OPEN PROBLEMS IN THE WILD MCKAY CORRESPONDENCE AND RELATED FIELDS

TAKEHIKO YASUDA

Abstract. The wild McKay correspondence is a form of McKay correspondence in terms of stringy invariants that is generalized to arbitrary characteristics. It gives rise to an interesting connection between the geometry of wild quotient varieties and arithmetic on extensions of local fields. The principal purpose of this article is to collect open problems on the wild McKay correspondence, as well as those in related fields that the author believes are interesting or important. It also serves as a survey on the present state of these fields.

Contents

1. Introduction 1
Convention 3
2. Brief guide to the wild McKay correspondence and stringy motives 3
2.1. Wild actions and wild quotients 3
2.2. Stringy motives 4
2.3. The wild McKay correspondence 5
3. Log-terminal singularities 6
4. Crepant resolutions 7
5. Rationality 9
6. Duality 9
7. The $v$-function/Fröhlich’s module resolvent 10
8. The moduli space $\Delta_G$ 11
9. Non-linear actions 12
10. Generalization in various directions 13
11. Relation to other theories on wild ramification 14
12. The derived wild McKay correspondence 14
13. The McKay correspondence over global fields 14
14. Stringy motives and quasi-étale Galois coverings 16
References 16

1. Introduction

This article is a substantial update to the author’s paper [Yas15b] written in 2015, which has been posted on his personal webpage. The principal purpose is to collect open

We would like to thank Editage (www.editage.com) for English language editing. This work was supported by JSPS KAKENHI Grant Number JP18H01112.
problems on the wild McKay correspondence, as well as those in related fields that the author believes are interesting or important. It also serves as a survey on the present state of these fields. Certain problems in this article appeared earlier than [Yas15b] but are scattered in a number of different papers. Thus, collecting them here in one place would be meaningful, in particular, for graduate students or young researchers looking for problems to work on. The author also wrote similar survey articles [Yas17a, Yas18] in 2013 and 2016, respectively. Since then, there has been considerable progress in the field; while some problems have been solved, new problems have also emerged. Therefore, it is worth producing the present paper for the sake of an update.

The wild McKay correspondence refers to a generalization of the McKay correspondence via stringy invariants, which originated in the works of Batyrev [Bat99] and Denef and Loeser [DL02], to positive and mixed characteristics. The adjective “wild” refers to a situation in which the relevant finite group has an order divisible by the characteristic of the base field (or a residue field). As is well known, it is a general phenomenon that problems become harder due to the wildness. The first attempt in this direction was [Yas14b] in which the author studied the linear actions of the group $\mathbb{Z}/p\mathbb{Z}$ on affine spaces in characteristic $p$.

**Remark 1.1.** This text discusses various works on wild quotient singularities from different perspectives. First, the modular invariant theory studies the algebraic nature (such as depth, factoriality, and a set of generators) of the invariant subring $k[x_1, \ldots, x_n]^G$ associated with a linear action of a finite group on a polynomial ring. For a thorough treatment of the modular invariant theory, the reader is referred to [CW11]. A geometric approach to wild quotient singularities, especially in dimension two, perhaps dates back to Artin’s works [Art75, Art77], followed by Peskin’s [Pes83]. More recent works include [KL13, IS15, LS20], to mention a few.

The wild McKay correspondence gives rise to the interaction of the geometry of singularities (in particular, from the perspective of birational geometry) and arithmetic problems such as counting extensions of a local field and counting rational points over a finite field. Therefore, it would shed new light on both birational geometry and such arithmetic problems. In birational geometry, after the recent achievement in characteristic zero, there appears to be a trend toward positive/mixed characteristics. However, generally speaking, it is harder to treat low characteristics compared with dimension. For details, the reader is referred to [HW19] and the references therein. The wild McKay correspondence mainly concerns such a situation of low characteristics and brings hope that the birational geometry as we have in characteristic zero will be eventually carried out in arbitrary characteristics to a certain extent. As for the number theory, the wild McKay correspondence provides a new way of counting extensions of a local field. Namely, they had been counted with weights determined by classical invariants such as discriminants. The wild McKay correspondence produces numerous new weights of geometric origin (see Section 7). Expanding this story to global fields is also of great

\[\text{The article [Yas17a] is the second paper on the wild McKay correspondence after the first [Yas14b] and provides the grand design of the theory of motivic integration over wild Deligne-Mumford stacks and its potential applications. The article [Yas18] is a survey paper written in Japanese in 2016. Its English version will be published in the journal “Sugaku Expositions.”}\]
interest and leads to relating the Batyrev-Manin conjecture and the Malle conjecture (see Section \[13\]).

The remainder of this paper is organized as follows. Section 2 reviews stringy motives and known results on the wild McKay correspondence. Open problems are discussed starting from Section 3. Generally, the problems are defined first, some of which are very vague, and meaning and further details are provided subsequently. Section 3 discusses the problem when quotient singularities are log terminal. In Section 4 problems concerning crepant resolutions of quotient varieties are discussed, including their existence, construction, and Euler characteristics. Sections 5 and 6 are concerned with rationality and duality of stringy motives, respectively. The latter is related to the Poincaré duality and both are related to the existence of a resolution of singularities. Sections 7 and 8 respectively discuss problems with the \(v\)-function (Fröhlich’s module resolvent) and the moduli space of \(G\)-torsors over the punctured formal disk, which play important roles in the wild McKay correspondence. Section 9 is about non-linear actions, that is, general actions other than linear actions on affine spaces. Section 10 considers the generalization of the McKay correspondence and motivic integration to various directions. Section 11 briefly discusses the problems on the relation of the wild McKay correspondence and other theories in terms of wild ramification. Section 12 discusses the problem of generalizing the derived McKay correspondence to the wild case. Section 13 discusses the McKay correspondence over global fields, in particular, number fields, which is expected to relate the Batyrev-Manin conjecture and the Malle conjecture. Finally, Section 14 discusses the behavior of stringy motives under quasi-étale maps and application to local étale fundamental groups.

Convention. Throughout the paper, an algebraically closed field \(k\) is assumed unless specified otherwise. This is for simplicity, and most arguments and problems are valid over any base field. The characteristic of \(k\) is denoted by \(p\). When considering the action of a finite group \(G\) on an affine space \(\mathbb{A}^d_k\), it is supposed that the action is linear and has no pseudo-reflection, unless noted otherwise (see Section 2.3).

2. Brief guide to the wild McKay correspondence and stringy motives

2.1. Wild actions and wild quotients. Although the term “McKay correspondence” can have multiple mathematical interpretations today, the common theme shared by them is the relation between a linear representation of a finite group and the associated quotient variety. Sometimes, more generally, one considers a smooth variety with a finite group action instead of a linear representation. A finite group action is said to be wild if the base field has a positive characteristic \(p > 0\) and the order of the group (or orders of stabilizer/isotropy groups) is divisible by \(p\). Otherwise, an action is said to be tame. In particular, every action in characteristic zero is tame. As these terms suggest, wild actions are often much more difficult to study than tame ones. Similarly, wild quotient singularities, that is, singularities appearing on the quotient variety associated with a wild action, do not satisfy many good properties of tame quotient singularities.

\[\text{In fact, there are a number of definitions of tameness/wildness. See } [\text{KS09}].\]
2.2. **Stringy motives.** An approach to the McKay correspondence that originated in the works of Batyrev \cite{Bat99} and Denef-Loeser \cite{DL02} uses stringy invariants. Let \( X \) be a normal \( \mathbb{Q} \)-Gorenstein variety \( X \) over a field \( k \), and let \( J_\infty X \) denote its arc space, which parametrizes arcs on \( X \), that is, morphisms \( D := \text{Spec} \ k[t] \to X \). For a positive integer \( r \) such that \( rK_X \) is Cartier, we can define a function

\[
F_X : J_\infty X \to \frac{1}{r} \mathbb{Z}_{\geq 0} \cup \{ \infty \}.
\]

Roughly speaking, this function measures the difference in the two sheaves \( \bigwedge^{\dim X} \Omega_{X/k} \) and the invertible sheaf \( \omega^{[r]}_X = \mathcal{O}(rK_X) \) along each arc. The *stringy motive* of \( X \) is defined by the motivic integral,

\[
M_{\text{st}}(X) := \int_{J_\infty X} L^{F_X} d\mu_X,
\]

which is an element of some version of the complete Grothendieck ring of varieties (see \cite{Yas20}). As usual, \( L \) denotes the Lefschetz motive, that is, the class \( \{ \mathbb{A}^1_k \} \) of affine line in the Grothendieck ring of varieties. (The class of a variety \( Z \) has been denoted \( \{ Z \} \) instead of the more standard \([ Z ]\); Square brackets are reserved for quotient stacks.) This version of the complete Grothendieck ring of varieties needs to contain the fractional power \( L^{1/r} \) of \( L \). Note that the integral may diverge; in this case, we set \( M_{\text{st}}(X) := \infty \). For a constructible subset \( C \), we can also define the *stringy motive along \( C \)* as

\[
M_{\text{st}}(X)_C := \int_{(J_\infty X)_C} L^{F_X} d\mu_X,
\]

by restricting the domain of integral to the space \((J_\infty X)_C\) of arcs passing through \( C \).

Suppose that there exists a resolution of singularities, \( f : Y \to X \), such that the relative canonical divisor \( K_{Y/X} = K_Y - f^*K_X \) has simple normal crossing support. Let us write \( K_{Y/X} = \sum_{i \in I} a_i E_i \) with \( E_i \) exceptional prime divisors and \( a_i \) nonzero rational numbers. For each non-empty subset \( J \subset I \), we define

\[
E_J^\circ := \bigcap_{j \in J} E_j \setminus \bigcup_{i \in I \setminus J} E_i.
\]

For \( J = \emptyset \), we set \( E_\emptyset^\circ := Y \setminus \bigcup_{i \in I} E_i \). Then, we have the formula

\[
M_{\text{st}}(X)_C := \sum_{J \subset I} \{ E_J^\circ \cap f^{-1}(C) \} \prod_{j \in J} \frac{L - 1}{L^{1+a_j} - 1},
\]

if \( a_i + 1 > 0 \) for every \( j \) such that \( E_j \) meets \( f^{-1}(C) \). Otherwise, \( M_{\text{st}}(X)_C = \infty \). In particular, assuming the existence of such a resolution as above, we have the following equivalences:

\[
M_{\text{st}}(X) \neq \infty \iff M_{\text{st}}(X)_{X_{\text{sing}}} \neq \infty \iff X \text{ has only log-terminal singularities}.
\]

Here, \( X_{\text{sing}} \) denotes the singular locus of \( X \).
A resolution of singularities, \( f : Y \to X \), is called crepant if \( K_{Y/X} = 0 \). For a crepant resolution \( f : Y \to X \), we have

\[
\text{M}_{st}(X)_C = \{ f^{-1}(C) \}.
\]

In particular,

\[
\text{M}_{st}(X) = \{ Y \}.
\]

2.3. The wild McKay correspondence. Suppose that a finite group \( G \) acts on an affine space \( \mathbb{A}^d_k \) linearly and faithfully. For simplicity, suppose also that \( G \) contains no pseudo-reflection; \( g \in G \) is called a pseudo-reflection if the fixed-point locus \( (\mathbb{A}^d_k)^g \subset \mathbb{A}^d_k \) has codimension one. In the tame case, the McKay correspondence in terms of stringy motive is formulated as follows [Bat99, DL02].

\[
\text{M}_{st}(\mathbb{A}^d_k/G) = \sum_{[g] \in \text{Conj}(G)} L^{d - \text{age}(g)}
\]

Here, \( \text{Conj}(G) \) is the set of conjugacy classes of \( G \); \( \text{age}(g) \) is the age of \( g \), a basic invariant in the McKay correspondence. In particular, this equality shows that if \( Y \to \mathbb{A}^d_k/G \) is a crepant resolution, then we have

\[
\chi(Y) = \# \text{Conj}(G).
\]

Here, \( \chi(Y) \) is the Euler characteristic of \( Y \) defined with either singular cohomology (the case \( k = \mathbb{C} \)) or \( l \)-adic cohomology (the general case). Equality (2.4) is generalized to the wild case as follows [Yas20].

\[
\text{M}_{st}(\mathbb{A}^d_k/G) = \int_{\Delta_G} L^{d - v}
\]

Here, \( \Delta_G \) is the moduli space of \( G \)-torsors over the punctured formal disk, \( \text{Spec } k[t] \), and \( v \) is a locally constructible function \( \Delta_G \to \frac{1}{2\pi i} \mathbb{Z}_{\geq 0} \). Broadly, the space \( \Delta_G \) may be infinite-dimensional but is always the disjoint union of at-most countably many varieties, and the last integral is defined to be the sum

\[
\sum_{a \in \frac{1}{2\pi i} \mathbb{Z}_{\geq 0}} \{ v^{-1}(a) \} L^{d - a}.
\]

For more details regarding the definition of this integral, see [TY19a]. In the tame case, \( \Delta_G \) has only finitely many points and is identified with \( \text{Conj}(G) \), which allows us to regard (2.6) as a generalization of (2.4). We may think of the left-hand side of (2.6) as an invariant concerning the geometry of \( \mathbb{A}^d_k/G \). However, the right-hand side has a more arithmetic nature. The integral is an analog of weighted count of the Galois extensions of \( \mathbb{Q}_p \).

The same integral is also regarded as the stringy motive \( \text{M}_{st}([\mathbb{A}^d_k/G]) \) of the quotient stack \( [\mathbb{A}^d_k/G] \), and equality (2.6) is rephrased as

\[
\text{M}_{st}(\mathbb{A}^d_k/G) = \text{M}_{st}([\mathbb{A}^d_k/G]).
\]

Note that since \( [\mathbb{A}^d_k/G] \) is a smooth Deligne-Mumford stack and the morphism \( [\mathbb{A}^d_k/G] \to \mathbb{A}^d_k/G \) is quasi-finite, proper, and birational, this morphism is a crepant resolution in the category of Deligne-Mumford stacks. Equality (2.7) is then viewed as a special case
of the more general result that the stringy motive is invariant under crepant proper maps\cite{Yas20}: if $(\mathcal{Y}, E) \to (\mathcal{X}, D)$ is a crepant proper birational morphism of “stacky log pairs,” then

$$M_{st}(\mathcal{Y}, E) = M_{st}(\mathcal{X}, D).$$

3. Log-terminal singularities

**Problem 3.1.** When are wild quotient singularities log terminal (resp. terminal, canonical, log canonical)?

The four important properties of singularities in the minimal model program are terminal, canonical, log terminal, and log canonical. There are the following implications among them:

$$\text{terminal} \Rightarrow \text{canonical} \Rightarrow \text{log terminal} \Rightarrow \text{log canonical}$$

Tame quotient singularities are always log terminal. Wild quotient singularities are sometimes log terminal, but sometimes they are not. It is desirable to have useful criteria to decide whether the given wild quotient singularity is log terminal or not. Focusing on linear actions $G \acts \mathbb{A}^d_k$, we may ask the following more specific question:

**Problem 3.2.** Find a purely representation-theoretic criterion for whether $\mathbb{A}^d_k/G$ is log terminal (resp. terminal, canonical, log canonical).

Note that for terminal and canonical singularities, the problem makes sense in the tame case as well. The criterion in this case is known as the Reid–Shepherd-Barron–Tai criterion (e.g., see \cite[Cor. 6]{Yas06}), which uses ages of $g \in G\setminus\{1\}$, invariants determined by eigenvalues. In the wild case, we have a criterion for each of the above four classes of singularities in terms of the integral $\int_{\Delta_G} |L|^{-d-v}$ in (2.6) (see \cite[Cor 1.4]{Yas20}). However, to compute this integral, we have to compute the moduli space $\Delta_G$ and the function $v$ on it, which is generally complex and not representation-theoretic. When $G = \mathbb{Z}/p^n\mathbb{Z}$, we have an affirmative answer to the problem \cite{Yas14b, Yas19, TY20, Tan21}.

Yamamoto \cite{Yam21} proved that the quotient variety $\mathbb{A}^3_k/(\mathbb{Z}/3\mathbb{Z})^2$ in characteristic three is not log canonical, provided that the group has no pseudo-reflection. However, from any proper subgroup $G \subset \langle (\mathbb{Z}/3\mathbb{Z})^2$, based on \cite{Yas14b}, the quotient $\mathbb{A}^3_k/G$ is canonical. This example shows that we cannot reduce the problem to the case of cyclic groups by restricting the given representation to cyclic subgroups.

Problem 3.1 would be much harder when $G$ non-linearly acts on a smooth variety and when $G$ acts on a singular variety. Note that the tame action on a smooth variety is always locally linearizable; each point $x \in X$ has local coordinates for which the action of the stabilizer is linear. This is no longer the case for the wild action. Perhaps we should start with the actions of $\mathbb{Z}/p$ on $\text{Spec } k[x, y]$ fixing only the closed point. This action is never linear. Artin \cite{Art75} and Peskin \cite{Pes83} gave partial results to Problem 3.1 in characteristics two and three, respectively. Looking at their computation, there appear to be two important numerical invariants of an action. The first one is the number $r$ of Jordan blocks for the linear action on $(x, y)/(x, y)^2$. The second one is the largest integer $n$ such that, after a suitable choice of coordinates $x$ and $y$, the action on
$k[[x, y]]$ is linear modulo $(x, y)^{n+2}$; this is an analog of ramification jump in the context of local fields.

**Problem 3.3.** Does $(\text{Spec } k[[x, y]])/G$ being log terminal depend only on the above pair $(r, n)$?

It might be too naive to expect this to be the case. If that is the case, we may look for more numerical invariants to get an affirmative result.

**Problem 3.4.** Even when both sides of (2.6) diverge, we may obtain an equality of motivic elements (or an equality of more meaningful quantities instead of the equality $\infty = \infty$) by “renormalizing” them.

If $X$ is not log terminal, then $\text{M}_{\text{st}}(X) = \infty$ by definition. We can still define the dimension of $\text{M}_{\text{st}}(X)$ in any case; however, it is also infinite if $X$ is not log canonical. Equality (2.6) holds even if $\mathbb{A}^d_k$ is not log canonical. However, in this case, it only says that both sides are infinity and have infinite dimension. Can we extract certain finite values by renormalization? As a step in this direction, Veys’ work [Vey03, Vey04] on stringy invariants of varieties with non-log-canonical singularities in characteristic zero is notable.

**Remark 3.5.** A possible approach to this problem may be to use analytic continuation. Suppose that we have a linear action $V = \mathbb{A}^d_k$ of a finite group $G$ such that $\int_{\Delta} \mathbb{L}^{d-v} = \infty$. For an integer $s > 0$, the direct sum $V^{\oplus s} = \mathbb{A}^{sd}_k$ of $s$ copies of $G$ as a $G$-representation leads to $\int_{\Delta} \mathbb{L}^{d-su}$. The larger the value of $s$, the more likely the last integral converges. Now imagine that $s$ is allowed to vary as a complex number. We hope that the integral may still make sense and converge for $s$ with a sufficiently large real part. We then try to take its analytic continuation to the entire complex $s$-plane.

4. Crepant resolutions

**Problem 4.1.** When does there exist a crepant resolution of $\mathbb{A}^d_k/G$? Is it possible to construct a crepant resolution as a certain moduli space?

The McKay correspondence, (2.4) and (2.6), takes the simplest form when there exists a crepant resolution $Y \to \mathbb{A}^d_k/G$, owing to (2.3). We would like to know when this is the case. In characteristic zero, for a finite subgroup $G \subset \text{SL}_d(\mathbb{C})$ with $d \leq 3$, there exists a crepant resolution of $\mathbb{A}^d_k/G$. In dimension two, the minimal resolution is a crepant resolution. In dimension three, this was proved via a case-by-case analysis by Roan, Ito, and Markushevich (see [Roan96] and references therein). It was later proved in [Nak01, BKR01] that the $G$-Hilbert scheme becomes a crepant resolution in this case, thereby giving a moduli-theoretic construction of a crepant resolution.

For the wild case, the following is a list of all the examples that the author knows:

1. The symmetric product of an affine plane $\mathbb{A}^2_k/S_n = S^n A^2_k$ in any characteristic; the Hilbert scheme of $n$ points on $\mathbb{A}^2_k$, $\text{Hilb}^n(\mathbb{A}^2_k)$, gives a crepant resolution [KT01, BK05].

2. A quotient of $\mathbb{A}^2_k$ by the wreath product $(\mathbb{Z}/2\mathbb{Z}) \wr S_n$ in characteristic $\neq 2$, as in [WYT7].
(3) Quotients of $\mathbb{A}^3_k$ in characteristic three by groups $\mathbb{Z}/3\mathbb{Z}$ [Yas14b] and by more general groups, including $S_3, A_4, S_4$ [Yam18, Yam21a].

(4) Quotients of $\mathbb{A}^2_k$ in characteristic three by an action of $\mathbb{Z}/6\mathbb{Z}$ that has pseudo-reflections [CDG20].

When $G = \mathbb{Z}/p\mathbb{Z}$, there is an integer invariant denoted by $D_V$ of the given representation $V = \mathbb{A}^d_k$ (see [Yas14b]). A necessary condition for the existence of a crepant resolution of $X = \mathbb{A}^d_k/G$ is $D_V = p$ [Yas14b Cor. 6.21]. It is natural to ask:

**Problem 4.2.** Is the equality $D_V = p$ also a sufficient condition?

The simplest example with $D_V = p$ is the following:

**Problem 4.3.** Suppose that $G = \mathbb{Z}/p\mathbb{Z}$ acts on $\mathbb{A}^{2p}_k$ by the Jordan matrix

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \oplus p,$$

which satisfies $D_V = p$. Does the associated quotient $\mathbb{A}^{2p}_k/G$ have a crepant resolution?

The last quotient variety $\mathbb{A}^{2p}_k/G$ appears (at least as per the author) as the wild counterpart of the cyclic quotient singularity in characteristic zero of type $\frac{1}{r}(1, \ldots, 1)$ of dimension $l$. The latter admits a crepant resolution.

For the cyclic group $G = \mathbb{Z}/p^n\mathbb{Z}$ of order $p^n$, Tanno [Tan21] defined a series of invariants, $D_V^{(0)}, D_V^{(1)}, \ldots, D_V^{(n-1)}$, generalizing the above invariant $D_V$ for the group $\mathbb{Z}/p\mathbb{Z}$. He used them to address Problem 3.2 for this group.

**Problem 4.4.** Is there a necessary condition expressed by using $D_V^{(0)}, D_V^{(1)}, \ldots, D_V^{(n-1)}$ for the existence of a crepant resolution of $\mathbb{A}^d_k/(\mathbb{Z}/p^n\mathbb{Z})$?

In the tame case, the Euler characteristic of a crepant resolution of $\mathbb{A}^d_k/G$ is equal to the number of conjugacy classes of $G$ (see [2.5]). This is not true in the wild case, as Yamamoto’s examples show [Yam18, Yam21b]. There is also a similar example such that the group has pseudo-reflections [CDG20].

**Problem 4.5.** Find a formula for the Euler characteristic of a crepant resolution of $\mathbb{A}^d_k/G$.

Note that from [2.3], the two crepant resolutions of the same variety have the same Euler characteristic. It would also be interesting to study the following two problems, on which little research has been carried out for the wild case.

**Problem 4.6** (cf. [Yam15]). Given a quotient variety $\mathbb{A}^d_k/G$, count crepant resolutions of it. When is it unique?

**Problem 4.7** (cf. [CI04]). Given two different crepant resolutions of the same quotient variety that are both constructed moduli-theoretically, can we connect them by wall-crossing?
5. Rationality

**Problem 5.1.** Suppose that $M_{st}(\mathbb{A}^d_k/G) \neq \infty$. Is $M_{st}(\mathbb{A}^d_k/G)$ a rational function in $\mathbb{L}^{1/\#G}$?

In the tame case, from Equation (2.6), $M_{st}(\mathbb{A}^d_k/G)$ is the sum of powers of $\mathbb{L}^{1/\#G}$, in particular, a polynomial in $\mathbb{L}^{1/\#G}$. In the wild case, it is not necessarily a polynomial (see computation in [Yas14b]). However, to the best of the author’s knowledge, it is always a rational function.

**Remark 5.2.** Even in the tame case, if $k$ is not algebraically closed, then $M_{st}(\mathbb{A}^d_k/G)$ is generally not a rational function in $\mathbb{L}^{1/\#G}$ (cf. [WY15, Th. 5.9 and Cor. 5.11]). However, a weaker rationality as in the next problem holds over any field in the tame case.

**Problem 5.3.** Suppose that $M_{st}(\mathbb{A}^d_k/G) \neq \infty$. Can $M_{st}(\mathbb{A}^d_k/G)$ always be written as a finite sum of the form

$$\sum_{i=1}^{n} \frac{\{Z_i\}}{\mathbb{L}^{an_i} - 1}$$

with $a_i \in \frac{1}{\#G} \mathbb{Z}_{\geq 0}$ and varieties $Z_i$?

If $\mathbb{A}^d_k/G$ has a log resolution, then formula (2.2) provides an affirmative answer to this question. In other words, if the last problem has a negative answer for some quotient variety, then this quotient variety does not admit any log resolution. The duality discussed in the next section is another example of properties that hold if some version of resolution of singularities is available.

6. Duality

**Problem 6.1.** Let $o \in \mathbb{A}^d_k/G$ be the image of the origin of $\mathbb{A}^d_k$ and let $M_{st}(\mathbb{A}^d_k/G)^\vee$ denote the “dual” of $M_{st}(\mathbb{A}^d_k/G)$. Does the following equality holds?

$$M_{st}(\mathbb{A}^d_k/G)_o = M_{st}(\mathbb{A}^d_k/G)^\vee \cdot \mathbb{L}^d$$

Specifically, we need to take certain realization of motives for the dual $M_{st}(\mathbb{A}^d_k/G)^\vee$ to make sense. For example, let us choose here the Poincaré polynomial realization. The Poincaré polynomial $P(X)$ of a variety $X$ is an element of the polynomial ring $\mathbb{Z}[T]$. It defines a ring map

$$P : K_0(\text{Var}_k) \to \mathbb{Z}[T],$$

which, in particular, sends $\mathbb{L}$ to $T^2$. For a smooth proper variety $X$, we have

$$P(X) = \sum_{i=0}^{2 \dim X} (-1)^i b_i T^i,$$

where $b_i$ is the $i$-th Betti number with respect to $l$-adic cohomology, and the Poincaré duality is expressed as the functional equation

$$P(X) = P(X)^\vee \cdot T^{2 \dim X}.$$
Here, the dual $P(X)^\vee$ is the Laurent polynomial obtained by substituting $T$ in $P(X)$ with $T^{-1}$. For a log-terminal variety $X$ admitting a log resolution, sending $M_{st}(X)$ by the natural extension of map (6.2) gives the “stringy Poincaré series”

$$P_{st}(X) \in \mathbb{Z}(T^{-1/r}).$$

This is a rational function in $T^{1/r}$, and we can define its dual $P_{st}(X)^\vee$. Batyrev’s argument [Bat98] shows

$$P_{st}(X) = P_{st}(X)^\vee \cdot T^{2\dim X}. \tag{6.3}$$

The duality in Problem 6.1 is related to the Poincaré duality for the quotient space $(X - \{o\})/\mathbb{G}_m$. See [Yas14b, WY17] for more details.

Note that, similar to equality (2.6), the stringy motive $M_{st}(A^d_k/G)_o$ at $o$ is expressed as an integral $\int_{\Delta G} \mathbb{L}^w$ with a function $w$ that is similar to $v$. In the tame case, the duality (6.1) is a direct consequence of the following equality, which easily follows from the definition of age:

$$\text{age}(g^{-1}) = \text{codim}((A^d_k)_o, A^d_k) - \text{age}(g).$$

Here, $(A^d_k)_o$ denotes the fixed-point locus for the action of $g$. However, in the wild case, the duality, if true, would be related to deeper natures of the moduli space $\Delta_G$ and of functions $v$ and $w$ on it.

### 7. The $v$-function/Fröhlich’s Module Resolvent

We now explain the background and motivation for the $v$-function. This function, which appears in the wild McKay correspondence, (2.6), is important, since it is a common generalization of the two fundamental invariants, age and the Artin conductor [WY15]. It is also a special case of Fröhlich’s module resolvent [Frö76] (see also [TY19a, Remark 9.4]).

The $v$-function is a function $\Delta_G \to 1/2\mathbb{Z}_{\geq 0}$ associated with a representation $G \to \text{GL}_d(k[[t]])$ over $k[[t]]$, in particular, to a representation over $k$. Here, $\Delta_G$ is the moduli space of $G$-torsors over the punctured formal disk $D^* = \text{Spec } k[[t]]$ (see Section 8). The function is defined by using the tuning module associated with the given representation and a $G$-torsor over $D^*$ [WY15, Def. 3.1] (cf. [Yas16, Rem. 4.1]).

The $v$-function can be viewed as a measure of how much the (normal) $G$-cover of the formal disk $D = \text{Spec } k[[t]]$ corresponding to a given $G$-torsor over $D^*$ is ramified, and one can use it as a weight when counting $G$-covers of $D$. In fact, the “$w$” in the $w$-function, an elder sibling of the $v$-function, stands for “weight” (see [Yas17a, Def. 6.5]), and “$v$” was chosen because it is the alphabet next to “$w$.” See Section 13 for further details on this viewpoint of counting applied to global fields. Later studies [WY15, WY17, Yas16, Yas20] gradually showed that $v$ is more fundamental than $w$.

**Problem 7.1.** Find a “nice” formula for the function $v$ in terms of the basic invariants of the representation $\rho$ and $G$-torsors.

As mentioned above, the function $v$ is the same as age in the tame case and is expressed in terms of the Artin conductor or the discriminant in the case where $\rho$ is a permutation representation. However, it is complex to compute this function for
a general representation. When $G = \mathbb{Z}/p^n\mathbb{Z}$, we have a formula for $v$ in terms of ramification filtration $[\text{Yas14b}]$ $[\text{TY20a}]$, though it is rather involved. However, even in the case of group $(\mathbb{Z}/p\mathbb{Z})^2$, the function is not determined by ramification filtration $[\text{Yam21c}]$. We can consider the same problem as above in mixed characteristic. There has been little research on this case, except the tame case and the case of permutation representations (see also Problem 10.1).

Problem 7.2. Is the $v$-function upper-semicontinuous?

To make this problem precise, we first need to decide what geometric structure to give to the moduli space $\Delta_G$. Thus, the problem is related to the construction of this moduli space, which we discuss in the next section.

The $w$-function mentioned above is used in the following variant of the wild McKay correspondence:

$$M_{\text{st}}(\mathbb{A}^d_k/G) = \int_{\Delta_G} \mathbb{L}^w$$

Cf. Section 6. This version is not explicitly stated in the literature, but its point-counting version appears in $[\text{Yas17b}]$. We can prove the above equality by combining arguments of $[\text{Yas17b}]$ and the proof of (2.6) in $[\text{Yas20}]$. In the above equality, we need to use the definition of $w$ in $[\text{Yas17b}]$, which is slightly different from the original one in $[\text{Yas17a}]$ $[\text{WY15}]$. The two definitions coincide for a few important cases (see $[\text{Yas17b}]$, Rem. 8.2 and Lem. 8.3). However, no example in which they are, indeed, different functions has been found so far.

Problem 7.3. Do the two definitions of $w$ eventually coincide? If not, construct a counterexample.

8. The moduli space $\Delta_G$

Problem 8.1. Can we construct the moduli space $\Delta_G$ of $G$-torsors over the punctured formal disk $D^* = \text{Spec } k[\langle t \rangle]$ as an inductive limit of Deligne-Mumford stacks of finite type?

Roughly speaking, for a finite group $G$, we consider the moduli functor

$$(\text{Affine schemes }/k) \rightarrow (\text{Sets})$$

$$\text{Spec } R \mapsto \{G\text{-torsors over Spec } R[\langle t \rangle]\}$$

and ask whether or not this is representable by a scheme or a generalization of it. Since $G$-torsors have nontrivial automorphisms, as is usual in the moduli problem, the fine moduli space should be a stack if it exists. When group $G$ is a semidirect product $H \rtimes C$ of a $p$-group $H$ and a tame cyclic group $C$, it was proved in $[\text{TY20b}]$ that the moduli stack $\Delta_G$ is an inductive limit of Deligne-Mumford stacks of finite type. For a general finite group, we can construct the “moduli space” of $\Delta_G$ as a rough geometric structure called $P$-scheme $[\text{TY19a}]$. However, the above problem itself is open.

Problem 8.2. Construct the moduli space $\Delta_G$ for a local field of characteristic zero such as $\mathbb{Q}_p$. 
Note that the moduli space should be defined over the residue field rather than the local field itself. In other words, we are interested in the following moduli functor:

\[(\text{Affine schemes}/k) \rightarrow \text{(Sets)}\]

\[\text{Spec } R \mapsto \{ G\text{-torsors over } \text{Spec}(K \otimes_{W(k)} W(R)) \} / \cong \]

Here, \(K\) is the given local field, \(k\) is its residue field, and \(W(\cdot)\) denotes the ring of the Witt vectors.

**Problem 8.3.** Construct the moduli space \(\Delta_G\) for more general group schemes \(G\).

For example, for \(G = \alpha_p\), the moduli space should be the ind-pro limit of finite dimensional spaces (see [TY19b]). This would add extra theoretical complexity in applying the moduli space to the McKay correspondence and motivic integration (see Problem 10.2).

9. **Non-linear actions**

In the tame case, every finite group action on a smooth variety is locally linear, as was mentioned in Section 3. This is no longer true in the wild case. The wild McKay correspondence can be generalized to the non-linear case as follows.

Suppose that a finite group \(G\) faithfully acts on a normal variety \(V\) of dimension \(d\). Let \(A\) be a \(\mathbb{Q}\)-divisor on \(V/G\) such that \(K_{V/G} + A\) is \(\mathbb{Q}\)-Cartier. For each \(G\)-torsor \(E\) over \(\text{Spec } k[t]\), we can construct the *normalized untwisting scheme*, \(V|_{E|,\nu}\), which is a \(k[t]\)-scheme of relative dimension \(d\) with a natural morphism \(V|_{E|,\nu} \rightarrow (V/G) \otimes_k k[t]\) and carries a natural \(\mathbb{Q}\)-divisor \(B_E\). The McKay correspondence can then be formulated as

\[(9.1) \quad \text{M}_\text{st}(V/G, A) = \int_{\Delta_G} \text{M}_\text{st}(V|_{E|,\nu}, B_E)/\text{Aut}(E),\]

where \(\text{Aut}(E)\) is the automorphism group of \(E\) as a \(G\)-torsor. See [Yas16, Conj. 1.3] and [Yas20, Th. 13.3 and Cor. 13.4].

**Problem 9.1.** Compute the right-hand side of (9.1). In particular, compute normalized untwisting schemes \(V|_{E|,\nu}\).

The only case where these have been performed is the one where \(V\) is a smooth curve in characteristic two, and the group \(G\) has order two and acts on \(V\) in the mildest way [Yas16, Section 10]. In this case, there appeared an infinite series of rational double points on surfaces \(V|_{E|,\nu}\) from Artin’s classification [Art77] as \(G\)-torsor \(E\) varies.

**Problem 9.2.** If \(V\) is a smooth curve, then what singularities do surfaces \(V|_{E|,\nu}\) have? Does every type of rational double point appear on some \(V|_{E|,\nu}\)?

**Problem 9.3.** Suppose that a finite group \(G\) acts on a smooth variety \(V\) and that \(V/G\) has log-terminal singularities. Does \(\text{M}_\text{st}(V/G)\) have the rationality like (5.1)? If \(V\) is proper, then does the quotient variety satisfy the Poincaré duality (6.3)?
We can ask the same question, more generally, for varieties with log-terminal singularities. However, if we look for pathological phenomena, especially, a counterexample to resolution of singularities, then quotient varieties associated with non-linear actions might be good places to search (cf. [KS13]).

10. GENERALIZATION IN VARIOUS DIRECTIONS

Problem 10.1. Develop motivic integration for Deligne-Mumford stacks over a complete discrete valuation ring of mixed characteristics and apply it to the proof of McKay correspondence over such a ring.

Sebag [Seb04] (see also [CLNS18]) developed motivic integration for (formal) schemes over a complete discrete valuation ring, which may have mixed characteristic like $p$-adic integers $\mathbb{Z}_p$. In [Yas04, Yas06, Yas20], the author developed motivic integration for (formal) Deligne-Mumford stacks over the power series ring $k[[t]]$. The wild McKay correspondence also holds over $k[[t]]$. It is natural to look for a further generalization to the complete discrete valuation ring possibly of mixed characteristic. Note that the $p$-adic integration in a similar setting was developed in [Yas17b] (cf. [GWZ20]), which includes the case of mixed characteristic. The remaining nontrivial part appears to be the construction of moduli spaces used in the theory, in particular, the moduli space $\Delta_G$ of $G$-torsors over $\text{Spec } k[[t]]$ or its variants (Problem 8.2).

Problem 10.2. Generalize the McKay correspondence to more general groups/group schemes such as algebraic groups (of positive dimension), abelian varieties, non-reduced finite group schemes, and pro-finite groups.

The case of the group scheme $\alpha_p$ was discussed in [TY19b] (cf. Problem 8.3). This problem is closely related to:

Problem 10.3. Generalize the motivic integration to Artin stacks.

Balwe [Bal15] has worked in this direction, but his results do not appear to be applicable to the context of the McKay correspondence. A recent work by Satriano and Usatine [SU20] also addresses the above problem with the view toward McKay-correspondence type results. However, studies on Problems 10.2 and 10.3 are still at an early stage, and there is considerable work that remains.

Problem 10.4. Generalize the works [GWZ20, LW21] on the Hitchin fibration and stringy invariants twisted by $\mu_n$-gerbes to the wild case.

In [GWZ20], Groechenig-Wyss-Ziegler proved the topological mirror symmetry conjecture of Hausel and Thaddeus by means of $p$-adic integration on Deligne-Mumford stacks. This conjecture is stated in terms of stringy Hodge numbers twisted by $\mu_n$-gerbes. Later, this result was refined and proved at the motivic level by Loeser-Wyss [LW21]. It would be interesting to incorporate wild actions into research in this direction.
11. Relation to other theories on wild ramification

Wild ramification is an important subject in number theory and arithmetic geometry. The reader is referred to [Sai11, XZ15] and references therein for more discussion on the subject. An approach to wild ramification in dimensions $\geq 2$ is to restrict ramification to various curves. The wild McKay correspondence as well as the motivic integration over wild Deligne-Mumford stacks can be viewed as yet another theory concerning wild ramification in higher dimensions via the restriction-to-curves approach.

**Problem 11.1.** Relate the wild McKay correspondence and motivic integration over wild Deligne-Mumford stacks with other theories on wild ramification.

For example, non-liftability of arcs along a wildly ramified finite cover, which played important roles in works of arithmetic geometers [Kat16, KS09], also played a crucial role in application of stringy invariants to the local étale fundamental group [CRY21]. See Section 14.

The Grothendieck-Ogg-Shafarevich formula [SGA5, Exposé X] is a classical result on wild ramification, and there are many works on generalizing it (see [Sai11, XZ15]).

**Problem 11.2.** Can we show a Grothendieck-Ogg-Shafarevich type formula by using motivic integration and/or a version of stringy invariant?

12. The derived wild McKay correspondence

**Problem 12.1.** Generalize the McKay correspondence at the level of derived categories as studied in [KV00, BKR01] to the wild case.

Suppose that there exists a crepant resolution $Y \rightarrow \mathbb{A}^d_k / G$. The derived McKay correspondence denotes the equivalence

$$D^b(\text{Coh}(Y)) \cong D^b(\text{Coh}^G(\mathbb{A}^d_k)),$$

between the bounded derived category of coherent sheaves on $Y$ and the one of equivariant coherent sheaves on $\mathbb{A}^d_k$. This was proved under some conditions in characteristic zero. The category $\text{Coh}^G(\mathbb{A}^d_k)$ always has infinite global dimension in the wild case [Yi94], while it has global dimension $d$ in the tame case. Therefore, the above equivalence never holds in the wild case. We would need to find an alternative to $\text{Coh}^G(\mathbb{A}^d_k)$ possibly from various notions of higher structures developed especially in noncommutative algebraic geometry.

13. The McKay correspondence over global fields

**Problem 13.1.** Explore the McKay correspondence over global fields, in particular, number fields.

The McKay correspondence in terms of stringy motive, equality (2.6), can be regarded as the McKay correspondence over a local field, as both sides of the equality have something to do with the “local field” $k[[t]]$. More precisely, the left side can be seen as the volume of the set of integral points (i.e., $k[[t]]$-points) on the quotient variety $\mathbb{A}^d_{k[[t]]} / G$, while the right side can be seen as the volume of the set of $G$-torsors over the punctured
formal disk. This point of view is more apparent in the point-counting version of (2.6) (see [WY15, Yas17]). It is natural to look for a counterpart over a global field, in particular, a number field. A prime candidate for such a theory would concern an interplay of the Batyrev-Manin conjecture [FMT89, BM90] and the Malle conjecture [Mal04, Mal02], as was discussed in [Yas14a, Yas15a]. The former discusses the number of rational points with bounded height on a Fano-type variety over a number field, while the latter discusses the number of Galois extensions with bounded discriminant. Both conjectures predict these numbers increase like

\[(13.1) \quad c \cdot B^a \cdot (\log B)^b\]

for some constants \(c > 0, a > 0, b \geq 0\), as the bound \(B\) tends to infinity. There are close relations among these constants in both contexts (see [Yas14a, Sec. 4] and [Yas15a, Prop. 8.5]), which are also considered as versions of the McKay correspondence. For example, in the Malle conjecture for a transitive subgroup \(G \subset S_n\), the exponent \(a\) of \(B\) is the reciprocal of the index, denoted by \(\text{ind}(G)\), induced from the \(G\)-action on \(\{1, \ldots, n\}\). The index is related to the discrepancy of a quotient variety, a basic invariant of singularities [Yas14a, Prop. 4.4]. The constants \(b\) in the two conjectures are related to the number of \(K\)-conjugacy classes with minimal index and to the number of crepant divisors over a quotient variety (see [Yas15a, Conj. 5.6]) respectively. They are related by [Yas14a, Prop. 4.5] (cf. [Yas15a, Prop. 8.5]). As for the constant \(c\), on the Malle side, we have the conjectural formula of Bhargava for \(G = S_n\) [Bha07, Conj. A], which he derived from his own formula for the local counterpart. On the Batyrev-Manin side, we have Peyre’s constant [Pey95] (see also [Pey03, BT98]), which is related to a volume of adelic points.

**Problem 13.2.** Prove one of the above two conjectures for some special case from the known case of the other conjecture.

Le Rudulier’s thesis [LR14] is the first such successful attempt. It would also be possible to interpret works of Schmidt, Gao, and Guignard [Sch95, Gao96, Gui17] on counting algebraic points on a projective space as counting rational points on a quotient variety via the correspondence of points (see [Yas14a, Sec. 4]).

**Problem 13.3.** Count Galois extensions of a number field using a new weight function derived from tuning modules.

Another importance of the McKay-correspondence viewpoint is to provide new weights of the geometric origin for counting global fields, which correspond to the \(v\)-function for local fields (see Section 7). Dummit [Dum14] defined \(\rho\)-discriminant as such a weight using the tuning module and studied the counting number fields with bounded \(\rho\)-discriminant. Similarly, Yasuda [Yas15a] considered \(V\)-discriminant and extended \(V\)-discriminant to discuss the relation between the Batyrev-Manin conjecture and the Malle conjecture. Later, Ellenberg, Satriano, and Zureick-Brown [ESZB21] announced a conjecture on asymptotics of the number of rational points on an algebraic stack with bounded height, which unifies the above two conjectures. For this purpose, they introduced height functions associated with vector bundles. In this new notion of height, the \(v\)-function appears as the contribution of each non-Archimedean place (see also
Darda [Dar21] also defined height on stacks in connection with the conjectures of Batyrev-Manin and Malle. The work [OR20] of O’Desky and Rosen is also considered as one aiming toward the same direction; they constructed the moduli space of pairs of a Galois \( \mathbb{Q} \)-algebra and a normal element of it as an open subscheme of a certain quotient variety.

14. **Stringy motives and quasi-étale Galois coverings**

Let \( Y \to X \) be a Galois quasi-étale (i.e., étale in codimension 1) cover of normal \( \mathbb{Q} \)-Gorenstein varieties with only log-terminal singularities. Suppose that the Galois group \( G \) fixes some point \( y \) of \( Y \). Then, we can define the \( G \)-quotient \( M_{st}(Y)_y/G \) of \( M_{st}(Y)_y \). If \( x \in X \) denotes the image of \( y \), then we have the inequality

\[
M_{st}(X)_x \geq M_{st}(Y)_y/G.
\]

Here, we compare these invariants by taking the Poincaré polynomial realization and comparing their coefficients lexicographically.

**Problem 14.1.** Is this inequality always strict unless the cover is trivial?

In [CRY21], the authors proved that the answer is affirmative in dimension two as well as in dimension three with the condition that either \( Y \) has rational singularities or \( G \) is a \( p \)-group. The difference between the two sides of (14.1) arises from the arcs on \( X \) that do not lift to \( Y \). To show the strict inequality, we have to show that these non-liftable arcs form a subset of the arc space of nonzero measure.

**Problem 14.2.** Consider a tower

\[
X \leftarrow Y_1 \leftarrow Y_2 \leftarrow \cdots
\]

of Galois quasi-étale coverings as above and let \( y_i \in Y_i \) and \( G_i \) be the point and Galois group as above for the Galois covering \( Y_i \to X \). Does the sequence

\[
M_{st}(Y_1)_{y_1}/G_1 \geq M_{st}(Y_2)_{y_2}/G_2 \geq \cdots
\]

satisfy the descending chain condition (DCC)?

In the paper mentioned above, it is proved that the answer to this problem is also affirmative in dimension two. By combining it with the strictness of inequality, it can be observed that the local étale fundamental group of a log-terminal surface singularity is finite (in any characteristic).

**References**

[SGA5] Cohomologie l-adique et fonctions L. Lecture Notes in Mathematics, Vol. 589. Springer-Verlag, Berlin-New York, 1977.

[Art75] M. Artin. Wildly ramified \( \mathbb{Z}/2 \) actions in dimension two. *Proceedings of the American Mathematical Society*, 52:60–64, 1975.

[Art77] M. Artin. Coverings of the Rational Double Points in Characteristic \( p \). In T. Shioda and W. L. Jr Baily, editors, *Complex Analysis and Algebraic Geometry: A Collection of Papers Dedicated to K. Kodaira*, pages 11–22. Cambridge University Press, Cambridge, 1977.

[Bal15] Chetan Balwe. \( p \)-adic and Motivic Measure on Artin \( n \)-stacks. *Canadian Journal of Mathematics*, 67(6):1219–1246, December 2015.
[Bat98] Victor V. Batyrev. Stringy Hodge numbers of varieties with Gorenstein canonical singularities. In Integrable systems and algebraic geometry (Kobe/Kyoto, 1997), pages 1–32. World Sci. Publ., River Edge, NJ, 1998.

[Bat99] Victor V. Batyrev. Non-Archimedean integrals and stringy Euler numbers of log-terminal pairs. Journal of the European Mathematical Society, 1(1):5–33, 1999.

[Bha07] Manjul Bhargava. Mass Formulae for Extensions of Local Fields, and Conjectures on the Density of Number Field Discriminants. International Mathematics Research Notices, 2007(rnm052), 2007.

[BK05] Michel Brion and Shrawan Kumar. Frobenius splitting methods in geometry and representation theory, volume 231 of Progress in Mathematics. Birkhäuser Boston, Inc., Boston, MA, 2005.

[BKR01] Tom Bridgeland, Alastair King, and Miles Reid. The McKay correspondence as an equivalence of derived categories. Journal of the American Mathematical Society, 14(3):535–554, 2001.

[BM90] V. V. Batyrev and Yu. I. Manin. Sur le nombre des points rationnels de hauteur borné des variétés algébriques. Mathematische Annalen, 286(1):27–43, 1990.

[BT98] Victor V. Batyrev and Yuri Tschinkel. Tamagawa numbers of polarized algebraic varieties. In Astérisque, number 251, pages 299–340. 1998.

[CDG20] Yin Chen, Rong Du, and Yun Gao. Modular quotient varieties and singularities by the cyclic group of order 2p. Communications in Algebra, 48(12):5490–5500, 2020.

[CI04] Alastair Craw and Akira Ishii. Flops of G-Hilb and equivalences of derived categories by variation of GIT quotient. Duke Mathematical Journal, 124(2):259–307, 2004.

[CLNS18] Antoine Chambert-Loir, Johannes Nicaise, and Julien Sebag. Motivic Integration. Progress in Mathematics. Birkhäuser Basel, 2018.

[CRY21] Javier Carvajal-Rojas and Takehiko Yasuda. On the behavior of stringy motives under Galois quasi-étale covers. arXiv:2105.05214, 2021.

[CW11] H. E. A. Eddy Campbell and David Wehlau. Modular Invariant Theory. Encyclopaedia of Mathematical Sciences, Invariant Theory and Algebraic Transformation Groups. Springer-Verlag, Berlin Heidelberg, 2011.

[Dar21] Ratko Darda. Rational points of bounded height on weighted projective stacks. arXiv:2106.10120, 2021.

[DL02] Jan Denef and François Loeser. Motivic Integration, Quotient Singularities and the McKay Correspondence. Compositio Mathematica, 131(3):267–290, 2002.

[Dum14] Evan P. Dummit. Counting Number Field Extensions of Given Degree, Bounded Discriminant, and Specified Galois Closure. PhD thesis, University of Wisconsin-Madison, 2014.

[ESZB21] Jordan S. Ellenberg, Matthew Satriano, and David Zureick-Brown. Heights on stacks and a generalized Batyrev-Manin-Malle conjecture. arXiv:2106.11340, 2021.

[FMT89] Jens Franke, Yuri I. Manin, and Yuri Tschinkel. Rational points of bounded height on Fano varieties. Inventiones mathematicae, 95(2):421–435, 1989.

[Frö76] A. Fröhlich. Module Conductors and Module Resolvents. Proceedings of the London Mathematical Society, s3-32(2):279–321, 1976.

[Gao96] Xia Gao. On Northcott’s theorem. phd, University of Colorado at Boulder, USA, 1996. UMI Order No. GAX96-20624.

[Gui17] Quentin Guignard. Counting algebraic points of bounded height on projective spaces. Journal of Number Theory, 170:103–141, 2017.

[GWZ20] Michael Groechenig, Dimitri Wyss, and Paul Ziegler. Mirror symmetry for moduli spaces of Higgs bundles via p-adic integration. Inventiones Mathematicae, 221(2):505–596, 2020.

[HW19] Christopher Hacon and Jakub Witaszek. The Minimal Model Program for threefolds in characteristic five. arXiv:1911.12893, 2019.

[IS15] Hiroyuki Ito and Stefan Schröer. Wild quotient surface singularities whose dual graphs are not star-shaped. Asian Journal of Mathematics, 19(5):951–986, 2015.

[Kat16] Hiroki Kato. Wild Ramification and Restrictions to Curves. arXiv:1611.07643, 2016.
Franz Király and Werner Lütkebohmert. Group actions of prime order on local normal rings. *Algebra & Number Theory*, 7(1):63–74, 2013.

Moritz Kerz and Alexander Schmidt. On different notions of tameness in arithmetic geometry. *Mathematische Annalen*, 346(3):641, 2009.

Moritz Kerz and Shuji Saito. Cohomological Hasse principle and resolution of quotient singularities. *New York Journal of Mathematics*, 19:597–645, 2013.

Shrawan Kumar and Jesper Funch Thomsen. Frobenius splitting of Hilbert schemes of points on surfaces. *Mathematische Annalen*, 319(4):797–808, 2001.

M. Kapranov and E. Vasserot. Kleinian singularities, derived categories and Hall algebras. *Mathematische Annalen*, 316(3):565–576, 2000.

Aaron Landesman. *A thesis of minimal degree: two*. PhD thesis, Stanford University, 2021.

Cécile Le Rudulier. *Points algébriques de hauteur bornée*. These de doctorat, Rennes 1, 2014.

Dino Lorenzini and Stefan Schröer. Moderately ramified actions in positive characteristic. *Mathematische Zeitschrift*, 295(3):1095–1142, 2020.

François Loeser and Dimitri Wyss. Motivic integration on the Hitchin fibration. *Algebraic Geometry*, 8(2):196–230, 2021.

Gunter Malle. On the Distribution of Galois Groups, II. *Experimental Mathematics*, 13(2):129–136, 2004.

Iku Nakamura. Hilbert schemes of abelian group orbits. *Journal of Algebraic Geometry*, 10(4):757–779, 2001.

Andrew O’Desky and Julian Rosen. The moduli space of $G$-algebras. [arXiv:2011.07716](https://arxiv.org/abs/2011.07716) 2020.

Barbara R Peskin. Quotient-singularities and wild $p$-cyclic actions. *Journal of Algebra*, 81(1):72–99, 1983.

Emmanuel Peyre. Hauteurs et mesures de Tamagawa sur les variétés de Fano. *Duke Mathematical Journal*, 79(1):101–218, July 1995.

Emmanuel Peyre. Points de hauteur bornée, topologie adélique et mesures de Tamagawa. *Journal de Théorie des Nombres de Bordeaux*, 15(1):319–349, 2003.

Shi-shyr Roan. Minimal resolutions of Gorenstein orbifolds in dimension three. *Topology*, 35(2):489–508, 1996.

Takeshi Saito. Wild Ramification of Schemes and Sheaves. In *Proceedings of the International Congress of Mathematicians 2010 (ICM 2010)*, pages 335–356, Hindustan Book Agency, 2011.

Wolfgang M. Schmidt. Northcott’s theorem on heights. II. The quadratic case. *Acta Arithmetica*, 70(4):343–375, 1995.

Julien Sebag. Intégration motivique sur les schémas formels. *Bulletin de la Société Mathématique de France*, 132(1):1–54, 2004.

Matthew Satriano and Jeremy Usatine. Stringy invariants and toric Artin stacks. [arXiv:2009.04585](https://arxiv.org/abs/2009.04585) 2020.

Mathito Tanno. On Convergence of Stringy Motives of Wild $p^n$-Cyclic Quotient Singularities. [arXiv:2101.06971](https://arxiv.org/abs/2101.06971) 2021.

Fabio Tonini and Takehiko Yasuda. Moduli of formal torsors II. [arXiv:1909.09276v2](https://arxiv.org/abs/1909.09276v2), 2019, to appear in Annales l’Institut Fourier.

Fabio Tonini and Takehiko Yasuda. Notes on the motivic McKay correspondence for the group scheme $\alpha_p$. In *Singularities – Kagoshima 2017*, pages 215–233. WORLD SCIENTIFIC, 2019.

Mahito Tanno and Takehiko Yasuda. The wild McKay correspondence for cyclic groups of prime power order. [arXiv:2006.12048](https://arxiv.org/abs/2006.12048) 2020, to appear in Illinois Journal of Mathematics.
[TY20b] Fabio Tonini and Takehiko Yasuda. Moduli of formal torsors. *Journal of Algebraic Geometry*, 29:753–801, 2020.

[Vey03] Willem Veys. Stringy zeta functions for $\mathfrak{g}$-Gorenstein varieties. *Duke Mathematical Journal*, 120(3):469–514, 2003.

[Vey04] Willem Veys. Stringy invariants of normal surfaces. *Journal of Algebraic Geometry*, 13(1):115–141, 2004.

[WY15] Melanie Matchett Wood and Takehiko Yasuda. Mass formulas for local Galois representations and quotient singularities. I: a comparison of counting functions. *International Mathematics Research Notices, IMRN*, (23):12590–12619, 2015.

[WY17] Melanie Wood and Takehiko Yasuda. Mass formulas for local Galois representations and quotient singularities II: Dualities and resolution of singularities. *Algebra & Number Theory*, 11(4):817–840, 2017.

[XZ15] L. Xiao and I. Zhukov. Ramification of higher local fields, approaches and questions. *St. Petersburg Mathematical Journal*, 26(5):695–740, 2015.

[Yam15] Ryo Yamagishi. Crepant Resolutions of a Slodowy Slice in a Nilpotent Orbit Closure in $\mathfrak{sl}_n(\mathbb{C})$. *Publications of the Research Institute for Mathematical Sciences*, 51(3):465–488, 2015.

[Yam18] Takahiro Yamamoto. A counterexample to the McKay correspondence in positive characteristic. Master’s thesis, Osaka University, 2018.

[Yam21a] Takahiro Yamamoto. Pathological Quotient Singularities in Characteristic Three Which Are Not Log Canonical. *Michigan Mathematical Journal*, Advance Publication:1–14, 2021.

[Yam21b] Takahiro Yamamoto. Crepant resolutions of quotient varieties in positive characteristics and their Euler characteristics. *arXiv:2106.11526*, 2021.

[Yam21c] Takahiro Yamamoto. The $v$-function in the wild McKay correspondence is not determined by the ramification filtration. *arXiv:2107.03511*, 2021.

[Yas04] Takehiko Yasuda. Twisted jets, motivic measures and orbifold cohomology. *Compositio Mathematica*, 140(2):396–422, 2004.

[Yas06] Takehiko Yasuda. Motivic integration over Deligne-Mumford stacks. *Advances in Mathematics*, 207(2):707–761, 2006.

[Yas14a] Takehiko Yasuda. Densities of rational points and number fields. *arXiv:1408.3912*, 2014.

[Yas14b] Takehiko Yasuda. The $p$-cyclic McKay correspondence via motivic integration. *Compositio Mathematica*, 150(7):1125–1168, 2014.

[Yas15a] Takehiko Yasuda. Manin’s conjecture vs. Malle’s conjecture. *arXiv:1505.04555*, 2015.

[Yas15b] Takehiko Yasuda. Open problems in the wild McKay correspondence. 2015.

[Yas16] Takehiko Yasuda. Wilder McKay correspondences. *Nagoya Mathematical Journal*, 221(1):111–164, 2016.

[Yas17a] Takehiko Yasuda. Toward motivic integration over wild Deligne-Mumford stacks. In *Higher dimensional algebraic geometry—in honour of Professor Yujiro Kawamata’s sixtieth birthday*, volume 74 of Adv. Stud. Pure Math., pages 407–437. Math. Soc. Japan, Tokyo, 2017.

[Yas17b] Takehiko Yasuda. The wild McKay correspondence and $p$-adic measures. *Journal of the European Mathematical Society (JEMS)*, 19(12):3709–3734, 2017.

[Yas18] Takehiko Yasuda. The wild McKay correspondence via motivic integration. *Mathematical Society of Japan. Sūgaku (Mathematics)*, 70(2):159–183, 2018.

[Yas19] Takehiko Yasuda. Discrepancies of $p$-cyclic quotient varieties. *The University of Tokyo. Journal of Mathematical Sciences*, 26(1):1–14, 2019.

[Yas20] Takehiko Yasuda. Motivic integration over wild Deligne-Mumford stacks. *arXiv:1908.02932*, 2020.

[Yi94] Z. Yi. Homological Dimension of Skew Group Rings and Crossed Products. *Journal of Algebra*, 164(1):101–123, 1994.
