GROMOV–WITTEN THEORY
OF DELIGNE–MUMFORD STACKS

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1. INTRODUCTION

Gromov–Witten theory of orbifolds was introduced in the symplectic setting in [CR]. In [AGV] we adapted the theory to algebraic geometry, using Chow rings and the language of stacks. The latter work is, to a large extent, a research announcement, as detailed proofs were not given. The main purpose of this paper is to complete that work and lay the algebro-geometric theory on a sounder footing.

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It appears from the emerging literature that the language of stacks – whether algebraic or differential – is imperative in making serious computations in orbifold Gromov–Witten theory (see, e.g. \cite{C2}, \cite{Ts}). It can therefore be hoped that some of the material here should be useful in the symplectic setting as well.

The fact that a few years have passed since the release of our paper \cite{AGV} may be one cause for a regretful slow development of applications. On the other hand, we believe recent developments have enabled us to set the foundations in a much better way than was possible at the time of the paper \cite{AGV}. Most important among these are Olsson’s papers \cite{O1} and \cite{O2}.

1.1. **Twisted stable maps.** Algebro-geometric treatments of Gromov–Witten theory of a smooth projective variety $X$ rely on Kontsevich’s moduli stacks $\mathcal{M}_{g,n}(X,\beta)$, parametrizing $n$-pointed stable maps from curves of genus $g$ to $X$ with image class $\beta \in H_2(X,\mathbb{Z})$, see \cite{BM}. When one replaces the manifold $X$ with an orbifold, one needs to replace the curves in the stable maps by orbifold curves - this is a phenomenon discussed in detail elsewhere (see \cite{AV1}). The stack of stable maps is thus replaced by the stack of *twisted stable maps*, denoted $\mathcal{K}_{g,n}(X,\beta)$ in \cite{AV}, where it was constructed.

The proof of the main theorem in \cite{AV} is not ideal as it relies on ad-hoc arguments and requires verifying Artin’s axioms one by one. An alternative approach which is much more conceptual is given in Olsson’s papers \cite{O1} and \cite{O2}. Artin’s axioms still need to be verified, but in a more general and cleaner situation. In Olsson’s paper \cite{O2} one also finds a direct construction of the stack of twisted pre-stable curves, which is an important tool in the theory developed here. See Section 4 (and Appendix B.2) for a quick review.

An analogous space of *orbifold stable maps* was constructed by W. Chen and Y. Ruan in \cite{CR} using very different methods. In spirit the two constructions describe the same thing, and one expects that the resulting Gromov–Witten numbers are identical.

For quotient stacks by a finite group $\mathcal{X} = [V/G]$, Jarvis, Kaufmann and Kimura considered in \cite{JKK} maps of pointed admissible $G$-covers to $V$. This was revisited in \cite{AGOT}. Lev Borisov recently discovered in his mail archives two letters from Kontsevich, dated from July 1996, where Gromov–Witten theory of a quotient stack by a finite group is outlined precisely using admissible $G$-covers. See \cite{A} for a reproduced text.

When the target stack $\mathcal{X}$ is smooth, the stack $\mathcal{K}_{g,n}(\mathcal{X},\beta)$ admits a perfect obstruction theory in the sense of \cite{BF} and so one gets a virtual fundamental class $[\mathcal{K}_{g,n}(\mathcal{X},\beta)]^{vir}$, which then gives rise to a Gromov-Witten theory.

1.2. **Gromov–Witten classes - manifold case.** We follow a formalism for Gromov–Witten theory suited for Chow rings developed in \cite{GP} (a paper...
longer overdue than this one). A similar formalism was given in [EK, Section 3]. Consider classes \( \gamma_1, \ldots, \gamma_n \in A^*(X) \). Define the associated Gromov–Witten classes to be

\[
\langle \gamma_1, \ldots, \gamma_n, * \rangle_{g, \beta} = e_{n+1} \ast (e_1^* \gamma_1 \cup \ldots \cup e_n^* \gamma_n \cap [\overline{M}_{g,n+1}(X, \beta)]^{vir}) \in A^*(X).
\]

An important part of Gromov–Witten theory is concerned with relations these classes satisfy. In particular, in genus 0, they satisfy the Witten–Dijkgraaf–Verlinde–Verlinde (WDVV) equation. These classes are convenient for defining the quantum product on either the Chow ring or cohomology ring. The associativity of that product is a consequence of the WDVV equation.

A fundamental ingredient of Gromov–Witten theory is a description of the “boundary” of the moduli stack of maps. The locus of nodes \( \Sigma \) on the universal curve of the stack of stable maps \( \overline{M}_{0,n}(X, \beta) \) is a partial normalization of this boundary, and has the beautiful description

\[
\Sigma = \bigotimes_{A \sqcup B = \{1, \ldots, n\}} \overline{M}_{0, A \sqcup \bullet}(X, \beta_1) \times_X \overline{M}_{0, B \sqcup \bullet}(X, \beta_2).
\]

Here the fibered product is taken with respect to the evaluation maps \( e_\bullet : \overline{M}_{0, A \sqcup \bullet}(X, \beta_1) \rightarrow X \) and \( e_\bullet : \overline{M}_{0, B \sqcup \bullet}(X, \beta) \rightarrow X \).

At the bottom of this is the fact that if \( C = C_1 \sqcup_p C_2 \) is a nodal curve with node \( p \) separating it into two subcurves \( C_1 \) and \( C_2 \), then \( C \) is a coproduct of \( C_1 \) and \( C_2 \) over \( p \). It follows from the universal property of a coproduct that

\[
\text{Hom}(C, X) = \text{Hom}(C_1, X) \times_{\text{Hom}(p, X)} \text{Hom}(C_2, X).
\]

Since \( \text{Hom}(p, X) = X \) we get the familiar formula

\[
\text{Hom}(C, X) = \text{Hom}(C_1, X) \times_X \text{Hom}(C_2, X).
\]

The decomposition of the boundary is immediate from this.

1.3. Boundary of moduli - orbifold case. When analyzing the orbifold case something new happens. The source curve is a twisted curve \( C = C_1 \sqcup G C_2 \) with node \( G \) which is a gerbe banded by \( \mu_r \) for some \( r \). We show in Proposition A.1.1 that the coproduct \( C_1 \sqcup G C_2 \) of \( C_1 \) and \( C_2 \) over \( G \) exists, and it is immediate that \( C_1 \sqcup G C_2 \rightarrow C \) is an isomorphism. It follows again that

\[
\text{Hom}(C, \mathcal{X}) = \text{Hom}(C_1, \mathcal{X}) \times_{\text{Hom}(G, \mathcal{X})} \text{Hom}(C_2, \mathcal{X}),
\]

see Appendix A.2. Now the data of a gerbe with a map to \( \mathcal{X} \) is not a point of \( \mathcal{X} \), but of a fascinating gadget \( \mathcal{I}_{\mu}(\mathcal{X}) \) we call the rigidified cyclotomic inertia stack, see Section 3. (A different notation \( \check{X}_1 \) was used in [AGV], but
we were convinced that the present notation, as used by Cadman \[C1, C2\], is more appropriate). We get the formula
\[
\text{Hom}(\mathcal{C}, \mathcal{X}) = \text{Hom}(\mathcal{C}_1, \mathcal{X}) \times \text{Hom}(\mathcal{C}_2, \mathcal{X}).
\]
The fibered product uses a natural morphism \(\text{Hom}(\mathcal{C}_2, \mathcal{X}) \to \mathcal{I}_\mathcal{X}(\mathcal{X})\) corresponding to \(G \to \mathcal{X}\), and a twisted map \(\text{Hom}(\mathcal{C}_1, \mathcal{X}) \to \mathcal{I}_\mathcal{X}(\mathcal{X})\) corresponding to \(G \to \mathcal{X}\) with the band inverted. This is necessary since the glued curve \(\mathcal{C}\) is balanced. The resulting map of moduli stacks
\[
\bigoplus_{\beta_1 + \beta_2 = \beta} K_{0, A, \beta_1} \times K_{0, B, \beta_2} \to K_{0, A \cup B, \beta} \times \mathcal{I}_\mathcal{X}(\mathcal{X})^\text{vir}
\]
is crucial for Gromov–Witten theory of stacks. Here the fibered product is taken using an evaluation map
\[
e_i : K_{0, B, \beta_2} \to \mathcal{I}_\mathcal{X}(\mathcal{X})
\]
and a twisted evaluation map
\[
\tilde{e}_i : K_{0, A, \beta_1} \to \mathcal{I}_\mathcal{X}(\mathcal{X})
\]
necessary to make the glued curves balanced, see Section \[4.4\].

1.4. Gromov–Witten classes - orbifold case. We can now define the orbifold Gromov–Witten classes by integrating along evaluation maps. We use the formalism of Chow rings, though any cohomology theory satisfying some reasonable assumptions works.

Since evaluation maps land naturally in \(\mathcal{I}_\mathcal{X}(\mathcal{X})\), the correct cohomological theory to use is not that of \(\mathcal{X}\) but rather of \(\mathcal{I}_\mathcal{X}(\mathcal{X})\). The fact that something like \(A^*(\mathcal{I}_\mathcal{X}(\mathcal{X}))_\mathbb{Q}\) or \(H^*(\mathcal{I}_\mathcal{X}(\mathcal{X}), \mathbb{Q})\) is of interest was recognized by physicists (see e.g. \[DHVW, Z\]) where “twisted sectors” and “orbifold Euler characteristics” were considered. Some mathematical reasons were discussed in \[AGV\]. Also, in Kontsevich’s remarkable messages to Borisov the same phenomenon occurs. But all these are reasoned by analogy, “delicious reciprocal reflections, furtive caresses, inexplicable quarrels” \[We\]. Alas, in orbifold Gromov–Witten theory all this becomes mundane once one understands that it all follows from the fact that \(\mathcal{C}\) is a coproduct.

Now, given \(\gamma_1, \ldots, \gamma_n \in A^*(\mathcal{I}_\mathcal{X}(\mathcal{X}))_\mathbb{Q}\) we can consider the class
\[
\tilde{e}_{n+1} \ast (e_1^* \gamma_1 \cup \ldots \cup e_n^* \gamma_n \cap [K_{g, n+1}(\mathcal{X}, \beta)]^\text{vir}) \in A^*(\mathcal{I}_\mathcal{X}(\mathcal{X}))_\mathbb{Q},
\]
where \(e_i : K_{g, n+1}(\mathcal{X}, \beta) \to \mathcal{I}_\mathcal{X}(\mathcal{X})\) are the evaluation maps of Section \[4.4\]. It turns out that in the orbifold theory we need to multiply this by the function \(r : \mathcal{I}_\mathcal{X}(\mathcal{X}) \to \mathbb{Z}\) describing the index of the gerbe (but see the second part of Proposition \[6.1.4\] for a way out of that annoyance). We thus define the Gromov–Witten classes to be
\[
\langle \gamma_1, \ldots, \gamma_n, * \rangle_{g, \beta} = r \cdot e_{n+1} \ast (e_1^* \gamma_1 \cup \ldots \cup e_n^* \gamma_n \cap [K_{g, n+1}(\mathcal{X}, \beta)]^\text{vir}).
\]
With this definition Gromov–Witten theory goes through almost as in the manifold case, though the proofs require some interesting changes. The main result is Theorem 6.2.1, in which the WDVV equation is proven.

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2. Chow rings, cohomology and homology of stacks

2.1. Intersection theory on Deligne–Mumford stacks. Throughout the paper we work over a fixed base field $k$ of characteristic 0. Let $\mathcal{X}$ be a separated Deligne–Mumford stack of finite type over $k$, $\pi : \mathcal{X} \to X$ its moduli space.

The rational Chow group $A_*(\mathcal{X})_\mathbb{Q}$ is defined as in [M], [G] and [V]. One defines the group of cycles on $\mathcal{X}$ as the free abelian group on closed integral substacks of $\mathcal{X}$, then divides by rational equivalence. There is also an integral version of the theory, developed in [EG] for quotient stacks and in [K] in general, but we will not need it.

These groups are covariant for proper morphisms of Deligne–Mumford stacks. The pushforward $\pi_* : A_*(\mathcal{X})_\mathbb{Q} \to A_*(X)_\mathbb{Q}$ is given by the following formula. Let $V$ be a closed integral substack of $\mathcal{X}$, and call $V$ its moduli space; this is an integral scheme of finite type over $k$. Because of the hypothesis on the characteristic, the natural morphism $V \to X$ is a closed embedding, and we have

$$
\pi_*[V] = \frac{1}{r}[V],
$$

where $r$ is the order of the stabilizer of a generic geometric point of $V$.

The homomorphism $\pi_*$ is an isomorphism. In what follows we will always identify $A_*(\mathcal{X})_\mathbb{Q}$ and $A_*(X)_\mathbb{Q}$ via $\pi_*$. If $\mathcal{X}$ is proper, we denote by

$$
\int_{\mathcal{X}} : A_*(\mathcal{X})_\mathbb{Q} \to \mathbb{Q}
$$

the pushforward $A_*(\mathcal{X})_\mathbb{Q} \to A_*(\text{Spec}k)_\mathbb{Q} = \mathbb{Q}$.

If $f : \mathcal{X} \to \mathcal{Y}$ is an l.c.i. morphism of constant codimension $k$, then for any morphism $\mathcal{Y}' \to \mathcal{Y}$ we have a Gysin homomorphism $f^! : A_*(\mathcal{Y}')_\mathbb{Q} \to A_*(\mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}')_\mathbb{Q}$, of degree $-k$. These commute with proper pushforwards, and among themselves ([V]).

As in [F], Chapter 17, with every such stack $\mathcal{X}$ we can associate a bivariant ring $A^*(\mathcal{X})_\mathbb{Q}$, that gives a contravariant functor from the 2-category of algebraic stacks of finite type over $k$ to the category of commutative rings. The product of two classes $\alpha$ and $\beta$ in $A^*(\mathcal{X})_\mathbb{Q}$ will be denoted by $\alpha \beta$, or by $\alpha \cup \beta$. By definition, $A_*(\mathcal{X})_\mathbb{Q}$ is a module over $A^*(\mathcal{X})_\mathbb{Q}$; we indicate the result
of the action of a bivariant class \( \alpha \in A^*(\mathcal{X})_\mathbb{Q} \) on a class of cycles \( \xi \in A_1(\mathcal{X})_\mathbb{Q} \) as a cup product \( \alpha \cap \xi \in A_{j-i}(\mathcal{X})_\mathbb{Q} \). The pullback ring homomorphism \( A^*(\mathcal{X})_\mathbb{Q} \to A^*(\mathcal{Y})_\mathbb{Q} \) is also an isomorphism. The projection formula holds: if \( f : \mathcal{X} \to \mathcal{Y} \) is a proper homomorphism, \( \beta \in A^*(\mathcal{Y})_\mathbb{Q} \) and \( \xi \in A_*(\mathcal{X})_\mathbb{Q} \), then

\[
f_*(f^* \beta \cap \xi) = \beta \cap f_* \xi.
\]

If \( \mathcal{E} \) is a vector bundle on \( \mathcal{X} \), then there are Chern classes \( c_i(\mathcal{E}) \in A^i(\mathcal{X}) \), satisfying the usual formal properties.

Suppose that \( \mathcal{X} \) is smooth: then the homomorphism \( A^*(\mathcal{X})_\mathbb{Q} \to A_*(\mathcal{X})_\mathbb{Q} \) defined by \( \alpha \mapsto \alpha \cap [\mathcal{X}] \) is an isomorphism. In this case we can identify \( A^*(\mathcal{X})_\mathbb{Q} \) with \( A_*(\mathcal{X})_\mathbb{Q} \).

The moduli space \( X \) will not be smooth. However, assuming for the moment that \( \mathcal{X} \) is connected, hence irreducible, and we call \( r \) the order of the automorphism group of a generic geometric point of \( \mathcal{X} \), then by the projection formula we have

\[
\pi_*(\pi^* \alpha \cap [\mathcal{X}]) = \alpha \cap \pi_*[\mathcal{X}] = \frac{1}{r} \alpha \cap [X]
\]

for any \( \alpha \in A^*(X)_\mathbb{Q} \); hence the homomorphism \( A^*(X)_\mathbb{Q} \to A_*(X)_\mathbb{Q} \) defined by \( \alpha \mapsto \alpha \cap [X] \) is also an isomorphism. The same holds without assuming that \( \mathcal{X} \) is connected, because if \( \mathcal{X}_i \) are the connected components of \( \mathcal{X} \) and \( X_i \) is the moduli space of \( \mathcal{X}_i \), then \( X = \coprod X_i \). Hence \( A_*(X)_\mathbb{Q} \) inherits a ring structure from that of \( A^*(X)_\mathbb{Q} \), even though \( X \) is in general singular.

However, one should be careful: the isomorphism \( \pi_* : A_*(\mathcal{X})_\mathbb{Q} \to A_*(X)_\mathbb{Q} \) is not a homomorphism of rings, unless \( \mathcal{X} \) is generically a scheme, because in general the identity \( [\mathcal{X}] \) of \( A_*(\mathcal{X})_\mathbb{Q} \) is not carried into the identity \( [X] \) of \( A_*(X)_\mathbb{Q} \).

Suppose that \( \mathcal{X} \) is smooth and proper: then \( X \) is a complete variety with quotient singularities. We say that an element of the group \( A_1(X)_\mathbb{Q} \) of 1-dimensional cycles is \textit{numerically equivalent to 0} if \( \int_X \alpha \cup \xi = 0 \) for all \( \alpha \in A^1(X)_\mathbb{Q} \). The elements of \( A_1(X) \) whose images in \( A_1(X)_\mathbb{Q} \) are numerically equivalent to 0 form a subgroup; we denote by \( N(X) \) the quotient group. This is finitely generated. Furthermore we denote by \( N^+(X) \) the submonoid of \( N(X) \) consisting of effective cycles.

Let \( \mathcal{E} \) be a vector bundle on \( \mathcal{X} \). If \( \xi \in A_1(X) \) is an integral 1-dimensional class, we denote

\[
c_1(\mathcal{E}) \cdot \xi := \int_\mathcal{X} c_1(\mathcal{E}) \cap \xi',
\]

where \( \xi' \) is the class in \( A_1(\mathcal{X})_\mathbb{Q} \) such that \( \pi_* \xi' \) equals the image of \( \xi \) in \( A_1(X)_\mathbb{Q} \).

Notice the following fact. If \( \alpha \) is the class in \( A^1(X)_\mathbb{Q} \) such that \( \pi^* \alpha = c_1(\mathcal{E}) \), then

\[
c_1(\mathcal{E}) \cdot \xi = \int_\mathcal{X} c_1(\mathcal{E}) \cap \xi'.
\]
\[
= \int_X \pi_*(c_1(\mathcal{E}) \cap \xi') \\
= \int_X c_1(\mathcal{E}) \cdot \xi;
\]
hence \(c_1(\mathcal{E}) \cdot \xi\) only depends on the class of \(\xi\) in \(N(X)\). This allows us define the rational number \(c_1(\mathcal{E}) \cdot \beta\) for a class \(\beta \in N(X)\). It is easy to see that the denominators in \(c_1(\mathcal{E}) \cdot \beta\) are uniformly bounded, but the following proposition gives a natural bound:

**Proposition 2.1.1.** Assume that \(\mathcal{X}\) is proper. For each geometric point \(p : \text{Spec} \, \overline{k} \to \mathcal{X}\) denote by \(e_p\) the exponent of the automorphism group of \(p\), and call \(e\) the least common multiple of the \(e_p\) for all geometric points of \(\mathcal{X}\). Then

\[c_1(\mathcal{E}) \cdot \beta \in \frac{1}{e} \mathbb{Z}\]

for any vector bundle \(\mathcal{E}\) on \(\mathcal{X}\) and any \(\beta \in N(X)\).

**Proof.** First note that \(c_1(\mathcal{E}) \cdot \beta \in \mathbb{Z}\) when \(\mathcal{E} = \pi^* \mathcal{M}\) for a bundle \(\mathcal{M}\) on \(X\). Indeed, in this case

\[c_1(\mathcal{E}) \cdot \beta = \int_X \pi^* c_1(\mathcal{M}) \cap \beta' = \int_X c_1(\mathcal{M}) \cap \beta \in \mathbb{Z}.
\]

We may substitute \(\mathcal{E}\) with its determinant, and assume that \(\mathcal{E}\) is a line bundle.

The following is a standard fact; we include a proof below for lack of a suitable reference.

**Lemma 2.1.2.** Let \(\mathcal{L}\) be a line bundle on \(\mathcal{X}\). The line bundle \(\mathcal{L}^\otimes e\) on \(\mathcal{X}\) is the pullback of a line bundle \(\mathcal{M}\) on \(X\).

To conclude the proof of the proposition,

\[c_1(\mathcal{E}) \cdot \beta = \frac{1}{e} c_1(\mathcal{L}^\otimes e) \cdot \beta = \frac{1}{e} \pi^* c_1(\mathcal{M}) \cdot \beta \in \frac{1}{e} \mathbb{Z}\]

as required.

**Proof of the lemma.** Observe that this is equivalent to the statement that \(\pi_* \mathcal{L}^\otimes e\) is a line bundle on \(X\), and the adjunction homomorphism \(\pi^* \pi_* \mathcal{L}^\otimes e \to \mathcal{L}^\otimes e\) is an isomorphism. This is a local statement in the étale topology.

Let \(p : \text{Spec} \, \overline{k} \to \mathcal{X}\) be a geometric point, \(G_p\) its automorphism group. By definition, the exponent of \(G_p\) divides \(e\). The action of \(G_p\) on the fiber \(\mathcal{L}_p\) of \(\mathcal{L}\) at \(p\) is given by a 1-dimensional character \(\chi : G_p \to \mathbb{C}^*\), and therefore
\(\chi^e\) is the trivial character. This implies that the action of \(G_p\) on the fiber of \(L^\otimes e\) is trivial.

There is an étale neighborhood \(\text{Spec} \, \mathbb{k} \to U \to X\) of the image of \(p\) in \(X\), such that the pullback \(U \times_X \mathcal{X}\) is isomorphic to the quotient \([V/G_p]_u\), where \(V\) is a scheme on which \(G_p\) acts with a fixed geometric point \(q : \text{Spec} \, \mathbb{k} \to V\), with an invariant morphism \(V \to U\) mapping \(q\) to \(p\); then the pullback \(L_{[V/G_p]}\) corresponds to a \(G_p\)-equivariant locally free sheaf \(L_V\) on \(V\). We may assume that \(U\) is affine. Then \(V\) is also affine; since the characteristic of the base field is 0, we can take an invariant non-zero element the in fiber of \(L^\otimes e\), and extend it to an invariant section of \(L_V^\otimes e\). By restricting \(V\) we may also assume that this section does not vanish anywhere: and then \(L_V^\otimes e\) is trivial as a \(G_p\)-equivariant line bundle. This implies that the restriction \(L^\otimes e_{[V/G_p]}\) is trivial, and so \(L^\otimes e_{[V/G_p]}\) is the pullback of the trivial line bundle on \(U\). This concludes the proof of the Lemma and of the Proposition.

2.2. Homology and cohomology of smooth Deligne–Mumford stacks.

In this section the base field will be the field \(\mathbb{C}\) of complex numbers. If \(\mathcal{X}\) is an algebraic stack of finite type over \(\mathbb{C}\), then we can define the classical homology and cohomology of \(\mathcal{X}\) as the homology and cohomology of the simplicial scheme associated with a smooth presentation of \(\mathcal{X}\). More precisely, if \(X_1 \to X_0\) is a smooth presentation of \(\mathcal{X}\), we can obtain from it a simplicial scheme \(X^\bullet\) in the usual fashion. To this we associate a simplicial space by taking the classical topology on each \(X_i\). The homotopy type of the realization of this simplicial space is by definition the homotopy type of \(\mathcal{X}\), and its homology and cohomology are the homology and cohomology of \(\mathcal{X}\).

Alternatively, one can define the classical site of \(\mathcal{X}\), and define cohomology as the sheaf-theoretic cohomology of a constant sheaf on this site. Homology can be defined by duality, as usual.

In this paper we are only going to need the cohomology and homology with rational coefficients of a proper Deligne–Mumford stack (mostly in the smooth case) \(\mathcal{X}\). In this case a much more elementary approach is available.

Let \(\mathcal{X}\) be a separated Deligne–Mumford stack of finite type over \(\mathbb{C}\), and let \(\pi : \mathcal{X} \to X\) be its moduli space. We define the homology and cohomology \(H_*(\mathcal{X}, \mathbb{Q})\) and \(H^*(\mathcal{X}, \mathbb{Q})\) as \(H_*(X, \mathbb{Q})\) and \(H^*(X, \mathbb{Q})\), respectively.

If \(f : \mathcal{X} \to \mathcal{Y}\) is a morphism of separated Deligne–Mumford stacks of finite type over \(\mathbb{C}\) with moduli spaces \(\pi : \mathcal{X} \to X\) and \(\rho : \mathcal{Y} \to Y\), this induces a morphism \(g : X \to Y\) of algebraic varieties over \(\mathbb{C}\). We define the pushforward \(f_* : H_*(\mathcal{X}, \mathbb{Q}) \to H_*(\mathcal{Y}, \mathbb{Q})\) and the pullback \(f^* : H^*(\mathcal{X}, \mathbb{Q}) \to H^*(\mathcal{Y}, \mathbb{Q})\) as

\[
g_* : H_*(X, \mathbb{Q}) \to H_*(Y, \mathbb{Q}) \quad \text{and} \quad g^* : H^*(X, \mathbb{Q}) \to H^*(Y, \mathbb{Q})
\]

respectively.
In particular, the pushforward
\[ \pi_* : H_*(\mathcal{X}, \mathbb{Q}) \to H_*(X, \mathbb{Q}), \]
is the identity function \( H_*(\mathcal{X}, \mathbb{Q}) \to H_*(X, \mathbb{Q}) \), and the pullback
\[ \pi^* : H^*(X, \mathbb{Q}) \to H^*(\mathcal{X}, \mathbb{Q}) \]
is the identity \( H^*(X, \mathbb{Q}) \to H^*(\mathcal{X}, \mathbb{Q}) \).

With these definitions, \( H_* \) becomes a covariant functor from the 2-category of separated stacks of finite type over \( \mathbb{C} \) to the category of graded abelian \( \mathbb{Q} \)-vector spaces. Similarly, \( H^* \) becomes a contravariant functor from the same 2-category to the category of graded-commutative \( \mathbb{Q} \)-algebras.

Also, the cap product \( \cap : H^*(X, \mathbb{Q}) \otimes H_*(X, \mathbb{Q}) \to H_*(X, \mathbb{Q}) \) can be interpreted as a cap product \( \cap : H^*(\mathcal{X}, \mathbb{Q}) \otimes H_*(\mathcal{X}, \mathbb{Q}) \to H_*(\mathcal{X}, \mathbb{Q}) \). If \( f : \mathcal{X} \to \mathcal{Y} \) is a morphism of stacks, the projection formula
\[ f_*(f^* \alpha \cap \xi) = \alpha \cap f_* \xi \]
for any \( \alpha \in H^*(\mathcal{Y}, \mathbb{Q}) \) and any \( \xi \in H_*(\mathcal{Y}, \mathbb{Q}) \), holds.

Now assume that \( \mathcal{X} \) is proper. We define the cycle homomorphism
\[ \text{cyc}_X : A_*(\mathcal{X})_{\mathbb{Q}} \to H_*(\mathcal{X}, \mathbb{Q}) \]
as the composition
\[ A_*(\mathcal{X})_{\mathbb{Q}} \xrightarrow{\pi_*} A_*(X)_{\mathbb{Q}} \xrightarrow{\text{cyc}_X} H_*(X, \mathbb{Q}). \]

It is easy to see that the cycle homomorphism gives a natural transformation of functors from the 2-category of proper Deligne–Mumford stacks over \( \mathbb{C} \) to the category of graded \( \mathbb{Q} \)-vector spaces. If \( V \) is a closed substack of \( \mathcal{X} \), this has a fundamental class \([V]\) in \( A_*(\mathcal{X})_{\mathbb{Q}} \); we denote its image in \( H_*(\mathcal{X}, \mathbb{Q}) \) also by \([V]\), and call it the homology fundamental class of \( V \).

Now assume that \( \mathcal{X} \) is smooth and proper. Then \( X \) is a variety with quotient singularities, hence it is a rational homology manifold; thus we have Poincaré duality, that is, the homomorphism
\[ \text{PD}_X := - \cap [X] : H^*(X, \mathbb{Q}) \to H_*(X, \mathbb{Q}) \]
is an isomorphism. From this we get Poincaré duality on \( \mathcal{X} \); however, one should be a little careful here, because the fundamental class \([\mathcal{X}]\), that we want to use to define Poincaré duality on \( \mathcal{X} \), does not coincide with the fundamental class \([X]\). In fact, if \( \mathcal{X}_i \) are the connected components of \( \mathcal{X} \), then their moduli spaces \( X_i \) are the connected components of \( X \). We have that \( \pi_*[\mathcal{X}_i] = \frac{1}{r_i}[X_i] \), if \( r_i \) is the order of the automorphism of a generic geometric point of \( \mathcal{X}_i \); hence we get the formula
\[ [\mathcal{X}] = \sum_i \frac{1}{r_i}[X_i] \]
in $A_*(\mathcal{X})_\mathbb{Q}$, and hence also in $H_*(\mathcal{X}, \mathbb{Q})$. We define the Poincaré duality homomorphism on $\mathcal{X}$ as 
\[ \text{PD}_\mathcal{X} := - \cap [\mathcal{X}] : H^*(\mathcal{X}, \mathbb{Q}) \to H_*(\mathcal{X}, \mathbb{Q}); \]
this is an isomorphism, because it is an isomorphism for every connected component of $\mathcal{X}$.

### 3. The cyclotomic inertia stack and its rigidification

#### 3.1. Cyclotomic inertia

Let $\mathcal{X}$ be a finite type Deligne–Mumford stack over $k$.

**Definition 3.1.1.** We define a category $I_{\mu_r}(\mathcal{X})$, fibered over the category of schemes, as follows:

1. **Objects** $I_{\mu_r}(\mathcal{X})(T)$ consist of pairs $(\xi, \alpha)$ where $\xi$ is an object of $\mathcal{X}$ over $T$, and
   \[ \alpha : (\mu_r)_T \to \text{Aut}_T(\xi) \]
   is an injective morphism of group-schemes. Here $(\mu_r)_T$ is $\mu_r \times T$.
2. **Arrows** from $(\xi, \alpha)$ over $T$ to $(\xi', \alpha')$ over $T'$ is an arrow $F : \xi \to \xi'$ making the following diagram commutative:

\[
\begin{array}{ccc}
(\mu_r)_T & \to & (\mu_r)_{T'} \\
\alpha \downarrow & & \alpha' \downarrow \\
\text{Aut}_T(\xi) & \to & \text{Aut}_{T'}(\xi') \\
\downarrow & & \downarrow \\
T & \to & T'
\end{array}
\]

where $(\mu_r)_T \to (\mu_r)_{T'}$ is the projection and $\text{Aut}_T(\xi) \to \text{Aut}_{T'}(\xi')$ is the map induced by $F$.

It is evident that this category is fibered in groupoids over the category of schemes. There is an obvious morphism $I_{\mu_r}(\mathcal{X}) \to \mathcal{X}$ which sends $(\xi, \alpha)$ to $\xi$.

**Proposition 3.1.2.** The category $I_{\mu_r}(\mathcal{X})$ is a Deligne–Mumford stack, and the functor $I_{\mu_r}(\mathcal{X}) \to \mathcal{X}$ is representable and finite.

**Proof.** We verify that $I_{\mu_r}(\mathcal{X}) \to \mathcal{X}$ is representable and finite, which implies that $I_{\mu_r}(\mathcal{X})$ is a Deligne–Mumford stack. Consider a morphism $T \to \mathcal{X}$ corresponding to an object $\xi$. The fibered product $I_{\mu_r}(\mathcal{X}) \times_{\mathcal{X}} T$ is an open and closed subscheme of the finite $T$-scheme $\text{Hom}_T((\mu_r)_T, \text{Aut}_T(\xi))$.

**Proposition 3.1.3.** Given an isomorphism $\mu_r \xrightarrow{\cong} \mathbb{Z}/r\mathbb{Z}$ there is an induced isomorphism $I_{\mu_r}(\mathcal{X}) \cong I(\mathcal{X}, r)$, where $I(\mathcal{X}, r) \hookrightarrow I(\mathcal{X})$ is the open and
closed substack of the inertia stack of $\mathcal{X}$, consisting of pairs $(\xi, g)$ with $g \in \text{Aut}(\xi)$ of order $r$ at each point.

Proof. The data of an element $g \in \text{Aut}(\xi)$ over $T$ of order $r$ at each point is equivalent to an injective group-scheme homomorphism $(\mathbb{Z}/r\mathbb{Z})_T \to \text{Aut}_T(\xi)$. Composing with $\phi: \mathbb{Z}/r\mathbb{Z} \to \mathbb{Z}/r\mathbb{Z}$ we get the result. ♣

Corollary 3.1.4. When $\mathcal{X}$ is smooth, $\mathcal{I}_{\mu_r}(\mathcal{X})$ is smooth as well.

Proof. It suffices to check this after extension of base field, so we may assume there exists an isomorphism $\phi: \mathbb{Z}/r\mathbb{Z} \to \mathbb{Z}/r\mathbb{Z}$, and by the Proposition it suffices to check the result for $\mathcal{I}(\mathcal{X})$, which is well known. ♣

Definition 3.1.5. We define $\mathcal{I}_{\mu_r}(\mathcal{X}) = \bigsqcup_r \mathcal{I}_{\mu_r}(\mathcal{X})$, and we name it the cyclotomic inertia stack of $\mathcal{X}$.

Note that, since $\mathcal{X}$ is of finite type, $\mathcal{I}_{\mu_r}(\mathcal{X})$ is empty except for finitely many $r$, so $\mathcal{I}_{\mu}(\mathcal{X})$ is also of finite type, and in fact finite over $\mathcal{X}$ by Proposition 3.1.2.

3.2. Alternative description of cyclotomic inertia. There is another less evident description of $\mathcal{I}_{\mu_r}(\mathcal{X})$, which we give in the following definition.

Definition 3.2.1. We define a category $\mathcal{I}_{\mu_r}(\mathcal{X})'$ over the category of schemes as follows.

1. Objects over a scheme $T$ consist of representable morphisms $\phi: (B\mu_r)_T \to \mathcal{X}$.
2. An arrow $\phi \to \phi'$ over $f: T \to T'$ is a 2-morphism $\rho: \phi \to \phi' \circ f_*$ making the following diagram commutative:

\[
\begin{array}{ccc}
(B\mu_r)_T & \xrightarrow{f_*} & (B\mu_r)_{T'} \\
\downarrow^\phi & & \downarrow^\phi' \\
\mathcal{X} & & \mathcal{X} \\
\end{array}
\]

It is clear that this category is fibered in groupoids over the category of schemes.

Definition 3.2.2. We define a morphism of fibered categories

$\mathcal{I}_{\mu_r}(\mathcal{X})' \to \mathcal{I}_{\mu_r}(\mathcal{X})$

as follows:
(1) Given an object

\[ \mathcal{B}(\mu_r)_T \xrightarrow{\phi} \mathcal{X} \]

we obtain a pair \((\xi, \alpha)\) as follows: \(\xi\) is obtained by composing \(\phi\) with the section \(T \to \mathcal{B}(\mu_r)_T\) associated with the trivial \(\mu_r\)-torsor \(1_{\mu_r,T}\). The homomorphism \(\alpha\) is the associated map of automorphisms

\[ (\mu_r)_T = \text{Aut}_T(1_{\mu_r,T}) \to \text{Aut}_T(\xi), \]

which is injective since \(\phi\) is representable.

(2) Given an arrow \(\rho\) as above, we obtain \(F: \phi(1_{\mu_r,T}) \to \phi'(1_{\mu_r,T'})\) by completing the following diagram:

\[
\begin{array}{ccc}
\phi(1_{\mu_r,T}) & \xrightarrow{\rho} & \phi'(1_{\mu_r,T'}) \\
\downarrow \rho & & \downarrow F \\
(\phi' \circ f_*)(1_{\mu_r,T}) & & \phi'(1_{\mu_r,T'}) \\
\downarrow \phi'(1_{\mu_r,T}) & & \downarrow \phi'(1_{\mu_r,T'}) \\
\end{array}
\]

**Proposition 3.2.3.** The morphism \(\mathcal{I}_{\mu_r}(\mathcal{X})' \to \mathcal{I}_{\mu_r}(\mathcal{X})\) is an equivalence of fibered categories.

**Proof.** It is enough to show that for each scheme \(T\), the induced functor on the fiber \(\mathcal{I}_{\mu_r}(\mathcal{X})(T) \to \mathcal{I}_{\mu_r}(\mathcal{X})(T)\) is an equivalence. In the following proof we will write the action of a group on a torsor on the left, not on the right, as is more customary.

**Step 1: The Functor is Faithful.** Given two objects \(\phi, \phi': \mathcal{B}(\mu_r)_T \to \mathcal{X}\), suppose that \(\alpha: \phi \to \phi'\) is a 2-arrow. We need to show that for any \(T\)-scheme \(f: U \to T\) and any \(\mu_r\)-torsor \(P \to U\), the arrow \(\alpha_{P \to U} : \phi(P \to U) \to \phi'(P \to U)\) is uniquely determined by \(\alpha_{1_{\mu_r,T}} : \phi(1_{\mu_r,T}) \to \phi'(1_{\mu_r,T})\). Let \(\{U_i \to U\}\) be an étale covering, such that the pullbacks \(P_i \to U_i\) are all trivial. Then the pullbacks of \(\phi(P \to U)\) and \(\phi'(P \to U)\) to \(\mathcal{X}(U_i)\) are \(\phi(P_i \to U_i)\) and \(\phi'(P_i \to U_i)\), respectively, and, since \(\mathcal{X}\) is a stack, the restrictions of a morphism \(\phi(P \to U) \to \phi'(P \to U)\) is determined by its restriction to \(\phi(P_i \to U_i) \to \phi'(P_i \to U_i)\). So we may assume that \(P \to U\) is trivial. But then there is a cartesian arrow \(P \to 1_{\mu_r,T}\), and an induced
diagram
\[
\begin{array}{ccc}
\phi(P \to U) & \xrightarrow{\alpha_{P \to U}} & \phi'(P \to U) \\
\downarrow & & \downarrow \\
\phi(1_{\mu_r,T}) & \xrightarrow{\alpha_{1_{\mu_r,T}}} & \phi'(1_{\mu_r,T})
\end{array}
\]
that proves what we want.

\textbf{Step 2: The functor is fully faithful.} Assume that \(\beta : \phi(1_{\mu_r,T}) \to \phi'(1_{\mu_r,T})\) is an arrow in \(\mathcal{X}(T)\), commuting with the actions of \(\mu_r\). First consider the case of a trivial \(\mu_r\)-torsor \(P \to U\) over a \(T\)-scheme. Choose a trivialization that induces a cartesian arrow \(P \to 1_{\mu_r,T}\). Then, by definition of cartesian arrow, there is a unique dotted arrow in \(\mathcal{X}(U_i)\) that we can insert in the diagram
\[
\begin{array}{ccc}
\phi(P \to U) & \xrightarrow{\alpha_{P \to U}} & \phi'(P \to U) \\
\downarrow & & \downarrow \\
\phi(1_{\mu_r,T}) & \xrightarrow{\beta} & \phi'(1_{\mu_r,T})
\end{array}
\]
making it commutative. This arrow \(\alpha_{P \to U}\) is independent of the chosen trivialization, because \(\beta\) commutes with the actions of \(\mu_r\), and two trivializations differ by a morphism \(U \to \mu_r\).

If \(P \to U\) is not necessarily trivial, choose a covering \(\{U_i \to U\}\) such that the pullbacks \(P_i \to U_i\) are trivial. We have arrows \(\alpha_{P \to U_i} : \phi(P_i \to U_i) \to \phi'(P_i \to U_i)\) in \(\mathcal{X}(U_i)\), and their pullbacks to \(U_i \times_U U_j\) coincide; hence they glue together to give an arrow \(\alpha_{P \to U} : \phi(P \to U) \to \phi'(P \to U)\). It is easy to see that \(\alpha_{P \to U}\) does not depend on the covering, and defines a 2-arrow \(\phi \to \phi'\) whose image in \(\mathcal{T}_\mu(\mathcal{X})'(r)\) coincides with \(\beta\).

\textbf{Step 3: The functor is essentially surjective.} Let there be given an object \((\xi, \alpha)\) of \(\mathcal{T}_{\mu_r}(\mathcal{X})(T)\), and let us construct a morphism \(\phi : \mathcal{B}(\mu_r)_T \to \mathcal{X}\) of fibered categories, whose image in \(\mathcal{T}_\mu(\mathcal{X})'(r)(T)\) is isomorphic to \((\xi, \alpha)\).

Let \(P \to U\) be a \(\mu_r\)-torsor, where \(U\) is a \(T\)-scheme: we will define an object \(\eta\) of \(\mathcal{X}(U)\), that is a twisted version of the pullback \(\xi_U\) to \(U\), by descent theory, using the action of \(\mu_r\) on \(\xi\). The facts that we are going to use are all in [VI], Sections 3.8 and 4.4.

Consider the pullback \(\xi_P\) of \(\xi\) to \(P\). The morphism \(\mu_r \times P \to P \times_U P\) defined as a natural transformation via Yoneda’s Lemma by the usual rule \((\zeta, p) \mapsto (\zeta p, p)\) is an isomorphism. The pullbacks of \(\xi\) to \(P \times_U P = \mu_r \times P\) along the first and second projection coincide with the pullback \(\xi_{\mu_r \times P}\). On the other end, the group scheme \(\mu_r\) acts on \(\xi\), so the projection \(\mu_r \times P \to \mu_r\) induces an automorphism of \(\xi_{\mu_r \times P}\), that gives descent data for \(\xi_P\) along the étale covering \(P \to U\). These descent data are effective, and define an object \(\eta\) of \(\mathcal{X}(U)\). So we have assigned to every object of \(\mathcal{B}(\mu_r)_T(U)\) an object
of $\mathcal{X}(U)$; this is easily seen to extend to a morphism of fibered categories $\phi : B(\mu_r)_T \to \mathcal{X}$.

Let $P = (\mu_r)_T \to T$ be the trivial torsor. We claim that the object $\eta \overset{\text{def}}{=} \phi(P \to T)$ of $\mathcal{X}(T)$ is isomorphic to $\xi$. In fact, the object with descent data defining $\xi$ is $\xi_P$, with the descent data given by the identity on $\xi_{P \times_T P}$. Then the projection $P \to \mu_r$ defines an automorphism of $\xi_P$ in $\mathcal{X}(P)$, that is easily seen to descend to an isomorphism $\xi \simeq \eta$ in $\mathcal{X}(T)$. This isomorphism is $\mu_r$-equivariant, because $\mu_r$ is commutative, hence the image of $\phi$ in $\mathcal{I}(\mathcal{X})(T)$ is isomorphic to $(\xi, \alpha)$, as we wanted. ♣

3.3. The stack of gerbes in $\mathcal{X}$. We introduce a stack $\mathcal{I}(\mathcal{X})$ closely related to $\mathcal{I}(\mathcal{X})$, which will play an important role below. It will be defined in terms of morphisms of gerbes. Recall that a gerbe over a scheme $X$ is an fppf stack $F$ over $X$ such that

- there exists an fppf covering $\{X_i \to X\}$ such that $F(X_i)$ is not empty for any $i$, and
- given two objects $a$ and $b$ of $F(T)$, where $T$ is an $X$-scheme, there exists a covering $\{T_i \to T\}$ such that the pullbacks $a_{T_i}$ and $b_{T_i}$ are isomorphic in $F(T_i)$.

Slightly more generally, a stack $F$ over a stack $X'$ is a gerbe if for any morphism $V \to X'$ with $V$ a scheme, $F_V \to V$ is a gerbe.

If $G$ is a sheaf of abelian groups over $X$, we say that $F$ is banded by $G$ if for each object $a$ of $F(T)$ we have an isomorphism of sheaves of groups $\text{Aut}_T(a)$ with $G_T$. This should be functorial, in the obvious sense.

If $G$ is not abelian, one can still define a gerbe banded by $G$, but the definition is more subtle. If $G$ is a group scheme of finite type, it is not hard to see that every gerbe banded by $G$ is an algebraic stack.

We first need the following definition.

**Definition 3.3.1.** Define a 2-category $\mathcal{I}(\mathcal{X})^{(2)}$ with a functor to the category of schemes as follows:

1. An object over a scheme $T$ is a pair $(G, \phi)$, where $G \to T$ is a gerbe banded by $\mu_r$, and $G \overset{\phi}{\to} \mathcal{X}$ is a representable morphism.
2. A morphism $(F, \rho) : (G, \phi) \to (G', \phi')$ consists of a morphism $F : G \to G'$ over some $f : T \to T'$, compatible with the bands, and a 2-morphism $\rho : \phi \to \phi' \circ F$ making the following diagram commutative:

$$
\begin{array}{ccc}
G & \xrightarrow{F} & G' \\
\phi & \downarrow & \phi' \\
\mathcal{X} & \xrightarrow{\phi'} & \mathcal{X}
\end{array}
$$
(3) A 2-arrow \((F, \rho) \to (F_1, \rho_1)\) is a usual 2-arrow \(\sigma : F \to F_1\) compatible with \(\rho\) and \(\rho_1\) in the sense that the following diagram is commutative:

\[
\begin{array}{ccc}
\phi & \to & \phi' \\
\rho \downarrow & & \rho_1 \downarrow \\
\phi' \circ F & \to & \phi' \circ F_1
\end{array}
\]

**Lemma 3.3.2.** The 2-category \(\mathcal{I}^{(2)}_{\mu_r}(\mathcal{X})\) is equivalent to a category.

**Proof.** Since all 2-arrows are isomorphisms, it suffices to show that the automorphism group of any 1-arrow is trivial. This is the content of the following general lemma.

**Lemma 3.3.3.** Suppose given a diagram

\[
\begin{array}{ccc}
G & \to & G' \\
\phi \downarrow & & \phi' \downarrow \\
\mathcal{X} & \to & \mathcal{X}'
\end{array}
\]

where \(G, G'\) and \(\mathcal{X}\) are categories fibered in groupoids over a base category, with a 2-arrow \(\rho : \phi \to \phi' \circ F\) making the diagram commutative. Assume \(\phi'\) is faithful. Then an automorphism of \(F\) compatible with \(\rho\) is the identity.

**Proof.** Take an object \(\xi\) of \(G\) over some \(T\) in the base category. We get a diagram

\[
\begin{array}{ccc}
\phi(\xi) & \to & \phi'(\xi) \\
\rho_\xi \downarrow & & \rho_\xi \downarrow \\
\phi'(F(\xi)) & \to & \phi'(F(\xi))
\end{array}
\]

But \(\phi'(\sigma_\xi)\) lies over the identity \(T \to T\). Since \(\mathcal{X}\) is fibered in groupoids, it follows that \(\phi'(\sigma_\xi)\) is the identity. Since \(\phi'\) is faithful, \(\sigma_\xi\) is the identity, which is what we wanted.

**Definition 3.3.4.** We define \(\mathcal{I}_{\mu_r}(\mathcal{X})\) to be the category associated with the 2-category \(\mathcal{I}^{(2)}_{\mu_r}(\mathcal{X})\), where arrows in \(\mathcal{I}_{\mu_r}(\mathcal{X})\) are 2-isomorphism classes of 1-arrows in \(\mathcal{I}^{(2)}_{\mu_r}(\mathcal{X})\).

We note that \(\mathcal{I}_{\mu_r}(\mathcal{X})\) is a category fibered in groupoids over the category of schemes.

**Remark 3.3.5.** There is a tautological morphism \(\mathcal{I}_{\mu_r}(\mathcal{X}) \to \mathcal{I}^{(2)}_{\mu_r}(\mathcal{X})\), defined as follows. An object of \(\mathcal{I}_{\mu_r}(\mathcal{X})(T)\) corresponds to a representable morphism \((\mathcal{B}_{\mu_r})_T \to \mathcal{X}\). Since \((\mathcal{B}_{\mu_r})_T\) is a gerbe over \(T\) banded by \(\mu_r\), this
gives an object of $I_{\mu_r}(\mathcal{X})$. An arrow in $I_{\mu_r}(\mathcal{X})$ gives an arrow in $I_{\mu_r}(\mathcal{X})$ in the obvious way.

We can also define a fibered category, an object of which is an object of $I_{\mu_r}(\mathcal{X})$, together with a section of the gerbe. There is an obvious forgetful functor into $I_{\mu_r}(\mathcal{X})$, which exhibits it as the universal gerbe over $I_{\mu_r}(\mathcal{X})$. Since a gerbe over $\mathcal{T}$ banded by $\mu_r$ with a section has a canonical isomorphism with $(B_{\mu_r})_{\mathcal{T}}$, this universal gerbe is evidently isomorphic to $I_{\mu_r}(\mathcal{X}) \rightarrow I_{\mu_r}(\mathcal{X})$.

In the next section we give an alternative description of $I_{\mu_r}(\mathcal{X})$, which in particular shows that it is a Deligne–Mumford stack.

**Definition 3.3.6.** We define $I_{\mu}(\mathcal{X}) = \bigsqcup_r I_{\mu_r}(\mathcal{X})$, and we name it the stack of cyclotomic gerbes in $\mathcal{X}$.

### 3.4. The rigidified cyclotomic inertia stack

Consider the stack $I_{\mu_r}(\mathcal{X})$. By definition, for every object $((\xi, \alpha), \mathcal{T})$ of $I_{\mu_r}(\mathcal{X})$ we have a canonical central embedding $(\mu_r)_{\mathcal{T}}(\xi, \alpha) \rightarrow Aut_{\mathcal{T}}(\xi, \alpha)$. By [ACV], Theorem 5.1.5 (see also Appendix C), there exists a rigidification denoted there $I_{\mu_r}(\mathcal{X}) \rightarrow I_{\mu_r}(\mathcal{X})_{\mu_r}$. We are adopting the better notation proposed by Romagny in [Ro], and denote this by $I_{\mu_r}(\mathcal{X})//\mu_r$.

**Proposition 3.4.1.** We have an equivalence of fibered categories

$I_{\mu_r}(\mathcal{X})//\mu_r \rightarrow I_{\mu_r}(\mathcal{X})$

so that the following diagram is commutative:

\[
\begin{array}{ccc}
I_{\mu_r}(\mathcal{X}) & \longrightarrow & I_{\mu_r}(\mathcal{X}) \\
\downarrow & & \downarrow \\
I_{\mu_r}(\mathcal{X})//\mu_r & \longrightarrow & I_{\mu_r}(\mathcal{X}) \\
\end{array}
\]

The right diagonal arrow is described in Remark 3.3.5.

**Proof.** We will use the modular interpretation $I_{\mu_r}(\mathcal{X})//\mu_r$ of the rigidified stack $I_{\mu_r}(\mathcal{X})//\mu_r$ given in Proposition C.2.1. There is an obvious morphism of fibered categories $I_{\mu_r}(\mathcal{X})//\mu_r \rightarrow I_{\mu_r}(\mathcal{X})$, defined as follows. An object of $I_{\mu_r}(\mathcal{X})//\mu_r(T)$ consists of a gerbe $\mathcal{G} \rightarrow T$ banded by $\mu_r$, and a $\mu_r$-equivariant morphism $\mathcal{G} \rightarrow I_{\mu_r}(\mathcal{X})$, in the sense of Section C.2; this can be composed with the projection $I_{\mu_r}(\mathcal{X}) \rightarrow \mathcal{X}$ to get a representable morphism $\mathcal{G} \rightarrow \mathcal{X}$. This function on objects can be extended to a function on arrows, and it defines the desired functor.

We claim that this an equivalence: let us construct an inverse $I_{\mu_r}(\mathcal{X}) \rightarrow I_{\mu_r}(\mathcal{X})//\mu_r$. Consider an object of $I_{\mu_r}(\mathcal{X})$, consisting of a gerbe $\mathcal{G} \rightarrow T$ banded by $\mu_r$, and a representable morphism $\phi : \mathcal{G} \rightarrow \mathcal{X}$. Given an object $\xi$ of $\mathcal{G}(U)$, where $U$ is a $T$-scheme, the morphism $\phi$ induces a homomorphism of
group-schemes \( \alpha : H_U = Aut_{U,G}(\xi) \to Aut_{U,X}(\phi \xi) \); and this homomorphism is an embedding, because a representable morphism is also faithful, so \((\xi, \alpha)\) is an object of \( I_{\mu}(X) \). This function from objects of \( \mathcal{G} \) to objects of \( I_{\mu}(X) \) extends naturally to a morphism \( \mathcal{G} \to I_{\mu}(X) \); and this morphism is \( \mu_r \)-equivariant, by definition.

We leave it to the reader to extend this to a functor \( I_{\mu}(X) \to I_{\mu}(X) \), and show that it gives an inverse to the functor \( I_{\mu}(X)\mu_r \to I_{\mu}(X) \) above. The commutativity of the diagram is straightforward.

**Corollary 3.4.2.** If \( \mathcal{X} \) is smooth, \( I_{\mu}(\mathcal{X}) \) is smooth as well. If \( \mathcal{X} \) is proper, \( I_{\mu}(\mathcal{X}) \) is proper as well.

**Proof.** This follows from Proposition 3.1.2 and Corollary 3.1.4, since the morphism \( I_{\mu}(\mathcal{X}) \to I_{\mu}(\mathcal{X}) \) is proper, étale and surjective.

3.5. **Changing the band by a group automorphism.** There is an involution \( \iota : I_{\mu}(\mathcal{X}) \to I_{\mu}(\mathcal{X}) \) defined as follows: given a gerbe \( \mathcal{G} \to T \) banded by a group-scheme \( \mathcal{G} \) and an automorphism \( \tau : G \to G \), we can change the banding of the gerbe through the automorphism \( \tau \). Applying this procedure to the gerbe \( \mathcal{G} \to \mathcal{X} \) of \( I_{\mu_r}(X)(T) \), we get another object \( \tau^* \mathcal{G} \to \mathcal{X} \) of \( I_{\mu_r}(X)(T) \). When \( \tau : \mu_r \to \mu_r \) is the inversion automorphism \( \zeta \mapsto \zeta^{-1} \), this induces an involution of \( I_{\mu_r}(X) \). Applying this to each piece of \( I_{\mu}(\mathcal{X}) \) separately, we obtain the desired involution of \( \iota : I_{\mu}(\mathcal{X}) \to I_{\mu}(\mathcal{X}) \).

3.6. **The tangent bundle lemma.**

**Lemma 3.6.1.** Let \( S \) be a scheme, and let \( f : S \to I_{\mu}(\mathcal{X}) \) be a morphism. Let

\[
\begin{array}{ccc}
\mathcal{G} & \xrightarrow{F} & \mathcal{X} \\
\pi \downarrow & & \downarrow \\
S & & I_{\mu}(\mathcal{X})
\end{array}
\]

be the associated diagram. Then there is a canonical isomorphism between \( \pi^*(\pi^*(T\mathcal{X})) \) and \( f^*(T_{I_{\mu}(\mathcal{X})}) \).

**Proof.** Consider the universal gerbe \( I_{\mu}(\mathcal{X}) \to I_{\mu}(\mathcal{X}) \) and the diagram of smooth stacks

\[
\begin{array}{ccc}
I_{\mu}(\mathcal{X}) & \xrightarrow{F_1} & \mathcal{X} \\
\downarrow & & \downarrow \\
I_{\mu}(\mathcal{X}) & & I_{\mu}(\mathcal{X})
\end{array}
\]

is an embedding, because a representable morphism is also faithful, so \((\xi, \alpha)\) is an object of \( I_{\mu}(X) \). This function from objects of \( \mathcal{G} \) to objects of \( I_{\mu}(X) \) extends naturally to a morphism \( \mathcal{G} \to I_{\mu}(X) \); and this morphism is \( \mu_r \)-equivariant, by definition.

We leave it to the reader to extend this to a functor \( I_{\mu}(X) \to I_{\mu}(X) \), and show that it gives an inverse to the functor \( I_{\mu}(X)\mu_r \to I_{\mu}(X) \) above. The commutativity of the diagram is straightforward.

**Corollary 3.4.2.** If \( \mathcal{X} \) is smooth, \( I_{\mu}(\mathcal{X}) \) is smooth as well. If \( \mathcal{X} \) is proper, \( I_{\mu}(\mathcal{X}) \) is proper as well.

**Proof.** This follows from Proposition 3.1.2 and Corollary 3.1.4, since the morphism \( I_{\mu}(\mathcal{X}) \to I_{\mu}(\mathcal{X}) \) is proper, étale and surjective.

3.5. **Changing the band by a group automorphism.** There is an involution \( \iota : I_{\mu}(\mathcal{X}) \to I_{\mu}(\mathcal{X}) \) defined as follows: given a gerbe \( \mathcal{G} \to T \) banded by a group-scheme \( \mathcal{G} \) and an automorphism \( \tau : G \to G \), we can change the banding of the gerbe through the automorphism \( \tau \). Applying this procedure to the gerbe \( \mathcal{G} \to \mathcal{X} \) of \( I_{\mu_r}(X)(T) \), we get another object \( \tau^* \mathcal{G} \to \mathcal{X} \) of \( I_{\mu_r}(X)(T) \). When \( \tau : \mu_r \to \mu_r \) is the inversion automorphism \( \zeta \mapsto \zeta^{-1} \), this induces an involution of \( I_{\mu_r}(X) \). Applying this to each piece of \( I_{\mu}(\mathcal{X}) \) separately, we obtain the desired involution of \( \iota : I_{\mu}(\mathcal{X}) \to I_{\mu}(\mathcal{X}) \).

3.6. **The tangent bundle lemma.**

**Lemma 3.6.1.** Let \( S \) be a scheme, and let \( f : S \to I_{\mu}(\mathcal{X}) \) be a morphism. Let

\[
\begin{array}{ccc}
\mathcal{G} & \xrightarrow{F} & \mathcal{X} \\
\pi \downarrow & & \downarrow \\
S & & I_{\mu}(\mathcal{X})
\end{array}
\]

be the associated diagram. Then there is a canonical isomorphism between \( \pi^*(\pi^*(T\mathcal{X})) \) and \( f^*(T_{I_{\mu}(\mathcal{X})}) \).

**Proof.** Consider the universal gerbe \( I_{\mu}(\mathcal{X}) \to I_{\mu}(\mathcal{X}) \) and the diagram of smooth stacks

\[
\begin{array}{ccc}
I_{\mu}(\mathcal{X}) & \xrightarrow{F_1} & \mathcal{X} \\
\downarrow & & \downarrow \\
I_{\mu}(\mathcal{X}) & & I_{\mu}(\mathcal{X})
\end{array}
\]
Given a morphism \( f : S \to \mathcal{I}_\mu(\mathcal{X}) \) as in the lemma, we have a fiber diagram

\[
\begin{array}{ccc}
G & \xrightarrow{g} & \mathcal{I}_\mu(\mathcal{X}) \\
\downarrow & & \downarrow \pi \\
S & \xrightarrow{f} & \mathcal{I}_\mu(\mathcal{X}).
\end{array}
\]

Since \( \pi \) is flat and \( \pi_* \) is exact on coherent sheaves, for any locally free sheaf \( H \) on \( \mathcal{I}_\mu(\mathcal{X}) \) we have \( f^* \pi_* H = \pi_* g^* H \). Therefore it suffices to check that \( \pi_* (F^1_\mu(T_X)) \) and \( T_{\mathcal{I}_\mu(\mathcal{X})} \) are canonically isomorphic.

We have a natural morphism of sheaves \( T_{\mathcal{I}_\mu(\mathcal{X})} \to F^1_\mu(T_X) \), giving a morphism \( \pi_* T_{\mathcal{I}_\mu(\mathcal{X})} \to \pi_* F^1_\mu(T_X) \). Since \( \mathcal{I}_\mu(\mathcal{X}) \to \mathcal{I}_\mu(\mathcal{X}) \) is an étale gerbe, we have \( \pi_* T_{\mathcal{I}_\mu(\mathcal{X})} \cong T_{\mathcal{I}_\mu(\mathcal{X})} \), giving a morphism \( T_{\mathcal{I}_\mu(\mathcal{X})} \to \omega_* F^1_\mu(T_X) \). We can check that this is an isomorphism by pulling back to geometric points.

Over a geometric point \( y \) of \( \mathcal{I}_\mu(\mathcal{X}) \), we can identify the fiber of \( \pi \) with \( B_{\mu_r} \). This gives a lift of \( y \) to \( \mathcal{I}_\mu(\mathcal{X}) \), and so the point \( y \) maps to a geometric point \( x \) of \( \mathcal{X} \), with stabilizer \( G \). We can locally describe \( \mathcal{X} \) around \( x \) as \( \mathcal{X} = \mathcal{X}/G \). The pullback \( T \) of the tangent space of \( \mathcal{X} \) to \( y \) has a natural action of \( \mu_r \), and the fiber of \( \pi_* (F^1_\mu(T_X)) \) at \( y \) is naturally the space of invariants \( T^{\mu_r} \). Given a local chart of \( \mathcal{X} \) of the form \( \mathcal{X}/G \), the stack \( \mathcal{I}_\mu(\mathcal{X}) \) has a local chart given by \( \mathcal{X}/\mu_r \). Since \( T_{\mathcal{I}_\mu(\mathcal{X})} \to T^\mu_r \), we obtain a natural isomorphism \( T_{\mathcal{I}_\mu(\mathcal{X})} \cong T^\mu_r \), which is what we need.

4. Twisted curves and their maps

The foundation of the theory of stable maps to an orbifold rests on the notion of a twisted curve. Over an algebraically closed field, a twisted curve is a connected, one-dimensional Deligne–Mumford stack which is étale locally a nodal curve, and which is a scheme outside the marked points and the singular locus. Moreover, we will always include the condition that the nodes be balanced, that is, formally locally near a node, the stack is isomorphic to \( \left[ \text{Spec}(k[x,y]/(xy)) / \mu_r \right] \).

where the action of \( \mu_r \) is given by \( \zeta(x,y) = (\zeta \cdot x, \zeta^{-1} \cdot y) \). In particular, the coarse moduli space of a twisted curve is always a nodal curve.

The notion of a family of twisted curves is straightforward, but involves one novelty. Although the relative coarse moduli scheme of a family of twisted pointed curves is a family of prestable curves — and hence comes with sections corresponding to the marked points — a family of twisted curves need not have sections. The marked point on each fiber instead gives rise to a gerbe banded by \( \mu_r \) where \( r \) is the order of the inertia group at the twisted point.
4.1. The stack of twisted curves. If we define a groupoid $\mathcal{M}_{g,n}^{\text{tw}}$ whose $S$ points are given by families of twisted curves over $S$, that is morphisms $\pi: C \to S$ which are flat, together with a collection of $n$ disjoint gerbes over $S$ embedded in $C$, such that the geometric fibers of $\pi$ are $n$ pointed twisted curves of genus $g$, then the following results are proven in [O2].

1. $\mathcal{M}_{g,n}^{\text{tw}}$ is a smooth algebraic stack, locally of finite type.
2. If we bound the topological type, including the twisting at marked points and nodes, we get a stack of finite type.
3. A formal deformation space $\Delta^{\text{tw}}$ of a twisted curve $C$ with nodes $q_1, \ldots, q_s$ of indices $r_1, \ldots, r_s$ can be obtained from a given formal deformation space $\Delta = \text{Spf} R$ of the coarse curve $C$ as follows: Let $D_i$ be the divisor in $\Delta$ corresponding to $q_i$, with defining equations $f_i$. Then $\Delta^{\text{tw}} = \text{Spf} R[[x_1, \ldots, x_s]]/(x_1^{r_1} - f_1, \ldots, x_s^{r_s} - f_s)$ is a formal deformation space of $C$.

4.2. The smooth locus of a twisted pointed curve. The theory in this section is due to the authors and to C. Cadman independently ([C1], Section 2). The reader is advised to read Appendix B.2 for the notion of root stacks.

Suppose that $C \to S$ is an $n$-pointed nodal curve; call $s_i: S \to C$ the sections, $S_i \subseteq C$ their images, $\sigma_i$ the canonical section of $\mathcal{O}(S_i)$. Given positive integers $d_1, \ldots, d_n$, we define a stack over $S$ as the fibered product

$$C[d_1, \ldots, d_n] = \sqrt[d_1]{\mathcal{O}(S_1), \sigma_1}/C \times \cdots \times \sqrt[d_n]{\mathcal{O}(S_n), \sigma_n}/C.$$ 

See Appendix B.2 for an explanation of the notation. The locus where the projection

$$\sqrt[d_i]{\mathcal{O}(S_i), \sigma_i}/C \to C$$

is not an isomorphism coincides with $S_i$ when $d_i > 1$; if $d_i = 1$ the projection is an isomorphism. Outside of the locus $S_i$ the stack $C[d_1, \ldots, d_n]$ is isomorphic to $C$; over $S_i$ we have an embedding

$$\sqrt[d_i]{\mathcal{O}(S_i), \sigma_i}/C|_{S_i} \hookrightarrow \sqrt[d_i]{\mathcal{O}(S_i), \sigma_i}/C,$$

since we have that for $j \neq i$ the morphism $\sqrt[d_i]{\mathcal{O}(S_i), \sigma_i}/C \to C$ is an isomorphism in a neighborhood of $S_i$, we also have an embedding

$$\sqrt[d_i]{\mathcal{O}(S_i)}|_{S_i} \hookrightarrow C[d_1, \ldots, d_n].$$

It is easily checked that, after taking the $\sqrt[d_i]{\mathcal{O}(S_i)}|_{S_i}$ as markings, the stack $C[d_1, \ldots, d_n]$ is a twisted curve over $S$ with moduli space equal to $C$.

The following theorem shows that the smooth part of a twisted curve is uniquely characterized by its moduli space and its indices along the markings. If $C \to S$ is a twisted curve, we will denote its smooth part by $C_{\text{sm}}$.
Theorem 4.2.1. Let \( C \rightarrow S \) be an \( n \)-pointed twisted curve with moduli space \( \pi : C \rightarrow C \). Assume that its index at the \( i \)-th section is constant for all \( i \), and call it \( d_i \). Denote by \( \Sigma_i \) the \( i \)-th marking, by \( s_i : S \rightarrow C \) the section corresponding to \( \Sigma_i \), and by \( N_i \) the normal bundle to \( S \) along \( s_i \).

(1) There is a canonical isomorphism of twisted curves
\[
C_{\text{sm}} \simeq C_{[d_1, \ldots, d_n]_{\text{sm}}}
\]
inducing the identity on \( C_{\text{sm}} \).

(2) There is a canonical isomorphism of gerbes over \( S \)
\[
\Sigma_i \simeq \mu_{d_i}
\]

In particular, \( \Sigma_i \) is canonically banded by \( \mu_{d_i} \).

Proof. For part (1), notice that we have an equality of a Cartier divisor \( \pi^* S_i = d_i \Sigma_i \) on \( C \); this induces a morphism \( C \rightarrow C_{[d_1, \ldots, d_n]} \). This is an isomorphism outside of the marked points and the nodes, and is easily seen to be representable outside of the nodes. To check that the restriction \( C_{\text{sm}} \rightarrow C_{[d_1, \ldots, d_n]_{\text{sm}}} \) is an isomorphism it is enough to restrict to the geometric fibers of \( S \), because \( C \) and \( C_{[d_1, \ldots, d_n]} \) are flat over \( S \); but the morphism \( C_{\text{sm}} \rightarrow C_{[d_1, \ldots, d_n]_{\text{sm}}} \) restricted to a geometric fiber is representable, finite and birational, and the stacks appearing are smooth, hence it is an isomorphism.

Part (2) follows immediately from part (1).

4.3. Twisted stable maps. Let \( \mathcal{X} \) be a Deligne–Mumford stack, \( g, n \) non-negative integers and \( \beta \) a curve class on the coarse moduli space \( \mathcal{X} \). Associated with this data we have the stack \( \mathcal{K}_{g,n}(\mathcal{X}, \beta) \) of \( n \)-pointed twisted stable maps into \( \mathcal{X} \) of genus \( g \) and class \( \beta \). This classifies stable maps from \( n \)-pointed twisted genus \( g \) curves to \( \mathcal{X} \) of degree \( \beta \). Precisely, an object of this stack over a scheme \( T \) consists of the data of a family of twisted curves \( C \rightarrow T \), \( n \) gerbes \( \Sigma_i \subset C \), and a representable morphism \( C \rightarrow \mathcal{X} \), such that the induced maps of underlying coarse moduli spaces give a family of \( n \) pointed genus \( g \) stable maps to \( X \). We refer the reader to [AV] for a construction and a more detailed discussion of this stack.

Let \( C \rightarrow T \) be a twisted \( n \)-pointed curve with \( T \) connected, and let \( 1 \leq i \leq n \). Assume that the index of the \( i \)-th marking is the integer \( r \). Note that, by Theorem 4.2.1 part (2) we have that \( \Sigma_i^C \) is canonically banded by \( \mu_r \).

4.4. Evaluation maps.

Definition 4.4.1.

(1) Assume given a twisted stable map \( f : C \rightarrow \mathcal{X} \) over a base \( T \). We define
\[
e_i(f) \in T_{\mu_r}(\mathcal{X})(T)
\]
to be the object associated with the diagram

\[
\begin{array}{ccc}
\Sigma_c^i & \xrightarrow{f_i \circ c} & \mathcal{X} \\
\downarrow & & \downarrow \\
T & & \\
\end{array}
\]

By Definition 3.3.4, this defines a morphism

\[ e_i : \mathcal{K}_{g,n}(\mathcal{X}, \beta) \to \mathcal{F}_\mu(\mathcal{X}), \]

which we call the \textit{i-th evaluation map}.

(2) The morphism \( \tilde{e}_i := \iota \circ e_i \), where \( \iota : \mathcal{F}_\mu(\mathcal{X}) \to \mathcal{F}_\mu(\mathcal{X}) \) is the involution defined in section 3.5, is called the \textit{i-th twisted evaluation map}.

4.5. The virtual fundamental class. The key technical point in developing Gromov-Witten theory for \( \mathcal{X} \) is the construction of the \textit{virtual fundamental class} \( [\mathcal{K}_{g,n}(\mathcal{X}, \beta)]_{vir} \) in \( A_*(\mathcal{K}_{g,n}(\mathcal{X}, \beta)) \). By [BF] and [LT], what is needed to construct this class is a perfect obstruction theory on this moduli stack. Following the methods of [BF], we will mean by this a morphism in the derived category

\[ \phi : E \to \mathcal{L}_{\mathcal{K}_{g,n}(\mathcal{X}, \beta)/\mathcal{M}_{tw}^{g,n}} \]

such that

- \( E \) is locally equivalent to a two term complex of locally free sheaves, and
- \( H^0(\phi) \) is an isomorphism and \( H^{-1}(\phi) \) is surjective.

As in the case of ordinary stable maps, there is a natural perfect obstruction theory with \( E = R\pi_*((f^*T\mathcal{X})^\vee) \). The proof of this is identical to the proof for ordinary stable maps since what is needed are formal properties of the cotangent complex and Illusie’s results [I] relating these to the deformation theory of morphisms. Since the theory of the cotangent complex for Artin stacks has been developed in [LMB] and corrected in [O3], and since Illusie explicitly works in the general setting of ringed topoi, all the necessary generalizations have already been established.

Specifically, in the discussion of [B] page 604, immediately after Proposition 4, one relies on the claim that

\[ \phi : R\pi_*((f^*T\mathcal{X})^\vee) \to \mathcal{L}_{\mathcal{K}_{g,n}(\mathcal{X}, \beta)/\mathcal{M}_{tw}^{g,n}} \]

is a perfect \textit{relative} obstruction theory. This relative case, discussed in section 7 of [BF] (page 84 onward), reworks the absolute case discussed earlier in that paper. The crucial result in [BF] is Proposition 6.3, where \( \mathcal{C} \) is assumed to be a Gorenstein and \textit{projective} curve. As explained above, projectivity is not necessary for deformation theory (i.e. [BF] Theorem 4.5) - it works just as well for a proper Deligne–Mumford stack. Both in [BF] Proposition 6.3
and in [BF] Lemma 6.1 on which it relies, one also needs relative duality, which is “well known” for proper Gorenstein Deligne–Mumford stacks; for a twisted curve $C$ with a projective coarse moduli space it can be shown using a finite flat Galois covering $D \to C$, ramified over auxiliary sections, which can be constructed locally over the base.

We remark that one additional feature of $E$ that is required in [BF] is that $E$ admit a global resolution (see discussion before [BF], Proposition 5.2). Kresch’s work on intersection theory for Artin stacks [K] (see specifically section 5.2 there) has removed the need for this hypothesis.

5. The boundary of moduli

5.1. Boundary of the stack of twisted curves. We will need to study the geometry of the moduli stack of pre-stable twisted curves, $\mathcal{M}_{g,n}^{tw}$, as described in [O2]. In particular we are interested in the structure of the boundary. We consider the following category

$$\mathcal{D}^{tw}(g_1; A | g_2; B)$$

fibered in groupoids over the category of schemes. Informally, this category parametrizes nodal twisted curves, with a distinguished node separating the curve in two connected components, one of genus $g_1$ containing the markings in a subset $A \subset \{1, \ldots, n\}$, the other, of genus $g_2$, containing the markings in the complementary set $B$. More formally the objects over a scheme $S$ consist of commutative diagrams

$$\xi = \left( \begin{array}{ccc} \mathcal{G}_1 & \mathcal{G}_2 \\ \downarrow & & \downarrow \\ \mathcal{C}_1 & \mathcal{C}_2 \\ \uparrow & & \uparrow \\ S & \\ \downarrow & \downarrow \\ \mathcal{C}_1 & \mathcal{C}_2 \end{array} \right)$$

where

1. $C_1 \to S$ is a pre-stable twisted curve of genus $g_1$ with marking in $A \sqcup \bullet$,
2. $C_2 \to S$ is a pre-stable twisted curve of genus $g_2$ with marking in $B \sqcup \bullet$,
3. $\mathcal{G}_1$ and $\mathcal{G}_2$ are the markings on $C_1$ and $C_2$ corresponding to $\bullet$ and $\bullet$, respectively, and
4. $\alpha$ is an isomorphism inverting the band.
An arrow of $\mathcal{D}^{tw}(g_1; A \mid g_2; B)$ is a fiber diagram

This, in particular, includes the data of a 2-isomorphism in the following square:

**Remark 5.1.1.** In any reasonable framework of 2-stacks, $\mathcal{D}^{tw}(g_1; A \mid g_2; B)$ should be the fibered product

$$
\mathcal{M}^{tw}_{g_1; A,t} \times_{\mathcal{B}\mathcal{M}_{\mu}} \mathcal{M}^{tw}_{g_2; B,t}.
$$

Here $\mathcal{B}\mathcal{M}_{\mu}$ is the classifying 2-stack of $\mathcal{B}\mathcal{M}_{\mu}$, which parametrizes gerbes banded by $\mu$, and the the first morphism underlying the product has the band inverted. Since this 2-stack occurs only as the basis of the fibered product, the result is still a 1-category.

There is a similar construction for non-separating nodes, which we will not describe explicitly here.

**Proposition 5.1.2.** The category $\mathcal{D}^{tw}(g_1; A \mid g_2; B)$ is a smooth algebraic stack, locally of finite type over the base field $k$.

**Proof.** Consider the universal gerbes $\mathcal{G}_1$ and $\mathcal{G}_2$ corresponding to the markings $\bullet$ and $\bullet$ over the product $\mathcal{M}^{tw}_{g_1; A, \bullet} \times \mathcal{M}^{tw}_{g_2; B, \bullet}$. By Theorem 1.1 of [O1], there exists an algebraic stack

$$\text{Isom}^{tw}_{g_1; A, \bullet} \times_{\mathcal{M}^{tw}_{g_2; B, \bullet}} (\mathcal{G}_1, \mathcal{G}_2)$$

parametrizing isomorphisms between $\mathcal{G}_1$ and $\mathcal{G}_2$. The “change of band” isomorphism $\mu_r \rightarrow \mu_r$ induced by such an isomorphism $\mathcal{G}_1 \rightarrow \mathcal{G}_2$ is locally constant, and $\mathcal{D}^{tw}(g_1; A \mid g_2; B)$ is the locus where it is the inversion isomorphism. Since the $\mathcal{G}_i$ are étale gerbes, the smoothness of the Isom stack follows immediately from that of $\mathcal{M}^{tw}_{g, n}$. $\clubsuit$
Proposition 5.1.3. We have a natural representable morphism
\[ gl : \mathcal{D}^{tw}(g_1; A \mid g_2; B) \to \mathfrak{M}^{tw}_{g_1+g_2, A\sqcup B} \]
induced by gluing the two families of curves over \( \mathcal{D}^{tw} \) into a family of reducible curves with a distinguished node.

**Proof.** Fix an object of \( \mathcal{D}^{tw}(g_1; A \mid g_2; B) \) over \( S \). By Proposition [A.1.1] applied to the diagram
\[
\begin{array}{ccc}
G_1 & \hookrightarrow & C_2 \\
\downarrow & & \downarrow \\
C_1 & & \\
\end{array}
\]
we have an associated family of nodal curves \( C := C_1 \cup G_1 C_2 \). Representability follows from a straightforward comparison of isotropy groups. ♣

**Definition 5.1.4.** We define the locally constant function
\[ r : \mathcal{D}^{tw}(g_1; A \mid g_2; B) \to \mathbb{Z} \]
(this is a Gothic “r”) which takes a nodal twisted curve to the index of the node.

5.2. **Boundary of the stack of twisted stable maps.** There is an analogous gluing map on the spaces of morphisms.

**Proposition 5.2.1.**

1. Consider the evaluation morphisms
\[ \hat{e} \bullet : K_{g_1, A\sqcup \bullet}(\mathcal{X}, \beta_1) \to \mathcal{T}_\mu(\mathcal{X}) \]
and
\[ e \bullet : K_{g_2, B\sqcup \bullet}(\mathcal{X}, \beta_2) \to \mathcal{T}_\mu(\mathcal{X}) \]
There exists a natural representable morphism
\[ K_{g_1, A\sqcup \bullet}(\mathcal{X}, \beta_1) \times_{\mathcal{T}_\mu(\mathcal{X})} K_{g_2, B\sqcup \bullet}(\mathcal{X}, \beta_2) \to K_{g_1+g_2, A\sqcup B}(\mathcal{X}, \beta_1 + \beta_2). \]

2. Consider the evaluation morphisms
\[ \hat{e} \bullet \times e \bullet : K_{g-1, A\sqcup \bullet}(\mathcal{X}, \beta) \to \mathcal{T}_\mu(\mathcal{X})^2 \]
There exists a natural representable morphism
\[ K_{g-1, A\sqcup \bullet}(\mathcal{X}, \beta) \times_{\mathcal{T}_\mu(\mathcal{X})^2} \mathcal{T}_\mu(\mathcal{X}) \to K_{g, A}(\mathcal{X}, \beta). \]

**Proof.** We prove the first statement, the second being similar, replacing Proposition [A.1.1] with Corollary [A.1.2].
We give the morphism on the level of \( S \)-valued points. We have an identification of objects

\[
\mathcal{K}_{g_1, A + B} (\mathcal{X}, \beta_1) \times_{\mathcal{I}_\mu (\mathcal{X})} \mathcal{K}_{g_2, B + B} (\mathcal{X}, \beta_2) (\mathcal{S}) = \begin{cases} 
\mathcal{G}_1 \ar[r]^\alpha \ar[d] & \mathcal{G}_2 \ar[d] \\
\mathcal{C}_1 \ar[r] \ar[d] & \mathcal{X} \ar[d] \ar[r] & \mathcal{C}_2 \\
S & & \end{cases}
\]

where the diagram is a 1-commutative diagram of stacks, \( \mathcal{G}_1 \subset \mathcal{C}_i \) are the markings corresponding to \( \bullet \) and \( \circ \), respectively, and \( \alpha : \mathcal{G}_1 \to \mathcal{G}_2 \) is the isomorphism inverting the band induced by \( \bar{e}_\bullet \) and \( e_\bullet \).

By Proposition A.1.1 applied to the diagram

\[
\mathcal{C}_1 \ar[d] \ar[r] & \mathcal{C}_2 \\
\mathcal{G}_1 \ar[r] & \mathcal{C}_2
\]

we have an associated family of nodal curves \( \mathcal{C} := \mathcal{C}_1 \cup_{\mathcal{G}_1} \mathcal{C}_2 \). Since Diagram (1) is commutative, we are given a 2-isomorphism between the two resulting maps \( \mathcal{G}_1 \to \mathcal{X} \). Therefore, by the universal property of \( \mathcal{C} \), we have a morphism \( \mathcal{C} \to \mathcal{X} \), which is clearly a twisted stable map over \( S \).

**Proposition 5.2.2.** We have a cartesian diagram

\[
\prod_{\beta_1 + \beta_2 = \beta} \mathcal{K}_{g_1, A + B} (\mathcal{X}, \beta_1) \times_{\mathcal{I}_\mu (\mathcal{X})} \mathcal{K}_{g_2, B + B} (\mathcal{X}, \beta_2) \ar[r] \ar[d] & \mathcal{K}_{g_1 + g_2, A + B} (\mathcal{X}, \beta) \\
\mathcal{D}^{tw} (g_1; A \mid g_2; B) \ar[r]^{g_{I}} & \mathfrak{M}^{tw}_{g_1 + g_2, A + B}
\]

**Proof.** For convenience of notation we will use the shorthand

\[
\prod_{\beta_1 + \beta_2 = \beta} \mathcal{K}_{1} (\beta_1) \times_{\mathcal{I}_\mu (\mathcal{X})} \mathcal{K}_{2} (\beta_2) \ar[r] \ar[d] & \mathcal{K} (\beta) \\
\mathcal{D}^{tw} \ar[r] \ar[d] & \mathfrak{M}^{tw}
\]

The diagram gives a morphism

\[
\prod_{\beta_1 + \beta_2 = \beta} \mathcal{K}_{1} (\beta_1) \times_{\mathcal{I}_\mu (\mathcal{X})} \mathcal{K}_{2} (\beta_2) \to \mathcal{K} (\beta) \times_{\mathfrak{M}^{tw}} \mathcal{D}^{tw}.
\]
We construct a morphism in the reverse direction as follows. It suffices to restrict attention to points over a connected base scheme $S$. In this case an object on the right hand side is a triple $(f : C \to X, \xi, \phi)$, where $\xi$ is an object of $D^{tw}(S)$ and $\phi$ is an isomorphism between the resulting objects in $M^{tw}$, namely between $C$ and $C_1 \cup G_1 C_2$. Since $f : C \to X$ is stable, so are the resulting morphisms $C_i \to X$, with the additional markings taken into account. By the connectedness of the base, these maps have constant image classes $\beta_1, \beta_2$. With this, the diagram describing $\xi$ is completed to a diagram as in the description of an object on the left hand side, which is what we needed.

5.3. Gluing and virtual fundamental classes. For Gromov-Witten theory, one of the key points is that a similar statement holds for the virtual fundamental class. First, a crucial fact is that fundamental classes exist for the stacks of twisted curves we need. This is because, as mentioned in Section 4.1 away from a closed substack of arbitrarily high codimension, the stack of twisted curves of bounded indices is of finite type. We will show below (Lemma 6.2.4) that $gl : D^{tw} \to M^{tw}$ is a finite unramified morphism. By [K], Section 4.1 it induces a pull-back homomorphism on Chow groups

$$gl^! : A_*(K(\beta)) \to \bigoplus_{\beta_1 + \beta_2 = \beta} A_*(K_1(\beta_1) \times_{I_\mu(X)} K_2(\beta_2)).$$

Pulling back the virtual class on $K(\beta)$ gives us a candidate for the virtual fundamental class of the boundary. There is another natural Chow class living on the fibered product spaces coming from the pull back by the diagonal morphism $\Delta : I_\mu(X) \to I_\mu(X)^2$. The splitting axiom in Gromov-Witten theory identifies these two classes.

**Proposition 5.3.1.**

$$gl^![K_{g_1 \cup B}(X, \beta)]_{vir} = \sum_{\beta_1 + \beta_2 = \beta} \Delta^\dagger([K_{g_1, A \cup B}(X, \beta_1)]_{vir} \times [K_{g_2, B \cup \{\cdot\}}(X, \beta_2)]_{vir}).$$

**Proof:** This proof is very similar to the proof of the splitting axiom for Gromov-Witten theory of schemes. First, we observe that the left hand side of our equation is the pullback of a relative virtual fundamental class under a change of base. It follows by Proposition 7.2 of [BF] that this left hand side is the relative virtual fundamental class of $K_1 \times_{I_\mu(X)} K_2$ over $D^{tw}$ with respect to the relative perfect obstruction theory $R\pi_*(f^*T X)$. We need to compare this to the right hand side. Here we use the basic compatibility result for virtual fundamental classes. The class $[K_1]_{vir} \times [K_2]_{vir}$ is the virtual fundamental class associated with the relative perfect obstruction theory on $K_1 \times K_2$ given by $R\pi_{1*}(f_1^*T X) \oplus R\pi_{2*}(f_2^*T X)$. This is an
immediate consequence of Proposition 5.7 of [BF]. By considering the normalization sequence for the family of nodal curves with a distinguished node over $K_1 \times_\underline{\mathcal{T}_\mu(\mathcal{X})} K_2$, we get the following distinguished triangle:

$$R\pi_*(f^*T\mathcal{X}) \to R\pi_1^*(f_1^*T\mathcal{X}) \oplus R\pi_2^*(f_2^*T\mathcal{X}) \to R\pi_{\Sigma}^*(f_{\Sigma}^*T\mathcal{X})$$

where $f_{\Sigma}$ denotes the restriction of $f$ to the gerbe which is the intersection of $C_1$ with $C_2$. By Proposition 5.10 of [BF], in order to prove the equality we want, we just need to identify $R\pi_{\Sigma}^*(f_{\Sigma}^*T\mathcal{X})$ with the normal bundle of the map $\Delta$. The normal bundle of $\Delta$ is obviously $T_{I^\mu}(\mathcal{X})$. Applying Lemma 3.6.1 to $S = K_1 \times_\underline{\mathcal{T}_\mu(\mathcal{X})} K_2$ with the morphism $F = f_{\Sigma}$ gives our result. ♣

An identical argument yields the analogous splitting axiom for a non-separating node.

**Proposition 5.3.2.**

$$gl^![K_{g,A}(\mathcal{X},\beta)]_{vir} = \Delta^![K_{g-1,A\cup{\bullet,\tilde{\bullet}}}(\mathcal{X},\beta_1)]_{vir}.$$

### 6. Gromov–Witten classes

#### 6.1. Algebraic Gromov–Witten classes.

**Definition 6.1.1.** We define a locally constant function $r : \underline{\mathcal{T}_\mu(\mathcal{X})} \to \mathbb{Z}$ by evaluating on geometric points: $(x, G) \mapsto r$, where $G$ is a gerbe banded by $\mu_r$. We view $r$ as an element in $A^0(\underline{\mathcal{T}_\mu(\mathcal{X})})$.

We now define Gromov–Witten Chow classes:

**Definition 6.1.2.** Fix integers $g, n$, Chow classes $\gamma_i \in A^*(\underline{\mathcal{T}_\mu(\mathcal{X})})_Q$, and a curve class $\beta$. We define a class in $A_*(\underline{\mathcal{T}_\mu(\mathcal{X})})_Q$ by the formula

$$\langle \gamma_1, \ldots, \gamma_n, * \rangle_{g,\beta}^X = r \cdot e_{n+1} \cdot \left( \prod_{i=1}^n e_i^* \gamma_i \right) \cap [K_{g,n+1}(\mathcal{X},\beta)]_{vir}.$$

We will suppress the superscript $\mathcal{X}$ when the target is clear, and the genus $g$ when $g = 0$. We will write $\langle \gamma_1, \gamma_2, \delta_A, * \rangle_{g,\beta}^X$ for an expression of the type $\langle \gamma_1, \gamma_2, \delta_{i_1}, \ldots, \delta_{i_m}, * \rangle_{g,\beta}^X$ with $A = \{i_1, \ldots, i_m\}$ subject to the convention $i_1 < \cdots < i_m$. The factor $r$ comes very naturally in the proof, though see Proposition 6.1.3 below for a way to avoid this factor.

#### 6.1.3. Alternative formalism.

There is at least one other way to define Gromov–Witten classes introduced in [AGV], and it is necessary to compare them. In that paper we defined

$$\underline{N}_{g,n}(\mathcal{X},\beta) = \Sigma_{\kappa_{g,n}(\mathcal{X},\beta)} \times \cdots \times \Sigma_{\kappa_{g,n}(\mathcal{X},\beta)} \Sigma_{\kappa_{g,n}(\mathcal{X},\beta)}^c.$$
the moduli stack of twisted stable maps with sections at the markings. It has degree \((\prod r_i)^{-1}\) over \(\mathcal{K}_{g,n}(\mathcal{X}, \beta)\). There are clearly natural evaluation maps

\[e_i^M : \mathcal{M}_{g,n}^{\text{vir}}(\mathcal{X}, \beta) \to \mathcal{I}_\mu(\mathcal{X})\]

to the non-rigidified cyclotomic inertia stack. Given \(n\) Chow classes \(\tilde{\gamma}_i \in A^*(\mathcal{I}_\mu(\mathcal{X}))_\mathbb{Q}\) on the non-rigidified cyclotomic inertia stack, we defined classes

\[\langle \tilde{\gamma}_1, \ldots, \tilde{\gamma}_n, \ast \rangle_{g,\beta}^{\mathcal{X}} = \tilde{e}_{n+1}^M \left( \prod_{i=1}^{n} e_i^M \ast \tilde{\gamma}_i \right) \cap \left[ \mathcal{M}_{g,n+1}(\mathcal{X}, \beta) \right]^{\text{vir}}.\]

This formalism is used in the work of Tseng [Ts].

There is another way to write down the same classes without need to introduce the stack \(\mathcal{M}_{g,n}(\mathcal{X}, \beta)\). Even though liftings \(\tilde{e}_i : \mathcal{K}_{g,n}(\mathcal{X}, \beta) \to \mathcal{I}_\mu(\mathcal{X})\) do not necessarily exist,

\[\begin{array}{ccc}
\mathcal{I}_\mu(\mathcal{X}) & \xrightarrow{\bar{e}_i} & \mathcal{K}_{g,n}(\mathcal{X}, \beta) \\
\mathcal{K}_{g,n}(\mathcal{X}, \beta) & \xrightarrow{e_i} & \mathcal{I}_\mu(\mathcal{X})
\end{array}\]

the isomorphism between the rational Chow groups (or cohomology groups) of \(\mathcal{I}_\mu(\mathcal{X})\) and \(\mathcal{I}_\mu(\mathcal{X})\) enables us to move from one to the other on the intersection theory level. A lifting \(\bar{e}_i \ast\) of \(e_i \ast\) is obtained by composing with the non-multiplicative isomorphism

\[(\varpi_\ast)^{-1} : A^*(\mathcal{I}_\mu(\mathcal{X}))_\mathbb{Q} \to A^*(\mathcal{I}_\mu(\mathcal{X}))_\mathbb{Q}.
\]

Note \((\varpi_\ast)^{-1} = r \cdot \varpi^\ast\), so

\[\bar{e}_i \ast = (\varpi_\ast)^{-1} \circ e_i \ast = r \varpi^\ast \circ e_i \ast.\]

Similarly we define

\[\bar{e}^\ast_i = e^\ast_i \circ (\varpi^\ast)^{-1}.\]

Since \((\varpi^\ast)^{-1} = r \cdot \varpi^\ast\) we can also write \(\bar{e}^\ast_i = r \cdot e^\ast_i \circ \varpi^\ast\). (We remark that the corresponding formula in [AGV] has \(r\) mistakenly replaced by \(r^{-1}\).)

The basic comparison result is

**Proposition 6.1.4.**

1. For Chow classes \(\gamma_i \in A^*(\mathcal{I}_\mu(\mathcal{X}))_\mathbb{Q}\), we have

\[\varpi^\ast(\gamma_1, \ldots, \gamma_n, \ast)_{g,\beta}^{\mathcal{X}} = (\varpi^\ast \gamma_1, \ldots, \varpi^\ast \gamma_n, \ast)_{g,\beta}^{\mathcal{X}}\]

2. For Chow classes \(\bar{\gamma}_i \in A^*(\mathcal{I}_\mu(\mathcal{X}))_\mathbb{Q}\), we have

\[\langle \bar{\gamma}_1, \ldots, \bar{\gamma}_n, \ast \rangle_{g,\beta}^{\mathcal{X}} = (\varpi_\ast \bar{e}_{n+1} \ast) \left( \prod_{i=1}^{n} \bar{e}_i \ast \bar{\gamma}_i \right) \cap \left[ \mathcal{K}_{g,n+1}(\mathcal{X}, \beta) \right]^{\text{vir}}.\]
The first part shows that, if one identifies 
\( A^\ast(\mathcal{I}_\mu(\mathcal{X}))_Q \) and 
\( A^\ast(\mathcal{I}_\mu(\mathcal{X}))_Q \)
using the multiplicative homomorphism \( \varpi^\ast \), the Gromov–Witten classes are unchanged. The second part says that, if one is willing to pretend a lifting \( \tilde{e}_i : K_{g,n}(\mathcal{X}, \beta) \rightarrow \mathcal{I}_\mu(\mathcal{X}) \) exists, all the factors of \( r \) are removed from the formalism. While the rigidified inertia stack and evaluation map \( e_i : K_{g,n}(\mathcal{X}, \beta) \rightarrow \mathcal{I}_\mu(\mathcal{X}) \) is what arises naturally, the formalism using \( \tilde{e}_i \) and \( \mathcal{I}_\mu(\mathcal{X}) \) is probably the most convenient one to work with, and has been used in the work of Cadman \([C1],[C2]\). We will use this formalism in our example in section 9. A direct comparison was carried out in the example in \([A]\).

**Proof.** This is immediate using the non-cartesian commutative diagram

\[
\begin{array}{ccc}
\mathcal{M}_{g,n+1}(\mathcal{X}, \beta) & \xrightarrow{\tilde{e}_i^M} & \mathcal{I}_\mu(\mathcal{X}) \\
\rho \downarrow & & \downarrow \varpi \\
K_{g,n+1}(\mathcal{X}, \beta) & \xrightarrow{e_i} & \mathcal{I}_\mu(\mathcal{X}),
\end{array}
\]

where \( \deg \rho = (\prod r_i)^{-1} \) and \( \deg \varpi = 1/r_i \).

### 6.2. The WDVV Equation.

In genus 0 we have the *Witten–Dijkgraaf–Verlinde–Verlinde* (WDVV) equation:

**Theorem 6.2.1.**

\[
\sum_{\beta_1 + \beta_2 = \beta} \sum_{A \cup B = \{1, \ldots, n\}} \left\langle \gamma_1, \gamma_2, \delta_A, * \right\rangle_{\beta_1} \gamma_3, \delta_B, * \right\rangle_{\beta_2} = \\
\sum_{\beta_1 + \beta_2 = \beta} \sum_{A \cup B = \{1, \ldots, n\}} \left\langle \gamma_1, \gamma_3, \delta_A, * \right\rangle_{\beta_1} \gamma_2, \delta_B, * \right\rangle_{\beta_2}
\]

Note that the two lines differ precisely in the positions of \( \gamma_2 \) and \( \gamma_3 \).

**Proof.** Consider the stabilization morphism

\( st : K_{0,n+4}(\mathcal{X}, \beta) \rightarrow \overline{\mathcal{M}}_{0,4} \)

corresponding to forgetting the map to \( \mathcal{X} \), passing to coarse curves, and forgetting all the marking except the last four.

We show the equality by showing that both sides equal

\[
\Psi = r \cdot \tilde{e}_{n+4}^\ast \left( \left( st^\ast [pt] \cup \prod_{i=1}^n e_i^\ast \delta_i \cup \prod_{j=1}^3 e_{n+j}^\ast \gamma_j \right) \cap [K_{0,n+4}(\mathcal{X}, \beta)]^\text{vir} \right).
\]
Write $\hat{A} = A \sqcup \{n + 1, n + 2\}$ and $\hat{B} = B \sqcup \{n + 3, n + 4\}$. Consider the following diagram:

\[ \begin{array}{ccc}
\bigodot K_0 \times_{\mathcal{K}} \mathcal{K}_0 \times_{\mathcal{K}} \mathcal{K}_0 \times_{\mathcal{K}} \mathcal{K}_0 & \longrightarrow & \mathcal{K} \\
\bigodot \mathcal{D}^{tw}(\hat{A} \mid \hat{B}) & \xrightarrow{\phi} & \mathcal{M}_{0,n+4}^{tw} \\
\mathcal{D}^{tw}(12|34) & \xrightarrow{i_\phi} & \mathcal{O} \\
\mathcal{M}_{0,3} \times \mathcal{M}_{0,3} & \xrightarrow{i_\phi} & \mathcal{O} \\
\{pt\} & \xrightarrow{st} & \mathcal{M}_{0,4} \\
\end{array} \]

Proposition 6.2.2.

\[ st^*[pt] \cap [\mathcal{K}]^{vir} = l_* (e^* r \cdot g^! [\mathcal{K}]^{vir}). \]

**Proof.** We expand the diagram into the following cartesian diagram:

\[ \begin{array}{ccc}
\bigodot K_0 \times_{\mathcal{K}} \mathcal{K}_0 \times_{\mathcal{K}} \mathcal{K}_0 \times_{\mathcal{K}} \mathcal{K}_0 & \longrightarrow & \mathcal{K} \\
\bigodot \mathcal{D}^{tw}(\hat{A} \mid \hat{B}) & \xrightarrow{\phi} & \mathcal{M}_{0,n+4}^{tw} \\
\mathcal{D}^{tw}(12|34) & \xrightarrow{i_\phi} & \mathcal{O} \\
\mathcal{M}_{0,3} \times \mathcal{M}_{0,3} & \xrightarrow{i_\phi} & \mathcal{O} \\
\{pt\} & \xrightarrow{st} & \mathcal{M}_{0,4} \\
\end{array} \]

Lemma 6.2.3. All the morphisms in

\[ \mathcal{M}_{0,n+4}^{tw} \longrightarrow \mathcal{M}_{0,4}^{tw} \longrightarrow \mathcal{M}_{0,4} \longrightarrow \mathcal{M}_{0,4} \]

are flat.

**Proof.** The first arrow just forgets the first $n$ markings which is smooth. The second is locally a finite morphism of smooth stacks. The third is a dominant morphism from a smooth stack to a smooth curve.

Lemma 6.2.4. All horizontal arrows in this diagram are finite and unramified.
Proof. We first show the result for the arrow \( \mathcal{M}_{0,3} \times \mathcal{M}_{0,3} \rightarrow \mathcal{M}_{0,4} \). Consider the universal curve \( f_{0,4} : \mathcal{C}_{0,4} \rightarrow \mathcal{M}_{0,4} \), which is clearly a proper and representable morphism. There is a closed substack \( \text{Sing}(f_{0,4}) \subset \mathcal{C}_{0,4} \) consisting of the nodes of the universal curve, schematically defined by the first Fitting ideal of \( \Omega^1_{\mathcal{C}_{0,4}/\mathcal{M}_{0,4}} \). Since \( \text{Sing}(f_{0,4}) \rightarrow \mathcal{M}_{0,4} \) is representable, quasi-finite and proper, it is a finite morphism.

Inside \( \text{Sing}(f_{0,4}) \) we have an open and closed substack \( \Sigma \subset \text{Sing}(f_{0,4}) \) consisting of the nodes of the universal curve, schematically defined by the first Fitting ideal of \( \Omega^1_{\mathcal{C}_{0,4}/\mathcal{M}_{0,4}} \). Since \( \text{Sing}(f_{0,4}) \rightarrow \mathcal{M}_{0,4} \) is representable, quasi-finite and proper, it is a finite morphism.

We claim that there is an isomorphism \( \Sigma \cong \mathcal{M}_{0,3} \times \mathcal{M}_{0,3} \) over \( \mathcal{M}_{0,4} \), which proves the required property.

We construct a morphism \( \Sigma \rightarrow \mathcal{M}_{0,3} \times \mathcal{M}_{0,3} \) as follows. The pullback \( \mathcal{C}_\Sigma \rightarrow \Sigma \) of the universal pre-stable curve has a canonical section landing at the appropriate node. The normalization of \( \mathcal{C}_\Sigma \) is the disjoint union of two families of 3-pointed curves, obtained by separating the node marked by the section above. This gives the required morphism.

A morphism in the other direction can be constructed as follows. We have two universal families \( \mathcal{C}_1 = \mathcal{C}_{0,3} \times \mathcal{M}_{0,3} \) and \( \mathcal{C}_2 = \mathcal{M}_{0,3} \times \mathcal{C}_{0,3} \). We have a non-cartesian diagram

\[
\begin{array}{ccc}
\mathcal{C}_1 \sqcup \mathcal{C}_2 & \longrightarrow & \mathcal{C}_{0,4} \\
\downarrow & & \downarrow \\
\mathcal{M}_{0,3} \times \mathcal{M}_{0,3} & \longrightarrow & \mathcal{M}_{0,4}
\end{array}
\]

where the diagonal arrow is the gluing map of the third marking of \( \mathcal{C}_1 \) with the first marking of \( \mathcal{C}_2 \). Composing the third section \( \mathcal{M}_{0,3} \times \mathcal{M}_{0,3} \rightarrow \mathcal{C}_1 \) with this diagonal arrow (or, for that matter, using the first section of \( \mathcal{M}_{0,3} \times \mathcal{M}_{0,3} \rightarrow \mathcal{C}_2 \)), we get a morphism \( \mathcal{M}_{0,3} \times \mathcal{M}_{0,3} \rightarrow \mathcal{C}_{0,4} \), which obviously lands in \( \Sigma \). It is a simple local computation to check that the two arrows are inverses of each other.

The arrow \( i_P \) is the embedding of the reduced substack in \( \mathcal{P} \). All other arrows arise by base change.

As a consequence, we can make sense of the following lemma.

Lemma 6.2.5.

\[
i_\Omega^* \left[ \mathcal{M}_{0,3} \times \mathcal{M}_{0,3} \right] = [\Omega]
\]

and, using Definition 5.1.3

\[
i_P^* \left( \tau \cdot [\mathcal{D}^{tw}(12|34)] \right) = [\mathcal{P}]
\]

The first statement is well-known—see [5], Proposition 8. The second follows from the deformation theory of twisted curves, as mentioned in section 4.1. Proposition 6.2.2 now follows from the lemmas using the projection formula and the observation that \( e^*r = e^*r = \phi^*r \).
Corollary 6.2.6.

\begin{align*}
\Psi &= r \cdot e_{n+4}^* \left( \prod_{i=1}^{n} e_i^* \delta_i \cup \prod_{j=1}^{3} e_{n+j}^* \gamma_j \right) \cap \iota_* (e_* r \cdot gl^l [K]^{\text{vir}}) \\
&= r \cdot (\tilde{e}^* \circ \iota)_* \left( e_* r \cdot \prod_{i=1}^{n} l_i e_i^* \delta_i \cup \prod_{j=1}^{3} l_* e_{n+j}^* \gamma_j \cap gl^l [K]^{\text{vir}} \right)
\end{align*}

To simplify notation, we fix $K_1 = K_{0,\mathcal{A}*}(\mathcal{X}, \beta_1)$, $K_2 = K_{0,\mathcal{B}*}(\mathcal{X}, \beta_2)$, and $K = K_{0,n+4}(\mathcal{X}, \beta)$ and

$\eta_1 = e_{n+1}^* \gamma_1 \cup e_{n+2}^* \gamma_2 \cup \prod_{i \in A} e_i^* \delta_i$ and $\eta_2 = e_{n+3}^* \gamma_3 \cup \prod_{i \in B} e_i^* \delta_i$.

Consider the following diagram.

\[ \begin{array}{ccc}
K_1 \times \mathcal{I}_\mu(\mathcal{X}) & \xrightarrow{p_2} & K_2 \\
\downarrow \iota & & \downarrow r_* \\
K_1 & \xrightarrow{\tilde{\iota}} & \mathcal{I}_\mu(\mathcal{X})
\end{array} \]

The expression $\left\langle \gamma_1, \gamma_2, \delta_A, *, \gamma_3, \delta_B, * \right\rangle_{\beta_1, \beta_2}$ is by definition

\[ r \cdot \tilde{e}_{n+4}^* (\eta_2 \cup e_*^* \left( r \cdot \tilde{e}_{n+4}^* (\eta_1 \cap [K_1]^{\text{vir}}) \right) \cap [K_2]^{\text{vir}}) \]

By the following lemma, this expression equals

\[ r \cdot \tilde{e}_{n+4}^* (e_*^* r \cdot p_2^* \eta_2 \cup p_1^* \eta_1 \cap \Delta^l ([K_1]^{\text{vir}} \times [K_2]^{\text{vir}})). \]

Applying Proposition 5.3.1 and summing over $A, B$ and $\beta_1, \beta_2$ gives the Theorem.

\begin{lemma}
Lemma 6.2.7.

\[ \eta_2 \cup e_*^* \left( \tilde{e}_{n+4}^* (\eta_1 \cap [K_1]^{\text{vir}}) \right) \cap [K_2]^{\text{vir}} = p_2^* (p_1^* \eta_2 \cup p_1^* \eta_1 \cap \Delta^l ([K_1]^{\text{vir}} \times [K_2]^{\text{vir}})). \]

\end{lemma}

\begin{proof}
Set $\xi_i = \eta_i \cap [K_i]^{\text{vir}}$; then the left hand side of the equality is

\[ e_*^* \left( \tilde{e}_{n+4}^* (\xi_1) \right) \cap \xi_2, \]

while the right hand side is

\[ p_2^* \Delta^l (\xi_1 \times \xi_2). \]
\end{proof}
Consider the following cartesian diagram

\[
\begin{array}{ccc}
\mathcal{K}_1 \times \mathcal{I}_\mu (\mathcal{X}) & \xrightarrow{p_2} & \mathcal{K}_2 \\
\downarrow & & \downarrow \Delta \\
\mathcal{K}_1 \times \mathcal{K}_2 & \xrightarrow{\tilde{\epsilon} \times \text{id}} & \mathcal{I}_\mu (\mathcal{X}) \times \mathcal{K}_2 \\
\pi_\mathcal{K}_1 & & \pi_{\mathcal{K}_2} \times \epsilon_* \\
\mathcal{K}_1 & \xrightarrow{\tilde{\epsilon}^*} & \mathcal{I}_\mu (\mathcal{X})
\end{array}
\]

We have

\[
\tilde{\epsilon}^* (\tilde{\epsilon}^* (\xi_1) \cap \xi_2) = \Gamma_{\epsilon_*}^* \left( (\pi_{\mathcal{I}_\mu (\mathcal{X})}^* \tilde{\epsilon}^* \xi_1) \cap \xi_2 \right)
= \Gamma_{\epsilon_*}^* (\tilde{\epsilon}^* \times \text{id})_* ((\xi_1 \times \xi_2))
= p_{2*} \Gamma_{\epsilon_*}^* (\xi_1 \times \xi_2)
= p_{2*} \Delta_{\epsilon_*}^* (\xi_1 \times \xi_2).
\]

6.3. Topological Gromov–Witten classes. In this section the base field will be \( k = \mathbb{C} \). The definition of a Gromov–Witten cohomology class is the same as in Definition 6.1.2.

**Definition 6.3.1.** Fix integers \( g, n \), classes \( \gamma_i \in H^* (\mathcal{I}_\mu (\mathcal{X}))_Q \), and a curve class \( \beta \in N^+ (\mathcal{X}) \). We define

\[
\langle \gamma_1, \ldots, \gamma_n, * \rangle_{g, \beta}^\mathcal{X} = r \cdot \tilde{\epsilon}_{n+1*} \left( \prod_{i=1}^n e_i^* \gamma_i \right) \cap [\mathcal{K}_{g,n+1}(\mathcal{X}, \beta)]^{\text{vir}}
\]

where by \([\mathcal{K}_{g,n+1}(\mathcal{X}, \beta)]^{\text{vir}}\) we mean the homology class in \( H_* (\mathcal{K}_{g,n+1}(\mathcal{X}, \beta), \mathbb{Q}) \) corresponding to the virtual fundamental class

\([\mathcal{K}_{g,n+1}(\mathcal{X}, \beta)]^{\text{vir}} \in A_* (\mathcal{K}_{g,n+1}(\mathcal{X}, \beta))_\mathbb{Q} \).

In genus 0 we have again the WDVV equation. To state it precisely we need a sign convention.

We restrict the discussion to cohomology classes \( \gamma_i \) and \( \delta_i \) which are homogeneous with respect to the usual grading in \( H^* (\mathcal{I}_\mu (\mathcal{X})) \) - in fact homogeneous parity suffices; the formulas extend to the inhomogeneous case, but are less clean. For \( A \sqcup B = \{1, \ldots, n\} \) we can write \( \delta_A = \delta_{i_1} \wedge \cdots \wedge \delta_{i_m} \), where \( A = \{i_1, \ldots, i_m\} \) subject to the ordering convention \( i_1 < \cdots < i_m \), and similarly for \( \delta_B \). We define signs \((-1)^{e_i(A)}\) and \((-1)^{e_2(A)}\) so that

\[
(\gamma_1 \wedge \gamma_2 \wedge \gamma_3) \wedge (\delta_1 \wedge \cdots \wedge \delta_n) = (-1)^{e_1(A)} (\gamma_1 \wedge \gamma_2 \wedge \delta_A) \wedge (\gamma_3 \wedge \delta_B)
\]

and

\[
(\gamma_1 \wedge \gamma_2 \wedge \gamma_3) \wedge (\delta_1 \wedge \cdots \wedge \delta_n) = (-1)^{e_2(A)} (\gamma_1 \wedge \gamma_3 \wedge \delta_A) \wedge (\gamma_2 \wedge \delta_B),
\]
Of course the products could vanish, but the signs are formally well defined depending only on the parity of the classes $\gamma_i$ and $\delta_i$. With these conventions we have:

**Theorem 6.3.2.**

$$
\sum_{\beta_1+\beta_2=\beta} \sum_{A\sqcup B=\{1,\ldots,n\}} (-1)^{\varepsilon_1(A)} \left( \langle \gamma_1, \gamma_2, \delta_A, \ast \rangle_{\beta_1}, \gamma_3, \delta_B, \ast \rangle_{\beta_2} \right) = \\
\sum_{\beta_1+\beta_2=\beta} \sum_{A\sqcup B=\{1,\ldots,n\}} (-1)^{\varepsilon_2(A)} \left( \langle \gamma_1, \gamma_3, \delta_A, \ast \rangle_{\beta_1}, \gamma_2, \delta_B, \ast \rangle_{\beta_2} \right) .
$$

The proof is identical to that of WDVV in the algebraic case.

We stress that the degrees used in determining the signs above are the standard degrees in cohomology, which may be different from those in the age grading defined in the next section.

### 6.4. Gromov–Witten numbers.

The usual definition of Gromov–Witten numbers works without changes in our context.

**Definition 6.4.1.** Fix integers $g, n$, classes $\gamma_i \in H^*(\overline{\calI}_\mu(\calX))_Q$, and a curve class $\beta \in N^+(\calX)$. We define

$$
\langle \gamma_1, \ldots, \gamma_n \rangle_{g, \beta}^X = \int_{K_{g,n}(\calX, \beta)} \left( \left( \prod_{i=1}^n \epsilon_i^* \gamma_i \right) \cap \left[ K_{g,n}(\calX, \beta) \right]_{\text{vir}} \right)
$$

The two definitions are connected by the following Proposition. First, some notation: let $\alpha_1, \ldots, \alpha_M$ be a basis for the cohomology of $\overline{\calI}_\mu(\calX)$. We write $g_{ij} = \int_{\overline{\calI}_\mu(\calX)} \alpha_i \cap \iota^* \alpha_j$ and denote by $g^{-1}$ the inverse matrix.

**Proposition 6.4.2.**

1. $\langle \gamma_1, \ldots, \gamma_n \rangle^X_{g, \beta} = \int_{\overline{\calI}_\mu(\calX)} \frac{1}{r} \langle \gamma_1, \ldots, \gamma_{n-1}, \ast \rangle^X_{g, \beta} \cap \iota^*(\gamma_n)$.

2. $\langle \gamma_1, \ldots, \gamma_{n-1}, \ast \rangle^X_{g, \beta} = r \cdot \sum_{i,j=1}^M \langle \gamma_1, \ldots, \gamma_{n-1}, \alpha_i \rangle^X_{g, \beta} g^{ij} \alpha_j$

**Proof.** This is immediate from the projection formula, noting that $\tilde{e}_n = \iota \circ e_n$.

As in Section 6.1.3 and Proposition 6.1.4 we can again make these formulas appear even more analogous to the usual manifold case (i.e. we can remove the factors of $r$) by multiplying the intersection form on $H^*(\overline{\calI}_\mu(\calX))$ by $1/r$. We will denote this modified intersection form by $\tilde{g}_{ij}$, and correspondingly define $\tilde{g}^{ij} = r \cdot g^{ij}$. This is equivalent to pulling back the classes to $\calI_\mu(\calX)$ (or considering directly classes on $\calI_\mu(\calX)$) and doing the intersection there.

We can now state the WDVV equation in its classical form:
Theorem 6.4.3.

\[ \sum_{\beta_1 + \beta_2 = \beta} \sum_{i,j=1}^M (-1)^{c_i(A)} \langle \delta_A, \gamma_1, \gamma_2, \alpha_i \rangle_{\beta_1 \beta_2} \tilde{g}^{ij} \langle \alpha_j, \delta_B, \gamma_3, \gamma_4 \rangle_{\beta_2} = \]

\[ \sum_{\beta_1 + \beta_2 = \beta} \sum_{i,j=1}^M (-1)^{c_2(A)} \langle \delta_A, \gamma_1, \gamma_3, \alpha_1 \rangle_{\beta_1} \tilde{g}^{ij} \langle \alpha_j, \delta_B, \gamma_2, \gamma_4 \rangle_{\beta_2}. \]

7. The age grading

7.1. The age of a sheaf. Consider the group-scheme \( \mu_r \) over a field, with its representation ring \( R_{\mu_r} \). Each character \( \lambda: \mu_r \to G_m \) is of the form \( t \mapsto t^k \) for a unique integer \( k \) with \( 0 \leq k \leq r - 1 \); following M. Reid (see e.g. [IR]), we define the age of \( \lambda \) as \( k/r \). Since these characters form a basis for the representation ring of \( \mu_r \), this extends to a unique additive homomorphism \( \text{age}: R_{\mu_r} \to \mathbb{Q} \).

Now let \( \mathcal{G} \to T \) be a gerbe banded by \( \mu_r \), and let \( \mathcal{E} \) be a locally free sheaf on \( \mathcal{G} \). There is an étale covering \( \{ T_i \to T \} \) with sections \( T_i \to \mathcal{G} \), inducing an isomorphism \( \mathcal{G}_{T_i} \simeq B(\mu_r)_{T_i} \). Then the pullback of \( \mathcal{E} \) to \( \mathcal{G}_{T_i} \) becomes a locally free sheaf \( \mathcal{E}_{T_i} \) on \( T_i \) with an action of \( \mu_r \); and the age of each fiber is a locally constant invariant. Furthermore, this invariant is independent of the section, and the age of \( \mathcal{E} \) is a locally constant function on \( T \).

Consider a connected scheme \( T \), and an object of \( \mathcal{I}_{\mu}(X)(T) \), consisting of a gerbe \( \mathcal{G} \to T \) and a representable morphism \( f: \mathcal{G} \to X \). Then the age of the object is a rational number defined to be the age of the locally free sheaf \( f^* T_X \). This number only depends on the connected component of \( \mathcal{I}_{\mu}(X) \) containing the image of \( T \).

Definition 7.1.1. The age of a connected component \( \Omega \) of \( \mathcal{I}_{\mu}(X) \) is the age of any object of \( \Omega(T) \), where \( T \) is a connected scheme.

The age is called the degree-shifting number in [CR].

7.2. Riemann-Roch for twisted curves. Let \( \mathcal{C} \) be a balanced twisted curve over an algebraically closed field, \( \mathcal{E} \) a coherent sheaf on \( \mathcal{C} \), that is locally free at the nodes of \( \mathcal{E} \) (for the applications needed in this paper, the case of a locally free sheaf is sufficient; however, the added generality helps with the proof). Call \( \pi: \mathcal{C} \to C \) the coarse curve, \( p_1, \ldots, p_n \) the marked points on \( C \). For each \( i \) call \( r_i \) the index of \( \mathcal{C} \) at \( p_i \). For each \( i \) choose a section \( p_i \to \Sigma_i \) of the marking gerbe \( \Sigma_i \to p_i \), inducing an isomorphism \( \Sigma_i \simeq B(\mu_r)_{p_i} \). If \( \mathcal{E} \) is locally free at \( p_i \), we have defined above the age of \( \mathcal{E} \) at \( p_i \) as \( \text{age}_{p_i}(\mathcal{E}) = \text{age}(\mathcal{E} |_{\Sigma_i}) \).
In the general case, when $\mathcal{E}$ has torsion, this is not the correct definition. Consider the embedding 

$$\iota_i : E_{\mu_i} \cong \Sigma_i \hookrightarrow C;$$

it induces a pullback in the K-theory of coherent sheaves of finite projective dimension 

$$\iota^*_i : K_0(C) \longrightarrow K_0(E_{\mu_i}) = R\mu_{\mu_i};$$

via the usual formula 

$$\iota^*_i \mathcal{E} = [E \otimes O_C O_{\Sigma_i}] - [\text{Tor}_1^{O_C}(\mathcal{E}, O_{\Sigma_i})];$$

the correct general definition is 

$$\text{age}_{\mu_i}(\mathcal{E}) = \text{age}(\iota^*_i \mathcal{E}).$$

This gives an additive homomorphism 

$$\text{age}_{\mu_i} : K_0(C) \longrightarrow \mathbb{Q}.$$

By $\chi(\mathcal{E})$ we denote as usual the Euler characteristic of $\mathcal{E}$ on $C$. For each $i$ we have $H^i(C, \mathcal{E}) = H^i(C, \pi_* \mathcal{E})$, so the Euler characteristic is finite. In particular, since $\pi_* O_C = O_C$, we have 

$$\chi(O_C) = 1 - g$$

where $g$ is the arithmetic genus of $C$.

We define the degree of $\mathcal{E}$ as follows. First assume that $C$ is smooth and irreducible. Take an ordinary connected smooth curve $D$ with a finite morphism $\phi : D \to C$ (it is not hard to see that this exists). Call $d$ the degree of $\phi$. Then we set 

$$\deg C \mathcal{E} = \frac{1}{d} \deg_D \phi^* \mathcal{E}.$$

It is easy to see that the degree is independent of the choice of $\phi$: if $\phi' : D' \to C$ is another choice, call $D''$ a component of the normalization of the fibered product $D \times_C D'$. This dominates both $D$ and $D'$, and then one uses standard properties of the degree.

If $C$ is not irreducible, take the normalization $\nu : \overline{C} \to C$, pull back $\mathcal{E}$, and sum the degrees over the irreducible components of $\overline{C}$.

The degree is a rational number. If $\mathcal{E}$ is locally free, then it follows from the definition that the degree of $\mathcal{E}$ is also the degree of $\det \mathcal{E}$, as usual; and we also have the formula 

$$\deg C \mathcal{E} = c_1(\mathcal{E}) \cdot [C] - \int_C c_1(\mathcal{E}) \cdot [C].$$

**Theorem 7.2.1.** We have 

$$\chi(\mathcal{E}) = \text{rk}(\mathcal{E}) \chi(O_C) + \deg \mathcal{E} - \sum_{i=1}^n \text{age}_{\mu_i}(\mathcal{E}).$$
It is worth noticing that the stack structure at the nodes does not intervene in the formula. This is due to the fact that the curve is balanced.

This theorem can be deduced from Toën’s Riemann-Roch theorem for stacks, but as it is not too much harder, we give a direct argument.

**Proof.** In the following we will use the fact that Euler characteristic, rank, degree and age are all additive invariants in $K_0(C)$.

Assume that $C$ is smooth.

First of all, assume that there exists a coherent sheaf $F$ on $C$ such that $\pi^* F = E$. Then the adjunction homomorphism $F \to \pi_* \pi^* F$ is an isomorphism, because $\pi$ is flat. Then $\chi(E) = \chi(F)$, $\deg_C E = \deg_C F$, and $\text{age}_{p_i} E = 0$ for all $i$; hence the formula follows from ordinary Riemann–Roch applied to $F$.

The kernel and cokernel of the adjunction homomorphism

$$\pi^* \pi_* E \to E$$

are torsion, supported at the marked points of $C$; hence by additivity it is enough to prove the formula for torsion sheaves supported in the stack locus. Each such sheaf is an extension of sheaves of the form $\iota_i^* L_k$, where $0 \leq k \leq r_i - 1$, $\iota_i : B\mu_{r_i}$ is the inclusion, and $L_k$ is the 1-dimensional representation of $\mu_{r_i}$ defined by the character $\mu_{r_i} \to \mathbb{G}_m$, $t \mapsto t^k$. So it suffices to prove the formula for the sheaves $\iota_i^* L_k$.

It is easy to see that

$$\deg_{\iota_i} L_k = \frac{1}{r_i}$$

for any $k$. Also we see that

$$\chi(\iota_i^* L_k) = \begin{cases} 1 & \text{if } k = 0 \\ 0 & \text{if } k \neq 0. \end{cases}$$

To complete the calculation let us compute $\text{age}_{p_i, \iota_i} L_k$. Let $\mathcal{I}_{\Sigma_i}$ be the sheaf of ideals of $\Sigma_i$ in $C$; by with tensoring $\iota_i^* L_k$ the sequence

$$0 \to \mathcal{I}_{\Sigma_i} \to \mathcal{O}_C \to \mathcal{O}_{\Sigma_i} \to 0$$

we obtain that

$$\iota_i^* \iota_i^* L_k = [L_k] - [\mathcal{T}_{\Sigma_i} \otimes L_k]$$

where $\mathcal{T}_{\Sigma_i}$ is the cotangent space to $\Sigma_i$ in $C$ (this is a very particular case of the self-intersection formula in K-theory). But by definition of the isomorphism $\Sigma_i \cong B\mu_{r_i}$, the tangent space to $\Sigma_i$ is $L_1$, hence $\mathcal{T}_{\Sigma_i}$ is $L_{r_i-1}$. From this we obtain the remarkable formula

$$\text{age}_{p_i, \iota_i} L_k = \begin{cases} \frac{r_i - 1}{r_i} & \text{if } k = 0 \\ \frac{1}{r_i} & \text{if } k \neq 0. \end{cases}$$
Now we plug in the values of the invariants and check the formula for the sheaves $\iota_* L_k$. This completes the proof when $C$ is smooth.

In the general case, let $q_1, \ldots, q_s$ be the nodes of $C$. Call $l_i$ the index of $C$ at the node $q_i$; call $\Theta_i$ the residual gerbe of $C$ at the unique point living over $q_i$. Consider the normalization $\nu : \overline{C} \to C$; the moduli space $\overline{C}$ of $C$ is the normalization of $C$. Call $q'_i$ and $q''_i$ the two inverse images of $q_i$ in $\overline{C}$, and call $\Theta'_i$ and $\Theta''_i$ the residual gerbe of $C$ over the unique point over $q'_i$ and $q''_i$ respectively. The natural morphisms $\Theta_i \to \Theta'_i$ and $\Theta_i \to \Theta''_i$ are isomorphisms of gerbes. Furthermore, $\Theta'_i$ and $\Theta''_i$ are isomorphic to $B\mu_{l_i}$, where the isomorphisms are chosen so that the action of $\mu_{l_i}$ on the tangent space to $C$ at the gerbe is given by the embedding $\mu_{l_i} \subseteq \mathbb{G}_m$. We give $\overline{C}$ the structure of a twisted curve by using these gerbes as a marking.

The fact that $C$ is balanced has the following important consequence: the two bandings by $\mu_{l_i}$ on $\Theta_i$ that we obtain via the two isomorphisms $\Theta_i \cong \Theta'_i$ and $\Theta_i \cong \Theta''_i$ are opposite. Consider the pullback $\nu^* \mathcal{E}$; its restrictions to $\Theta'_i$ and $\Theta''_i$ are isomorphic to the restriction of $\mathcal{E}$ to $\Theta_i$, and they give dual representations of $\mu_{l_i}$. This implies that

$$\text{age}_{q'_i} \nu^* \mathcal{E} + \text{age}_{q''_i} \nu^* \mathcal{E} = c_i,$$

where $c_i$ is the codimension of the space of invariants of the restriction of $\nu^* \mathcal{E}$ to $\Theta'_i$ or $\Theta''_i$.

We have an exact sequence

$$0 \to \mathcal{O}_C \to \nu_* \mathcal{O}_{\overline{C}} \to \bigoplus_{i=1}^s \mathcal{O}_{\Theta_i} \to 0,$$

from which we deduce that

$$\chi(\mathcal{O}_{\overline{C}}) = \chi(\nu_* \mathcal{O}_{\overline{C}}) = \chi(\mathcal{O}_C) + s.$$

By tensoring with $\mathcal{E}$, and keeping in mind that $\mathcal{E}$ is locally free at the nodes of $\overline{C}$, we get a sequence

$$0 \to \mathcal{E} \to \nu_* \nu^* \mathcal{E} \to \bigoplus_{i=1}^s (\mathcal{E} \otimes_{\mathcal{O}_C} \mathcal{O}_{\Theta_i}) \to 0.$$

By taking invariants, and then using Riemann–Roch on the smooth twisted curve $\overline{C}$ together with the formulas above, we get

$$\chi(\mathcal{E}) = \chi(\nu_* \nu^* \mathcal{E}) - \sum_{i=1}^s \dim_k H^0(\mathcal{E} \otimes_{\mathcal{O}_C} \mathcal{O}_{\Theta_i})$$

$$= \chi(\nu^* \mathcal{E}) - \sum_{i=1}^s (\text{rk} \mathcal{E} - c_i)$$

$$= \text{rk}(\mathcal{E}) \chi(\mathcal{O}_{\overline{C}}) + \deg(\nu^* \mathcal{E}) - \sum_{i=1}^n \text{age}_{p_i} \mathcal{E} - \sum_{i=1}^s (\text{age}_{q'_i} \nu^* \mathcal{E} + \text{age}_{q''_i} \nu^* \mathcal{E})$$
− \sum_{i=1}^{n} (\text{rk} \mathcal{E} - \text{age}_{q_1} \nu_1^* \mathcal{E} - \text{age}_{q_2} \nu_2^* \mathcal{E})
\hspace{1cm} = \text{rk}(\mathcal{E}) \chi(\mathcal{O}_C) + \text{deg}(\mathcal{E}) - \sum_{i=1}^{n} \text{age}_{p_i}(\mathcal{E}) - s \text{rk} \mathcal{E}
\hspace{1cm} = \text{rk}(\mathcal{E}) \chi(\mathcal{O}_C) + \text{deg}(\mathcal{E}) - \sum_{i=1}^{n} \text{age}_{p_i}(\mathcal{E}).

This concludes the proof.

7.3. The stringy Chow group and its grading. Let \( \mathcal{X} \) be a smooth Deligne–Mumford stack as in the introduction. We define the rational stringy Chow group of \( \mathcal{X} \) to be

\[
A^*_st(\mathcal{X})_Q := \bigoplus \mathbb{A}^{a - \text{age}(\Omega)}(\Omega)_Q,
\]

where the sum is taken over all connected components \( \Omega \) of \( \mathcal{I}^\mu(\mathcal{X}) \).

7.4. The small quantum Chow ring. The small quantum Chow ring is an algebra over the completed monoid-algebra \( \mathbb{Q}[N^+(\mathcal{X})] \), where we denote the monomial corresponding to a class \( \beta \) by \( q^\beta \). As a group, the quantum Chow ring is

\[
Q \mathbb{A}^*(\mathcal{X}) := A^*_st(\mathcal{X})_Q[N^+(\mathcal{X})].
\]

We define a product on \( Q \mathbb{A}^*(\mathcal{X}) \) by specifying the product of monomials, as follows:

\[
\gamma_1 \ast \gamma_2 = \sum_{\beta \in N^+(\mathcal{X})} \langle \gamma_1, \gamma_2, * \rangle_{0, \beta} q^\beta.
\]

We define the degree of \( q^\beta \) to be \( \beta \cdot c_1(\mathcal{T}_\mathcal{X}) \).

**Theorem 7.4.1.** The product defined above makes \( Q \mathbb{A}^*(\mathcal{X}) \) into a commutative, associative “pro-Q-graded” ring, in the sense that the product of two homogeneous elements of degrees \( a, b \) is homogeneous of degree \( a + b \).

**Proof.** Commutativity is immediate. Associativity is, as usual, a consequence of Theorem 6.2.1. It remains to check the claim about grading.

Consider the summand \( \langle \gamma_1, \gamma_2, * \rangle_{0, \beta} q^\beta \) in the formula above, where we assume that \( \gamma_1 \) and \( \gamma_2 \) are each supported on a single component \( \Omega_1 \), respectively \( \Omega_2 \subset \mathcal{T}_\mu(\mathcal{X}) \), of corresponding ages \( a_1, a_2 \). We need to show that it has degree

\[
\deg \gamma_1 + \deg \gamma_2 = (\dim_{\Omega_1} \gamma_1 + a_1) + (\dim_{\Omega_2} \gamma_2 + a_2).
\]
Similarly, it is enough to calculate the degree of a single term
\[ \langle \gamma_1, \gamma_2, \bullet \rangle_{0, \beta, \Omega_3} q^\beta \]
of \( \langle \gamma_1, \gamma_2, \bullet \rangle_{0, \beta} q^\beta \) lying in the Chow group of a component \( \Omega_3 \subset \overline{T}_\mu (X) \) having age \( \tilde{a}_3 \).

Given a stable map \( f : C \to X \) corresponding to a geometric point of a component \( K \subset K_{0,3}(X, \beta) \) with evaluations \( e_1 : K \to \Omega_1, e_2 : K \to \Omega_2 \) and \( \tilde{e}_3 : K \to \Omega_3 \), the bundle \( f^* T X \) has ages \( a_1, a_2 \) and \( a_3 \) at the three markings, with
\[ a_3 + \tilde{a}_3 = \dim X - \dim \Omega_3. \]
This is explained in [CR], the point being that \( \dim_{\xi, \alpha} \Omega_3 = \rank (T^\mu_{\xi, \alpha}(\Omega_3)) = \dim X - (a_3 + \tilde{a}_3) \), as noted in the last section. We denote the class of \( f_* C \) by \( \beta' \), and clearly we have \( \pi_\ast \beta' = \beta \).

We can now calculate dimensions at \( f : C \to X \) using Riemann-Roch:
first, the dimension of \( [K]_{\text{vir}} \) is given by
\[ \chi (f^* T X) = c_1 (T X) \cdot \beta' + \dim X - \deg \gamma_1 - \deg \gamma_2 - a_3. \]
Therefore we have
\[ \dim (\langle e_1^* \gamma_1 \cup e_2^* \gamma_2 \rangle \cap [K]_{\text{vir}}) = \deg q^\beta + \dim X - \deg \gamma_1 - \deg \gamma_2 - a_3 \]
\[ - \codim_{\Omega_1} \gamma_1 - \codim_{\Omega_2} \gamma_2 \]
\[ = \deg q^\beta + \dim X - \deg \gamma_1 - \deg \gamma_2 - \tilde{a}_3. \]

Pushing forward by \( \tilde{e}_3 \) we obtain
\[ \codim (\tilde{e}_3^* (\langle e_1^* \gamma_1 \cup e_2^* \gamma_2 \rangle \cap [K]_{\text{vir}})) = \dim \Omega_3 - (\deg q^\beta + \dim X - \deg \gamma_1 - \deg \gamma_2 - a_3) \]
\[ = \deg \gamma_1 + \deg \gamma_2 + (a_3 + \dim \Omega_3 - \dim X) - \deg q^\beta \]
\[ = \deg \gamma_1 + \deg \gamma_2 - \tilde{a}_3 - \deg q^\beta \]
and therefore
\[ \deg (\tilde{e}_3^* (\langle e_1^* \gamma_1 \cup e_2^* \gamma_2 \rangle \cap [K]_{\text{vir}})) = \deg \gamma_1 + \deg \gamma_2 - \deg q^\beta. \]
It follows that
\[ \deg (\tilde{e}_3^* (\langle e_1^* \gamma_1 \cup e_2^* \gamma_2 \rangle \cap [K]_{\text{vir}}) q^\beta) = \deg \gamma_1 + \deg \gamma_2, \]
as required.

We note that, while the grading has rational degrees, the denominators which appear are bounded. The fact that the \( q^\beta \) have degrees with bounded denominators follows from the fact that the ring is finitely generated, and more explicitly through Proposition 2.1.1 and the denominators in the ages of a connected component are bounded by the exponent of the automorphism group of a geometric point of \( X \).
An identical construction using the topological Gromov-Witten classes, gives us a definition of the small quantum cohomology ring $QH^*(X)$. This is an interesting ring structure on $H^*(\mathcal{I}_\mu(X)) [N(X)^+]$. Its associativity follows from Theorem 6.3.2.

7.5. **Stringy Chow ring.** By setting the $q$'s to zero in $QA^*(X)$, we get an interesting product structure on $A^*_{\text{st}}(X)$ which we refer to as the stringy Chow ring. This is the Chow analogue of the orbifold cohomology or Chen-Ruan cohomology ring, which arises by doing exactly the same thing to the small quantum cohomology ring of $X$. In [AGV] we show that it is possible to define these products with integral coefficients, provided one works with $A^*(\mathcal{I}_\mu(X))$ instead of $A^*(\mathcal{I}_\mu(X))$.

7.6. **The big quantum cohomology ring.** As in the Gromov-Witten theory of a manifold, the full topological WDVV equation, Theorem 6.4.3, is essentially equivalent to the associativity of a big quantum cohomology ring whose multiplication is defined in terms of the basis $\{\alpha_i\}$ of $H^*(\mathcal{I}_\mu(X))$ as

$$\mu * \nu = \sum_{ij} \sum_n \frac{1}{n!} \langle \mu, \nu, T^n, \alpha_i \rangle \beta q^i \tilde{g}^{ij} \alpha_j.$$ 

Here $T = \sum \alpha_i x_i$ should be expanded formally, so that the resulting product is a power series in the $x_i$ and $q_i$ whose coefficients record every genus zero Gromov-Witten invariant including the classes $\mu$ and $\nu$. The same arguments as in the previous subsection show that this gives rise to a pro-$Q$-graded, associative, commutative ring structure on $H^*(\mathcal{I}_\mu(X)) [N^+(X)][[x_1, \ldots, x_M]].$

8. **A few useful facts**

We collect here some useful facts analogous to standard properties of Gromov-Witten invariants for manifolds. While this will not constitute an exhaustive list, we hope that readers will see how standard facts and techniques from Gromov-Witten theory carry over to this context. Most sources of difference come from the need to systematically replace $X$ with $\mathcal{I}_\mu(X)$ at various points in formulating the theory, or from a difference in the relationship between the spaces of twisted stable maps and their universal curves, which we explain now.

8.1. **Universal curve.** A critical fact in ordinary Gromov-Witten theory is that $\overline{M}_{g,n+1}(X, \beta)$ is the universal curve over $\overline{M}_{g,n}(X, \beta)$. While this is not true for the orbifold theory, we have a similar result.

**Proposition 8.1.1.** The universal curve over $K_{g,n}(X, \beta)$ is naturally identified with the open and closed substack $\mathcal{U} \subseteq \overline{K}_{g,n+1}(X, \beta)$ for which the
(n + 1)st marked point is untwisted. Moreover, if we consider the flat morphism \( \pi : \mathcal{U} \to \mathcal{K}_{g,n}(X, \beta) \), then we have an equality of virtual fundamental classes \([\mathcal{U}]^{\text{vir}} = \pi^*[\mathcal{K}_{g,n}(X, \beta)]^{\text{vir}}\), where \([\mathcal{U}]^{\text{vir}}\) denotes the restriction of \([\mathcal{K}_{g,n+1}(X, \beta)]^{\text{vir}}\) to \(\mathcal{U}\).

Another way to say this, which is useful in practice, is that the universal curve \(\mathcal{U}\) fits in the cartesian square:

\[
\begin{array}{ccc}
\mathcal{U} & \rightarrow & \mathcal{X} \\
\downarrow & & \downarrow i \\
\mathcal{K}_{g,n+1}(X, \beta) & \rightarrow & \mathcal{I}_\mu(X)
\end{array}
\]

where \(i\) denotes the inclusion of \(X \cong \mathcal{I}_{\mu_1}(\mathcal{X})\) into the inertia stack.

**Proof.** The identification of the universal curve is essentially Corollary 9.1.3 in [AV]. The identification of the virtual classes follows by the same reasoning as the analogous statement in [B]. ♣

### 8.2. Degree zero invariants

The identification of the \((n + 1)\)-pointed space with the universal curve in ordinary Gromov-Witten theory implies that the degree zero invariants carry very little information. In particular, it implies that degree zero, genus zero invariants all vanish with the exception of the three point invariants, which simply compute the classical cohomology ring. This is not at all true for the orbifold theory, which is precisely why the ring structure on orbifold cohomology is interesting. However, one consequence of this fact does continue to hold. If we let 1 denote the fundamental class of the identity component of \(\mathcal{I}_\mu(X)\), then we have the following fact.

**Proposition 8.2.1.** The Gromov-Witten number \(\langle 1, \delta_1, \ldots, \delta_n \rangle_{g, \beta}\) vanishes unless \(g = 0\) and \(n = 2\), in which case we have

\[
\langle 1, \delta_1, \delta_2 \rangle_{0,0} = \int_{\mathcal{I}_\mu(X)} \frac{1}{r} \delta_1 \cup i^* \delta_2.
\]

**Proof.** Given Proposition 8.1.1, the vanishing portion follows from the same arguments as in ordinary Gromov-Witten theory (see [KM]). The precise formula comes from the easy fact that \(e_1^{-1}(\mathcal{I}_\mu(X))\) in \(\mathcal{K}_{0,3}(X, 0)\) is naturally isomorphic to \(\mathcal{I}_\mu(X)\) and we can identify \(e_2\) with the standard rigidification map. Using this identification, \(e_3\) is then identified with the composition of the rigidification map with \(\iota\) and the result follows. ♣

This has the immediate consequence that 1 is the identity element in both the small and big quantum cohomology rings. (A completely analogous lemma shows that 1 is the identity in the quantum Chow ring.)
8.3. **Gravitational descendants.** In studying higher genus Gromov–Witten theory it is important to include the descendant classes. These are analogues of the Mumford–Morita–Miller classes on $\overline{M}_{g,n}$ involving the Chern classes of the normal bundles of the $n$ sections.

One can define these classes in the orbifold theory in much the same way as in the ordinary theory. On $K_{g,n}(\mathcal{X},\beta)$, one defines $n$ tautological line bundles, $L_i$. There is more than one way to define these, but the most straightforward is to take $L_i$ to be the bundle whose fiber at a point is the cotangent space to the corresponding coarse curve at the $i$th marked point, in other words, the pullback via the $i$-th section on the universal coarse curve of the sheaf of relative differentials. We remark that while one might like to use the cotangent space to the twisted curve, these will not form a line bundle on $K_{g,n}(\mathcal{X},\beta)$, but only on the gerbe corresponding to the $i$th marking. However, one could decide to push down the Chern class of that “twisted” bundle, which would simply be $1/r$ times the Chern class of the $L_i$ as we are defining them here. We will not use that convention.

Let $\psi_i = c_1(L_i)$. Then given classes $\delta_1, \ldots, \delta_n$ in $H^*(\mathcal{M}(\mathcal{X}))$, we define the following invariants:

$$\langle \tau_{a_1}(\delta_1) \cdots \tau_{a_n}(\delta_n) \rangle_{g,\beta} = \int_{K_{g,n}(\mathcal{X},\beta)^{\text{vir}}} e_1^*(\delta_1) \cup \psi_1^{a_1} \cup \cdots \cup e_n^*(\delta_n) \cup \psi_n^{a_n}. $$

For simplicity of statements we define any class involving $\tau_{-1}$ in the formulas below to be 0.

With these definitions, the standard equations among descendants for manifolds (cf. [P, Section 1.2]) hold in the orbifold setting with no changes (keeping in mind that 1 means the fundamental class of the identity component of the inertia stack).

**Theorem 8.3.1.** Assume $(\beta, g, n)$ is not any of $\beta = 0, g = 0, n < 3$ or $\beta = 0, g = 1, n = 0$. Then

1. **(Puncture or String Equation)**
   $$\langle \tau_{a_1}(\gamma_1) \cdots \tau_{a_n}(\gamma_n) \tau_0(1) \rangle_{g,\beta} = \sum_{i=1}^n \langle \tau_{a_1}(\gamma_1) \cdots \tau_{a_{i-1}}(\gamma_{i-1}) \tau_{a_i-1}(\gamma_i) \tau_{a_{i+1}}(\gamma_{i+1}) \cdots \tau_{a_n}(\gamma_n) \rangle_{g,\beta}$$

2. **(Dilaton Equation)**
   $$\langle \tau_{a_1}(\gamma_1) \cdots \tau_{a_n}(\gamma_n) \tau_1(1) \rangle_{g,\beta} = (2g - 2 + n) \langle \tau_{a_1}(\gamma_1) \cdots \tau_{a_n}(\gamma_n) \rangle_{g,\beta}$$

3. **(Divisor Equation)** For $\gamma$ in $H^2(X) \subset H^2_{\text{orb}}(X)$ (but not for an arbitrary element of $H^2_{\text{orb}}(X)$), we have
   $$\langle \tau_{a_1}(\gamma_1) \cdots \tau_{a_n}(\gamma_n) \tau_0(\gamma) \rangle_{g,\beta}$$
\[
\left( \int_{\gamma} \right) \cdot (\tau_1(\gamma_1) \cdots \tau_n(\gamma_n))_{g,\beta} \\
+ \sum_{i=1}^{n} (\tau_i(\gamma_1) \cdots \tau_{i-1}(\gamma_{i-1}) \tau_{i-1}(\gamma_i) \tau_{i+1}(\gamma_{i+1}) \cdots \tau_n(\gamma_n))_{g,\beta}
\]

**Proof.** We reduce to the untwisted case. In all these equations the intersection happens on the open and closed substack \( U \subset \mathcal{K}_{g,n+1}(X, \beta) \) of Proposition 8.1.1. The key commutative diagram is the following:

\[
\begin{array}{ccc}
U & \xrightarrow{\mu_{n+1}} & \overline{\mathcal{M}}_{g,n+1}(X, \beta) \\
\pi_k \downarrow & & \downarrow \pi_M \\
\mathcal{K}_{g,n}(X, \beta) & \xrightarrow{\mu_n} & \overline{\mathcal{M}}_{g,n}(X, \beta)
\end{array}
\]

This is not quite a fiber diagram, but \( U \) has the same moduli space as the fibered product, so the projection formula still holds. If \( \alpha \) is a cohomology class on \( \mathcal{K}_{g,n}(X, \beta) \) and \( \psi \) is a cohomology class on \( \overline{\mathcal{M}}_{g,n+1}(X, \beta) \), we have by proposition 8.1.1

\[
\int_{\beta[\mathcal{M}]} \pi^*_K \alpha \cup \mu^*_{n+1} \psi = \int_{[\mathcal{K}_{g,n}(X, \beta)]^{vir}} \alpha \cup \mu^*_n \pi_{\mathcal{M}} \psi.
\]

So any identity which holds for \( \pi_{\mathcal{M}} \psi \) (keeping in mind that \( X \) may be singular) can be used here.

For all the equations use \( \alpha = \prod_{i=1}^{n} e_i^* \gamma_i \). For the Puncture Equation use \( \psi = \prod_{i=1}^{n} \psi_i^{a_i} \) and the equation \( \pi_{\mathcal{M}} \psi = \sum_{i=1}^{n} (\prod_{i=1}^{n} \psi_i^{a_i})/\psi_i \). For the Dilaton Equation use \( \psi = (\prod_{i=1}^{n} \psi_i^{a_i})\psi_{n+1} \) with \( \pi_{\mathcal{M}} \psi = (2g - 2 + n) \prod_{i=1}^{n} \psi_i^{a_i} \). For the Divisor equation use \( \psi = (\prod_{i=1}^{n} \psi_i^{a_i}) \cup e_{\mathcal{M}_{n+1,n+1}}^* \gamma \), with the equation \( \pi_{\mathcal{M}} \psi = \int_{\beta} \gamma \cdot \prod_{i=1}^{n} \psi_i^{a_i} + \sum_{i=1}^{n} (\prod_{i=1}^{n} \psi_i^{a_i})/\psi_i \cap e_{\mathcal{M}_{n+1,n+1}}^* \gamma \), noticing that \( e_i^* \gamma_i \cup u_n^* e_{\mathcal{M}_{n+1,n+1}}^* \gamma = e_i^* (\gamma_i \cup \gamma) \).

**Remark 8.3.2.** There is also a Topological Recursion Relation valid in this context, treated in [13], section 2.5.5.

9. **AN EXAMPLE: THE WEIGHTED PROJECTIVE LINE**

We conclude by giving a nontrivial calculation of the quantum Chow ring of a stack. We consider one of the simplest possible classes of examples – the weighted projective lines. We fix two positive integers, \( a \) and \( b \), and consider the one dimensional weighted projective space \( X = \mathbb{P}(a, b) \) which is the stack quotient of a punctured two dimensional affine space by the action of \( \mathbb{G}_m \) with weights \( a \) and \( b \). The coarse moduli space of \( \mathbb{P}(a, b) \) is always \( \mathbb{P}^1 \). If \( a \) and \( b \) are relatively prime, then \( \mathbb{P}(a, b) \) is a twisted curve. Otherwise this stack has a generic stabilizer. We will denote by \( 0 \), the point with stabilizer \( \mu_a \) and by \( \infty \) the point with stabilizer \( \mu_b \).
For the convenience of the reader, we collect the basic facts about this stack here. A morphism from a scheme $Z$ to $\mathbb{P}(a,b)$ is given by choosing a line bundle $L$ on $Z$ together with sections $s_1 \in \Gamma(Z, L^{\otimes a})$ and $s_2 \in \Gamma(Z, L^{\otimes b})$ with no common zeroes. 2-morphisms are given by morphisms between line bundles which take the sections to the sections. Note that if $a = b = 1$ we get the usual description of $\mathbb{P}^1$ and there are no nontrivial 2-automorphisms.

By descent, we get the same description of maps from a stack to $\mathbb{P}(a,b)$. We let $\mathcal{O}(1)$ denote the line bundle on $\mathbb{P}(a,b)$ corresponding to the identity morphism. Then $\text{Pic}(\mathbb{P}(a,b)) = \mathbb{Z}\mathcal{O}(1)$ and there are sections of $\mathcal{O}(a)$ and $\mathcal{O}(b)$ vanishing at $\infty$ and 0 respectively. The degree of $\mathcal{O}(1)$ is $\frac{1}{ab}$. Finally, $T\mathbb{P}(a,b) \cong \mathcal{O}(a + b)$.

The inertia stack of this stack is straightforward to describe. Note that since $\mathbb{G}_m$ is an abelian group, the quotient presentation of $\mathcal{X}$ endows the inertia group of each point of $\mathbb{P}(a,b)$ with an embedding in $\mathbb{G}_m$. (This character of the isotropy group is also its action on the fiber of $\mathcal{O}(1)$.) Because of this, each irreducible component of $I_\mu \mathcal{X}$ is canonically associated with the unit in $\mathbb{Z}/r\mathbb{Z}$ which relates that fixed embedding to the one obtained by composition with the embedding of $\mu_r$ in $\mathbb{G}_m$. Below we will use the obvious convention which identifies the set of elements of $\mathbb{Z}/d\mathbb{Z}$ with the set of units in $\mathbb{Z}/k\mathbb{Z}$ for all positive integers $k$ dividing $d$.

Let $d = \gcd(a,b)$. For each element of $\mathbb{Z}/d$, there is a one dimensional component of $I_\mu \mathcal{X}$ which is isomorphic to $\mathbb{P}(a,b)$. For each element of $\mathbb{Z}/a\mathbb{Z}$ which is not divisible by $a/d$, there is a zero dimensional component of $I_\mu \mathcal{X}$ lying over the point 0. Each of these components is isomorphic to $B\mu_a$. Similarly, for each element of $\mathbb{Z}/b\mathbb{Z}$ not divisible by $b/d$ there is a component lying over the point $\infty$ which is isomorphic to $B\mu_b$. We hope that the following picture of the inertia stack of $\mathbb{P}(4,6) \cong \overline{\mathcal{M}}_{1,1}$ will make this labeling system clear:

To give a presentation of the quantum Chow ring we need to choose generators. A convenient way to make this choice is as follows. Choose
integers \( m \) and \( n \) such that \( ma + nb = d \). Set \( A = a/d \) and \( B = b/d \).

We take \( \zeta \) to be the fundamental class of the one dimensional component of \( \mathcal{I}_{\mu}(X) \) corresponding to \( 1 \in \mathbb{Z}/d\mathbb{Z} \) (if \( d = 1 \), take \( \zeta = 1 \)), we let \( x \) be the fundamental class of the component lying over 0 which corresponds to \( n \in \mathbb{Z}/a\mathbb{Z} \), and we let \( y \) be the fundamental class of the component lying over \( \infty \) corresponding to \( m \in \mathbb{Z}/b\mathbb{Z} \). In the example of \( \mathbb{P}(4, 6) \) above, we chose \( n = 1, m = -1 \) and indicated the components where the resulting \( x, y \) as well as \( \zeta \) serve as fundamental classes. (If \( d = a \) or \( d = b \) some of these zero dimensional components don't exist. We will ignore this case in what follows, but the results all hold with essentially identical proofs if we take \( x \) or \( y \) to be the fundamental class of the appropriate zero dimensional substack of the inertia stack associated with \( m \) or \( n \) in \( \mathbb{Z}/d\mathbb{Z} \).)

One consequence of choosing \( x \) and \( y \) in this manner is that they have minimal age. Under this convention, we find that \( \text{deg}(x) = \frac{1}{A} \) and \( \text{deg}(y) = \frac{1}{B} \), while \( \text{deg}(\zeta) = 0 \). Following reasoning similar to that at the end of [AGV] it is easy to calculate that the stringy Chow ring of \( \mathbb{P}(a, b) \) is \( \mathbb{Q}[\zeta, x, y] / \langle R_1, R_2, R_3 \rangle \).

(Note that the factor \( \zeta^{n-m} \) in the second relation is missing in [AV1].)

The Néron-Severi group of \( \mathbb{P}(a, b) \) has rank one, so to compute the quantum Chow ring, we need to introduce one further generator, \( q \). We normalize this by selecting our generator of \( N(\mathbb{P}(a, b)) \) to correspond to the minimal positive degree map from a twisted curve to \( \mathbb{P}(a, b) \). If \( \gcd(a, b) = 1 \), then this minimal degree map can be taken to be the identity map, if not, then the presence of a generic stabilizer of order \( d \) forces any map from a twisted curve to have degree divisible by \( d \), we will see that there does exist a map of degree exactly \( d \). In other words, we take the generator of \( N(\mathbb{P}(a, b)) \) to be \( d \) times the fundamental class.

Since we know that \( \text{QH}^*(X) \) is free as a \( \mathbb{Q}[q] \) module, it follows that there is a presentation for \( \text{QH}^*(X) \) of the form

\[
\mathbb{Q}[q][\zeta, x, y] / \langle R_1, R_2, R_3 \rangle
\]

where we have

\[
R_1 \equiv xy \mod q
\]

\[
R_2 \equiv Ax^A - By^B\zeta^{n-m} \mod q
\]

\[
R_3 \equiv \zeta^d - 1 \mod q.
\]

Since the degree of \( q \) is \( 1/A + 1/B \), no monomial in the generators containing a \( q \) can possibly have degree equal to the degrees of \( R_2 \) or \( R_3 \). Hence all that remains to compute is the quantum product of \( x \) with \( y \). By degree considerations, the only possibility for the form of \( R_1 \) is then

\[
xy = q(c_0 + c_1\zeta + \cdots + c_{d-1}\zeta^{d-1})
\]

where the \( c_i \) are rational numbers. We will establish that \( c_0 = 1 \) and that the other \( c_i \) all vanish. As \( c_0 \) is the coefficient of the fundamental class in \( x * y \), it is determined by considering the moduli space of maps from a 3 pointed stacky \( \mathbb{P}^1 \) which actually has only two stacky points of indices \( a \) and
b. We let $C_{a,b}$ denote this curve. $\text{Pic}(C_{a,b})$ is generated by line bundles $L_0$ and $L_{\infty}$ of degrees $\frac{a}{n}$ and $\frac{b}{n}$ satisfying the single relation $L_0^{\otimes a} = L_{\infty}^{\otimes b}$. In particular, when $a$ and $b$ are not relatively prime, so that $C_{a,b} \not\cong \mathbb{P}(a,b)$ we find that there is torsion in the Picard group. Also, the restriction map $r_0 : \text{Pic}(C_{a,b}) \rightarrow \text{Pic}(\mathcal{B}_{\mu_a})$ takes $L_0$ to the standard generator. (A possibly confusing point here is that even when $a$ and $b$ are relatively prime so that $C_{a,b} \cong \mathbb{P}(a,b)$, our identification of the isotropy groups of the substacks supported at 0 with $\mathbb{P}^1$ then in order for the first point to evaluate to the correct component of $\mathcal{O}(1)$ as the standard representation, whereas on $C_{a,b}$ we are using the action on the tangent space.)

We are looking for a map of degree $d$ from $C_{a,b}$ to $\mathbb{P}(a,b)$. Since we can compute the degree of a morphism $f : C_{a,b} \rightarrow \mathbb{P}(a,b)$ by comparing the degree of $\mathcal{O}(1)$ to the degree of $f^*\mathcal{O}(1)$ we see that we need to find a line bundle $L$ of degree $\frac{d}{n}$ such that $L^{\otimes a}$ has a section vanishing at $\infty$ and $L^{\otimes b}$ has a section vanishing at 0. If we denote this line bundle $L = L_0^{\otimes a} \otimes L_{\infty}^{\otimes b}$ then in order for the first point to evaluate to the correct component of the inertia stack, we need that $z_0 \equiv n \mod a$ and for the second point we have the analogous condition that $z_{\infty} \equiv m \mod b$. It follows that we must have $L \cong L_0^m \otimes L_\infty^n$. Thus the relevant space of morphisms from $C_{a,b}$ to $\mathbb{P}(a,b)$ can be identified with the space of pairs $s_1 \in \Gamma(L^{\otimes a}), s_2 \in \Gamma(L^{\otimes b})$ modulo the action of $\mathbb{C}^*$ acting by scalar multiplication on $L$. Since we are assuming that $d \neq a$ and $d \neq b$, there is a unique section of both $L^{\otimes a}$ and $L^{\otimes b}$ by degree considerations. We conclude that the space of 3 pointed maps with irreducible source curve is the quotient of $\mathbb{G}_m \times \mathbb{G}_m$ by the linear action of $\mathbb{G}_m$ with weights $a$ and $b$. This is simply $\mathbb{G}_m \times \mu_d$. There are two additional points where the source is reducible with a component collapsed over either zero or infinity. The reader may verify that the full moduli space is isomorphic to $C_{a,b} \times \mathcal{B}_{\mu_d}$, but to compute the relevant pushforward, the exact structure of the compactification is irrelevant.

Since this space has the expected dimension, the virtual fundamental class is the ordinary fundamental class, which pushes forward to the fundamental class of $\mathbb{P}(a,b)$ and we conclude that $c_0 = 1$. To see that the other $c_i$ vanish we just need to verify that there are no representable morphisms of minimal degree from a three pointed twisted genus zero curve where the first two points are as before, but the third twisted point has nontrivial stacky structure. To find such a map, we would need to find an integer $D/d$, and a line bundle of degree $\frac{d}{D}$ on $C_{a,b,D}$ satisfying all of the conditions as before as well as being nontrivial when restricted to $\mathcal{B}_{\mu_D}$. This is obviously impossible.

We conclude that

$$\text{QG}^*(\mathbb{P}(a,b)) = \mathbb{Q}[q][\zeta, x, y] / \langle xy - q, Ax^A - By^B\zeta^{n-m}, \zeta^d - 1 \rangle.$$
Appendix A. Gluing of algebraic stacks along closed substacks

A.1. We introduce a gluing construction for Artin stacks.

**Proposition A.1.1.** Let $Z, X_1, X_2$ be algebraic stacks, and assume given

\[
\begin{array}{ccc}
Z & \xleftarrow{i_1} & X_1 \\
\downarrow{i_2} & & \downarrow \ \\
X_2 & \xrightarrow{} & X
\end{array}
\]

where $i_1, i_2$ are closed embeddings. Then there exists an algebraic stack $X$ and a diagram

\[
\begin{array}{ccc}
Z & \xleftarrow{i_1} & X_1 \\
\downarrow{i_2} & & \downarrow \ \\
X_2 & \xrightarrow{} & X
\end{array}
\]

such that the diagram is co-cartesian, namely, for any algebraic stack $M$, the natural functor

\[
\text{Hom}(X, M) \to \text{Hom}(X_1, M) \times_{\text{Hom}(Z, M)} \text{Hom}(X_2, M)
\]

is an equivalence of categories. Such $X$ is unique up to a unique isomorphism.

**Corollary A.1.2.** Let $Z, Y$ be algebraic stacks and $i_1, i_2 : Z \to Y$ closed embeddings with disjoint images. Then there exists

1. an algebraic stack $X$,
2. a morphism $\pi : Y \to X$, and
3. a 2-isomorphism $\alpha : \pi \circ i_1 \to \pi \circ i_2$

such that $(X, \pi, \alpha) = \varinjlim(Z \rightrightarrows Y)$.

In other words, we can glue together the two copies of $Z$ in $Y$, obtaining $X$.

A.2. Gluing of schemes and algebraic spaces. Given a scheme $Z$ together with a pair of closed embeddings of $Z$ in schemes $X_1$ and $X_2$, we can define a new scheme $X_1 \cup_Z X_2$. It is determined by the universal property that a morphism from $X_1 \cup_Z X_2$ to a scheme $W$ is given by a morphism from $X_1$ to $W$ and a morphism from $X_2$ to $W$ whose restrictions to $Z$ agree. To construct this scheme, we can just do the construction for affines, where this amounts to taking a fibered product of rings.

In the following we use the fact that a similar construction exists in the category of algebraic spaces. One way to prove that it exists is to use our proof below, replacing “algebraic space” by “scheme” and “algebraic stack” by “algebraic space”.
Our first lemma shows that the universal property of the gluing of algebraic spaces is preserved in the 2-category of stacks.

**Lemma A.2.1.** Let $\mathcal{M}$ be an algebraic stack, and let

$$
\begin{array}{c}
\begin{array}{c}
Z \\ \downarrow i_1 \\
X_1 \\
\downarrow \\
X_2 \\
\downarrow i_2 \\
X
\end{array}
\end{array}
$$

be a co-cartesian diagram of algebraic spaces, where $i_1, i_2$ are closed embeddings. Then the natural functor

$$
\mathcal{M}(X) \rightarrow \mathcal{M}(X_1) \times_{\mathcal{M}(Z)} \mathcal{M}(X_2)
$$

is an equivalence, where the fibered product is taken in the sense of categories.

**Proof.** We construct a functor inverse to the given one. Let $\rightarrow U$ be a presentation of $\mathcal{M}$. Assume given an object of the fibered product on the right hand side, namely

1. objects $f_i \in \mathcal{M}(X_i)$, and
2. an isomorphism $\alpha : i_1^* f_1 \rightarrow i_2^* f_2$.

Explicitly in terms of the presentation, we are given $U_i = f_i^* U$ and $R_i = f_i^* R$ and morphisms of groupoids

$$
\begin{array}{c}
\begin{array}{c}
R_i \\
\downarrow \\
U_i
\end{array}
\end{array} 
\rightarrow
\begin{array}{c}
\begin{array}{c}
R \\
\downarrow \\
U
\end{array}
\end{array}
$$

Moreover, $\alpha$ gives an isomorphism

$$
i_1^*(R_1 \rightarrow U_1) \rightarrow i_2^*(R_2 \rightarrow U_2).
$$

Since $R_i$ and $U_i$ are algebraic spaces, we can form a groupoid in algebraic spaces $R_X \rightarrow U_X$ presenting $X$, by gluing $U_1, U_2$ along the morphism $\alpha : i^* U_1 \rightarrow i^* U_2$, and similarly for $R_X$. We obtain a morphism of groupoids

$$
\begin{array}{c}
\begin{array}{c}
R_X \\
\downarrow \\
U_X
\end{array}
\end{array} 
\rightarrow
\begin{array}{c}
\begin{array}{c}
R \\
\downarrow \\
U
\end{array}
\end{array}
$$

giving a morphism $X \rightarrow \mathcal{M}$. The construction of the functor on the level of arrows, and the fact that the functors are inverses, is left to the reader. ♦
A.3. Extension of atlases.

**Lemma A.3.1.** Let \( Z \subset X \) be a closed embedding of schemes. Assume \( U_Z \to Z \) is a smooth morphism. Then there exists a Zariski open covering \( U'_Z \to U_Z \) and a smooth morphism \( U'_X \to X \) with \( U'_X \times_X Z \cong U'_Z \).

**Proof.** We may assume \( U_Z, Z \) and \( X \) are affine, and embed \( U_Z \subset A^n_\mathbb{Z} \) for some \( n \). The subscheme \( U_Z \) is locally a complete intersection in \( A^n_\mathbb{Z} \), so for every point \( u \in U_Z \) we can find elements \( (g_1, \ldots, g_N) \subset O_Z[x_1, \ldots, x_n] \) and a Zariski neighborhood \( V \subset A^n_\mathbb{Z} \) such that \( U'_Z := U_Z \cap V \subset V \) is the complete intersection of the zero schemes of \( g_i \). Choose \( \tilde{g}_i \subset O_X[x_1, \ldots, x_n] \) lifting \( g_i \), and let \( W \) be the zero scheme of \( (\tilde{g}_1, \ldots, \tilde{g}_N) \) in \( A^n_X \). Clearly \( W \cap V = U'_Z \), and so \( W \to X \) is smooth along points of \( U'_Z \). There exists a neighborhood \( U_X \subset W \) of \( u \) containing \( U'_Z \) which is smooth over \( X \). We can take

\[
U'_Z = \bigcup_u U'_Z, \quad U'_X = \bigcup_u U'_X.
\]

**Lemma A.3.2.** Let \( i_j : Z \to X_j \) be closed embeddings of algebraic stacks, \( j = 1, 2 \). There exist schemes \( U_j \) and smooth surjective morphisms \( U_j \to X_j \) such that

\[
Z \times_{X_j} U_1 \cong Z \times_{X_2} U_2.
\]

**Proof.** Let \( V_j \to X_j \) be smooth and surjective. We have a pullback diagram

\[
\begin{array}{ccc}
\hat{V} & \to & V_j \\
\downarrow & & \downarrow \\
Z & \to & X_j.
\end{array}
\]

Consider the fibered product \( \hat{V} = i_1^* V_1 \times_Z i_2^* V_2 \). Applying the previous lemma with \( Z \subset X \) replaced by the closed embedding \( i_1^* V_1 \subset V_1 \), and \( U_Z \to Z \) replaced by \( \hat{V} \to i_1^* V_1 \), there is a Zariski-open covering \( \hat{V}_1 \to \hat{V} \) and a smooth \( \hat{U}_1 \to X_1 \) with \( i_1^* \hat{U}_1 \cong \hat{V}_1 \). Applying the same procedure for \( i_2^* V_2 \subset V_2 \), we obtain a Zariski-open covering \( \hat{V}_2 \to \hat{V} \) and a smooth \( \hat{U}_2 \to X_2 \) with \( i_2^* \hat{U}_2 \cong \hat{V}_2 \). Replacing \( \hat{V}_1 \) and \( \hat{V}_2 \) by a common Zariski-refinement \( \hat{V}_{12} \), and replacing \( \hat{U}_j \) by a suitable Zariski-refinements \( U_j \) lifting \( \hat{V}_{12} \), the lemma is proven.

\[ \Box \]

A.4. The construction.

**Proof of A.4.** By the previous lemma, there is a choice of schemes \( U_i \) and smooth surjective morphisms \( U_i \to X_i \) with \( i_1^* U_1 \cong i_2^* U_2 = U_Z \). Set \( R_i = U_i \times_X U_i \), so that \( R_i \implies U_i \) is a presentation of \( X_i \), and \( i_1^* R_1 \cong i_2^* R_2 = U_Z \).
$R_Z \to U_Z$ is a presentation of $Z$. Set $U = U_1 \cup_{U_Z} U_2$ and $R = R_1 \cup_{R_Z} R_2$. We have a groupoid $R \to U$, with a diagram of groupoids

\[
\begin{array}{ccc}
R_Z & \\ \downarrow & \searrow & \downarrow \\
R_1 & \rightarrow & R_2 \\
U_Z & \downarrow & \downarrow \\
U_1 & \rightarrow & U_2 \\
R & \downarrow & \downarrow \\
U & \rightarrow & U_1 \cup U_2
\end{array}
\]

Let $X = [R \to U]$ be the quotient. We claim this is the desired stack.

Let $\mathcal{M}$ be an algebraic stack. By definition, we have

\[
\text{Hom}(X, \mathcal{M}) = \lim_{\leftarrow} \left( \mathcal{M}(U) \to \mathcal{M}(R) \right).
\]

By Lemma A.2.1, we have

\[
\mathcal{M}(U) = \mathcal{M}(U_1) \times_{\mathcal{M}(U_Z)} \mathcal{M}(U_2) = \lim_{\leftarrow} \left( \mathcal{M}(U_1 \sqcup U_2) \to \mathcal{M}(U_Z) \right)
\]

and

\[
\mathcal{M}(R) = \mathcal{M}(R_1) \times_{\mathcal{M}(R_Z)} \mathcal{M}(R_2) = \lim_{\leftarrow} \left( \mathcal{M}(R_1 \sqcup R_2) \to \mathcal{M}(R_Z) \right).
\]

Taking $\mathcal{M}$-valued points in diagram (2), we get

\[
\begin{array}{ccc}
\mathcal{M}(R_Z) & \\ \downarrow & \searrow & \downarrow \\
\mathcal{M}(R_1) & \rightarrow & \mathcal{M}(R_2) \\
\mathcal{M}(U_Z) & \downarrow & \downarrow \\
\mathcal{M}(U_1) & \rightarrow & \mathcal{M}(U_2) \\
\end{array}
\]

We thus have

\[
\text{Hom}(X, \mathcal{M}) = \lim_{\leftarrow} \left( \lim_{\leftarrow} \left( \mathcal{M}(U_1 \sqcup U_2) \to \mathcal{M}(U_Z) \right) \to \lim_{\leftarrow} \left( \mathcal{M}(R_1 \sqcup R_2) \to \mathcal{M}(R_Z) \right) \right)
\]

\[= \lim_{\leftarrow} \left( \text{diagram (3)} \right) \]
\[
\frac{\lim_{\to \left(M(U_1) \to M(R_1)\right)} \times \left(\lim_{\to \left(M(U_2) \to M(R_2)\right)} \left(\lim M(U_1) \to M(R_1)\right) \times \lim M(U_2) \to M(R_2)\right)}{\Hom(X_1, M) \times \Hom(Z, M)} \times \Hom(X_2, M).
\]

**Proof of corollary A.1.2.** Consider the morphisms

\[ j_1 := i_1 \sqcup i_2 : Z \sqcup Z \to Y \quad \text{and} \quad j_2 := i_2 \sqcup i_1 : Z \sqcup Z \to Y. \]

These morphisms are closed embeddings by the empty intersection hypothesis. Let

\[ \tilde{X} = Y \cup_{Z \sqcup Z} Y. \]

There is a canonical free \( \mathbb{G} = \mathbb{Z}/2\mathbb{Z} \)-action on \( \tilde{X} \) arising from the action on the gluing diagram. We claim that

\[ X = \left[ \tilde{X} / G \right] \]

is the desired colimit. Indeed, \( X \) is the colimit of the following diagram:

\[
\begin{array}{ccc}
G \times (Z \sqcup Z) & \longrightarrow & Z \sqcup Z \\
\downarrow & & \downarrow \\
G \times (Y \sqcup Y) & \longrightarrow & Y \sqcup Y.
\end{array}
\]

**Appendix B. Taking roots of line bundles**

The following construction is due independently to the authors and to C. Cadman ([C1], Section 2).

**B.1. The root of a line bundle.** If \( L \) is a line bundle on a scheme \( S \) and \( d \) is a positive integer, we will denote by \( \sqrt[d]{L/S} \) the stack over \( S \) of \( d \)-th roots of \( L \); an object of \( \sqrt[d]{L/S} \) over \( T \to S \) is a line bundle \( M \) over \( T \), together with an isomorphism of \( M \otimes \mathbb{G}_m \) with the pullback \( L_T \) of \( L \) to \( T \). The arrows are defined in the obvious way. This stack \( \sqrt[d]{L/S} \) is a gerbe over \( S \) banded by \( \mu_d \); its cohomology class in the flat cohomology group \( H^2(S, \mu_d) \) is obtained from the class \( [L] \in H^1(S, \mathbb{G}_m) \) via the boundary homomorphism \( \partial : H^1(S, \mathbb{G}_m) \to H^2(S, \mu_d) \) obtained from the Kummer exact sequence

\[ 0 \to \mu_d, S \to \mathbb{G}_m, S \to \mathbb{G}_m, S \to 0. \]

It is clear that if \( T \to S \) is a morphism of schemes, then \( \sqrt[d]{L_T/T} = \sqrt[d]{L/S} \times_S T \). The stack \( \sqrt[d]{L/S} \) can be described directly as the quotient stack \([L^0/\mathbb{G}_m]\). Here \( L^0 \) is the total space of the \( \mathbb{G}_m \)-bundle associated with
where the fibered product is taken with respect to the classifying morphism $[L] : S \to B\mathbb{G}_m$ on the left and the $d$-th power map $B\mathbb{G}_m \to B\mathbb{G}_m$ on the right. There is a universal line bundle $L$ over $d\sqrt{L/S}$, which is the quotient $[A_1 \times L_0/G_m]$ by the action defined by $\alpha(u, x) = (\alpha u, \rho(\alpha)x)$.

**B.2. The root of a line bundle with a section.** Now, given a section $\sigma : S \to L$ we can define a variant of this, which we denote by $d\sqrt{(L, \sigma)/S}$, in which the objects over a scheme $T \to S$ consist of triples $(M, \phi, \tau)$

1. $M$ is a line bundle over $T$,

2. $\phi : M^{\otimes d} \simeq L_T$ is an isomorphism, and

3. $\tau$ is a section of $M$ such that $\phi(\tau^m) = \sigma$.

If $Y$ is the scheme-theoretic zero locus of $\sigma$, then the restriction of the stack $d\sqrt{(L, \sigma)/S}$ to $S \times Y$ is equal to $S \times \mathbb{G}_m$. Its restriction to $Y$ is more interesting; it does not coincide with $d\sqrt{L_Y/Y}$, because an object of the stack $d\sqrt{(L, \sigma)/S}|_Y$ over $T \to Y$ consists of a $d$-th root $M$ of $L_T$ on a scheme $T$ over $Y$ to the same $M$ together with the zero section is a closed embedding, defined by a nilpotent sheaf of ideals on $d\sqrt{(L, \sigma)/S}|_Y$.

Thus $d\sqrt{(L, \sigma)/S}|_Y$ contains a canonical gerbe banded by $\mu_d$, supported over the zero scheme of $\sigma$. The forgetful map $d\sqrt{(L, \sigma)/S}|_Y \to d\sqrt{L_Y/Y}$ identifies $d\sqrt{(L, \sigma)/S}|_Y$ as the $d$-th infinitesimal neighborhood of $d\sqrt{L_Y/Y}$ in its universal line bundle.

The stack $d\sqrt{(L, \sigma)/S}$ can also be described as a quotient stack. Consider the universal line bundle $\mathcal{L} = [A_1 \times L^0/G_m]$ described above, and the morphism $\Phi : \mathcal{L} \to L$ induced by the $G_m$-invariant morphism $A_1 \times L^0 \to L$ defined by $(u, x) \mapsto u^d x$; then

$$d\sqrt{(L, \sigma)/S} = \Phi^{-1}\sigma(S).$$

In other words: if we call $V_{\sigma} \subseteq A_1 \times L^0$ the inverse image in $A_1 \times L^0$ of the embedding $\sigma : S \to L$; then

$$d\sqrt{(L, \sigma)/S} = [V_{\sigma}/G_m].$$

In particular, assume that $L = \mathcal{O}$, so that $\sigma$ is a regular function on $S$. In this case $L^0 = S \times \mathbb{G}_m$, and if we denote by $W_{\sigma} = A_1 \times S$ the subscheme
defined by the equation \( t^d - \sigma(x) = 0 \), where \( t \) is a coordinate on \( \mathbb{A}^1 \), then there is an isomorphism \( W_{\sigma} \times \mathbb{G}_m \cong V_{\sigma} \).

An equivalent description exists here too: Let \( \mathcal{U} = [\mathbb{A}^1/\mathbb{G}_m] \), the classifying stack for line bundles with section. Then

\[
\sqrt{(L, \sigma)/S} = S \times_{\mathcal{U}} \mathcal{U},
\]

where the map on the left is the classifying map and the map \( \mathcal{U} \to \mathcal{U} \) on the right is the \( d \)-th power map.

**Appendix C. Rigidification**

**C.1. The setup.** We recall the concept of rigidification of an algebraic stack, as presented in [ACV], see also the related treatment in [Ro].

Let \( H \) be a flat finitely presented separated group scheme over a base scheme \( S \), \( X \) an algebraic stack over \( S \). We say that \( X \) has an \( H \)-2-structure if for each object \( \xi \in X(T) \) there is an embedding

\[
i_\xi: H(T) \hookrightarrow \text{Aut}_T(\xi),
\]

which is compatible with pullback, in the following sense: given two objects \( \xi \in X(T) \) and \( \eta \in X(T) \), and an arrow \( \phi: \xi \to \eta \) in \( X \) over a morphism of schemes \( f: S \to T \), the natural pullback homomorphisms

\[
\phi^*: \text{Aut}_T(\eta) \to \text{Aut}_S(\xi)
\]

and

\[
f^*: H(T) \to H(S)
\]

commute with the embedding, that is, \( i_\xi f^* = \phi^* i_\eta \).

This condition can also be expressed as follows. Let \( \phi: \xi \to \eta \) be an arrow in \( X \) over a morphism of schemes \( f: S \to T \), and \( g \in H(T) \). Then the diagram

\[
\begin{array}{ccc}
\xi & \xrightarrow{\phi} & \eta \\
\downarrow{f^*g} & & \downarrow{g} \\
\xi & \xrightarrow{\phi} & \eta
\end{array}
\]

commutes. In particular, by taking \( \xi = \eta \) and \( \phi \) to be in \( \text{Aut}_S(\xi) \), we see that \( H(S) \) must be in the center of \( \text{Aut}_S(\xi) \); in particular, \( H \) is an abelian group scheme.

The simplest example of such a situation is when \( X \to T \) is a gerbe banded by \( H \); in this case the embedding \( H(S) \hookrightarrow \text{Aut}_S(\xi) \) is an isomorphism of group schemes.

Then we have the following result.

**Theorem C.1.1.** ([ACV, Theorem 5.1.5]) There is a smooth surjective finitely presented morphism of algebraic stacks \( X \to X//H \) satisfying the following properties:
For any object \( \xi \in X(T) \) with image \( \eta \in X \sslash H(T) \), we have that \( H(T) \) lies in the kernel of \( \text{Aut}_T(\xi) \to \text{Aut}_T(\eta) \).

(2) The morphism \( X \to X \sslash H \) is universal for morphisms of stacks \( X \to Y \) satisfying (1) above.

(3) If \( T \) is the spectrum of an algebraically closed field, then in (1) above, we have

\[
\text{Aut}_T(\eta) = \text{Aut}_T(\xi) / H(T).
\]

(4) A moduli space for \( X \) is also a moduli space for \( X \sslash H \).

Furthermore, if \( X \) is a Deligne–Mumford stack, then \( X \sslash H \) is also a Deligne–Mumford stack and the morphism \( X \to X \sslash H \) is étale.

The notation in [ACV] is \( X^H \); here we adopt the better notation \( X \sslash H \) proposed by Romagny in [Ro]. This stack \( X \sslash H \) is called the \( H \)-rigidification of \( X \). For example, if \( G \to T \) is a gerbe banded by \( H \), then \( G \sslash H \) is isomorphic to \( T \).

The stack \( X \sslash H \) is obtained as the fppf stackification of a prestack \( X^H_{\text{pre}} \), that has the same objects as \( X \); this has the property that for any object \( \xi \) of \( X \) over an \( S \)-scheme \( T \), the sheaf of automorphisms of \( \text{Aut}_{T,X}^H(\xi) \) in \( X^H_{\text{pre}} \) is the quotient sheaf of \( \text{Aut}_{T,X}(\xi) \) by the normal subgroup sheaf \( H_T \).

C.2. Moduli interpretation of the stack \( X \sslash H \). The construction of rigidification is functorial, in the sense described below. First let \( \phi : X \to Y \) be a morphism of algebraic stacks endowed with \( H \)-2-structures. We say that \( \phi \) is \( H \)-2-equivariant if for each \( S \)-scheme \( T \) and object \( \xi \) of \( X(T) \), the homomorphism of group-schemes

\[
H(T) \hookrightarrow \text{Aut}_{T,X}(\xi) \longrightarrow \text{Aut}_{T,Y}(\phi(\xi))
\]

defined by \( \phi \) coincides with the given embedding \( H(T) \hookrightarrow \text{Aut}_{T,Y}(\phi(\xi)) \).

Define a 2-category \( X \sslash_2 H \) over the category of schemes over \( S \), as follows.

(1) An object over a scheme \( T \) is a pair \( (G, \phi) \), where \( G \to T \) is a gerbe banded by \( H \), and \( G \times^H T \to X \) is an \( H \)-2-equivariant morphism of fibered categories.

(2) A morphism \( (F, \rho) : (G, \phi) \to (G', \phi') \) consists of a morphism \( F : G \to G' \) over some \( f : T \to T' \), compatible with the bands, and a 2-morphism \( \rho : \phi \to \phi' \circ F \) making the following diagram commutative:

\[
\begin{array}{ccc}
G & \xrightarrow{F} & G' \\
\phi \downarrow & & \downarrow \phi' \\
X & \xrightarrow{\phi'} & \end{array}
\]
(3) A 2-arrow \((F, \rho) \to (F_1, \rho_1)\) is a usual 2-arrow \(\sigma : F \to F_1\) compatible with \(\rho\) and \(\rho_1\) in the sense that the following diagram is commutative:

\[
\begin{array}{ccc}
\phi & \xrightarrow{\rho} & \phi' \\
\downarrow{\rho'} & & \downarrow{\phi'(\sigma)} \\
\phi' \circ F & \xrightarrow{\phi' \circ \rho} & \phi' \circ F_1
\end{array}
\]

From Lemma 3.3.3 we see that \(X \!/ H\) is equivalent to a 1-category, that we denote by \(X \!/_1 H\). This is easily checked to be a category fibered in groupoids over the category of schemes over \(S\).

**Proposition C.2.1.** There is an equivalence of fibered categories between \(X \!/ H\) and \(X \!/_1 H\).

**Proof.** Given an \(H\)-2-equivariant morphism \(G \to X\), where \(G \to T\) is a gerbe banded by \(H\), there is an induced morphism \(T = G \!/ H \to X \!/ H\). This gives a function from the objects of \(X \!/ H\) to the objects of \(X \!/ H\), that extends to a functor in the obvious way.

In the other direction, given an object \(\xi\) of \(X \!/ H(T)\), consider the fibered product \(G := T \times_{\!/ H} X \to T\). If \(U\) is a scheme over \(T\), an object of \(G(U)\) is a pair \((\zeta, \alpha)\), where \(\zeta\) is an object of \(X(U)\), and \(\alpha\) is an isomorphism between \(\zeta\) and the pullback \(\xi_U\) in \(X \!/ H(U)\). We claim that \(G\) is a gerbe over \(T\), and there is a unique banding of \(G\) by \(H\) making the projection \(G \to X\) \(H\)-2-equivariant.

Both statements are fppf local on \(T\), so we may assume that the given object \(\xi\) of \(X \!/ H(T)\) comes from an object \(\xi\) of \(X(T)\). Then the pair \((\xi_U, \text{id})\) is an object of \(G(U)\), showing the existence of local sections. Two objects \((\zeta_1, \alpha_1)\) and \((\zeta_2, \alpha_2)\) in \(G(U)\) are locally isomorphic, because the morphism of sheaves of sets \(\text{Hom}_{U,X}((\zeta_1, \zeta_2) \to \text{Hom}_{U,X}((\zeta_1, \zeta_2)\) is an \(H\)-torsor.

Also, given an object \((\zeta, \alpha)\) of \(G(U)\), its automorphism group in \(G(U)\) is the kernel of the homomorphism \(\text{Aut}_{U,X}(\zeta) \to \text{Aut}_{U,X}((\zeta)\), that is exactly \(H(U) \subseteq \text{Aut}_{U,X}(\zeta)\).

So \(G\) is a gerbe banded by \(H_T\). This function from the objects of \(X\) to the objects of \(X \!/_1 H\) extends to a functor in the obvious way.

It is immediate to see that the composition \(X \!/ H \to X \!/_1 H \to X \!/ H\) is isomorphic to the identity. Let us show that the composition \(X \!/ H \to X \!/ H \to X \!/ H\) is also isomorphic to the identity. Given a gerbe \(G \to T\) and an \(H\)-2-equivariant morphism \(G \to X\), the induced morphism \(G \to T \times_{\!/ H} X\) is a morphism of gerbes banded by \(H\); and any such morphism is an isomorphism.

♣

C.3. **The rigidification as a quotient.** Here is another way to think of the rigidification. Given the collection of data \(\iota_\xi : H(T) \to \text{Aut}_T(\xi)\) for all objects \(\xi\) of \(X\) required for forming the rigidification, we have an associated
action
\[ \mathcal{B}H \times \mathcal{X} \rightarrow \mathcal{X}, \]
defined as follows. First, given a scheme \( T \), an object of \( (\mathcal{B}H \times \mathcal{X}) (T) \) consists of a principal \( H \)-bundle \( P \rightarrow T \) together with an object \( \xi \in \mathcal{X}(T) \).
We need to form a new object \( P * \xi \in \mathcal{X}(T) \). We do this as follows: the pullback \( \xi_P \) of \( \xi \) to \( P \) admits a left diagonal action of \( H \) coming from the two actions on \( P \) (inverted, as to make it into a left action) and on \( \xi \).
By the descent axiom for \( \mathcal{X} \) there is a quotient object on \( T \) which we call \( P * \xi \in \mathcal{X}(T) \). The assumptions on \( \iota_\xi \) guarantee that the formation of \( P * \xi \) is functorial, giving the required morphism \( \mathcal{B}H \times \mathcal{X} \rightarrow \mathcal{X} \). The particular case \( \mathcal{X} = \mathcal{B}H \) shows that \( \mathcal{B}H \) is indeed a group stack, and in general one shows the morphism \( \mathcal{B}H \times \mathcal{X} \rightarrow \mathcal{X} \) is an action. The morphism \( \mathcal{X} \rightarrow \mathcal{X} \sslash H \) is easily seen to be invariant. From the fact that \( \iota_\xi \) is injective one obtains that
\[ \mathcal{B}H \times \mathcal{X} \rightarrow \mathcal{X} \times \mathcal{X} \sslash H \]
is an isomorphism, so this action is free and, whatever the 2-categorical quotient should mean, up to equivalence we obtain
\[ \mathcal{X} / \mathcal{B}H \simeq \mathcal{X} \sslash H. \]

This is certainly in agreement with our previous moduli interpretation of \( \mathcal{X} \sslash H \): a principal \( \mathcal{B}H \)-bundle \( \mathcal{G} \rightarrow T \) is simply a gerbe banded by \( H \), and saying that \( \mathcal{G} \rightarrow \mathcal{X} \) is equivariant translates to our requirement on the map of automorphisms to coincide with the homomorphisms \( \iota_\xi \).

The case of non-central actions is more complicated and is worked out in the appendix to [AOV].

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