THE RELATIVE COMMUTANT OF SEPARABLE
C*-ALGEBRAS OF REAL RANK ZERO

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ABSTRACT. We answer a question of E. Kirchberg (personal communication): does the relative commutant of a separable C*-algebra in its ultrapower depend on the choice of the ultrafilter?

All algebras and all subalgebras in this note are C*-algebras and C*-subalgebras, respectively, and all ultrafilters are nonprincipal ultrafilters on \( \mathbb{N} \). Our C*-terminology is standard (see e.g., [2]).

In the following \( \mathcal{U} \) ranges over nonprincipal ultrafilters on \( \mathbb{N} \). With \( A^\mathcal{U} \) denoting the (norm, also called C*)-ultrapower of a C*-algebra \( A \) associated with \( \mathcal{U} \) we have

\[
F_\mathcal{U}(A) = A' \cap A^\mathcal{U},
\]

the relative commutant of \( A \) in its ultrapower. This invariant plays an important role in [8] and [7].

**Theorem 1.** For every separable infinite-dimensional C*-algebra \( A \) of real rank zero the following are equivalent.

1. \( F_\mathcal{U}(A) \cong F_\mathcal{V}(A) \) for any two nonprincipal ultrafilters \( \mathcal{U} \) and \( \mathcal{V} \) on \( \mathbb{N} \).
2. \( A^\mathcal{U} \cong A^\mathcal{V} \) for any two nonprincipal ultrafilters \( \mathcal{U} \) and \( \mathcal{V} \) on \( \mathbb{N} \).
3. The Continuum Hypothesis.

The equivalence of [3] and [2] in Theorem 1 for every infinite-dimensional C*-algebra \( A \) of cardinality \( 2^{\aleph_0} \) that has arbitrarily long finite chains in the Murray-von Neumann ordering of projections was proved in [6, Corollary 3.8], using the same Dow's result from [4] used here.

We shall prove (1) implies (3) and (2) implies (3) in Corollary 10 below. The reverse implications are well-known consequences of countable saturatedness of ultrapowers associated with nonprincipal ultrafilters on \( \mathbb{N} \) (see [1, Proposition 7.6]). The implication from (3) to (1) holds for every separable C*-algebra \( A \) and the implication from (3) to (2) holds for every C*-algebra \( A \) of size \( 2^{\aleph_0} \). The point is that if \( A \) is separable then the isomorphism between diagonal copies of \( A \) extends to an isomorphism between

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the ultrapowers. Countable saturation of $A^U$ can be proved directly from its analogue, due to Keisler, in classical model theory. This also follows from the argument in [6, Theorem 3.2 and Remark 3.3].

While the Continuum Hypothesis implies that any two ultrapowers of $\mathcal{B}(H)$ associated with nonprincipal ultrafilters on $\mathbb{N}$ are isomorphic, it does not imply that the relative commutants of $\mathcal{B}(H)$ in those ultrapowers are isomorphic. As a matter of fact, it implies the opposite (see [5]).

For a C*-algebra $A$ let $\mathcal{P}(A) = \{p : p \in A \text{ is a projection}\}$ ordered by $p \leq q$ if and only if $pq = p$. Our proof depends on the analysis of types of gaps in $\mathcal{P}(A' \cap A^U)$ (see Definition 4). Gaps in $\mathcal{P}(\mathbb{N})/\text{Fin}$ and related quotient structures are well-studied; for example, analysis of such gaps is very important in the consistency proof of the statement ‘all Banach algebra automorphisms of $C(X)$ into some Banach algebra are continuous’ (see [3]). It was recently discovered that the gap-spectrum of $\mathcal{P}(C(H))$ (where $C(H)$ is the Calkin algebra, $\mathcal{B}(H)/\mathcal{K}(H)$)) is much richer than the gap-structure of $\mathcal{P}(\mathbb{N})/\text{Fin}$ ([12]).

**Notational convention.** We denote elements of ultraproducts by boldface Roman letters such as $p$ and their representing sequences by $p(n)$, for $n \in \mathbb{N}$. We shall follow von Neumann’s convention and identify a natural number $n$ with the set $\{0, \ldots, n - 1\}$. The symbol $\omega$ is used for ultrafilters in the operator algebra literature and it is reserved for the least infinite ordinal in the set-theoretic literature. I will avoid using it in this note.

By $\sigma(a)$ we denote the spectrum of a normal operator $a$. Lemma 2 below is well-known. A sharper result can be found e.g., in [9, Lemma 2.5.4] but we include a proof for reader’s convenience.

**Lemma 2.** For a self-adjoint $a$ and a projection $r$, if $\|a - r\| < \varepsilon < 1$ then $\sigma(a) \subseteq (-2\sqrt{\varepsilon}, 2\sqrt{\varepsilon}) \cup (1 - 2\sqrt{\varepsilon}, 1 + 2\sqrt{\varepsilon})$. If in addition $\varepsilon < 1/16$ then there is a projection $r'$ in $C^*(a)$ such that $\|r' - a\| < 2\sqrt{\varepsilon}$.

**Proof.** Since $\|a\| < 1 + a < 2$, we have $\|a^2 - a\| \leq \|a(a - r)\| + \|r(a - r)\| + \|a - r\| < 4\varepsilon$. Thus $|a(x - 1)| < 4\varepsilon$ for all $x \in \sigma(a)$ and in turn $|x| < 2\sqrt{\varepsilon}$ or $|1 - x| < 2\sqrt{\varepsilon}$.

Now assume $\varepsilon < 1/16$. In this case $1/2 \notin \sigma(a)$. Define a continuous function $f$ with domain $\sigma(a)$ as follows. Let $f(t) = 0$ for $-\infty < t < 1/2$ and $f(t) = 1$ for $1/2 \leq t < \infty$. Since $|f(t) - t| < 2\sqrt{\varepsilon}$ for all $t \in \sigma(a)$, $f(a)$ is a projection in $C^*(a)$ as required. \hfill $\Box$

A representing sequence $p(n)$ of a projection $p$ in an ultrapower can be chosen so that each $p(n)$ is a projection (see [6, Proposition 2.5 (1)]), this also follows immediately from [10, Lemma 4.2.2] or [9, Lemma 2.5.5]).

**Lemma 3.** For projections $p, q$ in $A^U$ the following are equivalent.

1. $p \leq q$.
2. There is a representing sequence $p'(i)$, for $i \in \mathbb{N}$, of $p$ such that $p'(i) \leq q(i)$ for all $i$. 
(3) There is a representing sequence $q'(i)$, for $i \in \mathbb{N}$, of $q$ such that $p(i) \leq q'(i)$ for all $i$.

Proof. Both (3) implies (1) and (2) implies (1) are trivial. We shall prove (1) implies (2). Assume $p \leq q$. For every $n \geq 1$ the set

$$X_n = \{j : \|q(j)p(j)q(j) - p(j)\| < 1/(4n)\}$$

belongs to $\mathcal{U}$. We may assume $\bigcap_n X_n = \emptyset$. Let $p'(j) = 0$ if $j \notin X_0$. If $j \in X_n \setminus X_{n+1}$ then Lemma 2 implies there is a projection $p'(j) \in \mathcal{C}^*(a(j))$ such that $\|p'(j) - a(j)\| < 1/(2\sqrt{n})$. Then $p'(j) \leq q(j)$ and $\|p'(j) - p(j)\| < 1/\sqrt{n}$ for all $j \in X_n$. Therefore $p'(j)$, for $j \in \mathbb{N}$, is a representing sequence of $p$ as required.

In order to prove (1) implies (3) apply the above to $1 - p \geq 1 - q$ in the ultrapower of the unitization of $A$ to find an appropriate representing sequence for $1 - q$. □

By $\mathbb{N}_{/\mathbb{N}}$ we denote the set of all nondecreasing functions $f$ from $\mathbb{N}$ to $\mathbb{N}$ such that $\lim f(n) = \infty$, ordered pointwise. Write $f \leq_U g$ if $\{n : f(n) \leq g(n)\} \in \mathcal{U}$ and denote the quotient linear ordering by $\mathbb{N}_{/\mathbb{N}}/\mathcal{U}$.

Following [4], for an ultrafilter $\mathcal{U}$ we write $\kappa(\mathcal{U})$ for the cointiality of $\mathbb{N}_{/\mathbb{N}}/\mathcal{U}$, i.e., the minimal cardinality of $X \subseteq \mathbb{N}_{/\mathbb{N}}$ such that for every $g \in \mathbb{N}_{/\mathbb{N}}$ there is $f \in X$ such that $f \leq_U g$. (It is not difficult to see that this is equal to $\kappa(\mathcal{U})$ as defined in [4], Definition 1.3.)

Definition 4. Let $\lambda$ be a cardinal. An $(\aleph_0, \lambda)$-gap in a partially ordered set $\mathbb{P}$ is a pair consisting of a $\leq_{\mathbb{P}}$-increasing family $a_m$, for $m \in \mathbb{N}$, and a $\leq_{\mathbb{P}}$-decreasing family $b_\gamma$, for $\gamma < \lambda$, such that $a_m \leq_{\mathbb{P}} b_\gamma$ for all $m$ and $\gamma$ but there is no $c \in \mathbb{P}$ such that $a_m \leq_{\mathbb{P}} c$ for all $m$ and $c \leq_{\mathbb{P}} b_\gamma$ for all $\gamma$.

Assume $r^0(n) \leq r^1(n) \leq \cdots \leq r^{l(n) - 1}(n)$ are projections in $A$ and $\lim_{n \to \infty} l(n) = \infty$. For $h: \mathbb{N} \to \mathbb{N}$ define $r^h$ via its representing sequence (let $r^i(n) = r^{l(n) - 1}(n)$ for $i \geq l(n)$)

$$r^h(n) = r^{h(n)}(n).$$

Let $p_m = r^m$, where $m(j) = m$ for all $j$.

Lemma 5. With notation from the previous paragraph, for every projection $s$ in $\mathcal{A}\mathcal{U}$ such that $p_m \leq s$ for all $m$ there is $h: \mathbb{N} \to \mathbb{N}$ such that $p_m \leq r^h$ for all $m$ and $r^h \leq s$.

Proof. Since $p_m \leq s$, for each $m \in \mathbb{N}$ the set

$$X_m = \{i : \|r^m(i)s(i) - r^m(i)\| < 1/m\}$$

belongs to $\mathcal{U}$. Since the value of $\|r^m(i)s(i) - r^m(i)\|$ is increasing in $m$ we have $X_m \supseteq X_{m+1}$. We may assume $\bigcap_m X_m = \emptyset$. Define $h: \mathbb{N} \to \mathbb{N}$ by letting $h(i) = 0$ for $i \notin X_0$ and for $i \in X_m \setminus X_{m+1}$ let $h(i) = m$.

For each $m$ and $i \in X_m$ we have $h(i) \geq m$ and therefore $r^h \geq p_m$. Also, $i \in X_m$ implies $\|r^h(i)s(i) - r^h(i)\| < 1/m$ hence $r^h \leq s$. □
The proof of Proposition 4 was inspired by Alan Dow’s [1] Proposition 1.4. Dow’s result was independently proved by Saharon Shelah and can be found in [11].

By $A_{\leq 1}$ we denote the unit ball of a C*-algebra $A$.

**Proposition 6.** Assume $A$ is a separable C*-algebra and there are finite self-adjoint sets $F_0 \subseteq F_1 \subseteq F_2 \subseteq \cdots \subseteq A_{\leq 1}$ whose union is dense in $A_{\leq 1}$ and such that for each $n$ there is a $\leq$-increasing chain $C_n$ of projections in $B_n = F_n \cap A$ of length at least $n$.

Then for every nonprincipal ultrafilter $\mathcal{U}$ on $\mathbb{N}$ and every cardinal $\lambda$ there is an $(\aleph_0, \lambda)$-gap in $\mathcal{P}(A' \cap A^\mathcal{U})$ if and only if $\kappa(\mathcal{U}) = \lambda$.

**Proof.** First we prove the converse implication. Assume $g_\gamma$, for $\gamma < \lambda = \kappa(\mathcal{U})$, is a $\leq_{\mathcal{U}}$-decreasing and $\leq_{\mathcal{U}}$-unbounded below chain of functions in $\mathbb{N}^{/\mathbb{N}}$. Let $0 = r^0(n) \leq r^1(n) \leq \cdots \leq r_{n-1}(n)$ be an enumeration of $C_n$.

**Claim 7.** For all $f, g$ in $\mathbb{N}^{/\mathbb{N}}$ the following are equivalent.

1. $f \leq_{\mathcal{U}} g$,
2. $r^f \leq r^g$.

**Proof.** Assume $f \leq_{\mathcal{U}} g$. Then $X = \{j : f(j) \leq g(j)\} \in \mathcal{U}$ and $r^f(j) \leq r^g(j)$ for all $j \in X$ hence (2) follows. If $f \not\leq_{\mathcal{U}} g$ then $X = \{j : f(j) > g(j)\} \in \mathcal{U}$ and for all $j \in X$ we have $\|r^f(i) r^g(i) - r^g(i)\| = 1$, hence $r^f \not\leq r^g$. 

Let $q_\gamma = r^g$, for $\gamma < \lambda$. By Claim 7 we have

$p_m \leq p_{m+1} \leq q_\delta \leq q_\gamma$

for all $m$ and all $\gamma < \delta < \lambda$. All of $p_m$ and $q_\gamma$ belong to $A' \cap A^\mathcal{U}$.

We shall show that this family forms a gap in $\mathcal{P}(A^\mathcal{U})$ (and therefore it forms a gap in $\mathcal{P}(A' \cap A^\mathcal{U})$). Assume $s \in A^\mathcal{U}$ is such that $s \leq q_\gamma$ for all $\gamma$. By Lemma 5 there is $h$ such that $p_m \leq r^h \leq s$ for all $m$. By Claim 7 we have $h \leq_{\mathcal{U}} g$, for all $\gamma$ and $m \leq_{\mathcal{U}} h$ for all $m$, a contradiction.

In order to prove the direct implication, assume that $p_m$, $q_\gamma$ form an $(\aleph_0, \lambda)$-gap in $\mathcal{P}(A' \cap A^\mathcal{U})$. By successively using Lemma 5 for $m = 1, 2, \ldots$ find representing sequences $p_m(i)_{i \in \mathbb{N}}$, for $p_m$ such that $p_m(i) \leq p_{m+1}(i)$ for all $i$. Choose an increasing sequence $0 = m_0 < m_1 < m_2 < \ldots$ such that the following holds for all $k$.

(*) for all $j < m_k$ and all $a \in F_{m_k}$, if $l \geq m_{k+1}$ then $\|[p_j(l), a]\| < 1/k$.

For $n \in \mathbb{N}$ and $i$ such that for some $k$ we have $i < m_k$ and $m_{k+1} \leq n$ let $r^i(n) = p_i(n)$. Thus we have projections

$r^0(n) \leq r^1(n) \leq \cdots \leq r^{m_k}(n)$

whenever $n \geq m_{k+1}$. For $h : \mathbb{N} \to \mathbb{N}$ define $r^h$ as in the paragraph before Lemma 5 by its representing sequence (let $r^f(n) = r^{m_k}(n)$ if $i \geq m_k$)

$r^h(n) = r^{h(n)}(n)$.

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1I could not find it, but it should be somewhere in Chapter VI.
Claim 8. If $h: \mathbb{N} \to \mathbb{N}$ then $r^h \in A' \cap A^\mathcal{U}$.

Proof. Fix any $b$ in the unit ball of $A$ and $\varepsilon > 0$. If $k > 1/\varepsilon$ and there is $b' \in F_{2k}$ satisfying $\|b - b'\| < \varepsilon/2$ then for $i > n_{2k}$ in $Y$ we have that $\|p_j(i), b'\| < \varepsilon/2$ and therefore $\|r^h(i), b\| < \varepsilon$ for $\mathcal{U}$-many $i$. □

Using Lemma 5 for each $q_i$ find $h_\gamma$ such that $r^\gamma = r^{h_\gamma}$ satisfies $p_i \leq r^\gamma$ for all $i$. Since $\mathbb{N}/\mathcal{U}$ is a linear ordering and $\lambda$ is a regular cardinal, we can find a cofinal subset $Z$ of $\lambda$ such that for all $\gamma < \delta$ in $Z$ we have $r^\delta \leq r^{\gamma}$. By reenumerating we may assume $Z = \lambda$ and then $r^{\gamma}$, for $\gamma \in Z$, together with $p_i$, for $i \in \mathbb{N}$, form an $(\aleph_0, \lambda)$-gap. However, $r^\delta \leq r^{\gamma}$ is equivalent to $h_\delta \leq_U h_\gamma$, and therefore $h_\gamma$, for $\gamma < \lambda$, form a $\leq_U$-decreasing and $\leq_U$-unbounded below sequence in $\mathbb{N}/\mathcal{U}$, and therefore $\lambda = \kappa(\mathcal{U})$. □

The proof of Proposition 8 can be modified (by removing some of its parts) to a proof of the following.

Proposition 9. Assume $A$ is a separable $C^*$-algebra and $\mathcal{P}(A)$ has arbitrarily long finite chains. Then for every nonprincipal ultrafilter $\mathcal{U}$ on $\mathbb{N}$ and every cardinal $\lambda$ there is an $(\aleph_0, \lambda)$-gap in $\mathcal{P}(A^\mathcal{U})$ if and only if $\kappa(\mathcal{U}) = \lambda$. □

Corollary 10. Assume the Continuum Hypothesis fails. If $A$ is an infinite-dimensional separable $C^*$-algebra of real rank zero then there are nonprincipal ultrafilters $\mathcal{U}$ and $\mathcal{V}$ on $\mathbb{N}$ such that $F_\mathcal{U}(A) \not\cong F_\mathcal{V}(A)$ and $A^\mathcal{U} \not\cong A^\mathcal{V}$.

Proof. By [4, Theorem 2.2] we can find $\mathcal{U}$ and $\mathcal{V}$ so that $\kappa(\mathcal{U}) = \aleph_1$ and $\kappa(\mathcal{V}) = \aleph_2$ (here $\aleph_1$ and $\aleph_2$ are the least two uncountable cardinals; all that matters for us is that they are both less or equal than $2^{\aleph_0}$ and different). Therefore $\mathcal{P}(A' \cap A^\mathcal{U})$ has an $(\aleph_0, \aleph_1)$-gap while $\mathcal{P}(A' \cap A^\mathcal{V})$ does not, and $A' \cap A^\mathcal{U}$ and $A' \cap A^\mathcal{V}$ cannot be isomorphic.

It remains to prove that if $A$ is an infinite-dimensional $C^*$-algebra of real rank zero then $\mathcal{P}(A)$ has an infinite chain of projections. We may assume $A$ is unital. Recursively find a decreasing sequence $r_n$ for $n \in \mathbb{N}$ in $\mathcal{P}(A)$ so that $r_n A_n r_n$ is infinite-dimensional for all $n$. Assume $r_n$ has been chosen. Since $A$ has real rank zero, in $r_n A_n r_n$ we can fix a projection $q \notin \{0, r_n\}$. If $q A_n q$ is infinite-dimensional then let $r_{n+1} = q$. Otherwise, let $r_{n+1} = r_n - q$ and note that $r_{n+1} A_n r_{n+1}$ is infinite-dimensional. □

It is likely that Theorem 1 and Corollary 10 can be extended to all infinite-dimensional separable $C^*$-algebras (possibly by considering the Cuntz ordering of positive elements instead of $\mathcal{P}(A)$).

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