Possible Knot-type Time-dependent Quantum-mechanically Dynamical System

Zotin K.-H. Chu
3/F, 4, Alley 2, Road Xiushan, Leshanxinchun, Xujiahui 200030, China

Abstract

We illustrate a possible traversing along the path of trefoil-type and 8_{18} knots during a specific time period by considering a quantum-mechanical system which satisfies a specific kind of phase dynamics of quantum mechanics. This result is relevant to the composite particle which is present in the initial or final configuration.

1 Introduction

Solutions of the time-dependent Schrödinger equation have been of considerable interest since quantum mechanics was established [1-5]. Driven by the need to solve theoretical and practical problems throughout physical fields, such as atomic physics, condensed matter physics, perturbative and nonperturbative methods were proposed and developed. One approach: Dirac’s perturbation theory [1-3], being one of the most successful methods in dealing with time-dependent quantum systems, caused seemingly unnecessary concerns. Meanwhile, knots themselves are macroscopic physical phenomena in three-dimensional space, occurring in rope, vines, telephone cords, polymer chains, DNA, certain species of eel and many other places in the natural and man-made world [6-13] (to have knots at all, that is configurations of a curve falling into a certain class which cannot be transformed into other classes (i.e. different knots, including no knot), one must have infinite or closed curves [6]). For example, the static and dynamic behavior of single polymer chains, such as DNA, and multichain systems like gels and rubbers, is strongly influenced by knots and permanent entanglements [6,12-13]. The study of topological invariants of knots leads to relationships with statistical mechanics and quantum physics [6-13]. This is a remarkable and deep situation where the study of certain (topological) aspects of the macroscopic world is entwined with theories developed for the subtleties of the microscopic world.

In this short paper, we shall illustrate the possible link between the study of time-dependent quantum system and the application of the relevant phase-part result to the simple knot, like the trefoil and 8_{18}-type.
2 Formulation

According to the basic formalism of quantum mechanics, we can formally construct a unitary operator to obtain the dynamical wavefunction $\Psi(t)$ from the initial wavefunction $\Psi(t_0)$, namely, we have

$$\Psi(t) = \exp\left[\frac{-i}{\hbar} \int_{t_0}^{t} H(\tau)d\tau\right]\Psi(t_0),$$

where $H(\tau)$ is the time-dependent Hamiltonian [1-3]. We first discretize the entire time span from $t_0$ to $t$ into $N$ intervals as $\Delta t_1 = t_1 - t_0$, $\Delta t_2 = t_2 - t_1$, · · · , $\Delta t_N = t - t_{N-1}$. We thus have

$$\Psi(t) = \exp\left[\frac{-i}{\hbar} \int_{t_{N-1}}^{t} H(\tau)d\tau\right] \cdots \exp\left[\frac{-i}{\hbar} \int_{t_0}^{t_1} H(\tau)d\tau\right]\Psi(t_0)$$

or

$$\Psi(t) = \exp\left[\frac{-i}{\hbar} \int_{t_{j-1}}^{t_j} H(\tau)d\tau\right]\Psi(t_{j-1}), \quad j = 1, 2, \cdots$$

We then explore the possibility of replacing the Hamiltonian $H(t)$ by its stepwise time varying approximation $\hat{H}(t)$ [14-15] which is defined as

$$\hat{H}_j = \frac{1}{\Delta t_j} \int_{t_{j-1}}^{t_j} H(\tau)d\tau, \quad t_{j-1} < t < t_j$$

or

$$\hat{H}_j \equiv H(\hat{t}_j), \quad \text{with} \quad \hat{t}_j = \frac{t_{j-1} + t_j}{2}.$$

Now, during each of the time intervals expressed above, the newly defined Hamiltonian $\hat{H}_j$ is independent of time and the typical intermediate state becomes

$$\Psi(t_j) \approx \exp\left[\frac{-i}{\hbar} \hat{H}_j \Delta t_j\right]\Psi(t_{j-1})$$

which implies

$$\Psi(t) = \lim_{N \to \infty} \exp\left[\frac{-i}{\hbar} \hat{H}_N \Delta t_N\right] \cdots \exp\left[\frac{-i}{\hbar} \hat{H}_1 \Delta t_1\right]\Psi(t_0).$$

For the Hilbert space related to $\hat{H}_j$ (each $\hat{H}_j$ defines a Hilbert space and there are $N$ Hilbert spaces), we have

$$\hat{H}_j \Psi_n^j(r) = E_n^j \Psi_n^j(r),$$

where $E_n^j$ and $\Psi_n^j$ are the $n$th-eigenenergy and $n$th normalized eigenfunction during $t_j < t < t_{j-1}$. Once the wavefunction $\Psi(t_{j-1})$ is known, the wavefunction $\Psi(t_j)$ can be expressed by

$$\Psi(t_j) = \sum C_n^j \exp\left[\frac{-i}{\hbar} E_n^j \Delta t_j\right]\Psi_n^j(r)$$

where $C_n^j$ is determined by a projection

$$C_n^j = \int \Psi(t_{j-1}) W_n^j(r) dr.$$
Using the Dirac notation, above expressions become
\[
\Psi(t_j) = \sum_n \exp[-i\omega_n^j \Delta t_j] |t_j, n\rangle \langle t_j, n|\Psi(t_{j-1})\rangle,
\]
where
\[
\omega_n^j = E_n^j / \hbar.
\]
It means for a short time interval a dynamical system and its corresponding stationary system evolve in almost the same way.

If the initial wavefunction \(\Psi(t_{j-1})\) is expanded in terms of the eigenfunctions during \(\Delta t_{j-1}\)
\[
\Psi(t_{j-1}) = \sum_l C_l |t_{j-1}, l\rangle
\]
we then have
\[
\Psi(t_j) = \sum_n C_n |t_j, n\rangle
\]
where
\[
C_n = \exp[-i\omega_n^j \Delta t_j] \sum_l C_l |t_j, n|t_{j-1}, l\rangle.
\]
In fact we can have
\[
\Psi(t) = \sum_{k, \ldots, l, n} \exp[-i\theta_{k, \ldots, l, n}] |t_N, k\rangle \langle t_N, k| \cdots |t_2, l\rangle \langle t_2, l| |t_1, n\rangle \langle t_1, n|\Psi(t_0)\rangle,
\]
with
\[
\exp[-i\theta_{k, \ldots, l, n}] = \exp[-i(\omega_N^N \Delta N + \cdots + \omega_2^2 \Delta t_2 + \omega_1^1 \Delta t_1)].
\]
Above results also mean that the multi-projection component \(|t_N, k\rangle \cdots |t_2, l\rangle |t_1, n\rangle \langle t_1, n|\Psi(t_0)\rangle\), as a part of the initial wavefunction, indeed passes through the energy states labelled \(n, l, \cdots, k\) in the defined time-division sequence and should eventually get the phase factor \(\exp[-i\theta_{k, \cdots, l, n}]\) (the real Hamiltonian is replaced by its stepwise time-varying counterparts; cf. [4-5]). Noting \(\sum_n |t_j, n\rangle \langle t_j, n| \equiv 1\), we thus find an interesting and important fact that if all phase factors of the form \(\exp(i\theta)\) disappeared from Eq. (9), the wavefunction would not change at all. Note also that, as to above approach, it is well known that the function \(\exp[i f(t, r)]\Psi(t, r)\) where \(f(t, r)\) is an arbitrary function of \(t\) and \(r\), is also the system’s wavefunction provided that the gauge in question (the gauge choice should be allowable in view of the fact that a certain gauge transformation can always make the longitudinal vector field vanish [14]) is allowed to be arbitrary. We remind the readers that a phase factor of the form \(\exp[i f(t, r)]\) can be a nonuniformly continuous function and there are mathematical complications in dealing with nonuniformly continuous function [15], we are convinced that the gauge arbitrariness related to \(\exp[i f(t, r)]\Psi(t, r)\) should be excluded.

The objective of above presentation is to illustrate that these principles are indeed workable in terms of solving the time-dependent Schrödinger equation and to show that the evolution of a quantum system can be characterized almost entirely by phase dynamics.
3 Results and Discussion

We are now to consider the relevant issue related to the simple knots [18-19] which is the our main focus here. The main assumptions are: The final Hamiltonian is the same as the initial Hamiltonian, the perturbation is relatively small and the action time of the perturbation is relatively short (above approach is similar to the path-integral approach as mentioned already and is related to the phase dynamics of quantum mechanics). In particular, considering this case: a harmonic oscillator disturbed for a time $0 < t < T$ and having the Hamiltonian

$$H = \frac{p^2}{2} + x^2 \frac{S(t)}{2},$$

where $S(t)$ is a stepwise function: $S = 1$ for $t \leq 0$, $S(t) = \chi > 0$ for $0 < t < T$, $S = 1$ for $t \geq T$.

We can then obtain the system’s wavefunction during $0 < t < T$ based on above approach

$$\Psi(t) = \sum_n e^{-iE_{n,\chi}t}|n, \chi\rangle\langle n, \chi|0\rangle,$$

where $E_{n,\chi} = \omega_{\chi}(n + 1/2)$, $\omega_{\chi} = \sqrt{\chi}$ ($\hbar = 1$). Meanwhile after (time) $t = T$,

$$\Psi(t) = \sum_{n,m} \exp(-iE_{n,\chi}T)\exp[-iE_{m,1}(t - T)]|m, 1\rangle\langle m, 1|n, \chi\rangle\langle n, \chi|0\rangle.$$

Now, an interesting observation occurs, considering the phase character of this evolving system, once we assume that the system is initially in the ground state and let $T$ be equal to $4\pi/\omega_{\chi}$. This system will return the same ground eigenstate after $t = T$ (due to $\exp(-iE_{n,\chi}T) = 1$ and $\sum_n |n, \chi\rangle\langle n, \chi| = 1$; cf. equations above)

$$\Psi(t) = \sum_m e^{-iE_{m,1}(t-T)}|m, 1\rangle\langle m, 1|0\rangle,$$

and the only effect of the disturbance for the additional phase factor could be tuned by choices of $\chi$. To be specific, the system seems to be completely frozen during $0 < t < T$ (this resembles the phenomena predicted by the quantum Zeno effect; cf., e.g., [16-17]).

With above results and $\omega_{\chi} = 1$ we then have for the case of simple knot: Trefoil [18-19] as schematically shown in Fig. 1. A trefoil is the simplest example of torus-knots, which are obtained by winding around a torus in both directions [10,12,17,20]. We can observe the evolution of certain system along the trefoil-like trajectory during $0 < t < T$ with respect to any initial site or state, say, located at U or R or L (cf. Fig. 1). The system evolves starting from U, (as $T = 4\pi$ for this kind of trefoil-like paths traversing w.r.t. the vertex U), after $t = T$, the system will return to U again. The above mentioned results could thus be applied to this kind of path-traversing ($T = 4\pi$) for any topological changes of trefoil-type knots [7,18-19].

Similarly, the 8_18 knot (cf., e.g., [21]) can also be realized by selecting $\omega_{\chi} = 2/3$ ($T = 6\pi$; say, starting from the vertex K). The illustration is shown in Fig. 2. To give a brief summary, we already illustrated a possible traversing along the path of trefoil-type and 8_18 knots during
a characteristic time period by considering a quantum-mechanic system which satisfies a specific kind of phase dynamics of quantum mechanics. We believe this presentation is relevant to the composite particle which is present in the initial or final configuration. Other complicated knot-type evolutions [22-26], the unparticle stuff [27] for the specific quantum-mechanic system introduced above as well as the role of Chern-Simons classes [28-29] based on the present application will be investigated in the future. 

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Fig. 1 Schematic (diagram) of a loosely hard trefoil [7,18-19]. Those points of solid circle : U, R, L could be prescribed as vertices or lattice sites.

Fig. 2 Schematic (diagram) of a loosely hard 8_{18}-knot [21].

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