The Exact Asymptotic Form of Bayesian Generalization Error in Latent Dirichlet Allocation

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Abstract

Latent Dirichlet allocation (LDA) obtains essential information from data by using Bayesian inference. It is applied to knowledge discovery via dimension reducing and clustering in many fields. However, its generalization error had not been yet clarified since it is a singular statistical model where there is no one to one map from parameters to probability distributions. In this paper, we give the exact asymptotic form of its generalization error and marginal likelihood, by theoretical analysis of its learning coefficient using algebraic geometry. The theoretical result shows that the Bayesian generalization error in LDA is expressed in terms of that in matrix factorization and a penalty from the simplex restriction of LDA’s parameter region.

1 Introduction

Latent Dirichlet allocation (LDA) is one of topic models which is a ubiquitous statistical model used in many research areas. Text mining, computer vision, marketing research, and geology are such examples. LDA had been originally proposed for natural language processing and it can extract an essential information from documents by defining the topics of the words. The topics are formulated as one-hot vectors subject to categorical distributions which depend on each document. The parameters of those categorical distributions express the topic proportion and they are the object of inference. In addition, the words are also formulated as one-hot vectors generated by other categorical distributions whose parameters represent appearance probability of the words in each document. This appearance probability is also estimated.

In the standard inference algorithms such as Gibbs sampling and variational Bayesian method, LDA requires setting the number of topics (the dimension of the topic one-hot vector) in advance. The optimal number of topics in the ground truth is unknown, thus researchers and practitioners face to typical model selection problem; the chosen number of topics may be larger than the optimal one. In this situation, the estimated parameter cannot be uniquely determined. From the theoretical point of view, LDA is non-identifiable, i.e. a map from a parameter set to a probability density function set is not injective. Besides, LDA has a degenerated Fisher information matrix and its likelihood and posterior distribution cannot be approximated by any normal distribution. Such models are called singular statistical models and LDA is one of them.

If a map from parameters to probability density functions is injection in a statistical model, then the model is called a regular statistical model. In a regular statistical model, its expected generalization error is asymptotically equal to \( d/2n + o(1/n) \), where \( d \) is the dimension of the parameter and \( n \) is the sample size.
Figure 1: (a) In this paper, we give the exact value of the learning coefficient of LDA $\lambda$. The learning coefficient is smaller than half of the parameter dimension, since LDA is a singular statistical model. The dotted blue line drawn by the circles in this figure represents the learning coefficients of LDA when the number of topics $H$ is increased. If LDA was a regular statistical model, its learning coefficient would be the dotted yellow line drawn by the squares. The behavior of them are so different.

(b) This figure shows the theoretical learning curve of LDA and that of a regular statistical model whose parameter dimension is same as LDA. The former is the solid blue line and the latter is the dashed yellow line. The vertical axis means the expected generalization error $E[G_n]$ and the horizontal one is the sample size $n$. This is based on Eq. (1) and the exact value of $\lambda$ which is clarified by our result.

Moreover, its negative log marginal likelihood (a.k.a. free energy) has asymptotic expansion represented by $nS_n + (d/2) \log n + O_p(1)$, where $S_n$ is the empirical entropy [23]. On the other hand, in the general case, by using resolution of singularity [19], Watanabe had proved that the asymptotic forms of its generalization error $G_n$ and marginal likelihood $Z_n$ are the followings [25, 26, 27]:

\[
E[G_n] = \frac{\lambda}{n} - \frac{m-1}{n \log n} + o\left(\frac{1}{n \log n}\right),
\]

\[
-\log Z_n = nS_n + \lambda \log n - (m - 1) \log \log n + O_p(1),
\]

where $\lambda$ is a positive rational number, $m$ is a positive integer, and $E[\cdot]$ is an expectation operator on the overall datasets. The constant $\lambda$ is called a learning coefficient since it is dominant in the leading term of the above forms which represent learning curves (Fig. 1b). The above forms hold not only in the case the model is regular but also in the case that is singular. In the regular case, $\lambda = d/2$ and $m = 1$ hold. However, in the singular case, they are depend on the model. Let $K(w)$ be the Kullback-Leibler (KL) divergence between the true distribution to the statistical model, where $w$ is a parameter of the model. The function $K(w)$ is non-negative and analytic. The constants $\lambda$ and $m$ are characterized by a set of the zero points of the KL divergence: $K^{-1}(0)$. $K^{-1}(0)$ is an analytic set (a.k.a. algebraic variety). $\lambda$ is called a real log canonical threshold (RLCT) and $m$ is called a multiplicity in algebraic geometry. In LDA, if the number of topics changes, then $K^{-1}(0)$ does (Fig. 1a). A model selection method, called sBIC which uses RLCTs of statistical models, has been proposed by Drton and Plummer [9]. Drton and Imai have empirically verified that sBIC is precise to select the optimal and minimal model if the exact values or tight bounds of RLCTs are clarified [9, 8, 17]. Other application of RLCTs is a design procedure for exchange probability in replica Monte Carlo method by Nagata [19]. To determine $\lambda$ and $m$, we should consider resolution of singularity for those concrete varieties. In general, we should find RLCTs to a family of functions to clarify a learning coefficient of a singular statistical model. There is no standard method to find RLCTs to a given collection.
of functions; thus, researchers study RLCTs with considering different procedures for each statistical model. In fact, RLCTs of several models has been analyzed in both statistics and machine learning. For instance, the RLCTs had been studied in Gaussian mixture model [30], Poisson mixture model [21], reduced rank regression [4], three-layered neural networks [26], naive Bayesian networks [20], Bayesian networks [31], Boltzmann machines [33, 2, 3], Markov models [35], hidden Markov models [32], Gaussian latent tree and forest models [8], and non-negative matrix factorization [14, 13, 12]. Note that clarifying the exact value of the RLCT in the all case is challenging problem. Whereas we would like to emphasize that this is not to deny the value and novelty of these researches, in deed, they cannot have clarified the exact value except for Aoyagi’s result in 2005 [4].

In this paper, we derive the exact asymptotic form of the Bayesian generalization error by determination of the exact RLCT in LDA. This article is divided to four parts. First, we introduced background of this research in the above. Second, we describe the framework of Bayesian inference and relationship between its theory and algebraic geometry. Third, we state the Main Theorem. Lastly, we discuss about this theoretical result and we conclude this paper. We prove the Main Theorem in Appendix.

2 Framework of Bayesian Inference and its Theory

Let \(X^n = (X_1, \ldots, X_n)\) be a sample (a.k.a. dataset: a collection of random variables) of \(n\) independent and identically distributed from a data generating distribution (a.k.a. true distribution). The densities of the true distribution and a statistical model is denoted by \(q(x)\) and \(p(x|w)\), respectively. These domain \(\mathcal{X}\) is a subset of a finite-dimensional real Euclidean or discrete space. Let \(\varphi(w)\) be a probability density of a prior distribution. The KL divergence between the true distribution to the statistical model is denoted by

\[
K(w) = \int \! dx q(x) \log \frac{q(x)}{p(x|w)},
\]

(3)

As technical assumptions, we suppose the parameter set \(W \subset \mathbb{R}^d\) is sufficiently wide compact and the prior is positive and bounded on \(K^{-1}(0)\): \(0 < \varphi(w) < \infty\) for any \(w \in K^{-1}(0)\). Moreover, \(\varphi(w)\) is a \(C^\infty\)-function on with the compact support \(W\). We define a posterior distribution as the following density function on \(W\):

\[
\varphi^*(w|X^n) = \frac{1}{Z_n} \varphi(w) \prod_{i=1}^n p(X_i|w),
\]

(4)

where \(Z_n\) is a normalizing constant to satisfy the condition \(\int \varphi^*(w|X^n) dw = 1:\)

\[
Z_n = \int \! dw \varphi(w) \prod_{i=1}^n p(X_i|w).
\]

(5)

This is called a marginal likelihood or a partition function. Its negative log value is called a free energy \(F_n = -\log Z_n\). Note that the marginal likelihood is a probability density function of a dataset. The free energy appears in a leading term of the difference between the true distribution and the model in the sense of dataset generating process. An entropy of the true distribution and an empirical one are denoted by

\[
S = -\int \! dx q(x) \log q(x),
\]

(6)

\[
S_n = -\frac{1}{n} \sum_{i=1}^n \log q(X_i).
\]

(7)

By definition, \(X_i \sim q(x)\) and \(X^n \sim \prod_{i=1}^n q(x_i)\) hold; thus let \(\mathbb{E}[^]\) be an expectation operator on overall dataset defined by

\[
\mathbb{E}[^] = \int \! dx^n \prod_{i=1}^n q(x_i)[^].
\]

(8)
\( \mathbb{E}[S_n] \) is \( S \). Then, we have the following KL divergence

\[
\int dx^n \prod_{i=1}^{n} q(x_i) \log \frac{\prod_{i=1}^{n} q(x_i)}{Z_n} = -\mathbb{E}[nS_n] - \mathbb{E}[^\log Z_n] \\
= -nS + \mathbb{E}[F_n].
\]

The expected free energy is an only term depend on the model and the prior. For this reason, the free energy is used as a criterion to select the model.

A predictive distribution is defined by the following density function on \( X \):

\[
p^*(x|X^n) = \int dw \varphi^*(w|X^n)p(x|w).
\]

This is a probability distribution of a new data. It is also important for statistics and machine learning to evaluate the dissimilarity between the true and the model in the sense of a new data generating process. A generalization error \( G_n \) is defined by a KL divergence between the true distribution and the predictive one:

\[
G_n = \int dx q(x) \log \frac{q(x)}{p^*(x|X^n)}.
\]

Bayesian inference is defined by inferring that the true distribution may be the predictive one. For an arbitrary finite \( n \), by the definition of the marginal likelihood [5] and the predictive distribution (11), we have

\[
p^*(X_{n+1}|X^n) = \frac{1}{Z_n} \int dw \varphi(w) \prod_{i=1}^{n} p(X_i|w)p(X_{n+1}|w)
\]

\[
= \frac{1}{Z_n} \int dw \varphi(w) \prod_{i=1}^{n+1} p(X_i|w)
\]

\[
= \frac{Z_{n+1}}{Z_n}.
\]

Considering expected negative log values of both sides, we get

\[
\mathbb{E}[ - \log p^*(X_{n+1}|X^n) ] = \mathbb{E}[ - \log Z_{n+1} - ( - \log Z_n ) ]
\]

\[
\mathbb{E}[G_n] + S = \mathbb{E}[F_{n+1}] - \mathbb{E}[F_n].
\]

Hence, \( G_n \) and \( F_n \) are important random variables in Bayesian inference. In mathematical theory of Bayesian statistics (a.k.a. singular learning theory), we consider how they asymptotically behave in the general case [29]. To establish this theory, resolution of singularity in algebraic geometry has been needed.

Now, we briefly explain the relationship between singular learning theory and algebraic geometry. Considering \( K(w) \) in Eq. (5) and its zero points \( K^{-1}(0) \), we use the following analytic form by [5] of the singularities resolution theorem [16]. Atiyah has derived this form of the singularities resolution theorem in order to analyze the relationship between a division of distributions (hyperfunctions) and local type zeta functions [5]. Watanabe has proved that it is useful for constructing singular learning theory [25, 26, 27].

**Theorem 2.1 (Singularities Resolution Theorem).** Let \( K \) be a non-negative analytic function on \( W \subset \mathbb{R}^d \). Assume that \( K^{-1}(0) \) is not an empty set. Then, there are an open set \( W' \), a \( d \)-dimensional smooth manifold \( \mathcal{M} \), and an analytic map \( g : \mathcal{M} \to W' \) such that \( g : \mathcal{M} \setminus g^{-1}(K^{-1}(0)) \to W' \setminus K^{-1}(0) \) is isomorphic and

\[
K(g(u)) = u_1^{2k_1} \ldots u_d^{2k_d},
\]

\[
| \det g'(u) | = b(u)|u_1^{h_1} \ldots u_d^{h_d}|
\]

hold for each local chart \( U \ni u \) of \( \mathcal{M} \), where \( k_j \) and \( h_j \) are non-negative integer for \( j = 1, \ldots, d \), \( \det g'(u) \) is the Jacobian of \( g \) and \( b : \mathcal{M} \to \mathbb{R} \) is strictly positive analytic: \( b(u) > 0 \).
By using Theorem 2.1, the following theorem is proved [5, 6, 22].

**Theorem 2.2.** Let \( K : \mathbb{R}^d \rightarrow \mathbb{R} \) be an analytic function of a variable \( w \in \mathbb{R}^d \). \( a : W \rightarrow \mathbb{R} \) is denoted by a \( C^\infty \)-function with compact support \( W \). Then

\[
\zeta(z) = \int_W |K(w)|^z a(w) \, dw
\]

is a holomorphic function in \( \text{Re}(z) > 0 \). Moreover, \( \zeta(z) \) can be analytically continued to a unique meromorphic function on the entire complex plane \( \mathbb{C} \). Its all poles are negative rational numbers.

Applying Theorem 2.1 to the KL divergence in Eq. (3), we have

\[
K(g(u)) = u_1^{2k_1} \ldots u_d^{2k_d},
\]

\[
| \det g'(u) | = b(u)|u_1^{h_1} \ldots u_d^{h_d} |.
\]

Suppose the prior density \( \phi(w) \) has the compact support \( W \) and the open set \( W' \) satisfies \( W \subset W' \). By using Theorem 2.2, we can define a zeta function of learning theory.

**Definition 2.1 (Zeta Function of Learning Theory).** Let \( K(w) \geq 0 \) be the KL divergence mentioned in Eq. (3) and \( \phi(w) \geq 0 \) be a prior density function which satisfies the above assumption. A zeta function of learning theory is defined by the following univariate complex function

\[
\zeta(z) = \int_W K(w)^z \phi(w) \, dw.
\]

**Definition 2.2 (Real Log Canonical Threshold).** Let \( \zeta(z) \) be a zeta function of learning theory represented in Definition 2.1. Consider an analytic continuation of \( \zeta(z) \) from Theorem 2.2. A real log canonical threshold (RLCT) is defined by the negative maximum pole of \( \zeta(z) \) and its multiplicity is defined by the order of the maximum pole.

Here, we describe how to determine the RLCT \( \lambda > 0 \) of the model corresponding to \( K(w) \). We apply Theorem 2.1 to the zeta function of learning theory. For each local coordinate \( U \), we have

\[
\zeta(z) = \int_U K(g(u))^z \phi(g(u)) | \det g'(u) | \, du
\]

\[
= \int_U u_1^{2k_1 z + h_1} \ldots u_d^{2k_d z + h_d} \phi(g(u)) b(u) \, du.
\]

The functions \( \phi(g(u)) \) and \( b(u) \) are strictly positive in \( U \); thus, we should consider the maximum pole of

\[
\int_U u_1^{2k_1 z + h_1} \ldots u_d^{2k_d z + h_d} \, du = \frac{C_1(z)}{2k_1 z + h_1} \ldots \frac{C_d(z)}{2k_d z + h_d},
\]

where \( (C_j(z))_{j=1}^d \) are non-zero functions of \( z \in \mathbb{C} \). Hence, we give the RLCT in the local chart \( U \) as follows

\[
\lambda_U = \min_{j=1}^d \left\{ \frac{h_j + 1}{2k_j} \right\}.
\]

By considering the duplication of indices, we can also find the multiplicity \( m \). Therefore, we can determine the RLCT as \( \lambda = \min_U \lambda_U \). The RLCT is equal to the learning coefficient because of Eq. (1) and (2). That is why we need resolution of singularity to clarify the behavior of the Bayesian generalization error \( G \) and the free energy \( F \) via determination of the learning coefficient.
3 Main Theorem

In this section, we state the Main Theorem: the exact value of the RLCT of LDA.

**Definition 3.1** (Stochastic Matrix). A stochastic matrix is defined by a matrix whose columns are in simplices, i.e. the sum of the entries in each column is equal to one and each entries are non-negative.

An $M \times N$ matrix $C = (c_{ij})_{i=1,j=1}^{M,N}$ is stochastic if and only if $c_{ij} \geq 0$ and $\sum_{j=1}^{N} c_{ij} = 1$ hold.

In the following, the parameter $w = (A, B)$ is a pair of stochastic matrices and the data $x$ is a one-hot vector. By definition, a set of stochastic matrices is compact. Let $\text{Onehot}(N) := \{w = (w_j) \in \{0, 1\}^N \mid \sum_{j=1}^{N} w_j = 1\}$ be a set of $N$-dimensional one-hot vectors and $\Delta_N := \{c = (c_j)_{j=1}^{N} \mid \sum_{j=1}^{N} c_j = 1\}$ be an $N$-dimensional simplex. In LDA terminology, the number of documents and the vocabulary size is denoted by $N$ and $M$, respectively. Let $H_0$ be the optimal (or true) number of topics and $H$ be the chosen one. In this situation, the sample size $n$ is the number of words in all of the given documents. See also Table I (this table is quoted and modified from our previous study [15]).

Table 1: Description of Variables in LDA Terminology

| Variable | Description | Index |
|----------|-------------|-------|
| $b_j = (b_{kj}) \in \Delta_H$ | topic proportion of topic $k$ in document $j$ | for $k = 1, \ldots, H$ |
| $a_k = (a_{ik}) \in \Delta_M$ | appearance probability of word $i$ in topic $k$ | for $i = 1, \ldots, M$ |
| $x = (x_i) \in \text{Onehot}(M)$ | word $i$ is defined by $x_i = 1$ | for $i = 1, \ldots, M$ |
| $y = (y_k) \in \text{Onehot}(H)$ | topic $k$ is defined by $y_k = 1$ | for $k = 1, \ldots, H$ |
| $z = (z_j) \in \text{Onehot}(N)$ | document $j$ is defined by $z_j = 1$ | for $j = 1, \ldots, N$ |
| $*_{0}$ and $*_{0}^{*}$ | optimal or true variable corresponding to $*$ | - |

Let $A = (a_{ik})_{i=1,k=1}^{M,H}$ and $B = (b_{kj})_{k=1,j=1}^{H,N}$ be $M \times H$ and $H \times N$ stochastic matrix, respectively. Assume that the pair of stochastic matrices $(A_0, B_0)$ is one of optimal parameters, where $A_0 = (a_{ik})_{i=1,k=1}^{M,H}$ and $B_0 = (b_{kj})_{k=1,j=1}^{H,N}$.

Here, we define the RLCT of LDA in the below. This definition is also quoted and modified from the statement in [15].

**Definition 3.2** (LDA). Assume that $M \geq 2$, $N \geq 2$, and $H \geq H_0 \geq 1$. Let $q(x|z)$ and $p(x|z, A, B)$ be conditional probability mass functions of $x \in \text{Onehot}(M)$ given $z \in \text{Onehot}(N)$ as the following:

$$q(x | z) = \prod_{j=1}^{N} \left( \sum_{k=1}^{H} c_{kj} \prod_{i=1}^{M} \sum_{c_i}^{z_j} a_{ik} \right), \quad (24)$$

$$p(x | z, A, B) = \prod_{j=1}^{N} \left( \sum_{k=1}^{H} b_{kj} \prod_{i=1}^{M} \sum_{c_i}^{z_j} a_{ik} \right). \quad (25)$$

The prior density function is denoted by $\varphi(A, B)$. The conditional mass $q(x|z)$ and $p(x|z, A, B)$ represent the true distribution of LDA and the statistical model of that, respectively.

These distributions are the marginalized ones of the followings which contain the true topics $y^0 \in \text{Onehot}(H_0)$ and the one of the model $y \in \text{Onehot}(H)$:

$$q(x, y^0 | z) = \prod_{j=1}^{N} \left[ \sum_{k=1}^{H} \left( b_{kj} \prod_{i=1}^{M} \sum_{c_i}^{z_j} a_{ik} \right) y^0_{kj} \right]^{z_j},$$

$$p(x, y | z, A, B) = \prod_{j=1}^{N} \left[ \sum_{k=1}^{H} \left( b_{kj} \prod_{i=1}^{M} \sum_{c_i}^{z_j} a_{ik} \right) y_{kj} \right]^{z_j} .$$

In practical cases, the topics are not observed; thus, we use Eq. (24) and (25) as the definition of LDA.
**Definition 3.3 (RLCT of LDA).** Let $K(A, B)$ be the KL divergence between $q(x|z)$ and $p(x|z, A, B)$:

$$K(A, B) = \sum_{z \in \text{Onehot}(M)} \sum_{z' \in \text{Onehot}(N)} q(x|z)q'(z) \log \frac{q(x|z)}{p(x|z, A, B)},$$

where $q'(z)$ is the true distribution of the document. In LDA, $q'(z)$ is not observed and assumed that it is positive and bounded. Assume that $\varphi(A, B) > 0$ is positive and bounded on $K^{-1}(0) \ni (A_0, B_0)$. Then, the zeta function of learning theory in LDA is the holomorphic function of univariate complex variable $z$ (Re($z$) > 0)

$$\zeta(z) = \iint K(A, B)^2 dAdB$$

and it can be analytically continued to a unique meromorphic function on the entire complex plane $\mathbb{C}$ and all of its poles are negative rational numbers. The RLCT of LDA is defined by $\lambda$ if the largest pole of $\zeta(z)$ is $(-\lambda)$. Its multiplicity $m$ is defined as the order of the maximum pole.

The RLCT of LDA is depend on $(M, N, H, H_0)$; thus, we clearly write it as $\lambda = \lambda(M, N, H, H_0)$. The main result of this paper is the following theorem.

**Theorem 3.1 (Main Theorem).** Suppose $M \geq 2$, $N \geq 2$, and $H \geq H_0 \geq 1$. The RLCT of LDA $\lambda$ and its multiplicity $m$ are as follows.

1. If $N + H_0 < M + H$ and $M + H_0 < N + H$ and $H + H_0 < M + N$,
   (a) and if $M + N + H + H_0$ is even, then
   $$\lambda = \frac{1}{8}(2(H + H_0)(M + N) - (M - N)^2 - (H + H_0)^2} - \frac{1}{2} N, \ m = 1.$$
   (b) and if $M + N + H + H_0$ is odd, then
   $$\lambda = \frac{1}{8}(2(H + H_0)(M + N) - (M - N)^2 - (H + H_0)^2 + 1} - \frac{1}{2} N, \ m = 2.$$

2. Else if $M + H < N + H_0$, then
   $$\lambda = \frac{1}{2}(MH + NH_0 - HH_0 - N), \ m = 1.$$

3. Else if $N + H < M + H_0$, then
   $$\lambda = \frac{1}{2}(NH + MH_0 - HH_0 - N), \ m = 1.$$

4. Else (i.e. $M + N < H + H_0$), then
   $$\lambda = \frac{1}{2}(MN - N), \ m = 1.$$

To prove Main Theorem, we use the RLCT of matrix factorization (MF).

**Definition 3.4 (RLCT of MF).** Let $U$, $V$, $U_0$ and $V_0$ be $M \times H$, $H \times N$, $M \times r$ and $r \times N$ real matrix, respectively. Set $M \geq 1$, $N \geq 1$, $H \geq r \geq 0$. Assume that they in a compact subset $W$ of $(M + N)H$-dimensional Euclidean space. The RLCT of MF $\lambda_{MF} = \lambda_{MF}(M, N, H, r)$ is defined by the negative maximum pole of the following zeta function

$$\zeta_{MF}(z) = \iint_W \|UV - U_0V_0\|^2 dUdV.$$

Its multiplicity $m_{MF}$ is defined as the order of the maximum pole.
The exact value of $\lambda_{\text{MF}}$ had been clarified as that of reduced rank regression (a.k.a. three-layered linear neural network) by Aoyagi in [4]. By making the RLCT of LDA come down to that of MF, we prove Main Theorem. The rigorous proof of Main Theorem is described in Appendix.

Sketch of Proof. Let $\zeta_{\text{SMF}}(z)$ be a zeta function of learning theory in stochastic MF:

$$\zeta_{\text{SMF}}(z) = \int \int \|AB - A_0B_0\|^2 \, dAdB.$$  

Let $\lambda_{\text{SMF}}$ and $m_{\text{SMF}}$ be the negative maximum pole and its order of $\zeta_{\text{SMF}}(z)$, respectively. According to [15],

$$\lambda = \lambda_{\text{SMF}}, \ m = m_{\text{SMF}}$$  

hold; thus, we only have to consider $\lambda_{\text{SMF}}$ and $m_{\text{SMF}}$.

Developing $\|AB - A_0B_0\|^2$, performing several changes of variables and considering the integral range of transformed variables in the zeta function, we have

$$\lambda_{\text{SMF}} = \frac{M-1}{2} + \lambda_{\text{MF}}(M-1,N-1,H-1,H_0-1), \ m_{\text{SMF}} = m_{\text{MF}}.$$  

We calculate $\lambda_{\text{MF}}$ and $m_{\text{MF}}$ by following [4], then we obtain Main Theorem.

4 Discussion and Conclusion

In this paper, we described how the exact RLCT (i.e. learning coefficient) of LDA is determined in the general case. Using this result, we also clarified the exact asymptotic forms of the Bayesian generalization error and the marginal likelihood in LDA.

The RLCT of LDA can be represented by using that of MF. Namely, Main Theorem can be interpreted as that the learning coefficient of LDA is that of the unconstrained MF minus the penalty due to the simplex constraint. In fact, it can be proved that

$$\lambda(M,N,H,H_0) = \lambda_{\text{MF}}(M,N,H,H_0) - \frac{N}{2}$$  

holds (see also the rigorous proof of Main Theorem in Appendix). The dimension of the stochastic matrix $AB$ with the degrees of freedom is $(M-1)N = MN - N$. The subtracted $N$ is the dimension of the parameter that can be uniquely determined from the parameters of the other $(M-1)N$ dimensions in the matrix $AB$. This part can be regarded as an $N$-dimensional regular statistical model, whose RLCT is $N/2$. This is the reason of the above statement. Note that Main Theorem and its proof are not trivial. A hermeneutic explanation cannot be a mathematical proof. In addition, the actual parameter dimension is $(M-1)H + (H-1)N = (M + N - 1)H - N$ because we have to consider the matrices $A$ and $B$ rather than $AB$. We cannot reach the result of this paper simply by maintaining consistency of the degrees of freedom. Algebraic geometrical methods are used to solve this problem in learning theory: what the learning coefficient of LDA is.

Since LDA is a knowledge discovery method, marginal-likelihood-based model selection often tends to be preferred. However, BIC [23] cannot be used for LDA because it is a singular statistical model. Although Gibbs sampling is usually used for full Bayesian inference of LDA, it is difficult to achieve a tempered posterior distribution; thus, we need other Markov chain Monte Carlo method (MCMC) to calculate WBIC [28] and $W_b$BIC [17]. The result of this study allows us to perform a rigorous model selection of LDA with sBIC [9], which is MCMC-free. Even when the marginal likelihoods are computed directly by the replica Monte Carlo method, our result is useful for the design of the exchange probability [19]. Furthermore, it may be possible to evaluate how precise MCMC approximates the posterior, by comparing Imai’s estimator of the RLCT [17] with the exact values of that [29, 17].

One may use BIC for model selection of LDA; however, using it causes that too small models are chosen. This is because there exists a large difference in values and behaviors between $d/2$ and $\lambda$. In a regular
statistical model, the learning coefficient is half of the parameter dimension $d/2$. In LDA, $d/2 = (M + N - 1)H/2 - N/2$ holds; hence, it linearly increases as the number of topics $H$ does. On the other hand, the RLCT of LDA $\lambda$ does not. In addition, $\lambda$ is much smaller than $d/2$. Fig. 1a shows how the RLCT of LDA $\lambda$ behaves when the number of topics $H$ increases, with $\lambda$-value in the vertical axis and $H$-value in the horizontal axis. However, in fact, $\lambda$ is given by Main Theorem and its curve is obviously non-linear (the circles dotted plot in Fig. 1a). Hence, their values and behaviors are very different. BIC is based on $d/2$ from the asymptotics of regular statistical models. In contrast, the foundation of sBIC is singular learning theory; thus, it uses $\lambda$ instead of $d/2$. That is why sBIC is theoretically recommended for LDA.

We can draw the theoretical learning curve like the solid line in Fig. 1b with $\mathbb{E}[G_n]$-value in the vertical axis and $n$-value in the horizontal axis. We also namely draw a curve like the dashed line in Fig. 1b. This dashed curve is not only an upper bound of the learning curve of LDA in Bayesian inference but also a lower bound of that in maximum likelihood or posterior estimation methods. Let $G_n^{\text{MAP}}$ and $\mu$ be the generalization error and the learning coefficient of LDA in maximum posterior methods, respectively. This is well-defined, i.e. $\mathbb{E}[G_n^{\text{MAP}}] = \mu/n + o(1/n)$ holds. On the basis of the same prior distribution, Watanabe proved the following inequality [29]:

$$\lambda < d/2 < \mu.$$  \hspace{1cm} (30)

This means $\mathbb{E}[G_n^{\text{MAP}}] > \mathbb{E}[G_n] + o(1/n)$ and the leading term of these difference is $(\mu - \lambda)/n > (d - 2\lambda)/2n$. Owing to Main Theorem, we immediately have the exact value of $d - 2\lambda$. Therefore, our result shows at least how much Bayesian inference improves the generalization performance of LDA compared to maximum posterior method. If the prior distribution is a uniform one, then $\mu$ equals the learning coefficient of LDA in maximum likelihood estimation. Hence, the above consideration can be applied to maximum likelihood estimation.

One of future works is finding simultaneous resolution of singularities when the prior is a Dirichlet distribution. A density function of a non-uniform Dirichlet distribution has zero or diverged points; thus, it may affect the learning coefficient. Another future aim is verifying the numerical behavior of our result.

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A Proof of Main Theorem

The structure of the proof of Main Theorem is as follows. First, we summarize terms in \( \|AB - A_0B_0\|^2 \) and consider degeneration of a polynomial ideal. Second, we resolve the non-negative restriction by variable transformations which are isomorphic maps. Third, we verify that the problem can be came down to finding the RLCT of reduced rank regression. Lastly, we calculate the concrete value of the RLCT in each case.

Proof of Main Theorem. As mentioned Sketch of Proof of Main Theorem in Section 3, we only have to consider the analytic set defined by
\[
\{(A, B) \mid \|AB - A_0B_0\|^2 = 0, A \text{ and } B \text{ are stochastic matrices.}\}
\]
to determine the RLCT of LDA \( \lambda = \lambda(M, N, H, H_0) \) and its multiplicity \( m = m(M, N, H, H_0) \).

The first part is same as the first half of the proof of Appendix A in our previous research [15]. For the sake of self-containedness, we write down the process of developing the terms in the above paper. Let \( \sim \) be a binomial relation such that the functions \( K_1(w) \) and \( K_2(w) \) have same RLCT if \( K_1(w) \sim K_2(w) \). Summarizing the terms, we have
\[
\|AB - A_0B_0\|^2 \\
= \sum_{j=1}^{N} \sum_{i=1}^{M-1} \left\{ \sum_{k=1}^{H_0-1} \left( a_{ik}b_{kj} - a^0_{ik}b^0_{kj} \right) + a_{iH_0}b_{H_0j} - a^0_{iH_0}b^0_{H_0j} + \sum_{k=H_0+1}^{H-1} a_{ik}b_{kj} + a_{iH}b_{Hj} \right\}^2 \\
+ \sum_{j=1}^{N} \left\{ \sum_{k=1}^{H_0-1} \left( a_{Mk}b_{kj} - a^0_{Mk}b^0_{kj} \right) + a_{MH_0}b_{H_0j} - a^0_{MH_0}b^0_{H_0j} + \sum_{k=H_0+1}^{H-1} a_{Mk}b_{kj} + a_{MH}b_{Hj} \right\}^2.
\] (31)


Put

\[ K_{ij} := \sum_{k=1}^{H_0-1} (a_{ik}b_{kj} - a_{ik}^0b_{kj}^0) + a_iH_0b_{H_0j} - a_i^0b_{H_0j}^0 + \sum_{k=H_0+1}^{H-1} a_{ik}b_{kj} + a_iHb_{Hj}, \]

\[ L_j := \sum_{k=1}^{H_0-1} (a_{Mk}b_{kj} - a_{Mk}^0b_{kj}^0) + a_MH_0b_{H_0j} - a_M^0b_{H_0j} + \sum_{k=H_0+1}^{H-1} a_{Mk}b_{kj} + a_MHb_{Hj}, \]

then we get

\[ \|AB - A_0B_0\|^2 = \sum_{j=1}^{N} \sum_{i=1}^{M-1} K_{ij}^2 + \sum_{j=1}^{N} L_j^2. \]

Using \( a_{Mk} = 1 - \sum_{i=1}^{M-1} a_{ik}, \ b_{Hj} = 1 - \sum_{k=1}^{H-1} b_{kj}, \ a_iM = 1 - \sum_{i=1}^{M-1} a_i^0, \) and \( b_{H_0j} = 1 - \sum_{k=1}^{H_0-1} b_{kj}^0, \) we have

\[ \sum_{i=1}^{M-1} K_{ij} = \sum_{i=1}^{M-1} \sum_{k=1}^{M-1} (a_{ik} - a_{ik}^0)b_{kj} - \sum_{i=1}^{M-1} \sum_{k=1}^{M-1} (a_{ik}^0 - a_i^0)b_{kj}^0 + \sum_{i=1}^{M-1} (a_iH - a_i^0), \]

\[ L_j = -\sum_{i=1}^{M-1} \sum_{k=1}^{M-1} (a_{ik} - a_{ik}^0)b_{kj} + \sum_{i=1}^{M-1} \sum_{k=1}^{M-1} (a_{ik}^0 - a_i^0)b_{kj}^0 - \sum_{i=1}^{M-1} (a_iH - a_i^0), \]

thus

\[ L_j^2 = \left( \sum_{i=1}^{M-1} K_{ij} \right)^2. \]

Therefore

\[ \|AB - A_0B_0\|^2 = \sum_{j=1}^{N} \sum_{i=1}^{M-1} K_{ij}^2 + \sum_{j=1}^{N} L_j^2 \\
= \sum_{j=1}^{N} \sum_{i=1}^{M-1} K_{ij}^2 + \sum_{j=1}^{N} \left( \sum_{i=1}^{M-1} K_{ij} \right)^2. \]

Since the polynomial \( \sum_{i=1}^{M-1} K_{ij} \) is contained in the ideal generated from \( (K_{ij})_{i=1, j=1}^{M-1,N} \), we have

\[ \|AB - A_0B_0\|^2 \sim \sum_{j=1}^{N} \sum_{i=1}^{M-1} K_{ij}^2, \]

i.e.

\[ \|AB - A_0B_0\|^2 \\
\sim \sum_{j=1}^{N} \sum_{i=1}^{M-1} \left\{ \left( \sum_{k=1}^{H-1} (a_{ik} - a_{ik}^0)b_{kj} - \sum_{k=1}^{H_0-1} (a_{ik}^0 - a_i^0)b_{kj}^0 + (a_iH - a_i^0) \right)^2 \right\} \\
= \sum_{j=1}^{N} \sum_{i=1}^{M-1} \left\{ \left( \sum_{k=1}^{H_0-1} (a_{ik} - a_{ik}^0)b_{kj} - (a_{ik}^0 - a_i^0)b_{kj}^0 \right) + \sum_{k=H_0}^{H-1} (a_{ik} - a_iH)b_{kj} + (a_iH - a_i^0) \right\}^2. \]

Let \( \{ a_{ik} = a_{ik} - a_{ik}^0, \ k < H \}

\( \{ c_i = a_iH - a_i^0, \}

\( \{ b_{kj} = b_{kj} \) (32) \)
and put $a_{ik}^0 = a_{ik}^0 - a_{iH_0}^0$. Then we have
\[
\|AB - A_0B_0\|^2 \\
\sim \sum_{j=1}^N \sum_{i=1}^{M-1} \left[ \sum_{k=1}^{H_0-1} (a_{ik} - a_{iH})b_{kj} - (a_{ik}^0 - a_{iH_0}^0)b_{kj}^0 \right] + \sum_{k=H_0}^{H-1} (a_{ik} - a_{iH})b_{kj} + (a_{iH} - a_{iH_0}^0) \right] \right] \\
= \sum_{j=1}^N \sum_{i=1}^{M-1} \left\{ \sum_{k=1}^{H_0-1} (a_{ik}b_{kj} - a_{ik}^0b_{kj}^0) + \sum_{k=H_0}^{H-1} a_{ik}b_{kj} + c_i \right\}. \tag{33}
\]
This is the end of the common part to Appendix A of [15]. We had derived an upper bound of $\lambda$ by using some inequalities of Frobenius norm and the exact value of $\lambda$ in special cases [15]. However, in this paper, we use changes of variables which resolve non-negative restrictions and find the RLCT in the all cases.

The transformation \([32]\) resolves the non-negative restrictions of $a_{ik}(k < H)$ and $c_i$ for $i = 1, \ldots, M - 1$. The changed variables $a_{ik}(k < H)$ and $c_i$ can be negative. We call the determinant of the Jacobian matrix Jacobian for the sake of simplicity. The Jacobian of the transformation \([32]\) equals one.

\[
\begin{align*}
&\text{Let } a_{ik} = a_{ik}, \\
&\quad x_i = c_i + \sum_{k=1}^{H_0-1} (a_{ik}b_{k1} - a_{ik}^0b_{k1}^0) + \sum_{k=H_0}^{H-1} a_{ik}b_{k1}, \\
&\quad b_{kj} = b_{kj}. \tag{34}
\end{align*}
\]

It is immediately derived that the Jacobian of this map is equal to one. About the transform \([34]\), for $j = 2, \ldots, N$, we have
\[
\sum_{k=1}^{H_0-1} (a_{ik}b_{kj} - a_{ik}^0b_{kj}^0) + \sum_{k=H_0}^{H-1} a_{ik}b_{kj} + c_i \]
\[
= x_i - \sum_{k=1}^{H_0-1} (a_{ik}b_{k1} - a_{ik}^0b_{k1}^0) - \sum_{k=H_0}^{H-1} a_{ik}b_{k1} + \sum_{k=H_0}^{H-1} (a_{ik}b_{kj} - a_{ik}^0b_{kj}^0) + \sum_{k=H_0}^{H-1} a_{ik}b_{kj}. \tag{35}
\]
Substituting this for $\sum_{k=1}^{H_0-1} (a_{ik}b_{kj} - a_{ik}^0b_{kj}^0) + \sum_{k=H_0}^{H-1} a_{ik}b_{kj} + c_i$ in Eq. \([33]\), we have
\[
\|AB - A_0B_0\|^2 \\
\sim \sum_{i=1}^{M-1} \left\{ \sum_{k=1}^{H_0-1} (a_{ik}b_{k1} - a_{ik}^0b_{k1}^0) + \sum_{k=H_0}^{H-1} a_{ik}b_{k1} + c_i \right\}^2 \\
+ \sum_{j=2}^N \sum_{i=1}^{M-1} \left\{ \sum_{k=1}^{H_0-1} (a_{ik}b_{kj} - a_{ik}^0b_{kj}^0) + \sum_{k=H_0}^{H-1} a_{ik}b_{kj} + c_i \right\}^2 \\
= \sum_{i=1}^{M-1} x_i^2 + \sum_{j=2}^N \sum_{i=1}^{M-1} \left\{ \sum_{k=1}^{H_0-1} (a_{ik}b_{k1} - a_{ik}^0b_{k1}^0) - \sum_{k=H_0}^{H-1} a_{ik}b_{k1} \right\}^2 \\
+ \sum_{k=1}^{H_0-1} (a_{ik}b_{kj} - a_{ik}^0b_{kj}^0) + \sum_{k=H_0}^{H-1} a_{ik}b_{kj} \right\}^2 \\
= \sum_{i=1}^{M-1} x_i^2 + \sum_{j=2}^N \sum_{i=1}^{M-1} \left[ x_i + \sum_{k=1}^{H_0-1} \left\{ a_{ik}(b_{kj} - b_{k1}) - a_{ik}^0(b_{kj}^0 - b_{k1}^0) \right\} + \sum_{k=H_0}^{H-1} a_{ik}(b_{kj} - b_{k1}) \right] \right\}^2. \tag{36}
\]
Put
\[
g_{ij} = \sum_{k=1}^{H_0-1} \left\{ a_{ik}(b_{kj} - b_{k1}) - a_{ik}^0(b_{kj}^0 - b_{k1}^0) \right\} + \sum_{k=H_0}^{H-1} a_{ik}(b_{kj} - b_{k1}).
\]
From Eq. (36), we have
\[ \| AB - A_0 B_0 \| ^2 \sim \sum_{i=1}^{M-1} x_i^2 + \sum_{j=2}^{N} \sum_{i=1}^{M-1} (x_i + g_{ij})^2. \] (37)

Let \( J \) be a polynomial ideal \( \langle (x_i)_{i=1}^{M-1}, (g_{ij})_{i=1, j=2}^{M-1, N} \rangle \). On account of \( x_i + g_{ij} \in J \), we have
\[ \sum_{j=2}^{N} \sum_{i=1}^{M-1} (x_i + g_{ij})^2 \sim \sum_{j=2}^{N} \sum_{i=1}^{M-1} (x_i^2 + g_{ij}^2), \]
i.e.
\[ \| AB - A_0 B_0 \| ^2 \sim \sum_{i=1}^{M-1} x_i^2 + \sum_{j=2}^{N} \sum_{i=1}^{M-1} g_{ij}^2 \]
\[ = \sum_{i=1}^{M-1} x_i^2 + \sum_{j=2}^{N} \sum_{i=1}^{M-1} \left\{ a_{ik}(b_{kj} - b_{k1}) - a_{ik}^0 (b_{kj}^0 - b_{k1}^0) \right\} + \sum_{k=H_0}^{H-1} a_{ik}(b_{kj} - b_{k1}) \right\}^2. \] (38)

Let \[
\begin{cases}
  a_{ik} = a_{ik}, & k < H \\
  x_i = x_i, & \\
  b_{k1} = b_{k1}, & \\
  b_{kj} = b_{kj} - b_{k1} & j > 1.
\end{cases}
\] (39)

For \( k = 1, \ldots, H - 1 \) and \( j = 2, \ldots, N \), non-negative restrictions of \( b_{kj} \) can be resolved. The Jacobian of the transformation (39) is one. Apply this map to Eq. (38) and put \( b_{k1}^0 = b_{kj} - b_{k1}^0 \). Then, we have
\[ \| AB - A_0 B_0 \| ^2 \sim \sum_{i=1}^{M-1} x_i^2 + \sum_{j=2}^{N} \sum_{i=1}^{M-1} \left\{ a_{ik} b_{kj} - a_{ik}^0 b_{k1} - a_{ik}^0 b_{kj} \right\} + \sum_{k=H_0}^{H-1} a_{ik} b_{kj} \right\}^2. \] (40)

There are not \( b_{k1} \) \( (k = 1, \ldots, H - 1) \) in the right hand side; thus, we can regard the non-negative restrictions of the all variable are resolved after applying the transformation (39).

Real matrices \( U \), \( V \), \( U_0 \), and \( V_0 \) are denoted by \( U := (u_{ik})_{i=1, k=1}^{M-1, H-1} \), \( V := (v_{kl})_{k=1, l=1}^{H-1, N-1} \), \( U_0 := (u_{ik}^0)_{i=1, k=1}^{M-1, H-1} \), and \( V_0 := (v_{kl}^0)_{k=1, l=1}^{H-1, N-1} \), respectively. Here, we have \( u_{ik} = a_{ik} \), \( v_{kl} = v_{k(l-1)} = b_{kj} \), \( u_{ik}^0 = a_{ik}^0 \), and \( v_{kl}^0 = v_{k(l-1)}^0 = b_{kj}^0 \) for \( i = 1, \ldots, M - 1 \), \( k = 1, \ldots, H - 1 \) and \( j = 2, \ldots, N \).

Now, let us start coming down from LDA to reduced rank regression.
\[ \| U V - U_0 V_0 \| ^2 = \sum_{i=1}^{N} \sum_{j=1}^{M-1} \left( \sum_{k=1}^{H-1} u_{ik} v_{kl} - \sum_{k=1}^{H_0-1} u_{ik}^0 v_{kl} \right) \right\}^2 \]
\[ = \sum_{j=2}^{N} \sum_{i=1}^{M-1} \left( \sum_{k=1}^{H-1} a_{ik} b_{kj} - \sum_{k=1}^{H_0-1} a_{ik}^0 b_{kj} \right) \right\}^2. \] (41)
holds; thus, from Eq. (10) and (11), we have
\[ \|AB - A_0 B_0\|^2 \sim \sum_{i=1}^{M-1} x_i^2 + \|UV - U_0 V_0\|^2. \] (42)

Let \((\lambda_1, m_1)\) and \((\lambda_2, m_2)\) be pairs of the RLCT and its multiplicity of the first and the second term, respectively. There is no intersection between \(\{(x_i)^M\}_{i=1}^M\) and \(\{(U, V)\}\); hence, we have
\[ \lambda = \lambda_1 + \lambda_2, \] (43)
\[ m = m_1 + m_2 - 1. \] (44)

By simple calculation, \(\lambda_1 = (M - 1)/2\) and \(m_1 = 1\) hold. Besides, the entries of the matrices \(U\) and \(V\) can be real as well as non-negative. Thus, \(\lambda_2\) is the RLCT of non-restricted MF, i.e. that of reduced rank regression [4]. The same is true for the multiplicity \(m_2\). Therefore, we obtain
\[ \lambda(M, N, H, H_0) = \frac{M-1}{2} + \lambda_{MF}(M-1, N-1, H-1, H_0-1), \] (45)
\[ m(M, N, H, H_0) = m_{MF}(M-1, N-1, H-1, H_0-1). \] (46)

Finally, we concretely calculate \(\lambda(M, N, H, H_0)\) and \(m(M, N, H, H_0)\). According to [4], the RLCT and its multiplicity of MF are as follows.

1. If \(N + H_0 < M + H\) and \(M + H_0 < N + H\) and \(H + H_0 < M + N\) and \(M + N + H + H_0\) is even, then
\[ \lambda_{MF}(M-1, N-1, H-1, H_0-1) = \frac{1}{8}\{(2(H + H_0 - 2)(M + N - 2) - (M - N)^2 - (H + H_0 - 2)^2\}, \]
\[ m_{MF}(M-1, N-1, H-1, H_0-1) = 1. \]

2. Else if \(N + H_0 < M + H\) and \(M + H_0 < N + H\) and \(H + H_0 < M + N\) and \(M + N + H + H_0\) is odd, then
\[ \lambda_{MF}(M-1, N-1, H-1, H_0-1) = \frac{1}{8}\{(2(H + H_0 - 2)(M + N - 2) - (M - N)^2 - (H + H_0 - 2)^2 + 1\}, \]
\[ m_{MF}(M-1, N-1, H-1, H_0-1) = 2. \]

3. Else if \(M + H < N + H_0\), then
\[ \lambda_{MF}(M-1, N-1, H-1, H_0-1) = \frac{1}{2}\{(M-1)(H-1) + (N-1)(H_0-1) - (H-1)(H_0-1)\}, \]
\[ m_{MF}(M-1, N-1, H-1, H_0-1) = 1. \]

4. Else if \(N + H < M + H_0\), then
\[ \lambda_{MF}(M-1, N-1, H-1, H_0-1) = \frac{1}{2}\{(N-1)(H-1) + (M-1)(H_0-1) - (H-1)(H_0-1)\}, \]
\[ m_{MF}(M-1, N-1, H-1, H_0-1) = 1. \]

5. Else (i.e. \(M + N < H + H_0\)), then
\[ \lambda_{MF}(M-1, N-1, H-1, H_0-1) = \frac{1}{2}(M-1)(N-1), \]
\[ m_{MF}(M-1, N-1, H-1, H_0-1) = 1. \]

Since the multiplicity is clear, we find the RLCT. We develop the terms in each case by using Eq. (45).
In the case (1), we have

\[
\lambda(M, N, H, H_0) = \frac{M - 1}{2} + \frac{1}{8}(2(H + H_0 - 2)(M + N - 2) - (M - N)^2 - (H + H_0 - 2)^2)
\]

\[
= \frac{M - 1}{2} + \frac{1}{8}(2H + H_0)(M + N) - 4(M + N + H + H_0) + 8
\]

\[
- (M - N)^2 - (H + H_0)^2 + 4(H + H_0) - 4
\]

\[
= \frac{1}{8}(2H + H_0)(M + N) - (M - N)^2 - (H + H_0)^2 + \frac{M - 1}{2} - \frac{M + N - 1}{2}
\]

\[
= \frac{1}{8}(2(H + H_0)(M + N) - (M - N)^2 - (H + H_0)^2) - \frac{N}{2}.
\]

In the case (2), by the same way as the case (1), we have

\[
\lambda(M, N, H, H_0) = \frac{1}{8}(2H + H_0)(M + N) - (M - N)^2 - (H + H_0)^2 + 1 - \frac{N}{2}.
\]

In the case (3), we have

\[
\lambda(M, N, H, H_0) = \frac{M - 1}{2} + \frac{1}{2}\{(M - 1)(H - 1) + (N - 1)(H_0 - 1) - (H - 1)(H_0 - 1)\}
\]

\[
= \frac{M - 1}{2} + \frac{1}{2}\{MH - (M + H) + 1 + NH_0 - (N + H_0) + 1 - HH_0 + (H + H_0) - 1\}
\]

\[
= \frac{1}{2}(MH + NH_0 - HH_0) + \frac{M - 1}{2} - \frac{M + N - 1}{2}
\]

\[
= \frac{1}{2}(MH + NH_0 - HH_0) - \frac{N}{2}.
\]

In the case (4), by the same way as the case (3), we have

\[
\lambda(M, N, H, H_0) = \frac{1}{2}(MH + NH_0 - HH_0) - \frac{N}{2}.
\]

In the case (5), we have

\[
\lambda(M, N, H, H_0) = \frac{M - 1}{2} + \frac{1}{2}(M - 1)(N - 1)
\]

\[
= \frac{1}{2}(M - 1)N
\]

\[
= \frac{1}{2}MN - \frac{N}{2}.
\]

From the above, Main Theorem is proved. Comparing the RLCT of MF [4], we also obtain

\[
\lambda(M, N, H, H_0) = \lambda_{MF}(M, N, H, H_0) - \frac{N}{2}.
\]