Massive and Massless Monopoles and Duality

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Abstract

I review some aspects of BPS magnetic monopoles and of electric-magnetic duality in theories with arbitrary gauge groups. When the symmetry is maximally broken to a U(1)$^r$ subgroup, all magnetically charged configurations can be understood in terms of $r$ species of massive fundamental monopoles. When the unbroken group has a non-Abelian factor, some of these fundamental monopoles become massless and can be viewed as the duals to the massless gauge bosons. Rather than appearing as distinct solitons, these massless monopoles are manifested as clouds of non-Abelian field surrounding one or more massive monopoles. I describe in detail some examples of solutions with such clouds.

1 Introduction

Elementary treatments of quantum field theory usually introduce particles as the quanta of the small oscillation of a weakly coupled field about the vacuum. There is one particle species for each elementary field, with the masses of the particles entering as parameters in the Lagrangian. However, it turns out that particles can also arise in a very different fashion. In many cases, the classical field equations of the theory possess finite energy solutions that are localized in space. These solitons also give rise to one-particle states in the quantum theory. To lowest approximation, the mass of the soliton is equal to the energy of the classical solution and is typically of the form $M_{\text{soliton}} \sim m/\lambda$, where $m$ is an elementary particle mass and $\lambda$ is some small coupling.

At first sight, these two types of particles seem radically different. The elementary excitations appear to be point particles with no substructure and no internal degrees of freedom. The solitons, on the other hand, are extended objects characterized by a classical field profile $\phi(x)$ that is meaningful in the quantum theory because its spatial extent ($\sim 1/m$) is much greater than the Compton wavelength ($\sim \lambda/m$) of the soliton.

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On closer inspection, however, this distinction is less clearcut. On the one hand, in the presence of interactions the “elementary” particles turn out to be not entirely structureless; instead, they have a partonic substructure that evolves with momentum scale. On the other hand, the internal structure of the soliton appears somewhat simpler when it is analyzed in terms of normal modes rather than directly in terms of the classical solution. Most of the small fluctuation modes are non-normalizable modes, with continuum eigenvalues, that are most naturally understood as scattering states of the elementary quanta in the presence of the soliton. There may also be normalizable modes with nonzero eigenvalues; if so, these correspond to states with elementary quanta bound to the soliton. This leaves only a small number of normalizable zero eigenvalue modes, whose quantization requires the introduction of collective coordinates. It is only these that correspond to fundamental degrees of freedom of the soliton.

These observations suggest that the one-particle states built from solitons and those based on the elementary quanta might not differ in any essential way, and that the apparent differences between the two may be simply artifacts of the weak coupling regime. If so, then what happens as the coupling is increased? For small coupling, where the semiclassical treatment of the soliton is still valid, we know that the ratio of the soliton mass to the elementary particle mass decreases, as indicated in Fig. 1. However, such perturbative results are no longer reliable by the time the couplings have become of order unity, the region indicated by the question mark. Nevertheless,

**Figure 1**: Possible evolution of elementary (E.P.) and soliton (Sol.) masses with increasing coupling strength.

one can speculate. For example, it might happen, as indicated in the figure, that the soliton mass continues to decrease until, at some large coupling, the “soliton” is much lighter than the “elementary particle”. Of course, as suggested by the quotation marks, in this large coupling regime
there is no reason to expect the corresponding states to have any of the characteristics that they
had at small coupling.

The symmetry between the left and right sides of this figure suggests a particularly intriguing
possibility. Could the theory be reformulated so that the roles of the “soliton” and the “elementary
particle” are interchanged? In some cases, the answer is positive: There is a dual formulation of
the theory in terms of a new set of fields whose elementary excitations give rise to the soliton states
of the original formulation, while the former elementary particles correspond either to solitons or
to bound states. Weak coupling for this new formulation corresponds to strong coupling for the
original. This duality can by summarized by

$$\text{strong coupling } L_1(\phi) \leftrightarrow \text{weak coupling } L_2(\chi)$$

$$\text{elementary } \phi \leftrightarrow \text{soliton}$$

$$\text{soliton} \leftrightarrow \text{elementary } \chi$$

Note that there is no reason for the dual Lagrangian $L_2$ to have the same form as the original
Lagrangian $L_1$. For example, in the classic illustration of field theory duality, that between the
sine-Gordon and the massive Thirring models [1], there seems to be no resemblance at all between
the two theories. However, it could happen that $L_1$ and $L_2$ have the same functional form. As I
will describe in more detail below, it is believed that certain supersymmetric Yang-Mills theories
have a self-duality of this kind that interchanges electric and magnetic charges and that can be
seen as a generalization of the duality symmetry of Maxwell’s equations [4].

Whether or not there is a dual formulation of the theory, these correspondences between the
elementary particle and the soliton states of the quantum theory have implications for the structure
and properties of the classical soliton solutions. Quite often, these solutions are associated with
a conserved topological charge. In many such cases, one finds not only a solution with unit topo-
logical charge, but also families of multiply-charged classical solutions. If the soliton one-particle
states of the quantum theory are not fundamentally different from those built from the elementary
excitations, then one might expect that the multiply-charged solutions would correspond to multi-
particle states. This expectation is clearly borne out in a number of examples, including, as I will
describe in these lectures, the magnetic monopole solutions in many spontaneously broken gauge
theories. However, we will also see that in some cases — those where the unbroken gauge group
has a non-Abelian component — where this correspondence is less clearcut. In these theories the
elementary particle sector includes massless particles carrying electric-type charges. The duals of
these should be massless magnetically charged objects. Such massless monopoles do not appear
as isolated classical solutions. Instead, they lose their individual identity and are manifested as “clouds” of non-Abelian fields that surround the massive monopoles. Understanding the nature of these clouds may well provide insight into the properties of non-Abelian gauge theories.

The remainder of these lectures is organized as follows. In Sec. 2, I describe the classic example of field theory duality, that between the sine-Gordon and massive Thirring models in two spacetime dimensions. I then move on to four dimensions and the conjectured duality between magnetic monopoles and electrically charged fundamental excitations. I begin in Sec. 3 by reviewing the essential properties of the ‘t Hooft-Polyakov monopole of SU(2) gauge theory. In Sec. 4, I describe the Bogomolny-Prasad-Sommerfield, or BPS, limit and its relevance for electric-magnetic duality. Next, in Sec. 5, I discuss the solutions with higher magnetic charge in an SU(2) theory and show how these can be understood as corresponding to multiparticle states. Section 6 describes how these methods can be extended to theories with larger gauge groups, concentrating initially on cases where the symmetry is maximally broken to an Abelian subgroup. A very important tool for carrying this out is the moduli space approximation, described in Sec. 7. In Sec. 8, I describe the special issues that arise when the symmetry breaking is nonmaximal and the unbroken group is non-Abelian. It is here that one finds the massless monopoles clouds. I describe in detail two examples of this, a relatively simple one in an SO(5) theory in Sec. 9, and a more complex family of solutions in Sec. 10. Section 11 contains some concluding remarks and discussion.

2 The Sine-Gordon – Massive Thirring Model Duality

The classic example of duality between two quantum field theories is the famous equivalence between the sine-Gordon and massive Thirring models. The sine-Gordon model is a theory of a single scalar field in (1 + 1)-dimensional spacetime, with Lagrangian density

\[ \mathcal{L} = \frac{1}{2} \left( \partial_\mu \phi \right)^2 + \frac{m^2}{\beta^2} \cos \beta \phi + \frac{m^2}{2} \phi^2 + \frac{m^2 \beta^2}{24} \phi^4 + \cdots \]  

(2.1)

There are an infinite number of degenerate vacua corresponding to the minima of the scalar field potential at \( \phi = 2\pi n / \beta \). Expanding about one of these (say \( \phi = 0 \)), we obtain

\[ \mathcal{L} = \frac{1}{2} \left( \partial_\mu \phi \right)^2 + \frac{m^2}{\beta^2} \phi^2 + \frac{m^2 \beta^2}{24} \phi^4 + \cdots \]  

(2.2)

from which we see that \( \beta \) is a dimensionless coupling constant, while \( m \) is the lowest-order approximation to the mass of the elementary \( \phi \) boson.

The existence of discrete degenerate vacua gives rise to a conserved topological charge

\[ Q_{\text{top}} = \frac{\beta}{2\pi} [\phi(\infty) - \phi(-\infty)] \]  

(2.3)
that takes on integer values. There is a classical static soliton solution — the “kink” — that approaches two adjacent vacua as \( x \to \pm \infty \), and thus carries \( Q_{\text{top}} = 1 \); there is also an antikink, with \( Q_{\text{top}} = -1 \). The corresponding one-particle states of the quantum theory have masses

\[
M_{\text{kink}} = \frac{8m}{\beta^2} \left( 1 - \frac{\beta^2}{8\pi} \right). \quad (2.4)
\]

(The prefactor is the classical energy, while the quantity in brackets is the result of quantum correction.\(^2\))

In addition, there are periodic time-dependent classical solutions, known as breathers. These resemble a kink and an antikink bound together and oscillating back and forth. Using WKB methods, Dashen, Hasslacher and Neveu \(^3\) showed that in the quantum theory these breathers give rise to a series of states with masses

\[
M_n = 2M_{\text{kink}} \sin \left[ \frac{n\pi}{2} \frac{\beta^2/8\pi}{1 - \beta^2/8\pi} \right] \quad (2.5)
\]

where \( n \) is a positive integer obeying

\[
n < \frac{8\pi}{\beta^2} - 1. \quad (2.6)
\]

At this point, it might seem as if there are three classes of states — the elementary \( \phi \), the kink and antikink, and the breather states. Actually, the elementary \( \phi \) is the same as the lowest breather state, as is suggested by the small-\( \beta \) expansion

\[
M_n = nm \left[ 1 - \frac{n^2\beta^4}{1536} + O(\beta^6) \right]. \quad (2.7)
\]

Thus, for small \( \beta \) the particle spectrum includes an elementary boson with mass \( M_{\phi} = M_1 \) and \( Q_{\text{top}} = 0 \), two solitons with mass \( M_{\text{kink}} \gg M_{\phi} \) and \( Q_{\text{top}} = \pm 1 \), and a series of \( Q_{\text{top}} = 0 \) states with masses \( M_{\phi} < M_n < 2M_{\text{kink}} \). As \( \beta \) is increased, the soliton mass decreases, while those of the elementary \( \phi \) and of the higher breather states all increase. The breather states disappear in succession as their masses become greater than twice the soliton mass. Finally, the elementary \( \phi \) disappears when \( \beta = \sqrt{4\pi} \), leaving only the \( Q_{\text{top}} = \pm 1 \) states.

It is instructive to examine the behavior of the \( \phi \) mass as \( \beta \to \sqrt{4\pi} \). If we write

\[
\frac{\beta^2}{4\pi} = \frac{1}{1 + \delta/\pi} \quad (2.8)
\]

the \( \phi \) mass is

\[
M_1 = M_{\text{kink}} \left[ 2 - \delta^2 + \frac{4\delta^3}{\pi} + O(\delta^4) \right]. \quad (2.9)
\]

\(^2\)I am neglecting here technical points associated with the renormalization of parameters and the definitions of composite operators. For a more rigorous treatment of these, see \(^3\).
This formula suggests that in this large-$\beta$ regime it might be more natural to regard the kink and antikink — which are now the lightest particles in the theory — as elementary objects, and the $\phi$ as being a kink and an antikink bound together by a weak interaction with a strength characterized by $\delta$. (Note that in this regime the terms “kink” and “antikink” are somewhat misleading, since the semiclassical correspondence between field profile and quantum state is no longer valid at such strong coupling.)

Let us turn now to the massive Thirring model. This is a theory of a fermion field $\psi$, also in $(1 + 1)$-dimensional spacetime, with Lagrangian density

$$L = \bar{\psi}(i\gamma^\mu\partial_\mu - M)\psi - \frac{g}{2}(\bar{\psi}\gamma^\mu\psi)^2. \quad (2.10)$$

The particle spectrum includes an elementary fermion and its antiparticle, each with mass $M$. If $g$ is positive, so that the four-fermion interaction is attractive, there are also fermion-antifermion bound states. The lowest (and, for sufficiently weak coupling, the only) one of these has a mass

$$M_{\text{Bound}} = M\left[2 - g^2 + \frac{4g^3}{\pi} + O(g^4)\right]. \quad (2.11)$$

The similarity between Eqs. (2.9) and (2.11) suggests that the massive Thirring model might indeed be a dual theory in which the interpretation of the $\phi$ as a kink-antikink bound state is realized. Let us therefore identify $g$ and $\delta$, so that

$$\frac{\beta^2}{4\pi} = \frac{1}{1 + g/\pi}. \quad (2.12)$$

Weak coupling ($\beta \to 0$) for the sine-Gordon theory thus corresponds to strong coupling ($g \to \infty$) for the massive Thirring model, while $g \to 0$ in the latter corresponds to the strong coupling limit $\beta \to \sqrt{4\pi}$ of the former. We would then have the equivalences

kink $\leftrightarrow$ elementary $\psi$

antikink $\leftrightarrow$ elementary $\bar{\psi}$

elementary $\phi$ $\leftrightarrow$ $\bar{\psi} - \psi$ bound state

between the particle states. These, in turn, imply the identification

topological charge $\leftrightarrow$ fermion number

between the conserved charges.
The remarkable fact is that one can beyond these correspondences between the mass spectra and rigorously establish an equivalence between the two theories. Making the operator identifications

\[
\frac{\beta}{2\pi} \epsilon^{\mu \nu} \partial_\nu \phi = -\bar{\psi} \gamma^\mu \psi \\
\frac{m^2}{\beta^2} \cos \beta \phi = -M \bar{\psi} \psi
\]

(with appropriate renormalization and normal ordering of the composite operators), one finds that the matrix elements of the two theories agree to all orders of perturbation theory. Furthermore, the fermion field operator \(\psi(x)\) of the massive Thirring model can be written \([10]\) as a nonlocal function of the sine-Gordon boson field \(\phi(x)\).

3 Magnetic monopoles

Let us now go on to four spacetime dimensions, where the most important examples of solitons arise in spontaneously broken gauge theories. If the vacuum expectation value of the Higgs field breaks a gauge group \(G\) down to a subgroup \(H\), there are topologically stable solitons if the second homotopy group \(\Pi_2(G/H)\) is nonzero. These are generally referred to as magnetic monopoles although, strictly speaking, the term is only appropriate when the unbroken group \(H\) is the \(U(1)\) of electromagnetism.

The archetypical example \([4]\) of these occurs in an \(SU(2)\) theory with a triplet Higgs field \(\phi\) whose expectation value \(\langle \phi \rangle = v\) breaks the symmetry to \(U(1)\). If the scalar potential is

\[
V(\phi) = \frac{\lambda}{4} (\phi^2 - v^2)^2
\]

the elementary excitations of the theory are a massless “photon”, two vector mesons \(W^\pm\) with mass \(m_W = ev\) and electric charges \(\pm e\) (where \(e\) is the gauge coupling), and a neutral scalar with mass \(m_H = \sqrt{2\lambda} v\).

Because \(\Pi_2[SU(2)/U(1)] = Z\), there is a single additive topological charge \(n\) that can take on any integer value. It is related to the magnetic charge by

\[
Q_M = \frac{4\pi n}{e}.
\]

\(^3\)In these units, the Dirac quantization condition would require that \(n\) be either integer or half-integer. The more restrictive condition here follows from requiring that the Higgs field be smooth at spatial infinity. It is ultimately related to the fact that one could add isospinor fields to the theory. After symmetry breaking, these would have electric charges \(\pm e/2\). The existence of such charges would modify the Dirac condition so as to require integer \(n\).
The classical solution corresponding to the unit monopole, with \( Q_M = 4\pi/e \), is most often written in the spherically symmetric form

\[
A^a_i = \epsilon_{iak} \hat{r}_k \left( \frac{1 - u(r)}{er} \right), \\
\phi^a = \hat{r}_a h(r).
\]  

(Here superscripts are SU(2) indices.) Requiring that the solution be nonsingular at the origin imposes the boundary conditions \( u(0) = 1 \) and \( h(0) = 0 \), while finiteness of the energy requires that \( u(\infty) = 0 \) and \( h(\infty) = v \). Substituting this ansatz into the field equations leads to a pair of second order equations that can be solved numerically. The solution has a core region, of radius \( R_{\text{core}} \sim 1/ev \). Outside of this core \( u \) and \( v - h \) fall exponentially fast and the field strength is, up to exponentially small terms, given by

\[
F^{a}_{ij} = \epsilon_{ijk} \hat{r}_a \hat{r}_i \frac{1}{er^2}.
\]  

(3.4)

The factor of \( \hat{r}_a \) indicates that in internal SU(2) space this is parallel to \( \phi \), and thus purely electromagnetic. From the remaining factors we see that this is just the Coulomb magnetic field of a magnetic monopole with \( Q_M = 4\pi/e \). The classical energy of the solution, which gives the leading approximation to the monopole mass, can be written as

\[
M = \frac{4\pi v}{e} f(\lambda/e^2)
\]  

(3.5)

where \( f(s) \) increases monotonically between from 0 to 1.787 \[11\] as \( s \) varies from 0 to \( \infty \).

Although the nontrivial topology of the Higgs field is most apparent when the solution is written in the radial gauge form of Eq. (3.3), a number of other aspects of the solution are clarified by applying a singular gauge transformation that makes the SU(2) orientation of the Higgs field uniform. Choosing \( \phi^a = \delta^{a3} h(r) \), we may identify the third isospin component with electromagnetism and write

\[
A_{i}^{\text{EM}} = A^3_i = \frac{1}{er} \left( \frac{1 - \cos \theta}{\sin \theta} \right) u(r) \frac{1}{er} e^{i\alpha}, \\
W_i = \frac{A^1_i + iA^2_i}{\sqrt{2}} = \hat{n}_i(\theta, \phi) u(r) e^{i\alpha}.
\]  

(3.6)

The first of these is just the U(1) vector potential for a point monopole of magnetic charge \( 4\pi/e \), with the Dirac string singularity lying along the negative z-axis. In the second equation, \( \hat{n}_i \) is a complex unit vector whose precise form is not needed for the present discussion. This gives \( u(r)/er \) a simple interpretation as the magnitude of the massive \( W \) field. Thus, the 't Hooft-Polyakov monopole is essentially a point Dirac monopole surrounded by a core of massive vector field, with
the components of the latter arranged to give a singular magnetic dipole density that cancels the singularity in the Coulomb magnetic energy density. Finally, note the arbitrary overall phase $\alpha$ that appears in the expression for $W_i$. This reflects the unbroken global U(1) symmetry that survives even after the gauge has been fixed.

This monopole configuration is not an isolated solution, but rather one of a many-parameter family of configurations, all with the same energy. Infinitesimal variations of these parameters correspond to zero-eigenvalue modes in the spectrum of small fluctuations about the monopole solution. Three of these modes correspond to spatial translations of the monopole; the corresponding parameters are most naturally chosen to be the spatial coordinates of the center of the monopole. Because this is a gauge theory, there are also an infinite number of zero modes that simply reflect the freedom to make local gauge transformations. To eliminate these, we must impose a gauge condition. However, this still leaves one zero mode, corresponding to gauge transformations that do not vanish at infinity; roughly speaking, this should be understood as a global gauge transformation in the unbroken subgroup. For example, if we impose the gauge conditions that $\phi^1 = \phi^2 = 0$ and that $A^{EM} = A^3$ satisfy the Coulomb gauge condition $\nabla \cdot A^{EM} = 0$, the surviving gauge zero mode corresponds simply to shift of the U(1) phase $\alpha$ in Eq. (3.6).

When the system is quantized, a collective coordinate and a conjugate momentum must be introduced for each of the zero modes. For the translation zero modes, the conjugate momentum is just the linear momentum $P$ of the monopole. For the global gauge zero mode, the collective coordinate is a U(1) phase variable, and the conjugate momentum is the U(1) electric charge $Q_E$, which is proportional to the time derivative of this phase; the quantization of $Q_E$ follows from the periodicity of the U(1) phase. Thus, allowing the collective coordinates to vary linearly in time gives solutions with nonzero momentum and electric charge and an energy of the form

$$E = M + \frac{P^2}{2M} + \frac{Q_E^2}{2I} + O(P^4, Q_E^4)$$

where the quantity $I$ can be expressed as a spatial integral involving the square of the $W$ field. The fact that the coefficient of $P^2$ turns out to be $1/2M$ is a nontrivial consequence of the underlying Lorentz invariance of the theory.

4 The BPS limit and electric-magnetic duality

In general, the classical field equations governing the monopole can only be solved numerically. Matters are different, however, if one works in the Bogomolny-Prasad-Sommerfield, or BPS, limit
Although initially introduced as a means of simplifying the classical equations, it turns out to have a much deeper significance.

At the simplest level, the BPS limit is obtained by taking $\lambda$ and $m_H$ to zero with $v$ held fixed. Since the field equations now depend only on a single dimensionless constant that can be absorbed by a rescaling of fields and distances, one might hope that they would become analytically tractable. Indeed, simply by guesswork one can find the solution

$$u(r) = \frac{v}{\sinh(evr)}$$

$$h(r) = v \coth(evr) - \frac{1}{er}.$$  (4.1)

The $1/er$ tail of $h(r)$ reflects the fact that the scalar field becomes massless in the BPS limit. Such a massless field can mediate a long-range attractive force, a fact that is of great importance.

Deeper insight is gained by examining the energy functional. To simplify the algebra, let us restrict ourselves to configurations with vanishing electric charge and $A_0$ identically zero. The energy for a static configuration can then be written as

$$E = \int d^3x \left[ \frac{1}{2} \text{Tr} B_i^2 + \frac{1}{2} \text{Tr} (D_i \phi)^2 + V(\phi) \right]$$

$$= \int d^3x \left[ \frac{1}{2} \text{Tr} (B_i \pm D_i \phi)^2 \pm \text{Tr} (D_i \phi) B_i + V(\phi) \right]$$  (4.2)

where $B_i \equiv \frac{1}{2} \epsilon_{ijk} F_{jk}$. The potential term vanishes in the BPS limit. A partial integration, together with the Bianchi identity, gives

$$\int d^3x \text{Tr} (D_i \phi) B_i = \int d^3x \left[ \partial_i \text{Tr} (\phi B_i) - \text{Tr} \phi D_i B \right] = \int d^2S \text{Tr} \phi B_i = Q_M v$$  (4.3)

where the surface integral after the second equality is over a sphere at spatial infinity. Substituting this result back into Eq. (4.2) and using the upper (lower) sign for positive (negative) magnetic charge gives the bound

$$E \geq v|Q_M| = \left| n \right| \left( \frac{4\pi v}{e} \right).$$  (4.4)

This bound is achieved by configurations that satisfy the first-order equations

$$B_i = \pm D_i \phi.$$  (4.5)

Because any static configuration that minimizes the energy is a stationary point of the theory, solutions of Eq. (4.5) are guaranteed to also be solutions of the full set of second-order field equations;

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1 Here, and henceforth, I write the gauge and Higgs fields as elements of the Lie algebra of the group. For the case of SU(2), these are related to the component fields used in the previous section by $A_i = A_a^i \tau^a / 2i$, where the $\tau^a$ are the Pauli matrices.
such solutions are referred to as BPS solutions. For the remainder of these lectures I will assume that the magnetic charge is positive, and so will always use Eq. (4.7) with the positive sign.

This argument is easily modified to include the possibility of dyons, carrying nonzero electric charge $Q_E$ in addition to their magnetic charge. The BPS equations become

$$B_i = \cos \gamma D_i \phi$$
$$E_i \equiv F_{0i} = \sin \gamma D_i \phi$$
$$D_0 \phi = 0 \quad (4.6)$$

where $\gamma = \tan^{-1}(Q_E/Q_M)$. The solutions of these have energy

$$E = v \sqrt{Q_M^2 + Q_E^2} \quad (4.7)$$

Although classically all values of $Q_E$ are possible, semiclassical quantization about the monopole shows that the electric charge must be an integral multiple of $e$. Hence, for weak coupling the lowest dyonic states have $Q_E \ll Q_M \sim 1/e$, and Eq. (4.7) can be expanded to give

$$E = vQ_M \left(1 + \frac{Q_E^2}{2Q_M^2} + \cdots\right) \quad (4.8)$$

which is in agreement with Eq. (3.7).

The fact that the energy is strictly proportional to the magnetic charge suggests that static multimonopole solutions might exist, since the energy of such a solution would be just the sum of the masses of the component monopoles and would be independent of their relative positions. From another point of view, this possibility, which is actually realized, occurs because the magnetic repulsion between a pair of static monopoles is exactly balanced by their mutual attraction via the long-range scalar force.

A similar reduction of second-order field equations to a set of first-order equations, with the energy of the solutions being determined by the conserved charges, is found in a number of other systems. These include the Abelian Higgs model with parameters chosen so that the scalar and vector masses are equal (corresponding to the Type I-Type II boundary in superconductivity) [12], Chern-Simons-Higgs theory with a specially chosen scalar potential [13], and the Yang-Mills instantons; all of these admit static multisoliton solutions.

In all of these cases the theory can be trivially generalized to be not only supersymmetric, but to also have extended supersymmetry. Extended supersymmetry algebras include central charges. These turn out to be related to the charges carried by the solitons, so that the BPS mass relations...
can be rewritten in terms of the generators of the superalgebra. For example, Eq. (4.7) is equivalent to

\[ H = \sqrt{Z_1^2 + Z_2^2} \quad (4.9) \]

where \( H \) is the Hamiltonian and \( Z_1 \) and \( Z_2 \) are central charges. The theory of supersymmetry representations shows that states obeying such relations lie in “short” supermultiplets with fewer components than otherwise. (This generalizes the distinction between massless and massive supermultiplets in \( N = 1 \) supersymmetry.) Since small quantum corrections cannot change the size of a multiplet, these mass relations must be exact and hold even when one passes from the classical limit to the full quantum theory \([14]\). Furthermore, states obeying relations such as Eq. (4.9) are invariant under a subset of the supersymmetry algebra. At the classical level, the vanishing of the corresponding supercharges reduces to first-order equations such as Eq. (4.5).

In particular, the BPS monopole solutions arise naturally in the context of \( N = 2 \) or \( N = 4 \) supersymmetric Yang-Mills theory. In addition to the gauge field, these contain \( N/2 \) Dirac fermion fields and \( 2N - 2 \) real scalar fields \( \phi_p \), all in the adjoint representation. The Lagrangian is

\[
\mathcal{L} = \frac{1}{2} \text{Tr} F_{\mu\nu}^2 + \frac{1}{2} \text{Tr} (D_\mu \phi_p)^2 - \frac{e^2}{4} \text{Tr} ([\phi_p, \phi_q])^2 + \text{fermion terms}. \quad (4.10)
\]

If the \( \phi_p \) with \( p > 1 \) are set identically to zero, the bosonic parts of the classical field equations reduce to the BPS equations for \( A_\mu \) and \( \phi_1 \).

| Mass |
|------|
| \text{photon} | 0 |
| \phi | 0 |
| \( W^\pm \) | \( ev \) | \( \pm e \) | 0 |
| Monopole | \( \frac{4\pi v}{e} \) | 0 | \( \pm \frac{4\pi v}{e} \) |

**Table 1:** The particle masses and charges in the BPS limit of the SU(2) theory.

Table 1 summarizes the masses and charges of particles of the nonsupersymmetric SU(2) theory in the BPS limit. Montonen and Olive \([2]\) noted that this spectrum is invariant under the transformation

\[
e \leftrightarrow \frac{4\pi}{e} \quad Q_E \leftrightarrow Q_M .
\]

This led them to conjecture that the theory might be self-dual, with the role of the solitons and of the elementary excitations being interchanged under this transformation.
An obvious difficulty with this conjecture is the fact that the massive $W$’s have spin one, whereas the classical monopole and antimonopole solutions are spherically symmetric and therefore lead to spinless particles after quantization. A resolution to this puzzle is found when fermion fields are added to make the theory supersymmetric. Each adjoint representation Dirac fermion field has two normalizable zero modes in the presence of the singly charged monopole \cite{[15]}. These lead to a degenerate multiplet of monopole states that differ only in the occupation numbers of these modes. The theory with $N$-extended supersymmetry has $N/2$ Dirac fermions, hence $N$ zero modes and $2^N$ degenerate states. For $N = 2$ these form a spin-1/2 doublet and two spin-0 states, but not the spin-1 states that are needed to match the $W$ bosons. For $N = 4$, there are 16 states, including a spin-1 triplet, four spin-1/2 doublets, and five spin-0 states. This not only gives the desired spin-1 states, but also exactly matches \cite{[16]} the complete supermultiplet structure of the electrically-charged elementary excitations.\footnote{Supersymmetry is also essential for ensuring that the spectrum of dyonic states is consistent with duality. For example, semiclassical quantization of the U(1) zero mode about the unit monopole leads to a tower of states with unit magnetic charge but multiple electric charge. Duality then requires a tower of states with unit electric charge and multiple magnetic charge. Sen\cite{[17]} has shown how these can arise as zero energy bound states in the supersymmetric theory; in Sec. I will describe how similar methods resolve some duality puzzles in a theory with a large gauge group.}

Further support for the duality conjecture is obtained by considering the low-energy scattering of electrically charged particles \cite{[3]}. The existence of classical static multimonopoles solutions in the BPS limit was noted above. The obvious dual to these would be states containing several electrically charged elementary particles at rest. Since we cannot construct a semiclassical approximation to these states in the weak coupling limit, it is hard to study them directly. Instead, we can examine the behavior of scattering states containing two like-charged particles as their relative momentum goes to zero, and look for the cancelation between the Higgs and electromagnetic forces in the static limit. As an example, consider the scattering of the massive positively charged fermions that are the superpartners of the massive gauge bosons. There are two graphs that contribute at tree level, one with a massless scalar exchanged between the fermions and one where they exchange a photon. (Note that the tree level scattering amplitude is the same in the $N = 2$ and $N = 4$ theories.) Let us work in the center of mass frame, with initial momenta

\[ p = (E, P) \quad k = (E, -P) \]

and final momenta

\[ p' = (E', P') \quad k' = (E', -P') \]
The contribution to the scattering amplitude from scalar exchange is

\[ \mathcal{M}_S = -\frac{ie^2}{q^2} [\bar{u}(p')u(p)] [\bar{u}(k')u(k)] \]  

(4.13)

where \( q \equiv p' - p = k - k' = (0, P' - P) \), while that from photon exchange is

\[ \mathcal{M}_V = ie^2 q^2 g_{\mu\nu} [\bar{u}(p')\gamma^\mu u(p)] [\bar{u}(k')\gamma^\nu u(k)] . \]  

(4.14)

Now use the fact that

\[ \bar{u}(p')\gamma^\mu u(p) = \bar{u}(p')\left]\left[\frac{(p + p')^\mu}{2m} + \frac{i\sigma^{\mu\nu}q_\nu}{2m}\right]\right] u(p) . \]  

(4.15)

For \( \mu = 1, 2, \) or \( 3 \), the quantity sandwiched between the spinors is manifestly of order \( P \). For \( \mu = 0 \), if we write \( E = m + P^2/2m + \cdots \) and recall that \( \bar{u}(p')\sigma^{0i}u(p) \) vanishes if \( P = 0 \), we find that

\[ \bar{u}(p')\gamma^0 u(p) = \bar{u}(p')[m + O(P^2)]u(p) . \]  

(4.16)

After substituting this into Eq. (4.14) and then combining the result with Eq. (4.13), we see that the scattering amplitude does indeed vanish in the static limit \( P \to 0 \).

5 \( SU(2) \) Multimonopole solutions and index theorems

Let us return now to the classical \( SU(2) \) monopole solutions, focusing in particular on those with higher magnetic charge, \( n \geq 2 \). It was argued previously that one might expect to find solutions corresponding to several unit monopoles at rest. One could also imagine that there might be additional solutions, corresponding to new species of monopoles with larger magnetic charges, that do not have multiparticle interpretations. A useful tool for exploring this possibility is the use of index theory methods to count the normalizable zero modes, and thus determine the number of collective coordinates needed to characterize a solution [15].

Consider an arbitrary charge \( n \) solution with gauge and Higgs fields \( A_i \) and \( \phi \). We are interested in small perturbations \( \delta A_i \) and \( \delta \phi \) that preserve the BPS equation. If we write \( A_i \) and \( \phi \) as antihermitean matrices in the adjoint representation of \( SU(2) \) and \( \delta A_i \) and \( \delta \phi \) as three-component column vectors, the equations resulting from the variation of Eq. (4.5) take the form

\[ 0 = \delta(B_j - D_j \phi) \]

\[ = D_j \delta \phi - e\phi \delta A_j - \epsilon_{jkl}D_k \delta A_l \]  

(5.1)

where \( D_j \) is the covariant derivative with respect to the unperturbed gauge field. We are not interested in zero modes that simply correspond to local gauge transformations, although we do
want to keep those due to global gauge transformations, since excitations of these give rise to electric charges. To eliminate the unwanted modes, we require that the perturbation be orthogonal to all local gauge transformations, and hence satisfy

\[ \int d^3x \left[ (\delta A_j)^t D_j \Lambda + ie(\delta \phi)^t \phi \Lambda \right] \]  \hspace{1cm} (5.2)

for any gauge function \( \Lambda(x) \) that vanishes at spatial infinity. By an integration by parts (valid only because \( \Lambda \) vanishes at large distance), this is equivalent to the background gauge condition

\[ 0 = D_j \delta A_j + e\phi \delta \phi . \]  \hspace{1cm} (5.3)

Our task is to determine the number of linearly independent normalizable solutions of Eqs. (5.1) and (5.3).

It is convenient to begin by defining

\[ \psi = I \delta \phi + i\sigma_j \delta A_j . \]  \hspace{1cm} (5.4)

This allows Eqs. (5.1) and (5.3) to be combined into a single Dirac equation

\[ 0 = (-i\sigma_j D_j + ie\phi)\psi \equiv \mathcal{D}\psi . \]  \hspace{1cm} (5.5)

It should be kept in mind that solutions of Eq. (5.5) that differ by multiplication by \( i \) correspond to linearly independent solutions for \( \delta A \) and \( \delta \phi \), so the number of bosonic zero modes is equal to twice the number of normalizable fermionic solutions this equation. Note also that Eq. (5.5) is preserved by transformations of the form

\[ \psi \rightarrow \psi e^{i\mathbf{v} \cdot \mathbf{\sigma}} . \]  \hspace{1cm} (5.6)

Given one bosonic zero mode, one can construct three more by means of such transformations. This relation among zero modes will be of significance later.

Now define

\[ \mathcal{I} = \lim_{M^2 \to 0} \frac{\text{Tr}}{\mathbb{M}^2} \left( \frac{M^2}{\mathcal{D}^\dagger \mathcal{D} + M^2} - \text{Tr} \left( \frac{M^2}{\mathcal{D} \mathcal{D}^\dagger + M^2} \right) \right) . \]  \hspace{1cm} (5.7)

This counts the difference between the numbers of zeroes of the operators \( \mathcal{D}^\dagger \mathcal{D} \) and \( \mathcal{D} \mathcal{D}^\dagger \) or, equivalently, the difference between the numbers of zeros of \( \mathcal{D} \) and \( \mathcal{D}^\dagger \). Because

\[ \mathcal{D} \mathcal{D}^\dagger = -\mathcal{D}^2 + \phi^2 \]  \hspace{1cm} (5.8)

is positive definite and has no normalizable zero modes, \( 2\mathcal{I} \) would seem to be precisely the quantity we want, with the factor of 2 taking into account the difference between the counting of bosonic
and fermionic modes. However, the presence of massless particles in the theory means that there will be a continuum spectrum with eigenvalues extending down to zero. If the continuum density of states were to be sufficiently singular near zero, it could give a nonzero contribution to $\mathcal{I}$. A careful examination of the large distance behavior of the various operators shows that this is not the case\footnote{Although the presence of the continuum spectrum does not affect the correspondence between $\mathcal{I}$ and the counting of normalizable zero modes, it does manifest itself by making the trace in Eq. (5.7) $M$-dependent; had the spectra of $D^*D$ and $DD^*$ been purely discrete, this trace would have been independent of $M$ and no limit would have been needed.} One can evaluate $\mathcal{I}$ by showing that it can be written as the integral over all space of the divergence of a current $J_i$. Gauss’s theorem converts this to a surface integral at spatial infinity that is determined by the long-range behavior of the Higgs and magnetic fields. Evaluating this integral and taking the limit $M^2 \to 0$ yields

$$2\mathcal{I} = 4n.$$ (5.9)

Thus, any SU(2) BPS solution carrying $n$-units of magnetic charge belongs to a $4n$-dimensional space of solutions, or moduli space. The corresponding collective coordinates are just what would be expected for a configuration of $n$ independent unit monopoles, each with three position variables and one U(1) phase. Allowing these to become time-dependent would yield independent nonzero linear momenta and electric charges for each of the individual monopoles. Together with the simple additive nature of the BPS mass formula, this strongly suggests that all $n \geq 2$ classical solutions should be interpreted as multimonopole solutions and that the corresponding states in the quantum theory are multiparticle states.

## 6 Fundamental monopoles in larger gauge groups

Let us now see what happens in the case of a larger simple gauge group $G$, of rank $r > 1$, with an adjoint Higgs field $\phi$. Recall that the generators of a Lie algebra can be taken to be $r$ commuting quantities $H_i$ that generate the Cartan subalgebra, together with a raising or lowering operator $E_{\alpha}$ for each of the $r$-component root vectors $\alpha$. By making an appropriate choice of basis, any element of the Lie algebra can be brought into the Cartan subalgebra. In particular, we can choose to write the asymptotic Higgs field in some reference direction to be of the form

$$\phi_0 = h \cdot H.$$ (6.1)

The $r$-component vector $h$ determines the nature of the symmetry breaking. If it has nonzero inner product with all of the $\alpha$, then $G$ is broken maximally, to U(1)$^r$. If instead there are some roots
orthogonal to \( h \), then these form the root diagram for some subgroup \( K \) of rank \( k \) and the unbroken gauge group is \( K \times U(1)^{r-k} \).

Because the long-range part of the magnetic field must commute with the asymptotic Higgs field, it too can be brought into the Cartan subalgebra. This allows us to define a second vector \( g \), characterizing the magnetic charge, by requiring that the asymptotic magnetic field in the direction used to define \( \phi_0 \) be of the form

\[
B_k = \frac{\hat{r}_k}{4\pi r^2} g \cdot H + O(r^{-3}) .
\]  

(6.2)

The topological quantization condition on the magnetic charge can then be written as

\[
e^{ig \cdot H} = 1 .
\]  

(6.3)

Let us now concentrate on the case of maximal symmetry breaking, returning later to the case where the unbroken symmetry contains a non-Abelian factor. To start, recall that a basis for the root diagram of \( G \) is given by \( r \) simple roots \( \beta_a \) with the property that any other root can be written as a linear combination of the \( \beta_a \) with coefficients that are either all positive or all negative. The relative angles and lengths of the simple roots characterize the Lie algebra (and are encoded in the Dynkin diagram), but the choice of the simple roots is not unique, with the various allowed choices being related by elements of the Weyl group. However, the vector \( h \) uniquely determines a preferred set of simple roots that satisfy the condition

\[
h \cdot \beta_a \geq 0
\]

for all \( a \). The solution to the quantization condition can be written in terms of these as

\[
g = \frac{4\pi}{e} \sum_{a=1}^{r} n_a \beta_a^* .
\]  

(6.5)

where the dual of a root is defined by \( \alpha^* = \alpha/\alpha^2 \). The coefficients \( n_a \) are integers and are the \( r \) topological charges corresponding to the homotopy group \( \Pi_2[SU(2)/U(1)^r] = Z^r \). For a BPS solution these must all be of the same sign; without loss of generality, we can take these to be positive, corresponding to the upper sign in Eq. (4.5).

The BPS mass formula becomes

\[
M = \left( \frac{4\pi}{e} \right) \sum_{a=1}^{r} n_a h \cdot \beta_a = \sum_{a=1}^{r} n_a m_a ,
\]  

(6.6)

An index calculation \[21\] similar to that for the SU(2) theory shows that the number of normalizable zero modes is

\[
2\mathcal{I} = 4 \sum_{a=1}^{r} n_a .
\]  

(6.7)
(Again, a detailed analysis shows that the continuum spectrum has no effect on this result.) These results suggest that any higher-charged arbitrary solution should be viewed as a multimonopole solution containing appropriate numbers of \( r \) different species of fundamental monopoles, with the \( a \)th species of fundamental monopole having topological charges \( n_b = \delta_{ab} \), mass \( m_a \), and four collective coordinates, three of which are position variables while the fourth is a phase in the \( a \)th \( \text{U}(1) \). In fact, one can easily construct the classical solutions corresponding to these fundamental monopoles. To do this, note that each simple root \( \beta_a \) defines an \( \text{SU}(2) \) subgroup of \( G \) with generators

\[
\begin{align*}
t^1 &= \frac{1}{\sqrt{2\beta_a^2}}(E_{\beta_a} + E_{-\beta_a}) \\
t^2 &= -\frac{i}{\sqrt{2\beta_a^2}}(E_{\beta_a} - E_{-\beta_a}) \\
t^3 &= \beta_a^* \cdot H. \tag{6.8}
\end{align*}
\]

The corresponding fundamental monopole solution is obtained by embedding the \( \text{SU}(2) \) unit monopole solution, appropriately rescaled, in this subgroup, and adding constant terms to the Higgs field so that the asymptotic \( \phi \) has the required eigenvalues. If \( A_i^s(r; v) \) and \( \phi^s(r; v) \) \((s = 1, 2, 3)\) are the fields for the \( \text{SU}(2) \) solution corresponding to a Higgs expectation value \( v \), then the embedded solution is

\[
\begin{align*}
A_i(r) &= \sum_{s=1}^{3} A_i^s(r; h \cdot \beta_a)t^s \\
\phi(r) &= \sum_{s=1}^{3} \phi^s(r; h \cdot \beta_a)t^s + (h - h \cdot \beta_a^* \beta_a) \cdot H. \tag{6.9}
\end{align*}
\]

It is easily verified that this has mass \( m_a \) and that there are precisely four normalizable zero modes.

To make this more concrete, let us consider the case of \( \text{SU}(3) \) broken to \( \text{U}(1) \times \text{U}(1) \). If the asymptotic Higgs field is taken to be diagonal with its eigenvalues decreasing along the diagonal, the \( \text{SU}(2) \) subgroups generated by the simple roots \( \beta_1 \) and \( \beta_2 \) lie in the upper left and lower right \( 2 \times 2 \) blocks, respectively. Embedding the \( \text{SU}(2) \) monopole in these subgroups gives a pair of fundamental monopoles, each with four normalizable zero modes. The first has topological charges \( n_1 = 1 \) and \( n_2 = 0 \) and mass \( m_1 \), while the second has \( n_1 = 0 \), \( n_2 = 1 \), and mass \( m_2 \). There is also a third \( \text{SU}(2) \) subgroup, corresponding to the composite root \( \beta_1 + \beta_2 \), lying in the four corner elements of the \( 3 \times 3 \) \( \text{SU}(3) \) matrix. Using this subgroup to embed the \( \text{SU}(2) \) unit monopole gives a third solution which, like the previous two, is spherically symmetric. It has topological charges \( n_1 = n_2 = 1 \) and mass \( m_1 + m_2 \). Unlike the two other embedding solutions, it has not
four, but instead eight, zero modes. The extra modes correspond to the fact that this solution can be continuously deformed into one containing two widely separated fundamental monopoles, one of each type. Despite its spherical symmetry, it is just one of a family of two-monopole solutions. After quantization, these give rise to a set of two-particle states, not to a new type of one-particle state.

Let us now consider these results in light of the Montonen-Olive conjecture. Table 2 lists the elementary vector particles, together with the masses and the values of their electric-type charges under the two unbroken U(1) factors. (The remaining elementary excitations are related to these by supersymmetry and follow the same pattern.) The two massless excitations, which carry no charge, should be self-dual in the same sense as the photon of the SU(2) → U(1) case. The duals to the W-bosons corresponding to \( \beta_1 \) are clearly the states built upon the \( \beta_1 \)-embedding of the SU(2) monopole and antimonopole, and similarly for the \( \beta_2 \)-W boson. What are the duals to the W’s corresponding to \( \beta_1 + \beta_2 \)? One might have thought that these would be obtained from the \( (\beta_1 + \beta_2) \)-embedding of the SU(2) solution. However, we have just seen that this corresponds to a two-particle state and so cannot give the dual to an electrically charged single-particle state. Instead the dual state must be some kind of zero-energy bound state involving the two fundamental monopoles. To explore this possibility, we need to know more about the dynamics of the monopoles. A very useful tool for doing this is the moduli space approximation.

|       | Mass | \( Q_{E1} \) | \( Q_{E2} \) |
|-------|------|--------------|--------------|
| \( \gamma_1 \) | 0    | 0            | 0            |
| \( \gamma_2 \) | 0    | 0            | 0            |
| \( \beta_1 \)-W | \( m_1 \) | \( \pm e \) | 0            |
| \( \beta_2 \)-W | \( m_2 \) | 0            | \( \pm e \) |
| \( (\beta_1 + \beta_2) \)-W | \( m_1 + m_2 \) | \( \pm e \) | \( \pm e \) |

**Table 2:** The particle masses and electric charges of the elementary vector particles in the maximally broken SU(3) theory. The two gauge bosons corresponding to the unbroken generators are denoted by \( \gamma_1 \) and \( \gamma_2 \); \( Q_{E1} \) and \( Q_{E2} \) are the electric charges in the corresponding U(1) subgroups.
The moduli space approximation

The essential assumption of the moduli space approximation is that if the monopoles are moving sufficiently slowly, the evolution of the field configurations can be approximated as motion on the moduli space of BPS solutions. Thus, let $A_a^{\text{BPS}}(r, z)$ be a complete family of gauge-inequivalent BPS solutions for a given magnetic charge, with $a = 1, 2, 3$ referring to the spatial components of the gauge potential and $A_4 \equiv \phi$, while $z$ denotes the various collective coordinates. In the $A_0 = 0$ gauge, the moduli space approximation amounts to assuming that the field configuration at any time $t$ is of the form

$$A_a(r, t) = U^{-1}(r, t) A_a^{\text{BPS}}(r, z(t)) U(r, t) - \frac{i}{e} U^{-1}(r, t) \partial_a U(r, t).$$

(7.1)

(Here $\partial_4 = 0$.) Differentiating with respect to time gives

$$\dot{A}_a = \dot{z}_j \left[ \frac{\partial A_k}{\partial z_j} + D_a \epsilon_j \right] \equiv \dot{z}_j \delta_j A_a$$

(7.2)

where the infinitesimal gauge transformations generated by the $\epsilon_j$ arise from differentiation of the factors of $U$. Because they correspond to variations on the space of BPS solutions, the $\delta_j A_a$ are zero modes about the monopole solution at a given time. The gauge functions $\epsilon_j$ are fixed by Gauss’s law, which must be imposed as a constraint when working in $A_0 = 0$ gauge. This gives

$$0 = -D_a F^{a0} = \dot{z}_j D_a \delta_j A_a$$

(7.3)

which implies that the $\delta_j A_a$ must obey the background gauge condition, Eq. (5.3).

If we now substitute these results into the $A_0 = 0$ gauge field theory Lagrangian, we obtain

$$L = \frac{1}{2} \int d^3 r \text{Tr} \left[ \dot{A}_i^2 + \dot{\phi}^2 + B_i^2 + (D_i \phi)^2 \right].$$

(7.4)

With the fields given by Eq. (7.1), the integral of the last two terms is just the energy of the static BPS solution. This is completely determined by the magnetic charge and is independent of the $z_j$; i.e., it is a constant term having no effect on the dynamics. The remaining terms are a quadratic form in the $\dot{z}_j$, giving the moduli space Lagrangian

$$L_{\text{MS}} = \frac{1}{2} g_{ij}(z) \dot{z}_i \dot{z}_j + \text{constant}$$

(7.5)

where

$$g_{ij}(z) = \int d^3 r \left[ \delta_i A_k \delta_j A_k + \delta_i \phi \delta_j \phi \right]$$

(7.6)

In this context, “slowly moving” means not only that the spatial velocities must be small, but also that the electric charges, which are proportional to the time derivatives of the U(1) phases, must also be small.
can be interpreted as a metric on the moduli space. With this interpretation, the solutions of the equations of motion are simply geodesic motions on the moduli space.

We have thus reduced the field theory Lagrangian to one involving only a finite number of degrees of freedom. To make use of this reduced Lagrangian, we need to determine the moduli space metric. There are at least three methods for doing this:

1. If a complete family of BPS solutions $A^\text{BPS}_a(r, z)$ is known explicitly, then one can solve for the background gauge zero modes and then substitute these into Eq. (7.6) to obtain the metric.

2. In some cases, the mathematical conditions that the moduli space metric must obey are so constraining as to essentially determine the metric.

3. The original reason for introducing the moduli space approximation was to obtain information about the low-energy interactions between monopoles. In some cases, this can be turned the other way around, and knowledge of low-energy monopole dynamics can be used to infer the moduli space metric.

The last of these strategies can be used to obtain the metric for the portion of an $N$-monopole moduli space that corresponds to widely-separated monopoles [22]. Consider first two BPS monopoles (or rather dyons) in the SU(2) theory, with positions $x_i$, spatial velocities $v_i = \dot{x}_i$, U(1) electric charges $q_i$, and U(1) magnetic charges $g_i = 4\pi/e$ ($i = 1, 2$). In order that the moduli space approximation be valid, the $v_i$ and $q_i$ should be small. When the separation between these is large (i.e., $|x_1 - x_2| \gg m_W^{-1}$), the only nonnegligible interactions between them are their mutual electromagnetic forces and the long-range force scalar force mediated by the Higgs field. The effect of these on monopole 1 are described by the Lagrangian

$$L_{\text{SU}(2)}^{(1)} = \sqrt{g_1^2 + q_1^2} \left| \phi_0 + \Delta \phi^{(2)}(x_1) \right| \sqrt{1 - \mathbf{v}_1^2} + q_1 \left[ \mathbf{v}_1 \cdot \mathbf{A}^{(2)}(x_1) - A_0^{(2)}(x_1) \right] + g_1 \left[ \mathbf{v}_1 \cdot \mathbf{\tilde{A}}^{(2)}(x_1) - \mathbf{\tilde{A}}_0^{(2)}(x_1) \right].$$

(7.7)

The first term includes the effect of the Higgs field, which is manifested through the effective reduction in the mass of monopole 1 due to the $1/r$ tail of the Higgs field of monopole 2, denoted here by $\Delta \phi^{(2)}$. The next term describes the effect on the electric charge $q_1$ of moving in the vector potential $\mathbf{A}^{(2)}$ and scalar potential $A_0^{(2)}$ of monopole 2, while the last term describes the effect on the magnetic charge $g_1$ of moving in the dual potentials generated by monopole 2. Expanding up to terms quadratic in the $v_i$ and $q_i$ gives

$$L_{\text{SU}(2)}^{(1)} = -m_1 \left( 1 - \frac{1}{2} \mathbf{v}_1^2 + \frac{q_1^2}{2g_1^2} \right) - g_1 g_2 \frac{8\pi r_{12}}{2g_1^2} \left[ (\mathbf{v}_1 - \mathbf{v}_2)^2 - \left( \frac{q_1}{g_1} - \frac{q_2}{g_2} \right)^2 \right]$$

$$- \frac{1}{4\pi} (g_1 g_2 - g_2 g_1) (\mathbf{v}_2 - \mathbf{v}_1) \cdot \mathbf{w}_{12}$$

(7.8)
where \( \mathbf{w}_{12} \) denotes the Dirac vector potential at \( \mathbf{x}_1 \) due to a unit magnetic charge at \( \mathbf{x}_2 \).

Generalizing this to the case of many particles, but still in the SU(2) theory, gives the many-particle Lagrangian

\[
L_{\text{SU}(2)} = \frac{1}{2} M_{ij} \left( \mathbf{v}_i \cdot \mathbf{v}_j - \frac{q_i q_j}{g^2} \right) + \frac{g}{4 \pi} q_i \mathbf{W}_{ij} \cdot \mathbf{v}_j
\]

where

\[
M_{ij} = \begin{cases} 
  m - \sum_{k \neq i} \frac{g^2}{4 \pi r_{ik}}, & i = j \\
  \frac{g^2}{4 \pi r_{ij}}, & i \neq j
\end{cases}
\]

and

\[
\mathbf{W}_{ij} = \begin{cases} 
  -\sum_{k \neq i} \mathbf{w}_{ik}, & i = j \\
  \mathbf{w}_{ij}, & i \neq j
\end{cases}
\]

with \( \mathbf{w}_{ij} \) being the value at \( \mathbf{x}_i \) of the Dirac potential due to the \( j \)th monopole. This is almost, but not quite, what we need. The moduli space Lagrangian is written in terms of the generalized velocities \( \dot{\mathbf{z}}_j \). However, the \( q_j \) are momenta, not velocities. We must therefore introduce U(1) phase variables \( \xi_j \) conjugate to the \( q_j \) and perform a Legendre transformation

\[
\mathcal{L}_{\text{SU}(2)}(\mathbf{x}_j, \xi_j) = \mathcal{L}_{\text{SU}(2)}(\mathbf{x}_j, q_j) + \sum_j \dot{\xi}_j q_j / e
\]

\[
= \frac{1}{2} M_{ij} \mathbf{v}_i \cdot \mathbf{v}_j + \frac{g^4}{2(4\pi)^2} (M^{-1})_{ij} \left( \dot{\xi}_i + \mathbf{W}_{ik} \cdot \mathbf{v}_k \right) \left( \dot{\xi}_j + \mathbf{W}_{jl} \cdot \mathbf{v}_l \right)
\]

to obtain a new Lagrangian that is of the form of Eq. (7.5). From this we can immediately read off the asymptotic metric describing widely-separated monopoles:

\[
ds_{\text{asym}}^2 = \frac{1}{2} M_{ij} d\mathbf{x}_i \cdot d\mathbf{x}_j + \frac{g^4}{2(4\pi)^2} (M^{-1})_{ij} (d\xi_i + \mathbf{W}_{ik} \cdot d\mathbf{x}_k) (d\xi_j + \mathbf{W}_{jl} \cdot d\mathbf{x}_l)
\]

The extension of this analysis to the case of many widely-separated fundamental monopoles in an arbitrary maximally broken gauge group is surprisingly simple [23]. Each monopole is associated with a fundamental root \( \beta_i \), and has an electric charge \( q_i \) and a phase \( \xi_i \) corresponding to the U(1) generated by \( \beta_i \cdot \mathbf{H} \). The long-range forces between a pair of monopoles are all proportional to the inner products of the corresponding simple roots. Since the inner product of any two simple roots is always less than or equal to zero, this means that the long-range interactions between two different fundamental monopoles are either vanishing (if the monopoles correspond to orthogonal simple roots), or else of the opposite sign from those between two monopoles of the same species.
The only effect on the metric is to insert factors of $\beta_i \cdot \beta_j$, replacing Eqs. (7.10) and (7.11) by

$$M_{ij} = \begin{cases} m - \frac{g^2 \beta_i^* \cdot \beta_k^*}{4\pi r_{ik}}, & i = j \\ \frac{g^2 \beta_i^* \cdot \beta_j^*}{4\pi r_{ij}}, & i \neq j \end{cases}$$

(7.14)

and

$$W_{ij} = \begin{cases} -\sum_{k \neq i} \beta_i^* \cdot \beta_k w_{ik}, & i = j \\ \beta_i^* \cdot \beta_j^* w_{ij}, & i \neq j . \end{cases}$$

(7.15)

Could this asymptotic form for the metric be in fact exact? For the case of two monopoles in the SU(2) theory, the answer is clearly no. The matrix $M$ that appears in the asymptotic metric is

$$M = \begin{pmatrix} m - \frac{g^2}{4\pi r} & \frac{g^2}{4\pi r} \\ \frac{g^2}{4\pi r} & m - \frac{g^2}{4\pi r} \end{pmatrix}$$

(7.16)

where $m$ is the mass of a single monopole. The determinant of $M$ vanishes, implying a singularity in the metric, at $r = g^2/(2\pi m) = 1/(2m_W)$. It seems quite unlikely that the interactions of two monopoles at such a distance would lead to such singular behavior. Indeed, there are short-range interactions, due to the massive fields in the monopole cores, that were not taken into account in the asymptotic analysis. If one works in the singular gauge of Eq. (3.6), these interactions can be characterized by the gauge-invariant quantity $\Re [W_{(1)}^* \cdot W_{(2)}]$.

Neither of these two objections applies in the case of a larger gauge group, provided that the monopoles correspond to different fundamental roots. There are no interactions at all, and the moduli space is flat, if the roots are orthogonal, so we need only consider the case where $\beta_1 \cdot \beta_2 < 0$. The presence of this negative inner product leads to a sign change in $M$ and removes the singularity. Furthermore, the obvious generalization of the bilinear interaction term, $\text{Tr} W_{(1)}^\dagger \cdot W_{(2)}$, vanishes identically. (The vanishing of this bilinear is a consequence of the two unbroken U(1) symmetries.)

While it thus seems possible that the asymptotic metric might be exact in this case, further analysis is required to show that this is actually so. To begin, we need several properties of the moduli space that hold whether or not the monopoles are of distinct species. From the counting of zero modes, we know that an $N$-monopole moduli space is a $4N$-dimensional manifold. Three of the coordinates can be taken to be the position of the center-of-mass. Because the center-of-mass motion is constant and independent of the behavior of the other variables, the moduli space

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*It is the deformations due to these interactions that prevent solutions with two separated SU(2) monopoles from being axial symmetric.*
can be factored as the product of a flat three-dimensional space spanned by the center-of-mass coordinates and a \((4N - 5)\)-dimensional manifold. The latter can itself be factored, at least locally, into the product of a flat \(R^1\), corresponding to an overall phase in the unbroken \(U(1)\) generated by \(h \cdot H\), and a \((4N - 4)\)-dimensional manifold depending only on relative coordinates and phases.

Furthermore, the relation between the zero modes expressed in Eq. (5.6) implies a relationship between infinitesimal motions on the moduli space that induces a quaternionic structure that makes it a hyper-Kähler manifold. (The hyper-Kähler property can also be inferred from the extended supersymmetry that underlies the BPS structure.) Finally, the moduli space must have a rotational isometry that reflects the rotational symmetry of physical space.

Taken together, these properties are rather restrictive. This is particularly the case for the two-monopole case, where the relative moduli space is four-dimensional. Any four-dimensional hyper-Kähler manifold must be a self-dual Einstein space. When combined with the requirement of a rotational isometry, this leaves only four possibilities:

1. Four-dimensional Euclidean space
2. The Eguchi-Hanson gravitational instanton \[24\]
3. The Atiyah-Hitchin geometry \[25\]
4. Taub-NUT space

The first of these can be ruled out, since it is flat and would imply that there were no interactions between the monopoles. The asymptotic behavior of the second does not match that inferred from the long-range monopole interactions, and so it too must be discarded. The Atiyah-Hitchin manifold asymptotically approaches the behavior described by Eq. (7.13), although it differs at short distance and thus avoids the singularity at \(r = g^2/(2\pi m)\); it is the relative moduli space for two SU(2) monopoles \[27\] or, more generally, for two identical monopoles with an gauge group. The only remaining possibility for the case of distinct fundamental monopoles is Taub-NUT space. Not only does this turn out to agree asymptotically with the metric found above, but it is actually identical to it for all values of the monopole separation \[26, 27, 28\].

The exact moduli space metric for than two SU(2) monopoles is not known, although it is clear that it must deviate from the asymptotic metric. On the other hand, the asymptotic metric for many fundamental monopoles, all corresponding to different simple roots of a large group \(G\), remains nonsingular for all values of the intermonopole separations. This fact led to the conjecture \[23\] that the asymptotic metric might also be exact for this case, a result that has since been proven.

\[Globally, the moduli space can be written as the product of an \(R^1\) times the \((4N - 4)\)-dimensional relative manifold divided by a discrete normal subgroup.\]
This discussion of the moduli space approximation was motivated by the need to find a zero energy bound state that would provide the missing state required by the Montonen-Olive duality conjecture. Let us now briefly return to this point. As we have seen, the natural context for this duality is \( N = 4 \) extended supersymmetry. This supersymmetry must be taken into account when deriving a low-energy approximation to the theory. This leads to a generalized moduli space Lagrangian that involves not only the bosonic coordinates \( z^i \) that span the moduli space, but fermionic coordinates \( \psi^j \) as well. For the \( N = 4 \) case, one obtains [31]

\[
L = \frac{1}{2} g_{ij}(z) \left( \dot{z}^i \dot{z}^j + i \bar{\psi}^i \gamma^0 D_t \psi^j \right) + \frac{1}{6} R_{ijkl}(\bar{\psi}^i \psi^j)(\bar{\psi}^k \psi^l) \tag{7.17}
\]

where \( R_{ijkl} \) is the Riemannian curvature on the moduli space and \( D_t \) is a covariant derivative.

When this theory is quantized, the bosonic coordinates are treated as usual. The fermionic coordinates give rise to a number of discrete modes whose occupation numbers are either 0 or 1. A state can be described by a multicomponent wave function of the form

\[
f^{(0)}(z)|\Omega\rangle + f^{(1)}_i(z)|\psi_i\rangle + f^{(2)}_{ij}(z)|\psi_{ij}\rangle + \cdots \tag{7.18}
\]

where the labeling of the kets indicates the occupied fermionic modes. The antisymmetry of the fermionic variables implies that \( f^{(n)}(z) \) is antisymmetric in its indices and so can be viewed as an \( n \)-form on the moduli space; a bound state corresponds to a normalizable form. The requirement that a state have zero energy turns out to be equivalent to requiring that the forms in its wave function be harmonic [32]. The duality conjecture predicts a single supermultiplet of zero energy bound states, and hence a single normalizable harmonic form; since the dual of a harmonic form is also harmonic, this form must be either self-dual or anti-self-dual. Once the explicit form of the two-monopole moduli space metric has been determined, it turns out to be fairly straightforward to identify the normalizable form required for self-duality in the maximally broken SU(3) theory, and to show that this form is unique [26, 27].

8 Nonmaximal symmetry breaking

Let us now return to the case of a nonmaximally broken symmetry, with the unbroken subgroup being of the form \( K \times U(1)^{r-k} \), where \( K \) is a semisimple group of rank \( k \). As in the maximally broken case, we choose a set of simple roots \( \beta_a \) whose inner products with \( h \) are nonnegative. However, we now must distinguish between the \( r-k \) roots \( \tilde{\beta}_i \) for which \( h \cdot \tilde{\beta}_i > 0 \) and the remaining \( k \) roots \( \gamma_i \) that are orthogonal to \( h \); the latter form a set of simple roots for \( K \). Furthermore, this set of
simple roots is not uniquely determined. Instead, there are several possible choices, all related by Weyl reflections that correspond to gauge transformations of $K$. Consider, for example, the case of $\text{SU}(3)$ broken to $\text{SU}(2) \times \text{U}(1)$, with the roots labeled as in Fig. 2 and the unbroken $\text{SU}(2)$ having roots $\pm \beta_2 = \pm \gamma$. The simple roots can be chosen as in the maximally broken case discussed in Sec. 6, with $\tilde{\beta} = \beta_1$ and $\gamma = \beta_2$, or they can instead be chosen to be $\tilde{\beta}' = \beta_1 + \beta_2$ and $\gamma' = -\beta_2$. The two choices are related by an $\text{SU}(2)$ gauge transformation.

![Figure 2: The root diagram of SU(3), with the Higgs vector $h$ oriented to give symmetry breaking to SU(2) x U(1).](image)

The quantization condition on the magnetic charge gives

$$
g = \frac{4\pi}{e} \left[ \sum_{a=1}^{r-k} \tilde{n}_a \tilde{\beta}_a^{\ast} + \sum_{j=1}^{k} q_j \gamma_j \right]. \quad (8.1)
$$

The $r-k$ integers $\tilde{n}_a$ are the conserved topological charges corresponding to the homotopy group $\Pi_2[G/(K \times \text{U}(1)^{r-k})]$. They are gauge-invariant, and hence independent of the choice of simple roots. The $q_i$ are also integers, but are neither gauge invariant nor conserved. As before, the energy of a BPS solution is determined by the magnetic charge, with

$$
M = \left( \frac{4\pi}{e} \right) \sum_{a=1}^{r-k} \tilde{n}_a h \cdot \tilde{\beta}_a \equiv \sum_{a=1}^{r-k} \tilde{n}_a m_a. \quad (8.2)
$$

Note that only the topological charges $\tilde{n}_a$, and not the gauge-variant $q_i$, appear in this formula.

In the maximally broken case we were able to identify $r$ species of fundamental monopoles, each carrying one unit of a single topological charge, that could be obtained by simple embeddings of the $\text{SU}(2)$ unit monopole. Both the mass formula and the counting of zero modes suggested that all BPS solutions should be interpreted as being composed of an appropriate number of fundamental monopoles. Can these ideas be extended to the case with nonmaximal breaking? Before addressing
this question, and indeed even before discussing the counting of zero modes, it may be helpful to return to the SU(3) example discussed in Sec. 3.

Recall that in the maximally broken case there were three spherically symmetric solutions that could be obtained by SU(2) embeddings. The two fundamental monopoles, corresponding to the simple roots $\beta_1$ and $\beta_2$, each had four zero modes. The third embedding solution, which corresponded to the composite root $\beta_1 + \beta_2$, had a mass equal to the sum of the other two masses, had eight zero modes, and was most naturally interpreted as a particularly symmetric member of a family of two-monopole solutions.

Now let us consider these embedding solutions for the case of SU(3) broken to SU(2) \times U(1), with the unbroken SU(2) corresponding to $\beta_2$ (i.e., lying in the lower left block). As before, the $\beta_1$-embedding gives a massive monopole solution. Explicit solution of the zero mode equations shows that there are precisely four normalizable zero modes. The $\beta_2$-embedding, on the other hand, does not give a monopole solution. Instead, Eq. (6.9), together with the fact that $h \cdot \beta_2 = 0$, shows that this embedding simple gives the vacuum. Finally, the third embedding, using $\beta_1 + \beta_2$, gives a solution that is gauge-equivalent to the $\beta_1$-embedding, and hence also has four zero modes.

This last result is particularly puzzling if we think of the nonmaximal breaking as a limiting case of maximal symmetry breaking. How are eight zero modes suddenly converted into four? One way to understand this is to follow the behavior of the three embedding solutions for the maximally broken case as $h \cdot \beta_2$ approaches zero. The $\beta_1$-solution is independent of $h \cdot \beta_2$, and neither it nor its normalizable zero modes are affected. The $\beta_2$-solution exists (with four normalizable zero modes) for any finite value of $h \cdot \beta_2$, but its core radius steadily increases, while the fields at any fixed point tend toward their vacuum values, as $h \cdot \beta_2 \to 0$. Although the mass and core radius of the ($\beta_1 + \beta_2$)-solution approach the values for the $\beta_1$-monopole, nothing particularly dramatic happens to the solution itself. The zero modes are another matter. Four of them [the three translation modes and an overall U(1) mode] lie entirely within the SU(2) subgroup defined by $\beta_1 + \beta_2$ and remain normalizable. The other four modes correspond to spatial separations of the two component monopoles and hence grow in spatial extent as the radius of the $\beta_2$-monopole increases; it is this growth, with the consequent divergence at spatial infinity, that makes these modes nonnormalizable in the limit of nonmaximal breaking.

How many zero modes would symmetry considerations have led us to expect? Usually, there is one zero mode for each symmetry of the vacuum that is not a symmetry of the soliton. Since the $\beta_1$-monopole is not invariant under the unbroken SU(2), one might at first expect to find three SU(2) modes in addition to the three translational and one U(1) mode of the maximally broken
fundamental monopole, for a total of seven. Because there is a U(1) subgroup, generated by a linear combination of one SU(2) generator and the original U(1) generator [i.e., by the $\lambda_8$ of the SU(3)], that leaves the monopole invariant, this should be reduced to six. However, it is clear that these SU(2) modes cannot be normalizable, since an SU(2) rotation affects the $1/r$ tail of the vector potential.

We can also use try to use index theory methods to count zero modes. A generalization of the SU(2) calculation gives $2I = 6$ for either the $\beta_1$- or the $(\beta_1 + \beta_2)$-monopole, in agreement with the naive symmetry arguments. However, $2I$ is equal to the number of normalizable zero modes only if the contribution of the continuum spectrum to Eq. (5.7) vanishes. In the maximally broken case, analysis of the large-distance behavior of the zero mode equations showed that the continuum contribution vanished. These arguments break down here, essentially because of terms involving the $1/r$ tail of the vector potential, thus allowing for a nonzero continuum contribution that accounts for the discrepancy between $2I$ and the actual number of normalizable modes.

The absence of normalizable zero modes associated with SU(2) transformations means that there is no need to introduce collective coordinates that specify the SU(2) orientation of the solution. As a result, the “chromodyons” — solutions carrying SU(2) charges associated with time-varying SU(2) collective coordinates — that one might have expected to find do not exist \textsuperscript{32}. At a deeper level, the absence of the chromodyons can be traced to the fact that one cannot define “global color”; i.e., there is a topological obstruction to smoothly choosing a triplet of SU(2) generators over the sphere at spatial infinity \textsuperscript{33}.

Thus, from many different approaches, we see anomalous aspects to these solutions, always associated with the slow falloff of the non-Abelian gauge fields at large distance. This long-range tail is a direct consequence of the fact that the SU(3) solutions that we have considered all have magnetic charges with non-Abelian components; i.e., their Coulomb magnetic fields are not invariant under the unbroken SU(2). This suggests that solutions whose magnetic charges commute with the unbroken subgroup might be better behaved.

For an arbitrary gauge group broken to $K \times U(1)^{r-k}$, requiring that the magnetic charge commute with the unbroken subgroup is equivalent to requiring that $g$ be orthogonal to all the roots of $K$,

$$g \cdot \gamma_i = 0 .$$

(8.3)

Many of the anomalies described above are absent when this condition holds. The zero modes associated with the action of $K$ are normalizable, and there is no obstruction to a global definition.
of “K-color”. The continuum contribution to $2\mathcal{I}$ vanishes, and so index methods can be used to count the normalizable zero modes \[34\]. One finds that there are

$$2\mathcal{I} = 4 \left[ \sum_{a=1}^{r-k} \tilde{n}_a + \sum_{j=1}^{k} q_k \right]$$  \tag{8.4}$$

such modes.\[11\]

All this suggests that in studying the case of nonmaximal symmetry breaking we should concentrate on configurations that obey Eq. (8.3) and thus have purely Abelian long-range magnetic fields. Imposing this constraint should not cause any essential loss of generality, since any additional monopoles needed to satisfy this condition can be placed arbitrarily far from the monopoles of interest. It also turns out to be helpful to treat this problem as a limiting case of maximal symmetry breaking; i.e., to start with maximal symmetry breaking and then consider the limit as some of the $h \cdot \beta_a$ tend toward zero. We will see that the moduli space for the maximal broken case appears to behave smoothly in this limit, and that the limiting value of the metric is indeed the metric for the case of nonmaximal symmetry breaking. Further, we will see that even though some fundamental monopoles may become massless, their degrees of freedom are not lost.

9 An SO(5) example

It is instructive to illustrate the case of nonmaximal symmetry breaking with an example. The simplest examples with unbroken non-Abelian subgroups occur when a rank-two group $G$ is broken to $SU(2) \times U(1)$. The choice $G = SU(3)$ is perhaps the first that comes to mind. With this choice, Eq. (8.3) requires that $\tilde{n}_1 = 2q_1$. The simplest possibility, $\tilde{n}_1 = 2$, $q_1 = 1$, would thus involve two massive monopoles and have 12 zero modes.

A somewhat simpler example \[3\] is obtained by taking the gauge group to be SO(5), whose root diagram is shown in Fig. 3. If $h$ is oriented as in Fig. 3a, the symmetry is maximally broken to $U(1) \times U(1)$, while if it is orthogonal to $\gamma$, as in Fig. 3b, the unbroken subgroup is $SU(2) \times U(1)$. Now consider the family of configurations with

$$g = \frac{4\pi}{e} (\beta^* + \gamma^*) \ .$$  \tag{9.1}$$

For nonmaximal symmetry breaking with $h$ as in Fig. 3b, these have $g \cdot \gamma = 0$, and thus satisfy Eq. (8.3). According to Eq. (8.4), there is an 8-parameter family of solutions. As I will describe

\[11\]As was noted above, the $q_i$ are gauge-variant, and so the sum appearing in this equation is in general gauge-invariant. However, one can show that for magnetic charges satisfying Eq. (8.3) this sum, and hence the expression for $2\mathcal{I}$, is gauge-invariant. If the magnetic charge does not obey Eq. (8.3), this expression for $2\mathcal{I}$ — which in any case is no longer equal to the number of normalizable zero modes — is not valid.
Figure 3: The root diagram of SO(5). With the Higgs vector $h$ oriented as in (a) the gauge symmetry is broken to $U(1) \times U(1)$, while with the orientation as in (b) the breaking is to $SU(2) \times U(1)$.

shortly, these solutions can be found explicitly, and these solutions can be used to directly obtain the moduli space metric. If instead the symmetry breaking is maximal, these solutions form a family of two-monopole solutions. Since the two component monopoles are distinct fundamental monopoles, the moduli space metric is given by the results of Sec. 7. By taking the limit $h \cdot \gamma \to 0$, we will be able to compare the limit of the maximally broken metric with the metric obtained directly from the solutions with nonmaximal breaking.

I will begin by describing the solutions for the case where SO(5) is broken to $SU(2) \times U(1)$. Of the eight parameters entering these solutions, three clearly correspond to spatial translation and just specify the position of the center of the solution. Four more must correspond to global $SU(2) \times U(1)$ transformations. This leaves only a single parameter whose interpretation is not immediately obvious. However, the solutions must be spherically symmetric, since otherwise there would be at least two rotational zero modes and hence two more parameters. This spherical symmetry can be used to reduce the field equations to a set of ordinary differential equations that can be solved explicitly \[7\]. The extra parameter then appears as an integration constant in this solution.

To write down the result, recall that the gauge field $A_i$ and the Higgs field $\phi$ are both elements of the 10-dimensional Lie algebra of SO(5). Any element $P$ of this Lie algebra can be decomposed into a pair of three-component vectors $P_{(1)}$ and $P_{(2)}$ and a $2 \times 2$ matrix $P_{(3)}$ obeying $P_{(3)}^* = -\tau_2 P_{(3)} \tau_2$
according to

\[ P = P_{(1)} \cdot t(\alpha) + P_{(2)} \cdot t(\gamma) + \text{tr} P_{(3)} \mathcal{M} \]  

(9.2)

where \( t(\alpha) \) and \( t(\gamma) \) are defined by Eq. (6.8) and

\[ \mathcal{M} = \frac{i}{\sqrt{\beta^2}} \begin{pmatrix} E\beta & -E\mu \\ E\mu & E\beta^* \end{pmatrix} . \]  

(9.3)

Under the unbroken SU(2) corresponding to \( \gamma \), \( P_{(1)}, P_{(2)}, \) and \( P_{(3)} \) transform as three singlets, a triplet, and a complex doublet, respectively.

Using this notation, the solution can be written as

\[
A_{i(1)}^a = \epsilon_{aim} \hat{r}_m A(r) \\
\phi_{(1)}^a = \hat{r}_a h(r) \\
A_{i(2)}^a = \epsilon_{aim} \hat{r}_m G(r,b) \\
\phi_{(2)}^a = \hat{r}_a G(r,b) \\
A_{i(3)} = \tau_i F(r,b) \\
\phi_{(3)} = -i F(r,b) .
\]  

(9.4)

The SU(2) singlet components \( A_{i(1)}^a \) and \( \phi_{(1)}^a \) are equal to the monopole fields of Eq. 4.1, with \( A(r) = [1 - u(r)]/er \) and \( v = \mathbf{h} \cdot \alpha \), while

\[ F(r) = \frac{v}{\sqrt{8 \cosh(evr/2)}} L(r,b)^{1/2} \]  

(9.5)

\[ G(r) = A(r) L(r,b) \]  

(9.6)

with

\[ L(r,b) = [1 + (r/b) \coth(evr/2)]^{-1} . \]  

(9.7)

The quantity \( b \) that enters through the function \( L \) is the new parameter; it can take on any positive real value. Its significance can be seen by examining the large-distance behavior of the solution. As with the SU(2) monopole, there is a core of radius \( \sim 1/ev \). Outside this core, \( A(r) \) falls as \( 1/r \), producing a \( 1/r^2 \) Coulomb magnetic field in the unbroken U(1), while \( F(r) \) falls exponentially. The behavior of \( G(r) \) depends on the relative size of \( r \) and \( b \). Because

\[
L \approx \begin{cases} 
1, & 1/ev \lesssim r \lesssim b \\
b/r, & r \gtrsim b
\end{cases}
\]  

(9.8)

\( G(r) \) falls as \( 1/r \) when \( 1/ev \lesssim r \lesssim b \). This corresponds to a \( 1/r^2 \) magnetic field in the unbroken SU(2), so in this region the solution appears to carry both U(1) and SU(2) magnetic charge. However, the \( 1/r \) falloff of \( L \) for \( r \gtrsim b \) implies that the SU(2) component of the magnetic field must fall at least as fast as \( 1/r^3 \) at large distance, so in actuality there is only a U(1) magnetic charge.
Thus, one can think of these solutions as being composed of a massive $\beta$-monopole, with a core of radius $\sim 1/ev$, surrounded by a “cloud” of non-Abelian fields of radius $b$. The eighth, nonsymmetry-related, zero mode corresponds to the fact that the energy of these solutions is independent of the value of the “cloud parameter” $b$.

The metric for the eight-dimensional moduli space of these solutions is easily obtained. First, the terms corresponding to the translation and U(1) coordinates follow immediately from the mass formula for a dyon. Next, the $b$-zero mode can be obtained by variation of the explicit solutions. Because this zero mode turns out to already satisfy the background gauge condition, it can be immediately substituted into Eq. (7.6) to give $g_{bb}$. Finally, by proceeding as described below Eq. (5.6), one can obtain three more modes from this zero mode. These correspond to global SU(2) transformations and, after conversion to the standard normalization, give the remaining components of the metric. The result is that

$$ds^2_{SU(2) \times U(1)} = Mdx^2 + \frac{16\pi^2}{M}d\chi^2 + k \left[ \frac{db^2}{b} + b \left( d\alpha^2 + \sin^2 \alpha d\beta^2 + (d\gamma + \cos \alpha d\beta)^2 \right) \right]$$

where $M$ is the total mass of the solution, $\alpha$, $\beta$, and $\gamma$ are SU(2) Euler angles, $\chi$ is a U(1) phase angle, and $k$ is a normalization constant whose value is not important for our purposes.

This should be compared with the metric for moduli space of solutions with one massive $\beta$-monopole and one massive $\gamma$-monopole in the maximally broken case. Equations (7.13–7.15) lead to

$$ds^2_{U(1) \times U(1)} = Mdx^2_{cm} + \frac{16\pi^2}{M}d\chi^2_{tot} + \left( \mu + \frac{k}{r} \right) \left[ dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2) \right]$$

$$+ k^2 \left( \mu + \frac{k}{r} \right)^{-1} (d\psi + \cos \theta d\phi)^2.$$ (9.10)

Here $M$ is the sum of the two monopole masses, $\mu$ is the reduced mass, and $r$, $\theta$, and $\phi$ specify the relative position vector $r = r_1 - r_2$. The overall U(1) phase (corresponding to the subgroup generated by $h \cdot H$) is $\chi$, while the relative U(1) phase is $\psi$. Finally, $k$ is the same constant as in Eq. (9.9).

Nonmaximal breaking corresponds to the limit $\mathbf{h} \cdot \gamma \to 0$ in which the $\gamma$-monopole becomes massless and the reduced mass $\mu$ vanishes. Setting $\mu = 0$ in Eq. (9.10) gives

$$ds^2_{U(1) \times U(1)} = Mdx^2_{cm} + \frac{16\pi^2}{M}d\chi^2_{tot} + k \left[ \frac{dr^2}{r} + r \left( d\theta^2 + \sin^2 \theta d\phi^2 + (d\psi + \cos \theta d\phi)^2 \right) \right].$$ (9.11)

This is exactly the same as Eq. (9.3), except for the change in notation.
Thus, at the level of the moduli space Lagrangian, the degrees of freedom of the $\gamma$-monopole survive even as the monopole becomes massless in the $h \cdot \gamma \to 0$ limit. However, the relation of these degrees of freedom to the classical solution changes. First, as its mass $m_\gamma$ decreases, the core of the $\gamma$-monopole spreads out until it becomes a cloud surrounding the $\beta$-monopole, with the intermonopole separation becoming the cloud size. Second, the directional angles of the $\beta$-monopole combine with the relative phase to give the Euler angles specifying the SU(2) orientation of the cloud. A consequence of the latter fact is a certain ambiguity in the position of the massless monopole: Initial configurations of massive monopoles that differ only in the relative direction of the two monopoles become gauge-equivalent in the $m_\gamma \to 0$ limit.

10 More complex examples with massless monopoles

We can gain further insight into the meaning of these massless monopoles by considering some more complicated examples. In particular, let us consider the case of SU($N$) broken to $U(1) \times SU(N-2) \times U(1)$, with the unbroken SU($N-2$) corresponding to the middle $N-3$ roots of the Dynkin diagram in Fig. 4. As before, we are interested in solutions that satisfy Eq. (8.3), so that their asymptotic magnetic fields are purely Abelian. All solutions of this equation can be written as sums of the following irreducible solutions:

1) $N-2$ massive and $(N-2)(N-3)/2$ massless monopoles, with

$$\tilde{n}_1 = 0$$

$$\tilde{n}_2 = N-2$$

$$q_j = j, \quad j = 1,2,\ldots,N-3$$

(10.1)

2) $N-2$ massive and $(N-2)(N-3)/2$ massless monopoles, with

$$\tilde{n}_1 = N-2$$

$$\tilde{n}_2 = 0$$

$$q_j = N-2-j, \quad j = 1,2,\ldots,N-3$$

(10.2)
3) Two massive and $N - 3$ massless monopoles, with

$$
\begin{align*}
\tilde{n}_1 &= 1 \\
\tilde{n}_2 &= 1 \\
q_j &= 1, \quad j = 1, 2, \ldots, N - 3
\end{align*}
$$

(10.3)

Figure 4: The Dynkin diagram of SU(N), with the labeling of the simple roots corresponding to symmetry breaking to U(1) $\times$ SU($N - 2$) $\times$ U(1).

In each of the first two cases Eq. (8.4) gives a total of $2(N - 2)(N - 1)$ zero modes. The positions and U(1) phases of the massive monopoles account for $4(N - 2)$ of these, while global SU($N - 2$) transformations give $(N - 2)^2 - 1$ more. This leaves $(N - 3)^2$ modes that presumably correspond to parameters describing gauge-invariant aspects of the non-Abelian cloud, showing that the structure of these clouds can be much more complex than in the simple SO(5) example of the previous section. It would be enormously instructive to examine these solutions in detail. Unfortunately, these solutions are not yet known explicitly, although there has been considerable progress [36] for the simplest case ($N = 4$), which can be viewed as SU(3) $\rightarrow$ SU(2) $\times$ U(1).

The third case turns out to be more tractable [8]. Here there are $4(N - 1)$ parameters, of which eight specify the positions and U(1) phases of the two massive monopoles. The number of remaining parameters is clearly less than the dimension of SU($N - 2$). This is explained by the fact that any solution with this magnetic charge can be written as an embedding of an SU(4) $\rightarrow$ U(1) $\times$ SU(2) $\times$ U(1) solution. Hence, there is a U($N - 4$) subgroup of the unbroken group that leaves any given solution invariant, so the number of global gauge zero modes is $4N - 13$, which is the dimension of SU($N - 2$)/U($N - 4$). The one remaining zero mode corresponds to the single cloud parameter, which I will again denote by $b$.

Even in this last case, the solutions are too complicated to be found by a direct attack on the field equations. However, they can be obtained by making use of a construction due to Nahm [35]. In this construction, one begins with a triplet of matrices $T_i(s)$ (where $s$ is a real variable associated with the eigenvalues of the asymptotic Higgs fields) that obey a nonlinear differential equation. These are then used to construct a linear equation that is obeyed by a quantity $v(r, s)$.
Finally, the gauge and Higgs fields for the monopole solution are obtained from integrals involving $v$ and its first derivative.

To be explicit, consider an SU($N$) theory where the asymptotic Higgs field is of the form

$$
\phi = \text{diag} (s_1, s_2, \ldots, s_N) \quad (10.4)
$$

with $s_1 \leq s_2 \leq \cdots \leq s_N$, while the magnetic charge is

$$
Q_M = \frac{4\pi}{e} \text{diag} (n_1, n_2 - n_1, n_3 - n_2, \ldots, -n_{N-1}) \quad (10.5)
$$

Define the stepwise continuous function $k(s)$ to be equal to $n_j$ on the interval $n_j < s < n_{j+1}$. The $T_i$ are required to have dimension $k(s) \times k(s)$, and to satisfy

$$
\frac{dT_i}{ds} = i \epsilon_{ijk} [T_j, T_k] + \sum_P (\alpha_P)_i \delta(s - s_P) \quad (10.6)
$$

Here $s_P$ denotes any of the points $s_j$ such that $n_{j-1} = n_j$, and the $(\alpha_P)_i$ are constant matrices of dimension $k(s_P) \times k(s_P)$.

The matrix $v(r, s)$ is of dimension $2k(s) \times N$, and obeys

$$
0 = \left[ -\frac{d}{ds} + (T_I + r_I I) \otimes \sigma_i \right] v(r, s) + \sum_P a^+_P S_P(r) \delta(s - s_P) \quad (10.7)
$$

together with the normalization condition

$$
I = \int ds v^+(r, s)v(r, s) + \sum_P S^+_P(r)S_P(r) \quad (10.8)
$$

Here $S_P(r)$ is an $N$-component row vector, while $a^+_P$ is a $2k(s_P)$-component column vector with the property that $a^+_P a_P = (\alpha_P)_i \sigma_i - i(\alpha_P)_0 I$ has rank one.\footnote{These two equations are preserved if $v$ and the $S_P$ are multiplied on the right by $N \times N$ unitary matrices; such transformations correspond to gauge transformations of the spacetime fields.}

Finally, the gauge and Higgs fields for the monopole solution are obtained from

$$
\mathbf{A}(r) = i \int ds v^+(r, s)\nabla v(r, s) + \sum_P S^+_P(r)\nabla S_P(r)
$$

$$
\phi(r) = \int ds s v^+(r, s)v(r, s) + \sum_P s_P S^+_P(r)S_P(r) \quad (10.9)
$$

In our case, the $n_j$ are all equal to unity and so the $T_i(s)$ are all $1 \times 1$. Equation (10.6) reduces to a trivial equation whose solution is that the $T_i(s)$ form a piecewise constant vector

$$
\mathbf{T}(s) = -x_a, \quad s_a < s < s_{a+1} \quad (10.10)
$$

\footnote{For the remainder of this section, I will set $e = 1.$}
With the $s_j$ all different, so that the symmetry breaking is maximal, it is clear that the $x_a$ should be interpreted as the positions of the $N - 1$ massive monopoles.

With the SO(5) example in mind, one might expect to find some ambiguity in the positions of the “massless monopoles” when the middle $N - 2$ eigenvalues of $\phi$ are set equal to give an unbroken $U(1) \times SU(N - 2) \times U(1)$. Tracing through the steps that lead from the $T_i$ to the space-time fields $A$ and $\phi$, one finds that this is indeed the case. The space-time fields are unaffected by any transformation of the massless monopoles positions $x_2, x_3, \ldots, x_{N-2}$ that leaves invariant the sum of distances

$$
\sum_{a=1}^{N-2} |x_{a+1} - x_a| \equiv 2b + |x_{N-1} - x_1| = 2b + R.
$$

(10.11)

Just as in the SO(5) example, all but one of the massless monopole coordinates are transformed into gauge-orientation parameters, leaving only a single gauge-invariant cloud parameter $b$.

Because of the particularly simple form of the $T_i$ here, Eq. (10.7) can be solved in closed form. After rescaling the solution so that the normalization condition Eq. (10.8) is satisfied, it is straightforward, although perhaps a bit tedious, to substitute the result into Eq. (10.9) and then integrate to obtain explicit expressions for $A$ and $\phi$; these involve only rational and hyperbolic functions. Because of the lack of spherical symmetry, these expressions are naturally more complex than in the SO(5) case. Nevertheless, they have some simplifying features that are reminiscent of the SO(5) solution. The fields can be decomposed into three pieces that correspond (at least at large distances) to terms transforming under the singlet, the fundamental, and the adjoint representations of the unbroken $SU(N - 2)$. Only the latter two depend on $b$, and in both cases this dependence is through a single function $L$, which is now a matrix. In addition, the Higgs fields for the fundamental and adjoint pieces are given in terms of the same spacetime functions as the gauge fields.

In fact, it is possible to choose the gauge so that, apart from a constant contribution to $\phi$, all nonzero terms in these fields lie in a $4 \times 4$ block. This shows that the solution is essentially an embedded SU(4) solution, and thus invariant under a U($N - 4$) subgroup, as was claimed above.

The details of these solutions inside the massive monopole cores are not very illuminating. On the other hand, insight into the nature of the non-Abelian cloud can be obtained by examining the asymptotic behavior of the fields well outside the cores. I will display the form that these take for the SU(4) case; the extension to $N > 4$ is straightforward.

Consider first the case $b \gg R$. If the distances $y_L$ and $y_R$ from a point $r$ to the two massive
monopoles are both much less than \( b \), the Higgs field and magnetic field can be written in the form

\[
\phi(r) = U_1^{-1}(r) \begin{pmatrix}
  t_4 - \frac{1}{2y_R} & 0 & 0 & 0 \\
  0 & t_2 + \frac{1}{2y_R} & 0 & 0 \\
  0 & 0 & t_2 - \frac{1}{2y_L} & 0 \\
  0 & 0 & 0 & t_1 + \frac{1}{2y_L}
\end{pmatrix} U_1(r) + \cdots \quad (10.12)
\]

\[
B(r) = U_1^{-1}(r) \begin{pmatrix}
  \frac{\hat{y}_R}{2y_R^2} & 0 & 0 & 0 \\
  0 & -\frac{\hat{y}_R}{2y_R^2} & 0 & 0 \\
  0 & 0 & \frac{\hat{y}_L}{2y_L^2} & 0 \\
  0 & 0 & 0 & -\frac{\hat{y}_L}{2y_L^2}
\end{pmatrix} U_1(r) + \cdots \quad (10.13)
\]

where \( U_1(r) \) is an element of SU(4) and the dots represent terms that are suppressed by powers of \( R/b, y_L/b, \) or \( y_R/b \). These are the fields that one would expect for two massive monopoles, each of whose magnetic charges has both a U(1) component and a component in the unbroken SU(2) that corresponds to the middle \( 2 \times 2 \) block. If instead \( y \equiv (y_L + y_R)/2 \gg b \),

\[
\phi(r) = U_2^{-1}(r) \begin{pmatrix}
  t_4 - \frac{1}{2y} & 0 & 0 & 0 \\
  0 & t_2 & 0 & 0 \\
  0 & 0 & t_2 & 0 \\
  0 & 0 & 0 & t_1 + \frac{1}{2y}
\end{pmatrix} U_2(r) + O(b/y^2) \quad (10.14)
\]

\[
B(r) = U_2^{-1}(r) \begin{pmatrix}
  \frac{\hat{y}}{2y^2} & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 \\
  0 & 0 & 0 & -\frac{\hat{y}}{2y^2}
\end{pmatrix} U_2(r) + O(b/y^3) . \quad (10.15)
\]

Thus, at distances large compared to \( b \) the non-Abelian part of the Coulomb magnetic field is cancelled by the cloud in a manner similar to that which we saw for the SO(5) case.
In the opposite limit, $b = 0$, the solutions are essentially embeddings of $\text{SU}(3) \rightarrow \text{U}(1) \times \text{U}(1)$ solutions. At large distances, one finds that

$$\phi(r) = U_3^{-1}(r) \begin{pmatrix} t_1 - \frac{1}{2y_R} & 0 & 0 & 0 \\ 0 & t_2 - \frac{1}{2y_L} + \frac{1}{2y_R} & 0 & 0 \\ 0 & 0 & t_2 & 0 \\ 0 & 0 & 0 & t_1 + \frac{1}{2y_L} \end{pmatrix} U_3(r) + \cdots \quad (10.16)$$

$$B(r) = U_3^{-1}(r) \begin{pmatrix} \hat{y}_R \frac{1}{2y_R} & 0 & 0 & 0 \\ 0 & \hat{y}_L \frac{1}{2y_L} - \hat{y}_R \frac{1}{2y_R} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\hat{y}_L \frac{1}{2y_L^2} \end{pmatrix} U_3(r) + \cdots . \quad (10.17)$$

Viewed as SU(3) solutions, the long-range fields are purely Abelian. Viewed as SU(4) solutions, the long-range part is non-Abelian in the sense that the unbroken SU(2) acts nontrivially on the fields. However, because of the alignment of the fields of the two massive monopoles, the non-Abelian part of the field is a purely dipole field that falls as $R/y^3$ at large distances.

11 Concluding remarks

I began these lectures by arguing that the one-particle states built from solitons should not differ in any essential way from those based on the elementary quanta. The massless monopoles that arise when there is nonmaximal symmetry breaking appear to present a challenge to this point of view. When viewed in terms of classical solutions, they do not seem very particle-like: They do not exist as isolated classical solutions, and in multimonopole configurations they coalesce into “clouds” of arbitrary size rather than appearing as localized objects with well-defined positions. Furthermore, while their degrees of freedom are preserved in the moduli space Lagrangian as one goes over from the massive to the massless case, the natural interpretation of these in terms of particle properties are lost.

Of course, the presumed duals to these massless monopoles are the massless elementary gauge bosons (“gluons”) carrying non-Abelian charges. These also differ significantly from their massive counterparts, especially in the low-energy regime. Like the massless monopoles, these can never
be at rest, so it is perhaps not so strange that there are no static solutions corresponding to a single isolated massless monopole. Further, the gluon field surrounding a massive particle carrying non-Abelian charge can be seen as somewhat analogous to the massless monopole clouds. It would be very desirable to be able to make these correspondences more precise, perhaps including scattering calculations along the lines of those described in Sec. 4.

Much remains to be learned about the properties and dynamics of these massless monopoles. Further investigation of these offers the promise of deeper insight into the nature of non-Abelian gauge theories.

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