ON MARGINAL MARKOV PROCESSES OF QUANTUM QUADRATIC STOCHASTIC PROCESSES

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Abstract. In the paper it is defined two marginal Markov processes on von Neumann algebras $\mathcal{M}$ and $\mathcal{M} \otimes \mathcal{M}$, respectively, corresponding to given quantum quadratic stochastic process (q.q.s.p.). It is proved that such marginal processes uniquely determines the q.q.s.p. Moreover, certain ergodic relations between them are established as well.

1. Introduction

It is known that Markov processes are well-developed field of mathematics which have various applications in physics, biology and so on. But there are some physical models which cannot be described by such processes. One of such models is a model related to population genetics. Namely, this model is described by quadratic stochastic processes (see \cite{8, 2, 11}). To define it, we denote

$$\ell^1 = \{x = (x_n) : \|x\|_1 = \sum_{n=1}^{\infty} |x_n| < \infty; \ x_n \in \mathbb{R}\},$$

$$S^\infty = \{x \in \ell^1 : x_n \geq 0; \|x\|_1 = 1\}.$$

Hence this process is defined as follows (see \cite{2},\cite{11}): Consider a family of functions $\{p^{[s,t]}_{ij,k} : i, j, k \in \mathbb{N}, \ s, \ t \in \mathbb{R}^+, \ t - s \geq 1\}$. Such a family is said to be quadratic stochastic process (q.s.p.) if for fixed $s, t \in \mathbb{R}^+$ it satisfies the following conditions:

(i) $p^{[s,t]}_{ij,k} = p^{[s,t]}_{ji,k}$ for any $i, j, k \in \mathbb{N}$.

(ii) $p^{[s,t]}_{ij,k} \geq 0$ and $\sum_{k=1}^{\infty} p^{[s,t]}_{ij,k} = 1$ for any $i, j, k \in \mathbb{N}$.

(iii) An analogue of Kolmogorov-Chapman equation; here there are two variants:

- for the initial point $x^{(0)} \in S$, $x^{(0)} = (x_1^{(0)}, x_2^{(0)}, \cdots)$ and $s < r < t$ such that $t - r \geq 1, r - s \geq 1$

(iiiA)

$$p^{[s,t]}_{ij,k} = \sum_{m,l=1}^{\infty} p^{[s,r]}_{ij,m} p^{[r,t]}_{ml,k} x^{(r)}_k,$$

where $x^{(r)}_k$ is given by

$$x^{(r)}_k = \sum_{i,j=1}^{\infty} p^{[0,r]}_{ij,k} x^{(0)}_i x^{(0)}_j;$$

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It is said that the q.s.p. \( \{p_{ij,k}^{[s,t]}\} \) is of type (A) or (B) if it satisfies the fundamental equations (iii_A) or (iii_B), respectively. In this definition the functions \( p_{ij,k}^{[s,t]} \) denotes the probability that under the interaction of the elements \( i \) and \( j \) at time \( s \) the element \( k \) comes into effect at time \( t \). Since for physical, chemical and biological phenomena a certain time is necessary for the realization of an interaction, we shall take the greatest such time to be equal to 1 (see the Boltzmann model [5] or the biological model [8]). Thus the probability \( p_{ij,k}^{[s,t]} \) is defined for \( t - s \geq 1 \).

It should be noted that such quadratic stochastic processes are related to the notion of a quadratic stochastic operator, which was introduced in [1], in the same way as Markov processes are related to linear transformations (i.e. Markov operators). The problem of studying the behaviour of trajectories of quadratic stochastic operators was stated in [14]. The limit behaviour and ergodic properties of trajectories of such operators were studied in (see for example [6, 8, 9, 15].

We note that quadratic stochastic processes describe physical systems defined above, but they do not occupy the cases in quantum level. So, it is naturally to define a concept of quantum quadratic processes. In [3, 10] quantum (noncommutative) quadratic stochastic processes (q.q.s.p.) were defined on a von Neumann algebra and studied certain ergodic properties ones. In [3] it is obtained necessary and sufficient conditions for the validity of the ergodic principle for q.q.s.p. From the physical point of view this means that for sufficiently large values of time the system described by such a process does not depend on the initial state of the system. It has been found relations between quantum quadratic stochastic processes and non-commutative Markov processes. In [10] an expansion of q.q.s.p. into a so-called fibrewise Markov process is given, and it is proved that such an expansion uniquely determines the q.q.s.p. As an application, it is given a criterion (in terms of this expansion) for the q. q. s. p. to satisfy the ergodic principle. Using such a result, it is proved that a q.q.s.p. satisfies the ergodic principle if and only if the associated Markov process satisfies this principle. It is natural to ask: is the defined Markov process determines the given q.q.s.p. uniquely, or how many Markov processes are needed to uniquely determine the q.q.s.p.? In this paper we are going to affirmatively solve the problem. Namely, we shall show that there two non-stationary Markov processes defined on different von Neumann algebras \( \mathcal{M} \) and \( \mathcal{M} \otimes \mathcal{M} \), respectively, called marginal Markov processes, which uniquely determine the given to quantum quadratic stochastic process. Such a description allows us to investigate other properties of q.q.s.p. by means of Markov processes. Moreover, certain ergodic relations between them are established as well.

2. Preliminaries

Let \( \mathcal{M} \) be a von Neumann algebra acting on a Hilbert space \( H \). The set of all continuous (resp. ultra-weak continuous) functionals on \( \mathcal{M} \) is denoted by \( \mathcal{M}^\ast \) (resp. \( \mathcal{M}_u^\ast \)), and put \( \mathcal{M}_{+,+} = \mathcal{M}_+ \cap \mathcal{M}_u^\ast \), here \( \mathcal{M}_+^\ast \) denotes the set of all positive linear functionals. By \( \mathcal{M} \otimes \mathcal{M} \) we denote tensor product of \( \mathcal{M} \) in into itself. By \( S \) and \( S^2 \) we denote the set of all normal states on \( \mathcal{M} \) and \( \mathcal{M} \otimes \mathcal{M} \) respectively. Recall a
mapping \( U : M \otimes M \to M \otimes M \) is a linear operator such that \( U(x \otimes y) = y \otimes x \) for all \( x, y \in M \). Let \( \varphi \in S \) be a fixed state. We define the conditional expectation operator \( E_\varphi : M \otimes M \to M \) on elements \( a \otimes b, a, b \in M \) by
\[
E_\varphi(a \otimes b) = \varphi(a)b
\]
and extend it by linearity and continuity to \( M \otimes M \). Clearly, such an operator is completely positive and \( E_\varphi \mathbb{1}_{M \otimes M} = \mathbb{1}_M \) (more details on von Neumann algebras we refer to [13]).

Consider a family of linear operators \( \{P^{s,t} : M \to M \otimes M, s, t \in \mathbb{R}_+, t-s \geq 1\} \).

**Definition 2.1.** We say that a pair \( \{P^{s,t}, \omega_0\} \), where \( \omega_0 \in S \) is an initial state, forms a quantum quadratic stochastic process \( \text{q.q.s.p.} \), if every operator \( P^{s,t} \) is ultra-weakly continuous and the following conditions hold:

1. Each operator \( P^{s,t} \) is a unital completely positive mapping with \( UP^{s,t} = P^{s,t} \);
2. An analogue of Kolmogorov-Chapman equation is satisfied: for initial state \( \omega_0 \in S \) and arbitrary numbers \( s, \tau, t \in \mathbb{R}_+ \) such that \( \tau-s \geq 1, t-\tau \geq 1 \) one has
   
   \[
   \begin{align*}
   (i)_A & \quad P^{s,t}x = P^{s,\tau}(E_\omega(P^{\tau,t}x)), \quad x \in M \\
   (ii)_B & \quad P^{s,t}x = E_\omega P^{s,\tau} \otimes E_\omega P^{s,\tau}(P^{\tau,t}x), \quad x \in M,
   \end{align*}
   \]

where \( \omega_0(x) = \omega_0 \otimes \omega_0(P^{0,\tau}x) \), \( x \in M \).

If for q.q.s.p. the fundamental equations (ii)_A or (ii)_B are held then we say that q.q.s.p. has type (A) or type (B), respectively.

**Remark 2.2.** By using the q.q.s.p., we can specify a law of interaction of states. For \( \varphi, \psi \in S \), we set
\[
V^{s,t}(\varphi, \psi)(x) = \varphi \otimes \psi(P^{s,t}x), \quad x \in M.
\]
This equality gives a rule according to which the state \( V^{s,t}(\varphi, \psi) \) appears at time \( t \) as a result of the interaction of states \( \varphi \) and \( \psi \) at time \( s \). From the physical point of view, the interaction of states can be explained as follows: Consider two physical systems separated by a barrier and assume that one of these systems is in the state \( \varphi \). and the other one is in the state \( \psi \). Upon the removal of the barrier, the new physical system is in the state \( \varphi \otimes \psi \) and, as a result of the action of the operator \( P^{s,t} \), a new state is formed. This state is exactly the result of the interaction of the states \( \varphi \) and \( \psi \).

**Remark 2.3.** If \( M \) is an \( \ell^\infty \), i.e., \( M = \ell^\infty \), then a q.q.s.p. \( \{P^{s,t}\}, \omega_0 \) defined on \( \ell^\infty \) coincides with a quadratic stochastic process. Indeed, we set
\[
P^{[s,t]}_{ij,k} = P^{s,t}(\chi_A)(i, j), \quad i, j, k \in \mathbb{N},
\]
where \( \chi_A \) is the indicator of the set \( A \in \mathcal{B} \). Then, by Definition 2.1, the family of functions \( p^{[s,t]}_{ij,k} \) forms a quadratic stochastic process.

Conversely, if we have a quadratic stochastic process \( \{p^{[s,t]}_{ij,k}\}, \mu^{(0)} \) then we can define a quantum quadratic stochastic process on \( \ell^\infty \) as follows:
\[
(P^{s,t}f)(i, j) = \sum_{k=1}^\infty f_k p^{[s,t]}_{ij,k}, \quad f = \{f_k\} \in \ell^\infty.
\]
As the initial state, we take the following state
\[ \omega_0(\mathfrak{F}) = \sum_{k=1}^{\infty} f_k \mu_k^{(0)}. \]

One can easily check the conditions of Definition 2.1 are satisfied. Thus, the notion of quantum quadratic stochastic process generalizes the notion of quadratic stochastic process.

**Remark 2.4.** Certain examples of q.q.s.p were given in [3].

Let \( \{P_{s,t}\}, \omega_0 \) be a q.q.s.p. Then by \( P_{s,t}^* \) we denote the linear operator mapping from \( (\mathcal{M} \otimes \mathcal{M})_* \) into \( \mathcal{M}_* \) given by
\[ P_{s,t}^*(\varphi)(x) = \varphi(P_{s,t} x), \quad \varphi \in (\mathcal{M} \otimes \mathcal{M})_*, \quad x \in \mathcal{M}. \]

**Definition 2.5.** A q.q.s.p. \( \{P_{s,t}\}, \omega_0 \) is said to satisfy the ergodic principle, if for every \( \varphi, \psi \in S^2 \) and \( s \in \mathbb{R}_+ \)
\[ \lim_{t \to \infty} \|P_{s,t}^* \varphi - P_{s,t}^* \psi\|_1 = 0, \]
where \( \| \cdot \|_1 \) is the norm on \( \mathcal{M}^* \).

Let us note that Kolmogorov was the first who introduced the concept of an ergodic principle for Markov processes (see, for example, [7]). For quadratic stochastic processes such a concept was introduced and studied in [12, 4].

### 3. Marginal Markov Processes and Ergodic Principle

In this section we are going to consider relation between q.q.s.p. and Markov processes.

First recall that a family \( \{Q^{s,t}_s : \mathcal{M} \to \mathcal{M}, f, \cup \in \mathbb{R}_+, \cup - f \geq \infty\} \) of unital completely positive operators is called Markov process if
\[ Q^{s,t} = Q^{s,\tau} Q^{\tau,t} \]
holds for any \( s, \tau, t \in \mathbb{R}_+ \) such that \( t - \tau \geq 1, \tau - s \geq 1 \).

A Markov process \( \{Q^{s,t}\} \) is said to satisfy the ergodic principle if for every \( \varphi, \psi \in S^2 \) and \( s \in \mathbb{R}_+ \) one has
\[ \lim_{t \to \infty} \|Q^{s,t}_s \varphi - Q^{s,t}_s \psi\|_1 = 0. \]

Let \( \{P_{s,t}\}, \omega_0 \) be a q.q.s.p. Then define a new process \( Q^{s,t}_P : \mathcal{M} \to \mathcal{M} \) by
\[ Q^{s,t}_P = E_{\omega_s} P_{s,t}. \] (3.1)

Then according to Proposition 4.3 [10] \( \{Q^{s,t}_P\} \) is a Markov process associated with q.q.s.p. It is evident that the defined process satisfies the ergodic principle if the q.q.s.p. satisfies one. An interesting question is about the converse. Corollary 4.4 [10] states the following important result:

**Theorem 3.1.** Let \( \{P_{s,t}\}, \omega_0 \) be a q.q.s.p. on a von Neumann algebra \( \mathcal{M} \) and let \( Q^{s,t}_P \) be the corresponding Markov process. Then the following conditions are equivalent

(i) \( \{P_{s,t}\}, \omega_0 \) satisfies the ergodic principle;
(ii) \( \{Q^{s,t}_P\} \) satisfies the ergodic principle;
(iii) There is a number \( \lambda \in [0, 1) \) such that, given any states \( \varphi, \psi \in S^2 \) and a number \( s \in \mathbb{R}_+ \) one has
\[
\|Q_{P,s}^{t}\varphi - Q_{P,s}^{t}\psi\|_1 \leq \lambda \|\varphi - \psi\|_1
\]
for at least one \( t \in \mathbb{R}_+ \).

3.1. Case type A. In this subsection we assume that q.q.s.p. \( \{P_{s,t}\}, \omega_0 \) has type (A).

Now define another process \( \{H_{s,t}^{s,t} : \mathcal{M} \otimes \mathcal{M} \to \mathcal{M} \otimes \mathcal{M}, f, \psi \in \mathbb{R}_+, f - \psi \geq \infty \} \) by
\[
H_{P,s}^{s,t}x = P_{s,t}E_{\omega_t}x, \quad x \in \mathcal{M} \otimes \mathcal{M}.
\]
(3.2)

It is clear that every \( H_{P,s}^{s,t} \) is a unital completely positive operator. It turns out that \( \{H_{s,t}^{s,t}\} \) is Markov process. Indeed, using (ii) of Def. 2.1 one has
\[
H_{P,s}^{s,t}x = P_{s,t}E_{\omega_t}x = P_{s,t}E_{\omega_t}(P_{t,r}E_{\omega_{\tau}}(x)) = H_{P,s}^{s,t}H_{P,s}^{s,t}x,
\]
which is the assertion.

The defined two Markov processes \( Q_{P,s}^{t} \) and \( H_{P,s}^{s,t} \) are related with each other by the following equality
\[
E_{\omega_t}(H_{P,s}^{s,t}x) = E_{\omega_t}(P_{s,t}(E_{\omega_t}(x))) = Q_{P,s}^{t}(E_{\omega_t}(x))
\]
for every \( x \in \mathcal{M} \otimes \mathcal{M} \). Moreover, \( H_{P,s}^{s,t} \) has the following properties
\[
H_{P,s}^{s,t}x = P_{s,t}(E_{\omega_t}(x)) = P_{s,t}E_{\omega_t}E_{\omega_t}(x) = H_{P,s}^{s,t}(E_{\omega_t}(x) \otimes 1)
\]
(3.3)

\[
UH_{P,s}^{s,t} = H_{P,s}^{s,t}, \quad H_{P,s}^{s,t}(x \otimes 1) = P_{s,t}x, \quad x \in \mathcal{M}.
\]

from (3.3) one gets \( H_{P,s}^{s,t}(1 \otimes x) = \omega_t(x)1 \otimes 1 \). Here we can represent
\[
\omega_t(x) = \omega_0 \otimes \omega_0(P_{0,t}x) = \omega_0(Q_{P,s}^{t}x),
\]
\[
= \omega_0 \otimes \omega_0(H_{P,s}^{s,t}(x \otimes 1)).
\]

Now we are interested in the following question: can such kind of two Markov processes (i.e. with above properties) determine uniquely a q.q.s.p.? To answer to this question we need to introduce some notations.

Let \( \{Q_{s,t} : \mathcal{M} \to \mathcal{M} \} \) and \( \{H_{s,t}^{s,t} : \mathcal{M} \otimes \mathcal{M} \to \mathcal{M} \otimes \mathcal{M} \} \) be two Markov processes with an initial state \( \omega_0 \in S \). Denote
\[
\varphi_t(x) = \omega_0(Q_{0,t}x), \quad \psi_t(x) = \omega_0 (H_{0,t}(x \otimes 1)).
\]

Assume that the given processes satisfy the following conditions:

(i) \( UH_{s,t} = H_{s,t} \),
(ii) \( E_{\varphi_t}H_{s,t}x = Q_{s,t}^{t}E_{\varphi_t}x \) for all \( x \in \mathcal{M} \otimes \mathcal{M} \);
(iii) \( H_{s,t}^{s,t}x = H_{s,t}^{s,t}(E_{\varphi_t}(x) \otimes 1) \).

First note that if we take \( x = 1 \otimes x \) in (iii) then we get
\[
H_{s,t}^{s,t}(1 \otimes x) = H_{s,t}^{s,t}(E_{\varphi_t}(1 \otimes x) \otimes 1) = H_{s,t}(\varphi_t(x)1 \otimes 1)
= \psi_t(x)1 \otimes 1
\]
(3.4)
Now from (ii) and (3.4) we have
\[
E_{\psi_s} H^{s,t}(\mathbf{I} \otimes x) = E_{\psi_s}(\psi_t(x) \mathbf{I} \otimes \mathbf{I}) = \psi_t(x) \mathbf{I} = Q^{s,t} E_{\psi_t}(\mathbf{I} \otimes x) = \varphi_t(x) \mathbf{I}.
\] (3.5)

This means that \( \varphi_t = \psi_t \), therefore in the sequel we denote \( \omega_t := \varphi_t = \psi_t \).

Now we are ready to formulate the result.

**Theorem 3.2.** Let \( \{Q^{s,t}\} \) and \( \{H^{s,t}\} \) be two Markov Processes with (i)-(iii). Then by the equality \( P^{s,t}x = H^{s,t}(x \otimes \mathbf{I}) \) one defines a q.q.s.p. of type (A). Moreover, one has
\[
\text{(a) } P^{s,t} = H^{s,\tau} P^{\tau,t} \text{ for any } \tau - s \geq 1, \ t - \tau \geq 1,
\]
\[
\text{(b) } Q^{s,t} = E_{\omega_s} P^{s,t}.
\]

**Proof.** We have to check only the condition (ii)\textsubscript{A} of Def. 2.1. Take any \( \tau - s \geq 1, \ t - \tau \geq 1 \). Then using the assumption (iii) we have
\[
P^{s,\tau} E_{\omega_\tau}(P^{\tau,t}x) = H^{s,\tau}(E_{\omega_\tau} H^{\tau,t}(x \otimes \mathbf{I}) \otimes \mathbf{I}) = H^{s,\tau} H^{\tau,t}(x \otimes \mathbf{I}) = H^{s,t}(x \otimes \mathbf{I}) = P^{s,t}x, \ x \in M.
\]

From Markov property of \( H^{s,t} \) we immediately get (a).

If we put \( x = x \otimes \mathbf{I} \) to (iii) then from (2.1) one finds
\[
E_{\omega_s} P^{s,t}x = E_{\omega_s} H^{s,t}(x \otimes \mathbf{I}) = Q^{s,t} E_{\omega_t}(x \otimes \mathbf{I}) = Q^{s,t}x.
\]
This completes the proof. \( \square \)

These two \( \{Q^{s,t}\} \) and \( \{H^{s,t}\} \) Markov processes are called *marginal Markov processes* associated with q.q.s.p. \( \{P^{s,t}\} \). So, according to Theorem 3.2 the marginal Markov processes uniquely define q.q.s.p.

Now define another process \( \{Z^{s,t} : \mathcal{M} \otimes \mathcal{M} \to \mathcal{M} \otimes \mathcal{M}\} \) by
\[
Z^{s,t}x = E_{\omega_s} H^{s,t}(x \otimes \mathbf{I}), \ x \in \mathcal{M} \otimes \mathcal{M}.
\] (3.6)

From (ii) one gets \( Z^{s,t}x = Q^{s,t} E_{\omega_s} x \otimes \mathbf{I} \). In particular,
\[
Z^{s,t}(x \otimes \mathbf{I}) = Q^{s,t}x \otimes \mathbf{I},
\]
\[
Z^{s,t}(\mathbf{I} \otimes x) = \omega_t(x) \mathbf{I} \otimes \mathbf{I}.
\]

**Proposition 3.3.** The process \( \{Z^{s,t}\} \) is a Markov one.

**Proof.** Take any \( \tau - s \geq 1, \ t - \tau \geq 1 \). Then using the assumption (iii) and markovianity of \( H^{s,t} \) we have
\[
Z^{s,\tau} Z^{\tau,t}x = E_{\omega_s} H^{s,\tau}(E_{\omega_\tau} H^{\tau,t}(x \otimes \mathbf{I}) \otimes \mathbf{I}) = E_{\omega_s} H^{s,\tau} H^{\tau,t}(x \otimes \mathbf{I}) = E_{\omega_s} H^{s,t}(x \otimes \mathbf{I}) = Z^{s,t}x,
\]
for every \( x \in \mathcal{M} \otimes \mathcal{M} \), which is the assertion. \( \square \)
Remark 3.4. Consider a q.q.s.p. \( \{P^{s,t}\}, \omega_0 \) of type (A). Let \( H^{s,t}, Z^{s,t} \) be the associated Markov processes. Take any \( \varphi \in S^2 \) then from (3.2) with taking into account (2.1), one concludes that
\[
\varphi(H^{s,t}x) = P^{s,t}_* \varphi(E_{\omega_t}(x)) = P^{s,t}_* \varphi \otimes \omega_t(x),
\] (3.7)
for any \( x \in \mathcal{M} \otimes \mathcal{M} \).

Similarly, using (3.6), for \( Z^{s,t} \) we have
\[
Z^{s,t}_*(\sigma \otimes \psi) = \psi(1)P^{s,t}_*(\sigma \otimes \omega_s) \otimes \omega_t,
\] (3.8)
for every \( \sigma, \psi \in \mathcal{M}_s \).

From Theorem 3.1 and using (3.7),(3.8) one can prove the following

Corollary 3.5. Let \( \{P^{s,t}\}, \omega_0 \) be a q.q.s.p. of type (A) on \( \mathcal{M} \) and let \( \{Q^{s,t}\}, \{H^{s,t}\} \) be its marginal processes. Then the following conditions are equivalent
(i) \( \{P^{s,t}\}, \omega_0 \) satisfies the ergodic principle;
(ii) \( \{Q^{s,t}\} \) satisfies the ergodic principle;
(iii) \( \{H^{s,t}\} \) satisfies the ergodic principle;
(iii) \( \{Z^{s,t}\} \) satisfies the ergodic principle;

3.2. Case type B. Now suppose that a q.q.s.p. \( \{P^{s,t}\}, \omega_0 \) has type (B).

Like (3.2) let us define a process \( h^{s,t}_P : \mathcal{M} \otimes \mathcal{M} \to \mathcal{M} \otimes \mathcal{M} \) by
\[
h^{s,t}_P x = P^{s,t}E_{\omega_t}x, \quad x \in \mathcal{M} \otimes \mathcal{M}.
\] (3.9)

The defined process \( \{h^{s,t}_P\} \) is not Markov, but satisfies other equation. Namely, using (ii)\(_B\) of Def. 2.1 and (3.1) we get
\[
h^{s,t}_P x = E_{\omega_s}P^{s,\tau} \otimes E_{\omega_\tau}P^{s,\tau}(P^{s,\tau}E_{\omega_t}x) = Q^{s,\tau}_P \otimes Q^{s,\tau}_P(h^{\tau,t}_P x),
\]
where \( x \in \mathcal{M} \otimes \mathcal{M} \).

Note that the process \( \{h^{s,t}_P\} \) has the same properties like \( \{H^{s,t}_P\} \).

Similarly to Theorem 3.2 we can formulate the following

Theorem 3.6. Let \( \{Q^{s,t}\} \) be a Markov process and \( \{h^{s,t}\} \) be another processes, which satisfy (i)-(iii) and
\[
h^{s,t} = Q^{s,\tau} \otimes Q^{s,\tau} \circ h^{\tau,t}
\] (3.10)
for any \( \tau - s \geq 1 \), \( t - \tau \geq 1 \). Then by the equality \( P^{s,t}x = h^{s,t}(x \otimes 1) \) one defines a q.q.s.p. of type (B). Moreover, one has \( Q^{s,t} = E_{\omega_s}P^{s,t} \).

Proof. We have to check only the condition (ii)\(_B\). Note that the assumption (iii) implies that
\[
E_{\omega_s}h^{s,t}_s(x \otimes 1) = Q^{s,t}E_{\omega_t}(\cdot \otimes 1) = Q^{s,t}x, \quad x \in \mathcal{M}.
\]

Using this equality with (3.10) for any \( \tau - s \geq 1 \), \( t - \tau \geq 1 \) we have
\[
E_{\omega_s}P^{s,\tau} \otimes E_{\omega_\tau}P^{s,\tau}(P^{\tau,t}x) = E_{\omega_s}h^{s,\tau}_\tau(\cdot \otimes 1) \otimes E_{\omega_\tau}h^{s,\tau}_\tau(\cdot \otimes 1)(h^{\tau,t}_s(x \otimes 1))
= Q^{s,\tau} \otimes Q^{s,\tau}(h^{\tau,t}_s(x \otimes 1))
= h^{s,t}_s(x \otimes 1)
= P^{s,t}x
\]
for any \( x \in \mathcal{M} \).

This completes the proof. \( \square \)
These two processes \( \{Q^{s,t}\} \) and \( \{h^{s,t}\} \) we call *marginal processes* associated with q.q.s.p. \( \{P^{s,t}\} \).

Now define process \( \{z^{s,t} : M \otimes M \to M \otimes M\} \) by
\[
z^{s,t}x = E_{\omega_s} h^{s,t}(x) \otimes 1, \quad x \in M \otimes M.
\] (3.11)

For this process (3.7) also holds.

**Proposition 3.7.** The process \( z^{s,t} \) is a Markov one.

**Proof.** First from Theorem 3.6 and Propsoition 4.3 [10] we conclude that
\[
E_{\omega_s} Q^{s,t} = E_{\omega_t}.
\] (3.12)

Let us take any \( s, \tau, t \in \mathbb{R}_+ \) with \( \tau - s \geq 1, t - \tau \geq 1 \). Then from (3.11) with (3.10),(3.12) one gets
\[
z^{s,t}x = E_{\omega_s} (Q^{s,\tau} \otimes Q^{s,\tau}(h^{\tau,t}(x)) \otimes 1
\]
\[
= Q^{s,\tau} E_{\omega_s} (h^{r,t}(x)) \otimes 1
\]
\[
= Q^{s,\tau} E_{\omega_{\tau}} (h^{r,t}(x)) \otimes 1.
\] (3.13)

On the other hand, using conditions (ii),(iii) we obtain
\[
z^{s,\tau} z^{r,t}x = E_{\omega_s} h^{s,\tau}(E_{\omega_r} h^{r,t}(x) \otimes 1) \otimes 1
\]
\[
= E_{\omega_s} h^{s,\tau} h^{r,t}(x) \otimes 1
\]
\[
= Q^{s,\tau} E_{\omega_r} h^{r,t}(x) \otimes 1
\]
for every \( x \in M \otimes M \). This relation with (3.13) proves the assertion. \( \square \)

**Corollary 3.8.** Let \( \{P^{s,t}\}, \omega_0 \) be a q.q.s.p. of type (B) on \( M \) and let \( \{Q^{s,t}\}, \{h^{s,t}\} \) be its marginal processes. Then the following conditions are equivalent

(i) \( \{P^{s,t}\}, \omega_0 \) satisfies the ergodic principle;
(ii) \( \{Q^{s,t}\} \) satisfies the ergodic principle;
(iii) \( \{h^{s,t}\} \) satisfies the ergodic principle;
(iv) \( \{z^{s,t}\} \) satisfies the ergodic principle;

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