Cylindrically polarized nondiffracting optical pulses

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Abstract
We extend the concept of radially and azimuthally polarized optical beams to the polychromatic domain by introducing cylindrically polarized nondiffracting optical pulses. In particular, we discuss in detail the case of cylindrically polarized X-waves, both in the paraxial and nonparaxial regime. The explicit expressions for the electric and magnetic fields of cylindrically polarized X-waves are also reported.

Keywords: X-waves, singular optics, physical optics, polarisation, optical pulses

(Some figures may appear in colour only in the online journal)

1. Introduction

Cylindrically polarized vector beams, i.e., solutions of Maxwell’s equation that are radially or azimuthally polarized, have attracted a lot of interest in the past decade, as they exhibit peculiar properties, such as the ability of producing a smaller focus [1], which opened new possibilities for spectroscopy [2], microscopy [3], particle manipulation [4], material processing [5, 6], propagation of linear and nonlinear waves in crystals [7–10], quantum information [11], metrology [12] and sensing [13]. This vast range of applications motivated the development of different ways to efficiently generate such modes [14–18]. A detailed theoretical discussion on the properties and applications of these beams for the paraxial case can be found in [19]. Inspired by these developments, many groups have tried to extend the properties and structure of cylindrically polarized vector beams to the non-paraxial case, by investigating strongly focused fields [20], complex dipole sources [21], elegant [22] and decentered [23] Laguerre–Gauss beams, and vector Bessel beams [24]. Recently, cylindrically polarized Bessel–Gauss beams have also been proposed theoretically [25]. Moreover, the concept of radial and azimuthal polarization has been investigated experimentally also for optical pulses, leading to interesting results concerning material processing [26], focusing of femtosecond pulses [27] and electron–photon interactions [28]. A fully theoretical description of cylindrically polarized optical pulses is, however, still elusive.

It is the aim of this work to fill this gap. As suggested in [19], to properly construct cylindrically polarized modes, one requires states of the electromagnetic field with one unit of orbital angular momentum (OAM) and one unit of spin angular momentum, combined in such a way that the total angular momentum of the field will remain zero. In the paraxial case, this is achieved by using linearly polarized Hermite–Gaussian beams [19], while in the nonparaxial domain, suitable superpositions of Bessel beams are used to mimic the structure of the corresponding paraxial modes [29]. Here, we use the latter approach to extend the definition of cylindrically polarized modes to the domain of optical pulses. Polychromatic superpositions of Bessel beams, commonly called X-waves, in fact, have been studied quite extensively in the past years [31], and very recently, OAM-carrying X-waves have been proposed [30]. Firstly introduced in acoustics [32, 33], X-waves have provided many interesting results in many different areas of physics, like nonlinear optics [34, 35], condensed matter [36], quantum optics [37], integrated optics [38, 39] and optical communications [40]. We believe that the interplay between the nondiffracting character typical of X-waves and the properties of cylindrically polarized beams can open new possibilities for fundamental and applied research.
This work is organized as follows: in section 2, we briefly review how to obtain scalar and vector X-waves carrying OAM starting from Bessel beams. In section 3, we introduce radially and azimuthally polarized optical pulses, and use these results in section 4 to discuss the special case of cylindrically polarized X-waves. Finally, conclusions are drawn in section 5.

2. X waves carrying OAM

In this section, we briefly discuss how to construct nondiffracting optical pulses carrying OAM. To do that, we essentially follow the method presented in [41]. Let us consider \( \psi(\mathbf{r}; k) \) to be a solution of the scalar Helmholtz equation, i.e.

\[
(\nabla^2 + k^2) \psi(\mathbf{r}; k) = 0,
\]

where \( k = \omega/c \) is the vacuum wave vector, \( \omega \) is the frequency and \( c \) is the speed of light in vacuum. From this solution, it is possible to construct a solution of the scalar wave equation \( (\nabla^2 - \frac{1}{c^2} \partial_t^2) \phi(\mathbf{r}, t) = 0 \), in the following way:

\[
\phi(\mathbf{r}, t) = \int_0^\infty dk f(k) e^{i k c t} \psi(\mathbf{r}; k),
\]

where \( f(k) \) is an arbitrary spectral function. If \( \psi^B(\mathbf{r}; k) \) are Bessel beams, i.e., nondiffracting solutions of the Helmholtz equation, then, according to [42], the solution to equation (1) can be written as

\[
\psi^B_m(\mathbf{r}; k) = J_m(k \sin \vartheta_0 R) e^{i m \vartheta_0} e^{i k z \cos \vartheta_0},
\]

where \( \vartheta = \arctan(y/x) \), \( R = \sqrt{x^2 + y^2} \) \( (x, y \) being the coordinates in the transverse plane orthogonal to the propagation direction \( z \), \( J_m(\cdot) \) is the Bessel function of the first kind of order \( m \) [43] and \( \vartheta_0 \) is the Bessel cone angle, that is, the beam’s characteristic parameter. X-waves are defined (according to equation (2)) as polychromatic superpositions of Bessel beams [31]:

\[
\phi_m(\mathbf{r}, t) = \int_0^\infty dk f(k) J_m(k \sin \vartheta_0 R) e^{i k c t} e^{i m \vartheta_0},
\]

where

\[
\zeta = z \cos \vartheta_0 - c t
\]

is the co-moving coordinate attached to the nondiffracting pulse itself. For the case \( m = 0 \), the above equation describes a scalar nondiffracting wave carrying \( m \) units of OAM [30].

2.1. Generalization to the vector case

The results presented above are only valid in the scalar case. To obtain vectorial X-waves that are exact solutions to Maxwell’s equation, we can use the following simple vectorialization procedure, based on the Hertz potential method [44]. If we introduce the monochromatic Hertz potential \( \mathbf{P}(\mathbf{r}, t; k) = \psi(\mathbf{r}; k) \exp(-ikc t) \hat{\mathbf{f}} \) (where \( \hat{\mathbf{f}} \) is an arbitrary unit vector), in fact, the corresponding monochromatic electric and magnetic fields \( \mathbf{E}(\mathbf{r}, t; k) \) and \( \mathbf{B}(\mathbf{r}, t; k) \) are defined via the following relations [45]:

\[
\mathbf{E}(\mathbf{r}, t; k) = \nabla \times \nabla \times \mathbf{P}(\mathbf{r}, t; k),
\]

\[
\mathbf{B}(\mathbf{r}, t; k) = \frac{1}{c^2} \frac{\partial}{\partial t} [\nabla \times \mathbf{P}(\mathbf{r}, t; k)].
\]

It is then easy to verify that the above fields are exact solutions of Maxwell’s equations. Once these fields are known, the electric and magnetic field corresponding to the nondiffracting optical field given by equation (2) are given by

\[
\mathbf{E}(\mathbf{r}, t) = \int_0^\infty dk f(k) \mathbf{E}(\mathbf{r}, t; k),
\]

\[
\mathbf{B}(\mathbf{r}, t) = \int_0^\infty dk f(k) \mathbf{B}(\mathbf{r}, t; k).
\]

Note that, alternatively, one could directly define the polychromatic Hertz potential \( \mathbf{P}(\mathbf{r}, t) = \phi_m(\mathbf{r}, t) \hat{\mathbf{f}} \) and use equation (6) with \( \mathbf{P} \) instead of \( \mathbf{P} \) to directly generate the polychromatic electric and magnetic fields \( \mathbf{E}(\mathbf{r}, t) \) and \( \mathbf{B}(\mathbf{r}, t) \), respectively.

2.2. Fundamental X-waves with OAM

As an explicit example of optical pulse carrying OAM, we consider the so-called fundamental X-waves [31], which possess the following exponentially decaying spectrum [50]:

\[
f(k) = k^\alpha e^{-\alpha k},
\]

where \( \alpha \) is a constant, which essentially accounts for the spectrum width, and \( n \in \mathbb{N} \). The spectrum in equation (8) can be obtained from the generating function \( \exp(-\alpha k) \) by subsequent differentiations with respect to \( \alpha \), i.e.,

\[
k^n \exp(-\alpha k) = (-1)^n \partial^m_{\alpha} \exp(-\alpha k).
\]

For the sake of simplicity, here we will only consider the case \( n = 0 \). The higher order fields corresponding to \( n \neq 0 \) can be then obtained from our fundamental solution by simple differentiation with respect to \( \alpha \). Substituting equation (8) with \( n = 0 \) into equation (4) and using the formula 6.621.1 of [47], gives the following result:

\[
\phi_m(\mathbf{r}, t) = \frac{1}{2^n} \frac{\xi^n}{(\alpha - i \zeta)^n} F_2 \left( m + 1, \frac{m}{2} + \frac{1}{2}; m + 1; -\xi \right),
\]

where \( F_2(a, b; c; x) \) is Gauss’ hypergeometric function [43] and

\[
\zeta = R \sin \vartheta_0 (\alpha - i \zeta).
\]

The higher-order solutions corresponding to \( n \neq 0 \) in equation (8) are obtained by taking the \( n \)th derivative of equation (9) with respect to \( (\alpha - i \zeta) \).

If we introduce the beam waist \( w(\zeta) \) (in analogy with the paraxial beams [48]) as

\[
w(\zeta) = w_0 \left( 1 - i \frac{\zeta}{\alpha} \right),
\]

where \( w_0 \equiv w(\zeta = 0) = \alpha / \sin \vartheta_0 \), and define the normalized radial variable \( \rho \equiv \rho(\zeta) = R/w(\zeta) \), equation (9) can be
rewritten in the following form:

\[
\phi_{0m}(\rho, \theta, \zeta) = \frac{\rho^{m+1}}{2^{m} \sin \theta_0} e^{\text{im} \theta_0} \sum_{n} 2F_1 \left( \frac{m + 1}{2}, \frac{m + 2}{2}; m + 1; -\rho^2 \right). \tag{12}
\]

Equation (12) represents a scalar nondiffracting optical pulse carrying \( m \) units of OAM. The vector electric and magnetic fields for fundamental X-waves with OAM can be calculated with equation (6) by choosing \( P(r, t) = \psi_0^m(r, k) \exp(-i k r) \hat{f} \) and using the spectrum (8) with \( n = 0 \). Their explicit expression can be found in [41].

3. Cylindrically polarized nondiffracting optical pulses: general aspects

Cylindrically polarized beams are a special subset of vector beams with non-uniform polarization. They can be thought of being defined as linear combinations of basis vector of a four-dimensional vector space \( \mathcal{H} = \mathcal{H}_0 \otimes \mathcal{H}_P \), defined as the cartesian product of two 2-dimensional vector spaces, namely the space \( \mathcal{H}_0 \), spanned by the two mode functions \( \{ \Psi_{00}(r), \Psi_0(r) \} \) and the space \( \mathcal{H}_P \), spanned by the two polarization vectors \( \{ \hat{x}, \hat{y} \} \) [19]. Therefore, \( \mathcal{H} = \{ \Psi_{00}(r) \hat{x}, \Psi_0(r) \hat{y}, \Psi_{00}(r) \hat{y}, \Psi_0(r) \hat{x} \} \). The explicit form of the mode functions \( \Psi_{00}(r) \) and \( \Psi_0(r) \), however, depends on whether we consider paraxial or nonparaxial beams. For the paraxial case, the mode functions are simply the Hermite–Gauss beams \( H_{nm}(r) \) of order \( N = n + m = 1 \) [48], namely \( \{ \Psi_{00}(r), \Psi_0(r) \} \equiv \{ HG_{10}(r), HG_{01}(r) \} \) [19]. In the nonparaxial case, instead, the mode functions \( \Psi_{00}(r) \) and \( \Psi_0(r) \) can be written in terms of superpositions of Bessel beams with \( m = \pm 1 \), as follows [29]:

\[
\Psi_{00}(r, k) = \frac{1}{\sqrt{2}} [\psi_0^B(r, k) + \psi_1^B(r, k)], \tag{13a}
\]

\[
\Psi_0(r, k) = -\frac{i}{\sqrt{2}} [\psi_1^B(r, k) - \psi_0^B(r, k)]. \tag{13b}
\]

Linear combinations of the four basis vectors spanning \( \mathcal{H} \) give then rise to radially (R) and azimuthally (A) polarized fields, as follows:

\[
\hat{u}^\pm R(r, k) = \frac{1}{\sqrt{2}} [\pm \Psi_{00}(r) \hat{x} + \Psi_0(r) \hat{y}]. \tag{14a}
\]

\[
\hat{u}^\mp A(r, k) = \frac{1}{\sqrt{2}} [\mp \Psi_{00}(r) \hat{x} + \Psi_0(r) \hat{y}]. \tag{14b}
\]

where \( \pm \) determine the co-rotating and counter-rotating solutions, respectively [19]. The results presented above are valid for monochromatic fields only. To generalize the mode functions \( \Psi_{00}(r) \) and \( \Psi_0(r) \) to the polychromatic case, we first substitute equation (13) into (2), thus obtaining

\[
\psi_{00}(r, t) = \int_0^\infty d k f(k)e^{-ikr}\Psi_{00}(r, k). \tag{15a}
\]

Next, we use the scalar pulses defined above into equation (14) and find the cylindrically polarized vector pulses:

\[
\hat{\mathbf{U}}^\pm R(r, t) = \int_0^\infty d k f(k)e^{-ikr}\hat{u}^\pm R(r, k) = \frac{1}{\sqrt{2}} [\pm \Psi_{00}(r, t) \hat{x} + \Psi_0(r, t) \hat{y}]. \tag{16a}
\]

\[
\hat{\mathbf{U}}^\mp A(r, t) = \int_0^\infty d k f(k)e^{-ikr}\hat{u}^\mp A(r, k) = \frac{1}{\sqrt{2}} [\mp \Psi_{00}(r, t) \hat{x} + \Psi_0(r, t) \hat{y}]. \tag{16b}
\]

The equations above are the first main result of our work. They are, essentially, a generalization of the corresponding radially and azimuthally polarized vector modes (introduced in [19, 29]) to the case of optical pulses. By choosing the form of \( \psi_{00}(r) \) and \( \psi_0(r) \) within the paraxial or nonparaxial regime, equation (16) define paraxial and nonparaxial cylindrically polarized pulses, respectively. As discussed in detail in [29], however, in the nonparaxial regime the vector fields in equation (16) do not constitute an exact solution to Maxwell’s equations, as they fail to satisfy the transversality condition \( \nabla \cdot \mathbf{U}^\pm R,A = 0 \). To overcome this problem, we can employ the vectorialization procedure described in section 3. If we choose \( P(r, t) = \mathbf{u}_{R,A}(r, k) \exp(-i k t) \) as Hankel potentials, then the electric and magnetic fields of a cylindrically polarized optical pulse can be written, in the general case, as follows:

\[
\mathbf{E}^\pm R,A(r, t) = \int_0^\infty d k f(k)\mathcal{E}^\pm R,A(r, t, k), \tag{17a}
\]

\[
\mathbf{B}^\pm R,A(r, t) = \int_0^\infty d k f(k)\mathcal{B}^\pm R,A(r, t, k), \tag{17b}
\]

where, according to equation (6), \( \mathcal{E}(r, t, k) = \nabla \times \mathbf{P}(r, t, k) \) and \( c^2 \mathcal{B}(r, t, k) = \frac{n_0}{n} [\nabla \times \mathbf{P}(r, t, k)] \). The set of equations above represents the electric and magnetic fields of a general cylindrically polarized, nondiffracting optical pulse, and it is the second result of our paper. Equation (17) describe a whole class of cylindrically polarized pulses, whose properties are defined by the choice of the spectrum \( f(k) \).

4. Cylindrically polarized X-waves

We now specify our analysis to the case of fundamental X-waves. In this case, equation (15) have the following explicit form:

\[
\psi_{00}(\rho, \theta, \zeta) = \frac{\rho \cos \theta}{\sqrt{2} w_0^2(z)} \sum_{n} 2F_1 \left( 1; \frac{3}{2}; -\xi^2 \right). \tag{18a}
\]

\[
\psi_{00}(\rho, \theta, \zeta) = \frac{\rho \sin \theta}{\sqrt{2} w_0^2(z)} \sum_{n} 2F_1 \left( 1; \frac{3}{2}; -\xi^2 \right). \tag{18b}
\]
A comparison between the mode \( \Phi_{10}(\rho, \theta, \zeta) \) and its corresponding monochromatic paraxial (Hermite–Gauss) and nonparaxial (Bessel) counterparts is reported in figure 1. As it can be seen, the pulse \( \Phi_{10} \) retains the same symmetry of its monochromatic counterparts, and can be then taken as a good candidate to realize cylindrically polarized modes. A similar argument also holds for the mode \( \Phi_{01}(\theta, \zeta) \). Substituting these expressions into equation (16) leads to the following result:

\[
U^\pm_0(\rho, \theta, \zeta) = \frac{\rho}{2w(\zeta)} \, _2F_1\left(1, \frac{3}{2}; 2; -\zeta^2\right) (\pm \cos \theta \hat{x} + \sin \theta \hat{y}), \quad (19a)
\]

\[
U^\pm_1(\rho, \theta, \zeta) = \frac{\rho}{2w(\zeta)} \, _2F_1\left(1, \frac{3}{2}; 2; -\zeta^2\right) (\mp \sin \theta \hat{x} + \cos \theta \hat{y}). \quad (19b)
\]

The components of the vector electric field generated by \( U_\lambda(\rho, \theta, \zeta) \) can be calculated by using equation (17a) in co-moving cylindrical coordinates \( \{\rho(\zeta), \theta, \zeta\} \). For the co-rotating radial polarized field we find:

\[
E^\rho_{R+}(\mathbf{R}, \zeta) = \frac{3\rho \cot^2 \vartheta_0}{w^2(\zeta)} \, _2F_1\left(2, \frac{5}{2}; 2; -\rho^2\right). \quad (20a)
\]

\[
E^\rho_{R-}(\mathbf{R}, \zeta) = 0, \quad (20b)
\]

\[
E^\rho_{K+}(\mathbf{R}, \zeta) = \frac{2i \cot \vartheta_0}{\sin \vartheta_0w^3(\zeta)} \, _2F_1\left(\frac{3}{2}, 2; 1; -\rho^2\right). \quad (20c)
\]

while for the counter-rotating radially polarized field, we have instead

\[
E^\rho_{R-}(\mathbf{R}, \zeta) = \frac{\cos(2\theta)}{\rho^2w^3(\zeta)} \left[ 2 \sin \vartheta_0 \, _2F_1\left(1, \frac{3}{2}; 1; -\zeta^2\right) - 2 \vartheta_0 \left(\frac{1}{2}, 1; 1; -\rho^2\right) - 3 \rho^2 \cos^2 \vartheta_0 \, _2F_1\left(2, \frac{5}{2}; 1; -\zeta^2\right) \right], \quad (21a)
\]

\[
E^\rho_{R+}(\mathbf{R}, \zeta) = \frac{\sin(2\theta)}{\rho^2w^3(\zeta)} \left[ 2 \sin \vartheta_0 \, _2F_1\left(1, \frac{3}{2}; 1; -\rho^2\right) - 2 \vartheta_0 \left(\frac{1}{2}, 1; 1; -\rho^2\right) + 3 \rho^2 \, _2F_1\left(2, \frac{5}{2}; 1; -\rho^2\right) \right], \quad (21b)
\]

\[
E^\rho_{K-}(\mathbf{R}, \zeta) = \frac{2i \cot \vartheta_0 \cos(2\theta)}{\sin \vartheta_0w^3(\zeta)} \, _2F_1\left(\frac{3}{2}, 2; 1; -\rho^2\right). \quad (21c)
\]

For the co-rotating azimuthally polarized field we find

\[
E^\phi_{\Lambda+}(\mathbf{R}, \zeta) = 0, \quad (22a)
\]

\[
E^\phi_{\Lambda+}(\mathbf{R}, \zeta) = \frac{3\rho}{w^2(\zeta)\sin^2 \vartheta_0} \, _2F_1\left(2, \frac{5}{2}; 2; -\rho^2\right). \quad (22b)
\]

\[
E^\phi_{\Lambda+}(\mathbf{R}, \zeta) = 0, \quad (22c)
\]

and, finally, for the counter-rotating azimuthally polarized field we have

\[
E^\phi_{\Lambda-}(\mathbf{R}, \zeta) = \frac{2\sin(2\theta)}{\rho^2w^3(\zeta)} \left[ -\sin \vartheta_0 \, _2F_1\left(1, \frac{3}{2}; 1; -\rho^2\right) + 2 \vartheta_0 \left(\frac{1}{2}, 1; 1; -\rho^2\right) + 3 \rho^2 \cos^2 \vartheta_0 \, _2F_1\left(2, \frac{5}{2}; 1; -\rho^2\right) \right], \quad (23a)
\]
Figure 2. Polarization pattern (arrows) and transverse intensity distribution \( I(x, y) = |E_x(x, y, 0)|^2 + |E_y(x, y, 0)|^2 \) calculated at \( \zeta = 0 \) (underlying doughnut) of the electric field of (a) a co-rotating radially polarized X-wave, as defined in equation (20), and (b) a counter-rotating radially polarized X-wave, as defined in equation (21). The axes of both graphs span the interval \([-1.5, 1.5]\) in units of the equivalent beam waist \( w_0 = \alpha / \sin \theta_0 \).

For the co-rotating azimuthally polarized field we find

\[
E^\varphi_{A+}(\mathbf{R}, \zeta) = \frac{2 \cos(2\theta)}{\rho w^2(\zeta)} \sin \theta_0 \left[ \begin{array}{l}
2 F_1 \left( 1, \frac{3}{2}; 1; -\rho^2 \right) \\
- 2 F_1 \left( 1, 1; 1; -\rho^2 \right) \\
+ \frac{3}{2} \rho^2 2 F_1 \left( 2, \frac{5}{2}; 1; -\rho^2 \right) \end{array} \right].
\]

(23b)

Here, \( \mathbf{R} = (\rho(\zeta) \hat{r}, \theta \hat{\theta}) \). In a similar manner, using the second of equation (17), the components of the magnetic field of the cylindrically polarized fundamental X-waves can be calculated using equation (17b). For the co-rotating radially polarized field we have

\[
cB^\rho_{A+}(\mathbf{R}, \zeta) = 0,
\]

(24a)

\[
cB^\rho_{A+}(\mathbf{R}, \zeta) = \frac{3 \rho \cot \theta_0}{w^2(\zeta)} 2 F_1 \left( 2, \frac{5}{2}; 2; -\rho^2 \right).
\]

(24b)

while the counter-rotating radially polarized field we have

\[
cB^\rho_{A-}(\mathbf{R}, \zeta) = \frac{3 \rho \cot \theta_0 \sin(2\theta)}{\sin \theta_0 w^2(\zeta)} 2 F_1 \left( 2, \frac{5}{2}; 2; -\rho^2 \right).
\]

(25a)

\[
cB^\rho_{A-}(\mathbf{R}, \zeta) = \frac{3 \rho \cot \theta_0 \cos(2\theta)}{\sin \theta_0 w^2(\zeta)} 2 F_1 \left( 2, \frac{5}{2}; 2; -\rho^2 \right).
\]

(25b)

\[
cB^\rho_{A-}(\mathbf{R}, \zeta) = \frac{3 \rho \cot \theta_0 \sin(2\theta)}{w(\zeta)} 2 F_1 \left( \frac{5}{2}; 3; 3; -\rho^2 \right).
\]

(25c)

For the co-rotating azimuthally polarized field we find

\[
cB^\varphi_{A+}(\mathbf{R}, \zeta) = - \frac{3 \rho \cot \theta_0}{\sin \theta_0 w^2(\zeta)} 2 F_1 \left( 2, \frac{5}{2}; 2; -\rho^2 \right).
\]

(26a)

\[
cB^\varphi_{A+}(\mathbf{R}, \zeta) = 0,
\]

(26b)

\[
cB^\varphi_{A-}(\mathbf{R}, \zeta) = - \frac{3 \rho \cot \theta_0 \sin(2\theta)}{\sin \theta_0 w^2(\zeta)} 2 F_1 \left( 2, \frac{5}{2}; 2; -\rho^2 \right).
\]

(26c)

and, finally, for the counter-rotating azimuthally polarized field we have

\[
cB^\varphi_{A-}(\mathbf{R}, \zeta) = - \frac{3 \rho \cot \theta_0 \cos(2\theta)}{\sin \theta_0 w^2(\zeta)} 2 F_1 \left( 2, \frac{5}{2}; 2; -\rho^2 \right).
\]

(27a)

\[
cB^\varphi_{A-}(\mathbf{R}, \zeta) = \frac{3 \rho \cot \theta_0 \sin(2\theta)}{w(\zeta)} 2 F_1 \left( \frac{5}{2}; 3; 3; -\rho^2 \right).
\]

(27b)

The polarization and intensity distribution patterns for the electric field of these cylindrically polarized pulses at \( \zeta = 0 \) are displayed in figures 2 and 3. It is worth noticing that the counter-rotating radially polarized pulse depicted in figure 2(b) and the azimuthally polarized pulse depicted in figure 3(b) may appear, at a first look, very similar. In particular, it seems that the two modes are linked by a rotation of 45° of their polarization profile. This resemblance, however, is only apparent and the physical properties of these two pulses are quite different, as the first represents a radially polarized state, while the second one represents an azimuthally polarized state of light. Moreover, these two states of light are mutually orthogonal, and together with the co-rotating
radially and azimuthally polarized pulses, depicted in figures 2(a) and 3(a) respectively, they form a complete basis in the four-dimensional Hilbert space \( \mathbb{R}^4 \) \( \{ \hat{\mathbf{x}}, \hat{\mathbf{y}} \} \otimes \{ \Phi_{10}(\rho, \theta, \zeta), \Phi_{01}(\rho, \theta, \zeta) \} \). An extensive discussion on the origin and the properties of the counter-rotating modes is reported in [19], for the case of paraxial beams.

Due to their intrinsically nonparaxial nature, these modes admit a nonzero longitudinal component, whose intensity distribution for \( \zeta = 0 \) is displayed in figure 4 for the case of radial polarization (upper panel) and azimuthal polarization (lower panel) respectively.

4.1. Paraxial limit

According to [29] the nonparaxial modes \( \Psi_{10} \) and \( \Psi_{01} \) admit (in \( z = 0 \)) the Hermite–Gaussian modes \( \text{HG}_{10} \) and \( \text{HG}_{01} \) as paraxial limit:

\[
\lim_{w_0 \to \infty} \Psi_{10}(R, \theta, 0; k) = \text{HG}_{10}(R, \theta, 0; k),
\]

\[
\lim_{w_0 \to \infty} \Psi_{01}(R, \theta, 0; k) = \text{HG}_{01}(R, \theta, 0; k).
\]

Analogously, the paraxial limit of the pulses \( \Phi_{10,01}(\rho, \theta, \zeta) \) in \( z = 0 \) is given by the corresponding Hermite–Gaussian modes. We have in fact

\[
\Phi_{10}^{(\par)}(R, \theta, t) \equiv \lim_{w_0 \to \infty} \Phi_{10}(\rho(z = 0), \theta, \zeta = -ct) = \int_0^\infty dk \int_0^\infty d\rho f(k) e^{-ikc} \lim_{w_0 \to \infty} \Psi_{10}(R, \theta, 0; k)
\]

\[
= \int_0^\infty dk f(k) e^{-ikc} \lim_{w_0 \to \infty} \Psi_{10}(R, \theta, 0; k) = \int_0^\infty dk f(k) e^{-ikc} \text{HG}_{10}(R, \theta, 0; k).
\]

A similar expression can be also obtained for \( \Phi_{01}^{(\par)}(R, \theta, t) \).

Notice that for \( z = 0 \), the expression of \( \text{HG}_{10}(R, \theta, 0; k) \) is independent on \( k \), as it is given by [48]

\[
\text{HG}_{10}(R, \theta, 0) = \frac{2\sqrt{2} R \cos \theta e^{-\frac{R^2}{w_0^2}}}{\sqrt{\pi} w_0^2}.
\]

We can use this result to explicitly calculate the last integral in equation (30) to obtain:

\[
\Phi_{10}^{(\par)}(R, \theta, t) = \frac{1}{\alpha - i ct} \text{HG}_{10}(R, \theta, 0),
\]

\[
\Phi_{01}^{(\par)}(R, \theta, t) = \frac{1}{\alpha - i ct} \text{HG}_{01}(R, \theta, 0).
\]

The radially and azimuthally polarized paraxial modes can be then built following the rules derived in section 3. In \( z = 0 \), therefore, the spatiotemporal features of the X-waves become separable, and the spatial structure of the cylindrically polarized X-waves is fully equivalent to their monochromatic paraxial counterpart. For \( z \neq 0 \), instead, the integrals in equation (30) do not admit a closed form solution, mainly because of the complicated \( k \)-dependence of the Hermite–Gaussian modes. If one wants to study the propagation properties of such modes, a numerical estimation of the integral (30) is therefore mandatory. In alternative, one could take directly the expression of the scalar pulses \( \Phi_{10,01}(r, t) \) (or, equivalently, of the corresponding vector electric and magnetic fields) and study their propagation properties as \( \nu_0 \to 0 \).

4.2. Polarisation patterns across the pulse area

The results presented above show the polarization patterns of the four cylindrically polarized pulses \( E_{z=0}^\pm(R, \zeta) \) for \( \zeta = 0 \), corresponding to the peak of these pulses. The peak of an optical pulse, however, is not the only interesting area of the pulse. In some cases, in fact, the properties of its tails are as important as the properties of its peak [49]. For this reason, it is therefore important to study how the polarization pattern of
Figure 4. Normalized intensity distributions for the longitudinal component $E_z(r, t)$ of the electric field of a radially (upper row) and azimuthally (lower row) polarized X-wave. Panels (a) and (c) display the co-rotating modes (equation (20c) and (22c), respectively), while panels (b) and (d) display the counter-rotating ones (equation (21c) and (23c), respectively). These figures are plotted for $\zeta = 0$, corresponding to $\varepsilon = 0 = t$. The transverse coordinates have been normalized to the beam waist $w_0 = \alpha / \sin \vartheta_0$.

Figure 5. Left panel: spatiotemporal profile of a co-rotating radially polarized X-wave as defined in equation (20) as a function of the normalized co-moving coordinate $\zeta / \alpha$. Right panel: real (red arrows) and imaginary (blue arrows) part of the polarization pattern and the intensity distribution (underlying doughnut) of a co-rotating radially polarized X-wave for various values of $\zeta$. Panels (a) and (d) correspond to the pulse tails ($\zeta / \alpha = \pm 1$, respectively), while panels (b) and (c) correspond to the FWHM of the pulse ($\zeta / \alpha = \pm 1/2$, respectively). The point (e) on the left panel corresponds to the case $\zeta = 0$, depicted in figure 2(a). The transverse coordinates have been normalized to the beam waist $w_0 = \alpha / \sin \vartheta_0$. 
the cylindrically polarized pulses presented in equation (20)–(23) changes across the pulse area, i.e., from its tails to its peak. To clarify this point, in figure 5 we consider explicitly the case of the co-rotating radially polarized X-wave $E_\phi(R, \phi)$. As it can be seen, while globally the character of the polarization remains always the same (in this particular case, for example, it is always radial), locally the polarization changes from linear at the peak (see figure 2(a)) to elliptical at the tails, where the polarization vector at each single point is a complex quantity. Moreover, as the real part of the polarization (red arrows) always points outside the transverse intensity pattern, its imaginary part (blue arrows) changes sign while passing from the left to the right tail and it becomes more dominant as one moves from the peak to the tail of the pulse. This behavior is similar to all four cylindrically polarized pulses. The visualisation of the polarization pattern for the other three modes is left to the reader, as it is straightforward obtained from equation (21)–(23).

5. Conclusions

In this work, we have generalized the method used in [19] to construct cylindrically polarized modes to the domain of optical pulses. To do that, we employed nondiffracting X-waves carrying OAM. We have shown that these vector fields retain all the typical properties of their monochromatic counterpart, and analyzed their behavior both in the paraxial and nonparaxial regime and presented the evolution of their polarization pattern across the pulse area.

We believe that our work is significant both for fundamental studies and for applications. In the latter case, the interplay between the nondiffracting character of X-waves and the cylindrical polarization could shine a new light on possible applications, such as material processing and particle manipulation. The fact that beams with cylindrical polarization may be tightly focalized [50] may open very interesting possibilities for spatially resolved pump–probe Raman spectroscopy and related fields. Moreover, as monochromatic cylindrically polarized beams have been recently used to prove alignment-free quantum communication protocols [51], the generalisation of these concepts to the pulsed domain could surely have a significative impact in quantum communication science. On the fundamental side, as cylindrically polarized modes are often associated with classical entanglement [52, 53], the extension of this concept to the non-monochromatic domain could open new horizons for the realization of classical and quantum hyper-entangled states.

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