UNIVERSALITY OF \( L \)-FUNCTIONS OVER FUNCTION FIELDS

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ABSTRACT. We prove that the Dirichlet \( L \)-functions associated to Dirichlet characters in \( \mathbb{F}_q[x] \) are universal. That is, given a modulus of high enough degree, \( L \)-functions with characters to this modulus can be found that approximate any given nonvanishing analytic function arbitrarily closely.

1. INTRODUCTION

Let \( s = \sigma + it \) and let \( \zeta(s) \) denote the Riemann zeta function. In 1975 S. M. Voronin \([14]\) proved the following remarkable “universality” theorem.

**Theorem 1** (Voronin). Let \( D_r \) be the open disc of radius \( 0 < r < 1/4 \) centered at \( s = 3/4 \) and let \( F(s) \) be a function that is analytic in \( D_r \), continuous on \( \overline{D}_r \), the closure of \( D_r \), and nonvanishing on \( \overline{D}_r \). Then for any \( \epsilon > 0 \) there exist real numbers \( \tau \to \infty \) such that

\[
\max_{s \in D_r} |\zeta(s + i\tau) - F(s)| < \epsilon. \tag{1}
\]

This was soon extended in various directions by A. Good \([6]\), S. M. Gonek \([4]\), Bagchi \([2, 3]\), Reich \([10, 11]\), and Voronin \([15, 16, 17, 18]\) himself. For instance, Gonek and Bagchi independently showed that the disc \( \overline{D}_r \) could be replaced by an arbitrary compact subset \( K \) in the strip \( \frac{1}{2} < \sigma < 1 \) whose complement is connected. Another direction was to prove universality for Dirichlet \( L \)-functions, Hurwitz and Epstein zeta functions, and a variety of other Dirichlet series. Implicit in the proofs of these results, though not always stated, was that the set of translation numbers \( \tau \) has positive proportion. That is, the measure of the set of \( \tau \in [0, T] \) satisfying (1) is \( \geq cT \) for all sufficiently large \( T \), where \( c \) is a positive constant.

Another version of universality, first proved by Gonek and independently soon after by Bagchi is

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Theorem 2 (Gonek, Bagchi). Let $C$ be a compact subset in the strip $\frac{1}{2} < \sigma < 1$ whose complement is connected, and let $F(s)$ be a nonvanishing function that is analytic in the interior of $C$ and continuous on $C$. Then for any $\epsilon > 0$ and sufficiently large integer $q$, there exists a Dirichlet character $\chi \pmod{q}$ such that the Dirichlet $L$-function $L(s, \chi)$ satisfies

$$\max_{s \in K} |L(s, \chi) - F(s)| < \epsilon.$$  \hspace{1cm} (2)

Here too it is implicit in the proofs that a positive proportion of the $\phi(q)$ characters $\chi \pmod{q}$ satisfy (2).

Over the years, a large number of papers on various aspects of universality have appeared. Good general surveys are Matsumoto’s paper [9] and the monographs by Laurinčikas [8] and by Steuding [13]. The work of Liza Jones [7] on a random matrix analog of Voronin’s theorem for characteristic polynomials associated to unitary, orthogonal and symplectic random matrices is also relevant. As far as we are aware, however, no one has extended the concept of universality to $L$-functions over function fields. Our goal here is to prove such a result. It will be obvious to those familiar with the area of universality that numerous extensions of our theorem are possible.

We first introduce some of the basic notation for function fields. Let $\mathbb{F}_q$ be a finite field with $q$ elements, where $q$ is odd, and let $\mathbb{F}_q[x]$ be the polynomial ring over $\mathbb{F}_q$ in the variable $x$. If $f$ is a nonzero polynomial in $\mathbb{F}_q[x]$, we define the norm of $f$ to be $|f| = q^{|\text{deg}(f)|}$. If $f = 0$, we set $|f| = 0$. A monic irreducible polynomial $P$ is called a prime polynomial or simply a prime. The $L$-function corresponding to an odd Dirichlet character $\chi \pmod{Q}$ is given by the Euler product

$$L(s, \chi) = \prod_{P \text{ prime}} (1 - \chi(P)Q^{-s})^{-1} \quad \text{Re}(s) > 1,$$

where $s$ is a complex variable and $Q$ is a monic polynomial. Multiplying out, we obtain the Dirichlet series representation

$$L(s, \chi) = \sum_{f \text{ monic}} \frac{\chi(f)}{|f|^s} \quad \text{Re}(s) > 1.$$  \hspace{1cm} (3)

It is sometimes convenient to work with the equivalent functions written in terms of the variable $u = q^{-s}$, namely,

$$L(u, \chi) = \prod_{P \text{ prime}} (1 - \chi(P)u^{\text{deg}(P)})^{-1} \quad |u| < 1/q$$  \hspace{1cm} (3)

and
\[ \mathcal{L}(u, \chi) = \sum_{f \text{ monic}} \chi(f) u^{\deg f} \quad |u| < 1/q. \] (4)

We refer to these pairs of equivalent expressions as the \( s \)-forms and \( u \)-forms of the \( \mathcal{L} \)-function, respectively.

It turns out that \( \mathcal{L}(u, \chi) \) is actually a polynomial of degree \( 2 \deg Q - 1 \) (see Rosen [12], Proposition 4.3), and it satisfies a Riemann hypothesis (see Weil [19]). That is, all its zeros lie on the circle \( |u| = q^{-\frac{1}{2}} \). It follows that we may write

\[ \mathcal{L}(u, \chi) = \prod_{j=1}^{2 \deg Q - 1} (1 - \alpha_j u), \]

where the \( \alpha_j = q^{\frac{1}{2}} e(-\theta_j) \), \( j = 1, 2, \ldots, 2 \deg Q - 1 \) are the reciprocals of the roots \( u_j = q^{-\frac{1}{2}} e(\theta_j) \) of \( \mathcal{L}(u, \chi) \). (Throughout we write \( e(x) \) to denote \( e^{2\pi ix} \).) In particular, the restriction \( |u| < 1/q \) in (4) (but not in (3)) may be deleted.

We may now state our first version of the universality theorem. Recall that if \( f \in \mathbb{F}_q[x] \), Euler’s phi function \( \phi(f) \) is the number of non-zero polynomials of degree less than \( \deg(f) \) and relatively prime to \( f \).

**Theorem 3.** Let \( \mathcal{U} = \{ u : q^{-1} < |u| < q^{-\frac{1}{2}} \} \) and let \( \mathcal{C} \) be a compact subset of \( \mathcal{U} \) whose complement in \( \mathcal{U} \) is connected. Let \( F(u) \) be a nonvanishing analytic function on the interior of \( \mathcal{C} \) that is continuous on the boundary of \( \mathcal{C} \). Then for any \( \epsilon > 0 \) and monic polynomial \( Q \) of large enough degree, there exists a Dirichlet character \( \chi \pmod{Q} \) such that

\[ \max_{u \in \mathcal{C}} |\mathcal{L}(u, \chi) - F(u)| < \epsilon. \] (5)

Moreover, (5) holds for a positive proportion of the \( \phi(Q) \) characters \( \chi \pmod{Q} \).

The theorem may also be formulated in terms of the variable \( s \).

**Theorem 4.** Let \( \mathcal{U} \) be the open rectangle with vertices \( \frac{1}{2}, 1, 1 + i \frac{2\pi}{\log q}, \frac{1}{2} + i \frac{2\pi}{\log q} \) and let \( \mathcal{C} \) be a compact subset of \( \mathcal{U} \) whose complement in \( \mathcal{U} \) is connected. Let \( F(s) \) be a nonvanishing function that is analytic on the interior of \( \mathcal{C} \) and continuous on the boundary of \( \mathcal{C} \). Then for any \( \epsilon > 0 \) and monic polynomial \( Q \) of large enough degree, there exists a Dirichlet character \( \chi \pmod{Q} \) such that

\[ \max_{s \in \mathcal{C}} |L(s, \chi) - F(s)| < \epsilon. \] (6)

Moreover, (6) holds for a positive proportion of the \( \phi(Q) \) characters \( \chi \pmod{Q} \).
2. Basic Lemmas on \( L \)-functions and Arithmetic in Function Fields

In this section we introduce several lemmas we require for the proof of Theorem 3. A version of our first lemma appears in [1]. The proof of the statement here is virtually identical (see Remark 2 of [1]).

Lemma 5 (The hybrid formula for \( L(u, \chi) \)). Let \( K \geq 0 \) be an integer and let

\[
\mathcal{P}_K(u, \chi) = \exp \left( \sum_{k=1}^{K} \sum_{\text{monic } f, \deg f = k} \Lambda(f) \chi(f) u^k \right),
\]

where \( \Lambda(f) = \deg P \) if \( f = P^n \) for some prime polynomial \( P \), and \( \Lambda(f) = 0 \) otherwise. Also set

\[
\mathcal{Z}_K(u, \chi) = \exp \left( - \sum_{j=1}^{2 \deg Q-1} \sum_{k>Q} \frac{(\alpha_j u)^k}{k} \right).
\]

Then for \( \chi \neq \chi_0 \) and \(|u| \leq q^{-1/2}\),

\[
L(u, \chi) = \mathcal{P}_K(u, \chi) \mathcal{Z}_K(u, \chi).
\]

Lemma 6. Let \( Q \) be a monic polynomial of degree greater than zero, \( \chi \) a character (mod\( Q \)), and \( K \geq 1 \) an integer. If \( \sigma_0 > \frac{1}{2} \), then uniformly for \( \sigma \geq \sigma_0 \) we have

\[
L(s, \chi) = P_K(s, \chi) \left( 1 + O \left( \frac{\deg Q}{K} q^{(1-\sigma)K} \right) \right),
\]

where

\[
P_K(s, \chi) = \exp \left( \sum_{1 \leq \deg P \leq K} \sum_{j=1}^{[K/\deg P]} \frac{\chi(P^j)}{j|P^j|^s} \right).
\]

\( \Lambda(f) = \deg P \) if \( f = P^n \) for some prime polynomial \( P \), and \( \Lambda(f) = 0 \) otherwise. The constant implicit in the \( O \)-term depends at most on \( q \) and \( \sigma_0 \).

Proof. It is not difficult to see that if \( u \) is replaced by \( q^{-s} \) in (7), we obtain \( P_K(s, \chi) \). Making this substitution in (8) also, and assuming that \( \sigma > \frac{1}{2} \), we find that
\[ |\log Z_K(u, \chi)| = \left| \sum_{j=1}^{2 \deg Q - 1} \sum_{k>K} \frac{(\alpha_j u)^k}{k} \right| \leq 2 \deg Q \sum_{k>K} \frac{q^{\frac{1}{2} - \sigma} k}{k} \]
\[ \leq 2 \deg Q \frac{q^{\frac{1}{2} - \sigma} K}{K(1 - q^{\frac{1}{2} - \sigma})} \ll_{\sigma_0, q} \frac{\deg Q q^{\frac{1}{2} - \sigma} K}{K}. \]

Equation (9) follows from this. □

**Lemma 7.** Let \( Q \) be a monic polynomial of degree greater than one. Then Euler’s phi function satisfies the bound

\[ \phi(Q) \gg \frac{|Q|}{\log_q(\deg Q)}, \quad (11) \]

where \( \log_q \) is the logarithm to the base \( q \).

**Proof.** We have

\[ \phi(Q) = |Q| \prod_{P|Q} \left( 1 - \frac{1}{|P|} \right). \]

As a function of \( |Q| \), this will clearly be smallest when \( Q \) is a product of all the primes of degree 1, times all the primes of degree 2, and so on up to the primes of degree \( n \), say. In that case we have

\[ \frac{\phi(Q)}{|Q|} = \prod_{j \leq n} \left( 1 - \frac{1}{q^j} \right)^{\pi_q(j)} = \exp \left( \sum_{j \leq n} \pi_q(j) \log(1 - q^{-j}) \right). \]

The sum in the exponential function equals

\[ -\sum_{j \leq n} \left( \frac{q^j}{j} + O\left(\frac{q^{j/2}}{j}\right) \right) \left( q^{-j} + O(q^{-2j}) \right) = -\sum_{j \leq n} \frac{1}{j} + O(q^{-1/2}) \]
\[ = -\log n + O(1). \]

Hence,

\[ \frac{\phi(Q)}{|Q|} \gg \frac{1}{n}. \quad (12) \]

Now
\[ |Q| = \prod_{j \leq n} q^{\pi_j(j)} = \prod_{j \leq n} q^j (q^{j/2} + O(q^{1/2}/j)) \]
\[ = q^n (1 + O(q^{-1})). \]

Thus, \( \text{deg } Q = q^n (1 + O(q^{-1})) \) and \( \log_q (\text{deg } Q) = n + O(1/(q \log q)) \). Using this in (12) we obtain the assertion of the lemma.

**Lemma 8.** Let \( \mathcal{N} \) be a set of distinct polynomials and \( Q \) a monic polynomial. If the norm of each element of \( \mathcal{N} \) is less than \( |Q| \), then

\[ \sum_{\chi \pmod{Q}} \left| \sum_{N \in \mathcal{N}} a_N \chi(N) \right|^2 = \phi(Q) \sum_{N \in \mathcal{N}} |a_N|^2. \]

This follows from the orthogonality relations for Dirichlet characters in function fields as presented in [12, Proposition 4.2].

### 3. Lemmas on Diophantine Approximation

The lemmas in this section are of similar nature as those proved in Gonek and Montgomery [5]. The first is exactly as in [5], the others are character analogues of lemmas there.

**Lemma 9.** Let \( K \) be a positive integer, and suppose that \( 0 < \delta \leq 1/2 \). There is a trigonometric polynomial \( f(\theta) \) of the form

\[ f(\theta) = \sum_{k=0}^{K} c_k e(k\theta) \]

such that \( \max_{\theta} |f(\theta)| = f(0) = 1 \) and \( |f(\theta)| \leq 2e^{-\pi K\delta} \) for \( \delta \leq \theta \leq 1 - \delta \).

Let \( Q \) be a monic polynomial and let \( \mathcal{P}_Q \) be a finite set of primes \( P \) coprime to \( Q \). For given real numbers \( 0 \leq \theta_P < 1 \), \( P \in \mathcal{P}_Q \), we want to show that if \( |Q| \) is large enough, there exist Dirichlet characters \( \chi \pmod{Q} \) such that

\[ \left\| \frac{\text{arg} \chi(P)}{2\pi} - \theta_P \right\| < \delta \quad (P \in \mathcal{P}_Q). \]

To this end we set

\[ g(\chi) = \prod_{P \in \mathcal{P}_Q} \left| f\left( \frac{\text{arg} \chi(P)}{2\pi} - \theta_P \right) \right|^2, \]
where \( f \) is defined as in Lemma 9.

**Lemma 10.** Let \( \mathcal{P}_Q \) be a fixed set of primes coprime to \( Q \), and for \( P \in \mathcal{P}_Q \) let numbers \( \theta_P \in [0, 1) \) be given. If \( g(\chi) \) is defined as in (14), and \( \kappa = \int_0^1 |f(\theta)|^2 d\theta \), then

\[
\sum_{\chi \neq \chi_0 \pmod{Q}} g(\chi) = \left( \phi(Q) + O(|\mathcal{P}_Q|^K) \right) \kappa |\mathcal{P}_Q|.
\]  

(15)

**Proof.** From (13) we see that

\[
\prod_{P \in \mathcal{P}_Q} f \left( \frac{\arg \chi(P)}{2\pi} - \theta_P \right) = \prod_{P \in \mathcal{P}_Q} \left( \sum_{k=0}^K c_k \chi(P^k) e(-k\theta_P) \right) = \sum_{N \in \mathcal{N}_Q} a_N \chi(N),
\]

where \( \mathcal{N}_Q \) is the set of monic polynomials composed entirely of primes in \( \mathcal{P}_Q \), with multiplicities not exceeding \( K \) and

\[
a_N = \prod_{P \in \mathcal{P}_Q \atop P^k || N} c_k e(-k\theta_P).
\]

Here the product is extended over all members of \( \mathcal{P}_Q \), not just those dividing \( N \). By Lemma 8

\[
\sum_{\chi \pmod{Q}} \left| \sum_{N \in \mathcal{N}_Q} a_N \chi(N) \right|^2 = \phi(Q) \sum_{N \in \mathcal{N}_Q} |a_N|^2.
\]

We wish to remove the principal character from the sum on the left. By the Cauchy-Schwarz inequality, its contribution is

\[
\left| \sum_{N \in \mathcal{N}_Q} a_N \right|^2 \leq |\mathcal{N}_Q| \sum_{N \in \mathcal{N}_Q} |a_N|^2.
\]

The cardinality of \( \mathcal{N}_Q \) is clearly \( \leq |\mathcal{P}_Q|^K \). Hence,

\[
\sum_{\chi \neq \chi_0 \pmod{Q}} \left| \sum_{N \in \mathcal{N}_Q} a_N \chi(N) \right|^2 = \left( \phi(Q) + O(|\mathcal{P}_Q|^K) \right) \sum_{N \in \mathcal{N}_Q} |a_N|^2.
\]

Now
\[
\sum_{N \in \mathcal{A}_Q} |a_N|^2 = \left( \sum_{k=0}^{K} |c_k|^2 \right)^{\mathcal{P}_Q} = \kappa^{\mathcal{P}_Q},
\]
so we obtain the stated result. \(\square\)

To apply Lemma 10 we need estimates for the parameter

\[
\kappa = \int_{0}^{1} |f(\theta)|^2 d\theta.
\]

Since

\[
1 = |f(0)|^2 = \left| \sum_{k=0}^{K} c_k \right|^2 \leq (K + 1) \sum_{k=0}^{K} |c_k|^2,
\]
by Cauchy’s inequality, we find that

\[
\frac{1}{K + 1} \leq \kappa \leq 1. \tag{16}
\]

Suppose next that the primes in \(\mathcal{P}_Q\) are all of degree at most \(\rho\). In order that the main term in (15) should be larger than the error term, we take

\[
\rho = \log_q(\deg Q), \quad K = \left[ \frac{\deg Q}{2 \log_q(\deg Q)} \right], \quad \delta = \frac{(\log_q(\deg Q))^2}{\deg Q}. \tag{17}
\]

Then

\[
|\mathcal{P}_Q|^K \ll (q^\rho / \rho)^K \ll |Q|^{1/2}. \tag{18}
\]

The function \(g(\chi)\) is large when the numbers \(\|\arg \chi(P) - \theta_P\|\) are small, but we need a peak function that is positive only when all of these numbers are < \(\delta\). We therefore define

\[
h(\chi) = \prod_{P \in \mathcal{P}_Q} \left| f\left( \frac{\arg \chi(P)}{2\pi} - \theta_P \right) \right|^2 - \varepsilon \sum_{P_1 \in \mathcal{P}_Q} \prod_{P \in \mathcal{P}_Q \setminus P_1} \left| f\left( \frac{\arg \chi(P)}{2\pi} - \theta_P \right) \right|^2,
\]
where \(\varepsilon = 4q^{-2\pi K\delta}\). Note that if for some \(\chi\) there is a prime \(P_1 \in \mathcal{P}_Q\) for which
\[
\| \frac{\arg \chi(P_1)}{2\pi} - \theta_{P_1} \| > \delta,
\]
then
\[
|f\left(\frac{\arg \chi(P_1)}{2\pi} - \theta_{P_1}\right)|^2 \leq \varepsilon
\]
by Lemma 9, and so \( h(\chi) \leq 0 \).

**Lemma 11.** With \( f(\theta) \) defined as in Lemma 9, \( h(\chi) \) defined as in (19), and parameters chosen as in (17),
\[
\sum_{\chi \neq \chi_0 \pmod{Q}} h(\chi) = (1 + O(1/\deg Q))\kappa^{\lvert \mathcal{P}_Q \rvert} \phi(Q) . \tag{20}
\]

**Proof.** By Lemma 10 and (18) we see that the sum above is
\[
= (\phi(Q) + O(|Q|^{1/2}))\kappa^{\lvert \mathcal{P}_Q \rvert} + O(\varepsilon |\mathcal{P}_Q| \phi(Q) \kappa^{\lvert \mathcal{P}_Q \rvert - 1}).
\]
Since \(|\mathcal{P}_Q| \ll q^\rho / \rho \leq \deg Q, 1/\kappa \ll \deg Q\) by (16) and (17), and \( \varepsilon < (\deg Q)^{-3} \), it follows that this last error term is \( \ll \phi(Q) \kappa^{\lvert \mathcal{P}_Q \rvert} / \deg Q \). Using (11) to estimate the first \( O \)-term, we obtain (20). \( \square \)

By means of these lemmas we see that there are characters \( \chi \pmod{Q} \) at which the primes \( P \) with \( \deg P \leq \rho \) and \( P \) coprime to \( Q \) behave as we want. However, for our application to universality we also need to know that for these “good” characters, the primes of larger degree, say those with \( \rho < \deg P \leq z \), do not always make too large a contribution. To eliminate this possibility we establish one final lemma.

**Lemma 12.** Let \( g(\chi) \) be defined as in (14) where \( \mathcal{P}_Q \) is the set of all primes with \( \deg P \) not exceeding \( \rho \) that are coprime to \( Q \). For the primes with \( \rho < \deg P \leq z \), let the functions \( b_P(s) \) have the property that \( |b_P(s)| \leq 1/|P|^\sigma \), where \( \sigma > 1/2 \). Then
\[
\sum_{\chi \pmod{Q}} g(\chi) \sum_{\rho < \deg P \leq z} b_P(s) \chi(P) \leq \phi(Q) \kappa^{\lvert \mathcal{P}_Q \rvert} q^{\rho(1-2\sigma)}/\rho .
\]

**Proof.** Let \( \mathcal{N}_Q \) and \( a_N \) be defined as in the proof of Lemma 10, and let the functions \( C_M(s) \) be determined by the identity


\[ \left( \sum_{N \in \mathcal{N}_Q} a_N \chi(N) \right) \left( \sum_{x < \deg P \leq z} b_P(s) \chi(P) \right) = \sum_M C_M(s) \chi(M). \]

If we assume the norm of \( Q \) is greater than the norm of every \( M \), then by Lemma 8

\[ \sum_{\chi \mod Q} \left| \sum_M C_M(s) \chi(M) \right|^2 dt = \phi(Q) \sum_M |C_M(s)|^2. \]

We note that a monic polynomial \( M \) has at most one decomposition \( M = NP \) with \( N \in \mathcal{N}_Q \) and \( \rho < \deg P \leq z \). In the main term we have

\[ \sum_M |C_M(s)|^2 = \left( \sum_{N \in \mathcal{N}_Q} |a_N|^2 \right) \left( \sum_{\rho < \deg P \leq z} |b_P(s)|^2 \right). \]

Here the sum over \( N \) is \( \kappa_{|\varphi_0|} \), and the sum over \( P \) is

\[ \ll \sum_{\rho < \deg P \leq z} |P|^{-2\sigma} \ll \frac{q^{\rho(1-2\sigma)}}{\rho}. \]

Thus we have the result. \( \square \)

4. Lemmas on approximating analytic functions by Dirichlet polynomials

We remind the reader that a multiset is a set in which each element may be repeated a finite number of times. Our next lemma is a minor modification of Lemma 2.2 of Gonek \[4\].

**Lemma 13.** Let \( \Lambda \) be an infinite multiset of real numbers whose counting function satisfies

\[ N_\Lambda(x) = \sum_{\lambda \in \Lambda, \lambda \leq x} 1 \ll e^x, \quad (21) \]

and for any fixed \( c > 0 \)

\[ N_\Lambda(x + c/x^2) - N_\Lambda(x) \gg \frac{e^x}{x^3}. \quad (22) \]

Let \( C \) be a simply connected compact set in the strip \( 1/2 < \sigma_1 < \sigma_2 < 1 \) and suppose that \( F(s) \) is continuous on \( C \) and analytic in the interior of \( C \). Then for each \( \mu > 0 \) there is a \( \rho_0 > \mu \) such that if \( \rho \geq \rho_0 \), there are numbers \( \theta_\lambda \in [0,1) \) such that
\[
\max_{s \in C} \left| F(s) - \sum_{\substack{\lambda \in \Lambda \\ \mu < e^\lambda \leq \rho}} e(\theta_\lambda) e^{-\lambda s} \right| \ll \mu^{-1/2}.
\] (23)

Here \( \rho_0 \) depends at most on \( \sigma_1, \sigma_2, C, \Lambda, F(s), \) and \( \mu, \) while the implied constant in the last inequality depends on at most \( \sigma_1, \sigma_2, C, \) and \( \Lambda. \)

In [4] this lemma is stated for sets rather than multisets. However, by adding small distinct shifts to identical elements of \( \Lambda, \) we can apply the original version of the lemma without affecting either (21) or (22). Then, letting the shifts tend to zero, we recover (23) by continuity.

From now on we let \( \mathcal{D} \) denote either

(a) the set of all prime polynomials \( P, \) or

(b) the set of all polynomials of the form \( P_1^{a_1} P_2^{a_2} \cdots P_r^{a_r}, \) where \( a_i \in \mathbb{N} \cup \{0\} \)
and \( P_1, \ldots, P_r \) are a fixed set of prime polynomials.

For \( Q \in \mathcal{D}, \) we apply Lemma 13 to the multiset

\[ \Lambda_Q = \{ \log |P| : P \nmid Q \}. \]

To see that \( N_{\Lambda_Q}(x) \) satisfies the conditions of the lemma, first note that

\[
N_{\Lambda_Q}(x) = \sum_{\log |P| \leq x} 1 \leq \sum_{\deg P \leq x / \log q} 1 = \sum_{j=1}^{[x / \log q]} \pi_q(j) 
\]

\[
\ll \sum_{j=1}^{[x / \log q]} q^j \ll \frac{e^x}{x}.
\]

Second, if \( \omega(Q) \) is the number of distinct primes dividing \( Q, \) then we have
\[ N_{\Lambda Q}(x + c/x^2) - N_{\Lambda Q}(x) = \sum_{x < \log |P| \leq x + c/x^2 \atop P \not| Q} 1 \geq \left( \sum_{x < \log q < \deg P \leq x + c/x^2} 1 \right) - \omega(Q) \]

\[ = \left( \sum_{x < \log q < j \leq x + c/x^2} \pi_q(j) \right) - \omega(Q) \]

\[ \gg \left( \sum_{x < \log q < j \leq x + c/x^2} \frac{q^j}{j} \right) - \omega(Q) \gg e^{x/x^3} - \omega(Q). \]

Note that for \( Q \in \mathcal{Q} \) this is \( \gg e^{x/x^3} \) because \( \omega(Q) \) is bounded. Thus, we have

**Lemma 14.** Let \( Q \) be as above. Let \( C \) be a simply connected compact set in the strip \( 1/2 < \sigma_1 < \sigma_2 < 1 \) and suppose that \( f(s) \) is continuous on \( C \) and analytic in the interior of \( C \). Then for each \( \mu > 0 \) there is a \( \rho_0 > \mu \) such that if \( \rho \geq \rho_0 \), there are numbers \( \theta_P \in [0, 1) \) such that for all \( Q \in \mathcal{Q} \),

\[ \max_{s \in C} \left| f(s) - \sum_{\mu < |P| \leq \rho \atop P \not| Q} e^{\theta_P} \right| |P|^s \ll q^{-\mu/2}. \]

Here \( \rho_0 \) depends at most on \( \sigma_1, \sigma_2, C, \Lambda, f(s) \), and \( \mu \), while the implied constant in the last inequality depends on at most \( \sigma_1, \sigma_2, C, \) and \( \Lambda \).

5. PROOF OF THEOREM 4

We recall that \( C \) is a compact subset of the open rectangle \( U \) with vertices \( \frac{1}{2}, 1, 1 + i \frac{2\pi}{\log q}, \frac{1}{2} + i \frac{2\pi}{\log q} \), whose complement in \( U \) is connected. Also, \( F(s) \) is analytic on the interior of \( C \), continuous on the boundary, and nonvanishing on \( C \). It follows that there is an analytic branch of logarithm of \( F(s) \) on the interior of \( C \) that is continuous on the boundary of \( C \). We denote it by \( f(s) \). We wish to show that for every \( \epsilon > 0 \) and monic polynomial \( Q \) of large enough degree, a positive proportion of the \( \phi(Q) \) Dirichlet characters \( \chi \) (mod \( Q \)) satisfy

\[ \max_{s \in C} |L(s, \chi) - e^{f(s)}| < \epsilon. \]

Let \( \sigma_0 = 1/2 + d \), where \( d = \min_{s \in C} (\sigma - \frac{1}{2}) > 0 \). By Lemma 6 with the parameters \( K \) and \( Q \) large and to be chosen later, we have
\[ \log L(s, \chi) = \log P_K(s, \chi) + O\left(\frac{\deg Q}{K} q^{-dK}\right), \tag{24} \]

where \( P_K(s, \chi) \) is defined in (10) and the implied constant depends on \( \sigma_0 \) and \( q \).

Next let \( 1 \leq \mu < \rho \) with \( \rho < K \) and write

\[ \log P_K(s, \chi) = \sum_{\deg P \leq \mu} \frac{[K/\deg P]}{P} \sum_{j=1}^{\deg P} \frac{\chi(P^j)}{j|P|^j s} + \sum_{\mu < \deg P \leq \rho} \frac{\chi(P)}{|P|^s} + \sum_{\rho < \deg P \leq K} \frac{\chi(P)}{|P|^s} \]

\[ = f_1(s, \chi) + f_2(s, \chi) + f_3(s, \chi) + f_4(s, \chi). \tag{25} \]

Clearly, for \( s \in C \) we have

\[ f_4(s, \chi) \ll \sum_{\mu < \deg P \leq K} \frac{1}{|P|^{2\sigma_0}} = \sum_{\mu < k \leq K} \frac{\pi_q(k)}{|Q|^{2k\sigma_0}} \ll \sum_{\mu < k \leq K} \frac{1}{k|Q|^{k(2\sigma_0 - 1)}} \]

\[ \ll \mu^{-1} |Q|^{-2d\mu}. \]

Thus, combining this with (24) and (25), we see that

\[ \log L(s, \chi) = f_1(s, \chi) + f_2(s, \chi) + f_3(s, \chi) + O(\mu^{-1} |Q|^{-2d\mu}) + O\left(\frac{\deg Q}{K} q^{-dK}\right). \tag{26} \]

Now define

\[ f_1(s) = \sum_{\deg P \leq \mu} \sum_{P \mid Q} \frac{[K/\deg P]}{P} \frac{1}{j|P|^j s} \]

and apply Lemma [14] to the function \( f(s) - f_1(s) \), which is also analytic on the interior of \( C \) and continuous on \( C \). Then we see that for each \( \mu > 0 \) there is a \( \rho_0 > \mu \) such that if \( \rho \geq \rho_0 \), there are numbers \( \theta_P \in [0, 1) \) such that for all \( Q \in \mathcal{Q} \),

\[ \max_{s \in C} \left| f(s) - f_1(s) - \sum_{\mu < \deg P \leq \rho} \frac{e(\theta_P)}{|P|^s} \right| \ll q^{-\mu/2}. \tag{27} \]
Here $\rho_0$ depends at most on $\sigma_0, C, \Lambda, f(s)$, and $\mu$, while the implied constant in the last inequality depends at most on $\sigma_0, C,$ and $\Lambda$. Combining (26) and (27), we obtain

$$\log L(s, \chi) - f(s) = (f_1(s, \chi) - f_1(s)) + \left( f_2(s, \chi) - \sum_{\mu < \deg P \leq \rho \atop P \mid Q} \frac{e(\theta_P)}{|P|^s} \right) + f_3(s, \chi)$$

$$+ O(\mu^{-1}|Q|^{-2d\mu}) + O(q^{-\mu/2}) + O\left(\frac{\deg Q}{K} q^{-dK}\right).$$

(28)

Suppose next that there exists a Dirichlet character $\chi \pmod Q$ such that for $P \in \mathcal{P}_Q, P \nmid Q$

$$\begin{cases} \frac{\arg \chi(P)}{2\pi} < \delta & \text{if } \deg P \leq \mu, \\ \frac{\arg \chi(P)}{2\pi} - \theta_P < \delta & \text{if } P \in \mathcal{P}_Q, \mu < \deg P \leq \rho. \end{cases}$$

(29)

Since $|e^{2\pi i \theta} - 1| \leq \|\theta\|$ for all real numbers $\theta$, the two terms preceding $f_3$ on the right-hand side of (28) are

$$\ll \delta \sum_{\deg P \leq \mu} \sum_{j=1}^{[K/\deg P]} \frac{1}{|P|^{j\sigma_0}} + \delta \sum_{\mu < \deg P \leq \rho} \frac{1}{|P|^{\sigma_0}}$$

$$\ll \delta \sum_{1 \leq m \leq \rho} \sum_{\deg P = m} \frac{1}{q^m} \ll \delta \sum_{1 \leq m \leq \rho} \frac{q^m}{mq^{m\sigma_0}}$$

$$\ll \delta \frac{q^{(\rho+1)(1-\sigma_0)}}{\rho}.$$

From this and (28) it follows that if $\chi$ satisfies (29), then for $s \in C$,

$$\log L(s, \chi) - f(s) = f_3(s, \chi) + O(\mu^{-1}|Q|^{-2d\mu}) + O(q^{-\mu/2}) + O\left(\delta \frac{q^{(\rho+1)(1-\sigma_0)}}{\rho}\right)$$

$$+ O\left(\frac{\deg Q}{K} q^{-dK}\right).$$

With the choice of the parameters $\rho, K$ and $\delta$ in (17), we see that for $\chi$ as above and $s \in C$ we can make the first two $O$-terms each less than $\epsilon/8$, say, by taking $\mu$ sufficiently large. The third $O$-term is
\[ \ll \sqrt{\frac{q}{\deg Q}}, \]

and the fourth is

\[ \ll \log_q(\deg Q)q^{-d\deg Q/\log_q(\deg Q)}. \]

Clearly each of these can be made less than \( \epsilon/8 \) by taking \( \deg Q \) sufficiently large. Thus, for \( \chi \) as above,

\[ \max_{s \in C} |\log L(s, \chi) - f(s) - f_3(s, \chi)| < \epsilon/2. \] (30)

Next, we set

\[ M(\chi) = \max_{s \in C} |f_3(s, \chi)| \]

and use Lemma 12 and (17) to see that

\[ \sum_{\chi \not\equiv \chi_0 \pmod{Q}} g(\chi)|M(\chi)|^2 \ll \phi(Q)\kappa_{|\mathfrak{p}Q|}q^{\phi(1-2\sigma_0)}q^{-d\rho} \ll \phi(Q)\kappa_{|\mathfrak{p}Q|}q^{-2d\rho} \] (31)

\[ = \frac{\phi(Q)\kappa_{|\mathfrak{p}Q|}}{(\deg Q)^{2d}}. \]

Setting \( h^+(\chi) = \max(0, h(\chi)) \), we see that for all \( \chi \)

\[ h(\chi) \leq h^+(\chi) \leq g(\chi). \]

Thus, by Lemmas 10 and 11

\[ \sum_{\chi \not\equiv \chi_0 \pmod{Q}} h^+(\chi) = (1 + O(1/\deg Q))\kappa_{|\mathfrak{p}Q|}\phi(Q), \] (32)

and by (31)

\[ \sum_{\chi \not\equiv \chi_0 \pmod{Q}} h^+(\chi)|M(\chi)|^2 \ll \frac{\phi(Q)\kappa_{|\mathfrak{p}Q|}}{(\deg Q)^{2d}}. \] (33)

It follows from the last two estimates that there exists a \( \chi(\pmod{Q}) \) and a positive constant \( c \) such that
By taking $\deg Q$ large enough we can make this less than $\epsilon/2$. It then follows from (30) that for this $\chi$

$$\max_{s \in C} |\log L(s, \chi) - f(s)| < \epsilon.$$  

This implies that

$$\max_{s \in C} |\log |L(s, \chi)/F(s)|| < \epsilon,$$

or

$$e^{-\epsilon} < \max_{s \in C} |L(s, \chi)/F(s)| < e^\epsilon.$$

From this and the fact that $F(s)$ has a positive minimum value on $C$, we obtain (6) on changing the value of $\epsilon$.

So far we know there exists at least one $\chi$ such that (6) holds. To complete the proof of Theorem 4 we must show that (6) holds for a positive proportion of the characters $\chi(\mod Q)$. To do this we argue as follows.

Define

$$\mathcal{G} = \{\chi \neq \chi_0(\mod Q) : M(\chi) \leq (\deg Q)^{-d/2}\}$$

and

$$\mathcal{B} = \{\chi \neq \chi_0(\mod Q) : M(\chi) > (\deg Q)^{-d/2}\}.$$  

Then $|\mathcal{G} \cup \mathcal{B}| = \phi(Q) - 1$ and, by (33),

$$\sum_{\chi \in \mathcal{G}} h^+(\chi)|M(\chi)|^2 + \sum_{\chi \in \mathcal{B}} h^+(\chi)|M(\chi)|^2 \ll \frac{\phi(Q)k|\mathcal{P}|}{(\deg Q)^{2d}}.$$

Thus

$$(\deg Q)^{-d} \sum_{\chi \in \mathcal{B}} h^+(\chi) < \sum_{\chi \in \mathcal{B}} h^+(\chi)|M(\chi)|^2 \ll \frac{\phi(Q)k|\mathcal{P}|}{(\deg Q)^{2d}},$$

and so
\[\sum_{\chi \in \mathcal{B}} h^+(\chi) \ll \frac{\phi(Q)\kappa|\mathcal{P}_Q|}{(\deg Q)^{d}}.\]

From this and (32)

\[\sum_{\chi \in \mathcal{G}} h^+(\chi) = (1 + O(1/(\deg Q)^d))\kappa|\mathcal{P}_Q|\phi(Q),\]

since \(0 < d < 1/2\). Now \(h^+(\chi) \leq 1\), so

\[|\mathcal{G}| \geq \sum_{\chi \in \mathcal{G}} h^+(\chi) = (1 + O(1/(\deg Q)^d))\kappa|\mathcal{P}_Q|\phi(Q).\]

Thus, we have \(M(\chi) \leq (\deg Q)^{-d/2}\) for at least \((1 + o(1))\kappa|\mathcal{P}_Q|\phi(Q)\) characters \(\chi\), that is, for a positive proportion of the \(\phi(Q)\) characters \(\chi(\mod Q)\). This completes the proof of Theorem 4.

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