Spin-Orbit Coupling and Topological States in $F = \frac{3}{2}$ Cold Fermi Gas

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In this work we study the possible occurrence of topological insulators for 2D fermions of high spin. They can be realized in cold fermion systems with ground-state atomic spin $F > \frac{1}{2}$, if the optical potential is properly designed, and spin-orbit coupling is relevant. The latter is shown to be induced by letting the fermions interact with a specially tuned arrangement of polarized laser beams. When the system is subject to a perpendicular magnetic field, time reversal symmetry is broken but the ensuing Hamiltonian is still endowed with a mirror symmetry.

Topological insulators for fermions of higher spins are fundamentally distinct from those pertaining to spin $\frac{1}{2}$. The underlying physics reveals a plethora of positive and negative mirror Chern numbers, respectively corresponding to chiral and anti-chiral edge states. Here, for simplicity, we concentrate on the case $F = \frac{3}{2}$ (which is suitable for $^6$Li or $^2$H atoms) but extension to higher spins (such as $^{40}$K whose ground-state spin is $F = \frac{3}{2}$), is straightforward.

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I. INTRODUCTION

Spin-orbit coupling (SOC) in cold-atom systems are now realizable, employing laser radiation impinging on a gas of cold atoms$^1$. Furthermore, spin-orbit coupling with equal Rashba and Dresselhaus strengths is synthetically induced by applying Raman lasers on atomic gases with hyperfine spin degrees of freedom$^{4,5}$, whose scheme was discussed theoretically in several earlier works$^{4,6}$. This scheme has recently been implemented using both cold boson$^{7,8}$ and fermionic degenerate gases$^{9,10}$. In addition, schemes for creating general Rashba and Dresselhaus SOC or three-dimensional (3D) analogue to Rashba SOC have theoretically been proposed$^{10-12}$, and experimental realization of 2D SOC has been demonstrated$^{13,14}$.

In a manner similar to solid state physics, SOC in cold-atom systems provides interesting non-trivial phenomena such as anomalous quantum Hall and quantum spin Hall effects$^{15}$. An $s$-wave superfluid of cold atoms may host Majorana fermions in the presence of SOC and a Zeeman magnetic field$^{16-18}$, Moreover, Rashba mechanism of SOC in 2D cold atomic system is under intensive study$^{19-21}$.

In solid state physics, except for a few materials$^{22-26}$, SOC pertaining to a degenerate Fermi gas is analyzed for electrons (or holes) whose intrinsic spin is $s = \frac{1}{2}$. In cold atom systems, SOC is mostly studied for atoms with atomic spin $F = \frac{1}{2}$. However, one of the advantages of studying cold atom systems (as compared with solid-state systems) is the possibility to explore the physics of degenerate Fermi gases in which the fermions have atomic spin $F > \frac{1}{2}$. For example, in such systems, the Kondo physics is expected to be richer than it is for degenerate Fermi gases with spin $F = \frac{1}{2}$, because it might lead to over-screening and hence to a non-Fermi liquid ground-state$^{29,30}$.

The purpose of the present work is to employ this godsend of controlling Fermi gases of higher spin fermions for exploring SOC in such systems in general, and, in particular, systems exhibiting topological properties such as topological insulators. To achieve this goal, we consider a model wherein a degenerate Fermi gas of spin $F = \frac{3}{2}$ atoms (for example, $^6$Li or $^2$H atoms) occupies a 2D optical lattice and is subject to SOC. The Bloch spectrum is composed of two bands, and the bulk system can be described by an $8 \times 8$ Hamiltonian (two bands and four spin states) with finite gap between them. This optical potential can be generated by a specially designed pattern of polarized laser fields. In the present work we concentrate on the case where the system is also subject to a uniform perpendicular magnetic field of strength $B$, so that time reversal symmetry (TRS) is broken. (Note that because the atoms are neutral, the magnetic field acts only on the spin degrees of freedom). In a future communication we will study the system in the absence of magnetic field. The ensuing Hamiltonian contains a real parameter $\Delta_0$ that, together with the external magnetic field $B > 0$, controls the shape of the gap. Crossing into the region $B > \frac{\Delta_0}{2} \Theta(\Delta_0)$ in the half plane ($B > 0, \Delta_0$) drives the insulator to be topologically non-trivial.

As it turns out, this Hamiltonian has additional symmetries beyond those listed in the Altland-Zirnbauer (AZ) classification scheme$^{31}$. Moreover, one of these symmetries is not broken when the system is opened to have one edge (for example, when the 2D system is constrained in a half plane). Hence it is susceptible for an analysis within the theory of topological insulators with additional symmetries (with respect to those listed by AZ), developed by one of us$^{32}$. As such, it is shown to support $Z \times Z$ topological indices, and the bulk-edge correspondence is beautifully demonstrated after the mirror Chern numbers in the bulk are calculated together with the edge state spectrum.
The main achievements reported here are as follows:

1. The Hamiltonian describing the topology of spin $F = \frac{n+1}{2}$ fermions with SOC for $n > 0$ is derived and found to be fundamentally distinct from that for $n = 0$. In particular, the former cannot be expressed solely in terms of spin operators that belong to the $2F + 1$ irrep of $SU(2)^{27}$. Thus, although the analysis detailed in the main text starting from the long-wave $8 \times 8$ Hamiltonian is self consistent, the reader who is interested in justification and derivation of the Hamiltonian will find it in Appendix A. Finally, Appendix B is devoted to the numerical solutions of Eqs. (42) and (48) and in Appendix C we elaborate upon the main properties of the optical potential defined through Eq. (A6).

II. THE LONG-WAVE MODEL HAMILTONIAN

Following the detailed analysis described in Appendix A, the model Hamiltonian of cold atoms subject to SOC and an external magnetic field $B = Be_z$ can be written in a matrix form as,

$$\hat{H} = \sum_{k} \hat{\Psi}_{k} \hat{H}_{k} \hat{\Psi}_{k},$$

where

$$\hat{H}_{k} = \hat{H}_{k}^{(0)} + \hat{H}_{k}^{(SO)}.$$  

The operator $\hat{\Psi}_{k}$ is

$$\hat{\Psi}_{k} = \begin{pmatrix} \hat{\Psi}_{\xi,k} \\ \hat{\Psi}_{\nu,k} \end{pmatrix}, \quad \hat{\Psi}_{\xi,k} = \begin{pmatrix} c_{\xi,k} \xi \tau \frac{1}{2} \\ c_{\xi,k} \xi \tau \frac{1}{2} \\ c_{\xi,k} \xi \tau \frac{1}{2} \\ c_{\xi,k} \xi \tau \frac{1}{2} \end{pmatrix},$$

where $\hat{\Psi}_{\xi,k}$ and $\hat{\Psi}_{\nu,k}$ are annihilation and creation operators for atom in the conduction and valence band with wave vector $k$. Here $c_{\xi,k} \xi \tau \frac{1}{2}$ is the 2D wave vector of length $k = |k|$. Here $c_{\xi,k} \xi \tau \frac{1}{2}$ is the 2D wave vector of length $k = |k|$. Here $c_{\xi,k} \xi \tau \frac{1}{2}$ and $c_{\xi,k} \xi \tau \frac{1}{2}$ are annihilation and creation operators for atom in the conduction ($\xi = c$) or valence ($\xi = v$) band with wave vector $k$ and magnetic quantum number $f = \pm \frac{1}{2}$, $\pm \frac{1}{2}$. The notations $f = -f$ is used throughout.

The first term on the RHS of eq. (2) describes the kinetic energy of the atoms that moves in an external magnetic field $B = Be_z$. Explicitly

$$\hat{H}_{k}^{(0)} = \Delta_{k} \hat{\tau} \hat{z} \otimes \hat{F}^{\otimes 0} + B \hat{\tau} \hat{z} \otimes \hat{F}^{\hat{z}}.$$  

The first term on the RHS of Eq. (4) describes atoms in the conduction or valence band with dispersion $\pm \Delta_{k}$, where

$$\Delta_{k} = \Delta_{0} + \frac{\hbar^{2} k^{2}}{2M_{0}}.$$  

The real parameter $-\infty < \Delta_{0} < \infty$ determines the shape of the gap, and plays a central rôle in driving the system through a topological transition. The second term on the RHS of Eq. (4) is Zeeman interaction of the atoms with...
the external magnetic field. It is assumed that $M_0 > 0$, and $B > 0$.

The spin-orbit interaction is encoded by the second term on the RHS of eq. (2),
\[
\hat{H}^{(SO)}_k = 2 \hat{\tau} \otimes (d_k \cdot \hat{\mathbf{F}}), \quad d_k = hvk, \quad (v > 0),
\]
where
\[
\hat{\tau} = (\hat{\tau}_x, \hat{\tau}_y, \hat{\tau}_z)
\]
is a vector of Pauli matrices acting in the isospin space of the conduction and valence bands, while
\[
\hat{\mathbf{F}} = (\hat{F}^x, \hat{F}^y, \hat{F}^z)
\]
is a vector of the spin $\frac{3}{2}$ tensors. Nontrivial matrix elements of $\hat{F}^\alpha$ ($\alpha = x, y, z$) are
\[
\begin{align*}
F^x_{f,f+1} &= F^x_{f+1,f} = \frac{1}{2} \mathcal{L}_f, \\
F^y_{f,f+1} &= -F^y_{f+1,f} = \frac{i}{2} \mathcal{L}_f, \\
F^z_{f,f} &= f,
\end{align*}
\]
where
\[
\mathcal{L}_f = \sqrt{(F-f)(F+1+f)}.
\]
$\hat{\tau}^0$ and $\hat{\mathbf{F}}^0$ are $2 \times 2$ and $4 \times 4$ identity matrices acting in the isospin and spin spaces.

1. Mirror symmetry

As an $8 \times 8$ matrix, the Hamiltonian (2) is similar to a matrix that has two $4 \times 4$ matrices on its diagonal. Here we briefly construct the similarity transformation matrix. In the next section, the physical origin of this symmetry will be elaborated upon. In brief, it is related to the mirror reflection symmetry of the Hamiltonian with respect to the $x-y$-plane. Consequently, the $8 \times 8$ matrix Hamiltonian commutes with the mirror operator $\hat{M}_z$ defined as,
\[
\hat{M}_z = -i \hat{\tau}^z \otimes \hat{R}^z,
\]
where $\hat{R}^z$ is given by
\[
\hat{R}^z = \text{diag}(1, -1, 1, -1).
\]
Therefore, performing a unitary transformation $\hat{U}$
\[
\hat{U} = \frac{1}{2} \left\{ \hat{\tau}^0 \otimes (\hat{\mathbf{F}}^0 + \hat{\mathbf{R}}^z) + \hat{\tau}^z \otimes (\hat{\mathbf{F}}^0 - \hat{\mathbf{R}}^z) \right\},
\]
which transforms $\hat{M}_z$ as,
\[
\hat{U} \hat{M}_z \hat{U}^\dagger = -i \hat{\tau}^z \otimes \hat{\mathbf{F}}^0,
\]
one obtains a block diagonal form of the Hamiltonian
\[
\hat{\mathcal{H}}_k = \hat{\mathcal{U}} \hat{H}_k \hat{\mathcal{U}}^\dagger = \begin{pmatrix}
\hat{H}_{k,1} & 0 \\
0 & \hat{H}_{k,1}
\end{pmatrix},
\]
with
\[
\hat{H}_{k,\eta} = \eta \Delta_k \hat{R}^z + B\hat{\mathbf{F}}^z + 2(d_k \cdot \hat{\mathbf{F}}).
\]
Here $\eta = \pm 1$ denotes the eigenvalue of $i\hat{M}_z$. Note that for high atomic spin (i.e., for $F \geq \frac{1}{2}$), the operator $\hat{R}^z$ (8) is not a generator of the $2\mathbb{F} + 1$ dimensional irrep of the SU(2) group, an important distinction from the model for atoms with spin $\frac{1}{2}$. When the atomic spin is $F = \frac{1}{2}$, the operator $\hat{R}^z = 2\hat{\mathbf{F}}^z$ is a generator of the SU(2) group.

The $4 \times 4$ matrix Hamiltonians $\hat{H}_{k,\eta}$ in the mirror sub-sector can be analytically diagonalized. The resultant energies depend on three quantum numbers: The mirror quantum number $\eta = \pm 1$, the band quantum number $\xi = \pm 1 = c$ (conductance), $v$ (valence), and $s = \frac{1}{2}, \frac{3}{2}$ (the positive possible values of a pseudo spin). Explicitly,
\[
\varepsilon_{\xi, s, \eta}(k) = \xi \sqrt{A_{\eta}(k^2) + 2(s-1)B_{\eta}(k^2)},
\]
with
\[
A_{\eta}(k^2) = \Delta_k^2 + \frac{5}{4}B^2 - 5d_k^2 + 2B\eta\Delta_k,
\]
\[
B_{\eta}(k^2) = \sqrt{(B^2 + 2B\eta\Delta_k + 4d_k^2)^2 - 24B\eta\Delta_k d_k^2},
\]
where $d_k \equiv |d_k| = hvk$ is the SOC contribution defined in Eq. (6). Due to rotation symmetry around the $z$-axis, $\hat{\mathbf{F}}^z$ becomes a good quantum number at $k = 0$ and $s$ reduces to $s = |f|$ (Recall that $-F \leq f \leq F$ is the bare magnetic quantum number defined after Eq. (3)). The energy dispersion (11) of the bulk system (without edges) is shown in Fig. 1 for $\Delta_0 = 0$ and $B = 2q_0$. The energy gap $D_\eta$ separating the conduction and valence bands is,
\[
D_\eta = \varepsilon_{c, \frac{1}{2}, \eta}(0) - \varepsilon_{v, \frac{3}{2}, \eta}(0) = 2\epsilon_0,
\]
with the notations
\[
\epsilon_0 = hvk_0, \quad k_0 = \frac{M_0v}{\hbar}.
\]

III. SYMMETRY AND TOPOLOGY OF $\hat{H}_k$

In this section we discuss some symmetries of the $8 \times 8$ Hamiltonian $\hat{H}_k$ defined through Eqs. (2,4,6). There are discrete as well as continuous symmetries that endow the Hamiltonian with a rich topological structure and enable its classification above that implied within the Altland-Zirnbauer scheme.

For the sake of completeness we very briefly discuss in subsection III A also the case where the magnetic field
Energy dispersion (11) for in addition, there is point group symmetry where optical lattice, which is generated by inversion . The pattern of winding numbers is discussed in subsection III E and that of the mirror Chern numbers is discussed in subsection III D.

**A. Symmetries for** $B = 0$

Consider first symmetries of the system in the absence of the external magnetic field, that is, $B = 0$. In this case, the system is (effectively) time-reversal invariant

$$T \hat{H}_k T^{-1} = \hat{H}_{-k}, \quad T = e^{i\hat{F}y} K,$$

where $K$ denotes the operation of complex conjugation. In addition, there is point group symmetry $D_{2h}$ of the optical lattice, which is generated by inversion $P = \tau_z$. $P \hat{H}_k P^{-1} = \hat{H}_{-k}$, and two-fold rotations $C_{2i} = e^{i\hat{F}i\pi}$ around the $i$-axis ($i = x, y, z$),

$$C_{2x} \hat{H}_k C_{2x}^{-1} = \hat{H}_{(k_x, -k_y)};$$

$$C_{2y} \hat{H}_k C_{2y}^{-1} = \hat{H}_{(-k_x, k_y)};$$

$$C_{2z} \hat{H}_k C_{2z}^{-1} = \hat{H}_{-k}.$$

In the long wavelength approximation, the two-fold rotations becomes the full rotation,

$$U(\theta) \hat{H}_k U^\dagger(\theta) = \hat{H}_{R(\theta)k},$$

where $U(\theta) = e^{i\theta \hat{F}y}$ is the rotation operator by $\theta$ and $R(\theta)$ is the corresponding rotation matrix for $k$. Consequently, the eigenvalues depend only on $k^2$.

A somewhat less evident symmetry (referred to as chiral symmetry) reads,

$$\hat{F}y \hat{H}_k (\hat{F}y)^{-1} = -\hat{H}_k,$$

which constitutes a relation between the valence and conduction bands.

**B. Symmetries for** $B \neq 0$

Now consider the system under a finite (perpendicular) magnetic field $B$, applied in the $z$-direction. Inversion $P$ and two-fold rotation $C_{2z}$ around the $z$-axis are still valid symmetries. While the other symmetries are broken, some of their combinations may survive: Both time-reversal and two-fold-rotation around the $y$-axis flip $B$, so their combination is preserved as a magnetic rotation symmetry,

$$[C_{2y} T] \hat{H}_k [C_{2y} T]^{-1} = \hat{H}_{(k_x, -k_y)},$$

which implies

$$\hat{H}_k^* = \hat{H}_{(k_x, -k_y)}.$$  

Furthermore, the combination of chiral symmetry and $C_{2y}$ leads to the relation,

$$[C_{2y} \hat{F}y] \hat{H}_k [C_{2y} \hat{F}y]^{-1} = -\hat{H}_{(-k_x, k_y)}.$$  

In Eqs. (19) and (21), two-fold rotation around the $y$-axis can be replaced with that around the $x$-axis.

By combining inversion $P$ and two-fold rotation $C_{2z}$ around the $z$-axis, we have mirror reflection symmetry $M_z$ with respect to the $xy$-plane. Since the Hamiltonian is independent of $k_z$, the mirror operator $M_{xy}$ commutes with the Hamiltonian. We have already used the mirror reflection symmetry to block diagonalize the Hamiltonian in Eq. (9). For the resultant $4 \times 4$ Hamiltonians $\hat{H}_{k, \eta}$, the symmetries in Eqs. (19) and (21) reduce to

$$T' \hat{H}_{k_x, k_y, \eta} (T')^{-1} = \hat{H}_{k'_x, k'_y, \eta}; \quad T' = K,$$

$$Q \hat{H}_{k_x, k_y; \eta} Q^{-1} = -\hat{H}_{-k_x, k_y; \eta},$$

where $Q$ is given by

$$Q = \sigma_x \otimes \sigma_x = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$
C. Topological Structure

To check whether a symmetry transformation induces a non-trivial topological structure, we need to confirm that this symmetry is not broken when the system has an edge. Note that we have to check the result of the symmetry operation on the Hamiltonian when it is written in configuration space, that is, $\hat{H}_N(x,y)$ (the Fourier transform of $\hat{H}_{k,\eta}$). So far, the above symmetries have been considered in the bulk 2D system. An important feature of the pertinent Hamiltonian is that some symmetries are valid also when the system is opened. In particular, mirror reflection symmetry with respect to the $z$-axis is retained even in the presence of an edge. Therefore, we shall later on employ the $4 \times 4$ Hamiltonians $\hat{H}_{k,\eta}$ in the mirror subsector for for the analysis of edge states.

For definiteness, it is assumed that the edge is normal to the $y$-axis, and the open system is defined on the half plane $y \geq 0$. In that case, the symmetries defined in Eq. (22) are not broken. Moreover, using rotation symmetry around the $z$-axis, we also have similar symmetries consistent with a boundary normal to the $x$-axis.

Now we specify possible topological phases in $\hat{H}_{k,\eta}$ by employing some techniques borrowed from $K$-theory. The idea is to deform the Hamiltonian $\hat{H}_{k,\eta}$ into a Dirac form Hamiltonian, while keeping the symmetries in Eq. (22) intact and assuring that the system remains gapped throughout the deformation. Explicitly, for a given $\eta$ we consider the deformation,

$$\hat{H}_{k,\eta} \rightarrow \hat{H}_{k,\eta}^{\text{Dirac}} = k_x \gamma_x + k_y \gamma_y + m \gamma_z,$$

$$\{\gamma_i, \gamma_j\} = 2 \delta_{i,j}, \quad (i, j = x, y, z).$$

(24)

Since $\hat{H}_{k,\eta}^{\text{Dirac}}$ is adiabatically connected to the original Hamiltonian $\hat{H}_{k,\eta}$, we can identify candidate topological phases in $\hat{H}_{k,\eta}$ by examining them in $\hat{H}_{k,\eta}^{\text{Dirac}}$. A possible topological phase in the latter can be specified when it contains a mass term. To identify such mass term, we examine algebraic structures required by symmetry.

The symmetries specified in Eq. (22) impose the following relations on the gamma matrices $\{\gamma_i\}$,

$$[T', \gamma_x] = [T', \gamma_z] = [Q, \gamma_x] = [T', Q] = 0,$$

$$\{T', \gamma_y\} = [Q, \gamma_y] = [Q, \gamma_z] = 0,$$

$$T'^2 = Q^2 = 1, \quad [T', i] = 0, \quad [Q, i] = 0.$$  

(25)

where the last two equations are due to the anti-unitarity of $T'$ and the unitarity of $Q$.

Let $\mathcal{C}_{l,p,q}$ denote the Clifford algebra $\{e_1, \ldots, e_p, e_{p+1}, \ldots, e_{p+q}\}$ obeying $\{e_i, e_j\} = 0$ ($i \neq j$), $e_i^2 = -1$ ($i = 1, \ldots, p$), and $e_i^2 = 1$ ($i = 1, \ldots, q$). With the following identification,

$$e_1 = i \gamma_x, \quad e_2 = i \gamma_z, \quad e_3 = i T', \quad e_4 = e_5 = \gamma_y, \quad S = \gamma_x Q.$$  

(26)

Eq. (25) is commensurate with the Clifford algebra $\mathcal{C}_{l,2,3}$ $\{e_1, \ldots, e_5\}$ with the commutating operator $S$

$$[e_i, S] = 0 \quad (i = 1, \ldots, 5), \quad S^2 = 1.$$  

(27)

On the other hand, if the mass term $m \gamma_z$ is absent, we encounter the $\mathcal{C}_{l,1,3}$ algebra with $S$, since $e_2$ is missing. Therefore, specifying a possible mass term of $\hat{H}_{k,\eta}^{\text{Dirac}}$ is identical to specifying a possible extension of the Clifford algebra from $\mathcal{C}_{l,1,3}$ to $\mathcal{C}_{l,2,3}$ with $S$. From the $K$-theory, the latter extension is found to define the classification space $\mathbb{Z}$, which is endowed with topologically distinct subspaces characterized by $\pi_0(R_0 \times R_0) = \mathbb{Z} \times \mathbb{Z}$. Therefore, we have topologically distinct mass terms in $\hat{H}_{k,\eta}^{\text{Dirac}}$, and correspondingly, topologically different phases with a $\mathbb{Z} \times \mathbb{Z}$ number. For the Hamiltonian $\hat{H}_{k,\eta}$, the $\mathbb{Z} \times \mathbb{Z}$ number is given by the pair $(C_\eta, w_\eta)$ composed of a mirror Chern number $C_\eta$ and a one-dimensional mirror winding number $w_\eta$. These two numbers are defined below, where it is also shown that the parities of $C_\eta$ and $w_\eta$ coincide, that is, $(-1)^{C_\eta} = (-1)^{w_\eta}$.

D. Mirror Chern Numbers

The mirror Chern numbers $\{C_\eta\}$ are defined (in the standard way) in terms of the eigenfunctions $\{\psi_{\xi,s,\eta}(k)\}$ that obey the Schrödinger equation in the bulk system

$$\hat{H}_{k,\eta} \psi_{\xi,s,\eta}(k) = \varepsilon_{\xi,s,\eta}(k) \psi_{\xi,s,\eta}(k).$$  

(28)

Recall that the "isospin" quantum number $\xi = c_i(v)$ refers to the conduction (valence) band, $s = \frac{1}{2}, \frac{3}{2}$ is a pseudo-spin quantum number and $\eta$ is the block number. Explicitly,

$$C_\eta = \frac{1}{2 \pi} \sum_s \int F_{v,s,\eta}(k) \, d^2 k.$$  

(29)

Here the Berry curvature is

$$F_{v,s,\eta}(k) = i \, (e_z \cdot \Omega_{v,s,\eta}),$$  

(30)

where

$$\Omega_{v,s,\eta} = \nabla_k \times \langle \psi_{v,s,\eta}(k) | \psi_{v,s,\eta}(k) \rangle,$$

$$\langle \psi_{v,s,\eta}(k) | \nabla_k \psi_{v,s,\eta}(k) \rangle.$$  

(31)

(32)

As shown in Appendix D, there are five relevant domains in the half-plane $\Delta_0 B$ with $B > 0$, illustrated in Fig. 2. We derive the mirror Chern numbers for each of the domains. Explicitly, the mirror Chern numbers $C_\eta$ for the domains (1) – (5) are:

1. $C_1 = C_\bar{1} = 0$.  
2. $C_1 = 0$ and $C_\bar{1} = 3$.  
3. $C_1 = 1$ and $C_\bar{1} = 3$.  
4. $C_1 = 1$ and $C_\bar{1} = 2$.  
5. $C_1 = -2$ and $C_\bar{1} = 2$.  


Relevant domains in the half-plane $\Delta_0 B$ with $B > 0$, see details in the text. The purple lines are $B = \pm 2\Delta_0$ and $B = -\frac{4}{3}\Delta_0$. $\epsilon_0$ is given by eq. (13). We illustrate formation of the edge states for the points $P_1$, $P_2$, $P_3$, $P_4$ and $P_5$.

These results indicate that the topological phases are driven by both $\Delta_0$ and $B$. For $B < \frac{2}{3}\Delta_0$ the mirror Chern numbers occur only for $\Delta_0 < 0$, but for $B > \frac{2}{3}\Delta_0$ there are non-zero mirror Chern numbers also for $\Delta_0 > 0$. From the bulk-boundary correspondence, when $C_\eta$ is nonzero, $\hat{H}_{k_x,k_y;\eta}$ should host $|C_\eta|$ chiral edge modes on a boundary normal to the $y$-axis. This result will be confirmed in the next section.

**E. Mirror Winding Numbers**

The one-dimensional mirror winding number $w_\eta$ is defined by

$$w_\eta = \frac{i}{4\pi} \int dk_y \text{tr} \left[ Q \hat{H}^{-1}_{0,k_y;\eta} \partial_{k_y} \hat{H}_{0,k_y;\eta} \right].$$

This number can be calculated analytically. Like the mirror Chern numbers, their values depend on the pertinent regions in the half plane $\Delta_0-B$ with $B > 0$ shown in Fig. 2. Thus, the mirror winding numbers for the domains (1) – (5) are:

1. $w_1 = w_\bar{1} = 0$.
2. $w_1 = 0$ and $w_\bar{1} = 1$.
3. $w_1 = w_\bar{1} = 1$.
4. $w_1 = 1$ and $w_\bar{1} = 0$.
5. $w_1 = w_\bar{1} = 0$.

The generalized index theorem implies that when $w_\eta$ is nonzero, there exist $|w_\eta|$ zero modes with $k_x = 0$ on a boundary normal to the $y$-axis. As will be confirmed below, these zero modes are realized as the $k_x = 0$ part of chiral edge states.

**IV. EDGE STATES**

Having defined and exposed the mirror Chern and winding numbers in the bulk, and their values in the five domains in the $(B,\Delta_0)$ plane, the next task is to analyze the corresponding pattern of edge states, whose existence is a consequence of the bulk-edge correspondence scenario. For this purpose, it is convenient to consider a system with single edge, such as a 2D half plane $-\infty < x < \infty$, $0 < y < \infty$. After deriving and solving the equations for the edge state wave functions in subsection IV A, we expose their dispersion curves in the five domains specified in Fig. 2.

**A. Equations for the Edge States**

Let us then consider an optical lattice occupying the half plane $y > 0$. In this case, $k_x$ is still a good quantum number, but $k_y$ is not. Rather, it becomes an operator $k_y \rightarrow -i\partial_y$. The edge states wave functions are the solutions of the Schrödinger equation,

$$\hat{H}(k_x, -i\partial_y, \eta) \Psi_{k_x,\eta}(y) = \varepsilon_\eta(k_x) \Psi_{k_x,\eta}(y),$$

that satisfy the boundary conditions,

$$\Psi_{k_x,\eta}(0) = 0,$$

$$\lim_{y \to \infty} \Psi_{k_x,\eta}(y) = 0, \quad \text{(exponential decay)}.$$

The general solution of eq. (34) that satisfies the boundary condition at $y \to \infty$ (but not the boundary condition at $y = 0$) is a four component (pseudo) spinor,

$$\Psi_{k_x,\eta}(y) = \begin{pmatrix} \chi_{\frac{1}{2}} \\ \chi_{\frac{1}{2}} \\ \chi_{\frac{3}{2}} \\ \chi_{\frac{3}{2}} \end{pmatrix} e^{-\kappa y},$$

where the components $\{\chi_{\ell}\}$ and the exponent $\kappa$ are (generically complex) constants. The boundary condition (35b) requires $\text{Re}(\kappa) > 0$. Substituting eq. (36) into eq. (34), we get the following set of equations,

$$
\left( \varepsilon_{\eta}(0, k_x, \kappa) - \varepsilon_{\eta}(k_x) \right) \chi_{\frac{3}{2}} + h_{\frac{1}{2},\frac{3}{2}}(k_x, \kappa) \chi_{\frac{3}{2}} = 0, \quad (37a)
$$

$$
\left( \varepsilon_{\eta}(0, k_x, \kappa) - \varepsilon_{\eta}(k_x) \right) \chi_{\frac{1}{2}} + h_{\frac{1}{2},\frac{1}{2}}(k_x, \kappa) \chi_{\frac{1}{2}} + h_{\frac{3}{2},\frac{1}{2}}(k_x, \kappa) \chi_{\frac{3}{2}} = 0, \quad (37b)
$$

$$
\left( \varepsilon_{\eta}(0, k_x, \kappa) - \varepsilon_{\eta}(k_x) \right) \chi_{\frac{3}{2}} + h_{\frac{1}{2},\frac{3}{2}}(k_x, \kappa) \chi_{\frac{3}{2}} + h_{\frac{3}{2},\frac{3}{2}}(k_x, \kappa) \chi_{\frac{3}{2}} = 0, \quad (37c)
$$

$$
\left( \varepsilon_{\eta}(0, k_x, \kappa) - \varepsilon_{\eta}(k_x) \right) \chi_{\frac{1}{2}} + h_{\frac{1}{2},\frac{1}{2}}(k_x, \kappa) \chi_{\frac{1}{2}} + h_{\frac{1}{2},\frac{3}{2}}(k_x, \kappa) \chi_{\frac{3}{2}} = 0. \quad (37d)
$$

FIG. 2: (color online) Relevant domains in the half-plane $\Delta_0 B$ with $B > 0$, see details in the text. The purple lines are $B = \pm 2\Delta_0$ and $B = -\frac{4}{3}\Delta_0$. $\epsilon_0$ is given by eq. (13). We illustrate formation of the edge states for the points $P_1$, $P_2$, $P_3$, $P_4$ and $P_5$. 
Here \( \varepsilon_{f,\eta}^{(0)}(k_x, i\kappa) \) are,
\[
\varepsilon_{f,\eta} = fB - (1 - F^2) \eta \Delta_{k_x, i\kappa},
\]
where \( F = \frac{3}{2}, f = -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2} \), and
\[
\Delta_{k_x, i\kappa} = M_0 v^2 + \frac{h^2 k_x^2}{2M_0} + \frac{h^2 \kappa^2}{2M_0}.
\]
The nontrivial matrix elements \( h_{f,f'}(k_x, i\kappa) \) are
\[
\begin{align*}
    h_{3/2, 1/2}(k_x, i\kappa) &= h_{1/2, 3/2}(k_x, i\kappa) = \sqrt{3} A k^-, \\
    h_{3/2, -1/2}(k_x, i\kappa) &= h_{-1/2, 3/2}(k_x, i\kappa) = \sqrt{3} A k^+, \\
    h_{3/2, 1/2}(k_x, i\kappa) &= 2 Ak^-, \\
    h_{3/2, -1/2}(k_x, i\kappa) &= 2 Ak^+,
\end{align*}
\]
where
\[
k^\pm = k_x \mp \kappa.
\]
The set of equations (37) has nontrivial solutions when the corresponding determinant vanishes, that is,
\[
\det (\tilde{H}_{k_x, i\kappa, \eta} - \varepsilon_{\eta}(k_x) \tilde{F}^0) = 0,
\]
where \( \tilde{F}^0 \) is the 4 \times 4 unit matrix. This equality yields the following equation for \( \kappa \),
\[
\left( \varepsilon_{\eta}^2(k_x) - A_{\eta}(k_x^2 - \kappa^2) \right)^2 - B_{\eta}^2(k_x^2 - \kappa^2) = 0,
\]
where \( A_{\eta}(k_x^2) \) and \( B_{\eta}(k_x^2) \) are given by eq. (12). Taking into account that both \( A_{\eta}(k_x^2 - \kappa^2) \) and \( B_{\eta}(k_x^2 - \kappa^2) \) are second order polynomials of \( \kappa^2 \), implies that eq. (42) is a quartic equation for \( \kappa^2 \). Consequently, it yields four solutions \( \{\kappa_n, n = 1, 2, 3, 4\} \) with Re(\( \kappa_n \)) > 0. These solutions are retained, as they satisfy the boundary conditions (35b). The other four solutions with Re(\( \kappa \)) < 0 diverge as \( y \to \infty \).

The remaining task is to form combinations of these basic solutions that satisfy also the boundary condition at \( y = 0 \). Knowing the four exponent \( \{\kappa_n\} \) the four solutions of eq. (42), we can express the constants \( \{\chi_{f}^{(n)}\} \) in terms of normalization constants \( N_n \),
\[
\chi_{f}^{(n)} = (-1)^{F-f} D_{f}^{(n)} N_n,
\]
where \( F = \frac{3}{2} \) and,
\[
D_{f}^{(n)} = \det \left( \tilde{M}_{f,\eta}(k_x, i\kappa_n) \right).
\]
The 3 \times 3 matrices \( \tilde{M}_{f,\eta}(k_x, i\kappa_n) \) are,
\[
\begin{align*}
\tilde{M}_{3/2, \eta}(k_x, i\kappa_n) &= \begin{pmatrix}
    h_{3/2, 1/2}(k_x, i\kappa_n) & 0 & 0 \\
    \varepsilon_{3/2, \eta}(k_x, i\kappa_n) - \varepsilon_{\eta}(k_x) & h_{1/2, 3/2}(k_x, i\kappa_n) & 0 \\
    h_{1/2, 1/2}(k_x, i\kappa_n) & \varepsilon_{1/2, \eta}(k_x, i\kappa_n) - \varepsilon_{\eta}(k_x) & h_{1/2, -1/2}(k_x, i\kappa_n)
\end{pmatrix}, \\
\tilde{M}_{1/2, \eta}(k_x, i\kappa_n) &= \begin{pmatrix}
    \varepsilon_{3/2, \eta}(k_x, i\kappa_n) - \varepsilon_{\eta}(k_x) & 0 & 0 \\
    h_{1/2, 1/2}(k_x, i\kappa_n) & \varepsilon_{1/2, \eta}(k_x, i\kappa_n) - \varepsilon_{\eta}(k_x) & h_{1/2, 3/2}(k_x, i\kappa_n) \\
    0 & h_{1/2, -1/2}(k_x, i\kappa_n) & \varepsilon_{1/2, \eta}(k_x, i\kappa_n) - \varepsilon_{\eta}(k_x)
\end{pmatrix}, \\
\tilde{M}_{-3/2, \eta}(k_x, i\kappa_n) &= \begin{pmatrix}
    h_{3/2, -1/2}(k_x, i\kappa_n) & 0 & 0 \\
    \varepsilon_{3/2, \eta}(k_x, i\kappa_n) - \varepsilon_{\eta}(k_x) & h_{-1/2, 3/2}(k_x, i\kappa_n) & 0 \\
    h_{-1/2, -1/2}(k_x, i\kappa_n) & \varepsilon_{-1/2, \eta}(k_x, i\kappa_n) - \varepsilon_{\eta}(k_x) & h_{-1/2, 1/2}(k_x, i\kappa_n)
\end{pmatrix}, \\
\tilde{M}_{-1/2, \eta}(k_x, i\kappa_n) &= \begin{pmatrix}
    \varepsilon_{3/2, \eta}(k_x, i\kappa_n) - \varepsilon_{\eta}(k_x) & 0 & 0 \\
    h_{-1/2, 1/2}(k_x, i\kappa_n) & \varepsilon_{-1/2, \eta}(k_x, i\kappa_n) - \varepsilon_{\eta}(k_x) & h_{-1/2, 3/2}(k_x, i\kappa_n) \\
    0 & h_{-1/2, -1/2}(k_x, i\kappa_n) & \varepsilon_{-1/2, \eta}(k_x, i\kappa_n) - \varepsilon_{\eta}(k_x)
\end{pmatrix}.
\end{align*}
\]
The wave function describing the edge state is a combination,
\[
\Psi_{k_x, \eta}(y) = \sum_{n=1}^{4} \mathcal{N}_n \hat{\chi}_n e^{-\kappa_n y},
\]
where
\[
\hat{\chi}_n = \left( \chi_{3/2}^{(n)}, \chi_{1/2}^{(n)}, \chi_{-1/2}^{(n)}, \chi_{-3/2}^{(n)} \right)^T,
\]
while \( \chi_{f}^{(n)} \) is given by eq. (43). The boundary condition
(35a) implies,
\[ \sum_{n=1}^{4} \chi_{f}^{(n)} \mathcal{N}_{n} = 0. \quad (47) \]
This is a system of linear homogeneous equations for \( \mathcal{N}_{n} \).
[Recall that \( f = \pm \frac{1}{2}, \pm \frac{3}{2} \), and \( n = 1, 2, 3, 4 \), so that the matrix \( \| \chi_{f}^{(n)} \| \) is a square \( 4 \times 4 \) matrix.] A nontrivial solution of Eq. (47) obtains when
\[ F_{\eta}(\varepsilon_{\eta}(k_{x})) = \det \| \chi_{f}^{(n)} \| = 0. \quad (48) \]
Since the matrix elements \( \chi_{f}^{(n)} \) are functions of the energy \( \varepsilon_{\eta}(k_{x}) \), eq. (48) yields the energies of the (topological) edge states (provided they exist). As we shall see below, there may be several solutions, denoted as \( \varepsilon_{\eta,a}(k_{x}) \).

In order to compute the energies for which solutions of eq. (48) exist, we apply a special numerical technique that is developed and explained in Appendix B. The edge state spectrum will be displayed below for \( \eta = 1 \) and \( \eta = -1 \) at the five points \( P_{i} \), \( i = 1, 2, 3, 4, 5 \) in the \( (B > 0, \Delta_{0}) \) half plane, representing the five domains as marked in Fig.2.

Since the atoms are neutral, the magnetic field acts only on the spin, so, unlike the electronic version, the direction of propagation of the edge states is not determined by the Lorentz force, that has a classical origin. In other words, when TRS is broken solely through the Zeeman effect, the chirality cannot be predicted a-priori. In fact, we shall see that there is a scenario where edge states propagate in both directions. To distinguish between them we refer to the states with negative (averaged) group velocity as chiral whereas states with positive (averaged) group velocity are referred to as anti-chiral. The bulk-edge correspondence implies that the positive mirror Chern numbers are equal to the number of chiral states and the negative mirror Chern numbers are equal to the number of anti-chiral states.

\[ B. \quad \text{Domain (1):} \quad (C_{1}, w_{1}) = (0, 0) \]

\[ \text{Fig. 3: (color online) Energy dispersion for } \eta = 1 \text{ [panel (a)] and } \eta = 1 \text{ [panel (b)]. For both panels: } \Delta_{0} = 0.987087 \epsilon_{0} \text{ and } B = 0.320364 \epsilon_{0}, \text{ corresponding to point } P_{1} \text{ in Fig. 2. The red and blue areas denote the conduction and valence bands, eq. (11). } \epsilon_{0} \text{ and } k_{0} \text{ are given by eq. (15).} \]

\[ C. \quad \text{Domain (2):} \quad (C_{1}, w_{1}) = (0, 0), (C_{1}, w_{1}) = (3, 1) \]

\[ \text{Fig. 4: (color online) Same as in Fig. 3 albeit with } \Delta_{0} = 0.850651 \epsilon_{0} \text{ and } B = 1.05146 \epsilon_{0}, \text{ corresponding to point } P_{2} \text{ in Fig. 2.} \]

The energy dispersion for domain (2) is displayed in Fig. 4 for \( \Delta_{0} = 0.850651 \epsilon_{0} \) and \( B = 1.05146 \epsilon_{0} \) [point \( P_{2} \) in Fig. 2]. For this domain, the mirror Chern and winding numbers are zero and there are no (gap closing) edge states. The reason is clear: For \( \Delta_{0} > 0 \) the topology phase driven solely by the magnetic field, but in the first domain, the magnetic field is too weak.

\[ D. \quad \text{Domain (3):} \quad (C_{1}, w_{1}) = (1, 1), (C_{1}, w_{1}) = (3, 1) \]

\[ \text{Fig. 5: (color online) Same as in Fig. 3 albeit with } \Delta_{0} = 0 \text{ and } B = 2 \epsilon_{0}, \text{ corresponding for point } P_{3} \text{ in Fig. 2.} \]

The energy dispersion for domain (3) is displayed in Fig. 5 for \( \Delta_{0} = 0 \) and \( B = 2 \epsilon_{0} \) [point \( P_{3} \) in Fig. 2]. For
this domain, the mirror Chern numbers are \(C_1 = 1\), and \(C_1' = 3\). Thus, there is a single chiral mode for \(\eta = 1\) [green curve in Fig. 5(a)], and three chiral modes associated with \(\eta = 1\) [green curves in Fig. 5(b)]. The properties of the latter three modes are the same as discussed in Fig. 4(b), except that here \(k_1 \approx 0.5k_0\) and \(k_2 \approx 3.2k_0\). In addition to the symmetries exposed in Eq. (49) (relevant to \(\bar{\eta} = -1\)), the symmetry (22) also implies
\[
\varepsilon_1(k_x) = -\varepsilon_1(-k_x). \quad (50)
\]

**E. Domain (4):** \((C_1, w_1) = (1, 1), (C_1, w_1) = (2, 0)\)

The energy dispersion for domain (4) is displayed in Fig. 6 for \(\Delta_0 = -0.850651\epsilon_0\) and \(B = 1.05146\epsilon_0\) [point \(P_4\) in Fig. 2]. For this domain, the mirror Chern numbers are \(C_1 = 1\), and \(C_1' = 2\). There is a single chiral mode for the block \(\eta = 1\) [green curve in Fig. 6(a)], and two chiral modes for \(\eta = 1\) [green curves in Fig. 6(b)]. As noted for states \(b, c\) in the analysis of Figs. 4(b) and 5(b), the energies of the two edge states for \(\bar{\eta} = -1\), displayed in Fig. 6(b) are not zero at \(k_x = 0\). Moreover, the corresponding energies \(\varepsilon_{\eta,b}(k_x)\) and \(\varepsilon_{\eta,c}(k_x)\) are not monotonic as function of \(k_x\). This result does not contradict the symmetry (22), according to which
\[
\varepsilon_{\eta,c}(-k_x) = -\varepsilon_{\eta,b}(k_x).
\]
However, this pattern of non-monotonicity implies that the group velocity changes sign twice inside the gap. The symmetry (22) implies property (50) of \(\varepsilon_1(k_x)\).

**F. Domain (5):** \((C_1, w_1) = (-2, 0), (C_1, w_1) = (2, 0)\)

Energy dispersion for the domain (5) is displayed in Fig. 7 for \(\Delta_0 = -0.987087\epsilon_0\) and \(B = 0.320364\epsilon_0\) [point \(P_5\) in Fig. 2]. For this domain, the mirror Chern numbers are \(C_1 = -2\), and \(C_1' = 2\). Thus, there are two edge states denoted as \(b, c\) for each value of \(\eta = \pm 1\) [green curves in Fig. 7(a) and Fig. 7(b) respectively]. Here, as in the dispersion curves displayed in Fig. 6(b), the energies of the edge states do not vanish at \(k_x = 0\). Indeed, following Eq. (22), these energies satisfy the relation,
\[
\varepsilon_{\eta,b}(-k_x) = -\varepsilon_{\eta,c}(k_x),
\]
which does not constrain them to vanish at \(k_x = 0\).

The occurrence of two chiral and two anti-chiral edge states in domain 5 is somewhat expected. In this domain, \(\Delta_0 < 0\) and \(B\) is rather small. The obvious question is what happens when \(B \to 0\). Since \(\Delta_0 < 0\) guarantees the occurrence of a topological phase also for \(B = 0\) we have a TRS conserving topological insulator where the fermions have spin \(F = \frac{1}{2}\). As will be shown in a future publication, in this case for fermion of spin \(F\), there are \(F = \frac{1}{2}\) Kramers pairs of helical edge states. The two helical states in each pair carry a pseudo-spin magnetic quantum numbers \(\pm s\), and different pairs have the quantum numbers \(\pm s, \pm s, \pm s, \ldots, \pm F\). All states with positive pseudo-spin magnetic quantum numbers have the same Kramers parity (they propagate along the same direction along the edge), but still they cannot scatter each other since they are protected by the quantum number \(s\). Moreover, the energies of all helical states are antisymmetric functions of \(k_x\) (so that they vanish at \(k_x = 0\)) and obey the symmetry
\[
\varepsilon_{\eta,b}(k_x) = \varepsilon_{\eta,b}(-k_x).
\]
Moreover, near \(k_x = 0\) the dispersion relations are proportional to odd powers of \(k_x\), that is, \(\varepsilon_{\eta,b}(k_x) \propto k_x^{2|s|}\). For \(F = \frac{1}{2}\) the number of Kramers pairs is \(N = 2\). As \(B \to 0\), the chiral and anti chiral edge states shown in Fig. 7 of the TRS breaking system in region 5 of Fig. 2, fuse smoothly into the two Kramers pairs of the TRS conserving system discussed above.

**G. Summary of section IV**

The results presented in this section underline the profound distinction between the nature of topological insulators wherein the fermions have spin \(S = \frac{1}{2}\) (e.g electrons or holes encountered in solid state physics), and topological insulators wherein the fermion have spin \(S > \frac{1}{2}\) (e.g cold atoms with spin \(F = \frac{1}{2}\)). On the fundamental level, it should be mentioned that in the former case, the SO coupling is a direct consequence of the Pauli equation that is derived from the Dirac theory of the electron. In the latter case, in contrast, SO coupling is not derived.
from a relativistic equation. Rather, it is due to an interaction between a polarized laser field and the atomic spin.

Once this interaction is justified and formulated, it is evidently clear that the pattern of edge states exposed in this section is much richer than that encountered in solid state physics. In order to highlight the main features of this pattern, it is crucial to stress that it pertains to edge state physics. In order to highlight the main features of this section is much richer than that encountered in solid state physics. In order to highlight the main features of this section is much richer than that encountered in solid state physics. In order to highlight the main features of this section is much richer than that encountered in solid state physics. In order to highlight the main features of this section is much richer than that encountered in solid state physics. In order to highlight the main features of this section is much richer than that encountered in solid state physics. In order to highlight the main features of this section is much richer than that encountered in solid state physics. In order to highlight the main features of this section is much richer than that encountered in solid state physics. In order to highlight the main features of this section is much richer than that encountered in solid state physics. In order to highlight the main features of this section is much richer than that encountered in solid state physics. In order to highlight the main features of this section is much richer than that encountered in solid state physics. In order to highlight the main features of this section is much richer than that encountered in solid state physics. In order to highlight the main features of this section is much richer than that encountered in solid state physics. In order to highlight the main features of this section is much richer than that encountered in solid state physics. In order to highlight the main features of this section is much richer than that encountered in solid state physics. In order to highlight the main features of this section is much richer than that encountered in solid state physics. In order to highlight the main features of this section is much richer than that encountered in solid state physics. In order to highlight the main features of this section is much richer than that encountered in solid state physics.

1. Depending on the domain of parameters in the \((B, \Delta_0) (B > 0)\) half plane, the number of edge states varies between 0 and 4.

2. Edge states may cross the middle of the gap at \(k_x \neq 0\). The only restriction is imposed by the symmetry (22).

3. As function of \(k_x\), the edge state dispersion curves need not be monotonic. Physically, it means that the group velocity of the edge states might change sign at certain values of \(k_x\).

4. For the same value of the magnetic field, it is possible to have chiral and anti-chiral edge states propagating on the same edge.

5. The total number of edge states is equal to \(C = |C_1| + |C_2|\) and the number of anti-chiral edge states appears with negative sign.

6. Properties 2, 4 and 5 are intimately related to the question whether the edge states are even or odd eigenstates of the parity operator \(Q\) defined in Eq. (23).

7. For \(\Delta_0 < 0\) and \(B \to 0\) the chiral and anti-chiral edge states fuse smoothly into Kramers pairs of helical states. The number of Kramers pairs is \(F + \frac{1}{2}\). Two different helical states of the same Kramers parity are protected by the pseudo-spin magnetic quantum number \(s\). The dispersion relation of helical edge states near \(k_x = 0\) is proportional to an odd power \(k_x^{2s}\).

V. CONCLUSIONS

We have developed an ab-initio theoretical framework for studying the physics of topological insulators in 2D gas of fermionic atoms in which the ground-state spin is \(F > \frac{1}{2}\). These systems can be realized if SOC is relevant and if the pertinent optical potential is properly tuned to have two bands (conduction and valence), and its gap dispersion has a Mexican hat shape. It is suggested that SOC can be induced by irradiating the gas of trapped cold fermionic atoms with a specially designed arrangement of four polarized laser beams. Within this construction the SOC term has a Dresselhaus form displayed in Eq. (6). In the long wavelength approximation the bare Hamiltonian, Eq. (1) is a matrix of dimension \((2n_F) \times (2n_F)\) where \(n_F = 2F + 1\). The occurrence of mirror symmetry enables its two-block decomposition as in Eq. (10). For \(F > \frac{1}{2}\), the SOC couples conduction and valence bands in a peculiar way. Thus, although each \(n_F \times n_F\) block Hamiltonian describes a quasiparticle with (pseudo) spin \(F\), it is not possible to express it in terms of generators of the \(n_F\) dimensional irrep of SU(2). This implies that the underlying physics is qualitatively distinct from that of spin \(\frac{1}{2}\) topological insulators.

In this work we concentrated on the case \(F = \frac{3}{2}\) and assumed that the 2D fermion gas is subject to an external magnetic field. Analysis of the topological properties revealed a rich pattern of (mirror) Chern numbers that, depending on the strength of the magnetic field and the gap parameter \(\Delta_0\), can be either zero, positive or negative integers. The corresponding edge states are either chiral or anti-chiral, as explained in more details in section IV. In particular, it is possible to have chiral and anti-chiral edge states on the same edge. This is not possible for fermions with atomic spin \(F = \frac{1}{2}\).

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Appendix A: Justification of the model

In this (relatively long) appendix we address, in some details, the question of whether the model analyzed above is realizable in a system of cold atoms. First, in subsection A 1 the design of the pertinent optical potential is explained. Then, in subsection A 2, the tight-binding picture is developed ab-initio. It follows by the introduction of spin-orbit interaction in subsection A 3, and concludes in subsection A 4 that describes the long wave approximation.

1. Designing the Optical Lattice

The key question of generating SOC in cold atom systems is addressed in this subsection. The basic idea is to subject the 2D fermion gas to a specific pattern of polarized laser beams. While completing this task we have learnt of a recent papers addressing this problem1,2,8–11,13–15,20,36–41.

Consider alkali atoms interacting with a configuration of laser beams producing an amplitude modulated electric field \( \mathbf{E}(r, t) \), as displayed in Fig. 8,

\[
\mathbf{E}(r, t) = \sum_{j=1}^{4} \mathbf{E}_j(r, t) e^{i \omega t}. \tag{A1}
\]

Here

\[
\mathbf{E}_j(r, t) = \left\{ E_0 + E_1 e^{i \omega t} \right\} e^{i \mathbf{q}_j \cdot \mathbf{r}} \times \left\{ \sqrt{1-\beta^2} \mathbf{e}_z + \beta (\mathbf{e}_x \times \mathbf{e}_j) \right\}, \tag{A2}
\]

where \( \beta, E_0 \) and \( E_1 \) are real parameters. Hereafter we assume that \( 0 < \beta \ll \beta_0 = 1/\sqrt{2} \) and \( E_0 \gg E_1 > 0 \). All the lasers produce light with the same wavelength \( \lambda_0 \) and wave number \( q_0 = 2\pi/\lambda_0 \). The wave vectors \( \mathbf{q}_j \) (\( j = 1, 2, 3, 4 \)) are,

\[
q_j = \mathbf{q}_j \quad \text{where the unit vectors} \quad \mathbf{e}_j \quad \text{are}
\]

\[
e_1 = -e_x \cos \phi - e_y \sin \phi, \quad e_2 = -e_x \cos \phi + e_y \sin \phi,
\]

where \( e_x, e_y \) and \( e_z \) are unit vectors parallel to the \( x-, y- \), and \( z \)-axes, the angle \( \phi \) is close to \( \pi/4 \).

We assume that detuning of the light frequency from the resonant frequency \( \omega \), of the \( ^3S_1/2 \) and \( ^3P_3/2 \) is much smaller than fine splitting between the levels \( ^3P_3/2 \) and \( ^3P_1/2 \). Therefore hereafter we are interested just in the states \( ^3S_1/2 \) and \( ^3P_3/2 \). Corresponding wave functions are \( |g, \sigma \rangle \) and \( |e, \mu \rangle \), where \( \sigma = \uparrow, \downarrow \) and \( \mu = \pm \frac{1}{2}, \pm \frac{3}{2} \). From the other hand, the detuning of the light frequency \( \omega \) from the resonance frequency \( \omega_0 \) is large with respect to the radiative width of the excited states, therefore spontaneous emission is suppressed and we can adiabatically eliminate the excited states by writing an effective Hamiltonian which involves only the ground states15,

\[
H_{\text{eff}} = \sum_{\sigma, \sigma'} V_{\sigma \sigma'}(r, t) \mathbf{X}^{\sigma \sigma'},
\]

where \( \mathbf{X}^{\sigma \sigma'} = |g, \sigma \rangle \langle g, \sigma' | \) is Hubbard operator. The effective interaction \( V_{\sigma \sigma'}(r) \) is,

\[
V_{\sigma \sigma'}(r, t) = \sum_{\mu} \frac{1}{\hbar (\omega - \omega_0)} \langle g, \sigma | \mathbf{E}(r, t) \cdot \mathbf{d} | e, \mu \rangle \times \langle e, \mu | \mathbf{E}^* (r, t) \cdot \mathbf{d} | g, \sigma' \rangle,
\]

where \( \mathbf{E}(r, t) \) is the electric field (A1) and \( \mathbf{d} \) is the dipole moment operator of the atom.

The total electronic orbital moment of atom in the ground state is \( \mathbf{J} = \frac{1}{2} \). Then the effective interaction \( V_{\sigma \sigma'}(r, t) \) can be written in terms of an artificial magnetic field \( \mathbf{B}(r) \) coupled to the total electronic orbital momentum operator \( \mathbf{J} \)

\[
V_{\sigma \sigma'}(r, t) = \mathbf{V}(r, t) \delta_{\sigma \sigma'} + \mathbf{B}(r, t) \cdot \mathbf{J}^{\sigma \sigma'} \tag{A3}
\]

Here the scalar potential \( \mathbf{V}(r, t) \) is proportional to the local light intensity, while the vectorial field \( \mathbf{B}(r, t) \) is proportional to the local electromagnetic spin:

\[
\mathbf{V}(r, t) = -\alpha_0 \mathbf{E}^*(r, t) \cdot \mathbf{E}(r, t), \tag{A4}
\]

\[
\mathbf{B}(r, t) = -i \alpha_1 \mathbf{E}^*(r, t) \times \mathbf{E}(r, t), \tag{A5}
\]

where \( \alpha_0 \) and \( \alpha_1 \) are scalar and vector dynamical polarizabilities of the atoms.

Taking into account eqs. (A1) and (A2), we get the scalar potential (A4) in the form

\[
\mathbf{V}(r, t) = -\left\{ V_0 + V_1 \cos(\omega t) \right\} \mathbf{V}(r),
\]

\[
\mathbf{V}(r) = 16 \left\{ 1 - 2 \beta^2 \cos^2 \left( \frac{\pi x}{a_x} \right) \cos^2 \left( \frac{\pi y}{a_y} \right) + \beta^2 \sin^2 \phi \cos^2 \left( \frac{\pi x}{a_x} \right) + + \beta^2 \cos^2 \phi \cos^2 \left( \frac{\pi y}{a_y} \right) \right\}, \tag{A6}
\]
where

\[ V_0 = \alpha_0 \left( E_0^2 + E_1^2 \right), \quad V_1 = 2\alpha_0 E_0 E_1, \]

\[ a_x = \frac{\lambda_0}{2\cos \phi}, \quad a_y = \frac{\lambda_0}{2\sin \phi}. \]  

(A7)

The potential \( V(r) \) satisfies the periodic conditions,

\[ V(r, t) = V(r + a_x, t) = V(r + a_y, t), \]  

(A8)

where the lattice vectors \( a_x \) and \( a_y \) are,

\[ a_x = a_x e_x, \quad a_y = a_y e_y, \]  

(A9)

\[ a_x = |a_x| \] and \( a_y = |a_y| \) are given by eq. (A7).

The artificial magnetic field \( B \) lies in the \( x-y \) plane,

\[ B(r) = B_x(r) e_x + B_y(r) e_y. \]

Explicitly \( B_x(r) \) and \( B_y(r) \) are

\[ B_x(r) = B_0 \cos \phi \sin \left( \frac{2\pi x}{a_x} \right) \cos^2 \left( \frac{\pi y}{a_y} \right), \]  

(A10a)

\[ B_y(r) = B_0 \sin \phi \sin \left( \frac{2\pi y}{a_y} \right) \cos^2 \left( \frac{\pi x}{a_x} \right). \]  

(A10b)

where

\[ B_0 = 16 \alpha_1 \{ E_0^2 + E_1^2 \} \beta \sqrt{1 - \beta^2}. \]  

(A11)

We assume that \( \beta \ll 1 \) and \( E_1 \ll E_0 \), so that the time-dependent part of the artificial magnetic field which is proportional to \( \alpha_1 E_0 E_1 \beta \) is neglected hereafter.

The artificial magnetic field \( B(r) \) satisfies the periodic conditions similar to eq. (A8),

\[ B(r) = B(r + a_x) = B(r + a_y), \]  

(A12)

where the lattice vectors \( a_x \) and \( a_y \) are given by Eq. (A9).

Optical potential (A6) and artificial magnetic field (A10) are displayed in Fig. 9 for \( \beta = 0.1 \) and \( \phi = 7\pi/30 \). It is seen that \( V(r) \) has minima \( V_{\text{min}} = -16V_0(1 - \beta^2) \) at \( r = n \) given by

\[ n = n_x a_x + n_y a_y, \]  

(A13)

\( n_x \) and \( n_y \) are integers.

First Brillouin zone of the rectangular lattice is given by the conditions,

\[ |k_x| \leq \frac{Q_x}{2}, \quad |k_y| \leq \frac{Q_y}{2}, \]

where

\[ Q_x = \frac{2\pi}{a_x}, \quad Q_y = \frac{2\pi}{a_y}. \]

(A14)

When \( W_0 \gg \mathcal{E}_Q \), the atoms are localized near the places of stable equilibrium and just hope from place to place. In this case, we can describe the gas in framework of the tight binding approximation.

In order to construct a tight binding Hamiltonian, we derive wave functions and energy levels of atom localized near a minimum position of the optical potential (A6). Then we estimate tunneling rates.

2. Tight Binding Approximation

The model considered here has two parameters of energy, the depth \( W_0 \) of the potential wells and the recoiling energy \( \mathcal{E}_Q \),

\[ W_0 = -16V_0(1 - \beta^2), \]  

(A15)

\[ \mathcal{E}_Q = \frac{\hbar^2(Q_x^2 + Q_y^2)}{2M}. \]

When \( W_0 \gg \mathcal{E}_Q \), the atoms are localized near the places of stable equilibrium and just hope from place to place. In this case, we can describe the gas in framework of the tight binding approximation.

In order to construct a tight binding Hamiltonian, we derive wave functions and energy levels of atom localized near a minimum position of the optical potential (A6). Then we estimate tunneling rates.

a. Quantum States of Localized Atoms

When \( 0 < \beta < \min(\beta_{c,x}, \beta_{c,y}) \) (which is assumed hereafter), the optical potential (A6) has minima at \( r = n \),

\[ V(r) = V_0 - 2V_0 \beta \cos \phi \left( \frac{x}{a_x} \right) \cos \left( \frac{\pi y}{a_y} \right), \]

(A16)
Energy levels \( A_21 \) for \( r \) close to one of the minimum points [say, \( r = (0, 0) \)], \( V(r) \) can be approximated as,
\[
V(r) = -W_0 + \frac{1}{2} \{ K_x x^2 + K_y y^2 \} + O(r^4). \tag{A16}
\]

Here
\[
K_x = K_0 \cos^2 \phi \left( 1 - \frac{3 \beta^2}{2} - \frac{\beta^2}{2} \cos (2\phi) \right), \tag{A17}
\]
\[
K_y = K_0 \sin^2 \phi \left( 1 - \frac{3 \beta^2}{2} + \frac{\beta^2}{2} \cos (2\phi) \right), \tag{A18}
\]
where
\[
K_0 = 16 V_{00}^2.
\]

Then the wave functions of atom localized near the point \( r = (0, 0) \) are,
\[
\Phi^{(2D)}_{\nu_x,\nu_y}(r) = \Phi^{(x)}_{\nu_x}(x) \Phi^{(y)}_{\nu_y}(y), \tag{A19}
\]
where
\[
\Phi^{(x)}_{\nu_x}(x) = \frac{N_{\nu_x} \sqrt{2^{\nu_x} \nu_x!}}{\sqrt{\pi} (K_0 M)^{1/4}} \exp(-\xi_x^2/2). \tag{A20}
\]
Here \( \nu_x \) is a Cartesian index, \( \nu_x = 0, 1, 2, \ldots \),
\[
N_{\nu_x} = \left( \frac{K_0 M}{\pi^2 \hbar^2} \right)^{1/8}, \quad \xi_x = x/\left( \frac{K_0 M \hbar^2}{h^2} \right)^{1/4}.
\]

Corresponding energy levels (measured from \( -W_0 \)) are,
\[
E_{\nu_x,\nu_y} = E_{x,\nu_x} + E_{y,\nu_y},
\]
\[
E_{\nu_x,\nu_y} = \hbar \sqrt{K_y M} \left( \frac{K_0 M}{h^2} \right)^{1/4}. \tag{A21}
\]

Energy levels (A21) are displayed in Fig. 10 for \( \beta = 0.1 \) and \( \phi = 7\pi/30 \).

![FIG. 10: (color online) Energy levels (A21) for \( \beta = 0.1 \) and \( \phi = 7\pi/30 \).]

Energy levels (A21) are displayed in Fig. 10 for \( \beta = 0.1 \) and \( \phi = 7\pi/30 \). We are interested in the close lying energy levels \( E_{1,0} \) and \( E_{0,1} \). Indeed, the difference \( E_{1,0} - E_{0,1} \) is,
\[
E_{1,0} - E_{0,1} = \frac{\hbar}{\sqrt{M}} \sqrt{K_x - K_y}. \tag{A22}
\]

When \( \phi = \pi/4 \), then
\[
K_x^{(0)} = K_y^{(0)} = \frac{K_0}{2} \left\{ 1 - \frac{3 \beta^2}{2} \right\},
\]

and
\[
E_{1,0}^{(0)} = E_{0,1}^{(0)} = 2 \hbar \sqrt{K_0/2M} \left\{ 1 - \frac{3 \beta^2}{2} \right\}. \tag{A23}
\]

For \( \phi = \frac{7\pi}{4} - \delta \) with \( |\delta| \ll 1 \), we can write,
\[
K_x = \frac{K_0}{2} \left\{ 1 - \frac{3 \beta^2}{2} \right\} + K_0 \delta \left\{ 1 - 2 \beta^2 \right\},
\]
\[
K_y = \frac{K_0}{2} \left\{ 1 - \frac{3 \beta^2}{2} \right\} - K_0 \delta \left\{ 1 - 2 \beta^2 \right\}. \tag{A24}
\]

Therefore \( E_{1,0} \) and \( E_{0,1} \) are
\[
E_{1,0} = E_{1,0}^{(0)} - \varepsilon_0, \quad E_{0,1} = E_{1,0}^{(0)} + \varepsilon_0,
\]
where
\[
\varepsilon_0 = \frac{\hbar \sqrt{K_0}}{\sqrt{M}} \frac{\delta(1 - 2 \beta^2)}{\sqrt{2 - 3 \beta^2}}. \tag{A25}
\]

b. Energy dispersion beyond the harmonic approximation

Let us consider bands with high band number \( \nu_x \) and \( \nu_y = 0 \) or 1, such that the harmonic approximation is still good for description of the motion of the atom in the \( y \)-direction, and falls for description of the motion in the \( x \)-direction. The optical potential can be approximated as,
\[
V(r) \approx V_{1D}(x) + V_{harm}(y), \tag{A26}
\]
where
\[
V_{1D}(x) = -16 V_0 \left\{ [1 - \beta^2 (1 + \cos^2 \phi)] \times \cos^2 \left( \frac{\pi x}{a_x} \right) + \beta^2 \cos^2 \phi \right\}, \tag{A27}
\]
\[
V_{harm}(y) = \frac{K_0 y^2}{a_y^2}. \tag{A28}
\]

We assume here that \( V_1 = 0 \). The effect of \( V_1 \) on the energy dispersion is calculated below.

The 1D potential \( V_{1D}(x) \) is periodic with period \( a_x \), therefore the motion in the \( x \)-direction is described by
two quantum numbers, the band number $\nu_x$ and the wave number $k_x$ belonging to the first Brillouin zone,

$$|k_x| < \frac{\pi}{a_x}.$$  

Then the wave function is a product of the two wave functions, depending on $x$ and $y$,

$$\Psi^{(2D)}_{\nu_x,\nu_y, k_x}(r) = \Psi_{\nu_x, k_x}(x) \Psi^{(y)}_{\nu_y}(y).$$

The wave function $\Psi^{(y)}_{\nu_y}(y)$ ($\nu_y = 0, 1$) is given by Eq. (A20), whereas the wave function $\Psi_{\nu_x}(x)$ are found from the equation,

$$-\frac{\hbar^2}{2M} \Psi''_{\nu_x, k_x}(x) + V_{1D}(x) \Psi_{\nu_x, k_x}(x) = \mathcal{E}_{x,\nu_x, k_x} \Psi_{\nu_x, k_x}(x).$$  \hspace{1cm} (A26)

![FIG. 11: (color online) Energy levels calculated numerically from Eq. (A26) for $\nu_0 = 0.2 \varepsilon_0$, $\beta = 0.1$ and $\phi = 7\pi/30$. The solid blue and red lines correspond to even and odd $\nu_x$. The dashed blue and red lines describe $\mathcal{E}_{x,0} + \hbar\omega_1$ and $\mathcal{E}_{x,1} + \hbar\omega_1$.](image)

Energy dispersion calculated numerically from Eq. (A26) is shown in Fig. 11 for $\nu_0 = 0.2 \varepsilon_0$, $\beta = 0.1$ and $\phi = 7\pi/30$. It is seen that $\mathcal{E}_{x,\nu_x, k_x}$ with $\nu_x = 0, 1$ does not depend on $k_x$ and can be approximated by Eq. (A21).

In similar way, we can consider motion of the atoms with $\nu_x = 0, 1$ and any $\nu_y$ and find dispersion $\mathcal{E}_{y,\nu_y, k_y}$. The Schrödinger equation for $\mathcal{E}_{y,\nu_y, k_y}$ can be obtained from Eq. (A26) by replacing $V_{1D}(x)$ by $V_{1D}(y)$ and $V_{\text{harm}}(y)$ by $V_{\text{harm}}(x)$, where

$$V_{1D}(y) = -16V_0 \left[ 1 - \beta^2 \left( 1 + \sin^2 \phi \right) \right] \times \cos^2 \left( \frac{\pi y}{a_y} \right) + \beta^2 \sin^2 \phi,$$

$$V_{\text{harm}}(x) = \frac{K_x x^2}{a_x^2}.$$  

A 2D energy $\varepsilon^{(0)}_{\nu_x, k_x}$ of the trapped atoms can be written as,

$$\varepsilon^{(0)}_{\nu_x, k_x} = \varepsilon^{(0)}_{x,\nu_x, k_x} + \varepsilon^{(0)}_{y,\nu_y, k_y}.\quad (A27)$$

c. Tunneling due to the time dependent potential

We now calculate corrections from the time-dependent potential on the energy dispersion relation $\varepsilon^{(0)}_{\nu, k}$ with $\nu = (0, 1)$ and $(1, 0)$. We assume here that $|V_1|$ is very small with respect to the excitation energies $|\varepsilon^{(0)}_{\nu', k} - \varepsilon^{(0)}_{\nu, k}|$, where $\varepsilon^{(0)}_{\nu', k}$ are excited energy levels. In this case the quantum transitions between the energy levels are forbidden. In order to calculate corrections to the energies due to elastic scattering of the light, we apply time-dependent perturbation theory. Within this theory, the wave functions of the state $\nu = (\nu_x, \nu_y)$ are,

$$\Psi_{\nu, k}(r, t) = \sum_n A_{\nu', k}^{(n)}(t) \Phi^{(2D)}_{\nu', k}(r, t),$$  \hspace{1cm} (A28)

where

$$\Phi^{(2D)}_{\nu', k}(r, t) = \Phi^{(2D)}_{\nu'}(r) e^{i k r - \frac{\varepsilon^{(0)}_{\nu'} k}{\hbar^2}}.$$  

and $\Phi^{(2D)}_{\nu'}(r)$ is given by Eq. (A19).

The functions $A_{\nu', k}(t)$ satisfy the equation,

$$i \hbar \dot{A}_{\nu', k}(t) = V_1 \cos(\omega_1 t) \sum_{\nu''} V_{\nu', \nu''} A_{\nu'', k}(t) \times \exp \left[ \frac{i t}{\hbar} \left( \varepsilon^{(0)}_{\nu', k} - \varepsilon^{(0)}_{\nu''} \right) \right].$$  \hspace{1cm} (A29)

where $\nu = (\nu_x, \nu_y)$, $k = (k_x, k_y)$,

$$V_{\nu', \nu''} = \int \Phi^{(2D)}_{\nu'}(r) \Phi^{(2D)}_{\nu''}(r) d^2 r.$$  \hspace{1cm} (A30)

In the zeroth approximation, the coefficients $A^{(0)}_{\nu', k}$ are,

$$A^{(0)}_{\nu', k} = \delta_{\nu', \nu'},$$

where $\delta_{\nu', \nu''}$ is the Kronecker delta. Equation for the coefficients $A^{(1)}_{\nu', k}$ in the first approximation can be get from Eq. (A29) by substituting in the right hand side $A^{(0)}_{\nu', k}$ instead of $A_{\nu', k}(t)$,

$$i \hbar \dot{A}_{\nu', k}(t) = V_1 \cos(\omega_1 t) V_{\nu', \nu''} e^{-i \Omega_{\nu', k} t},$$  \hspace{1cm} (A31)

where the resonant frequencies are

$$\Omega_{\nu', k} = \frac{1}{\hbar} \left( \varepsilon^{(0)}_{\nu', k} - \varepsilon^{(0)}_{\nu'} \right).$$

Solving eq. (A31),

$$A^{(1)}_{\nu', k}(t) = \frac{V_1 \nu', \nu'' e^{-i (\Omega_{\nu', k} + \omega_1) t}}{2 \left( \varepsilon^{(0)}_{\nu', k} - \varepsilon^{(0)}_{\nu'} + \hbar \omega_1 \right)} + \frac{V_1 \nu', \nu'' e^{-i (\Omega_{\nu', k} - \omega_1) t}}{2 \left( \varepsilon^{(0)}_{\nu', k} - \varepsilon^{(0)}_{\nu'} - \hbar \omega_1 \right)}.$$  \hspace{1cm} (A32)
The energy dispersion relation of the trapped atoms calculated in the second order perturbation theory with $V_1$ is

$$\varepsilon_{\nu,k} = \varepsilon^{(0)}_{\nu,k} + \frac{V_1^2}{2\hbar} \sum_{\nu' \neq \nu} \frac{\Omega_{\nu',\nu,k}^2 |V_{\nu',\nu}|^2}{\Omega_{\nu',\nu,k}^2 - \omega_n^2}. \quad (A33)$$

We are interested here in the energy bands $\nu = (0,1)$ and $(1,0)$. We perform the numerical calculations for $V_0 = 0.2E_0$, $V_1 = 0.028\hbar E_0$ and $h\omega_1 = 1.79\hbar E_0$. Non-perturbed energies $\varepsilon^{(0)}_{\alpha,\nu,k_a} [\alpha = x,y]$ (A27) are shown in Fig. 11, solid curves. The dashed curves show $\varepsilon^{(0)}_{\alpha,\nu,k_a} + h\omega_1$. It is clear that when the atoms are strongly localized at the bottom of the wells of the optical lattice potential, the probability to jump into a next adjacent site is negligible. This means that a tight-binding description is a good approximation. By the other worlds, the energies $\varepsilon^{(0)}_{\alpha,\nu,k_a}$ can be fitted by a simple equation as,

$$\varepsilon^{(0)}_{\alpha,\nu,k_a} \approx \varepsilon^{(0)}_{\nu,k_a} + \Lambda_{\nu,k_a} \cos(k_a a_0), \quad (A35)$$

and find the parameters $\varepsilon^{(0)}_{\nu,k_a}$ and $\Lambda_{\nu,k_a}$ from the condition of minimal standard deviation,

$$\sum_k \left( \varepsilon^{(\text{num})}_{\alpha,\nu,k_a} - \varepsilon^{(\text{fit})}_{\alpha,\nu,k_a} \right)^2,$$

where $\varepsilon^{(\text{num})}_{\alpha,\nu,k_a}$ are the energies calculated numerically from Eq. (A34), and $\varepsilon^{(\text{fit})}_{\alpha,\nu,k_a}$ are calculated from Eq. (A35). The fitting parameters calculated numerically are

$$\varepsilon_0 = -2.6315 E_0, \quad \Lambda_0 = -0.1061 E_0, $$
$$\varepsilon_1 = -0.8758 E_0, \quad \Lambda_1 = 0.1056 E_0.$$

![FIG. 12: (color online) Energy levels calculated numerically from Eq. (A26) for $V_0 = 0.2E_0$, $V_1 = 0.028E_0$, $\beta = 0.1$ and $\phi = \pi/30$. The solid blue and red lines correspond to even and odd $\nu_x$.](image)

Tight binding energies $\varepsilon_{0,\nu,k_a}$ and $\varepsilon_{1,\nu,k_a}$ (A35) are shown in Fig. 12, solid curves. The solid and dotted curves are close one to another, so that the approximation (A35) is good.

From the other side, the dispersion (A35) can be obtained from the tight-binding model, where $\Lambda_0$ and $\Lambda_1$ are the tunneling rates of the atoms at the level $\nu = 0$ and $1$. Taking into account that $\Lambda_0 \approx \Lambda_1$, we conclude that the tight-binding model is described by a single parameter, the tunneling rate $\Lambda \approx \Lambda_0 \approx \Lambda_1$.

The wave functions $\Phi_{1,0}^{(2D)}(r)$ and $\Phi_{0,1}^{(2D)}(r)$ satisfy the following properties,

$$\Phi_{1,0}^{(2D)}(-x,y) = -\Phi_{1,0}^{(2D)}(x,y), \quad (A36a)$$
$$\Phi_{0,1}^{(2D)}(-x,y) = \Phi_{0,1}^{(2D)}(x,y), \quad (A36b)$$
$$\Phi_{1,0}^{(2D)}(x,-y) = \Phi_{1,0}^{(2D)}(x,y), \quad (A36c)$$
$$\Phi_{0,1}^{(2D)}(x,-y) = -\Phi_{0,1}^{(2D)}(x,y). \quad (A36d)$$

When an atom tunnels from the site $n$ to the site $n + a_x$, the parity with respect to $y \rightarrow -y$ inversion
is a good quantum number. Similarly, when an atom tunnels from the site \( n \) to the site \( n + a_y \), the parity with respect to \( x \to -x \) inversion is a good quantum number. Because of this, tunneling of atoms from the quantum state \( \Phi_{1,0}^{(2D)}(r-n) \) in the site \( n \) to the quantum state \( \Phi_{0,1}^{(2D)}(r-n-a_\alpha) \) is forbidden. Similarly, tunneling of atoms from the quantum state \( \Phi_{1,0}^{(2D)}(r-n) \) to the quantum state \( \Phi_{1,0}^{(2D)}(r-n-a_\alpha) \) is forbidden.

d. Tight Binding Hamiltonian

Summarising the results of subsections A 2a, A 2b and A 2c, we are ready to construct a tight binding Hamiltonian \( H_0 \). Let \( \psi_{\xi,f}(n) \) and \( \psi^\dagger_{\xi,f}(n) \) be annihilation and creation operators of atom in the quantum state \( \Phi_{1,0}^{(2D)}(r-n) \) in the site \( n \) with magnetic quantum number \( f \). Similarly, \( \psi_{\nu,f}(n) \) and \( \psi^\dagger_{\nu,f}(n) \) are annihilation and creation operators of atom in the quantum state \( \Phi_{0,1}^{(2D)}(r-n) \). Here the index \( c \) and \( v \) means “conduction” and “valence” band. Then the tight binding Hamiltonian is

\[
H_0 = \varepsilon_0 \sum_{f,n} \left\{ \psi^\dagger_{c,f}(n) \psi_{c,f}(n) - \psi^\dagger_{\nu,f}(n) \psi_{\nu,f}(n) \right\} + \\
+ \sum_{f,\xi} \sum_{n,n'} \Lambda_\xi(a) \psi^\dagger_{\xi,f}(n) \psi_{\xi,f}(n'). 
\]

(A37)

Here \( \langle n,n' \rangle \) denotes neighboring sites, \( a = n' - n \) and

\[
\Lambda_\nu(\pm a_x) = \Lambda, \quad \Lambda_\nu(\pm a_y) = -\Lambda, \\
\Lambda_c(\pm a_x) = -\Lambda, \quad \Lambda_c(\pm a_y) = \Lambda.
\]

\( \varepsilon_0 \) is given by eq. (A22). We chose the Fermi energy to be zero.

In order to diagonalize \( H_0 \), we apply Fourier transformations,

\[
\psi_{\xi,f}(n) = \frac{1}{\sqrt{N}} \sum_k c_{\xi,k,f} e^{ikn - i\pi n y}, 
\]

(A38)

where \( N \) is the number of sites of the optical lattice. Then \( H_0 \) takes the form,

\[
H_0 = \sum_{k,f} \varepsilon_k \left\{ c^\dagger_{c,k,f} c_{c,k,f} - c^\dagger_{\nu,k,f} c_{\nu,k,f} \right\},
\]

(A39)

where

\[
\varepsilon_k = \varepsilon_0 - 2\Lambda \left\{ \cos(k_x a_x) + \cos(k_y a_y) \right\}. 
\]

(A40)

It should be noted the following: The lowest energy of the “conduction” band is \( \Delta_0 \), whereas the highest energy of the “valence” band is \(-\Delta_0\), where

\[
\Delta_0 = \varepsilon_0 - 4\Lambda.
\]

(A41)

Overlapping of the conduction and valence bands depends either \( \Delta_0 \) is positive or negative.

- When \( \Delta_0 \) is positive, there is a gap \( 2\Delta_0 \) separating the conduction and valence bands and the optical lattice is an insulator. In this case we need to apply external magnetic field to get topological edge states.

- When \( \Delta_0 \) is negative, there is overlapping of the conduction and valence bands. Spin-orbital interaction (considered in subsection A 3 below) opens a gap in the bulk of the optical lattice and keep the edge states gapless. In this case we get edge states even without magnetic field.

![FIG. 13: (color online) Energy spectrum (A40) for \( \Delta_0 = 0.2\varepsilon_0 \) [panel (a)] and \( \Delta_0 = 0.3\varepsilon_0 \) [panel (b)]. The red and blue filled areas denote the conduction and valence bands, the yellow filled area in the panel (b) denotes overlapping of the bands.](image-url)

The cases \( \Delta_0 > 0 \) and \( \Delta_0 < 0 \) are illustrated in Fig. 13.

3. Spin-Orbit Interaction

Spin orbit interaction Hamiltonian is

\[
H_{SO} = \frac{1}{2} \sum_{\xi:\xi',f,f'} \sum_{(n,n')} S_{\xi,\xi';f,f'}(a) \times \\
\psi^\dagger_{\xi',f'}(n') \psi_{\xi',f'}(n'), 
\]

(A42)

where \( \langle n,n' \rangle \) denotes neighboring sites, \( a = n' - n \),

\[
S_{\xi,\xi';f,f'}(a) = \frac{1}{2I+1} \int d^2 r \Phi_{\xi}(r) \Phi_{\xi'}(r-a) \times \\
\left( B(r) \cdot F_{f,f'} \right),
\]

(A43)

where \( B(r) \) is given by eq. (A10). Here we use the expression

\[
\sum_{f,f'} |f\rangle \langle f| \hat{J}^x \langle f' | = \frac{\hat{F}}{2I+1},
\]

which is true for the case \( F = I \pm \frac{1}{2} \), where \( J = \frac{1}{2} \) is the electronic orbital moment and \( J \) is the nuclear spin. Here we use the notations \( \Phi_c(r) \) and \( \Phi_v(r) \) for \( \Phi_{1,0}^{(2D)}(r) \) and \( \Phi_{0,1}^{(2D)}(r) \).
Let us consider in details tunneling of atoms between the sites \( n = 0 \) and \( n' = a_x \). The matrix element (A43) can be written as

\[
S_{c,\xi',f,f'}(a_x) = S_{\xi,\xi'}^c(a_x) F_{f,f'} + S_{\xi,\xi'}^v(a_x) F_{f,f'},
\]

where

\[
S_{\xi,\xi'}^c(a_x) = \frac{1}{2I + 1} \int d^2r \Phi_\xi(r) \Phi_\xi'(r - a_x) B^c(r),
\]

\[
S_{\xi,\xi'}^v(a_x) = \frac{1}{2I + 1} \int d^2r \Phi_\xi(r) \Phi_\xi'(r - a_x) B^v(r).
\]

The integration is restricted by the interval

\[ |y| \lesssim y_0, \quad x_0 < x < a_x - x_0, \]

where \( x_0 \) and \( y_0 \) are classical turning points for atom at the energy level \( \xi_1,0 \) or \( \xi_0,1 \). Consider \( S_{\xi,\xi'}^c(a_x) \). It is important that \( B_{\xi}(r) \) is even with respect to the inversion \( y \to -y \) and odd with respect to the glide inversion \( x \to a_x - x \). The functions \( \Phi_\xi(r) \Phi_\xi'(r - a_x) \) and \( \Phi_\xi(r) \Phi_\xi'(r - a_x) \) are odd with respect to the \( y \to -y \) inversion. Therefore

\[
S_{c,c}^c(a_x) = S_{c,c}^v(a_x) = 0.
\]

Similarly, we can derive \( S_{c,v}^v(a_x) \) and for the tunneling between the sites \( n = 0 \) and \( n' = a_y \), only \( S_{c,c}^c(a_y) \) and \( S_{c,v}^v(a_y) \) are nontrivial,

\[
S_{c,v}^v(a_y) = \frac{B_0 \cos \phi}{2I + 1} \int_{-x_0}^{x_0} \Phi_0(x) \Phi_1(x) \cos \left( \frac{\pi y}{a_y} \right) \cos \left( \frac{\pi x}{a_x} \right) dy.
\]

Taking into account that \( \Phi_1(y) \) is odd with respect to the inversion \( y \to -y \), we can see that

\[
S_{c,v}^c(a_y) = -S_{c,v}^c(a_y).
\]

When the angle \( \phi \) is close to \( \pi/4 \), then

\[
S_{c,v}^c(a_y) \approx S_{c,v}^v(a_x) = \lambda_{SO}.
\]

Here

\[
\lambda_{SO} \approx \frac{\pi^3 B_0}{2\sqrt{2} (2I + 1)} \left( \frac{2\pi^2 \hbar^2}{K_0 M (2 - 3\beta^2)} \right)^{1/4} \exp \left( -\frac{a_0^2}{4x_0^2} \right),
\]

where \( a_x \approx a_y \approx a_0 \).
Applying Fourier transformations (A38), we can write $H_{\text{SO}}$ in the form,

$$H_{\text{SO}} = \sum_{k,f,f'} (\lambda_k \cdot F_{f,f'}) \left\{ \hat{c}_c^\dagger_{k,f} c_{v,k,f'} + \text{h.c.} \right\},$$

where

$$\lambda_k = 2\lambda_{\text{SO}} \left\{ \sin(k_y a_y) \mathbf{e}_x - \sin(k_x a_x) \mathbf{e}_y \right\}.$$  

Finally, we can make the unitary transformations,

$$c_{v,k,f} \rightarrow i c_{v,k,f},$$

and applying proper unitary transformations of the matrices $\tilde{F}_{x,y,z}$, we get a standard form for the Dresselhaus spin-orbit interaction,

$$H_{\text{SO}} = \sum_{k,f,f'} (\lambda_k \cdot F_{f,f'}) \left\{ c_{c,k,f}^\dagger c_{v,k,f'} + \text{h.c.} \right\},$$

where

$$\lambda_k = 2\lambda_{\text{SO}} \left\{ \sin(k_x a_x) \mathbf{e}_x + \sin(k_y a_y) \mathbf{e}_y \right\}.$$  

### 4. Long Wave Approximation

Let us consider quantum states with $ka_0 \ll 1$ (hereafter we assume that $a_x \approx a_y \approx a_0$ and use $a_0$ instead of $a_x$ and $a_y$) and apply the long wave approximation. In this approximation, we can write

$$\varepsilon_k \approx \Delta_0 + \frac{\hbar^2 k^2}{2M_0}, \quad \lambda_k \approx \hbar v (k \cdot \mathbf{S}),$$

where $\Delta_0$ is given by eq. (A41),

$$M_0 = \frac{\hbar^2}{2\lambda_0 \varepsilon_0}, \quad v = \frac{2}{\hbar} \lambda_{\text{SO}} \varepsilon_0.$$  

Then the Hamiltonian $H = H_0 + H_{\text{SO}}$ of the atoms in the optical lattice is written as,

$$H_0 = \sum_{\xi,k,s} \xi \left( \Delta_0 + \frac{\hbar^2 k^2}{2M_0} \right) c_{\xi,k,s}^\dagger c_{\xi,k,s},$$

$$H_{\text{SO}} = \hbar v \sum_{\xi,k,s,s'} (k \cdot \mathbf{S}_{s,s'}) c_{\xi,k,s}^\dagger c_{\xi,k,s'},$$

where $\xi = c,v$ or $\xi = \pm 1$ for the “conduction” or “valence” bands.

### Appendix B: Numerical Calculations

Here we apply numerical calculations to solve eqs. (42) and (48) for $\Delta_0 = 0$, $B = 2\varepsilon_0$ and $k_x = 0.1k_0$. We consider the cases $\eta = \pm 1$ in turn.

1. **Numerical Calculations for $\eta = 1$**

Substituting $\Delta_0 = 0$, $B = 2\varepsilon_0$ and $k_x = 0.1k_0$ to eq. (42) and solving it, we get $\kappa_1$, $\kappa_2$, $\kappa_3$ and $\kappa_4$ as functions of $\varepsilon_1(k_x)$. We substitute $\kappa_n$ ($n = 1, 2, 3, 4$) into eq. (45) and get the matrices which depends just on energy $\varepsilon_1(k_x)$. Then with eqs. (44), and (48), we get an equation for $\varepsilon_1(k_x)$. The function $\mathcal{F}_1(\varepsilon_1)$ is displayed in Fig. 14.

![FIG. 14: (color online) Function $\mathcal{F}_1(\varepsilon_1)$, eq. (48), for $\Delta_0 = 0$, $B = 2\varepsilon_0$, $k_x = 0.1k_0$ and $\eta = 1$. $\varepsilon_0$ and $k_0$ are given by eq. (13).](image1)

Fig. 14. Here $\varepsilon_1(k_x)$ is restricted by the condition,

$$|\varepsilon_1(k_x)| < \varepsilon_{c,\frac{1}{2},1}(k_x) = 1.00016 \varepsilon_0.$$  

Solving eq. (48) numerically, we get

$$\varepsilon_1 = -0.213735 \varepsilon_0.$$  

2. **Numerical Calculations for $\eta = -1$**

Consider now topological edge states for $\Delta_0 = 0$, $B = 2\varepsilon_0$ and $k_x = 0.1k_0$ and $\eta = -1$. Solving eq. (42), we get $\kappa$’s as functions of $\varepsilon_1(k_x)$. It should be noted that $\kappa_1$ and $\kappa_2$ are complex.

![FIG. 15: (color online) Function $\mathcal{F}_1(\varepsilon_1)$, eq. (48), for $\Delta_0 = 0$, $B = 2\varepsilon_0$, $k_x = 0.1k_0$ and $\eta = -1$. $\varepsilon_0$ and $k_0$ are given by eq. (13).](image2)

We substitute $\kappa_n$ ($n = 1, 2, 3, 4$) into eq. (45) and get the matrices which depends just on energy $\varepsilon_1(k_x)$. Then with eqs. (44), (48) and (48), we get an equation for $\varepsilon_1(k_x)$. The function $\mathcal{F}_1(\varepsilon_1)$ is displayed in Fig. 15. Here $\varepsilon_1(k_x)$ is restricted by the condition,

$$|\varepsilon_1(k_x)| < \varepsilon_{c,\frac{1}{2},1}(k_x) = 1.00982 \varepsilon_0.$$
Solving eq. (48) numerically, we get
\[ \varepsilon_1 = -0.131167e_6. \]

**Appendix C: Minima and Maxima Points of \( V(r) \)**

We consider properties of the optical potential (A6) for different values of \( \beta \) within the interval
\[ 0 < \beta < 1. \]
The potential \( V(r) \) satisfies the periodic conditions (A8).
When an atom is placed in external potential \( V(r) \), the force \( \mathbf{F}(r) \) acting on the atom is,
\[ \mathbf{F}(r) = -\nabla V(r). \] (C1)
At equilibrium points, the force acting at the atom vanishes. Therefore we have a set of equations,
\[
\sin\left(\frac{2\pi x}{a_x}\right) \left\{ 1 - \beta^2 \left[ 1 + \cos\left(2\phi\right) \right] + \left[ 1 - 2\beta^2 \right] \cos\left(\frac{2\pi y}{a_y}\right) \right\} = 0, \]
\[
\sin\left(\frac{2\pi y}{a_y}\right) \left\{ 1 - \beta^2 \left[ 1 - \cos\left(2\phi\right) \right] + \left[ 1 - 2\beta^2 \right] \cos\left(\frac{2\pi x}{a_x}\right) \right\} = 0, \] (C2a, b)
where \( a_x \) and \( a_y \) are given by eq. (A7).
Note that there are values \( \beta_{c,x} \) and \( \beta_{c,y} \) of \( \beta \)
\[
\beta_{c,x} = \sqrt{\frac{2}{3 + \cos(2\phi)}}, \] (C3a)
\[
\beta_{c,y} = \sqrt{\frac{2}{3 - \cos(2\phi)}}, \] (C3b)
such that for \( \beta < \beta_{c,x} \), the equation
\[
1 - \beta^2 \left[ 1 + \cos\left(2\phi\right) \right] + \left[ 1 - 2\beta^2 \right] \cos\left(\frac{2\pi x}{a_x}\right) = 0, \] (C4a)
has no solutions. Similarly, for \( \beta < \beta_{c,y} \), the equation
\[
1 - \beta^2 \left[ 1 - \cos\left(2\phi\right) \right] + \left[ 1 - 2\beta^2 \right] \cos\left(\frac{2\pi y}{a_y}\right) = 0, \] (C4b)
has no solutions. In this case the equilibrium positions are given by the equations
\[
\sin\left(\frac{2\pi x}{a_x}\right) = 0, \] (C5a)
\[
\sin\left(\frac{2\pi y}{a_y}\right) = 0. \] (C5b)
For \( \beta > \beta_{c,x} \) and \( \beta > \beta_{c,y} \), eqs. (C4) and (C5) have nontrivial solutions.
In what following, we assume that \( 0 < \phi < \pi/4 \), and therefore
\[
\frac{1}{\sqrt{2}} < \beta_{c,x} \leq \beta_{c,y} < 1. \]
In order to answer the question is the equilibrium point \( r_0 = (x_0, y_0) \) stable, unstable or saddle, we investigate the following matrix,
\[
\mathcal{M}(r_0) = \frac{1}{16V_0q_0^2} \begin{pmatrix} V_{x,x}(r_0) & V_{x,y}(r_0) \\ V_{y,x}(r_0) & V_{y,y}(r_0) \end{pmatrix}, \] (C6)
where
\[
V_{x,x}(r_0) = \frac{\partial^2 V(r_0)}{\partial x_0^2}, \quad V_{x,y}(r_0) = \frac{\partial^2 V(r_0)}{\partial x_0 \partial y_0}, \quad V_{y,x}(r_0) = \frac{\partial^2 V(r_0)}{\partial y_0^2}. \]
Explicitly, they are
\[
V_{x,x}(r) = \frac{1}{16V_0q_0^2} \cos\left(\frac{2\pi x}{a_x}\right) \left\{ 1 - \beta^2 \left[ 1 + \cos(2\phi) \right] + \left[ 1 - 2\beta^2 \right] \cos\left(\frac{2\pi y}{a_y}\right) \right\} \cos^2\phi, \]
\[
V_{y,y}(r) = \frac{1}{16V_0q_0^2} \cos\left(\frac{2\pi y}{a_y}\right) \left\{ 1 - \beta^2 \left[ 1 - \cos(2\phi) \right] + \left[ 1 - 2\beta^2 \right] \cos\left(\frac{2\pi x}{a_x}\right) \right\} \sin^2\phi, \]
\[
V_{x,y}(r) = -\left[ 1 - 2\beta^2 \right] \sin\left(\frac{2\pi x}{a_x}\right) \sin\left(\frac{2\pi y}{a_y}\right) \times \cos\phi \sin\phi. \]
There are two eigenvalues of \( \mathcal{M}(r_0) \),
\[
\mathcal{M}_{\pm}(r_0) = \mathcal{A}(r_0) \pm \sqrt{\mathcal{B}(r_0) + \mathcal{C}(r_0)}, \] (C7)
where
\[
\mathcal{A}(r_0) = \frac{V_{x,x}(r_0) + V_{y,y}(r_0)}{32V_0q_0^2}, \]
\[
\mathcal{B}(r_0) = \frac{V_{x,x}(r_0) - V_{y,y}(r_0)}{32V_0q_0^2}, \]
\[
\mathcal{C}(r_0) = \frac{V_{x,y}(r_0)}{16V_0q_0^2}. \]
There are three cases,
- When \( \mathcal{M}_{+}(r_0) > 0 \) and \( \mathcal{M}_{-}(r_0) > 0 \), the equilibrium point \( r_0 \) is stable.
- When \( \mathcal{M}_{+}(r_0) < 0 \) and \( \mathcal{M}_{-}(r_0) < 0 \), the equilibrium point \( r_0 \) is unstable.
- When \( \mathcal{M}_{+}(r_0) > 0 \) and \( \mathcal{M}_{-}(r_0) < 0 \), the equilibrium point \( r_0 \) is saddle.

We investigate the equilibrium points for two intervals, \( \beta < \beta_{c,x} \) and \( \beta > \beta_{c,y} \), in turn [where \( \beta_{c,x} \) and \( \beta_{c,y} \) are given by eq. (C3)].
1. The Case $0 < \beta < \beta_{c,x}$

When $\beta < \beta_{c,x}$, the equilibrium points are given by eq. (C5). Explicitly, they are

$$\alpha_{nx,ny} = \frac{n_x a_x}{2} + \frac{n_y a_y}{2},$$

(C8)

where $a_x$ and $a_y$ are given by eq. (A9), $n_x$ and $n_y$ are integers.

The sign of $\mathcal{M}_+(\alpha_{nx,ny})$, eq. (C7), depend either $n_x$ and $n_y$ are even or odd. There are four cases:

- When $n_x$ and $n_y$ are even,
  $$\alpha_{nx,ny} = 2m_x, \quad n_y = 2m_y,$$
  then
  $$\mathcal{M}_+(\alpha_{2m_x,2m_y}) = 2s^2(\phi, \beta) \left( 1 - \frac{\beta^2}{\beta_{c,x}^2} \right) > 0,$$
  $$\mathcal{M}_-(\alpha_{2m_x,2m_y}) = 2s^2(\phi, \beta) \left( 1 - \frac{\beta^2}{\beta_{c,y}^2} \right) > 0,$$

where

\( c(\phi, \beta) = \begin{cases} \cos \phi & \text{for } 0 < \beta < \frac{1}{\sqrt{2}}, \\ \sin \phi & \text{for } \frac{1}{\sqrt{2}} < \beta < \beta_{c,x}, \end{cases} \)

\( s(\phi, \beta) = \begin{cases} \sin \phi & \text{for } 0 < \beta < \frac{1}{\sqrt{2}}, \\ \cos \phi & \text{for } \frac{1}{\sqrt{2}} < \beta < \beta_{c,x}. \end{cases} \)

Therefore, $\alpha_{2m_x,2m_y}$ are stable equilibrium points.

The potential energy (A6) at $r = \alpha_{2m_1,2m_2}$ is,

$$V(\alpha_{2m_x,2m_y}) = -16 \left( 1 - \beta^2 \right) V_0,$$

- When $n_x$ and $n_y$ are odd,
  $$n_x = 2m_x + 1, \quad n_y = 2m_y + 1,$$
  then
  $$\mathcal{M}_+(\alpha_{2m_1+1,2m_2+1}) = -\frac{\beta^2}{2} \sin(2\phi) < 0,$$
  $$\mathcal{M}_-(\alpha_{2m_1+1,2m_2+1}) = -\frac{\beta^2}{2} \sin(2\phi) < 0.$$

Therefore, $\alpha_{2m_1+1,2m_2+1}$ are unstable equilibrium points.

The potential energy (A6) at $r = \alpha_{2m_1+1,2m_2+1}$ is,

$$V(\alpha_{2m_1+1,2m_2+1}) = 0.$$

- When $n_x$ is even and $n_y$ is odd,
  $$n_x = 2m_x, \quad n_y = 2m_y + 1,$$
  then
  $$\mathcal{M}_+(\alpha_{2m_x,2m_y+1}) = \frac{\beta^2}{2} \sin^2(2\phi) > 0,$$
  $$\mathcal{M}_-(\alpha_{2m_x,2m_y+1}) = -2 \sin^2 \phi \left( 1 - \frac{\beta^2}{\beta_{c,y}^2} \right) < 0.$$

Therefore, $\alpha_{2m_x,2m_y+1}$ are saddle equilibrium points. The potential energy (A6) at $r = \alpha_{2m_x,2m_y+1}$ is,

$$V(\alpha_{2m_x,2m_y+1}) = -16\beta^2 V_0 \sin^2 \phi.$$

- When $n_x$ is odd and $n_y$ is even,
  $$n_x = 2m_x + 1, \quad n_y = 2m_y,$$
  then
  $$\mathcal{M}_+(\alpha_{2m_1+1,2m_2}) = \frac{\beta^2}{2} \sin^2(2\phi) > 0,$$
  $$\mathcal{M}_-(\alpha_{2m_1+1,2m_2}) = -2 \cos^2 \phi \left( 1 - \frac{\beta^2}{\beta_{c,x}^2} \right) < 0.$$

Therefore, $\alpha_{2m_1+1,2m_2}$ are saddle equilibrium points. The potential energy (A6) at $r = \alpha_{2m_1+1,2m_2}$ is,

$$V(\alpha_{2m_1+1,2m_2}) = -16\beta^2 V_0 \cos^2 \phi.$$

![FIG. 16: (color online) The potential energy (A6) for $\beta = 0.4$ [panel (a)] and $\beta = 0.9$ [panel (b)]. For both panels, $\phi = \frac{\phi_{max}}{\pi}$.

The purple, blue, sky blue, orange and red curves denote $V(r) = 0.04V_{min}, 0.27V_{min}, 0.5V_{min}, 0.73V_{min}$ and $0.96V_{min}$, where $V_{min}$ is the minimal value of $V(r)$ for given $\beta$. The potential energy (A6) is displayed in Fig. 16(a) for $\phi = \frac{\phi_{max}}{\pi}$ and $\beta = 0.4$ (which lies within the interval $0 < \beta < \beta_{c,x}$). The minima points $\alpha_{2m_x,2m_y}$ and the maxima points $\alpha_{2m_1+1,2m_2+1}$ are well seen.](image-url)
2. The Case $\beta > \beta_{c,y}$

When $\beta_{c,y} < \beta < 1$, the equilibrium points are given by eqs. (C4) and (C5). Explicitly, they are

$$\alpha_{n_x,n_y} = \frac{n_x a_x + n_y a_y}{2}, \quad (C9a)$$

$$\beta_{\xi, n_x, \xi, n_y} = \sum_{j=x,y} \zeta_j \left( \frac{\varphi_{\beta,j}}{\pi} + n_j \right) a_j. \quad (C9b)$$

Here $n_x$ and $n_y$ are integers, $n_j = \pm 1$ ($j = x, y$), and

$$\varphi_{\beta,x} = \arccos \left( \frac{1 - \beta^2 [1 + \cos(2\phi)]}{2\beta^2 - 1} \right),$$

$$\varphi_{\beta,y} = \arccos \left( \frac{1 - \beta^2 [1 - \cos(2\phi)]}{2\beta^2 - 1} \right).$$

Note that for any $\beta$ within the interval $\beta_{c,y} \leq \beta \leq 1$,

$$0 \leq 1 - \frac{1 - \beta^2 [1 \pm \cos(2\phi)]}{2\beta^2 - 1} \leq 1,$$

and therefore

$$0 \leq \varphi_{\beta,j} \leq \frac{\pi}{2}, \quad j = x, y.$$

We investigate now the sign of $M_{\pm}(\alpha)$, eq. (C7), for all equilibrium points (C9).

First, we consider sign of $M_{\pm}(\alpha_{n_1,n_2})$, where $\alpha_{n_1,n_2}$ is given by eq. (C9a). There are four cases, when $n_1$ and $n_2$ are even or odd. We consider all these cases in turn.

- When $n_x$ and $n_y$ are even,
  $$n_x = 2m_x, \quad n_y = 2m_y,$$
  then
  $$M_{+}(\alpha_{2m_x,2m_y}) = -2 \sin^2 \phi \left( \frac{\beta^2}{\beta_{c,y}^2} - 1 \right) < 0,$$
  $$M_{-}(\alpha_{2m_1,2m_2}) = -2 \cos^2 \phi \left( \frac{\beta^2}{\beta_{c,x}^2} - 1 \right) < 0.$$

Therefore, $\alpha_{2m_x,2m_y}$ are unstable equilibrium points. The potential energy $(A6)$ at $r = \alpha_{2m_x,2m_y}$ is,

$$V(\alpha_{2m_x,2m_y}) = -16 (1 - \beta^2) V_0.$$

- When $n_x$ and $n_y$ are odd,
  $$n_x = 2m_x + 1, \quad n_y = 2m_y + 1,$$
  then
  $$M_{+}(\alpha_{2m_x+1,2m_y+1}) = -\frac{\beta^2}{2} \sin^2 (2\phi) < 0,$$
  $$M_{-}(\alpha_{2m_x+1,2m_y+1}) = -\frac{\beta^2}{2} \sin^2 (2\phi) < 0.$$

Therefore, $\alpha_{2m_x+1,2m_y+1}$ are unstable equilibrium points. The potential energy $(A6)$ at $r = \alpha_{2m_x+1,2m_y+1}$ is,

$$V(\alpha_{2m_x+1,2m_y+1}) = 0.$$

- When $n_x$ is even and $n_y$ is odd,
  $$n_x = 2m_x, \quad n_y = 2m_y + 1,$$
  then
  $$M_{+}(\alpha_{2m_x,2m_y+1}) = \frac{\beta^2}{2} \sin^2 (2\phi) > 0,$$
  $$M_{-}(\alpha_{2m_x,2m_y+1}) = 2 \sin^2 \phi \left( \frac{\beta^2}{\beta_{c,y}^2} - 1 \right) > 0.$$

Therefore, $\alpha_{2m_1,2m_2+1}$ are stable equilibrium points. The potential energy $(A6)$ at $r = \alpha_{2m_1,2m_2+1}$ is,

$$V(\alpha_{2m_1,2m_2+1}) = -16\beta^2 V_0 \sin^2 \phi.$$

- When $n_x$ is odd and $n_y$ is even,
  $$n_x = 2m_x + 1, \quad n_y = 2m_y,$$
  then
  $$M_{+}(\alpha_{2m_x+1,2m_y}) = \frac{\beta^2}{2} \sin^2 (2\phi) > 0,$$
  $$M_{-}(\alpha_{2m_x+1,2m_y}) = 2 \cos^2 \phi \left( \frac{\beta^2}{\beta_{c,x}^2} - 1 \right) > 0.$$

Therefore, $\alpha_{2m_x+1,2m_y}$ are stable equilibrium points. The potential energy $(A6)$ at $r = \alpha_{2m_x+1,2m_y}$ is,

$$V(\alpha_{2m_x+1,2m_y}) = -16\beta^2 V_0 \cos^2 \phi.$$
Appendix D: Mirror Chern Numbers

Eigenfunctions $|\psi_{\xi,s,\eta}(k)\rangle$ of the Hamiltonian (10) are,

$$|\psi_{\xi,s,\eta}(k)\rangle = \sum_f \chi_{\xi,s,f,\eta}(k) e^{-if\phi} |f\rangle.$$  \hspace{1cm} (D1)

Here

$$\chi_{\xi,s,f,\eta}(k) = (-1)^{\frac{s+f}{2}} N_{\xi,s,\eta}(k) \det(\hat{M}_{\xi,s,f,\eta}(k)),$$

where $N_{\xi,s,\eta}(k)$ is a normalization constant, and the matrices $\hat{M}_{\xi,s,f,\eta}(k)$ are,

$$\hat{M}_{\xi,s,\eta}(k) = \begin{pmatrix} \sqrt{3} d_k & 0 & 0 \\ g_{\xi,s,\eta}(k) & 2d_k & 0 \\ 2d_k & g_{\xi,s,\eta}(k) & \sqrt{3} d_k \end{pmatrix},$$

where

$$g_{\xi,s,f,\eta}(k) = h_{\eta,f}(k) - \varepsilon_{\xi,s,\eta}(k),$$

$$h_{\eta,f}(k) = (-1)^{\frac{s+f}{2}} \eta \Delta_k + fB.$$  

Here $\Delta_k$ is given by Eq. (5) and $d_k \equiv |d_k| = \hbar v_k$.

Wave functions (D1) are described by the following quantum numbers:

- Iso-spin quantum number $\xi = c, v$ for the conduction or valence band.
- Sub-band quantum number $s = \frac{1}{2}$ and $\frac{3}{2}$.
- Block quantum number $\eta = \pm 1$.

Note that the sub-band quantum number $s$ takes just positive values. This is because of the following reason:

- When $\eta = 1$, then the modes with $f = \frac{3}{2}$ and $f = -\frac{1}{2}$ belong to the conduction band, whereas the modes with $f = -\frac{3}{2}$ and $f = \frac{1}{2}$ belong to the valence band.
- When $\eta = -1$, then the modes with $f = -\frac{3}{2}$ and $f = \frac{1}{2}$ belong to the conduction band, whereas the modes with $f = \frac{3}{2}$ and $f = -\frac{1}{2}$ belong to the valence band.

Therefore we describe the modes of the conduction or valence band by the positive $s = |f|$ and additional quantum number $\xi$. Existence of the topological edge states can be checked from the wave functions (D1). For this purpose, we consider mirror Chern numbers $C_\eta$, which is given by eq. (29).

It is convenient to use polar coordinates,

$$k_x = k \cos \phi, \quad k_y = k \sin \phi.$$  

The vector $|\tilde{\psi}_{\eta,s,\xi}(k)\rangle$, Eq. (32), in the polar coordinates is,

$$|\tilde{\psi}_{\eta,s,\xi}(k)\rangle = e_k \frac{\partial |\psi_{\xi,s,\eta}(k)\rangle}{\partial k} + \frac{e_\phi}{k} \frac{\partial |\psi_{\xi,s,\eta}(k)\rangle}{\partial \phi}.$$  \hspace{1cm} (D2)

Then the Berry curvature (30) takes the form,

$$F_{\eta,s,\xi}(k) = i \left( \partial_k A_{\xi,s,\eta,\xi}(k) - \partial_{\phi} A_{\xi,s,\eta,k}(k) \right),$$  \hspace{1cm} (D3)

where

$$A_{\xi,s,\eta,\xi}(k) = \left\langle \psi_{\xi,s,\eta}(k) | \partial_k |\psi_{\xi,s,\eta}(k)\rangle \right\rangle,$$  \hspace{1cm} (D4a)

$$A_{\xi,s,\eta,\phi}(k) = \left\langle \psi_{\xi,s,\eta}(k) | \partial_{\phi} |\psi_{\xi,s,\eta}(k)\rangle \right\rangle.$$  \hspace{1cm} (D4b)

Taking into account eq. (D1), we can write

$$F_{\eta,s,\xi}(k) = \frac{1}{k} \sum_f \partial_k \left( \chi_{\xi,s,f,\eta}^2(k) \right).$$  \hspace{1cm} (D5)

Note that the Berry curvature (D5) depends just on $k$ but not on $\phi$.

Expression (D5) allows us to derive the following expression the mirror Chern numbers (29),

$$C_\eta = \sum_s \left\{ \lim_{k \to \infty} \left\langle S^z_{\xi,s,\eta}(k) \right\rangle - \lim_{k \to 0} \left\langle S^z_{\xi,s,\eta}(k) \right\rangle \right\},$$  \hspace{1cm} (D6)

where

$$\left\langle S^z_{\xi,s,\eta}(k) \right\rangle = \sum_f \chi_{\xi,s,f,\eta}^2(k).$$  \hspace{1cm} (D7)

Eq. (D6) shows that in order to find the mirror Chern numbers, we need just wave functions (D1) for $k = 0$ and $k \to \infty$.

Consider $\langle S^z_{\xi,s,\eta}(k) \rangle$ for $k \to \infty$. Taking into account that the spin-orbital interaction is linear with $k$, and the kinetic energy is quadratic with $k$, we can neglect the
spin-orbital interaction and get diagonal Hamiltonian. Then we can write
\[
\langle S_v, \pm \eta, \eta (\infty) \rangle = \frac{\eta}{2}, \quad \langle S_v, \pm \eta, \eta^{(0)}(\infty) \rangle = \frac{3\eta}{2} . \tag{D8}
\]

The values \(\langle S_v, \pm \eta, k \rangle\) for \(k \to 0\) depends on the values of \(\Delta_0\) and \(B\). There are five relevant intervals in the half-plane \(\Delta_0, B (B > 0)\) displayed in Fig. 2. We derive \(\langle S_v, \pm \eta, (0) \rangle\) and the mirror Chern numbers (D6) for each of the intervals.

**Interval (1):** When \(\frac{2}{3}\Delta_0 > B > 0\), then
\[
\langle S_v, \pm \frac{1}{2}, (0) \rangle = \frac{1}{2}, \quad \langle S_v, \pm \frac{3}{2}, (0) \rangle = \frac{3}{2}, \\
\langle S_v, \pm \frac{1}{2}, (0) \rangle = \frac{1}{2}, \quad \langle S_v, \pm \frac{1}{2}, (0) \rangle = \frac{1}{2}.
\]

Taking into account eq. (D8), we get
\[C_1 = C_1 = 0,\]
and we have no topological edge states.

**Interval (2):** When \(2\Delta_0 > B > \frac{2}{3}\Delta_0 > 0\), then
\[
\langle S_v, \pm \frac{1}{2}, (0) \rangle = \frac{1}{2}, \quad \langle S_v, \pm \frac{3}{2}, (0) \rangle = \frac{3}{2}, \\
\langle S_v, \pm \frac{1}{2}, (0) \rangle = \frac{1}{2}, \quad \langle S_v, \pm \frac{1}{2}, (0) \rangle = \frac{1}{2}.
\]

Taking into account eq. (D8), we get
\[C_1 = 0, \quad C_1 = 3,\]
and we have no topological edge states with \(\eta = 1\), and there are three chiral topological modes with \(\eta = -1\).

**Interval (3):** When \(B > 2|\Delta_0| > 0\), then
\[
\langle S_v, \pm \frac{1}{2}, (0) \rangle = \frac{1}{2}, \quad \langle S_v, \pm \frac{3}{2}, (0) \rangle = \frac{3}{2}, \\
\langle S_v, \pm \frac{1}{2}, (0) \rangle = \frac{1}{2}, \quad \langle S_v, \pm \frac{1}{2}, (0) \rangle = \frac{1}{2}.
\]

Taking into account eq. (D8), we get
\[C_1 = -2, \quad C_1 = 2,\]
and we have two chiral modes with \(\eta = 1\), and two chiral modes with \(\eta = -1\).

Taking into account eq. (D8), we get
\[C_1 = 1, \quad C_1 = 3,\]
and we have a chiral mode with \(\eta = 1\), and three chiral modes with \(\eta = -1\).

**Interval (4):** When \(-2\Delta_0 > B > -\frac{2}{3}\Delta_0 > 0\), then
\[
\langle S_v, \pm \frac{1}{2}, (0) \rangle = -\frac{3}{2}, \quad \langle S_v, \pm \frac{3}{2}, (0) \rangle = -\frac{1}{2}, \\
\langle S_v, \pm \frac{1}{2}, (0) \rangle = \frac{1}{2}, \quad \langle S_v, \pm \frac{3}{2}, (0) \rangle = -\frac{3}{2}.
\]

Taking into account eq. (D8), we get
\[C_1 = 1, \quad C_1 = 2,\]
and we have a chiral mode with \(\eta = 1\), and two chiral modes with \(\eta = -1\).

**Interval (5):** When \(-\frac{2}{3}\Delta_0 > B > 0\), then
\[
\langle S_v, \pm \frac{1}{2}, (0) \rangle = -\frac{1}{2}, \quad \langle S_v, \pm \frac{3}{2}, (0) \rangle = \frac{3}{2}, \\
\langle S_v, \pm \frac{1}{2}, (0) \rangle = \frac{1}{2}, \quad \langle S_v, \pm \frac{3}{2}, (0) \rangle = -\frac{3}{2}.
\]

Taking into account eq. (D8), we get
\[C_1 = -2, \quad C_1 = 2,\]
and we have two chiral modes with \(\eta = 1\), and two chiral modes with \(\eta = -1\).

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It has been shown that spin 3/2 systems can be described by 4 × 4 SO(5) Γ matrices\textsuperscript{27,28}. However, our Hamiltonian is a 8x8 matrix, so it has an SO(7) structure rather than SO(5). In the mirror sub-sector, the Hamiltonian reduces to 4x4 blocks, so it can be written in terms SO(5) Γ matrices, but in that case, half of the degrees of freedom comes from the band index, so it is not directly related to the spin 3/2 SO(5) matrix.