UNIFORMLY BOUNDED REPRESENTATIONS OF SL(2, R)

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Abstract. We compute the “norm” of irreducible uniformly bounded representations of SL(2, R). We show that the Kunze–Stein version of the uniformly bounded representations has minimal norm in the similarity class of uniformly bounded representations.

1. Introduction

We begin by summarising our results briefly. A representation π, by which we always mean a continuous representation of a locally compact group G on a Hilbert space $\mathcal{H}_\pi$, is said to be uniformly bounded if $\pi(x)$ is a bounded operator on $\mathcal{H}_\pi$ for each $x \in G$, and there is a constant $C$, necessarily no less than 1, such that

$$C^{-1} \|\xi\|_{\mathcal{H}_\pi} \leq \|\pi(x)\xi\|_{\mathcal{H}_\pi} \leq C \|\xi\|_{\mathcal{H}_\pi} \quad \forall x \in G, \forall \xi \in \mathcal{H}_\pi;$$

the two inequalities are equivalent because $\pi$ is a representation. We write $\|\pi(x)\|_{op}$ for the operator norm of $\pi(x)$ and define the norm of $\pi$, written $\|\pi\|_{ub}$, to be the smallest possible value of $C$ in this inequality.

Suppose that $\pi$ and $\sigma$ are uniformly bounded representations of $G$. A linear operator from $\mathcal{H}_\pi$ to $\mathcal{H}_\sigma$ such that $\sigma(x)T = T \pi(x)$ for all $x \in G$ is called an intertwiner. We say that $\pi$ and $\sigma$ are similar if there is an intertwiner that is bounded with bounded inverse, and unitarily equivalent if there is a unitary intertwiner. Similarity and unitary equivalence are equivalence relations. Similar uniformly bounded representations may have different norms and hence not be unitarily equivalent. In general, little seems to be known about similarity classes of uniformly bounded representations, or about finding uniformly bounded representations in an equivalence class with minimal norm. Of course, if a uniformly bounded representation is similar to a unitary representation, then the unitary representation has minimal norm in the equivalence class.

In 1955, L. Ehrenpreis and F. Mautner [2, 3] showed that SL(2, R) has two analytic families of representations $\pi_{\lambda,\varepsilon}$, where $\lambda \in \mathbb{C}$ and $\varepsilon$ is either 0 or 1. These representations have bounded $K$-finite matrix coefficients (here $K$ is SO(2)) if and only if $|\text{Re}(\lambda)| \leq \frac{1}{2}$, and they are uniformly bounded when $|\text{Re}(\lambda)| < \frac{1}{2}$; most of them are not similar to unitary representations. Shortly after, R.A. Kunze and E.M. Stein [8] found a use for these uniformly bounded representations, first realising them on the same Hilbert space, and then using them to prove what is now called the Kunze–Stein phenomenon for SL(2, R).

We will define families of Hilbert spaces $\mathcal{H}_\alpha$ and $\mathcal{H}_{\alpha,a,b}$, where $\alpha \in (-\frac{1}{2}, \frac{1}{2})$ and $a, b \in \mathbb{R}^+$; the spaces $\mathcal{H}_\alpha$ are homogeneous fractional Sobolev spaces, and $\mathcal{H}_{\alpha,a,b}$ and $\mathcal{H}_\alpha$ have equivalent norms, so they coincide as spaces of (generalised) functions. The Kunze–Stein uniformly bounded representations $\pi_{\lambda,\varepsilon}$ act on the spaces $\mathcal{H}_\alpha$, where $\alpha = \text{Re} \lambda$; the
same representation, but with the Hilbert space equipped with the $H_{\alpha,a,b}$ norm, will be denoted by $\pi_{\lambda,\varepsilon,a,b}$.

Our main theorem is about this family of representations: it shows that the Kunze–Stein representations are optimal, in the sense of having minimal norms, and gives sharp estimates for these norms.

**Theorem 1.1.** Suppose that $\sigma$ is a uniformly bounded representation of $SL(2,\mathbb{R})$ that is similar to $\pi_{\lambda,\varepsilon}$, where $|\Re\lambda| < \frac{1}{2}$ and $\varepsilon \in \{0, 1\}$. Then there exists an equivalent Hilbert norm on $H_\sigma$ such that $\tau$, the representation $\sigma$ acting on the equivalent Hilbert space, is unitary on the subgroup of lower triangular matrices, and

$$\|\sigma\|_{ub} \geq \|\tau\|_{ub}.$$  

Further, there exist $a, b \in \mathbb{R}^+$ such that $\tau$ is unitarily equivalent to $\pi_{\lambda,\varepsilon,a,b}$ and

$$\|\tau\|_{ub} = \|\pi_{\lambda,\varepsilon,a,b}\|_{ub}.$$  

Further, when $a \neq b$ and $(\lambda, \varepsilon) \neq (0, 1)$,

$$\|\pi_{\lambda,\varepsilon,a,b}\|_{ub} > \|\pi_{\lambda,\varepsilon}\|_{ub}$$

and, when $|\Im\lambda|$ is large,

$$\|\pi_{\lambda,\varepsilon}\|_{ub} \simeq (1 + |\Im\lambda|)^{|\Re\lambda|} \left(\frac{1}{2} - |\Re\lambda|\right).$$

The expression $A(\lambda) \simeq B(\lambda)$ for all $\lambda$ in a subset $E$ of the domains of $A$ and of $B$ means that there exist (positive) constants $C$ and $C'$ such that

$$C A(\lambda) \leq B(\lambda) \leq C' A(\lambda) \quad \forall \lambda \in E.$$  

We now provide more context for our results. The history of uniformly bounded representations and their role in harmonic analysis is now quite extensive, and we just outline some of the most important work that we have not already mentioned.

Around 1950, a number of researchers looked at uniformly bounded representations in their studies of amenability. Once it was known that every uniformly bounded representation of an amenable group is unitarizable, that is, similar to a unitary representation, J. Dixmier \[1\] asked whether this was true in general or whether this characterized amenability. As already mentioned, the work of Ehrenpreis and Mautner showed that the former possibility does not hold; the status of the latter is still unresolved. Considerable effort has gone into the construction of uniformly bounded representations. Apart from their fundamental paper \[8\], Kunze and Stein \[9, 10, 11\], as well as several other authors, constructed analytic families of uniformly bounded representations for many noncompact semisimple Lie groups in the 1960s and 1970s. In the 1970s and 1980s, uniformly bounded representations were constructed for other groups; for example, A. Figà-Talamanca and M.A. Picardello \[5\] and shortly after T. Pytlik and R. Szwarc \[14\] found uniformly bounded representations of the noncommutative free groups.

Comparatively recently, G. Pisier \[12, 13\] has studied uniformly bounded representations, on the one hand taking giant strides towards the solution of the Dixmier similarity problem and on the other developing the links between uniformly bounded representations and multipliers of the Fourier algebra. Very recently, K. Juschenko and...
P.W. Nowak [7] linked uniformly bounded representations with the exactness of discrete groups.

In this paper, we return to the roots of all this, and study the uniformly bounded representations of SL(2, \(\mathbb{R}\)) in detail, to help to further clarify the nature of these still mysterious objects.

Our paper is structured as follows. In Section 2 we review a few general facts on uniformly bounded representations; in Section 3 we describe the representations of the group SL(2, \(\mathbb{R}\)); and in Section 4 we give precise estimates of the Kunze–Stein representations and prove Theorem 1.1.

2. Background

We include here a few results about uniformly bounded and unitary representations needed later.

Suppose that \(\pi\) is a uniformly bounded representation of a locally compact group \(G\). We may produce a new uniformly bounded representation \(\rho\) from \(\pi\) by putting an equivalent Hilbert norm on the representation space \(H^\pi\). When we do this, \(\rho\) and \(\pi\) are similar; indeed, the identity map from \(H^\pi\) with the original norm to \(H^\pi\) with the new norm is a similarity. In the next lemma we show that when \(G\) has a closed amenable subgroup, there is a clever choice for the equivalent norm.

**Lemma 2.1.** Suppose that \(G\) is a locally compact group and \(H\) is a closed amenable subgroup of \(G\). Suppose also that \(\pi\) is a uniformly bounded representation of \(G\). Then there is an equivalent Hilbert space norm on \(H^\pi\) relative to which \(H\) acts unitarily. Further, the norm of \(\pi\) relative to the new norm is no greater than that relative to the old norm.

**Proof.** Take a right invariant mean \(m_H\) on \(H\). We define a new inner product on \(H^\pi\) by the formula

\[
\langle \xi, \eta \rangle_{H^\pi} = m_H(h \mapsto \langle \pi(h)\xi, \pi(h)\eta \rangle_{H^\pi}),
\]

and then \(\langle \pi(h)\xi, \pi(h)\eta \rangle_{H^\pi} = \langle \xi, \eta \rangle_{H^\pi}\) for all \(h \in H\) trivially. From (1.1),

\[
\|\pi\|^{-2}_{ub} \|\xi\|_{H^\pi}^2 \leq m_H\left(h \mapsto \|\pi(h)\xi\|_{H^\pi}^2\right) \leq \|\pi\|^2_{ub} \|\xi\|^2_{H^\pi},
\]

and so

\[
\|\pi\|^{-1}_{ub} \|\xi\|_{H^\pi} \leq \|\xi\|_{H} \leq \|\pi\|_{ub} \|\xi\|_{H^\pi};
\]

moreover,

\[
\|\pi(x)\xi\|_{H} = m_H\left(h \mapsto \|\pi(h)\pi(x)\xi\|_{H^\pi}^2\right)^{1/2}
= m_H\left(h \mapsto \|\pi(hxh^{-1})\xi\|_{H^\pi}^2\right)^{1/2}
\leq m_H\left(h \mapsto \|\pi\|^2_{ub} \|\pi(h)\xi\|^2_{H^\pi}\right)^{1/2}
= \|\pi\|_{ub} \|\xi\|_{H},
\]

as required. \(\square\)

The next result is well known, but we include a proof for completeness. It states that similar unitary representations are in fact unitarily equivalent.
Lemma 2.2. Suppose that \( \pi \) and \( \sigma \) are irreducible unitary representations of a group \( G \), and \( T : \mathcal{H}_\pi \to \mathcal{H}_\sigma \) is a bounded operator with bounded inverse that intertwines \( \pi \) and \( \sigma \), that is, \( \sigma(x)T = T\pi(x) \) for all \( x \in G \). Then there exist \( a \in \mathbb{R}^+ \) and a unitary map \( U : \mathcal{H}_\pi \to \mathcal{H}_\sigma \) such that \( T = aU \).

Proof. By taking adjoints, we see that \( T^*\sigma(x) = \pi(x)T^* \) for all \( x \in G \), and hence \( T^*T\pi(x) = \pi(x)T^*T \) for all \( x \in G \). By Schur’s lemma, \( T^*T \) is a scalar operator; we take \( T^*T \) to be multiplication by \( a^2 \), where \( a > 0 \). Now \( a^{-1}T \) is unitary. \( \square \)

We are going to use techniques of classical analysis. We denote by \( \| \cdot \|_p \) the usual norm on the Lebesgue space \( L^p(\mathbb{R}) \), where \( 1 \leq p \leq \infty \), and we define the Fourier transform \( \hat{f} \) of a function \( f \) on \( \mathbb{R} \) by
\[
\hat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-ix\xi} \, dx \quad \forall \xi \in \mathbb{R}.
\]
Then the Fourier transform extends to a multiple of a unitary operator on \( L^2(\mathbb{R}) \), and more precisely,
\[
\| \hat{f} \|_2 = \sqrt{2\pi} \| f \|_2 \quad \forall f \in L^2(\mathbb{R}).
\]

3. The group \( SL(2, \mathbb{R}) \)

We now describe \( SL(2, \mathbb{R}) \), abbreviated to \( G \) for convenience, and various decompositions and representations thereof. We present an approach that the second-named author learnt from Kunze many years ago. First, define subgroups \( K, M, A, N \) and \( \bar{N} \) of \( G \) as follows:
\[
K = \{ k_\theta : \theta \in \mathbb{R} \} \quad M = \{ m_\pm \} \quad A = \{ a_s : s \in \mathbb{R}^+ \}
\]
\[
N = \{ n_t : t \in \mathbb{R} \} \quad \bar{N} = \{ \bar{n}_t : t \in \mathbb{R} \},
\]
where
\[
k_\theta = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \quad m_\pm = \begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix} \quad a_s = \begin{pmatrix} s & 0 \\ 0 & s^{-1} \end{pmatrix}
\]
\[
n_t = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} \quad \bar{n}_t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}.
\]
we will write \( w \) for the rotation \( k_{\pi/2} \).

Consider \( \mathbb{R}^2 \) as a space of row vectors, and \( G \) acting on \( \mathbb{R}^2 \) by right multiplication. Then \( G \) fixes the origin and acts transitively on \( \mathbb{R}^2 \setminus \{(0, 0)\} \). Write \( b \) for the “base point” \( (1, 0) \), and \( B \) for the space \( \mathbb{R}^2 \setminus \{(0, 0)\} \). The subgroup \( N \) is the stabiliser of the point \( b \), and so \( B \) may be identified with the coset space \( N \setminus G \). The polar decomposition in \( B \) leads to the Iwasawa decomposition of \( G \): every element \( x \) of \( G \) may be expressed uniquely in the form
\[
x = n ak
\]
where \( n \in N, a \in A \) and \( k \in K \). Indeed, \( bx \in B \), and if we choose \( s = \| bx \| \) and \( \theta = \arg(bx) \), then \( ba_s k_\theta = bx \); there is therefore an element \( n \) of \( N \) such that \( x = na_s k_\theta \); further, since \( k_\theta \) and \( a_s \) are uniquely determined, \( n \) is also unique. We may describe the Bruhat decomposition in similar terms: \( B \) is the disjoint union of the real axis (minus the origin) and \( \mathbb{R}^2 \) minus the real axis, and this corresponds to writing \( G \) as the disjoint union \( (NAM) \sqcup (NAMwNAM) \).
We now consider the space $V_{\lambda, \varepsilon}$, where $\lambda \in \mathbb{C}$ and $\varepsilon$ is either 0 or 1, of smooth functions on $B$ that satisfy
\[
f(\delta v) = |\delta|^{2\lambda-1} \operatorname{sgn}(\delta) \varepsilon f(v) \quad \forall v \in B \quad \forall \delta \in \mathbb{R} \setminus \{0\},
\]
equipped with the topology of locally uniform convergence of all partial derivatives. Since $G$ acts on $B$ and commutes with scalar multiplication, $G$ acts on $V_{\lambda, \varepsilon}$ by the formula
\[
\pi_{\lambda, \varepsilon}(x)f(v) = f(vx) \quad \forall v \in B \quad \forall x \in G.
\]
We obtain the “compact picture” of the representation by restricting $v$ to lie in the circle $bK$, and observing that
\[
\pi_{\lambda, \varepsilon}(x)f(v) = |vx|^{2\lambda-1} f(|vx|^{-1} vx).
\]
The “noncompact picture” is obtained similarly, by restricting $v$ to lie on the line $b\bar{N}$, and observing that
\[
(3.1) \quad \pi_{\lambda, \varepsilon}(x)f(1, t) = \operatorname{sgn}^t(a + tc) |a + tc|^{2\lambda-1} f(1, x \cdot t) \quad \forall t \in \mathbb{R},
\]
where
\[
x = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{and} \quad x \cdot t = \frac{b + dt}{a + ct}.
\]
Clearly some care is required “at infinity” in this version of the representation.

By completing $V_{\lambda, \varepsilon}$ in an appropriate norm, we may obtain representations of $G$ on Hilbert or Banach spaces. First note that $|f|$ is even, and so
\[
\int_{-\pi/2}^{\pi/2} |f(\cos \theta, \sin \theta)| d\theta = \frac{1}{2} \int_{-\pi}^{\pi} |f(\cos \theta, \sin \theta)| d\theta.
\]
Now observe that if $p(\Re \lambda - \frac{1}{2}) = 1$, then
\[
\int_{\mathbb{R}} |f(b\bar{n}_t)|^p dt = \int_{\mathbb{R}} |f(1, t)|^p dt = \int_{-\pi/2}^{\pi/2} |f(\cos \theta, \sin \theta)|^p d\theta = \int_{-\pi/2}^{\pi/2} |f(bk_t)|^p d\theta.
\]
Indeed, $f(1, \tan \theta) = (1 + \tan^2 \theta)^{\lambda-1/2} f(\cos \theta, \sin \theta)$, and the formula above is a consequence of setting $t = \tan \theta$ and changing variables. Taking $p$th roots, we see that
\[
\left( \int_{\mathbb{R}} |f(b\bar{n}_t)|^p dt \right)^{1/p} = \left( \frac{1}{2} \int_{-\pi}^{\pi} |f(bk_t)|^p d\theta \right)^{1/p}.
\]
Now the left hand integral is trivially unchanged if we replace $f$ by $\pi_{\lambda, \varepsilon}(\bar{n}_a)f$, while the right hand integral is trivially unchanged if we replace $f$ by $\pi_{\lambda, \varepsilon}(k_\phi)f$. Since the smallest subgroup of $G$ that contains both $\bar{N}$ and $K$ is $G$, it follows that in fact the integrals above are unchanged if we replace $f$ by $\pi_{\lambda, \varepsilon}(x)f$ for any $x \in G$.

The representations $\pi_{\lambda, \varepsilon}$, where $\Re \lambda = 0$, are isometric on a Hilbert space, and hence unitary. These representations are irreducible, except when $(\lambda, \varepsilon) = (0, 1)$; they are known as the unitary principal series.

Take $f \in V_{\lambda, \varepsilon}$ and $g \in V_{\mu, \varepsilon}$, where $\lambda + \mu = 0$. By a simple variant of the argument above, we may show that
\[
\int_{\mathbb{R}} f(1, t) g(1, t) dt = \int_{-\pi/2}^{\pi/2} f(\cos \theta, \sin \theta) g(\cos \theta, \sin \theta) d\theta.
\]
We define \((f, g)\) to be either of the above integrals, then the bilinear form \((\cdot, \cdot)\) is well defined and \(G\)-invariant, and exhibits the canonical duality between \(\mathcal{V}_{\lambda, \varepsilon}\) and \(\mathcal{V}_{-\lambda, \varepsilon}\).

When \(\lambda \in (-1/2, 0)\), we may find a Hilbert norm such that \(\pi_{\lambda, 0}\) acts isometrically, and hence unitarily. Since
\[
(tan \theta - tan \phi) cos \theta cos \phi = sin(\theta - \phi),
\]
it follows that if \(\lambda \in (-1/2, 0)\) and \(f \in \mathcal{V}_{\lambda, 0}\), then
\[
\int_{-\pi/2}^{\pi/2} \int_{-\pi/2}^{\pi/2} f(1, t) \tilde{f}(1, u) |t - u|^{-(1+2\lambda)} dt du
= \int_{-\pi/2}^{\pi/2} \int_{-\pi/2}^{\pi/2} f(1, \tan \theta) \tilde{f}(1, \tan \phi) \frac{|tan \theta - tan \phi|^{-(1+2\lambda)}}{cos^2 \theta cos^2 \phi} d\theta d\phi
= \int_{-\pi/2}^{\pi/2} \int_{-\pi/2}^{\pi/2} f(\cos \theta, \sin \theta) \tilde{f}(\cos \phi, \sin \phi) \frac{|\sin(\theta - \phi)|^{-(1+2\lambda)}}{\cos 2\theta \cos 2\phi} d\theta d\phi
= \frac{1}{4} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(\cos \theta, \sin \theta) \tilde{f}(\cos \phi, \sin \phi) |\csc(\theta - \phi)|^{1+2\lambda} d\theta d\phi;
\]
all integrals converge absolutely. Taking square roots, and introducing some notation, we see that
\[
\|f\|_{\mathcal{H}_{\lambda, \varepsilon}} := C_{\lambda} \left( \int_{-\pi/2}^{\pi/2} \int_{-\pi/2}^{\pi/2} f(1, t) \tilde{f}(1, u) |t - u|^{-(1+2\lambda)} dt du \right)^{1/2}
= C_{\lambda} \left( \frac{1}{4} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(\cos \theta, \sin \theta) \tilde{f}(\cos \phi, \sin \phi) |\csc(\theta - \phi)|^{1+2\lambda} d\theta d\phi \right)^{1/2}
=: \|f\|_{\mathcal{H}_{\lambda, \varepsilon}},
\]
say; the positive constant \(C\lambda\) is chosen such that
\[
(3.2) \quad \|f\|_{\mathcal{H}_{\lambda, \varepsilon}} = \left( \int_{\mathbb{R}} |r|^\lambda \tilde{f}(1, r)^2 dr \right)^{1/2},
\]
where the Fourier transform acts in the second variable only. Thus the first norm is a homogeneous fractional Sobolev norm on \(\mathbb{R}\); the second norm is equivalent to an inhomogeneous fractional Sobolev norm on even functions on the circle. Much as before, the norms above are \(G\)-invariant, and so \(\pi_{\lambda, 0}\) acts unitarily on the completion of \(\mathcal{V}_{\lambda, 0}\) in this norm, which may be identified with a space of distributions on \(B\). The duality described in the preceding paragraph enables us to find a Hilbert norm so that \(\pi_{\lambda, 0}\) acts unitarily when \(\lambda \in (0, 1/2)\); this norm is also given by the formula (3.2). The representations \(\pi_{\lambda, 0}\) and \(\pi_{-\lambda, 0}\) are unitarily equivalent. The family of representations \(\pi_{\lambda, 0}\), where \(\lambda \in (-\frac{1}{2}, 0) \cup (0, \frac{1}{2})\) is called the complementary series.

Recall that when \(\lambda\) is purely imaginary, the representations \(\pi_{\lambda, \varepsilon}\) act unitarily on the completion of \(\mathcal{V}_{\lambda, 0}\) in the norm \(\|\cdot\|_{\mathcal{H}_{0, \varepsilon}}\), giving us the unitary principal series. For completeness, we mention that \(G\) has some additional irreducible unitary representations, namely, the trivial representation, the discrete series of representations (which appear as
Uniformly bounded representations

Producing uniformly bounded representations is more difficult than producing unitary representations. In this section, we do this, and compute the norms of the Kunze–Stein representations and of more general uniformly bounded representations. We denote the Cartesian form of a complex number \( \lambda \) by \( \alpha + i \beta \); we suppose throughout that \( |\alpha| < \frac{1}{2} \).

Since our analysis is carried out in the noncompact picture, we shall more simply write \( f(t) \) instead of \( f(1,t) \). Moreover we shall realise the representation \( \pi_{\lambda,\varepsilon} \) on the completion \( \mathcal{H}_\alpha \) of the group \( \mathcal{H}_\alpha \) of smooth functions with compact support in the norm

\[
\|f\|_{\mathcal{H}_\alpha} = \left( \int_{0}^{\infty} |r|^{\alpha} \hat{f}(r)|^2 \, dr + \int_{-\infty}^{0} |r|^{\alpha} \hat{f}(r)|^2 \, dr \right)^{1/2}.
\]

The following lemma, observed by Kunze and Stein [8], is one of the keys to our approach.

**Lemma 4.1.** Suppose that \( |\alpha| < \frac{1}{2} \). Then \( \pi_{\lambda,\varepsilon} \) is a uniformly bounded representation of the group \( G \) on \( \mathcal{H}_\alpha \), and

\[
\|\pi_{\lambda,\varepsilon}\|_{\text{ub}} = \|\pi_{\lambda,\varepsilon}(w)\|_{\text{op}}.
\]

**Proof.** It is easy to see that the subgroup \( \bar{NAM} \) of \( G \) acts unitarily on \( \mathcal{H}_\alpha \). Now the Bruhat decomposition \( G = \bar{NAM} \cup \bar{NAM} w \bar{NAM} \) implies that \( G \) acts uniformly boundedly on this space if and only if \( \pi_{\lambda,\varepsilon}(w) \) is bounded thereon, and the uniformly bounded norm of the representation is the norm of the single operator \( \pi_{\lambda,\varepsilon}(w) \). \( \square \)

Kunze and Stein [8] just estimated the operator norm \( \|\pi_{\lambda,\varepsilon}(w)\|_{\text{op}} \); in this paper we compute it exactly. First however we construct some more general uniformly bounded representations.

Observe that for all \( f \) in \( C_c^{\infty}(\mathbb{R}) \),

\[
\pi_{\lambda,\varepsilon}(w)f(t) = \text{sgn}^\varepsilon(-t)|t|^{2(\alpha+i\beta)-1} f(-1/t) \quad \forall t \in \mathbb{R}.
\]

We may produce a new uniformly bounded representation by defining an equivalent norm \( \|\cdot\|_{\mathcal{H}_{\alpha,a,b}} \) on \( \mathcal{H}_\alpha \), thus:

\[
\|f\|_{\mathcal{H}_{\alpha,a,b}} = \left( a \int_{0}^{\infty} |r|^{\alpha} \hat{f}(r)|^2 \, dr + b \int_{-\infty}^{0} |r|^{\alpha} \hat{f}(r)|^2 \, dr \right)^{1/2},
\]

where \( a, b > 0 \), and considering \( \pi_{\lambda,\varepsilon} \) acting on \( \mathcal{H}_\alpha \) with this new norm. This representation is uniformly bounded because the new norm is equivalent to the old norm. Further, when \( a = b \), the norm on the space \( \mathcal{H}_{\alpha,a,b} \) is a multiple of the norm on \( \mathcal{H}_\alpha \); the uniformly bounded norms of \( \pi_{\lambda,\varepsilon} \) on the spaces \( \mathcal{H}_{\alpha,a,b} \) and \( \mathcal{H}_\alpha \) therefore coincide in this case.

We write \( \pi_{\lambda,\varepsilon,a,b} \) for the representation \( \pi_{\lambda,\varepsilon} \) on \( \mathcal{H}_{\alpha,a,b} \).

**Lemma 4.2.** Suppose that \( \sigma \) is a uniformly bounded representation of \( G \) that is similar to \( \pi_{\lambda,\varepsilon} \). Then there exist \( \tau \), obtained from \( \sigma \) by renorming the representation space, and \( a \) and \( b \) in \( \mathbb{R}^+ \) such that \( \tau \) and \( \pi_{\lambda,\varepsilon,a,b} \) are unitarily equivalent. Further,

\[
\|\sigma\|_{\text{ub}} \geq \|\tau\|_{\text{ub}} = \|\pi_{\lambda,\varepsilon,a,b}\|_{\text{ub}}.
\]
Proof. Take a uniformly bounded representation $\sigma$ of $G$, similar to $\pi_{\lambda, \varepsilon}$. Since $MAN$ is amenable, by Lemma 2.2 there exists an equivalent Hilbert norm on $H_{\sigma}$ such that $\tau$, the representation $\sigma$ acting on the equivalent Hilbert space, is unitary on $MAN$. Moreover $\|\sigma\|_{ub} \geq \|\tau\|_{ub}$.

Since also $\tau$ and $\pi_{\lambda, \varepsilon}$ are similar, there exists a bounded map $T : H_\alpha \to H_\tau$ with bounded inverse such that $T\pi_{\lambda, \varepsilon}(x) = \tau(x)T$ for all $x \in G$.

The representation space $H_\alpha$ splits into two complementary unitarily inequivalent subspaces, $H_\alpha^\pm$ say, given by

$$H_\alpha^\pm = \{ f \in H_\alpha : \hat{f}\big|_{N^\pm} = 0 \},$$

on both of which $\pi_{\lambda, \varepsilon}|_{MAN}$ acts irreducibly. We define $H_\tau^\pm = TH_\alpha^\pm$; then the unitary representation $\tau|_{MAN}$ acts irreducibly on $H_\tau^+$ and $H_\tau^-$, whence $H_\tau = H_\tau^+ \oplus H_\tau^-$. By Lemma 2.2 applied to each irreducible component, $T|_{H\alpha^\pm} : H_\alpha^\pm \to H_\tau^\pm$ is a multiple of a unitary map, and so for the right choice of $a$ and $b$, the intertwining operator is unitary from $H_{\alpha,a,b}$ to $H_\tau$, and we are done. \( \square \)

The rest of this section will be devoted to computing $\|\pi_{\lambda, \varepsilon,a,b}\|_{ub}$, where $a, b \in \mathbb{R}^+$. When $a = b$ the norm $\|\pi_{\lambda, \varepsilon,a,b}\|_{ub}$ reduces to the norm of the Kunze-Stein representation $\pi_{\lambda, \varepsilon}$ and we will simply write $\|\pi_{\lambda, \varepsilon}\|_{ub}$.

Lemma 4.3. Suppose that $a, b \in \mathbb{R}^+$. Define $\theta = (a - b)/(a + b)$,

$$m_{0,\alpha}(u) = \sqrt{2\pi} \, 2^{\alpha + iu} \frac{\Gamma\left(\frac{3}{4} + \frac{a + iu}{2}\right)}{\Gamma\left(\frac{1}{4} - \frac{a + iu}{2}\right)} \quad m_{1,\alpha}(u) = i \sqrt{2\pi} \, 2^{\alpha + iu} \frac{\Gamma\left(\frac{3}{4} + \frac{a + iu}{2}\right)}{\Gamma\left(\frac{1}{4} - \frac{a + iu}{2}\right)},$$

and

$$q_{0,\varepsilon} = q_{0,\varepsilon,\alpha + i\beta} = \frac{m_{0,\alpha}(2\beta - \cdot)}{m_{\varepsilon,\alpha}} \quad q_{1,\varepsilon} = q_{1,\varepsilon,\alpha + i\beta} = -\frac{m_{1,\alpha}(2\beta - \cdot)}{m_{1,\varepsilon,\alpha}}.$$

Then $\|\pi_{\lambda, \varepsilon,a,b}\|_{ub}^2$ is equal to

\begin{equation}
\sup \left\{ \frac{\|h_\varepsilon q_{0,\varepsilon}\|_2^2 + \|h_{1-\varepsilon} q_{1,\varepsilon}\|_2^2 + 2\theta \text{Re}<h_\varepsilon q_{0,\varepsilon}, h_{1-\varepsilon} q_{1,\varepsilon}>_{L^2}}{\|h_0\|_2^2 + \|h_1\|_2^2 + 2\theta \text{Re}<h_0, h_1>_{L^2}} : \|h_0\|_2^2 + \|h_1\|_2^2 \neq 0 \right\}. \tag{4.2}
\end{equation}

For brevity, when the dependence on the parameter $\alpha$ is not important, we shall omit it in subscripts, for example, we shall simply write $m_0$ instead of $m_{0,\alpha}$.

Proof. First, we need an efficient way to compute $\|f\|_{H_{\alpha,a,b}}$. We will decompose $f$ into its even and odd parts and use the Mellin transform. We write

$$f(t) = \int_\mathbb{R} c_0(u) |t|^{\alpha - \frac{1}{2} + iu} du + \int_\mathbb{R} c_1(u) \text{sgn}(t) |t|^{\alpha - \frac{1}{2} - iu} du.$$

As proved in [13] p. 160 and [6] p. 173 formulae (12) and (13)], the Fourier transform of $|t|^{\alpha - \frac{1}{2} + iu}$ is given by $m_0(u) |\cdot|^{\alpha - \frac{1}{2} - iu}$, where

$$m_0(u) = m_{0,\alpha}(u) = \sqrt{2\pi} \, 2^{\alpha + iu} \frac{\Gamma\left(\frac{3}{4} + \frac{a + iu}{2}\right)}{\Gamma\left(\frac{1}{4} - \frac{a + iu}{2}\right)}.$$
and the Fourier transform of \( \text{sgn}(\cdot) |\cdot|^{-\frac{1}{2}+iu} \) is given by \( m_1(u) |\cdot|^{-\frac{1}{2}+iu} \text{sgn}(\cdot) \), where

\[
m_1(u) = m_{1,\alpha}(u) = i\sqrt{2\pi} \frac{2^{\alpha+iu} \Gamma \left( \frac{3}{4} + \frac{\alpha+iu}{2} \right)}{\Gamma \left( \frac{3}{4} - \frac{\alpha+iu}{2} \right)}.
\]

Therefore

\[
|\xi|^\alpha \hat{f}(\xi) = |\xi|^{-\frac{1}{2}} \int_\mathbb{R} (c_0(u) m_0(u) + \text{sgn}(\xi) c_1(u) m_1(u)) |\xi|^{-iu} du.
\]

In light of the definition of \( ||\cdot||_{\mathcal{H}_a,b} \) above and the Plancherel Theorem,

\[
||f||^2_{\mathcal{H}_a,b} = a \int_0^{+\infty} \left| \int_\mathbb{R} (c_0(u) m_0(u) + c_1(u) m_1(u)) |\xi|^{-iu} du \right|^2 \frac{d\xi}{|\xi|} + b \int_{-\infty}^0 \left| \int_\mathbb{R} (c_0(u) m_0(u) - c_1(u) m_1(u)) |\xi|^{-iu} du \right|^2 \frac{d\xi}{|\xi|}
\]

\[
= a \int_\mathbb{R} \left| \int_\mathbb{R} (c_0(u) m_0(u) + c_1(u) m_1(u)) e^{-itu} du \right|^2 dt + b \int_\mathbb{R} \left| \int_\mathbb{R} (c_0(u) m_0(u) - c_1(u) m_1(u)) e^{-itu} du \right|^2 dt
\]

\[
= 2\pi \left( a \int_\mathbb{R} |c_0(u) m_0(u) + c_1(u) m_1(u)|^2 du \\
+ b \int_\mathbb{R} |c_0(u) m_0(u) - c_1(u) m_1(u)|^2 du \right)
\]

\[
= 2\pi \left( (a + b) \int_\mathbb{R} |c_0(u) m_0(u)|^2 + |c_1(u) m_1(u)|^2 du \\
+ 2(a - b) \int_\mathbb{R} \text{Re} (c_0(u) m_0(u) \bar{c}_1(u) \bar{m}_1(u)) du \right),
\]

\[
= 2\pi (a + b) \left( \|c_0 m_0\|^2_2 + \|c_1 m_1\|^2_2 + 2\theta \text{Re}(c_0 m_0, c_1 m_1)_{L^2} \right),
\]

where \( \theta = (a - b)/(a + b) \). Since \( a \) and \( b \) are positive, \(-1 < \theta < 1\).

As in the proof of Lemma 4.11 since \( \tilde{N} AM \) acts unitarily on \( \mathcal{H}_{a,b} \),

\[
||\pi_{\lambda,\varepsilon,a,b}||_{\text{op}} = ||\pi_{\lambda,\varepsilon,a,b}(w)||_{\text{op}}.
\]

Suppose that

\[
f(t) = \int_\mathbb{R} c_0(u) |t|^{\frac{1}{2}+iu} du + \int_\mathbb{R} c_1(u) \text{sgn}(t) |t|^{\frac{1}{2}+iu} du
\]

\[
= \int_\mathbb{R} [c_0(u) + c_1(u) \text{sgn}(t)] |t|^{\frac{1}{2}+iu} du.
\]
Then, from (4.1) and linearity,

\[
[\pi_{\lambda,\epsilon,a,b}(w)f](t) = \int_{\mathbb{R}} \left[ c_0(u) + c_1(u) \sgn(-1/t) \sgn^\epsilon(-t) \right] |t|^{2\lambda-1} \left| 1/t \right|^a \frac{1}{2} + i \mu \, du \\
= (-1)\epsilon \int_{\mathbb{R}} \left[ c_0(u) - c_1(u) \sgn(t) \right] \sgn^\epsilon(t) \, |t|^{2\lambda-1 - \frac{3}{2} + (2\epsilon - u)} \, du \\
= (-1)\epsilon \int_{\mathbb{R}} \left[ c_0(u) - c_1(u) \sgn(t) \right] \sgn^\epsilon(t) \, |t|^{\alpha - \frac{3}{2} + (2\epsilon - u)} \, du \\
= (-1)\epsilon \int_{\mathbb{R}} \left[ c_\epsilon(2\beta - u) - c_1(2\beta - u) \sgn(t) \right] \sgn^\epsilon(t) \, |t|^{\alpha - \frac{3}{2} + (2\epsilon - u)} \, du \\
= \int_{\mathbb{R}} [c_\epsilon(2\beta - u) - c_1(2\beta - u) \sgn(t)] \, |t|^{\alpha - \frac{3}{2} + i \mu} \, du.
\]

Hence, when \( f \neq 0, \)

\[
\|\pi_{\lambda,\epsilon,a,b}(w)f\|_{\mathcal{H}_{\alpha,a,b}}^2 \\
= \frac{\|f\|_{\mathcal{H}_{\alpha,a,b}}^2}{\|f\|_{\mathcal{H}_{\alpha,a,b}}^2} \\
= \frac{\|c_\epsilon(2\beta - \cdot) m_0\|_{L^2}^2 + \|c_1(2\beta - \cdot) m_1\|_{L^2}^2 - 2\theta \Re \langle c_\epsilon(2\beta - \cdot) m_0, c_1(2\beta - \cdot) m_1 \rangle_{L^2}}{\|c_0 m_0\|_{L^2}^2 + \|c_1 m_1\|_{L^2}^2 + 2\theta \Re \langle c_0 m_0, c_1 m_1 \rangle_{L^2}} \\
= \frac{\|m_0(2\beta - \cdot) c_\epsilon\|_{L^2}^2 + \|m_1(2\beta - \cdot) c_1 - \cdot \|_{L^2}^2 - 2\theta \Re \langle m_0(2\beta - \cdot) c_\epsilon, m_1(2\beta - \cdot) c_1 - \cdot \rangle_{L^2}}{\|c_0 m_0\|_{L^2}^2 + \|c_1 m_1\|_{L^2}^2 + 2\theta \Re \langle c_0 m_0, c_1 m_1 \rangle_{L^2}}.
\]

To find the norm of the operator \( \pi_{\lambda,\epsilon,a,b}(w) \), we need to take the supremum of the last expression as \( f \) varies over \( \mathcal{H}_{\alpha,a,b} \). The denominator of the last expression is the square of the \( \mathcal{H}_{\alpha,a,b} \)-norm of \( f \), so to take an arbitrary \( f \) in \( \mathcal{H}_{\alpha,a,b} \), we may replace the function \( c_k m_k \) by \( h_k \), and take arbitrary \( h_k \in L^2(\mathbb{R}) \). Hence

\[
sup \left\{ \frac{\|\pi_{\lambda,\epsilon}(w) f\|_{\mathcal{H}_{\alpha,a,b}}^2}{\|f\|_{\mathcal{H}_{\alpha,a,b}}^2} : f \in \mathcal{H}_{\alpha,a,b}, f \neq 0 \right\} \\
= \sup \left\{ \frac{\|h_0 q_{0,\epsilon}\|_{L^2}^2 + \|h_1 - \epsilon q_{1,\epsilon}\|_{L^2}^2 + 2\theta \Re \langle h_\epsilon q_{\epsilon,\epsilon}, h_1 - \epsilon q_{\epsilon,\epsilon} \rangle_{L^2}}{\|h_0\|_{L^2}^2 + \|h_1\|_{L^2}^2 + 2\theta \Re \langle h_\epsilon, h_1 \rangle_{L^2}} : \|h_0\|_{L^2}^2 + \|h_1\|_{L^2}^2 \neq 0 \right\},
\]

where \( q_{0,\epsilon} = q_{0,\epsilon,\epsilon + \beta} = \frac{m_{\epsilon,\alpha}(2\beta - \cdot)}{m_{\epsilon,\alpha}} \) and \( q_{1,\epsilon} = \frac{m_{1,\epsilon,\epsilon + \beta}}{m_{1,\epsilon,\epsilon}} \).

**Corollary 4.4.** For all \( a, b \in \mathbb{R}^+ \),

\[
\|\pi_{\lambda,\epsilon,a,b}\|_{ub} \geq \|\pi_{\lambda,\epsilon}\|_{ub} = \max\{\|q_{0,\epsilon}\|_{\infty}, \|q_{1,\epsilon}\|_{\infty}\}.
\]

**Proof.** On the one hand, from the cases where \( h_0 = 0 \) or \( h_1 = 0 \) in (4.2), we see that

\[
\|\pi_{\lambda,\epsilon,a,b}\|_{ub} \geq \max\{\|q_{0,\epsilon}\|_{\infty}, \|q_{1,\epsilon}\|_{\infty}\} \quad \forall a, b \in \mathbb{R}^+.
\]

On the other hand, taking \( a = b \) in (4.2), we deduce that

\[
\|\pi_{\lambda,\epsilon,a,a}\|_{ub} \leq \max\{\|q_{0,\epsilon}\|_{\infty}, \|q_{1,\epsilon}\|_{\infty}\},
\]

and we have already observed that \( \|\pi_{\lambda,\epsilon,a,a}\|_{ub} = \|\pi_{\lambda,\epsilon}\|_{ub} \).
This corollary says that the Kunze–Stein representation \( \pi_{\lambda, \varepsilon} \) has minimal norm in the sense explained in the introduction. We are now going to show that when \( a \neq b \) the inequality in Corollary 4.4 is strict. To do this, we will use the following general lemma.

**Lemma 4.5.** Consider the Hilbert space \( L^2(\mathbb{R}) \oplus L^2(\mathbb{R}) \), equipped with the norm
\[
\| (h_0, h_1) \|_\theta = \| h_0 \|_2^2 + \| h_1 \|_2^2 + 2\theta \text{Re}(h_0, h_1)_{L^2},
\]
where \(-1 < \theta < 1\), and the linear operator \( T : (h_0, h_1) \mapsto (q_0 h_0, q_1 h_1) \) on this space, where \( q_0, q_1 \in L^\infty(\mathbb{R}) \). If \( \theta = 0 \), then \( \| T \|_{\text{op}} = \max\{\| q_0 \|_\infty, \| q_1 \|_\infty\} \). If \( \theta \neq 0 \) and \( \| q_0 q_1 \|_\infty < \min\{\| q_0 \|_\infty, \| q_1 \|_\infty\}^2 \), then \( \| T \|_{\text{op}} > \max\{\| q_0 \|_\infty, \| q_1 \|_\infty\} \).

**Proof.** Without loss of generality, we suppose that \( \| q_0 \|_\infty = \max\{\| q_0 \|_\infty, \| q_1 \|_\infty\} = 1 \).

Taking first \( h_0 = 0 \) and then \( h_1 = 0 \) we see that \( \| T \|_{\text{op}} \geq 1 \).

When \( \theta = 0 \), it is evident that \( \| T \|_{\text{op}} = 1 \). Indeed,
\[
\| q_0 h_0 \|_2^2 + \| q_1 h_1 \|_2^2 \leq \| h_0 \|_2^2 + \| h_1 \|_2^2,
\]
whence \( \| T \|_{\text{op}} \leq 1 \).

Suppose now \( \theta \neq 0 \) and \( \| q_0 q_1 \|_\infty = 1 - \varepsilon \), where \( \varepsilon > 0 \). We consider \( T(\chi_{E(t)}, t\chi_{E(t)}) \) when \( t \) is very small, and \( E(t) \) is chosen such that \( |q_0(u)| > 1 - t^2 \) for all \( u \in E(t) \). On the one hand,
\[
\| (\chi_{E(t)}, t\chi_{E(t)}) \|_\theta = \left( \int_R \chi_{E(t)}(u) \, du + \int_R t^2 \chi_{E(t)}(u) \, du + 2\theta \int_R t\chi_{E(t)}(u) \, du \right)^{1/2}
\]
\[
= \| \chi_{E(t)} \|_2 (1 + 2\theta t + t^2)^{1/2}
\]
\[
= \| \chi_{E(t)} \|_2 (1 + \theta t + O(t^2)),
\]
so
\[
\frac{1}{\| (\chi_{E(t)}, t\chi_{E(t)}) \|_\theta} = \frac{1 - \theta t + O(t^2)}{\| \chi_{E(t)} \|_2}.
\]

On the other hand, \( \| (q_0 \chi_{E(t)}, t q_1 \chi_{E(t)}) \|_\theta \) is equal to
\[
\left( \int_R |q_0(u)|^2 \chi_{E(t)}(u) \, du + \int_R t^2 |q_1(u)|^2 \chi_{E(t)}(u) \, du + 2\theta \text{Re} \int_R t q_0(u) \bar{q}_1(u) \chi_{E(t)}(u) \, du \right)^{1/2}.
\]
Now
\[
\int_R |q_0(u)|^2 \chi_{E(t)}(u) \, du = (1 + O(t^2)) \| \chi_{E(t)} \|_2^2,
\]
while
\[
\int_R t^2 |q_1(u)|^2 \chi_{E(t)}(u) \, du = O(t^2) \| \chi_{E(t)} \|_2^2.
\]
It follows that
\[
(4.4) \quad \frac{\| (q_0 \chi_{E(t)}, t q_1 \chi_{E(t)}) \|_\theta}{\| (\chi_{E(t)}, t\chi_{E(t)}) \|_\theta} \geq 1 - \theta t + \theta \text{Re} \int_R q_0(u) \bar{q}_1(u) \chi_{E(t)}(u) \, du + O(t^2).
\]
Lemma 4.6. Suppose that

\[ \sup_{|\alpha| < \frac{1}{2}} \left| \frac{\int_{\mathbb{R}} q_0(u) \tilde{q}_1(u) \chi_E(u) \, du}{\int_{\mathbb{R}} \chi_E(u) \, du} \right| \leq 1 - \varepsilon, \]

when \( t \) is very small and \( \theta \neq 0 \), the right hand side of (4.1) is strictly greater than 1, so \( \|T\|_{op} > 1 \).

In light of Lemma 4.3, we need to analyse the functions \( q_{0,\varepsilon}, q_{1,\varepsilon} \) defined in Lemma 4.3.

**Lemma 4.6.** Suppose that \( 0 < |\alpha| < \frac{1}{2} \) and \( \varepsilon + \beta^2 \neq 0 \), and define \( q_{0,\varepsilon} \) and \( q_{1,\varepsilon} \) as in Lemma 4.3. Then

\[ \|q_{0,\varepsilon} q_{1,\varepsilon}\|_{\infty} < \max\{\|q_{0,\varepsilon}\|_{2}^2, \|q_{1,\varepsilon}\|_{2}^2\}. \]

**Proof.** Define \( t_\alpha(u) = i \tan(\pi(\frac{1}{4} + \frac{\alpha + im}{2})) \). Then by the reflection formula (see [4, p. 3, formula (6)]),

\[
m_{1,\alpha}(u) = i \sqrt{2\pi} 2^{\alpha + im} \frac{\Gamma\left(\frac{3}{4} + \frac{\alpha + im}{2}\right)}{\Gamma\left(\frac{1}{4} - \frac{\alpha + im}{2}\right)} \sin \left(\frac{\pi}{2} \frac{\alpha + im}{2}\right) \sin \left(\frac{\pi}{2} \frac{\alpha + im}{2}\right) = m_{0,\alpha}(u) t_\alpha(u). \]

We now prove the desired inequality. Note that \( \lim_{\varepsilon \to \infty} |q_{0,\varepsilon}(u) q_{1,\varepsilon}(u)| = 1 \), so

\[ \|q_{0,\varepsilon} q_{1,\varepsilon}\|_{\infty} \geq 1. \]

Since \( q_{0,\varepsilon} q_{1,\varepsilon} \) is a continuous function and \( |q_{0,\varepsilon} q_{1,\varepsilon}(\beta)| = 1 \), there exists \( \bar{u} \) in \( \mathbb{R} \) such that

\[ \|q_{0,\varepsilon} q_{1,\varepsilon}\|_{\infty} = |q_{0,\varepsilon}(\bar{u}) q_{1,\varepsilon}(\bar{u})|. \]

Suppose now that \( \varepsilon = 1 \) and write

\[
|q_{0,1}(\bar{u}) q_{1,1}(\bar{u})| = |q_{0,1}(\bar{u})|^2 |t_\alpha(2\beta - \bar{u}) t_\alpha(\bar{u})| \\
|q_{0,1}(\bar{u}) q_{1,1}(\bar{u})| = |q_{1,1}(\bar{u})|^2 \frac{1}{|t_\alpha(2\beta - \bar{u}) t_\alpha(\bar{u})|}. 
\]

Since \( \tan(x + iy) = \frac{\cos(2y) - \cos(2x)}{\cos(2y) + \cos(2x)} \), we obtain that

\[ |t_\alpha(u)|^2 = \frac{\cosh(\pi u) + \sin(\pi \alpha)}{\cosh(\pi u) - \sin(\pi \alpha)} \quad \forall u \in \mathbb{R}, \]

therefore \( |t_\alpha(u)| > 1 \) when \( \alpha > 0 \) and \( |t_\alpha(u)| < 1 \) when \( \alpha < 0 \). We easily conclude that

\[ \|q_{0,1} q_{1,1}\|_{\infty} < \max\{\|q_{0,1}\|_{2}^2, \|q_{1,1}\|_{2}^2\}. \]

In the case where \( \varepsilon = 0 \), we write

\[
q_{0,0}(\bar{u}) q_{1,0}(\bar{u}) = \left(\frac{m_{0,\alpha}(2\beta - \bar{u})}{m_{0,\alpha}(\bar{u})}\right)^2 t_\alpha(2\beta - \bar{u}) t_\alpha(\bar{u}) = q_{0,0}(\bar{u}) \frac{t_\alpha(2\beta - \bar{u})}{t_\alpha(\bar{u})} \\
q_{0,0}(\bar{u}) q_{1,0}(\bar{u}) = \left(\frac{m_{1,\alpha}(2\beta - \bar{u})}{m_{1,\alpha}(\bar{u})}\right)^2 t_\alpha(2\beta - \bar{u}) t_\alpha(\bar{u}) = q_{1,0}(\bar{u}) \frac{t_\alpha(2\beta - \bar{u})}{t_\alpha(\bar{u})},
\]
and note that

\[
\left| \frac{t_\alpha(2\beta - u)}{t_\alpha(u)} \right|^2 = \frac{\cosh(\pi(2\beta - u)) + \sin(\pi\alpha) \cosh(\pi u) - \sin(\pi\alpha)}{\cosh(\pi(2\beta - u)) - \sin(\pi\alpha) \cosh(\pi u) + \sin(\pi\alpha)}
\]

\[
= 1 + \frac{2\sinh(\pi\beta) \sinh(\pi(\beta - u))}{\cosh(\pi u) + \sin(\pi\alpha)}
\]

Suppose that \( \alpha \in (0, \frac{1}{2}) \) and \( \beta > 0 \), so that \( \sin(\pi\alpha) > 0 \). Then the fraction

\[
\left| \frac{t_\alpha(2\beta - \bar{u})}{t_\alpha(\bar{u})} \right|
\]

is less than 1 if \( \beta - \bar{u} > 0 \), is equal to 1 if \( \beta - \bar{u} = 0 \), and is greater than 1 if \( \beta - \bar{u} < 0 \). Therefore if \( \bar{u} \neq \beta \),

\[
\|q_{0,0} q_{1,0}\|_\infty < \max\{\|q_{0,0}\|_\infty^2, \|q_{1,0}\|_\infty^2\}.
\]

Finally, if \( \bar{u} = \beta \), we conclude that \( \|q_{0,0} q_{1,0}\|_\infty = 1 \). We claim that

\[
\|q_{0,0}\|_\infty > 1,
\]

from which the result follows easily. To prove the claim, suppose that \( |q_{0,0}(u)| = 1 \) for all \( u \in \mathbb{R} \). Then

\[
|m_0(2\beta - u)| = |m_0(u)| = |m_0(-u)| \quad \forall u \in \mathbb{R},
\]

that is, \( |m_0| \) is \( 2\beta \)-periodic. But if \( \alpha > 0 \), \( \lim_{u \to -\infty} |m_0(u)| = \lim_{u \to -\infty} \sqrt{2\pi 2^\alpha} |u|^\alpha = +\infty \), which contradicts the periodicity of \( |m_0| \). Therefore there exists some \( \bar{u} \) such that \( |q_{0,0}(\bar{u})| \neq 1 \). Note that

\[
q_{0,0}(u) = \frac{1}{q_{0,0}(2\beta - u)},
\]

therefore we conclude that

\[
\|q_{0,0}\|_\infty \geq \max\{|q_{0,0}(\bar{u})|, |q_{0,0}(2\beta - \bar{u})|\} = \max\{1/|q_{0,0}(\bar{u})|, 1/|q_{0,0}(\bar{u})|\} > 1.
\]

The cases where \( \beta < 0 \) or \( \alpha < 0 \) may be treated similarly. \( \square \)

**Corollary 4.7.** Suppose that \( \alpha \neq \beta \), that \( |\alpha| < \frac{1}{2} \) and that \( \lambda, \varepsilon \neq (0,1) \). Then

(4.5)

\[
\|\pi_{\lambda,\varepsilon, a, b}\|_{ub} > \|\pi_{\lambda,\varepsilon}\|_{ub}.
\]

**Proof.** In the case where \( \alpha \neq 0 \) and \( \varepsilon + \beta^2 = 0 \), the inequality follows from Lemmas 4.3 and 4.6. In the remaining cases, the representation \( \pi_{\lambda,\varepsilon} \) is unitary and irreducible. If equality holds in (4.5), then \( \pi_{\lambda,\varepsilon, a, b} \) must also be unitary. Then the identity map from \( \mathcal{H}_a \) to \( \mathcal{H}_{a,a,b} \), which is a similarity, is a multiple of a unitary operator, by Lemma 2.2 which implies that \( a = b \). \( \square \)

To complete the proof of Theorem 1.1, we find sharp estimates of \( \|q_{0,\varepsilon}\|_\infty \) and \( \|q_{1,\varepsilon}\|_\infty \). For this purpose we shall use the following technical lemma for the gamma function, in which we denote by \( S \) the set \( \{ z \in \mathbb{C} : \text{Re}(z) \in [-\frac{1}{2}, 1] \} \).
Lemma 4.8. There exist constants $C$ and $C'$ such that

$$C \leq \frac{|(x + iy) \Gamma(x + iy)|}{e^{-\frac{1}{2} \pi |y|} (1 + |y|)^{\frac{x}{2} + \frac{1}{2}}} \leq C' \quad \forall x + iy \in S.$$ 

Proof. Take $x + iy \in S$. Since $\Gamma(x - iy) = \overline{\Gamma(x + iy)}$, we may suppose that $y > 0$ and note that

$$1 < e^{y \arctan \frac{x}{y}} < e^{x+1}.$$ 

By using Stirling’s asymptotic expansion (see [4, p. 47, formula (2)]),

$$\Gamma(\zeta) = e^{-\zeta} e^{(\zeta - \frac{1}{2}) \ln \zeta} \sqrt{2\pi} (1 + O(\zeta^{-1})) \quad \zeta \to \infty,$$

where $\zeta = x + 1 + iy$, we obtain

$$\left| e^{-\zeta} e^{(\zeta - \frac{1}{2}) \ln \zeta} \right| = \left| e^{-(x+1)} e^{(x+\frac{1}{2}) \ln |\zeta| - y \arctan \frac{y}{x}} \right| = \left| e^{-(x+1)} |\zeta|^{x+\frac{1}{2}} e^{-\frac{\pi y}{2}} \arctan \frac{y}{x} \right| \simeq (1 + y)^{x+\frac{1}{2}} e^{-\frac{\pi y}{2}}.$$

The result follows observing that $\Gamma(\zeta) = (x + iy) \Gamma(x + iy)$. □

Lemma 4.9. Suppose that $\lambda = \alpha + i \beta$, where $\beta \neq 0$ and $-\frac{1}{2} < \alpha < \frac{1}{2}$. Then, with the constants $C$ and $C'$ of Lemma 4.8,

$$(\frac{C}{C'})^2 \left| \frac{1}{2} - |\alpha| + 2i\beta \right| \frac{1}{2} + |\alpha| \leq \frac{||q_{0,0,\lambda}||_{\infty}}{(1 + |\beta|)^{\alpha}} \leq \left( \frac{C'}{C'} \right)^2 \frac{1}{\frac{1}{2} - |\alpha|}$$

$$\left( \frac{C}{C'} \right)^2 \frac{||q_{1,0,\lambda}||_{\infty}}{(1 + |\beta|)^{\alpha}} \leq 2 \left( \frac{C'}{C'} \right)^2$$

$$(\frac{C}{C'})^2 \left| \frac{1}{2} - \alpha + 2i\beta \right| \frac{1}{2} + \alpha \leq \frac{||q_{0,1,\lambda}||_{\infty}}{(1 + |\beta|)^{\alpha}} \leq 2 \left( \frac{C'}{C'} \right)^2 \frac{1}{\frac{1}{2} + \alpha}$$

$$(\frac{C}{C'})^2 \left| \frac{1}{2} + \alpha + 2i\beta \right| \frac{1}{2} - |\alpha| \leq \frac{||q_{1,1,\lambda}||_{\infty}}{(1 + |\beta|)^{\alpha}} \leq 2 \left( \frac{C'}{C'} \right)^2 \frac{1}{\frac{1}{2} - \alpha}.$$
Proof. By Lemma 4.8

\[ \|q_{0,0,\alpha+i\beta}\|_\infty = \sup_{u \in \mathbb{R}} \left| \frac{\Gamma \left( \frac{1}{4} + \frac{\alpha+i(2\beta-u)}{2} \right)}{\Gamma \left( \frac{1}{4} - \frac{\alpha+2i}{2} \right)} \right| \leq \left( \frac{C'}{C} \right)^2 \sup_{u \in \mathbb{R}} \left| 1 + |\beta - u| \right|^\alpha \cdot \sup_{u \in \mathbb{R}} \left| \frac{\Gamma \left( \frac{1}{4} - \frac{\alpha}{2} - i(\beta-u) \right)}{\Gamma \left( \frac{1}{4} - \frac{\alpha}{2} - i\beta \right)} \frac{1 + \alpha + 2i(\beta-u)}{1 + \alpha + 2i\beta} \right| \]

\[ \leq \left( \frac{C'}{C} \right)^2 (1 + |\beta|)^{\alpha} \left( \sup_{u \in \mathbb{R}} \left( \frac{1}{4} + \frac{\alpha}{2} \right)^2 + (\beta-u)^2 \right)^{1/2} \cdot \sup_{u \in \mathbb{R}} \left( \frac{1}{4} + \frac{\alpha}{2} \right)^2 + u^2 \]

\[ = \left( \frac{C'}{C} \right)^2 (1 + |\beta|)^{\alpha} \frac{1}{2} - |\alpha|. \]

On the other hand,

\[ \|q_{0,0,\alpha+i\beta}\|_\infty \geq \max \{ |q_{0,0,\alpha+i\beta}(0)|, |q_{0,0,\alpha+i\beta}(2\beta)| \} \]

\[ \geq \left| \frac{\Gamma \left( \frac{1}{4} + \frac{\alpha}{2} + i\beta \right)}{\Gamma \left( \frac{1}{4} - \frac{\alpha}{2} - i\beta \right)} \right| \]

\[ \geq \left( \frac{C}{C'} \right)^2 (1 + |\beta|)^{\alpha} \left| \frac{1}{4} + \frac{\alpha}{2} + i\beta \right| \cdot \left| \frac{1}{4} - \frac{\alpha}{2} - i\beta \right|, \]

concluding the proof of the estimate for \( \|q_{0,0,\alpha+i\beta}\|_\infty \).

The other computations are similar and we leave the details to the reader. \( \square \)

Corollary 4.10. Suppose that \( \lambda = \alpha + i\beta \), where \( -\frac{1}{2} < \alpha < \frac{1}{2} \). Then when \( \beta \) is large,

\[ \|\pi_{\lambda,\varepsilon}\|_{ub} \simeq \frac{(1 + |\beta|)^{\alpha}}{2 - |\alpha|}. \]

Proof. By Corollary 4.4

\[ \|\pi_{\lambda,\varepsilon}\|_{ub} = \max \{ \|q_{0,\varepsilon}\|_{\infty}, \|q_{1,\varepsilon}\|_{\infty} \}, \]

and our estimates follow easily. \( \square \)

We conclude with the observation that some parts of the proof of Theorem 1.1 generalize easily to \( SO(n,1) \), with \( n > 2 \), and other parts with more difficulty. We believe that everything may be extended, with different formulae, and plan to return to this soon.

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