Experimental studies of light meson decays are important guides to our understanding of how QCD works in the nonperturbative regime. In this context, the $\pi\pi$ and $\pi K$ interactions at low energies have been the subject of investigations for a few decades. In $\pi\pi$ interaction, one of the prominent features is the loop contribution to the $\pi\pi$ scattering: the $S$-wave charge-exchange rescattering $\pi^+\pi^- \rightarrow \pi^0\pi^0$ (as shown in Fig. 1) causes a prominent cusp at the center of mass energy corresponding to the summed mass of two charged pions. The cusp effect can shed light on the fundamental properties of QCD at low energies, by determining the strength of the $S$-wave $\pi\pi$ interaction [1–6]. Six decades ago this effect was predicted to be seen in $K^+ \rightarrow \pi^0\pi^0\pi^+$ [7], and it was finally observed in 2006 [8] by the NA48/2 experiment and studied further [9]. These results inspired theoretical predictions for the cusp in other decays, such as $K_L \rightarrow 3\pi^0$ [3, 5, 10] and $\eta \rightarrow 3\pi^0$ [11, 12], which were experimentally investigated: it was observed in the decay of $K_L \rightarrow 3\pi^0$ by KTeV [13], while no clear evidence was seen in $\eta \rightarrow 3\pi^0$ decay [14–17].

Another process where the cusp effect is expected to have a sizable contribution is the hadronic decay $\eta' \rightarrow \eta\pi^0\pi^0$ [18];
this has been experimentally investigated by BESIII [19], with
5.6×10^4 η' → ηπ^0π^0 events, and no evidence was seen,
while the A2 experiment [20] accumulated about 1.24×10^5
η' → ηπ^0π^0 decays and reported a deviation with a signi-
ficance of about 2.5σ and it is also studied in chiral pertur-
bation theory [21]. Therefore, it is essential to further investigate this
case with higher precision.

The recently available data of ten billion J/ψ events [22] at
BESIII imply an increased data sample of η' decays by nearly
an order of magnitude, offering a unique opportunity for fur-
nher investigations of the cusp effect. In this Letter, we present
the first evidence of the cusp effect in η' → ηπ^0π^0 and the
corresponding measurement of the ππ scattering length based
on the nonrelativistic effective field theory (NREFT) [18].

The BESIII detector [23, 24] records symmetric e^+e^- col-
lision events provided by the BEPCII storage ring [25]. The cylin-
drical core of the BESIII detector covers 93% of the full solid
angle and consists of a helium-based multilayer drift cham-
ber (MDC), a plastic scintillator time-of-flight (TOF) system,
and a CsI(Tl) electromagnetic calorimeter (EMC), which are
all enclosed in a superconducting solenoidal magnet provid-
ing a 1.0 T (0.9 T in 2012) magnetic field. The end cap TOF
system was upgraded in 2015 using multigap resistive plate
chamber technology [26–28].

To reconstruct events of J/ψ → γη' with η' → ηπ^0π^0, the
π^0 and η are selected by π^0/η → 2γ process. The charged
tracks are reconstructed from hits in the MDC. The polar an-
gle with respect to the MDC symmetry axis should be in the
range |cosθ| < 0.93. The distance away from the interaction
point should be less than 10.0 cm in the beam direction
and 1.0 cm in the radial direction. The photon candi-
dates are reconstructed using clusters of energy deposited in
the EMC. The energy deposited in the nearby TOF system is
included in EMC measurements to improve the reconstruc-
tion efficiency and the energy resolution. Photon candidates
are required to have a deposited energy larger than 25 MeV
in the barrel region (|cosθ| < 0.80) and 50 MeV in the end cap
regions (0.86 < |cosθ| < 0.92). A requirement on the EMC
cluster timing with respect to the most energetic photon,
−500 < T < 500 ns, is used to suppress electronic noise
and energy deposits unrelated to the event. The events with at
least seven photon candidates and no charged tracks are kept
for further analysis.

For each candidate event, the photon with the maximum en-
ergy is assumed to be the radiative photon originating from
the decay of J/ψ, while the remaining photons are used to recon-
struct π^0/η candidates. A one-constraint (1C) kinematic fit
is performed by constraining the invariant mass of photon pairs
to the π^0 or η mass, and the χ^2 for this fit is required to be
less than 25. Since the π^0 decays into two photons isotropically
in its rest frame, the angle of one photon in the π^0 rest frame
with respect to the π^0 momentum direction is required to satis-
fy |cosθ_π^0| < 0.95. Afterward, an eight-constraint (8C)
kinematic fit is performed for the γηπ^0π^0 combinations, re-
quiring energy-momentum conservation and constraining the
invariant masses of the three photon pairs to the nominal π^0/η
masses and of the ηπ^0π^0 combinations to the η' mass. If more
than one γηπ^0π^0 combination is found, only the one with the
least χ^2 is retained. After the requirement of χ^2 < 100,
432295 candidate events are accepted for further analysis; the
corresponding Dalitz plot is shown in Fig. 2.

![FIG. 2. Dalitz plot of η' → ηπ^0π^0.](image)

To investigate the background contamination, a 6C kinema-
tic fit, instead of the 8C fit, is performed on candidate
events, in which the constraints on the masses of η and η' are
removed. Figure 3 shows the ηπ^0π^0 invariant mass distribu-
tion of the data sample, after requiring χ^2 < 100 and η mass
window cut |M(γγ) − M_π^0| < 30 MeV/c^2 on the uncon-
strained photon pair, a clear η' peak is observed. In addition,
a ten billion J/ψ inclusive decay Monte Carlo (MC) sample
generated with LUNDCHARM [29, 30] is used to check pos-
sible background sources; the surviving events mainly con-
sist of the peaking background η' → 3π^0 decay channel and
the flat contribution from J/ψ → ωη, with ω → γπ^0 and
η → 3π^0. The background contamination rate is estimated
about 0.82%, and its shape on π^0π^0 and ηπ^0 mass spec-
trum is smooth; therefore, it is neglected in the further anal-
ysis.

Using an unbinned maximum likelihood method, we fit the
Dalitz plot of M^2(π^0π^0) versus M^2(ηπ^0) within the frame-
work of NREFT. (More details are given in the Supplemental
Material [31].) The resolution effect and detection efficiency
are studied by MC simulation and taken into account in the fit.

In the simplest case (fit I), only the tree level contribu-
tion is included in which the final state interaction effect is ig-
nored. In this case, the amplitude is the same as the general
parametrization used in Ref. [19]. The projections of the fit res-
tult to the Dalitz coordinates X and Y are shown in Figs. 4(a)
and 4(b), and they indicate a good description of data. The fit-
ted parameter values, shown in Table I, are consistent with the
previous BESIII measurement and the statistical uncertainties
are about one third of the previous results [19]. In Figs. 4(c)
and 4(d) the comparisons between data and the fit projections
of the ηπ^0 and π^0π^0 invariant mass distributions divided by

the phase space are presented. The discrepancy between data and fit result below the charged pion mass threshold corresponds to the cusp effect. Therefore, we perform alternative fits by including the loop contributions within the framework of NREFT to evaluate this effect, and fit I is taken as the baseline for the further loop level fits.

At the loop level amplitude, only ππ scattering is considered while ηπ scattering is ignored; the S-wave ππ scattering lengths \( a_0 \) and \( a_2 \) are included in the loop level amplitude by matching between NREFT amplitude and partial wave decomposition,

\[
\begin{align*}
C_{00} &= \frac{16\pi}{3}(a_0 + 2a_2)(1 - \xi), \\
C_x &= \frac{16\pi}{3}(a_2 - a_0)(1 + \frac{\xi}{3}), \\
C_{+-} &= \frac{8\pi}{3}(2a_0 + a_2)(1 + \xi), \\
\xi &= \frac{M_{\pi^0\pi^0}^2 - M_{\pi^+\pi^-}^2}{M_{\pi^0\pi^0}^2},
\end{align*}
\]

where \( C_x \) denotes the coupling coefficient of the cusp term \( \pi^+\pi^- \rightarrow \pi^0\pi^0 \), and \( C_{00} \) and \( C_{+-} \) are the coupling coefficients of noncusp terms \( \pi^0\pi^0 \rightarrow \pi^0\pi^0 \) and \( \pi^+\pi^- \rightarrow \pi^+\pi^- \), which are defined in Ref. [18].

The distribution of \( M^2(\eta\pi^0) \) is determined by the whole amplitude and all five parameters \( a, b, d, a_0, \) and \( a_2 \), while the distribution of \( M^2(\eta^\prime \pi^0) \) is mainly determined by parameter \( d \), where \( a, b \), and \( d \) are coefficients in tree level amplitude.

To verify the prediction of NREFT and evaluate the scattering length combination \( a_0 - a_2 \), we perform many unbinned maximum likelihood fits in different cases after including the contributions from the amplitudes at one- and two-loop levels.

In the case when all the parameters are free (fit II), the fit quality is improved, and we obtain a statistical significance of \( 3.8\sigma \) compared to fit I. In Fig. 5, the comparison between the fit and data for the projections in different variables, as well as the pull distributions, shows that the fit provides a good description, in particular, for the region below the charged

| Parameters | Fit I | Fit II | Fit III | Fit IV |
|------------|------|-------|---------|-------|
| \( a \)    | \(-0.075 \pm 0.003 \pm 0.001 \) | \(-0.207 \pm 0.013 \) | \(-0.143 \pm 0.010 \) | \(-0.077 \pm 0.003 \pm 0.001 \) |
| \( b \)    | \(-0.073 \pm 0.005 \pm 0.001 \) | \(-0.051 \pm 0.014 \) | \(-0.038 \pm 0.006 \) | \(-0.066 \pm 0.006 \pm 0.001 \) |
| \( d \)    | \(-0.066 \pm 0.003 \pm 0.001 \) | \(-0.068 \pm 0.004 \) | \(-0.067 \pm 0.003 \) | \(-0.068 \pm 0.004 \pm 0.001 \) |
| \( a_0 - a_2 \) | \(-0.174 \pm 0.006 \) | \(0.225 \pm 0.062 \) | \(0.226 \pm 0.060 \pm 0.013 \) |
| \( a_0 \)  | \(-0.497 \pm 0.094 \) | \(- \) | \(- \) | \(- \) |
| \( a_2 \)  | \(-0.322 \pm 0.129 \) | \(- \) | \(- \) | \(- \) |
| Statistical significance | \(-3.4\sigma \) | \(3.7\sigma \) | \(3.6\sigma \) |
pion mass threshold. However, the correlation between the four parameters $a$, $b$, $a_0$, and $a_2$ is very large, as shown in Eq. (2). This strong correlation between $a$, $b$, $a_0$, and $a_2$ may be caused by the loop level amplitude contribution to noncusp terms. The scattering length combination is calculated to be $a_0 - a_2 = 0.174 \pm 0.066$.

$$
\begin{pmatrix}
  a & b & d & a_0 & a_2 \\
  0.831 & 0.189 & -0.966 & -0.789 \\
  0.348 & -0.918 & -0.839 \\
  -0.257 & -0.210 & 0.872 \\
\end{pmatrix}
$$

To reduce the correlations between parameters, we also made an attempt (fit III) by fixing $a_0 + 2a_2 = 0.1312$ according to the theoretical values $a_0 = 0.220 \pm 0.005$ and $a_2 = -0.0444 \pm 0.0010$ [18] and setting $a_0 - a_2$ as a free parameter, since only $C_x$ contributes to the cusp effect. The fit result presented in Fig. 5 shows a good agreement with data, also in the region below the charged pion mass threshold. The fitted parameter values are summarized in Table I, and the corresponding correlations are shown in Eq. (3); the obtained value of $a_0 - a_2 = 0.225 \pm 0.062$ is in agreement with the theoretical value $0.2644 \pm 0.0051$ [18]. We also test by changing the value of $a_0 + 2a_2$ or fixing both $a_0 + 2a_2$ and $2a_0 + a_2$ to theoretical values, the fit results are consistent with the result of fit III, and $a_0 - a_2$ is not sensitive to fixed value.

$$
\begin{pmatrix}
  a & b & d & a_0 - a_2 \\
  -0.560 & -0.046 & -0.955 \\
  0.249 & 0.457 & -0.032 \\
\end{pmatrix}
$$

Comparing to the tree level amplitude, the loop contributions with $C_{00}$ and $C_{+-}$ are expected to be small. Additionally, we performed an alternative fit (fit IV) by ignoring non-cusp terms with $C_{00}$ and $C_{+-}$ and only introducing the decay amplitude with $C_x$ for the description of the cusp effect. In this case, the fitted values of different parameters, summarized in Table I, are in agreement with those of fit I, the correlations shown in Eq. (4) are reduced and the statistical significance of the cusp effect is $3.6\sigma$, while the scattering length combination $a_0 - a_2 = 0.226 \pm 0.060$ is consistent with that in Ref. [18]. In addition, we found that the log-likelihood value of fit IV is very close to those of fit II fit III, which implies that the introduction of the loop contributions with $C_{00}$ and $C_{+-}$ has little impact on the improvement of the fit quality and the cusp effect, but significantly increases the correlations between the different parameters. Therefore, in this analysis, it is reasonable to ignore these loop contributions in fitting data,

$$
\begin{pmatrix}
  a & b & d & a_0 - a_2 \\
  -0.363 & -0.253 & 0.126 \\
  0.257 & 0.237 & -0.107 \\
\end{pmatrix}
$$

The systematic uncertainties for the Dalitz plots analysis are listed in Table II. We calculate the total systematic uncertainty by assuming that all the contributions are independent and adding them in quadrature.

The photon detection efficiency is studied with the control sample of $J/\psi \rightarrow \rho^0 \pi^0$ events. To evaluate the impact from the slight discrepancy between data and MC simulation, we perform a correction on the photon detection and the change of the fit results is considered as the systematic uncertainty.

To estimate the uncertainties from the 1C kinematic fit for $\pi^0$ and $\eta$, we selected as control samples $J/\psi \rightarrow \pi^+ \pi^- \pi^0$ and $J/\psi \rightarrow \gamma \eta'$ with $\eta' \rightarrow \eta \pi^+ \pi^-$, without kinematic fit. After taking into account the discrepancy between data and MC simulation, repeating the fit with the weighted events leads to changes of the parameter values, which are assigned as the systematic uncertainties.

To check if the photon miscombinations can effect the fitted parameters, we generate a MC sample based on NREFT amplitude and tag miscombination events by matching the truth and the reconstructed value of photon momentum. Two fits are performed to the sample with and without miscombination events, and the change of the results is taken as the systematic uncertainty.

To evaluate the uncertainty associated with the efficiency parametrization, we change the Dalitz plot variables to $M^2(\eta\pi)$ and $\cos \theta$, where $\theta$ is the angle between the directions of the two $\pi$s in the rest frame of $\eta\pi$. We repeat the fit based on the newly defined Dalitz plot variables, and the change of the resulting parameters with respect to the nominal results is assigned as the systematic uncertainty.

The uncertainty of the $8C$ kinematic fit mainly comes from the inconsistency of the photon resolution between data and MC simulation. We adjust the energy resolution in the reconstructed photon error matrix to ensure that the MC simulation provides a good description of data. Afterward, an alternative fit is performed and the change of the fitted parameters with respect to the nominal result is taken as the systematic uncertainty.

To estimate the uncertainty from resolution effect, we vary the resolution by $\pm10\%$ and perform alternative fits. The maximum change with respect to the nominal result is taken as the systematic uncertainty.

In summary, using ten billion $J/\psi$ events collected with the BESIII detector, we select a $\eta' \rightarrow \eta \pi^0 \pi^0$ sample 8 times larger than that previously analyzed by BESIII, and perform a Dalitz plot analysis within the framework of nonrelativistic effective field theory. The fit with tree level amplitude shows a discrepancy below the charged pion mass threshold, which implies the existence of the cusp effect. To describe the data
in this region, the contributions at one- and two-loop level are introduced in the decay amplitude. We perform alternative analyses by taking into account the cusp effect and the results are summarized in Table I. For each case, the amplitude provides a good description of the structure around the charged pion mass threshold and the statistical significance is found to be around $3.5\sigma$. The scattering length combination $a_0 - a_2$ is measured to be $0.226 \pm 0.060 \pm 0.013$, which is in good agreement with the theoretical value of $0.2644 \pm 0.0051$ [18] within the uncertainties. The observation of the evidence of the cusp effect in $\eta' \rightarrow \eta\pi^0\pi^0$ decay demonstrates the excellent potential to investigate the underlying dynamics of light mesons at the BESIII experiment. The prospects [32] for the precise measurements are very promising at the planned Super Tau-Charm Factories [33, 34].

The BESIII Collaboration thanks the staff of BEPCII and the IHEP computing center for their strong support. This work is supported in part by National Key R&D Program of China under Contracts No. 2020YFA0406300, No. 2020YFA0406400; National Natural Science Foundation of China (NSFC) under Contracts No. 11635010, No. 11735014, No. 11835012, No. 11935015, No. 11935016, No. 11935018, No. 1196141012, No. 12005195, No. 12022510, No. 12025502, No. 12035009, No. 12035013, No. 12192260, No. 12192261, No. 12192262, No. 12192263, No. 12192264, No. 12192265, No. 12225509; the Chinese Academy of Sciences (CAS) Large-Scale Scientific Facility Program; Joint Large-Scale Scientific Facility Funds of the NSFC and CAS under Contract No. U1832207; CAS Key Research Program of Frontier Sciences under Contract No. QYZDJ-SSW-SLH040; 100 Talents Program of CAS; INPAC and Shanghai Key Laboratory for Particle Physics and Cosmology; ERC under Contract No. 758462; European Union’s Horizon 2020 research and innovation program under Marie Sklodowska-Curie Grant Agreement under Contract No. 894790; German Research Foundation DFG under Contracts No. 443159800, Collaborative Research Center CRC 1044, GRK 2149; Istituto Nazionale di Fisica Nucleare, Italy; Ministry of Development of Turkey under Contract No. DPT2006K-120470; National Science and Technology fund; STFC (United Kingdom); The Royal Society, UK under Contracts No. DH140054, No. DH160214; The Swedish Research Council; U. S. Department of Energy under Award No. DE-FG02-05ER41374.

FIG. 5. The fit result projections divided by phase space of different models to variable (a) $M^2(\eta\pi^0)$ and (b) $M^2(\pi^0\pi^0)$. The black dots with error bars are from data. The solid lines are fit results from the corresponding models. The red dashed line indicates the charged pion mass threshold. The cusp region is also shown in the inset.

| Parameterization | Fit I | Fit IV |
|------------------|------|-------|
| Source           | $a$  | $b$  | $d$  | $a_0 - a_2$ | $a_0 - a_2$ |
| Photon detection | 0.7  | 0.4  | 1.0  | 0.6  | 0.4  | 0.9  | 1.8  |
| $\eta$ IC kinematic fit | 0.1  | 0.6  | 0.0  | 0.1  | 0.7  | 0.0  | 0.2  |
| $\pi^0$ IC kinematic fit | 0.1  | 0.2  | 1.0  | 0.1  | 0.2  | 0.9  | 0.3  |
| Photon miscombination | 0.0  | 0.2  | 1.1  | 0.0  | 0.2  | 1.1  | 1.6  |
| Efficiency presentation | 0.7  | 1.0  | 0.4  | 0.7  | 0.9  | 0.4  | 1.9  |
| Kinematic fit     | 0.5  | 1.3  | 0.7  | 0.4  | 0.9  | 0.8  | 4.2  |
| Resolution        | 0.0  | 0.0  | 0.0  | 0.1  | 0.3  | 0.0  | 2.0  |
| Total             | 1.1  | 1.8  | 2.0  | 1.0  | 1.6  | 1.9  | 5.6  |
Supplemental Material: A Brief Description of NREFT amplitude of $\eta' \to \eta \pi^0 \pi^0$ Decay

This supplemental material is based on Ref. [18]. In the $\eta' \to \eta \pi^0 \pi^0$ decay

$$\eta'(P_{\eta'}) \to \pi^0(p_1) \pi^0(p_2) \eta(p_3),$$  

the kinematical variables $s_i$ are defined as $s_i = (P_{\eta'} - p_i)^2$, $i = 1, 2, 3$, and $s_1 + s_2 + s_3 = M_{\eta'}^2 + M_1^2 + 2M_2^2$. The Dalitz plot distribution of this decay can be expanded by kinematical variables $X$ and $Y$

$$X = \frac{\sqrt{3}|s_1 - s_2|}{2M_{\eta'}Q_{\eta'}}, \quad Y = \frac{(M_\eta + 2M_{\eta'})[M_{\eta'} - M_\eta]^2 - s_3}{2M_{\eta'}M_{\eta'}Q_{\eta'}} - 1$$

where $T_i$ denote kinetic energy of mesons in the rest frame of $\eta'$, and $Q_{\eta'} = M_{\eta'} - M_\eta - 2M_{\eta'}$. The Dalitz plot distribution can be expanded by $X$ and $Y$ around the center of the Dalitz plot

$$|\mathcal{M}(X, Y)|^2 = |\mathcal{N}|^2(1 + aY + bY^2 + cX + dX^2 + \cdots),$$

which is known as general parameterization. Here $\mathcal{N}$ is a normalization factor and parameter $c$ is fixed at 0 since two $\pi^0$s are identical bosons. The general parameterization can be also expressed as

$$\mathcal{M}(X, Y) = \mathcal{N}\{1 + \frac{a}{2}Y + \frac{b}{2}(\frac{a^2}{4}Y^2 + \frac{d}{2}X^2 + \cdots)\}.$$  

The NREFT amplitude of $\eta' \to \eta \pi \pi$ can be decomposed to

$$\mathcal{M}_{\eta' \to \eta \pi \pi} = \mathcal{M}_{\eta' \to \eta \pi^0 \pi^0} + \mathcal{M}_{\eta' \to \eta \pi \pi} + \mathcal{M}_{\eta' \to \eta \pi \pi} + \cdots,$$

where $s_i$ are the low-energy coupling coefficients of $\eta' \to \eta \pi^0 \pi^0$ decay and $H_i$ is for $\eta' \to \eta \pi^0 \pi^0$ decay. The charged decay mode is introduced for the further description of loop level amplitude and we assume $H_i = -G_i$ according to the isospin limit [18–20]. $G_i$

can be evaluated by matching to the general parameterization

$$G_0 = \mathcal{N}\{1 - \frac{a}{2} + \frac{1}{2}(b - \frac{a^2}{4})\},$$
$$G_1 = \mathcal{N}\{\frac{a}{2} - (b - \frac{a^2}{4})\} \frac{M_{\eta}}{M_{\eta} + 2M_{\eta'}},$$
$$G_2 = \mathcal{N}(b - \frac{a^2}{4}) \frac{(M_{\eta} + 2M_{\eta'})^2}{2M_{\eta}^2Q_{\eta'}},$$
$$G_3 = \mathcal{N}\frac{3d}{2Q_{\eta'}}.$$  

The loop level amplitude of $\eta' \to \eta \pi^0 \pi^0$ decay are

$$\mathcal{M}^{\text{one-loop}}_{\eta' \to \eta \pi^0 \pi^0}(s_1, s_2, s_3) = B_{N1}(s_3)J_{++}(s_3)$$
$$+ B_{N2}(s_3)J_{00}(s_3),$$

$$\mathcal{M}^{\text{two-loop}}_{\eta' \to \eta \pi^0 \pi^0}(s_1, s_2, s_3) = C_{00}(s_3)B_{N2}(s_3)J_{00}(s_3)J_{++}(s_3)$$
$$+ C_{00}(s_3)B_{N1}(s_3)J_{00}(s_3)J_{++}(s_3)$$
$$+ C_{20}(s_3)B_{C2}(s_3)J_{00}(s_3)J_{++}(s_3),$$

with one-loop function

$$J_{ab} = \frac{ig_{ab}(s)}{8\pi\sqrt{s}},$$

$$q_{ab}^2(s) = \frac{\lambda(s, M_a^2, M_b^2)}{4s},$$

$$\lambda(a, b, c) = a^2 + b^2 + c^2 - 2(ab + ac + bc),$$

and neutral channel polynomials

$$B_{N1}(s_3) = 2C_x(s_3)\left\{\sum_{i=0}^{2} H_i X_i^3 + H_3 \frac{4Q_3^2}{3s_3} q_{++}(s_3)\right\},$$

$$B_{N2}(s_3) = C_{00}(s_3)\left\{\sum_{i=0}^{2} G_i X_i^3 + G_3 \frac{4Q_3^2}{3s_3} q_{00}(s_3)\right\},$$

and charged channel polynomials

$$B_{C1}(s_3) = 2C_{++}(s_3)\left\{\sum_{i=0}^{2} H_i X_i^3 + H_3 \frac{4Q_3^2}{3s_3} q_{++}(s_3)\right\},$$

$$B_{C2}(s_3) = C_x(s_3)\left\{\sum_{i=0}^{2} G_i X_i^3 + G_3 \frac{4Q_3^2}{3s_3} q_{00}(s_3)\right\},$$

where

$$C_{bc}(s_a) = C_{bc} + 4D_{bc}q_{bc}(s_a) + 16F_{bc}q_{bc}(s_a),$$

$$Q_{a}^2 = \frac{\lambda(M_a^2, \sqrt{s_a}, s_a)}{4M_a^2}.$$  

The parameters $C_i$, $D_i$ and $F_i$ are coupling coefficients of $\pi \pi$ interaction and are evaluated by matching to the effective range expansion of $\pi \pi$ scattering, where $i$ represent different
\( \pi \pi \) rescattering channels: 

\[(00) \pi^0 \pi^0 \rightarrow \pi^0 \pi^0, (x) \pi^+ \pi^- \rightarrow \pi^0 \pi^0, (\pm) \pi^+ \pi^- \rightarrow \pi^+ \pi^-.
\]

\[
C_{00} = \frac{16\pi}{3} (a_0 + 2a_2)(1 - \xi),
\]

\[
C_x = \frac{16\pi}{3} (a_2 - a_0)(1 + \frac{\xi}{3}),
\]

\[
C_{+-} = \frac{8\pi}{3} (2a_0 + a_2)(1 + \xi),
\]

\[
\xi = \frac{M_x^{2} - M^{2}_{\pi\eta}}{M^{2}_{\pi\pi}},
\]

\[
D_{00} = \frac{4\pi}{3} (b_0 + 2b_2),
\]

\[
D_x = \frac{4\pi}{3} (b_0 - b_2),
\]

\[
D_{+-} = \frac{2\pi}{3} (2b_0 + b_2),
\]

\[
F_{00} = \frac{\pi}{3} (f_0 + 2f_2),
\]

\[
F_x = \frac{\pi}{3} (f_2 - f_0),
\]

\[
F_{+-} = \frac{\pi}{6} (2f_0 + f_2),
\]

where \( a_i, b_i \) and \( f_i \) are S-wave scattering length, effective ranges and shape parameters of isospin 0 and 2, respectively, \( a_0 \) and \( a_2 \) are taken as free or fixed parameters in different cases in our study, and \( b_1 \) are fixed to theoretical value \( b_0 = (0.276 \pm 0.006) \times M^{-2}_{\pi\pi}, b_2 = (-0.0803 \pm 0.0012) \times M^{-2}_{\pi\pi} \), and \( f_i \) are fixed to 0. The \( \pi \eta \) scattering terms are ignored because the \( \pi \eta \) scattering is much weaker than the \( \pi \pi \) scattering.

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