Charging a Double Kerr Solution in 5D
Einstein–Maxwell–Kalb–Ramond Theory

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Abstract

We consider the low–energy effective action of the 5D Einstein–Maxwell–Kalb–Ramond theory. After compactifying this truncated model on a two–torus and switching off the $U(1)$ vector fields of this theory, we recall a formulation of the resulting three–dimensional action as a double Ernst system coupled to gravity. Further, by applying the so–called normalized Harrison transformation on a generic solution of this double Ernst system we recover the $U(1)$ vector field sector of the theory. Afterward, we compute the field content of the generated charged configuration for the special case when the starting Ernst potentials correspond to a pair of interacting Kerr black holes, obtaining in this way an exact field configuration of the 5D Einstein–Maxwell–Kalb–Ramond theory endowed with effective Coulomb and dipole terms with momenta. Some physical properties of this object are analyzed as well as the effect of the normalized Harrison transformation on the double Kerr seed solution.

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1 Introduction

Recently some natural interest has been shown to the study of field configurations that describe interacting black holes coupled to some matter fields, both in the framework of General Relativity [1]–[2] and string theory [3]–[5]. One of the reasons for such an interest is the development reached in the statistical approach to physics of single black holes (for a review, see for instance, [6]–[7]) and its possible generalization to more complicated systems of interacting black holes coupled to matter.

In this paper we construct a charged field configuration that consists of a pair of interacting sources of black hole type coupled to an anti–symmetric Kalb–Ramon tensor field and a set of Abelian gauge fields in the framework of the truncated five–dimensional Einstein–Maxwell–Kalb–Ramond (EMKR) theory. The construction is carried out by applying the normalized Harrison charging symmetry, which acts on the target space of the effective three–dimensional heterotic string theory and preserves the asymptotic properties of the starting field configurations, on a seed solution that corresponds to a double Ernst system in the framework of the toroidally reduced five–dimensional Einstein–Kalb–Ramond (EKR) theory. Several interesting results have been achieved regarding the physical properties of five–dimensional black objects [8]; it turns out that the BPS bound of rotating black holes is saturated precisely in five or more dimensions.

The paper is organized as follows: in Section 2 we briefly review the matrix Ernst potential (MEP) formalism for the effective field theory of the heterotic string (for an arbitrary number of dimensions) and its formal analogy to the stationary Einstein–Maxwell (EM) system. It turns out that after setting to zero the dilaton and all $U(1)$ vector fields, and considering the compactification of this theory on a two–torus, the resulting three–dimensional subsystem admits a Kähler representation which is defined by two vacuum Ernst potentials.

In Section 3 the parametrization which gives rise to this double Ernst system is pointed out and a discrete transformation between the metric and Kalb–Ramon degrees of freedom is established. In Section 4 we recall the normalized Harrison transformation (NHT) and apply it on a generic seed solution of the EKR theory which corresponds to two complex Ernst potentials in order to get a charged field configuration and recover, in this way, the $U(1)$ vector field sector of the EMKR theory.

Further, in Section 5 we reduce the system to two effective dimensions (dependence of just two dynamical coordinates) in order to be able to consider as seed solution a pair of Ernst potentials which correspond to interacting Kerr black holes. In this case, the 5–dimensional line element explicitly depends on the Ernst potentials and the resulting field configuration contains a Kalb–Ramond dipole hidden inside a horizon. In Section 6 we explicitly compute the generated charged solution, study its asymptotical behaviour and give an interpretation of the field configuration. Finally, we sketch our conclusions and discuss on the further development of the present work.
2 Matrix Ernst Potential Formalism

In this Section we review the MEP formalism for the $D$–dimensional effective field theory of the heterotic string and indicate an algorithm for generating a charged solution of the double Ernst system starting from a neutral one by making use of a matrix Lie–Bäcklund transformation of Harrison type.

We consider the effective action of the heterotic string theory at tree level

$$S^{(D)} = \int d^Dx |G^{(D)}|^2 e^{-\phi^{(D)}} \left(R^{(D)} + \phi^{(D)}_{,\mu} \phi^{(D)}_{,\mu} - \frac{1}{12} H^{(D)}_{MNP} H^{(D)MNP} - \frac{1}{4} F^{(D)I}_{MN} F^{(D)IMN} \right),$$

where

$$F^{(D)I}_{MN} = \partial_M A^{(D)I}_N - \partial_N A^{(D)I}_M, \quad H^{(D)}_{MNP} = \partial_M B^{(D)}_{NP} - \frac{1}{2} A^{(D)I}_M F^{(D)I}_{NP} + \text{cycl perms of } M, N, P.$$  

Here $G^{(D)}_{MN}$ is the metric, $B^{(D)}_{MN}$ is the anti–symmetric Kalb-Ramond field, $\phi^{(D)}$ is the dilaton, $A^{(D)I}_M$ is a set of $U(1)$ vector fields $(I = 1, 2, \ldots, n)$, $D$ is the original number of space–time dimensions; capital letters $M, N, \ldots, P$ are related to the whole set of space–time coordinates, lowercase letters $m, n$ label the extra dimensions, whereas Greek letters $\mu, \nu$ stand for the non–compactified coordinates. In the consistent critical case $D = 10$ and $n = 16$, but we shall leave these parameters arbitrary for the time being and will fix them later in Section 3. In [9]–[10] it was shown that after the compactification of this model on a $D – 3 = d$–torus, the resulting three–dimensional theory possesses the $SO(d + 1, d + n + 1)$ symmetry group that later was identified as $U$–duality [11] and describes gravity through the metric tensor

$$g_{\mu\nu} = e^{-2\phi} \left(G^{(D)}_{\mu\nu} - G^{(D)}_{m+n+3,\mu} G^{(D)}_{n+3,\nu} G^{mn} \right),$$

coupled to the following set of three–dimensional fields:

a) scalar fields

$$G = G^{(D)}_{mn} = G^{(D)}_{m+n+3}, \quad B = B^{(D)}_{mn} = B^{(D)}_{m+n+3}, \quad A = A^{(D)I}_m = A^{(D)I}_{m+3}, \quad \phi = \phi^{(D)} - \frac{1}{2} \ln|\det G|, \quad (2)$$

b) tensor field

$$B_{\mu\nu} = B^{(D)\mu\nu} - 4 B_{mn} A^m_{\mu} A^n_{\nu} - 2 \left(A^m_{\mu} A^{m+d}_{\nu} - A^m_{\nu} A^{m+d}_{\mu} \right), \quad (3)$$

c) vector fields

$$A^{(a)}_\mu = \left( (A_1)_\mu^m, (A_2)_\mu^{m+d}, (A_3)_\mu^{2d+1} \right)$$

$$\left( A_1 \right)_\mu^m = \frac{1}{2} G^{mn} G^{(D)}_{n+3,\mu}, \quad \left( A_3 \right)_\mu^{I+2d} = -\frac{1}{2} A^{(D)I}_{\mu} + A^I_{\mu} A^{n}_{\mu}, \quad \left( A_2 \right)_\mu^{m+d} = \frac{1}{2} B^{(D)}_{m+n+3,\mu} - B_{mn} A^m_{\mu} + \frac{1}{2} A^I_{\mu} A^{I+2d}_{\mu} \quad (4)$$

where the subscripts $m, n = 1, 2, \ldots, d$; and $a = 1, \ldots, 2d + n$. In this letter we set $B_{\mu\nu} = 0$ since an anti–symmetric tensor has no dynamical degrees of freedom in three dimensions; this
is equivalent to removing the effective cosmological constant which, in general, is included in
the spectrum of the three–dimensional effective theory.

We dualize all vector fields on–shell with the aid of the pseudoscalar fields $u$, $v$ and $s$ as
follows:

\[ \nabla \times \overrightarrow{A_1} = \frac{1}{2} e^{2\phi} G^{-1} \left( \nabla u + (B + \frac{1}{2} AA^T) \nabla v + A \nabla s \right), \]
\[ \nabla \times \overrightarrow{A_3} = \frac{1}{2} e^{2\phi} \left( \nabla s + A^T \nabla v \right) + A^T \nabla \times \overrightarrow{A_1}, \]
\[ \nabla \times \overrightarrow{A_2} = \frac{1}{2} e^{2\phi} G \nabla v - (B + \frac{1}{2} AA^T) \nabla \times \overrightarrow{A_1} + A \nabla \times \overrightarrow{A_3}. \]

Thus, the effective three–dimensional theory describes gravity $g_{\mu\nu}$ coupled to the scalars $G$, $B$, $A$, $\phi$ and pseudoscalars $u$, $v$, $s$. In [12] it was shown that all these matter fields can be arranged in the following pair of MEP

\[ \mathcal{X} = \begin{pmatrix} -e^{-2\phi} + v^T X v + v^T A s + \frac{1}{2} s^T s & v^T X - u^T \\ X v + u + A s & X \end{pmatrix}, \quad \mathcal{A} = \begin{pmatrix} s^T + v^T A \\ A \end{pmatrix}, \]

where $X = G + B + \frac{1}{2} AA^T$, in such a way that they reproduce the field equations of the three–dimensional theory. These matrices have dimensions $(d+1) \times (d+1)$ and $(d+1) \times n$, respectively.

In terms of the MEP the effective three–dimensional theory adopts the form

\[ ^3S = \int d^3x \sqrt{|g|} \{ -R + \text{Tr} \frac{1}{4} \left( \nabla \mathcal{X} - \nabla AA^T \right) G^{-1} \left( \nabla \mathcal{X}^T - A \nabla A^T \right) G^{-1} + \frac{1}{2} \nabla A^T G^{-1} \nabla A \}, \]

where $\mathcal{X} = \mathcal{G} + \mathcal{B} + \frac{1}{2} AA^T$, then $\mathcal{G} = \frac{1}{2} \left( \mathcal{X} + \mathcal{X}^T - AA^T \right)$ and

\[ \mathcal{G} = \begin{pmatrix} -e^{-2\phi} + v^T G v & v^T G \\ G v & G \end{pmatrix}, \quad \mathcal{B} = \begin{pmatrix} 0 & v^T B - u^T \\ B v + u & B \end{pmatrix}. \]

In [12] it also was shown that there exist a map between the stationary actions of the heterotic string and EM theories. The map reads

\[ \mathcal{X} \longleftrightarrow -E, \quad \mathcal{A} \longleftrightarrow F, \]

\[ \text{matrix transposition} \longleftrightarrow \text{complex conjugation}, \]

where $E$ and $F$ are the conventional complex Ernst potentials of the stationary EM theory [13]. This map allows us to extrapolate the results obtained in the EM theory to the heterotic string realm using the MEP formulation.
2.1 The normalized Harrison transformation

In the language of the MEP the three–dimensional action (7) possesses a set of symmetries which has been classified according to their charging properties in [14]. Among them one finds the matrix Ehlers and Harrison transformations [15], which are symmetries that change the properties of the spacetime in a non–trivial way; they represent the matrix counterpart of the Bäcklund transformation of the sine–Gordon equation in the realm of the stationary heterotic string theory. For instance, the so–called normalized Harrison transformation allows us to construct charged string vacua from neutral ones preserving the asymptotical values of the three–dimensional seed fields. Namely, the matrix transformation

\[
A \rightarrow \left( 1 + \frac{1}{2} \Sigma \lambda \lambda^T \right) \left( 1 - A_0 \lambda + \frac{1}{2} X_0 \lambda \lambda^T \right)^{-1} \left( A_0 - X_0 \lambda \right) + \Sigma \lambda,
\]

\[
X \rightarrow \left( 1 + \frac{1}{2} \Sigma \lambda \lambda^T \right) \left( 1 - A_0 \lambda + \frac{1}{2} X_0 \lambda \lambda^T \right)^{-1} \left[ A_0 + \left( A_0 - \frac{1}{2} X_0 \lambda \right) \lambda^T \Sigma \right] + \frac{1}{2} \Sigma \lambda \lambda^T \Sigma,
\]

where \( \Sigma = \text{diag}(-1,-1,1,\ldots,1) \) stands for the signature that the MEP adopts at spatial infinity and \( \lambda \) is an arbitrary constant \((d+1) \times n\)–matrix, generates charged string solutions (with non–zero potential \( A \)) from neutral ones if we start from the seed potentials

\( A_0 \neq 0, \quad A_0 = 0. \)

The parameters that enter the matrix \( \lambda \) can be interpreted as electromagnetic charges that couple to the original seed object. It is precisely with the aid of this Bäcklund transformation that we shall charge the double 5D Ernst system in the next Section.

3 5D Einstein-Kalb-Ramond vs double Ernst system

In this Section we present a formulation of the resulting three–dimensional model, upon toroidal compactification of the 5D EKR theory, as a double Ernst system by means of a complete parametrization of the matrices \( G \) and \( B \) in terms of the real and imaginary parts of a pair of complex Ernst potentials.

Let us begin by setting to zero all the \( U(1) \) gauge fields which correspond to the winding modes of the three–dimensional theory (this is equivalent to dropping the matrix \( A \) in (7)). Thus, we obtain the following action in terms of the MEP \( \mathcal{X} \)

\[
\mathcal{X} = G + B, \quad G = \frac{1}{2} \left( \mathcal{X} + \mathcal{X}^T \right) \quad \text{and} \quad J^X = \nabla \mathcal{X} G^{-1}.
\]

There are two physically different effective theories that can be expressed by the action (11), and hence admit a double Ernst formulation. On the one hand we have the \( D = 5 \) EKR model, where the dilaton field is set to zero as well [3]. On the other hand we have the \( D = 4 \)
bosonic string theory, for which a charged pair of rotating interacting black holes coupled to dilaton and Kalb–Ramond fields was constructed in [5] and its charged dual string vacua were studied in [16]. Here we will consider again the 5D EKR theory in order to apply the NHT on a neutral family of field configurations that correspond to the double Ernst system.

Thus, we start with the five–dimensional truncated action

$$\begin{align*}
5S &= \int d^5x \left| 5G \right|^{\frac{1}{2}} \left( 5R - \frac{1}{12}5H^2 \right),
\end{align*}$$

where $5R$ is the Ricci scalar constructed on the 5–dimensional metric $5G_{\mu\nu}$ and

$$\begin{align*}
5H_{MNP} &= \partial_M 5B_{NP} + \text{cyc. perms. of } M, N, P.
\end{align*}$$

It is worth noticing that we are considering a truncation which imposes the following condition on the Kaluza–Klein and Kalb–Ramond vector fields

$$\begin{align*}
5G_{\mu,n+2} = 5B_{\mu,n+2} = 0;
\end{align*}$$

this implies that the vector fields $A_1$ and $A_2$ must vanish identically, and hence, the pseudoscalar fields $u$ and $v$ also vanish (see (5)). Such a restriction does not provide any constraint on the remaining dynamical variables and can be considered as a consistent non–trivial ansatz for the EKR theory.

After the Kaluza–Klein reduction on $T^2$ we get the stationary effective action (11) (see, for instance, [3], [10]) with the matter field spectrum of the theory encoded in the $(2 \times 2)$–matrices $\mathcal{G} \equiv G$ and $\mathcal{B} \equiv B$ which can be parametrized in the following form

$$\begin{align*}
\mathcal{G} &= \frac{p_1}{p_2} \begin{pmatrix} 1 & q_2 \\ q_2 & p_2^2 + q_2^2 \end{pmatrix}, \\
\mathcal{B} &= q_1 \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = q_1 \sigma_2,
\end{align*}$$

where $\sigma_2$ is the Pauli matrix. Under such assumptions, the five–dimensional interval reads

$$ds_5^2 = g_{\mu\nu}dx^\mu dx^\nu + \mathcal{G}_{mn}dx^mdx^n.$$

By substituting (15) into (11) the action of the “matter fields” adopts the form

$$\begin{align*}
3S_m &= \frac{1}{2} \int d^3x | g |^{\frac{1}{2}} \left\{ p_1^{-2} \left[ (\nabla p_1)^2 + (\nabla q_1)^2 \right] + p_2^{-2} \left[ (\nabla p_2)^2 + (\nabla q_2)^2 \right] \right\},
\end{align*}$$

which allows us to introduce two independent Ernst potentials

$$\begin{align*}
\epsilon_1 &= p_1 + iq_1, \\
\epsilon_2 &= p_2 + iq_2.
\end{align*}$$

In terms of these field variables, the action of the system can be rewritten as a double Ernst system in the Kähler form [17]:

$$\begin{align*}
3S &= \int d^3x | g |^{\frac{1}{2}} \left\{ -3R + 2 \left( J^{\epsilon_1 J^{\overline{\epsilon_1}}} + J^{\epsilon_2 J^{\overline{\epsilon_2}}} \right) \right\},
\end{align*}$$
where \( J^{\epsilon_1} = \nabla \epsilon_1 (\epsilon_1 + \tau_1)^{-1} \) and \( J^{\epsilon_2} = \nabla \epsilon_2 (\epsilon_2 + \tau_2)^{-1} \).

A mathematically equivalent, but physically different \( 2 \times 2 \)-matrix representation arises from (12) by making use of the discrete symmetry \( p_1 \leftrightarrow p_2, \ q_1 \leftrightarrow q_2 \). This fact allows us to define new matrices

\[
\mathcal{G}' = \frac{p_2}{p_1} \begin{pmatrix} 1 & q_1 \\ q_1 & p_1^2 + q_1^2 \end{pmatrix}, \quad \mathcal{B}' = q_2 \sigma_2
\]

and, hence, \( \mathcal{X}' = \mathcal{G}' + \mathcal{B}' \) and to write down the action that corresponds to these magnitudes:

\[
^3S = \int d^3x \left| g \right|^\frac{1}{2} \left\{ -R + \frac{1}{4} \text{Tr} \left( J^{X'} J^{X'^\tau} \right) \right\} = \int d^3x \left| g \right|^\frac{1}{2} \left\{ -R + 2 \left( J^{\epsilon_1'} J^{\epsilon_1'^\tau} + J^{\epsilon_2'} J^{\epsilon_2'^\tau} \right) \right\},
\]

where similarly \( J^{X'} = \nabla X' \mathcal{G}'^{-1} \), \( J^{\epsilon_1'} = \nabla \epsilon_1' (\epsilon_1' + \tau_1)^{-1} \), \( J^{\epsilon_2'} = \nabla \epsilon_2' (\epsilon_2' + \tau_2)^{-1} \), \( \epsilon_1' = p_2 + iq_2 \) and \( \epsilon_2' = p_1 + iq_1 \).

In terms of the MEP the above-mentioned discrete transformation reads

\[
\mathcal{X} \leftrightarrow \mathcal{X'};
\]

thus, the matrices \( \mathcal{G}' \) and \( \mathcal{B}' \) must be interpreted as new Kaluza–Klein and Kalb–Ramond fields, respectively. This symmetry mixes the gravitational and matter degrees of freedom of the theory. It recalls the Bonnor transformation of the EM theory [18], but in the bosonic string realm. It can be used to generate new solutions starting, for instance, from pure Kaluza–Klein string vacua (see [16] as well).

### 4 Applying the NHT on the Double Ernst System

Let us now proceed to apply the NHT on the neutral double Ernst system. This will generate a non–zero electromagnetic potential \( A \) which accounts for non–trivial Abelian \( U(1) \) gauge fields. In order to achieve this aim, we must consider the following seed MEP

\[
\mathcal{X}_0 = \begin{pmatrix} \frac{p_1}{p_2} & \frac{p_1 q_2 - p_2 q_1}{p_2} \\ \frac{p_1 q_2 + p_2 q_1}{p_2} & \frac{p_1}{p_2} (p_1^2 + q_1^2) \end{pmatrix}, \quad \mathcal{A}_0 = 0.
\]

In this case, the charge matrix \( \lambda \) that parametrizes the NHT has the form

\[
\lambda = \begin{pmatrix} \lambda_{11} & \lambda_{12} & \cdots & \lambda_{1n} \\ \lambda_{21} & \lambda_{22} & \cdots & \lambda_{2n} \end{pmatrix},
\]

where \( n \geq 2 \) for consistency. Thus after applying the NHT on this double Ernst seed solution, the transformed MEP read

\[
\mathcal{X}'_{11} = \frac{1}{\Xi} \left[ \left( 4 + \Lambda^2 \right) |\epsilon_2|^2 \Re \epsilon_1 + 2 \left( \lambda_{1j}^2 + \lambda_{2j}^2 |\epsilon_1|^2 \right) \Re \epsilon_2 + 4 \lambda_{1j} \lambda_{2j} \Re \epsilon_2 \imag \epsilon_1 \right].
\]
Thus, we can write the line element in the Lewis–Papapetrou form making use of the Weyl effective theory under consideration in order to use as seed solution a pair of Kerr black holes.

In this section, following [3] we impose one more symmetry on the fields of the three–dimensional system.

5 Double Kerr seed solution

where \( \Lambda^2 = 4 \lambda_{ij}^2 - (\lambda_{1j} \lambda_{2j})^2 \), \( \Gamma_+ = 4 + 2 \lambda_{ij}^2 - 2 \lambda_{2j}^2 - \Lambda^2 \), \( \Gamma_- = 4 - 2 \lambda_{1j}^2 + 2 \lambda_{2j}^2 - \Lambda^2 \) and the non–trivial character of the matrix \( \mathcal{A} \) is evident. The fields configurations corresponding to these potentials live now in the 5D EMKR theory since we have recovered the \( U(1) \) vector fields of the system.

5 Double Kerr seed solution

In this section, following [3] we impose one more symmetry on the fields of the three–dimensional effective theory under consideration in order to use as seed solution a pair of Kerr black holes. Thus, we can write the line element in the Lewis–Papapetrou form making use of the Weyl coordinates as follows

\[
\frac{5 ds^2}{\Xi} = \mathcal{G}_{mn} dx^m dx^n + e^{2\gamma} \left( d\rho^2 + dz^2 \right) - \rho^2 d\tau^2
\]

where \( \mathcal{G}_{mn} \) and \( \gamma \) are \( \tau \)-independent. Thus, a solution of our system can be constructed using the solutions of the double vacuum Einstein equations written in the Ernst form in terms of \( \epsilon_k \) and \( \gamma^{\epsilon_k} \) (\( k = 1, 2 \))

\[
\nabla (\rho J^{\epsilon_k}) = \rho J^{\epsilon_k} (J^{\epsilon_k} - J^{\epsilon_k}), \\
\partial_z \gamma^{\epsilon_k} = \rho \left[ (J^{\epsilon_k})_z (J^{\epsilon_k})_\rho + (J^{\epsilon_k})_z (J^{\epsilon_k})_\rho \right], \\
\partial_\rho \gamma^{\epsilon_k} = \rho \left[ (J^{\epsilon_k})_\rho (J^{\epsilon_k})_z - (J^{\epsilon_k})_z \right],
\]

where

\[
\mathcal{X}_{12} = \frac{1}{\Xi} \left\{ \Gamma_+ (Re \epsilon_1 Im \epsilon_2 - Re \epsilon_2 Im \epsilon_1) + 2 \lambda_{1j} \lambda_{2j} \left[ (1 - |\epsilon_1|^2) Re \epsilon_2 - (1 - |\epsilon_2|^2) Re \epsilon_1 \right] \right\},
\]

\[
\mathcal{X}_{21} = \frac{1}{\Xi} \left\{ \Gamma_- (Re \epsilon_1 Im \epsilon_2 + Re \epsilon_2 Im \epsilon_1) + 2 \lambda_{1j} \lambda_{2j} \left[ (1 - |\epsilon_2|^2) Re \epsilon_1 + (1 - |\epsilon_1|^2) Re \epsilon_2 \right] \right\},
\]

\[
\mathcal{X}_{22} = \frac{1}{\Xi} \left\{ (\lambda^2 + 4 |\epsilon_2|^2) Re \epsilon_1 + 2 \left( \lambda_{1j}^2 + \lambda_{1j}^2 |\epsilon_1|^2 \right) Re \epsilon_2 - 4 \lambda_{1j} \lambda_{2j} Re \epsilon_1 Im \epsilon_2 \right\},
\]

\[
\mathcal{A}_{1j} = \frac{2}{\Xi} \left\{ \left[ (2 - \lambda_{1j}^2 |\epsilon_1|^2) Re \epsilon_2 - (2 - \lambda_{1j}^2 |\epsilon_2|^2) Re \epsilon_1 + \lambda_{1j} \lambda_{2j} \left( Re \epsilon_1 Im \epsilon_2 - Re \epsilon_2 Im \epsilon_1 \right) \right] \lambda_{1j} - \right\}
\]

\[
\left[ (2 + \lambda_{1j}^2) (Re \epsilon_1 Im \epsilon_2 - Re \epsilon_2 Im \epsilon_1) + \lambda_{1j} \lambda_{2j} \left( Re \epsilon_1 - |\epsilon_1|^2 Re \epsilon_2 \right) \right] \lambda_{2j} \right\},
\]

\[
\mathcal{A}_{2j} = \frac{-2}{\Xi} \left\{ \left[ (2 + \lambda_{1j}^2) (Re \epsilon_1 Im \epsilon_2 + Re \epsilon_2 Im \epsilon_1) + \lambda_{1j} \lambda_{2j} \left( Re \epsilon_1 - |\epsilon_1|^2 Re \epsilon_2 \right) \right] \lambda_{1j} - \right\}
\]

\[
\left[ (\lambda_{1j}^2 - 2 |\epsilon_2|^2) Re \epsilon_1 + (2 - \lambda_{1j}^2 |\epsilon_1|^2) Re \epsilon_2 + \lambda_{1j} \lambda_{2j} \left( Re \epsilon_1 Im \epsilon_2 + Re \epsilon_2 Im \epsilon_1 \right) \right] \lambda_{2j} \right\},
\]

\[
\Xi = 2 \left( \lambda_{1j}^2 + \lambda_{2j}^2 |\epsilon_2|^2 \right) Re \epsilon_1 + \left( 4 + \Lambda^2 |\epsilon_1|^2 \right) Re \epsilon_2 + 4 \lambda_{1j} \lambda_{2j} Re \epsilon_1 Im \epsilon_2,
\]

where \( \Lambda^2 = \lambda_{1j}^2 \lambda_{2j}^2 - (\lambda_{1j} \lambda_{2j})^2 \), \( \Gamma_+ = 4 + 2 \lambda_{1j}^2 - 2 \lambda_{2j}^2 - \Lambda^2 \), \( \Gamma_- = 4 - 2 \lambda_{1j}^2 + 2 \lambda_{2j}^2 - \Lambda^2 \) and the non–trivial character of the matrix \( \mathcal{A} \) is evident. The fields configurations corresponding to these potentials live now in the 5D EMKR theory since we have recovered the \( U(1) \) vector fields of the system.
if one identifies the function $\gamma$ that accounts for the general relativistic interaction between de black holes, in the following way $\gamma \equiv \gamma^{e_1} + \gamma^{e_2}$.

For instance, we can take as seed solution a double Kerr system consisting of a pair of rotating interacting black holes. In the framework of General Relativity, the Ernst potentials corresponding to two Kerr solutions with sources in different points of the symmetry axis read:

$$\epsilon_k = 1 - \frac{2m_k}{r_k + i\alpha_k \cos \theta_k}. \quad (34)$$

where $m_k$ and $\alpha_k$ are constant parameters which define the masses and rotations of the sources of the Kerr field configurations. Weyl and Boyer–Lindquist coordinates are related through

$$\rho = [(r_k - m_k)^2 - \zeta_k^2 \sin^2 \theta_k], \quad z = z_k + (r_k - m_k) \cos \theta_k, \quad (35)$$

where the sources are located at $z_k$ and $\zeta_k^2 = m_k^2 - a_k^2$. Thus, for the function $\gamma_k$ we have

$$e^{2\gamma_k} = \frac{P_k}{Q_k}, \quad (36)$$

where $P_k = \Delta_k - \alpha_k^2 \sin^2 \theta_k$, $Q_k = \Delta_k + \zeta_k^2 \sin^2 \theta_k$ and $\Delta_k = r_k^2 - 2m_k r_k + \alpha_k^2$.

We would like to make a remark at this point: when parameterizing the 5D interval (32) in the Lewis–Papapetrou form, one could choose a completely spatial (Euclidean) three–dimensional interval and require that the signature of the matrix $\mathcal{G}$ to be negative definite, i.e., $\mathcal{G}|_\infty = -I_2$ (the same signature holds for the matrix $\mathcal{X}|_\infty$). Thus, the five–dimensional metric would possess a signature with two time–like coordinates. Such kind of models have been studied in [19] and represent another line of investigation within this approach. It is clear that in order to fulfill this condition either $\epsilon_1$ or $\epsilon_2$ must adopt the asymptotic value $-1$, since when both potentials have the same asymptotic behaviour (with the same sign) the signature of the matrix $\mathcal{G}$ is positive definite.

Thus, in the language of the complex Ernst potentials the field configuration adopt the form

$$ds_5^2 = e^{2\gamma}(d\rho^2 + dz^2) - \rho^2 dr^2 + \frac{\epsilon_1 + \bar{\epsilon}_1}{\epsilon_2 + \bar{\epsilon}_2} |du + i\bar{\epsilon}_2 dv|^2, \quad (37)$$

$$\mathcal{B} = \frac{\epsilon_1 - \bar{\epsilon}_1}{2i} \sigma_2, \quad (38)$$

where $u = x^4$, $v = x^5$ and $\gamma = \gamma^{e_1} + \gamma^{e_2}$ as it was pointed out above.

In the case when the Ernst potentials correspond to two interacting Kerr black holes, the symmetric matrix $\mathcal{G}$ is determined by the following relations

$$\mathcal{G}_{uu} = \frac{(r_1^2 - 2m_1 r_1 + \alpha_1^2 \cos^2 \theta_1)(r_2^2 + \alpha_2^2 \cos^2 \theta_2)}{(r_1^2 - 2m_2 r_2 + \alpha_2^2 \cos^2 \theta_2)(r_1^2 + \alpha_1^2 \cos^2 \theta_1)}.$$
\[ G_{uv} = \frac{2m_2\alpha_2 \cos \theta_2 (r_1^2 - 2m_1 r_1 + \alpha_1^2 \cos^2 \theta_1)}{(r_1^2 - 2m_2 r_2 + \alpha_2^2 \cos^2 \theta_2)(r_1^2 + \alpha_1^2 \cos^2 \theta_1)} \] \hspace{1cm} (39)

\[ G_{vv} = \frac{(r_1^2 - 2m_1 r_1 + \alpha_1^2 \cos^2 \theta_1)(r_2^2 - 4m_2 r_2 + 4m_2^2 + \alpha_2^2 \cos^2 \theta_2)}{(r_1^2 - 2m_2 r_2 + \alpha_2^2 \cos^2 \theta_2)(r_1^2 + \alpha_1^2 \cos^2 \theta_1)} \]

the factor \( e^{2\gamma} \) reads

\[ e^{2\gamma} = \frac{(r_1^2 - 2m_1 r_1 + \alpha_1^2 \cos^2 \theta_1)(r_2^2 - 4m_2 r_2 + 4m_2^2 + \alpha_2^2 \cos^2 \theta_2)}{(r_1^2 - 2m_1 r_1 + \alpha_1^2 \cos^2 \theta_1 + m_1^2 \sin^2 \theta_2)(r_2^2 - 2m_2 r_2 + \alpha_2^2 \cos^2 \theta_2 + m_2^2 \sin^2 \theta_2)} \] \hspace{1cm} (40)

and the Kalb–Ramond matrix \( B \) is defined as

\[ B = \frac{2m_1 \alpha_1 \cos \theta_1}{r_1^2 + \alpha_2^2 \cos^2 \theta_1} \sigma_2 \] \hspace{1cm} (41)

and can be interpreted as a matrix Kalb–Ramond dipole configuration with momentum \( m_1 \alpha_1 \) located at \( z_1 \) and hidden inside the horizon \( r_1 = m_1 + \sqrt{m_1^2 - \alpha_1^2} \) of the metric (37). Simultaneously, the \( G_{uv} \) metric component also constitutes a dipole configuration but possesses momentum \( m_2 \alpha_2 \) and is located at \( z_2 \), hidden inside the horizon \( r_2 = m_2 + \sqrt{m_2^2 - \alpha_2^2} \).

## 6 Charged Field Configurations in 5D EMKR Theory

After applying the NHT on the double Ernst seed solution we get the following field configurations:

\[ G_{uu} = \chi_{11} - \frac{1}{2} A_{1j}^2, \quad G_{uv} = \frac{1}{2} (\chi_{12} + \chi_{21} - A_{1j} A_{2j}), \quad G_{vv} = \chi_{22} - \frac{1}{2} A_{2j}^2; \] \hspace{1cm} (42)

\[ B = \frac{1}{2} (\chi_{21} - \chi_{12}) \sigma_2, \quad A \equiv A = \left( \frac{A_{1j}}{A_{2j}} \right); \] \hspace{1cm} (43)

where the appearance of the electromagnetic potential is obvious. By substituting the Ernst potentials \( \epsilon_k \) by the corresponding double Kerr black hole system we obtain the following charged field configuration

\[ G_{uu} = \frac{DQ \Delta_1 \Delta_2 + 4m_1 (L^2 r_1 + 2m_1 \lambda_2^2 + 2 \lambda_{1j} \lambda_{2j} \alpha_1 \cos \theta_1) \Delta_2 + 2m_2 [(4 - \Lambda^2) r_2 + 2 \Lambda^2 m_2] \Delta_1}{DQ \Delta_1 \Delta_2 + 4m_2 (L^2 r_2 + 2m_2 \lambda_2^2 + 2 \lambda_{1j} \lambda_{2j} \alpha_2 \cos \theta_2) \Delta_1 + 2m_1 [(4 - \Lambda^2) r_1 + 2 \Lambda^2 m_1] \Delta_2} \]

\[ \times \frac{8 (h_1 \lambda_{1j} - h_2 \lambda_{2j})^2 + 2 \Lambda^2 (h_2 \lambda_{1j} - h_3 \lambda_{2j})^2 - 8 \Lambda^2 (h_1 h_3 - h_2^2)}{DQ \Delta_1 \Delta_2 + 4m_2 (L^2 r_2 + 2m_2 \lambda_2^2 + 2 \lambda_{1j} \lambda_{2j} \alpha_2 \cos \theta_2) \Delta_1 + 2m_1 [(4 - \Lambda^2) r_1 + 2 \Lambda^2 m_1] \Delta_2}, \] \hspace{1cm} (44)
\[
G_{uv} = \frac{4m_1 [2\lambda_{ij}\lambda_{2j}(r_1 - m_1) - L^2\alpha_1 \cos \theta_1] \Delta_2 + 2(4 - \Lambda^2)m_2\alpha_2 \cos \theta_2 \Delta_1}{DQ\Delta_1 \Delta_2 + 4m_2\left(L^2 r_2 + 2m_1 \lambda_{2j} + 2\lambda_{ij}\lambda_{2j}\cos \theta_2\right) \Delta_1 + 2m_1\left[(4 - \Lambda^2)r_1 + 2\Lambda^2 m_1\right] \Delta_2}
\]

\[
4(l_1^2 h_4 + l_2^2 h_6) + 2\Lambda^2 \left[(2 + \lambda_{1j})h_2 h_5 - (2 + \lambda_{2j})h_3 h_4\right] - 2\lambda_{ij}\lambda_{2j} [4h_1 h_6 + \Lambda^2 h_3 h_5 + (4 - \Lambda^2)h_2 h_4]
\]

\[
\left[ DQ\Delta_1 \Delta_2 + 4m_2\left(L^2 r_2 + 2m_1 \lambda_{2j} + 2\lambda_{ij}\lambda_{2j}\cos \theta_2\right) \Delta_1 + 2m_1\left[(4 - \Lambda^2)r_1 + 2\Lambda^2 m_1\right] \Delta_2 \right]^2
\]

\[
G_{uv} = \frac{DQ\Delta_1 \Delta_2 - 2m_2[(4 - \Lambda^2)r_2 - 8m_2]\Delta_1 - 4m_1\left(L^2 r_1 - 2m_1 \lambda_{ij} + 2\lambda_{ij}\lambda_{2j}\alpha_1 \cos \theta_1\right) \Delta_2}{DQ\Delta_1 \Delta_2 + 4m_2\left(L^2 r_2 + 2m_1 \lambda_{2j} + 2\lambda_{ij}\lambda_{2j}\cos \theta_2\right) \Delta_1 + 2m_1\left[(4 - \Lambda^2)r_1 + 2\Lambda^2 m_1\right] \Delta_2}
\]

\[
B_{uv} = \frac{2(4 - \Lambda^2)m_1\alpha_1 \cos \theta_1 \Delta_2 + 4m_2\left[2\lambda_{ij}\lambda_{2j}(r_2 - m_2) - L^2\alpha_2 \cos \theta_2\right] \Delta_1}{DQ\Delta_1 \Delta_2 + 4m_2\left(L^2 r_2 + 2m_1 \lambda_{2j} + 2\lambda_{ij}\lambda_{2j}\cos \theta_2\right) \Delta_1 + 2m_1\left[(4 - \Lambda^2)r_1 + 2\Lambda^2 m_1\right] \Delta_2}
\]

\[
A_{1j} = \frac{2\left[2h_1 - \lambda_{2j}^2 h_3 + \lambda_{1j}\lambda_{2j} h_2\right] \lambda_{1j} + 2\left[\lambda_{1j}\lambda_{2j} h_3 - (2 + \lambda_{1j}^2) h_2\right] \lambda_{2j}}{DQ\Delta_1 \Delta_2 + 4m_2\left(L^2 h_2 + 2m_1 \lambda_{2j} + 2\lambda_{ij}\lambda_{2j}\cos \theta_2\right) \Delta_1 + 2m_1\left[(4 - \Lambda^2)r_1 + 2\Lambda^2 m_1\right] \Delta_2}
\]

\[
A_{2j} = \frac{-2\left[(2 + \lambda_{2j}^2) h_4 + \lambda_{1j}\lambda_{2j} h_5\right] \lambda_{1j} + 2\left[\lambda_{2j}^2 h_5 + \lambda_{1j}\lambda_{2j} h_4 + 2h_6\right] \lambda_{2j}}{DQ\Delta_1 \Delta_2 + 4m_2\left(L^2 h_2 + 2m_1 \lambda_{2j} + 2\lambda_{ij}\lambda_{2j}\cos \theta_2\right) \Delta_1 + 2m_1\left[(4 - \Lambda^2)r_1 + 2\Lambda^2 m_1\right] \Delta_2}
\]

where

\[
h_1 = 2(m_1 r_1 \Delta_2 - m_2 r_2 \Delta_1), \quad h_2 = 2(m_2 \alpha_2 \cos \theta_2 \Delta_1 - m_1 \alpha_1 \cos \theta_1 \Delta_2),
\]

\[
h_3 = 4\left(m_1^2 \Delta_2 - m_2^2 \Delta_1\right) - h_1, \quad h_4 = 2(m_1 \alpha_1 \cos \theta_1 \Delta_2 + m_2 \alpha_2 \cos \theta_2 \Delta_1),
\]

\[
h_5 = 2[m_1(r_1 - 2m_1) \Delta_2 + m_2 r_2 \Delta_1], \quad h_6 = 2[m_1 \Delta_2 + m_2(r_2 - 2m_2) \Delta_1],
\]

\[
L^2 = \lambda_{1j}^2 - \lambda_{2j}^2, \quad l_1^2 = 2\lambda_{1j}^2 + \Lambda^2, \quad \Delta_k = r_k^2 - 2m_k r_k + \alpha_k^2 \cos^2 \theta_k, \quad (k = 1, 2) \text{ and, finally,}
\]

\[
DQ = 4 + 2\lambda_{1j}^2 + 2\lambda_{2j}^2 + \Lambda^2.
\]

A consistency checking of the generated solution consists of setting the parameters \(\lambda_{1j}\) and \(\lambda_{2j}\) to zero in order to recover the starting field configuration (39)-(41). It is straightforward to verify that this is indeed the case.
The asymptotical behaviour of the generated three–dimensional field configurations read

$$G_{uu}\big|_{\infty} \sim 1 - \frac{2\Gamma_{-}(m_1 - m_2)}{DQ r} + \frac{8\lambda_{1j}\lambda_{2j} (m_1 \alpha_1 \cos \theta_1 - m_2 \alpha_2 \cos \theta_2)}{DQ r^2} + O(r^{-2}) , \quad (51)$$

$$G_{uv}\big|_{\infty} \sim \frac{8\lambda_{1j}\lambda_{2j} m_1}{DQ r} - \frac{4L^2 m_1 \alpha_1 \cos \theta_1 - 2(4 - \Lambda^2) m_2 \alpha_2 \cos \theta_2}{DQ r^2} + O(r^{-2}) , \quad (52)$$

$$G_{vv}\big|_{\infty} \sim 1 - \frac{2\Gamma_{+}(m_1 + m_2)}{DQ r} - \frac{8\lambda_{1j}\lambda_{2j} (m_1 \alpha_1 \cos \theta_1 + m_2 \alpha_2 \cos \theta_2)}{DQ r^2} + O(r^{-2}) , \quad (53)$$

$$B_{uv}\big|_{\infty} \sim \frac{8\lambda_{1j}\lambda_{2j} m_2}{DQ r} + \frac{2(4 - \Lambda^2) m_1 \alpha_1 \cos \theta_1 - 4L^2 m_2 \alpha_2 \cos \theta_2}{DQ r^2} + O(r^{-2}) , \quad (54)$$

$$A_{1j}\big|_{\infty} \sim \frac{4[(2 + \lambda_2^2)\lambda_{1j} - \lambda_{1i}\lambda_{2i}\lambda_{2j}] (m_1 - m_2)}{DQ r} - \frac{4[\lambda_{1i}\lambda_{2i}\lambda_{1j} - (2 + \lambda_1^2)\lambda_{2j}](m_1 \alpha_1 \cos \theta_1 - m_2 \alpha_2 \cos \theta_2)}{DQ r^2} + O(r^{-2}) , \quad (55)$$

$$A_{2j}\big|_{\infty} \sim -\frac{4[\lambda_{1i}\lambda_{2i}\lambda_{1j} - (2 + \lambda_1^2)\lambda_{2j}](m_1 + m_2)}{DQ r} + \frac{4[(2 + \lambda_2^2)\lambda_{1j} - \lambda_{1i}\lambda_{2i}\lambda_{2j}](m_1 \alpha_1 \cos \theta_1 + m_2 \alpha_2 \cos \theta_2)}{DQ r^2} + O(r^{-2}) . \quad (56)$$

From this analysis it is clear that under the NHT, all the generated fields (gravitational, Kalb–Ramond and electromagnetic) effectively develop both Coulomb and dipole terms. Thus, from one side, the $G_{uu}$ component of the constructed metric possesses mass terms defined by $M_{uu} = \Gamma_{-} m_1/DQ$ at $z_1$ and $M_{uu} = \Gamma_{-} m_2/DQ$ at $z_2$, and from the other side, it acquires dipole sources with masses $\tilde{M}_{uu} = 8\lambda_{1j}\lambda_{2j} m_1/DQ$, $\tilde{M}_{uu} = -8\lambda_{1j}\lambda_{2j} m_2/DQ$, and their corresponding momenta $\tilde{M}_{uu}\alpha_1$, $\tilde{M}_{uu}\alpha_2$, located at $z_1$ and $z_2$, respectively. In a similar way the $G_{uv}$ component of the metric has masses $M_{uv} = \Gamma_{+} m_1/DQ$ at $z_1$ and $M_{uv} = \Gamma_{+} m_2/DQ$ at $z_2$; indeed, it possesses as well the massive dipole terms defined by $\tilde{M}_{uv} = -8\lambda_{1j}\lambda_{2j} m_1/DQ$ at $z_1$ and $\tilde{M}_{uv} = -8\lambda_{1j}\lambda_{2j} m_2/DQ$ at $z_2$ with the momenta $\tilde{M}_{uv}\alpha_1$ and $\tilde{M}_{uv}\alpha_2$, respectively.

The transformed Kalb–Ramond tensor field also acquires a Coulomb term determined by the charge $M_B = 8\lambda_{1j}\lambda_{2j} m_2/DQ$ located at $z_2$ and now possesses two dipole sources with masses $M_{B1} = (4 - \Lambda^2) m_1/DQ$, $M_{B2} = -4L^2 m_2/DQ$ and momenta $M_{B1}\alpha_1$, $M_{B2}\alpha_2$, located
at \( z_1 \) and \( z_2 \), respectively. The same situation exactly takes place for the \( G_{uv} \) component of the metric, which usually corresponds to the rotation of the gravitational field. Thus, the generated gravitational potential \( G_{uv} \) has a Coulomb source with mass \( M_{uv} = 8 \lambda_1 \lambda_2 m_1 / DQ \) located at \( z_1 \) and dipole sources with masses \( \tilde{M}_{uv_1} = -2L^2 m_1 / DQ \), \( \tilde{M}_{uv_2} = (4 - \Lambda^2) m_2 / DQ \) and momenta \( \tilde{\alpha}_1, \tilde{\alpha}_2 \), located at \( z_1 \) and \( z_2 \), respectively.

At this point we would like to point out that the discrete symmetry (22) which relates gravitational and Kalb–Ramond degrees of freedom is still present asymptotically and is quite evident in the language of the masses and charges of the components \( G_{uv} \) and \( B_{uv} \), since one can clearly see that these components transform into each other under the interchange of the respective masses and charges, even after the implementation of the nonlinear NHT.

Finally, the generated field configuration possesses an evidently non–trivial electromagnetic sector and its asymptotic structure reveals its usual Coulomb form, defining in this way the effective electromagnetic charges of the system. These fields also have effective dipole sources. Thus, the electromagnetic fields \( A_{1j} \) possess momenta defined by the expressions

\[
4 \left[ \lambda_1 \lambda_2 \lambda_{1j} - (2 + \Lambda^2) \lambda_{2j} \right] m_1 \alpha_1 / DQ \quad \text{and} \quad -4 \left[ \lambda_1 \lambda_2 \lambda_{1j} - (2 + \Lambda^2) \lambda_{2j} \right] m_2 \alpha_2 / DQ,
\]

whereas the respective momenta for the electromagnetic fields \( A_{2j} \) read

\[
4 \left[ (2 + \lambda^2_2) \lambda_{1j} - \lambda_1 \lambda_2 \lambda_{2j} \right] m_1 \alpha_1 / DQ \quad \text{and} \quad 4 \left[ (2 + \lambda^2_2) \lambda_{1j} - \lambda_1 \lambda_2 \lambda_{2j} \right] m_2 \alpha_2 / DQ.
\]

Thus, under the NHT, the double Kerr seed solution does not acquire just the electromagnetic charges, but it develops as well effective Coulomb and dipole terms for all the fields of the field configuration: gravitational, Kalb–Ramond and electromagnetic fields.

7 Conclusion and Discussion

In this paper we have obtained a charged field configuration of the five–dimensional EMKR theory starting from a neutral one that corresponds to a double Ernst (double Kerr, in particular) system. The generation of the new charged solution was carried out via a matrix Lie–Bäcklund transformation of Harrison type that preserves the asymptotical values of the seed fields.

An interesting novel feature of the generated exact solution is that all the fields of the field configuration develop effective Coulomb and dipole terms asymptotically. Thus, after applying the NHT, the 5D double Kerr seed solution acquires effective Coulomb terms and dipole sources with momenta. This is in contrast with the effect that the NHT produces on a neutral seed solution in the framework of the general theory of relativity where it just endows the initial field configuration with a set of electromagnetic charges.

The statistical analysis of such a configuration is an appealing direction to conduct the present research. The equilibrium properties of the generated solution is of interest as well and would generalize to the 5D case some previous results obtained in the framework of the four–dimensional general relativity [2], [20]–[21]. These issues are under current investigation.
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