Quantum diffusion in the quasiperiodic kicked rotor

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Abstract. – We study the mechanisms responsible for quantum diffusion in the quasiperiodic kicked rotor. We report experimental measurements of the diffusion constant on the atomic version of the system and develop a theoretical approach (based on the Floquet theorem) explaining the observations, especially the “sub-Fourier” character of the resonances observed in the vicinity of exact periodicity, i.e. the ability of the system to distinguish two neighboring driving frequencies in a time shorter than the inverse of the difference of the two frequencies.

Quantum chaos is the study of quantum systems whose classical limit is chaotic. A major challenge of quantum chaos is to understand the mechanisms that make quantum chaos different from classical chaos. An important difference between classical and quantum systems is the existence in the latter of interferences between various paths. At long times, a large number of complicated trajectories interfere. One could expect the contributions of the various paths to have uncorrelated phases, so that the interference terms vanish in the average after some time, implying that quantum and classical transport should be identical. This simple expectation is however too naive, because phases of the various paths are actually correlated; this is for example the case for the kicked rotor.

The quantum kicked rotor has been extensively studied experimentally in recent years [1–4]. In its atomic version it consists of a cloud of laser-cooled atoms exposed to short pulses of a far detuned, standing laser wave, corresponding to the Hamiltonian (for the external motion of the atoms)

$$H_0 = \frac{p^2}{2} + \frac{K}{T} \cos \theta \sum_n \delta(t - nT).$$ (1)

where $\theta$ is the $2\pi$–periodic position of the rotor, $p$ the conjugate momentum, $T$ the period and $K$ is proportional to the strength of the kicks. In the classical limit, this system is chaotic for $K \gtrsim 1$ [5], and the motion is an ergodic diffusion in momentum space for $K > 5$. Dynamical localization (DL) is the suppression of such a diffusion in the quantum system by subtle quantum interference effects [6].

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Fig. 1 - (a) Averaged squared momentum $\langle p^2 \rangle$ ($p$ is in recoil momentum units) vs. number of double kicks. All curves show DL after 5-10 kicks. After this break-time, the dynamics is frozen for the periodic system ($r=1$, solid curve), but a residual quantum diffusion is observed in the quasi-periodic case, for $r = 1 + 0.00111$ and $r = 1 - 0.00111$ (dotted curves) and $r = 1 \pm 0.00222$ (dashed curves). (b) The quantum diffusion constant (in units of the classical diffusion constant) versus $r$, from data shown in (a). It (almost) vanishes at $r = 1$ because of DL, and rapidly increases symmetrically on both sides, displaying a triangular cusp.

DL is expected to disappear if the time periodicity is broken [7], as experimentally observed in [3]. This can be done simply by adding a second series of kicks with a different period $rT$, giving

$$H(r, \lambda) = \frac{p^2}{2} + \frac{K}{T} \cos \theta \sum_n [\delta(t - nT) + \delta(t - nrT - \lambda T)]$$

(2)

where $\lambda$ is the initial phase between the two kick sequences. If $r$ is rational, the system is strictly time-periodic and DL takes place, but it is rapidly destroyed around any rational number. One way of characterizing this sensitivity to the time-periodicity is to measure the average kinetic energy $\langle p^2 \rangle$ of the system as a function of time (or number of kicks), which allows to deduce its diffusion constant. In this paper, we present new experimental results and a theoretical interpretation of the physical mechanism responsible for the destruction of DL in the vicinity of rational $r$.

Our experimental setup is described in [8]. Cold cesium atoms are produced in a magneto-optical trap, the trap is turned off, and a series of short pulses of a far-detuned (13.5 GHz $\sim$ 2600 $\Gamma$) standing wave (around 65 mW in each direction) is applied. At the end of the pulse series, pulses of counter-propagating phase-coherent beams perform velocity-selective Raman stimulated transitions between the hyperfine ground state sublevels $F_g = 4$ and $F_g = 3$. A resonant probe beam is then used to measure the fraction of transferred atoms, which corresponds to the population in a velocity class. Repeated measurements allow to reconstruct the atomic momentum distribution $P(p)$. In the periodic case, one observes, for $t > t_\ell$, where $t_\ell$ is the so-called localization time, two manifestations of DL: (i) $P(p)$ is frozen in a characteristic exponential shape $P(p) \sim \exp(-|p|/\ell)$, where $\ell$ is the localization length, and (ii) the average kinetic energy of the system tends to a constant value, or, equivalently, the diffusion constant $D = \lim_{t \to \infty} \langle p^2 \rangle / t$ vanishes.

Experimental measurements of $\langle p^2 \rangle$ and the corresponding diffusion constant are displayed in Fig. 1. In the periodic case $r = 1$ (solid line), after an initial linear increase, the kinetic energy saturates to a constant value – corresponding to $D(r = 1) \approx 0$ in Fig. 1(b). Due
to residual spontaneous emission by the kicked atoms \cite{9}, \( D(r = 1) \) is not strictly zero. In the quasiperiodic cases, there is a residual diffusion due to the quasiperiodicity, and Fig. 1(b) shows that \( D(r) \propto |r - 1| \).

Another way to characterize the observed residual diffusion is to measure the population \( P_0 \) of the zero-momentum class \footnote{As the total number of atoms is constant, measuring \( P(p = 0) \) is equivalent to measure the width of the momentum distribution: \( P(p = 0) \propto \langle p^2 \rangle^{-1/2} \).}, as a function of \( r \). Fig. 2 shows the experimentally measured \( P_0 \) after 20 and 100 double kicks. This “localization resonance” displays a sharp peak at \( r = 1 \), indicating the presence of DL, and decreases rapidly on both sides, evidencing the residual diffusion. The plot presents two surprising features: (i) The resonance is very narrow: after \( N \) kicks, it could be argued that the two quasi-periods can be distinguished only if they differ by \( 1/N \) (in relative value). This would predict a width of the order of \( \Delta r = 1/N = 0.01 \) for 100 kicks, whereas we experimentally observe a width five times smaller, 0.0018 \footnote{Sub-Fourier factors of 1/37 have been reported in \cite{10}, and factors of 1/60 have been experimentally observed by us.}. (ii) The “sub-Fourier” resonance is not smooth, but has a marked cusp at the maximum. The understanding of the underlying mechanism behind the quantum behavior of the system shall also allow us to explain these features.

Our analysis of the residual diffusion takes the periodic case as the reference system, because, being periodic, it can be analyzed using the Floquet theorem. A Floquet state (FS) \(| \varphi_k \rangle \) is defined as an eigenstate of the unitary evolution operator \( U(T) \) of \( H_0 \) over one period \( T \): 
\[
U(T)|\varphi_k \rangle = \exp(-i\epsilon_k)|\varphi_k \rangle
\]
\( \epsilon_k \) is called the eigenphase. The temporal evolution of any state \(| \psi \rangle \) after \( n \) periods is 
\[
|\psi(nT) \rangle = \sum_k c_k e^{-in\epsilon_k} |\varphi_k \rangle \text{ with } c_k = \langle \varphi_k | \psi(0) \rangle.
\]
The evolution of any quantity can be calculated using the basis of FS, for example:
\[
\langle p^2(nT) \rangle = \sum_{k,k'} c_k c_{k'}^* e^{-in(\epsilon_k - \epsilon_{k'})} \langle \varphi_{k'} | p^2 | \varphi_k \rangle.
\] (3)

The FS of the chaotic kicked rotor are well-known: they are on the average exponentially
Fig. 3 – Eigenphases $\epsilon_k$ of the evolution operator over one period for the doubly-kicked rotor, corresponding to the experimental conditions, versus the phase $\lambda$. The finite duration of the kicks (800 ns) is taken into account. We have only plotted initially populated Floquet states having a significant weight: $|\langle \psi(0)|\varphi_k(\lambda_0)\rangle|^2 > 10^{-4}$ (thin line), or $> 10^{-2}$ (thick line). States appear (disappear) as their weights go above (below) the threshold. Note that Floquet states rapidly change when avoided crossings with other states are encountered.

localized in momentum space around a most probable momentum $p_k$, with a characteristic localization length $\ell$ [11, 12]. Such a localization – from which DL originates – is far from obvious and is closely related to the Anderson localization in time-independent disordered one-dimensional systems [13].

The initial state is supposed to be localized in momentum space around zero-momentum, with a width much smaller than the width $\ell$ of the FS (which is the case in the experiment). Hence, only FS with roughly $|p_k| \lesssim \ell$ will play a significant role in the dynamics; we shall call such states initially populated FS. Eq. (3) is a coherent sum over FS. However, as times goes on, non-diagonal interference terms accumulate larger and larger phases. In a typical chaotic system, these phases will be uncorrelated at long enough times, leaving an incoherent sum over FS:

$$\langle p^2 \rangle \approx \sum_k |c_k|^2 \langle \varphi_k |p^2|\varphi_k \rangle$$

This equation is valid once DL is established (i.e., for $t > t_\ell$). How long does it take for the phases $n(\epsilon_k - \epsilon_{k'})$ to be scrambled? This can be simply estimated from the level spacing between initially populated FS, and turns out to be roughly $t_\ell = tT$, while $\langle p^2 \rangle$ saturates to a value $\propto \ell^2$.

A key point is to realize that, if $r$ is close enough to 1, the sequence of kicks is very similar to a periodic doubly-kicked rotor for $r = 1$, except that the phase $\lambda$ between the two sequences slowly drifts along the sequence. A small part of the kick sequence around the $n$th kick will seem “instantaneously” periodic, with a phase difference $\lambda(n) = \lambda_0 + n(r-1)$, where $\lambda_0$ is the initial phase between the two kick sequences. The evolution operator of the quasiperiodic system from time $(n-1)T$ to $nT$ is thus given by the evolution operator $U(\lambda) = \exp(-ip^2X_T/\hbar) \exp(-iK \cos \theta/hT) \exp(-ip^2(1-\lambda)T/\hbar) \exp(-iK \cos \theta/hT)$ of the periodic doubly-kicked rotor. The total evolution operator can thus be written as a product of the “instantaneous” evolution operators $\prod_{n=1}^N U(\lambda_0 + n(r - 1))$. For small enough $|r - 1|$, the adiabatic approximation [14] applies: if the system is initially in a FS of $U(\lambda_0)$, it remains
in the corresponding FS of the “instantaneous” evolution operator as \( \lambda \) changes. This is illustrated in Fig. 3, which shows the Floquet spectrum of the periodic doubly-kicked rotor, obtained by numerically diagonalizing \( U(\lambda) \). When \( \lambda \) is varied the eigenergies evolve along complicated “spaghetti”, characteristic of quantum-chaotic systems, with a large number of avoided crossings (AC).

In the experiment, the initial state is a linear combination of FS. The adiabatic approximation implies that it remains a linear combination of the “instantaneous” FS with the same weights (the phases evolve, but the squared moduli remain constant). As discussed above, the coherences between FS vanish after \( t_\ell \), which implies that Eq. 4 remains valid close to the resonance, provided one uses the “instantaneous” Floquet eigenbasis:

\[
\langle p^2(\lambda) \rangle \approx \sum_k |c_k|^2 \langle \varphi_k(\lambda)|p^2|\varphi_k(\lambda)\rangle.
\]

The adiabatic approximation implies that there are no population exchanges among Floquet eigenstates. This means that the weights \( |c_k|^2 \) are constant all along the evolution; they are evaluated at the initial time, corresponding to the initial wavefunction \( |\psi(0)\rangle \) and to \( \lambda = \lambda_0 \):

\[
|c_k|^2 = |\langle \psi(0)|\varphi_k(\lambda_0)\rangle|^2,
\]

where \( |\varphi_k(\lambda_0)\rangle \) are the “instantaneous” eigenstates corresponding to the initial time. The evolution of the average kinetic energy \( \langle p^2 \rangle \) is thus entirely due to the evolution of the “instantaneous” Floquet spectrum with the parameter \( \lambda \), which evolves adiabatically from its initial value \( \lambda_0 \) to the final value \( \lambda = \lambda_0 + N(r-1) \) corresponding to the end of the kick sequence. Thus, two distinct types of localization properties come into Eq. 4: those of the initial Floquet spectrum, present in the constants \( |c_k|^2 \), and those of the “instantaneous” Floquet spectra, present in the “instantaneous” eigenstates \( |\varphi_k(\lambda)\rangle \). The \( |c_k|^2 \) coefficient gives important weights to the FS represented in the initial distribution (that we shall call “initially populated” FS), which, because the initial momentum distribution is very sharp compared to the localization length, are mostly localized around zero momentum. This limits the range of the sum in Eq. 5 to eigenstates \( |\varphi_k(\lambda_0)\rangle \) centered at momenta \( |p_k(\lambda_0)| < \ell \). As \( \lambda \) moves away from \( \lambda_0 \), the center \( p_k(\lambda) \) of \( |\varphi_k(\lambda)\rangle \) moves away from zero momentum. As these eigenstates keep the same weight in the sum, this enlarges the momentum distribution.

This is the mechanism at the origin of the quantum diffusion responsible for the growth of the average kinetic energy in the quasi-periodic case. One thus expects \( \langle p^2(\lambda) \rangle \) to have a minimum at \( \lambda = \lambda_0 \) and to rapidly increase as the kicks are applied, that is, as \( \lambda \) evolves. Since \( \lambda - \lambda_0 \) is proportional to \( r-1 \), \( \langle p^2 \rangle \) shall also present a sharp minimum at \( r-1 \), which implies that the population in the zero-momentum class shall present a sharp maximum at \( r-1 \), as experimentally observed in Fig. 3.

What determines the linewidth of the resonance? A FS may considerably change because of its AC with other FS, as tiny AC may be crossed diabatically. On the average, the typical variation \( \lambda - \lambda_0 \) of the phase between kick series for which \( |\varphi_k(\lambda)\rangle \) loses the localization property of \( |\varphi_k(\lambda_0)\rangle \), is the distance \( \Delta \lambda_c \) to the next AC. We immediately deduce that the full width \( \Delta f \) (in frequency) of the sub-Fourier line is such that:

\[
2\Delta \lambda_c = (NT)\Delta f.
\]

If the classical dynamics is regular, the Floquet eigenstates evolve smoothly with the parameter \( \lambda \); a change in \( \lambda \) of the order of one is thus required to significantly modify the Floquet states: \( 2\Delta \lambda_c \approx 1 \), which corresponds to \( \Delta f = 1/(NT) \), that is, the Fourier limit. In a classically chaotic system, however, the level dynamics displays plenty of AC, see Fig. 4; \( \Delta \lambda_c \) is then much smaller than unity, leading to sub-Fourier resonances. Eq. 6 also predicts the linewidth to be inversely proportional to the temporal length of the kick sequence beyond the localization.
time (i.e. the sub-Fourier character is independent of $N$), as numerically observed \[10\]. The critical value $\Delta \lambda_c$ depends on the detailed dynamics of the system. It can be roughly estimated by visual inspection of the quasi-energy level dynamics, Fig. 3 to be of the order of 0.05 and a “factor 10” sub-Fourier line, about twice the experimentally observed factor (the additional experimental broadening is due to the transverse profile of the laser mode leading to spatial inhomogeneities in $K$).

From Eq. \[3\] and the previous analysis, it is expected that $\langle p^2 \rangle$ significantly increases from its minimum value at $\lambda = \lambda_0$. AC between FS localized around the same momentum are rather large (this is what determines $\Delta \lambda_c$) whereas AC between states localized a distance $L \gg \ell$ apart in momentum space are typically much smaller and scale like $\exp(-L/\ell)$, as a consequence of the exponential localization of the FS. There is thus a very broad distribution of AC widths, with a large number of tiny AC. A tiny AC typically extends over a small $\lambda$ interval and thus tends to produce small values of $\Delta \lambda_c$. The increase of $\langle p^2 \rangle$ with $\lambda$ thus depends on the number of small AC encountered. In the presence of exponential localization, the AC density scales with size $C$ as $1/C$ for $C \to 0$, and $\langle p^2(\lambda) \rangle - \langle p^2(\lambda_0) \rangle$ shall behave like $|\lambda - \lambda_0|$ \((^3)\), producing the cusp experimentally observed in the resonance line, Fig. 2b. The large number of extremely small AC is responsible for the singularity of the sub-Fourier resonance line. Another consequence is the diffusive behavior observed in the vicinity of the resonance, see Fig. 1b. Indeed, $\langle p^2(\lambda) \rangle - \langle p^2(\lambda_0) \rangle$ increases linearly with $|\lambda - \lambda_0|$, itself a linear function of time and of $|r - 1|$. Thus, our model correctly predicts two non-trivial properties: $\langle p^2 \rangle$ increases linearly with time and the corresponding diffusion constant is proportional to $|r - 1|$. This is distinct from the prediction of Random Matrix Theory, which implies an increase of $\langle p^2 \rangle$ initially quadratic in $\lambda - \lambda_0$ \[15\].

There will always be some degree of nonadiabaticity. Whatever small $|r - 1|$ is, tiny enough AC will be crossed diabatically. This puts a lower bound on the size of the AC effectively participating in the quantum dynamics and produces a rounding of the top of the sub-Fourier resonance line, too small to be seen in the experiment after 100 kicks, but easily visible after 20 kicks, Fig. 2.

Our approach concentrates on the immediate vicinity of the resonance. What happens in the wings of the sub-Fourier line? Eq. \[6\] indicates that this depends on the residual correlation between $|\varphi_k(\lambda_0)\rangle$ and $|\varphi_k(\lambda)\rangle$ for $|\lambda - \lambda_0| > \Delta \lambda_c$. A quantitative answer to this question is difficult. However, it seems clear that the quantum diffusion constant does not exceed the classical one, which corresponds to vanishingly small interference terms. Random Matrix Theory tells us that this type of parametric correlation usually decays algebraically with $\lambda - \lambda_0$. We thus propose the following ansatz:

$$\langle p^2(nT) \rangle = \langle p^2 \rangle_{DL} + D_{cl} \frac{|r - 1|}{|r - 1| + \Delta \lambda/\ell} nT$$

where $D_{cl}$ is the classical diffusion constant and $\langle p^2 \rangle_{DL}$ the saturation value of $p^2$ due to DL. This equation fits (using $P(p = 0) \propto \langle p^2 \rangle^{-1/2}$) very well the experimental curves in Fig. 2 reproducing the linear behavior at the center of the resonance and the classical diffusion in the wings. The parameters are however fitted, not extracted from the previous analysis. The reason for that is that, in the experiment, the standing wave laser beams creating the kicking potential have a gaussian profile whose width is only a few times the width of the spatial atomic distribution. This means that atoms away from the center of the beam see a smaller value of $K$ than the atoms at the center of the beam. There is thus an averaging effect over

\footnote{This basically comes from the fact that the probability to encounter a “bad” AC which expels the atoms from the zero-momentum region is proportional to the length of the interval $|\lambda - \lambda_0|$.}
K to which the dynamics is sensitive, unfortunately preventing us from directly comparing the fitted values to the theoretical values of the parameters.

In summary, we have developed a theoretical approach for the mechanism of sub-Fourier resonances, which correctly predicts the unexpected observed features. In particular, our approach evidences the role of chaotic dynamics for producing narrow sub-Fourier resonances. Deviations from exact periodicity are treated in the framework of the adiabatic approximation for the Floquet spectrum of the system, which thus goes beyond a pertubative approach. The dynamics is governed by instantaneous Floquet eigenstates, which are non-trivial objects, as they need to have very well defined internal (i.e., between the various parts of the wavefunction for each Floquet state) phase coherence to be stationary states of the periodic system, but at the same time, the inter-state coherences do not play any role (beyond the localization time, Floquet states effectively add incoherently). The dynamics is dominated by an incoherent sum of internally extremely coherent states. It shows that the role of interferences in quantum mechanics is far from obvious, and can produce unexpected behaviors.

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REFERENCES

[1] Moore F. L., Robinson J. C., Bharucha C. F., Williams P. E. and Raizen M. G., Phys. Rev. Lett., 73 (1994) 2974.
[2] Ammann H., Gray R., Shvarchuck I. and Christensen N., Phys. Rev. Lett., 80 (1998) 4111.
[3] Ringot J., Szriftgiser P., Garreau J. C. and Delande D., Phys. Rev. Lett., 85 (2000) 2741.
[4] D’Arcy M. B., Godun R. M., Oberthaler M. K., Cassetari M. K. and Summy G. S., Phys. Rev. Lett., 87 (2001) 74102.
[5] Chirikov B. V., Phys. Rep., 52 (1979) 263.
[6] Casati G., Chirikov B.V., Ford J. and Izrailev F.M., Lect. Notes Phys., 93 (1979) 334; Wimberger S. and Buchleitner A., J. Phys. A, 34 (2001) 7181.
[7] Casati G., Guarneri I. and Shepelyansky D. L., Phys. Rev. Lett., 62 (1989) 345; Glück M., Kolovsky A. R. and Korsch H. J., Europhys. Lett., 51 (2000) 255; Buchleitner A., Sirklo L. and Walter H., Europhys. Lett., 16 (1990) 35.
[8] Szriftgiser P., Lignier H., Ringot J., Garreau J. C. and Delande D., Commun. Nonlin. Sci. Num. Simul., 8 (2003) 301.
[9] Klappauf B. G., Oskay W. H., Steck D. A. and Raizen M. G., Phys. Rev. Lett., 81 (1998) 1203.
[10] Szriftgiser P., Ringot J., Delande D. and Garreau J. C., Phys. Rev. Lett., 89 (2002) 224101.
[11] Casati G., Guarneri I., Izrailev F. M. and Scharf R., Phys. Rev. Lett., 64 (1990) 5.
[12] Casati G., Guarneri I. and D.L. Shepelyansky, IEEE J. Quantum Electron., 24 (1988) 1420.
[13] Grempel D. R., Prange R. E. and Fishman S., Phys. Rev. A, 29 (1984) 1639.
[14] Landau L. and Lifchitz E., Mécanique Quantique (Mir, Moscou) 1966.
[15] Simons B. D. and Altshuler B. L., Phys. Rev. Lett., 70 (1993) 4063.