Computing Equilibria of Semi-algebraic Economies Using Triangular Decomposition and Real Solution Classification

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Abstract

In this paper, we are concerned with the problem of determining the existence of multiple equilibria in economic models. We propose a general and complete approach for identifying multiplicities of equilibria in semi-algebraic economies, which may be expressed as semi-algebraic systems. The approach is based on triangular decomposition and real solution classification, two powerful tools of algebraic computation. Its effectiveness is illustrated by two examples of application.

Key words: equilibrium, semi-algebraic economy, semi-algebraic system, triangular decomposition, real solution classification

1. Introduction

The equilibria of an economy are states where the quantity demanded and the quantity supplied are balanced. In other words, the values of the variables at the equilibria in the economic model remain stable (when there is no external influence). For example, a market equilibrium refers to a condition under which a market price is established through competition such that the amount of goods or services sought by buyers is equal to that produced by sellers. Equilibrium models have been used in various branches of economics such as macroeconomics, public finance, and international trade [1].

When analyzing equilibrium models, economists usually assume the global uniqueness of competitive equilibria. However, the rationality of this assumption is not yet convincing. For instance, in “realistically calibrated” models, it is still an open question whether or not the phenomenon of multiple equilibria is likely to appear. For this question, Gjerstad [2] has achieved some results: he pointed out that the multiplicity of equilibria is prevalent in a pure exchange economy which has CES utility functions with elasticities of substitution above 2. Moreover, from a practical point of view, sufficient

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assumptions for the global uniqueness of competitive equilibria are usually too restrictive to be applied to realistic economic models.

In the economic context, the multiplicity of equilibria of an economy refers to the number of equilibria of the economic model. Detecting the multiplicities of equilibria of economies is an important issue, as multiplicities may cause serious mistakes in the analysis of economics models and the prediction of economic trends. Moreover, that the known sufficient conditions for uniqueness are not satisfied does not imply that there must be several competitive equilibria. This means that the existing theories and results for the models may remain useful when the sufficient conditions are not satisfied.

Traditional approaches for computing equilibria are almost all based on numerical computation. They have several shortcomings: first, numerical computation may encounter the problem of instability, which could make the results completely useless; second, most numerical algorithms only search for a single equilibrium and are nearly infeasible for multiplicity detection. Thus it is desirable to develop methods which can detect exactly all the equilibria of applied economic models.

Recently, Kubler and Schmedders [3] have considered a special kind of standard finite Arrow–Debreu exchange economies with semi-algebraic preferences, which are called semi-algebraic exchange economies. Following the terminology used by Kubler and Schmedders, by semi-algebraic economies we mean economic models (including for example competitive models and equilibrium models with strategic interactions) whose equilibria can be described as real solutions of semi-algebraic systems, say of the form

\[
\begin{align*}
P_1(u_1, \ldots, u_d, x_1, \ldots, x_n) &= 0, \\
& \vdots \\
P_n(u_1, \ldots, u_d, x_1, \ldots, x_n) &= 0, \\
Q_1(u_1, \ldots, u_d, x_1, \ldots, x_n) &\leq 0, \\
& \vdots \\
Q_r(u_1, \ldots, u_d, x_1, \ldots, x_n) &\leq 0,
\end{align*}
\]

where the symbol \(\leq\) stands for any of \(>, \geq, <, \leq, \) and \(\neq\), and \(P_i, Q_j\) are polynomials over the field \(\mathbb{R}\) of real numbers, with \(u_1, \ldots, u_d\) as their parameters and \(x_1, \ldots, x_n\) as their variables. Note that systems from realistic economies should be zero-dimensional, i.e., their zeros \((\bar{x}_1, \ldots, \bar{x}_n)\) must be finite in number under any meaningful specialization of the parameters \(u_1, \ldots, u_d\).

Thus for semi-algebraic economies the problem of computing equilibria may be reduced to that of dealing with the semi-algebraic system \(\mathbf{I}\). For example, the multiplicity of equilibria can be detected by determining whether or not the corresponding system \(\mathbf{I}\) has multiple real solutions. This problem has been solved partially by Kubler and Schmedders [3, 4] using Gröbner bases. The main idea that underlies the remarkable work of Kubler and Schmedders is to use the method of Gröbner bases to transform the equation part of system \(\mathbf{I}\) into an equivalent set of new equations in a much simpler form, where only one equation, say \(G_1 = 0\), is nonlinear, yet it is univariate, and to count the real solutions of the equation part by using Strum’s sequence of \(G_1\).

On the other hand, Datta [5, 6] has compared the methods of Gröbner bases and homotopy continuation for computing all totally mixed Nash equilibria in games. Chatterji and Gandhi [7] have applied computational Galois theory to the problem of computing
Nash equilibria of a subclass of generic finite normal form games, i.e., the rational payoff games with irrational equilibria.

The work presented in this paper is based on our observation that triangular decomposition of polynomial systems [8, 9] and real solution classification of semi-algebraic systems [10, 11] may serve as a good alternative to Gröbner bases and Strum sequences for the computation of equilibria of semi-algebraic economies. This alternative may lead to new approaches that are theoretically more general and practically more effective than the approaches developed by Kubler, Schmedders, and others. The aim of the present paper is to propose one such approach, which is general and complete, for identifying the multiplicity of equilibria in semi-algebraic economies. The proposed approach takes inequalities into consideration and can give a tighter bound or even precise number of equilibria, depending on whether the economy is exactly described by (1), than the existing approaches mentioned above, which only compute an upper bound for the number of equilibria because inequality constraints from realistic economies are usually ignored for simplicity.

The key step of our approach is to decompose the semi-algebraic system in question into several triangularized semi-algebraic systems, with the total number of solutions unchanged. Consider for example the system

\[
\begin{align*}
    P_1 &= x_2x_3 - 1 = 0, \\
    P_2 &= x_4^2 + x_1x_2x_3 = 0, \\
    P_3 &= x_1x_2x_4 + x_3^2 - x_2 = 0, \\
    P_4 &= x_1x_3x_4 - x_3 + x_2^2 = 0
\end{align*}
\]

Under the variable ordering \(x_1 < \cdots < x_4\), triangular decomposition of the polynomial set \(P = \{P_1, \ldots, P_4\}\) results in two triangular sets

\[
T_1 = [x_1^3 + 4, x_2^3 + 1, x_2x_3 - 1, 2x_4 + x_1^2], \quad T_2 = [x_1, x_3^3 - 1, x_2x_3 - 1, x_4],
\]

such that the union of the zero sets of \(T_1\) and \(T_2\) is same as the zero set of \(P\). The zeros of the triangular sets \(T_1\) and \(T_2\) may be computed successively, which is easier than computing the zeros directly from \(P\). Triangular decomposition as such is used in the first stage of our approach to pre-process the equation part of (1).

The rest of the paper is structured as follows. In Section 2 we first show how to count equilibria of semi-algebraic economies without parameters by means of a simple example and then describe a complete method for the counting. In Section 3 a method based on real solution classification is presented to deal with semi-algebraic economies with parameters. In Section 4 we demonstrate the effectiveness of our methods using two examples of application. The paper is concluded with some remarks in Section 5.

2. Economies Without Parameters

From now on we denote by \(u\) and \(x\) the parameters \(u_1, \ldots, u_d\) and the variables \(x_1, \ldots, x_n\) respectively in system (1). In this section, we consider the simpler case when \(u\) do not occur in (1).

**Problem A.** Assume that the parameters \(u\) are not present in system (1). Count all the distinct real solutions of (1).
The method that we will present for solving this problem extends the approach of Kubler and Schmedders \[3, 4\]. It can systematically handle economies with inequality conditions (which are fairly prevalent in practical applications).

2.1. Triangular Decomposition Revisited

We recall some standard notations and algorithms used for triangular decomposition of polynomial systems, which play a fundamental role in our approach to be proposed.

Let the variables be ordered as $x_1 < \cdots < x_n$. An ordered set $[T_1, \ldots, T_r]$ of non-constant polynomials is called a triangular set if the leading variable of $T_i$ is smaller than that of $T_j$ for all $i < j$, where the leading variable of $T_i$ is the variable with biggest index occurring in $T_i$. For example, $[x_1 - 2, (x_1^2 - 4)x_3 - x_2]$ is a triangular set.

Let $\mathcal{P}$ and $\mathcal{Q}$ be two sets of multivariate polynomials with coefficients in the field $\mathbb{Q}$ of rational numbers. We denote by $\text{Zero}(\mathcal{P})$ the set of all common zeros (in some extension field of $\mathbb{Q}$) of the polynomials in $\mathcal{P}$ and by $\text{Zero}(\mathcal{P}/\mathcal{Q})$ the subset of $\text{Zero}(\mathcal{P})$ whose elements do not annihilate any polynomial in $\mathcal{Q}$.

Any multivariate polynomial can be viewed as a univariate polynomial in its leading variable. We use $\text{ini}(\mathcal{P})$ to denote the set of leading coefficients of all the polynomials in $\mathcal{P}$, viewed as univariate polynomials in their leading variables. Such leading coefficients are called initials.

**Theorem 2.1.** There are algorithms which can decompose any given polynomial set $\mathcal{P}$ into finitely many triangular sets $T_1, \ldots, T_k$ with different properties such that

$$\text{Zero}(\mathcal{P}) = \bigcup_{i=1}^{k} \text{Zero}(T_i/\text{ini}(T_i)).$$

Among the algorithms pointed out by the above theorem, the best known is Wu–Ritt’s algorithm based on the computation of characteristic sets, developed by Wu \[12, 13\] from the work of Ritt \[14\] in differential algebra. For example, the polynomial set

$$\mathcal{P} = [xy^2 + z^2, xz + y]$$

may be decomposed by using Wu–Ritt’s algorithm with $x < y < z$ into four triangular sets

$$T_1 = [(x^3 + 1)y^2, xz + y], \quad T_2 = [x^2 - x + 1, xz + y],$$
$$T_3 = [x + 1, z - y], \quad T_4 = [x, y, z^2].$$

(4)

It is not guaranteed that $\text{Zero}(T/\text{ini}(T)) \neq \emptyset$ for all triangular set $T$. For example, with

$$T = [x^2 - u, y^2 + 2xy + u, (x + y)z + 1]$$

and $u < x < y < z$, it can be easily proved that $\text{Zero}(T/\text{ini}(T)) = \emptyset$. We can impose additional conditions to obtain triangular sets of other kinds with better properties. Typical examples of such triangular sets are regular sets \[15\] (also known as regular chains \[16\] and proper ascending chains \[17\]), simple sets \[15\], and irreducible triangular sets \[12, 8\].
A triangular set \([T_1, \ldots, T_r]\) is said to be regular or called a regular set if no regular zero of \(T_i\) annihilates the initial of \(T_{i+1}\) for all \(i = 1, \ldots, r-1\), where \(T_i = [T_1, \ldots, T_i]\) and a regular zero of \(T_i\) is such a zero of \(T_i\) in which the variables other than the leading variables of \(T_1, \ldots, T_i\) are not specialized to concrete values. For example, with \(u < x < y\) the triangular set \(\hat{T} = [x^2 - u^2, (x + u)y + 1]\) is not regular, while \(\ddot{T} = [x^2 - u^2, xy + 1]\) is. Observe that the initials of the second polynomials in both \(\hat{T}\) and \(\ddot{T}\) vanish at the zeros of \(x^2 - u^2\) when \(u\) is specialized to 0. The reader may refer to [15, 8, 9] for formal definitions of regular sets and [16, 17, 15, 19] for effective algorithms that decompose any given polynomial set into finitely many regular sets.

A regular set \([T_1, \ldots, T_r]\) is called a simple set or an irreducible triangular set, respectively, if every \(T_i\) is squarefree or irreducible at any regular zero of \([T_1, \ldots, T_{i-1}]\) for \(i = 1, \ldots, r\). For example, the regular set \(\ddot{T}\) given above is also a simple set, but it is not irreducible. Simple sets and irreducible triangular sets have many nice properties (of which some are about their saturated ideals, see [8, 9, 20]). Algorithms for decomposing polynomial sets into simple sets or irreducible triangular sets may be found in [12, 18, 8, 20, 21].

The algorithms for triangular decomposition proposed by the second author [22, 15, 18] appear to be more general than other available ones. They can be used to decompose any given polynomial system \([P, Q]\) into finitely many triangular systems \([T_1, S_1], \ldots, [T_k, S_k]\) with different properties such that

\[
\text{Zero}(P/Q) = \bigcup_{i=1}^{k} \text{Zero}(T_i/S_i),
\]

where \([T_i, S_i]\) could be fine triangular systems [22], regular systems [15], or simple systems [18], corresponding to triangular sets, regular sets, or simple sets respectively. The interested reader may consult the above-cited references for formal definitions, properties, and algorithms.

Triangular decomposition discussed above may be effectively used in our approach for counting real solutions of semi-algebraic systems. Note that for the decomposition [8] or [9], there is no guarantee that

\[
\text{Zero}(T_i/\text{ini}(T_i)) \cap \text{Zero}(T_j/\text{ini}(T_j)) = \emptyset \quad \text{or} \quad \text{Zero}(T_i/S_i) \cap \text{Zero}(T_j/S_j) = \emptyset
\]

for \(i \neq j\). For the triangular sets in [11], it is easy to verify that \((\sqrt[3]{3}, 0, 0)\) is in both \(\text{Zero}(T_i/\text{ini}(T_i))\) and \(\text{Zero}(T_j/\text{ini}(T_j))\). This problem may cause some trouble for counting distinct real zeros, but it can be solved, e.g., by using the technique given in [22] (see also [18]).

A triangular set in which all polynomials other than the first are linear with respect to (w.r.t.) their corresponding leading variables is said to be quasi-linear. For example, in \([T_1, T_2, T_3]\), \(T_1, T_2\), and \(T_3\) are all quasi-linear, but \(T_4\) is not. The (real) zeros of quasi-linear triangular sets may be determined by analyzing essentially the first polynomials in the triangular sets. A triangular system \([T, S]\) is said to be quasi-linear if \(T\) is quasi-linear. Quasi-linearization is a key step in transforming an arbitrary semi-algebraic system into an equivalent semi-algebraic system in which the equation polynomials form a quasi-linear triangular set.
Theorem 2.2. Let $\mathcal{T} = [T_1(u, y_1), \ldots, T_r(u, y_1, \ldots, y_r)]$ be a regular set in $\mathbb{Q}[u, y_1, \ldots, y_r]$ and $c_2, \ldots, c_r$ be a sequence of $r-1$ randomly chosen integers. Then the polynomial set $\mathcal{T}^*$ obtained from $\mathcal{T}$ by replacing $y_1$ with $y_1 + c_2y_2 + \cdots + c_r y_r$ can be decomposed over $\mathbb{Q}(u)$, with probability 1, into finitely many quasi-linear triangular sets $\mathcal{T}_1, \ldots, \mathcal{T}_k$ w.r.t. the variable ordering $y_1 < \cdots < y_r$, such that

$$\text{Zero}(\mathcal{T}^*) = \bigcup_{i=1}^k \text{Zero}(\mathcal{T}_i/\text{ini}(\mathcal{T}_i)).$$

Proof. Let $\mathcal{T}^*$ be decomposed into $k$ simple sets

$$\mathcal{T}_i^* = [T_{i1}^*(u, y_1), \ldots, T_{ir}^*(u, y_1, \ldots, y_r)], \quad i = 1, \ldots, k,$

according to Theorem 2.1. By means of normalization (using, e.g., [24, Algorithm 4]) and pseudo-division, each $\mathcal{T}_i^*$ may be transformed into a normal and reduced simple set $\mathcal{T}_i = [T_{i1}(u, y_1), \ldots, T_{ir}(u, y_1, \ldots, y_r)]$ such that $\langle \mathcal{T}_i^* \rangle = \langle \mathcal{T}_i \rangle$ and the degree of $\mathcal{T}_i^*$ in $y_1$ remains unchanged (see [8] for the definitions of normal triangular set and reduced triangular set), where $\langle \mathcal{T}_i \rangle$ denotes the ideal in $\mathbb{Q}(u)[y_1, \ldots, y_r]$ generated by $\mathcal{T}_i^*$. Let

$$\mathcal{T}_i = [T_{i1}^*/\text{ini}(T_{i1}^*), \ldots, T_{ir}^*/\text{ini}(T_{ir}^*)].$$

Note that under the lexicographical term order, the leading monomials of any two different polynomials in $\mathcal{T}_i$ are relatively prime. Thus $\mathcal{T}_i$ is a Gröbner basis of $\langle \mathcal{T}_i^* \rangle$ by Proposition 4 in [23, section 2.9]. The monicness of the polynomials in $\mathcal{T}_i$ is obvious. As $\mathcal{T}_i$ is reduced, $\mathcal{T}_i$ is the reduced Gröbner basis of $\langle \mathcal{T}_i \rangle$. Furthermore, from [21, Theorem 3.3] we know that $\langle \mathcal{T}_i \rangle$ is a radical ideal (because $\mathcal{T}_i$ is a simple set).

Let $(\overline{y}_{11}, \ldots, \overline{y}_{1s}), \ldots, (\overline{y}_{k1}, \ldots, \overline{y}_{ks})$ be all the (distinct) zeros of $\langle \mathcal{T} \rangle$ in the algebraic closure $K$ of $\mathbb{Q}(u)$. Then for any $\mu \neq \nu$,

$$H(z_r, \ldots, z_2) = (\overline{y}_{r\mu} - \overline{y}_{r\nu})z_r + \cdots + (\overline{y}_{2\mu} - \overline{y}_{2\nu})z_2 + (\overline{y}_{1\mu} - \overline{y}_{1\nu}) = 0$$

defines a hyperplane or an empty set of $K^{r-1} = \{(z_r, \ldots, z_2) \mid z_i \in K\}$. As $c_r, \ldots, c_2$ are randomly chosen integers, the probability that $H(c_r, \ldots, c_2) \neq 0$ (i.e., $(c_r, \ldots, c_2)$ is not among the integer points in the hyperplane) is 1. Note that

$$\overline{y}_{1\mu} + c_2\overline{y}_{2\mu} + \cdots + c_r \overline{y}_{r\mu} = \overline{y}_{1\mu}' \quad \overline{y}_{1\nu} + c_2\overline{y}_{2\nu} + \cdots + c_r \overline{y}_{r\nu} = \overline{y}_{1\nu}'$$

are the $y_1$-coordinates of two zeros of some $\langle \mathcal{T}_i \rangle$ and $\langle \mathcal{T}_j \rangle$. It is thus with probability 1 that $\overline{y}_{1\mu}' - \overline{y}_{1\nu}' \neq 0$ for any $\mu \neq \nu$. Therefore, with probability 1 the $y_1$-coordinates of the zeros of all $\langle \mathcal{T}_i \rangle$ are distinct.

By the Shape Lemma [26], each $\mathcal{T}_i$ must be of the form $[G_{i1}(y_1), y_2 - G_{i2}(y_1), \ldots, y_r - G_{ir}(y_1)]$, where $G_{ij}$ are polynomials over $\mathbb{Q}(u)$. Thus $\mathcal{T}_i$ is quasi-linear and the proof is complete.

Triangular sets produced by triangular decomposition are often, but not always, quasi-linear. Among the four triangular sets in (4), only $\mathcal{T}_4$ is not quasi-linear. One may obtain quasi-linear triangular sets by means of linear transformation with randomly chosen integers $c_2, \ldots, c_r$ according to the above theorem. As the probability of success with one trial is 1, the quasi-linearization technique is effective.
2.2. Illustrative Example

Before describing the method, we provide a simple example to illustrate our general approach for solving Problem A. Consider the semi-algebraic system

\[
\begin{align*}
  x^3 - 20 y^2 &= 0, \\
  y^2 - 2x - 1 &= 0, \\
  x - y &\neq 0, \\
  2x - y &\geq 0, \\
  y &> 0.
\end{align*}
\]

It is easy to see that the number of distinct real solutions of this system is equal to the sum of those of the following two systems:

\[
\begin{align*}
  x^3 - 20 y^2 &= 0, \\
  y^2 - 2x - 1 &= 0, \\
  x - y &\neq 0, \\
  2x - y &\geq 0, \\
  y &> 0, \quad (6)
\end{align*}
\]

These two systems may be treated similarly, so we only consider (6) in what follows. The number of distinct real solutions of the system can be determined in five steps.

Step A1. Let the variables be ordered as \( x < y \) and decompose the set of equation polynomials

\[ \mathcal{F} = \{ x^3 - 20 y^2, y^2 - 2x - 1 \} \]

into triangular sets. The process of triangular decomposition for \( \mathcal{F} \) is trivial and one can easily obtain a triangular set

\[ \mathcal{T} = [x^3 - 40x - 20, y^2 - 2x - 1] \quad (8) \]

such that \( \text{Zero}(\mathcal{T}) = \text{Zero}(\mathcal{F}) \).

Substituting \( x \) in \( \mathcal{T} \) by \( x + y \) and decomposing the resulting set \( \mathcal{T}^* \) into simple sets under the same ordering \( x < y \), we obtain \( \mathcal{T}_1 = [T_1, T_2] \) with

\[ T_1 = x^6 - 83x^4 - 360x^3 + 1083x^2 + 1320x + 359, \quad T_2 = Iy + J, \]

\[ I = 3x^2 + 8x - 35, \quad J = x^3 + 6x^2 - 33x - 18. \]

Now \( T_1 \) is univariate, \( T_2 \) is linear in \( y \), and \( \text{Zero}(\mathcal{T}_1/\{I\}) = \text{Zero}(\mathcal{T}^*) \).

The linear transformation \( x \to x + y \) above transforms the inequality constraints \( x - y \neq 0, 2x - y > 0, \) and \( y > 0 \) in (6) into \( x \neq 0, 2x + y > 0, \) and \( y > 0 \) respectively.
Hence the number of real solutions of (7) is equal to that of the following system

\[
\begin{align*}
T_1 &= 0, \\
T_2 &= 0, \\
I &\neq 0, \\
x &\neq 0, \\
2x + y &> 0, \\
y &> 0.
\end{align*}
\]

**Step A2.** Solving \( T_2 = 0 \) for \( y \) yields \( y = -J/I \). Substituting this solution into the inequality constraints in the above system, we obtain the following system

\[
\begin{align*}
T_1 &= 0, \\
I &\neq 0, \\
x &\neq 0, \\
2x - J/I &> 0, \\
-J/I &> 0.
\end{align*}
\]

The constraint \(-J/I > 0\) can be replaced by the equivalent inequality \( G = -JI > 0 \). Similarly, \( 2x - J/I > 0 \) can be replaced by \( H = 2xI^2 - JI > 0 \).

**Step A3.** Further replace the constraints \( G > 0 \) and \( H > 0 \) respectively by

\[ G' = \text{rem}(G, F) = -3x^5 - 26x^4 + 86x^3 + 528x^2 - 1011x - 630 > 0, \]

and

\[ H' = \text{rem}(H, F) = 15x^5 + 70x^4 - 296x^3 - 592x^2 + 1439x - 630 > 0, \]

where \( \text{rem}(P, F) \) denotes the remainder of \( P \) divided by \( F \). Then the problem is reduced to counting the real solutions of the following semi-algebraic system in a single variable \( x \):

\[
\begin{align*}
F &= 0, \\
H' &= 0, \\
G' &= 0,
\end{align*}
\]

(9)

where \( F \) and \( G' \) have no common zeros, and so do \( F \) and \( H' \).

**Step A4.** In order to count the real solutions of (9), we isolate the real zeros of \( G' \cdot H' \) by rational intervals using an available algorithm. For instance, application of the modified Uspensky algorithm [27] may yield the following sorted sequence of intervals

\([-16, -8], [-5, -5], [-9/2, -4], [-1, -1/2], [1/2, 3/4], [1, 3/2], [2, 5/2], [3, 4].\]

These closed intervals do not intersect with each other, and each of them contains one and only one distinct real zeros of \( G' \) or \( H' \). Moreover, by Sturm’s theorem [26] or simply using the \texttt{sturm} function in Maple, we can prove that all the intervals cover no real zero of \( F \).

**Step A5.** The real zeros of \( F \) must be in

\[ (-\infty, -16), (-8, -5), (-5, -9/2), (-4, -1), \]

\[ (-1/2, 1/2), (3/4, 1), (3/2, 2), (5/2, 3), (4, +\infty), \]
the complement of the intervals given in the preceding step. In each of these open intervals, the signs of $G'$ and $H'$ are invariant, and can be determined by simply testing them at a sample point in the interval. For example, to determine the sign of $G'$ on $(-\infty, -16)$, we may compute $G'(-16 - 1)$ to get $1834656 > 0$. Thus $G'$ is positive at every point in $(-\infty, -16)$. Proceeding in this way for each interval, we can conclude that $G'$ and $H'$ are positive on and only on $(-5, -9/2), (5/2, 3)$.

Finally, applying the Sturm function to count the real zeros of $F$ on the above two open intervals, one finds that their numbers are 0 and 1 respectively. In conclusion, the original semi-algebraic system has only one real solution.

2.3. General Method

In this section, we formulate the steps of the illustrative example to a general method for counting equilibria of semi-algebraic economies without parameters. As we have explained, the problem may be reduced to counting distinct real solutions of (1), where the parameters $u$ are not present. Since an inequality constraint like $P \geq 0$ can be split into $P = 0$ or $P > 0$, we only need to consider semi-algebraic systems of the form

$$
\begin{align*}
F_1(x) &= 0, \ldots, F_n(x) = 0, \\
N_1(x) &\neq 0, \ldots, N_s(x) \neq 0, \\
P_1(x) &> 0, \ldots, P_t(x) > 0.
\end{align*}
$$

(10)

Let $F = \{F_1, \ldots, F_n\}$ and $N = \{N_1, \ldots, N_s\}$ and the variables be ordered as $x_1 < \cdots < x_n$. The method that provides an effective and complete solution to the problem consists of the following steps.

**STEP A1.** Decompose the polynomial system $[F, N]$ into finitely many triangular systems $[T_1, S_1], \ldots, [T_k, S_k]$ such that

$$
\text{Zero}(F/N) = \bigcup_{i=1}^k \text{Zero}(T_i/S_i) \quad \text{and} \quad \text{Zero}(T_i/S_i) \cap \text{Zero}(T_j/S_j) = \emptyset, \ i \neq j.
$$

We may assume that all $T_i$ are quasi-linear, for otherwise $T_i$ may be made quasi-linear by the quasi-linearization process using appropriate linear transformations (see Theorem 2.2 and remarks thereafter). Then the problem is reduced to counting the distinct real zeros in each $\text{Zero}(T_i/S_i)$ which satisfy $P_1 > 0$.

**STEP A2.** For each $i$, let

$$
T_i = [T_{i1}(x_1), \ldots, T_{in}(x_1, \ldots, x_n)]
$$

and $S_i$ be the product of all polynomials in $S_i$. Solve $T_{ij} = 0$ for $x_j, j = n, \ldots, 2$, and substitute the solutions successively into $S_i$ and $P_1$ to obtain rational functions $A_i/A'_i$ and $B_{il}/B'_{il}$ respectively, where $A_i, A'_i, B_{il}, B'_{il}$ are all univariate polynomials in $x_1$. The
problem is further reduced to counting the distinct real solutions of the semi-algebraic system
\[
\begin{cases}
T_{i1} = 0, \\
A_i \neq 0, \\
B_{il}^* = B_{il} \cdot B_{il}' > 0, \ l = 1, \ldots, t,
\end{cases}
\]
in one variable $x_1$ for $i = 1, \ldots, k$.

**STEP A3.** Simplify system (11), for example, by removing from $T_{i1}$ its common factors with every $A_i$ and $B_{il}^*$ to obtain $T_{i1}'$ and then replacing each $B_{il}^*$ with $C_{il} = \text{rem}(B_{il}, T_{i1}')$. In this way, we arrive at the system
\[
T_{i1}' = 0, \quad C_{il} > 0, \ l = 1, \ldots, t,
\]
which is equivalent to (11) for $i = 1, \ldots, k$.

**STEP A4.** Isolate the real zeros of each $C_{il}$ in (12) using, e.g., the modified Uspensky algorithm [27] to obtain a sequence of closed intervals $[a_1, b_1], \ldots, [a_m, b_m]$, such that
- $a_i, b_i$ are all rational numbers,
- $a_1 \leq b_1 < a_2 < b_2 < \cdots < a_m \leq b_m$,
- $[a_i, b_i] \cap [a_j, b_j] = \emptyset$ for $i \neq j$,
- each $[a_i, b_i]$ contains one and only one real zero of some $C_{il}$,
- every $[a_i, b_i]$ covers no real zero of $T_{i1}'$.

**STEP A5.** In each connected subset of the complement $(-\infty, a_1) \cup (b_1, a_2) \cup \cdots \cup (b_m, +\infty)$ of the above intervals, the sign of each $C_{il}$ is invariant and can be determined by computing the value of $C_{il}$ at a sample point in the subset. From the complement, select open intervals on which all the $C_{il}$ are positive. Finally, apply Sturm’s theorem to count the real zeros of $T_{i1}'$ on those selected intervals, and sum them up.

The correctness of the above method is quite obvious. Although the method is effective and may be easily understood and implemented, the process of quasi-linearization is time-consuming, in particular for systems with polynomials of high degree. Xia and Hou [28] proposed a direct method, which has the same functionality as ours, but does not need to make triangular sets quasi-linear. The method of Xia and Hou is also based on triangular decomposition and it works by recursively computing the so-called near roots of polynomials in triangular sets.

3. Economies with Parameters

If parameters appear in system (1), the number of real solutions of the system may change along with the variation of parameters. In this case, the method presented in the previous section cannot be directly applied to the analysis of equilibria. The problem of our concern is formulated as follows.
Problem B. Assume that the parameters $u$ are present in system (1). For any given non-negative integer $k$, determine the condition on $u$ for system (1) to have exactly $k$ distinct real solutions.

This is the problem of real solution classification for (1). We show how to solve the problem by first using triangular decomposition with quasi-linearization to reduce (1) to semi-algebraic systems in a single variable with parameters and then determining the numbers of distinct real solutions of such systems at sample points in regions of the parameter space decomposed by the border polynomials of the semi-algebraic systems in one variable.

3.1. Preliminaries

The main purpose of this section is to define the border polynomial of a semi-algebraic system in one variable. We first introduce some notations.

Let $F = \sum_{i=0}^{m} a_i x^i$, $G = \sum_{j=0}^{l} b_j x^j$ be two univariate polynomials in $x$ with coefficients $a_i, b_j$ in the field $\mathbb{C}$ of complex numbers, and $a_m, b_l \neq 0$. The determinant

$$\begin{vmatrix} a_m & a_{m-1} & \cdots & a_0 \\ \vdots & \vdots & \ddots & \vdots \\ b_l & b_{l-1} & \cdots & b_0 \\ \vdots & \vdots & \ddots & \vdots \\ b_l & b_{l-1} & \cdots & b_0 \end{vmatrix}$$

is called the Sylvester resultant (or simply resultant) of $F$ and $G$, and denoted by $\text{res}(F, G)$. The following lemma reveals the relation between the common zeros and the resultant of two polynomials.

**Lemma 3.1** ([29]). Two univariate polynomials $F$ and $G$ have common zeros in $\mathbb{C}$ if and only if $\text{res}(F, G) = 0$.

Let $dF/dx$ denote the derivative of $F$ w.r.t. $x$. The resultant of $F$ and $dF/dx$, $\text{res}(F, dF/dx)$, is called the discriminant of $F$ and denoted by $\text{discr}(F)$. The following proposition may be easily proved by definition.

**Proposition 3.2** ([29]). A univariate polynomial $F$ has multiple zeros in $\mathbb{C}$ if and only if $\text{discr}(F) = 0$.

Denote by $\text{NZero}(\ast)$ the number of distinct real zeros or solutions of $\ast$, where $\ast$ may be a polynomial, a polynomial set, or a semi-algebraic system. Consider the following semi-algebraic system in $x$ with parameters $u$:

$$S = \begin{cases} P(u, x) = 0, \\ Q_1(u, x) > 0, \ldots, Q_s(u, x) > 0, \end{cases}$$
where \( P(u, x) = \sum_{i=0}^m a_i(u) x^i \). It is easy to see that \( \text{NZero}(P) \) may change when the leading coefficient \( a_m(u) \) or the discriminant \( \text{discr}(P) \) goes from non-zero to zero and vice versa. Moreover, if \( \text{res}(P, Q_i) \) goes across zero, then the zeros of \( P \) will pass through the boundaries of the intervals determined by \( Q_i > 0 \), which means that \( \text{NZero}(S) \) may change. This motivates the following definition.

**Definition 3.1 (Border Polynomial).** The product
\[
a_m(u) \cdot \text{discr}(P) \cdot \prod_{i=1}^s \text{res}(P, Q_i)
\]
is called the *border polynomial* of \( S \) and denoted by \( \text{BP}(S) \).

Based on the above discussions, the proof of the following theorem is obvious.

**Theorem 3.3.** Let \( A, B \) be two points in the space of parameters \( u \), which do not annihilate \( \text{BP}(S) \). If there exists a real path \( C \) from \( A \) to \( B \) such that \( C \cap \text{Zero}(\text{BP}(S)) = \emptyset \), then \( \text{NZero}(S|A) = \text{NZero}(S|B) \).

### 3.2. Illustrative Example

Now we use an example to explain our general approach for solving Problem B. Consider the semi-algebraic system
\[
\begin{aligned}
  x^3 - uy^2 &= 0, \\
y^2 - 2x - 1 &= 0, \\
x - y &\neq 0, \\
y + s &> 0,
\end{aligned}
\]
where \( s, u \in \mathbb{R} \) are parameters. The following four steps permit us to decompose the parameter space into regions such that on each of them the number of distinct real solutions of \((13)\) is invariant and computable.

**Step B1.** Decomposing \( P = [x^3 - uy^2, y^2 - 2x - 1] \) under the variable ordering \( u < x < y \), we obtain three regular systems \([T_1, \{S_1\}], [T_2, \emptyset], [T_3, \emptyset]\) with
\[
\begin{aligned}
  T_1 &= [-x^3 + 2ux + u, -y^2 + 2x + 1], \\
  S_1 &= u(32u - 27), \\
  T_2 &= [u, x, y^2 - 1], \\
  T_3 &= [32u - 27, 8x^2 - 6x - 9, -y^2 + 2x + 1].
\end{aligned}
\]

It follows that for any given values \( \sigma, \tau \) of the parameters \( s, u \):

- if \( S_1|_{(\sigma, \tau)} \neq 0 \), then \( \text{Zero}(P|_{(\sigma, \tau)}) = \text{Zero}(T_1|_{(\sigma, \tau)}) \);
- if \( \sigma = 0 \), then \( \text{Zero}(P|_{(0, \tau)}) = \text{Zero}([x, y^2 - 1]) \);
- if \( 32\tau - 27 = 0 \), then \( \text{Zero}(P|_{(0, \tau)}) = \text{Zero}([8x^2 - 6x - 9, -y^2 + 2x + 1]) \),
where $S_1|_{(s,u)}, P|_{(s,u)},$ and $T_1|_{(s,u)}$ denote the results of $S_1, P,$ and $T_1$ after substitution of $(s, u)$ by $(s, u)$ respectively.

We consider only the main partition $\{(s, u) \mid S_1 \neq 0\}$ of the parameter space, which is of the same dimension as $\mathbb{R}^2$. Triangular systems corresponding to main partitions are called main branches of the triangular decomposition.

Under the same variable ordering $u < x < y$, $T_1|_{x=x+y}$ may be decomposed into four simple systems. The only main branch is $[[T_1, T_2], \{S_2\}]$ with

\[
T_1 = x^6 - (4u + 3)x^4 - 18ux^3 + (4u^2 - 26u + 3)x^2 \\
+ (4u^2 - 14u)x + u^2 - 2u - 1,
\]

\[T_2 = Iy + J,\]

\[I = -3x^2 - 8x + 2u - 5,\]

\[J = -x^3 - 6x^2 + (2u - 7)x + u - 2,\]

\[S_2 = u(32u^2 - 67u + 64).
\]

**Step B2.** Solving $T_2 = 0$ for $y$ yields $y = -J/I$. Substituting this solution into the inequality constraint $y + s > 0$ in (13), we obtain $-J/I + s > 0$, which is equivalent to $P = (-J + Is)I > 0$. On the other hand, the linear transformation $x \rightarrow x + y$ above transforms the constraint $x - y \neq 0$ into $x \neq 0$. Thus, when $S_1 \neq 0$ and $S_2 \neq 0$, the number of real solutions of system (13) is the same as that of

\[
\begin{cases}
T_1 = 0, \\
x \neq 0, \\
P > 0.
\end{cases}
\]

By Lemma 3.1 $T_1$ and $x$ have no common zero if $\text{res}(T_1, x) = u^2 - 2u - 1 \neq 0$. Hence, in the case when $S_1S_2(u^2 - 2u - 1) \neq 0$, the problem is reduced to that of real solution classification for the following semi-algebraic system in one variable $x$:

\[
U = \begin{cases}
T_1 = 0, \\
P > 0.
\end{cases}
\]

**Step B3.** The border polynomial of $U$ is

\[
\text{BP}(U) = 64u^{10}(32u - 27)^2(32u^2 - 67u + 64)^6(s^6 - 3s^4 - 8us^2 + 3s^2 - 1),
\]

whose zero set (consisting of algebraic curves) divide the parameter space $\mathbb{R}^2$ into 9 separated regions (see Figure 1). By Theorem 3.3 for all the points in each region, $\text{NZero}(U)$ is invariant. We choose 9 sample points

\[
A_1 = (-1, -1), \ A_2 = (0, -1), \ A_3 = (1, -1), \ A_4 = (-2, 1/2), \\
A_5 = (0, 1/2), \ A_6 = (2, 1/2), \ A_7 = (-3, 1), \ A_8 = (0, 1), \ A_9 = (3, 1)
\]

as shown in Figure 1. Let $A_i$ also denote the corresponding region of the parameter space.

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Figure 1: Partitions of the parameter space and sample points

**Step B4.** The number of real solutions of $\mathbb{U}$ for each region $A_i$ can be determined by counting $\text{NZero}(\mathbb{U}|_{A_i})$. For example, the result of $\mathbb{U}$ specialized at the sample point $A_1$ is

$$\mathbb{U}|_{A_1} = \begin{cases} 
    x^6 + x^4 + 18 x^3 + 33 x^2 + 18 x + 2 = 0, \\
    (x^3 + 9 x^2 + 17 x + 10)(-3 x^2 - 8 x - 7) > 0.
\end{cases}$$

Using the approach presented in Section 2.3, one can verify that the above parameter-free system has no real solution. So $\mathbb{U}$ has no real solution in region $A_1$ (without its border). In other regions $A_2, \ldots, A_9$, the numbers of distinct real solutions of $\mathbb{U}$ can be similarly computed; they are 1, 2, 0, 1, 2, 0, 1, 2 respectively.

Thus, provided that $S_1 S_2 (u^2 - 2 u - 1) \neq 0$ and $\text{BP}(\mathbb{U}) \neq 0$, or simply

$$N = u(32 u - 27)(u^2 - 2 u - 1) R \neq 0,$$

where $R = s^6 - 3 s^4 - 8 u s^2 + 3 s^2 - 1$, the number of distinct real solutions of system (13) is

- 0 if and only if $R < 0$ and $s < 0$ (i.e., $(s, u) \in A_1 \cup A_4 \cup A_7$);
- 1 if and only if $R > 0$ (i.e., $(s, u) \in A_2 \cup A_5 \cup A_8$);
- 2 if and only if $R < 0$ and $s > 0$ (i.e., $(s, u) \in A_3 \cup A_6 \cup A_9$).

In the above result, the additional polynomial $s$ is a must because the left $(A_1, A_4, A_7)$ and right $(A_3, A_6, A_9)$ regions cannot be distinguished by using the sign of $R$ only. For this simple example, $s$ can be easily observed from Figure 1. However, in general it is challenging to find polynomials to distinguish different regions described by the same inequality. It is pointed out by Yang and others [10] that such polynomials are contained in the so-called generalized discriminant list and can be picked out by repeated trials.
For the case when \( N = 0 \), one may add the equation to (13) and apply the above approach similarly. The difference is that only one of \( u, s \) is now viewed as parameter. Repeating the process, one can finally obtain the real solution classification of system (13) for all the points in the parameter space.

3.3. General Method

The problem of analyzing equilibria of an economy with parameters can be reduced to that of real solution classification of the following semi-algebraic system

\[
\begin{align*}
F_1(u, x) &= 0, \ldots, F_n(u, x) = 0, \\
N_1(u, x) &\neq 0, \ldots, N_m(u, x) \neq 0, \\
P_1(u, x) &> 0, \ldots, P_s(u, x) > 0, \\
P_{s+1}(u, x) &\geq 0, \ldots, P_{s+t}(u, x) \geq 0.
\end{align*}
\]

Let \( F = [F_1, \ldots, F_n], N = [N_1, \ldots, N_m] \), and the parameters and variables be ordered as \( u_1 < \cdots < u_d < x_1 < \cdots < x_n \). Our general method for solving the problem of real solution classification of (14) consists of the following main steps.

**STEP B1.** Decompose the polynomial system \([F, N]\) into finitely many regular systems \([T_1, S_1], \ldots, [T_k, S_k]\) such that

\[
\text{Zero}(F/N) = \bigcup_{i=1}^k \text{Zero}(T_i/S_i),
\]

where the zero sets of different main branches do not intersect with each other. Without loss of generality, suppose that the first \( r \) regular systems \([T_1, S_1], \ldots, [T_r, S_r]\) are the main branches. We may also suppose that \( T_1, \ldots, T_r \) are all quasi-linear, for otherwise they may be made quasi-linear by quasi-linearization using appropriate linear transformations.

**STEP B2.** For each \( i = 1, \ldots, r \), let

\[
T_i = [T_{i1}(u, x_1), \ldots, T_{in}(u, x_1, \ldots, x_n)]
\]

and \( S_i \) be the product of all the polynomials (in \( u \)) in \( S_i \). Solve \( T_{ij} = 0 \) for \( x_j, j = n, \ldots, 2 \), and substitute the solutions successively into \( P_l, l = 1, \ldots, s + t \), to obtain rational functions \( A_{il}/A_{il}' \) respectively, where \( A_{il}, A_{il}' \) are all polynomials in \( x_1 \) with parameters \( u \). Then under the assumption that the parameters \( u \) satisfy \( S_i \neq 0 \) and \( \text{res}(A_{il}, T_{i1}) \neq 0 \), the problem is further reduced to that of real solution classification of the semi-algebraic system

\[
U_i = \begin{cases} 
T_{i1} = 0, \\
A_{il}/A_{il}' > 0, & l = 1, \ldots, s + t.
\end{cases}
\]

**STEP B3.** For each \( i = 1, \ldots, r \), construct the border polynomial \( BP(U_i) \), whose real zero set decomposes the parameter space into separated regions. By Theorem 3.3, \( N\text{Zero}(U_i) \) is invariant in any region. Choose a sample point from each region (which can be done automatically by using, e.g., the method of partial cylindrical algebraic decomposition (PCAD) [30]).
STEP B4. For each region, determine $\text{NZero}(U_i)$ by counting the number of distinct real solutions of $U_i$ at the sample point. Finally, combining the computed results, we obtain the necessary and sufficient conditions on $u$ for system (14) to have any given number of distinct real solution, provided that

$$S_i \cdot \text{BP}(U_i) \prod_{l=1}^{s+t} \text{res}(A_{il}, T_{il}) \neq 0, \quad i = 1, \ldots, r.$$ 

STEP B5. Determine the numbers of distinct real solutions of the semi-algebraic systems corresponding to the regular systems $[T_{r+1}, S_{r+1}], \ldots, [T_k, S_k]$ similarly by regarding some of the parameters as variables. Treat each of the cases in which $S_i = 0$, or $\text{BP}(U_i) = 0$, or $\text{res}(A_{il}, T_{il}) = 0$ for $l = 1, \ldots, s + t$ and $i = 1, \ldots, r$ by adding the equality constraint to the original semi-algebraic system and by taking one of the parameters as variable.

The correctness of the above method is guaranteed by Theorem 3.3, and the termination is obvious. Yang and others [10] proposed a more direct method for real solution classification of semi-algebraic systems with parameters. Their method avoids the process of quasi-linearization of triangular sets, which is costly when the degrees of the involved polynomials are high.

4. Experimental Results

4.1. Arms Race Game with Cheap Talk

The arms race game is originally proposed by Baliga and Sjöström [31]. In this game, two players simultaneously and independently choose between building new weapons ($B$) and not building new weapons ($N$). The payoffs of the $i$th player are described as follows:

|       | $B$ | $N$ |
|-------|-----|-----|
| $B$   | $-c_i$ | $m-c_i$ |
| $N$   | $-d$   | 0   |

In this table, $c_i > 0$ is the cost of acquiring new weapons, $m > 0$ represents the gain of a player who chooses $B$ while his or her enemy chooses $N$, and $d > 0$ is the loss of a player when he or she chooses $N$ and his or her opponent chooses $B$.

The cost $c_i$ is the private information of player $i$, called the type of player $i$. Let each $c_i$ be independent and identically distributed (i.i.d.) with a continuous cumulative distribution function $F$, which has compact support $[0, \overline{c}]$ with $\overline{c} < d$. In addition, $F$ satisfies $F(0) = 0$, $F(\overline{c}) = 1$, and $F'(c) > 0$ for $0 < c < \overline{c}$.

To avoid the outcome of arms race, Baliga and Sjöström introduced the mechanism of cheap talk to the game, which consists of three stages. In stage zero, nature chooses the types $c_1$ and $c_2$. In stage one, each player simultaneously announces a message, conciliatory or aggressive. What the players plan to do in the future may be read from their messages. In the final stage, the two players make their decisions ($B$ or $N$) simultaneously according to the received messages.

The following lemma provides a basis for the analysis of equilibria (see the original paper [31] by Baliga and Sjöström for its proof).
Lemma 4.1. Suppose that $F(c)d \geq c$ for all $c \in [0, \tau]$. For any sufficiently small $m > 0$, there exists a triple $(c_L, c_*, c_H)$ such that

\[
\begin{align*}
1 & > c_H > c_* > c_L > m > 0, \\
d & > 0.
\end{align*}
\]

Moreover, if $m \to 0$, then $c_H \to 0$.

Informally speaking, the value of $c_*$ represents the cut-off where a player is indifferent between $B$ and $N$ in stage one if both players send a conciliatory message. The value of $c_L$ and $c_H$ indicate the critical point of the type space where a player changes its message in stage one. Therefore, if both players have a type exceeding $c_H$, then both of them send a conciliatory message in stage one and play $N$ in stage two.

One can observe that if $c_H$ tends to 0, then the probability of a player with type greater than $c_H$ tends to 1. By Lemma 4.1, the arms race may be avoided with high probability if the parameter $m$ is sufficiently small. This is the main result of [31]. We restate it as the following theorem.

Theorem 4.2. Suppose that $F(c)d \geq c$ for all $c \in [0, \tau]$. Then for any $\delta > 0$, there is an $m > 0$ such that for any $m$ $(0 < m < \overline{m})$, the arms race game with cheap talk has a perfect Bayesian equilibrium, where $N$ is played with probability at least $1 - \delta$.

Baliga and Sjöström also pointed out that if $m$ is small enough, then there may exist another equilibrium with cut-off $(c_L, c_*, c_H)$ satisfying (15). For this equilibrium, $c_L \to 0$ and $c_H \to c_M$ as $m \to 0$, where $c_M$ is determined by $F(c_M) = 1/2$. No statement is made in [31] about whether there are other equilibria.

In order to use the computational techniques presented in this paper for the analysis of equilibria, $F(c)$ must be replaced by a concrete polynomial function. For the convenience of comparison, we use the same setting as Kubler and Schmedders [4], i.e., $F(c) = c$ and $\tau = 1$. Then the conditions which the cut-off $(c_L, c_*, c_H)$ satisfies are reduced to a semi-algebraic system

\[
\begin{align*}
P_1 & = (c_H - c_L)c_L - (1 - c_H)m = 0, \\
P_2 & = (1 - 2c_H + 2c_L)c_H - c_Ld = 0, \\
P_3 & = (1 - c_H)(m - c_*) - c_Lc_* + c_Ld = 0, \\
d & > 0,
\end{align*}
\]

where $m, d$ are parameters.
4.1.1. Analyzing Equilibria Using Triangular Decomposition

Decomposing the set of the equation polynomials $P_1, P_2, P_3$ in (16) into regular systems under the variable ordering $d < m < c_L < c_* < c_H$, one may obtain 8 branches

$$[T_1, S_1] = [[T_1, T_2, T_3], \{m, m + 1, d - 2 m - 1\}],$$
$$[T_2, S_2] = [[m, c_L, c_* - 2 c_H^2 - c_L], \{d + 1\}],$$
$$[T_3, S_3] = [[d - 1, m, c_* - c_L, c_H - c_L], \{c_L\}],$$
$$[T_4, S_4] = [[m + 1, c_L - 1, d e^2 - 2 c_L^2 - 3 d e_* - c_* + 2 d^2 - 1, \{c_{c_L} + c_H - 2 c_* + d - 1\}, \{d + 2, d + 1\}],$$
$$[T_5, S_5] = [[d - 2 m - 1, 2 d e^2 + c_L^2 + d - 2 m - 1, \{d - 1, d + 1, 2 d + 1\}],$$
$$[T_6, S_6] = [[d + 2, m + 1, c_L - 1, 5 c_* + 7, 2 c_H + 1], \{0\}],$$
$$[T_7, S_7] = [[d + 1, m^2 + m, c_L + m, c_* - m, 2 c_H + 2 m c_H - c_H + m], \{0\}],$$
$$[T_8, S_8] = [[2 d + 1, 4 m + 3, 7 c_L - 6, 14 c_* + 9, 7 c_H + 1], \{0\}],$$

with

$$T_1 = (d - 2 m - 1) c_1 + (2 m d + m) c_1^2 + (d m^2 - 2 m^2 - m) c_L + m^2,$$
$$T_2 = (-m - 1) c_* - m c_L + d e^2 + d m + m,$$
$$T_3 = (-c_L - m) c_H + c_L^2 + m.$$

The main branch $[T_1, S_1]$ is of our concern: if $d, m$ are specialized such that $m \neq 0$, $m + 1 \neq 0$ and $d - 2 m - 1 \neq 0$, then the zero set of $\{P_1, P_2, P_3\}$ is the same as that of $T_1 = [T_1, T_2, T_3]$. One can see that $T_1$ is quasi-linear, so the number of real zeros of $T_1$ equals to that of $T_1$.

Let $P = a_m x^m + \cdots + a_1 x + a_0$ be a polynomial in $x$ with $a_i \in \mathbb{R}$ and $a_m \neq 0$. By Descartes’ rule [26], the number of sign changes of the coefficient sequence $a_m, \ldots, a_0$ gives an upper bound for the number of real positive zeros of $P$. The coefficient sequence of $T_1$ is

$$d - 2 m - 1, 2 m d + m, m^2 d - 2 m^2 - m, m^2.$$

The zero set of the polynomials in this sequence decomposes the parameter space into regions as shown in Figure 2. Since both $m$ and $d$ are required to be positive real numbers, only the first quadrant of the parameter space need be considered.

In any given region, the number of sign changes of the coefficient sequence is constant. It is of interest mainly in the case when $m$ is sufficiently small (i.e., the attraction of building new weapons is not big). For our setting, the pre-condition $F(c) \geq c$ in Lemma 11 and Theorem 12 is reduced to $d \geq 1$. Here we only consider region $A$ (see Figure 3) of the parameter space, which can be described by

$$d - 2 m - 1 > 0, 2 m d + m > 0, m^2 d - 2 m^2 - m < 0, m^2 > 0.$$

The number of sign changes of $T_1$’s coefficient sequence in region $A$ is 2, which is an upper bound for the number of positive real solutions of system 10. Therefore, there exist at most two equilibria under our setting if $m$ is sufficiently small and $d \geq 1$ (or
This demonstrates that the result of Baliga and Sjöström on the number of equilibria in the arms-race game is complete.

4.1.2. Analyzing Equilibria Using Real Solution Classification

By the method described in Section 3, the original system (16) can be reduced to a semi-algebraic system in one variable. The squarefree part of the border polynomial of the reduced system is

\[ B = dm(d - 1)(m + 1)(2d - m - 1)(d - 2m - 1)R_1, \]

where

\[ R_1 = 8d^3m^2 - 48d^2m^2 + 96dm^2 - 64m^2 - 71d^2m + 104dm - 32m + 4d - 4. \]

Note that \( R_1 > 0 \) corresponds to two different regions as shown in Figure 3 (with red color). Another polynomial \( R_2 \) is needed for distinguishing the two regions, where

\[ R_2 = 16d^4m^4 - 64dm^4 + 64m^4 + 32d^3m^3 - 20d^2m^3 - 78dm^3 + 64m^3 + 16d^4m^2 - 36d^3m^2 + 144d^2m^2 - 240dm^2 + 116m^2 + 3d^4m - 100d^3m + 247d^2m - 206dm + 56m - 8d^3 + 24d^2 - 24d + 8. \]

It is easy to see from Figure 3 that in the first quadrant \( R_2 < 0 \) and \( R_2 > 0 \) contain the left and right part of \( R_1 > 0 \) respectively.

In summary, provided that \( B \neq 0 \) we can classify the number of equilibria as follows:

- 1 equilibrium if and only if \( d - 1 < 0, 2d - m - 1 > 0, R_1 < 0 \);
• 2 equilibria if and only if \( d - 1 > 0, R_1 > 0, R_2 < 0; \)
• 3 equilibria if and only if \( d - 1 < 0, R_1 > 0. \)

Assuming that \( d \geq 1 \) and \( m \) is sufficiently small, Baliga and Sjöström showed the existence of two possible equilibria. The above result confirms the conclusion of Baliga and Sjöström. Moreover, we can assert that there is no other equilibrium if \( m \) is small enough and \( d > 1 \) under our setting.

More subtly, we can further compute the region in which there is only one equilibrium such that \( c_H \) is less than a given small number \( a \). We only need to add the inequality \( c_H < a \) into system (16) and similarly compute the region in which real solutions exist. Figure 4 shows the cases in which \( a \) equals to \( 1/10, 1/20, \) and \( 1/30 \). One can see that the corresponding region shrinks to the \( d \)-axis as \( a \to 0 \). This confirms the main result of Baliga and Sjöström, i.e., Theorem 4.2 in [31]. Moreover, the pre-condition \( F(c) d \geq c \) (\( d \geq 1 \)) is crucial. For our setting, if \( d \geq 1 \) is not satisfied, it can be observed from Figure 4 that no equilibrium with a given small \( c_H \) exists even if \( m \) is close enough to 0, which means that the arms race cannot be avoided with large probability.

4.2. Exchange Economy with Quadratic Utility

Consider the Arrow–Debreu exchange model with 2 agents and 2 commodities, studied first in [3]. Let \( u_{hl} \) denote the utility functions for agent \( h \) and commodity \( l \), where

\[
\begin{align*}
u_{11}(c) &= 9c - 1/2c^2, \quad u_{12}(c) = 29/4c - 7/16c^2, \\
u_{21}(c) &= 116c - 13c^2, \quad u_{22}(c) = 24c - 2c^2.
\end{align*}
\]

Suppose that the endowments of the two agents are \((e_1, 0)\) and \((0, e_2)\) respectively, where \( e_1, e_2 \) are parameters of the economy. In order to obtain computational results in reasonable time, we restrict the values of the endowments by \( 0 < e_h \leq 10 \).
Figure 4: Region in which one equilibrium exists with \( c_H < a \)

Let \( p_1 \) and \( p_2 \) be the prices of the two goods. Moreover, \((c_{11}, c_{12})\) and \((c_{21}, c_{22})\) represent the allocations of the commodities. Then \((p_1, p_2, c_{11}, c_{12}, c_{21}, c_{22})\) is a competitive equilibrium of the economy if the allocations maximize the above utilities under the conditions

\[
\begin{align*}
p_1 c_{11} + p_2 c_{12} &\leq p_1 e_1, \\
p_1 c_{21} + p_2 c_{22} &\leq p_2 e_2, \\
c_{11} + c_{21} &\leq e_1, \\
c_{12} + c_{22} &\leq e_2. 
\end{align*}
\]

An interior Walrasian equilibrium is a solution \((p_1, p_2, c_{11}, c_{12}, c_{21}, c_{22}, \lambda_1, \lambda_2)\) of the semi-algebraic system

\[
\begin{align*}
u_{11}'(c_{11}) - \lambda_1 p_1 &= 0, \\
u_{12}'(c_{12}) - \lambda_1 p_2 &= 0, \\
u_{21}'(c_{21}) - \lambda_2 p_1 &= 0, \\
u_{22}'(c_{22}) - \lambda_2 p_2 &= 0, \\
p_1 c_{11} + p_2 c_{12} - p_1 e_1 &= 0, \\
p_1 c_{21} + p_2 c_{22} - p_2 e_2 &= 0, \\
c_{11} + c_{21} - e_1 &= 0, \\
p_1 + p_2 - 1 &= 0, \\
p_1 > 0, p_2 > 0, \lambda_1 > 0, \lambda_2 > 0, \\
c_{hl} > 0, 10 \geq c_h > 0, 
\end{align*}
\]

where \( u_{hl}' \) (the derivatives of \( u_{hl} \)) are the marginal utility functions.

Fix a variable ordering, e.g., \( e_1 < e_2 < p_1 < p_2 < c_{11} < c_{12} < c_{21} < c_{22} < \lambda_1 < \lambda_2 \), and decompose the set of equation polynomials into regular systems. The result is
somewhat complicated and annoying for reading and thus is not reproduced here. Among the obtained regular systems, the main branch is quasi-linear and the first polynomial in the regular set is of degree 4 in $p_1$. Next compute the equivalent semi-algebraic system in one variable and construct its border polynomial. The squarefree part of the border polynomial is of degree 25 with 249 terms.

The final analytical result that can be derived is: multiple (exactly three) equilibria appear in the trade economy if and only if $R < 0$, where

$$R = 14336 e_1^4 - 2489600 e_2^3 + 3153968 e_1^2 e_2^2 - 75973600 e_1 e_2^2 + 603410000 e_2^2$$
$$- 73508800 e_1 e_2 + 1369715000 e_1 e_2 - 8810812500 e_2 + 106496 e_1^4$$
$$- 12416000 e_1^3 + 925640000 e_1^2 - 13045500000 e_1 + 60315234375,$$

provided that the border polynomial is not annihilated.

For any given values $e_1, e_2 \in (0, 10]$, the multiplicity of equilibria can be easily obtained by determining the sign of $R(e_1, e_2)$. For example, Kubler and Schmedders [3] showed that the economy has 3 equilibria when $e_1 = 10, e_2 = 10$. This result can be confirmed by using our approach since $R(10, 10) = -11390625$.

Figure 5 shows the region of the parameter space described by $R < 0, 0 < e_1, e_2 \leq 10$. It may be observed that the possibility for the existence of multiple equilibria is very small. Furthermore, one can see that multiple equilibria may not appear when either of the endowment parameters $e_1, e_2$ is sufficiently small.

5. Conclusion

Determining the existence of multiple equilibria is an important issue in both theoretical and practical studies of economic models. Equilibria of semi-algebraic economies may
be characterized by semi-algebraic systems. We have proposed an approach that allows systematic identification of the multiplicities of equilibria of semi-algebraic economies.

Two problems of identifying multiplicities of equilibria, for semi-algebraic economies without or with parameters, are addressed. The basic idea of solving the problems is to first transform the underlying semi-algebraic systems in several variables into those in a single variable and then analyze the real solutions of the resulting systems. The methods we have presented are different from those based on numerical computation. They can be used to establish exact and rigorous results and thus are more adequate for the theoretical study of economic models.

Compared to the method of Kubler and Schmedders [3, 4], ours can better handle models with inequality constraints, which are fairly prevalent in practice. Moreover, for parametric economies, necessary and sufficient conditions for the existence of multiple equilibria can be automatically generated and explicitly given by using our methods.

A shortcoming of our methods is their low efficiency for large problems. We expect to work out specialized and efficient techniques (e.g., by combining triangular decomposition with PCAD) to improve the performance of our methods for certain classes of large economic models. It is hoped that the approach introduced in this paper can be refined, extended, and further developed to become a potentially powerful alternative or complement to the widely used numerical approaches for computational economics.

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