THE GLOBAL FORMULATION OF GENERALIZED EINSTEIN-SCALAR-MAXWELL THEORIES

C. LAZAROIU AND C. S. SHAHBAZI

ABSTRACT. We summarize the global geometric formulation of Einstein-Scalar-Maxwell theories twisted by flat symplectic vector bundle which encodes the duality structure of the theory. We describe the scalar-electromagnetic symmetry group of such models, which consists of flat unbased symplectic automorphisms of the flat symplectic vector bundle lifting those isometries of the scalar manifold which preserve the scalar potential. The Dirac quantization condition for such models involves a local system of integral symplectic spaces, giving rise to a bundle of polarized Abelian varieties equipped with a symplectic flat connection, which is defined over the scalar manifold of the theory. Generalized Einstein-Scalar-Maxwell models arise as the bosonic sector of the effective theory of string/M-theory compactifications to four-dimensions, and they are characterized by having non-trivial solutions of “U-fold” type.

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1. Introduction

Supergravity theories [1, 2] are supersymmetric theories of gravity which extend general relativity and gauge theory and arise in the low energy limit of string/M-theory and of their compactifications. It is known that the construction of such theories involves interesting structures, such as Kähler-Hodge and special Kähler manifolds, symmetric spaces, exceptional Lie groups, generalized complex structures, differential cohomology and differential K-theory etc. However, the global formulation of these theories is not yet fully understood. This note is part of a larger project (see also [4–6]) aimed at obtaining the full mathematical formulation of supergravity theories (in the generality required by their relation to string theory) and at studying the global geometry of their solutions.

Supergravity theories are classical theories of gravity coupled to matter, formulated using systems of “fields” defined on a manifold $M$ of the appropriate dimension and subject to certain partial differential equations, known as the “equations of motion”. An unambiguous formulation of such theories requires that one specifies the global nature of the fields and of the partial differential operators arising in the equations of motion. Currently, however, the supergravity literature gives only local descriptions$^1$ of the fields and of these differential operators. The globalization problem is the problem of giving globally-unambiguous mathematical definitions of such theories which reduce locally to the local description found in the supergravity literature. The solution of this problem is non-unique since there can be many global definitions of “fields” subject to globally-defined partial differential equations which reduce to a given local description.

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$^1$Descriptions which are valid only if one restricts all fields to sufficiently small open subsets of $M$. 

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Since supergravity theories are supersymmetric, they require spinors for their formulation. In this note, we simplify the globalization problem by ignoring the spinor field content and the supersymmetry conditions, thus considering only the so-called universal bosonic sector. This sector arises in any supergravity theory, though it is subject to increasingly stringent supplementary constraints (not discussed in this paper) as the number of supersymmetries present in the theory increases. In addition, we focus exclusively on the case when $M$ is a four-manifold.

In four dimensions, the universal bosonic sector is the so-called Einstein-Scalar-Maxwell (ESM) model defined on a four-manifold $M$, which involves gravity (modeled globally by a Lorentzian metric on $M$), a finite number of real scalar fields (modeled globally by a smooth map from $M$ to a manifold $\mathcal{M}$ of arbitrary dimension) and a finite number of Abelian gauge fields, whose field strengths can be modeled locally as 2-forms defined on $M$. While the local form of ESM theories is well-known, their precise global formulation was systematically addressed only recently [3]. It turns out that the naive globalization of the local formulation fails to capture the classical limit of certain string theory backgrounds known as “U-folds” and hence is insufficient for the application of such models to string theory. The geometric description of the classical limit of U-fold backgrounds [4] requires that one globalizes ESM models by including a “twist” of the Abelian gauge field sector rather different global behavior. The naive globalization corresponds to using a trivial flat symplectic vector bundle on $M$.

The global mathematical formulation of generalized ESM models given in [3] is summarized in this note. We follow the notations and conventions of loc. cit.; in particular, all manifolds considered are smooth and connected and all bundles and maps considered are smooth. In this note, a Lorentzian metric is a smooth metric of signature $(3,1)$ defined on a four-manifold.

2. Generalized Einstein-Scalar-Maxwell theories

2.1. Scalar structures and related notions.

**Definition 2.1.** A scalar structure is a triplet $\Sigma = (\mathcal{M}, \mathcal{G}, \Phi)$, where $(\mathcal{M}, \mathcal{G})$ is a Riemannian manifold (called the scalar manifold) and $\Phi \in \mathcal{C}^\infty(\mathcal{M}, \mathbb{R})$ is a smooth real-valued function defined on $\mathcal{M}$ (called the scalar potential).

Let $\Sigma = (\mathcal{M}, \mathcal{G}, \Phi)$ be a scalar structure. Let $M$ be an oriented four-dimensional smooth manifold (which need not be compact).

**Definition 2.2.** The modified density of a smooth map $\varphi \in \mathcal{C}^\infty(M, \mathcal{M})$ relative to a Lorentzian metric $g \in \text{Met}_{3,1}(M)$ and to the scalar structure $\Sigma$ is the following smooth real-valued map defined on $M$:

$$e_\Sigma(g, \varphi) \overset{\text{def}}{=} \frac{1}{2} \text{Tr}_g \varphi^\ast(g) + \Phi^\varphi \in \mathcal{C}^\infty(M, \mathbb{R})$$

where $\Phi^\varphi \overset{\text{def}}{=} \Phi \circ \varphi$ and $\text{Tr}_g$ denotes trace taken with respect to $g$.

**Definition 2.3.** The modified tension field of a smooth map $\varphi \in \mathcal{C}^\infty(M, \mathcal{M})$ relative to the Lorentzian metric $g \in \text{Met}_{3,1}(M)$ and to the scalar structure $\Sigma$ is the section of the pulled-back bundle $(TM)^\varphi$ defined through:

$$\theta_\Sigma(g, \varphi) \overset{\text{def}}{=} \theta_\varphi(g, \varphi) - (\text{grad}_g \Phi)^\varphi \in \Gamma(M, (TM)^\varphi)$$

Here $\text{grad}_g \Phi \in \mathcal{X}(M)$ is the gradient vector field of $\Phi$ with respect to $\mathcal{G}$ and $\theta_\varphi(g, \varphi)$ is the tension field of $\varphi$ relative to $g$ and $\mathcal{G}$ [7]:

$$\theta_\varphi(g, \varphi) \overset{\text{def}}{=} \text{Tr}_g \nabla_\varphi \in \Omega^0(M, (TM)^\varphi)$$

where $\nabla_\varphi \in \Omega^1(M, (TM)^\varphi)$ denotes the $(TM)^\varphi$-valued one-form associated to the differential $d_\varphi : TM \to TM$ and $\nabla$ is the connection induced on $(TM)^\varphi$ by the Levi-Civita connections of $g$ and $\mathcal{G}$.
2.2. Duality structures. Let $N$ be a manifold.

Definition 2.4. A duality structure on $N$ is a flat symplectic vector bundle $\Delta = (S, D, \omega)$ defined over $N$, where $\omega$ denotes the symplectic pairing on the vector bundle $S$ and $D$ denotes the $\omega$-compatible flat connection on $S$.

Definition 2.5. Let $\Delta_i = (S_i, D_i, \omega_i)$ with $i = 1, 2$ be two duality structures defined on $N$. A morphism of duality structures from $\Delta_1$ to $\Delta_2$ is a based morphism of vector bundles $f \in \text{Hom}(S_1, S_2)$ such that $\omega_2(f \otimes f) = \omega_1$ and such that $D_2 \circ f = (\text{id}_{\Omega^1(N)} \otimes f) \circ D_1$.

With this notion of morphism, duality structures on $N$ form a category which we denote by $\text{DS}(N)$. Let $\Delta = (S, D, \omega)$ be a duality structure defined on $N$ such that $\text{rk} S = 2n$. Let $\text{Symp}$ denote the category of finite-dimensional symplectic vector spaces over $\mathbb{R}$ and linear symplectic morphisms. Let $\text{Symp}^x$ denote the unit groupoid of this category and $\Pi_1(N)$ denote the fundamental groupoid of $N$. Let $T^\Delta_\gamma$ denote the parallel transport of $D$ along a path $\gamma : [0, 1] \to N$.

Definition 2.6. The parallel transport functor of $\Delta$ is the functor $T^\Delta : \Pi_1(N) \to \text{Symp}^x$ which associates to any point $x \in N$ the symplectic vector space $T^\Delta(x) = (S_x, \omega_x)$ and to any homotopy class $c \in \Pi_1(N)$ with fixed initial point $x$ and fixed final point $y$ the invertible symplectic morphism $T^\Delta(c) = T^\Delta_\gamma : (S_x, \omega_x) \simto (S_y, \omega_y)$, where $\gamma \in \mathcal{P}(N)$ is any path which represents the class $c$.

Notice that $T^\Delta_\gamma$ can be viewed as a $\text{Symp}^x$-valued local system defined on $N$. The map which takes $\Delta$ into $T^\Delta_\gamma$ is an equivalence between the category $\text{DS}(N)$ and the functor category $[\Pi_1(N), \text{Symp}^x]$. This implies that duality structures on $N$ are classified up to isomorphism by the symplectic characteristic:

\begin{equation}
C_{\pi_1(N)}(\text{Sp}(2n, \mathbb{R})) \overset{\text{def}}{=} \text{Hom}(\pi_1(N), \text{Sp}(2n, \mathbb{R}))/\text{Sp}(2n, \mathbb{R}).
\end{equation}

Definition 2.7. A duality frame of $\Delta$ is a $D$-flat symplectic frame $\mathcal{E} \overset{\text{def}}{=} (e_1, \ldots, e_n; f_1, \ldots, f_n)$ of $(S, \omega)$ defined on an open subset $U \subset N$.

Definition 2.8. The duality structure $\Delta$ is called trivial if it is trivial as a flat symplectic vector bundle.

Remark 2.9. A duality structure is trivial iff it admits a globally-defined duality frame. If $N$ is simply connected, then any duality structure on $N$ is trivial.

2.3. Electromagnetic structures. Let $N$ be a manifold.

Definition 2.10. An electromagnetic structure defined on $N$ is a quadruplet $\Xi \overset{\text{def}}{=} (S, D, J, \omega)$, where $(S, D, \omega)$ is a duality structure defined on $N$ and $J$ is a taming of the symplectic vector bundle $(S, \omega)$.

Remark 2.11. Notice that we do not require $J$ to be compatible with $D$. Together with $\omega$, $J$ defines an Euclidean scalar product $Q$ on $S$ given by $Q(\cdot, \cdot) \overset{\text{def}}{=} \omega(J \cdot, \cdot)$.

Definition 2.12. Let $\Xi_1 = (S_1, D_1, J_1, \omega_1)$ and $\Xi_2 = (S_2, D_2, J_1, \omega_1)$ be two electromagnetic structures defined on $N$. A morphism of electromagnetic structures from $\Xi_1$ to $\Xi_2$ is a morphism of duality structures $f : (S_1, D_1, \omega_1) \to (S_2, D_2, \omega_2)$ such that $J_2 \circ f = f \circ J_1$.

With this definition of morphism, electromagnetic structures defined on $N$ form a category which we denote by $\text{ES}(N)$. This fibers over the category of duality structures $\text{DS}(N)$; the fiber at a duality structure $\Delta = (S, D, \omega)$ can be identified with the set $\mathfrak{fr}(S, \omega)$ of tamings of $(S, \omega)$, which is a contractible topological space. Accordingly, the set of isomorphism classes of $\text{ES}(N)$ fibers over the disjoint union of character varieties $\sqcup_{n \geq 0} C_{\pi_1(N)}(\text{Sp}(2n, \mathbb{R}))$. Let $\Xi = (S, D, J, \omega)$ be an electromagnetic structure defined on $N$ and $h = Q + i\omega$ be the Hermitian scalar product defined by $\omega$ and $J$ on $S$. 

\[ \]
Definition 2.13. The fundamental form of $\Xi$ is the $\text{End}(S)$-valued one-form on $N$ defined through:

$$\Theta_\Xi \overset{\text{def}}{=} \text{ad}(J) \overset{\text{def}}{=} D \circ J - J \circ D \in \Omega^1(N, \text{End}(S)) .$$

The electromagnetic structure $\Xi$ is called unitary if $\Theta_\Xi = 0$, i.e. if $J$ is parallel with respect to $D$.

If $\Xi$ is unitary, then $D$ is a unitary connection on the Hermitian vector bundle $(\mathcal{S}, J, h)$. In this case, we have $\text{Hol}_D^2 \subset U(S_x, J_x, h_x)$ for all $x \in N$. The category of unitary electromagnetic structures defined on $N$ is the full sub-category of $\text{ES}(N)$ whose objects are the unitary electromagnetic structures. This is equivalent with the category of Hermitian vector bundles defined on $N$ and endowed with a flat $\mathbb{C}$-linear Hermitian connection. In particular, isomorphism classes of unitary electromagnetic structures are in bijection with the points of the character variety:

$$C_{\pi_1(N)}(U(n)) \overset{\text{def}}{=} \text{Hom}(\pi_1(N), U(n))/U(n) ,$$

where $U(n)$ acts by conjugation.

2.4. Scalar-duality and scalar-electromagnetic structures.

Definition 2.14. A scalar-duality structure is an ordered system $(\Sigma, \Xi)$, where:

1. $\Sigma = (\mathcal{M}, \mathcal{G}, \Phi)$ is a scalar structure
2. $\Xi = (\mathcal{S}, D, \omega)$ is a duality structure defined on $\mathcal{M}$.

A scalar-electromagnetic structure is an ordered system $\mathcal{D} = (\Sigma, \Xi)$, where:

1. $\Sigma = (\mathcal{M}, \mathcal{G}, \Phi)$ is a scalar structure
2. $\Xi = (\mathcal{S}, D, J, \omega)$ is an electromagnetic structure defined on $\mathcal{M}$.

In this case, the system $\mathcal{D}_0 \overset{\text{def}}{=} (\Sigma, \Xi_0)$ is called the underlying scalar-duality structure, where $\Xi_0 \overset{\text{def}}{=} (\mathcal{S}, D, \omega)$ is the duality structure underlying $\Xi$.

Let $\mathcal{D}$ be a scalar-electromagnetic structure as in the definition.

Definition 2.15. The fundamental field of the scalar-electromagnetic structure $\mathcal{D}$ is defined through:

$$\Psi_\mathcal{D} \overset{\text{def}}{=} (\lambda_\mathcal{G} \otimes \text{id}_{\text{End}(S)})(\Theta_\Xi) \in \Gamma(\mathcal{M}, \mathcal{T}\mathcal{M} \otimes \text{End}(S)) .$$

2.5. Pulled-back electromagnetic structures. Let $\mathcal{D} = (\Sigma, \Xi)$ be a scalar-electromagnetic structure with underlying scalar structure $\Sigma = (\mathcal{M}, \mathcal{G}, \Phi)$ and underlying electromagnetic structure $\Xi = (\mathcal{S}, D, J, \omega)$. Let $M$ be a four-manifold and $\varphi \in \mathcal{C}^\infty(M, \mathcal{M})$ be a smooth map from $M$ to $\mathcal{M}$.

Definition 2.16. The $\varphi$-pullback of the electromagnetic structure $\Xi$ defined on $\mathcal{M}$ is the electromagnetic structure $\Xi^{\varphi} \overset{\text{def}}{=} (\mathcal{S}^{\varphi}, D^{\varphi}, J^{\varphi}, \omega^{\varphi})$ defined on $M$.

The Hodge operator $*_{\varphi} : \wedge^*T^*M \to \wedge^*T^*M$ of $(M, g)$ induces the endomorphism $* \overset{\text{def}}{=} *_{\varphi} \overset{\text{def}}{=} *_{\varphi} \otimes \text{id}_{\mathcal{S}^{\varphi}}$ of the bundle $\wedge_M(\mathcal{S}^{\varphi}) \overset{\text{def}}{=} \wedge^*T^*M \otimes \mathcal{S}^{\varphi}$.

Definition 2.17. The twisted Hodge operator of $\Xi^{\varphi}$ is the bundle endomorphism $* : = *_{g, J^{\varphi}} \in \text{End}(\wedge M, \wedge^*T^*M \otimes \mathcal{S}^{\varphi})$ defined through:

$$*_{g, J^{\varphi}} \overset{\text{def}}{=} *_{g} \otimes J^{\varphi} = *_{g} \circ J^{\varphi} = J^{\varphi} \circ *_{g} .$$

Let $\alpha \overset{\text{def}}{=} \bigoplus_{k=0}^4 (-1)^k \text{id}_{\wedge^k T^*M}$ be the main automorphism of $\wedge^*T^*M$. We have:

$$*^2 = \alpha \otimes \text{id}_{\mathcal{S}^{\varphi}} .$$

The operator $*_{g, J^{\varphi}}$ preserves the sub-bundle $\wedge_M^2(\mathcal{S}^{\varphi}) = \wedge^2T^*M \otimes \mathcal{S}^{\varphi}$, on which it squares to plus the identity. Accordingly, we have a direct sum decomposition:

$$\wedge^2T^*M \otimes \mathcal{S}^{\varphi} = (\wedge^2T^*M \otimes \mathcal{S}^{\varphi})^+ \oplus (\wedge^2T^*M \otimes \mathcal{S}^{\varphi})^- ,$$

where $(\wedge^2T^*M \otimes \mathcal{S}^{\varphi})^\pm$ are the sub-bundles of eigenvectors of $*$ corresponding to the eigenvalues $\pm 1$. 

The global formulation of generalized Einstein-Scalar-Maxwell theories

**Definition 2.18.** An $S^2$-valued two-form $\eta \in \Omega^2(M, S^2)$ defined on $M$ is called **positively polarized** with respect to $g$ and $J^2$ if it is a section of the vector bundle $(\wedge^2 T^* M \otimes S^2)^+$, which amounts to the requirement that it satisfies the **positive polarization condition:**

$$
\ast_g \cdot J^2 \eta = \eta \quad \text{i.e.} \quad \ast_g \eta = - J^2 \eta .
$$

For any open subset $U$ of $M$, let $g_U \overset{\text{def}}{=} g|U$, $\varphi_U \overset{\text{def}}{=} \varphi|U$ and let $\Omega^{\Xi, g, \varphi}$ be the sheaf of smooth sections of the bundle $(\wedge^2 T^* M \otimes S^2)^+$. Globally-defined and positively-polarized $S^2$-valued forms are the global sections of this sheaf. Notice that $\eta \in \Omega^2(M, S^2)$ is positively polarized iff $\ast \eta$ is.

**2.6. The mathematical formulation of generalized ESM theories.** Let $M$ be a four-manifold and $D = (\Sigma, \Xi)$ be a scalar-electromagnetic structure with underlying scalar structure $\Sigma = (M, G, \Phi)$ and underlying electromagnetic structure $\Xi = (\mathcal{S}, \mathcal{D}, J, \omega)$. The $\varphi$-pullback $Q^\varphi$ of the Euclidean scalar product $Q$ induced by $\omega$ and $J$ on $\mathcal{S}$ is a Euclidean scalar product on $S^2$. Let $\otimes_g : \otimes^4 T^* M \to \otimes^2 T^* M$ be the bundle morphism given by $g$-contraction of the two middle indices. This is uniquely determined by the condition:

$$
(\omega_1 \otimes \omega_2) \otimes (\omega_3 \otimes \omega_4) = (\omega_2 \otimes \omega_3) \otimes (\omega_1 \otimes \omega_4 \forall \omega_1, \omega_2, \omega_3, \omega_4 \forall \omega \in \Omega^1(M) ,
$$

where $(\ , \ , \ )$ is the pseudo-Euclidean metric induced by $g$ on $\wedge^2 T^* M$. Viewing $\wedge^2 T^* M$ as the sub-bundle of antisymmetric 2-tensors inside $\otimes^2 T^* M$, this restricts to a morphism of vector bundles $\otimes_g : \wedge^2 T^* M \otimes \wedge^2 T^* M \to \otimes^2 T^* M$, which we call the **inner $g$-contraction of 2-forms.**

**Definition 2.19.** The **twisted inner contraction** of $S^2$-valued 2-forms is the unique morphism of vector bundles $\mathcal{O} := \mathcal{O}_{\otimes g, J, \omega, \varphi} : \wedge^2_M(S^2) \times_M \wedge^2_M(S^2) \to \otimes^2(T^* M)$ which satisfies:

$$
(\rho_1 \otimes \xi_1) \otimes (\rho_2 \otimes \xi_2) = Q^\varphi(\xi_1, \xi_2) \rho_1 \otimes_g \rho_2
$$

for all $\rho_1, \rho_2 \in \Omega^2(M)$ and all $\xi_1, \xi_2 \in \Gamma(M, S^2)$.

Let $\Psi \overset{\text{def}}{=} \Psi_D \in \Gamma(M, T^* M \otimes \text{End}(\mathcal{S}))$ be the fundamental field of $D$ and let $\Psi^\varphi \in \Gamma(M, (T^* M)^{g, \varphi})$ be its pullback through $\varphi$. Let $(\ , \ )$ be the pseudo-Euclidean scalar product induced by $g$ and $Q^\varphi$ on the vector bundle $\wedge^2_M(S^2)$ defined on $M$, we extend this trivially to a $\mathcal{T}$-valued pairing (denoted by the same symbol) between the bundles $\mathcal{T} \otimes \wedge^2_M(S^2)$ and $\wedge^2_M(S^2)$. Similarly, we trivially extend the twisted wedge product $\wedge_\omega$ defined in Appendix C of reference [3] to a $\mathcal{T} \otimes \wedge^2_M(M)$-valued pairing (denoted by the same symbol) between the bundles $\mathcal{T} \otimes \wedge^2_M(S^2)$ and $\wedge^2_M(S^2)$.

**Definition 2.20.** The sheaf of ESM configurations $\text{Conf}_D$ determined by $D$ is the sheaf of sets defined on $M$ through:

$$
\text{Conf}_D(U) \overset{\text{def}}{=} \{(g, \varphi, \mathcal{V})| g \in \text{Met}_{3,1}(U), \varphi \in C^\infty(U, M), \mathcal{V} \in \Omega^{\Xi, g, \varphi}(U)\}
$$

for all open subsets $U \subset M$, with the obvious restriction maps. An element $(g, \varphi, \mathcal{D}) \in \text{Conf}_D(U)$ is called a **local ESM configuration** of type $D$ defined on $U$. The **set of global configurations of type $D$** is the set:

$$
\text{Conf}_D(M) \overset{\text{def}}{=} \{(g, \varphi, \mathcal{V})| g \in \text{Met}_{3,1}(M), \varphi \in C^\infty(M, M), \mathcal{V} \in \Omega^{\Xi, g, \varphi}(M)\} .
$$

of global sections of this sheaf. An element $(g, \varphi, \mathcal{D}) \in \text{Conf}_D(M)$ is called a **global ESM configuration** of type $D$.

**Definition 2.21.** The **generalized ESM theory** associated to $D$ is defined by the following set of partial differential equations on $M$ with unknowns $(g, \varphi, \mathcal{V}) \in \text{Conf}_D(M)$:

1. The **Einstein equation:**

$$
G(g) = \kappa T(g, \varphi, \mathcal{V}) ,
$$

with energy-momentum tensor $T_D$ given by:

$$
T_D(g, \varphi, \mathcal{V}) \overset{\text{def}}{=} g e\Sigma(g, \varphi) + 2 \mathcal{V} \mathcal{V} - \varphi^*(\mathcal{G}) .
$$
2. The scalar equations:

\[ \theta_S(g, \varphi) - \frac{1}{2}(\ast \varphi, \Psi \varphi \varphi) = 0 \, . \]

3. The twisted electromagnetic equations:

\[ d_{D^g} \varphi = 0 \, , \]

where \( d_{D^g} : \Omega^k(M, \mathcal{S}^\varphi) \to \Omega^{k+1}(M, \mathcal{S}^\varphi) \) is the de Rham differential of \( M \) twisted by the pulled-back flat connection \( D^\varphi \).

A local ESM solution of type \( D \) defined on \( U \) is a smooth solution \((g, \varphi, \varphi)\) of these equations which is defined on \( U \). A global ESM solution of type \( D \) is a smooth solution of these equations which is defined on \( M \). The sheaf of local ESM solutions \( \text{Sol}_D \) of type \( D \) is the sheaf of sets defined on \( M \) whose sections on an open subset \( U \subset M \) is the set of all local solutions defined on \( U \).

Remark 2.22. It is shown in [3] that a generalized ESM model is locally indistinguishable from an ordinary ESM model, in the sense that the global partial differential equations (7), (9) and (10) reduce locally to those used in the supergravity literature (see for example reference [1]) upon choosing a local flat symplectic frame of the duality structure \( \Delta = (\mathcal{S}, D, \omega) \). The supergravity literature tacitly assumes that the local formulas globalize trivially, which amounts to working with a trivial duality structure; this assumption implies existence of a globally-defined duality frame. Generalized ESM models with a non-trivial duality structure are globally quite different from the models used in the supergravity literature, since a non-trivial duality structure does not admit global duality frames. Due to this fact, global solutions of generalized ESM models afford a geometric description of a certain type of classical U-folds, thereby realizing the proposal of [4].

2.7. Sheaves of scalar-electromagnetic configurations and solutions. Let \( M \) and \( D \) be as above and fix a metric \( g \in \text{Met}_{3,1}(M) \).

Definition 2.23. The sheaf of local scalar-electromagnetic configurations \( \text{Conf}^g_D \) relative to \( g \) is the sheaf of sets defined on \( M \) whose set of sections on an open subset \( U \subset M \) is defined through:

\[ \text{Conf}^g_D(U) \overset{\text{def}}{=} \{ (\varphi, \varphi) \mid \varphi \in \mathcal{C}^\infty(U, \mathcal{M}), \varphi \in \Omega^\varphi(g, \varphi)(U) \} \, . \]

The set of global scalar-electromagnetic configurations relative to \( g \) is the set \( \text{Conf}^g_D(M) \) of global sections of this sheaf.

Definition 2.24. The sheaf of local scalar-electromagnetic solutions relative to \( g \) is the sheaf of sets defined on \( M \) whose set of sections \( \text{Sol}^g_D(U) \) on an open subset \( U \subset M \) is defined as the set of all solutions of the scalar and twisted electromagnetic equations (9) and (10) defined on \( U \). The set of global scalar-electromagnetic solutions relative to \( g \) is the set \( \text{Sol}^g_D(M) \) of global sections of \( \text{Sol}^g_D \).

Since it will be of use later, we define:

\[ \text{Conf}^g_{D_0}(M) \overset{\text{def}}{=} \bigcup_{J \in \text{J}(\mathcal{S}, \omega)} \text{Conf}^g_{\{D_0, J\}}(M) \, , \]

where \( D_0 \) is a scalar-duality structure.

2.8. Electromagnetic field strengths.

Definition 2.25. An electromagnetic field strength on \( M \) with respect to \( D \) and relative to \( g \in \text{Met}_{3,1}(M) \) and to the map \( \varphi \in \mathcal{C}^\infty(M, \mathcal{M}) \) is \( \mathcal{S}^\varphi \)-valued 2-form \( \varphi \in \Omega^2(M, \mathcal{S}^\varphi) \) which satisfies the following two conditions:

1. \( \varphi \) is positively polarized with respect to \( J^\varphi \), i.e. we have \( \ast_{g, \varphi} \varphi = \varphi \).
2. \( \varphi \) is \( d_{D^\varphi} \)-closed, i.e.:

\[ d_{D^\varphi} \varphi = 0 \, . \]

The second condition is called the electromagnetic equation.

\[ ^2 \text{Notice however that we use different conventions.} \]
For any open subset $U$ of $M$, let:

\[(13)\quad \Omega_{\text{cl}}^{\Xi,\varphi}(U) \overset{\text{def}}{=} \{ \mathcal{V} \in \Omega^{\Xi,\varphi}(U) \mid d_{\mathcal{V}} \varphi = 0 \}\]
denote the set of electromagnetic field strengths defined on $U$, which is an (infinite-dimensional) subspace of the $\mathbb{R}$-vector space $\Omega^{\Xi,\varphi}(U)$. This defines a sheaf of electromagnetic field strengths $\Omega_{\text{cl}}^{\Xi,\varphi}$ relative to $\varphi$ and $g$, which is a locally-constant sheaf of $\mathbb{R}$-vector spaces defined on $M$.

3. Scalar-electromagnetic dualities and symmetries

Let $\Delta = (S, D, \omega)$ be a duality structure on $M$ and $J$ be a taming of $(S, \omega)$. Let $\Xi = (S, D, J, \omega)$ be the corresponding electromagnetic structure with underlying duality structure $\Delta = (S, D, \omega)$.

**Definition 3.1.** An unbased automorphism $f \in \text{Aut}^{\text{ub}}(S)$ is called:

1. A *symmetry of the duality structure* $\Delta$, if $f$ is symplectic with respect to $\omega$ and covariantly constant with respect to $D$.
2. A *symmetry of the electromagnetic structure* $\Xi$, if $f$ is complex with respect to $J$ and is a symmetry of the duality structure $\Delta$.

Let $\text{Aut}^{\text{ub}}(\Delta) = \text{Aut}^{\text{ub}}(S, D, \omega)$ and $\text{Aut}^{\text{ub}}(\Xi) = \text{Aut}^{\text{ub}}(S, D, J, \omega)$ denote the groups of symmetries of $\Delta$ and $\Xi$. We have:

\[
\text{Aut}^{\text{ub}}(\Xi) = \text{Aut}^{\text{ub}}(\Delta) \cap \text{Aut}(S, J) = \text{Aut}^{\text{ub}}(S, D) \cap \text{Aut}^{\text{ub}}(S, J, \omega) \\
\text{Aut}^{\text{ub}}(\Delta) = \text{Aut}^{\text{ub}}(S, \omega) \cap \text{Aut}^{\text{ub}}(S, D).
\]

Given a symplectic automorphism $f \in \text{Aut}^{\text{ub}}(S, \omega)$, the endomorphism $\text{Ad}(f)(J)$ is again a taming of $(S, \omega)$, where $\text{Ad}(f)$ denotes the adjoint action of $f$ on ordinary sections of the bundle $\text{End}(S)$ (see [3]). Hence for any electromagnetic structure $\Xi = (S, D, J, \omega)$ having $\Delta$ as its underlying duality structure, the quadruplet:

\[(14)\quad \Xi_f \overset{\text{def}}{=} (S, D, \text{Ad}(f)(J), \omega)\]
is again an electromagnetic structure having $\Delta$ as its underlying duality structure. This defines a left action of the group $\text{Aut}^{\text{ub}}(S, \omega)$ on the set $\mathcal{E}_{S_\Delta}(M)$ of all electromagnetic structures whose underlying duality structure equals $\Delta$.

Let $M$ be a four-manifold and $D = (\Sigma, \Xi)$ be a scalar-electromagnetic structure with underlying scalar structure $\Sigma = (M, G, \Phi)$ and underlying electromagnetic structure $\Xi = (S, D, J, \omega)$. Let $D_0 = (\Sigma, \Delta)$ be the scalar-duality structure underlying $D$, where $\Delta = (S, D, \omega)$. Let $g \in \text{Met}_{3,1}(M)$ be a Lorentzian metric on $M$. Let:

\[ \text{Aut}(\Sigma) \overset{\text{def}}{=} \{ \psi \in \text{Iso}(M, G) \mid \Phi \circ \psi = \Phi \}, \]

where $\text{Iso}(M, G)$ denote the isometry group of $(M, G)$.

**Definition 3.2.** The *scalar-electromagnetic duality group* of $D_0$ is the following subgroup of $\text{Aut}^{\text{ub}}(\Delta)$:

\[ \text{Aut}(D_0) \overset{\text{def}}{=} \{ f \in \text{Aut}^{\text{ub}}(\Delta) \mid f_0 \in \text{Aut}(\Sigma) \} .\]

An element of this group is called a *scalar-electromagnetic duality*. The *duality action* is the action of $\text{Aut}(D_0)$ on the set $\text{Conf}^{\text{ub}}_{D_0}(M)$ given by:

\[ f \circ (\varphi, \mathcal{V}) \overset{\text{def}}{=} (f_0 \circ \varphi, \hat{f}^\varphi(\mathcal{V})) , \quad \forall f \in \text{Aut}(D_0) , \]

where $f_0 \in \text{Diff}(M)$ is the projection of $f$ to $M$ and $\hat{f} : S \to S^{f_0}$ is the based isomorphism of vector bundles induced by $f$. 


Theorem 3.3. For any \( f \in \text{Aut}(D_0) \), we have:
\[
(15) \quad f \circ \text{Sol}_D^g(M) = \text{Sol}_{D_f}^g(M),
\]
where:
\[
D_f \overset{\text{def.}}{=} (\Sigma, \Xi_f).
\]

Definition 3.4. The scalar-electromagnetic symmetry group of \( D \) is the following subgroup of \( \text{Aut}(D_0) \):
\[
\text{Aut}(D) \overset{\text{def.}}{=} \{ f \in \text{Aut}(D_0) | \text{Ad}(f)(J) = J \} = \{ f \in \text{Aut}^\text{ub}(\Xi) | f_0 \in \text{Aut}(\Sigma) \}.
\]
An element of this group is called a scalar-electromagnetic symmetry.

Corollary 3.5. For all \( f \in \text{Aut}(D) \), we have:
\[
f \circ \text{Sol}_D^g(M) = \text{Sol}_{D_f}^g(M).
\]
Thus \( \text{Aut}(D) \) consists of symmetries of the scalar-electromagnetic equations (9) and (10), for any fixed Lorentzian metric \( g \in \text{Met}_{3,1}(M) \).

We have short exact sequences:
\[
1 \to \text{Aut}(\Delta) \to \text{Aut}(D_0) \to \text{Aut}^\Delta(\Sigma) \to 1
\]
\[
1 \to \text{Aut}(\Xi) \to \text{Aut}(D) \to \text{Aut}^\Xi(\Sigma) \to 1,
\]
where \( \text{Aut}(\Delta) \overset{\text{def.}}{=} \text{Hom}_{\text{DS}(N)}(\Delta, \Delta) \) and \( \text{Aut}(\Xi) \overset{\text{def.}}{=} \text{Hom}_{\text{ES}(N)}(\Xi, \Xi) \) are the groups of based automorphisms of \( \Delta \) and \( \Xi \) and the groups \( \text{Aut}^\Delta(\Sigma) \) and \( \text{Aut}^\Xi(\Sigma) \) consist of those automorphisms of the scalar structure \( \Sigma \) which admit lifts to scalar-electromagnetic duality transformations and scalar-electromagnetic symmetries, respectively. Let \( \text{Hol}_D^g \) be the holonomy group of \( D \) at a point \( p \in \mathcal{M} \). Then we can identify \( \text{Aut}(\Delta) \) with the commutant of \( \text{Hol}_D^g \) inside the group \( \text{Sp}(\mathcal{S}_p, \omega_p) \cong \text{Sp}(2n, \mathbb{R}) \).

4. Twisted Dirac quantization

Let \( N \) be a manifold.

4.1. Integral duality structures and integral electromagnetic structures.

Definition 4.1. Let \( \Delta = (\mathcal{S}, D, \omega) \) be a duality structure of rank \( 2n \) defined on \( N \). A Dirac system for \( \Delta \) is a fiber sub-bundle \( \Lambda \subset \mathcal{S} \) which satisfies the following conditions:
1. For any \( x \in X \), the triple \( (\mathcal{S}_x, \omega_x, \Lambda_x) \) is an integral symplectic space, i.e. \( \Lambda_x \) is a full lattice in \( \mathcal{S}_x \) and \( \omega_x(\Lambda_x, \Lambda_x) \subset \mathbb{Z} \).
2. \( \Lambda \) is invariant under the parallel transport of \( D \), i.e. the following condition is satisfied for any path \( \gamma \in \mathcal{P}(N) \):
\[
T^\Delta_\gamma(\Lambda_\gamma(0)) = \Lambda_\gamma(1).
\]
For every \( x \in N \), the lattice \( \Lambda_x \subset \mathcal{S}_x \) is called the Dirac lattice defined by \( \Lambda \) at the point \( x \).

Definition 4.2. An integral duality structure defined on \( N \) is a pair \( \Delta \overset{\text{def.}}{=} (\Delta, \Lambda) \), where \( \Delta \) is a duality structure defined on \( N \) and \( \Lambda \) is a Dirac system for \( \Delta \).

Relation (16) implies that the type \( t \) (the ordered list of elementary divisors) of the integral symplectic space \( (\mathcal{S}_x, \omega_x, \Lambda_x) \) does not depend on the point \( x \in N \). This quantity is denoted \( t(\Delta) \) and called the type of \( \Delta \).

Definition 4.3. Let \( \Delta = (\Delta_1, \Lambda_1) \) and \( \Delta_2 = (\Delta_2, \Lambda_2) \) be two integral duality structures defined on \( N \). An morphism of integral duality structures from \( \Delta_1 \) to \( \Delta_2 \) is a morphism of duality structures \( f : \Delta_1 \to \Delta_2 \) such that \( f(\Lambda_1) \subset \Lambda_2 \).
Remark 4.4. The set of isomorphism classes of integral duality structures of type $t$ defined on $N$ is in bijection with the character variety:

$$C_{x_1(N)}(\text{Sp}_t(2n,\mathbb{Z})) = \text{Hom}(\pi_1(N), \text{Sp}_t(2n,\mathbb{Z}))/\text{Sp}_t(2n,\mathbb{Z}) ,$$

where $\text{Sp}_t(2n,\mathbb{Z})$ is the modified Siegel modular group of type $t$.

Let $\Delta \overset{\text{def}}{=} (\mathcal{S}, D, \omega, \Lambda)$ be an integral duality structure or rank $2n$ and type $t$, defined on $N$. For any $x \in N$, the integral symplectic space $(\mathcal{S}_x, \omega_x, \Lambda_x)$ defines a symplectic torus $X_x(\mathcal{S}_x, \omega_x, \Lambda_x)$. These tori fit into a fiber bundle $X_x(\Delta)$, endowed with a complete flat Ehresmann connection $\mathcal{H}_\Delta$ induced by $D$. The Ehresmann transport of this connection is through isomorphisms of symplectic tori so it preserves the group structure and symplectic form of the fibers; in particular, the holonomy group of $\mathcal{H}_\Delta$ is contained in $\text{Sp}_t(2n,\mathbb{Z})$.

Definition 4.5. The pair $(X_x(\Delta), \mathcal{H}_\Delta)$ is called the flat bundle of symplectic tori defined by the integral duality structure $\Delta$.

Definition 4.6. An integral electromagnetic structure defined on $N$ is a pair $\Xi = (\Delta, J)$, where $\Xi = (\Delta, J)$ is an electromagnetic structure defined on $N$ and $\Delta$ is a Dirac system for the underlying duality structure $\Delta = (\mathcal{S}, D, \omega, \Lambda)$ of $\Xi$. The type of the integral duality structure $\Delta = (\mathcal{S}, D, \omega, \Lambda)$ is called the type $t(\Xi)$ of $\Xi$:

$$t(\Xi) \overset{\text{def}}{=} t(\Delta) .$$

Let $\Xi = (\mathcal{S}, D, J, \omega, \Lambda)$ be an integral electromagnetic structure of real rank $2n$ and type $t$, with underlying duality structure $\Delta = (\mathcal{S}, D, \omega)$. For every $x \in N$, the fiber $(\mathcal{S}_x, J_x, \omega_x, \Lambda_x)$ of $\Xi$ is an integral tamed symplectic space which defines a polarized Abelian variety $X_h(\mathcal{S}_x, J_x, \omega_x, \Lambda_x)$ of type $t$, whose underlying symplectic torus is given by $X_x(\mathcal{S}_x, \omega_x, \Lambda_x)$. These polarized Abelian varieties fit into a smooth fiber bundle $X_h(\Xi)$. As above, the connection $D$ induces a complete integrable Ehresmann connection $\mathcal{H}_\Xi \overset{\text{def}}{=} \mathcal{H}_\Delta$ on this bundle, whose transport proceeds through isomorphisms of symplectic tori, so it preserves the Abelian group structure and symplectic form of the fibers but not their complex structure.

Definition 4.7. The pair $(X_h(\Xi), \mathcal{H}_\Xi)$ is called the bundle of polarized Abelian varieties defined by the integral electromagnetic structure $\Xi$.

4.2. The twisted Dirac quantization condition. Let $(M, g)$ be a Lorentzian four-manifold and $(M, g)$ be a Riemannian manifold. Let $\varphi \in C^\infty(M, M)$ be a fixed smooth map from $M$ to $M$. Let $\Xi = (\Xi, \Lambda)$ be an integral electromagnetic structure defined on $M$, with underlying electromagnetic structure $\Xi = (\mathcal{S}, D, J, \omega)$ and underlying duality structure $\Delta = (\mathcal{S}, D, \omega)$. Then the system $\Xi^{\varphi} = (\Xi^{\varphi}, \Lambda^{\varphi})$ is an electromagnetic structure on $M$, where $\Lambda^{\varphi}$ is the $\varphi$-pullback of the fiber sub-bundle $\Lambda \subset \mathcal{S}$; this has underlying duality structure $\Delta^{\varphi} = (\mathcal{S}^{\varphi}, D^{\varphi}, \omega^{\varphi})$. Let $\Delta^{\varphi} \overset{\text{def}}{=} (\Delta^{\varphi}, \Lambda^{\varphi})$ denote the integral duality structure underlying $\Xi^{\varphi}$. Let $\text{Symp}_0$ denote the category of finite-dimensional integral symplectic vector spaces. Let $H^*(M, \Delta^{\varphi})$ denote the total twisted singular cohomology group of $M$ with coefficients in the $\text{Symp}_0$-valued local system $T_{\Delta^{\varphi}}$ and let $H^*(M, \Delta^{\varphi})$ denote the total twisted singular cohomology space of $M$ with coefficients in the $\text{Symp}_0$-valued local system $T_{\Delta^{\varphi}}$. The latter can be identified with the total cohomology space $H^*_{\Delta^{\varphi}}(M, \mathcal{S}^{\varphi})$ of the twisted de Rham complex $(\Omega^*(M, \mathcal{S}^{\varphi}), d_{\Delta^{\varphi}})$. Since $\mathcal{S}^{\varphi} = \Lambda^{\varphi} \otimes \mathbb{R}$, the coefficient sequence gives a map $j_\ast : H^*(M, \Delta^{\varphi}) \rightarrow H^*(M, \Delta^{\varphi})$, whose image $H^*_{\Delta^{\varphi}}(M, \Delta^{\varphi}) \overset{\text{def}}{=} j_\ast(H^*(M, \Delta^{\varphi}))$ is a graded subgroup of the graded additive group of $H^*(M, \Delta^{\varphi})$.

Definition 4.8. An electromagnetic field $\mathcal{V} \in \Omega^2(M, \mathcal{S}^{\varphi})$ is called $\Lambda^{\varphi}$-integral if its $D^{\varphi}$-twisted cohomology class $[\mathcal{V}] \in H^2_{\Delta^{\varphi}}(M, \mathcal{S}^{\varphi}) \equiv H^2(M, \Delta^{\varphi})$ belongs to $H^2_{\Delta^{\varphi}}(M, \Delta^{\varphi})$:

$$[\mathcal{V}] \in H^2_{\Delta^{\varphi}}(M, \Delta^{\varphi}) = j_\ast(H^2(M, \Delta^{\varphi})) .$$
The condition that $V$ be $Λ_ϕ$-integral is called the twisted Dirac quantization condition defined by the Dirac structure $Λ$. This should be viewed as a condition constraining semiclassical Abelian gauge field configurations; a mathematical model for such configuration can be given using a certain version of twisted differential cohomology.

4.3. Integral scalar-electromagnetic duality and symmetry groups.

**Definition 4.9.** An integral scalar-duality structure is a pair $D_0 \overset{\text{def}}{=} (D_0, Λ)$, where $D_0 = (Σ, Δ)$ is a scalar-duality structure and $Λ$ is a Dirac system for $Δ$. An integral scalar-electromagnetic structure is a pair $D = (D, Λ)$, where $D = (Σ, Ξ)$ is a scalar-electromagnetic structure and $Λ$ is a Dirac system for the underlying duality structure of the electromagnetic structure $Ξ$.

Let $D = (D, Λ)$ be an integral scalar-electromagnetic structure with underlying scalar-electromagnetic structure $D = (Σ, Ξ)$, where $Σ = (M, G, Φ)$ and $Ξ \overset{\text{def}}{=} (S, D, J, ω)$. Let $Δ = (S, D, ω)$ be the underlying duality structure and let $Δ = (Δ, Λ)$ and $Ξ = (Ξ, Λ)$ be the underlying integral duality structure and integral electromagnetic structure. Let $D₀ = (Σ, Δ)$ be the underlying scalar-duality structure and $D₀ = (D₀, Λ)$ be the underlying integral scalar-duality structure.

**Definition 4.10.** The integral scalar-electromagnetic duality group defined by the integral scalar-duality structure $D₀$ is the following subgroup of the scalar-electromagnetic duality group $\text{Aut}(D₀)$:

$$\text{Aut}(D₀) \overset{\text{def}}{=} \{ f \in \text{Aut}(D₀) \mid f(Λ) = Λ \} \subset \text{Aut}(D₀).$$

Elements of this group are called integral scalar-electromagnetic dualities. The integral scalar-electromagnetic symmetry group defined by the integral scalar-electromagnetic structure $D$ is the following subgroup of the scalar-electromagnetic symmetry group $\text{Aut}(D)$:

$$\text{Aut}(D) \overset{\text{def}}{=} \{ f \in \text{Aut}(D) \mid f(Λ) = Λ \} \subset \text{Aut}(D).$$

Elements of this group are called integral scalar-electromagnetic symmetries.

Notice that $\text{Aut}(D)$ is a subgroup of $\text{Aut}(D₀)$.

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**References**

[1] T. Ortin, *Gravity and Strings*, Cambridge Monographs on Mathematical Physics, 2nd edition, 2015.
[2] D. Z. Freedman, A. Van Proeyen, *Supergravity*, Cambridge Monographs on Mathematical Physics, 2012.
[3] C. I. Lazaroiu, C. S. Shahbazi, *Generalized Einstein-Scalar-Maxwell theories and locally geometric U-folds*, arXiv:1609.05872.
[4] C. I. Lazaroiu, C. S. Shahbazi, *Geometric U-folds in four dimensions*, arXiv:1603.03095
[5] C. I. Lazaroiu, C. S. Shahbazi, *Real pinor bundles and real Lipschitz structures*, arXiv:1606.07894.
[6] C. I. Lazaroiu, C. S. Shahbazi, *On the spin geometry of supergravity and string theory*, arXiv:1607.02103.
[7] P. Baird, J. C. Wood, *Harmonic morphisms between Riemannian manifolds*, Clarendon Press, Oxford, 2003.