Bogoliubov Dynamics and Higher-order Corrections for the Regularized Nelson Model

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October 4, 2021

Abstract

We study the time evolution of the Nelson model in a mean-field limit in which \(N\) nonrelativistic bosons weakly couple (w.r.t. the particle number) to a positive or zero mass quantized scalar field. Our main result is the derivation of the Bogoliubov dynamics and higher-order corrections. More precisely, we prove the convergence of the approximate wave function to the many-body wave function in norm, with a convergence rate proportional to the number of corrections taken into account in the approximation. We prove an analogous result for the unitary propagator. As an application, we derive a simple system of PDEs describing the time evolution of the first- and second-order approximation to the one-particle reduced density matrices of the particles and the quantum field, respectively.

MSC class: 35Q40, 35Q55, 81Q05, 81T10, 81V73, 82C10

1 Introduction

The Nelson model describes the interaction between a relativistic scalar field and nonrelativistic particles. Since its introduction in the community of mathematical physics by E. Nelson [53] it has served as a playground for a rigorous study of many features of quantum field theory, in a simple but nonetheless realistic model. It was originally introduced to study the interaction of nonrelativistic spinless nucleons with a scalar meson field but has also been used in condensed matter physics and quantum optics to describe either particles in interaction with phonons in a crystal, or particles interacting with radiation (in a linearized approximation). In this paper, we focus on the regular Nelson Hamiltonian, whose interaction is cut off for high momenta by an ultraviolet regularization, as this analysis serves as a starting point for the more interesting case of a renormalized interaction without ultraviolet regularization.

The Nelson model has been studied from different points of view (see, e.g., [1, 16, 27, 57, 44, 48, 35] and references therein). Here, we consider \(N \gg 1\) nonrelativistic particles coupled to the quantum field, and focus on its effective description in a regime in which the nonrelativistic particles are close to a Bose–Einstein condensate and the scalar field is macroscopic, and thus behaving classically (with \(N^{-1/2}\) quantifying the “degree of quantumness”, as a semiclassical \(\hbar\) parameter). This regime has already been studied using semiclassical techniques [2, 3], Fock space methods [18] or a combination...
of the coherent state approach and the method of counting [39]. In [15, 32, 60] it has, moreover, been shown that the effect of the quantum field on the particles can in certain situations be approximated by a direct pair interaction. Let us also mention further works that focus on related models [37, 38, 40], a partial limit [9, 12, 13, 14] and the strong coupling limit of the polaron [19, 20, 21, 26, 36, 41, 46].

In this paper we take a different route, drawing inspiration by a series of works by several authors [4, 8, 10, 22, 23, 24, 25, 28, 29, 30, 31, 34, 42, 47, 49, 50, 51, 52, 55, 58] on systems of many nonrelativistic bosons in direct pair interaction. In particular, we follow the route from [5] (see also [6] for a related result in the static setting). We construct a Bogoliubov theory, and next-order corrections, for the Nelson Hamiltonian in the mean-field/semiclassical regime described above. In addition to the inherent interest of building up such a theory for this model, this also allows to strengthen the previously available propagation of chaos results from weak-* to strong convergence, at least for a suitable class of initial states: the wave functions converge, at any time and with an explicit rate of convergence, in Hilbert space norm and not only through the expectation of suitable observables (of course, provided that strong convergence holds at a given initial time). Moreover, we provide a series expansion that approximates the two-parameter group generated by the Nelson Hamiltonian on a subspace of the excitation Fock space in the strong sense to arbitrary precision. As an application we show that the time evolution of the first and second-order approximation of one-particle and one-field boson reduced density matrices are determined by a self-contained system of PDEs.

The article is organized as follows. In the rest of this section we introduce the main notations and mathematical definitions used throughout the paper, providing a more detailed introduction to the problem at hand. In Section 2 we state the main convergence results of this paper, viz., Theorems 2.1, 2.5, and 2.6. In Section 3 we discuss the usefulness of our results by explaining how Bogoliubov theory can be used for computations, and by deriving the next-order PDEs for the reduced density matrices. Section 4 discusses the Bogoliubov dynamics, and its well-posedness, following mainly [18, 42, 49]. Finally, in Section 5 we provide the technical details that lead to the proof of Theorems 2.1, 2.5, and 2.6.

We assume that the reader is familiar with the basic objects of Fock spaces such as creation and annihilation operators, the number operator, and sectors with a fixed number of particles. For an introduction to the topic we refer to [17].

1.1 Basic definitions and notations

The Nelson model we are considering is set up as follows. Let us consider a fixed number $N$ of nonrelativistic bosons, coupled to a quantized scalar field with an ultraviolet-regular interaction. The Hilbert space of such a theory is

$$H_N = (L^2(\mathbb{R}^3))^\bigotimes N \otimes F,$$

where $F$ is the bosonic Fock space over $L^2(\mathbb{R}^3)$, and $\otimes_s$ is the symmetric tensor product. Any wave function $\Psi_N(t)$, where we have made explicit the dependence on the number of nonrelativistic particles $N$ that we are going to choose very large, evolves with the Schrödinger equation

$$i\partial_t \Psi_N(t) = H_{N,\text{Nelson}} \Psi_N(t),$$

generated by the cutoff Nelson Hamiltonian with mean-field scaling,

$$H_{N,\text{Nelson}} = -\sum_{j=1}^N \Delta_j + \int dk \omega(k)a^*(k)a(k) + N^{-1/2} \sum_{j=1}^N \Phi(x_j).$$

Here, $a^*$ and $a$ are the bosonic creation and annihilation operators satisfying the canonical commutation relations

$$[a(k), a^*(\ell)] = \delta(k-\ell), \quad [a(k), a(\ell)] = 0 = [a^*(k), a^*(\ell)].$$
and the field operator is defined as

$$\hat{\Phi}(x) = \int dk \eta(k)e^{-2\pi ikx} \left( a^*(k) + a(-k) \right)$$  \hspace{1cm} (5)

with

$$\eta(k) = \frac{g(k)}{\sqrt{2\omega(k)}},$$  \hspace{1cm} (6)

for some real cutoff function $g$, satisfying $g(k) = g(-k)$ and $\eta \in L^2$, with dispersion $\omega(k) = \sqrt{k^2 + m^2}$; $m \geq 0$. We also assume $\frac{g}{\sqrt{\omega}} \in L^2$, which makes $H^N_{Nelson}$ self-adjoint on $D(H^N_{Nelson,0})$, where $H^N_{Nelson,0} = -\sum_{j=1}^{N} \Delta_j + \int dk \omega(k)a^*(k)a(k)$ is the non-interacting part of the Nelson Hamiltonian.

In order to approximate the time evolution of the Nelson model we introduce the classical field

$$\Phi(t,x) = \int dk \eta(k)e^{-2\pi ikx} \left( \alpha(t,k) + \alpha(t,-k) \right)$$  \hspace{1cm} (7)

and the effective equations

$$i\partial_t u(t) = (-\Delta + \Phi(t,\cdot))u(t),$$  \hspace{1cm} (8a)

$$i\partial_t \alpha(t) = \omega\alpha(t) + \eta|u(t)|^2,$$  \hspace{1cm} (8b)

with $u(t), \alpha(t) \in L^2(\mathbb{R}^3)$ for all $t$ (and where $|u(t)|^2$ denotes the Fourier transform of $|u(t)|^2$). Here, $\alpha$ and $\hat{\alpha}$ should be considered as the classical counterparts of $a$ and $a^*$. These equations are called Schrödinger–Klein–Gordon equations, since taking a second time derivative of Equation (8b) yields

$$(\partial_t^2 - \Delta + m^2)\Phi(t) = -\tilde{g}^2 \ast |u(t)|^2,$$  \hspace{1cm} (9)

a Klein–Gordon equation with source term ($g^2$ denotes the inverse Fourier transform of $g^2$).

Given a solution $(u(t), \alpha(t))$ of the Schrödinger–Klein–Gordon equations, let us define the wave function $\varphi(t)$ by

$$\varphi(t,x) = e^{i\mu(t)}u(t,x),$$  \hspace{1cm} (10)

with the time-dependent phase $\mu(t)$ defined by

$$\mu(t) := \frac{1}{2} \int dx \Phi(t,x)|u(t,x)|^2.$$  \hspace{1cm} (11)

The Schrödinger–Klein–Gordon system is globally well-posed in several function spaces. In this work, we rely on the following well-posedness lemma. For $\ell \in \mathbb{N}$, let $H^\ell(\mathbb{R}^3)$ be the Sobolev space of order $\ell$ and $L^2_\ell(\mathbb{R}^3)$ be a weighted $L^2$-space with norm $||\alpha||_{L^2_\ell(\mathbb{R}^3)} = ||(1 + |\cdot|^2)^{\ell/2}||_{L^2(\mathbb{R}^3)}$

**Lemma 1.1.** Let $(u_0, \alpha_0) \in H^2(\mathbb{R}^3) \times L^2_2(\mathbb{R}^3)$. Then there is a continuous map $t \mapsto (u(t), \alpha(t))$ from $\mathbb{R}$ to $H^2(\mathbb{R}^3) \times L^2_2(\mathbb{R}^3)$ that satisfies (8a)-(8b) with initial condition $(u(0), \alpha(0)) = (u_0, \alpha_0)$.

In [18], Appendix B ($N = \delta = 1$), Lemma [14] was proved for $g(k) = 1_{|k| \leq \Lambda}(k)$ with $\Lambda > 0$. The proof, however, can easily be extended to a larger class of cutoff functions. We remark that a similar result was shown in [18], and we refer the interested reader to [11, 54] for a dedicated analysis of well-posedness for the Schrödinger–Klein–Gordon system without ultraviolet cutoff.

For convenience, let us denote

$$h(t) = -\Delta + \Phi(t,\cdot) - \mu(t),$$  \hspace{1cm} (12)

$^1$The choice of a real cutoff function $g$ is taken only to simplify some formulas in the presentation. The generalization to an arbitrary complex $g$ is straightforward.
Moreover, let us define the Weyl operator

\[ W(f) = \exp \left( \int dk \left( f(k) a^*(k) - f(k) a(k) \right) \right) \]

whose action on the vacuum of \( \mathcal{F} \) creates a coherent state with mean particle number \( \|f\|_{L^2_{\mathbb{R}^3}}^2 \) (see, e.g., [12] for a more detailed introduction).

In this work, we are interested in the evolution of a Bose–Einstein condensate of particles with initial condensate wave function \( \phi_0 \) and a coherent state \( W(\sqrt{N} \phi_0) |\Omega\rangle \) of field bosons with mean particle number \( N \|\phi_0\|_{L^2_{\mathbb{R}^3}}^2 \) (here, \( |\Omega\rangle \) denotes the vacuum vector). We will prove the persistence of the condensate and the coherent structure of the field during the time evolution and show that they are described by \( (\phi(t), \alpha(t)) \) evolving according to (13a) and (13b) with initial condition \( (\phi(t), \alpha(t))|_{t=0} = (\phi_0, \alpha_0) \). Moreover, we will give an explicit description of the fluctuations around the Schrödinger–Klein–Gordon equations. For this purpose it is convenient to embed the Hilbert space of the particles into a second Fock space, to factor out the condensate as well as the coherent state and to look at the corresponding excitation Fock spaces (for a similar strategy without second quantization see [20, 12]).

The excitation Fock space of the particles is given by

\[ \mathcal{F}_b = \bigoplus_{k=0}^{\infty} \mathcal{F}_b^{(k)} \quad \text{with} \quad \mathcal{F}_b^{(k)} = (\phi(t)^\perp)_{\otimes, k}. \]  

It describes bosonic particles, each with a wave function orthogonal to the reference state \( \phi(t) \). The excitation space for the phonons is defined in a slightly different fashion as

\[ \mathcal{F}_a = W^* (\sqrt{N} \alpha(t)) \mathcal{F}_b. \]  

This space is a unitary “rotation” of the original space; however, it is the coherent state \( W(\sqrt{N} \alpha(t)) |\Omega\rangle \), \( |\Omega\rangle \) being the Fock vacuum vector, that now plays the role of a new vacuum \( |\Omega_a\rangle \) for \( \mathcal{F}_a \). The combined excitation space is then given by the double Fock space

\[ \mathcal{G} = \mathcal{F}_b \otimes \mathcal{F}_a. \]  

Let us remark that, even though we omit this in our notation, both spaces \( \mathcal{F}_b \) and \( \mathcal{F}_a \) depend on time. Let us denote the creation and annihilation operators on \( \mathcal{F}_b^{(k)} \) by \( b^*(x) \) and \( b(x) \), and on \( \mathcal{F}_a \), as before, by \( a^*(k) \) and \( a(k) \). We call the respective number operators \( N_b \) and \( N_a \). We also introduce the operator that counts the total number of excitations,

\[ N = N_a + N_b. \]  

We now consider the decomposition

\[ \Psi_N(t) = W(\sqrt{N} \alpha(t)) \sum_{k=0}^{N} \phi(t)_{\otimes (N-k)} \otimes_a \chi_{\leq N}^{(k)}(t) \]  

where \( \chi_{\leq N}^{(k)}(t) \in \mathcal{F}_b^{(k)} \otimes \mathcal{F}_a \). For given \( (\phi(t), \alpha(t)) \), this establishes a unitary map\(^2\) between \( \mathcal{H}_N \) and the \( N \)-particle excitation space

\[ \mathcal{G}_{\leq N} := \left( \bigoplus_{k=0}^{N} \mathcal{F}_b^{(k)} \right) \otimes \mathcal{F}_a. \]  

\(^2\)The unitary map and some of its properties are discussed in greater detail in Appendix A.
The inverse of \( \mathcal{G} \) is
\[
\chi^{(k)}_{\leq N}(t) = \left( \frac{N}{k} \right)^{1/2} \prod_{i=1}^{k} q_i(t) \langle \varphi(t) \rangle \otimes (\mathbb{R}^{N-k}), \quad W^* (\sqrt{N} \alpha(t)) \Psi_N(t) \rangle_{L^2(\mathbb{R}^{d(N-k)})}, \quad k = 0, 1, \ldots, N, \tag{21}
\]
where we take a partial inner product w.r.t. the coordinates \( x_{k+1}, \ldots, x_N \), and \( q_i(t) = 1 - p_i(t) \) with \( p_i(t) = |\varphi(t) \rangle \langle \varphi(t)|_i \), that is, the projector onto the state \( \varphi(t) \) in the \( x_i \) coordinate. We can thus equally express the Schrödinger equation \( \ref{eq:schrodinger} \) as
\[
i \hat{\epsilon} \chi_{\leq N}(t) = H_{\leq N}(t) \chi_{\leq N}(t), \tag{22}
\]
where
\[
\chi_{\leq N}(t) = \left( \chi^{(k)}_{\leq N}(t) \right)_{k=0}^N \in \mathcal{G}_{\leq N}.
\tag{23}
\]
The Hamiltonian \( H_{\leq N}(t) \) can be written (see Appendix A) as the restriction of a Hamiltonian \( H(t) \) (defined on \( \mathcal{G} \)) to \( \mathcal{G}_{\leq N} \), i.e., \( H_{\leq N}(t) = H(t)|_{\mathcal{G}_{\leq N}} \), with\(^8\)
\[
H(t) = \int dx b^*(x) h(t) b(x) + \int dk \omega(k) a^*(k) a(k)
+ \int dx \int dk K(t, k, x) (a^*(k) + a(-k)) b^*(x) \left[ 1 - N^{-1} \eta_0 \right]^{1/2} + \text{h.c.}
+ N^{-1/2} \int dx b^*(x) \left( q(t) \hat{\Phi} q(t) - \langle \varphi(t), \hat{\Phi} \varphi(t) \rangle \right) b(x),
\tag{24}
\]
where \( [x]_+ \) denotes the positive part of \( x \), h.c. denotes the Hermitian conjugate of the preceding term, and \( q(t) \) is the operator with integral kernel
\[
q(t, x, y) = \delta(x - y) - \varphi(t, x) \bar{\varphi}(t, y).
\tag{25}
\]
Moreover,
\[
K(t, k, x) = \int dy q(t, x, y) \tilde{K}(t, k, x) \quad \text{with} \quad \tilde{K}(t, k, x) = \eta(k) e^{-2\pi i k x} \varphi(t, x),
\tag{26}
\]
and we denote by \( K(t) \) the operator with operator kernel \( K(t, k, x) \), i.e., \( K(t) = \tilde{K}(t) q(t) \) where \( \tilde{K}(t) \) has kernel \( \tilde{K}(t, k, x) \) and \( q(t) \) has the kernel given by \( \ref{eq:kernel} \). Note that if an operator \( A \) has integral kernel \( A(x, y) \), we write \( \bar{A} \) for the operator with integral kernel \( \bar{A}(x, y) \).

The Bogoliubov approximation is to disregard all terms with more than two \( a^*, a, b^* \) or \( b \) operators in \( \ref{eq:hamiltonian} \). This leads us to define the Bogoliubov Hamiltonian
\[
H_0(t) = \int dx b^*(x) h(t) b(x) + \int dk \omega(k) a^*(k) a(k)
+ \int dx \int dk K(t, k, x) (a^*(k) + a(-k)) b^*(x) + \text{h.c.}.
\tag{27}
\]
While the Hamiltonian \( H(t) \) maps \( \mathcal{G}_{\leq N} \) to itself (because of the appearance of the square root in the second line of \( \ref{eq:hamiltonian} \)), this does not hold for \( H_0(t) \). The corresponding time evolution, usually referred to as Bogoliubov equation, must therefore be defined on the double Fock space \( \mathcal{F} \). It reads
\[
i \hat{\epsilon} \chi_0(t) = H_0(t) \chi_0(t),
\tag{28}
\]
where \( \chi_0(t) \in \mathcal{G} \). Its well-posedness is discussed in Section 4.

Let us remark that for states \( \chi \) in \( \mathcal{G} \) and \( \mathcal{G}_{\leq N} \) we shall always use the notation \( \chi^{(k)} \) to indicate the component in the \( k \) particle sector w.r.t. the particle excitations, i.e.,
\[
\chi^{(k)} \in \mathcal{F}^{(k)}_p \otimes \mathcal{F}_a.
\tag{29}
\]
In particular, every \( \chi \in \mathcal{G} \) corresponds to a sequence \( (\chi^{(k)})_{k \geq 0} \).

\(^8\) For one-body operators on \( L^2(\mathbb{R}^3) \) with kernel \( A(x, y) \) we use the usual shorthand notation \( \int dx b^*(x) A(x, y) b(y) = \int dx dy b^*(x) A(x, y) b(y) \).
2 Main Results

In this section we state the main results of this paper. We start with Bogoliubov theory, and the first result on norm convergence. Then, we focus on higher-order corrections, and the refinement of the rate of convergence for initial states that admit a power series expansion in the parameter $N \gg 1$. As our last result, we provide a similar statement for the unitary propagator.

2.1 Bogoliubov Theory

Our first result shows that Bogoliubov theory approximates the microscopic dynamics well, up to an error of order $N^{-1/2}$.

Theorem 2.1. Let $\Psi_N(t)$ be the solution to the Schrödinger equation (2) with initial condition

$$\Psi_N(0) = W(\sqrt{Na}(0)) \sum_{k=0}^{N} \varphi(0) \otimes (N-k) \otimes \chi_0^{(k)}(0) \in \mathcal{H}_N,$$

where $(\varphi(0), a(0)) \in H^2(\mathbb{R}^3) \times L^2_0(\mathbb{R}^3)$ and $\chi_0(0) \in \mathcal{G}$ satisfy $\|\varphi(0)\| = 1$ and $\|\chi_0\| = 1$. We assume that there is a constant $C > 0$ such that

$$\|(N + 1)^{3/2} \chi_0(0)\| \leq C.$$  

Then there are constants $C_1, C_2 > 0$ such that

$$\|\Psi_N(t) - \Psi_N^{(0)}(t)\| \leq C_1 e^{C_2 t} N^{-1/2},$$

where

$$\Psi_N^{(0)}(t) = W(\sqrt{Na}(t)) \sum_{k=0}^{N} \varphi(t) \otimes (N-k) \otimes \chi_0^{(k)}(t),$$

with $\varphi(t), a(t)$, and $\chi_0(t)$ solutions to Equations (13a), (13b), and (28), respectively.

Remark 2.2. By density of $D(N^{3/2}) \cap \mathcal{G}$ in $\mathcal{G}$, the above statement extends to any $N$-independent state $\chi_0(0) \in \mathcal{G}$ if we omit the explicit rate of convergence in (32). More precisely, for $\Psi_N(0)$ as in (30) with $\chi_0(0)$ normalized but not necessarily satisfying (31), it follows that

$$\lim_{N \to \infty} \|\Psi_N(t) - \Psi_N^{(0)}(t)\| = 0.$$  

Remark 2.3. Since we choose $\chi_0(0) = (\chi_0^{(k)}(0))_{k \geq 0}$ normalized to one, this does not necessarily hold for $\Psi_N(0)$. However, it is easy to verify that $\|\Psi_N(t)\| \to 1$ as $N \to \infty$.

2.2 Higher-Order Corrections

Formally, one can expand the Hamiltonian (24) by using the Taylor series $\sqrt{1 - x} = \sum_{n=0}^{\infty} c_n x^n$ (see (17) for the definition of $c_n$). This yields

$$H(t) = \int dx b^* (x) h(t) b(x) + \int dk \omega(k) a^* (k) a(k)$$

$$+ \sum_{n=0}^{\infty} N^{-n} c_n \int dx \int dk K(t, k, x) (a^* (k) + a(-k)) b^* (x) \mathcal{N}_b^n + \text{h.c.}$$

$$+ N^{-1/2} \int dx b^* (x) (q(t) \Phi (x) q(t) - \langle \varphi(t), \Phi \varphi(t) \rangle) b(x)$$

$$= \sum_{\ell=0}^{\infty} N^{-\ell/2} H_\ell(t)$$

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The rigorous expansion with explicit remainder estimates is given in Lemma 5.2. In addition, we formally expand the wave function \( \chi(t) \in \mathcal{G} \) in a power series in \( N^{-1/2} \), i.e.,

\[
\chi(t) = \sum_{\ell=0}^{\infty} N^{-\ell/2} \chi_{\ell}(t), \quad \chi_{\ell}(t) \in \mathcal{G}.
\]

Later, we will only consider this power series truncated at some \( r \in \mathbb{N}_0 \), since we do not expect the series to converge. Then the Schrödinger equation

\[
i \partial_t \chi(t) = H(t) \chi(t)
\]

leads in each order in \( N^{-1/2} \) to the equations

\[
i \partial_t \chi_{\ell}(t) = H_0(t) \chi_{\ell}(t) + \sum_{m=0}^{\ell-1} H_{\ell-m}(t) \chi_m(t).
\]

Let us denote by \( U_0(t,s) \) the unitary time evolution generated by the Bogoliubov Hamiltonian \( H_0(t) \). Then Equation (38) in integral form reads

\[
\chi_{\ell}(t) = U_0(t,0)\chi_{\ell}(0) - i \sum_{m=0}^{\ell-1} \int_0^t ds U_0(t,s) H_{\ell-m}(s) \chi_m(s)
\]

\[
= U_0(t,0)\chi_{\ell}(0) - i \sum_{m=0}^{\ell-1} \int_0^t ds \tilde{H}_{\ell-m}(s,t) U_0(t,s) \chi_m(s),
\]

where

\[
\tilde{H}_m(s,t) = U_0(t,s) H_m(s) U_0(s,t).
\]

This iteration can be solved by a straightforward computation (see also [5, Proposition 3.2]). We find

\[
\chi_{\ell}(t) = U_0(t,0)\chi_{\ell}(0) + \sum_{m=0}^{\ell-1} \sum_{k=1}^{\ell-m} \sum_{|s|=\ell-m} (-i)^k \int_{\Delta_k} ds^{(k)} \prod_{i=1}^k \tilde{H}_{\alpha_i}(s_i,t) U_0(t,0) \chi_{m}(0),
\]

where we abbreviated \( ds^{(k)} = ds_1 \cdots ds_k \), and \( \Delta_k \) is the region \([0,t]^k\) with \( s_{i+1} \leq s_i \) for all \( i = 1, \ldots, k-1 \). Our main assumption on the initial data is the following.

**Assumption 2.4.** Let \( r \geq 1 \) and assume that for each \( \ell \in \{0, \ldots, r\} \), the state \( \chi_{\ell}(0) \in \mathcal{G} \) is in the domain of any power of the number operator \( \mathcal{N} \) with uniform bounds as \( N \to \infty \), that is, for all \( \ell \in \{0, \ldots, r\} \) and \( n \in \mathbb{N}_0 \) there are \( C(\ell, n) > 0 \) such that

\[
\| (\mathcal{N} + 1)^n \chi_{\ell}(0) \| \leq C(\ell, n)
\]

for all \( N \geq 1 \). Moreover, let \( (\varphi(0), \alpha(0)) \in H^2(\mathbb{R}^3) \times L^2(\mathbb{R}^3) \).

Note that naturally one would consider \( N \)-independent coefficients \( \chi_{\ell}(0) \), such that the uniformity in \( N \) in [22] is given; however, the following theorem remains true if the \( \chi_{\ell}(0) \) depend on \( N \) but satisfy (42) uniformly in \( N \). For mean-field bosons interacting via a class of two-body potentials, it was shown in [3] that low-energy eigenstates for particles in a suitable trap can indeed be expanded into a series, with \( N \)-independent \( \chi_{\ell}(0) \), and we expect that a similar result holds true for the Nelson model.

Our main result on the higher order corrections is the following.
Theorem 2.5. Let $\Psi_N(t)$ be the solution to the Schrödinger equation with initial condition $\Psi_N(0) \in \mathcal{H}_N$. Let $r \in \mathbb{N}_0$, and

$$\Psi^{(r)}_N(0) = W(\sqrt{N}\alpha(0)) \sum_{k=0}^N \varphi(0) \otimes (N-k) \otimes_s \left( \sum_{\ell=0}^r N^{-\ell/2} \chi^{(k)}_\ell(0) \right),$$

(43)

and let Assumption 2.4 hold. Then for all $\Psi_N(0)$ with

$$\|\Psi_N(0) - \Psi^{(r)}_N(0)\| \leq C_r N^{-(r+1)/2}$$

(44)

for some $C_r > 0$, there is a $\tilde{C}_r > 0$ such that

$$\|\Psi_N(t) - \Psi^{(r)}_N(t)\| \leq \tilde{C}_r e^{\tilde{C}_r t} N^{-(r+1)/2},$$

(45)

where

$$\Psi^{(r)}_N(t) = W(\sqrt{N}\alpha(t)) \sum_{k=0}^N \varphi(t) \otimes (N-k) \otimes_s \left( \sum_{\ell=0}^r N^{-\ell/2} \chi^{(k)}_\ell(t) \right),$$

(46)

with $(\varphi(t), \alpha(t))$ satisfying (49) and $\chi(t)$ being defined as in Equation (41).

To formulate our last result, we consider the unitary propagator $U(t,s)$ being defined by

$$i\partial_t U(t,s) = H(t)U(t,s).$$

(47)

With the formal ansatz

$$U(t,s) = \sum_{\ell=0}^\infty N^{-\ell/2} U_\ell(t,s)$$

(48)

we obtain similarly to (41) that

$$U_\ell(t,s) = \sum_{k=1}^\ell \sum_{s_{1:k}|=\ell} (-i)^k \int_{\Delta_k(s)} ds^{(k)} \prod_{i=1}^k \tilde{H}^{(\alpha_i)}(s_i, t) U_0(t,s),$$

(49)

where $\Delta_k(s)$ is the region $[s, t]^k$ with $s_{i+1} \leq s_i$ for all $i = 1, \ldots, k-1$. When $\chi(0) = \sum_{\ell=0}^\infty N^{-\ell/2} \chi^{(0)}(0)$, we recover the expression (41) via $\chi(t) = \sum_{m=0}^\ell U_{\ell-m}(t, 0) \chi_m(0)$. Note that $U(t,s)$ and $U_0(t,s)$ are unitary two-parameter groups, but $U_\ell(t,s)$ for $\ell \geq 1$ is generally not unitary. Rather, $U(t,t) = 1 = U_0(t,t)$ yields $U_\ell(t,t) = 0$, and the group property $U(t,s)U(s,r)$ yields $U_\ell(t,r) = \sum_{k=0}^\ell U_k(t,s)U_{\ell-k}(s,r)$.

Theorem 2.6. Let $U(t,s)$ be the unitary two-parameter group generated by $H(t)$, and let $\chi \in \mathcal{G}$ (possibly $N$-dependent) be such that

$$\forall n \in \mathbb{N}_0 : \sup_{N \geq 1} \langle \chi, (N+1)^n \chi \rangle < \infty.$$  

(50)

Then for all $r \in \mathbb{N}_0$ and $t,s \in \mathbb{R}$,

$$\left\| \left( U(t,s) - \sum_{\ell=0}^r N^{-\ell/2} U_\ell(t,s) \right) \chi \right\| \leq \tilde{C}_r e^{\tilde{C}_r(|t|+|s|)} N^{-(r+1)/2},$$

(51)

for some $\tilde{C}_r > 0$, with $U_\ell(t,s)$ as defined in Equation (49).
3 Application of Main Results

In this section we outline two important applications of our main results. In the first part, we use that the Bogoliubov Hamiltonian is a quadratic operator which allows us to explicitly compute the action of the Bogoliubov time evolution on the creation and annihilation operators. In particular, this yields a simple method to compute correlation functions within the Bogoliubov approximation. The explicit form of the higher order terms from (41) allows us to approximate correlation functions also to higher order. We use this in the second part, where we apply Theorem 2.5 to obtain a set of coupled PDEs that describe the next-order correction to the time evolution of the one-particle reduced density matrices.

3.1 Explicit action of the Bogoliubov time evolution

To determine how the Bogoliubov time evolution acts on the creation and annihilation operators, it is convenient to start by writing the Bogoliubov Hamiltonian as an operator that is quadratic in one type of (two-component) creation/annihilation operators. To this end, we introduce the Fock spaces

\[ \mathcal{L} = \bigoplus_{n=0}^{\infty} \mathcal{h}^0_2 \approx_n \quad \text{and} \quad \mathcal{L}_\perp = \bigoplus_{n=0}^{\infty} \mathcal{h}^0_\perp \varphi(t) \subset \mathcal{L}, \]

with

\[ \mathcal{h}_2 = L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3) \quad \text{and} \quad \mathcal{h}_\perp \varphi(t) = \{ \varphi(t) \} \perp L^2(\mathbb{R}^3) \subset \mathcal{h}_2. \]

We denote the creation and annihilation operators on these spaces by

\[ \mathcal{Z}_\varphi \quad \text{and} \quad \mathcal{Z}_\varphi^*. \]

We note that the space \( \mathcal{L}_\perp \) is unitarily equivalent to \( \mathcal{G} \), that is, there is a unitary (time-independent) map \( V \) such that

\[ V \Omega_{\mathcal{G}} = \Omega_{\mathcal{L}_\perp} \quad \text{and} \quad V \mathcal{Z}_\varphi = \mathcal{Z}_\varphi V \]

(see also [45, Chapter 4.2]). As usual, we define the second quantization of a linear operator \( A \) on \( \mathcal{h}_2 \) by

\[ d\Gamma(A) = \sum_{n,m=0}^{\infty} \langle u_n, Au_m \rangle Z^*(u_n)Z(u_m) \]

(55)

where \( (u_n)_{n \in \mathbb{N}} \subset \mathcal{h}_2 \) is a suitable ONB. The number operator, for instance, is given by \( \mathcal{N} = d\Gamma(1) \). The action of the Bogoliubov Hamiltonian on \( \mathcal{L}_\perp \) can then be written as

\[ H_0(t) = VH_0(t)V^* \]

\[ = d\Gamma(A(t)) + \frac{1}{2} \left( \sum_{n,m} \langle u_n, B(t) \overline{u_m} \rangle Z^*(u_n)Z(u_m) + \sum_{n,m} \langle u_n, B(t) \overline{u_m} \rangle Z(u_n)Z(u_m) \right) \]

(56)

where

\[ A(t) = A_1 + A_2(t) = \left( \begin{array}{cc} \h(t) & \overline{K^\pm(t)} \\ K_{\pm}(t) & \omega \end{array} \right) = \left( \begin{array}{cc} -\Delta & 0 \\ 0 & \omega \end{array} \right) + \left( \begin{array}{cc} \Phi(t) - \mu(t) & \overline{K^\pm(t)} \\ K_{\pm}(t) & 0 \end{array} \right) \]

(57)

and

\[ B(t) = \left( \begin{array}{cc} 0 & \overline{K^\pm(t)} \\ K(t) & 0 \end{array} \right) \]

(58)
are defined as operators on $h_2$, $(u_n)_{n \in \mathbb{N}} \subset D(A_1)$ is an ONB of $h_2$ and $K_-(t)$ is the operator on $L^2(\mathbb{R}^3)$ with kernel $K_-(t, k, x) = K(t, -k, x)$. We can now apply well-known results about the time evolution of quadratic Hamiltonians to our model (56); we refer to [59] and [1] for an introduction to the topic and an overview of some of the results. For

$$F = f \oplus Jg = f_1 \oplus f_2 \oplus \overline{g_1} \oplus \overline{g_2} \in h_2 \oplus h_2,$$  

(59)

let the generalized creation and annihilation operators $A^*(F)$ and $A(F)$ be defined by

$$A^*(F) = Z^*(f) + Z(g) = A(JF), \quad A(F) = Z(f) + Z^*(g),$$  

(60)

where the operator

$$J = \begin{pmatrix} 0 & J \cr J & 0 \end{pmatrix}$$

acts on $h_2 \oplus h_2$ and $J : h_2 \to h_2$, $J(g_1(k), g_2(x)) = (\overline{g_1(k)}, \overline{g_2(x)})$ is the complex conjugation map. We further define $S$ on $h_2 \oplus h_2$ by

$$S = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$  

(62)

The generalized creation and annihilation operators satisfy the commutation relations

$$[A(F), A^*(G)] = \langle F, SG \rangle_{h_2 \oplus h_2}, \quad [A(F), A(G)] = \langle F, S J G \rangle_{h_2 \oplus h_2}.$$  

(63)

Next we recall the notion of Bogoliubov maps and Bogoliubov transformations. A bounded operator

$$V : h_2 \to h_2 \rightarrow h_2 \oplus h_2$$

(64)

is called a Bogoliubov map if it satisfies

$$V^*SV = S = VSV^*, \quad JVJ = V.$$  

(65)

Equivalently, $V$ is a Bogoliubov map if it has the block form

$$V = \begin{pmatrix} u & v^* \\ \overline{v} & \overline{u} \end{pmatrix} \quad \text{with} \quad u, v : h_2 \to h_2, \quad \overline{v} = JV, \quad \overline{u} = JuJ,$$  

(66)

and $u, v$ satisfy the conditions $u^*u = 1 + v^*v$, $uv^* = 1 + \overline{v^*v}$, $v^*\overline{v} = v^*\overline{v}$, $uv^* = v^*\overline{v}$. A well-known result about Bogoliubov maps states that there exists a unitary transformation $U_V : \mathcal{L} \to \mathcal{L}$ satisfying

$$U_V^*A(F)U_V = A(VF) \quad \forall F \in h_2 \oplus h_2,$$  

(67)

if and only if it satisfies the Shale–Stinespring condition

$$\|v\|_{HS}^2 = \text{Tr}_{h_2} |v^*v| < \infty.$$  

(68)

In this case, $V$ is called unitarily implementable and its implementation $U_V$ will be called the Bogoliubov transformation associated to $V$.

It is a fundamental property of the dynamics generated by (suitable) quadratic Hamiltonians on Fock space that they are equivalent to time-dependent Bogoliubov transformations. More precisely, in our case, the unitary propagator $U_0(t, t_0) : \mathcal{G} \to \mathcal{G}$ that corresponds to the Schrödinger equation
We now express the reduced densities in terms of the excitation vector $\mathbf{V}$ and of the field as

$$U(t, t_0) = V^*U(t, t_0)V,$$

where the time-dependent Bogoliubov map $\mathbf{V}(t, t_0)$ solves the PDE

$$\begin{cases}
    i\partial_t \mathbf{V}(t, t_0) &= \mathbf{A}(t)\mathbf{V}(t, t_0) \\
    \mathbf{V}(t_0, t_0) &= 1,
\end{cases} \quad (69)$$

with

$$\mathbf{A}(t) = \begin{pmatrix} A(t) & -B(t) \\ B(t) & -A(t) \end{pmatrix},$$

and $A(t)$ and $B(t)$ defined by \[77\] and \[85\], respectively. Note that Equation \[89\] reduces the Bogoliubov equation to a PDE on $\mathfrak{h}_2 \oplus \mathfrak{h}_2$ and that

$$VU_0(t, 0)^*V^*A(F)VU_0(t, 0)V^* = A(V(t, 0)F) \quad \forall F \in \mathfrak{h}_2 \oplus \mathfrak{h}_2. \quad (71)$$

Equation \[74\] is our main result of this section. It is very convenient since it allows one to explicitly compute expectation values of a large class of observables in the state $\chi_0(t)$ or, equivalently, in the state $\Psi_N(0)(t)$ defined by \[33\].

### 3.2 Equations for the reduced density and next-order corrections

The expansion of the wave function in Theorem \[25\] implies a corresponding expansion of correlation functions, as has been discussed in \[5\] for pair-interacting bosons. Here, we provide the next-order expansion of the wave function in Theorem 2.5 implies a corresponding expansion of correlation functions.

Equation \[72\] is unitarily equivalent to the Bogoliubov transformation $U(t, t_0) : \mathcal{L}_+ \to \mathcal{L}_+$, i.e., $U_0(t, t_0) = V^*U(t, t_0)V$, where the time-dependent Bogoliubov map $\mathbf{V}(t, t_0)$ solves the PDE.

The expansion of the wave function in Theorem 2.5 implies a corresponding expansion of correlation functions, as has been discussed in \[5\] for pair-interacting bosons. Here, we provide the next-order correction to the reduced one-particle density matrices, as one important application of Theorem \[25\].

The one-particle reduced density matrix of the condensate particles is defined as

$$\mu_{\Psi_N(t)}^{\text{part}} = \text{tr}_{2, \ldots, N} |\Psi_N(t)\rangle \langle \Psi_N(t)|,$$

and of the field as

$$\mu_{\Psi_N(t)}^{\text{field}}(k, k') = N^{-1} \langle \Psi_N(t), \alpha^*(k')\alpha(k)\Psi_N(t) \rangle_{\mathcal{H}_N}. \quad (73)$$

These reduced densities can be used to compute expectation values of bounded one-body operators. We now express the reduced densities in terms of the excitation vector $\chi_{\mathbf{k}}(t)$ by using the unitary $U_N(t) = \tilde{U}_N(t) \otimes W^*(\sqrt{N}\alpha(t))$ that was explicitly defined in \[24\] (see Appendix A for more details). A straightforward computation (using Lemma A.1) shows that

$$\begin{align*}
\mu_{\Psi_N(t)}^{\text{part}}(x, x') &= N^{-1} \langle \chi_{\mathbf{k}}(t), \tilde{U}_N(t)b^*(x')b(x)\tilde{U}_N(t)\chi_{\mathbf{k}}(t) \rangle_{\mathcal{H}_N} \\
&= \varphi(t, x)\bar{\varphi}(t, x') + N^{-1/2} \left( \varphi(t, x)\beta_{\chi_{\mathbf{k}}(t)}(x') + \varphi(t, x')\beta_{\chi_{\mathbf{k}}(t)}(x) \right) \\
&\quad + N^{-1} \left( \gamma_{\chi_{\mathbf{k}}(t)}(x, x') - \varphi(t, x)\bar{\varphi}(t, x') \text{tr} \gamma_{\chi_{\mathbf{k}}(t)}^{\text{part}} \right),
\end{align*} \quad (74)$$

where, for any $\Phi \in \mathcal{G}$,

$$\begin{align*}
\beta_\Phi^{\text{part}}(x) &= \langle \Phi, \sqrt{1 - N^{-1}N_{\mathbf{k}}^\dagger b(x)\Phi} \rangle_{\mathcal{G}}, & \gamma_\Phi^{\text{part}}(x, x') &= \langle \Phi, b^*(x')b(x)\Phi \rangle_{\mathcal{G}}.
\end{align*} \quad (75)$$

and

$$\begin{align*}
\mu_{\Psi_N(t)}^{\text{field}}(k, k') &= N^{-1} \langle \chi_{\mathbf{k}}(t), W^*(\sqrt{N}\alpha(t))\alpha^*(k')\alpha(k)W(\sqrt{N}\alpha(t))\chi_{\mathbf{k}}(t) \rangle_{\mathcal{H}_N} \\
&= \alpha(t, k)\bar{\alpha}(t, k') + N^{-1/2} \left( \alpha(t, k)\beta_{\chi_{\mathbf{k}}(t)}^{\text{field}}(t, k') + \beta_{\chi_{\mathbf{k}}(t)}^{\text{field}}(t, k)\alpha(t, k') \right) \\
&\quad + N^{-1} \gamma_{\chi_{\mathbf{k}}(t)}^{\text{field}}(k, k'),
\end{align*} \quad (76)$$

$^4$Existence and further properties of $U_0(t, t_0)$ will be discussed in Section 4.
where, for any $\Phi \in \mathcal{G}$,

$$\beta_{\Phi}^\text{field}(k) := \langle \Phi, a(k)\Phi \rangle_{\mathcal{G}}, \quad \gamma_{\Phi}^\text{field}(k, k') := \langle \Phi, a^*(k')a(k)\Phi \rangle_{\mathcal{G}}. \quad (77)$$

Equations (74) and (76) tell us how the reduced densities of the particles and the quantum field are related to reduced densities and one-point functions of the excitations.

The leading orders of the reduced density matrices of the excitations, $\gamma_{\chi_{0}(t)}^\text{part}$ and $\gamma_{\chi_{0}(t)}^\text{field}$, can be computed via Bogoliubov theory. It is easier to directly compute the time evolution of the generalized one-particle density matrix $\Gamma_{\chi_{0}(t)}$, defined via

$$\langle F_{1}, \Gamma_{\chi_{0}(t)}(F_{2}) \rangle_{\mathcal{B}_{2}\otimes \mathcal{B}_{2}} := \langle \chi_{0}(t), V^*A^*(F_{2})A(F_{1})V\chi_{0}(t) \rangle_{\mathcal{G}}$$

for any $F_{1}, F_{2} \in \mathcal{B}_{1,\varphi}(t) \oplus \mathcal{B}_{1,\varphi}(t)$, where $A^*, A$ are the generalized creation and annihilation operators defined in (69). Using the fact that $\chi_{0}(t) = V^*\Upsilon_{t,0}V\chi_{0}(0)$ with the Bogoliubov map $\Upsilon(t,0)$ that solves (59), a direct computation yields

$$\Gamma_{\chi_{0}(t)} = \mathcal{S}\Upsilon(t,0)\mathcal{S}^{\dagger}V_{\chi_{0}(0)}\mathcal{S}^{\dagger}V^{\ast}(t,0)\mathcal{S}, \quad (79)$$

or

$$i\partial_{t}\Gamma_{\chi_{0}(t)} = A(t)^{\dagger}\Gamma_{\chi_{0}(t)} - \Gamma_{\chi_{0}(t)}A(t). \quad (80)$$

From $\Gamma_{\chi_{0}(t)}$ we can read off all possible two-point functions of $\chi_{0}(t)$, in particular $\gamma_{\chi_{0}(t)}^\text{part}$ and $\gamma_{\chi_{0}(t)}^\text{field}$.

Let us next consider the one-point functions $\beta_{\chi_{0}(t)}^\text{part}$ and $\beta_{\chi_{0}(t)}^\text{field}$. The leading order of $\beta_{\chi_{0}(t)}^\text{part}$ is

$$\beta_{\chi_{0}(t)}^\text{part}(t, x) := \langle \chi_{0}(t), b(x)\chi_{0}(t) \rangle_{\mathcal{G}}. \quad (81)$$

If we assume that $\chi_{0}(0) = V^*\Upsilon_{t,0}\Omega$ for some Bogoliubov map $\Upsilon$ (as defined in the previous section), i.e., that $\chi_{0}(0)$ is quasi-free, then $\chi_{0}(t) = V^*\Upsilon_{t,0}U_{t}\Upsilon_{0}\Omega$ is also quasi-free, as the composition of two Bogoliubov transformations defines again a Bogoliubov transformation. Therefore, $\beta_{\chi_{0}(0)}^\text{part} = 0$ due to Wick’s rule [6].

**Remark 3.1.** We could more generally assume that $\chi_{0}(0) = V^*\Upsilon_{t,0}Z^{\ast}(f_{1})\cdots Z^{\ast}(f_{n})\Omega$ for some orthonormal $f_{1}, \ldots, f_{n} \in \mathcal{B}_{1,\varphi}(0)$, i.e., that $\chi_{0}(0)$ is a state with a fixed number $n$ of excitations. Then $\beta_{\chi_{0}(0)}^\text{part}$ is a $(2n + 1)$-point function of a quasi-free state, i.e., still $\beta_{\chi_{0}(0)}^\text{part} = 0$ due to Wick’s rule. Such states are the prediction of Bogoliubov theory for the low-energy excited states of trapped systems; see [6] for references in the case of bosons with two-body interaction.

The structure of trapped initial data $\chi_{\ell}(0)$ for $\ell \geq 1$ follows from time-independent perturbation theory, which was proven in [6] for a class of two-body interactions. We expect similar results to hold for our model, in particular that

$$\chi_{\ell}(0) = V^* \sum_{m, \ell \text{ even}}^{3\ell} \sum_{m = 0}^{m} \sum_{\mu = 0}^{m} G(t)_{m, \mu} V\chi_{0}(0), \quad (82)$$

where $G(t)_{m, \mu}$ is the quantization of a bounded operator $g_{m, \mu} : (\mathcal{B}_{1,\varphi}(0))^{m-\mu} \to (\mathcal{B}_{1,\varphi}(0))^{\mu}$, i.e.,

$$G(t)_{m, \mu} = \int dx^{(\mu)} \int dy^{(m-\mu)} g_{m, \mu}(x^{(\mu)}; y^{(m-\mu)}) Z^{\ast}(x_{1}) \cdots Z^{\ast}(x_{\mu})Z(y_{1}) \cdots Z(y_{m-\mu}), \quad (83)$$

with $x^{(\mu)} := (x_{1}, \ldots, x_{\mu})$. So for even $\ell + n$ (n being the number of excitations in $\chi_{0}(0)$) the state $\chi_{\ell}(0)$ has only an even number of excitations, whereas for odd $\ell + n$ it has an odd number. Note that the time evolution (71) indeed preserves the structure of Equation (81), i.e., also

$$\chi_{\ell}(t) = V^* \sum_{m, \ell \text{ even}}^{3\ell} \sum_{m = 0}^{m} \sum_{\mu = 0}^{m} G(t)_{m, \mu}(t) V\chi_{0}(t) \quad (84)$$

for some $G(t)_{m, \mu}(t)$. Assuming (81), all correlation functions of the type $\langle \chi_{\ell}(t), Z_{1}^{\ast} \cdots Z_{k}^{\ast} \chi_{m}(t) \rangle$, where $Z^{\ast} \in \{Z, Z^{\ast}\}$, vanish for $\ell + m + k$ odd due to Wick’s rule.
With the assumption from Remark 3.1 all terms of order one and of order $N^{-1}$ vanish in the expansion of $\beta_{\chi,N}^{\text{part}}$, so

$$\beta_{\chi,N}^{\text{part}} = N^{-1/2} \beta_{01}^{\text{part}}(t) + O(N^{-3/2}), \quad \text{with} \quad \beta_{01}^{\text{part}}(t, x) = \langle \chi_0(t), b(x)\chi_1(t) \rangle_G + \langle \chi_1(t), b(x)\chi_0(t) \rangle_G. \tag{84}$$

In the same way,

$$\beta_{\chi,N}^{\text{field}} = N^{-1/2} \beta_{01}^{\text{field}}(t) + O(N^{-3/2}), \quad \text{with} \quad \beta_{01}^{\text{field}}(t, k) = \langle \chi_0(t), a(k)\chi_1(t) \rangle_G + \langle \chi_1(t), a(k)\chi_0(t) \rangle_G. \tag{85}$$

A direct computation shows that $\beta_{01}^{\text{part}}$ and $\beta_{01}^{\text{field}}$ solve the coupled PDEs

$$i\partial_t \beta_{01}^{\text{part}}(t, x) = h(t) \beta_{01}^{\text{part}}(t, x) + \int dt \left( K(t, k, x) \overline{\beta_{01}^{\text{field}}(t, k)} + K(t, -k, x) \beta_{01}^{\text{field}}(t, k) \right) + \int dt \, dy \, F(t, x, k, y) \langle \chi_0(t), (a(k)b(y) + a(-k)b(y))\chi_0(t) \rangle_G \tag{86a}$$

$$i\partial_t \beta_{01}^{\text{field}}(t, k) = \omega(k) \beta_{01}^{\text{field}}(t, k) + \int dx \left( K(t, k, x) \overline{\beta_{01}^{\text{part}}(t, x)} + \beta_{01}^{\text{part}}(t, x) K(t, -k, x) \right) + \int dx \, dy \, F(t, x, k, y) \langle \chi_0(t), b^*(x)b(y)\chi_0(t) \rangle_G, \tag{86b}$$

where

$$F(t, x, k, y) := \eta(k) \int dz \, (q(t, x, z)e^{-2\pi i k z}g(t, z) - \delta(x - y)e^{-2\pi i k z}|\varphi(t, z)|^2). \tag{87}$$

The two-point functions such as $\langle \chi_0(t), a(k)b^*(y)\chi_0(t) \rangle_G$ can be read off from the corresponding entry of $\Gamma_{\chi_0(t)}$.

To summarize, for initial data as in Remark 3.1 we have found that

$$\mu_{\chi,N}^{\text{part}} = |\varphi(t)\rangle\langle \varphi(t) | + N^{-1} \left( |\varphi(t)\rangle\langle \beta_{01}^{\text{part}}(t) | + |\beta_{01}^{\text{part}}(t)\rangle\langle \varphi(t) | + \gamma_{\chi_0(t)}^{\text{part}} - |\varphi(t)\rangle\langle \varphi(t) | \text{tr} \gamma_{\chi_0(t)}^{\text{part}} \right) + O(N^{-2}), \tag{88}$$

and

$$\mu_{\chi,N}^{\text{field}} = |\alpha(t)\rangle\langle \alpha(t) | + N^{-1} \left( |\alpha(t)\rangle\langle \beta_{01}^{\text{field}}(t) | + |\beta_{01}^{\text{field}}(t)\rangle\langle \alpha(t) | + \gamma_{\chi_0(t)}^{\text{field}} \right) + O(N^{-2}), \tag{89}$$

where the one-point functions $\beta_{01}^{\text{part}}(t)$ and $\beta_{01}^{\text{field}}(t)$ solve Equations (86a) and (86b), and the two-point functions $\gamma_{\chi_0(t)}^{\text{part}}$ and $\gamma_{\chi_0(t)}^{\text{field}}$ can be read off from $\Gamma_{\chi_0(t)}$, the solution of Equation (81).

## 4 Well-posedness of the Bogoliubov Equation

Let us now discuss the well-posedness of the Cauchy problem

$$\begin{cases}
i\partial_t \chi(t) = \mathbb{H}_0(t)\chi(t) \\
\chi(t_0) = \chi_0 \in \mathcal{L}.
\end{cases} \tag{90}$$

Since $H_0(t)$ is unitarily equivalent to $\mathbb{H}_0(t)$ (see (50)), this provides the well-posedness of the Bogoliubov equation (28). The main ingredient in the proof is a regularity estimate for the time-dependent part of $H_0(t)$; more precisely, below we shall show that $A_2(t)$ and $B(t)$, defined in (57) and (58), satisfy

$$\sup_{t \in K} \left( \|A_2(t)\| + \left\| \frac{d}{dt} A_2(t) \right\|_{\text{HS}} + \|B(t)\|_{\text{HS}} + \left\| \frac{d}{dt} B(t) \right\|_{\text{HS}} \right) < \infty \tag{91}$$
The commutation yields $q(t) \in L^2(\mathbb{R}^3)$ with initial datum

$$(q(t_0), G(t_0)) \in H^2(\mathbb{R}^3) \times L^2(\mathbb{R}^3).$$

Then there exists a two-parameter unitary group $(U_0(t, t_0))_{t, t_0 \in \mathbb{R}}$ on $\mathcal{L}$ such that for any $t, t_0 \in \mathbb{R}$,

$$\chi_{t, t_0}(t) = U_0(t, t_0)\chi$$

is the unique solution to [90] in the following sense. For any $\xi, \chi \in Q(d\Gamma(1 + A_1)) \subseteq \mathcal{L}$ (where $Q(\cdot)$ denotes the quadratic form domain), we have that $U_0(\xi, \chi) = U_0(t, t_0, t)\xi \in Q(d\Gamma(1 + A_1))$, and therefore the function $\langle \xi, U_0(t, t_0)\chi \rangle_{\mathcal{L}}$ is differentiable with respect to both $t$ and $t_0$, with

$$i\partial_t \langle \xi, U_0(t, t_0)\chi \rangle_{\mathcal{L}} = \langle \xi, \mathcal{H}_0(t)U_0(t, t_0)\chi \rangle_{\mathcal{L}}$$

$$i\partial_{t_0} \langle \xi, U_0(t, t_0)\chi \rangle_{\mathcal{L}} = -\langle \xi, U_0(t, t_0)\mathcal{H}_0(t)\chi \rangle_{\mathcal{L}}.$$  

Moreover, $U_0(t, t_0)$ satisfies the following properties.

(i) For any $r \in \mathbb{R}$ there exists a constant $C_r > 0$, such that

$$\|(d\Gamma(1 + 1)^{-r})U_0(t, t_0)(d\Gamma(1 + 1)^{-r})\| \leq e^{C_r |t - t_0|}$$

for all $t, t_0 \in \mathbb{R}$.

(ii) For $\chi \in \mathcal{L}_1(t_0)$, we have $\chi_{t_0}(t) \in \mathcal{L}_1(t)$.

Proof. The existence of the two-parameter unitary group satisfying (94a) and (94b) follows from a general result about the evolution generated by a time-dependent Hamiltonian [12 Theorem 8]. That [12 Theorem 8] can be applied in our setting under condition (91) is shown in analogy to the first part of the proof of [49 Proposition 7]. For our purpose, the bound (95) is crucial. It can be proved by a Gronwall argument, following the same strategy used in the proof of [18 Proposition 4.2]. (For a similar bound, see also [12 Theorem 8].) In the following, we first outline the argument of the proof of property (i), then of property (ii), and finally we prove the bound (95).

We consider $\chi_{t_0}(t)$, with $\chi \in Q(d\Gamma(1 + A_1))$. Its derivative is an element of the Hilbert space obtained from completing $\mathcal{L}$ w.r.t. the scalar product

$$\langle \cdot, (d\Gamma(1 + A_1))^{-1} \cdot \rangle_{\mathcal{L}}.$$  

Hence, $M(t, t_0) := \langle \chi_{t_0}(t), \chi_{t_0}(t) \rangle_{\mathcal{L}}$ is differentiable w.r.t. both $t$ and $t_0$. Analogously, for any $r \geq 1$,

$$M_r(t, t_0) := \langle \chi_{t_0}(t), (d\Gamma(1 + A_1)^{-r})\chi_{t_0}(t) \rangle_{\mathcal{L}}$$

is differentiable w.r.t. both $t$ and $t_0$ as well. We have that

$$i\partial_t M_r(t, t_0) = \langle \chi_{t_0}(t), [(d\Gamma(1 + A_1)^{-r})\mathcal{H}_0(t)\chi_{t_0}(t) \rangle_{\mathcal{L}}.$$  

The commutation yields

$$|\partial_t M_r(t, t_0)| \leq 2C\langle \chi_{t_0}(t), ((d\Gamma(1 + A_1)^{-r} - (d\Gamma(1 + A_1)^{-r} - d\Gamma(1 + A_1)^{-r}\chi_{t_0}(t) \rangle_{\mathcal{L}}\right),$$

where $C = \sup_{\tau \in [t_0, t]}\|B(\tau)\|_{HS}$. Now, by spectral calculus the inequality

$$((n + 1)^{-r} - (n + 3)^{-r})n \leq 2r(n + 1)^{-r},$$
Gronwall’s lemma gives
\[
\| (d\Gamma(1) + 1)^{-r/2}\chi_{t_0}(t) \|_{L^2}^2 \leq e^{4C r[t-t_0]} \| (d\Gamma(1) + 1)^{-r/2}\chi \|_{L^2}^2 .
\]

The result is extended to \( 0 \leq r \leq 1 \) by interpolation, the case \( r = 0 \) being trivial. In addition, the bound is extended to any \( \chi \in \mathcal{L} \) by a density argument. Therefore, we can choose \( \chi = (d\Gamma(1) + 1)^{r/2}\xi \), for some \( \xi \in Q(d\Gamma(1)^r) \). In this case, the result reads
\[
\| (d\Gamma(1) + 1)^{-r/2}U_0(t, t_0)(d\Gamma(1) + 1)^{r/2}\xi \|_{L^2}^2 \leq e^{4C r[t-t_0]} \| \xi \|_{L^2}^2 ,
\]
for any \( t, t_0 \in \mathbb{R} \). Again by density the result is extended to any \( \xi \in \mathcal{L} \). Hence,
\[
\| (d\Gamma(1) + 1)^{-r/2}U_0(t, t_0)(d\Gamma(1) + 1)^{r/2}\| \leq e^{2C r[t-t_0]} .
\]

Since \( U_0(t, t_0)^* = U_0(t_0, t) \), the bound follows, setting \( C_r = 4C|r| \).

Next, we prove property (ii). One computes
\[
\frac{d}{dt} \| Z(\varphi(t) \oplus 0)\chi_{t_0}(t) \| = 2\text{Im}\langle Z(\varphi(t) \oplus 0)\chi_{t_0}(t), (Z((\varphi(t) + 0)) \oplus 0) + Z(\varphi(t) \oplus 0)\chi_{t_0}(t) \rangle .
\]

Using
\[
Z(\varphi(t) \oplus 0)\chi_{t_0}(t) = \chi_{t_0}(t)Z(\varphi(t) \oplus 0) + Z(h(t)\varphi(t) \oplus 0),
\]
one finds
\[
\frac{d}{dt} \| Z(\varphi(t) \oplus 0)\chi_{t_0}(t) \| = 2\text{Im}\langle Z(\varphi(t) \oplus 0)\chi_{t_0}(t), \chi_{t_0}(t)Z(\varphi(t) \oplus 0)\chi_{t_0}(t) \rangle = 0,
\]
and thus \( \| Z(\varphi(t) \oplus 0)\chi_{t_0}(t) \| = 0 \) for all \( \chi_0 \) satisfying \( \| Z(\varphi(t_0) \oplus 0)\chi_0 \| = 0 \). This implies (ii).

It remains to show (ii). Since
\[
|\Phi(t, x)| \leq 2 \| \eta \|_{L^2} |\alpha(t)|_{L^2} , \quad |\mu(t)| \leq |\eta|_{L^2} |\alpha(t)|_{L^2} , \quad \| K(t, \cdot, \cdot) \|_{L^2(\mathbb{R}^3)} \leq 2 \| \eta \|_{L^2} ,
\]
we have \( \sup_{t \in K} (\| A_2(t) \| + \| B(t) \|_{L^2} ) < \infty \) for any compact interval \( K \subset \mathbb{R} \).

Using the Schrödinger–Klein–Gordon equations, we further find
\[
i\hat{\partial}_t \Phi(x, t) = \int dk \eta(k) \omega(k)e^{2\pi ikx}(\alpha(t, k) - \overline{\alpha(t, -k)}),
\]
and hence, \( \sup_{x \in \mathbb{R}^3} |\hat{\partial}_t \Phi(x, t)| \leq 2 |\eta|_{L^2} |\omega(t)|_{L^2} \), where \( |\omega(t)|_{L^2} \) is bounded by Lemma 1. With \( \| \partial_x \|_{L^2} = 1 \), we further get
\[
|\hat{\partial}_t \mu(t)| \leq \sup_{x \in \mathbb{R}^3} \left( |\hat{\partial}_t \Phi(t, x)| + |\Phi(t, x)| \| h(t)\varphi(t) \|_{L^2} \right).
\]

The last norm is bounded by estimating
\[
\sup_{t \in K} |h(t)\varphi(t)| \leq \sup_{t \in K} \left( |\Delta \varphi(t)| + (|\Phi(t, x)| + |\mu(t)|) \| \varphi(t) \| \right) < \infty .
\]

To prove the bound for \( \hat{\partial}_t K(t) \), we recall \( K(t) = \tilde{K}(t)(1 - p(t)) \) and use
\[
\hat{\partial}_t K(t) = (\hat{\partial}_t \tilde{K}(t))q(t) - \tilde{K}(t)\hat{\partial}_t p(t) .
\]
Then we proceed with \( \sup_{t \in K} \| \partial_t p(t) \|_{HS} \leq C \) which follows from \( i \partial_t p(t) = |h(t, x)\varphi(t)\rangle\langle \varphi(t)| - \text{h.c.} \) in combination with (111). Hence we can estimate
\[
\left\| \hat{K}(t) \partial_t p(t) \right\|^2_{HS} = \int dk \int dx \left| \int dy \hat{K}(t, k, y) \left( \langle h \varphi(y) \rangle \bar{\varphi}(x) - \varphi(y) \bar{h} \varphi(x) \right) \right|^2 \\
\leq 2 \| \varphi(t) \|^2_{L^2} \left\| \hat{K}(t, \cdot, \cdot) \right\|^2_{L^2} \| h(t) \varphi(t) \|^2_{L^2},
\]
and similarly,
\[
\left\| (\partial_t \hat{K}(t)) q(t) \right\|^2_{HS} = \int dk \int dx \left| \int dy \partial_t \hat{K}(t, k, y) (\delta(x - y) - \varphi(y) \bar{h} \varphi(x)) \right|^2.
\]
In the last expression we insert \( \hat{K}(t, k, x) = \eta(k)e^{-2\pi ikx} \varphi(t, x) \) to get
\[
\left\| (\partial_t \hat{K}(t)) q(t) \right\|^2_{HS} = \int dk \int dx \left| \int dy \eta(k) e^{-2\pi iky} (\partial_t \varphi(t, y)) (\delta(x - y) - \varphi(y) \bar{h} \varphi(x)) \right|^2 \\
\leq 4 \| \eta \|^2_{L^2} \| h(t) \varphi(t) \|^2_{L^2}.
\]
Together, this implies \( \sup_{t \in K} \left( \| A_2(t) \| + \| B(t) \|_{HS} \right) < \infty. \)

5 Proofs

5.1 Preliminary Lemmas

Let us start by discussing the power series expansion of the Hamiltonian \( H(t) \) in more detail. We prove two lemmas in this subsection: one on the growth of the coefficients, and one on remainder estimates. Recall that on \([0, 1]\) we have the Taylor expansion
\[
\sqrt{1 - x} = \sum_{n=0}^{k} c_n x^n + \tilde{R}_k(x),
\]
with the \( c_n \) given by
\[
c_n := (-1)^n \left( \frac{\frac{1}{n}}{n} \right) := (-1)^n \frac{1}{n!} \left( -\frac{1}{2} \right) \left( -\frac{3}{2} \right) \cdots \left( \frac{1}{2} - (n - 1) \right). \quad (117)
\]
Defining \( \tilde{R}_k(x) := \sqrt{1 - x} - \sum_{n=0}^{k} c_n x^n \) for all \( x \geq 0 \), an expansion of the Hamiltonian \( H(t) \) yields
\[
H(t) = \sum_{\ell=0}^{r} N^{-\ell/2} H_\ell(t) + S_N^{(r)}(t)
\]
with \( H_{0}(t) \) given by (12c), and
\[
H_1(t) = \int dx b^* (x) \left( \langle q(t) \hat{\Phi} q(t) - \langle \varphi(t) , \hat{\Phi} \varphi(t) \rangle \right) b(x),
\]
\[
H_{2n}(t) = c_n \int dx \int dk K(t, k, x) (a^* (k) + a(-k)) b^* (x) N_0^2 + \text{h.c.} \quad \forall n \geq 1,
\]
\[
H_{2n+1}(t) = 0 \quad \forall n \geq 1,
\]
\[
S_N^{(0)}(t) = N^{-1/2} H_1(t) + \left( \int dx \int dk K(t, k, x) (a^* (k) + a(-k)) b^* (x) \tilde{R}_0 \left( \frac{N_0}{N} \right) + \text{h.c.} \right),
\]
(119d)
\[ S_N^{(1)}(t) = S_N^{(0)} - N^{-1/2} H_1(t), \]
\[ S_N^{(2r)}(t) = \int dx \int dk K(t, k, x)(a^*(k) + a(-k))b^*(x)\hat{R}_r \left( \frac{N_b}{N} \right) + \text{h.c.} \quad \forall r \geq 1, \]
\[ S_N^{(2r+1)}(t) = S_N^{(2r)}(t) \quad \forall r \geq 1. \]

Note that we write (118) as an expansion in \( N^{-\ell/2} \) instead of \( N^{-\ell} \) due to the appearance of \( H_1 \).

The next lemma provides estimates that are needed to bound the above operators relative to number operators.

**Lemma 5.1.** Let \( n \in \mathbb{N}_0 \) and \( a^\# \in \{a, a^*\} \). Then there exist positive constants \( C \) and \( C(n) \), such that
\[
\left\| \int dx \int dk b^*(x)\eta(k) \left( (q(t)e^{-2\pi i k}q(t) - \langle \varphi(t), e^{-2\pi i k} \varphi(t) \rangle) a^\#(\pm k) \right) b(x)\phi \right\| \leq C \left( (\hat{N}_a + 1)^{1/2}(\hat{N}_b + 1) \phi \right) \]
and
\[
\left\| \int dx \int dk K(t, k, x)a^\#(\pm k)b^*(x)\eta^{\pm} + \text{h.c.} \right\| \leq C(n) \left( (\hat{N}_a + 1)^{1/2}(\hat{N}_b + 1)^{n+1} \phi \right) \]
for all \( \phi \in \mathcal{F} \).

**Proof.** To derive (120), let us consider the contribution from the \( q(t)e^{-2\pi i k}q(t) \) term first. Straightforwardly estimating the creation and annihilation operators in terms of number operators as in (5) Lemma 5.1 gives
\[
\left\| \int dx \int dk b^*(x)\eta(k)(q(t)e^{-2\pi i k}q(t)) a^\#(k) b(x)\phi \right\| \leq \|f\|_{\mathcal{H} \rightarrow \mathcal{H} \otimes \mathcal{H}} \left( (\hat{N}_a + 1)^{1/2} \hat{N}_b \phi \right),
\]
where \( \mathcal{H} = L^2(\mathbb{R}^3) \) and \( f : \mathcal{H} \rightarrow \mathcal{H} \otimes \mathcal{H} \) is the operator with kernel
\[
f(x, k; y) = \int dz \eta(k)q(x, z)e^{ikz}q(z, y).
\]
Thus, \( \|f\|_{\mathcal{H} \rightarrow \mathcal{H} \otimes \mathcal{H}} \leq \|\eta\| \). The second term inside the norm on the left side of (120) can be estimated in complete analogy. Since (121) is obtained in a similar way, we omit the details (here one needs to use \( \|K\|_{HS} \leq \|\eta\| \)). \( \square \)

The remainders in Equation (118) can be estimated in terms of number operators.

**Lemma 5.2.** For any \( r \geq 1 \) there is a constant \( C(r) \geq 0 \), such that
\[
\|S_N^{(r)}(t)\phi\| \leq C(r)N^{-(r+1)/2} \left( (\hat{N}_a + 1)^{1/2}(\hat{N}_b + 1)^{(r+2)/2} \phi \right)
\]
for all \( \phi \in \mathcal{F} \).

**Proof.** For \( x \in [0, 1] \), Taylor’s theorem gives \( \sqrt{1-x} = \sum_{n=0}^{k} c_n x^n + \tilde{R}_k(x) \), with rest term
\[
\tilde{R}_k(x) = (k+1)c_{k+1} \int_0^x (1-y)^{-k-1/2}(x-y)^k \, dy = c_{k+1}(1-x)^{-k-1/2}x^{k+1}
\]
for some \( \xi \in (0, x) \). For \( x \in [0, 1] \), we thus find that
\[
|\tilde{R}_k(x)| \leq C(k)x^{k+1}. \]
For $x \geq 1$, we have $\tilde{R}_k(x) := \sqrt{1 - x^2} - \sum_{n=0}^{k} c_n x^n = -\sum_{n=0}^{k} c_n x^n$, therefore

$$|\tilde{R}_k(x)| \leq C(k) x^k,$$

and hence $|\tilde{R}_k(x)| \leq C(k) x^{k+1}$ for all $x \geq 0$. Moreover, combining (126) and (127) provides also $|\tilde{R}_k(x)| \leq C(k) x^{k+\frac{1}{2}}$ for all $x \geq 0$.

Thus, with Lemma 5.1 and $|\tilde{R}_0(x)| \leq C(0)x^{\frac{1}{2}}$, we find

$$\|S_N^{(0)}(t)\| \leq N^{-1/2} \|H(t)\| + \left( \int dx \int dk K(t, k, x) (a^*(k) + a(-k)) b^*(x) \tilde{R}_0 \left( \frac{N_b}{N} \right) + \text{h.c.} \right) \phi \leq CN^{-1/2} \left( (N_a + 1)^{1/2} (N_b + 1) \right),$$

and, with $|\tilde{R}_0(x)| \leq C(0)x$, we get

$$\|S_N^{(1)}(t)\| = \left( \int dx \int dk K(t, k, x) (a^*(k) + a(-k)) b^*(x) \tilde{R}_0 \left( \frac{N_b}{N} \right) + \text{h.c.} \right) \phi \leq CN^{-1} \left( (N_a + 1)^{1/2} (N_b + 1)^{3/2} \phi \right).$$

Then, for $r$ even, we use Lemma 5.1 and $|\tilde{R}_k(x)| \leq C(k) x^{k+\frac{1}{2}}$, and find

$$\|S_N^{(r)}(t)\| = \left( \int dx \int dk K(t, k, x) (a^*(k) + a(-k)) b^*(x) \tilde{R}_{r/2} \left( \frac{N_b}{N} \right) + \text{h.c.} \right) \phi \leq N^{-(r+1)/2} C(r) \left( (N_a + 1)^{1/2} (N_b + 1)^{(r+2)/2} \phi \right).$$

For $r$ odd, we employ $S_N^{(r)} = S_N^{(r-1)}$ and $|\tilde{R}_k(x)| \leq C(k) x^{k+1}$ to obtain

$$\|S_N^{(r)}(t)\| = \|S_N^{(r-1)}(t)\| = \left( \int dx \int dk K(t, k, x) (a^*(k) + a(-k)) b^*(x) \tilde{R}_{(r-1)/2} \left( \frac{N_b}{N} \right) + \text{h.c.} \right) \phi \leq N^{-(r+1)/2} C((r + 1)/2) \left( (N_a + 1)^{1/2} (N_b + 1)^{(r+2)/2} \phi \right).$$

\[\square\]

### 5.2 Propagation of Moments of Number Operators

We now prove that moments of the number operator $N = N_a + N_b$ with respect to the state $\chi_t(0)$ can be propagated in time.

**Lemma 5.3.** Let Assumption 2.4 hold. Then for all $n \in \mathbb{N}_0$ and $\ell \in \{0, ..., r\}$ there is a $C(n, \ell) > 0$ such that

$$\| (N + 1)^n \chi_t(0) \| \leq C(n, \ell) e^{C(n, \ell)t}. \tag{132}$$

**Proof.** By the definition of $\chi_t(0)$ from (111),

$$\| (N + 1)^n \chi_t(0) \| \leq \| (N + 1)^n U_0(t, 0) \chi(0) \| + \sum_{m=0}^{\ell-1} \sum_{k=1}^{\ell-m} \int d\alpha \left[ (N + 1)^n \right] \prod_{i=1}^{k} \tilde{H}_{\alpha}(s_i, t) U_0(t, 0) \chi_m(0). \tag{133}$$

Then by the number operator bound in (128) and by Assumption 2.4,

$$\| (N + 1)^n U_0(t, 0) \chi(0) \| \leq C(n, \ell) e^{C(n) t}. \tag{134}$$
Next, let us estimate the term
\[
\| (\mathcal{N} + 1)^n \tilde{H}_{2j}(s) U_0(t, 0) \phi \| = \| (\mathcal{N} + 1)^n U_0(t, s) H_{2j}(s) U_0(s, 0) \phi \| \tag{135}
\]
for any \( j = \frac{1}{2} \) or \( j \in \mathbb{N} \), and \( \phi \in \mathcal{F} \). In the following, we use \( \text{Lemma 5.1} \) in the first step, then commute \( H_{2j}(s) \) with the number operators and use \( \text{Lemma 5.1} \) in the second step, and use \( \text{(95)} \) again in the third step; we find
\[
\| (\mathcal{N} + 1)^n H_{2j}(s) U_0(s, 0) \phi \| \leq C(n, j) e^{C(n)|t-s|} \| (\mathcal{N} + 1)^{n+j+1} U_0(s, 0) \phi \|
\]
\[
\leq C(n, j) e^{C(n)|t-s|} e^{C(n,j)|s|} \| (\mathcal{N} + 1)^{n+j+1} \phi \|. \tag{136}
\]
Finally, note that the term with the highest number of creation and annihilation operators in the last line of \( \text{(133)} \) for given \( m \) comes from \( k = \ell - m \), i.e., \( \alpha = (1, 1, \ldots, 1) \). Thus,
\[
\sum_{m=0}^{\ell-1} \sum_{k=1}^{\ell-m} \sum_{|\alpha|=\ell-m} \int \Delta_k \| \mathcal{N} + 1 \|^n \prod_{i=1}^{k} \tilde{H}_{\alpha_i}(s_i, t) U_0(t, 0) \chi_m(0) \| \\
\leq \sum_{m=0}^{\ell-1} C(n, m) e^{C(n,m)|t|} \| \mathcal{N} + 1 \|^{|n+3(\ell-m)/2} \chi_m(0) \|, \tag{137}
\]
and the lemma is proven by Assumption \text{2.4} \]

### 5.3 Proof of the Theorems

#### Proof of Theorems \text{2.1} and \text{2.5}

We define the difference
\[
\chi^\text{rest}_r(t) := \chi_{\leq N}(t) - \sum_{\ell=0}^{r} N^{-\frac{\ell}{2}} \chi_\ell(t) \in \mathcal{G}, \tag{138}
\]
where we extend the state \( \chi_{\leq N} \in \mathcal{G}_{\leq N} \) to a state in \( \mathcal{G} \) by setting \( \chi_{\leq N}^{(k)} = 0 \) for all \( k \geq N + 1 \). Then
\[
\| \Psi_N(t) - \Psi_N^{(r)}(t) \| = \| \chi_{\leq N}(t) - \sum_{\ell=0}^{r} N^{-\frac{\ell}{2}} \chi_\ell(t) \|_{\mathcal{G}_{\leq N}} = \| \chi^\text{rest}_r(t) \|_{\mathcal{G}_{\leq N}}, \tag{139}
\]
where \( \| \phi \|_{\mathcal{G}_{\leq N}} := \| \phi \|_{\mathcal{G}_{\leq N}} \) (which defines only a semi-norm). Now note that
\[
\chi^\text{rest}_r(t) = U_0(t, 0) \chi^\text{rest}_r(0) - i \int_0^t ds U_0(t, s) F(s), \tag{140}
\]
with
\[
F(s) := (H^{\leq N}(s) - H_0(s)) \chi_{\leq N}(s) - \sum_{\ell=0}^{r} N^{-\frac{\ell}{2}} \sum_{m=1}^{\ell} H_m(s) \chi_{\ell-m}(s)
\]
\[
= (H^{\leq N}(s) - H_0(s)) \chi^\text{rest}_r(s) + \sum_{\ell=0}^{r} N^{-\frac{\ell}{2}} \left( (H^{\leq N}(s) - H_0(s)) \chi_\ell(s) - \sum_{m=1}^{\ell} H_m(s) \chi_{\ell-m}(s) \right)
\]
\[
= (H^{\leq N}(s) - H_0(s)) \chi^\text{rest}_r(s) + \sum_{\ell=0}^{r} N^{-\frac{\ell}{2}} \left( H^{\leq N}(s) - \sum_{m=0}^{\ell-\frac{\ell}{2}} N^{-m/2} H_m(s) \right) \chi(s), \tag{141}
\]

\[
19
\]
where we reordered the summation in the last step. Then
\[
\| \chi_r^{\text{rest}}(t) \|^2_{\mathcal{G}_{\leq N}} = \| \chi_r^{\text{rest}}(0) \|^2_{\mathcal{G}_{\leq N}} + 2 \text{Im} \int_0^t ds \left\langle U_0(s,0) \chi_r^{\text{rest}}(0), F(s) \right\rangle_{\mathcal{G}_{\leq N}} + \int_0^t ds \int_0^t d\tilde{s} \left\langle U_0(0,\tilde{s}) F(\tilde{s}), U_0(0,s) F(s) \right\rangle_{\mathcal{G}_{\leq N}}
\]
\[
= \| \chi_r^{\text{rest}}(0) \|^2_{\mathcal{G}_{\leq N}} + 2 \text{Im} \int_0^t ds \left\langle \chi_r^{\text{rest}}(s), F(s) \right\rangle_{\mathcal{G}_{\leq N}} - 2 \text{Re} \int_0^t ds \int_0^t d\tilde{s} \left\langle U_0(0,\tilde{s}) F(\tilde{s}), U_0(0,s) F(s) \right\rangle_{\mathcal{G}_{\leq N}}
\]
\[
+ \int_0^t ds \int_0^t d\tilde{s} \left\langle U_0(0,\tilde{s}) F(\tilde{s}), U_0(0,s) F(s) \right\rangle_{\mathcal{G}_{\leq N}}
\]
\[
= \| \chi_r^{\text{rest}}(0) \|^2_{\mathcal{G}_{\leq N}} + 2 \text{Im} \int_0^t ds \left\langle \chi_r^{\text{rest}}(s), F(s) \right\rangle_{\mathcal{G}_{\leq N}}
\]
using self-adjointness of \( H^{\leq N}(s) - H^{(0)}(s) \) in the last step. By the definition of the rest term and using the Cauchy–Schwarz inequality, we find
\[
\| \chi_r^{\text{rest}}(t) \|^2_{\mathcal{G}_{\leq N}} = \| \chi_r^{\text{rest}}(0) \|^2_{\mathcal{G}_{\leq N}} + 2 \sum_{\ell=0}^{r} N^{-\ell/2} \text{Im} \int_0^t ds \left\langle \chi_r^{\text{rest}}(s), S^{(r-\ell)}(s) \chi(\ell) \right\rangle_{\mathcal{G}_{\leq N}}
\]
\[
\leq \| \chi_r^{\text{rest}}(0) \|^2_{\mathcal{G}_{\leq N}} + 2 \sum_{\ell=0}^{r} N^{-\ell/2} \int_0^t ds \left\| \chi_r^{\text{rest}}(s) \right\|_{\mathcal{G}_{\leq N}} \left\| S^{(r-\ell)}(s) \chi(\ell) \right\|_{\mathcal{G}_{\leq N}}.
\] (142)

Now we use the estimate of the remainder in terms of number operators from Lemma 5.2 and the estimate of moments of number operators from Lemma 5.3, which yield
\[
\| \chi_r^{\text{rest}}(t) \|^2_{\mathcal{G}_{\leq N}} - \| \chi_r^{\text{rest}}(0) \|^2_{\mathcal{G}_{\leq N}} \leq 2 N^{-(r+1)/2} \int_0^t ds \left\| \chi_r^{\text{rest}}(s) \right\|_{\mathcal{G}_{\leq N}} \sum_{\ell=0}^{r} C(r-\ell) \left( (N_a + 1)^{1/2} (N_b + 1)^{(r-\ell+2)/2} \right) \chi(\ell)
\]
\[
\leq N^{-(r+1)/2} \int_0^t ds \left\| \chi_r^{\text{rest}}(s) \right\|_{\mathcal{G}_{\leq N}} C(r) e^{C(r)s}
\]
\[
\leq \int_0^t ds \left( \frac{1}{2} N^{r-1} C(r)^2 e^{2C(r)s} + \frac{1}{2} \left\| \chi_r^{\text{rest}}(s) \right\|^2_{\mathcal{G}_{\leq N}} \right).
\] (144)

Then Grönwall’s lemma implies
\[
\| \chi_r^{\text{rest}}(t) \|^2_{\mathcal{G}_{\leq N}} \leq C(r) e^{C(r)t} \left( \| \chi_r^{\text{rest}}(0) \|^2_{\mathcal{G}_{\leq N}} + N^{r-1} \right).
\] (145)

\textbf{Proof of Theorem 2.7} We abbreviate
\[
\tilde{\chi}_r^{\text{rest}}(t,s) = U(t,s) \chi - \sum_{\ell=0}^{r} N^{-\ell/2} U_\ell(t,s) \chi.
\] (146)

A computation analogous to (142) and (143) yields
\[
\| \tilde{\chi}_r^{\text{rest}}(t,s) \|^2_{\mathcal{G}_{\leq N}} \leq 2 \sum_{\ell=0}^{r} N^{-\ell/2} \text{Im} \int_0^t d\tilde{s} \left\langle \tilde{\chi}_r^{\text{rest}}(\tilde{s},s), S_N^{(r-\ell)}(\tilde{s}) U_\ell(\tilde{s},s) \chi \right\rangle
\]
\[
\leq 2 \sum_{\ell=0}^{r} N^{-\ell/2} \int_0^t d\tilde{s} \left\| \tilde{\chi}_r^{\text{rest}}(\tilde{s},s) \right\| \left\| S_N^{(r-\ell)}(\tilde{s}) U_\ell(\tilde{s},s) \chi \right\|.
\] (147)
The rest term is bounded in terms of number operators according to Lemma [5.2]. Furthermore, by the same computations as in the proof of Lemma [5.3], we deduce that

$$\| (N + 1)^n U_\ell(t, s) \chi \| \leq C(n, t) e^{C(n, t)|t| + |s|} ,$$

(148)

for all $n \in \mathbb{N}_0$ and for $\chi$ satisfying our assumption (140). Then the proof is concluded as in (141) and (145) in the proof of Theorem 2.9.

## A More details on the excitation Fock spaces

In this appendix we give further details about the unitary, defined by (19), and the derivation of the excitation Hamiltonian from (24). To this end, we closely follow [42, Chapter 4.1] and [43, Chapter 2.3]. First note that the unitary Weyl operator $W^* (\sqrt{N} \alpha(t))$ maps the Fock space $\mathcal{F} = \bigoplus_{n=0}^\infty (L^2(\mathbb{R}^3))^\otimes_n$ into itself. Under this mapping the coherent state $W(\sqrt{N} \alpha(t)) | \Omega \rangle$ is mapped to the vacuum state. Second we recall that any function $\Psi \in (L^2(\mathbb{R}^3))^\otimes N$ can be written as

$$\Psi = \psi^{(0)} \varphi(t)^\otimes N + \psi^{(1)} \otimes_s \varphi(t)^\otimes (N-1) + \psi^{(2)} \otimes_s \varphi(t)^\otimes (N-2) + \ldots + \psi^{(N)} .$$

(149)

One way to obtain this decomposition is to introduce for $k \in \{0, 1, \ldots, N\}$ the operators

$$P_{N,k} = \sum_{a \in \{0,1\}} \prod_{i=1}^N p_i^{a_i - q_i^{a_i}} = \frac{1}{k!(N-k)!} \sum_{\sigma \in S_N} q^{\sigma(k+1)} \ldots q^{\sigma(N)}$$

(150)

with $p_i(t) = |\varphi(t)\rangle \langle \varphi(t)|_i$ and $q_i(t) = 1 - p_i(t)$ satisfying the identity $\sum_{k=0}^N P_{N,k} = 1_{L^2(\mathbb{R}^{3N})}$ (see, e.g., [33, Section 3.3.1]). Then $\Psi = \sum_{k=0}^N P_{N,k} \Psi$ where $P_{N,k} \Psi = \psi^{(k)} \otimes_s \varphi^{\otimes (N-k)}$ with

$$\psi^{(k)} = \left( \begin{array}{c} N \\ k \end{array} \right)^{1/2} \prod_{i=1}^k q_i \langle \varphi^{(N-k)} , \psi \rangle_{L^2(\mathbb{R}^{3(N-k)})} .$$

(151)

The mapping $\tilde{U}_N(t) : (L^2(\mathbb{R}^3))^\otimes N \rightarrow \bigoplus_{k=0}^N (\varphi(t)^\perp)^\otimes_k$, $\Psi \mapsto \left( \psi^{(k)}(t) \right)_{k=0}^N$ is unitary because

$$\left\langle \psi^{(k)} \otimes_s \varphi(t)^\otimes (N-k) , \psi^{(l)} \otimes_s \varphi(t)^\otimes (N-l) \right\rangle_{L^2(\mathbb{R}^{3(N-k)})} = \delta_{k,l} \left\langle \psi^{(k)} , \psi^{(l)} \right\rangle_{L^2(\mathbb{R}^{3k})}$$

(152)

and therefore $\| \Psi \|^2_{L^2(\mathbb{R}^{3N})} = \| \psi^{(0)} \|^2 + \sum_{k=1}^N \| \psi^{(k)} \|^2_{L^2(\mathbb{R}^{3k})}$.

Let $U_N(t) : \mathcal{H}_N \rightarrow \left( \bigoplus_{k=0}^N (\varphi(t)^\perp)^\otimes_k \right) \otimes \mathcal{F}$ be the unitary operator defined by $U_N(t) = \tilde{U}_N(t) \otimes W^* (\sqrt{N} \alpha(t))$. In analogy to [43, Chapter 4] and [32, Chapter 4.1] we can use the inclusions $\mathcal{H}_N = (L^2(\mathbb{R}^3))^\otimes N \otimes \mathcal{F} \subset \mathcal{F} \otimes \mathcal{F}$ and $\left( \bigoplus_{k=0}^N (\varphi(t)^\perp)^\otimes_k \right) \otimes \mathcal{F}_a \subset \mathcal{F} \otimes \mathcal{F}$ to represent $U_N(t)$ and its adjoint in terms of annihilation and creation operators. To this end, we use $b(f) \otimes 1_F , b^*(f) \otimes 1_F , N_b \otimes 1_F$ and $1_F \otimes 1_F$ as the basic annihilation, creation and number of particles operators on the first and second Fock space of $\mathcal{F} \otimes \mathcal{F}$.

**Lemma A.1.** The operators $U_N(t)$ and $U_N(t)^*$ can equivalently be written as

$$U_N(t) \Psi_N = W^* (\sqrt{N} \alpha(t)) \left( \bigoplus_{k=0}^N q(t)^\otimes_k \frac{b(\varphi(t))^{N-k}}{\sqrt{N-k}} \right) \Psi_N ,$$

(153)

$$U_N(t)^* \left( \bigoplus_{k=0}^N \chi^{(k)}_\in \right) = W \left( \sqrt{N} \alpha(t) \right) \sum_{k=0}^N \frac{b^*(\varphi(t))^{N-k}}{\sqrt{N-k}} \chi^{(k)}_\in$$

(154)
for all \( \Psi \in \mathcal{H}_N \) and \( \chi^{(k)}_{\leq N} \in (\varphi(t)^\perp)^{\otimes k} \otimes \mathcal{F} \), \( k = 0, \ldots, N \). On \( \left( \bigoplus_{k=0}^N (\varphi(t)^\perp)^{\otimes k} \right) \otimes \mathcal{F} \) we have

\[
U_N(t)b^*(\varphi(t))b(\varphi(t))U_N(t)^* = N - (\mathcal{N}_b(t))_+ 
\]

\( (155a) \)

\[
U_N(t)b^*(f)b(\varphi(t))U_N(t)^* = b^*(f) \left[ N - (\mathcal{N}_b(t))_+ \right]^{1/2}.
\]

\( (155b) \)

\[
U_N(t)b^*(\varphi(t))b(f)U_N(t)^* = \left[ N - (\mathcal{N}_b(t))_+ \right]^{1/2} b(f),
\]

\( (155c) \)

\[
U_N(t)b^*(f)b(g)U_N(t)^* = b^*(f)b(g),
\]

\( (155d) \)

for all \( f, g \in \{\varphi(t)^\perp\} \) and \( (\mathcal{N}_b(t))_+ = \mathcal{N}_b - b^*(\varphi(t))b(\varphi(t)) \). Moreover,

\[
U_N(t) a(h) U_N(t)^* = a(h) + \sqrt{\mathcal{N}} \langle \psi, h \rangle_{L^2(\mathbb{R}^3)},
\]

\( (156a) \)

\[
U_N(t) a^*(h) U_N(t)^* = a^*(h) + \sqrt{\mathcal{N}} \langle \alpha(h), h \rangle_{L^2(\mathbb{R}^3)},
\]

\( (156b) \)

for all \( h \in L^2(\mathbb{R}^3) \).

**Proof.** The first part of the Lemma is a direct consequence of \( \tilde{U}_N(t) \otimes 1_F, 1_F \otimes W^* \left( \sqrt{\mathcal{N}} \alpha(t) \right) = 0 \) and Proposition 4.2. The relations \( (156a) \) and \( (156b) \) follow from the shifting property of the Weyl operators. \( \square \)

In the following, we briefly explain how one derives the Schrödinger equation \( (22) \). To this end, we need to compute the time derivative of the unitary mapping.

**Lemma A.2.** Let \( \{\varphi(t), \alpha(t)\} \) be a sufficiently regular trajectory on \( L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3) \) satisfying \( \|\varphi(t)\|_{L^2(\mathbb{R}^3)} = \|\varphi(0)\|_{L^2(\mathbb{R}^3)} \) for all \( t \geq 0 \). Then the time derivative of \( U_N(t) \) is

\[
i \dot{U}_N(t) = \left[ b^*(\varphi(t)) b(q(t)i\dot{\varphi}(t)) - \sqrt{N} - (\mathcal{N}_b(t))_+ b(q(t)i\dot{\varphi}(t)) - b^*(q(t)i\dot{\varphi}(t)) \sqrt{N} - (\mathcal{N}_b(t))_+ - \langle i\dot{\varphi}(t), \varphi(t) \rangle (N - (\mathcal{N}_b(t))_+) - \mathcal{N} Re \langle i\dot{\alpha}(t), \alpha(t) \rangle - \sqrt{\mathcal{N}} a^* (i\dot{\alpha}(t)) - \sqrt{\mathcal{N}} a^* (i\dot{\alpha}(t)) \right] U_N(t).
\]

\( (157) \)

**Proof.** From \( \text{[12] Lemma 6} \) we know that

\[
i \dot{U}_N(t) \otimes 1_F = \left[ b^*(\varphi(t)) b(q(t)i\dot{\varphi}(t)) - \sqrt{N} - (\mathcal{N}_b(t))_+ b(q(t)i\dot{\varphi}(t)) - b^*(q(t)i\dot{\varphi}(t)) \sqrt{N} - (\mathcal{N}_b(t))_+ - \langle i\dot{\varphi}(t), \varphi(t) \rangle (N - (\mathcal{N}_b(t))_+) \right] \tilde{U}_N(t) \otimes 1_F.
\]

\( (158) \)

Combining this with

\[
i \tilde{U}_N(t) \otimes 1_F \left( \sqrt{\mathcal{N}} \alpha(t) \right) = - \left[ \mathcal{N} Re \langle i\dot{\alpha}(t), \alpha(t) \rangle + \sqrt{\mathcal{N}} a^* (i\dot{\alpha}(t)) \right] 1_F \otimes W^* \left( \sqrt{\mathcal{N}} \alpha(t) \right)
\]

\( (159) \)

and using that both unitary mappings commute shows the claim. The last equality can be obtained by \( \text{[20] Lemma A.3.} \) \{\( \alpha = 1, a = b \) and \( a^* = b^* \)\} and the fact that \( W^* (f) = W(-f) \) for \( f \in L^2(\mathbb{R}^3) \). \( \square \)

Now, let \( \Psi_N(t) \) and \( \{\varphi(t), \alpha(t)\} \) be solutions of \( \text{[22]} \) and \( \text{[13a]–[13b]} \) such that \( \|\varphi(0)\|_{L^2(\mathbb{R}^3)} = 1 \).

Then \( \chi_{\leq N}(t) \in \left( \bigoplus_{k=0}^N (\varphi(t)^\perp)^{\otimes k} \right) \otimes \mathcal{F} \subset \mathcal{F} \otimes \mathcal{F} \), given by \( \chi_{\leq N}(t) = U(t) \Psi_N(t) \) satisfies

\[
i \dot{\chi}_{\leq N}(t) = \left( U_N(t) H_N^{\text{Nelson}} U_N(t)^* + i \dot{U}_N(t) U_N(t)^* \right) \chi_{\leq N}(t).
\]

\( (160) \)
The Schrödinger equation (22) is then obtained by means of Lemma A.2 the Schrödinger–Klein–Gordon equations (13a)–(13b),
\[ b^*(\varphi(t))b(q(t)h(t)\varphi(t)) = \int dx b^*(x)\left(h(t) - h(t)p(t) - q(t)h(t)q(t)\right)b(x) \quad (161) \]
and
\[
U_N(t)H_N^{\text{Nelson}}U_N(t)^* = N\left\|\sqrt{\omega}\alpha(t)\right\|^2 + H_f + N^{1/2}\left\{a(\omega\alpha(t)) + a^*(\omega\alpha(t))\right\} \\
+ \left\langle\varphi(t), h(t)\varphi(t)\right\rangle + \mu(t)\left(N - (\mathcal{N}_b(t))^+_+\right) \\
+ \int dx b^*(x)q(t)(-\Delta + \Phi(t, \cdot))q(t)b(x) \\
+ N^{-1/2}\int dx b^*(x)\left(q(t)\hat{\Phi}q(t) - \langle\varphi(t), \hat{\Phi}\varphi(t)\rangle\right)b(x) \\
+ N^{-1/2}\langle\varphi(t), \hat{\Phi}\varphi(t)\rangle\left(N - (\mathcal{N}_b(t))^+ + N\right) \\
+ \left\{b^*(q(t)(-\Delta + \Phi(t, \cdot))\varphi(t))\left[N - (\mathcal{N}_b(t))^+_+\right]^{1/2} + \text{h.c.}\right\} \\
+ \int dx \int dk K(t, k, x)(a^*(k) + a(-k))b^*(x)[1 - N^{-1}(\mathcal{N}_b(t))^+_+]^{1/2} + \text{h.c.} \quad (162) \]

In order to obtain Equation (162) one needs to write $H_N^{\text{Nelson}}$ in the second quantized form on $\mathcal{F} \otimes \mathcal{F}$ and proceed in a similar way as in [43, Chapter 4] and [12, Appendix B]. Moreover, note that $(\mathcal{N}_b(t))^+_+ \chi \leq N = \mathcal{N}_b\chi \leq N$ holds for all $\chi \leq N \in \left(\bigoplus_{k=0}^{N}(\varphi(t)\downarrow)\otimes k\right) \otimes \mathcal{F}$. For notational convenience, we define the Fock spaces $\mathcal{F}_a$, $\mathcal{F}_b$ and $\mathcal{G}_{\leq N}$ as in [14], [16] and [20] and view $U_N(t)$ as a mapping from $\mathcal{H}_N$ to $\mathcal{G}_{\leq N}$.

Acknowledgments. We would like to thank Phan Thanh Nam for helpful remarks regarding the well-posedness of the Bogoliubov time evolution. N.L. gratefully acknowledges support from the SNSF Eccellenza project PCEFP2 181153 and the NCCR SwissMAP. M.F. has been partially supported by the European Research Council (ERC) under the European Union’s Horizon 2020 research and innovation programme (ERC CoG UniCoSM, grant agreement n.724939).

References

[1] Z. Ammari. Asymptotic completeness for a renormalized nonrelativistic Hamiltonian in quantum field theory: the Nelson model. *Math. Phys. Anal. Geom.* 3(3), 217–285 (2000).

[2] Z. Ammari and M. Falconi. Wigner measures approach to the classical limit of the Nelson model: convergence of dynamics and ground state energy. *J. Stat. Phys.* 157(2), 330–362 (2014).

[3] Z. Ammari and M. Falconi. Bohr’s correspondence principle for the renormalized Nelson model. *SIAM J. Math. Anal.* 49(6), 5031–5095 (2017).

[4] C. Boccato, S. Cenatiempo, and B. Schlein. Quantum many-body fluctuations around nonlinear Schrödinger dynamics. *Ann. Henri Poincaré*, 18(1):113–191 (2016).

[5] L. Boßmann, S. Petrat, P. Pickl, and A. Soffer. Beyond Bogoliubov dynamics. *Pure Appl. Anal.*, to appear. Preprint, arXiv:1912.11004 (2019).

[6] L. Boßmann, S. Petrat and R. Seiringer. Asymptotic expansion of low-energy excitations for weakly interacting bosons. *Forum Math. Sigma* 9, E28 (2021).
[7] O. Bratteli and D. W. Robinson. Operator algebras and quantum-statistical mechanics. II. Equilibrium states. Models in quantum-statistical mechanics. Texts and Monographs in Physics. Springer-Verlag, New York-Berlin (1981).

[8] C. Brennecke, P. T. Nam, M. Napiórkowski and B. Schlein. Fluctuations of N-particle quantum dynamics around the nonlinear Schrödinger equation. Ann. Inst. H. Poincaré Anal. Non Linéaire 36(5), 1201–1235 (2019).

[9] R. Carlone, M. Correggi, M. Falconi and M. Olivieri. Microscopic derivation of time-dependent point interactions. SIAM J. Math. Anal. 53(4), 4657–4691 (2021).

[10] J. Chong. Derivation of large boson systems with attractive interaction and a derivation of the cubic focusing NLS in $\mathbb{R}^3$. Preprint arXiv:1608.01615 (2016).

[11] J. Colliander, J. Holmer and N. Tzirakis. Low regularity global well-posedness for the Zakharov and Klein–Gordon–Schrödinger systems. Trans. Amer. Math. Soc. 360(9), 4619–4638 (2008).

[12] M. Correggi and M. Falconi. Effective potentials generated by field interaction in the quasi-classical limit. Ann. Henri Poincaré 19(1), 189-235 (2018).

[13] M. Correggi, M. Falconi and M. Olivieri. Quasi-classical dynamics. J. Eur. Math. Soc., to appear. Preprint arXiv:1909.13313 (2019).

[14] M. Correggi, M. Falconi and M. Olivieri. Ground state properties in the quasi-classical regime. Preprint arXiv:2007.09442 (2020).

[15] E. B. Davies. Particle-boson interactions and the weak coupling limit. J. Math. Phys. 20, 345–351 (1979).

[16] J. Dereziński and C. Gérard. Asymptotic completeness in quantum field theory. Massive Pauli–Fierz Hamiltonians. Rev. Math. Phys. 11(4), 383–450 (1999).

[17] J. Dereziński and C. Gérard. Mathematics of Quantization and Quantum Fields (Cambridge Monographs on Mathematical Physics). Cambridge: Cambridge University Press (2013).

[18] M. Falconi. Classical limit of the Nelson model with cutoff. J. Math. Phys. 54(1), 012303 (2013).

[19] D. Feliciangeli, S. Rademacher and R. Seiringer. Persistence of the spectral gap for the Landau–Pekar equations. Lett. Math. Phys. 111(19) (2021).

[20] R.L. Frank and Z. Gang. Derivation of an effective evolution equation for a strongly coupled polaron. Anal. PDE 10(2), 379–422 (2017).

[21] R.L. Frank and B. Schlein. Dynamics of a strongly coupled polaron. Lett. Math. Phys. 104, 911–929 (2014).

[22] J. Ginibre and G. Velo. The classical field limit of scattering theory for non-relativistic many-boson systems. I. Commun. Math. Phys. 66(1), 37–76 (1979).

[23] J. Ginibre and G. Velo. The classical field limit of scattering theory for non-relativistic many-boson systems. II. Commun. Math. Phys. 68(1), 45–68 (1979).

[24] J. Ginibre and G. Velo. The classical field limit of non-relativistic bosons. II. Asymptotic expansions for general potentials. Ann. Inst. H. Poincaré Physique théorique 33(4), 363–394 (1980).

[25] J. Ginibre and G. Velo. The classical field limit of nonrelativistic bosons. I. Borel summability for bounded potentials. Ann. Phys. 128(2), 243–285 (1980).
[26] M. Griesemer. On the dynamics of polarons in the strong-coupling limit. *Rev. Math. Phys.* 29(10), 1750030 (2017).

[27] M. Griesemer and A. Wünsch. On the domain of the Nelson Hamiltonian. *J. Math. Phys.* 59(4), 042111 (2018).

[28] M. Grillakis and M. Machedon. Pair excitations and the mean field approximation of interacting Bosons. I. *Commun. Math. Phys.*, 324(2):601–636 (2013).

[29] M. Grillakis and M. Machedon. Pair excitations and the mean field approximation of interacting Bosons. II. *Commun. PDE* 42(2), 24–67 (2017).

[30] M. Grillakis, M. Machedon and D. Margetis. Second-order corrections to mean field evolution of weakly interacting bosons. I. *Commun. Math. Phys.* 294(1), 273 (2010).

[31] M. Grillakis, M. Machedon and D. Margetis. Second-order corrections to mean field evolution of weakly interacting bosons. II. *Adv. Math.* 228(3), 1778–1815 (2011).

[32] F. Hiroshima. Weak coupling limit with a removal of an ultraviolet cutoff for a Hamiltonian of particles interacting with a massive scalar field. *Infin. Dimens. Anal. Qu.* 1, 407–423 (1998).

[33] A. Knowles and P. Pickl. Mean-field dynamics: singular potentials and rate of convergence. *Comm. Math. Phys.* 298, 101–138 (2009).

[34] E. Kuz. Exact evolution versus mean field second-order correction for bosons interacting via short-range two-body potential. *Differ. Integral Equ.** 30(7/8), 587–630 (2017).

[35] J. Lampart, J. Schmidt, S. Teufel and R. Tumulka. Particle creation at a point source by means of interior-boundary conditions. *Math. Phys. Anal. Geom.* 21, 12 (2018).

[36] N. Leopold, D. Mitrouskas, S. Rademacher, B. Schlein and R. Seiringer. Landau–Pekar equations and quantum fluctuations for the dynamics of a strongly coupled polaron. *Pure Appl. Anal* (in press).

[37] N. Leopold, D. Mitrouskas and R. Seiringer. Derivation of the Landau–Pekar equations in a many-body mean-field limit. *Arch. Ration. Mech. Anal.* 240, 383–417 (2021).

[38] N. Leopold and S. Petrat. Mean-field dynamics for the Nelson model with fermions. *Ann. Henri Poincaré* 20(10), 3471–3508 (2019).

[39] N. Leopold and P. Pickl. Mean-field limits of particles in interaction with quantized radiation fields. In: D. Cadamuro, M. Duell, W. Dybalski, and S. Simonella (eds) *Macroscopic Limits of Quantum Systems*, volume 270 of Springer Proceedings in Mathematics & Statistics, 185–214 (2018).

[40] N. Leopold and P. Pickl. Derivation of the Maxwell–Schrödinger equations from the Pauli–Fierz Hamiltonian. *SIAM J. Math. Anal.* 52(5), 4900–4936 (2020).

[41] N. Leopold, S. Rademacher, B. Schlein and R. Seiringer. The Landau–Pekar equations: adiabatic theorem and accuracy. *Anal. & PDE* (in press).

[42] M. Lewin, P. T. Nam, and B. Schlein. Fluctuations around Hartree states in the mean-field regime. *Am. J. Math.*, 137(6):1613–1650 (2015).

[43] M. Lewin, P. T. Nam, S. Serfaty, J. P. Solovej. Bogoliubov spectrum of interacting Bose gases. *Comm. Pure Appl. Math.* 68(3), 413–471 (2015).
[44] O. Matte and J.S. Møller. Feynman-Kac formulas for the ultra-violet renormalized Nelson model. *Astérisque* 404 (2018).

[45] A. Michelangeli, P.T. Nam and A. Olgiati. Ground state energy of mixture of Bose gases. *Rev. Math. Phys.*, 31(2) (2019).

[46] D. Mitrouskas. A note on the Fröhlich dynamics in the strong coupling limit. *Lett. Math. Phys.* 111, 45 (2021).

[47] D. Mitrouskas, S. Petrat, and P. Pickl. Bogoliubov corrections and trace norm convergence for the Hartree dynamics. *Rev. Math. Phys.*, 31(8) (2019).

[48] J.S. Møller. On the essential spectrum of the translation invariant Nelson model. *Mathematical physics of quantum mechanics*, 179–195. *Lecture Notes in Physics*, 690, Springer, Berlin (2006).

[49] P.T. Nam and M. Napiórkowski. A note on the validity of Bogoliubov correction to mean-field dynamics. *J. Math. Pures Appl.*, 108(5):662–688 (2017).

[50] P.T. Nam and M. Napiórkowski. Bogoliubov correction to the mean-field dynamics of interacting bosons. *Adv. Theor. Math. Phys.*, 21(3):683–738 (2017).

[51] P.T. Nam and M. Napiórkowski. Norm approximation for many-body quantum dynamics and Bogoliubov theory. In: A. Michelangeli, and G. Dell’Antonio, editors, *Advances in Quantum Mechanics: contemporary trends and open problems*. Springer-INdAM Series, 18 (2017).

[52] P.T. Nam and M. Napiórkowski. Norm approximation for many-body quantum dynamics: focusing case in low dimensions. *Adv. Math.* 350, 547–587 (2019).

[53] E. Nelson. Interaction of nonrelativistic particles with a quantized scalar field. *J. Math. Phys.* 5, 1190 (1964).

[54] H. Pecher. Some new well-posedness results for the Klein–Gordon–Schrödinger system. *Diff. Int. Equations* 25(1/2), 117–142 (2012).

[55] S. Petrat, P. Pickl, and A. Soffer. Derivation of the Bogoliubov time evolution for a large volume mean-field limit. *Ann. Henri Poincaré* 21(2), 461–498 (2020).

[56] P. Pickl. A simple derivation of mean field limits for quantum systems. *Lett. Math. Phys.* 97, 151–164 (2011).

[57] A. Pizzo. Scattering of an Infraparticle: the one particle sector in Nelson’s massless model. *Ann. Henri Poincaré* 6(3), 553-606 (2005).

[58] I. Rodnianski and B. Schlein. Quantum fluctuations and rate of convergence towards mean field dynamics. *Commun. Math. Phys.* 291(1), 31–61 (2009).

[59] J.P. Solovej. Many Body Quantum Mechanics. *Lecture notes*, http://web.math.ku.dk/~solovej/MANYBODY/mbnotes-ptn-5-3-14.pdf, 2007.

[60] S. Teufel. Effective N-body dynamics for the massless Nelson model and adiabatic decoupling without spectral gap. *Ann. Henri Poincaré* 3, 939–965 (2002).