Equation of motion for incompressible mixed fluid driven by evaporation and its application to online rankings

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ABSTRACT

We give a unique classical solution to initial value problem for a system of partial differential equations for the densities of components of one dimensional incompressible fluid mixture driven by evaporation.

Motivated by the known fact that the solution appears as an infinite particle limit of stochastic ranking processes, which is a simple stochastic model of time evolutions of e.g., Amazon Sales Ranks, we collected data from the web and performed statistical fits to our formula. The results suggest that the fluid equations and solutions may have an application in the analysis of online rankings.

Keywords: evaporation driven fluid; non-linear wave; stochastic ranking process; long tail; Pareto distribution;

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1 Introduction.

Let \( f_i > 0, i = 1, 2, \ldots \), be positive constants, and consider the following system of non-linear partial differential equations for the functions \( u_i(y, t), i = 1, 2, \ldots \), and \( v(y, t) \), defined on \((y, t) \in [0, 1) \times \mathbb{R}_+\):

\[
\frac{\partial u_i(y, t)}{\partial t} + \frac{\partial (v(y, t) u_i(y, t))}{\partial y} = -f_i u_i(y, t), \quad i = 1, 2, \ldots, \tag{1}
\]

\[
\sum_j u_j(y, t) = 1. \tag{2}
\]

We consider initial value problems for smooth non-negative initial data

\[
u_i(y, 0) \geq 0, \quad 0 \leq y < 1, \quad i = 1, 2, \ldots,
\]

with the boundary conditions at \( y = 0 \) and \( y = 1 \):

\[
v(1 - 0, t) = 0, \tag{3}
\]

\[
u_i(0, t) = \frac{f_i \rho_i}{\sum_j f_j \rho_j}, \quad i = 1, 2, \ldots, \tag{4}
\]

for \( t \geq 0 \), where, for each \( i \),

\[
\rho_i = \int_0^1 u_i(z, 0) \, dz, \tag{5}
\]

and we assume

\[
\sum_j f_j \rho_j < \infty. \tag{6}
\]

Note that adding up (1) over \( i \) and applying (2) we have

\[
\frac{\partial v(y, t)}{\partial y} = -\sum_j f_j u_j(y, t), \tag{7}
\]

which, with (3), determines \( v \) in terms of \( u_i \).

Given the positive constants \( f_i \) and the initial data \( u_i(y, 0) \), the set of equations (1) (2) (3) (4) defines the evolution of our system. The following arguments and results hold both for finite components \((i = 1, 2, \ldots, N)\) and infinite components. (In fact, we can extend the system and the solution to a case with any probability space \( \Omega \), by replacing \( u_i(y, t) \) with a measure \( \mu(\omega, y, t) \). See [4] for probability theoretic arguments.)

A physical meaning of the system is as follows. We are considering a motion of incompressible fluid mixture in an interval with length normalized to 1, where \( u_i(y, t) \) is the density of \( i \)-th component at space-time point \((y, t)\). (2) implies that we normalize \( u_i \) so that it represents the ratio of \( i \)-th component. We are naturally interested in the non-negative solutions \( u_i(y, t) \geq 0 \). \( v(y, t) \) is the velocity field of the fluid. (1) is the equation of continuity, and the right-hand-side implies that each component evaporates with rate \( f_i \) per unit time and unit mass. Note that the set of equations (7) and (3) is equivalent to

\[
v(y, t) = \sum_j f_j \int_y^1 u_j(z, t) \, dz, \tag{8}
\]
which implies that the velocity field, or the motion of the fluid, is caused solely by filling the amount of fluid which evaporated from the right side of the point \( y \). In particular, we have no flux through the boundary \( y = 1 \) \((v(1 - 0, t) = 0)\).

The boundary condition (4) at \( y = 0 \) is so tuned by the initial data that the loss of mass by evaporation is compensated by the immediate re-entrance at \( y = 0 \) as liquidized particles, so that the total mass of each fluid component in the interval \([0, 1)\) is conserved over time:

\[
\int_0^1 u_i(z, t) \, dz = \rho_i, \quad t \geq 0. \tag{9}
\]

In fact, (9) is equivalent to (4), under the conditions (1) (2) (3) (5), if \( \sup_i f_i < \infty \). (See Appendix A.)

Let

\[
y_C(y, t) = 1 - \sum_j e^{-f_j t} \int_y^1 u_j(z, 0) \, dz, \quad 0 \leq y < 1, \quad t \geq 0. \tag{10}
\]

For each \( t \geq 0 \), \( y_C(\cdot, t) : [0, 1) \rightarrow [y_C(0, t), 1) \) is a continuous, strictly increasing, onto function of \( y \), and its inverse function \( \hat{y}(\cdot, t) : [y_C(0, t), 1) \rightarrow [0, 1) \) exists:

\[
1 - y = \sum_j e^{-f_j t} \int_{\hat{y}(y, t)}^1 u_j(z, 0) \, dz, \quad y_C(0, t) \leq y < 1, \quad t \geq 0. \tag{11}
\]

\( y_C(y, t) \) denotes the position of a fluid particle at time \( t \) (on condition that it does not evaporate up to time \( t \)) whose initial position is \( y \). \( \hat{y}(y, t) \) denotes the initial position of a fluid particle located at \( y \geq y_C(0, t) \) at time \( t \).

With slight abuse of notations, we will often write \( y_C(t) \) for \( y_C(0, t) \):

\[
y_C(t) = y_C(0, t) = 1 - \sum_j \rho_j e^{-f_j t}. \tag{12}
\]

\( y_C : [0, \infty) \rightarrow [0, 1) \) is a continuous, strictly increasing, onto function of \( t \), and its inverse function \( t_0 : [0, 1) \rightarrow [0, \infty) \) exists:

\[
y_C(t_0(y)) = y, \quad 0 \leq y < 1. \tag{13}
\]

In this paper, we prove the following.

**Theorem 1** There exists a unique classical solution to the initial value problem for the system of partial differential equations defined by (1)–(5), which is explicitly given by (8) and

\[
u_i(y, t) = \begin{cases} 
  e^{-f_i t_0(y)} f_i \rho_i, & y < y_C(t), \\
  \sum_j e^{-f_j t_0(y)} f_j \rho_j, & y = y_C(t), \\
  \sum_j e^{-f_j t_0(\hat{y}(y, t), 0)} u_j(\hat{y}(y, t), 0), & y > y_C(t), 
\end{cases} \tag{14}
\]

Note the unique feature of the solution that for \( y < y_C(t) \) the solution is stationary:

\[
\frac{\partial u_i}{\partial t}(y, t) = 0, \quad y < y_C(t), \tag{15}
\]

while initial conditions affect \( y > y_C(t) \) part only through wave propagation.
In natural phenomena where evaporation is active, such as producing salt out of sea water, viscosity, surface tension, and external forces such as gravitational forces dominate, and the effect of evaporation on the motion of fluid would be relatively too small to observe. Thus the equation and the solution we consider in this paper may not have attracted much attention. However, there are phenomena on the web for which our formulation may work as a simplified mathematical model, such as the time evolutions of rankings of book sales in the online booksellers. Such possibility is theoretically based on a result that (14) appears as an infinite particle limit of the stochastic ranking process [4], which is a simple model of the time evolutions of e.g., the number known as the Amazon Sales Rank. (We note that this number has mathematically little to do with the perhaps more popular notion of Google Page Ranks.) We collected data of the time evolution of the numbers from the web, and performed statistical fits of the data to (12). Considering the simplicity of our model and formula, the fits seem good, which suggest that there is a new application of our results in the analysis of online rankings.

The plan of this paper is as follows. In Section 2 we give a proof of Theorem 1. In Section 3 we give results of fits to (12) of data from the web.

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2 Proof of the main theorem.

Put

\[ U_i(y, t) = \int_y^1 u_i(z, t) \, dz, \quad i = 1, 2, \ldots. \]  (16)

With (16), the system of equations (1)–(5) is equivalent to the following: \( U_i(y, t) \) is decreasing in \( y \) and \( U_i(1-0, t) = 0 \), and

\[
\frac{\partial U_i}{\partial t}(y, t) + v(y, t) \frac{\partial U_i}{\partial y}(y, t) = -f_i U_i(y, t), \quad i = 1, 2, \ldots, \]  \hspace{1cm} (17)

\[ \sum_j U_j(y, t) = 1 - y, \]  \hspace{1cm} (18)

\[ v(y, t) = \sum_j f_j U_j(y, t), \]  \hspace{1cm} (19)

for \( 0 \leq y < 1, t \geq 0 \) (note (5)), and noting (9),

\[ U_i(0, t) = \rho_i \quad i = 1, 2, \ldots, \]  \hspace{1cm} (20)

Now, for each \( 0 \leq y_0 < 1 \) let \( y_B = y_B(y_0; t) \) be a solution to an ODE

\[
\frac{d y_B}{d t}(y_0; t) = v(y_B(y_0; t), t), \quad t \geq 0, \quad y_B(y_0; 0) = y_0, \]  \hspace{1cm} (21)

and put

\[ \phi_i(t) = U_i(y_B(y_0; t), t). \]  \hspace{1cm} (22)
With (21) and (17) it follows that
\[
\frac{d \phi_i}{dt}(t) = \frac{\partial U_i}{\partial t}(y_B(y_0; t), t) + v(y_B(y_0; t), t) \frac{\partial U_i}{\partial y}(y_B(y_0; t), t) = -f_i U_i(y_B(y_0; t), t) = -f_i \phi_i(t),
\]
hence, with \(\phi_i(0) = U_i(y_0, 0)\), \(\phi_i\) is uniquely solved as
\[
\phi_i(t) = U_i(y_0, 0) e^{-f_i t}. \tag{23}
\]
With (19) and (21), we then find
\[
\frac{dy_B}{dt}(y_0; t) = \sum_j f_j U_j(y_0, 0) e^{-f_j t},
\]
hence, using (18) and (24), we have
\[
y_B(y_0; t) = 1 - \sum_j U_j(y_0, 0) e^{-f_j t} = 1 - \sum_j \int_{y_0}^t u_j(z, 0) dz e^{-f_j t} = y_C(y_0, t), \tag{24}
\]
where \(y_C\) is defined in (10). With (11), (22) and (23) we uniquely obtain
\[
U_i(y, t) = U_i(y_A(y, t), 0) e^{-f_i t}. \tag{25}
\]
Differentiating by \(y\) and using (16) and (24)
\[
u_i(y, t) = \left(\frac{\partial y_B}{\partial y}(\hat{y}(y, t), 0)\right)^{-1} u_i(\hat{y}(y, t), 0) e^{-f_i t} = \sum_j u_i(\hat{y}(y, t), 0) e^{-f_i t},
\]
where \(\frac{\partial y_B}{\partial y_0}\) is the derivative of \(y_B = y_B(y_0; t)\) with respect to the parameter \(y_0\). This proves (14) for \(y > y_C(t)\).

Next let \(y < y_C(t)\) and put \(t_1 = t - t_0(y) \in (0, t)\), where \(t_0\) is as in (13). Let \(y_A\) be a solution to an ODE
\[
y_A'(s) = v(y_A(s), s), \quad s \geq t_1, \quad y_A(t_1) = 0, \tag{25}
\]
and put
\[
\phi_i(s) = U_i(y_A(s), s), \quad s \geq t_1. \tag{26}
\]
Note that (20) implies \(\phi_i(t_1) = U_i(0, t_1) = \rho_i\), hence, as below (21), \(\phi_i\) is uniquely solved as
\[
\phi_i(s) = \rho_i e^{-f_i(s-t_1)}. \tag{27}
\]
With (19) and (25), we then find
\[
y_A'(s) = \sum_j f_j \rho_j e^{-f_j(s-t_1)},
\]
Note that (5) and (2) imply \(\sum_j \rho_j = 1\). Hence,
\[
y_A(s) = 1 - \sum_j \rho_j e^{-f_j(s-t_1)} = y_C(s - t_1), \quad s \geq t_1. \tag{28}
\]
where $y_C$ is defined in (12).

Putting $s = t$ in (26) and (27), using (28), and recalling that $t_1 = t - t_0(y)$, we have, with (13),

$$U_i(y, t) = U_i(y_C(t_0(y)), t) = \rho_i e^{-f_i t_0(y)}.$$  

Differentiating by $y$ and using (16), (13) and (12),

$$u_i(y, t) = \left(\frac{d y_C}{d t}(t_0(y))\right)^{-1} \rho_i f_i e^{-f_i t_0(y)} = \sum_j \rho_j f_j e^{-f_j t_0(y)},$$  

which proves (14) for $y < y_C(t)$. This completes a proof of Theorem 1.

Before closing this section, we give a couple of examples of the solution to the equation of motion.

**Example 1 (One particle type).** For a pure fluid, $\rho_1 = 1$, hence we have $u_1(y, t) \equiv 1$. Since there is only one type of incompressible fluid, the density is constant. However, there is a flow driven by evaporation even in this case, and we actually have $y_C(t) = 1 - e^{-f_1 t}$.

**Example 2 (Two particle types with $f_2 = 0$).** Consider a mixture of 2 components with the ratios satisfying $\rho_1 > 0$ and $\rho_2 = 1 - \rho_1 > 0$. $f_2 = 0$ means no evaporation, so the situation is a salty sea with salt density $\rho_2$, where evaporation and flow from a river balance. In Theorem 1 we assumed $f_2 > 0$, but we can calculate explicitly for $f_2 \geq 0$. We have

$$y_C(t) = \rho_1 (1 - e^{-f_1 t}), \quad t_0(y) = -\frac{1}{f_1} \log(1 - \frac{y}{\rho_1}).$$

The expressions of $u_i(y, t)$ for $y < y_C(t)$ become simple:

$$v(y, t) = f_1 (\rho_1 - y), \quad u_1(y, t) \equiv 1, \quad u_2(y, t) \equiv 0.$$  

The pure water from the river comes in up to $y < y_C(t)$. (Note that we are considering a fictitious 1-dimensional case where no spacial mixing such as turbulence occurs and we have no other dynamics such as diffusion.)

If, furthermore, the initial distribution is uniform on $[0, 1)$: $u_i(y, 0) = \rho_i, \ i = 1, 2$, then the expressions for $y > y_C(t)$ are also simple:

$$y_C(y, t) = 1 - (1 - y)(\rho_1 e^{-f_1 t} + \rho_2), \quad \dot{y}(y, t) = 1 - \frac{1 - y}{\rho_1 e^{-f_1 t} + \rho_2},$$

and consequently,

$$v(y, t) = (1 - y)f_1 \frac{\rho_1 e^{-f_1 t}}{\rho_1 e^{-f_1 t} + \rho_2}, \quad u_1(y, t) = \frac{\rho_1 e^{-f_1 t}}{\rho_1 e^{-f_1 t} + \rho_2}, \quad u_2(y, t) = \frac{\rho_2}{\rho_1 e^{-f_1 t} + \rho_2},$$

for $y > y_C(t)$. (In general, the formulas are dependent on initial data in complex ways, and we do not have explicit formula.)
3 Possible application to rankings on the webs.

3.1 Pareto distribution.

An idea that the time evolution of ranking numbers such as in Amazon.co.jp sales rank may be explained by evaporation-driven fluid motion, is theoretically based on a result \[ \text{(14)} \] which says that \( \text{(14)} \) appears as a time evolution of empirical distributions of the stochastic ranking processes in an infinite particle limit. Roughly speaking, the stochastic ranking process is a simple model of the time evolution of a list of rankings, such as a ranking of book sales in an online bookseller, or a table of page titles in a collected web bulletin board. A particle (a book, in the case of online booksellers) in the ranking jumps randomly to the rank 1 (each time the book is sold), and increases the ranking number by 1 each time some other particle of larger ranking number jumps to rank 1. Because an increase in ranking number is a result of jumps of very large number of particles in the tail side of the ranking, the particle effectively moves on the ranking queue in a deterministic way, even though each jump occurs at a random time. The jump corresponds to the evaporation in the fluid model. We can therefore predict the time evolutions of rankings appearing on the web rankings based on our model. See \[ \text{(4)} \] for details of the stochastic ranking processes.

Here we try to see how a trajectory of a particle \( \text{(12)} \) could be observed in the web rankings. In applying \( \text{(12)} \) to the actual rankings, we need to choose the distribution of evaporation rates \( \{(\nu_i, \rho_i) \mid i = 1, 2, \cdots, N\} \). In the case of social or economic studies such as online booksellers, this corresponds to choosing the distribution of activities or transactions. In the case of online booksellers, we have to chose the distribution of sales rates over books. The Pareto distribution (also called log-linear distribution in social studies, or power-law in physics literatures) is traditionally used as a basic model distribution for various social rankings, perhaps a most well-known example is the ranking of incomes. Let \( N \) be the total size of population, and for \( i = 1, 2, \cdots, N \), denote by \( f_i \) the income of the \( i \)-th wealthiest person. If

\[
 f_i = a \left( \frac{N}{i} \right)^{1/b}, \quad \rho_i = \frac{1}{N}, \quad i = 1, 2, 3, \cdots, N,
\]

holds for some positive constants \( a \) and \( b \), then the distribution of incomes is said to satisfy the Pareto distribution. (The Pareto distribution assumes all the constituents to have distinct \( f_i \), which leads to equal weight \( \rho_i = 1/N \) in our notation.) The constant \( a \) corresponds to the smallest income, and the exponent \( b \) reflects a social equality of incomes: in fact the ratio of the largest income to the smallest is \( f_1/f_N = N^{1/b} \), which is close to 1 if \( b \) is large (a fair society), while is large (society is in monopoly) if \( b \) is small. (Our \( b \) corresponds to \( \alpha \) in a standard textbook on statistics, \( \theta \) in \[ \text{(3)} \], and \( -1/\beta_2 \) in \[ \text{(2)} \].)

Substituting \( \text{(29)} \) in \[ \text{(12)} \] of Section 2 and approximating the summation by integration, we have, after a change of variable,

\[
y_C(t) = 1 - b(at)^b \Gamma(-b, at) + O(N^{-1}),
\]

where \( \Gamma(z, p) = \int_p^\infty e^{-w}w^{z-1} dw \) is the incomplete Gamma function. \( y_C(t) \) is a relative ranking normalized by \( N \), so the time evolution of ranking \( x_C(t) \) is

\[
x_C(t) = 1 + N y_C(t).
\]

The \( O(N^{-1}) \) contribution in \[ \text{(30)} \] is (by a careful calculation) seen to be non-negative and bounded by \( \frac{1}{N} e^{-at} \leq \frac{1}{N} \), leading to a difference of at most 1 in the ranking \( x_C(t) \), which is insignificant for our applications below, so we will ignore it.
Note that $\Gamma(-b, at) \to \infty$ as $t \to 0$, for $b > 0$. This divergence is harmless because it is cancelled by $t^b$ in (30), but for numerical and asymptotic analysis, it is better to perform a partial integration on the right-hand side to find

$$y_C(t) = 1 - e^{-at} + (at)^b \Gamma(1 - b, at),$$

which, with (31), leads to

$$x_C(t) = N (1 - e^{-at} + (at)^b \Gamma(1 - b, at)) + 1.$$  

The constant $a$, which denotes the lowest income in the Pareto distribution, has a role of a time constant in (33). In particular, the short time behavior of $x_C(t)$ for $0 < b < 1$ is

$$x_C(t) = c t^b + O(t),$$

where

$$c = Na^b \Gamma(1 - b).$$

For $1 < b < 2$ we need a partial integration once more for a better expression;

$$y_C(t) = 1 - e^{-at}(1 - \frac{at}{b - 1}) - \frac{(at)^b}{b - 1} \Gamma(2 - b, at) = \frac{ab}{b - 1} t - \Gamma(2 - b) \frac{a^b}{b - 1} t^b + O(t^2).$$

Note that for $0 < b < 1$ the leading short time behavior is $y_C(t) = O(t^b)$, which is tangential to the $y$ axis at $t = 0$, while for $b > 1$ (the case $b \geq 2$ can be handled similarly) the linear dependence $y_C(t) = O(t)$ is dominant for small $t$.

### 3.2 2ch.net bulletin board thread index listings.

2ch.net is one of the largest collected web bulletin boards in Japan. Each category (‘board’) has an index listing of the titles of ‘threads’ or the web pages in the board. The titles are ordered by “the last written thread at the top” principle; if one writes an article (‘response’) to a thread, the title of that thread in the index listing jumps to the top instantaneously, and the titles of other threads which were originally nearer to the top are pushed down by 1 in the listing accordingly. We can extract the exact time that a thread jumped to rank 1, because the time of each response in a thread is recorded together with the response itself. All these features of the 2ch.net index listing match the definition of the stochastic ranking process in [4].

As a first attempt to apply our theoretical results to online data, we collected data of the time evolution of the index listing and performed statistical fits of the data to (12). The actual properties of jump rates would be more involved than the models; for example, the distribution of jump rates (namely, the distribution of the frequencies of responses of the threads) would be more complex than the Pareto distribution, and looking into actual data would provide a test to the applicability of our simple model.

We note that we can use deterministic (non-stochastic) formula such as (33) if $N$, the total number of threads in the board is large. When we are keeping track of the ranking of a thread, it goes down (number increases) when and only when other threads at the tail side of the ranking jumps to the top (i.e., someone writes a response to one of these other threads), and though each jump occurs randomly, since there are $O(N)$ threads on the tail side of the thread in question (unless it is extremely near the tail), a law-of-large-numbers like mechanism works (as rigorously proved in [4]), and the time evolution of the ranking for each thread becomes deterministic as in
In the case of 2ch.net, $N$ is roughly about 700 to 800, so we would expect fluctuation of a few percent, and up to that accuracy, we expect a time evolution predicted by (33).

We also note that the time evolutions are independent of which thread one is looking at, because the changes in the ranking are caused by the collective motion of the threads towards the tail; popular threads jump back to the top ranking more frequently than the less popular ones, but as long as the threads remain in the queue (i.e., before the next jump), both a popular thread and an unpopular thread should behave in the same way, depending only on their position in the ranking.

![Fig 1: Record of ranking changes in an afternoon for 12 threads in a board of 2ch.net. Points from a thread are joined by line segments to guide the eye.](image1)

Fig. 1 is a plot of the threads in a board which jumped to the rank 1 during active hours one day and stayed in the queue without jumps until midnight. There were 12 such threads. The ranking is obviously monotone function of time between jumps, and there are no overtaking, so that the lines do not cross in the figure. Fig. 2 is a plot of same data as in Fig. 1 except that, for the horizontal axis the time is so shifted for each thread that the ranking of the thread is 1 at time 0. Though each thread starts at rank 1 on different time of the day, Fig. 2 shows that time evolutions after rank 1 are on a common curve. $N$ is the total number of threads, which is $N = 795$ at the time.

![Fig 2: Collection of 12 threads in a board of 2ch.net, same as in Fig. 1. For each thread, time is shifted so that the rank of the thread is 1 at time 0. The curve is $x(t)$ of (33) with the best fit $a = a^*$ and $b = b^*$ to the data. Horizontal and vertical axes are the hours and ranking, respectively.](image2)
of observation for Fig. 2, and \( a \) and \( b \) are positive constant parameters to be determined from the data. We performed a least square fit to (33) of \( n_d = 117 \) data points shown in Fig. 2. The best fit for the parameter set \((a, b)\) is \((a^*, b^*) = (3.3425 \times 10^{-4}, 0.6145)\) \((\sqrt{\chi^2/n_d} \simeq 1.8)\). In particular, we see a rather clear behavior close to the origin that the plotted points are on a curve tangential to \( y \) axis, indicating \( x_C(t) = O(t^b) \) with \( b < 1 \) as in (31). Considering the simplicity of our model and formula, the fits seem good, suggesting a possibility of new application of fluid dynamics in the analysis of online rankings.

### 3.3 Amazon.co.jp book sales rankings.

We next turn to the ranking in the Amazon.co.jp online book sales. In this century of expanding online retail business, the economic impact of internet retails has attracted much attention, and there are studies using the sales rankings which appear on the webs of online booksellers such as Amazon.com \[2,3\]. We will study Amazon.co.jp, a Japanese counterpart of Amazon.com, which seems to be not studied (and easier to access for the authors). Basic structures of web pages for individual books are similar for Amazon.com and Amazon.co.jp; on a web page for a book there are the title, price and related information such as shipping, brief description of the book, the sales ranking of the book, customer reviews and recommendations.

We should note that Amazon.co.jp, as well as Amazon.com, does not disclose exactly how it calculates rankings of books. In fact, there are observations \[6\] that Amazon.com defines the rankings for the top sales in a rather involved way. Therefore, it would be non-trivial and interesting if we could observe in the data behaviors similar to those of our simple model such as (33). See \[5\] for economic implications of the ranking numbers in Amazon.co.jp.

According to observation, Amazon.co.jp, as well as Amazon.com, updates their rankings once per hour, in contrast to the 2ch.net where the update procedure is instantaneous. This implies a limit of short time observational precision of 1 hour. On the other hand, for the long time observations, we have to consider a fact that the total number of books \( N \) is not constant. It is said that each year about \( 5 \times 10^4 \) books are published in Japan, or about 5.7 books per hour. Certainly not all of the books are registered on Amazon.co.jp, so the increase of \( N \) per hour must be less than this value. Speed of ranking change decreases in the very tail side of the listing, and these practical changes in \( N \) will affect validity of applying (33) to the data in the very tail regime of the ranking. This gives a practical limit to long time analysis. Fortunately, at ranking as far down as \( 6.5 \times 10^5 \), we still observe about 200 ranking change per hour, which makes an increase by 5.7 books negligible, so we expect a chance of applicability of our theory for long time data.

We will now summarize our results. The plotted 77 points in Fig. 3 show the result of our observation of a Japanese book rankings data, taken between the end of May, 2007 and mid August, 2007. As seen in the figure, the ranking number falls very rapidly near the top position (about 200 thousands in 5 days). The solid curve is a least square fit of these points to (33). Amazon.co.jp announces the total number of Japanese books in their list, which is a few times \( 10^6 \), but we suspect that this number includes a large number of books which are registered but never sell (so that we should discard in applying our theory). Therefore in addition to \( a \) and \( b \) in (33) we include \( N \) as a parameter to be fit from the data. Also, Amazon does not disclose the exact point of sales of each book, unlike 2ch.net where the exact jump time of a thread is recorded, so that the jump time to rank 1 of a book is also a parameter. The best fit for the parameter set \((N, a, b)\) is:

\[
(N^*, a^*, b^*) = (8.57 \times 10^5, 3.939 \times 10^{-4}, 0.6312), \quad (\sqrt{\chi^2/n_d} \simeq 1.4 \times 10^4).
\]

Incidentally, we note in Fig. 3 a small jump at about 300 hours. We suspect this as a result of inventory control such as unregistering books out of print. Obviously, these controls need man-
Fig 3: A long time sequence of data from Amazon.co.jp. The solid curve is a theoretical fit. Horizontal and vertical axes are the hours and ranking, respectively.

power, so that they appear only occasionally, making it a kind of unknown time dependent external source for our analysis.

All in all, we think it an impressive discovery that a simple formula as (33) could explain the data for more than 2 months. Our way of extracting basic sociological parameters such as the Pareto exponent $b$ from the ranking data on the web has advantages over previous methods such as in [3], in that because the time development of the ranking of a book is a result of sales of very large, $O(N)$, number of books, the book moves on the ranking queue in a deterministic way though each sale is a stochastic process. The fluctuations of sales (randomness about who buys what and when) are suppressed through a law-of-large-numbers type mechanism [4]. By looking at the time development of the ranking of a single book, we are in fact looking at the total sales of the books on the tail side of the book.

The theory of ‘long-tail economy’ says [1] that each product might sell only a little, but because of the overwhelming abundance in the species of the products the total sales will be of economic significance: It is not any specific single book but the total of books on the long-tail that matters. Our analysis on accumulated effect of products each with random and small sales, is particularly suitable in analyzing the new and rapidly expanding economic possibility of online retails, and moreover, is natural from the long-tail philosophy point of view.

Among the parameters to be fit in the Pareto distribution there is an exponent $b$ which is of importance in the studies of economy. For example, in the case of distribution of incomes, which is usually quantitatively analyzed by the Pareto distribution, small $b$ means that a few people of high incomes hold most of the wealth (the so called ‘20–80 law’ is a nickname for the Pareto distribution with $b = 1$), while for large $b$ the society is more equal. In the case of ranking of online booksellers, large $b$ means that there are many books (books in the ‘long-tail’ [1] regime), each of which do not sell much but the total sales of which is significant, further implying strong impacts of online retails...
to economy \[3, 2\], while small \( b \) favors dominance of traditional business model of ‘greatest hits’. Our studies on the 2ch.net bulletin board and the Amazon.co.jp online bookseller both consistently give the Pareto exponent \( b \approx 0.6 \). Existing studies on online booksellers \[3, 2\] adopt the value of \( b = 1.2 \) and \( b = 1.148 \), respectively. These references also quote values from other studies, most of which satisfy \( b > 1 \). Note that \[6\] discovers, apparently based on extensive observations, that in the long-tail regime the sales are worse than in the head and intermediate regime, and gives the Pareto exponent \( b = 0.4 \) in the long-tail regime. Our method gives the total effect of intermediate and long-tails, so our value \( b = 0.6 \) could be more or less consistent with the observation of \[6\].

\section*{A Proof of equivalence of (9) and (4).}

Here we prove that, if \( \sup_i f_i < \infty \), (9) and (4) are equivalent, under the equations (11) (7) (2) (3) (5), with positive constants \( f_i, \rho_i, i = 1, 2, \ldots \). (The extra condition on boundedness of \( f_i \) is of course irrelevant for the finite component cases.)

First assume (9). Then with (38) we have \( v(0,t) = \sum f_i \rho_i \). On the other hand, integrating (1) by \( y \) from 0 to 1 and using (9) and (3) we have \( v(0,t) u_i(0,t) = f_i \rho_i \). The two equations imply (4).

Next assume (4) and let

\[ \mathbf{U} = (U_1, U_2, \ldots); \quad U_i(t) = \int_0^1 u_i(z,t) \, dz, \quad i = 1, 2, \ldots. \]  

(37)

Then (38) implies

\[ v(0,t) = \sum_j f_j U_j(t). \]  

(38)

Integrating (1) by \( y \) from 0 to 1, and using (3) (1) (38), we have

\[ \frac{d}{dt} \mathbf{U}(t) = (A \mathbf{U})(t), \quad t \geq 0, \]  

(39)

with

\[ (A \mathbf{U})_i(t) = \sum_j f_{ij} \rho_j \sum_j f_j U_j(t) - f_i U_i(t), \quad i = 1, 2, 3, \ldots, \quad t \geq 0. \]  

(40)

The definition (3) implies \( U_i(0) = \rho_i \), hence \( U_i(t) = \rho_i, \quad t \geq 0 \), is a solution, implying (9). Uniqueness of the solution to (39), a linear differential equation with constant bounded coefficients, proves (9).

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