The Secretary Recommendation Problem

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Abstract

In this paper we revisit the basic variant of the classical secretary problem. We propose a new approach, in which we separate between an agent that evaluates the secretary performance and one that has to make the hiring decision. The evaluating agent (the sender) signals the quality of the candidate to the hiring agent (the receiver) who must make a decision. Whenever the two agents’ interests are not fully aligned, this induces an information transmission (signaling) challenge for the sender. We study the sender’s optimization problem subject to incentive-compatibility constraints of the receiver for several variants of the problem.

Our results quantify the loss in performance for the sender due to online arrival. We provide optimal or near-optimal incentive-compatible mechanisms, which recover at least a constant fraction of a natural utility benchmark for the sender. The separation of evaluation and decision making can have a substantial impact on the approximation results. While in some variants, the techniques and results closely mirror the conditions in the standard secretary problem, we also exhibit variants leading to very different characteristics.

1 Introduction

In the classical secretary problem a sequence of \(n\) candidates with unknown valuations arrives sequentially. Upon arrival, a decision maker observes the value of a candidate and must make an irreversible decision of whether to hire the candidate or observe the subsequent one. The lion’s share of this literature studies optimal or near-optimal algorithms for the decision maker and provides many elegant results on how well he can perform (see, e.g., [4, 7, 8, 11, 14, 15, 17, 18, 21, 23–25] and many more).

In this problem, the decision maker is tasked with two responsibilities. He has to evaluate the candidate and he must also make the decision of whether or not to hire her. In many settings, these two tasks are separated and the evaluation is done separately from the hiring decision. In this paper, we initiate the study of (near-)optimal incentive-compatible mechanisms for the secretary problem when the evaluation is done by one agent while the decision is taken by another. Following tradition in the literature, we shall often refer to the latter as a ‘receiver’ and to the former as a ‘sender’. The separation between the evaluation and the hiring decision introduces incentive constraints that are immaterial to the original setting.

The aforementioned separation is wide-spread in applications. Consider, for example, the original motivating story of hiring an employee (e.g., a secretary). It is often the case that firms separate between the evaluation process, often led by a specialized HR department, and the hiring individual (the prospective boss). Another setting that has such a separation is in financial projects and investment opportunities. Many firms hire third-party consultants to evaluate a sequence of financial opportunities and use the consultant’s report in their decision process. An opportunity not seized is often an opportunity foregone.

To capture the incentive discrepancies between the sender and receiver, we endow each candidate with two valuations, one that is associated with the sender and one with the receiver. Candidates

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arrive according to a uniform random order. Upon arrival, the valuation pair is disclosed to the sender, who then makes some recommendation to the receiver. The receiver in turn takes one of two irrevocable actions, to hire or not to hire. We study several variants that differ in terms of the a-priori and a-posteriori information given to the two players as well as their objectives.

The first scenario we study is primarily studied for didactic reasons – we assume that the valuation pairs of all candidates are known to both players in advance. We use this scenario as benchmark for the scenarios with the same respective objectives but different information available to the sender and the receiver. Hence, this is our basic or benchmark scenario. The second scenario is termed the secretary scenario as it is more reminiscent of the original secretary problem in the sense that the valuation pairs are unknown to the two players and are set by an adversary.

Additionally, we study both scenarios in two different variants: In the first one, the receiver only receives the sender’s signals; in the second one, he is also informed about the valuations of rejected candidates. Thus, in the second case the sender’s informational advantage is diminished. We refer to this variant as a scenario with disclosure. For motivation, one can think of candidates that are foregone as candidates that go on the market which, inter-alia, extracts and makes public all the relevant information about the candidate. In the example with financial investments, one can think of a sequence of investment opportunities given to some prominent investor (e.g., a top-ten private equity fund) before they go on the market.

We study all four scenarios (basic and secretary, both with and without disclosure) with two objectives that are standard in the literature on the secretary problem, namely ordinal and cardinal objectives, both for sender and receiver. If a player has ordinal utility, she strives to maximize the probability that her best candidate is hired. With cardinal utility, the player strives to maximize the expected utility of the hired candidate. Overall, we examine a total of sixteen scenarios.

Our main results are incentive-compatible mechanisms that recover a constant fraction of the optimal utility that can be achieved by the sender in a corresponding benchmark. For most scenarios, we show that our bounds are asymptotically optimal with respect to any incentive-compatible mechanism. Our results quantify the loss in performance for the sender due to online arrival. Moreover, we prove that the separation of evaluation and decision making can have a substantial impact. We identify scenarios that yield optimal bounds of \(1/e\) as in the standard secretary problem, but also show scenarios that can lead to (provably) different optimal bounds. In some cases, this difference can be severe (constant vs. \(1/n\)). For example, in the secretary scenario with disclosure the approximation ratios of \(1/4\) and \(1/n\) (for ordinal and cardinal receiver utility, resp.) are asymptotically optimal and achieved by our mechanisms.

1.1 Related Literature

Our work can be seen as an extension of the celebrated secretary problem which first appeared in print in Martin Gardner’s 1960 Scientific American column [16] (but apparently originated much earlier, see [15]). The problem gained considerable popularity, and subsequently various extensions have been studied (c.f. [15] for an early survey and [8] for recent extensions). In all of the body of literature we are familiar with, the separation between the task of evaluating the candidate and the hiring decision is non-existent.

In our model the sender has commitment power and hence our work contributes to the burgeoning literature on ‘Bayesian Persuasion’, which originated in Aumann and Maschler [3] and more recently enjoyed a renaissance through the work of Gentzkow and Kamenica [20]. In particular, one can think of our model as an online Bayesian persuasion model, where the state of nature is revealed in a round-wise fashion to the sender who in turn sends a signal to the receiver in each round. Whereas dynamic Bayesian persuasion problems have been studied by various authors (e.g., [2, 12, 13]), we are not familiar with an online variant similar to what we study. Ely [12] studies the case that the sender observes the current state and sends a message to the receiver. Using this information, the receiver’s
beliefs are updated and he takes an action according to his current belief. The state of nature evolves according to a stochastic process, namely Poisson transitions over states, in contrast to our model with a constant state which is revealed over time. Au [2] studies an approach, where in each round the receiver either takes an action or goes on to the next round. Unlike our model, the action is the same one throughout. Once the action has been taken, the process ends and both parties get their respective utilities. The sender extracts positive utility from the fact that the receiver has taken the action, regardless of the state of nature. Hence, the sender’s objective is persuading the receiver to take the action. The receiver’s utility on the other hand depends on the state. A somewhat different approach is taken by Ely et al. [13] where the authors study the notions of suspense and surprise, i.e. the variance in the next update of belief and the distance of two consecutive beliefs, respectively. Thus, the sender uses these objectives in order to design the disclosure policies.

In the algorithmic literature, Dughmi and Xu [9] study the offline variant of Bayesian persuasion where candidates’ valuations are drawn independently from known distributions. In [5,10], the authors study algorithms and approximation for Bayesian persuasion with multiple receivers, each with a binary choice of actions. In contrast, our approach extends the Bayesian persuasion model with a single sender and a single receiver in a different direction, by studying an online approach with adversarial valuations and random-order arrival.

Somewhat related is previous work in the context of delegation [19]. Here, a principal delegates a search problem to an agent. Both have individual interests which might be misaligned. The principal can either accept or reject the agent’s proposed solution and will do so according to a specified mechanism. A key difference between the two models lies in the role of commitment power and the resulting mechanism designer. In our model, the party that observes the information designs the mechanism while in delegation, the mechanism is designed by the principal who only has a-priori information. Kleinberg and Kleinberg [22] recently studied how techniques and results from stochastic online optimization can be used to approximately solve delegation problems. In contrast to our work, they study application of these techniques to approximately solve an offline delegation problem.

1.2 Model

We consider a Bayesian persuasion problem with online arrival. There is a sender $S$ and a receiver $R$. A set of candidates arrives sequentially in $n$ rounds over time in uniform random order. In round $t$, a candidate arrives and a state of nature $\theta_t \in \Theta$ is revealed to $S$. We use $\theta = (\theta_1, \ldots, \theta_n)$ to denote the vector of the states of nature revealed in all rounds $t \in \{1, \ldots, n\} = [n]$. Associated with $\theta_t$ is a pair of non-negative values $\xi(\theta_t), \rho(\theta_t) \geq 0$, where $\rho(\theta_t) (\xi(\theta_t))$ is the utility of $R$ ($S$) when the candidate gets hired. We use $\rho$ and $\xi$ to denote the vectors of all possible utility values of all candidates in all states of nature, and $\rho_{\max} = \max_{\theta} \rho(\theta_t)$ and $\xi_{\max} = \max_{\theta} \xi(\theta_t)$ as well as $c_S = \arg \max_{\theta} \xi(\theta_t)$ and $c_R = \arg \max_{\theta} \rho(\theta_t)$.\footnote{For simplicity, we assume w.l.o.g. that for all candidates all utility values $\rho$, and $\xi$, are mutually disjoint.} Having observed $\theta_t$ and the corresponding values, $S$ transmits a signal $\sigma_t$ to $R$. Based on the signal, $R$ then decides whether to hire the current candidate in round $t$ or not. Every decision is final and cannot be revoked later on. If $R$ decides to hire, the process ends. Otherwise, round $t + 1$ starts and the next candidate arrives. The process ends by the end of round $n$ at the latest. The goal of both $S$ and $R$ is to maximize their respective objective. We consider two different goals, for each $S$ and $R$: The ordinal objective, where a player wants to maximize the success probability, i.e., the probability that the hired candidate is $c_S$ ($c_R$), resp.; and the cardinal objective, where a player strives to maximize the expected utility. All probabilities/expectations are with respect to input randomization and internal randomization of the mechanism. Our mechanisms for $S$ are truthful-in-expectation mechanisms, where we strive to maximize the sender’s objective.

Following the approaches in [9, 20], we assume that there is commitment power, i.e., $S$ shall commit a-priori on a signaling strategy $\phi$, mapping each partially revealed vector of states of nature
Table 1: Approximation guarantees of IC mechanisms for ordinal receiver utility discussed in this paper. All bounds stated without lower-order terms. Results indicated in bold have asymptotically matching upper bounds.

|                | without Disclosure | with Disclosure |
|----------------|--------------------|-----------------|
| Basic          | $1$ (Prop. 19)     | $1$ (Prop. 22)  |
| Secretary      | $1/e$ (Thm. 20)    | $1/4$ (Thm. 24, 25) |

Table 2: Approximation guarantees of IC mechanisms for cardinal receiver utility discussed in this paper. All bounds stated without lower-order terms. Results indicated in bold have asymptotically matching upper bounds.

|                | without Disclosure | with Disclosure |
|----------------|--------------------|-----------------|
| Basic          | Optimal mechanism (Prop. 1) | $1/2$ (Thm. 11, 14) |
| Secretary      | $1/4$ (Thm. 6)     | $\Theta(1/n)$ (Thm. 16) |

$(\theta_1, \ldots, \theta_t)$ to a signal $\sigma_t$, for all $t \in [n]$. The order of events is (1) $S$ commits to a signaling strategy $\phi$, (2) first candidate arrives, (3) $S$ learns $\theta_1$ and sends signal $\phi(\theta_1)$ to $R$, (4) $R$ decides to hire or not, (5) repeat from step (2) with $\theta_2$ and $\phi(\theta_1, \theta_2)$ etc., until a candidate gets hired or all $n$ candidates arrived.

Applying a revelation-principle style argument [1, 20], $S$ can restrict herself to a signaling strategy $\phi$ that is direct and persuasive. For direct signals, $S$ directly recommends one candidate for hire, i.e., $\sigma_t \in \{\text{HIRE, NOT}\}$. A persuasive strategy $\phi$ is an incentive-compatible mechanism, i.e., a recommendation strategy that $R$ finds in his interest to follow. We assume that ties are broken in favor of $S$, i.e., only when $R$ can strictly increase his success probability/expected utility, there is an incentive to deviate from the signals of $\phi$.

1.3 Results

We consider two different scenarios regarding the sender’s initial knowledge of the candidates as well as two scenarios regarding the information revealed to $R$ during the process.

In the basic scenario, all utility values are known beforehand, whereas in the secretary scenario, the utility values are only revealed to $S$ once the candidates arrive. In both scenarios, the order of arrival is unknown (but it is known that the arrival order is determined uniformly at random). In a scenario with disclosure, the receiver is informed of the rejected candidates. In a scenario without disclosure, the sender’s signals are the only information for $R$ in each round.

Note that in the secretary scenarios studied in this paper, $S$ faces a problem with the main characteristics of the secretary problem, i.e., unknown candidate values and random-order arrival. Formally, the candidate values $(\rho_i, \xi_i)$, $i \in [n]$ are determined upfront and are unknown to both $S$ and $R$. The candidates arrive in uniform random order. This scenario leaves some ambiguity in how to evaluate whether a deviation from a mechanism $\phi$ is profitable for $R$, since the benefit of such a deviation might depend on the unknown candidate values.

As a robust approach, we design mechanisms that satisfy a pointwise notion of truthfulness-in-
expectation. In such mechanisms, \( R \) finds it in his best interest to follow the mechanism even in a hypothetical scenario where he knows the set of candidates a-priori (similar to the corresponding basic scenario). Consequently, the secretary scenario only decreases the informational advantage of the sender. As such, the optimal (cardinal or ordinal) utility that can be achieved by \( S \) in the secretary scenario is clearly upper bounded by the utility that can be obtained in the corresponding basic scenario. Similarly, the corresponding scenarios with disclosure even further decrease the informational advantage of the sender by revealing rejected candidates.

In Table 1, we summarize the approximation ratios achieved by our polynomial-time algorithms in the case \( R \) has ordinal utility (i.e., strives to maximize the probability of hiring his best candidate). In Table 2, we give the respective approximation guarantees when \( R \) has cardinal utility (i.e., strives to maximize expected value of the hired candidate). In both tables, \( S_{\text{Card}} \) and \( S_{\text{Ord}} \) indicate whether \( S \) applies the cardinal or ordinal objective, respectively.

We view the basic scenario without disclosure as our benchmark scenario. Hence, all other entries in both Tables 1 and 2 represent approximation ratios with respect to the optimal value for \( S \) that can be obtained in this case.

When \( R \) has ordinal utility in the benchmark scenario, we obtain incentive-compatible mechanisms for \( S \) such that \( c_S \) gets hired with probability \( 1 - o(1) \). Hence, for ordinal receiver utility, our benchmark is simply the standard benchmark of the secretary problem – the (value \( \xi_{\text{max}} \) of the) best candidate \( c_S \) for \( S \). In contrast, when \( R \) has cardinal utility, the optimal expected utility achievable by \( S \) in the benchmark scenario can differ substantially from \( \xi_{\text{max}} \), and the optimal success probability for \( S \) can be significantly lower than 1. In fact, both these values can depend on the complete candidate set, and there is no immediate closed-form expression. Hence, for cardinal receiver utility, we will use \( \text{OPT} \) to denote the optimal value achievable in the corresponding instance in the benchmark scenario. In particular, for ordinal sender utility \( \text{OPT} \) is the optimal success probability in the benchmark scenario. Throughout the paper, all asymptotics will be in \( n \), the number of candidates.

Observe that for almost all our ratios we provide asymptotically tight upper bounds. Some of these upper bounds are trivial (e.g., 1 in the basic scenario), others mirror the conditions in the standard secretary problem (e.g., \( 1/e \) in the secretary scenario with ordinal receiver utility). However, we also prove several new upper bounds, such as 1/2 in the basic scenario with disclosure, or 1/4 and \( O(1/n) \) in the secretary scenario with disclosure. These bounds highlight the structural differences between our signaling scenarios and the standard variant of the secretary problem.

Additionally, for cardinal receiver utility, we give a characterization of the optimal mechanism in the basic scenario with disclosure using an exponential family of nested linear programs.

1.4 Techniques

Observe that the secretary scenarios studied in this paper strictly generalize the classic secretary problem. In particular, if \( \xi_i = \rho_i \) for every candidate \( i \in [n] \), the incentives of both players are perfectly aligned. In all basic scenarios, when \( S \) knows all candidates upfront, she would simply choose \( c_S = c_R \) and recommend it for hire. Thus, the benchmark in this case is success probability 1 and expected utility \( \xi_{\text{max}} \).

Since the benchmark for instances with \( \xi_i = \rho_i \) is the (value of the) best candidate for \( S \), it is easy to see that in such instances no incentive-compatible mechanism in any of the secretary scenarios (with or without disclosure, with ordinal or cardinal utilities) can beat the \( 1/e - o(1) \) guarantee from the classic optimal algorithm [11].

Interestingly, when the interests of both agents are not aligned, the techniques from the classical setting carry over for some scenarios (with appropriate adaptations), while in other scenarios new techniques are called for. For example, in the secretary scenario without disclosure, and a receiver with ordinal utility, we mix the optimal algorithm for \( S \) and for \( R \) and decide randomly upfront which version is applied. Although intuitive, it requires some effort to show that the mix retains incentive
compatibility for $R$ even when it strongly favors $S$.

In contrast, in the secretary scenario with disclosure, such an approach is not sufficient to incentivize $R$ to follow the mechanism. The first round, in which a best-so-far candidate for $R$ is rejected and revealed, $R$ learns that the sender-optimal version is used. This can give $R$ an incentive to ignore any upcoming HIRE signal.

Instead, we propose to run the two optimal algorithms in parallel and hire the first candidate that either algorithm would hire (first-come first-serve). We term this the First-Opt algorithm, and the success probability for $S$ becomes at least $1/4 - o(1)$. Our main insight is that for negatively correlated utilities, this algorithm and the guarantee are indeed optimal, which we prove via a generalized dynamic programming technique.

In contrast, for a receiver with cardinal utility, our mechanisms depart substantially from the classic template. Here, every incentive-compatible mechanism must guarantee an expected utility for the receiver of $\mu^R = \frac{1}{n} \sum \rho_i$. Our Pareto mechanism (Algorithm 1) optimizes the sender utility under this constraint. For secretary and disclosure variants, this technique is adjusted and applied w.r.t. growing and shrinking sets of already arrived or yet-to-arrive candidates, respectively. Perhaps surprisingly, this suffices to generate an extremely regular distribution of HIRE signals, which leaves $R$ with no additional information beyond the guarantee of expected utility $\mu^R$ (and, possibly, disclosure of rejected candidates). For the basic scenario with disclosure and ordinal sender utility, our mechanism achieves an approximation ratio of $1/2 - o(1)$. We show that this is tight for any incentive-compatible mechanism by giving a suitable class of instances. For the most challenging secretary scenario with disclosure, the informational advantage of the sender essentially vanishes. We show that there are cases in which no incentive-compatible mechanism can obtain a success probability of more than $1/n$ – while in the benchmark scenario the success probability is $1/2$. Thus, even in terms of approximation of OPT, no incentive-compatible mechanism can have a better approximation ratio than $2/n – 1/n$ is always achieved by the trivial strategy of recommending a candidate uniformly at random.

1.5 Outline

The paper is organized as follows. Section 2 focuses on the case where the receiver maximizes his expected utility. The techniques in this section depart from the classical secretary problem. We then move to discuss the ordinal receivers in Section 3. Each section discusses 8 variants of the problem: basic vs. secretary scenarios, disclosure vs. non-disclosure, ordinal vs. cardinal sender utility. We open each section with the benchmark case (basic, non-disclosure) and conclude with the most complex one (secretary with disclosure). Each scenario is analyzed for ordinal and cardinal sender utility, respectively.

2 Cardinal Utility for $R$

Let us first consider the scenario of cardinal receiver utility, in which $R$ wants to maximize the expected utility of the hired candidate.

To capture the sender’s value (i.e., her success probability/expected utility), we use the Pareto frontier, a geometric interpretation of the candidate set. It consists of the upper boundary of the convex hull of the candidates’ valuations. Hence, for increasing receiver-values, the Pareto frontier is non-increasing in the sender-values. This means that the value for $S$ of the Pareto curve at $\rho = \mu^R$ is an obvious upper bound on the sender’s expected utility.

Note that we can apply two normalization steps to the instance, both without loss of generality: First, we can scale all values such that $\xi_{\text{max}} = 1$ and $\rho_{\text{max}} = 1$. Second, in case of ordinal sender utility, we assume $\xi_i = 0$ for all other candidates $i' \neq c_S$. This way, the two scenarios (distinguishing two types of sender, ordinal and cardinal) collapse into a single scenario.
The Pareto mechanism is an optimal incentive-compatible mechanism.

**Proposition 1.** The Pareto mechanism is an optimal incentive-compatible mechanism.

**Proof.** Consider the Pareto mechanism and the event that \( R \) gets a signal \( \sigma_t = \text{HIRE} \) in round \( t \). We denote by \( x \) the distribution over candidates in round \( t \) conditioned on this event. Clearly \( x \) is the same distribution, no matter in which round \( t \) \( R \) gets a signal to hire. Now denote by \( y \) the distribution over candidates in round \( t' \) conditioned on a HIRE signal in round \( t \neq t' \). Clearly, \( y \) is the same distribution, for any \( t, t' \) with \( t \neq t' \).

For \( R \), the expected utility of following a HIRE signal is \( x^\top \rho \geq \mu^R \) by construction of the mechanism. By simply hiring in round 1 deterministically, \( R \) gets a value of \( \mu^R \) and a HIRE signal.
Algorithm 2: Growing Pareto Mechanism

Input: Number of candidates $n$, sample size $s$

1. $A_0 \leftarrow \emptyset$

2. for $t = 1$ to $n - 1$ do
   3. $A_t \leftarrow A_{t-1} \cup \{\theta_t\}$
   4. if $t \leq s$ then Signal NOT
   5. else
      6. $c_t \leftarrow$ candidate chosen by Pareto mechanism on the set $A_t$
      7. if $c_t = \theta_t$ then Signal HIRE and end mechanism; else Signal NOT

8. Signal HIRE on $n$-th candidate.

with probability $1/n$. Hence, $(x + (n-1)y)^T \rho = n \mu^R$. Consequently, the expected utility of $\mathcal{R}$ for hiring upon a NOT signal is $y^T \rho \leq \mu^R$. Thus, the mechanism is incentive compatible.

Now consider any incentive-compatible mechanism $\phi$ used by $\mathcal{S}$. Let $x_i$ denote the probability that (over random arrival and randomization in $\phi$) candidate $i \in [n]$ is the first one with signal HIRE. We use $x$ to denote the vector. Truthfulness of $\phi$ implies the following constraints: (1) $x^T \rho \geq \mu^R$, since this expected utility can be achieved by $\mathcal{R}$ simply by hiring in any fixed round; (2) $\|x\|_1 \leq 1$, since the mechanism can be assumed to recommend at most one candidate; (2) $\|x\|_1 \geq 1$, since all $\rho_i \geq 0$ at least one candidate must be recommended (otherwise, $\mathcal{R}$ will deviate from $\phi$ by hiring in the last round). Hence, the distribution $x$ resulting from the optimal incentive-compatible mechanism is a feasible solution to the following maximization problem for the expected sender utility

$$\begin{align*}
\text{Max.} & \quad x^T \xi \\
\text{s.t.} & \quad x^T \rho \geq \mu^R \\
& \quad \|x\|_1 = 1 \\
& \quad x_i \geq 0 \quad \text{for all } i = 1, \ldots, n
\end{align*}$$

(1)

It is straightforward to see that the Pareto mechanism computes a distribution that represents an optimum solution to the above LP. Hence, it is an optimal incentive-compatible mechanism.

2.2 Secretary Scenario without Disclosure

The unknown values for $\mathcal{S}$ represent a decrease of informational advantage over $\mathcal{R}$. To cope with this challenge, we apply the Pareto mechanism adaptively. Our mechanism $\phi(s)$ first samples a number of rounds and signals $\sigma_t = \text{NOT}$ for rounds $t = 1, \ldots, s$. Then, in each round $t = s + 1, \ldots, n - 1$ let $A_t$ be the set of all previously arrived candidates, including the one currently under consideration in round $t$. The mechanism invokes the Pareto mechanism on the set $A_t$. It signals $\sigma_t = \text{HIRE}$ if and only if the candidate chosen by the Pareto mechanism has arrived in the current round $t$.

When $\sigma_t = \text{HIRE}$ and $\mathcal{R}$ deviates by refusing to hire, the mechanism signals NOT in every subsequent round. If $t = n$ and the mechanism has not signaled HIRE so far, it sets $\sigma_n = \text{HIRE}$ deterministically in the last round. Due to the dependence on a growing candidate set $A_t$, we term this mechanism the Growing Pareto mechanism (see Algorithm 2).

We proceed to show a number of properties for the Growing Pareto mechanism. Consider round $t$. We follow [21] and rephrase the generation of the candidate arriving in round $t$ as follows: First choose the subset $A_t$ of candidates that arrived in rounds $1, \ldots, t$ uniformly at random from $[n]$, then pick the candidate arriving in round $t$ uniformly at random from $A_t$.

**Lemma 2.** Consider a given round $t$ and a given subset of candidates $A_t$ that arrived up to round $t$. A
In the Growing Pareto mechanism

\[
\Pr \left[ \bigwedge_{i=1}^{t-1} \sigma_i = \text{NOT} \mid A_{t-1} \right] = \begin{cases} 
1 & t = 2, \ldots, s + 1 \\
\frac{t - 3}{t - 1} & t = s + 2, \ldots, n
\end{cases}
\]

and

\[
\Pr[\sigma_t = \text{HIRE} \mid A_t] = \begin{cases} 
0 & t = 1, \ldots, s \\
\frac{t - 1}{t} & t = s + 1, \ldots, n - 1 \\
\frac{s}{n - 1} & t = n
\end{cases}
\]

Proof. Given that the set \(A_t\) of candidates arrived up to round \(t\), we draw all possible arrival sequences of \(A_t\) in the first \(t\) rounds in a reverse fashion. For \(A_t\), the Pareto mechanism singles out the candidates \(a\) and \(b\). The candidate \(i_t \in [n]\) that arrives in round \(t\) is chosen uniformly at random from \(A_t\). Thus, the probability is \(\frac{1}{t}\) that the candidate is in \(\{a, b\}\) and that in addition the Pareto mechanism applied in round \(t\) would signal HIRE. For a signal \(\sigma_t = \text{HIRE}\) of the Growing Pareto mechanism, however, it also needs to hold that \(\sigma_t = \text{NOT}\) for all \(i \leq t - 1\).

Given candidate set \(A_t\) arrives in the first \(t\) rounds and candidate \(i_t\) in round \(t\), consider the signal in round \(t - 1\). Now given the set \(A_{t-1} = A_t \setminus \{i_t\}\), the Pareto mechanism singles out some candidates \(a\) and \(b\). Since the candidate \(i_{t-1}\) arriving in round \(t - 1\) is chosen uniformly at random from \(A_{t-1}\), the probability that the Pareto mechanism in round \(t - 1\) signals NOT is \(\frac{t - 2}{t - 1}\).

Suppose in round \(t - 1\) a candidate \(i_{t-1}\) arrives and the Pareto mechanism signals \(\sigma_{t-1} = \text{NOT}\). Then, we can apply the same argument for round \(t - 2\) and candidate set \(A_{t-2} = A_t \setminus \{i_t, i_{t-1}\}\). Since we need a NOT signal in all previous rounds \(i \in [t - 1]\), we continue to apply the argument pointwise for all subsets \(A_t\). Note that for \(i \leq s\) the probability \(\Pr[\sigma_i = \text{NOT}] = 1\) always.

Hence, for every set \(A_{t-1}\) we obtain

\[
\Pr \left[ \bigwedge_{i=1}^{t-1} \sigma_i = \text{NOT} \mid A_{t-1} \right] = \frac{t - 2}{t - 1} \cdot \frac{t - 3}{t - 2} \cdots \frac{s}{s + 1} = \frac{s}{t - 1}
\]

for rounds \(t = s + 2, \ldots, n\). Thus, for rounds \(t = s + 1, \ldots, n - 1\)

\[
\Pr[\sigma_t = \text{HIRE} \mid A_t] = \mathbb{E}_{i_t} \left[ \Pr[\sigma_t = \text{HIRE} \mid A_t, i_t] \cdot \Pr \left[ \bigwedge_{i=1}^{t-1} \sigma_i = \text{NOT} \mid A_t \setminus \{i_t\} \right] \right] = \frac{1}{t} \cdot \frac{s - 1}{t - 1}
\]

and, analogously, for \(t = n\)

\[
\Pr[\sigma_t = \text{HIRE} \mid A_t] = 1 \cdot \frac{s}{n - 1}
\]

Lemma 3. The Growing Pareto mechanism is incentive compatible.

Proof. Suppose the Growing Pareto mechanism sets \(\sigma_t = \text{HIRE}\) in some round \(t \in \{s + 1, \ldots, n\}\) and \(\sigma_t = \text{NOT}\) for all \(t' \in [t - 1]\). Note \(A_t\) is a subset chosen uniformly at random. For every \(i \in A_t\), the probability to receive only signals NOT for \(A_t \setminus \{i\}\) in rounds \(1, \ldots, t - 1\) is the same. Hence, by following the HIRE signal, \(\mathcal{R}\) obtains an expected utility of

\[
\mathbb{E}[\rho_t \mid \sigma_t = \text{HIRE}] \geq \mathbb{E}_{A_t} \left[ \sum_{i \in A_t} \frac{\rho_i}{t} \right] = \mu^\mathcal{R}.
\]

Consider the event that \(\mathcal{R}\) sees only NOT in all rounds up to \(t - 1\). By Lemma 2, this event has the same probability for all subsets \(A_{t-1}\). Hence, even conditioned on this event, the candidate arriving
in round \( t \) is uniformly distributed. Thus, \( \mathbb{E} \left[ \rho_t \left| \bigwedge_{t'=1}^{t-1} \sigma_{t'} = \text{NOT} \right. \right] = \mu^R. \) Therefore, the expected utility of hiring when \( \text{NOT} \) is received also in round \( t \) is \( \mathbb{E} \left[ \rho_t \left| \bigwedge_{t'=1}^{t} \sigma_{t'} = \text{NOT} \right. \right] \leq \mu^R. \)

Now consider the event that \( \mathcal{R} \) observes \( \sigma_t = \text{HIRE} \). For all possible subsets \( A_t \) this event has the same probability by Lemma 2. Hence, for any round \( r > t \), candidate \( \theta_r \) is uniformly distributed and gives expected value of \( \mu^R. \) Thus, no \( r > t \) yields a profitable deviation. The lemma follows. \( \square \)

Let \( \rho_{2nd} \) denote the second highest utility of any candidate for \( \mathcal{R} \). Let \( L = \{ i \in [n] \mid \rho_i \leq \mu^R \} \) and \( H = \{ i \in [n] \mid \rho_i > \mu^R \} = [n] \setminus L \) be a partition of the candidates into the ones with low receiver utility and high receiver utility, respectively. Let \( d = |L| \) denote the cardinality of \( L \). Without loss of generality we assume \( \rho_{\max} = 1 \) and \( \xi_{\max} = 1 \).

We first concentrate on the case with ordinal sender utility. Consider the Pareto mechanism in the benchmark scenario. We apply the normalization steps indicated in the beginning of the section and apply the Pareto mechanism to this instance. If \( \rho_{c_S} \geq \mu^R \), then \( a = b \). The mechanism will wait for this particular candidate and recommend HIRE deterministically. Otherwise, \( a \neq b \), and OPT is simultaneously the expected utility for \( \mathcal{S} \) and the success probability for hiring \( c_S \).

For the Growing Pareto mechanism, in each round \( t \) we apply the Pareto mechanism to the set \( A_t \) of arrived candidates. Since we assume ordinal sender utility, we apply the normalization appropriately in each round to \( A_t \). Formally, we scale receiver utilities such that the best known candidate for \( \mathcal{R} \) in \( A_t \) has receiver utility 1. Also, we assume that the sender utility is 1 for the best known candidate for \( \mathcal{S} \) in \( A_t \), and 0 otherwise.

For any subset of candidates \( M \subseteq [n] \), we use \( \text{OPT}_{-M} \) to denote the expected utility for \( \mathcal{S} \) in the benchmark scenario with candidate set \( [n] \setminus M \). \( \text{OPT}_{-M} \) is the probability of hiring the best candidate for \( \mathcal{S} \) in \( [n] \setminus M \). Similar to \( \text{OPT}_{-\{i\}} \), we define \( \mu^R_i = \frac{1}{n-1} \left( \sum_{j=1}^n \rho_j - \rho_i \right) \) for \( i \in [n] \).

**Lemma 4.** Let \( \text{OPT} \) and \( \text{OPT}_{-\{i\}} \) denote the expected utility in the benchmark scenario for candidate sets \([n]\) and \([n] \setminus \{i\}\), respectively. Then, the following holds:

\[
\sum_{i \notin c_S, c_R} \text{OPT}_{-\{i\}} + \text{OPT} \cdot \text{OPT}_{-\{c_R\}} \geq \text{OPT} \left( n - 2 - \frac{1}{n-1} \right). 
\]

**Proof.** For the first case, we assume \( \rho_{c_S} \leq \mu^R \leq \rho_{2nd} \). Note that in this case, the Pareto mechanism chooses \( a = c_S \) and \( b = c_R \). The resulting value \( \text{OPT}_{-\{i\}} \) is different depending on the candidate that gets removed from the pool. If a candidate \( i \in L \) with \( \rho_i \leq \mu^R \) is removed, then \( \mu^R_i \geq \mu^R \), which implies that \( \text{OPT}_{-\{i\}} \leq \text{OPT} \). Note that upon removal of \( b \), we have \( \text{OPT}_{-\{b\}} = 1 \) if \( \mu^R_b \leq \rho_a \).

Otherwise, the new optimum point is located at \( \mu^R_{-b} \) and has value \( \text{OPT}_{-\{b\}} = \frac{\rho_{2nd} - \mu^R_b}{\rho_{2nd} - \rho_a} \). Overall, we see that

\[
\sum_{i \notin c_S, c_R} \text{OPT}_{-\{i\}} + \text{OPT} \cdot \text{OPT}_{-\{c_R\}} \\
= \sum_{i \in L \setminus \{a\}} \text{OPT}_{-\{i\}} + \sum_{i \in H \setminus \{b\}} \text{OPT}_{-\{i\}} + \text{OPT} \cdot \min \left\{ 1, \frac{\rho_{2nd} - \mu^R_{-b}}{\rho_{2nd} - \rho_a} \right\} \\
= \text{OPT} \left[ (d - 1) - \frac{1}{(n-1)(1-\mu^R)} \sum_{i \in L \setminus \{a\}} (\mu^R - \rho_i) + (n - d - 1) \right. \\
+ \sum_{i \in H \setminus \{b\}} \min \left\{ \frac{\rho_i - \mu^R}{(n-1)(1-\mu^R)}, \frac{\mu^R - \rho_a}{1-\mu^R} \right\} + \frac{\rho_{2nd} - \mu^R_{-b}}{\rho_{2nd} - \rho_a} \left. \right]
\]

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\[ \begin{align*}
&\geq \text{OPT} \left[ (n - 2) - \frac{1}{(n - 1)(1 - \mu^R)} \sum_{i \in L} (\mu^R - \rho_i) \\
&\quad + \sum_{i \in H \setminus \{b\}} \min \left\{ \frac{\rho_i - \mu^R}{(n - 1)(1 - \mu^R)}, \frac{\mu^R - \rho_a}{1 - \mu^R} \right\} + \frac{\rho_{2nd} - \mu^R}{\rho_{2nd} - \rho_a} \right] \\
&= \text{OPT} \left[ (n - 2) + \frac{\rho_{2nd} - \mu^R}{\rho_{2nd} - \rho_a} - \frac{1}{n - 1} \\
&\quad - \frac{1}{(n - 1)(1 - \mu^R)} \sum_{i \in H \setminus \{b\}} \left( (\rho_i - \mu^R) - \min \left\{ \rho_i - \mu^R, (\mu^R - \rho_a)(n - 1) \right\} \right) \right].
\end{align*} \]

We consider two subcases.

**Subcase** $\mu^R < \frac{\rho_{2nd} + (n - 1)\rho_a}{n}$: Here, we see

\[
\sum_{i \neq c, \not\in cR} \text{OPT} \cdot \text{OPT} - \{i\} + \text{OPT} \cdot \text{OPT} - \{cR\} \\
\geq \text{OPT} \left[ (n - 2) + \frac{\rho_{2nd} - \mu^R}{\rho_{2nd} - \rho_a} - \frac{1}{n - 1} \\
&\quad - \frac{1}{(n - 1)(1 - \mu^R)} \sum_{i \in H \setminus \{b\}} \left( (\rho_i - \mu^R) - \min \left\{ \rho_i - \mu^R, (\mu^R - \rho_a)(n - 1) \right\} \right) \right]
\]

For the penultimate line we used that $\rho_i \leq 1$ and the min is at least 0. In the last inequality, we used the assumption that $\mu^R < \frac{\rho_{2nd} + (n - 1)\rho_a}{n}$ and thus

\[ \frac{\mu^R - \rho_a}{\rho_{2nd} - \rho_a} \geq -\frac{\rho_{2nd} - \rho_a}{n(\rho_{2nd} - \rho_a)} = -\frac{1}{n}. \]

**Subcase** $\mu^R \geq \frac{\rho_{2nd} + (n - 1)\rho_a}{n}$: In this case, we see

\[
\sum_{i \in H \setminus \{b\}} \left( (\rho_i - \mu^R) - \min \left\{ \rho_i - \mu^R, (\mu^R - \rho_a)(n - 1) \right\} \right) = 0
\]
as $\rho_i \leq \rho_{2nd}$ for all $i \in H \setminus \{b\}$ and $\rho_{2nd} - \mu^R \leq (\mu^R - \rho_a)(n - 1)$. Thus,

\[
\sum_{i \neq c, \not\in cR} \text{OPT} \cdot \text{OPT} - \{i\} + \text{OPT} \cdot \text{OPT} - \{cR\} \\
\geq \text{OPT} \left[ (n - 2) + \frac{\rho_{2nd} - \mu^R}{\rho_{2nd} - \rho_a} - \frac{1}{n - 1} \\
&\quad - \frac{1}{(n - 1)(1 - \mu^R)} \sum_{i \in H \setminus \{b\}} \left( (\rho_i - \mu^R) - \min \left\{ \rho_i - \mu^R, (\mu^R - \rho_a)(n - 1) \right\} \right) \right]
\]
\[ \begin{align*}
&= \text{OPT} \left[ (n - 2) + \frac{\rho_{2nd} - \mu^R}{\rho_{2nd} - \rho_a} - \frac{1}{n - 1} \right] \\
&\geq \text{OPT} \left[ n - 2 - \frac{1}{n - 1} \right].
\end{align*} \]

Now, consider the second case in which \( \rho_{cs}, \rho_{2nd} \leq \mu^R \). Again, in this case the Pareto mechanism chooses \( a = c_s \) and \( b = c_R \). We can bound as follows
\[
\sum_{i \neq c_s, c_R} \text{OPT}_{-i} + \text{OPT} = \sum_{i \in L \setminus \{a\}} \text{OPT}_{-i} + \sum_{i \in H \setminus \{c_s, c_R\}} \text{OPT}_{-i} + \text{OPT} \\
\geq \text{OPT} \left( d - \frac{1}{(n - 1)(1 - \mu^R)} \sum_{i \in L} (\mu^R - \rho_i) \right) + \sum_{i \in H \setminus \{c_s, c_R\}} \text{OPT} + \text{OPT} \\
\geq \text{OPT} \left( n - 1 - \frac{1}{(n - 1)(1 - \mu^R)} (n - d) (1 - \mu^R) \right) \\
\geq (n - 2) \text{OPT} \geq \left( n - 2 - \frac{1}{n - 1} \right) \text{OPT}.
\]

Finally, consider the third case, \( \rho_{cs} > \mu^R \). Then, the Pareto mechanism chooses \( a = b = c_s \), and \( \text{OPT}_{-c_R} = 1 \). We can bound as follows
\[
\sum_{i \neq c_s, c_R} \text{OPT}_{-i} + \text{OPT} = \sum_{i \in L} \text{OPT}_{-i} + \sum_{i \in H \setminus \{c_s, c_R\}} \text{OPT}_{-i} + \text{OPT} \\
\geq \text{OPT} \left( d - \frac{1}{(n - 1)(1 - \mu^R)} \sum_{i \in L} (\mu^R - \rho_i) \right) + \sum_{i \in H \setminus \{c_s, c_R\}} \text{OPT} + \text{OPT} \\
\geq \text{OPT} \left( n - 1 - \frac{1}{(n - 1)(1 - \mu^R)} (n - d) (1 - \mu^R) \right) \\
\geq (n - 2) \text{OPT} \geq \left( n - 2 - \frac{1}{n - 1} \right) \text{OPT}.
\]

\[\square\]

For the basic scenario, let \( \text{OPT}_t \) denote the success probability of the Pareto mechanism when applied to the benchmark scenario with random subset \( A_t \).

**Corollary 5.** For all \( t \in [n] \) it holds that \( \text{OPT}_t \geq \frac{(t-2)(t-1)}{(n-2)(n-1)} \text{OPT} \).

**Proof.** The random subset \( A_t \) can be generated by starting from the candidate set \([n]\) and iteratively removing a candidate picked uniformly at random. In a single step
\[
\text{OPT}_{n-1} = \frac{1}{n} \sum_{i \in [n]} \text{OPT}_{-i} = \frac{1}{n} \sum_{i \neq c_s, c_R} \text{OPT}_{-i} + \frac{1}{n} \text{OPT}_{-\{c_R\}} \\
\geq \frac{1}{n} \sum_{i \neq c_s, c_R} \text{OPT}_{-i} + \frac{1}{n} \text{OPT} \cdot \text{OPT}_{-\{c_R\}} \geq \frac{n - 2 - \frac{1}{n - 1}}{n} \cdot \text{OPT}
\]
due to \( \text{OPT}_{-\{c_R\}} = 0 \), \( \text{OPT} \leq 1 \), and Lemma 4. Repeated application of this property implies
\[
\text{OPT}_t \geq \text{OPT} \prod_{k=t+1}^{n} \frac{k - 2 - \frac{1}{k - 1}}{k} \geq \text{OPT} \prod_{k=t+1}^{n} \frac{k - 3}{k - 1} \\
= \text{OPT} \cdot \frac{t - 2}{t} \cdot \frac{t - 1}{t + 1} \cdot \frac{t}{t + 2} \cdot \ldots \cdot \frac{n - 4}{n - 2} \cdot \frac{n - 3}{n - 1} = \text{OPT} \cdot \frac{(t-2)(t-1)}{(n-2)(n-1)}. 
\]

\[\square\]
For ordinal sender utility, we obtain the following theorem.

**Theorem 6.** Using \( s = \lfloor n/2 \rfloor \), the Growing Pareto mechanism yields a success probability of at least \( \left( \frac{1}{2} - o(1) \right) \) times the success probability in the corresponding benchmark scenario.

**Proof.** In a given round \( t = s + 1, \ldots, n - 1 \), the Growing Pareto mechanism obtains a success probability of at least \( \Pr[\sigma_t = \text{HIRE}] \cdot \text{OPT}_t \). We underestimate the success probability in round \( n \) by 0. Using linearity of expectation and choosing a sample size \( s = \lfloor c \cdot n \rfloor \) with a constant \( c \), we obtain

\[
\sum_{t=s+1}^{n-1} \frac{1}{t} \cdot \frac{s}{t-1} \cdot \frac{(t-1)(t-2)}{(n-1)(n-2)} \cdot \text{OPT} = \text{OPT} \cdot \frac{s}{(n-1)(n-2)} \sum_{t=s+1}^{n-1} \frac{t-2}{t} \cdot \frac{2s \cdot (H_{s-1} - H_{s+1})}{(n-1)(n-2)} = \text{OPT} \cdot \left( c - c^2 - o(1) \right).
\]

Here, \( H_k = \sum_{i=1}^{k} 1/i \) is the \( k \)-th harmonic number. The last expression is maximized at \( c = 1/2 \), therefore we choose \( s = \lfloor n/2 \rfloor \). The theorem follows. 

Now consider the case when \( S \) has cardinal utility and aims to maximize her expected utility. We show that the Growing Pareto mechanism provides a constant-factor approximation to the optimum in the corresponding benchmark scenario.

**Theorem 7.** Using \( s = \lfloor n/\sqrt{3} \rfloor \), the Growing Pareto mechanism yields a \( \left( \frac{1}{3\sqrt{3}} - o(1) \right) \)-approximation of the optimal expected utility in the corresponding benchmark scenario.

The proof uses the following Lemma 8, an adapted version of Lemma 4. We use the same notation.

**Lemma 8.** Consider an instance of the benchmark scenario with candidate set \([n]\). Let \( a \) and \( b \) be the candidates as determined in the Pareto mechanism. It holds that

\[
\sum_{i \neq a, b} \text{OPT}_{-\{i\}} \geq (n - 3) \text{OPT}.
\]

**Proof.** Consider the signaling scheme computed by the Pareto mechanism. If we adapt the instance and set \( \xi_j = 0 \) for all \( j \neq a, b \), this does not change \( \text{OPT} \), but it can only worsen the optima \( \text{OPT}_{-\{i\}} \). Hence, in the new instance, the Pareto frontier of the convex hull of \( P \) consists only of the (at most) two points for candidates \( a, b \), and a point for candidate \( (\max_i \rho_i, 0) \) and the point \( (0, \max_i \xi_i) = (0, \xi_a) \) (see Fig. 2).

![Figure 2: Solid: Pareto frontier in the adapted instance. Dashed: Lower bound on the Pareto frontier when \( a \) and \( b \) remain in the candidate set](image)

Similar to Lemma 4, we partition the candidate set into the sets \( L \) and \( H \) and set \( d = |L| \).
The resulting value $\OPT_{\{i\}}$ is different depending on the candidate that gets removed from the pool. We underestimate the Pareto frontier with a line to $(\rho_{\max}, 0)$ (see the dashed line in Fig. 2). More formally, to estimate the loss in expected utility, use the slope $\OPT_{\rho_{\max} - \mu^R}$ from the point $(\mu^R, \OPT)$ representing the optimum to the point with maximal receiver value. The loss in sender utility caused by the change $\mu^R_i - \mu^R = \sum_{j \in [n] \setminus \{i\}} \rho_j n^{-1} - \mu^R$ can be bounded by

$$\sum_{i \in L \setminus \{a\}} \OPT - \OPT_{\{i\}} \leq \frac{\OPT}{\rho_{\max} - \mu^R} \sum_{i \in L} \left[ \sum_{j \in [n] \setminus \{i\}} \rho_j n^{-1} - \mu^R \right]$$

$$= \frac{1}{n-1} \cdot \frac{\OPT}{\rho_{\max} - \mu^R} \sum_{i \in L} \left[ \sum_{j \in [n] \setminus \{i\}} \rho_j + \sum_{j \in H} (\rho_j - (n-1) \mu^R) \right]$$

$$= \frac{\OPT}{(n-1)(\rho_{\max} - \mu^R)} \sum_{j \in H} (\rho_j - \mu^R)$$

$$\leq \frac{\OPT}{(n-1)(\rho_{\max} - \mu^R)} \sum_{j \in H} (\rho_{\max} - \mu^R)$$

$$= \frac{n-d}{(n-1)} \OPT \leq \OPT.$$

If a candidate $i \in H$ arrives, $\mu^R_i \leq \mu^R$, and thus the expected payoff for $S$ can only increase. We can lower bound the payoff by $\OPT$, c.f. Fig. 2, i.e.

$$\sum_{i \in H \setminus \{b\}} \OPT - \OPT_{\{i\}} \leq 0.$$

Overall, this implies

$$(n-2) \cdot \OPT - \sum_{i \in [n] \setminus \{a,b\}} \OPT_{\{i\}} \leq \OPT$$

which implies the lemma.

Let $\OPT_t$ be the expected value of the Pareto mechanism when applied to the benchmark scenario composed of the random subset $A_t$. Note that $\OPT_n = \OPT$, the optimum in the benchmark scenario.

**Corollary 9.** For $t \geq 3$ it holds that

$$\OPT_t \geq \prod_{k=t+1}^{n} \left( 1 - \frac{3}{k} \right) \OPT = \frac{t(t-1)(t-2)}{n(n-1)(n-2)} \OPT.$$  

**Proof.** We can generate the random set $A_t$ by starting with $[n]$ and iteratively removing a random candidate. Note that for $t = n - 1$ we have by Lemma 8

$$\OPT_{n-1} = \frac{1}{n} \sum_{i \in [n]} \OPT_{\{i\}} \geq \frac{1}{n} \sum_{i \in [n] \setminus \{a,b\}} \OPT_{\{i\}} \geq \frac{n-3}{n} \OPT.$$  

The result for $t < n - 1$ follows by repeated application.

**Proof of Theorem 7.** By combining the insights of Corollary 9 and Lemma 2, we see that in a given round $t = s + 1, \ldots, n - 1$ the Growing Pareto mechanism obtains an expected utility for $S$ of at least $\Pr[\sigma_t = \text{HIRE}] \cdot \OPT_t$. For simplicity, we underestimate the utility in the last round $t = n$ by

$$\OPT_n = \frac{n(n-1)(n-2)}{n(n-1)(n-2)} \OPT = \OPT.$$
0. For the expected utility of $S$, we use linearity of expectation over all rounds and set $s = \lfloor c \cdot n \rfloor$ for a constant $c$:

$$
\sum_{t=s+1}^{n-1} \frac{1}{t} \cdot \frac{s}{t-1} \cdot \frac{t(t-1)(t-2)}{n(n-1)(n-2)} \text{OPT} = \text{OPT} \cdot \frac{s}{n(n-1)(n-2)} \sum_{t=s+1}^{n-1} (t-2) = \text{OPT} \cdot \frac{s}{n(n-1)(n-2)} \left( \frac{n(n-1)}{s} - \frac{s(s+1)}{2} - 2(n-1-s) \right) = \text{OPT} \cdot \left( \frac{s}{n-2} - \frac{s^2(s+1)}{2n(n-1)(n-2)} - \frac{2s(n-1-s)}{n(n-1)(n-2)} \right) = \text{OPT} \cdot (c - c^3 - o(1)).
$$

The last expression is maximized at $c = \frac{1}{\sqrt{3}}$, so we set $s = \lfloor n/\sqrt{3} \rfloor$. The theorem follows.

2.3 Basic Scenario with Disclosure

Now, consider the basic scenario in which rejected candidates are revealed to $R$. First, we give a characterization of the optimal mechanism using backwards induction and show that there are instances in which no online mechanism can achieve more than a fraction of $1/2$ of the success probability/expected utility obtained in the benchmark scenario. Again, we utilize the results obtained in the case of cardinal sender utility to express bounds for the ordinal sender case. By setting $\xi_{\text{max}} = 1$ and $\xi_i = 0$ for all $i \neq cS$, success probability and expected utility are the same value. Hence, the following result holds for both sender objectives. For simplicity, we state and prove the theorem for cardinal sender utility.

An optimal mechanism for the scenario can be computed by backward induction and solution of a series of linear programs. These programs express the optimal decision given a subset $C \subseteq [n]$ of candidates and given the optimal decision policy computed for each subset $C \setminus \{i\}$, for every $i \in C$.

**Theorem 10.** An optimal incentive-compatible mechanism for the sender’s expected utility in the basic scenario with disclosure can be computed by solving $2^n$ linear programs.

**Proof.** Suppose we reach the beginning of round $t$ with a set $C$ of $n - t + 1$ remaining candidates. For every candidate $i \in C$, suppose the sender sees candidate $i$ in round $t$ and has computed the signaling policy for an optimal incentive-compatible mechanism for rounds $t + 1, \ldots, n$ and subset $C \setminus \{i\}$.

Clearly, in round $t = n$ the sender should set $\sigma_n = \text{HIRE}$ with probability 1. This is the only incentive-compatible signaling scheme. Now suppose we are in round $t < n$. We denote by

$$
x^C_t = \Pr[\sigma_t = \text{HIRE} \mid i \in C \text{ arrives in round } t].
$$

In case $i$ is not hired although $S$ signaled HIRE, $S$ will never signal HIRE again. Let $u^S_{C \setminus \{i\}}$ be the expected utility of $S$ from the optimal incentive-compatible mechanism applied from round $t + 1$ onwards with candidate set $C \setminus \{i\}$. Similarly, we define $u^R_{C \setminus \{i\}}$ for $R$. Assuming the signaling scheme $\phi$ determined by $x$ is incentive compatible, then the expected utility obtained by $S$

$$
\frac{1}{|C|} \sum_{i \in C} (x^C_i \xi_i + (1 - x^C_i) \cdot u^S_{C \setminus \{i\}})
$$

is the objective function of a linear program. In this LP we have the obvious constraints $x^C_t \in [0,1]$, and two constraints on $x$ that ensure incentive compatibility. Suppose $R$ gets signal HIRE. This happens with total probability $\Pr[\sigma_t = \text{HIRE} \mid C] = \sum_{i \in C} x^C_i / |C|$. We assume w.l.o.g. that this probability is positive, otherwise there is nothing to prove. The probability that the recommended candidate is $i$ is $p^R_i = \frac{x^C_i}{\sum_{j \in C} x^C_j}$. Upon compliance with the HIRE signal, $R$ gets a utility of $\sum_{i \in C} p^R_i \rho_i$. 

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Upon deviation, the candidate is rejected and $S$ stops to signal HIRE. Thus, the expected utility of $R$ becomes $\sum_{i \in C} p_i^h \cdot \frac{1}{|C|-1} \sum_{j \in C \setminus \{i\}} \rho_j$. Hence, incentive compatibility requires

$$\sum_{i \in C} p_i^h \rho_i \geq \sum_{i \in C} p_i^h \cdot \frac{1}{|C|-1} \sum_{j \in C \setminus \{i\}} \rho_j$$

$$\sum_{i \in C} x_i^C \rho_i \geq \sum_{i \in C} x_i^C \cdot \frac{1}{|C|-1} \sum_{j \in C \setminus \{i\}} \rho_j$$

(2)

Similarly, $R$ should not have an incentive to accept a candidate upon the event of $\sigma_t = \text{NOT}$. We again assume that the probability of this event is positive. The probability that upon a NOT signal the sender sees candidate $i$ is $p_i^{nh} = \frac{1-x_i^C}{\sum_{j \in C} (1-x_j^C)}$. Upon deviation and hiring the candidate, $R$ gets utility $\sum_{i \in C} p_i^{nh} \cdot \rho_i$. Upon following the signal, $R$ gets a utility of $\sum_{i \in C} p_i^{nh} u_{C \setminus \{i\}}^R$. Hence, incentive compatibility requires

$$\sum_{i \in C} p_i^{nh} u_{C \setminus \{i\}}^R \geq \sum_{i \in C} p_i^{nh} \cdot \rho_i$$

$$\sum_{i \in C} (1-x_i^C) u_{C \setminus \{i\}}^R \geq \sum_{i \in C} (1-x_i^C) \cdot \rho_i$$

(3)

Claim 1. Constraint (3) is redundant and implied by (2).

We show the claim below. Thus, we get the following family of linear programs to determine $u_i^S$:

$$\max \quad \frac{1}{|C|} \sum_{i \in C} (x_i^C \xi_i + (1-x_i^C) \cdot u_{C \setminus \{i\}}^S)$$

(4a)

s.t. $$\sum_{i \in C} x_i^C \left[ \rho_i - \frac{1}{|C|-1} \sum_{j \in C \setminus \{i\}} \rho_j \right] \geq 0$$

(4b)

$$x_i^C \in [0, 1] \text{ for all } i \in C$$

(4c)

Proof of Claim 1. Every solution $x$ that satisfies (3) for some values for $u_{C \setminus \{i\}}^R$ also fulfills it for pointwise larger values. This is clear since with expected utility in subsequent rounds becoming larger, $R$ does not get incentive to deviate from a NOT signal and hire in round $t$.

Recall that $R$ can always ignore all signals and hire a random candidate. Hence, clearly, $u_{C \setminus \{i\}}^R \geq \frac{1}{|C|-1} \sum_{j \in C \setminus \{i\}} \rho_j$. The pointwise smallest values $u_{C \setminus \{i\}}^R = \frac{1}{|C|-1} \sum_{j \in C \setminus \{i\}} \rho_j$ imply for the constraint

$$\sum_{i \in C} (1-x_i^C) \cdot \frac{1}{|C|-1} \sum_{j \in C \setminus \{i\}} \rho_j \geq \sum_{i \in C} (1-x_i^C) \cdot \rho_i$$

$$\sum_{i \in C} x_i^C \left[ \rho_i - \frac{1}{|C|-1} \sum_{j \in C \setminus \{i\}} \rho_j \right] \geq 0$$

(5)

What reduction of utility does $S$ suffer when rejected candidates are announced to $R$? The following theorem shows an upper bound of $\frac{1}{3}$.
Theorem 11. For every $\varepsilon > 0$, there is an instance such that every incentive-compatible mechanism in the basic scenario with disclosure guarantees at most a fraction of $\left(\frac{1}{2} + \varepsilon\right)$ of the optimum in the benchmark scenario. This holds for both cardinal as well as ordinal sender utility.

Proof. Consider the following class of instances. Candidate 1 has value $(\rho_1, \xi_1) = \left(\frac{n-2}{n-1}, 1\right)$, candidate 2 has value $(\rho_2, \xi_2) = (0, 0)$, and candidates $i = 3, \ldots, n$ have values $(\rho_i, \xi_i) = (1, 0)$. Hence, this can be seen as an instance for ordinal as well as cardinal sender utility. In the remainder of the proof, we will only use “expected utility”.

In the benchmark case, the Pareto mechanism would pick $a = b = 1$ and signal HIRE if and only if candidate 1 arrives. This would yield an expected utility of 1 for $S$ and $\frac{n-2}{n-1}$ for $R$.

For the announcement case, constraint (4b) implies that $x_2^C = 0$ is optimal for all $C \ni 2$, i.e., an optimal mechanism never recommends to hire 2. $S$ has no value for candidates other than 1, so w.l.o.g. the optimal mechanism will always try to recommend 1 with $x_1^C = 1$ for every $C \ni 1$. For every subset $C \not\ni 1$, the utility is 0, and we can assume that $x_i^C = 1$ for all $i \in C, i \not= 2$.

Let $k = \sqrt{n}$, and consider the event $E$ that the following two conditions are satisfied: (1) at least one of candidates $\{1, 2\}$ arrives within the first $n - k$ rounds, and (2) candidate 2 arrives before candidate 1. Suppose we draw two arrival rounds for candidates 1 and 2 uniformly at random. The probability that we draw both rounds from the last $k$ candidate 1. Suppose we draw two arrival rounds for candidates 1 and 2 uniformly at random. The probability that 2 arrives in the earlier one is $1/2$. Thus, the overall expected probability that we draw both rounds from the last $k$ candidate 1. Suppose we draw two arrival rounds for candidates 1 and 2 uniformly at random. The probability that 2 arrives in the earlier one is $1/2$. Thus,

$$\Pr[E] = \frac{1}{2} - \frac{k(k - 1)}{2n(n - 1)}.$$

Suppose candidate 2 arrives in round $t$. Since 2 is always rejected, consider any subsequent round $t + 1$. The set $C$ of remaining candidates consists only of candidate 1 and a subset of candidates $3, \ldots, n$. Since the latter are symmetric, the optimal mechanism uses the same value $x_1^C = x_2^C = x_{C-1}^C$ for all candidates $i, j \in C \setminus \{1\}$. The optimal mechanism needs to satisfy (4b)

$$(|C| - 1) \cdot x_{C-1}^C \cdot \left(1 - \frac{1}{|C| - 1} \cdot \left(|C| - 2 + \frac{n - 2}{n - 1}\right)\right) + x_1^C \cdot \left(\frac{n - 2}{n - 1} - \frac{1}{|C| - 1} \cdot (|C| - 1)\right) \geq 0.$$

Using the assumption that $x_1^C = 1$ and solving for $x_{C-1}^C$ implies

$$x_{C-1}^C \geq 1.$$

Hence, after candidate 2 is rejected in round $t$, there is an optimal mechanism that signals HIRE with probability 1 in round $t + 1$. The probability that candidate 1 is hired in this round is only $1/|C| \leq 1/(k - 1)$. Note that this upper bound holds even conditioned on event $E$, since the round in which 1 remains uniformly distributed among the remaining ones. Thus, the overall expected utility of the optimal mechanism in the disclosure case is upper bounded by

$$(1 - \Pr[E]) \cdot 1 + \Pr[E] \cdot \frac{1}{k - 1} \leq \frac{1}{2} + \frac{k(k - 1)}{2n(n - 1)} + \frac{1}{2(k - 1)} - \frac{k}{2n(n - 1)} = \frac{1}{2} + o(1)$$

since $k = \sqrt{n}$.

Our polynomial time approximation mechanism for this scenario is another variant of the Pareto mechanism, the Shrinking Pareto mechanism (see Algorithm 3).

Lemma 12. The Shrinking Pareto mechanism is incentive compatible.

Proof. Recall from our analysis above that the probabilities for a signal $\sigma_t = \text{HIRE}$ in the Pareto mechanism are chosen as a linear combination, which yields at least an expected utility of $\mu^R$ to
In the following, we denote by $SP_1$ signal is HIRE with probability $b$.

Consider a Pareto mechanism for the instance $SP_1$. By Lemma 13, the utility obtained in a round depends on the current candidate. With probability $R_1\cdot\cdot\cdot n$ candidates. Suppose the mechanism sends a HIRE signal. Then, the expected utility for $R_1\cdot\cdot\cdot n$ thus

$SP_1 = \{ t \mid \sum_{i \in R_t} x_i^{R_t} = 1 \}$, therefore

$SP_1 = \{ t \mid \sum_{i \in R_t} x_i^{R_t} \geq \frac{1}{|R_t|-1} \sum_{i \in R_t} x_i^{R_t} \cdot \sum_{j \in R_t \setminus \{i\}} \rho_j \}$.

The signaling probabilities $x_i^{R_t}$ of the Shrinking Pareto mechanism represent a feasible solution for every linear program (4). Hence, the mechanism is incentive compatible.

We use this result to bound the utility achieved by the Shrinking Pareto mechanism from below.

**Lemma 13.** By $SP$ we denote the expected utility from the Shrinking Pareto mechanism. Then, the following holds:

$$SP \geq OPT \cdot \left( \frac{1}{2} - \frac{1}{2n} \right).$$

**Proof.** The utility obtained in a round depends on the current candidate. With probability $\frac{1}{n}$, the current candidate is $a$. The signal HIRE is sent with probability $\alpha = OPT$. With probability $1 - OPT$, NOT is sent. Analogously, if the current candidate is $b$, NOT is sent with probability OPT while the signal is HIRE with probability $1 - OPT$. Any other candidate is guaranteed to get a signal NOT.

In the following, we denote by $SP_{-M}$ for any subset $M$ of candidates the expected utility from the Shrinking Pareto mechanism for the instance $[n] \setminus M$. We apply Lemma 4 recursively and obtain

$$SP = \frac{1}{n} OPT + \frac{1}{n} OPT \cdot SP_{-\{c_R\}} + \frac{1}{n} \sum_{i \notin c_R} SP_{-\{i\}}$$

$$= \frac{OPT}{n} + \frac{OPT}{n} \left[ \frac{1}{n-1} OPT_{-\{c_R\}} + \frac{1}{n-1} OPT_{-\{c_R\}} \cdot SP_{-\{c_R\}} \right] + \frac{1}{n-1} \sum_{i \notin c_R} SP_{-\{c_R, i\}}$$

$$+ \frac{1}{n} \sum_{i \notin c_R} \left[ \frac{1}{n-1} OPT_{-\{i\}} + \frac{1}{n-1} OPT_{-\{i\}} \cdot SP_{-\{i\}} \right] + \frac{1}{n-1} \sum_{j \notin c_R, i} SP_{-\{i, j\}}$$

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Theorem 14. The Shrinking Pareto mechanism yields a success probability of at least \( \frac{1}{2} - o(1) \).

Now, consider cardinal sender utility. Again, the Shrinking Pareto mechanism guarantees a constant approximation of the optimal utility achieved in the benchmark scenario. In contrast to the ordinal case, it is an interesting open question to prove a tight bound in this scenario.

Theorem 15. The Shrinking Pareto mechanism scenario obtains a \( \frac{1}{3} - o(1) \)-approximation of the optimum in the corresponding benchmark scenario.

Proof. Consider an instance of the benchmark scenario with \( n \) candidates. We again use \( \text{OPT} \) to denote the optimal expected utility of \( S \) achieved by the Pareto mechanism. Consider the same instance in the public announcement scenario, and let \( \text{SP} \) be the expected utility of \( S \) achieved by the Shrinking Pareto mechanism.

We denote by \( \theta_t \) the candidate arriving in round \( t \). Consider the first round \( t = 1 \) and the application of the Pareto mechanism. Obviously, \( \sigma_1 = \text{HIRE} \) with probability \( \frac{1}{n} \), and the expected utility \( \mathbb{E}[\xi | \sigma_1 = \text{HIRE}] = \text{OPT} \).

Now suppose \( \sigma_1 = \text{NOT} \). For every subset \( S \subseteq [n] \), let \( \text{OPT}_{-S} \) and \( \text{SP}_{-S} \) be the expected utility of \( S \) achieved by the (Shrinking) Pareto mechanism on the benchmark (disclosure) instance with candidates \( [n] \setminus S \), respectively. For the expected value of the Shrinking Pareto mechanism we can derive the following recursive lower bound

\[
\frac{1}{n} \sum_{i \in [n]} \Pr[\theta_1 = i, \sigma_1 = \text{HIRE}] \cdot \xi_i + \Pr[\theta_1 = i, \sigma_1 = \text{NOT}] \cdot \text{SP}_{-\{i\}} = \frac{\text{OPT}}{n} + \frac{1}{n} \left( (1 - \alpha) \text{SP}_{-\{i\}} + \alpha \text{SP}_{-\{b\}} + \sum_{i \neq a, b} \text{SP}_{-\{i\}} \right) \geq \frac{\text{OPT}}{n} + \frac{1}{n} \sum_{i \neq a, b} \text{SP}_{-\{i\}}
\]
\[
\geq \frac{\text{OPT}}{n} + \frac{1}{n} \sum_{i \neq a, b} \left( \frac{1}{n-1} \sum_{j \in [n] \setminus \{i\}} \Pr[\theta_2 = j, \sigma_2 = \text{HIRE}] \cdot \xi_j + \Pr[\theta_2 = j, \sigma_2 = \text{NOT}] \cdot \text{SP}_{\{i,j\}} \right)
\]

\[
\geq \frac{\text{OPT}}{n} + \frac{1}{n} \left( \sum_{i \neq a, b} \frac{\text{OPT}_{\{i\}}}{n-1} + \frac{1}{n-1} \sum_{i \in [n] \setminus \{a,b\}} \sum_{j \in [n] \setminus \{i\}} \Pr[\theta_2 = j, \sigma_2 = \text{NOT}] \cdot \text{SP}_{\{i,j\}} \right)
\]

\[
\geq \frac{\text{OPT}}{n} + \sum_{i \neq a, b} \frac{\text{OPT}_{\{i\}}}{n(n-1)} + \sum_{i \in [n] \setminus \{a,b\}} \sum_{j \neq a_i, b_i} \frac{\text{OPT}_{\{i,j\}}}{n(n-1)(n-2)} + \ldots
\]

where \(a_i, b_i\) are the candidates \(a\) and \(b\) identified by the Pareto mechanism when applied to the candidate set \([n] \setminus \{i\}\). Applying the lower bound on \(\sum_{i \neq a, b} \text{OPT}_{\{i\}}\) from Lemma 8 repeatedly in the formula above, we obtain

\[
\frac{\text{OPT}}{n} + \sum_{i \neq a, b} \frac{\text{OPT}_{\{i\}}}{n(n-1)} + \sum_{i \in [n] \setminus \{a,b\}} \sum_{j \neq a_i, b_i} \frac{\text{OPT}_{\{i,j\}}}{n(n-1)(n-2)} + \ldots
\]

\[
\geq \frac{\text{OPT}}{n} + \frac{n-3}{n(n-1)} \text{OPT} + \sum_{i \in [n] \setminus \{a,b\}} \frac{n-4}{n(n-1)(n-2)} \text{OPT}_{\{i\}} + \ldots
\]

\[
\geq \text{OPT} \cdot \frac{1}{n(n-1)(n-2)} \left( \sum_{i=1}^{n-3} \left( \frac{n-i}{n-i-1} \right) \right)
\]

\[
= \text{OPT} \cdot \left( \frac{1}{3} - \frac{2}{n^3 - 3n^2 + 2n} \right).
\]

\[\square\]

2.4 Secretary Scenario with Disclosure

Finally, consider the secretary scenario with disclosure and cardinal receiver utility. Again, \(S\) does not have any information on the candidates’ valuations and rejected candidates are revealed to \(R\). In contrast, \(R\) knows all candidate values upfront, but does not know which candidate arrived in the current round \(t\).

Obviously, we can apply the trivial mechanism: Recommending a candidate chosen uniformly at random. This mechanism is incentive compatible. It achieves a success probability \(1/n\) and an expected utility of \(1/n \cdot \xi_{\text{max}}\), a trivial lower bound.

We complement this insight with a strong upper bound for a sender with ordinal utility.

**Theorem 16.** There is no incentive-compatible mechanism that guarantees \(S\) a success probability greater than \(\frac{2}{n} \cdot \text{OPT}\), where \(\text{OPT}\) is the success probability in the corresponding benchmark instance.

The lower bound follows from the following two different instances. In both instances, there is one candidate \(a\) with value-pair \((\rho_a, \xi_a) = (0,1)\) and \(n-2\) indistinguishable candidates (each termed \(c\)) with value-pair \((1/2, 0)\). In instance I, there is one additional candidate \(b\) with value-pair \((1,0)\). In instance II, there is just another \(c\)-candidate with \((1/2,0)\).

Hence, the only uncertainty for \(S\) in this set of instances is about the existence of \(b\).

Now consider an arbitrary incentive-compatible mechanism. Clearly, we can assume that the mechanism issues exactly one HIRE signal in exactly one of the rounds \(t \in [n]\). If this signal is not followed, it just issues NOT signals. This can only increase the incentive for \(R\) to follow the HIRE signal. We always assume that the mechanism issues a HIRE signal in the last round if it has not done
so before. This is just a standard assumption. Note that in the last round, the arriving candidate is perfectly known to \( \mathcal{R} \).

Now consider round \( t = 3, \ldots, n - 1 \). Suppose the mechanism has only sent NOT signals so far. Based on the set of candidates \( A_{t-1} \) that arrived up to and including round \( t - 1 \) (and whose arrival is now known to both \( \mathcal{S} \) and \( \mathcal{R} \)), we distinguish eight cases.

| \( \theta_t \) | \( a \) | \( b \) | \( c \) | \( a \) | \( b \) | \( c \) | \( a \) | \( b \) | \( c \) |
|----------------|------|------|------|------|-----|------|------|-----|------|
| Candidates in \( A_{t-1} \) | only \( c \) | \( b, c \) | only \( c \) | only \( c \) | \( a, c \) | \( b, c \) | \( a, b, c \) |
| \( \Pr[\sigma_t = \text{HIRE} \mid A_{t-1}, \theta_t] \) | \( p_{a,c}^t \) | \( p_{a,b}^t \) | \( p_{b,c}^t \) | \( p_{b,a}^t \) | \( p_{c,c}^t \) | \( p_{c,a}^t \) | \( p_{c,b}^t \) | 1 |

Note that, obviously, in the last case when both \( a \) and \( b \) arrived and were rejected, it is clear to both \( \mathcal{S} \) and \( \mathcal{R} \) that the underlying instance is instance I and there are only candidates \( c \) to come in the subsequent rounds. Hence, in this case we can w.l.o.g. assume that \( \mathcal{S} \) issues a HIRE signal directly. In round \( t = 2 \), this case does not exist:

| \( \theta_2 \) | \( a \) | \( b \) | \( c \) | \( a \) | \( b \) | \( c \) |
|----------------|------|------|------|------|-----|------|
| Candidate \( \theta_1 \) | \( c \) | \( b \) | \( c \) | \( a \) | \( c \) | \( a \) |
| \( \Pr[\sigma_2 = \text{HIRE} \mid \theta_1, \theta_2] \) | \( p_{a,c}^2 \) | \( p_{a,b}^2 \) | \( p_{b,c}^2 \) | \( p_{b,a}^2 \) | \( p_{c,c}^2 \) | \( p_{c,a}^2 \) | \( p_{c,b}^2 \) |

Finally, for round \( t = 1 \), there are only three cases depending on whether \( \theta_1 \) is \( a \), \( b \) or \( c \). We denote the probabilities for a HIRE signal by \( p_a^1 \), \( p_b^1 \) and \( p_c^1 \), respectively.

We now derive necessary constraints for incentive compatibility.

**Lemma 17.** For every mechanism that is incentive compatible in both instances I and II, it must hold for round \( t = 1 \)

\[
p_b^1 \geq p_a^1 \quad \text{and} \quad p_c^1 \geq p_a^1
\]

and for every round \( t = 2, \ldots, n - 1 \)

\[
p_{b,c}^t \geq p_{a,c}^t \quad p_{b,a}^t \geq p_{c,a}^t \quad p_{c,b}^t \geq p_{a,b}^t \quad p_{c,c}^t \geq p_{a,c}^t
\]

**Proof.** Suppose there is a HIRE signal in round \( t \). \( \mathcal{R} \) must find it in his interest to follow the signal. More precisely, in what follows we condition on \( \sigma_t = \text{HIRE} \). Now, for incentive compatibility the expected utility from \( \theta_t \) must exceed the expected utility from \( \theta_{t+1} \).

First, consider round \( t = 1 \):

**Instance I:** We must have \( \mathbb{E}[\rho_0 \mid \sigma_1 = \text{HIRE}] \geq \mathbb{E}[\rho_0 \mid \sigma_1 = \text{HIRE}] \), which implies

\[
p_a^1 \cdot 0 + p_b^1 \cdot 1 + (n - 2) \cdot p_c^1 \cdot 1/2 \geq p_a^1 \cdot \frac{(n - 2) \cdot 1/2 + 1}{n - 1} + p_b^1 \cdot \frac{(n - 2) \cdot 1/2}{n - 1} + (n - 2) \cdot p_c^1 \cdot \frac{(n - 3) \cdot 1/2 + 1}{n - 1}
\]

The constraint becomes \( p_b^1 \geq p_a^1 \).

**Instance II:** We must have \( \mathbb{E}[\rho_0 \mid \sigma_1 = \text{HIRE}] \geq \mathbb{E}[\rho_0 \mid \sigma_1 = \text{HIRE}] \), which implies

\[
p_a^1 \cdot 0 + (n - 1) \cdot p_c^1 \cdot 1/2 \geq p_a^1 \cdot 1/2 + (n - 1) \cdot p_c^1 \cdot \frac{(n - 2) \cdot 1/2}{n - 1}
\]

The constraint becomes \( p_c^1 \geq p_a^1 \).
Hence, the mechanism must be more likely to signal HIRE in round 1 for each $c$ and $b$ than for $a$. Technically, these two constraints also hold for round $t = n$, since the probability to signal HIRE is 1 for every candidate. The two constraints also emerge in the remaining rounds $t = 2, \ldots, n - 1$:

**Instance I, $A_{t-1}$ contains only $c$:** $E[\rho_{t_i} \mid \sigma_t = \text{HIRE}] \geq E[\rho_{t_{i+1}} \mid \sigma_t = \text{HIRE}]$ implies

$$p_{a,c}^t \cdot 0 + p_{b,c}^t \cdot 1 + (n - t - 1) \cdot p_{c,c}^t \cdot 1/2 \geq p_{a,c}^t \cdot \frac{(n - t - 1) \cdot 1/2 + 1}{n - t} + p_{b,c}^t \cdot \frac{(n - t - 1) \cdot 1/2}{n - t} + (n - t - 1) \cdot p_{c,c}^t \cdot \frac{(n - t - 2) \cdot 1/2 + 1}{n - t}$$

Observe that the $p_{b,c}^t$ terms cancel, and the constraint becomes $p_{b,c}^t \geq p_{a,c}^t$.

**Instance I, $A_{t-1}$ contains $a$:** $E[\rho_{t_i} \mid \sigma_t = \text{HIRE}] \geq E[\rho_{t_{i+1}} \mid \sigma_t = \text{HIRE}]$ implies

$$p_{b,a}^t \cdot 1 + (n - t) \cdot p_{c,a}^t \cdot 1/2 \geq p_{b,a}^t \cdot 1/2 + (n - t) \cdot p_{c,a}^t \cdot \frac{(n - t - 1) \cdot 1/2 + 1}{n - t}$$

The constraint becomes $p_{b,a}^t \geq p_{c,a}^t$.

**Instance I, $A_{t-1}$ contains $b$:** $E[\rho_{t_i} \mid \sigma_t = \text{HIRE}] \geq E[\rho_{t_{i+1}} \mid \sigma_t = \text{HIRE}]$ implies

$$p_{a,b}^t \cdot 0 + (n - t) \cdot p_{c,b}^t \cdot 1/2 \geq p_{a,b}^t \cdot 1/2 + (n - t) \cdot p_{c,b}^t \cdot \frac{(n - t - 1) \cdot 1/2}{n - t}$$

The constraint becomes $p_{c,b}^t \geq p_{a,b}^t$.

**Instance I, $A_{t-1}$ contains $a$ and $b$:** For $t = 2$ this case does not occur. For $t \geq 3$, both players are aware that there are only $c$-candidates left. $\mathcal{R}$ follows any signaling strategy that guarantees a HIRE signal in the remaining rounds.

**Instance II, $A_{t-1}$ contains only $c$:** $E[\rho_{t_i} \mid \sigma_t = \text{HIRE}] \geq E[\rho_{t_{i+1}} \mid \sigma_t = \text{HIRE}]$ implies

$$p_{a,c}^t \cdot 0 + (n - t) \cdot p_{c,c}^t \cdot 1/2 \geq p_{a,c}^t \cdot 1/2 + (n - t) \cdot p_{c,c}^t \cdot \frac{(n - t - 1) \cdot 1/2}{n - t}$$

The constraint becomes $p_{c,c}^t \geq p_{a,c}^t$.

**Instance II, $A_{t-1}$ contains $a$:** In this case $\mathcal{R}$ knows that there are only $c$-candidates left. $\mathcal{R}$ follows any signaling strategy that guarantees a HIRE signal in the remaining rounds.

The lemma shows that incentive compatibility implies constraints on the probability for a HIRE signal. It must be higher for $b$ and $c$ than for $a$, in every round $t = 1, \ldots, n$. The proof of the theorem now follows by analyzing the approximation factor when applying such a mechanism in instance I.

**Proof of Theorem 16.** Consider instance I. Instance I in the benchmark scenario yields an optimal success probability of $1/2$ (recommend $a$ or $b$ each with probability $1/2$). Let us analyze the performance of a mechanism on instance I that satisfies Lemma 17. Such a mechanism is much more careful to recommend $a$. If candidate $b$ would not exist, there would be a lack of interest of $\mathcal{R}$ (that knows instance II is present) to accept a recommendation of $a$. This risk arising from the potential presence of instance II is captured by the constraints in Lemma 17.

Formally, if the mechanism reaches round $n$ without a previous HIRE signal, then, obviously, it is optimal to signal HIRE in round $n$ with probability $1$. We will prove by induction that there is an optimal mechanism $\phi$ with the following property. If $\phi$ reaches any round $t$ without a previous HIRE
signal, it is optimal to hire in round $t$ with probability 1. Thus, there is an optimal mechanism that hires in round 1 with probability 1. Obviously, this mechanism has success probability $1/n$.

Suppose the inductive assumption is true for rounds $t+1, \ldots, n$, i.e., if the mechanism reaches round $t+1$ without a previous HIRE signal, it is optimal to hire any candidate in that round with probability 1.

Now consider round $t \geq 2$.

$A_{t-1}$ contains only $c$: If $a$ arrives in round $t$, then the success probability is $p_{a,c}^t$. If $b$ arrives, the success probability is $(1 - p_{b,c}^t) \cdot \frac{1}{n-t}$, since by assumption the mechanism hires every candidate in the next round, which due to random arrival is $a$ with uniform probability. If $c$ arrives, the success probability is $(1 - p_{c,c}^t) \cdot \frac{1}{n-t}$ by similar arguments. Overall, we want to maximize

$$p_{a,c}^t + \frac{1 - p_{b,c}^t}{n-t} + (n-t-1) \cdot \frac{1 - p_{c,c}^t}{n-t}.$$

Due to Lemma 17 $p_{b,c}^t \geq p_{a,c}^t$ and $p_{c,c}^t \geq p_{a,c}^t$. Hence, the expression is maximized if both constraints hold with equality, in which case it becomes

$$p_{a,c}^t + \frac{1 - p_{a,c}^t}{n-t} + (n-t-1) \cdot \frac{1 - p_{a,c}^t}{n-t} = 1,$$

i.e., independent of the value of $p_{a,c}^t$. Thus, $p_{a,c}^t = p_{b,c}^t = p_{c,c}^t = 1$ is an optimal choice.

$A_{t-1}$ contains $a$: If $a$ has arrived and been rejected, the success probability is 0. Setting $p_{b,a} = p_{c,a} = 1$ is an optimal choice.

$A_{t-1}$ contains $b$: If $a$ arrives in round $t$, then the success probability is $p_{a,b}^t$. If $c$ arrives, the success probability is $(1 - p_{c,b}^t) \cdot \frac{1}{n-t}$, since by assumption the mechanism hires every candidate in the next round, which due to random arrival is $a$ with uniform probability. Overall, we want to maximize

$$p_{a,b}^t + (n-t) \cdot \frac{1 - p_{c,b}^t}{n-t}.$$

Due to Lemma 17 $p_{c,b}^t \geq p_{a,b}^t$. Hence, the expression is maximized when the constraints holds with equality, in which case it becomes

$$p_{a,b}^t + (n-t) \cdot \frac{1 - p_{a,b}^t}{n-t} = 1.$$

Thus, $p_{a,b}^t = p_{c,b}^t = 1$ is an optimal choice.

$A_{t-1}$ contains $a$, and $b$: If $a$ has arrived and been rejected, the success probability is 0. We already observed above that in this case we can directly hire in round $t$ with probability 1.

For round $t = 1$ we make a final similar observation. The success probability is $p_{a}^1$ if $a$ arrives, $(1 - p_{b}^1) \cdot \frac{1}{n-1}$ if $b$ arrives and $(1 - p_{c}^1) \cdot \frac{1}{n-1}$ if $c$ arrives. Overall, we want to maximize

$$p_{a}^1 + \frac{1 - p_{b}^1}{n-1} + (n-2) \cdot \frac{1 - p_{c}^1}{n-1}.$$

Due to Lemma 17 $p_{b}^1 \geq p_{a}^1$ and $p_{c}^1 \geq p_{a}^1$. Hence, the expression is maximized when both constraints hold with equality, in which case it becomes

$$p_{a}^1 + (n-1) \cdot \frac{1 - p_{a}^1}{n-1} = 1.$$

Thus, $p_{a}^1 = p_{b}^1 = p_{c}^1 = 1$ is an optimal choice.

This proves the theorem. □
Corollary 18. There is no incentive-compatible mechanism that guarantees $S$ an expected utility of more than $2n \cdot \text{OPT}$, where $\text{OPT}$ is the optimal expected utility in the corresponding benchmark instance.

Proof. Observe that the proof of Theorem 16 can be applied literally by simply replacing “success probability” with “expected utility” for $S$, since in both instances I and II the best candidate $a$ is the only one that yields positive utility for $S$.

3 Ordinal Utility for $R$

Let us now consider the problem when $R$ is only interested in hiring his (unique) best candidate, i.e., the case of ordinal receiver utility.

3.1 Basic Scenario without Disclosure

In this scenario, utility values are known a-priori to both parties. While $S$ observes the candidates, $R$ only observes the signals sent by $S$. $R$ is indifferent about all candidates which are not his best. As such, for an incentive-compatible mechanism for $S$, we can restrict to signal HIRE for exactly one of the two optimal candidates $c_R$ and $c_S$.

In the elementary mechanism, we decide upfront whether to signal HIRE for either $c_R$ or $c_S$: Draw $x \sim \text{Unif}[0,1]$. If $x \leq 1/n$, signal HIRE upon arrival of $c_R$, otherwise, signal HIRE upon arrival of $c_S$. Signal NOT for any other candidate.

Proposition 19. The elementary mechanism is incentive compatible. It yields a success probability of $(1-o(1))$ and an expected utility of $(1-o(1)) \cdot \xi_{\text{max}}$ for $S$.

Proof. First, we prove incentive compatibility.

In round $t$ with $\sigma_t = \text{HIRE}$ we have $\Pr[\theta_t = c_R] = \frac{1/n}{1/n + (n-1)/n} = 1/n$. Suppose $R$ deviates to HIRE in a later round $r > t$. Since $S$ will not signal HIRE again, $R$ must choose $r$ without additional information. Then $\Pr[\theta_r = c_R] = (1 - 1/n) \cdot 1/(n-1) = 1/n$. Thus, it is optimal for $R$ to hire $\theta_t$.

If $\sigma_t' = \text{NOT}$ for all $t' \in [t]$, then $\Pr[\theta_t = c_R] = \frac{\frac{n-1}{n-2+\frac{1}{n}}}{\frac{n-1}{n-2+\frac{1}{n}}} = \frac{1}{n}$. Hence, it is optimal for $c_R$ to wait for the round with signal HIRE.

Additionally, the elementary mechanism clearly implies $c_S$ is hired with probability $1-o(1)$ and yields an expected utility of $(1-o(1))\xi_{\text{max}}$ for $S$.

3.2 Secretary Scenario without Disclosure

If the candidates’ valuations are unknown, $S$ can use the following mechanism which relies on the classic secretary algorithm due to Dynkin [11].

The simple secretary mechanism decides once in the beginning whether to run the classic algorithm based either on $\xi$-values or $\rho$-values. In the classic algorithm, we sample the first $s = \lfloor n/e \rfloor$ candidates. Then we signal HIRE for the first candidate that is the best one so far (in terms of either $\xi$- or $\rho$-values, depending on the variant that is used). The classic algorithm hires the best candidate with probability $1/e - o(1)$. For our mechanism, with probability $1 - e/n$, we run the classic algorithm using the $\xi$-values of $S$, with probability $e/n$ the $\rho$-values of $R$. By assumption, if the mechanism decided to signal $\sigma_t = \text{NOT}$ in all rounds $t = 1, \ldots, n-1$, it sets $\sigma_n = \text{HIRE}$.

Theorem 20. The simple secretary mechanism is incentive compatible. It yields a success probability of $1/e - o(1)$ and an expected utility of $(1/e - o(1)) \cdot \xi_{\text{max}}$ for $S$.

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Proof. We begin the proof by showing incentive compatibility.

Due to random order and the symmetric structure of the algorithm, the algorithm produces the same distribution of HIRE signals, both for the sender-optimal or the receiver-optimal version of the secretary algorithm. In particular, for every round \( t > \lfloor n/e \rfloor \), and for any set \( A_t \) of arrived candidates in the rounds \( 1, \ldots, t \), the mechanism sends a hire signal in round \( t \) when the best candidate from \( A_t \) arrives in round \( t \) and the second best arrived in the sample phase. Hence, \( \Pr[\sigma_t = \text{HIRE} | A_t] = \frac{1}{t} \cdot \frac{e}{t-1} \), which is the same for every set \( A_t \) and for both the sender-optimal or receiver-optimal variant. There is no disclosure, so \( \mathcal{R} \) cannot distinguish which variant is used.

We first show incentive compatibility for negatively correlated utilities, i.e., when the \( i \)-th best candidate for \( \mathcal{S} \) is the \((n - i + 1)\)-th best candidate for \( \mathcal{R} \).

Consider a round \( t \in [n - 1] \).

Case \( \sigma_t = \text{HIRE} \): With probability \( e/n \) the receiver-optimal version is run and the candidate is the best so far in terms of \( \rho \). The probability that this candidate is actually \( c_\mathcal{R} \) is \( t/n \). If the sender-optimal version is run, there is no HIRE signal for \( c_\mathcal{R} \). Hence, overall in this case \( \Pr[\theta_t = c_\mathcal{R}] = et/n^2 \).

If \( \mathcal{R} \) decides to deviate to a later round, he does not receive additional information from \( \mathcal{S} \). Due to random order arrival and the independence of the HIRE signal of the set of arrived candidates, in any given round \( r > t \) we have \( \Pr[\theta_r = c_\mathcal{R}] = \frac{n-e}{n} \cdot \frac{1}{n-t} = \frac{1}{n} \). Since \( t \geq s + 1 > n/e \) we have \( et/n^2 > 1/n \). Hence, following the HIRE signal is optimal.

Case \( t > s \) and \( \sigma_i = \text{NOT} \) for all \( i \leq t \): With probability \( e/n \) the receiver-optimal version is run and the current candidate in round \( t \) is not the best so far in terms of \( \rho \). Then it cannot be \( c_\mathcal{R} \). With probability \( 1 - e/n \) the sender-optimal version is run and the candidate in round \( t \) is not the best so far in terms of \( \xi \). Then, \( c_\mathcal{R} \) has probability \( t/n \) to be among the set \( A_t \) of the first \( t \) candidates. Given that \( c_\mathcal{R} \in A_t \), it arrives in round \( t \) with probability \( \frac{1}{t-1} \) – there have been only NOT signals, so the best candidate from \( A_t \) must be in the sample phase. Hence, overall in this case \( \Pr[\theta_t = c_\mathcal{R}] = (1 - \frac{e}{n}) \cdot \frac{t}{n(t-1)} = \frac{(n-e)t}{n^2(t-1)} \).

If instead \( \mathcal{R} \) follows the mechanism, then in round \( r > t \) the candidate is best so far with probability \( \frac{1}{r} \cdot \frac{s/n}{s/t} = \frac{1}{r-1} \). Note that in this case, we condition on NOT signals in all rounds \( \leq t \). Hence, candidate \( \theta_r \) is best so far with probability \( \frac{s/(r(r-1))}{s/t} \). This candidate is \( c_\mathcal{R} \) with probability \( er/n^2 \) as noted above.

Moreover, in the last round, there is always a HIRE signal. The probability that no candidate after the sample phase is ever the best so far is \( \frac{s}{n} \). Since we condition on NOT signals in all rounds \( \leq t \), this gives a probability of \( \frac{s/n}{s/t} \). This candidate can be \( c_\mathcal{R} \) only when we run the sender-optimal variant. Hence, the probability that this candidate is \( c_\mathcal{R} \) is \( (1 - \frac{e}{n}) \cdot \frac{1}{n-t} \). Hence, in this case, following the mechanism \( \mathcal{R} \) obtains \( c_\mathcal{R} \) with probability

\[
\sum_{r=t+1}^{n} \frac{s/(r(r-1))}{s/t} \cdot \frac{er}{n^2} + \frac{s/n}{s/t} \left(1 - \frac{e}{n}\right) \cdot \frac{1}{n-1} = \frac{t}{n^2} \left(\frac{n-e}{n-1} + \sum_{r=t+1}^{n} \frac{e}{r-1}\right)
\]

Thus, deviation by hiring in round \( t \) is unprofitable if

\[
\frac{n-e}{t-1} \leq \frac{n-e}{n-1} + e \cdot \sum_{r=t+1}^{n} \frac{1}{r-1}
\]

or, equivalently

\[
\frac{1}{t-1} - \frac{1}{n-1} \leq \frac{e}{n} \left(\frac{1}{t-1} - \frac{1}{n-1} + \sum_{r=t+1}^{n} \frac{1}{r-1}\right)
\]
and, thus,
\[
\frac{n - t}{(n - 1)(t - 1) \left( \sum_{r=t-1}^{n-2} \frac{1}{r} \right)} \leq \frac{e}{n}.
\]  
(7)

**Claim 2.** The lower bound in equation (7) is monotonically decreasing in \( t \). Hence, the strongest lower bound arises in round \( t = s + 1 \).

**Proof.** We want to prove that
\[
\frac{n - t}{(n - 1)(t - 1) \left( \sum_{r=t-1}^{n-2} \frac{1}{r} \right)} \leq \frac{n - t + 1}{(n - 1)(t - 2) \left( \sum_{r=t-2}^{n-2} \frac{1}{r} \right)}
\]

We drop the terms \((n - 1)\), bring the summations up, and cluster the non-summation terms on the right side. This yields
\[
\sum_{r=t-2}^{n-2} \frac{1}{r} \leq \left(1 + \frac{n - 1}{(n - t)(t - 2)}\right) \cdot \left( \sum_{r=t-1}^{n-2} \frac{1}{r} \right)
\]
which implies
\[
\frac{1}{t - 2} \leq \frac{n - 1}{(n - t)(t - 2)} \cdot \left( \sum_{r=t-1}^{n-2} \frac{1}{r} \right)
\]
and, hence
\[
n - t \leq (n - 1) \left( \sum_{r=t-1}^{n-2} \frac{1}{r} \right).
\]
This is true for \( t = n - 1 \). Suppose it is true for some value of \( t \), then it is also true for \( t' = t - 1 \) – the left-hand side increases by 1, the right-hand side increases by \( \frac{n - 1}{t - 2} > 1 \). This proves the claim.

Using the claim, we can restrict attention to
\[
\frac{e}{n} \geq \frac{n - s - 1}{(n - 1)s \left( \sum_{r=s}^{n-2} \frac{1}{r} \right)}
\]
Recall that \( s = \lfloor n/e \rfloor \). For small \( n \leq 8 \), the bound follows by direct inspection. For \( n \geq 9 \), we use the following inequality (cf [26]), where \( \gamma \) is the Euler-Mascheroni constant and \( H_n \) the \( n \)-th harmonic number:
\[
\frac{1}{2(n + 1)} < H_n - \ln n - \gamma < \frac{1}{2n}
\]
and bound
\[
\sum_{t=\lfloor n/e \rfloor}^{n-2} \frac{1}{t} = H_{n-2} - H_{\lfloor n/e \rfloor - 1} \geq \frac{1}{2(n - 1)} + \ln(n - 2) - \frac{1}{2(\lfloor n/e \rfloor - 1)} - \ln \left( \left\lfloor \frac{n}{e} \right\rfloor - 1 \right).
\]  
(8)

Further, we have \( n \geq 3 + 2e \) and thus \( 2n - 2 - 2e \geq n + 1 \). Hence, the following holds:
\[
1 \geq \frac{n + 1}{2(n - (1 + e))} = \frac{n^2 + n}{2n^2 - 2(1 + e)n}
\]
\[ \geq \frac{2n^2(\frac{1}{e} - \frac{1}{e^2}) + n(e + \frac{2}{e} - 3)}{2n^2 - 2(1 + e)n + 2e} = \frac{2n(1 - \frac{1}{e})}{2(n - 1)e} - \frac{1}{2(n - 1)} + \frac{1}{2(\frac{n}{e} - 1)} \]

Combining this with (8) and the fact that \( n - 2 \geq e \left( \left\lfloor \frac{n}{e} \right\rfloor - 1 \right) \) and hence \( \ln \frac{n - 2}{\left\lfloor \frac{n}{e} \right\rfloor - 1} \geq 1 \), we get

\[ \ln \frac{n - 2}{\left\lfloor \frac{n}{e} \right\rfloor - 1} + \frac{1}{2(n - 1)} - \frac{1}{2(\left\lfloor \frac{n}{e} \right\rfloor - 1)} \geq \frac{n(1 - \frac{1}{e})}{(n - 1)e} \]

and therefore

\[ \frac{e}{n} \geq \frac{n - \left\lfloor \frac{n}{e} \right\rfloor - 1}{(n - 1) \left[ \frac{n}{e} \right] \sum_{t=\left\lfloor \frac{n}{e} \right\rfloor}^{n-2} \frac{1}{t}} \]

as desired.

**Case \( t \leq s \):** Finally, suppose \( R \) deviates and hires a candidate during round \( t \in [s] \). The probability of hiring \( c_R \) is \( 1/n \), no matter which variant of the algorithm is running.

If we want to bound the probability to hire \( c_R \) when following the mechanism, we can apply the bound in equation (6). However, since \( t \leq s \), we always have NOT signals until round \( t \). We remove the conditioning on having NOT signals until round \( t \), and the probability becomes

\[ \sum_{r=s+1}^{n} \frac{s}{r(r-1)} \cdot \frac{e^r}{n^r} + \frac{s}{n} \cdot \frac{n-e}{n(n-1)} = \frac{s}{n^2} \left( \frac{n-e}{n-1} + \sum_{r=s+1}^{n} \frac{e}{r-1} \right) \]

Thus, accepting the candidate in round \( t \leq s \) is unprofitable if

\[ \frac{n}{s} \leq \frac{n-e}{n-1} + \sum_{r=s+1}^{n} \frac{e}{r-1} \cdot \]

This implies

\[ \frac{e}{n} \left( \sum_{r=s}^{n-2} \frac{1}{r} \right) \geq \frac{1}{s - \frac{1}{n}} \]

and, thus,

\[ \frac{e}{n} \geq \frac{n-s-1}{(n-1)s \left( \sum_{r=s}^{n-2} \frac{1}{r} \right)} \]

which we proved true in the previous case.

Hence, the mechanism is incentive compatible if utilities are negatively correlated.

If \( c_R \) is not the worst candidate for \( S \), the following changes to the probabilities occur.

Obviously, the probabilities of hiring \( c_R \) when running the receiver-optimal algorithm are the same. When running the sender-optimal algorithm, the probability of getting \( c_R \) upon a signal HIRE increases. Suppose the sender-rank of \( c_R \) is \( x < n \). This means that there are \( x-1 \) candidates which \( S \) prefers over \( c_R \). Thus, a signal HIRE for \( c_R \) implies that all \( x-1 \) candidates have to arrive after the current round. If they arrive during the sample phase, \( c_R \) will not get a HIRE signal, if they arrive between sample phase and the current round, they would have gotten the signal HIRE instead of \( c_R \).
Thus, instead of 0, the probability of getting \( c_R \) upon a HIRE signal in the sender-optimal variant is
\[
\frac{t}{n} \cdot \frac{n-t}{n-1} \cdot \frac{n-t-1}{n-2} \cdot \ldots \cdot \frac{n-t-(x-2)}{n-(x-1)} > 0.
\]

Further, a candidate that is not recommended for hire has an even smaller probability of being \( c_R \). The exact probability amounts to
\[
\frac{t}{n} \cdot \frac{1}{n-1} \cdot \left( 1 - \frac{n-t}{n-1} \cdot \frac{n-t-1}{n-2} \cdot \ldots \cdot \frac{n-t-(x-2)}{n-(x-1)} \right) \leq \frac{n-t}{n(t-1)}.
\]

The probabilities of getting \( c_R \) when hiring in a round during the sampling phase or in a round after the HIRE signal remain 1/n by similar calculations as above.

Overall, the incentive to follow the algorithm weakly increases for decreasing rank \( x \). The mechanism is incentive compatibility when \( c_R \) is the worst candidate for \( S \), and also of \( c_R \) is at any rank \( 1 \leq x < n \) in the preference of \( S \).

Together with the fact that the mechanism clearly hires \( c_S \) with probability \( 1/e - o(1) \) and thus yields an expected utility of \((1/e - o(1))\xi_{\text{max}}\) for \( S \), this proves the theorem.

3.3 Basic Scenario with Disclosure

In this section, we consider the basic scenario with disclosure, i.e., rejected candidates are revealed to \( R \). If all candidate values are known to both sender and receiver a-priori, the additional information obtained by disclosure does not significantly help \( R \).

In round \( t \), the adaptive elementary mechanism signals as follows. If there has been no HIRE signal in any earlier round and \( c_S \) arrives in round \( t \), it signals HIRE. If there has been no HIRE signal in any earlier round and \( c_R \) arrives in round \( t \), it signals HIRE with probability \( 1/(n-t) \). It signals NOT in any other case.

**Lemma 21.** The adaptive elementary mechanism is incentive compatible.

**Proof.** If \( c_R \) has already arrived and was rejected, \( R \) will not get his best candidate. Recall that the rejected candidates are revealed. Since he is indifferent among the remaining candidates, it is optimal to follow the remaining signals of \( S \). Hence, for the remainder of the incentive-compatibility argument, we assume that \( c_R \) has not been rejected.

If the signal in round \( t \) is HIRE, the current candidate is \( c_R \) with probability \( \frac{1}{n-t} \cdot \left( 1 + \frac{t-1}{n-t} \right) = \frac{1}{n-t+1} \). If \( R \) decides to deviate and hire in some later round \( r \), he will not get any additional information. Thus, due to random-order arrival, the probability of hiring \( c_R \) in round \( r \) is \( \left( 1 - \frac{1}{n-r+1} \right) \cdot \frac{1}{n-t} = \frac{r}{n-t+1} \). Hence, it is optimal to follow the signal.

Now suppose up to and including round \( t \) there have been only NOT signals. If \( R \) deviates and hires the current candidate in round \( t \), he hires \( c_R \) with probability \( \frac{n-t}{n-t} \cdot \left( 1 + \frac{t}{n-t} \right) = \frac{1}{n-t+1} \). If instead \( R \) follows the mechanism, then suppose the HIRE signal comes in round \( r > t \). We showed above that this gives a success probability of \( \frac{1}{n-r+1} \) - if \( c_R \) was not rejected in one of the rounds \( i = t, \ldots, r-1 \). By the previous paragraph, in round \( i \) with signal \( \sigma_i = \text{NOT} \), candidate \( c_R \) is getting rejected with probability \( \frac{1}{n-r+1} \). Thus, if \( R \) follows the mechanism, the success probability is \( \prod_{i=t}^{r-1} \left( 1 - \frac{1}{n-i+1} \right) \cdot \frac{1}{n-r+1} = \frac{1}{n-r+1} \).

Hence, it is optimal for \( R \) to follow the mechanism.

**Proposition 22.** The adaptive elementary mechanism yields a success probability of \( 1 - o(1) \) and an expected utility of \( (1 - o(1)) \cdot \xi_{\text{max}} \) for \( S \).

**Proof.** Conditional on the fact that \( c_S \) has not shown up in rounds \( 1, \ldots, t-1 \), \( c_S \) arrives in round \( t \) with probability \( 1/(n-t+1) \). A similar probability holds for \( c_R \) (i.e., conditional on \( c_R \) not having arrived in an earlier round, the current candidate being \( c_R \)).

We first bound the success probability if \( R \) follows the mechanism. Let \( A_j \) be the probability of hiring \( c_S \) if \( j \geq 2 \) candidates remain and \( c_S \) and \( c_R \) are both among those remaining candidates. The
We can express $A_j$ by the following recursion:

$$A_j = \frac{1}{j} \cdot 1 + \frac{1}{j} \cdot \frac{j-2}{j-1} \cdot A_{j-1}$$

Solving the recursion gives us

$$A_n = \frac{1}{n} + \frac{1}{n} \cdot \frac{n-2}{n-1} + \frac{2}{n} \cdot A_{n-2} = \frac{2n-3}{n(n-1)} + \frac{2n-5}{n(n-1)} + \frac{n-3}{n-1} \cdot A_{n-2}$$

$$= \frac{2n-3}{n(n-1)} + \frac{2n-5}{n(n-1)} + \frac{2n-7}{n(n-1)} + \frac{(n-4)(n-3)}{n(n-1)} \cdot A_{n-3}$$

$$= \sum_{i=1}^{n-2} \frac{2(n-i)-1}{n(n-1)} + \frac{2}{n(n-1)} \cdot \frac{1}{2} = 1 - \frac{1}{n}.$$

This proves the approximation guarantee. \qed

### 3.4 Secretary Scenario with Disclosure

In the secretary scenario with disclosure, $R$ sees in each round $t$ a signal $\sigma_t$. When $R$ decides to reject the candidate in round $t$, he gets to see its identity $\theta_t$. Recall that we assume $R$ knows the set of candidates upfront, while $S$ only learns the values of a candidate when it arrives. We consider the First-Opt mechanism (see Algorithm 4). It rejects the first $s$ candidates. We will choose $s = \lfloor n/2 \rfloor$. Subsequently, it recommends for hire the first candidate that is best among the arrived candidates, either for $S$ or for $R$. In the last round, $S$ always signals HIRE.

**Lemma 23.** The First-Opt mechanism is incentive compatible.

**Proof.** If $c_R$ has already arrived and was rejected (and, thus, revealed), $R$ is indifferent among the remaining candidates. As such, it is optimal to follow the remaining signals of $S$. For the remainder of the proof of incentive compatibility, we assume $c_R$ has not been rejected.

Suppose $\sigma_t = \text{HIRE}$ in round $t \leq n-1$, then $\Pr[\theta_t = c_R \mid \sigma_t = \text{HIRE}] \geq \frac{1}{n-t+1}$. If $R$ decides to deviate and hire in some later round $r > t$, he will not get any additional information from $S$. Due to random-order arrival, the success probability is $\Pr[\theta_r = c_R \mid \sigma_t = \text{HIRE}] \leq \left(1 - \frac{1}{n-t+1}\right) \frac{1}{n-t} = \frac{1}{n-t+1}$. Hence, it is optimal to follow the signal.

Now suppose $\sigma_{t'} = \text{NOT}$ for all $t' \in [t]$. If $t \geq s + 1$ after the sample phase, the candidate is not the best one among the arrived ones, so $\theta_t \neq c_R$. It is optimal for $R$ to wait for a round
with a HIRE signal. Otherwise, if \(t \leq s\) during the sample phase, then \(\Pr[\theta_t = c_R] = \frac{1}{n-t+1}\) (note that we assume \(c_R\) has not been rejected). In contrast, if \(R\) follows the mechanism, then 
\[
\Pr[\theta_t, \theta_{t+1}, \ldots, \theta_s \neq c_R] = \prod_{t'=t}^{s} \left(1 - \frac{1}{n-t'+1}\right) = \frac{n-s}{n-t+1}. 
\]
Then, the sample phase ends and, due to disclosure, \(R\) can determine all remaining candidates that are better (in terms of \(\xi, \rho\), or both) than all arrived (and revealed) candidates during the sample phase. All remaining \(n-s\) candidates might fulfill this condition, with \(c_R\) being one of them. If \(R\) follows the mechanism, he will end up hiring a random one of these candidates. Hence, the success probability when following the mechanism is at least \(\frac{n-s}{n-t+1} \cdot \frac{1}{n-s} = \frac{1}{n-t+1}\). Hence, it is optimal for \(R\) to follow the mechanism.

**Theorem 24.** Using \(s = \lfloor n/2 \rfloor\), the First-Opt mechanism yields a success probability of at least \(1/4 - o(1)\) and an expected utility of at least \((1/4 - o(1)) \cdot \xi_{\max}\) for \(S\).

**Proof.** Let \(A_t\) denote the random set of candidates observed up to round \(t\). The first \(s\) candidates get a NOT signal. For each round \(t \geq s+1\), we can determine the candidate in round \(t\) by first drawing randomly the set \(A_t\), and then drawing a uniform random candidate from \(A_t\) to arrive in round \(t\). In \(A_t\) there are (one or) two candidates that can potentially generate a HIRE signal in round \(t\), the best one for \(S\) and the one best for \(R\) (could be the same one). Hence, the probability of signal HIRE is at most \(\frac{s}{n}\). Overall, the probability \(\Pr[\sigma_t = \text{NOT} \mid A_t]\) is at least

\[
\Pr[\sigma_t = \text{NOT} \mid A_t] \geq \begin{cases} 
\frac{1}{t-2} & t = 1, \ldots, s \\
\frac{1}{t-2} & t = s + 1, \ldots, n - 1 
\end{cases}.
\]

The success probability for \(S\) in round \(t > s\) conditioned on \(A_t\) is

\[
\Pr[\theta_t = c_S \mid A_t] = \Pr[c_S \in A_t] \cdot \Pr[\theta_t = c_S \mid c_S \in A_t] \cdot \Pr[\sigma_1, \ldots, \sigma_{t-1} = \text{NOT} \mid A_{t-1}] 
\]

\[
= \frac{t}{n} \cdot \frac{1}{t} \cdot \prod_{k=s+1}^{t-1} \Pr[\sigma_k = \text{NOT} \mid A_k] \geq \frac{1}{n} \prod_{k=s+1}^{t-1} \frac{k-2}{k} 
\]

\[
= \frac{(s-1)s}{n (t-2)(t-1)},
\]

and, since these probabilities are independent of \(A_t\), the overall success probability for \(S\) is at least

\[
\sum_{t=s+1}^{n} \frac{1}{n} \cdot \frac{(s-1)s}{(t-2)(t-1)} = \frac{(s-1)s}{n} \sum_{t=s+1}^{n} \left( \frac{1}{t-2} - \frac{1}{t-1} \right) = \frac{s}{n} \left( 1 - \frac{s-1}{n-1} \right).
\]

The expression is optimized for \(s = \lfloor n/2 \rfloor\) and yields a success probability of \(\frac{1}{4} - o(1)\). The mechanism is entirely symmetric in terms of sender and receiver, so the same result applies to the success probability of \(R\).

We now show that the First-Opt mechanism is indeed optimal, in the sense that there are cases in which no mechanism can achieve a better success probability. We study negatively correlated utilities, i.e., the candidate with rank \(i\) for \(S\) has rank \(n-i+1\) for \(R\) for all \(i \in [n]\). Observe that, beyond defining the ranking, cardinal utility values are irrelevant for the objectives of \(S\) and \(R\). Moreover, given the ranking of known candidates, \(S\) can infer no additional information from their values about the position of the current candidates in the overall ranking. As such, we will assume that \(S\) ignores the cardinal values.

**Theorem 25.** If utilities of sender and receiver are negatively correlated, the First-Opt mechanism maximizes the success probability for \(S\) among all incentive-compatible mechanisms.
We first show the structural Lemmas 26-28. We concentrate on the following class of randomized Best-So-Far mechanisms. A Best-So-Far mechanism signals HIRE in rounds $1, \ldots, n - 1$ with probability $> 0$ only if the candidate in the current round is best so far for either $S$ or $R$. A Best-So-Far mechanism always signals HIRE in the last round if it has not done so before. We show that we can turn any mechanism into a Best-So-Far mechanism. This yields higher success probabilities for $S$ and does not hurt the incentives for $R$, since hiring non-best-so-far candidates is unprofitable for both $S$ and $R$.

**Lemma 26.** If utilities of sender and receiver are negatively correlated, then for every incentive-compatible mechanism there is an incentive-compatible Best-So-Far mechanism with weakly higher success probability for $S$.

**Proof.** Consider an incentive-compatible mechanism $\phi$ for $S$. We construct a new Best-So-Far mechanism $\phi'$ as follows. The new mechanism runs $\phi$. Whenever $\phi$ decides to signal HIRE on a candidate that is neither best so far for $S$ nor $R$, $\phi'$ changes the signaling strategy. We assume for the moment that in this case $\phi'$ sends to $R$ a separate transition signal. $\phi'$ then signals NOT in all rounds $t, \ldots, n - 1$ and HIRE in round $n$.

Clearly, for any HIRE signal received in rounds $1, \ldots, n - 1$, $\phi'$ yields a higher conditional probability that the recommended candidate is $c_R$. If the transition signal is received, the mechanism switches to a deterministic HIRE signal in the last round. This only increases both, the probability to hire $c_S$ and $c_R$, since in this case $\phi$ would have hired a suboptimal candidate for both $S$ and $R$.

If $\phi$ comes to round $n - 1$ without a HIRE signal, $\phi'$ signals HIRE in round $n$. This, too, weakly increases both the probabilities of hiring $c_S$ and of hiring $c_R$.

Hence, given that $\phi$ is incentive compatible, $\phi'$ is incentive compatible as well. It increases the success probability for both $S$ and $R$. Finally, $\phi'$ remains incentive compatible even when we omit the transition signal, since $R$ is given only less information. □

For the rest of the argument, we concentrate on Best-So-Far mechanisms. Note that Best-So-Far mechanisms are not always incentive-compatible. For example, simply flipping a coin and running the sender-optimal or the receiver-optimal secretary algorithm throughout is a Best-So-Far mechanism.

In the secretary scenario without revelation this was our incentive compatible approach, but here it might give $R$ incentives to deviate: By the first time $R$ sees that a currently best candidate for him was rejected, he knows the sender-optimal algorithm is running. Then he might have an incentive to deviate, since the sender-optimal algorithm never recommends $c_R$.

We now enlarge the class of mechanisms under consideration. Suppose we are given any history $A_{t-1}$ of arrived candidates until round $t - 1$. In round $t$, let $p^S_t$ be the probability of signaling $\sigma_t = \text{HIRE}$ if $\theta_t$ is the best candidate so far for $S$. We define $p^R_t$ accordingly.

**Lemma 27.** If a Best-So-Far mechanism is incentive compatible for negatively correlated utilities, it satisfies $p^R_t \geq p^S_t$ for all rounds $t \in [n]$ and all histories $A_{t-1}$.

**Proof.** Consider the beginning of round $t \leq n - 1$, where $n - t + 1$ candidates are still to arrive. Since $R$ knows all candidates upfront and sees all rejected ones, he can determine which candidates would qualify in round $t$ as best so far for $S$ or $R$. Suppose $c_R$ has not arrived yet. We construct a worst-case scenario for $R$ as follows: The second-best candidate for $R$ has arrived, so $c_R$ is the only one that generates a HIRE signal in favor of $R$. All other $n - t$ candidates are the top $n - t$ candidates for $S$.

In this worst-case scenario, if the probability $p^S_t > p^R_t$, then a HIRE signal in round $t$ leads to hiring of $c_R$ with probability of $p^R_t / (p^R_t + (n-t)p^S_t) < 1/(n-t+1)$. In contrast, if a NOT signal is sent and $R$ deviates, he gets $c_R$ with probability of $(1-p^S_t) / (1-p^R_t + (n-t)(1-p^S_t)) > 1/(n-t+1)$. Hence, the mechanism is clearly not incentive compatible.
\( S \) (unlike \( R \)) is entirely unaware of whether the situation in the current round represents a worst-case scenario or not. In fact, in every round, every history of arrived candidates could give rise to such a worst-case scenario. As such, in order to guarantee incentive compatibility, the mechanism needs to satisfy \( p_t^R \geq p_t^S \) for all rounds \( t \in \{n-1\} \) and all histories of arrived candidates.

Consider the class of Best-So-Far mechanisms with \( p_t^R \geq p_t^S \). It could potentially extend slightly beyond incentive-compatible ones. We now optimize the success probability for \( S \) within this class. By the previous two lemmas, this gives an upper bound on the success probability of any incentive-compatible mechanism. For larger values of \( p_t^R \), we obviously have smaller success probability for \( S \). Thus, the best mechanism in the class satisfies \( p_t = p_t^R = p_t^S \) in each round and for each history. Moreover, we can show that \( p_t \in \{0,1\} \).

**Lemma 28.** There is an optimal incentive-compatible Best-So-Far mechanism for negatively correlated utilities such that \( p_t \in \{0,1\} \) for all \( t \in \{n-1\} \).

**Proof.** The proof is a simple backwards induction. Given round \( n \), we assume by definition that a Best-So-Far mechanism has \( p_n = 1 \). Consider round \( n-1 \) and condition on the event that the current candidate is best so far for either \( S \) or \( R \).\(^2\) Now there are two options, hire in this round or hire in round \( n \). \( S \) can determine probabilities of hiring \( c_S \) for both these options. It is then optimal for \( S \) to decide deterministically for the option (i.e., either \( p_{n-1} = 1 \) or \( p_{n-1} = 0 \)), whichever gives higher success probability.

Thus, starting in round \( n-1 \), the optimal policy uses \( p_n, p_{n-1} \in \{0,1\} \). In round \( t < n-1 \), we have a similar situation. If the candidate is best so far, there are two options - either hire in this round or reject and invoke the optimal policy for rounds \( t+1, t+2, \ldots, n \). \( S \) can determine probabilities of hiring \( c_S \) for both these options. It is then optimal for \( S \) to decide deterministically for the option (i.e., either \( p_t = 1 \) or \( p_t = 0 \)), whichever gives higher success probability.

Hence, by induction, the optimal Best-So-Far mechanism with \( p_t = p_t^R = p_t^S \) has all \( p_t \in \{0,1\} \).

**Proof of Theorem 25.** Based on Lemma 28, our goal is to determine optimal values \( p_t \in \{0,1\} \). Our proof generalizes ideas for the standard secretary problem [6]. Let \( B_t \) be the event that the candidate in round \( t \) is best so far for either \( S \) or \( R \). Let \( R_t \) be the event that candidate \( \theta_t \) is rejected. Let \( R_{t-1} \) be the event that all candidates \( \theta_1, \ldots, \theta_{t-1} \) were rejected. Let \( u_t = \Pr[c_S \text{ hired in rounds } t, \ldots, n \mid B_t, R_{t-1}] \) and \( v_t = \Pr[c_S \text{ hired in rounds } t+1, \ldots, n \mid R_t, R_{t-1}] \).

To reach round \( t+1 \), we must have \( R_{t-1} \) and either \( R_t \), or \( B_t \) and \( p_t = 0 \). In both cases, consider the conditional probability of \( B_{t+1} \) i.e., \( \Pr[B_{t+1} \mid R_t, R_{t-1}] \) and \( \Pr[B_{t+1} \mid B_t, R_{t-1}] \).

Note that due to random order arrival, conditioned on every set \( A_t \) of candidates arrived in rounds \( k = 1, \ldots, t \), we have the same probabilities of generating \( B_k \) and \( R_k \) events and, thus, the same probability of \( B_t, R_{t-1} \) and \( R_t \). Now, conditioned on \( A_{t+1} \), the probability to have \( B_{t+1} \) in round \( t + 1 \) is \( \Pr[B_{t+1} \mid A_{t+1}] = 2/t+1 \), which is the same for every \( A_{t+1} \). Since \( \Pr[R_{t-1} \mid A_t] \) and \( \Pr[R_t \mid A_t] \) are independent of the set \( A_t \), we also have \( \Pr[B_{t+1} \mid A_{t+1}, R_t, R_{t-1}] = \Pr[B_{t+1} \mid R_t, R_{t-1}] = 2/t+1 \). Moreover, since \( \Pr[B_t \mid A_t] \) is the same for every \( A_t \), we have \( \Pr[B_{t+1} \mid A_{t+1}, B_t, R_{t-1}] = \Pr[B_{t+1} \mid B_t, R_{t-1}] = 2/t+1 \).

Thus, we see that

\[
v_t = \frac{2}{t+1} \cdot u_{t+1} + \frac{t-1}{t+1} \cdot v_{t+1} \tag{10}
\]

Now suppose \( \theta_t \) is best so far, i.e., condition on \( B_t \). If we hire \( \theta_t \), we get \( c_S \) with probability \( \frac{t}{2t+1} \). Otherwise, if we reject it, we get \( c_S \) with probability \( v_t \). Thus, an optimal Best-So-Far mechanism

\(^2\)Note that since \( p_{n-1} = p_{n-1}^R = p_{n-1}^S \), we may not distinguish between the cases best so far for \( S \) or \( R \).
will choose the better alternative and obtain

\[ u_t = \max \left\{ \frac{t}{2n}, v_t \right\} . \]  \hspace{1cm} (11)

We resolve the recurrences for \( u_t \) and \( v_t \) by backwards induction. The base cases are \( v_n = 0 \) and \( u_n = \frac{1}{t} \). Note that \( v_{n-1} = \frac{n-1}{n} \cdot v_n + \frac{2}{n} \cdot u_n = \frac{1}{t} \), which yields \( u_{n-1} = \max \left\{ \frac{n-1}{2n}, v_{n-1} \right\} = \frac{n-1}{2n} \). Repeating this argument, we have \( u_t > v_t \) as long as \( \frac{n}{2} > v_t \), and thus should set \( p_t = 1 \). If on the other hand \( v_t \geq \frac{n}{2n} \), it is optimal to wait and set \( p_t = 0 \), even though the current candidate is best so far. More generally, \( u_t \) and \( v_t \) can be given as follows which we prove below.

**Lemma 29.** \( u_t \) and \( v_t \) are given by

\[
\begin{align*}
   u_t &= \begin{cases} 
   \frac{t}{2n} & t \geq \frac{n+1}{2}, n \text{ even}, \\
   \frac{n}{4(n-1)} & t < \frac{n+1}{2}, n \text{ odd}, 
   \end{cases} \\
   v_t &= \begin{cases} 
   \frac{(n-t)}{n(n-1)} & t \geq \frac{n+1}{2}, \\
   \frac{n}{4(n-1)} & t < \frac{n+1}{2}, n \text{ even}, \\
   \frac{n+1}{4n} & t < \frac{n+1}{2}, n \text{ odd}.
   \end{cases}
\end{align*}
\]

Hence, it is optimal to signal NOT in the first \( s = \lfloor n/2 \rfloor \) rounds and then signal HIRE for the first candidate that is best so far, for either \( R \) or \( S \). This is the First-Opt mechanism, and Theorem 25 is proved.

**Proof of Lemma 29.** First, consider the case \( t \geq \frac{n+1}{2} \) and start with \( t = n \). Due to the base cases, \( v_n = 0 = \frac{n(n-n)}{n(n-1)} \) and \( u_n = \frac{1}{t} = \max \left\{ \frac{n}{2n}, 0 \right\} \).

Assume the lemma holds for \( t+1 \geq \frac{n+1}{2} + 1 \). We show that it holds for \( t \) as well. First, observe that \( \frac{t}{2n} \geq \frac{(n-t)}{n(n-1)} \) for \( t \geq \frac{n+1}{2} \) and thus \( u_t = \frac{t}{2n} \). For \( v_t \) we get the following:

\[
\begin{align*}
   v_t &= \frac{t-1}{t+1} \cdot v_{t+1} + \frac{2}{t+1} \cdot u_{t+1} \\
   &= \frac{t-1}{n} \cdot \frac{1}{n} \cdot \frac{(t+1)(n - (t+1))}{n-1} + \frac{2}{t+1} \cdot u_{t+1} \\
   &= \frac{1}{n} \cdot \frac{(t-1)(n-t-1) + (n-1)}{n-1} \\
   &= \frac{1}{n} \cdot \frac{t(n-t)}{n-1}.
\end{align*}
\]

Now consider the second case in which \( t < \frac{n+1}{2} \). The base case is \( t = \frac{n}{2} \) or \( t = \frac{n+1}{2} \), depending on the parity of \( n \).

**Case n even:** It holds

\[
\begin{align*}
   v_t^n &= \frac{n/2 - 1}{n/2 + 1} \cdot v_{t+1}^n + \frac{2}{n/2 + 1} \cdot u_{t+1}^n \\
   &= \frac{n-2}{n+2} \cdot \frac{1}{n} \cdot \frac{(n+2)(n-2)}{4(n-1)} + \frac{2}{n/2 + 1} \cdot \frac{n/2 + 1}{2n} \\
   &= \frac{1}{n} \cdot \frac{(n-2)^2 + 4(n-1)}{4(n-1)} \\
   &= \frac{1}{4n(n-1)} \cdot n^2 - 4n + 4n - 4 \\
   &= \frac{n}{4n(n-1)}.
\end{align*}
\]

Since \( t = \frac{n}{2} < \frac{n+1}{2} \) and thus \( \frac{t}{n} = \frac{1}{t} < \frac{n}{n(t+1)} = u_t \), we now have \( u_t = v_t \). This obviously leads to \( v_t = v_{t+1} \) and thus \( u_t = v_t \) for all \( t < \frac{n+1}{2} \).
Case \( n \) odd: The case is similar:

\[
v_{\frac{n-1}{2}} = \frac{(n-1)/2 - 1}{(n+1)/2} \cdot v_{\frac{n+1}{2}} + \frac{2}{(n+1)/2} \cdot u_{\frac{n+1}{2}}
\]

\[
= \frac{n-3}{n+1} \cdot \frac{1}{n} \cdot \frac{(n+1)/2((n-1)/2)}{n-1} + \frac{2}{(n+1)/2} \cdot \frac{(n+1)/2}{2n}
\]

\[
= \frac{1}{n} \cdot \frac{1}{4(n-1)} \cdot ((n-3)(n-1) + 4(n-1))
\]

\[
= \frac{1}{n} \cdot \frac{1}{4(n-1)} \cdot (n^2 - 4n + 3 + 4n - 4)
\]

\[
= \frac{1}{n} \cdot \frac{1}{4(n-1)} \cdot (n+1)(n-1)
\]

\[
= \frac{n+1}{4n}
\]

Again, \( t = \frac{n-1}{2} < \frac{n+1}{2} \) and thus \( \frac{n-1}{2n} = \frac{1}{4} - \frac{1}{4n} < \frac{n+1}{4n} = v_t \). Hence, \( u_t = v_t \) for all \( t < \frac{n+1}{2} \).

This proves the lemma. \( \square \)

References

[1] I. Arieli and Y. Babichenko. Private Bayesian persuasion. 2016.

[2] P. H. Au. Dynamic information disclosure. The RAND Journal of Economics, 46(4):791–823, 2015.

[3] R. Aumann and M. Maschler. Game theoretic aspects of gradual disarmament. Report of the US Arms Control and Disarmament Agency, 80:1–55, 1966.

[4] M. Babaioff, N. Immorlica, D. Kempe, and R. Kleinberg. Online auctions and generalized secretary problems. SIGecom Exchanges, 7(2), 2008.

[5] Y. Babichenko and S. Barman. Algorithmic aspects of private Bayesian persuasion. In Proc. 8th Symp. Innovations in Theoret. Comput. Sci. (ITCS), pages 34:1–34:16, 2017.

[6] M. J. Beckmann. Dynamic programming and the secretary problem. Computers & Mathematics with Applications, 19(11):25 – 28, 1990.

[7] N. Chen, M. Hoefer, M. Künemann, C. Lin, and P. Miao. Secretary markets with local information. In Proc. 42nd Intl. Coll. Automata, Languages & Programming (ICALP), volume 2, pages 552–563, 2015.

[8] M. Dinitz. Recent advances on the matroid secretary problem. SIGACT News, 44(2):126–142, 2013.

[9] S. Dughmi and H. Xu. Algorithmic Bayesian persuasion. In Proc. 48th Symp. Theory of Comput. (STOC), pages 412–425, 2016.

[10] S. Dughmi and H. Xu. Algorithmic persuasion with no externalities. In Proc. 18th Conf. Econom. Comput. (EC), pages 351–368, 2017.

[11] E. Dynkin. The optimum choice of the instant for stopping a Markov process. In Sov. Math. Dokl, volume 4, pages 627–629, 1963.
[12] J. Ely. Beeps. *Amer. Econ. Rev.*, 107(1):31–53, January 2017.

[13] J. Ely, A. Frankel, and E. Kamenica. Suspense and surprise. *J. Political Economy*, 123(1):215–260, 2015.

[14] M. Feldman, O. Svensson, and R. Zenklusen. A simple $O(\log \log(\text{rank}))$-competitive algorithm for the matroid secretary problem. *Math. Oper. Res.*, 43(2):638–650, 2018.

[15] T. Ferguson. Who solved the secretary problem? *Statistical Science*, 4(3):282–289, 1989.

[16] M. Gardner. *New Mathematical Diversions from Scientific American*. Simon & Schuster, 1966. Reprint of the original column published in February 1960 with additional comments.

[17] O. Göbel, M. Hoefer, T. Kesselheim, T. Schleiden, and B. Vöcking. Online independent set beyond the worst-case: Secretaries, prophets and periods. In *Proc. 41st Intl. Coll. Automata, Languages & Programming (ICALP)*, volume 2, pages 508–519, 2014.

[18] M. Hoefer and B. Kodric. Combinatorial secretary problems with ordinal information. In *Proc. 44th Intl. Coll. Automata, Languages & Programming (ICALP)*, pages 133:1–133:14, 2017.

[19] B. R. Holmstrom. On the theory of delegation. In M. Boyer and R. E. Kihlstrom, editors, *Bayesian Models in Economic Theory*, pages 115–141. Elsevier, 1984.

[20] E. Kamenica and M. Gentzkow. Bayesian persuasion. *Amer. Econ. Rev.*, 101(6):2590–2615, October 2011.

[21] T. Kesselheim, K. Radke, A. Tönnis, and B. Vöcking. An optimal online algorithm for weighted bipartite matching and extensions to combinatorial auctions. In *Proc. 21st European Symp. Algorithms (ESA)*, pages 589–600, 2013.

[22] J. Kleinberg and R. Kleinberg. Delegated search approximates efficient search. In *Proc. 19th Conf. Econom. Comput. (EC)*, 2018.

[23] R. Kleinberg. A multiple-choice secretary algorithm with applications to online auctions. In *Proc. 16th Symp. Discrete Algorithms (SODA)*, pages 630–631, 2005.

[24] R. Reiffenhäuser. An optimal truthful mechanism for the online weighted bipartite matching problem. In *Proc. 30th Symp. Discrete Algorithms (SODA)*, pages 1982–1993, 2019.

[25] A. Rubinstein. Beyond matroids: Secretary problem and prophet inequality with general constraints. In *Proc. 48th Symp. Theory of Comput. (STOC)*, pages 324–332, 2016.

[26] R. Young. Euler’s constant. *Math. Gaz*, pages 187–190, 1991.