ASYMPTOTIC PROBABILITY OF ENERGY INCREASING SOLUTIONS TO THE HOMOGENEOUS BOLTZMANN EQUATION

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Abstract. Weak solutions to the homogeneous Boltzmann equation with increasing energy have been constructed by Lu and Wennberg. We consider an underlying microscopic stochastic model with binary collisions (Kac’s model) and show that these solutions are atypical. More precisely, we prove that the probability of observing these paths is exponentially small in the number of particles and compute the exponential rate. This result is obtained by improving the established large deviation estimates in the canonical setting. Key ingredients are the extension of Sanov’s theorem to the microcanonical ensemble and large deviations for the Kac’s model in the microcanonical setting.

1. Introduction

The derivation of the Boltzmann equation from an underlying microscopic dynamics of $N$ interacting particles is a paradigmatic problem in non-equilibrium statistical mechanics. It is based on the validity of the Stosszahlansatz with probability one in the limit $N \to +\infty$. At a more refined level, it is possible to analyze the corresponding large deviations, whose derivation is related to the validity of the Stosszahlansatz with probability super-exponentially close to one for $N$ large.

In this perspective, the most challenging case of Newtonian dynamics of hard spheres in the Boltzmann-Grad limit has been recently discussed in [3]. Nevertheless, also the case of stochastic dynamics presents interesting features. The first result in this setting has been obtained in [10], where a large deviation upper bound is derived in the space homogeneous case. A complete large deviation principle has been obtained in [17] for a space inhomogeneous model with a finite set of velocities. In [2] a large deviation upper bound is achieved for a homogeneous model which conserves momentum but not energy, while the matching lower bound is obtained for a restricted class of paths. A similar result, in the case of energy and momentum conservation, has been proven in [7]. In this case the upper and lower bound match for a subset of paths for which energy is conserved.

For energy preserving microscopic dynamics with unbounded velocities, a main obstacle to a complete proof of large deviations is the occurrence of macroscopic paths with finite rate function that violate the conservation of the energy. In particular, as discussed in [7], a class of such paths is given by the solutions to the homogeneous Boltzmann equations constructed by Lu and Wennberg in [12], for which the energy is increasing. Another example of large deviation asymptotic for

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non-conserving energy path has been constructed in [1], for a Kac-like microscopic dynamics with discrete energies. More precisely, as proven in [7], the upper bound rate function derived in [10] vanishes on Lu and Wennberg solutions, while their asymptotic probability is $e^{-cN}$, which implies the upper bound rate function in [10] is not optimal.

The homogeneous Boltzmann equation with hard sphere cross-section reads as

$$
\partial_t f_t(v) = \frac{1}{2} \int_{\mathbb{R}^d} dv_\star \int_{S_{d-1}} d\omega \left| (v - v_\star) \cdot \omega \right| (f_t(v') f_t(v'_\star) - f_t(v) f_t(v_\star)),
$$

(1.1)

where $S_{d-1}$ is the sphere in $\mathbb{R}^d$. The associate Cauchy problem has a unique solution in the class of function with constant energy [13]. Let us discuss how the Lu and Wennberg solutions, with increasing energy, can be constructed in the special case in which the energy has a unique jump at time zero. Consider a sequence of initial densities $f_0^n$ such that $f_0^n$ converges weakly to $f_0$ but $e := \lim_n \int f_0^n(v)^2 \, dv > \int f_0(v)^2 \, dv$, namely a fraction of energy evaporates at infinity. Denoting by $f^n_t$ the unique energy conserving solution of the homogeneous Boltzmann equation with initial datum $f_0^n$, we then have that $f^n_t$, $t \geq 0$, converges to a solution to the homogeneous Boltzmann equation, with initial datum $f_0$, but the energy has a positive jump at time 0. Observe that this construction does not yield a jump in the energy if the total cross section is bounded, in fact in this case there is uniqueness of the solution without the requirement of energy conservation. A model with this feature has been analyzed in [6].

A main result of this paper is the proposal of a rate function that improves the one in [10], being strictly positive on Lu and Wennberg solutions. In particular we consider a Kac walk with the hard sphere cross section, and prove the large deviation upper bound with such rate function. The matching large deviation lower bound is achieved for both Lu and Wennberg solutions and the same restricted class of path as in [2] [7].

As it is clear from the previous construction, Lu and Wennberg solutions can be produced from a microscopic model only if there exists a fluctuation of the initial energy and then following the typical behavior. To introduce the improved rate function we consider first the case in which the initial velocities are sampled from the microcanonical ensembles, namely the total energy and momentum are not random, and given by $(eN, uN)$. After [1, 2], we consider as empirical observable the pair $(\pi^N, Q^N)$ where $\pi^N$ is the empirical distribution of velocities, while the empirical flux $Q^N$ records the collision times together the incoming and outgoing velocities. The microcanonical rate function reads

$$
I_{e,u}(\pi, Q) = H_{e,u}(\pi_0) + J_{e,u}(\pi, Q),
$$

(1.2)

where $H_{e,u}$ takes into account the fluctuation of the initial data, while $J_{e,u}$ is the dynamical contribution, that is defined as follows. Set $dQ^\pi := \frac{1}{2} d\pi \otimes d\tau B \, d\omega \, dt$, with $B = B(v - v_\star, \omega) = \frac{1}{2} |(v - v_\star) \cdot \omega|$, and let $J(\pi, Q)$ be the relative entropy of $Q$ with respect to $Q^\pi$, namely

$$
J(\pi, Q) = \int \left\{ dQ \log \frac{dQ}{dQ^\pi} - dQ + dQ^\pi \right\}.
$$

(1.3)

Then, by the microcanonical constraint, $J_{e,u}(\pi, Q)$ is equal to $J(\pi, Q)$ if the energy of $\pi$ does not exceed $e$ and its momentum is equal to $u$, while $J_{e,u}(\pi, Q) = +\infty$ otherwise. The functional $H_{e,u}$ will be derived by extending Sanov’s theorem to
the microcanonical ensemble. In particular, $H_{e,u}(\pi_0)$ is infinite when the energy of $\pi_0$ exceed $e$, but it can be finite when the energy is below $e$. Namely, loss of energy at time 0 occurs with exponentially small probability. According to (1.2), the asymptotic probability of Lu and Wennberg solutions is then $\exp(-NH_{e,u}(\pi_0))$.

We then analyze the case in which the initial velocities are sampled from the canonical ensemble, namely are i.i.d. $m$-distributed random variables. The canonical rate function can then be obtained from (1.2) as follows

$$I(\pi, Q) = \inf_{e,u} (A(e,u) + I_{e,u}(\pi, Q)), \quad (1.4)$$

where $A$ is the rate function for the energy and momentum of the sum of i.i.d. $m$-distributed random variables, given by Cramér’s theorem. The rate function introduced in [10] and further analyzed in [7] is given by

$$I(\pi, Q) = \text{Ent}(\pi_0|m) + J(\pi, Q),$$

where $\text{Ent}(\pi_0|m)$ is the relative entropy. In particular, $I$ vanishes on Lu and Wennberg solutions. We show that $I$ defined in (1.4) is larger than $\mathcal{I}$ and vanishes only on the unique energy conserving solution to (1.1). Moreover, we compute explicitly its value on the Lu and Wennberg solutions, which is given by $c\Delta E$, where $c$ is a strictly positive constant depending on the tail of initial distribution $m$ and $\Delta E$ is the total gain of the energy. Hence, the asymptotic probability of Lu and Wennberg solutions is $e^{-cN\Delta E}$.

The present work is organized as follows. In Section 2 we consider the static case, by analyzing the large deviations of the empirical measure when the velocities are sampled from the microcanonical ensemble. As discussed before, we show that the large deviation functional is finite on probability measures with energy evaporation. In Section 3 we state the large deviation principle for the Kac model with hard sphere cross section and microcanonical initial data. The corresponding proof is carried out in Sections 4, 5. In Section 6 we derive the large deviation asymptotic for the Kac model with canonical initial distribution. Section 7 is finally devoted to the asymptotic probability of Lu and Wennberg solutions.

2. **Sanov theorem for microcanonical ensemble**

Sanov’s theorem, that describes the asymptotic behavior of the empirical measures associated to a sequence of $N$ i.i.d. random variables, is a basic result in the theory of large deviations. A natural question is to replace the independence assumption by some dependency structure. For instance, the case of the empirical measure associated to Markov chains is the content of the classical Donsker-Varadhan theorem. We here analyze the case in which the underlying sequence of random variables is sampled according to a microcanonical ensemble, that can be realized by conditioning i.i.d. random variables to the sum of their squares, i.e. to the total kinetic energy in physical interpretation. A particular case of this situation has been previously discussed in [8]; there it is in fact analyzed the case where $N$ real random variables are sampled according to the uniform measure on the sphere of radius $\sqrt{N}$ on $\mathbb{R}^N$ and corresponds to the microcanonical ensemble associated to i.i.d. Gaussians. A peculiar feature of this setting is the possibility of observing – at the large deviations level – probabilities that violate the microcanonical constraint. More precisely, while for each $N$ the law of the empirical measure is supported by the probabilities with fixed second moment, the large deviations rate function is
finite also for probabilities with second moment strictly smaller than the prescribed value. In view of the application to homogeneous Boltzmann equations, we shall next consider microcanonical ensembles that are obtained by conditioning both to the total energy and to the total momentum.

Fix hereafter $d \geq 2$ and denote by $\mathcal{P}(\mathbb{R}^d)$ the set of probability measures on $\mathbb{R}^d$ equipped with the topology induced by the weak convergence and the associated Borel $\sigma$-algebra. Let $\zeta: \mathbb{R}^d \mapsto [0, +\infty) \times \mathbb{R}^d$ be the map given by $\zeta = (\zeta_0, \zeta)(v) = (|v|^2/2, u)$. We shall consider probabilities $m \in \mathcal{P}(\mathbb{R}^d)$ satisfying the following conditions.

**Assumption 2.1.** There exists $\gamma_0^* \in (0, +\infty]$ such that

(i) $m$ is absolutely continuous with respect to the Lebesgue measure and $m$ is strictly positive on open sets;
(ii) $m(e^{\gamma_0^*}) < +\infty$ for any $\gamma_0 \in (-\infty, \gamma_0^*)$, and $\lim_{\gamma_0 \uparrow \gamma_0^*} m(e^{\gamma_0^*}) = +\infty$;
(iii) for each $\gamma = (\gamma_0, \gamma) \in (-\infty, \gamma_0^*) \times \mathbb{R}^d$ the Fourier transform of $\frac{dm}{d\gamma}$ belongs to $L^1(\mathbb{R}^d)$;
(iv) there exists $c > 0$ such that $\frac{dm}{d\gamma} \geq \frac{1}{c} \exp\{-c|v|^2\}$.

Condition (iv) is mainly technical, and will be used only to derive the lower bound for the dynamical rate function.

We observe that the map $(-\infty, \gamma_0^*) \times \mathbb{R}^d \ni \gamma \mapsto \log m(e^{\gamma})$ is strictly convex. Set $Z = \{(e, u) \in (0, +\infty) \times \mathbb{R}^d : e > |u|^2/2\}$, then $\nabla \log m(e^{\gamma})$ is a bijection from $(-\infty, \gamma_0^*) \times \mathbb{R}^d$ to $Z$. We denote by $(e, u) \mapsto \gamma(e, u)$ the inverse map and by $m_{e,u}$ the probability on $\mathbb{R}^d$ defined by

$$m_{e,u}(dv) := \frac{e^{\gamma(e,u)} \zeta(v)}{m(e^{\gamma(e,u)})} m(dv).$$

In words, $m_{e,u}$ is the exponential tilt of $m$ such that $m_{e,u}(\zeta) = (e, u)$. Namely, $u$ and $e$ are the average values of velocity and total energy, respectively. Note that $m_{e,u}$ satisfies the conditions in Assumption 2.1 with $\gamma_0^*$ replaced by $\gamma_0^* - \gamma_0(e, u)$. We denote by $U$ the internal energy defined by the relation $e = U + |u|^2/2$, so that $U$ is the expected value of $|v - u|^2/2$.

Let $\Sigma_N := (\mathbb{R}^d)^N$ be the configuration space for $N$ velocities in $\mathbb{R}^d$. Given $(e, u) \in Z$, we denote by

$$\Sigma_{e,u} := \{v \in (\mathbb{R}^d)^N : \frac{1}{N} \sum_{i=1}^N \zeta(v_i) = (e, u)\}$$

(2.2) the set of configurations with total momentum $Nu$ and total energy $Ne$.

Let $\mu_N$, be the probability on $\Sigma_N$ given by $m^\otimes N$, interpreted as the canonical ensemble. Let also $(e, u) \mapsto \nu_{e,u}^N$ be a regular version of the probability $\mu_N$ conditioned to $\frac{1}{N} \sum_{i=1}^N \zeta(v_i)$. In particular, $\nu_{e,u}^N$, interpreted as the microcanonical ensemble, is the probability supported by $\nu_{e,u}^N$ informally given by $\nu_{e,u}^N = \mu_N(\cdot | \Sigma_{e,u})$. As $N \to \infty$ the one-marginal of $\nu_{e,u}^N$ converge to $m_{e,u}$ (equivalence of ensembles in the thermodynamic limit), see [3, §1.5] and [16]. Our aim is to describe the corresponding large deviations asymptotic. In order to apply this result to Kac’s walk with a canonical initial distribution of the velocities, the large deviation principle will be proven uniformly for $(e, u)$ in compact subsets of $Z$. 

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We define the empirical measure as the map \( \pi^N : \Sigma^N \to \mathcal{P}(\mathbb{R}^d) \) given by
\[
\pi^N(u) = \frac{1}{N} \sum_i \delta_{u_i}.
\] (2.3)

Given two probabilities \( m_1, m_2 \), recall that the relative entropy \( \text{Ent}(m_2|m_1) \) is defined as \( \text{Ent}(m_2|m_1) = \int \text{d}m_1 \rho \log \rho \), where \( \text{d}m_2 = \rho \text{d}m_1 \), understanding that \( \text{Ent}(m_2|m_1) = +\infty \) if \( m_2 \) is not absolutely continuous with respect to \( m_1 \).

Given \((e,u) \in Z\) set
\[
C_{e,u} := \{ \pi \in \mathcal{P}(\mathbb{R}^d) : \pi(\zeta) = u, \pi(\zeta_0) \leq e \} \tag{2.4}
\]
that is a compact and convex subset of \( \mathcal{P}(\mathbb{R}^d) \). Note that \( C_{e,u} \) is the closure in \( \mathcal{P}(\mathbb{R}^d) \) of the set of probabilities \( \pi \) satisfying the microcanonical constraint \( \pi(\zeta) = (e,u) \).

**Theorem 2.2.** Fix \((e,u) \in Z\) and a sequence \((e_N,u_N) \to (e,u)\). If \( m \) satisfies item (i)–(iii) in Assumption \( \mathbb{Z} \) then the family of probabilities \( \{\nu_{e_N,u_N}^{N} \circ (\pi^N)^{-1}\} \)

on \( \mathcal{P}(\mathbb{R}^d) \) satisfies a large deviation principle with good and convex rate function \( H_{e,u} : \mathcal{P}(\mathbb{R}^d) \to [0, +\infty] \) given by
\[
H_{e,u}(\pi) = \begin{cases} 
\text{Ent}(\pi|m_{e,u}) + [\gamma_0^* - \gamma_0(e,u)] [e - \pi(\zeta_0)] & \text{if } \pi \in C_{e,u}, \\
+\infty & \text{otherwise.} 
\end{cases}
\] (2.5)

The rate function \( H_{e,u} \) can be understood as the canonical rate function \( \text{Ent}(-m_{e,u}) \) with an extra penalization for violations of the energy constraint. When \( m \) is the standard Gaussian on \( \mathbb{R} \) and the momentum constraint is dropped this result reduces to the one obtained in \( [8] \).

While the arguments in \( [8] \) rely on the representation of the uniform measure on the spheres in terms of i.i.d. Gaussian, the proof of the above theorem will be achieved by applying the Gärtner-Ellis theorem, which provides the large deviation rate function as the Legendre transform of the log-moment generating function. To this end, for \( \phi \in C_b(\mathbb{R}^d) \) set
\[
dm_{e,u}^\phi := \frac{\text{d}m_{e,u} e^\phi}{m_{e,u} \left(e^\phi\right)},
\] (2.6)
and
\[
\Lambda_{e,u}(\phi) := -\gamma \cdot (e,u) + \log m_{e,u}(e^\phi + \gamma \cdot \zeta),
\] (2.7)
where \( \gamma = \gamma(\phi) \) is chosen so that
\[
\frac{m_{e,u}^\phi(\exp(\gamma \cdot \zeta))}{m_{e,u}(\exp(\gamma \cdot \zeta))} = (e,u),
\] (2.8)

namely, it is chosen in order that the exponential tilt of \( m^\phi \) has average energy and momentum \( (e,u) \).

**Lemma 2.3.** For each \( \phi \in C_b(\mathbb{R}^d) \),
\[
\lim_{N \to \infty} \frac{1}{N} \log \nu_{e_N,u_N}^N \left(e^{N\pi^N(\phi)}\right) = \Lambda_{e,u}(\phi).
\] (2.9)

**Proof.** As simple to check,
\[
\left| \frac{1}{N} \log \nu_{e_N,u_N}^N \left(e^{N\pi^N(\phi_2)}\right) - \frac{1}{N} \log \nu_{e_N,u_N}^N \left(e^{N\pi^N(\phi_1)}\right) \right| \leq \sup_{v \in \mathbb{R}^d} |\phi_2(v) - \phi_1(v)|.
\]

By a density argument, it is therefore enough to prove the statement for smooth \( \phi \).
Observing that \( m^{\otimes N}(|\Sigma^N_{e,u}|) = m^{\otimes N}(|\Sigma^N_{e,u}|) \), for \( \gamma \in (-\infty, \gamma^*_{0}) \times \mathbb{R}^d \) we write

\[
\frac{1}{N} \log \nu^N_{e,u,N}(e^{N\pi_N(\phi)}) = -\gamma \cdot (e_N, u_N) + \frac{1}{N} \log m^{\otimes N}_{e,u,N}(e^{N\pi_N(\phi+\gamma \zeta)}).
\]

By a direct computation (cfr. Lemma 3.5 in [2]) for any \( \psi \in C_b(\mathbb{R}^d) \)

\[
m_{e,N,u,N}^{\otimes N}(e^{N\pi_N(\psi)}|\Sigma^N_{e,N,u,N}) = (m_{e,N,u,N}(e^{\psi}))^N \frac{f_N^\psi(e_N,u_N)}{f_N(e_N,u_N)},
\]

where \( f_N^\psi, f_N \) are the densities of the random vector \( \frac{1}{N} \sum_{i=1}^N \zeta(v_i) \), in which \( \{v_i\} \) are i.i.d. with law \( m_{e,N,u,N}, m_{e,N,u,N} \) respectively. Observe that, as we assumed that \( m \) is strictly positive on open set, the law of \( \frac{1}{N} \sum_{i=1}^N \zeta(v_i) \) is absolutely continuous for \( N > 2 \). Choosing \( \psi = \phi + \gamma \cdot \zeta \), with \( \gamma = \gamma_N^*(\phi) \) such that (2.8) holds with \( (e,u) \) replaced by \( (e_N,u_N) \), by the local central limit theorem (see e.g. [15]) we deduce

\[
\lim_{N \to \infty} \frac{1}{N} \log \frac{f_N^{\phi+\gamma \zeta}(e_N,u_N)}{f_N(e_N,u_N)} = 0.
\]

Note indeed that the local central limit holds in view of Assumption 2.1 and the smoothness of \( \phi \). Gathering the above computations, the statement follows. \( \square \)

**Lemma 2.4.** Let \( \Lambda^*_{e,u}(\pi) := \sup_{\phi} \{ \pi(\phi) - \Lambda_{e,u}(\phi) \} \) be the Legendre transform of \( \Lambda_{e,u} \). Then \( \Lambda^*_{e,u} = H_{e,u} \).

**Proof.** Note that \( \Lambda^*_{e,u}(\pi) < +\infty \) implies \( \pi(\zeta_0) < +\infty \). For such \( \pi \), recalling (2.5),

\[
\Lambda^*_{e,u}(\pi) = \sup_{\phi} \{ \pi(\phi) + \gamma \cdot (e,u) - \log m_{e,u}(e^{\phi+\gamma \zeta}) \}
\]

\[
= \sup_{\gamma < \zeta_0 - \zeta_0(e,u)} \sup_{\phi} \{ \pi(\phi) + \gamma \cdot (e,u) - \log m_{e,u}(e^{\phi+\gamma \zeta}) \}
\]

\[
= \sup_{\gamma < \zeta_0 - \zeta_0(e,u)} \sup_{\phi} \{ \pi(\phi + \gamma \cdot \zeta) + \gamma \cdot (e - \pi(\zeta_0), u - \pi(\zeta)) - \log m_{e,u}(e^{\phi+\gamma \zeta}) \}
\]

\[
= \text{Ent}(\pi|m_{e,u}) + \sup_{\gamma < \zeta_0 - \zeta_0(e,u)} \{ \gamma \cdot (e - \pi(\zeta_0), u - \pi(\zeta)) \} = H_{e,u}(\pi)
\]

that concludes the proof. \( \square \)

**Proof of Theorem 2.2.** For \( \delta > 0 \) let \( C^\delta_{e,u} \) be the compact subset of \( \mathcal{P}(\mathbb{R}^d) \) given by \( C^\delta_{e,u} := \{ \pi \in \mathcal{P}(\mathbb{R}^d) : \pi(\zeta_0) \leq e + \delta, |\pi(\zeta) - u| \leq \delta \} \). By the very definition of \( \nu^N_{e,u,N} \), definitely in \( N \) we have \( \nu^N_{e,u,N}(\pi^N \in C^\delta_{e,u}) = 1 \), which implies the exponential tightness of the sequence \( \nu^N_{e,u,N} \circ (\pi^N)^{-1} \).

Since the map \( \pi \mapsto H_{e,u}(\pi) \) is strictly convex, in the terminology of convex analysis used in [20], Thm. 4.5.20], every \( \pi \in C^\delta_{e,u} \) is an exposed point of \( H_{e,u} \). Therefore the statement follows from Lemmata 2.3 and 2.4 by the abstract Gärtner-Ellis theorem. \( \square \)

**Large deviations from total probability formula.** We next show how the Sanov’s theorem for i.i.d. random variables can be recovered from Theorem 2.2. While this route is overcomplicated in the present context, it will be crucial to deduce the large deviations for Kac’s walks with canonical initial distribution of the velocities.

We first state a general argument to deduce the large deviation principle from the total probability formula. Let \( X \) be a Hausdorff topological space and \( \mu^u \) be a sequence of probabilities on \( X \). Let also \( Y \) be a locally compact Polish space, \( Y \).
be a $\mathcal{Y}$-valued random variable on $\mathcal{X}$ and denote by $\nu_n$ its law. Letting $y \mapsto \nu^n_y$ be a regular version of the conditional probability of $\mu^n$ given $Y$ we have the disintegration

$$\mu^n = \int p_n(dy) \nu^n_y. \quad (2.10)$$

We will deduce the large deviation of $\mu^n$ from the large deviations of $p_n$ and the large deviations on $\nu^n_y$, that will be assumed to hold uniformly for $y$ in compact subsets of $\mathcal{Y}$.

**Proposition 2.5.** Assume:

(i) the family $\{p_n\}$ is exponentially tight and satisfies a large deviation principle with good rate function $I: \mathcal{Y} \to [0, +\infty]$;

(ii) for each compact $K \subset \subset \mathcal{Y}$ there exists a sequence of compacts $H_\ell \subset \subset \mathcal{X}$ such that $\sup_{y \in K} \nu^n_y \left(H_\ell^c\right) \leq e^{-n\ell}$;

(iii) for each $y \in \mathcal{Y}$ and each sequence $y_n \to y$ the family $\{\nu^n_y\}$ satisfies a large deviation principle with good rate function $F_y: \mathcal{X} \to [0, +\infty]$.

Then the family $\{\mu^n\}$ is exponentially tight and satisfies a large deviation principle with good rate function $I: \mathcal{X} \to [0, +\infty]$ given by

$$I(x) = \inf_{y \in \mathcal{Y}} \{A(y) + F_y(x)\}. \quad (2.11)$$

**Proof.**

**Step 1. Exponential tightness.** As follows from (2.10), for each compact $K \subset \subset \mathcal{Y}$ and each compact $H \subset \subset \mathcal{X}$

$$\mu^n(H^c) \leq \sup_{y \in K} \nu^n_y(H^c) + p_n(K^c).$$

The assumptions on $\{p_n\}$ and $\{\nu^n_y\}$ thus yield the exponential tightness of $\{\mu^n\}$.

**Step 2. Lower semicontinuity of the rate function.** Since $A$ is lower semicontinuous, the lower semicontinuity of $I$ in (2.11) is implied by the (joint) lower semicontinuity of the map $\mathcal{X} \times \mathcal{Y} \ni (x, y) \mapsto F_y(x)$ that we next deduce. Since $\mathcal{Y}$ is Polish, the joint lower semicontinuity of $F$ is in fact equivalent to the following statement. For each $(x, y) \in \mathcal{X} \times \mathcal{Y}$, each sequence $y_k \to y$, and each $\delta > 0$ there exists an open neighborhood $\mathcal{N} \ni x$ such that

$$\liminf_k \inf_{x' \in \mathcal{N}} F_{y_k}(x') \geq F_y(x) - \delta. \quad (2.12)$$

Fix $(x, y) \in \mathcal{X} \times \mathcal{Y}$, a sequence $y_k \to y$, and $\delta > 0$. By the lower semicontinuity of $\mathcal{X} \ni x \mapsto F_y(x)$, there exists an open neighborhood $\mathcal{N}' \ni x$ such that

$$\inf_{x' \in \mathcal{N}'} F_y(x') \geq F_y(x) - \delta. \quad (2.13)$$

Denoting by an over-line the closure, let now $\mathcal{N}$ be an open neighborhood such that $x \in \mathcal{N} \subset \overline{\mathcal{N}} \subset \mathcal{N}'$. We then claim that the bound (2.12) holds. In order to show it, by passing to a not relabeled subsequence, we may assume that $\lim \inf_{x \in \mathcal{N}} F_{y_k}(x') = \lim_k \inf_{x' \in \mathcal{N}} F_{y_k}(x')$. For $k$ fixed, by the lower bound for the sequence $\{\nu^n_{y_k}\}$,

$$\lim_{n \to \infty} \frac{1}{n} \log \nu^n_{y_k}(\mathcal{N}) \geq - \inf_{x' \in \mathcal{N}'} F_{y_k}(x').$$
which, by taking the inferior limit in $k$, implies

$$
\lim \lim_{k \to +\infty} \frac{1}{n} \log \nu_{y_k}^n (\mathcal{N}) \leq - \inf_{x \in \mathcal{N}} F_{y_k} (x').
$$

By a diagonal argument, there exists a sequence $n_k \uparrow +\infty$ such that

$$
\lim \lim_{k \to +\infty} \frac{1}{n} \log \nu_{y_k}^n (\mathcal{N}) = \lim_{k \to +\infty} \frac{1}{n_k} \log \nu_{y_k}^{n_k} (\mathcal{N}) \leq - \inf_{x \in \mathcal{N}} F_{y_k} (x') \leq - \frac{[F_{y_k} (x) - \delta]}{\epsilon / n}
$$

where we used the large deviations upper bound for the sequence $\{\nu_{y_k}^{n_k}\}$ and $\{\nu_{y_k}^{n_k}\}$ in the last step. Comparing the two last displayed equations the bound follows.

**Step 3. Lower bound.** It is enough to show that for each $x \in \mathcal{X}$ and each open neighborhood $\mathcal{N} \ni x$

$$
\lim_{n \to +\infty} \frac{1}{n} \log \mu^n (\mathcal{N}) \geq - I(x).
$$

Fix a metric inducing the topology of $\mathcal{Y}$ and, given $y \in \mathcal{Y}$ and $\delta > 0$, let $B_\delta (y)$ the corresponding open ball of radius $\delta$ centered in $y$. In order to show (2.14), fix $y \in \mathcal{Y}$. By the large deviations lower bound of the sequence $\{p_n\}$, for each $\delta > 0$ we then have

$$
\lim_{n \to +\infty} \frac{1}{n} \log p_n (B_\delta (y)) \geq - A(y).
$$

Therefore, by a diagonal argument, there exists a sequence $\delta_n \downarrow 0$ such that

$$
\lim_{n \to +\infty} \frac{1}{n} \log p_n (B_{\delta_n} (y)) \geq - A(y).
$$

From the disintegration (2.10) we then obtain

$$
\mu^n (\mathcal{N}) \geq \int_{B_{\delta_n} (y)} p_n (dy') \nu_{y'}^n (\mathcal{N}) \geq p_n (B_{\delta_n} (y)) \inf_{y' \in B_{\delta_n} (y)} \nu_{y'}^n (\mathcal{N}).
$$

Whence, for a suitable sequence $y'_n \to y$,

$$
\lim_{n \to +\infty} \frac{1}{n} \log \mu^n (\mathcal{N}) \geq \lim_{n \to +\infty} \frac{1}{n} \log p_n (B_{\delta_n} (y)) + \lim_{n \to +\infty} \frac{1}{n} \log \nu_{y'_n}^n (\mathcal{N}) \geq - [A(y) + F_y (x)]
$$

where we used the large deviations lower bound for the family $\{\nu_{y'_n}^n\}$. By optimizing over $y \in \mathcal{Y}$ and recalling (2.11) we then deduce (2.14).

**Step 4. Upper bound for compacts.** Fix a compact set $H \subset \subset \mathcal{X}$, $\ell > 0$, $\epsilon > 0$, and observe that, by the joint lower semicontinuity of $F$ proven in Step 2 above, the map $\mathcal{Y} \ni y \mapsto \inf_{x \in H} F_y (x)$ is lower semicontinuous. By the exponential tightness of $\{p_n\}$, there exists a compact $K_\ell \subset \subset \mathcal{Y}$ such that $p_n (K_\ell^c) \leq e^{-\epsilon \ell}$. For each $y \in K_\ell$, by the lower semicontinuity of $A$ and the previous observation, there exists $\delta > 0$ such that $A(y') \geq A(y) - \epsilon / 2$ and $\inf_{x \in H} F_{y'} (x) \geq \inf_{x \in H} F_y (x) - \epsilon / 2$ for any $y' \in B_{\delta} (y)$. By the local compactness of $\mathcal{Y}$, possibly by decreasing $\delta$, we can assume that $B_{\delta} (y)$ is relatively compact. Furthermore, by the compactness of $K_\ell$, there exists a finite family $\{B_{\delta_i} (y_i)\}_{i=1}^{r} \subset \bigcup_i B_{\delta_i} (y_i)$. In view of
\( (2.10) \),

\[
\mu^n(H) \leq \sum_{i=1}^{r} \int_{B_{\delta_i}(y_i)} p_n(dy') \nu^n_{y'}(H) + p_n(K^r_i)
\]

\[
\leq \sum_{i=1}^{r} p_n(\overline{B}_{\delta_i}(y_i)) \sup_{y' \in B_{\delta_i}(y_i)} \nu^n_{y'}(H) + e^{-n\ell}.
\]

(2.15)

Since the sets \( B_{\delta_i}(y_i) \) are relatively compact, by passing if necessary to a not relabeled subsequence, for each \( i = 1, \ldots, r \) there exist \( \bar{y}_i \in B_{\delta_i}(y_i) \) and a sequence \( y^n_i \to \bar{y}_i \) such that

\[
\lim_n \frac{1}{n} \log \nu^n_{y^n_i}(H) = \lim_n \frac{1}{n} \log \nu^n_{\bar{y}_i}(H).
\]

Letting \( a \lor b := \max\{a, b\} \) and using the large deviation upper bound both for \( \{p_n\} \) and \( \{\nu^n_{y^n_i}\} \) in (2.15) we thus get

\[
\lim_n \sup_{y' \in B_{\delta_i}(y_i)} \frac{1}{n} \log \mu^n(H) \leq \max_{i=1, \ldots, r} \left\{ \inf_{y' \in \overline{B}_{\delta_i}(y_i)} A(y') - \inf_{x \in H} F_{\bar{y}_i}(x) \right\} \lor (-\ell)
\]

\[
\leq - \min_{i=1, \ldots, r} \left\{ A(y_i) + \inf_{x \in H} F_{\bar{y}_i}(x) - \epsilon \right\} \lor (-\ell)
\]

\[
\leq - \inf_{x \in H} \inf_{y' \in y_i} \left\{ A(y) + F_{\bar{y}_i}(x) - \epsilon \right\} \lor (-\ell).
\]

Recalling (2.11), we conclude by taking the limits \( \epsilon \downarrow 0 \) and \( \ell \uparrow +\infty \).

Let \( v \in \Sigma^N \) be sampled according to the product probability \( \mu^N = m^\otimes N \) and denote by \( p_N \) the law of \( \frac{1}{N} \sum_i \zeta(v_i) \). We then have the disintegration

\[
\mu^N = \int p_N \left( d(e, u) \right) \nu^N_{e,u}.
\]

Moreover the sequence \( \{p_N\} \) satisfies a large deviations principle with rate function \( A: (0, +\infty) \times \mathbb{R}^d \to [0, +\infty) \) given by

\[
A(e, u) = \sup_{\gamma} \left\{ \gamma \cdot (e, u) - \log m(e^\gamma \zeta) \right\}.
\]

This follows from the multidimensional Cramér’s theorem in [4, Thm. 2.3.6]. Indeed, in the terminology of convex analysis used in [3], the function \( \gamma \mapsto \log m(e^\gamma \zeta) \) is steep. Namely \( |\nabla \log m(e^\gamma \zeta)| \) diverges when \( \gamma_0 \to \gamma^*_0 \). This follows from item (ii) in Assumption 2.1.

In view of Proposition 2.5 and the following remark, Sanov’s theorem for i.i.d. random variables can be deduced from Theorem 2.2.

**Remark 2.6.** We have

\[
\text{Ent}(\pi|m) = \inf_{(e,u)} \left\{ A(e, u) + H_{e,u}(\pi) \right\}.
\]

In fact, by a direct computation, the infimum is achieved for \( (e, u) = \pi(\zeta) \).
3. Large deviations for Kac model with microcanonical initial data

The model. Recall that \( \Sigma^N = (\mathbb{R}^d)^N \). We consider the Kac walk given by the Markov process on the configuration space \( \Sigma^N \), whose generator acts on bounded continuous functions \( f: \Sigma^N \to \mathbb{R} \) as
\[
\mathcal{L}_N f(v) = \frac{1}{N} \sum_{\{i,j\}} L_{i,j} f(v),
\]
where the sum is carried over the unordered pairs \( \{i,j\} \subset \{1,\ldots,N\}, i \neq j \), and
\[
L_{i,j} f(v) = \int_{S_{d-1}} \mathrm{d}\omega \, B(v_i - v_j, \omega) [f(T^\omega_{i,j} v) - f(v)].
\]
Here \( S_{d-1} \) is the sphere in \( \mathbb{R}^d \) and
\[
(T^\omega_{i,j} v)_k = \begin{cases} v_i + (\omega \cdot (v_j - v_i)) \omega & \text{if } k = i \\ v_j - (\omega \cdot (v_j - v_i)) \omega & \text{if } k = j \\ v_k & \text{otherwise,} \end{cases}
\]
and the collision kernel \( B \) is given by
\[
B(v - v_*, \omega) = \frac{1}{2} |(v - v_*) \cdot \omega|. \tag{3.2}
\]
Observe that the dynamics preserves energy and momentum, i.e. can be restricted to the set \( \Sigma^N_{\pi^N_v} \) as defined in (2.2). We denote by \( (v(t))_{t \geq 0} \) the Markov process generated by \( \mathcal{L}_N \).

Fix hereafter \( T > 0 \). Given a probability \( \nu \) on \( \Sigma^N \) we denote by \( \mathbb{P}^N_\nu \) the law of this process on the time interval \([0,T]\). Observe that \( \mathbb{P}^N_\nu \) is a probability on the Skorokhod space \( D([0,T]; \Sigma^N) \). As usual if \( \nu = \delta_v \) for some \( v \in \Sigma^N \), the corresponding law is simply denoted by \( \mathbb{P}^N_v \).

Empirical observables. Recall that \( \mathcal{P}(\mathbb{R}^d) \) is the set of probability measures \( \pi \) on \( \mathbb{R}^d \) equipped with the weak topology and the corresponding Borel \( \sigma \)-algebra. Let \( D([0,T]; \mathcal{P}(\mathbb{R}^d)) \) the set of \( \mathcal{P}(\mathbb{R}^d) \)-valued càdlàg paths endowed with the Skorokhod topology and the corresponding Borel \( \sigma \)-algebra. Recalling the empirical measure \( \pi^N \) defined in (2.2), with a slight abuse of notation we denote also by \( \pi^N \) the map from \( D([0,T]; \Sigma^N) \) to \( D([0,T]; \mathcal{P}(\mathbb{R}^d)) \) defined by \( \pi^N(v) := \pi^N(v(t)), t \in [0,T] \).

We denote by \( \mathcal{M} \) the subset of the finite measures \( Q \) on \([0,T] \times \mathbb{R}^{2d} \times \mathbb{R}^{2d} \) that satisfy \( Q(dt; dv, dv_*, dv', dv_*') = Q(dt; dv_*, dv, dv', dv_*') = Q(dt; dv, dv_*, dv', dv_*') \). We consider \( \mathcal{M} \) endowed with the weak topology and the corresponding Borel \( \sigma \)-algebra. By definition, the weak topology is the weakest topology such that the map \( Q \mapsto Q(F) \) is continuous for each \( F \) in \( C_b([0,T] \times \mathbb{R}^{2d} \times \mathbb{R}^{2d}) \).

The empirical flow is the map \( Q^N: D([0,T]; \Sigma^N) \to \mathcal{M} \) defined by
\[
Q^N(v)(F) := \frac{1}{N} \sum_{\{i,j\}} \sum_{k \geq 1} F\left( \tau_{k}^{i,j}; v_i(\tau_{k}^{i,j} -), v_j(\tau_{k}^{i,j} -), v_i(\tau_{k}^{i,j}), v_j(\tau_{k}^{i,j}) \right) \tag{3.3}
\]
where \( F: [0,T] \times \mathbb{R}^{2d} \times \mathbb{R}^{2d} \to \mathbb{R} \) is continuous, bounded, and satisfies \( F(t; v, v_*, v', v_*') = F(t; v_*, v, v', v_*') \) for \( F(t; v, v_*, v', v_*') \) and \( (\tau_{k}^{i,j})_{k \geq 1} \) are the jump times of the pair \( (v_i, v_j) \). Here, \( v_i(t\cdot) = \lim_{s \uparrow t} v_i(s) \). In view of the conservation of the energy and momentum, the measure \( Q^N(dt; \cdot) \) is supported on \( E := \{ \zeta(v) + \zeta(v_*) = \zeta(v') + \zeta(v_*') \} \subset \mathbb{R}^{2d} \times \mathbb{R}^{2d} \).
Let $S$ be the subset of $D([0, T]; \mathcal{P}(\mathbb{R}^d) \times \mathcal{M}$ given by elements $(\pi, Q)$ that satisfies the balance equation

$$
\pi_T(\phi_T) - \pi_0(\phi_0) - \int_0^T dt \pi_t(\partial_t \phi_t) + \int Q(\partial_t dv; dv_*; dv', dv'_*) \left[ \phi_t(v) + \phi_t(v_*) - \phi_t(v') - \phi_t(v'_*) \right] = 0
$$

(3.4)

for each $\phi \in C_b([0, T] \times \mathbb{R}^d)$ continuously differentiable in $t$, with bounded derivative. For each $v \in \Sigma^N$, with $\mathbb{P}^N_v$ probability one, the pair $(\pi^N, Q^N)$ belongs to $S$.

**The rate function.** Given $(e, u) \in Z$, recall $C_{e,u} := \{ \mu \in \mathcal{P}(\mathbb{R}^d) : \mu(\zeta_0) \leq e, \mu(\zeta) = u \}$, and set

$$
\mathcal{E}_{e,u} := \{ \pi \in C([0, T], \mathcal{P}(\mathbb{R}^d) : \pi_t \in C_{e,u}, t \in [0, T] \},
$$

(3.5)

that is a closed subset of $C([0, T], \mathcal{P}(\mathbb{R}^d)$. For notation convenience, let $r(v, v^*, \cdot)$ be the measure on $\mathbb{R}^{2d}$ supported on $\{ \zeta(v) + \zeta(v^*) = \zeta(v') + \zeta(v'_*) \}$ such that

$$
r(v, v_*, dv', dv'_*) = d\omega B(v - v_*, \omega),
$$

where $v'$ and $v'_*$ are related to $\omega$ by the collision rules, as in (3.1). For $\pi \in D([0, T]; \mathcal{P}(\mathbb{R}^d))$ let $Q^\pi$ be the measure defined by

$$
Q^\pi(dt; dv, dv_*; dv', dv'_*) := \frac{1}{2} dt \pi_t(dv_*) \pi_t(dv) r(v, v_*; dv', dv'_*)
$$

(3.6)

and observe that $Q^\pi(dt, \cdot)$ is supported on $\mathcal{E}$.

**Definition 3.1.** Let $S^\text{ac}_{e,u}$ be the subset of $S$ given by the elements $(\pi, Q)$ that satisfy the following conditions:

(i) $\pi \in \mathcal{E}_{e,u}$;
(ii) $Q \ll Q^\pi$.

Observe that if $(\pi, Q) \in S^\text{ac}_{e,u}$ then $Q^\pi$ is a finite measure. The dynamical rate function $J_{e,u} : S \to [0, +\infty]$ is defined by

$$
J_{e,u}(\pi, Q) := \left\{ \begin{array}{ll}
\int dQ^\pi \left[ \frac{dQ}{dQ^\pi} \log \frac{dQ}{dQ^\pi} - (\frac{dQ}{dQ^\pi} - 1) \right] & \text{if } (\pi, Q) \in S^\text{ac}_{e,u} \\
+\infty & \text{otherwise}
\end{array} \right.
$$

(3.7)

Recalling $H_{e,u}$ has been defined in (2.5), the microcanonical rate function is

$$
I_{e,u}(\pi, Q) := H_{e,u}(\pi_0) + J_{e,u}(\pi, Q).
$$

(3.8)

Let also $\hat{S}$ be the subset of $S$ given by the pair $(\pi, Q)$ such that

$$
\int_{[0, T] \times \mathbb{R}^d} dQ \left[ \zeta_0(v) + \zeta_0(v_*) + \zeta_0(v') + \zeta_0(v'_*) \right] < +\infty.
$$

Observe that for $(\pi, Q) \in \hat{S}$ the balance equation (3.4) holds for $\phi = \zeta_0$, therefore $\pi_t(\zeta_0) = \pi_0(\zeta_0)$ for every $t \in [0, T]$.
Lemma 4.1. follows from the next two lemmata.

Then definitely in $N$ momentum $e$.

Theorem 3.2. Assume $m$ satisfies condition (i)–(iii) in Assumption [27] fix $(e, u) \in Z$, a sequence $(e_N, u_N) \to (e, u)$, and let $\nu_{e_N, u_N}^N$ be the microcanonical probabilities as in Section 3. The family $\mathbb{P}_{e_N, u_N}^N \circ (\pi_N, Q_N)^{-1}$ satisfies a large deviation upper bound with good rate function $I_{e,u} : \mathcal{S} \to [0, +\infty]$, namely $I_{e,u}$ has compact level sets and for each closed $C \subset \mathcal{S}$

$$\lim_{N \to +\infty} \frac{1}{N} \log \mathbb{P}_{e_N, u_N}^N \left( (\pi_N, Q_N) \in C \right) \leq -\inf_C I_{e,u}. \quad (3.9)$$

Moreover, if $m$ satisfies also condition (iv) in Assumption [24] then for each open $O \subset \mathcal{S}$

$$\lim_{N \to +\infty} \frac{1}{N} \log \mathbb{P}_{e_N, u_N}^N \left( (\pi_N, Q_N) \in O \right) \geq -\inf_{O \cap \mathcal{S}} I_{e,u}. \quad (3.10)$$

4. Proof of the upper bound

The proof follows the same strategy as in [2] and in [7]. For the reader convenience we here provide the details. The upper bound is achieved by an established pattern in large deviation theory. We first prove the exponential tightness, which allows us to reduce to compacts. By an exponential tilting of the measure, we prove an upper bound for open balls and finally we use a mini-max argument to conclude.

The basic observation is the following. Given a bounded measurable function $F : [0, T] \times \mathbb{R}^d \to \mathbb{R}$ such that $F(t; v, u) = F(t; v, v', v') = F(t; v, v, v') = F(t; v, v, v')$, set

$$\lambda^F(t; v) = \int r(v, v'; dv', dv')e^{\int F(t; v, v', v')} \quad (4.1)$$

If $F = 0$ we drop it from the notation. Denoting by $Q^N_{[0,t]}$ the restriction of the measure $Q^N$ on $[0, t]$, and using that $\lambda(v, v) = \lambda^F(t, v, v) = 0$ the process

$$\mathcal{M}^F_t = \exp \left\{ N \left( Q^N_{[0,t]}(F) - \frac{1}{2} \int_0^t ds \pi^N_s \otimes \pi^N_s (\lambda^F - \lambda) \right) \right\} \quad (4.2)$$

is a $\mathbb{P}_N^N$ positive martingale for each $v \in \Sigma^N$, see e.g. [9] App. 1, Prop. 2.6.

For any $\delta > 0$ we also define the compact set $C^\delta_{e,u} := \{ \mu \in \mathcal{P}(\mathbb{R}^d) : \mu(\zeta_0) \leq e + \delta, |\mu(\zeta) - u| \leq \delta \}$. For $\delta > 0$, by the conservation of the energy and the momentum

$$\mathbb{P}_{e_N, u_N}^N (\pi_t^N \in C^\delta_{e,u}, t \in [0, T]) = 1, \quad (4.3)$$

definitely in $N$. By Ascoli-Arzelà and Prohorov theorems, the exponential tightness follows from the next two lemmata.

Lemma 4.1. Set

$$\tilde{F}(v, v, v', v') = \log(1 + \zeta_0(v) + \zeta_0(v') + \zeta_0(v'') + \zeta_0(v''')).$$

Then

$$\lim_{t \to +\infty} \lim_{N \to +\infty} \frac{1}{N} \log \mathbb{P}_{e_N, u_N}^N \left( Q^N(\tilde{F}) > \hat{t} \right) = -\infty. \quad (4.4)$$

Lemma 4.2. For each $\varepsilon > 0$ and $\phi \in C_b(\mathbb{R}^d)$

$$\lim_{\eta \downarrow 0} \lim_{N \to +\infty} \frac{1}{N} \log \mathbb{P}_{e_N, u_N}^N \left( \sup_{t,s \in [0,T]: |t-s| \leq \eta} \left| \pi^N_t(\phi) - \pi^N_s(\phi) \right| > \varepsilon \right) = -\infty. \quad (4.5)$$
Proof of Lemma 4.1. Observe that by the conservation of the energy, there exists a constant \( c > 0 \), depending on \( e \), such that for any \( N \) the bound \( Q^N(e^{\frac{t}{2}}) \leq c \) holds with \( P_{\nu_N}^N \) probability one.

Let \( M_T \) be the exponential martingale in (4.2) with \( F = \frac{1}{2} \bar{F} \). Then, for each \( \ell > 0 \),
\[
\mathbb{P}^N_{\nu_N} \left( Q^N(F) > \ell \right) = \mathbb{E}^N_{\nu_N} \left( M_T (M_T)^{-1} 1_{Q^N(F) > \ell} \right) \leq \exp\{-N\ell/2 + cN\}.
\]

\[\square\]

Proof of Lemma 4.2. In view of the balance equation (3.4) it is enough to show that there exists a function \( c: (0, 1) \rightarrow \mathbb{R}_+ \) with \( c(\eta) \uparrow +\infty \) as \( \eta \downarrow 0 \) such that, for any \( \varepsilon > 0 \)
\[
\mathbb{P}^N_{\nu_N} \left( \sup_{t \in [0, T]} Q^N_{t, t+\eta}(1) > \varepsilon \right) \leq e^{-Nc(\eta)}.
\]

By a straightforward inclusion of events, the previous bound follows from
\[
\frac{1}{\eta} \sup_{t \in [0, T]} \mathbb{P}^N_{\nu_N} \left( Q^N_{t, t+\eta}(1) > \varepsilon \right) \leq e^{-Nc(\eta)}.
\]

Consider the super-martingale (4.2) with \( F = \gamma 1_{[t, t+\eta]} \), \( \gamma > 0 \). Using the same argument of the previous lemma we deduce
\[
\mathbb{P}^N_{\nu_N} \left( Q^N_{t, t+\eta}(1) > \varepsilon \right) \leq \exp\{-N[\gamma \varepsilon - \eta (e^\gamma - 1)C(1+e)]\}.
\]

The proof is concluded by choosing \( \gamma = \log(1/\eta) \).

\[\square\]

Upper bound on compacts. Recalling the set \( C^\delta_{e,u} \) defined above (4.6), let \( C^\delta_{e,u} \) be the closed subset of \( C([0, T]; \mathbb{P}(\mathbb{R}^d)) \) defined as
\[
C^\delta_{e,u} := \bigcap_{t \in [0, T]} \{ \pi : \pi_t \in C^\delta_{e,u} \}.
\]

By Urysohn’s lemma, for each \( \eta > 0 \) there exists \( \psi_{\delta, \eta} : C([0, T]; \mathbb{P}(\mathbb{R}^d)) \rightarrow [0, 1] \) continuous such that
\[
\psi_{\delta, \eta}(\pi) = \begin{cases} 0 & \text{if } \pi \in C^\delta_{e,u} \\ 1 & \text{if } \text{dist}(\pi, C^\delta_{e,u}) \geq \eta, \end{cases}
\]

where dist is the uniform distance. Moreover, for \( \pi \in D([0, T], \mathbb{P}(\mathbb{R}^d)) \), we extend it to a function defined on \( \mathbb{R} \) by setting \( \pi_t = \pi_0 \) if \( t < 0 \), \( \pi_t = \pi_T \) if \( t > T \). Let \( \psi \) be the a smooth approximation of the \( \delta \) function, and denote by \( \psi \ast \pi \) the time convolution of \( \pi \).

Lemma 4.3. Fix a measurable subset \( B \subset \mathcal{S} \). For any \( (\phi, F) \in C_b(\mathbb{R}^d) \times C_b(\mathbb{R}^+ \times (\mathbb{R}^d)^2 \times (\mathbb{R}^d)^2) \) such that and \( F(t; v, v_s, v'_s) = F(t; v, v, v'_s) = F(t; v, v_s, v, v'_s) = F(t; v, v_s, v', v'_s) \), and any \( \delta, \eta, \varepsilon, \alpha > 0 \),
\[
\lim_{N \rightarrow \infty} \frac{1}{N} \log \mathbb{P}^N_{\nu_N} \left( (\pi^N, Q^N) \in B \right) \leq - \inf_{(\pi, Q) \in B} \{ I_{\phi, F}(\pi, Q) + \alpha \psi_{\delta, \eta}^N(\psi \ast \pi) \},
\]

where
\[
I_{\phi, F}(\pi, Q) := \pi_0(\phi) - A_{e,u}(\phi) + Q(F) - \frac{1}{2} \int_0^T dt \pi_t \otimes \pi_t (\lambda F - \lambda).
\]
Proof. Let \( \tilde{\nu}^N_{e,u} \) be the probability on \( \Sigma^N \) defined by
\[
\frac{d\tilde{\nu}^N_{e,u}}{d\nu^N_{e,u}} = \exp \left\{ N \pi^N(\phi) - \log \nu^N_{e,u} \right\}.
\]
Recalling (4.3) and the definition of the martingale \( \mathbb{M}^F_t \) in (4.2), we write
\[
\mathbb{P}^N_{\nu^N_{e,u}} \left( (\pi^N, Q^N) \in B \right) = \int \nu^N_{e,u} (d\nu) \mathbb{E}^N \left( e^{-N \alpha \psi^N \pi^N (\phi)} \mathbb{I}(\pi^N, Q^N) \right)
\]
\[
= \int \nu^N_{e,u} \frac{d\nu^N_{e,u}}{d\nu^N_{e,u}} \mathbb{P}^N \left( e^{-N \alpha \psi^N \pi^N (\phi)} \mathbb{M}^F_t^{-1} \mathbb{I}(\pi^N, Q^N) \right)
\]
We get
\[
\mathbb{P}^N_{\nu^N_{e,u}} \left( (\pi^N, Q^N) \in B \right) \leq \sup_{(\pi, Q) \in B} \exp \left\{ -N \left[ \pi_0(\phi) - \frac{1}{N} \log \nu^N_{e,u} \left( e^{N \pi^N(\phi)} \right) + \alpha \psi^N \pi^N (\phi) \right]
\]
\[
+ Q(F) - \frac{1}{2} \int_0^T dt \, \pi_t \otimes \pi_t (\lambda^F - \lambda) \right\},
\]
where we used that \( \mathbb{E}^N_{\nu^N_{e,u}} (\mathbb{M}^F_T) = 1 \). The statement follows from Lemma 2.7. □

Lemma 4.4 (Variational characterization of the dynamical rate functional). For any pair \((\pi, Q) \in \mathcal{S}\) such that \( \pi \in C([0, T]; \mathcal{P}(\mathbb{R}^d)) \)
\[
J_{e,u}(\pi, Q) = \sup_{F, \alpha, \delta, \eta, \varepsilon} \left\{ Q(F) - \frac{1}{2} \int_0^T dt \, \pi_t \otimes \pi_t (\lambda^F - \lambda) + \alpha \psi^N \pi^N (\pi_t \otimes \pi_t (\lambda^F - \lambda) \right\}
\]
where the supremum is carried out over all continuous and bounded \( F : [0, T] \times (\mathbb{R}^d)^2 \times (\mathbb{R}^d)^2 \to \mathbb{R} \) such that \( F(t; v, u, v', u') = F(t; v, u, v', u') = F(t; v, u, v', u') \), and \( \alpha, \delta, \eta, \varepsilon > 0 \).

Proof. By monotonicity
\[
\sup_{\alpha, \delta, \eta, \varepsilon} \alpha \psi^N \pi^N (\pi_t \otimes \pi_t (\lambda^F - \lambda) = \left\{ \begin{array}{ll}
0 & \text{if } \pi \in C_{e,u} \\
+\infty & \text{otherwise}
\end{array} \right.
\]
where we have used that \( (\pi_t \otimes \pi_t) \in C_{e,u} \), for any \( t \in [0, T] \) and \( \varepsilon > 0 \), then \( \pi_t \in C_{e,u} \) for any \( t \in [0, T] \). To complete the proof, it remains to show that for \( \pi \in C_{e,u} \)
\[
J_{e,u}(\pi, Q) = \sup_F \left\{ Q(F) - \frac{1}{2} \int_0^T dt \, \pi_t \otimes \pi_t (\lambda^F - \lambda) \right\}.
\]
Recall the definition of \( Q^\pi \) in (3.6) and observe that
\[
\frac{1}{2} \int_0^T dt \, \pi_t \otimes \pi_t (\lambda^F - \lambda) = Q^\pi (e^F - 1).
\]
This implies that if \( \sup_F \left[ Q(F) - \frac{1}{2} \int_0^T dt \, \pi_t \otimes \pi_t (\lambda^F - \lambda) \right] \) is finite, then \( Q \) is absolutely continuous with respect to \( Q^\pi \). The proof is now completed by a direct computation. □

Proof of Theorem 3.2 upper bound. In view of (4.3), Lemma 4.1 and Lemma 4.2 imply the exponential tightness of the family \( \{\mathbb{P}^N_{\nu^N_{e,u}} \circ (\pi^N, Q^N)^{-1} \} \). Moreover, Lemma 4.2 implies that if the large deviation upper bound rate function is finite
then \(\pi \in C([0,T], \mathcal{P}(\mathbb{R}^d))\). Therefore it is enough to show the statement for compacts. In view of Lemma 4.3 and the mini-max argument in [9, App.2, Lemma 3.2], the statement follows from Lemma 4.4 and Lemma 4.1.

\[\square\]

### 5. Proof of the lower bound

In this section we adapt the strategy in [2] to the Kac model, where the kernel \(B\) is not strictly positive. We shall first prove the lower bound for open neighborhoods of “nice” \((\pi, Q)\), and then use a density argument. As in [2] and [7] we will restrict to \(Q\) with bounded second moment, but we will not require, as in [7], that \(B \geq c > 0\).

**Perturbed Kac walks.** We start by the following law of large numbers for a class of perturbed Kac’s walks. Consider perturbed time dependent collision kernels \(\tilde{\pi}\) that are continuous and satisfy

\[
\sup_{t,v,v_s} \tilde{\lambda}_t(v,v_s) = \sup_{t,v,v_s} \int \tilde{B}_t(v,v_s,\omega) \, d\omega \leq C, \tag{5.1}
\]

for some \(C < +\infty\). Fix \((e,u) \in Z\), a sequence \((e_N,u_N) \to (e,u)\), and let \(\nu_{e_N,u_N}^N\) be the family of probabilities on \(\Sigma^N\) as in Section 2, and denote by \(\tilde{\mathbb{P}}_\nu^N\) the law of the perturbed Kac walk with initial datum \(\nu_{e_N,u_N}^N\).

**Lemma 5.1.** As \(N \to +\infty\), the pair \((\pi^N, Q^N)^N\) converges, in \(\tilde{\mathbb{P}}_\nu^N\) probability, to \((f dv, q dt dr dv, d\omega)\), where \(q_t(v,v_s,\omega) = \frac{1}{2} f_t(v)f_t(v_s) \tilde{B}_t(v,v_s,\omega)\) and \(f \in C([0,T]; L^1(\mathbb{R}^d))\) is the unique solution to the perturbed Kac’s equation

\[
\begin{aligned}
\partial_t f_t(v) &= \int dv_s \omega \left[ \tilde{B}_t(v',v_s',\omega)f_t(v')f_t(v_s') - \tilde{B}_t(v,v_s,\omega)f_t(v)f_t(v_s) \right], \\
f_0(\cdot) &= \frac{dm_{e,u}}{dv}.
\end{aligned} \tag{5.2}
\]

Here we understand that (5.2) holds by integrating against continuous, bounded test functions which are continuous differentiable in time.

The proof follow from the fact the large deviation upper bound holds also for the perturbed Kac’s walk, and the uniqueness of the solution due (5.1), see proof of Lemma 4.1 in [2] for the details.

The following specifies the collection of “nice” \((\pi, Q)\). Recall \(\mathcal{S}_e^e\) in Definition 3.1.

**Definition 5.2.** Let \(\hat{\mathcal{S}}_{e,u}\) be the collection of elements \((\pi, Q) \in \mathcal{S}_e^e\) whose densities \((f,q)\) are continuous and such that

\[
\sup_{t,v,v_s} \frac{q_t(v,v_s,\omega)}{f_t(v)f_t(v_s)} < +\infty, \tag{5.3}
\]

and

\[
\sup_{t,v,v_s} \frac{q_t(v,v_s,\omega)}{f_t(v)f_t(v_s)B(v,v_s,\omega)} < +\infty. \tag{5.4}
\]

Given \((\pi, Q) \in \hat{\mathcal{S}}_{e,u}\), denote by \(\tilde{B}_t\) the time dependent perturbed kernel defined by

\[
\tilde{B}_t(v,v_s,\omega) = 2 \frac{q_t(v,v_s,\omega)}{f_t(v)f_t(v_s)}, \tag{5.5}
\]

that meets (5.1).
The next statement provides the large deviation lower bound for neighborhood of elements in \( \hat{S}_{e,u} \).

**Proposition 5.3.** Let \((\tau, Q) \in \hat{S}_{e,u}\). Assume that \(\tau_0\) satisfies items (iii) in Assumption 2.1 and suppose \(\tau_0(dv) = e^\phi m(dv)/m(e^\phi)\) for some \(\phi\) bounded and continuous. Fix a sequence \((e_N, u_N) \to (e, u)\), and denote by \(\tilde{\nu}_{e_N,u_N}^N\) the regular version of the probability \(\tau_0\) conditioned to \((\frac{1}{N} \sum_{i=1}^N \frac{1}{2}|v_i|^2, \frac{1}{N} \sum_{i=1}^N v_i)\) evaluated at \((e_N, u_N)\). Then

\[
\lim_{N \to \infty} \frac{1}{N} \text{Ent}\left(\tilde{\nu}_{e_N,u_N}^N || \nu_{e_N,u_N}^N\right) = I_{e,u}(\tau, Q).
\]

We premise the following Lemma.

**Lemma 5.4.** If \(F \in C_b([0, T] \times \mathbb{R}^d \times \mathbb{R}^d \times S_{d-1})\), then

\[
\lim_{N \to \infty} \sup_{N \to \infty} \tilde{\nu}_{e_N,u_N}^N (Q^N(F)^2) < +\infty.
\]

**Proof.** Set

\[
\tilde{M}_t^N := Q^N_{0,t}(F) - \frac{1}{N^2} \sum_{\{i,j\}} \int_0^t ds \int dw \tilde{B}_s(v_i, v_j, \omega) F_s(v_i, v_j, \omega),
\]

that it is a \(\tilde{\nu}_{e_N,u_N}^N\) martingale with predictable quadratic variation

\[
\langle \tilde{M}^N \rangle_t = \frac{1}{N^2} \sum_{\{i,j\}} \int_0^t ds \int dw \tilde{B}_s(v_i, v_j, \omega) F_s(v_i, v_j, \omega)^2.
\]

In view of (5.1), the random variable \(\langle \tilde{M}^N \rangle_T\) is uniformly bounded in \(N\), which implies the statement. \(\square\)

**Proof of Proposition 5.3.** By using Theorem 2.2 it is enough to show that

\[
\lim_{N \to \infty} \frac{1}{N} \text{Ent}\left(\tilde{\nu}_{e_N,u_N}^N || \nu_{e_N,u_N}^N\right) = J_{e,u}(\tau, Q). \quad (5.6)
\]

In view of the assumptions on \(\tilde{B}\), the value at time \(T\) of the martingale defined in (4.2) with \(F_1 = \log(\tilde{B}/B)\) is the Radon-Nykodim derivative of \(\tilde{\nu}_{e_N,u_N}^N\) with respect to \(\nu_{e,N,u}^N\). Since \(\lambda^F = \tilde{\lambda}_t\),

\[
\frac{1}{N} \text{Ent}\left(\tilde{\nu}_{e_N,u_N}^N || \nu_{e_N,u_N}^N\right) = \tilde{E}_{\nu_{e_N,u_N}^N}^N \left(Q^N_{0,T}(F) - \frac{1}{2} \int_0^T ds \pi_s^N \otimes \pi_s^N (\tilde{\lambda}_s - \lambda)\right).
\]

Now observe that, by Lemma 5.3 \((\pi^N, Q^N)\) converges to \((\tau, Q)\) in \(\tilde{\nu}_{e_N,u_N}^N\) probability. By definition of \(\hat{S}_{e,u}\), \(F\) satisfies the assumption of Lemma 5.3 then the sequence \(Q^N_{0,T}(F)\) is uniformly integrable with respect to \(\tilde{\nu}_{e_N,u_N}^N\). By (5.1), \(\pi_s^N \otimes \pi_s^N (\tilde{\lambda}_s)\) converges to \(\pi_s \otimes \pi_s (\tilde{\lambda}_s)\) for almost all \(s \in [0, T]\). Moreover, by conservation of energy, \(\lambda\) is uniformly integrable with respect to \(ds \pi_s^N \otimes \pi_s^N\). Therefore (5.6) follows. \(\square\)
Approximating paths. Recall that the set $\hat{S}$ has been defined above Theorem 3.2.

**Theorem 5.5.** For each $(\pi, Q) \in \hat{S}$ such that $I_{e,u}(\pi, Q) < +\infty$ there exists a sequence $\{((\pi_n, Q_n)) \subset S_{e,u} \cap \hat{S}$ satisfying $(\pi_n, Q_n) \to (\pi, Q)$ and $I_{e,u}(\pi_n, Q_n) \to I(\pi, Q)$.

**Proof.** The proof is achieved by combining the following three steps and a standard diagonal argument. In particular, in Step 1 we construct positive regular approximating probability paths, in Step 2 we regularize in time, and in Step 3 we perform a truncation argument as in [2], adapted to the hard-sphere kernel.

**Step 1. Velocity convolution.** Since $I_{e,u}(\pi, Q) < +\infty$ and $(\pi, Q) \in \hat{S}$, $\pi_t(\zeta) = u$, $\pi_t(\zeta_0) = \pi_0(\zeta_0) = U + |u|^2/2 \in (0, e]$, where $U = \frac{1}{2} \int \pi_t(dv)|v - u|^2$ is the internal energy.

Let $(f, q)$ be the densities of $(\pi, Q)$. Given $0 < \delta < 1$, let $g_\delta$ be the Gaussian kernel on $\mathbb{R}^d$ with variance $\delta$ and define

$$
\begin{align*}
&f^\delta_t(v) = \alpha(g_\delta * f_t)(\alpha(v - u) + u) \\
&q^\delta_t(v, v_\ast, \omega) = \alpha^2(g_\delta \otimes g_\delta \otimes \text{id} * q)(\alpha(v - u) + u, \alpha(v_\ast - u) + u, \omega)
\end{align*}
$$

(5.7)

where $\text{id}$ is the identity function and $\alpha = \alpha(\delta) > 0$ is chosen such that $\int dv f^\delta_t(v)|v - u|^2/2 = U$. Observe that for any $\alpha > 0$, $\int dv f^\delta_t(v)v = u$.

Let $(\pi^\delta, Q^\delta)$ be the pair with densities $(f^\delta_t, q^\delta_t)$, which satisfies the balance equation. In order to prove the convergence of the rate function, we first observe that, by item (ii) in Assumption 2.1, we can write

$$
\text{Ent}(\pi^\delta_0|m_{e,u}) = \int f^\delta_0 \log f^\delta_0 + \int f^\delta_0 \log \frac{1}{m_{e,u}}.
$$

Since $\alpha(\delta) \to 1$ as $\delta \to 0$, by Jensen inequality and item (ii) in Assumption 2.1

$$
\lim_{\delta \to 0} \text{Ent}(\pi^\delta_0|m_{e,u}) \leq \text{Ent}(\pi_0|m_{e,u}).
$$

By the choice of $\alpha$, $f^\delta_0$ has the same energy as $f_0$. Therefore

$$
\lim_{\delta \to 0} H_{e,u}(\pi^\delta_0) \leq H_{e,u}(\pi_0).
$$

We will conclude the proof showing that $\lim J_{e,u}(f^\delta, Q^\delta) \leq J_{e,u}(f, q)$. We first observe that by a straightforward approximation argument we can choose $F = \log 1/B$ in (4.10), and deduce

$$
Q(\log \frac{1}{B}) \leq J_{e,u}(f, q) + \frac{1}{2} \int_0^T dt \int dv dv_\ast d\omega f f_\ast B(\frac{1}{2} - 1) < \infty.
$$

(5.8)

We prove in Appendix A that $Q^\delta(\log 1/B)$ is bounded and converges to $Q(\log 1/B)$ as $\delta \to 0$. Therefore

$$
J_{e,u}(\pi^\delta, Q^\delta) = \int_0^T dt \int dv dv_\ast d\omega q^\delta \log \frac{2d^\delta}{f^\delta t_2} + Q^\delta \left( \log \frac{1}{B} \right) - Q^\delta(1) + Q^\pi(1).
$$

Since the map $[0, +\infty)^2 \ni (a, b) \mapsto a \log(a/b)$ is one-homogeneous and convex, by (5.7) and Jensen’s inequality the first term on the r.h.s. is bounded by $Q(\log \frac{2d^\delta}{f^\delta t_2})$.

Moreover, $Q^\delta(1) = Q(1)$, while, since $B = \frac{1}{2}(|v - v_\ast| \cdot \omega)$, $Q^\pi(1) = \frac{1}{2}Q^\pi(1)$.

**Step 2. Time convolution.** Consider $(\pi, Q) \in \hat{S}$ such that $I_{e,u}(\pi, Q) < +\infty$, and denote with $(f, q)$ their densities. Assume that $f$ and $q$ are smooth in the velocities,
and $f > 0$. Observe that approximating path constructed in Step 1 meets these requirements.

Extend $[0, T] \ni t \mapsto (f_t, q_t)$ to a function defined on $(-\infty, T]$ by setting $(f_t, q_t) = (f_0, 0)$ if $t < 0$. Let $\eta_\varepsilon$ be the a smooth approximation of the $\delta$ function, with support in $(-\varepsilon, 0)$, and denote by $(\pi^\varepsilon, Q^\varepsilon)$ the path with densities $(f^\varepsilon, q^\varepsilon) = \eta_\varepsilon \ast (f, q)$; here we understand the convolution in time. The pair $(\pi^\varepsilon, Q^\varepsilon)$ converges to $(\pi, Q)$ and satisfies the balance equation (3.4). Observe that $f_0^\varepsilon = f_0$ and, since $(\pi, Q) \in \mathcal{S}$, $\pi_t(\zeta) = \pi_0(\zeta)$ for any $t \in [0, T]$, so that $\pi^\varepsilon_t(\zeta) = \pi_0(\zeta)$ for any $t \in [0, T]$.

We claim that $\lim_{\varepsilon \to 0} J_{e,u}(\pi^\varepsilon, Q^\varepsilon) = J_{e,u}(\pi, Q)$. To this hand, as $H_{e,u}(\pi^\varepsilon_0) = H_{e,u}(\pi_0)$, by lower semi-continuity it is enough to show that $\lim_{\varepsilon \to 0} J_{e,u}(\pi^\varepsilon, Q^\varepsilon) \leq J_{e,u}(\pi, Q)$.

Let $g_1$ be the standard Gaussian density on $\mathbb{R}^d$. We observe that, by standard approximation argument, we can choose $F = \log g_1 / f$ in the variational formula (4.7), and deduce that $\int q \log q < +\infty$ is finite. Since $J_{e,u}(\pi, Q)$ is bounded, using (6.9), we then deduce that $\int q \log q < +\infty$.

By Jensen inequality $\int q^\varepsilon \log q^\varepsilon \leq \int q \log q < +\infty$. On the other hand, by convexity, the maps $q \mapsto \int q \log q$ is lower semi-continuous, therefore we conclude that

$$\lim_{\varepsilon \to 0} \int q^\varepsilon \log q^\varepsilon = \int q \log q.$$ 

We write

$$J_{e,u}(\pi^\varepsilon, Q^\varepsilon) = -\int q^\varepsilon \log 2q^\varepsilon + \int q^\varepsilon \log \frac{2q^\varepsilon}{f^\varepsilon} + \int q^\varepsilon \log \frac{2q^\varepsilon}{f^\varepsilon} + \int q^\varepsilon \log \frac{1}{B} - 1 + \int f^\varepsilon f^\varepsilon B.$$ 

As already stated, the first term on the right-hand-side converges. By Jensen inequality the second term is bounded by $\int q \log (2q / f)$ and the third by $\int q \log (2q / f_e)$. Moreover, the fourth does not depend on $\varepsilon$. The convergence of the last term follows from the fact that, since the energy is uniformly bounded and $\pi \in C([0,T],\mathcal{P}(\mathbb{R}^d))$, the map $[0,T]^2 \ni (s, s') \mapsto \int \omega f(v) g(v) B(v - v_e, \omega)$ is continuous.

**Step 3.** Truncation. Consider $(\pi, Q) \in \mathcal{S}$ with $I_{e,u}(\pi, Q) < +\infty$, with densities $(f, q)$. We denote by $q^{(i)}$, $i = 1, \ldots, 4$ the marginal of $q_t$ respectively on $v, v_e, v' e, v'$. Then $q^{(1)} = q^{(2)}$, $q^{(3)} = q^{(4)}$, and the balance equation is the weak version of the identity

$$\partial_t f_t = 2(q^{(3)}_t - q^{(1)}_t).$$

In the sequel we assume $(f, q)$ smooth, $f$ strictly positive, and $q^{(3)}_t \in L^2([0,T] \times \mathbb{R}^d)$. Observe that the approximating path defined by applying sequentially Step 1 and 2 meets the above conditions. Indeed, the last condition above follows by Young inequality for convolutions.

Given $\ell > 0$, let $\chi^\ell(v, v_e, \omega) \in [0,1]$ be a continuous function such that

$$\chi^\ell(v, v_e, \omega) = \begin{cases} 1 & \text{if } |v|^2 + |v_e|^2 < \ell \text{ and } |(v - v_e) \cdot \omega| > 1/\ell \\ 0 & \text{if } |v|^2 + |v_e|^2 \geq (\ell + 1) \text{ or } |(v - v_e) \cdot \omega| \leq 1/(\ell + 1) \end{cases}$$
We start by proving that $c$ convergence the first term on the right-hand-side of (5.12) converges to $\text{Ent}(\bar{\nu})$. By (5.9),

$$
\phi
$$

where $h_m$ and denote by ($\tilde{\nu}$, $\tilde{\nu}$) the probability measure satisfying

$$
\int_0^t ds \left( \tilde{q}_s^{\ell,(3)} - \tilde{q}_s^{\ell,(1)} \right) + 2 \int_0^T ds \left( q_s^{(3)} - \tilde{q}_s^{\ell,(3)} \right)
$$

We define ($\tilde{\nu}$, $\tilde{\nu}$) by:

$$
\tilde{f}_t = f_0 + 2 \int_0^t ds \left( \tilde{q}_s^{\ell,(3)} - \tilde{q}_s^{\ell,(1)} \right) + 2 \int_0^T ds \left( q_s^{(3)} - \tilde{q}_s^{\ell,(3)} \right)
$$

Observe that $q_t^{\ell} \leq q_t$. Moreover $\tilde{f}_t \geq f_t$, since

$$
\int_0^t ds \left( \tilde{q}_s^{\ell,(3)} - \tilde{q}_s^{\ell,(1)} \right) + \int_0^T ds \left( q_s^{(3)} - \tilde{q}_s^{\ell,(3)} \right)
$$

Set

$$
c_t^{-1} = 1 + 2 \int_0^T ds \int dv \left( q_s^{(3)} - \tilde{q}_s^{\ell,(3)} \right),
$$

and denote by $(c_t, u_t)$ the energy and momentum of the probability $c_t \tilde{f}_t dv$. Note that $(c_t, u_t)$ does not depend on $t$ since $(\pi, Q) \in \mathcal{S}$. We define $(f^t, q^t)$ by:

$$
f^t(v) = c_t f^t(\alpha(v - u) + u_t), \quad q^t(v, v, \omega) = \alpha^2 c_t q^t(\alpha(v - u) + u_t, \alpha(v - u) + u_t)
$$

where $\alpha = \alpha_t > 0$ is chosen such that $\int dv f^t_0(v) = \int dv f_0(v)$. Observe that the pair $(f^t, q^t)$ satisfies the balance equation. As $t \to +\infty$, $c_t \to 1$, $u_t \to u$, $\alpha_t \to 1$, therefore $(f^t, q^t)$ converges to $(f, q)$.

We claim that

$$
\lim_{t \to +\infty} I_{c_t,u}(\pi^t, Q^t) \leq I_{c_t,u}(\pi, Q).
$$

We start by proving that

$$
\lim_{t \to +\infty} H_{c_t,u}(\pi^t_0) \leq H_{c_t,u}(\pi_0).
$$

(5.11)

Let $m^t$ be the probability measure satisfying

$$
\int m^t(dv) \varphi(v) = \int m_{c_t,u}(dv) \alpha \varphi(\alpha(v - u) + u_t),
$$

for any $\varphi \in C_b(\mathbb{R}^d)$, and let $\rho^t$ be its density. By a change of variable

$$
\text{Ent}(\pi^t_0|m_{c_t,u}) = \text{Ent}(c_t \tilde{f}_t^0 dv|m^t).
$$

(5.12)

By (5.9),

$$
c_t \tilde{f}_t^0 = c_t f_0 + (1 - c_t) \tilde{h}^t,
$$

where $h^t = 2 \int_0^T ds \left( q_s^{(3)} - \tilde{q}_s^{\ell,(3)} \right)$ and $\tilde{h}^t = h^t/\int h^t$. By convexity

$$
\text{Ent}(c_t \tilde{f}_t^0 dv|m^t) \leq c_t \text{Ent}(\pi_0|m^t) + (1 - c_t) \text{Ent}(\tilde{h}^t dv|m^t).
$$

Since $c_t \to 1$, $u_t \to u$, in view of item (iv) in Assumption 2.1 by dominated convergence the first term on the right-hand-side of (5.12) converges to $\text{Ent}(\pi_0|m)$. We now show that the second term vanishes. Observe that

$$
(1 - c_t) \text{Ent}(\tilde{h}^t dv|m^t) = c_t \int h^t \log h^t + (1 - c_t) \log \frac{c_t}{1 - c_t} - c_t \int h^t \log \rho^t.
$$

Since, by assumption on $q^{(3)}$, $h^t \in L^2$ and it converges to zero pointwise, the first term vanishes. The second term vanishes since $c_t \to 1$. Finally, using item (iv) of Assumption 2.1 the last term vanishes by dominated convergence. Since $\pi_0^t(\zeta) = \pi_0(\zeta)$, (5.11) follows.
We conclude the proof by showing that
\[ \lim_{\ell \to +\infty} J_{e,u}(\pi^\ell, Q^\ell) = J_{e,u}(\pi, Q). \]

By a change of variables,
\[ J_{e,u}(\pi^\ell, Q^\ell) = c_\ell \int \tilde{q}^\ell \log \frac{2\tilde{q}^\ell}{c_\ell \tilde{f}^\ell \tilde{f}_\ast B} + c_\ell \int \tilde{q}^\ell - c_\ell \int \tilde{q}^\ell + \frac{c^2}{\alpha} \int \tilde{f}^\ell \tilde{f}_\ast B. \]

Since \( \tilde{q}^\ell \leq q, \tilde{f}^\ell \geq f, \) and \( c_\ell \to 1, \) by dominated convergence the first term on the right-hand-side converges to \( \int q \log(2q/f f_\ast B). \) Since \( \int q^\ell \to \int q \) and \( \alpha \to 1, \) the second term tends to 0, and the third converges to \( Q(1). \) Finally, since \( \int q^{(3)} \zeta_0 < +\infty, \) \( B \) is uniformly integrable with respect to \( \tilde{f}^\ell \tilde{f}_\ast, \) therefore the last term converges to \( Q^\pi(1). \) \( \square \)

6. Large deviations for Kac model with canonical initial data

In this section we consider the Kac model with canonical initial data, namely when the initial velocities are i.i.d. sampled from a given probability \( m. \) In view of the abstract Proposition 2.5, the large deviation principle for the pair empirical measure and flow can be deduced from the large deviation principle of the Kac model with microcanonical initial data.

The canonical rate function is given by
\[ I(\pi, Q) = \inf_{(e,u) \in Z} (A(e,u) + I_{e,u}(\pi, Q)), \]
where \( A, \) as defined in (2.16), is the rate function relative to the sum of i.i.d. random variables given by Cramér’s theorem.

In order to compare this rate function with the one in [10, 7], consider the dynamical function as in (3.7), but without the microcanonical constraint, namely
\[ J(\pi, Q) = \int dQ \pi \left[ \frac{dQ}{dQ^\pi} \log \frac{dQ}{dQ^\pi} - \left( \frac{dQ}{dQ^\pi} - 1 \right) \right]. \]

Then functional in [10, 7] reads
\[ \mathcal{I}(\pi, Q) = \text{Ent}(\pi_0|m) + J(\pi, Q). \]

By Remark 2.1, for any \((\pi, Q) \in S\) we have \( \mathcal{I}(\pi, Q) \leq I(\pi, Q). \) For some path \((\pi, Q)\) this inequality is strict because, as discussed in detail in the next section, \( \mathcal{I} \) vanishes on Lu and Wennberg solutions, while \( I \) is strictly positive.

**Theorem 6.1.** Let \( m \) be a probability measure in \( \mathbb{R}^d \) and set \( \mu^N = m \otimes m. \) If \( m \) satisfies item (i)–(iii) in Assumption 2.1 then the family \( \mathbb{P}_{\mu^N} \circ (\pi^N, Q^N)^{-1} \) satisfies a large deviation upper bound with good rate function \( I : S \to [0, +\infty], \) namely \( I \) has compact level sets and for each closed \( C \subset S \)
\[ \lim_{N \to +\infty} \frac{1}{N} \log \mathbb{P}_{\mu^N} \left( (\pi^N, Q^N) \in C \right) \leq -\inf_C I. \tag{6.3} \]

Moreover, if \( m \) satisfies also condition (iv) in Assumption 2.1, then for each open \( O \subset S \)
\[ \lim_{N \to +\infty} \frac{1}{N} \log \mathbb{P}_{\mu^N} \left( (\pi^N, Q^N) \in O \right) \geq -\inf_{O \cap \delta} I. \tag{6.4} \]
Proof. By the definition of the microcanonical ensemble \( \nu_{e,u}^N \) given below equation (2.2), we have
\[
\mathbb{P}_{\mu^N}^N = \int p_N(d(e,u)) \mathbb{P}_{\nu_{e,u}^N},
\]
where \( p_N \) is the law of \( \frac{1}{N} \sum \zeta(v_i) \) with \( v \) sampled according to \( \mu^N \). By Cramér’s theorem, as discussed before remark 2.6, \( p_N \) satisfies a large deviation principle with rate function \( A \). The proof is thus essentially achieved by combining Theorem 3.2 with the abstract Proposition 2.5. However, since in the large deviation result with microcanonical initial data the upper and lower bound rate function may differ, we need a replacement for Step 2 in the proof of Proposition 2.5.

Upper bound. The argument in Step 4 in the proof of Proposition 2.5 applies, provided we show that the map \( Z \times S \ni (e,u,\pi,Q) \mapsto I_{e,u}(\pi,Q) \) is lower semicontinuous.

Recall the set \( C_{e,u} \) defined in (3.5), and let \( C \) be the subset of \( Z \times S \) defined by
\[
C := \{ (e,u,\pi,Q) : \pi \in C_{e,u} \}.
\]
By the lower semicontinuity of the map \( \pi \mapsto \pi(\zeta_0) \), and the continuity of the map \( \pi \mapsto \pi(\zeta) \) when the energy of \( \pi \) is uniformly bounded, we deduce that \( C \) is closed.

By the variational representation (4.10), this implies the joint lower semicontinuity of \( J_{e,u}(\pi,Q) \).

By Theorem 2.2 and Step 2 in the proof of Proposition 2.5, we also deduce the joint lower semicontinuity of \( H_{e,u}(\pi_0) \), that conclude the proof.

Lower bound. Fix \( (\pi,Q) \in \hat{S} \). By Step 3 in the proof of proposition 2.5, we deduce that for any open neighborhood \( \mathcal{N} \) of \( (\pi,Q) \) we have
\[
\lim_{N \to +\infty} \frac{1}{N} \log \mathbb{P}_{\mu^N}^N \left( (\pi^N,Q^N) \in \mathcal{N} \right) \geq -I(\pi,Q),
\]
that implies the statement.

\[ \square \]

7. Asymptotic probability of Lu and Wennberg solutions

We start by observing that the balance equation (3.4) for a pair \( (\pi,Q) \) with \( Q = Q^\pi \) is equivalent to the statement that \( \pi \) is a weak solution (1.1). Recalling that the functional \( J \), as defined in (3.2), vanishes if and only if \( Q = Q^\pi \), then we deduce that the zero level set of \( J \) are the weak solutions to the homogeneous Boltzmann equation (1.1). As we next state, the zero level set of both the functional \( I_{e,u} \) and \( I \) respectively defined in (3.8), (6.1) is a singleton. As a consequence the large deviation upper bound stated in theorems (3.2) and (6.1) implies the convergence of the empirical measure to the unique energy solution to the homogeneous Boltzmann equation (1.1) with an exponential bound on the error.

Theorem 7.1.

(i) \( I_{e,u}(\pi,Q) = 0 \) if and only if \( \pi = f dv, Q = Q^\pi \) and \( f \) is the unique energy conserving solution to the Cauchy problem associated to (1.1) with initial datum \( \frac{dm_{e,u}}{dt} \) as defined in (2.1).

(ii) \( I(\pi,Q) = 0 \) if and only if \( \pi = f dv, Q = Q^\pi \) and \( f \) is the unique energy conserving solution to the Cauchy problem associated to (1.1) with initial datum \( \frac{dm}{dt} \).
Proof. We prove only the first statement. By definition of $I_{e,u}$ if $f$ is an energy conserving solution to the Cauchy problem associated to (1.1) with initial datum $\frac{dm_{e,u}}{dv}$ then $\pi = f \, dv$ and $Q = Q^\pi$ belong to the zero level set of $I_{e,u}$. To prove the converse, we observe that, by the very definition (3.7), $J_{e,u}(\pi,Q) = 0$ implies that $Q = Q^\pi$ and $\pi_t(\zeta_0) \leq e$ for any $t \in [0,T]$. Since $H_{e,u}(\pi_0) = 0$ implies that $\pi_0 = m_{e,u}$ we deduce $\pi_t = f_t \, dv$ where $f$ is a weak solution to the Cauchy problem associated to (1.1) with initial datum $\frac{dm_{e,u}}{dv}$ and non increasing energy. Since for any weak solution to (1.1) the energy can not decrease in time (see [11, 13]), $f_t$ is the unique energy conserving solution.

□

Fix a non-decreasing piecewise constant, left-continuous profile $E : [0,T] \to \mathbb{R}_+$, with finite, non zero, number of jumps.

**Definition 7.2.** A Lu and Wennberg solution to the Cauchy problem associated to the homogeneous Boltzmann equation with initial datum $f_0$ and energy profile $E$ is a measurable function $f : [0,T] \times \mathbb{R}^d \to [0,\infty)$ such that

1. the map $t \mapsto f_t(v) \, dv =: \pi_t$ in $C([0,T]; \mathcal{P}(\mathbb{R}^d))$;
2. $f$ is a weak solution to the homogeneous Boltzmann equation;
3. $\pi_t(\zeta_0) = E(t)$, $t \in [0,T]$.

Observe that for any $e \geq E(T)$, for $\pi = f \, dv$, with $f$ a Lu and Wennberg solution, $J_{e,u}(\pi,Q^\pi) = 0$. Hence

$$I_{e,u}(\pi,Q^\pi) = \text{Ent}(\pi_0|m_{e,u}) + [\gamma^*_0 - \gamma_0][e - E(0)],$$

namely the Lu and Wennberg solutions contribute to the rate function only at time zero. We remark that these pairs $(\pi,Q^\pi)$ do not belong to the set $\mathcal{S}$ for which the upper and lower bound in Theorem 3.2 is proven to match. In the next theorem we will show they actually match also for a suitable class of Lu and Wennberg solutions.

**Theorem 7.3.** Fix $(e,u) \in Z$ and a sequence $(e_N,u_N) \to (e,u)$. For each energy profile $E$ with $E(T) < e$ and each $f_0$ with energy $E(0)$, there exists a Lu and Wennberg solution $f$ with energy profile $E$ such that for every open neighborhood $A$ of $(\pi,Q^\pi)$, $\pi = f \, dv$,

$$\lim_{N \to +\infty} \frac{1}{N} \log \mathbb{P}^N_{e_N,u_N} \left( (\pi^N,Q^N) \in A \right) \geq -I_{e,u}(\pi,Q^\pi). \quad (7.1)$$

Observe that, by the upper bound in Theorem 3.2

$$\lim_{N \to +\infty} \frac{1}{N} \log \mathbb{P}^N_{e_N,u_N} \left( (\pi^N,Q^N) \in \bar{A} \right) \geq -\inf_A I_{e,u},$$

which, together with (7.1), identifies the asymptotic probability of Lu and Wennberg solutions.

As in [12], the Lu and Wennberg solutions will be constructed as a limit of a suitable sequence. In particular we will consider a sequence $f^n$ which conserve the energy and such that $t \mapsto f^n_t(v) \, dv \in \mathcal{P}(\mathbb{R}^d)$ is continuous. We start with the static result.

**Lemma 7.4.** Consider $\rho \in \mathcal{P}(\mathbb{R}^d)$ such that $H_{e,u}(\rho)$ is finite and $e_0 := \rho(\zeta_0) < e$. Given $e_1 \in (e_0,e]$ and $n \in \mathbb{N}$, let $g_n = m_{n(e_1-e_0),u}$ be the exponential tilt of $m$
with energy \( n(e_1 - e_0) \) and momentum \( u \). Set \( \rho_n = (1 - \frac{1}{n})\rho + \frac{1}{n}g_n \), so that \( \rho_n(\zeta_0) = e_1 - e_0 \), then

\[
\lim_{n \to \infty} H_{e,u}(\rho_n) = H_{e,u}(\rho). \tag{7.2}
\]

**Proof.** By the lower semicontinuity of \( H_{e,u} \), it is enough to show that \( \lim H_{e,u}(\rho_n) \leq H_{e,u}(\rho) \). By the convexity of \( H_{e,u} \) and Jensen inequality

\[
H_{e,u}(\rho_n) \leq (1 - \frac{1}{n})H_{e,u}(\rho) + \frac{1}{n}H_{e,u}(g_n).
\]

Let \( \lambda^n \) such that

\[
g_n(dv) = \frac{e^{\lambda_n \cdot \zeta}m(dv)}{m(e^{\lambda_n \cdot \zeta})} = \frac{e^{\lambda_n \cdot (\lambda - \gamma(e,u))\cdot \zeta}m_{e,u}(e^{\lambda_n \cdot (\lambda - \gamma(e,u))\cdot \zeta})}{m_{e,u}(e^{\lambda_n \cdot (\lambda - \gamma(e,u))\cdot \zeta})}m_{e,u}(dv),
\]

where we used (2.1). Observe that \( \lambda^n_0 \uparrow \gamma^*_0 \) as \( n \to +\infty \). Since \( g^n \) has energy \( n(e_1 - e_0) \) we get

\[
\lim_{n \to +\infty} \frac{1}{n} \text{Ent}(g^n|m_{e,u}) \leq \lim_{n \to +\infty} (\lambda^n_0 - \gamma_0(e,u))(e_1 - e_0) = (\gamma^n_0 - \gamma_0(e,u))(e_1 - e_0),
\]

which concludes the proof. \( \square \)

For any probability density \( h \) with finite energy let \( \mathcal{U}_t(h), t \geq 0 \), be the unique energy conserving solution to the Cauchy problem associated to the homogeneous Boltzmann equation with initial datum \( h \). In the following statement we collect the result on moment estimate in [13, 14].

**Lemma 7.5.** Let \( h \) be a probability density on \( \mathbb{R}^d \) with finite energy and entropy. Then

(i) For each \( p > 2 \) and \( t > 0 \) there exists a real \( C > 0 \) depending only on \( p \), \( t \) and the initial energy, such that

\[
\int dv \mathcal{U}_t(h)(v)|v|^p \leq C.
\]

(ii) For each \( p > 2 \), if \( \int dv h(v)|v|^p < +\infty \), then

\[
\sup_{t \in [0,T]} \int dv \mathcal{U}_t(h)(v)|v|^p < +\infty.
\]

Fix an energy profile \( \mathcal{E} : [0, T] \to \mathbb{R}^+ \) and denote by \( 0 \leq t_1 < \ldots < t_k < T \) the discontinuity set of \( \mathcal{E} \). Given \( f_0 \) with finite entropy and energy \( \mathcal{E}(0) \), let \( h^n_0 \) be a sequence weakly convergent to \( f_0 \) satisfying the following requirements. The energy of \( h^n_0 \) is independent on \( n \) and equal to \( \mathcal{E}(0) \), its entropy converges the entropy of \( f_0 \), and it has finite (\( n \)-dependent) \( p \)-moment for some \( p \geq 3 \). For \( n \geq 1 \) and \( i = 1, \ldots, k \), set \( e_{n,i} = nk[\mathcal{E}(t^+_i) - \mathcal{E}(t_i)] \) and define \( g^n_i \) as the density of the tilted probability \( m_{e_n,i,u} \). Define

\[
f^n_t = \begin{cases} 
(1 - \frac{1}{nk})\mathcal{U}_t(h^n_0) + \frac{1}{nk} \sum_{i=1}^k g^n_i & t \in [0, t_1] \\
(1 - \frac{1}{nk})\mathcal{U}_{t-t_1}^i(h^n_i) + \frac{1}{nk} \sum_{i=2}^k g^n_i & t \in (t_1, t_2) \\
\ldots & \ldots \\
(1 - \frac{1}{nk})\mathcal{U}_{t-t_{k-1}}(h^n_{k-1}) + \frac{1}{nk} g^n_k & t \in (t_{k-1}, t_k) \\
\mathcal{U}_{t-t_k}(h^n_k) & t \in (t_k, T),
\end{cases} \tag{7.3}
\]
where $h^n_i$ are recursively defined so that $t \mapsto f^n_t(v)\,dv$ is continuous, namely

$$h^n_i = \frac{1}{1-n}\left[f^n_t - \frac{1}{nk}\sum_{j=i+1}^k g^n_j\right].$$

Let also $q^n_i(v, v_*, \omega)$ be such that, for $t \in (t_i, t_{i+1})$,

$$q^n_i(v, v_*, \omega) = \left(1 - \frac{k-i}{n}\right)\mathcal{U}_{t-t_i}(h^n_i)(v)\mathcal{U}_{t-t_i}(h^n_i)(v_*)B(v, v_*, \omega).$$

Here $i = 0, ..., k$, with $t_0 = 0$ and $t_{k+1} = T$. Observe that, by construction, the pair $(\pi^n, \mathcal{Q}^n)$ with densities $(f^n, q^n)$ satisfies the balance equation (6.4). Furthermore, by definition of $h^n_0$ and item (ii) in Lemma 7.5, for each $n$ the pair $(\pi^n, \mathcal{Q}^n) \in \hat{\mathcal{S}}$.

**Lemma 7.6.** The sequence $\{(\pi^n, \mathcal{Q}^n)\}$ is relatively compact in $\mathcal{S}$. Any cluster point $(\pi, \mathcal{Q})$ is such that $Q = \mathcal{Q}^\pi$, $\pi = f \,dv$, where $f$ is a Lu and Wennberg solution with initial datum $f_0$ and energy profile $\mathcal{E}$. Moreover

$$\lim_{n \to \infty} I_{e,u}(\pi^n, \mathcal{Q}^n) = H_{e,u}(f_0 \,dv). \quad (7.4)$$

**Proof.** We start by proving (7.4). Observe that $\int dv h^n_i \zeta_0 = \mathcal{E}(t_i^+)$, for $i, 1, ..., k$. Then by Lemma 7.4 and Jensen inequality,

$$\lim_{n \to \infty} H_{e,u}(\pi^n_0) = \text{Ent}(f_0 \,dv|m_{e,u}) + (\gamma_0^s - \gamma(e, u))\left[e - \mathcal{E}(T) + \sum_{i=1}^k (\mathcal{E}(t_i^+) - \mathcal{E}(t_i))\right]$$

$$= H_{e,u}(f_0 \,dv).$$

We now show that

$$\lim_{n \to \infty} J_{e,u}(\pi^n, \mathcal{Q}^n) = 0. \quad (7.5)$$

By definition, for $t \in (t_i, t_{i+1})$, $i = 0, ..., n$, we have

$$f^n_t \geq \left(1 - \frac{k-i}{n}\right)\mathcal{U}_{t-t_i}(h^n_i).$$

Hence the the contribution to $J_{e,u}(\pi^n, \mathcal{Q}^n)$ in the time window $(t_i, t_{i+1}]$ is bounded by

$$\int_{t_i}^{t_{i+1}} dt \int dv \,dv_* \,d\omega \left\{g^n_i(v, v_*, \omega) \log \left(1 - \frac{k-i}{n}\right)^{-1} - q^n_i(v, v_*, \omega) + f^n_t(v)f^n_t(v_*)B(v, v_*, \omega)\right\}.$$
\( \mathcal{U}_{t-i}(h_n^i), p > 2, \) is bounded uniformly in \( n, \) therefore \( \zeta_0 \) is uniformly integrable with respect to \( \mathcal{U}_{t-i}(h_n^i), \) then
\[
\int dv f_i \zeta_0 = \lim_{n \to +\infty} \int dv \mathcal{U}_{t-i}(h_n^i) \zeta_0 = \mathcal{E}(T) - \sum_{j=i+1}^{k} (\mathcal{E}(t_j^+) - \mathcal{E}(t_j)) = \mathcal{E}(t).
\]

\[\square\]

**Theorem 7.7.** Let \( m \) be a probability measure satisfying Assumption 2.1, and set \( \mu^N := m^\otimes N. \) For each energy profile \( \mathcal{E} \) with \( \mathcal{E}(0) = m(\zeta_0) \) there exists a Lu and Wennberg solution \( f \) with \( f_0 = m \) and energy profile \( \mathcal{E} \) such that for every open neighborhood \( A \) of \( (\pi, \mathcal{Q}^\pi), \) \( \pi = f \, dv, \)
\[
\lim_{N \to +\infty} \frac{1}{N} \log \mathbb{P}^N_{\mu^N}(\{\pi^N, \mathcal{Q}^N \in A\}) \geq -I(\pi, \mathcal{Q}^\pi) = \gamma_0^*(\mathcal{E}(T) - \mathcal{E}(0)). \tag{7.6}
\]

**Proof.** The proof of the inequality in (7.6) follows the same arguments of the proof of Theorem 7.3. We here discuss the equality. Since \( \pi = f \, dv \) is a weak solution to (1.11), \( J_{e,u}(\pi, \mathcal{Q}) = 0 \) if \( e \geq \mathcal{E}(T), \) otherwise is infinity. Then, by definition (6.1) and Theorem 2.2 we have that
\[
I(\pi, \mathcal{Q}) = \inf_{e \geq \mathcal{E}(T)} (A(e, u) + \text{Ent}(m|m_{e,u}) + (\gamma_0^* - \gamma_0(e, u))(e - \mathcal{E}(0))
\]
where \( u = m(\zeta). \) The supremum in the definition (2.16) of \( A_{e,u} \) is achieved in \( \gamma = \gamma(e, u). \) By definition of the relative entropy
\[
\text{Ent}(m|m_{e,u}) = -\gamma(e, u) \cdot m(\zeta) + \log m(e^{\gamma(e, u)} \cdot \zeta).
\]
Then, by direct computation, \( I(\pi, \mathcal{Q}) = \inf_{e \geq \mathcal{E}(T)} \gamma_0^*(e - \mathcal{E}(0)) = \gamma_0^*(\mathcal{E}(T) - \mathcal{E}(0)). \)
\[\square\]

**Appendix A.**

It is sufficient to prove that \( Q^\delta(\lceil \log 1/B \rceil^+) \) converges to \( Q(\lceil \log 1/B \rceil^+) \) as \( \delta \to 0, \) since the result for the negative part easily follows from the fact that \( \lceil \log 1/B \rceil^+ \) is sublinear in \( |v - v_*|, \) and \( (\pi, \mathcal{Q}) \in \hat{\delta}. \) We indicate with \( g_3^\delta \) the Gaussian kernel in one dimension, and note that
\[
(g_3^\delta \otimes g_3^\delta \otimes \text{id}) \left( \lceil \log \frac{1}{2B} \rceil^+ (v, v_*, \omega) = \int_{\mathbb{R}} g_3^\delta(w \cdot \omega - y) \left( \lceil \log \frac{1}{|y|} \rceil^+ \right) dy, \right.
\]
where \( w = (v - v_*)/\sqrt{2}. \) We now prove that there exist some constants \( c_1, c_2 > 0 \) such that
\[
\int_{\mathbb{R}} g_3^\delta(x - y) \left( \lceil \log \frac{1}{|y|} \rceil^+ \right) dy \leq c_1 \left( \lceil \log \frac{1}{|x|} \rceil^+ \right) + c_2,
\]
which implies that
\[
(g_3^\delta \otimes g_3^\delta \otimes \text{id}) \left( \lceil \log \frac{1}{B} \rceil^+ (v, v_*, \omega) \leq c_1 \left( \lceil \log \frac{1}{B} \rceil^+ (v, v_*, \omega) + c_2. \right.
\]
Using this fact and that \( Q(\lceil \log 1/B \rceil^+) < +\infty, \) we achieve the convergence result by using Fubini-Tonelli theorem and dominate convergence.

We denote by \( z \) a standard Gaussian stochastic variable and note that
\[
\int_{\mathbb{R}} g_3^\delta(x - y) \left( \lceil \log \frac{1}{|y|} \rceil^+ \right) dy = \mathbb{E} \left( \left( \lceil \log \frac{1}{|x - \delta z|} \rceil^+ \right) \leq \log \frac{1}{\delta} + \mathbb{E} \left( \left( \lceil \log \frac{1}{|x/\delta - z|} \rceil^+ \right). \right.
\]
Since $[\log 1/|y|]^+$ is summable, by the Young’s inequality the second term is uniformly bounded, so that, if $|x| \leq \sqrt{\delta}$ we have
\[
E\left(\log 1/|x - \delta z|\right)^+ \leq 2 \log 1/|x| + c.
\]
To handle the case $|x| \geq \sqrt{\delta}$, we use the Jensen inequality:
\[
E\left(\log 1/|x - \delta z|\right)^+ = 2 \log e^{E\left(\log 1/\sqrt{|x - \delta z|\wedge 1}\right)} \leq 2 \log E\left(1/\sqrt{|x - y|\wedge 1}\right)
\]
We estimate
\[
E\left(1/\sqrt{|x - y|\wedge 1}\right) = \int_{\mathbb{R}} g^\delta(y) 1/\sqrt{|x - y|\wedge 1} \, dy
\]
by noticing that in the region $|y| < |x|/2$ or $|y| > 2|x|$ we have $1/\sqrt{|x - y|\wedge 1} \leq \sqrt{2}/\sqrt{|x|\wedge 1}$. Therefore
\[
E\left(1/\sqrt{|x - \delta z|\wedge 1}\right) \leq c 1/\sqrt{|x|\wedge 1} + g^\delta(|x|/2) \int_{|x|/2}^{2|x|} 1/\sqrt{|x - y|\wedge 1} \, dy.
\]
We conclude the proof observing that the last term is estimate by $ce^{-1/8\delta}(1 + 1/\delta)$, which is uniformly bounded in $\delta$.

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