q-Fourier Transform and its Inversion-Problem

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Abstract. Tsallis’ q-Fourier transform is not generally one-to-one. It is shown here that, if we eliminate the requirement that $q$ be fixed, and let it instead “float”, a simple extension of the $F_q$—definition, this procedure restores the one-to-one character.

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1. Introduction

Nonextensive statistical mechanics (NEXT) [1, 2, 3], a current generalization of the Boltzmann-Gibbs (BG) one, is actively studied in diverse areas of Science. NEXT is based on a nonadditive (though extensive [4]) entropic information measure characterized by the real index $q$ (with $q = 1$ recovering the standard BG entropy). It has been applied to variegated systems such as cold atoms in dissipative optical lattices [5], dusty plasmas [6], trapped ions [7], spinglasses [8], turbulence in the heliosheath [9], self-organized criticality [10], high-energy experiments at LHC/CMS/CERN [11] and RHIC/PHENIX/Brookhaven [12], low-dimensional dissipative maps [13], finance [14], galaxies [15], Fokker-Planck equation’s applications [16], etc.

NEXT can be advantageously expressed via q-generalizations of standard mathematical concepts (the logarithm and exponential functions, addition and multiplication, Fourier transform (FT) and the Central Limit Theorem (CLT) [17, 22, 25]). The q-Fourier transform $F_q$ exhibits the nice property of transforming q-Gaussians into q-Gaussians [17]. Recently, plane waves, and the representation of the Dirac delta into plane waves have been also generalized [18, 19, 20, 21].

A serious problem afflicts $F_q$. It is not generally one-to-one. A detailed example is discussed below. In this work we show that by recourse to a rather simple but efficient stratagem that consists in

- eliminating the requirement that $q$ be fixed and instead
2. Generalizing the q-Fourier transform

We define, following [17], a q-Fourier transform of \( f(x) \in L^1(\mathbb{R}) \), \( f(x) \geq 0 \) as

\[
F(k,q) = [H(q-1) - H(q-2)]
\times \int_{-\infty}^{\infty} f(x)\{1 + i(1 - q)kx[f(x)]^{(q-1)}\}^{\frac{1}{1-q}} \, dx
\tag{2.1}
\]

where \( H(x) \) is the Heaviside step function.

The only difference between this definition and that given in [17] is that \( q \) is not fixed and varies within the interval \([1,2)\). Herein lies the hard-core of our presentation. This simple change of perspective makes it easy to find the inversion-formula for (2.1) by recourse to the inverse Fourier transform

\[
f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[ \lim_{\epsilon \to 0^+} \int_{1-\epsilon}^{2} F(k,q)\delta(q - 1 - \epsilon) \, dq \right] e^{-ikx} \, dk.
\tag{2.2}
\]

As a consequence, we see that this q-Fourier transform is one-to-one, unlike what happens in [23],[24]. The link between Eqs. (2.1)- (2.2) is discussed in more detail in the illustrative example presented below (next Section).

3. Example

As an illustration we discuss the example given by Hilhorst in Ref. ([22]). Let \( f(x) \) be

\[
f(x) = \begin{cases} 
\left( \frac{\lambda}{x} \right)^\beta ; & x \in [a,b] ; \ 0 < a < b ; \ \lambda > 0 \\
0 ; & x \text{ outside } [a,b].
\end{cases}
\tag{3.1}
\]

The corresponding q-Fourier transform is

\[
F(k,q) = \lambda^\beta \int_{a}^{b} x^{-\beta}\{1 + i(1 - q)k\lambda^\beta(q-1)x^{1-\beta(q-1)}\}^{\frac{1}{1-q}} \, dx.
\tag{3.2}
\]

Effecting the change of variables

\[
y = x^{1-\beta(q-1)},
\]

we have for (3.2)

\[
F(k,q) = [H(q-1) - H(q-2)]
\times \frac{\lambda^\beta}{1 - \beta(q-1)} \int_{a^{1-\beta(q-1)}}^{b^{1-\beta(q-1)}} y^{\beta(q-2)}\{1 + i(1 - q)k\lambda^\beta(q-1)y\}^{\frac{1}{1-q}} \, dy.
\tag{3.3}
\]
Now, (3.3) can be rewritten in the useful form

\[ F(k, q) = [H(q - 1) - H(q - 2)] \times \left\{ H(q - 1) - H \left[ q - \left( 1 + \frac{1}{\beta} \right) \right] \right\} \]

\[ \times \frac{\lambda^\beta}{1 - \beta(q - 1)} \lim_{\epsilon \to 0^+} \int_{a^{1-\beta(q-1)}}^{b^{1-\beta(q-1)}} y^{-\frac{\beta(2-q)}{1-\beta(q-1)}} \{1 + i(1 - q)k \lambda^{q(1-\beta)} \} \frac{1}{1-q} dy \]

\[ + \left\{ H \left[ q - \left( 1 + \frac{1}{\beta} \right) \right] - H(q - 2) \right\} \]

\[ \times \frac{\lambda^\beta}{\beta(q - 1) - 1} \lim_{\epsilon \to 0^+} \int_{b^{1-\beta(q-1)}}^{a^{1-\beta(q-1)}} y^{-\frac{\beta(q-2)}{1-\beta(q-1)}} \{1 + i(1 - q)k \lambda^{q(1-\beta)} \} \frac{1}{1-q} dy \} \]. \quad (3.4)\]

Taking into account that the involved integrals are defined in a finite interval, we can cast (3.4) as

\[ F(k, q) \]

\[ = [H(q - 1) - H(q - 2)] \times \left\{ H(q - 1) - H \left[ q - \left( 1 + \frac{1}{\beta} \right) \right] \right\} \]

\[ \times \frac{\lambda^\beta}{1 - \beta(q - 1)} \lim_{\epsilon \to 0^+} \int_{a^{1-\beta(q-1)}}^{b^{1-\beta(q-1)}} y^{-\frac{\beta(2-q)}{1-\beta(q-1)}} \{1 + i(1 - q)k + i\epsilon \lambda^{q(1-\beta)} \} \frac{1}{1-q} dy \]

\[ + \left\{ H \left[ q - \left( 1 + \frac{1}{\beta} \right) \right] - H(q - 2) \right\} \]

\[ \times \frac{\lambda^\beta}{\beta(q - 1) - 1} \lim_{\epsilon \to 0^+} \int_{b^{1-\beta(q-1)}}^{a^{1-\beta(q-1)}} y^{-\frac{\beta(q-2)}{1-\beta(q-1)}} \{1 + i(1 - q)k + i\epsilon \lambda^{q(1-\beta)} \} \frac{1}{1-q} dy \} \]. \quad (3.5)\]

We now use results of the Integral’s table [26] to evaluate (3.5) and get

\[ \lim_{\epsilon \to 0^+} \int_{a^{1-\beta(q-1)}}^{\infty} y^{-\frac{\beta(2-q)}{1-\beta(q-1)}} \{1 + i(1 - q)k + i\epsilon \lambda^{q(1-\beta)} \} \frac{1}{1-q} dy \]

\[ = \frac{(q - 1)[1 - \beta(q - 1)]a^{\frac{q-2}{1-q}}}{(2 - q)[(1 - q)i(k + i0)\lambda^{q(1-\beta)} \frac{2}{1-q}]}

\times F \left( \frac{1}{q - 1}, \frac{2 - q}{(q - 1)[1 - \beta(q - 1)]}; 1, \frac{1}{q - 1} + \frac{\beta(2-q)}{1 - \beta(q-1)}; \right)

\[ - \frac{1}{(1 - q)i(k + i0)\lambda^{q(1-\beta)} a^{1-\beta(q-1)}} \]. \quad (3.6)\]
and 
\[
\lim_{\epsilon \to 0^+} \int_0^{\gamma(\beta-1)} y^{\beta(2-q)} \{1 + i(1-q)(k + i\epsilon)\lambda^{\beta(q-1)}y\}^{1/q} dy \\
= \frac{[\beta(q-1) - 1]a^{1-\beta}}{\beta - 1} \times F\left(\frac{1}{q-1}, \frac{\beta - 1}{\beta(q-1) - 1}, \frac{\beta q - 2}{\beta(q-1) - 1}; (q-1)i(k + i0)\lambda^{\beta(q-1)}a^{1-\beta(q-1)}\right)
\]
where \(F(a, b, c; z)\) is the hypergeometric function. Thus we obtain for \(F(k, q)\)
\[
F(k, q) = [H(q-1) - H(q-2)] \times \left\{ \left\{ H(q-1) - H\left[q - \left(1 + \frac{1}{\beta}\right)\right] \right\} \right\}^{\frac{(q-1)\lambda^\beta}{(2-q)[(1-q)i(k + i0)\lambda^\beta]^{\frac{1}{q-1}}}} \\
\times \left\{ a^{\frac{q-2}{q-1}} F\left(\frac{1}{(q-1)[1-\beta(q-1)]}, \frac{1}{q-1} + \frac{\beta(2-q)}{1-\beta(q-1)}; \right) \\
- b^{\frac{q-2}{q-1}} F\left(\frac{1}{(q-1)[1-\beta(q-1)]}, \frac{1}{q-1} + \frac{\beta(2-q)}{1-\beta(q-1)}; \right) \\
+ \left\{ H\left[q - \left(1 + \frac{1}{\beta}\right)\right] - H(q-2) \right\}^{\frac{\lambda^\beta}{\beta - 1}} \\
\times \left\{ a^{1-\beta} F\left(\frac{1}{(q-1)[1-\beta(q-1)]}, \frac{1}{q-1} + \frac{\beta(2-q)}{1-\beta(q-1)}; \right) \\
- b^{1-\beta} F\left(\frac{1}{(q-1)[1-\beta(q-1)]}, \frac{1}{q-1} + \frac{\beta(2-q)}{1-\beta(q-1)}; \right) \right\}. \tag{3.8}
\]
As we can see from (3.8), \(F(k, q)\) depends on \(a\) and \(b\), and, as consequence, is one-to-one, as shown in Section 2.

However, and this is the crucial issue, if we fix \(q\) and select \(\beta = 1/(q-1)\) (3.8) simplifies and adopts the appearance
\( F(k, q) = \frac{1}{q - 1} \frac{q - 1}{2 - q} [H(q - 1) - H(q - 2)] \)

\[
\times \left[ a^{\frac{q-2}{q-1}} F\left( \frac{1}{q-1}, \frac{2-q}{q-1}, \frac{2-q}{q-1}; (q-1)i(k+i0)\lambda \right) - b^{\frac{q-2}{q-1}} F\left( \frac{1}{q-1}, \frac{2-q}{q-1}, \frac{2-q}{q-1}; (q-1)i(k+i0)\lambda \right) \right].
\] (3.9)

With the help of the result given in [27] for
\( F(-a, b, b, -z) = (1+z)^a \),
we obtain for (3.9):
\[
F(-a, b, b, -z) = (1+z)^a,
\]

\[
F(k, q) = \lambda^{\frac{1}{q-1}} \frac{q - 1}{2 - q} [H(q - 1) - H(q - 2)] \left( a^{\frac{q-2}{q-1}} - b^{\frac{q-2}{q-1}} \right) [1 + (1-q)ik\lambda]^{\frac{1}{1-q}}.
\] (3.10)

Using now the expression for \( \lambda \) of [22], i.e.,
\[
\lambda = \left[ \left( \frac{q - 1}{2 - q} \right) \left( a^{\frac{q-2}{q-1}} - b^{\frac{q-2}{q-1}} \right) \right]^{1-q},
\]
we have, finally,
\[
F(k, q) = [H(q - 1) - H(q - 2)] [1 + (1-q)ik\lambda]^{\frac{1}{1-q}},
\] (3.11)

which is the result given by Hilhorst in [22], that is independent of the values adopted by \( a, b \). Such independence is evidence that \( F(k, q) \) is not one-to-one. All infinite \( F(k, q, a, b) \) associated to each possible pair \( a, b \) coalesce now in a infinitely degenerate solution \( F(k, q) \). As a conclusion we can say that for fixed \( q \) the q-Fourier transform is NOT one-to-one for fixed \( q \). On the contrary, as we have shown in section 2, when \( q \) is NOT fixed, the q-Fourier transform is indeed one-to-one.

**Conclusions**

In the present communication we have discussed the NOT one-to-one nature of the q-Fourier transform \( F_q \). We have shown that, if we eliminate the requirement that \( q \) be fixed and let it “float” instead, such simple extension of the \( F_q \)-definition restores the desired one-to-one character.

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**References**

[1] C. Tsallis, J. Stat. Phys. 52 (1988) 479.

[2] M. Gell-Mann, C. Tsallis (Eds.), Nonextensive Entropy Interdisciplinary Applications, Oxford University Press, New York, 2004; C. Tsallis, Introduction to Nonextensive Statistical Mechanics Approaching a Complex World, Springer, New York, 2009.

[3] A. R. Plastino, A. Plastino, Phys. Lett A 177 (1993) 177.
[4] C. Tsallis, M. Gell-Mann, Y. Sato, Proc. Natl. Acad. Sci. USA 102 (2005) 15377; F. Caruso, C. Tsallis, Phys. Rev. E 78 (2008) 021102.

[5] P. Douglas, S. Bergamini, F. Renzoni, Phys. Rev. Lett. 96 (2006) 110601; G.B. Bagci, U. Tiranakli, Chaos 19 (2009) 033113.

[6] B. Liu, J. Goree, Phys. Rev. Lett. 100 (2008) 055003.

[7] R.G. DeVoe, Phys. Rev. Lett. 102 (2009) 063001.

[8] R.M. Pickup, R. Cywinski, C. Pappas, B. Farago, P. Fouquet, Phys. Rev. Lett. 102 (2009) 097202.

[9] L.F. Burlaga, N.F. Ness, Astrophys. J. 703 (2009) 311.

[10] F. Caruso, A. Pluchino, V. Latora, S. Vinciguerra, A. Rapisarda, Phys. Rev. E 75 (2007) 055101(R); B. Bakar, U. Tiranakli, Phys. Rev. E 79 (2009) 040103(R); A. Celikoglu, U. Tiranakli, S.M.D. Queiros, Phys. Rev. E 82 (2010) 021124.

[11] V. Khachatryan, et al., CMS Collaboration, J. High Energy Phys. 1002 (2010) 041; V. Khachatryan, et al., CMS Collaboration, Phys. Rev. Lett. 105 (2010) 022002.

[12] Adare, et al., PHENIX Collaboration, Phys. Rev. D 83 (2011) 052004; M. Shao, L. Yi, Z.B. Tang, H.F. Chen, C. Li, Z.B. Xu, J. Phys. G 37 (8) (2010) 085104.

[13] M.L. Lyra, C. Tsallis, Phys. Rev. Lett. 80 (1998) 53; E.P. Borges, C. Tsallis, G.F.J. Ananos, P.M.C. de Oliveira, Phys. Rev. Lett. 89 (2002) 254103; G.F.J. Ananos, C. Tsallis, Phys. Rev. Lett. 93 (2004) 020601; U. Tiranakli, C. Beck, C. Tsallis, Phys. Rev. E 75 (2007) 040106(R); U. Tiranakli, C. Tsallis, C. Beck, Phys. Rev. E 79 (2009) 056209.

[14] L. Borland, Phys. Rev. Lett. 89 (2002) 098701.

[15] A. R. Plastino, A. Plastino, Phys. Lett A 174 (1993) 834.

[16] A. R. Plastino, A. Plastino, Physica A 222 (1995) 347.

[17] S. Umarov, C. Tsallis, S. Steinberg, Milan J. Math. 76 (2008) 307; S. Umarov, C. Tsallis, M. Gell-Mann, S. Steinberg, J. Math. Phys. 51 (2010) 033502.

[18] M. Jauregui, C. Tsallis, J. Math. Phys. 51 (2010) 063304.

[19] A. Chevreuil, A. Plastino, C. Vignat, J. Math. Phys. 51 (2010) 093502.

[20] M. Mamode, J. Math. Phys. 51 (2010) 123509.

[21] A. Plastino and M.C.Rocca: J. Math. Phys 52, (2011) 103503.

[22] H.J.Hilhorst: J. Stat. Mech. (2010) P10023

[23] M.Jauregui and C.Tsallis: Phys. Lett. A 375, (2011) 2085.

[24] M.Jauregui, C.Tsallis and E.M.F. Curado: J. Stat. Mech. P10016 (2011).

[25] M. Jauregui, C, Tsallis, Phys. Lett. A 375 (2011) 2085.

[26] L. S. Gradshtein and I. M. Ryzhik : Table of Integrals, Series, and Products. Fourth edition, Academic Press (1965) 3.194 1 and 3.194 2 pages 284 and 285.

[27] M.Abramowitz and I.A.Stegun: Handbook of Mathematical Functions. National Bureau of Standards. Applied Mathematical Series 55 Tenth Printing (1972), 15.1.8 page 556.
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