Optimal estimate of the life span of solutions to the heat equation with a nonlinear boundary condition

Kotaro Hisa

Abstract

Consider the heat equation with a nonlinear boundary condition

\[
\begin{aligned}
\partial_t u &= \Delta u, & x \in \mathbb{R}_+^N, & t > 0, \\
\partial_\nu u &= u^p, & x \in \partial \mathbb{R}_+^N, & t > 0, \\
u(x,0) &= \kappa \psi(x), & x \in D := \mathbb{R}_+^N,
\end{aligned}
\]

where \( N \geq 1, p > 1, \kappa > 0 \) and \( \psi \) is a nonnegative measurable function in \( \mathbb{R}_+^N := \{ y \in \mathbb{R}^N : y_N > 0 \} \). Let us denote by \( T(\kappa \psi) \) the life span of solutions to this problem. We investigate the relationship between the singularity of \( \psi \) at the origin and \( T(\kappa \psi) \) for sufficiently large \( \kappa > 0 \) and the relationship between the behavior of \( \psi \) at the space infinity and \( T(\kappa \psi) \) for sufficiently small \( \kappa > 0 \). Moreover, we give an optimal estimate to \( T(\kappa \psi) \), as \( \kappa \to \infty \) or \( \kappa \to +0 \).

Address:
K. H.: Mathematical Institute, Tohoku University, Aoba, Sendai 980-8578, Japan
E-mail: kotaro.hisa.s5@dc.tohoku.ac.jp

MSC: 35A01; 35B44; 35K05; 35K60.
Keywords: Life span, Heat equation, Nonlinear boundary condition, Blow-up.
1 Introduction and main results

1.1 Introduction

Consider the heat equation with a nonlinear boundary condition

\[
\begin{cases}
\partial_t u = \Delta u, & x \in \mathbb{R}^N_+, \ t > 0, \\
\partial_{\nu} u = u^p, & x \in \partial \mathbb{R}^N_+, \ t > 0,
\end{cases}
\]

with the initial condition

\[ u(x, 0) = \kappa \psi(x), \ x \in D := \mathbb{R}^N_+. \]

where \( N \geq 1, \ p > 1, \ \kappa > 0 \) and \( \psi \) is a nonnegative measurable function in \( \mathbb{R}^N_+ := \{ y \in \mathbb{R}^N : y_N > 0 \} \). The aim of this paper is to obtain an optimal estimate of the life span \( T(\kappa \psi) \) of solutions to problem (1.1) with (1.2), as \( \kappa \to \infty \) or \( \kappa \to +0 \). In general, the life span \( T(\kappa \psi) \) is complicated and this may be a reason why the research on the life span \( T(\kappa \psi) \) have fascinated many mathematicians.

Problem (1.1) can be physically interpreted as a nonlinear radiation law and it has been studied in many papers (see e.g., [1, 2, 3, 5, 6, 7, 10, 13, 14, 15] and references therein). Among others, the author of this paper and Ishige [10] obtained the necessary conditions and the sufficient conditions for the solvability of problem (1.1) and identified the strongest singularity. It follows from these conditions that the behavior of the life span \( T(\kappa \psi) \) as \( \kappa \to \infty \) depends on the singularity of \( \psi \) and that of the life span \( T(\kappa \psi) \) as \( \kappa \to +0 \) depends on that of \( \psi \) at the space infinity. In this paper, we investigate these relationships and give an estimate to the life span \( T(\kappa \psi) \) as \( \kappa \to \infty \) and \( \kappa \to +0 \) (See Subsection 1.4).

The main idea is to apply the necessary conditions and the sufficient conditions for the solvability, which have been proved in [10] (see Section 2, in which we review these conditions). Unfortunately, since these conditions have many parameters and are complicated, careful calculation is required to apply them.

1.2 Preliminaries

Before stating the main results of this paper, we have to define the life span \( T(\kappa \psi) \) of solutions to (1.1) with (1.2) strictly. To do that, we formulate the definition of solutions to (1.1). Let \( G = G(x, y, t) \) be the Green function for the heat equation on \( \mathbb{R}^N_+ \) with the homogeneous Neumann boundary condition.

**Definition 1.1** Let \( u \) be a nonnegative and continuous function in \( D \times (0, T) \), where \( 0 < T < \infty \).

- Let \( \varphi \) be a nonnegative measurable function in \( \mathbb{R}^N_+ \). We say that \( u \) is a solution to (1.1) in \( [0, T] \) with \( u(0) = \varphi \) if \( u \) satisfies

\[
     u(x, t) = \int_D G(x, y, t) \varphi(y) \, dy + \int_0^t \int_{\mathbb{R}^{N-1}} G(x, y', 0, t - s) u(y', 0, s)^p \, dy' \, ds
\]

for \( (x, t) \in D \times (0, T) \).
We say that \( u \) is a minimal solution to \( (1.1) \) in \([0, T)\) with \( u(0) = \varphi \) if \( u \) is a solution to \( (1.1) \) in \([0, T)\) with \( u(0) = \varphi \) and satisfies

\[
u(x,t) \leq w(x,t) \quad \text{in} \quad D \times (0,T)
\]

for any solution \( w \) to \( (1.1) \) in \([0, T)\) with \( w(0) = \varphi \).

Since the minimal solution is unique, we can define the life span \( T(\kappa \psi) \) as following:

**Definition 1.2** The life span \( T(\kappa \psi) \) of solutions to \( (1.1) \) with \( (1.2) \) is defined by the maximal existence time of the minimal solution to \( (1.1) \) with \( (1.2) \).

Next, we set up notation. Throughout this paper, \( p^* \) is given by

\[
p^* := 1 + \frac{1}{N}.
\]

For any \( x \in \mathbb{R}^N \) and \( r > 0 \), set

\[
B_+(x, r) := \{r \in \mathbb{R}^N : |x - y| < r\} \cap D.
\]

For any set \( E \), let \( \chi_E \) be the characteristic function which has value 1 in \( E \) and value 0 outside \( E \).

### 1.3 Main results

Now we are ready to state the main results of this paper. In Theorem 1.1 we obtain the relationship between the singularity of \( \psi \) and the life span \( T(\kappa \psi) \) as \( \kappa \to \infty \) and give an optimal estimate to the life span as \( \kappa \to \infty \). Subsection 1.4 contains a brief summary of Theorem 1.1 (See Tables 1, 2 and 3).

**Theorem 1.1** Assume that

\[
\psi(x) := |x|^A \left[ \log \left( e + \frac{1}{|x|} \right) \right]^{-B} \chi_{B_+(0,1)}(x) \in L^1(\mathbb{R}^N_+) \setminus L^\infty(\mathbb{R}^N_+),
\]

where \(-N \leq A \leq 0\) and

\[
B > 0 \quad \text{if} \quad A = 0, \quad B \in \mathbb{R} \quad \text{if} \quad -N < A < 0, \quad B > 1 \quad \text{if} \quad A = -N.
\]

Then \( T(\kappa \psi) \to 0 \) as \( \kappa \to \infty \) and following holds:

(i) \( T(\kappa \psi) \) behaves

\[
T(\kappa \psi) \sim \begin{cases} 
[\kappa (\log \kappa)^{-B}]^{-\frac{2(p-1)}{N(p-1)+1}} & \text{if} \quad A > -\min\left\{N, \frac{1}{p-1}\right\}, \\
[\kappa (\log \kappa)^{-B+1}]^{-\frac{2(p-1)}{N(p-1)+1}} & \text{if} \quad 1 < p < p^*, \quad A = -N, \quad B > 1,
\end{cases}
\]

and

\[
|\log T(\kappa \psi)| \sim \begin{cases} 
\kappa \frac{1}{p} & \text{if} \quad p > p^*, \quad A = -\frac{1}{p-1}, \quad B > 0, \\
\kappa \frac{1}{p-1} & \text{if} \quad p = p^*, \quad A = -N, \quad B > N + 1,
\end{cases}
\]

as \( \kappa \to \infty \).
(ii) Let \( p > p_\ast \). If, either
\[
A < -1/(p-1) \quad \text{and} \quad B \in \mathbb{R} \quad \text{or} \quad A = -1/(p-1) \quad \text{and} \quad B < 0,
\]
then problem (1.1) with (1.2) possesses no local-in-time solutions for all \( \kappa > 0 \). If
\[
A = -1/(p-1) \quad \text{and} \quad B = 0,
\]
then problem (1.1) with (1.2) possesses no local-in-time solutions for sufficiently large \( \kappa > 0 \);

(iii) Let \( p = p_\ast \). If
\[
A = -N \quad \text{and} \quad B < N + 1,
\]
then problem (1.1) with (1.2) possesses no local-in-time solutions for all \( \kappa > 0 \). If
\[
A = -N \quad \text{and} \quad B = N + 1,
\]
then problem (1.1) with (1.2) possesses no local-in-time solutions for sufficiently large \( \kappa > 0 \).

We remark that when \( \psi \) is as in Theorem 1.1, \( \psi \) satisfies (1.3) if and only if \( \psi \in L^1_{\text{loc}}(\mathbb{R}^N_+) \).
It is obvious that \( T(\kappa \psi) = 0 \) for all \( \kappa > 0 \) if (1.3) does not hold.

**Remark 1.1** Ishige and Sato [14] obtained following: if \( \psi \) satisfies
\[
\psi(x) = |x|^A
\]
in a neighborhood of the origin, where
\[
-N < A \leq 0 \quad \text{if} \quad 1 < p < p_\ast \quad \text{and} \quad -\frac{1}{p-1} < A \leq 0 \quad \text{if} \quad p \geq p_\ast,
\]
then
\[
T(\kappa \psi) \sim \kappa^{\frac{-2(p-1)}{A(p-1)+1}}
\]
for sufficiently large \( \kappa > 0 \). Compare with Theorem 1.1.

Theorem 1.2 gives an optimal estimate to the life span \( T(\kappa \psi) \) as \( \kappa \to 0^+ \) with \( \psi \) behaving like \( |x|^{-A}(A > 0) \) at the space infinity. Subsection 1.4 contains a brief summary of Theorem 1.2 (See Tables 4 and 5).

**Theorem 1.2** Let \( A > 0 \) and \( \psi(x) = (1 + |x|)^{-A} \). Then \( T(\kappa \psi) \to \infty \) as \( \kappa \to 0 \) and following holds:

1. Let \( 1 < p < p_\ast \) or \( 0 < A < 1/(p-1) \). Then
\[
T(\kappa \psi) \sim \begin{cases} 
\kappa \left( \frac{1}{p-1} \cdot \frac{1}{\min\{A,N\}} \right)^{-1} & \text{if} \quad A \neq N, \\
\left( \frac{\kappa^{-1}}{\log(\kappa^{-1})} \right) \left( \frac{1}{p-1} \cdot \frac{N}{p} \right)^{-1} & \text{if} \quad A = N,
\end{cases}
\]
as \( \kappa \to +0 \);
(2) Let \( p = p_* \) and \( A \geq 1/(p - 1) \). Then
\[
\log T(\kappa \psi) \sim \begin{cases} 
\kappa^{-(p-1)} & \text{if } A > N, \\
\kappa^{-\frac{p-1}{p}} & \text{if } A = N,
\end{cases}
\]
as \( \kappa \to +0 \);

(3) Let \( p > p_* \) and \( A \geq 1/(p - 1) \). Then problem (1.1) with (1.2) possesses a global-in-time solution if \( \kappa > 0 \) is sufficiently small.

Remark 1.2 An optimal estimate of the life span \( T(\kappa \psi) \) as \( \kappa \to +0 \) have been already obtained in some cases. Specifically, if \( \psi \) satisfies
\[
\psi(x) = (1 + |x|)^{-A} \quad (A > 0)
\]
for all \( x \in D \), then the following holds:
\[
T(\kappa \psi) \sim \begin{cases} 
\kappa^{-\frac{1}{2(p-1)} - \frac{A}{2}} & \text{if } p \geq p_*, \ 0 \leq A < 1/(p - 1), \\
\kappa^{-\frac{1}{2(p-1)} \cdot \frac{1}{2} \min\{A, N\}} & \text{if } p < p_*, \ A \neq N, \\
\left( \frac{\kappa^{-1}}{\log(\kappa^{-1})} \right) \left( \frac{1}{2(p-1)} - \frac{A}{2} \right)^{-1} & \text{if } p < p_*, \ A = N,
\end{cases}
\]
for sufficiently small \( \kappa > 0 \) (See also [14]).

Finally, we show that \( \lim_{\kappa \to 0} T_\kappa = \infty \) does not necessarily hold for problem (1.1) if \( \psi \) has an exponential growth as \( x_N \to \infty \).

Theorem 1.3 Let \( p > 1, \lambda > 0 \) and \( \psi(x) := \exp(\lambda x_N^2) \). Then
\[
\lim_{\kappa \to +0} T(\kappa \psi) = (4\lambda)^{-1}.
\] (1.4)

1.4 Summary of Theorems 1.1 and 1.2

By Theorem 1.1 we obtain following tables. These tables show the behavior of the life span \( T(\kappa \psi) \) as \( \kappa \to \infty \) when \( \psi \) is as in Theorem 1.1. For simplicity of notation, we write \( T_\kappa \) instead of \( T(\kappa \psi) \).

Table 1: In the case of \( 1 < p < p_* \) (as \( \kappa \to \infty \))

| \( B \) | \( A \) | \( A > -N \) | \( A = -N \) |
|-------|-------|-----------|-----------|
| \( B > 1 \) | \( T_\kappa \sim \left[ (\log \kappa)^{-B} \right]^{-\frac{2(p-1)}{A(p-1)+1}} \) | \( T_\kappa \sim \left[ (\log \kappa)^{-B+1} \right]^{-\frac{2(p-1)}{A(p-1)+1}} \) | \( T_\kappa = 0 \) |
| \( B \leq 1 \) | \( T_\kappa \sim \left[ (\log \kappa)^{-B} \right]^{-\frac{2(p-1)}{A(p-1)+1}} \) | \( T_\kappa = 0 \) |
Table 2: In the case of \( p > p_* \) (as \( \kappa \to \infty \))

| \( B \) | \( A > -\frac{1}{p-1} \) | \( A = -\frac{1}{p-1} \) | \( -N \leq A < -\frac{1}{p-1} \) |
|-------|-----------------|-----------------|-----------------|
| \( B > 0 \) | \( T_\kappa \sim [\kappa(\log \kappa)^{-B}]^{-\frac{2(p-1)}{A(p-1)+1}} \) | \( \log T_\kappa \sim \kappa^\frac{1}{2} \) | \( T_\kappa = 0 \) |
| \( B = 0 \) | \( T_\kappa \sim [\kappa(\log \kappa)^{-B}]^{-\frac{2(p-1)}{A(p-1)+1}} \) | \( T_\kappa = 0 \) | \( T_\kappa = 0 \) |
| \( B < 0 \) | \( T_\kappa \sim [\kappa(\log \kappa)^{-B}]^{-\frac{2(p-1)}{A(p-1)+1}} \) | \( T_\kappa = 0 \) | \( T_\kappa = 0 \) |

Table 3: In the case of \( p = p_* \) (as \( \kappa \to \infty \))

| \( B \) | \( A > -\frac{1}{p-1} \) | \( A = -\frac{1}{p-1} \) |
|-------|-----------------|-----------------|
| \( B > N + 1 \) | \( T_\kappa \sim [\kappa(\log \kappa)^{-B}]^{-\frac{2(p-1)}{A(p-1)+1}} \) | \( \log T_\kappa \sim \kappa^\frac{1}{B-N-1} \) |
| \( B = N + 1 \) | \( T_\kappa \sim [\kappa(\log \kappa)^{-B}]^{-\frac{2(p-1)}{A(p-1)+1}} \) | \( T_\kappa = 0 \) |
| \( B < N + 1 \) | \( T_\kappa = 0 \) | \( T_\kappa = 0 \) |

By Theorem 1.2 we obtain following tables. These tables show the behavior of the life span \( T(\kappa \psi) \) as \( \kappa \to +0 \) when \( \psi \) is as in Theorem 1.2.

Table 4: In the case of \( A \neq N \) (as \( \kappa \to +0 \))

| \( p \) | \( A < \frac{1}{p-1} \) | \( A = \frac{1}{p-1} \) | \( A > \frac{1}{p-1} \) |
|-------|-----------------|-----------------|-----------------|
| \( p < p_* \) | \( T_\kappa \sim \kappa^{-\left(\frac{1}{2(p-1)}\cdot\frac{1}{2}\cdot\min\{A,N\}\right)^{-1}} \) | \( T_\kappa \sim \kappa^{-\left(\frac{1}{2(p-1)}\cdot\frac{N}{2}\right)^{-1}} \) | \( T_\kappa \sim \kappa^{-\left(\frac{1}{2(p-1)}\cdot\frac{N}{2}\right)^{-1}} \) |
| \( p = p_* \) | \( T_\kappa \sim \kappa^{-\left(\frac{1}{2(p-1)}\cdot\frac{N}{2}\right)^{-1}} \) | none | log \( T_\kappa \sim \kappa^{-(p-1)} \) |
| \( p > p_* \) | \( T_\kappa \sim \kappa^{-\left(\frac{1}{2(p-1)}\cdot\frac{N}{2}\right)^{-1}} \) | \( T_\kappa = \infty \) | \( T_\kappa = \infty \) |
Table 5: In the case of $A = N$ (as $\kappa \to +0$)

| $p$       | $A = N$               |
|-----------|-----------------------|
| $p < p^*$ | $T_\kappa \sim \left( \frac{\kappa^{-1}}{\log(\kappa^{-1})} \right)^{\frac{1}{2(p-1)}}$ |
| $p = p^*$ | $\log T_\kappa \sim \kappa^{-1} p$ |
| $p > p^*$ | $T_\kappa = \infty$ |

The rest of this paper is organized as follows. In Section 2, we review some of the facts on the solvability of problem (1.1), which have been already proved in [10]. In Section 3, we give an upper estimate and a lower estimate to the life span $T(\kappa \psi)$ as $\kappa \to \infty$ (See Proposition 3.1 and 3.2). By combining these estimates, we can prove Theorem 1.1. In Section 4, we prove Theorem 1.2 by same method as in Section 3 (See Proposition 4.1 and 4.2), and prove Theorem 1.3. Section 5 deals with the life span in the case of the semilinear heat equation in $\mathbb{R}^N$.

2 Necessary conditions and sufficient conditions for the solvability of problem (1.1)

For any $L \geq 0$, we set

$$
D_L := \{(x', x_N) : x' \in \mathbb{R}^{N-1}, x_N \geq L^\frac{1}{2}\},
$$

$$
D'_L := \{(x', x_N) : x' \in \mathbb{R}^{N-1}, 0 \leq x_N < L^\frac{1}{2}\}.
$$

Now, we review the necessary conditions for the solvability of problem (1.1), which has been proved in [10].

**Theorem 2.1** Let $p > 1$ and $u$ be a solution to (1.1) in $[0, T)$ with $u(0) = \varphi$, where $0 < T < \infty$. Then for any $\delta > 0$, there exists $\gamma_1 = \gamma_1(N, p, \delta) > 0$ such that

$$
\sup_{x \in \mathbb{R}^N} \exp\left(- (1 + \delta) \frac{x_N^2}{4\sigma^2}\right) \int_{B_\sigma(x, \sigma)} \varphi(y) dy \leq \gamma_1 \sigma^{N-\frac{1}{p-1}}
$$

(2.1)

for $0 < \sigma \leq T^{1/2}$. In particular, in the case of $p = p^*$, there exists $\gamma'_1 = \gamma'_1(N, \delta) > 0$ such that

$$
\sup_{x \in \mathbb{R}^N} \exp\left(- (1 + \delta) \frac{x_N^2}{4\sigma^2}\right) \int_{B_\sigma(x, \sigma)} \varphi(y) dy \leq \gamma'_1 \left[\log \left(e + \frac{T^\frac{1}{2}}{\sigma}\right)\right]^{-N}
$$

(2.2)

for $0 < \sigma \leq T^{1/2}$.

**Remark 2.1** If $1 < p \leq p^*$ and $\mu \not\equiv 0$ in $D$, then problem (1.1) possesses no nonnegative global-in-time solutions. See [3] and [7].
Next, we review the sufficient conditions for the solvability of problem (1.1), which have been proved also in [10]. For any measurable function $\phi$ in $\mathbb{R}^N$ and any Borel set $E$, we set
\[
\int_E \phi(y) \, dy = \frac{1}{|E|} \int_E \phi(y) \, dy, \quad \phi_E(x) := \phi(x) \chi_E(x),
\]
where $|E|$ is the Lebesgue measure of $E$.

**Theorem 2.2** Let $1 < p < p_*$, $T > 0$ and $\delta \in (0,1)$. Set $\lambda := (1 - \delta)/4T$. Then there exists $\gamma_2 = \gamma_2(N, p, \delta) > 0$ with the following property:
- If $\varphi$ is a nonnegative measurable function in $\mathbb{R}_+^N$ satisfying
  \[
  \sup_{x \in D} \int_{B_+(x,T^{1/2})} e^{-\lambda y^2} \varphi(y) \, dy \leq \gamma_2 T^{-\frac{1}{2(\sqrt{p_1}-1)}}, \tag{2.3}
  \]
  then there exists a solution $u$ to (1.1) in $[0,T)$ with $u(0) = \varphi$.

**Theorem 2.3** Let $p > 1$, $a \in (1,p)$, $T > 0$ and $\delta \in (0,1)$. Let $\varphi$ be a nonnegative measurable function in $\mathbb{R}_+^N$. Set $\varphi_1 := \varphi_{D_1}$, $\varphi_2 := \varphi_{D_2}$ and $\lambda := (1 - \delta)/4T$. Then there exists $\gamma_3 = \gamma_3(N, p, a, \delta) > 0$ with the following property:
- Assume that $\varphi_1$ satisfies
  \[
  \sup_{x \in D} \int_{B_+(x,T^{1/2})} e^{-\lambda y^2} \varphi_1(y) \, dy \leq \gamma_3 T^{-\frac{1}{2(\sqrt{p_1}-1)}}, \tag{2.4}
  \]
  Furthermore, assume that $\varphi_2$ satisfies
  \[
  \sup_{x \in D_1} \left[ \int_{B_+(x,T^{1/2})} \varphi_2(y)^a \, dy \right]^\frac{1}{a} \leq \gamma_3 \sigma^{-\frac{1}{p-1}} \quad \text{for } 0 < \sigma \leq T^{\frac{1}{2}}. \tag{2.5}
  \]
  Then there exists a solution $u$ to (1.1) in $[0,T)$ with $u(0) = \varphi$.

**Theorem 2.4** Let $p = p_*$, $T > 0$ and $\delta \in (0,1)$. Let $\varphi$ be a nonnegative measurable function in $\mathbb{R}_+^N$. Set $\varphi_1 := \varphi_{D_1}$, $\varphi_2 := \varphi_{D_2}$, $\lambda := (1 - \delta)/4T$ and
\[
\Phi(s) := s \left[ \log(e + s) \right]^N, \quad \rho(s) := s^{-N} \left[ \log \left( e + \frac{1}{s} \right) \right]^{-N} \quad \text{for } s > 0. \tag{2.6}
\]
Then there exists $\gamma_4 = \gamma_4(N, \delta) > 0$ with the following property:
- Assume that $\varphi_1$ satisfies
  \[
  \sup_{x \in D_1} \int_{B_+(x,T^{1/2})} e^{-\lambda y_2^2} \, d\mu_1(y) \leq \gamma_4 T^{-\frac{1}{2(\sqrt{p_1}-1)}}, \tag{2.7}
  \]
  Furthermore, assume that $\varphi_2$ satisfies
  \[
  \sup_{x \in D_1} \Phi^{-1} \left[ \int_{B_+(x,T^{1/2})} \Phi \left( T^{\frac{1}{2(\sqrt{p_1}-1)}} \rho(y) \right) \, dy \right] \leq \gamma_4 \rho(\sigma T^{-\frac{1}{2}}) \quad \text{for } 0 < \sigma \leq T^{\frac{1}{2}}. \tag{2.8}
  \]
  Then there exists a solution $u$ to (1.1) in $[0,T)$ with $u(0) = \varphi$. 

8
3 Proof of Theorem 1.1

For simplicity of notation, we write $T_\kappa$ instead of $T(\kappa \psi)$. Let $\kappa > 0$ and $\psi$ be a nonnegative measurable function in $D$. In this section we study the behavior of $T_\kappa$ as $\kappa \to \infty$ and prove Theorem 1.1. In the following two propositions we study the relationship between the behavior of the life span $T_\kappa$ for sufficiently large $\kappa > 0$ and the singularity of $\psi$ at $0 \in \partial D$. In order to prove Theorem 1.1, we suffice to prove these propositions. Proposition 3.1 gives an upper estimate of $T_\kappa$ as $\kappa \to \infty$.

**Proposition 3.1** Let $\psi$ be a nonnegative measurable function in $D$ such that

$$\psi(y) \geq |y|^A \log \left( e + \frac{1}{|y|} \right)^{-B}, \quad y \in B_+(0,1), \quad (3.1)$$

where $-N \leq A \leq 0$ and $B$ is as in (1.3). Then $\lim_{\kappa \to \infty} T(\kappa \psi) = 0$. Furthermore, the following holds:

(i) Let $1 < p < p_*$. Then there exists $\gamma > 0$ such that

$$T(\kappa \psi) \leq \gamma (\kappa (\log \kappa)^B)^{-\frac{2(p-1)}{A(p-1)+1}} \quad \text{if } A > -N, B \in \mathbb{R}, \quad (3.2)$$

$$T(\kappa \psi) \leq \gamma (\kappa (\log \kappa)^{-B+1})^{-\frac{2(p-1)}{A(p-1)+1}} \quad \text{if } A = -N, B > 1, \quad (3.3)$$

for sufficiently large $\kappa > 0$;

(ii) Let $p > p_*$. If, either

$$A < -1/(p-1) \quad \text{and} \quad B \in \mathbb{R} \quad \text{or} \quad A = -1/(p-1) \quad \text{and} \quad B < 0, \quad (3.4)$$

then problem (1.1) with (1.2) possesses no local-in-time solutions for all $\kappa > 0$. If

$$A = -1/(p-1) \quad \text{and} \quad B = 0,$$

then problem (1.1) with (1.2) possesses no local-in-time solutions for sufficiently large $\kappa > 0$. Furthermore,

(a) if $A > -1/(p-1)$, then (3.2) holds:

(b) if $A = -1/(p-1)$ and $B > 0$, then there exists $\gamma' > 0$ such that

$$T(\kappa \psi) \leq \exp(-\gamma' \kappa^\frac{1}{p})$$

for sufficiently large $\kappa > 0$;

(iii) Let $p = p_*$. If

$$A = -N \quad \text{and} \quad B < N + 1,$$

then problem (1.1) with (1.2) possesses no local-in-time solutions for all $\kappa > 0$. If

$$A = -N \quad \text{and} \quad B = N + 1,$$

then problem (1.1) with (1.2) possesses no local-in-time solutions for sufficiently large $\kappa > 0$. Furthermore,

(c) if $A > -N$, then (3.2) holds:
if \( A = -N \) and \( B > N + 1 \), then there exists \( \gamma'' > 0 \) such that
\[
T(\kappa \psi) \leq \exp(-\gamma'' \kappa^{-\frac{1}{N+1}})
\]
for sufficiently large \( \kappa > 0 \).

**Proof.** For any \( p > 1 \), by (2.1) and (3.1) we can find a constant \( \gamma_1 > 0 \) such that
\[
\gamma_1 \sigma^{N-\frac{1}{p-1}} \geq \kappa \int_{B_+(0, \sigma)} \psi(y) \, dy \geq \kappa \int_{B_+(0, \sigma)} |y|^A \left[ \log \left( e + \frac{1}{|y|} \right) \right]^{-B} \, dy > 0
\]
for \( 0 < \sigma \leq T_\kappa^{1/2} \). Firstly, we show that \( \lim_{\kappa \to \infty} T_\kappa = 0 \) by contradiction. Assume that there exist \( \{\kappa_j\}_{j=1}^\infty \) and \( c_* > 0 \) such that
\[
\lim_{j \to \infty} \kappa_j = \infty, \quad T_{\kappa_j} > c_*^2 \quad \text{for all } j = 1, 2, \ldots.
\]
Applying Theorem 2.1 with \( \sigma = c_* \), by (3.5) we have
\[
\gamma_1 \sigma^{N-\frac{1}{p-1}} \geq \kappa_j \int_{B_+(0, c_*)} |y|^A \left[ \log \left( e + \frac{1}{|y|} \right) \right]^{-B} \, dy > 0, \quad j = 1, 2, \ldots,
\]
where \( \gamma_1 \) is a constant independent of \( \kappa_j \). Since \( \lim_{j \to \infty} \kappa_j = \infty \), we have a contradiction. Since \( c_* \) is arbitrary, we have
\[
\lim_{\kappa \to \infty} T_\kappa = 0.
\]
Without loss of generality we can assume that \( T_\kappa > 0 \) is sufficiently small.

We prove assertion (i). Let \( 1 < p < p_* \), \( A > -N \) and \( B \in \mathbb{R} \). For any \( p > 1 \), by (3.5) we have
\[
\gamma_1 \sigma^{N-\frac{1}{p-1}} \geq \kappa \int_{B_+(0, \sigma)} |y|^A \left[ \log \left( e + \frac{1}{|y|} \right) \right]^{-B} \, dy > 0, \quad j = 1, 2, \ldots
\]
for \( 0 < \sigma \leq T_\kappa^{1/2} \) and sufficiently large \( \kappa > 0 \), where \( L \geq e \) is a sufficiently large constant. We notice that for any \( a_1 > 0 \) and \( a_2 \in \mathbb{R} \),
\[
\Psi(\tau) := \tau^{a_1} [\log(L + \tau^{-1})]^{a_2} \text{ is increasing for sufficiently small } \tau > 0
\]
and \( \Psi^{-1} \) behaves like
\[
\tau^{\frac{1}{a_1}} [\log(L + \tau^{-1})]^{\frac{a_2}{a_1}}
\]
for sufficiently small \( \tau > 0 \). We consider the case where \( A > -N \) and \( B \in \mathbb{R} \). Set
\[
a_1 := A + \frac{1}{p-1} > 0 \quad \text{and} \quad a_2 := -B.
\]
By (3.6), (3.7) and (3.8) we have
\[
\sigma \leq C \Psi^{-1}(C \gamma_1 \kappa^{-1})
\leq C(C \gamma_1 \kappa^{-1})^{(A+\frac{1}{p-1})^{-1}} [\log(L + (C \gamma_1 \kappa^{-1})^{-1})]^{B(A+\frac{1}{p-1})^{-1}}
\leq C[\kappa(\log \kappa)^{-B}]^{-\frac{p-1}{4(p-1)+1}}
\]
for $0 < \sigma \leq T_{\kappa}^{1/2}$. Setting $\sigma = T_{\kappa}^{1/2}$, we obtain (3.2). Similarly, we can obtain (3.3). Thus assertion (i) follows.

We prove assertion (ii). Let $p > p_\ast$. In the case of (3.4), we can assume that

$$A > -N \quad \text{and} \quad B \in \mathbb{R} \quad \text{or} \quad A = -N \quad \text{and} \quad B > 1 \quad (3.9)$$

since $-N < -1/(p - 1)$. If $A$ and $B$ do not satisfy (3.9), then

$$\int_{B_+ (0, \sigma)} |y|^A \left[ \log \left( e + \frac{1}{|y|} \right) \right]^{-B} \, dy = \infty$$

for all $\sigma > 0$. By (3.5), this implies that $T_{\kappa} = 0$ for all $\kappa > 0$. By condition (3.9), we have (3.6). Since $A$ and $B$ satisfy (3.4), the right hand side of (3.6) goes to infinity as $\sigma \to +0$. This implies that $T_{\kappa} = 0$ for all $\kappa > 0$. In the case where $A = -1/(p - 1)$ and $B = 0$, it follows from (3.6) that

$$\gamma_1 \geq C \kappa$$

for $0 < \sigma \leq T_{\kappa}^{1/2}$. This implies that $T_{\kappa} = 0$ for sufficiently large $\kappa > 0$. Furthermore, if $A > -1/(p - 1) > -N$, we obtain (3.3) by a similar argument to the proof of assertion (i). Then we obtain (a). It remains to consider the case where $A = -1/(p - 1)$ and $B > 0$. Since $T_{\kappa} > 0$ is sufficiently small, by (3.6) we have

$$\gamma_1 \geq C \kappa$$

for sufficiently large $\kappa > 0$ and (b) follows. Thus assertion (ii) is proved.

Finally, we prove assertion (iii). Let $A = -N$. Since $p = p_\ast$ and $B > 1$, by (2.2) we have

$$\gamma_1 \left[ \log \left( e + \frac{T_{\kappa}^{1/2}}{\kappa} \right) \right]^{-N} \geq \kappa \int_{B_+ (0, \sigma)} \psi(y) \, dy \geq \kappa \int_{B_+ (0, \sigma)} |y|^A \left[ \log \left( e + \frac{1}{|y|} \right) \right]^{-B} \, dy \quad (3.10)$$

for $0 < \sigma \leq T_{\kappa}^{1/2}$. Now we assume that $T_{\kappa} > 0$. In the case where $B < N + 1$, we see that (3.10) does not hold for sufficiently small $\sigma > 0$. This implies that $T_{\kappa} = 0$ for all $\kappa > 0$. In the case of $B = N + 1$, it follows from (3.10) with $\sigma = T_{\kappa} < T_{\kappa}^{1/2}$ that

$$\gamma_1 \left[ \log (e + T_{\kappa}^{-1}) \right]^{-N} \geq C \kappa \left[ \log (e + T_{\kappa}^{-1}) \right]^{-N}.$$

Since this inequality does not hold for sufficiently large $\kappa > 0$, we have a contradiction. This implies that $T_{\kappa} = 0$ for sufficiently large $\kappa > 0$. In the case of $A > -N$, since (3.6) holds, we obtain (3.3) by a similar argument to the proof of assertion (i). In the case where $A = -N$ and $B > N + 1$, since $T_{\kappa} > 0$ is sufficiently small, by (3.10) with $\sigma = T_{\kappa}$ we have

$$C_{\gamma_1} \kappa^{-N} \leq \left[ \log (e + T_{\kappa}^{-1}) \right]^{-B+N+1} \leq \left[ \log (T_{\kappa}^{-1}) \right]^{-B+N+1}. $$

Since $B - N - 1 > 0$, this implies that there exists a constant $\gamma'' > 0$ such that

$$T_{\kappa} \leq \exp(-\gamma'' \kappa^{1/\gamma''})$$

11
for sufficiently large $\kappa > 0$. Thus assertion (iii) follows and the proof of Proposition 3.1 is complete. □

In Proposition 3.2 we give an lower estimate of $T_\kappa$ as $\kappa \to \infty$ and show the optimality of the estimate of $T_\kappa$ in Proposition 3.1.

**Proposition 3.2** Let $\psi$ be a nontrivial nonnegative measurable function in $D$ such that $\text{supp } \psi \subset B(0,1)$ and

$$\psi(y) \leq |y|^A \left[ \log \left( e + \frac{1}{|y|} \right) \right]^{-B}, \quad y \in B_+(0,1),$$

where $-N \leq A \leq 0$ and $B$ is as in (1.3).

(i) Let $1 < p < p^\ast$. Then there exists $\gamma > 0$ such that

$$T(\kappa \psi) \geq \gamma \left[ \kappa \left( \log \kappa \right) - B \right]^{\frac{2(p-1)}{1-\frac{1}{p}}}$$

if $A > -N$, $B \in \mathbb{R}$, \hspace{1cm} (3.12)

$$T(\kappa \psi) \geq \gamma \left[ \kappa \left( \log \kappa \right) - B + 1 \right]^{\frac{2(p-1)}{1-\frac{1}{p}}}$$

if $A = -N$, $B > 1$.

(ii) Let $p > p^\ast$.

(a) If $A > -1/(p-1)$, then (3.12) holds:

(b) If $A = -1/(p-1)$ and $B > 0$, then there exists $\gamma' > 0$ such that

$$T(\kappa \psi) \geq \exp\left( -\gamma' \kappa^\frac{1}{p} \right)$$

for sufficiently large $\kappa > 0$;

(iii) Let $p = p^\ast$.

(c) If $A > -N$, then (3.12) holds:

(d) If $A = -N$ and $B > N + 1$, then there exists $\gamma'' > 0$ such that

$$T(\kappa \psi) \geq \exp\left( -\gamma'' \kappa^\frac{1}{p-1-N-1} \right)$$

for sufficiently large $\kappa > 0$.

**Proof.** We first consider the case where $p > p^\ast$ and $A > -1/(p-1)$. Let $a \in (1,p)$ be such that $aA > -N$. By (3.11), (3.7) and the Jensen inequality, we have

$$\sigma^{\frac{1}{p-1}} \sup_{x \in D} \int_{B_+(x,\sigma)} \kappa \psi(y) dy \leq \sigma^{\frac{1}{p-1}} \sup_{x \in D} \left[ \int_{B_+(x,\sigma)} [\kappa \psi(y)]^a dy \right]^\frac{1}{a}$$

$$\leq C \kappa \sigma^{\frac{1}{1-p}} \left[ \int_{B_+(0,\sigma)} |y|^A \left( \log \left( L + \frac{1}{|y|} \right) \right)^{-aB} dy \right]^{\frac{1}{a}}$$

for sufficiently small $\sigma > 0$. Let $c$ be a sufficiently small positive constant and set

$$\tilde{T}_\kappa := c \left[ \kappa \left( \log \kappa \right) - B \right]^{\frac{2(p-1)}{4(p-1)+1}}.$$
Since $A > -1/(p - 1)$, taking a sufficiently small $c > 0$ if necessary, we have
\[ C \kappa^{\frac{1}{p-1}+A} \left[ \log \left( L + \frac{1}{\sigma} \right) \right]^{-B} \leq C \kappa^{\frac{1}{p-1}+A} \left[ \log \left( L + \frac{1}{\sigma} \right) \right]^{-B} \bigg|_{\sigma = T_{\kappa}^{1/2}} \]
\[ \leq C e^{\frac{1}{2(p-1)}} \leq \gamma_3 \]
for $0 < \sigma \leq T_{\kappa}^{1/2}$ and sufficiently large $\kappa > 0$, where $\gamma_3$ is as in Theorem 2.3. Then (3.13) and (3.14) yield (2.4) and (2.5). Applying Theorem 2.3, we see that (1.1) with (1.2) has a solution in $[0, \tilde{T}_{\kappa})$ and
\[ T_{\kappa} \geq \tilde{T}_{\kappa} = c [\kappa (\log \kappa)^{-B}]^{-2(p-1)/(p-1)+1} \]
for sufficiently large $\kappa > 0$. So we have (a). Similarly, we have (b) and (c). Furthermore, we can prove (3.12) by using the above argument with $\alpha = 1$ and applying Theorem 2.2.

Next we consider the case where $1 < p < p_*$, $A = -N$ and $B > 1$, let $c$ be a sufficiently small positive constant and set
\[ \tilde{T}_{\kappa}' := c [\kappa (\log \kappa)^{-B+1}]^{-2(p-1)/(p-1)+1} \]
\[ \sup_{x \in D} \int_{B_+(x, \tilde{T}_{\kappa}')^{1/2}} \kappa \psi(y) dy \]
\[ \leq C \kappa^{\frac{1}{p-1}} \left( \frac{1}{\tilde{T}_{\kappa}^{1/2}} \right)^{-B+1} \leq C e^{\frac{1}{2(p-1)}} \leq \gamma_2 \]
for sufficiently large $\kappa > 0$, where $\gamma_2$ is as in Theorem 2.2. Then (3.15) yields (2.3). Applying Theorem 2.2 we see that (1.1) with (1.2) has a solution in $[0, \tilde{T}_{\kappa}')$ and
\[ T_{\kappa} \geq \tilde{T}_{\kappa}' = c [\kappa (\log \kappa)^{-B+1}]^{-2(p-1)/(p-1)+1} \]
for sufficiently large $\kappa > 0$. So we have assertion (i).

It remains to prove (d). Let $p = p_*$, $A = -N$ and $B > N + 1$. Let $c$ be a sufficiently small positive constant and set
\[ \hat{T}_{\kappa} := \exp(-c^{-1} \kappa^{\frac{1}{p-1}+1}) \]
for sufficiently large $\kappa > 0$. Similarly to (3.13), we have
\[ \sup_{x \in D} \int_{B_+(x, \kappa)} \frac{1}{\tilde{T}_{\kappa}'^{1/2}} \kappa \psi(y) dy \leq \sup_{x \in D} \Phi^{-1} \left( \int_{B_+(x, \sigma)} \Phi \left( \frac{1}{\tilde{T}_{\kappa}'^{1/2}} \kappa \psi(y) \right) dy \right) \]
\[ \leq \Phi^{-1} \left( \int_{B_+(0, \sigma)} \Phi \left( \frac{1}{\tilde{T}_{\kappa}'^{1/2}} \kappa \psi(y) \right) dy \right) \right) \]
\[ \leq \Phi^{-1} \left( \int_{B_+(0, \sigma)} \Phi \left( \frac{1}{\tilde{T}_{\kappa}'^{1/2}} \kappa \psi(y) \right) dy \right) \]
\[ \leq \Phi^{-1} \left( \int_{B_+(0, \sigma)} \Phi \left( \frac{1}{\tilde{T}_{\kappa}'^{1/2}} \kappa \psi(y) \right) dy \right) \]
\[ \leq \Phi^{-1} \left( \int_{B_+(0, \sigma)} \Phi \left( \frac{1}{\tilde{T}_{\kappa}'^{1/2}} \kappa \psi(y) \right) dy \right) \]
for all $\sigma > 0$, where $\Phi$ is as in (2.6). Since
\[
\log \left[ e + \hat{T}_\kappa^N \kappa |y|^{-N} \left[ \log \left( e + \frac{1}{|y|} \right) \right]^{-B} \right] \\
\leq \log \left[ \left( e + \hat{T}_\kappa^N \kappa |y|^{-N} \right) \left( e + \left[ \log \left( e + \frac{1}{|y|} \right) \right]^{-B} \right) \right] \\
\leq \log \left[ C \hat{T}_\kappa^N \kappa |y|^{-N} \right] \leq C \log \left[ \hat{T}_\kappa^N \kappa |y|^{-N} \right] \leq C \log \frac{1}{|y|}
\]
for $y \in B_+(0, \sigma)$, $0 < \sigma \leq \hat{T}_\kappa^{1/2}$ and sufficiently large $\kappa$, we have
\[
\int_{B_+(0, \sigma)} \Phi \left( \hat{T}_\kappa^{N} \kappa |y|^{-N} \left[ \log \left( e + \frac{1}{|y|} \right) \right]^{-B} \right) \, dy \\
\leq C \hat{T}_\kappa^{N} \kappa \int_{B_+(x, \sigma)} |y|^{-N} \left[ \log \frac{1}{|y|} \right]^{-B+N} \, dy \leq C \kappa \sigma^{-N} \hat{T}_\kappa^{N} \left[ \left( C \kappa \sigma^{-N} \left[ \log \frac{1}{\sigma} \right]^{-B+N+1} \right) \right]^{-N}
\]
for $0 < \sigma \leq \hat{T}_\kappa^{1/2}$ and sufficiently large $\kappa > 0$. This together with (3.16) implies that
\[
\sup_{x \in D} \int_{B_+(x, \sigma)} \hat{T}_\kappa^{N} \kappa |y|^{-N} \kappa \psi(y) \, dy \leq \sup_{x \in D} \Phi^{-1} \left[ \int_{B_+(x, \sigma)} \Phi \left( \hat{T}_\kappa^{N} \kappa |y|^{-N} \kappa \psi(y) \right) \, dy \right] \\
\leq C \kappa \sigma^{-N} \hat{T}_\kappa^{N} \left[ \left( \log \frac{1}{\sigma} \right)^{-B+N+1} \right] \left( \log \left[ \hat{T}_\kappa^{N} \kappa \sigma^{-N} \left( \log \frac{1}{\sigma} \right)^{-B+N+1} \right] \right) \left[ \left( \log \frac{1}{\sigma} \right)^{-B+N+1} \right]^{-N}
\]
(3.17)
for $0 < \sigma \leq \hat{T}_\kappa^{1/2}$ and sufficiently large $\kappa > 0$. On the other hand, it follows from (2.6) that
\[
\rho(\sigma \hat{T}_\kappa^{\frac{1}{2}}) = \sigma^{-N} \hat{T}_\kappa^{N} \left[ \log \left( e + \frac{\hat{T}_\kappa^{\frac{1}{2}}}{\sigma} \right) \right]^{-N} \geq \sigma^{-N} \hat{T}_\kappa^{N} \left[ \log \frac{1}{\sigma} \right]^{-N}
\]
(3.18)
for $0 < \sigma \leq \hat{T}_\kappa^{1/2}$ and sufficiently large $\kappa$. Since
\[
\kappa \left[ \log \frac{1}{\sigma} \right]^{-B+1+N} \leq C \kappa \left[ \log \frac{1}{\hat{T}_\kappa^N} \right]^{-B+1+N} = C \kappa \left[ \frac{1}{\hat{T}_\kappa^N} \right]^{-B+1+N}
\]
taking a sufficiently small $c > 0$ if necessary, (3.17) and (3.18) yield (2.7) and (2.8). Applying Theorem 5.6 we see that
\[
T_\kappa \geq \hat{T}_\kappa = \exp(-c^{-1} \kappa \left[ \frac{1}{\hat{T}_\kappa^N} \right]^{-B+1+N})
\]
for sufficiently large $\kappa > 0$. This implies (d). The proof of Proposition 3.2 is complete. □

4  Proofs of Theorem 1.2 and Theorem 1.3

We state two results on the behavior of $T_\kappa$ as $\kappa \to +0$. If $\psi$ is a bounded function in $\mathbb{R}^N$, then $T_\kappa \to \infty$ as $\kappa \to +0$ and the behavior of $T_\kappa$ depends on that of $\psi$ at the space infinity. In order to prove Theorem 1.2 we suffice to prove following propositions. In Proposition 4.1 we give an upper estimate of $T_\kappa$ as $\kappa \to +0$. 14
Proposition 4.1 Let $N \geq 1$ and $p > 1$. Let $A > 0$ and $\varphi$ be a nonnegative $L^\infty(D)$-function such that $\psi(x) \geq (1 + |x|)^{-A}$ for $x \in D$.

(i) Let $p = p_*$ and $A \geq 1/(p - 1) = N$. Then there exists $\gamma > 0$ such that

$$\log T(\kappa \psi) \leq \begin{cases} \gamma \kappa^{-(p - 1)} & \text{if } A > N, \\ \gamma \kappa^{-\frac{1}{p}} & \text{if } A = N, \end{cases}$$

for sufficiently small $\kappa > 0$.

(ii) Let $1 < p < p_*$ or $A < 1/(p - 1)$. Then there exists $\gamma' > 0$ such that

$$T(\kappa \psi) \leq \begin{cases} \gamma' \kappa - \frac{1}{2} \min(A, N) & \text{if } A \neq N, \\ \gamma' \left( \frac{\kappa^{-1}}{\log(\kappa^{-1})} \right)^{-\frac{1}{2}} & \text{if } A = N, \end{cases}$$

for sufficiently small $\kappa > 0$.

Proof. Since $\psi \in L^\infty(D)$, by Theorem 2.3 we have

$$T_\kappa \geq C \kappa^{-(p - 1)}$$

for sufficiently small $\kappa > 0$. This implies that $\lim_{\kappa \to 0} T_\kappa = \infty$. Without loss of generality, we can assume that $T_\kappa > 0$ is sufficiently large. For any $p > 1$, we see that

$$\int_{B_+(0, \sigma)} \kappa \psi(y) \, dy \geq \kappa \int_{B_+(0, \sigma)} (1 + |y|)^{-A} \, dy \geq \begin{cases} C \kappa & \text{if } \sigma > 1, A > N, \\ C \kappa \log(e + \sigma) & \text{if } \sigma > 1, A = N, \\ C \kappa \sigma^{N - A} & \text{if } \sigma > 1, A < N, \end{cases}$$

for $\sigma > 1$ and sufficiently small $\kappa > 0$. In the case of $p = p_*$, it follows from (2.2) that

$$\int_{B_+(0, \sigma)} \kappa \psi(y) \, dy \leq \gamma' \left[ \log \left( e + \frac{T_\kappa^{\frac{1}{2}}}{\sigma} \right) \right]^{-N}$$

for $0 < \sigma \leq T_\kappa^{1/2}$ and sufficiently small $\kappa > 0$. This implies that

$$\int_{B_+(0, T_\kappa^{1/4})} \kappa \psi(y) \, dy \leq C \gamma'_1 [\log T_\kappa]^{-N}, \quad (4.2)$$

$$\int_{B_+(0, T_\kappa^{1/2})} \kappa \psi(y) \, dy \leq C \gamma'_1, \quad (4.3)$$

for sufficiently small $\kappa > 0$. By (4.1) and (4.2) with $\sigma = T_\kappa^{1/4}$ we obtain assertion (i). Furthermore, by (4.1) and (4.3) with $\sigma = T_\kappa^{1/2}$ we obtain assertion (ii) in the case where $p = p_*$ and $A < 1/(p - 1)$.

We prove assertion (ii) in the case of $1 < p < p_*$. By (2.1) we see that

$$\int_{B_+(0, T_\kappa^{1/2})} \kappa \psi(y) \, dy \leq \gamma_1 T_\kappa^{\frac{N - 1}{2(p - 1)}}. \quad (4.4)$$
By (4.1) and (4.3), we obtain assertion (ii) in the case of $1 < p < p_*$. Similarly, we obtain assertion (ii) in the case of $p > p_*$. Thus Proposition 4.1 follows. □

In Proposition 4.2 we give an lower estimate of $T_\kappa$ as $\kappa \to +0$ and show the optimality of the estimate of $T_\kappa$ in Proposition 4.1.

**Proposition 4.2** Let $N \geq 1$ and $p > 1$. Let $A > 0$ and $\psi$ be a nonnegative measurable function in $D$ such that supp $\psi \subset D$ and $0 \leq \psi(x) \leq (1 + |x|)^{-A}$ for $x \in D$.

(i) Let $p = p_*$ and $A \geq 1/(p-1) = N$. Then there exists $\gamma > 0$ such that

$$\log T(\kappa \psi) \geq \begin{cases} \gamma \kappa^{-\frac{p-1}{p}} & \text{if } A > N, \\ \gamma \kappa^{-\frac{p-1}{p}} & \text{if } A = N, \end{cases}$$

for sufficiently small $\kappa > 0$.

(ii) Let $1 < p < p_*$ or $A < 1/(p-1)$. Then there exists $\gamma' > 0$ such that

$$T(\kappa \psi) \geq \begin{cases} \gamma' \kappa^{-\frac{1}{2(p-1)} - \frac{1}{2} \min(A,N)^{-1}} & \text{if } A \neq N, \\ \gamma' \left( \frac{\kappa^{-1}}{\log(\kappa^{-1})} \right)^{\frac{1}{2(p-1)} - \frac{1}{2} \min(A,N)^{-1}} & \text{if } A = N, \end{cases}$$

for sufficiently small $\kappa > 0$.

**Proof.** Let $p = p_*$ and $A > N$. Let $c$ be a sufficiently small positive constant and set

$$\hat{T}_\kappa := \exp(c \kappa^{-\frac{p-1}{p}}) = \exp(c \kappa^{-\frac{1}{N}}).$$

Similarly to (4.17), we have

$$\sup_{x \in D} \int_{B_+(x,\sigma)} \hat{T}_\kappa^{\frac{1}{2(p-1)}} \kappa \psi(y) dy \leq \sup_{x \in D} \Phi^{-1} \left[ \int_{B_+(x,\sigma)} \Phi \left( \hat{T}_\kappa^{\frac{1}{2(p-1)}} \kappa \psi(y) \right) dy \right]$$

$$\leq \Phi^{-1} \left[ \int_{B_+(0,\sigma)} \Phi \left( \hat{T}_\kappa^\frac{N}{\kappa} (1 + |y|)^{-A} \right) dy \right]$$

(4.5)

for all $\sigma > 0$. Since

$$\log \left[ e + \hat{T}_\kappa^\frac{N}{\kappa} (1 + |y|)^{-A} \right] \leq \log(C \hat{T}_\kappa^\frac{N}{\kappa}) \leq C \kappa^{-\frac{1}{N}}$$

(4.6)

for sufficiently small $\kappa > 0$, we have

$$\int_{B_+(0,\sigma)} \Phi \left( \hat{T}_\kappa^\frac{N}{\kappa} (1 + |y|)^{-A} \right) dy \leq Cc^N \hat{T}_\kappa^\frac{N}{\kappa} \int_{B_+(0,\sigma)} (1 + |y|)^{-A} dy \leq Cc^N \hat{T}_\kappa^\frac{N}{\kappa} \sigma^{-N}$$

(4.7)

for $0 < \sigma \leq \hat{T}_\kappa^{1/2}$ and sufficiently small $\kappa > 0$. This together with (4.5) implies that

$$\sup_{x \in D} \int_{B_+(x,\sigma)} \hat{T}_\kappa^{\frac{1}{2(p-1)}} \kappa \psi(y) dy \leq \sup_{x \in D} \Phi^{-1} \left[ \int_{B_+(x,\sigma)} \Phi \left( \hat{T}_\kappa^{\frac{1}{2(p-1)}} \kappa \psi(y) \right) dy \right]$$

$$\leq Cc^N \sigma^{-N} \hat{T}_\kappa^\frac{N}{\kappa} \left( \log \left[ e + Cc^N \hat{T}_\kappa^\frac{N}{\kappa} \sigma^{-N} \right] \right)^{-N} \leq Cc^N \sigma^{-N} \hat{T}_\kappa^\frac{N}{\kappa} \left( \log \left[ e + \frac{\hat{T}_\kappa^\frac{1}{2}}{\sigma} \right] \right)^{-N}$$

$$= Cc^N \rho(\sigma \hat{T}_\kappa^\frac{1}{2})$$
for $0 < \sigma \leq \hat{T}_\kappa^{1/2}$ and sufficiently small $\kappa > 0$. Therefore, taking a sufficiently small $c > 0$ if necessary, we apply Theorem 5.6 to see that (1.1) with (1.2) has a solution in $[0, \hat{T}_\kappa)$ and

$$T_\kappa \geq \hat{T}_\kappa = \exp(c\kappa^{-(p-1)})$$

for all sufficiently small $\kappa > 0$.

In the case of $A = N$, setting

$$\tilde{T}_\kappa := \exp\left(c\kappa - \frac{p - 1}{4}\right),$$

similarly to (4.6) and (4.7), we have

$$- \int_{B_{\hat{T}_\kappa^{1/2}}} \frac{\Phi\left(\frac{\tilde{T}_\kappa}{\kappa}, (1 + |y|)^{-A}\right)}{\kappa \psi(y)} dy \leq C\kappa \tilde{T}_\kappa^N \int_{B_{\hat{T}_\kappa^{1/2}}} (1 + |y|)^{-N} dy \leq C\kappa \tilde{T}_\kappa^N \sigma^{-N} (\log \tilde{T}_\kappa)^{N+1} = Cc^{N+1} \tilde{T}_\kappa^N \sigma^{-N}$$

for $0 < \sigma \leq \hat{T}_\kappa^{1/2}$ and sufficiently small $\kappa$. Then we apply the same argument as in the case of $A > N$ to see that

$$T_\kappa \geq \tilde{T}_\kappa = \exp\left(c\kappa - \frac{p - 1}{4}\right)$$

for sufficiently small $\kappa$. Thus assertion (i) follows.

We show assertion (ii). Let $1 < p < p_*$ and $0 < A < N$. Let $c$ be a sufficiently small positive constant and set

$$\tilde{T}_\kappa := c\kappa^{-\left(\frac{1}{p-1} - \frac{1}{2}\right)^{-1}}.$$

Then

$$\sup_{x \in \mathcal{D}} \int_{B_{\hat{T}_\kappa^{1/2}}} \kappa \psi(y) dy \leq C\kappa \int_{B_{\hat{T}_\kappa^{1/2}}} (1 + |y|)^{-A} dy \leq C\kappa \tilde{T}_\kappa^{-\frac{4}{2(p-1)}} \tilde{T}_\kappa^{-\frac{1}{2(p-1)}}$$

for sufficiently small $\kappa > 0$. Then we have assertion (ii) in the case where $1 < p < p_*$ and $0 < A < N$. Similarly we can prove assertion (ii) in the other cases and assertion (ii) follows. Thus the proof of Proposition 4.2 is complete. 

Finally, we show that $\lim_{\kappa \to 0} T_\kappa = \infty$ does not necessarily hold for problem (1.1) if $\varphi$ has an exponential growth as $x_N \to \infty$.

**Proof of Theorem 1.3.** Let $\kappa > 0$ and $\delta > 0$. It follows from Theorem 2.1 that

$$\gamma_1 T_\kappa^{-\frac{N}{2}} \geq (1 + \delta) x_N^2 4T_\kappa^{-\frac{1}{2(p-1)}} \int_{B_{\hat{T}_\kappa^{1/2}}} \kappa \psi(y) dy \geq C \exp\left(\frac{1}{4T_\kappa} \left(\lambda x_N - T_\kappa^{1/2}\right)^2\right) \exp\left(-2\lambda T_\kappa^{1/2} x_N + \lambda T_\kappa\right)$$

for all $x \in D_{T_\kappa}$, where $\gamma_1$ is as in Theorem 2.1. Letting $x_N \to \infty$, we see that $\lambda - (1 + \delta)/4T_\kappa \leq 0$. Since $\delta > 0$ is arbitrary, we obtain

$$\lim \sup_{\kappa \to +0} T_\kappa \leq (4\lambda)^{-1}. \quad (4.8)$$
On the other hand, it follows that
\[
\int_{B_r(x,T^*_\delta/2)} \exp \left( -\left(1 - \frac{y^2}{4T^*_\delta} \right) \kappa \exp (\frac{\lambda y^2}{4N}) \right) dy = \kappa, \quad x \in D_{T^*_\delta},
\]
where \(T^*_\delta := \frac{(1 - \delta)}{4\lambda}\). Then we deduce from Theorem 2.3 that \(T_\kappa \geq T^*_\delta\) for sufficiently small \(\kappa > 0\). Since \(\delta > 0\) is arbitrary, we obtain \(\lim \inf_{\kappa \to +0} T_\kappa \geq (4\lambda)^{-1}\). This together with (4.8) implies (1.4). Thus Theorem 1.3 follows.

We remark that, if \(\psi(x) = \exp (\lambda x^2 N)\), then
\[
\int_D G(x,y,t)\psi(y) dy = (1 - 4\lambda t)^{-\frac{1}{2}} \exp \left( \frac{\lambda x^2 N}{1 - 4\lambda t} \right)
\]
and it does not exist after \(t = \frac{1}{4\lambda}\).

5 Life span of the solution to the semilinear heat equation

5.1 Motivation and known results

Let \(v\) be a nonnegative solution to semilinear parabolic equation
\[
\partial_t v + (-\Delta)^\alpha v = v^q, \quad x \in \mathbb{R}^N, \quad t > 0,
\]
with the initial condition
\[
v(x,0) = \kappa \psi(x), \quad x \in \mathbb{R}^N,
\]
where \(N \geq 1\), \(\alpha \in (0, 1) \cup \mathbb{N}\), \(q > 1\), \(\kappa > 0\) and \(\psi\) is a nonnegative measurable functions in \(\mathbb{R}^N\). In the case of \(\alpha \in (0, 1)\), the operator \((-\Delta)^\alpha\) is defined by
\[
(-\Delta)^\alpha \phi(x) := \mathcal{F}^{-1} \left| \xi \right|^{2\alpha} \mathcal{F}[\phi](\xi)(x)
\]
for any \(x \in \mathbb{R}^N\) and \(\phi \in S(\mathbb{R}^N)\), where \(\mathcal{F}[\phi]\) is the Fourier transform of \(\phi\). In this section, we consider the life span \(T'(\kappa \psi)\) of nonnegative solutions to problem (5.1) with (5.2), as \(\kappa \to \infty\) or \(\kappa \to +0\) and obtain analog estimates of Theorem 1.1 and 1.2.

The research on the life span \(T'(\kappa \psi)\) has been studied in many papers (See e.g. [4, 8, 9, 12, 16, 17, 18, 19, 20, 22, 21] and references therein). Among others, in 1992 Lee and Ni [16] gave an optimal estimate to the life span \(T'(\kappa \psi)\) as \(\kappa \to +0\) in the case of \(\alpha = 1\) when \(\psi\) behaves like \(|x|^{-A}\) (\(A > 0\)) at the space infinity. Subsequently, the author of this paper and Ishige [9] extended Lee and Ni’s work [16] to the case of \(0 < \alpha < 1\), that is, they proved following:

Let \(A > 0\) and \(\psi(x) = (1 + \left| x \right|)^{-A}\). Then \(T'(\kappa \psi) \to \infty\) as \(\kappa \to 0\) and following holds:

1. Let \(1 < q < 1 + 2\alpha/N\) or \(0 < A < 2\alpha/(q - 1)\). Then

\[
T'(\kappa \psi) \sim \left\{ \begin{array}{ll} \kappa^{-\left(\frac{1}{q-1} - \frac{1}{2\alpha} \min\{A,N\} \right)^{-1}} & \text{if } A \neq N, \\
\left(\frac{\kappa^{-1}}{\log(\kappa^{-1})}\right)^{-\left(\frac{1}{q-1} - \frac{N}{2\alpha}\right)^{-1}} & \text{if } A = N,
\end{array} \right.
\]
as \(\kappa \to +0\);
(2) Let \( p = 1 + 2\alpha/N \) and \( A \geq 2\alpha/(q-1) \). Then

\[
\log T'(\kappa\psi) \sim \begin{cases} 
\kappa^{-(q-1)} & \text{if } A > N, \\
\kappa^{-\frac{q-1}{q}} & \text{if } A = N,
\end{cases}
\]

as \( \kappa \to +0 \);

(3) Let \( q > 1 + 2\alpha/N \) and \( A \geq 2\alpha/(q-1) \). Then, problem (5.1) with the initial data \( v(0) = \kappa\psi \) possesses a global-in-time solution if \( \kappa > 0 \) is sufficiently small.

The main idea is to apply the necessary conditions and the sufficient conditions for the solvability. In the study of the solvability of problem (5.1), the author of this paper and Ishige [9] (in the case of \( \alpha \in (0,1) \)) and Ishige, Kawakami and Okabe [11] (in the case of \( \alpha \in \mathbb{N} \)) obtained the necessary conditions and the sufficient conditions. In subsection 5.3 we review these conditions.

5.2 Main results

Before stating our main results, we formulate the definition of solutions to (5.1) and the life span \( T'(\kappa\psi) \) of solutions to (5.1). In Definition 5.1 (i) is the definition of solutions to (5.1) in the case \( \alpha \in (0,1) \) and (ii) is that of in the case \( \alpha \in \mathbb{N} \). Let \( G_{\alpha} = G_{\alpha}(x,t) \) be the fundamental solution to

\[
\partial_t w + (-\Delta)^\alpha w = 0 \quad \text{in} \quad \mathbb{R}^N \times (0,\infty),
\]

where \( \alpha \in (0,1) \).

**Definition 5.1** Let \( v \) be a nonnegative measurable function in \( \mathbb{R}^N \times (0,T) \), where \( 0 < T \leq \infty \) and \( \varphi \) be a nonnegative measurable function in \( \mathbb{R}^N \).

(i) In the case of \( \alpha \in (0,1) \), we say that \( v \) is a solution to problem (5.1) in \( [0,T) \) with \( v(0) = \varphi \) if \( v \) satisfies

\[
v(x,t) = \int_{\mathbb{R}^N} G_{\alpha}(x-y,t)\varphi(y)dy + \int_0^t \int_{\mathbb{R}^N} G_{\alpha}(x-y,t-s)v(y,s)^qdyds
\]

for almost all \( x \in \mathbb{R}^N \) and \( t \in (0,T) \).

(ii) In the case of \( \alpha \in \mathbb{N} \), we say that \( v \) is a solution to problem (5.1) in \( [0,T) \) with \( v(0) = \varphi \) if \( v \) satisfies

\[
\int_0^T \int_{\mathbb{R}^N} v(-\partial_t \eta + (-\Delta)^\alpha \eta)dxdt = \int_0^T v^q \eta dxdt + \int_{\mathbb{R}^N} \varphi \eta dxdt
\]

for \( \eta \in C_0^\infty(\mathbb{R}^N \times [0,T)) \).

(iii) We say that \( v \) is a minimal solution to (5.1) in \( [0,T) \) with \( v(0) = \varphi \) if \( v \) is a solution to (5.1) in \( [0,T) \) with \( v(0) = \varphi \) and satisfies

\[
v(x,t) \leq w(x,t) \quad \text{for almost all } x \in \mathbb{R}^N \text{ and } t \in (0,T)
\]

for any solution \( w \) to (5.1) in \( [0,T) \) with \( w(0) = \varphi \).
(iv) The life span \( T'(\varphi) \) of solutions to (5.1) with \( v(0) = \varphi \) in \( \mathbb{R}^N \) is defined by the maximal existence time of the minimal solution to (5.1) with \( v(0) = \varphi \) in \( D \).

Now we are ready to state our main results. We state them without the proofs, since the results on the solvability of (5.1) (which will be stated later) can yield them in a similar way to the proofs Theorems 1.1 and 1.2. In Theorem 5.1 we obtain the relationship between the singularity of \( \psi \) and the life span \( T'(\kappa \psi) \) as \( \kappa \to \infty \) and give an optimal estimate to the life span \( T'(\kappa \psi) \) as \( \kappa \to \infty \). In what follows, we set \( q_\alpha := 1 + 2\alpha/N \).

**Theorem 5.1** Assume that

\[
\psi(x) := |x|^A \left[ \log \left( e + 1/|x| \right) \right]^{-B} \chi_{B(0,1)}(x) \in L^1(\mathbb{R}^N) \setminus L^\infty(\mathbb{R}^N),
\]

where \( -N \leq A \leq 0 \) and

\[
B > 0 \quad \text{if} \quad A = 0, \quad B \in \mathbb{R} \quad \text{if} \quad -N < A < 0, \quad B > 1 \quad \text{if} \quad A = -N. \tag{5.3}
\]

Then \( T'(\kappa \psi) \to 0 \) as \( \kappa \to \infty \) and following holds:

- \( T'(\kappa \psi) \) behaves

\[
T'(\kappa \psi) \sim \begin{cases} 
\kappa (\log \kappa)^{-B} & \text{if} \quad A > -\min \left\{ N, \frac{2\alpha}{q-1} \right\}, \\
\kappa (\log \kappa)^{-B+1} & \text{if} \quad 1 < q < q_\alpha, \; A = -N, \; B > 1,
\end{cases}
\]

and

\[
|\log T'(\kappa \psi)| \sim \begin{cases} 
\kappa^{B} & \text{if} \quad q > q_\alpha, \; A = \frac{-2\alpha}{q-1}, \; B > 0, \\
\kappa^{(B-1-\frac{N}{2\alpha})^{-1}} & \text{if} \quad q = q_\alpha, \; A = -N, \; B > \frac{N}{2\alpha} + 1,
\end{cases}
\]

as \( \kappa \to \infty \).

- Let \( q > q_* \). If, either

\[
A < -2\alpha/(q-1) \quad \text{and} \quad B \in \mathbb{R} \quad \text{or} \quad A = -2\alpha/(q-1) \quad \text{and} \quad B < 0,
\]

then problem (5.1) with (5.2) possesses no local-in-time solutions for all \( \kappa > 0 \). If

\[
A = -2\alpha/(q-1) \quad \text{and} \quad B = 0,
\]

then problem (5.1) with (5.2) possesses no local-in-time solutions for sufficiently large \( \kappa > 0 \);

- Let \( q = q_* \). If

\[
A = -N \quad \text{and} \quad B < \frac{N}{2\alpha} + 1,
\]

then problem (5.1) with (5.2) possesses no local-in-time solutions for all \( \kappa > 0 \). If

\[
A = -N \quad \text{and} \quad B = \frac{N}{2\alpha} + 1,
\]

then problem (5.1) with (5.2) possesses no local-in-time solutions for sufficiently large \( \kappa > 0 \).
We remark that (5.3) implies that $\psi \in L^1_{\text{loc}}(\mathbb{R}^N)$ if $\psi$ is as in Theorem 5.1.

Theorem 5.2 gives an optimal estimate to the life span $T'(\kappa \psi)$ with $\psi$ behaving like $|x|^{-A} (A > 0)$ at the space infinity as $\kappa \to +0$. As mentioned above, in the case of $0 < \alpha \leq 1$ an optimal estimate has been already obtained (see [16] and [9]).

**Theorem 5.2** Let $A > 0$ and $\psi(x) = (1 + |x|)^{-A}$. Then $T'(\kappa \psi) \to \infty$ as $\kappa \to 0$ and following holds:

1. Let $1 < q < q_\alpha$ or $0 < A < 2\alpha/(q - 1)$. Then
   
   \[
   T'(\kappa \psi) \sim \begin{cases} 
   \kappa^{-\left(\frac{1}{q-1} - \frac{1}{2\alpha} \min\{A,N\}\right)} & \text{if } A \neq N, \\
   \left(\frac{1}{\log(\kappa^{-1})}\right)^{-\left(\frac{1}{q-1} - \frac{N}{2\alpha}\right)} & \text{if } A = N,
   \end{cases}
   \]

   as $\kappa \to +0$;

2. Let $q = q_\alpha$ and $A \geq 2\alpha/(p - 1)$. Then
   
   \[
   \log T'(\kappa \psi) \sim \begin{cases} 
   \kappa^{-(q-1)} & \text{if } A > N, \\
   \kappa^{-\frac{q-1}{q}} & \text{if } A = N,
   \end{cases}
   \]

   as $\kappa \to +0$;

3. Let $q > q_\alpha$ and $A \geq 2\alpha/(p - 1)$. Then, problem (5.1) with (5.2) possesses a global-in-time solution if $\kappa > 0$ is sufficiently small.

### 5.3 Solvability of problem (5.1)

In this subsection we review the results on the solvability of problem (5.1), which have been proved in [9] and [11]. These results can yield Theorems 5.1 and 5.2. For any $x \in \mathbb{R}^N$ and $r > 0$, let $B(x,r) := \{y \in \mathbb{R}^N : |x - y| < r\}$. We consider the solvability of problem (5.1) with the initial condition

\[
\begin{aligned}
v(x,0) &= \varphi(x), & x \in \mathbb{R}^N, \\
\end{aligned}
\tag{5.4}
\]

where $\varphi$ is a nonnegative measurable function in $\mathbb{R}^N$.

In Theorem 5.3 we obtain the necessary conditions for the solvability of (5.1) with (5.4).

**Theorem 5.3** Let $N \geq 1$, $\alpha \in (0,1) \cup \mathbb{N}$ and $q > 1$. Let $v$ be a solution to (5.1) with (5.4) in $\mathbb{R}^N \times (0,T)$, where $0 < T < \infty$. Then there exists $\gamma_1 > 0$ depending only on $N$, $\alpha$ and $q$ such that

\[
\begin{aligned}
\sup_{x \in \mathbb{R}^N} \int_{B(x,\sigma)} \varphi(y) \, dy &\leq \begin{cases} 
\gamma_1 \sigma^{-\frac{2\alpha}{p-1}}, & \text{if } q \neq q_\alpha, \\
\gamma_1 \left(\log \left(e + \frac{T \sigma^N}{\sigma}\right)\right)^{-\frac{N}{2\alpha}}, & \text{if } q = q_\alpha,
\end{cases}
\end{aligned}
\]

for all $0 < \sigma \leq T^{\frac{1}{q_\alpha}}$.

Next, we review the results on sufficient conditions for the solvability of problem (5.1), which have been proved also in [9] and [11].
Theorem 5.4 Let $N \geq 1$, $\alpha \in (0, 1) \cup \mathbb{N}$ and $1 < q < q_\alpha$. Then there exists $\gamma_2 > 0$ such that, if $\varphi$ is a nonnegative measurable function in $\mathbb{R}^N$ satisfying
\[ \sup_{x \in \mathbb{R}^N} \int_{B(x, T^{1/(2\alpha)})} \varphi(y) \, dy \leq \gamma_2 T^{\frac{N}{2\alpha} - \frac{1}{q-1}} \text{ for some } T > 0, \]
then problem (5.1) with (5.4) possesses a solution in $\mathbb{R}^N \times [0, T]$.

Theorem 5.5 Let $N \geq 1$, $\alpha \in (0, 1) \cup \mathbb{N}$ and $1 < a < q$. Then there exists $\gamma_3 > 0$ such that, if $\varphi$ is a nonnegative measurable function in $\mathbb{R}^N$ satisfying
\[ \sup_{x \in \mathbb{R}^N} \left[ \int_{B(x, \sigma)} \varphi(y)^a \, dy \right]^\frac{1}{a} \leq \gamma_3 \sigma^{-\frac{2a}{q-1}}, \quad 0 < \sigma \leq T^{\frac{1}{2\alpha}}, \]
for some $T > 0$, then problem (5.1) with (5.4) possesses a solution in $\mathbb{R}^N \times [0, T]$.

Theorem 5.6 Let $N \geq 1$, $\alpha \in (0, 1) \cup \mathbb{N}$ and $q = q_\alpha$. For $s > 0$, set
\[ \Psi_\alpha(s) := s [\log(e + s)]^\frac{N}{2\alpha}, \quad \rho_\alpha(s) := s^{-N} \left[ \log\left( \frac{e + 1}{s} \right) \right]^{-\frac{N}{2\alpha}}. \]
Then there exists $\gamma_4 > 0$ such that, if $\varphi$ is a nonnegative measurable function in $\mathbb{R}^N$ satisfying
\[ \sup_{x \in \mathbb{R}^N} \Psi_\alpha^{-1} \left[ \int_{B(x, \sigma)} \Psi_\alpha(T^{1/(q-1)} \varphi(y)) \, dy \right] \leq \gamma_4 \rho_\alpha(\sigma T^{-\frac{1}{2\alpha}}), \quad 0 < \sigma \leq T^{\frac{1}{2\alpha}}, \]
for some $T > 0$, then problem (5.1) with (5.4) possesses a solution in $\mathbb{R}^N \times [0, T]$.

Acknowledgments. The author of this paper is grateful to Professor K. Ishige for mathematical discussions and proofreading the manuscript. I am also grateful to Professor S. Okabe for carefully proofreading the manuscript.

References

[1] J. M. Arrieta, A. N. Carvalho and A. Rodríguez-Bernal, Parabolic problems with nonlinear boundary conditions and critical nonlinearities, J. Differential Equations 156 (1999), 376–406.

[2] J. M. Arrieta and A. Rodríguez-Bernal, Non well posedness of parabolic equations with supercritical nonlinearities, Commun. Contemp. Math. 6 (2004), 733–764.

[3] K. Deng, M. Fila and H. A. Levine, On critical exponents for a system of heat equations coupled in the boundary conditions, Acta Math. Univ. Comenianae 63 (1994), 169–192.

[4] F. Dickstein, Blowup stability of solutions of the nonlinear heat equation with a large life span. J. Differ. Equ. 223, 303–328 (2006)
[5] J. Fernández Bonder and J. D. Rossi, Life span for solutions of the heat equation with a nonlinear boundary condition, Tsukuba J. Math. 25 (2001), 215–220.

[6] J. Filo and J. Kačur, Local existence of general nonlinear parabolic systems. Nonlinear Anal. 24 (1995), 1597–1618.

[7] V. A. Galaktionov and H. A. Levine, On critical Fujita exponents for heat equations with nonlinear flux conditions on the boundary, Israel J. Math. 94 (1996), 125–146.

[8] C. Gui and X. Wang, Life spans of solutions of the Cauchy problem for a semilinear heat equation. J. Differ. Equ. 115 (1995), 166–172

[9] K. Hisa and K. Ishige, Existence of solutions for a fractional semilinear parabolic equation with singular initial data, Nonlinear Anal. 175 (2018), 108–132.

[10] K. Hisa and K. Ishige, Solvability of the heat equation with a nonlinear boundary condition, SIAM J. Math. Anal. 51 (2019), no. 1, 565–594.

[11] K. Ishige, T. Kawakami and S. Okabe, Existence of solutions for a higher-order semilinear parabolic equation with singular initial data, arXiv:1909.05492v1.

[12] M. Ikeda and M. Sobajima, Sharp upper bound for lifespan of solutions to some critical semilinear parabolic, dispersive and hyperbolic equations via a test function method, Nonlinear Anal. 182 (2019), 57–74.

[13] K. Ishige and T. Kawakami, Global solutions of the heat equation with a nonlinear boundary condition, Calc. Var. Partial Differential Equations 39 (2010) 429–457.

[14] K. Ishige and R. Sato, Heat equation with a nonlinear boundary condition and uniformly local $L^r$ spaces, Discrete Contin. Dyn. Syst. 36 (2016), 2627–2652.

[15] K. Ishige and R. Sato, Heat equation with a nonlinear boundary condition and growing initial data, Differential Integral Equations 30 (2017), no. 7-8, 481–504.

[16] T. Y. Lee and W. M. Ni, Global existence, large time behavior and life span of solutions of a semilinear parabolic Cauchy problem, Trans. Amer. Math. Soc. 333 (1992), 365–378.

[17] N. Mizoguchi and E. Yanagida, Blowup and life span of solutions for a semilinear parabolic equation, SIAM J. Math. Anal. 29 (1998), no. 6, 1434–1446.

[18] N. Mizoguchi and E. Yanagida, Life span of solutions with large initial data in a semilinear parabolic equation, Indiana Univ. Math. J. 50 (2001), no. 1, 591–610.

[19] T. Ozawa and Y. Yamauchi, Life span of positive solutions for a semilinear heat equation with general non-decaying initial data, J. Math. Anal. Appl. 379 (2011) 518–523.

[20] S. Sato, Life span of solutions with large initial data for a superlinear heat equation, J. Math. Anal. Appl. 343 (2008), 1061–1074.

[21] M. Yamaguchi and Y. Yamauchi, Life span of positive solutions for a semilinear heat equation with non-decaying initial data, Differential Integral Equations 23 (2010) 1151–1157.
[22] Y. Yamauchi, Life span of solutions for a semilinear heat equation with initial data having positive limit inferior at infinity, Nonlinear Anal. 74 (2011), no. 15, 5008–5014.