Smith-Purcell radiation from a charge moving above a finite-length grating with rectangular profiles

Weiwei Li*, Weihao Liu², Zhigang He¹, Qika Jia¹, and Lin Wang¹
¹National Synchrotron Radiation Laboratory, University of Science and Technology of China, Hefei, Anhui, 230029, China
²College of Electronic and Information Engineering, Nanjing University of Aeronautics and Astronautics, Nanjing, Jiangsu, 211106, People’s Republic of China

(Dated: 12 April 2020)

Abstract

Smith-Purcell radiation is generated by a charged particle beam passing close to the surface of a diffraction grating which has a strong dependency of the emitted radiation intensity on the form of the grating profile. For relativistic electron beam, it is important to take into account the number of grating periods in practical SPR setups. In this paper, the theoretical investigations of Smith-Purcell radiation from a three-dimensional bunch of relativistic electrons that moves at constant speed parallel to an electrically perfectly conducting grating with finite rectangular grooves are carried out by using the modal matching method. This model may offer a new efficient tool for terahertz production by SPR interaction and for nondestructive bunch-length measurements by SPR.

Introduction

Smith-Purcell radiation is originated when a charged particle travels parallel to a plane with diffraction grating. Recent renewed interest in this problem is caused by different applications. Among these applications are length determination for short electron bunches, creation of monochromatic light source in the various spectral regions.

Various theoretical models were proposed for describing the SPR. However, the general theory of SPR has not been created yet and at present there exist a number of different approaches, which sometimes are in agreement and sometimes contradict each other. The rigorous solution by van den Berg of the SPR from an infinitely long grating is obtained by solving an integral equation having a periodic Green’s function. For the non-relativistic electron energies, the approach developed by van den Berg ensures a reasonable agreement with experiments. In principle, an arbitrary tooth profile can be analyzed with this approach. In practice, extensive numerical computation is generally required to approach the asymptotic value for the radiation intensity, however for particular case of rectangular grating, the modal matching method is recommended since it agrees with integral method, but takes shorter computational time.

For relativistic electron beam, it is important to take into account the number of grating periods in practical SPR setups. A model based on the finite-difference time-domain (FDTD) method, in which a total-field scattered-field technique combined with a near- to far-field projection, is described in. A frequency-domain model based on the electric-field integral equation (EFIE) extends van den Berg’s model for practical gratings of finite length and agrees with the FDTD model. Both models are adaptable to arbitrary grating geometries but requires extensive numerical computation. Accurate far-field measurements of SPR with a 15 MeV beam have demonstrated good agreement between measured power and the predictions of EFIE theory.

* email address: liwei@ustc.edu.cn
In this paper, we will develop the modal matching method for the calculations of finite-grating with rectangular grooves, which is one of the most popular grating types. Although the EFIE theory is also still applicable for this case, the modal matching method will take much less computation. In the previous theoretical work $^{[7,11,12]}$ which applied the modal matching method on SPR from the rectangular gratings, the periodic grating conditions are used, but operating the SPR under the periodic regime driving by the relativistic electrons would require a large number of grating grooves which is difficult to fulfill in the practical experiments.

**Formula Derivation**

![Fig. 1 The SPR scheme. An electron bunch is moving above a grating with rectangular grooves](image)

In the formula derivations, we primarily consider the case that the electron beam excites an electrically perfectly conducting grating with finite rectangular grooves, and the schematic diagram of which is illustrated in Fig. 1.

The incident filed can be given by the Fourier transform of the free space field produced by an electron beam with charge of $Q$ moving along the $z$ direction at a relativistic velocity $\beta c$:

$$H^m_y (x, z; k_y, \omega) = -\frac{Q}{2} F_{coh} \text{sign}(x-x_0) \exp(jk_{x_0}(z-z_0)) \exp(jk_x|x-x_0|)$$

$$E^m_x (x, z; k_y, \omega) = \frac{Q}{2} F_{coh} \frac{\mu_0 \epsilon_0}{\epsilon_0} k_x k_{x_0} \exp(jk_{x_0}(z-z_0)) \exp(jk_x|x-x_0|)$$

where $x_0$ and $z_0$ are the initial charge coordinates, $k_{x_0} = k/\beta$ and $k_{x_0} = \sqrt{k^2 - k_{x_0}^2}$ and the $F_{coh}$ is the coherence bunch factor. Assuming uncorrelated electron distributions, $\rho(x, y, t) = X(x)Y(y)T(t)$, the coherent integral is given by:

$$F_{coh} = \int_{-\infty}^{\infty} X(x) \exp(jk_{x_0}(x-x_0)) dx \int_{-\infty}^{\infty} Y(y) \exp(jk_{y_0}(y-y_0)) dy \int_{-\infty}^{\infty} T(t) \exp(j\omega t) dt$$

The reflected waves in the region above the grating can be expanded in continuous series,

$$E'_{y} (x, z; k_y, \omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk_z E_{\omega x} (k_z, k_y, \omega) \exp(jk_z z + jk_x x)$$

$$H'_{y} (x, z; k_y, \omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk_z H_{\omega x} (k_z, k_y, \omega) \exp(jk_z z + jk_x x)$$

The radiation directions can be defined in terms of the angular coordinate in relation to the wave-numbers

$$k_z = k_{x_0} \sin \eta; k_y = k_{x_0} \cos \eta \sin \zeta; k_x = k_{x_0} \cos \eta \cos \zeta$$

The field in the grooves can be expanded into a series of open mouth waveguide modes $^{[11]}$
where \( z_p, a_p \) and \( h_p \) are the center longitudinal position, width and depth of \( p \)-th groove respectively. The number of the grooves is \( N_p \). It should be noted that the groove sizes can be set to be different from each other.

Using boundary conditions on the grating and continuity conditions between the solutions of the upper half space and those in the grooves, we get a set of algebraic equations for the unknowns coefficients. The system for the unknown waveguide amplitude are

\[
B_{pm} \frac{1+\delta_{00}}{2} (\Gamma_{pm} + 1) - \frac{1}{2\pi} \int_{0}^{\infty} \frac{\sum_{q=1}^{N_q} a_{q} \kappa_{pq} B_{pq} (\Gamma_{pq} - 1)}{k_{0}} \Phi(q, n, k_{0}) \Phi^{*}(p, m, k_{0}) = H_{0} \Phi^{*}(p, m, k_{0}) \tag{6}
\]

and

\[
\frac{1}{2} C_{pm} \kappa_{pq} (1+\Gamma_{pq}) + \int_{-\infty}^{\infty} \frac{d k_{z}}{2\pi} \sum_{q=1}^{N_q} C_{pq} \Psi(q, n, k_{0}) [1-\Gamma_{pq}] \Psi^{*}(p, m, k_{0}) = E_{0} k_{z} \Psi^{*}(p, m, k_{0}) \tag{7}
\]

with

\[
\begin{aligned}
H_{0} &= Q F_{\text{ins}} \exp(-jk_{z} z + jk_{z} x_{0}) \\
E_{0} &= Q F_{\text{ins}} \sqrt{\frac{\mu_{0}}{\varepsilon_{0}}} \frac{k_{z} k_{z0}}{k_{0} k_{z0}} \exp(-jk_{z} z_{0} + jk_{z} x_{0}) \\
\Phi(p, m, k_{0}) &= \exp(-jk_{z} z) \frac{1}{a_{p}} \int_{-a_{p}}^{a_{p}} \exp(-jk_{z} z) \cos \frac{m \pi z}{a_{p}} + \frac{m \pi}{2} \, dz \\
\Psi(p, m, k_{0}) &= \exp(-jk_{z} z) \frac{1}{a_{p}} \int_{-a_{p}}^{a_{p}} \exp(-jk_{z} z) \sin \frac{m \pi z}{a_{p}} + \frac{m \pi}{2} \, dz
\end{aligned}
\tag{8}
\]

The coefficients of the field in the upper half space can be then obtained by

\[
\begin{bmatrix}
-\pi k_{z} H_{0} \delta(k_{0} - k_{z}) + k_{z} H_{z0} = \sum_{p=1}^{N_p} \kappa_{pq} B_{pq} a_{p} \Phi(p, m, k_{0}) \bigl[1 + \Gamma_{pq}\bigr] \\
\pi E_{0} \delta(k_{0} - k_{z}) + E_{z0} = \sum_{p=1}^{N_p} \kappa_{pq} B_{pq} a_{p} \Psi(p, m, k_{0}) \bigl[1 - \Gamma_{pq}\bigr]
\end{bmatrix}
\tag{9}
\]

The radiated energy related to the amplitudes of the reflected waves can be expressed as

\[
\frac{dW}{d\Omega} = \frac{1}{4\pi^3} \frac{k_{z0}^2}{1 - k_{z0}^2/k_{z0}^2}
\tag{10}
\]

**Summary**

In this paper, we have carried out the modal matching method for the calculations of finite-grating with rectangular grooves. This method can be an alternative and equivalent method of the EFIE theory for the
specific situation. More characteristics about the SPR from finite-grating with rectangular grooves will be studied in the future work.

Acknowledgements

This work is supported by National Foundation of Natural Sciences of China (11705198)

References

[1] S. J. Smith and E. M. Purcell, “Visible Light from Localized Surface Charges Moving across a Grating,” Phys. Rev. 92, 1069 (1953).

[2] V. Blackmore et al., “First measurements of the longitudinal bunch profile of a 28.5 GeV beam using coherent Smith-Purcell radiation,” Phys. Rev. Accel. Beams 12, 032803 (2009).

[3] H. L. Andrews et al., “Reconstruction of the time profile of 20.35 GeV, subpicosecond long electron bunches by means of coherent Smith-Purcell radiation”, Phys. Rev. Accel. Beams 17, 052802 (2014).

[4] Y. Liang et al., “Selective excitation and control of coherent terahertz Smith-Purcell radiation by high-intensity period-tunable train of electron micro-bunches,” Appl. Phys. Lett. 112, 053501 (2018).

[5] D. Yu. Sergeeva, A. A. Tishchenko, and M. N. Strikhanov, “Conical diffraction effect in optical and x-ray Smith-Purcell radiation”, Phys. Rev. Accel. Beams 18, 052801 (2015).

[6] P. M. van den Berg, “Smith–Purcell radiation from a point charge moving parallel to a reflection grating,” J. Opt. Soc. Am. 63, 1588 (1973).

[7] O. Haeberle et al., “Calculations of Smith-Purcell radiation generated by electrons of 1-100 MeV,” Phys. Rev. E 49, 3340 (1994).

[8] A. S. Kesar et al., “Time- and frequency-domain models for Smith-Purcell radiation from a two-dimensional charge moving above a finite length grating,” Phys. Rev. E 71, 016501 (2005).

[9] A. S. Kesar, “Smith-Purcell radiation from a charge moving above a finite-length grating,” Phys. Rev. ST Accel. Beams 8, 072801 (2005).

[10] A. S. Kesar, R. A. Marsh, and R. J. Temkin, “Power measurement of frequency-locked Smith-Purcell radiation,” Phys. Rev. ST Accel. Beams 9, 022801 (2006).

[11] Y. Shibata et al., “Coherent Smith-Purcell radiation in the millimeter-wave region from a short-bunch beam of relativistic electrons,” Phys. Rev. E 57, 1061 (1998).

[12] W. Li, Y. Liu and W. Liu “Spectrum discretization of Smith–Purcell radiation for efficient multicolor terahertz emission,” J. Phys. D: Appl. Phys. 53, 185107(2020).