Spin-statistics relation and robustness of braiding phase for anyons in fractional quantum Hall effect

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It is well-known that quasihole excitations in fractional quantum Hall (FQH) systems are anyons exhibiting fractional statistics, but there are many caveats in formulating an appropriate spin-statistics relation. Here we propose the proper definition of the anyon intrinsic spin, which allows us to interpret braiding as an adiabatic process of self-rotation, and thus formulate the spin-statistics theorem for anyons with arbitrary internal structures. This in particular includes deformed anyons, which are both theoretically and experimentally important, since anyons are not point particles in contrast to elementary particles like electrons. We show the intrinsic spin of an anyon comes from part of the intrinsic angular momentum of the particle within a well-defined conformal Hilbert space. It is equal to the topological spin for undeformed anyons. For anyons deformed by local potentials or disorder, the intrinsic spin can be a continuous real number and a measurable quantity in certain braiding schemes, and we discuss its experimental ramifications.

Anyons are theoretically proposed particles in two dimensions with fractional charges and statistics. The adiabatic exchange of two anyons leads to a phase between 0 and π, so the statistics of anyons can be anywhere between a boson and a fermion[1–3]. This possibility has potential applications in quantum computing because the exchange phase, being topological, is robust against local perturbations[3–5]. One of the most promising platform to realize anyons is the fractional quantum Hall (FQH) systems, where anyons are elementary excitations[6–10]. The Laughlin state provides the simplest example: the quasiholes of a Laughlin state are Abelian quasiholes with an exchange phase of νπ where ν is the filling factor. After this was demonstrated by analytical calculation of the adiabatic phase from the model many-body wavefunctions[2, 6], there were efforts to define the anyonic “topological spin” that accounts for this fractional statistics, satisfying the spin-statistics theorem in a manner simillar to fermions and bosons[7, 10–15]. However, a complete understanding of this spin-statistics relation remained elusive[7, 10]. Moreover, the scarcity and the difficulty of accurate measurements in experiments of these fractional braiding phases[16–20] raise the question of how robust this statistical phase is in realistic experimental conditions.

The proper understanding of the anyonic spin-statistics relations is important both theoretically and experimentally. Formally, such relationship for elementary particles requires the full machinery of the relativistic quantum field theory (r-QFT)[21–23]. In its most simplified form, Lorentz invariance (for bosons) and the additional requirement of energy being bounded from below (for fermions) are needed[21]. On the other hand, nonrelativistic explanations employ the intuitive picture that exchanging two particles involves particle self-rotation[3, 24]. This argument relies on a “string attachment”[24] as shown in Fig. 1a. While the existence of such a string cannot be physically justified for point particles in condensed matter systems, quasiparticles (e.g. anyons) are not point particles[25] (see Fig. 1b). Furthermore, since Lorentz invariance is mostly irrelevant in such systems, we can in principle expect a nonrelativistic justification for the spin-statistics relation.

When the positions of two anyons (or quasiholes, we
will use these two terms interchangeably) in the FQH system are exchanged, the wavefunction gains a Berry phase that is the sum of two terms:

$$\gamma_{\text{Berry}} = \gamma_{\text{B-field}} + \gamma_{\text{stats}}$$

(1)

where the first term on the RHS is the Aharonov-Bohm phase from the background magnetic field, and the second term encodes the exchange statistics of the anyons. It is well-known that for the exchange of two quasiholes (each of charge $e/m$) of the Laughlin state at filling factor $1/m$, $\gamma_{\text{stats}} = \pi/m$. This result is generalized for two clusters of “$k$-stacked quasiholes” (from the insertion of $k$ magnetic fluxes at the same point), which have an exchange phase of $\gamma_{\text{stats}} = k^2\pi/m$. On the other hand, the intrinsic quasihole angular momentum can be extracted from the wavefunction on the sphere, either by reading off the clustering property of the wavefunction or by the Berry phase calculation. For a $k$-stacked quasihole the intrinsic angular momentum in the $n$-th Landau level is

$$s_k = -\nu k^2/2 + k/2 + n\nu k.$$  

(2)

It is clear that the standard spin-statistics relation, $\gamma_{\text{stats}} = 2\pi s_k$, is not satisfied if we take $s_k$ as the particle spin (e.g. for $k = \nu = 1$, $\gamma_{\text{stats}}$ is fermionic but $s_k = n$). One proposed remedy is to take the average spin of the quasihole and quasiparticle so that \( S = (S_{\text{ph}} + S_{\text{qe}})/2 = -\nu k^2/2 \) gives the appropriate exchange phase: $\gamma_{\text{stats}} = 2\pi S$. However it is not clear why quasiparticles need to be involved for the braiding of quasiholes, which are perfectly well-defined physical objects. On the disk the quasihole angular momentum can instead be calculated from the electron distribution in the many-body state. One should also note that if the statistical phase from the exchange comes entirely from the quasihoole spin defined in Eq.(2), then it is not compatible with the general argument from the flux attachment that the statistical phase of $k$ identical anyons should be $k^2\theta$, where $\theta$ is the statistical phase of a single anyon. Here, a modification to the standard spin-statistics relation is proposed, which can be generalized to braiding the $k$-stacked quasiholes. The resulting braiding phase reads:

$$\theta_{k,k'} = 2\pi(s_k + s_{k'} - s_{k+k'})$$

(3)

This relationship was only numerically verified for the case of $k = k'$, and it is not entirely clear how self-rotation, thus the intrinsic angular momentum of quasiholes are involved in this generalised relationship. Furthermore, it lacks a microscopic justification: many authors have pointed out that any definition of $s_k = -\nu k^2/2 + \lambda k$ for an arbitrary $\lambda$ will give the correct exchange phase when applied to Eq.(3). However in general (as we will show in subsequent sections) if one allow the quasiholes to take any shape, then the intrinsic angular momentum will not take such nice forms (it will generally be a real number). It is thus necessary to have a rigorous definition of an intrinsic spin for the anyons, that is compatible with both the self-rotation and the exchange phase of the particles. A full microscopic understanding will not only shed light on whether r-QFT is fundamentally necessary for the spin-statistics theorem, but also has experimental implications for the measurement of exchange phases, especially when the anyon shapes are deformed by disorder.

In this paper, we propose an intuitive and rigorous understanding for the relationship between the intrinsic spin and statistics for all types of Abelian anyons. These include not only the special cases of the fermions and bosons, but also anyons from $k$-stacked quasiholes that occupy a finite area, as well as those with an arbitrary shape (e.g. deformed by external potential or disorder). We show that one can treat the physical process of adiabatic exchange as a perturbation to a rotationally invariant Hamiltonian, and it is natural to define a physically relevant particle intrinsic spin (of which the topological spin is the special case) in the exchange process, so that the contribution to the exchange phase comes entirely from particle self-rotation. We use the same approach to compute the exchange and braiding phases for anyons that are deformed into any arbitrary shape. Such deformation is possible because anyons are no longer point particles, and we discuss the possible experimental consequences. While most of the arguments presented are analytic and valid in the thermodynamic limit, we also illustrate with numerical computations some simple examples.

$k$-stacked quasiholes on the sphere– It is useful to first look at a quantum Hall fluid realised on the spherical geometry with a magnetic monopole at the center, generalising the original arguments in Ref. 15. This is done by embedding the two-dimensional system on the surface of the sphere with a magnetic monopole of strength $2S$ placed at its center. Every particle on this system has angular momentum $L_z = -S, -S+1, ..., S-1, S$, where $S$ is an integer due to Dirac quantization, and we define $L_z = S$ as the north pole (so $L_z = -S$ is the south pole). An FQH ground state is always a highest weight state at $L_z = L^2 = 0$. Quasiholes can then be added to this ground state by inserting additional fluxes. In order to add localized quasiholes that are $L_z$ eigenstates, quasiholes are stacked at either the north pole or the south pole.

Starting with the ground state of a generic quantum Hall phase with $N_e$ electrons in a single LL, inserting $k$ magnetic fluxes yields the following relationship:

$$N_\theta = \nu^{-1}N_e - s_e - s_f + k$$

(4)

where $N_\theta$ is the total number of fluxes. Here $s_e, s_f$ are the cyclotron and guiding center topological shifts of
the FQH state, with \( s_f = 2n + 1 \), where \( n \) is the LL index, and \( \nu \) is the filling factor. Note that \( \nu \) and \( s_f \) are topological indices characterising the FQH topological phase (and \( s_f = 0 \) for integer \( \nu \)). If all \( k \) magnetic fluxes are inserted at the north pole, a \( k \)-stacked Abelian anyon is created with charge \( k\nu \) and the corresponding many-body wavefunction has quantum number \( L_z = -kN_e/2 \). An adiabatic rotation of the sphere about the \( z \)-axis by \( 2\pi \) thus gives a Berry phase of \( \gamma_1 = -kN_e\pi \). On the other hand, if we create the \( k \)-stacked anyon at the south pole, the same \( 2\pi \) rotation gives the Berry phase of \( \gamma_1 = kN_e\pi \). The difference between the two phases is given by:

\[
\Delta \gamma_{21} = \gamma_2 - \gamma_1 = 2\pi k\nu N_e + 4\pi (\gamma_{s_1} + \gamma_{s_2} + \gamma_{s_3}) \quad (5)
\]

\[
\gamma_{s_1} = \frac{\nu k}{2} s_e, \quad \gamma_{s_2} = \frac{\nu k}{2} s_f \quad \gamma_{s_3} = -\frac{\nu}{2} k^2 \quad (6)
\]

This phase difference captures the coupling of the \( k \)-stacked anyon with both the total magnetic flux through the sphere surface \((N_e)\), and the total Gaussian curvature of the sphere \([25]\), which is the solid angle of \( 4\pi \). The first term on the R.H.S of Eq.\( (5) \) is the usual Aharonov-Bohm phase from the coupling of the anyon charge \( ke \) to the magnetic flux. Note that for the Laughlin state, \( s_f = \nu^{-1} - 1 \) and the three terms in Eq.\( (5) \) sums up to the intrinsic angular momentum as given in Eq.\( (2) \), but now for an arbitrary topological phase. This is understood as the parallel transport on the sphere induces a self-rotation of the quasihole, contributing to the total Berry phase an amount proportional to its intrinsic angular momentum \([26, 27, 32]\).

The physical operation on the sphere can also be mapped onto the disk by a conformal stereographic projection (see Fig.\( (1) \)). The north pole of the sphere is mapped to the center of the disk while the south pole is mapped to the infinity. A \( 2\pi \) rotation of the quasiholes about the \( z \)-axis on the sphere is then mapped to the adiabatic dragging of the quasiholes in a circular loop on the disk. It is important to note that \( \gamma_1 \) on the sphere corresponds to the self-rotation of the quasiholes at the origin on the disk, but when the quasiholes are adiabatically dragged in a loop, only the disk counterpart of \( \Delta \gamma_{12} \) is measurable, either from the numerical calculation or from the physical Hamiltonian: it gives the total phase that is entirely the Aharonov-Bohm phase from the magnetic field enclosed in the loop (or the first term in Eq.\( (5) \)), since the curvature is zero everywhere on the disk.

This seemingly trivial observation is actually the underlying principle for most of the results in this work. Naively rotationally invariant quasiholes (or any objects) moving in a circular loop on the disk is equivalent to a rotation of the entire system about the origin, the associated Berry phase should be given by \( e^{2\pi i \langle L_z \rangle} \), where \( \langle L_z \rangle \) is the total angular momentum of the quantum state about the origin. Ref.\([17]\) pointed out that this quantity diverges in the thermodynamic limit, and proposed to subtract from it the angular momentum of the neutral ground state (with no quasihole). This gives the angular momentum of the quasihole, which is related to the local deficiency in electron density, also calculated in \([28]\). However, this quasihole angular momentum still does not give the right phase. Instead, the measurable quantity is the excess quasihole angular momentum as the deviation from the angular momentum of the quasihole at the origin.

The microscopic reason for this can be understood by considering a realistic system where the quasihole is trapped and rotated by some potential profile. Consider a physical Hamiltonian as follows:

\[
\hat{H}(\theta) = \hat{H}_0 + \hat{H}_1(\theta) \quad (7)
\]

which is a local trapping potential away from the origin with its position parameterized by the angle \( \theta \); \( \hat{H}_0 \) is a rotationally invariant part (e.g. a Delta function or a cylindrical well centered at the axis of rotation) and \( \hat{H}_1 \) can be considered as a \( \theta \)-dependent perturbation (e.g. the difference between two trapping potentials). Adiabatic tuning \( \theta \) in Eq.\( (7) \) drags the ground state \( |\psi(\theta)\rangle \) adiabatically in a circular loop. We can write:

\[
|\psi(\theta)\rangle = \lambda_0 |\psi_0\rangle + \lambda_1 |\psi_1(\theta)\rangle, \quad \langle \psi_0 | \psi_1(\theta) \rangle = 0 \quad (8)
\]

where \( |\psi_0\rangle \) is the rotationally invariant ground state of \( \hat{H}_0 \) and \( |\psi_1(\theta)\rangle \) is the component orthogonal to \( |\psi_0\rangle \). Tuning \( \theta \) in Eq.\( (7) \) only affects \( \lambda_1 \) and \( |\psi_1\rangle \), and one can thus show \([34]\) that the Berry connection is given by:

\[
A_\theta = - (\langle \psi(\theta) | L_z | \psi(\theta) \rangle - \langle \psi_0 | L_z | \psi_0 \rangle) \quad (9)
\]

giving the contribution only from the excess angular momentum. Intuitively, note that we cannot rotate a perfectly symmetric object on a flat surface, since it is an eigenstate of a rotationally invariant Hamiltonian. Only a deformed quasihole (e.g. by squeezing or dragging it away from the center of rotation) can be rotated, and the Berry phase comes from the change of the angular momentum from the deformation.

The derivation of the spin-statistics theorem—Let us now start with the quantum Hall ground state on the sphere, and insert \((k + k')\) fluxes at the north pole, creating a \((k + k')\)-stacked anyon there. A \( 2\pi \) rotation about the \( z \)-axis gives an adiabatic phase of \( \gamma_3 = -(k + k')N_e\pi \). Next we pull \( k \) of the fluxes to the south pole, leaving behind \( k' \) fluxes at the north pole. The \( 2\pi \) rotation of the resulting state gives the adiabatic phase of \( \gamma_4 = (k - k')N_e\pi \). The difference between these two phases is given by:

\[
\Delta \gamma_{43} = \gamma_4 - \gamma_3 = \Delta \gamma_{21} - 2\pi \nu kk' \quad (10)
\]

where \( \Delta \gamma_{21} \) is the single \( k \)-stacked anyon contribution given in Eq.\( (5) \), thus the extra term is the braiding phase.
due to the presence of the $k'$-stacked anyon at the north pole. We can rewrite $\Delta \gamma_{43}$ as follows:

$$\Delta \gamma_{43} = \Delta \gamma_{21} - 2\pi (s_k^{\text{topo}} + s_{k'}^{\text{topo}} - s_{k+k'}^{\text{topo}})$$

Here we have replaced $s_k$ in Eq. (3) with the more sensibly defined topological spin compatible with the clustering of anyons [10, 26]:

$$s_k^{\text{topo}} = -\frac{\nu k^2}{2}$$

Eq. (11) agrees with Eq. (3), but this calculation reveals the microscopic origin of each terms. We claim the first two terms, $s_k^{\text{topo}}$ and $s_{k'}^{\text{topo}}$, comes from the self-rotation of each quasi-hole stack, while the third term is due to the self-rotation of the $(k + k')$-stacked quasi-hole. Even though this topological spin does not capture the total angular momentum of the quasi-holes, it reflects the intrinsic spin as we will justify later on. A conformal mapping to the plane immediately implies Eq. (11) is valid on a flat surface. Thus for rotationally invariant quasihole, we go back to Eq. (6) where the intrinsic angular momentum of an anyon can be separated into three parts: $L_{cy} = \gamma_{s_1}$ is the cyclotron angular momentum associated with different LLs; $\gamma_{s_2}$ comes from the FQH topological shift, related to the dipole moment at the edge of the QH

Conformal Hilbert space angular momentum– To justify that $s_k^{\text{topo}}$ fully captures the Berry phase of the self-rotation of the rotationally invariant $k$-stacked quasihole, we go back to Eq. (6) where the intrinsic angular momentum of an anyon can be separated into three parts: $L_{cy} = \gamma_{s_1}$ is the cyclotron angular momentum associated with different LLs; $\gamma_{s_2}$ comes from the FQH topological shift, related to the dipole moment at the edge of the QH
fluid and vanishes for the IQHE. Here $L_{LL} = \gamma_{s_2} + \gamma_{s_3}$ is the total guiding center angular momentum within a single LL, and their relationship is illustrated in Fig 2.

The separation of $L_2$ into $L_{cy}$ and $L_{LL}$ is physically motivated, as $L_{LL}$ is well-defined within a sub-Hilbert space (a single LL) spanned by the ground state and quasihole states of the IQHE. For any physical operation within a single LL (e.g. the adiabatic braiding), only $L_{LL}$ is physically accessible. A single LL is an example of the conformal Hilbert space (CHS) introduced in Ref [33], which are Hilbert spaces spanned by the ground state and quasihole states of a particular QH phase. Thus $L_{LL}$ is the angular momentum defined within this conformal Hilbert space, characterised by the guiding center metric, or the “shape” of the CHS. We can deform the shape of a hole within a single LL with a local potential, thus changing the expectation value of $L_{LL}$, while keeping $L_{cy}$ invariant (i.e. in the limit of large magnetic field so there is no LL mixing). An example of this can be seen in Fig 2: the complicated bare electron density can only be resolved with measurements that mixes LLs (when probing with large energy); in the limit of large magnetic field, only the guiding center density can be measured.

Following the same spirit, we can further separate $L_{LL}$ into $\gamma_{s_2}$ and $\gamma_{s_3}$, where the latter is the angular momentum defined within a sub-CHS, the null space of $\tilde{V}_1^{2bdy}$ interaction spanned by the Laughlin ground state and quasiholes, denoted as $H_q$. If we are braiding anyons within $H_q$, as is the case in Eq. (11), then only $\gamma_{s_3}$ is physically relevant. If we deform a Laughlin quasihole with a local potential that is smaller than the incompressibility gap (i.e. $\tilde{V}_1^{2bdy}$ gives the dominant energy scale), only the metric characterising $\gamma_{s_3}$ will be deformed, while that of $\gamma_{s_2}$ will be invariant. An example is given in Fig 3, which shows the results of trapping a Laughlin quasihole with an elliptical potential well in the nullspace of $\tilde{V}_1^{2bdy}$. We see that only the interior of the quasihole is deformed (dark blue region) while the dipole [35] that is a characteristic of $\gamma_{s_2}$ remains circular (yellow region). In complete analogy to Fig 3, the CHS metric only deforms the region associated with $\gamma_{s_3}$.

For Abelian FQH phases, one can understand $\gamma_{s_3}$ more intuitively by a unitary transformation to the composite fermion (CF) basis. Using the Laughlin phase at $\nu = 1/(2q+1)$ as an example, the unitary transformation from the electron basis to the CF basis (where each electron is attached with 2q fluxes) maps the $k$-stacked Laughlin quasihole at the north pole to a product state of $k$ CF holes of the IQH of CFs (see Fig 2). By construction the CFs are particles within $H_q$, and the $k$-stacked CF holes give the CHS angular momentum of $\gamma_{s_3}$ [36].

We are now ready to explicitly write down the general spin-statistics relation:

$$\gamma_{k_1,n_1;k_2,n_2} = 2\pi \left( \tilde{S}_{k_1,n_1} + \tilde{S}_{k_2,n_2} - S_{k_1+k_2}^{\text{topo}} \right)$$  (13)

which generalises Eq.(11), and gives the phase obtained by adiabatically braiding a cluster of $k_1$ quasiholes, around a cluster of $k_2$ quasiholes. Here $\eta_1$ and $\eta_2$ parametrize the deformation of the quasiholes and $S_{k_1,n_1}$ denotes the intrinsic spin of the cluster $k_1$ with deformation $\eta_1$, coming from the CHS angular momentum. If the $k_1$-stack is rotationally invariant, we get $S_{k_1,n_1=0} = S_{k_1}^{\text{topo}}$ in Eq (12), but in general a non-trivial internal structure (e.g. deformed by the CHS metric) can give additional corrections: $S_{k_1,n_1} = S_{k_1}^{\text{topo}} + \delta S_{k_1}$. This can change the braiding phase if braiding also involves the rotation of the quasiholes about their center of mass, reflecting the intrinsic geometric degree of freedom within each CHS (see Fig 2-d).

**Braiding of hollow-core and deformed quasiholes** – The discussion above and Eq (13) are readily applicable to quasiholes of any shape, for which the intrinsic spins can be properly defined. We illustrate here for two cases: the “hollow-core quasihole”, which is a cluster of $k$-stacked
quasiholes that do not have the same angular momentum, and the squeezed quasihole, which is a single quasihole deformed by the relevant CHS metric (guiding center metric for the IQH hole, and the \( V_1 \) nullspace metric for the Laughlin quasihole). In the former case, we find the same braiding statistics for all types of \( k \)-stacked quasiholes as long as rotational invariance is preserved, even though they may have different intrinsic spin (just like fermions can have different half integer spins). This is explained by the generalised spin-statistics relation shown in Eq. (13). In the latter, both the intrinsic spin and the braiding statistics deviate from that of the regular quasihole, while their still obeying the proposed spin-statistics relation. This deviation of the braiding statistics has consequences in experimental measurements. However, only the topological spin of the deformed anyons contribute to the braiding phase, if the braiding does not involve self-rotation of the anyons (see Fig 3a).

An example of the “hollow-core” quasiholes is illustrated in Fig 3b. A cluster of \( k \) Laughlin quasiholes stacked at the north pole as \( 00...010001001001... \) (with \( k - 1 \) leading zeros) is obtained by applying \( L^+ \) on the state \( |00...00100010011...\) once. Consequently, this decreases the angular momentum of the state, and hence the intrinsic angular momentum of the quasihole decreases by 1. Since \( L^+ \) commutes with \( V_1 \), this action occurs within the \( V_1 \) nullspaces and affects only the \( V_1 \) angular momentum (guiding center angular momentum outside \( V_1 \) nullspace and cyclotron angular momentum remain unchanged). The topological spin of this species of hollow-core quasihole is then:

\[
 s_{k \backslash 1}^{topo} = -\frac{V_1^2}{2} - 1 \tag{14}
\]

Here we use the subscript \( k \backslash 1 \) to denote this species of quasihole cluster, with one quasihole detached from the \( k \)-stack. Eq. (14) can also be calculated directly on the sphere in the same manner as the calculation for the \( k \)-stack quasihole as shown in Eq. (13). Using this spin, the braiding property for this quasihole species can also be calculated directly according to Eq. (14), which yields the same result as braiding two regular quasihole stacks up to modulo \( 2\pi \). While this scenario does not give new braiding phase, it illustrates how the intrinsic spin can be determined by the internal structure of the quasiholes.

In the other case, the change in intrinsic spin due to the quasihole’s internal structure can have measurable consequences on the braiding property. We consider here the deformed quasiholes. As discussed above, each CHS comes with an intrinsic geometric degree of freedom, and the quasihole can be arbitrary deformed while still remaining inside the nullspace of the model Hamiltonian. In practice, this is done by the following Hamiltonian:

\[
 H = \lambda_1 V_{model} + \lambda_2 V_{trap}(z) \tag{15}
\]

in the limit \( \lambda_1 \gg \lambda_2 \). Here \( V_{model} \) is the model Hamiltonian for the FQH state (e.g. \( V_{12bdg} \) for the Laughlin state) and \( V_{trap}(z) \) is a local one-body potential that localizes the position of the quasihole. When \( V_{trap} \) does not take a rotationally-invariant profile, the resulting quasihole is deformed within the CHS. For the simplest case, we can look at the deformed \( V_{trap} \) characterised by a single modular metric.

We first consider the exchange of squeezed IQH holes at filling factor \( \nu = 1 \). Here there exists two possible exchange schemes. When the two holes are exchanged along a circle by pure translation, we observe a fermionic statistics expected of the \( \nu = 1 \) phase. However, when each hole self-rotates along the exchange path, there is additional contribution to the statistics. As shown in Fig. 3a, the amount of deviation from fermionic statistics exactly matches the deviation of the intrinsic spin from the topological spin, multiplied by \( 2\pi \). The implication of this is illustrated in Fig. 3b. When self-rotation is involved in the exchange procedure the exchange phase contains both a topological component that depends only on the topological indices of the FQH phase, and a self-rotation component that depends on the internal shape of the quasiholes. In order to observe the topological phase in real experiments, one must ensure that the quasiholes are not rotated by any additional potentials in the environment, such as disorder. When rotation is inevitable, the topological phase can be recovered by subtracting away the self-rotation phase from the measured exchange phase.

Due to the computational constraint, it is difficult to obtain a large enough separation to repeat the above calculation for the squeezed Laughlin quasiholes, since even the computation with first-quantized wavefunction using Monte-Carlo method is numerically expensive. Comparing to the IQHE, the physics should be completely analogous, when we take the intrinsic spin to be the \( V_1 \) nullspace angular momentum instead of the guiding center angular momentum. Numerically, the intrinsic spin can be computed from the fermionization process. Fig. 3c shows the Berry phase from self-rotation of a squeezed Laughlin quasihole agrees very well with its intrinsic spin. We also note that by analogy between the LLL and the \( V_1 \) nullspace, our understanding of the IQH hole from the results describe above can be immediately applied to predict properties of the Laughlin quasiholes. In particular, Fig 3d readily generalizes over all Abelian FQH phases.

Experimental ramifications— The fact that deformation of a quasihole can modify the braiding statistics can have consequences in real experiments. Here we illustrate how the connection between anyonic statistics and rotation can help us understand and improve experimental measurement. We consider a toy model consisting of two Dirac delta potential traps, each pinning a quasihole, on a disk with a potential “wall”. This wall approximates...
a potential trap used in real experiments to contain the entire quantum Hall droplet (see Fig.4a). The potential wall in our model increases linearly in the radial direction, from $V(|z| = R_{\text{disk}} - w) = 0$ to $V(|z| = R_{\text{disk}}) = mw$. (This potential wall is parametrized by the width $w$ and the slope $m$, see Fig.4a.) Two quasiholes are then pinned by Dirac delta potentials, which can be easily manipulated along an exchange path of radius $R$. When $R \sim R_{\text{disk}}$, this setup emulates experiments where the braiding anyons are carried by the edge current [19, 20]. The effect of the wall potential on the anyons’ exchange property can be seen in Fig.4c. Fig.4b shows how the density around the quasihole changes when the slope of the wall is increased. We see that the density increases on the left of the quasihole (positioned at $x = 5.7$ in the plot) and decreases on the right. This signifies a density shift along the $x$-direction, which results in a change in the hole’s intrinsic spin. As a result, even in regions where a fermionic statistics is expected, e.g. around $R = 4.5$, the statistical phase can deviate significantly when the wall potential is non-zero (see Fig.4). In real experiments, the ideal statistics when there is no potential traps (hence no deformation to the anyons) can be obtained by extrapolating measurement at different strengths to the limit $V_{\text{trap}} \to 0$. For our toy model, the deviation of measured statistics from expected statistics, plotted in Fig.4 at different points in which fermionic statistics is expected, shows excellent linear dependent on the strengths of the potential. Our findings imply that detection of anyonic properties is made difficult by the effect of microscopic details of the systems on the quasiholes shape; such effects must be investigated in order to establish a clear statistical phase of topological origin.

FIG. 4: (a) Modelling a 2DEG in a confining potential with a “wall” potential on the quantum Hall droplet. The two IQH holes (red solid circles) are exchanged along a circle of radius $R$ (dashed arrows). (b) Electron density in the region around the hole centered at $R = 5.70$ for a system with 28 electrons ($R_{\text{disk}} \approx 7.74$) with wall potential with $w = 3$ and $m = 200$. The center of the hole is marked with white cross. (c) Deviation in electron density compared to when there is no wall potential ($m = 0$) (d) Deviation from fermionic statistics at different radii of exchange path near the edge of the Hall droplet. Dashed lines show linear fit.

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Online supplementary material for “Spin-statistics relation and robustness of braiding phase for anyons in fractional quantum Hall effect”

We prove that the Berry phase from rotating any quasihole state by rotating potential traps equals $2\pi$ times the excess in angular momentum of the state compared to its rotationally invariant part (Eq. (9) in the main text). We also describe the numerical calculation that can be applied to calculating the Berry phase of any physical procedure within a CHS. This can be used to verify our results regarding the general spin-statistics relation. Finally a brief description of the composite fermionization process is described, which helps to numerically compute the intrinsic spin of the quasihole of any Abelian FQH state.

**ROTATING QUASIHOLE WITH A TRAPPING POTENTIAL**

**Perturbative method**

In an idealized scenario, a quasihole can be localized by applying a potential trap on an electron gas at a certain FQH phase. Any physical operation on the potential trap will affect the quasihole accordingly. Here we consider a quasihole rotating about its own center, which can be realized by rotating the potential trap. This rotation is only physically meaningful if the potential profile is not rotationally invariant. We can write the potential generally as

$$\hat{H}(\theta) = \hat{H}_0 + \lambda \hat{H}_1(\theta)$$

where $\hat{H}_0$ is a rotationally invariant function (e.g. a Dirac delta function or a cylindrical well) and $\lambda \hat{H}_1$ with $|\lambda| \ll 1$ is a small perturbation to break (continuous) rotationally symmetry. The Hamiltonian is parametrized with $\theta$ which denotes its orientation on the $x$-$y$ plane, and we can rotate the Hamiltonian by tuning $\theta$ from 0 to $2\pi$. We consider the result of this rotation on the ground state:

$$\hat{H}(\theta)\psi^{(0)}(\theta)) = E^{(0)}\psi^{(0)}(\theta))$$

(S16)

This rotation gives a Berry phase:

$$\gamma = \int \langle \psi(\theta) | \frac{d}{d\theta} | \psi(\theta) \rangle d\theta$$

(S17)

where by defining the Berry connection

$$A_\theta = i \langle \psi(\theta) | \frac{d}{d\theta} | \psi(\theta) \rangle$$

(S18)

one may note the symmetry of the problem and expect the total phase to be $2\pi A_\theta$ (i.e. the Berry connection is independent of $\theta$). Below we will calculate this Berry connection. Throughout this section we will assume the groundstates of both $\hat{H}_0$ and $\hat{H}(\theta)$ are non-degenerate.

Let $|\psi_0^{(n)}\rangle$ be the $n$-th eigenstate of $\hat{H}_0$:

$$\hat{H}_0 |\psi_0^{(n)}\rangle = E_0^{(n)} |\psi_0^{(n)}\rangle$$

(S20)

The ground state of $\hat{H}_0$ can be written as

$$|\psi^{(0)}\rangle = |\psi_0^{(0)}\rangle + \lambda |\psi_1^{(0)}\rangle$$

(S21)

$$|\psi_1^{(0)}\rangle = \sum_{n>0} \frac{|\psi_0^{(n)}\rangle \langle \psi_0^{(n)}| \hat{H}_0 |\psi_0^{(0)}\rangle}{E_0^{(n)} - E_0^{(0)}}$$

(S22)

where $|\lambda_1^{(0)}\rangle = |\lambda_1^{(0)}(\theta)\rangle$ inherit the $\theta$-dependence from $\hat{H}_1$. Since $|\psi_0\rangle$ is independent of $\theta$, the differential $d/d\theta$ in Eq. (S19) only acts on $|\psi_1\rangle$. To see its result, consider an infinitesimal rotation by $\delta \theta$ is described by the unitary operator $e^{i \delta \theta \hat{L}_z}$, where $\hat{L}_z$ is the angular momentum operator. $\hat{H}_1$ is rotated as

$$\hat{H}_1(\theta + \delta \theta) = e^{i \delta \theta \hat{L}_z} \hat{H}_1(\theta) e^{-i \delta \theta \hat{L}_z}$$

(S23)

$$= \hat{H}_1(\theta) + i \delta \theta [\hat{L}_z, \hat{H}_1(\theta)] + O(\delta \theta^2)$$

(S24)

As a result one finds

$$\frac{d}{d\theta} |\psi_1^{(0)}\rangle = i \sum_{n>0} \langle \psi_0^{(n)} | \langle \psi_0^{(n)} | \hat{L}_z, \hat{H}_1(\theta) | \psi_0^{(0)}\rangle$$

(S25)

Note that since $\hat{H}_0$ commutes with $\hat{L}_z$ (by definition of rotational invariance), every eigenstate of $\hat{H}_0$ is also an eigenstate of $\hat{L}_z$:

$$\hat{L}_z |\psi_0^{(n)}\rangle = \ell_n |\psi_0^{(n)}\rangle$$

(S26)

Thus Eq. (S25) simplifies to

$$\frac{d}{d\theta} |\psi_1^{(0)}\rangle = i \sum_{n>0} \frac{(\ell_n - \ell_0) \langle \psi_0^{(n)} | \hat{L}_z | \psi_0^{(n)}\rangle}{E_0^{(n)} - E_0^{(0)}}$$

(S27)

which gives

$$A_\theta = i |\lambda|^2 \langle \psi_1^{(0)} | \frac{d}{d\theta} | \psi_1^{(0)} \rangle$$

(S28)

$$= -|\lambda|^2 \left( \langle \psi_0^{(0)} | \hat{L}_z | \psi_0^{(0)}\rangle - \ell_0 \right)$$

(S29)

On the other hand, calculating the average angular momentum of Eq. (S25) yields (to the second order in $\lambda$):

$$\langle \psi^{(0)} | \hat{L}_z | \psi^{(0)} \rangle = \langle \psi_0^{(0)} | \hat{L}_z | \psi_0^{(0)}\rangle + |\lambda|^2 \langle \psi_0^{(0)} | \hat{L}_z | \psi_0^{(0)}\rangle$$

(S30)

$$= \ell_0 + |\lambda|^2 \langle \psi_1^{(0)} | \hat{L}_z | \psi_1^{(0)} \rangle$$

(S31)
Thus we get:

$$A_\theta = -\left(\langle \psi^{(0)} | \hat{L}_z | \psi^{(0)} \rangle - \langle \psi_0^{(0)} | \hat{L}_z | \psi_0^{(0)} \rangle\right)$$  \hspace{1cm} (S32)

In other words, the self-rotation phase of a quasihole state is proportional to its excess of angular momentum compared to the rotationally invariant part. Physically this reflects the fact that a perfectly symmetric quasihole cannot be rotated (the Berry connection vanishes as $\theta \to 0$). To observe a nontrivial self-rotational phase, a small perturbation that breaks rotational symmetry has to be added. This results in an additional angular momentum that can be detected as Berry phase.

### Nonperturbative method

In the main text we argue that the quasihole self-rotation discussed above generalizes to rotation of any quasihole state. In particular in the case of multiple quasiholes it gives rise to an additional contribution from the quasihole statistics. To shows this rigorously, in this section we generalize the above discussion to the case with finite $\lambda$.

When the addition to $\hat{H}_0$ is nonperturbative, the eigenstates of $\hat{H}_0$ still form a complete orthonormal basis for the Hilbert space. We can thus expand the ground state in Eq. (S17) as

$$|\psi^{(0)}\rangle = \lambda_0|\psi_0^{(0)}\rangle + \sum_{n>0} \lambda_n|\psi_0^{(n)}\rangle,$$

for $\lambda_0 \in \mathbb{R}$  \hspace{1cm} (S33)

where $\sum_{i \geq 0} |\lambda_i|^2 = 1$ normalizes the state. Here the restriction that $\lambda_0$ serves to fix the gauge of the ground state. Without this gauge fixing, any state of the form $e^{i\phi(\theta)}|\psi^{(0)}\rangle$ for some real function $\phi(\theta)$ would be an equally valid ground state of $\hat{H}$, but this would add an additional contribution to the Berry connection that is ultimately non-physical. Note that in the previous section this restriction was implicitly implemented as the coefficient of $|\psi_0^{(0)}\rangle$ in Eq. (S21) is always unity.

An infinitesimal rotation to $\hat{H}$ results in $(\theta + \delta \theta) = e^{i\delta \theta \hat{L}_z} \hat{H}(\theta)e^{-i\delta \theta \hat{L}_z}$. One can easily check that the ground state of this Hamiltonian is given by the ansatz

$$|\psi^{(0)}(\theta + \delta \theta)\rangle = e^{i\delta \theta(\hat{L}_z - \ell_0)}|\psi^{(0)}(\theta)\rangle$$  \hspace{1cm} (S34)

which also satisfies Eq. (S33). We can then write

$$\frac{d}{d\theta} |\psi^{(0)}\rangle = i \sum_{n>0} \lambda_n \ell_n |\psi_0^{(n)}\rangle$$  \hspace{1cm} (S35)

(note that here Eq. (S20) still applies). Thus the Berry connection can be calculated as

$$A_\theta = i\langle \psi^{(0)} | \frac{d}{d\theta} |\psi^{(0)}\rangle$$

$$= -\sum_{n>0} |\lambda_n|^2 (\ell_n - \ell_0)$$

$$= -\left(\langle \psi^{(0)} | \hat{L}_z | \psi^{(0)} \rangle - \langle \psi_0^{(0)} | \hat{L}_z | \psi_0^{(0)} \rangle\right)$$  \hspace{1cm} (S38)

Thus, we have proven that the Berry phase obtained by rotating any state is equal to the excess of its angular momentum compared to the rotationally invariant part. As discussed in the main text, this is the physical process that describe the Berry phase on the plane which is the stereographic counterpart of the derivation on the sphere. This in turn gives us the microscopic origin of the general spin-statistics relation.

### HOLLOW-CORE QUASIHOLE

We provide here more details of the hollow-core quasiholes discussed in the main text. We illustrate our results with an analytical calculation for the quasihole derivative of the $k$-stack.

#### One quasihole

The intrinsic angular momentum of a hollow-core quasihole can be calculated on the sphere in the same manner as for the $k$-stack in the main text. The Berry phase is given by $2\pi$ times the difference in angular momenta of the states with the quasihole in the south pole and the north pole. At the north pole, a $k$-stack quasihole has angular momentum $kN_c/2$ where $kN_c$ is the number of electrons. A hollow core quasihole is created by applying the $\hat{L}_z$ operator on this state; thus the angular momentum of the state is $kN_c/2 - 1$. Similarly, the hollow core at the south pole has angular momentum $-kN_c/2 + 1$. The total Berry phase is

$$\gamma_1 = 2\pi (\langle L_z \rangle_{\text{south}} - \langle L_z \rangle_{\text{north}})$$

$$= 4\pi (kN_c/2 - 1)$$  \hspace{1cm} (S39) \hspace{1cm} (S40)

On the sphere $N_c$ is related to the number of fluxes $N_\phi$ as

$$k + \nu \frac{1}{N_c} - s_f = \frac{N_\phi}{2} + s_c$$  \hspace{1cm} (S41)

where $s_f$ and $s_c$ are respectively the topological shift and cyclotron shift as discussed in the main text. Substituting this into Eq. (S40), one can see that the intrinsic angular momentum is

$$s_{k\downarrow} = -\frac{\nu k^2}{2} - 1 + \frac{\nu s f k}{2} + \frac{\nu s_c k}{2}$$  \hspace{1cm} (S42)

Thus the topological spin can be read off as

$$s^{\text{topo}}_{k\downarrow} = -\frac{\nu k^2}{2} - 1$$  \hspace{1cm} (S43)
Two quasiholes

Similarly, the total Berry phase for braiding two hollow-core quasiholes can be taken to be 2π times the difference in angular momentum of two states. The first is with each quasihole on one pole (which on the disk corresponds to braiding one quasiholes along an infinitely large circle around the other), and the second is with both quasiholes stacked on the north pole. Here there is an ambiguity in choosing this second state. Intuitively, one might view stacking two k \ 1 quasiholes as a single 2k \ 2 quasiholes. However, the 2k-stack is also a possible choice, as it corresponds to the ground state of a single rotationally invariant local potential \( H_0 \). However, we see that the angular momenta of these two choices differ by 2 (2k \ 2 quasiholes is obtained from a 2k-stack by applying the \( L^- \) operator twice). This corresponds to a difference of 4\( \pi \) in the calculated phase, which has no real physical significance.

Taking the 2k \ 2 quasiholes as the north pole stack, we find that the total Berry phase is

\[
\gamma_2 = -4\pi(Nc/2 - 1) \quad (S44)
\]

which is of similar form to Eq. \( S40 \), but here \( N_c \) and \( N_\phi \) are related as

\[
2k + \nu^{-1}N_c - s_f = N_\phi + s_c \quad (S45)
\]

Taking the difference in the total phase, we find that the braiding phase is

\[
\gamma_{br} \equiv \gamma_2 - \gamma_1 = 2\pi\nu q^2 \quad (S46)
\]

which is the same as braiding two regular \( k \)-stacks. This is rather unsurprising and demonstrate the argument that the braiding phase of two quasiholes comes from the deficiency in electron density within the enclosed area\[6, 27\]: both the \( k \)-stack and the hollow core quasiholes result in the same amount of deficiency, only with different shapes (see Fig. S1). However, comparing this braiding phase with the topological spin in Eq. \( S43 \), we emphasize that the general spin-statistics theorem discussed in the main text gives the right relation between the two quantities.

**NUMERICAL METHOD**

**Berry phase calculation**

The ground state of any Hamiltonian can be found by exact diagonalization up to a random phase. The Berry phase can be obtained by calculating

\[
\mathcal{D} e^{i\gamma(N)} = \prod_{i=1}^N \langle \psi(\theta_i) | \psi(\theta_{i+1}) \rangle, \quad N + 1 \equiv 1 \quad (S47)
\]

where \( |\psi(\theta_i)\rangle \) is the ground state of \( H(\theta_i) \) and 0 = \( \theta_1, \theta_2, ..., \theta_N = 2\pi \) are points along the closed loop in parameter space. The final term, \( \langle \psi(\theta_N) | \psi(\theta_1) \rangle \) is added to remove the random gauge that results from diagonalizing each \( H(\theta_i) \) independently so that the quantity in Eq. \( S47 \) is gauge-independent. In the limit \( N \rightarrow \infty \), \( \mathcal{D} \rightarrow 1 \) and \( \gamma(N) \rightarrow \gamma \), the total adiabatic phase accumulated. In practice, for a finite \( N \), the closeness of \( \mathcal{D} \) to unity indicates how well \( \gamma(N) \) approximates the Berry phase\[38\]. All numerics presented here are done on the disk geometry.

**One-body local potential**

Eq. \( S47 \) allows us to numerically calculate the statistical phase for any exchange and braiding scheme, so long as a Hamiltonian and its ground state can be constructed. The Hilbert space of the lowest Landau level (LLL) is spanned by many-body states which are product states of the single-particle wavefunction \( \phi_{m}(z) = z^m e^{z|z|^2/4\ell_B^2} \) where \( z = x + iy \) and \( \ell_B \) is the magnetic length. These many-body wavefunction can be written as anti-symmetric monomials of variables \( \{z\} \equiv (z_1, z_2, ..., z_N) \). We denote each state in this basis as \( |\psi_\lambda\rangle \) parametrized by some \( \lambda \) (which is a partition that characterizes the monomial\[39\]). Given a one-body potential profile \( V(z) \), its matrix representation in the monomial basis can be calculated as

\[
V_{\mu\nu} = \int d^2z V(z) \langle \psi_\mu | \hat{\rho}(z) | \psi_\nu \rangle \quad (S48)
\]

where \( \hat{\rho}(z) \) is the density operator. This matrix can in turn be used to construct the Hamiltonian in the conformal Hilbert space (CHS). A basis that span a CHS can be constructed numerically using either the Jack polynomials\[40\], or the local exclusion constrain formalism\[41\]. The integral in Eq. \( S48 \) is calculated with Monte-Carlo integration by uniform sampling over a disk.
FIG. S2: (a) Two different exchange/braiding schemes are possible when the quasiholes are not rotationally invariant. (b) Exchanging elliptical holes (IQH state at $\nu = 1$) with rotation shows deviation from the fermionic statistics, calculated for $N = 24$ particles.

of radius $R_{\text{disk}} = \sqrt{2N_o}$ where $N_o$ is the number of orbitals in the finite system. (In the numerics the magnetic length $\ell_B = \sqrt{\hbar/eB}$ is set to one.)

A “squeezed” quasihole can be constructed as the ground state of an elliptical pin:

$$V_{r_0,R}^{\text{ellipse}}(r) = \begin{cases} V_0 & d^a d^b g_{ab} < R, \mathbf{d} \equiv \mathbf{r} - \mathbf{r}_0 \\ 0 & \text{otherwise} \end{cases}$$

(S49)

which is non-zero only within an elliptical region with area $\pi R^2$ centered at $\mathbf{r}_0 = (x_0, y_0)$. The shape of the ellipse is parametrized by a unimodular metric $g_{ab}$:

$$g = \begin{pmatrix} \cosh \theta + \sinh \theta \cos \phi & \sinh \theta \sin \phi \\ \sinh \theta \sin \phi & \cosh \theta - \sinh \theta \cos \phi \end{pmatrix}$$

(S50)

Here $\theta$ parametrizes the squeezing ratio and $\phi$ parametrizes the rotation w.r.t. the $x$-axis. This potential can be used to numerically study the self-rotational phase of one quasihole and the exchange property of two quasiholes. In the case of exchanging two quasiholes, we construct a Hamiltonian

$$H(R, \theta) = V^{\text{ellipse}}_{(-x_0,-y_0),R} + V^{\text{ellipse}}_{(x_0,y_0),R}$$

$$x_0 = R_0 \cos \theta$$

$$y_0 = R_0 \sin \theta$$

(S51)

and tune $\theta$ between $\left(0, \pi\right)$. The Berry phase is calculated as Eq. (S47). In general, the exchange statistics is obtained at large separation distance: $2|r_0| \to \infty$ where $|r_0| = \sqrt{x_0^2 + y_0^2}$. However on a finite system this is not possible and one must ensure the two quasiholes are well-separated without falling off the edge of the quantum Hall droplet (Fig. S3). This drastically limit the possible cases that can be studied numerically even for simple states such as the Laughlin $1/3$ state. However we claim that the results from the study on the IQH state at $\nu = 1$ shown below generalizes to all Abelian FQH states.

Exchanging two elliptical holes

For the IQH state on the LLL, each quasihole is a hole with fermionic statistics. For two squeezed IQH holes, one can verify that the fermionic statistics is observed for any squeezing ratio provided that the exchange procedure does not rotate them. However, when the squeezed holes are exchanged with self-rotation, we see deviation from the fermionic statistics (see Fig. S3). Here the exchange statistics is taken to be the values of the plateau formed in the middle at separation distance between 6 and 8. A separation too small means the quasiholes are not well separated, while on a finite systems a separation too large puts the quasiholes too close to the edge of the quantum Hall droplet. In Fig S3 only figure b shows two quasiholes that are both separate from each other and far away from the edge of the disk. In numerics we are only interested in this region.

To tell the exchange statistics apart from such finite size effect, we plot the exchange phase (difference in the total phase from the enclosed area) against separation distance, as in Fig S2. On a large enough disk it is possible to observe a range of separation in which the exchange phase is relatively constant. In Fig S2 we take the exchange phase to be the values averaged within the range $R \in \left(6, 8\right)$. This value decreases from 0.5 as the squeezing ratio is increased. As shown in the main text, the amount of deviation agrees well with the phase gained by a single elliptical hole rotating about its center by $2\pi$. This lead us to conclude that nontrivial self-rotation phase results in additional contribution to the quasihole statistics apart from the topological spin. Both contributions are captured by the proposed intrinsic spin.
FIG. S3: Density of two squeezed IQH holes with squeezing ratio \( r = 2 \) at different separation distances: (a) \( 2|r_0| = 2 \) (b) \( 2|r_0| = 5 \) (c) \( 2|r_0| = 8 \)

**COMPOSITE FERMIONIZATION BETWEEN CONFORMAL HILBERT SPACES**

The CF formalism

In this section we will discuss a powerful tool of dealing with FQH phases, namely the composite fermionization method (fermionization for short)\[36\]. In particular we will focus on the isomorphism between the Hilbert spaces before and after the fermionization, which allows us to define the angular momentum within a specific conformal Hilbert space (CHS) as shown in the main text.

As a phenomenological theory, composite fermionization combines one electron with even number of magnetic fluxes into a new bound state named as composite fermion (CF), which will significantly decrease the difficulty of resolving the interactions between particles and transform a strong-coupling FQH phase of electrons to some effective weak-interacting phase (such as an IQH phase) of CFs. Besides the vivid electron-flux binding illustrations, the CF picture can also provides the microscopic wavefunction for many states. The validity of the whole CF theory is built upon the assumption that the flux quanta are attached to electrons with a specific ratio and the gap will not close during the whole process of flux binding, neither of which is easy to be proven generically, though. However, inspired by the idea of fermionization, one can also focus on some aspects of the Hilbert space to pursue a more rigorous definition of fermionization\[36\]. Below we will show one approach from the invariant counting pattern of \( L_z \) sectors.

Here we take the CF IQH state with \( \nu^* = 1 \) and the Laughlin state of electrons with \( \nu = 1/3 \) on the disk as an example. The generating function of the degeneracy of the \( L_z \) sector with \( N_e \) electrons and \( N_o^e \) orbitals within \( \mathcal{H}_1 \) (nullspace of \( \hat{V}_1 \)) is given by:

\[
G_e(q) = q^{N_e(N_e-1)/2} \left[ N_o^e - 2N_e + 2 \right] \tag{S52}
\]

Here \( q \) is just a dummy variable whose coefficient gives the counting of the sector with \( L_z \) equal to its power. Meanwhile the generating function with \( N_{CF} \) CFs and \( N_o^{CF} \) orbitals in the lowest CF level can be written as:

\[
G_{CF}(q) = q^{N_{CF}(N_{CF}-1)/2} \left[ N_o^{CF} \right] \tag{S53}
\]

Thus to make the Hilbert spaces isomorphic with respect to the \( L_z \) sectors, we should look for a specific relation between \( \{N_e, N_o^e\} \) and \( \{N_{CF}, N_o^{CF}\} \) that makes the following relation holds:

\[
G_e(q) = \mathcal{G}_{CF}(q) \cdot q^n, \quad n \in \mathbb{Z} \tag{S54}
\]

which means the \( L_z \) sector countings of electrons and CFs are identical up to a shift of \( L_z \). An easy observation is that if we have:

\[
N_e = N_{CF}; \quad N_o^{CF} = N_o^e - 2N_e + 2 \tag{S55}
\]

then the CHS of electrons and the corresponding CF Hilbert space will be isomorphic, i.e.

\[
\mathcal{H}_1(N_e, N_o^e) \cong \mathcal{H}_{LLL}(N_{CF}, N_o^{CF}) \tag{S56}
\]

Note that the second equation in Eq \[S55\] suggests that we should combine each electrons with two fluxes, which is exactly the same as the statement form the original CF theory. Moreover two additional orbitals should be added for the CFs, which will make the isomorphism holds for systems of any sizes rather than only the thermodynamic limit. This formalism can be used for other geometries as well, for example it can naturally show that the fermionization commutes with angular momentum on the sphere.

One of the simplest cases is the fermionization of the squeezed Laughlin state at \( \nu = 1/3 \) with a single quasi-hole. The squeezing will mix different \( L_z \) sectors so we can generically express this state as:

\[
|\psi_e\rangle = \sum_{m, \alpha} c_{m, \alpha} |m, \alpha\rangle_e \tag{S57}
\]
where \( m \) is the \( L_z \) quantum number and \( \alpha \) denotes the degeneracy in each \( L_z \) sector, which will be 1 within \( \mathcal{H}_1 \).

Thus there is a one-to-one correspondence between each \( L_z \) sector of electrons’ and CFs’ Hilbert spaces and all we need to do is to replace \( |m, 1\rangle_c \) with the corresponding \( |m, 1\rangle_{CF} \). For the general case of \( \alpha > 1 \), the fermionization is given by a unitary transformation between the Hilbert spaces of electrons and CFs.

### Calculating the topological spin

The topological spin of an FQH quasihole is defined in the main text as the component of the intrinsic angular momentum within its CHS. In practice it is difficult to calculate this quantity exactly, as compared to the cyclotron angular momentum \( \hat{L}_{cy} = -\hat{a}^\dagger \hat{a} + 1/2 \) or the guiding center momentum \( \hat{L}_{gc} = \hat{b}^\dagger \hat{b} + 1/2 \), there is no expression for the CHS angular momentum in terms of the single-particle cyclotron operators \( \hat{a}, \hat{a}^\dagger \) and guiding center operators \( \hat{b}, \hat{b}^\dagger \). (This fact is a simple example of how within a CHS the fundamental degree of freedom is no longer single electrons, but involves many-electron interactions.) However, for Abelian FQH states, the fermionization provides a simple way to numerically compute the CHS angular momentum by mapping it to the guiding center angular momentum of CFs.

Here we take the Laughlin quasihole as an example. The quasihole is ensured to reside completely within the CHS that satisfies: \[
\langle \psi | \hat{L}^V | \psi \rangle = \langle \psi | \hat{L}^C | \psi \rangle = \langle \psi | \hat{L}^G | \psi \rangle = \langle \psi | \hat{L}^H | \psi \rangle
\]

Eq. (S59) can be understood as a result of treating the CF as the fundamental degree of freedom within the nullspace of \( \hat{L}^H \). Therefore, rotating a Laughlin quasihole within \( \hat{L}^H \) nullspace can be viewed as rotating a CF hole within a single CF level, which is exactly analogous to rotating a hole within the single LL. With this understanding, the discussions of the braiding properties with numerical evidences on the IQH states presented in the main text and in the sections above can be generalized to all Abelian states. In particular, metric deformation within a CHS leads to a change in the intrinsic spin, which in turns affect the exchange statistics according to our generalized spin-statistics relation.

where \( \langle \psi | L_z^V | \psi \rangle = \langle \psi | L_z^C | \psi \rangle = \langle \psi | L_z^G | \psi \rangle = \langle \psi | L_z^H | \psi \rangle
\]

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