Manifestation of a non-abelian gauge field in a p-type semiconductor system

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Gauge theories, while describing fundamental interactions in nature, also emerge in a wide variety of physical systems. Abelian gauge fields have been predicted and observed in a number of novel quantum many-body systems, topological insulators, ultracold atoms and many others. However, the non-abelian gauge field, while playing the most fundamental role in particle physics, up to now has remained a purely theoretical construction in many-body physics. In the present paper we report the first observation of a non-abelian gauge field in a spin-orbit coupled quantum system. The gauge field manifests itself in quantum magnetic oscillations of a hole doped two-dimensional (2D) GaAs heterostructure. Transport measurements were performed in tilted magnetic fields, where the effect of the emergent non-abelian gauge field was controlled by the components of the magnetic field in the 2D plane.

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Gauge theories were originally conceived to describe elementary particles and their interactions. The concept of the emergent gauge field is relevant to a wide class of quantum systems whose initial formulation has no apparent relationship to gauge fields. Such emergent gauge fields arise naturally in many geometrical contexts and the idea that physical systems can be classified according to their geometrical properties has become an overarching paradigm of modern physics. One example of an abelian gauge theory in this context is the Berry phase, which is associated with the adiabatic evolution of a non-degenerate quantum state. The emergence of non-abelian gauge fields in degenerate quantum systems was first theoretically proposed by Wilczek and Zee shortly after the work of Berry.

While abelian gauge fields have been observed in systems ranging from optical fibers and semiconductor rings to Bose condensates of ultracold atoms, signs of non-abelian effects have so far only been observed in the nuclear quadrupole resonance of $^{35}$Cl in a single crystal of sodium chlorate. Non-abelian gauge fields have been theoretically predicted in a number of many-body systems including fractional quantum Hall liquids, spin-orbit coupled systems, and ensembles of ultracold atoms. In spite of the theoretical excitement and great interest all previous attempts to observe these fields were unsuccessful. This demonstrates the challenge involved in the experimental realization of emergent non-abelian gauge fields.

The idea of our experiment is partially based on previous theoretical work by Arovas and Lyanda-Geller as well as Murakami, Nagaosa and Zhang who proposed that effects relating to non-abelian gauge fields must be pronounced in hole-doped zinc blende semiconductors due to the strong spin-orbit coupling (SOC). In this context the gauge fields are closely associated with spin dynamics along curved trajectories: Ref. proposed the use of mesoscopic rings to bend the trajectory, whilst Ref. suggested use of an external electric field for the same purpose. In this work we use a 2D GaAs hole-doped heterostructure in a relatively small (fraction of a Tesla) magnetic field applied perpendicular to the 2D plane to curve the hole trajectories. In addition, we apply an in-plane magnetic field ($B_{||} \sim$ several Tesla), which allows us to control the magnitude of the spin-orbit coupling. The combination of the SOC and curved trajectories makes the non-abelian gauge field observable. The perpendicular magnetic field gives rise to quantum magnetic oscillations which are influenced by non-abelian spin dynamics. We measure the oscillations via the Shubnikov-de Haas (SdH) effect. The SdH effect has been measured previously in numerous experiments with 2D systems with strong spin-orbit interaction, see e.g. Refs. However in all previous studies, effects related to the non-abelian Berry phase are negligible, and what is measured is simply the densities of the spin-split subbands. One needs very special conditions to distinguish between the abelian and the non-abelian Berry phases, it is necessary to tune independently the spin precession, the orbital dynamics, and the spin-orbit interaction. To do so in our experiment we use the following crucial points. (i) We can tune the spin-orbit coupling over a wide range using the in-plane field $B_{||}$ while keeping the orbital dynamics fixed. (ii) We use a low symmetry crystal with highly anisotropic coupling to $B_{||}$, which allows us to control independently the Larmor and the spin precession frequencies. This is key to proving that the effects we observe cannot be due to abelian physics, nor due to differences between datasets taken at different carrier densities, gate biases, or even from different samples. (iii) We use a device where we can minimise the undesirable Rashba interaction, allowing a simple analytic theory to explain the data. These three factors allow us to report the first observation of the non-abelian gauge phase which was
elusive for 30 years since its theoretical prediction.

The spin dynamics of a particle moving around a circle in momentum space used in Onsager quantization is illustrated in Fig.1. The three panels in this figure correspond to three qualitatively different situations: (a) spin dynamics being absent, (b) abelian spin dynamics and (c) non-abelian spin dynamics. The spin (red arrows) is driven by a local effective magnetic field \( \mathbf{B}_{\text{eff}} \) (blue arrows), which is the sum of the external magnetic field \( \mathbf{B}_{\text{ext}} \) and the momentum-dependent spin-orbit field \( \mathbf{B}_{\text{soc}} \). Panel (a) depicts the trajectory of a non-relativistic electron in the absence of spin-orbit. In this case \( \mathbf{B}_{\text{eff}} \approx \mathbf{B}_{\text{soc}} \approx \mathbf{k} \). The precession of spin around the orbit generates a geometric Berry phase (abelian gauge field) which appears as the \( \pi \)-phase shift observed in magnetic oscillations. The non-abelian case addressed in the present work is illustrated in panel (c). Here, the driving field \( \mathbf{B}_{\text{eff}} \) is not collinear with spin and the noncollinearity is proportional to the non-abelian gauge field. Due to the non-abelian spin dynamics, the particle acquires a matrix-valued phase equal to the circulation of the gauge field along the trajectory in momentum space. The phase manifests itself in quantum magnetic oscillations.

Holes in GaAs originate from atomic \( p_{3/2} \) orbitals and hence posses an angular momentum \( J = 3/2 \). The electric quadrupole interaction leads to strong coupling between the angular momentum \( \mathbf{J} \) and the linear momentum \( \mathbf{k} \), which is described by the Luttinger Hamiltonian. The \( z \)-confinement in a 2D heterostructure enforces quantization of \( J \) along the \( z \)-axis. Therefore, a hole quantum state with a given in-plane momentum \( \mathbf{k} = (k_x, k_y) \) splits into two doublets with \( J_z = \pm 3/2 \) (heavy holes) and \( J_z = \pm 1/2 \) (light holes). Since light holes lie significantly higher in energy, we shall only consider heavy holes for the low energy dynamics.

The heavy-hole Kramers doublet can be described by an effective spin \( s = 1/2, \left| J_z = \pm 3/2 \right\rangle \equiv \left| \uparrow \right\rangle, \left| J_z = \pm 3/2 \right\rangle \equiv \left| \downarrow \right\rangle \). The Hamiltonian describing heavy holes consists of the kinetic energy, the Zeeman interaction and the SOC, \( H = H_K + H_Z + H_{\text{soc}} \),

\[
H_K = \epsilon(k) \\
H_Z = -\frac{\Delta}{2} \sigma_z, \quad \Delta = g\mu_B B_z \\
H_{\text{soc}} \equiv -\mathbf{\beta}(k) \cdot \sigma = -\frac{1}{2} \alpha \left[ \sigma_+ B_z \sigma_- + \sigma_- B_z \sigma_+ \right],
\]

where \( k = -i\hbar \nabla - eA; \sigma_{\pm} = \sigma_x \pm i\sigma_y, B_z = B_z \pm iB_y, k_{\pm} = k_x \pm ik_y; A \) is the in-plane vector potential created by \( B_z \), \( e \) is the elementary charge, \( \sigma_i \) are Pauli matrices describing the spin, \( \mu_B \) is Bohr magneton, \( g = g_z \) is the effective g-factor and \( \alpha \) is the SOC strength. Note that due to mixing between heavy holes states the dispersion \( \epsilon(k) \) can significantly differ from the simple quadratic form, see discussion in Appendix A. Note also that generally \( g \) and \( \alpha \) depend on \( k \), and in combination with nonquadratic dispersion \( \epsilon(k) \) this dependence results in a very complex fan diagram of Landau levels. However, according to the Landau theory of normal Fermi liquids this complexity is irrelevant to the problem we address. We do not need the full Landau level fan diagram. According to normal Fermi liquid theory only the values of the parameters at the Fermi energy are relevant. This statement is very general, and even includes hole-hole Coulomb interaction effects. We will fit the experimental data to obtain the parameters \( g \) and \( \alpha \) at \( \epsilon = \epsilon_F \).

A derivation of the spin-orbit interaction \( H_{\text{soc}} \) is presented in Appendix A although we shall make two comments here on its origin: (i) The spin-orbit coupling arises from a small mixing between heavy and light holes, where the mixing probability is 1-2\% (see Appendix A). (ii) The kinematic structure of \( H_{\text{soc}} \) in Eq. (1) is dictated by the fact that the Pauli matrices \( \sigma_z \) correspond to \( \Delta J_z = \pm 3 \).

If the perpendicular magnetic field is zero, \( A \propto B_z = 0 \), then the hole trajectories are straight lines and \( H_{\text{soc}} \) in Eq. (1) simply splits the doubly degenerate band, \( \epsilon_\pm \), into a pair of chiral bands. In presence of \( B_z \) the hole trajectory forms a circle, \( k = k_0 \cos \theta, \sin \theta \). Semiclassically, using the wave packet picture, the angle is \( \theta = -\omega_z t \) (the sign corresponds to \( B_z > 0 \)), where \( \omega_z = eB_z/m \) is the cyclotron frequency and \( m = k_m (\frac{\hbar}{2\pi})^{-1} \) is the effective cyclotron mass at the Fermi energy. The spin-orbit field \( \beta(k) \) varies along the trajectory. This variation can be removed by a local gauge transformation of the spinor wave function \( \psi \to \psi' = g^{-1}(k)\psi \). Taking \( g(k) = e^{-i\theta\sigma_z} \) we gauge out the angle dependence of the SOC,

\[
\beta'(k) \cdot \sigma = g^{-1} [\beta \cdot \sigma] g, \quad \beta' = \alpha k^2 B_{||}.
\]

Since our choice of \( g(k) \) ensures that we perform a transformation to the co-rotating frame of the hole, it follows that \( \beta' \) does not vary along the trajectory. The gauge transformation results in the covariant derivative \( \nabla \to \nabla - i\Omega_k \), where \( \Omega_k \) is the non-abelian gauge field possessing a vortex structure in 2D momentum space

\[
\Omega_k = i g^{-1} \nabla \sigma = \left( -\frac{k_y \sigma_z}{k^2}, \frac{k_z \sigma_x}{k^2} \right).
\]

The field tensor corresponding to this gauge field is zero, \( F_{\mu\nu} = \partial_\mu \Omega_\nu - \partial_\nu \Omega_\mu - i[\Omega_\mu, \Omega_\nu] = 0 \). However, the gauge field has a nonzero circulation along the hole trajectory

\[
\oint \Omega \cdot dk = 2\pi \sigma_z,
\]

and this circulation reveals itself in quantum magnetic oscillations.

To understand quantum magnetic oscillations we need to consider the impact of spin-orbit coupling upon the Landau level structure. For this analysis, we restrict ourselves to a semiclassical approximation, where the Landau levels are determined by the Onsager quantization
condition. Consider a hole traversing the circular trajectory, where the hole is initially prepared in a polarization state \( \psi(0) \). Under the combined action of \( H_{soc} \) and \( H_Z \) spin will precess along the trajectory, as shown in Fig.1. After a full cycle the spin wave function is 

\[
\psi(2\pi) = U \psi(0), \quad U \in SU(2) \text{ is a unitary evolution matrix.}
\]

In order to satisfy the semiclassical quantization condition, it is necessary for \( \psi(0) \) to be an eigenvector of \( U \), i.e., \( \psi(2\pi) = e^{\pm i\Phi} \psi(0) \). Here \( e^{\pm i\Phi} \) are the complex conjugate eigenvalues of \( U \). Hence, depending on the spin state, an additional phase \( \pm \Phi \) appears in the Onsager quantization condition due to spin dynamics.

SDH oscillations in the resistivity are given by the usual Lifshitz-Kosevich formula\(^{22}\). Accounting for the additional phase \( \Phi \) we obtain,

\[
\Delta \rho_{xx} = \rho_{xx}(B) - \rho_{xx}(0) = \mathcal{A}(B) \cos \Phi \cos \frac{\pi k_F^2}{eB_Z} . \tag{5}
\]

The amplitude factor depends on the hole scattering time \( \tau \), \( \mathcal{A}(B) \propto e^{-\frac{\pi k_F^2}{B_Z}} \). Spin dynamics enters only via the spin evolution phase factor \( \Phi \). For the semi-classical approximation approach we assume large filling factors \( \nu = \frac{k_F^2}{2\pi^2 B_{||}} \gg 1 \), hence only the lowest harmonic of magnetic oscillations is taken into consideration.

The matrix phase \( U \) may be explicitly expressed as a path-ordered exponential which can be calculated using the gauge transformation from Eq.(2)

\[
U = \mathcal{P} \exp \left\{ - \frac{i}{\omega_c} \oint \mathbf{B} \cdot \mathbf{d} \mathbf{\theta} \right\} = \exp \left\{ i \oint \mathbf{\Omega}_k \cdot \mathbf{d} \mathbf{k} - \frac{2\pi}{\omega_c} \left[ \mathbf{\beta} \cdot \mathbf{\sigma} + \frac{\Delta}{2} \mathbf{\sigma}_z \right] \right\} . \tag{6}
\]

Hence, using Eqs. (1) and (2) we find the prefactor in Eq.(2) for SDH oscillations, \( 2 \cos \Phi = \text{tr}U \),

\[
\Phi = \frac{2\pi}{\omega_e} \sqrt{\left( \omega_c - \frac{\Delta}{2} \right)^2 + |\alpha k_F^2|^2 (B_x^2 + B_y^2)} . \tag{7}
\]

Here the \( \omega_c \) term under the square root comes from the non-abelian gauge field. It is worth noting that the effect of the gauge field is somewhat analogous to Thomas precession in special relativity\(^{23}\). As previously mentioned, the gauge field cannot be observed without the in-plane magnetic field. This is evident from Eq.(7): if \( B_{||} = 0 \) the gauge contribution is exactly \( 2\pi \) and hence the phase shift is determined only by the Zeeman splitting, \( \text{tr}U = 2 \cos(\pi \Delta/\omega_c) \). The Zeeman splitting with \( B_{\parallel} \neq 0 \) is \( \Delta E_Z = \sqrt{\left( \frac{\Delta}{2} \right)^2 + |\alpha k_F^2|^2 (B_x^2 + B_y^2)} \). A naive expectation for the spin accumulated phase would be \( \Phi_{\text{naive}} = 2\pi \delta E_Z/\omega_c \), but Eq.(7) is different from this. A semi-naive expectation would take into account the abelian Berry phase \( \varphi_B \) on top of the Zeeman splitting. The phase \( \varphi_B \) is given by the first term of the square root expansion in (7) in powers of \( \omega_c \), yielding

\[
\Phi_{AB} = \frac{2\pi \delta E_Z}{\omega_c} + \varphi_B = \frac{2\pi \delta E_Z}{\omega_c} - \frac{\pi \Delta}{\delta E_Z} . \tag{8}
\]

The subscript “AB” in \( \Phi \) stands for “Abelian Berry”. The abelian Berry phase approach provides a good description for magneto-oscillations in Dirac fermion systems\(^{19,20}\), and for quantum interference in mesoscopic rings with strong spin-orbit coupling\(^{24}\). However in our case, both the “naive” \( \Phi_{\text{naive}} \) and the abelian Berry phase \( \Phi_{AB} \) approach are inconsistent with the data.

In our experiments the 2D hole system is formed in a 20nm wide symmetric GaAs quantum well, grown in a (311)A GaAs-Al\(_{0.33}\)Ga\(_{0.67}\)As heterostructure as indicated in Fig.2. Previous experiments on this system have shown that holes in a (311) oriented quantum well have a tensor g-factor with an unusual off-diagonal term \( g_{xz} \). Although tilted field measurements revealed the presence of the \( g_{xx} \) term, no comparison of the Shubnikov de Haas oscillations with theory was possible, as there was no theory available for 2D hole systems in tilted magnetic fields. We are now able to show that there is excellent qualitative agreement between the experimental data and the new theoretical model based on the non-abelian gauge field.

We use the coordinates \( x \parallel [233], y \parallel [0\bar{1}1], z \parallel [311] \) shown in Fig.2. The gyromagnetic tensor is not diagonal in the \( x, y \) and \( z \) axes, therefore the expression for \( \Delta \) presented in Eq.(1) and used elsewhere is now replaced

\[
\text{FIG. 1: Spin dynamics along the closed trajectory in momentum space (green circle) in three qualitatively different situations. The spin shown by red arrows is driven by a local effective magnetic field shown by blue arrows. (a) Absence of spin dynamics. This corresponds to the case of an electron moving within an external magnetic field in the absence of spin-orbit interaction. (b) Abelian spin dynamics. Spin is changing, but it remains parallel to the driving field \( \mathbf{B}_{\text{eff}} \approx \mathbf{k} \). (c) Non-abelian spin dynamics. The spin is parallel to the vector sum of the driving field \( \mathbf{B}_{\text{eff}} \) and the non-abelian gauge field.}
\]
FIG. 2: Magnetoresistance (SdH) oscillations in an external magnetic field $\mathbf{B}$ applied at an angle $\theta_{\text{tilt}}$ to the 2D heterostructure, $B_{||} = B_z \tan \theta_{\text{tilt}}, B_z > 0$. (a) The orientation of the magnetic field components relative to the crystal axes. (b) SdH oscillations plotted as a function of $1/B_z$ for an applied field of $B_y > 0$, $B_x = 0$ (green solid lines) and $B_y < 0$, $B_x = 0$ (purple dotted lines). Traces are offset for clarity. The amplitude of the oscillations was normalized by multiplying the data by $\rho_0$.

Corresponding to the change in sign of $\cos \Phi$ in Eq.(5).

Moreover, as we discuss below, it brings an additional confirmation of the non-abelian dynamics.

Note that the off-diagonal tensor component $g_{xx}$ makes the magnetic response different for three orientations of $B_{||}$: $B_{||} = B_x, B_{||} = -B_x$ and $B_{||} = B_y$. This triples the amount of data we can get from the same sample. Details of our experimental setup/method are presented in Appendices [B, C and D]

So far we have only considered the effect of the external magnetic field, however spin dynamics can also be influenced by additional couplings, such as the Rashba interaction (stemming from the asymmetry of the interface) and the Dresselhaus interaction (arising from the lack of inversion symmetry in the bulk GaAs crystal). We apply a voltage bias to the back-gate, to tune the symmetry of the GaAs quantum well such that the Rashba interaction is practically zero, see Appendix [D]. The Dresselhaus interaction is relatively weak, nevertheless it is important in some regimes. Moreover, as we discuss below, it brings an additional confirmation of the non-abelian dynamics.

The results of our measurements are presented in panels b and c of Fig.2 which plot resistivity versus $1/B_z$ where the in-plane field is altered by tilting the sample at an angle $\theta_{\text{tilt}}$ with respect to the applied field, such that $B_z = B_{||} \tan \theta_{\text{tilt}}, B_z > 0$. Panel b corresponds to tilting in the $yz$-plane ($B_x = 0$) and panel c corresponds to tilting in the $xz$-plane ($B_x = 0$). The data in panel b is symmetric with respect to $B_y \rightarrow -B_y$, whilst the data in panel c exhibits asymmetry with respect to $B_x \rightarrow -B_x$ due to nonvanishing $g_{xx}$ in Eq.(9). According to Eqs. (5) and (7), the normalized amplitude of resistivity oscillations, $\cos \Phi$, is a function only of $\theta_{\text{tilt}}$ and is independent of the magnitude of the total magnetic field $\mathbf{B}$. At tilt angles corresponding to changes in the sign of $\text{tr}U = 2 \cos \Phi$, the first harmonic of the SdH oscillations invert (i.e. maxima become minima, and vice versa). At these “coincidence” angles the phase $\Phi$ must coincide with a half-integer multiple of $\pi$. In the data, these coincidences are observed at the tilt angle $\theta_{\text{tilt}} = \pm (7 \pm 0.5^\circ)$ for the field applied along the $yz$-plane in Fig.2. For the field applied in the $xz$-plane (Fig.2c) there are multiple coincidence
angles at \( \theta_{\text{tilt}} = 18 \pm 1^\circ, 5.5 \pm 0.5^\circ, 3.5 \pm 0.25^\circ \) for \( B_x > 0 \) and only a single coincidence at \( \theta_{\text{tilt}} = -6.5 \pm 0.5^\circ \) for \( B_z < 0 \). The coincidence angles are plotted in Fig. 3, and are described by Eqs. (7) and (9). There are three independent device-specific parameters in these equations, which are \( g \), 2\( \lambda k^2_F/(g \mu_B) \), and \( g_{xz}/g \). We use the value \( m = 0.25m_e \) derived in Appendix A as our reference point and hence we are left with unknowns \( g \), \( \lambda = 2\lambda k^2_F/\mu_B \), and \( g_{xz} \) which we treat as free fitting parameters.

Altogether we have three fitting parameters to describe five coincidence angles. To compare the experimental coincidence angles to those of theory, we perform a least-squares fit to \( \Phi \) using the observed four coincidences angles for the orientations \( B_x > 0, B_y = 0 \) and \( B_y \neq 0, B_x = 0 \) (red and green symbols in Fig. 3a), and use the values obtained to predict the coincidence angles for the orientation \( B_x < 0, B_y = 0 \) (blue traces in Fig. 3b). The solid red and green curves in Fig. 3a show the calculated \( \Phi/\pi \) obtained from this fitting, with the following values of the fitting parameters: \( g = 7 \), \( |\lambda| = 0.88 \), and \( g_{xz} = -0.87 \). The solid blue line in Fig. 3b shows \( \Phi/\pi \) for the \( B_x < 0, B_y = 0 \) orientation calculated using the fitting parameters from Fig. 3a, which predicts that there will be only a single coincidence observed in the experimentally measured range of \( \theta_{\text{tilt}} \), in agreement with experiment (blue square). There is a reasonable agreement between the predicted coincidence angle of \( \theta_{\text{tilt}} = -4.5^\circ \) and that observed in the experiment of \( \theta_{\text{tilt}} \approx -6.5^\circ \pm 0.5^\circ \), although we will shortly discuss the origins of this 2° discrepancy.

To highlight the non-abelian dynamics we have attempted to fit the observed data using the abelian Berry formula Eq. (11) instead of Eq. (7). Using Eqs. (5), (9) we repeat the same procedure described above and fit \( \Phi_{AB} \) to the observed four coincidence angles for the orientations \( (B_x > 0, B_y = 0) \) and \( (B_y \neq 0, B_x = 0) \), as shown in Fig. 3c. The fitting parameters obtained are \( g = 9.5 \), \( |\lambda| = 0.54, g_{xz} = +0.81 \). These parameters were then used to predict the coincidence angles occurring for the orientation \( B_x < 0, B_y = 0 \), shown in Fig. 3d. The key point is that the abelian theory always predicts three coincidence angles in contrast to the single coincidence observed in experiment.

Although the number of coincidence angles is not a topological invariant (for example it depends on the range of tilt angles available in the experiment), it is robust within both the non-abelian and the abelian theories. As shown in Appendix C, although the precise tilt angles at which the coincidences occurred are sensitive to the fitting parameters, the number of coincidences could not be changed even after significant variation of the parameters.

Additional confirmation of the non-abelian dynamics comes from the Dresselhaus interaction, neglected so far because of its smallness. In the co-rotating frame the spin-orbit coupling given by Eq. (2) results in an energy splitting \( \Delta_{\uparrow \downarrow} = \omega_c \frac{e}{2} \) between the “up” and “down” spin states, see Eq. (7). The quantization axis for \( \Delta_{\uparrow \downarrow} \) is tilted with respect to \( z \). The Dresselhaus interaction in the co-rotating frame takes the form of a small periodic perturbation \( \sigma_z \cos \omega_c t \). Since the quantization axis is tilted, this perturbation drives transitions between the spin “up” and spin “down”. Because of the smallness of the perturbation the transitions are significant only close to resonance, \( \Delta_{\uparrow \downarrow} \approx \omega_c \). We use the amplitude of the Dresselhaus interaction as an additional fitting parameter, and find that it is close to the value known from the literature, see Appendix F. The effect of the Dresselhaus perturbation is shown by the dashed curves in Figs. 3a,b. The tiny difference from the red and green solid curves in Fig. 3a, which do not include the Dresselhaus interaction, show that the effect of the interaction is very weak. On the other hand, for \( B_x < 0 \) (Fig. 3b) the resonance condition \( \Delta_{\uparrow \downarrow} \approx \omega_c \) is satisfied and the Dresselhaus term now becomes significant. This causes a clear difference between the solid and dashed blue curves in Fig. 3b, which completely removes the small disagreement between experimental and theoretical values of the coincidence angle.

Of course, the inclusion of the Dresselhaus interaction does not influence the number of coincidence angles, which is a very robust number. Moreover the inclusion of the Dresselhaus term explains why the single coincidence for \( B_x < 0 \) is not sharp, but occurs over a much wider range of angles than for \( +B_x \) or \( \pm B_x \) (seen as the slow phase inversion and small amplitude of the SdH oscillations in the range \( 5^\circ < |\theta_{\text{tilt}}| < 10^\circ \) for blue traces in Fig. 2b). This non-sharp transition for \( B_x < 0 \) is explained by the inflection in the blue dashed curve in Fig. 3b, which is due to the Dresselhaus interaction. The “inflection” effect provides further confirmation of the non-abelian dynamics, since the small Dresselhaus perturbation is always insignificant in the abelian theory.

Finally we present in Figs. 2b and c theoretical SdH curves calculated with modified Lifshitz-Kosevich formula (5). The agreement between theory (including Dresselhaus interaction) and experiment is very good. Overall, our data on the number of coincidences, supported by the slow phase flip of the SdH oscillations for \( B_x < 0 \), provide unambiguous evidence for the non-abelian gauge field.

The non-abelian gauge field features centrally in theoretical proposals to exploit hole systems for spintronics and quantum information purposes, including the realization of the dissipationless spin Hall effect and non-abelian manipulation of hole qubits. The capacity of hole systems in this context is further enhanced by the suppression of decoherence due to absence of the hyperfine interaction. The observation of the non-abelian gauge field in a 2D hole system has positive implications for future studies of hole systems which rely on this concept.

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Expressing the envelope of the SdH oscillations as $\cos \Phi$, we find that $\Phi/\pi$ becomes a smooth function of tilt angle. This function is plotted for the experimental range of tilting angles. Angles at which $\Phi$ crosses a half-integer multiple of $\pi$ correspond to inversions of the SdH oscillations. Both theories contain three unknown parameters $g, g_{\pm}, \lambda$. In panels a, c we show the least squares fits of $\Phi$ to the observed coincidence angles for an applied magnetic field $B$. The plots of $\Phi$ in panels b, d show the predicted coincidences for $B_x < 0$, $B_y = 0$. For the non-abelian theory the solid curves do not include the Dresselhaus perturbation, whereas it is included for the dashed curves. In the abelian case the influence of the Dresselhaus perturbation is negligible. The non-abelian gauge theory predicts a single angle of coincidence blue solid line in panel b, consistent with the observed coincidence point (blue square). Including the Dresselhaus interaction (blue dashed line) provides quantitative agreement with experiment. In contrast, the abelian theory (panel d) predicts three coincidences whilst experimentally only one angle of coincidence was observed (solid blue square); the two coincidence angles marked with open symbols were not observed in experiment.

**Appendix A: Derivation of the spin-orbit interaction for heavy holes**

In a zinc blende semiconductor the hole wave function originates from atomic $p_{3/2}$ orbitals resulting in an angular momentum $J = 3/2$. In the long wavelength approximation, the effective Luttinger Hamiltonian for holes is quadratic in the hole momentum $k^{(2)}$ (see also Ref.29):

$$H_L = \left( \gamma_1 + \frac{5}{2} \gamma_2 \right) \frac{k^2}{2m_e} - \gamma_2 \frac{e}{m_e} (k_1^2 S_1^2 + k_2^2 S_2^2 + k_3^2 S_3^2) + \gamma_2 \frac{e}{m_e} \left( k_1 k_2 \{S_1, S_2 \} + k_2 k_3 \{S_2, S_3 \} + k_3 k_1 \{S_3, S_1 \} \right),$$

(A1)

1, 2, 3 are the crystal axes of the cubic lattice, $m_e$ is the electron mass, $\{ \}$ denotes the anticommutator, and $\gamma_1$, $\gamma_2$ and $\gamma_3$ are Luttinger parameters. In GaAs $\gamma_1 \approx 6.89$, $\gamma_2 \approx 2.1$, $\gamma_3 \approx 2.93$. The Hamiltonian (A1) can be rewritten as

$$H_L = \left( \gamma_1 + \frac{5}{2} \gamma_2 \right) \frac{k^2}{2m} - \frac{\gamma_2}{m} (k \cdot S)^2 + k_i k_j S_m S_n T_{ijmn}^{(4)},$$

where

$$\tau_2 = \frac{2\gamma_2 + 3\gamma_3}{5} \approx 2.6.$$  

The irreducible $4^{th}$ rank tensor $T_{ijmn}^{(4)}$ depends on the orientation of the cubic lattice, the tensor is proportional to $\gamma_3 - \gamma_2$. Neglecting $\gamma_3 - \gamma_2$ compared to $\gamma_2$, the Luttinger Hamiltonian can be approximated by the following rotationally invariant (independent of the lattice orientation) Hamiltonian

$$H_L \rightarrow H = \frac{\hbar^2}{2m_e} \left( \gamma_1 + \frac{5}{2} \gamma_2 \right) k^2 - 2\tau_2 (k \cdot J)^2 \right].$$

(A2)

Due to the confining potential $V(z)$, motion perpendicular to the 2D plane of the heterostructure is quantized, leading to the formation of 2D subbands, where only the lowest subband occupied in the low-temperature experimental regime. Assuming a square well confining potential of width $d$ we have $\langle k_z^2 \rangle = \frac{\hbar^2}{d^2}$. Since $\langle k_x^2 \rangle \gg k_F^2$, we may expand $- (k \cdot J)^2 = - k_x^2 f_x^2 + \ldots$, with the leading term becoming diagonal in a basis of states with $J_z$. Due to the sign of the interaction, states with $J_z = \pm \frac{3}{2}$ (heavy hole) are lower in energy, and the splitting between these states and states with $J_z = \pm \frac{1}{2}$ (light hole) at $k_x = k_y = 0$ becomes

$$\Delta_{hl} = \frac{e^2}{m_e d^2} \approx 9.6 \text{meV}.$$
Here we take $d = 20\text{nm}$. The splitting between the lowest and the next heavy hole band at $k_x = k_y = 0$ is

$$\Delta_{H12} \approx \frac{3}{2} (\gamma_1 - 2 \gamma_2) \frac{\pi^2 \hbar^2}{m_e d^2} \approx 4.6\text{meV} . \quad (A4)$$

Numerical diagonalization of the full Luttinger Hamiltonian $\mathbf{(A1)}$ using the NextNano++ package gives the energy levels (2D dispersions) plotted in Fig. 4. The HH1-HH2 splitting at $k_x = k_y = 0$ is pretty close to $\mathbf{(A1)}$ while the HH1-LH1 splitting in Fig. 4 is somewhat smaller than $\mathbf{(A3)}$ because of the $\sim (\gamma_3 - \gamma_2)$ tensor corrections. At hole density corresponding to our experiment, $n \approx 10^{11}\text{cm}^{-2}$, only the lowest band is populated. The lowest band dispersion $\epsilon(k)$ enters Eq. $\mathbf{(1)}$. We describe this band by the effective spin $s = 1/2$, $|J_z = +3/2 \rangle \equiv | \uparrow \rangle$, $|J_z = -3/2 \rangle \equiv | \downarrow \rangle$. The Fermi momentum is $k_F \approx 0.0079\AA^{-1}$ and the Fermi energy, $E_F \approx 1.3\text{meV}$, is shown in Fig. 4 by the red horizontal line. The heavy hole effective mass, $m = k \left( \frac{\hbar^2}{2m} \right)^{-1}$, follows from Fig. 4. At $k \rightarrow 0$ the mass is about $0.14m_e$ and at $k = k_F$ the mass is $m \approx 0.25m_e$. Obviously only the latter mass is relevant to our analysis.

The off-diagonal part of $(\mathbf{k} \cdot \mathbf{J})^2$ in the Hamiltonian $\mathbf{(A2)}$, $(\mathbf{k} \cdot \mathbf{J})^2 \rightarrow \frac{1}{3} (k_- J_+ + k_+ J_-)^2$, leads to heavy-light hole mixing.

$$|\mathbf{k}, \uparrow\rangle = \left[ \begin{array}{c} \frac{3}{2} + ak_+^2 \pm \frac{1}{2} \end{array} \right] e^{i\mathbf{k} \cdot \mathbf{r}}$$

$$|\mathbf{k}, \downarrow\rangle = \left[ \begin{array}{c} -\frac{3}{2} + ak_+^2 \pm \frac{1}{2} \end{array} \right] e^{i\mathbf{k} \cdot \mathbf{r}}$$

$$\alpha = \sqrt{\frac{3\gamma_3}{2m_e \Delta_{bl}}} = \frac{\sqrt{3}}{4(k_x^2)} . \quad (A5)$$

Taking the square well width $d = 20\text{nm}$ and the hole density $n = 10^{11}\text{cm}^{-2}$, we arrive at the following estimate for the mixing probability, $a^2 k_F^2 = \frac{3}{16} d^4 n^2 \approx 1.2 \times 10^{-2}$. This very small mixing, of order $1/1000$ in probability, is responsible for the SOC considered here.

The Zeeman interaction of a $J = 3/2$ hole with magnetic field $\mathbf{B}$ is\cite{17},

$$\delta H = -\frac{g_0}{3} \mu_B \mathbf{B} \cdot \mathbf{J} , \quad (A6)$$

where $g_0 \approx 7.2$. Taking the matrix element of $\delta H$ between states Eq. $\mathbf{(A5)}$ we find the effective matrix of $H_{soc}$$

$$\langle \downarrow | H_{soc} | \uparrow \rangle \equiv \langle \downarrow | \delta H | \uparrow \rangle = -\frac{g_0 \mu_B}{4(k_x^2)} B Z k_+^2 . \quad (A7)$$

Comparing this with $H_{soc}$ in Eq. $\mathbf{(1)}$ we determine the coefficient $\alpha$ in this equation to be

$$\alpha = \frac{g_0 \mu_B}{4(k_x^2)} . \quad (A8)$$

According to our fit of SdH data $|\lambda| = 2|\alpha|k_F^2 / \mu_B \approx 0.88$. Hence we find that $k_F^2 / k_x^2 \approx 0.25$ and the probability of the heavy-light hole mixing is $a^2 k_F^2 = \frac{3}{16} k_F^2 \approx 1.1 \times 10^{-2}$, which is remarkably consistent with the estimate presented after Eq. $\mathbf{(A5)}$. It is worth noting that Eq. $\mathbf{(A8)}$ is approximate, since one should expect a comparable contribution to $\alpha$ which is not accounted for by the calculations presented. So far, we have neglected the coupling to the vector potential created by $\mathbf{B}_0$, $(\mathbf{k} \cdot \mathbf{J})^2 \rightarrow ((\mathbf{k} - e \mathbf{A}) \cdot \mathbf{J})^2$. This coupling also gives a contribution to the coefficient $\alpha$, see Refs.\cite{28,31}. This contribution is highly sensitive to the exact shape of the confining potential and therefore cannot be reliably calculated\cite{31}. The kinematic form of $H_{soc}$ however remains unambiguous and we can fit the value of $\alpha$ to the experimental data.

Appendix B: Sample and transport measurements

The 2D hole system resides within a symmetrically doped 20nm-wide GaAs/Al$_{0.33}$Ga$_{0.67}$As quantum well, grown on the low symmetry plane (311) by molecular beam epitaxy. A heavily doped $n^+$ GaAs layer located 2.6μm below the quantum well, acts as an in situ back gate, allowing the 2D density to be tuned\cite{32}. At zero back-gate voltage, the density of the 2D hole system is $n = 1.33 \times 10^{11}\text{cm}^{-2}$ with a corresponding mobility of $\mu = 678,000\text{cm}^2\text{V}^{-1}\text{s}^{-1}$. Transport measurements were performed in a Kelvinox 100 dilution refrigerator within the bore of a 15T magnet at a base temperature of 25mK, using standard lock-in techniques, with a constant ac current of 10nA at a frequency of 5Hz. To perform tilted field measurements, the sample was mounted on a piezo-electric rotator which allowed for in situ rotation to be conducted with an accuracy of ±0.01°\cite{33}.
Initially the 2D device was rotated to $\theta_{\text{tilt}} = 90^\circ$, so the magnetic field lies perpendicular to the sample plane, $B_z \neq 0$, $B_{\parallel} = 0$, and the sample orientation confirmed by measuring the Hall plateaus as a function of perpendicular field, shown in Fig.5 (blue). The corresponding low-field oscillating longitudinal resistivity is shown in (red), with spin-splitting appearing for $B_z > 0.35\, \text{T}$. For the purposes of our analysis we are only interested in low field data between $B_z = 0.15\, \text{T}$ and $0.25\, \text{T}$.

Appendix C: Tilted field measurements

The coincidence method using tilted fields was first pioneered by Fang and Stiles in 1998 to study the Landé g-factor in 2D electron systems. Here we perform a similar set of tilted field transport measurements for a 2D hole system, taken along two crystal directions: the high symmetry [011] and the low symmetry [233], as depicted in Fig.4. To achieve this, the device was first mounted on the rotator such that it tilts between the crystal axes [011] and [001], where the 2D plane is fully perpendicular to the field at $\theta_{\text{tilt}} = 90^\circ$. The sample was then rotated towards the [011] direction till $\theta_{\text{tilt}} = 10^\circ$ to introduce a parallel field component $B_y$, and the total field $B$ swept, changing the sign of the in-plane field ±$B_y$. This procedure was repeated for a number of different $\pm \theta_{\text{tilt}}$ with increasing in-plane field components. The experiment was then repeated for equivalent $-\theta_{\text{tilt}}$ and the results plotted in Fig.5. During a second cooldown, the sample was re-oriented to perform tilted measurements along the [311] and low symmetry [233] crystal axes. The experiment was then repeated for both $\pm \theta_{\text{tilt}}$ and the results shown in Fig.6.

Appendix D: Tuning the confining potential with the back-gate voltage to compensate Rashba spin-orbit interaction

The electric potential across the quantum well was tuned via the in situ back gate, to adjust the confining potential. The presence of the Rashba SOC results in beatings of the SdH oscillations even without any tilting of the magnetic field. The Rashba interaction is sensitive to the back-gate voltage ($V_{\text{BG}}$), so by varying the applied bias-voltage, we can tune the system to minimize the amount of beatings and hence to eliminate the Rashba interaction. Fig.6 shows these beatings in detail, where the SdH oscillations at each back gate voltage are periodic in $\frac{1}{B_y}$ and the amplitudes of these oscillations normalized for clarity by multiplying the datasets by $e^{-\frac{E}{T}}$ to remove the envelope. The data are taken without any tilting, $\theta_{\text{tilt}} = 90^\circ$. From Fig.6 we select $V_{\text{BG}} = +1.50\, \text{V}$ as the final operating point with the least amount of beating in the SdH oscillations. We will show that the major part of the Dresselhaus interaction does not influence dynamics at $\theta_{\text{tilt}} = 90^\circ$. Hence, minimizing the beating we tune the Rashba interaction to be close.
to zero. This back-gate voltage is used as the operating point for the rest of the experiment. At this point the carrier density is \( n = 9.26 \times 10^{10} \text{cm}^{-2} \) and the mobility is 600,000 cm\(^2\)V\(^{-1}\)s\(^{-1}\).

### Appendix E: Sensitivity to Fitting Parameters

The comparison of the experimental result with possible theories is presented in Fig.3. Panels a and b show the non-abelian theory and panels c and d show the abelian theory. The non-abelian theory is consistent with experiment while the abelian theory is not consistent. Since the conclusions are based on our fits, a natural question which arises is ‘how sensitive is the number of coincidences with respect to variation in our fitting parameters?’ In Fig.7 we show the response of the non-abelian prediction, Eq.(7), as the fitting parameters are varied. The layout and colour scheme are similar to Fig.3: the top panels show the theoretical phase and the experimental coincidence points for the field orientation \( B_x < 0, B_y = 0 \) (blue). Solid lines in Fig.7 are identical to that in Fig.3a,b. In Fig.7 panels a & b correspond to \( \pm 10\% \) variation of \( \lambda \), the c & d panels correspond to \( \pm 30\% \) variation of \( g_{xz} \), and the e & f panels correspond to \( \pm 10\% \) variation of \( g \). From these plots, the presented deviations are larger than those accepted in Fig.3. The curves corresponding to the lower boundaries of the parameters (dotted lines) are too far away from the experimental points. On the other hand the curves corresponding to the upper boundaries of the parameters (dashed lines) demonstrate an additional coincidence point (panels b,d,f) which is not observed experimentally. This shows that the selected parameters, \( |\lambda| = 0.88, g_{xz} = -0.87, g = 7.0 \), provide the best fit to the data. The curves in Fig.7 do not account for the Dresselhaus interaction. There is no point to account for the interaction for purposes of the present analysis, since it hardly effects the red and green curves which are used to determine the fit parameters, and it does not change the number of coincidences. Dresselhaus only deforms the blue curves in panels b, d, and f exactly in the same way as in the panel b of Fig. 3.

A similar comparison for the abelian theory, Eq.(8), is presented in Fig.8 where once again the parameters \( \lambda, g_{xz}, m \) are varied. The top panels show the theoretical phase \( \Phi_{AB} \) and the experimental coincidence angles for orientations of the external field \( B_x > 0, B_y = 0 \) (red) and \( B_y \neq 0, B_x = 0 \) (green). The bottom panels show the phase and the experimental coincidence points for the field orientation \( B_x < 0, B_y = 0 \) (blue). Solid lines in Fig.8 are identical to that in Fig.3c and 3d. In Fig.8 panels a and b correspond to \( \pm 30\% \) variation of \( \lambda \), panels c and d correspond to \( \pm 30\% \) variation of \( g_{xz} \), and panels e and f correspond to \( \pm 10\% \) variation of \( g \). Despite the significant amount of variation in these parameters, the abelian theory always predicts three coincidence angles for \( B_x < 0, B_y = 0 \) (blue), whilst experimentally only one angle is observed. This discrepancy illustrates that the experimental data cannot be reconciled with the paradigm of Berry phases alone and renders our evidence for the non-abelian gauge field unambiguous.

### Appendix F: Accounting for the Dresselhaus interaction

Dresselhaus spin-orbit interaction arises due to the lack of inversion symmetry in the bulk GaAs crystal. The interaction is cubic in the momentum \( k \) and linear in the angular momentum \( J \), see Ref.28. In the coordinate system defined in Fig.2n the leading term of the Dresselhaus Hamiltonian is

\[
H_D \rightarrow -\frac{12\sqrt{22}}{121} b_D k^2 g_{xy} = -\frac{12\sqrt{22}}{121} b_D k^2 f \sigma_z \sin \theta .
\]  

(F1)
\[
\frac{\Phi_D}{\pi} = 1 + \sqrt{\left(\frac{\Phi}{\pi} - 1\right)^2 + D^2}
\]  
(F3)

\[
D = D_0 \frac{\alpha k_F^2 B_\parallel}{\sqrt{(\omega_c - \frac{\Delta}{2})^2 + |\alpha k_F^2 B_\parallel|^2}}
\]

where

Without tilting, \( \theta_{tilt} = 90^\circ \), the Dresselhaus perturbation \([F2]\) is zero, \( \Phi_D = \Phi \). Hence, our back-gate tuning performed at \( \theta_{tilt} = 90^\circ \) does not compensate the Dresselhaus interaction. The interaction becomes important at intermediate values of \( \theta_{tilt} \). The dashed curves in Figs. 3a,b display Eq.\([F3]\) calculated with \( D_0 = 0.5 \). We chose this value of \( D_0 \) to shift the coincidence angle in Fig.3b from \( \theta_{tilt} = -4.5^\circ \) to \( \theta_{tilt} \approx -6.5^\circ \). This is an additional fitting parameter. Because of weakness of the Dresselhaus interaction the dashed and the solid curves in Fig.3a are practically indistinguishable. On the other hand, because of the resonance, the effect of the Dresselhaus interaction in Fig.3b is significant. We stress again the point made in the main text, the correct value of \( D_0 \) necessarily leads to the extended range of \( \theta_{tilt} \) over which the phase of the SDH oscillation inverts, clearly seen in the blue traces in Fig 2c. The value of \( D_0 \) can also be calculated using Eq.\([F4]\). With \( B_z = 0.2 \) Tesla and with parameters of the system discussed in the paper, Eq.\([F4]\) gives \( D_0 \approx 0.8 \). So, the Dresselhaus interaction is weak and insignificant compared to the dominating magnetic field controlled spin orbit effects that drive the non-abelian Berry phase. The strength of the Dresselhaus interaction measured in our experiment is slightly smaller but comparable with theoretical estimates presented in Refs.\([28,36]\).

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