\( \mathcal{N} = 5 \) three-algebras and 5-graded Lie superalgebras

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Abstract

We discuss a generalization of \( \mathcal{N} = 6 \) three-algebras to \( \mathcal{N} = 5 \) three-algebras in connection to anti-Lie triple systems and basic Lie superalgebras of type II. We then show that the structure constants defined in anti-Lie triple systems agree with those of \( \mathcal{N} = 5 \) superconformal theories in three dimensions.
1 Introduction and summary

Since the pioneering work by Bagger and Lambert [1], and Gustavsson [2] (BLG), where three-algebras were used to construct the $\mathcal{N}=8$ superconformal theory in three dimensions, such structures have played an important rôles toward a better understanding of M-theory. This $\mathcal{N}=8$ theory allows only an SO(4) gauge symmetry. More general gauge groups, $\text{SU}(n) \times \text{SU}(n)$ and $\text{U}(n) \times \text{U}(n)$, were soon considered by Aharony, Bergman, Jafferis and Maldacena (ABJM) [3] in theories with $\mathcal{N}=6$ supersymmetries. The corresponding three-algebra construction was found in [4].

In [5] Hosomichi, Lee, Lee, Lee and Park constructed a superconformal three-dimensional theory with $\mathcal{N}=5$ supersymmetry and $\text{Sp}(2n) \times \text{O}(m)$ gauge symmetry. (Similar theories were constructed in [6].) It was described by embedding the corresponding Lie algebra into the Lie superalgebra $B(m, n) = \mathfrak{osp}(2m + 1|2n)$ or $D(m, n) = \mathfrak{osp}(2m|2n)$. When the same construction is applied to the Lie superalgebras $A(m, n) = \mathfrak{sl}(m + 1|n + 1)$ and $C(n + 1) = \mathfrak{osp}(2|2n)$, the supersymmetry enhances from $\mathcal{N}=5$ to $\mathcal{N}=6$ [5]. In the case of $A(m, n)$ this is the ABJM theory. Additional $\mathcal{N}=5$ theories, based on the exceptional Lie superalgebras, $F(4)$, $G(3)$ and $D(2,1;\alpha)$, were found in [7] by use of the embedding tensor approach.

The connection between Lie superalgebras and superconformal theories was first noticed in [8]. The odd part determines the representations for matter fields and the even part corresponds to the gauge group. It turned out that two important types of Lie superalgebras, called basic Lie superalgebras of type I and type II, are relevant for theories with $\mathcal{N}=6$ and $\mathcal{N}=5$ supersymmetry, respectively. Indeed, among the Lie superalgebras mentioned above, $A(m, n)$ and $C(n + 1)$ are the basic ones of type I, whereas $B(m, n)$, $D(m, n)$, $F(4)$, $G(3)$ and $D(2,1;\alpha)$, are those of type II. The connection between Lie superalgebras and superconformal theories has also been discussed in [9].

It was shown in [10] that basic Lie superalgebras of type I are in one-to-one correspondence with simple $\mathcal{N}=6$ three-algebras. These two algebraic structures were related by generalized Jordan triple systems, which were also considered in [11]. In an alternative approach [12] it was shown in [13] that basic Lie superalgebras are in one-to-one correspondence with simple quaternionic anti-Lie triple systems. The most general $\mathcal{N}=5$ theory was constructed from such a triple system in [14] (in $\mathcal{N}=1$ superfield language), and from a ‘symplectic three-algebra’ in [15]. Studying $\mathcal{N}=5$ theories based on an arbitrary triple system, Bagger and Bruhn determined in [16] the conditions that this triple system must satisfy. The resulting conditions are the same
as those defining a ‘symplectic three-algebra’ in [15]. Furthermore, this is indeed an anti-Lie triple system.

In this note, we generalize the \( N = 6 \) three-algebras defined in [10] to \( N = 5 \) three-algebras. We show that the \( N = 5 \) theories can be formulated in terms of anti-Lie triple systems which are obtained from basic Lie superalgebras and thus related to \( N = 5 \) three-algebras. We explain how the \( N = 5 \) three-algebras in turn lead to \( N = 6, 8 \) three-algebras by imposing further conditions in the definition. After reviewing the construction in [16] of \( N = 5 \) theories, we study the possible representations of the structure constants in section 5 and show that the representations agree with [7].

2 Graded Lie superalgebras

In this section we review some facts from the theory of Lie superalgebras. For details, we refer to [17, 18]. Any Lie superalgebra \( G \) can be written as a direct sum of two subspaces \( G_{(0)} \) and \( G_{(1)} \) (the even and odd part) such that

\[
\{ G_{(p)}, G_{(q)} \} \subseteq G_{(p+q)},
\]

(2.1)

where the subscripts are counted mod 2. A Lie superalgebra \( G \) may also be written as a sum of subspaces \( G_k \) for any integer \( k \), such that \( G_k \subseteq G_{(0)} \) if \( k \) is even and \( G_k \subseteq G_{(1)} \) if \( k \) is odd. Then \( G \) is said to have a consistent \( \mathbb{Z} \)-grading. When we henceforth talk about a 3-graded or 5-graded Lie superalgebra we refer to a consistent \( \mathbb{Z} \)-grading such that \( G_k = 0 \) for \( |k| \geq 2 \) or \( |k| \geq 3 \), respectively.

It follows from (2.1) that \( G_{(0)} \) is a subalgebra (which is an ordinary Lie algebra) and that \( G_{(1)} \) is a representation of \( G_{(0)} \). The Lie superalgebra \( G \) is said to be classical if this representation is completely reducible. Then there are two cases, which divide the classical Lie superalgebras into two types: type I and type II. The representation of \( G_{(0)} \) on \( G_{(1)} \) is either a direct sum of two irreducible representations (type I), or irreducible (type II). A classical Lie superalgebra \( G \) is said to be basic if it admits a non-degenerate bilinear form \( \kappa \) that is invariant, which means

\[
\kappa([x, y], z) = \kappa(x, [y, z])
\]

(2.2)

for all \( x, y, z \in G \). This bilinear form will furthermore satisfy

\[
\kappa(x, y) = \frac{1}{2}((-1)^p + (-1)^q) \kappa(y, x),
\]

(2.3)

for all \( x \in G_{(p)} \) and \( y \in G_{(q)} \). Thus it satisfies the requirements for an inner product [18].

We will occasionally write \( \kappa(x, y) = \langle x|y \rangle \).
Any basic Lie superalgebra admits a 3-grading if it is of type I, and a 5-grading if it is of type II. The inner product is such that $\kappa(x, y)$, where $x \in \mathcal{G}_i$ and $y \in \mathcal{G}_j$, is nonzero only if $i + j = 0$. There is also an antilinear map $\tau: \mathcal{G}_k \to \mathcal{G}_{-k}$ such that $\tau^2(x) = (-1)^p x$ for any $x \in \mathcal{G}_{(p)}$, and

$$\left[ \tau(x), \tau(y) \right] = \tau(\left[ x, y \right])$$

(2.4)

for any $x, y \in \mathcal{G}$. We call such a map $\tau$ a graded superconjugation. The antilinearity of $\tau$ means that $\tau(\alpha x) = \alpha^* \tau(x)$ if $\alpha^*$ is the conjugate of a complex number $\alpha$.

Let $\mathcal{G}$ be a basic Lie superalgebra with inner product $\kappa$ and graded superconjugation $\tau$. Let $M^m$ be a basis for $\mathcal{G}_{(0)}$ and $Q^a$ a basis for $\mathcal{G}_{(1)}$. Then we write

$$k^{mn} = k^{nm} = \kappa(M^m, M^n), \quad \omega^{ab} = -\omega^{ba} = \kappa(Q^a, Q^b).$$

(2.5)

Let $k_{mn}$ and $\omega_{ab}$ be the inverses of $k^{mn}$ and $\omega^{ab}$,

$$k_{mp} k_{pn} = \delta^m_n, \quad \omega^{ac} \omega_{cb} = -\delta^a_b.$$ 

(2.6)

We use these tensors to raise and lower indices (with the convention $X^a = \omega^{ab} X_b$ and $X_a = -\omega_{ab} X^b$). Now there are structure constants $(t_m)^{ab}$ and $(f_m)^{np}$ such that

$$[M^m, M^n] = (f_m)^{np} M^p, \quad [M^m, Q^a] = (t_m)^a_b Q^b, \quad \{Q^a, Q^b\} = (t_m)^{ab} M^m.$$ 

(2.7)

We will use these structure constants in the next section to construct the structure constants of an anti-Lie triple system.

### 3 Three-algebras and triple systems

With a triple system we here simply mean a complex vector space $V$ with a triple product $f: V \times V \times V \to V$ that is linear or antilinear in each argument. By imposing further conditions, one obtains different kinds of triple systems, some of which are called ‘three-algebras’.

The original notion of a three-algebra [1] was generalized in [4]. In [10] these triple systems were called $\mathcal{N} = 8$ three-algebras and $\mathcal{N} = 6$ three-algebras, respectively. We will follow the terminology in this note, but also generalize the notion further to triple systems that we call $\mathcal{N} = 5$ three-algebras.

#### 3.1 Three-algebras

An $\mathcal{N} = 5$ three-algebra is a triple system $V$ with a triple product $f: V \times V \times V \to V$ and an ‘inner product’ $h: V \times V \to \mathbb{C}$, such that
(i) the triple product \((xyz) \equiv f(x, y, z)\) is linear in \(x\) and \(z\) but antilinear in \(y\):

\[
\alpha(xyz) = ((\alpha x)y)z = (x(\alpha^* y)z) = (xy(\alpha z))
\]

for any complex number \(\alpha\) (where \(*\) is the complex conjugate),

(ii) the triple product satisfies

\[
(uv(xyz)) = ((uvx)yz) - (x(vuy)z) + (xy(uvz)),
\]

\[
K_{xy}(K_{uv}(z)) = (K_{xy}(v)uz) + (K_{xy}(u)vz),
\]

where \(K_{xy}(z) = (xzy) + (yzx)\),

(iii) the inner product \(\langle x, y \rangle \equiv h(x, y)\) is linear in \(x\) and antilinear in \(y\),

(iv) the inner product satisfies

\[
\langle w, (xyz) \rangle = \langle (ywx), x \rangle = \langle (wyx), z \rangle = \langle (yxw), z \rangle,
\]

\[
\langle x, y \rangle = \langle y, x \rangle^*,
\]

(v) the inner product is positive-definite.

By imposing further conditions one obtains \(\mathcal{N} = 6\) and \(\mathcal{N} = 8\) three-algebras. An \(\mathcal{N} = 6\) three-algebra is an \(\mathcal{N} = 5\) three-algebra with \(K_{xy} = 0\) for any \(x, y\). This means that the triple product is antisymmetric in the first and third arguments,

\[
(xy) = -(yzx).
\]

Thus (3.3) is trivially satisfied. An \(\mathcal{N} = 8\) three-algebra is an \(\mathcal{N} = 6\) three-algebra \(V\) with a conjugation \(C\) (an antilinear involution) such that the triple product satisfies

\[
(xC(y)z) = -(yC(x)z)
\]

and the inner product \(h\) is real. This implies that the triple product is totally antisymmetric and that the inner product is symmetric.

3.2 Anti-Lie triple systems

An anti-Lie triple system is a triple system with a triple product \([xyz]\) that is trilinear and satisfies

\[
[uv|xyz|] - [xy|uvz|] = [[uv|x]yz] + [x|uvy]z],
\]

\[
[xyz] = [yhx],
\]

\[
[xyz] + [yhx] + [zxh] = 0.
\]
With the opposite sign in (3.8) we would get a Lie triple system instead.

The anti-Lie triple systems in [15,16], furthermore, have a bilinear form such that

$$\langle w, [xyz] \rangle = \langle y, [zwx] \rangle,$$
$$\langle x, y \rangle = -\langle y, x \rangle.$$  \hspace{1cm} (3.10)

This is also true for the quaternionic anti-Lie triple systems considered in [13], but these have in addition a ‘quaternionic’ structure, which is a vector space automorphism \( J \) such that \( J^2 = -1 \) and

$$[J(x)J(y)J(z)] = J([xyz]).$$  \hspace{1cm} (3.11)

### 3.3 Connection to Lie superalgebras

For any Lie superalgebra \( G \), the odd subspace \( G_{(1)} \) is a triple system under the triple product

$$[XYZ] = \{\{X,Y\},Z\},$$  \hspace{1cm} (3.12)

where \( X,Y,Z \in G_{(1)} \). The general properties that such a triple product satisfies (by the Jacobi superidentity and the symmetries of the superbracket) are exactly those that define an anti-Lie triple system (in the same way as an ordinary Lie algebra leads to a Lie triple system). The structure constants can be obtained from (2.7),

$$g^{abcd} \equiv \kappa([\{Q^a, Q^b\},Q^c],Q^d) = (t_m)^{ab}(\ell_m)^{cd}.$$  \hspace{1cm} (3.13)

Conversely, any anti-Lie triple system gives rise to a Lie superalgebra \( G \) [19].

In the case of a basic Lie superalgebra \( G \) with a 3- or 5-grading, the graded superconjugation \( \tau \) becomes a quaternionic structure \( J \) on the anti-Lie triple system \( G_{(1)} \). We can use \( \tau \) to decompose each element \( X \in G_{(1)} \) into a sum \( X = x + \tau(y) \), where \( x,y \in G_{-1} \). Since \( G \) is either 3- or 5-graded, we have

$$[\{x,y\},z] = [\{\tau(x),\tau(y)\},\tau(z)] = 0$$  \hspace{1cm} (3.14)

for any \( x,y,z \in G_{-1} \). Then, by use of the Jacobi superidentity, any triple product (3.12), where \( X,Y,Z \in G_{(1)} \), decomposes into a sum of triple products

$$(xyz) = \{\{x,\tau(y)\},z\},$$  \hspace{1cm} (3.15)

where \( x,y,z \in G_{-1} \). It is straightforward to verify that \( G_{-1} \) now satisfies the definition above of an \( N = 5 \) three-algebra with the triple product (3.15) and the inner product

$$h(x,y) = \kappa(x,\tau(y)).$$  \hspace{1cm} (3.16)
Furthermore, when $G$ is 3-graded, the $\mathcal{N}=5$ three-algebra reduces to an $\mathcal{N}=6$ three-algebra. We have thus shown that an anti-Lie triple system obtained from a basic Lie superalgebra gives rise to an $\mathcal{N}=5$ three-algebra.

In this note we show that the anti-Lie triple systems that have been used in $\mathcal{N}=5$ theories can be obtained from basic Lie superalgebras of type II. Thus they give rise to $\mathcal{N}=5$ three-algebras that could be used instead of anti-Lie triple systems. Since the $\mathcal{N}=5$ three-algebras, unlike anti-Lie triple systems, are generalizations of the $\mathcal{N}=6$ three-algebras, they are (in our opinion) more natural to use in the construction of $\mathcal{N}=5$ theories. These could then be unified with the $\mathcal{N}=6$ theories in a way similar to the approach in [15]. Such a construction was performed in [20] showing explicitly that $\mathcal{N}=5$ three-algebras indeed lead to $\mathcal{N}=5$ theories.

4 Construction of the $\mathcal{N}=5$ theory based on anti-Lie triple systems

Here we review the construction on $\mathcal{N}=5$ theories by Bagger and Bruhn [16]. Keeping global symmetry to be $\text{Sp}(4)$ R-symmetry, they found non-trivial supersymmetry transformations\(^1\)

$$\begin{align*}
\delta Z^{Ad} &= i \xi^{AD} \Psi_D^a, \\
\delta \Psi_D^a &= \xi_{AD} \Psi_D^a + \frac{1}{2} g^{abcd} Z_a^A Z_b^B Z_c^C \xi_{DC} \omega_{AB} + g^{acbd} Z_a^A Z_b^B Z_c^C \xi_{AB} \omega_{DC}, \\
\delta A_{\mu}^a &= \frac{3i}{2} g^{bca} \omega^{BE} \xi_{EC} \gamma_\mu \Psi_B \Psi_D^a Z_C^c,
\end{align*}$$

(4.1)

where $Z^{Ad}, \Psi_D^a$ are the scalars and fermions, respectively, transforming in the bifundamental representation of the gauge group, and $A_{\mu}^a$ denote the gauge fields. The capital letters $A, B, \ldots$ are for $\mathfrak{sp}(4)$ R-symmetry indices, and lower case letters $a, b, \ldots$ for gauge group indices. The $\mathfrak{sp}(4)$ invariant tensor $\omega_{AB}$ satisfying $\omega^{AB} \omega_{BC} = -\delta^{AC}$ is used to raise and lower the indices, e.g., $X^A = \omega^{AB} X_B$ and $X_A = -\omega_{AB} X^B$. In the same way, the gauge indices are raised and lowered by an invariant tensor $J^{ab} J_{bc} = -\delta^a_c$. The structure constants

$$g^{abcd} = g^{bacd} = g^{cdab}$$

(4.2)

satisfy the cyclicity condition

$$g^{(abc)d} = 0.$$  

(4.3)

\(^1\)It is not difficult to see that the supersymmetry transformations (4.1) are equivalent to those in [5] by introducing an embedding tensor $\frac{4\pi}{3} (t_m)^{ab} (t^m)^{cd} = f^{abcd}$. 

7
and the fundamental identity
\[ g^{abhe} g_e^{fcd} + g^{abfe} g_e^{hcd} + g^{abce} g_e^{dhf} + g^{abde} g_e^{chf} = 0, \] (4.4)
which is of the same form as that for \( \mathcal{N} = 6 \) theories.

The supersymmetry algebras close on a translation and a gauge transformation. For instance, the supersymmetry transformations on the scalars are given by
\[ [\delta_1, \delta_2] Z^A_d = \frac{i}{2} \xi_2^BC \gamma^\mu \xi_1 BC D\mu Z^A_d + \delta_\Lambda Z^A_d, \] (4.5)
where \( \xi_{BC} \) are real antisymmetric supersymmetry transformation parameters and the gauge transformation is given by
\[ \delta_\Lambda Z^A_d = \Lambda_a^A Z^A_d = -\frac{3i}{4} \xi_2^{DF} \xi_1 BF Z^B_d Z^C_d \omega_{DC} g^{bca}_{\text{def}} Z^A_d. \] (4.6)

The representations of \( g^{abcd} \) are characterized by \( \mathfrak{so}(m) \) and \( \mathfrak{sp}(2n) \) algebras. Then the combinations that satisfy (4.3) and (4.4) are
\[ g^{aibjckdl} = (\delta^{ac} \delta^{bd} - \delta^{ad} \delta^{bc}) J^{ij} J^{kl} - (J^{ik} J^{jl} + J^{jk} J^{il}) \delta^{ab} \delta^{cd}, \] (4.7)
\[ g^{aibjckdl} = \delta^{ac} \delta^{bd} J^{ij} J^{kl} + \delta^{ad} \delta^{bc} J^{jk} J^{il}, \] (4.8)
where \( a, b, \ldots \) denote the indices for \( \mathfrak{so}(m) \) and \( i, j, \ldots \) for \( \mathfrak{sp}(2n) \). The first combination (4.7) leads to \( \mathfrak{sp}(2n) \oplus \mathfrak{so}(m) \) transformations
\[ \delta Z^{Adl} = -\frac{3i}{2} \xi_2^{DF} \xi_1 BF \omega_{DC} \left( Z^B_k Z^C_l Z^A_d + Z^B_d Z^C_l Z^A_k \right). \] (4.9)
The second combination (4.8) leads to \( \mathfrak{sp}(2mn) \) transformations
\[ \delta Z^{Adl} = 3i \xi_2^{DF} \xi_1 BF \omega_{DC} Z^B_k Z^C_l Z^{Adl} Z^A_k = \Lambda^{kl}_{bk} Z^A_k, \] (4.10)
since \( \Lambda^{kl}_{bk} = \Lambda^{kl}_{bk} \).

We introduce a map [4] \( g: \mathfrak{sp}(N) \rightarrow \mathfrak{sp}(N) \)
\[ g(\lambda)^a_d = \lambda_{bc} g^{bca}_{\text{def}} d, \] (4.11)
it then follows from the fundamental identity that
\[ [g(\lambda_1), g(\lambda_2)] = g(\lambda_3), \] (4.12)
where \( \lambda_{3bc} = -f(\lambda_1)^{e_c} \lambda_{2be} - f(\lambda_1)^{e_b} \lambda_{2ce} \). This means that the gauge transformations act as a matrix commutator, implying that they indeed form a Lie subalgebra. In what follows, we examine how these \( \mathcal{N} = 5 \) theories can be understood from the anti-Lie triple systems.
5 Connection to anti-Lie triple systems

We now relate the three-algebra constructions to the anti-Lie triple system that we discussed in section 3.2. First we introduce the basis of the Lie algebra of the gauge transformations such that the fields take the form of \( Z^A = Z^a T^a \). Then the anti-Lie triple product is given by

\[
[T^a T^b T^c] = -g^{abc} d T^d,
\]

where, by construction, the first two indices are symmetric \( g^{abcd} = g^{bacd} \). The structure constants are given by

\[
g^{abcd} = \langle [T^a T^b T^c] | T^d \rangle.
\]

The identity (3.7) implies that the gauge transformation acts as a derivation

\[
\delta[Z^A Z^B Z^C] = [\delta Z^A Z^B Z^C] + [Z^A \delta Z^B Z^C] + [Z^A Z^B \delta Z^C] + [Z^A Z^B \delta Z^C].
\]

It is straightforward to see that (3.7) is equivalent to the fundamental identity (4.4).

5.1 Basic Lie superalgebras of type II

The Lie superalgebras \( B(m, n) \) and \( D(m, n) \), are the algebras \( \mathfrak{osp}(m|2n) \) for \( m = 1 \) and \( m = 3, 4, \ldots \). When \( m = 2 \) we have instead \( \mathfrak{osp}(2|2n) = C(n+1) \), which is a basic Lie algebra of type I, and thus associated with an \( \mathcal{N} = 6 \) three-algebra, see [10].

As a first example we consider \( \mathfrak{osp}(1|2N) \). Its even subalgebra is \( \mathfrak{sp}(2N) \), spanned by \( N(2N+1) \) symmetric generators \( M^{IJ} \), and the odd subspace is spanned by \( 2N \) generators \( Q^I \), where \( I, J = 1, \ldots, 2N \). The commutation relations read

\[
\{Q^I, Q^J\} = M^{IJ},
\]

\[
[M^{IJ}, Q^K] = \Omega^{JK} Q^I + \Omega^{IK} Q^J,
\]

\[
[M^{IJ}, M^{KL}] = \Omega^{JK} M^{IL} + \Omega^{IK} M^{JL} + \Omega^{IL} M^{JK} + \Omega^{IJ} M^{LK},
\]

where \( \Omega^{IJ} \) is the antisymmetric invariant tensor of \( \mathfrak{sp}(2N) \) satisfying

\[
\Omega^{IJ} \Omega_{JK} = -\delta^I_K.
\]

The inner product is given by \( \langle Q^I | Q^J \rangle = \Omega^{IJ} \). Using this, we find that the structure constants of the anti-Lie triple system are

\[
\langle [Q^I Q^J Q^K]|Q^L \rangle = \langle [\{Q^I, Q^J\}, Q^K]|Q^L \rangle = \langle \Omega^{JK} Q^I + \Omega^{IK} Q^J|Q^L \rangle = \Omega^{JK} \Omega^{IL} + \Omega^{IK} \Omega^{JL}.
\]
Suppose that \( N = mn \) for some integers \( m, n \geq 1 \). We can then decompose each \( \mathfrak{sp}(2N) \) index into one \( \mathfrak{so}(m) \) index \((a, b, \ldots = 1, 2, \ldots, m)\) and one \( \mathfrak{sp}(2n) \) index \((i, j, \ldots = 1, 2, \ldots, 2n)\), so that

\[
\Omega^{ij} = \Omega^{ab} = \delta^{ab} J^{ij}, \tag{5.7}
\]

where \( \delta^{ab} \) and \( J^{ij} \) are the invariant tensors of the corresponding \( \mathfrak{so}(m) \) and \( \mathfrak{sp}(2n) \) subalgebras. Under this decomposition, the structure constants \((5.6)\) become

\[
\langle [Q^a_i Q^b_j Q^c_k] | Q^{dl} \rangle = \delta^{bc} \delta^{ad} J^{jk} J^{il} + \delta^{ac} \delta^{bd} J^{ik} J^{jl}, \tag{5.8}
\]

which are the structure constants \((4.8)\) leading to the \( \mathfrak{sp}(2mn) \) gauge transformations. Now we turn to \( \mathfrak{osp}(m|2n) \) for \( m = 3, 4, \ldots \) and \( n = 1, 2, \ldots \). The even subalgebra is \( \mathfrak{so}(m) \oplus \mathfrak{sp}(2n) \), spanned by the antisymmetric generators \( M^{ab} \) and the symmetric generators \( M^{ij} \). The odd subspace is spanned by \( 2mn \) elements \( Q^{ai} \). The commutation relations read

\[
[M^{ab}, M^{cd}] = \delta^{bc} M^{ad} - \delta^{bd} M^{ac} - \delta^{ac} M^{bd} + \delta^{ad} M^{bc},
\]

\[
[M^{ij}, M^{kl}] = J^{jk} M^{il} + J^{jl} M^{ik} + J^{ik} M^{jl} + J^{il} M^{jk},
\]

\[
[M^{ab}, Q^{ai}] = \delta^{bc} Q^{ai} - \delta^{ac} Q^{bi},
\]

\[
[M^{ij}, Q^{ak}] = J^{jk} Q^{ai} + J^{ik} Q^{aj},
\]

\[
\{Q^{ai}, Q^{bj}\} = v(J^{ij} M^{ab} + \delta^{ab} M^{ij}),
\]

where \( v \) is a normalization constant. The inner product is given by \( \langle Q^{ai} | Q^{bj} \rangle = \delta^{ab} J^{ij} \), and the structure constants of the anti-Lie triple system are then

\[
\langle [Q^{ai} Q^{bj} Q^{ck}] | Q^{dl} \rangle = \langle \{Q^{ai}, Q^{bj}\}, Q^{ck} \} | Q^{dl} \rangle
\]

\[
= v(\langle J^{ij} M^{ab} + \delta^{ab} M^{ij}, Q^{ck} \rangle | Q^{dl} \rangle)
\]

\[
= v(J^{ij} J^{kl} (\delta^{bc} \delta^{ad} - \delta^{bd} \delta^{ac}) + \delta^{ab} \delta^{cd} (J^{jk} J^{il} + J^{ik} J^{jl})). \tag{5.10}
\]

For \( v = -1 \), this agrees with \((4.7)\), yielding the \( \mathfrak{so}(m) \oplus \mathfrak{sp}(2n) \) gauge transformations.

### 5.2 Exceptional Lie superalgebras

The construction of anti-Lie triple systems from Lie superalgebras can be applied to the exceptional cases \( F(4), G(3), \) and \( D(2,1;\alpha) \). We end this note by showing that the construction reproduces the structure constants discussed in [7] as well as [16].
5.2.1 \( F(4) \)

The even part of the Lie superalgebra \( F(4) \) is \( \mathfrak{su}(2) \oplus \mathfrak{so}(7) \) spanned by the \( \mathfrak{su}(2) \) generators \( S^i \) \((i = 1, 2, 3)\) and the \( \mathfrak{so}(7) \) generators \( M^{ab} \) \((a, b = 1, \ldots, 7)\). The odd part is spanned by \( Q_{\alpha m} \) where \( \alpha = +1, -1 \) and \( m = 1, \ldots, 8 \). The commutation relations [18, 21] are

\[
[S^i, S^j] = i \epsilon^{ijk} S^k, \quad [S^i, M^{ab}] = 0, \\
[S^i, Q_{\alpha m}] = \frac{1}{2} \sigma^i_{\beta\alpha} Q_{\beta m}, \quad [M^{ab}, Q_{\alpha m}] = \frac{1}{2} (\Gamma^a \Gamma^b)_{nm} Q_{\alpha n},
\]

\[
[M^{ab}, M^{cd}] = \delta^{bc} M^{ad} - \delta^{bd} M^{ac} - \delta^{ac} M^{bd} + \delta^{ad} M^{bc}, \\
\{Q_{\alpha m}, Q_{\beta n}\} = v^2 \tilde{C}_{mn} (\mathcal{C}^{(2)}_{ij} \sigma^i)_{\alpha \beta} S^j + v^2 \frac{1}{3} \tilde{C}_{ij} (\tilde{C} \Gamma^a \Gamma^b)_{mn} M^{ab}, \quad (5.11)
\]

where \( v \) is a normalization constant, \( \sigma^i \) are the Pauli matrices, \( \mathcal{C}^{(2)} = i \sigma^2 \), \( \tilde{C} \) is the \( 8 \times 8 \) charge conjugation matrix with

\[
(\tilde{C})^T = \tilde{C}, \quad (\Gamma^a)^T = -\tilde{C} \Gamma^a, \quad (5.12)
\]

and \( \Gamma^a \) are 8-dimensional gamma matrices satisfying \( \{\Gamma^a, \Gamma^b\} = 2 \delta^{ab} \). The inner product is then \( \langle Q_{\alpha m} | Q_{\beta n} \rangle = \epsilon_{\alpha \beta} \tilde{C}_{mn} \) and the structure constants are

\[
g_{m \alpha n \beta p \gamma q \delta} = \langle [\{Q_{\alpha m}, Q_{\beta n}\}, Q_{\gamma p}] | Q_{\delta q} \rangle \\
= v \left( \tilde{C}_{mn} \tilde{C}_{pq} (\epsilon_{\alpha \gamma} \epsilon_{\beta \delta} - \epsilon_{\gamma \beta} \epsilon_{\alpha \delta}) + \frac{1}{6} \epsilon_{\alpha \beta} \epsilon_{\gamma \delta} (\tilde{C} \Gamma^a \Gamma^b)_{mn} (\tilde{C} \Gamma^a \Gamma^b)_{pq} \right), \quad (5.13)
\]

where we used the completeness relations of the Pauli matrices

\[
(\sigma^i)_{\alpha \beta} (\sigma^i)_{\gamma \delta} = 2 \delta_{\alpha \delta} \delta_{\beta \gamma} - \delta_{\alpha \beta} \delta_{\gamma \delta}, \quad (5.14)
\]

and the cyclicity condition

\[
\epsilon_{\alpha \beta} \epsilon_{\gamma \delta} + \epsilon_{\beta \gamma} \epsilon_{\alpha \delta} + \epsilon_{\gamma \alpha} \epsilon_{\beta \delta} = 0. \quad (5.15)
\]

One may take \( \tilde{C}_{mn} = \delta_{mn} \) and \( v = -1/2 \) to obtain

\[
g_{m \alpha n \beta p \gamma q \delta} = \frac{1}{2} \delta_{mn} \delta_{pq} (\epsilon_{\gamma \alpha} \epsilon_{\beta \delta} + \epsilon_{\gamma \beta} \epsilon_{\alpha \delta}) + \frac{1}{12} \epsilon_{\alpha \beta} \epsilon_{\gamma \delta} (\Gamma^a \Gamma^b)_{mn} (\Gamma^a \Gamma^b)_{pq}, \quad (5.16)
\]
5.2.2 \(G(3)\)

The even part of the Lie superalgebra \(G(3)\) is \(\mathfrak{su}(2) \oplus G_2\) spanned by the \(\mathfrak{su}(2)\) generators \(S^i (i = 1, 2, 3)\) and the \(G_2\) generators \(M^{ab} (a, b = 1, \ldots, 7)\) obeying \(\xi^{abc} M^{ab} = 0\) [18, 21], where \(\xi^{abc}\) is a totally anti-symmetric \(G_2\) invariant tensor whose nonvanishing components are

\[
\xi^{123} = \xi^{145} = \xi^{176} = \xi^{246} = \xi^{257} = \xi^{347} = \xi^{365} = 1.
\] (5.17)

The odd part is spanned by \(Q^{\alpha a}\) (\(\alpha = +1, -1\)). The commutations relations are

\[
[S^i, S^j] = i \epsilon^{ijk} S^k, \quad [S^i, M^{ab}] = 0, \quad [S^i, Q^{\alpha a}] = \frac{1}{2} \sigma^{ij\alpha} Q^{\beta a},
\]

\[
[M^{ab}, M^{cd}] = 3 \delta^{bc} M^{ad} - 3 \delta^{bd} M^{ac} - 3 \delta^{ac} M^{bd} + 3 \delta^{ad} M^{bc} - \xi^{abe} \xi^{cd} M^{ef},
\]

\[
[M^{ab}, Q^{\alpha c}] = 2 \delta^{ac} Q^{\alpha b} - 2 \delta^{bc} Q^{\alpha a} - \eta^{abcd} Q^{ad},
\]

\[
\{Q^m, Q^{\beta n}\} = 2 v \delta^{ab} (C(2) \sigma^m)_{\alpha\beta} - \frac{v}{2} C(2) \alpha\beta M^{mn},
\] (5.18)

where \(C(2) = i \sigma^2\), and \(\eta^{abcd}\) is a totally antisymmetric tensor whose nonvanishing components are

\[
\eta^{1247} = \eta^{1265} = \eta^{1364} = \eta^{1375} = \eta^{2345} = \eta^{2376} = \eta^{4576} = 1,
\] (5.19)

satisfying

\[
\eta^{abcd} = \delta^{ad} \delta^{bc} - \delta^{ac} \delta^{bd} + \xi^{abe} \xi^{cde}.
\] (5.20)

The inner product is defined as \(\langle Q^{\alpha a} | Q^{\beta b} \rangle = \epsilon^{\alpha\beta} \delta^{ab}\), and the structure constants are

\[
g^{\alpha a \beta b \gamma c \delta d} = \langle \{Q^{\alpha a}, Q^{\beta b}\}, Q^{\gamma c} \rangle | Q^{\delta d} \rangle
\]

\[
= v \left( \delta^{ab} \delta^{cd} \left( \epsilon^{\alpha\gamma} \epsilon^{\beta\delta} + \epsilon^{\alpha\delta} \epsilon^{\beta\gamma} \right) - \epsilon^{\alpha\beta} \epsilon^{\gamma\delta} \left( \delta^{ac} \delta^{bd} - \delta^{ad} \delta^{bc} - \frac{1}{2} \eta^{abcd} \right) \right).
\] (5.21)

5.2.3 \(D(2, 1; \alpha)\)

The even part of the Lie superalgebra \(D(2, 1; \alpha)\) is \(\mathfrak{so}(4) \oplus \mathfrak{sp}(2)\) spanned by the antisymmetric generators \(M^{ab} (a, b = 1, \ldots, 4)\) and the symmetric generators \(M^{ij}\) \((i = 1, 2)\). The odd part is spanned by generators \(Q^{ai}\). It has a free parameter \(\alpha \neq -1, 0\) [18]. The commutation relations are of the same form as (5.9) (with \(\epsilon^{ij}\) instead of \(J^{ij}\)) except for

\[
\{Q^{ai}, Q^{bj}\} = v \left( \epsilon^{ij} (M^{ab} + \frac{\beta}{2} \epsilon^{abcd} M^{cd}) + \delta^{ab} M^{ij} \right)
\] (5.22)
where $\varepsilon^{abcd}$ is the totally antisymmetric invariant tensor of $\mathfrak{so}(4)$ and $\beta$ is a free parameter. Comparing with the construction in [18] we find that the relation between $\alpha$ and $\beta$ is

$$\frac{\beta}{2} = 1 - \frac{1}{1+\alpha}.$$  

(5.23)

It follows from the inner product $\langle Q^{ai}|Q^{bj}\rangle = \delta^{ij}\delta^{ab}$ that

$$g^{abijkl} = \langle[[Q^{ai},Q^{bj}],Q^{ck}]|Q^{dl}\rangle = v\left((\delta^{ad}\delta^{bc} - \delta^{ac}\delta^{bd} - \beta \varepsilon^{abcd})\epsilon^{ij}\epsilon^{kl} + (\epsilon^{ik}\epsilon^{jl} + \epsilon^{jk}\epsilon^{il})\delta^{ab}\delta^{cd}\right).$$  

(5.24)

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### References

[1] J. Bagger and N. Lambert, *Gauge Symmetry and Supersymmetry of Multiple M2-Branes*, Phys. Rev. D77, 065008 (2008) [arXiv:0711.0955 [hep-th]].

[2] A. Gustavsson, *Algebraic structures on parallel M2-branes*, Nucl. Phys. B811, 66–76 (2009) [arXiv:0709.1260 [hep-th]].

[3] O. Aharony, O. Bergman, D. L. Jafferis and J. Maldacena, *N=6 superconformal Chern-Simons-matter theories, M2-branes and their gravity duals*, JHEP 10, 091 (2008) [arXiv:0806.1218 [hep-th]].

[4] J. Bagger and N. Lambert, *Three-Algebras and N=6 Chern-Simons Gauge Theories*, Phys. Rev. D79, 025002 (2009) [arXiv:0807.0163 [hep-th]].

[5] K. Hosomichi, K.-M. Lee, S. Lee, S. Lee and J. Park, *N=5,6 superconformal Chern-Simons theories and M2-branes on orbifolds*, JHEP 09, 002 (2008) [arXiv:0806.4977 [hep-th]].
[6] O. Aharony, O. Bergman and D. L. Jafferis, Fractional M2-branes, JHEP 11, 043 (2008) [arXiv:0807.4924 [hep-th]].

[7] E. A. Bergshoeff, O. Hohm, D. Roest, H. Samtleben and E. Sezgin, The superconformal gaugings in three dimensions, JHEP 09, 101 (2008) [arXiv:0807.2841 [hep-th]].

[8] D. Gaiotto and E. Witten, Janus Configurations, Chern-Simons Couplings, And The Theta-Angle in N=4 Super Yang-Mills Theory, JHEP 06, 097 (2010) [arXiv:0804.2907 [hep-th]].

[9] F.-M. Chen, Y.-S. Wu, Superspace Formulation in a Three-Algebra Approach to D = 3, N = 4, 5 Superconformal Chern-Simons Matter Theories, Phys. Rev. D 82, 106012 (2010) [arXiv:1007.5157 [hep-th]].

[10] J. Palmkvist, Three-algebras, triple systems and 3-graded Lie superalgebras, J. Phys. A43, 015205 (2010) [arXiv:0905.2468 [hep-th]].

[11] B. E. W. Nilsson and J. Palmkvist, Superconformal M2-branes and generalized Jordan triple systems, Class. Quant. Grav. 26, 075007 (2009) [arXiv:0807.5134 [hep-th]].

[12] P. de Medeiros, J. Figueroa-O’Farrill, E. Mendez-Escobar and P. Ritter, On the Lie-algebraic origin of metric 3-algebras, Commun. Math. Phys. 290, 871-902 (2009). [arXiv:0809.1086 [hep-th]].

[13] J. Figueroa-O’Farrill, Simplicity in the Faulkner construction, J. Phys. A42, 445206 (2009) [arXiv:0905.4900 [hep-th]].

[14] P. de Medeiros, J. Figueroa-O’Farrill, E. Mendez-Escobar, Superpotentials for superconformal Chern-Simons theories from representation theory, J. Phys. A A42, 485204 (2009). [arXiv:0908.2125 [hep-th]].

[15] F.-M. Chen, Symplectic Three-Algebra Unifying N = 5, 6 Superconformal Chern-Simons-Matter Theories, JHEP 08, 077 (2010) [arXiv:0908.2618 [hep-th]].

[16] J. Bagger and G. Bruhn, Three-Algebras in N = 5, 6 Superconformal Chern-Simons Theories: Representations and Relations, Phys. Rev. D 83, 025003 (2011) [arXiv:1006.0040 [hep-th]].
[17] V. G. Kac, *A sketch of Lie superalgebra theory*, Comm. Math. Phys. 53, 31–64 (1977).

[18] L. Frappat, A. Sciarrino and P. Sorba, *Dictionary on Lie algebras and superalgebras*. Academic Press, 2000.

[19] N. Cantarini and V. G. Kac, *Classification of linearly compact simple $N = 6\ 3$-algebras*, [1010.3599 [math.QA]].

[20] J. Palmkvist, *Unifying $N = 5$ and $N = 6\$, JHEP 1105, 088 (2011) [arXiv:1103.4860 [hep-th]].

[21] M. Scheunert, W. Nahm and V. Rittenberg, *Classification of all simple graded Lie algebras whose Lie algebra is reductive*, J. Math. Phys. 17, 1640 (1976).