A clean way to separate sets of surreals

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Abstract. Let surreal numbers be defined by means of sign sequences. We give
a proof that if $S < T$ are sets of surreals, then there is some surreal $w$ such that
$S < w < T$. The classical proof is simplified by observing that, for every set $S$ of
surreals, there exists a surreal $s$ such that, for every surreal $w$, we have $S < w$ if and
only if the restriction of $w$ to the length of $s$ is $\geq s$. Hence $S < w < T$ if and only if
$w$ satisfies the above condition, as well as its symmetrical version with respect to $T$.
It is now enough to check that if $S < T$, then the two conditions are compatible.

1. Introduction

We prove the following theorems (the reader who does not know the defini-
tions will find full details below).

Theorem 1.1. If $S$ is any set of surreals, then there exists a surreal $s$ such
that, for every surreal $z$, the following conditions are equivalent.

(1) $u < z$, for every $u \in S$;
(2) the restriction of $z$ to the length of $s$ is $\geq s$.

Theorem 1.2. If $S$ and $T$ are sets of surreals and $u < v$, for every $u \in S$
and $v \in T$, then there exists a surreal $w$ such that $u < w < v$, for every $u \in S$
and $v \in T$.

Theorems 1.1 and 1.2 follow from, respectively, Theorems 4.3 and 5.1 which
shall be proved below.

The present note is intended to be as much self-contained as possible. The
only prerequisite is a basic knowledge of ordinal numbers. The main facts
about ordinals are stated without proof in Subsection 2.1 below. Complete
details for the construction of surreals as sign sequences are given in Subsection
2.2. Subsection 2.3 contains the definition of an initial segment of a surreal,
and a useful lemma about it. Section 3 contains a brief discussion about the
usefulness of separating sets of surreals, as well as a few comments about the
subsequent proofs. Strengthenings of Theorems 1.1 and 1.2 will be proved in
Sections 4 and 5 respectively. In Section 5 the shortest surreal $w$ satisfying
Theorem 1.2 will be also evaluated. Section 6 contains a few further remarks.

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2. Preliminaries

2.1. Ordinals. Recall that a linearly ordered set \((W, \prec)\) is *well-ordered* if every nonempty subset of \(W\) has a minimum. If \(w \in W\), the subset \(W_{\prec w} = \{v \in W \mid v < w\}\) of \(W\) is called a (proper) *initial segment* of \(W\) and inherits from \(W\) the structure of a well-ordered set. We shall usually consider \(W\) itself as an (improper) initial segment of \(W\).

It turns out that any two well-ordered sets are comparable, in the sense that they are either isomorphic, or one is isomorphic to some proper initial segment of the other. Exactly one of the above possibilities occurs. One can choose a representative for any isomorphism class of well-ordered sets. These representatives are called *ordinals* and can be chosen in such a way that each ordinal is the set of all smaller ordinals and the order relation \(\prec\) coincides with the membership relation \(\in\). That is, if \(\beta \in \alpha\) and \(\alpha\) is an ordinal, then \(\beta\) is an ordinal which is an initial segment of \(\alpha\) and, conversely, if \(\beta\) is an ordinal which is a proper initial segment of the ordinal \(\alpha\), then \(\beta \in \alpha\). Ordinals shall be always denoted by small Greek letters such as \(\alpha, \beta, \gamma\ldots\) and any such symbol is always intended to represent an ordinal, even when not explicitly mentioned. The exact definition of an ordinal shall not be relevant here. We shall make heavy use of the assumption that an ordinal is well-ordered. For every ordinal \(\alpha\), there is the smallest ordinal \(\beta\) which is strictly larger than \(\alpha\). Such a \(\beta\) is denoted by \(\alpha + 1\) (one can actually define a sum between ordinals, but we shall not need it here). An ordinal \(\beta\) is a *successor ordinal* if it has the form \(\alpha + 1\), for some ordinal \(\alpha\). Otherwise, \(\beta\) is a *limit ordinal*. According to the above definition, the smallest ordinal \(0\) (the order-type of the empty order) is considered a limit ordinal\(^1\).

Full details about ordinals can be found in Jech [J] or in any textbook on set theory.

2.2. Surreals. A *surreal* is a function \(s\) from some ordinal \(\ell(s)\) to the set \(\{+, -\}\). The ordinal \(\ell(s)\) depends on \(s\) and is called the *length* of \(s\). Notice that we allow \(\ell(s) = 0\); in this case \(s\) is the empty sequence. The exact nature of the symbols + and − does not concern us, one can take any two distinct elements, say, \(+ = (0, 0)\) and \(- = (0, 0)\). Surreal numbers have been introduced by Conway [C]. See Gonshor [G] for a full presentation of the surreals considered, as above, as sequences of +’s and −’s. Notice that Conway introduced surreals in a different fashion. The above way of representing a surreal is called its *sign sequence* and is also due to Conway. Siegel [S] is another useful reference about surreals.

Recall that an ordinal is equal to the set of all smaller ordinals, thus \(\ell(s)\) is the first ordinal for which \(s\) is not defined. If \(\gamma \geq \ell(s)\) is an ordinal, we

\(^1\)In some cases, it is convenient to consider 0 to be neither limit nor successor, that is, the only one of his kind; however, here it will be more convenient to consider 0 as a limit ordinal.
say that $s(\gamma)$ is undetermined. If $t$ is another surreal, then, with some abuse of
terminology, we write $s(\gamma) = t(\gamma)$ to mean that $s(\gamma)$ and $t(\gamma)$ are both
undefined, or both defined and equal. Thus $s(\gamma) \neq t(\gamma)$ means that either $s(\gamma)$
and $t(\gamma)$ are both defined and are distinct, or exactly one between $s(\gamma)$ and $t(\gamma)$
is defined. In particular $s$ and $t$ are distinct surreals if and only if there is
some ordinal $\gamma$ such that $s(\gamma) \neq t(\gamma)$ in the above sense.\footnote{Strictly formally, it would be inappropriate to consider a surreal as a function from the
class of all ordinals to a set with three elements $\{+,$ undetermined, $-\}$. First, we would like a
surreal to be a set, rather than a proper class. The above difficulty could be overcome some
way. The point is that if $s(\gamma)$ is undefined, then $s(\delta)$ is undefined, too, for every $\delta \geq \gamma$. While
it might be intuitively simpler to think of a surreal as a function to $\{+, \text{undefined}, -\}$, it
should be clear that undetermined plays a very special role: everything should be undetermined
after the first occurrence of undefined. In spite of the above considerations, we shall be quite
informal and we shall write $s(\gamma)$, or speak of the value assumed by $s$ at $\gamma$ even when
$\gamma \geq \ell(s)$. When we want $s(\gamma)$ to be either $+$ or $-$, we shall explicitly say that $s(\gamma)$ is defined.}

We write $s < t$ to mean that $s \neq t$ and $s(\gamma) < t(\gamma)$, where $\gamma$ is the least ordinal such that $s(\gamma) \neq t(\gamma)$ and with the provision that $- < \text{undefined} < t$.
Without the above conventions and more formally, $s < t$ means that either

(i) there is some ordinal $\gamma$ such that $\gamma < \ell(s)$, $\gamma < \ell(t)$, $s(\gamma) \neq t(\gamma)$ and if
$\gamma$ is the least such ordinal, then $s(\gamma) = -$ and $t(\gamma) = +$, or

(ii) $\ell(s) < \ell(t)$ and $t(\ell(s)) = +$, that is, $t$ is longer than $s$ and $t$ assumes the
value $+$ at the first place at which $s$ is undefined, or

(iii) $\ell(s) > \ell(t)$ and $s(\ell(t)) = -$, that is, $s$ is longer than $t$ and $s$ assumes
the value $-$ at the first place at which $t$ is undefined.

The class of all surreals is linearly ordered by $<$.\footnote{Strictly formally, it would be inappropriate to consider a surreal as a function from the
class of all ordinals to a set with three elements $\{+,$ undetermined, $-\}$. First, we would like a
surreal to be a set, rather than a proper class. The above difficulty could be overcome some
way. The point is that if $s(\gamma)$ is undefined, then $s(\delta)$ is undefined, too, for every $\delta \geq \gamma$. While
it might be intuitively simpler to think of a surreal as a function to $\{+, \text{undefined}, -\}$, it
should be clear that undetermined plays a very special role: everything should be undetermined
after the first occurrence of undefined. In spite of the above considerations, we shall be quite
informal and we shall write $s(\gamma)$, or speak of the value assumed by $s$ at $\gamma$ even when
$\gamma \geq \ell(s)$. When we want $s(\gamma)$ to be either $+$ or $-$, we shall explicitly say that $s(\gamma)$ is defined.}

2.3. A useful lemma about initial segments. If $\gamma \leq \ell(s)$, we define the
$\gamma$-initial segment of $s$, in symbols, $s_\gamma$, as the surreal $t$ such that $\ell(t) = \gamma$ and
t($\delta$) = $s(\delta)$, for every $\delta < \gamma$. Sometimes we shall allow the possibility $\gamma > \ell(s)$;
in that case we put $s_\gamma = s$. If $t = s_\gamma$, for some $\gamma$, we say that $t$ is an initial
segment of $s$ and that $s$ is a prolongment of $t$. If $\gamma < \ell(s)$, we say that the
initial segment is proper. Intuitively, $t$ is an initial segment of $s$ if $t$ can be
"prolonged" to $s$ by setting further values as defined. The $\leq \gamma$-initial segment
$s_{\gamma \leq \gamma}$ of $s$ is defined to be $s_{\gamma (\gamma + 1)}$. In other words, the $\leq \gamma$-initial segment of $s$
is the surreal $t$ such that $\ell(t) \leq \gamma + 1$ and $t(\delta) = s(\delta)$, for every $\delta \leq \gamma$.

Lemma 2.1. Let $s$, $t$ and $u$ be surreals.

(1) Suppose that $s \neq t$ and $\gamma$ is the first ordinal such that $s(\gamma) \neq t(\gamma)$.

Then $s_{\gamma \gamma} = t_{\gamma \gamma}$. Moreover, both $s(\gamma')$ and $t(\gamma')$ are defined, for every
$\gamma' < \gamma$.

(2) If $s \leq t$, then $s_{\gamma \gamma} \leq t_{\gamma \gamma}$, for every ordinal $\gamma$.

(3) If $\gamma$ is an ordinal and $s_{\gamma \gamma} < t_{\gamma \gamma}$, then $s < t$. Actually, $s_{\delta \delta} < t_{\delta \delta}$, for
every $\delta \geq \gamma$.

(4) If $\alpha$ is an ordinal, $\ell(u) \leq \alpha$, $\ell(s) \leq \alpha$ and $u_{\gamma \leq \gamma} \leq s_{\gamma \leq \gamma}$, for every
$\gamma < \alpha$, then $u \leq s$.\footnote{Strictly formally, it would be inappropriate to consider a surreal as a function from the
class of all ordinals to a set with three elements $\{+,$ undetermined, $-\}$. First, we would like a
surreal to be a set, rather than a proper class. The above difficulty could be overcome some
way. The point is that if $s(\gamma)$ is undefined, then $s(\delta)$ is undefined, too, for every $\delta \geq \gamma$. While
it might be intuitively simpler to think of a surreal as a function to $\{+, \text{undefined}, -\}$, it
should be clear that undetermined plays a very special role: everything should be undetermined
after the first occurrence of undefined. In spite of the above considerations, we shall be quite
informal and we shall write $s(\gamma)$, or speak of the value assumed by $s$ at $\gamma$ even when
$\gamma \geq \ell(s)$. When we want $s(\gamma)$ to be either $+$ or $-$, we shall explicitly say that $s(\gamma)$ is defined.}
Proof. (1) It is trivial from the definitions that $s_{|\gamma} = t_{|\gamma}$. Thus, if by contradiction $s(\gamma')$ is undefined, for some $\gamma' < \gamma$, then also $t(\gamma') = s(\gamma')$ is undefined, hence, by a comment in footnote 2, both $s$ and $t$ are undefined at values larger than $\gamma'$, hence they cannot be different at $\gamma$, a contradiction.

(2) and (3) are trivial from the definitions.

Notice that if $s < t$, then it is not true that $s_{|\gamma} < t_{|\gamma}$, for every $\gamma$. Actually, if $\gamma = 0$, then $s_{|0}$ is the empty sequence, for every surreal $s$, thus $s_{|0} = t_{|0}$, for every pair $s$, $t$ of surreals.

(4) This is true if $u = s$. Otherwise, there is $\gamma$ such that $u(\gamma) \neq s(\gamma)$. We have $\gamma < \alpha$, for any such $\gamma$, since, $u(\gamma)$ and $s(\gamma)$ being different, at least one of them is defined, and $u$ and $s$ have both length $\leq \alpha$. Choose $\gamma$ minimal such that $u(\gamma) \neq s(\gamma)$. Then $u_{|\gamma} = s_{|\gamma}$, by (1); moreover, $u(\gamma) < s(\gamma)$, by the definition of the order on the surreals, since $u(\gamma) \neq s(\gamma)$ and $u_{|\gamma} \leq s_{|\gamma}$, by assumption. Then, again by the definition of the order on the surreals, $u_{|\gamma} \leq s_{|\gamma}$, that is, $u_{|\gamma+1} < s_{|\gamma+1}$. Since $\gamma < \alpha$, then $\gamma + 1 \leq \alpha$, hence we get $u = u_{|\alpha} < s_{|\alpha} = s$ from (3).

3. Separating sets of surreals

If $S$ and $T$ are sets of surreals, then $S < T$ means that $s < t$, for every $s \in S$ and $t \in T$. If $S = \{s\}$ is a singleton, we shall simply write $s < T$ and a similar convention applies if $T = \{t\}$. Notice that the shorthand is consistent with the notation $s < t$. If $S < u < T$, we shall say that $u$ separates $S$ and $T$ or simply that $u$ is a separating element, when $S$ and $T$ are clear from the context.

We now want to show that if $S < T$ are sets, then there actually exists some $u$ separating $S$ and $T$. This is a fundamental theorem, when surreals are defined as above as sign sequences, since many constructions (e. g., the definitions of the sum of surreals and a big deal of other functions and operations) rely on the existence of such an $u$. When surreals are defined “a la Conway”, the existence of such an $u$ is, in a sense, part of the definition itself of the surreals. However, Theorems like 5.5 below are useful also in such a framework, since they provide a description of some appropriate $u$ in terms of the sign expansions of the elements of $S$ and $T$.

The existence of some $u$ separating $S$ and $T$ is well-known, e. g., [G, Theorem 2.1]. However, we believe that our proof is particularly clean. In the theory of surreals defined as sign sequences most proofs proceed by considering several cases— frequently, a very large number of them. It seems that we have reduced the number of cases in the construction of a separating element to the minimum. In fact, the definition of the shortest separating element $\hat{s}_{|\gamma}$ in Theorem 5.3 is not given by cases. In Theorem 5.1 a simpler proof of the existence of some separating element (possibly, not the shortest) is provided; the definition of $\hat{s}$ given in that proof is not by cases, either.
It should be pointed out, however, that the constructions in Theorems 5.1 and 5.3 rely on Definition 4.1, which furnishes the shortest element \( s \) satisfying Theorem 1.1. Definition 4.1 does indeed consider two cases. This looks quite natural, anyway, since the cases to be considered are when \( S \) has a maximum and when \( S \) has no maximum. The number of cases (i.e., two!) is surely small, in comparison with the usual habit in the theory of surreals. There is also the possibility of giving Definition 4.1 in a uniform way not involving a division into cases. See Remark (e) in Section 6. All the remaining parts of the proofs in the present note essentially rely only on the trivial monotonicity properties of restrictions, as stated in Lemma 2.1.

Finally, let us notice two curious facts. Though, of course, we want the strict inequalities \( s < u < t \), for every \( t \in T \) and \( s \in S \), the proofs eventually involve non-strict inequalities expressed in terms of \( \leq \). This turns out to be not particularly surprising, however, after one looks at Theorem 4.3 below. Moreover, our proofs in Section 5 simplify if we consider certain surreals which turn out to be longer than necessary, sometimes strictly longer than any other surreal involved. See Theorems 5.1 and 5.3. Without using this technique, we would be forced to resort to the usual divisions by cases. See Definition 5.4 and Theorem 5.5. The above observation suggests that the technique of using prolonged surreals might have further applications.

4. A canonical bounding element

We are now going to prove Theorem 1.1. Actually, we shall give an explicit description of an element \( s \) whose existence follows from 1.1 and then derive some useful properties of such an \( s \).

If \( u \) is a surreal, let \( u^+ \) be obtained by adding a + at the top of the string \( u \). Formally, if \( \ell(u) = \alpha \), then \( u^+ \) is the surreal \( s \) defined by: \( \ell(s) = \alpha + 1 \), \( s(\alpha) = + \) and \( s(\gamma) = u(\gamma) \), for \( \gamma < \alpha \). The surreal \( u^- \) is defined similarly, by adding a - at the top.

Definition 4.1. For every set \( S \) of surreals, define \( \sup^* S \) in the following way. If \( S \) has some maximum \( u \), let \( \sup^* S = u^+ \).

If \( S \) has no maximum, let \( \sup^* S \) be the surreal \( s \) defined as follows. If \( \gamma \) is an ordinal, let \( s(\gamma) \) be defined if and only if there is \( u \in S \) such that \( u(\gamma) \) is defined and, for every \( u' \geq u \) such that \( u' \in S \), it happens that \( u'|_{\leq \gamma} = u|_{\leq \gamma} \). If this is the case, let \( s(\gamma) = u(\gamma) \). Of course, by the assumption, the definition does not depend on the choice of \( u \). Notice also that if \( s(\gamma) \) is defined, then \( s(\gamma') \) is defined, for every \( \gamma' \leq \gamma \), hence the definition actually provides a surreal.

The surreal \( \inf^* S \) is defined in the symmetrical way, namely, if \( S \) has some minimum \( u \), let \( \inf^* S = u^- \). If \( S \) has no minimum, let \( \inf^* S \) be the surreal \( t \) such that \( t(\gamma) \) is defined if and only if there is \( v \in T \) such that \( v(\gamma) \) is defined.
and, for every \( v' \leq v, v' \in T \), it happens that \( v'_{\leq \gamma} = v_{\leq \gamma} \), and, if this is the case, then let \( t(\gamma) = v(\gamma) \).

An alternative definition of \( \sup^* S \) and \( \inf^* T \) can be given by using the surreal limit introduced in Mezö [M] for sequences of length \( \omega \) and in [L] for sequences indexed by an arbitrary linearly ordered set. If \( S \) has no maximum and the elements of \( S \) are ordered as a strictly increasing sequence \((s_i)_{i \in I}\), then \( \sup^* S = \lim_{i \in I} s_i \) in the notation of [L].

Notice that Definition 4.1 makes sense also in case \( S \) is the empty set. In this case, \( \sup^* S \) is the empty sequence, the surreal of length 0.

**Proposition 4.2.** Suppose that \( S \) is a set of surreals, \( s = \sup^* S \) and \( \ell(s) = \alpha \). Then \( u|_\alpha < s \), for every \( u \in S \). In particular, \( S < s \).

**Proof.** The last statement is immediate from the previous one, by Lemma 2.1(3), since \( s = s|_\alpha \), by assumption.

If \( S \) has a maximum \( w \), then \( w|_\alpha = w < s \), by construction. If \( u \in S \), then \( u \leq w \), hence, by Lemma 2.1(2), \( u|_\alpha \leq w|_\alpha < s \).

Suppose that \( S \) has no maximum. First we show the following statement.

(*) \( u|_\alpha \leq s \), for every \( u \in S \).

Indeed, for every \( \gamma < \alpha \), we have \( u|_{\leq \gamma} = s|_{\leq \gamma} \), for all sufficiently large \( w \in S \), by the definition of \( s = \sup^* S \). In particular, we can choose \( w \geq u \), hence \( u|_{\leq \gamma} \leq w|_{\leq \gamma} = s|_{\leq \gamma} \). Since this holds for every \( \gamma < \alpha \), we have \( u|_\alpha \leq s \), by Lemma 2.1(4) with \( u|_\alpha \) in place of \( u \) and since \( (u|_\alpha)|_{\leq \gamma} = u|_{\leq \gamma} \), if \( \gamma < \alpha \).

Having proved (*), suppose by contradiction that \( u|_\alpha = s \), for some \( u \in S \). If \( w \in S \) and \( w \geq u \), then \( s \geq w|_\alpha \geq u|_\alpha = s \), by (*) with \( w \) in place of \( u \) and by Lemma 2.1(2), hence \( s = w|_\alpha = u|_\alpha \). Now \( w(\alpha) \) can assume only a finite number of values (+, − and possibly undefined). Since \( w|_\alpha \) is constant, for \( w \geq u \), then, for \( w \) sufficiently large, the value of \( w(\alpha) \) stabilizes, as \( w \) increases. But then, by the definition of \( s = \sup^* S \), \( w(\alpha) \) should stabilize to the value of \( s(\alpha) \), which is undefined, by the assumption \( \ell(s) = \alpha \). Then \( s = w \), for some sufficiently large \( w \in S \), hence such a \( w \) would be the maximum of \( S \), contrary to the assumption that \( S \) has no maximum. We have reached a contradiction, hence \( u|_\alpha < s \).

**Theorem 4.3.** Suppose that \( S \) is a set of surreals, \( s = \sup^* S \) and \( \ell(s) = \alpha \). Then, for every surreal \( z \),

\[ S < z \text{ if and only if } s \leq z|_\alpha. \]

**Proof.** Suppose that \( S < z \) and \( S \) has a maximum. Say, \( u \) is the maximum of \( S \). Then \( u < z \). Let \( \gamma \) be smallest ordinal such that \( u(\gamma) < z(\gamma) \), hence \( \gamma < \alpha \), by the second statement in Lemma 2.1(1), since \( \ell(u) < \alpha \). Indeed, if \( \ell(u) = \beta \), then Definition 4.1 gives \( \alpha = \ell(s) = \beta + 1 \). If \( \gamma < \beta \), then, by the definition of \( s = \sup^* S \), we have \( s(\gamma) = u(\gamma) < z(\gamma) \), hence \( s = s|_\alpha < z|_\alpha \), by Lemma 2.1(3). If \( \gamma = \beta \), then \( u|_\gamma = z|_\gamma \), by Lemma 2.1(1); moreover, \( u(\gamma) \) is undefined, hence \( z(\gamma) = + \), since \( u(\gamma) < z(\gamma) \). Thus \( z|_\alpha = s \). In any case, \( s \leq z|_\alpha \).
Suppose that $S < z$ and $S$ has no maximum. By the definition of $s = \sup^* S$, we have that, for every $\gamma < \alpha$, there is $u \in S$ such that $u_{\mid \leq \gamma} = s_{\mid \leq \gamma}$. Since $u < z$, we get $s_{\mid \leq \gamma} = u_{\mid \leq \gamma} \leq z_{\mid \leq \gamma}$, by Lemma 2.1(2). Since this holds for every $\gamma < \alpha$, we get $s = s_{\mid \leq \alpha} \leq z_{\mid \leq \alpha}$, by Lemma 2.1(4) with $s$ in place of $u$ and $z_{\mid \leq \alpha}$ in place of $s$.

Conversely, suppose that $s \leq z_{\mid \leq \alpha}$. By Proposition 4.2, $u_{\mid \leq \alpha} < s = s_{\mid \leq \alpha} \leq z_{\mid \leq \alpha}$, for every $u \in S$. But then $u < z$, by Lemma 2.1(3). □

Definition 1.1 and Theorem 1.3 not only furnish a useful “bounding element” relative to some set $S$ of surreals. They also provide significant information about the possible lengths of separating elements of some pair of sets of surreals.

**Corollary 4.4.** Suppose that $S$, $T$ are sets of surreals and $S < T$. Let $s = \sup^* S$, $t = \inf^* T$, $\alpha = \ell(s)$ and $\beta = \ell(t)$. If $\varepsilon \geq \max\{\alpha, \beta\}$ and $w$ is any surreal number, then $w$ separates $S$ and $T$ if and only if $w_{\mid \leq \varepsilon}$ separates $S$ and $T$.

**Proof.** Immediate from Theorem 1.3 and its symmetrical version. Indeed, from, say, $\varepsilon \geq \alpha$, it follows $(w_{\mid \leq \varepsilon})_{\mid \leq \alpha} = w_{\mid \leq \alpha}$. □

5. A separating element

We now have at our disposal all the tools necessary in order to prove Theorem 1.2 and its improvements. However, we shall first discuss an example. One might expect that if $S < T$, $s = \sup^* S$ and $t = \inf^* T$, then $s < t$, or at least $s \leq t$. Were this the case, then $u < s$, for every $u \in S$, by the last statement in Proposition 4.2. Symmetrically, $t < v$, for every $v \in T$, hence Theorem 1.2 would follow, since then both $s$ and $t$ (in fact, any intermediate surreal) would separate $S$ and $T$. However, it is not necessarily the case that $S < T$ implies $\sup^* S < \inf^* T$. Actually, it might happen that $\sup^* S > \inf^* T$.

Indeed, for $i < \omega$, let $s_i = ++ . . . +$ be a sequence consisting of $i$ pluses, and let $S = \{s_i \mid i \in \omega\}$, thus $s = \sup^* S = ++ . . . +$ is a sequence of $\omega$ pluses. For $i < \omega$, let $t_i = ++ . . . +$ be a sequence of $\omega$ pluses followed by $i$ minuses. If $T = \{t_i \mid i \in \omega\}$, then $t = \inf^* T = ++ . . . +$. . . . . is the sequence consisting of $\omega$ pluses followed by $\omega$ minuses. The above example shows that it might happen that $S < T$ and, nevertheless, $\sup^* S > \inf^* T$. However, by using Theorem 1.3 we can easily show that there exists some element separating $S$ and $T$, actually, either $s$ or $t$ works (see Corollary 5.2 below). But it is not necessarily the case that both $s$ and $t$ work. In the above example, $t$ separates $S$ and $T$, but $s$ does not separate $S$ and $T$. To the opposite! We have $t_i < s$, for every nonzero $i < \omega$, rather than $s < t_i$.

However, we can show that some prolongment of $s$ (and, symmetrically, some prolongment of $t$) does separate $S$ and $T$.

**Theorem 5.1.** If $S$ and $T$ are sets of surreals such that $S < T$, then some prolongment of $\sup^* S$ separates $S$ and $T$. 
Proof. Let \( s = \sup^* S \) and \( \alpha = \ell(s) \). By assumption, \( S < v \), for every \( v \in T \), hence \( s \leq v_{\alpha} \), for every \( v \in T \), by Theorem 4.3. Choose some ordinal \( \eta \) such that \( \eta > \ell(v) \), for every \( v \in T \) and \( \eta \geq \alpha = \ell(s) \). Let \( \hat{s} \) be the prolongment of \( s \) of length \( \eta \) obtained by adding only minuses.

Let \( v \in T \). We have \( \hat{s} \neq v \), since by construction they have different length. If \( \gamma \) is the first ordinal such that \( \hat{s}(\gamma) \neq v(\gamma) \), then \( \hat{s}(\gamma) < v(\gamma) \); this is trivial by construction if \( \gamma \geq \alpha \) and follows from \( s \leq v_{\alpha} \) if \( \gamma < \alpha \). Hence \( \hat{s} < v \).

Since, by construction, \( \hat{s}_{|\alpha} = s \), we have \( u < \hat{s} \), for every \( u \in S \), by Theorem 4.3. Hence \( \hat{s} \) separates \( S \) and \( T \).

Using Corollary 4.4, we can dramatically cut down the length either of the element constructed in the proof of 5.1, or of the element constructed in the symmetrical way.

**Corollary 5.2.** Suppose that \( S \) and \( T \) are sets of surreals and \( S < T \). If \( s = \sup^* S \) and \( t = \inf^* T \), then either \( s \) or \( t \) separates \( S \) and \( T \).

In fact, if \( \ell(s) = \ell(t) \) then both \( s \) and \( t \) separate \( S \) and \( T \). If \( \ell(s) \neq \ell(t) \), then the longer one between \( s \) and \( t \) is a separating element.

Proof. Suppose that \( \alpha = \ell(s) \geq \ell(t) = \beta \). The prolongment \( \hat{s} \) of \( s \) constructed in the proof of Theorem 5.1 separates \( S \) and \( T \). By Corollary 4.3, \( \hat{s}_{|\alpha} \) separates \( S \) and \( T \), since, by assumption, \( \alpha \geq \beta \). By construction, \( \hat{s}_{|\alpha} = s \).

Symmetrically, if \( \ell(t) \geq \ell(s) \), then \( t \) separates \( S \) and \( T \).

It is evident from Corollary 5.2 that the separating element given by the proof of Theorem 5.1 might be longer, in general, than the shortest possible separating element. As another example, if all the elements of \( S \) are negative surreals and all the elements of \( T \) are positive surreals, then 0 is a separating element (hence it is the shortest one!), but the proof of 5.1 furnishes a surreal longer than every member of \( T \)—in principle, a surreal as long as any prescribed ordinal.

Hence it is quite surprising to discover that if we combine the proof of Theorem 5.1 with the symmetric argument, we are quickly led to the shortest separating surreal. Actually, as we mentioned, the construction proceeds plainly without divisions into cases. To the contrary, if we work out an explicit description of the shortest separating element, then four cases emerge. See Definition 5.4 and Theorem 5.3.

**Theorem 5.3.** Suppose that \( S \) and \( T \) are sets of surreals and \( S < T \). Let \( s = \sup^* S \), \( t = \inf^* T \), \( \alpha = \ell(s) \), \( \beta = \ell(t) \) and \( \varepsilon = \max\{\alpha, \beta\} \). Let \( \hat{s} \) be the prolongment of \( s \) to length \( \varepsilon + 1 \) obtained by adding only minuses and, symmetrically, let \( \hat{t} \) be the prolongment of \( t \) to length \( \varepsilon + 1 \) obtained by adding only pluses.

Then both \( \hat{s} \) and \( \hat{t} \) separate \( S \) and \( T \). Moreover, \( \hat{s} < \hat{t} \).

If \( \gamma \) is the shortest ordinal such that \( \hat{s}(\gamma) \neq \hat{t}(\gamma) \), then \( \hat{s}_{|\gamma} = \hat{t}_{|\gamma} \) is the shortest surreal separating \( S \) and \( T \). Any other separating surreal is a prolongment of \( \hat{s}_{|\gamma} \).
Proof. Consider the surreal \( \hat{s} \) constructed in the proof of Theorem 5.1 taking \( \eta > \varepsilon \). It might happen that \( \hat{s} \) is longer than \( \hat{t} \); however, since \( \varepsilon + 1 \geq \alpha \), we get from Corollary 4.3 that \( \hat{s} \) still separates \( S \) and \( T \). This is the same argument as the one in the proof of Corollary 4.2 (so far, we could have done with prolongments of length \( \varepsilon \), rather than \( \varepsilon + 1 \). Adding 1 simplifies the subsequent argument). Symmetrically, \( \hat{t} \) separates \( S \) and \( T \).

By Theorem 4.3 we have \( \hat{s}_1 = s < \hat{t}_1 = t \) and, symmetrically, \( \hat{s}_n < \hat{t}_n \). Hence, if \( \delta = \min\{\alpha, \beta\} \), then \( \hat{s}_1 \leq \hat{t}_1 \), by Lemma 2.1(2). Moreover, \( \hat{s} \neq \hat{t} \), since the last element of \( s \) is \( - \) and the last element of \( t \) is \( + \) (here we are using the assumption that the prolongments \( \hat{s} \) and \( \hat{t} \) have length \( > \max\{\alpha, \beta\} \)). Let \( \gamma \) be the smallest ordinal such that \( \hat{s}(\gamma) \neq \hat{t}(\gamma) \). If \( \gamma < \delta \), then \( \hat{s}(\gamma) < \hat{t}(\gamma) \), since \( \hat{s}_1 \leq \hat{t}_1 \). If \( \gamma \geq \delta \), then, by construction, either \( \hat{s}(\gamma) = - \) or \( \hat{t}(\gamma) = + \), hence \( \hat{s}(\gamma) < \hat{t}(\gamma) \) in this case, as well. In both cases, we get \( \hat{s} < \hat{t} \).

By Lemma 2.1(1) and the above paragraph we have \( \hat{s} \leq \hat{s}_\gamma = \hat{t}_\gamma \), since both \( \hat{s} \) and \( \hat{t} \) separate \( S \) and \( T \), then \( \hat{s}_\gamma \) separates \( S \) and \( T \). Suppose that \( u \) is any other separating element. By Theorem 4.3 \( s \leq u_\alpha \). Since the prolongment \( \hat{s} \) is obtained by adding minuses to \( s \), we also have \( \hat{s} \leq u_\varepsilon \). Symmetrically, \( u_\varepsilon \leq \hat{t} \). By Lemma 2.1(2) and since \( \gamma \leq \varepsilon \), we get \( \hat{s}_\gamma < s_\varepsilon < \hat{t}_\varepsilon \), hence \( u_\varepsilon = \hat{s}_\gamma \), that is, \( u \) is a prolongment of \( \hat{s}_\gamma \).

It is now not difficult to evaluate explicitly the shortest surreal separating \( S \) and \( T \).

Definition 5.4. If \( (s, t) \) is an ordered pair of surreals, the (ordered) separator \( \text{sep}(s, t) \) of \( s \) and \( t \) is defined in the following way.

(a) If \( s = t \), we let \( \text{sep}(s, t) = s = t \).

(b) If there is some \( \gamma \) such that \( s(\gamma) \neq t(\gamma) \) are both defined, choose \( \gamma \) minimal and let \( \text{sep}(s, t) = s_\gamma = t_\gamma \). Notice that \( s_\gamma = t_\gamma \) by Lemma 2.1(1).

(c) If \( t \) is a (proper) prolongment of \( s \), say, \( t(s) = \alpha \), consider the first ordinal \( \varepsilon \geq \alpha \) such that \( t(\varepsilon) = + \) and let \( \text{sep}(s, t) = t_\varepsilon \). If no such \( \varepsilon \) exists, that is, \( t(\varepsilon) = - \), for every \( \varepsilon \geq \alpha \), then let \( \text{sep}(s, t) = t \).

(d) Symmetrically, if \( s \) is a prolongment of \( t \) with \( t(\alpha) = \beta \), let \( \text{sep}(s, t) = s_\varepsilon \), for the least \( \varepsilon \geq \beta \) such that \( s(\varepsilon) = - \), if such a \( \varepsilon \) exists, and let \( \text{sep}(s, t) = s \) otherwise.

Notice that, because of Lemma 2.1(1), the above definition covers all possible cases, that is, \( \text{sep}(s, t) \) is defined for every pair \( (s, t) \) of surreals.

Notice that, due to the last two clauses, it might happen that \( \text{sep}(s, t) \neq \text{sep}(t, s) \). For example, if \( s = + \) and \( t = + + + \), then \( \text{sep}(s, t) = + + + \), while \( \text{sep}(t, s) = + + . \)

\footnote{Actually, we have \( \hat{s} < \hat{t} \), since \( \gamma \leq \varepsilon \), hence both \( \hat{s}(\gamma) \) and \( \hat{t}(\gamma) \) are defined, hence \( \hat{s}(\gamma) = - \) and \( \hat{t}(\gamma) = + \), since they are distinct. However we do not need the strict inequality in the proof.}
**Theorem 5.5.** If $S < T$, then $w = \text{sep}(\sup^* S, \inf^* T)$ is the shortest surreal separating $S$ and $T$. Moreover, any $u$ separating $S$ and $T$ is a prolongment of $w$.

**Proof.** Just follow the proof of 5.3.

If $s = t$, then $\hat{s}$ and $\hat{t}$, as constructed in the proof of 5.3, differ only by the last element, that is, $\gamma = \varepsilon$ and the shortest separating element turns out to be $s = t$.

If $\gamma < \delta = \inf\{\alpha, \beta\}$, then case (b) in Definition 5.4 applies.

If $\alpha \leq \beta$ and $t$ is a prolongment of $s$, then the proof of Theorem 5.3 asks for the smallest $\gamma$ such that $\hat{s}(\gamma)$ and $\hat{t}(\gamma)$ differ. This is the first ordinal $\gamma \geq \alpha$ such that $\hat{t}(\gamma) = +$, since $\hat{s}$ assumes always the value $-$ above $\alpha$, $\hat{s}$ and $\hat{t}$ have equal length and they are identical up to $\alpha$. The present case incorporates the case $s = t$, which we have treated separately for clarity.

The case when $\alpha \geq \beta$ and $s$ is a prolongment of $t$ is symmetrical. □

6. Remarks

(a) The surreal $\sup^* S$ is not the only surreal for which Theorem 4.3 is satisfied. In fact, every prolongment $s'$ of $\sup^* S$ obtained by adding minuses will obviously satisfy Theorem 4.3 in case $\alpha$ is taken to be the length of $s'$. However, $\sup^* S$ is the shortest surreal for which Theorem 4.3 holds.

(b) Not every surreal $w$ can be obtained as $\sup^* S$, for some set $S$ of surreals. In fact, if $w$ has some “tail” consisting only of minuses, then $w$ cannot be obtained as $\sup^* S$, for some $S$. Actually, this is an if and only if condition. If $w$ has $+$ as a last element, then $w = w^\sim +$ for some surreal $u$, hence $w$ has the form $\sup^* S$, for $S = \{u\}$. If $w$ has limit length and is not eventually $-$, then $w = \sup^* S$, where $S$ is the set of all initial segments of $w$ which are “cut just below some $+$”.

(c) Remarks (a) and (b) above show that, anyway, for every surreal $s$ there is some set $S$ such that the conclusion of Theorem 4.3 holds.

(d) Under the assumptions in Theorem 4.3 we have that, for every surreal $z$, the following conditions are equivalent.

1. $S \leq z$,
2. either $s \leq z_{i\alpha}$, or $z$ is a maximum of $S$.

Indeed, $S \leq z$ if and only if either $S < z$, or $z$ is the maximum of $S$.

(e) As we mentioned shortly after Definition 4.1 the definition of $\sup^* S$ can be obtained as a special case of the s-limit $\lim M L$, when $S$ has no maximum.

It is possible to give a uniform definition of $\sup^*$ which takes into account also the case in which $S$ has a maximum. If the elements of $S$ are ordered in increasing order as $(s_i)_{i \in I}$, then it is easy to see that $\sup^* S = \lim(s_i^\sim +)$. In other words, if $S$ is a set of surreals, let $S^\sim + = \{s^\sim + \mid s \in S\}$. Then $\sup^* S$ is obtained by taking the s-limit of the elements of $S^\sim +$, ordered in increasing order. A symmetrical remark applies to $\inf^* T$. 
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