Neutrino Mass: Mechanisms and Models

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In these lectures (at the 2007 Summer School in Akyaka, Mugla, Turkey), I discuss the various mechanisms for obtaining small Majorana neutrino masses, as well as specific models of varying complexity, in the context of the standard model and beyond.
I. INTRODUCTION

In the Standard Model (SM) of particle interactions based on the gauge group $SU(3)_C \times SU(2)_L \times U(1)_Y$, all fermions (and gauge bosons) owe their masses to its one Higgs scalar doublet $\Phi = (\phi^+, \phi^0)$. In particular, the charged-lepton mass comes from

$$f_i(\bar{\nu}_L \phi^+ + \bar{l}_L \phi^0) \rightarrow m_l = f_i \langle \phi^0 \rangle.$$  \hfill (1)

The lone exception is the neutrino because the singlet $\nu_R$ is trivial under the SM gauge group, i.e. $\nu_R \sim (1, 1, 0)$, so it is not required to be part of the SM. Thus the minimal SM has zero neutrino mass, which is of course not realistic, in the face of established neutrino-oscillation data in the last decade. In the following lectures, I will discuss the generic mechanisms for obtaining small Majorana neutrino masses, and specific models which realize them in the context of the SM and beyond. I will also discuss $A_4$ briefly for understanding tribimaximal neutrino mixing.

A. TYPE I SEESAW

The most prevalent idea for obtaining a neutrino mass is to add $\nu_R$, then

$$f_\nu(\bar{\nu}_L \phi^0 - \bar{l}_L \phi^-) \nu_R \Rightarrow m_D = f_\nu \langle \phi^0 \rangle$$  \hfill (2)

is a fermion Dirac mass just like $m_l$. However, since $\nu_R$ is a gauge singlet, it can have a Majorana mass $M$, so that the $2 \times 2$ mass matrix linking $\bar{\nu}_L$ to $\nu_R$ is of the form

$$M_\nu = \begin{pmatrix} 0 & m_D \\ m_D & M \end{pmatrix},$$  \hfill (3)

with eigenvalues $m_{1,2} = M/2 \mp \sqrt{(M/2)^2 + m_D^2}$. There are two interesting limits, as discussed below.

(a) If $M = 0$, then $m_{1,2} = \mp m_D$ and $\nu_L$ pairs with $\nu_R$ to form a Dirac fermion with additive lepton number $L = 1$, which is exactly conserved. This also shows that a neutral Dirac fermion may be regarded as two mass-degenerate Majorana fermions of opposite $CP$. It is a perfectly acceptable explanation of neutrino mass, but it requires a very tiny $f_\nu$ in Eq. (2), of order $10^{-11}$ or less.

(b) Since $M$ is an invariant mass term, it is presumably very large, corresponding to the scale of new physics responsible for its existence. In that case, $m_D \ll M$, and $m_1 \approx$
−m_D^2/M, m_2 \simeq M$. This is the famous canonical seesaw mechanism \[1\]. Theoretically, (b) is considered much more *natural* than (a) because the former requires the *imposition* of an exactly conserved global U(1) symmetry, i.e. lepton number. Consequently, (b) dominated the thinking on neutrino mass for many years until somewhat recently.

### B. TYPE II SEESAW

Another just as *natural* way to obtain a small Majorana neutrino mass is to add a Higgs triplet ($\xi^{++}, \xi^+, \xi^0$) which couples directly to the symmetric triplet combination of two $(\nu, l)_L$ doublets, i.e.

$$
\frac{h_\nu}{2} \left[ \nu \nu \xi^0 - \frac{(\nu l + l \nu)}{\sqrt{2}} \xi^+ + ll \xi^{++} \right] \Rightarrow m_\nu = h_\nu \langle \xi^0 \rangle,
$$

with $\langle \xi^0 \rangle \ll \langle \phi^0 \rangle$. It is often *mistakenly* assumed that this requires extreme fine tuning and is thus not very natural. To see how this mechanism really works \[2\], consider the most general Higgs potential of $\Phi$ and $\xi$:

$$
V = m^2 \Phi^\dagger \Phi + M^2 \xi^\dagger \xi + \frac{1}{2} \lambda_1 (\Phi^\dagger \Phi)^2 + \frac{1}{2} \lambda_2 (\xi^\dagger \xi)^2 + \lambda_3 (\xi^\dagger \xi^\dagger) (\xi \xi) + f_1 (\Phi^\dagger \Phi) (\xi^\dagger \xi) + f_2 (\xi^\dagger \Phi)(\Phi^\dagger \xi) + [\mu \xi^\dagger \Phi \Phi + H.c.]
$$

Let $\langle \phi^0 \rangle = v$, $\langle \xi^0 \rangle = u$, then

$$
v [m^2 + \lambda_1 v^2 + (f_1 + f_2) u^2 + 2 \mu u] = 0, \quad u [M^2 + \lambda_2 u^2 + (f_1 + f_2) v^2] + \mu v^2 = 0.
$$

If lepton number is imposed on $V \[3\]$, then $\mu = 0$, and for both $v$ and $u$ to be nonzero, they must be given by

$$
v^2 = -\frac{\lambda_2 m^2 + (f_1 + f_2) M^2}{\lambda_1 \lambda_2 - (f_1 + f_2)^2}, \quad u^2 = \frac{(f_1 + f_2) M^2 - \lambda_1 M^2}{\lambda_1 \lambda_2 - (f_1 + f_2)^2}.
$$

Since $u$ has to be tiny, extreme fine tuning is required. In addition, this model breaks lepton number spontaneously, which implies the existence of a massless Goldstone particle, the majoron, i.e. $\sqrt{2} \text{Im} \xi^0$. Now the mass of $\sqrt{2} \text{Re} \xi^0$ is of order $u$, hence the invisible decay $Z \to \sqrt{2} \text{Re} \xi^0 + \sqrt{2} \text{Im} \xi^0$ is expected and its rate is equivalent to that of two neutrino pairs. This has been ruled out experimentally for more than 20 years.

For $\mu \neq 0$, a completely *new* and *naturally small* solution for $u$ appears:

$$
v^2 \simeq -\frac{m^2}{\lambda_1}, \quad u \simeq \frac{-\mu v^2}{M^2 + (f_1 + f_2) v^2}.
$$
This is now commonly called the Type II seesaw. It works because the spontaneous breaking of electroweak symmetry is already accomplished by \( \langle \phi^0 \rangle \), hence \( \langle \xi^0 \rangle \) may be small, as long as \( m_2^2 \) is positive and large. The parameter \( \mu \) (which has the dimension of mass) may also be *natu rally* small, because its absence enhances the symmetry of \( V \). In the past ten years, this mechanism is being appreciated more, and is now competitive with the Type I seesaw.

**II. SIX GENERIC MECHANISMS**

In 1979, Weinberg showed [4] that in the Minimal Standard Model, there is only one effective dimension-five operator, i.e.

\[
L_5 = \frac{f_{\alpha\beta}}{2\Lambda}(\nu_\alpha \phi^0 - l_\alpha \phi^+)(\nu_\beta \phi^0 - l_\beta \phi^+),
\]

and it generates a small Majorana neutrino mass given by \( f_{\alpha\beta}v^2/\Lambda \), where \( \Lambda \) is a large effective mass. This shows that all Majorana neutrino masses in the SM are necessarily *seesaw*: for \( v \) fixed, \( m_\nu \) goes down as \( \Lambda \) goes up.

In 1998, I showed [5] that there are three and only three ways to obtain the Weinberg operator at tree level, as shown in FIG. 1, and that there are three generic mechanisms in one-loop order.

![FIG. 1: Three tree-level realizations of seesaw Majorana neutrino mass.](image)

**A. Type I**

This is the canonical seesaw. Instead of using the \( 2 \times 2 \) matrix of Eq. (3) which you learned in text books, consider the Feynman diagram of FIG. 1. Just read off the neutrino mass from the two couplings of \( \nu_L \) to \( \phi^0 \), each multiplied by \( \langle \phi^0 \rangle \), and divided by the large Majorana mass of the neutral fermion singlet \( N \). The insertion of \( N \) is obvious in the explicit structure of Eq. (9).
B. Type II

This is obtained by coupling two lepton doublets to a scalar triplet \((\xi^{++}, \xi^+, \xi^0)\) as shown in Eq. (4), and linked to \(\Phi \Phi\) as shown in Eq. (5). It generates the Weinberg operator as well because

\[
(\nu_\alpha \phi^0 - l_\alpha \phi^+)(\nu_\beta \phi^0 - l_\beta \phi^+) = \nu_\alpha \nu_\beta \phi^0 \phi^0 - (\nu_\alpha l_\beta + l_\alpha \nu_\beta) \phi^+ \phi^0 + l_\alpha l_\beta \phi^+ \phi^+. \tag{10}
\]

Note that the decay branching fractions of \(\xi^{++} \rightarrow l_\alpha^+ l_\beta^+\) would be proportional to the entries of the neutrino mass matrix \((\mathcal{M}_\nu)_{\alpha\beta}\), and may be verifiable \([6]\) at the Large Hadron Collider (LHC).

C. Type III

Replace the singlet \(N\) by the triplet \((\Sigma^+, \Sigma^0, \Sigma^-)\) \([7]\), then this is again obtained with the Weinberg operator because

\[
(\nu_\alpha \phi^0 - l_\alpha \phi^+)(\nu_\beta \phi^0 - l_\beta \phi^+) = -2\nu_\alpha \phi^+ l_\beta \phi^0 + (\nu_\alpha \phi^0 + l_\alpha \phi^+)(\nu_\beta \phi^0 + l_\beta \phi^+) - 2l_\alpha \phi^0 \nu_\beta \phi^+. \tag{11}
\]

This mechanism was largely neglected until recently. For a recent review, see Ref. \([8]\).

D. Type IV, V, VI

There are also three generic one-loop mechanisms \([5]\). Consider a loop linking \(\nu_L\) to \(\nu_L\). It should have an internal fermion line, as well as an internal scalar line. The two external \(\phi^0\) lines of the Weinberg operator may then be chosen to be attached in three different ways: one to the fermion line and one to the scalar line (Type IV), two to the scalar line (Type V), and two to the fermion line (Type VI). Almost all models of one-loop neutrino mass are of Type IV, the most well-known of which is the Zee model \([9]\). Type V models used to be quite rare. Since 2006, the idea \([10]\) that neutrino mass comes from dark matter in one loop (scotogenic) requires precisely this mechanism. Type VI models are unknown, presumably because they are rather complicated to realize.

III. MINIMAL MODELS

\footnote{The canonical approach is to add three neutral singlet fermions \(N_{1,2,3}\) without imposing additive lepton number. Majorana neutrino masses are then obtained via mechanism (I) and...}
a conserved multiplicative $Z_2$ lepton number $(-)^L$ emerges naturally. The most important consequence is the occurrence of neutrinoless double beta decay:

$$d \rightarrow ue^-\{\nu_e\nu_e\}e^-u \leftarrow d.$$  

In the $3 \times 3$ Majorana neutrino mass matrix in the $(e, \mu, \tau)$ basis, the effective neutrino mass $m_{ee}$ can be read off as the $\{\nu_e\nu_e\}$ entry.

(2) If $N_{1,2,3}$ are added with the imposition of additive lepton number $L$, then $m_{ee} = 0$. Of course, if $L$ is violated by other interactions, there will be a contribution to $m_{ee}$, as well as to neutrinoless double beta decay, but the former may not be the dominant cause of the latter.

(3) Instead of $N_{1,2,3}$, one Higgs scalar triplet $(\xi^+, \xi^+, \xi^0)$ is added with $L = -2$. Here $\xi^0$ couples directly to $\nu_L\nu_L$, and a Majorana neutrino mass is obtained if $\langle \xi^0 \rangle \neq 0$. However, $L$ is then broken spontaneously, resulting in a massless Goldstone boson, i.e. the triplet majoron $\sqrt{2}{\text{Im}}\xi^0$. At the same time, the scalar boson $\sqrt{2}{\text{Re}}\xi^0$ has a mass of order $\langle \xi^0 \rangle$ so that the decay width of $Z \rightarrow \sqrt{2}{\text{Im}}\xi^0 + \sqrt{2}{\text{Re}}\xi^0$ is equal to that of 2 neutrinos, thus ruled out by the well-measured invisible width of the $Z$.

(4) If $(\xi^+, \xi^+, \xi^0)$ is added with $(-)^L = +$, then $\langle \xi^0 \rangle$ does not break $(-)^L$ and all neutrino masses are Majorana via mechanism (II). If $M_\xi$ is of order 1 TeV, the decay $\xi^{++} \rightarrow l_i^+l_j^+$ will map out the relative magnitudes of all elements of the neutrino mass matrix.

(5) Instead of $N$, Majorana triplet fermions $(\Sigma^+, \Sigma^0, \Sigma^-)$ are added. Neutrino masses are then obtained via mechanism (III). Mixing occurs between $l$ and $\Sigma$ as well.

(6) If just one $N$ is added, then a particular linear combination of $\nu_{e,\mu,\tau}$, call it $\nu_1$, will couple to $N$ and gets a seesaw mass. The other two linear combinations are massless at tree level, but since there is no symmetry which prevents it, they will become massive in two loops, to be discussed in the next section.

(7) Consider the addition of one $N$ and a second scalar doublet $(\eta^+, \eta^0)$ without any symmetry restriction. Define $\Phi$ to be the scalar doublet with vacuum expectation value and $\eta$ the one without, then the Yukawa terms $f_{11}\bar{\nu}_1LN_R\eta^0 + f_{12}\bar{\nu}_1LN_R\eta^0 + f_{22}\bar{\nu}_2LN_R\eta^0$ imply $\nu_1$ gets a tree-level mass, $\nu_2$ gets a radiative mass via mechanism (V), and $\nu_3$ gets a two-loop
mass mentioned in \{6\}.

\{8\} Consider \footnote{13} the addition of $N_{1,2,3}$ with $L = 0$ and $(\eta^+, \eta^0)$ with $L = -1$. Then $(\nu \phi^0 - l \phi^+) N$ is forbidden but $(\nu \eta^0 - l \eta^+) N$ is allowed. Let $L$ be broken softly in the scalar sector by the unique term $\mu^2 (\Phi^\dagger \eta + \eta^\dagger \Phi)$. The Higgs potential is then given by

$$V = m_1^2 \Phi^\dagger \Phi + m_2^2 \eta^\dagger \eta + \frac{1}{2} \lambda_1 (\Phi^\dagger \Phi)^2 + \frac{1}{2} \lambda_2 (\eta^\dagger \eta)^2 + \lambda_3 (\Phi^\dagger \Phi)(\eta^\dagger \eta) + \lambda_4 (\Phi^\dagger \eta)(\eta^\dagger \Phi) + \mu^2 (\Phi^\dagger \eta + \eta^\dagger \Phi).$$  \hspace{1cm} (12)

Let $\langle \phi^0 \rangle = v$ and $\langle \eta^0 \rangle = u$, then

$$v[m_1^2 + \lambda_1 v^2 + (\lambda_3 + \lambda_4)u^2] + \mu^2 u = 0, \hspace{0.5cm} u[m_2^2 + \lambda_2 u^2 + (\lambda_3 + \lambda_4)v^2] + \mu^2 v = 0.$$  \hspace{1cm} (13)

For $m_1^2 < 0$ and $m_2^2 > 0$ and large, the solution is

$$v^2 \approx -\frac{m_1^2}{\lambda_1}, \hspace{0.5cm} u \approx -\frac{\mu^2 v}{m_2^2 + (\lambda_3 + \lambda_4)v^2}.$$  \hspace{1cm} (14)

The Majorana neutrino mass is still given by the seesaw formula $m_\nu \approx -m_D^2 / m_N$, but $m_D$ is now proportional to $u$ which is small because $\mu^2$ is a $L$ violating parameter and can be chosen to be small naturally. For example, if $u \sim 1 \text{ MeV}$, $m_N \sim 1 \text{ TeV}$, then $u^2 / m_N \sim 1 \text{ eV}$ is the neutrino mass scale.

**IV. RADIATIVE MODELS**

Neutrino masses may be generated in one loop or more, depending on the assumed particle content beyond the minimal SM, and additional possible symmetries. Here I discuss four examples, three early and one recent.

**A. Generic 2-W Mechanism in the SM**

The minimal model for all neutrinos to acquire mass is to add just one $N$ as discussed in \{6\} of the previous section. In that case, only one linear combination of $\nu_{e,\mu,\tau}$, call it $\nu_1$, gets a tree-level Majorana mass. The others appear to be massless, but that cannot be so, because if $\nu_1$ spans all three flavors, there is no remaining symmetry which can keep them massless. On the other hand, only SM particles are available, so how in the world can they acquire mass? The answer was provided by Ref. \footnote{11} where it was shown that these
masses appear in two loops, from the exchange of two $W$ bosons, as shown in FIG. 2. This diagram is doubly suppressed by the Glashow-Ilioupoulos-Maiani mechanism \[14\] and yields extremely small neutrino masses.

**B. Zee/Wolfenstein Model**

\[
\begin{align*}
\phi_2^0 & \\
\chi^- & \\
\phi_1^+ & \\
\end{align*}
\]

FIG. 3: One-loop radiative neutrino mass.

The most well-known radiative model \[9\] uses the fact that the invariant combination of two (different) lepton doublets couples to a charged scalar singlet, i.e. \((\nu_i l_j - l_i \nu_j)\chi^+\). By the same token, two different scalar doublets are also needed, i.e. \((\phi_1^+ \phi_2^0 - \phi_1^0 \phi_2^+ )\chi^-\). In that case, radiative neutrino masses are obtained in one loop via mechanism (IV), where $\Phi_2$ has been assumed not to couple to leptons \[15\], as shown in FIG. 3. The $3 \times 3$ neutrino mass matrix is then of the form

\[
M_\nu = \begin{pmatrix}
0 & f_{\mu e}(m_\mu^2 - m_e^2) & f_{\tau e}(m_\tau^2 - m_e^2) \\
-f_{\mu e}(m_\mu^2 - m_e^2) & 0 & f_{\tau \mu}(m_\tau^2 - m_\mu^2) \\
-f_{\tau e}(m_\tau^2 - m_e^2) & -f_{\tau \mu}(m_\tau^2 - m_\mu^2) & 0
\end{pmatrix}
\] (15)
This model was studied intensively, but it is now ruled out by data.

C. Zee/Babu Model

![Diagram of Two-loop radiative neutrino mass.]

**FIG. 4:** Two-loop radiative neutrino mass.

In the previous model, if the second Higgs doublet is replaced by a doubly charged singlet $\zeta^{++}$, a two-loop neutrino mass is obtained [16, 17], using the additional interactions $\zeta^{++} \chi^- \chi^-$ and $l_i l_j \zeta^{- -}$. Note that it is doubly suppressed by lepton masses as in Eq. (15). However, nonzero diagonal entries are now allowed in the neutrino mass matrix and there are enough free parameters not to be ruled out. Processes such as $\mu \rightarrow eee$ and $\tau \rightarrow \mu \mu \mu, \mu \mu e, \mu e e, eee$ are possible at tree level and act as constraints as well as opportunities for discoveries.

D. Scotogenic Neutrino Mass

A recent new development [10, 18] is to connect the origin of neutrino mass to the existence of dark matter, i.e. scotogenic. The idea is very simple. Let the SM be extended to include three $N$’s and a second scalar doublet $(\eta^+, \eta^0)$ [19] which are odd under a new exactly conserved $Z_2$ discrete symmetry, whereas all SM particles are even. In that case, the usual Yukawa term $(\nu \phi^0 - l \phi^+)N$ is forbidden, but $(\nu \eta^0 - l \eta^+)N$ is allowed. However, unlike the case {8} discussed in Section 3, $\langle \eta^0 \rangle = 0$ here because of the conserved $Z_2$. Hence there is no $m_D$ linking $\nu$ and $N$. However, $\nu$ gets a radiative Majorana mass (of Type V) directly as shown in FIG. 5.

Specifically, this diagram is exactly calculable from the exchange of $\sqrt{2} \text{Re} \eta^0$ and $\sqrt{2} \text{Im} \eta^0$, i.e.

$$ (M_\nu)_{ij} = \sum_k h_{ik} h_{jk} M_k \left[ \frac{m_R^2}{m_R^2 - M_k^2} \ln \frac{m_R^2}{M_k^2} - \frac{m_I^2}{m_I^2 - M_k^2} \ln \frac{m_I^2}{M_k^2} \right]. \quad (16) $$

If $\sqrt{2} \text{Re} \eta^0$ or $\sqrt{2} \text{Im} \eta^0$ is the lightest particle of odd $Z_2$, then it is a possible dark-matter candidate [10, 20, 21, 22] and may be searched for at the LHC [23].
V. GENERIC CONSEQUENCES OF NEUTRINO MASS

(A) Once neutrinos have mass and mix with one another, the radiative decay $\nu_2 \rightarrow \nu_1 \gamma$ happens in all models, but is usually harmless as long as $m_\nu < \text{few eV}$, in which case it will have an extremely long lifetime, many orders of magnitude greater than the age of the Universe.

(B) The analogous radiative decay $\mu \rightarrow e \gamma$ also happens in all models, but is only a constraint for some models where $m_\nu$ is radiative in origin.

(C) Neutrinoless double beta decay occurs, proportional to the $\{\nu_e \nu_e\}$ entry of the Majorana neutrino mass matrix.

(D) Leptogenesis is possible from $N \rightarrow l^+ \phi^- (l^- \phi^+) \text{ or } \xi^{++} \rightarrow l^+ l^+ (\phi^+ \phi^+)$. There may also be other possibilities.

(E) New particles at the 100 GeV mass scale exists in some models. They can be searched for at the LHC and beyond.

(F) Lepton-flavor changing processes at tree level may provide subdominant contributions to neutrino oscillations.

(G) Lepton-number violating interactions at the TeV mass scale may erase any pre-existing $B$ or $L$ asymmetry of the Universe.

VI. LEPTON NUMBER IN SUPERSYMMETRY

In the SM (without $\nu_R$), neutrinos are massless and four global $U(1)$ symmetries are automatically conserved: $B$, $L_e$, $L_\mu$, $L_\tau$. In its supersymmetric extension, the lepton doublet
superfields $L_i = (\nu_i, l_i)$ transform exactly like one of the two Higgs superfields, i.e. $\Phi_1 = (\phi^0_1, \phi^-_1)$. To tell them apart, lepton number has to be imposed, so that $L_i$ have $L = 1$, $l_i$ have $L = -1$, and $\Phi_{1,2}$ have $L = 0$. Again, neutrinos are massless and $B, L_e, L_\mu, L_\tau$ are conserved. Thus $R$ parity, defined as $(-)^{3B + L + 2j}$, is also conserved and the model is known as the Minimal Supersymmetric Standard Model (MSSM).

The terms in the superpotential which conserve $B$ but not $L$ may be organized to allow for 5 generic scenarios [24]. Let

$$W^{(1)} = h_i \Phi_1 L_i l_i^c + h^{\mu} \Phi_2 Q_i u_i^c + h^{\nu} \Phi_1 Q_i d_i^c + \mu_0 \Phi_1 \Phi_2,$$  
$$W^{(2)} = f_{\epsilon \mu} L_3 L_1 l_1^c + f_{\mu} L_3 L_2 l_2^c + f_{\nu} L_3 Q_i d_i^c + \mu_3 L_3 \Phi_2,$$  
$$W^{(3)} = f_{\epsilon \mu} L_1 L_2 l_3^c,$$  
$$W^{(4)} = f_{\epsilon \mu} L_3 L_1 l_1^c + f_{\mu \mu} L_3 L_2 l_2^c,$$  
$$W^{(5)} = f_{\epsilon \mu} L_2 L_1 l_1^c + f_{\mu} L_2 L_3 l_3^c + f_{\nu} L_2 Q_i d_i^c + \mu_2 L_2 \Phi_2,$$

then the following 5 models are possible (each with 3 obvious permutations):

(A) $W^{(1)} + W^{(2)} \Rightarrow L_e, L_\mu$ conserved, but $\nu_\tau$ mixes with the other 4 neutralinos, and gets a seesaw mass:

$$m_{\nu_\tau} = \frac{\mu_3^2}{2 \mu_0 \tan \beta} \left[ 1 - \frac{\mu_0 M_1 M_2}{M_Z^2 (c^2 M_1 + s^2 M_2) \sin 2\beta} \right]^{-1}.$$  

(B) $W^{(1)} + W^{(3)} \Rightarrow L_e, L_\mu, L_\tau$ conserved, with $L_\tau = L_e + L_\mu$. This is the simplest new model with just one term in $W^{(3)}$. Neutrinos remain massless. The $\tilde{W}^+$ gaugino will decay into $\epsilon^+ \mu^- \tau^-$ via $\tilde{\nu}_e$ and $\tilde{\nu}_\mu$.

(C) $W^{(1)} + W^{(2)} + W^{(3)} \Rightarrow L_e, L_\mu(\mu_e = -L_e)$ conserved.

(D) $W^{(1)} + W^{(2)} + W^{(4)} \Rightarrow L_e, L_\mu(\mu_\mu = L_\epsilon)$ conserved.

(E) $W^{(1)} + W^{(2)} + W^{(5)} \Rightarrow L_e$ conserved only. One neutrino gets a tree-level mass as in (A),(C),(D), another gets a radiative mass.

VII. U(1) GAUGE SYMMETRIES

The SM may be extended to include an extra $U(1)_X$ gauge symmetry. This requires the absence of quantum anomalies:

(A) Mixed gravitational-gauge anomaly: The sum of $U(1)_X$ charges should be zero.

(B) Global SU(2) anomaly: The number of SU(2) fermion doublets should be even.
(C) Axial-vector-vector-vector anomaly: The sum over $L^3 - R^3$ charges, i.e. $[SU(3)_C]^2 U(1)_X$, $[SU(2)_L]^2 U(1)_X$, $[U(1)_Y]^2 U(1)_X$, $U(1)_Y[U(1)_X]^2$, and $[U(1)_X]^3$, should be zero.

Given the particle content of the SM (without $\nu_R$), there are 3 often neglected possible $U(1)_X$ gauge extensions \[25\]: $L_e - L_\mu$, $L_e - L_\tau$, $L_\mu - L_\tau$. If 3 $\nu_R$’s are added with $L = 1$, then $U(1)_X = B - L$ is possible. The anomaly conditions are satisfied as follows:

\[
X : (3)(2) \left[ \frac{1}{3} - \frac{1}{3} \right] + (2)[(-1) - (-1)] = 0. \tag{23}
\]

\[
C^2 X : \frac{1}{2} (2) \left[ \frac{1}{3} - \frac{1}{3} \right] = 0. \tag{24}
\]

\[
L^2 X : \frac{1}{2} \left[ (3) \left( \frac{1}{3} \right) + (-1) \right] = 0. \tag{25}
\]

\[
Y^2 X : 2 \left( \frac{1}{6} \right)^2 - \left( \frac{2}{3} \right)^2 - \left( -\frac{1}{3} \right)^2 \left( \frac{1}{3} \right) + \left[ 2 \left( -\frac{1}{2} \right)^2 - (-1)^2 \right] (-1) = 0. \tag{26}
\]

\[
Y X^2 : (3) \left[ 2 \left( \frac{1}{6} \right) - \left( \frac{2}{3} \right) - \left( -\frac{1}{3} \right) \left( \frac{1}{3} \right)^2 + \left[ 2 \left( -\frac{1}{2} \right) - (-1) \right] (-1)^2 \right] = 0. \tag{27}
\]

\[
X^3 : (3)(2) \left[ \left( \frac{1}{3} \right)^3 - \left( \frac{1}{3} \right)^3 \right] + (2)[(-1)^3 - (-1)^3] = 0. \tag{28}
\]

Neutrino mass may thus be a hint of $U(1)_{B-L}$ and point to $SU(3)_C \times SU(2)_L \times SU(2)_R \times U(1)_{B-L}$ and $SO(10)$.

\[\text{VIII. } B - 3L_\tau\]

If just one $\nu_R$ with $L_\tau = 1$ is added, then $U(1)_X = B - 3L_\tau$ is anomaly-free and can be gauged \[26, 27\]. To break $U(1)_X$ spontaneously, a neutral scalar singlet $\chi^0 \sim (1, 1, 0; 6)$ is used, which also gives $\nu_R$ a large Majorana mass, thereby making $\nu_\tau$ massive. The $X$ boson decays into quarks and $\tau$ but not $e$ or $\mu$. If a scalar doublet $(\eta^+, \eta^0) \sim (1, 2, 1/3; -3)$ and a scalar singlet $\chi^- \sim (1, 1, -1; -3)$ are added, then one linear combination of $\nu_e, \nu_\mu, \nu_\tau$ gets a tree-level mass, and the others get radiative masses via the radiative mechanism of Type IV.

The $X$ boson is not constrained to be very heavy because it does not couple to $e$ or $\mu$. It can be produced easily at the LHC because it has quark couplings. Its decay into $\tau^+\tau^-$ is also a good signature. Realistic neutrino masses and mixing are possible, with additional $U(1)_X$ scalars. It is well-known that $B - L$ may come from $SU(4) \times SU(2)_L \times SU(2)_R$ with $Q = T_{3L} + T_{3R} + (B - L)/2$ and $SU(4)$ breaking to $SU(3)_C \times U(1)_{B-L}$. Analogously, $B - 3L_\tau$
may come from $SU(10) \times SU(2)_L \times U(1)_Y$, with $Q = T_{3L} + Y' + (B - 3L_\tau)/5$ and $SU(10)$ breaking to $[SU(3)_C]^3 \times U(1)_{B-3L_\tau}$.

**IX. U(1)$_\Sigma$**

Instead of using the Type I seesaw for neutrino mass, consider Type III by adding 3 copies of the fermion triplet $(\Sigma^+, \Sigma^0, \Sigma^-)_R \sim (1, 3, 0)$. Is there a $U(1)$ gauge symmetry like $B - L$ as in the case of $\nu_R$? The answer is yes [28, 29, 30, 31]. Call this $U(1)_\Sigma$ and let $(u, d)_L \sim n_1$, $u_R \sim n_2$, $d_R \sim n_3$, $(\nu, e)_L \sim n_4$, $e_R \sim n_5$, and $\Sigma_R \sim n_6$. Then the 6 conditions for $U(1)_\Sigma$ to be anomaly-free, including the highly nontrivial

$$6n_1^3 - 3n_2^3 - 3n_3^3 + 2n_4^3 - n_5^3 - 3n_6^3 = 0,$$

are satisfied with

$$4n_2 = 7n_1 - 3n_4, \quad 4n_3 = n_1 + 3n_4, \quad 4n_5 = -9n_1 + 5n_4, \quad 4n_6 = 3n_1 + n_4. \quad (30)$$

This is a very remarkable result.

There is thus a family of solutions defined by $n_4 = \lambda n_1$. If $\lambda = -3$ and $n_1$ is chosen to be 1/6 for convenience, then $U(1)_\Sigma = U(1)_Y$, but if $\lambda \neq -3$, then $U(1)_\Sigma$ is new. Two Higgs doublets are required for fermion masses: $(\phi^+_1, \phi^0_1) \sim n_1 - n_3 = n_2 - n_1 = n_6 - n_4 = 3(n_1 - n_4)/4$ couples to quarks and $\Sigma$, and $(\phi^+_2, \phi^0_2) \sim n_4 - n_5 = (9n_1 - n_4)/4$ couples to $e$. Since $2\Sigma_R^2 + \Sigma^0_R + \Sigma^{-}_R$ is an SM invariant, $\Sigma_R$ may obtain a large Majorana mass just as $\nu_R$. It mixes necessarily with $(\nu, e)_L$ through $\Phi_1$, so that $\Sigma_i^- \rightarrow e^-_j Z$ and $\nu_j W^-$ are possible signals at the LHC.

**X. SUPERSYMMETRIC U(1)$_X$**

If the SM is extended to include supersymmetry, 3 well-known issues spring up. (A) $m_\nu = 0$ as in the SM. (B) $B$ and $L_i$ are conserved only if imposed. (C) The allowed term $\mu \hat{\phi}_1 \hat{\phi}_2$ in the superpotential must be adjusted with $\mu \sim M_{SUSY}$, i.e. the supersymmetry breaking scale. Each has a piecemeal solution, but is there one unifying explanation using $U(1)_X$? The answer is again yes [32]. Here all superfields must be considered in the anomaly-free conditions. Under $U(1)_X$, let there be 3 copies of

$$(\hat{u}, \hat{d}) \sim (3, 2, 1/6; n_1), \quad \hat{u}^c \sim (3^*, 1, -2/3; n_2), \quad \hat{d}^c \sim (3^*, 1, 1/3; n_3), \quad (31)$$

$$(\hat{v}, \hat{e}) \sim (1, 2, -1/2; n_4), \quad \hat{e}^c \sim (1, 1, 1; n_5), \quad \hat{N}^c \sim (1, 1, 0; n_6), \quad (32)$$
and 1 copy of
\[ \hat{\phi}_1 \sim (1, 2, -\frac{1}{2}; -n_1 - n_3), \quad \hat{\phi}_2 \sim (1, 2, \frac{1}{2}; -n_1 - n_2), \]  
with \( n_1 + n_3 = n_4 + n_5 \) and \( n_1 + n_2 = n_4 + n_6 \), so that quarks and leptons obtain masses through the two scalar superfields as in the MSSM. The Higgs singlet superfield
\[ \hat{\chi} \sim (1, 1, 0; 2n_1 + n_2 + n_3) \]  
is then added, so that \( \mu \hat{\phi}_1 \hat{\phi}_2 \) is replaced by \( \hat{\chi} \hat{\phi}_1 \hat{\phi}_2 \) and \( \langle \chi \rangle \neq 0 \) breaks \( U(1)_X \). Two copies of singlet up quark superfields
\[ \hat{U} \sim (3, 1, \frac{2}{3}; n_7), \quad \hat{U}^c \sim (3^*, 1, -\frac{2}{3}; n_8), \]  
and one copy of singlet down quark superfields
\[ \hat{D} \sim (3, 1, -\frac{1}{3}; n_7), \quad \hat{D}^c \sim (3^*, 1, \frac{1}{3}; n_8), \]  
are added with \( n_7 + n_8 = -2n_1 - n_2 - n_4 \) so that \( \hat{\chi} \hat{U} \hat{U}^c \) and \( \hat{\chi} \hat{D} \hat{D}^c \) are allowed, with \( M_{U,D} \) appearing also at the \( U(1)_X \) breaking scale. So far there are 8 numbers and 3 constraints, resulting in 5 independent numbers. Consider first
\[ [SU(3)]^2 U(1)_X : 2n_1 + n_2 + n_3 + n_7 + n_8 = 0. \]  
This is already satisfied. Consider then \([SU(2)]^2 U(1)_X\) and \([U(1)_Y]^2 U(1)_X\) respectively:
\[ 3(3n_1 + n_4) + (-n_1 - n_3) + (-n_1 - n_3) = 7n_1 - n_2 - n_3 + 3n_4 = 0, \]  
\[ -n_1 + 7n_2 + n_3 + 3n_4 + 6n_5 + 6n_7 + 6n_8 = -7n_1 + n_2 + n_3 - 3n_4 = 0. \]
These two conditions are identical, resulting in the elimination of one number. Using \( n_1, n_2, n_4, n_7 \) as independent, consider \( U(1)_Y [U(1)_X]^2\):
\[ 3n_1^2 - 6n_2^2 + 3n_3^2 - 3n_4^2 + 3n_5^2 + 3n_7^2 - 3n_8^2 - (n_1 + n_3)^2 + (n_1 + n_2)^2 = 6(3n_1 + n_4)(2n_1 - 4n_2 - 3n_7) = 0, \]  
which factors exactly and has two solutions. If \( 3n_1 + n_4 = 0 \), \( U(1)_X = U(1)_Y \) as expected, so the condition \( 2n_1 - 4n_2 - 3n_7 \) is chosen from now on. Using \( n_1, n_4, n_6 \) as independent, the other 5 numbers are
\[ n_2 = -n_1 + n_4 + n_6, \quad n_3 = 8n_1 + 2n_4 - n_6, \quad n_5 = 9n_1 + n_4 - n_6, \]
\[ n_7 = 2n_1 - \frac{4}{3}n_4 - \frac{4}{3}n_6, \quad n_8 = -11n_1 - \frac{5}{3}n_4 + \frac{4}{3}n_6. \]  
\[ (41) \]
The most nontrivial condition is

\[
[U(1)_X]^3 : 3[6n_1^3 + 3n_2^3 + 3n_3^3 + 2n_4^3 + n_5^3 + n_6^3] + 3(3n_7^3 + 3n_8^3) \\
+ 2(-n_1 - n_3)^3 + 2(-n_1 - n_2)^3 + (2n_1 + n_2 + n_3)^3 \\
= -36(3n_1 + n_4)(9n_1 + n_4 - 2n_6)(6n_1 - n_4 - n_6) = 0.
\]

(42)

The sum of 11 cubic terms has been factorized exactly! Two possible solutions are

(A) \( n_6 = \frac{1}{2}(9n_1 + n_4) \), \quad (B) \( n_6 = 6n_1 - n_4 \).

(43)

To obtain \( L \) conservation automatically, the solutions are (A) \( 9n_1 + 5n_4 \neq 0 \), or (B) \( 3n_1 + 4n_4 \neq 0 \). To obtain \( B \) conservation automatically, the conditions are (A) \( 7n_1 + 3n_4 \neq 0 \), or

(B) \( 3n_1 + 2n_4 \neq 0 \). If (A)=(B), then

\[
n_1 = n_4 = 1, \quad n_2 = n_3 = n_5 = n_6 = 5, \quad n_7 = n_8 = -6,
\]

(44)

and \( U(1)_X \) is orthogonal to \( U(1)_Y \). However, there is still the mixed gravitational-gauge anomaly, i.e. the sum of \( U(1)_X \) charges = 6(3n_1 + n_4) \neq 0 \). To cancel this without affecting the other conditions, add singlet superfields with charge in units of \( (3n_1 + n_4) \): one with charge 3, three (\( \hat{S}^c \)) with charge \(-2\), and three (\( \hat{N} \)) with charge \(-1\), so that \( 3 + 3(-2) + 3(-1) = -6 \) and \( 27 + 3(-8) + 3(-1) = 0 \). Consider now the neutrino mass. Since \( L \) is conserved, this mass is Dirac, coming from the pairing of \( \nu \) with \( N^c \). However, if \( n_6 = 3n_1 + n_4 \), then the singlets \( S^c \) and \( N \) are exactly right to allow the neutrinos to acquire small seesaw Dirac masses. In the basis \((\nu, S^c, N, N^c)\), the \( 12 \times 12 \) neutrino mass matrix is

\[
\mathcal{M}_\mu = \begin{pmatrix}
0 & 0 & 0 & m_1 \\
0 & 0 & m_2 & 0 \\
0 & m_2 & 0 & M \\
m_1 & 0 & M & 0
\end{pmatrix},
\]

(45)

with \( m_\nu = -m_1m_2/M \). Since \( m_1 \) comes from electroweak symmetry breaking and \( m_2 \) from \( U(1)_X \) breaking, and \( M \) is an invariant mass, this is a natural explanation of the smallness of \( m_\nu \) just as in the seesaw Majorana case.

**XI. NEUTRINO TRIBIMAXIMAL MIXING**

From neutrino-oscillation data in the past decade, it is now established that the neutrino mixing matrix \( U_{\mu\nu} \) takes a particular form which is approximately tribimaximal. Here I show
how it can be understood in terms of an underlying non-Abelian discrete symmetry $A_4$. In 1978, soon after the putative discovery of the third family of leptons and quarks, it was conjectured by Cabibbo \[33\] and Wolfenstein \[34\] independently that

$$U_{l^C \nu}^{CW} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix}, \tag{46}$$

where $\omega = \exp(2\pi i/3) = -1/2 + i\sqrt{3}/2$. This should dispel the myth that everybody expected small mixing angles in the lepton sector as in the quark sector. In 2002, after much neutrino oscillation data have been established, Harrison, Perkins, and Scott \[35\] proposed the tribimaximal mixing matrix, i.e.

$$U_{l^C \nu}^{HPS} = \begin{pmatrix} \sqrt{2/3} & 1/\sqrt{2} & 0 \\ -1/\sqrt{6} & 1/\sqrt{3} & -1/\sqrt{2} \\ -1/\sqrt{6} & 1/\sqrt{3} & 1/\sqrt{2} \end{pmatrix} \sim (\eta_8, \eta_1, \pi^0), \tag{47}$$

where the 3 columns are reminiscent of the meson nonet. In 2004, I discovered \[36\] the simple connection:

$$U_{l^C \nu}^{HPS} = (U_{l^C \nu}^{CW})^\dagger \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & i \end{pmatrix}. \tag{48}$$

This means that if

$$M_l = U_{l^C \nu}^{CW} \begin{pmatrix} m_e & 0 & 0 \\ 0 & m_\mu & 0 \\ 0 & 0 & m_\tau \end{pmatrix} (U_{l^C \nu}^{l_R})^\dagger \tag{49}$$

and $M_\nu$ has $2-3$ reflection symmetry, with zero $1-2$ and $1-3$ mixing, i.e

$$M_\nu = \begin{pmatrix} a + 2b & 0 & 0 \\ 0 & a - b & d \\ 0 & d & a - b \end{pmatrix}, \tag{50}$$

$U_{l^C \nu}^{HPS}$ will be obtained, but how? Tribimaximal mixing means that

$$\theta_{13} = 0, \quad \sin^2 2\theta_{23} = 1, \quad \tan^2 \theta_{12} = 1/2. \tag{51}$$

In 2002 (when HPS proposed it), world data were not precise enough to test this idea. In 2004 (when I derived it), SNO data implied $\tan^2 \theta_{12} = 0.40 \pm 0.05$, which was not so encouraging. Then in 2005, revised SNO data obtained $\tan^2 \theta_{12} = 0.45 \pm 0.05$, and tribimaximal mixing became a household word, unleashing a glut of papers.
XII. TETRAHEDRAL SYMMETRY $A_4$

For 3 families, one should look for a group with a $3$ representation, the simplest of which is $A_4$, the group of the even permutation of 4 objects. It has 12 elements, divided into 4 equivalence classes, and 4 irreducible representations: $1, 1', 1'', \text{ and } 3$, with the multiplication rule

$$3 \times 3 = \begin{cases} 1 (11 + 22 + 33) + 1' (11 + \omega^2 22 + \omega 33) + 1'' (11 + \omega 22 + \omega^2 33) \\ + 3 (23, 31, 12) + 3 (32, 13, 21). \end{cases}$$ (52)

$A_4$ is also the symmetry group of the regular tetrahedron, one of the 5 perfect geometric solids in 3 dimensions and identified by Plato as “fire” [37]. It is a subgroup of both SO(3) and SU(3). The latter also has 2 sequences of finite subgroups which are of interest: $\Delta(3n^2)$ has $\Delta(12) \equiv A_4$ and $\Delta(27); \Delta(3n^2 - 3)$ has $\Delta(24) \equiv S_4$.

There are two ways to achieve Eq. (49). The original proposal [38, 39] is to assign $(\nu_i, l_i) \sim 3, l_i^c \sim 1, 1', 1''$, then with $(\phi^0_1, \phi^+_1) \sim 3$.

$$M_l = \begin{pmatrix} h_1 v_1 & h_2 v_1 & h_3 v_1 \\ h_1 v_2 & h_2 \omega v_2 & h_3 \omega^2 v_2 \\ h_1 v_3 & h_2 \omega^2 v_3 & h_3 \omega v_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix} \begin{pmatrix} h_1 v & 0 & 0 \\ 0 & h_2 v & 0 \\ 0 & 0 & h_3 v \end{pmatrix},$$ (53)

if $v_1 = v_2 = v_3 = v$. This is the starting point of most subsequent $A_4$ models. More recently, I discovered [40] that Eq. (49) may also be obtained with $(\nu_i, l_i) \sim 3, l_i^c \sim 3$ and $(\phi^0_1, \phi^+_1) \sim 1, 1, 1$, in which case

$$M_l = \begin{pmatrix} h_0 v_0 & h_1 v_3 & h_2 v_2 \\ h_2 v_3 & h_0 v_0 & h_1 v_1 \\ h_1 v_2 & h_2 v_1 & h_0 v_0 \end{pmatrix} = U_{l\nu}^{CW} \begin{pmatrix} m_e & 0 & 0 \\ 0 & m_\mu & 0 \\ 0 & 0 & m_\tau \end{pmatrix} (U_{l\nu}^{CW})^\dagger,$$ (54)

if $v_1 = v_2 = v_3 = v$. Either way, $U_{l\nu}^{CW}$ has been derived. To obtain $U_{l\nu}^{HPS}$, let $M_\nu$ be Majorana and come from Higgs triplets: $(\xi^{++}, \xi^+, \xi^0)$, then [36]

$$M_\nu = \begin{pmatrix} a + b + c & f & e \\ f & a + \omega b + \omega^2 c & d \\ e & d & a + \omega^2 b + \omega c \end{pmatrix},$$ (55)

where $a$ comes from $1, b$ from $1'$, $c$ from $1''$, and $(d, e, f)$ from $3$. To obtain Eq. (50), we simply let $b = c$ and $e = f = 0$. Note that the tribimaximal mixing matrix does not depend on the neutrino mass eigenvalues $a - b + d, a + 2b, -a + b + d$, nor the charged-lepton masses. This implies the existence of residual symmetries [41, 42].
Since $1'$ and $1''$ are unrelated in $A_4$, the condition $b = c$ is rather *ad hoc*. A very clever solution was proposed by Altarelli and Feruglio [43]: they eliminated both $1'$ and $1''$ so that $b = c = 0$. In that case, $m_1 = a + d$, $m_2 = a$, $m_3 = -a + d$. This is the simplest model of tribimaximal mixing, with the prediction of normal ordering of neutrino masses and the sum rule [44]

$$|m_{\nu_e}|^2 \simeq |m_{ee}|^2 + \Delta m^2_{atm}/9. \quad (56)$$

Babu and He [45] proposed instead to use 3 heavy neutral singlet fermions with $M_D$ proportional to the identity and $M_N$ of the form of Eq. (50) with $b = 0$. In that case, the resulting $M_\nu$ has $b = c$ and $d^2 = 3b(b - a)$. This scheme allows both normal and inverted ordering of neutrino masses.

The technical challenge in all such models is to break $A_4$ spontaneously along 2 incompatible directions: (1,1,1) with residual symmetry $Z_3$ in the charged-lepton sector and (1,0,0) with residual symmetry $Z_2$ in the neutrino sector. There is also a caveat. If $\nu_2 = (\nu_e + \nu_\mu + \nu_\tau)/\sqrt{3}$ remains an eigenstate, i.e. $e = f = 0$, but $b \neq c$ is allowed, then the bound $|U_{e3}| < 0.16$ implies [36] $0.5 < \tan^2 \theta_{12} < 0.52$, away from the preferred experimental value of $0.45 \pm 0.05$.

**XIII. BEYOND $A_4$ [$S_4$, $\Delta(27)$, $\Sigma(81)$, $Q(24)$]**

The group of permutation of 4 objects is $S_4$. It contains both $S_3$ and $A_4$. However, since the $1'$ and $1''$ of $A_4$ are now combined into the $\mathbf{2}$ of $S_4$, tribimaximal mixing is achieved only with Eq. (54). Furthermore, $h_1 \neq h_2$ in $M_l$ now requires both $\mathbf{3}$ and $\mathbf{3}'$ Higgs representations. No advantage appears to have been gained.

The group $\Delta(27)$ has the interesting decomposition $\mathbf{3} \times \mathbf{3} = \overline{\mathbf{3}} + \overline{\mathbf{3}} + \mathbf{3}$, which allows

$$M_\nu = \begin{pmatrix} x & f z & f y \\ f z & y & f x \\ f y & f x & z \end{pmatrix}. \quad (57)$$

Using $\tan^2 \theta = 0.45$ and $\Delta m^2_{atm} = 2.7 \times 10^{-3}$ eV$^2$, this implies [46] $m_{ee} = 0.14$ eV.

The subgroups $\Sigma(3n^3)$ of $U(3)$ may also be of interest. $\Sigma(81)$ has 17 irreducible representations and may be applicable [47] to the Koide lepton mass formula

$$m_e + m_\mu + m_\tau = (2/3) (\sqrt{m_e} + \sqrt{m_\mu} + \sqrt{m_\tau})^2; \quad (58)$$

as well as neutrino tribimaximal mixing [48].

Since $A_4$ is a subgroup of $SO(3)$, it has a spinorial extension which is a subgroup of $SU(2)$. This is the binary tetrahedral group, which has 24 elements with 7 irreducible
representations: $\mathbf{1}$, $\mathbf{1}'$, $\mathbf{1}''$, $\mathbf{2}$, $\mathbf{2}'$, $\mathbf{2}''$, $\mathbf{3}$. It is also isomorphic to the quaternion group $Q(24)$ whose 24 elements form the vertices of the self-dual hyperdiamond in 4 dimensions. There have been several recent studies [49, 50, 51, 52] involving $Q(24)$, which may be useful for extending the success of $A_4$ for leptons to the quark sector. Note the peculiar fact that $A_4$ is not a subgroup of $Q(24)$.

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