The relativistic approach to electroweak properties of two-particle composite systems developed in Ref. [1] is generalized here to the case of nonzero spin. In developed technique the parametrization of matrix elements of electroweak current operators in terms of form factors is a realization of the Wigner–Eckart theorem on the Poincaré group and form factors are reduced matrix elements. The ρ meson charge form factor is calculated as an example.

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A new relativistic approach to electroweak properties of composite systems has been proposed in our recent paper [1]. The approach is based on the use of the instant form (IF) of relativistic Hamiltonian dynamics (RHD). The detailed description of RHD can be found in the review [2]. Some other references as well as some basic equations of RHD approach are given in Ref. [1].

Now our aim is to generalize the approach to composite systems of two particles of spin 1/2 with nonzero values of total angular momentum, total orbital momentum and total spin. The main problem is a construction of electromagnetic current operator satisfying standard conditions (see, e.g., Refs. [1,3]).

The basic point of our approach [1] to the construction of the electromagnetic current operator is the general method of relativistic invariant parameterization of local operator matrix elements proposed as long ago as in 1963 by Cheshkov and Shirokov [4]. This canonical parametrization of local operators matrix elements was generalized to the case of composite systems of free particles in Refs. [5,6]. This parametrization is a realization of the Wigner–Eckart theorem for the Poincaré group and so it enables one for given matrix element of arbitrary tensor dimension to separate the reduced matrix elements (form factors) that are invariant under the Poincaré group.

Physical approximations that we use in our approach are formulated in terms of reduced matrix elements, for example, the well known relativistic impulse approximation. In our method this approximation does not violate the standard conditions for the current.

In the present paper we propose a general formalism for the operators diagonal in the total angular momentum. The details of calculations can be found in [3].

Let us consider the operator $j_\mu = j_\mu(0)$ that describes a transition between two states of a composite two–constituent system. Let us neglect temporarily for simplicity the conditions of self–adjointness, conservation law and parity conservation. The Wigner–Eckart decomposition of the matrix element has the form [4]:

$$
\langle \tilde{p}_c, m_{jc}; j_\mu | \tilde{p}_c', m_{jc}' \rangle = \langle m_{jc}| D^{Lc}(p_c, p'_c) \times \left[ F_1^c K_\mu^c + F_2^c \Gamma_\mu(p'_c) + F_3^c \Gamma_\mu(p'_c) K_\mu^c \right] | m_{jc}' \rangle .
$$

(1)

Here $K_\mu = (p_c - p'_c)_\mu = q_\mu, K_\mu' = (p_c + p'_c)_\mu, R_\mu = \epsilon_{\mu\nu\lambda\sigma} p_\nu^c p'_\nu \Gamma^\sigma(p'_c); (p_c - p'_c)^2 = -Q^2, p_\mu^2 = p'_\mu^2 = M_c^2, M_c, J_c$ are the mass and spin of the composite particle, $m_{jc}$ is spin projection, $\Gamma^\sigma(p'_c)$ is the spin four–vector defined with the use of the Pauli–Lubansky vector [1], $f_{\mu n}$ are reduced matrix elements, $\epsilon_{\mu\nu\lambda\sigma}$ is a completely antisymmetric pseudo-tensor in four dimensional space-time with $\epsilon_{0123} = -1$.

In the frame of RHD the form factors of composite systems $f_{\mu n}$ are to be expressed in terms of RHD wave functions and constituents form factors.

In RHD a state of two particle interacting system is described by a vector in the direct product of two one–particle Hilbert spaces (see, e.g., Ref. [1]). So, the matrix element in RHD can be decomposed in the basis [1]

$$
| \tilde{P}, \sqrt{s}, J, l, S, m_J \rangle .
$$

(3)

Here $P_\mu = (p_1 + p_2)_\mu, P_\mu^2 = s, \sqrt{s}$ is the invariant mass of the two-particle system, $l$ is the orbital angular momentum in the center–of–mass frame (c.m.), $S$ is the total spin in the c.m., $J$ is the total angular momentum with the projection $m_J$. 

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\[
\langle \vec{p}_c, m_J | j_{\mu} | \vec{p}_c', m'_{Jc} \rangle = \sum \int \frac{d\vec{P}}{N_{CG}} \frac{d\vec{P}'}{N_{CG}'} d\sqrt{s} d\sqrt{s'} \times \langle \vec{p}_c, m_J | \hat{P} | \sqrt{s}, J, l, S, m_J \rangle \\
\times \langle \vec{P}, \sqrt{s}, J, l, S, m_J | j_{\mu} | \vec{P}', \sqrt{s'}, J', l', S', m_{J'} \rangle \\
\times \langle \vec{P}', \sqrt{s'}, J', l', S', m_{J'} | \vec{p}_c', m'_{Jc} \rangle .
\]

Here the sum is over variables \( J, J', l, l', S, S', m_J, m_{J'} \), and \( \langle \vec{P}', \sqrt{s'}, J', l', S', m_{J'} | \vec{p}_c', m'_{Jc} \rangle \) is the wave function in the sense of IF RHD.

\[
\langle \vec{P}, \sqrt{s}, J, l, S, m_J | \vec{p}_c \rangle = N_c \delta(\vec{P} - \vec{p}_c) \delta_{J,J_c} \delta_{m_J,m_J_c} \varphi_{JS}(k). \tag{5}
\]

Here \( k = \sqrt{\lambda(s, M_1^2, M_2^2)/(2\sqrt{s})} \), \( M_1, M_2 \) are masses of constituents, \( \lambda(a, b, c) = a^2 + b^2 + c^2 - 2(ab + bc + ac) \).

The RHD wave function of constituents relative motion with fixed total angular momentum is defined as

\[
\varphi_{JS}(k) = \sqrt{\sqrt{s}(1 - \eta^2/s^2)} u_{JS}(k), \tag{6}
\]

and is normalized by the condition

\[
\sum_{J, S} \int u_{JS}^2(k) k^2 dk = 1. \tag{7}
\]

Here \( \eta = M_1^2 - M_2^2 \), \( u_{JS}(k) \) is a model wave function.

The main difficulty arising in this case is the following. In the expression (1) we were dealing with the parametrization of local operator matrix elements in the case when the transformations of the state vectors and of the operators were defined by one and the same representation of the quantum mechanical Poincaré group.

A different situation takes place in the case of the matrix element in the r.h.s. of Eq. (4). The operator describes the system of two interacting particles and transforms following the representation with Lorentz boosts generators depending on the interaction [1]. The state vectors physically describe the system of two free particles and present the basis of a representation with interaction–independent generators. So, the Wigner–Eckart decomposition can not be applied directly to the matrix element in the integrand in the r.h.s. of Eq. (4). This is caused by the fact that it is impossible to construct 4–vectors describing the matrix element transformation properties under the action of Lorentz boosts from the variables entering the state vectors (contrary to the case of, e.g., Eq. (1)). In fact, the possibility of matrix element representation in the form (1) is based on the following fact. Let us act by Lorentz transformation on the operator \( U^{-1}(\Lambda) j^{\mu} U(\Lambda) \) = \( j^{\mu} \). We obtain the following chain of equalities:

\[
\langle \vec{p}_c, m_J | j_{\mu} | \vec{p}_c', m'_{Jc} \rangle = \langle \vec{p}_c, m_Jc | \hat{U}^{-1}(\Lambda) j^{\mu} | \vec{p}_c', m'_{Jc} \rangle \\
= \sum_{m_{Jc}} \langle m_{Jc} | [D^{Jc}(R_\Lambda)]^{-1} | m_{Jc} \rangle \langle \vec{p}_c, m_{Jc} | j_{\mu} | \vec{p}_c', m'_{Jc} \rangle \\
\times \langle \vec{p}_c, m_{Jc} | \hat{U}^{-1}(\Lambda) j^{\mu} | \vec{p}_c', m'_{Jc} \rangle.
\]

Here \( D^{Jc}(R_\Lambda) \) is rotation matrix realizing the angular momentum transformation under the action of Lorentz transformations. The equalities (8) show that the transformation properties of the current as a 4–vector can be described using the 4–vectors of the initial and the final states. This means that the canonical parameterization [4] is the realization of the Wigner–Eckart theorem on the Poincaré group.

In the case of the current matrix element in the r.h.s. of Eq. (4) the relations (8) are not valid and direct application of the Wigner–Eckart theorem is impossible.

However, it can be shown that for the matrix element in Eq. (4) considered as a generalized function (distribution of the Wigner–Eckart theorem is impossible.

Let us consider the matrix element in question as a regular Lorentz covariant generalized function (see, e.g., Ref. [7]). Using Eq. (5) let us rewrite Eq. (4) in the following form:

\[
\langle \vec{p}_c, m_{Jc} | j_{\mu} | \vec{p}_c', m'_{Jc} \rangle = \sum_{l, l', S, S'} \int N d\sqrt{s} d\sqrt{s'} \varphi_{JS}(s) \varphi_{JS'}^{l', l}(s') \times \langle \vec{p}_c, \sqrt{s}, J, l, S, m_J | j_{\mu} | \vec{p}_c', \sqrt{s'}, J', l', S', m_{J'} \rangle . \tag{9}
\]

Here it is taken into account that the current operator \( j_{\mu} \) is diagonal in total angular momentum of the composite system, \( N' = N_c N_{CG} N_{CG} \).

Let us make use of the fact that the set of the states (3) is complete:

\[
\hat{I} = \sum \int \frac{d\vec{P}}{N_{CG}} \times | \vec{P}, \sqrt{s}, J, l, S, m_J \rangle \langle \vec{P}, \sqrt{s}, J, l, S, m_J | . \tag{10}
\]

Here the sum is over the discrete variables of the basis (3).

Under the integral the matrix element of the transformed current satisfies the following equalities ((5) and (10) are taken into account):

\[
\sum \int N d\sqrt{s} d\sqrt{s'} \varphi_{JS}(s) \varphi_{JS'}^{l', l}(s') \times \langle \vec{p}_c, \sqrt{s}, J, l, S, m_J | j_{\mu} | \vec{p}_c', \sqrt{s'}, J', l', S', m_{J'} \rangle.
\]
variables \((R\) and Eq. (9) can be rewritten as a functional in function.\(A\) (see Eq. (6), too))

So now it is possible to use the parameterization under the integral, that is to use the Wigner–Eckart theorem in the weak sense. The r.h.s. of Eq. (9) can be written as a functional on the space of test functions of the form (see Eq. (6), too) \(\psi^{llSS}_{s,s'}(s, s') = u_{lS}(k(s)) u_{lS'}(k(s'))\), and Eq. (9) can be rewritten as a functional in \(R^2\) with variables \((s, s'):\n\)

\[
\langle \hat{p}_c, m_{jc}|\hat{p}_c'(0)|\hat{p}_c', m_{jc}' \rangle = \sum_{l',l,S,S'} \int d\mu(s, s') N \psi^{llSS}_{s,s'}(s, s') \times (\langle \hat{p}_c, \sqrt{s}, J_c, l, S, m_{jc}|\hat{p}_c', \sqrt{s'}, J_c', l', S', m_{jc}' \rangle) .
\] (12)

Here the measure is chosen with the account of the relativistic density of states, subject to the normalization (6), (7):

\[
d\mu(s) = 16 \theta(s - (M_1 + M_2)^2) \theta(s' - (M_1 + M_2)^2) \times \sqrt{s(1 - \eta^2/s^2)} \sqrt{s'(1 - \eta^2/s'^2)} d\mu(s) d\mu(s') .
\] (13)

Here \(d\mu(s) = (1/4) k d\sqrt{s}\).

The sums over discrete invariant variables can be transformed into integrals by introducing the adequate delta–functions. The obtained expressions are functionals in \(R^2\).

The functional in the r.h.s. of Eq. (12) defines a Lorentz covariant generalized function, generated by the current operator matrix element.

Taking into account Eq. (11) we decompose the matrix element in the r.h.s. of Eq. (12) into the set of linearly independent scalars entering the r.h.s. of Eq. (1):

\[
N(\hat{p}_c, \sqrt{s}, J_c, l, S, m_{jc}|\hat{p}_c', \sqrt{s'}, J_c', l', S', m_{jc}') = \langle m_{jc}| D^{J_c} p_c, p_c' \rangle \sum_{n=0}^{2J_c} (ip_{cm} \Gamma^{m}(p_c'))^n \times A^{llSS}_{s,s'}(s, Q^2, s') |m_{jc}' \rangle .
\] (14)

Making use of Eq. (14) and comparing the r.h.s. of Eq. (1) with Eq. (12) we obtain:

\[
\sum_{l',l,S,S'} \int d\mu(s, s') \psi^{llSS}_{s,s'}(s, s') \times (m_{jc}|A^{llSS}_{s,s'}(s, Q^2, s') |m_{jc}' \rangle = \langle m_{jc}| \left[ f_{1n} K'_\mu + f_{2n} \Gamma_\mu(p_c') + f_{3n} R_\mu + f_{4n} K_\mu \right] |m_{jc}' \rangle .
\] (15)

All the form factors in the r.h.s. of Eq. (15) are nonzero if the generalized function \(A\) contains parts that are diagonal \((A_1)\) and non-diagonal \((A_2)\) in \(m_{jc}, m_{jc}'\). For the diagonal part we have from Eq. (15):

\[
\sum_{l',l,S,S'} \int d\mu(s, s') \psi^{llSS}_{s,s'}(s, s') \times (m_{jc}|A^{llSS}_{1i\mu}(s, Q^2, s') |m_{jc}' \rangle = \langle m_{jc}| \left[ f_{1n}^{(l)} K'_\mu + f_{4n}^{(l)} |K_\mu \right] |m_{jc}' \rangle .
\] (16)

The notation \(f_{jn}^{(l)}\) in the r.h.s. emphasizes the fact that form factors of composite systems are functionals on the wave functions of the intrinsic motion and so, on the test functions.

Let the equality (16) be valid for any test function \(\psi^{llSS}_{s,s'}(s, s')\). When the test functions (the intrinsic motion wave functions) are changed the vectors in the r.h.s. are not changed because according to the essence of the parametrization (1) they do not depend on the model for the particle intrinsic structure. So, when the test functions are varied the vector of the r.h.s. of Eq. (16) remains in the hyperplane defined by the vectors \(K'_\mu, K'_\mu\).

When test functions are varied arbitrarily the vector in l.h.s. of Eq. (16) can take, in general, an arbitrary direction. So, the requirement of the validity of Eq. (16) in the whole space of our test functions is that the l.h.s. of Eq. (16) remains zero. Then this generalized function have the form:

\[
A^{llSS}_{1i\mu}(s, Q^2, s') = K'_\mu G^{llSS}_{1n}(s, Q^2, s') + K_\mu G^{llSS}_{4n}(s, Q^2, s') .
\] (17)

Here \(G^{llSS}_{1n}(s, Q^2, s')\) \(, i = 1, 4\) are Lorentz invariant generalized functions. Substituting Eq. (17) in Eq. (16) and taking into account Eqs. (6) and (13) we obtain the following integral representations:

\[
f_{jn}^{(l)}(Q^2) = \sum_{l',l,S,S'} \int d\sqrt{s} d\sqrt{s'} \psi^{ll}_{s,s'}(s) \psi^{llSS}_{s,s'}(s') .
\] (18)

for \(i = 1, 4\). In the case of matrix element in Eq. (15) non-diagonal in \(m_{jc}, m_{jc}'\) we can proceed in an analogous
way and obtain an analogous integral representations for $f_i^\nu(Q^2)$, $i = 2, 3$.

So, the matrix element in the r.h.s. of Eq. (12) considered as Lorentz covariant generalized function can be written as the following decomposition of the type of Wigner–Eckart decomposition:

$$\langle \vec{p}, \sqrt{s}, J_c, l, S, m_{J_c}; j_\mu | \vec{p}_e, \sqrt{s}, J_c, l', S', m_{J_c} \rangle_C = \frac{1}{N} \langle m_{J_c} | D^{J_c}(p_e, p'_e) | F_1 K'_\mu + F_2 \Gamma^{\nu}(p'_e) + F_3 R_\mu + F_4 K_\mu \rangle | m'_{J_c} \rangle.$$  (19)

$$F_i = \sum_{n=0}^{2J_c} G_i^{n,SS'}(s, Q^2, s')(i \rho^\nu \Gamma^n(p'_e))^n.$$  (20)

![Graph](image)

**FIG. 1.** The results of the calculations of the $\rho$-meson charge form factor with different model wave functions [1,3]. The solid line represents the relativistic calculation with the wave function of harmonic oscillator, the dashed line – with the power–law wave function for $n = 3$, dash–dot–line – with the wave function with linear confinement, dotted line – with the power–law wave function for $n = 2$, dot–dot–dash–line – the non-relativistic calculation with the wave function of harmonic oscillator. The wave functions parameters are obtained from the fitting of $\rho$ – meson MSR. The sum of quark anomalous magnetic moments is taken as $\kappa_u + \kappa_d = 0.09$ in natural units. The quark mass is $M = 0.25$ GeV.

In Eqs. (19), and (20) the form factors $G_i^{n,SS'}(s, Q^2, s')$ contain all the information about the physics of the transition described by the operator $j_\mu$. They are connected with the composite particle form factors (1), and (2) through Eq. (18). In particular, physical approximations are formulated in our approach in terms of form factors $G_i^{n,SS'}(s, Q^2, s')$ (see Ref. [1] for details). The matrix element transformation properties are given by the 4–vectors in the r.h.s. of Eq. (19).

It is worth to emphasize that it is necessary to consider the composite system form factors as the functionals generated by the Lorentz invariant generalized functions $G_i^{n,SS'}(s, Q^2, s')$.

Now let us impose the conditions of self–adjointness, conservation law and parity conservation on the matrix elements in Eqs. (1), and (19). The r.h.s. of equalities (1) and (19) contain the same 4–vectors and the same sets of Lorentz scalars (2) and (20), so, to take into account the additional conditions it is necessary to redefine these 4–vectors and functions $G_i^{n,SS'}(s, Q^2, s')$. For example, the conservation law gives $F_4 = 0$ and $F_4 = 0$.

Let us write the parameterization (19), (20) for the particular case of composite particle electromagnetic current with quantum numbers $J = J' = S = S' = 1$, which is realized, for example, in the case of deuteron. Separating the quadrupole form factor and using Eqs. (19), and (20) we obtain the following form:

$$\langle \vec{p}, \sqrt{s}, J_c, l, S, m_{J_c}; j_\mu | \vec{p}_e, \sqrt{s}, J_c, l', S', m_{J_c} \rangle_C = \frac{1}{N} \langle m_{J_c} | D^{J_c}(p_e, p'_e) | \tilde{F}_1 K'_\mu + \frac{i}{M_c} \tilde{F}_3 R_\mu \rangle | m'_{J_c} \rangle.$$  (21)

$$\tilde{F}_1 = \tilde{G}_{10}(s, Q^2, s') + \tilde{G}_{12}(s, Q^2, s') \left\{ |i \rho^\nu \Gamma^n(p'_e)|^2 \right\}^{\frac{1}{3}} \frac{2}{\mbox{Sp}[\rho^\nu \Gamma^n(p'_e) |^2]},$$

$$\tilde{F}_3 = \tilde{G}_{30}(s, Q^2, s').$$  (22)

We have taken into account that the equation $\tilde{G}_{10}(s, Q^2, s') = 0$ is valid in weak sense. The parametrization (1), (2) takes the form (21), (22) with $\tilde{G}_{10}(s, Q^2, s') \to \tilde{f}_{10}(Q^2)$. It is easy to see that for the redefined form factors the equality (18) remains valid. Form factors $\tilde{f}_{10}(Q^2)$ are connected with charge, quadrupole and magnetic Sachs form factors: $G_C(Q^2) = \tilde{f}_{10}(Q^2), G_Q(Q^2) = (2 M^2/Q^2) \tilde{f}_{12}(Q^2), G_M(Q^2) = -M_c \tilde{f}_{30}(Q^2)$. The modified impulse approximation (MIA) can be formulated in terms of form factors $\tilde{G}_{10}(s, Q^2, s')$. The physical meaning of this approximation is considered in detail in Ref. [1,3].

The results of calculations for the $\rho$ – meson charge form factor in MIA ($l = l' = 0$) are represented in Fig.1.

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