A binary partition of a positive integer $n$ is a partition of $n$ in which each part has size a power of two. Let $b(n)$ denote the number of binary partitions of $n$. Since a binary partition either has a part of size 1 or else is twice a binary partition of $n/2$, $b(n)$ satisfies the recurrence

$$
\begin{align*}
    b(n) &= b(n - 1), & n \text{ odd;} \\
    b(n) &= b(n - 1) + b(n/2), & n \text{ even.}
\end{align*}
$$

The generating function $\sum b(n)x^n = 1/\prod(1 - x^{2^k})$ and small values of $b(n)$ were written down by Euler [2, §50], and Mahler [3] and de Bruijn [1] gave increasingly good asymptotics for $b(n)$ and for partitions into powers of $r$ other than 2; this is sometimes called “Mahler’s partition problem.”

In this note we first use a variation on the recurrence (1) to construct a Gray sequence on the set of binary partitions themselves. This is an ordering of the set of binary partitions of each $n$ (or of all $n$) such that adjacent partitions differ by one of a small set of elementary transformations; here the allowed transformations are replacing $2^k + 2^k$ by $2^{k+1}$ or vice versa (or addition of a new +1). Next we give a purely local condition for finding the successor of any partition in this sequence; the rule is so simple that successive transitions can be performed in constant time. Finally we show how to compute directly the bijection between $k$ and the $k$th term in the sequence.

Thanks to Donald Knuth for requesting Theorem 1, and to Richard Stanley for responding to an early draft by asking about the possibility of Theorem 3.

1. Construction of Gray sequences

Theorem 1. For each positive integer $n$, the binary partitions of $n$ can be arranged in a sequence $B(n)$ such that adjacent partitions differ by an operation of the form

$$
\ldots + 2^k + 2^k + \ldots \longleftrightarrow \ldots + 2^{k+1} + \ldots
$$

Moreover the sequence runs first through all partitions ending with $\ldots + 1$, and next through all partitions ending with $\ldots + 2$.

Proof. We give a recursive construction for $B(n)$. We let $Q(n)$ and $S(n)$ denote the first and last partitions in the sequence (reserving $R$ for later use); $Q(n)$ will always be $1 + 1 + \cdots + 1$.

$$
\begin{align*}
    n \equiv 1 \mod 2 : & \quad Q(n - 1) + 1, \ldots, S(n - 1) + 1 \\
    n \equiv 0 \mod 4 : & \quad Q(n - 1) + 1, \ldots, S(n - 1) + 1, Q(\frac{n}{2}) \times 2, \ldots, S(\frac{n}{2}) \times 2 \\
    n \equiv 2 \mod 4 : & \quad Q(n - 1) + 1, \ldots, S(n - 1) + 1, S(\frac{n}{2}) \times 2, \ldots, Q(\frac{n}{2}) \times 2
\end{align*}
$$

1 open problem Ex. 59 in TAOC 7.2.1.4 pre-fascicle 2D, 28 November 2002.
Here $\times 2$ indicates doubling each part of a partition. Note that in the 2 mod 4 case, the $\frac{n}{2}$ sequence appears in reverse order.

The same logic that justified formula (1) for $b(n)$ shows that the above sequences certainly contain all binary partitions. By induction, we only need to check that the two points of concatenation obey the adjacency condition.

For $n \equiv 0 \mod 4$, $S(n-1) + 1 = S(n-2) + 1 + 1 = Q\left(\frac{n}{2} - 1\right) \times 2 + 1 + 1$ (since $n - 2 \equiv 2 \mod 4$), which is $2 + \cdots + 2 + 1 + 1$, since $Q(m)$ is the all 1’s partition for all $m$. Meanwhile, $Q\left(\frac{n}{2}\right) \times 2$ is $2 + \cdots + 2 + 2$, and is thus connected to $S(n-1) + 1$ by a $1 + 1 \leftrightarrow 2$ move.

For $n \equiv 2 \mod 4$, $S(n-1) + 1$ is again $S(n-2) + 1 + 1$, but now $n - 2 \equiv 0 \mod 4$, so this is $S\left(\frac{n}{2} - 1\right) \times 2 + 1 + 1$. On the other hand, $S\left(\frac{n}{2}\right) \times 2$ is $S\left(\frac{n}{2} - 1\right) \times 2 + 2$ (because $\frac{n}{2}$ is odd), so $S(n-1) + 1$ and $S\left(\frac{n}{2}\right) \times 2$ are also connected by a $1 + 1 \leftrightarrow 2$ move, and we are done.

As $n$ increases, the head of $B(n)$ remains unchanged aside from adding $+1$s to each partition. So there is a single infinite sequence $B = B_1, B_2, \ldots$, beginning $\emptyset, 2, 22, 4, 42, 222, 2222, 442, 444, 8, 82, 442, 4222, 222222, 422222, 44222, 822, 84, 444, \ldots$ such that each $B(n)$ is just the initial substring of $B$ of partitions summing to $\leq n$, padded with the appropriate number of 1s.

$B$ is a list of all binary partitions with only even parts, so halving each one gives a Gray sequence $B/2$ of the binary partitions of all $n$. Here the notion of a legal transition must be expanded to include the operation $P \rightarrow P + 1$, the remnant of the transition $1 + 1 \rightarrow 2$ after dropping all 1s and then halving. By the construction, the subsequence of partitions in $B/2$ with constant sum $n$ is identical to $B(n)$ above if $n$ is even, and is the reverse of $B(n)$ if $n$ is odd.

2. Stepping through the sequence

Given a partition $P$ in the sequence $B$, we can calculate the partition which comes before or after $P$ easily. We will give explicit maps $\phi_+$ and $\phi_-$ which take a binary partition and return which of the rules $2^k + 2^k \leftrightarrow 2^{k+1}$ transforms $P$ into its successor or predecessor in $B$.

Looking back at the construction of $B$ from the $B(n)$, we see that for most $P$, the adjacent partitions $\phi_{\pm}(P)$ are the same size as $P$; by “size” or $|P|$ we mean the sum of the parts of $P$ when viewed as an element of $B$, so after discarding any parts of size 1. The only exception is when we try to apply $\phi_+$ to the last partition of size $n$, which we called $S(n)$, or to apply $\phi_-$ to the first partition in $B$ of size $n$, which we now name $R(n)$. In the construction of $B(n)$ this is the first partition which results from a $\times 2$ operation.

From the construction of the $B(n)$, we can calculate:

$$
\begin{align*}
\text{For } n \equiv 0 \mod 4, \quad R(n) & = 2 + \cdots + 2, \\
\text{For } n \equiv 2 \mod 4, \quad S(n) & = 2 + \cdots + 2, \\
\text{For } n \equiv 0 \mod 4, \quad S(n) & = S(n/2) \times 2 = S(n/4) \times 4 = \ldots \\
& = 2^n + \cdots + 2^n, \text{ where } n = 2^a b \text{ with } b \text{ odd}, \\
\text{For } n \equiv 2 \mod 4, \quad R(n) & = S(n - 2) + 2.
\end{align*}
$$

(3)

Conveniently, $S(n)$ for $n \equiv 2 \mod 4$ fits the 0 mod 4 pattern as well.
So the special cases are easily identified, and we already know which transformation rules to apply to them:

\[
\begin{align*}
\phi_+(S(n)) &= 1 + 1 \rightarrow 2 \\
\phi_-(R(n)) &= 2 \rightarrow 1 + 1
\end{align*}
\]

Of course, all parts of size 1 are suppressed in \( B \), so the effect of the rule \( 1 + 1 \rightarrow 2 \) is that a part of size 2 appears from nowhere, and the apparent size of the partition increases.

Aside from these special cases, all other transformations can be determined recursively:

\[
\phi_\pm(P) = \begin{cases} 
2 \times \phi_\pm(\lfloor P/2 \rfloor), & |P| \equiv 0 \mod 4 \\
2 \times \phi_\mp(\lfloor P/2 \rfloor), & |P| \equiv 2 \mod 4
\end{cases}
\]

Here \( \lfloor \cdot \rfloor \) denotes deleting all parts of size 1 to again get an even binary partition, and the \( \mp \) in the \( 2 \mod 4 \) case accounts for the reversal of \( B(n/2) \) in the definition of \( B(n) \). The multiplication by 2 acts on the rule returned, so \( 2 \times (1 + 1 \rightarrow 2) \) is \( 2 + 2 \rightarrow 4 \).

Unravelling the recursion leads to a startlingly quick way to step through \( B \).

**Theorem 2.** Suppose you are given a binary partition \( P \) written \( \ldots d_3d_2d_1d_0 \), where the the digit \( d_k \) is the number of parts of \( P \) of size \( 2^k \). Since we are working in \( B \), we will ignore the value of the \( 1 \)s place \( d_0 \), except to assume it is at least two if the transformation \( 1 + 1 \rightarrow 2 \) is needed.

- Let \( i \) be maximal with \( d_i > 0 \) (\( i = 0 \) if \( P = \emptyset \)).
- Let \( j \) be the second largest integer with \( d_j > 0 \) (or \( j = 0 \) if none).
- Let \( \epsilon \) be \((-1)^{\sum d_k} \) summing over \( 1 \leq k \leq i - 1 \).

Then the transformations \( \phi_\pm \) act on \( P \) according to the following rules:

| If \((d_i, d_j, \epsilon)\) is... | then \( \phi_\pm \) says... |
|-----------------------------|-----------------------------|
| (a) \((1, *, \mp 1)\)       | \(d_i \downarrow, d_{i-1} \uparrow\), split a largest part |
| (b) \(\text{any other (odd, *, \mp 1)}\) | \(d_{i+1} \uparrow, d_i \downarrow\), merge two largest parts |
| (c) \((\text{odd}, 1, \pm 1)\) | \(d_j \downarrow, d_{j-1} \uparrow\), split a 2nd-largest part |
| (d) \(\text{any other (odd, *, \pm 1)}\) | \(d_{j+1} \uparrow, d_j \downarrow\), merge two 2nd-largest parts |
| (e) \((\text{even, *, \mp 1)}\) | \(d_i \downarrow, d_{i-1} \uparrow\), split a largest part |
| (f) \((\text{even, *, \pm 1)}\) | \(d_{i+1} \uparrow, d_i \downarrow\), merge two largest parts |

Here \( \uparrow, \downarrow \) indicate an increment or decrement by one, and \( \uparrow\uparrow, \downarrow\downarrow \) by two.

Note in particular that every transition in \( B \) involves merging or splitting one of the largest two sizes of parts in \( P \)! As a result, successive updates can be done in place in constant time, in a well-chosen data type (where you keep track of \( \epsilon \), and you never need to search for \( d_j \), e.g. a linked list of nonzero \( d_k \).

**Proof.** The recursive part of the definition of \( \phi_\pm \) is handled trivially in this notation. The operation \( |P| \) just shifts each digit to the right and forgets \( d_0 \), and the value of \( |P| \mod 4 \) depends only on the parity of the 2s digit \( d_1 \). So we keep dropping rightmost digits until we arrive at one of the base cases \( \phi_+(S(n)) \) or \( \phi_-(R(n)) \),
and just remember $\epsilon$ to know if we’ve reversed direction an even or odd number of times.\footnote{The fact that we ignore $d_0$ in the definition of $\epsilon$ perhaps reflects a moral imperfection in our definition of $\mathcal{B}(n)$: maybe for odd $n$ it should be the reverse of $\mathcal{B}(n-1)$, which then eliminates the reversal in the 2 mod 4 case.}

The rest of the proof consists of identifying which of the base cases listed in equations (3) and (4) is the destination of each partition:

- Rule (d) is for partitions that end at $\phi_+(S(n))$. The (odd) lead digit $d_i$ is the (odd) number of identical parts in $S(n) = 2^n + \cdots + 2^n$. We reach the base case as soon as the second-largest nonzero digit $d_j$ is deleted, and the transformation $1 + 1 \rightarrow 2$ therefore joins two $2^j$s into a $2^{j+1}$.
- Rule (c) takes care of the exception where we run into $\phi_-(R(2n+2))$ one step before we would otherwise reach $\phi_+(S(n))$.
- Rule (e) is for partitions that end at $\phi_-(R(n))$ when $n \equiv 0 \mod 4$, an even-length sum $2 + \cdots + 2$.
- Rule (a) is for the somewhat special case $\phi_-(R(2))$. $R(2) = 2$, the only time $R(n)$ is a partition with an odd number of parts, since $S(0) = \emptyset$.
- Rules (b) and (f) are the fall-through cases: the recursion avoids all base cases until it gets to $\phi_+(\emptyset)$, to which the transformation $1+1 \rightarrow 2$ is applied.\hfill $\Box$

For example, let us compute the next several partitions in $\mathcal{B}$ beginning with $P = 256^{5}32^{4}16^{4}8^{4}4^{2}2^{3}1$ (so $|P| = 1382$).

\begin{tabular}{cccccccccc}
1024 & 512 & 256 & 128 & 64 & 32 & 16 & 8 & 4 & 2 & 1 & $\epsilon$ \\
5 & 0 & 0 & 2 & 1 & 0 & 4 & 3 & 0 & +1 & Rule (d) \\
5 & 0 & 1 & 0 & 1 & 0 & 4 & 3 & 0 & -1 & Rule (b) \\
1 & 3 & 0 & 1 & 0 & 1 & 0 & 4 & 3 & 0 & +1 & Rule (d) \\
2 & 1 & 0 & 1 & 0 & 1 & 0 & 4 & 3 & 0 & +1 & Rule (f) \\
1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 4 & 3 & 0 & +1 & Rule (c) \\
1 & 0 & 0 & 2 & 1 & 0 & 1 & 0 & 4 & 3 & 0 & -1 & Rule (a) \\
2 & 0 & 2 & 1 & 0 & 1 & 0 & 4 & 3 & 0 & -1 & Rule (e) \\
1 & 2 & 2 & 1 & 0 & 1 & 0 & 4 & 3 & 0 & -1 & \ldots \\
\end{tabular}

Note that $\epsilon$ changes sign under rules (b) and (c), and under (d) unless $i = j + 1$.

3. Calculating Individual Terms

The recursive definition leading to the sequence $\mathcal{B} = \mathcal{B}_1, \mathcal{B}_2, \ldots$ allows us to explicitly compute the bijection $k \mapsto \mathcal{B}_k$. First we introduce an alternate notation for even binary partitions.

**Definition.** Let $P$ be an even binary partition (that is, with no parts of size 1). Define the trail of $P$, $\tau(P) = \tau = \tau_0, \tau_1, \tau_2, \ldots$ by

$$\tau_i = \lfloor |P/2^i| \rfloor$$

where again $\lfloor \cdot \rfloor$ indicates deleting any parts of size 1 or smaller to again get an even binary partition.

In other words, $\tau_i$ is the size of the partition after $i$ iterations of the map “halve all parts and delete parts of size 1.” For example, if $P = 88422$, then $|P/2| = 442$ and $|P/4| = 22$, so $\tau(P) = 24, 10, 4, 0, \ldots$, and we henceforth omit the trailing 0s.
The partition $P$ is easily recovered from $\tau(P)$: the number of parts of $P$ of size $2^i$ is $\tau_{i-1}/2 - \tau_i$, the number of parts of size 1 dropped on the $i$th iteration.

**Theorem 3.** Given just the integer $k$, the trail of $B_k$ can be determined as follows:

(i) $|B_k|$ is the smallest $n$ such that $k \leq b(n)$.

(ii) With $n = |B_k|$ as above, $|B_k/2| = B_{\ell}$, where

$$
\ell = k - b(n - 2), \quad n \equiv 0 \mod 4,
\ell = b(n) + 1 - k, \quad n \equiv 2 \mod 4.
$$

Conversely, the trail $\tau = \tau_0, \tau_1, \tau_2, \ldots$ corresponds to the $k$th partition $B_k$ if its truncation $\tau_1, \tau_2, \tau_3, \ldots$ corresponds to the $\ell$th, where

$$
k = b(\tau_0 - 2) + \ell, \quad \tau_0 \equiv 0 \mod 4,
k = b(\tau_0) + 1 - \ell, \quad \tau_0 \equiv 2 \mod 4.
$$

This recurrence and the base case $\tau = 0, 0, \ldots$ corresponding to $B_{b(0)} = B_1 = \emptyset$ suffices to determine the location in $B$ of the binary partition with any given trail.

For example, partition 88422 with trail 24, 10, 4 appears at position $b(22) + b(10) + 1 - (b(2) + b(0)) = 86$. And to find $B_{123456789}$, we calculate that

$$
123456789 \in (b(646), b(648)) \quad \text{note } 648 \equiv 0 \mod 4,
123456789 - b(646) \in (b(304), b(306)) \quad \text{note } 306 \equiv 2 \mod 4,
b(306) + 1 - (123456789 - b(646)) \in (b(120), b(122)) \quad \text{etc},
$$

and its trail is 648, 306, 122, 58, 28, 14, so the partition is $64^732^616^38^34^{31}2^{18}$. This amounts to a writing of 123456789 in terms of values of $b(n)$ with $n \equiv 2 \mod 4$:

$$
b(646) + b(306) + 1 - (b(122) + 1 - (b(58) + 1 - (b(26) + b(14) + 1 - (b(0)))))
$$

It is possible to work with this representation directly, using sign changes to track the mod 4 behavior, but statements tend to be inelegant.

Both maps rely heavily on the values of the function $b(n)$ which counts the number of binary partitions of $n$, and for which we have no closed form. From a computational complexity point of view this is inevitable, since from the map $k \rightarrow B_k$ the values of $b(n)$ can be determined easily. In practice the computation of $b(n)$ by recurrence (1) is inexpensive.

**Proof.** Since the trail of any even binary partition $P$ begins with $\tau_0 = |P|$, the given maps $k \rightarrow \tau$ and $\tau \rightarrow k$ are clearly inverse, so it suffices to show either direction.

The map $k \rightarrow \tau(B_k)$ follows directly from the construction of the sequences $B(n)$. Suppose $n$ is even. After deleting all parts of size 1 from the partitions in all $B(n)$, the first $b(n-2)$ terms of $B(n)$ are exactly $B(n-2)$, and the remaining $b(n/2)$ terms are the even binary partitions of $n$. So these partitions occur in $B$ as a solid block of terms with indices in $(b(n-2), b(n)]$, justifying (i).

For (ii), recall that each of the terms in this block was obtained by doubling some term in $B(n/2)$; the $|\cdot|$ operation deletes all parts of size 1 and allows us to work in $B$ instead. If $n \equiv 0 \mod 4$ then we doubled $B_\ell$ to get the $\ell$th term in the block, $B_{b(n-2)+\ell}$, while if $n \equiv 2 \mod 4$ then $B(n/2)$ was reversed, and we doubled $B_\ell$ to get the $\ell$th term from the end, $B_{b(n)+1-\ell}$. 

\[ \square \]
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