A GENERALIZATION OF PURELY EXTENDING MODULES RELATIVE TO A TORSION THEORY

SEMRA DOĞRUOZ AND AZİME TARHAN

Abstract. In this work we introduce a new concept, namely, $\tau_s$-extending modules (rings) which is torsion-theoretic analogues of extending modules and then we extend many results from extending modules to this new concept. For instance we show that for any ring $R$ with unit, if $R$ is purely $\tau_s$-extending then every cyclic $\tau$-nonsingular $R$-module is flat and we show that this fact is true over a principal ideal domain as well. Also, we make a classification for the direct sums of the rings to be purely $\tau_s$-extending.

1. Introduction

Injective modules have been intensively studied in the 1960s and 1970s in module theory and more generally in algebra. As a generalization of injective modules extending modules (CS), that is every closed submodule is a direct summand, have been studied widely in last three decades. In general setting Chatters and Hajarnavis [6], Harmancı and Smith [21], Kamal and Muller [22] and their schools can be mentioned involving studies of extending modules.

Recently, torsion-theoretic analogues of extending modules has been an interest to extend many results and concepts from extending modules to a torsion theory such primarily studies as Asgari and Haghany [3], Berktaş, Doğruöz and Tarhan [5], Crivei [10], Çeken and Alkan [11], Doğruöz [12]. Clark [7] defined a module $M$ is purely extending if every submodule of $M$ is essential in a pure submodule of $M$, equivalently every closed submodule of $M$ is pure in $M$. Al-Bahrani [1] generalized purely extending modules as a purely $y$-extending module using $s$-closed submodules which was defined by Goodearl [19] such as a submodule $N$ of a module $M$ is $s$-closed in $M$ if $M/N$ is non-singular. So a module $M$ is called purely $y$-extending if every $s$-closed submodule of $M$ is pure in $M$. In fact Al-Bahrani [1] belike misused the terminology of $s$-closed submodules. They used the term $y$-closed (purely $y$-extending) instead of $s$-closed (purely $s$-extending) respectively. In this study we use $s$-closed submodule and purely $s$-extending module instead of $y$-closed submodule and purely $y$-extending module in the sense of Al-Bahrani [1].

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We use the concept 'purity' in the sense of Cohn [9] (as in [7]) which implies definition of Anderson and Fuller [2], that is, a submodule \( N \) of an \( R \)-module \( M \) is called pure submodule in \( M \) in case \( IN = N \cap IM \) for each finitely generated right ideal \( I \) of the ring \( R \) (see also [23]). In the present paper we introduce purely \( \tau_s \)-extending modules and then we extend many results from [1], [7] and [19] to this new concept.

For instance we show that:

**Theorem 18:** Let \( R \) be a \( \tau \)-torsion ring and \( M \) be an \( R \)-module. Let \( E(M) \) be an injective hull of \( M \). Then \( M \) is a purely \( \tau_s \)-extending module if and only if \( A \cap M \) is pure in \( M \) for every direct summand \( A \) of \( E(M) \) such that the submodule \( A \cap M \) is \( \tau_s \)-closed in \( M \).

**Proposition 26:** Let \( R \) be a ring with identity. If \( R \) is purely \( \tau_s \)-extending then every cyclic \( \tau \)-nonsingular \( R \)-module is flat.

and

**Theorem 31:** Let \( R \) be a commutative integral domain. Then the following properties are equivalent:

1. \( R \) is a semi-hereditary ring.
2. \( R \oplus R \) is an extending module.
3. \( R \oplus R \) is a purely extending module.
4. \( R \oplus R \) is a purely s-extending module.
5. \( R \oplus R \) is a purely \( \tau_s \)-extending module.
6. For each \( n \in \mathbb{N} \), \( \bigoplus_n R \) is an extending module.
7. For each \( n \in \mathbb{N} \), \( \bigoplus_n R \) is a purely extending module.
8. For each \( n \in \mathbb{N} \), \( \bigoplus_n R \) is a purely s-extending module.
9. For each \( n \in \mathbb{N} \), \( \bigoplus_n R \) is a purely \( \tau_s \)-extending module.

which is a torsion-theoretic analogue of [7, Proposition 1.6].

Throughout the work \( R \) will be an associative ring with identity and all \( R \)-modules will be unitary left \( R \)-modules unless otherwise stated. \( R-\text{Mod} \) will be the category of unitary left \( R \)-modules, and all modules and module homomorphisms will belong to \( R-\text{Mod} \). Let \( \tau : (\mathcal{T}, \mathcal{F}) \) be a torsion theory on \( R-\text{Mod} \). The modules in \( \mathcal{T} \) are called \( \tau \)-torsion modules and the modules in \( \mathcal{F} \) are called \( \tau \)-torsion-free modules. Let \( M \in R-\text{Mod} \). Then the \( \tau \)-torsion submodules of \( M \), denoted by \( \tau(M) \), is defined to be the sum of all \( \tau \)-torsion submodules of \( M \). Thus \( \tau(M) \) is the unique largest \( \tau \)-torsion submodule of \( M \) and \( \tau(M/\tau(M)) = 0 \) for an \( R \)-module \( M \). The torsion class \( \mathcal{T} \) is given \( \mathcal{T} := \{ M \in R-\text{Mod} | \tau(M) = M \} \) and \( \mathcal{F} \) is refered to as torsion-free class and given by \( \mathcal{F} := \{ M \in \text{Mod} - R | \tau(M) = 0 \} \). In our study \( \tau \) will be a hereditary torsion theory on \( R-\text{Mod} \) and we mean \( R \) is \( \tau \)-torsion ring if \( R R \) is \( \tau \)-torsion.

Let \( M \) be an \( R \)-module. A submodule \( N \) of \( M \) is called \( \tau \)-essential in \( M \) (\( N \leq_{\tau} M \)) if \( N \) is essential in \( M \) and \( M/N \) is \( \tau \)-torsion (see [17], originally defined by Tsai in 1965 [26]). Define the set \( Z_{\tau}(M) = \{ m \in M | \text{Ann}(m) \leq_{\tau} R \} \). If \( Z_{\tau}(M) = M \)
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then $M$ is called \( \tau \)-singular module and if $Z_{\tau}(M) = 0$ then $M$ is called \( \tau \)-nonsingular module \((13)\).

For elementary, additional and unexplained terminology the reader is referred to \([2]\) or \([27]\) for module and ring theory, \([17]\) and \([25]\) for torsion theory, \([13]\) for extending modules and \([23]\) for homological algebra.

2. Purely \( \tau_{s} \)-Extending Modules

Let $M$ be an $R$-module and $N$ be a submodule of $M$. We call $N$ is \( \tau_{s} \)-closed submodule of $M$ if the factor module $M/N$ is a \( \tau \)-nonsingular and it is denoted by $N \leq_{\tau_{s}c} M$.

Definition 1. Let $M$ be an $R$-module. If every \( \tau_{s} \)-closed submodule of $M$ is pure in $M$ then we call $M$ is a purely \( \tau_{s} \)-extending module. It is denoted briefly \( p_{\tau_{s}} \)-extending.

Lemma 2. Let $R$ be a \( \tau \)-torsion ring. If $N$ is \( \tau_{s} \)-closed in $M$ then $N$ is closed in $M$.

Proof. Let $N$ be a \( \tau_{s} \)-closed submodule of $M$. Then the factor module $M/N$ is \( \tau \)-nonsingular i.e., $Z_{\tau}(M/N) = 0$. Since $R$ is \( \tau \)-torsion, $Z_{\tau}(M/N) = Z(M/N)$. Assume $N$ is not closed in $M$. Then there exists a submodule $K$ of $M$ such that $K$ contains $N$ as an essential submodule. So the factor module $K/N$ is singular \([19]\). Hence $Z(K/N) = K/N$. On the other hand since $Z(K/N)$ is a submodule of $Z(M/N)$, we have $Z(K/N) = 0$. Hence $K/N$ is nonsingular. But since $K/N$ is singular, it must be zero (i.e. $K/N = 0$). Therefore $N = K$ and so $N$ is closed submodule of $M$. \( \square \)

Corollary 3. Let $R$ be a \( \tau \)-torsion ring. If $M$ is a purely extending module then $M$ is purely \( \tau_{s} \)-extending.

Proof. Let $M$ be a purely extending module and $N$ be a \( \tau_{s} \)-closed submodule of $M$. Since $R$ is \( \tau \)-torsion $N$ is closed in $M$ by Lemma 2. From \([7]\) Lemma 1.1 every closed submodule of $M$ is pure in $M$. So $N$ is pure in $M$. Therefore $M$ is purely \( \tau_{s} \)-extending module. \( \square \)

As in general extending module theory we have some of the fundamental properties of purely \( \tau_{s} \)-extending modules as follows:

Lemma 4. Let $M = M_1 \oplus M_2$ be a purely \( \tau_{s} \)-extending module then $M_1$ and $M_2$ are also purely \( \tau_{s} \)-extending modules i.e., any direct summand of a purely \( \tau_{s} \)-extending module is purely \( \tau_{s} \)-extending.

Proof. $M = M_1 \oplus M_2$ be a purely \( \tau_{s} \)-extending module and let $N_1$ be a \( \tau_{s} \)-closed submodule of $M_1$. Then $Z_{\tau}(M_1/N_1) = 0$. For the proof we want to show that $N_1$ is pure in $M_1$. First let us show that $N_1$ is \( \tau_{s} \)-closed in $M$ i.e., $(M/N_1)$ is \( \tau \)-nonsingular.
Assume $M/N_1$ is not $\tau$-nonsingular module. Thus $Z_\tau(M/N_1) \neq 0$. Then there exists an element $N_1 \neq m + N_1 \in M/N_1$ such that $Ann(m + N_1) \leq \tau_m R$. On the other hand, since $m \in M = M_1 \oplus M_2$, $m_1 \in M_1$ and $m_2 \in M_2$, the writing $m = m_1 + m_2$ is unique. Thus

$$Ann(m + N_1) = Ann((m_1 + m_2) + N_1) = Ann(m_1 + N_1 + m_2 + N_1) = Ann(m_1 + N_1) \cap Ann(m_2 + N_1)$$

(see [2] Proposition 2.16). In addition since $Ann(m + N_1) \leq \tau_m R$, we have $Ann(m_1 + N_1) \cap Ann(m_2 + N_1) \leq \tau_m R$. Since $Ann(m_1 + N_1) \cap Ann(m_2 + N_1) \subseteq Ann(m_1 + N_1) \subseteq R$, we have $Ann(m_1 + N_1) \leq \tau_m R$. But this contradicts with $Z_\tau(M/N_1) \neq 0$. Hence $Z_\tau(M/N_1) = 0$ i.e., $N_1$ is a $\tau_s$-closed submodule of $M$. By the hypothesis $N_1$ is pure in $M$ since $M$ is purely $\tau_s$-extending module. By [15] Proposition 1.2 (2) $N_1$ is pure in $M_1$. Thus $M_1$ is purely $\tau_s$-extending module. Similarly it can be shown that $M_2$ is also purely $\tau_s$-extending module.

**Corollary 5.** Let $M = \bigoplus_{i \in I} M_i$ be a purely $\tau_s$-extending module where $I$ is a finite index set. Then for every $i \in I$, $M_i$ is purely $\tau_s$-extending.

**Proof.** It is clear from Lemma

**Lemma 6.** Let $C$ be an $R$-module. Then $C$ is a $\tau$-nonsingular module if and only if for every $\tau$-singular $R$-module $A$, $\text{Hom}_R(A,C) = 0$.

**Proof.** Let $f : A \rightarrow C$ be an $R$-module homomorphism where $C$ is a $\tau$-nonsingular module and $A$ is a $\tau$-singular $R$-module. Then $f(A) = f(Z_\tau(A))$. We show $f(Z_\tau(A)) \subseteq Z_\tau(C)$. If $x \in f(Z_\tau(A))$ then there is an element $a \in Z_\tau(A)$ such that $x = f(a)$. So $Ann(a) \leq \tau_m R$. If $r \in Ann(a)$, then $rx = rf(a) = f(ra) = 0$ i.e., $r \in Ann(x)$. Since $Ann(a) \leq Ann(x) \leq R$, we have $Ann(x) \leq \tau_m R$ i.e., $x \in Z_\tau(C)$. By the hypothesis, since $Z_\tau(C) = 0$, $f = 0$ and thus $\text{Hom}_R(A,C) = 0$.

For the converse let $\text{Hom}_R(A,C) = 0$ for every $\tau$-singular $R$-module $A$. Specially $\text{Hom}_R(Z_\tau(C),C) = 0$. So the inclusion map $Z_\tau(C) \hookrightarrow C$ is zero. Hence $Z_\tau(C) = 0$ and so $C$ is $\tau$-nonsingular module.

**Lemma 7.** The class of $\tau$-nonsingular modules is closed under extensions by short exact sequences.

**Proof.** Let $C$ and $A$ be $\tau$-nonsingular modules and consider the following short exact sequence

$$0 \longrightarrow C \longrightarrow B \longrightarrow A \longrightarrow 0$$

For every $\tau$-singular $R$-module $M$, using Lemma we have $\text{Hom}_R(M,C) = 0$ and $\text{Hom}_R(M,A) = 0$. Then the following short exact sequence

$$0 \longrightarrow \text{Hom}_R(M,C) \longrightarrow \text{Hom}_R(M,B) \longrightarrow \text{Hom}_R(M,A) \longrightarrow 0$$
yields $\text{Hom}_R(M, B) = 0$. Again by Lemma 8, the $R$-module $B$ must be $\tau$-nonsingular.

\[\square\]

Next we can show $\tau_s$-closed submodules have transitivity property.

**Lemma 8.** Let $M$ be an $R$-module and let $K$ and $N$ be submodules of $M$ such that $K \subseteq N$. If $K$ is $\tau_s$-closed submodule of $N$ and $N$ is $\tau_s$-closed submodule of $M$, then $K$ is $\tau_s$-closed submodule of $M$.

**Proof.** Since $K$ is $\tau_s$-closed submodule of $N$ and $N$ is $\tau_s$-closed submodule of $M$, $Z_T(N/K) = 0$ and $Z_T(M/N) = 0$. We must show that $Z_T(M/K) = 0$. Consider the following short exact sequence

\[
0 \longrightarrow N/K \longrightarrow M/K \longrightarrow M/N \longrightarrow 0
\]

By Lemma 8, the class of $\tau$-nonsingular modules are closed under extensions by short exact sequences. Since $N/K$ and $M/N$ are both $\tau$-nonsingular, $M/K$ is $\tau$-nonsingular. Hence $Z_T(M/K) = 0$. Thus $K$ is $\tau_s$-closed submodule of $M$. \[\square\]

Now we have some basic properties as follows.

**Lemma 9.** Any $\tau_s$-closed submodule of a purely $\tau_s$-extending module is purely $\tau_s$-extending.

**Proof.** Let $M$ be a purely $\tau_s$-extending module and let $N$ be a $\tau_s$-closed submodule of $M$. Then $M/N$ is $\tau$-nonsingular. Let $K$ be a $\tau_s$-closed submodule of $N$. Then by Lemma 8, $K$ is a $\tau_s$-closed submodule of $M$. Since $M$ is purely $\tau_s$-extending module, $K$ is pure in $M$. By [15] Proposition 1.2 (2), $K$ is pure in $N$. So $N$ is purely $\tau_s$-extending module. \[\square\]

There exist submodules $K, L$ of a module $M$ such that $K$ and $L$ both closed submodules of $M$ but $K \cap L$ is not closed in $K, L$ or $M$ (see [19] Example 1.6). But we have the following in our case.

**Proposition 10.** Let $M$ be an $R$-module and $N, K$ be $\tau_s$-closed submodules of $M$. Then $N \cap K$ is a $\tau_s$-closed submodule of $M$.

**Proof.** Let $M$ be an $R$-module and $N$, $K$ be $\tau_s$-closed submodules of $M$. Then $M/K$ and $M/N$ are $\tau$-nonsingular, i.e., $Z_T(M/N) = 0$ and $Z_T(M/K) = 0$. Assume $Z_T(M/(N \cap K)) \neq 0$. Then there is a $(N \cap K) \neq \bar{m} \in M/(N \cap K)$ such that $\text{Ann}(\bar{m}) \subseteq \tau_s R$. Now for $\bar{m} = m + (N \cap K)$, $m \notin N \cap K$. On the other hand for $m \in M$, choose the elements $\hat{m} = m + N \in M/N$ and $\bar{m} = m + K \in M/K$. Then we have $\text{Ann}(\bar{m}) \subseteq \text{Ann}(\hat{m})$ and $\text{Ann}(\bar{m}) \subseteq \text{Ann}(\hat{m})$. Indeed, now let $0 \neq r \in \text{Ann}(\bar{m})$. Then $r\bar{m} = 0$ and so $r\bar{m} + (N \cap K) = N \cap K$. Hence $r\bar{m} \in N \cap K$. So we have $r\bar{m} \in N$ and $r\bar{m} \in K$. Thus $r\bar{m} + N = N$ and $r\bar{m} + K = K$, i.e. $r\bar{m} = 0$ and $r\bar{m} = 0$. Consequently $r \in \text{Ann}(\bar{m})$ and $r \in \text{Ann}(\hat{m})$. Hence $\text{Ann}(\bar{m}) \subseteq$
Ann(\(\tilde{m}\)) and Ann(\(\tilde{m}\)) \(\subseteq\) Ann(\(\tilde{m}\)). On the other hand since Ann(\(\tilde{m}\)) \(\leq\) \(\tau_s\) R we have Ann(\(\tilde{m}\)) \(\leq\) \(\tau_s\) R and Ann(\(\tilde{m}\)) \(\leq\) \(\tau_s\) R. Then by hypothesis \(Z_\tau(M/N) = 0\) and \(Z_\tau(M/K) = 0\), we have \(m \in N\) and \(m \in K\) and so \(m \in N \cap K\). Hence \(\tilde{m} = m + (N \cap K) = N \cap K\). This is a contradiction. Thus \(Z_\tau(M/(N \cap K)) = 0\). Therefore \(N \cap K\) is a \(\tau_s\)-closed submodule of \(M\).

**Corollary 11.** Any intersection of \(\tau_s\)-closed submodules is also \(\tau_s\)-closed.

**Proof.** It is an evident result of Proposition 10.

**Lemma 12.** Let \(M\) be an \(R\)-module and let \(K, L\) be submodules of \(M\) such that \(K \leq L\). If \(L\) is a \(\tau_s\)-closed submodule of \(M\) then \(L/K\) is a \(\tau_s\)-closed submodule of \(M/K\).

**Proof.** Let \(L\) be a \(\tau_s\)-closed submodule of \(M\). Then \(Z_\tau(M/L) = 0\). On the other hand \((M/K)/(L/K) \cong M/L\) and since \(\tau\)-nonsingular modules are closed under isomorphisms, \(Z_\tau((M/K)/(L/K)) = 0\). Hence \(L/K\) is \(\tau_s\)-closed in \(M/K\).

**Lemma 13.** Let \(M\) be an \(R\)-module and let \(K, L\) be submodules of \(M\) such that \(K \leq L\). If the factor module \(L/K\) of \(M/K\) is \(\tau_s\)-closed then \(L\) is a \(\tau_s\)-closed submodule of \(M\).

**Proof.** Since \(L/K\) is a \(\tau_s\)-closed submodule of \(M/K\), \(Z_\tau((M/K)/(L/K)) = 0\). Since \((M/K)/(L/K) \cong M/L\) and \(\tau\)-nonsingular modules are closed under isomorphisms, \(Z_\tau(M/L) = 0\). Hence \(L\) is a \(\tau_s\)-closed submodule of \(M\).

**Proposition 14.** Let \(M\) be a purely \(\tau_s\)-extending \(R\)-module and \(N\) be a \(\tau_s\)-closed submodule of \(M\). Then the factor module \(M/N\) is purely \(\tau_s\)-extending.

**Proof.** Let \(M\) be a purely \(\tau_s\)-extending \(R\)-module and \(N\) be a \(\tau_s\)-closed submodule of \(M\). By the definition of purely \(\tau_s\)-extending module, \(N\) is pure in \(M\). For \(N \leq K \leq M\) let \(K/N\) be \(\tau_s\)-closed in \(M/N\). Now \((M/N)/(K/N) \simeq M/K\) and since the \(\tau\)-nonsingular modules are closed under isomorphisms, \(Z_\tau(M/K) = 0\). So \(K\) is \(\tau_s\)-closed submodule of \(M\). Since \(M\) is purely \(\tau_s\)-extending, \(K\) is pure in \(M\). By Proposition 1.2 (3) \(K/N\) is pure in \(M/N\). Thus \(M/N\) is purely \(\tau_s\)-extending.

Let \(M\) be an \(R\)-module. For an arbitrary submodule \(N\) of \(M\) by Zorn’s Lemma there is a maksimal submodule \(K\) of \(M\) such that \(N\) is essential in \(K\). The submodule \(K\) is called closure of \(N\) in \(M\) ([24]).

Now we give another generalization of closures relative to a torsion theory as follows:

**Definition 15.** Let \(M\) be an \(R\)-module and let \(N\) be a submodule of \(M\). The smallest \(\tau_s\)-closed submodule \(K\) of \(M\) which is containing \(N\) is called \(\tau_s\)-closure of \(N\) in \(M\). The \(\tau_s\)-closure of \(N\) is denoted by \(N^{\tau_s}\).

**Lemma 16.** Every submodule \(N\) of an \(R\)-module \(M\) has a \(\tau_s\)-closure in \(M\).
Proof. Let $M$ be an $R$-module and $N$ be a submodule of $M$. Now define the set $S = \{ K \leq M \mid N \subseteq K \text{ and } K \leq \tau_s \ M \}$. Since $Z_s(M/M) = 0$, $M$ is $\tau_s$-closed in $M$ and so $M \in S$. Then $S$ is non-empty. Let $C$ be a chain in $S$. Take $C = \bigcap_{K_i \in C} K_i$. By Corollary 11, $C$ is a $\tau_s$-closed submodule of $M$. Then $C \in S$. By Zorn’s Lemma there is a minimal element in $S$. If we call this element such as $H$ then $H$ is $\tau_s$-closure of $N$ in $M$. Thus every submodule $N$ of $M$ has a $\tau_s$-closure in $M$. □

Proposition 17. An $R$-module $M$ is a purely $\tau_s$-extending if and only if the $\tau_s$-closure of $N$ (i.e., $N^{-\tau_s}$) is pure in $M$ for every submodule $N$ of $M$.

Proof. Let $M$ be a purely $\tau_s$-extending module. Then every $\tau_s$-closed submodule of $M$ is pure in $M$. By Zorn’s Lemma every submodule $N$ of $M$ has a $\tau_s$-closure in $M$. By the definition of $\tau_s$-closure, the submodule $N^{-\tau_s}$ is $\tau_s$-closed in $M$ and by the hypothesis the submodule $N^{-\tau_s}$ is pure in $M$.

Conversely, let $K$ be a $\tau_s$-closed submodule in $M$. By the definition of $\tau_s$-closure, $K^{-\tau_s} = K$. By the hypothesis $K^{-\tau_s}$ i.e. $K$ is a pure submodule in $M$. Then any $\tau_s$-closed submodule of $M$ is pure in $M$. Thus $M$ is a purely $\tau_s$-extending module.

Theorem 18. Let $R$ be a $\tau$-torsion ring, let $M$ be an $R$-module and let $E(M)$ be the injective hull of $M$. Then, $M$ is a purely $\tau_s$-extending module if and only if $A \cap M$ is pure in $M$ for every direct summand $A$ of $E(M)$ such that the submodule $A \cap M$ is $\tau_s$-closed in $M$.

Proof. Let $R$ be a $\tau$-torsion ring, $M$ be an $R$-module, $E(M)$ be the injective hull of $M$ and $M$ be a purely $\tau_s$-extending module. Then for every direct summand $A$ of $E(M)$ such that $A \cap M$ is a $\tau_s$-closed submodule of $M$ it is clear that $A \cap M$ is pure in $M$.

Conversely, let $A$ be a $\tau_s$-closed submodule of $M$ and let $B$ be a complement of $A$ in $M$. Then $A \oplus B$ is essential in $M$ [19, Proposition 1.3]. Now it is clear that $A \oplus B$ is essential in $E(M)$. Hence $E(A) \oplus E(B) = E(A \oplus B) = E(M)$ [20]. Since $A = A \cap M \leq s E(A) \cap M \leq (E(A) \cap M)/A$ is singular (see [19]). Moreover since $R$ is a $\tau$-torsion ring $(E(A) \cap M)/A$ is $\tau$-singular. On the other hand since $(E(A) \cap M)/A \leq M/A$ and $A$ is $\tau_s$-closed submodule of $M$, $M/A$ is $\tau$-nonsingular and thus $(E(A) \cap M)/A$ is $\tau$-nonsingular. Therefore $(E(A) \cap M)/A = 0$ and so $E(A) \cap M = A$. Since $A$ is $\tau_s$-closed in $M$, $E(A) \cap M$ is also $\tau_s$-closed in $M$. Since $E(A)$ is a direct summand of $E(M)$ by the hypothesis $E(A)/M$ is a pure submodule of $M$. Hence $A$ is pure in $M$. Thus $M$ is a purely $\tau_s$-extending module. □

Theorem 19. Let $R$ be a $\tau$-torsion ring, let $M$ be an $R$-module and let $E(M)$ be the injective hull of $M$. Assume $A + M$ be a flat module for every direct summand $A$ of $E(M)$ with $A \cap M$ is $\tau_s$-closed submodule of $M$. Then $M$ is a purely $\tau_s$-extending module.

Proof. Let $A$ be a direct summand of $E(M)$ such that $A \cap M$ is $\tau_s$-closed in $M$. Consider the following short exact sequences of $R$-modules
where \( i_1, i_2 \) are inclusion maps and \( f_1, f_2 \) are natural epimorphisms. Since \( A \) is a direct summand of \( E(M) \), there is a submodule \( A' \) of \( E(M) \) such that \( E(M) = A \oplus A' \). Thus \( A \) is also a direct summand of \( A + M \). Then the short exact sequence

\[
0 \rightarrow A \rightarrow A + M \rightarrow (A + M)/A \rightarrow 0
\]

splits. Therefore \((A + M)/A\) is flat since it is isomorphic to a direct summand of \( A + M \). On the other hand since \( M/(A \cap M) \cong (A + M)/A \) the factor module \( M/(A \cap M) \) is again flat. By \cite{[15]} Theorem 1.7 \( A \cap M \) is pure in \( M \). Hence by Theorem \cite{[15]} \( M \) is a purely \( \tau_s \)-extending module. □

3. Purely \( \tau_s \)-Extending Rings

If the ring \( R \) is purely \( \tau_s \)-extending as an \( R \)-module over itself then \( R \) is called purely \( \tau_s \)-extending.

3.1. Multiplication Modules. Let \( R \) be a commutative ring and \( M \) be an \( R \)-module. For every submodule \( N \) of \( M \) if there exists an ideal \( I \) of \( R \) such that \( N = IM \) then \( M \) is called a multiplication module. For every submodule \( N \) of \( M \) let us define

\[
(N : M) = \{ r \in R \mid rM \subseteq N \}.
\]

Then \( M \) is an multiplication \( R \)-module if and only if \( N = (N : M)M \) \cite{[4]}.

Definition 20. \cite{[8]} Let \( M \) be an \( R \)-module and \( N \) be a submodule of \( M \). If

\[
N = \text{Hom}(M, N) = \sum \{ \varphi(N) \mid \varphi : M \rightarrow N \}
\]

then \( N \) is called an idempotent submodule of \( M \). If every submodule of \( M \) is idempotent then \( M \) is called a fully idempotent module.

Theorem 21. \cite{[14]} Theorem 2.11] Let \( M \) be a multiplication \( R \)-module and \( M = M_1 \oplus M_2 \), is a direct sum of fully idempotent submodules \( M_1 \) and \( M_2 \). Then \( M \) is a fully idempotent module.
Lemma 22. [14] Lemma 2.13] Let $M$ be a fully idempotent $R$-module, $N$ be a submodule of $M$ and $I$ be an ideal of $R$. Then $N \cap MI = NI$, i.e., $N$ is pure in $M$.

Now we can give the following theorem by using fully idempotent submodules:

Theorem 23. Let $R$ be a commutative ring and let $M = M_1 \oplus M_2$ be a multiplication $R$-module with fully idempotent submodules $M_1$, $M_2$ of $M$. Then $M$ is a purely $\tau_s$-extending module.

Proof. Let $M$ be a multiplication $R$-module and $N$ be a $\tau_s$-closed submodule of $M$. By Theorem 21 $M$ is fully idempotent $R$-module and by Lemma 22 the $\tau_s$-closed submodule $N$ of $M$ is pure in $M$. Hence $M$ is purely $\tau_s$-extending. □

Now we can give a characterization of a purely $\tau_s$-extending $R$-module with a ring as follows:

Proposition 24. Let $R$ be a commutative ring and let $M$ be a faithful multiplication $R$-module. If $R/R$ is purely $\tau_s$-extending module then $M$ is also purely $\tau_s$-extending module.

Proof. Let $N$ be a $\tau_s$-closed submodule of $M$. Since $M$ is multiplication $R$-module, we can write $N = (N : M)M$. Claim: $(N : M)$ is $\tau_s$-closed submodule in $R$. Assume $(N : M)$ is not $\tau_s$-closed submodule in $R$. Then $R/(N : M)$ is not $\tau$-nonsingular i.e., $Z(R/(N : M)) \neq 0$. Then there exists at least one non-zero element $\bar{r}$ of $R/(N : M)$ such that $Ann(r + (N : M))$ is $\tau$-essential in $R$. So $\bar{r} = r + (N : M) \neq (N : M)$. Then there is an element $0 \neq m_0 \in M$ such that $rm_0 \notin N$. Now $Ann(r + (N : M)) \subseteq Ann(rm_0 + N)$. If $s \in Ann(r + (N : M))$, then $sr + (N : M) = (N : M)$. Hence we have $sr \in (N : M)$ so it is easy to check that $(sr)M \subseteq N$ (*). Let us show that $s \in Ann(rm_0 + N)$. Now $s(rm_0 + N) = srm_0 + N$ but since $(sr)M \subseteq N$ and by (*) for $m_0 \in M$, $srm_0 \in N$, i.e., $srm_0 + N = N$. So $s \in Ann(rm_0 + N)$. Hence we have $Ann(r + (N : M)) \subseteq Ann(rm_0 + N)$. On the other hand, since $N$ is $\tau_s$-closed in $M$ it is clear that $M/N$ $\tau$-nonsingular. So $r + (N : M)$ and $N = N$ but it contradicts with $rm_0 \notin N$. Hence $(N : M)$ must $\tau_s$-closed in $R$. Moreover since $R$ is purely $\tau_s$-extending $(N : M)$ is pure in $R$. Now for the finitely generated ideal $I$ of $R$ we have $IN = I(N : M)M = I \cap (N : M)M = IM \cap (N : M)M = IM \cap N$ ([1]). Therefore the $\tau_s$-closed submodule $N$ of $M$ is pure in $M$. Hence $M$ is a purely $\tau_s$-extending module. □

Remark 25. [23] Proposition 3.46] Let $R$ be an arbitrary ring. The left $R$-module $R$ is a flat left $R$-module.

In the sequel we use the flat ring in the sense of Rotman [23] Proposition 3.46], i.e the ring $R$ is flat if $R/R$ is flat.

Proposition 26. Let $R$ be an arbitrary ring. If $R/R$ is purely $\tau_s$-extending then every cyclic $\tau$-nonsingular $R$-module is flat.
Proof. Let \( R R \) be a purely \( \tau_s\)-extending module. Let \( M = Ra \) be a cyclic \( \tau\)-nonsingular \( R\)-module which is generated by a where \( a \in R \). Define the map \( f : R \rightarrow M \) with \( f(r) = ra \). Clearly \( f \) is an epimorphism and \( \text{Ker}(f) = \text{Ann}(a) \). So \( R/\text{Ker}(f) = R/\text{Ann}(a) \cong Ra \). Moreover, since \( Ra \) is a \( \tau\)-nonsingular module and the class of \( \tau\)-nonsingular modules is closed under isomorphisms \( R/\text{Ann}(a) \) is \( \tau\)-nonsingular. Hence \( \text{Ann}(a) \) is \( \tau_s\)-closed in \( R \). By the hypothesis \( \text{Ann}(a) \) is pure in \( R \). Since \( R \) is flat and \( \text{Ann}(a) \) is pure in \( R \), \( R/\text{Ann}(a) \) is flat by \cite{1} Lemma 19.18. Therefore \( Ra \) is flat.

Proposition 27. Let \( R \) be a principal ideal domain (for short PID). If every cyclic \( \tau\)-nonsingular \( R\)-module is flat then \( R R \) is purely \( \tau_s\)-extending.

Proof. Let \( K \) be a \( \tau_s\)-closed ideal of \( R \). Then \( R/K \) is \( \tau\)-nonsingular. Since \( R \) is PID the factor ring \( R/K \) is also PID. Hence \( R/K \) is cyclic. By hypothesis \( R/K \) is flat. Thus by \cite{1} Lemma 19.18, \( K \) is pure in \( R \). Then \( R \) is purely \( \tau_s\)-extending. \( \square \)

3.2. Semi-hereditary Rings. Let \( R \) be a ring with unit element. If every left (right) ideal of \( R \) is projective then \( R \) is called a left (right) hereditary ring. If every finitely generated left (right) ideal of \( R \) is projective then \( R \) is called a left (right) semi-hereditary ring \((25)\).

When the ring \( R \) is semi-hereditary we have Proposition 26 with its converse also as follows.

Theorem 28. Let \( R \) be a semi-hereditary ring. Then \( R R \) is purely \( \tau_s\)-extending if and only if every cyclic \( \tau\)-nonsingular \( R\)-module is flat.

Proof. Let \( R \) be a purely \( \tau_s\)-extending semi-hereditary ring and let \( M \) be a cyclic \( \tau\)-nonsingular \( R\)-module generated by an element \( a \) of \( R \) \((M = Ra \) is \( \tau\)-nonsingular). Take the homomorphism \( f : R \rightarrow Ra \) with \( f(r) = ra \). It is clear that \( f \) is an epimorphism. Since \( R/\text{Ker}(f) = R/\text{Ann}(a) \cong Ra \) and \( \tau\)-nonsingular modules are closed under isomorphisms, \( R/\text{Ker}(f) \) is \( \tau\)-nonsingular. Then \( \text{Ker}(f) \) is a \( \tau_s\)-closed submodule of \( R R \). By the hypothesis \( \text{Ker}(f) \) is pure in \( R \). On the other hand since \( R \) is a semi-hereditary ring, every finitely generated ideal of \( R \) is projective and so \( R/\text{Ker}(f) \) is projective. Since projective modules are flat, \( R/\text{Ker}(f) \) is flat and thus \( Ra \) (and so \( M \)) is flat.

Conversely let \( C \) be a \( \tau_s\)-closed ideal of \( R \). Then \( R/C \) is \( \tau\)-nonsingular and also \( R/C \) is cyclic. Hence by the hypothesis \( R/C \) is flat. By \cite{13} Theorem 1.7 we have \( C \) is pure in \( R \). Thus \( R R \) is a purely \( \tau_s\)-extending module. \( \square \)

Theorem 29. Let \( R \) be a left semi-hereditary ring. Then \( R \oplus R \) is purely \( \tau_s\)-extending if and only if every \( \tau\)-nonsingular 2-generated \( R\)-module is flat.

Proof. Let \( M = Rm_1 + Rm_2 \) be a \( \tau\)-nonsingular \( R\)-module. Define the map \( f : R \oplus R \rightarrow M \) with \( f(r_1, r_2) = r_1 m_1 + r_2 m_2 \). Now it is clear that \( f \) is an epimorphism. Hence \( (R \oplus R)/\text{Ker}(f) \cong M \). Since \( (R \oplus R)/\text{Ker}(f) \) is \( \tau\)-nonsingular, \( \text{Ker}(f) \) is a \( \tau_s\)-closed submodule of \( R \oplus R \). By the hypothesis \( \text{Ker}(f) \) is pure in \( R \oplus R \). Since
$R$ is a semi-hereditary ring, $R$ is flat. Because of the direct sum of flat modules is flat $R \oplus R$ is flat \cite{19}. Thus by \cite{15} Proposition 1.3 (3), we have the $R$-module $M$ is flat.

For the converse, let $C$ be a $\tau_s$-closed submodule of $R \oplus R$. Then $(R \oplus R)/C$ is $\tau$-nonsingular. On the other hand since $R \oplus R$ is a 2-generated $\tau$-nonsingular, $R$-module $(R \oplus R)/C$ is also 2-generated $\tau$-nonsingular $R$-module. By the hypothesis $(R \oplus R)/C$ is flat. Then by \cite{15} Theorem 1.7 we get $C$ is pure in $R \oplus R$. Thus $R \oplus R$ is purely $\tau_s$-extending. □

**Corollary 30.** Let $R$ be a left semi-hereditary ring and $I$ be a finite index set. Then $\bigoplus I R$ is purely $\tau_s$-extending if and only if every $\tau$-nonsingular $I$-generated $R$-module is flat.

Now we can give the following generalized characterization of purely $\tau_s$-extending modules.

**Theorem 31.** Let $R$ be a commutative integral domain. Then the following properties are equivalent:

1. $R$ is a semi-hereditary ring.
2. $R \oplus R$ is an extending module.
3. $R \oplus R$ is a purely extending module.
4. $R \oplus R$ is a purely $s$-extending module.
5. $R \oplus R$ is a purely $\tau_s$-extending module.
6. For each $n \in \mathbb{N}$, $\bigoplus_n R$ is an extending module.
7. For each $n \in \mathbb{N}$, $\bigoplus_n R$ is a purely extending module.
8. For each $n \in \mathbb{N}$, $\bigoplus_n R$ is a purely $s$-extending module.
9. For each $n \in \mathbb{N}$, $\bigoplus_n R$ is a purely $\tau_s$-extending module.

**Proof.** The equivalence of (1), (2) and (6) are given in \cite{13} Corollary 12.10.

In addition the equivalence of (1), (2), (3), (6) and (7) are given in \cite{7} Proposition 1.6.

(3) \iff (4). Every $s$-closed submodule of a module $M$ is closed in $M$. But converse is true if $M$ is nonsingular \cite{19} Proposition 2.4. Here since $R$ is commutative integral domain, $R$ is nonsingular. Therefore closed submodule and $s$-closed submodule implies each other. Thus the proof is clear by \cite{7} Lemma 1.1 in fact Lemma 1.1 is originally given by Fuchs \cite{16}.

(7) \iff (8). It can be easily checked be like (3) \iff (4).

Now we show (5) \Rightarrow (4). Let $K$ be a $s$-closed submodule of $R \oplus R$. Then $(R \oplus R)/K$ is nonsingular. Since any nonsingular module is $\tau$-nonsingular. $(R \oplus R)/K$ is a $\tau$-nonsingular. By the hypothesis $K$ is pure in $R \oplus R$. Hence $R \oplus R$ is a purely $s$-extending module.

The implication of (9) \Rightarrow (8) is a generalization of (5) \Rightarrow (4).

Now it’s left to show (1) \Rightarrow (5). For this let $K$ be a $\tau_s$-closed submodule of $R \oplus R$. Since $R$ is a semi-hereditary ring, as a finitely generated $R$-module $(R \oplus R)/K$ is
projective and so \((R \oplus R)/K\) is flat (see \[23\] Proposition 3.46). By \[15\] Proposition 1.3 \(K\) is pure in \(R \oplus R\). Hence \(R \oplus R\) is a purely \(\tau_s\)-extending module.

Finally \((1) \Rightarrow (9)\) is also similar to \((1) \Rightarrow (5)\). This completes the proof. \(\Box\)

**Example 32.** Let \(\mathbb{Z}\) be the ring of integers. Then \(\mathbb{Z}\) is a purely \(\tau_s\)-extending \(\mathbb{Z}\)-module over itself.

**Proof.** Since \(\mathbb{Z}\) is a principal ideal domain (PID), every ideal of \(\mathbb{Z}\) is free and so it is projective. Therefore \(\mathbb{Z}\) is a hereditary ring. Moreover \(\mathbb{Z}\) is a semi-hereditary ring. By Theorem \[31\] \(((1) \Rightarrow (5))\) we have \(\mathbb{Z} \oplus \mathbb{Z}\) is purely \(\tau_s\)-extending. Additionally, by Lemma \[9\] since the direct summands of purely \(\tau_s\)-extending modules are purely \(\tau_s\)-extending, we have \(\mathbb{Z}\) is a purely \(\tau_s\)-extending module. \(\Box\)

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*Current address:* Semra Doğruöz: Adnan Menderes University, Aydın, Turkey  
*Email address:* sdogruoz@adu.edu.tr  
*URL: http://orcid.org/0000-0002-7928-301X*

*Current address:* Azime Tarhan: Adnan Menderes University, Aydın, Turkey  
*Email address:* a.tarhan89@hotmail.com  
*URL: http://orcid.org/0000-0002-5363-1936*