CYCLE SPACE OF GRAPHS OF POLYTOPES

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Abstract. It is folklore that the cycle space of graphs of polytopes is generated by the cycles bounding the 2-faces. We provide a proof of this result that bypass homological arguments, which seem to be the most widely known proof. As a corollary, we obtain a result of Blind & Blind (1994) stating that graphs of polytopes are bipartite if and only if graphs of every 2-face are bipartite.

1. Introduction

A (convex) polytope is the convex hull of a finite set $X$ of points in $\mathbb{R}^d$. The dimension of a polytope in $\mathbb{R}^d$ is one less than the maximum number of affinely independent points in the polytope, and a polytope of dimension $d$ is referred to as a $d$-polytope. A face of a polytope $P$ in $\mathbb{R}^d$ is $P$ itself, or the intersection of $P$ with a hyperplane in $\mathbb{R}^d$ that contains $P$ in one of its closed halfspaces. A face of dimension 0, 1, and $d-1$ in a $d$-polytope is a vertex, an edge, and a facet, respectively. The set of vertices and edges of a polytope or a graph are denoted by $V$ and $E$, respectively. The graph $G(P)$ of a polytope $P$ is the abstract graph with vertex set $V(P)$ and edge set $E(P)$.

Let $G$ be a graph, and let $G'$ and $G''$ be two spanning subgraphs of $G$. The symmetric difference $\Delta$ of $G'$ and $G''$ is the spanning subgraph of $G$ whose edge set is the symmetric difference of $E(G')$ and $E(G'')$. A spanning subgraph $G'$ of a graph $G$ is an even subgraph if every vertex of $G'$ has even degree. Every cycle in $G$ can be regarded as an even subgraph if enough isolated vertices of $G$ are added. The set of all even subgraphs of a graph $G$ forms a vector space $Z(G)$ over the field $GF(2)$, the 2-element field, with respect to the symmetric difference of spanning subgraphs; the space $Z(G)$ is called the cycle space of $G$. And the cycles of $G$, viewed as even subgraphs, span $Z(G)$. Thus, each even subgraph is the symmetric difference of cycles. See, for instance, in Diestel (2017, Sec. 1.9).

A cycle in a plane graph or a graph of a polytope is facial if it bounds a face of the plane graph or a 2-face of the polytope. It is well known that every cycle...
in a 2-connected plane graph is the symmetric difference of facial cycles and that a 2-connected plane graph is bipartite if and only if every facial cycle is bipartite; see, for instance, Mohar & Thomassen (2001, Thm. 2.2.3, Cor. 2.4.6). We provide elementary proofs of extensions of these results to graphs of polytopes of all dimensions. These extensions read as follows.

**Theorem 1.** For \( d \geq 2 \), every even subgraph in the graph of a \( d \)-polytope is the symmetric difference of facial cycles. In particular, every cycle in the graph of a \( d \)-polytope is the symmetric difference of facial cycles.

A first corollary of Theorem 1 is immediate.

**Corollary 2.** For \( d \geq 2 \), the cycle space of the graph of a \( d \)-polytope is spanned by the facial cycles of the polytope.

As a second corollary of Theorem 1, we obtain a characterisation of bipartite polytopal graphs, which was proved in Blind & Blind (1994, Sec. 3) via shellings of polytopes; this proof is the inspiration for our proof of Theorem 1. We remark that this characterisation was known to Coxeter (1973, p. 154) but his proof was incorrect, as pointed out in Blind & Blind (1994, Sec. 3).

**Corollary 3 (Blind & Blind 1994).** For \( d \geq 2 \), the graph of a \( d \)-polytope is bipartite if and only if every 2-face of the polytope is bipartite.

**Proof.** If the graph of a polytope is bipartite, then every facial cycle must have even length and so it is bipartite. If a graph \( G \) of a polytope is nonbipartite, then \( G \) has a cycle \( C \) of odd length. By Corollary 2, the cycle \( C \) (as an even subgraph) is the symmetric difference of facial cycles of \( G \). Accordingly, one of these facial cycles must have odd length. \( \square \)

Theorem 1, Corollary 2, and Corollary 3 are known to the community of discrete geometry. Theorem 1 is usually proved by homological arguments (N Harvey, 2022), and Corollary 3 follows from it, as we illustrated it.

2. **Proofs**

A *polytopal complex* \( C \) is a finite, nonempty collection of polytopes in \( \mathbb{R}^d \) where the faces of each polytope in \( C \) all belong to \( C \) and where polytopes intersect only at faces. The *boundary complex* \( B(P) \) of a polytope \( P \) is the set of faces of \( P \) other than \( P \) itself, while the *complex* \( C(P) \) of \( P \) is the set of faces of \( P \).

Let \( C \) be a *pure polytopal complex*; that is, each of the faces of \( C \) is contained in some facet. A *shelling* of \( C \) is a linear ordering \( F_1, \ldots, F_s \) of its facets such that either \( \dim C = 0 \), in which case the facets are vertices, or it satisfies the following:

(i) The boundary complex of \( F_1 \) has a shelling.

(ii) For \( 2 \leq j \leq s \), the intersection

\[
F_j \cap \left( \bigcup_{i=1}^{j-1} F_i \right) = R_1 \cup \cdots \cup R_r
\]
is nonempty and the beginning $R_1,\ldots,R_r$ of a shelling $R_1,\ldots,R_r,R_{r+1},\ldots,R_t$ of $B(F)$. 

Bruggesser & Mani (1971) proved that every polytope admits a shelling.

A pure polytopal complex $C$ is strongly connected if every pair of facets $F$ and $F'$ is connected by a path $F_1\ldots F_n$ of facets in $C$ such that $F_i \cap F_{i+1}$ is a ridge of $C$ for $1 \leq i \leq n-1$, $F_1 = F$ and $F_n = F'$. This definition implies the following two assertions.

**Proposition 4.** For $d \geq 1$, the graph of a strongly connected $d$-complex is connected.

For a set $\{P_1,\ldots,P_n\}$ of polytopes, we denote by $C(P_1 \cup \cdots \cup P_n)$ the polytopal complex $C(P_1) \cup \cdots \cup C(P_n)$.

**Proposition 5.** Let $P$ be a $d$-polytope, and let $F_1,\ldots,F_s$ be a shelling of $P$. Then the complex $C(F_1 \cup \cdots \cup F_i)$ is a strongly connected $(d-1)$-complex, for each $1 \leq i \leq s$.

**Proof of Theorem 4** For $d = 2$, the theorem is trivially true. So assume that $d \geq 3$ and proceed by induction on $d$.

Let $S := F_1,\ldots,F_s$ be a shelling of $P$, let

$$C_n := C(F_1 \cup \cdots \cup F_n),$$

and let $G_n$ be the graph of $C_n$. For $1 \leq i \leq s-1$, each facet $F_i$ is a $(d-1)$-polytope, and so the induction hypothesis ensures that the even subgraphs in each $G(F_i)$ are generated by facial cycles in such a graph.

Since all the edges of $P$ lie in $G_{s-1}$—and in particular, $G_{s-1} = G$—it suffices to prove that every even subgraph in each $G_n$ is spanned by the facial cycles of $G_n$, for $1 \leq n \leq s-1$. We further proceed by induction on $n$. The case $n = 1$ holds, because $G_1 = G(F_1)$. We then assume that each even subgraph of $G_{n-1}$, where $1 \leq n-1 \leq s-2$, is spanned by the facial cycles in $G_{n-1}$ and that some problematic even subgraph in $G_n$ is not. Because $G_n = G_{n-1} \cup G(F_n)$, each problematic even subgraph in $G_n$ is contained in neither $G_{n-1}$ nor $G(F_n)$. Let $C$ be a problematic even subgraph in $G_n$ with a smallest number of edges in $G_{n-1} \setminus G(F_n)$ among the problematic even subgraphs in $G_n$. Each even subgraph is the symmetric difference of cycles; and consequently, we have that $C$ is a cycle of $G_n$.

A path from a vertex $x$ to a vertex $y$ in a graph is an $x - y$ path, and for a path $L := x_0\ldots x_n$ and for $0 \leq i \leq j \leq n$, we write $x_iLx_j$ to denote the subpath $x_i\ldots x_j$.

Think of $C$ as a cycle directed from in $G_{n-1} \setminus G(F_n)$ to $G(F_n)$, starting at a vertex in $G_{n-1} \setminus G(F_n)$; the addition of $G(F_n)$ to $G_{n-1}$ may introduce new edges and no new vertices. The cycle $C$ intersects $G_{n-1} \cap G(F_n)$ in the distinct vertices $x_1,\ldots,x_\ell$ found in this order as we traverse $C$. That is, $x_1$ is the first vertex on $C$ that touches $G_{n-1} \cap G(F_n)$, $x_\ell$ is the last vertex on $C$ that touches $G_{n-1} \cap G(F_n)$,
and the directed subpath $L := x_j C x_1$ of $C$ lies in $G_{n-1} \setminus G(F_n)$ except for $x_1$ and $x_j$. If $G_{n-1} \cap G(F_n) \cap C$ consists of one vertex, then $C$ would be a union of cycles with a cycle in $G_{n-1}$ and a cycle in $G(F_n)$, a contradiction to $C$ being a cycle. Therefore $j \geq 2$.

The graph $G_{n-1} \cap G(F_n)$ is connected. Since $S$ is a shelling, the complex $B(F_n) \cap C_{n-1}$ is the beginning of a shelling of $F_n$, which implies that $B(F_n) \cap C_{n-1}$ is a strongly connected $(d-2)$-complex (Proposition 5); the connectivity of $G_{n-1} \cap G(F_n)$ now follows from Proposition 4.

From the connectivity of $G_{n-1} \cap G(F_n)$ follows the existence of an $x_1 - x_j$ path $M$ in $G_{n-1} \cap G(F_n)$. Concatenating the paths $L$ and $M$, we form a cycle $C_1$ in $G_{n-1}$. By the induction hypothesis on $n$, this cycle $C_1$ is the symmetric difference of facial cycles in $G_{n-1}$. Let $L'$ be the directed subpath $L'$ of $C$ from $x_1$ to $x_j$. Then $C = L \cup L'$ and $V(L) \cap V(L') = \{x_1, x_j\}$. In addition, let $W := L' \triangle M$; here we understand $L'$ and $M$ as spanning subgraphs of $G_n$. It follows that $W$ is an even subgraph of $G$, since $x_1$ and $x_j$ have each degree one in both $L'$ and $W$, and every other vertex in $W$ has even degree in both $L'$ and $W$. It is also the case that $W$ has fewer edges in $G_{n-1} \setminus G(F_n)$ than $C$, and so it is the symmetric difference of facial cycles in $G_n$.

The cycle $C$ is the symmetric difference of $C_1$ and $W$, and as a consequence, it is the symmetric difference of facial cycles in $G_n$. This contradiction ensures that the cycle $C$ does not exist, which amounts to saying that every even subgraph in $G_n$ is spanned by facial cycles in $G_n$. Hence the induction is complete, and so is the proof of the theorem. □

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