TANGENT LIE ALGEBRA OF A DIFFEOMORPHISM GROUP AND APPLICATION TO HOLONOMY THEORY

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Abstract. In this paper we introduce the notion of tangent space \( T_eG \) of a (not necessarily smooth) subgroup \( G \) of the diffeomorphism group \( \text{Diff}^\infty(M) \) of a manifold \( M \). We prove that \( T_eG \) is a Lie subalgebra of the Lie algebra of smooth vector fields on \( M \). The construction can be generalized to subgroups of any (finite or infinite dimensional) Lie groups. The tangent Lie algebra \( T_eG \) introduced this way is a generalization of the classical Lie algebra in the smooth cases. As a working example we discuss in detail the tangent structure of the holonomy group and fibered holonomy group of Finsler manifolds.

1. Introduction

Important geometric objects, structures or properties can often be investigated through algebraic structures. In many interesting cases, these algebraic structures are groups, where the group operations are smooth maps. Such groups became indispensable tools for modern geometry, analysis, and theoretical physics. Lie groups and diffeomorphism groups are the most important examples for such structures.

Considering a Lie group \( G_L \), it is well known that most of the important information about it is captured in its tangent object, the Lie algebra \( g_L \). Naturally, if \( G \) is a Lie subgroup of \( G_L \), then its Lie algebra \( g \) is a Lie subalgebra of \( g_L \). The Lie subalgebra \( g \subset g_L \) can be used to obtain information or eventually to determine the subgroup \( G \). In many relevant geometric situations, however, this framework is not general enough because of two factors: Firstly, \( G_L \) is not a (finite dimensional) Lie group but the (infinite dimensional) diffeomorphism group \( \text{Diff}^\infty(M) \) of some manifold \( M \). Secondly, the subgroup \( G \) is not necessarily a Lie subgroup of \( \text{Diff}^\infty(M) \). Nevertheless, natural questions arise: can we introduce a tangential property and tangent objects to the subgroup \( G \) in this situation? Does the set of tangent elements possess a special algebraic structure? Can this algebraic structure be used to get information about the subgroup and thus about geometric properties?

In this paper we answer these questions.

We introduce the notion of tangent vector fields to a subgroup \( G \) of the diffeomorphism group \( \text{Diff}^\infty(M) \). Denoting by \( T_eG \) the set of tangent vector fields to \( G \) at the identity, we prove that \( T_eG \) is a Lie subalgebra of the Lie algebra of smooth vector fields on \( M \) (Theorem 3.4). It follows that subalgebras of \( T_eG \) inherit the tangential properties, therefore the elements of a subalgebra generated by vector fields tangent to the subgroup \( G \) are tangent to \( G \) (Corollary 3.6). This property can be particularly interesting when the Lie bracket of two tangent vector fields to \( G \) generates a new direction: the tangential property will be satisfied in this new direction as well. As we show in Theorem 3.10, the group generated by the exponential image of \( T_eG \) is a subgroup of the closure of \( G \) in \( \text{Diff}^\infty(M) \) which can give important information about the group \( G \) itself, especially in the infinite dimensional cases.

We note, that a similar tangential property was already introduced in [10, Definition 2], but we also remark that the concept had two major defects: the tangent property

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introduced in [10] is not preserved under the bracket operation, therefore in that approach it is not true that tangent vector fields to a subgroup \( \mathcal{G} \) generate a tangent Lie algebra to \( \mathcal{G} \). Secondly, [10] was not able to guaranty the existence of the tangent Lie algebra \( T_G \) associated to \( \mathcal{G} \). We note that with our approach we are able to overcome both deficiencies.

The main reason to investigate the tangent structure of a subgroup \( \mathcal{G} \) of the diffeomorphism group is that it can provide valuable information about the group \( \mathcal{G} \) itself. This method can be very effective when \( \mathcal{G} \) is for example a symmetry group, the holonomy group, etc. We note that in many cases the determination of \( T_G \) or its subalgebras can be highly nontrivial, especially in the infinite dimensional cases. As working examples, we consider the holonomy group and the fibered holonomy group of Finsler manifolds. The holonomy group is the transformation group generated by parallel translations with respect to the canonical connection along closed curves. For Riemannian manifolds it has been extensively studied and now the complete classification is known [2, 1, 3, 6]. In particular, it is well known, that the holonomy group of a simply connected Riemannian manifold is a closed Lie subgroup of the special orthogonal group \( SO(n) \). Despite the analogues in the construction, Finslerian holonomy groups can be much more complex and up to now, we do not know much about them: For special spaces the holonomy can be a finite dimensional Lie group (see [17] and [7]), but recent results show that there are Finsler manifolds with infinite dimensional holonomy group [12, 13, 14]. These latter results show the difficulties: on cannot use the well understood principal bundle machinery in the investigation because the structure group should be infinite dimensional. In [9] P. Michor proposed a general setting for the study of infinite dimensional holonomy groups and holonomy algebras which was the motivation for Z. Muzsnay and P.T. Nagy to start investigating the tangent objects to a subgroup of the diffeomorphism group [10]. In this paper we are able to step forward: using the results of Chapter 3 we are able to introduce the notion of holonomy algebra and fibered holonomy algebra for Finslerian manifolds. By improving the results of [10] we also prove that the curvature and the infinitesimal holonomy algebras (resp. their restrictions) are Lie subalgebras of the fibered holonomy algebras (resp. the holonomy algebra). We are confident that in the future, the tools described above can be used successfully in the investigation of geometric structures in general and in the holonomy theory in particular.

2. Preliminaries

In this chapter we introduce the basic notions and concepts of Finsler geometry which are necessary to understand in Chapter 4.1 and Chapter 4.2 the nontrivial application of the theory discussing the tangent structure of a subgroup of the diffeomorphism group. These notions are not necessary to understand Chapter 3 therefore the reader who is not particularly interested in these applications, can jump directly to the next chapter.

In this paper, \( M \) denotes a \( C^\infty \)-smooth \( n \)-dimensional manifold, \( \mathfrak{X}^\infty(M) \) is the Lie algebra of \( C^\infty \) vector fields and \( \text{Diff}^\infty(M) \) is the group of \( C^\infty \) diffeomorphisms of \( M \). We will denote by \( TM \) the tangent manifold and by \( \tilde{TM} = TM \setminus \{0\} \) the slit tangent manifold. Local coordinate charts \( (U, x^i) \) on \( M \) induce local coordinate charts \( (\pi^{-1}(U), (x^i, y^j)) \) on \( TM \), where \( \pi : TM \to M \) is the canonical projection. The vertical distribution \( \mathcal{V}TM \subset TTM \) on \( TM \) is given by \( \mathcal{V}TM = \ker \pi_* \).

2.1. Finsler manifold.

A Finsler manifold is a pair \( (M, \mathcal{F}) \), where the norm \( \mathcal{F} : TM \to \mathbb{R}_+ \) is a positively 1-homogeneous continuous function, which is smooth on \( \tilde{TM} \) and the symmetric bilinear form

\[
g_{x,y} : (u, v) \mapsto g_{ij}(x, y)u^iv^j = \frac{1}{2} \frac{\partial^2 \mathcal{F}_x^2(y + su + tv)}{\partial s \partial t} \big|_{t=s=0}
\]
is positive definite at every \( y \in T_x M \). The indicatrix \( I_x M \) at \( x \in M \) is a hypersurface of \( T_x M \) defined by
\[
I_x M = \{ y \in T_x M : F(y) = 1 \}.
\]

Geodesics of \((M, F)\) are determined by a system of second order ordinary differential equations \( \ddot{x}^i + 2G^i(x, \dot{x}) = 0, \; i = 1, \ldots, n \) in a local coordinate system \((x^i, y^i)\) of \( TM \), where \( G^i(x, y) \) are determined by the formula \( 4G^i = g^{il}(\partial g_{jk} - \partial g_{jl}) y^j y^k \).

2.2. Parallel translation.

A vector field \( X(t) = X^i(t) \frac{\partial}{\partial x^i} \) along a curve \( c : [0, 1] \to M \) is called parallel if \( D_c X(t) = 0 \) where the covariant derivative is defined as
\[
D_c X(t) = \left( \frac{dX^i(t)}{dt} + G^i_j(c_t, X(t)) \dot{c}^j \right) \frac{\partial}{\partial x^i},
\]
with \( G^i_j = \frac{\partial G^i}{\partial y^j} \). Clearly, for any \( X_0 \in T_{c(0)} M \) there is a unique parallel vector field \( X(t) \) along the curve \( c \) such that \( X_0 = X(0) \). Moreover, if \( X(t) \) is a parallel vector field along \( c \), then \( \lambda X(t) \) is also parallel along \( c \) for any \( \lambda \geq 0 \). Then the homogeneous (nonlinear) parallel translation along a curve \( c(t) \)
\[
\mathcal{P}^i_c : T_{c(0)} M \to T_{c(t)} M
\]
is defined by the positive homogeneous map \( \mathcal{P}^i_c : X_0 \to X_t \) given by the value \( X_t = X(t) \) of the parallel vector field with initial value \( X(0) = X_0 \). We remark that (3) preserves the Finslerian norm, therefore it can be considered as a map between the indicatrices
\[
\mathcal{P}^i_c : I_{c(0)} M \to I_{c(t)} M.
\]
Moreover, since the parallel translation is a homogeneous map, (3) and (4) determine each other.

2.3. Holonomy.

The holonomy group \( \text{Hol}_p(M) \) of a Finsler manifold \((M, F)\) at a point \( p \in M \) is the group generated by parallel translations along piece-wise differentiable closed curves starting at \( p \). Considering the parallel translation (4) on the indicatrix, a holonomy element is a diffeomorphism \( \mathcal{P}_c : I_p \to I_p \), therefore the holonomy group \( \text{Hol}_p(M) \subset \text{Diff}^\infty(I_p) \) is a subgroup of the diffeomorphism group of the indicatrix \( I_p \).

In the particular case, when \((M, F)\) is a simply connected Riemann manifold, the holonomy group is a closed Lie subgroup of the special orthogonal group \( SO(n) \). Finslerian holonomy groups can, however, be much more complex: in [12, 13, 14] one can find examples of Finsler manifolds with infinite dimensional holonomy groups. Until now it is not known if the Finsler holonomy groups are (finite of infinite dimensional) Lie groups or not.

2.4. Horizontal lift and curvature.

The parallel translation on a Finsler manifold can also be introduced by considering the associated Ehresmann connection (cf. [18]): the horizontal distribution is determined by the horizontal lift \( T_x M \to T(x, y) TM \) defined in the local basis as
\[
\left( \frac{\partial}{\partial x^i} \right)^h = \frac{\partial}{\partial x^i} - G^i_j(x, y) \frac{\partial}{\partial y^j},
\]
where \( y \in T_x M \). Since the horizontal distribution is complementary to the vertical distribution we have the decomposition \( T_y TM = \mathcal{H}_y \oplus \mathcal{V}_y \) with canonical projectors \( h : TTM \to \mathcal{H} \) and \( v : TTM \to \mathcal{V} \). The image \( \mathcal{H} \subset TTM \) is the horizontal distribution of the manifold. The horizontal lift of a curve \( c : [0, 1] \to M \) with initial condition \( X_0 \in T_{c(0)} M \) is a curve
We have $X \in \mathfrak{X}^\infty(M)$ such that $\pi \circ \varphi_t = c$, $\varphi_t = (t) \in G$ and $\varphi_t(0) = X_0$. Then the parallel translation can be geometrically obtained as $\mathcal{P}_h^t(X_0) = \varphi_t$. We remark that the horizontal lift $\varphi_t$ of the flow $\varphi_t$ of a vector field $X \in \mathfrak{X}^\infty(M)$ is the flow of the horizontal lift of the vector field $X_0 \in \mathfrak{X}^\infty(TM)$. Therefore the parallel translation along the integral curves of $X$ can be calculated in terms of the horizontal lift of the flow:

$$\mathcal{P}_h^t = \varphi_t^h.$$  

The horizontal distribution $\mathcal{H}TM$ is, in general, non-integrable. The obstruction to its integrability is given by the curvature tensor $R = \frac{1}{2}[h, h]$ which is the Nijenhuis torsion of the horizontal projector $[4]$.

3. Tangent Lie algebra of a subgroup of the diffeomorphism group

In this paragraph we investigate the tangential property and tangential structure of subgroups of the diffeomorphism group. Let $G$ be a subgroup of $Diff^\infty(M)$ where $M$ is a differentiable manifold. We do not suppose any special property on $G$, in particular, we do not suppose that $G$ is a Lie subgroup of $Diff^\infty(M)$. Questions that we consider: can we introduce a tangential property and tangent object to the subgroup $G$? Does the set of tangent elements possess a special algebraic structure? Can this algebraic structure be used to get information about the subgroup? In this paragraph, we answer all these questions.

A smooth curve $c: I \rightarrow M$ on the manifold $M$ has a $(k-1)^{\text{st}}$-order singularity at $t = 0$, if its derivatives vanish up to order $k-1$, $(k \geq 0)$. It is well known that if a curve $c$ has a $(k-1)^{\text{st}}$-order singularity at $0 \in \mathbb{R}$ then its $k^{\text{th}}$ order derivative $c^{(k)}(0) = X_p$ is a tangent vector at $p = c(0)$. In that case, the curve $c$ is called a $k^{\text{th}}$-order integral curve of the vector field $X_p \in T_pM$. Extending this concept to vector fields, we can introduce the following

**Definition 3.1.** A $C^\infty$—smooth curve in the diffeomorphism group $\varphi: I \rightarrow Diff^\infty(M)$, $t \rightarrow \varphi_t$, is called an integral curve of the vector field $X \in \mathfrak{X}^\infty(M)$ if

1. $\varphi_0 = id_M$,
2. there exists $k \in \mathbb{N}$ such that for any point $p \in M$ the curve $t \rightarrow \varphi_t(p)$ is a $k^{\text{th}}$-order integral curve of $X(p) \in T_pM$.

This $k \in \mathbb{N}$ is called the order of the integral curve $\varphi_t$ of the vector field $X$.

In particular, the flow $\varphi^X_t$ of $X \in \mathfrak{X}^\infty(M)$ is a $1^{\text{st}}$-order integral curve of $X$. Moreover, if $k > 1$ and $t \rightarrow \varphi_t$ is a $k^{\text{th}}$—order integral curve of the vector field $X$ then we have

$$\varphi_0 = id_M, \quad \frac{\partial \varphi_t}{\partial t} |_{t=0} = 0, \ldots, \frac{\partial^{k-1} \varphi_t}{\partial t^{k-1}} |_{t=0} = 0, \quad \frac{\partial^k \varphi_t}{\partial t^k} |_{t=0} = X.$$  

Let $G \subset Diff^\infty(M)$ be an arbitrary subgroup of the diffeomorphism group $Diff^\infty(M)$. Using the terminology of Definition 3.1 we introduce the following

**Definition 3.2.** A vector field $X \in \mathfrak{X}^\infty(M)$ is called tangent to a subgroup $G \subset Diff^\infty(M)$ of the diffeomorphism group if there exists an integral curve of $X$ in $G$. The set of tangent vector fields of $G$ is denoted by $\mathfrak{T}_G$.

**Remark 3.3.** We have $X \in \mathfrak{T}_G$ if and only if there exists a $C^\infty$—smooth curve $\varphi: I \rightarrow Diff^\infty(M)$ such that

1. $\varphi_t \in G$,
2. $\varphi_0 = id_M$,
3. there exists $k \in \mathbb{N}$ such that equation (7) is satisfied.
One can observe that in Definition 3.2 we do not suppose that \( \mathcal{G} \) is a Lie subgroup of \( \text{Diff}^{\infty}(M) \). Indeed, we use the differential structure of the later to formulate the smoothness condition on the curve in \( \mathcal{G} \). Nevertheless, we have the following

**Theorem 3.4.** If \( \mathcal{G} \) is a subgroup of \( \text{Diff}^{\infty}(M) \), then \( \mathcal{T}_c \mathcal{G} \) is a Lie subalgebra of \( \mathcal{X}^{\infty}(M) \).

**Proof.** To prove the theorem, we have to show that

\[
(8a) \quad X, Y \in \mathcal{T}_c \mathcal{G} \Rightarrow [X, Y] \in \mathcal{T}_c \mathcal{G},
\]

\[
(8b) \quad X, Y \in \mathcal{T}_c \mathcal{G} \Rightarrow X + Y \in \mathcal{T}_c \mathcal{G},
\]

\[
(8c) \quad \lambda \in \mathbb{R}, \ X \in \mathcal{T}_c \mathcal{G} \Rightarrow \lambda X \in \mathcal{T}_c \mathcal{G}.
\]

Indeed, let \( X, Y \in \mathcal{T}_c \mathcal{G} \), that is \( X, Y \in \mathcal{X}^{\infty}(M) \) tangent to \( G \). According to Definition 3.1 there exist \( k, l \in \mathbb{N} \) such that \( \varphi_t, \psi_t \in \mathcal{G} \) are integral curves of \( X \) and \( Y \) respectively. Let us suppose that \( \varphi_t \) is a \( k \)-th-order integral curve of \( X \) and \( \psi_t \) is an \( l \)-th-order integral curve of \( Y \) \((k, l \geq 1)\). Then

\[
\varphi_0 = id_M, \quad \left\{ \frac{\partial^i \varphi_t}{\partial t^i} \bigg|_{t=0} = 0 \right\}_{1 \leq i < k} \quad \frac{\partial^k \varphi_t}{\partial t^k} \bigg|_{t=0} = X,
\]

and

\[
\psi_0 = id_M, \quad \left\{ \frac{\partial^j \psi_t}{\partial t^j} \bigg|_{t=0} = 0 \right\}_{1 \leq j < l} \quad \frac{\partial^l \psi_t}{\partial t^l} \bigg|_{t=0} = Y.
\]

**• Proof of \((8a)\).** The computation is similar to that of \([5]\): Considering the group theoretical commutator

\[
[\varphi_t, \psi_s] := \varphi_t^{-1} \circ \psi_s^{-1} \circ \varphi_t \circ \psi_s,
\]

we get a two-parameter family of diffeomorphisms such that if one of the parameters \( s \) or \( t \) is zero then \((11)\) is the identity transformation. From \((8a)\) and \((8b)\) we also know that the first, potentially nonzero derivative is the \((k + l)\)-th order mixed derivative:

\[
\frac{\partial^{k+l} [\varphi_t, \psi_s]}{\partial t^k \partial s^l} \bigg|_{(0,0)} (p) = \frac{\partial^j}{\partial s^j} \bigg|_{s=0} \left( \frac{\partial^k (\varphi_s^{-1} \circ \psi_t^{-1} \circ \varphi_s \circ \psi_t(p))}{\partial t^k} \bigg|_{t=0} \right)
\]

\[
= \frac{\partial^j}{\partial s^j} \bigg|_{s=0} \left( d (\varphi_s^{-1})_{\varphi_s(p)} \circ \frac{\partial^k \psi_t^{-1}}{\partial t^k} \bigg|_{t=0} (\psi_s(p)) \right),
\]

where \( d (\varphi_s^{-1})_{\varphi_s(p)} \) denotes the tangent map (or Jacobi operator) of \( \varphi_s^{-1} \) at the point \( \varphi_s(p) \).

Since \( d (\varphi_s^{-1})_{\varphi_s(p)} = id \), the above formula can be written in the form

\[
d \left( \frac{\partial^j \varphi_s^{-1}}{\partial s^j} \bigg|_{s=0} \right)_p \frac{\partial^k \psi_t^{-1}(p)}{\partial t^k} \bigg|_{t=0} + d \left( \frac{\partial^k \psi_t^{-1}}{\partial t^k} \bigg|_{t=0} \right)_p \frac{\partial^j \varphi_s(p)}{\partial s^j} \bigg|_{s=0}.
\]

From \( \varphi_t \circ \varphi_t^{-1} = id \) we get

\[
0 = \frac{\partial^k}{\partial t^k} \bigg|_{t=0} (\varphi_t \circ \varphi_t^{-1}) = X + \frac{\partial^k (\varphi_t^{-1})}{\partial t^k} \bigg|_{t=0},
\]

which yields

\[
\frac{\partial^k (\varphi_t^{-1})}{\partial t^k} \bigg|_{t=0} = -X.
\]

Therefore we get that \((13)\) can be written as

\[
d \left( \frac{\partial^j \varphi_s}{\partial s^j} \bigg|_{s=0} \right)_p \frac{\partial^k \psi_t(p)}{\partial t^k} \bigg|_{t=0} - d \left( \frac{\partial^k \psi_t}{\partial t^k} \bigg|_{t=0} \right)_p \frac{\partial^j \varphi_s(p)}{\partial s^j} \bigg|_{s=0}.
\]
which is the Lie bracket of the vector fields $X$ and $Y$, that is
\begin{equation}
\frac{\partial^{k+l} [\varphi_t, \psi_t]}{\partial t^k \partial s^l} \bigg|_{(0,0)} = [Y, X].
\end{equation}

From (10) we get that $t \to [\varphi_t, \psi_t]$ is a $(k+l)$th-order integral curve of $[X, Y]$ in $\mathcal{G}$. Therefore $[X, Y] \in T_e \mathcal{G}$ which proves (8a).

- **Proof of (8b).**

For any $c_1, c_2, m_1, m_2 \in \mathbb{R}$, $\phi_t = \varphi_{c_1 t^{m_1}} \circ \psi_{c_2 t^{m_2}}$ is a smooth curve in $\mathcal{G}$ with $\phi_0 = \varphi_0 \circ \psi_0 = id_M$. Moreover, if $r$ denotes the least common multiple of $k$ and $l$ and

\[ m_1 = \frac{r}{k}, \quad m_2 = \frac{r}{l}, \quad c_1 = \left( m_1^l (r-k)! \right)^{-1/r}, \quad c_2 = \left( m_2^l (r-l)! \right)^{-1/r}, \]

one gets
\begin{equation}
\frac{\partial^r \phi_t}{\partial t^r} \bigg|_{t=0} = \frac{\partial^r}{\partial t^r} \bigg|_{t=0} (\varphi_{c_1 t^{m_1}} \circ \psi_{c_2 t^{m_2}}) = X + Y,
\end{equation}

showing that $\psi_t$ is an $r$th-order integral curve of $X + Y$ in $\mathcal{G}$, therefore $X + Y$ is tangent to $\mathcal{G}$.

- **Proof of (8c).**

It is clear that in the case when $\lambda \geq 0$, one can reparametrize the integral curve of $X$, and using that the lower order terms are zero, we get
\begin{equation}
\frac{\partial^k \varphi^{\lambda X}}{\partial t^k} \bigg|_{t=0} = \lambda X.
\end{equation}

In the case when $\lambda < 0$ one can use (14) and we get
\begin{equation}
\frac{\partial^k}{\partial t^k} \bigg|_{t=0} \left( \frac{\varphi^{-1}}{\varphi^{\lambda X}} \right) = -|\lambda| X = \lambda X
\end{equation}

From (21) and (22) we get that $\lambda X$ is tangent to $G$, that is $\lambda X \in T_e \mathcal{G}$, and from 11b) and 11c) we get that any linear combinations of $X$ and $Y$ are in $T_e \mathcal{G}$.

Motivated by the results of Theorem 3.4 we propose the following

**Definition 3.5.** $T_0 \mathcal{G}$ is called the tangent Lie algebra of the subgroup $\mathcal{G} \subset Diff^\infty(M)$.

As a direct consequence of Theorem 3.4 we get the following

**Corollary 3.6.** Let $\mathcal{G}$ be a subgroup of $Diff^\infty(M)$ and $S$ be a subset of $X^\infty(M)$ such that the elements of $S$ are tangent to $\mathcal{G}$. Then the Lie subalgebra $\langle S \rangle_{\text{Lie}}$ of $X^\infty(M)$ generated by the elements of $S$ is also tangent to $\mathcal{G}$, that is

\[ S \subset T_e \mathcal{G} \implies \langle S \rangle_{\text{Lie}} \subset T_e \mathcal{G}. \]

**Remark 3.7.** Slightly different tangent properties of vector fields to a subgroup $\mathcal{G}$ of the diffeomorphism group were already introduced in [10]. We will refer to the property [10, Definition 2.] as the weak tangent property and to [10, Definition 4.] as the strong tangent property. Our language is justified by the following proposition which is clarifying the relationship between the tangent property introduced in Definition 3.1 and the tangent properties introduced in [10]:

**Proposition 3.8.** Let $\mathcal{G}$ be a subgroup of $Diff^\infty(M)$ and $X \in X^\infty(M)$. Using the terminology of Remark 3.7:

(i) if $X$ is strongly tangent to $\mathcal{G}$, then $X \in T_e \mathcal{G}$.
(ii) if $X \in T_e \mathcal{G}$, then it is weakly tangent to $\mathcal{G}$. 

Proof. (i) If \( X \in \mathcal{X}^\infty(M) \) is a strongly tangent vector field to the subgroup \( \mathcal{G} \subset \text{Diff}^\infty(M) \), there exists a \( k \)-parameter commutator like family of diffeomorphisms \( \phi_{t_1,...,t_k} \in \mathcal{G} \) which is \( C^\infty \)-smooth in \( \text{Diff}^\infty(M) \), \( \phi_{t_1,...,t_k} = id_M \) whenever one of its parameters is zero and
\[
X = \frac{\partial^{k} \phi_{t_1,...,t_k}}{\partial t_1 \cdots \partial t_k}(0...,0).
\]
Consequently, if we consider the map \( t \to \varphi_t \) where \( \varphi_t = \phi_{t,...,t} \), we get a 1-parameter family of diffeomorphisms which satisfies the conditions of Definition 3.2. Therefore, the tangent field \( X \) is tangent to \( \mathcal{G} \).

To prove (ii), let us suppose that \( \varphi_t \) is a \( k \)-th order integral curve of \( X \). Then we have (9) and one can write \( \varphi_t(p) \) as
\[
\varphi_t(p) = p + \frac{1}{k!} t^k (X(p) + \omega(p,t))
\]
where \( \lim_{t \to 0} \omega(p,t) = 0 \). The reparametrization \( t \to \psi_t := \varphi_k \circ \varphi_t \) gives a \( C^1 \)-differentiable 1-parameter family of diffeomorphisms in \( \text{Diff}^\infty(M) \) such that \( \psi_0 = id_M \) and
\[
\frac{\partial \psi_t}{\partial t} \Big|_{t=0}(p) = \frac{\partial \varphi_k \circ \varphi_t}{\partial t} \Big|_{t=0}(p) = X(p),
\]
which proves (ii).

Remark 3.9. One may wonder why to introduce a new tangent property when there are already two, the weak and the strong tangent property (using the terminology of Remark 3.7) introduced in the literature. As an answer we point out that, the concept in [10] has already two, the weak and the strong tangent property (using the terminology of Remark 3.9).

Theorem 3.10. Let \( \mathcal{G} \) be a subgroup of \( \text{Diff}^\infty(M) \) and \( \overline{\mathcal{G}} \) its topological closure with respect to the \( C^\infty \) topology. Then the group generated by the exponential image of the tangent Lie algebra \( \mathcal{T}_e \mathcal{G} \) with respect to the exponential map \( \exp: \mathcal{X}^\infty(M) \to \text{Diff}^\infty(M) \) is a subgroup of \( \overline{\mathcal{G}} \).

Proof. From the proof of Proposition 3.8 we know that for any element \( X \in \mathcal{T}_e \mathcal{G} \) there exists a \( C^1 \)-differentiable 1-parameter family \( \{ \psi_t \} \subset \mathcal{G} \) of diffeomorphisms of \( M \) such that \( \psi_0 = id_M \) and \( X = \frac{\partial \psi_t}{\partial t} \Big|_{t=0} \). Then, using the argument of [10] Corollary 5.4, p. 84] on \( \psi_t \) we get that
\[
\psi^m \left( \frac{t}{n} \right) = \psi \left( \frac{t}{n} \right) \circ \cdots \circ \psi \left( \frac{t}{n} \right)
\]
in \( \mathcal{G} \), as a sequence of \( \text{Diff}^\infty(M) \), converges uniformly in all derivatives to \( \exp(tX) \). It follows that
\[
\{ \exp(tX) \mid t \in \mathbb{R} \} \subset \overline{\mathcal{G}},
\]
for any \( X \in \mathcal{T}_e \mathcal{G} \). Therefore, one has \( \exp(\mathcal{T}_e \mathcal{G}) \subset \overline{\mathcal{G}} \) and if we denote by \( \langle \exp(\mathcal{T}_e \mathcal{G}) \rangle \) the group generated by the exponential image of \( \mathcal{T}_e \mathcal{G} \) we get
\[
\langle \exp(\mathcal{T}_e \mathcal{G}) \rangle \subset \overline{\mathcal{G}},
\]
which proves Theorem 3.10. \( \square \)
The concept worked out in Definition 3.2 and Theorem 3.4 can be adapted not only for subgroups of the diffeomorphism group but for any subgroup of any (finite or infinite dimensional) Lie group:

**Definition 3.11.** Let $G_L$ be a Lie group, $e \in G_L$ is the unit element of $G_L$ and $g_L := T_e G_L$ the Lie algebra of $G_L$. If $G \subset G_L$ is a subgroup of $G_L$, then $X \in g_L$ is called tangent to $G$ if there exist a $C^\infty$-smooth curve $\varphi : I \to G_L$ such that

1. $\varphi(t) \in G$,
2. $\varphi(0) = e$,
3. there exists $k \in \mathbb{N}$ such that $t \to \varphi(t)$ is a $k$th order integral curve of $X$.

The set of tangent vector of $G$ is denoted by $T_e G$.

Then, adapting the proof of Theorem 3.4 and Theorem 3.10 we can get the following

**Theorem 3.12.** If $G$ is a subgroup of a Lie group $G_L$, then $T_e G$ is a Lie subalgebra of $g_L$. The group $\langle \exp(T_e G) \rangle$ generated by the exponential image of $T_e G$ with respect to the exponential map $\exp : g_L \to G_L$ is a subgroup of the topological closure $\overline{G}$ of $G$ in $G_L$.

It is clear that in the case when $G$ is a Lie subgroup of $G_L$, then $T_e G = g$ is just the usual Lie subalgebra of $g_L$ associated to the Lie subgroup $G$. Therefore Definition 3.11 generalizes the classical notion of the Lie subalgebra associated to a Lie subgroup.

### 4. Application: the holonomy algebra

The notion of the holonomy group was already introduced in Chapter 2.3. It is well known that in the particular case when $(M, F)$ is a Riemann manifold, the holonomy group is a compact Lie subgroup of the orthogonal group $O(n)$ and its Lie algebra is a Lie subalgebra of $\mathfrak{o}(n)$. It is also clear that the holonomy group of a linear connection is a subgroup of the linear group $GL(n)$ and its Lie algebra is a Lie subalgebra of $\mathfrak{gl}(n)$. However, the situation for a Finsler manifold or in a more general context the holonomy of a homogeneous connection can be much more complex. Examples show that in some cases the holonomy group can not be a finite dimensional Lie group [11, 12, 13]. Until now it is not known if the Finsler holonomy groups are (finite or infinite dimensional) Lie groups or not. Nevertheless, the theory developed in Chapter 3 allows us to consider its tangent Lie algebra, the holonomy algebra.

#### 4.1. The fibered holonomy algebra and its Lie subalgebras.

Let $(M, F)$ be a Finsler manifold. The notion of fibered holonomy group $\mathcal{H}ol_f(M)$ appeared in [10]: It is the group generated by fiber preserving diffeomorphisms $\Phi$ of the indicatrix bundle $(\mathcal{I}M, \pi, M)$, such that for any $p \in M$ the restriction $\Phi_p = \Phi|_{\mathcal{I}p}$ is an element of the holonomy group $\mathcal{H}ol_p(M)$. It is obvious that

$$\mathcal{H}ol_f(M) \subset \mathcal{D}iff^\infty(\mathcal{I}M),$$

where $\mathcal{H}ol_f(M)$ is a subgroup of the diffeomorphism group of the indicatrix bundle. Until now it is not known whether or not $\mathcal{H}ol_f(M)$ is a Lie subgroup of $\mathcal{D}iff^\infty(\mathcal{I}M)$. The set of tangent vector fields to the group $\mathcal{H}ol_f(M)$ denoted as

$$\mathfrak{h}ol_f(M) := T_0(\mathcal{H}ol_f(M)).$$

**Definition 4.1.** $\mathfrak{h}ol_f(M)$ is called the fibered holonomy algebra of the Finsler manifold $(M, F)$.

From Theorem 3.4 one can obtain the following
Theorem 4.2. The fibered holonomy algebra $\mathfrak{hol}_f(M)$ is a Lie subalgebra of the Lie algebra of smooth vector fields $\mathfrak{X}^\infty(\mathcal{I}M)$.

In the sequel we will investigate the two most important Lie subalgebras of $\mathfrak{hol}_f(M)$ which can be introduced with the help of the curvature tensor (see Paragraph 2.3) of a Finsler manifold: the curvature algebra and the infinitesimal holonomy algebra.

Definition 4.3. A vector field $\xi \in \mathfrak{X}^\infty(\mathcal{I}M)$ is called a curvature vector field if there exist vector fields $X, Y \in \mathfrak{X}^\infty(M)$ such that $\xi = R(X^h, Y^h)$. The Lie subalgebra $\mathfrak{R}$ of vector fields generated by curvature vector fields is called the curvature algebra.

It is easy to see that from the definition of the curvature tensor that a curvature vector field can be calculated as

$$\xi = R(X^h, Y^h) = [X^h, Y^h] - [X, Y]^h,$$

and from the definition we have also $\mathfrak{R} \subset \mathfrak{X}^\infty(\mathcal{I}M)$. Moreover, we have the following

Proposition 4.4.

(1) The elements of the curvature algebra are tangent to the group $\mathcal{Hol}_f(M)$.

(2) The curvature algebra $\mathfrak{R}$ is a Lie subalgebra of $\mathfrak{hol}_f(M)$.

To prove the first part of the proposition, we have to show that the curvature vector fields are tangent to the fibered holonomy group $\mathcal{Hol}_f(M)$, that is they are elements of $\mathfrak{hol}_f(M)$. Let $\xi \in \mathfrak{X}^\infty(\mathcal{I}M)$ be a curvature vector field and $X, Y \in \mathfrak{X}^\infty(M)$ such that $\xi = R(X^h, Y^h)$. We denote by $\varphi$ and $\psi$ the integral curves of $X$ and $Y$ respectively. Define

$$\alpha_{t,s} := \begin{cases} 
\varphi_s, & 0 \leq s \leq t, \\
\psi_{s-t}\varphi_t, & t \leq s \leq 2t, \\
\varphi_{2t-s}\psi_t\varphi_t, & 2t \leq s \leq 3t, \\
\psi_{3t-s}\varphi_t\psi_t, & 3t \leq s \leq 4t.
\end{cases}$$

and

$$\beta_{t,s} := \psi_s\varphi_s\psi_s\varphi_s, \quad 0 \leq s \leq t.$$

Then, for every $p \in M$ and fixed $t$ the map $\alpha_t(p) : s \rightarrow \alpha_{t,s}(p)$ and $\beta_t(p) : s \rightarrow \beta_{t,s}(p)$ are parametrized curves: $\alpha_t(p) : s \rightarrow \alpha_{t,s}(p)$ is a (not necessarily closed) parallelogram and $\beta_t(p)$ joins the endpoints of $\alpha_t(p)$. Indeed, for every $p \in M$ and fixed $t$ the endpoint of $\alpha_t(p)$ coincides with the endpoint of $\beta_t(p)$ and consequently the curve $\alpha_t(p) * \beta_t^{-1}(p)$ defined as going along the curve $\alpha_t(p)$ then continuing along $\beta_t^{-1}(p)$ (which is the curve $\beta_t(p)$ with opposed orientation) is a closed curve that starts and ends at $p \in M$. Let us consider

$$h_{t,p} := \mathcal{P}_{\alpha_t(p)\beta_t^{-1}(p)} = \mathcal{P}_{\alpha_t(p)} \circ \mathcal{P}_{\beta_t^{-1}(p)},$$

the parallel translation along $\alpha_t(p) * \beta_t^{-1}(p)$. We have the following

Lemma 4.5. For any $p \in M$

(1) $h_{t,p} \in \mathcal{Hol}_p(M)$.

(2) $t \rightarrow h_{t,p}$ is a second order integral curve of the vector field $\xi_p := \xi|_{\mathcal{I}_p}$ ($\xi_p \in \mathfrak{X}^\infty(\mathcal{I}_p)$).

Proof. Indeed, for every $p \in M$ and sufficiently small $t$ the curve $\alpha_t(p) * \beta_t^{-1}(p)$ is a closed loop starting and ending at $p$, therefore the parallel transport $h_{t,p} : \mathcal{I}_p \rightarrow \mathcal{I}_p$ is a holonomy transformation at $p$ and we get (1) of the lemma.

To show (2) we first remark that $\alpha_0(p)$ and $\beta_0(p)$ are the trivial curves ($s \rightarrow \alpha_{0,s}(p) = \beta_{0,s}(p) \equiv p$), therefore the parallel translation along them is the identity transformation and

$$h_{0,p} = id_{\mathcal{I}_p}. $$
On the other hand, as we have seen in Chapter 2 the parallel transport along a curve is determined by the horizontal lift of the curve. Consequently, the parallel transport along the integral curves of the vector fields $X$ and $Y$ can be expressed with the flows of the horizontal lifts $X^h$ and $Y^h$. Let us consider first the parallel transport along the curve $\alpha_t(p)$: the parallel transport of a vector $v \in \mathcal{I}_p$ along the curve $\alpha_t(p)$ is

$$
\mathcal{P}_{\alpha_t(p)}(v) = \begin{cases} 
\varphi^X_s(v), & 0 \leq s \leq t, \\
\varphi^{Y}_{s-t} \varphi^X_t(v), & t \leq s \leq 2t, \\
\varphi^X_{(s-2t)} \varphi^{Y}_{t} \varphi^X_t(v), & 2t \leq s \leq 3t, \\
\varphi^X_{(s-3t)} \varphi^{Y}_{t} \varphi^X_t(v), & 3t \leq s \leq 4t.
\end{cases}
$$

Therefore, $\mathcal{P}_{\alpha_t(p)}$ corresponds to the infinitesimal (not necessarily closed) parallelogram having as sides the integral curves of the horizontal lifts $X^h$ and $Y^h$. From the well known properties of the Lie brackets (see for example [10] p.162) we get that

$$
\frac{d}{dt} \bigg|_{t=0} \mathcal{P}_{\alpha_t} = 0, \quad \text{and} \quad \frac{d^2}{dt^2} \bigg|_{t=0} \mathcal{P}_{\alpha_t} = 2 \left[ X^h, Y^h \right]_v.
$$

On the other hand, the parallel transport of a vector $w \in \mathcal{I}_{\alpha_t(p)}$ along $\beta^{-1}_t(p)$ can be calculated with the help of it’s horizontal lift $\mathcal{P}_{\beta^{-1}_t} = \mathcal{P}_{\beta^{-1}_t}(w) = ((\beta^h(t))^{-1}(w)$, where by the definition of the horizontal lift $\pi \circ (\beta) = ((\beta^h(t))^{-1}(w).$ Since $\frac{d}{dt} \big|_{t=0} \beta_t^h(p) = 0,$ and $\frac{d^2}{dt^2} \big|_{t=0} \beta_t^h(p)(v) = 2 [X, Y]_p,$ we obtain

$$
\frac{d}{dt} \bigg|_{t=0} \mathcal{P}_{\beta_t} = 0 \quad \text{and} \quad \frac{d^2}{dt^2} \bigg|_{t=0} \mathcal{P}_{\beta_t} = -2 \left[ X^h, Y^h \right]_v,
$$

thus, from the two equations of (26) and the two equations of (27) we get

$$
\frac{d}{dt} \bigg|_{t=0} h_t(v) = 0 \quad \text{and} \quad \frac{d^2}{dt^2} \bigg|_{t=0} h_t(v) = 2 \left( [X^h, Y^h] - [X, Y]^h \right)_v.
$$

where we also used (23). To summarize, we get from (25) and (28):

$$
\frac{d}{dt} \bigg|_{t=0} h_t = \text{id}|\mathcal{I}_p, \quad \frac{d}{dt} \bigg|_{t=0} h_{t,p} = 0, \quad \frac{1}{2} \frac{d^2}{dt^2} \bigg|_{t=0} h_{t,p} = \xi_p,
$$

which means that the reparametrized map $t \rightarrow h_t$ is a second order integral curve of the curvature vector field $\xi_p \in \mathfrak{X}^\infty(\mathcal{I}_p)$ and proves point (2) of the lemma.

**Proof of Proposition 4.4.** Let us consider the map $h_t : TM \rightarrow TM$ on the indicatrix bundle, where $h_t|\mathcal{I}_p := h_{t,p}$. From Lemma 1.5 we get (by dropping the variable $p \in M$) that

1. $h_t \in \text{Hol}_f(M)$,
2. $t \rightarrow h_t$ is a second order integral curve of the vector field $\xi \in \mathfrak{X}^\infty(\mathcal{I})$.

which shows that the curvature vector field $\xi$ is tangent to $\text{Hol}_f(M)$ and proves the first part of the proposition. Applying Corollary 4.4 we get that the Lie algebra generated by the curvature vector field is tangent to $\text{Hol}_f(M)$ which proves the second part of the proposition.

**We remark that (1) of Proposition 4.4 is an improvement of Proposition 3. and Corollary 2. of [10].** Indeed, the tangent property proved in [10] is weaker: $C^1$ instead of $C^\infty$ smoothness. Moreover, [10] uses a very strong topological restriction on the manifold $M$ supposing it is diffeomorphic to the $n$-dimensional euclidean space. In Proposition 4.4 we presented a natural geometric construction without any constraints on the topology of the manifold $M$. 

Definition 4.6. The infinitesimal holonomy algebra $\mathfrak{hol}^*(M)$ of a Finsler manifold $(M, F)$ is the smallest Lie algebra on the indicatrix bundle which satisfies the following properties:

1) Every curvature vector field $\xi$ is an element of $\mathfrak{hol}^*(M)$,
2) if $\xi, \eta \in \mathfrak{hol}^*(M)$, then $[\xi, \eta] \in \mathfrak{hol}^*(M)$,
3) if $\xi \in \mathfrak{hol}^*(M)$ and $X \in \mathcal{X}^\infty(M)$, then the horizontal Berwald covariant derivative $\nabla_X \xi$ is also an element of $\mathfrak{hol}^*(M)$.

We have the following

Proposition 4.7.

1) The elements of the infinitesimal holonomy algebra $\mathfrak{hol}^*(M)$ are tangent to $\mathcal{H}o\mathcal{l}_f(M)$. 
2) The infinitesimal holonomy algebra $\mathfrak{hol}^*(M)$ is a Lie subalgebra of $\mathcal{H}o\mathcal{l}_f(M)$.

Proof. From Proposition 4.4 we know that the curvature vector fields are tangent to the fibered holonomy group. Moreover, from [10, Proposition 4] and from (i) of Remark 3.8 we get that the horizontal Berwald covariant derivative of tangent vector fields to $\mathcal{H}o\mathcal{l}_f(M)$ are also tangent to $\mathcal{H}o\mathcal{l}_f(M)$ which proves the first part of the proposition. As a consequence, the infinitesimal holonomy algebra is generated by tangent vector fields and, according to Corollary 3.6, it is tangent to $\mathcal{H}o\mathcal{l}_f(M)$ proving the second part of the proposition. □

We remark that the first part of Proposition 4.7 is an improvement of [10, Theorem 2], because in Proposition 4.7 the strong topology condition on the manifold $M$ is dropped.

4.2. Holonomy algebra and its Lie subalgebras.

Let $(M, F)$ be an $n$-dimensional Finsler manifold. At any points $p \in M$ the indicatrix defined in (1) is an $(n-1)$-dimensional compact manifold in $T_p M$. Therefore, the diffeomorphism group $\text{Diff}^\infty(I_p)$ is an infinite dimensional Fréchet Lie group whose Lie algebra is $\mathcal{X}^\infty(I_p)$, the Lie algebra of smooth vector fields on $I_p$. As it was introduced in Chapter 2.3 the holonomy group

\begin{equation}
\mathcal{H}o\mathcal{l}_p(M) \subset \text{Diff}^\infty(I_p M),
\end{equation}

is a subgroup of the diffeomorphism group $\text{Diff}^\infty(I_p M)$. The set of tangent vector fields to the group $\mathcal{H}o\mathcal{l}_p(M)$, denoted as $\mathfrak{hol}_p(M) := T_0(\mathcal{H}o\mathcal{l}_p(M))$.

Definition 4.8. $\mathfrak{hol}_p(M)$ is called the holonomy algebra of the Finsler manifold $(M, F)$ at $p \in M$.

From Theorem 3.4 one can obtain

Theorem 4.9. The holonomy algebra $\mathfrak{hol}_p(M)$ of a Finsler manifold $(M, F)$ at $p \in M$ is a Lie subalgebra of $\mathcal{X}^\infty(I_p)$.

In the sequel we identify two important Lie subalgebras of the holonomy algebra of Finsler manifolds.

Definition 4.10. A vector field $\xi_p \in \mathcal{X}^\infty(I_p)$ on the indicatrix $I_p \subset T_p M$ is called a curvature vector field at $p \in M$ if there exist tangent vectors $X_p, Y_p \in T_p M$ such that $\xi_p = R(X^h_p, Y^h_p)$. The Lie subalgebra $\mathfrak{R}_p$ of vector fields generated by curvature vector fields at $p \in M$ is called the curvature algebra at $p$. 
The relationship between the curvature algebra $R_p$ at $p \in M$ and the curvature algebra $\mathfrak{R}$ introduced in Definition 4.3 is:

$$R_p = \{ \xi_p = \xi|_{\mathcal{I}_p} \mid \xi \in \mathfrak{R} \},$$

that is $R_p$ is the restriction of $\mathfrak{R}$ to the indicatrix $\mathcal{I}_p$. We have

**Proposition 4.11.** The elements of the curvature algebra $R_p$ at $p \in M$ are tangent to the group $\text{Hol}_p(M)$ and the curvature algebra $R_p$ is a Lie subalgebra of the holonomy algebra $\mathfrak{hol}_p(M)$.

The proof is a direct consequence of the computation of Proposition (4.4). Moreover, by localizing the infinitesimal holonomy algebra at a point we can obtain

**Definition 4.12.** The Lie algebra $\mathfrak{hol}_p^*(M) := \{ \xi|_{\mathcal{I}_p} \mid \xi \in \mathfrak{hol}^*(M) \}$ of vector fields on the indicatrix $\mathcal{I}_p$ is called the infinitesimal holonomy algebra at the point $p \in M$.

From Proposition 4.7 we get

**Proposition 4.13.** The elements of the infinitesimal holonomy algebra $\mathfrak{hol}_p^*(M)$ are tangent to the group $\text{Hol}_p(M)$ and the infinitesimal holonomy algebra $\mathfrak{hol}_p^*$ is a Lie subalgebra of the holonomy algebra $\mathfrak{hol}_p(M)$.

We note that by the construction of the infinitesimal holonomy algebra, the curvature vector fields are elements of $\mathfrak{hol}_p^*(M)$, therefore we have the sequence of the Lie algebras

$$R_p(M) \subset \mathfrak{hol}_p^*(M) \subset \mathfrak{hol}_p(M) \subset \mathfrak{X}^\infty(\mathcal{I}_p).$$

We also remark that the first parts of the statement of Proposition 4.11 and 4.13 are improvements of the results of [10] because the tangential property of the Lie algebra is improved: we can guarantee $C^\infty$-smoothness instead of $C^1$-smoothness.

**Concluding remarks.**

Many interesting geometric results can be obtained on the holonomy structure from the Lie algebras (31) through the tangent property. Indeed, by using Theorem 3.10, one can find examples where, in contrast to the Riemannian case, the holonomy group $\text{Hol}_p(M)$ is not a compact Lie group [11, 12, 13], or where the closure of the holonomy group is the infinite dimensional Lie group $\text{Diff}^\infty_+(\mathcal{I}_p)$ of the orientation preserving diffeomorphism group of the indicatrix [3, 13]. All these results were obtained by using the tangent property of the curvature algebra $R_p(M)$ and the infinitesimal holonomy algebra $\mathfrak{hol}_p^*(M)$. The method developed in Chapter 3 however, allows us to introduce in a natural and canonical way a potentially larger Lie algebra, the holonomy algebra, which is the tangent Lie algebra of the holonomy group. This Lie algebra gives the best linear approximation of the holonomy group. The technique can be applied in other fields of geometry as well. We are convinced that the method, exploring the tangential property of a group associated with a geometric structure, can be used successfully to investigate various geometric properties.

**References**

[1] M. Berger. Sur les groupes d’holonomie homogène des variétés à connexion affine et des variétés riemanniennes. *Bull. Soc. Math. France*, 83:279–330, 1955.

[2] A. Borel and A. Lichnerowicz. Groupes d’holonomie des variétés riemanniennes. *C. R. Acad. Sci. Paris*, 234:1835–1837, 1952.

[3] R. Bryant. Recent advances in the theory of holonomy. *Astérisque*, 266:Exp. No. 861, 5, 351–374, 2000. Séminaire Bourbaki, Vol. 1998/99.

[4] J. Grifone. Structure presque-tangente et connexions. I. *Ann. Inst. Fourier (Grenoble)*, 22(1):287–334, 1972.
References

[5] B. Hubicska and Z. Muzsnay. The holonomy groups of projectively flat Randers two-manifolds of constant curvature. preprint, 2017.
[6] D. D. Joyce. Compact manifolds with special holonomy. Oxford Mathematical Monographs. Oxford University Press, Oxford, 2000.
[7] L. Kozma. On holonomy groups of Landsberg manifolds. Tensor (N.S.), 62(1):87–90, 2000.
[8] M. Mauhart and P. W. Michor. Commutators of flows and fields. Arch. Math. (Brno), 28(3-4):229–236, 1992.
[9] P. W. Michor. Gauge theory for fiber bundles, volume 19 of Monographs and Textbooks in Physical Science. Lecture Notes. Bibliopolis, Naples, 1991.
[10] Z. Muzsnay and P. T. Nagy. Tangent Lie algebras to the holonomy group of a Finsler manifold. Commun. Math., 19(2):137–147, 2011.
[11] Z. Muzsnay and P. T. Nagy. Finsler manifolds with non-Riemannian holonomy. Houston J. Math., 38(1):77–92, 2012.
[12] Z. Muzsnay and P. T. Nagy. Witt algebra and the curvature of the Heisenberg group. Commun. Math., 20(1):33–40, 2012.
[13] Z. Muzsnay and P. T. Nagy. Characterization of projective finsler manifolds of constant curvature having infinite dimensional holonomy group. Publ. Math. Debrecen, 84(1-2):17–28, 2014.
[14] Z. Muzsnay and P. T. Nagy. Finsler 2-manifolds with maximal holonomy group of infinite dimension. Differential Geom. Appl., 39:1–9, 2015.
[15] H. Omori. Infinite-dimensional Lie groups, volume 158 of Translations of Mathematical Monographs. American Mathematical Society, Providence, RI, 1997. Translated from the 1979 Japanese original and revised by the author.
[16] M. Spivak. A comprehensive introduction to differential geometry. Vol. I. Publish or Perish, Inc., Wilmington, Del., second edition, 1979.
[17] Z. I. Szabó. Positive definite Berwald spaces. Structure theorems on Berwald spaces. Tensor (N.S.), 35(1):25–39, 1981.
[18] J. Szilasi, R. L. Lovas, and D. C. Kertész. Connections, sprays and Finsler structures. Hackensack, NJ: World Scientific, 2014.

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