Screening the fifth force in the Horndeski’s most general scalar-tensor theories

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We study how the Vainshtein mechanism operates in the most general scalar-tensor theories with second-order equations of motion. The field equations of motion, which can be also applicable to most of other screening scenarios proposed in literature, are generally derived in a spherically symmetric space-time with a matter source. In the presence of a field coupling to the Ricci scalar, we clarify conditions under which the Vainshtein mechanism is at work in a weak gravitational background. We also obtain the solutions of the field equation inside a spherically symmetric body and show how they can be connected to exterior solutions that accommodate the Vainshtein mechanism. We apply our general results to a number of concrete models such as the covariant/extended Galileons and the DBI Galileons with Gauss-Bonnet and other terms. In these models the fifth force can be suppressed to be compatible with solar-system constraints, provided that non-linear field kinetic terms coupled to the Einstein tensor do not dominate over other non-linear field self-interactions.

I. INTRODUCTION

Motivated by the dark energy problem, there have been numerous attempts to modify General Relativity (GR) at large distances (see Refs. 1 for reviews). Generally, such modifications give rise to new degrees of freedom associated with the breaking of gauge symmetries of GR. In f(R) gravity, for example, the Lagrangian including the non-linear terms of the Ricci scalar R brings a scalar degree of freedom called “scalarons” in the gravitational sector 2. Those scalar degrees of freedom can freely propagate to mediate a long-range fifth force with baryonic matter. Since the gravitational experiments within the solar system agree with GR in high precision, we need to screen the fifth force at small distances while realizing the cosmic acceleration on large scales.

In modified gravitational theories there are several different ways to recover the General Relativistic behavior in local regions. One is the so-called chameleon mechanism 3, under which the effective mass m_{eff}(\phi) of a scalar field \phi is different depending on the surrounding matter density. In the regions of high density with large m_{eff}(\phi), a spherically symmetric body can have a thin-shell to suppress the coupling between the field and non-relativistic matter outside the body. In fact, the chameleon mechanism was applied to dark energy models based on f(R) theories 4 and Brans-Dicke theories 5. There exists a similar screening scenario called the symmetron mechanism 6. The choices of the field potential and the matter coupling for symmetrons are different from those for chameleons. Unfortunately, the energy scale of the simplest potential of symmetrons is too small to act for dark energy.

While the existence of the field potentials is crucial for the success of the chameleon and symmetron mechanisms, there is another screening scenario called the Vainshtein mechanism 7 based on derivative self-interactions of a scalar degree of freedom. The Vainshtein mechanism was originally discovered in the context of massive gravity (spin-2 Pauli-Fierz theory 8). The helicity-0 mode of massive gravitons does not decouple from matter in a linear approximation 9, but the derivative self-interactions of the helicity-0 mode allow to suppress the matter coupling 2. The Vainshtein mechanism can be at work in the Dvali-Gabadadze-Poratti (DGP) braneworld model 10 where the cosmic acceleration is realized by a gravitational leakage to the extra dimension. In the DGP model, a non-linear field self-interaction of the form (\partial \phi)^2 \Box \phi, which arises due to the mixture of the longitudinal and transverse gravitons, can lead to the recovery of GR within the so-called Vainshtein radius \ri 11 12.

In flat (Minkowski) space-time, the non-linear field Lagrangian (\partial \phi)^2 \Box \phi gives rise to the field equation of motion respecting the Galilean symmetry \partial_{\mu} \phi \rightarrow \partial_{\mu} \phi + b_{\mu}. Imposing this symmetry in flat space-time, the field Lagrangian is restricted to have only five terms including \chi \equiv -(\partial \phi)^2/2 and \chi \Box \phi 15. The covariant generalization of this “Galileon” theory in curved space-time was carried out in Refs. 16. The Lagrangian of the covariant Galileon is constructed to keep the equations of motion at second order, while recovering the Galilean symmetry in the limit of Minkowski space-time. The application of covariant Galileon theory to dark energy has been extensively studied in Refs. 17 18.

If we consider a probe brane embedded in a five-dimensional Minkowski bulk, all the non-linear self-interactions of Galileons naturally arise from the brane tension, induced curvature, and the Gibbons-Hawking-York boundary terms of the bulk contributions 19. Moreover, the coupling to gravity is straightforward by taking the induced metric on the brane in the form g_{\mu \nu} = q_{\mu \nu} + \partial_{\mu} \phi \partial_{\nu} \phi, where q_{\mu \nu} is an arbitrary four-dimensional metric. In the non-relativistic limit this approach nicely recovers the Lagrangian of covariant Galileons derived in Refs. 10. The constructions of...
more general Galileon theories in the framework of branes in a co-dimensional (or maximally symmetric) bulk and in supersymmetric theories have been carried out in Refs. 20.

Since the field equations of motion following from the action of the covariant Galileon are kept up to second order in time and spatial derivatives, this theory can avoid the Ostrogradski’s instability 21 associated with the appearance of the Hamiltonian unbounded from below. The four-dimensional action of the most general scalar-tensor theories with second-order equations of motion was first found by Horndeski in 1974 22. The same action was re-derived by Deffayet et al. 23 with a more convenient form in a general D-dimensional space-time (see also Refs. 24, 25). The four-dimensional Horndeski’s theory is closely related to the effective field theory of inflation 26 or dark energy 27 in that the latter covers the former with extra spatial derivatives higher than second order at the level of linear cosmological perturbations 28. The Horndeski’s theory was applied to the dark energy cosmology in Refs. 29, 30.

In the presence of the covariant Galileon Lagrangian $M^{-3}X \Box \phi$ and non-relativistic matter coupled to $\phi$, the Vainshtein mechanism works to recover the General Relativistic behavior at short distances. The Vainshtein radius $r_V$ depends on the mass scale $M$. In the DGP model, $r_V$ can be as large as $10^{20}$ cm for $M$ related to dark energy and the Schwarzschild radius $r_g$ of the Sun 12. For the distance $r$ satisfying $r_g \ll r \ll r_V$ there is the solution $\phi'(r) \propto r^{-1/2}$ responsible for the suppression of the fifth force within the solar system. A similar suppression also occurs for the extended Galileon Lagrangian $g(\phi)M^{1-4n}X^n\Box \phi$ ($n \geq 1$) with a non-minimal coupling $F(\phi)R$ 31, 32, where $g(\phi)$ and $F(\phi)$ are slowly varying functions with respect to $\phi$.

In the context of the Horndeski’s theory the Vainshtein screening effect was studied by Kimura et al. 33 in the spherically symmetric configurations on the cosmological background. For the theory in which non-linear field derivatives couple to the Einstein tensor (i.e., $G_{5, X} \neq 0$ in the Lagrangian [3] given below), Kimura et al. claimed that the Newton gravity is not recovered at short distances. It is not clear however that in such models the solution of the field in the regime $r_g \ll r \ll r_V$ does not really connect to another solution which appears in the region of high density (i.e., inside a spherically symmetric body). In this paper we shall address this issue in detail by taking into account the variation of the matter density inside the body.

In the Horndeski’s theory we derive the field equations of motion for a spherically symmetric metric characterized by two gravitational potentials $\Psi$ and $\Phi$. Our analysis is general enough to address the Vainshtein mechanism for most of modified gravitational models proposed in literature (see Refs. 34, 35, for the study of the Vainshtein mechanism in related models). Not only we clarify conditions under which the Vainshtein mechanism can be at work, but we apply our results to a number of concrete models such as covariant/extended Galileons and the Dirac-Born-Infeld (DBI) Galileons with Gauss-Bonnet and other terms. We obtain the solutions of the scalar field and the gravitational potentials inside the Vainshtein radius, paying particular attention to the matching of solutions around the surface of a spherically symmetric body.

This paper is organized as follows. In Sec. II the full equations of motion are derived in a spherically symmetric space-time with a matter source. On the weak gravitational background we reduce the equations of the field and gravitational potentials to simpler forms. In Sec. III we obtain a general formula of the Vainshtein radius in the presence of a non-minimal field coupling $e^{-2\phi/M_{Pl}}$ with the Ricci scalar $R$ and discuss conditions for the existence of solutions that accommodate the Vainshtein mechanism. In Sec. IV we study in details how the screening mechanism operates in the presence of all the covariant Galileon terms. In Sec. V we apply our general results to a number of concrete models which are mostly the extension of the covariant Galileon. Section VI is devoted to conclusions.

II. FIELD EQUATIONS OF MOTION

The Lagrangian in the most general scalar-tensor theories in four dimensions is described by 22, 25

$$\mathcal{L} = \sum_{i=2}^{5} \mathcal{L}_i ,$$

where

$$\mathcal{L}_2 = K(\phi, X) ,$$
$$\mathcal{L}_3 = -G_3(\phi, X) \Box \phi ,$$
$$\mathcal{L}_4 = G_4(\phi, X) R + G_{4,X} [ (\Box \phi)^2 - (\nabla_{\mu} \nabla_{\nu} \phi) (\nabla^{\mu} \nabla^{\nu} \phi) ] ,$$
$$\mathcal{L}_5 = G_5(\phi, X) G_{\mu \nu} (\nabla^{\mu} \nabla^{\nu} \phi) - \frac{1}{6} G_{5,X} [ (\Box \phi)^3 - 3 (\Box \phi)(\nabla_{\mu} \nabla_{\nu} \phi)(\nabla^{\mu} \nabla^{\nu} \phi) + 2 (\nabla^{\mu} \nabla_{\alpha} \phi)(\nabla^{\alpha} \nabla_{\beta} \phi)(\nabla^{\beta} \nabla_{\mu} \phi) ] .$$

Here $K(\phi, X)$ and $G_i(\phi, X)$ ($i = 3, 4, 5$) are functions with respect to a scalar field $\phi$ and its kinetic energy $X = -g^{\mu \nu} \partial_{\mu} \phi \partial_{\nu} \phi/2$ ($g^{\mu \nu}$ is the metric tensor), $R$ is the Ricci scalar, and $G_{\mu \nu}$ is the Einstein tensor. We use the notations...
where the momentum tensor of matter is derived from \( g_{\mu\nu} \), and \( L_m \) is the Lagrangian of the matter fields \( \Psi_m \). The energy-momentum tensor of matter is derived from \( L_m \), as \( T_{\mu\nu} = -(2/\sqrt{-g})\delta L_m/\delta g^{\mu\nu} \). In terms of the energy density \( \rho_m \) and the pressure \( P_m \) of matter, we have that \( T_{\mu\nu} = (\rho_m, P_m, P_m, P_m) \). We do not introduce the direct coupling between the field \( \phi \) and matter. In scalar-tensor theories in which the function \( G_i \) is a function of \( \phi \), the conformal transformation to the Einstein frame gives rise to a matter coupling with \( \phi \) (as we will discuss later).

Let us consider a spherically symmetric space-time with the distance \( r \) from the center of symmetry. The line element of such a background is

\[
ds^2 = -e^{2\Psi(r)}dt^2 + e^{2\Phi(r)} dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2),
\]

where \( \Psi(r) \) and \( \Phi(r) \) are functions of \( r \). On the weak gravitational background (\(|\Psi| \ll 1, |\Phi| \ll 1\)), the metric (47) approximately reduces to that of the Newtonian gauge. For the general metric (47) we derive the equations of motion valid on the strong gravitational background as well. The (00), (11) and (22) components of the equations of motion following from the action (12) are given, respectively, by

\[
\left( A_1 + \frac{A_2}{r} + \frac{A_3}{r^2} \right) \Phi' + A_4 + \frac{A_5}{r^2} = e^{2\Phi} \rho_m, \\
\left( A_1 + \frac{A_2}{r} + \frac{A_3}{r^2} \right) \Psi' + A_7 + \frac{2A_1}{r} + \frac{A_2 + 2A_8}{2r^2} = e^{2\Phi} P_m, \\
\left( -e^{-2\Phi} A_8 + \frac{A_9}{r} \right) (\Psi'' + \Phi'') - \left[ \left( \frac{A_2}{2} + \frac{A_3 + e^{2\Phi} A_9}{r} \right) \Phi' + A_1 + \frac{A_2}{2r} \right] \Phi' \nonumber - \left( \frac{A_5}{2} + \frac{A_6 - A_10}{r} \right) \Psi' - A_4 - \frac{A_5}{2r} = e^{2\Phi} P_m,
\]

where a prime represents the derivative with respect to \( r \). The coefficients \( A_i \) \((i = 1, 2, \cdots, 10)\) are

\[
A_1 = -2 \phi' X G_{3, X} + 2 \phi' (G_{4, \phi} + 2 X G_{4, \phi X}), \\
A_2 = 4 G_4 - 16 X (G_{4, X} + X G_{4, XX}) + 4 X (3 G_{5, \phi} + 2 X G_{5, \phi X}), \\
A_3 = 2 \phi' (5 e^{-2\Phi} - 1) X G_{5, X} + 4 \phi' e^{-2\Phi} X^2 G_{5, XX}, \\
A_4 = 2 \phi' (3 G_{4, \phi} - 2 X G_{4, X}) - 4 \phi' e^{-2\Phi} (G_{4, X} - G_{5, \phi} - X G_{5, \phi X}) + 4 \phi' X G_{5, \phi X}, \\
A_6 = -2 (1 - e^{2\Phi}) G_4 + 4 X G_{4, X} - 2 X \left\{ (1 + e^{2\Phi}) G_{5, \phi} - 2 X G_{5, \phi X} \right\} + 2 \phi' X \left\{ (1 - 3 e^{-2\Phi}) G_{5, X} - 2 e^{-2\Phi} X G_{5, XX} \right\}, \\
A_7 = -e^{2\Phi} (K - 2 X K, X + 2 X G_{5, \phi}), \\
A_8 = -2 e^{2\Phi} (G_4 - 2 X G_{4, X} + X G_{5, \phi}), \\
A_9 = 2 \phi' e^{-2\Phi} X G_{5, X}, \\
A_{10} = 2 e^{2\Phi} (G_4 - 2 X G_{5, \phi}) + 2 \phi' X G_{5, X},
\]

where \( X = -e^{-2\phi} \phi''/2 \). The matter fluid satisfies the continuity equation

\[
P'_m + \Psi'(\rho_m + P_m) = 0.
\]

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1. Recently, Koyama et al. [13] expanded the action (6) up to second order of perturbations around the Minkowski background by imposing that the scalar action respects the Galilean symmetry \( \delta_1 \phi \to \delta_1 \phi + b_\mu \). In our work we derive the equations of motion from the original Horndeski’s action without putting any restriction on the functional forms of \( K \) and \( G_i \) \((i = 3, 4, 5)\) from the beginning. As a result, unlike Ref. [43], our general equations of motion can be used not only for other screening mechanisms such as chameleons [3] and symmetrons [6] but also for the study of field configurations on the strong gravitational background [43].
Varying the action \( \mathcal{L} \) with respect to \( \phi \), we obtain the equation of motion for the scalar field. Taking the \( r \) derivative of Eq. (9) and substituting it into Eq. (12) with Eqs. (8) and (10), we can derive the same field equation of motion.

On the weak gravitational background (\( |\Phi| \ll 1 \) and \( |\Psi| \ll 1 \)) the dominant contribution to the l.h.s. of Eq. (8) is of the order of \( (G_4/r^2)\Phi \). For the comparison between each term in Eqs. (8) and (10) relative to \( G_4/r^2 \), we introduce the following quantities

\[
\varepsilon_K = \frac{2 \phi^4 K X r^2}{2G_4}, \quad \varepsilon_{K\phi} = \frac{2 \phi^4 K \phi', r^3}{2G_4}, \quad \varepsilon_{KXX} = -\frac{\varepsilon_{X}}{2G_4}, \quad \varepsilon_{Pm} = \frac{\varepsilon_{2\phi} P_m r^2}{2G_4}, \quad \varepsilon_{G3\phi} = -\frac{\varepsilon_{2\phi} X G_3, \phi r^2}{2G_4},
\]

\[
\varepsilon_{G3X} = -\frac{X G_3, X \phi', r}{G_4}, \quad \varepsilon_{G4\phi} = \frac{r \phi', G_4, \phi}{G_4}, \quad \varepsilon_{G4X} = \frac{2 X G_4, X}{G_4}, \quad \varepsilon_{G5\phi} = \frac{X G_5, \phi}{G_4}, \quad \varepsilon_{G5X} = \frac{\varepsilon_{2\phi} X G_5, \phi', r}{2G_4 r},
\]

which are required to be much smaller than 1 to recover the General Relativistic behavior inside the solar system. As long as the Vainshtein mechanism is at work one can confirm that \( |\varepsilon_i| \ll 1 \), after deriving the solutions to the field equation \( \mathcal{L} \). We also define

\[
\lambda_{K\phi} = \frac{K_{\phi}, \phi', r}{K, X}, \quad \lambda_{KXX} = \frac{X K, X, X}{K, X}, \quad \lambda_{G3\phi} = \frac{G_3, \phi', r}{G_3, \phi}, \quad \lambda_{G3XX} = \frac{X G_3, \phi, X}{G_3, \phi}, \quad \lambda_{G3X} = \frac{X G_3, X}{G_3, X},
\]

\[
\lambda_{G4\phi} = \frac{G_4, \phi, \phi', r}{G_4, \phi}, \quad \lambda_{G4XX} = \frac{X G_4, X, X}{G_4, X}, \quad \lambda_{G5\phi} = \frac{G_5, \phi, \phi', r}{G_5, \phi}, \quad \lambda_{G5XX} = \frac{X G_5, X, X}{G_5, X}, \quad \lambda_{G5X} = \frac{X G_5, X}{G_5, X},
\]

which are not generally smaller than the order of 1.

For the rest of the paper we use the approximation under which all the quantities in Eq. (13) are much smaller than 1 on the weak gravitational background. From Eq. (8) the matter density \( \rho_m \) is of the order of \( (G_4/r^2)\Phi \). The continuity equation (12) shows that \( \rho_m / \rho_m \sim \Psi \) in the weak gravitational background and hence \( \varepsilon_{Pm} \sim \Psi^2 \). In what follows we neglect the terms coming from the gravitational potentials higher than first order (such as \( \Psi^2 \) and \( \Phi^2 \)) relative to the parameters \( \varepsilon_i \) defined in Eq. (13). We only keep the first-order terms of \( \varepsilon_i \). We deal with the terms \( \varepsilon_i \) multiplied by \( \lambda_j \) given in Eq. (14) as first-order terms.

Eliminating the terms \( \Psi' \) and \( \Phi' \) from Eqs. (8)-(10), we obtain

\[
\square = \mu_1 \rho_m + \mu_2 \Box \phi + \mu_3,
\]

where \( \Box \equiv d^2/dr^2 + (2/r)(d/dr) \), and

\[
\mu_1 \simeq \frac{1}{8G_4} \left[ 2 + 6 \phi + \varepsilon_K + \varepsilon_{KX} - \varepsilon_{G3\phi} - \varepsilon_{G3X} - \varepsilon_{G4\phi} - (\lambda_{G4\phi} X - 2 \lambda_{G4XX} - 3) \varepsilon_{G4X} - 8 \varepsilon_{G5\phi} (2 \lambda_{G5\phi} - 4 \lambda_{G5XX} - 12) \varepsilon_{G5X} \right],
\]

\[
\mu_2 \simeq -\frac{\varepsilon_{G3XX} + \varepsilon_{G4XX} + \lambda_{G4XX} \varepsilon_{G4X} - 4 (1 + \lambda_{G5XX} \varepsilon_{G5X})}{2 \phi'}
\]

\[
\mu_3 \simeq \frac{2 \varepsilon_K + \varepsilon_{KX} - \lambda_{G4\phi} \varepsilon_{G4\phi} + 2 \lambda_{G4XX} \varepsilon_{G4X} + 4 (\lambda_{G5\phi} - 2 \lambda_{G5XX} - 2) \varepsilon_{G5X}}{2 \phi'}.
\]

If we rewrite Eq. (12) explicitly by using Eqs. (8) and (9), we find that two Laplacian terms \( \Box \) and \( \Box \phi \) are present. Combining this equation with Eq. (15), we can eliminate the term \( \Box \) to derive the closed-form equation of \( \phi \). Using the approximation \( e^{2\phi} \simeq 1 \), it follows that

\[
\Box \phi = \mu_4 \rho_m + \mu_5.
\]
The representative example having a non-zero value of the gravitational constant $G$ is such that the gravitational potentials are not affected by the presence of the field $\phi$ where $G = 4\pi G_{\text{eff}}$. The deviation of $G$ is of the order of 1, the deviation of $\phi$ is $\propto 1$. The concrete, let us consider the theories described by the action $\mathcal{S} = \int d^4x \sqrt{-g} \left[ \frac{M_{\text{pl}}^2}{2} F(\phi) R + \omega(\phi) X - V(\phi) \right] + \int d^4x \mathcal{L}_m(g_{\mu\nu}, \Psi_m)$, where $\omega(\phi) = \frac{1}{2} \frac{\partial \phi}{\partial \phi}$ is the dilatonic coupling given by $G_{\phi} = \frac{M_{\text{pl}}^2}{2} e^{-2Q\phi/M_{\text{pl}}}$, where $Q$ is a coupling constant of the order of unity. If we consider a canonical massless field, i.e., $K = X$ and $G_3 = 0$, we have that $G_{\text{eff}} \simeq G(1 + 2Q^2 [1 + \omega(\phi)/(2Q M_{\text{pl}})] + 3\Phi + \mathcal{O}(\varepsilon_i))$ in the regime $|\phi/M_{\text{pl}}| \ll 1$. For $|Q|$ of the order of 1, the deviation of $G_{\text{eff}}$ from $G$ is significant due to the presence of the term $2Q^2$. In such cases we need to resort to some mechanism to suppress the propagation of the fifth force.

Provided that the term $\mu_2 \Box \phi$ in Eq. (13) is suppressed relative to other terms, the General Relativistic behavior can be recovered at short distances. There are several known mechanisms to screen the fifth force: (i) the chameleon mechanism [3], (ii) the symmetron mechanism [6], and (iii) the Vainshtein mechanism [7]. All of them are covered in our general set-up.

Both the chameleon and symmetron mechanisms are based on the presence of the field potential $V(\phi)$. To be more concrete, let us consider the theories described by the action

$$S = \int d^4x \sqrt{-g} \left[ \frac{M_{\text{pl}}^2}{2} F(\phi) R + \omega(\phi) X - V(\phi) \right] + \int d^4x \mathcal{L}_m(g_{\mu\nu}, \Psi_m),$$

where

$$\mu_4 \simeq -\frac{r}{4G_4 \beta} \left[ 2 G_{4,\phi} + 4 X G_{4,\phi X} - 2 X G_{3,\phi X} - \phi' \beta - \frac{4 X (G_{5,\phi} + X G_{5,XX})}{r^2} \right],$$

$$\mu_5 \simeq -\frac{1}{r^2 \beta} \left[ (K_{\phi} - 2 X K_{\phi X} + 2 X G_{3,\phi X}) r^2 - 4 X (K_{XX} - 2 G_{3,\phi X} + 2 G_{4,\phi X}) \phi' r - 4 X (3 G_{3,\phi X} + 4 X G_{3,XX} - 9 G_{4,\phi X} - 10 X G_{4,\phi XX} + X G_{5,\phi X}) + \frac{8 X \phi' (G_{4,\phi XX} + 2 X G_{4,XXX} - 2 G_{5,\phi X} - X G_{5,XX})}{r} \right].$$

After the linear expansion with respect to the parameters $\varepsilon_i$, we reverted to use the original functions $K$ and $G_i$. Eliminating the term $\Box \phi$ from Eqs. (15) and (18), we obtain the modified Poisson equation

$$\Box \Psi = 4\pi G_{\text{eff}} \rho_m + \mu_2 \mu_5 + \mu_3,$$

where

$$G_{\text{eff}} \equiv \frac{1}{4\pi} (\mu_2 \mu_4 + \mu_1) = \frac{1}{16\pi G_4} \left[ 1 + \frac{r}{G_4 \beta} \alpha \left( \alpha - \frac{1}{2} \phi' \beta \right) + 3\Phi + \mathcal{O}(\varepsilon_i) \right],$$

$$\alpha \equiv G_{4,\phi} + 2 X G_{4,\phi X} - X G_{3,\phi X} - \frac{2 X (G_{5,\phi} + X G_{5,XX})}{r^2}.$$
where $F(\phi)$ and $\omega(\phi)$ are functions of $\phi$. The Brans-Dicke theory with the potential $V(\phi)$ corresponds to $F(\phi) = e^{-2Q\phi/M_{pl}}$ and $\omega(\phi) = (1 - 6Q^2)e^{-2Q\phi/M_{pl}}$. The coupling $Q$ is related to the Brans-Dicke parameter $\omega_{BD}$, as $3 + 2\omega_{BD} = 1/(2Q^2)$ [3, 5]. The metric $f(R)$ gravity and the dilaton gravity are the sub-class of Brans-Dicke theory with the parameters $\omega_{BD} = 0$ ($Q^2 = 1/6$) [40] and $\omega_{BD} = -1$ ($Q^2 = 1/2$) [40], respectively. In this case the field equation of motion (19) reads

$$\Box \phi = \frac{1}{\omega} \left( -\frac{M_{pl}^2 F_{\phi}}{2M_{pl}^2 F} \rho_m - \phi' \omega_T - \rho_m \omega_{\phi} + V_{,\phi} \right).$$

(29)

The coupling such as $F(\phi) = e^{-2Q\phi/M_{pl}}$ gives rise to a matter coupling term $Q\rho_m/M_{pl}$ inside the parenthesis of Eq. (29). In order to have the description of a canonical field coupled to matter, it is convenient to transform the action (28) to that in the Einstein frame by a conformal transformation $\hat{g}_{\mu\nu} = F(\phi)g_{\mu\nu}$ [50]. The action in the Einstein frame is given by

$$S = \int d^4x \sqrt{-g} \left[ \frac{M_{pl}^2}{2} \hat{R} \right. - \frac{1}{2} \hat{g}^{\mu\nu} \partial_\mu \chi \partial_\nu \chi - \hat{V}(\chi) \bigg] + \int d^4x L_m(A^2(\chi)\hat{g}_{\mu\nu},\Psi_m),$$

(30)

where

$$\chi \equiv \int d\phi \sqrt{\frac{3}{2} \left( \frac{M_{pl}^4 F_{,\phi}}{F} \right)^2 + \frac{\omega}{F}} + \frac{\hat{V}(\chi)}{F^2}, \quad \hat{V}(\chi) \equiv \frac{V}{F^2}, \quad A^2(\chi) \equiv F^{-1}(\phi).$$

(31)

Variation of the action (30) with respect to the canonical field $\chi$ gives $\hat{\Box} \chi = \hat{V}_{eff,\chi} - (1/\sqrt{-g})(\partial L_m/\partial \chi)$. The field $\chi$ couples to matter, as $\partial L_m/\partial \chi = (A^2(\chi)/\sqrt{-g}) T$, where $T = -\rho_m + 3P_m \simeq -\rho_m$ is the trace of non-relativistic matter in the Einstein frame. The energy density $\rho_m$ in the Jordan frame is related to $\hat{\rho}_m$ via $\hat{\rho}_m = A^4 \rho_m$. Using the conserved energy density $\rho'_m = A^3 \rho_m = \hat{\rho}_m/A$ in the Einstein frame, the field equation reads [3]

$$\hat{\Box} \chi = \hat{V}_{eff,\chi}, \quad \hat{V}_{eff}(\chi) \equiv \hat{V}(\chi) + A(\chi) \rho'_m.$$

(32)

In Brans-Dicke theory with the functions $F(\phi) = e^{-2Q\phi/M_{pl}}$ and $\omega(\phi) = (1 - 6Q^2)e^{-2Q\phi/M_{pl}}$, the field $\chi$ is equivalent to $\phi$ and hence $A(\chi) = e^{Q\chi/M_{pl}}$. For a run-away potential $\hat{V}(\chi)$, the effective potential $\hat{V}_{eff}(\chi)$ can have a minimum at $\hat{V}_{eff,\chi}(\chi_M) = 0$ due to the presence of the matter coupling $e^{Q\chi/M_{pl}}\rho'_m$. The effective mass $m_\chi$ of the field at $\chi = \chi_M$ depends on the matter density $\rho'_m$. Provided that the mass $m_\chi$ is large in the region of high density and that a spherically symmetric body has a thin shell around its surface, the propagation of the fifth force is suppressed outside the body. This is the screening effect of the chameleon mechanism. The solution of Eq. (32) and the resulting local gravity constraints on concrete potentials [such as $V(\phi) = M^{4+}\phi^{-n}$ and $V(\phi) = M^4 \exp(M^n/\phi^n)$] were studied in detail in Refs. [3, 51], so we do not repeat them here. The application of the chameleon mechanism to dark energy models based on $f(R)$ gravity and Brans-Dicke theory was carried out in Refs. [4, 5].

In the symmetron mechanism the choices of the coupling $A(\chi)$ and the potential $\hat{V}(\chi)$ are different from those in the chameleon mechanism. They are given by [6]

$$A(\chi) = 1 + \frac{\chi^2}{2M^2}, \quad \hat{V}(\chi) = -\frac{1}{2} \hat{\mu}^2 \chi^2 + \frac{1}{4} \lambda \chi^4,$$

(33)

where $M$, $\mu$, $\lambda$ are constants. In this case the effective potential (32) reads

$$\hat{V}_{eff}(\chi) = \frac{1}{2} \left( \frac{\rho'_m}{M^2} - \hat{\mu}^2 \right) \chi^2 + \frac{1}{4} \lambda \chi^4,$$

(34)

up to an irrelevant constant. For large $\rho'_m$ the effective potential is $\hat{V}_{eff} \simeq \rho'_m \chi^2/(2M^2)$ and hence the field is nearly frozen around $\chi = 0$. For small $\rho'_m$ the $Z_2$ symmetry is spontaneously broken, so that the field has a vacuum expectation value $\chi_0 = \mu/\sqrt{\lambda}$ in the limit $\rho'_m \to 0$. Since the propagation of the fifth force is suppressed in the region of high density, it is possible for the symmetron field to pass local gravity constraints. The experimental bounds and the cosmological implication of symmetrons were studied in detail in Refs. [52].

In both the chameleon and symmetron mechanisms the $\hat{\Box} \chi$ term in Eq. (32) is suppressed in the regions of high density, in which case the field-dependent term $\hat{\mu}^2 \hat{\Box} \chi$ in Eq. (15) is effectively decoupled from gravity. The viability of these two mechanisms heavily depends on the choice of the field potentials. If the chameleon field is responsible...
for the cosmic acceleration today, the potential needs to be carefully designed to satisfy both cosmological and local gravity constraints \(^{[3]}\). It is also known that the energy scale of the simplest symmetron potential \(^{[33]}\) is too small to be used for dark energy \(^{[6]}\).

The Vainshtein mechanism, which is the main topic of our paper, is based on non-linear field self-interactions like \(X \Box \phi\). In such cases, the last two terms inside the parentheses on the r.h.s. of Eqs. (21) and (22) provide the dominant contributions for the distance smaller than the so-called Vainshtein radius \(r_V\). In this regime we have \(|\mu_4 \rho_m| \ll |\mu_5|\) and hence \(\Box \phi \simeq \mu_5 \Box \phi\). For the choice \(G_3 \propto X\) and the coupling (27), the solution to the field equation is given by \(\phi'(r) \propto r^{-1/2}\), so that the terms \(\mu_2 \mu_4\) and \(\mu_2 \mu_5\) in Eq. (24) are suppressed relative to other terms.

In this paper we study the effects of other non-linear field self-interactions such as those coming from \(G_4(\phi, X)\) and \(G_5(\phi, X)\) in addition to the term \(G_3(\phi, X)\). For concreteness we choose the field coupling of the form (27) and non-linear field derivative couplings. The coupling (27) is sufficiently general in that it covers a wide variety of theories (the field \(\phi\) characterizes the size of compact space or the position of a probe brane) \(^{[19, 53, 54]}\).

Unlike the chameleon and symmetron scenarios, the Vainshtein mechanism can be at work even without the field potential \(V(\phi)\). We adopt the k-essence Lagrangian of the form \(K(\phi, X) = f_2(\phi)g_2(X)\) \(^{[55]}\) without an explicit potential \(V(\phi)\), where the function \(g_2(X)\) includes the non-linear terms of \(X\). For the functions \(G_i(\phi, X)\) \((i = 3, 4, 5)\) we also consider the couplings of the form \(f_i(\phi)g_i(X)\). In summary we focus on the theories characterized by

\[
K(\phi, X) = f_2(\phi)g_2(X), \quad G_3(\phi, X) = f_3(\phi)g_3(X),
G_4(\phi, X) = \frac{M_{pl}^2}{2}e^{-2Q\phi/M_{pl}} + f_4(\phi)g_4(X), \quad G_5(\phi, X) = f_5(\phi)g_5(X).
\]

For concreteness we take the exponential couplings of the form

\[
f_i(\phi) = e^{-\lambda_i\phi/M_{pl}}, \quad (i = 2, 3, 4, 5),
\]

where \(\lambda_i\)’s are constants. The choice of (36) is motivated by the dilatonic couplings appearing in low-energy effective string theory. The constants \(\lambda_i\) and \(Q\) are assumed to be at most of the order of unity. Provided that the Vainshtein mechanism is at work, the field can stay in the regime \(|\phi/M_{pl}| \ll 1\). In most cases the models with constant \(f_i(\phi)\) do not exhibit significant differences from those with the exponential couplings (36), but there are some specific models in which the presence of the field-dependent couplings can change the behavior of solutions (such as those discussed in Sec. V.B.1). In Secs. IV and V we clarify this issue in detail.

The non-linear self-interaction \(g_3(X)\) proportional to \(X\) arises in the DGP braneworld scenario \(^{[10, 11]}\) and in the Kaluza-Klein theory with a higher-dimensional Gauss-Bonnet term \(^{[54]}\). The covariant Galileon \(^{[16]}\), whose Lagrangian arises as a non-relativistic limit for a probe brane embedded in a five-dimensional bulk \(^{[19]}\), corresponds to the choice \(g_3(X) = X\), \(g_3(X) = X\), \(g_4(X) = X^2\), and \(g_5(X) = X^2\), with constant \(f_i(\phi)\) (i.e., \(\lambda_i = 0\)). The extended Galileon \(^{[30, 51]}\) has more general powers \(p_i\) of the derivative terms, i.e., \(g_i(X) = X^{p_i}\). The four-dimensional Gauss-Bonnet coupling \(-f(\phi)R^2_{	ext{GB}}\) can be also accommodated in Eq. (35) for specific choices of \(f_i(\phi)\) and \(g_i(X)\) \(^{[22]}\).

### III. VAINShteIN MECHANISM

Let us study how the Vainshtein mechanism generally works for the theories given by the functions (35). We assume that the coupling \(Q\) is of the order of unity. The field non-linear self-interactions \(f_i(\phi)g_i(X)\) in \(G_i(\phi, X)\) \((i = 3, 4, 5)\) can be responsible for the suppression of the fifth force within the so-called Vainshtein radius \(r_V\). In the following we study general solutions of the field equation of motion (19) in the regimes (A) \(r \gg r_V\), (B) \(r_a \ll r \ll r_V\), and (C) \(r < r_a\), separately, where \(r_a\) is the Schwarzschild radius of a star with the radius \(r_s\). In Sec. III.D we apply our results to a concrete model to discuss the matching of solutions in three different regimes. In the same model we also derive the explicit expression of the gravitational potentials inside the Vainshtein radius.

#### A. \(r \gg r_V\)

For the distance \(r\) much larger than \(r_V\), the non-linear field-self interactions are suppressed in Eqs. (20) – (22). In Eq. (20) this means that the term \(2G_{4, \phi}\) is the dominant contribution, i.e.,

\[
|2G_{4, \phi}| \gg |4XG_{4, \phi} - 2XG_{3, \phi} - \phi' \beta - 4X(G_{5, \phi} + XG_{5, XX})/r^2|.
\]
For the function $G_4(\phi, X)$ given in Eq. (35), the following condition should be satisfied
\[ M_{pl}^2 e^{-2Q\phi/M_{pl}}/2 \gg e^{-\lambda_4\phi/M_{pl}} g_4(X). \] (38)

Moreover, we assume that the field is in the range
\[ |\phi/M_{pl}| \ll 1, \] (39)
which can be justified after deriving the solution to Eq. (19).

The function $g_2(X)$ inside $K(\phi, X)$ may be written in terms of the sum of the polynomials, as $g_2(X) = \sum_{n=1}^{\infty} c_n \mu^4 (X/\mu^4)^n$, where $c_n$’s are dimensionless constants and $\mu$ is another constant having a dimension of mass. In the following we focus on the model in which the first term in $g_2(X)$ dominates over the other terms, i.e., $g_2(X) \simeq c_1 X$. Without loss of generality we can choose the coefficient to be $c_1 = 1$, so that the function $K(\phi, X)$ is
\[ K(\phi, X) = e^{-\lambda_2\phi/M_{pl}} X. \] (40)

Since the term $K X r$ should be the dominant contribution in Eq. (22), we have
\[ r \gg 2(G_{3,\phi} + XG_{3,\phi X})r + 4(G_{3,X} + XG_{3,XX} - 3G_{4,\phi X} - 2XG_{4,\phi XX})\phi' - 2(3G_{4,XX} + 2XG_{4,XXX} - 2G_{5,\phi X} - XG_{5,\phi XX})\phi'^2/r]. \] (41)

We are also in the regime where the matter-coupling term $\mu_4 \rho_m$ dominates over another term $\mu_5$, i.e.,
\[ |\mu_4| |\rho_m| \gg |\mu_5|. \] (42)

Since $\mu_4 \simeq Q/M_{pl}$ under the above conditions, Eq. (19) reads
\[ \frac{d}{dr}(r^2 \phi') \simeq QM_{pl} \frac{d \rho_m}{dr}, \] (43)
where the Schwarzschild radius $r_g$ is defined by
\[ r_g \equiv \frac{1}{M_{pl}^2} \int_0^r \rho_m r^2 d\bar{r}. \] (44)

Integration of Eq. (42) gives the following solution
\[ \phi'(r) = \frac{QM_{pl} r_g}{r^2} \quad (r \gg r_V). \] (45)

This gives rise to the fifth force of the order of $|\phi'(r)/M_{pl}| = |Q|r_g/r^2$, by which the gravitational law is significantly modified. On using the boundary condition $\phi(\infty) \to 0$, we obtain
\[ \phi(r) = -\frac{QM_{pl} r_g}{r}. \] (46)

Then, the condition (39) translates into $r \gg r_g$ for $|Q| = O(1)$. Since we are now in the regime $r \gg r_V$, it can be interpreted as
\[ r_V \gg r_g. \] (47)

For a given model (i.e., for given functions of $G_{3,4,5}$) we need to confirm whether the conditions (37), (38), (41), and (42) are satisfied. In Sec. III D we confirm those conditions for a concrete model.

B. $r_g \ll r \ll r_V$

For the distance much smaller than $r_V$, the non-linear field self-interactions are the dominant contribution in Eq. (19). The Vainshtein radius is characterized by the distance at which the field self-interactions become comparable to the term $K X r$, i.e.,
\[ r_V = |2(G_{3,\phi} + XG_{3,\phi X})r_V + 4(G_{3,X} + XG_{3,XX} - 3G_{4,\phi X} - 2XG_{4,\phi XX})\phi'(r_V) - 2(3G_{4,XX} + 2XG_{4,XXX} - 2G_{5,\phi X} - XG_{5,\phi XX})\phi'^2(r_V)/r_V|]. \] (48)
For a given model, the Vainshtein radius is explicitly known by employing the solution 45. The regime \( r < r_V \) corresponds to the opposite inequality of Eq. (11), i.e.,

\[
-2(3G_{4,\phi,XX} + 2XG_{4,\phi,XX} - 2G_{5,\phi,XX} - XG_{5,\phi,XX})/|r|.
\]

The terms inside Eq. (21) should satisfy the following condition

\[
|\frac{1}{\xi_2} \cdot \frac{d}{dr} (r^2 \phi')| \ll |4(3G_{4,\phi,XX} + XG_{4,\phi,XX} - 10G_{4,\phi,XX} + XG_{5,\phi,XX}) - 8\phi'(3G_{4,\phi,XX} + 2XG_{4,\phi,XX} - 2G_{5,\phi,XX} - XG_{5,\phi,XX})/|r|.
\]

As long as the Vainshtein mechanism is at work, the matter coupling term \( \mu_4 \rho_m \) should be suppressed relative to the term \( \mu_5 \), i.e.,

\[
|\mu_4 \rho_m| \ll |\mu_5|.
\]

Under the conditions (19), 20, 21 the field equation (19) reads

\[
\frac{d}{dr} (r^2 \phi') \approx \frac{\xi_1}{\xi_2} r \phi',
\]

where

\[
\xi_1 \equiv r(3G_{3,\phi,XX} + 4XG_{3,\phi,XX} - 9G_{4,\phi,XX} - 10G_{4,\phi,XX} + XG_{5,\phi,XX}) - 2\phi'(3G_{4,\phi,XX} + 2XG_{4,\phi,XX} - 2G_{5,\phi,XX} - XG_{5,\phi,XX}),
\]

\[
\xi_2 \equiv 2\phi'(3G_{3,\phi,XX} + 4XG_{3,\phi,XX} - 8G_{4,\phi,XX}) - \phi'(3G_{4,\phi,XX} + 2XG_{4,\phi,XX} - 2G_{5,\phi,XX} - XG_{5,\phi,XX}).
\]

For a given model, the solution to \( \phi'(r) \) is known by integrating Eq. (52). After deriving the solution, we need to confirm whether the conditions (49)-(51) are satisfied in the regime \( r < r_V \). In Sec. IIIB we study the solution of Eq. (52) for a concrete model to understand the consistency of the conditions used above. For the validity of the solution we typically require that the distance \( r \) is much larger than \( r_s \). This is related to the fact that inside a spherically symmetric body the matter density \( \rho_m \) becomes large, so that the condition (51) tends to be violated.

C. \( r < r_s \)

If the condition (51) is violated inside a star with the radius \( r_s \), we can no longer use the solution to the field equation (52). At the origin we generally impose the following boundary condition

\[
\phi'(0) = 0.
\]

If all the non-linear terms \( f_i(\phi) g_i(X) \) in Eq. (35) are suppressed relative to the term \( G_4 = M_\text{pl}^2 e^{-2Q/\rho_m}/(2M_\text{pl}) \), the field equation (19) reduces to the same form as Eq. (43), i.e., \( d(r^2 \phi)/dr \approx Q \rho_m r/M_\text{pl} \). Assuming that \( \rho_m \) is nearly constant inside the body, we obtain the integrated solution \( \phi'(r) \approx Q \rho_m r/(3M_\text{pl}) \). In fact, this satisfies the boundary condition (54). However, if the solution inside the body corresponds to this type, it gives rise to a large modification of gravity around the surface of the star because it is the analogue of (14) outside the star.

In the presence of the non-linear field self-interactions, the solutions to the field equation (62) inside the body are different from \( \phi'(r) \approx Q \rho_m r/(3M_\text{pl}) \). As we will see in the following sections, the solutions depend on the choice of the functions \( G_i(\phi, X) \). As long as the Vainshtein mechanism operates both inside and outside the star, we will show that the interior and exterior solutions smoothly connect each other. In order to study the matching of the solutions properly, we need to assume the density profile of the star. In the numerical simulations given in the following sections, we employ the profile

\[
\rho_m = \rho_c \exp \left( -r^2/r_1^2 \right),
\]

where \( \rho_c \) is the central density of the body with the radius \( r_s \). The density \( \rho_m \) starts to decrease significantly around the distance \( r_1 \). We confirmed that the different choices of the density profile do not affect our main results.
Let us consider the covariant Galileon model \([16]\) in the presence of the term \(G_4\) alone, i.e.,

\[
G_4(\phi, X) = \frac{M_\text{pl}^2}{2} e^{-2Q\phi/M_\text{pl}} + \frac{c_4}{M^6} X^2, \quad G_3 = G_5 = 0,
\]

where \(c_4\) is a dimensionless constant of the order of 1, and \(M\) is another constant having the dimension of mass. Substituting the solution \([45]\) into Eq. \([48]\), we obtain the Vainshtein radius

\[
r_V = \left(12 |c_4| \right)^{1/6} \left(\frac{|Q|M_\text{pl} r_g}{M} \right)^{1/3} \approx \left(\frac{|Q|M_\text{pl} r_g}{M} \right)^{1/3}.
\]

The field equation \([52]\) reduces to

\[
\frac{d}{dr} (r^2 \phi') = 2r \phi'.
\]

The solution to this equation is simply given by

\[
\phi'(r) = C,
\]

where \(C\) is a constant. Matching \([59]\) with \([45]\) at \(r = r_V\), it follows that

\[
\phi'(r) = \frac{Q M_\text{pl} r_g}{r_V^3} \quad (r_g \ll r \ll r_V).
\]

Using the boundary condition \(\phi(r_V) = -Q M_\text{pl} r_g / r_V\), we obtain the following solution

\[
\phi(r) = \frac{Q M_\text{pl} r_g}{r_V} \left( \frac{r}{r_V} - 2 \right).
\]

This shows that even in the regime \(r_g \ll r \ll r_V\), the condition \([39]\) is satisfied for \(r_V \gg r_g\). Now we check the consistency of several other conditions used in Secs. \([11A]\) and \([11B]\). Let us first consider the regime \(r \gg r_V\) with the solution \([45]\). The condition \([41]\) simply corresponds to \(r \gg r_V\), where \(r_V\) is given by Eq. \([67]\). Since \(\beta \approx r\) in this case the condition \([57]\) translates to \(r \gg r_g\), which is equivalent to \([47]\). The condition \([38]\) reduces to \((r/r_V)^{6} \gg Q^2 (r_g/r_V)^3/24\), which is automatically satisfied for \(r_V \gg r_g\). Since \(\mu_4 \approx Q/M_\text{pl}\) and \(\mu_5 \approx 24c_4 Q^3/M_\text{pl}^3 r_g^3/(M_\text{pl}^3 r_g^3)\) for \(r\) not away from \(r_V\), the condition \([42]\) corresponds to

\[
r \gg r_* \equiv \left( 24 |c_4| Q^2 \frac{M_\text{pl}^4 r_g^3}{M^6 \rho_m} \right)^{1/9} \approx \left( \frac{Q^2 M_\text{pl}^4 r_g^3}{M^6 \rho_m} \right)^{1/9}.
\]

The critical radius \(r_*\) depends on the mass scale \(M\) and the density profile \(\rho_m\). The ratio between \(r_*\) and \(r_V\) is given by

\[
\frac{r_*}{r_V} \approx \left( \frac{M^3 M_\text{pl}}{|Q| \rho_m} \right)^{1/9}.
\]

If the same model is responsible for the late-time cosmic acceleration, the mass scale \(M\) is related to the today’s Hubble parameter \(H_0\), as \(M^3 \approx M_\text{pl} H_0^2\). Since the critical density \(\rho_0 \approx 10^{-29} \text{g/cm}^3\) has the relation \(\rho_0 \approx M_\text{pl}^2 H_0^2\), the ratio \([63]\) can be estimated as \(r_* / r_V \approx (\rho_0 / \rho_m)^{1/9}\) for \(|Q| = O(1)\). If \(\rho_m\) is close to \(\rho_0\), then \(r_* \approx r_V\). The Schwarzschild radius of the Sun is \(r_g \approx 3 \times 10^8\) cm, in which case the Vainshtein radius can be estimated as \(r_V \approx (r_g H_0^{-2})^{1/3} \approx 10^{20}\) cm for \(M^3 = M_\text{pl} H_0^2\). This radius is much larger than the solar-system scale. Even by taking the mean density \(\rho_m \approx 10^{-24}\) g/cm\(^3\) of our galaxy, \(r_*\) is the same order as \(r_V\). The above discussion shows that all the conditions \([37], [38], [41], [42]\) are satisfied in the regime \(r \gg r_V\).

We proceed to the regime \(r_g \ll r \ll r_V\) characterized by the solution \([60]\). The condition \([49]\) exactly corresponds to \(r \ll r_g\). For \(|\lambda_2| = O(1)\) the condition \([50]\) translates to \((r/r_V)^3 \ll r_g / r_V\), which is automatically satisfied for \(r_V \gg r_g\). In the regime \(r \gg r_g\) we have that \(\mu_4 \approx Q(r/r_V)^2/(12c_4 M_\text{pl})\) and \(\mu_5 \approx 2QM_\text{pl} r_g / (r_V^2 r)\). Then, the condition \([51]\) can be interpreted as \(r \ll \tilde{r}_* \equiv (M_\text{pl}^2 r_g / \rho_m)^{1/3}\). Since \(\tilde{r}_*/r_V \approx (M^3 M_\text{pl} / \rho_m)^{1/3}\), \(\tilde{r}_*\) is close to \(r_V\) for
\( M^2 \approx M_{\text{in}} H_0^2 \) and \( \rho_m \approx \rho_0 \). Thus, all the conditions used to derive the solution (60) are consistently satisfied. We can also check that the quantities \( \varepsilon \), defined in Eq. (13) remain much smaller than the order of 1.

Picking up the dominant terms of Eqs. (5) and (9) in the regime \( r_g \ll r \ll r_V \), it follows that

\[
\frac{d}{dr}(r\Phi) \simeq -\frac{2Q\phi'r^2}{M_{\text{pl}}} + \frac{\rho_mr^2}{2M_{\text{pl}}}.
\]

(64)

\[
\Psi' \simeq \frac{\Phi}{r} + \frac{2Q\phi'}{M_{\text{pl}}}.
\]

(65)

Using (60), we obtain the following integrated solutions

\[
\Phi \simeq \frac{r_{\text{in}}}{2r} \left[ 1 - 2Q^2 \left( \frac{r}{r_V} \right)^2 \right], \quad \Psi \simeq \frac{r_{\text{in}}}{2r} \left[ 1 - 2Q^2 \left( \frac{r}{r_V} \right)^2 \right].
\]

(66)

We define the post-Newtonian parameter \( \gamma \), as

\[
\gamma \equiv \frac{\Phi}{\Psi},
\]

(67)

whose experimental bound is \( |\gamma - 1| < 2.3 \times 10^{-5} \). From Eq. (60) we have \( \gamma \simeq 1 \) in the present model. The higher-order terms we neglected to derive the solutions (60) are even much smaller than the term \( 2Q^2(r/r_V)^2 \). Hence the experimental bound of \( \gamma \) is well satisfied inside the Vainshtein radius.

For \( r \) close to 0 the solution to Eq. (19) is different from Eq. (60). In this regime there is the solution

\[
\phi'(r) = CM^3 r,
\]

(68)

where \( C \) is a dimensionless constant determined below. We assume that \( \rho_m \) approaches a constant value \( \rho_c \) as \( r \to 0 \). Since \( \mu_4 \simeq \frac{Q}{M_{\text{pl}}(1 + 12c_4^2)} \) and \( \mu_5 \simeq 24c_4^3 M^3/(1 + 12c_4^2) \), integration of Eq. (19) gives the relation

\[
3C_4 (4c_4^2 + 1) \simeq Q \rho_c / (M^3 M_{\text{pl}}).
\]

If we consider the mass scale \( M^3 \approx M_{\text{pl}} H_0^2 \approx \rho_0 / M_{\text{pl}} \), then \( |Q \rho_c / (M^3 M_{\text{pl}})| \approx |Q \rho_c / \rho_0| \gg 1 \) for the Sun. Since \( |4c_4^2| \gg 1 \) in this case, the constant \( C \) reduces to

\[
C \simeq \left( \frac{Q \rho_c}{12c_4 M_{\text{pl}}^3} \right)^{1/3},
\]

(69)

by which the solution is given by

\[
\phi'(r) \simeq \left( \frac{Q \rho_c}{12c_4 M_{\text{pl}}} \right)^{1/3} M^2 r, \quad (r \sim 0).
\]

(70)

This satisfies the boundary condition (54). The sign of (70) should be the same as (60) for the matching of two solutions, in which case we require

\[
c_4 > 0.
\]

(71)

For the star with constant density \( \rho_c \) the solution (70) should be valid up to the radius \( r_s \). In this case the Schwarzschild radius is \( r_s = \rho_c r_s^3 / (3M_{\text{pl}}^2) \) from Eq. (14). Using the Vainshtein radius (57), the constant \( C \) in Eq. (69) can be estimated as \( |C| \simeq r_V / r_s \) for \( c_4 = O(1) \). Then the solution inside the star is given by \( |\phi'(r_s)| \simeq (r_V / r_s) M^3 r \), by which \( |\phi'(r_s)| \approx r_M H_0^2 \) around the surface. Since the solution outside the star is \( \phi \simeq Q M_{\text{pl}} r / r_V^2 \), we find that \( |\phi'(r)| / |\phi'(r_s)| \simeq 1 \) by using Eq. (67). Hence the two solutions smoothly connect each other around the surface of the body. In other words, the Vainshtein mechanism is at work for the solution (70) inside the body.

In order to see how the matching of the two solutions (60) and (70) occurs for the varying matter density, we solve the field equation (19) numerically for the density profile (55). We introduce the following dimensionless variables

\[
x = \frac{r}{r_s}, \quad y = \frac{M_{\text{pl}}}{M_0} \rho_c r_s^3 \phi'(r), \quad z = \frac{\phi}{M_{\text{pl}}}, \quad \beta_1 = \frac{r}{r_s}, \quad b_1 \equiv \left( \frac{\rho_c r_s^3}{M_{\text{pl}}} \right)^{1/3}, \quad b_2 \equiv \left( \frac{M^3 r_s^2}{M_{\text{pl}}} \right)^{1/3}.
\]

(72)

The parameter \( b_1 \) can be estimated as \( b_1 \approx 0.1 \) for the Sun (\( \rho_c \approx 100 \text{ g/cm}^3, r_s \approx 7 \times 10^{10} \text{ cm} \)) and \( b_1 \approx 10^{-3} \) for the Earth (\( \rho_c \approx 10 \text{ g/cm}^3, r_s \approx 6 \times 10^8 \text{ cm} \)). The parameter \( b_2 \) depends on the mass scale \( M \). If we take
the mass \( M^3 \approx M_{pl}H_0^2 \) relevant to dark energy, we have \( b_2 \approx 10^{-12} \) for the Sun and \( b_2 \approx 10^{-13} \) for the Earth. The field equation \((19)\) can be rewritten in terms of the dimensionless variables \((72)\). In realistic situations it is a good approximation to neglect the terms including \( r \), in which case Eq. \((19)\) reads

\[
\frac{dy(x)}{dx} \approx \frac{1}{4c_4} x \left[ Qx + 6c_4 b_1^2 e^{2Qz(x)} y(x) \right] e^{-x^2/\beta^2}.
\]  

The variable \( z(x) \) obeys the differential equation

\[
\frac{dz(x)}{dx} = b_1^2 y(x)^{1/3}.
\]  

For \( x \) close to 0, the solution \((70)\) corresponds to \( y(x)^{1/3} \approx [Q/(12c_4)]^{1/3} x \), in which case the first term on the r.h.s. of Eq. \((73)\) dominates over the second term. In the regime \( r_\gamma \ll r \ll r_V \) the solution is given by Eq. \((60)\), i.e., \( y(x)^{1/3} = Qr_\gamma r_\beta / (b_1^2 r_V^2) \). For \( M^3 \approx M_{pl}H_0^2 \) the order of \((60)\) is estimated as \( y(x)^{1/3} = O(0.1) \) for both the Sun and the Earth. From Eq. \((73)\) the variation of \( z(x) \) is negligibly small, so that \( e^{2Qz(x)} \approx 1 \) in Eq. \((73)\). The second term in the square bracket of Eq. \((73)\) becomes comparable to the term \( Qx \) for the distance \( x > (2b_1^{-3})^{1/2} \), which translates into the condition \( r \gg r_s \) for both the Sun and the Earth. Outside the star the r.h.s. of Eq. \((73)\) starts to decrease rapidly by the exponential factor \( e^{-x^2/\beta^2} \). Then, for \( r \gg r_s \), the solution should be described by \( y(x) = \text{constant} \), i.e., \((60)\).

Numerically we solve Eqs. \((73)\) and \((74)\) with the boundary conditions \( y(0) = 0 \) and \( z(0) = 0 \). Although the terms including \( b_2 \) are neglected in Eq. \((73)\), we checked that using the full field equation of motion gives practically identical results for \( b_2 \ll 1 \). Figure 1 shows \( y^{1/3} \) versus \( r/r_s \) for three different values of \( r_\gamma/r_s \). Clearly the solution \((70)\) smoothly connects with another solution \((60)\). For smaller \( r_\gamma/r_s \) the term \( \mu_4 \rho_m \) starts to be suppressed at shorter distances, so that the transition to the solution \((60)\) occurs at smaller \( r \). The numerical values of the asymptotically constant solution are typically of the order of \( y^{1/3} = O(0.1) \).

Substituting the solution \((70)\) into Eqs. \((8)\) and \((9)\), we find that the corrections from the field derivative to \( \Phi \) and \( \Psi \) are proportional to \( r^2 \). For the star with a nearly constant density the leading-order contributions to the gravitational potentials have the \( r \)-dependence: \( \Phi \propto \Psi \propto r^2 \). The ratio between the corrections and the leading-order terms is of the order of \( |CQ M^3 M_{pl}/\rho_c| \approx Q^2/C^2 \approx Q^2(r_s/r_V)^2 \ll 1 \), so that the corrections are suppressed for \( r \ll r_s \ll r_V \).
IV. COVARIANT GALILEONS

The covariant Galileon \([12]\) is characterized by the Lagrangian

\[
G_3(X) = \frac{c_3}{M^3} X, \quad G_4(\phi, X) = \frac{M_{\text{pl}}^2}{2} e^{-2Q \phi/M_{\text{pl}}} + \frac{c_4}{M^6} X^2, \quad G_5(X) = \frac{c_5}{M^9} X^2, \tag{75}
\]

where \(c_{3,4,5}\) are dimensionless constants, and \(M\) is a constant having a dimension of mass. Since \(G_5(X)\) does not have a \(\phi\)-dependence, the terms such as \(G_{5,\phi X}\) and \(G_{5,\phi X X}\) in Eqs. (21) and (22) vanish. This means that \(G_5(X)\) alone does not accommodate the Vainshtein mechanism. This situation is different in the presence of the terms \(G_3(X)\) and \(G_4(\phi, X)\) given above.

From Eq. (43) the Vainshtein radius \(r_V\) is given by

\[
\frac{M^3 r_V^3}{Q M_{\text{pl}} r_g} = c_V, \tag{76}
\]

where \(c_V = 2c_3 \pm \sqrt{4c_3^2 - 12c_4}\) or \(c_V = -2c_3 \pm \sqrt{4c_3^2 + 12c_4}\). The signs inside \(c_V\) should be chosen to have a real value of \(c_V\) consistent with the l.h.s. of Eq. (76). For \(c_3 = c_4 = 1\) and \(Q > 0\), it follows that \(r_V = (2QM_{\text{pl}} r_g)^{1/3}/M\). In the limit that \(c_4 \to 0\) and \(c_3 \to 0\) we have \(M^3 r_V^3/(QM_{\text{pl}} r_g) = \pm 4c_3\) and \(M^3 r_V^3/(QM_{\text{pl}} r_g) = \pm 2\sqrt{3}c_4\), respectively. In the following we study the case in which \(r_V\) is of the order of \((Q|M_{\text{pl}} r_g)^{1/3}/M\), i.e., \(|c_V| \sim 1\).

A. \(c_5 = 0\)

Let us first consider the case in which the term \(G_5(X)\) is absent. Using the solution (45) in the regime \(r \gg r_V\), one can show that the conditions (37), (38), and (41) are satisfied for \(r \sim r_g\). For the distance \(r\) not away from \(r_V\) \((r \gtrsim r_V)\), the quantities \(\mu_4\) and \(\mu_5\) can be estimated as

\[
\mu_4 \approx \frac{Q}{M_{\text{pl}}}, \quad \mu_5 \approx -\frac{6Q^2 M_{\text{pl}}^2 r_g^2}{M^3 r^6} \left( c_4 - \frac{4c_4 Q M_{\text{pl}} r_g}{M^3 r^3} \right). \tag{77}
\]

When \(c_3 = 0\), the distance \(r_*\) at which \(|\mu_4|/\mu_m = |\mu_5|\) is given by Eq. (62). If \(c_4 = 0\), then we obtain \(r_* = (6|c_3||Q|M_{\text{pl}} r_g^2/(M^3 |\rho_m|))^{1/6} \approx |M^3 M_{\text{pl}}/(Q|\rho_m|)|^{1/6} r_V\) for \(|c_3| \sim 1\). For \(M^3 M_{\text{pl}} H_0^2 \approx \rho_0/M_{\text{pl}}\) and \(|Q| = \mathcal{O}(1)\) it follows that \(r_* \approx (\rho_0/|\rho_m|)^{1/6} r_V\). Since the second term in the parenthesis of \(\mu_5\) in Eq. (77) is of the order of \(|c_4|(r_V/r)^3\), the distance \(r_*\) in the case \(|c_3| \sim |c_4| \sim 1\) is similar to that discussed above for \(\rho_m\) not significantly different from \(\rho_0\).

In the regime \(r_g \ll r \ll r_V\) the field equation of motion (42) reads

\[
\phi''(r) + \frac{\phi'(r)}{2r} \left[ 1 - \frac{3\alpha_{43} \phi'(r)}{M^3 r} \right]^{-1} = 0, \quad \text{where} \quad \alpha_{43} = \frac{c_4}{c_3}. \tag{78}
\]

This is integrated to give

\[
r \phi'^2(r) - \frac{2\alpha_{43}}{M^3} \phi^3(r) = C, \tag{79}
\]

where \(C\) is an integration constant determined by matching (79) with the solution (43) at \(r = r_V\). Then, the implicit solution (79) reads

\[
r \phi'^2(r) - \frac{2\alpha_{43}}{M^3} \phi^3(r) = \frac{(Q M_{\text{pl}} r_g)^2}{r_V^2} \left( 1 - \frac{2\alpha_{43}}{c_V} \right) \quad (r \ll r_V). \tag{80}
\]

In the limit \(\alpha_{43} \to 0\) we have \(\phi'(r) \propto r^{-1/2}\), whereas for \(|\alpha_{43}| \to \infty\) there is the solution \(\phi'(r) = \text{constant}\). The latter corresponds to the one derived in Eq. (59). The behavior of solutions changes at the radius \(r_{43}\) satisfying

\[
r_{43} = 2|\alpha_{43} \phi'(r_{43})|/M^3. \tag{81}
\]

In the following we study two qualitatively different cases separately.
• (i) \( r_{43} \gg r_V \)

Let us first consider the case \( r_{43} \gg r_V \). Substituting the solution (45) into Eq. (81), we obtain

\[
 r_{43} = \left( \frac{2|\alpha_{43}|Q|\mu_g|}{M} \right)^{1/3} \approx \left( \frac{2|\alpha_{43}|}{|c_V|} \right)^{1/3} r_V. \tag{82}
\]

Since \(|c_V| \sim 1\) the condition \( r_{43} \gg r_V \) translates to \(|\alpha_{43}| \gg 1\), i.e., \(|c_3| \ll 1\) and \(|c_4| \sim 1\). For the distance \( r \ll r_V \), the dominant contribution to the l.h.s. of Eq. (80) is the second term. The first term of Eq. (80) can be treated as a perturbation to the leading-order solution. In this way we can derive the approximate solution

\[
\phi' (r) \approx \frac{QMplr_g}{rV} \left[ 1 - \frac{c_V}{6\alpha_{43}} \left( 1 - \frac{r}{r_V} \right)^{3/2} \right] \quad (r \ll r_V \ll r_{43}). \tag{83}
\]

This case is similar to what we discussed in Sec. (III D) but there is a correction coming from the term \( G_3 (X) \). On using \(|\phi'' (r)| \approx |c_V\phi' (r)/(6\alpha_{43}r_V)|\), one can show that this correction is negligibly small in Eqs. (80) and (81). Then the gravitational potentials are approximately given by Eq. (80), so that local gravity constraints are satisfied deep inside the Vainshtein radius.

• (ii) \( r_{43} \ll r_V \)

In another case \( r_{43} \ll r_V \), the first term on the l.h.s. of Eq. (80) is the dominant contribution for the distance \( r_{43} \ll r \ll r_V \). Dealing with the second term of Eq. (80) as a perturbation to the leading-order solution of \( \phi' (r) \), it follows that

\[
\phi' (r) \approx \frac{QMplr_g}{rV^{3/2}} \left[ 1 - \frac{\alpha_{43}}{c_V} \left( 1 - \frac{r}{r_V} \right)^{3/2} \right] \quad (r_{43} \ll r \ll r_V). \tag{84}
\]

The leading-order solution \( \phi' (r) = QMplr_g/(rV^{3/2}) \) is the same as that derived in Ref. (82) in the presence of the term \( G_3 \) alone. From Eq. (81) the distance \( r_{43} \) can be estimated as

\[
r_{43} \approx \left( \frac{2|\alpha_{43}|Q|\mu_g|}{M^{2/3}r_V} \right)^{2/3} = \left( \frac{2|\alpha_{43}|}{|c_V|} \right)^{2/3} r_V. \tag{85}
\]

The condition \( r_{43} \ll r_V \) translates to \(|\alpha_{43}| \ll 1\), i.e., \(|c_4| \ll 1\) and \(|c_3| \sim 1\). In the regime \( r \ll r_{43} \) the dominant contribution to the l.h.s. of Eq. (80) is the second term. Taking into account the first term of Eq. (80) as a perturbation, we obtain the following solution

\[
\phi' (r) \approx \frac{M(QMplr_g)^{2/3}}{\left( -2\alpha_{43} \right)^{1/3}r_V} \left[ 1 - \frac{2\alpha_{43}}{3c_V} - \frac{1}{3} \left( \frac{c_V}{-2\alpha_{43}} \right)^{2/3} \frac{r}{r_V} \right] \quad (r \ll r_{43}). \tag{86}
\]

From Eq. (85) the last term in the parenthesis of Eq. (86) is of the order of \( r/(3r_{43}) \), so that it is suppressed in the regime \( r \ll r_{43} \). The sign of \( \phi' (r) \) should not change around \( r = r_{43} \), so that we require the following condition

\[
\alpha_{43} Q < 0. \tag{87}
\]

One can confirm that Eqs. (81) and (80) satisfy the conditions (89), (50), and (51).

On using Eq. (81) in the regime \( r_{43} \ll r \ll r_V \), Eqs. (8) and (9) are approximately given by

\[
\frac{d}{dr} (r\Phi) \approx - \frac{3Q\phi' r}{2Mpl} + \frac{\rho_m r^2}{2Mpl^2}, \tag{88}
\]

\[
\Psi' \approx \frac{\Phi}{r} + \frac{2Q\phi'}{Mpl}. \tag{89}
\]

Substituting the approximate solution \( \phi' (r) \approx MQplr_g/(rV^{3/2}) \) into Eqs. (88) and (89), the integrated solutions are

\[
\Phi \approx \frac{r_a}{2r} \left[ 1 - 2Q^2 \left( \frac{r}{r_V} \right)^{3/2} \right], \quad \Psi \approx \frac{r_a}{2r} \left[ 1 - 4Q^2 \left( \frac{r}{r_V} \right)^{3/2} \right] \quad (r_{43} \ll r \ll r_V). \tag{90}
\]
For the distance $r$ close to $r_{43}$ the correction term from $\alpha_{43}$ in Eq. (84) tends to be important, but this does not change the order of the estimation (90). Since $2Q^2(r/r_V)^{3/2} \ll 1$ deep inside the Vainshtein radius, the deviation of the post-Newtonian parameter $\gamma = -\Phi/\Psi$ from 1 is much smaller than unity.

Employing the solution (86) in the regime $r \ll r_{43}$, the gravitational potentials approximately satisfy Eqs. (64) and (65) with $\phi'(r) \simeq M(QM_p r_g)^{3/2}/(-2\alpha_{43})^{1/3} r_V \simeq QM_p r_g/(r_V r_{43}^{3/2})$ for $r \gg r_g$, where we used the condition (87). Then it follows that

$$\Phi \simeq -\frac{r_g}{2r} \left( 1 - 2Q^2 \frac{r^2}{r_V^3/2 r_{43}^{1/2}} \right), \quad \Psi \simeq -\frac{r_g}{2r} \left( 1 - 2Q^2 \frac{r^2}{r_V^3/2 r_{43}^{1/2}} \right) \quad (r \ll r_{43}),$$

from which $\gamma \simeq 1$. The correction terms in Eq. (86) only give the contributions much smaller than the term $2Q^2 r^2/(r_{43}^3 r_V)$ (≪ 1), so that the experimental bound of $\gamma$ is well satisfied.

If $|c_3| \sim |c_4| \sim 1$, then $r_{43}$ is the same order as $r_V$. In this case the regime in which the $G_3$ term contributes to the field equation for $r < r_V$ is narrow, so that the solution is described by the $G_4$-dominant one ($\phi'(r) = \text{constant}$) for most of $r$ smaller than $r_V$. The gravitational potentials within the Vainshtein radius can be estimated by taking the limit $r_{43} \to r_V$ in Eq. (91), i.e., Eq. (66).

As we studied in Sec. III D, both (83) and (86) can connect with another solution (70) around the surface of the star. There is an extreme case $r_{43} \to 0$, in which the field derivative is given by Eq. (83) even for small $r$ down to the radius of the star. In such a case we discuss the matching of solutions for more general models in Sec. IV A.

B. $c_5 \neq 0$

We estimate the effect of the term $G_5(X) = c_5 X^2/M^9$ on the solutions discussed in Sec. IV A. We study two qualitatively different cases: (1) $|c_4| \sim 1, |c_3| \ll 1$, and (2) $|c_3| \sim 1, |c_4| \ll 1$.

1. $|c_4| \sim 1, |c_3| \ll 1$

This corresponds to the case (i) studied in Sec. IV A. In the following we focus on the case in which the effect of the $c_3$ term is practically absent, i.e., the limit $|\alpha_{43}| \to \infty$. At small $r$ the term $[4X(G_5 X + X G_5 X X)/r^2]$ in Eq. (20) gets larger than the other term $|2G_4|$, $\phi'(r) \simeq QM_p r_g^2/r_V^5$. Employing the solution $\phi'(r) \simeq QM_p r_g^2/r_V^5$, this region is estimated as $r \ll (2|c_5|r_g r_V/|c_3|^{1/2})^{1/2} \simeq (r_g r_V)^{1/2}$ for $|c_3| \sim 1$. For the mass scale $M^3 \sim M_p H_0^2$, this condition translates to $r \lesssim 10^{13}$ cm for the Sun and $r \lesssim 10^9$ cm for the Earth. In the regime $r \lesssim (r_g r_V)^{1/2}$ we have that $\mu_4 \simeq c_5 Q^2 r_g^2/(6c_4^2 M^3 r_V^4)$ and $\mu_5 \simeq 2QM_p r_g^2/(r_V^2 r_g)$. Since the density $\rho_m(r)$ grows around the surface of the star, the condition $|\mu_4| \rho_m(r) < |\mu_5|$ can be violated for

$$r_{\rho_m}(r) > \frac{1}{M^3 M_p r_g^2 r_V^2} \simeq \frac{M_p}{r_V},$$

where the second approximate equality holds for $|c_5| \sim 1$. Around the surfaces ($r = r_s$) of the Sun and the Earth the term $r_\rho_m(r_s)$ becomes the same order as $M_p/r_V$ for $\rho_m(r_s) \approx 10^{-5}$ g/cm$^3$ and $\rho_m(r_s) \approx 0.1$ g/cm$^3$, respectively. Inside these stars we have $|\mu_4| \rho_m(r) > |\mu_5|$, so that the solution $\phi'(r) \simeq QM_p r_g^2/r_V^5$ is subject to change.

Outside the star ($r > r_s$), let us estimate the effect of the $c_5$ term on the gravitational potentials. Substituting the solution $\phi'(r) \simeq QM_p r_g^2/r_V^5$ into Eqs. (8) and (9), we obtain

$$\Phi \simeq -\frac{r_g}{2r} \left[ 1 - 2Q^2 \left( \frac{r}{r_V} \right)^2 \frac{1 - 3c_5 Q^2 r_g^2}{c_3^2 r_V^2 r_g^2} \right], \quad \Psi \simeq -\frac{r_g}{2r} \left[ 1 - 2Q^2 \left( \frac{r}{r_V} \right)^2 \frac{1 - 3c_5 Q^2 r_g^2}{c_3^2 r_V^2 r_g^2} \right],$$

where we used the fact that the dominant contribution to $\Phi$ is $-r_g/(2r^2)$. The corrections to $\Phi$ and $\Psi$ coming from the $c_5$ term are negligibly small for $r > r_s$ (at most of the order of $10^{-21}$ for the Sun).

Inside the star we study the solution of Eq. (19) to see whether the matching with another solution $\phi'(r) \simeq QM_p r_g^2/r_V^5$ can be done properly. We use the density profile (55) together with the dimensionless variables defined in Eq. (72). Under the approximation that the terms including $b_2$ are negligible, the field equation (19) reads

$$\frac{dy(x)}{dx} \simeq \frac{1}{4c_4} \left[ Q x^2 + 6c_5 b_5 e^{2Qz(x)} x y(x) + 2c_5 (b_1^2/2) e^{2Qz(x)} y(x)^4/3 \right] e^{-x^2/\beta^2},$$

(94)
where the variable $z(x)$ satisfies the same equation as $[74]$. In the following we study the case $c_4 > 0$ and $Q > 0$. Around $x \sim 0$ the solution of Eq. (94) is given by $y(x)^{1/3} \approx \sqrt[3]{Q/(12c_4)}x$. The third term on the r.h.s. of Eq. (94) dominates over the first one for $x^2 > x_s^2 \equiv (12c_4)^{4/3}b_2/2|c_5|Q^{1/3}b_1^3$. As long as $x_s$ is larger than 1, the effect of the $c_5$ term does not manifest itself inside the star. This demands the following condition

$$|c_5| < \frac{(12c_4)^{4/3}b_2}{2Q^{1/3}b_1^3} \approx \frac{10b_2}{b_1^3} = \frac{10M M_{pl}^{7/3}}{r_s^2 M_s^{14/3}}.$$  \hspace{1cm} (95)$$

If $M^3 \approx M_{pl}H_0^2$, then we have $|c_5| \lesssim 1$ for the Earth ($b_1 \approx 10^{-3}$ and $b_2 \approx 10^{-13}$) and $|c_5| \lesssim 10^{-7}$ for the Sun ($b_1 \approx 0.1$ and $b_2 \approx 10^{-12}$). Thus the upper bound of $|c_5|$ depends on the density and the radius of the star.

Numerically we solve the field equation of motion (19) without neglecting the terms including $b_2$. In the left panel of Fig. 2 we plot the field derivative $y^{1/3}$ versus $r/r_s$ for $b_1 = 1.6 \times 10^{-3}$, $b_2 = 1.6 \times 10^{-13}$, and $\beta_i = 0.7$ with three different values of $c_5$. This case mimics the density profile of the Earth. Even for the cases $c_5 = 1$ and $c_5 = -1$, the solution inside the star smoothly connects to the exterior solution $y^{1/3} = O(0.1)$.

The right panel of Fig. 2 corresponds to the model parameters $b_1 = 0.1$, $b_2 = 3.0 \times 10^{-12}$, and $\beta_i = 0.35$, in which case the density profile is similar to that of the Sun. For the values of $c_5$ satisfying the condition (95), e.g., the case (b) in Fig. 2, the matching of the interior and exterior solutions occurs smoothly. If $c_5 \sim 1$, however, the third term on the r.h.s. of Eq. (94) dominates over the first term for $r \ll r_s$. This leads to the rapid growth of $\phi'(r)$ at small $r$, in which case the matching with another solution $y^{1/3} = O(0.1)$ outside the star does not occur. In Fig. 2 this behavior is clearly seen in the case $c_5 = 1$.

For negative values of $c_5$ satisfying the condition $|c_5| \gg 10b_2/b_1^3$, we numerically confirmed that the first term on the r.h.s. of Eq. (94) almost balances with the third term, i.e., $y(x)^{1/3} \approx \sqrt[3]{Qb_2/(2|c_5|b_1^3)}x^1/2 \ll x^{1/2}$ for $r \lesssim r_s$. For larger $|c_5|$ the field derivative $y(x)^{1/3}$ gets smaller around the surface of the star. As we see in the case (c) of Fig. 2, we have $y(1)^{1/3} = O(10^{-2})$ for $c_5 \sim -1$. This is by one order of magnitude smaller than the exterior solution $y^{1/3} = O(0.1)$ and hence there is a problem of the matching of two solutions.

In summary, for the values of $c_5$ satisfying the condition (95), the solution inside the star smoothly connects to the exterior solution $\phi'(r) \approx QM_{pl}r_s/r_s^2$ around the surface. The upper bound of $|c_5|$ depends on the radius and density of the star.
This belongs to the case (ii) discussed in Sec. [IV.A] We study the case in which the effect of the $c_4$ term is practically absent, i.e., $\alpha_{43} \to 0$, so that the solution in the regime $r_g \ll r \ll r_V$ is given by $\phi'(r) \simeq Q M_{pl} r_g / (r_V^{3/2} r^{1/2})$. The term $|4X(G_{5,X} + XG_{5,XX})/r^2| = 4|c_5|\phi'^2/(M^9 r^2)$ in Eq. (20) becomes larger than the other term $|2G_{4,\phi}|$ for the distance $r < (2|c_5| r_g r_V^2 / c_5^3)^{1/4}$. For the Sun with $|c_5| \sim 1$ and $M^3 \approx M_\odot H_0^2$, this condition translates to $r \lesssim 10^{17}$ cm. Using the solution $\phi'(r) \simeq Q M_{pl} r_g / (r_V^{3/2} r^{1/2})$ in the regime $r < (r_g r_V^{3/2})^{1/4}$, we have that $\mu_4 \approx -c_5 Q r_g^3 / (2c_3 r_V^2 M_{pl} r^{5/2})$ and $\mu_5 \approx 3Q M_{pl} r_g / (r_V^{3/2} r^{3/2})$. Then, the condition $|\mu_4| r_m < |\mu_5|$ is satisfied for the distance

$$r > r_5 \equiv \frac{\rho_m c_5}{2|c_4| \rho_m c_3^{2/3} M_{pl}^2 r} \approx \frac{\rho_m c_5}{\rho_0} \left( \frac{r_V}{H_0^{-1}} \right)^{2/3} r_V,$$

where the second and third approximate equalities are valid for $|c_5| \sim 1$. If the Vainshtein radius is $r_V \approx 10^{20}$ cm, it follows that $r_5 \approx 10^4 (\rho_m / \rho_0)$ cm for $|c_5| \sim 1$. The lower bound of $r$ depends on the density profile of the star. If we use the mean density $\rho_m \approx 10^{-24}$ g/cm$^3$ of our galaxy, the condition (96) translates to $r > 10^9$ cm (whose lower bound is of the same order as the radius of the Earth). Around the Sun the density $\rho_m$ is much larger than $10^{-24}$ g/cm$^3$, so that the condition (96) translates to $r \gg 10^7$ cm. This suggests that the condition $|\mu_4| r_m < |\mu_5|$ can be violated in the solar system.

In order to understand how the effect of the $G_5(X)$ term manifests itself in the regime $|\mu_4| r_m < |\mu_5|$, i.e., for the radius $r_5 \lesssim r \ll r_V$, we estimate the behavior of the gravitational potentials by employing the solution $\phi'(r) \simeq Q M_{pl} r_g / (r_V^{3/2} r^{1/2})$. Since the leading-order gravitational potentials are given by $\Phi \simeq r_g / (2r)$ and $\Psi \simeq -r_g / (2r)$, the $G_5(X)$-dependent term inside $A_5$ of Eq. (8) provides a much larger contribution relative to the term $A_3 \Phi' / r^2$ for $r \gg r_g$. Then the r.h.s. of Eq. (88) gets corrected by the term $-c_5 Q r_g r_V^{3/2} / (c_3 r^{7/2})$, whereas Eq. (89) is unchanged. Integration of these equations gives

$$\Phi \simeq \frac{r_g}{2r} \left[ 1 - 2Q^2 \left( \frac{r}{r_V} \right)^{3/2} + \frac{4c_5 Q^2 r_g r_V^{3/2}}{5c_3^2 M_{pl} r^{5/2}} \right], \quad \Psi \simeq -\frac{r_g}{2r} \left[ 1 - 4Q^2 \left( \frac{r}{r_V} \right)^{3/2} + \frac{8c_5 Q^2 r_g r_V^{3/2}}{35c_3^2 M_{pl} r^{5/2}} \right].$$

(97)

The third terms in Eq. (97) dominate over the leading-order contribution for the distance

$$r < \left( \frac{|c_5| Q^2 r_g r_V^{3/2}}{c_3^2 r^{3/2}} \right)^{2/5} \approx (c_3 r_g r_V^{3/2})^{1/5}.$$

(98)

For the Sun with $|c_5| \sim 1$ and $M^3 \approx M_\odot H_0^2$, the condition (98) corresponds to $r < 10^{14}$ cm. Then the experimental bound of the post-Newtonian parameter $\gamma$ is not satisfied in the solar system. Hence the presence of the term $G_5(X) = c_5 X^2 / M^9$ disrupts the Vainshtein mechanism induced by the field self-interaction $G_3(X) = c_3 X / M^3$. For the consistency with local gravity constraints we require that $|c_5|$ is very much smaller than 1.

V. APPLICATION TO OTHER MODELS

In this section we study how the Vainshtein mechanism is at work for several concrete models such as (A) extended Galileons, (B) Galileons with dilatonic couplings, and (C) DBI Galileons with Gauss-Bonnet and other terms.

A. Extended Galileons

The extended Galileon model [30, 31] is given by the Lagrangian

$$G_3(X) = c_3 M^{1-4p_3} X^{p_3}, \quad G_4(\phi, X) = \frac{M_{pl}^2}{2} e^{-2Q\phi/M_{pl}} + c_4 M^{2-4p_4} X^{p_4},$$

(99)

where $p_3$ and $p_4$ are integers satisfying $p_3 \geq 1$ and $p_4 \geq 2$. We do not take into account the term $G_5(X) = c_5 M^{-1-4p_5} X^{p_5}$ ($p_5 \geq 2$) because its effect is similar to what we studied in Sec. [IV.B]

2. $|c_3| \sim 1, |c_4| \ll 1$
From Eq. (48) the Vainshtein radius can be estimated as

\[
\rho \approx \sqrt{(|Q|M_{pl}r_g)^{2p_3-1}}/M, \quad (|c_3| \sim 1, c_4 = 0),
\]

(100)

\[
\rho \approx \sqrt{(|Q|M_{pl}r_g)^{2p_3-1}}/M, \quad (|c_4| \sim 1, c_3 = 0).
\]

(101)

If \( p_4 = 2p_3 \), then both (100) and (101) are the same. For \( p_3 \ll 1 \) and \( p_4 \ll 1 \) it follows that \( \rho \sim (|Q|M_{pl}r_g)^{1/2}/M. \)

In the regime \( r_g \ll r \ll \rho \), integration of Eq. (62) leads to the following implicit solution

\[
\rho' = \left( QM_{pl}r_g \right)^{2p_3-1/43} \left[ -2p_3p_4 - p_4 - 1 \right] \alpha_{43} M_{pl}^{1+4p_3-4p_4} \rho(\rho)^{2p_4-1}
\]

(102)

where \( \alpha_{43} = c_4/c_3 \) and we used Eq. (43) to match the solutions at \( r = \rho \). For the large distance (or the limit \( |\alpha_{43}| \to 0 \)) the solution behaves as \( \rho' \sim r^{-1/(2p_3)} \), whereas for small \( r \) (or the limit \( |\alpha_{43}| \to \infty \)) we have \( \rho' \sim r \). The behavior of \( \rho' \) changes at the distance \( r_{43} \) satisfying

\[
r_{43} = p_4^{-1} p_4^{-1} M_{pl}^{1+4p_3-4p_4} \left[ -2p_3p_4 - p_4 - 1 \right] \alpha_{43} \rho_{43}^{2p_4-1}.
\]

(103)

If \( r_{43} \gg \rho \), i.e. \( |\alpha_{43}| \gg 1 \), then both (100) and (101) reduce to the result (66). This shows that, for larger \( p_3 \), the deviation from GR tends to be smaller.

Inside the star the above solutions are subject to change. Let us consider the limit \( r_{43} \to 0 \), i.e., the case in which the solution is given by Eq. (105) for small \( r \) down to the surface of the star. We consider the density profile (62) of the star and introduce the following dimensionless variables

\[
y = M_{pl} M_{4p_3-1} \phi(2p_3), \quad b_2 = \left( \frac{M_{4p_3-1}^{-1} b_1}{M_{pl}^{4p_3-1}} \right)^{1/4},
\]

(109)

where \( x, z, \beta_1, \) and \( b_1 \) are the same as those defined in Eq. (72). For the mass scale \( M \) relevant to dark energy, we have \( b_2 \ll 1 \) for both the Sun and the Earth. Neglecting the contribution of the terms including \( b_2 \), the field equation of motion (10) reads

\[
\frac{dy}{dx} \approx \frac{1}{4c_3^3 M_{pl} x} \left[ (z)^{2p_3} Q + c_4 p_3 (1 + 4p_3) b_1^2 y \right] e^{-z^2/\beta_1^2} - 4c_3 p_3 y,
\]

(110)

where we used the approximation \( e^{2Qz(x)} \approx 1 \). The variable \( z(x) \) satisfies the differential equation \( dz(x)/dx = (b_1^2 b_2 y(x))^{1/(2p_3)} \). For small \( x \) the second term on the r.h.s. of Eq. (110) can be neglected relative to other two terms. Using the approximation \( e^{-z^2/\beta_1^2} \approx 1 \) in this regime, we obtain the following solution

\[
y(x) \approx \left( \frac{-2p_3 Q}{12c_3^3 p_3} \right) x^2,
\]

(111)
Vainshtein mechanism with a proper matching of solutions around the surface of the star. The corrections to the leading-order gravitational potentials are suppressed for most of the region inside the star. Connecting with another solution inside the star for an appropriate choice of the sign of the field derivative to $\Phi$ and $\Psi$ are suppressed under the condition $r \geq r_s$. In the case where the field derivative is given by either (104) or (106) outside the star, we confirmed that it smoothly connects with another solution inside the body for an appropriate choice of the sign of $c_4$ (i.e., $c_4 < 0$ for odd $p_4$ and $c_4 > 0$ for even $p_4$). Hence the extended Galileon model with $p_3 \geq 1$ and $p_4 \geq 2$ can successfully accommodate the Vainshtein mechanism with a proper matching of solutions around the surface of the star.

B. Galileons with dilatonic couplings

We proceed to the model in which the $G_{3,4,5}$ terms have the dilatonic coupling of the form

$$G_3(\phi, X) = \frac{c_3}{M^3} e^{-\lambda_3 \phi/M_p} X, \quad G_4(\phi, X) = \frac{M_p^4}{2} e^{-2Q\phi/M_p} + \frac{c_4}{M^6} e^{-\lambda_4 \phi/M_p} X^2, \quad G_5(\phi, X) = \frac{c_5}{M^3} e^{-\lambda_5 \phi/M_p} X^2,$$

with $\lambda_{3,4,5}$ being dimensionless constants of the order of unity. The dilatonic coupling of the above form arises not only in low-energy effective string theory but also in the conformal Galileon model characterized by a probe brane moving in an Anti-de Sitter throat. In the following we study two different cases: (i) $c_5 \neq 0, c_3 = 0, c_4 = 0$, and (ii) $c_3 \neq 0, c_4 \neq 0, c_5 = 0$, separately.
Since the terms such as $G_{5,\phi X}$ and $G_{5,\phi XX}$ in Eqs. (21) and (22) do not vanish for the function $G_5$ involving the field $\phi$, it seems to be possible for the term $G_5(\phi, X) = c_3 e^{-\lambda_5 \phi/M_{pl}^2} X^2/M^9$ alone to accommodate the Vainshtein mechanism. In the following we study this possibility by assuming that $|\lambda_5|$ is of the order of 1.

From Eq. (18) the Vainshtein radius can be estimated as $r_V = \left(6c_3 \lambda_5 Q^3 M_{pl}^4 r_g^2 / M^9\right)^{1/10}$. For $r$ smaller than $r_V$ we have $\xi_1/\xi_2 \simeq 2$ in Eq. (22), so that the field derivative is the same as that derived in Sec. III.D i.e.,

$$\phi_{\text{in}}'(r) = \frac{Q M_{pl} r_g}{r_V^2}.$$  

(113)

The term $|4X(G_{5,X} + X G_{5,XX})/r^2|$ in Eq. (20) gets larger than the other term $|2G_{4,\phi}|$ for $r < r_V/\sqrt{3|\lambda_5 Q|} \approx r_V$. Thus in the regime $r \ll r_V$ we have that $\mu_4 \simeq -1/(3\lambda_5 M_{pl})$ and $\mu_5 \simeq 2QM_{pl} r_g/(r_V^2 r)$. The condition $|\mu_4|\rho_m(r) < |\mu_5|$ is violated for

$$\rho_m(r) > 6|Q|\frac{M_{pl}^2 r_g}{r_V^2} \approx \frac{M_{pl}^2 r_g}{r_V^2}.$$  

(114)

Around the surfaces ($r = r_s$) of the Sun and the Earth with $M^3 \sim M_{pl} H_0^2$, the above inequality is satisfied for $\rho_m(r_s)$ larger than $10^{-17}$ g/cm$^3$ and $10^{-16}$ g/cm$^3$, respectively.

The solution (113) is subject to change inside the star. Around the very vicinity of the center of the star the term $\mu_4$ can be estimated as $\mu_4 \simeq -G_4/\beta$ with $\beta \simeq r$, so that the field equation (19) reads $\Box \phi \simeq Q\rho_m/\beta$. Assuming that $\rho_m$ approaches a constant $\rho_c$ for $r \rightarrow 0$, the solution to this equation is given by $\phi'(r) \simeq Q\rho_c r/(3M_{pl})$.

The last term in the square bracket of Eq. (20) dominates over the $2G_{4,\phi}$ term for $r > r_c \equiv \sqrt{81 M^9 M_{pl}^5 / (2|c_5| Q^3 \rho_c^4)}$. If $\rho_c \sim 100$ g/cm$^3$ and $M^3 \sim M_{pl} H_0^2$, then $r_c \approx 10^{-5}$ cm. For the distance $r$ larger than $r_c$ the field equation (19) is simplified as

$$\frac{d}{dr} (r^2 \phi') \simeq -\frac{\rho_m}{3\lambda_5 M_{pl}} r^2 + 2r \phi'.$$  

(115)

Under the approximation that $\rho_m$ is nearly constant inside the star, we obtain the following solution for $r \ll r_c$:

$$\phi_{\text{in}}'(r) \simeq -\frac{\rho_m}{3\lambda_5 M_{pl}} r.$$  

(116)

In order to match this with another solution (113), we require the condition $\lambda_5 Q < 0$. Since the Schwarzschild radius can be estimated as $r_g \simeq \rho_m r_s^2 / (3M_{pl}^2)$ from Eq. (44), the ratio between $\phi_{\text{in}}'(r)$ and $\phi_{\text{out}}'(r)$ around $r = r_s$ is approximately given by

$$\left| \frac{\phi_{\text{in}}'(r_s)}{\phi_{\text{out}}'(r_s)} \right| \simeq \frac{1}{|\lambda_5|} \left( \frac{r_V}{r_s} \right)^2 \approx \left( \frac{r_V}{r_s} \right)^2.$$  

(117)

For the matching of two solutions we require that $r_V \approx r_s$, but the Vainshtein mechanism works outside the star only for $r_V \gg r_s$. Hence the interior solution (116) does not connect to the exterior solution (113) that accommodates the Vainshtein mechanism. In other words, if we integrate the field equation outwards with the boundary condition $\phi'(0) = 0$, the field derivative becomes too large to be compatible with local gravity constraints around the surface of the star. This is an example where the Vainshtein mechanism does not operate inside the star.

While the above discussion corresponds to the case of nearly constant $\rho_m$ inside the body, we also solved Eq. (19) numerically for the density profile (53). We confirmed that the interior and exterior solutions given above do not match with each other for $r_V \gg r_s$.

2. $c_3 \neq 0, c_4 \neq 0, c_5 = 0$

This case corresponds to the extension of the covariant Galileon model studied in Sec. [V.A] Using the solution (45) in the regime $r \gg r_V$, one can show that the term $|2(G_{3,\phi} + X G_{3,\phi XX})|$ in Eq. (48) is of the order of $r_g/r_V$ and that the term $|4(-3G_{4,\phi X} - 2X G_{4,\phi XX})\phi'(r_V)|$ is suppressed by the factor $r_g/r_V$ relative to the last term of Eq. (48). Then, the Vainshtein radius is practically the same as Eq. (76).
In the regime $r_g \ll r \ll r_V$ the field equation of motion \cite{52} reads

$$\phi''(r) + \frac{\phi'(r)}{2r} \left[1 - \frac{3\alpha_4 \phi'(r)}{M^4 r} - 5u(r)\right]^{-1} [1 - u(r)] \simeq 0,$$

(118)

where $\alpha_4 = c_4/c_3$ and $u(r) = \lambda_4 \alpha_4 \phi'(r)^2/(M^3 M_\mu)$. Integration of Eq. (118) gives

$$r \phi'(r)^2 [1 - u(r)]^{-1} - 2 \frac{\alpha_4}{M^2} \phi'(r)^3 \left[1 - \frac{9}{5} u(r) + \frac{9}{7} u^2(r) - \frac{1}{3} u^3(r)\right] = C,$$

(119)

where $C$ is an integration constant determined by substituting the solution $\phi'(r) = Q M_\mu r_g/r_V^2$ at $r = r_V$. In the limit $|\alpha_4| \ll 1$, the leading-order solution to Eq. (119) is the same as \cite{54}, i.e., $\phi'(r) \approx Q M_\mu r_g/(r^2/2r_{1/2})$. In this case we have $u(r) \approx (\lambda_4 Q \alpha_4 c_4/r_V^2)(r_g/r_V)$, so that the correction from the non-zero $\lambda_4$ to the leading-order solution is very small. When $|\alpha_4| \sim 1$, the leading-order solution is given by $\phi'(r) \approx Q M_\mu r_g/r_V^2$ for most of $r$ smaller than $r_V$. Since $u(r) \approx (\lambda_4 Q \alpha_4 c_4/r_V^2)(r_g/r_V)$ in this case, the correction is suppressed as well. In the limit $|\alpha_4| \gg 1$ we have $|u(r)| \gg 1$ and hence Eq. (119) reduces to $\phi'(r)^3[\phi(r) + 2M_\mu/(3\lambda_4)] \approx 2M_\mu \phi'(r)^3/(3\lambda_4)$. Here we used the relation $|r_V \phi'(r_V)| \ll [2M_\mu/(3\lambda_4)]$ to determine the integration constant. We then obtain the solution $\phi'(r) \approx (Q M_\mu r_g/r_V^2)[1 - \lambda_4 Q r_g/(br_V^2)],$ which shows that the correction from the non-zero $\lambda_4$ is very small.

Inside the star, the correction from the $\lambda_4$ term to the leading-order solution is also suppressed. For the theory with $c_3 = 0$ and the density profile \cite{55}, the variable $y$ defined in Eq. (73) obeys the following approximate equation

$$\frac{dy(x)}{dz} \approx \frac{1}{4c_4} x Q e^{\lambda_4 z(x)} + 6c_4 b_3 e^2 Q z(x) y(x) e^{-\lambda_4 z(x)}$$

(120)

which is valid for $b_2 \ll 1$. Since the variable $z$ satisfies the same equation as (73), the variation of $z$ is very tiny for $b_2 \ll 1$ and hence $e^{\lambda_4 z(x)} \simeq 1$. In this case, Eq. (120) reduces to Eq. (73). The similar property also holds for the theory with $c_4 = 0$.

Thus the model \cite{112} with $c_5 = 0$ can successfully accommodate the Vainshtein mechanism. We note that the Vainshtein mechanism is also at work for extended Galileons with dilatonic couplings characterized by the Lagrangians $G_3(\phi, X) = c_3 M^{1 - 4p_3} e^{-\lambda_3 \phi/M_\mu} X^{p_3} (p_3 \geq 1)$ and $G_4(\phi, X) = (M_\mu^2/2) e^{-2Q \phi/M_\mu} + c_4 M^{2 - 4p_4} e^{-\lambda_4 \phi/M_\mu} X^{p_4} (p_4 \geq 2)$.

### C. DBI Galileons with Gauss-Bonnet and other terms

In higher-dimensional theories there appears a scalar degree of freedom associated with the size of compact space or with the position of a probe brane in large extra dimensions. In the set-up of a relativistic probe brane embedded in a five-dimensional bulk, de Rham and Tolley \cite{19} showed that all the Galileon self-interactions and its generalizations arise from the brane tension, induced curvature, and the Gibbons-Hawking-York boundary terms. If we consider a Gauss-Bonnet term in a higher-dimensional space-time, the dimensional reduction on a compact space gives rise to a self-interaction $X \Box \phi$ of a scalar field $\phi$ (corresponding to the size of the extra dimensions) as well as other interactions with $\phi$ \cite{53, 54}. In order to accommodate such scenarios, let us consider the following four-dimensional action

$$S = \int d^4x \sqrt{-g} \left[\frac{M_\mu^2}{2} e^{-2Q \phi/M_\mu} R - f_1(\phi)^2 \mu^4 \left(1 - \frac{2}{\mu^4} f_1(\phi)^{-1} X - 1\right) + f_2(\phi) X^2 \right]$$

$$+ f_3(\phi) \frac{X}{M_\mu^2} \Box \phi + f_4(\phi) c_{\text{GB}} R^2_{\text{GB}} + f_5(\phi) \frac{1}{m^2} G_{\mu\nu} \partial_\mu \phi \partial_\nu \phi,$$

(121)

where $\mu$, $M$, $c_{\text{GB}}$, $m$ are constants, and $R^2_{\text{GB}}$ is the Gauss-Bonnet term defined by

$$R^2_{\text{GB}} \equiv R^2 - 4 R_{\mu\nu} R^{\mu\nu} + R_{\alpha \beta \gamma \delta} R^{\alpha \beta \gamma \delta}.$$

(122)

For the functions $f_i(\phi)$ ($i = 1, \ldots, 5$) we assume that, without loss of generality, they are all equivalent to

$$f(\phi) = e^{-\lambda \phi/M_\mu},$$

(123)

where $\lambda$ is a constant of the order of 1. The second term in Eq. (121) corresponds to the DBI term appearing in the relativistic set-up of a probe brane moving in an Anti-de Sitter throat \cite{19}. The last four terms in Eq. (121) arise after the dimensional reduction of a higher-dimensional Gauss-Bonnet theory \cite{54} or as higher-order $r'$-corrections to
the low-energy bosonic string action \cite{57}. In the case of \( \alpha' \)-corrections, the mass scales \( \mu, M \), and \( m \) are usually much higher than those related to dark energy. In the following we do not put some restriction on the mass scales from the beginning, but we constrain those scales from the demand of realizing the Vainshtein mechanism. In the action \cite{121} we can also take into account other dilatonic Galileon self-interactions, but those effects are similar to what we studied in Sec. \[13\].

The Gauss-Bonnet coupling \( f(\phi)R_{\text{GB}}^2 \) gives rise to the same equations of motion as those derived from the Horndeski’s action \cite{59} for the choice \( K = 8f^{(4)}(\phi)X^2[3 - \ln(X/\mu^4)] \), \( G_3 = 4f^{(3)}(\phi)X[7 - 3\ln(X/\mu^4)] \), \( G_4 = 4f^{(2)}(\phi)X[2 - \ln(X/\mu^4)] \), and \( G_5 = -4f^{(1)}(\phi)\ln(X/\mu^4) \), where \( f^{(n)}(\phi) \equiv d^n f/d\phi^n \). \cite{25}. The last term in Eq. \cite{121} is equivalent to \( G_5(\phi)C^{\mu\nu}(\nabla_\mu \nabla_\nu \phi) \) with \( G_5(\phi) = M_{\text{pl}}f(\phi)/(\lambda m^2) \) after integration by parts\(^3\). In the language of the Horndeski’s action \cite{59}, the theory \cite{121} corresponds to

\[
K(\phi, X) = -f(\phi)\mu^4 \left( 1 - \frac{2f(\phi)^{-1}X}{\mu^4} - 1 \right) + f(\phi)\frac{X^2}{\mu^4} + 8c_{\text{GB}}f^{(4)}(\phi)X^2 \left[ 3 - \ln \left( \frac{X}{\mu^4} \right) \right],
\]

\[
G_3(\phi, X) = -f(\phi)\frac{X}{M^3} + 4c_{\text{GB}}f^{(3)}(\phi)X \left[ 7 - 3\ln \left( \frac{X}{\mu^4} \right) \right],
\]

\[
G_4(\phi, X) = \frac{M_{\text{pl}}^2}{2}e^{-2Q/M_{\text{pl}}} + 4c_{\text{GB}}f^{(2)}(\phi)X \left[ 2 - \ln \left( \frac{X}{\mu^4} \right) \right],
\]

\[
G_5(\phi, X) = \frac{M_{\text{pl}}}{\lambda m^2}f(\phi) - 4c_{\text{GB}}f^{(1)}(\phi)\ln \left( \frac{X}{\mu^4} \right).
\]

Provided that the conditions \( |X/\mu^4| \ll 1 \) and \( |X/(M^3M_{\text{pl}})| \ll 1 \) are satisfied, the function \cite{124} reduces to the form \( K(\phi, X) \approx f(\phi)X + 8c_{\text{GB}}f^{(4)}(\phi)X^2[3 - \ln(X/\mu^4)] \). The contribution of the Gauss-Bonnet term vanishes for the term \( \beta \) in Eq. \cite{22}. Then, the Vainshtein radius is known from Eq. \cite{48} as

\[
r_V \simeq (4Q|M_{\text{pl}}r_g|)^{1/3}/M,
\]

where we used Eq. \cite{45}. In the regime \( r \gg r_V \) we recall that the solution \( \phi'(r) = QM_{\text{pl}}r_g/r^2 \) is valid under several conditions presented in Sec. \[11A\]. Now we have \( \beta \simeq f(\phi)r \simeq r \) for \( r \gg r_V \), where the second approximate equality is valid for \( r_V \gg r_g \) i.e., equivalent to \cite{29}. For \( r_V \gg r_g \) we have \( |X/(M^3M_{\text{pl}})| \ll 1 \). Provided that

\[
|c_{\text{GB}}X/M_{\text{pl}}|^4 \ll 1, \quad |X|/\mu^4 | \ll 1,
\]

both \cite{37} and \cite{53} are met in the regime \( r \gg r_V \), so that \( \mu_4 \simeq Q/M_{\text{pl}} \) and \( \mu_5 \simeq -12X(1 - \phi'M^3r/\mu^4)/(M^3r^2) \). Under the condition

\[
\mu^4 \gg |Q|M^3M_{\text{pl}}r_g/r_V,
\]

it follows that \( \mu_5 \simeq -12X(M^3r^2) \). Then, the distance \( *_r \) at which \( |\mu_4|/\rho_m \) becomes the same order as \( \mu_5 \) can be estimated as \( r_* \), \( r \gg r_V \approx [M^3M_{\text{pl}}/(|Q|\rho_m)]^{1/6} \). For \( M^3 \approx M_{\text{pl}}H_0^2 \) and \( \rho_m \) close to \( \rho_0 \), \( *_r \) is the same order as \( r_V \).

In the regime \( r_g \ll r \ll r_V \), the terms in Eq. \cite{53} are simply given by \( \xi_1 = -3re^{-\lambda\phi/M_{\text{pl}}}/M^3 \) and \( \xi_2 = -2re^{-\lambda\phi/M_{\text{pl}}}/M^3 \). Then the solution to the field equation \cite{22}, after matching at \( r = r_V \), is

\[
\phi'(r) \simeq QM_{\text{pl}}r_g/r_V^{3/2}r_V^{1/2},
\]

Using this solution as well as the conditions \cite{129} and \cite{130}, one can show that Eqs. \cite{49}, \cite{50}, and \cite{51} are satisfied. If the condition

\[
m^2 \gg \frac{|\phi''(r)|}{|Q|M_{\text{pl}}},
\]

is met in addition to \cite{129} and \cite{130}, Eqs. \cite{8} and \cite{9} of the gravitational potentials approximately reduce to the same equations as \cite{88} and \cite{89} respectively. Then the gravitational potentials in the regime \( r_g \ll r \ll r_V \) are given by Eq. \cite{90}, so that the fifth force is suppressed deep inside the Vainshtein radius.

\[^{3}\text{The model characterized by } G_5(\phi) \propto \phi \text{ corresponds to the one studied in Refs. } 58.\]
Inside the star ($r < r_s$) the solution to the field equation is subject to change. As long as the conditions (129), (130), and (132) are satisfied, the situation is similar to what we studied in Sec. VA for $p_3 = 1$. Around the radius of the star the solution (131) smoothly connects to another solution $\phi'(r) \propto r$ (see Fig. 3).

Since $|\phi'(r)|$ reaches a maximum around the surface of the star, we can substitute the values $\phi'(r_s)$ and $\phi''(r_s)$ into Eqs. (129) and (132) to derive the bounds of $c_{GB}$, $\mu$, and $m$, as

$$|c_{GB}| \ll \frac{r_V^3 r_s M_{pl}^2}{Q^2 r_s^2 g}, \quad \mu \gtrsim \left( \frac{Q^2 M_{pl}^2 g^2}{r_V^2 r_s^4} \right)^{1/4}, \quad m \gtrsim \left( \frac{r_s^2 g}{r_V^3 r_s^4} \right)^{1/4}. \quad (133)$$

The condition (130) gives a weaker bound on $\mu$ than the second of Eq. (133). If we demand that the experimental bound of the post-Newtonian parameter (i.e., $|\gamma - 1| \approx Q^2 (r/r_V)^{3/2} < 2.3 \times 10^{-5}$) is satisfied up to the scales $r = 10$ Au $\approx 10^{14}$ cm, then $r_V$ needs to be larger than $10^{17}$ cm for $|Q| = O(1)$. On using Eq. (128), this corresponds to the mass scale $M \lesssim 10^{-18}$ GeV. In the case of the Sun with the Vainshtein radius $r_V = 10^{20}$ cm, for example, the conditions (133) translate to $|c_{GB}| \ll 10^{124}$, $\mu \gtrsim 10^{-13}$ GeV, and $m \gtrsim 10^{-34}$ GeV. In particular, the Gauss-Bonnet coupling with $|c_{GB}| \sim 1$ does not give rise to any modification to the Vainshtein mechanism. It is worthy of mentioning that in the field equation (119) the effect of the Gauss-Bonnet coupling appears only in the $G_4$ term of Eq. (20).

VI. CONCLUSIONS

In this paper we have studied the Vainshtein mechanism in the most general second-order scalar-tensor theories given by the action (1). We derived the full equations of motion (1-12) for a spherically symmetric metric (7) characterized by two gravitational potentials $\Psi$ and $\Phi$. Under the weak gravity approximation the equations of motion for the field $\phi$ and the gravitational potential $\Psi$ reduce to fairly simple forms (16) and (20), respectively. These equations can be used to study the Vainshtein screening effect as well as the chameleon and symmetron mechanisms.

In the presence of a non-minimal coupling $e^{-2Q\phi/M_{pl}}$ with the Ricci scalar $R$, we clarify conditions under which the Vainshtein mechanism operates due to the field non-linear self-interactions. The Vainshtein radius $r_V$ is implicitly given by the formula (13), from which $r_V$ is known explicitly for a given model. For the distance $r$ larger than $r_V$ the non-linear field self-interactions are suppressed, so that the solution to Eq. (19) is $\phi'(r) = Q M_{pl} g r_s^2/r^2$. For the validity of this solution we require that all the conditions (37), (40), (41), and (42) are satisfied. For the distance characterized by $r_g \ll r \ll r_V$, the field equation (19) reduces to (32) under the conditions (42), (50), and (61). This is the regime in which the Vainshtein mechanism works to suppress the propagation of the fifth force. Inside a spherically symmetric body ($r < r_s$), the solution is different from that in the regime $r_g \ll r \ll r_V$. For the smooth matching of two solutions the Vainshtein mechanism needs to be at work inside the body as well.

The covariant Galileon model characterized by $G_4 = M_{pl}^2 e^{-2Q\phi/M_{pl}}/2 + c_4 X^2/M^6$ and $G_3 = G_5 = 0$ is a prototype that accommodates the Vainshtein mechanism successfully. In this model there is the solution $\phi'(r) = Q M_{pl} g r_s^2/r_V^2 = constant$ for the distance $r_g \ll r \ll r_V$. In this regime the gravitational potentials are given by Eq. (63), in which case local gravity constraints are well satisfied. In Sec. III V we confirmed that all the conditions to derive the solutions (15) and (16) are consistently satisfied. For the star with a constant density there is the solution $\phi'(r) \propto r$ with which the Vainshtein mechanism is at work inside the body. For the varying density characterized by the profile (55) we numerically showed that the interior solution smoothly connects with the exterior solution (60). This result is insensitive to the choice of the density profile, so that the Vainshtein mechanism operates successfully both outside and inside the body.

In Sec. IV we studied the covariant Galileon model in which all the non-linear derivative terms in $G_{3,4,5}$ exist. In the absence of the term $G_5 = c_5 X^2/M^3$ we showed that the Vainshtein mechanism is at work to suppress the fifth force inside the Vainshtein radius. However, if the term $G_5 = c_5 X^2/M^3$ is present, this modifies the solution of the field equation (19) due to the appearance of the term $4X(G_5 X + X G_5 X)/r^2$ in Eq. (20). For the model with $|c_4| \sim 1$ and $|c_3| \ll 1$, unless the coefficient $c_5$ satisfies the condition (15), the Vainshtein mechanism does not operate inside the star and hence there is a problem of matching the solutions around the surface of the body. For the model with $|c_4| \sim 1$ and $|c_4| \ll 1$, unless $|c_5|$ is much smaller than 1, we showed that local gravity constraints are not satisfied within the solar system. These results are consistent with those of Kimura et al. [33] and Koyama et al. [43], but we derived the conditions for the success of the Vainshtein mechanism more precisely in the presence of all the covariant Galileon terms.

In Sec. V we applied our results to several models such as extended Galileons, covariant Galileons with dilatonic couplings, and DBI Galileons with Gauss-Bonnet and other terms. As long as the non-linear derivative terms coupled to the Einstein tensor ($G_5 = c_5 M^{-1-p_5} e^{-\lambda_0 \phi/M_{pl}} X^{p_5}$ with $p_5 \geq 2$) do not dominate over other non-linear field self-interactions, we showed that the Vainshtein mechanism is at work both inside and outside the star. In short, the dominance of the terms such as $G_3 = c_3 M^{1-p_3} e^{-\lambda_0 \phi/M_{pl}} X^{p_3}$ ($p_3 \geq 1$) and
shown that the time variation of the Newton "constant" $G$ cosmologically large scales at which the time variations of physical quantities are non-negligible. In Ref. [59] it was shown that the time variation of the Newton "constant" $G_N$ can put tight constraints on scalar-tensor theories when the matter-scalar coupling is of the order of unity. In order to address this point, we need to discuss solutions of the field equations in the spherically symmetric configurations on the time-dependent cosmological background (along the line of Ref. [33]). It will be certainly of interest to study whether there exist dark energy models based on the Horndeski's theory which can be compatible with both local gravity constraints and the bounds of the time variation of $G_N$. We leave this for a future work.

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