[ \frac{1}{h} \left( \mathcal{I}_p(h) \int_0^t \int_0^s \mathcal{I}_p(u-s) dudL(s) - \int_0^t \int_0^s \mathcal{I}_p(u-s) dudL(s) \right) \]

\[ = \frac{1}{h} \int_0^t \int_0^s \mathcal{I}_p(u+h-s) dudL(s) - \frac{1}{h} \int_0^t \int_0^s \mathcal{I}_p(u-s) dudL(s) \]

\[ = \int_0^t \frac{1}{h} \int_{s+h}^s \mathcal{I}_p(v) dv - \frac{1}{h} \int_0^t \mathcal{I}_p(v-s) dL(s) \]

\[ = \int_0^t \frac{1}{h} \int_{s+h}^s \mathcal{I}_p(v) dv - \frac{1}{h} \int_0^t \mathcal{I}_p(v-s) dL(s) \]

\[ = \int_0^t \frac{1}{h} \int_0^h \mathcal{I}_p(u) du \mathcal{I}_p(t-s) - \frac{1}{h} \int_0^t \mathcal{I}_p(u) dL(s) \]

\[ = \frac{1}{h} \int_0^t \mathcal{I}_p(u) du \left( \int_0^h \mathcal{I}_p(t-s) dL(s) - L(t) \right). \]

By the fundamental theorem of calculus for Bochner integrals, \( (1/h) \int_0^t \mathcal{I}_p(u) du \to \text{Id} \) when \( h \downarrow 0 \). Therefore, the limit exists and the claim follows. From this we find that

\[ X(t) = \mathcal{P}_t \mathcal{I}_p(t) \mathbf{Z}_0 + \mathcal{P}_t \mathcal{I}_p^\flat \int_0^t \mathcal{I}_p(u-s) \mathcal{P}_p^* dL(s) \]

\[ = \mathcal{P}_t \mathcal{I}_p(t) \mathbf{Z}_0 + \mathcal{P}_t \int_0^t \mathcal{P}_p(u-s) \mathcal{P}_p^* dL(s) du. \]

In the last equality, we applied the stochastic Fubini Theorem (see e.g. [18] Thm. 8.14). Hence, the result follows.

Note that if \( \mathbf{Z}_0 \in \text{Dom}(\mathcal{C}_p) \), then by [12, Ch. II, Lemma 1.3] \( t \mapsto \mathcal{P}_t \mathcal{I}_p(t) \mathbf{Z}_0 \) are differentiable. Assuming that \( \int_0^t \mathcal{I}_p(t-s) \mathcal{P}_p^* dL(s) \in \text{Dom}(\mathcal{C}_p) \), it follows from the Proposition above that the paths \( t \mapsto X(t), t \geq 0 \) of \( X \) is differentiable, with

\[ X'(t) = \mathcal{P}_t \mathcal{I}_p^\flat \mathcal{I}_p(t) \mathbf{Z}_0 + \mathcal{P}_t \mathcal{I}_p^\flat \int_0^t \mathcal{I}_p(t-s) \mathcal{P}_p^* dL(s), \tag{23} \]

for \( t \geq 0 \). The stochastic integral in (23) has RCLL (cadlag) paths, and therefore the \( H_1 \)-valued CAR(\( p \))-processes for \( p > 1 \) have differentiable paths being RCLL.
If \( L = W \), an \( H_p \)-valued Wiener process, then the stochastic integral has continuous paths and the paths of \( X \) become continuously differentiable. We point out that \( p > 1 \) is very different from \( p = 1 \) in this respect, as the Ornstein-Uhlenbeck process

\[
X(t) = \mathcal{S}_1(t)Z_0 + \int_0^t \mathcal{S}_1(t-s)dL(s)
\]

\[
= \mathcal{S}_1(t)Z_0 + L(t) + \int_0^t \int_0^u \mathcal{S}_1(u-s)dL(s)du,
\]

does not have differentiable paths except in the trivial case when the Lévy process is simply a drift. It is straightforward to define an \( H_p \)-valued Lévy process

\[
\mathcal{L} = \mathcal{L}_g \quad \text{for} \quad g \in \text{Dom}(A_1) \cap \text{Dom}(I_2).
\]

We apply the proposition above to give a recursive description of the \( C_0 \)-semigroup \( \{ \mathcal{S}_p(t) \}_{t \geq 0} \) with \( C_p \) as generator, where we recall \( C_p \) from (8). The following result is known as the variation-of-constants formula (see e.g. [12, Appendix B.1.1 and Thm. B.5]) and turns out to be convenient when expressing the semigroup for \( C_p \).

**Proposition 4.** Let \( \mathcal{A} \) be a linear operator on \( H \) being the generator of a \( C_0 \)-semigroup \( \{ \mathcal{S}_t \}_{t \geq 0} \). Assume that \( \mathcal{B} \in L(H) \). Then the operator \( \mathcal{A} + \mathcal{B} : \text{Dom}(\mathcal{A}) \rightarrow H \) is the generator of the \( C_0 \)-semigroup \( \{ \mathcal{S}_t \}_{t \geq 0} \) defined by

\[
\mathcal{S}_t = \mathcal{S}_t + \mathcal{B}(t),
\]

where

\[
\mathcal{B}(t) = \sum_{n=1}^\infty \mathcal{B}_n(t),
\]

and

\[
\mathcal{B}_{n+1}(t) = \int_0^t \mathcal{S}_s(t-s)\mathcal{B}_n(s)ds,
\]

for \( n = 0, 1, 2, \ldots \), with \( \mathcal{B}_0(t) = \mathcal{S}_t(t) \).

We apply the proposition above to give a recursive description of the \( C_0 \)-semigroup of \( C_p \).

**Proposition 5.** Given the operator \( C_p \) defined in (8) for \( p \in \mathbb{N} \), where \( C_1 = A_1 \) is a densely defined linear operator on \( H_p \) (possibly unbounded) with \( C_0 \)-semigroup
For \( p > 1 \), assume that \( I_p \in L(H_2, H_1) \), \( A_p \in L(H_1, H_p) \) and \( \mathcal{C}_{p-1} \) is a densely defined operator on \( H_2 \times \cdots \times H_p \) with a \( C_0 \)-semigroup \( \{ \mathcal{S}_{p-1}(t) \}_{t \geq 0} \), then

\[
\mathcal{S}_p(t) = \mathcal{S}_{p-1}^+(t) + \sum_{n=1}^{\infty} \mathcal{R}_{n,p}(t),
\]

where \( \mathcal{R}_{0,p}(t) = \mathcal{S}_{p-1}^+(t) \) and for \( n = 1, 2, \ldots \),

\[
\mathcal{R}_{n+1,p}(t) = \int_0^t \mathcal{S}_{p-1}^+(t-s) \mathcal{B}_p \mathcal{R}_{n,p}(s) \, ds.
\]

Here, \( \mathcal{B}_p \in L(H) \) is

\[
\mathcal{B}_p = \begin{bmatrix}
0 & I_p & 0 & \ldots & 0 \\
0 & 0 & 0 & \ldots & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots \\
0 & \ldots & \ldots & \ldots & 0 \\
A_p & 0 & \ldots & 0 & 0
\end{bmatrix}
\]

and \( \mathcal{S}_{p-1}^+ \in L(H) \)

\[
\mathcal{S}_{p-1}^+ = \begin{bmatrix}
\text{Id} & 0 & \ldots & 0 \\
0 & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots \\
0 & \ldots & \ldots & \ldots & 0 \\
0 & \ldots & \ldots & \ldots & 0
\end{bmatrix},
\]

for \( \text{Id} \) being the identity operator on \( H_1 \).

Proof. By assumption, \( I_p \in L(H_2, H_1) \) and \( A_p \in L(H_1, H_p) \), and thus \( \mathcal{B}_p \in L(H) \). Define

\[
\mathcal{A}_p = \begin{bmatrix}
0 & 0 & \ldots & 0 \\
0 & 0 & \ldots & 0 \\
\vdots & \ddots & \ddots & \ddots \\
0 & \ldots & \ldots & \ldots & 0 \\
A_p & 0 & \ldots & 0 & 0
\end{bmatrix}.
\]

Then, \( \mathcal{A}_p + \mathcal{B}_p = \mathcal{C}_p \). Moreover, \( \{ \mathcal{S}_{p-1}^+(t) \}_{t \geq 0} \) is the \( C_0 \)-semigroup of \( \mathcal{A}_p \). Hence, the result follows from Proposition 4.

As an example, consider \( p = 3 \). Then we have

\[
\mathcal{C}_3 = \begin{bmatrix}
0 & I_3 & 0 \\
0 & 0 & I_2 \\
A_3 & A_2 & A_1
\end{bmatrix}.
\]

First, \( \mathcal{C}_1 = A_1 \) is a (possibly unbounded) operator on \( H_3 \), with \( C_0 \)-semigroup \( \{ \mathcal{S}_1(t) \}_{t \geq 0} \subset L(H_3) \). Next, let

\[
\mathcal{B}_2 = \begin{bmatrix}
0 & I_2 \\
A_2 & 0
\end{bmatrix}.
where we assume \( I_2 \in L(H_1, H_2) \) and \( A_2 \in L(H_2, H_3) \) to have \( D_2 \in L(H_2 \times H_3) \). With
\[
\mathcal{S}_1^+(t) = \begin{bmatrix} \text{Id} & 0 \\ 0 & \mathcal{S}_1(t) \end{bmatrix},
\]
which defines a \( C_0 \)-semigroup on \( L(H_2 \times H_3) \) with generator
\[
\mathcal{A}_1 = \begin{bmatrix} 0 & 0 \\ 0 & A_1 \end{bmatrix},
\]
we obtain
\[
\mathcal{S}_2(t) = \mathcal{S}_1^+(t) + \sum_{n=1}^{\infty} \mathcal{R}_{n,2}(t)
\]
for \( \mathcal{R}_{0,2} = \mathcal{S}_1^+(t) \) and
\[
\mathcal{R}_{n+1,2}(t) = \int_0^t \mathcal{S}_1^+(t-s) \mathcal{R}_{n,2}(s) ds, n = 1, 2, \ldots.
\]
We note that \( \{ \mathcal{S}_2 \}_{t \geq 0} \subset L(H_2 \times H_3) \) is the \( C_0 \)-semigroup with generator \( \mathcal{C}_2 \) densely defined on \( H_2 \times H_3 \). Finally, let
\[
\mathcal{B}_3 = \begin{bmatrix} 0 & I_3 & 0 \\ 0 & 0 & 0 \\ A_3 & 0 & 0 \end{bmatrix}
\]
which is a bounded operator on \( H \) after assuming \( I_3 \in L(H_2, H_1) \) and \( A_3 \in L(H_1, H_3) \). With
\[
\mathcal{S}_3(t) = \begin{bmatrix} \text{Id} & 0 \\ 0 & \mathcal{S}_3(t) \\ 0 & \mathcal{C}_2 \end{bmatrix},
\]
which is a \( C_0 \)-semigroup on \( L(H) \) with generator
\[
\mathcal{A}_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \mathcal{C}_2 \\ 0 & \mathcal{C}_2 \end{bmatrix},
\]
we conclude with
\[
\mathcal{S}_3(t) = \mathcal{S}_2^+(t) + \sum_{n=1}^{\infty} \mathcal{R}_{n,3}(t),
\]
where \( \mathcal{R}_{0,3}(t) = \mathcal{S}_2^+(t) \) and
\[
\mathcal{R}_{n+1,3}(t) = \int_0^t \mathcal{S}_2^+(t-s) \mathcal{B}_3 \mathcal{R}_{n,3}(s) ds.
\]
From this example we see that $A_2, A_3, L$ and $I_1$ must all be bounded operators, while only $A_1$ is allowed to be unbounded. By recursion in Proposition 5 we see that we must have $L_i \in L(H_{p+2-i}, H_{p+1-i})$ and $A_i \in L(H_{p+1-i}, H_p), i = 2, 3, \ldots, p$. and $A_1 : Dom(A_1) \to H_p$ can be an unbounded operator with densely defined domain $Dom(A_1) \subset H_p$.

We remark that Ch. III in [12] presents a deep theory for perturbations of generators $\mathcal{A}$ by operators $\mathcal{B}$. Matrix operators of the kind $\mathcal{B}_p$ for $p = 2$ has been analysed in, for example [21], where conditions for analyticity of the semigroup $\{\mathcal{S}_2(t)\}_{t \geq 0}$ is studied.

4 Applications of CARMA processes

Recall Proposition 1 and let $t_i := i \cdot \delta$ for $i = 0, 1, \ldots$ and a given $\delta > 0$. Define further $z_i := Z(t_i)$. By the semigroup property of $\{\mathcal{S}_p(t)\}_{t \geq 0}$ it holds,

$$z_{t_{i+1}} = \mathcal{S}_p(t_{i+1})z_0 + \int_{t_i}^{t_{i+1}} \mathcal{S}_p(t_{i+1} - s)p^*_dL(s)$$

$$= \mathcal{S}_p(\delta) \mathcal{S}_p(t_i)z_i + \int_{t_i}^{t_{i+1}} \mathcal{S}_p(t_{i+1} - s)p^*_dL(s)$$

$$+ \mathcal{S}_p(t_i - s)p^*_dL(s)$$

$$= \mathcal{S}_p(\delta)z_i + \mathcal{E}_i,$$

with

$$\mathcal{E}_i := \int_{t_i}^{t_{i+1}} \mathcal{S}_p(t_{i+1} - s)p^*_dL(s).$$

The process above has the form of a discrete-time AR(1) process. Obviously, $\mathcal{S}_p(\delta) \in L(H)$ and by the independent increment property of the $H_p$-valued Lévy process $L$, $\{\mathcal{E}_i\}_{i \geq 0}$ is a sequence of independent $H$-valued random variables. Furthermore, $\mathbb{E}[\mathcal{E}_i] = 0$ due to the zero-mean hypothesis of $L$. Finally, we can compute the covariance operators of $\mathcal{E}_i$ by appealing to the Itô isometry (cf. [18, Cor. 8.17])

$$\mathbb{E}[\langle \mathcal{E}_i, x \rangle \langle \mathcal{E}_j, y \rangle] = \int_{t_i}^{t_{i+1}} \langle Q P_p \mathcal{S}_p(t_{i+1} - s)p^*_1 x, P_p \mathcal{S}_p(t_{i+1} - s)p^*_1 y \rangle ds$$

$$= \int_0^\delta \langle \mathcal{P}_1 \mathcal{S}_p(s) P_p Q P_p \mathcal{S}_p(s) P_p \mathcal{S}_p(s) \mathcal{P}_1^* x, y \rangle ds,$$

where $x, y \in H$. Thus, $\mathcal{E}_i$ has covariance operator $\mathcal{Q}_\mathcal{E}$ independent of $i$ given by

$$\mathcal{Q}_\mathcal{E} = \int_0^\delta \mathcal{P}_1 \mathcal{S}_p(s) P_p Q P_p \mathcal{S}_p(s) \mathcal{P}_1^* ds.$$
Therefore, \( \{ e_i \}_{i=0}^\infty \) is an iid sequence of \( H \)-valued random variables. Hence, the \( H \)-valued time series \( \{ z_i \}_{i=0}^\infty \) is a so-called linear process according to [10].

Let us now focus on the \( H_1 \)-valued CAR\( (p) \) dynamics in Def. 2 and see how this process can be related to a times series in \( H_1 \). To this end, recall the operator-valued polynomial \( Q_p(\lambda) \) introduced in (14) and the formal \( p \)th-order stochastic differential equation in (15). Let \( \Delta \delta \) be the forward differencing operator with time step \( \delta > 0 \). Moreover, we assume \( \Delta \delta_n \) to be the \( n \)th order forward differencing, defined as

\[
\Delta \delta_n f(t) = \sum_{k=0}^{n} \binom{n}{k} (-1)^k f(t + (n-k)\delta)
\]

for a function \( f \) and \( n \in \mathbb{N} \). Obviously, \( \Delta \delta_1 = \Delta \delta \). Introduce the discrete time grid \( t_i := i\delta, i = 0, 1, 2, \ldots \), and observe that

\[
\frac{1}{\delta} \Delta \delta I_p \cdots I_2 L(t_i) = \frac{1}{\delta} I_p \cdots I_2(L(t_{i+1}) - L(t_i)).
\]

Assuming that the increments of \( L \) belongs to the domain of \( I_p \cdots I_2 \), we find that

\[
e_i := \frac{1}{\delta} I_p \cdots I_2(L(t_{i+1}) - L(t_i)) \tag{24}
\]

for \( i = 0, 1, 2, \ldots \) define an iid sequence of \( H_1 \)-valued random variables. We remark that this follows from the stationarity hypothesis of a Lévy process saying that the increments \( L(t_{i+1}) - L(t_i) \) are distributed as \( L(\delta) \). The random variables \( e_i,i = 0, 1, \ldots \), will be the numerical approximation of the formal expression \( I_p \cdots I_2 L(t_i) \). Finally, we define (formally) a time series \( \{ x_i \}_{i=0}^\infty \) in \( H_1 \) by

\[
Q_p \left( \frac{\Delta \delta}{\delta} \right) x_i = e_i. \tag{25}
\]

In this definition, we use the notation \( x_i = x(t_i) \) when applying the forward differencing operator \( \Delta \delta \). The polynomial \( Q_p \) involves the linear operators \( B_1, \ldots, B_p \) that may not be everywhere defined. We define the domain \( \text{Dom}(B) \subset H_1 \) by

\[
\text{Dom}(B) := \text{Dom}(B_1) \cap \cdots \cap \text{Dom}(B_p), \tag{26}
\]

which we assume to be non-empty. This will form the natural domain for the time series \( \{ x_i \}_{i=0}^\infty \).

**Proposition 6.** Assume that for any \( y_1, \ldots, y_p \in \text{Dom}(B) \), \( B_1 y_1 + \cdots + B_p y_p \in \text{Dom}(B) \). If \( \{ e_i \}_{i=0}^\infty \subset \text{Dom}(B) \) with \( e_i \) defined in (24) and \( x_0, \ldots, x_{p-1} \in \text{Dom}(B) \), then \( \{ x_i \}_{i=0}^\infty \) is an AR\( (p) \) process in \( H_1 \) with dynamics

\[
x_{i+p} = \sum_{q=1}^{p} B_q x_{i+(p-q)} + \delta^p e_i
\]

where
\[ \bar{B}_q = (-1)^{q+1} \left( p \right) \mathbb{I}d + \sum_{k=1}^{q} \delta^k B_k (-1)^{q-k} \left( p-k \right) \left( q-k \right), q = 1, \ldots, p, \]

and \(\mathbb{I}d\) is the identity operator on \(H_1\).

**Proof.** First we observe that the assumption \(B_1 y_1 + \cdots + B_p y_p \in \text{Dom}(B)\) for any \(y_1, \ldots, y_p \in \text{Dom}(B)\) is equivalent with \(\bar{B}_1 y_1 + \cdots + \bar{B}_p y_p \in \text{Dom}(B)\) for any \(y_1, \ldots, y_p \in \text{Dom}(B)\) since \(\bar{B}_q\) is a linear combination of \(B_1, \ldots, B_q\). Thus, by the assumptions, we see that \(x_i \in \text{Dom}(B)\) for all \(i = 0, 1, 2, \ldots\) and the recursion for the time series dynamics is well-defined.

We next show that the time series \(\{x_i\}_{i=0}^\infty\) is indeed given by the recursion in the Proposition. From the definition of \(Q_p\) and the forward differencing operators, we find after isolating \(x_{i+p}\) on the left hand side and the remaining terms on the right hand side in the definition in Eq. (25) that

\[ x_{i+p} = -\sum_{q=1}^{p} \left( -1 \right)^q \left( \begin{array}{c} p \\ q \end{array} \right) x_{i+q(p-q)} + \sum_{q=1}^{p-1} \delta^q B_q \left( \sum_{k=1}^{p-q} \left( -1 \right)^k \left( \begin{array}{c} p-k \\ k \end{array} \right) x_{i+k(p-q-k)} \right) \]

\[ \quad \quad \quad \quad \quad \quad + \delta^p B_p x_i + \delta^p \varepsilon_i. \]

Identifying terms for \(x_{i+(p-1)}, x_{i+(p-2)}, \ldots, x_i\) yields the result.

The time series \(\{x_i\}_{i=0}^\infty\) defined in (25) can be viewed as the numerical approximation of the \(H_1\)-valued CAR\((p)\) process \(X(t)\). Notice that for small \(\delta\) we find that \(\mathcal{F}_p(\delta) \approx \delta \varepsilon_p + \mathbb{I}d\). Using this approximation in the explicit representation of \(Z(t)\) in Prop. 1 will yield the same conclusion as in our discussion above.

We remark that if the operators \(B_1, \ldots, B_p\) are bounded, then \(\text{Dom}(B) = H_1\). In this case, the time series \(\{x_i\}_{i=0}^\infty\) will be everywhere defined on \(H_1\), and we do not need to impose any additional "domain preservation" hypothesis.

Let us consider an example where \(p = 3\), and \(H_1 = H_2 = H_3\). Suppose that \(I_l = \mathbb{I}d\) for \(i = 1, 2, 3\) and recall from the discussion in Section 2 that in this case \(B_q = A_q\) for \(q = 1, 2, 3\). Using Prop. 6 yields that

\[ x_{i+3} = (3 \mathbb{I}d + A_1) x_{i+2} + (A_2 - 2A_1 - 3 \mathbb{I}d) x_{i+1} + (\mathbb{I}d + A_1 - A_2 + A_3) x_i + \varepsilon_i \]

when \(\delta = 1\). Here, \(\varepsilon_i = L(t_{i+1}) - L(t_i)\) and thus being distributed as \(L(1)\). This formula is the analogy of Ex. 10.2 in [2]. Indeed, Prop. 6 is the generalization of [8, Eq. (4.17)] to Hilbert space.

The \(H_1\)-valued AR\((p)\)-process in Prop. 6 is called a **functional autoregressive** process of order \(p\) (or, in short-hand notation, FAR\((p)\)-process) by [10]. For example, [13] apply such models in a functional data analysis of Eurodollar futures, where they find statistical evidence for a FAR\((2)\) dynamics. At this point, we would also like to mention that the stochastic wave equation considered in Sect. 1 will be an AR\((2)\) process with values in \(H_1\) (or a FAR\((2)\)-process). Indeed, since in this case \(A_1 = 0, I_2 = \mathbb{I}d\) and \(A_2 = \Delta\), the Laplacian, we find that \(B_1 = 0\) and \(B_2 = \Delta\), and hence,

\[ x_{i+2} = 2 \mathbb{I}d x_{i+1} - (\mathbb{I}d - \delta^2 \Delta) x_i + \delta^2 \varepsilon_i, \]
for \( i = 0, 1, 2, \ldots \). Obviously, this recursion is obtained by approximating the wave equation by the discrete second derivative in time.

Recalling from [19] the semigroup \( \{ \mathcal{S}_t \}_{t \geq 0} \) of the wave equation, we see from (11) that it has the representation (with initial condition \( Z_0 = 0 \))

\[
X(t) = \int_0^t (-\Delta)^{-1/2} \sin((-\Delta)^{1/2}(t-s))dL(s).
\]

Following the analysis in [6], \( X \) will be a Hilbert-valued ambit field. Ambit fields have attracted a great deal of attention as random fields in time and space suitable for modelling turbulence, say (see the seminal paper [5] on ambit fields and turbulence). As \( L \) is a \( L^2(0,1) \)-valued \( \text{Lévy} \) process, one can represent it in terms of the basis \( \{ e_n \}_{n=1}^\infty \), where \( e_n(x) = \sqrt{2} \sin(\pi nx) \), as

\[
L(t,x) = \sum_{n=1}^\infty \ell_n(t)e_n(x),
\]

with

\[
\ell_n(t) := \langle L(t,\cdot), e_n \rangle, \quad n = 1, 2, \ldots
\]

being real-valued square-integrable \( \text{Lévy} \) processes with zero mean (see [18 Sect. 4.8]). Thus, the stochastic wave equation has the representation

\[
X(t,x) = \sum_{n=1}^\infty \frac{\sqrt{\pi}}{\pi n} \int_0^t \sin(\pi n(t-s))d\ell_n(s) \sin(\pi nx)
\]

which becomes an example of an ambit field. Hilbert-valued CARMA\((p,U,U)\)-processes provide us with a rich class of ambit fields, as real-valued CARMA processes are specific cases of \( \text{Lévy} \) semistationary processes (see e.g. [4, 8, 9]).

Acknowledgements Financial support from the project FINEWSTOCH, funded by the Norwegian Research Council, is gratefully acknowledged.

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