Extra Observables in Gauged WZW Models

Nobuyuki Ishibashi

Department of Physics
University of California
Santa Barbara, CA 93106

Abstract

It is known that Liouville theory can be represented as an \( SL(2,\mathbb{R}) \) gauged WZW model. We study a two dimensional field theory which can be obtained by analytically continuing some of the variables in the \( SL(2,\mathbb{R}) \) gauged WZW model. We can derive Liouville theory from the analytically continued model, (which is a gauged \( SL(2,\mathbb{C})/SU(2) \) model,) in a similar but more rigorous way than from the original gauged WZW model. We investigate the observables of this gauged \( SL(2,\mathbb{C})/SU(2) \) model. We find infinitely many extra observables which cannot be identified with operators in Liouville theory. We concentrate on observables which are \((1,1)\) forms and the correlators of their integrals over two dimensional spacetime. At a special value of the coupling constant of our model, the correlators of these integrals on the sphere coincide with the results from matrix models.

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*Address after Nov. 1: Department of Physics, University of California, Davis, CA 95616*
1 Introduction

Polyakov's discovery of the existence of $SL(2)$ current algebra in two dimensional quantum gravity in the light-cone gauge\cite{1} and the subsequent success of the derivation of scaling exponents\cite{2}, suggested that two dimensional quantum gravity could be rewritten in such a way that the $SL(2)$ current algebra is more apparent. The authors of \cite{3}\cite{4} showed that two dimensional quantum gravity in the light cone gauge could be represented by an $SL(2, \mathcal{R})$ WZW model with the constraint
\begin{equation}
J^- = 1.
\end{equation}

The "soldering" procedure in \cite{6} and the study of $SL(2, \mathcal{R})$ Chern Simons theory explored in \cite{7}, exposed the relation between two dimensional quantum gravity and $SL(2, \mathcal{R})$ gauge theory consisting of zweibein and spin connection. These works imply that the $SL(2, \mathcal{R})$ structure is not a peculiarity of the light cone gauge but a more fundamental feature of two dimensional quantum gravity.

Although the light cone gauge reveals the $SL(2, \mathcal{R})$ structure, the most convenient gauge of two dimensional quantum gravity is conformal gauge. In conformal gauge, two dimensional quantum gravity is described by Liouville theory, which is simpler and more useful than the complicated light cone gauge action. In \cite{5}, it was shown that Liouville theory could also be represented as a constrained $SL(2, \mathcal{R})$ WZW model classically. This time the $SL(2)$ currents should be constrained as
\begin{equation}
J^+ = J^- = \sqrt{\mu}.
\end{equation}

In \cite{8}\cite{9}, a gauged WZW model realizing the above constraints was analyzed. Liouville theory was derived from this $SL(2, \mathcal{R})$ gauged WZW model at the quantum level. Therefore, in this formulation, we can see the $SL(2, \mathcal{R})$ structure is also hidden in Liouville theory.

However the analysis of \cite{8}\cite{9} is somewhat formal because they are dealing with WZW model of a noncompact group. In this paper, we will propose another model which represents Liouville theory. This new model, which is a gauged $SL(2, \mathcal{C})/SU(2)$ model, is obtained by analytically continuing some of the variables in the $SL(2, \mathcal{R})$ gauged WZW model. The analytic continuation does not spoil the left and right $SL(2)$ current algebras in the WZW model. Because of this analytic continuation, the arguments of \cite{8}\cite{9} go through more rigorously in this model. We will study the observables and their correlation functions in this model.

The organization of this paper is as follows. In section 2, we first review the analysis of \cite{8}\cite{9}. In order to make their arguments more rigorous, we analytically continue the $SL(2, \mathcal{R})$ gauged WZW model to the gauged $SL(2, \mathcal{C})/SU(2)$ model. Liouville theory is derived from this gauged $SL(2, \mathcal{C})/SU(2)$ model in a more rigorous way than in the derivation in \cite{8}\cite{9}. In section 3, observables in the gauged $SL(2, \mathcal{C})/SU(2)$ model are discussed. It is natural to expect that all the observables correspond to Liouville theory operators. However, we find that there exist infinitely many extra observables, which cannot be identified with operators in Liouville theory. We concentrate on the observables with conformal weight $(1, 1)$ and their
integrals over two dimensional spacetime. The correlators of such integrals are calculated on the sphere. If one chooses the coupling constant of our model so that it corresponds to Liouville theory induced by \( c = -2 \) conformal field theory, these correlators coincide with the correlators of the observables at the first critical point of the one matrix model. In section 4, we present a brief discussion of our results. Appendix is devoted to definitions and some useful formulas about \( SL(2, \mathbb{C})/SU(2) \) model.

2 Gauged WZW Model and Liouville Theory

Let us consider the \( SL(2, \mathbb{R}) \) gauged WZW model with gauge fields \( A^+_z \) and \( A^-_{\bar{z}} \) following [5],

\[
I = kS_{WZW}(g) + \frac{k}{\pi} \int d^2x \{ A^-_{\bar{z}}(tr(t^+ \partial gg^{-1}) - \sqrt{\mu}) + A^+_z(tr(t^- g^{-1} \bar{\partial}g) - \sqrt{\mu}) + A^+_zA^-_{\bar{z}} tr(t^+ g t^- g^{-1}) \}.
\] (3)

Here \( g \in SL(2, \mathbb{R}) \) and \( S_{WZW}(g) \) is the action of the WZW model

\[
S_{WZW}(g) = -\frac{1}{2\pi} \int d^2x \partial g \bar{\partial}g^{-1} + \frac{i}{12\pi} \int_B d^3x tr(g^{-1}dg)^3,
\] (4)

and \( t^\pm = t^1 \pm t^2 \) are the generators of \( SL(2, \mathbb{R}) \) with positive and negative roots respectively. The action \( I \) is invariant under the gauge transformations \( \delta g = -\epsilon^- t^+ g - g\epsilon^+ t^- \), \( \delta A^+_z = \partial \epsilon^+, \delta A^-_{\bar{z}} = \bar{\partial} \epsilon^- \).

In [9], it was shown that Liouville theory can be deduced from this gauged WZW model. Let us review their derivation of Liouville theory. \( g \in SL(2, \mathbb{R}) \) can be parametrized via the Gauss decomposition

\[
g = \begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^\phi & 0 \\ 0 & e^{-\phi} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \bar{v} & 0 \end{pmatrix}.
\] (5)

In these coordinates, the action becomes

\[
I = \frac{k}{\pi} \int d^2x \{ \partial \phi \bar{\partial} \phi + e^{-2\phi} \bar{\partial}v \bar{\partial} \bar{v} \\
+ A^+_z(e^{-2\phi} \partial \bar{v} - \sqrt{\mu}) + A^-_{\bar{z}}(e^{-2\phi} \bar{\partial}v - \sqrt{\mu}) + e^{-2\phi} A^+_zA^-_{\bar{z}} \}.
\] (6)

The authors in [9] derived Liouville theory from this field theory of \( \phi, v, \bar{v} \) and \( A^\pm \)'s. \footnote{As was stressed in [5], the Gauss decomposition is possible for an element near the identity in the group manifold. In this case, the \( SL(2, \mathbb{R}) \) matrix \( g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \), with \( d = 0 \) cannot be represented by eq.(5). Therefore, there is a subtlety in representing the \( SL(2, \mathbb{R}) \) WZW model as the field theory of \( \phi, v \) and \( \bar{v} \).}
Liouville theory appears if one integrates out \( v, \bar{v} \) and \( A \)'s in the partition function
\[
Z = \int \frac{[d\phi d\bar{\phi} dA]}{Vol.} e^{-I}. \tag{7}
\]
Here \( Vol. \) denotes the gauge volume of the gauge transformation
\[
\delta \bar{v} = -\epsilon^+, \quad \delta A^+_z = \partial \epsilon^+, \\
\delta v = -\epsilon^-, \quad \delta A^-_z = \bar{\partial} \epsilon^-.
\tag{8}
\]
Integration over the \( v \)'s and \( A \)'s was done by choosing the gauge \( v = \bar{v} = 0 \) or equivalently shifting the integration variable \( A \)'s by \( A^+_z \rightarrow A^+_z + \bar{\partial} \bar{v}, A^-_z \rightarrow A^-_z + \partial v \). After doing so, we are left with the following expression for the partition function:
\[
Z = \int \frac{[d\phi d\bar{\phi} dA]}{Vol.} \exp \left\{ -\frac{k}{\pi} \int d^2x (\partial \phi \bar{\partial} \phi + e^{-2\phi} A^+_z A^-_z - \sqrt{\mu} A^+_z - \sqrt{\mu} A^-_z) \right\}. \tag{9}
\]
The \( v, \bar{v} \) integration merely corresponds to an overall constant. Since the functional integration measure for \( \phi, v, \bar{v} \) is defined by the norm
\[
\| \delta g \|^2 = \int d^2x tr(g^{-1} \delta g)^2 = 2 \int d^2x \{ (\delta \phi)^2 + e^{-2\phi} \delta v \delta \bar{v} \}, \tag{10}
\]
The \( v, \bar{v} \) integration divided by the gauge volume \( Vol. \) gives us the factor arising from the determinant which we formally write as \( \prod_x e^{-2\phi} \). The measure for \( A^+_z \) and \( A^-_z \) is defined by the norm
\[
\| \delta A \|^2 = \int d^2x \delta A^+_z \delta A^-_z. \tag{11}
\]
The integration over \( A^+_z, A^-_z \) contributes the inverse of \( \prod_x e^{-2\phi} \). Naively this cancels the determinant coming from the \( v \) integration. Thus we obtain,
\[
Z = \int [d\phi] \exp \left\{ -\frac{k}{\pi} \int d^2x (\partial \phi \bar{\partial} \phi - \mu e^{2\phi}) \right\}. \tag{12}
\]
The partition function therefore is the same as the partition function of Liouville theory. However, notice that we are comparing the determinants of the operator \( e^{-2\phi} \) acting by multiplication on \( v \) with the same operator acting by multiplication on \( A \). This is a situation analogous to the one we encounter in the ghost number anomaly in string theory. In that case, one should compare the determinant of an operator acting by multiplication on the ghost with that of the same operator acting on the antighost. Since the spins of the ghost and the antighost are different, there is a nontrivial difference between the two determinants. Since the spins of \( v \) and \( A \) are different, we expect that there is also a nontrivial difference.

\( \overline{Vol.} \) is defined by the functional integration over the gauge parameter \( \epsilon \) with the norm \( \| \epsilon \|^2 = \int d^2 x e^+ \epsilon^- \).
between the two determinants in our case. Assuming that the difference of the two determinants changes eq. (12) into the form,

$$Z = \int [d\phi] \exp\left\{ -\frac{k'}{\pi} \int d^2x (\partial \phi \bar{\partial} \phi - Q \phi \bar{\partial} \sigma - \mu' e^{2\phi + \sigma}) \right\}, (13)$$

where the spacetime metric is $ds^2 = e^\sigma dz d\bar{z}$, we can determine the values of $k'$ and $Q$ à la DDK\cite{10} as $k' = k - 2, Q = \frac{k - 1}{k - 2}$. The Virasoro central charge of the Liouville theory is $c = \frac{3}{k - 2} + 6k - 2$. This is exactly the relation between the level of the $SL(2, R)$ current algebra and the Virasoro central charge in \cite{2}.

In this way, the authors of \cite{8, 9} showed that the partition function of the gauged WZW model coincides with that of Liouville theory. Here, the following two remarks are in order.

1) In the Gauss decomposition eq. (3) of $g \in SL(2, R), v$ and $\bar{v}$ are real. Therefore the system of bosons $v$ and $\bar{v}$ involves a negative signature kinetic term and the norm eq. (10) is not positive definite. The origin of such a negative kinetic term and norm is the noncompactness of $SL(2, R)$. Accordingly the norms of $A$'s and the gauge parameter $\epsilon$ fail to be positive definite. Therefore, strictly speaking, the functional integral over $v$'s and $A$'s discussed above is not well defined. One way to make the functional integral over $v$ and $\bar{v}$ well defined is to continue $v$ and $\bar{v}$ so that the norm and the kinetic term become positive definite. If $k > 0$ (which we assume in the following), this amounts to regarding $v$ and $\bar{v}$ (and accordingly $A_+^+ \text{ and } A_-^-$) as complex conjugate to each other. As is shown in the appendix, the action eq. (4) with $v$ and $\bar{v}$ complex conjugate to each other, can be considered as a gauged version of $SL(2, C)/SU(2)$ model \cite{11}. Therefore, strictly speaking, we should do the analytic continuation in order to make the above calculation rigorous. The analytic continuation seems to be legitimate, if what we are dealing with is the field theory of $\phi, v$ and $\bar{v}$. However, considering that we are dealing with an $SL(2, R)$ gauged WZW model, this analytic continuation seems subtle, because the negative kinetic term of $v$'s stems from the noncompactness of $SL(2, R)$ which is an essential feature of the group $SL(2, R)$.

2) In the usual $SL(2, R)$ WZW model (or $SL(2, C)/SU(2)$ model), the variables $v, \bar{v}$ and $\phi$ are all scalars and $A$ is a vector field. However, because of the presence of the terms proportional to $\sqrt{\mu}$ in eq. (7), the action is not even rotationally invariant under such spin assignments. In order to make the theory conformally invariant, we should change the assignments. We have to take the left and right conformal weights of $\phi, v$ and $\bar{v}$ so that the currents $e^{-2\phi} \partial v, e^{-2\phi} \partial \bar{v}$ have the left and right conformal weights $(0, 0)$. This can be achieved by “twisting” the model. Namely, as was done in \cite{3, 4, 5}, we add to the stress tensor a derivative of the zeroth component of the chiral $SL(2, R)$ current:

$$T'_{zz} = T_{zz} + \partial J^0_z, \; T'_{\bar{z}\bar{z}} = T_{\bar{z}\bar{z}} + \bar{\partial} J^0_{\bar{z}}. \quad (14)$$

This corresponds to shifting the conformal weights of $\phi, v$ and $\bar{v}$ to $(0, 0), (1, 0)$ and $(0, 1)$ respectively. The conformal weights of the $A$'s are taken to be $(1, 1)$. Accordingly we should
modify the action as

\[ I = \frac{k}{\pi} \int d^2x \left\{ \partial \phi \bar{\partial} \bar{\phi} - \phi \bar{\partial} \partial \sigma + e^{-2\phi - \sigma} \partial v \bar{\partial} \bar{v} \\
+ A^+ ( e^{-2\phi - \sigma} \bar{\partial} \bar{v} - \sqrt{\mu} ) + A^- ( e^{-2\phi - \sigma} \partial v - \sqrt{\mu} ) \\
+ e^{-2\phi - \sigma} A^+ A^- \right\}. \tag{15} \]

In \[8\]\[9\], such a twisting was not mentioned at the stage of considering the gauged WZW action eq.(3). However, we should start the discussion from this twisted action in order to define the gauged WZW model to be rotationally invariant. This modified action depends explicitly on the conformal factor \(\sigma\) of the metric. However it is invariant under the Weyl transformation \(\sigma \rightarrow \sigma + \epsilon, \phi \rightarrow \phi - \frac{1}{2} \epsilon\) as in the case of a Feigin-Fuchs boson.

The arguments above show that the analysis of [8][9] described in the first part of this section is somewhat formal. In order to make it more rigorous, we should start from the action eq.(15), with \(v\) and \(\bar{v}\) complex conjugate to each other, instead of eq.(7). However the analytic continuation does not seem to be legitimate, considering that we are dealing with the \(SL(2,\mathbb{R})\) WZW model. Therefore we would rather propose this (twisted) gauged \(SL(2,\mathbb{C})/SU(2)\) model eq.(15) as a new model related to the \(SL(2,\mathbb{R})\) gauged WZW model. In this model, the analysis of [8][9] goes through more rigorously. Here, instead, we will use an alternative method to deduce Liouville theory starting from this gauged \(SL(2,\mathbb{C})/SU(2)\) model eq.(15). In our method, we can derive eq.(13) directly without any assumption, and it gives a more rigorous derivation of Liouville theory from the gauged \(SL(2,\mathbb{C})/SU(2)\) model eq.(15).

Let us consider the partition function of the gauged \(SL(2,\mathbb{C})/SU(2)\) model  \[4\]

\[ Z = \int \frac{[d\phi dv dA]}{Vol.} e^{-I}. \tag{16} \]

Now \(I\) is the action in eq.(15) and \(v\) and \(\bar{v}\) are complex conjugate to each other. \(Vol.\) denotes the volume of the gauge transformation eq.(8), with \(\epsilon^+\) and \(\epsilon^-\) being complex conjugate to each other. We will integrate out \(v, \bar{v}\) and \(A\) in eq.(16) and obtain Liouville theory. After the twisting mentioned above, the functional integration measures for these variables are defined by the norm

\[ \|\delta v\|^2 = \int d^2x e^{-2\phi} \delta v \delta \bar{v}, \]
\[ \|\delta A\|^2 = \int d^2x e^{-\sigma} \delta A^+ \delta A^- \tag{17} \]

In order to integrate out \(v\) and \(A\), one should somehow take care of the gauge invariance. Essentially, what we will do here is to fix the gauge as \(A^+ = A^- = 0\). The gauge fixed action

\[ For\ notational\ simplicity,\ we\ will\ discuss\ this\ model\ on\ the\ sphere\ with\ the\ conformal\ metric\ \text{d}s^2 = e^\sigma dzd\bar{z}. \]
then becomes
\[
I = \frac{k}{\pi} \int d^2x \{ \partial \phi \bar{\partial} \phi - \phi \partial \bar{\partial} \sigma + e^{-2\phi - \sigma} \bar{\partial}v \bar{\partial}v \} \\
+ \frac{1}{\pi} \int d^2x (b \bar{\partial}c + \bar{b} \partial \bar{c}),
\]
and the theory becomes a system consisting of the twisted $\text{SL}(2,\mathbb{C})/\text{SU}(2)$ model and ghosts. In this form, our model is solvable using the current algebra technique. However there is one thing one has to notice with such a gauge choice. One cannot choose such a gauge globally on a compact Riemann surface. Indeed, by expanding the $A$’s in terms of the eigenfunctions of the Laplacian on the surface, one can show that $A$’s can be decomposed as
\[
A^+_z = \partial \bar{\Lambda} + a_0 e^\sigma, \\
A^-_z = \bar{\partial} \Lambda + \bar{a}_0 e^\sigma.
\]
(19) Here $\Lambda$ and $\bar{\Lambda}$ are $(1,0)$ and $(0,1)$ forms respectively and $a_0$ and $\bar{a}_0$ are constants. The second terms in eq.(19) cannot be gauged away. Therefore the gauge eq.(18) is possible only locally. Of course, such global obstructions do not matter when one is canonically quantizing the system and calculating commutation relations of operators. Therefore, quantities such as anomalous dimensions of operators can be reliably computed using the current algebra technique available in the gauge eq.(18). In order to derive Liouville theory, we will construct an alternative form of the action depending explicitly on the moduli $a_0$.

Let us change variables from $A$ to $\Lambda$, $\bar{\Lambda}$, $a_0$ and $\bar{a}_0$ in the functional integration eq.(16). The partition function becomes
\[
Z = \int \frac{[d\phi dv d\Lambda da_0]}{Vol.} \det'(\Delta) e^{-I}.
\]
(20) The integration measure for $\Lambda$ and $a_0$ are defined by the norm
\[
\|\delta \Lambda\|^2 = \int d^2x \delta \Lambda \delta \bar{\Lambda}, \\
\|\delta a_0\|^2 = \int d^2x e^\sigma \delta a_0 \delta \bar{a}_0.
\]
(21) \(\Delta\) denotes the Laplacian $-\bar{\partial} e^{-\sigma} \partial$ on $(0,1)$ forms. $\det'\Delta$ is the Jacobian for the change of variables $A \to \Lambda$, $\bar{\Lambda}$. The action $I$ is written as
\[
I = \frac{k}{\pi} \int d^2x \{ \partial \phi \bar{\partial} \phi - \phi \partial \bar{\partial} \sigma + e^{-2\phi - \sigma} \bar{\partial}(v + \Lambda) \bar{\partial}(\bar{v} + \bar{\Lambda}) \\
+ a_0 (e^{-2\phi} \bar{\partial}(v + \Lambda) - \sqrt{\mu} e^\sigma) + \bar{a}_0 (e^{-2\phi} \partial(\bar{v} + \bar{\Lambda}) - \sqrt{\mu} e^\sigma) \\
+ e^{-2\phi + \sigma} a_0 \bar{a}_0 \}. \]
(22) Since the functional integration measure for $v$ defined by eq.(17) is invariant under the transformation $v \to v + \Lambda$, $\bar{v} \to \bar{v} + \bar{\Lambda}$, we can factorize the $\Lambda$ integration in eq.(20), which
cancels the gauge volume $Vol.$ Eventually, one obtains the following expression of the partition function

$$Z = \int [d\phi dvda_0] det'(\Delta) e^{-I_f},$$

(23)\

$$I_f = \frac{k}{\pi} \int d^2 x \{ \partial \phi \bar{\partial} \phi - \phi \bar{\partial} \bar{\partial} \sigma + e^{-2\phi - \sigma} \bar{\partial} v \bar{\partial} \bar{v} + a_0 (e^{-2\phi} \bar{v} - \sqrt{\mu} \bar{e}^\sigma) + \bar{a}_0 (e^{-2\phi} \bar{v} - \sqrt{\mu} e^\sigma) + e^{-2\phi + \sigma} a_0 \bar{a}_0 \}.$$ 

(24)

$det'(\Delta)$ can be expressed by a ghost $c$ ( (1, 0) form ) and an antighost $b$ ( (0, 0) form ) and their complex conjugates as usual. The sum of $I_f$ and the ghost action $I_{gh} = \frac{1}{\pi} \int d^2 x (b \bar{\partial} c + \bar{b} \partial \bar{c})$ gives us a gauge fixed action with explicit moduli dependence. Locally it is possible to gauge away the moduli $a_0$ in $I_f$ to obtain eq.(18). We can construct the BRST charges

$$Q = \oint dz J_{BRST} = \oint dz c (J^+_z - k \sqrt{\mu})$$

$$\bar{Q} = \oint d\bar{z} \bar{J}_{BRST} = \oint d\bar{z} \bar{c} (J^-_{\bar{z}} - k \sqrt{\mu}),$$

(25)

where

$$J^+_z = k (e^{-2\phi - \sigma} \bar{\partial} v + a_0 e^{-2\phi})$$

$$J^-_{\bar{z}} = k (e^{-2\phi - \sigma} \bar{\partial} v + \bar{a}_0 e^{-2\phi}).$$

(26)

The functional integration over $v, a_0$ and ghosts in eq.(23) will be done using the following trick. Let us further decompose the moduli $a_0 e^\sigma$ and $\bar{a}_0 e^\sigma$ as

$$a_0 e^\sigma = \partial \bar{f} + f_0 e^{2\phi + \sigma}$$

$$\bar{a}_0 e^\sigma = \bar{\partial} f + \bar{f}_0 e^{2\phi + \sigma}.$$ 

(27)

This can be done by considering the nondegenerate bilinear form

$$\|\omega\|^2 = \int d^2 x e^{-2\phi - \sigma} \omega \bar{\omega},$$

(28)

on (1, 1) forms and a Laplacian $-\bar{\partial} \partial e^{-2\phi - \sigma}$, which is self-adjoint with respect to this bilinear form. Eqs.(27) amount to the orthogonal decomposition of the (1, 1) form $e^\sigma$ into the zero mode and nonzero modes of this Laplacian and its complex conjugate. Nonzero mode parts are written as derivatives of a (1, 0) form $f$ and (0, 1) form $\bar{f}$. The coefficients of the zero mode $e^{2\phi + \sigma}$ are

$$f_0 = a_0 \frac{\int d^2 x e^\sigma}{\int d^2 x e^{2\phi + \sigma}}, \quad \bar{f}_0 = \bar{a}_0 \frac{\int d^2 x e^\sigma}{\int d^2 x e^{2\phi + \sigma}}.$$ 

(29)
Inserting this decomposition into eq. (24), we obtain
\[ I_f = k \left( \frac{d^2 x e^\sigma}{\pi} \right)^2 \oint \left\{ \partial \bar{\phi} \partial \phi - \partial \bar{\phi} \partial \sigma + e^{-2\phi - \sigma} \bar{v}' \partial \bar{v}' - \mu e^{2 \phi + \sigma} \right\} \]
\[ + \frac{k (\int d^2 x e^\sigma)^2}{\pi \int d^2 x e^{2\phi + \sigma}} \bar{a}'_0 \bar{a}_0', \tag{30} \]
where
\[ v' = v + f, \quad a'_0 = a_0 - \sqrt{\mu} \int d^2 x e^{2\phi + \sigma} \]
\[ \bar{v}' = \bar{v} + \bar{f}, \quad \bar{a}'_0 = \bar{a}_0 - \sqrt{\mu} \int d^2 x e^{2\phi + \sigma}. \tag{31} \]

A good thing about this form of the action is that \( \bar{a}'_0 \) and \( \bar{v}' \) decouple from each other. We can do the integration over \( v \) and \( a_0 \) separately. The \( a_0 \) integration is just a simple Gaussian integration
\[ \int [da_0] \exp\left\{ -k (\int d^2 x e^\sigma)^2 a'_0 \bar{a}_0' \right\} = \text{const.} \times \frac{\int d^2 x e^{2\phi + \sigma}}{\int d^2 x e^\sigma}. \tag{32} \]
The integration over \( v \) can be evaluated by the standard anomaly calculation [11]:
\[ \int [dv] \exp\left\{ -\frac{k}{\pi} \int d^2 x e^{-2\phi - \sigma} \partial \bar{v}' \partial v' \right\} \]
\[ = \frac{\int d^2 x e^\sigma}{\int d^2 x e^{2\phi + \sigma}} \text{det}'(\Delta)^{-1} \exp\left\{ \frac{2}{\pi} \int d^2 x \partial \bar{\phi} \partial \phi - \frac{1}{\pi} \int d^2 x \partial \bar{\phi} \partial \sigma \right\}. \tag{33} \]
Putting all these together, we obtain
\[ Z = \int [d\phi dv da_0] \text{det}'(\Delta) e^{-I_f} \]
\[ = \text{const.} \int [d\phi] e^{-I_{\text{Liou.}}} \]
\[ I_{\text{Liou.}} = \frac{k - 2}{\pi} \int d^2 x \partial \bar{\phi} \partial \phi - k - 1 \int d^2 x \partial \bar{\phi} \partial \sigma + \mu' \int d^2 x e^{2\phi + \sigma}. \tag{34} \]
Eq. (34) is the partition function of Liouville theory with the cosmological constant \( \mu' = -\frac{\mu}{\pi} \).

Thus, the partition function of the \( SL(2, \mathbb{C})/SU(2) \) model coincides with that of Liouville theory.

We would like to conclude this section by several comments.

For \( k > 0 \), \( \mu \) should be negative for the functional integral to be well defined. This implies that the action in eq. (3) has an imaginary part proportional to \( \sqrt{\mu} \). It does not cause a serious problem in our analysis, because eventually the \( \sqrt{\mu} \) term is relevant only in the gaussian integration eq. (32).
The Liouville theory obtained in eq. (34) is not always relevant to two dimensional quantum gravity. In two dimensional quantum gravity the cosmological constant $\mu'$ should be coupled to the lowest dimensional operator in the matter theory dressed by gravity. Therefore, eq. (34) is relevant to quantum gravity for only special values of $k$.

It is intriguing to observe that in the above derivation, the existence of the moduli $a_0$ is essential to generate the cosmological term $\int d^2x e^{2\phi+\sigma}$. This moduli also play an essential role in the next section.

3 Extra Observables

In this section, we would like to discuss the observables and their correlation functions in the gauged $SL(2,\mathbb{C})/SU(2)$ model proposed in the previous section. In the light of its relation to Liouville theory, it is natural to expect that every observable in this model corresponds to an operator in Liouville theory. However we will find that there exist infinitely many extra observables which do not correspond to Liouville theory operators.

Let us consider the gauged $SL(2,\mathbb{C})/SU(2)$ model in the gauge fixed form eq. (24). The observables in this gauge are determined by the usual BRST procedure using the BRST charges in eq. (25). The only singular operator product expansions of $J^+_z(z)v$ and $J^-_{\bar{z}}(\bar{z})\bar{v}$ with $\phi$, $v$, and $\bar{v}$ are given by

$$J^+_z(z)v(w,\bar{w}) \sim \frac{1}{z-w},$$

$$J^-_{\bar{z}}(\bar{z})\bar{v}(w,\bar{w}) \sim \frac{\bar{v}}{\bar{z}-\bar{w}}. \quad (35)$$

Therefore, operators made out of only $\phi$ are BRST closed. It can be proved that correlation functions of such operators $V_i$ ($i=1,\ldots,n$) in the gauged $SL(2,\mathbb{C})/SU(2)$ model reduce to correlation functions in Liouville theory

$$< V_1 \cdots V_n > = \int [d\phi d\bar{\phi} da_0] det(\Delta) e^{-I_f} V_1 \cdots V_n$$

$$= \text{const.} \int [d\phi] e^{-I_{\text{Liou.}}} V_1 \cdots V_n, \quad (36)$$

following the same procedure in the previous section.

The operators of the form $e^{-2l\phi}$ are important in the application of Liouville theory to two dimensional quantum gravity. In the gauged $SL(2,\mathbb{C})/SU(2)$ model such operators are highest weight operators of the $SL(2,\mathbb{R})$ current algebra:

$$J^+_z(z)e^{-2l\phi}(w) \sim \text{regular},$$

$$J^0_z(z)e^{-2l\phi}(w) \sim \frac{l}{z-w} e^{-2l\phi}. \quad (37)$$
The conformal weights of such highest weight operators are \(-\frac{(l+1)}{k-2} - l\) which of course coincide with the values evaluated in Liouville theory. When \(l = -1\), the conformal weight is \((1, 1)\), which is consistent with the fact that this operator corresponds to the volume element in quantum gravity.

Therefore, every operator in Liouville theory can be represented as a BRST invariant operator in the gauged \(SL(2,\mathbb{C})/SU(2)\) model. If such an operator is a null observable in the gauged \(SL(2,\mathbb{C})/SU(2)\) model, it will decouple from the other operators in Liouville theory as can be seen in eq.(36). Hence, if all the observables in the gauged \(SL(2,\mathbb{C})/SU(2)\) model are made out of \(\phi\), we can have a complete correspondence between the nontrivial operators in Liouville theory and the observables in the gauged \(SL(2,\mathbb{C})/SU(2)\) model. However, there exist observables which consist not only of \(\phi\) but also of other fields in the gauged \(SL(2,\mathbb{C})/SU(2)\) model.

We can construct such extra observables starting from the following observation. The ghost fields \(b, c\) in the gauge fixed action are used to express the determinant \(\det'(\Delta)\) (on the sphere, for example,)

\[
\det'(\Delta) = \int \langle dbdc\rangle \bar{b}b(z_0)e^{-I_{gh}}. \tag{38}
\]

Here, a pair of antighosts is inserted to soak up the zero mode in the ghost path integral. The insertion point \(z_0\) can be taken arbitrarily. Since the antighosts are \((0, 0)\) forms, they have one zero mode on a surface of any genus. Such a zero mode does not appear in the action \(I_{gh}\) or in the BRST charge eq.(25). Therefore we eliminate it from the theory by inserting \(\bar{b}b\) as above. This situation is analogous to the treatment of the \(\xi\) zero mode of the superghost bosonization in superstring theory\cite{12}. This leads an analogue of “picture changing”

\[
O \rightarrow \{Q, bO\}, \tag{39}
\]

for the left moving sector along with the right moving one. By this operation, we can construct a new observable \(\{Q, bO\}\) from an observable \(O\), if \(O\) contains no \(c\). The new observable \(\{Q, bO\}\) is in the form of a BRST exact operator. However since it is an anticommutator of BRST operator with an operator including the antighost zero mode, it does not necessarily decouple from the other observables as in the case of the picture changing in superstring theory.

Notice that this picture changing operation does not change the conformal weight of the observable, because \(b\) is a \((0, 0)\) form field and \(Q\) commutes with the Virasoro operators. Therefore we obtain a new observable with the same conformal weight by this operation. In superstring theory, the picture changing operation generates infinitely many equivalent expressions of one observable. However, as we will see, in our case the picture changing operation generates an infinite number of distinct observables. By applying this picture changing operation to the observables like \(e^{-2l\phi}\), we can obtain infinitely many observables which contain not only \(\phi\) but also \(v\)'s and \(a_0\)'s. We are not sure if the observables constructed in such a way exhaust the observables of our model.

\footnote{To be precise, this suggests that \(e^{-l(2\phi+\sigma)}\) is a \((-\frac{l(l+1)}{k-2} - l, -\frac{l(l+1)}{k-2} - l)\) form.}
Let us consider the correlation functions of such observables on the sphere:

\[
<V_1 \cdots V_N> = \int [d\phi dv da_0 db dc] e^{-I_f - I_{gh} \bar{b}\bar{b}(z_0)} V_1 \cdots V_N
\]  

Such correlation functions can be calculated as follows. As in the previous section, it is convenient to rewrite everything in terms of \(a'_0\) and \(v'\) in eq.(31). While \(a'_0\) and \(v'\) decouple from each other in the action,

\[
I_f = \frac{k}{\pi} \int d^2x \{ \partial \phi \bar{\partial} \phi - \phi \partial \bar{\partial} \sigma + e^{-2\phi - \sigma} \bar{\partial} v' \partial v' - \mu e^{2\phi + \sigma} \} + \frac{k (\int d^2xe^\sigma)^2}{\pi} \int d^2xe^{2\phi + \sigma} a'_0 \bar{a}'_0, \quad (41)
\]

there appears the nontrivial interaction term \(\int d^2xe^{2\phi + \sigma}\). This term can be taken care of by the method employed in \([13][14]\), namely, by integrating over the \(\phi\) zero mode first. Since the \(\phi\) zero mode is coupled to \(v'\) and \(a'_0\) in our model, we will proceed as follows. Let us introduce a spacetime independent integration variable \(\phi_0\) and couple it to \(I_f\) as

\[
I'_f = \frac{k}{\pi} \int d^2x \{ \partial \phi \bar{\partial} \phi - (\phi + \phi_0) \partial \bar{\partial} \sigma + e^{-2\phi - \sigma} \bar{\partial} v' \partial v' - \mu e^{2\phi + 2\phi_0 + \sigma} \} + \frac{k (\int d^2xe^\sigma)^2}{\pi} \int d^2xe^{2\phi + \sigma} a'_0 \bar{a}'_0 - 2\phi_0. \quad (42)
\]

The transformation \(\delta \phi = \epsilon, \; \delta v = \epsilon v, \; \delta \bar{v} = \epsilon \bar{v}, \; \delta a'_0 = \epsilon a'_0, \; \delta \bar{a}'_0 = \epsilon \bar{a}'_0, \; \delta \phi_0 = -\epsilon\) leaves

\[
[d\phi dv da_0 d\phi_0] e^{-I'_f}
\]  

invariant. Suppose there exists \(\alpha_i\) such that \(e^{-2\alpha_i \phi_0} V_i\) is invariant under the above transformation. Then the correlation function can be written as

\[
<V_1 \cdots V_N> = \int [d\phi dv da_0 d\phi_0 db dc] \frac{e^{-I_f - I_{gh} \bar{b}\bar{b}(z_0)}}{V} \prod_i e^{-2\alpha_i \phi_0} V_i, \quad (44)
\]

where \(V\) is the volume of the above continuous symmetry of the path integral. This formula is easily proved, if one fixes the symmetry by setting \(\phi_0 = 0\). Here we will fix the symmetry by the condition \(\int d^2xe^\sigma \phi = 0\), which kills the zero mode of \(\phi\). \(\phi_0\) plays the role of the \(\phi\) zero mode. Then, after integrating over \(\phi_0\), one obtains the following expression for the correlation function

\[
<V_1 \cdots V_N> = \text{const.} \times (\mu')^{k-1+\Sigma\alpha_i} \Gamma(-(k-1+\Sigma\alpha_i)) 
\]

\[
\times \int [d\phi dv da_0 db dc] \delta(\int e^\sigma \phi) e^{-I_0 - I_{gh} \bar{b}\bar{b}(z_0)} V_1 \cdots V_N (\int d^2xe^{2\phi + \sigma})^{k-1+\Sigma\alpha_i}, \quad (45)
\]

where

\[
I_0 = \frac{k}{\pi} \int d^2x \{ \partial \phi \bar{\partial} \phi - \phi \partial \bar{\partial} \sigma + e^{-2\phi - \sigma} \bar{\partial} v' \partial v' \} + \frac{k (\int d^2xe^\sigma)^2}{\pi} \int d^2xe^{2\phi + \sigma} a'_0 \bar{a}'_0. \quad (46)
\]
The $\alpha_i$ are the analogs of the scaling dimensions in Liouville theory. Since

$$Q = k \int c(e^{-2\phi-\sigma} \partial \bar{v} + \frac{\int d^2x e^{\sigma}}{\int d^2x e^{2\phi+\sigma} \bar{a}_0'})$$

the picture changing operation eq.(39) changes the scaling dimension $\alpha$ by $-\frac{1}{2}$. The scaling dimension is one of the physical quantum numbers in our theory, on which the correlation functions crucially depend. Therefore the picture changing operation produces an infinite number of distinct observables.

If $k-1+\Sigma\alpha_i$ is not a positive integer, eq.(48) is not well defined. One needs the analytic continuation as was done in [14] to define it. Since we are not sure if there exist any justifications for that in our case, we will restrict ourselves to the case when $k-1+\Sigma\alpha_i$ is a positive integer. If $k-1+\Sigma\alpha_i$ is a positive integer, the factor $\Gamma(-(k-1+\Sigma\alpha_i))$ is divergent. This divergence comes from the volume of $\phi_0$. We will replace $(\mu')^{k-1+\Sigma\alpha_i}\Gamma(-(k-1+\Sigma\alpha_i))$ by $\log \frac{1}{\mu}$ and interpret $\log \frac{1}{\mu}$ as the volume of $\phi_0$ as was suggested in [15].

In principle one can compute any correlation function of observables by performing the functional integral in eq.(45), which amounts to successive Gaussian integrals. Here we will concentrate on the following extra observables. Starting from $O_0 = e^{2\phi+\sigma}$, let us define $O_n$ inductively as

$$O_{n+1}(w, \bar{w}) = \frac{1}{k^n} \{Q, [Q, \bar{b}bO_n(w, \bar{w})]\}$$

$$= \frac{1}{k^n} \int d\bar{z} J_{\text{BRST}} \int dw J_{\text{BRST}} \bar{b}bO_n(w, \bar{w}).$$

(48)

$O_n \propto (J_{\bar{z}} - k\sqrt{\bar{m}})^n (J_z - k\sqrt{m})^n e^{2\phi+\sigma}$ and $O_n$ does not contain $c$. Therefore eq.(48) is a well defined picture changing operation. Since the conformal weight of $O_0 = e^{2\phi+\sigma}$ is $(1,1)$, all of the $O_n$ are $(1,1)$ operators. In the rest of this section, we will show that it is possible to compute explicitly correlation functions of $\sigma_n = \int d^2x O_n$,

$$<\sigma_{n_1} \cdots \sigma_{n_N}> = \int [d\phi d\psi da_0 db dc] e^{-I_{1+10}\bar{b}b(z_0)} \int d^2x O_{n_1} \cdots \int d^2x O_{n_N}.$$  

(49)

One should drop the integrals of three of $O_n$’s in order to fix the $SL(2,\mathbb{C})$ invariance in the above correlation function. We are interested in these observables, because $\sigma_n$ are in a sense “descendants” of $\int d^2x e^{2\phi+\sigma}$. The area of spacetime $\int d^2x e^{2\phi+\sigma}$ is one of the most important observables, in the application of Liouville theory to two dimensional gravity. Also $\sigma_n$ can be added to the action as a marginal perturbation. Hence if one knows all the correlators of $\sigma_n$’s, one is able to solve such perturbed field theories exactly.

Since $O_0 = e^{-2\phi+\sigma}$ has scaling dimension $\alpha = -1$, $O_n$ has $\alpha = n-1$. In order for $k-1+\Sigma\alpha_i$ to be a positive integer, $k$ should be an integer. For later convenience, we will restrict $k$ to be an integer satisfying $k \geq 4$. 

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Correlation functions of $O_n$'s have a remarkable property which originates from their definition eq.(48). Namely they satisfy

$$<O_n(x)O_m(y)\cdots>=<O_{n-1}(x)O_{m+1}(y)\cdots> \quad (n>0).$$

The proof is given using the same argument as one uses for the demonstration of Bose sea equivalence in superstring theory[12]. Writing $O_n(x)=\frac{1}{k\pi}\oint d\bar{z}_{BRST}\oint J_{BRST}b\bar{b}O_{n-1}(x)$, the left hand side of the above formula becomes

$$\int [d\phi d\alpha d\beta d\alpha' d\beta'] e^{-\int_{-\infty}^{\infty} d\phi d\beta d\alpha d\beta' b\bar{b}(z_0)} \frac{1}{k\pi} \oint x_{BRST} \oint J_{BRST} b\bar{b}O_{n-1}(x)O_m(y)\cdots.$$ (51)

Since the point $z_0$ at which the antighost is inserted is arbitrary, we will take it to coincide with $y$. Then by using the BRST invariance of the other observables, we move the integration contours of BRST currents so that they surround only $y$. Thus we obtain the right hand side of eq.(50).

Eq.(50) is useful in reducing correlation functions of $\sigma_n$'s to a form in which they are easily calculated. Eq.(43) and the interpretation of the divergent gamma function suggest that the following equation between the correlation functions holds,

$$<\int d^2xO_{n_1}\cdots \int d^2xO_{n_i}> = \frac{(\mu')^{k-1+\Sigma-i}}{(k-1+\Sigma-i)!} <\int d^2xO_{n_1}\cdots \int d^2xO_{n_i} (\int d^2xO_0)^{k-1+\Sigma-i}>,$$

where $\Sigma = \sum n_i$. Eq.(50) implies

$$<\int d^2xO_{n_1}\cdots \int d^2xO_{n_i} (\int d^2xO_0)^{k-1+\Sigma-i} > = <\int \int d^2xO_{n_i} (\int d^2xO_0)^{k-1} >.$$ (53)

Therefore the computation of correlation functions of $\sigma_n$'s is reduced to that of the correlation functions of $\sigma_1$'s and $\sigma_0$'s only.

Now we are going to evaluate the right hand side of eq.(53). We should fix the $SL(2,C)$ invariance to define this correlation function. Suppose we fix the positions of three of the $O_0$'s. This is possible, since $k \geq 4$. $O_1$ has the form

$$\frac{1}{k\pi} \oint dz_{BRST} \oint dz J_{BRST} e^{2\phi+\sigma} = \frac{1}{\pi} \{Q, \bar{b}(\partial \bar{v}'+ \int d^2x e^\sigma a_0' a_0 e^{2\phi+\sigma})\}. \quad (54)$$

Hence $\int d^2xO_1$ is written as

$$\int d^2xO_1 = \frac{1}{\pi} \{Q, \int d^2x \bar{b}\partial \bar{v}'\} + \frac{k}{\pi} (\int d^2x e^\sigma)^2 \int d^2x e^{2\phi+\sigma} a_0' a_0.$$ (55)

The first term is a commutator of the BRST operator and an operator which does not contain the antighost zero mode. It decouples from the other observables in the correlation function. Hence the right hand side of eq.(53) is equal to

$$<\frac{k}{\pi} (\int d^2x e^\sigma)^2 \int d^2x e^{2\phi+\sigma} a_0' a_0 a_0' a_0>.$$ (56)
It is annoying to have $\int d^2xe^{2\phi+\sigma}$ in the denominator, but they all disappear after the $a_0$ integration. The integration over $a_0$ is easily done. We find

$$< \left( \frac{k}{\pi} \int d^2xe^{2\phi+\sigma} a_0 a_0' \right)^{\Sigma} \left( \int d^2xe^{2\phi+\sigma} \right)^{k-1} > = \Sigma! \left( \int d^2xe^{2\phi+\sigma} \right)^{k-1} .$$

(57)

Using eq.(52), one finally obtains

$$< \sigma_{n_1} \cdots \sigma_{n_l} > = \left( \frac{k}{\pi} \int d^2xe^{2\phi+\sigma} a_0 a_0' \right)^{\Sigma-l} \frac{\Sigma!(k-1)!}{(k-1+\Sigma-l)!} Z .$$

(58)

Since the $\sigma_n$'s can be added to the original action as a perturbation, this result makes it possible to calculate various quantities exactly in such a perturbed theory.

The $k=4$ case is extremely interesting. Since $Z \propto \left( -\mu' \right)^3 \log \frac{1}{\mu'}$,

$$< \sigma_{n_1} \cdots \sigma_{n_l} > \sim \left( -\mu' \right)^{3+\Sigma-l} \frac{\Sigma!}{(3+\Sigma-l)!} \log \frac{1}{\mu'} .$$

(59)

Up to a constant and the Liouville volume $\log \frac{1}{\mu'}$, these correlation functions precisely coincide with the correlation functions of the first critical point of one matrix model on the sphere $\Sigma$.

This coincidence is very suggestive. $k=4$ is exactly the point that our $SL(2,C)/SU(2)$ model realizes the Liouville theory which is induced by the matter theory with $c=-2$. The first critical point of the one matrix model is supposed to correspond to the quantum gravity coupled to $c=-2$ matter theory. $k=4$ is the only integer greater than three, for which the corresponding Liouville theory is relevant to two dimensional quantum gravity. For other values of $k$, $\mu'$ does not couple to $\int d^2xe^{2\phi+\sigma}$ in two dimensional quantum gravity.

Unfortunately it seems that such coincidence does not exist for the correlators of $\int d^2xO_n$ on higher genus surfaces. In general, matrix model correlators are not compatible with eq.(50).

4 Conclusions and Discussions

In this paper, we have studied the relationship between the gauged $SL(2,C)/SU(2)$ model and Liouville theory. One of the crucial points in our analysis is that there exists moduli $a_0$ on any compact Riemann surfaces. This is because the gauge field in our model is a $(1,1)$ form. The existence of the moduli $a_0$ is essential to obtain the cosmological term in Liouville theory. Then we investigated the observables in the gauged $SL(2,C)/SU(2)$ model. Although the partition function of this model coincides with that of Liouville theory, the gauged $SL(2,C)/SU(2)$ model possesses more observables than Liouville theory. The existence of the zero modes of antighosts (which is of course deeply related to the existence of $a_0$, ) was important in the construction of such observables. We have calculated the
correlators of some of these extra observables. They look quite similar to correlators in the matrix models and coincide with them when $k = 4$.

We are not sure if our results have any implications for two dimensional quantum gravity. The extra observables studied in section 3 do not exist in Liouville theory. And their correlators do not appear to coincide with the matrix model results on higher genus surfaces. Still it is possible to conceive that some modified version of the gauged $SL(2,\mathbb{C})/SU(2)$ model would reproduce the matrix model results completely and elucidate the importance of the $SL(2,\mathbb{R})$ structure in two dimensional quantum gravity.

It is straightforward to extend our analysis to the case of more general constrained WZW models. For example, $SL(N,\mathbb{R})$ WZW model with constraints similar to $SL(2,\mathbb{R})$ case was shown to be relevant to Toda field theories. Gauged WZW models based on certain super Lie groups yield super Toda field theories. In both of these cases, there exist $(1,1)$ gauge fields, and $(0,0)$ antighosts when one fixes the gauge. It is possible to consider the construction of observables by the “picture changing operation” using these antighost fields. The generalizations to these cases will be reported elsewhere.

## Appendix

In this appendix we will give the definition and some useful properties and formulas of the $SL(2,\mathbb{C})/SU(2)$ model and its gauged version.

The $SL(2,\mathbb{C})/SU(2)$ model is defined by the action

$$I = k S_{WZW}(g g^\dagger),$$

where $g \in SL(2,\mathbb{C})$ and $S_{WZW}$ is the WZW action eq.(4). This model describes the induced gauge theory which is obtained by integrating the matter part in $G/H$ gauged WZW model, when $H$ is $SU(2)$. The gauge transformation of such an induced gauge theory corresponds to

$$g \longrightarrow gh, \ h \in SU(2).$$

In order to define the functional integral, we should fix this invariance. This can be done most conveniently by taking $g$ to be

$$g = \begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^{\frac{\phi}{2}} & 0 \\ 0 & e^{-\frac{\phi}{2}} \end{pmatrix},$$

and

$$gg^\dagger = \begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^{\phi} & 0 \\ 0 & e^{-\phi} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \bar{v} & 0 \end{pmatrix}. \quad (63)$$

Here $\phi$ is real, $v$ is complex, and $\bar{v}$ is the complex conjugate of $v$. Eq.(63) is similar to the Gauss decomposition eq.(5). Contrary to the Gauss decomposition eq.(5), however, $gg^\dagger$ for
any $g \in SL(2,\mathbb{C})$ can be represented as in eq. (63). Inserting this parametrization of $g$ into eq. (60), one obtains the action,

$$I = \frac{k}{\pi} \int d^2 x \{ \partial \phi \bar{\partial} \phi + e^{-2\phi} \bar{v} v \bar{\partial} \bar{v} \}, \quad (64)$$

with $v$ and $\bar{v}$ complex conjugate to each other. The $SL(2,\mathbb{C})/SU(2)$ model is exactly described by this action.

This theory possesses $SL(2)$ chiral currents:

$$J^+_z = \frac{k}{\sqrt{2}} e^{-2\phi} \bar{v}$$
$$J^0_z = k (\partial \phi + e^{-2\phi} v \partial \bar{v})$$
$$J^-_z = \frac{k}{\sqrt{2}} (\bar{v} - 2 \partial \phi v - e^{-2\phi} v^2 \partial \bar{v}),$$

and

$$J^-_\bar{z} = \frac{k}{\sqrt{2}} e^{-2\phi} \partial \bar{v}$$
$$J^0_\bar{z} = k (\bar{\partial} \phi + e^{-2\phi} \bar{v} \partial v)$$
$$J^+_{\bar{z}} = \frac{k}{\sqrt{2}} (\partial \bar{v} - 2 \bar{\partial} \bar{v} \bar{\partial} v - e^{-2\phi} v^2 \bar{\partial} v). \quad (65)$$

These currents correspond to the transformation

$$g \rightarrow hg, \ h \in SL(2,\mathbb{C}), \quad (66)$$

and satisfy the left and right $SL(2)$ current algebras. They are important in constructing the BRST charge in $G/SU(2)$ gauged WZW models. The form of the currents in terms of $\phi$, $v$ and $\bar{v}$ are the same as the chiral $SL(2)$ currents in the $SL(2,\mathbb{R})$ WZW model. However, since $v$ and $\bar{v}$ are complex conjugate to each other, the left and right currents are related via complex conjugation in a different way in this model. The stress tensor can be written in terms of these $SL(2)$ currents in the Sugawara form.

We will define the functional integral measure for $\phi$ and $v$ by the norm

$$\int d^2 x e^\sigma \{ (\delta \phi)^2 + e^{-2\phi} \delta v \delta \bar{v} \}, \quad (67)$$

which is invariant under eq. (66). If one performs the $v$ integration first in the $\phi$ background and then do the $\phi$ integration, the functional integral is

$$\int [d\phi dv] e^{-I}, \quad (68)$$

Since this system has a global symmetry $g \rightarrow hg$, one should divide by the volume of such a global symmetry to define this functional integral.
is evaluated essentially by successive Gaussian integrals.

The $v$ integration in the $\phi$ background yields the following partition function \[11\]

$$
\int [dv] \exp\left\{- \frac{k}{\pi} \int d^2 x e^{-2\phi} \partial \bar{v} \partial v\right\} = \left(\det' (\Delta) \int d^2 x e^{\sigma}\right)^{-1} \exp\left\{ \frac{2}{\pi} \int d^2 x \partial \bar{\phi} \partial \phi + \frac{1}{\pi} \int d^2 x \phi \partial \bar{\phi} \partial \sigma\right\}. \quad (69)
$$

where the measure $[dv]$ is defined by the norm $\|\delta v\|^2 = \int d^2 x e^{-2\phi+\sigma} \delta v \delta \bar{v}$. Correlation functions of $v$'s are calculable using Wick’s theorem. After the $v$ integration, the functional integral eq.(68) becomes a theory of boson $\phi$ with Feigin-Fuchs type action. Hence, in principle, one can calculate any correlation function in this model. Indeed, computation of some of the correlation functions in this model in this way was done in \[11\], and the results were consistent with the Knizhnik-Zamolodchikov equations derived from the $SL(2)$ current algebras satisfied by eq.(65).

We can gauge $SL(2,\mathbb{C})/SU(2)$ model without any problem in a similar way to the $SL(2,\mathbb{R})$ case in eq.(3). We obtain exactly the action eq.(7) with $v$ and $\bar{v}$ complex conjugate to each other. Also one can twist the $SL(2,\mathbb{C})/SU(2)$ model as was done in section two. In this case, the action should be modified as follows

$$
I = \frac{k}{\pi} \int d^2 x \{ \partial \phi \partial \bar{\phi} - \phi \partial \bar{\phi} \partial \sigma + e^{-2\phi-\sigma} \partial \bar{v} \partial \bar{v}\}, \quad (70)
$$

where now $v$ ($\bar{v}$) is a $(1,0)$ ($\bar{v},1)$ form. The $SL(2)$ currents in eq.(65) with a little modification (changing all $\phi$’s in the expression to $\phi + \frac{1}{2} \sigma$) still satisfy the $SL(2)$ Kac Moody algebra, after the twisting. The Virasoro generators are written in terms of these currents as in eq.(14). The twisted $SL(2,\mathbb{C})/SU(2)$ model is also solvable by successive functional Gaussian integration in the same way as in the untwisted case.

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