$L^p$-Kato class measures and their relations with Sobolev embedding theorems

Takahiro Mori

Abstract

In this paper, we discuss relationships between the continuous embeddings of Dirichlet spaces $(\mathcal{F}, \mathcal{E}_1)$ into Lebesgue spaces and the integrability of the associated resolvent kernel $r_\alpha(x, y)$. For a positive measure $\mu$, we consider the following two properties; the first one is that the Dirichlet space $(\mathcal{F}, \mathcal{E}_1)$ is continuously embedded into $L^2_p(E; \mu)$ (which we write as $(\text{Sob})_p$), and the second one is that the family of 1-order resolvent kernels $\{r_1(x, y)\}_{x \in E}$ is uniformly $p$-th integrable in $y$ with respect to the measure $\mu$ (which we write as $(\text{Dyn})_p$).

Under some assumptions, for a measure $\mu$ satisfying $(\text{Dyn})_1$, we prove $(\text{Dyn})_p$ implies $(\text{Sob})_p$ for $1 \leq p < p' < \infty$, and prove $(\text{Sob})_p$ implies $(\text{Dyn})_p$ for $1 \leq p < p' < \infty$. To prove these results we introduce $p$-Kato class, an $L^p$-version of the set of Kato class measures, and discuss its properties. As an application, we discuss the continuity of intersection measures in time.

Keywords: Dirichlet form; Sobolev embedding theorem; Kato class; Resolvent kernel

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1 Introduction

In this paper, we discuss relationships between the continuous embeddings of Dirichlet spaces into Lebesgue spaces and the integrability of the associated resolvent kernel.

The prototype of the relationships we are focusing on is the classical Dirichlet integral $(\frac{1}{2}D, H^1(\mathbb{R}^d))$ on $\mathbb{R}^d$ and the associated resolvent kernel $r_\alpha(x, y), x, y \in \mathbb{R}^d, \alpha > 0$, that is, $H^1(\mathbb{R}^d)$ is the Sobolev space on $\mathbb{R}^d$,

$$D(u, v) = \sum_{i=1}^{d} \int_{\mathbb{R}^d} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} dx \quad \text{for } u, v \in H^1(\mathbb{R}^d)$$

and

$$r_\alpha(x, y) = \frac{1}{(2\pi)^{d/2}} \int_0^{\infty} \frac{1}{t^{d/2}} \exp\left\{-\left(\alpha t + \frac{|x-y|^2}{2t}\right)\right\} \, dt \quad \text{for } x, y \in \mathbb{R}^d, \alpha > 0.$$ 

The classical Sobolev embedding theorem on $\mathbb{R}^d$ is well known:

- $H^1(\mathbb{R}^d)$ is continuously embedded into $L^{2p}(\mathbb{R}^d)$ only for $p \in [1, \infty)$ with $d - p(d-2) \geq 0$ (hence, $2p \leq 2d/(d-2)$ when $d \geq 2$).

By an elementary calculation, it holds that

$$\sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} r_1(x, y)^p \, dy < \infty \text{ if and only if } d - p(d-2) > 0.$$ 

(1.2)

Note that $d - p(d-2)$ appears in both conditions (1.1) and (1.2). This means that there is a relation between the Sobolev embedding theorem and the integrability of the resolvent.

*Research Institute for Mathematical Sciences, Kyoto University, Kyoto, 606-8502, JAPAN. tmori@kurims.kyoto-u.ac.jp
The main purpose of this paper is to generalize such relations from the perspective of the Dirichlet form theory. Let \( E \) be a locally compact separable metric space, \( m \) be a Radon measure on \( E \) with \( \text{supp}[m] = E \), and let \((\mathcal{E}, \mathcal{F})\) be a Dirichlet form on \( L^2(E; m) \) with the associated resolvent kernel \( r_\alpha(x, y) \) with respect to \( m \). Suppose \( \mu \) is a Radon measure on \( E \). For \( p \in [1, \infty) \), we consider two properties;

\[(\text{Sob})_p \] the Hilbert space \((\mathcal{F}, \mathcal{E}_1)\) is continuously embedded into \( L^{2p}(E; \mu) \), that is, there exists a positive constant \( C > 0 \) such that \( ||u||^{2p}_{L^{2p}(E; \mu)} \leq C \mathcal{E}_1(u, u) \) for all \( u \in \mathcal{F} \),

\[(\text{Dyn})_p \] it holds that \( \sup_{x \in E} \int_{E} r_\alpha(x, y)^p \mu(dy) < \infty \).

The property \((\text{Dyn})_p\) is named after Dynkin, which is equivalent to \( D^p(X) \) defined later. The aim of this paper is to show the following: under some conditions, for any measure \( \mu \) satisfying \((\text{Dyn})_1\),

- if \((\text{Dyn})_{p'}\) holds for some \( p' \in [1, \infty) \), then \((\text{Sob})_p\) holds for all \( 1 \leq p \leq p' \), \hspace{1cm} (1.3)
- if \((\text{Sob})_{p'}\) holds for some \( p' \in [1, \infty) \), then \((\text{Dyn})_p\) holds for all \( 1 \leq p < p' \). \hspace{1cm} (1.4)

See Sections 4.1 and 4.2 for precise statements and proofs. We will prove (1.3) in Proposition 4.1 and the following Corollary 4.2. Regarding (1.4), Theorem 4.6 gives a stronger result, that is, we obtain the order of decay for the quantity \( \sup_{x \in E} \int_{E} r_\alpha(x, y)^p \mu(dy) \) as \( \alpha \uparrow \infty \).

By (1.3) and (1.4), (1.2) follows from (1.1) and (1.1) with \( d - p(d - 2) > 0 \) follows from (1.2).

The Sobolev inequality has been studied for various settings; Euclidean space, Riemannian manifolds, Lie groups, and so on (see [SC02, BCLSC95] for example). It is known that the Sobolev inequality is equivalent to the ultra-contractivity of the associated transition semigroup [Var85], the Nash type inequality [CKS87], and the capacity isoperimetric inequality [Kai92, FU03].

When \( p = p' = 1 \), our result (1.3) is related to the theory of the Kato class of measures. Kato class is introduced to analyse the Schrödinger semigroups and analyse integral kernels of semigroups given by Feynman-Kac functionals (see [AS82] and [ABM91] for example). The set of measures satisfying \((\text{Dyn})_1\) is so-called the Dynkin class (1-order version of Green-bounded measures). The embedding result (1.3) for \( p = p' = 1 \) is proved by Stollmann and Voigt [SV96] via the operator theory, and later Shiozawa and Takeda [ST05] proved it in terms of Dirichlet forms.

The organization of the paper is as follows. In Section 2.1, we give the framework. In Section 2.2, we introduce a \( p \)-Dynkin class, which is equivalent to \((\text{Dyn})_p\), and a \( p \)-Kato class, which is an \( L^p \)-analogy of the classical Kato class. We will give equivalent conditions of these classes in terms of heat kernels, so we can check that a measure is in the classes once an upper bound of the heat kernel such as the (sub-)Gaussian estimate (see (2.12)) or the jump type estimate (see (2.13)) holds for a short time. These estimates are established for many processes. Regarding the (sub-)Gaussian estimate, it is obtained for Brownian motion on a manifold [LY86], Brownian motion on a metric measure space with Riemannian curvature dimension condition [Stu06], Brownian motion on the Sierpiński gasket [BP88] and other diffusions on fractals [Bar98], and so on. Regarding the jump-type estimate, stable-like processes on \( d \) sets [CK03] are studied for example.

In Section 2.3, we introduce a subclass of the \( p \)-Kato class (denoted by \( K^{\delta}(X) \)), which has additional information on the order of decay for the quantity \( \sup_{x \in E} \int_{E} r_\alpha(x, y)^p \mu(dy) \) as \( \alpha \uparrow \infty \). Similarly to the \( p \)-Dynkin and the \( p \)-Kato classes, this subclass can be characterized via the heat kernel estimate, and the above example satisfies the estimate.

In Section 3, some relation between \( p \)-Kato classes for a process and for its time changed process is discussed. This section plays a key role later to prove (1.4). Sections 4.1 and 4.2 are devoted to proving (1.3) and (1.4) as mentioned above.

Section 5 is an application to the intersection of the paths of independent stochastic processes. Analysis of the intersection of Brownian paths was initiated by Dvoretzky, Erdős, Kakutani [DEK50,
DEK54] and Dvoretzky, Erdős, Kakutani and Taylor [DEKT57]. They gave the following dichotomy: for \( p \) independent Brownian motions \( B^{(1)}, \ldots, B^{(p)} \) on \( \mathbb{R}^d \),

the paths intersect, i.e., \( B^{(1)}(0,\infty) \cap \cdots \cap B^{(p)}(0,\infty) \neq \emptyset \) almost surely if

\[
d - p(d - 2) > 0,
\]

and does not intersect almost surely if \( d - p(d - 2) \leq 0 \). \hspace{1cm} (1.5)

Note that the same condition appears in (1.1) and (1.2).

Motivated by problems in statistical physics such as the configurations of interacting polymers, a random measure called the intersection local time has been introduced; see [LG92] for example. In this paper, we consider the occupation measure of the set of intersections for independent processes \( X^{(1)}, \ldots, X^{(p)} \) with the same distribution \( X \) which is formally written as

\[
\ell^\text{IS}_t(A) = \left\{ \int_A \prod_{i=1}^p \int_0^{t_i} \delta_x(X^{(i)}(s_i)) \, ds_i \right\} m(dx) \quad \text{for} \ A \in \mathcal{B}(E)
\]

and for \( t = (t_1, \ldots, t_p) \in [0,\infty)^p \), where \( \delta_x \) is the Dirac measure at \( x \), \( m \) is the reference measure of the processes and \( \mathcal{B}(E) \) is the family of Borel sets in \( E \). We call the measure as the (mutual) intersection measure named after König and Mukherjee [KM13]. Here and in the following, the superscript “IS” means “InterSection”.

In Theorem 5.1 we prove the following: if the reference measure \( m \) belongs to the subclass \( \mathcal{K}^{p,\delta}(X) \) introduced in Section 2.3, then the measure-valued process \( t \mapsto \ell^\text{IS}_t(dx) \) has a continuous modification, and the real-valued process \( t \mapsto \langle f, \ell^\text{IS}_t \rangle \) has a Hölder continuous modification for each bounded Borel function \( f \). This is a generalization of [Che10, Section 2.2], in which the results are obtained for independent Brownian motions.

2 \( p \)-Kato class and its variant

In this section, we give the framework and introduce the \( p \)-Kato class and its variants.

2.1 Framework

Let \( E \) be a locally compact, separable metric space and let \( m \) be a Radon measure on \( E \) with \( \text{supp}[m] = E \). Suppose \((\mathcal{E}, \mathcal{F})\) is a regular Dirichlet form on \( L^2(E; m) \) and \( X = (\Omega, X_t, \zeta, \mathbb{P}_x) \) is an associated \( m \)-symmetric Hunt process. For \( \alpha > 0 \) and \( u \in \mathcal{F} \), we simply write \( \mathcal{E}_\alpha(u, u) = \|u\|^2_{L^2} := \mathcal{E}(u, u) + \alpha \int_E u^2 \, dm \). In this paper, we always take the quasi-continuous version of the element \( u \) of \( \mathcal{F} \) (see [FOT11, Section 2] for example).

Throughout this paper, we assume the transition kernel \( (P_t)_{t>0} \) of \( X \) satisfies the absolute continuity condition:

\[
P_t(x, dy) \text{ is absolutely continuous with respect to } m(dy) \text{ for each } t > 0 \text{ and } x \in E.
\]

Note that the condition (2.1) implies the measurability of the heat kernel (see [Yan88, Theorem 2] for example):

\[
(P_t)_{t>0} \text{ admits a heat kernel } p_t(x, y) \text{ which is jointly measurable on } (0,\infty) \times E \times E \text{ such that } p_t(x, y) = p_t(y, x) \text{ and } p_{t+s}(x, y) = \int_E p_s(x, z)p_t(z, y) \, m(dz) \text{ for all } s, t > 0, x, y \in E.
\]

Remark 2.1 \hspace{1cm} We may consider a slightly weaker condition than (2.1):

There exists a Borel properly exceptional set \( N \) such that \( P_t(x, dy) \) is absolutely continuous with respect to \( m(dy) \) for each \( t > 0 \) and \( x \in E \setminus N \). \hspace{1cm} (2.3)

It is known that the Sobolev inequality implies (2.3) (see [FOT11, Theorem 4.27] for example). Under the condition, we can obtain the corresponding results of this paper by replacing \( \sup_{x \in E} \) by \( \inf_{\text{Cap}(N)=0} \sup_{x \in E \setminus N} \) as in [ABM91, (3.3)]. In this paper, we do not give detailed calculations under the assumption (2.3) and we always impose (2.1) for simplicity.
For each $\alpha > 0$, write the $\alpha$-order resolvent kernel of $X$ by $r_\alpha(x,y) = \int_0^\infty e^{-\alpha t}p_t(x,y)dt$.

We denote by $S_{00}(X)$ the set of positive Borel measures $\mu$ such that $\mu(E) < \infty$ and $R_1\mu(x) := \int_E r_1(x,y)\mu(dy)$ is uniformly bounded in $x \in E$. A positive Borel measure $\mu$ on $E$ is said to be smooth in the strict sense if there exists a sequence $\{E_n\}_{n=1}^\infty$ of Borel sets increasing to $E$ such that $1_{E_n} \cdot \mu \in S_{00}(X)$ for each $n$ and

$$\mathbb{P}_x\left(\lim_{n \to \infty} \sigma_{E \setminus E_n} \geq \zeta\right) = 1, \quad \text{for all } x \in E,$$

where $\sigma_{E \setminus E_n}$ is the first hitting time of $E \setminus E_n$. The totality of smooth measures in the strict sense is denoted by $S_1(X)$.

### 2.2 The class $\mathcal{K}^p(X)$

In this section, we introduce the $L^p$-version of the Kato class measures.

**Definition 2.2.** Let $p \in [1, \infty)$. For a positive Radon measure $\mu$ on $E$, $\mu$ is said to be of $p$-Kato class with respect to $X$ (write $\mu \in \mathcal{K}^p(X)$) if

$$\limsup_{\alpha \to \infty} \alpha \int_E r_\alpha(x,y)^p \mu(dy) = 0 \quad (2.4)$$

and $\mu$ is said to be of $p$-Dynkin class with respect to $X$ (write $\mu \in \mathcal{D}^p(X)$) if

$$\sup_{x \in E} \int_E r_\alpha(x,y)^p \mu(dy) < \infty \quad \text{for some } \alpha > 0. \quad (2.5)$$

Clearly $\mathcal{K}^p(X) \subset \mathcal{D}^p(X)$. The condition (Dyn)$_p$, which we introduced in Section 1 is nothing else the $p$-Dynkin class.

**Remark 2.3.**

(i) $\mathcal{K}^1(X)$ and $\mathcal{D}^1(X)$ are so-called the set of Kato and Dynkin class measures, respectively. The reference measure $m$ always belongs to $\mathcal{K}^1(X)$.

(ii) If $\mu(E) < \infty$, Hölder’s inequality gives that $\mu \in \mathcal{K}^p(X)$ implies $\mu \in \mathcal{K}^q(X)$ for $1 \leq p < p'$.

**Example 2.4** (Brownian motion on $\mathbb{R}^d$). Suppose $E = \mathbb{R}^d$, $m$ is the Lebesgue measure on $\mathbb{R}^d$ and $X$ is a Brownian motion on $\mathbb{R}^d$. Let $p \in [1, \infty)$ with $d - p(d - 2) > 0$ and $\mu$ be a positive Radon measure on $\mathbb{R}^d$. By the same way as the proof of [AS82, Theorem 4.5], $\mu \in \mathcal{K}^p(X)$ if and only if

$$\limsup_{\alpha \to 0} \int_{|x-y| < \alpha} \frac{\mu(dy)}{|x-y|^{d-2}} = 0, \quad d \geq 3,$$

$$\limsup_{\alpha \to 0} \int_{|x-y| < \alpha} (- \log |x-y|)^p \mu(dy) = 0, \quad d = 2,$$

$$\sup_{x \in \mathbb{R}^d} \int_{|x-y| \leq 1} \mu(dy) < \infty, \quad d = 1.$$  

In particular, when $d = 1$, $\mathcal{K}^1(X) = \mathcal{K}^p(X)$ for any $p > 1$.

By the above characterization, we may give a sufficient condition for $\mathcal{K}^p(X)$. If a Borel function $f$ on $\mathbb{R}^d$ satisfies $\sup_{x \in \mathbb{R}^d} \int_{|x-y| \leq 1} |f(x)|^r \mu(dy) < \infty$ for some $r > d/(d - p(d - 2))$ for $d \geq 2$, or $r = 1$ for $d = 1$, then the measure $f(x)dx$ is in the class $\mathcal{K}^p(X)$. This gives an extension of [AS82, Theorem 1.4 (iii)], in which the result is obtained for $p = 1$. (See also [KT07], in which such results are obtained under more general heat kernel estimates.) In particular, $|x|^{-\beta}dx \in \mathcal{K}^p(X)$ if $\beta < d - p(d - 2)$ for $d \geq 2$, and $\beta < 1$ for $d = 1$.

When $d \geq 3$, Schechter [Sch71] introduced related classes $M_{\alpha,r}$ ($\alpha > 0, r > 1$) of functions $V$ given by

$$\sup_{x \in \mathbb{R}^d} \int_{|x-y| \leq 1} \frac{|V(y)|^r}{|x-y|^{d-\alpha}} dy < \infty,$$

and [AS82] studied relations between $M_{\alpha,r}$ and the classical Kato class $\mathcal{K}^1(X)$. By Hölder’s inequality, we have $M_{\alpha,r} \subset \mathcal{K}^p(X)$ if $r > \alpha/(d - p(d - 2))$. This is an extension of [AS82, Proposition 4.1, 4.2],
in which the result are obtained for \( p = 1 \). (Note that there are typos in [AS82]; \( \beta > 2 \) in Proposition 4.1 (resp. \( \alpha > 2p \) in Proposition 4.2) should be \( \beta < 2 \) (resp. \( \alpha < 2p \)).

The following proposition is the \( L^p \)-version of [ABM91, Proposition 3.8] in some sense.

**Proposition 2.5.** Let \( p \in [1, \infty) \). It holds that
\[
\mathcal{D}^p(X) \subset S_1(X).
\]

**Proof.** Suppose \( \mu \in \mathcal{D}^p(X) \) and assume first \( \mu(E) < \infty \). As in Remark 2.3, we have \( \mu \in \mathcal{D}^1(X) \), that is, \( \sup_{x \in E} R_1 \mu(x) < \infty \). This means \( \mu \) is of \( S_0(X) \) and hence is in \( S_1(X) \).

When \( \mu \in \mathcal{D}^p(X) \) may not be a finite measure, take a sequence \( \{E_n\}_{n=1}^{\infty} \) of relatively compact open sets that go to \( E \) as \( n \uparrow \infty \). By the above, we have \( 1_{E_n} \cdot \mu \in S_0(X) \). Set \( \sigma = \lim_{n \to \infty} \sigma_{E_n \setminus E_n} \). For each \( x \in E \), the quasi-left-continuity of the Hunt process \( X \) (see for example, [FOT11, Appendix A.2]) implies that
\[
\lim_{n \to \infty} X_{\sigma_{E_n \setminus E_n}} = X_\sigma, \quad \mathbb{P}_x\text{-a.s. on } \{ \sigma < \infty \}
\]
and then \( \mathbb{P}_x(\sigma \geq \zeta) = 1 \), which concludes \( \mu \in S_1(X) \).

The next two propositions characterize the \( p \)-Dynkin class and the \( p \)-Kato class in terms of the heat kernel.

**Proposition 2.6.** Let \( p \in [1, \infty) \). For a Radon measure \( \mu \) on \( E \), the following are equivalent:

(i) \( \mu \in \mathcal{D}^p(X) \),

(ii) \( \sup_{x \in E} \int_E r_\alpha(x, y)^p \mu(dy) < \infty \) for all \( \alpha > 0 \),

(iii) \( \sup_{x \in E} \int_E \left( \int_0^t p_s(x, y) ds \right)^p \mu(dy) < \infty \) for some \( t > 0 \),

(iv) \( \sup_{x \in E} \int_E \left( \int_0^t p_s(x, y) ds \right)^p \mu(dy) < \infty \) for all \( t > 0 \).

**Proof.** Trivially (ii) implies (i) and (iv) implies (iii).

Assume \( \mu \in \mathcal{D}^p(X) \) and take \( \beta > 0 \) such that \( \sup_{x \in E} \int_E r_\beta(x, y)^p \mu(dy) < \infty \). For \( \alpha > \beta \), the monotonicity of the resolvent clearly implies that \( \sup_{x \in E} \int_E r_\alpha(x, y)^p \mu(dy) < \infty \). For \( 0 < \alpha < \beta \), fix \( x \in E \) and set \( F(\cdot) := \int_E r_\alpha(x, z) z r_\beta(z, \cdot) m(dz) \). Then we have by Hölder’s inequality,
\[
\int_E F(y)^p \mu(dy) = \int_E \left( \int_E F(y)^{p-1} r_\beta(z, y) \mu(dy) \right) r_\alpha(x, z) m(dz) \\
\leq \int_E \left( \int_E F(y)^{p} \mu(dy) \right)^{\frac{1}{p}} \left( \int_E r_\beta(z, y)^{p} \mu(dy) \right)^{\frac{1}{p}} r_\alpha(x, z) m(dz) \\
\leq \frac{1}{\alpha} \left( \int_E F(y)^{p} \mu(dy) \right)^{\frac{1}{p}} \left( \sup_{z \in E} \int_E r_\beta(z, y)^{p} \mu(dy) \right)^{\frac{1}{p}},
\]
which implies that
\[
\left( \int_E F(y)^{p} \mu(dy) \right)^{\frac{1}{p}} \leq \frac{1}{\alpha} \left( \sup_{z \in E} \int_E r_\beta(z, y)^{p} \mu(dy) \right)^{\frac{1}{p}}.
\]

Hence the resolvent equation \( r_\alpha(x, y) = r_\beta(x, y) + (\beta - \alpha) \int_E r_\alpha(x, z) r_\beta(z, y) \mu(m(dz)) \) yields that
\[
\left( \int_E r_\alpha(x, y)^p \mu(dy) \right)^{\frac{1}{p}} \leq \left( \int_E r_\beta(x, y)^p \mu(dy) \right)^{\frac{1}{p}} + (\beta - \alpha) \left( \int_E F(y)^p \mu(dy) \right)^{\frac{1}{p}} \\
\leq \frac{\beta}{\alpha} \left( \sup_{z \in E} \int_E r_\beta(z, y)^p \mu(dy) \right)^{\frac{1}{p}} < \infty,
\]

(2.6)
which concludes (ii). Moreover, we have for any \( t > 0, x \in E \) and \( \alpha > 0 \),
\[
\int_E \left( \int_0^t p_s(x,y)ds \right)^p \mu(dy) \leq e^{\alpha t} \sup_{x \in E} \int_E r_\alpha(x,y)^p \mu(dy),
\] (2.7)
which concludes (iv).

Next, assume (iii). Take \( t_0 > 0 \) such that \( \sup_{x \in E} \int_E \left( \int_0^{t_0} p_s(x,y)ds \right)^p \mu(dx) < \infty \). For any \( t \leq t_0 \) and \( \alpha > 0 \), we have
\[
\int_E \left( \int_0^{\alpha + t} p_s(x,y)ds \right)^p \mu(dy) \leq \sup_{z \in E} \int_E \left( \int_0^t p_s(x,y)ds \right)^p \mu(dy).
\] (2.8)
Indeed, the Chapman-Kolmogorov equation gives that the left-hand side equals
\[
\int_E \left\{ \int_0^t \int_E p_s(x,z)p_s(z,y)m(dz)ds \right\} \mu(dy) = \int_E \left\{ \int_0^t p_s(y,z)m(dz) \right\} \mu(dy).
\]
Applying Hölder’s inequality with the measure \( p_s(x,z)m(dz) \), the above equation is bounded from above by
\[
\int_E \left\{ \int_0^t \int_E p_s(x,z)m(dz)ds \right\}^p \mu(dy) \leq \int_E \left\{ \int_0^t p_s(y,z)m(dz) \right\} \mu(dy) \leq \sup_{x \in E} \int_E \left( \int_0^t p_s(x,y)ds \right)^p \mu(dy),
\]
where we used \( P_1 1 \leq 1 \) in the last two lines. This proves (2.8).

Now, suppose \( t > 0 \). By taking large \( N \) such that \( Nt_0 \geq t \), we have from (2.8)
\[
\left\{ \int_E \left( \int_0^t p_s(x,y)ds \right)^p \mu(dy) \right\}^{\frac{1}{p}} \leq \sum_{n=0}^{N-1} \left\{ \int_E \left( \int_{nt_0}^{(n+1)t_0} p_s(x,y)ds \right)^p \mu(dy) \right\}^{\frac{1}{p}} \leq N \left\{ \sup_{x \in E} \int_E \left( \int_0^{t_0} p_s(x,y)ds \right)^p \mu(dy) \right\}^{\frac{1}{p}},
\]
which concludes (iv).

Further, for any \( \alpha > 0, x \in E \) and \( t \leq t_0 \), we have from the triangle inequality,
\[
\left( \int_E r_\alpha(x,y)^p \mu(dy) \right)^{\frac{1}{p}} = \left\| \int_0^\infty e^{-\alpha s} p_s(x,\cdot)ds \right\|_{L^p(E;\mu)} \leq \sum_{n=0}^{\infty} \left\| \int_{nt}^{(n+1)t} e^{-\alpha s} p_s(x,\cdot)ds \right\|_{L^p(E;\mu)}
\]
and by (2.8), the right-hand side is bounded from above by
\[
\sum_{n=0}^{\infty} e^{-\alpha nt} \sup_{x \in E} \left\{ \int_E \left( \int_0^t p_s(x,y)ds \right)^p \mu(dy) \right\}^{\frac{1}{p}} \leq \frac{1}{1 - e^{-\alpha t}} \sup_{x \in E} \left\{ \int_E \left( \int_0^t p_s(x,y)ds \right)^p \mu(dy) \right\}^{\frac{1}{p}},
\] (2.9)
which concludes (ii).

\[
\square
\]

**Corollary 2.7.** Let \( p \in [1, \infty) \). Then, for a Radon measure \( \mu \) on \( E \), the following are equivalent:
(i) \( \mu \in K^p(X) \),
(ii) \( \lim \sup_{t \downarrow 0} \int_E \left( \int_0^t p_s(x,y)ds \right)^p \mu(dy) = 0 \).

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Proof. (i) implies (ii) by letting $t \downarrow 0$ and then $\alpha \uparrow \infty$ in (2.7). Conversely, (ii) implies (i) by letting $\alpha \uparrow \infty$ and then $t \downarrow 0$ in (2.9).

Remark 2.8. Let $Y$ be the 1-subprocess of $X$, that is, the $m$-symmetric Markov process with transition probability $e^{-t}p_t(x,y)m(dy)$. Clearly $Y$ satisfies the absolute continuity condition (2.1). We claim that $\mathcal{K}^p(X) = \mathcal{K}^p(Y)$. Indeed, we have the inclusion $\mathcal{K}^p(X) \subset \mathcal{K}^p(Y)$ since the $\alpha$-order resolvent kernel of $Y$ is $r_{1+\alpha}(x,y)$ and the inequality $r_{1+\alpha}(x,y) \leq r_{\alpha}(x,y)$ holds. By applying (2.6) with $\beta = \alpha + 1$, we have the converse inclusion $\mathcal{K}^p(Y) \subset \mathcal{K}^p(X)$. In the same way, we also have $\mathcal{D}^p(X) = \mathcal{D}^p(Y)$.

2.3 The class $\mathcal{K}^{p,\delta}(X)$

In this section, we introduce a subclass of $p$-Kato class, which has additional information on the order of decay of the quantities $\sup_{x \in E} \int_E r_{\alpha}(x,y)^p \mu(dy)$ (they have been introduced in Definition 2.2) as $\alpha \uparrow \infty$.

Definition 2.9. Let $p \in [1, \infty)$ and $\delta \in (0, 1]$. For a positive Radon measure $\mu$ on $E$, $\mu$ is said to be of $p$-Kato class with order $\delta$ (write $\mu \in \mathcal{K}_{p,\delta}(X)$) if

$$\sup_{x \in E} \left( \int_E r_{\alpha}(x,y)^p \mu(dy) \right)^{\frac{1}{p}} = O(\alpha^{-\delta}) \quad \text{as} \quad \alpha \to \infty.$$  

That is, there exist constants $C > 0$ and $\alpha_0 > 0$ such that the left-hand side is bounded from above by $C\alpha^{-\delta}$ for all $\alpha > \alpha_0$. Clearly $\mathcal{K}_{p,\delta}(X) \subset \mathcal{K}^p(X)$.

Similarly to Proposition 2.6 and Corollary 2.7, we can characterize the set of $p$-Kato class measures with order $\delta$ in terms of the heat kernel.

Proposition 2.10. Let $p \in [1, \infty)$ and $\delta \in (0, 1]$. For a Radon measure $\mu$ on $E$, the following are equivalent:

(i) $\mu \in \mathcal{K}_{p,\delta}(X)$,

(ii) $\sup_{x \in E} \left( \int_E \left( \int_0^t p_s(x,y)ds \right)^p \mu(dy) \right)^{\frac{1}{p}} = O(t^\delta)$ as $t \to 0$.

Proof. By setting $\alpha t = 1$, (i) implies (ii) from (2.7) and (ii) implies (i) from (2.9). □

In the following, we write

$$\gamma(\alpha) := \sup_{x \in E} \left( \int_E r_{\alpha}(x,y)^p \mu(dy) \right)^{\frac{1}{p}} \tag{2.10}$$

and write

$$\eta(t) := \sup_{x \in E} \left( \int_E \left( \int_0^t p_s(x,y)ds \right)^p \mu(dy) \right)^{\frac{1}{p}}. \tag{2.11}$$

Remark 2.11.

(i) As we see in Corollary 2.7, if $\mu \in \mathcal{K}^p(X)$ then $\eta(t) < \infty$ for all $t > 0$. Hence, under (2.2), $\mu \in \mathcal{K}_{p,\delta}(X)$ if and only if $\sup_{t < 0} \left\{ t^{-\delta} \eta(t) \right\} < \infty$ for some (also for all) $T > 0$.

(ii) Let $Y$ be the 1-subprocess of $X$. By the same way as in Remark 2.8, one can show that $\mathcal{K}_{p,\delta}(X) = \mathcal{K}_{p,\delta}(Y)$.

Example 2.12 (Brownian motion). We continue with Example 2.4. The heat kernel of the Brownian motion on $\mathbb{R}^d$ is $p_t(x-y)$, where

$$p_t(x) = (2\pi t)^{-\frac{d}{2}} e^{-\frac{|x|^2}{2t}} \quad t > 0, x \in \mathbb{R}^d.$$
Then, as on page 34 of [Che10], we have for $t > 0$,
\[
\int_{\mathbb{R}^d} \left( \int_0^t p_s(x) ds \right)^p dx \leq (2\pi)^{\frac{d(p-1)}{2}} p^{-\frac{d}{2}} \left( \frac{2p}{2p-d(p-1)} \right)^{p-\frac{2p-d(p-1)}{2}},
\]
whenever $2p - d(p-1) = d - p(d - 2) > 0$. Hence, the Lebesgue measure on $\mathbb{R}^d$ is in $K^{\frac{d-p(d-2)}{2p}}(X)$ if $d - p(d - 2) > 0$.

**Example 2.13** (Sub-)Gaussian heat kernel estimate. We fix a metric $\rho$ on $E$. Denote $\text{diam}(E) := \sup\{\rho(x, y) : x, y \in E\}$ the diameter of $E$ and denote $B(x, r)$ the open ball with center $x \in E$ and radius $r > 0$. Assume that $\text{diam}(E) = 1, m(E) < \infty$ and there exist constants $c_1, c_2 > 0$ and $d_t \geq 1$ such that $c_1 r^{d_t} \leq m(B(x, r)) \leq c_2 r^{d_t}$ for all $x \in E, r \in (0, 1]$. We also assume that $p_t(x, y)$ enjoys the (sub-)Gaussian heat kernel upper estimate: there exist constants $c_3, c_4 > 0$ and $d_w \geq 2$ such that
\[
p_t(x, y) \leq c_3 t^{-\frac{d_t}{d_w}} \exp\left\{ -c_4 \frac{d(x, y)^{d_w}}{t} \right\} \quad \text{for all } x \in E, r \in (0, 1]. \tag{2.12}
\]
Then, a straightforward calculation gives that $m \in K^{p,\delta}(X)$ if $pd_w\delta < d_t - p(d_t - d_w)$, i.e., $\delta < (d_s - p(d_s - 2))/2p$ by setting $d_s := 2d_t/d_w$. $d_t, d_w$ and $d_s$ are the so-called fractal dimension of $E$ and walk dimension and spectrum dimension of the process $X$, respectively.

**Example 2.14** (Jump-type heat kernel estimate). Under the setting of Example 2.13 we assume that $p_t(x, y)$ enjoys the jump-type heat kernel upper estimate: there exist constants $c_3 > 0$ and $d_w \geq 2$ such that
\[
p_t(x, y) \leq c_3 \left\{ t^{-\frac{d_t}{d_w}} \wedge \frac{t}{d(x, y)^{d_t+d_w}} \right\} \quad \text{for all } x \in E, r \in (0, 1]. \tag{2.13}
\]
Then, a straightforward calculation gives that $m \in K^{p,\delta}(X)$ if $pd_w\delta < d_t - p(d_t - d_w)$, i.e., $\delta < (d_s - p(d_s - 2))/2p$ by setting $d_s := 2d_t/d_w$.

In [Mor20, Section 1.3], these computations are made in relation to the intersection measure for such processes.

### 3 $p$-Kato class with respect to the time changed process

In this section, we discuss a relation between $p$-Kato classes with respect to $X$ and with respect to its time changed process. The goal of this section is to prove Proposition 3.2.

We first introduce the notation about the time changed processes of $X$ (for detail, see [FOT11, Section 6] for example). Let $\{\mathcal{F}_t\}_{t \geq 0}$ is the augmented filtration of $X$ and $\theta_t$ be the translation operator on $\Omega$. A stochastic process $\{A_t\}_{t \geq 0}$ is said to be a positive continuous additive functional in the strict sense (PCAF in abbreviation) if the following conditions hold:

(i) $A_t(\cdot)$ is $\mathcal{F}_t$-measurable for all $t \geq 0$,

(ii) there exists a set $\Lambda \in \mathcal{F}_\infty = \sigma(\bigcup_{t \geq 0} \mathcal{F}_t)$ such that $P_x(\Lambda) = 1$ for all $x \in E$, $\theta_t \Lambda \subset \Lambda$ for all $t > 0$, and for each $\omega \in \Lambda$, $A_t(\omega)$ is a real-valued continuous function satisfying the following:
\[
A_0(\omega) = 0, \quad A_t(\omega) = A_{t_s}(\omega) \quad \text{for } t \geq s, \quad \text{and } A_{t+s}(\omega) = A_t(\omega) + A_s(\theta_t \omega) \quad \text{for } t, s \geq 0.
\]

It is known that there is a one-to-one correspondence between $S_1(X)$ and the family of PCAF's (Revuz correspondence) as follows: for each $\mu \in S_1(X)$, there exists a unique PCAF $\{A_t\}_{t \geq 0}$ such that for any non-negative Borel function $f$ on $E$ and $\gamma$-excessive function $h(\gamma > 0)$, it holds that
\[
\int_E f(x) h(x) \mu(dx) = \lim_{t \downarrow 0} \frac{1}{t} \mathbb{E}_{h,m} \left[ \int_0^t f(X_s) dA_s \right],
\]
where $\mathbb{E}_{h,m}[\cdot] = \int_E \mathbb{E}_x[\cdot] h(x) \mu(dx)$ (see [FOT11, Theorem 5.1.7] for example). We denote by $A^\mu$ the PCAF corresponding to $\mu \in S_1(X)$. We write the fine support of $\mu \in S_1(X)$ by $f\text{-supp}[\mu] = \{x \in E : P_x(\tau = 0) = 1\}$, $\tau = \inf\{t > 0 : A^\mu_t > 0\}$.
From now on, we assume that the fine support is identical to the topological support $\text{supp}[\mu]$, and write this as $F$.

For $\mu \in S^1(X)$, denote $\tilde{X} = (\Omega, \tilde{X}_t, \tilde{\zeta}, \mathbb{P}_x)$ the time changed process of $X$ with respect to the PCAF $A^\mu$, that is,

$$\tilde{X}_t = X_{\tau_t}, \quad \tau_t = \inf\{s > 0 : A^\mu_s > t\}, \quad \tilde{\zeta} = A^\mu_{\tau_t}.$$  

Write the $\alpha$-order resolvent of $\tilde{X}$ by

$$\hat{R}_\alpha f(x) = \mathbb{E}_x \left[ \int_0^\infty e^{-\alpha t} f(\tilde{X}_t) dt \right], \quad f \in \mathcal{B}_b(F), x \in F,$$

where $\mathcal{B}_b(F)$ is the set of bounded Borel functions on $F$.

We note that $\tilde{X}$ also satisfies the absolute continuity condition $\eqref{eq:abs_cont}$.  

**Lemma 3.1.** Let $\mu$ be a measure in $S_1(X)$ whose fine support is identical to the topological support. Then the time changed process $\tilde{X}$ satisfies the absolute continuity condition $\eqref{eq:abs_cont}$.

**Proof.** Suppose $\mu(N) = 0$. By the definition of $S_1(X)$, we can take a sequence $\{E_n\}_{n=1}^\infty$ of Borel sets increasing to $E$ such that $1_{E_n} \cdot \mu \in S_{00}(X)$ for each $n$ and

$$\mathbb{P}_x \left( \lim_{n \to \infty} \sigma_{E_n} \geq t \right) = 1, \quad \text{for all } x \in E. \quad \tag{3.1}$$

For each $n$, the Revuz correspondence (see [FOT11, Theorem 5.1.6] for example) implies that

$$\mathbb{E}_x \left[ \int_0^\infty e^{-\alpha t} 1_{N \cap E_n}(X_t)dA^\mu_t \right] = R_\alpha[1_{N \cap E_n} \cdot \mu](x) = 0$$

for all $\alpha > 0$ and $x \in E$. By letting $\alpha \downarrow 0$, we have

$$\mathbb{E}_x \left[ \int_0^\infty 1_{N \cap E_n}(X_t)dA^\mu_t \right] = 0.$$

Since the inclusion $\{t < \sigma_{E_n} \cup \{X_t \in E_n\} \} \subseteq \{X_t \in E_n\}$ holds, the above equality and $\eqref{eq:abs_cont}$ give that

$$\mathbb{E}_x \left[ \int_0^\infty 1_N(X_t)dA^\mu_t \right] = \lim_{n \to \infty} \mathbb{E}_x \left[ \int_0^{\sigma_{E_n} \cap E_n} 1_N(X_t)dA^\mu_t \right] \leq \mathbb{E}_x \left[ \int_0^\infty 1_{N \cap E_n}(X_t)dA^\mu_t \right] = 0.$$

Now, we have for every $x \in F$

$$\hat{R}_\alpha 1_N(x) = \mathbb{E}_x \left[ \int_0^\infty e^{-\alpha t} 1_N(\tilde{X}_t) dt \right] \leq \mathbb{E}_x \left[ \int_0^\infty 1_N(X_t) dt \right] = \mathbb{E}_x \left[ \int_0^\infty 1_N(X_t)dA^\mu_t \right] = 0,$$

which concludes the absolute continuity condition for $\tilde{X}$ (see [FOT11, Theorem 4.2.4] for example). \hfill $\square$

The next proposition makes a key role later to prove Theorem 4.6, one of our main results. Roughly it means that, if $\mu$ is $p$-Kato with respect to the time changed process, then $\mu$ is $p$-Kato with respect to the original process.

**Proposition 3.2.** Let $p \in (1, \infty)$ and let $\mu$ be a measure in $S_1(X)$ whose fine support is identical to the topological support. Then, $\mu \in \mathcal{D}^1(X)$ and $\mu \in \mathcal{D}^p(\tilde{X}) (K^p(\tilde{X}), K^{p,\delta}(\tilde{X}), \text{resp.})$ imply $\mu \in \mathcal{D}^p(X) (K^p(X), K^{p,\delta}(X), \text{resp.})$

To prove this, we first consider the transient version of Proposition 3.2.

**Definition 3.3.** When $(\mathcal{E}, \mathcal{F})$ is transient, $\mu$ is said to be Green-bounded (write $\mu \in \mathcal{D}_0(X)$) if

$$\sup_{x \in E} \int_E r_0(x,y)\mu(dy) < \infty, \quad \tag{3.2}$$

where $r_0(x,y) = \lim_{\alpha \downarrow 0} r_\alpha(x,y)$.

**Lemma 3.4.** Suppose $(\mathcal{E}, \mathcal{F})$ is transient. Let $p \in (1, \infty)$ and $\mu$ be a smooth measure in $S_1(X)$ whose fine support is identical to the topological support. Then, $\mu \in \mathcal{D}_0(X)$ and $\mu \in \mathcal{D}^p(\tilde{X}) (K^p(\tilde{X}), K^{p,\delta}(\tilde{X}), \text{resp.})$ imply $\mu \in \mathcal{D}^p(X) (K^p(X), K^{p,\delta}(X), \text{resp.})$.

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Proof of Lemma 3.4. Denote \( \tilde{r}_0(x, y) \) the \( \alpha \)-order resolvent kernel of \( \check{X} \). First, we claim that for all \( x \in F \),

\[
\tilde{r}_0(x, y) = r_0(x, y) \quad \text{for } \mu\text{-a.e. } y \in F.
\] (3.3)

Take a sequence \( \{E_n\}_{n=1}^{\infty} \) of Borel sets increasing to \( E \) as in the definition of \( S_1(X) \). Let \( f \) be a non-negative Borel function on \( F \). By a similar argument as the proof of Lemma 3.1, we have

\[
\mathbb{E}_x \left[ \int_0^\infty 1_{\bigcup_{n=1}^{\infty} E_n}(X_t) f(X_t) dA_t^\mu \right] = R_0[f \cdot \mu](x)
\]
for all \( x \in E \). The left-hand side of the above equation is equal to \( \mathbb{E}_x \left[ \int_0^\infty f(X_t) dA_t^\mu \right] \). Indeed, by the inclusion \( \{t \leq \sigma_{E \setminus E_n}\} \subset \{X_t \in \bigcup_{n=1}^{\infty} E_n\} \) we have

\[
\mathbb{E}_x \left[ \int_0^\infty f(X_t) dA_t^\mu \right] = \lim_{n \to \infty} \mathbb{E}_x \left[ \int_0^{\sigma_{E \setminus E_n}} f(X_t) dA_t^\mu \right]
\]
for all \( x \in E \) and from the equality

\[
\mathbb{E}_x \left[ \int_0^\infty f(X_t) dA_t^\mu \right] = \mathbb{E}_x \left[ \int_0^\infty f(X_t^\mu) dt \right] = \int_F \tilde{r}_0(x, y) \mu(dy)
\]
for \( x \in F \) which is obtained from the change of variables.

Next, we recall the resolvent equation

\[
\tilde{r}_0(x, y) = \tilde{r}_\alpha(x, y) + \alpha \int_F \tilde{r}_0(x, z) \tilde{r}_\alpha(z, y) \mu(dz), \quad \alpha > 0, x, y \in F.
\]

A similar calculation as (2.6) gives that, for \( x \in F \)

\[
\left( \int_F \tilde{r}_0(x, y)^p \mu(dy) \right)^{\frac{1}{p}} \leq \left( 1 + \frac{1}{\alpha} \sup_{x \in F} \int_F \tilde{r}_0(x, y) \mu(dy) \right) \left( \sup_{x \in F} \int_F \tilde{r}_\alpha(x, y)^p \mu(dy) \right)^{\frac{1}{p}}.
\]

By combining this with (3.3), we have

\[
\left( \sup_{x \in F} \int_F r_0(x, y)^p \mu(dy) \right)^{\frac{1}{p}} \leq \left( 1 + \frac{1}{\alpha} \sup_{x \in E} \int_E r_0(x, y) \mu(dy) \right) \left( \sup_{x \in F} \int_F \tilde{r}_\alpha(x, y)^p \mu(dy) \right)^{\frac{1}{p}}.
\] (3.4)

The right-hand side of (3.4) is finite because of the assumptions \( \mu \in \mathcal{D}_0(X) \) and \( \mu \in \mathcal{D}_0^{\check{X}} \).

Now we will show that

\[
\sup_{x \in E} \int_F r_0(x, y)^p \mu(dy) = \sup_{z \in F} \int_F r_0(z, y)^p \mu(dy).
\] (3.5)

Let \( x \in F \). Define the 0-order hitting distribution \( H^0_F(x, dz) \) by

\[
H^0_F(x, A) = \mathbb{E}_x[1_A(X_{\sigma_F}); \sigma_F < \infty], \quad \text{for } x \in E, \ A \in \mathcal{B}(E).
\]

Then we can see that

\[
r_0(x, y) = \int_F r_0(z, y) H^0_F(x, dz) \quad \text{q.e. } y \in E.
\]

Indeed, for a non-negative Borel function \( f \) on \( E \) we have from the strong Markov property,

\[
\int_E r_0(x, y) f(y) m(dy) = \mathbb{E}_x \left[ \int_0^\infty f(X_t) dt \right] = \mathbb{E}_x \left[ \int_{\sigma_F}^\infty f(X_t) dt \right] = \mathbb{E}_x \left[ R_0 f(X_{\sigma_F}); \sigma_F < \infty \right] = \int_E \left( \int_F r_0(z, y) H^0_F(x, dz) \right) f(y) m(dy)
\]
and hence the equality holds for \( m\text{-a.e. } x \in E \). The desired equality for q.e. \( x \in E \) follows from the fact that the functions on both hand sides are 0-excessive in \( y \in E \). By applying Hölder’s inequality to the measure \( H^0_F(x, dz) \), we have
Proposition 4.1. We also give a Rellich-Kondrachov type compact embedding theorem (Corollary 4.3). \(\mu \in D\) consequence, Corollary 4.2 proves the assertion (1.3) introduced in Section 1, that is, for any \(u \in \mathcal{F}\),

\[
\int_E r_0(x,y)^p \mu(dy) = \int_F r_0(x,y)^{p-1} \left( \int_F r_0(z,y) H_F^0(x,dz) \right) \mu(dy)
\]

\[
= \int_F \int_F r_0(x,y)^{p-1}r_0(z,y) \mu(dy) H_F^0(x,dz)
\]

\[
\leq \int_F \left( \int_F r_0(x,y)^p \mu(dy) \right)^{\frac{p-1}{p}} \left( \int_F r_0(z,y)^p \mu(dy) \right)^{\frac{1}{p}} H_F^0(x,dz)
\]

and the right-hand side is bounded from above by

\[
\left( \int_F r_0(x,y)^p \mu(dy) \right)^{\frac{p-1}{p}} \left( \sup_{z \in F} \int_F r_0(z,y)^p \mu(dy) \right)^{\frac{1}{p}}
\]

because of \(H_F^0(x,F) \leq 1\). Hence we obtain (3.5).

By combining (3.4) with (3.5), we have

\[
\left( \sup_{x \in E} \int_F r_\alpha(x,y)^p \mu(dy) \right)^{\frac{1}{p}} \leq \left( \sup_{x \in E} \int_F r_0(x,y)^p \mu(dy) \right)^{\frac{1}{p}} \leq \left( 1 + \frac{1}{\alpha} \sup_{x \in F} \int_F r_0(x,y) \mu(dy) \right) \left( \sup_{x \in F} \int_F r_\alpha(x,y)^p \mu(dy) \right)^{\frac{1}{p}},
\]

which completes the proof. \(\Box\)

We now prove Proposition 3.2.

Proof of Proposition 3.2. We only prove the case \(\mu \in \mathcal{D}^p(\hat{X})\) (the other cases can be proved similarly). Let \(Y\) be the 1-subprocess of \(X\) defined in Remark 2.8. We can find that \(\mu\) is in \(S_1(Y)\) and its fine support with respect to \(Y\) is identical to the topological support.

We can also find that the assumptions \(\mu \in \mathcal{D}(X)\) and \(\mu \in \mathcal{D}^p(\hat{X})\) imply \(\mu \in \mathcal{D}_0(Y)\) and \(\mu \in \mathcal{D}^p(\hat{Y})\). Since \(Y\) is transient, Lemma 3.4 gives that \(\mu \in \mathcal{D}^p(Y)\). The conclusion follows from the equality \(\mathcal{D}^p(X) = \mathcal{D}^p(Y)\), which is already seen in Remark 2.8. \(\Box\)

4 Main result

This is the main part of this paper. In this section, we give relations between the \(p\)-Kato classes and the Sobolev embeddings as we introduced in (1.3) and (1.4).

4.1 \(p\)-Kato implies the Sobolev embedding

In this section, we first give an \(L^p\)-version of the Stollmann-Voigt inequality (Proposition 4.1). As a consequence, Corollary 4.2 proves the assertion (1.3) introduced in Section 1, that is, for a measure \(\mu \in D^1(X)\) and \(1 \leq p \leq p'\), \(\mu \in \mathcal{D}^p(X)\) implies that \((\mathcal{F}, \mathcal{E}_1)\) is continuously embedded into \(L^{2p}(E; \mu)\). We also give a Rellich-Kondrachov type compact embedding theorem (Corollary 4.3).

Proposition 4.1. Let \(p \in [1, \infty)\). Then, for \(\mu \in \mathcal{D}^p(X)\) it holds that

\[
\|u\|_{L^{2p}(E; \mu)}^2 \leq \left( \sup_{x \in E} \int_E r_\alpha(x,y)^p \mu(dy) \right)^{\frac{1}{p}} \mathcal{E}_\alpha(u, u) \tag{4.1}
\]

for any \(u \in \mathcal{F}\) and \(\alpha > 0\). In particular, the Hilbert space \((\mathcal{F}, \mathcal{E}_1)\) is continuously embedded into \(L^{2p}(E; \mu)\).

Proof of Proposition 4.1. The case \(p = 1\) is exactly the Stollmann-Voigt inequality (see [ST05] and [SV96]), so we assume \(p > 1\). By the regularity of \((\mathcal{E}, \mathcal{F})\), it suffices to prove (4.1) for \(u \in \mathcal{F} \cap C_0(E)\), where \(C_0(E)\) is the set of continuous functions on \(E\) with compact support. Fix \(u \in \mathcal{F} \cap C_0(E)\) and
\(\alpha > 0\). Define a finite measure \(\nu\) on \(E\) by \(\nu(dy) = u^{2p-2}(y)\mu(dy)\). By Hölder’s inequality, we have for \(x \in E\),
\[
R_\alpha \nu(x) = \int_E r_\alpha(x, y) u^{2p-2}(y)\mu(dy) \leq \left( \sup_{x \in E} \int_E r_\alpha(x, y)\mu(dy) \right) \left( \int_E u^{2p}\mu(dy) \right)^{\frac{p-1}{p}}.
\]
By applying the inequality (4.1) with \(p = 1\) and \(\nu \in S_1(X)\), we have
\[
\int_E u^2 d\mu = \int_E u^2 d\nu \\
\leq \|R_\alpha\nu\|_\infty \mathcal{E}_\alpha(u, u) \\
\leq \left( \sup_{x \in E} \int_E r_\alpha(x, y)\mu(dy) \right) \left( \int_E u^{2p}\mu(dy) \right)^{\frac{p-1}{p}} \mathcal{E}_\alpha(u, u),
\]
which concludes (4.1).

By combining Proposition 4.1 with Hölder’s inequality, we have the following corollary.

**Corollary 4.2.** Let \(p' \in [1, \infty)\). Then, for any measure \(\mu \in \mathcal{D}^1(X) \cap \mathcal{D}^{p'}(X)\) the Hilbert space \((\mathcal{F}, \mathcal{E}_1)\) is continuously embedded in \(L^{2p}(E; \mu)\) for all \(1 \leq p \leq p'\).

At the end of this section, we give a Rellich-Kondrachov type compact embedding theorem. The following corollary is a generalization of [Tak19, Corollary 4.5], where the statement is proved for \(p = 1\) and \(\mu = m\).

**Corollary 4.3.** Assume \(X\) satisfies

- (resolvent strong Feller property) \(R_1(\mathcal{B}_b(E)) \subset C_b(E)\), where \(C_b(E)\) is the set of bounded continuous functions on \(E\), and
- (tightness) for any \(\varepsilon > 0\), there exists a compact set \(K \subset E\) such that \(\sup_{x \in E} R_1 1_{K^c(x)} < \varepsilon\).

Let \(p \in [1, \infty)\) and suppose \(\mu \in \mathcal{K}^p(X)\). Then the Hilbert space \((\mathcal{F}, \mathcal{E}_1)\) is compactly embedded into \(L^2(E; m)\) and \(L^{2p}(E; \mu)\).

**Proof.** Suppose \(\{u_n\}_{n=1}^\infty \subset \mathcal{F}\) is bounded in \((\mathcal{F}, \mathcal{E}_1)\). By [Tak19, Corollary 4.5], \((\mathcal{F}, \mathcal{E}_1)\) is compactly embedded into \(L^2(E; m)\). (We remark that the irreducibility assumption is not needed to prove [Tak19, Corollary 4.5].) Hence we can take \(u \in L^2(E; m)\) and a subsequence \(\{u_{n(k)}\}_{k=1}^\infty\) such that \(u_{n(k)}\) converges to \(u\) in \(L^2(E; m)\) as \(k \to \infty\).

Recall the notation \(\gamma(\alpha)\) introduced in (2.10). By Proposition 4.1, we have
\[
\|u_{n(k)} - u_{n(l)}\|^2_{L^{2p}(E; \mu)} \leq \gamma(\alpha) \|u_{n(k)} - u_{n(l)}\|^2_{\mathcal{E}_\alpha} \\
= \gamma(\alpha) \mathcal{E}(u_{n(k)} - u_{n(l)}, u_{n(k)} - u_{n(l)}) + \alpha \gamma(\alpha) \|u_{n(k)} - u_{n(l)}\|^2_{L^2(E; m)} \\
\leq 4 \gamma(\alpha) \sup_n \|u_n\|^2_{\mathcal{E}_1} + \alpha \gamma(\alpha) \|u_{n(k)} - u_{n(l)}\|^2_{L^2(E; m)}.
\]
By letting \(k, l \to \infty\), and then \(\alpha \to \infty\), we find that \(\{u_{n(k)}\}\) is Cauchy in \(L^{2p}(E; \mu)\), and this completes the proof.

**Remark 4.4.** Assumption (A5) of [Mor20] means that the reference measure \(m\) belongs to the \(p\)-Kato class \(\mathcal{K}^p(X)\). Hence, by Corollary 4.3 we may drop the assumption (A4) in [Mor20].

**Example 4.5** (Killed Brownian motion in a domain \(D\)). Suppose \(D \subset \mathbb{R}^d\) be a domain with a smooth boundary satisfying
\[
\lim_{x \in D, |x| \to \infty} m(D \cap B(x, 1)) = 0,
\]
where $m$ is the Lebesgue measure on $\mathbb{R}^d$. Let $\partial$ be a point added to $D$ so that $D_0 := D \cup \{\partial\}$ is the one-point compactification of $D$. A killed Brownian motion $X$ in $D$ is the process given by

$$X_t = \begin{cases} B_t, & t < \tau_D, \\ \partial, & t \geq \tau_D, \end{cases}$$

where $B$ is a Brownian motion on $\mathbb{R}^d$ and $\tau_D = \inf\{t > 0 : B_t \notin D\}$ is the exit time of $B$ from $D$. Its Dirichlet form is $(\frac{1}{2}D, H^1_0(D))$, where $H^1_0(D)$ is the Sobolev space with zero boundary values. It is known that $X$ satisfies the resolvent strong Feller property and the tightness property (see [TTT17, Lemma 3.3] for example).

Hence, by combining Example 2.12 with Corollary 4.3, we can see that $H^1_0(D)$ is compactly embedded into $L^{2p}(D)$ for $p \in [1, \infty)$ with $d - p(d - 2) > 0$. This is exactly the classical Rellich-Kondrachov embedding theorem.

### 4.2 Sobolev embedding implies $p$-Kato

In this section, we prove (1.4) introduced in Section 1, that is, for a measure $\mu \in \mathcal{D}^1(X)$ and $1 \leq p < p'$, if $(\mathcal{F}, \mathcal{E}_1)$ is continuously embedded in $L^{2p'}(E; \mu)$, then $\mu \in \mathcal{D}^{0}(X)$. Furthermore, we prove that $\mu$ is also in $\mathcal{K}^\delta(X)$ for a suitable $\delta$.

**Theorem 4.6.** Let $p' \in (1, \infty)$ and let $\mu$ be a measure in $S_1(X)$ whose fine support is identical to the topological support. Suppose $\mu \in \mathcal{D}^1(X)$ and the following Sobolev type inequality: there exists a constant $S > 0$ such that

$$\|u\|^2_{L^{2p'}(E;\mu)} \leq S\mathcal{E}_1(u, u) \quad \text{for all } u \in \mathcal{F}.$$  \hfill (4.2)

Then $\mu \in \mathcal{K}^\delta(X)$ for any $p \in [1, p')$ with $\delta = 1 - \frac{p'}{p} - \frac{p-1}{p}.$

To prove this, we first consider the case $\mu = m$ and consider the assertion that $\mathcal{E}_1$ is replaced by $\mathcal{E}$:

**Lemma 4.7.** Let $p' \in (1, \infty)$ and suppose that the following Sobolev type inequality holds: there exists a constant $S > 0$ such that

$$\|u\|^2_{L^{2p'}(E;m)} \leq S\mathcal{E}(u, u) \quad \text{for all } u \in \mathcal{F}.$$  \hfill (4.3)

Then $\mu \in \mathcal{K}^\delta(X)$ for any $p \in [1, p')$ with $\delta = 1 - \frac{p'}{p-1} - \frac{p-1}{p}.$

**Proof of Lemma 4.7.** First, by [Var85], (4.3) implies the ultra-contractivity, that is, there exists $C > 0$ such that

$$\|P_t\|_{L^1 \to L^\infty} \leq Ct^{-\frac{p'}{p-1}}$$

for all $t > 0$.

where $\| \cdot \|_{L^r \to L^q}$ is the operator norm from $L^q(E; m)$ to $L^r(E, m)$. By Jensen’s inequality, we then have

$$\|P_t\|^\alpha_{L^{\frac{p'}{p-1}} \to L^\infty} \leq C' \alpha \frac{\|P_t\|_{L^{\frac{p'}{p-1}}}}{t^{\frac{p'}{p-1} - \frac{p-1}{p}}} \quad \text{for all } t > 0.$$

Fix $\alpha > 0$. For any non-negative Borel function $f$ with $f \in L^1(E; m) \cap L^\infty(E; m)$, we have

$$R_\alpha f(x) = \int_0^\infty e^{-\alpha t} P_t f(x) dt$$

$$\leq \int_0^\infty e^{-\alpha t} \|P_t f\|_{L^\infty(E; m)} dt$$

$$\leq C' \frac{\|f\|_{L^{\frac{p'}{p-1}}(E; m)}}{t^{\frac{p'}{p-1} - \frac{p-1}{p}}} \int_0^\infty e^{-\alpha t} \frac{\|P_t f\|_{L^{\frac{p'}{p-1}}(E; m)}}{t^{\frac{p'}{p-1} - \frac{p-1}{p}}} dt = C' \alpha^{-\delta} \|f\|_{L^{\frac{p'}{p-1}}(E; m)}$$

for $m$-a.e. $x \in E$, where $C' > 0$ is another constant and $\delta = 1 - \frac{p'}{p-1} - \frac{p-1}{p} > 0$. Since $R_\alpha f = \int_E r_\alpha(x, y)f(y)m(dy)$ is $\alpha$-excessive and the absolute continuity condition (2.1) holds, we have for every $x \in E$,
\[ R_\alpha f(x) = \lim_{\varepsilon \downarrow 0} e^{-\alpha \varepsilon E_x[R_\alpha f(X_\varepsilon)]]} = \lim_{\varepsilon \downarrow 0} e^{-\alpha \varepsilon} \int_E p_t(x,y) R_\alpha f(y) m(dy) \]
\[ \leq C' \alpha^{-\delta} \| f \|_{L^{p-1} \left( E; m \right)}^\frac{p-1}{p}. \tag{4.4} \]

Now, fix \( x \in E, M > 0 \) and a compact set \( K \subset E \). By applying (4.4) with \( f = 1_K(\cdot)(r_\alpha(x, \cdot) \wedge M)^{p-1} \), we have
\[
\int_K (r_\alpha(x, y) \wedge M)^p m(dy) \leq \int_K (r_\alpha(x, y) \wedge M)^{p-1} r_\alpha(x, y) m(dy) \\
\leq C' \alpha^{-\delta} \left( \int_K (r_\alpha(x, y) \wedge M)^p m(dy) \right)^\frac{p-1}{p},
\]
which means that
\[
\left( \int_K (r_\alpha(x, y) \wedge M)^p m(dy) \right)^\frac{1}{p} \leq C' \alpha^{-\delta}.
\]
Let \( M \uparrow \infty \) and \( K \uparrow E \). Hence we have the conclusion \( m \in K^{p, \delta}(X) \) by the dominated convergence theorem.

We now prove Theorem 4.6.

**Proof of Theorem 4.6.** Let \( Y \) be the 1-subprocess of \( X \). As we have seen in the proof of Proposition 3.2, \( \mu \) is in \( S_1(Y) \) and its fine support with respect to \( Y \) is identical to the topological support. We also have \( \mu \in D^1(Y) \).

Denote \( \tilde{Y} \) the time changed process of \( Y \) with respect to the PCAF of \( Y \) with Revuz measure \( \mu \) and denote its Dirichlet form as \( (\tilde{\mathcal{E}}, \tilde{\mathcal{F}}) \). Since the Dirichlet form of \( Y \) is \( (\mathcal{E}, \mathcal{F}) \), (4.2) implies that
\[
\|u\|_{L^{2p'}(\mu)}^2 \leq S \tilde{\mathcal{E}}(u, u) \quad \text{for all} \quad u \in \tilde{\mathcal{F}}.
\]
Let \( p \in [1, p'] \) and \( \delta = 1 - \frac{p'}{p-1} \frac{p-1}{p} \). By Lemma 4.7 we have \( \mu \in K^{p, \delta}(\tilde{Y}) \), and then by Proposition 3.2 we have \( \mu \in K^{p, \delta}(Y) \). The conclusion follows from Remark 2.11 (ii).

**Example 4.8** (Brownian motion). We continue with Example 2.12. As in Section 1, the classical Sobolev embedding theorem on \( \mathbb{R}^d \) gives that, \( H^1(\mathbb{R}^d) \) is continuously embedded into \( L^{2p}(\mathbb{R}^d) \) for \( p \in [1, \infty) \) with \( d - p(d-2) \geq 0 \). By combining this with Theorem 4.6, the Lebesgue measure on \( \mathbb{R}^d \) is of \( K^{p, \delta}(X) \) for \( p \in [1, \infty) \) with \( d - p(d-2) > 0 \) and \( \delta = \frac{d-p(d-2)}{2} \). This is exactly the same conclusion as Example 2.12.

5 **Application: Continuity in time of the intersection measure**

In this section, we give an application of the \( p \)-Kato class to the continuity of the intersection measure in time. Throughout this section, we assume that \( p \geq 2 \) is an integer and that the reference measure \( m \) is in the \( p \)-Dynkin class. Let \( X^{(1)}, \ldots, X^{(p)} \) be independent Hunt processes with distribution \( X \).

We write \( \zeta^{(1)}, \ldots, \zeta^{(p)} \) as their life times and write \( x_0^{(1)}, \ldots, x_0^{(p)} \) as their starting points, respectively.

First, we review the construction of the intersection measure. For detail, see [Che10, Mor20] for example. Fix bounded Borel sets \( J^{(1)}, \ldots, J^{(p)} \subset [0, \infty) \) and write \( J = \prod_{i=1}^p J^{(i)} \). For each \( \varepsilon > 0 \), we define the approximated (mutual) intersection measure \( \ell_{J, \varepsilon}^{\mathbb{S}} \) of \( X^{(1)}, \ldots, X^{(p)} \) with respect to the (multi-parameter) time interval \( J \) by
\[
(\ell_{J, \varepsilon}^{\mathbb{S}}, f) = \int_E f(x) \left[ \prod_{i=1}^p \int_{J^{(i)}} p_\varepsilon(x, X^{(i)}_s) ds \right] m(dy),
\]
for \( f \in B_b(E) \), where, for convenience we regard \( p_\varepsilon(x, X^{(i)}_s) = 0 \) when \( s \geq \zeta^{(i)} \). Then, there exists a random measure \( \ell_{J}^{\mathbb{S}} \) on \( E \) such that, \( \ell_{J, \varepsilon}^{\mathbb{S}} \) converges vaguely to \( \ell_{J}^{\mathbb{S}} \) in \( \mathcal{M}(E) \) and that
for any integer \( k \geq 1 \) and \( f \in C_0(E) \), where \( \mathcal{M}(E) \) is the set of Radon measures on \( E \) equipped with the vague topology. We call the limit \( \ell_{t}^{IS} \) as the \((\text{mutual})\) intersection measure of \( X^{(1)}, \ldots, X^{(p)} \) with respect to \( J \). For \( t = (t_1, \ldots, t_p) \in [0, \infty)^p \) we simply denote the approximated intersection measure and the intersection measure with respect to \( [0, t] \) as \( \ell_{t}^{IS} \) and \( \ell_{t}^{IS} \), respectively.

The intersection measure \( \ell_{t}^{IS} \) enjoys the so-called Le Gall’s moment formula: for any \( f \in \mathcal{B}_b(E) \) with compact support and for any integer \( k \geq 1 \), it holds that
\[
\mathbb{E}[\langle f, \ell_{t}^{IS} \rangle^k] = \int_{E^k} f(x_1) \cdots f(x_k) \prod_{i=1}^{p} \sum_{\sigma \in \mathfrak{S}_k} \prod_{j=1}^{k} \mathbb{P}_{s_j-s_{j-1}}(x_{\sigma(j-1)}(t_{j-1}), x_{\sigma(j)}(t_j)) \, ds_1 \cdots ds_k \, m(dx_1) \cdots m(dx_k),
\]
(5.1)
where \( (J^{(i)})_s := \{(s_1, \ldots, s_k) \mid (J^{(i)})_s \leq \{s_1 < \cdots < s_k\} \) and \( \mathfrak{S}_k \) is the set of permutations of \( \{1, \ldots, k\} \).

The goal of this section is to justify \( \{\ell_{t}^{IS} : t \in [0, \infty)^p\} \) as a measure-valued continuous stochastic process. That is,

**Theorem 5.1.** Assume \( \mu \in K^{p, \delta}(X) \) for some \( \delta \in (0, 1) \). Then it holds that

(i) the \( \mathcal{M}(E) \)-valued process \( \{\ell_{t}^{IS} : t \in [0, \infty)^p\} \) has a continuous modification,

(ii) for any \( f \in \mathcal{B}_b(E) \), the real-valued process \( \{\langle f, \ell_{t}^{IS} \rangle : t \in [0, \infty)^p\} \) has a modification whose paths are locally \( \gamma \)-Hölder continuous of every order \( \gamma \in (0, \delta) \).

**Proof.** Our proof is based on that of [Che10, Lemma 2.2.4]. First, we claim the following estimate: for any integer \( k \geq 1 \), positive constant \( T > 0 \) and \( s, t \in [0, T]^p \), it holds that
\[
\mathbb{E}[\langle f, \ell_{t}^{IS}, \ell_{s}^{IS} \rangle^k] \leq (kl)^k (2^p \|f\|^{k+1}) \sup_{0 \leq t \leq T} \{L^{-\delta}(t)\} \|t-s\|^k,
\]
(5.2)
where \( L(t) \) is introduced in (2.11) and \( |t-s| \) is the Euclidean distance between \( t \) and \( s \) in \([0, \infty)^p\).

Fix \( s, t \in [0, T]^p \). Define the family \( \{J_i\}_{i=1}^{p-1} \) consisting of products of intervals as
\[
\{J_i\}_{i=1}^{p-1} = \left\{ \prod_{i=1}^{p} [a_i, b_i] \mid [a_i, b_i] = [0, s_i \land t_i] \text{ or } [s_i \land t_i, s_i \lor t_i], 1 \leq i \leq p \right\} \setminus \left\{ \prod_{i=1}^{p} [0, s_i \land t_i] \right\}.
\]
Then we have \([0, t] \triangle [0, s] \subset \bigcup J_i \) and then
\[
\|\langle f, \ell_{t}^{IS}, \ell_{s}^{IS} \rangle - \langle f, \ell_{t}^{IS}, \ell_{s}^{IS} \rangle\| \leq \sum_{i=1}^{2^{p-1}} \|\langle f, \ell_{t}^{IS}, \ell_{s}^{IS} \rangle\|, \quad \mathbb{P}\text{-a.s.}
\]
By Le Gall’s moment formula, we have
\[
\mathbb{E}[\langle f, \ell_{t}^{IS}, \ell_{s}^{IS} \rangle^k] = \int_{E^k} |f(x_1)| \cdots |f(x_k)| \prod_{i=1}^{p} \sum_{\sigma \in \mathfrak{S}_k} \mathbb{P}_{s_i-s_{i-1}}(x_{\sigma(i-1)}, x_{\sigma(i)}) \, ds_1 \cdots ds_k \, m(dx_1) \cdots m(dx_k),
\]
where
\[
H_{a_i, b_i}(x_1, \ldots, x_p) := \int_{E} \int_{[0, \infty)^p} \{\sum_{j=1}^{k} r_j \leq b_i - a_i\} \prod_{j=1}^{k} p_{r_j}(x_{j-1}, x_j) \, dr_1 \cdots dr_k \, \nu_{a_i}(dx_0)
\]
and \( \nu_{a_i}(dx_0) = \mathbb{P}_{x_0}(X_{a_i}^{(i)} \in dx_0) \). Since \( \|H_{a_i, b_i}\|_{L^p(E^k, m^{\otimes k})} \leq \eta(b_i - a_i)^k \), we have by Hölder’s inequality
\[
\mathbb{E}[\langle f, \ell_{t}^{IS}, \ell_{s}^{IS} \rangle^k] \leq (kl)^k \|f\|_p \prod_{i=1}^{p} \|H_{a_i, b_i}\|_{L^p(E^k, m^{\otimes k})}
\]
and
\[
15
\]
\[(k!)^p \|f\|_\infty^k \prod_{i=1}^p \eta(b_i - a_i)^k \leq (k!)^p \|f\|_\infty^k (\eta(T) + 1)^p \eta(|t - s|) \].

In the last inequality, we used the fact that \([a_i, b_i] = [s_i \wedge t_i, s_i \vee t_i]\) holds for at least one \(i\) because of the definition of \(J_i\). Therefore

\[E\left[\langle f, \ell_\mathcal{IS}^t \rangle - \langle f, \ell_\mathcal{IS}^s \rangle\right] \leq \sum_{l=1}^{2^p - 1} E\left[\langle |f|, \ell_\mathcal{IS}^t \rangle \right]^{1/k} \leq (k!)^p \left(2^p \|f\|_\infty \eta(T) + 1\right)^p \eta(|t - s|)^k.\]

The desired estimate (5.2) follows from \(|t - s| \leq pT\).

Now, \((\delta k - p)/k\) increases to \(\delta\) as \(k \to \infty\). By applying Kolmogorov’s continuity theorem (see [Kal02, Theorem 3.23] for example) to (5.2), we can find that the process \([0, T]^p \ni t \mapsto \langle f, \ell_\mathcal{IS}^t \rangle \in \mathbb{R}\) has a continuous modification whose paths are \(\gamma\)-Hölder continuous of every order \(\gamma \in (0, \delta)\). Since \(T > 0\) is arbitrary, we obtain (ii).

To prove (i), take a dense subset \(\{\phi_n\}_{n=1}^\infty\) in \(C_0^+ (E)\), the family of non-negative continuous functions with compact support equipped with the uniform metric. Then \(\mathcal{M}(E)\) is homeomorphic to a subset of \(\mathbb{R}^\infty\) by the mapping \(\mathcal{M}(E) \ni \mu \mapsto \{\langle \mu, \phi_n \rangle\}_{n=1}^\infty \in \mathbb{R}^\infty\), and hence the process \([0, \infty]^p \ni t \mapsto \ell_\mathcal{IS}^t \in \mathcal{M}(E)\) has a continuous version by (ii). Therefore we complete the proof.

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