Symmetry properties of SU3 vector coupling coefficients

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Abstract

A presentation of the problem of calculating the vector coupling coefficients for $SU3 \supset SU2 \otimes U1$ is made, in the spirit of traditional treatments of SU2 coupling. The coefficients are defined as the overlap matrix element between product states and a coupled state with good SU3 quantum numbers. A technique for resolution of the outer degeneracy problem, based upon actions of the infinitesimal generators of SU3 is developed, which automatically produces vector coupling coefficients with symmetries under exchange of state labels which parallel the familiar symmetries of the SU2 case. An algorithm for efficient computation of these coefficients is outlined, for which an ANSI C code is available.
I. Introduction

The algebra associated with the irreducible representations (hereafter called “irreps”) of the special unitary group in two dimensions, SU2, has proven itself an invaluable tool in describing the angular momentum properties of quantum states\(^1,2,3\). In an early manifestation of group symmetry as an elementary particle classification scheme, SU2 was recognized also to be the appropriate group for formalization of the implications of charge invariance through the concept of isospin. The vector coupling coefficients, \(^4\) 3-j, 6-j, and 9-j symbols, and recoupling techniques have become part of the standard arsenal for dealing with quantum amplitudes in particle, nuclear, atomic, and molecular physics.

The group SU3 was recognized in the late 1950’s to be reflective of the symmetries inherent in the properties of the known “elementary” hadrons\(^5\). Nearly simultaneously, its utility in classification of rotational states in non-spherical nuclei (due to the fact that the three dimensional harmonic oscillator potential has SU3 symmetry) was exercised\(^6\). The introduction of the color quantum number to explain the absence of ”exotic” hadronic states in the quark model, leading to Quantum Chromodynamics which has proven successful in explaining the dynamics as well as the mass spectra of elementary particles, has made further use of SU3 as the color symmetry of QCD. Despite the subsequent discovery of additional quark flavors, due to the extreme symmetry breaking brought on by the heavy masses of the top and bottom quarks, it is still often practical to use SU3 flavor symmetry in calculations based upon quark structure of hadrons.

While the coupling and recoupling algebra of SU2 has been long known, nearly standardized, and universally utilized, that for SU3 (despite its long history) is much more poorly known, and is often underutilized in problems where it is potentially quite useful. The reasons are several.
The algebra, as one would expect, is more complicated than that for SU2. The irreducible representations are labeled by two integers (the set $p, q$ analogous to $j$ for SU2), and the basis states within an irrep are labeled by three integers ($k, l, m$ with $p + q \geq k \geq q, q \geq l \geq 0,$ and $k \geq m \geq l$ in place of $j \geq m \geq -j$ for SU2). Most of the work done on the SU3 case has been presented in a highly abstract language and notation rather unfamiliar to most physicists, and it has often been published in journals not typically seen by the majority of practitioners. The problem known as “outer multiplicity” does not occur in the SU2 case, further complicates the SU3 case, and inserts another point in the algebra where some convention must be chosen to present unambiguous results. As is typically the case, this has lead to several competing conventions, each of which has been created for the particular convenience or world view of the author. This presentation is intended to introduce a set of conventions which will be of greatest convenience to those wishing to use SU3 as a tool in the understanding of quantum state classification.

The notation used herein has been chosen to reflect the similarities of the SU3 problem to the familiar SU2 classification of angular momentum states. For this reason, for example, a fully specified SU3 state will be denoted by a ket, $|p, q; k, l, m>$ in analogy to the angular momentum state $|j; m>$ rather than the Gel’fand-Weyl notation frequently found in the mathematical physics literature

$$|p, q; k, l, m>.$$ 

The variables $p$ and $q$ distinguish the various irreducible representations of SU3, and will here be referred to as irrep labels; $k, l,$ and $m$ enumerate the basis states within the vector space acted upon by the irrep, and will be referred to as subspace labels. In elementary particle classification of states through flavor SU3, the subspace labels $k, l, m$ are related to the quantum numbers
for total isospin \((I)\), its projection \((I_z)\), and hypercharge \((Y)\), as follows:

\[
I = \frac{1}{2}(k - l) \quad (1)
\]
\[
I_z = m - \frac{1}{2}(k + l) \quad (2)
\]
\[
Y = k + l - \frac{2}{3}(p + 2q). \quad (3)
\]

In the use of SU3 to classify nuclear rotational states, a variable \(\Lambda\) and its projection \(M_\Lambda\) play roles analogous to \(I\) and \(I_z\), and the variable \(\epsilon\) corresponds to \(-3Y\). Letting \(n_i\) represent the number of harmonic oscillator quanta in the \(i\) direction,

\[
M_\Lambda = (n_1 - n_2)/2, \quad (4)
\]
\[
\epsilon = 2n_3 - n_1 - n_2. \quad (5)
\]

When it helps to clarify meaning, the elementary particle notation will be used herein, and states will appear as \(|p, q; I, I_z, Y>\).

II. Description of the problem

At the heart of the coupling/recoupling problem for a particular symmetry group is the definition of the vector coupling coefficient (for SU3, this will be at times hereafter be abbreviated VCC). Considering the direct product of two states, each classified according to SU3 quantum numbers, it will be possible to express this as a weighted sum of SU3 classified states of the composite system:

\[
|p_1, q_1; k_1, l_1, m_1 > |p_2, q_2; k_2, l_2, m_2 > = \sum_i C_i |p, q; k, l, m > . \quad (6)
\]

The weights in this sum are known as the vector coupling coefficients – they depend upon all fifteen variables describing the three states involved, and will be denoted by

\[
C_i = (p_1, q_1; k_1, l_1, m_1 : p_2, q_2; k_2, l_2, m_2|p, q; k, l, m) .
\]
A choice of phases for the SU3 states can be made which will insure that the vector coupling coefficients are real. This fact, and the orthonormality of the state kets allows the above relation to be inverted, so as to express a composite SU3 state as a sum over product states, as

\[ |p, q; k, l, m > = \sum_{k_1, l_1, m_1, k_2, l_2, m_2} (p_1, q_1; k_1, l_1, m_1 : p_2, q_2; k_2, l_2, m_2 |p, q; k, l, m) |p_1, q_1; k_1, l_1, m_1 > |p_2, q_2; k_2, l_2, m_2 > \]  

(7)

Operators which map the space of these ket states into itself can also be classified by their transformation properties under the group SU3. It can be shown that the matrix elements of such operators between SU3 states is proportional to the vector coupling coefficients. This relationship is known as the Wigner-Eckart Theorem:

\[ < p_f, q_f; k_f, l_f, m_f |T_{p_i,q_i; k,l,m}| p_i, q_i; k_i, l_i, m_i > = \]

\[ (p, q; k, l, m : p_i, q_i; k_i, l_i, m_i |p_f, q_f; k_f, l_f, m_f) < p_f, q_f \|T_{p_i,q_i}\| p_i, q_i >, \]  

(8)

where \( T \) is the SU3 classified operator, and

\[ < p, q \|T_{p_1,q_1}\| p_2, q_2 >, \]

the "reduced matrix element" is a complex number which depends only upon the irrep labels. It is possible, therefore, to define the VCC’s as matrix elements of a set of unit tensor operators. Determination of a set of unit tensor operators is then tantamount to the determination of the vector coupling coefficients.

The requirement in equation 7 that all kets be eigenstates of the appropriate group operators imposes severe restrictions on the vector coupling coefficients. The subspace labels must satisfy relations usefully thought of as
1) conservation of hypercharge \((Y_1 + Y_2 = Y)\)

\[ k_1 + l_1 - \frac{2}{3}(p_1 + 2q_1) + k_2 + l_2 - \frac{2}{3}(p_2 + 2q_2) = k + l - \frac{2}{3}(p + 2q); \]  

(9)

2) conservation of charge \((I_{1z} + I_{2z} = I_z)\)

\[ m_1 - \frac{1}{2}(k_1 + l_1) + m_2 - \frac{1}{2}(k_2 + l_2) = m - \frac{1}{2}(k + l); \]  

(10)

3) triangularity relation for isospin \((\vec{I}_1 + \vec{I}_2 = \vec{I})\)

\[ k_1 - l_1 + k_2 - l_2 \geq k - l, \]

\[ |k_1 - l_1 - k_2 + l_2| \leq k - l. \]  

(11)

As well, the ”betweeness” relations

\[ p + q \geq k \geq q , \ q \geq l \geq 0 \ , \text{ and } k \geq m \geq l \]  

(12)

must be satisfied by each set of \(k, l, m\)’s.

There are, as well, restrictions on which sets of irrep labels correspond to non-vanishing vector coupling coefficients, in analogy to the triangularity condition for angular momentum vectors for SU2, but rather more complicated. The Clebsch Gordan series is a formal expression of which irrep products contribute to a particular composite irrep. Letting \((p, q)\) represent the irreducible representation labeled by indices \(p\) and \(q\), and \(\alpha\) represent the number of times a particular composite irrep appears in the outer product of irreps \((p_1, q_1)\) and \((p_2, q_2)\),

\[ (p_1, q_1) \otimes (p_2, q_2) = \sum_{p,q} \alpha(p_1, q_1, p_2, q_2, p, q) (p, q), \]  

(13)

where

\[ \alpha = \max(\alpha'(p_1, q_1, p_2, q_2, p, q), 0), \]
\[ \alpha'(p_1, q_1, p_2, q_2, p, q) = 1 + \min(q_1, q_2, p, p_1 + q_1 - p, q_2 - p) \]

with

\[ \gamma = \frac{1}{3} \max(p_1 + p_2 - p, q_1 + q_2 - q) \]

\[ \sigma = \frac{1}{3} \min(p_1 + p_2 - p, q_1 + q_2 - q) \]

The multiplicity \( \alpha \) is equal zero for "forbidden" couplings for which all associated VCC's vanish identically, and is a positive integer in all remaining cases, signifying the possibility of non-vanishing VCC's (depending upon the values of the subspace labels). The above expression for the multiplicity results from the betweeness conditions (equation \([12]\)) for the subspace labels, and the conservation laws required by the group symmetry (equations \([9], [10], [11]\)). Other equivalent expressions for this quantity, less useful in the present context, appear in the literature \([8]\).

In the case of SU2 vector coupling coefficients, the multiplicity is either one or zero. In contrast, the multiplicity in the SU3 case can be arbitrarily large. The existence of multiplicities greater than one is the circumstance typically referred to as the outer multiplicity problem. For a coupling of degeneracy \( \alpha \), the implication is as follows: for two quantum states put in combination leading to a SU3 classified composite state \( |p, q; k, l, m> \) there are \( \alpha \) such composite states which are distinct in their physical properties specified by non-SU3 variables. In effect, there is an \( \alpha \) dimensional subspace of states with the same SU3 description, and one must go beyond the requirements of SU3 symmetry to chose \( \alpha \) distinct states in this subspace to serve as a basis for the degenerate subspace. Only then are the vector coupling coefficients determined (to within a sign.) Here, the user must chose a convention, so that the resulting
states are as free of arbitrary choice as possible, and which are as useful as possible for his or her purposes.
III. Prior solutions to the problem

An early published work on the SU3 vector coupling coefficients, which grew from the requirements of the quark model \cite{9}, gave a clear and concise description of the problem and related mathematical tools. Values for a limited set of VCC’s are presented there in tables, including cases of multiplicity two. One such case was appropriate to the problem of pion-nucleon scattering – the coupling of an 8 dimensional representation \((p = 1, q = 1)\) to another 8. The Clebsch Gordan series for this product is

\[
(1, 1) \otimes (1, 1) = (0, 0) \oplus 2 (1, 1) \oplus (3, 0) \oplus (0, 3) \oplus (2, 2).
\]

The twofold appearance of \((1, 1)\) in the product is the simplest case in SU3 where outer multiplicity appears, but evaluation of VCC’s for this case does not require a general resolution of this issue: a simple parity condition (coefficients either even or odd with respect to interchange of particles one and two) suffices to define the coefficients within a sign. The same set of coefficients for this case result from several quite distinct choices of resolution criteria, which produce very different results in other cases.

Formal work on the general problem of outer multiplicity was undertaken \cite{10,11} and the problem was formally resolved as was exhibited in a 1970 paper \cite{12} which derived results for general matrix elements of the \((p, q) = (2, 2)\) tensor operator, which admits one case of threefold multiplicity. Further development and clarification of this solution has continued \cite{13,14,15}.

A simplified summary of the agenda of Biedenharn, Hecht, Louck and collaborators for resolution of the outer degeneracy problem is illustrative. This agenda will henceforth be referred to as the \textit{canonical} resolution, adopting the language of those authors. The focus is on the determination of unit tensor operators which can be classified according to the SU3 quantum numbers \(p, q; k, l, m\) according to their behavior under the action of the group generators. The
general nature of the Clebsch Gordan series shows that such an operator has non-vanishing matrix elements between states from several different irreps. Further restriction of the operators is made to those which produce a specific set of shifts – changes in the irrep labels of a state \((p, q)\) resulting from the action of the operator thereon. In cases of non-trivial outer degeneracy, even these restricted operators produce ambiguous results, since there is more than one version of a particular set of irrep variables which results from the application of the operator on a state \((p, q)\). The complete resolution of the problem is to define in these cases a unique set of operators with the same SU3 labels which produce the same shifts, and which produce a unique single version of the shifted state.

It was shown that such a completely restricted tensor must be labeled by eight integers: \(p, q, k, l, m\) representing the SU3 transformation properties of the operator, and a further set \(\kappa, \lambda, \mu\) which properly imply the restrictions required. This triplet of integers, called the operator pattern, satisfies betweenness conditions like those for the \(k, l, m: p + q \geq \kappa, q \geq \lambda \geq 0, \kappa \geq \mu \geq \lambda\). The shifts,

\[
\Delta p = 2\mu - \kappa - \lambda \\
\Delta q = 2\kappa + 2\lambda - \mu - 2p - q
\]

indicate the effect of the operator on the irrep indices of the state upon which it operates: in terms of the operator’s reduced matrix element,

\[
< p_f, q_f | T_{p,q; k, l, m; \kappa, \lambda, \mu} | p_i, q_i >= 0
\]

except in cases when

\(p_f - p_i = \Delta p\) and \(q_f - q_i = \Delta q\).

The remaining freedom in the operator pattern labels, that is the range of values of \(\kappa\) and \(\lambda\)
which obey betweeness and have a fixed sum (thus producing always the same shifts) precisely
labels the degeneracy of the coupling of \((p, q)\) to \((p_i, q_i)\) to produce \((p_f, q_f)\).

Distinctions among operators having the same SU3 labels and shifts is made using the
concept of a *characteristic null space*. This refers to the union of all irreps \((p_i, q_i)\) which when
operated upon by the tensor operator \(T\) yield a zero result. In the case of a tensor operator
without degeneracy \((\alpha = 1)\), the null space is precisely determined by the restrictions the group
imposes upon the operator and the state upon which it operates. The only other operator which
has the same group labels and shifts, and a different null space than the previous one is the null
operator. For degenerate operators, this is not the case, and the various operators of a given
set of group labels and shifts can be identified with a set of operators with null spaces, each
larger than the previous and completely containing it. Construction of operators with precisely
these null space properties was accomplished through a build-up process using SU3 operators
without degeneracy, whose matrix elements had algebraic expressions \(^{16}\). Continuing work
to make explicit the operators and vector coupling coefficients so generated has produced a
mapping of the tensor operators onto an SU3 invariant norm, called the denominator function,
for which explicit expressions involving ratios of polynomials have been derived \(^{13,14,15}\).

A complete implementation of the canonical operator build-up scheme exists in the form of
computer codes generated by Draayer et.al. \(^{17,18,19,20}\). They exhibit that each tensor operator of
multiplicity greater than one can be built up from an operator of multiplicity one (whose matrix
elements are simply computable) by multiple products with a special set of tensor operators of
\((p, q) = (1, 1)\) with shifts \(\Delta p = \Delta q = 0\) which are the infinitesimal generators of SU3. Their
results are shown to generate precisely the canonical vector coupling coefficients of Biedenharn
et. al.. The current manifestation of this computer work is a 3600 line set of Fortran codes
which computes SU3 vector coupling coefficients (both for the SU2xU1 decomposition and the R3 decomposition) as well as 6-j and 9-j symbols constructed therefrom.

An alternative scheme for outer multiplicity, but similar in spirit to the canonical scheme just described, was recently presented \(^{21}\). It, too, relates the VCC’s to matrix elements of tensor operators, and relies heavily upon prior work on the so-called Bargmann tensors. It is shown how known Bargmann tensors can be used to build the desired tensor operators for SU3, from which the VCC’s are easily computed. A direct (non-recursive) way of computing matrix elements of the Bargmann tensors of interest is given, and from that, the VCC’s are produced using a Gramm-Schmidt orthogonalization procedure. The tensors here differ from those of the canonical scheme in their null space properties, and as a consequence, the Wigner-Eckart theorem involves a sum over the outer multiplicity label. This necessitates the Gram-Schmidt procedure to extract the VCC’s from the matrix elements of the unit tensor operators. The ordering of the tensors which is necessary to specify the orthogonalization algorithm is based upon a conjecture due to Braunschweig \(^{22}\) which is straightforwardly verified in the work presented below.

### IIIa. Symmetry properties of coefficients

The physical process which corresponds to the coupling of a particular SU3 classified quantum state to another is the observation of the two particles in combination, either as a bound state or an intermediate state of a scattering event or reaction. The probability of creation of a particular such state is proportional to the square of the associated vector coupling coefficient. The order of coupling (which physical state is associated with state ”1” and which with state ”2”) is not observable, so in order that the coupled state have an interpretation as a possible physical state, it is necessary that the vector coupling coefficients
and \((p_1, q_1; k_1, l_1, m_1 : p_2, q_2; k_2, l_2, m_2|p, q; k, l, m)\) and \((p_2, q_2; k_2, l_2, m_2 : p_1, q_1; k_1, l_1, m_1|p, q; k, l, m)\) differ by, at most, a sign. A similar symmetry regarding the interchange of the first and last set of indices would allow the definition of a proportional coefficient with at most a sign change under interchange of any of the three sets of SU3 quantum numbers – the 3-j symbol. This forms the basis for straightforward definitions of 6-j and 9-j recoupling symbols with high degrees of symmetry under interchange of indices. The SU2 vector coupling coefficients have these symmetries. As well, all SU3 VCC’s with multiplicity one have them as well. When a method of resolution of the outer multiplicity degeneracy is prescribed, there is no guarantee that these desirable properties will be possessed by the resulting VCC’s. The coefficients of the canonical resolution, as calculated by the Draayer codes, do not have these interchange symmetries. Because the alternative resolution of Le Blanc and Rowe, like the canonical one, resolves the degeneracy by consideration of properties of the operator (in the Wigner-Eckart theorem) which places it in an unsymmetrical relationship with the initial state tensor properties, it seems quite likely that the exchange symmetries are also missing in those results. The existence of VCC’s with such symmetries for arbitrary groups has been investigated and it was shown that while it was not, in general, possible (it is impossible, for example, for S6 – the symmetric group on six symbols), in the particular case of SU3, such VCC’s indeed existed. A scheme for the generation of SU3 VCC’s with interchange symmetry was devised by Pluhar, et. al. in 1986. These authors create a labelling operator constructed from representation generators and show that finding basis states within the degenerate subspace of a multiplicity > 1 coupling, which were eigenstates of the labelling operator, would guarantee interchange symmetry. The eigenvalues of the labelling operator become the missing label. The labels, in general, are irrational numbers. This implies that the VCC’s which result cannot be
represented as the square root of ratios of integers, a useful property familiar from the SU2 case, and which has been proven to be the case for the coefficients of the canonical scheme \(^{13}\).

IV. Symmetric resolution of outer degeneracy

For many theoretical and experimental physicists, the coupling and recoupling Racah algebra for SU2 is a tool required for even routine calculations. Those parts of the algebra necessary for this kind of calculation are typically learned in courses in graduate school from advanced quantum mechanics texts like Messiah \(^{27}\), or specialized monographs like Edmunds \(^{2}\), or Rose \(^{3}\). There, the vector coupling coefficients are introduced as elements of a transformation matrix from a set of basis states (of a two particle system) which are product of single particle states,

\[
|j_1; m_1 > |j_2; m_2 >
\]

each simultaneously an eigenvector of the angular momentum operator and the z-component of angular momentum operator; to a set of states

\[
|j_1, j_2, J; M >
\]

which is an eigenvector of the total (two particle) angular momentum operator and its corresponding z-component operator. The connection of the VCC’s to matrix elements of tensor operators is presented after the coefficients are fully explicated, through the Wigner-Eckart Theorem. Properties of the VCC’s and explicit formulae for their evaluation are achieved by means of \(J_+\) and \(J_-\), the angular momentum raising and lowering operators, two of the infinitesimal generators of the group SU2.

The motivation for the present work is twofold: to present the SU3 Clebsch Gordan problem in terms as similar as possible to this familiar agenda from the SU2 problem, and to produce
in this language a straightforward means of resolving the outer degeneracy problem which will preserve certain desirable and familiar properties of symmetry under interchange of particle labels.

For a system whose Hamiltonian is invariant under the operations of the SU3 group, the degeneracy structure of its energy spectrum is labeled by the irreducible representations of the group. Allowed energy eigenvalues can be labeled by two integers \( p, q \) which are related to the eigenvalues of the Casimir operators of the group. Each energy \( E_{p,q} \) is shared by exactly 
\[
d = \frac{1}{2}(p + 1)(q + 1)(p + q + 2)
\]
distinct eigenvectors, where \( d \) is the dimension of the irrep \( (p, q) \). An orthonormal set of basis states for the irrep can be chosen by picking states which are simultaneously eigenvalues of the three commuting operators \( \hat{T}_3, \hat{Y} \), and \( \hat{T}^2 = \frac{1}{2}(\hat{T}_+\hat{T}_- + \hat{T}_-\hat{T}_+) - \hat{T}_3^2 \). The operators \( \hat{T}_+, \hat{T}_-, \hat{T}_3, \hat{Y} \) are four of eight infinitesimal generators of the group, and are defined by their mutual commutation relations. The relationships among these generators and their relation to the canonical generators can be found in a classic elementary presentation of the theory from the point of view of elementary particle classification, by Gasiorowicz and Glashow \(^{28}\), whose notation is adopted here. The states of the orthonormal basis for the irrep \( (p, q) \) are defined by the eigenvalue equations:

\[
\hat{T}_3|p, q; k, l, m > = (m - \frac{1}{2}(k + l))|p, q; k, l, m >= I_z|p, q; I, I_z, Y >
\]

\[
\hat{Y}|p, q; k, l, m > = (k + l - \frac{2}{3}(p + 2q))|p, q; k, l, m >= Y|p, q; I, I_z, Y >
\]

\[
\hat{T}^2|p, q; k, l, m > = \frac{1}{4}(k - l)(k - l + 2)|p, q; k, l, m >= I(I + 1)|p, q; I, I_z, Y >
\]

The remaining four infinitesimal generators, along with two already mentioned \( (\hat{T}_+, \hat{T}_-) \), transform among states within an irrep, playing a role akin to the angular momentum raising and
lowering operators:

\[
\hat{T}_+ | p, q; k, l, m > = \sqrt{(k-m)(m-l+1)} | p, q; k, l, m + 1 > 
\]

\[
\hat{T}_- | p, q; k, l, m > = \sqrt{(k-m+1)(m-l)} | p, q; k, l, m - 1 > 
\]

\[
\hat{V}_+ | p, q; k, l, m > = \sqrt{(k+2)(m-l+1)(k-q+1)(p+q-k)} | p, q; k+1, l, m + 1 > 
\]

\[
+ \sqrt{(l+1)(k-m)(q-l)(p+q-l+1)} | p, q; k, l+1, m + 1 > 
\]

\[
\hat{V}_- | p, q; k, l, m > = \sqrt{(k+1)(m-l)(k-q)(p+q-k+1)} | p, q; k-1, l, m - 1 > 
\]

\[
+ \sqrt{l(k-m+1)(q-l+1)(p+q-l+2)} | p, q; k, l-1, m - 1 > 
\]

\[
\hat{U}_+ | p, q; k, l, m > = \sqrt{(k+2)(k-m+1)(k-q+1)(p+q-k)} | p, q; k+1, l, m > 
\]

\[
- \sqrt{(m-l)(l+1)(q-l)(p+q-l+1)} | p, q; k, l+1, m > 
\]

\[
\hat{U}_- | p, q; k, l, m > = \sqrt{(k+1)(k-m)(k-q)(p+q-k+1)} | p, q; k-1, l, m > 
\]

\[
- \sqrt{l(m-l+1)(q-l+1)(p+q-l+2)} | p, q; k, l-1, m > 
\]

The operators \( \hat{T}_+ \) and \( \hat{T}_- \) move up and down in the variable \( I_z \) at constant value of \( I \) and \( Y \).

The remaining operators move simultaneously to neighboring states in the \( I, I_z, \) and \( Y \) grid in such a way that appropriate products and sums of these six operators allow one to generate any state in a multiplet (of constant \( p, q \)) from any other in the same multiplet.

These operators simplify the outer degeneracy resolution problem, in that if one can find a means to produce a single state within a set of irrep basis states, properly resolved and labeled according to outer multiplicity, all other states within the multiplet (and thus all VCC’s with the same value of \( (p, q) \)) follow immediately. This advantage has been frequently utilized in the prior work cited above. Here, a precise algorithm for degeneracy resolution will be generated.
for the "state of highest weight" (SHW) for a particular coupling, which is defined to be the composite state with \( k = m = p + q, l = 0 \). This state has the largest values of \( I \) and \( I_z \) of all states in the multiplet. Knowledge of this state implies knowledge of all the VCC’s of the form

\[
(p_1, q_1; k_1, l_1, m_1 : p_2, q_2; k_2, l_2, m_2|p, q; p + q, 0, p + q)
\]

The remaining VCC’s for the same values of the irrep labels can be deduced iteratively from relations generated by use of the raising and lowering operators above.

In order to determine the product states which will be represented in the composite SHW,

\[
|p, q; p + q, 0, p + q > \equiv |p, q; SHW >
\]

necessary and sufficient conditions for product states with non-vanishing VCC’s come from the betweenness conditions (equation 12) for states 1 and 2, and the conservation laws for the subspace labels, which read for this state as

\[
\begin{align*}
    k_1 + l_1 - \frac{2}{3}(p_1 + 2q_1) + k_2 + l_2 - \frac{2}{3}(p_2 + 2q_2) &= \frac{1}{3}(p - q) \\
    m_1 - \frac{1}{2}(k_1 + l_1) + m_2 - \frac{1}{2}(k_2 + l_2) &= \frac{1}{2}(p + q) \\
    k_1 - l_1 + k_2 - l_2 &\geq p + q \\
    |k_1 - l_1 - k_2 + l_2| &\leq p + q.
\end{align*}
\]

A set of conditions for the determination of the VCC’s in this sum (equation 24) come from the action of selected operators on \(|p, q; SHW >\). By inspection of this sum, which defines the state of highest weight, and the properties of the raising and lowering operators equations 17, 19 and 22, it is easily seen that

\[
\hat{T}_+ |SHW > = 0
\]
\[ \hat{V}_+ |SHW > = 0, \text{and} \]
\[ \hat{U}_- |SHW > = 0. \] (24)

**IV.a. Multiplicity one**

For couplings of irreps which lead to a multiplicity of one (equation [13]), the above conditions (equation [24]) are sufficient to determine all the VCC’s. Each of the operators is linear, so that the operator for the composite case is the sum of the operators for each of the single particle states, for example

\[ \hat{V}_+ |SHW > = (\hat{V}_1^+ + \hat{V}_2^+)|SHW > \]
\[ = \sum_{p_1, q_1; k_1, l_1, m_1; p_2, q_2; k_2, l_2, m_2} (p_1, q_1; k_1, l_1, m_1 > |p_2, q_2; k_2, l_2, m_2 > + |p_1, q_1; k_1, l_1, m_1 > (\hat{V}_2^+ |p_2, q_2; k_2, l_2, m_2 >)) \] (25)

The actions of the operators upon the single particle states are as in equation [19]. The result is a sum over different product states, with coefficients which are linear combinations of VCC’s and coefficients from equation [19]. Because product states with any difference in their indices are orthogonal, the coefficient of each unique product state in the sum (equation [23]) must vanish, producing a set of equations which must be satisfied by the VCC’s. The result may be represented by a four term recursion relation for VCC’s of various indices. Similarly, the action of \( \hat{U}_- \) on the SHW produces a distinct four term recursion relation, and that of \( \hat{T}_+ \) produces a two term recursion relation. These relationships are not linearly independent, and the use of the \( \hat{V}_+ \) and \( \hat{U}_- \) equations alone give maximal information.

The requirement that \( |p, q; SHW > \) be normalized, implies that the sum of squares of all the VCC’s involved in the summations of equation [24] must be 1. Finally, the sign of one of
the VCC’s in equation [24] must be chosen: for reasons which will become clearer in the next section, the sign convention chosen is to require that

\[
(p_1, q_1; p_1 + q_1, 0, p_1 + q_1 : p_2, q_2; k_2^*, l_2^*, m_2^*|p, p + q, 0, p + q) \equiv (p_1, q_1; SHW : p_2, q_2; k_2^*, l_2^*, m_2^*|p, q; SHW) > 0,
\]

(26)

where

\[
k_2^* = \frac{1}{3}(p - p_1 + 2p_2 - q + q_1 + 4q_2)
\]

\[
l_2^* = 0
\]

\[
m_2^* = \frac{1}{3}(2p - 2p_1 + p_2 + q - q_1 + 2q_2).
\]

in cases where \(p_1 + p_2 - p \geq q_1 + q_2 - q\); otherwise

\[
k_2^* = p_2 + q_2
\]

\[
l_2^* = \frac{1}{3}(q_1 + q_2 - q - p_1 - p_2 + p)
\]

\[
m_2^* = \frac{1}{3}(2p - 2p_1 + p_2 + q - q_1 + 2q_2).
\]

The combination of conditions represented by equations [24], the normalization condition, and the sign convention, completely determine all the VCC’s for any coupling of irreps with multiplicity one. While judicious handling of the recursion relations might yield an algebraic closed form expression for any single VCC, the complexity of the resulting formula would render it in most cases useless but for computer evaluation: coding the equations and letting the computer carry out the required recursion should be less error-prone, more comprehensible, and usually as efficient.
IV.b. Multiplicity greater than one

When the values of the six irrep variables for a particular coupling produce a value of the multiplicity ($\alpha$) which is greater than one, the combined conditions of betweenness, equations and normalization do not totally determine the state $|SHW>$. They instead give ($\alpha - 1$) fewer conditions than there are unknown VCC’s in the state $|SHW>$. The remaining conditions, necessary to resolve the outer degeneracy, must come from outside the requirements of SU3 symmetry. Here they are chosen to produce the symmetry under particle exchange properties mentioned earlier, and to simplify computation of the VCC’s.

A description of this outer multiplicity resolution technique will be described which applies only to cases such that $p_1 + p_2 - p \geq q_1 + q_2 - q$. For the remaining cases, a technique which differs only in the details applies. There will be presented later a symmetry property for the resulting VCC’s which will relate each VCC with $p_1 + p_2 - p < q_1 + q_2 - q$ to one with $p_1 + p_2 - p > q_1 + q_2 - q$ with the same magnitude and a predictable sign, so the restriction above is of little practical consequence.

It can be shown in a straightforward manner that each member of the following series of VCC’s does not violate any previously given conditions, and therefore does not necessarily vanish:

$$ (p_1, q_1; SHW : p_2, q_2; k_2^* - j, j, m_2^* | p, q; SHW) ,$$

where, as before,

$$ k_2^* = \frac{1}{3} (p - p_1 + 2p_2 - q + q_1 + 4q_2) $$

$$ m_2^* = \frac{1}{3} (2p - 2p_1 + p_2 + q - q_1 + 2q_2). $$

The integer variable $j$ ranges in steps of 1 from 0 to $j_{max}$, a maximum variable limited only by
the betweeness conditions for state 2, which are

\[ k_2^* - j \geq m_2^* \]
\[ k_2^* - j \geq q_2 \]
\[ j \geq q_2 \]
\[ j \geq m_2^*. \]

This leads simply to the expression

\[ j_{\text{max}} = \min(\gamma + 2\sigma, p_2 + \sigma - \gamma, q_2, p_2 + q_2 - 2\gamma - \sigma), \]

where, in this case, \( \gamma = \frac{1}{3}(p_1 + p_2 - p) \) and \( \sigma = \frac{1}{3}(q_1 + q_2 - q) \). Inspection of equation 13 for the degeneracy, shows that the number of members of the series of equation 27, \( j_{\text{max}} + 1 \), is at least as large as the multiplicity. This proves what has been referred to in the literature as Braunschweig’s conjecture \(^{22}\), which has been important in several previous investigations into the SU3 VCC’s \(^{21}\).

Each member of the series given in equation 27 can be seen as representative of a family of all the VCC’s for a particular coupling, with a fixed value of the scalar sum

\[ S \equiv I_1 + I_2 = \frac{1}{2}(k_1 - l_1 + k_2 - l_2). \]

The largest value, \( S_{\text{max}} \), corresponding to \( j = 0 \), is

\[ S_{\text{max}} = \frac{1}{3}(p + 2p_1 + 2p_2 - q + 4q_1 + 4q_2); \]

for the \( j \)-th member of the series, the related family all have

\[ S = S_{\text{max}} - j. \]
The linear relations among the VCC’s implied by the conditions of the last two of equations 24 involve four VCC’s – two with \( I_1 + I_2 = S \) and two with \( S + 1 \). For the family of VCC’s with \( S = S_{\text{max}} \), these become two-term recursion relations which connect all the VCC’s in the \( S_{\text{max}} \) family. Should one of the \( S = S_{\text{max}} \) VCC’s vanish, all will vanish; if so, then the VCC’s with \( S = S_{\text{max}} - 1 \) will be connected by proportionalities for the same reason. Generalizing, if each member of the series (equation 27) for \( 0 \leq j \leq j^* \) vanishes, then all VCC’s with \( S \geq S_{\text{max}} - j^* \) will vanish: as a result, all VCC’s with \( S = S_{\text{max}} - j^* - 1 \) will be proportional to one another so that knowledge of one fixes the values of all others.

To resolve the outer degeneracy, therefore, for a case of multiplicity \( \alpha (> 1) \), follow the following steps:

1) set

\[
(p_1, q_1; SHW : p_2, q_2; k_2^z - j, j, m_2^z | p, q; SHW) = 0
\]  

for \( j = 0, 1, \ldots, \alpha - 2 \).

2) set

\[
(p_1, q_1; SHW : p_2, q_2; k_2^z - \alpha + 1, \alpha - 1, m_2^z | p, q; SHW) = c_1 > 1.
\]

This will allow direct calculation via recursion of all VCC’s in the \( j = \alpha - 1 \) family, and all remaining non-zero VCC’s, in terms of the constant \( c_1 \). After choosing the value of the constant \( c_1 \) to satisfy the normalization condition and sign convention for the VCC’s, a set of VCC’s appropriate to the involved irrep labels is completely determined. These VCC’s will carry the outer degeneracy label \( j = \alpha - 1 \).
3) to determine a second, orthogonal, set of VCC’s, with outer degeneracy label $j = \alpha - 2$, set

$$(p_1, q_1; SHW : p_2, q_2; k_2^s - j, j, m_2^s | p, q; SHW) = 0$$  \quad (30)$$

for $j = 0, 1, \ldots, \alpha - 3$.

4) set

$$(p_1, q_1; SHW : p_2, q_2; k_2^s - \alpha + 2, \alpha - 2, m_2^s | p, q; SHW) = c_2,$$  \quad (31)$$

and

$$(p_1, q_1; SHW : p_2, q_2; k_2^s - \alpha + 1, \alpha - 1, m_2^s | p, q; SHW) = c_1.$$  \quad (32)$$

The first condition allows calculation of all VCC’s in the $j = \alpha - 2$ family in terms of the constant $c_2$, and the second allows a recursive evaluation of all remaining non-zero VCC’s in terms of $c_1$ and $c_2$. The two constants are determined by the condition of normalization and the requirement that this set of VCC’s must be orthogonal to the set previously determined, i.e.

$$\sum_{k,l,m,s} (p_1, q_1; k_1, l_1, m_1 : p_2, q_2; k_2, l_2, m_2 | p, q; k, l, m [j])$$

$$\cdot (p_1, q_1; k_1, l_1, m_1 : p_2, q_2; k_2, l_2, m_2 | p, q; k, l, m [j']) = 0,$$  \quad (33)$$

for $j \neq j'$.

5) this process is repeated, each time setting one fewer of the VCC’s in the $j$ series equal zero, setting one more from this series equal to a yet-undetermined constant, and ultimately determining all these constants by the sign convention, normalization condition, and the
requirement that each of the various sets of \( j \)-labeled VCC’s must be orthogonal to all the remaining (equation 33). The procedure is similar to the familiar Gram-Schmidt process.

The sign convention chosen for each of the \( j \)-labeled sets of VCC’s is a generalization of that for the multiplicity one case;

\[
(p_1, q_1; SHW : p_2, q_2; k^*_2 - j, j, m^*_2 | p, q; SHW) > 0.
\]

The result of carrying this out to its conclusion is the determination of exactly \( \alpha \) distinct sets of VCC’s which satisfy all the SU3 requirements for the involved irreps, which correspond to mutually orthogonal and normalized kets \( | p, q; SHW > \), and which satisfy a set of symmetry relations which will be described in the next section.

V. Symmetries of the Vector Coupling Coefficients

The vector coupling coefficients of SU2 possess three symmetries, which aid in their numerical evaluation, inspire the creation of the 3-j symbol with its more natural symmetries, and greatly simplify the evaluation and manipulation of the 6-j and 9-j recoupling coefficients. In the case where angular momentum states \( | j_1; m_1 > \) and \( | j_2; m_2 > \) are coupled together to a good final angular momentum state \( | j_1, j_2, j; m > \), the symmetries are as follows:

1) upon interchange of the values of \( j_1, m_1 \) with \( j_2, m_2 \), the VCC changes by, at most, a sign;

2) upon replacement of each \( j_i, m_i \) pair with its conjugate, \( j_i, -m_i \), the VCC changes by, at most, a sign;
3) upon interchange of the values of $j_1, m_1$ with $j, -m$, the VCC changes in magnitude by the square root of the ratio of the dimension of the multiplet $j_1$ to that of the multiplet $j$, i.e. $\sqrt{(2j_1 + 1)/(2j + 1)}$; and can change in sign.

The exact expressions for the sign changes resulting from these alterations depends on the choice of sign convention, and varies from author to author. The symmetry under interchange of $j_2, m_2$ with $j, -m$ can be deduced from the symmetries above.

The SU3 VCC's defined in the last section, including those with multiplicity greater than one, can be shown to have a set of symmetries which entirely parallel those for SU2. Consider, for example, the interchange of the set of indices $p_1, q_1, k_1, l_1, m_1$ with $p_2, q_2, k_2, l_2, m_2$, and the steps involved in the evaluation of the VCC. The product basis states are invariant under the interchange. The conditions which specify which product states will correspond to non-vanishing VCC's – the betweenness conditions, and the conservation of $I, I_z$, and $Y$ – are also invariant under the interchange. The conditions which completely specify the values of the VCC's in the case of multiplicity one (equations 24) are also invariant, since each operator is the scalar sum of the operator for state 1, and that for state 2. In the case of higher multiplicity, it is straightforward to show that the procedure is invariant, since for every non-zero VCC

$$(p_1, q_1; SHW : p_2, q_2; k_2^* - j, j, m_2^* | p, q; SHW)$$

there is a corresponding non-zero VCC

$$(p_2, q_2; SHW : p_1, q_1; k_1^* - j, j, m_1^* | p, q; SHW)$$

which has the same value of $S = I_1 + I_2$. The VCC itself is not invariant under interchange solely because the sign convention is not invariant. The state which is chosen to be positive in the two cases $((p_1, q_1) \otimes (p_2, q_2) \rightarrow (p, q)$ and $(p_2, q_2) \otimes (p_1, q_1) \rightarrow (p, q))$ is in each case in
the $S = S_{\text{max}}$ family. VCC's in this family are related by a two term recursion relation with alternating signs, and it is straightforward to predict the sign change upon interchange:

\[
(p_1, q_1; k_1, l_1, m_1 : p_2, q_2; k_2, l_2, m_2 | p, q; k, l, m | j) = (-1)^{2\gamma + \sigma + j} (p_2, q_2; k_2, l_2, m_2 : p_1, q_1; k_1, l_1, m_1 | p, q; k, l, m | j)
\]

where $j$ is the degeneracy index, and $\gamma, \sigma$ are the maximum and minimum, respectively, of $\frac{1}{3} (p_1 + p_2 - p)$ and $\frac{1}{3} (q_1 + q_2 - q)$.

The Hermitian conjugation operation, which maps the state $|j, m \rangle$ into $|j, -m \rangle$ in SU2, maps $|p, q; k, l, m \rangle$ into $|q, p; \tilde{k}, \tilde{l}, \tilde{m} \rangle$ (where $\tilde{k} = p+q-l$, $\tilde{l} = p+q-k$, $\tilde{m} = p+q-m$) in SU3, changing not only the subgroup indices, but the irrep indices as well. Otherwise stated, if an operator $T$ transforms like an SU3 tensor of indices $p, q; k, l, m$, then its Hermitian conjugate, $T^\dagger$, transforms like one of indices $q, p; \tilde{k}, \tilde{l}, \tilde{m}$.

If all three sets of SU3 indices in a VCC are simultaneously conjugated, the numerical value of the VCC changes by, at most, a sign. To establish the phase in this relationship, it is necessary to examine the VCC’s for cases such that $q_1 + q_2 - q > p_1 + p_2 - p$, since simultaneous conjugation of all states transforms a VCC with $p_1 + p_2 - p > q_1 + q_2 - q$ into one with the opposite inequality. The steps described in the last section for determining the VCC’s and resolving outer degeneracy (if non-trivial) are followed as well in cases such that $q_1 + q_2 - q > p_1 + p_2 - p$, with one major change: the series of VCC’s which characterize the families of constant $I_1 + I_2$ and which fix the sign convention, consists of

\[
(p_1, q_1; \text{SHW} : p_2, q_2; p_2 + q_2 - j, l_2^* + j, m_2^* | p, q; \text{SHW})
\]

It is easily seen that under conjugation, the various terms in the infinitesimal generators transform among one another in a way that enables a straightforward albeit tedious derivation of
the phase relationship between a VCC and one for conjugated states:

\[(p_1, q_1; k_1, l_1, m_1 : p_2, q_2; k_2, l_2, m_2|p, q; k, l, m[j]) = (-1)^{\gamma+2\sigma+j} \]

\[ \cdot (q_1, p_1; ˜k_1, ˜l_1, ˜m_1 : q_2, p_2; ˜k_2, ˜l_2, ˜m_2|q, p; ˜k, ˜l, ˜m[j]) \]

It is tedious to justify the relationship between a VCC and one in which the final state is interchanged with either of the first two. Even in the case of SU2, the corresponding formula is often justified as being a consequence of analytic expressions for the VCC’s, rather than proven from the properties of the SU2 generators. In the SU3 case, it is straightforward to show that such permutations leave the \( j \) degeneracy label unchanged; inspection of the ladder operators show they transform among one another; the sign convention fixes the signs of all coefficients; and a numerical multiplication factor is necessary to preserve normalization. The resulting relation is

\[(p_1, q_1; k_1, l_1, m_1 : p_2, q_2; k_2, l_2, m_2|p, q; k, l, m[j]) = (-1)^{\gamma+2\sigma+m_2+j} \sqrt{\frac{(p + 1)(q + 1)(p + q + 2)}{(p_1 + 1)(q_1 + 1)(p_1 + q_1 + 2)}} \]

\[ \cdot (q, p; ˜k, ˜l, ˜m : p_2, q_2; k_2, l_2, m_2|q_1, p_2; ˜k_1, ˜l_1, ˜m_1[j]) \]

VI. Conclusion

In the previous sections, a presentation has been made of a technique for derivation of a consistent set of vector coupling coefficients for SU3. They are discussed in a fashion as near as possible to the standard language of SU2 as it is used in angular momentum theory. These VCC’s have symmetry properties under state interchange which parallels the familiar relations of SU2, and should thus be immediately useful for particle state classification uses and should lead to 6-j and 9-j recoupling coefficients with familiar properties of symmetry which should also simplify their evaluation. The algorithm described above is easily implemented.
for computing, and such a program has been written in ANSI standard C, and will be made available to interested users. A quite similar code, which generates the SU3 VCC’s using the canonical resolution scheme, has been put in the public domain by the author \(^{29}\). Only minor modifications of this code are necessary to implement the scheme of this paper.

The recursion relations for the VCC’s for states of highest weight, mentioned earlier, suggest a much more efficient computer procedure for evaluation of VCC’s than the existing one is possible. Implementation of this is presently underway. The increased computer efficiency which this scheme will afford will make possible quite prompt evaluation of SU3 6-j and 9-j symbols. This efficiency provides another commendation of this scheme in comparison to many previous degeneracy resolution procedures, which require diagonalization of quite large matrices to determine the VCC’s \(^{30}\).

**Acknowledgements**

The author would like to acknowledge the useful guidance received during several conversations and correspondences with L. C. Biedenharn, and express appreciation for this help. Thanks are also due to M. Danos for careful reading of an early draft of this work, and many useful suggestions. This work was partially supported by a research grant from Washington and Lee University.
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