Hadrons as Skyrmions in the presence of isospin chemical potential

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The stability of the Skyrmion solution in the presence of finite isospin chemical potential \( \mu \) is considered, showing the existence of a critical value \( \mu_c = 222.8 \) MeV where the Skyrmion mass vanishes. Using the Hamiltonian formulation, in terms of collective variables, we discuss the behavior of different skyrmionic parameters as function of the isospin chemical potential (\( \mu \)), such as the energy density, the isoscalar radius and the isoscalar magnetic radius. We found that the radii start to grow very fast for \( \mu \geq 140 \) MeV, suggesting the occurrence of a phase transition.

The skyrmion picture has attracted the attention of many authors as a possible way for understanding hadronic dynamics and the hadronic phase structure. The behavior of hadrons in a media, i.e. taking into account temperature and/or density effects, can be analyzed according to this perspective.

The Skyrme lagrangian is

\[
\mathcal{L} = \frac{F^2}{16} Tr \left[ \partial_{\mu} U \partial^{\mu} U^\dagger \right] + \frac{1}{32\pi^2} Tr \left[ (\partial_{\mu} U) U^\dagger, (\partial_{\nu} U) U^\dagger \right]^2,
\]

where \( F_\pi \) is the pion decay constant and \( e \) is a numerical parameter. The isospin chemical potential is introduced as a covariant derivative of the form \( \xi_{\mu} \)

\[
\partial_{\mu} U \rightarrow D_{\mu} U = \partial_{\mu} U - \frac{i\mu}{2} [\sigma^3, U] g_{\mu.0}.
\]

The \( U \) field matrix, for the static case, can be parameterized in the standard way

\[
U = U_0 = \exp \left( -i\xi \hat{\sigma} \cdot \hat{n} \right) = \cos \xi - i(\hat{\sigma} \cdot \hat{n}) \sin \xi,
\]

where \( \hat{\sigma} \) is the sigma matrix vector and \( \hat{n}^2 = 1 \). This ansatz has a “Hedgehog” shape obeying the boundary conditions \( \xi(\hat{r}) = \xi(r), \quad \hat{n} = \hat{r}, \quad \xi(0) = \pi, \quad \xi(\infty) = 0. \)

The mass of the Skyrmion, for static solutions, develops a dependence on the Isospin Chemical potential as well as on the temperature. Defining \( \hat{r} = eF_\pi r \), the mass of the Skyrmion will be given by

\[
M_\mu = M_{\mu = 0} - \frac{\mu^2}{4e^2F_\pi} I_2 - \frac{\mu^2}{32e^2F_\pi} I_4,
\]

where \( M_{\mu = 0} \) is the zero chemical potential contribution. Notice that the chemical potential terms contribute with opposite sign. This implies that the solution will become unstable above certain value of \( \mu \). In the previous equation

\[
M_{\mu = 0} = \frac{F_\pi}{4e} \left[ 4\pi \int_0^{\infty} d\hat{r} \left( \frac{\hat{r}^2}{2} \left( \frac{d\xi}{d\hat{r}} \right)^2 + \sin^2(\xi) \right) \right] + 4\pi \int_0^{\infty} d\hat{r} \sin^2(\xi) \times \left[ 4\hat{r}^2 \left( \frac{d\xi}{d\hat{r}} \right)^2 + 2 \sin^2(\xi) \right].
\]

Assuming a radial profile \( \xi = \xi(r) \), the integrals \( I_2 \) and \( I_4 \) are given by

\[
I_2 = \frac{4\pi}{3} \int d\hat{r} \hat{r}^2 \sin^2 \xi, \quad I_4 = \frac{32\pi}{3} \int d\hat{r} \hat{r}^2 \left[ \sin^2 \xi \left( \frac{d\xi}{d\hat{r}} \right)^2 + \frac{4}{\hat{r}^2} \sin^2 \xi \right].
\]

In order to minimize the mass, we use a variational procedure which leads us to the following condition for the radial profile...
in an hadronic approach. As usual, the SU$_2$ critical value shows the chemical potential dependence of the mass.

It turns out that this procedure is limited only for small values of $(\mu - 100 \text{ MeV})$, been in this region quite in agreement with the numerical solution.

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To characterize different hadronic states, in an hadronic approach. As usual, the SU(2) collective coordinates $A(t)$ are introduced as

$$U = A(t)U_0A^\dagger(t).$$  \hspace{1cm} (9)
and considering the canonical quantization procedure
\[ p_i \rightarrow \hat{p}_i = -i\delta /\delta a_i, \]
we get
\[ H = M_\mu - \frac{1}{8\lambda} \delta^2 - 2\mu \hat{I}_3, \]  \hspace{1cm} \text{(16)}
where \( \hat{I}_3 \) is the third component of the isospin operator 2.

Following the usual procedure, we may associate a wave function to the Skyrme Hamiltonian. In order to identify baryons in this model, these wave functions have to be odd, i.e. \( \psi(A) = -\psi(-A) \). In particular, nucleons correspond to linear terms in the \( a \)'s, whereas the quartet of \( \Delta \)'s are given by cubic terms.

The energy spectra of nucleons as function of \( \mu \) is shown in figure 3. We can see that an energy splitting between neutrons and protons is induced.

The baryonic charge density for the Skyrmion is given by
\[ \rho_B = 4\pi r^2 B^0(r) = -\frac{2}{\pi} \sin^2 F(r) F'(r). \]  \hspace{1cm} \text{(19)}

Obviously, \( \int_0^\infty dr \rho_B = 1 \), independently of the shape of the skyrmionic profile. The isoscalar mean square radius is defined by
\[ \langle r^2 \rangle_{I=0} = \int_0^\infty dr r^2 \rho_B. \]  \hspace{1cm} \text{(20)}

This radius seems to be quite stable up to the value of \( \mu \approx 120 \text{ MeV} \), starting then to grow dramatically. Although we do not have a formal proof that this radius diverges at a certain critical \( \mu = \mu_c \), the numerical evidence supports such claim, as it is shown in figure 4.

Divergent behavior for several radii, associated to different currents, has also been observed in different hadronic effective couplings as function of temperature in the frame of thermal QCD sum rules 6. Similar behavior is found for the mean square radius associated to the isoscalar magnetic density
\[ \rho_M^{I=0}(r) = \frac{r^2 F' \sin^2 F}{\int dr r^2 F' \sin^2 F}, \]  \hspace{1cm} \text{(21)}

The divergent behavior of the the effective radii, suggests the occurrence of a phase transition, in reference 7, the behavior of the nucleons magnetic moments is also presented, which also have a divergent behavior.

\textbf{Acknowledgements:} We acknowledge support from Fondecyt under grants 1051067 and 1060653.
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