Kronecker multiplicities in the \((k, \ell)\) hook are polynomially bounded

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Abstract. The problem of decomposing the Kronecker product of \(S_n\) characters is one of the last major open problems in the ordinary representation theory of the symmetric group \(S_n\). Here we prove upper and lower polynomial bounds for the multiplicities of the Kronecker product \(\chi^\lambda \otimes \chi^\mu\), where for some fixed \(k\) and \(\ell\) both partitions \(\lambda\) and \(\mu\) are in the \((k, \ell)\) hook, \(\lambda\) and \(\mu\) are partitions of \(n\), and \(n\) goes to infinity.

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1 Introduction

We assume that the characteristic of the base field is zero: \(\text{char}(F) = 0\). As usual \(S_n\) is the \(n\)-th symmetric group. Let \(\lambda\) be a partition of \(n\), \(\lambda \vdash n\), then \(\lambda\) corresponds to the irreducible \(S_n\) character \(\chi^\lambda\), and all the irreducible \(S_n\) characters are of that form \(\chi^\lambda\) [5, 7, 9, 10]. Let \(\varphi, \psi\) be two \(S_n\) characters (same \(n\)). Their Kronecker – or inner tensor – product \(\varphi \otimes \psi\) is defined via \((\varphi \otimes \psi)(\sigma) = \varphi(\sigma) \cdot \psi(\sigma)\) where \(\sigma \in S_n\). Then \(\varphi \otimes \psi\) is an \(S_n\) character, and since \(\text{char}(F) = 0\), \(\varphi \otimes \psi\) is a (non-negative) integer combination of the irreducibles \(\chi^\lambda\). In fact, the same construction and decomposition problem exist – for any finite group.

Definition 1.1. Let \(\lambda, \mu \vdash n\), let \(\chi^\lambda \otimes \chi^\mu\) denote the Kronecker product of \(\chi^\lambda\) and \(\chi^\mu\) and write

\[
\chi^\lambda \otimes \chi^\mu = \sum_{\rho \vdash n} \kappa(\lambda, \mu, \rho) \cdot \chi^\rho.
\]  

(1)

This equation defines the multiplicities \(\kappa(\lambda, \mu, \rho)\). Thus \(\kappa(\lambda, \mu, \rho)\) is the multiplicity of \(\chi^\rho\) in \(\chi^\lambda \otimes \chi^\mu\). We call the coefficients \(\kappa(\lambda, \mu, \rho)\) the Kronecker multiplicities.

Algorithms for calculating the multiplicities \(\kappa(\lambda, \mu, \rho)\) are given for example in [4, 5]. However, in the general case these algorithms become extremely involved. We remark that the problem of computing these Kronecker multiplicities \(\kappa(\lambda, \mu, \rho)\) – or obtaining significant
quantitative information about them – is one of the last major open problems in the ordinary representation theory of the symmetric groups.

In this paper we consider the case where the partitions $\lambda$ and $\mu$ are in the $k$-strip $H(k,0)$, and more generally – in the $(k,\ell)$ hook $H(k,\ell)$. The partitions in the $k$-strip are denoted $H(k,0) = \bigcup_n H(k,0;n)$.

Similarly, the partitions in the $(k,\ell)$-hook are denoted $H(k,\ell) = \bigcup_n H(k,\ell;n)$.

We later apply the fact that as a function of $n$, the cardinality $|H(k,\ell;n)|$ is polynomially bounded, see for example [1, Theorem 7.3].

We mention here that these two distinct subsets of partitions, $H(k,0)$ and $H(k,\ell)$, play an important role in representation theory: By Schur’s Double Centralizer Theorem, the partitions in $H(k,0)$ parametrize the irreducible polynomial representations of the General Linear Lie Group $GL(k,\mathbb{C})$. And a similar role is played by the partitions in $H(k,\ell)$ and the irreducible representations of the General Linear Lie superalgebra $pl(k,\ell)$ [1].

The main results in this paper are Theorem 1.2, proved in Section 4, and Theorem 1.3 which is proved in Section 5. Theorem 1.2 is a special case of Theorem 1.3.

**Theorem 1.2.** Given $0 < k \in \mathbb{Z}$, there exist $a = a(k)$, $b = b(k)$, satisfying the following condition: For any $n$ and any partitions $\lambda, \mu \in H(k,0;n)$ and $\rho \vdash n$, the multiplicities $\kappa(\lambda,\mu,\rho)$ of $\chi_{\lambda} \otimes \chi_{\mu}$ satisfy $\kappa(\lambda,\mu,\rho) \leq a \cdot n^b$. Namely, in the $k$-strip these multiplicities are polynomially bounded.

In Section 4.2 we prove a lower bound for some multiplicities $\kappa(\lambda,\lambda,\nu)$, a lower bound which grows as a polynomial of a rather large degree.

In Section 5 we hook-generalize Theorem 1.2 to the following theorem.

**Theorem 1.3.** Given $0 \leq k, \ell \in \mathbb{Z}$, there exist $a = a(k,\ell)$, $b = b(k,\ell)$, satisfying the following condition: For any $n$ and any partitions $\lambda, \mu \in H(k,\ell;n)$ and $\rho \vdash n$, the multiplicities $\kappa(\lambda,\mu,\rho)$ of $\chi_{\lambda} \otimes \chi_{\mu}$ satisfy $\kappa(\lambda,\mu,\rho) \leq a \cdot n^b$. Namely, in the $(k,\ell)$-hook these multiplicities are polynomially bounded.

One of the main tools for proving Theorem 1.2 is a recursive formula for computing the multiplicities $\kappa(\lambda,\mu,\rho)$, a formula due to Dvir [3, Theorem 2.3], and which yields a convenient upper bound for the Kronecker multiplicities. To prove Theorem 1.3 we also need – and we prove – a conjugate version of that theorem of Dvir.

The *outer* tensor product $\chi_{\lambda} \otimes \chi_{\mu}$, together with the Littlewood-Richardson multiplicities $r(\lambda,\mu,\nu)$, are introduced in Remark 2.1.2. Another key tool in proving Theorems 1.2 and 1.3
is the fact that in the strip and in the hook, the multiplicities \( r(\lambda, \mu, \nu) \) are polynomially bounded. These properties are proved in Sections 3 and 5. We remark that Dvir’s formula [3, Theorem 2.3] connects the Littlewood-Richardson and the Kronecker multiplicities, see (6).

In Section 6 we show that outside the hook the above Theorems 1.2 and 1.3 fail. In fact we show that outside the hook some multiplicities \( \kappa(\lambda, \mu, \rho) \) can grow at least as fast as \( \sqrt{n!} \).

Finally we remark that it is of some interest to find out if similar phenomena – of the multiplicities being polynomially bounded – hold when the characteristic of the base field is finite.

2 Preliminaries

The form \(< \lambda, \mu > = < \chi^\lambda, \chi^\mu >\) equals 1 if \( \lambda = \mu \), equals 0 otherwise, and is extended to all characters of the symmetric groups by bi-linearity [7, pg 114].

We list some facts that will be needed later.

Remark 2.1. 1. Let \( \lambda = (\lambda_1, \lambda_2, \ldots) \) be a partition. Then \( \ell(\lambda) = k \) if \( \lambda_k > 0 \) and \( \lambda_{k+1} = 0 \). For example, \( \ell(\lambda) \leq k \) if and only if \( \lambda \in H(k,0) \).

2. (a) Let \( \varphi \) be a character of \( S_m \), \( \psi \) a character of \( S_n \), with possibly \( m \neq n \). The outer tensor product \( \varphi \hat{\otimes} \psi \) is defined as follows: \( \varphi \times \psi \) is a character of \( S_m \times S_n \), which is a subgroup of \( S_{m+n} \). Inducing up, we have

\[
\varphi \hat{\otimes} \psi = (\varphi \times \psi) \uparrow_{S_m \times S_n}^{S_{m+n}}.
\]

Let now \( \varphi = \chi^\lambda \) and \( \psi = \chi^\mu \). Then \( \chi^\lambda \hat{\otimes} \chi^\mu \) is a character of \( S_{m+n} \), and since \( char(F) = 0 \), by complete reducibility

\[
\chi^\lambda \hat{\otimes} \chi^\mu = \sum_{\nu \vdash n+m} r(\lambda, \mu, \nu) \cdot \chi^\nu.
\]  

This equation defines the multiplicities \( r(\lambda, \mu, \nu) \).

(b) The evaluation of the multiplicities \( r(\lambda, \mu, \nu) \) is given by the celebrated Littlewood-Richardson rule, hence we call \( r(\lambda, \mu, \nu) \) the Littlewood-Richardson multiplicities. In the special case that \( \mu = (m) \), the decomposition of \( \chi^\lambda \hat{\otimes} \chi^{(m)} \) is given by the “horizontal” Young rule [3, 7, 9]. The decomposition of \( \chi^\lambda \hat{\otimes} \chi^{(1^m)} \) is given by the analogue “vertical” Young rule.

(c) The “horizontal” Young rule. Let \( \lambda = (\lambda_1, \lambda_2, \ldots) \) be a partition and \( m \geq 0 \) an integer. Let \( Par(\lambda, m) \) denote the following set of partitions of \( |\lambda| + m \):

\[
Par(\lambda, m) = \{ \mu = (\mu_1, \mu_2, \ldots) \mid |\lambda| + m \mid \mu_1 \geq \lambda_1 \geq \mu_2 \geq \lambda_2 \geq \cdots \}.
\]

then

\[
\chi^\lambda \hat{\otimes} \chi^{(m)} = \sum_{\mu \in Par(\lambda, m)} \chi^\mu.
\]
3. Let \( \alpha, \lambda \) be partitions, \( \alpha \subseteq \lambda \) and consider the skew shape \( \lambda/\alpha \). The corresponding \( S_{|\lambda|-|\alpha|} \) character \( \chi^{\lambda/\alpha} \) is defined as follows \cite{7}: Let \( \nu \) be a partition of \( |\lambda| - |\alpha| \), then
\[
< \chi^{\lambda/\alpha}, \chi^{\nu} > = < \chi^{\lambda}, \chi^{\alpha} \otimes \chi^{\nu} >. \tag{3}
\]
Write \( \chi^{\alpha} \otimes \chi^{\nu} = \sum_{\lambda} r(\lambda, \alpha, \nu) \cdot \chi^{\lambda} \), then \( \chi^{\lambda/\alpha} \) implies that \( \chi^{\lambda/\alpha} = \sum_{\nu} r(\lambda, \alpha, \nu) \cdot \chi^{\nu} \).

Thus the Littlewood-Richardson multiplicities \( r(\lambda, \alpha, \nu) \) also yield the decomposition of \( \chi^{\lambda/\alpha} \).

4. If \( \chi^{\rho} \) is a component of \( \chi^{\alpha} \otimes \chi^{\nu} \) then \( \ell(\alpha), \ell(\nu) \leq \ell(\rho) \leq \ell(\alpha) + \ell(\nu) \). If \( \alpha \subseteq \lambda \) and \( \chi^{\nu} \) is a component of \( \chi^{\lambda/\alpha} \) then \( r(\lambda, \alpha, \nu) \neq 0 \) so \( \chi^{\lambda} \) is a component of \( \chi^{\alpha} \otimes \chi^{\nu} \), hence \( \ell(\nu) \leq \ell(\lambda) \).

5. Let \( \lambda = (\lambda_1, \lambda_2, \ldots) \) and \( \mu = (\mu_1, \mu_2, \ldots) \) be two partitions. Then \( \lambda \cap \mu \) is the partition obtained by intersecting the two corresponding Young diagrams. Thus
\[
(\lambda \cap \mu)_i = \min \{ \lambda_i, \mu_i \}, \quad i = 1, 2, \ldots.
\]
The following is a consequence of Young’s rule.

**Lemma 2.2.** Let \( \varphi \) be an \( S_m \) character supported on \( H(k-1,0) \):
\[
\varphi = \sum_{\mu \in H(k-1,0;m)} c_{\mu} \cdot \chi^{\mu},
\]
and assume the multiplicities \( c_{\mu} \) satisfy \( c_{\mu} \leq M \). Let \( 0 < u \) and write
\[
\varphi \otimes \chi^{(u)} = \sum_{\nu \in H(k,0;m+u)} d_{\nu} \cdot \chi^{\nu}.
\]
Then the multiplicities \( d_{\nu} \) satisfy \( d_{\nu} \leq M \cdot (u+1)^k \).

**Proof.** We have
\[
\varphi \otimes \chi^{(u)} = \sum_{\mu \in H(k-1,0;m)} c_{\mu} \cdot (\chi^{\mu} \otimes \chi^{(u)}) = \sum_{\nu \in H(k,0;m+u)} d_{\nu} \cdot \chi^{\nu}.
\]
Let \( \nu \in H(k,0;m+u) \) and denote by \( L \) the number of partitions \( \mu \in H(k-1,0;m) \) such that \( \chi^{\nu} \in \chi^{\mu} \otimes \chi^{(u)} \). Then the multiplicity \( d_{\nu} \) equals the sum of \( L \) multiplicities \( c_{\mu} \). By Young’s rule
\[
L = (\nu_1 - \nu_2 + 1)(\nu_2 - \nu_3 + 1) \cdots (\nu_k - \nu_{k+1} + 1) \quad \text{and} \quad u \geq \sum_i (\nu_i - \nu_{i+1}).
\]
(where \( \nu_{k+1} = 0 \)). Now each \( \nu_i - \nu_{i+1} \leq u \), so \( L \leq (u+1)^k \) and \( d_{\nu} \leq M \cdot L \leq M \cdot (u+1)^k \). \( \square \)
We shall need the following properties.

**Remark 2.3.** Let \( \lambda \in H(k, \ell; n) \). Then the number of sub-partitions \( \alpha \subseteq \lambda \) is \( \leq (n+1)^{k+\ell} \).

In particular, if \( \lambda \in H(k, 0; n) \) then the number of sub-partitions \( \alpha \subseteq \lambda \) is \( \leq (n+1)^k \).

**Proof.** For each \( 1 \leq i \leq k \) there are \( \leq n + 1 \) possible values for \( \alpha_i \), namely the values \( 0, 1, \ldots, n \). Similarly for the possible values of \( \alpha'_j, 1 \leq j \leq \ell \), where \( \alpha' \) is the conjugate partition of \( \alpha \).

**Proposition 2.4.** [1, Theorem 3.26.a] Let \( \lambda \in H(k_1, \ell_1; n) \), \( \mu \in H(k_2, \ell_2; n) \), and let \( k = k_1 \ell_1 + k_2 \ell_2 \) and \( \ell = k_1 \ell_2 + k_2 \ell_1 \). Then \( \chi^\lambda \otimes \chi^\mu \) is supported on \( H(k, \ell; n) \):

\[
\chi^\lambda \otimes \chi^\mu = \sum_{\nu \in H(k, \ell; n)} \kappa(\lambda, \mu, \nu) \cdot \chi^\nu.
\]

(4)

In particular

\[
\chi^\lambda \otimes \chi^\mu = \sum_{\nu \in H(h, 0; n)} \kappa(\lambda, \mu, \nu) \cdot \chi^\nu \quad \text{where} \quad h = \ell(\lambda) \cdot \ell(\mu).
\]

(5)

For an interesting refinement of (5) – see [2].

### 3 Polynomial bounds in the strip for the Littlewood-Richardson multiplicities

In this section we prove polynomial bounds for the Littlewood-Richardson multiplicities in the strip. In section [3] we indicate how to extend this to the hook.

#### 3.1 Polynomial upper bounds in the strip

**Lemma 3.1.** Given non-negative integers \( k_1, k_2 \), there exist \( a = a(k_1, k_2), b = b(k_1, k_2) \) satisfying the following condition:

Let \( \lambda \in H(k_1, 0), \mu \in H(k_2, 0) \), and as in Equation (2), the Littlewood-Richardson multiplicities \( r(\lambda, \mu, \nu) \) are defined by

\[
\chi^\lambda \otimes \chi^\mu = \sum_{\nu \vdash |\lambda| + |\mu|} r(\lambda, \mu, \nu) \cdot \chi^\nu \quad \text{(so} \ \nu \in H(k_1 + k_2, 0): r(\lambda, \mu, \nu) = 0 \text{ if} \ \ell(\nu) > k_1 + k_2).\]

Then these multiplicities satisfy \( r(\lambda, \mu, \nu) \leq a \cdot (|\lambda| + |\mu|)^b \). Namely, in the strip, the multiplicities of \( \chi^\lambda \otimes \chi^\mu \) are polynomially bounded.
Proof. The proof is by double induction: the first case is \((k_1, k_2)\) where \(k_1\) is arbitrary and \(k_2 = 0\). Then in the general case \(k_2 > 0\), we assume true for the pair \((k_1, k_2 - 1)\), and prove for \((k_1, k_2)\).

Note first that in the case \(k_1\) arbitrary and \(k_2 = 0\), these multiplicities \(r(\lambda, \mu, \nu)\) are 1 and 0, so there is nothing to prove. Indeed, \(k_2 = 0\) implies that \(\mu\) is the empty partition \(\mu = \emptyset\), then \(\chi^\lambda \hat{\otimes} \chi^\mu = \chi^\lambda\), hence \(r(\lambda, \emptyset, \lambda) = 1\) and \(r(\lambda, \emptyset, \nu) = 0\) if \(\nu \neq \lambda\).

We proceed with the general case.

By induction on the pair \((k_1, k_2 - 1)\), there exist \(a_2, b_2 > 0\) satisfying the following condition: Let \(\ell(\lambda) \leq k_1\) and \(\ell(\rho) \leq k_2 - 1\) and write \(\chi^\lambda \hat{\otimes} \chi^\rho = \sum_\theta r(\lambda, \rho, \theta) \cdot \chi^\theta\), then the multiplicities \(r(\lambda, \rho, \theta)\) satisfy \(r(\lambda, \rho, \theta) \leq a_2(|\lambda| + |\rho|)^{b_2}\).

It is given that \(\lambda \in H(k_1, 0)\) and \(\mu \in H(k_2, 0)\). Denote \(k_2 = k\), so \(\mu = (\mu_1, \ldots, \mu_k)\), and denote \(\bar{\mu} = (\mu_1, \ldots, \mu_{k-1})\). We have \(\chi^\lambda \hat{\otimes} \chi^{\bar{\mu}} = \sum_\theta r(\lambda, \bar{\mu}, \theta) \cdot \chi^\theta\), then by induction all \(r(\lambda, \bar{\mu}, \theta) \leq a_2 \cdot (|\lambda| + |\bar{\mu}|)^{b_2}\).

As in \((\ref{2.2})\), we write

\[
\chi^\lambda \hat{\otimes} \chi^\mu = \sum_\nu r(\lambda, \mu, \nu) \cdot \chi^\nu
\]

and we also denote \(\chi^\lambda \hat{\otimes} \chi^{\bar{\mu}} \hat{\otimes} \chi^{(\mu_k)} = \sum_\nu w(\lambda, \mu, \nu) \cdot \chi^\nu\).

By Young’s rule \(\chi^\mu\) is a component of \(\chi^{\bar{\mu}} \hat{\otimes} \chi^{(\mu_k)}\) and therefore \(r(\lambda, \mu, \nu) \leq w(\lambda, \mu, \nu)\). Apply now Lemma \((\ref{2.2})\) with \(\varphi = \chi^\lambda \hat{\otimes} \chi^{\mu}, u = \mu_k\), and \(M = a_2 \cdot (|\lambda| + |\bar{\mu}|)^{b_2}\). Each component \(\chi^\nu\) of \(\chi^\lambda \hat{\otimes} \chi^{\bar{\mu}}\) is of length \(\ell(\nu) \leq \ell(\lambda) + \ell(\bar{\mu}) \leq k_1 + k_2 - 1\). By Lemma \((\ref{2.2})\) the multiplicities \(w(\lambda, \mu, \nu)\) satisfy \(w(\lambda, \mu, \nu) \leq a_2 \cdot (|\lambda| + |\bar{\mu}|)^{b_2} \cdot (\mu_k + 1)^{k_1 + k_2}\).

For any non-negative integers \(c, d, r\) and \(s\), \(r^c \cdot (s + 1)^d \leq (r + s)^{c + d}\), hence

\[
r(\lambda, \mu, \nu) \leq w(\lambda, \mu, \nu) \leq a_2 \cdot (|\lambda| + |\bar{\mu}|)^{b_2} \cdot (\mu_k + 1)^{k_1 + k_2} \leq a_2 \cdot (|\lambda| + |\bar{\mu}|)^{b_2 + k_1 + k_2} = a_2 \cdot (|\lambda| + |\bar{\mu}|)^{b_2 + k_1 + k_2}.
\]

With \(a = a_2\) and \(b = b_2 + k_1 + k_2\) the proof of the lemma is now complete. 

\[\square\]

Corollary 3.2. Given \(k\), there exist \(a = a(k), b = b(k)\) satisfying the following condition: Let \(\lambda \in H(k, 0)\), let \(\alpha \subseteq \lambda\) and write \(\chi^{\lambda/\alpha} = \sum_\rho r(\lambda, \alpha, \rho) \cdot \chi^\rho\), then all \(r(\lambda, \alpha, \rho) \leq a \cdot |\lambda|^b\). Namely the multiplicities \(r(\lambda, \alpha, \rho)\) are polynomially bounded. Moreover \(r(\lambda, \alpha, \rho) = 0\) if \(\ell(\rho) > \ell(\lambda)\) or similarly if \(\rho_1 > \lambda_1\). In particular \(\chi^{\lambda/\alpha} = \sum_{\rho \in H(k, 0)} r(\lambda, \alpha, \rho) \cdot \chi^\rho\).

Proof. Let \(k = k_1 = k_2\), let \(a(k, k)\) and \(b(k, k)\) as in Lemma \((\ref{3.1})\) and let \(a = a(k, k)\) and \(b = b(k, k)\).

We show first that if \(\ell(\rho) > \ell(\lambda)\) then \(r(\lambda, \alpha, \rho) = 0\). Indeed, let \(\chi^{\lambda/\alpha} = \sum_\rho r(\lambda, \alpha, \rho) \cdot \chi^\rho\), then

\[
r(\lambda, \alpha, \rho) = < \chi^{\lambda/\alpha}, \chi^\rho > = < \chi^\lambda, \chi^{\alpha \hat{\otimes} \rho} >.
\]

If \(\chi^\theta\) is a component of \(\chi^{\alpha \hat{\otimes} \rho}\) then by Remark \((\ref{2.1})\) \(\ell(\theta) \geq \ell(\rho) > \ell(\lambda)\), so \(\theta \neq \lambda\;\text{hence} < \chi^\lambda, \chi^\theta > = 0\). So \(0 = < \chi^\lambda, \chi^{\alpha \hat{\otimes} \rho} > = r(\lambda, \alpha, \rho)\), namely \(r(\lambda, \alpha, \rho) = 0\). Thus we can
assume that \( \rho \in H(k,0) \). Since \( \alpha \subseteq \lambda \), also \( \alpha \in H(k,0) \). The multiplicities \( r(\lambda, \alpha, \rho) \) are also the multiplicities of the irreducibles \( \chi^\lambda \) in \( \chi^{\alpha \otimes \rho} \), and since \( \alpha, \rho \in H(k,0) \), by Lemma 3.1

\[
r(\lambda, \alpha, \rho) \leq a \cdot (|\alpha| + |\rho|)^b = a \cdot (|\lambda|)^b.
\]

3.2 Polynomial lower bounds in the strip

The following is an example of a polynomial lower bound:

Let \( \lambda = \mu = (2m, m) \), so \( n = 3m \), and let \( \nu = (3m, 2m, m) \). By direct calculations with the Littlewood–Richardson rule one deduces that \( r(\lambda, \lambda, \nu) \geq m + 1 \), which is a polynomial lower bound.

This indicates that Lemma 3.1 essentially, cannot be improved.

4 Polynomial bounds for the Kronecker multiplicities in the strip

We now prove a polynomial upper bound for the Kronecker multiplicities \( \kappa(\lambda, \mu, \rho) \) (see (1)), where \( \lambda, \mu \in H(k,0;n) \) and \( \rho \vdash n \). In Section 5 we extend this to the hook \( H(k, \ell;n) \). We also prove here a polynomial lower bound for some \( \kappa(\lambda, \mu, \rho) \).

4.1 A polynomial upper bound in the strip

Theorem 2.3 in [3] gives a recursive formula for calculating \( \kappa(\lambda, \mu, \rho) \). Discarding the negative term in that formula in [3], it implies the following upper bound for \( \kappa(\lambda, \mu, \rho) \).

**Theorem 4.1.** Let \( \lambda, \mu, \rho \vdash n \), and as in (1) let \( \chi^\lambda \otimes \chi^\mu = \sum_{\rho \vdash n} \kappa(\lambda, \mu, \rho) \cdot \chi^\rho \), then

\[
\kappa(\lambda, \mu, \rho) \leq \sum_{\alpha \vdash \rho_1, \alpha \subseteq \lambda \cap \mu} \langle \chi^{\lambda/\alpha} \otimes \chi^{\mu/\alpha}, \chi^{(\rho_2, \rho_3, \ldots)} \rangle > . \tag{6}
\]

**The proof of Theorem 4.1.**

**Proof.** By assumption \( \lambda, \mu \in H(k,0;n) \) and \( \rho \vdash n \). By Proposition 2.4 \( \kappa(\lambda, \mu, \rho) = 0 \) if \( \ell(\rho) > k^2 \), hence in (6) we can assume that \( \ell(\rho) \leq k^2 \), namely \( \rho = (\rho_1, \ldots, \rho_{k^2}) \). By Remark 2.3 since \( \lambda \in H(k,0;n) \), in (6) the number of sub-partitions \( \alpha, \alpha \subseteq \lambda \cap \mu \subseteq \lambda \) is \( \leq (n+1)^k \), which is polynomial. Hence suffices to show that each summand

\[
\langle \chi^{\lambda/\alpha} \otimes \chi^{\mu/\alpha}, \chi^{(\rho_2, \rho_3, \ldots, \rho_{k^2})} \rangle >
\]

is polynomially bounded.
By Corollary 3.2 for each skew shape \( \lambda/\alpha \), \( \chi^{\lambda/\alpha} \) is a sum \( \chi^{\lambda/\alpha} = \sum_{\pi} r(\lambda, \alpha, \pi) \cdot \chi_{\pi} \) where the \( r(\lambda, \alpha, \pi) \) are polynomially bounded (polynomial in \( n = |\lambda| \)). Moreover by Remark 2.1.4, in that sum \( \chi^{\lambda/\alpha} = \sum_{\pi} r(\lambda, \alpha, \pi) \cdot \chi_{\pi} \), \( \pi \in H(k, 0; n - |\alpha|) = H(k, 0; n - \rho_1) \). So in particular, since \(|H(k, 0; n)|\) is polynomially bounded [11, Theorem 7.3], that sum has at most polynomially many summands (polynomial in \( n = |\lambda| \)), and the multiplicities \( r(\lambda, \alpha, \pi) \) in that sum are polynomially bounded. Similarly for the skew shape \( \mu/\alpha \): \( \chi^{\mu/\alpha} \) is a sum of \( \leq \) polynomially many irreducible characters \( \chi^\theta \); again, polynomial in \( n = |\lambda| \), with polynomially bounded multiplicities.

Therefore it suffices to show that for partitions \( \pi, \theta \in H(k, 0; n - \rho_1) \), each Kronecker coefficient \( \kappa(\pi, \theta, (\rho_2, \ldots, \rho_k)) \) is polynomially bounded. Repeating one more step, deduce that it suffices to show that for any two partitions \( \tau, \omega \in H(k, 0; n - (\rho_1 + \rho_2)) \), the multiplicity \( \kappa(\tau, \omega, (\rho_3, \ldots, \rho_k)) \) is polynomially bounded.

Continue! After at most \( k^2 \) steps we arrive at at-most polynomially many summands \( \kappa(\emptyset, \emptyset, \emptyset) = 1 \), and the proof is complete. \( \square \)

### 4.2 A polynomial lower bound in the strip

**Lemma 4.2.** Let \( n = k \cdot w \), \( \lambda = (w, w, \ldots, w) = (w^k) \in H(k, 0; n) \), fix \( k \) and let \( w \) go to infinity (hence also \( n \) goes to infinity). Let \( \varepsilon > 0 \). Then there exist partitions \( \nu \vdash n \) such that

\[
\kappa(\lambda, \lambda, \nu) \geq n^{(k^2 - 4)(k^2 - 1)/4 - \varepsilon}.
\]

**Proof.** Since \( k \) is fixed and \( w \) goes to infinity, by Stirling’s formula, for some constant \( A \)

\[
f^\lambda \simeq A \cdot \left( \frac{1}{n} \right)^{(k^2 - 1)/2} \cdot k^n, \quad \text{hence} \quad (f^\lambda)^2 \simeq A^2 \cdot \left( \frac{1}{n} \right)^{(k^2 - 1)/2} \cdot k^{2n}. \tag{7}
\]

By [5] \( \chi^\lambda \otimes \chi^\lambda \) is supported on \( H(k^2, 0) \), therefore

\[
\chi^\lambda \otimes \chi^\lambda = \sum_{\nu \in H(k^2, 0; n)} \kappa(\lambda, \lambda, \nu) \cdot \chi^\nu, \quad \text{so taking degrees we have} \quad (f^\lambda)^2 = \sum_{\nu \in H(k^2, 0; n)} \kappa(\lambda, \lambda, \nu) \cdot f^\nu.
\]

Denote \( g = (k^2 - 4)(k^2 - 1)/4 - \varepsilon \) and assume all \( \kappa(\lambda, \lambda, \nu) < n^g \). Then

\[
(f^\lambda)^2 < n^g \cdot \sum_{\nu \in H(k^2, 0; n)} f^\lambda.
\]

By [8]

\[
\sum_{\nu \in H(k^2, 0; n)} f^\lambda \simeq B \cdot \left( \frac{1}{n} \right)^{k^2(k^2 - 1)/4} \cdot k^{2n},
\]

for some constant \( B \), so

\[
(f^\lambda)^2 < n^g \cdot B \cdot \left( \frac{1}{n} \right)^{k^2(k^2 - 1)/4} \cdot k^{2n}. \tag{8}
\]
Combining (7) and (8), deduce that for sufficiently large $n$,
\[ A^2 \cdot \left( \frac{1}{n} \right)^{k^2-1} \cdot k^{2n} \cdot B \cdot \left( \frac{1}{n} \right)^{k^2(k^2-1)/4} \cdot k^{2n} < n^g \cdot B \cdot n^{(k^2-4)(k^2-1)/4} \cdot k^{2n}. \]

Forming l.h.s./r.h.s, deduce that for the constant $C = A^2 / B$
\[ C \cdot n^{(k^2-4)(k^2-1)/4-g} = C \cdot n^g < 1 \]
for all large $n$, which is a contradiction.

\[ \square \]

5 Polynomial upper bounds for the Kronecker multiplicities in the $(k, \ell)$-hook

In this section we prove Theorem 1.3. The proof is a hook generalization of the proof of Theorem 1.2.

Both Lemma 3.1 and Corollary 3.2 hold in the vertical strip $H(0, \ell)$, by essentially the same – but conjugate – arguments. To prove 3.1 we decomposed $\mu$ into $(\mu_1)$ and $\bar{\mu} = (\mu_2, \mu_3, \ldots)$ (namely – first row, then the rest of $\mu$), then applied Young’s rule and the fact that $\chi^\mu$ is a component of $\chi^{\bar{\mu}} \otimes \chi^{(1 \mu_1')}$. To prove the “vertical” version of Lemma 3.1, decompose $\mu$ into its first column and the rest: $(1 \mu_1')$ and $\tilde{\mu} = (\mu_1 - 1, \mu_2 - 1, \ldots)$, then by the vertical Young rule $\chi^\mu$ is a component of $\chi^{\tilde{\mu}} \otimes \chi^{(1 \mu_1')}$. The rest of the arguments are the same, yielding the vertical versions of Lemma 3.1 and of Corollary 3.2. For example, Corollary 5.1 below is the vertical version of Corollary 3.2.

**Corollary 5.1.** Given $\ell$, there exist $a = a(\ell), b = b(\ell)$ satisfying the following condition: Let $\lambda \in H(0, \ell)$, let $\alpha \subseteq \lambda$ and write $\chi^{\lambda/\alpha} = \sum_\rho r(\lambda, \alpha, \rho) \cdot \chi^\rho$, then all $r(\lambda, \alpha, \rho) \leq a \cdot |\lambda|^b$, namely the multiplicities $r(\lambda, \alpha, \rho)$ are polynomially bounded. Moreover $r(\lambda, \alpha, \rho) = 0$ if $\ell(\rho) > \ell(\lambda)$, or similarly if $\rho_1 > \lambda_1$. In particular $\chi^{\lambda/\alpha} = \sum_{\rho \in H(0, \ell)} r(\lambda, \alpha, \rho) \cdot \chi^\rho$.

Combining the horizontal and the vertical versions, we deduce the $(k, \ell)$ hook versions. We now state the $(k, \ell)$ hook version of Corollary 3.2.

**Corollary 5.2.** Given $k, \ell$, there exist $a = a(k, \ell), b = b(k, \ell)$ satisfying the following condition: Let $\lambda \in H(k, \ell)$, let $\alpha \subseteq \lambda$ and write $\chi^{\lambda/\alpha} = \sum_\rho r(\lambda, \alpha, \rho) \cdot \chi^\rho$, then all $r(\lambda, \alpha, \rho)$ satisfy $r(\lambda, \alpha, \rho) \leq a \cdot |\lambda|^b$, namely the multiplicities $r(\lambda, \alpha, \rho)$ are polynomially bounded. Moreover $r(\lambda, \alpha, \rho) = 0$ if $\rho_i > \lambda_i$ or $\rho'_i > \lambda'_i$ for some $i$. In particular, since $\lambda \in H(k, \ell)$ this implies that $\chi^{\lambda/\alpha} = \sum_{\rho \in H(k, \ell)} r(\lambda, \alpha, \rho) \cdot \chi^\rho$.

We shall need the vertical version of the inequality (6), which is inequality (9) below. Note that the proof of [3, Theorem 2.3] was based on the decomposition of $\rho$ into its first row.
(\rho_1) and the rest of the rows (\rho_2, \rho_3, \ldots). The second main ingredient in the proof of \textit{Theorem 1.2} was the horizontal Young rule, which implied that \chi^{\rho} is a component of \chi^{(\rho_1)} \otimes \chi^{(\rho_2, \rho_3, \ldots)}. Conjugate: Decompose \rho as first column (1^{\rho_1}) and the rest of the columns (\rho_1 - 1, \rho_2 - 1, \ldots). By the vertical Young rule \chi^{\rho} is a component of \chi^{(1^{\rho_1})} \otimes \chi^{(\rho_1, \rho_2, \ldots)}. The same arguments that proved Theorem \textit{4.1} now prove the following theorem.

**Theorem 5.3.** Let \lambda, \mu, \rho \vdash n, and as in \textit{1} let \chi^{\lambda} \otimes \chi^{\mu} = \sum_{\rho \vdash n} \kappa(\lambda, \mu, \rho) \cdot \chi^{\rho}. Then

\[ \kappa(\lambda, \mu, \rho) \leq \sum_{\alpha \vdash \rho_1', \alpha \subseteq \lambda \cap \mu} < \chi^{\lambda/\alpha} \otimes \chi^{\mu/\alpha} \cdot \chi^{(\rho_1 - 1, \rho_2 - 1, \ldots)} >. \]  

(9)

We later use the obvious fact that if \rho \in H(0, \ell) then (\rho_1 - 1, \rho_2 - 1, \ldots) \in H(0, \ell - 1).

We also need the fact that if \lambda, \mu \in H(k, \ell) then all the components of \chi^{\lambda} \otimes \chi^{\mu} are in the \((k^2 + \ell^2, 2k\ell)\) hook; namely \kappa(\lambda, \mu, \rho) = 0 if \rho \not\in H(k^2 + \ell^2, 2k\ell), see \textit{[1, Theorem 3.26.a]}. Analyze the proof of Theorem \textit{4.2}. That proof describes one step (which we call a "D-step") in which Theorem \textit{4.1} is applied to replace \kappa(\pi, \theta, (\rho_2, \rho_3, \ldots)) by at most polynomially many summands of the form \kappa(\pi, \theta, (\rho_1 - 1, \rho_2 - 1, \ldots)). Then D-steps are applied repeatedly, removing more and more rows of \rho. Since \rho \in H(k^2, 0), after at most \( k^2 \) D-steps the process stops and we are done.

Similarly, by applying Theorem \textit{5.3} we have the (conjugate) D’-step: this replaces the term \kappa(\lambda, \mu, (\rho_1, \rho_2, \ldots)) by at most polynomially many summands of the form \kappa(\pi, \theta, (\rho_1 - 1, \rho_2 - 1, \ldots)). Thus, given \kappa(\lambda, \mu, \rho), a D’-step removes the first column of \rho. The crucial fact here is, that the condition of polynomially bounded is preserved in ether a D or a D’ step.

**The proof of Theorem \textit{1.3}**

Proof. Start with \kappa(\lambda, \mu, \rho). Since \lambda, \mu \in H(k, \ell), by Proposition \textit{2.4} we can assume that \rho \in H(k^2 + \ell^2, 2k\ell). Apply D-steps repeatedly until \rho is replaced by \rho^* where \rho^* \in H(0, 2k\ell). Since \rho \in H(k^2 + \ell^2, 2k\ell), this happens after at most \( k^2 + \ell^2 \) D-steps. Now apply D’-steps repeatedly until \kappa(\emptyset, \emptyset, \emptyset) = 1 is reached. Since \rho^* \in H(0, 2k\ell), this happens after at most \( 2k\ell \) D’-steps. Thus, after a total of at most \((k^2 + \ell^2) + 2k\ell = (k + \ell)^2\) steps of types D and D’, we arrive at at-most polynomially many summands, each equals \kappa(\emptyset, \emptyset, \emptyset) = 1, and that completes the proof.

6 **Outside the hook**

We now give examples outside the hook, were the Littlewood-Richardson and the Kronecker multiplicities are \textit{not} polynomially bounded.
6.1 A $\sqrt{n!}$ lower bounds for some Kronecker multiplicities

**Example 6.1.** Here we show that outside the hook, as $n$ goes to infinity some Kronecker multiplicities grow at least as fast as $(n/e)^{n/2}$ namely as $\sqrt{n!}$. In particular these multiplicities are not polynomially bounded. So let $\varepsilon > 0$, assume that as $n$ goes to infinity all Kronecker multiplicities $\kappa(\lambda, \mu, \nu)$ satisfy

$$\kappa(\lambda, \mu, \nu) < \left(\frac{n}{e}\right)^{\frac{n}{2}(1-\varepsilon)},$$

and we derive a contradiction.

As usual we denote $\deg(\chi^\lambda) = f^\lambda$. Let $\lambda = \mu \vdash n$ be the Vershik-Kerov Logan-Shepp partition which maximizes $f^\lambda$ [6], [11]. It is known that for these partitions $\lambda$ there exist constants $c_0, c_1 > 0$ such that

$$e^{-c_1 \sqrt{n}} \cdot \sqrt{n!} \leq \deg(\chi^\lambda) \leq e^{-c_0 \sqrt{n}} \cdot \sqrt{n!},$$

and by a slight abuse of notations we write

$$\deg(\chi^\lambda) \simeq e^{-c_0 \sqrt{n}} \cdot \sqrt{n!}.$$  \hspace{1cm} (11)

By squaring we similarly have

$$\deg(\chi^\lambda \otimes \chi^\lambda) = (\deg(\chi^\lambda))^2 \simeq e^{-C\sqrt{n}} \cdot n!,$$  \hspace{1cm} (12)

$C = 2c > 0$ a constant.

On the other hand assumption [10] implies that

$$\deg(\chi^\lambda \otimes \chi^\lambda) < \left(\frac{n}{e}\right)^{\frac{n}{2}(1-\varepsilon)} \sum_{\nu \vdash n} \deg(\chi^\nu).$$  \hspace{1cm} (13)

It follows from the RSK correspondence [10] that the sum $\sum_{\nu \vdash n} \deg(\chi^\nu)$ equals $T_n$, the number of involutions in $S_n$. It was proved in [2] that

$$T_n \simeq \frac{(n/e)^{n/2}}{\sqrt{2} \cdot e^{1/4}} \cdot e^{\sqrt{n}}.$$ 

By Stirling’s formula

$$\left(\frac{n}{e}\right)^n \simeq \frac{1}{\sqrt{\pi n}} \cdot n!$$

hence

$$T_n = \sum_{\nu \vdash n} \deg(\chi^\nu) \simeq \frac{e^{\sqrt{n}} \cdot \sqrt{n!}}{(\pi n)^{1/4} \cdot q} \quad \text{where} \quad q = \sqrt{2} \cdot e^{1/4}.$$
so

\[ T_n < e^{\sqrt{n}} \cdot \sqrt{n!} \]  \hspace{1cm} (14)

By (12), (13) and (14)

\[ e^{-C \sqrt{n}} \cdot n! < \left( \frac{n}{e} \right)^{\frac{n}{2}} \cdot e^{\sqrt{n}} \cdot \sqrt{n!} \quad \text{so} \quad \sqrt{n!} < \left( \frac{n}{e} \right)^{\frac{n}{2}} \cdot e^{(C+1) \sqrt{n}}. \]  \hspace{1cm} (15)

From Stirling's formula deduce that \((n/e)^{n/2} < \sqrt{n!}\), therefore (15) implies that

\[ \left( \frac{n}{e} \right)^{n/2} \cdot e^{(C+1) \sqrt{n}}, \quad \text{hence} \quad \left( \frac{n}{e} \right)^{\frac{n}{2}} \cdot e^{(C+1) \sqrt{n}}. \]  \hspace{1cm} (16)

This is a contradiction since the right hand side is sub-exponential, while the left hand side is essentially \((\sqrt{n!})^c\), which grows to infinity faster than any exponential.

We conclude this section with the following conjecture.

**Conjecture 6.2.** For partitions of \( n \) \( \lambda, \mu, \rho \vdash n \), all Kronecker multiplicities \( \kappa(\lambda, \mu, \rho) \) are bounded above by \( \sqrt{n!} \).

### 6.2 Exponential lower bound for some Littlewood-Richardson multiplicities

**Example 6.3.** Based on that same Vershik-Kerov Logan-Shepp partitions \( \lambda \), we now give an example where the Littlewood-Richardson multiplicities are not bounded by exponential growth \( a^n \) for any \( a < 2 \). Assume that as \( n \) goes to infinity, the Littlewood-Richardson multiplicities in \( \lambda \hat{\otimes} \lambda \) indeed are bounded by \( a^n \) for some \( a < 2 \). Then for large \( n \)

\[ \deg(\chi^\lambda \hat{\otimes} \chi^\lambda) < a^n \sum_{\nu \vdash 2n} \deg(\chi^\nu). \]

Replacing \( n \) by \( 2n \) in (14) we have

\[ \sum_{\nu \vdash 2n} \deg(\chi^\nu) \simeq \frac{e^{\sqrt{2n}} \cdot \sqrt{(2n)!}}{(2\pi n)^{1/4} \cdot q} \quad \text{where} \quad q = \sqrt{2} \cdot e^{1/4}. \]  \hspace{1cm} (17)

Thus for large \( n \)

\[ \deg(\chi^\lambda \hat{\otimes} \chi^\lambda) < a^n \cdot e^{\sqrt{2n}} \cdot \sqrt{(2n)!}. \]  \hspace{1cm} (18)

In general (see for example [5]), if \( \varphi \) is an \( S_m \) character and \( \psi \) is an \( S_n \) character, then

\[ \deg(\varphi \hat{\otimes} \psi) = \binom{m+n}{n} \deg(\varphi) \deg(\psi). \]
Therefore
\[ \deg(\chi^\lambda \hat{\otimes} \chi^\lambda) = (\deg(\chi^\lambda))^2 \cdot \binom{2n}{n}. \quad (19) \]

By (11) \( \deg(\chi^\lambda) \simeq e^{-c\sqrt{n}} \cdot \sqrt{n!} \) hence
\[ \deg(\chi^\lambda \hat{\otimes} \chi^\lambda) \simeq e^{-2c\sqrt{n}} \cdot n! \cdot \binom{2n}{n} = e^{-2c\sqrt{n}} \cdot \frac{(2n)!}{n!}. \]

Thus (18) and (19) imply that
\[ e^{-2c\sqrt{n}} \cdot \frac{(2n)!}{n!} < a^n \cdot e^{n \cdot \sqrt{n}} \cdot \sqrt{(2n)!}, \]
so
\[ \frac{(2n)!}{n! \cdot \sqrt{(2n)!}} < a^n \cdot e^{(2c+2)\sqrt{n}}. \quad (20) \]

Squaring both sides we get that
\[ \binom{2n}{n} < (a^2)^n \cdot e^{4(c+1)\sqrt{n}}. \quad (21) \]

By Stirling’s formula
\[ \binom{2n}{n} \simeq \frac{\sqrt{2}}{\sqrt{\pi n}} \cdot 4^n, \]
hence (21) implies that
\[ \left( \frac{4}{a^2} \right)^n < \frac{\pi n}{2} \cdot e^{4(c+1)\sqrt{n}}. \]

Since \( a^2 < 4 \), the left hand side grows exponentially with \( n \), while the right grows sub-exponentially, hence a contradiction.

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