Higher analytic torsion and cohomology of
diffeomorphism groups

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Abstract

We consider a closed odd-dimensional oriented manifold $M$ together with an
acyclic flat hermitean vector bundle $F$. We form the trivial fibre bundle with fi-
bre $M$ over the manifold of all Riemannian metrics on $M$. It has a natural flat
connection and a vertical Riemannian metric. The higher analytic torsion form of
Bismut/Lott associated to the situation is invariant with respect to the connected
component of the identity of the diffeomorphism group of $M$. Using that the space
of Riemannian metrics is contractible we define continuous cohomology classes of
the diffeomorphism group and its Lie algebra. For the circle we compute this classes
in degree 2 and show that the group cohomology class is non-trivial, while the Lie
algebra cohomology class vanishes.

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1 The form T

Let $M$ be a closed odd-dimensional oriented manifold. By $\text{Met}(M)$ we denote the space of all Riemannian metrics on $M$. It is a convex open subset of $C^\infty(M, S^2T^*M)$ and therefore a Fréchet manifold.

By $\text{Diff}(M)$ we denote the group of diffeomorphisms of $M$. It is a Fréchet Lie group, and its Lie algebra is $\mathfrak{X}(M) = C^\infty(M, TM)$. The group $\text{Diff}(M)$ acts on $\text{Met}(M)$ by

$$(f, g) \in \text{Diff}(M) \times \text{Met}(M) \mapsto w(f)g := (f^{-1})^*g \in \text{Met}(M).$$

By $\text{Diff}(M)^0$ we denote the connected component of the identity of $\text{Diff}(M)$.

Let $F \to M$ be a flat hermitian vector bundle. By $\text{Diff}(M, F)$ we denote the group of its automorphisms. $\text{Diff}(M, F)$ is again a Fréchet Lie group, and we denote by $\text{Diff}(M, F)^0$ its connected component of the identity. There is a natural surjection $q_F : \text{Diff}(M, F)^0 \to \text{Diff}(M)^0$.

**Assumption 1.1** We assume that $\mathcal{F}$ is acyclic, i.e., that $H^*(M, \mathcal{F}) = 0$, where $\mathcal{F}$ is the sheaf of parallel sections of $F$.

We are going to define a closed and $\text{Diff}(M)^0$-invariant form

$$T = T_0 + T_2 + T_4 \ldots, \quad T_{2i} \in \Omega^{2i}(\text{Met}(M))$$

by specializing the higher real analytic torsion introduced by Bismut/Lott [1].

Let $\Omega^*(M, F)$ denote the space of $F$-valued forms on $M$. If we choose a Riemannian metric $g \in \text{Met}(M)$, then we can define an $L^2$-scalar product on $\Omega^*(M, F)$. For $X \in \mathcal{X}(M)$ let $i(X) \in \text{End}(\Omega^*(M, F))$ denote the insertion of $X$. We put $e(X) := i(X)^*$ and define

$$c(X) := e(X) - i(X), \quad \hat{c}(X) := e(X) + i(X).$$

Let $\nabla$ denote the connection on $\Lambda^*T^*M \otimes F$ induced by the Levi-Civita connection associated to $g$ and the flat connection on $F$. 
We identify \( T_g \text{Met}(M) \cong C^\infty(M, S^2T^*M) \) in the natural way. Then we define \( S \in \text{Hom}(T_g \text{Met}(M), \text{End}(\Omega^*(M, F))) \) by
\[
S(h) := h(e_i, e_j)c(e_i)\hat{c}(e_j), \quad h \in T_g \text{Met}(M),
\]
where \( \{e_i\} \) denotes a local orthonormal frame on \( M \).

We put \( A^k := \text{Hom}(\Lambda^k T_g \text{Met}(M), \text{End}(\Omega^*(M, F))) \) and \( A := \oplus_k A^k \). Then there is a natural product \( A^k \otimes A^l \to A^{k+l} \).

For \( t > 0 \) we define
\[
D_t := -\sqrt{t}\hat{c}(e_i)\nabla_{e_i} + S \in A.
\]
(1)

By \( N \in A^0 \) we denote the \( \mathbb{Z} \)-grading on \( \Omega^*(M, F) \). We have \( D_t^2 := -t\Delta(\mod A^{>0}) \), where \( \Delta \) is the Laplacian on \( \Omega^*(M, F) \) defined with the Riemannian metric \( g \). Hence for any \( i \in \mathbb{N} \) and for \( t > 0 \) the term \( [(1 + 2D_t^2)e_i^{D_t^2}]_i \in A^i \) has values in the trace class operators on \( \Omega(M, F) \).

**Lemma 1.2** For all \( i \in 2\mathbb{N}_0 \) and \( h \in \Lambda^i T_g \text{Met}(M) \) the integral
\[
T_i(h) := -(\frac{1}{2\pi i})^{i/2} \int_0^{\infty} \text{Tr}_s N[(1 + 2D_t^2)e_i^{D_t^2}]_i(h) \frac{dt}{t}
\]
(2)
converges. Moreover \( T := T_0 + T_2 + T_4 + \ldots \) is a closed form in \( \Omega^\text{ev}(\text{Met}(M)) \).

**Proof.** We show how this can be deduced from the results of [1] by specialization. We consider the trivial fibre bundle \( p : E := \text{Met}(M) \times M \to \text{Met}(M) \). Let \( q : E \to M \) denote the projection onto the second factor. Then \( q^*F \to E \) is a flat hermitean vector bundle over \( E \). We choose the tautological vertical Riemannian metric on \( E \) such that the fibre \( E_g \) carries the metric \( g \in \text{Met}(M) \). There is a natural choice of a horizontal distribution \( T^H \to E \) given by the kernel of \( dq \).

The tensor \( T \) given in [1], (3.11), vanishes since \( T^H \) is integrable. Since \( q^*F \) is flat as a hermitean vector bundle the tensor \( \psi \) introduced in [1], (d), vanishes, too. The symmetric tensor \( \omega_{\alpha kj} \) defined in [1], (3.21), is can be identified with the linear map \( h \in T_g \text{Met}(M) \to \omega(h)_{ij} = h(e_i, e_j) \). Thus \( D_1 \) coincides with \( 2X \) defined in [1], (3.41). One can now check that \( D_t \) is just twice the operator given in [1], (3.50).

Convergence of the integral (2) follows from [1], Thm. 3.21. By [1], Cor. 3.25 the form \( T \) is closed. \( \square \)
Lemma 1.3 The form $T$ is invariant under $\text{Diff}(M)^0$.

Proof. Via the homomorphism $q_F : \text{Diff}(M,F)^0 \to \text{Diff}(M)^0$ the group $\text{Diff}(M,F)^0$ acts on $M$, $\text{Met}(M)$, and thus on $E$ such that the projections $p, q$ are equivariant. Since $q$ is equivariant, the horizontal distribution $T^HE$ is $\text{Diff}(M,F)^0$ invariant. The tautological vertical Riemannian metric is invariant with respect to $\text{Diff}(M,F)^0$, too. Now $\text{Diff}(M,F)^0$ acts on $q^*F \to E$ by automorphisms of flat hermitean vector bundles. Since all structures used to define $T$ are $\text{Diff}(M,F)^0$-invariant, we conclude that $T$ is invariant with respect to $\text{Diff}(M,F)^0$, too (compare [1]. Thm. A 1.1). Since $q_F : \text{Diff}(M,F)^0 \to \text{Diff}(M)^0$ is surjective, $T$ is invariant under $\text{Diff}(M)^0$. $\square$

2 Group cohomology classes of $\text{Diff}(M)^0$

Let $T$ be a closed $p$-form on the convex subset $\text{Met}(M)$ of $C^\infty(M,S^2T^*M)$ which is invariant under $\text{Diff}(M)^0$. We fix a base point $g_b \in \text{Met}(M)$. If $f_0, \ldots, f_p \in \text{Diff}(M)^0$, then we define a smooth map

$$s(f_0, \ldots, f_p) : \Delta_p \to \text{Met}(M)$$

from the standard $p$-simplex $\Delta_p$ to $\text{Met}(M)$ by $s(f_0, \ldots, f_p)(t) := \sum_{i=0}^p t_i w(f_i)g_b$, where $t = (t_0, \ldots, t_p) \in \Delta_p$, $\sum_{i=0}^p t_i = 1$. If $f \in \text{Diff}(M)^0$, then $s(ff_0, \ldots, ff_p) = w(f)s(f_0, \ldots, f_p)$.

Let $C^p(\text{Diff}(M)^0)$ be the space of real alternating group $p$-cochains, and let $d : C^p(\text{Diff}(M)^0) \to C^{p+1}(\text{Diff}(M)^0)$ be the usual differential [3]. Ch 1.5. Then $\text{Diff}(M)^0$ acts on $C^p(\text{Diff}(M)^0)$ by $(fc)(f_0, \ldots, f_p) := c(f^{-1}f_0, \ldots, f^{-1}f_p)$. We define the cochain $c_T$ by

$$c_T(f_0, \ldots, f_p) := \int_{\Delta_p} s(f_0, \ldots, f_p)^*T.$$ 

Lemma 2.1 $c_T$ is a $\text{Diff}(M)^0$-invariant cocycle.

Proof. Let $f \in \text{Diff}(M)^0$. Then

$$(fc_T)(f_0, \ldots, f_p) = c_T(f^{-1}f_0, \ldots, f^{-1}f_p)$$

$$= \int_{\Delta_p} s(f^{-1}f_0, \ldots, f^{-1}f_p)^*T$$
\[ \begin{align*}
\Delta_p^*(w(f^{-1}) \circ s(f_0, \ldots, f_p))^* T &= \int_{\Delta_p} s(f_0, \ldots, f_p)^* w(f^{-1})^* T \\
&= \int_{\Delta_p} s(f_0, \ldots, f_p)^* T \\
&= c_T(f_0, \ldots, f_p).
\end{align*} \]

We have by Stokes Lemma

\[ (dc_T)(f_0, \ldots, f_{p+1}) = \sum_{i=0}^{p+1} (-1)^p c_T(f_0, \ldots, \hat{f}_i, \ldots, f_{p+1}) \]

\[ = \sum_{i=0}^{p+1} (-1)^p \int_{\Delta_p} s(f_0, \ldots, \hat{f}_i, \ldots, f_{p+1})^* T \\
= \int_{\partial \Delta^{p+1}} s(f_0, \ldots, f_{p+1})^* T \\
= \int_{\Delta^{p+1}} d(s(f_0, \ldots, f_{p+1})^* T) \\
= \int_{\Delta^{p+1}} s(f_0, \ldots, f_{p+1})^* dT \\
= 0. \]

Thus \( c_T \) defines a cohomology class \( h_T \in H^*(\text{Diff}(M)^0, \mathbb{R}) \).

**Lemma 2.2** \( h_T \) does not depend on the choice of the base point \( g_0 \).

**Proof.** Let \( g'_0 \in \text{Met}(M) \) be another base point and \( c'_T \) be defined using \( g'_0 \). Then we define the chain \( u \in C^{p-1}(\text{Diff}(M)^0) \) by

\[ u(f_0, \ldots, f_{p-1}) = \int_{\Delta^{p-1} \times I} U_{p-1}(f_0, \ldots, f_{p-1})^* T, \]

where \( U_{p-1}(f_0, \ldots, f_{p-1}) : \Delta^{p-1} \times I \to \text{Met}(M) \) is given by \( U_{p-1}(f_0, \ldots, f_{p-1})(t, s) := \sum_{i=0}^{p-1} t_i (s g_0 + (1 - s) g'_0) \). Now we have

\[ c'_T(f_0, \ldots, f_p) - c_T(f_0, \ldots, f_p) - du(f_0, \ldots, f_p) = \int_{\partial(\Delta^p \times I)} U_p(f_0, \ldots, f_p)^* T \\
= \int_{\Delta^p \times I} U_p(f_0, \ldots, f_p)^* dT \\
= 0. \]
Recall that $\mathcal{D}iff(M)^0$ is a Fréchet Lie group. Let $C^*_c(\mathcal{D}iff(M)^0) \subset C^*(\mathcal{D}iff(M)^0)$ denote the subcomplex of smooth cochains and $H^*_c(\mathcal{D}iff(M), \mathbb{R})$ be its cohomology. Then $u, c_t \in C^*_c(\mathcal{D}iff(M)^0)$, and $h_T \in H^p_c(\mathcal{D}iff(M), \mathbb{R})$ is well-defined independently of the choice of $g_b$.

**Definition 2.3** In the special case that $T$ is the higher analytic torsion form associated to the closed odd-dimensional oriented manifold $M$ and the acyclic locally constant sheaf of Hilbert spaces $\mathcal{F}$ defined in Section 1 we denote the class $h_T$ by $T(M, \mathcal{F})$.

### 3 Lie algebra cohomology classes of $\mathcal{X}(M)$

Recall that $\mathcal{X}(M)$ is the Lie algebra of $\mathcal{D}iff(M)^0$. Let $C^*(\mathcal{X}(M)) := \text{Hom}(\Lambda^*\mathcal{X}(M), \mathbb{R})$ denote the complex of continuous Lie algebra cochains with differential $d$ (see [4], Ch. 1.3). There is a natural map of cochain complexes $\mathcal{D} : C^*_c(\mathcal{D}iff(M)^0) \to C^*(\mathcal{X}(M))$ given by

$$(\mathcal{D}c)(X_1, \ldots, X_p) := p! \frac{d}{dt_1 t_1=0} \cdots \frac{d}{dt_p t_p=0} c(1, e^{t_1 X_1}, \ldots, e^{t_p X_p}) ,$$

c \in C^p_c(\mathcal{D}iff(M)^0), X_i \in \mathcal{X}(M).$$

Let $\mathcal{D}_* : H_c(\mathcal{D}iff(M), \mathbb{R}) \to H^*(\mathcal{X}(M), \mathbb{R})$ denote the induced map.

If $X \in \mathcal{X}(M)$, then we define

$$h_X := \frac{d}{dt} w(e^{tX}) g_b \in T_{g_b} \text{Met}(M) .$$

**Lemma 3.1** Let $T$ be a closed $p$-form on $\text{Met}(M)$. We have

$$(\mathcal{D}c_T)(X_1, \ldots, X_p) = \frac{1}{(p-1)!} T(h_{X_1} \wedge \ldots \wedge h_{X_p}) .$$

**Proof.** We have

$$s(1, e^{t_1 X_1}, \ldots, e^{t_p X_p})(t) = t_0 g_b + \sum_{i=1}^{p} t_i e^{t_i X_i}(g_b)$$


The restriction of $\mathcal{T}$ to $\mathcal{D}$ is acyclic.

We further identify $\Omega^{\infty} = \bigcup_{l=0}^{\infty} \Omega^{l}$, where $\Omega^{l} = \bigcup_{i=1}^{p} \bigoplus_{i=1}^{p} \Omega^{l} \mathcal{V}$.

In the following we fix the metric $\text{d}x$ using the form $\text{d}x = \text{d}x_{1} \wedge \text{d}x_{2} \wedge \text{d}x_{3}$.

We make the identification $\Omega^{p}(S^{1}, \mathcal{F}) \cong \{ f \in C^{\infty}(\mathcal{R}) \mid f(x + 1) = e^{2\pi a} f(x) \} := \mathcal{H}$ for $p = 0, 1$ using the form $\text{d}x$ in the case $p = 1$. We further identify $C^{\infty}(S^{1}, S^{2}T^{*}S^{1})$ with $C^{\infty}(S^{1})$ using the metric $\text{d}x^{2}$.

Let $g_{b}$ be the standard metric $\text{d}x^{2}$ on $S^{1}$ of volume 1. Then $w(\mathcal{D}iff(S^{1})^{0})\text{d}x^{2} = \mathcal{M}et_{1}(S^{1})$, where $\mathcal{M}et_{1}(S^{1}) = \{ g \in \mathcal{M}et(S^{1}) \mid \text{vol}_{g}(S^{1}) = 1 \}$. In order to compute the restriction of $T$ to $\mathcal{M}et_{1}(S^{1})$ it is therefore sufficient to compute $T(\text{d}x^{2})$.

In the following we fix the metric $\text{d}x^{2}$. We put $i := i(\partial_{x})$, $e := e(\partial_{x}) = \text{d}x \wedge$. Then
Let $V \subset T_{d\Omega^2}\text{Met}(S^1)$ be a finite-dimensional subspace and $\text{Gr}(V^*)$ be the Grassmann algebra generated by $V^*$. Then $\text{Gr}(V^*)$ is a $\mathbb{Z}$-graded algebra, and we denote by $\text{Gr}(V^*)_p$ the subspace of elements of degree $p$. We choose a base $\{h_\alpha\}$ of $V$ and let $\{E^\alpha\}$ denote the dual base. Then we define

$$D := -\hat{c}\partial_x - h_\alpha E^\alpha z$$

acting on $\mathcal{H} \otimes \mathbb{C}^2$, where

$$\hat{c} := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$h_\alpha \in C^\infty(S^1)$ acts as multiplication operator on $\mathcal{H}$, and $\{E^\alpha, \hat{c}\} = 0 = [E^\alpha, z]$.

The operator $D$ corresponds to $D_1$ in (1) under the identifications above. We obtain $D_t$ by rescaling. For $t > 0$ let $\Psi_t : V \to V$ be multiplication by $\frac{1}{\sqrt{t}}$ and $\Psi^*_t : \text{Gr}(V^*) \to \text{Gr}(V^*)$ be the induced automorphism. Then we have $D_t = \Psi^*_t \sqrt{t} D$.

We have

$$D^2 = \partial_x^2 + \{\hat{c}\partial_x, h_\alpha E^\alpha z\} + h_\alpha h_\beta E^\alpha E^\beta$$

$$= \partial_x^2 + 2\hat{c}h_\alpha E^\alpha z\partial_x + \hat{c}(\partial_x h_\alpha) E^\alpha z.$$

We can write $\text{Tr}_s N(1 + 2D^2_t)e^{D^2_t} = \Psi^*_t(1 + 2\frac{d}{dt})\text{Tr}_s Ne^{D^2}$, where

$$N = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$  

We want to compute $[e^{tD^2}]_2 \in \text{Gr}(V^*)_2 \otimes \text{End}(\Omega^*(S^1, F))$. Let $R_\alpha := -2\hat{c}h_\alpha z\partial_x - \hat{c}(\partial_x h_\alpha) z$. Then we have $D^2 = \partial_x^2 + E^\alpha R_\alpha$. By Duhamel’s formula we have

$$[e^{tD^2}]_2 = -t^2 E^\alpha E^\beta \int_{\Delta^2} e^{t\sigma_0 \partial_\sigma^2} R_\alpha e^{t\sigma_1 \partial_\sigma^2} R_\beta e^{t\sigma_2 \partial_\sigma^2} d\sigma$$

$$= t^2 E^\alpha E^\beta \int_{\Delta^2} e^{t\sigma_0 \partial_\sigma^2} (2h_\alpha \partial_x + \partial_x h_\alpha) e^{t\sigma_1 \partial_\sigma^2} (2h_\beta \partial_x + \partial_x h_\beta) e^{t\sigma_2 \partial_\sigma^2} d\sigma,$$

$$\text{Tr}_s N[e^{tD^2}]_2 = -t^2 E^\alpha E^\beta \text{Tr} \int_{\Delta^2} e^{t\sigma_0 \partial_\sigma^2} (2h_\alpha \partial_x + \partial_x h_\alpha) e^{t\sigma_1 \partial_\sigma^2} (2h_\beta \partial_x + \partial_x h_\beta) e^{t\sigma_2 \partial_\sigma^2} d\sigma,$$

where the operator in the last line acts on $\mathcal{H}$.

For $k \in \mathbb{Z}$ let $f_k(x) := e^{2\pi i(k + a)x}$. Then $\partial^2_x f_k = -4\pi^2(k + a)^2 f_k$. Furthermore, for $\alpha \in \mathbb{Z}$ let $h_\alpha(x) := e^{2\pi i\alpha x}$. Let $V$ be spanned by a finite number of these $h_\alpha$. Then we
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We conclude that if \( \alpha + \beta \neq 0 \), then

\[
\text{Tr} e^{t_\sigma \partial_2^2} (2h_\alpha \partial_x + \partial_x h_\alpha) e^{t_\sigma_1 \partial_2^2} (2h_\beta \partial_x + \partial_x h_\beta) e^{t_\sigma_2 \partial_2^2} f_k = 0.
\]

If \( \alpha = -\beta \), then

\[
\text{Tr} e^{t_\sigma_0 \partial_2^2} (2h_\alpha \partial_x + \partial_x h_\alpha) e^{t_\sigma_1 \partial_2^2} (2h_\beta \partial_x + \partial_x h_\beta) e^{t_\sigma_2 \partial_2^2} f_k = -4\pi^2 \sum_{k \in \mathbb{Z}} e^{-4\pi^2(k+a)^2 t_\sigma_0} (2(k + a) - \alpha) e^{-4\pi^2(k+a-\alpha)^2 t_\sigma_1} (2(k + a) - \alpha) e^{-4\pi^2(k+a)^2 t_\sigma_2}.
\]

Now

\[
\int_{\Delta^2} f(\sigma_1)d\sigma = \int_0^1 f(\sigma_1) \int_0^{\sigma_1} d\sigma_2 d\sigma_1 = \int_0^1 f(\sigma_1) \sigma_1 d\sigma_1.
\]

We conclude

\[
\text{Tr} \int_{\Delta^2} e^{t_\sigma_0 \partial_2^2} (2h_\alpha \partial_x + \partial_x h_\alpha) e^{t_\sigma_1 \partial_2^2} (2h_\beta \partial_x + \partial_x h_\beta) e^{t_\sigma_2 \partial_2^2} d\sigma
\]

\[
= -4\pi^2 \sum_{k \in \mathbb{Z}} \int_0^1 (2(k + a) - \alpha)^2 e^{-4\pi^2(k+a)^2 t_\sigma} e^{4\pi^2\alpha(2(k+a)-\alpha)t_\sigma} \sigma d\sigma
\]

\[
= -4\pi^2 \sum_{k \in \mathbb{Z}} e^{-4\pi^2(k+a)^2 t} \frac{\int_{0}^{2(k+a)-\alpha} e^{4\pi^2\alpha t u} u du}{4\pi^2 \alpha t}
\]

\[
= -4\pi^2 \sum_{k \in \mathbb{Z}} e^{-4\pi^2(k+a)^2 t} \left( \frac{e^{4\pi^2\alpha t(2(k+a)-\alpha)}(2(k + a) - \alpha)}{4\pi^2 \alpha t} - \frac{e^{4\pi^2\alpha t(2(k+a)-\alpha)} - 1}{(4\pi^2 \alpha t)^2} \right)
\]

\[
= -4\pi^2 \sum_{k \in \mathbb{Z}} \left( e^{-4\pi^2(k+a-\alpha)^2 t} \frac{2(2(k + a) - \alpha)}{4\pi^2 \alpha t} - \frac{1}{(4\pi^2 \alpha t)^2} \right) + e^{-4\pi^2(k+a)^2 t} \left( \frac{2(2(k + a) - \alpha)}{4\pi^2 \alpha t} - \frac{1}{(4\pi^2 \alpha t)^2} \right).
\]
Since $E^\alpha E^{-\alpha} = -E^{-\alpha} E^\alpha$ we obtain by some resummation

\[ -t^2 E^\alpha E^{-\alpha} \text{Tr}_s \int_{\Delta^2} e^{i\sigma_0 \partial_z^2} R_\alpha e^{i\sigma_1 \partial_z^2} R_{-\alpha} e^{i\sigma_2 \partial_z^2} d\sigma = \frac{2t}{\alpha} E^\alpha E^{-\alpha} \sum_{k \in \mathbb{Z}} (k + a) e^{-4\pi^2 (k+a)^2 t}. \]

Using

\[ t^{-2} (1 + 2t \frac{d}{dt}) = (5 + 2t \frac{d}{dt}) t^{-2} \]

we obtain

\[ -E^\alpha E^{-\alpha} t^{-1} \Psi_t (1 + 2t \frac{d}{dt}) t^2 \text{Tr}_s \int_{\Delta^2} e^{i\sigma_0 \partial_z^2} R_\alpha e^{i\sigma_1 \partial_z^2} R_{-\alpha} e^{i\sigma_2 \partial_z^2} d\sigma = (5 + 2t \frac{d}{dt}) \frac{2}{t\alpha} E^\alpha E^{-\alpha} \sum_{k \in \mathbb{Z}} (k + a) e^{-4\pi^2 (k+a)^2 t}. \]

We employ

\[ e^{-4\pi^2 (k+a)^2 t} = \frac{1}{2\pi^{1/2}} \int_{-\infty}^{\infty} e^{-z^2/4} e^{-2\pi i(k+a)t^{1/2} z} dz \]

in order to write

\[ (5 + 2t \frac{d}{dt}) \frac{2}{t\alpha} \sum_{k \in \mathbb{Z}} (k + a) e^{-4\pi^2 (k+a)^2 t} \]

\[ = (5 + 2t \frac{d}{dt}) \frac{2}{t\alpha} \sum_{k \in \mathbb{Z}} \frac{1}{2\pi^{1/2}} \int_{-\infty}^{\infty} e^{-z^2/4} - \frac{1}{2\pi^{1/2}} \frac{d}{dz} e^{-2\pi i(k+a)t^{1/2} z} dz \]

\[ = (5 + 2t \frac{d}{dt}) \frac{l}{4t^{1/2} \alpha \pi^{3/2}} \sum_{k \in \mathbb{Z}} \int_{-\infty}^{\infty} z e^{-z^2/4} e^{-2\pi i(k+a)t^{1/2} z} dz \]

\[ = (5 + 2t \frac{d}{dt}) \frac{l}{4t^{1/2} \alpha \pi^{3/2}} \int_{-\infty}^{\infty} e^{-2\pi i t^{1/2} z} \sum_{k \in \mathbb{Z}} e^{-2\pi i t^{1/2} z k} z e^{-z^2/4} dz \]

\[ = (5 + 2t \frac{d}{dt}) \frac{l}{4t^{1/2} \alpha \pi^{3/2}} \int_{-\infty}^{\infty} e^{-2\pi i t^{1/2} z} \sum_{m \in \mathbb{Z}} \delta(t^{1/2} z - m) z e^{-z^2/4} dz \]

\[ = (5 + 2t \frac{d}{dt}) \frac{l}{4t^{1/2} \alpha \pi^{3/2}} \sum_{m \in \mathbb{Z}} me^{-m^2/4t} e^{-2\pi i m} \]

\[ = \frac{l}{2t^{3/2} \alpha \pi^{3/2}} \sum_{m \in \mathbb{Z}} m e^{-m^2/4t} e^{-2\pi i m} \]

\[ = \frac{l}{8t^{7/2} \alpha \pi^{3/2}} \sum_{m \in \mathbb{Z}} m^3 e^{-m^2/4t} e^{-2\pi i m}. \]

Integrating from 0 to $\infty$ with respect to $t$ and substituting $t = m^2/z$ we obtain

\[ \int_0^{\infty} \frac{l}{8t^{7/2} \alpha \pi^{3/2}} \sum_{m \in \mathbb{Z}} m^3 e^{-m^2/4t} e^{-2\pi i m} dt \]
\[ \int_{0}^{\infty} z^{3/2} e^{-z^2/4} \, dz = \frac{1}{4\alpha^3/2} \int_{0}^{\infty} z^{3/2} e^{-z^2/4} \, dz \sum_{m \geq 1} \frac{1}{m^2} \sin(2\pi am). \]

Note that

\[ \int_{0}^{\infty} z^{3/2} e^{-z^2/4} \, dz = \frac{1}{2} \int_{0}^{\infty} u^{1/4} e^{-u/4} \, du = 2^{3/2} \int_{0}^{\infty} v^{5/4 - 1} e^{-v} \, dv = 2^{3/2} \Gamma(5/4). \]

We conclude

**Lemma 4.1** Let \( V \) be spanned by \( h_\alpha, \) \( |\alpha| \leq R. \) Then

\[ (T_2)|_V = \frac{1}{21/2 \pi^{5/2}} \Gamma(5/4) \left( \sum_{m \geq 1} \frac{1}{m^2} \sin(2\pi am) \right) \sum_{a=1}^{R} \alpha^{-1} E^\alpha E^{-\alpha}. \]

Next we compute \( D_c T_2(S^1, \mathcal{F}). \) For \( k \in \mathbb{Z} \) let \( X_k(x) := e^{2\pi i k x} \partial_x \in \mathcal{X}(S^1). \) Then \( h_{X_k}(x) = -4\pi i k e^{2\pi i k x} \partial_x. \) By Lemma 3.1 and 4.1 we have

\[ D_c T_2(X_k, X_h) = T_2(h_{X_k}, h_{X_h}) = \frac{1}{21/2 \pi^{5/2} 2^{1/2}} \Gamma(5/4) (-4\pi i k e^{2\pi i k}) (-4\pi i h e^{2\pi i h}) \delta_{k+h} \sum_{m \geq 1} \frac{1}{m^2} \sin(2\pi am) \]

\[ = 2^{7/2} \pi^{-1/2} k \Gamma(5/4) \delta_{k+h} \sum_{m \geq 1} \frac{1}{m^2} \sin(2\pi am). \]

Define \( u \in C^1(A(S^1)) \) by \( u(f \partial_x) := \int_{S^1} f(x) \, dx. \) Then \( du(f_1 \partial_{x_1}, f_2 \partial_{x_2}) = u([f_1 \partial_{x_1}, f_2 \partial_{x_2}]). \) In particular,

\[ du(X_k, X_h) = 2\pi i (h - k) \int_{S^1} e^{2\pi i (k+h)} \, dx = 2\pi i (h - k) \delta_{h+k} = -4\pi i k \delta_{h+k}. \]

We conclude that

\[ D_c T_2 = -2^{3/2} \pi^{3/2} \Gamma(5/4) \left( \sum_{m \geq 1} \frac{1}{m^2} \sin(2\pi am) \right) \, du. \]

Thus we have shown the following
Lemma 4.2

\[ D_*(T_2(S^1, F)) = 0. \]

Finally we compute \( T_2(S^1, F) \in H^2_*(Diff(S^1)^0, \mathbb{R}) \). By [4], Thm. 3.4.4, the ring \( H^*(Diff(S^1)^0, \mathbb{R}) \) is generated by two classes \( \alpha, \beta \in H^2_*(Diff(S^1)^0, \mathbb{R}) \), where \( \beta^2 = 0 \), \( D_* \beta = 0 \), and \( D_* \alpha \neq 0 \). Thus \( T_2(S^1, F) = c \beta \) for some constant \( c \in \mathbb{R} \). It remains to describe \( \beta \) and to determine the constant \( c \). As explained in [4], there is a commutative diagram

\[
\begin{array}{cccc}
0 & \to & H^1_{top}(Diff(S^1)^0, \mathbb{R}) & \xrightarrow{a} & H^2_{e}(Diff(S^1)^0, \mathbb{R}) & \to & H^2(A(S^1), \mathbb{R}) & \to & 0 \\
& & \downarrow \cong & & w \downarrow & & & & \\
0 & \to & H^1_{top}(SL(2, \mathbb{R}), \mathbb{R}) & \xrightarrow{v} & H^2_{e}(SL(2, \mathbb{R}), \mathbb{R}) & \to & 0 & &
\end{array}
\]

where the vertical maps are induced by the inclusion \( SL(2, \mathbb{R}) \hookrightarrow Diff(S^1)^0 \) which is a homotopy equivalence of topological spaces, \( H^*_{top} \) denotes cohomology of topological spaces, and we have used that \( H^1(sl(2, \mathbb{R}), \mathbb{R}) = 0, H^2(sl(2, \mathbb{R}), \mathbb{R}) = 0, H^2(A(S^1), \mathbb{R}) = 0. \)

The Cartan decomposition gives a decomposition \( SL(2, \mathbb{R}) = SO(2) \times \mathbb{R}^2 \) as topological space, where \( SO(2) \cong S^1 \) is a maximal compact subgroup. Let \( f : SL(2, \mathbb{R}) \to SO(2) \) be projection onto the first factor.

We fix the orientation of \( SO(2) \) such that

\[
\begin{pmatrix}
0 & -1 \\
1 & 0
\end{pmatrix} \in so(2)
\]

points into the positive direction. Let \( \delta \in H^1_{top}(SL(2, \mathbb{R}), \mathbb{R}) \) be such that \( \langle \delta, [SO(2) \times \{0\}] \rangle = 1 \). Then \( \beta := b \circ a^{-1}(\delta) \).

We determine the constant \( c \) using the identity \( w(T_2(S^1, F)) = cv(\delta) \). By [4], Thm. 3.4.3., there is a unique sheet of \( \text{Arg} \) such that \( \tilde{c} : SL(2, \mathbb{R}) \times SL(2, \mathbb{R}) \to \mathbb{R} \) given by \( \tilde{c}(g, h) := \frac{1}{2\pi} \text{Arg}(f(gh) - f(h) - f(g)) \) satisfies \( \tilde{c}(1, 1) = 0 \). The function \( \tilde{c} \) represents \( v(\delta) \) using the description [4], Ch. 4.1 (II), of \( H^2_{e}(SL(2, \mathbb{R}), \mathbb{R}) \). As explained in [4], p. 264, the class \( v(\delta) \) is also represented by the cocycle \( \langle g, h \rangle \mapsto \frac{1}{2\pi} \text{vol}_{H^2}(o, go, gho) \), where \( o \) is the origin in the hyperbolic plane \( H^2 = SL(2, \mathbb{R})/SO(2) \) of constant sectional curvature \(-1\) and \( \text{vol}_{H^2}(o, go, gho) \) denotes the oriented volume of the geodesic triangle \( \Delta(g, h) \) spanned by \( o, go, gho \in H^2 \).

Note that \( w(T_2(S^1, F)) \) is represented by \( \tilde{c}T_2(g, h) := cT_2(1, g, gh) \). Since \( SO(2) \) stabilizes the metric \( dx^2 \), the inclusion \( SL(2, \mathbb{R}) \hookrightarrow Diff(S^1)^0 \) induces an inclusion \( i : H^2 \hookrightarrow Met_1(S^1) \) such that \( i(o) = dx^2 \).
Let $I$ denote the unit interval $[0, 1]$. If $g \in SL(2, \mathbb{R})$, then we define $j(g) : I^2 \mapsto Met(S^1)$ by $j(g)(s, t) = si(\gamma_t) + (1 - s)(tdx^2 + (1 - t)u(g)dx^2)$, where $\gamma : I \mapsto H^2$ is the geodesic path joining $o$ and $go$. Furthermore we define the cochain $u \in C^1_c(Diff(S^1)^0)$ by $u(g) := \int_{I^2} j(g)^* T_2$. Then we have $c^t T_2(g, h) - \int_{\Delta(g, h)} i^* T_2 = u(gh) - u(g) - u(h)$. It follows that $w(T_2(S^1, F))$ is represented by the cocycle $(g, h) \mapsto \int_{\Delta(g, h)} i^* T_2$. Since $i$ is $SL(2, \mathbb{R})$-equivariant and $T_2$ is $Diff(S^1)^0$-invariant, $i^* T_2$ is $SL(2, \mathbb{R})$-invariant and hence proportional to the volume form $\text{vol}_{H^2}$, thus $i^* T_2 = \frac{c^2}{2\pi} \text{vol}_{H^2}$.

Let $A := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $N := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

Let $A^2, N^2 \in \mathcal{X}(H^2)$ denote the corresponding fundamental vector fields. Then

$$\text{vol}_{H^2}(A^2(o), N^2(o)) = -2.$$ 

Let $A^*, N^* \in \mathcal{X}(S^1)$ denote the fundamental vector fields corresponding to $A, N$. Then we have

$$A^*(x) = \frac{1}{2\pi} (e^{2\pi ix} - e^{-2\pi ix}) \partial_x, \quad N^*(x) = \frac{1}{4\pi} (e^{2\pi ix} + e^{-2\pi ix} + 2) \partial_x.$$ 

We have

$$i^* T_2(A^2, N^2) = \frac{d}{ds}_{s=0} \frac{d}{dt}_{t=0} c T_2(1, e^{tA}, e^{tA} e^{sN}) = Dc T_2(A^*, N^*) = -\frac{1}{2^{5/2} \pi^{9/2}} \Gamma(5/4) \sum_{m \geq 1} \frac{1}{m^2} \sin(2\pi am) .$$

We conclude that

$$w(T_2(S^1, F)) = -\frac{1}{2^{5/2} \pi^{7/2}} \Gamma(5/4) \sum_{m \geq 1} \frac{1}{m^2} \sin(2\pi am) v(\delta) .$$

We have shown the following

**Lemma 4.3**

$$T_2(S^1, F) = -\frac{1}{2^{5/2} \pi^{7/2}} \Gamma(5/4) \left( \sum_{m \geq 1} \frac{1}{m^2} \sin(2\pi am) \right) \beta .$$
5 Concluding remarks

1. The one-dimensional example shows that $\mathcal{T}(M, \mathcal{F})$ is nontrivial in general.

2. In [2] Bott gave a construction of cocyles for $\mathcal{D}iff(M)$ given by integration of locally computable quantities. In our example the class $\alpha$ can be represented in this way. Since $\mathcal{T}(S^1, \mathcal{F})$ depends non-trivially on $\mathcal{F}$ there is no local representation for $\mathcal{T}(M, \mathcal{F})$ in general.

3. Is there any easy way to compute $\mathcal{T}_{2j}(S^1, \mathcal{F})$ for $j \geq 2$?

4. Can $\mathcal{D}_* \mathcal{T}(M, \mathcal{F})$ be non-trivial?

5. Give a differential topological interpretation of $\mathcal{T}(M, \mathcal{F})$?

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