Controlling sudden transition from classical to quantum decoherence via non-equilibrium environments

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Abstract
We investigate the freezing and sudden transition in the dynamical behavior of quantum and classical correlations in a system composed of two identical non-interacting qubits locally subjected to their own non-equilibrium environments. In contrast to the equilibrium case, one can observe striking results when a bipartite quantum system couples with the non-equilibrium dephasing environment with non-stationary and non-Markovian features. Remarkably, the finite time interval in which the quantum correlation remains impervious to decoherence can be further prolonged as the environment deviates from equilibrium. This reveals that the non-equilibrium parameter provides an alternative tool to efficiently control the appearance of a sudden transition in the decay rates of correlations and their immunity towards the decoherence. Furthermore, for certain initial states, the appearance of another time-interval over which quantum correlation remains constant and the revival of classical correlation not only depends on the non-Markovianity but also on the non-equilibrium parameter.

1. Introduction
Quantum discord is considered as a significant trait of quantumness [1, 2] and has potential applications in quantum information processing [3], remote state preparation [4], quantum metrology [5], microwave quantum illumination [6], quantum cryptography [7], quantum biology [8] and quantum phase transition [9]. Unlike entanglement, quantum discord characterizes all sort of non-classical correlations and interestingly, it has non-zero value even for the separable states. However, the inevitable couplings of the open quantum system with the ambient environment lead to the rapid destruction of quantum correlations, which is a fundamental hurdle for the realization of quantum technologies [10, 11]. Consequently, understanding the dynamical nature of quantum correlations in the presence of decoherence is crucial for both the foundation of quantum theory and their applications in quantum information science. For this aim, several studies have been devoted to investigate the dynamical behaviors of the quantum correlations subjected to various decoherence environments [12–15]. It was proved that the quantum discord under both Markovian and non-Markovian noises is more robust than entanglement [13, 15].

One of the most striking phenomena in the dynamics of an open quantum system is the sudden change between classical and quantum decoherence [16]. Particularly, this peculiar phenomenon was observed by Mazzola et al., in both Markovian [17] and non-Markovian [18] dephasing environments. It was shown that for a certain class of initial states there exists a time-interval in which quantum correlation (measured by quantum discord) can be completely shielded from the detrimental effects induced by environments and only classical correlations decay in that period. After a critical time, the discord experiences a sudden decay during the dynamics while classical correlations become constant with a non-zero value. This fascinating behavior was named the phenomenon of sudden transition from classical to quantum decoherence. These intriguing behaviors of freezing and sudden transition were further explored theoretically and verified experimentally.
[19–23]. Consequently, it was demonstrated that the type of initial states which exhibit such as non-trivial behaviors rely upon the feature of the decohering environment being considered. Furthermore, in the literature, some schemes are used to enhance the time-interval in which quantum correlations remain constant and control the appearance of sudden transition time e.g. dynamical decoupled method and non-Markovianity [24–26]. Unfortunately, these controlled schemes have some restrictions, for example, in the short-time interactions, the dynamical decoupling is an inappropriate technique because of the lack of memory issue [27, 28]. Similarly, the simultaneous application of memory effects and dynamical decoupling protocol are harmful for the preservation of quantum coherence [29].

It is worth noting that in all the aforementioned studies environments are assumed to be in equilibrium with stationary statistical properties. On the other hand, there are several situations where non-equilibrium feature plays a vital role in the dynamics of open quantum systems. For instance, in physical systems, the transient and ultra-fast dynamical processes can happen on sufficiently short periods such that the initial non-equilibrium state induced by system-environment coupling may not have the ability to return to equilibrium [30–36]. Moreover, in the recent years, it was shown that non-equilibrium feature of environment with non-stationary statistical properties gives rise to frequency shift which has a dominant effect on the reducing of decoherence effects, quantum speed limits and geometric phase of the quantum systems [37–40]. To the best of our knowledge, phenomena of freezing discord and sudden transition from classical to quantum decoherence in the presence of non-equilibrium environments have not been studied yet. Therefore, several important questions arise: Does the phenomenon of freezing discord followed by the sudden transition between classical and quantum decoherence for the certain Bell diagonal states still appear in the non-equilibrium environments? Is it possible to control and prolong the finite time-interval in which the quantum discord is not affected by the decoherence via the non-equilibrium feature of environments?

To answer these questions, here we study the phenomena of freezing and sudden transition by assuming the case when two identical non-interacting qubits locally coupled to their own dephasing non-equilibrium environments with non-stationary and non-Markovian properties. We show for the first time that the phenomena of freezing and sudden transition still occurs even when the quantum system interacts with a non-equilibrium environment in both weak and strong coupling regimes for certain Bell diagonal states. Interestingly, unlike the application of two-flip dynamical decoupling pulse sequence technique [24–26], here the time interval in which the quantum correlation is unaffected by the decoherence can be further extended only by allowing the environment to deviate from equilibrium. This reflects that the non-equilibrium feature offers an efficient alternative protocol to delay the occurrence of the sudden transition time and combat against the environmental noise. In contrast with the aforementioned studies, we show that the appearance of the second non-decay interval for quantum correlations and the revival of classical correlations depend on both non-Markovianity and non-equilibrium features of the environment. Additionally, depending on the initial state one can observe multiple sudden transitions during the dynamics of quantum and classical correlations due to the memory effects of the equilibrium environment [18]. However, we show that these multiple sudden transitions do not appear in the presence of the non-equilibrium nature of the environment.

The article is organized as follows. Physical model and its exact solution are given in section 2. Section 3 is devoted to the dynamics of mutual information, classical and quantum correlations. Finally, a summary of our results is presented in section 4.

2. The physical model and its solution

The quantum system under consideration consists of two separated, non-interacting qubits A and B, each qubit is independently coupled with their relevant non-equilibrium dephasing environments. In this scenario, the time evolution of the bipartite quantum system can be calculated straightforwardly from the single qubit’s dynamics [41]. Therefore, first, we focus on the time-evolution of qubit A.

Let us consider a two-level quantum system i.e. qubit A, coupled to a pure dephasing environment with the non-equilibrium feature. The environmental effects induce stochastic fluctuations in the intrinsic frequency of the qubit i.e. \( \omega(t) = \omega_0 + \eta(t) \) [42, 43]. Here, the parameter \( \omega_0 \) represents the intrinsic transition frequency between excited \( |e\rangle \) and ground \( |g\rangle \) states while \( \eta(t) \) characterizes the environmental dichotomic noise with both non-Markovian and non-stationary properties. Consequently, the Hamiltonian of the single qubit (A) can be written as [42–45]

\[
H_A(t) = \frac{\hbar}{2} (\omega_0 + \eta(t)) \sigma_A^z.
\]  

(1)

where, \( \sigma_A^z \) denotes the Pauli operator acting on the subspace of qubit A. Dynamics of the total system i.e. subsystem A plus environment, can be described by the well-known Liouville master equation
\[ \frac{\partial}{\partial t} \rho(t; \eta(t)) = -\frac{i}{\hbar} [H^A(t), \rho(t; \eta(t))]. \]  

(2)

Here, \( \rho(t; \eta(t)) \) represents the total density matrix which depends on the non-stationary non-Markovian noise induced by the non-equilibrium environment i.e. \( \eta(t) \). Elements of the total density matrix in the basis \( \{|e\}, \{|g\} \} \), obey the following stochastic differential equations

\[ \frac{\partial}{\partial t} \rho_{ee}(t; \eta(t)) = 0, \]
\[ \frac{\partial}{\partial t} \rho_{gg}(t; \eta(t)) = i[\omega_0 + \eta(t)] \rho_{ge}(t; \eta(t)), \]

(3)

with \( \rho_{gg}(t; \eta(t)) = 1 - \rho_{ee}(t; \eta(t)) \) and \( \rho_{ge}(t; \eta(t)) = \rho_{eg}(t; \eta(t)) \). Now by taking the integration of equation (3), we can obtain

\[ \rho_{ee}(t; \eta(t)) = \rho_{ee}(0; \eta(0)), \]
\[ \rho_{ge}(t; \eta(t)) = \exp\left[i\omega_0 t + \int_0^t \eta(t) \, dt\right] \rho_{ge}(0; \eta(0)). \]

(4)

We suppose that initially there is no correlation between the qubit A and its corresponding non-equilibrium environment. To obtain the reduced density matrix for qubit A, one needs to take statistical mean over environmental noise \( \eta(t) \) i.e. \( \rho^A(t) = \langle \rho(t; \eta(t)) \rangle = \int d\eta(t) \rho_{AA}(t \eta(t) P(\eta, t), \) where \( \rho_{AA}(t) = \rho(t; \eta(t)) \) defines the conditional density matrix when the stochastic process takes the value \( \eta \) at time \( t \) with probability \( P(\eta, t) \)[39]. Hence the final reduced density matrix can be written in the form

\[ \rho^A(t) = \begin{pmatrix} \rho_{ee}(0) & \rho_{ge}(0) e^{-i\omega_0 t F^A(t)} \\ \rho_{ge}(0) e^{-i\omega_0 t F^A(t)} & \rho_{gg}(0) \end{pmatrix}, \]

(5)

The parameter \( F^A(t) = \left\langle \exp\left[i \int_0^t \eta(t) \, dt\right] \right\rangle \) represents the decoherence factor of qubit A and (…) shows the stochastic means over the environmental noise \( \eta(t) \). Here the environmental noise \( \eta(t) \) is modeled by a non-stationary and non-Markovian random telegraph process which can be described by the non-equilibrium parameter \( \alpha \) and memory damping rate \( \kappa \), respectively. The amplitude of this noisy process randomly jumps between \( \pm \epsilon \) and \( -\epsilon \) with switching rate \( \Omega \). Under this non-equilibrium dephasing environment, the exact analytical result of the decoherence factor for the qubit A is given by (for the details calculations see appendix [37, 39])

\[ F^A(t) = \mathcal{L}^{-1}[F^A(s)], \]

(6)

with

\[ F^A(s) = \frac{s^2 + (\kappa + i\alpha \epsilon)s + \kappa (2\Omega + i\alpha \epsilon)}{s^3 + \kappa s^2 + (2\kappa \Omega + \epsilon^2)s + \kappa \epsilon^2}, \]

(7)

where \( \mathcal{L}^{-1} \) represents the inverse Laplace transformation. Unlike the equilibrium case, here decoherence factor is a complex time-dependent function because of the non-stationary nature of the noise. Interestingly, for \( a = 0 \), the environment becomes in equilibrium and \( \eta(t) \) returns to stationary dichotomic noise.

At this point, it is worth to mention that the above model was used to study the dynamical dephasing of the two-level system subjected to non-equilibrium environment with non-stationary non-Markovian noise [37, 39]. Particularly, it was showed that non-equilibrium feature can effectively suppress the dephasing effect in the quantum system and also decreases the coherent backaction from surroundings. This motivated us to extend the aforementioned model to a two-qubit system and investigate the role of non-equilibrium nature on the intriguing phenomena of freezing discord and sudden transition in decay rates of classical and quantum correlations (see section 3). Therefore, by using the method discussed in [41], one can easily construct the reduced density matrix for two qubits \( \rho^{AB}(t) \) in the standard product basis \( \{|e_1e_2\}, \{|e_1g_2\}, \{|g_2e_2\}, \{|g_2g_2\}\} \) as,

\[ \rho^{AB}(t) = \begin{pmatrix} \rho^{11}_{ee}(0) & \rho^{12}_{ee}(0) & \rho^{13}_{ee}(0) & \rho^{14}_{ee}(0) \\ \rho^{12}_{ee}(0) & \rho^{22}_{ee}(0) & \rho^{23}_{ee}(0) & \rho^{24}_{ee}(0) \\ \rho^{13}_{ee}(0) & \rho^{23}_{ee}(0) & \rho^{33}_{ee}(0) & \rho^{34}_{ee}(0) \\ \rho^{14}_{ee}(0) & \rho^{24}_{ee}(0) & \rho^{34}_{ee}(0) & \rho^{44}_{ee}(0) \end{pmatrix}. \]

(8)

This reduced density matrix characterizes the dynamics of two qubits quantum system where \( F^A(t) (F^B(t)) \) represents dephasing factor of qubit A(B). Furthermore, we assume that both qubits A and B are initially in the class of states with maximally mixed marginals \( (\rho^{AB} = I^{AB}/2) \), called Bell-diagonal states i.e.
\[ \rho^{AB}(0) = \frac{1}{4} \left( I^{AB} + \sum_{j=1}^{3} \sigma_j^A \sigma_j^B \right), \]  

where, \( I^{AB} \) is the identity operator of the composite system and \( \sigma_j^{A(B)} \) denotes \( j \)th component of the Pauli operator acting on subspace \( A(B) \). Similarly, \( c_j \) shows the initial states (\( \rho^{AB}(0) \)) parameters, taking real values such that \( 0 \leq |c_j| \leq 1 \). Interestingly, this class of initial states contain both the Werner (\( |c| = |c| = c \)) and the Bell (\( |c| = |c| = |c| = 1 \)) states.

### 3. Dynamics of quantum and classical correlations

Before starting to present our main results, let us first briefly introduce the quantum and classical correlations. The total amount of correlations (including both quantum and classical) in quantum system can be quantified by quantum mutual information. Particularly, if \( \rho^{AB} \) represents the density matrix of a composite quantum system and \( \rho^{A}(\rho^{B}) \) characterizes the reduced density matrix of the subsystem \( A(B) \), then quantum mutual information can be calculated as \( I(\rho^{AB}) = S(\rho^{A}) + S(\rho^{B}) - S(\rho^{AB}) \), with \( S(\rho) = -Tr(\rho \log_2 \rho) \) the von Neumann entropy and the index \( \text{i} \) shows either the the composite system \( AB \) or subsystem \( A(B) \). On the other hand, the classical correlations in two-qubit system \( \rho^{AB} \) can be measured as, \( C(\rho^{AB}) = S(\rho^{A}) - \min_{\{I_A\}} \{S(\rho_{AB}^{A}(I_B))\} \). Here, minimum is taken over the entire set of projective measurements \( \{I_A\} \), and \( S(\rho_{AB}^{A}(I_B)) \) denotes the conditional entropy of the subsystem \( A \), given the information about subsystem \( B \), with \( I_B = \text{Tr}(\rho_{AB}^{A}(I_B))/P_B \) and \( P_B = \text{Tr}(\rho^{AB}I_B) \). Now, the quantum discord that quantifies the amount quantum correlation in bipartite system can be written as \( D(\rho^{AB}) = I(\rho^{AB}) - C(\rho^{AB}) \).

Here, we investigate the time evolution of the quantum and classical correlations in the two-qubit system which is initially in the Bell diagonal states \( \rho^{AB}(0) \). Therefore, the density matrix \( \rho^{AB}(t) \) given by equation (8) can also be written in the form [17]

\[ \rho^{AB}(t) = \frac{1}{4} \left( \mu^{A}(t) + \mu^{B}(t) \right) + \mu^{A}(t) + \mu^{B}(t) + \mu^{A}(t) + \mu^{B}(t), \]  

with

\[ \mu^{A}(t) = \frac{1}{4} \left[ 1 + \alpha(t) + \alpha(t) + \alpha(t) \right], \]  

\[ \mu^{B}(t) = \frac{1}{4} \left[ 1 + \alpha(t) + \alpha(t) + \alpha(t) \right], \]  

where, \( |\psi^+\rangle = \frac{1}{\sqrt{2}} (|00\rangle \pm |11\rangle) \) and \( |\phi^\pm\rangle = \frac{1}{\sqrt{2}} (|01\rangle \pm |10\rangle) \) are the well-known Bell states. The time dependent parameters in equations (11) and (12) are \( \alpha(t) = \alpha(t) = \alpha(t) = \alpha(t) = \alpha(t) = \alpha(t) \). It is important to note that \( \alpha(t) \) and \( \alpha(t) \) exhibit various dynamical behaviors depending on the dephasing factors of both qubits while \( \alpha(t) \) is constant through the dynamics. Based on equations (10)–(12), the analytical expressions for the quantum mutual information \( I(\rho^{AB}) \), and the classical \( C(\rho^{AB}) \) correlations are given by [46]

\[ I(\rho^{AB}) = 2 + \sum_{lm} (\mu^{A}(t) \log_2 (\mu^{B}(t))), \]  

and

\[ C(\rho^{AB}) = \sum_n 1 + \frac{(-1)^n \beta(t)}{2} \log_2 [1 + (-1)^n \beta(t)], \]  

with, \( \beta(t) = \max(|\alpha(t)|, |\alpha(t)|, |\alpha(t)|) \) and the index \( m = \psi, \phi \) while \( l = \pm \). The quantum correlation is the difference between mutual information and classical correlation i.e. \( D(\rho^{AB}) = I(\rho^{AB}) - C(\rho^{AB}) \).

Now first, we focus on initial family of states for which \( c_1 = \pm 1 \) and \( c_2 = 0 \) with \( |c| < 1 \). For these initial states equation (13) can take the following form [17, 18]

\[ I(\rho^{AB}) = 2 - \sum_n 1 + (-1)^n c_1 \log_2 [1 + (-1)^n c_1] \]

\[ + \sum_n 1 + (-1)^n c_2 \log_2 [1 + (-1)^n c_2]. \]  

It is quite obvious from equations (14) and (15) that the dynamical behaviors of mutual information, classical and quantum correlations depend on the parameters \( |\alpha(t)|, |\alpha(t)| \) and \( |\alpha(t)| \). For example, initially \( |\alpha(t)| > |\alpha(t)| \), \( |\alpha(t)| \), hence \( \beta(t) \) is equal to \( |\alpha(t)| \), resulting in decaying of classical correlations (see equation (14)) while at the same time quantum discord remains constant which is given by first term in equation (15). After a certain time where \( |\alpha(t)| = |\alpha(t)| = |\alpha(t)| \), a sudden change occurs at this point in the dynamics of correlations. One immediately sees after this sudden transition point that \( |\alpha(t)| = |\alpha(t)| = c_2 \) gets bigger than \( |\alpha(t)| \) and \( |\alpha(t)| \), therefore now \( \beta(t) \) is equal to \( c_3 \) which is constant in time. Hence, in this regime, classical
correlations become constant while mutual information and quantum correlations start decaying which can be seen straightforwardly in equation (14) and (15). Next, we shall study how the non-equilibrium feature of the environment can prolong the finite time-interval for the frozen discord and harness the sudden transition time.

As was demonstrated, the non-equilibrium nature of the environment can suppress the dephasing effects in a quantum system and also decrease the coherent backaction from the environment \([37, 39]\). Therefore, in light of these results, we show that the decay rates of the parameters \(\alpha_1(t)\) and \(\alpha_2(t)\) can be reduced by a non-equilibrium feature which results in enhancing the finite time-interval over which the quantum discord remains constant and also can control the sudden transition time. Specifically, in figure 1, we display the dynamics of mutual information, classical and quantum correlations for various values of non-equilibrium parameter \(a\) in weak coupling (non-Markovian) regime i.e. \(\epsilon/\Omega < 1\), with initial state parameters \(c_1 = 1, c_2 = -0.6\) and \(c_3 = 0.6\). It can be seen that when the environment starts deviation from the equilibrium i.e. \(a\) departs from zero, the decay rates of mutual information and classical correlations slow down as illustrated by the green dashed and red dotted–dashed curves in figure 1. Consequently, the time interval over which quantum

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**Figure 1.** The dynamics of mutual information (dashed green curves), classical correlation (dotted–dashed red curves), and the quantum correlation (solid blue curves) for different values of non-equilibrium parameters: (a) \(a = 0\), (b) \(a = \pm 0.7\) and (c) \(a = \pm 1\), in the weak coupling regime \(\epsilon/\Omega = 0.8\) with damping rate \(\kappa = \Omega\) and initial state \(c_1 = 1, c_2 = -0.6, c_3 = 0.6\). In each panel, the vertical dashed gray lines manifest the transition times.
correlations measured by quantum discord remains frozen can further prolong as the environment moves away from the equilibrium i.e. $\epsilon < \Omega^\pm_{\epsilon=0} < \Omega^\pm_{\epsilon=\pm0.7} < \Omega^\pm_{\epsilon=\pm1}$, as shown by the positions of vertical dashed gray lines in figures 1(a)–(c), respectively. This implies that the non-equilibrium feature of the environments plays a constructive role in suppressing the decay rates of mutual information, classical and quantum correlations. Furthermore, it also reflects the non-equilibrium nature provides an efficient tool to control the occurrence of sudden transition in the behavior of the quantum correlation and its classical counterpart.

Next, for the same initial state, in figure 2, we display the result for mutual information, classical and quantum correlations in the strong coupling (non-Markovian) regime i.e. $\epsilon/\Omega > 1$, with different values of the non-equilibrium parameter ($a = 0, \pm0.7, \pm1$). Comparing the panels (a)–(c), in figure 2, one can see a finite period in which quantum discord is completely shielded from decoherence even in the presence of non-equilibrium environment with non-Markovian and non-stationary characteristics. Subsequently, both quantum correlation and its classical counterpart exhibit the phenomenon of sudden transition in their decay rates. Similarly, in the strong coupling regime we also found that duration of the time interval over which quantum correlation remains constant can be extended by increasing the value of non-equilibrium parameter.
and resulting in the delay of sudden transition time as shown by the positions of vertical dashed gray lines in figures 2(a), (b) and (c). Furthermore, in case of equilibrium environment i.e. $a = 0$, after the sudden transition, the classical correlation becomes constant while mutual information and quantum correlation start damping oscillations in the same fashion as displayed by dotted–dashed red, dashed green and blue solid curves, respectively, in figure 2(a). Interestingly, the non-monotonic oscillatory feature of the mutual information and quantum discord decreases as the environment diverges from the equilibrium. Consequently, quantum correlation revives without the dark periods when the environment moves away from the equilibrium as illustrated by the solid blue curves in figures 2(b) and (c). This means that the non-equilibrium property of the environment can minimize the non-Markovian feature in the dephasing dynamics of the composite quantum system and also may suppress coherent backaction from the surrounding environment.

Moreover, if the dephasing environment is far away from the equilibrium i.e. $a = \pm 1$, then we observe a second finite period over which quantum discord is completely protected from the decay. As a result one can see another sudden change in the dynamical behavior of the classical and quantum correlations as displayed by the two vertical dashed purple lines in figure 2(c). This implies that the occurrence of the second non-decay time interval of quantum discord and the phenomenon of multiple sudden changes from classical to quantum decoherence not only depend on the memory effect but also the value of the non-equilibrium parameter.

Furthermore, in figure 3, we depict the dynamics of mutual information, classical and quantum correlation in the strong coupling (non-Markovian) regime for the different initial condition $c_1 = 0.35, c_2 = -0.3, c_3 = 0.1$, with different values of the non-equilibrium parameter ($a = 0, a = \pm 0.7$ and $a = \pm 1$). Particularly, in the non-Markovian regime when the environment is in equilibrium i.e. $a = 0$, we can observe multiple sudden transition points in the dynamical behavior of the classical and quantum correlations without the frozen discord which coincides with the results obtained in [18], as displayed in figure 3(a). In [18], the authors showed that these multiple transitions depend on the geometry of the initial states and memory effect of the environments. On the other hand, for the same initial condition, we found no multiple transitions phenomenon in the dynamics of quantum and classical correlations when the environment starts deviation from the equilibrium as illustrated in figures 3(b) and (c). The reason for this is that the non-equilibrium behavior of the environments reduces the non-Markovianity in system dynamics [37]. Nevertheless, the decay rates of the correlations can be significantly reduced due to the non-equilibrium feature as shown in figures 3(b) and (c).

4. Conclusions

In conclusion, we studied the evolution of classical and quantum correlations between two qubits, each qubit is independently subjected to their own phase damping non-equilibrium environments with both non-Markovian and non-stationary statistical features. For a specific family of initial states, we have shown that these correlations exhibit the freezing and sudden transitions dynamical behaviors even in the presence of the non-equilibrium environment. Interestingly, the finite time period over which the quantum correlations are not destroyed by the decoherence can be further increased as the environment departs from the equilibrium without the bang-bang pulses or any other protocol. This means that the non-equilibrium parameter offers an alternative and efficient tool to delay the occurrence of a sudden transition in the decay rates of correlations and their immunity towards decoherence. It is worth mentioning that, for certain initial condition in strong coupling regime, we have observed a second non-decay interval for quantum correlation and also the revival of classical correlations which depend on both the non-Markovian and non-equilibrium features of the environment. Furthermore, for certain initial states, we also discussed the physical reason for the absence of multiple sudden transitions when the quantum system is driven by the non-equilibrium environment. We would like to comment that as recently, in some experiments, the non-equilibrium nature of the dichotomic environmental noise has been observed [47, 48]. Therefore, it will be quite interesting to experimentally realize our study which may be helpful in the in-depth understanding of these peculiar phenomena and the advancement of the quantum information and computation science.

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Appendix

Here we present some details calculations leading to equations (6) and (7). The dynamical evolution of the conditional probability for the non-stationary and non-Markovian dichotomic noise process is governed by the following master equation [49]

\[
\frac{\partial}{\partial t} P(\epsilon, \eta; t) = -\int_{t'}^{\epsilon} d\tau K(t - \tau)\left[\Omega P(\epsilon, \eta; t') - \Delta P(-\epsilon, \eta; t')\right],
\]

\[
\frac{\partial}{\partial t} P(-\epsilon, \eta; t) = \int_{t'}^{\epsilon} d\tau K(t - \tau)\left[\Omega P(-\epsilon, \eta; t') - \Delta P(\epsilon, \eta; t')\right],
\]

(A.1)

with initial condition \( P(\eta; t|\eta'; t') = \delta_{\eta,\eta'} \) for \( \eta = \pm \epsilon \). Here \( K(t) = \kappa e^{-\kappa t} \) is generally used an exponential form of the memory kernel with damping rate \( \kappa \). By solving the above equation (A.1), we can obtain the conditional probability as

Figure 3. Time evolution of mutual information (dashed green curves), classical correlation (dotted–dashed red curves), and the quantum correlation (solid blue curve), for different values of non-equilibrium parameters: (a) \( a = 0 \), (b) \( a = \pm 0.7 \) and (c) \( a = \pm 1 \), in the strong coupling regime with \( \epsilon/\Omega = 3 \) with damping rate \( \kappa = \Omega \) and initial state \( c_1 = 0.35, c_2 = -0.3, c_3 = 0.1 \). In panel (a), the arrow represents the sudden transition point.
\[ P(\eta, t; \eta', t') = \frac{1}{2} \left\{ 1 - \frac{1}{2\Lambda_1} [\Lambda_2 e^{-\Lambda_2(t-t')} - \Lambda_3 e^{-\Lambda_3(t-t')} ] \right\} \delta_{\eta, \eta'} \\
+ \frac{1}{2} \left\{ 1 + \frac{1}{2\Lambda_1} [\Lambda_2 e^{-\Lambda_2(t-t')} - \Lambda_3 e^{-\Lambda_3(t-t')} ] \right\} \delta_{\eta, \eta'}, \tag{A.2} \]

with \( \Lambda_1 = \sqrt{\frac{k^2}{4} - 2\kappa \Omega}, \Lambda_2 = \frac{1}{2} \kappa + \Lambda_1 \) and \( \Lambda_3 = \frac{1}{2} \kappa - \Lambda_1 \). The single-time probability distribution with non-stationary feature obeys

\[ P(\eta, t) = \frac{1}{2} \left\{ 1 - \frac{a}{2\Lambda_1} (\Lambda_2 e^{-\Lambda_2 t} - \Lambda_3 e^{-\Lambda_3 t}) \right\} \delta_{\eta, \eta'} \\
+ \frac{1}{2} \left\{ 1 + \frac{a}{2\Lambda_1} (\Lambda_2 e^{-\Lambda_2 t} - \Lambda_3 e^{-\Lambda_3 t}) \right\} \delta_{\eta, \eta'}, \tag{A.3} \]

with initial non-stationary distribution of \( \eta(t) \) is \( P(\eta_0, 0) = \frac{1}{2}(1-a) \delta_{\eta_0, \eta} + \frac{1}{2}(1+a) \delta_{\eta_0, \epsilon} \) and \( |a| \leq 1 \) denotes the non-equilibrium parameter.

Decoherence factor \( F^A(t) \) can be characterized via the Dyson series expansion as

\[ F^A(t) = \left\{ \exp \left[ i \int_0^t \eta(t') dt' \right] \right\} = 1 + \sum_{n} i^n \int_0^{t_1} dt_1 \int_0^{t_2} dt_2 \cdots \int_0^{t_n} dt_n \langle \eta(t_1) \cdots \eta(t_n) \rangle, \tag{A.4} \]

with time-ordering \( t > t_1 > \cdots > t_n > 0 \). For this type of non-stationary non-Markovian noise process, the first and second may calculate as

\[ \langle \eta(t) \rangle = \frac{a \epsilon}{2\Lambda_1} (\Lambda_2 e^{-\Lambda_2 t} - \Lambda_3 e^{-\Lambda_3 t}) \]

\[ \langle \eta(t) \eta(t) \rangle = \frac{\epsilon^2}{2\Lambda_1} (\Lambda_2 e^{-\Lambda_2 (t-t_1)} - \Lambda_3 e^{-\Lambda_3 (t-t_1)}). \tag{A.5} \]

Based on Bayes’ theorem \cite{49, 50}, one can easily show that the higher-order moments satisfy the recurrence relation

\[ \langle \eta(t) \cdots \eta(t_n) \rangle = \langle \eta(t) \eta(t_1) \rangle \langle \eta(t_2) \cdots \eta(t_n) \rangle, \tag{A.6} \]

for all the time instants \( t > t_1 > \cdots > t_n > 0 \) such \( n \geq 2 \).

For the non-stationary non-Markovian noise dichotomic noise environment, we can obtain an exact analytical expression with the help of a closed differential equation of \( F^A(t) \). To this aim, first, we combine equations (A.5) and (A.6) and then differentiate with respect to time to get

\[ \frac{\partial^2}{\partial t^2} \langle \eta(t) \cdots \eta(t_n) \rangle = -\kappa \frac{\partial}{\partial t} \langle \eta(t) \cdots (t_n) \rangle - 2\kappa \Omega \langle \eta(t) \cdots (t_n) \rangle, \tag{A.7} \]

for each order moment. Now taking the time-derivative of equation (A.4) and substitute equation (A.7), consequently, we can get the following third-order differential equation for \( F^A(t) \)

\[ \frac{d^3 F^A(t)}{dt^3} + \frac{\kappa}{\epsilon} \frac{d^2 F^A(t)}{dt^2} + (2\kappa \Omega + \epsilon^2) \frac{dF^A(t)}{dt} + \kappa \epsilon^2 F^A(t) = 0, \tag{A.8} \]

with \( F^A(0) = 1, \frac{dF^A(0)}{dt} = i \epsilon \) and \( \frac{d^2 F^A(0)}{dt^2} = -\epsilon^2 \) are the initial conditions. The decoherence factor in equation (A.8) contains all the information about the environmental noise \( \eta(t) \), without any approximation. Therefore, the exact analytical solution of the above equation (A.8) can be obtained by using the Laplace transformation (see equations (6) and (7), in section 2).

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