Errors and difficulties in solving algebraic procedures in secondary school students

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Abstract. The research presented was aimed at determining the errors and difficulties presented by secondary school students in a Colombian public educational institution when faced with the solution of algebraic procedures. A contribution is sought to contribute to studies that monitor the performance of the teaching and learning processes of students. The methodology was based on the quantitative paradigm with a descriptive design, for which two instruments were applied to detect errors and difficulties in students. The results allowed to detect that, of the 12 errors that were taken as dimensions, six are significantly replicated, the most significant being the simple calculation error, unfinished procedure and the errors when performing arithmetic-algebraic operations. Participants allege associated difficulties, mostly to teaching problems, which generated guidelines aimed at teachers in order to raise didactic situations related to error as a source of learning.

1. Introduction
The investigations that are being developed in the teaching of mathematics, arise, probably, from the observable fact that the student is wrong when asked to perform certain tasks. Then, researchers in this area of knowledge study when the learner provides erroneous answers, with respect to an evaluation pattern, or simply is not able to give any response; considering that errors and difficulties do not occur in a random or unpredictable way.

Various investigations in the international context, as Castro points out [1]. They support the introduction of algebraic content in the primary and high school curriculum. However, this same diversity is one of the reasons why such research has had difficulties for its implementation. In this regard, it is recognized that there is not a sufficiently explicit position on the nature of algebraic reasoning at the elementary levels of primary and secondary education, nor a conception of what could be considered as algebra at these levels. But, if in the framework of the teaching of mathematics, algebra is an important discipline, since it constitutes a thematic axis that contains essential knowledge for the educational and social life of the student.

The research seeks an approach to the phenomenon of the problems presented by secondary school students of a public educational institution in San José de Cúcuta, Colombia, when facing the resolution of algebraic procedures. A contribution is sought to contribute to studies that monitor the performance of the teaching and learning processes of students.

2. Theoretical framework
Given that the study seeks to determine the errors and difficulties presented by students of the tenth grade when faced with the solution of algebraic procedures, it is based on the premise that leads to see
errors as obvious opportunities to improve, to grow and to perfect. Therefore, the theoretical paradigm that underpins this research is anchored in constructivism, because to say Pozo, et al. [2] this approach suggests that the mistake made by students, when solving problems or any educational task, should be a source of learning.

Then, it can be thought that the so-called error in the learning of an object of knowledge is nothing other than an element for the acquisition of socially constructed knowledge. In this sense, Abrate, Pochulu, and Vargas [3] mention that throughout the history of the progress of scientific knowledge it can be found that error is a factor that has contributed to the advancement of different sciences and is an integral part of knowledge human.

Then, it is necessary to think about the correction processes, because the error is seen by the teachers of mathematics, as a punitive aspect that is punished with the qualifications. In this regard, Ribas and D’Aquino [4] report that the correction processes should not penalize the student's work nor should the error be explicitly indicated. The teacher must retake part of his speech, reformulates it correctly and integrates it into the class, which follows its course. The error should not be marked as negative, because it exposes the student to the group and away from significant learning.

For this reason, it is important to raise some characterizing aspects of the error and the correction from the point of view of the constructivist approach. In this sense, aspects characterizing the error are mentioned taking into consideration the contributions of Pozo and others [2], Ribas and D’Aquino [4], Astolfi [5] Torijano [6], Arrabal, Laporte and Rivas [7], of their works they took ideas to make the corresponding adaptations to the present investigation. Among the characterizing aspects were selected, the following: see errors as a learning opportunity; recognize the individual differences of the students; starting from the previous knowledge of the students; propose evaluations the achievement of specific goals and develop research on appropriate teaching methodologies to change the concept of error. In this regard, it is important to see the classroom as a research space.

2.1. Type of error, definition and characterization
The beliefs of the educator about the errors of the students obey to their own conceptions, in this case, about mathematics. To change the paradigms on the error of the Algebra of educators implies a certain change in the relation of said teacher with respect to the teaching of mathematics. Well, the way he directs the teaching will have a type of learning in school. In a behavioral teaching, the error must be corrected, while in a constructivist teaching error is part of the learning process.

In this order of ideas, Brousseau [8] expresses: the error is not only the effect of ignorance, uncertainty, chance, as is believed in empirical or behavioral theories of learning, but the effect of previous knowledge, which had its interest, its achievements, but now reveals false, or simply inadequate. Errors of this kind are not erratic or unpredictable but constitute obstacles. Both in the functioning of the teacher and the student, the error is integral to the sense of knowledge acquired.

The error procedure is a matter of interest in the framework of the teaching processes at present. This topic is not only considered normal but necessary for learning, although this is often not expressed in the classroom. Consequently, according to Gil and Cortés [9] the teacher must be sensitive to the previous ideas of the students and should use the techniques of cognitive conflict to achieve progress in learning.

On the other hand, Rico and Sierra [10] points out that errors can contribute positively in the learning process. It indicates that errors do not appear by chance but arise in a conceptual framework established on previously acquired knowledge. It argues the need for any theory of instruction to modify the tendency to condemn errors by blaming the students, according to teachers' beliefs, replacing them with the forecast of errors and their consideration in the learning process. In this order of ideas, it can be said that from their mistakes, if the teacher has a constructive conception, the student can learn different properties of a concept of which he was not previously aware. By making a mistake, the student expresses the incomplete nature of his knowledge and allows the classmates or the teacher to help him complete the additional knowledge or make him understand for himself what was wrong.
In the context of the classification of errors, this research takes the typology presented by García [11] as it adapts to the objectives of the same. In such a way, that a classification with twelve types of errors is assumed: Incorrect elimination of denominators; errors when performing arithmetic-algebraic operations; unfinished procedure; incorrect own problems and invalid inference; partial application of factoring rule by common factor; incorrect association of remarkable products; use of basic arithmetic ignoring the rules of algebra; error in determining the power of another power; additive resolution of the power of a binomial; incorrect application of the bin rule of a binomial; error when making polynomial products and simple calculation error.

2.2. Definition, characterization and typology of the difficulties
The term difficulty, in the pedagogical scope indicates the greater or lesser degree of success of the students before an activity or subject of study. In this sense, Ruano, Socas and Palarea [12] and refers that if the percentage of errors is high it is said that the difficulty is high, while if this percentage is low, the difficulty is low.

For the study of the typology of difficulties, the postulates proposed by Di Blasi, et al. [13] cited by Abrate, Pochulu and Vargas [3] who group them into the following topics related to difficulties associated with: the complexity of mathematical objects; mathematical thinking processes; teaching processes; the cognitive development of students; affective and emotional attitudes.

3. Methodology
The inquiry developed was supported by the quantitative paradigm. The research was based on a descriptive, field design.

3.1. Population and sample object of study
For the present investigation, the population was formed by five groups of students of the Seventh grade of Secondary Basic Education of the Educational Institution General Santander School, of public character, located in the metropolitan area of the city of San José de Cúcuta, Colombia, which make up 40 students per group, for a total of 200 sample subjects. From this population the group of 7A was intentionally selected where one of the researchers is the group's owner.

3.2. Techniques and instruments for data collection
Starting from the need to characterize the errors and difficulties in the resolution of algebraic problems, the mathematics questions of the “Instituto Colombiano para la Evaluación de la Educación (ICFES)” state test booklet for the years 2017 and 2018 were taken as a support for the instrument. They selected 10 questions that were organized in development items, of which they were adjusted for the progress of the research. In addition, we used an interview with 5 questions that sought to gather information about the difficulties that students present.

4. Results
4.1. Diagnostic results
When analyzing the errors and difficulties in the solution of algebraic procedures in the students of the seventh grade degree, Table 1 is shown where the items are evidenced with the percentage of occurrence of the error, the type of error proposed and the main difficulty mentioned the subjects participating in the investigation in each item. This allows an analysis of those typical errors proposed by Garcia [11] and that were significantly replicated in this population and their relationship with the most common difficulties based on the proposal of Di Blasi, et al. [13].

When reviewing the Table 1, it is possible to see with respect to the items where a high percentage of incorrect answers can be detected, and that are related to the proposal of Garcia [11], it can be inferred that in this study they are replicated significantly six of the twelve errors raised this author. The most frequent ones were the simple calculation error and the unfinished procedure, followed using basic
arithmetic ignoring the rules of algebra, errors when performing arithmetic-algebraic operations, incorrect own procedures and invalid inferences and to a lesser degree the incorrect elimination of denominators. In this sense, this author mentions that in the course of various investigations, error can be found as part of human knowledge, an error that allows us to learn and understand the reason for these wrong answers.

### Table 1. Relationship between errors and type of difficulty.

| Item | % occurrence of the error | Type of error by García [11] | Typology of difficulties associated with the type of error of Di Blasi, et al. [13] |
|------|---------------------------|------------------------------|----------------------------------------------------------------------------------|
| 1    | 100.0%                    | Incorrect elimination of denominators | Difficulties associated with the cognitive development of students. |
| 2    | 25.0%                     | Unfinished procedure           | Difficulties associated with teaching processes.                                |
| 3    | 67.8%                     | Unfinished procedure           | Complexity of mathematical objects.                                              |
| 4    | 20.0%                     | Errors when performing arithmetic-algebraic operations | Cognitive development of students.                                               |
| 5    | 25.0%                     | Unfinished procedure           | Difficulties associated with teaching processes.                                |
| 6    | 12.5%                     | Simple calculation error       | Difficulties associated with teaching processes.                                |
| 7    | 30.0%                     | Use of basic arithmetic ignoring the rules of algebra | Difficulties of teaching processes.                                           |
| 8    | 67.9%                     | Unfinished procedure           | Difficulty referred to affective and emotional attitudes.                        |
| 9    | 3.6%                      | Simple calculation error       | Difficulty associated with teaching processes.                                  |
| 10   | 35.0%                     | Incorrect own procedures and invalid inferences | Difficulties with the teaching processes and cognitive development of the student. |

Similarly, among the errors obtained through the observation of the response sheet and the interview with the students, we can mention some: they do not look at the denominators when they are operations with fractions, they solve them as if they were heterogeneous fractions, others that they did not know how to solve it; they are not clear that it is a similar term to be able to solve the addition and subtraction of algebraic expressions.

Regarding the mistakes made, García [11] and Socas [14] mention that these can also be achieved in student productions. This leads to infer that the teaching of algebra is a determining factor to explain the persistence of these errors with participants of different ages. These authors refer that this, too, may be due to the fact that in his previous training different ways of teaching are used and this can cause confusion in the students when they are presented with other contexts in which the proposed exercises take on other meanings as specific values, values of generalized or variable numbers.
However, the students who had greater problems to solve the exercises, mentioned in their interviews, difficulties associated with the cognitive development of the students, the teaching processes and the affective and emotional attitudes. In this regard, and given that a difficulty, is referred to difficulties associated with the cognitive development of students, then, the approaches of Suarez [15], are interesting, as it raises that the student is comfortable in circumstances where he understands the contents. Therefore, it could be said that in the educational process when the student understands what he studies, he feels comfortable when he must give answers to evaluations.

Regarding the cognitive development of the student, Pozo, et al. [2], and Shunk [16] argue that the development of cognition is a construction of the human being, each person perceives reality, organizes it and gives it meaning in the form of constructs, that is way each person perceives reality in a particular way depending on their physical capacities and the emotional state they are in, as well as their social and cultural conditions. The way to solve a mathematical exercise changes from one student to another.

In connection with the difficulties associated with the teaching processes, Goleman's ideas [17] who said that the teaching environment, because it is emotional and attitudinal, has two aspects; a general one, common to all education, that encompasses a set of constructs that influence learning, such as motivation, self-concept, self-control, self-efficacy, interests, attitudes and social skills. Another that involves attitudinal and emotional elements, related to the specialized, disciplinary contents of the different subjects.

In the same way, Sánchez, Segovia, and Miñán [18], and Aragonés and Cuervo [19], refer that mathematics allows to contribute with the configuration of attitudes and values in students. All this creates, according to Damasio [20], a conscious and favorable disposition to undertake actions that lead to the solution of the problems that they face every day, for which the teaching processes must start from the situations that the students already they know, in addition to their previous knowledge.

### 4.2. Results of the guidelines addressed to teachers

At the conclusion of the discussion of the variables associated with the subject under study, it is necessary to propose guidelines, simple but with definite theoretical support, aimed at the mathematics teachers, with the purpose of making contributions to contribute to the teaching of mathematics, since the results show a high rate of errors and difficulties, probably due to the way in which the error committed by the student is used, which is taken only to penalize it and not as a source of new learning.

It is expected that these guidelines, which are based on the theoretical assumptions, will constitute a contribution for the reflection on the pedagogical practices of the professors that will allow them to see the errors of the students as interesting elements for the teaching process.

Starting from the previous knowledge of the students. Much has been said about this theoretical construct and little has been done. Therefore, Pozo, et al. [2] and Torijano [6], refer to the importance of the knowledge of apprentices for better and greater learning. In this regard, they point out that the constructivist contributions agree in affirming that knowledge is not a copy of the pre-existing reality, but rather it is an interactive process through which external information is interpreted by the mind that progressively builds more and more complex models and powerful, for which it is necessary the previous knowledge.

See mistakes as a learning opportunity. In this sense, Arrabal, Laporte and Rivas [7] and Abrate, Pochulu and Vargas [3] refer that one should not fear error, since they constitute opportunities that refine and polish the discovery of truth. Therefore, we must know how to use the mistakes of students as enriching elements that the teacher has every day in front of his students.

Recognize the individual differences of the students. In this regard, Pozo, et al. [2], says that to develop innovations in pedagogical practices, it is important to take into consideration the student, with its strengths and weaknesses. There arises another reason more for the educating mission, since it has been easier to create doors, locks and locks than to grant keys. The constructivist teaching starts from a process that allows us to discover the hidden language of the answers and reveal the capacities, the soul and the dreams of the learners.
Propose evaluations the achievement of specific goals. On this aspect, Ribas and D’Aquino [4] and Astolfi [5], refer that the evaluations must be carried out according to the needs of the students. One of the difficulties with the evaluations and the questioning of the same by students is that they must submit to the system without further questioning or opinion about it. Therefore, it is important that evaluative practices that favor a constructive rather than punitive approach that respect diversity and that favor the development of competencies should be a matter that leads the student to achieve specific goals. It must be watched that the evaluation guarantees the appropriation of the concepts or learning, even with the errors that are committed in the evaluation process.

Develop research on appropriate teaching methodologies to change the concept of error. In this regard, it is important to see the classroom as a research space. Pozo, et al. [2] refers that constructivism, as an educational approach, is a broad movement that defends the idea that the teacher should be a researcher of their own reality. Well, as Alsina [21] say the student should be seen not as a mere product of the sociocultural environment, nor a simple result of internal dispositions of a biological nature, since it is necessary for teachers to focus on the acquisition of all knowledge new occurs through the mobilization, on the part of the subject of an ancient knowledge. The fact of considering that prior knowledge facilitates learning is an essential feature of constructivism and that sustains meaningful learning, for which the research from the didactic allows reviewing these premises.

5. Conclusions
From the results of the investigation, one can reflect on the role of the teacher in the teaching of mathematics. Which, not necessarily, must be a transmitter of knowledge, but a mediator among its students, activities and knowledge so that students with high levels of algebra learning can be trained. One might think, according to the theoretical aspects, particularly that the errors demonstrated by the students during the diagnosis, may be due to the poor application of correct strategies by the teachers, because, in addition, some students expressed this during the interview. Likewise, it is thought that the lack of execution of new educational situations accordingly, by the teacher, represents one of the causes of the raised problem, where it stands out, lack of a planning that allows to see the errors as useful elements for new learnings.

With respect to the objective that allowed to determine the errors and the difficulties that the students present, it can be said that the errors that emerged in the sample are related, mostly, to the unfinished procedure, simple calculation error, then errors appear when performing arithmetic-algebraic operations, followed by incorrect elimination of denominators and incorrect own problems and invalid inferences and use of basic arithmetic ignoring the rules of algebra.

In general, and based on the sample, students make a set of errors in the solution of algebraic problems. From the results obtained, with the application of the instrument to the students, it is possible to infer that the errors that are evident, could very well be the product of their learning experiences that they have had in their previous studies. Errors are noticed in students' activities, especially when faced with knowledge that forces them to review or restructure what they already know.

About the difficulties, the one with the greatest appearance was related to the teaching processes. In this sense, the teacher of the area of mathematics must play an active role in learning and understand the purpose that students learn in a meaningful way. Learning is more effective when students know that their mistakes will serve for new learning. It is possible to deepen the study of the difficulties that the students have in the learning of the algebraic problems, especially, looking for new ways of observing the errors.

Also, it has been concluded that a good alternative to start this process of improvement, is to affect the practices of teachers using innovative strategies. Well, the role of the teacher in teaching, particularly in terms of error and correction, should not necessarily be a transmitter of knowledge, but a mediator among its students, activities and knowledge so that errors are sources of knowledge and learning.
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