The energetic efficiencies of rocked ratchets reported in the literature typically lie in the sub-percent range. We discuss the problem of optimization of the energetic efficiency of a ratchet, and show that considerably higher efficiencies can be achieved; however this assumes a fine-tuning of the parameters of the system. The domain of parameters corresponding to high efficiencies is typically narrow.

I.M. Sokolov
Institut für Physik, Humboldt-Universität zu Berlin, Invalidenstr. 110, D-10115 Berlin, Germany

(Dated: November 13, 2018)

The energetic efficiency of vegetative ratchets, inspired by biological applications (where ratchets serve as gears of molecular motors powering cells and subcellular units, see Ref. [1] for an introductory review), stays strong for already a decade. The simplest model (over-simplified compared to any biological system, but physically still not elementary) is a so-called rocked ratchet. The model corresponds to a particle moving in a spatially asymmetric potential under the influence of an external field, either periodic or stochastic, of strong friction and of thermal noise. This model was one of the first ones discussed by physicists, and is investigated to a great detail, see Refs. [2, 3, 4, 5] for comprehensive reviews. The energetic efficiency of one of the simplest (and deepest) thermodynamic characteristics of such systems is now under extensive investigation, see Ref. [6] for a review. Even if this efficiency is not a crucial parameter in biological systems or in nanomechanical appliances, it is still of primary importance since it determines the heat production under operation and thus the overall heat regime. The energetic efficiency of a rocked ratchet is notoriously low: The Refs. [7, 8] discussing this issue give numerical values of the efficiencies in a sub-percent domain; Ref. [9] discussing a related discrete model presents similarly low values. On the other hand, the values of parameters of the ratchets discussed in these works are arbitrary, so that these efficiencies may be low just by chance. In what follows we discuss this issue in detail and show that rather high efficiencies can be attained; however the domain of parameters, where the efficiencies are high, is narrow. Thus, a ratchet, as a technical device, needs scrupulous optimization if the efficient performance is aimed.

Let us first discuss the parameters of the ratchet gear. A rocked ratchet is defined by a Langevin equation,

\[ \dot{x} = \mu \left[ F(x) + f(x, t) \right] + \xi(t), \]

where \( \mu \) is the mobility of the particles, and \( \xi(t) \) is a \( \delta \)-correlated Gaussian Langevin force with zero mean and with \( \langle \xi^2(t) \rangle = 2\theta \mu \), where \( \theta \) is the energetic temperature. Here \( F(x) \) is a force corresponding to the ratchet potential and \( f(x, t) = f_0 + f_1(t) \) is a sum of the external see-saw force \( f_1(t) \) with zero mean and of the constant force \( f_0 \) against which the useful work is done by pumping particles uphill. The temporal evolution of the particles’ distribution is given by a Fokker-Planck equation,

\[ \frac{\partial p(x, t)}{\partial t} = \frac{\partial}{\partial x} \left( D \frac{\partial p(x, t)}{\partial x} + \mu p(x, t) \frac{\partial}{\partial x} U(x, t) \right), \]

where \( U(x) = V(x) + f(t)x \) is the overall potential, and \( D = \theta \mu \) is the diffusion coefficient. In our work, we take \( f(x) \) to be piecewise-constant, so that the potential \( V(x) \) is a saw-tooth function:

\[ V(x) = \begin{cases} \frac{Vx}{a} & \text{for } 0 < x \leq a \\ \frac{V(L-x)}{(L-a)} & \text{for } a < x \leq L. \end{cases} \]

The mobility \( \mu \) is set to unity. Apart from the time-dependence (protocol) of the external force, which is a function, the parameters characterizing the situation are: the geometrical parameters \( L \) and \( a \), characterizing the ratchet, the amplitude of the saw-tooth potential \( V \), the energetic temperature \( \theta \), and value \( f_0 \) of a constant force against which the work is done. The geometrical parameters and \( V \) characterize the appliance itself, \( \theta \) characterizes the external conditions, and \( f_0 \) will be tuned in order to achieve the maximal efficiency under other conditions fixed. Although some parameters can be absorbed into dimensionless combinations, their overall number is still too large to allow for simple optimization. In what follows we will use the same ratchet as in Refs. [3, 4, 5], which in our notation corresponds to \( L = 1 \), \( a = 0.8 \) and \( V = 1 \). Moreover, in order to be able to compare the results, we use the same protocol of the external force, switching between the values \( f_1 \) and \( -f_1 \), and having a period \( T \). The overall force \( f(t) \) meanders in this case between the values \( f = f_0 - f_1 \) and \( f = f_0 + f_1 \). Under homogeneous in space forcing field and load force, the energetic efficiency \( \eta = W/A \) (where \( W \) is the useful work and \( A \) is the input work) is given by \[ \eta = \frac{\tilde{I}(f_0 + f_1) f_0}{\tilde{I}(f_0 + f_1) f_1}, \]

where \( \tilde{I} \) is the overall current through the system, and the mean values are taken over the period of forcing. Let us first discuss the efficiency under the adiabatic mode of the operation, just as it was done in Ref. [6].
so that $T \to \infty$. Now, only 3 parameters are left: the amplitude $f$ of the external forcing, the force $f_0$ (external load), and the temperature $\theta$. Under a very slowly changing force, the system can simply be described as a rectifier (nonlinear element) whose ”Volt-Ampere” characteristics can easily be calculated in the adiabatic approximation. The current $I(f, \theta)$ is the given by \[ \mu I^{-1} = \frac{a}{f-V/a} + \frac{L-a}{f+V/(L-a)} + \left( \exp \left( \frac{V-fa}{\theta} \right) - 1 \right) \left( \frac{f-V/(L-a)}{f-V/a} \right)^2 \left( \frac{f(V/(L-a)+V)}{f+V/(L-a)} - 1 \right)^2 \left( \frac{\exp \left( \frac{f(V/(L-a)+V)}{\theta} \right) - 1}{\exp \left( - \frac{V-fa}{\theta} \right)} \right)^{-1}. \] From this expression the limiting forms for $\theta \to \infty$ and for $\theta \to 0$ readily follow. Thus for $\theta \to \infty$ one has $I = \mu f + O(\theta^{-2})$: the nonlinearities vanish, and the rectifier doesn’t work, on the other hand, for $\theta \to 0$ the current is given by

$$I(f) = \begin{cases} 0 & \text{for } -V/(L-a) < f < V/a, \\ \mu \left( \frac{a}{f-V/a} + \frac{L-a}{f+V/(L-a)} \right)^{-1} & \text{otherwise}. \end{cases} \quad (6)$$

The interval $-V/(L-a) < f < V/a$ where the current is zero is termed the mobility gap. This expression corresponds to the current in the deterministic mode of operation, which was discussed in detail in Ref. [10]. We use this expression to plot the efficiency as a function of $f_-$ and $f_+$ in upper panel of Fig.1. Here only positive values of efficiencies are plotted; the efficiency in the regimes where the useful work is negative, is set to zero. According to Ref. [11], where the optimization of a deterministic ratchet was discussed, the maximal efficiency in the deterministic regime under symmetric forcing depends only on the ratchet’s geometry and is given by

$$\eta_{\text{sym}} = |1 - 2a/L|. \quad (7)$$

The maximum is achieved under

$$f_{0, \text{max}} = \frac{-V(L-2a)}{2a(L-a)}, \quad |f_{1, \text{max}}| = \frac{VL}{2a(L-a)}, \quad (8)$$

i.e. under $f_- = -V/(L-a)$ and $f_+ = V/a$, corresponding to the mobility thresholds.

Note that the maximal efficiency under adiabatic forcing in the deterministic regime for the ratchet considered in Ref. [11] would be $\eta_{\text{max}} = 0.6$, three orders of magnitude higher than the efficiency reported in this work even for very low temperatures.

In the middle panel of Fig.1 we plot the efficiency $\eta$ at very low energetic temperature $\theta = 0.01$, where now the

\[ \eta \to \max, \quad \text{for } \theta \to 0. \]

This expression corresponds to the current in the deterministic mode of operation, the middle panel presents the results for a thermal ratchet at $\theta = 0.01$, and the lower panel corresponds to $\theta = 0.1$. The dashed lines denote the boundary of the region in which the deterministic ratchet produces positive work.
The overall dependence of $\eta_{\text{max}}$ on $\theta$ is shown in Fig. 3 for $0 \leq \theta \leq 0.2$. We see that in the adiabatic regime it is a monotonous function of temperature, as discussed in Ref.[8]. The numerical results for higher temperatures are less reliable, since the maxima get to be very sharp. Note that for $\theta = 0.1$ the maximal efficiency under adiabatic operation is still around $\eta_{\text{max}} = 0.154$ and that, according to Fig. 2 maximal efficiency is achieved for the values of $f_-$ and $f_+$ lying near the boundary of the domain where the deterministic ratchet produces positive work.

We also note that the adiabatic ratchet works irreversibly, even under the limiting transition $a \to L$, when $\eta \to 1$ [10]. Such a ratchet, in which one side of the saw-tooth is very steep, is essentially an ideal rectifier. It performs extremely good in a broad temperature range, due to the possibility of using strongly negative $f_-$, but still loses efficiency for $\theta \geq V$. The reason of the irreversibility is easy to understand when following the particle’s path along the ratchet: for the positive overall force the particle moves infinitesimally slow when sliding along the flatter side of the saw-tooth and falls with a constant velocity along the steeper side, thus producing heat and losses.

So far we have shown that adiabatic ratchets may perform much better than supposed, provided the temperatures are low enough. In Ref.[10] it is shown, that each deterministic ratchet can also work reversibly, thus achieving the efficiency $\eta = 1$ under the mode of operation which is quasistatic but not adiabatic. Thus, the reversible regime under the deterministic mode of operation implies the synchronization of the external force and particle’s position: the external force must change its sign when the particle passes the apex of the potential, so that the particle’s velocity stays infinitesimally small. In a piecewise-linear potential, Eq.(3), and for piecewise-constant, symmetric force $f_1(t)$ of period $T$ this corresponds to such a choice of the forces that particle passes the distances $a$ and $L - a$, during the respective half-periods. The velocities thus are: $v_1 = 2a/T$ and $v_2 = 2(L - a)/T$. This gives us the optimal values of the forces

$$f_{0,\text{max}} = -\frac{V(2a - L)}{2a(L - a)} + \frac{L}{\mu T}, \quad (9)$$

$$|f_{1,\text{max}}| = \frac{V L}{2a(L - a)} + \frac{2a - L}{\mu T}.$$
and the maximal efficiency corresponding to
\[
\eta_{\text{max}} = \frac{-f_{0,\text{max}}L}{-f_{0,\text{max}}L + 2(a^2 + (L-a)^2)/\mu T}.
\] (10)

Note that the possibility of the reversible mode of operation is closely connected to the fine time-tuning of the external force, and that these high efficiencies follow as a kind of a non-linear resonance: even slight changes of temporal properties of the force (keeping the amplitudes constant) lead to a dramatic drop in efficiency. As an example let us consider our system with the values of forces which optimize the efficiency for \( T = 40 \), which are \( f_0 = -1.85, f_1 = 3.14 \) and correspond to \( \eta = 0.982 \). Taking now a period \( T = 40.04 \) (relative detuning from the “resonance” of the order of \( 10^{-3} \)) we get (through the numerical solution of Eq.(1) with \( \xi = 0 \) and numerical evaluation of \( \eta \) using Eq.(4) and time-averaging over \( 10^4 \) periods of the field) that the value of \( \eta \) drops to 0.60, only slightly higher that the value corresponding to the adiabatic operation, which for this values of the forces is \( \eta = 0.578 \). To understand the situation it is enough to examine the trajectories of the particle’s motion: even at tiny detuning trajectories develop parts corresponding to the particle’s falling down the steep potentials, which correspond to high losses. The same dramatic drops to the values which are only slightly higher than the adiabatic ones are seen when increasing the temperature (switching of \( \xi = \sqrt{2} \)).

The active synchronization (“the Humphrey Potter’s solution”), where the see-saw force is triggered when the particle actually passes the apex [10], helps to save the situation and stabilize high efficiencies: posing the “switches”, say, at distance \( \delta \) to the left from the bottom of the potential and to the right of its maximum, one can achieve the efficiency of \( \eta = 0.961 \) for \( \delta = 10^{-4} \) and \( \eta = 0.790 \) for \( \delta = 10^{-2} \) under deterministic regime, but the drops in efficiency when increasing temperature are still dramatic. The effective regimes close to reversibility seem not to survive under elevated temperatures.

We note that ratchets in a finite-time mode of operation can show the increase of efficiency with increasing temperature, Ref.[3][4]. No such regimes were found in the vicinity of the modes of operation, corresponding to high efficiencies (either reversible or irreversible). Probably, such situations are pertinent to the outer regions of the domain where the work is positive, where the overall efficiency is low.

Let us summarize our findings. The efficiencies of the rocked ratchets reported in the literature are notoriously low; however, since the parameters of the systems discussed are chosen at random, this could only mean that the domain of parameters corresponding to high efficiencies is rather narrow. We show that it is indeed the case, and that the efficiencies of the rocked ratchets at moderate temperatures can be reasonably high.

The author is indebted to Prof. P. Hänggi for useful discussions, and to the Fonds der Chemischen Industrie for partial financial support.

[1] E. Frey, ChemPhysChem, 3, 270 (2002)
[2] M.O. Magnasco, Phys. Rev. Lett. 71, 1477 (1993); ibid. 72, 2656 (1994)
[3] F. Jülicher, A. Ajdari and J. Prost, Rev. Mod. Phys. 69, 1269 (1997)
[4] P. Reimann, Phys. Rep. 361, 57 (2002)
[5] P. Reimann and P. Hänggi, Appl. Phys. A 75, 169 (2002)
[6] J.M.R. Parrondo and B.J. de Cisneros, Appl. Phys. A 75, 179 (2002)
[7] H. Kamegawa, T. Hondou and F. Takagi, Phys. Rev. Lett. 80, 5251 (1998)
[8] K. Sumithra and T. Sintes, Physica A, 297, 1 (2001)
[9] I.M. Sokolov and A. Blumen, Chem. Phys. 235, 39 (1998)
[10] I.M. Sokolov, Phys. Rev. E 63, 021107 (2001)