Spin and Gauge Systems on Spherical Lattices

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We present results for 2D and 4D systems on lattices with topology homotopic to the surface of a (hyper) sphere $S^2$ or $S^4$. Finite size scaling is studied in situations with phase transitions of first and second order type. The Ising and Potts models exhibit the expected behaviour; for the 4D pure gauge $U(1)$ theory we find consistent scaling indicative of a second order phase transition with critical exponent $\nu \simeq 0.36(1)$.

1. INTRODUCTION

Some time ago it was suggested to study certain systems on lattices with trivial first homotopy group \[1\]. The D-dimensional boundaries of (D+1)-dimensional hypercubes have this property, being homotopic to $S^D$. Here we give a brief summary of recent results obtained by various subgroups of the present authors for such studies for compact $U(1)$ pure gauge theory with mixed action \[2\], for the 2D Ising model \[3\] and for the 2D Potts model \[4\].

The original motivation was to check the possible relevance of periodic boundary conditions for the dynamics of looplike excitations (like monopole loops in compact U(1) gauge theory). In the thermodynamic limit at the critical point the manifold becomes flat and should lead to a universal field theory independent of the details of the boundary conditions. Other 2D studies on similar lattices include random as well as tetrahedral lattices \[5,6\] and a recent investigation of the 7-vertex model (the strong coupling Schwinger model) for a pillow-like topology \[7\].

We studied the following lattices:

- $T[L]$ periodic b.c., i.e. torus topology for the $L^D$ hypercubic lattice.
- $SH[L]$ the surface of a $D + 1$-dimensional $L^{D+1}$ hypercube, or its dual lattice $SH'$ (where points of $SH$ are identified with $(D-1)$-dimensional objects of $SH'$ etc.)
- $S[L]$ like $SH$ but with weight factors correcting for a spherelike, smooth distribution of the curvature over the lattice.
- $PI[L]$ a pillow-like shape, like the surface of a $L^D \times 1$ hypercube; may be visualized as two copies of a $L^D$ hypercube glued together at their boundaries.

In the finite size scaling (FSS) analyses we use $V^{1/D}$ as the generic lattice size parameter, where the volume $V$ denotes the number of sites or plaquettes on 2D or 4D lattices, respectively.

2. MONTE CARLO SIMULATIONS

The Monte Carlo updating details depend on the studied system and are discussed below. In all cases bulk quantities were obtained at various values of the coupling and combined with the multihistogram method \[8\], giving

$$Z_L(\beta) = \sum_E \rho_L(E) \exp(-\beta E).$$

(1)

This allows us to obtain continuous curves for the observables and to determine the complex positions of the partition function zeroes. Also the combined data for the “multihistogram” provides
for a reliable identification of possible 2-state signals in the distribution.

We studied the peak positions (pseudocritical couplings $\beta_{c,V}$) and peak values of the cumulants

$$C_V(\beta, L) = \frac{1}{V} \langle (E - \langle E \rangle)^2 \rangle, \quad (2)$$

$$V_{BCL}(\beta, L) = -\frac{1}{3} \frac{\langle (E^2 - \langle E^2 \rangle)^2 \rangle}{\langle E^2 \rangle^2}, \quad (3)$$

$$U_4(\beta, L) = \frac{\langle (E - \langle E \rangle)^4 \rangle}{\langle (E - \langle E \rangle)^2 \rangle^2}, \quad (4)$$

and the positions $z_0$ of the Fisher zeroes closest to the real axis.

For 1st order transitions one expects the FSS behaviour

$$C_{V,\max} \rightarrow \frac{1}{V} (\epsilon_o - \epsilon_d)^2, \quad (5)$$

$$V_{BCL,\min} \rightarrow -\frac{1}{12} (\epsilon_o \epsilon_d)^2, \quad (6)$$

$$U_{4,\min} \rightarrow 1 \quad (7)$$

$$\beta_{c,V} = \beta_c + aV^{-1} + O(V^{-2/D}) \quad (8)$$

with “surface” corrections of $O(V^{-2/D})$ due to the inhomogeneities of the boundary (curvature); in $D = 4$ this is the leading term $[1]$. The parameter $a$ in $[1]$ depends on the considered cumulant.

For 2nd order transitions we expect

$$C_{V,\min} \simeq V^{\alpha/D
u} \quad (9)$$

$$V_{BCL,\max} \simeq V^{\alpha/D\nu - 1} \quad (10)$$

$$\beta_{c,V} = \beta_c + aV^{-1/D\nu} + O(V^{-2/D}) \quad (11)$$

$$Im \; z_0(V) \simeq V^{-1/D\nu} \quad (12)$$

with Josephson’s law $\alpha = 2 - D\nu$ (for Gaussian systems $\nu = \frac{1}{2}, \alpha = 0$; for a discontinuous transition $\nu = 1/D$). Again in $\beta_{c,V}$ one expects “surface” corrections of $O(V^{-2/D})$.

### 3. 2D ISING MODEL

This was studied in $[3]$ in order to identify the FSS effects of the new topology in more detail. We used lattices of various size and of type $T$ (where there exists an analytic solution), $SH'\langle 16 \rangle \ldots SH'\langle 128 \rangle$, $S\langle 16 \rangle \ldots S\langle 128 \rangle$ and $P\langle 16 \rangle \ldots P\langle 128 \rangle$. The updating was with the Swendsen-Wang Cluster algorithm with typically $10^7$ measured configurations (with an autocorrelation time < 5) per lattice size.

The results were typical for the other systems as well. Since $\nu = 1$ one expects a leading FSS behaviour $O(1/L)$. The pseudocritical points for the spherelike topologies are – compared to the torus values – very close to their thermodynamic values and exhibited a very small contribution $O(1/L)$; the scaling behaviour is dominated by the non-leading term $O(1/L^2)$ which we interpret as coming from the boundary or curvature, respectively. This holds also for the real part of the closest Fisher zero.

![Figure 1. Ising model: closest Fisher zeroes.](image)

Both, the peak value of $C_V$ as well as $Im \; z_0$, however, show scaling in perfect agreement with $\nu = 1$. From fig. $[3]$ we find, that for the spherelike topology the zeroes approach the real axis in the complex coupling constant plane almost perpendicularly. Thus the real part (related to the pseudocritical points derived from the cumulants) is always very close to the thermodynamic
value with its scaling behaviour dominated by the $O(1/L^2)$ term.

4. 2D POTTS MODEL

In [4] the $q$-state Potts model for $q = 5, 7, 10$ at the (for $q = 5$ extremely weak) 1st order transition is studied with multicanonical updating [9], typically $10^7$ updates for each lattice size. The lattices were of type $T(20)\ldots T(50)$, $SH(10)\ldots SH(22)$ and $PI(16)\ldots PI(36)$.

Even for the weak signal of the $q = 5$ model the histogram for the energy distribution exhibits a clear double peak structure (cf. fig. 2). Thus the spherical lattices do no “wash out” weak 2-state signals, as might be suspected. Note that there is only one interface as compared to two for the periodic boundary conditions. This difference may be of some technical advantage in the determination of the interface tension.

5. 4D GAUGE THEORY

The study [1] of compact $U(1)$ pure gauge action in the Wilson formulation with an additional adjoint term (coupling $\gamma$) has been continued [2].

For $\gamma = 0$, studies for torus geometry find consistently stable 2-state signals, reconfirmed by the recent results of [10] and [11] (who in particular emphasized the role of monopole clusters for the phase transition). Already in 1985 a careful study of this action at positive $\gamma$ hinted at a tricritical point (TCP) [12] near $\gamma_{TCP} = -0.11$. However, 2-state signals have been observed even below that value [12]. On the other hand, studies for spherelike topology show no 2-state signals in the energy distribution at $\gamma \leq 0$ [1] for the studied lattices $L \leq 12$ (although they do show 2 states at e.g. $\gamma = 0.2$).

Independent of the issue of a possible 2-state signal for the Wilson action there are no generally agreed results and values for critical indices in the $U(1)$ gauge model.

In order to respond to a possible criticism of our original investigation of the model on $SH$-lattices – where the curvature was concentrated on the edges and corners of the hypercubic lattice – we have continued work on an “almost smooth sphere” $S(L)$. There we project sites, links and plaquettes of $SH(L)$ or $SH'(L)$ (dual) onto the surface of 4D spheres and introduce weight factors similar to those used by [13] in their study of random triangulated lattices,

$$S = -\sum_P \frac{A'_P}{A_P} [\beta \cos(U_P) + \gamma \cos(2U_P)] .$$

Here, $A_P$ and $A'_P$ denote the areas of the corresponding plaquette (and its dual, respectively) of the projected lattice.

More technical details of the simulation have been discussed [1] and will be discussed elsewhere [2]. We have worked with lattices $S(L)$ for $L$ ranging between 4 and 12. The couplings were chosen in the immediate neighbourhood of the critical values of $\beta$ for $\gamma = 0, -0.2, -0.5$. Since we never observed 2-state signals we did not implement multi-canonical updating. For each lattice size we typically accumulated $10^6$ updates (3-hit Metropolis, for $\gamma = 0$ with an additional overrelaxation step). Close to the phase transition we found quite large autocorrelation lengths (300-2000).

We find no indication of 2-state signals in the multi-histogram distributions. The values of the cumulants at their respective extrema are compatible with a continuous phase transition. Like
for the Ising model, also here we observe in general smaller FS corrections than for the torus geometry (periodic b.c.). The leading behaviour seems to be dominated again by the boundary or curvature term, i.e. suppressed by $O(1/L^2)$, which at first sight cannot be distinguished from a leading scaling behaviour for $\nu = 0.5$.

The peak values of $C_V$ show power law behaviour and in the table we give the range of values for $\alpha/\nu$ (and the resulting $\nu$, if we use Josephson’s law) from fits to data for all lattice sizes or without the largest and/or smallest lattices. These values appear to depend on $\gamma$. One has to keep in mind, however, that the FSS law used for this fit is asymptotic and that there is a regular contribution to the value of $C_V$ which may be non-negligible for the lattice sizes entering the fit.

Table 1

| $\gamma$ | $\alpha/\nu$ | $\nu$ | $\nu$ |
|----------|--------------|-------|-------|
| 0        | 1.50-1.54    | 0.35-0.37 | 0.35(1) |
| -0.2     | 1.10-1.38    | 0.37-0.39 | 0.37(1) |
| -0.5     | 0.86-1.02    | 0.40-0.41 | 0.36(1) |

If we study the closest Fisher zeroes (in the complex $\beta$-plane), we find a behaviour reminiscent to the discussed Ising results. Fig. 3 exhibits FSS consistent with $\nu = 0.36(1)$ for all three values of $\gamma$ (cf. the results of the fit in table 1). Once again Im $z_0$ gives the cleanest and most consistent result.

We should mention, that this value of $\nu$ is compatible with neither the Gaussian value 0.5 nor the “discontinuity” value 0.25; it is compatible with results obtained in the mid-80s with MCRG methods. More detailed analyses with improved statistics will be presented elsewhere.

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Figure 3. We find consistent FSS for Im$(z_0)$ for various values of $\gamma$