MILNOR ALGEBRAS COULD BE ISOMORPHIC TO MODULAR ALGEBRAS
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Abstract. We find and describe unexpected isomorphisms between two very different objects associated to hypersurface singularities. One object is the Milnor algebra of a function, while the other object is the local ring of the flatness stratum of the singular locus in a miniversal deformation, an invariant of the contact class of a defining function. Such isomorphisms exist for unimodal hypersurface singularities. However, for the moment it is not well understood, which principle causes these isomorphisms and how far this observation generalises. Here, we also provide an algorithmic approach for checking algebra isomorphy.

0. Introduction

Let $X_0 \subseteq \mathbb{C}^n$ be a germ of an isolated hypersurface singularity defined by an analytic function $f(x) = 0$, $f \in \mathbb{C}\{x\}$. An important topological invariant of the germ is the Milnor number, which can be computed as the $\mathbb{C}$-dimension of the so-called Milnor algebra $Q(f) = \mathbb{C}\{x\}/(\partial f/\partial x)$, [Mil68]. The Milnor algebra carries a canonical structure of a $\mathbb{C}[T]$-algebra defined by the multiplication with $f$. A special version of the Mather-Yau theorem states that the $R$-class (right-equivalence class) of the function $f(x)$ with isolated critical point is fully determined by the isomorphism class of $Q(f)$ as $\mathbb{C}[T]$-algebra [Sche83].

By computational experiments, we have found another occurrence of the Milnor algebra – this time connected with the $K$-class (the contact equivalence class) of $f(x)$, i.e. with the isomorphism-class of the germ $X_0$. Our observation concerns unimodal functions that are not quasi homogeneous. Here we consider a miniversal deformation $F : X \to S$ of the singularity $X_0$. It has a smooth base space of dimension $\tau$, with $\tau$ being the Tjurina number, i.e. the $\mathbb{C}$-dimension of the Tjurina algebra $T(f) := Q(f)/fQ(f)$. We consider the relative singular locus $\text{Sing}(X/S)$ of $X$ over $S$ and its flatness stratum $F := F_S(\text{Sing}(X/S)) \subseteq S$, which depends only on $X_0$, up to isomorphism. The flatness stratum is computable for sufficiently simple functions using a special algorithm [Mar02]. Surprisingly, the local ring of the flatness stratum of a unimodal singularity is isomorphic in all computed cases, either to the Milnor algebra of the defining function (in case $\dim(F) = 0$), or to the Milnor algebra of a 'nearby' function with non-isolated critical point, otherwise.
The notion of a modular stratum was developed by Palamodov, [Pal78], in order to find a moduli space for singularities. It coincides with the flatness stratum $\mathbb{F} = F_S(\text{Sing}(X/S))$ [Mar03], which has been described for unimodal functions in [Mar06]. Only for some singularities from the T-series the modular stratum has expected dimension 1 with smooth curves and embedded fat points as primary components. The combinatorial pattern of its occurrence was found and the phenomenon of a splitting singular locus along a $\tau$-constant stratum was discovered. Here we extend our observation that the modular stratum is the spectrum of the Milnor algebra of an associated non-isolated limiting singularity.

The modular stratum is a fat point of multiplicity $\mu$ isomorphic to $\text{Spec} \mathbb{Q}(f)$ in all other (computed) cases of T-series singularities. The same holds for all 14 exceptional and non-quasihomogeneous unimodal singularities. In the case of a quasihomogenous exceptional singularity, the modular stratum is a smooth germ, hence corresponding to a trivial Milnor algebra.

For completeness, we will first recall the basic results on modular strata and prove that they are algebraic. Second, we collect and complete results on the modular strata of unimodal functions, which are already found in [Mar06]. Subsequently, some of the non-trivial unexpected isomorphisms are presented. A further example of higher modality is discussed in section 4. Hypotheses towards a possible generalisation of these experimental results are formulated. Finally an algorithmic approach is outlined, how to perform a check of algebra isomorphy using a computer algebra system, knowing that there is no practical algorithm in general. All computations were executed in the computer algebra system SINGULAR [GPS05].

1. Characterisations of a modular germ

The definition of modularity was introduced by Palamodov, cf. for instance [Pal78] and [Pal93], and was simultaneously discussed by Laudal for the case of formal power series under the name prorepresentable substratum’. This notion can be considered for any isolated singularity with respect to several deformation functors or to deformations of other objects, cf. [HM05]. For simplicity we restrict ourselves to the case of a germ of an isolated complex hypersurface singularity $X_0 = f^{-1}(0) \subseteq \mathbb{C}^n$, or an isolated complete intersection singularity (ICIS).

A deformation of $X_0$ is a flat morphism of germs $\xi : X \to S$ with its special fibre isomorphic to $X_0$. It is called versal, if any other deformation of $X_0$ can be induced via a morphism of the base spaces up to isomorphism. It is called miniversal, if the dimension of the base space is minimal. Miniversal deformations exist for isolated singularities and are unique up to a non-canonical isomorphism. In case of a hypersurface, a miniversal deformation has a smooth base space, i.e. the deformations are unobstructed. It can be represented as an embedded deformation $\xi : X \subseteq \mathbb{C}^n \times S \to S$, $S = \mathbb{C}^\tau$, \[ \text{Spec} \mathbb{Q}(f) \]
\(X = F^{-1}(0), F(x, s) = f(x) + \sum_{i=1}^{r} s_{\alpha}m_{\alpha},\) where \(\{m_{1}, \ldots, m_{r}\} \subset \mathbb{C}\{x\}\) induces a \(\mathbb{C}\)-basis of the Tjurina algebra \(T(f)\).

Obviously, a miniversal deformation has not the properties of a moduli space, because there are always isomorphic fibres or even locally trivial subfamilies. Hence, the inducing morphism of another deformation is not unique. One can, however, look for subfamilies of a miniversal deformation having this universal property.

**Definition 1.1.** Let \(\xi : X \to S\) be a miniversal deformation of a complex germ \(X_{0}\). A subgerm \(M \subseteq S\) of the base space germ is called modular if the following universal property holds: If \(\varphi : T \to M\) and \(\psi : T \to S\) are morphisms such that the induced deformations \(\varphi^{*}(\xi|M)\) and \(\psi^{*}(\xi)\) over \(T\) are isomorphic, then \(\varphi = \psi\).

The union of two modular subgerms inside a miniversal family is again modular. Hence, a unique maximal modular subgerm exists. It is called modular stratum of the singularity. Note that any two modular strata of a singularity are isomorphic by definition.

**Example 1.2.** If \(X_{0}\) is an ICIS with a good \(\mathbb{C}^{*}\)-action, i.e. defined by quasihomogeneous polynomials, then its modular stratum coincides with the reduced \(\tau\)-constant stratum and is smooth, cf. [Ale85].

Palamodov’s definition of modularity is difficult to handle. It made it challenging to find non-trivial explicit examples. Even the knowledge of the basic characterisations of modularity in terms of cotangent cohomology, which were already discussed by Palamodov and Laudal, did not lead to more examples, cf. [Pal78, Thm. 6.2].

**Proposition 1.3.** Given a miniversal deformation \(\xi : X \to S\) of an isolated singularity \(X_{0}\), the following conditions are equivalent for a subgerm of the base space \(M \subseteq S\):

i) \(M\) is modular.

ii) \(M\) is infinitesimally modular, i.e. injectivity of the relative Kodaira-Spencer map \(T^{0}(S, \mathcal{O}_{M}) \to T^{1}(X/S, \mathcal{O}_{S})|_{M}\) holds.

iii) \(M\) has the lifting property of vector fields of the special fibre: \(T^{0}(X/S, \mathcal{O}_{S})|_{M} \to T^{0}(X_{0}, \mathcal{C})|_{M}\) is surjective.

Note that \(T^{0}\) corresponds to the module of associated vector fields, while \(T^{1}\) describes all infinitesimal deformations. It is represented in the hypersurface case by the (relative) Tjurina algebra \(T^{1}(X/S) = T(F) = \mathbb{C}\{x, s\}/(F, \partial_{x}F)\).

All geometric objects belong to the category of analytic germs. But an isolated singularity is algebraic, i.e. its defining equations can be chosen as polynomials. Ad hoc it is not clear whether the modular stratum is algebraic, too, and to our knowledge it has not been demonstrated. Here, we add a proof in case of ICIS for completeness.

**Lemma 1.4.** Let \(X_{0}\) be a germ of an isolated complete intersection singularity. Then its modular stratum \(M(X_{0}) \subset \mathbb{C}^{r}\) is an algebraic subgerm.
The proof uses the characterisation of modularity as flatness stratum of the Tjurina-module. A more general result holds under weaker assumptions than ICIS, too, cf. [Mar03, Prop. 2.1].

**Proposition 1.5.**

Let \( X_0 \subseteq \mathbb{C}^n \) be an isolated complete intersection singularity defined by \( p \) equations \( f \in \mathbb{C}\{x\}^p \) with miniversal deformation \( \xi : X \to S \). Then the modular space coincides with the flatness stratum of the relative Tjurina module \( T^1(X/S) = \mathcal{O}_X^p / (\partial F/\partial x)\mathcal{O}_X^p \) as \( \mathcal{O}_S \)-module, \( F \) being the equations of the deformation.

**Proof of the lemma.** We may choose the defining equations \( f = (f_1, \ldots, f_p) \) of the germ \( X_0 \) as polynomials by finite determination of isolated singularities. The affine variety defined by these polynomials \( V(f) \subset \mathbb{C}^n \) has in general other singularities than the zero point. But, we can choose an embedding such that \( \text{Sing}(V(f)) \) is concentrated at zero. This holds iff global and local Tjurina number are equal

\[
\dim_\mathbb{C}\{\mathbb{C}[x]^p/(f\mathbb{C}[x]^p, \partial f/\partial x)\} = \dim_\mathbb{C}\{\mathbb{C}[x]^p/(f\mathbb{C}[x]^p, \partial f/\partial x)\} = \tau.
\]

Consider the \( \mathbb{C}[s, x] \)-module \( B := \mathbb{C}[s, x]^p / (FC[s, x]^p, \partial F/\partial x) \). The module \( B \) is finite as \( \mathbb{C}[s] \)-module. Its flatness stratum over \( S \) at zero \( \mathbb{F}_{s_0}(B) \subset S, \ S := \mathbb{C}^r = \text{Spec}(\mathbb{C}[s]) \), is well defined by the fitting ideal of \( B \) as \( \mathbb{C}[s] \)-module. The \( \mathbb{C}[s, x] \)-module \( T^1(X/S, \mathcal{O}_S) \) is finite as \( \mathbb{C}[s] \)-module. Consider the modules \( B_0 := B/sB \) and \( T^1(X_0) = T^1(X/S, \mathcal{O}_S)_{s=0} \), then the localisation at \( x = 0 \) of \( B_0 \) and \( T^1(X_0) \) have identical module-structures which are both already given as \( \mathbb{C}[x]/(x)^n \)-modules: \( B_0 (x) = T^1(X_0) \), hence the germ at zero \( \mathbb{F}(B)_{(s,x)} \) coincides with the flattening stratum of \( T^1(X/S, \mathcal{O}_S) \).  

We add some remarks concerning the flatness criterion:

- The support of \( T^1(X/S, \mathcal{O}_S) \) is exactly the relative singular locus of the mapping germ \( F : \mathbb{C}^n \times S \to \mathbb{C}^p \times S \) over \( S \). In case of a hypersurface, i.e. \( p = 1 \), \( T^1(X/S, \mathcal{O}_S) \) coincides with the \( \mathcal{O}_S \)-algebra of the relative singular locus, that is the relative Tjurina-algebra \( T(F) = \mathcal{O}_{\text{Sing}(X/S)} \).
- The support of the flatness-stratum \( \mathbb{F}_{\mathcal{O}_S}(\text{Sing}(X/S)) \) is the locus where the finitely generated \( \mathcal{O}_S \)-module \( T^1(X/S, \mathcal{O}_S) \) has constant rank \( \tau \). The fact that \( M_{\text{red}} \) is the \( \tau \)-constant stratum was already explained by Palamodov, cf. [Pal73, Thm. 7.1]. Besides the case of a quasihomogeneous singularity, see example 1.2 a non-reduced structure of \( M \) is expected generally.
- It follows from a non-trivial result, cf. [LR76], that the finite map \( \text{Sing}(X/S) \to S \) is unramified over the \( \mu \)-constant stratum. But the analogous statement for the \( \tau \)-constant stratum does not hold, see below. This phenomenon we have called splitting singular locus over the \( \tau \)-constant stratum.
2. Computing the modular germs of unimodal singularities

Applying the algorithm for computing the flatness stratum, cf. [Mar02], we can compute the modular stratum of not too complicated singularities. More precisely, the output of the algorithm is the $k$-jet of the germ of the flatness stratum for some positive integer $k$. If the modular stratum is a fat point we are done with some big number $k$. We can not prove or even expect to end up with an algebraic representation generally. But, it does occur, as in all presented examples below.

The classification of singularities starts with the simple singularities, the ADE-singularities. These are all quasihomogeneous, their modular strata are all trivial, i.e. simple points. Following the classification of functions by Arnold [AGZV85] the next more complicated singularities are the unimodal ones. They are characterised by the fact, that in a neighbourhood of the function only $\mathcal{R}$-orbit families occur which depend on at most one parameter.

Recall the classification of unimodal functions: We have the $T$-series singularities and 14 so called exceptional unimodal singularities. We may restrict their representation to three variables up to stable equivalence. Any type is representing a one-parameter $\mu$-constant family of $\mathcal{R}$-equivalence classes. The exceptional ones, cf. appendix [A] are all semi-quasihomogeneous of a special type: The $\mu$-constant family can be written as

$$f_\lambda = f_0(x) + \lambda h_f(x), \quad \lambda \in \mathbb{C},$$

where $f_0$ is quasihomogeneous and $h_f(x) := \det(\frac{\partial^2 f_0}{\partial x_i \partial x_j})$ is the Hesse form of $f_0$. Such a family splits into exactly two $\mathcal{K}$-classes, one quasihomogeneous ($\lambda = 0$) with trivial modular stratum and one semi-quasihomogeneous ($\lambda \neq 0$, we call it of Hesse-type). Here $\tau(f_1) = \mu(f_1) - 1$ holds, the modular strata are fat points of multiplicity $\mu$, cf. [Mar06] Prop. 5.1.

The singularities of the $T$-series are defined by the equations

$$T_{p,q,r} : x^p + y^q + z^r + \lambda xyz, \quad \frac{1}{p} + \frac{1}{q} + \frac{1}{r} \leq 1.$$

In exactly three cases we have $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$. These singularities are quasihomogeneous. They are called the parabolic singularities $P_3$, $X_9$ and $J_{10}$ in Arnold’s notation or elliptic hypersurface singularities $\tilde{E}_6$, $\tilde{E}_7$ and $\tilde{E}_8$, in Saito’s paper [Sai74]:

$$\tilde{E}_6 = P_3 = T_{3,3,3} : x^3 + y^3 + z^3 + \lambda xyz, \quad \lambda^3 \neq -3^3, \quad \tau = \mu = 8;$$
$$\tilde{E}_7 = X_9 = T_{4,4,2} : x^4 + y^4 + z^2 + \lambda xyz, \quad \lambda^4 \neq 2^6, \quad \tau = \mu = 9;$$
$$\tilde{E}_8 = J_{10} = T_{6,3,2} : x^6 + y^3 + z^2 + \lambda xyz, \quad \lambda^6 \neq 2^4 3^3, \quad \tau = \mu = 10.$$

It is well known that their $\mathcal{K}$-classes (here equal to the $\mathcal{R}$-classes) are not determined by $\lambda$. The $\mathcal{K}$-equivalence induces a discrete equivalence relation on the $\lambda$-line with the indicated gaps. Its quotient is an affine line parametrised by the classical $j$-invariant of elliptic curves. Precise formulæ for $j$ can be found in Saito’s paper. Their modular strata are germs of a line.
All other $T$-series singularities are called \textit{hyperbolic}. Their $K$-class is independent of $\lambda$, $\lambda \neq 0$, the Newton boundary has three maximal faces and the singularity is neither quasihomogeneous nor semi-quasihomogeneous. We have $\tau(T_{p,q,r}) = \mu(T_{p,q,r}) - 1 = p + q + r - 2$. The modular strata of the hyperbolic singularities are more complicated. Some of them are 1-dimensional, others are just fat points. First computations are discussed in [Mar06].

Obviously, a hyperbolic singularity of type $T_{p,q,r}$ is adjacent to another $T$-series singularity, iff all three parameters $(p,q,r)$ are greater or equal to the parameters of the second. Hence, any hyperbolic singularity is adjacent to at least one parabolic singularity. Inspecting the list we find exactly six types of hyperbolic singularities which have the same Tjurina number as an adjacent parabolic singularity. They are candidates for modular strata of dimension 1, because in their miniversal deformation the associated parabolic singularity occurs as one dimensional $\tau$-constant family including the special fibre. Indeed, these six singularities are leading members of six sub-series of the hyperbolic singularities, which have a one-dimensional modular strata, cf. [Mar06, Prop.4.2, 4.3]. These exceptional sub-series are characterised by the fact, that two of the three indices coincide with that of a parabolic one.

\textbf{Proposition 2.1.} Any singularity of one of the following six exceptional sub-series contains a splitting line in their modular stratum:

\begin{align*}
(T_{k,3,3})_k, \; k \geq l = 4, & \quad (T_{k,4,2})_k, \; k \geq l = 5, & \quad (T_{k,4,4})_k, \; k \geq l = 3, \\
(T_{k,3,2})_k, \; k \geq l = 7, & \quad (T_{k,6,2})_k, \; k \geq l = 4, & \quad (T_{k,6,3})_k, \; k \geq l = 3.
\end{align*}

The families over the $\tau$-constant lines with index $k$ are given by

\[ f_t := x^{l-1}(x + t)^{k-l+1} + y^d + z^r + xyz. \]

The fibre singularities over $t \neq 0$ consist of one singularity of the associated parabolic type and of one singularity of type $A_{k-l-1}$ outside the zero section, if $k > l-1$. These lines, called \textit{splitting lines}, are components of the modular stratum, which mostly contains another embedded component (fat point) at zero. Exactly three of the singularities of the sub-series have two line components and two of them have three line components, cf. [Mar06, Cor. 4.4]. We call these five types the \textit{symmetric exceptions}, because they come from an additional symmetry in the equation or from a crossover of two exceptional sub-series. All other computed examples of modular strata of $T$-series singularities, not belonging to the above six exceptional sub-series, are fat points.

\textbf{Example 2.2.} $f_t := x^4 + y^3 + z^3 + xyz + tx^3$ is a $\tau$-constant deformation of $T_{4,3,3}$ with generic fibre type $P_8$. The modular deformation $f_t$ fits into the $\lambda$-line of $P_8(\lambda)$ at infinity: $f_t \sim_K P_8(t^{-1/3})$ for $t \neq 0$, i.e. we get a threefold covering of the $t$-line, $t \neq 0$, by the $\lambda$-line, $\lambda \neq 0$. We may think of a compactification of the modular $\lambda$-line of $P_8$ at infinity with a point corresponding to the $T_{4,3,3}$-singularity. The same holds for $T_{4,4,3}$ and $T_{5,4,2}$ with respect to $X_9$. This causes two different compactifications of the same
modular family over the punctured disc \(X_0(1/\lambda), \lambda > N\), at the special point zero to a modular family over the disc.

3. NEW EXPLICIT RESULTS ON UNIMODAL MODULAR STRATA

A careful inspection of our computations leads us to generic formulæ for the equations of the modular strata of the \(T\)-series singularities in terms of the index-parameters \((p, q, r)\). We confirm these equations by checking all modular strata up to \(\tau < 50\), giving a strong indication of their correctness. Moreover, it gives an explanation for the occurrence of line components in the modular strata of exceptional sub-series singularities. In addition we obtain the isomorphy of the modular algebra of a \(T\)-series singularity, not belonging to one of the six sub-series, to its Milnor algebra, the isomorphy of the modular algebra of sub-series singularities to the Milnor algebra of certain non-isolated functions, and the isomorphy of all modular strata from the same sub-series up to the five symmetric exceptions.

Example 3.1. Let \(X_0\) be the hyperbolic singularity defined by \(f = x^p + y^q + z^r + xyz\). Then

\[ F = f + t_1x^{p-1} + \ldots + t_{p-1}x + t_{p} + u_1y^{q-1} + \ldots + u_{q-1}y + v_1z^{r-1} + \ldots + v_{r-1}z \]

defines a miniversal deformation \(X \to S\) of \(X_0\), with \(O_S = \mathbb{C}\{t, u, v\}\).

We consider the following the ideal \(I(p, q, r) \subset O_S:\)

\[ I(p, q, r) = (f_2, \ldots, f_p, g_2, \ldots, g_{q-1}, h_1, \ldots, h_{r-1}, u_1v_1 - c_1t_1^{p-1}, t_1v_1 - d_1u_1^{q-1}, t_1u_1 - e_1v_1^{r-1}) \]

where

\[ f_i := a^2t_1 - c_1^i, \quad g_i := a^2u_1 - d_1^iu_1, \quad h_i := a^2v_1 - e_1^iv_1 \]

with coefficients

\[ a := a(p, q, r) := pqr(1 - \frac{1}{p} - \frac{1}{q} - \frac{1}{r}), \quad c_i := \prod_{k=1}^i a(p - k + 1, q, r), \]

\[ d_i := \prod_{k=1}^i a(p, q - k + 1, r), \quad e_i := \prod_{k=1}^i a(p, q, r - k + 1) \]

The coefficient \(c_p\) is zero iff \(\frac{1}{k} + \frac{1}{q} + \frac{1}{r} = 1\) for some \(1 \leq k \leq p\), and similarly for \(d_q\) and \(e_r\). So one of them vanishes exactly if \(T_{p,q,r}\) belongs to one of the six exceptional sub-series. More than one of the coefficients is zero exactly for the five symmetric exceptions, see proposition 3.5. Hence, the number of vanishing coefficients corresponds to the number of line components.

We have checked by computer.

Proposition 3.2. The ideal \(I(p, q, r)\) from example 3.1 defines the modular stratum of the singularity \(T = T_{p,q,r}\), if \(\tau(T) \leq 50\).
A linear diagonal transformation

\[ t_1 \mapsto \alpha t_1, \quad u_1 \mapsto \beta u_1, \quad v_1 \mapsto \gamma v_1. \]

induces isomorphy of the algebra defined by \( I(p, q, r) \) and the Milnor algebra of \( T_{p,q,r} \) in the case that none of the coefficients \( c_p, d_q \) and \( e_r \) vanishes.

**Proposition 3.3.** If \( T_{p,q,r} \) does not belong to one of the six exceptional sub-series, then

\[ Q(T_{p,q,r}) \cong \mathcal{O}/I(p, q, r). \]

In particular, for those singularities with \( \tau < 50 \) the Milnor algebra is isomorphic to the modular algebra of the singularity.

We associate to an exceptional sub-series \( (T_{k,q,r})_k \) the limit singularity \( T_{\infty,q,r} \), defined by the equation \( y^q + z^r + xyz \), which is non-isolated and belongs to the closure of the \( K \)-orbit of any member of the sub-series.

**Proposition 3.4.** If \( T_{\bullet} = T_{k,q,r} \) belongs to one of the six exceptional sub-series, then

\[ Q(T_{\infty,q,r}) \cong \mathcal{O}/I(k, q, r), \]

apart from the five symmetric exceptions. In particular, for those singularities \( T_{\bullet} \) with \( \tau < 50 \) the Milnor algebra \( Q(T_{\infty,q,r}) \) is isomorphic to the modular algebra of the singularity.

Again, the isomorphy to the Milnor algebra is obtained by multiplying \( t_1, u_1, v_1 \) with some constants. It remains to check the five symmetric exceptions.

**Proposition 3.5.** The local algebra of a modular stratum is isomorphic to the Milnor algebra of a non-isolated singularity for the five symmetric exceptions of T-series singularities:

- \( Q(xyz) \) for \( T_{4,4,4} \) and \( T_{6,3,3} \),
- \( Q(x^2 + xyz) \) for \( T_{6,4,2} \) and \( T_{6,6,2} \),
- \( Q(x^3 + xyz) \) for \( T_{6,6,3} \).

This follows from the equations in example 3.1 and proposition 3.2.

Next we investigate the modular strata of all 14 exceptional semi-quasihomogeneous unimodal singularities and obtain a similar result.

**Proposition 3.6.** All 14 exceptional semi-quasihomogeneous unimodal singularities fulfill: The local ring of their modular stratum is isomorphic to their Milnor algebra.

We list our results in a table at the end of the article. The isomorphisms are rather complicated. They are computed with the algorithm presented in appendix B. We omit the formulæ, but discuss only one case in detail.
Example 3.7 \((W_{12}, Z_{11} \text{ and } S_{11})\). We consider \(f = x^4 + y^5 + x^2y^3\) and choose \((b_1, \ldots, b_{11}) := (1, x, x^2, y, xy, x^2y, xy^2, x^3y, y^4)\) as representatives of a \(\mathbb{C}\)-basis of the Tjurina algebra \(T(f)\). Now, \(F = f + s_1b_1 + \ldots + s_{11}b_{11} \in \mathbb{C}\{x, y\} \otimes \mathcal{O}_S\) defines a miniversal deformation \(X \to S\) of \(X_0\).

The ideal \(I_M \subset \mathcal{O}_S\) of the maximal modular subgerm \(M \subset S\), computed with SINGULAR is given by the following completely interreduced generators:

\[
\begin{align*}
s_1 &= \frac{30445}{7392} s_1^2 s_2 + \frac{4240139}{1897280} s_1^3 s_2, \\
s_2 &= \frac{2696}{48125} s_1^3 s_2, \\
s_{11} &= \frac{11699}{144375} s_1 s_2, \\
s_{10} &= \frac{3904}{48125} s_1^3 s_2, \\
s_9 &= \frac{52}{625} s_1^3 - \frac{951}{7000} s_1 s_2^2 + \frac{592717}{8421875} s_2^2 s_2 - \frac{119567878949}{518775000000} s_1^3 s_2, \\
s_8 &= \frac{1304}{5775} s_1^3 - \frac{1411481}{18528125} s_1^2 s_2, \\
s_7 &= \frac{618}{1925} s_1^2 s_2 + \frac{1024869}{37056250} s_1 s_2, \\
s_6 &= \frac{6}{25} s_1^2 + \frac{3}{80} s_2^2 - \frac{21}{3125} s_1^3 + \frac{531}{20000} s_1 s_2^2 - \frac{31001023}{5390000000} s_1^2 s_2, \\
&\quad + \frac{25063327841}{20751500000000} s_1^3 s_2, \\
s_5 &= \frac{2}{25} s_1^3 + \frac{9}{16} s_1 s_2^2 - \frac{114057}{539000} s_1^2 s_2^2 + \frac{6306416817}{83006000000} s_1^3 s_2, \\
s_4 &= \frac{6}{7} s_1 s_2 + \frac{1227}{67375} s_1^2 s_2 - \frac{16557777}{2593937500} s_1^2 s_2, \\
s_3 &= \frac{2}{5} s_1^2 + \frac{9}{16} s_2^2 - \frac{9}{625} s_1^3 - \frac{621}{4000} s_1 s_2^2 + \frac{49325643}{10780000000} s_1^2 s_2, \\
&\quad - \frac{644553838881}{41503000000000} s_1^3 s_2.
\end{align*}
\]

\(\mathcal{O}_M\) is a zero-dimensional local algebra of embedding dimension 2. A minimal embedding is defined by the two polynomials printed in bold. The mapping

\[
\varphi : \mathcal{O}_M \to \mathbb{C}\{x, y\}/(0_{f_{xx}}, 0_{f_{yy}})
\]

\[
\begin{align*}
\bar{x}_1 &\mapsto \frac{2668050}{2051993} \sqrt[4]{\frac{11759762521875525}{25638801731506361}} \\
\bar{y} &\mapsto \frac{2134440}{2051993} \sqrt[6]{-6089} \cdot \bar{x}.
\end{align*}
\]
defines an isomorphism between this local algebra and the Milnor algebra of \( f \).

We give the isomorphism for \( Z_{11} \) and \( S_{11} \):

| Name   | Equation | Deformation | Isomorphism |
|--------|----------|-------------|-------------|
| \( f \) | \( f = y^2 z + x z^2 + x^4 + x^3 z \) | \( F = f + s_1 x^2 z + s_2 x^2 y + s_3 x^3 + s_4 x z + s_5 z + s_6 x y + s_7 y + s_8 x^2 + s_9 x + s_{10} \) | \( \begin{align*} S_1 & \mapsto - \frac{3^{657^{113} 12} x}{2^{6} 23^{2} 67^{4}} + \frac{3^{657^{113} 19} 163}{2^{12} 23^{4} 67^{4}} \frac{z}{z} \\ S_2 & \mapsto - \frac{3^{1357^{113} 15} x}{2^{10} 23^{2} 67^{4}} \\ S_3 & \mapsto \frac{3^{657^{113} 13} x}{2^{10} 23^{4} 67^{4}} + \frac{3^{11547^{113} 13} x^2}{2^{12} 23^{4} 67^{4}} - \frac{3^{657^{113} 41} 307 587 32677 569 187}{2^{28} 23^{4} 67^{4}} \frac{y^2}{y^2} \\ S_4 & \mapsto - \frac{3^{657^{113} 31} 22805 600 070 42079}{2^{30} 23^{4} 67^{4}} \frac{x}{x} \\
\end{align*} \) |

name: \( S_{11} \)  
name: \( Z_{11} \)

See appendix \( B \) for an algorithm that calculates these isomorphisms.

In the table in appendix \( A \) we list the relevant equations of the modular strata of all 14 unimodal exceptional singularities. The last monomial of the normal form induces in \( Q(f) \) the Hesse form of the quasihomogeneous leading form \( f_0 \). The representatives of a \( \mathbb{C} \)-basis of \( T(f) \) \( b_1, \ldots, b_r \) are given in the second column, which are used to fix a miniversal deformation \( F = f + \sum_i s_i b_i \). The equations in column three describe the modular stratum in the base of this deformation. Note, that we omit equations of the form \( s_i + O(s^2) \), because we may eliminate the variables occurring linearly in these equations. A minimal embedding with the remaining variables is given by the listed equations in the last column. In the miniversal deformation these remaining variables \( s_i \) correspond to the monomials printed in bold.

### 4. Further Examples and Questions

We have calculated modular strata for singularities of higher modality, too. The results raise hope that our observation generalises. We give one example of a bimodal singularity.

**Example 4.1.** We consider the hypersurface singularity given by the semi-quasihomogeneous singularity of Hesse type \( f = x^{10} + y^3 + x^4 y^2 \). A miniversal
deformation is defined by

\[
f = s_1 + s_2 x + s_3 x^2 + s_4 x^3 + s_5 x^4 + s_6 x^5 + s_7 x^6 + s_8 x^7 + s_9 x^8 + s_{10} y \]
\[
+ s_{11} x y + s_{12} x^2 y + s_{13} x^3 y + s_{14} x^4 y + s_{15} x^5 y + s_{16} x^6 y + s_{17} x^7 y
\]

The maximal modular subgerm \( M \) in the base this deformation is defined by an ideal generated by

\[
s_1 + \mathcal{O}(s^2),
\]
\[
\vdots
\]
\[
s_8 + \mathcal{O}(s^2),
\]
\[
s_9 = \frac{9}{256} s_{17} s_9 - \frac{29342801}{33890000} s_{17} s_9 - \frac{9963}{33890000} s_8 - \frac{831341832017399}{3283872972800000} s_9,
\]
\[
s_{10} + \mathcal{O}(s^2),
\]
\[
\vdots
\]
\[
s_{16} + \mathcal{O}(s^2),
\]
\[
s_9 = \frac{67372}{100029} s_{17} s_9.
\]

The local ring \( \mathcal{O}_M = \mathcal{O}_{17}/J_M \) is again isomorphic to \( Q(f) \) via

\[
\varphi : \mathcal{O}_M \to \mathbb{C}\{x, y\}/(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}),
\]
\[
s_{17} \mapsto a_1 x,
\]
\[
s_9 \mapsto a_2 y + a_3 x^4 + a_4 x^2 y + a_5 x^6 + a_6 x^4 y + a_7 x^8,
\]

with coefficients

\[
a_1 = 8 \sqrt[4]{\frac{17943573032}{1269497754275}},
\]
\[
a_2 = \frac{2261952}{84215} \sqrt[17]{\frac{17943573032}{1269497754275}},
\]
\[
a_3 = \frac{753984}{84215} \sqrt[17]{\frac{17943573032}{1269497754275}} + \frac{1291937258304}{1269497754275},
\]
\[
a_4 = \frac{920238621242928785663189881632}{1007265342568292675484765625},
\]
\[
a_5 = \frac{25742505984143872}{158687219284375} \sqrt[17]{\frac{17943573032}{1269497754275}} + \frac{30734128737462985254399600544}{1007265342568292675484765625},
\]
\[
a_6 = \frac{547510092238050506695293449974819328}{3777245034634697533306787109375} \sqrt[17]{\frac{17943573032}{1269497754275}},
\]
\[
a_7 = \frac{547510092238050506695293449974819328}{11341735403839292992066128125} \sqrt[17]{\frac{17943573032}{1269497754275}}.
\]

In all the examples we have considered a function \( f \) defining an isolated hypersurface singularity \( X_0 \), and related its modular stratum to the Milnor algebra of \( f \). If we take another \( \mathcal{K} \)-equivalent function \( f' \), the isomorphism-class of the modular stratum does not change by definition. While \( \mu(f) \) is an invariant of \( \mathcal{K} \)-class, this is in general not true for the isomorphism-class of the Milnor algebra, cf. [BY90].

Nevertheless, we have the following proposition for singularities with \( \tau = \mu - 1 \).
Proposition 4.2. Let be \( f \) an analytic function with isolated critical point with \( \tau(f) = \mu(f) - 1 \), then its Milnor algebra is \( \mathcal{K} \)-invariant.

This is a direct implication of the following two lemmas.

Lemma 4.3. [BY90] If \( m_f \subset m(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}) \) then \( RL \)-class and the \( K \)-class of \( f \) coincide.

Lemma 4.4. Let \( f \in \mathbb{C}\{x_1, \ldots, x_n\} \) be a function with isolated singularity and \( \delta \in \mathcal{T}_{\mathbb{C}^n} \) a vector field with \( \delta(f) \in (f) \) then \( \delta \in m\mathcal{T}_{\mathbb{C}^n} \).

Proof. This statement is only a slight generalisation of [Sai71, lemma 4.2] and we can use the same proof found there with \( gf = \delta(f) \) instead of \( f \). \( \square \)

Proof of the Proposition. Look at the exact sequence

\[
0 \to \text{Ann}(f) \to Q(f) \xrightarrow{T} Q(f) \to T(f) \to 0.
\]

Then \( \text{Ann}(f) \) has \( \mathbb{C} \)-dimension \( \mu - 1 \) and thus equals the maximal ideal \( m_{Q(f)} \). That means \( m_f \subset \langle \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \rangle \). Because of lemma 4.4 even \( m_f \subset m(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}) \) holds. By lemma 4.3 we know that the \( \mathcal{K} \)-class of \( f \) equals its \( RL \)-class, but \( Q(f) \) is \( RL \)-invariant. \( \square \)

Due to this result we can speak of the Milnor algebra of a hypersurface singularity in the case \( \tau = \mu - 1 \). Hence, we can state the following conjecture, motivated by our examples.

Hypothesis 4.5. Consider a hypersurface singularity \( f \) with \( \tau = \mu - 1 \). Then the local ring of the modular stratum \( \mathcal{O}_{\mathcal{M}(f)} \) is of Milnor type, i.e. there exists a germ of an analytic function \( f' \) such that \( Q(f') \cong \mathcal{O}_{\mathcal{M}} \). If \( f \) has an Artinian modular stratum, then the local ring of the modular stratum is isomorphic to the Milnor algebra of \( f \) itself.

Remark 4.6. We found the modular strata to be of Milnor type in all computed examples. So one could ask more generally: For which singularities is the modular stratum of Milnor type?

Up to now we were not able to proof the hypothesis. There are some indications that it is strongly connected with the restriction \( \mu - \tau = 1 \).
## Appendix A. Modular strata of exceptional unimodal singularities

| Singularity | $T(f)$-basis | Equations of Modular Stratum |
|-------------|--------------|-------------------------------|
| $E_{12}: x^3 + y^2 + z^2 + xy^3$ | $y^6, xy^3, y^5, y^3, y^2, x^2, y, x, 1$ | $s_1^{12} + 2168 s_2 + 2037 s_3 + 906 s_4 + 123807975 s_5 + 27064052350995 s_9 + 1114136598614359125 s_{11} + 2199792909213140034 s_{13} + 533624932061451711681218716 s_{17} + 55400848109045 s_{19} + 158948791742474 s_{17}$ |
| $E_{13}: x^3 + y^8 + z^2 + y^8$ | $y^6, y^3, y^2, x^2, y, x, 1$ | $s_2^9 + 11 s_2^8 + 27 s_2^7 + 45 s_2^6 + 47 s_2^5 + 27 s_2^4 + 12 s_2^3 + 2 s_2^2 + 1 s_2 + 1$ |
| $E_{14}: x^3 + y^8 + z^2 + x^2 y^6$ | $y^7, y^4, y^3, y^2, x^2, y, x, 1$ | $s_2^9 + 20 s_2^8 + 71 s_2^7 + 189 s_2^6 + 324 s_2^5 + 430 s_2^4 + 340 s_2^3 + 190 s_2^2 + 89 s_2 + 1$ |
| $Z_{11}: x^3 + y^2 + z^2 + z^4 + x^4$ | $y^4, x^4, y^3, x^2, y, x, 1$ | $s_1^{12} + 272095 s_2 + 152998 s_3 + 1039 s_4 + 1$ |
| $Z_{12}: x^3 + y^3 + z^3 + x^2 y^3$ | $y^5, y^4, y^3, y^2, x^2, y, x, 1$ | $s_1^{12} + 31447 s_2 + 20084 s_3 + 20084 s_4 + 22400 s_5 + 1$ |
| $Z_{13}: x^3 + y^6 + z^2 + x^2 y^3$ | $y^5, y^4, y^3, x^2, y, x, 1$ | $s_1^{12} + 15308 s_2 + 89999 s_3 + 619 s_4 + 1$ |
| $W_{12}: x^3 + y^9 + z^2 + x^2 y^3$ | $y^4, x^4, y^3, y^2, x^2, y, x, 1$ | $s_2^9 + 134435 s_2 + 506291 s_3 + 2240191 s_4 + 1380571 s_5 + 1$ |
| $W_{13}: x^4 + y^4 + z^2 + y^6$ | $y^5, x^2 y^3, y^2, y^3, x^2, y, x, 1$ | $s_1^{12} + 8323315 s_2 + 21341173563 s_3 + 21341173563 s_4 + 132300 s_5 + 1$ |
| $Q_{10}: x^3 + y^4 + z^2 + x^3 y^3$ | $y^3, y^2, x^2, y, x, 1$ | $s_1^{12} + 244 s_2 + 1971 s_3 + 1$ |
| $Q_{11}: x^3 + y^2 z + x z^3 + z^5 | | $s_1^{12} + 32415222 s_3 + 21341173563 s_4 + 132300 s_5 + 1$ |
| $Q_{12}: x^3 + y^4 + z^2 + x^3 y^3$ | $y^3, y^2, x^2, y, x, 1$ | $s_1^{12} + 244 s_2 + 1971 s_3 + 1$ |
| $S_{11}: x^4 + y^2 z + x z^2 + x^2 z^2$ | $x^2 y, x^2 y, x, 1$ | $s_1^{12} + 32415222 s_3 + 21341173563 s_4 + 132300 s_5 + 1$ |
| $S_{12}: x^4 + y^2 z + x z^3 + z^5 | | $s_1^{12} + 32415222 s_3 + 21341173563 s_4 + 132300 s_5 + 1$ |
| $U_{12}: x^3 + y^2 z + x z^2$ | $z^2, y z, x, z, x, 1$ | $s_1^{12} + 244 s_2 + 1971 s_3 + 1$ |
Appendix B. Finding isomorphisms between Artinian local rings

We have to look for isomorphisms between local rings of modular strata and algebras of Milnor type in order to test the hypothesis for the above examples. Using computer algebra and a direct approach we check, whether two Artinian local algebras $A := \mathbb{C}[x_1, \ldots, x_m]/I_A$ and $B := \mathbb{C}[x_1, \ldots, x_n]/I_B$ are isomorphic. A slightly generalised problem is considered. Let $A$ be arbitrary and $B$ Artinian. We will check if there is a surjective homomorphism $A \to B$. After eliminating variables we may assume that $n = \text{embdim }B$.

Choose representatives of a $\mathbb{C}$-basis of $B \{ f_1, \ldots, f_r \} \subset \mathbb{C}[x_1, \ldots, x_n]$ and make an ansatz as follows:

$$\varphi : \mathbb{C}[x_1, \ldots, x_m] \to \mathbb{C}[x_1, \ldots, x_n]$$

$$x_i \mapsto \sum_j \alpha_{ij} f_j$$

Our aim is to decide for which choice of $\alpha_{11}, \ldots, \alpha_{nr} \in \mathbb{C}$ $\varphi$ defines a surjection such that $\varphi(I_A) \subset I_B$. Put $R := \mathbb{C}[\alpha_{11}, \ldots, \alpha_{nr}]$ and

$$\tilde{\varphi} : \mathbb{C}[x_1, \ldots, x_n] \to R[x_1, \ldots, x_n]$$

$$x_i \mapsto \sum_j \alpha_{ij} f_j.$$

Fix generators $a_1, \ldots, a_k$ of $I_A$ and a local term ordering $>$ on $R[x]$ and compute the reduced normal forms $\tilde{a}_i := \text{NF}_>(\tilde{\varphi}(a_i), G_B)$ of $a_i$ with respect to a standard basis $G_B$ of $I_B R[x]$. Consider the ideal $J \subset R$, generated by the coefficients of the $\tilde{a}_i$. Any $p \in \mathbb{C}^{n-r}$ defines an evaluation homomorphism $e_p : R \to \mathbb{C}$, $\alpha \mapsto p$.

**Lemma B.1.** With the notation of above: $e_p \circ \tilde{\varphi}$ defines surjection $\varphi : \mathbb{C}[X] \to \mathbb{C}[X]$ with $\varphi(I_A) \subset I_B$ if and only if $p \in V(J) \setminus V(\text{minor}_n(\partial \tilde{\varphi}/\partial x|_{x=0}))$.

**Proof.** Obviously $e_p \circ \tilde{\varphi}$ defines indeed a surjection $\mathbb{C}[X] \to \mathbb{C}[X]$ iff not all $n$-minors of $(\partial \tilde{\varphi}/\partial x|_{x=0})$ vanish. The key observation is: $e_p(\tilde{a}_i)$ is a reduced normal form for $e_p(a_i)$ with respect to $G_B$. Hence, all coefficients of $\tilde{a}_i$ have to be mapped to zero by $e_p$ in order to have $(e_p \circ \tilde{\varphi})(a_i) \in I_B$. Thus, all coefficients must belong to ker $e_p = m_p$. □

For Artinian algebras $A$ and $B$ we first check that their $\mathbb{C}$-dimensions coincide. If the algorithm leads to a homomorphism $\varphi : A \to B$, this will be a surjective homomorphism of vector spaces of same dimension. Thus, $\varphi$ is injective, too. If the algorithm does not succeed in finding a surjective homomorphism, then $A$ and $B$ can not be isomorphic.
Algorithm 1 findSurjection($I_A, I_B$)

Input:

- $I_A = (a_1, \ldots, a_k)$ ideal in $\mathbb{C}[x_1, \ldots, x_m]$
- $I_B = (b_1, \ldots, b_l)$ ideal in $\mathbb{C}[x_1, \ldots, x_n]$

Output: $\varphi : \mathbb{C}[X] \rightarrow \mathbb{C}[X]$ such that $\varphi(I_A) \subset I_B$ or $\varphi = \text{nil}$

$G_B \leftarrow \text{standardBasis}(b_1, \ldots, b_l)$

$\{f_1, \ldots, f_r\} \leftarrow \mathbb{C}\text{-Basis}(B)$

$R := \mathbb{C}[\alpha_{11}, \ldots, \alpha_{nr}]$

$\tilde{\varphi} \leftarrow [\mathbb{C}[X] \rightarrow R[X], \ x_i \mapsto \sum \alpha_{ij} f_j]$

$J \leftarrow (0)$

for $i = 1$ to $k$ do

$\tilde{a}_i \leftarrow \text{NF}(\tilde{\varphi}(a_i), G_B)$

$J \leftarrow J + (\text{coefficients of } \tilde{a}_i)$

end for

if $\text{minor}_{n}(\frac{\partial \tilde{\varphi}}{\partial x_i}|_{x=0}) \not\subset J$ then

choose $p \in V(J) \setminus V(\text{minor}_{n}(\frac{\partial \tilde{\varphi}}{\partial x_i}|_{x=0}))$

$\varphi \leftarrow [\mathbb{C}[X] \rightarrow \mathbb{C}[X], \ x_i \mapsto \sum p_{ij} f_j]$

else

return nil

end if

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