Physical Principles Based on Geometric Properties

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Abstract
In this paper we present some results obtained in a previous paper about the Cartan’s approach to Riemannian normal coordinates and our conformal transformations among pseudo-Riemannian manifolds. We also review the classical and the quantum angular momenta of a particle obtained as a consequence of geometry, without postulates. We present four classical principles, identified as new results obtained from geometry. One of them has properties similar to the Heisenberg’s uncertainty principle and another has some properties similar to the Bohr’s principle. Our geometric result can be considered as a possible starting point toward a quantum theory without forces.

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1 Introduction

In a previous paper dedicated only to mathematical results [1], we show how n-dimensional pseudo-Riemannian spaces are related to each other by a conformal transformation. As a geometric consequence, we obtained the classical angular momentum and the quantum angular momentum operator of a particle, regardless of a physical theory. In this paper, suggested by geometric results, we build some new principles and consider the possibility of a new starting point toward a quantum theory without forces.

This paper is organized as follows. In Sec. 2 we present normal coordinates and elements of differential geometry. In Sec. 3 we show that all well-behaved n-dimensional pseudo-Riemannian metrics in origin and in its neighborhood, in normal coordinates, are conformal to an n-dimensional flat manifold and to an n-dimensional manifold of constant curvature. In Sec. 4, we make an embedding of all n-dimensional pseudo-Riemannian manifold of constant curvature in an n+1-dimensional flat manifold, obtaining, without postulates, the quantum angular momentum operator of a particle as a consequence of geometry. In Sec. 5, based on geometric properties, we present some physical principles. Section 6 is a continuation of section 5 with more concentration in quantum mechanics.
2 Normal Coordinates

In this section we briefly present normal coordinates and review some elements of differential geometry for an n-dimensional pseudo-Riemannian manifold, \([2], [3], [4]\).

Let us consider the line element

\[ ds^2 = G_{\Lambda\Pi} du^{\Lambda} du^{\Pi}, \]  

(2.1)

with

\[ G_{\Lambda\Pi} = E^{(A)}_{\Lambda} E^{(B)}_{\Pi} \eta_{(A)(B)}, \]  

(2.2)

where \( \eta_{(A)(B)} \) and \( E^{(A)}_{\Lambda} \) are flat metric and vielbein components respectively. We choose each \( \eta_{(A)(B)} \) as a plus or minus Kronecker’s delta function.

Let us give the 1-form \( \omega^{(A)} \) by

\[ \omega^{(A)} = du^{\Lambda} E^{(A)}_{\Lambda}. \]  

(2.3)

We now define Riemannian normal coordinates by

\[ u^{\Lambda} = v^{\Lambda} t, \]  

(2.4)

Substituting in (2.3)

\[ \omega^{(A)} = tdv^{\Lambda} E^{(A)}_{\Lambda} + dtv^{\Lambda} E^{(A)}_{\Lambda}. \]  

(2.5)

Let us define

\[ z^{(A)} = v^{\Lambda} E^{(A)}_{\Lambda}, \]  

(2.6)

so that

\[ \omega^{(A)} = dtz^{(A)} + tdz^{(A)} + tE^{\Pi(A)}_{\Pi} \frac{\partial E^{(B)}_{\Pi}}{\partial z^{(C)}} z^{(B)} dz^{(C)}. \]  

(2.7)

We now make

\[ A^{(A)(B)(C)} = tE^{\Pi(A)}_{\Pi} \frac{\partial E^{(B)}_{\Pi}}{\partial z^{(C)}}, \]  

(2.8)

then

\[ \varpi^{(A)} = tdz^{(A)} + A^{(A)(B)(C)} z^{(B)} dz^{(C)}, \]  

(2.9)

with

\[ \omega^{(A)} = dtz^{(A)} + \varpi^{(A)}. \]  

(2.10)
We have at $t = 0$

\[ A^{(A)(B)(C)}(t = 0, z^{(D)}) = 0, \]
\[ \omega^{(A)}(t = 0, z^{(D)}) = 0, \]  
and

\[ \omega^{(A)}(t = 0, z^{(D)}) = dt z^{(A)}. \]

Consider, at an $n+1$-manifold, a coordinate system given by $(t, z^{(A)})$. For each value of $t$ we have a hyper-surface, where $dt = 0$ on each of them. We are interested in the hyper-surface with $t = 1$, where we verify the following equality

\[ \omega^{(A)}(t = 1, z) = \omega^{(A)}(t = 1, z). \]

It is well known the following expression in a vielbein basis

\[ d\omega^{(A)} = -\omega^{(A)(B)} \wedge \omega^{(B)}. \]

Considering now the map $\Phi$, between two manifolds $M$ and $N$, and two subsets, $U$ of $M$ and $V$ of $N$, we have

\[ \Phi : U \longrightarrow V. \]

Defining now pull-back as follows, [3], [4],

\[ \Phi^* : F^p(V) \longrightarrow F^p(U), \]

so that $\Phi^*$ sends $p$-forms into $p$-forms.

It is well known that the exterior derivative commutes with pull-back, so that

\[ \Phi^*(d\omega^{(A)}) = d\Phi^*(\omega^{(A)}), \]

and

\[ \Phi^*(d\omega^{(A)}) = d\Phi^*(\omega^{(A)}). \]

We also have

\[ \Phi^*(\omega^{(A)}(B) \wedge \omega^{(B)}) = \Phi^*(\omega^{(A)}(B)) \wedge \Phi^*(\omega^{(B)}). \]

The equation (2.10) can be seen as pull-back,

\[ \Phi^*(\omega^{(A)}) = dt z^{(A)} + \omega^{(A)}. \]

It can be shown, by a simple calculation that

\[ \Phi^*(\omega^{(A)}(B)) = \omega^{(A)}(B). \]
By the exterior derivative of (2.22), we obtain
\[
d(\Phi^*(\omega^{(A)})) = d(dtz^{(A)} + \varpi^{(A)}) = dz^{(A)} \wedge (dt) + dt \wedge \frac{\partial(\varpi^{(A)})}{\partial(t)}
\]  
(2.23)

+ terms not involving \(dt\).

Making a pull-back of (2.15) and using (2.20) we have
\[
\Phi^*(d\omega^{(A)}) = \Phi^*(-\omega^{(A)} \wedge \omega^{(B)}) = -\Phi^*(\omega^{(A)}_B) \wedge \Phi^*(\omega^{(B)}).
\]  
(2.24)

Using (2.19), (2.22), (2.23) and (2.24) we have
\[
\frac{\partial(\varpi^{(A)})}{\partial(t)} = dz^{(A)} + \varpi^{(A)}_B z^{(D)}.
\]  
(2.25)

We can, by a similar procedure to (2.19), and using the Cartan’s second structure equation, obtain the following result
\[
\frac{\partial(\varpi^{(A)}_B)}{\partial(t)} = R^{(C)}(A)_B(C)(D) z^{(D)} \varpi^{(A)}.
\]  
(2.26)

Making a new partial derivative of (2.25), two partial derivatives of (2.9), comparing the results and using (2.26) we have the following equation
\[
\frac{\partial^2(A^{(A)}_C)}{\partial(t^2)} = tz^{(B)} R^{(A)}(B)_C(D) + z^{(L)} z^{(M)} R^{(A)}(L)(M)(N) A^{(P)}(C)(D) \eta^{(N)(P)}.
\]  
(2.27)

It is easy to show that
\[
A^{(A)}_C(D) + A^{(A)}_D(C) = 0
\]  
(2.28)

is the solution for all \(t\).

Then,
\[
A^{(A)}_C(D) = -A^{(A)}_D(C),
\]  
(2.29)
so that, we can rewrite (2.9) as

$$\omega^{(A)} = tdz^{(A)} + \frac{1}{2} A^{(A)(B)(C)} (z^{(B)}dz^{(C)} - z^{(C)}dz^{(B)}).$$  \hspace{1cm} (2.30)$$

Let us define

$$A^{(A)(C)(D)} = z^{(B)} B^{(A)(B)(C)(D)}. \hspace{1cm} (2.31)$$

The following result is obtained by substituting (2.31) in (2.27),

$$\frac{\partial^2 (B^{(A)(B)(C)(D)})}{\partial (t^2)} = t R_{(A)(B)(C)(D)} + z^{(L)} R_{(A)(B)(L)(N)} B_{(P)(M)(C)(D)} \eta^{(N)(P)}. \hspace{1cm} (2.32)$$

By a simple procedure we obtain the following solution

$$B^{(A)(B)(C)(D)} + B^{(B)(A)(C)(D)} = 0. \hspace{1cm} (2.33)$$

Using (2.29), (2.31) and (2.33) we conclude that $B^{(A)(B)(C)(D)}$ has the same symmetries as the Riemann curvature tensor

$$B^{(A)(B)(C)(D)} = -B^{(B)(A)(C)(D)} = -B^{(A)(B)(D)(C)}. \hspace{1cm} (2.34)$$

Using (2.29) and (2.31) we have

$$A^{(A)(C)(D)}dz^{(A)}z^{(C)}dz^{(D)} =$$

$$+ \frac{1}{4} B^{(A)(B)(C)(D)} \cdot$$

$$(z^{(B)}dz^{(A)} - z^{(A)}dz^{(B)}) \cdot$$

$$(z^{(C)}dz^{(D)} - z^{(D)}dz^{(C)}). \hspace{1cm} (2.35)$$

Now we can write the line element of the hyper-surface. We have

$$ds'^2 = t^2 \eta^{(A)(B)}dz^{(A)}dz^{(B)} +$$

$$+ \frac{1}{2} \left\{ t \epsilon^{(B)} B^{(A)(B)(C)(D)} +$$

$$+ \eta^{(M)(N)} A^{(M)(B)(A)} A^{(N)(C)(D)} \right\} \cdot$$

$$(z^{(B)}dz^{(A)} - z^{(A)}dz^{(B)})(z^{(C)}dz^{(D)} - z^{(D)}dz^{(C)}). \hspace{1cm} (2.36)$$
The line elements of the manifold and the hyper-surface are equal at \( t = 1 \), where \( u^\Lambda = v^\Lambda \),

\[
ds^2 = ds'^2, \quad (2.37)
\]

and

\[
ds^2 = \eta_{(A)(B)} dz^{(A)} dz^{(B)} + \\
+ \frac{1}{2} \left( \frac{1}{2} \epsilon(B) B_{(A)(B)(C)(D)} + \\
+ \eta^{(M)(N)} A_{(M)(B)(A)} A_{(N)(C)(D)} \right) \cdot (z^{(B)} dz^{(A)} - z^{(A)} dz^{(B)}) (z^{(C)} dz^{(D)} - z^{(D)} dz^{(C)}). \quad (2.38)
\]

In the next section we build, by a simple procedure, the conformal form of \( n \)-dimensional pseudo-Riemannian manifolds.

### 3 Conformal Form of Riemannian Metrics

We now write (2.38) as

\[
ds^2 = \eta_{(A)(B)} dz^{(A)} dz^{(B)} + \\
+ \frac{1}{2} \left( \frac{1}{2} \epsilon(B) B_{(A)(B)(C)(D)} + \\
+ \eta^{(M)(N)} A_{(M)(B)(A)} A_{(N)(C)(D)} \right) \cdot (z^{(B)} \frac{dz^{(A)}}{ds} - z^{(A)} \frac{dz^{(B)}}{ds}) (z^{(C)} \frac{dz^{(D)}}{ds} - z^{(D)} \frac{dz^{(C)}}{ds}) ds^2. \quad (3.1)
\]

It can also be written in the form

\[
[1 - \frac{1}{2} \epsilon(B) B_{(A)(B)(C)(D)} + \\
+ \eta^{(M)(N)} A_{(M)(B)(A)} A_{(N)(C)(D)} \cdot (z^{(B)} \frac{dz^{(A)}}{ds} - z^{(A)} \frac{dz^{(B)}}{ds}) (z^{(C)} \frac{dz^{(D)}}{ds} - z^{(D)} \frac{dz^{(C)}}{ds})] ds^2 \\
= \eta_{(A)(B)} dz^{(A)} dz^{(B)}. \quad (3.2)
\]
We now define the function
\[
L^{A(B)} = (z^{(B)} \frac{dz^{(A)}}{ds} - z^{(A)} \frac{dz^{(B)}}{ds}),
\]
which is the classical angular momentum of a free particle.

The line element (3.2) can assume the following form
\[
\{1 + \frac{1}{2} \left[ \epsilon(B) B_{(A)(B)(C)(D)} + \eta^{(M)(N)} A_{(M)(B)(A)} A_{(N)(C)(D)} \right] L^{(A)(B)} L^{(C)(D)} \} ds^2 = (\eta(A)B) dz^{(A)} dz^{(B)}.
\]

We now define the function
\[
\exp(-2\sigma) = \{1 + \frac{1}{2} \left[ \epsilon(B) B_{(A)(B)(C)(D)} + \eta^{(M)(N)} A_{(M)(B)(A)} A_{(N)(C)(D)} \right] L^{(A)(B)} L^{(C)(D)} \},
\]
so that, the line element assumes the form
\[
ds^2 = \exp(2\sigma) \eta(A)B dz^{(A)} dz^{(B)}.
\]

The metric (3.6) is conformal to a flat manifold, and we conclude that all n-dimensional pseudo-Riemannian metrics are conformal to flat manifolds, when, in normal coordinates, the transformations are well-behaved in the origin and in its neighborhood. It is important to pay attention to the fact that a normal transformation and its inverse are well-behaved in the region where geodesics are not mixed. Points where geodesics close or mix are known as conjugate points of Jacobi’s fields. Jacobi’s fields can be used for this purpose.

We can place (3.6) in the following form, [1],
\[
ds^2 = \{1 + \frac{1}{2} \left[ \epsilon(B) B_{(A)(B)(C)(D)} + \eta^{(M)(N)} A_{(M)(B)(A)} A_{(N)(C)(D)} \right] L^{(A)(B)} L^{(C)(D)} \}^{-1} \eta_{(A)B} d\Omega^a d\Omega^b.
\]
We now present the metric of a constant curvature manifold in the well known form
\[ ds'^2 = \left\{ 1 + \frac{K\Omega^\alpha\Omega^\beta\eta_{\alpha\beta}}{4} \right\}^{-2} d\Omega^\rho d\Omega^\sigma \eta_{\rho\sigma}. \] (3.8)

Because (3.7) and (3.8) are conformal to a flat manifold, there is a conformal transformation between them,
\[ g'_{\alpha\beta} = (\exp 2\psi) g_{\alpha\beta}. \] (3.9)

More specifically,
\[ \{1 + \frac{1}{2}(\epsilon^\rho B_{\rho\alpha\beta}) + \eta^\rho\sigma A_{\rho\alpha\beta} A_{\sigma\gamma\delta}) L^{\alpha\beta} L^{\gamma\delta}\} = (\exp 2\psi) \left(1 + \frac{K\Omega^\alpha\Omega^\beta\eta_{\alpha\beta}}{4}\right)^2. \] (3.10)

Note that (3.8) is an Einstein’s space with a constant curvature, where
\[ R'_{\alpha\beta} = \frac{R'}{n} g'_{\alpha\beta}, \] (3.11)
and \( R' \) is the scalar curvature. Spaces, as the Schwarzschild’s, where
\[ R_{\alpha\beta} = 0, \] (3.12)
are Einstein’s spaces and are not maximally symmetric.

Einstein’s spaces with a constant scalar curvature obey homogeneity and isotropy conditions. They are maximally symmetric spaces.

We will be using the following definitions, [5]
\[ \Delta_1 \psi = g^{\mu\nu} \psi_{,\mu} \psi_{,\nu}, \] (3.13)
\[ \psi_{\mu\nu} = \psi_{,\mu\nu} - \psi_{,\mu} \psi_{,\nu}, \] (3.14)
\[ \Delta_2 \psi = g^{\mu\nu} \psi_{,\mu\nu}. \] (3.15)
From (3.9), (3.13), (3.14), and (3.15) we obtain

\[ \psi_{\mu\nu} = \frac{1}{(n - 2)}(R_{\mu\nu}) \]

\[- \frac{1}{(2)(n - 1)(n - 2)}(g'_{\mu\nu}R' - g_{\mu\nu}R) \]

\[- \frac{1}{2}\triangle_1 \psi g_{\mu\nu}. \]

(3.16)

If \( g'_{\mu\nu} \) is a metric of an Einstein’s space, then (3.16) is simplified to

\[ \psi_{\mu\nu} = -\frac{1}{(n - 2)}R_{\mu\nu} + \]

\[ + \left( \frac{1}{(2)(n - 1)(n - 2)}R + \frac{1}{(2n)(n - 1)}R'(\exp 2\psi) - \frac{1}{2}\triangle_1 \psi \right)g_{\mu\nu}. \]

(3.17)

In the following we consider the Einstein’s equation

\[ R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi T_{\mu\nu}. \]

(3.18)

Spaces as (3.11) with a non-constant scalar curvature do not obey (3.18).
4 Embedding of Manifolds of Constant Curvatures in Flat Manifolds

In this section we embed the n-dimensional manifold (3.8) in an n+1-dimensional flat manifold obtaining, as a geometric result, without postulate, the quantum angular momentum of a particle. Other results will be presented in another section. We now consider a manifold (3.8) called $S$, embedded in an n+1-dimensional flat manifold. The following constraint is obeyed [6],

$$\eta_{\alpha\beta} x^\alpha x^\beta = K = \epsilon \frac{1}{R^2},$$  \hspace{1cm} (4.1)

where $K$ is the scalar curvature of the n-dimensional manifold (3.8), $\alpha, \beta = (1, 2, ..., n + 1)$ and $\epsilon = (+1, -1)$. For the special case of an n-sphere we use the following notation $S^n$ for (3.8).

It is convenient that we use a local basis $X^\beta = \frac{\partial}{\partial (x^\beta)}$.

We consider a constant vector $C$ in the n+1-dimensional manifold given by

$$\eta_{\alpha\beta} C^\alpha X^\beta = \eta_{\alpha\beta} C^\alpha X^\beta = C,$$  \hspace{1cm} (4.2)

where each $C^\alpha$ is constant and $N$ is a unitary and normal vector to $S$. We use the symbol $\langle , \rangle$ for the internal product in the n+1-dimensional flat manifold and $\langle, \rangle'$ for $S$.

A constant vector $C$ can be decomposed into two parts, one in $S$ and the other outside $S$ as follows

$$C = \bar{C} + \langle C, N \rangle N.$$  \hspace{1cm} (4.3)

From the definition of $N$ and (4.1) we obtain

$$N^\alpha = \frac{x^\alpha}{R}.$$  \hspace{1cm} (4.4)

Let us construct the covariant derivative of $C$. We have a local basis and a diagonal and unitary tensor metric, so that the Christoffel symbols are null. Then the covariant derivative of $C$ in the $Y$ direction is given by

$$\nabla_Y C = 0.$$  \hspace{1cm} (4.5)
It is easy to show that
\[ \nabla_Y N = Y \frac{\nabla}{R} . \] (4.6)

The Lie derivative of the metric tensor in S is given by [6],
\[ L_{\bar{U}} g' = 2 \lambda_U g' , \] (4.7)
where \( U \) is a constant vector in the flat manifold, and \( \lambda_U \) is the characteristic function. For S the characteristic function is given by
\[ \lambda_U = -\frac{1}{R} \langle U, N \rangle . \] (4.8)

Substituting (4.8) in (4.7) we have
\[ L_{\bar{U}} g' = -2 \frac{1}{R} \langle U, N \rangle g' . \] (4.9)

In the region of S where \( \langle U, N \rangle \) is not null, \( \bar{U} \) is a conformal Killing vector and in the region where \( \langle U, N \rangle \) is null, \( \bar{U} \) is a Killing vector.

We now consider another constant vector \( V \) in the flat space. The Lie derivative of its projection in S is given by
\[ L_{\bar{U}} g' = -2 \frac{1}{R} < U, N > g' . \] (4.10)

As we consider a local basis and constant vectors \( U \) and \( V \), the commutator is given by
\[ [U, V] = 0 . \] (4.11)

Then,
\[ L_{[\bar{U}, \bar{V}]} g' = -2 \frac{1}{R} < [U, V], N > g' = 0 . \] (4.12)

Regardless of \( \bar{U} \) and \( \bar{V} \) being conformal Killing vectors or Killing vectors, their commutator is a Killing vector. In the following we will show that the commutator \([\bar{U}, \bar{V}]\) is proportional to the quantum angular momentum of a
Using (3.3) in the following commutator of elements of the basis, we obtain

\[
[\bar{U}, \bar{V}] = U^\alpha V^\beta \left[ X_\alpha - <X_\alpha, N > N, X_\beta - <X_\beta, N > N \right] = U^\alpha V^\beta [X_\alpha, X_\beta].
\]

\hspace{1cm} (4.13)

We now calculate the commutator of elements of the basis, by parts. We have by simple calculation

\[<X_\alpha, N > N = \frac{1}{R} \eta_{\alpha \beta} x^\beta.\]

\hspace{1cm} (4.14)

Substituting (4.14) in (4.13) we obtain

\[
[\bar{X}_\alpha, \bar{X}_\beta] = [X_\alpha, X_\beta] - [X_\alpha, \frac{1}{R} \eta_{\beta \sigma} x^\sigma N] + [X_\beta, \frac{1}{R} \eta_{\alpha \sigma} x^\sigma N] + \frac{1}{R^2}\left[\eta_{\alpha \sigma} x^\sigma N, \eta_{\beta \sigma} x^\sigma N\right].
\]

\hspace{1cm} (4.15)

In a local basis we have

\[ [X_\alpha, X_\beta] = 0, \]

\hspace{1cm} (4.16)

\[ [\eta_{\alpha \sigma} x^\sigma N, \eta_{\beta \sigma} x^\sigma N] = 0. \]

\hspace{1cm} (4.17)

Substituting in (4.15) we obtain

\[
[\bar{X}_\alpha, \bar{X}_\beta] = \frac{1}{R^2} \left( \eta_{\alpha \sigma} x^\sigma \frac{\partial}{\partial (x^\beta)} - \eta_{\beta \sigma} x^\sigma \frac{\partial}{\partial (x^\alpha)} \right)
\]

\[= \frac{1}{R^2} \left( x_\alpha \frac{\partial}{\partial (x^\beta)} - x_\beta \frac{\partial}{\partial (x^\alpha)} \right)
\]

\[= -i \frac{1}{\hbar R^2} L_{\alpha \beta}. \]

\hspace{1cm} (4.18)
Multiplying $L_{\alpha\beta}$ by a vielbein basis we obtain

$$L_{(A)(B)} =$$

$$= (i\hbar)(R^2)R_{(A)(B)(C)(D)}x^{(D)}\eta^{(C)(M)}\frac{\partial}{\partial(x^M)},$$

(4.19)

where

$$\hat{p}_{(M)} = (i\hbar)\frac{\partial}{\partial(x^M)},$$

(4.20)

is the quantum momentum operator of a particle, and

$$R_{(A)(B)(C)(D)} =$$

$$= \frac{1}{R^2}[\eta_{(A)(D)}\eta_{(B)(C)} - \eta_{(A)(C)}\eta_{(B)(D)}],$$

(4.21)

is the curvature of S in the vielbein basis and $\eta_{(A)(C)}$ is diagonal. We consider as an important observation that the association between the quantum angular momentum operator and the constant curvature operator is allowed in an orthogonal vielbein basis of a Cartesian coordinate, regardless of having a curved or a flat manifold. We have used the embedding of an n-dimensional manifold S in an n+1-dimensional flat manifold, only to obtain the quantum angular momentum operator of a particle, without postulates. We can rewrite (4.19) as follows

$$L_{(A)(B)} =$$

$$= (i\hbar)[\eta_{(A)(D)}\eta_{(B)(C)} - \eta_{(A)(C)}\eta_{(B)(D)}].$$

$$x^{(D)}\eta^{(C)(M)}\frac{\partial}{\partial(x^M)},$$

(4.22)

Note that the coordinates in (3.7) are in the n+1-dimensional flat manifold and $L_{\alpha\beta} \subset S$, so that $L_{\alpha\beta} = 0$ for $\alpha$ or $\beta$ is equal to $n + 1$. Racah has shown that [7] the Casimir operators of any semisimple Lie group can be constructed from the quantum angular momentum (4.22). Each multiplet of semisimple Lie group can be uniquely characterized by the eigenvalues
of the Casimir operators.
We have built the quantum angular momentum operator from classical geo-
metric considerations. We can write the usual expression for the eigenstates
of Casimir operator without reference to quantum mechanics, as follows
\[ \hat{C} | \ldots \rangle = C | \ldots \rangle . \] (4.23)
This is simple and well known,[7] for SO(3). There is an interesting con-
struction from the group theoretical point of view to Dirac theory with or
without Dirac’s equation, [8].
In the following we calculate the Lie derivative of the so(p,n-p) algebra. For
the Lie group SO(p,n-p) we choose the signature
\( (p, n - p) = (-, -, -, ..., +, +, +) \), with the algebra
\[
[L_{(A)(B)}, L_{(C)(D)}] = -i(\eta_{(A)(C)}L_{(B)(D)} + \eta_{(A)(D)}L_{(C)(B)}
+ \eta_{(B)(C)}L_{(D)(A)} + \eta_{(B)(D)}L_{(A)(C)}).
\] (4.24)
Considering the Lie derivative
\[
L_{[L_{(A)(B)}, L_{(C)(D)}]}g' =
- R^4 < [X_{(A)}, X_{(B)}], [X_{(C)}, X_{(D)}] >, N > g' = 0,
\] (4.25)
where, for the orthogonal Cartesian coordinates, the vielbein is given by
\[
E^{(A)}_{\Lambda} = \delta^{(A)}_{\Lambda},
\] (4.26)
we have
\[
[X_{(A)}, X_{(B)}] = [X_\alpha, X_\beta] = 0.
\] (4.27)
Note that \( g' \) in \( S \) is form-invariant [9] in relation to the Killing’s vector \( \xi \),
and in relation to the algebra of SO(p,n-p), as well. We conclude that the
algebra of SO(p,n-p) is a Killing’s object.
The same is true for the algebra of the Lie group SO(n), where for SO(n) we could choose the following signature \((+, +, +, ..., +, +)\). The constraint (4.1) is invariant for many of the classical groups. For these groups it is possible to build operators, from the combination of the quantum angular momentum operators, which are Killing’s objects in relation to \(g'\). Therefore, the metric is form-invariant in relation to this algebra. It is interesting to see some of these groups in the Cartan’s list of irreducible Riemannian globally symmetric spaces, [4], and in [10].

Note that we start from a normal coordinate transformation. In other words, in the region where the transformation (2.4) is well-behaved, we can build (3.6) and by a conformal transformation we have (3.8), which was essential to obtain the quantum angular momentum operator from geometry.
5 Physical Principles Based on Geometric Properties

From a different point of view, Dirac [12] embedded the De Sitter space in a five-dimensional flat manifold. He has considered functions and fields living in a five-dimensional flat manifold and has built a procedure to project them in the De Sitter space. The Dirac procedure implies the need for the quantum momentum and the quantum angular momentum postulates. Other authors have used the Dirac’s idea or variants of it, as in [13]. To obtain the quantum angular momentum from geometric considerations, we have considered only constant vectors in an n+1-dimensional flat manifold. There are many procedures to define or introduce functions, fields and geometric objects in (3.8). In the following, we reconsider the qualitative analyzes of (4.9) made in section 4. In the region where \( < U, N > \) does not vanish \( \bar{U} \) is a conformal Killing vector and in the region where it vanishes, \( \bar{U} \) is a Killing vector. In other words, we have Killing and conformal Killing vectors living in (3.8). For our objective we need only Killing objects as the quantum angular momentum.

If we make the Lie derivative of (3.9) in relation to \( \xi \), we obtain

\[
L_\xi g' = (2\xi(\psi)g + L_\xi g)(\exp 2\psi).
\]

(5.1)

More specifically, (5.1) can be written as

\[
L_\xi\left[\left\{1 + \frac{K\Omega^\alpha\Omega^\beta\eta_{\alpha\beta}}{4}\right\}^{(-2)}\eta\right] = \\
L_\xi\left[(\exp 2\psi)\left\{1 + \frac{1}{2}\left(\epsilon_{\beta}B_{\alpha\beta\gamma\delta}\right) + \right.ight. \\
\left. + \eta^{\alpha\beta}A_{\rho\alpha\beta}A_{\sigma\gamma\delta}\right]L_{\alpha\beta}L_{\gamma\delta}\right\}^{(-1)}\eta].
\]

(5.2)

We now consider the following condition

\[
2\xi(\psi)g + L_\xi g = 0,
\]

(5.3)

which implies the following

\[
L_\xi g' = 0,
\]

(5.4)
which is a definition of a Killing vector. If $\xi$ obeys (5.3) we conclude that $\xi$ is a Killing vector in (3.8). Note that $\xi$ is a conformal Killing vector in (3.7). The equation (5.3) shows how a Killing vector in (3.8) will be in (3.7). We conclude that the best way toward our objective will be from a postulate as follows, associated to conditions of minimal energy: In (3.8) nature always choose Killing objects. Based on this postulate we will build four classical principles, where one of them is identified as a classical version of the Heisenberg’s uncertainty principle and another as a classical version of the Bohr’s non-radiation postulate. The third principle is not new and is associated to the electric neutrality of some stable systems. The fourth can be interpreted as an equivalence between two descriptions of a particle’s motion. The first one as the motion due to the presence of forces and the second as a consequence of geometry, as in Einstein’s gravitation. For this we assume only constant vectors in an $n+1$-dimensional flat manifold, where (3.8) will be embedded.

The equations (2.27) and (2.32) tell us that if the curvature is null, $A_{(A)(C)(D)}$ and $B_{(A)(B)(C)(D)}$ are null. In this case, the equation (3.4) implies a null angular momentum. We conclude that any free particle in a curved manifold will be always in movement, with angular momentum not null regardless of wether or not we consider a physical theory.

The equation (3.6) tells us that $ds^2$ is conformal to a flat manifold and to (3.8). An observer in (3.8) will see the space as being homogeneous and isotropic in the small region where the transformation (2.4) is well-behaved. With this condition, Ricci principal directions of space will be indeterminate so that in that region the position of the particle is uncertain. In the conjugate points of Jacobi’s fields, the transformation (2.4) fails because geodesics cross, mix or touch each other. Therefore, close to a conjugate point we will not have indetermination in the Ricci principal directions and the uncertainty in the position of the particle disappears. If (3.8) is valid in all points of the space, in each point there will be an indetermination of Ricci principal directions and consequently a total uncertainty in the position of the particle. This resembles a property of the Heisenberg’s uncertainty principle and could be seen as a classical version.

The metric (3.8) will be form-invariant for a displacement $\xi$ which is a Killing’s vector. In this metric [9], a scalar function will be constant or null, there will be Killing’s vectors only, and tensors will be a combination of the metric tensor. In these conditions, the electromagnetic fields will be
trivial and there will not be radiation. In the neighborhoods of the conjugate points the transformations in normal coordinates fail and we will not have an indetermination of Ricci principal directions and the electromagnetic fields will not be trivial, being a radiative field. This is similar to the Bohr’s postulate for radiation and could be seen as a classical version.

In the metric (3.8) there are no forces generated by fields in the region where the transformation (2.4) is well-behaved. Particles move free of forces. In the local system, there are ordinary forces generated by fields. If we consider a Kalusa-Klein theory, where gauge fields are present in the metric, particles are free in the local coordinates and in (3.6). In the local system, there are fields while in (3.8) there are not.

We can consider this as a principle, creating an equivalence between two descriptions of motion, which are possible by normal transformations. The first description, in local coordinates, is related to the conception of force generating fields. The second is related to the conception of motion without forces.

We believe that this principle is going toward the Einstein’s dream because it points to the possibility of thinking in physics without forces as in Einstein’s gravity.

We notice that the conjugate points of the Jacobi’s fields can be a consequence of geometric singularities, as it is in the origin of the Schwarzschild’s geometry, [11], where the curvature diverges, but it can also be due to the construction of the coordinates, as it is in the case of a maximally symmetric space, where the curvature is finite in all points. In the second case we have an indetermination of Ricci principal directions, and in the first we do not.

Considering the momentum-energy tensor of matter and electromagnetic fields,

\[ T_{\alpha\beta} = \frac{1}{4\pi} (F_{\alpha\sigma} F_{\beta}^{\sigma} - \frac{1}{4} g'_{\alpha\beta} F_{\varphi\sigma} F^{\varphi\sigma}) + t_{\alpha\beta}, \]

(5.5)

where \( t_{\alpha\beta} \) is associated to electric charges and \( T(F_{\alpha\sigma}) = T_{\alpha\beta}\text{em} \) to the electromagnetic fields, with

\[ T_{\alpha\beta}\text{em} = \frac{1}{4\pi} (F_{\alpha\sigma} F_{\beta}^{\sigma} - \frac{1}{4} g'_{\alpha\beta} F_{\varphi\sigma} F^{\varphi\sigma}). \]

(5.6)
Then,

\[ T_{\alpha\beta} = T_{\alpha\beta}^{\text{em}} + t_{\alpha\beta}. \]  

(5.7)

As \( g' \) is form-invariant in (3.8), the electromagnetic vector \( A_\mu \) will be null in (3.8),

\[ A_\mu = 0. \]  

(5.8)

From Maxwell’s equations we have

\[ F^{\rho\sigma}_{;\rho} = -J^\sigma. \]  

(5.9)

Using (5.9) in (5.10) we obtain

\[ J^\sigma = 0. \]  

(5.10)

We conclude that in (3.8) the sum of all charges is zero, as well as it is the sum of all currents.
6 Geometric Properties Based on Quantum Principles and Quantum Principles Based on Geometric Properties

In this section we analyze some results obtained in section 5 which resemble some postulates of quantum mechanics. Part of the development of our results is qualitative because only some applications, like SO(3) and the Casimir eigenvalues, for instance, can be easily calculated. Considerations of more complex systems are qualitative at the moment. Therefore, from a theoretical point of view, there is a gap.

We recall that in the region where there are no conjugate points of Jacobi’s fields, it is possible to build a transformation (2.4) between the ordinary metric and (3.6), and a conformal transformation between (3.6) and (3.8). Because $g'$ is form-invariant in the region where (3.8) is defined, there are no fields nor radiation. The quantum angular momentum, which is a Killing’s object, appears as a geometric consequence of embedding (3.8) in an $n+1$-dimensional flat manifold. Particles will be in a free motion, but confined in (3.8). In this context, where forces do not exist, the particle confinement is due to the manifold geometry. This resembles the Heisenberg’s principle of quantum mechanics.

We consider as an important observation, made in section 5, that the association between the quantum angular momentum operator and the constant curvature operator is allowed in an orthogonal vielbein basis of a Cartesian coordinate, even for a flat manifold. This suggests that, even in a flat spacetime, we can consider the intrinsic angular momentum, or spin $\frac{1}{2}$ of a free massive particle, as a quantum object living in an manifold of constant curvature, embedded in this flat spacetime.

From the geometric point of view, some operations with the quantum angular momentum, as sums and products, suggest the same operations with curvature. We can define some procedures in differential geometry by operations with quantum angular momenta. As an example, consider the algebra (4.24) of the group SO(p,n-p)

$$[L_{(A)}(B), L_{(C)}(D)] = -i(\eta_{(A)(C)}L_{(B)}(D) + \eta_{(A)(D)}L_{(C)}(B) + \eta_{(B)(C)}L_{(D)}(A) + \eta_{(B)(D)}L_{(A)}(C)).$$
We can substitute (4.19) in (6.1) obtaining a representation of the algebra in terms of curvature operators. Substituting (4.19) in (4.25) we will have the form-invariance of the metric tensor $g'$, in relation to the algebra $\text{so}(p, n-p)$, in terms of the curvature operators. Any other possible operation among quantum angular momenta, here defined, can be placed in terms of curvatures. This offers some curious procedures in differential geometry by simple operations with quantum angular momentum, which may not be possible by geometric methods. We do not know if this is known in the specialized literature. The association between the quantum angular momentum and differential geometry can be useful in geometry, as well as in physics.

In the following, we make more qualitative considerations. We know that mass and energy curve the spacetime. From the postulate of section 5, the existence of electric charges is allowed in (3.8) provided that the total sum is zero. For each charge there is an associated particle. The particles confinement in the metric (3.8) is not due to forces, it is a consequence of the form-invariance of $g'$. Obviously, in the region where there is a transition between well-behaved and not well-behaved transformations (2.4), we also have a transition from the condition where there are no forces to an ordinary description by forces. This can be seen as a small deformation in (3.8). The intensity of the deformation could be responsible for some polarization or emission.

For a stable and isolated system, like SO(3), we can write the eigenvalues of the Casimir operators by (4.23). Particles will be confined if the metric is (3.8). As we have seen, the postulate implies the electric neutrality of (3.8) and the confinement of the particles. We imagine that an incident particle curves and deforms (3.8), then, there will be a transition from the well-behaved to the not well-behaved transformation (2.4). This can be seen as a transition, emission or scattering. Mass and energy of each particle curve the spacetime, then particles in (3.8) will be responsible for the confinement.

We conclude that each particle contributes to the confinement, which is proportional to its mass if the resultant metric is (3.8). In this case a proton curves the spacetime with more intensity than an electron because its mass is bigger. As a consequence, the associated curvatures obey

$$K_p > K_e.$$  

(6.2)
But

\begin{equation}
K = \frac{1}{R^2},
\end{equation}

then

\begin{equation}
R_p < R_e.
\end{equation}

Note that the region of the electron confinement is bigger than that of the proton. This is compatible with the Heisenberg’s uncertainty principle and with the experimental evidence of nuclear and atomic dimensions. Protons live in nucleus and electrons live outside. Starting from the above point of view, if the metric deformation, caused by mass, energy and motion of some interacting particles, generates a resultant metric given by (3.8), there will not be any transition, emission, or scattering. If the metric deformation generates a different metric from (3.8), there will be transition, emission or scattering. This resembles the Bohr’s principle of quantum mechanics. In this paper we are using the classical point of view and Riemannian manifolds. Considering an ionized positive system, we conclude that the metric will not be (3.8) because the sum of all charges is zero only for (3.8). Although, the Einstein’s equation is well-behaved for an ionized system, we know that it is not valid for small regions. Obviously, in the limit of a classical distribution of charges, the Einstein’s equation is a good theory, with a negative curvature in the interior of a classical distribution of charges, and null curvature outside, obeying the weak energy condition. [14]. For coherence with this classical limit, we consider that a microscopic ionized positive system has a negative curvature, so that the Jacobi equation in a Fermi-Walker transported vielbein basis will have a form similar to an inverted oscillator, [15]. A positive charged incident particle will have in this space a trajectory which will be interpreted as a repulsive electric force. If we have a negative charged incident particle interacting with an ionized positive system, the sum of all charges will be zero and by the postulate above presented, another system with positive curvature will be in process. The conditions of minimal energy of (3.8) will be obeyed, resulting in energy loss (fotons) and a new electrically neutral system (3.8) will be formed. This process will be seen as an attractive electric force. If the new system is not electrically neutral, therefore having a net positive charge, internal parts of the system will be electrically neutral, shielding parts of the system. The resultant geometry can be very
complex. It will be impossible to understand geometric details only by geometric methods. However, the appropriate and usual quantum mechanics operations with angular momenta can be converted in geometric operations as we have seen in the beginning of this section. This quantum mechanics approach to geometry, associated to the Landau-Raychaudhuri equation [14] can be very useful.

The Heisemberg’s and the Bohr’s postulates are part of a theory involving force and interaction, known as quantum mechanics. We have not exchanged the Heisemberg’s postulate and the Bohr’s postulate by geometric properties of classical nature. Actually, we have made classical geometric descriptions with similar properties, as much as possible, to the quantum postulates.

We offered the possibility of obtaining a quantum mechanics without forces, from only one postulate, simple systems, and geometry. From our point of view this theory will preserve the essence of quantum mechanics, differing from the usual one in some aspects above presented. This would represent a program of very unexpected results and would not be attractive enough to researchers because forces are considered as spacetime deformations. Although we have presented a geometric alternative to electric forces, unfortunately this is an unsolvable problem at the moment, so that force will be present in many situations. The conceptions of force and non-force may be seen, provisionally, as complementary. We believe that our results can be considered as a possible starting point to a quantum theory without forces.
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