Spectral singularity and non-Hermitian $PT$-symmetric extension of an $A_{N-1}$-type Calogero model without confining potential

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Abstract

We consider non-Hermitian $PT$-symmetric deformation of an $A_{N-1}$-type Calogero model without confining potential to investigate the possible existence of spectral singularity. By considering the Wronskian between asymptotic incoming and outgoing scattering state wavefunctions, we found that there exists no spectral singularity in this model. We further explicitly show that the transmission coefficient vanishes and the reflection coefficient becomes unity for all values of the energy in such a momentum-dependent non-Hermitian $PT$-symmetric model.

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1. Introduction

The non-Hermitian extension of quantum theories has become an important topic of research in recent years [1], mainly due to the following reasons. Firstly, fully consistent quantum theories (complete real spectrum, probabilistic interpretation and unitary time evolution) have been developed for certain categories of a non-Hermitian system in a Hilbert space equipped with an appropriate inner product, and secondly because of the large number of application of such non-Hermitian models in quantum optics [2], open quantum systems [3], quantum field theories [4], quasi-exactly solvable (QES) models [5], etc. The non-Hermitian but combined parity ($P$) and time reversal ($T$) invariant extension of some exactly solvable many-particle quantum mechanical systems [6, 7] in one space dimension has also recently been investigated [8–14]. In particular, the $PT$-symmetric non-Hermitian deformation of the $A_{N-1}$ Calogero model (related to $A_{N-1}$ Lie algebra) and the $B_N$ Calogero model (related to $B_N$ Lie algebra) is considered by different groups. All these non-Hermitian models exhibit generalized exclusion statistics of Haldane type [15] and are very attractive due to their various applications in condensed matter physics [16]. It has been observed that the generalized exclusion and
exchange statistic parameter differ from each other in the presence of PT-symmetric non-Hermitian interaction [8–12]. Fring has addressed the important question of whether such extensions are meaningful for all remaining Lie algebra (Coxeter group) and if in addition one can make such extension of the models beyond the rational case to trigonometric, hyperbolic and elliptic models [13]. They have shown that all these deformed rational models are integrable and additional interactions are required to maintain the integrability for deformed non-rational models. Different issues related to solvability and/or integrability of the Calogero–Sutherland (CS) model with PT-symmetric non-Hermitian interactions are discussed in [14]. New QES deformation of the CS model has also been studied [14].

Lack of completeness in a non-Hermitian system is associated with exceptional points (EPs) or spectral singularity (SS) when two or more eigenvalues along with corresponding eigenfunctions coalesce [17–19]. These are the obstructions to develop fully consistent quantum theories with non-Hermitian Hamiltonians and hence should be investigated in such systems. Different signatures of EP/SS and their consequences have been studied in great detail mainly for single-particle systems. To the best of our knowledge, these singularities have not yet been investigated in the context of non-Hermitian-deformed many-particle systems. The purpose of this paper is to explore the possibility of the existence/non-existence of spectral singular points in the $A_{N−1}$ Calogero model with PT-symmetric non-Hermitian long-range interaction. Since we are interested in studying SS in such models, we would like to restrict our discussion to the PT-symmetric non-Hermitian-deformed $A_{N−1}$ Calogero model without confining the potential as it leads to the scattering states [11, 12].

By considering the Wronskian between asymptotic incoming and outgoing scattering states, we show that these states never become linearly dependent for any values of the energy. It suggests the non-existence of SS in this model. Our result is further supported by the explicit calculation of reflection and transmission coefficients in the presence of the $PT$-invariant long-range interacting potential. The transmission coefficient vanishes and the reflection coefficient becomes unity in this non-Hermitian case. These coefficients receive no modification due to such a non-Hermitian $PT$-symmetric deformation of the $A_{N−1}$ Calogero model.

Now we present the plan of the paper. In section 2, we discuss basic aspects of the $A_{N−1}$ Calogero model and its deformations. Scattering states of this model are constructed. The nonexistence of SS points in such a model is shown explicitly in section 3, and section 4 is kept for summary and discussions.

2. The $A_{N−1}$ Calogero model with non-Hermitian $PT$-long-range interaction

In this section, we briefly outline some of the basic features of the $A_{N−1}$-type Calogero model [6] and review our earlier works [11, 12] related to $PT$-symmetric non-Hermitian momentum-dependent deformation of such a model. The $A_{N−1}$ Calogero model containing $N$ particles on a line is described by the Hamiltonian

$$H = -\frac{1}{2} \sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} + \frac{g}{2} \sum_{j<k} \frac{1}{(x_j - x_k)^2} + \frac{\omega^2}{2} \sum_{j=1}^{N} x_j^2. \tag{1}$$

Here $g$ is the coupling of long-range interaction and $\omega$ is the coupling of harmonic confining interaction. The complete set of bound state energy eigenvalues is given as

$$E_{n_1,n_2,...,n_N} = \frac{N\omega}{2} [1 + (N - 1)\nu] + \omega \sum_{j=1}^{N} n_j, \tag{2}$$

where $n_j$ are the non-negative integer-valued quantum numbers with $n_j \leq n_{j+1}$ and $\nu$ is a real positive parameter related to a coupling constant $g = \nu^2 - \nu$. Scattering states with
The Hamiltonian in equation (1) is deformed by adding an extra term \( \delta \sum_{j \neq k} \frac{1}{(x_j - x_k)^2} \), which is non-Hermitian but symmetric under combined PT transformation. For a Hamiltonian containing \( N \) particles, the \( P \) and \( T \) transformations are evidently given by
\[
\begin{align*}
P : & \quad x_j \rightarrow -x_j, \quad p_j \rightarrow -p_j \\
T : & \quad x_j \rightarrow x_j, \quad p_j \rightarrow -p_j, \quad i \rightarrow -i
\end{align*}
\]
where \( j \in \{1, 2, \ldots, N\} \) and \( x_j \) (\( p_j \equiv -i \frac{\partial}{\partial x_j} \)) denotes the coordinate (momenta) operator of the \( j \)th particle. Hence, the extended Calogero model which we will be considering here is described by the Hamiltonian
\[
\mathcal{H}_{\text{ext}} = -\frac{1}{2} \sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} + \frac{g}{2} \sum_{j \neq k} \frac{1}{(x_j - x_k)^2} + \delta \sum_{j \neq k} \frac{1}{(x_j - x_k)} \frac{\partial}{\partial x_j}. \tag{3}
\]

The eigenvalue problem for the above Hamiltonian can be solved to obtain scattering states within a sector of configuration space corresponding to a definite ordering of particles like \( x_1 \geq x_2 \geq \cdots \geq x_N \). The zero energy ground state wavefunction of this model is given by
\[
\psi_{\text{gr}} = \prod_{j<k} (x_j - x_k)^{\nu'}, \tag{4}
\]
where the exponent \( \nu' \) is related to the coupling constants \( g \) and \( \delta \) through the relation
\[
g = \nu'^2 - \nu' (1 + 2\delta). \tag{5}
\]
For the purpose of obtaining the non-singular ground state eigenfunction at the limit \( x_i \rightarrow x_j \), \( \nu' \) should be a non-negative exponent. This condition restricts the ranges of coupling constants \( g \) and \( \delta \) as (i) \( \delta \geq -\frac{1}{2}, \quad 0 > g \geq -(\delta + \frac{1}{2})^2 \) and (ii) \( g \geq 0 \) with an arbitrary value of \( \delta \). The general eigenvalue equation associated with the Hamiltonian (3) is given by
\[
\mathcal{H}_{\text{ext}} \psi = \rho^2 \psi, \tag{6}
\]
where \( \rho \) is a real positive parameter. It is easy to see that the solutions of this eigenvalue equation can be written in the form \( \psi = \psi_{\text{gr}} \tau'(x_1, x_2, \ldots, x_N) \), where \( \psi_{\text{gr}} \) represents the modified ground state eigenfunction (4) and \( \tau'(x_1, x_2, \ldots, x_N) \) satisfies a differential equation like
\[
-\frac{1}{2} \sum_{j=1}^{N} \frac{\partial^2 \tau'}{\partial x_j^2} + (\nu' - \delta) \sum_{j \neq k} \frac{1}{(x_j - x_k)} \frac{\partial \tau'}{\partial x_j} = \rho^2 \tau'. \tag{7}
\]
Next \( \tau'(x_1, x_2, \ldots, x_N) \) is assumed to be factorized as
\[
\tau'(x_1, x_2, \ldots, x_N) = P_{k,q}(x) \chi'(r), \tag{8}
\]
where \( r \) is the radial variable defined as \( r^2 = \frac{1}{N} \sum_{i<j} (x_i - x_j)^2 \) and \( P_{k,q}(x) \) are translationally invariant, symmetric, \( k \)-th order homogeneous polynomials satisfying the differential equations
\[
\sum_{j=1}^{N} \frac{\partial^2 P_{k,q}(x)}{\partial x_j^2} + (\nu' - \delta) \sum_{j \neq k} \frac{1}{(x_j - x_k)} \left( \frac{\partial}{\partial x_j} - \frac{\partial}{\partial x_k} \right) P_{k,q}(x) = 0. \tag{9}
\]
The index \( q \) in \( P_{k,q}(x) \) can take any integral value ranging from 1 to \( g(N, k) \), where \( g(N, k) \) is the number of independent polynomials which satisfy equation (9) for a given \( N \) and \( k \) [6]. These polynomials are translationally invariant and satisfy the homogeneity property leading to the relation
\[
\sum_{j=1}^{N} \frac{\partial P_{k,q}(x)}{\partial x_j} = 0, \quad \sum_{j=1}^{N} x_j \frac{\partial P_{k,q}(x)}{\partial x_j} = k P_{k,q}(x). \tag{10}
\]
Substituting the factorized form equation (8) of $\tau(x_1, x_2, \ldots, x_N)$ in the differential equation (7) and making use of the properties of $P'_{k,q}(x)$, the equation satisfied by the ‘radial’ part of the wavefunction is obtained as

$$-\frac{d^2\chi'(r)}{dr^2} - \frac{1 + 2b'}{r}\frac{d\chi'(r)}{dr} = p^2\chi'(r),$$

where $b' = \frac{N-3}{2} + k + (v' - \delta)\frac{N(N-1)}{2}$. The solution of equation (11) can be expressed through the Bessel function: $\chi'(r) = r^{-b'}J_{b'}(pr)$.

In the above expression, $\psi$ is a product of a radial part and an angular part $\alpha_i$, i.e., $p_i = p\alpha_i$. By performing dimensional analysis, we obtain

$$\psi_{\text{gen}} = \prod_{j<k} (x_j - x_k)^{\nu'} \sum_{k=0}^{\infty} \sum_{q=1}^{g(N,k)} C'_{kq}r^{-b'}J_{b'}(pr)P'_{k,q}(x),$$

where $C'_{kq}$ are the expansion coefficients and depend on the particle momenta. Each of the momenta $p_i$ is a product of a radial part $p$ and an angular part $\alpha_i$, i.e. $p_i = p\alpha_i$. The solutions for a complex scattering problem in one dimension can be expressed in terms of an incoming wave ($\psi_+$) and an outgoing wave ($\psi_-$). For this purpose, one has to take the appropriate linear superposition of all degenerate eigenfunctions (with the eigenvalue $p^2$) of the form equation (12):

$$\psi = \prod_{j<k} (x_j - x_k)^{\nu'} \sum_{k=0}^{\infty} \sum_{q=1}^{g(N,k)} C'_{kq}r^{-b'}J_{b'}(pr)P'_{k,q}(x).$$

In the above expression, $A' = b' - k = \frac{N-3}{2} + (v' - \delta)\frac{N(N-1)}{2}$ and $n' = \frac{3-N}{2} + \frac{N(N-1)\delta}{2}$. By choosing the coefficients $\bar{C}_{kq}$ in a proper way, the incoming and outgoing wavefunctions can be written in the form of a plane wave like [12]

$$\psi_{\text{in}} = C \exp \left[ i \sum_{j=1}^{N} p_j x_j \right],$$

$$\psi_{\text{out}} = C \exp \left[ -i \sum_{j=1}^{N} p_j x_j \right].$$

where $p_j \leq p_{j+1}$, $p^2 = \sum_{j=1}^{N} p_j^2$ and $\sum_{j=1}^{N} p_j = 0$. These wavefunctions will be used in the following section for further calculations.

3. Spectral singularity and the $A_{N-1}$ Calogero model without confining potential

In this section, we investigate the possible existence of SS in this non-Hermitian many-particle system. If $\psi_{k,\pm}(x)$ denotes the solutions for a complex scattering problem in one dimension
having continuous positive energy, $H\psi(x) = K^2\psi(x)$, satisfying the asymptotic boundary conditions

$$\psi_{K\pm}(x) \to e^{iKx} \text{ as } x \to \pm \infty,$$

(i.e. the Jost solutions), then there will be an SS at $K = K_*$ only when $\psi_{K\pm}(x)$ are linearly dependent, at $K = K_*$ [18]. This implies that the Wronskian between the two asymptotic solutions $\psi_{K_+}$ and $\psi_{K_-}$ will vanish at spectral singular point $K = K_*$, i.e.

$$W[\psi_{K_+}, \psi_{K_-}] = \psi_{K_+}e^{-iKx} - \psi_{K_-}e^{iKx} = 0. \quad (16)$$

We use this well-known result to find the possible existence of SS points in an $N$-particle non-Hermitian system. For this purpose, we have to construct the general asymptotic wavefunction using equation (15) in terms of individual momenta $p_j, j = 1, 2, \ldots, N$ as

$$\psi_{\pm} = A_{\pm} \exp \left[ i \sum_{j=1}^{N} p_j x_j \right] + B_{\pm} e^{i\pi \phi} \exp \left[ i \sum_{j=1}^{N} p_j p_{N+1-j} \right] \text{ as } \{x_j\} \to \pm \infty, \quad (17)$$

where $\phi = -\pi \frac{N(N-1)}{2}$ and $A_{\pm}, B_{\pm}$ are possibly the $\{p_j\}$-dependent complex coefficients. The individual particle momenta are restricted as, $p_j \leq p_{j+1}, p^2 = \sum_{j=1}^{N} p_j^2$ and $\sum_{j=1}^{N} p_j = 0$.

The Jost solutions $\psi_{p_\pm}$ in terms of their asymptotic behavior for this system are given as

$$\psi_{p_+}(\{x_j\}) \to \exp \left[ i \sum_{j=1}^{N} p_j x_j \right] \text{ as } \{x_j\} \to \infty,$$

$$\psi_{p_-}(\{x_j\}) \to e^{i\pi \phi} \exp \left[ i \sum_{j=1}^{N} x_j p_{N+1-j} \right] \text{ as } \{x_j\} \to -\infty. \quad (18)$$

From equations (17) and (18) and with the help of the transfer matrix [18]

$$\begin{pmatrix} A_+ \\ B_+ \end{pmatrix} = M \begin{pmatrix} A_- \\ B_- \end{pmatrix}, \quad M \text{ is a } 2 \times 2 \text{ matrix,} \quad (19)$$

we write the Jost solutions $\psi_{p_+}(\{x_j\}), \psi_{p_-}(\{x_j\})$ for the asymptotic limit of $\{x_j\} \to \mp\infty$ as

$$\psi_{p_+}(\{x_j\}) = M_{12}(\{p_j\}) \exp \left[ i \sum_{j=1}^{N} p_j x_j \right] + M_{22}(\{p_j\}) e^{i\pi \phi} \exp \left[ i \sum_{j=1}^{N} p_j p_{N+1-j} \right]$$

as $\{x_j\} \to +\infty,$

$$\psi_{p_-}(\{x_j\}) = M_{22}(\{p_j\}) \exp \left[ i \sum_{j=1}^{N} p_j x_j \right] + M_{21}(\{p_j\}) e^{i\pi \phi} \exp \left[ i \sum_{j=1}^{N} x_j p_{N+1-j} \right] \det M(\{p_j\})$$

as $\{x_j\} \to -\infty. \quad (20)$

Now it is straightforward to calculate the Wronskian for $\{x_j\} \to +\infty$ using equations (18) and (20) as

$$W[\psi_{p_+}, \psi_{p_-}] = \psi_{p_+}e^{-iKx} - \psi_{p_-}e^{iKx} \equiv iM_{22}(\{p_j\})[p_{N+1} - p_{N+1-j}] e^{i\pi \phi} \exp \left[ i \sum_{j=1}^{N} x_j p_{N+1-j} \right] \exp \left[ i \sum_{j=1}^{N} p_j x_j \right]. \quad (21)$$

The prime in the above equation denotes differentiation with respect to $x_i$. Now the Wronskian will vanish if either $M_{22}(\{p_j\}) = 0$ or

$$p_i = p_{N+1-i}. \quad (22)$$
However, \( M_{22} \) is not equal to zero unless the reflection coefficient \( R \) is infinite \([18]\). We have shown explicitly toward the end of this section that in such a deformed PT-symmetric many-particle system \( R \) is always unity. This implies \( M_{22} \neq 0 \) and hence SS can occur in such systems if equation (22) is satisfied. However, equation (22) along with the restriction \( p_j \leq p_{j+1} \) has only solution \( p_j = 0 \) for all \( j \). The non-existence of SS in such a non-Hermitian many-particle system now solely depends on the behavior of the reflection coefficient \( R \) of the spectrum. The same conclusion can also be obtained by considering the asymptotic wavefunctions (equation (14)) in terms of radial and angular parts. Now to find the reflection coefficient \( R \) in such systems, for simplicity, we start with two-body scattering, i.e. \( N = 2 \).

The two-body scattering wavefunction in the presence of a non-Hermitian interaction can be written compactly as

\[
\psi_{(2)} = A_2 \hat{r} p^{n-1/2} J_0 (pr), \tag{23}
\]

where \( c = v' - v, \ n' = 1/2 + \delta \) and all the \( r \)-independent terms are included in \( A_2 \). The asymptotic behavior of the above wavefunction

\[
\psi_{(2)\pm} = A_1' r^{-1/2} p^{n-1/2} e^{\mp ipr}. \tag{24}
\]

We consider the incoming wavefunctions in the region \( r < r_− \) as

\[
\psi_{(2)\text{in}} = r^{-1/2} p^{n-1/2} (A e^{-ipr} + B e^{ipr}) \tag{25}
\]

and the outgoing wavefunction

\[
\psi_{(2)\text{out}} = D r^{-1/2} p^{n-1/2} e^{-ipr} \tag{26}
\]

is considered in the region \( r > r_+ \). \( r_\pm \) are some reference points where the wavefunctions satisfy the boundary conditions.

We calculate the constants \( A, B \) by putting the boundary conditions at \( r = r_− \), \( \psi_{(2)\text{in}} |_{r=r_−} = \psi_{(2)} \big|_{r=r_−} \) and \( \psi_{(2)\text{in}} |_{r=r_−} = \psi_{(2)} \big|_{r=r_−} \) as

\[
A = \frac{cr^{−2} + p^2}{p^{n−1/2}[2p^2 + 2ipcr^{−1}]} \left[ (c - 1/2)r^{−1/2} J_0 + r^{−1/2} J'_0 \right] e^{−ipr},
\]

\[
B = \frac{cr^{−2} + p^2}{p^{n−1/2}[2p^2 + 2ipcr^{−1}]} \left[ (c - 1/2)r^{−1/2} J_0 + r^{−1/2} J'_0 \right] e^{−ipr}.
\]  

The reflection coefficient \( R = \frac{|B|^2}{|A|^2} = 1 \) for all values of \( p \). Similarly, we calculate the constant \( D \) by satisfying the boundary conditions at \( r = r_+ \), \( \psi_{(2)\text{out}} \big|_{r=r_+} = \psi_{(2)} \big|_{r=r_+} = 0 \), \( \psi_{(2)\text{out}} \big|_{r=r_+} = \psi_{(2)} \big|_{r=r_+} \) as

\[
D = \frac{r_+ J_0^2 (pr_+)}{p^{n−1/2} e^{−ipr_+}}.
\]  

The transmission coefficient \( T = \frac{|D|^2}{|A|^2} \) vanishes for all values of \( p \) in the asymptotic limit \( r_− \to \infty \).

Now we generalize our result for \( N \)-particle scattering. We denote the \( N \)-body scattering wavefunction given in (equation (13)) as

\[
\psi_{(N)} = p^{n−1/2} F(r, \alpha_i), \tag{29}
\]

and the incoming scattering wavefunction is written using the asymptotic behavior of the wavefunction given in equation (14) as

\[
\psi_{(N)\text{in}} = p^{n−1/2} S(r, \alpha_i)(A_1 e^{-ipr} + B_1 e^{ipr}). \tag{30}
\]
$F(r, \alpha_i)$ and $S(r, \alpha_i)$, take care of all the $r$-dependence and other factors in equations (13) and (14), respectively. The above wavefunctions satisfy the boundary conditions at some reference point $r_\star$. $\psi_{(N)}$ in $|r_\star\rangle = |\psi_{(N)}\rangle_{r_\star}$ and $|r_\star\rangle = |\psi_{(N)}\rangle_{r_\star}$. These lead to the values of $A_1$ and $B_1$ as

$$A_1 = \frac{ipF(r_\star, \alpha_i)S(r_\star, \alpha_i) - F'(r_\star, \alpha_i)S(r_\star, \alpha_i) + S'(r_\star, \alpha_i)F(r_\star, \alpha_i)}{p^{\nu-1/2}\sqrt{p^2 S(r_\star, \alpha_i)} e^{ipr_\star}}$$

$$B_1 = \frac{ipF(r_\star, \alpha_i)S(r_\star, \alpha_i) + F'(r_\star, \alpha_i)S(r_\star, \alpha_i) - S'(r_\star, \alpha_i)F(r_\star, \alpha_i)}{p^{\nu-1/2}\sqrt{p^2 S(r_\star, \alpha_i)} e^{-ipr_\star}}. \quad (31)$$

Thus, the reflection coefficient $R = \frac{B_1}{A_1} = 1$ for the entire spectrum in all energies. This clearly indicates that $M_{22}$ can never be zero establishing the non-existence of SS in the spectrum of this non-Hermitian many-particle system.

We would like to point out here that the behavior of the transmission coefficient and that of the reflection coefficient are exactly the same as mentioned Calogero’s original work [6]. These coefficients receive no modifications due to the addition of the $PT$-symmetric non-Hermitian interaction that suggests the absence of SS points in such many-particle systems.

4. Summary and discussions

It is extremely important to search for spectral singular points and the EP which are obstacles to developing a fully consistent quantum theory with non-Hermitian interactions. There are several different methods to find these spectral singular points. We have studied the possible existence of such singular points in the $PT$-symmetric non-Hermitian $A_{N-1}$ Calogero model without a harmonic confining potential. We have shown explicitly that the asymptotic scattering states never become linearly dependent as the Wronskian between them is not zero. This suggests the non-existence of SS points in this model. Our results are further supported by the fact that the reflection coefficient becomes unity in this non-Hermitian model for all energy values. These scattering coefficients remain unaffected even in the presence of such $PT$-symmetric non-Hermitian interactions. The non-existence of SS points in the $A_{N-1}$ Calogero model with $PT$-invariant non-Hermitian momentum-dependent long-range interaction indeed favors the work of Fring [13], where it has been shown that such momentum-dependent non-Hermitian deformations of the many-particle system are in fact integrable classically. It will be interesting to extend this analysis to the non-Hermitian deformation of non-rational Calogero models, where additional interactions are required to maintain the integrability.

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