Restoration of $k_T$ factorization for low $p_T$ hadron hadroproduction

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We discuss the applicability of the $k_T$ factorization theorem to low-$p_T$ hadron production in hadron-hadron collision. It has been shown that the $k_T$ factorization for high-$p_T$ hadron hadroproduction is broken by soft gluons in the Glauber region, which are exchanged among a transverse-momentum-dependent (TMD) parton density and other subprocesses of the collision. We explain that the contour of a soft loop momentum can be deformed away from the Glauber region at low $p_T$, so the above soft gluons are factorized by means of the standard eikonal approximation. The $k_T$ factorization is then restored in the sense that a TMD parton density maintains its universality. Because the resultant soft factor is independent of hadron flavors, experimental constraints on its behavior are possible. The $k_T$ factorization can not be restored for the transverse single-spin asymmetry in hadron-hadron collision at low $p_T$, since the leading-order soft divergence is not factorizable.

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The $k_T$ factorization theorem has been widely applied to inclusive and exclusive processes in perturbative QCD. This theorem holds for simple processes, such as deeply inelastic scattering (DIS) and Drell-Yan production. Recently, it was found that the $k_T$ factorization breaks down for complicated processes like high-$p_T$ hadron production in hadron-hadron collision. The $k_T$ factorization, if applicable, gives the leading-power differential cross section

$$d\sigma_{i+\rightarrow k+l}/d\mathbf{p}_3^i d\mathbf{p}_4^j d\mathbf{p}_5^k d\mathbf{p}_6^l,$$

which involves the transverse-momentum-dependent (TMD) parton densities $f_{i/H}$, the fragmentation functions $d_{H/i}$, and the parton-level differential cross section $d\sigma_{i+\rightarrow k+l}$. The sum over the flavors and the integral over momenta of the partons are implicit in the above expression.

When factorizing the TMD parton density $f_{i/H}$, infrared divergences from the soft gluons were identified, which are exchanged among $f_{i/H}$ and other subprocesses of the collision. These divergences, violating the universality of $f_{i/H}$, break the $k_T$ factorization for the hadron hadroproduction. The source of the factorization breakdown is briefly explained below. Consider the one-loop diagrams in Figs. (a), (b), and (c) with radiative gluons being emitted by the spectator in $H_1$ and attaching to the active partons in $H_2$, $H_3$, and $H_4$, respectively. These active parton lines can be eikonalized, if focusing on the collection of infrared divergences. The eikonal line from $H_3$ corresponds to the Wilson line appearing in the operator definition of $f_{i/H}$. The infrared divergences in Figs. (a) and (b), which carry the information of the TMD parton density $f_{j/2}$, need to cancel in order to ensure the universality of $f_{i/H}$. However, the eikonal propagators associated with the partons in $H_2$ and $H_3$ are summed into an imaginary piece

$$\frac{1}{-l^+ + i\epsilon} + \frac{1}{l^+ + i\epsilon} = -2\pi i \delta(l^+),$$

where the first (second) term comes from Fig. (a) and (b), and $l$ denotes the loop momentum. The $\delta$-function leads to a residual soft divergence from the Glauber region with $l^+ = 0$. 

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This one-loop residual soft divergence may not cause trouble, if the leading-order (LO) amplitude $\mathcal{M}^{(0)}$ is real. The expansion of the differential cross section for the hadron hadroproduction up to next-to-leading order (NLO) gives

$$|\mathcal{M}|^2 = |\mathcal{M}^{(0)}|^2 + 2\text{Re}[\mathcal{M}^{(0)}\mathcal{M}^{(1)*}]$$

According to Eq. (3), the residual soft divergence will be purely imaginary, if $\mathcal{M}^{(0)}$ is real, so it does not survive in the second term $\text{Re}[\mathcal{M}^{(0)}\mathcal{M}^{(1)*}]$. At two loops, the residual soft divergence, arising from the real product $\delta(l_1^+)\delta(l_2^+)$, does exist in the $k_T$ factorization. Certainly, the product $\delta(l_1^+)\delta(l_2^+)$ does not break the collinear factorization, it has been explicitly demonstrated \cite{9} that the two-loop residual soft divergence cancels, when the parton transverse momenta are integrated out. The cancellation occurs between the diagrams with the two soft gluons on the different sides of the final-state cut and the diagrams with the two soft gluons on the same side. That is, the universality of $\mathcal{M}$ with respect to the virtual photon direction is maintained in the hadron-hadron collision. It was also found that the Glauber effect exists in complicated dijet production in hadron-hadron collision \cite{17}, but does not in simple Drell-Yan processes \cite{18}.

In the present work we shall have a closer look at the failure of the $k_T$ factorization for the hadron hadroproduction. If the residual soft divergences could be factorized from the collision, the universality of $f_{i/1}$ would be recovered at the price that Eq. (2) involves an additional soft factor. However, the soft gluons are characterized by momenta in the Glauber region as stated before, in which the spectator line of $H_1$ cannot be eikonalized \cite{7, 8}. Therefore, the Ward identity argument used in standard factorization proofs \cite{19, 20} does not apply. It will be shown that the $k_T$ factorization actually holds at low $p_T$ (e.g., few GeV at Tevatron), though it breaks down at high $p_T$ as claimed in \cite{8}. The argument is that the contour of a soft loop momentum can be deformed away from the Glauber region at low $p_T$, so the eikonal approximation applies to the spectator line of $H_1$. The soft gluons responsible for the factorization breakdown are then factorized into a soft factor, and the $k_T$ factorization is restored in the sense that $f_{i/1}$ maintains its universality. Since the soft factor is independent of hadron flavors, experimental constraints on its behavior from some processes are possible, based on which predictions for others can be made.

At low $p_T$, the dominant contribution to the hadron hadroproduction comes from the region with a small parton momentum fraction $x_1 \equiv k_1^+/p_1^+ \ll 1$, in which $k_{1T}$ is not negligible in the hard scattering either \cite{21}. Another process to which the $k_T$ factorization applies is the semi-inclusive DIS \cite{22}. In this case the small $x$ region is reached by lowering the square of the transverse momentum of the outgoing hadron with respect to the virtual photon direction. The residual soft divergence from Figs. (a) and (b) is collected by

$$T_L^{(1)} = 2\pi \lambda g^4 \int \frac{d^4l}{(2\pi)^4} \frac{2(p_1^+ - k_1^+)(k_2 + k_4) \cdot (k_1 + k_3)\delta(l^+)}{(k_1 - l)^2(p_1 - k_1 + l)^2(2k_1^+k_2^- - |k_{1T} - k_{3T} - l_T|^2)^4} \cdots,$$  \hspace{1cm} (5)

where only the piece relevant to the soft factorization is shown explicitly. $\lambda$ and $g$ denote the triple-scalar coupling and the gluon-scalar coupling, respectively \cite{41}. The transverse loop momentum $l_T$ in the numerator, being smaller than other terms, has been dropped \cite{21, 22}. The Glauber region involved in $T_L^{(1)}$ is defined by $l^+ = (l^+, l_T, l_T) \sim (0, \Lambda^2/E, \Lambda)$ with the center-of-mass energy $E$ and the small hadronic scale $\Lambda \sim k_{1T}$. Because of $l^2 = -l_T^2$, the diagrams with real gluon emissions were not included in Fig. 1 \cite{8}.  

FIG. 1: Some one-loop diagrams relevant to the factorization of $f_{i/1}$ in hadron hadroproduction. Hermitian conjugates of these graphs also contribute. Particles in the above diagrams do not carry colors in an abelian gauge group. In the unpolarized case, both the solid and dashed lines represent scalars. In the case of single-spin symmetry, the solid lines represent fermions and the dashed lines represent scalars.
In the Glauber configuration the spectator propagator \(1/(p_1 - k_1 + l)^2\) can not be eikonalized into \(1/[2(p_1 - k_1) \cdot l]\), since the term from the transverse momentum \(|k_{1T} - l_r|^2 \sim \Lambda^2\), being of the same order as \((p_1^+ - k_1^+)l^- \sim \Lambda^2\), should be kept. To make our statement explicit, we consider the two poles in the \(l^-\) complex plane for Eq. (5),

\[
l^- = k_1^- - \frac{|l_T - k_{1T}|^2}{2k_1^+} + i\varepsilon, \quad l^- = k_1^- + \frac{|l_T - k_{1T}|^2}{2(p_1^+ - k_1^+)} - i\varepsilon, \quad (6)
\]

with \(k_1^- \sim \Lambda^2/E\) being demanded by the on-shell condition \((p_1 - k_1)^2 = 0\). For a large \(x_1\), i.e., for \(k_1^- \sim p_1^+\), these two poles in different half planes are close to each other, such that the contour of \(l^-\) is pinched and must go through the region of \(l^- \sim \Lambda^2/E\) as shown in Fig. 2(a). Therefore, \(|k_{1T} - l_r|^2\) and \((p_1^+ - k_1^+)l^-\) are of the same order, and the eikonal approximation can be justified for the spectator line in \(H_1\). To verify our argument, we compare the result from Eq. (6) and that from the simplified integral with the eikonal approximation, \(2(p_1^- - k_1^-)/(p_1 - k_1 + l)^2 \approx 1/l^-\),

\[
T_L^{(1eik)} = \frac{2\pi i\lambda g^4}{(2\pi)^4} \int \frac{d^4 l_T}{(2\pi)^4} \frac{1}{|k_{1T} - l_r|^2 - 2k_1^+ k_3^-} \frac{(k_2 + k_4) \cdot (k_1 + k_3)}{(k_2 + k_4) \cdot (k_1 + k_3) \delta(l^+)} \cdots. \quad (7)
\]

For a small \(x_1\) or \(k_1^+ \ll p_1^+\), the first pole moves away from the origin and becomes located at \(l^- \sim \Lambda^2/k_1^+\), while the second one remains of \(O(\Lambda^2/E)\). One can then deform the contour of \(l^-\) in the complex plane, so that the region of \(l^- \sim \Lambda^2/E\) is avoided as shown in Fig. 2(b). We then have the hierarchy \((p_1^+ - k_1^+)l^- \sim (p_1^+ / k_1^+)\Lambda^2 \gg |k_{1T} - l_r|^2 \sim \Lambda^2\), and the eikonal approximation can be justified for the ladder diagrams. The diagrams with the Glauber gluons attaching to rung gluons generate the residual soft divergences. It has been known that the region with strong rapidity ordering for the ladder diagrams gives a dominant contribution, which has been summed into the Balitsky-Fadin-Kuraev-Lipatov evolution equation [24]. The strong rapidity ordering corresponds to \(k_1^- \ll p_1^+\) in the model discussed here, under which the two poles in Eq. (6) are far apart from each other, and the eikonal approximation holds. We stress that to justify the eikonal approximation, \(k_1^+\) needs not to be as small as \(k_{1T}\). Therefore, ordinary soft gluons exchanged between the active partons of, say, \(H_1\) and \(H_2\), can be still factorized using the eikonal approximation. In this case the parton, after emitting an ordinary soft gluon of the momentum \(l'' \sim (\Lambda, \Lambda, \Lambda)\), carries the momentum \(k_1 - l\). We then have the hierarchy \(k_1^+ l^- \gg |k_{1T} - l_r|^2\) for \(k_1^+ \gg k_{1T}, l_r\), which justifies the eikonal approximation \(1/(k_1 - l)^2 \approx 1/(2k_1 \cdot l)\).
One may point out that the Wilson line for a TMD parton density should be rotated away from the light cone in order to regularize the light-cone singularity [25, 26]. Below we examine the effect of replacing the Wilson line direction \( a^\mu = (0, 1, 0_T) \) by \( n^\mu = (n^+, n^-, 0_T) \) with \( n^2 < 0 \). The three denominators in Eq. (5), containing the terms \( 2(l^+ - k^+_1)l^- \), \( 2(l^+ + p^+_1 - k^+_1)l^- \), and \( 2l^+l^- \), lead to three poles in different \( l^- \) half planes for the range \( 0 > l^+ > -(p^+_1 - k^+_1) \):

\[
l^- = k^-_1 + \frac{|l^- - k^-_{1T}|^2}{2(l^+ - k^+_1)l^-} + i\epsilon, \tag{10}
\]

\[
l^- = k^-_1 + \frac{|l^- - k^-_{1T}|^2}{2(l^+ + p^+_1 - k^+_1)l^-} - i\epsilon, \tag{11}
\]

\[
l^- = \frac{p^+_1}{2l^+} + i\epsilon. \tag{12}
\]

Solving Eqs. (10), (11), and (12) together with the constraint \( n^+l^- + n^-l^+ = 0 \) from the \( \delta \)-function \( \delta(n \cdot l) \), we obtain

\[
t^+ = \frac{1}{2} \left[ k^+_1 - k^-_1 \frac{n^+}{n} - \sqrt{\left(k^+_1 + k^-_1 \frac{n^+}{n}\right)^2 - 2\frac{n^+}{n}|l^- - k^-_{1T}|^2}\right], \tag{13}
\]

\[
t^+ = \frac{1}{2} \left[ k^+_1 - p^+_1 - k^-_1 \frac{n^+}{n} + \sqrt{\left(p^+_1 - k^-_1 \frac{n^+}{n}\right)^2 - 2\frac{n^+}{n}|l^- - k^-_{1T}|^2}\right], \tag{14}
\]

\[
t^+ = -\sqrt{\frac{n^+}{2n^-}l_T}, \tag{15}
\]

respectively. It is then clear that the choice \( n^2 < 0 \) prevents complex solutions. We have checked that no solution of \( l^+ \) exists in the range \( k^+_1 > l^+ > 0 \), for which the three poles of \( l^- \) are also located in different half planes. Equations (10) and (13) imply the pole \( l^- \sim \Lambda^2/k^+_1 + i\epsilon \), if \( n^+ \) and \( n^- \) are of the same order of magnitude. Equations (11) and (14) and (12) imply the pole \( l^- \sim \Lambda^2/E - i\epsilon \) \( (l^- \sim \Lambda + i\epsilon) \). Hence, the contour of \( l^- \) can be deformed, such that \( l^- \) remains of \( O(\Lambda^2/k^+_1) \) in the contour integration. That is, the eikonal approximation for the spectator line of \( H_1 \) is justified even under the variation of the Wilson line direction.

We sum the residual soft corrections to all orders in the impact parameter space. The factors \( 1/|k^-_{1T} - l^-| \) and \( (k_2 + k_4) \cdot (k_1 + k_3)/(2k^-_1k^-_3 - |k^-_{1T} - k^-_{3T} - l^-|^2) \) are absorbed into the LO TMD parton density \( f_{i/1}^{(0)} \) and the LO parton-level differential cross section \( d\sigma_{ij}^{(0)} \), respectively. The contribution from Figs. 11a and 11b is then factorized into the convolution

\[
T_L^{(i)} \approx -i\frac{g^2}{(2\pi)^2} \int \frac{d^2l_T}{l_T} d\sigma_{ij+\rightarrow k+i}^{(0)}(k^-_{1T} - l^-_T - k^-_{3T}, k^-_{1T} - k^-_{3T}) f_{i/1}^{(0)}(k^-_{1T} - l^-_T, k^-_{1T}) \cdots, \tag{16}
\]

where the soft divergence from \( l^-_T \rightarrow 0 \) is obvious. The argument \( k^-_{1T} - l^-_T \) \( (k^-_{1T}) \) in \( f_{i/1}^{(0)} \) labels the parton transverse momentum in the hadron \( H_1 \) before (after) the final-state cut. The two arguments in \( d\sigma_{ij+\rightarrow k+i}^{(0)} \) indicate that two active partons participate the hard scattering actually: one parton corresponds to the valence scalar of \( H_1 \), and another to the Glauber gluon. Equation (16) is rewritten, in the impact parameter space, as

\[
T_L^{(i)} \approx \int \frac{d^2b_d}{2\pi^2} \frac{d^2b'_d}{2\pi^2} d^2b'_{d'} [-iS(b)] d\sigma_{ij+\rightarrow k+i}^{(0)}(b_d - b'_d, b_r - b'_r) f_{i/1}^{(0)}(b'_d, b'_r) e^{i\kappa_{3T}(b_d - b_r)} e^{-i\kappa_{3T}(b_d - b_r + b'_d + b'_r)} \cdots, \tag{17}
\]

with the one-loop soft factor

\[
S(b) = \frac{g^2}{(2\pi)^2} \int \frac{d^2l_T}{l_T^2} e^{-il_T \cdot b}. \tag{18}
\]

The factor \( -i \) has been made explicit, so that the soft factor \( S \) defined in Eq. (18) is real.

Here we give more explanation to the notations employed in Eq. (17). The variables \( b'_d \) and \( b'_r \) denote the transverse coordinates of the partons coming out of \( H_1 \), namely, the lower ends of the virtual gluons, before and after the final-state cut, respectively. The definition of \( f_{i/1} \) is given by the standard matrix element of the nonlocal operator

\[
\hat{f}_{i/1}(b'_d, b'_r) = \int \frac{dy^-}{2\pi} e^{-iy^+_1 p^+_1 y^-} \langle H_1 | \phi^+_1(y^-, b'_r)| W^-(y^-, b'_r; \infty)^\dagger W^-(0, b'_d; \infty) \phi_1(0, b'_d)| H_1 \rangle, \tag{19}
\]
where the parton momentum fraction $x_1$ has been omitted in Eq. (17). The factor $W_-$ denotes the Wilson line operator

$$W_-(y^-, b; \infty) = P \exp \left[ -i g \int_0^\infty d\lambda u_- \cdot A(y + \lambda u_-) \right], \quad (20)$$

with the coordinate $y = (0, y^-, b)$. It should be understood that the two Wilson lines $W_-(y^-, b'_i)\dagger$ and $W_-(0, b'_i)$ are connected by a link at infinity $[27]$. The translational invariance of $\hat{f}_{i/1}$ demands that it depends only on the difference $b'_i - b'_j$. The above notations correspond to the choice of the triple-scalar vertex in $H_1$ as the origin of the transverse coordinates. Choices of other vertices as the origin are certainly allowed. Accordingly, $b_i$ and $b_r$ denote the transverse coordinates of the upper ends of the virtual gluons before and after the final-state cut, respectively. The factor $S(b_i)$ in Eq. (17) then describes the gluon propagation in the transverse plane from the triple-scalar vertex to the upper end of the virtual gluon before the final-state cut.

If the soft gluons appear on the right-hand side of the final-state cut, we have

$$T_R^{(1)} \approx \int d^2 b_1 d^2 b_2 d^2 b'_1 d^2 b'_2 d\sigma^{(0)}_{i+j-k+i+1}(b_l - b'_i, b_r - b'_j) [i S(b_r)] \hat{f}_{i/1}^{(0)}(b'_i, b'_j) \times e^{i k_{iT} \cdot (b_1 - b_i)} e^{-i k_{3T} \cdot (b_1 - b_r - b'_i + b'_j)} \ldots. \quad (21)$$

At next-to-next-to-leading order, we have two radiative gluons either on each side or on the same side of the final-state cut. The former case leads to

$$T_L^{(1)} \approx \int d^2 b_1 d^2 b_2 d^{2} b'_1 d^{2} b'_2 [-i S(b_i)] d\sigma^{(0)}_{i+j-k+i+1}(b_l - b'_i, b_r - b'_j) [i S(b_r)] \hat{f}_{i/1}^{(0)}(b'_i, b'_j) \times e^{i k_{iT} \cdot (b_1 - b_i)} e^{-i k_{3T} \cdot (b_1 - b_r - b'_i + b'_j)} \ldots. \quad (22)$$

For the latter with the two radiative gluons on the left-hand side of the final-state cut, we derive the factorization of the residual soft divergence

$$T_L^{(2)} \approx \int d^2 b_1 d^2 b_2 d^{2} b'_1 d^{2} b'_2 \left[ -i \frac{g^2}{(2\pi)^2} \right]^2 \left[ \frac{d^2 l_{1T}}{l_{1T}^2} \right] \left[ \frac{d^2 l_{2T}}{l_{2T}^2} \right] d\sigma^{(0)}_{i+j-k+i+1}(k_{1T} - l_{1T} - l_{2T} - k_{3T}, k_{1T} - k_{3T}) \times \hat{f}_{i/1}^{(0)}(b'_i, b'_j) \ldots, \quad (23)$$

which can be rewritten as

$$T_L^{(2)} \approx \int d^2 b_1 d^2 b_2 d^{2} b'_1 d^{2} b'_2 \frac{1}{2} \left[ -i S(b_i) \right]^2 d\sigma^{(0)}_{i+j-k+i+1}(b_l - b'_i, b_r - b'_j) \hat{f}_{i/1}^{(0)}(b'_i, b'_j) \times e^{i k_{iT} \cdot (b_1 - b_i)} e^{-i k_{3T} \cdot (b_1 - b_r - b'_i + b'_j)} \ldots. \quad (24)$$

To achieve at Eq. (23) from the summation of all the relevant two-loop diagrams, the following relation from the eikonal approximation for the spectator line of $H_1$ must be employed

$$\frac{1}{l_1^2 + i\epsilon (l_1 + l_2^2)} + \frac{1}{l_2^2 + i\epsilon (l_1 + l_2^2)} = \frac{1}{l_1^2 + i\epsilon l_2^2} + \frac{1}{i\epsilon}. \quad (25)$$

On the left-hand side of Eq. (25), the second term is obtained by exchanging the gluon carrying the momentum $l_1$ and the gluon carrying $l_2$ in the first term. The above relation will not hold, if the corresponding transverse loop momenta are retained in the denominators. That is, the all-order factorization of the soft gluons from the process is not possible without the eikonal approximation.

Viewing Eqs. (23) and (24), it is clear why the all-order summation of the residual soft divergences can be facilitated in the $b$ convolution, instead of in the $k_T$ convolution. Applying the above procedure to higher loops, the diagrams with gluons being emitted by the spectator of the hadron $H_1$ and attaching to the active partons of $H_2$ and $H_4$ are summed into

$$T \approx \int d^2 b_1 d^2 b_2 d^{2} b'_1 d^{2} b'_2 e^{-i S(b_i)} d\sigma_{i+j-k+i+1}(b_l - b'_i, b_r - b'_j) e^{i S(b_r)} \hat{f}_{i/1}(b'_i, b'_j) \times e^{i k_{iT} \cdot (b_1 - b_i)} e^{-i k_{3T} \cdot (b_1 - b_r - b'_i + b'_j)} \ldots. \quad (26)$$
The collinear gluon exchanges of the type in Fig. (1c) can be factorized in the standard way, so the definitions of both $d\sigma_{i+j\rightarrow i+1}$ and $\delta_{i+1}$ have been extended to all orders. When the final-state hadron pair carries a net large transverse momentum, the $k_{T}$ dependence in the fragmentation functions are negligible. Integrating over $k_{1T}$, the $\delta$-function $\delta(b_{1} - b_{r})$ renders the soft factor vanish, and Eq. (26) reduces to a formula in the collinear factorization [28].

The operator definition of the soft factor is given by

$$e^{-is(b)} = \langle 0 | W_{-}(0, b; \infty) | W_{+}(0, 0; \infty) | W_{+}(0, 0; \infty) | 0 \rangle,$$

where the factor $W_{+}$ denotes another Wilson line operator

$$W_{+}(y^{+}, b; \infty) = P \exp \left[ -ig \int_{0}^{\infty} d\lambda u_{+} \cdot A(y + \lambda u_{+}) \right],$$

with the coordinate $y = (y^{+}, 0, b)$ and the dimensionless vector $u_{+} = (1, 0, 0)$. The net effect of $W_{-}(0, b; \infty) | W_{+}(0, 0; \infty)$ and $W_{+}(0, 0; \infty) | W_{+}(0, 0; \infty)$ demands the vanishing of the components $l^{+}$ and $l^{-}$ of the loop momentum, respectively. A Glauber gluon is then off-shell by $l_{T}^{2}$ as indicated in Eq. (18). It can be shown, by expanding the Wilson line operators order by order, that Eq. (27) reproduces the Feynman rules obtained here for the soft factor.

To derive the differential cross section, we integrate the momentum conservations $\delta(k_{1}^{+} - k_{2}^{+} - k_{3}^{+} - k_{4}^{+})$, $\delta(k_{2}^{+} - k_{3}^{+} - k_{4}^{+})$, and $\delta(k_{1T} + k_{2T} - k_{3T} - k_{4T})$ over $k_{1}^{+}$, $k_{2}^{+}$, and $k_{3T}$, respectively, and the on-shell conditions $\delta(k_{3}^{0})$ and $\delta(k_{2}^{0})$ over $k_{3}^{0}$ and $k_{4}^{0}$, respectively. The longitudinal parton momenta $k_{1}^{0}$ and $k_{2}^{0}$ are then related to $k_{3}$ and $k_{4}$. Substituting $k_{1T} = k_{3T} + k_{4T} - k_{2T}$ into the Fourier factor $\delta^{(1)}(k_{1T} - b_{r} - b_{s})$, and integrating over $k_{2T}$, $\delta_{j/2}(k_{2T})$ is transformed into the impact parameter space. At last, we arrive at the factorization formula modified by the soft factor associated with the hadron $H_{1}$

$$E_{3}E_{4} \frac{d\sigma}{d^{3}p_{3}d^{3}p_{4}} = \sum_{k_{3}} \int \frac{d^{3}k_{3} d^{3}k_{4}}{|k_{3}| |k_{4}|} q_{2} b_{2} d^{2}b_{r} d^{2}b_{s} \delta^{(1)}(k_{1T} - b_{r} - b_{s}) e^{-is(b_{r})} \delta_{i+j-k-i+1} \frac{d\sigma_{i+j-k-i+1}^{(1)}}{d^{3}p_{3}d^{3}p_{4}}(b_{1} - b_{2} - b_{r} - b_{s}) e^{is(b_{r})},$$

where the dependence on $k_{3}$ and $k_{4}$ in $\delta_{i+j-k-i+1}$ is not shown explicitly. The residual soft divergences associated with the hadron $H_{2}$ can be analyzed similarly, which are not discussed in this work.

Note that the soft factor in Eq. (27) does not carry the flavor indices $i$, $j$, $k$, and $l$, and is independent of the species of hadrons involved in the collision. This universality makes possible experimental constraints on its behavior from some processes (e.g., $HH \rightarrow \pi^{+}X$), and predictions from the $k_{T}$ factorization for other processes (e.g., $HH \rightarrow KK + X$). Equation (28) can also be employed to study the effect from the factorization breakdown: one can compare the numerical difference between the predictions from Eq. (28) and from the formula without the soft factor. The soft factor derived above differs from that in the simple Drell-Yan process [10], for which the $k_{T}$ factorization has been justified [13, 29]. That is, the infrared divergences studied in [11] arise from the ordinary (not Glauber) soft region, and can always be collected by means of the eikonalization. The appearance of the soft factor in [10] is attributed to the incomplete infrared cancellation between virtual corrections, where loop momenta do not flow through hard scattering, and real corrections, where loop momenta do.

We next discuss the failure of the $k_{T}$ factorization for the transverse single-spin asymmetry (SSA) in hadron-hadron collision [8, 29, 33] with $H_{1}$ being the transversely polarized hadron. For the SSA, the parton transverse momentum must be taken into account, and the imaginary part of the polarized TMD parton density contributes. Adopting a similar model field theory [34], which contains additional fermion fields, the $k_{T}$ factorization for the SSA was also shown to fail [7]. The mechanism is identical to that in the unpolarized hadron hadroproduction. This is the reason why the definitions of the TMD parton densities in the SSA and unpolarized cases were modified by including the same additional Wilson links in [3]. The sum of these additional Wilson links leads to the $\delta$-function in Eq. (3), which breaks the $k_{T}$ factorization at one loop.

The loop integral associated with the sum of Figs. (a) and (b) for the SSA, where the solid lines represent fermions, is written as

$$S^{(1)}_{L} = 2\pi \lambda \gamma \int \frac{d^{4}l}{(2\pi)^{4}} \frac{(k_{2} + k_{4}) \cdot (k_{1} + k_{3}) \delta(l^{+})}{(k_{3} - l)^{2}(p_{1} - k_{1} + l)^{2} l^{2} (2k_{1}^{+} k_{3}^{+} - |k_{1T} - k_{3T}| - l_{T}^{2})},$$

$$\times \frac{1}{2} Tr[(\not{p}_{1} + m_{H}) \gamma_{5} \not{s}(\not{p}_{1} - k_{1} + l) \gamma^{+}(\not{p}_{1} - k_{1})].$$

The spin vector $s$ is chosen in the transverse direction, and $m_{H}$ denotes the mass of the hadron $H_{1}$. The trace in the above expression gives, in the small $k_{1}$ region,

$$\frac{1}{2} Tr[(\not{p}_{1} + m_{H}) \gamma_{5} \not{s}(\not{p}_{1} - k_{1} + l) \gamma^{+}(\not{p}_{1} - k_{1})] \approx 2im_{H} \epsilon_{\alpha\beta}s^{\alpha}l^{3}p_{1}^{+},$$

(31)
where the $\epsilon$ tensor obeys $\epsilon_{12} = 1$, and $l^\beta$ picks up the transverse components. Working out the integrations over $l^+$ and then over $l^-$ in Eq. (30), the denominator $l^2(l_1 - l)^2$ becomes $l_1^2 |l_T - k_{1T}|^2$. Due to the existence of $l^\beta$ in Eq. (31), a residual soft divergence arises from the region with $l_T \rightarrow k_{1T}$ in this case.

Following the same reasoning, we may be able to eikonalize the spectator quark line of $H_1$ in the Glauber region

$$\frac{2p_1^+}{(p_1 - k_1 + l)^2 + ie} \approx \frac{p_1^+}{p_1 \cdot l + ie} = \frac{1}{l^+ + ie}, \quad (32)$$

where the numerator $2p_1^+$ comes from Eq. (31). However, the rest of the procedure for factorizing the residual soft divergence in the unpolarized hadron hadroproduction does not apply: the propagator $1/|l_T - k_{1T}|^2$ can not be absorbed into the LO polarized TMD parton density, which vanishes with the fermion trace

$$\frac{1}{2} Tr[(\not p_1 + m_H)\gamma_5 \not s(\not p_1 - \not k_1)] = 0. \quad (33)$$

It implies that the factorization of the residual soft divergence in Eq. (32) into the convolution of the one-loop soft factor, the LO polarized TMD parton density, and the LO hard kernel is not possible even at low $p_T$ in this model field theory. In QCD, the corresponding TMD parton density, namely, the Sivers function, also vanishes at LO in perturbation theory. Therefore, the universality of the Sivers function can not be restored by factorizing out a soft factor.

Our observation obtained in this work has wide applications. For example, the same residual soft divergences from the Glauber region have been identified in the color-suppressed tree amplitudes [35] of two-body nonleptonic $B$ meson decays in the perturbative QCD (PQCD) approach, which is based on the $k_T$ factorization [36–40]. A soft factor has been introduced into the PQCD factorization formulas, which enhances the color-suppressed tree amplitudes significantly, such that the known $\pi\pi$ and $\pi K$ puzzles were resolved [35].

In a forthcoming paper we shall discuss the $k_T$ factorization of the residual soft divergences from low-$p_T$ hadron hadroproduction in real QCD, where the spectator lines in Fig. 1 are replaced by infinitely many rung gluons. A difference arises from additional color degrees of freedom of quarks and gluons. Equation (3) is then a consequence of the summation over the attachments of the radiative gluon to the active parton lines in $H_2$ and $H_4$ and to the hard gluon line. To employ Eq. (25) in QCD, the diagrams with triple gluon vertices should be included too. We shall demonstrate, with the help of the eikonalization and the Ward identity, that the breakdown of the universality of a TMD parton density is suppressed in the large $N_c$ limit, $N_c$ being the number of colors, after summing over the attachments to all rung gluons. The $k_T$ factorization is also applicable to the $W$ boson plus jet production and the direct photon production, for which the momentum $k_2$ ($k_3$) is carried by a partonic gluon (gauge bosons). In these processes the $\delta$-function leading to the residual soft divergence appears after summing over the attachments to the partonic gluon, the quark carrying the momentum $k_4$, and the virtual quark.

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