Hilbert scales and Sobolev spaces defined by associated Legendre functions

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Abstract

In this paper we study the Hilbert scales defined by the associated Legendre functions for arbitrary integer values of the parameter. This problem is equivalent to study the left–definite spectral theory associated to the modified Legendre equation. We give several characterizations of the spaces as weighted Sobolev spaces and prove identities among the spaces corresponding to lower regularity index.

Key words: Legendre Functions, Sobolev Towers, Hilbert Spaces, Spherical Harmonics

MSC: 34B30, 46E35, 47B25, 33C55

1 Introduction and motivation

In this paper we deal with the Hilbert scales generated by the sets of Legendre functions. The concept of Hilbert scale is related (in many ways it is equivalent and leads to the same sets) to the left– and right–definite spectral theories for differential operators and to the construction of the so–called Sobolev towers. We will clarify this point in the following section. We first remark that the study of the Hilbert scales defined by Legendre polynomials (and actually by some other sets of orthogonal polynomials) has been subject of active research (see [3], [5], [10]). The aim is often the study of the spaces of the domain of definition of iterated powers (and also powers of the square root) of a given unbounded self–adjoint operator that stems from a differential equation, whose spectral set is a known sequence of orthogonal polynomials. Part of the interest of that study is being able to characterize the spaces as weighted Sobolev spaces.

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In this paper we extend the study of the Hilbert scales for Legendre polynomials to the sequence of Hilbert scales defined by associated Legendre functions, which are the eigenfunctions of the modified Legendre operator in \((-1, 1)\)

\[-((1 - t^2)u')' + \frac{m^2}{(1 - t^2)} u\]

for positive integer values of \(m\) (the case \(m = 0\) corresponds to the Legendre polynomials). Apart from the theoretical interest of extending this study to new families of spaces, including some new results where we will be able to describe the spaces in several different ways and to identify the ‘central part’ of the Hilbert scales for different values of \(m\), we are now going to try to motivate this study from the point of view of ongoing research in boundary integral operators of sphere-like bodies.

A basis of spherical harmonics can be built as follows. Consider the normalized associated Legendre functions \(Q^m_n\) (the precise definition is given in (1) and (2) in terms of the Legendre polynomials). Then, the functions

\[Y^m_n(\theta, \phi) := Q^{|m|}_n(\cos \theta) \exp(\pm m \phi), \quad n = 0, 1, \ldots, \quad -n \leq m \leq n\]

form an orthonormal basis of \(L^2(S)\), \(S\) being the three-dimensional sphere parametrized in spherical coordinates, \((\theta, \phi) \in [0, \pi] \times [0, 2\pi]\), which has \(\sin \theta \, d\theta \, d\phi\) as surface element. The Sobolev spaces on the sphere \(H^s(S)\) for \(s > 0\) can be described as the set of functions \(g\) for which

\[\sum_{n=0}^{\infty} \sum_{m=-n}^{n} (2n + 1)^{2s} |\hat{g}_{n,m}|^2 < \infty,\]

where \(\hat{g}_{n,m}\) are the Fourier coefficients of \(g\) with respect to the Hilbert basis \(Y^m_n\) of \(L^2(S)\). For negative values of \(s\), a norm can be defined by duality (pivoting around \(L^2(S)\)) or by completing the space of spherical harmonic polynomials (linear combinations of the \(Y^m_n\)) with respect to the norms above, which are well defined when the number of non-zero coefficients is finite. Most of the work related to integral operators (which is very relevant in scattering theory: [6], [7], [13]) uses the basis of spherical harmonics by grouping in terms of the eigenvalues. This means that we group the functions \(Y^m_n\) for \(-n \leq m \leq n\) and then consider all values of \(n\). This treatment gives an orthogonal decomposition of the spaces \(H^s(S)\) as a sum of finite dimensional spaces (with dimensions growing linearly in \(n\)). However, we can think of the spaces defined by closing span \(\{Y^m_n : n \geq |m|\}\) with the Sobolev norms above. This means that we group the terms in the norm differently like

\[\sum_{m=-\infty}^{\infty} \left( \sum_{n=|m|}^{\infty} (2n + 1)^{2s} |\hat{g}_{n,m}|^2 \right).\]

Apart from the exponential common factor \(\exp(\pm m \phi)\) in \(Y^m_n\), what we have to study are then spaces created by closing span \(\{Q^m_n(\cos \theta) : n \geq m\}\) for different values of \(m \geq 0\). This will give a different orthogonal decomposition of the Sobolev spaces on the sphere as a countable sum of infinite-dimensional spaces, that are basically spaces defined along the generatrix of the sphere (a half-circle) rotated and multiplied with the functions \(\exp(\pm im \phi)\). Finally, instead of working with the functions \(Q^m_n(\cos \theta)\) and including
the \( \sin \theta \) weight from the surface measure, we can directly study the spaces related to \( \operatorname{span} \{ Q_n^m : n \geq m \} \) as subspaces of \( L^2(-1,1) \) for different values of \( m \). This study is motivated by our wish to understand the behavior of the sequence of one-dimensional integral equations that arises when some particular numerical methods are applied to boundary integral equations of the sphere or on any smooth axisymmetric body in \( \mathbb{R}^3 \). 

The paper is structured as follows. In Section 2 we introduce the Hilbert scales defined by the associated Legendre functions after having shown how Hilbert scales can be defined in several equivalent ways. The spaces will be denoted \( H^s_m \) where \( m \geq 0 \) is the parameter in the Legendre function and \( s \in \mathbb{R} \) is the regularity index. The case \( m = 0 \) will correspond to the well-studied case of the left-definite spectral theory of Legendre’s equation. In Section 3, we show that the abstract construction of the Hilbert scales departing from the Legendre function is equivalent to the constructions that arise from both the weak and strong forms of the modified Legendre differential equation. In particular, we show how the spaces \( H^1_m \) coincide for all values of \( m \neq 0 \). We advance in the same direction in Section 4 to prove that all the spaces \( H^m_m \) for \( m \geq 2 \) are equal and that \( H^0_0 \) and \( H^2_0 \) are strict different supersets of them. In Section 5 we give an alternative expression of all the spaces \( H^k_m \) for non-negative integer value of \( k \) as weighted Sobolev spaces. We use this characterization in Section 6 to prove some useful additional properties of these spaces. Finally in Section 7 we show that given \( k \geq 0 \) all the spaces \( H^k_m \) are equal for \( m \geq k \) and study how the remaining ones behave. A first appendix is devoted to collect several purely technical lemmas and a second one to sketch how the particular case \( m = 0 \) (which had already been analyzed in the literature) can be studied with the techniques of this paper.

**Background material.** Throughout the paper we will be using the space

\[ \mathcal{D}(-1,1) = \{ \varphi \in C^\infty(-1,1) : \operatorname{supp} \varphi \subset (-1,1) \} \]

and the classical Sobolev spaces \( H^k(-1,1) \) and \( H^0_0(-1,1) \) for non-negative integer values of \( k \). For elementary properties of these spaces we refer the reader to any textbook or monograph on Sobolev theory or elliptic PDEs (for example, [2]). We will also use the spaces

\[ H^k_{\text{loc}}(-1,1) = \{ u : (-1,1) \to \mathbb{R} : u|_{(-a,a)} \in H^k(-a,a) , \ \forall a \in (0,1) \} \]

\[ = \{ u : (-1,1) \to \mathbb{R} : \varphi u \in H^k(-1,1) , \ \forall \varphi \in \mathcal{D}(-1,1) \} . \]

The \( L^1 \)-based Sobolev space,

\[ W^{1,1}(-1,1) := \{ u \in L^1(-1,1) : u' \in L^1(-1,1) \} \]

will also appear in the sequel. All along the paper, derivatives will be understood in the sense of distributions in \((-1,1)\). Whenever a derivative of a function appears, it will be implicit that from the conditions given to the function, it can be proved that the function is locally integrable in \((-1,1)\) and therefore it can be understood as a distribution. The measure in all integrals will be the Lebesgue measure and we will commonly shorten

\[ \int_{-1}^{1} f = \int_{-1}^{1} f(t) dt \]
to alleviate many expressions to come from the explicit presence of both the variable and the symbol for the Lebesgue measure.

We will also make repeated use of this elementary form of the integration by parts formula.

**Lemma 1.1** Assume that $g \in W^{1,1}_0(-1,1) = \{ u \in W^{1,1}(-1,1) : u(-1) = u(1) = 0 \}$ and that $g' = g_1 + g_2$ with $g_1, g_2 \in L^1(-1,1)$; then

$$\int_{-1}^{1} g_1 = -\int_{-1}^{1} g_2.$$

## 2 Definitions

**First construction of Hilbert scales.** The following construction is based on how Hilbert scales are commonly introduced in the literature of integral equations: see [14], which bases this part in [4]. Let $H^0$ be a separable real Hilbert space and $\{ \psi_n \}$ an orthonormal basis of the space. We consider a sequence of positive numbers $\lambda_n$ such that $\lambda_n \to \infty$. The following collection of norms for $s \in \mathbb{R}$

$$\| u \|_s := \left( \sum_{n=1}^{\infty} \lambda_n^{2s} |(u, \psi_n)_{H^0}|^2 \right)^{1/2},$$

is well defined in the set $T := \text{span} \{ \psi_n : n \geq 1 \}$. For $s > 0$ we can define

$$H^s := \{ u \in H^0 : \| u \|_s < \infty \}.$$

For negative $s$ we have two options: (a) take the completion of $T$ with the norm $\| \cdot \|_s$; (b) define $H^s$ as the representation of the dual space of $H^{-s}$ when $H^0$ is identified with its dual space. Both constructions lead to isometrically isomorphic definitions of the spaces $H^s$ for negative $s$. The resulting chain of spaces is what is known as a Hilbert scale (see [14] and [4]). Note that $H^r \subset H^s$ for all $r > s$ with compact and dense inclusion. We also have the direct estimate for the size of the norms

$$\| u \|_s \leq (\min_n \lambda_n)^{-\varepsilon} \| u \|_{s+\varepsilon}, \quad \forall u \in H^{s+\varepsilon}, \quad \forall \varepsilon > 0.$$

The set $T$ is dense in $H^s$ for all $s$. An element of $H^s$ can be written as a convergent series

$$u = \sum_{n=1}^{\infty} (u, \psi_n)_{H^0} \psi_n \quad \text{(convergent in $H^s$)},$$

where the coefficients $(u, \psi_n)_{H^0}$ are defined as $H^0$ inner products when $s > 0$ and as duality products for $s < 0$. Moreover the couple formed by $H^s$ and $H^{-s}$ is a dual pair, where each of the spaces can be understood as the dual space of the other one and its duality bracket is just the extension of the $H^0$ inner product. The spaces $H^s$ are interpolation spaces, so $[H^r, H^s]_\gamma = H^{(1-\gamma)r+\gamma s}$ for any $r \neq s$ and $\gamma \in (0,1)$.
**Second construction.** Assume now that $X$ is another Hilbert space such that $X \subset H^0$ with compact and dense inclusion. Then, we can define the operator $G : H^0 \to H^0$ that associates $u = Gf$, where $u$ solves

$$u \in X, \quad \text{s.t.} \quad (u,v)_X = (f,v)_{H^0}, \quad \forall v \in X.$$ 

Then $G$ is selfadjoint, compact and positive definite. Therefore by Hilbert–Schmidt’s theorem, we can write

$$Gf = \sum_{n=1}^{\infty} \lambda_n^{-2} (f, \phi_n)_{H^0} \phi_n,$$

where: (a) $\{\phi_n\}$ is an orthonormal basis of $H^0$; (b) the sequence $\{\lambda_n^2\}$ is non-decreasing and diverges to $+\infty$; (c) $\{\phi_n\}$ is complete orthogonal in $X$.

Note that $\|\phi_n\|_X = \lambda_n$ and that $(\phi_n; \lambda_n^2)$ are eigenpairs for the problem that defines $G$:

$$(\phi_n, v)_X = \lambda_n^2 (\phi_n, v)_{H^0} \quad \forall v \in X.$$ 

The sequence $\{(\phi_n; \lambda_n) : n \geq 1\}$ defines a Hilbert scale $H^s$ and we can see that: $H^1 = X$ with the same inner product; the space $H^s$ is the range of the $s/2$–th power of $G$

$$H^s = \mathcal{R}(G^{s/2}), \quad \forall s > 0.$$ 

The operator $G$ can be naturally extended (restricted when $s > 0$) to $G : H^s \to H^{s+2}$ and it is an isometric isomorphism between these pairs of spaces. Furthermore $H^{-1}$ is the representation of $X'$ that appears when we identify $(H^0)'$ with $H^0$, i.e.,

$$X = H^1 \subset H^0 \cong (H^0)' \subset H^{-1} = X'$$

is a Gelfand triple. We can restart the construction from the point of view of the unbounded selfadjoint operator $A : D(A) \subset H^0 \to H^0$, where $D(A) = \mathcal{R}(G)$ and $A = G^{-1}$ in this domain. From the point of view of $A$, $(\phi_n; \lambda_n^2)$ are just the eigenpairs of the associated Sturm–Liouville problem. The spaces for integer values constitute a Sobolev tower in the sense described in [9, Chapter 2]. If we start with a symmetric differential operator $A$, different choices of $D(A)$ will lead to different Hilbert scales: for instance, $Au := -u'' + u$ leads to the Fourier series of sines (by taking Dirichlet conditions), cosines (Neumann conditions) or sines and cosines (periodic conditions). Starting the construction directly from the differential operator, an adequate choice of boundary conditions leads to the concept of left–definite spectral theory, which is equivalent to this construction.

**Hilbert scales with Legendre functions.** Consider the Legendre polynomials

$$P_n(t) := \frac{(-1)^n}{2^n n!} \frac{d^n}{dt^n} ((1 - t^2)^n), \quad n \geq 0$$

and the associated Legendre functions

$$P_n^m(t) := (1 - t^2)^{m/2} P_n^{(m)}(t), \quad 0 \leq m \leq n, \quad (1)$$
normalized as
\[ Q_n^m := c_{n,m} P_n^m; \quad c_{n,m} := \left( \frac{2n + 1}{2} \frac{(n - m)!}{(n + m)!} \right)^{1/2}, \quad 0 \leq m \leq n. \]  
\[ \text{(2)} \]

Then, for any \( m \), \( \{Q_n^m : n \geq m\} \) is an orthonormal basis of \( L^2(-1,1) \). From now on we will write
\[ \omega(t) := \sqrt{1 - t^2}. \]

The modified Legendre differential operator
\[ L_m u := -(\omega^2 u')' + m^2 \omega^{-2} u \]
has \( Q_n^m \) as eigenfunctions:
\[ L_m Q_n^m = n(n + 1) Q_n^m. \]  
\[ \text{(3)} \]

Let \( H^s_m \) with \( s \in \mathbb{R} \) be the Hilbert scale defined by \( \{(Q_n^m; 2n + 1) : n \geq m\} \). Their respective norms will be denoted \( \| \cdot \|_{m,s} \). The upper index in all spaces will be a regularity index, taking values in all of \( \mathbb{R} \), while we use the lower index (the integer \( m \geq 0 \)) to mark the different scales. As a reminder, for \( \mathbb{R} \ni s \geq 0 \) and \( \mathbb{Z} \ni m \geq 0 \), elements of \( H^s_m \) are functions \( u : (-1,1) \to \mathbb{R} \) such that
\[ \| u \|_{m,s}^2 = \sum_{n=m}^{\infty} (2n + 1)^{2s} |(u, Q_n^m)_{L^2(-1,1)}|^2 < \infty. \]

Since
\[ 2 \leq \frac{2n + 1}{\sqrt{n(n + 1)}} \leq \frac{3}{\sqrt{2}}, \quad n \geq 1, \]
the scales can be defined equivalently with the pairs \( (Q_n^m; \sqrt{n(n + 1)}) \), with the only exception of the scale associated to \( m = 0 \), where we have to take \( (Q_0^0; 1) \) instead of \( (Q_0^0; 0) \) to avoid cancelation of the first coefficient. We will keep the first choice to avoid this singular case and also to fit into the frame of spherical harmonics which is the original motivation of this work. Modifications to use \( \sqrt{n(n + 1)} \) or the even simpler values \( n + 1 \) are simple, although they change the values of the constants in many of the inequalities to follow.

The space of univariate polynomials will be denoted by \( \mathbb{P} \), with \( \mathbb{P}_n \) denoting the space of polynomials of degree not greater than \( n \). A relevant set throughout will be
\[ \omega^m \mathbb{P} = \{ \omega^m p : p \in \mathbb{P} \} = \text{span}\{Q_n^m : n \geq m\} = \text{span}\{P_n^m : n \geq m\}. \]

For easy reference, let us write down two properties of the weight function \( \omega \):
\[ (\omega^\beta)' = -\beta t \omega^{\beta-2}, \]  
\[ \omega^\alpha \in L^2(-1,1) \iff \alpha > -1. \]  
\[ \text{(4)} \]
\[ \text{(5)} \]

In [1] we have denoted by \( t \) the monomial of degree one, that is, the function \( p(t) = t \). We will maintain this notation henceforth.
3 An alternative definition

The study of the spaces generated by Legendre polynomials, which corresponds to \( m = 0 \) in the present work, has already been undertaken in \cite{10} (see also \cite{3} for a previous study and \cite{5} for a more general theory covering some classical families of orthogonal polynomials). The analysis there is based on rewriting the inner product defined by the powers of the Legendre differential operator. Roughly speaking, by integrating by parts, this product is shown to be equal to a sum of weighted \( L^2 \) inner products of the derivatives of the functions. Hence, as a simple byproduct, these spaces are identified with weighted Sobolev spaces, namely,

\[
H^k_0 = \{ u \in L^2(-1,1) : \omega^k u^{(\ell)} \in L^2(-1,1), \quad 0 \leq \ell \leq k \}
\]

with equivalent norms. Note that this is going to be a particular case of the study we do in Section 5 (see Theorem 5.1), where we generalize the above result for any \( m \).

In this section, we are going to derive again the Hilbert scales \( H^*_m \) for \( m \geq 1 \) using the weak form of Legendre’s equation. This will serve us to start obtaining weighted Sobolev type expressions for the spaces for positive integer values of the regularity index and to conclude properties on how the scales coincide for small values of the regularity index.

The corresponding theory for Legendre polynomials (the case \( m = 0 \) in this paper) is already known (see \cite{3}). For the sake of completeness, we will sketch the basic results (in parallel to those of this section for \( m \geq 1 \)) in Appendix B.

Let

\[
Y := \{ u : (-1,1) \to \mathbb{R} : \omega^{-1} u, \quad \omega u' \in L^2(-1,1) \},
\]

endowed with the norm

\[
\left( \int_{-1}^1 \omega^2 |u'|^2 + \int_{-1}^1 \omega^{-2} |u|^2 \right)^{1/2}.
\]

It is simple to see that \( Y \) is a Hilbert space. Since \( u = \omega(\omega^{-1} u) \) and \( |\omega(t)| \leq 1 \), then \( Y \subset L^2(-1,1) \).

Proposition 3.1 \( Y \subset H^1_{loc}(-1,1) \subset \mathcal{C}(-1,1) \).

Proof. Let \( u \in Y \). Using the fact that \( \omega^{-1} \in \mathcal{C}^\infty(-1,1) \), it follows that \( u' = \omega^{-1}(\omega u') \in L^2_{loc}(-1,1) \) and therefore \( u \in H^1(-a,a) \subset \mathcal{C}[-a,a] \) for all \( 0 < a < 1 \).

Since the rule \( (u^2)' = 2u u' \) holds in \( H^1_{loc}(-1,1) \), it holds in \( Y \). Note also that \( \mathbb{P}_0 \cap Y = \{0\} \). The following set

\[
\mathcal{C}_0 := \{ u \in \mathcal{C}[-1,1] : u(-1) = u(1) = 0 \}
\]

will be relevant in the sequel.

Proposition 3.2 \( Y \subset \mathcal{C}_0 \) with continuous injection.

Proof. Using the definition we prove that if \( u \in Y \), then \( u u' = (\omega^{-1} u)(\omega u') \in L^1(-1,1) \). Therefore \( u^2 \in W^{1,1}(-1,1) \subset \mathcal{C}[-1,1] \) and \( u \in \mathcal{C}[-1,1] \). Since \( \omega^{-1} u \in L^2(-1,1) \) and \( u \) is continuous near the two singularities of \( \omega^{-1} \), necessarily \( u \) has to vanish in both of them. □
Lemma 3.3 If \( u \in H^1_0(-1, 1) \) then:

(a) \( \omega^{-1}u \in C_0 \),

(b) \( \omega u \in H^1_0(-1, 1) \).

Proof. Let \( u \in H^1_0(-1, 1) \). Note first that \( \omega^{-1}u \in C(-1, 1) \). We can write

\[
u(t) = \int_{-1}^t u'(s)ds
\]

and therefore

\[|u(t)| \leq \sqrt{1+t} \left( \int_{-1}^t |u'(s)|^2 ds \right)^{1/2}, \quad \forall t \in [-1, 1].\]

Using Lebesgue’s Theorem we prove that

\[|\omega^{-1}(t)u(t)| \leq \frac{1}{\sqrt{1+t}} \left( \int_{-1}^t |u'(s)|^2 ds \right)^{1/2} \to 0 \text{ as } t \to 1.\]

The limit for \( t \to 1 \) is obtained similarly.

To prove the second statement, note first that \( \omega u \in L^2(-1, 1) \) (because \( \omega \) is bounded) and

\[(\omega u)' = -t\omega^{-1}u + \omega u' \in L^2(-1, 1)\]

because of the first part of the Lemma. Finally, \( \omega u \in C[-1, 1] \) and the limits at both extreme points of the interval are zero because both \( \omega \) and \( u \) vanish there. \( \Box \)

Proposition 3.4 For all \( m \geq 0 \)

\[\{\omega^m u : u \in H^1_0(-1, 1)\} \subset Y.\]

Proof. From Lemma 3.3(a) and the boundedness of \( \omega \) it is clear that \( H^1_0(-1, 1) \subset Y \). By Lemma 3.3(b) it follows that if \( u \in H^1_0(-1, 1) \), then \( \omega^m u \in H^1_0(-1, 1) \) for \( m \geq 1 \), which finishes the proof. \( \Box \)

Lemma 3.5 If \( u \in Y \), \( p \) is a polynomial and \( m \geq 1 \), then

\[\omega^2(\omega^m p)'u \in H^1_0(-1, 1).\]

Proof. Assume first that \( m \) is even. Then \( \omega^m p \in \mathbb{P} \) and \( \omega^2(\omega^m p)' \in \mathbb{P} \), so \( \omega^2(\omega^m p)'u \in C[-1, 1] \subset L^2(-1, 1) \). Taking the derivative

\[(u\omega^2(\omega^m p)')' = u'(\omega^2(\omega^m p)') + u(\omega^2(\omega^m p)')'.\]

Note \( \omega u' \in L^2(-1, 1) \), \( \omega \) is bounded and \( (\omega^m p)' \) is a polynomial, as is \( \omega^2(\omega^m p)' \). This proves that \( (u\omega^2(\omega^m p)')' \in L^2(-1, 1) \).

If \( m \) is odd, then \( \omega(\omega^m p)' \in \mathbb{P} \) and therefore \( \omega^2(\omega^m p)'u \in C[-1, 1] \subset L^2(-1, 1) \). For the derivative we decompose

\[(u\omega^2(\omega^m p)')' = (u' \omega - tu \omega^{-1})\omega(\omega^m p)' + u\omega(\omega^m p)''.\]
and note that 
\[ u', u^{-1}, u \in L^2(-1, 1), \quad \omega^m p' \in \mathbb{P}, \]
so the derivative is in \( L^2(-1, 1) \).

Finally, in both cases the function is continuous and vanishes at the extremes of the interval. Therefore it is in \( H^1_0(-1, 1) \). \( \square \)

**Proposition 3.6** \( \omega^m p \) is dense in \( Y \) for all \( m \geq 1 \).

**Proof.** Suppose that
\[
\int_{-1}^{1} \omega^2 u'(\omega^m p)' + m^2 \int_{-1}^{1} \omega^{-2} u \omega^m p = 0, \quad \forall p \in \mathbb{P}.
\]
By Lemma 3.5 we can apply integration by parts and obtain
\[
\int_{-1}^{1} u \left( -(\omega^2 (\omega^m p)' + \frac{m^2}{\omega^2} \omega^m p) \right) = \int_{-1}^{1} u \mathcal{L}_m(\omega^m p) = 0, \quad \forall p \in \mathbb{P}.
\]
Taking \( p = P_{n}^{(m)} \) or equivalently \( \omega^m p = P_{n}^{m} \) and applying that
\[
\mathcal{L}_m P_{n}^{m} = n(n+1) P_{n}^{m},
\]
it follows that
\[
n(n+1) \int_{-1}^{1} u P_{n}^{m} = 0, \quad \forall m \geq 1.
\]
This implies that \( u = 0 \) since \( \{ P_{n}^{m} : n \geq m \} \) is an orthogonal basis of \( L^2(-1, 1) \). \( \square \)

**Theorem 3.7** \( \mathcal{D}(-1, 1) \) is dense in \( Y \).

**Proof.** Note that \( \omega^2 \mathbb{P} \subset H^1_0(-1, 1) \subset Y \) by Proposition 3.4. By Proposition 3.6 the first subset is dense in \( Y \). Moreover, \( \mathcal{D}(-1, 1) \) is dense in \( H^1_0(-1, 1) \) with the norm of this last space. \( \square \)

Take now \( m \geq 1 \). As a consequence of Theorem 3.7 the problem of finding \( \lambda \in \mathbb{R} \) and non–trivial \( u \in Y \) such that
\[
\mathcal{L}_m u = \lambda u \quad \text{in } (-1, 1) \tag{7}
\]
is easily seen to be equivalent to finding \( \lambda \in \mathbb{R} \) and non–trivial \( u \in Y \) satisfying
\[
(u, v)_m = \lambda(u, v)_{L^2(-1, 1)}, \quad \forall v \in Y,
\]
where
\[
(u, v)_m := \int_{-1}^{1} \left( \omega^2 u' v' + \frac{m^2}{\omega^2} u v \right).
\]
Consider now for \( m \geq 1 \) the set of functions \( \{ Q_{n}^{m} : n \geq m \} \). They form an orthonormal basis of \( L^2(-1, 1) \). On the other hand, because \( Q_{n}^{m} \) solves Legendre’s equation (7) with \( \lambda = n(n+1) \), then
\[
(Q_{n}^{m}, Q_{\ell}^{m})_m = n(n+1) \delta_{n,\ell}.
\]
This means that \( \{Q^m_n : n \geq m\} \) is orthogonal in \( Y \). By Proposition \( \text{3.6} \) \( \text{span} \{Q^m_n : n \geq m\} = \omega^m \mathbb{P} \) is dense in \( Y \), so \( \{Q^m_n : n \geq m\} \) is a complete orthogonal set in \( Y \) and we can characterize
\[
Y = \{ u \in L^2(-1, 1) : \sum_{n=m}^{\infty} n(n+1)|\langle u, Q^m_n \rangle_{L^2(-1,1)}|^2 < \infty \}.
\]
The injection of \( Y \) into \( L^2(-1, 1) \) is therefore compact. Let \( f \in L^2(-1, 1) \) and let \( G_m f := u \) be the solution of
\[
u \in Y, \quad \text{s.t.} \quad (u,v)_m = (f,v)_{L^2(-1,1)} \quad \forall v \in Y.
\]
Standard arguments show that \( G_m \) is self-adjoint, compact and injective. Its Hilbert-Schmidt decomposition is given by the orthonormal system \( \{Q^m_n : n \geq m\} \). Actually
\[
G_m f = \sum_{n=m}^{\infty} \frac{1}{n(n+1)} \langle f, Q^m_n \rangle_{L^2(-1,1)} Q^m_n.
\]
The set \( Y \) can be characterized as the range of \( G_m^{1/2} \). We thus have proved the following theorem:

**Theorem 3.8** For all \( m \geq 1 \), \( H^1_m = Y \). However, \( H^1_0 \neq Y \).

**Proof.** The first assertion has already been proved. Since \( \mathbb{P}_0 \subset \mathbb{P} \subset H^1_0 \), but \( \mathbb{P}_0 \cap Y = \{0\} \), it is clear that \( H^1_0 \) is different from the rest of the spaces.

The positive and negative powers of \( G_m \) define a new sequence of Hilbert scales, one for each value of \( m \geq 1 \). Since \( (n(n+1))^{1/2} \approx 2n + 1 \), as explained at the beginning of Section 2, these Hilbert scales are just \( H^s_m \) with equivalent but different norms: note that for positive values of \( s \), we have obtained the spaces
\[
\mathcal{R}(G_m^{s/2}) = \{ u \in L^2(-1, 1) : \sum_{n=m}^{\infty} (n(n+1))^s |\langle u, Q^m_n \rangle_{L^2(-1,1)}|^2 < \infty \}
\]
which are obviously the same as \( H^s_m \).

**Corollary 3.9** For all \( m, m' \geq 1 \) and \(-1 \leq s \leq 1\), \( H^s_m = H^s_{m'} \).

**Proof.** Since \( H^0_m = L^2(-1, 1) \) and \( H^1_m = Y \) for all \( m \geq 1 \), by interpolation we can prove that the scales \( H^s_m \) are independent of \( m \) for \( 0 \leq s \leq 1 \) (they have different but equivalent norms). For \(-1 \leq s < 0\) the result follows by duality. \( \square \)

### 4 The second set of identifications

**Proposition 4.1** For all \( m \geq 1 \),
\[
H^2_m = \{ u \in Y : \mathcal{L}_m u \in L^2(-1,1) \}
\]
and there exist \( c_m, C_m > 0 \) such that
\[
c_m \|u\|_{m,2} \leq \|\mathcal{L}_m u\|_{L^2(-1,1)} \leq C_m \|u\|_{m,2}, \quad \forall u \in H^2_m.
\]
Moreover, \( H^1_1 \neq H^2_m \) for all \( m \geq 2 \).
Proof. Using the fact that $H^2_m = \mathcal{R}(G_m)$ and the definition of $G_m$ given by solving problems (8), (10) follows readily. Since we can use the image norm of $\mathcal{R}(G_m)$ as an equivalent norm, we only need to prove that

$$\|\mathcal{L}_m u\|_{L^2(-1,1)}^2 = \sum_{n=m}^{\infty} (n(n+1))^2 |(u, Q_n^m)_{L^2(-1,1)}|^2, \quad \forall u \in \mathcal{R}(G_m).$$

This can be easily done in $\omega^m \mathbb{P} = \text{span} \{Q_n^m : n \geq m\}$ using the eigenvalue property for the Legendre functions (3), their $L^2(-1,1)$ orthonormality and the density of $\omega^m \mathbb{P}$ in $\mathcal{R}(G_m)$.

From (10) it follows that $\omega \in H^2_0$ ($\omega^\mathbb{P}$ is dense in $H^2_0$) but $\omega \not\in H^2_m$ for $m \geq 2$. To see that notice that if $u \in H^2_m \cap H^2_0$, we would need that $\omega^{-2} u \in L^2(-1,1)$ (compare the differential characterizations (10) of these spaces), which is not the case for $u = \omega$. □

Proposition 1.1 allows us to recommence the construction of the Hilbert scales for $m \geq 1$ in a different way, using the unbounded operator $\mathcal{L}_m : D(\mathcal{L}_m) \subset L^2(-1,1) \rightarrow L^2(-1,1)$ with domain

$$D(\mathcal{L}_m) := \{u \in Y : \mathcal{L}_m u \in L^2(-1,1)\},$$

with the operator $\mathcal{L}_m$ applied in the sense of distributions. Once $\mathcal{L}_m$ is shown to be selfadjoint, the Hilbert scale (or Sobolev tower as explained in [9], or left–definite spectral sets) can be constructed again. Note that is relevant that we demand $u \in Y$ as part of the conditions for a function to be in the domain of the operator but once that is done, we only require the image of the differential operator to be in $L^2(-1,1)$. In this sense, the cases $m \geq 1$ differ in an essential way from the case $m = 0$, where some additional boundary conditions appear in the domain of the differential operator that are not in the ‘energy space’ (see Appendix B and [5] and references therein).

Consider the space

$$Z := \{u : (-1,1) \rightarrow \mathbb{R} : \omega^{-2} u, u', \omega^2 u'' \in L^2(-1,1)\}$$

endowed with its natural norm

$$\|u\|_Z := \left( \int_{-1}^{1} \omega^{-4} |u|^2 + |u'|^2 + \omega^4 |u''|^2 \right)^{1/2}.$$

Note that the elements of $Z$ are in $H^2_{\text{loc}}(-1,1) \subset C^1(-1,1)$ (the argument is identical to that of Proposition 3.1). The remainder of this section is going to be devoted to proving the following result:

**Theorem 4.2** For all $m \geq 2$, $H^2_m = Z$ with equivalent norms. Moreover $Z \subset H^2_0 \cap H^2_1$ (it is a strict subset) and both $H^2_2 \setminus H^2_0$ and $H^2_0 \setminus H^2_1$ are non–empty.

We start the process of proving Theorem 4.2 by showing some key properties.

**Proposition 4.3** For $m \geq 2$, $\omega^m \mathbb{P} \subset Z \subset H^2_0(-1,1) \subset Y$. The injection of $Z$ in $H^2_0(-1,1)$ is continuous.
Proof. The assertion \( \omega^m \mathbb{P} \subset Z \) for \( m \geq 2 \) follows directly from the definition of \( Z \). Note also that \( Z \subset H^1(-1,1) \subset C[-1,1] \). From the definitions of \( Z \) and \( Y \) it is clear that \( Z \subset Y \), but Proposition 4.2 shows that elements of \( Y \) vanish in \( \pm 1 \), which proves that \( Z \subset H^1_0(-1,1) \).

Lemma 4.4 Let \( u \in Z \). Then \( \omega^{-1}u, \omega u' \) and \( uu' \) are in \( C_0 \).

Proof. By Proposition 4.3 \( Z \subset H^1_0(-1,1) \) and then by Lemma 3.3(a), \( \omega^{-1}u \in C_0 \).

Consider now the function \( v := \omega^2u' \). It is simple to check that \( v \in H^1(-1,1) \subset C[-1,1] \) by the conditions that define \( Z \). Also \( \omega^{-2}v = u' \in L^2(-1,1) \). These two properties imply that \( v(\pm 1) = 0 \) and therefore \( v \in H^1_0(-1,1) \). We now apply Lemma 3.3(a) to prove that \( \omega u' = \omega^{-1}v \in C_0 \).

Finally \( uu' = (\omega^{-1}u)(\omega u') \) and \( C_0 \) is an algebra and therefore the last assertion is a simple consequence of the first two.

Proposition 4.5 For all \( m \geq 2 \), there exist constants \( C_m, c_m > 0 \) such that

\[
\|L_m u\| \leq \|L_m u\|_{L^2(-1,1)} \leq C_m \|u\|_Z, \quad \forall u \in Z.
\]

For \( m = 1 \) there holds an upper bound \( \|L_1 u\|_{L^2(-1,1)} \leq C_1 \|u\|_Z \) for all \( u \in Z \).

Proof. It is simple to check that if \( u \in Z \), then \( L_m u \in L^2(-1,1) \). Elementary computations show that

\[
\|L_m u\|^2_{L^2(-1,1)} = m^4 \int_{-1}^1 \omega^4|u|^2 + \int_{-1}^1 |(\omega^2u')|^2 - 2m^2 \int_{-1}^1 \omega^{-2}u(\omega^2u')' \\
= m^4 \int_{-1}^1 \omega^4|u|^2 + \int_{-1}^1 4t^2|u'|^2 + \int_{-1}^1 \omega^4|u''|^2 - 4 \int_{-1}^1 t \omega^2u'u'' \\
-2m^2 \int_{-1}^1 u' u'' + 4m^2 \int_{-1}^1 t \omega^{-2}u' u'. \quad (11)
\]

We are now going to apply Lemma 1.1 three times: (a) \( 2t \omega^2(u')^2 \in L^1(-1,1) \cap C_0 \) by Lemma 4.4 and in the decomposition

\[
(2t \omega^2(u')^2)' = 4t \omega^2 u'' + (2 - 6t^2)(u')^2,
\]
both terms are in \( L^1(-1,1) \); (b) we have a similar behavior of \( uu' \in L^1(-1,1) \cap C_0 \) (again by Lemma 4.4) with

\[
(uu')' = |u'|^2 + uu'';
\]
(c) finally \( t \omega^{-2}u^2 \in L^1(-1,1) \cap C_0 \) (Lemma 4.4) and

\[
(t \omega^{-2}u^2)' = 2t \omega^{-2}u + (1 + t^2)\omega^{-4}u^2.
\]

The above justifies using integration by parts in the last three terms of \((11)\) to obtain

\[
\|L_m u\|^2_{L^2(-1,1)} = m^4 \int_{-1}^1 \omega^4|u|^2 + \int_{-1}^1 4t^2|u'|^2 + \int_{-1}^1 \omega^4|u''|^2 + \int_{-1}^1 (2 - 6t^2)|u'|^2 \\
+2m^2 \int_{-1}^1 |u'|^2 - 2m^2 \int_{-1}^1 (1 + t^2)\omega^{-4}|u|^2 \\
= \int_{-1}^1 q_m \omega^4|u|^2 + \int_{-1}^1 p_m |u'|^2 + \int_{-1}^1 \omega^4|u''|^2 \quad (12)
\]
with
\[ p_m(t) := 2(1 + m^2 - t^2), \quad q_m(t) := m^2(m^2 - 2 - 2t^2). \]

Note that
\[ 2m^2 \leq p_m(t) \leq 2 + 2m^2, \quad m^2(m^2 - 4) \leq q_m(t) \leq m^2(m^2 - 2), \quad -1 \leq t \leq 1. \]

This proves the result for any \( m \geq 3 \) and only the upper bound for \( m = 2 \) (the constant for the lower bound of \( q_m \) cancels). In this last case we apply integration by parts only to two of the three last terms of (11). Our starting point for the lower bound is then
\[
\|\mathcal{L}_2u\|_{L^2(-1,1)}^2 = 16 \int_{-1}^1 \omega^{-4}|u|^2 + \int_{-1}^1 p_2|u'|^2 + \int_{-1}^1 \omega^4|u''|^2 + 16 \int_{-1}^1 t\omega^{-2}u' u'
\]

Applying the inequality
\[
ab \leq \frac{1}{3}a^2 + \frac{3}{4}b^2
\]
we can bound
\[
16 \int_{-1}^1 (t u)(\omega^{-2}u') \geq -\frac{16}{3} \int_{-1}^1 t^2|u'|^2 - 12 \int_{-1}^1 \omega^{-4}|u|^2
\]
and thus
\[
\|\mathcal{L}_2u\|^2 \geq 4 \int_{-1}^1 \omega^{-4}|u|^2 + \int_{-1}^1 \left(10 - 2t^2 - \frac{16}{3}t^2\right)|u'|^2 + \int_{-1}^1 \omega^4|u''|^2
\]
\[
\geq 4 \int_{-1}^1 \omega^{-4}|u|^2 + \frac{8}{3} \int_{-1}^1 |u'|^2 + \int_{-1}^1 \omega^4|u''|^2.
\]

This completes the proof for \( m = 2 \). The upper bound for \( m = 1 \) is a direct consequence of (12). \( \square \)

**Proof of Theorem 4.2.** Using Propositions 4.3 and 4.5 and the characterization of \( H^2_m \) of Proposition 4.1 it follows that \( Z \subset H^2_m \) with continuous injection. By Propositions 4.1 and 4.3 \( Z \) is a closed subspace of \( H^2_m \) for every \( m \geq 2 \). However, by Proposition 4.3 we know that \( \omega^m \mathbb{P} \subset Z \) and at the same time \( \omega^m \mathbb{P} \) is dense in \( H^2_m \) by definition. This proves that \( Z = H^2_m \) for \( m \geq 2 \) with equivalent norms.

Note that Propositions 4.1 and 4.3 (the last assertion of this one) prove that \( Z \subset H^1_0 \). To see that \( Z \subset H^2_0 \), we need to characterize \( H^2_0 \) as in (6). This will be done in the next section (it is a particular case of Theorem 5.1), although the result follows from results in [5] and related references.

It is simple to see that \( p(t) \equiv 1 \) belongs to \( H^2_0 \) but not to \( Y \) (and therefore not to \( Z \subset Y \)). Also \( \omega \in H^2_1 \), but \( \omega \notin Z \) and \( \omega \notin H^2_0 \) by (6). \( \square \)

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5 Weighted Sobolev space characterization

The following section is devoted to giving a characterization of the spaces $H^k_m$ for positive integer values of $k$ (and all $m$) as weighted Sobolev spaces. This section is independent of Sections 3 and 4 and does not use any result that appears therein.

Consider the spaces

$$X^k_m := \{ u \in L^2(-1, 1) : \omega^{m+k}((\omega^{-m}u)^{(k)}) \in L^2(-1, 1) \},$$

endowed with their natural norms:

$$\|u\|_{X^k_m} := \left( \int_{-1}^{1} \omega^{2m+2k} |(\omega^{-m}u)^{(k)}|^2 + \int_{-1}^{1} |u|^2 \right)^{1/2}.$$

It is simple to observe that due to the fact that $\omega$ is bounded in $[-1, 1]$, multiplication by $\omega$ defines a bounded linear operator from $X^k_m$ into $X^k_{m+1}$ for all $k$ and $m$. The aim of this section is the proof of the following theorem:

**Theorem 5.1** For all $m \geq 0$ and $k \geq 1$,

$$H^k_m = X^k_m$$

and there exist $C_{m,k} > 0$ such that for all $u \in H^k_m$

$$C_{m,k} \|u\|^2_{m,k} \leq (m + \frac{1}{2})^{2k} \int_{-1}^{1} |u|^2 + \int_{-1}^{1} \omega^{2(m+k)} |(\omega^{-m}u)^{(k)}|^2 \leq 2^{1-2k} \|u\|^2_{m,k}.$$

As in Section 4, we first prove some key results that will allow us to show the identifications of spaces of Theorem 5.1 at the end of this section.

**Proposition 5.2** For all $m \geq 0$ and $k \geq 1$

$$X^k_m \subset H^k_{\text{loc}}(-1, 1) \subset C^{k-1}(-1, 1).$$

Moreover, if $v \in H^k(-1, 1)$, then $\omega^m v \in X^k_m$.

**Proof.** The first part uses the same arguments as the ones in Proposition 3.1. The second part is straightforward. \( \Box \)

**Proposition 5.3** For all $m \geq 0$ and $k \geq 1$, $X^k_m \subset X^{k-1}_m$ with continuous injection. Therefore

$$X^k_m = \{ u : (-1, 1) \rightarrow \mathbb{R} : \omega^{m+\ell}((\omega^{-m}u)^{(\ell)}) \in L^2(-1, 1), \ 0 \leq \ell \leq k \}.$$ 

**Proof.** Given $u \in X^k_m$, we consider the function

$$v := \omega^{m+k}((\omega^{-m}u)^{(k)}) \in L^2(-1, 1).$$

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There holds $(\omega^{-m}u)^{(k-1)} \in H^1_{\text{loc}}(-1, 1) \subset \mathcal{C}(-1, 1)$ and we can bound

$$j(u) := (\omega^{-m}u)^{(k-1)}(0), \quad |j(u)| \leq C_{m,k} \|u\|_{X^k_m}.$$ 

Therefore

$$(\omega^{-m}u)^{(k-1)}(t) = \int_0^t (\omega^{-m}u)^{(k)}(s)\,ds + j(u) = \int_0^t \omega^{-m-k}(s)v(s)\,ds + j(u). \quad (13)$$

For all $t \in (-1, 1)$

$$\left| \omega^{m+k-1}(t) \int_0^t \omega^{-m-k}(v(s))\,ds \right|^2 \leq \left| \omega^{2m+2k-2}(t) \int_0^t \omega^{-2m-2k}(s)\,ds \right| \int_0^t |v(s)|^2\,ds$$

$$= \left| g_{m+k-1}(t) \right| \int_0^t |v(s)|^2\,ds \leq \left| g_{m+k-1}(t) \right| \|v\|^2_{L^2(-1,1)},$$

$g_{m+k-1} \in L^1(-1,1)$ being one of the functions of Lemma A.4. Therefore, using (13), it follows that

$$\omega^{m+k-1}(\omega^{-m}u)^{(k-1)} \in L^2(-1,1)$$

and its norm is controlled by the one of $u$ in $X^k_m$. $\square$

**Corollary 5.4** Let $m, k \geq 0$. Multiplication by a fixed $v \in \mathcal{C}^\infty[-1,1]$ is a linear bounded operator from $X^k_m$ to itself.

**Proof.** Let $v \in \mathcal{C}^\infty[-1,1]$ and $u \in L^2(-1,1)$. Then

$$\omega^{m+k}((\omega^{-m}u)^{(k)}) = \sum_{j=0}^k \binom{k}{j} \omega^{k-j}v^{(k-j)} \omega^{m+j}(\omega^{-j}u)^{(j)}$$

and the result is a direct consequence of Proposition 5.3. Note that the result also holds if $v \in W^{\infty,k}(-1,1) := \{v : (-1,1) \to \mathbb{R} : v^{(j)} \in L^\infty(-1,1), \quad 0 \leq j \leq k\}$. $\square$

**Lemma 5.5** If $u \in X^k_m$ with $k \geq 1$ and $m \geq 0$, then $\omega^{2m+2k}(\omega^{-m}u)^{(k-1)} \in H^1_0(-1,1)$.

**Proof.** Note that if we prove that $v := \omega^{m+k+1}(\omega^{-m}u)^{(k-1)} \in H^1_0(-1,1)$, then by Lemma 3.3(b) the result follows readily.

First of all, it is clear that $v \in L^2(-1,1)$ and that

$$v' = -(m+k+1)t \omega^{m+k-1}(\omega^{-m}u)^{(k-1)} + \omega^{m+k+1}(\omega^{-m}u)^{(k)} \in L^2(-1,1),$$

that is, $v \in H^1(-1,1) \subset \mathcal{C}[-1,1]$. Besides, $\omega^{-2}v \in L^2(-1,1)$ and therefore $v(\pm 1) = 0$, which proves the result. $\square$

**Lemma 5.6** If $u \in X^k_m$ with $m \geq 0$ and $k \geq 1$, then

$$\int_{-1}^1 (\omega^{-m}u)^{(k)} \omega^{2m+2k}p = (-1)^k \int_{-1}^1 \omega^{-m}u(\omega^{2m+2k}p)^{(k)}, \quad \forall p \in \mathbb{P}.$$
Proof. The result is true for \( p \in C^k[-1,1] \) but the argument is simpler with polynomials. Note first that by Lemma 5.6
\[
\omega^{2m+2k}(\omega^{-m}u)^{(k-1)}p \in H^1_0(-1,1)
\]
and that in the expression
\[
\left((\omega^{-m}u)^{(k-1)}\omega^{2m+2k}p\right)' = (\omega^{-m}u)^{(k)}\omega^{2m+2k}p + (\omega^{-m}u)^{(k-1)}(\omega^{2m+2k}p)'
\]
both terms on the right hand side are integrable, so can apply integration by parts to

\[
\int_{-1}^{1} (\omega^{-m}u)^{(k)}\omega^{2m+2k}p = -\int_{-1}^{1} (\omega^{-m}u)^{(k-1)}(\omega^{2m+2k}p)'.
\]
However, \((\omega^{2m+2k}p)' = \omega^{2m+2k-2}q\) with \( q \in \mathbb{P} \) and \( u \in X_m^{k-1} \). These facts allow us to apply the same argument again to obtain, when \( k \geq 2 \)
\[
\int_{-1}^{1} (\omega^{-m}u)^{(k)}\omega^{2m+2k}p = -\int_{-1}^{1} (\omega^{-m}u)^{(k-1)}\omega^{2m+2k-2}q
\]
\[
= \int_{-1}^{1} (\omega^{-m}u)^{(k-2)}(\omega^{2m+2k-2}q)' = \int_{-1}^{1} (\omega^{-m}u)^{(k-2)}(\omega^{2m+2k}p)''.
\]
The statement follows by induction. \( \square \)

Theorem 5.7 For all \( m \geq 0 \) and \( k \geq 1 \), \( \omega^m\mathbb{P} \) is dense in \( X_m^k \).

Proof. Assume that \( u \in X_m^k \) is orthogonal to all elements of the form \( \omega^m p \) with \( p \) a polynomial. This means that
\[
\int_{-1}^{1} \omega^{2m+2k}(\omega^{-m}u)^{(k)}p(k) + \int_{-1}^{1} u \omega^m p = 0, \quad \forall p \in \mathbb{P}.
\]
We can apply Lemma 5.6 to prove that
\[
(-1)^k \int_{-1}^{1} \omega^{-m}u(\omega^{2m+2k}p(k)) + \int_{-1}^{1} u \omega^m p = 0, \quad \forall p \in \mathbb{P}. \quad (14)
\]
Consider now the operator
\[
\mathcal{L}_{m,k}p := (-1)^k \omega^{-2m}(\omega^{2m+2k}p(k)) + p
\]
This is a linear differential operator with polynomial coefficients and maps \( \mathbb{P}_n \) to itself. Furthermore, if \( p \in \mathbb{P} \) and \( \mathcal{L}_{m,k}p = 0 \), then taking \( u = \omega^m p \) in (14) we derive
\[
0 = \int_{-1}^{1} (\omega^{2m}p) (\mathcal{L}_{m,k}p) = \int_{-1}^{1} \omega^{2m+2k}|p(k)|^2 + \int_{-1}^{1} \omega^{2m}|p|^2.
\]
This means that \( \mathcal{L}_{m,k} : \mathbb{P} \to \mathbb{P} \) is injective and does not increase the degree. Therefore \( \mathcal{L}_{m,k} : \mathbb{P} \to \mathbb{P} \) is a bijection. Condition (14) is thus equivalent to
\[
\int_{-1}^{1} \omega^m u q = 0, \quad \forall q \in \mathbb{P},
\]
which implies that \( u = 0 \). Therefore \( \omega^m\mathbb{P} \) is dense in \( X_m^k \). \( \square \)
Proof of Theorem 5.1. Let $\| \cdot \|_{m,k,*}$ be the norm that appears in the inequality of the statement and $(\cdot, \cdot)_{m,k,*}$ the associated inner product. Note that $\{P_m^n : n \geq m\}$ is orthogonal in $H^k_m$ and that since $\omega^m \cdot P_m^n = \omega^{m+k}P_{m+k}^n = \begin{cases} P_{m+k}^n, & \text{if } n \geq m+k, \\ 0, & \text{if } m+k > n \geq m, \end{cases}$ then

$$(P_m^n, P_{\ell}^m)_{m,k,*} = (m + \frac{1}{2})^{2k} \int_{-1}^{1} P_m^n P_{\ell}^m + \int_{-1}^{1} P_{m+k}^n P_{\ell+k}^m,$$

so these functions are also orthogonal in $X^k_m$.

For all $n \geq m$

$$\|P_m^n\|_{m,k}^2 = c_{n,m}^{-2} \|Q_m^n\|_{m,k}^2 = c_{n,m}^{-2} (2n + 1)^{2k}.$$ If $m+k > n \geq m$, then

$$\|P_m^n\|_{m,k,*}^2 = (m + \frac{1}{2})^{2k} c_{n,m}^{-2},$$

whereas for $n \geq m+k$

$$\|P_m^n\|_{m,k,*}^2 = (m + \frac{1}{2})^{2k} c_{n,m}^{-2} + c_{n,m+k}^{-2} = c_{n,m}^{-2} \left( (m + \frac{1}{2})^{2k} + \left( \frac{c_{n,m}}{c_{n,m+k}} \right)^2 \right).$$

Using Lemma A.4 we can bound

$$\frac{(m+1)!}{(m+k+1)!} \frac{(2n+1)^{2k}}{2^{2k}} \leq \left( \frac{c_{n,m}}{c_{n,m+k}} \right)^2 \leq (m + \frac{1}{2})^{2k} + \left( \frac{c_{n,m}}{c_{n,m+k}} \right)^2 \leq 2 \frac{(2n+1)^{2k}}{2^{2k}},$$

which translates into

$$C_{m,k} \|P_m^n\|_{m,k}^2 \leq \|P_m^n\|_{m,k,*}^2 \leq \frac{2}{2^{2k}} \|P_m^n\|_{m,k}^2, \quad n \geq m,$$

where

$$C_{m,k} := \min \left\{ \frac{(m+1)!}{(m+k+1)!} \frac{1}{2^{2k}} \cdot \frac{(m+1/2)^{2k}}{(2m+2k-1)^{2k}} \right\}.$$ Note that $\omega^m \cdot = \operatorname{span} \{P_m^n : n \geq m\}$ is dense in both $X^k_m$ and $H^k_m$. Using Lemma A.6 we prove that $X^k_m$ can be identified with $H^k_m$ and that the inequalities for the norms of $P_m^n$ can be extended to any element of the spaces. □

Note that as a consequence of the fact that $H^1_m = Y$ for all $m \geq 1$, it follows that

$$\{u \in L^2(-1,1) : \omega^{m+1}(\omega^{-m}u)' \in L^2(-1,1)\} = \{u : \omega^{-1}u, \omega u' \in L^2(-1,1)\},$$

as long as $m \neq 0$. 17
6 Further properties

Theorem 6.1 (Embedding theorem) Let \( u \in H^s_m \) with \( s > m + 2k + 1 \). Then \( \omega^{-m} u \in C^k[-1, 1] \). Moreover, there exist \( C_{m,s} > 0 \) such that

\[
\max_{0 \leq \ell \leq k} \| (\omega^{-m} u)^{(\ell)} \|_{L^\infty(-1,1)} \leq C_{m,s} \| u \|_{m,s}, \quad \forall u \in H^s_m.
\]

Proof. Decomposing \( u \) in \( \{Q^m_n : n \geq m\} \), which is an orthonormal basis of \( L^2(-1,1) \), we observe that

\[
u = \sum_{n=m}^{\infty} u_n Q^m_n = \omega^m \sum_{n=m}^{\infty} u_n c_{n,m} P^{(m)}_n, \quad u_n := (u, Q^m_n)_{L^2(-1,1)}.
\]

Using Lemmas A.3 and A.5, we can bound term by term

\[
|c_{n,m} P^{(m+k)}_n(t)| \leq C_m \left(2n + 1\right)^{1/2 - m} D_{m+k} \left(2n + 1\right)^{2m+2k} = C_m D_{m+k} \left(2n + 1\right)^{1/2 + m+2k}, \quad \forall t \in [-1, 1].
\]

Note that if \( s > r + 1 \), then \( 2(r-s) + 1 < -1 \) and

\[
\sum_{n=1}^{\infty} |f_n| n^{r+1/2} \leq \left(\sum_{n=1}^{\infty} |f_n|^2 \right)^{1/2} \left(\sum_{n=1}^{\infty} n^{2(r-s)+1}\right)^{1/2} =: E_{s-r} \left(\sum_{n=1}^{\infty} |f_n|^2 \right)^{1/2}.
\]

Therefore

\[
\sum_{n=1}^{\infty} |u_n c_{n,m} P^{(m+k)}_n(t)| \leq C_m D_{m+k} \sum_{n=1}^{\infty} |u_n| \left(2n + 1\right)^{m+2k+1/2} \leq C_m D_{m+k} E_{s-m-2k} \left(\sum_{n=1}^{\infty} |u_n|^2 \left(2n + 1\right)^{2s}\right)^{1/2} = C_m D_{m+k} E_{s-m-2k} \| u \|_{m,s},
\]

for all \( t \in [-1, 1] \) and \( s > 2k + m + 1 \). Therefore the series

\[
\sum_{n=1}^{\infty} u_n c_{n,m} P^{(m+k)}_n
\]

converges uniformly to continuous functions, which proves that \( \omega^{-m} u \) has \( k \) continuous derivatives in \([-1, 1]\). □

Proposition 6.2

\[
\bigcap_{k=1}^{\infty} H^k_m = \{ \omega^m g : g \in C\infty[-1, 1]\}.
\]

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Proof. By Theorem 5.1 if \( g \in C^k[-1,1] \), then \( u := \omega^m g \in H^k_m \). Therefore elements of the form \( \omega^m g \) with \( g \in C^\infty[-1,1] \) belong to \( H^k_m \) for all \( k \). The converse statement is a direct consequence of Theorem 5.1. \( \square \)

The following result gives a simple identity regarding the \( m \)-weighted forms of the natural norms of \( X^1_m \) and \( Y \). Thanks to it we will be able to prove a first set of inclusions of the spaces \( X^k_m \) for different values of \( m \).

Proposition 6.3 For all \( m \geq 1 \) and \( u \in H^1_m = Y \)

\[
m(m+1) \int_{-1}^{1} |u|^2 + \int_{-1}^{1} \omega^{m+2}|(\omega^{-m}u')|^2 = m^2 \int_{-1}^{1} \omega^{-2}|u|^2 + \int_{-1}^{1} \omega^2|u'|^2.
\]

Proof. Note that \( H^1_m = X^1_m = Y \) for all \( m \geq 1 \) and we can use many integral properties of \( u \in H^1_m \). In particular \( u \in H^1_m \) and we can apply the product rule \((u^2)' = 2uu'\). Therefore

\[
m(m+1)u^2 + \omega^{m+2}(\omega^{-m}u')^2 = m(m+1)u^2 + m^2\omega^{-2}u^2 + \omega^2(u')^2 + 2mtu' \nonumber
\]

\[
= u^2m^2(1 + \omega^{-2}t^2) + \omega^2(u')^2 + m(u^2 + 2tuu') \nonumber
\]

\[
= m^2\omega^{-2}u^2 + \omega^2(u')^2 + m(tuu'). \nonumber
\]

Using the arguments of the proof of Proposition 3.2, we can show that

\[
\int_{-1}^{1} (t^2u')' = 0, \quad \forall u \in Y,
\]

from where we obtain the equality of norms. Note that the result can also be proved by comparison of the norms of the functions \( Q^m_n \) and using Lemma A.6. \( \square \)

Corollary 6.4 Let \( m \geq 1 \). Then

\[
m\|\omega^{m-1}v\|^2_{L^2((-1,1),1)} \leq (m+1)\|\omega^m v\|^2_{L^2((-1,1),1)} + \frac{1}{m}\|\omega^{m+1}v'\|^2_{L^2((-1,1),1)}, \quad \forall v \in H^1((-1,1)).
\]

Proof. Take \( u := \omega^m v \) and note that \( u \in X^1_m \) by Proposition 5.2. Applying Proposition 6.3 to \( u \), we obtain

\[
m(m+1)\|\omega^m v\|^2_{L^2((-1,1),1)} + \|\omega^{m+1}v'\|^2_{L^2((-1,1),1)} = m^2\|\omega^{m-1}v\|^2_{L^2((-1,1),1)} + \|\omega(vm)v'\|^2_{L^2((-1,1),1)} \nonumber
\]

\[
\geq m^2\|\omega^{m-1}v\|^2_{L^2((-1,1),1)},
\]

which proves the result. \( \square \)

Proposition 6.5 Let \( m, k \geq 0 \). Then \( X^k_{m+2} \subset X^k_m \) with continuous injection. The inclusion is strict if \( m < k \).
Proof. The result is trivial for $k = 0$ and has already been proved for $k = 1$, when $X^1_m = Y$ for $m > 0$ and $Y$ is a strict subspace of $X^1_0$.

The density of $\omega^{m+2}\mathbb{P}$ in $X^k_{m+2}$ (Theorem 5.7) reduces the proof to showing that
\[
\|\omega^{m+k}(\omega^{-m}\omega^{m+2}p^{(k)})\|_{L^2(-1,1)} = \|\omega^{m+k}(\omega^{2}p^{(k)})\|_{L^2(-1,1)} \leq C_{m,k}\|\omega^{m+2}p\|_{X^k_{m+2}}, \quad \forall p \in \mathbb{P}.
\]
(15)

By Proposition 5.2 and Theorem 5.7 we could equivalently prove it for $p \in H^k(-1,1)$. Note that
\[
\omega^{m+k}(\omega^{2}p^{(k)}) = \omega^{m+k+2}p^{(k)} - 2kt\omega^{m+k}p^{(k-1)} - k(k-1)\omega^{m+k}p^{(k-2)}.
\]
(16)

The three terms on the right hand side are bounded separately. First of all
\[
\|\omega^{m+k+2}p^{(k)}\|_{L^2(-1,1)} = \|\omega^{m+2+k}(\omega^{-m-2}\omega^{m+2}p)\|_{L^2(-1,1)} \leq \|\omega^{m+2}p\|_{X^k_{m+2}}.
\]
The third term of (16) is equally easy to bound
\[
\|\omega^{m+k}p^{(k-2)}\|_{L^2(-1,1)} = \|\omega^{m+2+k-2}(\omega^{-m-2}\omega^{m+2}p)\|_{L^2(-1,1)} \leq \|\omega^{m+2}p\|_{X^k_{m+2}} \leq C_{m,k}\|\omega^{m+2}p\|_{X^k_{m+2}}
\]
by Proposition 5.3. Finally, for the second term of (16) we use Corollary 6.4 and Proposition 5.3
\[
\|\omega^{m+k}p^{(k-1)}\|_{L^2(-1,1)} \leq C_{m,k}\left(\|\omega^{m+k+1}p^{(k-1)}\|_{L^2(-1,1)} + \|\omega^{m+k+2}p^{(k)}\|_{L^2(-1,1)}\right)
\]
\[
\leq C_{m,k}\left(\|\omega^{m+2}p\|_{X^k_{m+1}} + \|\omega^{m+2}p\|_{X^k_{m+2}}\right) \leq C_{m,k}\|\omega^{m+2}p\|_{X^k_{m+2}}
\]
Using these last three bounds and (16) we prove (15).

Finally, assume that $m < k$. It is clear that $\omega^m \notin X^k_m$ (this is true for all values of $m$ and $k$). We now prove that $\omega^m \notin X^k_{m+2}$. Using Lemma A.1 it follows that
\[
\omega^{m+2+k}(\omega^{-m-2}\omega^m) = \omega^{m-k}p_{-2,k}
\]
where $p_{-2,k} \in \mathbb{P}_k$ and $p_{-2,k}(\pm 1) \neq 0$. However, for $m < k$, the singularities of $\omega^{m-k}$ at $\pm 1$ do not allow it to be in $L^2(-1,1)$ (see (5)) and the result is proved. \hfill \square

In the following section we will go further to prove that the remainder inclusions of the above proposition are just equalities of sets.

7 The last set of identifications

For any non-negative integer $k$ we consider the Hilbert space
\[
Z_k := \left\{ u : (-1,1) \to \mathbb{R} : \omega^{-k+2\ell}u^{(\ell)} \in L^2(-1,1), \quad \ell = 0, \ldots, k \right\}
\]
endowed with its natural norm:
\[
\|u\|_{Z_k} := \left( \sum_{\ell=0}^{k} \int_{-1}^{1} \omega^{2(\ell-k)}|u^{(\ell)}|^2 \right)^{1/2}.
\]

Notice that $Z_1 = Y$ and $Z_2 = Z$ which have appeared in previous sections.
Proposition 7.1 For all \( k \geq 1 \), \( Z_k \subset Z_{k-1} \) with continuous injection.

Proof. It is a straightforward application of the definition of the spaces. \( \square \)

The aim of this section is to prove the following result:

Theorem 7.2 Let \( m \) and \( k \) be non negative integers:

(a) If \( m \geq k \), then \( Z_k = X_m^k = H_m^k \) with equivalent norms.

(b) \( Z_k \subset H_k^0 \cap H_1^k \cap \ldots \cap H_{k-1}^k \) and the inclusion is strict.

The cases \( k = 1 \) and \( k = 2 \) of Theorem 7.2(a) were proved in Theorems 3.8 and 4.2.

The different assertions of this main theorem will be broken down to a small collection of properties that we now proceed to state and prove. Because of Proposition 6.5 and the equality of the sets \( X_m^k = H_m^k \), when \( k \geq 2 \), the superset in Theorem 7.2(b) is just \( H_{k-2}^k \cap H_{k-1}^k \). We will end the section distinguishing the remaining sets \( H_m^k \) with \( m < k \).

Proposition 7.3 For all \( m, k \geq 0 \), \( Z_k \subset X_m^k = H_m^k \) with continuous injection. Moreover, if \( m < k \), the inclusion is strict.

Proof. Note first that

\[
\omega^{m+k}(\omega^{-m} u)^{(k)} = \sum_{\ell=0}^{k} \binom{k}{\ell} \omega^{2k-2\ell+m}(\omega^{-m})^{(k-\ell)} \omega^{2\ell-k} u^{(\ell)}.
\]  

(17)

The functions \( \omega^{2k-2\ell+m}(\omega^{-m})^{(k-\ell)} \) that appear in (17) are polynomials and we can bound

\[
\|\omega^{m+k}(\omega^{-m} u)^{(k)}\|_{L^2(-1,1)} \leq C_{m,k} \sum_{\ell=0}^{k} \|\omega^{2\ell-k} u^{(\ell)}\|_{L^2(-1,1)} \leq \sqrt{k} C_{m,k} \|u\|_{Z_k}.
\]

This inequality proves the first assertion of the result. Note now that \( \omega^m \in X_m^k = H_m^k \) (this follows from the definition and also from Proposition 6.2). However, if \( \omega^m \in Z_k \), then \( \omega^{m-k} \in L^2(-1,1) \), which requires (see 3) that \( m - k > -1 \). Therefore \( \omega_m \notin Z_k \) if \( m \leq k - 1 \) and the inclusion of \( Z_k \) in \( X_m^k \) is strict in these cases. \( \square \)

Note that so far we have proved Theorem 7.2(b) as well as one of the inclusions needed to prove Theorem 7.2(a).

Lemma 7.4 For all \( m \geq k \), there exist \( C_{m,k} > 0 \) such that

\[
\|\omega^m u\|_{Z_k} \leq C_{m,k} \sum_{\ell=0}^{k} \|\omega^{m-k+2\ell} u^{(\ell)}\|_{L^2(-1,1)}, \quad \forall u \in H^k(-1,1).
\]

Hence, \( \omega^m H^k(-1,1) \subset Z_k \) for all \( m \geq k \).
Proof. The $Z_k$-norm of $\omega^m u$ includes $L^2(-1, 1)$-norms of terms like:

$$\omega^{-k+2\ell}(\omega^m u)(\ell) = \sum_{j=0}^{\ell} \binom{\ell}{j} \omega^{2\ell-2j-m}(\omega^m)^{(\ell-j)} \omega^{-k+2j} u(j).$$

However, by (20), it follows that $\omega^{2\ell-2j-m}(\omega^m)^{(\ell-j)} = p_{m, \ell-j} \in \mathbb{P}$, which allows us to bound

$$\|\omega^{-k+2\ell}(\omega^m u)(\ell)\|_{L^2(-1, 1)} \leq C_{k, \ell, m} \sum_{j=0}^{\ell} \|\omega^{-k+2j} u(j)\|_{L^2(-1, 1)}.$$ 

Summing for $\ell = 0, \ldots, k$, the result is proven. \hfill \Box

**Proposition 7.5** For $m \geq k$, $X^k_m \subset Z_k$ with continuous injection.

**Proof.** Note that $\omega^m \mathbb{P}$ is a dense subset of $X^k_m$ (this is Theorem 5.7) and therefore by Proposition 5.2, so is $\omega^m H^k(-1, 1)$. This density result and Lemma 7.4 show that if we are able to prove the inequalities

$$\sum_{\ell=0}^{k} \|\omega^{-k+2\ell} v(\ell)\|_{L^2(-1, 1)} \leq C_{m, k} \sum_{\ell=0}^{k} \|\omega^{m+\ell} v(\ell)\|_{L^2(-1, 1)}, \quad \forall v \in H^k(-1, 1) \quad (19)$$

for any $m \geq k$ we will have proved the result. We will do this by induction in the pair $(k, m)$.

The inequality (19) is clearly true for $(0, m)$ and any $m \geq 0$. We now assume that that it holds for the pair $(k, m)$ and proceed to prove it for $(k+1, m+1)$. However, this is a simple consequence of Corollary 6.4 as we now show. Note that

$$\sum_{\ell=0}^{k+1} \|\omega^{-m+2(k+1)+2\ell} v(\ell)\|_{L^2(-1, 1)} = \|\omega^{-m+k+2} v(k+1)\|_{L^2(-1, 1)} + \sum_{\ell=0}^{k} \|\omega^{-m+k+2+2\ell} v(\ell)\|_{L^2(-1, 1)}$$

$$\leq \|\omega^{-m+k+2} v(k+1)\|_{L^2(-1, 1)} + C_{m, k} \sum_{\ell=0}^{k} \|\omega^{m+\ell} v(\ell)\|_{L^2(-1, 1)}$$

$$\leq \|\omega^{-m+k+2} v(k+1)\|_{L^2(-1, 1)} + C'_{m, k} \sum_{\ell=0}^{k+1} \|\omega^{m+\ell+1} v(\ell)\|_{L^2(-1, 1)}$$

$$= (1 + C'_{m, k}) \sum_{\ell=0}^{k+1} \|\omega^{m+\ell+1} v(\ell)\|_{L^2(-1, 1)},$$

where we have applied the induction hypothesis and Corollary 6.4 to the functions $v(\ell) \in H^1(-1, 1)$ (since $\ell \leq k$ and $v \in H^{k+1}(-1, 1)$). \hfill \Box

Note that Theorem 7.2(a) is a direct consequence of Propositions 7.3 and 7.5

**Proposition 7.6** Let $0 \leq m, m' \leq k$. Then $H^k_{m'} \subset H^k_m$ if and only if $m' - m$ is a non-negative even number. In that case, the inclusion is strict.
Proof. Note that one of the implications is part of Proposition 6.5. However what we are going to prove is that

$$\omega^{m'} \in X^k_m \iff \left\{ \begin{array}{l} m' \geq k, \\
 m' < k \text{ and } m' - m \text{ is a non-negative even number} \end{array} \right.$$

(19)

By Lemma A.1

$$v := \omega^{m+k}(\omega^{-m}\omega^{m'})(k) = \omega^{m-k}p_{m'-m,k}, \quad p_{m'-m,k} \in \mathbb{P}_k.$$ 

If $m' \geq k$ then $v \in L^2(-1,1)$. Consider now that $m' < k$. Then, if $m' - m$ is a non-negative even number, $\omega^{-m}\omega^{m'}$ is a polynomial of degree $m' - m < k$ and therefore $v \equiv 0$. Otherwise, $p_{m'-m,k}(\pm 1) \neq 0$ and then the local behavior of $v$ in the vicinity of $\pm 1$ shows that $v \notin L^2(-1,1)$. The result is now an easy consequence of (19). □

A Technical lemmas

A.1 Functions and bounds

Lemma A.1 For all integer $k \geq 0$ and $\alpha \in \mathbb{R}$

$$(\omega^{\alpha})^{(k)} = p_{\alpha,k} \omega^{\alpha-2k}, \quad p_{\alpha,k} \in \mathbb{P}_k.$$ 

(20)

Moreover

$$p_{\alpha,k}(-1) = (-1)^k p_{\alpha,k}(1) = \prod_{j=0}^{k-1} (\alpha - 2j).$$ 

(21)

Therefore $p_{\alpha,k}(\pm 1) \neq 0$ unless $\alpha$ is a non-negative even integer and $k \geq \alpha/2 + 1$. In this last case, $p_{\alpha,k} \equiv 0$.

Proof. The result is proved by induction in $k$ (the assertion being valid for all $\alpha \in \mathbb{R}$). Note that the case $k = 0$ is trivial (giving $p_{\alpha,0} \equiv 1$) and that the case $k = 1$ is just (4), giving $p_{\alpha,1}(t) = -\alpha t$.

Assume that the result is true for all integer values up to $k$. An application of the induction hypothesis and (4) yield

$$(\omega^{\alpha})^{(k+1)} = (-\alpha t \omega^{\alpha-2})^{(k)} = (-\alpha t)(\omega^{\alpha-2})^{(k-1)} - \alpha k (\omega^{\alpha-2})^{(k-1)}$$

$$= (-\alpha t)p_{\alpha-2,k}\omega^{\alpha-2-2k} - \alpha k p_{\alpha-2,k-1}\omega^{\alpha-2-2(k-1)}$$

$$= (-\alpha tp_{\alpha-2,k} - \alpha k \omega^2 p_{\alpha-2,k-1})\omega^{\alpha-2(k+1)}$$

which gives the result for $k + 1$ as well as the formula

$$p_{\alpha,k+1} = -\alpha tp_{\alpha-2,k} - \alpha k \omega^2 p_{\alpha-2,k-1}.$$ 

In particular $p_{\alpha,k+1}(\pm 1) = (\mp \alpha)p_{\alpha-2,k}(\pm 1)$ and (21) follows by another inductive argument. The final assertions of the lemma are straightforward. □
Lemma A.2 For \( n \geq 0 \) consider the functions

\[
g_n(t) := \omega^{2n}(t) \int_0^t \omega^{-2n-2}(s) \, ds = (1 - t^2)^n \int_0^t \frac{1}{(1 - s^2)^{n+1}} \, ds.
\]

Then

\[
g_0(t) = \frac{1}{2} \log \left( \frac{1 + t}{1 - t} \right)
\]

and \( g_n \in C^{n-1}[-1, 1] \) for \( n \geq 1 \). In particular \( g_n \in L^1(-1, 1) \) for all \( n \geq 0 \).

Proof. Note first the following computation:

\[
\left( (1 - t^2)^{-n} \right)' = (1 - t^2)^{-n} + 2nt^2(1 - t^2)^{-n-1}
\]

\[
= (1 - t^2)^{-n} + 2n \left( (1 - t^2)^{-n-1} - (1 - t^2)^{-n} \right)
\]

\[
= 2n(1 - t^2)^{-n-1} - (2n - 1)(1 - t^2)^{-n}.
\]

Therefore

\[
t(1 - t^2)^{-n} = 2n \int_0^t (1 - s^2)^{-n-1} \, ds - (2n - 1) \int_0^1 (1 - s^2)^{-n} \, ds,
\]

which after multiplication by \( \omega^{2n} \) and reordering of terms yields

\[
2ng_n = t + (2n - 1)\omega^2 g_{n-1}.
\]

The expression for \( n = 0 \) is straightforward and the increasing regularity of the sequence of functions follows by induction. \( \Box \)

Lemma A.3 Let \( c_{n,m} \) be the quantities defined in (2). For all \( n \geq m \geq 0 \)

\[
\left( \frac{2n + 1}{2} \right)^{1-2m} \leq c_{n,m}^2 \leq \left( \frac{2n + 1}{2} \right)^{1-2m} (m + 1)!
\]

Proof. Note that for \( m = 0 \) there is nothing to prove, so we will assume henceforth that \( m \geq 1 \). We can write

\[
c_{n,m}^2 = \frac{2n + 1}{2} \frac{1}{p_m(n)}, \tag{22}
\]

where

\[
p_m(x) := (x + m)(x + m - 1) \ldots (x - m + 1) = \prod_{j=-m}^{m-1} (x - j),
\]

is a polynomial of degree \( 2m \) with zeros in \( \{-m, -m+1, \ldots, m-1\} \). If \( x \geq m \), all factors in the definition of \( p_m(x) \) are positive and we can apply the comparison inequality between the geometric and the arithmetic mean to obtain

\[
p_m(x) \leq \left( \frac{1}{2m} \sum_{j=-m}^{m-1} (x - j) \right)^{2m} = \frac{1}{(2m)^{2m}} \left( \sum_{j=-m}^{m-1} (x - j) + \sum_{j=1}^{m} (x - j + 1) \right)^{2m}
\]

\[
= \frac{1}{(2m)^{2m}} \left( m(2x + 1) \right)^{2m} = \left( \frac{2x + 1}{2} \right)^{2m}.
\]

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If $j \leq -1$, then
\[ x - j \geq \frac{2x + 1}{2}, \quad \forall x, \]
whereas for $j \geq 0$
\[ x - j \geq \frac{1}{(j + 2)} \left( \frac{2x + 1}{2} \right), \quad \forall x \geq j + 1. \]
Therefore for $x \geq m(\geq 1)$
\[ p_m(x) \geq \left( \frac{2x + 1}{2} \right)^{m-1} \prod_{j=0}^{m-1} \frac{1}{j + 2} = \left( \frac{2x + 1}{2} \right)^{m-1} \frac{1}{(m+1)!}. \]
We can then apply the two bounds above to obtain
\[ \left( \frac{2n + 1}{2} \right)^{-2m} \leq \frac{1}{p_m(n)} \leq \left( \frac{2n + 1}{2} \right)^{-2m} (m+1)!, \quad \forall n \geq m \geq 1, \]
which together with the expression of $c_{n,m}$ in (22) proves the result. \hfill \Box

**Lemma A.4** For all $m, k \geq 0$ and $n \geq m + k$
\[ \frac{(m+1)!}{(m+k+1)!} \left( \frac{2n + 1}{2} \right)^{2k} \leq \left( \frac{c_{n,m}}{c_{n,m+k}} \right)^2 \leq \left( \frac{2n + 1}{2} \right)^{2k}. \]

**Proof.** For $k = 0$ there is nothing to prove, so we assume that $k \geq 1$. We follow similar steps to those applied when proving Lemma A.3. We first remark that
\[ \left( \frac{c_{n,m}}{c_{n,m+k}} \right)^2 = \prod_{j=m}^{m+k-1} \frac{(n - j)(n + j + 1)}{(n - j)(n + j + 1)} = q_{m,k}(n), \]
where $q_{m,k}$ is a polynomial of degree $2k$. Since all the factors are positive when $x \geq m + k$, we can bound
\[ q_{m,k}(x) \leq \frac{1}{(2k)^{2k}} \left( \sum_{j=m}^{m+k-1} (x - j) + \sum_{j=m}^{m+k-1} (x + j + 1) \right)^{2k} = \left( \frac{2x + 1}{2} \right)^{2k}. \]
Also, for $x \geq m + k$
\[ q_{m,k}(x) \geq \left( \frac{2x + 1}{2} \right)^{2k} \prod_{j=m}^{m+k-1} \frac{1}{j + 2} = \left( \frac{2x + 1}{2} \right)^{2k} \frac{(m+1)!}{(m+k+1)!}, \]
and the remaining inequality is proved. \hfill \Box

**Lemma A.5** For all $0 \leq k \leq n$
\[ |P_n^{(k)}(t)| \leq \frac{1}{4^k k!} (2n + 1)^{2k}, \quad \forall t \in [-1, 1]. \quad (23) \]
Proof. The result is true for $k = 0$ since $|P_n(t)| \leq 1$ for all $t \in [-1, 1]$. We will proceed by induction in $k$.

Elementary computations show that for all $n$

$$P'_n = (2n - 1)P_{n-1} + P'_{n-2}.$$  

Then we can proceed by induction to prove that

$$P'_n = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} (2n - 1 - 4j)P_{n-1-2j}.$$  

Note that all coefficients are positive for the given values of $j$. Assume that (23) holds for a given $k$. Using the previous expansion of the first derivative of the Legendre polynomials we can derive

$$P^{(k+1)}_n = \sum_{j=0}^{\lfloor \frac{n-k}{2} \rfloor} (2n - 1 - 4j)P^{(k)}_{n-1-2j} = \sum_{j=0}^{\lfloor \frac{n-k}{2} \rfloor} (2n - 1 - 4j)P^{(k)}_{n-1-2j}.$$  

Therefore

$$|P^{(k+1)}_n(t)| \leq \frac{1}{4^kk!} \sum_{j=0}^{\lfloor \frac{n-k}{2} \rfloor} (2n - 1 - 4j)(2(n - 1 - 2j) + 1)^{2k}$$

$$= \frac{1}{4^kk!} \sum_{j=0}^{\lfloor \frac{n-k}{2} \rfloor} (2n - 1 - 4j)^{2k+1} \leq \frac{1}{4^kk!} \sum_{j=0}^{\lfloor \frac{n-k}{2} \rfloor} \frac{1}{2} \int_{2n-1-4j}^{2n+1-4j} x^{2k+1} \, dx$$

$$\leq \frac{1}{4^kk!} \frac{1}{2} \int_{0}^{2n+1} x^{2k+1} \, dx = \frac{1}{4^kk!} \frac{(2n + 1)^{2k+2}}{2(2k + 2)}$$

and the result is proven for $k + 1$. □

A.2 An abstract lemma

Lemma A.6 Let $H, H_1$ and $H_2$ be Hilbert spaces such that $H_1 \subset H$ and $H_2 \subset H$ with continuous injections. If $\{\xi_n\}$ is an orthonormal basis of $H$ that is complete orthogonal in $H_1$ and $H_2$ and if

$$c\|\xi_n\|_{H_1} \leq \|\xi_n\|_{H_2} \leq C\|\xi_n\|_{H_1}, \quad \forall n,$$

(24)

then $H_1 = H_2$ and

$$c\|u\|_{H_1} \leq \|u\|_{H_2} \leq C\|u\|_{H_1}, \quad \forall u \in H_1 = H_2.$$  

Proof. We first prove that

$$H_1 = \left\{ u \in H : \sum_{n=1}^{\infty} \|\xi_n\|_{H_1}^2 |(u, \xi_n)_H|^2 < \infty \right\}$$

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and
\[ \|u\|_{H_1}^2 = \sum_{n=1}^{\infty} \|\xi_n\|_{H_1}^2 |(u, \xi_n)_{H_1}|^2, \quad \forall u \in H_1. \]

Since \( \xi_n/\|\xi_n\|_{H_1} \) is an orthonormal basis of \( H_1 \), we can decompose
\[ H_1 \ni u = \sum_{n=1}^{\infty} \frac{(u, \xi_n)_{H_1}}{\|\xi_n\|_{H_1}^2} \xi_n. \]

The series converges in \( H_1 \) and therefore in \( H \). Hence
\[ (u, \xi_n)_{H} = \frac{(u, \xi_n)_{H_1}}{\|\xi_n\|_{H_1}^2} \]
and
\[ \sum_{n=1}^{\infty} \|\xi_n\|_{H_1}^2 |(u, \xi_n)_{H_1}|^2 = \sum_{n=1}^{\infty} \frac{|(u, \xi_n)_{H_1}|^2}{\|\xi_n\|_{H_1}^2} = \|u\|_{H_1}^2. \]

A similar characterization of \( H_2 \) and comparison of the weights by means of (24) shows that \( H_1 = H_2 \). The inequality for the norms is a consequence of the series form of the norms in \( H_1 \) and \( H_2 \). □

B The case of Legendre polynomials

We can start the definition of the Hilbert scale defined by Legendre polynomials in the following way. In the space
\[ D_0 := \{ u \in L^2(-1,1) : \omega u' \in L^2(-1,1) \} \]
we consider its natural inner product
\[ (u,v)_{D_0} := \int_{-1}^{1} \omega^2 u' v' + \int_{-1}^{1} u v. \]

Because in this case \( \lambda = 0 \) is an eigenvalue of the differential operator \( \mathcal{L}_0 u := -(\omega^2 u')' \), to simplify the exposition we will simply translate the spectrum by adding an identity operator and we will study instead \( \tilde{\mathcal{L}}_0 u := \mathcal{L}_0 u + u \) as in [5] and related references. Note that \( H^1(-1,1) \subset D_0 \subset H_{loc}^1(-1,1) \) with continuous injection.

**Proposition B.1** \( \mathbb{P} \) is a dense subset of \( D_0 \).

**Proof.** Let us start by proving the following assertion: if \( u \in D_0 \), then \( \omega^2 u \in H^1_0(-1,1) \). The fact that \( \omega^2 u \in H^1(-1,1) \) is straightforward to prove using Leibniz’s rule and the definition of \( D_0 \). Hence \( \omega^2 u \in C[-1,1] \) and since \( u = \omega^{-2}(\omega^2 u) \in L^2(-1,1) \), necessarily \( \omega^2 u \in C_0 \).

It is clear that \( \mathbb{P} \subset D_0 \). If
\[ \int_{-1}^{1} \omega^2 u' p' + \int_{-1}^{1} u p = 0, \quad \forall p \in \mathbb{P}, \]
we can apply the integration by parts lemma (Lemma 1.1) and easily show that
\[
\int_{-1}^{1} u(-\omega^2 p')' + p = \int_{-1}^{1} u(\widetilde{L}_0 p) = 0, \quad \forall p \in \mathbb{P}.
\]

It suffices to take \( p = P_n \) for all values of \( n \) and use that Legendre polynomials form a Hilbert basis of \( L^2(-1, 1) \) to prove that \( u = 0 \). This proves that \( \mathbb{P} \) is dense in \( D_0 \). \( \square \)

Thanks to Proposition B.1 we know that \( \{ Q_n^0 : n \geq 0 \} \) is a complete orthogonal set in \( D_0 \) and that
\[
\| u \|_{D_0}^2 = \sum_{n=0}^{\infty} \left( n(n+1) \right) (u, Q_n^0)_{L^2(-1,1)}^2, \quad \forall u \in D_0.
\]

The corresponding Green’s operator
\[
G_0 u := \sum_{n=0}^{\infty} \frac{1}{n(n+1)+1} (u, Q_n^0)_{L^2(-1,1)} Q_n^0
\]
(note the translation of eigenvalues with respect to the operator defined in (9) due to the addition of the identity operator to \( \mathcal{L}_0 \) to translate the zero eigenvalue) is equivalent to the operator defined by \( G_0 f = u \), where \( u \) is the unique solution of
\[
u \in D_0, \quad (u, v)_{D_0} = (f, v)_{L^2(-1,1)}, \quad \forall v \in D_0. \quad (25)
\]

With help of this representation we will be able to describe the range \( \mathcal{R}(G_0) \) as a Sobolev type space in two different ways. Note that unlike in the cases \( m \geq 1 \), the range of \( \mathcal{R}(G_0) \) contains some generalized boundary conditions.

**Proposition B.2**

\[
\mathcal{R}(G_0) = \{ u \in L^2(-1,1) : \omega^2 u' \in H^1_0(-1,1) \} \quad (26)
\]
\[
= \{ u \in D_0 : \mathcal{L}_0 u \in L^2(-1,1), (\omega^2 u')(1) = 0 \}. \quad (27)
\]

**Proof.** Let \( A_1 \) and \( A_2 \) be the respective sets in the right-hand side of (26) and (27). Note that in \( A_1 \) the condition \( \omega u' \in L^2(-1,1) \) (that appears in the definition of \( D_0 \)) has been relaxed to \( \omega^2 u' \in L^2(-1,1) \). Therefore \( A_2 \subset A_1 \).

We next prove that \( \mathcal{R}(G_0) \subset A_2 \). If \( u = G_0 f \) with \( f \in L^2(-1,1) \), using (25) with a general \( \psi \in \mathcal{D}(-1,1) \subset D_0 \) we prove that \( \mathcal{L}_0 u = f \) in the sense of distributions. Note that this implies that \( \mathcal{L}_0 u = f - u \in L^2(-1,1) \). Substituting this differential equation in (25) we obtain that
\[
\int_{-1}^{1} (\omega^2 u') v + \int_{-1}^{1} (\omega^2 u')' v = 0, \quad \forall v \in D_0.
\]

If we consider the function \( \tilde{u} := \omega^2 u' \in H^1(-1,1) \), then the equality
\[
\int_{-1}^{1} (\tilde{u} v' + \tilde{u}' v) = 0, \quad \forall v \in H^1(-1,1)
\]

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(note that $H^1(-1, 1) \subset D_0$) is a weak form of the boundary conditions \( \bar{u}(-1) = \bar{u}(1) = 0 \).

This completes the proof that \( \mathcal{R}(G_0) \subset A_2 \).

We finally have to prove that \( A_1 \subset \mathcal{R}(G_0) \). Let now \( u \in A_1 \). Because \( \omega^2 u' \in H_0^1(-1, 1) \), then by Lemma 3.3(a), \( \omega u' \in C_0 \subset L^2(-1, 1) \), so \( u \in D_0 \). Also

\[
\int_{-1}^{1} \omega^2 u' v' + \int_{-1}^{1} (\omega^2 u')' v = 0, \quad \forall v \in H^1(-1, 1).
\]

If we define \( f := \tilde{\mathcal{L}}_0 u \in L^2(-1, 1) \), it follows readily that

\[
(u, v)_{D_0} = (f, v)_{L^2(-1, 1)}, \quad \forall v \in \mathbb{P}
\]

and by density we show that \( u = G_0 f \), which completes the proof. \( \square \)

**Proposition B.3** If \( u \in \mathcal{R}(G_0) \), then \( u' \in L^2(-1, 1) \).

**Proof.** By Theorem 5.1 (let us emphasize again that Section 5 is independent of the two sections that precede it), we have a different characterization of \( \mathcal{R}(G_0) = H_0^2 \subset X_0^2 \). If \( u \in \mathcal{R}(G_0) \), then

\[
2t u' = \omega^2 u'' - (\omega^2 u')' \in L^2(-1, 1).
\]

On the other hand \( u' \in L_{\text{loc}}^2(-1, 1) \), so we can divide by \( t \) and ensure that \( u' \in L^2(-1, 1) \). \( \square \)

This last result appears in [3], quoted as already been proved in the unpublished preprint [11].

**References**

[1] M. Abramowitz and I.A. Stegun. *Handbook of mathematical functions with formulas, graphs, and mathematical tables*, volume 55 of *National Bureau of Standards Applied Mathematics Series*, Washington, D.C., 1964.

[2] R.A. Adams. *Sobolev spaces*. Pure and Applied Mathematics, Vol. 65. Academic Press, New York-London, 1975.

[3] J. Arvesú, L. L. Littlejohn and F. Marcellán. On the right-definite and left-definite spectral theory of the Legendre polynomials. *J. Comput. Anal. Appl.*, 4(4):363–387, 2002.

[4] J.M. Berezanski. *Expansions in eigenfunctions of selfadjoint operators*. Translated from the Russian by R. Bolstein, J. M. Danskin, J. Rovnyak and L. Shulman. Translations of Mathematical Monographs, Vol. 17. American Mathematical Society, Providence, R.I., 1968.

[5] A. Bruder, L. L. Littlejohn, D. Tuncer and R. Wellman. Left-definite theory with applications to orthogonal polynomials. *J. Comput. Appl. Math.*, to appear.
[6] D. Colton and R. Kress. *Integral equation methods in scattering theory*. Pure and Applied Mathematics (New York). John Wiley & Sons, Inc., New York, 1983.

[7] D. Colton and R. Kress. *Inverse acoustic and electromagnetic scattering theory*. Second edition. Applied Mathematical Sciences, 93. Springer-Verlag, Berlin, 1998.

[8] V. Domínguez, N. Heuer and F.-J. Sayas. Boundary element methods on axisymmetric domains. (in preparation)

[9] K.J. Engel and R. Nagel. *A short course on operator semigroups*. Universitext. Springer, New York, 2006.

[10] W.N. Everitt, L.L. Littlejohn and R. Wellman. Legendre polynomials, Legendre-Stirling numbers, and the left-definite spectral analysis of the Legendre differential expression. *J. Comput. Appl. Math.*, 148(1):213–238, 2002.

[11] W. N. Everitt and V. Marić. Some remarks on the Legendre differential expression, (Unpublished manuscript, Novi Sad, 1988).

[12] C. Müller. *Spherical harmonics*, volume 17 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1966.

[13] J.C. Nédélec. *Acoustic and electromagnetic equations. Integral representations for harmonic problems*. Applied Mathematical Sciences, 144. Springer-Verlag, New York, 2001.

[14] S. Prössdorf and B. Silbermann. *Numerical analysis for integral and related operator equations*. Akademie-Verlag, Berlin, 1991.

[15] L. Schwartz. *Théorie des distributions*. Hermann, Paris, 1966.