On $\psi_{gs}$-closed Sets in Bitopological Spaces

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Abstract. In this paper, the properties of $\psi_{gs}$-closed sets in bitopological spaces are investigated. The relationships between $\psi_{gs}$-closed set and other closed sets in bitopological spaces are established and some properties of $\psi_{gs}$-closure and $\psi_{gs}$-interior are provided.

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1. Introduction

Over the years, many researchers have introduced different types of sets in topological spaces. One of these sets is the semi-open set, which was introduced and studied by Levine [12]. Thereafter, the notion of generalized closed sets (briefly, g-closed set) in topological spaces was introduced and investigated in [13]. In 2000, the concepts between closed sets and g-closed sets in topological spaces were studied in [10] and a few years later, the same author [11] studied $\psi$-closed sets in topological spaces. Ramya and Parvathi introduced a new concept of $\psi_{g}$-generalized closed (briefly, $\psi_{g}$-closed) sets in topological spaces. Recently, a new class of sets namely $\psi$ generalized semi-closed (briefly, $\psi_{gs}$-closed) sets were introduced in topological spaces and some of their basic properties were investigated.

Nowadays, a new concept coined from topological spaces is the so-called bitopological spaces (briefly, BTS). Bose [2], studied semi continuity and semi open mappings in BTS. Thereafter, in [7] and [8], the concepts on generalized closed and semi open sets in bitopological spaces were investigated.

The concepts of bitopological spaces have been widely investigated up to other types of spaces, like soft bitopological spaces. The researcher of this present study was inspired by the work of Şenel and Cagman where in [5] they studied soft closed sets on soft bitopological spaces. Thereafter in [6] they investigated soft topological subspaces. In addition, in [3] a new approach to Hausdorff space theory via the soft sets was investigated by Şenel and further studied soft topology generated by L-soft sets in [4]. With all these concepts in

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mind, we are motivated to define and introduce $\psi_{gs}$-closed sets in bitopological spaces and will intend to further study in other spaces such as soft bitopological spaces.

Moreover, we are interested to find the properties of $\psi_{gs}$-closed sets in BTS and their relationship to other existing sets and will intend to investigate the properties of $\psi_{gs}$-interior and $\psi_{gs}$-closure of a set. In general, this study establishes some properties of $\psi_{gs}$-closed set in bitopological spaces. Specifically, this study investigates some properties of $\psi_{gs}$-closed set in BTS; establishes the relationships between $\psi_{gs}$-closed set and other closed sets in BTS; and provides some properties of $\psi_{gs}$-closure and $\psi_{gs}$-interior in BTS.

The major contributions of this study are the original results on $\psi_{gs}$-closed set in BTS. The findings reveal that every $(i, j)$-$\psi$-closed set, $\tau_i$ closed set, regular-closed set, semi-closed set, $\alpha$-$\psi$-closed set, $\alpha_{gs}$-$\psi$-closed set in $(X, \tau_i)$ is $(i, j)$-$\psi_{gs}$-closed where $i, j \in \{1, 2\}$. Hence, $(i, j)$-$\psi_{gs}$-closed set is bigger than those of the mentioned sets. Also, it was found out that the intersection of $(i, j)$-$\psi_{gs}$-closed sets is $(i, j)$-$\psi_{gs}$-closed. Furthermore, the results on $\psi_{gs}$-closure and $\psi_{gs}$-interior in BTS are analogous to that in other spaces.

We are motivated to have the results or theorems since these results could also be applied in other spaces to come up with analogous results or theorems. This study could serve as a resource material for future researches and possible applications. This may encourage other mathematics enthusiasts to come up with more results and to establish possible research directions for further study.

2. Preliminaries

In this section, some basic definitions and some known results are provided. Examples are also given for a clearer understanding of several terms defined.

A collection $\tau$ of subsets of a nonempty set $X$ is a topology on $X$ if $\varnothing, X \in \tau$, $\{M_\omega : \omega \in \Omega\} \subseteq \tau$ implies $\bigcup_{\omega \in \Omega} M_\omega \in \tau$, and $A, B \in \tau$ implies $A \cap B \in \tau$. If $\tau$ is a topology on $X$, then $(X, \tau)$ is called a topological space, and the elements of $\tau$ are called $\tau$-open (or simply open) sets. A subset $F$ of $X$ is said to be $\tau$-closed (or simply closed) if its complement $X \setminus F$ is open.

The interior of $A$, denoted by $\text{int}(A)$, is the union of all open sets contained in $A$. That is, $\text{int}(A) = \bigcup\{O \in \tau : O \subseteq A\}$.

The closure of $A$, denoted by $\text{cl}(A)$, is the intersection of all closed sets containing $A$. That is, $\text{cl}(A) = \bigcap\{F \subseteq X : F \text{ is closed and } F \supseteq A\}$.

Now, if $\tau_1$ and $\tau_2$ are arbitrary topologies on $X$ then $(X, \tau_1, \tau_2)$ is called a bitopological space. The interior of $A$ and the closure of $A$ with respect to $\tau_i$ are denoted by $\text{int}_i(A)$ and $\text{cl}_i(A)$, respectively. Note that through out this context $i, j \in \{1, 2\}$ such that $i \neq j$.

**Definition 1.** Let $(X, \tau)$ be a topological space. A subset $A$ of $X$ is called

(i) semi-open set [12] if $A \subseteq \text{cl}(\text{int}(A))$;
(ii) regular-open set [19] if \( A = \text{int} (\text{cl} (A)) \);

(iii) \( \alpha \)-open set [16] if \( A \subseteq \text{int} (\text{cl} (\text{int} (A))) \);

(iv) semi-generalized closed (briefly, sg-closed) set [1] if \( \text{scl} (A) \subseteq U \) whenever \( A \subseteq U \) and \( U \) is semi-open in \((X, \tau)\);

(v) \( \alpha \)gs-closed set [18] if \( \alpha \text{cl} (A) \subseteq U \) whenever \( A \subseteq U \) and \( U \) is semi-open in \((X, \tau)\); and

(vi) \( \psi \)-closed set [11] if \( \psi \text{cl} (A) \subseteq U \) whenever \( A \subseteq U \) and \( U \) is semi-open in \((X, \tau)\); and

(vii) \( \psi \) generalised semi-closed (briefly, \( \psi \)gs-closed) set [9] if \( \psi \text{cl} (A) \subseteq U \) whenever \( A \subseteq U \) and \( U \) is semi-open in \((X, \tau)\).

The complement of semi-open (resp. regular-open, \( \alpha \)-open, gs-closed, \( \alpha \)gs-closed, \( \psi \)-closed, and \( \psi \)gs-closed) set is called semi-closed (resp. regular-closed, \( \alpha \)-closed, gs-open, \( \alpha \)gs-open, \( \psi \)-open, and \( \psi \)gs-open) set.

**Definition 2.** Let \((X, \tau_1, \tau_2)\) be a bitopological space. A subset \( A \) of \( X \) is called

(i) \((i, j)\)-semi open set [15] if \( A \subseteq \text{cl}_j (\text{int}_i (A)) \);

(ii) \((i, j)\)-semi generalized closed (briefly, \((i, j)\)-sg closed) set [17] if \((i, j)\)-scl \( A \) \( \subseteq U \) whenever \( A \subseteq U \) and \( U \) is \((i, j)\)-semi open; and

(iii) \((i, j)\)-\( \psi \)-closed set [20] if \((i, j)\)-scl \( A \) \( \subseteq U \) whenever \( A \subseteq U \) and \( U \) is \((i, j)\)-sg open.

The complement of \((i, j)\)-semi open (resp. \((i, j)\)-sg closed and \((i, j)\)-\( \psi \)-closed) set is called \((i, j)\)-semi closed (resp. \((i, j)\)-sg open and \((i, j)\)-\( \psi \)-open) set.

The next result was proven in [14].

**Lemma 1.** If a subset \( A \) of \( X \) is semi-open (respectively, semi-closed, sg-closed, gs-closed, g-closed, \( \psi \)-closed, and \( \psi \)gs-closed) set in \((X, \tau_i)\) for \( i \in \{1, 2\} \), then it is semi-open (respectively, semi-closed, sg-closed, gs-closed, g-closed, \( \psi \)-closed) set in \((X, \tau_1, \tau_2)\).

The next Theorem is a composition of several results from Gowsalya, S. and Balamani, N. in [9].

**Theorem 1.** Let \((X, \tau_1, \tau_2)\) be bitopological space. Then

(i) Every semi-closed set in \((X, \tau)\) is \( \psi \)gs-closed in \((X, \tau)\).

(ii) Every closed set in \((X, \tau)\) is \( \psi \)gs-closed set in \((X, \tau)\).

(iii) Every regular-closed set in \((X, \tau)\) is \( \psi \)gs-closed in \((X, \tau)\).

(iv) Every \( \alpha \)-closed set in \((X, \tau)\) is \( \psi \)gs-closed in \((X, \tau)\).

(v) Every \( \psi \)-closed set in \((X, \tau)\) is \( \psi \)gs-closed in \((X, \tau)\).

(vi) Every \( \alpha \)gs-closed set in \((X, \tau)\) is \( \psi \)gs-closed in \((X, \tau)\).
3. \(\psi gs\)-closed sets and its relationship to other closed sets in BTS

In this section, some properties of \(\psi gs\)-closed sets in BTS are investigated. Moreover, the relationship to some other existing closed sets in BTS is established.

**Definition 3.** A subset \(A\) of a bitopological space \((X, \tau_1, \tau_2)\) is called \((i, j)\)-\(\psi\) generalized semi-closed (briefly, \((i, j)\)-\(\psi gs\)-closed) set if \((i, j)\)-\(\psi cl(A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is \((i, j)\)-semi-open in \((X, \tau_1, \tau_2)\), \(i, j \in \{1, 2\}\) where \(i \neq j\). The complement of \((i, j)\)-\(\psi gs\)-closed set is called \((i, j)\)-\(\psi gs\)-open set.

**Example 1.** Let \((X, \tau_1, \tau_2)\) be a bitopological space such that \(X = \{a, b, c\}\), \(\tau_1 = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}\), and \(\tau_2 = \{\emptyset, X, \{a\}, \{a, b\}, \{a, c\}\}\) and \(A = \{b, c\}\). Note that \(X\) is the only \((1, 2)\)-semi-open set containing \(A = \{b, c\}\). Since \(A = \{b, c\}\) is a \(\psi\) closed set, it follows that \((i, j)\)-\(\psi cl(\{b, c\}) = \{b, c\} \subseteq X\). Thus \(A = \{b, c\}\) is a \((1, 2)\)-\(\psi\) generalized semi closed set. Similarly, \(\emptyset, \{b\}, \{c\}\) and \(X\) are \((1, 2)\)-\(\psi\) generalized semi closed sets.

Throughout this context, the open (resp., closed) set in \((X, \tau_1, \tau_2)\) is denoted by \((i, j)\)-open (resp., \((i, j)\)-closed) set.

**Proposition 1.** Every \((i, j)\)-\(\psi\)-closed set is \((i, j)\)-\(\psi gs\)-closed.

**Proof.** Let \(A\) be \((i, j)\)-\(\psi\)-closed set and \(U\) be \((i, j)\)-semi-open in \((X, \tau_1, \tau_2)\) such that \(A \subseteq U\). Then \((i, j)\)-\(\psi cl(A) = A \subseteq U\). Hence \(A\) is \((i, j)\)-\(\psi gs\)-closed in \((X, \tau_1, \tau_2)\).

**Theorem 2.** Every \(\psi gs\)-closed set in \((X, \tau_i)\) is \(\psi gs\)-closed set in \((X, \tau_1, \tau_2)\).

**Proof.** Let \(A\) be \(\psi gs\)-closed set in \((X, \tau_i)\). Then \(\psi cl(A) \subseteq U\) where \(U\) is semi-open in \(\tau_i\) such that \(A \subseteq U\). Since, for each \(i \in \{1, 2\}\), every semi-open in \((X, \tau_i)\) is semi-open in \((X, \tau_1, \tau_2)\) by Lemma 1, it follows that \(U\) is semi-open in \((X, \tau_1, \tau_2)\). Moreover, by Lemma 1, every \(\psi\)-closed in \(\tau_i\) is \(\psi\)-closed in \((X, \tau_1, \tau_2)\). Thus \((i, j)\)-\(\psi cl(A) \subseteq \psi cl(A) \subseteq U\). Hence \(A\) is a \(\psi gs\)-closed set in \((X, \tau_1, \tau_2)\).

The following corollary follows from Theorem 1 and Theorem 2.

**Corollary 1.** Let \((X, \tau_i)\) be a topological space and \((X, \tau_1, \tau_2)\) be bitopological space. Then the following statements hold.

(i) Every \(\tau_i\) closed set is \((i, j)\)-\(\psi gs\)-closed set.

(ii) Every regular-closed set in \((X, \tau_i)\) is \((i, j)\)-\(\psi gs\)-closed.

(iii) Every semi-closed set in \((X, \tau_i)\) is \((i, j)\)-\(\psi gs\)-closed.

(iv) Every \(\alpha\)-closed set in \((X, \tau_i)\) is \((i, j)\)-\(\psi gs\)-closed.

(v) Every \(\psi\)-closed set in \((X, \tau_i)\) is \((i, j)\)-\(\psi gs\)-closed.

(vi) Every \(\alpha gs\)-closed set in \((X, \tau_i)\) is \((i, j)\)-\(\psi gs\)-closed.
Theorem 3. Let $A$ be a $(i, j)$-$\psi$-closed set and $A \subseteq B \subseteq (i, j)$-$\psi\text{cl}(A)$. Then $B$ is also a $(i, j)$-$\psi\text{gs}$-closed set.

Proof. Let $A$ be a $(i, j)$-$\psi$-closed set and $A \subseteq (i, j)$-$\psi\text{cl}(A)$. Suppose $U$ is $(i, j)$-semi-open such that $B \subseteq U$. We want to show $(i, j)$-$\psi\text{cl}(B) \subseteq U$. Since $A \subseteq B$ and $B \subseteq U$, it follows that $A \subseteq U$. Also, by Proposition 1, $A$ is a $(i, j)$-$\psi\text{gs}$-closed set since $A$ is a $(i, j)$-$\psi$-closed set; and so $(i, j)$-$\psi\text{cl}(A) \subseteq U$. Note that $B \subseteq (i, j)$-$\psi\text{cl}(A)$ implies

$$(i, j)$-$\psi\text{cl}(B) \subseteq (i, j)$-$\psi\text{cl}((i, j)$-$\psi\text{cl}(A)) = (i, j)$-$\psi\text{cl}(A) \subseteq U.$$

\[ \square \]

Theorem 4. Let $\{A_k\}$ be $(i, j)$-$\psi\text{gs}$-closed set, $k \in \mathbb{N}$. Then $\bigcap_{k=1}^{\infty} A_k$ is $(i, j)$-$\psi\text{gs}$-closed set.

Proof. Suppose $U = \bigcap_{k=1}^{\infty} A_k$ is $(i, j)$-semi-open such that $\bigcap_{k=1}^{\infty} A_k \subseteq U$. We want to show that $(i, j)$-$\psi\text{cl}(\bigcap_{k=1}^{\infty} A_k) \subseteq U$. From our assumption, $A_k$ is $(i, j)$-$\psi\text{gs}$-closed set for each $k \in \mathbb{N}$. It follows that $(i, j)$-$\psi\text{cl}(A_k) \subseteq U_k$ for each $k$ such that $A_k \subseteq U_k$ where $U_k$ is $(i, j)$-semi-open. Now,

$$(i, j)$-$\psi\text{cl}(\bigcap_{k=1}^{\infty} A_k) \subseteq \bigcap_{k=1}^{\infty} (i, j)$-$\psi\text{cl}(A_k) \subseteq \bigcap_{k=1}^{\infty} U_k = U.$$ 

\[ \square \]

Note that if $A_k$ is $(i, j)$-$\psi\text{gs}$-closed set, then by Definition 3, $X \setminus A_k$ is $(i, j)$-$\psi\text{gs}$-open set. Moreover, by De Morgan’s law, $X \setminus (\bigcap_{k=1}^{\infty} A_k) = \bigcup_{k=1}^{\infty} (X \setminus A_k)$. Hence the following corollary follows.

Corollary 2. If $\{B_k\}$ be $(i, j)$-$\psi\text{gs}$-open set, $k \in \mathbb{N}$, then $\bigcup_{k=1}^{\infty} B_k$ is $(i, j)$-$\psi\text{gs}$-open set.

4. $\psi\text{gs}$-closure and $\psi\text{gs}$-interior in BTS

In this section, the $\psi\text{gs}$-closure and $\psi\text{gs}$-interior in bitopological spaces are introduced and some of their properties are explored.

Definition 4. Let $(X, \tau_1, \tau_2)$ be a bitopological space and $A \subseteq X$. An element $x \in A$ is called $(i, j)$-$\psi\text{gs}$-interior point of $A$ if there exists a $(i, j)$-$\psi\text{gs}$-open set $O$ such that $x \in O \subseteq A$. The set of all $(i, j)$-$\psi\text{gs}$-interior points of $A$ is called the $(i, j)$-$\psi\text{gs}$-interior of $A$ and is denoted by $(i, j)$-$\psi\text{gs}$-$\text{Int}(A)$.

Theorem 5. The $(i, j)$-$\psi\text{gs}$-interior of a subset $A$ of $X$ is the countable union of $(i, j)$-$\psi\text{gs}$-open sets contained in $A$, that is,

$$(i, j)$-$\psi\text{gs}$-$\text{Int}(A) = \bigcup\{O : O \text{ is (i, j)-}\psi\text{gs}\text{-open and } O \subseteq A\}.$$
Proof. Let $x \in (i,j)\text{-}\psi_{gs}\text{-}\text{Int}(A)$. Then there exists a $(i,j)\text{-}\psi_{gs}\text{-open set } O$ such that $x \in O \subseteq A$ implying $x \in \bigcup\{O : O \text{ is } (i,j)\text{-}\psi_{gs}\text{-open and } O \subseteq A\}$. Next, suppose $y \in \bigcup\{O : O \text{ is } (i,j)\text{-}\psi_{gs}\text{-open and } O \subseteq A\}$. Then there exists $(i,j)\text{-}\psi_{gs}\text{-open set } O_0 \subseteq A$ such that $y \in O_0$. Thus $y \in (i,j)\text{-}\psi_{gs}\text{-}\text{Int}(A)$.

The previous theorem implies that $(i,j)\text{-}\psi_{gs}\text{-}\text{Int}(A)$ is contained in $A$, where $A$ is any subset of $X$, being the union of all $(i,j)\text{-}\psi_{gs}\text{-open sets}$ contained in $A$. Now Corollary 2 entails that the arbitrary union of $(i,j)\text{-}\psi_{gs}\text{-open sets}$ is also $(i,j)\text{-}\psi_{gs}\text{-open}$, hence we can say that $(i,j)\text{-}\psi_{gs}\text{-}\text{Int}(A)$ is $(i,j)\text{-}\psi_{gs}\text{-open}$. Consequently, $(i,j)\text{-}\psi_{gs}\text{-}\text{Int}(A)$ is the largest $(i,j)\text{-}\psi_{gs}\text{-open set}$ contained in $A$, as stated in the following remark.

Remark 1. Let $(X, \tau_1, \tau_2)$ be a bitopological space and $A, B \subseteq X$. Then the following hold:

(i) $(i,j)\text{-}\psi_{gs}\text{-}\text{Int}(A) \subseteq A$;

(ii) $(i,j)\text{-}\psi_{gs}\text{-}\text{Int}(A)$ is $(i,j)\text{-}\psi_{gs}\text{-open set}$; and

(iii) If $B \subseteq A$ such that $B$ is $(i,j)\text{-}\psi_{gs}\text{-open set}$, then $B \subseteq (i,j)\text{-}\psi_{gs}\text{-}\text{Int}(A)$.

Theorem 6. Let $(X, \tau_1, \tau_2)$ be a bitopological space and $A \subseteq X$. $A$ is $(i,j)\text{-}\psi_{gs}\text{-open set}$, if and only if $(i,j)\text{-}\psi_{gs}\text{-}\text{Int}(A) = A$.

Proof. Let $A$ be $(i,j)\text{-}\psi_{gs}\text{-open set}$ and $x \in A$. Note that by Remark 1 (i), $(i,j)\text{-}\psi_{gs}\text{-}\text{Int}(A) \subseteq A$. Hence it suffices to show $A \subseteq (i,j)\text{-}\psi_{gs}\text{-}\text{Int}(A)$. Suppose $x \notin (i,j)\text{-}\psi_{gs}\text{-}\text{Int}(A)$. Then $x \notin O$ for all $(i,j)\text{-}\psi_{gs}\text{-open sets } O$ such that $A \subseteq O$. It follows that $x \in X-O$ such that $A \cap (X-O) = \emptyset$. This is a contradiction since $x \in A \cap X-O$. Thus $x \in (i,j)\text{-}\psi_{gs}\text{-}\text{Int}(A)$. Consequently, $(i,j)\text{-}\psi_{gs}\text{-}\text{Int}(A) = A$. Conversely, suppose $(i,j)\text{-}\psi_{gs}\text{-}\text{Int}(A) = A$. By Remark 1 (ii), $(i,j)\text{-}\psi_{gs}\text{-}\text{Int}(A)$ is $(i,j)\text{-}\psi_{gs}\text{-open set}$, and so $A$ is $(i,j)\text{-}\psi_{gs}\text{-open set}$.

Definition 5. Let $A \subseteq X$. Then $x \in X$ is $(i,j)\text{-}\psi_{gs}\text{-adherent to } A$ if $V \cap A \neq \emptyset$ for every $(i,j)\text{-}\psi_{gs}\text{-open set } V$ containing $x$. The set of all $(i,j)\text{-}\psi_{gs}\text{-adherent points of } A$ is called the $(i,j)\text{-}\psi_{gs}\text{-closure}$ of $A$ and is denoted by $(i,j)\text{-}\psi_{gs}\text{-Cl}(A)$.

Theorem 7. The $(i,j)\text{-}\psi_{gs}\text{-closure}$ of a subset $A$ of $X$ is the countable intersection of $(i,j)\text{-}\psi_{gs}\text{-closed sets}$ containing $A$, that is,

$$(i,j)\text{-}\psi_{gs}\text{-Cl}(A) = \bigcap\{F : F \text{ is } (i,j)\text{-}\psi_{gs}\text{-closed and } A \subseteq F\}.$$ 

Proof. Let $x \in (i,j)\text{-}\psi_{gs}\text{-Cl}(A)$. Then $O \cap A \neq \emptyset$ for every $(i,j)\text{-}\psi_{gs}\text{-open set } O$ containing $x$. Suppose $x \notin \bigcap\{F : F \text{ is } (i,j)\text{-}\psi_{gs}\text{-closed and } A \subseteq F\}$. It follows that there exists $(i,j)\text{-}\psi_{gs}\text{-closed } F_0$ such that $A \subseteq F_0$ and $x \in X \setminus F_0$. Note that $X \setminus F_0$ is $(i,j)\text{-}\psi_{gs}\text{-open set}$ containing $x$ such that $(X \setminus F_0) \cap A = \emptyset$, a contradiction. Thus $x \in \bigcap\{F : F \text{ is } (i,j)\text{-}\psi_{gs}\text{-closed and } A \subseteq F\}$. 


Next, let \( y \in \cap \{ F : F \text{ is } (i, j)\)-\(\psi\)gs-closed and \( A \subseteq F \} \). Then \( y \in F \) for all \((i, j)\)-\(\psi\)gs-closed such that \( A \subseteq F \). Suppose on the contrary, \( y \notin (i, j)\)-\(\psi\)gs-\(\text{Cl}(A) \). It implies that \( U \cap A = \emptyset \) for some \((i, j)\)-\(\psi\)gs-closed set \( U \) containing \( y \). Hence there exists \((i, j)\)-\(\psi\)gs-closed set \( X \setminus U \) such that \( y \notin X \setminus U \) and \( A \subseteq X \setminus U \), a contradiction. Consequently, \( y \in (i, j)\)-\(\psi\)gs-\(\text{Cl}(A) \). \( \square \)

Theorem 7 indicates that \((i, j)\)-\(\psi\)gs-\(\text{Cl}(A) \) contains \( A \) since the intersection of all \((i, j)\)-\(\psi\)gs-closed sets contains \( A \). Now Theorem 4 implies that the arbitrary intersection of \((i, j)\)-\(\psi\)gs-closed sets is also \((i, j)\)-\(\psi\)gs-closed, thus it follows that \((i, j)\)-\(\psi\)gs-\(\text{Cl}(A) \) is \((i, j)\)-\(\psi\)gs-closed. Hence, \((i, j)\)-\(\psi\)gs-\(\text{Cl}(A) \) is the smallest \((i, j)\)-\(\psi\)gs-closed set that contains \( A \), as stated in the following remark.

**Remark 2.** Let \((X, \tau_1, \tau_2) \) be a bitopological space and \( A, B \subseteq X \). Then the following hold:

(i) \( A \subseteq (i, j)\)-\(\psi\)gs-\(\text{Cl}(A) \);

(ii) \( (i, j)\)-\(\psi\)gs-\(\text{Cl}(A) \) is \((i, j)\)-\(\psi\)gs-closed set; and

(iii) If \( A \subseteq B \) such that \( B \) is \((i, j)\)-\(\psi\)gs-closed set, then \((i, j)\)-\(\psi\)gs-\(\text{Cl}(A) \subseteq B \).

**Theorem 8.** \( A \) is \((i, j)\)-\(\psi\)gs-closed set, if and only if \((i, j)\)-\(\psi\)gs-\(\text{Cl}(A) = A \).

*Proof.* Let \( A \) be \((i, j)\)-\(\psi\)gs-closed set and \( x \in (i, j)\)-\(\psi\)gs-\(\text{Cl}(A) \). Then for all \((i, j)\)-\(\psi\)gs-open set \( U \) containing \( x \), we have \( U \cap A \neq \emptyset \). Suppose on the contrary, \( x \notin A \). Then \( x \in X \setminus A \) where \( X \setminus A \) is \((i, j)\)-\(\psi\)gs-open and \((X \setminus A) \cap A = \emptyset \), a contradiction since \( x \in (i, j)\)-\(\psi\)gs-\(\text{Cl}(A) \). Thus \( x \in A \), and so \((i, j)\)-\(\psi\)gs-\(\text{Cl}(A) \subseteq A \). Note that by Remark 2 (i), \( A \subseteq (i, j)\)-\(\psi\)gs-\(\text{Cl}(A) \), and hence \((i, j)\)-\(\psi\)gs-\(\text{Cl}(A) = A \). Conversely, suppose \((i, j)\)-\(\psi\)gs-\(\text{Cl}(A) = A \). By Remark 2 (ii), \((i, j)\)-\(\psi\)gs-\(\text{Cl}(A) \) is \((i, j)\)-\(\psi\)gs-closed set, and so \( A \) is \((i, j)\)-\(\psi\)gs-closed set. \( \square \)

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