Composition Operators on Function Spaces on the Halfplane: Spectra and Semigroups

I. Chalendar¹ · J. R. Partington²

Received: 9 December 2022 / Accepted: 11 March 2023 / Published online: 8 April 2023
© The Author(s), under exclusive licence to Springer Nature Switzerland AG 2023

Abstract
This paper considers composition operators on Zen spaces (a class of weighted Bergman spaces of the right half-plane related to weighted function spaces on the positive half-line by means of the Laplace transform). Generalizations are given to work of Kucik on norms and essential norms, to work of Schroderus on (essential) spectra, and to work by Arvanitidis and the authors on semigroups of composition operators. The results are illustrated by consideration of the Hardy–Bergman space; that is, the intersection of the Hardy and Bergman Hilbert spaces on the half-plane.

Keywords  Composition operator · Hardy space · Bergman space · Spectrum · Essential spectrum · Operator semigroup

Mathematics Subject Classification  30H10 · 30H20 · 47B33 · 47D03

1 Introduction and Background Material

The subject of this paper involves properties of composition operators on holomorphic function spaces on the right half-plane \( \mathbb{C}_+ \), both as individual operators and as elements of one-parameter semigroups. One difficulty, even in the case of the Hardy space
$H^2(\mathbb{C}_+)$, is that not all composition operators on the spaces are bounded, although in many cases the bounded operators have been characterised, as we explain below. The literature on semigroups of composition operators is not as extensive as it is for the disc, although it has been an object of study since the work of Berkson and Porta [5].

In this note we shall concentrate on the so-called Zen spaces (weighted Hardy–Bergman spaces), which form a large class of spaces with applications in systems and control theory [11, 12]. For these information on the bounded composition operators is also available, thanks to Kucik [13].

The remainder of this section presents necessary background material. In Sect. 2 we provide new results on the norm and spectral radius of composition operators. Then, in Sect. 3, we consider the spectral theory of composition operators on the half-plane with linear fractional symbols, extending several results of Schroderus [15]. Finally, in Sect. 4, we consider semigroups of composition operators on Zen spaces, providing generalizations of results of Arvanitidis [3].

1.1 Zen Spaces

Kucik [13] considered composition operators on Zen spaces $A^2_\nu$, which are isometrically Laplace transforms of weighted Hardy spaces $L^2_w(0, \infty) = L^2(0, \infty, w(t)dt)$. For general background see [6], but we provide the basic facts now.

Let $\nu$ be a positive regular Borel measure on $[0, \infty)$ satisfying the doubling condition

$$R := \sup_{t > 0} \frac{\nu(0, 2t)}{\nu(0, t)} < \infty.$$ 

The Zen space $A^2_\nu$ is defined to consist of all analytic functions $F$ on $\mathbb{C}_+$ such that the norm, given by

$$\|F\|^2 = \sup_{\epsilon > 0} \int_{\mathbb{C}_+} |F(s + \epsilon)|^2 \nu(x) \, dy$$

is finite, where we write $s = x + iy$ for $x \geq 0$ and $y \in \mathbb{R}$.

The best-known examples here are:

1. For $\nu = \delta_0$, a Dirac mass at 0, we obtain the Hardy space $H^2(\mathbb{C}_+)$;
2. For $\nu$ equal to Lebesgue measure we obtain the Bergman space $A^2(\mathbb{C}_+)$.

Often we shall have $\nu(\{0\}) = 0$, in which case $\|F\|^2$ can be written simply as

$$\int_{\mathbb{C}_+} |F(s)|^2 \nu(x) \, dy.$$ 

**Theorem 1.1** [11] Suppose that $w$ is given as a weighted Laplace transform

$$w(t) = 2\pi \int_0^\infty e^{-2rt} \nu(r) \, dr, \quad (t > 0).$$
Then the Laplace transform provides an isometric map
\[ \mathcal{L} : L^2(0, \infty, w(t)dt) \rightarrow A_v^2. \]

The result also holds for Hilbert-space valued functions [1]. The following result generalizes the Elliott–Jury theorem in [8] and the Elliott–Wynn theorem in [9].

**Theorem 1.2** [13] The composition operator \( C_\phi \) is bounded on \( A_v^2 \) if and only if \( \phi \) has a finite nonzero angular derivative \( L = \angle \lim_{z \to \infty} z/\phi(z) \) at \( \infty \). In that case
\[
L \inf_{t > 0} \frac{w(t)}{w(Li)} \leq \| C_\phi \|^2 \leq L \sup_{t > 0} \frac{w(t)}{w(Li)}.
\]

(1)

This gives correctly \( \| C_\phi \| = \sqrt{L} \) in the Hardy case and \( \| C_\phi \| = L \) in the Bergman case.

### 1.2 Semigroups of Composition Operators

Berkson and Porta [5] give the following criterion for an analytic function \( G \) to generate a one-parameter semigroup of analytic mappings of the right half-plane \( \mathbb{C}_+ \) to itself; that is, solutions to
\[
\frac{\partial \phi_t(z)}{\partial t} = G(\phi_t(z)), \quad \phi_0(z) = z,
\]

(2)
in terms of the following condition:
\[
x \frac{\partial (\text{Re} G)}{\partial x} \leq \text{Re} G \text{ on } \mathbb{C}_+,
\]

(3)

where \( x = \text{Re} z \). The associated composition operators \( C_{\phi_t} \) are bounded on \( H^2(\mathbb{C}_+) \) if and only if the non-tangential limit
\[
L_t := \angle \lim_{z \to \infty} z/\phi_t(z)
\]
exists and is non-zero. It will be positive, and then \( \| C_{\phi_t} \| = L_t^{1/2} \) (see [8, 14]).

Arvanitidis [3] showed that a necessary and sufficient condition for boundedness of the composition operators is that \( \delta := \angle \lim_{z \to \infty} G(z)/z \) exists. In this case \( \| C_{\phi_t} \| = e^{-\delta t/2} \) and the semigroup is quasicontractive.

The following result is taken from [4].

**Theorem 1.3** For an operator \( A \) given by \( Af = Gf' \) on \( D(A) \subseteq H^2(\mathbb{C}_+) \), the following are equivalent:

(i) \( A \) generates a quasi-contractive \( C_0 \)-semigroup of bounded composition operators on \( H^2(\mathbb{C}_+) \);

(ii) Condition (3) holds and \( \angle \lim_{z \to \infty} G(z)/z \) exists.
2 The Essential Norm and Spectral Radius

We know from [13, Lem. 4] that in every $A^2_ν$ space the normalized reproducing kernels $k_n/\|k_n\|$ tend weakly to 0 as $n \to \infty$. Now we can adapt the proof that there is no compact composition operator as follows.

Using the isometry between $L^2(0, \infty; w(t) \, dt)$ and $A^2_ν$ we may pull back the reproducing kernel of $A^2_ν$ at a point $\lambda \in \mathbb{C}_+$ by the inverse Laplace transform to obtain $k_\lambda(t) := e^{-\Re \lambda t}/w(t)$ and its norm is $\langle k_\lambda, k_\lambda \rangle^{1/2}$ or

$$\left( \int_0^\infty \frac{e^{-2\Re \lambda t}}{w(t)} \, dt \right)^{1/2}.$$  

(4)

**Theorem 2.1** Let $C_\phi$ be a bounded composition operator on the Zen space $A^2_ν$ corresponding to a weight $w$ on $(0, \infty)$. Let $L$ denote the finite nonzero angular derivative $L = \angle \lim_{z \to \infty} z/\phi(z)$ at $\infty$. Then

$$L \inf_{t > 0} \frac{w(t)}{w(Lt)} \leq \|C_\phi\|_e^2 \leq \sup_{t > 0} \frac{w(t)}{w(Lt)}.$$  

(5)

**Proof** For every $\delta > 0$ there is a compact operator $Q$ on $A^2_ν$ such that

$$\|C_\phi\|_e + \delta \geq \|C_\phi - Q\| \geq \limsup_{n \to \infty} \|(C_\phi - Q) k_n\|/\|k_n\| = \limsup_{n \to \infty} \|k_{\phi(n)}\|/\|k_n\|$$

since $Q k_n/\|k_n\| \to 0$ in norm. Since this is true for all $\delta > 0$ we have

$$\|C_\phi\|_e^2 \geq \limsup_{n \to \infty} \left( \int_0^\infty \frac{e^{-2\Re \phi(n)t}}{w(t)} \, dt \right) / \left( \int_0^\infty \frac{e^{-2nt}}{w(t)} \, dt \right)$$

by (4). Hence, for every $0 < M < L$ we have

$$\|C_\phi\|_e^3 \geq \limsup_{n \to \infty} \left( \int_0^\infty \frac{e^{-2nt/M}}{w(t)} \, dt \right) / \left( \int_0^\infty \frac{e^{-2nt}}{w(t)} \, dt \right) = \limsup_{n \to \infty} \left( \int_0^\infty \frac{e^{-2un}}{w(Mu)} \, du \right) / \left( \int_0^\infty \frac{e^{-2nt}}{w(t)} \, dt \right) \geq M \inf_{u > 0} \frac{w(u)}{w(Mu)}.$$

Now, using Theorem 1.2, we have the estimate for the essential norm given in (5).
For the case of the Hardy and weighted Bergman spaces we therefore recover the formula \( \| C_\phi \| = \| C_\phi \|_e \) from [8, 9].

Note that the left and right hand sides of (1) and (5) may certainly differ, and as an example we shall consider the Hardy–Bergman space \( H^2(C_+)^{\cap} A^2(C_+) \). On the disc the Hardy space \( H^2(\mathbb{D}) \) is contained in the Bergman space \( A^2(\mathbb{D}) \) but no such inclusion either way exists on the half-plane, as we shall now explain. We may give \( H^2(C_+) \cap A^2(C_+) \) a Hilbert space structure with the norm defined by

\[
\| f \|_{H^2(C_+) \cap A^2(C_+)}^2 = \| f \|_{H^2(C_+)}^2 + \| f \|_{A^2(C_+)}^2.
\]

Since the Hardy space is isomorphic by means of the Laplace transform to \( L^2(0, \infty) \) and the Bergman space to \( L^2(0, \infty, dt/t) \) we see that neither space is contained in the other and that the Hardy–Bergman space \( H^2(C_+) \cap A^2(C_+) \) corresponds to \( L^2(0, \infty, w(t) dt) \) with the weight \( w(t) = 1 + 1/t \).

In this case \( w(t)/w(Lt) = \frac{t+1}{t+L} \), so that the sup and inf are 1 and 1/L in some order.

For the spectral radius \( \rho(C_\phi) \) we may apply (1) to get an estimate which is once more tight in the case of the Hardy and weighted Bergman spaces.

**Corollary 2.2** For a bounded composition \( C_\phi \) on a space \( A^2_v \) corresponding to a weight \( w \), the spectral radius \( \rho(C_\phi) \) satisfies

\[
L \limsup_{n \to \infty} \left( \inf_{t > 0} \frac{w(t)}{w(L^n t)} \right)^{1/n} \leq \| \rho(C_\phi) \| \leq L \liminf_{n \to \infty} \left( \sup_{t > 0} \frac{w(t)}{w(L^n t)} \right)^{1/n},
\]

where \( L = \angle \lim_{z \to \infty} z/\phi(z). \)

**Proof** For the iterate \( \phi^{(n)} \) the corresponding angular derivative \( L_n = \angle \lim_{z \to \infty} z/\phi^{(n)}(z) \) is simply \( L^n \), being the product of \( n \) terms each tending to \( L \). Thus

\[
L^n \inf_{t > 0} \frac{w(t)}{w(L^n t)} \leq \| C^n_\phi \| \leq L^n \sup_{t > 0} \frac{w(t)}{w(L^n t)},
\]

from which the result follows by the standard spectral radius formula. \( \square \)

### 3 Spectral Theory

For the spectral theory of composition operators on the half-plane, little seems to be known, although Schroderus [15] has determined the spectrum and essential spectrum in the case of linear fractional mappings \( \phi \) for Hardy and weighted Bergman spaces. The associated composition operator \( C_\phi \) is bounded if and only if \( \phi \) is a parabolic or hyperbolic mapping fixing \( \infty \); that is, \( \phi(s) = \mu s + s_0 \) with \( \mu > 0 \) and \( \text{Re}s_0 \geq 0 \) (we shall discuss this in detail below). The same applies in general Zen spaces, by Theorem 1.2.
Schroderus restricts herself to Hardy spaces and weighted Bergman spaces on the upper half-plane. Transforming to the right half-plane these are Zen spaces $A^2_\nu$ with measures $d\nu(x) = x^\alpha \, dx$ for $\alpha > -1$ (in the following the Hardy space is formally identified with the case $\alpha = -1$). Her results in this particular context are:

**Theorem 3.1** [15, Thm. 1.1, Thm. 1.2]

(1) In the parabolic case $\phi(s) = s + s_0$, where $\Re s_0 \geq 0$ and $s_0 \neq 0$, the spectrum and essential spectrum of $C_\phi$ coincide and equal

(a) $\mathbb{T}$ if $s_0 \in i\mathbb{R}$;
(b) $\{e^{-s_0 t} : t \geq 0\} \cup \{0\}$ if $s_0 \in \mathbb{C}_+$.

(2) In the hyperbolic case $\phi(s) = \mu s + s_0$, where $\mu \in (0, 1) \cup (1, \infty)$ and $\Re s_0 \geq 0$, the spectrum and essential spectrum of $C_\phi$ coincide and equal

(a) $\{\lambda \in \mathbb{C} : |\lambda| = \mu^{-(\alpha+2)/2}\}$ if $s_0 \in i\mathbb{R}$;
(b) $\{\lambda \in \mathbb{C} : |\lambda| \leq \mu^{-(\alpha+2)/2}\}$ if $s_0 \in \mathbb{C}_+$.

### 3.1 The Parabolic Case

For a general Zen space, part (1) of Theorem 3.1 is easy to prove.

**Proposition 3.2** Let $\phi(s) = s + s_0$ where $\Re s_0 \geq 0$ and $s_0 \neq 0$. Then the spectrum and essential spectrum of $C_\phi$ on $A^2_\nu$ coincide and equal

(a) $\mathbb{T}$ if $s_0 \in i\mathbb{R}$;
(b) $\{e^{-s_0 t} : t \geq 0\} \cup \{0\}$ if $s_0 \in \mathbb{C}_+$.

**Proof** The composition operator is seen to be unitarily equivalent to the multiplication operator on $L^2(0, \infty, w(t) \, dt)$ given by multiplication by the function $t \mapsto e^{-s_0 t}$. Thus it is a normal operator: its spectrum and essential spectrum equal the closure of $\{e^{-s_0 t} : t \geq 0\}$, as in (a) and (b).

### 3.2 The Hyperbolic Case

In this subsection we take $\phi(s) = \mu s + s_0$, where $\mu \in (0, 1) \cup (1, \infty)$ and $\Re s_0 \geq 0$.

Schroderus observed that we have the following simplifications:

For $s_0 = iy$, with $y \in \mathbb{R}$, the operator $C_\phi$ is similar to $C_\psi$ with $\psi(s) = \mu s$. Indeed, if $\rho(s) = s + iy/(\mu - 1)$ then

$$\rho^{-1} \circ \psi \circ \rho(s) = \mu(s + iy/(\mu - 1)) - iy/(\mu - 1) = \mu s + iy.$$  

Similarly, for $s_0 = x + iy$ with $x > 0$ and $y \in \mathbb{R}$, let $\psi(s) = \mu s + x$ and take the same $\rho$. Then

$$\rho^{-1} \circ \psi \circ \rho(s) = \mu(s + iy/(\mu - 1)) + x - iy/(\mu - 1) = \mu s + x + iy.$$
Thus, since $C_\rho$ is a unitary map on every $A^2_\nu$, we need only consider the spectrum of $C_\phi$ when $\phi(s) = \mu s + x$ for $\mu \in (0, 1) \cup (1, \infty)$ and $x \geq 0$.

In case (a) we can estimate the spectral radius of $C_\phi$ and $C_\phi^{-1}$, although this tells us only that the spectrum is contained in an annulus (which may be a circle). In case (b) we have that the spectrum is contained in a disc.

We showed in [6, Prop. 3.5] that the norm of the weighted composition operator $C_\psi$ corresponding to $\psi(s) = \mu s$ with $\mu > 0$ is exactly $\left( \frac{1}{\mu} \sup_{t > 0} \frac{w(\mu t)}{w(t)} \right)^{1/2}$. The method of [6] also shows that for all $x \geq 0$ and $\phi(s) = \mu s + x$ the composition operator has norm equal to the norm of the mapping $f \mapsto g$ in $L^2(0, \infty; w(t) \, dt)$, where

$$g(t) = \frac{1}{\mu} e^{-xt/\mu} f(t/\mu)$$

since

$$\int_0^\infty e^{-st} g(t) \, dt = \frac{1}{\mu} \int_0^\infty e^{-st} e^{-xt/\mu} f(t/\mu) \, dt$$

$$= \int_0^\infty e^{-\mu st} e^{-xt} f(\tau) \, d\tau$$

and

$$\|g\|^2 = \frac{1}{\mu^2} \int_0^\infty e^{-2xt/\mu} \|f(t/\mu)\|^2 w(t) \, dt = \frac{1}{\mu} \int_0^\infty e^{-2xt} \|f(\tau)\|^2 w(\mu \tau) \, d\tau,$$

whence

$$\|C_\phi\|^2 = \frac{1}{\mu} \sup_{t > 0} \frac{e^{-2xt} w(\mu t)}{w(t)}.$$ 

Now by looking at iterates we can obtain an explicit formula for the spectral radius of $C_\phi$ as the $n$th iterate of $\phi$ is given by $\phi_n(s) = \mu^n s + x_n$, where $x_n = \frac{\mu^n - 1}{\mu - 1} x$, and this has the same form as $\phi$.

That is,

$$\rho(C_\phi) = \frac{1}{\sqrt{\mu}} \lim_{n \to \infty} \left( \sup_{t > 0} \frac{e^{-2x_n t} w(\mu^n t)}{w(t)} \right)^{1/2n}. \quad (6)$$

In the time domain (that is, using the inverse Laplace transform), the operator $C_\phi$ corresponds to the operator $B_\phi$ on $L^2(0, \infty, w(t) \, dt)$ with

$$B_\phi f(t) = af(bt) e^{-ct}$$
such that we have
\[
\int_{0}^{\infty} a f(bt) e^{-ct} e^{-st} dt = \int_{0}^{\infty} a f(u) e^{-cu/b} e^{-su/b} du/b = (a/b)(L f)(s/b + c/b) = (L f)(\mu s + x),
\]
and so \( a = b = 1/\mu \) and \( c = x/\mu \).

With \( Af(t) = \frac{1}{\mu} f(t/\mu) e^{-xt/\mu} \) on \( L^2(0, \infty, w(t) dt) \) we find that
\[
\int_{0}^{\infty} \frac{1}{\mu} f(t/\mu) e^{-xt/\mu} \frac{1}{g(t)} w(t) dt = \int_{0}^{\infty} f(u) e^{-xu} \frac{w(\mu u)}{w(u)} w(u) du
\]
so that \( A^* g(u) = g(\mu u) e^{-xu} \frac{w(\mu u)}{w(u)} \).

### 3.2.1 The Case \( x = 0 \)

It is enough to consider the spectrum of \( C_\phi \) for \( \phi(s) = \mu s \) with \( \mu > 1 \) since the case \( 0 < \mu < 1 \) may be studied by taking inverses.

**Theorem 3.3** For the composition operator \( C_\phi \) on \( A^2_\nu \) where \( \phi(s) = \mu s \) with \( \mu > 1 \) and \( \nu \) determining the weight \( w \) on \( (0, \infty) \) we have
\[
\sigma(C_\phi) \subseteq \{ z \in \mathbb{C} : r \leq |z| \leq R \},
\]
where
\[
r = \frac{1}{\sqrt{\mu}} \lim_{n \to \infty} \left( \inf_{t > 0} \frac{w(\mu^n t)}{w(t)} \right)^{1/(2n)}
\]
and
\[
R = \frac{1}{\sqrt{\mu}} \lim_{n \to \infty} \left( \sup_{t > 0} \frac{w(\mu^n t)}{w(t)} \right)^{1/(2n)}.
\]

**Proof** This follows from (6). \( \square \)

With \( A^* f(t) = f(\mu t) \frac{w(\mu t)}{w(t)} \), we have eigenvalues of \( A^* \) given by \( \mu^\alpha \), with eigenvectors \( f(t) = t^\alpha / w(t) \), provided that these eigenvectors lie in the space \( L^2(0, \infty, w(t) dt) \). Indeed the eigenvalues then have infinite multiplicity, since \( f \chi_E \) is an eigenvector for any measurable \( E \subset \mathbb{R}_+ \) such that \( \mu E = E \), the notation \( \chi_E \) denoting the characteristic (indicator) function of a set \( E \). Clearly there are infinitely many distinct subsets \( E \subset \mathbb{R} \) such that \( \mu E = E \). For example, if \( \mu > 1 \) we may take
\[
E = \bigcup_{n=-\infty}^{\infty} \chi(\mu^n, \mu^n(1+\delta))
\]
for any $\delta$ with $0 < \delta < \mu - 1$. The same is true for $0 < \mu < 1$ working with $1/\mu$ instead of $\mu$; this will be required later.

Thus we have the following lower bound for the essential spectrum $\sigma_e(C_\phi)$:

**Theorem 3.4** For the composition operator $C_\phi$ on $A^2_\nu$ where $\phi(s) = \mu s$ with $\mu > 1$ and $\nu$ determining the weight $w$ on $(0, \infty)$ it holds that $\sigma_e(C_\phi)$ contains all $\alpha$ such that

$$\int_0^\infty |t^\alpha|^2/w(t) < \infty. \quad (7)$$

**Proof** The eigenvectors of $f(t) = t^\alpha/w(t)$ of the unitarily equivalent operator $A^*$ lie in the space $L^2(0, \infty, w(t)dt)$ if and only if (7) holds.

Note that the $\alpha$ occurring in Theorem 3.4 form an annulus (if the set is non-empty), and in some cases (e.g. the Hardy–Bergman space below) this enables us to find the spectrum exactly.

**3.2.2 The Case $x > 0$**

Here the calculations are necessarily more difficult, but we do have the spectral radius formula (6) to give a disc in which the spectrum is contained. In many cases all the points in the interior are eigenvalues, either of $A$ or $A^*$. Indeed, they are again eigenvalues of infinite multiplicity, and hence in the essential spectrum, as we see in the following two propositions,

**Proposition 3.5** For $\mu > 1$, if $\alpha \in \mathbb{C}$ and $\beta = x/(\mu - 1) > 0$ are such that the function $f(t) = t^\alpha e^{-\beta t}$ lies in $L^2(0, \infty, w(t)dt)$, then $A$ has the eigenvalue $1/\mu^{\alpha+1}$ and $f \chi_E$ is an eigenvector for any measurable $E \subset \mathbb{R}_+$ such that $\mu E = E$.

**Proof** This is an easy calculation since $Af(t) = \frac{1}{\mu} f(t/\mu)e^{-xt/\mu}$.

**Proposition 3.6** For $0 < \mu < 1$ if $\alpha \in \mathbb{C}$ and $\beta = x/(1 - \mu) > 0$ are such that the function $f(t) = t^\alpha e^{-\beta t}/w(t)$ lies in $L^2(0, \infty, w(t)dt)$, then $A^*$ has the eigenvalue $\mu\alpha$ and $f \chi_E$ is an eigenvector for any measurable $E \subset \mathbb{R}_+$ such that $\mu E = E$.

**Proof** This is similarly straightforward, since

$$A^*f(t) = f(\mu t)e^{-xt}w(\mu t)/w(t).$$

One technique from [15] is not available in general: for the Hardy and standard weighted Bergman spaces, the weighted composition operators with $\mu$ and $1/\mu$ are related by taking adjoints. This is not true unless $w(t)$ is a power of $t$.

However, as we shall see in the next section the information here is enough to allow us to determine the exact spectrum in an important new example (as well as the cases covered by Schroderus).
3.3 The Hardy–Bergman Space

As a significant example of a situation not covered by older results, let us look again at the Hardy–Bergman space $H^2(\mathbb{C}^+) \cap A^2(\mathbb{C}^+)$. To determine the spectrum we may take any equivalent norm, so we take $w(t) = 1 + 1/t$ as before.

**Theorem 3.7** Let $X = H^2(\mathbb{C}^+) \cap A^2(\mathbb{C}^+)$ be the Hardy-Bergman space and $C_\phi \in \mathcal{L}(X)$.

a) If $\varphi(s) = \mu s$ with $\mu > 0$, then $\sigma(C_\phi)$ is the annulus

$$\sigma(C_\phi) = \left\{ z \in \mathbb{C} : \min \left\{ \frac{1}{\mu}, \frac{1}{\sqrt{\mu}} \right\} \leq |z| \leq \max \left\{ \frac{1}{\mu}, \frac{1}{\sqrt{\mu}} \right\} \right\}.$$

b) If $\varphi(s) = \mu s + (x + iy)$ with $\mu, x > 0$ and $y \in \mathbb{R}$, then $\sigma(C_\phi)$ is the closed disc

$$\sigma(C_\phi) = \{ z \in \mathbb{C} : |z| \leq 1/\mu \}.$$

In all cases we have $\sigma_e(C_\phi) = \sigma(C_\phi)$.

The proof of this theorem follows from the two subsections detailed below.

3.3.1 The Case $x = 0$

For $\phi(s) = \mu s$ with $\mu > 0$ we have

$$\|C_\phi\|^2 = \frac{1}{\mu} \sup_{t > 0} \frac{w(\mu t)}{w(t)} = \frac{1}{\mu} \sup_{t > 0} \frac{1 + 1/(\mu t)}{1 + 1/t} = \frac{1}{\mu} \sup_{t > 0} \frac{\mu t + 1}{\mu t + \mu}.$$

This is

$$\begin{cases} 1/\mu^2 & \text{if } 0 < \mu < 1, \\ 1/\mu & \text{if } \mu > 1. \end{cases}$$

Thus for $\mu > 1$ we have

$$\|C^n_\phi\|^2 = \frac{1}{\mu^n}$$

while

$$\|C^{-n}_\phi\|^2 = \mu^{2n}.$$
We now show that the (essential) spectrum equals the whole annulus. To do this we use Theorem 3.4. The function \( t^\alpha / (1 + 1/t) \) lies in the space \( L^2(0, \infty, (1 + 1/t) \, dt) \) if and only if
\[
\int_0^\infty \frac{|t^\alpha|^2}{(1 + 1/t)^2} (1 + 1/t) \, dt < \infty,
\]
so that \(-1 < \text{Re}\alpha < -1/2\). From this we see that all points in the annulus \( \{ z \in \mathbb{C} : 1/\mu < |z| < 1/\sqrt{\mu} \} \) are eigenvalues of \( A^* \) so that for \( \mu > 1 \) we have
\[
\sigma(C_\phi) = \left\{ z \in \mathbb{C} : \frac{1}{\mu} < |z| < \frac{1}{\sqrt{\mu}} \right\}.
\]
By considering inverses, we see that for \( 0 < \mu < 1 \)
\[
\sigma(C_\phi) = \left\{ z \in \mathbb{C} : \frac{1}{\sqrt{\mu}} < |z| < \frac{1}{\mu} \right\}.
\]

3.3.2 The Case \( x > 0 \)

We begin with the spectral radius formula (6). Elementary calculus shows that the supremum in (6) is at \( t = 0 \) except if \( 2x_n < \mu^n - 1 \). This requires \( \mu > 1 \). If the supremum is at \( t = 0 \) it is \( 1/\mu^n \).

Otherwise (with \( \mu > 1 \)) we can estimate the supremum for each \( n \) by considering the two cases (recall that \( x_n \to \infty \) as \( n \to \infty \)):

1. \( t \geq \frac{n \log \mu}{2x_n} \), when the expression is clearly at most \( \mu^{-n} \) (the second factor is bounded by 1);
2. \( 0 \leq t \leq \frac{n \log \mu}{2x_n} \), when we may ignore the exponential and obtain an upper bound
\[
\frac{1 + 1/(\mu^n t)}{1 + 1/t} \leq \frac{1 + 2x_n/(n\mu^n \log \mu)}{2x_n/n \log \mu} \leq \frac{1 + C_1/n}{C_2\mu^{n-1}/n},
\]
where \( C_1 \) and \( C_2 \) are independent of \( n \).

Taking the maximum of these two estimates and then the \( 2n \)-th root gives a limit \( 1/\sqrt{\mu} \). That is \( \rho(C_\phi) \leq \frac{1}{\sqrt{\mu}} \times \frac{1}{\sqrt{\mu}} = \frac{1}{\mu} \) for all \( x > 0 \).

Now we use Propositions 3.5 and 3.6 to find eigenvectors.

For \( \mu > 1 \) the range of admissible \( \alpha \) is \( \{ \text{Re}\alpha > 0 \} \) and the eigenvalue is \( 1/\mu^{\alpha+1} \), so that every point of the disc \( \{ z \in \mathbb{C} : 0 < |z| < 1/\mu \} \) is an eigenvector of \( A \) and we deduce that \( \sigma(C_\phi) = \{ z \in \mathbb{C} : |z| \leq 1/\mu \} \).

Similarly, for \( 0 < \mu < 1 \) the range of admissible \( \alpha \) is \( \{ \text{Re}\alpha > -1 \} \) and the eigenvalue is \( \mu^\alpha \), so again we deduce that \( \sigma(C_\phi) = \{ z \in \mathbb{C} : |z| \leq 1/\mu \} \), and the same holds for \( \sigma_e(C_\phi) \).
4 Semigroups of Composition Operators

4.1 Norm Estimates

Our first task is to generalize the result of Arvanitidis [3] showing that a necessary and sufficient condition for boundedness of a semigroup \((C_{\phi_t})\) of composition operators on \(H^2(\mathbb{C}_+)\) with generator \(A : f \mapsto Gf'\) is that \(\delta := \angle \lim_{z \to \infty} G(z)/z\) exists. In this case \(\|C_{\phi_t}\|_{H^2(\mathbb{C}_+)} = e^{-\delta t/2}\). This relies on a theorem in [7] (which is stated for the disc), which gives the angular derivative, and hence the norm, in terms of \(G\). In particular, \(L_t = e^{-\delta t}\).

By virtue of Theorem 1.2 (Kucik) we know that the condition for boundedness does not depend on which Zen space we use, and all that changes is the norm of the operator. We may thus state the following easy theorem.

**Theorem 4.1** For a semigroup of analytic self-maps \((\phi_t)\) on \(\mathbb{C}_+\), the following are equivalent:

1. the non-tangential limit \(\delta := \angle \lim_{z \to \infty} G(z)/z\) exists;
2. the semigroup \((C_{\phi_t})_{t \geq 0}\) consists of bounded operators on \(A^2_{\nu}\).

In this case, we have

\[
L_t \inf_{x > 0} \frac{w(x)}{w(L_t x)} \leq \|C_{\phi_t}\|^2 \leq L_t \sup_{x > 0} \frac{w(x)}{w(L_t x)},
\]

where \(L_t = e^{-\delta t}\) for \(t \geq 0\).

**Proof** This follows from [3, Thm. 3.4] together with the estimate (1) from [13]. \(\square\)

For example, with the Hardy space \(H^2(\mathbb{C}_+)\) considered by Arvanitidis, we have \(\|C_{\phi_t}\| = e^{-\delta t/2}\), while for the standard Bergman space \(A^2(\mathbb{C})\) we have the apparently new result that \(\|C_{\phi_t}\| = e^{-\delta t}\).

In [2] there is a result applying to analytic function spaces, i.e., Banach spaces \(X\) of holomorphic functions on a domain \(\Omega\) for which point evaluations are continuous functionals. The result applies to such spaces satisfying the following supplementary condition:

\((E)\) If \((z_n)\) is a sequence in \(\Omega\) such that \(z_n \to z \in \overline{\Omega} \cup \{\infty\}\) and \(\lim_{n \to \infty} f(z_n)\) exists in \(\mathbb{C}\) for all \(f \in \text{dom}(A)\), then \(z \in \Omega\).

**Theorem 4.2** Suppose that \((T_t)_{t \geq 0}\) is a \(C_0\) semigroup on a function space \(X\) with property \((E)\) such that for some \(G \in \text{Hol}(\Omega)\) the generator \(A\) is the operator \(f \mapsto Gf'\) for all \(f \in \text{dom}(A)\). Then \((T_t)\) is a semigroup of composition operators.

Similar results are given in [10], for function spaces on the disc \(\mathbb{D}\). We may now apply Theorem 4.2 to our situation.

**Theorem 4.3** Suppose that \((T_t)_{t \geq 0}\) is a \(C_0\) semigroup on a Zen space \(A^2_{\nu}\) such that for some \(G \in \text{Hol}(\mathbb{C}_+)\) the generator \(A\) is the operator \(f \mapsto Gf'\) for all \(f \in \text{dom}(A)\). Then \((T_t)\) is a semigroup of composition operators.
Proof For the halfplane we need to find sequences \((z_n)\) in \(C_+\) tending to \(\infty\) or a point on the imaginary axis for which \(\lim f(z_n)\) does not exist for some \(f \in A^2_\nu\).

Now if \((z_n)\) is such that \((f(z_n))\) converges for all \(f \in A^2_\nu\) then by the uniform boundedness theorem the reproducing kernels at \((z_n)\) are uniformly bounded. But if \(\text{Re}z_n \to 0\) monotonically then the integrals in (4) with \(\lambda = z_n\) tend to \(\infty\) (note that \(w\) is a decreasing function). This shows that condition (E) holds for \(A^2_\nu\) (take sequences tending to either a point on the imaginary axis or to infinity with real parts tending to \(0\)). The result follows from Theorem 4.2.

Remark 4.4 We may also consider the question of groups of composition operators on \(A^2_\nu\). However, since we have same semigroups for all the spaces under consideration, with different norms, we have the same groups. Proposition 4.4 of [4] asserts that for a \(C_0\)-quasicontractive group of bounded composition operators on on \(H^2(C_+\) we have \(G(z) = ps + iq\) for real \(p\) and \(q\), and \(\phi_t(s) = e^{pt}s + \frac{iq}{p}(e^{pt} - 1)\) if \(p \neq 0\), or \(\phi_t(s) = s + iqt\) if \(p = 0\). The same will apply to any \(\text{Zen space} A^2_\nu\).

4.2 Semigroups of Linear Fractional Mappings

As in Sect. 3 we may discuss parabolic or hyperbolic mappings fixing \(\infty\). That is, \(\phi_t(s) = \mu_t s + s_t\), where \(\mu_t > 0\) and \(s_t \in \overline{C_+}\).

The semigroup relation \(\phi_{t+u} = \phi_t \circ \phi_u\) gives us \(\mu_t = e^{pt}\) for some \(p \in \mathbb{R}\), and

(1) \(s_t = \alpha(e^{pt} - 1)\) for some \(\alpha \in \overline{C_+}\) if \(p \neq 0\) (the hyperbolic case), or

(2) \(s_t = \alpha t\) for some \(\alpha \in \overline{C_+}\) if \(p = 0\) (the parabolic case).

Using (2) we see that \(G(z) = pz + p\alpha\) in the hyperbolic case and \(G(z) = \alpha\) in the parabolic case.

An immediate corollary of Theorem 3.7 is the following description of the spectrum.

Corollary 4.5 Let \(X = H^2(C_+) \cap A^2(C_+)\) be the Hardy-Bergman space and \((C_{\phi_t})_{t \geq 0} \subset \mathcal{L}(X)\). 

a) If \(\varphi_t(s) = e^{pt}s + \alpha(e^{pt} - 1)\) with \(p \neq 0\), then \(\sigma(C_{\varphi_t})\) is the annulus

\[
\sigma(C_{\varphi_t}) = \left\{ z \in \mathbb{C} : \min \left\{ \frac{1}{e^{pt}}, \frac{1}{e^{pt/2}} \right\} \leq |z| \leq \max \left\{ \frac{1}{e^{pt}}, \frac{1}{e^{pt/2}} \right\} \right\}.
\]

b) If \(\varphi_t(s) = s + \alpha t\), then \(\sigma(C_{\varphi_t})\) is the closed disc

\[
\sigma(C_{\varphi_t}) = \{ z \in \mathbb{C} : |z| \leq e^{-pt} \}
\]

In all cases the spectrum and essential spectrum coincide.

Author Contributions Isabelle Chalendar and Jonathan Partington wrote the manuscript.

Declarations

Conflict of interest The authors declare no competing interests.
References

1. Alajyan, A.E., Partington, J.R.: Weighted operator-valued function spaces applied to the stability of delay systems. Oper. Matrices 15(4), 1257–1266 (2021)
2. Arendt, W., Chalendar, I.: Generators of semigroups on Banach spaces inducing holomorphic semiflows. Isr. J. Math. 229(1), 165–179 (2019)
3. Arvanitidis, A.G.: Semigroups of composition operators on Hardy spaces of the half-plane. Acta Sci. Math. (Szeged) 81(1–2), 293–308 (2015)
4. Avicou, C., Chalendar, I., Partington, J.R.: Analyticity and compactness of semigroups of composition operators. J. Math. Anal. Appl. 437(1), 545–560 (2016)
5. Berkson, E., Porta, H.: Semigroups of analytic functions and composition operators. Mich. Math. J. 25(1), 101–115 (1978)
6. Chalendar, I., Partington, J.R.: Norm estimates for weighted composition operators on spaces of holomorphic functions. Complex Anal. Oper. Theory 8(5), 1087–1095 (2014)
7. Contreras, M.D., Díaz Madrigal, S., Pommerenke, Ch.: On boundary critical points for semigroups of analytic functions. Math. Scand. 98(1), 125–142 (2006)
8. Elliott, S., Jury, M.T.: Composition operators on Hardy spaces of a half-plane. Bull. Lond. Math. Soc. 44(3), 489–495 (2012)
9. Elliott, S.J., Wynn, A.: Composition operators on weighted Bergman spaces of a half-plane. Proc. Edinb. Math. Soc. 54, 373–379 (2011)
10. Gallardo-Gutiérrez, E.A., Yakubovich, D.V.: On generators of $C_0$-semigroups of composition operators. Isr. J. Math. 229(1), 487–500 (2019)
11. Jacob, B., Partington, J.R., Pott, S.: On Laplace-Carleson embedding theorems. J. Funct. Anal. 264(3), 783–814 (2013)
12. Jacob, B., Partington, J.R., Pott, S.: Applications of Laplace-Carleson embeddings to admissibility and controllability. SIAM J. Control Optim. 52, 1299–1313 (2014)
13. Kucik, A.S.: Weighted composition operators on spaces of analytic functions on the complex half-plane. Complex Anal. Oper. Theory 12(8), 1817–1833 (2018)
14. Matache, V.: Weighted composition operators on $H^2$ and applications. Complex Anal. Oper. Theory 2(1), 169–197 (2008)
15. Schroderus, R.: Spectra of linear fractional composition operators on the Hardy and weighted Bergman spaces of the half-plane. J. Math. Anal. Appl. 447(2), 817–833 (2017)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Springer Nature or its licensor (e.g. a society or other partner) holds exclusive rights to this article under a publishing agreement with the author(s) or other rightsholder(s); author self-archiving of the accepted manuscript version of this article is solely governed by the terms of such publishing agreement and applicable law.