Sandwiching dense random regular graphs between binomial random graphs

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Abstract

Kim and Vu made the following conjecture (Advances in Mathematics, 2004): if \( d \gg \log n \), then the random \( d \)-regular graph \( G(n, d) \) can asymptotically almost surely be “sandwiched” between \( G(n, p_1) \) and \( G(n, p_2) \) where \( p_1 \) and \( p_2 \) are both \((1 + o(1))d/n\). They proved this conjecture for \( \log n \ll d \leq n^{1/3 - o(1)} \), with a defect in the sandwiching: \( G(n, d) \) contains \( G(n, p_1) \) perfectly, but is not completely contained in \( G(n, p_2) \). The embedding \( G(n, p_1) \subseteq G(n, d) \) was improved by Dudek, Frieze, Ruciński and Šileikis to \( d = o(n) \). In this paper, we prove Kim–Vu’s sandwich conjecture, with perfect containment on both sides, for all \( d \) where \( \min\{d, n - d\} \gg n/\sqrt{\log n} \). The sandwich theorem allows translation of many results from \( G(n, p) \) to \( G(n, d) \) such as Hamiltonicity, the chromatic number, the diameter, etc. It also allows translation of threshold functions of phase transitions from \( G(n, p) \) to bond percolation of \( G(n, d) \). In addition to sandwiching regular graphs, our results cover graphs whose degrees are asymptotically equal. The proofs rely on estimates for the probability of small subgraph appearances in a random factor of a pseudorandom graph, which is of independent interest.

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1 Introduction

Random graph theory is one of the most important subjects in modern graph theory. Besides the rich theory in its own field of study, random graphs have many connections and applications in the general area of combinatorics. Many existence results in graph theory are proved by using and modifying random graphs. Today, random graphs are widely used in computer science, engineering, physics and other branches of sciences.

There are many random graph models. The most classical models $G(n, p)$ and $G(n, m)$ were introduced by Erdős and Rényi [7, 8] more than half a century ago. The binomial model $G(n, p)$ retains each edge of the complete graph $K_n$ independently with probability $p$. The uniform model $G(n, m)$ is simply $G(n, p)$ conditioned on having exactly $m$ edges. In other words, $G(n, m)$ is the random graph on $n$ vertices and $m$ edges with the uniform distribution. These two models are the best studied and understood. The independence between the occurrence of the edges makes $G(n, p)$ a relatively easier model compared to many others, for analysing its properties and for analysing algorithms on $G(n, p)$. Some algorithms depend on the degrees of vertices, and unavoidably the algorithms need to “expose” the degrees of the vertices as the algorithms proceed. For instance, the peeling algorithm [9, 13] for obtaining the $k$-core of a graph repeatedly deletes a vertex whose degree is below $k$.

An important property of $G(n, p)$ and $G(n, m)$ is that, by conditioning on the vector of degrees of $G(n, p)$ or $G(n, m)$ being $d = (d_1, \ldots, d_n)^T$, with $m = \frac{1}{2} \sum_{i=1}^n d_i$, the resulting random graph is exactly $G(n, d)$, the uniformly random graph with given degree sequence $d$. For the special case where $d = (d, \ldots, d)^T$ for some constant $d$, that is the random $d$-regular graph, we simply write $G(n, d)$.

The model $G(n, d)$ is among the most important in the study of random graphs and large networks. It is often referred to as the Molloy–Reed model [21] in the network community. Unlike for $G(n, p)$, probabilities of events in $G(n, d)$ such as two vertices $u$ and $v$ being adjacent are highly non-trivial to compute. The most common methods of analysis of $G(n, d)$ are the configuration model [2] for constant or slowly growing degrees, the switching method [18] for degrees bounded by a small power of $n$, and the complex-analytic method [11, 19] for very high degrees; see also the detailed survey by Wormald [24].

Nevertheless, many questions that deserve an affirmative answer remain open for $G(n, d)$ because the methods listed above have severe restrictions. For instance, is $G(n, d)$ Hamiltonian? What is the chromatic number of $G(n, d)$? What is the connectivity of $G(n, d)$? Using highly non-trivial switching arguments and enumeration results for $d$-regular graphs, these particular questions were answered [4, 15] for $G(n, d)$. Using similar techniques it may be possible to work out the answers for the more general model $G(n, d)$. However, it will be desirable to have simpler approaches.

This is the motivation of the sandwich conjecture, proposed by Kim and Vu in 2004. They conjectured that for every $d \gg \log n$, the random $d$-regular graph can be
sandwiched between two binomial random graphs \( \mathcal{G}(n, p_1) \) and \( \mathcal{G}(n, p_2) \), the former with average degree slightly less than \( d \), and the latter with average degree slightly greater. The formal statement is as follows. Recall that a coupling of random variables \( Z_1, \ldots, Z_k \) is a random variable \((\hat{Z}_1, \ldots, \hat{Z}_k)\) whose marginal distributions coincide with the distributions of \( Z_1, \ldots, Z_k \), respectively. With slight abuse of notation, we use \((Z_1, \ldots, Z_k)\) as a coupling of \( Z_1, \ldots, Z_k \).

**Conjecture 1** (Sandwich Conjecture [14]) For \( d \gg \log n \), there are probabilities \( p_1 = (1 - o(1))d/n \) and \( p_2 = (1 + o(1))d/n \) and a coupling \((G^L, G, G^U)\) such that \( G^L \sim \mathcal{G}(n, p_1) \), \( G^U \sim \mathcal{G}(n, p_2) \), \( G \sim \mathcal{G}(n, d) \) and \( \mathbb{P}(G^L \subseteq G \subseteq G^U) = 1 - o(1) \).

The condition \( d \gg \log n \) in the conjecture is necessary. When \( p = O(\log n/n) \), there probably exist vertices in \( \mathcal{G}(n, p) \) whose degrees differ from \( pn \) by a constant factor. Therefore, Conjecture 1 cannot hold for this range of \( d \). For \( \log n \ll d \ll n^{1/3}/\log^2 n \), Kim and Vu proved a weakened version of the sandwich conjecture where \( G \subseteq G^U \) is replaced by a bound on \( \Delta(G \setminus G^U) \) (see the precise statement in [14, Theorem 2]).\(^1\) Note that this weakened sandwich theorem already allows direct translation of many results from \( \mathcal{G}(n, p) \) to \( \mathcal{G}(n, d) \), including all increasing graph properties such as Hamiltonicity.

Dudek, Frieze, Ruciński and Šileikis [6] improved one side of Kim and Vu’s result, \( G^L \subseteq G \), to cover all degrees \( d \) such that \( \log n \ll d \ll n \) and also extended it to the hypergraph setting. In particular, this new embedding theorem allows them to translate Hamiltonicity from binomial random hypergraphs to random regular hypergraphs.

In the graph case, the embedding result of [6] relies on an estimate for the probability of edge appearances in a random \( t \)-factor (spanning subgraph with degrees \( t \)) of a nearly complete graph \( S \), where \( t \) is near-regular and sparse. This can be done using a switching argument, which has already appeared in several enumeration works; see for example [18]. Extending the results of [6] to \( d = \Theta(n) \) requires new proof methods beyond switchings: one needs to consider the case when \( S \) is no longer a nearly complete graph, and components of \( t \) are all linear in \( n \).

An immediate corollary of the sandwich conjecture, if it were true, is that one can couple two random regular graphs \( G_1 \sim \mathcal{G}(n, d_1) \) and \( G_2 \sim \mathcal{G}(n, d_2) \) such that asymptotically almost surely (a.a.s.) \( G_1 \subseteq G_2 \), if \( d_2 \) is sufficiently greater than \( d_1 \). In fact we conjecture that such a coupling exists as long as \( d_2 \gg d_1 \). However, the weakened versions of the sandwich conjecture, as proved in [14] and [6], are not strong enough to imply the existence of such a coupling, even when \( d_2 \) is much greater than \( d_1 \).

**Conjecture 2** Let \( 0 \leq d_1 \leq d_2 \leq n - 1 \) be integers, other than \((d_1, d_2) = (1, 2)\) or \((d_1, d_2) = (n - 3, n - 2)\). Assume \( d_1 n \) and \( d_2 n \) are both even. Then there exists a coupling \((G_1, G_2)\) such that \( G_1 \sim \mathcal{G}(n, d_1) \), \( G_2 \sim \mathcal{G}(n, d_2) \), and \( \mathbb{P}(G_1 \subseteq G_2) = 1 - o(1) \).

**Remark 3** This conjecture or some variant of it has already been the subject of speculation and discussion in the community, but we have not found any written work

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\(^1\) Vu has confirmed that \( \Delta(G^U \setminus G) \) in their theorem is a typo for \( \Delta(G \setminus G^U) \).
about it. The case when $d_1 = 1$ and $3 \leq d_2 \leq n - 1$ is simple, since almost all
$d_2$-regular graphs have perfect matchings, which follows from them being at least
$(d_2 - 1)$-connected [4, 15]. Generate a random $d_2$-regular graph $G_2$. If $G_2$ has any
perfect matchings, select one at random; otherwise select a random $1$-regular graph.
By symmetry, this gives a random $1$-regular graph which is a subgraph of $G_2$ with
probability $1 - o(1)$.

The two binomial random graphs in Conjecture 1 differ by $o(d/n)$ in edge density.
This gap gives enough room to sandwich a random graph with more relaxed degree
sequences. We propose a stronger sandwich conjecture stated as Conjecture 4 below.

Given a vector $d = (d_1, \ldots, d_n)^T \in \mathbb{R}^n$, let $\text{rng}(d)$ stand for the difference between
the maximum and minimum components of $d$. Denoting $\Delta(d) := \max_j d_j$, we can
also write $\text{rng}(d) := \Delta(d) + \Delta(-d)$. If $d(G)$ is the degree sequence of a graph $G$,
we will also use the notation $\Delta(G) := \Delta(d(G))$.

**Definition 1** A sequence $d(n) \in \{0, \ldots, n - 1\}^n$ is called near-regular as $n \to \infty$ if
\[
\text{rng}(d(n)) = o(\Delta(d(n))) \quad \text{and} \quad \text{rng}(d(n)) = o(n - \Delta(d(n))).
\]

**Conjecture 4** Assume $d = d(n)$ is a near-regular degree sequence such that $\Delta(d) \gg
\log n$. Then, there are $p_1 = (1 - o(1))\Delta(d)/n$ and $p_2 = (1 + o(1))\Delta(d)/n$ and a
coupling $(G^L, G, G^U)$ such that $G^L \sim \mathcal{G}(n, p_1)$, $G^U \sim \mathcal{G}(n, p_2)$, $G \sim \mathcal{G}(n, d)$ and
$\mathbb{P}(G^L \subseteq G \subseteq G^U) = 1 - o(1)$.

We categorise the family of near-regular degree sequences into the following three
classes.

**Definition 2** Let $d$ be a near-regular degree sequence. We say $d$ is sparse if $\Delta(d) =
o(n)$; dense if $\min\{\Delta(d), n - \Delta(d)\} = \Theta(n)$; and co-sparse if $n - \Delta(d) = o(n)$.

In this paper, we confirm Conjecture 4 for all dense near-regular $d$, and we confirm
Conjecture 1 for all $d$ where $\min\{d, n - d\} \gg n/\sqrt{\log n}$. This proves the sandwich
conjecture by Kim and Vu for asymptotically almost all $d$. We will treat sparse and
co-sparse near-regular degree sequences in a subsequent paper, as the proof techniques
used for those ranges are very different from this work.

## 2 Main results

Throughout the paper we assume that $d$ is a realisable degree sequence, i.e. $\mathcal{G}(n, d)$
is nonempty. This necessarily requires that $d$ has nonnegative integer coordinates and
even sum. All asymptotics in the paper refer to $n \to \infty$, and there is an implicit
assumption that statements about functions of $n$ hold when $n$ is large enough. For two
sequences of real numbers $a_n$ and $b_n$, we say $a_n = o(b_n)$ if $b_n \neq 0$ eventually and
$\lim_{n \to \infty} a_n/b_n = 0$. We say $a_n = O(b_n)$ if there exists a constant $C > 0$ such that
$|a_n| \leq C |b_n|$ for all (large enough) $n$. We write $a_n = o(b_n)$ or $a_n = \Omega(b_n)$ if $a_n > 0$
always and $b_n = o(a_n)$ or $b_n = O(a_n)$, respectively. If both $a_n$ and $b_n$ are positive
sequences, we will also write $a_n \ll b_n$ if $a_n = o(b_n)$, and $a_n \gg b_n$ if $a_n = \omega(b_n)$.

Our contribution towards Conjectures 1 and 4 are given by the following theorems.
Theorem 5 Conjecture 4 is true for all dense near-regular degree sequences.

Theorem 6 Conjecture 1 is true for all $d$ where $\min\{d, n - d\} \gg n/\sqrt{\log n}$.

Theorem 6 directly implies a weaker version of Conjecture 2.

Corollary 1 There is a coupling $(G_{d_1}, G_{d_2})$ such that $G_{d_1} \sim \mathcal{G}(n, d_1), G_{d_2} \sim \mathcal{G}(n, d_2)$ and $\mathbb{P}(G_{d_1} \subseteq G_{d_2}) = 1 - o(1)$, if $d_1, n - d_2 \gg n/\sqrt{\log n}$ and $d_2/d_1 = 1 + \Omega(1)$.

This section is organised as follows. First, we discuss some important properties that immediately translate from $\mathcal{G}(n, p)$ to random graphs with given degrees by our sandwich theorems. Then, we show that Theorem 5 and Theorem 6 follow from a more general and more accurate theorem for embedding $\mathcal{G}(n, p)$ inside $\mathcal{G}(n, d)$ (see Theorem 8 in Sect. 2.2). The proof of the embedding theorem is long and technical so we postpone it till Sect. 4. Nevertheless, in Sect. 2.3, we state its key ingredient, which may be of independent interest: the probability estimate for a random factor in a pseudorandom graph to contain or avoid a given small set of edges.

2.1 Translation from $\mathcal{G}(n, p)$ to $\mathcal{G}(n, d)$

Our sandwich theorem allows translation of many results from binomial random graphs to random graphs with dense near-regular degree sequences. Some of the translations can already be obtained from a one-sided sandwich, e.g. the monotone properties. For instance, we can immediately transfer properties of Hamiltonicity or containment of other subgraphs from $\mathcal{G}(n, p)$ to $\mathcal{G}(n, d)$. Other translations require sandwiching on both sides. For example, we can translate graph parameters such as chromatic number, diameter, and independence number. We refer the reader to the conference version of this work [10, Section 3] for these applications. In this section, we only give one example showing how to translate threshold functions of phase transitions from $\mathcal{G}(n, p)$ to those of the random graph obtained by edge percolation on $\mathcal{G}(n, d)$.

A graph property $\Gamma$ has threshold $f(n)$ in $\mathcal{G}(n, p)$ if

$$\lim_{n \to \infty} \mathbb{P}(\mathcal{G}(n, p) \in \Gamma) = \begin{cases} 0, & \text{if } p \ll f(n), \\ 1, & \text{if } p \gg f(n). \end{cases}$$

We say $\Gamma$ has a sharp threshold $f(n)$ in $\mathcal{G}(n, p)$ if for every fixed $\varepsilon > 0$,

$$\lim_{n \to \infty} \mathbb{P}(\mathcal{G}(n, p) \in \Gamma) = \begin{cases} 0, & \text{if } p < (1 - \varepsilon)f(n), \\ 1, & \text{if } p > (1 + \varepsilon)f(n). \end{cases}$$

The concept of (sharp) threshold extends naturally to other random graph models such as $\mathcal{G}(n, m), \mathcal{G}(n, d)$ and $\mathcal{G}(n, d)$ where $d$ is near-regular.

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2 [10, Theorem 3.3] claimed the chromatic number of $\mathcal{G}(n, d)$ for both dense $d$ and sparse $d$. The dense case follows directly from the tight sandwich theorem in the current paper. The sparse cases uses a non-tight sandwich theorem together with some manipulations which were not correct as in [10]. We will fix this issue in a subsequent paper by proving a tight sandwich theorem for sparse $d$. 
Theorem 7 (Percolation on $G(n, d)$) Assume $d$ is near-regular and dense. Let $G \sim G(n, d)$ and $G_p$ be the subgraph of $G$ obtained by independently keeping each edge with probability $p$. Let $Q$ be a monotone property and let $\text{th}(Q)$ denote a (sharp) threshold function of $Q$ in $G(n, p)$. Then $(n/\Delta(d)) \cdot \text{th}(Q)$ is a (sharp) threshold function of $Q$ in $G_p$.

We give one example of Theorem 7. A giant component in $G(n, p)$ is a component of size linear in $n$. Determining the sharp threshold of the emergence of a giant component in $G(n, p)$ is a remarkable landmark result in random graph theory. The emergence threshold of a giant component in other random graph models has also been extensively studied. For instance, the emergence threshold of a giant component in $G_p$ is known to be $1/(d - 1)$ in the special case where $G \sim G(n, d)$ with $d \geq 3$, following from a sequence of results [15, 16, 22]. Theorem 7 extends this result to near-regular degree sequences where $\Delta(d) = \Omega(n)$.

Corollary 2 (Giant component) Assume $d$ is near-regular and $\Delta(d) = \Omega(n)$. The emergence of a giant component in $G_p$ has a sharp threshold $1/\Delta(d)$.

2.2 Embedding theorem

Our sandwich theorems are corollaries of the following more general theorem for embedding $G(n, p)$ inside $G(n, d)$. In fact, the embedding theorem provides better $p_1$ and $p_2$ and sharper bounds on the probability of $G_L \subseteq G \subseteq G_U$ than Theorems 5 and 6.

Theorem 8 (The embedding theorem) Let $d = d(n) \in \mathbb{N}^n$ be a degree sequence and $\xi = \xi(n) > 0$ be such that $\xi(n) = o(1)$. Denote $\Delta = \Delta(d)$. Assume $n \cdot \text{rng}(d) \leq \xi \Delta(n - \Delta)$ and $n - \Delta \gg \xi \Delta \gg n/\log n$. Then there exist $p = (1 - O(\xi))\Delta/n$, and a coupling $(G_L, G)$ with $G_L \sim G(n, p)$ and $G \sim G(n, d)$

$$P(G_L \subseteq G) = 1 - e^{-\Omega(\xi^4 \Delta)} = 1 - e^{-o(n/\log n)}.$$

We prove Theorem 8 in Sect. 4 using the coupling procedure described in Sect. 3. The proposition below shows that the probability bound of Theorem 8 is tight up to an additional three powers of $\log n$ in the exponent.

Lemma 1 Assume is d-regular, i.e. all components equal to d. Let $(G_L, G)$ be any coupling such that $G_L \sim G(n, p)$ and $G \sim G(n, d)$, where $p(n - 1) \leq d$. Then

$$P(G_L \nsubseteq G) = \Omega\left((\frac{p(n-1)}{d})^{d+d^{1/2}}\right).$$

In particular, if $p$ is defined as in Theorem 8, then

$$P(G_L \nsubseteq G) = e^{-O(\xi d)} = e^{-o(\xi^4 d \log^3 n)}.$$
Proof. We have

\[ 1 - \mathbb{P}(G^L \subseteq G) \geq \mathbb{P}_\mathcal{G}(n, p)(v_1 \text{ has degree greater than } d) = \mathbb{P}(\text{Bin}(n - 1, p) \geq d + 1). \]

The given bound follows from comparing values \{d + 1, \ldots, d + d^{1/2}\} of Bin\((n - 1, p)\) to the same values of Bin\((n - 1, d/(n - 1))\). For the second part, observe that the assumptions imply \(\xi \gg 1/\log n\).

Now we prove that Theorem 5 follows from Theorem 8.

Proof (of Theorem 5) Let \(d' = (n - 1)1 - d\). Since \(d\) is near-regular and dense, there exists \(\xi = o(1)\) which satisfies all the conditions in Theorem 8 for both \(d\) and \(d'\). Hence, by Theorem 8 applied to \(d\), there exists \(p_1 = (1 - o(1))\Delta(d)/n\) and a coupling \(\pi\) which a.a.s. embeds \(G(n, p_1)\) into \(G(d)\). Similarly, by Theorem 8 applied to \(d'\), there exists \(p_2 = (1 + o(1))\Delta(d)/n\) and a coupling \(\tilde{\pi}\) which a.a.s. embeds \(G(n, 1 - p_2)\) into \(G(d')\). Let \(\pi'\) be the coupling obtained by complementing each component of \(\tilde{\pi}\). Clearly, \(\pi'\) a.a.s. imbeds \(G(d)\) into \(G(n, p_2)\).

We can now stitch \(\pi\) and \(\pi'\) together to construct a coupling \((G^L, G, G^U)\), where

\[ G^L \sim \mathcal{G}(n, p_1), G \sim \mathcal{G}(n, d) \text{ and } G^U \sim \mathcal{G}(n, p_2). \]

First uniformly generate a graph \(G \in \mathcal{G}(n, d)\). Then, conditional on \(G\), generate \(G^L\) under \(\pi\) and generate \(G^U\) under \(\pi'\). This yields \((G^L, G, G^U)\) with the desired marginal distributions. Moreover, a.a.s.

\[ G^L \subseteq G \subseteq G^U. \]

Based on Theorem 8 we also establish the following result, which covers Theorem 6.

Theorem 9 Assume the degree sequence \(d\) is near-regular, with \(\Delta(d), n - \Delta(d) \gg n/\sqrt{\log n}\) and \(\text{rng}(d) = O(\Delta(d)/\log n)\). Then, Conjecture 4 holds for \(d\).

Proof. Let \(\xi = 1/\sqrt{\log n}\) and let \(d' := (n - 1)1 - d\) as in the proof of Theorem 5. Then the conditions of Theorem 8 are satisfied by both \(d\) and \(d'\). Hence, we have the embedding of \(\mathcal{G}(n, p)\) inside \(\mathcal{G}(n, d)\) where \(p = (1 - O(\xi))\Delta(d)/n = (1 - o(1))\Delta(d)/n\). We also have embedding of \(\mathcal{G}(n, p')\) inside \(\mathcal{G}(n, d')\) where

\[ p' = (1 - O(\xi))\frac{n - \Delta(d) + \text{rng}(d)}{n} \]

\[ = 1 - (1 + O(\xi))\frac{\Delta(d)}{n} + O\left(\frac{\xi + \text{rng}(d)}{n}\right) = 1 - (1 + o(1))\frac{\Delta(d)}{n}. \]

Complementing gives an embedding of \(\mathcal{G}(n, d)\) inside \(\mathcal{G}(n, 1 - p')\), where \(1 - p' = (1 + o(1))\Delta/n\) as required. Note that the error \(O(\xi + \text{rng}(d)/n)\) in the last equation is absorbed because of the condition on the range of \(\Delta(d)\). This proves that Conjecture 4 is true for near-regular \(d\) where \(\Delta(d), n - \Delta(d) \gg n/\sqrt{\log n}\) and \(\text{rng}(d) = O(\Delta(d)/\log n)\).

2.3 Forced and forbidden edges in a random factor

As explained later, see Question 1 in Sect. 3, a key step towards proving Theorem 8 is to estimate, to the desired accuracy, the edge probability in a random \(t\)-factor \(S_t\) of a graph \(S\), where \(t = (t_1, \ldots, t_n)^T\) is a degree sequence.
We will estimate the edge probabilities by enumerating $t$-factors of $S$, using a complex-analytic approach which is presented in detail in Sect. 5. Here, we just give a quick overview. Given $S$, the generating function for subgraphs of $S$ with given degrees is $\prod_{j,k \in S} (1 + z_j z_k)$. Using Cauchy’s integral formula, we find that the number $N(S, t)$ of $t$-factors of $S$ is given by

$$N(S, t) = \frac{1}{(2\pi i)^n} \oint \cdots \oint \prod_{j,k \in S} (1 + z_j z_k) \frac{dz_1 \cdots dz_n}{z_1^{t_1+1} \cdots z_n^{t_n+1}}.$$ 

We will derive an asymptotic expression of $N(S, t)$ using a multidimensional variant of the saddle-point method. The integral is split into two parts. The first part corresponds to the neighbourhood of saddle points. Using the Laplace approximation, we need to estimate the moment-generating function of a polynomial with complex coefficients of an $n$-dimensional Gaussian random vector. To do this, we apply the general theory based on complex martingales developed in [11]. The second part consists of the integral over the other regions and has a negligible contribution. See Sect. 5 and Theorem 11 for these calculations.

Estimating both parts of the integral is highly non-trivial and this analysis was previously done in the literature only for the case when $S$ is the complete graph $K_n$ or not far from it, see [1, 11, 19, 20]. Extending these results to a general graph $S$ requires significant improvements of known techniques. Our enumeration result (see Theorem 11) gives an asymptotic value of $N(S, t)$ for $S$ such that every pair of vertices have $\Theta(\Delta^2(S)/n)$ common neighbours and under some technical conditions on $t$.

We also investigate the connection between the random graph $S_t$ and the so-called $\beta$-model which belongs to the exponential family of random graphs. We show that the probability of containing/avoiding a prescribed small set of edges is asymptotically the same for both models; see Sect. 6 and Theorem 12.

We would like to note that, even for $S = K_n$, Theorem 11 and Theorem 12 extend previously known results. Here, we state a special case of Theorem 12 for the case when the degree sequence $t$ is approximately proportional to the degree sequence of $S$. Here, and throughout the paper, we use $\|\cdot\|_p$ for $p \in \{1, 2, \infty\}$ to denote the customary vector norms and their induced matrix norms.

**Theorem 10** Let $S$ be a graph with vertex set $[n]$ and degree sequence $(s_1, \ldots, s_n)$. For a degree sequence $t = (t_1, \ldots, t_n)$, let

$$\lambda := \frac{t_1 + \cdots + t_n}{s_1 + \cdots + s_n}.$$ 

Let $H^+$ and $H^-$ be disjoint subgraphs of $S$, and let $h$ be the degree sequence of $H^+ \cup H^-$. Assume that the following conditions hold:

(A1) for any two distinct vertices $j$ and $k$ and some constant $\gamma > 0$, we have

$$\gamma \Delta^2(S) \leq |\{\ell : j \ell \in S \text{ and } k \ell \in S\}| \leq \frac{\Delta^2(S)}{\gamma n};$$

(A2) $\lambda (1 - \lambda) \Delta(S) \gg \frac{n}{\log n}$.
(A3) \( \|t - \lambda s\|_\infty \|h\|_1 \ll \lambda (1 - \lambda) \Delta(S) \);
(A4) \( \|h\|_2^2 \ll \lambda (1 - \lambda) \Delta(S) \).

Let \( S_t \) be a random \( t \)-factor of \( S \). Then, for any \( \varepsilon > 0 \),
\[
\Pr(H^+ \subseteq S_t \text{ and } H^- \cap S_t = \emptyset) = \left(1 + O(n^{-1/2+\varepsilon} + \frac{\|t - \lambda s\|_\infty \|h\|_1 + \|h\|_2^2}{\lambda (1 - \lambda) \Delta(S)})\right) \lambda^{m(H^+)} (1 - \lambda)^{m(H^-)},
\]
where \( m(H^+) \) and \( m(H^-) \) denote the number of edges in \( H^+ \) and \( H^- \), respectively.

Theorem 10 will be sufficient for us to prove the embedding theorem. In fact, for Theorem 8, we only need to consider the case when \( H^+ \) is an edge and \( H^- \) is empty, for which (A4) is trivial; see Sect. 4. We prove Theorem 10 in Sect. 6.3.

### 3 Embedding \( \mathcal{G}(n, p) \) inside \( \mathcal{G}(n, d) \)

To prove our embedding theorem, we will use a procedure called \( \text{Coupling}() \) which constructs a joint distribution of \( (G^L, G) \) where \( G^L \subseteq G \) a.a.s. and their marginal distributions follow \( \mathcal{G}(n, p) \) and \( \mathcal{G}(n, d) \) respectively. The procedure is given in Fig. 1.

#### 3.1 The coupling procedure

Procedure \( \text{Coupling}() \) takes a graphical degree sequence \( d \), a positive integer \( \mathcal{I} \) and a positive real \( \zeta < 1 \) as an input, and outputs three random graphs \( G_\zeta, G, G_0 \), all on \([n]\), such that \( G \sim \mathcal{G}(n, d) \) and \( G_\zeta \subseteq G_0 \). Roughly speaking, the procedure constructs \( (G_{\zeta}^{(t)}, G^{(t)}, G_0^{(t)}) \) by sequentially adding edges to the three graphs, and \( G_{\zeta}^{(t)} \subseteq G_0^{(t)} \) is maintained up to step \( \mathcal{I} \). The outputs \( G_\zeta \) and \( G_0 \) of \( \text{Coupling}() \) will be \( G_\zeta^{(\mathcal{I})} \) and \( G_0^{(\mathcal{I})} \), ignoring some technicality. The output \( G \) will be a “proper” completion of \( G^{(\mathcal{I})} \) into a graph with degree sequence \( d \). For a careful choice of \( \mathcal{I} \) and \( \zeta \), procedure \( \text{Coupling}() \) typically produces an outcome that \( G_\zeta \subseteq G \) and \( G - G_0 \) is “small”. Moreover, if \( \mathcal{I} \) is chosen randomly according to a suitable distribution, which we specify later in this section, then \( G_\zeta \sim \mathcal{G}(n, p_\zeta) \) and \( G_0 \sim \mathcal{G}(n, p_0) \), where \( p_\zeta \approx p_0 \) for small \( \zeta > 0 \). (See the definition of \( p_\zeta \) in (2).) Even though we only need the coupling \( (G^L, G) \) with \( G^L = G_\zeta \) for our purposes, it will be convenient to include \( G_0 \) in our coupling construction in order to deduce certain properties of \( G \) required for our proofs.

In rare cases, if certain parameters become too large, \( \text{Coupling}() \) calls another procedure \( \text{IndSample}() \). Procedure \( \text{IndSample}() \) also generates three random graphs \( G_\zeta \sim \mathcal{G}(n, p_\zeta), G \sim \mathcal{G}(n, d) \) and \( G_0 \sim \mathcal{G}(n, p_0) \) but the relation \( G_\zeta \subseteq G \) is not a.a.s. guaranteed. In fact, \( G \) will be independent of \( (G_\zeta, G_0) \). The main challenge will be to show that the probability for \( \text{Coupling}() \) to call \( \text{IndSample}() \) is rather small.

If \( M \) is a multigraph, we write \( G \preceq M \) if \( G \) is the simple graph obtained by suppressing multiple edges in \( M \) into single edges. With a slight abuse of notation, we write \( jk \in \mathcal{G}(n, d) \) for the event that \( jk \) is an edge in a graph randomly chosen from...
Procedure Coupling($d, \mathcal{S}, \zeta$):
Let $M_\zeta^{(0)}$, $G^{(0)}$ and $M_0^{(0)}$ be the empty multigraphs on vertex set $[n]$.
For every $1 \leq i \leq \mathcal{S}$:

- Uniformly at random choose an edge $jk$ from $K_n$;
- If $jk \in G^{(i-1)}$ then
  - $M_0^{(i)} = M_0^{(i-1)} \cup \{jk\}$;
  - $M_\zeta^{(i)} = G^{(i-1)}$ with probability $\zeta$, or
  - $M_\zeta^{(i)} = M_\zeta^{(i-1)} \cup \{jk\}$ otherwise;
- If $jk \notin G^{(i-1)}$, define $\eta_{jk}^{(i)} = 1 - \max_{rg \in G^{(i-1)}} \frac{\mathbb{P}(\{jk\in \mathcal{G}(n,d)\}|G^{(i-1)})}{\mathbb{P}(rg \in \mathcal{G}(n,d)|G^{(i-1)})}$;
  - If $\eta_{jk}^{(i)} > \zeta$ then Return IndSample($d, M_\zeta^{(i-1)}, M_0^{(i-1)}, \mathcal{S}, G^{(i-1)}$);
  - Otherwise, generate $a \in [0,1]$ uniformly randomly;
    - If $a \in (\zeta, 1]$ then $G^{(i)} = G^{(i-1)} \cup \{jk\}$ and $M_\zeta^{(i)} = M_\zeta^{(i-1)} \cup \{jk\}$;
    - If $a \in [\eta_{jk}^{(i)}, \zeta]$ then $G^{(i)} = G^{(i-1)} \cup \{jk\}$ and $M_\zeta^{(i)} = M_\zeta^{(i-1)}$;
    - If $a \in [0, \eta_{jk}^{(i)}]$ then $G^{(i)} = G^{(i-1)}$ and $M_\zeta^{(i)} = M_\zeta^{(i-1)}$;

For $t \geq \mathcal{S} + 1$, while $G^{(i-1)}$ has fewer edges than $\mathcal{G}(n,d)$ repeat:
- Pick an edge $uv \notin G^{(i-1)}$ with probability proportional to $\mathbb{P}(uv \notin \mathcal{G}(n,d) | G^{(i-1)})$,
- $G^{(i)} = G^{(i-1)} \cup \{uv\}$;
- Assign $G = G^{(i)}$.
Return $(G_\zeta, G, G_0)$, where $G_\zeta < M_\zeta^{(\mathcal{S})}$ and $G_0 < M_0^{(\mathcal{S})}$.

Procedure IndSample($d, M_\zeta, M_0, 1, \mathcal{S}, \zeta$):
Let $M_\zeta^{(1-1)} = M_\zeta$ and $M_0^{(1-1)} = M_0$; and let $G$ be sampled from $\mathcal{G}(n,d)$.
For every $1 \leq \mathcal{S} \leq \mathcal{S}$:

- Uniformly at random choose an edge $jk$ from $K_n$;
- $M_0^{(i)} = M_0^{(i-1)} \cup \{jk\}$;
- $M_\zeta^{(i)} = M_\zeta^{(i-1)}$ with probability $\zeta$, or
- $M_\zeta^{(i)} = M_\zeta^{(i-1)} \cup \{jk\}$ otherwise;

Return $(G_\zeta, G, G_0)$ where $G_\zeta < M_\zeta^{(\mathcal{S})}$ and $G_0 < M_0^{(\mathcal{S})}$.

Fig. 1 Procedures Coupling() and IndSample()
Lemma 2 Let \( \mathcal{I} \sim \text{Po}(\mu) \) and \((G_\xi, G, G_0)\) be the output of Coupling\((d, \mathcal{I}, \xi)\). Then \(G_0 \sim \mathcal{G}(n, p_0)\) and \(G_\xi \sim \mathcal{G}(n, p_\xi)\), where
\[
p_0 := 1 - e^{-\mu/N} \quad \text{and} \quad p_\xi := 1 - e^{-(1-\xi)/N}.
\]

**Proof** By the definition of Coupling() and IndSample(), whether IndSample() is called or not, the construction for \(G_\xi\) and \(G_0\) lasts exactly \(\mathcal{I}\) steps. In each step \(1 \leq t \leq \mathcal{I}\), a uniformly random edge \(jk\) from \(K_n\) is chosen. Then \(jk\) is added to \(M_0(t)\) always, and \(jk\) is added to \(M_\xi(t)\) with probability \(1 - \xi\).

Let \(e_1, \ldots, e_N\) be an enumeration of the edges of \(K_n\). For \(1 \leq z \leq N\), let \(X_z\) denote the number of times that edge \(e_z\) is chosen during these \(\mathcal{I}\) iterations. Clearly,
\[
\mathbb{P}(X_z = 0) = \sum_{m=0}^{\infty} e^{-\mu} \frac{\mu^m}{m!} (1 - 1/N)^m = e^{-\mu + \mu(1-1/N)} = e^{-\mu/N}.
\]

Moreover, the probability generating function for the random vector \(X = (X_z)_{z \in [N]}\) is
\[
\sum_{j_1, \ldots, j_N} \mathbb{P}(X_1 = j_1, \ldots, X_N = j_N) x_1^{j_1} \cdots x_N^{j_N} = \sum_{m=0}^{\infty} e^{-\mu} \frac{\mu^m}{m!} \left( \frac{\sum_{1 \leq j \leq N} x_j}{N} \right)^m
\]
\[
= \exp \left( -\mu + \mu \left( \frac{\sum_{1 \leq j \leq N} x_j}{N} \right) \right) = \prod_{1 \geq j \geq N} \exp \left( -\mu/N + \frac{\mu x_j}{N} \right).
\]

This implies that the components of \(X\) are independent. Hence, each edge of \(K_n\) is included in \(G\) independently with probability \(\mathbb{P}(X_z \geq 1) = 1 - e^{-\mu/N}\). This verifies that \(G_0 \sim \mathcal{G}(n, p_0)\).

Next we consider the distribution of \(G_\xi\). By the definition of Coupling\((d, \mathcal{I}, \xi)\), for every \(1 \leq t \leq \mathcal{I}\), the chosen edge \(e_z\) is added to \(M_\xi(t)\) with probability \(1 - \xi\). Let \(Y_z\) denote the multiplicity of \(e_z\) in \(M_\xi(\mathcal{I})\). Observe that the distribution of \(Y = (Y_z)_{z \in [N]}\) is similar to the distribution of \(X\) but with \(\mathcal{I}\) replaced by \(\mathcal{I}' \sim \text{Bin}(\mathcal{I}, 1 - \xi)\). It is also straightforward to verify that \(\mathcal{I}' \sim \text{Po}(\lambda')\) where \(\lambda' = \mu(1 - \xi)\). Thus, we conclude that \(G_\xi \sim \mathcal{G}(n, p_\xi)\). \(\Box\)

If \(G \sim \mathcal{G}(n, d)\) and \(m \leq \frac{1}{2} \sum_j d_j = |E(G)|\), let \(\mathcal{G}(n, d, m)\) denote the probability space of all subgraphs of \(G\) containing exactly \(m\) edges with the uniform distribution. In the next lemma, we verify the marginal distribution of \(G^{(i)}\) during the coupling procedure. Define \(m^{(i)}\) to be the number of edges in \(G^{(i)}\).

**Lemma 3** Suppose IndSample() was not called during the first \(i\) iterations of procedure Coupling(). Then \(G^{(i)} \sim \mathcal{G}(n, d, m^{(i)})\).

**Proof** With a slight abuse of notation, let \(G^{(i)}\) be the graph where edges are labelled with \([m^{(i)}]\) in the order that they are added by Coupling(). We will prove by induction
that \( G^{(i)} \) has the same distribution as the graph obtained by uniformly labelling edges in \( \mathcal{G}(n, d, m^{(i)}) \) with \( [m^{(i)}] \). This is obviously true for \( i = 1 \).

Without loss of generality, assume \( G^{(t-1)} \) has \( m^{(i)} - 1 \) edges and has the claimed distribution, and assume that \( G^{(i)} \) contains \( m^{(i)} \) edges. Let \( \mathcal{L}(G^{(t-1)}) \) be the set of edge-labelled graphs with degree sequence \( d \) which contain \( G^{(t-1)} \) as an edge-labelled subgraph. For every \( jk \notin G^{(t-1)} \), let \( \mathcal{L}(G^{(t-1)}, jk) \) be the set of edge-labelled \( d \)-regular graphs in \( \mathcal{L}(G^{(t-1)}) \) which contains \( jk \) as an edge labelled with \([m^{(t)}] \).

Define \( \mathcal{U}(G^{(t-1)}) \) and \( \mathcal{U}(G^{(t-1)}, jk) \) similarly except that edges not in \( G^{(t-1)} \) are not labelled. Since every graph in \( \mathcal{U}(G^{(t-1)}, jk) \) corresponds to exactly \((M - m^{(i)})!\) edge-labelled graphs in \( \mathcal{L}(G^{(t-1)}, jk) \), and every graph in \( \mathcal{U}(G^{(t-1)}) \) corresponds to exactly \((M - m^{(i)} + 1)!\) edge-labelled graphs in \( \mathcal{U}(G^{(t-1)}) \), where \( M = \frac{1}{2} \sum_{j=1}^{n} d_j \), we have

\[
\frac{\left| \mathcal{U}(G^{(t-1)}, jk) \right|}{\left| \mathcal{U}(G^{(t-1)}) \right|} = (M - m^{(i)} + 1) \frac{\left| \mathcal{L}(G^{(t-1)}, jk) \right|}{\left| \mathcal{L}(G^{(t-1)}) \right|}.
\]

Since

\[
\frac{\left| \mathcal{U}(G^{(t-1)}, jk) \right|}{\left| \mathcal{U}(G^{(t-1)}) \right|} = \mathbb{P}(jk \in \mathcal{G}(n, d) \mid G^{(t-1)}),
\]

it follows that \( \frac{\left| \mathcal{L}(G^{(t-1)}, jk) \right|}{\left| \mathcal{L}(G^{(t-1)}) \right|} \) is proportional to the conditional probability \( \mathbb{P}(jk \in \mathcal{G}(n, d) \mid G^{(t-1)}) \). Hence, the random graph \( G^{(i)} \) also has the claimed distribution.

The above immediately implies the statement of the lemma for the non-edge-labelled \( G^{(i)} \), since there are exactly \( m^{(i)}! \) ways to label edges of \( G^{(i)} \) for any realisation of \( G^{(i)} \) with \( m^{(i)} \) edges.

Lemma 3 immediately yields the following corollary.

**Corollary 3** If \((G_\xi, G, G_0)\) is the output of \( \text{Coupling}(d, \mathcal{I}, \xi) \), then \( G \sim \mathcal{G}(n, d) \).

Thus, procedure \( \text{Coupling}(d, \mathcal{I}, \xi) \) with \( \mathcal{I} \sim \text{Po}(\mu) \) always produces a random triple of graphs with suitable marginal distributions. Next, we need to choose parameters \( \mu \) and \( \xi \) in such a way that \( p_\xi \) approximates the density of \( \mathcal{G}(n, d) \) reasonably well and the probability of \( G_\xi \not\subseteq G \) is small. Note that \( G_\xi \subseteq G \) could only be violated when \( \text{IndSample}(\iota) \) is called, in which case \( G_\xi \) and \( G \) are generated independently.

Define \( \mathcal{I}^* \) to be the value of \( \iota \) when \( \text{IndSample}(\iota) \) is called, otherwise \( \mathcal{I}^* = \mathcal{I} + 1 \). Then we have

\[
\mathbb{P}(G_\xi \not\subseteq G) \geq \mathbb{P}(\mathcal{I}^* \leq \mathcal{I}) = \mathbb{P}\left( \exists \iota \leq \mathcal{I}^* - 1 : \eta^{(\iota+1)}_{jk} > \xi \right) \\
\geq \mathbb{P}\left( \exists \iota \leq \mathcal{I}^* - 1 : \min_{jk \notin G^{(i)}} \mathbb{P}(jk \in \mathcal{G}(n, d)) < 1 - \xi \right). 
\]

(1)

For each \( 0 \leq \iota \leq \mathcal{I}^* - 1 \), define

\[
S^{(i)} := K_n - G^{(i)},
\]
and let $g^{(i)}$ be the degree sequence of $G^{(i)}$. (We use $A \setminus B$ for set subtraction. For simplicity we use $G - H$ to denote $E(G) \setminus E(H)$ for graphs $G$ and $H$ defined on the same set of vertices.) Denote by $H(t)$ the set of spanning subgraphs of a graph $H$ with degree sequence $t$. Thus, $K_n(d)$ is the set of graphs with degree sequence $d$, and $S^{(i)}(d - g^{(i)})$ is the set of graphs disjoint from $G^{(i)}$ whose union with $G^{(i)}$ is a graph in $\mathcal{G}(n, d)$. Note that in each step of the algorithm, a new edge $jk$ is added to $G^{(i)}$ only if $\mathbb{P}(jk \in G^{(i)} | G^{(i)}(\cdot - 1))$ is non-zero. Inductively that implies that the set of subgraphs of $S^{(i)}$ with degree sequence $d - g^{(i)}$ is never empty, for every $i$, as stated in the following observation.

**Observation 1** For $0 \leq i \leq I^* - 1$ in $\text{Coupling}()$, $S^{(i)}(d - g^{(i)}) \neq \emptyset$.

We find that

$$
\mathbb{P}(jk \in \mathcal{G}(n, d) | G^{(i)}) = \frac{|\{G \in K_n(d) : G^{(i)} \cup \{jk\} \subseteq G\}|}{|\{G \in K_n(d) : G^{(i)} \subseteq G\}|} = \frac{|\{G \in S^{(i)}(d - g^{(i))} : jk \in G\}|}{|S^{(i)}(d - g^{(i))}|},
$$

(2)

where the second equation above holds since the denominators are nonzero by Observation 1. Thus, (1) and (2) motivate the following question.

**Question 1** Let $S_t$ be a uniform random $t$-factor (spanning subgraph with degree sequence $t$) of a graph $S$. Under which assumptions on $S$ and $t$ can one guarantee that

$$
\frac{\mathbb{P}(z \in S_t)}{\mathbb{P}(z' \in S_t)} \approx 1
$$

for any two edges $z, z'$ of $S$?

Having an accurate estimate of the above probability ratio is crucial in our approach towards solving the sandwich conjecture, and tightening the density gap between the two binomial random graphs that sandwich $\mathcal{G}(n, d)$. Theorem 10 answers Question 1 for dense pseudorandom graph $S$ and dense near-regular $t$. Resolving the full sandwich conjecture requires solving Question 1 for pseudorandom $S$ with near-regular $t$ in all density regimes: $S$ can be sparse or dense and $t$ can be sparse or dense relative to $S$. We will address that in the subsequent paper. Some partial solutions have been presented in the conference version [10].

4 Proof of theorem 8

We continue using all notations introduced in Sect. 3. In this paper, all graphs are defined on the vertex set $[n]$. When we do algebraic operations on graphs, we always operate on the edge sets of the graphs. In particular, for graphs $G$ and $H$, $G - H$ denotes $E(G) \setminus E(H)$, $G \cup H$ denotes $E(G) \cup E(H)$, and $G \cap H$ denotes $E(G) \cap E(H)$.

In this section we show how to choose $\mu$ and $\zeta$ such that procedure $\text{Coupling}()$ produces a desirable outcome. As explained before (in particular, see (1) and (2)) it
is important that all edges of $S^{(i)} = K_n - G^{(i)}$ are approximately equally likely to appear in the uniform random subgraph of $S^{(i)}$ with degree sequence $d - g^{(i)}$, where $g^{(i)}$ denotes the degree sequence of $G^{(i)}$. We will employ Theorem 10 for this purpose.

4.1 Preliminaries

We will need the following bounds.

Lemma 4 Let $Y \sim \text{Bin}(K, p)$ for some positive integer $K$ and $p \in [0, 1]$.

(a) For any $\varepsilon \geq 0$, we have $\mathbb{P}(|Y - pK| \geq \varepsilon pK) \leq 2e^{-\frac{\varepsilon^2}{2\pi} pK}$.

(b) If $p = m/K$ for some integer $m \in (0, K)$, then $\mathbb{P}(Y = m) \geq \frac{1}{2}(p(1-p)K)^{-1/2}$.

(c) Let $\mathcal{J} \sim \text{Po}(\mu)$ for some $\mu > 0$. Then, for any $\varepsilon \geq 0$, $\mathbb{P}(\mathcal{J} \geq \mu(1+\varepsilon)) \leq e^{-\frac{\varepsilon^2}{2\pi}\mu}$.

Proof Bound (a) follows combining the upper and lower Chernoff bounds in multiplicative form. For (b), we use the bounds $\sqrt{2\pi k} \left(\begin{array}{c} k \\ \varepsilon \end{array}\right)^k \leq k! \leq \sqrt{2\pi k} \left(\begin{array}{c} k \\ \varepsilon \end{array}\right)^k e^{1/12}$ to estimate the factorials in the expression $\mathbb{P}(Y = m) = \frac{K!}{m!} \frac{m^m}{(K-m)^{K-m}}$. Bound (c) comes from approximating $\text{Po}(\mu)$ with $\text{Bin}(K, \mu/K)$ as $K \to \infty$ and using the upper Chernoff bound.

The next lemma will assist us in verifying assumption (A1) in Theorem 10.

Lemma 5 Let $S \sim \mathcal{G}(n, m)$ for some integer $m \gg n^{3/2}(\log n)^{1/2}$. Then, with probability $1 - e^{-\Omega(m^2/n^3)}$, assumption (A1) of Theorem 10 is satisfied with $\gamma = \frac{1}{8}$.

Proof Let $\tilde{S} \sim \mathcal{G}(n, p)$ where $p = m/N$. Observe that the degrees of $\tilde{S}$ are distributed according to $\text{Bin}(n-1, p)$. Also, the number of common neighbours of any two vertices in $\tilde{S}$ is distributed according $\text{Bin}(n-2, p^2)$. Observing that $np^2 \gg \log n$ and combining Lemma 4(a) and the union bound, we get that, with probability $e^{-\Omega(p^2n)}$,

$$\frac{\Delta(\tilde{S})}{pn} \in \left[\frac{1}{2}, 2\right] \quad \text{and} \quad \frac{|\{\ell : j \ell \in \tilde{S} \text{ and } k \ell \in \tilde{S}\}|}{p^2n} \in \left[\frac{1}{2}, 2\right]$$

for all pairs of distinct vertices $j$ and $k$. This implies that

$$\frac{\Delta^2(\tilde{S})}{8n} \leq |\{\ell : j \ell \in \tilde{S} \text{ and } k \ell \in \tilde{S}\}| \leq 8\frac{\Delta^2(\tilde{S})}{n}.$$

Note that $S$ has the same distribution as $\tilde{S}$ conditioned on the event that $\tilde{S}$ has exactly $m$ edges. From Lemma 4(b), we know that $\mathbb{P}(|E(\tilde{S})| = m) = \Omega(m^{-1/2})$. Then observing that $e^{-\Omega(p^2n)}/\mathbb{P}(|E(\tilde{H})| = m) = e^{-\Omega(m^2/n^3)}$ completes the proof.

4.2 Estimates for $S^{(i)}$ and $g^{(i)}$

Recall that

$$N = \binom{n}{2}, \quad M = \frac{1}{2} \sum_{j=1}^{n} d_j.$$
Lemma 4 is sufficient to extract some information about the density of $S^{(i)}$ and the sequence $d - g^{(i)}$, as described in the following lemma. Recall the definition of $p_0$ and $p_\xi$ from Lemma 2 and that $m^{(i)}$ denotes the number of edges in $G^{(i)}$. Define

$$p^{(i)} := (M - m^{(i)})/M.$$ 

**Lemma 6** Let $\xi \in (0, \frac{1}{3})$ be such $\Delta = \Delta(d) \gg \xi^{-4} \log n$. Suppose $n - \Delta \gg \xi \Delta$. Take $\mathcal{I} \sim \text{Po} (\mu)$, where $\mu$ is such that $p_0 = 1 - e^{-\mu/N} \leq (1 - \xi) M/N$.

Suppose $0 \leq i \leq \mathcal{I}^* - 1$. Then,

(a) $p^{(i)} \geq \xi/2$ with probability $1 - e^{-\Omega(\xi^2 M)}$;
(b) $\|d - g^{(i)} - p^{(i)} d\|_\infty \leq \min\{\xi p^{(i)} \Delta, \xi (n - \Delta)\}$ with probability $1 - e^{-\Omega(\xi^4 \Delta)}$.

**Proof** By the assumption that $\text{IndSample}()$ was not called during the first $\mathcal{I}$ steps, and using Lemma 2, we have that $G^{(i)} \subseteq G_0 \sim G(n, p_0)$. Therefore, $m^{(i)} = |E(G^{(i)})| \leq |E(G_0)| \sim \text{Bin}(N, p_0)$. Applying Lemma 4(a), we find that

$$\mathbb{P}(p^{(i)} \leq \xi/2) = \mathbb{P}(M - m^{(i)} \leq \xi M/2) = e^{-\Omega(\xi^2 M)}.$$ 

Since $e^{-\Omega(\xi^2 M)} = e^{-\Omega(\xi^4 \Delta)}$ we can proceed conditioned on the event that $p^{(i)} \geq \xi/2$. Take $G \sim G(n, d)$ and let $h = (h_1, \ldots, h_n)$ denote the degree sequence of the random graph $G_{p^{(i)}}$ obtained by independently keeping every edge from $G$ with probability $p^{(i)}$. By Lemma 3, the sequence $d - g^{(i)}$ has exactly the same distribution as $h$ conditioned on the event $|E(G_{p^{(i)}})| = M - m^{(i)}$, therefore

$$\mathbb{P}(\|h - p^{(i)} d\|_\infty \geq \xi p^{(i)} \Delta) \leq \frac{\mathbb{P}(\|h - p^{(i)} d\|_\infty \geq \xi p^{(i)} \Delta)}{\mathbb{P}(|E(G_{p^{(i)})})| = M - m^{(i)})}.$$ 

Observing $h_j \sim \text{Bin}(d_j, p^{(i)})$ and using Lemma 4(a), we find that

$$\mathbb{P}(\|h - p^{(i)} d\|_\infty \geq \xi p^{(i)} \Delta) \leq 2^n \sum_{j=1}^n \exp \left( - \frac{\xi \Delta}{2d_j + \xi \Delta} \xi p^{(i)} \Delta \right) = ne^{-\Omega(\xi^2 p^{(i)} \Delta)} = e^{-\Omega(\xi^4 \Delta)}.$$ 

If $n - \Delta > p^{(i)} \Delta$ then

$$\mathbb{P}(\|h - p^{(i)} d\|_\infty \geq \xi (n - \Delta)) \leq \mathbb{P}(\|h - p^{(i)} d\|_\infty \geq \xi p^{(i)} \Delta) = e^{-\Omega(\xi^4 \Delta)}.$$
Otherwise, if \( n - \Delta \leq p^{(i)} \Delta \) then, using Lemma 4(a) and the assumption \( n - \Delta \gg \xi \Delta \), we find that

\[
P(\| h - p^{(i)} d \|_\infty \geq \xi (n - \Delta)) = ne^{-\Omega\left(\frac{\xi^2 (n-\Delta)^2}{p^{(i)} \Delta}\right)} = ne^{-\Omega(\xi^4 \Delta)} = e^{-\Omega(\xi^4 \Delta)}.
\]

Finally, Lemma 4(b) gives a polynomial lower bound on \( P(| E(G_{p^{(i)}}) | = M - m^{(i)}) \), which is absorbed by the main error term by virtue of the assumption \( \Delta(d) \gg \xi^{-4} \log n \). This completes the proof. \( \square \)

Alas, there is not much structural information available about the graphs \( S^{(i)} \). In fact, by virtue of Lemma 3, such questions are similar in some sense to investigating the model \( G(n, d) \) that is the problem we started with. Nevertheless, it turns out that the following trivial observation will be sufficient for our purposes:

\[
K_n - G^{(i)}_0 \subseteq S^{(i)} \subseteq K_n - G^{(i)}_{\xi}, \quad \text{for } i \leq \mathcal{I}^* - 1
\]

where

\[
G^{(i)}_0 \triangleq M^{(i)}_0 \quad \text{and} \quad G^{(i)}_{\xi} \triangleq M^{(i)}_{\xi}.
\]

**Lemma 7** Assume \( i \leq \mathcal{I}^* - 1 \). Let \( m^{(i)}_0 = |E(G^{(i)}_0)| \) and \( m^{(i)}_{\xi} = |E(G^{(i)}_{\xi})| \). Then

\[
G^{(i)}_0 \sim \mathcal{G}(n, m^{(i)}_0), \quad \text{and} \quad G^{(i)}_{\xi} \sim \mathcal{G}(n, m^{(i)}_{\xi}).
\]

If \( \zeta = o(1) \) and \( i \zeta = o(N) \) then we have

\[
(1 - \zeta - 3i \zeta / N)(N - m^{(i)}_0) \leq N - m^{(i)}_0 \leq N - m^{(i)}_{\xi} \leq (1 + \zeta + 3i \zeta / N)(N - m^{(i)})
\]

with probability at least \( 1 - e^{-\Omega(\zeta^2 (N-M)^2/i)} \).

**Proof** The distributions of \( G^{(i)}_0 \) and \( G^{(i)}_{\xi} \) follow directly from the definition. For the second part, it is sufficient to bound \( P(m^{(i)}_0 - m^{(i)}_{\xi} \geq (\zeta + 3i \zeta / N)(N - m^{(i)}) \) because

\[
N - m^{(i)}_0 \leq N - m^{(i)} \leq N - m^{(i)}_{\xi}.
\]

By the construction of \( G^{(i)}_0 \) and \( G^{(i)}_{\xi} \), we have that

\[
\mathbb{E}\left( m^{(i)}_0 - m^{(i)}_{\xi} \right) = N \left( 1 - \frac{1 - \zeta}{N} \right)^i - N \left( 1 - \frac{1}{N} \right)^i
\]

\[
= \left( N - \mathbb{E}m^{(i)}_0 \right) \left( (1 + \zeta / (N - 1))^i - 1 \right)
\]

\[
\leq \left( N - \mathbb{E}m^{(i)}_0 \right) 2i \zeta / N.
\]

\( \square \) Springer
Note that $m^{(i)}_0$ and $m^{(i)}_\xi$ are functions of $2i$ independent random variables (corresponding for the edge choices and rejections). The change of any of these random variables affects the values of $m^{(i)}_0$ and $m^{(i)}_\xi$ by at most 1. Using McDiarmid’s concentration inequality [17], we get that

$$
P\left(\left|m^{(i)}_0 - \mathbb{E}m^{(i)}_0\right| \geq \xi(N - M)/2\right) \leq e^{-\Omega(n^2(N-M)^2/i)},$$

$$
P\left(\left|m^{(i)}_\xi - \mathbb{E}m^{(i)}_\xi\right| \geq \xi(N - M)/2\right) \leq e^{-\Omega(n^2(N-M)^2/i)}.$$  

(4)

Using $\xi = o(1)$ and $\iota \xi = o(N)$, the inequalities $\left|m^{(i)}_0 - \mathbb{E}m^{(i)}_0\right| \leq \xi(N - M)/2$, $\left|m^{(i)}_\xi - \mathbb{E}m^{(i)}_\xi\right| \leq \xi(N - M)/2$, and $m^{(i)} \leq M$ together with (3) imply that

$$N - m^{(i)} = (1 + o(1))\left(N - \mathbb{E}m^{(i)}_0\right) \geq \frac{2}{3}\left(N - \mathbb{E}m^{(i)}_0\right) \geq \frac{N}{3\iota\xi}\mathbb{E}\left(m^{(i)}_0 - m^{(i)}_\xi\right).$$

Thus, by (4), we get

$$
P\left(m^{(i)}_0 - m^{(i)}_\xi \geq (\xi + 3\iota\xi/N)(N - m^{(i)}_\xi)\right) \leq e^{-\Omega(n^2(N-M)^2/i)},$$

completing the proof.

Lemma 7 implies that $N - m^{(i)}_0, N - m^{(i)}_\xi = (1 + o(1))|E(S^{(i)})|$ with high probability provided $\iota \xi \ll N$. This enables us to derive all the necessary structural properties about $S^{(i)}$ from the well-studied model $G(n, m)$.

4.3 Specifying $\xi$ and $\mu$

Take $\mathcal{I} \sim \text{Po}(\mu)$, where $\mu$ is the unique solution of

$$
(1 - \xi)\frac{M}{N} = p_0 = 1 - e^{-\mu/N}.
$$

(5)

Let $\xi = C\xi$ for some sufficiently large constant $C > 0$ (which depends only on the implicit constant in $O(\cdot)$ of Theorem 10 with $\gamma = \frac{1}{9}$ and $\varepsilon = \frac{1}{4}$).

4.4 Completing the proof of theorem 8

First, by the assumptions, observe that

$$
\xi^4 \Delta \geq \frac{\xi^4 A^4}{n^4} \gg \frac{n}{(\log n)^4}.
$$
Thus, it is sufficient to prove the assertion with probability $1 - e^{-\Omega(\xi^4 \Delta)}$.

We prove that if $\text{IndSample}()$ was not called during the first $\iota$ steps of $\text{Coupling}(d, \mathcal{I}, \zeta)$ then the probability that it is called in step $\iota + 1$ is $e^{-\Omega(\xi^4 \Delta)}$. Then our assertion holds by taking the union bound over the $\mathcal{I}$ steps which, with probability at least $1 - e^{-\Omega(\xi^4 \Delta)}$, is bounded by $n^2$. Suppose $\text{IndSample}()$ was not called during the first $\iota$ steps of $\text{Coupling}(d, \mathcal{I}, \zeta)$. To bound the probability that it is called at the next iteration, we use Theorem 10 with

$$S := \mathcal{S}^{(\iota)}, \quad t := d - g^{(\iota)}, \quad H^+ := \{jk\}, \quad H^- := \emptyset.$$  

By Observation 1, the set of $t$-factors of $S$ is not empty. By the assumptions of Theorem 8, for all $j$,

$$\Delta \geq d_j \geq \Delta - \frac{\xi (n-\Delta)}{n}. \quad (6)$$

Using Lemma 6, we get that, with probability $1 - e^{-\Omega(\xi^4 \Delta)}$,

$$p^{(\iota)} \geq \frac{\xi}{2} \quad \text{and} \quad t_j = p^{(\iota)} \Delta + O(\xi) \min\{p^{(\iota)} \Delta, n - \Delta\}. \quad (7)$$

Let $s$ denote the degree sequence of $\mathcal{S}^{(\iota)}$ and $\lambda = \frac{t_1 + \ldots + t_n}{s_1 + \ldots + s_n}$. Then, by (6) and the two bounds of $t_j$ in (7),

$$s_j = n - 1 + t_j - d_j = (n - \Delta + p^{(\iota)} \Delta) + O(\xi) \left( \min\{p^{(\iota)} \Delta, n - \Delta\} + \frac{\Delta (n-\Delta)}{n} \right).$$

Consequently, letting $\tilde{t} = \|t\|_1/n$ and $\tilde{s} = \|s\|_1/n$ we have

$$t_j - \lambda s_j = t_j - \frac{\tilde{t}}{\tilde{s}} s_j = (t_j - \tilde{t}) - \frac{\tilde{t}}{\tilde{s}} (s_j - \tilde{s}) = O\left(|t_j - \tilde{t}| + \lambda |s_j - \tilde{s}|\right).$$

Using the two bounds for $t_j$ in (7) and the corresponding bounds for $s_j$, we have

$$t_j - \lambda s_j = O(\xi) \left( \min\{p^{(\iota)} \Delta, n - \Delta\} + \frac{\lambda \Delta (n-\Delta)}{n} \right), \quad \text{for } j \in [n]. \quad (8)$$

Note that

$$\lambda = \frac{p^{(\iota)} M}{N-M+p^{(\iota)} M} = (1 + o(1)) \frac{p^{(\iota)} \Delta}{p^{(\iota)} \Delta + n - \Delta}. $$

Therefore,

$$\frac{\lambda \Delta (n-\Delta)}{n} = O(\min\{p^{(\iota)} \Delta, n - \Delta\}). \quad (9)$$

Combining the bounds above, we get that

$$\lambda (1 - \lambda) \Delta (\mathcal{S}^{(\iota)}) = \frac{p^{(\iota)} M(N-M)}{(N-M+p^{(\iota)} M)^2} \Delta (\mathcal{S}^{(\iota)}) \geq \frac{2 p^{(\iota)} M(N-M)}{n(N-M+p^{(\iota)} M)} = (1 + o(1)) \frac{2 p^{(\iota)} (n-\Delta)}{p^{(\iota)} \Delta + n - \Delta}.$$
Next, we prove that
\[
\frac{\|t - \lambda s\|_\infty}{\lambda(1 - \lambda)\Delta(S^{(i)})} = O(\xi), \quad \frac{n/\log n}{\lambda(1 - \lambda)\Delta(S^{(i)})} = o(1). \tag{10}
\]
For the first inequality in (10), note that
\[
\frac{2p^{(i)}\Delta(n - \Delta)}{p^{(i)}\Delta + n - \Delta} \geq \min\{p^{(i)}\Delta, n - \Delta\} = \Omega\left(\xi^{-1}\|t - \lambda s\|_{\infty}\right), \tag{11}
\]
where the equality in (11) holds by (8) and (9). The second equality in (10) follows by (11), the bound \(p^{(i)} \geq \xi/2\) from (7) and the theorem assumption that \(n - \Delta \gg \xi \Delta \gg n/\log n\). It follows from (10) that \(\lambda(1 - \lambda)\Delta(S^{(i)}) \gg \|t - \lambda s\|_\infty + n/\log n\), and thus assumptions (A2) and (A3) of Theorem 10 are satisfied. Assumption (A4) is also immediate since \(H = H^+ \cup H^-\) consists of one edge.

Next, observe that \(1 - p_0 \geq \xi\), so \(\mu \leq N \log \frac{1}{\xi}\). From Lemma 4(c) we get that
\[
P\left(\mathcal{I} > 2N \log \frac{1}{\xi}\right) = e^{\Omega(N \log \frac{1}{\xi})} = e^{\Omega(\xi^{4}\Delta)}. \tag{12}
\]
Then, given that \(\mathcal{I} \leq 2N \log \frac{1}{\xi}\), using \(\iota \leq \mathcal{I}^* - 1 \leq \mathcal{I}\) and \(\zeta = O(\xi)\) from its definition, we get that \(\iota \zeta = O(N \xi \log \frac{1}{\xi}) \ll N\). For each \(\iota \leq \mathcal{I}^* - 1\), by Lemma 7, we get that
\[
N - m_0^{(i)} = (1 + o(1))(N - m^{(i)}), \quad N - m_\zeta^{(i)} = (1 + o(1))(N - m^{(i)})
\]
with probability at least \(1 - e^{-\Omega(\xi^{2}(N-M)^{2}/\iota)}\). Combining Lemma 5, the assumption \(n - \Delta \gg \xi \Delta\), using the monotonicity of the number of common neighbours, we find that, with probability at least
\[
1 - e^{-\Omega(\xi^{4}\Delta)} + e^{-\Omega(\xi^{2}(N-M)^{2}/\iota)} - e^{-\Omega((N-m)^{2}/n^{3})}
\]
\[
= 1 - e^{-\Omega(\xi^{4}\Delta)} - e^{-\Omega(\xi^{2}n^{(n-\Delta)^{2}/N \log \frac{1}{\xi}})} - e^{-\Omega((n-\Delta)^{2}/n)}
\]
\[
= 1 - e^{-\Omega(\xi^{4}\Delta)} - e^{-\Omega(\xi^{3}n\Delta/\log \frac{1}{\xi})} - e^{-\Omega(\xi \Delta(n-\Delta)/n)} = 1 - e^{-\Omega(\xi^{4}\Delta)},
\]
assumption (A1) of Theorem 10 holds for \(S = S^{(i)}\) with \(\gamma = 1/9\).

Observing \(\xi \gg (\log n)^{-1}\) by the assumptions, and applying Theorem 10 with \(\varepsilon = 1/4\) and \(\gamma = 1/9\), and by (10), we get that, with probability \(1 - e^{-\Omega(\xi^{4}\Delta)}\)
\[
\frac{\mathbb{P}(jk \in \mathcal{G}(n,d)\mid G^{(i)})}{\mathbb{P}(j^*k^* \in \mathcal{G}(n,d)\mid G^{(i)})} = 1 + O\left(n^{-1/4} + \frac{\|t - \lambda s\|_{\infty}}{\lambda(1 - \lambda)\Delta(S^{(i)})}\right) = 1 + O(\xi) > 1 - \xi
\]
for any \(jk, j^*k^* \notin G^{(i)}\), where the last inequality holds by choosing sufficiently large \(C\) in the definition of \(\zeta\). Applying the union bound for all such \(jk, j^*k^*\) we get that the probability that \(\text{IndSample}(\cdot)\) is called at step \(\iota + 1\) is \(e^{-\Omega(\xi^{4}\Delta)}\).
Using (1) and (12), we conclude that procedure Coupling\((d, I, \xi)\) produces a “bad” output \((G_\xi, G, G_0)\) with probability

\[ P(G_\xi \not\subseteq G) = O(N \log \frac{1}{\xi}) e^{-\Omega(\xi^4 \Delta)} = e^{-\Omega(\xi^4 \Delta)}. \]

To complete the proof we take \((G^L, G) = (G_\xi, G)\) and \(p = p_\xi\), recall that \(G \sim \mathcal{G}(n, d)\) by Corollary 3, and \(G_\xi \sim \mathcal{G}(n, p_\xi)\) by Lemma 2, where

\[
p_\xi = 1 - e^{-\mu(1-\xi)/N} = 1 - e^{-\mu/N} + e^{-\mu/N} (1 - e^{\mu \xi/N})
= p_0 + O\left(\xi (1 - p_0) \log(1 - p_0)\right) = (1 - O(\xi)) p_0 = (1 - O(\xi)) \Delta/n.
\]

The last equation follows by (5) and the assumption that \(d\) is near-regular. \(\Box\)

5 Enumeration of factors

In this section we establish an asymptotic formula for the number of factors (subgraphs with given degree sequence) of a graph in the dense case.

Let \(S\) be a simple graph. We start from the observation that \(\prod_{jk \in S} (1 + z_j z_k)\) is the generating function for subgraphs of \(S\) with powers of \(z_1, \ldots, z_n\) corresponding to degrees. In particular, the number \(N(S, t)\) of \(t\)-factors of \(S\) is given by

\[ N(S, t) = [z_1^{t_1} \cdots z_n^{t_n}] \prod_{jk \in S} (1 + z_j z_k), \]

where \([\cdot]\) denotes coefficient extraction. Using Cauchy’s integral formula, it follows that

\[ N(S, t) = \frac{1}{(2\pi i)^n} \oint \cdots \oint \frac{\prod_{jk \in S} (1 + z_j z_k)}{z_1^{t_1+1} \cdots z_n^{t_n+1}} dz_1 \cdots dz_n. \]

Let

\[ U_n(\rho) = \{\theta = (\theta_1, \ldots, \theta_n) \in \mathbb{R}^n : \|\theta\|_{\infty} \leq \rho\}. \]

Substituting \(z_j = e^{\beta_j + i \theta_j}\), we get that

\[
N(S, t) = \frac{1}{(2\pi)^n} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \frac{\prod_{jk \in S} (1 + e^{\beta_j + \beta_k + i(\theta_j + \theta_k)})}{e^{\sum_{j=1}^{n} (\theta_j + \theta_k)}} d\theta_1 \cdots d\theta_n
= \frac{\prod_{jk \in S} (1 + e^{\beta_j + \beta_k})}{(2\pi)^n e^{\sum_{j=1}^{n} \theta_j}} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \frac{\prod_{jk \in S} (1 + e^{\beta_j + \beta_k + i(\theta_j + \theta_k)})}{e^{\sum_{j=1}^{n} \theta_j}} d\theta_1 \cdots d\theta_n
= \frac{\prod_{jk \in S} (1 + e^{\beta_j + \beta_k})}{(2\pi)^n e^{\sum_{j=1}^{n} \theta_j}} \int_{U_n(\pi)} F_{S, t}(\theta) d\theta,
\]

where \(\mathcal{S}\) is Springer.
where
\[ F_{S,t}(\theta) := \prod_{jk \in S} \frac{(1+\lambda_{jk}(e^{i(\theta_j+\theta_k)}-1))}{e^{\sum_{j=1}^{n} t_j \theta_j}} \]

and
\[ \lambda_{jk} = \lambda_{jk}(\beta) := \frac{e^{\beta_j+\beta_k}}{1+e^{\beta_j+\beta_k}}, \text{ for } jk \in S. \tag{14} \]

The choice of parameters \( \beta = (\beta_1, \ldots, \beta_n) \) will be specified later.

The values \( \lambda_{jk} \) defined in (14) have an interesting property: if we consider a random subgraph \( S(\lambda_{jk}) \) of \( S \) with independent adjacencies where, for each \( jk \in S \), the probability that vertices \( j \) and \( k \) are adjacent in \( S(\lambda_{jk}) \) equals \( \lambda_{jk} \), then the probability of each outcome depends only on its degree sequence \( t = (t_1, \ldots, t_n) \). In other words, the conditional distribution of \( S(\lambda_{jk}) \) with respect to given \( t \) is uniform. The random model of \( S(\lambda_{jk}) \) is referred as the \( \beta \)-model and it is a special case of the exponential family of random graphs, see [3, 11] for more details. A further connection between \( S(\lambda_{jk}) \) and \( S_t \) is established in Sect. 6.

The exact value of the integral (13) can be found very rarely. Instead, we will approximate it. The complex-analytical approach consists of the following steps:

(i) estimate the contribution of critical regions around concentration points, where the integrand achieves its maximum value,

(ii) show that other regions give a negligible contribution.

The maximum absolute value of \( F_{S,t}(\theta) \) is 1. It is achieved at points \((0, \ldots, 0)\) and \((\pm\pi, \ldots, \pm\pi)\). If \( S \) does not contain a bipartite component then \(|F_{S,t}(\theta)|\) is strictly less than 1 at any other point of \( U_n(\pi) \) because there will be at least one pair \( jk \in S \) such that \( e^{i(\theta_j+\theta_k)} \neq 1 \). Since \( t \) is a degree sequence, we have that \( t_1 + \cdots + t_n \) is even. Then the contributions of neighbourhoods of \((0, \ldots, 0)\) and \((\pm\pi, \ldots, \pm\pi)\) to the integral (13) are identical because \( F_{S,t}(\theta) \) is 2\(\pi\)-periodic with respect to each component of \( \theta \) and

\[ F_{S,t}(\theta_1 + \pi, \ldots, \theta_n + \pi) = e^{i(t_1+\cdots+t_n)\pi} F_{S,t}(\theta_1, \ldots, \theta_n) = F_{S,t}(\theta_1, \ldots, \theta_n). \tag{15} \]

Thus, we can focus on estimates around the origin and then multiply by 2.

By Taylor’s theorem, for \( a \in [0, 1] \) and \( x \in [-\pi/4, \pi/4] \), we have
\[
1 + a(e^{ix} - 1) = \exp \left( iax - \frac{1}{2}a(1-a)x^2 - \frac{1}{6}a(1-a)(1-2a)x^3 + \frac{1}{24}a(1-a)(1-6a+6a^2)x^4 + O(x^5) \right).
\]

Using this to expand the multipliers of \( F_{S,t}(\theta) \), we find that
\[
F_{S,t}(\theta) = \exp \left( -i \sum_{j=1}^{n} \theta_j t_j + i \sum_{jk \in S} \lambda_{jk}(\theta_j + \theta_k) - \theta^T Q \theta + u(\theta) - i v(\theta) + O(\|\theta\|^5 |E(S)|) \right), \tag{16}
\]
where the \( n \times n \) symmetric matrix \( Q \) is defined by

\[
\theta^T Q \theta = \frac{1}{2} \sum_{jk \in S} \lambda_{jk} (1 - \lambda_{jk}) (\theta_j + \theta_k)^2
\]

and the multivariable polynomials \( u \) and \( v \) are defined by

\[
u(\theta) := \frac{1}{24} \sum_{jk \in S} \lambda_{jk} (1 - \lambda_{jk}) (1 - 6 \lambda_{jk} + 6 \lambda_{jk}^2) (\theta_j + \theta_k)^4,
\]

\[
u(\theta) := \frac{1}{6} \sum_{jk \in S} \lambda_{jk} (1 - \lambda_{jk}) (1 - 2 \lambda_{jk}) (\theta_j + \theta_k)^3.
\]

Observe that \( \theta^T Q \theta \geq 0 \), so \( Q \) is a positive semidefinite matrix. Moreover, it is positive definite if \( S \) does not contain a bipartite component.

The optimal choice for \( \beta \) is such that the linear part in (16) disappears, which corresponds to the case when our contours in the complex plane pass through the saddle point. Thus, we get the following system of equations:

\[
t_j = \sum_{k : jk \in S} \lambda_{jk} = \sum_{k : jk \in S} \frac{e^{\beta_j + \beta_k}}{1 + e^{\beta_j + \beta_k}} \quad \text{for all } 1 \leq j \leq n.
\]

For the case \( S = K_n \), the existence and the uniqueness of the solution was studied in [1, 3, 23]: the necessary and sufficient condition is that \( t \) lies in the interior of the polytope defined by the Erdős-Gallai inequalities. When \( S \) is the complete graph, it is also known that system (19) is equivalent to (i) maximisation of the likelihood with respect to the parameters of the \( \beta \)-model given observations of the degrees (ii) finding the random model with independent adjacencies and given expected degrees that maximises the entropy. Unfortunately, analogs of these results are not available for general \( S \) even though the methods used in the literature will certainly carry over. Since such results are not needed for our purposes here, we leave these questions for a subsequent paper.

Denote

\[
\lambda := \frac{\sum_{jk \in S} \lambda_{jk}}{|E(S)|}
\]

If system (19) holds then we have \( \lambda = \frac{t_1 + \cdots + t_n}{2|E(S)|} \), which is the relative density of a \( t \)-factor in \( S \). We are ready to state our main result of this section.

**Theorem 11** Let \( \varepsilon, \gamma \) and \( c \) be fixed positive constants. Suppose a graph \( S \) on \( n \) vertices and a degree sequence \( t \) satisfy the following assumptions:

(B1) for any two distinct vertices \( j \) and \( k \), we have

\[
\frac{\gamma \Delta^2(S)}{n} \leq |\{ \ell : j \ell \in S \text{ and } k \ell \in S \}| \leq \frac{\Delta^2(S)}{\gamma n};
\]

(B2) there exists a solution \( \beta \) of system (19) such that \( \text{rng}(\beta) \leq c \);
Let $X$ be a random variable with the normal density $\pi^{-n/2}|Q|^{1/2}e^{-x^TQx}$. Then,

$$N(S, t) = \frac{2\prod_{j,k \in S}(1+e^{\beta_j+\beta_k})}{(4\pi)^{n/2}|Q|^{1/2}} \prod_{j=1}^{n} e^{\beta_j} \exp\left(\mathbb{E}u(X) - \frac{1}{2}\mathbb{E}v^2(X) + O(n^{-1/2+\varepsilon})\right),$$

where the constant implicit in $O()$ depends on $\gamma$, $\varepsilon$ and $c$ only.

There is a vast literature on asymptotic enumeration of dense subgraphs with given degrees in the case when $S$ is the complete graph or not far from it, see, for example, [1, 11, 19, 20] and references therein. An important advantage of Theorem 11 with respect to the previous results is that it allows $S$ to be essentially different from $K_n$ and it holds for a very wide range of degrees. Theorem 11 follows immediately from Eqs. (13), (15), Lemma 9 and Corollary 4.

5.1 The integral in the critical regions

For given $S$ and $t$, denote

$$\Lambda := \lambda(1 - \lambda) \quad \text{and} \quad \Delta := \Delta(S).$$

In the following, we always assume that $\Lambda \Delta \gg n / \log n$ which is the assumption (B3) of Theorem 11. We also assume that (19) is satisfied. Let $\varepsilon$ be a fixed positive constant required to be sufficiently small in several places of the argument. Define

$$\eta := \frac{n^\varepsilon}{\Lambda \Delta} = o(1).$$

Given $x \in \mathbb{R}$, define

$$|x|_{2\pi} := \min\{|y| : y \equiv x \mod 2\pi\}.$$

It is easily seen that $| \cdot |_{2\pi}$ is a seminorm on $\mathbb{R}$ that induces a norm on $\mathbb{R}/(2\pi)$, the real numbers modulo $2\pi$. Our critical regions are

$$\mathcal{B}_0 := U_n(\eta) \quad \text{and} \quad \mathcal{B}_\pi := \{\theta \in \mathbb{R}^n : |\theta_j - \pi|_{2\pi} \leq \eta \text{ for all } j\}.$$

As explained above (see (15)), the contributions of these two regions to the integral in (13) are identical so we can focus on $\mathcal{B}_0$. From (16), we have

$$\int_{\mathcal{B}_0} F_{S,t}(\theta) \, d\theta = \int_{U_n(\eta)} e^{-\theta^TQ\theta + u(\theta) - iv(\theta) + h(\theta)} \, d\theta,$$

where $h(\theta) = O(n^{-1/2+6\varepsilon})$ uniformly for $\theta \in \mathcal{B}_0$. A general theory on the estimation of such integrals was developed in [11], based on the second-order approximation of
complex martingales. We will apply the tools from [11] here and, for the reader’s convenience, also quote them in the appendix, see Section A.2.

We will need the following bounds.

**Lemma 8** Let $\lambda$ be defined in (20). If $\text{rng}(\beta) \geq c$ for some fixed $c > 0$, then

(a) uniformly over all $j k \in S$, $\lambda_{j k} = \Theta(\lambda)$ and $1 - \lambda_{j k} = \Theta(1 - \lambda)$.

Furthermore, suppose $\Delta = \Omega(n^{1/2})$ and assumption (B1) of Theorem 11 holds. Then $Q$ is positive definite and the following hold.

(b) If $Q^{-1} = (\sigma_{j k})$, then $\sigma_{j k} = \begin{cases} \Theta \left( \frac{1}{\Lambda \Delta} \right) , & \text{if } j = k ; \\ O \left( \frac{1}{\Lambda \Delta^2} \right) , & \text{if } j k \in S ; \\ O \left( \frac{1}{\Lambda \Delta n} \right) , & \text{otherwise} \end{cases}$.

(c) There exists a real matrix $T$ such that $T^T Q T = I$ and

$$\| T \|_1 , \| T \|_\infty = O \left( (\Lambda \Delta)^{-1/2} \right) , \quad \| T^{-1} \|_1 , \| T^{-1} \|_\infty = O \left( (\Lambda \Delta)^{1/2} \right).$$

**Proof** Observe that $1 \leq \frac{1 + e^y}{1 + e^x} \leq e^{y-x}$ for any real $x \leq y$. Since all $\beta_j + \beta_k$ and $\beta_j' + \beta_k'$ are at most $2c$ apart, this implies that $\frac{\lambda_{j k}}{\lambda_{j' k'}} = \Theta(1)$ and $\frac{1 - \lambda_{j k}}{1 - \lambda_{j' k'}} = \Theta(1)$ for all $j k , j' k' \in S$. Recalling the definition (20), we have proved (a).

Note that assumption (B1) of Theorem 11 implies that $S$ is a connected non-bipartite graph. Thus, $Q$ is positive definite. Parts (b) and (c) follow from Lemma 20 (see appendix) applied to the scaled matrix $Q / \Lambda$.

We are ready to establish asymptotic estimates for the critical region $\mathcal{B}_0$. Note that in the next lemma we allow the components of $t$ to be non-integers.

**Lemma 9** Suppose a graph $S$ and a real vector $t \in \mathbb{R}^n$ satisfy assumptions (B1)–(B3) of Theorem 11. Then, for any sufficiently small fixed $\varepsilon > 0$, we have

$$\int_{U_n(\eta)} F_{S,t}(\theta) \, d\theta = \frac{\pi^{n/2}}{|Q|^{1/2}} \exp \left( \mathbb{E} u(X) - \frac{1}{2} \mathbb{E} v^2(X) + O(n^{-1/2 + 13\varepsilon}) \right),$$

where $X$ is a random vector in $\mathbb{R}^n$ with the normal density $\pi^{-n/2} |Q|^{1/2} e^{-x^T Q x}$. Furthermore,

$$\mathbb{E} u(X) = O \left( \frac{n}{\Lambda \Delta} \right), \quad \mathbb{E} v^2(X) = O \left( \frac{n}{\Lambda \Delta} \right),$$

and for any fixed $c > 0$

$$\int_{U_n(c\eta)} |F_{S,t}(\theta)| \, d\theta = \frac{\pi^{n/2}}{|Q|^{1/2}} e^{O \left( \frac{n}{\Lambda \Delta} \right)}.$$
**Proof** The proof is based on [11, Theorem 4.4] which is quoted as Theorem 13 for the reader’s convenience. Let

$$\Omega := U_n(\eta), \quad f(\theta) := u(\theta) - iv(\theta), \quad g(\theta) := u(\theta).$$

From Lemma 8, we know that $Q$ is positive definite. Let $T$ be the matrix from Lemma 8(c). Define

$$\rho_1 := \|T\|_\infty^{-1}\eta, \quad \rho_2 := \|T^{-1}\|_\infty\eta.$$

Then we have $U_n(\rho_1) \subseteq T^{-1}(\Omega) \subseteq U_n(\rho_2)$ and

$$\rho_2 \geq \rho_1 = \Omega\left(\frac{n^\varepsilon}{(\Lambda\Delta)^{1/2}}(\Lambda\Delta)^{1/2}\right) = \Omega(n^\varepsilon).$$

Thus, $\rho_1$ and $\rho_2$ satisfy assumption (a) of Theorem 13. Similarly, observe $\rho_2 = O(n^\varepsilon)$.

Next, we estimate the partial derivatives of $f(x)$. Recalling the definitions of $u$ and $v$ from (18) and using Lemma 8(a), we get that, provided $\|\theta\|_\infty \leq 1$

$$\frac{\partial f}{\partial \theta_j}(\theta) = \frac{1}{2} \sum_{k:jk \in S} \lambda_{jk}(1 - \lambda_{jk})(1 - 6\lambda_{jk} + 6\lambda^2_{jk})(\theta_j + \theta_k)^3$$

$$-\frac{i}{2} \sum_{k:jk \in S} \lambda_{jk}(1 - \lambda_{jk})(1 - 2\lambda_{jk})(\theta_j + \theta_k)^2 = O(\Lambda\Delta\|\theta\|_\infty^2),$$

and, if $jk \in S$,

$$\frac{\partial^2 f}{\partial \theta_j \partial \theta_k}(\theta) = \frac{1}{2} \lambda_{jk}(1 - \lambda_{jk})(1 - 6\lambda_{jk} + 6\lambda^2_{jk})(\theta_j + \theta_k)^2$$

$$-i\lambda_{jk}(1 - \lambda_{jk})(1 - 2\lambda_{jk})(\theta_j + \theta_k) = O(\Lambda\|\theta\|_\infty).$$

Again using Lemma 8(c), we find that assumption (b) of Theorem 13 holds with $\phi_1 = n^{-1/6+4\varepsilon}$. Exactly the same calculation shows that assumption (c)(ii) holds with $\phi_2 = n^{-1/6+4\varepsilon}$. Assumption (d) also holds because $u$ and $v$ are polynomials. Applying Theorem 13 to the integral of (21), we obtain that

$$\int_{U_n(\eta)} F_{S,t}(\theta) \, d\theta = (1 + K)\frac{\pi^{n/2}}{|Q|^{1/2}} \exp\left(\mathbb{E}f(X) + \frac{1}{2}\mathbb{E}(f(X) - \mathbb{E}f(X))^2\right), \quad (22)$$

where $K = O(n^{-1/2+12\varepsilon})e^{\frac{1}{2}\text{Var}v(X)}$. Similarly, using (16) and Theorem 13, we get

$$\int_{U_n(c\eta)} |F_{S,t}(\theta)| \, d\theta = (1 + O(n^{-1/2+6\varepsilon})) \int_{U_n(c\eta)} e^{-\theta^TQ\theta + u(\theta)} \, d\theta$$

$$= (1 + O(n^{-1/2+12\varepsilon}))\frac{\pi^{n/2}}{|Q|^{1/2}} \exp\left(\mathbb{E}u(X) - \frac{1}{2}\text{Var}(u(X))\right).$$

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Next, we need to estimate some moments of \( u(X) \) and \( v(X) \). Let \( \Sigma = (\sigma_{jk, \ell m}) \) denote the covariance matrix of the variables \( X_j + X_k \) for \( jk \in S \):

\[
\sigma_{jk, \ell m} := \text{Cov}(X_j + X_k, X_\ell + X_m). \tag{23}
\]

Since \( X \) is gaussian with density \( \pi^{-n/2}|Q|^{1/2}e^{-x^TQx} \), the covariances \( \text{Cov}(X_j, X_k) \) equal the corresponding entries of \( (2Q)^{-1} \). Using the bounds of Lemma 8(b), we find that

\[
\sigma_{jk, \ell m} = \begin{cases} 
O\left(\frac{1}{\Lambda\Delta}\right), & \text{if } \{j, k\} \cap \{\ell, m\} \neq \emptyset; \\
O\left(\frac{1}{\Lambda^{2}}\right), & \text{if } \{j, k\} \cap \{\ell, m\} = \emptyset \text{ and } \{j \ell, jm, k \ell, km\} \cap S \neq \emptyset; \\
O\left(\frac{1}{n\Lambda\Delta}\right), & \text{otherwise}.
\end{cases} \tag{24}
\]

The expectation of a polynomial of odd degree is zero (due to the symmetry of the distribution) so \( \text{Cov}(u(X), v(X)) = \mathbb{E}v(X) = 0 \). The following are special cases of Isserlis’ theorem (see [12]), which is also known as Wick’s formula in quantum field theory:

\[
\mathbb{E}(X_j + X_k)^4 = 3\sigma_{jk, jjk}, \\
\mathbb{E}(X_j + X_k)^6 = 15\sigma_{jk, jjk}^3, \\
\mathbb{E}(X_j + X_k)^3(X_\ell + X_m)^3 = 9\sigma_{jk, jjk} \sigma_{\ell m, \ell m} \sigma_{jk, \ell m} + 6\sigma_{jk, \ell m}^3, \\
\mathbb{E}(X_j + X_k)^4(X_\ell + X_m)^4 = 9\sigma_{jk, jjk}^2 \sigma_{\ell m, \ell m}^2 + 72\sigma_{jk, jjk} \sigma_{\ell m, \ell m} \sigma_{jk, \ell m}^2 + 24\sigma_{jk, \ell m}^4.
\]

Recalling (18) and using (24), we obtain that

\[
\mathbb{E}u(X) = \frac{1}{8} \sum_{jk \in S} \lambda_{jk}(1 - \lambda_{jk})(1 - 6\lambda_{jk} + 6\lambda_{jk}^2)\sigma_{jk, jjk}^2 = O\left(\frac{n}{\Lambda\Delta}\right). \tag{25}
\]

Similarly as above, we derive that

\[
\text{Var} v(X) = \mathbb{E}v^2(X) = O\left(\Lambda^2 \sum_{jk \in S} \sum_{\ell m \in S} (|\sigma_{jk, jjk} \sigma_{\ell m, \ell m} \sigma_{jk, \ell m}| + |\sigma_{jk, \ell m}|)^2\right)
\]

\[
= O\left(\Lambda^2 \sum_{jk \in S} \sum_{\ell m \in S} |\sigma_{jk, \ell m}| \right) \left(\frac{n\Lambda^2}{\Lambda\Delta^2} + \frac{n^2\Lambda^2}{\Lambda^2\Delta} \right) = O\left(\frac{n}{\Lambda\Delta}\right) \tag{26}
\]

and

\[
\text{Var} u(X) = \mathbb{E}u^2(X) - (\mathbb{E}u(X))^2
\]

\[
= O\left(\Lambda^2 \sum_{jk \in S} \sum_{\ell m \in S} (|\sigma_{jk, jjk} \sigma_{\ell m, \ell m} \sigma_{jk, \ell m}|^2 + \sigma_{jk, \ell m}^4)\right).
\]
\[
= O\left( \Lambda^2 \sum_{j \in S} \sum_{\ell \in S} |\sigma_{j,\ell}|^2 \right) = O\left( \frac{1}{\Lambda^2} \left( \frac{n \Lambda^2}{(\Lambda \Delta)^2} + \frac{n \Lambda^3}{(\Lambda \Delta)^2} + \frac{n^2 \Lambda^2}{(n \Lambda \Delta)^2} \right) = o\left( \frac{\log^2 n}{n} \right),
\]
noting that the leading term containing \(\sigma_{j,k}^2\) appears in both \(E u^2(X)\) and \((\mathbb{E} u(X))^2\) and gets cancelled from the subtraction. Substituting these bounds into (22) and bounding \(e^{\frac{1}{2} \text{Var} \, v(X)} = e^{o(\log n)} = n^{o(1)}\), the proof is complete. \(\Box\)

5.2 Estimates outside of the critical regions

In this section, we show that the contribution to the integral (13) of the remaining region \(B = U_n(\pi) - B_0 - B_\pi\) is negligible, where the critical regions \(B_0\) and \(B_\pi\) are defined in Sect. 5.1. Observe that

\[
|F_{S,t}(\theta)| = \prod_{j \in S} \left| 1 + \lambda_{j,k} (e^{i(\theta_j + \theta_k)} - 1) \right|
\]
depends on \(S\) and \((\lambda_{j,k})\) only but does not depend on \(t\). To bound the factors of \(|F_{S,t}(\theta)|\), we use the following inequality, whose uninteresting proof we omit.

Lemma 10 For \(x \in \mathbb{R}\) and \(a \in [0, 1]\), we have \(|1 + a(e^{ix} - 1)| \leq e^{-\frac{1}{2} a(1-a)x^2} \). Throughout this section, including the lemma statements, we always assume that the assumptions of Theorem 11 hold. Recall that

\[
B_0 = U_n(\eta), \quad \eta = \frac{n^\varepsilon}{(\Lambda \Delta)^{1/2}}.
\]

Lemma 8(a,b) implies that all the eigenvalues of \(Q\) are \(\Theta(\Lambda \Delta)\) (by bounding the 1-norms of \(Q\) and \(Q^{-1}\)). From Lemma 9, we find that

\[
J_0 := \int_{B_0} F_{S,t}(\theta) \, d\theta = \pi^{n/2} |Q|^{-1/2} e^{O\left( \frac{n}{\Lambda \Delta} \right)} \geq \exp\left( -\frac{1}{2} n \log n + O(n) \right). \quad (27)
\]

As a first step, we demonstrate as negligible the domain where many components of \(\theta \in U_n(\pi)\) lie sufficiently far from 0 and \(\pm \pi\). Define

\[
B' := \{ \theta \in U_n(\pi) : \text{more than } n^{1-\varepsilon} \text{ components } \theta_j \text{ satisfy } \eta/2 \leq |\theta_j|_{2\pi} \leq \pi - \eta/2 \}.
\]

The following lemma depends on a technical lemma (Lemma 18) which we present in the appendix.
Lemma 11  We have

\[ \int_{\mathcal{B}'} |F_{S,t}(\theta)| \, d\theta = e^{-\Omega(n^{1+\varepsilon})} J_0. \]

Proof  Without loss of generality, at least \( \frac{1}{2} n^{1-\varepsilon} \) components \( \theta_j \) lie in \([\eta/2, \pi - \eta/2]\). Denote \( U = \{ j : \theta_j \in [\eta/2, \pi - \eta/2]\} \). We estimate the number \( N_T(U) \) of triangles \( \{j, k, \ell\} \) (i.e. \( jk, j\ell, k\ell \in S \)) such that \( \{j, k, \ell\} \cap U \neq \emptyset \). Using Lemma 18(a), we find that the degree of any vertex of \( U \) is at least \( \gamma \Delta \). For any \( jk \in S \) and \( \{j, k\} \cap U \neq \emptyset \) there are at least \( \frac{\gamma \Delta^2}{n} \) common neighbours each of which gives rise to a triangle contributing to \( N_T(U) \). Since every triangle is counted at most 3 times, we get that

\[ N_T(U) \geq \frac{\gamma \Delta |U|}{2}. \quad \frac{\gamma \Delta^2}{3n} = \frac{\gamma^2 \Delta^3 |U|}{6n}. \]

For each such triangle \( \{j, k, \ell\} \) that \( j \in U \), observe that

\[ |\theta_j + \theta_k|_{2\pi} + |\theta_k + \theta_\ell|_{2\pi} + |\theta_\ell + \theta_j|_{2\pi} \geq |\theta_j + \theta_k - \theta_\ell - \theta_j|_{2\pi} \geq \eta. \]

Therefore, we can mark one edge \( j'k' \) from this triangle such that \( |\theta_j + \theta_k|_{2\pi} \geq \eta/3 \). Repeating this argument for all such triangles and observing that any edge is present in at most \( \frac{\Delta^2}{\gamma n} \) triangles, we show that at least \( \gamma^3 \Delta |U| / 6 \) edges were marked. Using Lemma 8(a) and Lemma 10, we get that

\[ |F_{S,t}(\theta)| \leq e^{-\Omega(\Delta \Delta |U| \eta^2)} = e^{-\Omega(n^{1+\varepsilon})}. \]

Multiplying by the volume of \( \mathcal{B}' \), which is less than \((2\pi)^n\), and comparing with (27), completes the proof. \( \square \)

If Lemma 11 doesn’t apply, we have at least \( n - \frac{1}{2} n^{1-\varepsilon} \) components of \( \theta \) lying in neighbourhoods of 0 and \( \pm \pi \). Next we will use a similar argument to show that most of these components lie in one of those two intervals (on a circle). Define

\[ \mathcal{B}'' := \{ \theta \in U_n(\pi) \setminus \mathcal{B} : |\theta_j| \leq \eta/2 \text{ holds for more than } n^{2\varepsilon} \text{ components } \theta_j \text{ and } |\theta_j - \pi|_{2\pi} \leq \eta/2 \text{ holds for more than } n^{2\varepsilon} \text{ components } \theta_j \}. \]

Lemma 12  We have

\[ \int_{\mathcal{B}''} |F_{S,t}(\theta)| \, d\theta = e^{-\Omega(n^{1+\varepsilon})} J_0. \]

Proof  Let \( U_1 = \{ j : |\theta_j| \leq \eta/2 \} \) and \( U_2 = \{ j : |\theta_j - \pi|_{2\pi} \leq \eta/2 \} \). Since \( \theta \notin \mathcal{B}' \), we have \( |U_1| + |U_2| \geq n - \frac{1}{2} n^{1-\varepsilon} \). For \( j \in U_1, k \in U_2 \) and any \( \ell \) such that \( j\ell, k\ell \in S \), we have

\[ |\theta_j + \theta_\ell|_{2\pi} + |\theta_k + \theta_\ell|_{2\pi} \geq |\theta_j + \theta_\ell - \theta_k - \theta_\ell|_{2\pi} \geq \pi - \eta. \]
Thus, we can mark some \(j'k' \in \{j \ell, k \ell\}\) that \(|\theta_j + \theta_k|_{2\pi} = \Omega(1)\). By the assumptions, the number of choices for \((j, k, \ell)\) is at least \(|U_1||U_2|^{\nu \Delta^2}/n^2\). Dividing by \(2\Delta\) to compensate for over-counting, we get that at least \(|U_1||U_2|^{\nu \Delta^2}/2n\) edges were marked. Using Lemma 8(a) and Lemma 10, we find that

\[
|F_{S,t}(\theta)| = e^{-\Omega(|U_1||U_2|\Delta/n)} = e^{-\Omega(n^{1+2\epsilon}/\log n)} = e^{-\Omega(n^{1+\epsilon})}.
\]

The proof now follows the same line as in the previous lemma. \(\square\)

Since adding \(\pi\) to each component is a symmetry, see (15), we can now assume that at least \(n - n^{1-\epsilon}\) components of \(\theta\) lie in \([-\eta/2, \eta/2]\). If \(\theta \notin B_0\) then we should have some components \(|\theta_j| > \eta\). Let \(B(m)\) denote the region of \(\theta \in B \setminus (B' \cup B'')\) such that exactly \(m\) components of \(\theta\) lie outside of \([-\eta, \eta]\), where \(1 \leq m \leq n^{1-\epsilon}\). Let

\[
J(m) = \int_{B(m)} |F_{S,t}(\theta)| \ d\theta.
\]

For notational simplicity, we first prove a bound for the integral over the region \(B^*(m) \subset B(m)\), where the set of \(m\) components of \(\theta\) lying outside of \([-\eta, \eta]\) is exactly \(\{\theta_1, \ldots, \theta_m\}\). Our bound will be actually independent of this choice of \(m\) components so then we just need to multiply it by \(\binom{n}{m} \leq n^m\).

Note that

\[
m \leq n^{1-\epsilon} = o\left(\frac{n}{\log^2 n}\right) = o\left(\frac{\Delta^2}{n}\right) = o(\Delta).
\] (28)

Take any \(j \leq m\). Using Lemma 18(a), we find that at least \(\gamma \Delta - n^{1-\epsilon} = \Theta(\Delta)\) vertices \(k\) such that \(jk \in S\) and \(|\theta_k|_{2\pi} \leq \eta/2\). For such \(k\), we have \(|\theta_j + \theta_k|_{2\pi} \geq \eta/2\). Similarly as before, by Lemma 10, for \(\theta \in B^*(m)\),

\[
\prod_{j=1}^{m} \prod_{k=m+1}^{n} \left|1 + \lambda_{jk}(e^{i(\theta_j + \theta_k)} - 1)\right| \geq e^{-\Omega(m \Lambda \Delta \eta^2)} = e^{-\Omega(m \eta^2)}.
\]

Thus, we can bound

\[
\int_{B^*(m)} |F_{S,t}(\theta)| \ d\theta \leq \int_{U_m(\pi)} e^{-\Omega(m \eta^2)} \left(\int_{U_{n-m}(\eta)} |F_{S',t'}(\theta^1)| \ d\theta^1\right) \ d\theta^2,
\]

where \(\theta^1 \in \mathbb{R}^{n-m}, \theta^2 \in \mathbb{R}^m\) and \(S'\) is obtained from \(S\) by deletion of the first \(m\) vertices. Recall that \(|F_{S',t'}(\theta^1)|\) does not depend on \(t'\), but we define \(t'\) anyway by

\[
t'_j := \sum_{j : \theta \in S'} \lambda_{jk} \text{ for all } j.
\]
By (28), $S'$ and $t'$ satisfy all the assumptions of Lemma 9 and Lemma 20. Thus,
\[
\int_{U_{n-m}(\eta)} |F_{S',t'}(\theta^1)| d\theta^1 = \frac{\pi^{(n-m)/2}}{|Q'|^{1/2}} e^{O\left(\frac{n}{\pi^2}\right)},
\]
where $Q'$ is the matrix of (17) for the graph $S'$ and $(\lambda_{jk})_{j,k \in S'}$. Applying Lemma 20(d) (see appendix) $m$ times for the scaled matrix $Q / \Lambda$, we find that
\[
|Q| / |Q'| = (\Lambda \Delta)^m e^{O(m)}.
\]
Allowing $nm$ for the choice of the set of $m$ big components and using (27), (28), we obtain that
\[
J(m) \leq n^m e^{-\Omega(mn^2 \varepsilon)} (\Lambda \Delta)^{m/2} e^{O(m \pi^{(n-m)/2} / |Q|^{1/2})} e^{O\left(\frac{n}{\pi^2}\right)} = e^{-\Omega(mn^2 \varepsilon)} J_0.
\]
Summing over $m$ and multiplying by 2 for the symmetry of $(0, \ldots, 0)$ and $(\pi, \ldots, \pi)$, we find that
\[
\int_{B \setminus (B' \cup B'')} |F_{S,t}(\theta)| d\theta \leq 2 \sum_{m=1}^{n^{1-\varepsilon}} J(m) = e^{-\Omega(n^2 \varepsilon)} J_0.
\]
Using Lemma 11 and Lemma 12, we conclude the following.

**Corollary 4** Under the assumptions of Theorem 11 and for sufficiently small $\varepsilon$,
\[
\int_{B} |F_{S,t}(\theta)| d\theta = e^{-\Omega(n^2 \varepsilon)} \int_{B_0} F_{S,t}(\theta) d\theta.
\]

### 6 Random $t$-factors and the beta model

In this section we establish a deep relation between $S_t$ (a uniform random element of the set of $t$-factors of $S$) and the corresponding $\beta$-model: the probabilities of any forced or forbidden small structure are asymptotically the same.

**Theorem 12** Suppose a graph $S$ and a degree sequence $t$ satisfy the assumptions of Theorem 11. Let $H^+$ and $H^-$ be disjoint subgraphs of $S$ such that $\|h\|_2 \ll (\Lambda \Delta)^{1/2}$, where $h$ is the degree sequence of $H^+ \cup H^-$. Then, for any $\varepsilon > 0$,
\[
\mathbb{P}(H^+ \subseteq S_t \text{ and } H^- \cap S_t = \emptyset) = \left(1 + O\left(n^{-1/2 + \varepsilon} + \frac{\|h\|_2}{\Lambda \Delta}\right)\right) \prod_{j \in H^+} \lambda_{jk} \prod_{j \in H^-} (1 - \lambda_{jk}).
\]

For the case $S = K_n$, the estimate of Theorem 12 was previously established by Isaev and McKay in [11, Theorem 5.2] under the additional constraint that $\|h\|_\infty = O(n^{1/6})$. When $S = K_n$ and $t$ is near-regular, a more precise formula for $\mathbb{P}(H^+ \subseteq S_t \text{ and } H^- \cap S_t = \emptyset)$ can be derived from [19, Theorem 1.3], provided $\|h\|_1 \leq$
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In this section, we first give some preliminary estimates for the solution of system (19). Then, we prove Theorem 12. Then, in Sect. 6.3, we show that Theorem 10 follows from Theorem 12.

6.1 The solution of the beta system

The following lemma will be useful for investigating system (19).

Lemma 13 Let \( r : \mathbb{R}^n \to \mathbb{R}^n \), \( \delta > 0 \), and \( U = \{ x \in \mathbb{R}^n : \| x - x^{(0)} \| \leq \delta \| r(x^{(0)}) \| \} \) and \( x^{(0)} \in \mathbb{R}^n \), where \( \| \cdot \| \) is any vector norm in \( \mathbb{R}^n \). Assume that

\[
\sup_{x \in U} \| J^{-1}(x) \| < \delta,
\]

where \( J \) denotes the Jacobian matrix of \( r \) and \( \| \cdot \| \) stands for the induced matrix norm. Then there exists \( x^* \in U \) such that \( r(x^*) = 0 \).

Proof Let \( y^{(0)} = r(x^{(0)}) \) and note that \( x^{(0)} \in U \). If \( y^{(0)} = 0 \) there is nothing to prove so we may assume otherwise. Using the Cauchy-Kovalevskaya theorem, define the curve \( x(t) \) by \( x(0) = x^{(0)} \) and \( \frac{dx(t)}{dt} = -J^{-1}(x(t))y^{(0)} \). Note that \( x(t) \) remains in \( U \) for \( 0 \leq t \leq 1 \), because

\[
x(t) - x(0) = -\int_0^t J^{-1}(x(\tau))y^{(0)}d\tau
\]

and \( \| x(t) - x(0) \| \leq t \sup_{x \in U} \| J^{-1}(x)y^{(0)} \| < \delta \| y^{(0)} \| \). Observe that \( \frac{d}{dt}r(x(t)) = -y^{(0)} \). Therefore, \( r(x(t)) = (1 - t)y^{(0)} \). Taking \( x^* = x(1) \) the proof is complete. \( \square \)

Using Lemma 13, we can estimate the difference between the exact solution of (19) and a given approximate solution \( \beta^{(0)} \) in terms of the deviation of the corresponding expected degrees.

Corollary 5 Let \( S \) satisfy assumption (B1) of Theorem 11 and \( \Delta = \Omega(n^{1/2}) \). For \( t \in \mathbb{R}^n \), let \( \lambda = \frac{t_1 + \ldots + t_n}{2|E(S)|} \). For \( \beta \in \mathbb{R}^n \), define \( r(\beta) = (r_1, \ldots, r_n) \) by

\[
r_j = r_j(\beta) := -t_j + \sum_{k : j \in S} \frac{e^{\beta_j + \beta_k}}{1 + e^{\beta_j + \beta_k}} \quad \text{for all } j.
\]

Suppose, for some \( \beta^{(0)} \), we have \( \text{rng}(\beta^{(0)}) \leq c \) and \( \| r(\beta^{(0)}) \|_\infty \ll \lambda(1 - \lambda)\Delta \). Then there exists a solution \( \beta^* \) of system (19) such that

\[
\| \beta^* - \beta^{(0)} \|_p = O\left(\frac{\| r(\beta^{(0)}) \|_p}{\lambda(1 - \lambda)\Delta}\right), \quad \text{for any } p \in \{1, 2, \infty\}.
\]
Proof Observe that
\[
\frac{\partial}{\partial \beta_j} \left( \frac{e^{\beta_j + \beta_k}}{1 + e^{\beta_j + \beta_k}} \right) = \frac{e^{\beta_j + \beta_k}}{1 + e^{\beta_j + \beta_k}} \left( 1 - \frac{e^{\beta_j + \beta_k}}{1 + e^{\beta_j + \beta_k}} \right) = \lambda_{jk} (1 - \lambda_{jk}).
\]
Therefore, the Jacobian matrix \( J(\beta) \) of \( r(\beta) \) coincides with \( 2A(\beta) \), where \( A(\beta) \) is the matrix defined in (17) for \( \beta \). Using the bounds of Lemma 8(b), for any \( \beta \in \mathbb{R}^n \) that \( \| \beta - \beta^{(0)} \|_\infty \leq c \), we have
\[
\| J^{-1}(\beta) \|_2 \leq \| J^{-1}(\beta) \|_1 = \| J^{-1}(\beta) \|_\infty = O \left( \frac{1}{\Lambda(\beta) \Delta} \right),
\]
where \( \Lambda(\beta) = \lambda(\beta)(1 - \lambda(\beta)) \) and \( \lambda(\beta) \) is defined according (20). Note that if \( \| r(\beta) \|_\infty \ll \lambda(1 - \lambda) \Delta \), then we get that \( \lambda(\beta) = \Theta(\lambda) \) and \( 1 - \lambda(\beta) = \Theta(1 - \lambda) \). Applying Lemma 13 with \( \delta = C/(\lambda(1 - \lambda) \Delta) \) where \( C > 0 \) is sufficiently large, the proof is complete.

6.2 Proof of theorem 12

Let \( S' = S - (H^+ \cup H^-) \) and \( t' \in \mathbb{N}^n \) be such that \( t - t' \) is the degree sequence of \( H^+ \). Then, by definition,
\[
\mathbb{P}(H^+ \subseteq S_t \text{ and } H^- \cap \{S_t\}) = \frac{N(S', t')}{N(S, t)}.
\]
Since \( h \) is an integer vector, we have that
\[
\| h \|_1 \leq \| h \|_2^2 \ll \Lambda \Delta, \quad \| h \|_\infty \leq \| h \|_2 \ll (\Lambda \Delta)^{1/2}.
\]
Using \( \beta \) as \( \beta^{(0)} \) in Corollary 5, we find a solution \( \beta' \) of system (19) for the graph \( S' \) and the vector \( t' \) such that
\[
\| \beta' - \beta \|_\infty \leq \| \beta' - \beta \|_2 = O \left( \frac{\| h \|_2}{\Lambda \Delta} \right) = o((\Lambda \Delta)^{-1/2}),
\]
\[
\| \beta' - \beta \|_1 = O \left( \frac{\| h \|_1}{\Lambda \Delta} \right) = O \left( \frac{\| h \|_2^2}{\Lambda \Delta} \right) = o(1).
\]
Observe that \( \text{rng}(\beta') = \text{rng}(\beta) + o(1) \) and \( \| h \|_\infty \ll (\Lambda \Delta)^{1/2} \ll \frac{\Delta^2}{n} \). Therefore, \( S' \) and \( t' \) also satisfy the assumptions of Theorem 11. Applying Theorem 11 twice, we find that
\[
\frac{N(S', t')}{N(S, t)} = \left( 1 + O(n^{-1/2+\varepsilon}) \right) \frac{|Q'|^{1/2} \exp \left( \frac{1}{2} \mathbb{E} u'(X') \right)}{|Q|^{1/2} \exp \left( \mathbb{E} u(X) - \frac{1}{2} \mathbb{E} u^2(X) \right)} R,
\]
\[\square\]
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where \( Q, u, v \) and \( Q', u', v' \) are matrices of (17) and polynomials of (18) for \( S, t \) and \( S', t' \), respectively, \( X \) and \( X' \) are the corresponding normally distributed vectors and

\[
R := \frac{\prod_{jk \in S'}(1 + e^{\beta_j + \beta_k'})}{\prod_{jk \in S}(1 + e^{\beta_j + \beta_k})} \prod_{j=1}^{n} e^{t_j \beta_j - t'_j \beta'_j}.
\]

**Lemma 14** Under the assumptions of Theorem 12, we have

\[
R = \left(1 + O\left(\frac{\|h\|_2^2}{\Lambda \Delta}\right)\right) \prod_{jk \in H^+} \lambda_{jk} \prod_{jk \in H^-} (1 - \lambda_{jk}).
\]

**Proof** Let \((\lambda_{jk})\) and \((\lambda'_{jk})\) be defined as in (14) for for \( S, t \) and \( S', t' \). From (30), we have

\[
\lambda'_{jk}(1 - \lambda'_{jk}) = (1 + O(\beta_j + \beta_k - \beta'_j - \beta'_k)) \lambda_{jk}(1 - \lambda_{jk}) = O(\Lambda).
\]

Applying Taylor’s theorem to \(\log(1 + e^x)\) and symmetry \(\lambda'_{jk} = \lambda'_{kj}\), we obtain that

\[
\sum_{jk \in S'} \log\left(\frac{1 + e^{\beta_j + \beta_k}}{1 + e^{\beta'_j + \beta'_k}}\right) = \sum_{jk \in S'} \left(\lambda'_{jk}(\beta_j + \beta_k - \beta'_j - \beta'_k)
\right)
+ O\left(\Lambda(|\beta_j - \beta'_j|^2 + |\beta_k - \beta'_k|^2)\right)
= \sum_{j=1}^{n} \sum_{k: jk \in S'} \left(\lambda'_{jk}(\beta_j - \beta'_j) + O\left(\Lambda(\|\beta_j - \beta'_j\|_2^2)\right)\right)
= O\left(\Lambda \Delta \|\beta' - \beta\|_2^2 + \sum_{j=1}^{n} t'_j (\beta_j - \beta'_j)\right).
\]

Then, using (30) again, we get that

\[
R = \left(1 + O\left(\Lambda \Delta \|\beta' - \beta\|_2^2\right)\right) \prod_{jk \in H^+} \lambda_{jk} \prod_{jk \in H^-} (1 - \lambda_{jk}).
\]

\(\square\)

To complete the proof of Theorem 11, it remains to be shown that

\[
\log \left(\frac{|Q|}{|Q'|}\right) + |\mathbb{E}u(X) - \mathbb{E}u'(X')| + |\mathbb{E}v^2(X) - \mathbb{E}v'^2(X')| = O\left(n^{-1/2 + \varepsilon} + \frac{\|h\|_2^2}{\Lambda \Delta}\right). \tag{32}
\]
Let $Q = (q_{jk})$ and $Q' = (q'_{jk})$. Using (31), we find that

$$q'_{jk} - q_{jk} = \begin{cases} 
O(\Delta) \left( h_j + \sum_{k:jk \in S'} |\beta_j + \beta_k - \beta'_j - \beta'_k| \right), & \text{if } j = k; \\
O(\Delta) |\beta_j + \beta_k - \beta'_j - \beta'_k|, & \text{if } jk \in S'; \\
O(\Delta), & \text{if } jk \in S - S'; \\
0, & \text{otherwise.}
\end{cases}$$

(33)

**Lemma 15** Under the assumptions of Theorem 12, we have

$$\log \left( \frac{|Q|}{|Q'|} \right) = O \left( \frac{\|h\|_2^2}{\Lambda \Delta} \right).$$

**Proof** If $U$, $V$ are symmetric positive definite matrices of equal size, then the matrix $UV$ has positive real eigenvalues. To see this, note that $UV$ is similar to the matrix $U^{-1/2}UVU^{1/2} = (V^{1/2}U^{1/2})^T(V^{1/2}U^{1/2})$, which is symmetric positive definite. In particular, $Q^{-1}Q'$ and $(Q')^{-1}Q$ have positive real eigenvalues, since $Q$ and $Q'$ are symmetric positive definite (see Lemma 8). Therefore, using $\log x \leq x - 1$, we can bound

$$\frac{|Q|}{|Q'|} \leq e^{\text{tr}(Q^{-1}Q') - n} = e^{\text{tr}(Q^{-1}(Q' - Q))}, \quad \frac{|Q|}{|Q'|} \leq e^{\text{tr}((Q')^{-1}Q) - n} = e^{\text{tr}((Q')^{-1}(Q - Q'))}.$$

Using (30), (33) and the bounds of Lemma 8(b), we get that

$$\text{tr}(Q^{-1}(Q' - Q)) = O(\Delta^{-1}) \sum_{j=1}^n \left( h_j + \sum_{k:jk \in S'} |\beta_j + \beta_k - \beta'_j - \beta'_k| \right) = O(\Delta^{-1} (\|h\|_1 + \Delta \|\beta' - \beta\|_1)) = O \left( \frac{\|h\|_2^2}{\Lambda \Delta} \right).$$

The same argument applies to $\text{tr}((Q')^{-1}(Q - Q'))$ and thus $\log \frac{|Q|}{|Q'|} = O \left( \frac{\|h\|_2^2}{\Lambda \Delta} \right)$. □

To show the remaining part of (32), we will use the following estimate

$$|\mathbb{E}u(X) - \mathbb{E}u'(X')| \leq |\mathbb{E}u(X) - \mathbb{E}u(X')| + |\mathbb{E}u(X') - \mathbb{E}u'(X')|$$

and similarly for $|\mathbb{E}v^2(X) - \mathbb{E}v^2(X')|$. 

**Lemma 16** Under the assumptions of Theorem 12, we have

$$|\mathbb{E}(u(X') - u'(X'))| + |\mathbb{E}v^2(X') - \mathbb{E}v^2(X')| = O(n^{-1/2+\varepsilon}).$$
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Proof Repeating the arguments of Lemma 9 (see (25) and (26)) and using (29), (30), (31), we derive that

$$\mathbb{E}(u(X) - u'(X')) = O \left( \lambda \|\beta' - \beta\|_{\infty} \sum_{jk \in S} \frac{1}{(A\Delta)^2} + \lambda \sum_{jk \in S - S'} \frac{1}{(A\Delta)^2} \right)$$

$$= O \left( n \frac{(A\Delta)^{-1/2}}{A\Delta} + \frac{\|h\|_{1}}{A\Delta} \right) = O(n^{-1/2+\varepsilon})$$

and

$$\mathbb{E}\left((v(X) - v'(X'))^2\right) = O \left( \frac{n}{A\Delta} \|\beta' - \beta\|_{\infty}^2 + \lambda^2 \frac{\|h\|_{2}^2}{(A\Delta)^2} \right) = O \left( \frac{n}{(A\Delta)^2} \right).$$

Observe also that (using the arguments of (26))

$$\mathbb{E}\left((v(X) + v'(X'))^2\right) \leq 2\mathbb{E}v^2(X) + 2\mathbb{E}v'^2(X') = O \left( \frac{n}{A\Delta} \right).$$

Applying the Cauchy-Schwartz inequality, we find that

$$\mathbb{E}v^2(X') - \mathbb{E}v'^2(X') = \mathbb{E}(v(X') - v'(X'))(v(X') + v'(X'))$$

$$= O \left( \sqrt{\frac{n}{(A\Delta)^2}} \cdot \frac{n}{A\Delta} \right) = O(n^{-1/2+\varepsilon}).$$

\[\Box\]

We complete the proof of (32) and of Theorem 12 with the following lemma.

Lemma 17 Under the assumptions of Theorem 12, we have

$$|\mathbb{E}u(X) - \mathbb{E}u'(X')| + |\mathbb{E}v^2(X) - \mathbb{E}v'^2(X')| = O(n^{-1/2+\varepsilon}).$$

Proof First, we need to establish a few more bounds on the difference of the covariance matrices of $X$ and $X'$. From (29), (30) and (33), we get that

$$q'_{jj} - q_{jj} = O(\lambda) \left( \|h\|_{\infty} + \Delta \|\beta - \beta'\|_{\infty} \right) = O(\|h\|_{2}) = O((A\Delta)^{1/2})$$

(34)

and, for $jk \in S'$,

$$q'_{jk} - q_{jk} = O(\lambda \|\beta - \beta'\|_{\infty}) = O \left( \frac{\|h\|_{2}}{\Delta} \right) = O \left( \frac{(\lambda\Delta)^{1/2}}{\lambda} \right).$$

(35)

Let $Q^{-1} = (\sigma_{jk})$ and $(Q')^{-1} = (\sigma'_{jk})$. Observe that

$$Q^{-1} - (Q')^{-1} = Q^{-1}(Q - Q)(Q')^{-1}.$$
Then, using (29), (33), (34), (35), and the bounds of Lemma 8(b) for $Q$ and $Q'$, we obtain that

$$\sigma'_{jj} - \sigma_{jj} = O\left(\frac{|q'_{jj} - q_{jj}|}{\Delta^2} + \sum_{k=1}^{n} \frac{|q'_{kk} - q_{kk}|}{\Delta^2} + \sum_{k \notin S} \frac{|q'_{kk} - q_{kk}|}{\Delta^2}\right)$$

$$= O\left(\frac{|q'_{jj} - q_{jj}|}{\Delta^2} + \sum_{k=1}^{n} \frac{|q'_{kk} - q_{kk}|}{\Delta^2} + \sum_{k \notin S'} \frac{|q'_{kk} - q_{kk}|}{\Delta^2} + \frac{\|h\|_1 A}{\Delta^2}\right)$$

$$= O\left((\Lambda \Delta)^{1/2} + \frac{n(\Lambda \Delta)^{1/2}}{\Lambda^2} + \frac{n\Delta(\Lambda \Delta)^{1/2}}{\Lambda^2} + \frac{\|h\|_1 A}{\Delta^2}\right)$$

$$= O(n^{-3/2+\epsilon}).$$

Similarly, for $jk \in S'$ or $jk \notin S$, we have

$$\sigma'_{jk} - \sigma_{jk} = O\left(\frac{|q'_{jk} - q_{jk}|}{\Delta^2} + \sum_{\ell=1}^{n} \left(\frac{|q'_{j\ell} - q_{j\ell}| + |q'_{\ell k} - q_{\ell k}|}{\Delta^2} + \frac{|q'_{\ell k} - q_{\ell k}|}{\Delta^2}\right) + \sum_{\ell m \in S} \frac{|q'_{\ell m} - q_{\ell m}|}{\Delta^2}\right).$$

Next, using (33) and (35), we estimate

$$\sum_{\ell=1}^{n} |q'_{j\ell} - q_{j\ell}| = \sum_{\ell, j \in S - S'} O(\Lambda) + \sum_{\ell, j \in S} O((\Lambda / \Delta)^{1/2}) = O(\|h\|_\infty A + \Delta(\Lambda / \Delta)^{1/2}).$$

The same bound holds for $\sum_{\ell=1}^{n} |q'_{\ell k} - q_{\ell k}|$. By (33), (34) and (35), we also have that

$$|q'_{jk} - q_{jk}| = O((\Lambda / \Delta)^{1/2}),$$

$$\sum_{\ell m \in S} |q'_{\ell m} - q_{\ell m}| = \sum_{\ell m \in S'} |q'_{\ell m} - q_{\ell m}| + O(\|h\|_1 A) = O(\Delta n(\Lambda / \Delta)^{1/2} + \|h\|_1 A).$$

Combining the above and using (29), (34), (35) we get, for any $jk \notin S - S'$,

$$\sigma'_{jk} - \sigma_{jk} = O\left(\frac{(\Lambda / \Delta)^{1/2}}{\Lambda^2} + \frac{\Delta n(\Lambda / \Delta)^{1/2}}{\Lambda^2} + \frac{\|h\|_\infty A}{\Lambda^2} + \frac{\|h\|_1 A}{\Delta^2}\right) = O(n^{-5/2+\epsilon}).$$

For random vectors $X$ and $X'$, define $(\sigma_{jk, \ell m})$ and $(\sigma'_{jk, \ell m})$ as in (23). From the above and Lemma 8(b), we obtain that

$$\sigma_{jk, \ell m} - \sigma'_{jk, \ell m} = \begin{cases} O(n^{-3/2+\epsilon}), & \text{if } \{j, k\} \cap \{\ell, m\} \neq \emptyset; \\ O(n^{-5/2+\epsilon}), & \text{if } \{j, k\} \cap \{\ell, m\} = \emptyset \\ \text{and } \{j \ell, jm, k \ell, km\} \cap (S - S') = \emptyset. & \end{cases}$$

Now, using the arguments of (25) and (26), we get that

$$\mathbb{E}u(X) - \mathbb{E}u(X') = O\left(\Lambda \sum_{jk \in S} |\sigma'_{jk, jk} - \sigma_{jk, jk}| \cdot |\sigma'_{jk, jk} + \sigma_{jk, jk}|\right) = O(n^{-1/2+\epsilon}).$$
Note that, if real \( x, y, z, x', y', z' \) admit bounds \(|x|, |x'| \leq a, |y|, |y'| \leq b \) and \(|z|, |z'| \leq c \) for some positive \( a, b, c \), then
\[
|x'y'z' - x' y' z'| \leq \left( \frac{|x-x'|}{a} + \frac{|y-y'|}{b} + \frac{|z-z'|}{c} \right) abc.
\]

Thus, using (24), (26) and (29) and for \((\sigma_{jk, \ell m}) \) and \((\sigma'_{jk, \ell m}) \), we find that
\[
E_v^2(X) - E_v^2(X') = O\left( \Lambda^2 \sum_{jk \in S} \sum_{\ell m \in S} |\sigma_{jk, \ell m} - \sigma'_{jk, \ell m}| + |\sigma_{jk, \ell m}^3 - (\sigma'_{jk, \ell m})^3| \right)
\]
\[
= O\left( \left( \frac{n^{-3/2+\epsilon}}{A} + \frac{n^{-5/2+\epsilon}}{A} \right) \frac{n}{A^2} + \Lambda^2 \sum_{jk \in S} \sum_{\ell m \in U_{jk}} \frac{1}{A^2} \right)
\]
\[
= O\left( n^{-1/2+\epsilon} + \frac{n A^2 \|h\|_\infty}{A^4} \right) = O(n^{-1/2+\epsilon}).
\]
where \( U_{jk} = \{ \ell m \in S : \{j, k\} \cap \{\ell, m\} = \emptyset \) and \( \{j \ell, j m, k \ell, k m\} \cap (S-S') \neq \emptyset \).

This completes the proof. \( \square \)

### 6.3 Proof of theorem 10

Since Theorem 10 is trivially true for \( h = 0 \), we assume otherwise.

Let \( \beta^{(0)} = (\beta^{(0)}_1, \ldots, \beta^{(0)}_n) \), where \( \beta^{(0)}_j \) is defined by
\[
e^{-2\beta^{(0)}_j} \frac{1}{1 + e^{2\beta^{(0)}_j}} = \lambda = \frac{t_1 + \cdots + t_n}{s_1 + \cdots + s_n}.
\]

By assumption (A3), we get that
\[
\|r(\beta^{(0)})\|_\infty = \|t - \lambda s\|_\infty \ll A\Delta,
\]
where \( r(\cdot) \) is defined in Corollary 5. Applying that corollary, we find a solution \( \beta \) of system (19) such that
\[
\|\beta - \beta^{(0)}\|_\infty = O\left( \frac{\|t - \lambda s\|_\infty}{A\Delta} \right). \tag{36}
\]
In particular, the assumptions of Theorem 10 and (36) imply that all the assumptions of Theorem 12 hold. By Taylor’s theorem, we have that
\[
\lambda_{jk} = (1 + O(\beta_{jk} - \beta^{(0)}_j) + O(\beta_k - \beta^{(0)}_j))\lambda,
\]
\[
1 - \lambda_{jk} = (1 + O(\beta_{jk} - \beta^{(0)}_j) + O(\beta_k - \beta^{(0)}_j))(1 - \lambda).
\]

Then, we get that
\[
\prod_{jk \in H^+} \lambda_{jk} \prod_{jk \in H^-} (1 - \lambda_{jk}) = \lambda^{m(H^+)}(1 - \lambda)^{m(H^-)} \exp\left( \sum_{j \in [n]} O((\beta_{j} - \beta^{(0)}_j)h_j) \right).
\]
Using (36), we find that
\[ \sum_{j \in [n]} |(\beta_j - \beta^{(0)})h_j| \leq \|\beta - \beta^{(0)}\|_\infty \|h\|_1 = O\left(\frac{\|t - \lambda s\|_\infty \|h\|_1}{\Lambda/\Delta}\right). \]

Thus, applying Theorem 12 gives the required probability bound. \(\square\)

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A Technical lemmas

Here we prove or cite the technical lemmas that are used in the proofs. This section is self-contained and does not rely on assumptions other than those stated.

A.1 When common neighbours are not rare

Here, we explore the properties of graphs for which any two vertices have sufficiently many common neighbours.

Lemma 18  Let \( G \) be a graph on \( n \) vertices and \( \gamma > 0 \) be fixed. Assume that any two distinct vertices have at least \( \gamma \Delta^2/2 \) common neighbours in \( G \), where \( \Delta = \Delta(G) \). Then the following hold.

(a)  The minimal degree of \( G \) is at least \( \gamma \Delta \).

(b)  For any \( x \in \mathbb{R}^n \), we have \( \sum_{j,k \in G} (x_j + x_k)^2 \geq \frac{\gamma^4}{576} \|x\|_2^2 \Delta \).

Proof  For a vertex \( j \), we count its common neighbours with other vertices. Note that any vertex is counted at most \( \Delta - 1 \) times (since it is already connected to \( j \)). Therefore, the degree of \( j \) is at least \( (n-1)\gamma \Delta^2/(\Delta-1)n \geq \gamma \Delta \) which proves (a). To help with the proof of (b), note that \( \gamma \leq 1 \) by condition (a).

Let \( Q_G \) denote the matrix defined by \( x^T Q_G x = \sum_{j,k \in G} (x_j + x_k)^2 \). The matrix \( Q_G \) is known as the signless Laplacian matrix. From [5, Theorem 3.2], we find that all eigenvalues of \( Q_G \) are bounded below by \( \frac{\psi(G)}{4\Delta} \), where

\[ \psi(G) := \min \left\{ \frac{\varepsilon_b(G[U]) + |\partial_G(U)|}{|U|} : \emptyset \neq U \subseteq V(G) \right\}, \]
where $G[U]$ denotes the induced subgraph, $e_b(G[U])$ is the minimal number of edges required to delete from the graph $G[U]$ to make it bipartite, and $\partial_G(U)$ is the set of edges of $G$ with exactly one end in $U$. Thus, to prove (b), it is sufficient to show $\psi(G) \geq \frac{\gamma^2 \Delta}{12}$.

First, consider the case $|U| \leq n(1 - \gamma/6)$. Observe that, for any common neighbour $\ell$ of two vertices $j \in U$ and $k \notin U$, either $j \ell$ or $jk$ contributes to $\partial_G(U)$. By the assumptions, the number of choices of $j, k$ and $\ell$ is at least $|U|(n - |U|)\frac{\gamma \Delta^2}{n}$. We need to divide by $2\Delta$ to allow for over-counting. Thus, we get

$$\frac{|\partial_G(U)|}{|U|} \geq \frac{(n-|U|)\gamma \Delta^2}{2n\Delta} \geq \frac{\gamma^2 \Delta}{12}.$$

Now, assume $|U| > n(1 - \gamma/6)$. Consider any partition $(W_1, W_2)$ of $U$ into two disjoint sets. We may assume $|W_1| \geq |W_2|$. If $|W_2| \leq \gamma n/3$ then, bounding the degrees of the vertices in $W_1$ from below by $\gamma \Delta$ (by part (a)) and degrees of vertices of $W_2$ above by $\Delta$, we get that

$$|\partial_G(U)| + |E(G[W_1])| \geq \gamma \Delta|W_1| - \Delta|W_2| \geq \gamma(1 - \gamma/6 - \gamma/3)\Delta n - \gamma \Delta n/3$$

$$\geq \frac{\gamma^2/3 - \gamma/6}{1 - \gamma/6} \Delta|U| \geq \frac{\gamma^2}{5} \Delta|U| \geq \frac{\gamma^2}{12} \Delta|U|.$$

If $|W_2| > \gamma n/3$, observe that, for any common neighbour $\ell$ of two vertices $j \in W_1$ and $k \in W_2$, at least one of $\{j \ell, k \ell\}$ contributes to $E(G[W_1]) \cup E(G[W_2]) \cup \partial_G(U)$. By the assumptions, the number of choices of $j, k$ and $\ell$ is at least $|W_1||W_2|\frac{\gamma \Delta^2}{n}$.

Dividing by $2\Delta$ to adjust for over-counting, we get

$$|E(G[W_1])| + |E(G[W_2])| + |\partial_G(U)| \geq \frac{|W_1||W_2|\gamma \Delta^2}{2\Delta n} \geq (|U| - \gamma n/3)\frac{\gamma^2 \Delta}{6}$$

$$\geq \left(1 - \frac{\gamma/3}{1 - \gamma/6}\right)\frac{\gamma^2}{6} \Delta|U| \geq \frac{\gamma^2}{12} \Delta|U|.$$

Combining the above cases, we conclude that $\frac{e_b(G[U]) + |\partial_G(U)|}{|U|} \geq \frac{\gamma^2}{12} \Delta$ always. Part (b) follows. \hfill \Box

### A.2 Integration theorem

Here, we quote the results from [11] that were used in Sect. 5. For a domain $\Omega \subseteq \mathbb{R}^n$ and a twice continuously differentiable function $q : \Omega \to \mathbb{C}$, define

$$H(q, \Omega) = (h_{jk}), \text{ where } h_{jk} := \sup_{x \in \Omega} \left| \frac{\partial^2 q}{\partial x_j \partial x_k} (x) \right|.$$

**Theorem 13** (Theorem 4.4 of [11]) Let $c_1, c_2, c_3, \epsilon, \rho_1, \rho_2, \phi_1, \phi_2$ be nonnegative real constants with $c_1, \epsilon > 0$. Let $Q$ be an $n \times n$ positive-definite symmetric real matrix and let $T$ be a real matrix such that $T^\top QT = I$. Let $\Omega$ be a measurable set such
that $U_n(\rho_1) \subseteq T^{-1}(\Omega) \subseteq U_n(\rho_2)$, and let $f : \mathbb{R}^n \to \mathbb{C}$ and $g : \mathbb{R}^n \to \mathbb{R}$ be twice continuously differentiable and let $h : \Omega \to \mathbb{C}$ be integrable. We make the following assumptions.

(a) $c_1 (\log n)^{1/2+\epsilon} \leq \rho_1 \leq \rho_2$.

(b) For $x \in T(U_n(\rho_1)), \ 2\rho_1 \| T \|_1 \left| \frac{\partial f}{\partial x_j}(x) \right| \leq \phi_1 n^{-1/3} \leq \frac{2}{3}$ for $1 \leq j \leq n$ and $4\rho_1^2 \| T \|_1 \| T \|_\infty \| H(f, T(U_n(\rho_1))) \|_\infty \leq \phi_1 n^{-1/3}$.

(c) For $x \in \Omega$, $\forall f(x) \leq g(x)$. For $x \in T(U_n(\rho_2))$, either

(i) $2\rho_2 \| T \|_1 \left| \frac{\partial g}{\partial x_j}(x) \right| \leq (2\rho_2)^{3/2} n^{-1/2}$ for $1 \leq j \leq n$, or

(ii) $2\rho_2 \| T \|_1 \left| \frac{\partial g}{\partial x_j}(x) \right| \leq \phi_2 n^{-1/3}$ for $1 \leq j \leq n$ and $4\rho_2^2 \| T \|_1 \| T \|_\infty \| H(g, T(U_n(\rho_2))) \|_\infty \leq \phi_2 n^{-1/3}$.

(d) $|f(x)|, |g(x)| \leq n^{c_3} e^{c_2 x^T Q x/n}$ for $x \in \mathbb{R}^n$.

Let $X$ be a random variable with normal density $\pi^{-n/2} |Q|^{-1/2} e^{-x^T Q x}$. Then, provided $\forall f(X) = E(f(X) - E f(X))^2$ and $\text{Var}(X)$ are finite and $h$ is bounded in $\Omega$,

$$
\int_{\Omega} e^{-x^T Q x + f(x) + h(x)} \, dx = (1 + K) \pi^{n/2} |Q|^{-1/2} e^{E f(X) + \frac{1}{2} \text{Var}(f(X)) + \frac{1}{2} \text{Var}(g(X)) - \text{Var}(f(X))},
$$

where, for some constant $C$ depending only on $c_1, c_2, c_3, \epsilon$,

$$
|K| \leq C e^{\frac{1}{2} \text{Var}(f(X))} \left( e^{\phi_1^3 + e^{-\rho_1^2/2}} - 1 + 2 (2 e^{\phi_3^3 + e^{-\rho_2^2/2}} - 2 + \sup_{x \in \Omega} |e^{h(x)} - 1|) e^{E g(X) - \text{Var}(f(X)) + \frac{1}{2} \text{Var}(g(X)) - \text{Var}(f(X))} \right).
$$

In particular, if $n \geq (1 + 2c_2)^2$ and $\rho_2^2 \geq 15 + 4c_2 + (3 + 8c_3) \log n$, we can take $C = 1$.

In order to apply Theorem 13, we need to verify that $T$ exists and satisfies all required conditions. The following lemma is a special case of [11, Lemma 4.9] (for trivial ker $Q$ and $\gamma = \mu_{\text{min}}/d_{\text{max}}$). Recall that $\| \cdot \|_{\text{max}}$ stands for the maximum of the absolute values of the elements of a given matrix.

**Lemma 19** Let $Q$ be an $n \times n$ real symmetric matrix with positive minimum eigenvalue $\mu_{\text{min}}$. Let $D$ be a diagonal matrix with diagonal entries in $[d_{\text{min}}, d_{\text{max}}]$ for $d_{\text{min}} > 0$. Suppose that $\| Q - D \|_{\text{max}} \leq rd_{\text{min}}/n$ for some $r$. Then

(a) $\| Q^{-1} - D^{-1} \|_{\text{max}} \leq \frac{r (rd_{\text{max}} + \mu_{\text{min}})}{\mu_{\text{min}} d_{\text{min}} n}$.

Furthermore, there exists a real matrix $T$ such that $T^TQT = I$ and

(b) $\| T \|_1, \| T \|_\infty \leq \frac{rd_{\text{max}}^{1/2} + \mu_{\text{min}}^{1/2}}{\mu_{\text{min}}^{1/2} d_{\text{min}}^{1/2}}$.

(c) $\| T^{-1} \|_1, \| T^{-1} \|_\infty \leq \frac{(r+1) (rd_{\text{max}} + \mu_{\text{min}}^{1/2} d_{\text{min}}^{1/2})}{\mu_{\text{min}}^{1/2}}$.
A.3 Weighted graphs and norm bounds

In the case when the matrix has a specific graph-related structure, the bounds of Lemma 19 can be improved. For a graph \( G \) on \( n \) vertices and weights \( W = (w_{jk}) \), define the symmetric matrix \( Q_W \) by

\[
x^TQ_Wx = \sum_{jk \in G} w_{jk}(x_j + x_k)^2.
\] (37)

Observe that if \( w_{jj} = 0 \) for all \( j \) then \( D = Q_W - W \) is the diagonal matrix with the same diagonal elements as \( Q_W \).

**Lemma 20** Let \( G \) be a graph on \( n \) vertices. Assume that \( \Delta = \Delta(G) = \Omega(n^{1/2}) \) and the number of common neighbours of any two vertices in \( G \) is \( \Theta(\Delta^2/n) \). Consider an \( n \times n \) matrix \( W = (w_{jk}) \) with nonnegative real entries such that \( w_{jk} = \Theta(1) \) uniformly if \( jk \in G \) and \( w_{jk} = 0 \) otherwise. Then the following hold with the implicit constant for each \( O() \) and \( \Theta() \) expression depending only on the implicit constants in the assumptions.

(a) The diagonal elements of \( Q_W \) are \( \Theta(\Delta) \).

(b) If \( Q_W^{-1} = (\sigma_{jk}) \) then \( \sigma_{jk} = \begin{cases} \Theta\left(\frac{1}{\Delta}\right), & \text{if } j = k; \\ O\left(\frac{1}{\Delta^2}\right), & \text{if } jk \in G; \\ O\left(\frac{1}{\Delta n}\right), & \text{otherwise}. \end{cases} \)

(c) There exists a real matrix \( T \) such that \( T^TQ_WT = I \) and

\[
\|T\|_1, \|T\|_\infty = O(\Delta^{-1/2}), \quad \|T^{-1}\|_1, \|T^{-1}\|_\infty = O(\Delta^{1/2}).
\]

(d) Let \( G' \) be the graph obtained by deleting vertex 1 from \( G \) and \( W' \) be formed by deleting the first row and column from \( W \). Define \( Q_W' \) to be the matrix of (37) for \( G' \) and \( W' \). Then \( |Q_W| = O(\Delta)|Q_W'| \).

**Proof** In Lemma 18(a) we proved that all degrees of \( G \) are \( \Theta(\Delta) \). Thus, the diagonal elements of \( Q_W \) are \( \Theta(\Delta) \), proving part (a).

From Lemma 18(b), we find that for any non-trivial \( x \in \mathbb{R}^n \)

\[
\frac{x^TQ_Wx}{\|x\|_2^2} = \Theta(1) \frac{\sum_{jk \in G} (x_j + x_k)^2}{\|x\|_2^2} = \Omega(\Delta). \quad (38)
\]

Therefore, the eigenvalues of \( Q_W \) are \( \Theta(\Delta) \). Let

\[
\tilde{Q} := (I - \frac{1}{2} WD^{-1})Q_W(I - \frac{1}{2} D^{-1} W) = D - \frac{3}{4} WD^{-1} W + \frac{1}{4} WD^{-1} WD^{-1} W.
\]

Using the upper bound on the number of common neighbours in \( G \), we find that the off-diagonal elements of \( WD^{-1} W \) are \( O\left(\frac{\Delta}{n}\right) \), while its diagonal elements are \( O(1) \).

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Also, all elements of $WD^{-1}WD^{-1}W$ are $O\left(\frac{\Delta}{n} + \frac{1}{\Delta}\right) = O\left(\frac{\Delta}{n}\right)$. Then we get that

$$\|\tilde{Q} - \tilde{D}\|_{\max} = O\left(\frac{\Delta}{n}\right) \quad \text{and} \quad \|\tilde{D} - D\|_{\max} = O(1),$$

where $\tilde{D}$ is the diagonal matrix with the same diagonal as $\tilde{Q}$. Next, observe,

$$
\begin{align*}
\|D - \frac{1}{2}W\|_{\infty}, & \|D - \frac{1}{2}W\|_{1} = O(\Delta), \\
\|(D - \frac{1}{2}W)^{-1}\|_{\infty}, & \|(D - \frac{1}{2}W)^{-1}\|_{1} = O(\Delta^{-1})
\end{align*}
$$

Using part (a) and (38), we find that all the eigenvalues of $\tilde{Q}$ are $\Theta(\Delta)$. Thus, we can apply Lemma 19(a) to matrix $\tilde{Q}$ to obtain that

$$\|\tilde{Q}^{-1} - \tilde{D}^{-1}\|_{\max} = O\left(\frac{1}{\Delta n}\right).$$

Observe that $\|\tilde{D}^{-1} - D^{-1}\|_{\max} = O(\Delta^{-2})$ and

$$Q_W^{-1} = D^{-1}(D - \frac{1}{2}W)\tilde{Q}^{-1}(D - \frac{1}{2}W)D^{-1}. $$

Since $\|XY\|_{\max} \leq \|X\|_{\infty}\|Y\|_{\max}$ and $\|XY\|_{\max} \leq \|X\|_{\max}\|Y\|_{1}$, we get from (39)

$$\|(I - \frac{1}{2}D^{-1}W)(\tilde{Q}^{-1} - \tilde{D}^{-1})(I - \frac{1}{2}WD^{-1})\|_{\max} = O\left(\frac{1}{\Delta n}\right).$$

Arguing as before to bound the entries of $D^{-1}W\tilde{D}^{-1}WD^{-1}$, part (b) follows. From Lemma 19(b,c), we find a real matrix $\tilde{T}$ such that $\tilde{T}^{T}\tilde{Q}\tilde{T} = I$ and

$$\|\tilde{T}\|_{1}, \|\tilde{T}\|_{\infty} = O(\Delta^{-1/2}), \quad \|\tilde{T}^{-1}\|_{1}, \|\tilde{T}^{-1}\|_{\infty} = O(\Delta^{1/2}).$$

Taking $T := D^{-1}(D - \frac{1}{2}W)\tilde{T}$ and using (39), we have proved part (c).

For (d), write the matrix $Q_W = (q_{jk})$ as follows.

$$Q_W = \begin{pmatrix} q_{11} & q^T \\ q & Q'_W + \text{diag}(q) \end{pmatrix},$$

where $q = (q_{12}, \ldots, q_{1n})$ and diag($q$) is a diagonal matrix with the elements of $q$ down the diagonal. Now perform the first step of Gaussian elimination by subtracting multiples of the first row from the other rows. The result is

$$\begin{pmatrix} q_{11} & q^T \\ 0 & Q'_W + \text{diag}(q) - q_{11}^{-1}qq^T \end{pmatrix}.$$

Consequently, $|Q_W| = q_{11}|Q'_W + \text{diag}(q) - q_{11}^{-1}qq^T| = q_{11}|Q'_W||I + B|$ where

$$B := (Q'_W)^{-1}(\text{diag}(q) - q_{11}^{-1}qq^T)$$

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Observe that $G'$ and $W'$ satisfy all the assumptions of Lemma 20.

From the definition of $Q_W$ and $q_{11} = \Theta(\Delta)$ we find that

\[
\left(\text{diag}(q) - q_{11}^{-1}qq^T\right)_{jk} = \begin{cases} 
\Theta(1), & \text{if } j = k \text{ and } 1k \in G, \\
\Theta(\Delta^{-1}), & \text{if } j \neq k \text{ and } 1j, 1k \in G, \\
0, & \text{otherwise.}
\end{cases}
\]

Applying part (b), we find that $B = (b_{jk})$ where

\[
b_{jk} = \begin{cases} 
O(\Delta^{-1}), & \text{if } j = k \text{ and } 1k \in G, \\
O(\Delta^{-2}), & \text{if } j \neq k \text{ and } 1k \in G, \\
0, & \text{otherwise.}
\end{cases}
\]

Since $\|B\|_\infty = O(\Delta^{-1})$, all the eigenvalues of $B$ are $O(\Delta^{-1})$, so we have $|I + B| = \exp(\text{tr} B + O(\|B\|_F^2))$, where $\|B\|_F$ is the Frobenius norm. We have $\text{tr}(B) = O(1)$ and $\|B\|_F^2 = \sum_{j,k=1}^n |b_{jk}^2| = O(\Delta^{-1})$, so $|I + B| = O(1)$ and the proof of part (d) is complete. \(\square\)

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