A Quantum Version of Sanov’s Theorem

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Abstract

We present a quantum extension of a version of Sanov’s theorem focusing on a hypothesis testing aspect of the theorem: There exists a sequence of typical subspaces for a given set $\Psi$ of stationary quantum product states asymptotically separating them from another fixed stationary product state. Analogously to the classical case, the exponential separating rate is equal to the infimum of the quantum relative entropy with respect to the quantum reference state over the set $\Psi$. However, while in the classical case the separating subsets can be chosen universal, in the sense that they depend only on the chosen set of i.i.d. processes, in the quantum case the choice of the separating subspaces depends additionally on the reference state.

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1 Introduction

In this article we present a natural quantum version of the classical Sanov's theorem as part of our attempt to explore basic concepts and results at the interface of classical information theory and stochastics from the point of view of quantum information theory.

Among those classical results a crucial role plays the Shannon-McMillan-Breiman theorem (SMB theorem) which clarifies the concept of typical subsets, yielding the rigorous background for asymptotically optimal lossless data compression. It says that a long $n$-block message string from an ergodic data source belongs most likely to a typical subset (of the generally very much larger set of all possible messages). The cardinality within the sequence of these typical sets grows with the length $n$ of the message at an exponential rate given by the Shannon entropy rate of the data source. As a general rule, when passing to the quantum situation, the notion of a typical subset has to be replaced by that of a typical subspace (of an entire Hilbert space describing the pure $n$-block states of the quantum data source), with the dimension of that subspace being the quantity growing exponentially fast at a rate given now by the von Neumann entropy rate as $n$ goes to infinity (cf. [11] in the i.i.d. situation, [1], [2] in the general ergodic case).

We know from the classical situation that typical subsets have even more striking properties, when chosen in the right way: For a given alphabet $A$ and a given entropy rate there is (a bit surprisingly) a universal sequence of typical subsets growing at the given rate for all ergodic sources which do not top the given entropy rate (this result has been generalized to the quantum context by Kaltchenko and Yang [12]). Moreover, for any ergodic data source $P$ we can find a sequence of typical subsets growing at the rate given by the entropy and at the same time separating it exponentially well from any i.i.d. (reference) data source $Q$ in the sense that the $Q$-probability of the entire $P$-typical subset goes to zero at an exponential rate given by the relative entropy rate $h(P, Q)$. Furthermore, the relative entropy is the best achievable (optimal) separation rate. This assertion which gives an operational interpretation of the relative entropy is Stein's lemma. We mention that the i.i.d. condition concerning the reference source cannot be weakened too much, since there are examples where even the relative entropy has no asymptotic rate, though the reference source is very well mixing (B-process, cf. [15]). A quantum generalization of this result can be found in [12] for the case that both sources are i.i.d., and in [5] for the case of a general ergodic quantum information source. This result was mainly inspired by [11], where complete ergodicity was assumed and optimality was still left open.

From the viewpoint of information theory or statistical hypothesis testing the essential assertion of Sanov's theorem is that it represents a universal version of Stein’s lemma by saying that for a set $\Omega$ of i.i.d. sources there exists a common choice of the typical set such that the probability with respect to the i.i.d. reference source $Q$ goes to zero at a rate given by $\inf_{P \in \Omega} h(P, Q)$. 
Originally Sanov’s theorem is of course a result on large deviations of empirical distributions (cf. [4], [8]). It is the information-theoretical viewpoint taken here which suggests to look at it as a large deviation principle for typical subsets. With the main topic of this paper being a quantum theorem of Sanov type, it is especially appealing to shift the focus from empirical distributions to typical subspaces, since the notion of an individual quantum message string is at least problematic, and as will be seen by an example, a reasonable attempt to define something like a quantum empirical distributions via partial traces leads to a separation rate worse than the relative entropy rate (see the last section).

Another aspect of the classical Sanov result has to be modified for the quantum situation: The typical subspace will no longer be universal for all i.i.d. reference sources, but has to be chosen in dependence of the reference source. So only ‘one half’ of universality is maintained when passing to quantum sources, namely that which refers to the set $\Omega$. This will be demonstrated by an example in the last section. The basic mechanism behind this no go result is - heuristically speaking: In the quantum setting even pure states cannot be distinguished with certainty, while classical letters can. In our forthcoming paper [4] we extend the results given here to the case where only stationarity is assumed for the states in $\Omega$.

## 2 A quantum version of Sanov’s theorem

Let $A$ be a finite set with cardinality $\#A = d$. By $\mathcal{P}(A)$ we denote the set of probability distributions on $A$. The relative entropy $H(P,Q)$ of a probability distribution $P \in \mathcal{P}(A)$ with respect to a distribution $Q$ is defined as usual:

$$H(P,Q) := \begin{cases} \sum_{a \in A} P(a)(\log P(a) - \log Q(a)), & \text{if } P \ll Q \\ \infty, & \text{otherwise,} \end{cases}$$

where $\log$ denotes the base 2 logarithm. For the base $e$ logarithm we use the notation $\ln$. The function $H(\cdot, Q)$ is continuous on $\mathcal{P}(A)$, if the reference distribution $Q$ has full support $A$. Otherwise it is lower semi-continuous. The relative entropy distance from the reference distribution $Q$ to a subset $\Omega \subseteq \mathcal{P}(A)$ is given by:

$$H(\Omega, Q) := \inf_{P \in \Omega} H(P,Q).$$

Our starting point is the classical Sanov’s theorem formulated from the point of view of hypothesis testing:

**Theorem 2.1 (Sanov’s Theorem)** Let $Q \in \mathcal{P}(A)$ and $\Omega \subseteq \mathcal{P}(A)$. There exists a sequence $\{M_n\}_{n \in \mathbb{N}}$ of subsets $M_n \subseteq A^n$ with

$$\lim_{n \to \infty} P^n(M_n) = 1, \quad \forall P \in \Omega,$$

where $P^n(M_n)$ denotes the probability of the subset $M_n$ under the distribution $P^n$. The theorem states that for sequences of empirical distributions, the probability of being in the typical set grows exponentially with the length of the sequence.

The quantum version of Sanov’s theorem extends this classical result to the quantum domain, where the typical sets are replaced by typical subspaces and the classical notion of typicality is replaced by a quantum version that takes into account the non-commutative nature of quantum states.

In our forthcoming paper [4] we extend the results given here to the case where only stationarity is assumed for the states in $\Omega$.
such that
\[ \lim_{n \to \infty} \frac{1}{n} \log Q^n(M_n) = -H(\Omega, Q). \]

Moreover, for each sequence of sets \( \{\tilde{M}_n\} \) fulfilling (4) we have
\[ \liminf_{n \to \infty} \frac{1}{n} \log Q^n(\tilde{M}_n) \geq -H(\Omega, Q), \]
such that \( H(\Omega, Q) \) is the best achievable separation rate.

We emphasize that in the above formulation we omitted the assertion that the sets \( M_n \) can be chosen independently from the reference distribution \( Q \). However, as will be shown in the last section, in the quantum case this universality feature is not valid any longer and Theorem 3.2.21 in [8] is the strongest version that has a quantum analogue. It is an immediate consequence of Lemma (2.3) and is related to the usual formulation of Sanov’s theorem ([14], see also Theorem 3.2.21 in [8]) in terms of empirical measures \( P_n := \frac{1}{n} \sum_{i=1}^n \delta_{x_i} \) for sequences \( x^n := \{x_1, \ldots, x_n\} \) as follows: By the strong law of large numbers, the sequence of empirical distributions \( \{P_n\} \) formed along an i.i.d. sequence \( \{x_1, x_2, \ldots\} \) of letters distributed according to a probability measure \( P \in \mathcal{P}(A) \) tends to \( P \) almost surely. Hence for any neighbourhood \( U \) of \( P \) we have the limit relation
\[ \lim_{n \to \infty} P^n(\{x^n : P_{x^n} \in U\}) = 1 \]
meaning that the sequence of sets \( \{x^n : P_{x^n} \in U\} \) is typical for \( P^n \). If \( \Omega \) is an open set, then we may choose \( U \) as \( \Omega \) and \( \{x^n : P_{x^n} \in U\} \) is universally typical for all \( P^n, P \in \Omega \). Now Sanov’s theorem in its traditional form says that
\[ -\frac{1}{n} \log Q^n(\{x^n : P_{x^n} \in \Omega\}) \to H(\Omega, Q). \]
So it says that the (explicitly specified) typical sets separate \( Q^n \) exponentially fast from all \( P^n, P \in \Omega \) with the given order \( H(\Omega, Q) \).

Passing to the quantum setting we substitute the set \( A \) by a \( C^* \)-algebra \( A \) with dimension \( \dim A = d < \infty \) and the cartesian product \( A^n := \prod_{i=1}^n A \) by the tensor product \( A^{(n)} := \bigotimes_{i=1}^n A \). We denote by \( S(A) \) the set of quantum states on the algebra of observables \( A \), i.e. \( S(A) \) is the set of positive functionals \( \varphi \) on \( A \) fulfilling the normalisation condition \( \varphi(1) = 1 \). For \( \varphi \in S(A) \) we mean by \( \varphi^{\otimes n} \) a product state on \( A^{(n)} \). The quantum relative entropy \( S(\psi, \varphi) \) of the state \( \psi \in S(A) \) with respect to the reference state \( \varphi \in S(A) \) is defined by:
\[ S(\psi, \varphi) := \begin{cases} \text{tr}_{\mathcal{A}} D_\psi(\log D_\psi - \log D_\varphi), & \text{if supp}(\psi) \leq \text{supp}(\varphi) \\ \infty, & \text{otherwise.} \end{cases} \]

Observe that in the case of a commutative \( C^* \)-algebra \( A \) the quantum relative entropy \( S \) coincides with the classical relative entropy \( H \) defined in (4), where the probabilities are defined as the expectations of minimal projectors in \( A \). The functional \( S(\cdot, \varphi) \) is continuous on \( S(A) \) only if the reference state \( \varphi \) is faithful, i.e. \( \text{supp}\varphi = 1_A \). otherwise it is lower semi-continuous. The relative entropy distance from the reference state \( \varphi \) to a subset \( \Psi \subset S(A) \) is given by:
\[ S(\Psi, \varphi) := \inf_{\psi \in \Psi} S(\psi, \varphi). \]

Now we are in the position to state our main result:
Theorem 2.2 (Quantum Sanov Theorem) Let $\varphi \in S(A)$ and $\Psi \subseteq S(A)$. There exists a sequence $\{p_n\}_{n \in \mathbb{N}}$ of orthogonal projections $p_n \in A^n$ such that
\[ \lim_{n \to \infty} \psi^\otimes n(p_n) = 1, \quad \forall \psi \in \Psi, \quad (7) \]
and
\[ \lim_{n \to \infty} \frac{1}{n} \log \varphi^\otimes n(p_n) = - \inf_{\psi \in \Psi} S(\psi, \varphi). \]
Moreover, for each sequence of projections $\{\tilde{p}_n\}$ fulfilling (7) we have
\[ \liminf_{n \to \infty} \frac{1}{n} \log \varphi^\otimes n(\tilde{p}_n) \geq - \inf_{\psi \in \Psi} S(\psi, \varphi), \]
such that $S(\Psi, \varphi)$ is the best achievable separation rate.

The proof of Theorem 2.2 will be based to a large extent on the following classical lemma, which is a stronger version of Theorem 2.1.

Lemma 2.3 Let $Q \in \mathcal{P}(A)$ and $\Omega \subseteq \mathcal{P}(A)$. For each sequence $\{\varepsilon_n\}_{n \in \mathbb{N}}$ satisfying $\varepsilon_n \searrow 0$ and $\frac{\log(n+1)}{\varepsilon_n} \to 0$ there exists a sequence $\{M_n\}_{n \in \mathbb{N}}$ of subsets $M_n \in A^n$ such that for each $P \in \Omega$ there is an $N(P) \in \mathbb{N}$ with
\[ P^n(M_n) \geq 1 - (n + 1)^\#A \cdot 2^{-n b \varepsilon_n^2}, \quad \forall n \geq N(P), \quad (8) \]
where $b$ is a positive number. Moreover we have:

1. $\liminf_{n \to \infty} \frac{1}{n} \log Q^n(M_n) \geq -H(\Omega, Q)$,
2. $Q^n(M_n) \leq (n + 1)^\#A \cdot 2^{-n I_n}$, \quad $\forall n \in \mathbb{N}$,
where $I_n \geq 0$ and for all $n \in \mathbb{N}$ fulfilling $\varepsilon_n \leq \frac{1}{n}$
\[ 0 \leq H(\Omega, Q) - I_n \leq \log(\#A)\varepsilon_n - \varepsilon_n \log \varepsilon_n - \varepsilon_n \log Q_{\min}, \quad (9) \]
holds with $Q_{\min} := \min\{Q(a) : Q(a) > 0, a \in A\}$.

Proof: Due to the classical Stein’s lemma any sequence of subsets $\{M_n\}_{n \in \mathbb{N}}$, which has asymptotically a non vanishing measure with respect to the product distributions $P^n$ satisfies:
\[ \liminf_{n \to \infty} \frac{1}{n} Q^n(M_n) \geq -H(P, Q), \quad (10) \]
where $Q \in \mathcal{P}(A)$ is the reference distribution. Then the lower bound (10) implies the first item of lemma 2.3.

We partition the set $\Omega$ into the set $\Omega_1$ consisting of probability distributions which are absolutely continuous w.r.t. $Q$ and its complement $\Omega_2$ within $\Omega$, i.e.
\[ \Omega_1 := \{P \in \Omega : H(P, Q) < \infty\}, \quad \text{and} \quad \Omega_2 := \Omega \setminus \Omega_1. \]
Observe that
\[ H(\Omega, Q) = \begin{cases} H(\Omega_1, Q), & \text{if } \Omega_1 \neq \emptyset \\ \infty, & \text{otherwise.} \end{cases} \] (11)
holds. We will treat these two sets separately.

It is obvious that we can ideally distinguish the distributions in \( \Omega_2 \) from \( Q \), we just have to set
\[ M_{2,n} := \{ x^n \in A^n : Q^n(x^n) = 0 \text{ and } P^n(x^n) > 0 \text{ for some } P \in \Omega_2 \}. \] (12)
Then we have for all \( n \in \mathbb{N} \)
\[ Q^n(M_{2,n}) = 0. \] (13)
Moreover we have for each \( P \in \Omega_2 \) and \( n \in \mathbb{N} \)
\[ P^n(M_{2,n}) = 1 - q_P \to 1 \quad (n \to \infty), \] (14)
where
\[ q_P := P(A_+), \]
with
\[ A_+ := \{ a \in A : Q(a) > 0 \}. \] (15)

Observe that the speed of convergence in (14) is exponential.

In treating the set \( \Omega_1 \) we may consider the restricted alphabet \( A_+ \) defined in (15) only. Note that \( H(\cdot, Q) \) is continuous as a functional on \( \mathcal{P}(A_+) \). Choose a sequence \( \varepsilon_n \searrow 0 \) with \( \frac{\log(n+1)}{n\varepsilon_n^2} \to 0 \) and define the following decreasing family of sets
\[ \Omega_n := \{ R \in \mathcal{P}(A_+ ) : \|R - P\|_1 \leq \varepsilon_n \text{ for at least one } P \in \Omega_1 \}. \]
Observe that \( \Omega_n \searrow \Omega_1 \). Moreover we set
\[ M_{1,n} := \{ x^n \in A^n_+ : P_{x^n} \in \Omega_n \}, \] (16)
where \( P_{x^n} \) denotes the empirical distribution or type of the sequence \( x^n \). Now, by type counting methods (cf. [7] section 12.1) and Pinsker’s inequality \( H(P_1, P_2) \geq \frac{1}{2 \ln 2} \|P_1 - P_2\|_1^2 \) we arrive at
\[ P^n(M_{1,n}^c) \leq (n + 1)^{\#A_+} 2^{-n\varepsilon_n^2} \to 0 \quad (n \to \infty), \] (17)
for each \( P \in \Omega_1 \) where \( b \) is a positive number and \( M_{n}^c \) denotes the complement of the set \( M_n \).

The upper bounds with respect to the distribution \( Q \) are a consequence of type
counting methods together with the (lower semi-) continuity of the functional $H(\cdot, Q)$ combined with (11) and the fact that $\Omega_n \searrow \Omega_1$:

$$Q^n(M_{1,n}) \leq (n+1)^{A+2^{-nH(\Omega_n, Q)}} \quad \text{(by type counting, cf. [7] sect. 12.1).}$$

We set $I_n := H(\Omega_n, Q)$. Observe that the sequence $(I_n)_{n \in \mathbb{N}}$ is increasing and $I_n \leq \min_{P \in \Omega_1} H(P, Q)$, for all $n \in \mathbb{N}$, since $\Omega_n \searrow \Omega_1$. Next, observe that

$$\widetilde{\Omega}_n \subseteq \{ R \in \mathcal{P}(A^+) : \| R - P \|_1 \leq \varepsilon_n \quad \text{for at least one } P \in \Omega_1 \}. \quad (19)$$

Let $R_n \in \widetilde{\Omega}_n$ be such that $H(R_n, Q) = \min_{R \in \Omega_n} H(R, Q)$. By the continuity we have $H(R_n, Q) = I_n$. According to (19) for each $n \in \mathbb{N}$ there is a distribution $P_n \in \Omega_1$ such that $\| R_n - P_n \|_1 \leq \varepsilon_n$. Using the inequality $|H(P) - H(R)| \leq \log(\#A)\| P - R \|_1 + \eta(\| P - R \|_1)$ valid for distributions $P, R$ with $\| P - R \|_1 \leq \frac{1}{2}$, where $\eta(t) := -t \log t$, and $Q_{\min} := \min \{ Q(a) : a \in A^+ \}$, we obtain finally

$$0 \leq H(\Omega, Q) - I_n = H(\Omega_1, Q) - I_n \quad \text{(by (11))}$$
$$= \min_{Q \in \Omega_1} H(P, Q) - I_n \quad \text{(by continuity)}$$
$$\leq H(P_n, Q) - H(R_n, Q)$$
$$= H(R_n) - H(P_n) + \sum_{a \in A^+} (R_n(a) - P_n(a)) \log P(a)$$
$$\leq \log(\#A+1)P_n - R_n\|_1 + \eta(\| P_n - R_n \|_1)$$
$$\leq \log(\#A)e_n - \varepsilon_n \log \varepsilon_n - \varepsilon_n \log Q_{\min}.$$

Now, setting $M_n := M_{1,n} \cup M_{2,n}$, we see by (14) and (15) that for all $n \in \mathbb{N}$ we have

$$Q^n(M_n) \leq Q^n(M_{1,n}) + Q^n(M_{2,n}) \leq (n+1)^{A+2^{-nI_0}}.$$

Moreover for each $P \in \Omega$ we may infer from (14) and (17) that for all sufficiently large $n \in \mathbb{N}$

$$P^n(M_n) \geq 1 - (n+1)^{A+2^{-nI_0}},$$

holds. \(\square\)

### 3 Proof of the quantum Sanov theorem

Before we prove the quantum Sanov theorem, we cite the here relevant known results. We define the maximal separating exponent

$$\beta_{\varepsilon,n}(\psi^{\otimes n}, \varphi^{\otimes n}) := \min \{ \log \varphi^{\otimes n}(q) : q \in A^{(n)} \text{ projection, } \psi^{\otimes n}(q) \geq 1 - \varepsilon \}.$$
Proposition 3.1  Let $\psi, \varphi \in S(A)$ with the relative entropy $S(\psi, \varphi)$. Then for every $\varepsilon \in (0, 1)$

$$\lim_{n \to \infty} \frac{1}{n} \beta_{\varepsilon,n}(\psi^\otimes n, \varphi^\otimes n) = -S(\psi, \varphi).$$

(20)

The assertion of Proposition 3.1 was shown by Ogawa and Nagaoka in [13]. A different proof based on the approach of Hiai and Petz in [10] was given in [6].

Proof of the main Theorem 2.2:

1. Proof of the lower bound: Due to the Proposition 3.1 any sequence of projections $\{p_n\}_{n \in \mathbb{N}}$, which has asymptotically a non vanishing expectation value with respect to the stationary product state $\{\psi^\otimes n\}_{n \in \mathbb{N}}$ satisfies:

$$\liminf_{n \to \infty} \frac{1}{n} \log \varphi^\otimes n(p_n) \geq -S(\psi, \varphi),$$

(21)

where $\varphi \in S(A)$ is a fixed reference state.

The lower bound (21) implies the lower bound

$$\liminf_{n \to \infty} \frac{1}{n} \log \varphi^\otimes n(p_n) \geq -S(\Psi, \varphi)$$

for any sequence $\{p_n\}_{n \in \mathbb{N}}$ of orthogonal projections $p_n \in \mathcal{A}^{(n)}$ satisfying condition (7) in Theorem 2.2.

Proof of the upper bound: To obtain the upper bound

$$\limsup_{n \to \infty} \frac{1}{n} \log \varphi^\otimes n(p_n) \leq -S(\Psi, \varphi),$$

where $\varphi$ is a fixed reference state, it is obviously sufficient to show that to each positive $\delta$ there exists a sequence $p_n$ such that

$$\lim_{n \to \infty} \psi^\otimes n(p_n) = 1, \quad \forall \psi \in \Psi,$$

(22)

and

$$\limsup_{n \to \infty} \frac{1}{n} \log \varphi^\otimes n(p_n) \leq -S(\Psi, \varphi) + \delta$$

is fulfilled for sufficiently large $n$. To show this we will apply the classical result, Lemma 2.3, to states restricted to appropriate abelian subalgebras approximating the quasi-local algebra $\mathcal{A}^\infty$.

Consider the spectral decomposition of the density operator $D_\varphi$:

$$D_\varphi = \sum_{i=1}^{d} \lambda_i e_i,$$
where $\lambda_i$ are the eigen-values and $e_i$ are the corresponding spectral projections. It follows a decomposition for $D_{\varphi^\otimes l} = D_{\varphi}^\otimes l$:

$$D_{\varphi}^\otimes l = \sum_{i_1, \ldots, i_l=1}^d \left( \prod_{j=1}^l \lambda_{i_j} \right) \otimes e_{i_1},$$

which leads to the spectral representation

$$D_{\varphi}^\otimes l = \sum_{i_1, \ldots, i_l \cdot \sum_i l = l}^d \left( \prod_{i=1}^l \lambda_i \right) e_{i_1, \ldots, i_l},$$

with the spectral projections

$$e_{i_1, \ldots, i_l} := \sum_{(i_1, \ldots, i_l) \in I_{i_1, \ldots, i_l}} \otimes e_{i_1},$$

where $I_{i_1, \ldots, i_l} := \{(i_1, \ldots, i_l) : \# \{j : i_j = k\} = l_k$ for $k \in [1, d]\}$. Let $\psi$ be a state on $A$ and $l \in \mathbb{N}$. We denote by $D_{l, \psi}$ the abelian subalgebra of $A^{(l)}$ generated by $\{e_{i_1, \ldots, i_d}\}_{i_1, \ldots, i_d} \cup \{e_{i_1, \ldots, i_d} D_{\varphi^\otimes l} e_{i_1, \ldots, i_d}\}_{i_1, \ldots, i_d}$. As a finite-dimensional abelian algebra, it has a representation

$$D_{l, \psi} = \bigoplus_{i_1=1}^{d_l} C \cdot f_{i_1},$$

where $\{f_{i_1}\}_{i_1=1}^{d_l}$ is a set of mutually orthogonal minimal projections in $D_{l, \psi}$. Hiai and Petz have shown that

$$S(\psi^\otimes l \circ E_l) - S(\psi^\otimes l) \leq d \log(l + 1), \quad (cf. \ [10], \ [6]) \quad (24)$$

which gives the lower bound

$$S(\psi^\otimes l \upharpoonright D_{l, \psi}, \varphi^\otimes l \upharpoonright D_{l, \psi}) \geq S(\psi^\otimes l, \varphi^\otimes l) - d \log(l + 1) \quad (25)$$
implying
\[
\lim_{l \to \infty} \frac{1}{l} S(\psi \otimes l \mid D_{l,\psi}, \varphi \otimes l \mid D_{l,\psi}) = S(\psi, \varphi).
\]

Next we consider a maximal abelian refinement \(B_{l,\psi}\) of \(D_{l,\psi}\) in the sense of the algebra \(A^{(l)}\):
\[
B_{l,\psi} := \bigoplus_{j,k=1}^{d_l} C \cdot g_{l,j,k},
\]
where \(g_{l,j,k}\) are one-dimensional projections in the algebra \(A^{(l)}\) such that \(f_{l,j} = \bigoplus_{k=1}^{d_l} C \cdot g_{l,j,k}\). This means that \(B_{l,\psi} \supseteq D_{l,\psi}\). It holds by monotonicity of the relative entropy and by the estimate
\[
S(\psi \otimes l \mid B_{l,\psi}, \varphi \otimes l \mid B_{l,\psi}) \geq S(\psi \otimes l \mid D_{l,\psi}, \varphi \otimes l \mid D_{l,\psi}) \geq l(S(\psi, \varphi) - \eta_l), \tag{26}
\]
where we used the abbreviation \(\eta_l := \frac{d \log(l+1)}{l}\) in the last line.

Due to the Gelfand isomorphism and the Riesz representation theorem the restricted states \(\psi \otimes l \mid B_{l,\psi}\) and \(\varphi \otimes l \mid B_{l,\psi}\) can be identified with probability measures \(P\) and \(Q\) on the compact maximal ideal space \(B_{l,\psi}\). The relative entropy of \(P\) with respect to \(Q\) is determined by:
\[
H(P, Q) = S(\psi \otimes l \mid B_{l,\psi}, \varphi \otimes l \mid B_{l,\psi}) \geq l(S(\psi, \varphi) - \eta_l).
\]

Similarly, the states \(\psi \otimes nl \mid B_{l,\psi}^{(n)}\) and \(\varphi \otimes nl \mid B_{l,\psi}^{(n)}\) correspond to the product measures \(P^n\) and \(Q^n\) on the product space \(B_{l,\psi}^n\).

We define
\[
S_l := \inf \{ S(\psi \otimes l \mid B_{l,\psi}, \varphi \otimes l \mid B_{l,\psi}) : \psi \in \Psi \}
\]
and fix an \(\psi_0 \in \Psi\). For any \(\psi \in \Psi\) and each \(l \in \mathbb{N}\) there exists a unitary operator \(U_\psi \in A^{(l)}\) that transforms the minimal projections spanning \(B_{l,\psi}\) into the minimal projections of \(B_{l,\psi_0}\) and that leaves the spectral subspaces of \(D_{\varphi \otimes l}\) invariant. Let us denote by \(\Omega_l(\Psi, \varphi)\) the set of unitaries having these properties. To each \(\psi \in \Psi\) denote by \(\tilde{\psi}^{(l)}\) the state on \(A^{(l)}\) with density operator \(U_\psi D_\psi \otimes l \otimes U_\psi^*\). Then we have
\[
S(\psi \otimes l \mid B_{l,\psi}, \varphi \otimes l \mid B_{l,\psi}) = S(\tilde{\psi}^{(l)} \mid B_{l,\psi_0}, \varphi \otimes l \mid B_{l,\psi_0}).
\]

Let \(\Omega_l\) be the set of probability measures on \(B_{l,\psi_0}\) corresponding to all \(\tilde{\psi}^{(l)} \mid B_{l,\psi_0}\), where \(\psi \in \Psi\). Further let the measure \(Q\) on \(B_{l,\psi_0}\) correspond to the restricted reference state \(\varphi \otimes l \mid B_{l,\psi_0}\). Then
\[
H(\Omega_l, Q) \geq S_l \geq l(S(\Psi, \varphi) - \eta_l), \tag{27}
\]
where the second inequality follows from (26). Due to the Lemma 2.3 there exists a sequence \( \{M_n\}_{n \in \mathbb{N}} \) of subsets \( M_n \in B^n_{l,\psi_0} \) (cf. (16)) such that
\[
\lim_{n \to \infty} P^n(M_n) = 1, \quad \forall P \in \Omega_l
\] (28)
and for every \( n \in \mathbb{N} \)
\[
Q^n(M_n) \leq (n + 1)^d 2^{-nI_n(l)},
\]
where \( I_n(l) \geq H(\Omega_l, Q) \) for \( n \to \infty \). Moreover, we know that
\[
I_n(l) \geq H(\Omega_l, Q) - \epsilon_n(log d^l - \log \epsilon_n - \log Q_{\min}(l)),
\]
where \( Q_{\min}(l) := \min\{Q(a) : a \in B_{l,\psi_0}\} \). We introduce the abbreviation
\[
\triangle_n(l) := \epsilon_n(log d^l - \log \epsilon_n - \log Q_{\min}(l)).
\]
It holds:
\[
\frac{1}{n} \log Q^n(M_n) \leq \frac{d^l \log(n + 1)}{n} - I_n \leq \frac{d^l \log(n + 1)}{n} - (H(\Omega_l, Q) - \triangle_n(l)).
\] (29)
To each \( M_n \in B^n_{l,\psi_0} \) there corresponds a projection \( p_{nl} \in A(l) \). For an arbitrary \( m \in \mathbb{N} \) such that \( m = nl + r \in \mathbb{N} \) with \( r \in \{0, \ldots, l - 1\} \) we define a projection \( p_m \in A(m) \) by
\[
p_m := p_{nl} \otimes 1_{[nl+1,nl+r]},
\]
where \( 1_{[nl+1,nl+r]} \) denotes the identity in the local algebra \( A_{[nl+1,nl+r]} \). It holds
\[
\tilde{\psi}^{(l) \otimes n}(p_{nl}) = P^n(M_n), \quad \forall \psi \in \Psi
\]
and
\[
\frac{1}{m} \log \varphi^{\otimes m}(p_m) \leq \frac{1}{nl} \log \varphi^{\otimes n}(p_{nl}) = \frac{1}{nl} \log Q^n(M_n).
\]
Using (28), (29) and (27) we conclude
\[
\lim_{n \to \infty} \psi^{\otimes n}(U_{\varphi}^{* \otimes n} p_{nl} U_{\varphi}^{\otimes n}) = 1, \quad \forall \psi \in \Psi
\] (30)
and
\[
\frac{1}{m} \log \varphi^{\otimes m}(p_m) = \frac{1}{nl} \log Q^n(M_n)
\]
\[
\leq \frac{d^l \log(n + 1)}{nl} - \frac{1}{l} I_n(l)
\]
\[
\leq \frac{d^l \log(n + 1)}{nl} - \frac{1}{l} (H(\Omega_l, Q) - \triangle_n(l))
\]
\[
\leq \frac{d^l \log(n + 1)}{nl} - S(\Psi, \varphi) + \eta_l + \frac{\triangle_n(l)}{l}. \quad (31)
\]
For fixed \( l \in \mathbb{N} \) we construct for each \( n \in \mathbb{N} \) the projection:
\[
\overline{p}_{nl} := \bigvee_{U \in U_{l}(\Psi, \varphi)} U^{* \otimes n} p_{nl} U^{\otimes n}.
\]
For an arbitrary number $m = nl + r$, $r \in \{0, \ldots, l-1\}$, we define
\[ \mathcal{P}_m := \mathcal{P}_{nl} \otimes 1_{[nl+1, nl+r]} . \]
It follows for arbitrary $\psi \in \Psi$ and each $m = nl + r \in \mathbb{N}$:
\[ \psi \otimes^m (\mathcal{P}_m) = \psi \otimes^{nl} (\mathcal{P}_{nl}) \geq \psi \otimes^{nl} (U_\psi^{*} \otimes^n p_{nl} U_\psi^{\otimes^n}) . \tag{32} \]
Using the estimate \[ \text{(30)} \] we obtain the general statement:
\[ \lim_{m \to \infty} \psi \otimes^m (\mathcal{P}_m) = 1, \quad \forall \psi \in \Psi. \]
Next we consider the expectation values $\phi \otimes^{nl} (U_\psi^{*} \otimes^n p_{nl} U_\psi^{\otimes^n})$ for any $U_\psi \in U_l (\Psi, \phi)$ and $n \in \mathbb{N}$. From the assumed invariance of $D_\phi^{\otimes^l}$ with respect to the unitary transformations given by elements of $U_l (\Psi, \phi)$ we conclude
\[ \phi \otimes^{nl} (U_\psi^{*} \otimes^n p_{nl} U_\psi^{\otimes^n}) = \phi \otimes^{nl} (p_{nl}), \quad \forall U_\psi \in U_l (\Psi, \phi). \tag{33} \]
The dimension of the symmetric subspace
\[ \text{SYM}(A^{(l)}, n) := \text{span}\{ A^{\otimes^n} : A \in A^{(l)} \} \]
is upper bounded by $(n+1)^{\dim A^{(l)}}$, which leads to the estimate
\[ \text{tr} \mathcal{P}_{nl} \leq (n+1)^{d^2_l} \cdot \text{tr} p_{nl}. \tag{34} \]
Using \[ \text{(33)}, \, \text{(34)} \] and \[ \text{(31)} \] we obtain
\[ \frac{1}{m} \log \phi \otimes^m (\mathcal{P}_m) \leq \frac{1}{nl} \log \phi \otimes^{nl} (\mathcal{P}_{nl}) \]
\[ \leq \frac{1}{nl} \log ((n+1)^{d^2_l} \cdot \phi \otimes^{nl} (p_{nl})) \]
\[ \leq \frac{(d^2_l + d^2_l) \log(n + 1)}{nl} - S(\Psi, \phi) + \eta_l + \frac{\Delta^{(l)}}{l} . \tag{35} \]
For fixed $l$ the upper bound above converges to $-S(\Psi, \phi) + \eta_l$, for $n \to \infty$. Choosing $l$ sufficiently large, $\eta_l$ becomes smaller than $\delta$. This proves the upper bound. \[ \Box \]

4 Two examples

1. Consider a quantum system where $\mathbb{C}^2$ is the underlying Hilbert space and let $v, w$ be two different non-orthogonal unit vectors in $\mathbb{C}^2$. Let $\psi^{\otimes^n}$ be the product state on $(\mathcal{B}(\mathbb{C}^2))^{\otimes^n}$ with the density operator $\rho_{w}^{\otimes^n}$, where $\rho_{w}$ is the projection onto the one-dimensional subspace in $\mathbb{C}^2$ spanned by $w$. Further let $\delta \geq 0$ and denote by $\varphi_\delta$ the state on $\mathcal{B}(\mathbb{C}^2)$ corresponding to the density operator $(1 - \delta)\rho_{w} + \delta \rho_{w}$. It seems rather clear that any reasonable attempt to define
empirical distributions (states) in quantum context should choose \( p_w \) in the case of \( \psi^\otimes n \) (or more general the underlying one-site state in the case of a stationary product state). So, when trying to define typical projectors via empirical states and to use these in analogy to the classical Sanov’s theorem, the \( n \)-block typical projector \( p(n) \) for the set \( \Psi = \{ \psi \} \subseteq \mathcal{S}(\mathcal{B}(\mathbb{C}^2)) \) would be expected to fulfil \( p(n) \geq p_w \otimes \delta \). Then we have \( \psi \otimes \delta(n)(p(n)) \geq \psi \otimes \delta(w) \otimes \delta(n) = (\delta + 1 - \delta)(w, w)^2 n \geq (v, w)^2 n \).

On the other hand, the relative entropy of the density operator \( p_w \) with respect to \( (1 - \delta)p_v + \delta p_w \) can be made arbitrary large by choosing \( \delta \) small but positive. This shows that, in contrast to the classical situation, when relying on empirical states the relative entropy rate is not an accessible separation rate (which can be at most \( -2\log(\|v, w\|) \)). We might simplify the argument by saying that though the relative entropy of \( \psi \) with respect to \( \varphi \) is infinite the separation rate using empirical distributions remains bounded. But choosing \( p(n) \) as \( p_{(v^\otimes n)} - \delta \), where \( (v^\otimes n)^\perp \) denotes the orthogonal complement of the vector \( v^\otimes n \) in \( (\mathbb{C}^2)^\otimes n \) yields \( \varphi \otimes v^\otimes n(p(n)) \equiv 0 \) and \( \psi \otimes v^\otimes n(p(n)) \to 1 \), hence the separation rate can in fact be made infinite when choosing the typical projector in another way.

2. A slightly more involved example shows that, again in contrast to the classical case, there is in general no universal choice of the separating projector, i.e. it has to depend upon the reference state \( \varphi \). This time we will refer directly to the infinite relative entropy case and leave the simple ‘smoothening’ argument which leads to a finite entropy example to the reader. Let \( v \) and \( w \) be two orthogonal unit vectors in \( \mathbb{C}^2 \). Let \( \Phi \) be the set of pure states \( \varphi_t \) on \( \mathcal{B}(\mathbb{C}^2) \) corresponding to the vectors \( v_t := \cos t \cdot v + \sin t \cdot w, t \in [-T, T], \frac{\pi}{2} > T > 0 \) and \( \Psi = \{ \psi \} \), where \( \psi \) is the pure state corresponding to \( w \). Assume there is a typical projector \( p(n) \) for \( \psi^\otimes n \) separating it from each \( \varphi_t^\otimes n \) super-exponentially fast. This should be valid for an universal projector since all the relative entropies \( S(\psi, \varphi_t) \) are infinite. Let \( \mathrm{SYM}(n) \subset (\mathbb{C}^2)^\otimes n \) be the symmetrical \( n \)-fold tensor product of \( \mathbb{C}^2 \). Without any loss of the generality we may choose \( p(n) \leq p_{\mathrm{SYM}(n)} \) since all \( \varphi_t^\otimes n \) as well as \( w^\otimes n \) belong to \( \mathrm{SYM}(n) \). Observe that the existence of \( p(n) \) (with the desired property) implies the existence of at least one (sequence of) unit vectors \( x_n \) in \( \mathrm{SYM}(n) \) such that \( (x_n, \varphi_t^\otimes n) \) tends to zero super-exponentially fast uniformly in \( t \). Choose an orthonormal basis in \( \mathrm{SYM}(n) \) by \( e_{n,k} := (n^k_k)^{1/2}/n! \sum_{\pi \in \mathrm{PERM}(n)} U_\pi (w^\otimes k \otimes w^{\otimes (n-k)}) \), where \( \mathrm{PERM}(n) \) is the group of \( n \)-Permutations and \( U_\pi \) is the unitary operator which interchanges the order in the tensor product according to \( \pi \). Representing \( \varphi_t^\otimes n \) in that basis yields the numerical vector \( ((n^k_k)^{1/2}(\cos t)^k(\sin t)^{n-k})_k=0 \). So the question is whether there exists a sequence of unit vectors \( x_n = (x_{n,k})_k \) such that \( \sup_{t \in [-T, T]} (\cos t)^n \sum_k x_{n,k} (n^k_k)^{1/2}(\tan t)^k \) tends to zero super-exponentially fast. Observe that the factor \( (\cos t)^n \) is bounded from below by \( (\cos T)^n \) and can be omitted since it goes to zero only exponentially fast. Moreover, if we replace \( x_n \) by \( \bar{x}_n = (x_{n,k}^2)^{-1/2} \) we change its norm only by an at most exponentially smaller factor (the maximum of binomial coefficient is of exponential order \( 2^n \)). So we may simplify the problem by asking whether there is a sequence of unit vectors \( x_n \) which has a super-exponentially decreasing
inner product with the numerical vectors \(((\tan t)^k)_{k=0,1,...,n}\), uniformly in \(t \in [-T,T]\). This can be excluded: Let \(n\) be uneven and consider the set of values \(t_m = \arctan((1-2m/n)\cdot \tan T)\), \(m = 0, 1, ..., n\). Even for this finite set of values we have necessarily \(\sup_m \sum_k x_{n,k}((1-2m/n)^k)(\tan T)^k\) tending to zero at most exponentially fast. In fact, the factor \((\tan T)^k\) can be omitted as before. Let \(V_n\) be the Vandermonde matrix \(((1-2m/n)^k\cdot n_{m,k})_{m,k=0}^n\). Then the \(L_\infty\)-norm of the vector \(V_n x_n\) can be estimated by a sub-exponential factor times its \(L_2\)-norm, and by [9], Example 6.1 the least singular value of \(V_n\) behaves like \(\pi e^{\frac{\pi}{4}e^{-n(\frac{\pi}{4}+\frac{1}{2}\ln 2)}}\).

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