Toeplitz Determinants for the Class of Functions with Bounded Turning

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Abstract. In this paper, we obtain the upper bounds of the Toeplitz determinants for the class of functions with bounded turning. We also present some consequences of our main results. Some estimates obtained on Toeplitz determinants are sharp.

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1. Introduction

Let \( A \) denote the class of all functions \( f(z) \) of the form

\[
f(z) = z + \sum_{n=2}^{\infty} a_n z^n,
\]

which are analytic in the open unit disk \( E = \{z \in \mathbb{C} : |z| < 1\} \). We denote by \( S \) the subclass of \( A \) consisting of univalent functions in \( E \).

Let \( P \) denote the class of positive real part functions \( p(z) \) of the form

\[
p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n,
\]

which satisfy \( \text{Re} p(z) > 0 \) for \( z \in E \). This class is also known as the class of Carathéodory functions.

Let \( G(\alpha, \delta) \) be the class of normalized functions \( f(z) \in A \) satisfying the condition \( \text{Re}(e^{i\alpha} f'(z)) > \delta, z \in E \), where \( |\alpha| < \pi, 0 \leq \delta < 1 \), and \( \cos \alpha > \delta \). This class was introduced by Mohamad [10].

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Remark 1. For the specific values of the parameters $\alpha$ and $\delta$, we obtain the special cases of $G(\alpha, \delta)$ as follows:

(i) If we let $\alpha = \delta = 0$, then we have the class $G(0, 0) \equiv R$ which satisfies $\text{Re} f'(z) > 0$. The functions from $R$ are said to be of bounded turning.

(ii) If we let $\alpha = 0$, then we have the class $G(0, \delta) \equiv R(\delta)$ which satisfies $\text{Re} (f'(z)) > \delta$. The class $R(\delta)$ is called the class of bounded turning functions of order $\delta$.

(iii) If we let $\delta = 0$, then we have the class $G(\alpha, 0) \equiv R(\alpha)$ which satisfies $\text{Re} (e^{i\alpha} f'(z)) > 0$.

Goel and Mehrok [8], Macgregor [9], and Silverman and Silvia [17] were among the first researchers to study the classes $R$, $R(\delta)$, and $R(\alpha)$. Recently, the investigation into the class of bounded turning functions and coefficient problems such as the Hankel determinant for the higher order has been extensively studied by other researchers, see for example [3, 4]. We may point interested readers to recent advances in the class of bounded turning functions connected to a three-leaf-shaped domain and Bernoulli’s lemniscate as well as their coefficient problems like the Hankel determinant, logarithmic coefficients, and the Hankel determinant with logarithmic coefficients, which point in a different direction than the current study, see [16, 23].

Finding estimates on the functional involving coefficients of $f(z) \in A$ has been a major research area in geometric function theory since the development of the Bieberbach conjecture. Toeplitz determinant, for example, whose elements are the coefficients of $f(z) \in A$ has been appealing to many researchers because it is related to the coefficient problems. Toeplitz determinant appeared in all branches of pure and applied mathematics, statistics and probability, image processing, quantum mechanics, queuing networks, signal processing, and time series analysis (see Ye and Lim [24] and references therein). Here we consider the symmetric Toeplitz determinant and it is defined by [21]

$$T_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_n & \cdots & a_{n+q-2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+q-1} & a_{n+q-2} & \cdots & a_n \end{vmatrix}, \ a_1 = 1.$$ 

The estimates of the Toeplitz determinant were obtained for different classes of univalent functions. For instance, Ali et al. [2] studied Toeplitz matrices whose elements are the coefficients of bounded turning, starlike, close-to-convex, and univalent functions, Radhika et al. [12] obtained sharp bounds for Toeplitz determinants for the class of bounded turning functions, Zhang et al. [25] considered Toeplitz determinants of starlike functions connected with the sine function, and Zulfiqar et al. [26] investigated the fourth-order Toeplitz determinant for convex functions connected with the sine function. Much of the recent history of the development of this problem can also be found in [1, 5, 11, 13–15, 18–20, 22]. Thus, inspired by these works, in this paper, we aim to investigate the upper
bounds of the second, third, and fourth-order Toeplitz determinants for functions of the class $G(\alpha, \delta)$. In particular, we find the bounds for the following determinants:

$$T_2(n) = \begin{vmatrix} a_n & a_{n+1} \\ a_{n+1} & a_n \end{vmatrix}, \quad n \geq 2,$$

(3)

$$T_3(1) = \begin{vmatrix} 1 & a_2 & a_3 \\ a_2 & 1 & a_2 \\ a_3 & a_2 & 1 \end{vmatrix},$$

(4)

$$T_3(2) = \begin{vmatrix} a_2 & a_3 & a_4 \\ a_3 & a_2 & a_3 \\ a_4 & a_2 & a_2 \end{vmatrix},$$

(5)

$$T_3(3) = \begin{vmatrix} a_3 & a_4 & a_5 \\ a_4 & a_3 & a_4 \\ a_5 & a_4 & a_3 \end{vmatrix},$$

(6)

and

$$T_4(1) = \begin{vmatrix} 1 & a_2 & a_3 & a_4 \\ a_2 & 1 & a_4 & a_3 \\ a_3 & a_4 & 1 & a_2 \\ a_4 & a_3 & a_2 & 1 \end{vmatrix},$$

(7)

where the elements are the coefficients of the functions $f(z)$ of the form (1) in $G(\alpha, \delta)$. Besides, we point out several special cases and the consequences of our results.

We shall need the following lemmas in order to prove our main results.

**Lemma 1.** ([6]) Let $p(z) \in P$ of the form $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$. Then

$$|p_n| \leq 2, \quad n \geq 1.$$

The inequality is sharp for the function $p(z) = \frac{1+z}{1-z}$.

**Lemma 2.** ([7]) Let $p(z) \in P$ of the form $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$ and $\mu \in \mathbb{C}$. Then

$$|p_n - \mu p_k p_{n-k}| \leq 2 \max \{1, |2\mu - 1|\}, \quad 1 \leq k \leq n - 1.$$

If $|2\mu - 1| \geq 1$, then the inequality is sharp for the function $p(z) = \frac{1+z}{1-z}$ or its rotations. If $|2\mu - 1| < 1$, then the inequality is sharp for the function $p(z) = \frac{1+z^2}{1-z^2}$ or its rotations.

2. Main Results

In this section, we state and prove the main results of our present investigation.
Theorem 1. Let \( f(z) \in G(\alpha, \delta) \) be of the form (1). Then
\[
|T_2(n)| \leq 4t_{\alpha\delta}^2 \left[ \frac{1}{n^2} + \frac{1}{(n+1)^2} \right],
\]
where \( t_{\alpha\delta} = \cos \alpha - \delta \). The inequality is sharp.

Proof. Let a function \( f(z) \in G(\alpha, \delta) \) given by (1). Then there exists a function \( p(z) \in P \) of the form (2) such that
\[
e^{i\alpha} f'(z) - i \sin \alpha - \delta t_{\alpha\delta} = p(z),
\]
where \( t_{\alpha\delta} = \cos \alpha - \delta \).
Rearranging (8) and hence using the series representations for \( f'(z) \) and \( p(z) \), we get
\[
1 + 2a_2 z + 3a_3 z^2 + 4a_4 z^3 + \cdots = e^{-i\alpha} [t_{\alpha\delta} (1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots) + i \sin \alpha + \delta].
\]
Equating the coefficients of like powers of \( z^n \), \( n \geq 1 \) yields
\[
a_n = \frac{t_{\alpha\delta} e^{-i\alpha} p_{n-1}}{n}, \quad n \geq 2.
\]
Then, applying Lemma 1, we get
\[
|a_n| = \frac{t_{\alpha\delta} |p_{n-1}|}{n} \leq \frac{2t_{\alpha\delta}}{n},
\]
and so
\[
|a_{n+1}| = \frac{t_{\alpha\delta} |p_n|}{n+1} \leq \frac{2t_{\alpha\delta}}{n+1}.
\]
Clearly from (3) leads to
\[
|T_2(n)| = |a_n^2 - a_{n+1}^2| \leq |a_n^2| + |a_{n+1}^2|.
\]
Thus, making use of (11) and (12) gives the desired inequality. The inequality is sharp for the function \( \frac{e^{i\alpha} f'(z) - i \sin \alpha - \delta}{t_{\alpha\delta}} = \frac{1+iz}{1-iz} \).

Theorem 2. Let \( f(z) \in G(\alpha, \delta) \) be of the form (1). Then
\[
|T_3(1)| \leq \frac{1}{9} \left( 9 + 18t_{\alpha\delta}^2 + 4t_{\alpha\delta}^2 \sqrt{9t_{\alpha\delta}^2 - 6t_{\alpha\delta} \cos \alpha + 1} \right),
\]
where \( t_{\alpha\delta} = \cos \alpha - \delta \). The inequality is sharp.
Proof. By making use of (10) for \( n = 2, 3 \), from (4), we obtain
\[
T_3(1) = 1 - 2a_2^2 + 2a_2^2a_3 - a_3^2
= 1 - 2\left(\frac{t_\alpha e^{-ia_1}}{2}\right)^2 + 2\left(\frac{t_\alpha e^{-ia_1}p_1}{2}\right)^2 \left(\frac{t_\alpha e^{-ia_2}p_2}{3}\right)^2 - \left(\frac{t_\alpha e^{-ia_2}p_2}{3}\right)^2
= \frac{1}{18}\left(18 - 9t_\alpha e^{-2ia_1}p_1^2 - 2t_\alpha e^{-2ia_2}p_2^2 + 3t_\alpha^3 e^{-3ia_1}p_1^2p_2^2\right).
\] (14)

Further, we can rearrange (14) as
\[
|T_3(1)| = \frac{1}{18}\left|18 - 9t_\alpha e^{-2ia_1}p_1^2 - 2t_\alpha e^{-2ia_2}p_2^2 + 3t_\alpha^3 e^{-3ia_1}p_1^2p_2^2\right|,
\] (15)
where \( \mu = \frac{3t_\alpha e^{-ia}}{2} \).

Thus, by the triangle inequality along with Lemma 1 and Lemma 2, we get
\[
|T_3(1)| \leq \frac{1}{9}\left(9 + 18t_\alpha^2 + 4t_\alpha^2 \sqrt{9t_\alpha^2 - 6t_\alpha\cos \alpha + 1}\right).
\] (16)

This inequality is sharp for the function \( \frac{e^{-i\beta f'(z)} - i\sin \alpha - \delta}{t_\alpha} = \frac{1 + iz}{1 - iz} \).

**Theorem 3.** Let \( f(z) \in G(\alpha, \delta) \) be of the form (1). Then
\[
|T_3(2)| \leq \frac{7t_\alpha^3}{3},
\]
where \( t_\alpha = \cos \alpha - \delta \).

Proof. Using (10) for \( n = 2, 3, 4 \), from (5), it follows that
\[
T_3(2) = a_3^3 - 2a_3a_4^2 + 2a_3^2a_4 - a_2a_4^2
= \left(\frac{t_\alpha e^{-ia_1}p_1}{2}\right)^3 - 2\left(\frac{t_\alpha e^{-ia_1}p_1}{2}\right)^2 \left(\frac{t_\alpha e^{-ia_2}p_2}{3}\right)^2 + 2\left(\frac{t_\alpha e^{-ia_2}p_2}{3}\right)^2 \left(\frac{t_\alpha e^{-ia_3}p_3}{4}\right)^2
- \left(\frac{t_\alpha e^{-ia_1}p_1}{2}\right)\left(\frac{t_\alpha e^{-ia_2}p_1}{3}\right)^2
= t_\alpha^3 e^{-3ia_2}\left(36p_1^3 - 32p_1p_2^2 + 16p_2^2p_3 - 9p_3p_2^2\right).
\] (17)

Rearranging the terms in (17) and hence applying the triangle inequality, then we can rewrite it as
\[
|T_3(2)| \leq \frac{t_\alpha^3}{288}\left[36|p_1|^3 + 32|p_1||p_2|^2 + 16|p_2|^2 + 16|p_3||p_4 - \eta_1 p_2|^2 + 16|p_3||p_4 - \eta_2 p_1 p_3|\right],
\] (18)
where \( \eta_1 = 1 \) and \( \eta_2 = \frac{9}{16} \).

Further, by implementing Lemma 1 and Lemma 2, thus we obtain
\[
|T_3(2)| \leq \frac{7t_\alpha^3}{3}.
\]
This concludes the proof.
Theorem 4. Let \( f(z) \in G(\alpha, \delta) \) be of the form (1). Then

\[
|T_3(3)| \leq \frac{112t_{\alpha \delta}^3}{135},
\]

where \( t_{\alpha \delta} = \cos \alpha - \delta \).

Proof. Using the values of \( a_3, a_4, \) and \( a_5 \) from (10) and in view of (6), it can be seen that

\[
T_3(3) = a_3^3 - 2a_3a_4^2 + 2a_4^2a_5 - a_3a_5^2
\]

\[
= \left( \frac{t_{\alpha \delta}e^{-i\alpha}p_2}{3} \right)^3 - 2 \left( \frac{t_{\alpha \delta}e^{-i\alpha}p_2}{3} \right) \left( \frac{t_{\alpha \delta}e^{-i\alpha}p_3}{4} \right)^2 + 2 \left( \frac{t_{\alpha \delta}e^{-i\alpha}p_3}{4} \right)^2 \left( \frac{t_{\alpha \delta}e^{-i\alpha}p_4}{5} \right)
\]

\[
= \frac{t_{\alpha \delta}^3e^{-3i\alpha}}{5400} \left( 200p_2^3 - 225p_2p_3^2 + 135p_3^2p_4 - 72p_2p_4^2 \right).
\]

After rearranging the terms and using triangular inequalities, (19) yields

\[
|T_3(3)| \leq \frac{t_{\alpha \delta}^3}{5400} \left[ 200|p_2|^2 + 225|p_2||p_3|^2 + 135|p_3|^2|p_4| - \eta_1p_3^2 + 135|p_4||p_6 - \nu p_2p_4| \right],
\]

where \( \eta_1 = 1 \) and \( \nu = \frac{72}{135} \).

Finally, by applying Lemma 1 and Lemma 2, we get

\[
|T_3(3)| \leq \frac{112t_{\alpha \delta}^3}{135}.
\]

This completes the proof.

Theorem 5. Let \( f(z) \in G(\alpha, \delta) \) be of the form (1). Then

\[
|T_4(1)| \leq \frac{1}{1296} \left( 1296 + 4392t_{\alpha \delta}^2 + 3456t_{\alpha \delta}^3 + 1921t_{\alpha \delta}^4 \right),
\]

where \( t_{\alpha \delta} = \cos \alpha - \delta \).
**Proof.** From the expansion of (7) and using the values of $a_2$, $a_3$, and $a_4$ from (10), we get

$$T_4(1) = 1 - 2a_2^2 + a_2^4 - 2a_3^2 + a_3^4 - 2a_4^2 + a_4^4 - 2a_2^2 a_3^2 - 2a_2^2 a_4^2$$

$$-2a_3^2 a_4^2 + 8a_2 a_3 a_4$$

$$= 1 - 2 \left( \frac{t_{\alpha \delta} e^{-i \alpha p_1}}{2} \right)^2 + \left( \frac{t_{\alpha \delta} e^{-i \alpha p_1}}{2} \right)^4 - 2 \left( \frac{t_{\alpha \delta} e^{-i \alpha p_2}}{3} \right)^2 + \left( \frac{t_{\alpha \delta} e^{-i \alpha p_2}}{3} \right)^4$$

$$-2 \left( \frac{t_{\alpha \delta} e^{-i \alpha p_3}}{4} \right)^2 + \left( \frac{t_{\alpha \delta} e^{-i \alpha p_3}}{4} \right)^4 - 2 \left( \frac{t_{\alpha \delta} e^{-i \alpha p_2}}{2} \right)^2 \left( \frac{t_{\alpha \delta} e^{-i \alpha p_3}}{4} \right)^2$$

$$-2 \left( \frac{t_{\alpha \delta} e^{-i \alpha p_3}}{4} \right)^2 \left( \frac{t_{\alpha \delta} e^{-i \alpha p_3}}{4} \right)^2 - 2 \left( \frac{t_{\alpha \delta} e^{-i \alpha p_2}}{3} \right)^2 \left( \frac{t_{\alpha \delta} e^{-i \alpha p_3}}{4} \right)^2$$

$$+8 \left( \frac{t_{\alpha \delta} e^{-i \alpha p_1}}{2} \right) \left( \frac{t_{\alpha \delta} e^{-i \alpha p_2}}{3} \right) \left( \frac{t_{\alpha \delta} e^{-i \alpha p_3}}{4} \right)$$

$$= \frac{1}{20736} \left( 20736 - 10368 t_{\alpha \delta}^2 e^{-2i \alpha} p_1^2 + 1296 t_{\alpha \delta}^4 e^{-4i \alpha} p_1^4 - 4608 t_{\alpha \delta}^2 e^{-2i \alpha} p_2^2 \right.$$

$$\left. + 256 t_{\alpha \delta}^4 e^{-4i \alpha} p_2^4 - 2592 t_{\alpha \delta}^2 e^{-2i \alpha} p_3^2 + 81 t_{\alpha \delta}^4 e^{-4i \alpha} p_3^4 - 1152 t_{\alpha \delta}^2 e^{-4i \alpha} p_1^2 p_2^2 \right.$$

$$- 648 t_{\alpha \delta}^4 e^{-4i \alpha} p_1^2 p_3^2 - 288 t_{\alpha \delta}^4 e^{-4i \alpha} p_2^2 p_3^2 + 6912 t_{\alpha \delta}^3 e^{-3i \alpha} p_1 p_2 p_3 \right).$$

(21)

Rearranging the terms in (21) and applying the triangle inequality, as well as some calculations, we can rewrite it in the following expression:

$$|T_4(1)| \leq \frac{1}{20736} \left[ 20736 + 10368 t_{\alpha \delta}^2 |p_1|^2 + 256 t_{\alpha \delta}^4 |p_2|^4 + 81 t_{\alpha \delta}^4 |p_3|^4 + 2592 t_{\alpha \delta}^2 |p_3|^2 \right.$$

$$\left. + 648 t_{\alpha \delta}^4 |p_1|^2 |p_3|^2 + 288 t_{\alpha \delta}^4 |p_2|^2 |p_3|^2 + 1296 t_{\alpha \delta}^4 |p_1|^2 |p_2|^2 - \eta_1 |p_1|^2 \right.$$

$$\left. + 4608 t_{\alpha \delta}^2 |p_2|^2 - v_1 p_1^2 + 6912 t_{\alpha \delta}^3 |p_1| |p_2|^2 - v_2 p_1 p_2 \right].$$

(22)

where $\eta_1 = 1$, $v_1 = \frac{9 t_{\alpha \delta}^2 e^{-2i \alpha}}{32}$, and $v_2 = t_{\alpha \delta} e^{-i \alpha}$.

Now, with the help of Lemma 1 and Lemma 2, we obtain

$$|T_4(1)| \leq \frac{1}{1296} \left( 1296 + 4392 t_{\alpha \delta}^2 + 3456 t_{\alpha \delta}^3 + 1921 t_{\alpha \delta}^4 \right).$$

Thus, this completes the proof.

### 3. Corollaries and Consequences

In this section, we shall give the consequences of our main results.

For $\alpha = 0$ and $\delta = 0$ in Theorem 1, Theorem 2, Theorem 3, Theorem 4, and Theorem 5, we get the estimates for the class $R$.

**Corollary 1.** Let $f(z) \in R$. Then
(i) \(|T_2(n)| \leq 4 \left[ \frac{1}{n^2} + \frac{1}{(n+1)^2} \right]\).

(ii) \(|T_3(1)| \leq \frac{35}{9}\).

(iii) \(|T_3(2)| \leq \frac{7}{3}\).

(iv) \(|T_3(3)| \leq \frac{112}{135}\).

(v) \(|T_4(1)| \leq \frac{11065}{1296}\).

For \(\alpha = 0\) in Theorem 1, Theorem 2, Theorem 3, Theorem 4, and Theorem 5, we obtain the estimates for the class \(R(\delta)\).

**Corollary 2.** Let \(f(z) \in R(\delta)\). Then

(i) \(|T_2(n)| \leq 4(1 - \delta)^2 \left[ \frac{1}{n^2} + \frac{1}{(n+1)^2} \right]\).

(ii) \(|T_3(1)| \leq \frac{9}{7} \left( 9 + 18(1 - \delta)^2 + 4(1 - \delta)^2 (3\delta - 2) \right)\).

(iii) \(|T_3(2)| \leq \frac{7(1-\delta)^3}{3}\).

(iv) \(|T_3(3)| \leq \frac{112(1-\delta)^3}{135}\).

(v) \(|T_4(1)| \leq \frac{1}{1296} \left( 1296 + 4392(1 - \delta)^2 + 3456(1 - \delta)^3 + 1921(1 - \delta)^4 \right)\).

By choosing \(\delta = 0\) in Theorem 1, Theorem 2, Theorem 3, Theorem 4, and Theorem 5, we obtain the estimates for the class \(R(\alpha)\).

**Corollary 3.** Let \(f(z) \in R(\alpha)\). Then

(i) \(|T_2(n)| \leq 4\cos^2\alpha \left[ \frac{1}{n^2} + \frac{1}{(n+1)^2} \right]\).

(ii) \(|T_3(1)| \leq \frac{9}{7} \left( 9 + 18\cos^2\alpha + 4\cos^2\alpha \sqrt{3\cos^2\alpha + 1} \right)\).

(iii) \(|T_3(2)| \leq \frac{7\cos^3\alpha}{3}\).

(iv) \(|T_3(3)| \leq \frac{112\cos^3\alpha}{135}\).

(v) \(|T_4(1)| \leq \frac{1}{1296} \left( 1296 + 4392\cos^2\alpha + 3456\cos^3\alpha + 1921\cos^4\alpha \right)\).

We remark that the inequalities in Corollary 1(i), Corollary 1(ii), and Corollary 1(iii) coincide with the results of Ali et al. [2]. It is also shown in [2] that the results in Corollary 1(i) and Corollary 1(ii) were sharp. In the existing literature, no bounds for \(|T_2(n)|, n \geq 2, |T_3(n)|, n = 1, 2, 3, and |T_4(1)|\) for the classes \(G(\alpha, \delta), R(\delta),\) and \(R(\alpha)\) were obtained. Additionally, the results on \(|T_3(3)|\) and \(|T_4(1)|\) for functions in the class \(R\) had never been studied before.
4. Conclusions

In the present paper, we have considered the Toeplitz determinants whose elements are coefficients of univalent functions. We have obtained the upper bounds of $|T_2(n)|$, $n \geq 2$, $|T_3(n)|$, $n = 1, 2, 3$, and $|T_4(1)|$ not only for functions of the class $G(\alpha, \delta)$, but also for some classes of functions with bounded turning namely $R$, $R(\delta)$, and $R(\alpha)$. Some results obtained are reduced to the estimates proven in [2] for specific choices of parameters $\alpha$ and $\delta$. For the class $G(\alpha, \delta)$, we have determined the sharp estimates for $|T_2(n)|$, $n \geq 2$ and $|T_3(1)|$. The results obtained perhaps give an opportunity for researchers to further investigate inequalities problems for functions of the class $G(\alpha, \delta)$ as well as other subclasses of $S$.

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References

[1] Saba N Al-Khafaji, Ali Al-Fayadh, Ahmed Hadi Hussain, and Sameer Annon Abbas. Toeplitz determinant whose its entries are the coefficients for class of non-bazilevič functions. In Journal of Physics: Conference Series, volume 1660, page 012091. IOP Publishing, 2020.

[2] Md Firoz Ali, DK Thomas, and A Vasudevarao. Toeplitz determinants whose elements are the coefficients of analytic and univalent functions. Bulletin of the Australion Mathematical Society, 97(2):253–264, 2018.

[3] Muhammad Arif, Mohsan Raza, Inayat Ullah, and Paweł Zaprawa. Hankel determinants of order four for a set of functions with bounded turning of order $\alpha$. Lithuanian Mathematical Journal, 62(2):135–145, 2022.

[4] Muhammad Arif, Inayat Ullah, Mohsan Raza, and Paweł Zaprawa. Investigation of the fifth hankel determinant for a family of functions with bounded turnings. Mathematica Slovaca, 70(2):319–328, 2020.

[5] Luminiţa-Ioana Cotîrîlă and Abbas Kareem Wanas. Symmetric toeplitz matrices for a new family of prestarlike functions. Symmetry, 14(7):1413, 2022.

[6] PL Duren. Univalent functions, vol. 259, 1983.

[7] Iason Efraimidis. A generalization of livingston’s coefficient inequalities for functions with positive real part. Journal of Mathematical Analysis and Applications, 435(1):369–379, 2016.
[8] RM Goel and Beant Singh Mehrok. A subclass of univalent functions. *Journal of the Australian Mathematical Society*, 35(1):1–17, 1983.

[9] Thomas H Macgregor. Functions whose derivative has a positive real part. *Transactions of the American Mathematical Society*, 104(3):532–537, 1962.

[10] Daud Mohamad. On a class of functions whose derivatives map the unit disc into a half plane. *Bulletin of The Malaysian Mathematical Sciences Society*, 23(2), 2000.

[11] Daud Mohamad and Nur Hazwani Aqilah Abdul Wahid. Bounds on Toeplitz determinant for starlike functions with respect to conjugate points. *International Journal of Analysis and Applications*, 19(3):477–493, 2021.

[12] V Radhika, S Sivasubramanian, G Murugusundaramoorthy, Jay M Jahangiri, et al. Toeplitz matrices whose elements are the coefficients of functions with bounded boundary rotation. *J. Complex Anal*, 4960704(4), 2016.

[13] Varadharajan Radhika, Jay M Jahangiri, Srikanth Sivasubramanian, and Gangadharan Murugusundaramoorthy. Toeplitz matrices whose elements are coefficients of Bazilevič functions. *Open Mathematics*, 16(1):1161–1169, 2018.

[14] C Ramachandran and S Annamalai. On Hankel and Toeplitz determinants for some special class of analytic functions involving conical domains defined by subordination. *Internat. J. Engrg. Res. Technol*, 5:553–561, 2016.

[15] C Ramachandran and D Kavitha. Toeplitz determinant for some subclasses of analytic functions. *Global Journal of Pure and Applied Mathematics*, 13(2):785–793, 2017.

[16] Lei Shi, Muhammad Arif, Mohsan Raza, and Muhammad Abbas. Hankel determinant containing logarithmic coefficients for bounded turning functions connected to a three-leaf-shaped domain. *Mathematics*, 10(16):2924, 2022.

[17] HERB Silverman and EM Silvia. On $\alpha$-close-to-convex functions. *Publ. Math. Debrecen*, 49(3-4):305–316, 1996.

[18] S Sivasubramanian, M Govindaraj, and G Murugusundaramoorthy. Toeplitz matrices whose elements are the coefficients of analytic functions belonging to certain conic domains. *Int. J. Pure Appl. Math.*, 109(10):39–49, 2016.

[19] Hari M Srivastava, Qazi Zahoor Ahmad, Nasir Khan, Nazar Khan, and Bilal Khan. Hankel and Toeplitz determinants for a subclass of q-starlike functions associated with a general conic domain. *Mathematics*, 7(2):181, 2019.

[20] Huo Tang, Shahid Khan, Saqib Hussain, and Nasir Khan. Hankel and Toeplitz determinant for a subclass of multivalent q-starlike functions of order $\alpha$. *AIMS Math*, 6:5421–5439, 2021.
[21] DK Thomas and S Abdul Halim. Retracted article: Toeplitz matrices whose elements are the coefficients of starlike and close-to-convex functions. *Bulletin of the Malaysian Mathematical Sciences Society*, 40(4):1781–1790, 2017.

[22] Nur Hazwani Aqilah Abdul Wahid and Daud Mohamad. Toeplitz determinant for a subclass of tilted starlike functions with respect to conjugate points. *Sains Malaysiana*, 50(12):3745–3751, 2021.

[23] Zhi-Gang Wang, Mohsan Raza, Muhammad Arif, and Khurshid Ahmad. On the third and fourth hankel determinants for a subclass of analytic functions. *Bulletin of the Malaysian Mathematical Sciences Society*, 45(1):323–359, 2022.

[24] Ke Ye and Lek-Heng Lim. Every matrix is a product of toeplitz matrices. *Foundations of Computational Mathematics*, 16(3):577–598, 2016.

[25] Hai-Yan Zhang, Rekha Srivastava, and Huo Tang. Third-order hankel and toeplitz determinants for starlike functions connected with the sine function. *Mathematics*, 7(5):404, 2019.

[26] Farah Zulfiqar, Sarfraz Nawaz Malik, Mohsan Raza, Md Ali, et al. Fourth-order hankel determinants and toeplitz determinants for convex functions connected with sine functions. *Journal of Mathematics*, 2022, 2022.