Gap soliton formation by nonlinear supratransmission in Bragg media

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A Bragg medium in the nonlinear Kerr regime, submitted to incident cw-radiation at a frequency in a band gap, switches from total reflection to transmission when the incident energy overcomes some threshold. We demonstrate that this is a result of nonlinear supratransmission, which allows to prove that: i) the threshold incident amplitude is simply expressed in terms of the deviation from the Bragg resonance, ii) the process is not the result of a shift of the gap in the nonlinear dispersion relation, iii) the transmission does occur by means of gap soliton trains, as experimentally observed [D. Taverner et al., Opt Lett 23 (1998) 328], iv) the required energy tends to zero close to the band edge.

Introduction. Light propagation in dielectric media with periodically varying index acquires intriguing properties when Kerr nonlinearities come into play. From a linear theory we know that such a medium is a photonic band gap structure, or Bragg medium, which is totally reflective when the frequency of the incoming wave falls in one of the gaps.

However, for sufficiently energetic irradiation, the Kerr effect becomes sensible and the Bragg medium may become transparent in the gap, switching from total reflection to high transmissivity, which has been predicted in 1979 by using the stationary coupled mode approach of [8], and which has been experimentally realized for the first time in 1992 [4].

In 1987 the existence of effective wave propagation in a gap has been associated to the existence of gap solitons derived from slowly varying envelope limits of the Maxwell equation in a periodic structure. For unidirectionnal propagation, fast Kerr nonlinearity and slowly varying envelope approximation, one of these limit models is the coupled mode system which describes the evolution of the envelopes of the contrapropagating electric fields. A rigorous derivation of from the anharmonic Maxwell-Lorentz equations by the method of multiple scales is given in [8]. The explicit soliton-like solution of the coupled mode equation has been discovered in 1989 [9], and at the same time generalized to a two-parameter family in [10]. This two-parameter gap soliton solution, that we use here, has motivated many experimental searches of the Bragg soliton.

After the pioneering experiments of in steady-state regime, the first experimental observations of nonlinear pulse propagation in a fiber Bragg grating have been performed in 1996 under laser pulse irradiation at a frequency near (but not inside) the photonic band gap. In that case the nonlinearity acts as a source of modational instability generating trains of pulses. Particularly interesting is the observation of pulse propagation at a velocity less than the light velocity. Later in 1998 a convincing experiment with a fiber Bragg grating demonstrated the repeated formation of gap soliton under quasi-constant wave irradiation inside a band gap, this is the problem we are interested in.

There has been a great deal of discussions concerning the exact role of the gap soliton in the switching to high transmissivity, see e.g. [14]. A rather natural intuition is that the nonlinearity shifts the gap and hence allows the medium to become transparent and to create a gap soliton. We shall discover that this intuition is wrong as the process at the origin of high transmissivity in Bragg media is the nonlinear supratransmission recently discovered in the sine-Gordon chain and experimentally realized (together with application to Josephson junctions arrays) in [17]. As a consequence, the switching to high transmissivity is indeed accomplished by means of gap soliton generation as a result of a fundamental instability which are effectively the objects experimentally detected in the output in [13].

This result allows us to predict analytically by formula the threshold of incident energy of a cw-radiation above which high transmissivity is reached, as an explicit simple function of the departure (denoted by the dimensionless angular frequency Ω) from the Bragg frequency (gap center). In particular we obtain that the process requires less energy for frequencies close to the band edge Ω = 1, a result which was previously attributed to pulse shape properties. Interestingly enough, in the limit Ω → 1 the required energy vanishes, opening the way to experimental realizations.

The model and its solution. Our starting point is the basic coupled mode system governing the forward E_f and backward E_b slowly varying envelopes of the electric field

\[ E(Z,T) = [E_f e^{ik_0 Z} + E_b e^{-ik_0 Z}] e^{-i\omega_0 T}. \]  

(1)

E(Z,T) is the transverse polarized component propagating in the direction Z at frequency close to the Bragg frequency ω0. Following we write the coupled mode
system in reduced units as
\[
\begin{align*}
&i\left[\frac{\partial e}{\partial t} + \frac{\partial e}{\partial z}\right] + f + \left(\frac{1}{2}|e|^2 + |f|^2\right)e = 0 , \\
&i\left[\frac{\partial f}{\partial t} - \frac{\partial f}{\partial z}\right] + e + \left(\frac{1}{2}|f|^2 + |e|^2\right)f = 0 .
\end{align*}
\]
(2)
The reduced units are \( z = \kappa Z \) and \( t = \kappa \tau \) where \( c \) is the light velocity in the medium and \( \kappa \) the coupling constant. The reduced field variables are \( e = E f \sqrt{2V / \kappa} \) and \( f = E_b \sqrt{2V / \kappa} \) where \( \Gamma \) is the nonlinear factor. The (linear) dispersion relation of \( \mathfrak{R} \), namely \( \omega^2 = 1 + k^2 \), possess the gap \([-1, +1]\).

The solitary wave solution of \( \mathfrak{R} \) given in \( \mathfrak{E} \) has a simple stationary expression
\[
\begin{align*}
e &= \sqrt{2/3} \lambda e^{-i((\Omega - \phi)z - \frac{i}{2}q)} , \\
f &= -\sqrt{2/3} \lambda e^{-i((\Omega - \phi)z + \frac{i}{2}q)} .
\end{align*}
\]
(3) (4)
The real valued parameters determining this ‘gap soliton’ are \( \Omega \) (frequency), \( \phi \) (initial phase) and \( z_0 \) (center), and we have
\[
\Omega^2 = 1 - \lambda^2 , \quad \Omega = \cos q .
\]
(5)
Then \( \lambda \) plays the role of the ‘wave number’ of the evanescent wave.

Nonlinear supratransmission threshold. For a Bragg medium, extending in the region \( z \in [0, L] \), and initially in the dark, we set the initial data
\[
\begin{align*}
e(z, 0) &= 0 , \quad f(z, 0) = 0 ,
\end{align*}
\]
(6)
The boundary value problem that mimics the scattering of an incident radiation at frequency \( \Omega_0 + \kappa \Omega \) on the medium in \( z = 0 \) is
\[
\begin{align*}
e(0, t) &= A e^{-i\Omega t} , \quad f(L, t) = 0 ,
\end{align*}
\]
(7)
where the second requirement means no backward wave incident from the right in \( z = L \). The constant \( A \) (in general complex valued) is the dimensionless amplitude of the incoming radiation. Equations \( \mathfrak{E} \) and \( \mathfrak{D} \) constitute a well posed initial-boundary value problem for the PDE \( \mathfrak{B} \).

In the linear case such a boundary forcing would simply generate the evanescent wave \( e(z, t) = A e^{-i\Omega t - \lambda z} \). In the nonlinear case however this boundary forcing generates the solution \( \mathfrak{E} \) for the value of \( \phi \) and \( z_0 < 0 \) such that the amplitude in \( z = 0 \) be precisely \( A \). Then, for each fixed forcing frequency \( \Omega \), there exists a maximum value \( A_s \) of \( A \) beyond which there is no solution \{ \( \phi, z_0 \) \}. This threshold \( A_s \) is given by \( |e(0, t)| \) from \( \mathfrak{E} \) evaluated at \( z_0 = 0 \), namely
\[
A_s = 2\sqrt{2/3} \sin\left(\frac{1}{2} \arccos \Omega \right) .
\]
(8)
This is the threshold amplitude of the incident envelope above which the system develops an instability and generates a propagating nonlinear mode.

FIG. 1: Plots of the amplitude \(|e(z, t)|\) of the right-going envelope at given time (\( t = 145 \)) for two values of the driving amplitude \( A \) at frequency \( \Omega = 0.8 \).

Nonlinear dispersion relation prediction. By seeking a solution of \( \mathfrak{E} \) \( e = a \exp[i(kz - \omega t)], \quad f = b \exp[i(kz - \omega t)] \) one gets a nonlinear algebraic system for the unknowns \( a \) (incident amplitude) and \( \omega \) (incident frequency) expressed in terms of the wave number \( k \) once the amplitude \( b \) has been eliminated. This gives the nonlinear dispersion relation \( \omega(a, k) \) as a solution of a third order algebraic equation. One of the 3 solutions must be discarded as it is singular in the linear limit \( a \to 0 \), the other two giving the deformations of the linear branches \( \omega(k) = \pm \sqrt{1 + k^2} \). It can be shown that they correspond to the relations \( b = -a \) (upper branch) and \( b = a \) (lower branch).

Being interested in the shift of the gap, it is sufficient to study the solutions at \( k = 0 \) where the system simplifies, and to stick with the upper branch (as we look at right-going incident waves) for which \( b = -a \). Then \( \mathfrak{K} \) readily provides the value \( \omega(0, a) = 1 - \frac{4}{9} a^2 \) of the gap opening.

For our purpose it is more convenient to express the value \( A_m \) of the incident amplitude \( a \) for which the incident frequency \( \Omega \) touches the nonlinear gap edge \( \omega(0, a) \), namely
\[
A_m = \sqrt{\frac{2}{3}(1 - \Omega)} .
\]
(9)
This formula provides a prediction of transparency by nonlinear shift of the gap. It is now compared to our prediction \( \mathfrak{U} \) by means of numerical simulations.

Numerical simulations. The system \( \mathfrak{B} \) is solved by means of an semi-implicit third order finite difference scheme and boundary values are taken into account at first order. Explicitly we first rewrite \( \mathfrak{B} \) for two functions \( u(z, t) \) and \( v(z, t) \) and replace the operators \( \partial_t, \partial_z \) by the set of differences for \( u_n(t) = u(nh, t) \)
\[
\begin{align*}
\dot{u}_1 + \frac{1}{h}(u_1 - u_0) , \quad &\dot{u}_2 + \frac{1}{2h}(u_2 - u_0) , \quad \cdots \\
\dot{u}_n + \frac{1}{h}\left[\frac{2}{3}(u_{n+1} - u_{n-1}) - \frac{1}{12}(u_{n+2} - u_{n-2})\right] , \quad \cdots \\
\dot{u}_{N-1} + \frac{1}{2h}(u_{N+1} - u_{N-1}) , \quad &\dot{v}_N + \frac{1}{h}(u_N - u_{N-1}) ,
\end{align*}
\]
so as for \( v_n(t) = v(nh, t) \). The length is \( L = hN \) and overdot means time differentiation. Equation \( \mathfrak{B} \) results as a system of 2N coupled ODE then solved through the
The subroutine `dsolve` of the MAPLES software package that uses a Fehlberg fourth/fifth order Runge-Kutta method. Finally the solution, e.g. \( e(z, t) \), is obtained from \( u_n(t) \) by
\[
e(z, t) = \frac{1}{2} [u_{n+1}(t) + u_n(t)]
\]
Although such code sends some numerical noise in the solution, it is ineffective for sufficiently small \( h \) depending on the required time of integration.

Next, in order to avoid initial shock, the boundary condition \( e(0, t) \) is smoothly turned on and smoothly turned off by assuming instead of \( e(0, t) = 0 \)
\[
e(0, t) = \frac{A}{2} \left[ \tanh(p(t - t_0)) - \tanh(p(t - t_1)) \right].
\]
A practical interest of such an incident wave is that it reproduces the quasi-constant wave irradiation of the experiments.

Most of the results presented here are obtained with \( N = 120 \) spatial mesh points, \( h = 0.05 \) grid spacing over (hence a length \( L = 6 \) normalize units) a time of integration \( t_m = 200 \). The parameters of the boundary field are a real-valued amplitude \( A \), a slope \( p = 0.2 \), an ignition time \( t_0 = 20 \) and an extinction time \( t_1 = 180 \).

We display in figure 1 an instance of two different numerical solutions (the modulus of the right-going envelope) for an incident frequency \( \Omega = 0.8 \) and amplitudes \( A = 0.51 \) (no transmission) and \( A = 0.52 \) (gap soliton generation) when the theoretical threshold predicted by \( 8 \) is \( A_s = 0.5164 \).

Using this simple diagnostic, the bifurcation predicted by expression \( 8 \) is numerically checked for a series of frequency values (in the range \( [0.1, 0.995] \)) and we obtain the figure 2 where the dots represent the smallest value (with absolute precision of \( 10^{-2} \)) of the amplitude \( A \) for which nonlinear supratransmission is seen to occur. The expression \( 8 \) is also plotted (dashed line) which shows that the nonlinear shift provides a wrong answer.

**FIG. 2:** Values of the threshold of nonlinear supratransmission observed on numerical simulations (crosses) as compared with expression \( 8 \) (solid line) and with \( 9 \) (dashed line).

**Bifurcation of transmitted energy.** The system (11) possess the conservation law
\[
\partial_t(|e|^2 + |f|^2) + \partial_z(|e|^2 - |f|^2) = 0.
\]
For a given boundary condition \( 11 \) (i.e. for fixed \( A \) and \( \Omega \)), we define the incident \( (I) \), reflected \( (R) \) and transmitted \( (T) \) energies at given arbitrary time \( t_m \) by
\[
I(A, \Omega) = \int_0^{t_m} dt |e(0, t)|^2,
\]
\[
R(A, \Omega) = \int_0^{t_m} dt |f(0, t)|^2,
\]
\[
T(A, \Omega) = \int_0^{t_m} dt |e(L, t)|^2.
\]
Then, upon integration of (11) on the length \( [0, L] \) and time \( [0, t_m] \) we obtain
\[
R + T - I + \int_0^L dz (|e(z, t_m)|^2 + |f(z, t_m)|^2) = 0.
\]
If \( t_m \) is larger than the irradiation duration, the energy injected eventually radiates out completely, namely \( e(z, t_m) \) and \( f(z, t_m) \) vanish, and we are left with the conservation relation \( R + T = I \) which can be written
\[
\rho(A, \Omega) + \tau(A, \Omega) = 1, \quad \rho = R/I, \quad \tau = T/I.
\]

The figure 3 shows a typical numerical simulation where, for \( \Omega = 0.95 \), we have computed the reflection and transmission factors, together with their sum, for 50 different values of the amplitude \( A \) in the range \([0.20, 0.45]\). These simulations show the sudden energy flow through the medium, as a result of nonlinear supratransmission. Note from the sum \( \rho + \tau \) before the bifurcation on figure 3 that the numerical code conserves the energy with a precision of \( 2\% \).

The transmissivity is due to the generation and propagation of light pulses shown figure 4 representing the energy density \( |e(L, t)|^2 \) measured at the output as a function of time. We have also plotted the input energy density given in 10 with parameters \( t_m = 100, t_0 = 20, t_1 = 90 \) and \( p = 2 \). We show now that these are gap solitons travelling at a fraction of the light velocity (slow light pulses).

**Travelling gap solitons.** Although the system (2) is not Lorentz invariant, the propagating solution of (12) can still be written in terms of the boosted variables (13).
As we are interested here only in the energy flux, we write the soliton solution $|e(z, t)|^2$ moving at velocity $v < 1$ with frequency $\cos q$ as

$$|e(z, t)|^2 = \frac{2\sin^2 q}{\cosh\left(\frac{\sin q}{\sqrt{1-v^2}}(z - z_0 - vt) - \frac{q}{2}\right)}.$$ 

Such an expression allows to fit a given simulation by seeking the two parameters $v$ and $q$, and the initial position $z_0$, that reproduce with the explicit solution the results of the numerical simulation.

An instance of such a fit is displayed in figure 5 that shows the analytic soliton $|e(z, t)|^2$ compared with the numerical simulation for two fixed values of time. This furnishes the following velocities

$$
\begin{array}{cccccccc}
0.95 & 0.9 & 0.8 & 0.7 & 0.6 & 0.5 \\
0.28 & 0.38 & 0.53 & 0.65 & 0.75 & 0.83 \\
0.25 & 0.34 & 0.42 & 0.50 & 0.58 & 0.66
\end{array}
$$

These are slow light pulses and naturally, small velocities are obtained close to the gap edge where the required energy is small.

**Conclusion.** The property of a Bragg medium in the nonlinear Kerr regime to sustain nonlinear supratransmission has allowed us to obtain the analytic expression (3) that fixes the threshold amplitude of an incident cw-beam in terms of its departure from the Bragg resonance.

This constitutes a practical tool to investigate switching properties of a Bragg medium and allows at the same time to understand the process at the origin of sudden transmissivity of the Bragg mirror. It is the nonlinear supratransmission that results from a nonlinear instability intrinsic to boundary value problems (15).

We expect our result to be useful for experiments by the ability of the Bragg medium to become partly transparent by gap soliton generation for an incident beam of low energy if its frequency is chosen close to the gap edge.

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