COSMOLOGICAL MODELS WITH ISOTROPIC SINGULARITIES

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In 1985 Goode and Wainwright devised the concept of an isotropic singularity. Since that time, numerous authors have explored the interesting consequences, in mathematical cosmology, of assuming the existence of this type of singularity. In this paper, we collate all examples of cosmological models which are known to admit an isotropic singularity, and make a number of observations regarding their general characteristics.

1 Introduction

In 1985 Goode and Wainwright introduced the concept of an isotropic singularity (IS) to the field of mathematical cosmology, in order to clarify what is meant by a “quasi-isotropic” singularity and a “Friedmann-like” singularity. Their definition of an IS is coordinate-independent, as well as being independent of the Einstein field equations (EFE), and hence of the source of the gravitational field. Scott has amended their definition to remove some redundancy, and it is this amended definition that will be used throughout this paper.

Definition 1 (Isotropic Singularity) A space-time \((M, g)\) is said to admit an isotropic singularity if there exists a space-time \((^*M, ^*g)\), a smooth cosmic time function \(T\) defined on \(^*M\), and a conformal factor \(\Omega(T)\) which satisfy

1. \(M\) is the open submanifold \(T > 0\),
2. \(^*g = \Omega^2(T) \ast g\) on \(M\), with \(^*g\) regular (at least \(C^3\) and non-degenerate) on an open neighbourhood of \(T = 0\),
3. \(\Omega(0) = 0\) and \(\exists b > 0\) such that \(\Omega \in C^0[0, b] \cap C^3(0, b)\) and \(\Omega(0, b] > 0\),
4. \(\lambda \equiv \lim_{T \to 0^+} L(T)\) exists, \(\lambda \neq 1\), where \(L \equiv \frac{\Omega''}{\Omega'}(\Omega)\) and a prime denotes differentiation with respect to \(T\).
The definition of an isotropic singularity is described pictorially in Figure (1).

Since \((\mathcal{M}, g)\) is the space-time in hand—in our case, typically, a cosmological solution of the Einstein field equations—it is usual to call it the physical space-time. The conformally related space-time, \((\mathcal{M}, ^*g)\), is then called the unphysical space-time. Although the definition of an IS, and the arrow in Figure (1), lead one to think of the unphysical space-time being “created” from the physical space-time, it is actually useful to consider the situation in the reverse fashion. In this way, the singularity in the physical space-time should be regarded as arising due to the vanishing of the conformal factor, \(\Omega(T)\), at the regular hypersurface \(T = 0\) in \(\mathcal{M}\).

The definition of an isotropic singularity, as it stands, allows cosmological models to admit an “isotropic” singularity whose singularities are, in no sense, actually isotropic, or quasi-isotropic, or Friedmann-like. For example, the exact viscous fluid FRW cosmology of Coley and Tupper can be shown to have an IS, and yet the shear and acceleration of the fluid flow are not dominated by its expansion as the singularity is approached, as one would expect with a Friedmann-like singularity. The fluid congruence is not regular at the IS, however, which motivated Goode and Wainwright to include the following additional definition relating to the fluid flow.

**Definition 2 (Fluid Congruence)** With any time-like congruence \(u\) in \(\mathcal{M}\) we can associate a time-like congruence \(^*u\) in \(^*\mathcal{M}\) such that

\[ ^*u = \Omega \ u \quad \text{in} \ \mathcal{M}. \tag{1} \]

(a) If we can choose \(^*u\) to be regular (at least \(C^3\)) on an open neighbourhood...
of \( T = 0 \) in \( ^*\mathcal{M} \), we say that \( u \) is regular at the isotropic singularity.

(b) If, in addition, \( ^*u \) is orthogonal to \( T = 0 \) we say that \( u \) is orthogonal to the isotropic singularity.

It is the requirement that the fluid flow be regular at the IS (condition (a) of Definition 2), which ensures that the appropriate kinematic quantities behave as one would expect as an “isotropic” singularity is approached.

A question which naturally follows from these two definitions is what cosmological models actually admit an isotropic singularity at which the fluid flow is regular? In Sections 2, 3, 4, and 5, we collect together the various examples which have appeared throughout the literature, and in Section 6, we discuss their general characteristics, and draw certain conclusions from them.

2 Kantowski-Sachs models

The Kantowski-Sachs models are irrotational, geodesic, perfect fluid models with a radiation equation of state \( p = \frac{1}{3} \mu \). These models are spatially homogeneous but not spatially isotropic. The manifold \( \mathcal{M} \) for these models is \( \mathbb{R}^4 \) and they have a metric \( g \), in comoving coordinates, of the form

\[
ds^2 = -Adt^2 + t \left[ A^{-1}dx^2 + A^2b^{-2}(dy^2 + f^2dz^2) \right], \quad t > 0, \tag{2}
\]

where \( A = 1 - \frac{4eb^2t}{9} \), \( b \) is a constant,

\[
f(y) = \begin{cases}
\sin y & \epsilon = +1 \\
\sinh y & \epsilon = -1
\end{cases},
\]

and the fluid flow \( u \) is

\[
u = A^{-\frac{1}{2}} \frac{\partial}{\partial t} . \tag{3}
\]

Now define a cosmic time function \( T \) by

\[
T(t) = \sqrt{2t}. \tag{4}
\]

Let \( ^*\mathcal{M} \) be the manifold \( \mathbb{R}^4 \) covered by the coordinate patch \((T, x, y, z)\), where \( T \in \mathbb{R} \). Thus \( \mathcal{M} \) is the open submanifold \( T > 0 \) of \( ^*\mathcal{M} \), thereby satisfying requirement (1) of the definition of an IS.

Now define a new coordinate \( \pi \) by \( \pi = \frac{1}{\sqrt{2}}x \), and a new constant \( \bar{b} \) by \( \bar{b} = \sqrt{2}b \). Hence

\[
dt = Td\pi \quad \text{and} \quad d\pi = \frac{1}{\sqrt{2}}dx . \tag{5}
\]
Rewriting the metric $g$ in terms of the new $T$ and $\bar{x}$ coordinates, we obtain

$$ds^2 = T^2 \left( -AdT^2 + A^{-1}d\bar{x}^2 + A^2(\bar{b})^{-2}(dy^2 + f^2dz^2) \right),$$  \hspace{1cm} (6)$$

and $u = A^{-\frac{1}{2}}T^{-1} \frac{\partial}{\partial T}$, where $A = 1 - \frac{e^{2\bar{b}}T^2}{9}$.  \hspace{1cm} (7)

Now define the conformal factor $\Omega$ by

$$\Omega(T) = T, \quad \text{where} \quad T \geq 0.$$  \hspace{1cm} (8)

Clearly this conformal factor satisfies requirements (3) and (4) of the definition of an IS, with $\lambda = 0$.

The conformally related metric $^*g$ is then given by

$$^*ds^2 = -AdT^2 + A^{-1}d\bar{x}^2 + A^2(\bar{b})^{-2}(dy^2 + f^2dz^2),$$  \hspace{1cm} (9)

and is certainly $C^3$ and non-degenerate on an open neighbourhood of $T = 0$, thereby satisfying requirement (2) of the definition of an IS. Thus, the Kantowski-Sachs models do indeed admit an isotropic singularity—this was first shown by Goode and Wainwright.\cite{GoodeWainwright}

From Definition (3), the unphysical fluid flow $^*u$ is given by

$$^*u = A^{-\frac{1}{2}} \frac{\partial}{\partial T}.$$  \hspace{1cm} (10)

It is clear that $^*u$ is $C^3$ on an open neighbourhood of $T = 0$ in $^*M$, and hence $u$ is regular at the isotropic singularity.

The Weyl tensor $C^a_{\,bcd}$ is bounded at the IS, with some components having a non-zero limit. Splitting the Weyl tensor up into its electric, $E_{ab}$, and magnetic, $H_{ab}$, parts, it is found that the magnetic part is everywhere zero and the electric part is bounded at the IS, with some components having a non-zero limit.

3 Szekeres models

The Szekeres models\cite{Szekeres} are irrotational, geodesic, pressure-free dust solutions. In comoving coordinates, the metric $g$ and fluid congruence $u$ for these models are given by

$$ds^2 = -dt^2 + t^\frac{4}{3}(dx^2 + dy^2 + Z^2dz^2), \quad t > 0, \quad u = \frac{\partial}{\partial t},$$  \hspace{1cm} (11)$$

where

$$Z = A + k_+t^{\frac{2}{3}} + k_-t^{-1}, \quad A = ax + by + c + \frac{5}{9}k_+(x^2 + y^2).$$  \hspace{1cm} (12)$$
and \( a, b, c, k_+, k_- \) are arbitrary smooth functions of \( z \).

We will only examine the Szekeres models in which the decaying mode is absent, i.e., when \( k_- = 0 \). This subclass has previously been shown by Goode and Wainwright\(^{10}\) to have a Friedmann-like singularity and was first explicitly shown to have an IS in a paper by Goode, Coley, and Wainwright.\(^{11}\) The manifold \( \mathcal{M} \) for these models is \( \mathbb{R}^4 \). Now define a cosmic time function \( T \) by

\[
T(t) = 3t^{\frac{4}{3}}, \quad \text{and hence,} \quad dt = \frac{T^2}{9}dT.
\]

Let \( \ast \mathcal{M} \) be the manifold \( \mathbb{R}^4 \) covered by the coordinate patch \((T, x, y, z)\), where \( T \in \mathbb{R} \). Thus \( \mathcal{M} \) is the open submanifold \( T > 0 \) of \( \ast \mathcal{M} \), thereby satisfying requirement (1) of the definition of an IS.

Rewriting the metric \( g \) in \((T, x, y, z)\) coordinates, we obtain

\[
ds^2 = \frac{T^4}{81}(-dT^2 + dx^2 + dy^2 + Z^2dz^2),
\]

where

\[
Z = A + k_+ \frac{T^2}{9}.
\]

Now define the conformal factor \( \Omega \) by

\[
\Omega(T) = \frac{T^2}{9}, \quad \text{where} \quad T \geq 0.
\]

Clearly this conformal factor satisfies requirements (3) and (4) of the definition of an IS, with \( \lambda = \frac{3}{2} \).

The conformally related metric \( \ast g \) and fluid flow \( \ast u \) are given by

\[
\ast ds^2 = -dT^2 + dx^2 + dy^2 + Z^2dz^2, \quad \ast u = \frac{\partial}{\partial T}.
\]

It is readily seen that condition (2) of the definition of an IS is satisfied and that the fluid flow \( u \) is regular at the IS. Thus the Szekeres models with \( k_- = 0 \) certainly admit an IS.

The Weyl tensor behaves qualitatively the same as the Kantowski-Sachs Weyl tensor. It is bounded at the IS, with some components having a non-zero limit, the magnetic part is everywhere zero, and the electric part is bounded at the IS, with some components having a non-zero limit.
4 Mars models

Mars has found three types of solution corresponding to a perfect fluid with an Abelian two-dimensional group of isometries acting orthogonally transitively on spacelike 2-surfaces and such that both Killing vectors are integrable. In comoving coordinates, the third type of solution has a metric $g$ of the form

$$ds^2 = -\frac{e^{at+\epsilon c^2e^{2a(t+x)}}}{1+\epsilon e^{-2at+\beta e^{-6at}}} dt^2 e^{at+\epsilon c^2e^{2a(t+x)}} dx^2$$

$$+ e^{a(t-x)+2\epsilon e^{ax}} dy^2 e^{a(t-x)-2\epsilon e^{ax}} dz^2,$$

(18)

where $a, c, \epsilon, \beta$ are constants with $a > 0, \epsilon = +1, \beta \geq 0$. The fluid flow $u$ is given by

$$u = \frac{\sqrt{1+\epsilon e^{-2at+\beta e^{-6at}}} \partial }{e^{at+\epsilon c^2e^{2a(t+x)}}}.$$  

(19)

The manifold $\mathcal{M}$ for these models is $\mathbb{R}^4$, and it is to be noted that they all have a big-bang singularity at $t = -\infty$. We can define a cosmic time function $T$ for these models by

$$T = e^{at}, \quad \text{and hence,} \quad dt = \frac{1}{aT} dT.$$

(20)

Let $^*\mathcal{M}$ be the manifold $\mathbb{R}^4$ covered by the coordinate patch $(T, x, y, z)$, where $T \in \mathbb{R}$. Thus $\mathcal{M}$ is the open submanifold $T > 0$ of $^*\mathcal{M}$, thereby satisfying requirement (1) of the definition of an IS.

Rewriting the metric $g$ in $(T, x, y, z)$ coordinates, we obtain

$$ds^2 = -T \left( \frac{e^{c^2T^2e^{2ax}}}{1+\epsilon T^{-2}+\beta T^{-6}} a^2 T^2 + e^{c^2T^2e^{2ax}} dx^2$$

$$+ e^{-ax+2\epsilon e^{ax}} dy^2 + e^{-ax-2\epsilon e^{ax}} dz^2 \right),$$

(21)

and the fluid flow $u$ becomes

$$u = aT \frac{\sqrt{1+\epsilon T^{-2}+\beta T^{-6}}} {Te^{c^2T^2e^{2ax}}} \frac{\partial }{\partial T}.$$  

(22)

For $\beta > 0$, there is a limiting $\gamma$-law equation of state, $p = (\gamma-1)\mu$, for the perfect fluid, with $\gamma = \frac{14}{3}$. If $\beta = 0$, then the fluid is a stiff fluid, i.e., it has an exact $\gamma$-law equation of state with $\gamma = 2$. 
The vorticity scalar, shear scalar, and acceleration for the fluid flow $u$ are given by

$$\omega^2 = 0, \quad (23)$$

$$\sigma^2 = \frac{a^2 e^2 c^4}{3} \left[ T^3 + e T + \frac{\beta}{T^3} \right] e^{-ec^2 T^2 e^{2ax} + 4ax}, \quad (24)$$

$$\dot{u}^b = a e c^2 T^2 e^{2ax} \delta^b_1. \quad (25)$$

It follows that the Mars models are irrotational, perfect fluid models which have non-geodesic fluid flows. Now restricting ourselves to the subclass of these models which satisfy $\beta = 0$, we can define the conformal factor $\Omega$ by

$$\Omega(T) = \sqrt{T}, \quad T > 0, \quad \Omega(0) = 0. \quad (26)$$

Clearly this conformal factor satisfies requirements (3) and (4) of the definition of an IS, with $\lambda = -1$.

The conformally related metric $^*g$ and fluid flow $^*u$ are given by

$$^*ds^2 = -\frac{e^{ec^2 T^2 e^{2ax}}}{a^2 (e + T^2)} dT^2 + e^{c^2 T^2 e^{2ax}} dx^2 + e^{-ax + 2ce^x} dy^2 + e^{-ax - 2ce^x} dz^2, \quad (27)$$

and

$$^*u = a \sqrt{\frac{e + T^2}{e^{c^2 T^2 e^{2ax}}}} \frac{\partial}{\partial T}. \quad (28)$$

It is readily seen that condition (2) of the definition of an IS is satisfied, and that the fluid flow $u$ is regular at the IS as well as being orthogonal to the IS. Thus the Mars models with $\beta = 0$ do admit an IS. This fact was stated by Mars but not shown explicitly until this paper.

The Weyl tensor $C^a_{bcd}$ is bounded at the IS, with some components having a non-zero limit. Splitting the Weyl tensor up into its electric, $E_{ab}$, and magnetic, $H_{ab}$, parts, it is found that the magnetic part limits to zero, as the IS is approached, and the electric part is bounded at the IS, with some components having a non-zero limit.

5 Other models

In this section, we list all other cosmological models which are known to admit an IS. For some models, the authors present the actual conformal structure which yields the IS, whilst for the others, the authors simply claim that the models admit an IS.
5.1 Collins 71 models - Bianchi type VIh

These models, found by Collins in 1971, are spatially homogeneous, irrotational, geodesic, perfect fluid cosmological models with a $\gamma$-law equation of state, $p = (\gamma - 1)\mu$. A subclass of these models was first shown to admit an IS in 1984 by Wainwright and Anderson. For this subclass, the fluid flow has non-zero shear, although the shear vanishes at the IS. The magnetic part of the Weyl tensor limits to zero as the IS is approached, while the electric part of the Weyl tensor is bounded at the IS, with some components having a non-zero limit.

5.2 Bondi models

The spherically symmetric dust models of Bondi are irrotational and geodesic, and the magnetic part of the Weyl tensor is everywhere zero. A subclass of these models was shown to admit an IS in 1992 by Tod. For this subclass, the fluid flow has non-zero shear, with the shear vanishing at the IS, and the electric part of the Weyl tensor is bounded at the IS, with some components having a non-zero limit.

5.3 Others

Goode, Coley, and Wainwright state that Wainwright and Hsu have shown that all Bianchi classes contain solutions which admit an IS, although the only known non-FRW solutions are those of Wainwright and Anderson.

The plane-symmetric, stiff matter solutions of Tabensky and Taub are irrotational and geodesic, and the magnetic part of the Weyl tensor is everywhere zero. Tod states that a certain subclass of these solutions (those with the coefficient $b$ set to zero) admit an IS. Rendall has actually shown that, for a specific case, the Tabensky-Taub solutions admit an IS.

Mimoso and Crawford have found a class of spatially homogeneous, irrotational, shear-free, geodesic cosmological models, with an imperfect fluid matter source, which they claim admits an IS. The matter source is, in fact, an anisotropic fluid without heat flux, and the magnetic part of the Weyl tensor is everywhere zero.

6 Discussion

In Sections 2, 3, 4, and 5, we have listed, and examined, in varying detail, all cosmological models which are known to admit an IS. We note that all models discussed have a perfect fluid source, except for the Mimoso-Crawford
models, which have an imperfect fluid source corresponding to an anisotropic fluid without heat flux. The conformal structure which yields an IS for the Mimoso-Crawford models has yet to be presented.

It is instructive, at this point, to place all the perfect fluid models in a table, categorised according to their physical attributes—the fluid vorticity, shear, and acceleration; the Weyl tensor; the electric and magnetic parts of the Weyl tensor; the equation of state for the perfect fluid.

| Models            | $\omega_{ab}$ | $\sigma_{ab}$ | $u^a$ | $C^{a}_{bcd}$ | $E_{ab}$ | $H_{ab}$ | $p = p(\mu)$ |
|-------------------|---------------|---------------|-------|---------------|----------|----------|--------------|
| FRW               | 0             | 0             | 0     | 0             | 0        | 0        | yes (a subclass) |
| Kantowski-Sachs   | 0 (a)         | 0             | (b)   | (b)           | 0        | 0        | yes ($p = \frac{2}{3}\mu$) |
| Szekeres          | 0 (a)         | 0             | (b)   | (b)           | 0        | 0        | yes ($p = 0$) |
| Bondi             | 0 (a)         | 0             | (b)   | (b)           | 0        | 0        | yes ($p = 0$) |
| Tabensky-Taub     | 0 (a)         | 0             | (b)   | (b)           | 0        | 0        | yes ($p = \mu$) |
| Collins 71        | 0 (a)         | 0             | (b)   | (b)           | (a)      | 0        | yes ($p = (\gamma-1)\mu$) |
| Mars 95           | 0 (a)         | (a)           | (b)   | (b)           | (a)      | 0        | yes ($p = \mu$) |

Table 1. Perfect fluid cosmological models with an IS: (a) means that the relevant tensor is non-zero away from the IS but vanishes as the IS is approached, (b) means that the relevant tensor components are bounded as the IS is approached, with some components having a non-zero limit.

A number of interesting characteristics are apparent from the table. Apart from the FRW models, all other perfect fluid models which are known to admit an IS have an exact $\gamma$-law equation of state. This poses the question: do there exist any non-FRW barotropic perfect fluid cosmological models which admit an IS, yet which do not have an exact $\gamma$-law equation of state?

All perfect fluid models in the table are irrotational. Indeed, the General Vorticity Result of Scott\(^4\) proves that a barotropic perfect fluid cosmological model with non-zero vorticity cannot admit an IS. We also note that there is one class of models in the table, namely the Mars models, which have a non-geodesic fluid flow. We deduce from this that, unlike irrotationality, geodesicity of the fluid flow is not a necessary condition for a barotropic perfect fluid cosmological model to satisfy in order to admit an IS.

It remains an important and open problem to produce a precise set of necessary and sufficient conditions which a barotropic perfect fluid cosmological model must satisfy in order to admit an IS at which the fluid flow is regular.

We also note from the table that the FRW models are the only models present with Weyl tensor components $C_{abcd}$ which all vanish as the IS is approached. This lends weight to what is known as the FRW conjecture.\(^1\)\(^1\)\(^1\)\(^1\)\(^1\)\(^1\)
Conjecture 1 (FRW) If a space-time is
1. a $C^3$ solution of the Einstein field equations with a barotropic perfect fluid
   source, and
2. the unit timelike fluid congruence is regular at an isotropic singularity,
   and
3. the Weyl Curvature Hypothesis holds,
then the space-time is necessarily a Friedmann-Robertson-Walker model.

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