SHAFAREVICH–TATE GROUPS OF HOLOMORPHIC LAGRANGIAN FIBRATIONS

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Abstract. Consider a Lagrangian fibration $\pi: X \to \mathbb{P}^n$ on a hyperkähler manifold $X$. There are two ways to construct a holomorphic family of deformations of $\pi$ over $\mathbb{C}$. The first one is known under the name Shafarevich–Tate family while the second one is the degenerate twistor family constructed by Verbitsky. We show that both families coincide. We prove that for a very general $X$ all members of the Shafarevich–Tate family are Kähler. There is a related notion of the Shafarevich–Tate group $\mathbb{III}$ associated to a Lagrangian fibration. The connected component of unity of $\mathbb{III}$ can be shown to be isomorphic to $\mathbb{C}/\Lambda$ where $\Lambda$ is a finitely generated subgroup of $\mathbb{C}$ and $\mathbb{C}$ is thought of as the base of the Shafarevich–Tate family. We show that for a very general $X$, projective deformations in the Shafarevich–Tate family correspond to the torsion points in $\mathbb{III}^0$. A sufficient condition for a Lagrangian fibration $X$ to be projective is existence of a holomorphic section. We find sufficient cohomological conditions for existence of a deformation in the Shafarevich–Tate family that admits a section.

Contents

1. Introduction
2. Preliminaries
   2.1. Lagrangian fibrations
   2.2. Shafarevich–Tate groups
   2.3. Degenerate twistor deformations
3. Shafarevich–Tate group of a Lagrangian fibration
   3.1. Structure of Shafarevich–Tate groups: first steps
   3.2. Shafarevich–Tate family
   3.3. Shafarevich–Tate twists are symplectic
   3.4. Degenerate twistor deformations as Shafarevich–Tate twists
   3.5. The period map of a Shafarevich–Tate family
4. Connected component of unity of a Shafarevich–Tate group
   4.1. The sheaf $\Gamma$
   4.2. The connected component of unity of Shafarevich–Tate groups
5. Kählerness and projectivity properties
   5.1. M-special Hodge structures
   5.2. Kählerness of degenerate twistor deformations
   5.3. Algebraic points in Shafarevich–Tate families
6. Sections of Lagrangian fibrations
   6.1. Obstruction for existence of a section
   6.2. Hard Lefschetz type theorems for higher direct images of $\mathbb{Q}_X$
   6.3. The discrete part of Shafarevich–Tate groups
Bibliography

1. Introduction

A compact connected Kähler manifold $X$ is called a hyperkähler manifold if it is simply connected and $H^0(X, \Omega^2_X)$ is generated by a holomorphic symplectic form $\sigma$. We can associate to a Lagrangian fibration $\pi: X \to \mathbb{P}^n$ a certain group called the Shafarevich–Tate group.
Definition. (see Definition 2.11) The Shafarevich–Tate group\(^1\) III of a Lagrangian fibration is the abelian group \(H^1(\mathbb{P}^n, \text{Aut}_{X/\mathbb{P}^n}^0)\), where \(\text{Aut}_{X/\mathbb{P}^n}^0\) is the sheaf on \(\mathbb{P}^n\) that is the image of the exponential map \(\pi_*T_{X/\mathbb{P}^n} \to \text{Aut}_{X/\mathbb{P}^n}\) (Subsection 2.2).

Choose an affine open cover \(\mathbb{P}^n = \bigcup U_i\). Denote \(U_i \cap U_j\) by \(U_{ij}\). A class \(s \in \text{III}\) can be represented by a Čech cocycle with coefficients in \(\text{Aut}_{X/\mathbb{P}^n}^0\). That is to say, for every pair of indices \(i,j\), we are given an automorphism \(s_{ij}\) of \(\pi^{-1}(U_{ij})\) that commutes with \(\pi\). Let us glue the manifolds \(\pi^{-1}(U_i)\) by the automorphisms \(s_{ij}\). We obtain a complex manifold \(X^s\) equipped with a holomorphic projection \(\pi^s: X^s \to \mathbb{P}^n\). The fibers of \(\pi^s\) are isomorphic to the fibers of \(\pi: X \to \mathbb{P}^n\). This new manifold is called the Shafarevich–Tate twist of \(X\) by \(s \in \text{III}\). Markman studied Shafarevich–Tate twists of Lagrangian fibrations on manifolds of \(K3^{[n]}\)-type in [Markman], and our work was notably influenced by his paper.

Assume that the fibration \(\pi\) has local analytic sections, i.e., every scheme-theoretic fiber has a smooth point. Define the sheaf \(\Gamma\) of finitely generated abelian groups by the exact sequence

\[
0 \to \Gamma \to \pi_*T_{X/\mathbb{P}^n} \to \text{Aut}_{X/\mathbb{P}^n}^0 \to 0
\]

This exact sequence induces the following long exact sequence of cohomology groups, which can be shown to be right exact.

\[
H^1(\mathbb{P}^n, \Gamma) \to H^1(\pi_*T_{X/\mathbb{P}^n}) \to \text{III} \to H^2(\mathbb{P}^n, \Gamma) \to 0
\]

The holomorphic symplectic form \(\sigma\) on \(X\) induces an isomorphism \(\pi_*T_{X/\mathbb{P}^n} \cong \Omega^1_{\mathbb{P}^n}\) (Theorem 2.6). Thus, \(H^1(\pi_*T_{X/\mathbb{P}^n}) \cong H^{1,1}(\mathbb{P}^n) = \mathbb{C}\). Let \(\text{III}^0\) be the image of \(H^1(\pi_*T_{X/\mathbb{P}^n})\) in \(\text{III}\). We conclude that \(\text{III}^0\) is isomorphic to a quotient of \(\mathbb{C}\) by the finitely generated abelian group \(\text{im}(H^1(\mathbb{P}^n, \Gamma))\).

In [Ver15] the author associates to a Lagrangian fibration \(\pi: X \to \mathbb{P}^n\) a family of deformations over \(\mathbb{C}\) called the degenerate twistor family. The construction goes as follows. Let \(\alpha\) be a closed \((1,1)\)-form on \(\mathbb{P}^n\). There exists a unique complex structure \(I_\alpha\) on \(X\) such that the form \(\omega + \pi^s\alpha\) is a holomorphic 2-form on \((X, I_\alpha)\) [Ver15, Thm. 3.5]. The manifold \((X, I_\alpha)\) is the degenerate twistor deformation of \(X\) with respect to the form \(\alpha\). Note that the construction of degenerate twistor deformations has differential geometric flavour while the construction of Shafarevich–Tate twists is of complex analytic nature. Nevertheless, these two constructions turn out to yield the same result.

Theorem A (Theorem 3.9). Assume that the Lagrangian fibration \(\pi\) admits local analytic sections outside a codimension two subset of the base. Pick a class \(s \in H^1(\pi_*T_{X/\mathbb{P}^n})\). Consider the twist \(X^s\) of \(\pi: X \to \mathbb{P}^n\) by the image of \(s\) in \(\text{III}\). Let \(\alpha\) be a closed \((1,1)\)-form on \(\mathbb{P}^n\) representing the same class in \(H^{1,1}(\mathbb{P}^n) \cong H^1(\pi_*T_{X/\mathbb{P}^n})\) as \(s\). Then the complex manifolds \(X^s\) and \((X, I_\alpha)\) are isomorphic as fibrations over \(\mathbb{P}^n\).

Are Shafarevich–Tate twists of a hyperkähler manifold hyperkähler themselves? We prove that indeed, they admit a holomorphic symplectic form (Subsection 3.3) but the question of Kählerness turns out to be trickier. We need to introduce a notion of \(M\)-special hyperkähler manifolds which is motivated by [Markman, Def. 1.1]. A hyperkähler manifold \(X\) is called \(M\)-special if the subspace \(H^{2,0}(X) + H^{0,2}(X) \subset H^2(X, \mathbb{C})\) contains a rational class. A very general hyperkähler manifold is not \(M\)-special.

Theorem B (Corollary 5.11, Theorem 5.19). Let \(\pi: X \to \mathbb{P}^n\) be a Lagrangian fibration on a not \(M\)-special projective hyperkähler manifold \(X\). Assume that \(\pi\) admits local analytic sections outside a codimension two subset of \(\mathbb{P}^n\). Then every Shafarevich–Tate twist \(X^s\)

\(^1\)It is sometimes called analytic Shafarevich–Tate group in the literature in order to distinguish it from arithmetic Shafarevich–Tate group.
of $X$ by an element $s \in \Pi^0$ is a hyperkähler manifold. Moreover, the set of $s \in \Pi^0$ such that the twist $X^s$ is a projective manifold forms a non-empty torsor over the group of torsion points of $\Pi^0$.

In order to prove Theorem B, we first study the sheaf $\Gamma$. It turns out that $\Gamma \otimes \mathbb{Q}$ is isomorphic to $R^1\pi_*\mathbb{Q}$ (Proposition 4.4). The Leray spectral sequence helps us to compute some cohomology groups of $R^1\pi_*\mathbb{Q}$. We derive from that a description of the group $\Pi^0$ solely in terms of the Hodge structure on $H^2(X, \mathbb{Z})$ (Corollary 4.8). As we have already seen, the group $\Pi^0$ is isomorphic to $\mathbb{C}/\Lambda$ for a finitely generated abelian group $\Lambda$. It turns out that $X$ is not $M$-special if and only if $\Lambda$ is a dense subgroup of $\mathbb{C}$. We use that the set of twists of $X$ that are Kähler manifolds is open and $\Lambda$-invariant to conclude that all Shafarevich–Tate twists of $X$ are Kähler. The statement about projectivity follows by carefully applying Huybrechts’ criterion [Huy01, Thm. 3.11].

In Section 6 we study whether there exists a Shafarevich–Tate twist $X^s$ of $\pi: X \rightarrow \mathbb{P}^n$ that admits a holomorphic section. We show that obstructions to existence of such a twist lie in $H^2(\mathbb{P}^n, \Gamma) = \Pi^0/\Pi^0$ (Corollary 6.2). Assume that the fibers of $\pi$ are reduced and irreducible. We show that the group $H^2(\mathbb{P}^n, \Gamma) \otimes \mathbb{Q}$ can be embedded into $H^2(X, \mathbb{Q})$ (Corollary 6.8). This implies the following theorem.

**Theorem C** (Theorem 6.3). Let $\pi: X \rightarrow \mathbb{P}^n$ be a Lagrangian fibration on a compact hyperkähler manifold over a smooth base. Assume that the following holds:

- the fibers of $\pi$ are reduced and irreducible;
- $H^3(X, \mathbb{Q}) = 0$;
- $H^2(\mathbb{P}^n, \Gamma)$ is torsion-free.

Then there exists a unique $s \in \Pi^0$ such that $\pi^s: X^s \rightarrow \mathbb{P}^n$ admits a holomorphic section. In particular, the fibration $\pi$ can be deformed to a fibration with a holomorphic section through the Shafarevich–Tate family.

At the core of the proof of Theorem C is a Hard Lefschetz-type result for fibers of Lagrangian fibrations. We believe that it may be of independent interest and state it here.

**Theorem D** (Corollary 6.6). Let $\pi: X \rightarrow \mathbb{P}^n$ be a Lagrangian fibration on a projective hyperkähler manifold $X$. Assume that $\pi$ admits local analytic sections outside a codimension two subset. Then there exists a sheaf $\mathcal{N}$ on $\mathbb{P}^n$ such that the sheaf $R^{2n-1}\pi_*\mathbb{Q}_X$ decomposes into the direct sum

$$R^{2n-1}\pi_*\mathbb{Q}_X \simeq R^1\pi_*\mathbb{Q}_X \oplus \mathcal{N}.$$

Theorem D allows us to prove that the differential $d_2: H^0(R^2\pi_*\mathbb{Q}) \rightarrow H^2(R^1\pi_*\mathbb{Q})$ in the Leray spectral sequence for $\mathbb{Q}_X$ vanishes. The argument is inspired by the one used by Deligne to prove that the Leray spectral sequence of a smooth projective submersion degenerates on the second page [Del].

The paper is organized as follows. The notions of a Shafarevich–Tate group and a degenerate twistor deformation, as well as basic facts about Lagrangian deformations are recalled in Section 2. The main result of Section 3 is a refined version of Theorem A. We also prove that Shafarevich–Tate twists are holomorphic symplectic in Subsection 3.3 and study the period map for the family of Shafarevich–Tate twists in Subsection 3.5. Section 4 is concerned with the description of the group $\Pi^0$ in terms of topological invariants of $X$. Its results are used in Section 5 to prove Theorem B. In Section 6 we discuss obstructions to existence of sections and prove Theorem C. Theorem D is proved there as an intermediate result.
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2. Preliminaries

2.1. Lagrangian fibrations. Let $X$ be a hyperkähler manifold. Fix a holomorphic symplectic form $\sigma$ on $X$. Consider a map $\pi: X \to B$ to a normal base $B$ of dimension half of that of $X$. A map $\pi: X \to B$ is called a Lagrangian fibration if it is surjective with connected fibers and the restriction of $\sigma$ to every smooth fiber vanishes. We always assume that the base $B$ of a Lagrangian fibration $\pi$ is isomorphic to $\mathbb{P}^n$. It is the case in all known examples (see e.g. [HM, Subsection 1.2]). We refer the reader to [Huy01] for basics on hyperkähler manifolds and Lagrangian fibrations.

Throughout this paper, we write $X$ for a compact hyperkähler manifold, $\sigma$ for a holomorphic symplectic form on $X$ and $\pi: X \to B$ for a Lagrangian fibration. For $b \in B$ we denote the fiber of $\pi$ over $b$ by $F_b$. The subset of the regular values of $\pi$ is denoted by $B^0$. This is a Zariski open subset of $B$ and $D := B \setminus B^0$ is known to be a divisor [HO, Prop. 3.1]. The divisor $D$ is called the discriminant of $\pi$.

Smooth fibers of a Lagrangian fibration $\pi: X \to B$ are complex tori [Markush86]. One can show that they are abelian varieties even if $X$ is not projective [Cam].

Let $X$ be a hyperkähler manifold. Then $H^2(X, \mathbb{Z})$ carries a non-degenerate symmetric bilinear form $q: H^2(X, \mathbb{Z}) \otimes H^2(X, \mathbb{Z}) \to \mathbb{Z}$ called the Beauville–Bogomolov–Fujiki form [Beauv, Fu85]. It satisfies the following properties:

1. The Hodge decomposition

$$H^2(X, \mathbb{C}) = H^{2,0}(X) \oplus H^{1,1}(X) \oplus H^{0,2}(X)$$

is orthogonal with respect to $q$. The restriction of $q$ to $H^{2,0}(X) \oplus H^{0,2}(X)$ is a positive Hermitian form and the restriction of $q$ to $H^{1,1}(X)$ has the signature $(1, h^{1,1}(X) - 1)$;

2. (Fujiki formula) There exists a non-zero constant $c_X$ such that $\forall \omega \in H^2(X)$

$$q(\omega, \omega)^n = c_X \int_X \omega^{2n}.$$ 

Let $\eta = \pi^*[H] \in H^2(X, \mathbb{Z})$ be the pullback of the class of a hyperplane. It follows from the Fujiki formula that $q(\eta, \eta) = 0$.

Proposition 2.1. ([O, Section 2]) Let $\pi: X \to B$ be a Lagrangian fibration, $\eta \in H^2(X, \mathbb{Z})$ as above. Then the restriction map $H^2(X, \mathbb{Q}) \to H^2(F_b, \mathbb{Q})$ has rank 1 for every smooth fiber $F_b = \pi^{-1}(b) \subset X$. Moreover, the kernel of this map is precisely

$$\eta^\perp := \{ v \in H^2(X, \mathbb{Q}) \mid q(v, \eta) = 0 \}.$$ 

Corollary 2.2. With the assumptions of Proposition 2.1, there is a Kähler class $[\omega]$ on $X$ that restricts to an integral class on every smooth fiber $F_b$. 
Proof: By Proposition 2.1 the restriction map $H^2(X, \mathbb{Q}) \to H^2(F_b, \mathbb{Q})$ for any $b \in B^\circ$ factors through the one-dimensional quotient $H^2(X, \mathbb{Q})/\eta$. This vector space is generated by an integral class. Every Kähler class $[\omega]$ on $X$ restricts to a non-zero class on a fiber. Therefore, the class $[\omega]$ restricts to an integral class after appropriate scaling. \qed

Let $E$ be an irreducible component of the discriminant $D$. Then $\pi^*E$ decomposes as $\sum d_i H_i$ where $H_i$ are irreducible components of $\pi^{-1}(E)$.

Definition 2.3. The Lagrangian fibration $\pi: X \to B$ is said to be without multiple fibers in codimension one if for every irreducible component $E$ of the discriminant we have $\gcd(d_i) = 1$.

Remark 2.4. By [HwO, Thm. 1.4] the condition above is equivalent to the following: if $x$ is a general point of the discriminant $D$, then $\pi$ admits a local section in a neighborhood of $x$. In other words, $\pi$ admits local analytic sections outside of a codimension two subset of the base.

We will need the following easy lemma:

Lemma 2.5. Let $f: Y \to Z$ be a proper map of normal varieties with equidimensional fibers. The pushforward of a reflexive sheaf on $Y$ is reflexive.

Proof: Consider a reflexive sheaf $\mathcal{E}$ on $Y$. We need to check that its pushforward $f_* \mathcal{E}$ is torsion-free and normal. The torsion-freeness is straightforward. Normality means that for an open subset $U \subset Z$ and a closed subset $W \subset U$ of codimension at least two, the restriction map $f_* \mathcal{E}(U) \to f_* \mathcal{E}(U - W)$ is surjective. This is implied by the normality of $\mathcal{E}$ because $f^{-1}(W)$ is of codimension at least two in $f^{-1}(U)$. \qed

Theorem 2.6. Let $\pi: X \to B$ be a Lagrangian fibration. Consider the isomorphism $\sigma: \Omega^1_X \cong T_X$ induced by the holomorphic symplectic form $\sigma$. Then the map $\sigma$ sends $\pi^* \Omega^1_B \subset \Omega^1_X$ into $T_{X/B} := \ker(d\pi: T_X \to \pi^*T_B)$. Assume that $\pi$ has no multiple fibers in codimension one. Then the induced map

$$\pi_*\sigma: \Omega^1_B \to \pi_*T_{X/B}$$

is an isomorphism of coherent sheaves on $B$.

Proof: Step 1: First, we will show that $\pi^* \Omega^1_B$ is mapped into $T_{X/B}$. For that, we need to check that the composition of maps

$$\pi^* \Omega^1_B \hookrightarrow \Omega^1_X \xrightarrow{\sigma} T_X \xrightarrow{d\pi} \pi^* T_B$$

vanishes. This is true generically. Since $\pi^* T_B$ is locally free, in particular torsion-free, a map to $\pi^* T_B$ that vanishes generically is necessarily zero.

Step 2: Observe that the map $\pi^* \Omega^1_B \to T_{X/B}$ is an isomorphism over the subset $X^\circ$ of $X$ where the map $\pi$ is smooth. The subset $B^\circ$ is clearly contained in $\pi(X^\circ)$. The fibration $\pi$ has no multiple fibers in codimension one, hence $\pi(X^\circ)$ is the complement of a closed subset of $B$ of codimension at least two (Remark 2.4).

Step 3: We will show that the map $\pi_*\sigma: \Omega^1_B \to \pi_* T_{X/B}$ is an isomorphism over $\pi(X^\circ)$. It is definitely an isomorphism over $B^\circ$. Take a point $b \in \pi(X^\circ)$. Choose a vector field $v \in \Gamma(\pi^{-1}(U), T_{X/B})$ for a neighborhood $U$ of $b$. There exists a unique 1-form $\alpha$ on $U \cap B^\circ$ such that $\sigma(\pi^*\alpha) = v|_{\pi^{-1}(U) \cap B^\circ}$. Moreover, there exists a unique 1-form $\beta \in \Gamma(\pi^{-1}(U) \cap X^\circ, \pi^* \Omega^1_B)$ such that $\sigma(\beta) = v|_{\pi^{-1}(U) \cap X^\circ}$ by Step 2. We have that $\beta|_{\pi^{-1}(U) \cap B^\circ} = \pi^* \alpha$. Let $s: U \to \pi^{-1}(U)$ be a local section of $\pi$. Its image is necessarily contained in $X^\circ$. The form $\gamma := s^* \beta$ is a 1-form on $U$ whose restriction to $U \cap B^\circ$ is $\alpha$. 

The sheaf $\pi_*T_{X/B}$ is torsion-free, hence $\pi_*\sigma(\gamma) = v$. We have proved that $\pi^*\sigma|_{\pi(X)}$ is a surjective map, hence an isomorphism.

**Step 4:** The sheaf $T_{X/B}$ is reflexive since it is the kernel of a map of locally free sheaves. Hence $\pi_*T_{X/B}$ is reflexive (Lemma 2.5). The map $\Omega^1_B \to \pi_*T_{X/B}$ is a map of reflexive sheaves which is an isomorphism outside of a closed subset of codimension at least two. Hence it is an isomorphism.

It follows that the sheaves $\Omega^1_B$ and $\pi_*T_{X/B}$ are isomorphic. In particular, the sheaf $\pi_*T_{X/B}$ is locally free.

Assume that $X$ is projective. Let $\omega$ be a closed $(1,1)$-form on $X$ that represents a rational class $[\omega] \in H^{1,1}_Q(X)$. The contraction of $\omega$ with holomorphic vector fields defines a map

$$\tilde{\omega}: \pi_*T_{X/B} \to R^1\pi_*\mathcal{O}_X.$$

We get the following maps by taking exterior powers of $\tilde{\omega}$:

$$\tilde{\omega}^i: \Omega^i_B \cong \Lambda^i(\pi_*T_{X/B}) \to \Lambda^iR^1\pi_*\mathcal{O}_X \to R^i\pi_*\mathcal{O}_X, \quad i = 1, \ldots, n.$$

**Proposition 2.7.** ([Mats, Thm. 1.2]) Let $\pi: X \to B$ be a Lagrangian fibration on a projective hyperkähler manifold $X$ over a smooth base $B$. Then the map

$$\tilde{\omega}^i: \Omega^i_B \to R^i\pi_*\mathcal{O}_X.$$

is an isomorphism for every $i = 1, \ldots, n$. In particular, the sheaves $R^i\pi_*\mathcal{O}_X$ are locally free.

**Remark 2.8.** The assumption of projectivity of $X$ can be dropped, as was shown in [SV]. Let $[\omega]$ be a Kähler class on $X$ as in Corollary 2.2. Then the restriction of the isomorphism $\pi_*T_{X/B} \to R^1\pi_*\mathcal{O}_X$ to $B^0$ is given by the contraction of $\omega$ with holomorphic vector fields [SV, Cor. 3.7].

**Corollary 2.9.** If $\pi: X \to B$ is a Lagrangian fibration without multiple fibers in codimension one, then the following cohomology groups are isomorphic:

$$H^1(B, \pi_*T_{X/B}) \cong H^1(B, \Omega^1_B) \cong H^1(B, R^1\pi_*\mathcal{O}_X) \cong \mathbb{C}.$$

**Proof:** The sheaves $\pi_*T_{X/B}$, $\Omega^1_B$, and $R^1\pi_*\mathcal{O}_X$ are isomorphic by Lemma 2.6 and Proposition 2.7. The last isomorphism holds since $h^{1,1}(\mathbb{P}^n) = 1$. \hfill $\square$

### 2.2. Shafarevich–Tate groups

Let $\pi: M \to S$ be a surjective holomorphic map of complex manifolds. Consider the sheaf $\text{Aut}_{M/S}$ of vertical automorphisms of $M$ over $S$. This is a sheaf of groups on $S$ defined as

$$\text{Aut}_{M/S}(U) := \{ \varphi: \pi^{-1}(U) \to \pi^{-1}(U) \mid \pi \circ \varphi = \pi \}.$$

Let us define the subsheaf $\text{Aut}^0_{M/S} \subset \text{Aut}_{M/S}$ as the image of the exponential map

$$\pi_*T_{M/S} \to \text{Aut}_{M/S}.$$

Pick a class $s \in H^1(S, \text{Aut}^0_{M/S})$. We can represent it by a Čech cocycle $s_{ij}$ for an open cover $S = \bigcup U_i$. We are given an automorphism $s_{ij}: \pi^{-1}(U_{ij}) \to \pi^{-1}(U_{ij})$ for each pairwise intersection $U_{ij} := U_i \cap U_j$. Let us glue a new complex manifold as

$$M^s := \bigcup \pi^{-1}(U_i) \Big/ x \in \pi^{-1}(U_i) \sim s_{ij}(x) \in \pi^{-1}(U_j).$$
The manifold $M^s$ is equipped with a natural fibration $\pi^s: M^s \to S$. The isomorphism class of the fibration $\pi^s: M^s \to S$ depends only on the class of $s$ in $H^1(S, Aut^0_{M/S})$. The manifold $M^s$ is called the twist of $M$ by the class $s \in H^1(S, Aut^0_{M/S})$.

Suppose from now that a general fiber of $\pi$ is a complex torus. Let $S^\circ \subset S$ be the set of values of $\pi$ for which the fiber $\pi^{-1}(b)$ is a complex torus.

**Lemma 2.10.** Let $\pi: M \to S$ be as above. Then the sheaf $Aut^0_{M/S}$ is a sheaf of commutative groups.

**Proof:** The sheaf $Aut^0_{M/S}$ is a sheaf of commutative groups at least over $S^\circ \subset S$. Indeed, for any point $b \in S^\circ$ the group $Aut^0(F_b)$ is a complex torus. In particular, $Aut^0(F_b)$ is commutative. Let $g, h \in Aut^0_{M/S}(U)$ be two local sections of $Aut^0_{M/S}$. Then $[g, h]|_{S^\circ \cap U}$ is trivial. Therefore, $[g, h]$ is itself trivial, because $S^\circ \cap U$ is dense in $U$. □

**Definition 2.11.** ([FM, Ch. I, Section 1.5.1]) The group $H^1(S, Aut^0_{M/S})$ is called the Shafarevich–Tate group of the fibration $\pi: M \to S$ and is denoted by $\Sha$.

See [DG, K, Markman] for applications of Shafarevich–Tate groups to elliptic and Lagrangian fibrations.

**Remark 2.12.** Let $\pi: M \to S$ be a fibration whose general fiber is a torus. Consider the subsheaf $Aut'_{M/S}$ of $Aut_{M/S}$ consisting of those vertical automorphisms $\varphi$ whose restriction to every fiber $F_b$ lies in $Aut^0(F_b)$. We have the following inclusion:

$$Aut^0_{M/S} \subset Aut'_{M/S}.$$ 

If $\pi$ is a Lagrangian fibration admitting local analytic sections, then the sheaves $Aut^0_{M/S}$ and $Aut'_{M/S}$ coincide [Markush96, Prop. 2.1]. We do not know if this is true in greater generality, for example, for Lagrangian fibrations without multiple fibers in codimension one.

2.3. Degenerate twistor deformations. Fix a holomorphic symplectic form $\sigma$ on $X$. The key observation here is that the complex structure of $X$ is determined by the form $\sigma$, viewed as a smooth $\mathbb{C}$-valued 2-form on $X$. Namely, the complex structure is uniquely determined by the subbundle of $(0, 1)$-vectors $T^{0,1}X \subset TX$. This subbundle can be recovered as $\ker \sigma|_{TX\otimes \mathbb{C}}$.

We can characterize all complex-valued smooth 2-forms on $X$ that can be realized as holomorphic symplectic forms for some complex structures.

**Definition 2.13.** ([BDV]) Let $M$ be a smooth manifold of real dimension $4n$. Consider a smooth complex-valued 2-form $\sigma \in \Gamma(\Lambda^2 T^* M \otimes \mathbb{C})$. It is called a c-symplectic structure if the following holds:

- $d\sigma = 0$;
- $\sigma^{n+1} = 0$;
- $\sigma^n \wedge \overline{\sigma^n} \neq 0$ at each point of $M$.

**Proposition 2.14.** ([Ver15, Prop. 3.1]) Let $\sigma$ be a c-symplectic structure on $M$. Define $T^{0,1}M := \ker \sigma \subset TM \otimes \mathbb{C}$ and $T^{1,0}M := \overline{T^{0,1}M}$. Then the following holds:

- $TM = T^{0,1}M \oplus T^{1,0}M$;
- $[T^{0,1}M, T^{0,1}M] \subseteq T^{0,1}M$.

Equivalently, the operator $I_\sigma: TM \to TM$ that acts as $i$ on $T^{1,0}M$ and as $-i$ on $T^{0,1}M$ is an integrable complex structure on $M$. The form $\sigma$ is holomorphic symplectic with respect to $I_\sigma$. 
Lemma 2.15. ([Ver15, Thm. 1.10]) Let $M$ be a complex manifold (not necessarily compact) and $\sigma$ a holomorphic symplectic 2-form on $M$. Consider a proper Lagrangian fibration $\pi: M \to S$ over a complex manifold $S$. Fix a closed 2-form $\alpha$ on $S$ of Hodge type $(2,0) + (1,1)$. Then the form $\sigma_\alpha := \sigma + \pi^*\alpha$ is a $c$-symplectic structure. Moreover, the projection $\pi: M \to S$ is holomorphic with respect to the complex structure $I_\alpha := I_{\sigma_\alpha}$. It is a Lagrangian fibration with respect to the holomorphic symplectic form $\sigma_\alpha$.

Suppose that the base $S$ of a Lagrangian fibration satisfies the condition $H^1(S, \Omega^S_S) = 0$. The next two statements will show that the isomorphism class of the complex manifold $(M, I_\alpha)$ depends only on the cohomology class of $\alpha$.

Lemma 2.16. ([SV, Thm. 2.7]) Let $(M, \sigma)$ be a holomorphic symplectic manifold and $\pi: M \to S$ a proper Lagrangian fibration. Assume that $H^1(S, \Omega^S_S) = 0$. Let $\alpha$ be an exact 2-form of type $(2,0) + (1,1)$ on $S$. Consider a family of $c$-symplectic structures $\sigma_t := \sigma + t\pi^*\alpha$.

Then there exists a flow of diffeomorphisms $\varphi_t$ on $M$ preserving the fibers of $\pi$ such that for each $t$

$$\varphi_t: (M, I_0) \to (M, I_t)$$

is a biholomorphism.

Corollary 2.17. Let $\pi: M \to S$ be as in Lemma 2.16. Consider two cohomologous 2-forms $\alpha$ and $\alpha'$ of Hodge type $(2,0) + (1,1)$ on $S$. Let $I$ and $I'$ be the complex structures on $M$ induced by the $c$-symplectic forms $\sigma + \pi^*\alpha$ and $\sigma + \pi^*\alpha'$ respectively. Then there exists a biholomorphism $\varphi: (M, I) \to (M, I')$ preserving the fibers of $\pi$.

Proof: Follows by applying Lemma 2.16 to the holomorphic symplectic manifold $(M, I)$ equipped with the holomorphic symplectic form $\sigma_\alpha := \sigma + \pi^*\alpha$. Indeed, in this case the form $\sigma + \pi^*\alpha'$ can be written as $\sigma_\alpha + \pi^*d\beta$ for some 1-form $\beta$. \hfill $\square$

Definition 2.18. ([Ver15, Def. 3.17]) Let $\pi: X \to B$ be a Lagrangian fibration and $\alpha$ a closed 2-form on $B$ of type $(2,0) + (1,1)$. Consider the family of $c$-symplectic structures $\sigma_t := \sigma + t\pi^*\alpha$, $t \in \mathbb{C}$. Let $I_t$ be the associated family of complex structures on $X$. The degenerate twistor family $\mathcal{X}_{deg, tw}(X, \pi)$ is the following manifold. Its underlying smooth manifold is $X \times \mathbb{C}$ and the almost complex structure is defined as

$$(I_{tw})_{(x,t)} := I_t \oplus I_{\mathbb{C}},$$

where $x \in X$, $t \in \mathbb{C}$, and $I_{\mathbb{C}}$ is the standard complex structure on $\mathbb{C}$.

One can show that the almost complex structure $I_{tw}$ from Definition 2.18 is integrable [Ver15, Thm. 3.18]. In other words, $\mathcal{X}_{deg, tw}$ is a complex manifold. The projection $\mathcal{X}_{deg, tw} \to \mathbb{C}$ is holomorphic and $\mathcal{X}_{deg, tw}$ is the total space of a holomorphic family of holomorphic symplectic manifolds. It is endowed with a holomorphic fibration $\Pi_{deg, tw}: \mathcal{X}_{deg, tw} \to B \times \mathbb{C}$ that restricts to a fiber $X_t \subset \mathcal{X}_{deg, tw}$ as $\pi$.

3. Shafarevich–Tate group of a Lagrangian fibration

3.1. Structure of Shafarevich–Tate groups: first steps. Let $X$ be a compact hyperkähler manifold and $\pi: X \to B$ a Lagrangian fibration, $B \simeq \mathbb{P}^n$. Recall that the Shafarevich–Tate group $\III$ of $\pi$ is defined to be $H^1(B, \text{Aut}^0_{X/B})$ (Definition 2.11).

Consider the exponential map $\pi_*T_{X/B} \to \text{Aut}^0_{X/B}$. Define the sheaf $\Gamma$ by the short exact sequence

$$0 \to \Gamma \to \pi_*T_{X/B} \to \text{Aut}^0_{X/B} \to 0.$$
It induces the following long exact sequence:

\[ H^1(B, \Gamma) \to H^1(B, \pi_* T_X/B) \to \Pi \to H^2(B, \Gamma) \to 0. \]

The sequence is exact on the right because \( H^2(B, \pi_* T_X/B) = H^{1,2}(\mathbb{P}^n) = 0 \). Indeed, the sheaf \( \pi_* T_X/B \) is isomorphic to \( \Omega^1_B \) by Lemma 2.6.

Write

\[ \widehat{\Pi} = \widehat{\Pi}(X, \pi) := H^1(B, \pi_* T_X/B). \]

Of course, the group \( \widehat{\Pi}(X, \pi) \) is (non-canonically) isomorphic to \( \mathbb{C} \) (Corollary 2.9). However, we will use this notation when we want to emphasize its relation to \( \Pi \).

**Lemma 3.1.** The sheaf \( \Gamma := \ker(\pi_* T_X/B \to Aut^0_X/B) \) is a sheaf of finitely generated torsion-free abelian groups.

**Proof:** The restriction of the sheaf \( \Gamma \) to \( B^\circ \) is a local system of torsion-free abelian groups of rank \( n = \frac{1}{2} \dim X \). For every open \( U \subset B \) the restriction map

\[ H^0(U, \Gamma) \to H^0(U \cap B^\circ, \Gamma) \]

is injective. Indeed, an element of \( H^0(U, \Gamma) \) is a vector field on \( \pi^{-1}(U) \). If a vector field vanishes on an open subset of \( \pi^{-1}(U) \), then it vanishes on \( \pi^{-1}(U) \). The group on the right-hand side of (2) is torsion-free of finite rank, hence so is the group on the left-hand side. \( \square \)

Lemma 3.1 implies that cohomology groups of \( \Gamma \) are finitely generated abelian groups. Let \( \Pi^0 \) denote the image of the map \( \widehat{\Pi} = H^1(B, \pi_* T_X/B) \to \Pi \) from the long exact sequence (1). The group \( \Pi \) fits into the short exact sequence

\[ 0 \to \Pi^0 \to \Pi \to H^2(B, \Gamma) \to 0. \]

Being a quotient of \( \widehat{\Pi} = \mathbb{C} \) by a subgroup, the group \( \Pi^0 \) inherits a structure of a connected topological group. Endow \( \Pi \) with the translation invariant topology such that \( \Pi^0 \) is the connected component of unity of \( \Pi \). View \( H^2(B, \Gamma) \) as a discrete topological group. Both maps in the exact sequence (3) are continuous maps of topological groups.

The following remark provides useful intuition for Shafarevich–Tate groups.

**Remark 3.2.** Let \( S \) be a complex manifold. Let \( Y := S \times \mathbb{C}^\times \). Let \( Aut^0_Y/S \subset Aut_Y/S \) be the subsheaf consisting of the automorphisms that lie in the connected component of unity. Then \( Aut^0_Y/S = \Theta_{\hat{S}} \) and \( H^1(S, Aut^0_Y/S) = \text{Pic}(S) \). Total spaces of twists by classes in \( H^1(S, Aut^0_Y/S) \) are holomorphic principal \( \mathbb{C}^\times \)-bundles over \( S \). They are in one-to-one correspondence with holomorphic line bundles.

Thus, the Shafarevich–Tate group \( \Pi \) serves as an analog of the Picard group, its subgroup \( \Pi^0 \) is an analog of \( \text{Pic}^0(S) \), and \( \Pi / \Pi^0 \) may be thought of as the Néron–Severi group \( \text{NS}(S) \).

### 3.2. Shafarevich–Tate family.

Consider an element \( s \in \widehat{\Pi} \). The twist \( X^s \) of \( X \) by \( s \) is defined to be the manifold \( X^{[s]} \) where \([s]\) is the image of \( s \) under the map \( \widehat{\Pi} \to \Pi \) (Subsection 2.2). The twist \( X^s \) is a complex manifold endowed with a holomorphic map \( \pi^s := \pi^{[s]} \) to \( B \).

**Proposition 3.3.** Let \( \pi: X \to B \) be a Lagrangian fibration without multiple fibers in codimension one. There exists a holomorphic family of complex manifolds

\[ \mathfrak{x}_{\Pi} \to \widehat{\Pi} \cong \mathbb{C} \]

such that the fiber over \( s \in \widehat{\Pi} \) is \( X^s \). This family is endowed with a holomorphic fibration

\[ \Pi_{\Pi} : \mathfrak{x}_{\Pi} \to B \times \widehat{\Pi}. \]
and the map $\Pi_{\text{III}}$ restricts to $\pi^*$ on each fiber $X^*$.

**Proof:** Set $\tilde{B} := B \times \tilde{\Pi}$ and $\tilde{X} := X \times \tilde{\Pi}$. Let $\tilde{\pi} := \pi \times id : \tilde{X} \to \tilde{B}$ be the projection. For any affine cover $B = \bigcup_i U_i$, define $\tilde{U}_i := U_i \times \tilde{\Pi}$. Of course, $\tilde{B} = \bigcup_i \tilde{U}_i$ is an affine covering of $\tilde{B}$.

Let $v \in \tilde{\Pi}$ be a non-zero class. Represent it by a Čech cocycle $v_{ij}$ on some open covering $U_{ij}$. Let $\tilde{v}_{ij}$ be the pullback of $v_{ij}$ to $\tilde{U}_{ij}$. Define a 1-cocycle on $\tilde{B}$ with coefficients in $\tilde{\pi}_* T_{\tilde{X}/\tilde{B}}$ as

$$w_{ij} := tv_{ij} \in \tilde{\pi}_* T_{\tilde{X}/\tilde{B}}(\tilde{U}_{ij})$$

where $t$ is the coordinate on $\tilde{\Pi} \cong \mathbb{C}$. The twist of $\tilde{\pi} : \tilde{X} \to \tilde{B}$ along the cocycle $\{v_{ij}\} := \{\exp(w_{ij})\}$ is the desired family $X_{\text{III}}$. In other words, the manifold $X_{\text{III}}$ is obtained as

$$X_{\text{III}} = \bigsqcup \left( \pi^{-1}(U_i) \times \mathbb{C} \right) \div \left( \pi^{-1}(U_j) \times \mathbb{C} \right) \ni x \sim \varphi_{ij}(x) \in \left( \pi^{-1}(U_j) \times \mathbb{C} \right).$$

\[ \square \]

**Definition 3.4.** The family $X_{\text{III}} \to \tilde{\Pi}$ constructed in Proposition 3.3 will be referred to as the Shafarevich–Tate family.

### 3.3. Shafarevich–Tate twists are symplectic

A twist of a Lagrangian fibration by an element of $\text{III}$ is a priori only a complex manifold. We will see in this Subsection that it is a holomorphic symplectic manifold.\(^2\)

**Lemma 3.5.** Let $(M, \sigma)$ be a (not necessarily compact) holomorphic symplectic manifold and $\pi : M \to S$ a proper Lagrangian fibration on $M$ over a smooth base $S$. Consider an automorphism $\varphi \in H^0(S, \text{Aut}^0_{M/S})$. Then

$$\varphi^* \sigma - \sigma = \pi^* \alpha$$

where $\alpha$ is a closed holomorphic 2-form on $S$.

**Proof:** The statement is local on $S$ so we can always shrink $S$ if necessary. For $S$ small enough, one can realize $\varphi$ as $\exp(v)$ for a vertical holomorphic vector field $v$. There exists a holomorphic 1-form $\beta$ on $S$ such that $\iota_v \sigma = \pi^* \beta$ (Lemma 2.6). Let $\varphi_t$ denote the flow of $v$. Then

$$\varphi^* \sigma - \sigma = \int_0^1 \frac{d(\varphi^*_t \sigma)}{dt} dt = \int_0^1 \varphi^*_t \iota_v \sigma dt = d \int_0^1 \varphi^*_t (\iota_v \sigma) dt = d \int_0^1 \varphi^*_t \pi^* \beta dt = \pi^* \beta.$$

The last identity holds because $\pi \circ \varphi_t = \pi$. Thus $\varphi^* \sigma - \sigma = \pi^* \alpha$ where $\alpha = d\beta$. \[ \square \]

**Theorem 3.6.** Let $\pi : X \to B$ be a Lagrangian fibration on a compact holomorphic symplectic manifold $X$. Denote by $\text{Aut}^{0,\sigma}_{X/B}$ the subsheaf of $\text{Aut}^{0}_{X/B}$ consisting of $\sigma$-symplectic automorphisms. Then the inclusion $\text{Aut}^{0,\sigma}_{X/B} \to \text{Aut}^{0}_{X/B}$ induces an isomorphism of cohomology groups $H^1(B, \text{Aut}^{0,\sigma}_{X/B}) \to H^1(B, \text{Aut}^{0}_{X/B})$ for $i = 0, 1$.

**Proof:** We will only prove that the map $H^1(B, \text{Aut}^{0,\sigma}_{X/B}) \to H^1(B, \text{Aut}^{0}_{X/B})$ is surjective. The proof of injectivity of this map, as well as the proof of the isomorphism $H^0(B, \text{Aut}^{0,\sigma}_{X/B}) \simeq H^0(B, \text{Aut}^{0}_{X/B})$ follow the same pattern and are left to the reader.

---

\(^2\)A twist of a Lagrangian fibration might not be Kähler. That is why we use the term holomorphic symplectic instead of hyperkähler.
Consider a class $\varphi \in H^1(B, \text{Aut}_X^0(B))$. It can be represented by a Čech 1-cocycle $\varphi_{ij} \in \text{Aut}_X^0(U_{ij})$ for an open covering $B = \bigcup U_i$. Lemma 3.5 implies that $\varphi_{ij}^*\sigma - \sigma = \pi^*\alpha_{ij}$ for a closed holomorphic 2-form $\alpha_{ij}$ on $U_{ij}$.

The collection of forms $\{\alpha_{ij}\}$ defines a Čech 1-cocycle on $B$ with coefficients in $\Omega^2_B$. As $H^1(B, \Omega^2_B)$ vanishes, the cocycle $\{\alpha_{ij}\}$ is exact. Consequently, there exist holomorphic 2-forms $\beta_i$ on $U_i$ such that $\alpha_{ij} = \beta_i|_{U_{ij}} - \beta_j|_{U_{ij}}$. The form $\alpha_{ij}$ is closed for every $i, j$, hence $d\beta_i|_{U_{ij}} = d\beta_j|_{U_{ij}}$. That means that the forms $d\beta_i$ glue to a globally defined holomorphic 3-form. There are no non-trivial holomorphic 3-forms on $B = \mathbb{P}^n$, hence $d\beta_i = 0$ for every $i$. Therefore, the forms $\beta_i$ are exact.

We can find a holomorphic 1-form $\gamma_i$ on $U_i$ such that $\beta_i = d\gamma_i$. Let $v_i$ be the vertical vector field on $\pi^{-1}(U_i)$ such that $\nabla v_i, \sigma = \pi^*\gamma_i$ and $\psi_i$ be the flow of $v_i$. Define $\tilde{\varphi}_{ij}$ to be the automorphism

$$\tilde{\varphi}_{ij} := \varphi_{ij} \circ \psi_i|_{U_{ij}} \circ \psi_j^{-1}|_{U_{ij}}.$$  

A direct computation shows that $\tilde{\varphi}_{ij}^*\sigma = \sigma$. At the same time $\{\tilde{\varphi}_{ij}\}$ define the same class in $H^1(B, \text{Aut}_X^0(B))$ as $\{\varphi_{ij}\}$. This proves the surjectivity of the map

$$H^1(B, \text{Aut}_X^0(B)) \to H^1(B, \text{Aut}_X^0(B)).$$

Corollary 3.7. Let $\pi: X \to B$ be a Lagrangian fibration and $\pi^*: X^* \to B$ be the twist of $\pi$ by an element $s \in \mathbb{N}$. Then $X^*$ is a holomorphic symplectic manifold and $\pi^*$ is a Lagrangian fibration.

Proof: Every element $s \in \mathbb{N}$ can be represented by a Čech cocycle $\varphi_{ij} \in \text{Aut}_X^0(U_{ij})$ (Theorem 3.6). The manifold $X^*$ is obtained by gluing the holomorphic symplectic manifolds $\pi^{-1}(U_i)$ by the automorphisms $\varphi_{ij}$. Since the automorphisms $\varphi_{ij}$ preserve $\sigma$, the forms $\sigma|_{\pi^{-1}(U_i)}$ are glued to a well-defined holomorphic symplectic form $\sigma^*$ on $X^*$. Locally $\sigma^*$ coincides with $\sigma$, therefore it is Lagrangian with respect to $\sigma^*$.

Remark 3.8. A similar result was obtained in [Markush96, Prop. 2.6] for families of commutative complex Lie groups whose general fiber is a torus.

3.4. Degenerate twistor deformations as Shafarevich–Tate twists. Let $\pi: X \to B$ be a Lagrangian fibration without multiple fibers in codimension one. Consider the Shafarevich–Tate family $X_{\text{HT}} \to \tilde{\mathbb{H}}$ (Proposition 3.3) and the degenerate twistor family $X_{\text{deg.tw}} \to \mathbb{C}$ (Definition 2.18) associated to $\pi$. Fix a holomorphic symplectic form $\sigma$ on $X$. It induces an isomorphism $\tilde{\mathbb{H}} \cong H^{1,1}(B)$. Let $[\alpha] \in H^{1,1}(B)$ denote the class of a hyperplane section of $B = \mathbb{P}^n$. The class $[\alpha]$ induces the natural isomorphism $H^{1,1}(B) \cong \mathbb{C}$.

Theorem 3.9. There exists an isomorphism of $X_{\text{HT}}$ and $X_{\text{deg.tw}}$ as families of complex manifolds that lifts the isomorphism $\tilde{\mathbb{H}} \cong \mathbb{C}$. In other words, we have the commutative diagram:

$$
\begin{array}{ccc}
X_{\text{deg.tw}} & \xrightarrow{\phi} & X_{\text{HT}} \\
\Pi_{\text{deg.tw}} \downarrow & & \downarrow \Pi_{\text{HT}} \\
\mathbb{C} & \xrightarrow{\simeq} & \tilde{\mathbb{H}}.
\end{array}
$$

Proof. Step 1: First, let us recall the relation between Dolbeault and Čech cohomology groups of the sheaf $\Omega^2_B$. The class $[\alpha]$ considered as a Dolbeault cohomology class, can be represented by a closed $(1,1)$-form $\alpha$ on $B$. Take an affine open cover $B = \bigcup U_i$. The class $[\alpha]$ may be represented by a Čech cocycle $\{a_{ij}\}$ where $a_{ij} \in H^0(U_{ij}, \Omega^2_B)$ are holomorphic 1-forms on $U_{ij}$. Given a closed $(1,1)$-form $\alpha$, one can construct $\{a_{ij}\}$ as
follows. There exists a \((1,0)\)-form \(a_i\) on \(U_i\) such that \(\omega|_{U_i} = da_i = \overline{\partial}a_i\) for every \(i\). We define the form \(a_{ij}\) as
\[
a_{ij} := a_i|_{U_i} - a_j|_{U_i}.
\]

**Step 2:** In this step we will construct an isomorphism from \(X_{\mathrm{III}}\) to \(X_{\mathrm{deg.tw.}}\) locally on the base \(B\). Consider the vertical vector fields \(w_i\) on \(\pi^{-1}(U_i) \times \mathbb{C} \subset X \times \mathbb{C}\) defined by the formula
\[
t_{w_i}\sigma = t\pi^*a_i
\]
Here \(t\) is the linear coordinate function on \(\mathbb{C}\) and \(\sigma\) is the pullback of the holomorphic symplectic form on \(X\) to \(\pi^{-1}(U_i) \times \mathbb{C}\). For any \(t \in \mathbb{C}\) we have the following identity on \(\pi^{-1}(U_i) \times \{t\}\):
\[
L_{w_i}\sigma = t\pi^*da_i = t\pi^*\alpha|_{\pi^{-1}(U_i)}.
\]
Let \(\varphi_i\) be the exponential of the vector field \(w_i\). This gives the following equality of forms on \(\pi^{-1}(U_i) \times \{t\}\):
\[
\varphi_i^*\sigma = \sigma + t\pi^*\alpha,
\]
so that for any \(t \in \mathbb{C}\) the map \(\varphi_i\) is a biholomorphism
\[
\varphi_i|_{\pi^{-1}(U_i) \times \{t\}} : (\pi^{-1}(U_i), I_t) \to \pi^{-1}(U_i),
\]
where \(I_t\) is the complex structure from Lemma 2.16. In fact, since every automorphism \(\varphi_i\) commutes with the projection \(\pi^{-1}(U_i) \times \mathbb{C} \to \mathbb{C}\), it induces a biholomorphism
\[
\varphi_i : (\pi^{-1}(U_i) \times \mathbb{C}, I_{tw}) \to \pi^{-1}(U_i) \times \mathbb{C}.
\]
Here \(I_{tw}\) is the complex structure introduced in Definition 2.18.

**Step 3:** It remains to prove that the maps \(\varphi_i\) can be glued together to a global isomorphism of families \(X_{\mathrm{deg.tw.}} \to X_{\mathrm{III}}\). Define the vector fields \(w_{ij}\) on \(\pi^{-1}(U_{ij}) \times \mathbb{C}\) by
\[
t_{w_{ij}}\sigma = t\pi^*(a_{ij}).
\]
Let \(\varphi_{ij}\) be the exponential of \(w_{ij}\). Then \(\varphi_{ij}\) is a holomorphic automorphism of \(\pi^{-1}(U_{ij}) \times \mathbb{C}\). Since \(a_{ij} = a_j - a_i\) one has
\[
\varphi_{ij} = \varphi_j \circ \varphi_i^{-1}.
\]
By Proposition 3.3 the manifold \(X_{\mathrm{III}}\) is the twist of \(X \times \widehat{\Pi}\) by the cocycle \(\{\varphi_{ij}\}\). Therefore the maps \(\varphi_i\) glue together to give rise to a global biholomorphism \(\Phi : X_{\mathrm{deg.tw.}} \to X_{\mathrm{III}}\). \(\square\)

3.5. **The period map of a Shafarevich–Tate family.** Here we will describe the period map of a Shafarevich–Tate family. That can be viewed as a variation on Markman’s result [Markman, Thm. 7.11].

The following definition is standard. Let \(X\) be a hyperkähler manifold. Its *period domain* \(D\) is defined as
\[
D := \{[x] \mid q(x, x) = 0 \text{ and } q(x, \overline{x}) > 0\} \subset \mathbb{P}(H^2(X, \mathbb{C}))
\]
where \(q\) is the BBF form. Let \(p : X \to T\) be a holomorphic family of hyperkähler manifolds over a connected simply connected base \(T\). One can associate the period map \(\mathcal{P}_T : T \to D\) to \(p\). This is the holomorphic map defined as
\[
t \mapsto [H^{2,0}(p^{-1}(t))].
\]

Its importance is manifested by the Torelli theorem. Let \(\text{Teich}(X)\) be the Teichmüller space of \(X\), i.e., the space of complex structures of hyperkähler type on \(X\) modulo isotopies. Then the period map
\[
\mathcal{P} : \text{Teich}(X) \to D
\]
is generically injective on connected components of $Teich(X)$ [Huy10].

**Proposition 3.10.** Let $\mathcal{X}_{\text{deg.tw}} \to \mathbb{C}$ be the degenerate twistor family of a Lagrangian fibration $\pi: X \to B$. Then its period map $\mathcal{P}_{\text{deg.tw}}: \mathbb{C} \to \mathcal{D}$ is injective. The image of $\mathcal{P}_{\text{deg.tw}}$ is the entire curve obtained as the intersection of $\mathcal{D}$ with a projective line lying on the quadric $\{q(x, x) = 0\}$.

**Proof:** Let $\alpha$ be a Kähler form on $B$. The period map of the degenerate twistor family is given by

$$\mathcal{P}_{\text{deg.tw}}(t) = [\sigma + t\pi^*\alpha]$$

Let $\eta$ denote the class of $\pi^*\alpha$ in $H^2(X, \mathbb{C})$. Consider the projective line $L := \mathbb{P}(\{\sigma, \eta\})$ lying in $\mathbb{P}(H^2(X, \mathbb{C}))$. The map $\mathcal{P}_{\text{deg.tw}}$ sends $\mathbb{C}$ isomorphically to $L \setminus \{\eta\}$. All points of this line satisfy $q(x, x) = 0$. The point corresponding to $\eta$ does not lie in the period domain $\mathcal{D}$ since $q(\eta, \eta) = 0$. Therefore, the image of $\mathcal{P}_{\text{deg.tw}}$ is $L \cap \mathcal{D}$. □

**Remark 3.11.** One may think of $\tilde{\mathcal{M}}$ as an entire curve in the Teichmüller space of $X$ and $\mathcal{M}^0$ as its image inside the moduli space $\mathcal{M}$. The moduli space $\mathcal{M}$ can be defined as $Teich(X)$ modulo the mapping class group of $X$. It is known that the action of the mapping class group on $Teich(X)$ is very non-discrete. Therefore, $\mathcal{M}$ is non-Hausdorff.

This phenomenon is often seen already at the level of Shafarevich–Tate groups (Theorem 5.7)

4. Connected Component of Unity of a Shafarevich–Tate Group

4.1. The sheaf $\Gamma$. Consider the exponential exact sequence

$$0 \to \mathbb{Z}_X \to \mathcal{O}_X \to \mathcal{O}_X^\times \to 0$$

Apply the higher direct image functor $R^i\pi_*(-)$ to it. Since the fibers of $\pi$ are connected and proper, we have $\pi_*\mathbb{Z}_X \simeq \mathbb{Z}_B$, $\pi_*\mathcal{O}_X \simeq \mathcal{O}_B$, and $\pi_*\mathcal{O}_X^\times \simeq \mathcal{O}_B^\times$. The sequence

$$0 \to \pi_*\mathbb{Z}_X \to \pi_*\mathcal{O}_X \to \pi_*\mathcal{O}_X^\times \to 0$$

is therefore exact. Hence we get a long exact sequence of sheaves:

$$0 \to R^1\pi_*\mathbb{Z}_X \to R^1\pi_*\mathcal{O}_X \to R^1\pi_*\mathcal{O}_X^\times \to R^2\pi_*\mathbb{Z}_X \to \ldots$$

**Proposition 4.1.** The isomorphism

$$\tilde{\omega}: R^1\pi_*\mathcal{O}_X \to \pi_*T_{X/B}$$

from Proposition 2.7 sends $R^1\pi_*\mathbb{Z}_X \subset R^1\pi_*\mathcal{O}_X$ into $\Gamma := \ker(\pi_*T_{X/B} \to Aut^0_{X/B})$.

**Proof:** We need to prove that the composition of the maps

$$R^1\pi_*\mathbb{Z} \xrightarrow{i} R^1\pi_*\mathcal{O}_X \xrightarrow{\tilde{\omega}} \pi_*T_{X/B} \xrightarrow{\varepsilon} Aut^0_{X/B}$$

vanishes. Let $b \in B^\circ$ and $F_b$ be a smooth fiber. Then $(R^1\pi_*\mathcal{O}_X)|_b = H^1(F_b, \mathcal{O}_{F_b})$, $(\pi_*T_{X/B})|_b = H^0(F_b, TF_b)$ and $\tilde{\omega}_b$ is the map

$$H^0(F_b) \to H^1(F_b, TF_b),$$

given by the polarization $[\omega]|_{F_b}$ on the abelian variety $F_b$. By Corollary 2.2 we can choose $\omega$ in such a way that this polarization is integral, so $\tilde{\omega}_b$ maps $H^1(F_b, \mathbb{Z}) \subset H^1(F_b, \mathcal{O}_{F_b})$ to $\Gamma_b = H^1(F_b, \mathbb{Z}) \subset H^0(F_b, TF_b)$.

It follows that the statement of the lemma holds after the restriction to $B^\circ$. Consider a local section $\tau$ of $R^1\pi_*\mathbb{Z}_X$ over an open subset $U \subset B$. Denote $U \cap B^\circ$ by $U^\circ$. We have seen that the restriction of the automorphism $(\varepsilon \circ \tilde{\omega} \circ i)(\tau)$ to $\pi^{-1}(U^\circ)$ is trivial. An automorphism that is trivial on a dense open subset is trivial. Hence, the vertical automorphism $(\varepsilon \circ \tilde{\omega} \circ i)(\tau)$ of $\pi^{-1}(U)$ is trivial. □
We have constructed a morphism of sheaves $\alpha : R^1\pi_*\mathcal{Z}_X \to \Gamma$. Note that it is injective, since it is a composition of an isomorphism $\tilde{\omega} : R^1\pi_*\mathcal{O}_X \to \pi_*\mathcal{T}_{X/B}$ and an embedding $i : R^1\pi_*\mathcal{Z}_X \to R^1\pi_*\mathcal{O}_X$.

There is the following diagram with exact rows:

$$
\begin{array}{ccccccccc}
0 & \to & \Gamma & \to & \pi_*\mathcal{T}_{X/B} & \xrightarrow{\varepsilon} & \text{Aut}_{X/B}^0 & \to & 0 \\
\uparrow{\alpha} && \uparrow{\tilde{\omega}} && \uparrow{\omega} & & & & \\
0 & \to & R^1\pi_*\mathcal{Z}_X & \xrightarrow{i} & R^1\pi_*\mathcal{O}_X & \to & R^1\pi_*\mathcal{O}_X^\omega & \to & 0 \\
\end{array}
$$

Recall the local invariant cycle theorem:

**Proposition 4.2.** ([BBDG, Cor. 6.2.9]) Let $f : X \to Y$ be a proper map of complex algebraic varieties with $X$ non-singular. For a subset $U \subset Y$, denote by $U^\circ$ the set of non-critical values of $f$ in $U$. Then for any $b \in Y$ there exists a small ball $U_b \subset Y$ with the center $b$ such that the following holds.

1. For any $i$ the embedding $f^{-1}(b) \to f^{-1}(U_b)$ induces an isomorphism:
   $$H^i(f^{-1}(b), \mathbb{Q}) \cong H^i(f^{-1}(U_b), \mathbb{Q});$$

2. For any point $b^0 \in U_b^\circ$ the cohomology group $H^i(f^{-1}(U_b), \mathbb{Q})$ surjects onto the local monodromy invariants $H^i(f^{-1}(b^0), \mathbb{Q})^{\pi_1(\mathcal{O}_{\mathbb{Q}})} = R^\pi\mathcal{O}_{\mathbb{Q}}(U_b^\circ)$. \hfill \blacksquare

**Remark 4.3.** Let $\pi : X \to B$ be a Lagrangian fibration on a hyperkähler manifold. For every $b \in B$ there exists an open subset $U \subset B$ containing $b$ such that $\pi^{-1}(U)$ is a smooth algebraic variety [SV, Cor. 3.4]. Therefore, the local invariant cycle theorem can be applied in this situation.

Let $\Gamma_{\mathbb{Q}}$ denote the sheaf $\Gamma \otimes \mathbb{Q}_{\mathbb{B}}$.

**Proposition 4.4.** The map $\alpha \otimes \mathbb{Q} : R^1\pi_*\mathbb{Q} \to \Gamma_{\mathbb{Q}}$ is an isomorphism.

**Proof:** Step 1. First, notice that the lemma holds after the restriction to the non-critical set $B^\circ \subset B$. Indeed, if $F_b = \pi^{-1}(b)$ is a smooth fiber, then

$$\alpha_b : (R^1\pi_*\mathcal{O}_X)_b = H^1(F_b, \mathbb{Q}) \to (\Gamma_{\mathbb{Q}})_b = H_1(F_b, \mathbb{Q})$$

is the map given by the polarization on the abelian variety $F$. Hence it is an isomorphism. Note that this does not hold in general with integral coefficients.

Step 2. Take a point $b \in B$ and let $U_b \subset B$ be as in Proposition 4.2. We need to prove that the map

$$\alpha_{U_b} : H^1(U_b, \mathbb{Q}) \to \Gamma_{\mathbb{Q}}(U_b)$$

is an isomorphism. We already know that it is injective, hence we only need to check surjectivity. Take a local section $\gamma \in \Gamma_{\mathbb{Q}}(U_b)$. Let $\gamma^0$ be its restriction to $U^0_b$. By Step 1 there exists $\beta^0 \in H^0(U^0_b, R^1\pi_*\mathcal{O}_X)$, such that $\alpha_{U^0_b}(\beta^0) = \gamma^0$. The section $\beta^0$ can be lifted to a section $\beta$ of $H^0(U^0_b, R^1\pi_*\mathcal{O}_X) = H^1(\pi^{-1}(U^0_b), \mathbb{Q})$ by the local invariant cycle theorem. The local sections $\alpha_{U_b}(\beta) \in \Gamma_{\mathbb{Q}}(U_b)$ and $\gamma \in \Gamma_{\mathbb{Q}}(U_b)$ coincide after restriction to $U^0_b$ by their construction. The sheaf $\Gamma_{\mathbb{Q}}$ is a subsheaf of the locally free sheaf $R^1\pi_*\mathcal{O}_X$, hence $\alpha_{U_b}(\beta)$ and $\gamma$ coincide. \hfill \blacksquare
4.2. The connected component of unity of Shafarevich–Tate groups. The goal of this Subsection is to describe the group $\Pi^0$, i.e., the connected component of unity of the group $\Pi$. This group fits into the short exact sequence

$$H^1(B, \Gamma) \to \tilde{\Pi} \to \Pi^0 \to 0.$$ 

In the previous Subsection, we constructed an injective map $\alpha : R^1\pi_*\mathbb{Z} \to \Gamma$ and proved that $\alpha$ is an isomorphism after tensoring with $\mathbb{Q}$ (Proposition 4.4). Although we are mostly concerned with the sheaf $\Gamma$, the sheaf $R^1\pi_*\mathbb{Z}$ is often much easier to understand.

Define the subspace $W \subset H^2(X, \mathbb{C})$ as the kernel of the restriction map

$$H^2(X, \mathbb{C}) \to H^0(B, R^2\pi_*\mathbb{C}_X)$$

This is a Hodge substructure in $H^2(X, \mathbb{Z})$ that can be described as

$$W = \bigcap_{b \in B} \ker(H^2(X, \mathbb{Z}) \to H^2(F_b, \mathbb{Z})).$$

Let us denote $W_\mathbb{Q} = W \bigcap H^2(X, \mathbb{Q})$.

As before, let $\eta \in H^2(X, \mathbb{Z})$ be the pullback of the class of a hyperplane section. Let $\eta^\perp$ be the orthogonal to $\eta$ with respect to the BBF form.

**Proposition 4.5.** Let $W$ be as above. Then we have the following chain of inclusions

$$\{\eta, [\sigma], [\tilde{\sigma}]\} \subset W \subset \eta^\perp.$$ 

**Proof:** The class $\eta$ is the Chern class of the line bundle $\pi^*\mathcal{O}_{\mathbb{P}^n}(1)$. This line bundle restricts trivially to every fiber of $\pi$, hence $\eta$ is contained in $W$. By Proposition 2.1 the kernel of the restriction map $H^2(X, \mathbb{C}) \to H^2(F_b, \mathbb{C})$ to a smooth fiber $F_b$ is $\eta^\perp$. We obtain that $W \subset \eta^\perp$.

We are left to prove that $[\sigma]$ is contained in $W$. Indeed, $W$ is a Hodge substructure, thus it will imply that $[\tilde{\sigma}]$ is in $W$ as well. The fibration $\pi$ is Lagrangian, hence the restriction of the 2-form $\sigma$ to every smooth fiber of $\pi$ vanishes. Therefore the class $[\sigma]$ in $H^2(X, \mathbb{C})$ restricts trivially to a smooth fiber.

Let $F_b$ be a singular fiber. By [Cl, Thm. 6.9] there is a neighborhood $U_b \subset B$ of $b$ and a smooth retraction of $\pi^{-1}(U_b)$ to $F_b$, i.e., a one-parameter family of diffeomorphisms $f_t$ such that

$$f_1 : \pi^{-1}(U_b) \to \pi^{-1}(U_b), \quad f_0 = \text{id}_{\pi^{-1}(U_b)}, \quad \text{im}(f_1) = F_b, \quad f_t|_{F_b} = \text{id}_{F_b} \quad \forall t.$$ 

One can see that $f_0^*\sigma = \sigma$ and $f_1^*\sigma = 0$. Let $\xi_t$ denote the tangent vector field to the one-parameter family $f_t$. Compute

$$\sigma|_{\pi^{-1}(U_b)} = -(f_1^*\sigma - f_0^*\sigma) = -\int_0^1 \frac{d}{dt} f_t^*\sigma = -\int_0^1 f_t^*\xi_t\sigma = -d \left( \int_0^1 f_t^*\xi_t\sigma \right).$$

We conclude that the form $\sigma$ becomes exact after the restriction to $\pi^{-1}(U_b)$. Therefore, the class of $\sigma$ is indeed contained in $W$. \qed

**Remark 4.6.** If all fibers of $\pi$ are irreducible, then $\ker(H^2(X, \mathbb{Q}) \to H^2(F_b, \mathbb{Q}))$ does not depend on $b$. In particular, $W = \eta^\perp$. We learned the proof of this fact from Claire Voisin. Let $\nu : \tilde{F}_b \to F_b$ be a resolution of singularities of $F_b$. Denote the inclusion of $F_b$ into $X$ by $\iota$. By [Vois92, Lemma 1.5, Remark 1.6], the kernel of the pullback map $(\iota \circ \nu)^* : H^2(X, \mathbb{Q}) \to H^2(\tilde{F}_b, \mathbb{Q})$ is equal to $\{\alpha \in H^2(X, \mathbb{Q}) \mid [F_b] \cdot \alpha = 0\}$. This subspace does not depend on the choice of $b$, hence it is equal to $\eta^\perp$.
The differential $d$ groups of the sheaf $O_X$.

This spectral sequence degenerates on the second page. It converges to the cohomology $E_d$ differentials starting in $p$. By [PS, Corollary 5.42] we have

$$\ker \nu^*: H^2(F) \to H^2(\bar{F}) = W_1 H^2(F).$$

The desired claim follows.

**Proposition 4.7.** Consider the map

$$H^1(B, R^1 \pi_* \mathbb{Z}) \to H^1(R^1 \pi_* O_X) \simeq \tilde{\Pi}$$

induced by the embedding $R^1 \pi_* \mathbb{Z} \to R^1 \pi_* O_X$. There are canonical isomorphisms

$$\tilde{\Pi} \simeq H^{0,2}(X)$$

and

$$H^1(B, R^1 \pi_* \mathbb{Z}) \simeq W_\mathbb{Z}/\eta$$

that fit into the commutative diagram

$$
\begin{array}{ccc}
H^1(B, R^1 \pi_* \mathbb{Z}) & \xrightarrow{\simeq} & \tilde{\Pi} \\
\downarrow & & \downarrow \simeq \\
W_\mathbb{Z}/\eta & \xrightarrow{p} & H^{0,2}(X)
\end{array}
$$

Here $p$ is induced by the Hodge projection $W_\mathbb{Z} \subset H^2(X, \mathbb{Z}) \to H^{0,2}(X)$.

**Proof:** Consider the Leray spectral sequence for the sheaf $O_X$. The terms of the second page of this spectral sequence can be computed using Proposition 2.7 as

$$E^{p,q}_2 = H^p(B, R^q \pi_* O_X) = H^p(B, \Omega^q_b) = \begin{cases} \mathbb{C} & \text{if } p = q, \\ 0 & \text{otherwise.} \end{cases}$$

This spectral sequence degenerates on the second page. It converges to the cohomology groups of the sheaf $O_X$. Therefore,

$$H^2(X, O_X) \simeq E^{1,1}_2 = H^1(R^1 \pi_* O_X) \simeq \tilde{\Pi}.$$

Let us now consider the second page of the Leray spectral sequence for the sheaf $\mathbb{Z}_X$. The differential $d_2: E^{1,1}_2 \to E^{3,0}_2$ vanishes because $E^{3,0}_2 = H^3(\mathbb{P}^n, \mathbb{Z}) = 0$. All the other differentials starting in $E^{1,1}$ vanish because their targets are in negative degrees. Hence,

$$E^{1,1}_\infty = E^{1,1}_2 = H^1(B, R^1 \pi_* \mathbb{Z}).$$

The filtration on the Leray spectral sequence induces a filtration $F^* H^2(X, \mathbb{Z})$ on the cohomology groups of $X$. We obtain the following short exact sequences:

$$0 \to \mathbb{Z} \xrightarrow{\pi_2} F^1 H^2(X, \mathbb{Z}) \to H^1(B, R^1 \pi_* \mathbb{Z}) \to 0$$

$$0 \to F^1 H^2(X, \mathbb{Z}) \to H^2(X, \mathbb{Z}) \to H^0(B, R^2 \pi_* \mathbb{Z}).$$

The second exact sequence implies that $F^1 H^2(X, \mathbb{Z}) = W_\mathbb{Z}$. We see from the first exact sequence that $H^1(B, R^1 \pi_* \mathbb{Z}_X) \simeq W_\mathbb{Z}/\eta$. □

**Corollary 4.8.** Let $\pi: X \to B$ be a Lagrangian fibration without multiple fibers in codimension one. Define the group $\Pi^0$ to be the cokernel of the map

$$W_\mathbb{Z}/\eta \xrightarrow{\pi} H^{0,2}(X).$$

Then the group $\Pi^0$ is a quotient of $\Pi^0$ by a finite group.
**Proof:** The map $\alpha: R^1\pi_*\mathbb{Z} \to \Gamma$ induces the map $H^1(\alpha): H^1(R^1\pi_*\mathbb{Z}) \to H^1(B, \Gamma)$. The map $H^1(\alpha)$ has a finite cokernel by Proposition 4.4. By Proposition 4.7 there is the following commutative diagram

$$
\begin{array}{ccc}
W^1_{\mathbb{Z}}/\eta & \longrightarrow & H^{0,2}(X) \\
\downarrow \cong & & \downarrow \cong \\
H^1(B, \Gamma) & \longrightarrow & \widehat{H}^0 \longrightarrow 0
\end{array}
$$

The kernel of the map $\widehat{H}^0 \to H^0$ is a quotient of $\text{coker}(H^1(\alpha))$ by the Snake lemma. Hence it is finite. \(\square\)

5. Kählerness and projectivity properties

5.1. M-special Hodge structures. In this Subsection, we summarize several results about Hodge structures of K3 type. In particular, we introduce and discuss M-special Hodge structures. A pure Hodge structure $W$ is said to be of K3 type if it is of weight 2 and $\dim W^{2,0} = 1$.

Let $W$ be a $\mathbb{Z}$-Hodge structure of weight 2 and $\mathbb{K} \subset \mathbb{C}$ a subring. We will write $W^{1,1} := W_{\mathbb{K}} \cap W^{1,1}$ and $W^{2,0+0,2} := W_{\mathbb{K}} \cap (W^{2,0} \oplus W^{0,2})$. We denote the Hodge projection by $p: W \to W^{0,2}$. Let $q$ be a polarization on $W$. If $W$ is a Hodge structure of K3 type, then the projection $p: W \to W^{0,2}$ is given by the map

$$
v \to q(v, \sigma)\sigma
$$

for an appropriate choice of $\sigma \in W^{2,0}$.

We define the transcendental Hodge substructure $T \subseteq W$ to be the orthogonal complement to $W^{1,1}$ of $W^{2,0+0,2}$. The lattice $T$ has the rank at least two because

$$W^{2,0+0,2} \subseteq T_{\mathbb{R}}.$$

Note that the restriction of the Hodge projection $p$ to $T$ is an isomorphism to its image.

The following definition is a straightforward generalization of [Markman, Def. 1.1].

**Definition 5.1.** Let $W$ be a Hodge structure of K3 type. It is said to be M-special if $W^{2,0+0,2} \neq 0$.

**Lemma 5.2.** Let $W$ be a polarized Hodge structure of K3 type. The following are equivalent.

1. The subspace $W^{1,1}$ is rational ("Picard rank is maximal").
2. The subspace $W^{2,0+0,2}$ is rational.
3. The rank of the transcendental Hodge substructure $T$ is two.
4. The image of $W_{\mathbb{Z}}$ under the Hodge projection $p: W_{\mathbb{Z}} \to W^{0,2}$ is a discrete subgroup of $W^{0,2}$.

The proof of this lemma is straightforward and is left to the reader. If one of the conditions of the lemma holds, then $W$ is M-special. However, the converse is not true.

**Lemma 5.3.** ([Markman, Lemma 5.5]) Let $W$ be a polarized Hodge structure of K3 type that is not M-special. Assume that $\text{rk} T \geq 3$. Then for every sublattice $T' \subset T$ of corank one, the group $p(T')$ generates $W^{0,2}$ as an $\mathbb{R}$-vector space.

**Proof:** Let $T' \subset T$ be a sublattice of corank one. Suppose that $p(T')$ does not generate $W^{0,2}$ over $\mathbb{R}$. In that case there exist real numbers $a, b$ such that

$$a \cdot q(\text{Re}(\sigma), t) + b \cdot q(\text{Im}(\sigma), t) = 0 \quad \forall t \in T'$$
Hence the vector \( l = a \cdot \text{Re}(\sigma) + b \cdot \text{Im}(\sigma) \in W^{2,0+0.2}_\mathbb{R} \) is orthogonal to \( T' \). Observe that \( l^\perp = (W^{1,1}_\mathbb{Z} + T') \otimes \mathbb{R} \) since \( (W^{1,1}_\mathbb{Z} + T') \otimes \mathbb{R} \subseteq l^\perp \) and the dimensions of the two spaces coincide. The subspace \( l^\perp \) is therefore rational. Hence, a multiple of \( l \) is rational. This contradicts the assumption that \( W \) is not M-special. \( \Box \)

**Lemma 5.4.** ([Markman, Lemma 5.5]) Let \( T \subset \mathbb{R}^2 \) be a finitely generated subgroup of \( \mathbb{R}^2 \) of rank at least three. Assume that every subgroup \( T' \subset T \) of corank one generates \( \mathbb{R}^2 \) over \( \mathbb{R} \). Then \( T \) is dense in \( \mathbb{R}^2 \).

**Theorem 5.5.** ([Markman, Lemma 5.4, 5.5]) Let \( W \) be a polarized \( \mathbb{Z} \)-Hodge structure of \( K3 \) type. Then the following are equivalent

1. The Hodge structure \( W \) is M-special.
2. The image of \( W^\perp \) under the Hodge projection \( p: W \to W^{0,2} \) is not dense in \( W^{0,2} \).

**Proof:** (1) \( \Rightarrow \) (2) Suppose that \( W \) is M-special. Choose a non-zero element \( l \in W^{2,0+0.2}_\mathbb{Z} \). There exists a unique element \( \sigma \in W^{2,0} \) such that \( \text{Re}(\sigma) = l \). The projection \( p: W \to W^{0,2} \) is identified with the map \( v \in W \mapsto q(\sigma, v) \) up to scaling. For every \( v \in W \) the number \( \text{Re}(q(\sigma, v)) = q(l, v) \) is an integer. Hence, there exists a constant \( \lambda \in \mathbb{R} \) such that \( \text{Re}(z) \in \mathbb{Z} \lambda \) for every vector \( z \) in \( p(W_\mathbb{Z}) \). Therefore, \( p(W_\mathbb{Z}) \) cannot be dense in \( W^{0,2} \).

(2) \( \Rightarrow \) (1) Suppose \( W \) is not M-special. In that case the rank of \( T \) is at least three (Lemma 5.2). Lemma 5.3 implies that for every sublattice \( T' \subset T \) of corank one, the group \( p(T') \) generates \( W^{0,2} \) over \( \mathbb{R} \). It follows from Lemma 5.4 that \( p(T_\mathbb{Z}) = p(W_\mathbb{Z}) \) is dense in \( W^{0,2} \). \( \Box \)

The next lemma describes how the properties of being of maximal Picard rank or M-special behave under subquotients.

**Lemma 5.6.** Let \( W \) be a polarized Hodge structure.

1. Consider a Hodge substructure \( U \subset W \) such that \( q|_U \) is non-degenerate and \( U^{0,2} \) is equal to \( W^{0,2} \). Then \( W \) is of maximal Picard rank (resp. M-special) if and only if \( U \) is of maximal Picard rank (resp. M-special).
2. Let \( N \subset W^{1,1} \) be a rational isotropic subspace. Suppose \( W \) has maximal Picard rank (resp. is M-special). Then the same is true for the polarized Hodge structure \( N^\perp/N \).
3. Let \( N \subset W^{1,1} \) be as above. Suppose that there exists a rational subspace \( L \subset W^{1,1} \) such that \( q|_L \) is non-degenerate and \( W = N^\perp \oplus L \). Then the converse of (2) holds. Namely if \( N^\perp/N \) is of maximal Picard rank (resp. M-special) then the same is true for \( W \).

**Proof:** (1) The assertion holds since \( U^{2,0+0.2} = W^{2,0+0.2} \).

(2) The subspace \( W^{2,0+0.2} \) is contained in \( N^\perp \) and it is mapped isomorphically to \( (N^\perp/N)^{2,0+0.2} \). The quotient map \( N^\perp \to N^\perp/N \) maps rational vectors to rational vectors, hence the claim.

(3) The direct sum decomposition \( W = N^\perp \oplus L \) implies that \( W = N \oplus L^\perp \). Therefore, \( N^\perp = N \oplus (L^\perp \cap N^\perp) \). The subspace \( L^\perp \cap N^\perp \) is a polarized Hodge substructure of \( W \) satisfying \( (L^\perp \cap N^\perp)^{0,2} = W^{0,2} \). By (1) the Hodge structure \( W \) is of maximal Picard rank (resp. M-special) if and only if so is \( (L^\perp \cap N^\perp) \). The quotient map \( N^\perp \to N^\perp/N \) induces an isomorphism of polarized Hodge structures \( L^\perp \cap N^\perp \to N^\perp/N \), hence the claim. \( \Box \)
5.2. Kählerness of degenerate twistor deformations. Let \( \pi: X \to B \) be a Lagrangian fibration. Recall that we defined the Hodge substructure \( W \subset H^2(X, \mathbb{Z}) \) as
\[
W := \ker (H^2(X, \mathbb{Z}) \to H^0(B, R^2\pi_*\mathbb{Z}_X)) .
\]
The Hodge structure \( W \) is of K3 type by Proposition 4.5. Let us denote the Hodge projection by \( p: H^2(X) \to H^{0,2}(X) \).

**Theorem 5.7.** Let \( \pi: X \to B \) be a Lagrangian fibration without multiple fibers in codimension one, \( \Pi^0 \) the connected component of unity of its Shafarevich–Tate group \( \Pi \). Then \( \Pi^0 \) is Hausdorff if and only if the Hodge structure \( \eta^+/\eta \) is of maximal Picard rank. In this case \( \Pi^0 \) is isomorphic to an elliptic curve.

**Proof:** The group \( \Pi^0 \) is Hausdorff if and only if the topological group \( \tilde{\Pi}^0 \) defined as \( H^{0,2}(X)/p(W_Z) \) is Hausdorff. Indeed, the latter group is a finite unramified cover of \( \Pi^0 \) (Corollary 4.8). We see that \( \Pi^0 \) is Hausdorff if and only if \( p(W_Z) \subset H^{0,2}(X) \) is a discrete subgroup. Apply Lemma 5.2 to the Hodge structure \( W/\eta \). We obtain that \( p(W_Z) \) is discrete in \( H^{0,2}(X) \) if and only if \( (W/\eta)^{2,0+0,2} \) is a rational subspace. This is equivalent to saying that \( \eta^+/\eta \) has maximal Picard rank by Lemma 5.6 (1) and Lemma 5.2. If \( p(W_Z) \) is discrete, then \( p(W_Z) \) is a lattice in \( H^{0,2}(X) \) of rank two and \( \Pi^0 \) is an elliptic curve. \( \Box \)

We are now ready to prove the main theorem of this Section.

**Theorem 5.8.** Let \( \pi: X \to B \) be a Lagrangian fibration without multiple fibers in codimension one and \( \mathfrak{X}_{\text{III}} \to \tilde{\Pi} \) the Shafarevich–Tate family (Subsection 3.2). Assume that the Hodge structure \( \eta^+/\eta \) is not M-special. Then the members of the family \( X^s \subset \mathfrak{X}_{\text{III}} \) are Kähler for each \( s \in \tilde{\Pi} \).

**Proof:** The zero fiber \( X^0 \) is Kähler by assumption. Kählerness is an open condition. Thus there exists an open subset \( U \subset \tilde{\Pi} \) such that for every \( s \in U \) the twist \( X^s \) is Kähler [Vois03, Thm. 9.23]. The manifolds \( X^s \) and \( X^{s+\lambda} \) are isomorphic for each \( \lambda \) in the subgroup \( \Lambda := \text{im}(H^1(B, \Gamma)) \subset \tilde{\Pi} \). The vector spaces \( p(W_Z) \otimes \mathbb{Q} \) and \( \Lambda \otimes \mathbb{Q} \) coincide (Proposition 4.4). By Lemma 5.6 (1) the Hodge structure \( W/\eta \) is not M-special. It follows from Theorem 5.5 that \( p(W_Z) \) is dense in \( \tilde{\Pi} \approx H^{0,2}(X) \). Hence \( \Lambda \) is dense in \( \tilde{\Pi} \). We obtain that for every \( s \in \tilde{\Pi} \) there exists \( s' \in U \) such that \( X^s \approx X^{s'} \). Consequently, \( X^s \) is Kähler for every \( s \). \( \Box \)

**Definition 5.9.** ([Markman, Def. 1.1]) Let \( X \) be a hyperkähler manifold. It is called M-special if
\[
(H^{0,0}(X) \oplus H^{0,2}(X)) \cap H^2(X, \mathbb{Z}) \neq \{0\}
\]
In other words, a hyperkähler manifold is M-special if the Hodge structure on its second cohomology is M-special (Definition 5.1).

**Lemma 5.10.** Let \( \pi: X \to B \) be a Lagrangian fibration on a projective hyperkähler manifold \( X \). Then \( X \) is of maximal Picard rank (resp. M-special) if and only if the Hodge structure \( \eta^+/\eta \) is of maximal Picard rank (resp. M-special).

**Proof:** Let \( l \in H^{1,1}(X)_\mathbb{Z} \) be an ample class. Since \( q(l, \eta) > 0 \) there is a direct sum decomposition \( H^2(X) = l \oplus \eta^+ \). The claim now follows from Lemma 5.6 (2) and (3). \( \Box \)

Assume that \( X \) is projective. Lemma 5.10 lets us restate Theorem 5.7 as follows: the group \( \Pi^0 \) is Hausdorff if and only if \( X \) has maximal Picard rank. Similarly, we obtain the following version of Theorem 5.8.
Corollary 5.11. Let $\pi : X \to B$ be a Lagrangian fibration without multiple fibers in codimension one and $X_{\text{III}} \to \widetilde{\text{III}}$ the Shafarevich–Tate family (Subsection 3.2). Assume that $X$ is projective and not $M$-special. Then the members of the family $X^s \subset X_{\text{III}}$ are Kähler for each $s \in \widetilde{\text{III}}$.

Remark 5.12. We do not know if the statement of Theorem 5.8 holds for $M$-special manifolds. Another open question is the following. Consider an element $s \in \text{III} \setminus \text{III}^0$. Is the twist $X^s$ a non-Kähler manifold? Note that from results of Subsection 3.3 we know that $X^s$ possesses a holomorphic symplectic form. It would be interesting to obtain examples of non-Kähler holomorphic symplectic manifolds, such as in [Gu, Bog96].

Remark 5.13. Being $M$-special is a restrictive condition. Hence the statement of Theorem 5.8 holds for most Lagrangian fibrations. Consider the set $\mathcal{D}_\eta$ of Hodge structures on $H^2(X, \mathbb{Z})$ such that $\eta \in H^{1,1}(X)$. It can be identified with a complex manifold of complex dimension $b_2 - 3$. The set of Hodge structures in $\mathcal{D}_\eta$ such that $H^2(X, \mathbb{Z})/\eta$ is $M$-special is a countable union of real analytic subvarieties of real dimension $b_2 - 3$.

5.3. Algebraic points in Shafarevich–Tate families. In this subsection we will describe the set of projective deformations in the Shafarevich–Tate family.

Lemma 5.14. Consider the degenerate twistor family $X_{\text{deg.tw}} \to \mathbb{C}$ and let $X = X^0$ be the fiber over $0 \in \mathbb{C}$. Let $\alpha$ be a class in $H^2(X, \mathbb{R})$. Then exactly one of the following holds:

1. $\alpha \in \eta^\perp$;
2. there exists a unique $s \in \mathbb{C}$ such that $\alpha \in H^{1,1}(X^s)$.

Moreover, $\bigcap_{s \in \mathbb{C}} H^{1,1}(X^s) = (\eta^\perp)^{1,1}$.

Proof: For each $s \in \mathbb{C}$ the complex structure on $X^s$ is defined by a $c$-symplectic form with cohomology class $[\sigma_c] = [\sigma_0] + s\eta$. A real class $\alpha$ lies in $H^{1,1}(X^s)$ if and only if it is orthogonal to $\sigma_c$, i.e., $q(\alpha, \sigma_c) = q(\alpha, \sigma_0) + sq(\alpha, \eta) = 0$. Suppose that $\alpha \in H^2(X, \mathbb{R}) \setminus \eta^\perp$. Then $s_\alpha := -q(\alpha, \sigma_0)/q(\alpha, \eta)$ is the unique number satisfying $q(\alpha, \sigma_{s_\alpha}) = 0$. If $\alpha$ is contained in $\eta^\perp$, then it is of type $(1, 1)$ for every degenerate twistor deformation.

Corollary 5.15. The set $\mathcal{R} := \{ s \in \widetilde{\text{III}} \mid X^s \text{ is algebraic} \}$ is at most countable. In particular, a very general member of the Shafarevich–Tate family is non-algebraic.

Proof: If $X^s$ is algebraic, it carries an ample divisor $L$ and $c_1(L) \in H^{1,1}_2(X)$. Moreover, we have the inequality $q(c_1(L), \eta) > 0$ in this case. For every $\alpha \in H^2(X, \mathbb{Z}) \setminus \eta^\perp$ there is only one point $s_\alpha$ such that $\alpha \in H^{1,1}(X^{s_\alpha})$ (Lemma 5.14). Thus, $\mathcal{R}$ is contained in the set

$\{ s_\alpha \mid \alpha \in H^2(X, \mathbb{Z}) \setminus \eta^\perp \}$.

It follows that $\mathcal{R}$ is at most countable.

See Theorem 5.19 below for a refinement of Corollary 5.15. A similar theorem holds for twistor families $X_{\text{tw}} \to \mathbb{P}^1$ ([Fu82]).

Lemma 5.16. Let $\pi : X \to B$ be a Lagrangian fibration on a hyperkähler manifold over $B = \mathbb{P}^n$. The following are equivalent:

1. $X$ is projective;
2. $\pi$ admits a rational multisection, i.e., there exists a subvariety $Z \subset X$ such that $\pi|_Z : Z \to B$ is a generically finite morphism;
3. there exists a class $\alpha \in H^{1,1}_0(X)$ such that $q(\alpha, \eta) \neq 0$;
4. there exists a class $\omega \in H^{1,1}_0(X)$ such that $q(\omega, \omega) > 0$. 
Theorem 5.19. Let \( q \in H^r(X, \mathbb{Z}) \) be a codimension one. Suppose that \( q \) are equivalent.

Proof: (1) \( \Rightarrow \) (2) Choose a holomorphic embedding \( i: X \hookrightarrow \mathbb{P}^N \). Pick a point \( x \in X \) outside of the singular locus of \( \pi \). Consider a general linear subspace \( L \subset \mathbb{P}^N \) of codimension \( n \) passing through \( x \) and transversal to the fiber over \( \pi(x) \) at \( x \). Let \( Z \) be a component of \( Z' := L \cap X \subset X \) which dominates \( B \). Then \( Z \) is rational multisection of \( \pi \).

(2) \( \Rightarrow \) (3) Let \( Z \subset X \) be a multisection. Let \( C_0 \subset B \) be a general smooth curve and let \( C := \pi^{-1}(C_0) \cap Z \) be its proper preimage in \( Z \). Consider the homology class \( [C] \in H_2(X, \mathbb{Z}) \) of the curve \( C \). The BBF form defines an isomorphism

\[
H^2(X, \mathbb{Q}) \cong H^2(X, \mathbb{Q})^* \cong H_2(X, \mathbb{Q})
\]

Let \( \alpha \in H^2(X, \mathbb{Q}) \) be the class corresponding to \( [C] \) under this isomorphism. Then \( q(\alpha, \eta) = \int_C \eta > 0 \).

(3) \( \Rightarrow \) (4) Let \( \alpha \) be a rational \((1,1)\)-class such that \( q(\alpha, \eta) \neq 0 \). We may assume that \( q(\alpha, \eta) > 0 \). Let us find a number \( t \) such that \( \omega := \alpha + t\eta \) satisfies \( q(\omega, \omega) > 0 \). We compute that

\[
q(\alpha + t\eta, \alpha + t\eta) = q(\omega, \omega) = q(\alpha, \alpha) + 2tq(\alpha, \eta),
\]

Therefore, the class \( \omega \) is rational and has a positive square for any rational \( t > \frac{-q(\alpha, \alpha)}{2q(\alpha, \eta)} \).

(4) \( \Rightarrow \) (1) See [Huy03, Thm. 3.11] \( \square \)

Remark 5.17. If a Kähler manifold is algebraic\(^3\), it is necessarily projective by the Moishezon theorem [Moish, Thm. 11]. Therefore, we can replace the first condition in the theorem above with the one that \( X \) is algebraic.

Denote by \( p: H^2(X, \mathbb{C}) \to H^{0,2}(X) \) the Hodge projection. Take \( \sigma \) to be the holomorphic symplectic form such that \( p(\tilde{\sigma}) \) is identified with the class of the Fubini-Study form under the isomorphism \( H^{1,1}(X) \cong \mathbb{H} \cong H^{1,1}(B) \) (Prop. 4.7, Cor. 2.9). In the following lemma we will give a characterization of torsion in \( \mathbb{III}^0 \) in terms of the BBF form.

Lemma 5.18. Let \( \pi: X \to B \) be a Lagrangian fibration without multiple fibers in codimension one. Consider a class \( \tilde{\sigma} \in H^{0,2}(X), t \in \mathbb{C} \). Let \([\tilde{\sigma}]\) denote its image in \( \mathbb{III}^0 \). Then \([\tilde{\sigma}]\) is torsion if and only if there exists a rational class \( l \in W_\mathbb{Q} \subset H^2(X, \mathbb{Q}) \) such that \( q(l, \sigma) = t \) (see formula (4) for the definition of \( W_\mathbb{Q} \)).

Proof: We know from the short exact sequence (1) that \( \mathbb{III}^0 = \mathbb{III}/\Lambda \) where \( \Lambda \) denotes the group \( \text{Im}(H^1(B, \Gamma) \to \mathbb{III}) \). Hence, the class \([\tilde{\sigma}]\) is torsion if and only if \( \tilde{\sigma} \) lies in the image of the group \( H^1(B, \Gamma) \otimes \mathbb{Q} \). This is equivalent to saying that \( \tilde{\sigma} \) lies in the image of \( W_\mathbb{Q} \subset H^2(X, \mathbb{Q}) \) under the Hodge projection \( H^2(X, \mathbb{Q}) \to H^{0,2}(X) \) (see Proposition 4.4, Proposition 4.7). Since \( H^{1,1}(X) \) is orthogonal to \( H^{2,0}(X) \oplus H^{0,2}(X) \), the projection \( H^2(X, \mathbb{C}) 	o H^{0,2}(X) \) is given by the formula

\[
v \mapsto q(v, \sigma)\tilde{\sigma}.
\]

Hence, \( t\tilde{\sigma} \) lies in the image of \( W_\mathbb{Q} \) under the Hodge projection if and only if there exists a class \( l \in W_\mathbb{Q} \) such that \( t = q(l, \sigma) \). \( \square \)

Theorem 5.19. Let \( \pi: X \to B \) be a Lagrangian fibration without multiple fibers in codimension one. Suppose that \( X \) is projective. Let \( s \) be an element of \( \mathbb{III}^0 \). The following are equivalent.

1. **The Shafarevich–Tate twist** \( X^s \) of \( X \) is projective.

\(^3\)i.e., the analytification of a smooth proper algebraic variety over \( \mathbb{C} \)
The manifold $X^s$ is Kähler and $s$ is a torsion element of III$^0$.

**Proof:** (2) ⇒ (1) Suppose that the element $s$ is torsion. Let $\tilde{s} = t\tilde{\sigma}$ denote a preimage of $s$ in $H^{0,2}(X) \cong \tilde{\Pi}$. There exists a class $l \in W_Q$ such that $q(l, \sigma) = t$ (Lemma 5.18). As $X$ is projective, there exists a class $\alpha \in H^2_Q(X)$ such that $q(\alpha, \eta) = 1$ (Lemma 5.16). The class $\alpha^s := \alpha - l$ satisfies $q(\alpha^s, \eta) = q(\alpha, \eta) = 1$ as $l$ is contained in $W_Q \subset \eta^\perp$. We claim that $\alpha^s$ lies in $H^1_Q(X^s)$. Indeed, $\alpha^s$ is orthogonal to $H^{2,0}(X^s)$ because $q(\alpha^s, \sigma + t\eta) = q(\alpha - l, \sigma + t\eta) = q(\alpha, \sigma) - tq(l, \eta) = t - t = 0$.

Lemma 5.16 implies that $X^s$ is projective.

(1) ⇒ (2) Suppose that $X$ and $X^s$ are both projective. Let $\tilde{s} = t\tilde{\sigma}$ denote a preimage of $s$ in $H^{0,2}(X)$ as before. There exist two rational classes $\alpha \in H^1_Q(X)$ and $\alpha^s \in H^2_Q(X^s)$ such that $q(\alpha, \eta) = q(\alpha^s, \eta) = 1$ (Lemma 5.16). Hence, the class $l := \alpha - \alpha^s$ is a rational class orthogonal to $\eta$. This class satisfies $q(l, \sigma) = q(l, \sigma + t\eta) = q(\alpha, \sigma + t\eta) = aq(\alpha, \eta) = t$.

We can not conclude directly that $[t\sigma]$ is torsion because $l$ might not lie in $W_Q$. Therefore, we need to adjust $l$.

Consider the subspace $W^\perp \subset H^2(X, \mathbb{Q})$. It follows from Proposition 4.5 that $\eta \in W^\perp \subset (\eta^\perp \cap H^{1,1}(X))$.

The BBF form $q$ has signature $(1, h^{1,1}(X) - 1)$ on $H^{1,1}(X)$ and $\eta$ is isotropic with respect to this form. Therefore, the restriction of $q$ to $W^\perp$ is semi-negative definite with kernel generated by $\eta$.

Let $U \subset W^\perp$ be a rational hyperplane in $W^\perp$ not containing $\eta$. The form $q|_U$ is negative definite, in particular, it is non-degenerate. Therefore there exists a unique rational vector $u \in U$ such that for every $v \in U$ the following holds $q(l, v) = q(u, v)$.

The vector $l - u$ is orthogonal to every vector in $W^\perp$. Hence $l - u$ is contained in $W_Q$. Since $u \in H^{1,1}(X)$, we have $q(l - u, \sigma) = q(l, \sigma) = t$. Lemma 5.16 concludes the proof. □

**Remark 5.20.** By [SV, Cor. 3.4] the set of $s \in \text{III}^0$ such that $X^s$ is projective is non-empty.

6. **Sections of Lagrangian fibrations**

6.1. **Obstruction for existence of a section.** We move on to study obstructions to existence of sections of Lagrangian fibrations. In this Subsection, we will always assume that $\pi: X \to B$ is a Lagrangian fibration with reduced irreducible fibers. Note that this is stronger than requiring no multiple fibers in codimension one.

In this case, the fibration $\pi: X \to B$ admits a local section in a neighborhood of every point $b \in B$. Consider an open cover $B = \bigcup U_i$ and choose a collection of local sections $s_i: U_i \to \pi^{-1}(U_i)$.

By [Markush96, Prop. 2.1 (iii)] for every pair $i, j$ there exists a unique automorphism $\varphi_{ij} \in \text{Aut}^0_{X/B}(U_{ij})$ such that $\varphi_{ij}(s_i|_{U_{ij}}) = s_j|_{U_{ij}}$. The collection of automorphisms $\{\varphi_{ij}\}$ satisfies the cocycle condition. Therefore, $\{\varphi_{ij}\}$ defines a class $\alpha(X, \pi) \in H^1(B, \text{Aut}^0_{X/B}) = \text{III}$.

We denote this class by $\alpha(X)$ when the structure of the Lagrangian fibration on $X$ is clear.
Lemma 6.1. (1) The class $\alpha(X)$ does not depend on the choice of local sections $s_i: U_i \to \pi^{-1}(U_i)$.
(2) For any element $s \in III$ we have $\alpha(X^s) = \alpha(X) + s$.
(3) The class $\alpha(X)$ vanishes if and only if the fibration $\pi: X \to B$ admits a section.

Proof: We will prove only the third statement. The proof of the first two ones follows the same lines. Suppose that $\alpha(X) = 0$. Then there exist automorphisms $\psi_i \in Aut^0_B(X_i)$ such that $\varphi_{ij} = \psi_j^{-1}\psi_i$. The sections $\psi_i(s_i)$ coincide on intersections, so they define a global section of $\pi$. The converse implication is straightforward.

Let $a(X)$ to be the image of $\alpha(X)$ in $III/III_0 = H^2(B, \Gamma)$.

Corollary 6.2. Let $\pi: X \to B$ be a Lagrangian fibration with reduced irreducible fibers. Then the class $\alpha(X)$ vanishes if and only if there exists a deformation $X^s$ of $X$ in the Shafarevich–Tate family that admits a holomorphic section. Moreover, in this case, the class of $s$ in $III_0$ is uniquely defined.

Proof: The class $\alpha(X) \in H^2(B, \Gamma)$ vanishes if and only if $\alpha := \alpha(X)$ lies in $III_0$. The class $\alpha(X^{-a})$ vanishes by Lemma 6.1 (2). Therefore

$$\pi^{-a}: X^{-a} \to B$$

admits a holomorphic section. Conversely, if $a \notin III_0$, then for every deformation $X^s$ in the Shafarevich–Tate family we have $a(X^s) \neq 0$.

It was proved in [BDV, Thm. 3.5] that a Lagrangian fibration admits a smooth section if and only if some of its degenerate twistor deformations admits a holomorphic section. Combined with Corollary 6.2 we get that $a(X)$ is a complete topological obstruction for existence of a smooth section on a Lagrangian fibration with reduced irreducible fibers.

In the rest of the paper we will be proving the following theorem.

Theorem 6.3. Let $\pi: X \to B$ be a Lagrangian fibration on a compact hyperkähler manifold over a smooth base. Let $\Gamma$ be the kernel of the exponential map $\pi_* T_{X/B} \to Aut^0_B X$. Assume that the following holds:

- the fibers of $\pi$ are reduced and irreducible;
- $H^3(X, \mathbb{Q}) = 0$;
- $H^2(B, \Gamma)$ is torsion-free.

Then there exists a unique deformation $(X^s, \pi^s)$ of $(X, \pi)$ in the Shafarevich–Tate family such that $\pi^s: X^s \to B$ admits a holomorphic section.

Note that the condition $H^3(X, \mathbb{Q}) = 0$ holds if $X$ is deformation equivalent to the Hilbert scheme of points on a K3 surface or to one of the exceptional O'Grady examples.

Unfortunately, we are not able to get rid of the condition on $H^2(B, \Gamma)$. However, we strongly believe, that the theorem should be true without this assumption. Therefore we pose the following conjecture.

Conjecture 1. Let $\pi: X \to B$ be a Lagrangian fibration with irreducible fibers. Assume that $b^3(X) = 0$. Then there exists a degenerate twistor deformation of $\pi$ admitting a holomorphic section.

6.2. Hard Lefschetz type theorems for higher direct images of $\mathcal{Q}_X$. Let $\pi: X \to B$ be a Lagrangian fibration without multiple fibers in codimension one. In this Subsection we prove a version of the Hard Lefschetz theorem for the sheaf $R^1 \pi_* \mathcal{Q}_X$ which will be used in the proof of the Theorem 6.3. Throughout this Subsection we assume that $X$ is projective. Let $l \in H^1_{\mathbb{Q}}(X)$ be an ample class. Abusing notation, we will denote by the same letter the induced section of $R^2 \pi_* \mathcal{Q}_X$. 
For each $p \geq 0$, multiplication by $l$ induces a map

$$L: R^p \pi_* Q_X \to R^{p+2} \pi_* Q_X.$$ 

We will refer to $L$ as the Lefschetz map.

**Lemma 6.4.** (Lefschetz decomposition for $R^2 \pi_* Q_X$) Let $\pi: X \to B$ be a Lagrangian fibration with $X$ projective. Assume that all fibers of $\pi$ are reduced and irreducible. Then the sheaf $R^2 \pi_* Q_X$ decomposes as

$$R^2 \pi_* Q_X = Q_B \cdot l \oplus (R^2 \pi_* Q_X)_{\text{prim}}$$

where $(R^2 \pi_* Q_X)_{\text{prim}}$ is the kernel of the map

$$L^{n-1}: R^2 \pi_* Q_X \to R^{2n} \pi_* Q_X.$$ 

**Proof:** Since the fibres are irreducible, we have $R^{2n} \pi_* Q_X \cong Q_B$. The restriction of $L^{n-1}: R^2 \pi_* Q_X \to R^{2n} \pi_* Q_X$ on the subsheaf generated by $l$ is an isomorphism, hence the claim. □

We move on to study the $(n-1)$-th power of the Lefschetz map on $R^1 \pi_* Q_X$, that is

$$L^{n-1}: R^1 \pi_* Q_X \to R^{2n-1} \pi_* Q_X.$$ (5)

First, note that multiplication by $l \in H^1_{\Omega^1_X}(X)$ induces maps on $R^p \pi_* \Omega^q_X$ as well:

$$L: R^p \pi_* \Omega^q_X \to R^{p+1} \pi_* \Omega^{q+1}_X.$$ 

for any $p, q = 0, \ldots, n$. Abusing notation, we will denote these maps also by $L$. Lefschetz maps commute with the natural morphisms $R^* \pi_* Q_X \to R^* \pi_* O_X$, the relative Hodge projections. In particular, the following diagram is commutative.

$$
\begin{array}{ccc}
R^1 \pi_* Q_X & \to & R^1 \pi_* O_X \\
L^{n-1} & & L^{n-1} \\
R^{2n-1} \pi_* Q_X & \to & R^n \pi_* \Omega^{n-1}_X
\end{array}
$$

The horizontal arrows in this diagram are Hodge projections. The holomorphic symplectic form $\sigma$ on $X$ induces the isomorphism $\Omega^1_X \cong T_X$. The composition of this isomorphism with the natural map $T_X \to \pi^* T_B$ gives us the map

$$\theta: \Omega^1_X \to \pi^* T_B.$$ 

For every pair of integers $p, q$ one can apply the functor $R^p \pi_* A^p(\cdot)$ and get the following map of sheaves on $B$:

$$R^p \pi_* (A^p \theta): R^p \pi_* \Omega^p_X \to R^p \pi_* (\pi^* A^p T_B).$$ 

The sheaf on the right is:

$$R^p \pi_* (\pi^* A^p T_B) \cong R^p \pi_* O_X \otimes A^p T_B \cong \Omega^p_B \otimes A^p T_B \cong R^p \pi_* O_X \otimes (R^p \pi_* O_X)^*.$$ 

The isomorphisms follow from the projection formula and Matsushita’s theorem (Proposition 2.7).

For each $p, q$ we have constructed a map

$$f_{p,q}: R^p \pi_* \Omega^p_X \to R^p \pi_* O_X \otimes (R^p \pi_* O_X)^*.$$ 

In particular, there are the following maps:

$$f_{0,1}: R^1 \pi_* O_X \to R^1 \pi_* O_X$$

$$f_{1,1}: R^1 \pi_* \Omega^1_X \to \delta n d (R^1 \pi_* O_X)$$

$$f_{n-1,n}: R^n \pi_* \Omega^{n-1}_X \to R^n \pi_* O_X \otimes (R^{n-1} \pi_* O_X)^* \cong R^1 \pi_* O_X.$$
The map \( f_{0,1} \) is the identity map by construction. It is easy to see that the image of \( l \) under \( f_{1,1} \) is the identity operator.

**Lemma 6.5.**  \( f_{n-1,n} \circ L^{n-1} = \text{id}_{R^1\pi_*\mathcal{O}_X} \).

**Proof:** The maps \( f_{p,q} \) commute with the multiplication of forms in the sense that the following diagram is commutative for any \( p_1, q_1, p_2, q_2 = 0, \ldots, n \).

\[
\begin{array}{ccc}
R^{n} \pi_*\mathcal{O}_X^p & \otimes & R^{p_2} \pi_*\mathcal{O}_X^{p_2} \\
\downarrow f_{p_1 \cdot q_1} \otimes f_{p_2 \cdot q_2} & & \downarrow f_{p_1 + p_2, q_1 + q_2} \\
R^{n} \pi_*\mathcal{O}_X \otimes (R^{p_1} \pi_*\mathcal{O}_X)^* & \otimes & R^{p_2} \pi_*\mathcal{O}_X \otimes (R^{p_2} \pi_*\mathcal{O}_X)^* \\
\end{array}
\]

It follows that the following diagram is commutative

\[
\begin{array}{ccc}
R^{1} \pi_*\mathcal{O}_X \otimes (R^{1} \pi_*\mathcal{O}_X)^{\otimes n-1} & \rightarrow & R^{n} \pi_*\mathcal{O}_X^{n-1} \\
\downarrow \text{id} \otimes (f_{1,1})^{\otimes n-1} & & \downarrow f_{n-1,n} \\
R^{1} \pi_*\mathcal{O}_X \otimes (\text{End}(R^{1} \pi_*\mathcal{O}_X))^{\otimes n-1} & \rightarrow & R^{1} \pi_*\mathcal{O}_X \\
\end{array}
\]

Since \( f_{1,1}(l) = \text{id}_{R^1\pi_*\mathcal{O}_X} \), we obtain that for every local section \( \alpha \) of \( R^1\pi_*\mathcal{O}_X \)
\[
f_{n-1,n}(\alpha \cdot L^{n-1}) = f_{n-1,n} \circ L^{n-1}(\alpha) = \alpha.
\]

\[\square\]

**Corollary 6.6.** Let \( \pi : X \rightarrow B \) be a Lagrangian fibration without multiple fibers in codimension one on a projective hyperkähler manifold \( X \). Then there exists a sheaf \( \mathcal{N} \) on \( B \) such that the sheaf \( R^{2n-1}\pi_*\mathcal{Q}_X \) decomposes into the direct sum

\[
R^{2n-1}\pi_*\mathcal{Q}_X \simeq R^1\pi_*\mathcal{Q}_X \oplus \mathcal{N}.
\]

The embedding of the first summand is given by the map \( (5) \).

**Proof:** The map (5) is an isomorphism after the restriction to \( B^0 \subseteq B \) by the Hard Lefschetz theorem. Together with Lemma 6.5 this implies that the map \( f_{n-1,n}|_{B^0} \) sends \( R^{2n-1}\pi_*\mathcal{Q}|_{B^0} \) isomorphically to \( R^1\pi_*\mathcal{Q}|_{B^0} \). A local section of \( R^1\pi_*\mathcal{Q}_X \) whose restriction to \( B^0 \) lies in \( R^1\pi_*\mathcal{Q} \) is necessarily a section of \( R^1\pi_*\mathcal{Q}_X \) by the proof of Proposition 4.4. Hence the map \( f_{n-1,n} \) descends to a map

\[
f_{n-1,n}|_{R^{2n-1}\pi_*\mathcal{Q}_X} : R^{2n-1}\pi_*\mathcal{Q} \rightarrow R^1\pi_*\mathcal{Q}.
\]

This map satisfies the following property (Lemma 6.5):

\[
f_{n-1,n}|_{R^{2n-1}\pi_*\mathcal{Q}_X} \circ L^{n-1} = \text{id}_{R^1\pi_*\mathcal{Q}_X}.
\]

The claim now follows. \[\square\]

6.3. **The discrete part of Shafarevich–Tate groups.** Let \( \text{III} \) be the Shafarevich–Tate group of a Lagrangian fibration \( \pi : X \rightarrow B \) without multiple fibers in codimension one. Recall that the group \( \text{III}/\text{III}^0 \) of connected components of \( \text{III} \) is isomorphic to \( H^2(B, \Gamma) \) (see the exact sequence (3)). We sometimes refer to \( \text{III}/\text{III}^0 \) as the discrete part of \( \text{III} \).

By Proposition 4.4, there is an isomorphism

\[
H^2(B, \Gamma) \otimes_{\mathbb{Z}} \mathbb{Q} \simeq H^2(B, R^1\pi_*\mathcal{Q}_X).
\]

The natural map \( E_{p,0}^n = H^p(B, \mathbb{Q}) \rightarrow H^p(X, \mathbb{Q}) \) is given by the pullback map \( \pi^* \). The pullback map on cohomology is injective for every surjective map of compact Kähler manifolds (see e.g. [Vois03, Lem. 7.28]). This implies that \( E_{2}^{p,0} = E_{\infty}^{p,0} \). Therefore, the differential \( d_2 : E_2^{2,1} \rightarrow E_2^{4,0} \) vanishes. All the higher differentials with the source in
implies that $E_{2}^{3,1} = E_{2}^{3,1}/\text{im}(d_{2})$ into $H^{3}(X, Q)$. Moreover, we have the following exact sequence of $\mathbb{Q}$-vector spaces

\[ H^{2}(X, Q) \xrightarrow{\partial} H^{0}(B, R^{2}\pi_{*}Q_{X}) \xrightarrow{d_{2}} H^{2}(B, R^{1}\pi_{*}Q_{X}) \xrightarrow{\partial} H^{3}(X, Q). \]

Our goal is to prove the following theorem:

**Theorem 6.7.** Let $\pi: X \to B$ be a Lagrangian fibration with reduced irreducible fibers. Then the differential

\[ d_{2}: H^{0}(B, R^{2}\pi_{*}Q_{X}) \to H^{2}(B, R^{1}\pi_{*}Q_{X}) \]

in the Leray spectral sequence of $\pi$ vanishes.

**Proof:** Assume that $X$ is projective. Lemma 6.4 implies that

\[ H^{0}(B, R^{2}\pi_{*}Q_{X}) = H^{0}(B, Q) \oplus H^{0}(B, (R^{2}\pi_{*}Q_{X})_{\text{prim}}) \]

where $(R^{2}\pi_{*}Q_{X})_{\text{prim}}$ is the kernel of the map $L^{n-1} : R^{2}\pi_{*}Q_{X} \to R^{2n}\pi_{*}Q_{X}$. The summand $H^{0}(B, Q)$ is generated by the image of the ample class $l$ of $X$ in $H^{0}(R^{2}\pi_{*}Q_{X})$. Hence $d_{2}|_{H^{0}(B, Q)}$ vanishes.

We are left to prove that $d_{2}|_{H^{0}((R^{2}\pi_{*}Q_{X})_{\text{prim}})}$ vanishes. The differentials in the Leray spectral sequence commute with the Lefschetz maps. In particular, the following diagram is commutative.

\[
\begin{array}{ccc}
H^{0}((R^{2}\pi_{*}Q_{X})_{\text{prim}}) & \xrightarrow{d_{2}} & H^{2}(R^{1}\pi_{*}Q_{X}) \\
\downarrow L^{n-1} & & \downarrow L^{n-1} \\
H^{0}(R^{2n}\pi_{*}Q_{X}) & \xrightarrow{d_{2}} & H^{2}(R^{2n-1}\pi_{*}Q_{X})
\end{array}
\]

The vertical arrow on the left-hand side vanishes by the definition of the primitive part of $R^{2}\pi_{*}Q_{X}$. The vertical map on the right-hand side is injective. Indeed, by Corollary 6.6 the map $L^{n-1}$ embeds $H^{2}(R^{1}\pi_{*}Q_{X})$ into $H^{2}(R^{2n-1}\pi_{*}Q_{X})$ as a direct summand. It follows that $d_{2}|_{H^{0}((R^{2}\pi_{*}Q_{X})_{\text{prim}})}$ must vanish.

**Step 2:** In the case when $X$ is only assumed to be Kähler there exists a degenerate twistor deformation $\pi': X' \to B$ of $X$ such that $X'$ is projective ([SV, Cor. 3.4]). The statement of the theorem holds for $X'$. Since topologically the maps $\pi$ and $\pi'$ coincide, the statement of the theorem holds for $X$ as well. 

**Corollary 6.8.** In the setting of Theorem 6.7, the following holds:

1. The restriction map $r: H^{2}(X, Q) \to H^{0}(B, R^{2}\pi_{*}Q_{X})$ is surjective.
2. The map $e: H^{2}(R^{1}\pi_{*}Q_{X}) \to H^{3}(X, Q)$ from the exact sequence (6) is injective.

**Proof:** Follows from Theorem 6.7 and the exact sequence (6).

**Proof of Theorem 6.3:** It follows from Corollary 6.8 (2) and Proposition 4.4 that there exists an embedding

\[ (\mathbb{Z}/2\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q} \hookrightarrow H^{3}(X, Q). \]

In particular, if $b_{3}(X)$ vanishes, $\mathbb{Z}/2\mathbb{Z}$ is finite. Since $\mathbb{Z}/2\mathbb{Z} \approx H^{3}(B, \Gamma)$, this finishes the proof.

**Remark 6.9.** If $\pi: X \to B$ is an elliptic fibration on a K3 surface, then $\mathbb{Z}/2\mathbb{Z}$ is trivial ([FM, I.1, Lemma 5.11]).
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