Series solutions of single-field models of inflation

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Abstract. We describe a generic method, first presented in Ref.[1], to find inflationary solutions without the need to be specific about the scalar potential. The method shows that single-field models of inflation can be classified in two groups according to their predictions of inflationary observables, we call these groups as Class I and Class II. We apply the method to the study of different scalar potentials and compare their results with recent observational data. We conclude that potentials belonging to Class I are in better agreement with observations mostly because they can provide large enough values of the spectral index of linear perturbations.

1. Introduction
What we actually call the standard model of Cosmology is a phenomenological description of the physics that seems to be at play to reproduce the observed Universe [2, 3, 4, 5]. The predictions of such model are in strikingly good agreement with a variety of observational constraints [6], and it is because of this reason that it is the fiducial model used in present and future observational forecasts. The standard cosmological model is also known as the ΛCDM model, for the initials of its main matter components at the present time: the cosmological constant Λ and Cold Dark Matter (CDM). The core of the model is based on a spatially-flat expanding Universe whose gravitational dynamics is governed by the equations of motion of General Relativity (for a pedagogical introduction to the state of the art of modern cosmology please see Ref. [7]).

Apart from this, the ΛCDM model also needs to set up the appropriate initial conditions so that the formation of structure can start as required by present observations of the distribution of galaxies. These seeds of structure formation must have been produced by some separate process in the early Universe. It is because of this that a period of inflation, which refers to an accelerated stage in the very early Universe, remains as one of the cornerstone ideas of modern Cosmology [8, 9, 10].

In the case of inflationary models with a single scalar field, the equations of motion can be simplified because of the assumption that the scalar field is the only matter field present in the early Universe. Another simplification is that we do not need to solve the equations of motion for linear perturbations, as there already exist general results for the features of the primordial perturbations that only require input from the dynamics in the background [8, 9, 10, 11].

In this paper we extend the study first presented in Ref. [1] to consider more examples of inflationary potentials, in particular some selected examples from the so-called Enciclopaedia
Inflationaris [12]. We focus our attention in those models that are able to provide an initial de Sitter phase of inflation, and calculate their inflationary predictions to compare them with the general solutions presented in Ref. [12].

A brief description of the paper is as follows. In Sec. 2 we describe the general mathematical framework to describe inflationary models and the series technique to find their solutions. We also present the results obtained for the inflationary numbers in the case of selected models and make a classification of them according to the classes suggested in Ref. [1]. Finally, Sec. 3 is devoted to conclusions.

2. Mathematical background

We will be concerned here with the inflationary dynamics of a (homogeneous) scalar field $\phi$ endowed with a scalar field potential $V(\phi)$. If we take the Friedmann-Robertson-Walker line element for a homogeneous and isotropic flat universe $ds^2 = -dt^2 + a^2(t)\delta_{ij}dx^idx^j$, where $a(t)$ is the scale factor, the equations of motion we have to solve are

$$H^2 = \frac{\kappa^2}{3} \left[ \frac{1}{2} \dot{\phi}^2 + V(\phi) \right], \quad \ddot{H} = -\kappa^2 \dot{\phi}^2, \quad \dddot{\phi} = -3H\dot{\phi} - \partial_{\phi}V. \quad (1)$$

Next, we define the following transformation of variables,

$$\frac{\kappa\dot{\phi}}{\sqrt{6}H} = \sin(\theta/2), \quad \frac{\kappa V^{1/2}}{\sqrt{3}H} = \cos(\theta/2), \quad y_1 = -2\sqrt{2}\frac{\partial_{\phi}V^{1/2}}{H}, \quad y_2 = -\frac{4\sqrt{3}}{\kappa} \frac{\partial_{\phi}^2V^{1/2}}{H}, \quad (2)$$

under which the dynamical equations in (1) can be written generically as

$$\theta' = -3\sin\theta + y_1, \quad y_1' = 3 \left( 1 - \cos\theta \right) y_1 + \sin(\theta/2)y_2. \quad (3)$$

Here, a prime denotes derivative with respect to the number of e-folds of expansion $N = \ln(a)$, and the Friedmann constraint appears as a consequence of the definitions in Eq. (2).

The system of equations (3) is a neat representation of the original equations of motion (1) that only requires input information about the second potential variable $y_2$. It must be noticed that the new angular variable $\theta$ has a direct interpretation in terms of the scalar field equation of state (EoS) $w_\phi$, which is defined as

$$w_\phi = \frac{\frac{1}{2}\dot{\phi}^2 - V(\phi)}{\frac{1}{2}\dot{\phi}^2 + V(\phi)} = -\cos\theta. \quad (4)$$

In terms of the EoS, the inflationary phase is represented by an evolution form $w_\phi \simeq -1$ to $w_\phi = -1/3$. The latter can be written in terms of the angular variable as $\theta : 0 \to \theta_{end}$, where $\theta_{end} = \arccos(1/3) = 2\arcsin(1/\sqrt{3}) \simeq 1.2309 \ldots$

Following the work in Ref. [1], we will try a series solution of the second expression in Eq. (3) with the following ansatz,

$$y_1 = \sum_{j=1} k_{1j} \theta^j, \quad (5)$$

where $k_{1j}$ are all constant coefficients. If the values of $k_{1j}$ can be calculated somehow, then a solution of Eqs. (3) can be found from the series expansion

$$\theta' = (k_{11} - 3)\theta + k_{12}\theta^2 + (k_{13} + 1/2)\theta^3 + \ldots. \quad (6)$$
Eq. (6) can then be solved at any order to provide a solution in the form \( \theta_N = \theta_N(k_{1j}, \theta_{\text{end}}, N) \), where the subscript \( N \) denotes the number of e-foldings before the end of inflation. Actually, the series solution up to the third order \( \theta^3 \) is

\[
(k_{11} - 3)N^{(3rd)} = \ln\left(\frac{\theta_{\text{end}}}{\theta_N}\right) + \frac{\theta_+ - \theta_-}{\theta_+ - \theta_-} \ln\left(\frac{\theta_{\text{end}} - \theta_+}{\theta_N - \theta_+}\right) - \frac{\theta_+ - \theta_-}{\theta_+ - \theta_-} \ln\left(\frac{\theta_{\text{end}} - \theta_-}{\theta_N - \theta_-}\right),
\]

where

\[
\theta_\pm = \frac{-k_{12} \pm \sqrt{k_{12}^2 - 2(k_{11} - 3)(2k_{13} + 1)}}{2k_{13} + 1}, \quad \theta_1 = \frac{k_{11} - 3}{k_{12}}.
\]

Eq. (7) is the expression that we will use to calculate the solutions and inflationary quantities for any given model. Approximate formulas can be obtained from Eq. (7) for special cases in which either or both of the following conditions \( k_{11} = 3 \) and \( k_{12} = 0 \) apply.

As shown in Ref. [1], the coefficients \( k_{1j} \) can be systematically determined if the scalar field potential allows a phase of the so-called de Sitter inflation, which is characterized by \( w_\phi \simeq -1 \); this is a regime that exists as long as the scalar field potential is the dominant term of the energy density at very early times. The coefficients \( k_{1j} \) are then calculated from appropriate combinations of the field derivatives of \( V(\phi) \) at the point of de Sitter inflation. In this form Eq. (7) gives the solution that corresponds to the given scalar field potential.

For the calculation of inflationary quantities, we choose the so-called Hubble slow-roll (HSR) variables [13], under which various inflationary observables can be written in terms of the so-called HSR parameters \( \epsilon_H \) and \( \eta_H \). For instance, the spectral index \( n_s \) and the tensor-to-scalar ratio \( r \) are given at first order in the HSR parameters as:

\[
1 - n_s = 4\epsilon_H - 2\eta_H = 12\sin^2(\theta/2) - 6 + y_1 \cot(\theta/2), \quad (9)
\]
\[
r = 16\epsilon_H = 48\sin^2(\theta/2), \quad (10)
\]

where the last expressions on the right hand side appear from the change to polar variables (2).

Taking into account the expansions in Eq. (5), the expressions in Eqs. (9) and (10) can be written alternatively as:

\[
1 - n_s \simeq 2(k_{11} - 3) + 2k_{12}\theta_N + (2k_{13} - k_{11}/6 + 3)\theta_N^2, \quad r \simeq 12\theta_N^2. \quad (11)
\]

In principle, Eqs. (7) and (11) is all what we need to find the values of the inflationary observables \( n_s \) and \( r \), once the values of the expansion coefficients \( k_{1j} \) have been determined by other means. Selected examples from the Encyclopaedia Inflationaris [12] are shown in Table 1, which extend the number of cases originally shown in Ref. [1], together with their corresponding values of the series coefficients \( k_{1j} \) and the obtained values of the inflationary quantities. The models are also classified as Class I (\( k_{12} = 0 \)) and Class II (\( k_{12} \neq 0 \)), which are the general classes suggested in Ref. [1].

To finish this section, we show in Fig. 1 the generic predictions for the observables \( n_s \) and \( r \) of the two classes of inflationary solutions, for \( N = 50 \), which is a typical number of e-folds before the end of inflation. When compared with the updated observational constraints reported in Ref. [14], see also Ref. [15]: \( n_s = 0.9783 \pm 0.0072 \) and \( r < 0.074 \), it can be seen that it is Class I, rather than Class II, the one set of models that can provide a good fit to the observational data. This is generically explained due to the fact that, for \( N = 50 \), \( n_s \leq 0.98 \) (\( n_s \leq 0.96 \)) for models in Class I (Class II). Even more, for the models in Class I a better agreement with observations is reached when \( k_{11} \to 3 \) and \( k_{13} \to \infty \), as in this case the prediction for the spectral index is just \( n_s = 1 - 1/N \), and then \( n_s = 0.98 \) if \( N = 50 \) (see Ref. [1] for more details).
Table 1. List of selected models of inflation taken from the Enciclopaedia Inflationaris [12]. The columns indicate, in order from left to right: 1) Name of the potential, 2) Functional form of the potential, 3) de Sitter point; 4-6) Series coefficients $k_{11}$, see Eq. (5); 7) Class of the model; 8) Spectral index $n_s$; and 9) Tensor-to-scalar ratio $r$, see Eq. (11).

| Model | Potential | $\phi_{4S}$ | $k_{11}$ | $k_{12}$ | $k_{13}$ | Class | $n_s$ | $r$ |
|-------|-----------|-------------|---------|---------|---------|-------|-------|-----|
| RCHI  | $1 - 2e^{\sqrt{2/3}\phi_{c}} - \frac{\kappa A_{1/6}}{16\pi^2\sqrt{6}}$ | $-\infty$ | 3       | 0       | 1       | I     | 0.970 | $8.0 \times 10^{-2}$ |
| MLFI  | $\kappa^2 \phi^2 (1 + \alpha \kappa^2 \phi^2)$ | $-\infty$ | 3       | 0       | -1/8    | I     | 0.941 | $3.1 \times 10^{-1}$ |
| HI    | $\left(1 - e^{\sqrt{2/3\kappa}\phi}\right)^2$ | $-\infty$ | 3       | 1       | $-1/12$ | II    | 0.962 | $4.6 \times 10^{-3}$ |
| ESI   | $1 - e^{0.1k_{11}\phi}$ | $-\infty$ | 3       | 0.122   | 0.995   | II    | 1.058 | $2.4 \times 10^{-1}$ |

Figure 1. General results on the plane $n_s$ vs $r$ for the Class I (left panel) and II (right panel) of the single field models of inflation, for different values of the coefficients $k_{11}$, $k_{12}$ and $k_{13}$. The plots show the results for $N = 50$. Notice that Class I is not completely ruled out by observations, mostly because it can provide low enough values of the tensor-to-scalar ratio $r$. The predictions from the typical Large Field Inflation for the potential $\phi^2$ [16], Natural Inflation (NI) [17] and Radiatively Corrected Higgs Inflation (RCHI)[12], are indicated on the left panel for comparison, whereas we do the same for the Higgs Inflation (HI) model [18] on the right panel. The predictions from other potentials listed in Table 1 lie outside of the regions shown in the figures, and are then also incompatible with the observational constraints. The grey shaded rectangle in each figure represents the observational constraints as reported recently in Ref. [14].

3. Conclusions

We have presented an extended study of single-field models of inflation following the prescription in Ref. [1]. In doing so, we have shown that the series solution and the general classification of models can be applied to models beyond those presented in the previous reference. However, we have also discovered that the series method may require the consideration of higher order solutions, mostly because the influence of some free parameters in the inflationary models is not present, in some instances, in the lowest order coefficients. This should be done specially for some models in Class I (with $k_{11} = 3$ and $k_{12} = 0$), for which one could even find an analytical solution up to the fifth order in Eq. (6). It is then expected that the higher-order solutions will provide more accurate predictions for the inflationary quantities. This is work in progress that we shall present elsewhere.
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