A random version of Mazur’s lemma

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Abstract

The purpose of this paper is to generalize the classical Mazur’s lemma from the classical convex analysis to the framework of locally $L^0$-convex modules. In this version an extra condition of countable concatenation is included. We provide a counterexample showing that this condition cannot be removed.

Keywords: lemma’s Mazur, $L^0$-modules, locally $L^0$-convex modules, gauge function, countable concatenation, countable concatenation closure.

Introduction

In recent years, works like [2, 3, 5, 7, 9, 10] have highlighted that the appropriate theoretical framework in which embed the theory of conditional risk measures is the theory of locally $L^0$-convex modules.

As [3] propose, to carry out this study it is necessary to bring tools from classical convex analysis fitting them to this new framework. Ultimately, to create a randomized generalization of convex analysis.

Due to difficulties deriving from working with scalars into the ring $L^0$ instead of $\mathbb{R}$, some obstacles must be overcome, namely, as shown in [3] and [9], mainly the fact that not all the non-zero elements possess a multiplicative inverse, i.e., $L^0$ is not a field, and that $L^0$ is not endowed with a total order.

Thus, some theorems from convex analysis will remain valid in the $L^0$-convex modules, but others will require additional conditions. So in [3]

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and \cite{9}, some notions known as countable concatenation properties are addressed.

An important tool of classical convex analysis is the classical Mazur’s lemma, which allows on many occasions changing weak topology by strong topology and vice versa working with normed spaces.

Then, the purpose of this article is to show a randomize version for $L^0$-normed modules. In addition, we will see that an extra condition of countable concatenation is needed.

This paper is structured as follow: We give a first section of preliminaries. In the second section we study some properties of the gauge function for $L^0$-modules. And finally, the third section is devoted to the Mazur’s lemma for $L^0$-modules, proving the main result and a counterexample showing that the extra condition of countable concatenation cannot be removed.

1 Preliminaries

Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, which will be fixed for the rest of this paper, we consider $L^0(\Omega, \mathcal{F}, \mathbb{P})$, the set of equivalence classes of real valued $\mathcal{F}$-measurable random variables, which will be denoted simply as $L^0$.

It is known that the triple $(L^0, +, \cdot)$ endowed with the partial order of the almost sure dominance is a lattice ordered ring.

We say “$X \geq Y$” if $\mathbb{P}(X \geq Y) = 1$. Likewise, we say “$X > Y$”, if $\mathbb{P}(X > Y) = 1$.

And, given $A \in \mathcal{F}$, we say that $X > Y$ (respectively, $X \geq Y$) on $A$, if $\mathbb{P}(X > Y | A) = 1$ (respectively , if $\mathbb{P}(X \geq Y | A) = 1$).

We also define

$$L^0_+ := \{ Y \in L^0; Y \geq 0 \}$$

$$L^0_{++} := \{ Y \in L^0; Y > 0 \}.$$ 

And denote by $\bar{L}^0$, the set of equivalence classes of $\mathcal{F}$-measurable random variables taking values in $\bar{\mathbb{R}} = \mathbb{R} \cup \{\pm \infty\}$, and extend the partial order
of the almost sure dominance to $\bar{L}^0$.

In A.5 of [4] is proved the proposition below

**Proposition 1.1.** Let $\phi$ be a subset of $L^0$, then

1. There exists $Y^* \in \bar{L}^0$ such that $Y^* \geq Y$ for all $Y \in \phi$, and such that any other $Y'$ satisfying the same, verifies $Y' \geq Y^*$.

2. Suppose that $\phi$ is directed upwards. Then there exists a increasing sequence $Y_1 \leq Y_2 \leq \ldots$ in $\phi$, such that $Y_n$ converges to $Y^*$ almost surely.

**Definition 1.1.** Under the conditions of the previous proposition, $Y^*$ is called essential supremum of $\phi$, and we write

$$\text{ess. sup } \phi = \text{ess. sup } Y := Y^*$$

The essential infimum of $\phi$ is defined as

$$\text{ess. inf } \phi = \text{ess. inf } Y := -\text{ess. sup } (-Y)$$

The order of the almost sure dominance also lets us define a topology on $L^0$. Let

$$B_\varepsilon := \{ Y \in L^0; |Y| \leq \varepsilon \}$$

the ball of radius $\varepsilon \in L^0_{++}$ centered at $0 \in L^0$. Then, for all $Y \in L^0$, $\mathcal{U}_Y := \{ Y + B_\varepsilon; \varepsilon \in L^0_{++} \}$ is a neighborhood base of $Y$. Thus, it can be defined a topology on $L^0$ that it will be known as the topology induced by $|\cdot|$ and $L^0$ endowed with this topology will be denoted by $L^0[|\cdot|]$.

Now, we are going to give the central concepts of the theory of locally $L^0$-convex modules.

**Definition 1.2.** A topological $L^0$-module $E[\tau]$ is a $L^0$-module $E$ endowed with a topology $\tau$ such that

1. $E[\tau] \times E[\tau] \rightarrow E[\tau], (X, X') \mapsto X + X'$ and
Definition 1.3. A topology \( \tau \) on a \( L^0 \)-module \( E \) is a locally \( L^0 \)-convex module if there is a neighborhood base of \( 0 \in E \) such that each \( U \in \mathcal{U} \) is

1. \( L^0 \)-convex, i.e. \( YX_1 + (1 - Y)X_2 \in U \) for all \( X_1, X_2 \in U \) and \( Y \in L^0 \) with \( 0 \leq Y \leq 1 \),
2. \( L^0 \)-absorbent, i.e. for all \( X \in E \) there is a \( Y \in L^0_{++} \) such that \( X \in YU \),
3. \( L^0 \)-balanced, i.e. \( YX \in U \) for all \( X \in U \) and \( Y \in L^0 \) with \( |Y| \leq 1 \).

And, as discussed in [13], we add an extra condition

4. \( U \) is closed under countable concatenations.

In this case, \( E[\tau] \) is a locally \( L^0 \)-convex module.

The notion of being closed under countable concatenations will be recalled later (see 2.2).

Definition 1.4. A function \( \| \cdot \| : E \to L^0_{++} \) is a \( L^0 \)-seminorm on \( E \) if:

1. \( \|YX\| = |Y| \|X\| \) for all \( Y \in L^0 \) and \( X \in E \).
2. \( \|X_1 + X_2\| \leq \|X_1\| + \|X_2\| \), for all \( X_1, X_2 \in E \).
   
   If, moreover
3. \( \|X\| = 0 \) implies \( X = 0 \),

Then \( \| \cdot \| \) is a \( L^0 \)-norm on \( E \)

Definition 1.5. Let \( \mathcal{P} \) be a family of \( L^0 \)-seminorms on a \( L^0 \)-module \( E \). Given \( Q \subset \mathcal{P} \) finite and \( \varepsilon \in L^0_{++} \), we define

\[
U_{Q,\varepsilon} := \left\{ X \in E : \sup_{\| \cdot \| \in Q} \|X\| \leq \varepsilon \right\}.
\]

Then for all \( X \in E \), \( \mathcal{U}_{Q,X} := \{ X + U_\varepsilon : \varepsilon \in L^0_{++}, \ Q \subset \mathcal{P} \) finite \} is a neighbourhood base of \( X \). Thereby, we define a topology on \( E \), which it will be
known as the topology induced by \( P \) and \( E \) endowed with this topology will be denoted by \( E[\mathcal{P}] \).

In addition, it is proved by the lemma 2.16 of [3] that \( E[\mathcal{P}] \) is a locally \( L^0 \)-convex module.

Furthermore, according to [13] a topological \( L^0 \)-module \( E[\tau] \) is a locally \( L^0 \)-convex module if, and only if, \( \tau \) is induced by a family of \( L^0 \)-seminorms.

**Definition 1.6.** Given a topological \( L^0 \)-module \( E[\tau] \), we denote by \( E[\tau]^* \), or simply by \( E^* \), the \( L^0 \)-module of continuous \( L^0 \)-linear functions \( \mu : E \to L^0 \).

We define

\[
\langle \cdot, \cdot \rangle : E \times E^* \to L^0 \\
\langle X, X^* \rangle := X^*(X)
\]

For each \( X^* \in E^* \) it holds that

\[
p_{X^*} : E \to L^+_0 \\
p_{X^*}(X) := |\langle X, X^* \rangle|
\]

is a \( L^0 \)-seminorm.

Now, consider the topology \( \sigma(E, E^*) \) induced by the family of \( L^0 \)-seminorms

\[
\{p_{X^*}; X^* \in E^* \}.
\]

Then, \( \sigma(E, E^*) \) is a locally \( L^0 \)-convex topology, which is called the weak topology of \( E \).

Likewise, for each \( X \in E \) it holds that

\[
p_X : E^* \to L^+_0 \\
p_X(X^*) := |\langle X, X^* \rangle|
\]

is a \( L^0 \)-seminorm.

And we have the \( L^0 \)-convex topology \( \sigma(E^*, E) \) induced by the family of \( L^0 \)-seminorms

\[
\{p_X; X \in E \},
\]

which is called the weak-* topology of \( E \).
2 The gauge function and the countable concatenation closure.

Let us write the notion of gauge function given in [3]:

**Definition 2.1.** Let $E$ be a $L_0$-module. The gauge function $p_K : E \to \bar{L}_0^+$ of a set $K \subset E$ is defined by

$$p_K(X) := \text{ess. inf } \{Y \in L_0^+; X \in YK\}.$$  

In addition, in [1] the properties below are proved:

**Proposition 2.1.** The gauge function $p_K$ of a $L_0$-convex and $L_0$-absorbent $K \subset E$ satisfies:

1. $1_A p_K(1_A X) = 1_A p(X)$, for all $A \in \mathcal{F}$ and $X \in E$.

2. $p_K(X) = \text{ess. inf } \{Y \in L_0^+; X \in YK\}$ for all $X \in E$.

3. $Y p_K(X) = p_K(Y X)$ for all $X \in E$ and $Y \in L_0^+$.

4. $p_K(X + Y) \leq p_K(X) + p_K(Y)$ for $X, Y \in E$.

5. For all $X \in E$ there exists a sequence $\{Z_n\}$ in $L_0^+$ such that $Z_n \downarrow p_K(X)$ almost surely and such that $X \in Z_n K$ for all $n$.

6. If in addition, $K$ is $L_0$-balanced then $p_K(Y X) = |Y| p_K(X)$ for all $Y \in L_0$ and $X \in E$.

In particular, $p_K$ is an $L_0$-seminorm.

Now we are going to give the notion of being closed under countable concatenations, which is based in the notion of countable concatenation property given in [9]. In [3] the authors work with two others notions of countable concatenation property, one for the topology and other for the family of $L_0$-seminorms, although both properties turn out to be the same.

The notion introduced in [9], and the one given below, are related to the $L_0$-module itself rather than the topology.

**Definition 2.2.** Let $E$ be a $L_0$-module and $K \subset E$ a subset, and denote by $\Pi(\Omega, \mathcal{F})$ the set of countable partitions on $\Omega$ to $\mathcal{F}$.
• Given a sequence \( \{X_n\}_{n \in \mathbb{N}} \) in \( E \) and a partition \( \{A_n\}_{n \in \mathbb{N}} \in \Pi(\Omega, \mathcal{F}) \), we define the set of countable concatenations of \( \{X_n\}_n \) and \( \{A_n\}_n \) as

\[
cc(\{A_n\}_n, \{X_n\}_n) := \{X \in E; 1_{A_n}X_n = 1_{A_n}X \text{ for each } n \in \mathbb{N}\}
\]

• We say that \( K \) is closed under countable concatenations, if for each sequence \( \{X_n\}_n \) in \( K \) and each partition \( \{A_n\}_n \in \Pi(\Omega, \mathcal{F}) \) it holds

\[
cc(\{A_n\}_n, \{X_n\}_n) \subset K
\]

• We call the countable concatenation closure of \( K \) the set defined below

\[
K^{cc} := \bigcup cc(\{A_n\}_n, \{X_n\}_n)
\]

where \( \{A_n\}_n \) runs through \( \Pi(\Omega, \mathcal{F}) \) and \( \{X_n\}_n \) runs through the sequences in \( K \). Or in another way written

\[
K^{cc} = \{X \in E; \exists \{A_n\}_{n \in \mathbb{N}} \in \Pi(\Omega, \mathcal{F}) \text{ with } 1_{A_n}X \in 1_{A_n}K \text{ for all } n\}
\]

It is clear that \( K \) is closed under countable concatenations if, and only if, \( K^{cc} = K \).

An useful result is the following

**Proposition 2.2.** If \( C \) is a bounded below (resp. above) closed under countable concatenations subset of \( L^0 \), then for each \( \epsilon \in L^0_{++} \) there exists \( Y_\epsilon \in C \) such that

\[
\text{ess. inf } C \leq Y_\epsilon < \text{ess. inf } C + \epsilon
\]

(resp., \( \text{ess. sup } C \geq Y_\epsilon > \text{ess. sup } C - \epsilon \))

In particular, given a \( L^0 \)-module \( E \) and \( K \subset E \) a \( L^0 \)-convex, \( L^0 \)-absorbent and closed under countable concatenations subset. We have that, for \( \epsilon \in L^0_{++} \), there exists \( Y_\epsilon \in L^0_{++} \) with \( X \in Y_\epsilon K \) such that

\[
p_K(X) \leq Y_\epsilon < p_K(X) + \epsilon
\]

**Proof.** Firstly, let us see that \( C \) is downwards directed. Indeed, given \( Y, Y' \in C \), define \( A := (Y < Y') \). Then, since \( C \) is closed under countable concatenations, \( 1_A Y + 1_{A'} Y' = Y \wedge Y' \in C \).
Therefore, by 1.1, for \( \varepsilon \in L^0_{++} \), there exists a decreasing sequence \( \{Y_k\}_k \) in \( C \) converging to ess. inf \( C \) almost surely.

Thereby, we consider the sequence of sets

\[
A_0 := \emptyset \text{ and } A_k := (Y_k < \text{ess. inf } C + \varepsilon) - A_{k-1} \text{ for } k > 0.
\]

Then \( \{A_k\}_{k \geq 0} \) is a partition of \( \Omega \).

We define

\[
Y_\varepsilon := \sum_{k \geq 0} 1_{A_k} Y_k.
\]

Thus \( Y_\varepsilon \in C \) as \( C \) is closed under countable concatenations.

For the second part, it suffices to see that if \( K \) is closed under countable concatenations then the set below

\[
\{ Y \in L^0_{++}; X \in YK \}
\]

is closed under countable concatenations as well.

Indeed, if \( \{A_k\}_k \in \Pi(\Omega, F) \) is a partition and \( \{Y_k\}_k \) is a sequence in \( L^0_{++} \) such that \( X \in Y_kK \) for each \( k \in \mathbb{N} \). Take \( Y := \sum_{n \in \mathbb{N}} Y_n 1_{A_n} \in L^0_{++} \). Then we have

\[
X/Y \in \text{cc}(\{A_k\}_k, \{X/Y_k\}_k)
\]

and

\[
X/Y_k \in K.
\]

Therefore \( X/Y \in K \), as \( K \) is closed under countable concatenations. \( \Box \)

Now, we have the following proposition

**Proposition 2.3.** Let \( E[\tau] \) a topological \( L^0 \)-module and \( C \subset E \) a \( L^0 \)-convex and \( L^0 \)-absorbent subset. Then the following are equivalent

1. \( p_C : E \to L^0 \) is continuous.

2. \( 0 \in \text{int } C \)

If in addition \( C \) is closed under countable concatenations

\[
\overline{C} = \{ X \in E; p_C(X) \leq 1 \}.
\]
Proof. The proof is exactly the same as the real case.

For the equality.

“⊆”: It is obtained from the continuity.

“⊇”: Let $X \in E$ be satisfying $p_C(X) \leq 1$. By proposition 2.2 we have that for every $\varepsilon \in L^0_{++}$ there exists $Y_\varepsilon \in L^0_{++}$ such that

$$1 \leq Y_\varepsilon < 1 + \varepsilon,$$

with $X \in Y_\varepsilon C$.

Then, $\{X/Y_\varepsilon\}_{\varepsilon \in L^0_{++}}$ is a net in $C$ converging to $X$. Thus $X \in \overline{C}$. And the proof is complete.

Let us see an example showing that for the equality proved in the last proposition it is necessary to take $C$ closed under countable concatenations.

Example 2.1. Given $\Omega = (0, 1)$, $\mathcal{E} = \mathcal{B}(\Omega)$ the $\sigma$-algebra of Borel, $A_n = [\frac{1}{2^n}, \frac{1}{2^{n+1}})$ with $n \in \mathbb{N}$, $\mathbb{P}$ the Lebesgue measure and $E := L^0(\mathcal{E})$ endowed with $| \cdot |$.

We define the set

$$U := \left\{ Y \in L^0; \exists I \subset \mathbb{N} \text{ finite}, |Y1_{A_i}| \leq 1 \forall i \in \mathbb{N} - I \right\}.$$  

Then, it is easily shown that $U$ is $L^0$-convex, $L^0$-absorbent and $\overline{U} = U$.

Nevertheless, it can be proved that $p_U(X) = 0$ for all $X \in L^0$, and therefore

$$\{X \in E; p_U(X) \leq 1\} = E.$$  

3 A random version of the Mazur’s lemma

Finally, in the theorem below we provide a version for $L^0$-modules of the classical Mazur’s lemma.

Theorem 3.1. Let $\{X_\alpha\}_{\alpha \in \Delta}$ a net in a $L^0$-normed module $(E, \|\cdot\|)$, which converges weakly to $X_\infty \in E$. Then, for any $\varepsilon \in L^0_{++}$, there exists

$$Z_\varepsilon \in \text{co}_{L^0}\{X_\alpha; \alpha \in \Delta\}^{cc}$$

such that $p_{Z_\varepsilon}(X) = 0$ for all $X \in L^0$, and therefore

$$\{X \in E; p_{Z_\varepsilon}(X) \leq 1\} = E.$$
such that
\[ \|X_\infty - Z\| \leq \varepsilon \]

**Proof.** Define
\[ M_1 := \text{co}_{\ell^0}\{X_\alpha; \alpha \in \Delta\}^\infty. \]

We may assume that \( 0 \in M_1 \), by replacing \( X_\infty \) by \( X_\infty - X_{\alpha_0} \) and \( X_\alpha \) by \( X_\alpha - X_{\alpha_0} \) for some \( \alpha_0 \in \Delta \) fixed and all \( \alpha \in \Delta \).

Following way of contradiction, suppose that for every \( X \in M_1 \) there exists \( A_X \in \mathcal{F} \) with \( \mathbb{P}(A_X) > 0 \) such that
\[ \|X_\infty - X\| > \varepsilon \quad \text{on } A_X \]  

Denote
\[ M := \bigcup_{X \in M_1} X + U_{\|\cdot\|, \varepsilon/2}. \]

Then \( M \) is a \( L^0 \)-convex, \( L^0 \)-absorbent and closed under countable concatenations neighbourhood of \( 0 \in E \) and such that for every \( Z \in M \) there exists \( B_Z \in \mathcal{F} \) with \( \mathbb{P}(B_Z) > 0 \) such that
\[ \|X_\infty - Z\| > \varepsilon/2 \quad \text{on } B_Z. \]

Then it is obvious that \( X_\infty \notin \overline{M} \).

Thus, by proposition 2.3 using the guage function, we have that there exists \( C \in \mathcal{F} \) with \( \mathbb{P}(C) > 0 \) such that
\[ p_M(X_\infty) > 1 \quad \text{on } C. \]

Further, given \( Y, Y' \in L^0 \) with \( YX_\infty = Y'X_\infty \) on \( C \), it holds that \( Y = Y' \) on \( C \).

Indeed, \( (Y - Y')P_M(X_\infty) \leq P_M((Y - Y')X_\infty) = 0 \). And, by swapping \( Y \) and \( Y' \), we conclude that \( |Y - Y'|P_M(X_\infty) = 0 \). And [3] yields \( Y = Y' \) on \( C \).

Then, we can define the following \( L^0 \)-linear application
\[ \mu_0 : \text{span}_{\ell^0}\{X_\infty\} \rightarrow L^0 \]
\[ \mu_0(YX_\infty) := Y1_{C}\text{PM}(X_\infty) \]
Thereby, we have that
\[ \mu_0(X) \leq p_M(X) \quad \text{for all } X \in \text{span}_{L^0}\{X_\infty\}. \]

Thus, by the Hahn-Banach extension theorem for \( L^0 \)-modules ([3], theorem 2.14), there exists a \( L^0 \)-linear extension \( \mu \) of \( \mu_0 \) defined on \( E \) such that
\[ \mu(X) \leq p_M(X) \quad \text{for all } X \in E. \]

Since \( M \) is a neighbourhood of 0, by proposition 2.3, the gauge function \( p_M \) is continuous on \( E \). Hence \( \mu \) is a continuous \( L^0 \)-linear function defined on \( E \).

Furthermore, we have that on \( C \)
\[
\text{ess. sup}_{X \in M_1} \mu(X) \leq \text{ess. sup}_{X \in M} \mu(X) \leq \text{ess. sup}_{X \in M} p_M(X) \leq 1 < p_M(X_\infty) = \mu(X_\infty)
\]

Therefore, \( X_\infty \) cannot be a weak accumulation point \( M_1 \) contrary to the hypothesis of \( X_\alpha \) converging weakly to \( X_\infty \).

Note that we can argue in a analogue way for a net in \( E^* \) and the weak-\( * \) topology.

We have the following corollaries

**Corolario 3.1.** Given \( K \subset E \) (resp. \( K \subset E^* \)) a \( L^0 \)-convex and closed under countable concatenations subset, we have that the closure in norm coincides with the closure in the weak (resp. weak-\( * \)) topology, i. e.
\[
\overline{K}_{\|\cdot\|} = \overline{K}_{\sigma(E,E^*)}
\]
(resp.
\[
\overline{K}_{\|\cdot\|}^* = \overline{K}_{\sigma(E^*,E)}
\]
).

For the next collorary we shall recall some notions for functions \( f : E \to L^0 \) brought from ([3], section 3). Namely,
• $f$ is proper if $f(X) > -\infty$ for all $X \in E$ and $\text{dom}(f) = \{ X \in E; f(X) \in L^0 \} \neq \emptyset$.

• A proper function $f$ is $L^0$-convex if $f(Y X_1 + (1 - Y)X_2) \leq Y f(X_1) + (1 - Y)f(X_2)$ for all $X_1, X_2 \in E$ and $Y \in L^0$ with $0 \leq Y \leq 1$.

• A proper function $f$ is local if $1_A f(X) = 1_A f(1_A X)$ for all $X \in E$ and $A \in F$.

• A proper function $f$ is lower semicontinuous if the level set $V(Y) := \{ X \in E; f(X) \leq Y \}$ is closed for all $Y \in L^0$.

In addition, in [3] it is proved that if $f$ is $L^0$-convex then $f$ is local.

**Corolario 3.2.** Let $f : E \to \bar{L}^0$ (resp. $f : E^* \to \bar{L}^0$) be a proper $L^0$-convex function. It holds that

if $f$ is continuous then $f$ is lower semicontinuous with the weak (resp. weak-*) topology.

**Proof.** Suppose $f$ continuous.

Since $f$ is $L^0$-convex, it follows that $f$ is local as well, and $V(Y)$ is $L^0$-convex.

Furthermore, it is easy to see that $V(Y)$ is closed under countable concatenations as $f$ is local.

Let us see that $V(Y)$ is weakly closed.

Let $\{ X_\alpha \}_{\alpha \in \Delta}$ be a net in $V(Y)$ such that $X_\alpha$ converges weakly to $X$. By Mazur’s lemma it holds that for each $\varepsilon \in L^0_+$ there exists $X_\varepsilon \in \text{co}_{L^0} \{ X_\alpha; \alpha \in \Delta \}$ with $\| X_\varepsilon - X \| \leq \varepsilon$. Since $V(Y)$ is $L^0$-convex closed under countable concatenations, we have that $X_\varepsilon \in V(Y)$. Finally, we conclude that $X \in V(Y)$ as $f$ is continuous.

Finally, we are going to provide an example showing that in the version of Mazur’s lemma proved, rather than just take $X_\varepsilon$ into the $L^0$-convex hull, we must take it into the countable concatenation closure of the $L^0$-convex hull. Namely, we shall give an example of a net weakly convergent to some limit, which is not a cluster point of the $L^0$-convex hull of that net.
Example 3.1. Given $\Omega = (0, 1)$, $\mathcal{E} = \mathcal{B}(\Omega)$ the $\sigma$-algebra of Borel, $A_n = \left(\frac{1}{2^n}, \frac{1}{2^{n-1}}\right)$ with $n \in \mathbb{N}$ and $\mathbb{P}$ the Lebesgue measure.

We define

$$\mathcal{F} := \sigma\{A_n; n \in \mathbb{N}\}$$

the $\sigma$-algebra generated by $\{A_n; n \in \mathbb{N}\}$.

Then, we take the $L^0(\mathcal{F})$-module

$$L^2_{\mathcal{F}}(\mathcal{E}) := L^0(\mathcal{F}) L^2(\mathcal{E})$$

and the $L^0(\mathcal{F})$-seminorm

$$\|X \mid \mathcal{F}\|_2 := \mathbb{E}\left[|X|^2 \mid \mathcal{F}\right]^{1/2}$$

as we can see defined in [3].

Then the following holds

$$L^0(\mathcal{F}) = \left\{ \sum_{n \in \mathbb{N}} \alpha_n 1_{A_n}; \alpha_n \in \mathbb{R} \right\}$$ (4)

$$L^2_{\mathcal{F}}(\mathcal{E}) = \left\{ \sum_{n \in \mathbb{N}} X_n 1_{A_n}; X_n \in L^2(\mathcal{E}) \right\}$$ (5)

$$\|X \mid \mathcal{F}\|_2^2 := \sum_{n \in \mathbb{N}} \frac{\|X_n 1_{A_n}\|_2^2}{2^n} 1_{A_n} \quad \text{for } X \in L^2_{\mathcal{F}}(\mathcal{E}).$$ (6)

Now, we are going to define a net in $L^2_{\mathcal{F}}(\mathcal{E})$ indexed with the set $\mathbb{N}^\mathbb{N}$.

Given $\{n_k\}_{k \in \mathbb{N}}$ we define

$$X_{\{n_k\}_{k \in \mathbb{N}}}(t) := \sum_{k \in \mathbb{N}} 1_{A_k} \text{sgn}[\sin 2\pi(2^k + n_k t - 1)] \quad \text{for } t \in (0, 1).$$

We shall show that this net converges weakly to 0 and that 0 is not a cluster point of $\text{co}_{L^0} \left\{X_{\{n_k\}}; \{n_k\} \in \mathbb{N}^\mathbb{N}\right\}$.

Indeed, by [3] or [10], we know that for each $X^* \in E^*$, there exists $Y \in L^2_{\mathcal{F}}(\mathcal{E})$ such that

$$\langle X, X^* \rangle = \mathbb{E}[XY \mid \mathcal{F}]$$
But, for every \( Y \in L^2_F (\mathcal{E}) \)

\[
\left| \mathbb{E} \left[ X_{\{n_k\}} Y | \mathcal{F} \right] \right| = \sum_{k \in \mathbb{N}} \frac{\mathbb{E}[1_{A_k} Y X_{n_k}]}{\mathbb{P}(A_k)} 1_{A_k} =
\]

\[
= \sum_{k \in \mathbb{N}} \left| \frac{1/2^{k-1}}{1/2^k} Y \text{sgn}[\sin 2\pi(2^{k+n_k} s - 1)] ds \right| 1_{1/2^k ; \mathbb{R}^+},
\]

and it can be proved that the later converges to 0 on \( A_k \) for \( k = 1, 2, \ldots \).

Hence, since \( Y \) is arbitrary we conclude that \( X_{\{n_k\}} \) converges weakly to 0.

On the other hand, let us see that 0 is not a cluster point of \( \text{co}_{L^0} \{ X_{\{n_k\}}; \{n_k\} \in \mathbb{N}^N \} \).

Indeed, given \( Y \in \text{co}_{L^0} \{ X_{\{n_k\}}; \{n_k\} \in \mathbb{N}^N \} \). We have that \( Y \) will be as follows

\[
Y = \sum_{k \in \mathbb{N}} 1_{A_k} \sum_{i=1}^N \alpha_k^i \text{sgn}[\sin 2\pi(2^{k+n_k} s - 1)]
\]

with \( N \in \mathbb{N}, \alpha_k^i \in \mathbb{R} \) and with \( \sum_{i=1}^N \alpha_k^i = 1 \) for all \( k \in \mathbb{N} \).

In addition, we have that can be proved that

\[
\left\| 1_{A_k} \sum_{i=1}^N \alpha_k^i \text{sgn}[\sin 2\pi(2^{k+n_k} s - 1)] \right\|_2^2 \geq
\]

\[
\geq \mathbb{P} \left( A_k \cap \bigcap_{j=1}^N \left( \text{sgn}[\sin 2\pi(2^{k+n_k} s - 1)] = 1 \right) \right) \geq \frac{1}{2^{N+K-1}}
\]

Therefore, using \( 6 \)

\[
\| Y | \mathcal{F} \|^2_2 = \sum_{k \in \mathbb{N}} \left\| 1_{A_k} \sum_{i=1}^N \alpha_k^i \text{sgn}[\sin 2\pi(2^{k+n_k} s - 1)] \right\|_2^2 1_{A_k} \geq
\]

\[
= \sum_{k \in \mathbb{N}} \frac{1/2^{N+k-1}}{1/2^k} 1_{A_k} = \sum_{k \in \mathbb{N}} \frac{1}{2^{N-1}} 1_{A_k}
\]

But, taking \( \varepsilon := \sum_{k \in \mathbb{N}} \frac{1}{2} 1_{A_k} \in L^0_{++}(\mathcal{F}) \) it is clear that for each \( Y \in \text{co}_{L^0} \{ X_{\{n_k\}}; \{n_k\} \in \mathbb{N}^N \} \) there exists \( A \in \mathcal{F} \) with \( \mathbb{P}(A) > 0 \) such that

\[
\| Y | \mathcal{F} \|^2_2 > \varepsilon \quad \text{on} \ A.
\]

Hence, 0 cannot be a cluster point of \( \text{co}_{L^0} \{ X_{\{n_k\}}; \{n_k\} \in \mathbb{N}^N \} \) as could be expected considering the classical Mazur’s lemma.
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