In memory of my Teacher,
Mark Iosifovich Vishik

Controllability implies mixing. I.
Convergence in the total variation metric

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Abstract. This paper is the first part of a project to study the interconnection between the controllability properties of a dynamical system and the large-time asymptotics of trajectories for the associated stochastic system. It is proved that the approximate controllability to a given point and the solid controllability from the same point imply the uniqueness of a stationary measure and exponential mixing in the total variation metric. This result is then applied to random differential equations on a compact Riemannian manifold. In the second part of the project, the solid controllability will be replaced by a stabilisability condition, and it will be proved that this is still sufficient for the uniqueness of a stationary distribution, whereas the convergence to it occurs in the weaker dual-Lipschitz metric.

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Keywords: controllability, ergodicity, exponential mixing.

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0. Introduction

It is well known in the theory of stochastic differential equations that the mixing character of a random flow is closely related to the controllability properties of the associated deterministic dynamics. To be precise, let us consider the following stochastic differential equation on a compact Riemannian manifold $X$ without boundary:

$$
\frac{du_t}{dt} = V_0(u_t) \, dt + \sum_{j=1}^{n} V_j(u_t) \circ d\beta_j, \quad u_t \in X, \quad (0.1)
$$

where $V_0, V_1, \ldots, V_n$ are smooth vector fields on $X$, $\{\beta_j\}$ are independent Brownian motions, and the equation is interpreted in the sense of Stratonovich. Along with (0.1), consider the control system

$$
\dot{u} = V_0(u) + \sum_{j=1}^{n} \zeta_j(t)V_j(u), \quad u \in X. \quad (0.2)
$$

Here the $\zeta_j$ are real-valued piecewise continuous (control) functions. Let us denote by $\Gamma(TX)$ the Lie algebra of smooth vector fields on $X$ and by $\text{Lie}(V_1, \ldots, V_n)$ the minimal Lie subalgebra containing $V_j$, $j = 1, \ldots, n$. We assume that the following conditions are fulfilled.

- **Hörmander condition.** The subalgebra $\text{Lie}(V_1, \ldots, V_n)$ has full rank at some point $\tilde{u} \in X$; that is,

$$
\{ V(\tilde{u}) : V \in \text{Lie}(V_1, \ldots, V_n) \} = T_{\tilde{u}}X, \quad (0.3)
$$

where $T_{\tilde{u}}X$ denotes the tangent space of $X$ at the point $\tilde{u}$.

- **Approximate controllability.** For any $u_0, u_1 \in X$ and $\varepsilon > 0$ there exist piecewise continuous functions $\zeta^j : [0, T] \rightarrow \mathbb{R}$ for some $T > 0$ such that the solution $u(t)$ of (0.2) beginning at $u_0$ belongs to the $\varepsilon$-neighbourhood of $u_1$ at time $T$.

Under the above hypotheses, the results established in [5] (see also [21]) imply that the diffusion process generated by (0.1) has a unique stationary measure. Thus, a sufficient condition for the uniqueness of a stationary distribution is expressed in terms of the control system (0.2): the first hypothesis is well known in control theory and ensures the accessibility of (0.2) (see §8.1 in [2], for example), while the second is nothing else but the global approximate controllability in finite time. We remark that both papers mentioned above use the regularity of the transition probabilities, which is based on this or that form of the theory of hypoelliptic partial differential equations (PDEs).

The aim of our project is twofold: first, to investigate the problem of ergodicity for (0.1) in the situation when the Brownian motions are replaced by other types of random processes (that need not be Gaussian, so that the tools related to hypoelliptic PDEs are not applicable), and second, to establish similar results for Markov processes corresponding to PDEs with a degenerate noise. The main emphasis is on the general principle according to which suitable controllability properties of the

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1The manifold $X$ is endowed with the natural distance associated with the Riemannian metric.
control system associated with the stochastic equation under study imply ergodicity of the latter. In this paper we consider the situation where the convergence to the unique stationary measure holds in the total variation metric, and our main example is a differential equation driven by vector fields with random amplitudes. We refer the reader to §1.2 for an exact formulation of our result on mixing and to §2.1 for an application of it to ordinary differential equations on a compact manifold.

We mention that the question of ergodicity for Markov processes is rather well understood, especially in the situation when the strong Feller property is satisfied; see the monographs [13], [17], and [15]. In the context of stochastic differential equations, the strong Feller property is often verified using the Malliavin calculus or the regularity of solutions for hypoelliptic PDEs (see §2.3 in [16], Chap. 12 in [6], §11.5 in [7], and §22.2 in [11]). In our approach, we do not use the Malliavin calculus or the regularity theory for PDEs, replacing them by a general result on the image of probability measures under a smooth map that possesses a controllability property. Finally, we emphasise that even though we confine ourselves to the case of a compact phase space, it is not difficult to extend our results to a more general setting of an unbounded space, assuming that the stochastic dynamics satisfies an appropriate dissipativity condition.

I am grateful to A. Agrachev for numerous discussions on controllability properties of non-linear systems and to S. Kuksin for suggesting a number of improvements.

**Notation.** Let $X$ be a Polish space with a metric $d$, let $E$ be a separable Banach space, and let $J \subset \mathbb{R}$ be a bounded closed interval. We shall use the following notation.

- $B_X(u, r)$ and $\dot{B}_X(u, r)$ denote, respectively, the closed and open ball in $X$ of radius $r$ centred at $u$.
- $\mathcal{B}(X)$ is the Borel $\sigma$-algebra on $X$.
- $C_b(X)$ is the space of bounded continuous functions $f: X \to \mathbb{R}$ with the norm $\|f\|_\infty = \sup_{u \in X} |f(u)|$.
- In the case when $X$ is compact we shall write $C(X)$.
- $\mathcal{P}(X)$ is the space of probability measures on $X$. It is endowed with the total variation metric defined in §1.1.
- $C(J, E)$ is the space of continuous functions $f: J \to E$ with the supremum norm.
- $L^2(J, E)$ is the space of Borel-measurable functions $f: J \to E$ such that $\|f\|_{L^2(J, E)} = \left(\int_J \|f(t)\|_E^2 \, dt\right)^{1/2} < \infty$.
- In the case $E = \mathbb{R}$ we write $L^2(J)$.
- If $\Phi: E \to F$ is a measurable map and $\mu \in \mathcal{P}(E)$, then $\Phi_*\mu$ denotes the image of $\mu$ under $\Phi$.  


For a set \( \Gamma \) we denote its indicator function by \( I_\Gamma \). If \( f \in C_b(X) \) and \( \mu \in \mathcal{P}(X) \), then we write
\[
(f, \mu) = \int_X f(u) \mu(du).
\]
In particular, we have \((I_\Gamma, \mu) = \mu(\Gamma)\).

\( \mathcal{D}(\xi) \) denotes the law (distribution) of a random variable \( \xi \).

1. Mixing in terms of controllability properties

1.1. General framework and definitions. Let \((X, d)\) be a compact metric space, let \(E\) be a separable Banach space, and let \(S : X \times E \to X\) be a continuous map. We consider the stochastic system
\[
u_k = S(u_{k-1}, \eta_k), \quad k \geq 1, \quad (1.1)
\]
with the initial condition
\[
u_0 = u, \quad (1.2)
\]
where \(\{\eta_k\}\) are independent identically distributed (i.i.d.) \(E\)-valued random variables and \(u\) is a random variable with values in \(X\) that is independent of \(\{\eta_k\}\).
In this case the trajectories of (1.1) form a discrete-time Markov process \((\nu_k, P_\nu)\), and we denote its transition function by \(P_k(u, \Gamma)\) and the corresponding Markov semigroups acting on the respective spaces \(C(X)\) and \(\mathcal{P}(X)\) by \(\{P_k\}\) and \(\{P_k^*\}\).

Recall that a measure \(\mu \in \mathcal{P}(X)\) is said to be stationary for \((\nu_k, P_\nu)\) if \(P_1^* \mu = \mu\).

Our aim in this section is to establish a sufficient condition for the uniqueness of a stationary measure and its (exponential) stability in the total variation metric
\[
\|\mu_1 - \mu_2\|_{\text{var}} = \sup_{\Gamma \in \mathcal{B}(X)} |\mu_1(\Gamma) - \mu_2(\Gamma)| = \frac{1}{2} \sup_{\|f\|_{\infty} \leq 1} |(f, \mu_1) - (f, \mu_2)|,
\]
where the second supremum is taken over all continuous functions whose \(L_\infty\) norm is bounded by 1.

Let us introduce some controllability properties associated with the stochastic system (1.1).

Approximate controllability to a given point. Given any initial point \(\hat{u} \in X\), we say that the system (1.1) is \textit{globally approximately controllable to} \(\hat{u}\) if for any \(\varepsilon > 0\) there exist a compact set \(\mathcal{K} = \mathcal{K}_\varepsilon \subset E\) and an integer \(m = m_\varepsilon \geq 1\) such that, for any initial point \(u \in X\), one can find \(\zeta_1, \ldots, \zeta_m \in \mathcal{K}\) satisfying
\[
d(S_m(u; \zeta_1, \ldots, \zeta_m), \hat{u}) \leq \varepsilon, \quad (1.3)
\]
where \(S_k(u; \eta_1, \ldots, \eta_k)\) stands for the trajectory of the system (1.1), (1.2).

Solid controllability. Following [3], §12, we say that (1.1) is \textit{solidly controllable from} \(\hat{u}\) if there exist a compact set \(Q \subset E\), a non-degenerate ball \(B \subset X\), and a number \(\varepsilon > 0\) such that, for any continuous map \(\Phi : Q \to X\) satisfying
\[
\sup_{\zeta \in Q} d(\Phi(\zeta), S(\hat{u}, \zeta)) \leq \varepsilon, \quad (1.4)
\]
we have \(\Phi(Q) \supset B\).
We shall also need a class of probability measures on $E$. A measure $\ell \in \mathcal{P}(E)$ is said to be decomposable if there exist two sequences of closed subspaces $\{F_n\}$ and $\{G_n\}$ in $E$ such that the following properties hold:

(i) we have $\dim F_n < \infty$ and $F_n \subset F_{n+1}$ for any $n \geq 1$, and the union $\bigcup_n F_n$ is dense in $E$;

(ii) the space $E$ can be represented as the direct sum of $F_n$ and $G_n$, the operator norms of the corresponding projections $P_n$ and $Q_n$ are bounded, and for any $n \geq 1$ the measure $\ell$ can be written as the product of its projections $P_n \ell$ and $Q_n \ell$.

We note that the boundedness of the projections $P_n$ is equivalent to the following property:

$$P_n \to I \quad \text{and} \quad Q_n \to 0 \quad \text{in the strong operator topology.}$$

Indeed, the fact that (1.5) implies the boundedness of the norms of $P_n$ and $Q_n$ follows immediately from Baire’s theorem. Conversely, suppose that the norms of the projections $P_n$ are bounded by a number $C$ and fix a vector $\zeta \in E$. In view of the denseness of $\bigcup_n F_n$, there are $\zeta_n \in F_n$ such that $\|\zeta - \zeta_n\|_E \to 0$ as $n \to \infty$. It follows that

$$\|\zeta - P_n \zeta\|_E \leq \|\zeta - \zeta_n\|_E + \|P_n (\zeta - \zeta_n)\|_E + \|\zeta_n - P_n \zeta_n\|_E \leq (C + 1)\|\zeta - \zeta_n\|_E,$$

where we have used the relation $P_n \zeta_n = \zeta_n$. This implies that $P_n \zeta \to \zeta$ as $n \to \infty$.

1.2. Exponential mixing in the total variation metric. Recall that we are considering a stochastic system (1.1) in which $S: X \times E \to X$ is a continuous map and $\{\eta_k\}$ is a sequence of i.i.d. random variables in $E$ whose law $\ell$ is a decomposable measure on $E$. We denote by $\ell_n$ the image of $\ell$ under the projection to the subspace $F_n$ (in the definition of a decomposable measure). We say that a stationary measure $\mu \in \mathcal{P}(X)$ for the Markov process $(u_k, P_u)$ associated with (1.1) is exponentially mixing if there are positive numbers $\gamma$ and $C$ such that

$$\|P^k \lambda - \mu\|_{\text{var}} \leq Ce^{-\gamma k} \quad \text{for} \ k \geq 0, \ \lambda \in \mathcal{P}(X).$$

**Theorem 1.1.** Let $X$ be a compact Riemannian manifold, and let $S(u, \cdot): E \to X$ be an infinitely Fréchet-differentiable map whose derivative $(D_\eta S)(u, \eta)$ is a continuous function of $(u, \eta)$. Suppose, in addition, that the system (1.1) is globally approximately controllable to some point $\tilde{u} \in X$ and solidly controllable\(^2\) from $\tilde{u}$, the law $\ell$ of the random variables $\eta_k$ is decomposable, and the measures $P_n \ell$ possess positive continuous densities $\rho_n$ with respect to the Lebesgue measure on $F_n$. Then the Markov process $(u_k, P_u)$ associated with (1.1) has a unique stationary measure $\mu \in \mathcal{P}(X)$, which is exponentially mixing.

**Proof.** We shall prove that the hypotheses of Theorem 3.1 (see Appendix) are fulfilled for the Markov process $(u_k, P_u)$. Thus, we need to show that (3.1) and (3.2) hold for some positive numbers $\varepsilon$, $\delta$, $p$, and $m$.

\(^2\)The importance of the concept of solid controllability was first noted by Agrachev and Sarychev in [3] (see also [4]). It was later used in [1] to establish the absolute continuity of finite-dimensional projections of laws for solutions of stochastic PDEs.
Step 1: recurrence. The global approximate controllability will immediately imply (3.1) once we have proved that the support of \( \ell \) coincides with \( E \). Indeed, fix any \( \delta > 0 \). In view of global approximate controllability, there exist an integer \( m \geq 1 \) and a compact set \( \mathcal{K} \subset E \) such that, given \( u \in X \), one can find vectors \( \zeta_1^u, \ldots, \zeta_m^u \in \mathcal{K} \) such that

\[
d_{X}(S_m(u; \zeta_1^u, \ldots, \zeta_m^u), \hat{u}) \leq \frac{\delta}{2}.
\]

By the uniform continuity of \( S_m \) on the compact set \( X \times \mathcal{K}^m \) (where \( \mathcal{K}^m \) stands for the \( m \)-fold product of the set \( \mathcal{K} \) with itself) one can find an \( \varepsilon > 0 \) independent of \( u \) such that \( S_m(u; \zeta_1, \ldots, \zeta_m) \in B_X(\hat{u}, \delta) \) for any vectors \( \zeta_1, \ldots, \zeta_m \in E \) satisfying \( \| \zeta_k - \zeta_k^u \|_E \leq \varepsilon \) for \( 1 \leq k \leq m \). The foregoing implies that

\[
P_m(u, B_X(\hat{u}, \delta)) \geq \mathbb{P}\{ \eta_k \in B_E(\zeta_k^u, \varepsilon), 1 \leq k \leq m \} = \prod_{k=1}^m \mathbb{P}\{ \eta_1 \in B_E(\zeta_k^u, \varepsilon) \}.
\]

The product on the right-hand side of this inequality is positive, because the support of the law of \( \eta_1 \) coincides with \( E \). It follows from the portmanteau theorem (see Theorem 11.1.1 in [8]) that the function \( u \mapsto P_m(u, B_X(\hat{u}, \delta)) \) on the compact space \( X \) is lower-semicontinuous and hence minorised by a positive number \( p \). This implies the required inequality (3.1).

We now prove that \( \supp \ell = E \). Since the union \( \bigcup_n F_n \) is dense in \( E \), the latter property will be established once we have shown that \( \supp \mathcal{D}(\eta_1) \supset F_m \) for any \( m \geq 1 \). Fix any integer \( m \geq 1 \), any vector \( \hat{\eta} \in F_m \), and any number \( \varepsilon > 0 \). It follows from (1.5) that the sequence \( \{ Q_n \eta_1 \} \) goes to zero almost surely, and therefore also in probability. Hence

\[
\mathbb{P}\left\{ \| Q_n \eta_1 \| > \frac{\varepsilon}{2} \right\} \to 0 \quad \text{as } n \to \infty.
\]

(1.7)

For any \( n \geq m \), we write

\[
\mathbb{P}\{ \eta_1 \in B_E(\hat{\eta}, \varepsilon) \} = \mathbb{P}\{ \| \eta_1 - \hat{\eta} \| \leq \varepsilon \} \geq \mathbb{P}\left\{ \| P_n \eta_1 - \hat{\eta} \| \leq \frac{\varepsilon}{2}, \| Q_n \eta_1 \| \leq \frac{\varepsilon}{2} \right\} = \mathbb{P}\{ \| P_n \eta_1 - \hat{\eta} \| \leq \frac{\varepsilon}{2} \} \mathbb{P}\{ \| Q_n \eta_1 \| \leq \frac{\varepsilon}{2} \}.
\]

The first factor on the right-hand side is positive, since \( \hat{\eta} \in \supp \mathcal{D}(P_n \eta_1) \) due to the positivity of the density \( \rho_n \). In view of (1.7) the second factor goes to 1 as \( n \to \infty \), so that the right-hand side is positive for sufficiently large \( n \). Since \( \varepsilon > 0 \) was arbitrary, we conclude that \( \hat{\eta} \in \mathcal{D}(\eta_1) \).

Step 2: coupling. We need to prove (3.2). To this end we first establish a lower bound for the measures \( P_1(u, \cdot) \) on a ball \( B_X(\hat{u}, \delta) \), where \( \delta > 0 \) is sufficiently small. This will be done with the help of Proposition 3.2.

By the hypothesis, (1.1) is solidly controllable from \( \hat{u} \). We denote by \( Q \subset E \) a compact subset such that the image of any map \( \Phi: Q \to X \) satisfying the inequality (1.4) with \( \varepsilon \ll 1 \) contains a ball in \( X \). It follows from (1.5) that

\[
\sup_{\zeta \in Q} \| P_n \zeta - \zeta \|_E \to 0 \quad \text{as } n \to \infty.
\]
Combining this with the uniform continuity of $S(u, \cdot): E \to X$ on $Q$, we see that (1.4) is satisfied for $\Phi(\zeta) = S(\tilde{u}, P_n \zeta)$ with sufficiently large $n \geq 1$. Thus, there exist an integer $n \geq 1$ and a ball $Q_n \subset F_n$ such that the image of the map $S(\tilde{u}, \cdot): Q_n \to X$ covers a ball in $X$. By the Sard theorem (see §II.3 in [20]) there exists a $\tilde{\zeta} \in Q_n$ such that the derivative $(D_n S)(\tilde{u}, \tilde{\zeta})$ has full rank. Proposition 3.2 implies that there exist $\delta > 0$ and a continuous function $\psi: B_X(\tilde{u}, \delta) \times X \to \mathbb{R}_+$ such that

$$\psi(\tilde{u}, \tilde{x}) > 0,$$

$$S(u, \cdot)_* \ell \geq \psi(u, x) \text{ vol}(dx) \quad \text{for } u \in B_X(\tilde{u}, \delta),$$

(1.8)

(1.9)

where $\tilde{x} = S(\tilde{u}, \tilde{\zeta})$ and $\text{vol}(\cdot)$ denotes the Riemannian measure on $X$. It follows from (1.8) that by taking smaller $\delta > 0$ if necessary we obtain

$$\psi(u, x) \geq \epsilon > 0 \quad \text{for } u \in B_X(\tilde{u}, \delta), \; x \in B_X(\tilde{x}, \delta).$$

(1.10)

Now note that $S_*(u, \ell) = P_1(u, \cdot)$. Combining (1.9) and (1.10), we get that

$$P_1(u, dx) \geq \epsilon I_{B_X(\tilde{x}, \delta)}(x) \text{ vol}(dx) \quad \text{for } u \in B_X(\tilde{u}, \delta),$$

where $I_{\Gamma}$ stands for the indicator function of a set $\Gamma$. It follows that

$$\|P_1(u, \cdot) - P_1(u', \cdot)\|_{\text{var}} \leq 1 - \epsilon \text{ vol}(B_X(\tilde{x}, \delta)).$$

This completes the proof of Theorem 1.1. □

2. Differential equations on a compact manifold

2.1. Main result. Let $X$ be a compact Riemannian manifold of dimension $d \geq 1$ without boundary. We consider the ordinary differential equation

$$\dot{u} = V_0(u) + \sum_{j=1}^{n} \eta^j(t)V_j(u), \quad u(t) \in X.$$  

(2.1)

Here the $V_j$, $j = 0, \ldots, n$, are smooth vector fields on $X$ and the $\eta^j(t)$ are real-valued random processes of the form

$$\eta^j(t) = \sum_{k=1}^{\infty} I_{[k-1, k)}(t) \eta^j_k(t-k+1),$$

(2.2)

where the $\eta^j_k$ are random variables in $L^2(J)$ with $J = [0, 1]$ such that the vector functions $\eta_k = (\eta^1_k, \ldots, \eta^n_k)$ are i.i.d. random variables in $E := L^2(J, \mathbb{R}^n)$. We denote by $\ell \in \mathcal{P}(E)$ the law of the $\eta_k$, $k \geq 1$.

Before formulating the main result of this section, we recall some well-known facts about equation (2.1). Let $\eta^j: \mathbb{R}_+ \to \mathbb{R}^n$ be measurable functions that are integrable on any compact subset of $\mathbb{R}_+$. Then for any $v \in X$ there is a unique absolutely continuous function $u: \mathbb{R}_+ \to X$ that satisfies (2.1) for almost every $t \geq 0$ and also the initial condition

$$u(0) = v.$$  

(2.3)
Moreover, if we denote by $S$ the map from $X \times E$ to $X$ that takes the pair $(v, \eta)$ to $u(1)$, where $u(t)$ is the solution of the problem (2.1)–(2.3) on $J$ with $(\eta^1, \ldots, \eta^n) = \eta$, then classical results in the theory of ordinary differential equations imply that $S$ is infinitely Fréchet differentiable. We let $u_k = u(k)$ and observe that

$$u_k = S(u_{k-1}, \eta_k), \quad k \geq 1. \tag{2.4}$$

Since $\{\eta_k\}$ are i.i.d. random variables, the family of all sequences $\{u_k\}$ satisfying (2.4) forms a discrete-time Markov process, which we denote by $(u_k, \mathbb{P}_u)$. We write $\mathcal{P}_k$ and $\mathcal{P}_k^*$ for the corresponding Markov semigroups.

We say that the control system (0.2) considered on $X$ satisfies the weak Hörmander condition at a point $\hat{u} \in X$ if there are $d$ vector fields in the family

$$\{V_j, j = 1, \ldots, n: [V_j, V_k], 0 \leq j, k \leq n; [[V_j, V_k], V_l], 0 \leq j, k, l \leq n; \ldots \}$$

that are linearly independent at $\hat{u}$. In other words, letting

$$V_\zeta = V_0 + \zeta^1 V_1 + \cdots + \zeta^n V_n \quad \text{for} \quad \zeta = (\zeta^1, \ldots, \zeta^n) \in \mathbb{R}^n, \tag{2.5}$$

we see that the weak Hörmander condition is equivalent to the hypothesis that the zero-time ideal$^3$ of the family $\{V_\zeta, \zeta \in \mathbb{R}^n\}$ has full rank at $\hat{u}$ (see §2.4 in [12]). We refer the reader to §2.3 in [16] and §2 in [9] for a discussion of this condition from the probabilistic point of view.

Let $X = C(J, X)$. The theorem below, which is proved in the next subsection, describes the large-time asymptotics of the laws of the trajectories of the system (2.1)–(2.3).

**Theorem 2.1.** In addition to the above hypotheses, assume that the following two conditions are satisfied:

(a) there is a point $\hat{u} \in X$ such that the system (1.1) is globally approximately controllable to $\hat{u}$, and the weak Hörmander condition holds at $\hat{u}$;

(b) the law $\ell$ is decomposable, and the measures$^4$ $\mathbb{P}_n \ll \ell$ possess positive continuous densities $\rho_n$ with respect to the Lebesgue measure on $F_n$.

Then there exist a unique measure $\mu \in \mathcal{P}(X)$ and positive numbers $\gamma$ and $C$ such that, for any $X$-valued random variable $v$ independent of $\{\eta_k\}$, the solution $u(t)$ of (2.1)–(2.3) satisfies the inequality

$$\|\mathcal{D}(u_k) - \mu\|_{\text{var}} \leq Ce^{-\gamma k}, \quad k \geq 1, \tag{2.6}$$

where $u_k$ denotes the restriction of $u(t)$ to the interval $[k-1, k]$, and $\|\cdot\|_{\text{var}}$ denotes the total variation norm on $\mathcal{P}(X)$.

**2.2. Proof of Theorem 2.1.** We begin with a simple remark reducing the proof of the theorem to the problem of exponential mixing for the discrete-time Markov process $(u_k, \mathbb{P}_u)$ associated with the system (2.4). Suppose that we have proved that (2.4) has a unique stationary measure $\mu \in \mathcal{P}(X)$ which is exponentially mixing.

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$^3$We do not use this concept in what follows, so readers unfamiliar with it may safely ignore this reformulation.

$^4$We denote by $F_n$ the finite-dimensional spaces in the definition of a decomposable measure, and by $\mathbb{P}_n$ the corresponding projections.
in the sense that (1.6) holds for the corresponding Markov semigroup. We denote by $S: X \times E \to X$ the map that takes $(v, \eta)$ to $(u(t), t \in J)$, where $u(t)$ is the solution of the problem (2.1)–(2.3) on $J$ with $(\eta^1, \ldots, \eta^n) = \eta$. It follows from the independence of the random variables $\{\eta_k\}$ that

$$\mathcal{D}(u_k) = S_*((\mathcal{P}^*_{k-1} \lambda) \otimes \ell) \quad \text{for any } k \geq 1,$$

(2.7)

where $\lambda = \mathcal{D}(v)$. Let $\mu = S_* (\mu \otimes \ell)$. Since the total variation distance does not increase under a measurable map, (2.7) implies that

$$\| \mathcal{D}(u_k) - \mu \|_{\text{var}} \leq \| ((\mathcal{P}^*_{k-1} \lambda) \otimes \ell - \mu \otimes \ell) \|_{\text{var}} = \| \mathcal{P}^*_{k-1} \lambda - \mu \|_{\text{var}}, \quad k \geq 1.$$

(2.8)

The required inequality (2.6) follows from (1.6) and (2.8).

We thus need to prove that $(u_k, \mathbb{P}_u)$ has a unique stationary measure, which is exponentially mixing in the total variation metric. To this end we show that the hypotheses of Theorem 1.1 are fulfilled. Namely, it suffices to check that the map $S(u, \cdot): E \to X$ is infinitely differentiable, $(D_\eta S)(u, \eta)$ is continuous on $X \times E$, and (1.1) is solidly controllable from $\hat{u}$. As we mentioned above, the first two properties hold due to classical results in the theory of ordinary differential equations. To prove the solid controllability from $\hat{u}$ we use a degree-theory argument (see §12.2 in [3] and §2.3 in [18]) and a well-known idea from control theory (see the proof of Theorem 3 in [12], Chap. 3, §1.2).

Step 1: reduction to continuous exact controllability. Given a closed ball $B = B_X(\hat{v}, r)$, we say that the system (1.1) is continuously exactly controllable from $\hat{u}$ to $B$ if there exists a continuous map $f: B \to E$ such that

$$S(\hat{u}, f(v)) = v \quad \text{for any } v \in B.$$

(2.9)

We claim that if (1.1) is continuously exactly controllable from $\hat{u}$ to some ball $B = B_X(\hat{v}, r)$, then it is solidly controllable from $\hat{u}$. Indeed, given a continuous map $\varphi: B \to X$ and a point $z \in X \setminus \varphi(\partial B)$, we denote the degree of $\varphi$ at $z$ by $\deg(\varphi, B, z)$. We choose $\varepsilon > 0$ so small that

$$\deg(\varphi, B, z) = \deg(I, B, z) = 1 \quad \text{for any } z \in B_X(\hat{v}, \varepsilon),$$

(2.10)

where $I: B \to B$ is the identity map and $\varphi: B \to X$ is an arbitrary continuous map such that

$$\sup_{v \in B} d_X(\varphi(v), v) \leq \varepsilon.$$

(2.11)

Let $Q = f(B)$, where $f$ is the map in (2.9), and consider any continuous map $\Phi: Q \to X$ satisfying (1.4). Then (2.11) is true for $\varphi = \Phi \circ f$, whence it follows that (2.10) holds. In particular, for any $z \in B_X(\hat{v}, \varepsilon)$ there exists a $y \in B$ such that $\Phi(f(y)) = z$. We have thus shown that $\Phi(Q) \supset B_X(\hat{v}, \varepsilon)$.

Step 2: extended system. To prove the continuous exact controllability of the system (1.1), we introduce the extended phase space $\tilde{X} = X \times \mathbb{R}$ with the natural Riemannian structure, and we consider the new control system

$$\dot{y} = \tilde{V}_0(y) + \sum_{j=1}^n \zeta_j(t) \tilde{V}_j(y), \quad y \in \tilde{X},$$

(2.12)
where \( y = (u, z) \), \( \tilde{V}_0(y) = (V_0(u), 1) \), and \( \tilde{V}_j(y) = (V_j(u), 0) \) for \( j = 1, \ldots, n \).

It is clear that if \( y(t) \) is a trajectory for (2.12), then the projection of \( y \) to \( X \) is a trajectory for (0.2), and conversely, any trajectory of (0.2) can be extended to a trajectory of (2.12) by adding to it the function \( z(t) = t + z_0 \), where \( z_0 \in \mathbb{R} \) is an arbitrary initial point. In what follows, given a vector \( \zeta = (\zeta^1, \ldots, \zeta^n) \in \mathbb{R}^n \), we shall write

\[
\tilde{V}_\zeta = \tilde{V}_0 + \zeta^1 \tilde{V}_1 + \cdots + \zeta^n \tilde{V}_n \quad \text{and} \quad V_\zeta = V_0 + \zeta^1 V_1 + \cdots + \zeta^n V_n
\]

and let \( \mathcal{V} = \{ V_\zeta, \zeta \in \mathbb{R}^n \} \) and \( \bar{\mathcal{V}} = \{ \tilde{V}_\zeta, \zeta \in \mathbb{R}^n \} \). Note that the last component of the vector \( \tilde{V}_\zeta \) is equal to 1 for any \( \zeta \in \mathbb{R}^n \).

**Step 3:** Lie algebra generated by \( \bar{\mathcal{V}} \). Denote by \( \text{Lie}(\bar{\mathcal{V}}) \) the Lie algebra generated by \( \bar{\mathcal{V}} \). We claim that \( \text{Lie}(\bar{\mathcal{V}}) \) has full rank at any point \((\bar{u}, z)\) with \( z \in \mathbb{R} \); that is, the space of values of the vector fields in \( \text{Lie}(\bar{\mathcal{V}}) \) at the point \((\bar{u}, z)\) coincides with the tangent space \( T_{(\bar{u}, z)}\bar{X} \). Indeed, it is straightforward to check that

\[
[\tilde{V}_i, \tilde{V}_j] = ([V_i, V_j], 0) \quad \text{for} \quad 0 \leq i, j \leq n,
\]

whence it follows that the derived algebra\(^5\) of \( \bar{\mathcal{V}} \) has the form

\[
\mathcal{D} = \{(W, 0), W \in \mathcal{D}\}, \tag{2.13}
\]

where \( \mathcal{D} \) denotes the derived algebra of \( \mathcal{V} \). The weak Hörmander condition implies that

\[
\text{span}\{V_1, \ldots, V_n, \mathcal{D}\}|_{\bar{u}} = T_{\bar{u}}X.
\]

Combining this with (2.13), we see that

\[
\text{span}\{\tilde{V}_1, \ldots, \tilde{V}_n, \mathcal{D}\}|_{(\bar{u}, z)} = T_{\bar{u}}X \times \{0\}.
\]

Recalling that \( \tilde{V}_0 = (V_0, 1) \), we obtain the required result.

**Step 4:** continuous exact controllability at a time \( \tau \in (0, 1) \). Given an interval \( J_\tau = [0, \tau] \) and a function \( \zeta \in L^2(J_\tau, \mathbb{R}^n) \), we denote by \( \mathcal{R}_\tau(\zeta) \) the value at time \( \tau \) of the solution of (0.2) with initial condition \( \bar{u} \). We claim that there exist a number \( \tau \in (0, 1) \), a closed ball \( B' \subset X \), and a continuous function \( g: B' \to L^2(J_\tau, \mathbb{R}^n) \) such that

\[
\mathcal{R}_\tau(g(v)) = v \quad \text{for any} \quad v \in B'. \tag{2.14}
\]

To prove this, consider the extended system (2.12) and, for \( \zeta \in \mathbb{R}^n \) and \( y_0 \in \bar{X} \), denote by \( e^{t\tilde{V}_\zeta}y_0 \) its solution corresponding to the initial condition \( y_0 \) and the control functions \( (\zeta^1, \ldots, \zeta^n) \equiv \zeta \). Suppose that we have found vectors \( \zeta_0, \ldots, \zeta_d \in \mathbb{R}^n \) and an open parallelepiped

\[
\Pi = \{ \alpha = (\alpha_0, \ldots, \alpha_d) \in \mathbb{R}^{d+1}: a_l < \alpha_l < b_l, \ 0 \leq l \leq d \} \subset [0, 1]^{d+1}
\]

---

\(^5\)Recall that the derived algebra of a family \( \mathcal{V} \) of vector fields is defined as the linear span of all possible (iterated) commutators of the elements of \( \mathcal{V} \).
such that \( \sum_j b_l < 1 \), and the map

\[
\tilde{F}: \Pi \to \tilde{X}, \quad \alpha \mapsto e^{\alpha_d \tilde{V}_{\alpha_d}} \circ \cdots \circ e^{\alpha_0 \tilde{V}_{\alpha_0}} (\tilde{u}, 0),
\]

is an embedding of \( \tilde{\Pi} \) into \( \tilde{X} \). For any \( \alpha \in \tilde{\Pi} \) we set

\[
T_\alpha = \alpha_0 + \cdots + \alpha_d
\]

and define \( \xi^\alpha: [0, T_\alpha] \to \mathbb{R}^n \) by the relation

\[
\xi^\alpha(t) = \zeta_l \quad \text{for } \alpha_0 + \cdots + \alpha_{l-1} < t < \alpha_0 + \cdots + \alpha_l,
\]

where \( l = 0, \ldots, d \), and the left-hand bound in the inequality is set equal to zero for \( l = 0 \). Then, denoting by \( F \) the projection of \( \tilde{F} \) to \( X \), we see that

\[
F(\alpha) = \mathcal{R}_{T_\alpha}(\xi^\alpha) \quad \text{for } \alpha \in \tilde{\Pi}.
\]

We now fix \( \tilde{\alpha} \in \tilde{\Pi} \) and denote by \( \Pi \) the intersection of \( \tilde{\Pi} \) with the \( d \)-dimensional hyperplane \( L_{\tilde{\alpha}} = \{ \alpha_0 + \cdots + \alpha_d = \tau \} \subset \mathbb{R}^{d+1} \), where \( \tau = T_{\tilde{\alpha}} \). Then \( \Pi \) is an open polyhedron in \( \tilde{L}_{\tilde{\alpha}} \). Since the last component of \( V \xi \) is equal to 1 for any \( \xi \in \mathbb{R}^n \), the last component of \( \tilde{F}(\alpha) \) is equal to \( \tau \) for any \( \alpha \in \Pi \), so that \( \tilde{F}(\Pi) \) lies in the set \( \{(u, z) \in \tilde{X} : z = \tau \} \). Combining this fact with (2.16), we see that \( \mathcal{R}_{\tau}: \Pi \to X \) is a diffeomorphism of \( \Pi \) onto its image. Denote its inverse by \( \mathcal{R}_{\tau}^{-1} \). Now let \( B' \subset \mathcal{R}_{\tau}(\Pi) \) be an arbitrary closed ball. Then the map

\[
g: B' \to L^2(J_\tau, \mathbb{R}^n), \quad g(v) = \xi^{\mathcal{R}_{\tau}^{-1}(v)},
\]

is continuous and satisfies the required relation (2.14).

Thus, it remains to find a parallelepiped \( \tilde{\Pi} \) such that \( \tilde{F} \) defined by (2.15) is an embedding. Even though this is a well-known result, for the reader’s convenience we outline the main idea, following the argument in the proof of Krener’s theorem (see Theorem 8.1 in [2], for instance).

It was proved in Step 3 that \( \text{Lie}(\tilde{Y}) \) has full rank at the point \( \tilde{y} = (\tilde{u}, 0) \). By continuity, there is an open set \( U \subset X \) containing \( \tilde{y} \) such that \( \text{Lie}(\tilde{Y}) \) has full rank at any point \( y \in U \). In the construction below we assume without mentioning it explicitly that all the points belong to \( U \). We shall construct vectors \( \xi_j \in \mathbb{R}^n, 0 \leq j \leq d \), and numbers \( 0 < a_j < b_j < 1 \) such that \( \sum_j b_j < 1 \), and the following conditions hold:

(i) the map \( F_j: (\alpha_0, \ldots, \alpha_j) \mapsto e^{\alpha_j \tilde{V}_{\alpha_j}} \circ \cdots \circ e^{\alpha_0 \tilde{V}_{\alpha_0}} (\tilde{u}, 0) \) defines an embedding of the open parallelepiped

\[
\Pi_j = \{(\alpha_0, \ldots, \alpha_j) \in \mathbb{R}^{j+1}: a_l < \alpha_l < b_l, \ 0 \leq l \leq j\} \subset [0, 1]^{j+1}
\]

into the manifold \( \tilde{X} \);

(ii) the vector field \( \tilde{V}_{\xi_j}(y) \) is transversal to \( Y_{j-1} \) at any point \( y \in Y_{j-1} \) (we denote the \( F_j \)-image of \( \Pi_j \) by \( Y_j \)).

Once this is established, one can take \( \Pi = \Pi_d \), thus completing the construction of \( g \). To prove the above properties, we proceed by induction. For \( j = 0 \) we take
any vector \(\zeta_0 \in \mathbb{R}^n\) such that \(\widetilde{V}_{\zeta_0}(\tilde{y}) \neq 0\). We then set \(a_0 = 0\) and choose \(b_0 \in (0, 1)\) so small that \(F_0(\alpha)\) is an embedding of \(\Pi_0\). The condition (ii) is trivial for \(j = 0\).

Assume that the vectors \(\zeta_j \in \mathbb{R}^n\) and the intervals \((a_l, b_l)\) have been constructed for \(0 \leq l \leq j - 1\). Since \(\text{Lie}(\tilde{\psi})\) has full rank at any point \(y \in Y_{j-1}\), we can find \(\zeta_j \in \mathbb{R}^n\) and \(y_j \in Y_{j-1}\) such that \(\widetilde{V}_{\zeta_j}(y_j)\) is transversal to \(Y_{j-1}\). By continuity, reducing the sizes of the intervals \((a_l, b_l)\) if necessary, we can assume that \(\widetilde{V}_{\zeta_j}(y)\) is transversal to \(Y_{j-1}\) at any point \(y \in Y_{j-1}\). We now set \(a_j = 0\) and choose \(b_j > 0\) so small that \(b_0 + \cdots + b_j < 1\) and \(F_j(\alpha_0, \ldots, \alpha_j)\) defines an embedding of \(\Pi_j\) into \(\tilde{X}\). We have thus established the required property.

**Step 5: Completion of the proof.** We can now easily prove the inequality (2.9) where \(B \subset X\) is a closed ball. To this end we define \(\psi: X \to X\) to be the map that takes \(w_0 \in X\) to \(w(1 - \tau)\), where \(w(t)\) is the solution of the equation \(\dot{w} = V_0(w)\) with initial condition \(w_0\). It is well known from the theory of ordinary differential equations that \(\psi\) is a diffeomorphism of \(X\). Given any \(v \in B'\), we extend the function \(g(v) \in L^2(J_\tau, \mathbb{R}^n)\) to the interval \((\tau, 1]\) by zero and note that, in view of (2.14),

\[
S(\tilde{u}, g(v)) = \mathcal{R}_1(g(v)) = (\psi \circ \mathcal{R}_\tau)(g(v)) = \psi(v) \quad \text{for } v \in B'.
\]

Defining \(f: \psi(B') \to L^2(J_\tau, \mathbb{R}^n)\) by \(f(v) = g(\psi^{-1}(v))\), we see that \(S(\tilde{u}, f(v)) = v\) for \(v \in \psi(B')\).

It remains to note that since \(\psi\) is a diffeomorphism, the set \(\psi(B')\) contains a non-degenerate closed ball \(B \subset X\), and hence (2.9) holds. This completes the proof of Theorem 2.1. \(\square\)

3. Appendix

3.1. Sufficient condition for mixing. Let \(X\) be a compact metric space and let \((u_k, \mathbb{P}_u)\) be a discrete-time Markov process in \(X\). Since \(X\) is compact, \((u_k, \mathbb{P}_u)\) has at least one stationary measure \(\mu \in \mathcal{P}(X)\). The following result gives a sufficient condition for the uniqueness of the stationary measure and for its exponential stability.

**Theorem 3.1.** Suppose that there exist a point \(\tilde{u} \in X\) and a number \(\delta > 0\) such that the following conditions are satisfied.

(i) Recurrence: there exist a \(p > 0\) and an integer \(m \geq 1\) such that

\[
P_m(u, B_X(\tilde{u}, \delta)) \geq p \quad \text{for any } u \in X.
\]

(ii) Coupling: there exists an \(\varepsilon > 0\) such that

\[
\|P_1(u, \cdot) - P_1(u', \cdot)\|_{\text{var}} \leq 1 - \varepsilon \quad \text{for any } u, u' \in B_X(\tilde{u}, \delta).
\]

Then \((u_k, \mathbb{P}_u)\) has a unique stationary measure \(\mu \in \mathcal{P}(X)\), which is exponentially mixing for the total variation metric in the sense that (1.6) holds for some positive numbers \(\gamma\) and \(C\).
Even though this theorem is a particular case of more general results established in [15], Chaps. 15 and 16, (see also [10]), we give a direct proof of it for the reader’s convenience.

**Proof.** We shall prove that the map

\[ \mathcal{P}^*_{m+1}: \mathcal{P}(X) \to \mathcal{P}(X) \]

is a contraction. This will imply all the required results.

**Step 1.** Recall that, given two measures \( \lambda, \lambda' \in \mathcal{P}(X) \), we can find \( \nu, \hat{\lambda}, \hat{\lambda}' \in \mathcal{P}(X) \) such that

\[ \lambda = (1 - d)\nu + d\hat{\lambda} \quad \text{and} \quad \lambda' = (1 - d)\nu + d\hat{\lambda}', \]  

where \( d = \|\lambda - \lambda'\|_{\text{var}} \) (see, for example, Corollary 1.2.25 in [14]). It follows that

\[ \|\mathcal{P}^*_{m+1}\lambda - \mathcal{P}^*_{m+1}\lambda'\|_{\text{var}} = d\|\mathcal{P}^*_{m+1}\hat{\lambda} - \mathcal{P}^*_{m+1}\hat{\lambda}'\|_{\text{var}} = d\|\mathcal{P}^*_1\mu - \mathcal{P}^*_1\mu'\|_{\text{var}}, \]

where we have set \( \mu = \mathcal{P}^*_m\hat{\lambda} \) and \( \mu' = \mathcal{P}^*_m\hat{\lambda}' \). We see that the required contraction will be proved once we have shown that

\[ \|\mathcal{P}^*_1\mu - \mathcal{P}^*_1\mu'\|_{\text{var}} \leq q < 1. \]  

**Step 2.** To prove (3.4), we first note that

\[ \mathcal{P}^*_1\mu - \mathcal{P}^*_1\mu' = \int_{X \times X} (P_1(u, \cdot) - P_1(u', \cdot)) \mu(du) \mu'(du'). \]

Taking the total variation norm and using (3.2), we get that

\[ \|\mathcal{P}^*_1\mu - \mathcal{P}^*_1\mu'\|_{\text{var}} \leq \int_{X \times X} \|P_1(u, \cdot) - P_1(u', \cdot)\|_{\text{var}} \mu(du) \mu'(du') \]

\[ \leq (\mu \otimes \mu')(G^c_\delta) + (1 - \varepsilon)(\mu \otimes \mu')(G_\delta) \]

\[ = 1 - \varepsilon(\mu \otimes \mu')(G_\delta), \]

where \( G_\delta = B_X(\hat{u}, \delta) \times B_X(\hat{u}, \delta) \) and \( G^c = (X \times X) \setminus G \). It remains to note that by (3.1),

\[ (\mu \otimes \mu')(G_\delta) \geq \mu(B_X(\hat{u}, \delta))\mu'(B_X(\hat{u}, \delta)) \geq p^2, \]

and therefore (3.4) holds with \( q = 1 - \varepsilon p^2 \) for any \( \lambda, \lambda' \in \mathcal{P}(X) \). This completes the proof of Theorem 3.1. \( \square \)

### 3.2. Image of measures under regular maps.

Let \( E \) be a separable Banach space, let \( X \) be a compact metric space, and let \( Y \) be a Riemannian manifold. We consider a continuous map \( f: X \times E \to Y \) and recall that the concept of a decomposable measure was defined in §1.1. The following proposition is a particular case of more general results established in Chap. 9 of [6].

**Proposition 3.2.** Assume that the map \( f(u, \cdot): E \to Y \) is Fréchet differentiable for any fixed \( u \in X \), the derivative \( (D_\eta f)(u, \eta) \) is continuous on \( X \times E \), the image of the linear operator \( (D_\eta f)(u_0, \eta_0) \) has full rank for some \( (u_0, \eta_0) \in X \times E \), and \( \ell \)
is a decomposable measure on $E$ such that $P_n \ast \ell$ possesses a positive continuous density with respect to the Lebesgue measure on $F_n$. Then there exist a ball $Q \subset X$ centred at $u_0$ and a non-negative continuous function $\psi(u, y)$ defined on $Q \times Y$ such that

$$\psi(u_0, y_0) > 0 \quad (3.5)$$

and

$$f(u, \cdot) \ast \ell \geq \psi(u, y) \mathrm{vol}(dy) \quad \text{for } u \in Q, \quad (3.6)$$

where $y_0 = f(u_0, \eta_0)$, and $\mathrm{vol}(\cdot)$ is the Riemannian measure on $Y$.

A simple direct proof of Proposition 3.2 can be found in [19] in the case when $Y$ is a finite-dimensional vector space (see Theorem 2.4). An extension to the case of a Riemannian manifold is straightforward.

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