Double-Cut of Scattering Amplitudes and Stokes’ Theorem

Pierpaolo Mastrolia
Theory Group, Physics Department, CERN, CH-1211 Geneva 23, Switzerland

Abstract

We show how Stokes’ Theorem, in the fashion of the Generalised Cauchy Formula, can be applied for computing double-cut integrals of one-loop amplitudes analytically. It implies the evaluation of phase-space integrals of rational functions in two complex-conjugated variables, which are simply computed by an indefinite integration in a single variable, followed by Cauchy’s Residue integration in the conjugated one. The method is suitable for the cut-construction of the coefficients of 2-point functions entering the decomposition of one-loop amplitudes in terms of scalar master integrals.
Unitarity and analyticity are well-known properties of scattering amplitudes \([1]\). Analyticity grants that amplitudes are determined by their own singularity-structure, while unitarity grants that the residues at the singular points factorize into products of simpler amplitudes. Unitarity and analyticity become tools for the quantitative determination of one-loop amplitudes \([2]\) when merged with the existence of an underlying representation of amplitudes as a combination of basic scalar one-loop functions \([3]\). These functions, known as Master Integrals (MI’s), are \(n\)-point one-loop integrals, \(I_n (1 \leq n \leq 4)\), with trivial numerator, equal to 1, characterised by external momenta and internal masses present in the denominator. Important improvements of unitarity-based numerical algorithms also make use of the general structure of one-loop integrands \([4, 5, 6, 7]\). In the context of unitarity-based algorithms, the issue of computing one-loop amplitudes can be addressed in two stages: the computation of the coefficients; and the actual evaluation of the MI’s themselives. The principle of a unitarity-based method is the extraction of the coefficients multiplying each MI by matching the multiparticle cuts of the amplitude onto the corresponding cuts of the MI’s.

Cutting a propagating particle in an amplitude amounts to applying the on-shell condition and replacing its Feynman propagator by the corresponding \(\delta\)-function, \((p^2 - m^2 + i0)^{-1} \to (2\pi i) \delta^{(n)}(p^2 - m^2)\). As a result, the original function is substituted by a simpler one, easier to compute, which, nevertheless, still carries non-trivial information. In fact, the \(n\)-particle cut of \(I_n\) appears in the \(0\)-transcedentality term (rational or irrational) of the corresponding cut-amplitude, multiplied by the same coefficient of \(I_n\) in the decomposition of the complete amplitude. Higher-transcedentality terms, such as logarithms, are associated to the cuts of higher-point MI’s.

In general, the fulfillment of multiple-cut conditions requires loop momenta with complex components. Since the loop momentum has four components, the effect of the cut-conditions is to fix some of them according to the number of the cuts. Any quadruple-cut \([8]\) fixes the loop-momentum completely, yielding the algebraic determination of the coefficients of \(I_n, (n \geq 4)\); the coefficient of 3-point functions, \(I_3\), are extracted from triple-cut \([9, 10, 11, 12, 13]\); the evaluation of double-cut \([14, 15, 16, 17, 12, 18, 19, 13]\) is necessary for extracting the coefficient of 2-point functions, \(I_2\) and finally, in processes involving massive particles, the coefficients of 1-point functions, \(I_1\), are detected by single-cut \([12, 20, 21]\). In cases where fewer than four denominators are cut, the loop momentum is not frozen: the free-components are left over as phase-space integration variables.

In this letter, we show a novel efficient method for the analytic evaluation of the coefficients of one-loop 2-point functions via double-cuts. Spun-off from the spinor-integration technique \([14, 15, 16, 17, 18, 19]\), the method hereby presented is an application of Stokes' Theorem. We analyze the double-cut of massless particles in four-dimensions, which also is the essential ingredient for the phase-space integration in the general case of one-loop massive amplitudes in dimensional-regularization \([16, 17, 18, 19, 9]\). Due to a special decomposition of the loop-momentum, the double-cut phase-space integral is written as parametric integration of rational function in two complex-conjugated variables. By applying Stokes’ Theorem, the integration is carried on in two simple steps: an indefinite integration in one variable, followed by Cauchy’s Residue Theorem in the conjugated one.

The coefficients of the 2-point scalar functions, being proportional to the rational term of the double-cut, can be directly extracted from the indefinite integration by Hermite Polynomial Reduction.

In a framework where factorization properties of scattering amplitudes are accessed via complex momenta, the double-cut integration presented here can be considered as the natural extension of the technique used to prove BCFW-recurrence relation for tree-level amplitudes \([22]\). In the latter case, scattering tree-amplitudes are holomorphic functions, depending only on one complex variable, and Cauchy’s Residue Theorem is sufficient for their complete determination. In the case of the double-cut of one-loop amplitudes, where the integrand depends on two complex-conjugated variables, Stokes’ Theorem in the fashion of Generalised Cauchy Formula, becomes the driving principle.
1. Double-Cut

- Phase-Space Parametrization. The starting point of our derivation is the spinorial parametrization of the Lorentz invariant phase-space (LIPS) in the $K^2$-channel \[23\] \[14\] \[15\].

\[
\int d^4\Phi = \int d^4\ell_1 \delta^{(+)}(\ell_1^2) \delta^{(+)}((\ell_1 - K)^2) = 
\int \frac{(\ell_1\ell_1')|d\ell\ell_1'}{(\ell_1|K|\ell_1')} \int dt \delta^{(+)} \left(t - \frac{K^2}{(\ell_1|K|\ell_1')}\right),
\]

(1)

obtained by rescaling the original loop-variable $\ell_1^\mu$ as,

\[
\ell^\mu_1 = \frac{\langle \ell_1|\gamma^\mu|\ell_1 \rangle}{2} \equiv t \ell^\mu = t\langle \ell|\gamma^\mu|\ell \rangle \frac{2}{2},
\]

(2)

with $\ell_2^2 = \ell_1^2 = 0$. In terms of spinor variables, the rescaling reads,

\[
|\ell_1| = \sqrt{t} |\ell|, \quad |\ell_1| = \sqrt{t} |\ell|,
\]

(3)

where $t$, the rescaling parameter, is frozen as a consequence of the (second of the) on-shell conditions, and $\ell_1$ becomes the new loop integration variable.

- Change of Variables. We take two massless momenta, say $p_\mu$ and $q_\mu$ fulfilling the conditions,

\[
p_\mu + q_\mu = K_\mu, \quad p^2 = q^2 = 0, \quad 2p \cdot q = 2p \cdot K = 2q \cdot K = K^2,
\]

(4)

and decompose $\ell_1\ell_\mu$ in a basis of four massless momenta constructed out of them,

\[
\ell_1\ell_\mu = p_\mu + z \bar{z} q_\mu + \frac{z}{2} \langle q|\gamma_\mu|p \rangle + \frac{\bar{z}}{2} \langle p|\gamma_\mu|q \rangle.
\]

(5)

Notice that the vectors $\frac{\langle q|\gamma_\mu|p \rangle}{2}$ and $\frac{\langle p|\gamma_\mu|q \rangle}{2}$ are trivially orthogonal to both $p_\mu$ and $q_\mu$. The above decomposition can be realized starting from the definition of $\ell_1\ell_\mu$ in terms of spinor variables, $\ell_\mu = \frac{\ell(\ell_1\ell_1')}{2}$, and performing the following spinor decomposition,

\[
|\ell| \equiv |p| + z|q|, \quad |\ell| \equiv |p| + \bar{z}|q|.
\]

(6)

By changing variables $\langle |\ell|, |\ell| \rangle \rightarrow (z, \bar{z})$ as in \[9\], and using \[4\], one can write,

\[
\langle \ell d\ell|d\ell_1 \rangle = K^2 dz d\bar{z}, \quad \langle \ell|K|\ell \rangle = K^2 (1 + z\bar{z}).
\]

(7)

\[
\langle \ell|K|\ell \rangle = K^2 (1 + z\bar{z}.
\]

(8)

Hence, the LIPS in \[1\] reduces to the novel form,

\[
\int d^4\Phi = \oint dz \int d\bar{z} \int dt \delta^{(+)} \left(t - \frac{1}{1 + z\bar{z}}\right), \quad (9)
\]

where $t$ is a positive quantity as assured by the argument of the $\delta$-function.

- Double-Cut Integration. The double-cut of a generic $n$-point amplitude in the $K^2$-channel is defined as

\[
\Delta \equiv \int d^4\Phi A^{\text{tree}}_{L,R}((\ell_1), |\ell_1|) A^{\text{tree}}_{L,R}((\ell_1), |\ell_1|),
\]

(10)

where $A^{\text{tree}}_{L,R}$ are the tree-level amplitudes sitting at the two sides of the cut, as in \[3\], and using expression \[9\] for the LIPS, one has,

\[
\Delta = \int d^4\Phi A^{\text{tree}}_{L}((t, |\ell|, |\ell|) A^{\text{tree}}_{R}(t, |\ell|, |\ell|)
\]

\[
\Delta = \oint dz \int d\bar{z} \int t^2 dt \delta \left(t - \frac{1}{1 + z\bar{z}}\right) \times
\]

\[
\ell_{\alpha L + \alpha R} A^{\text{tree}}_{L}(\ell_1, |\ell|) A^{\text{tree}}_{R}(\ell_1, |\ell|),
\]

(11)

where $\ell_{\alpha L,R}$ parametrizes the scaling behaviour of $A^{\text{tree}}_{L,R}$. The $t$-integration can be performed trivially, because of the presence of the $\delta$-function. Then, by using the decomposition \[5\] \[8\], the double-cut becomes a double-integral,

\[
\Delta = \oint dz \int d\bar{z} f(z, \bar{z}),
\]

(12)

where $f$ is a rational function of $z$ and $\bar{z}$. As such, it can be expressed as a ratio of two polynomials, say $P$ and $Q$,

\[
f(z, \bar{z}) = \frac{A^{\text{tree}}_{L}(z, \bar{z}) A^{\text{tree}}_{R}(z, \bar{z})}{(1 + z\bar{z})^{\alpha L + \alpha R + 1}} = \frac{P(z, \bar{z})}{Q(z, \bar{z})},
\]

(13)
with the following relations between their degrees,
\[ \deg_z Q = \deg_z P + 2 \quad , \quad \deg_z Q = \deg_z P + 2 \ . \] (14)
We remark that the double integration in \( z \)- and \( \bar{z} \)-variables appearing in Eq. [12] will be properly justified in Sec. [2]. For the moment, with abuse of notation, we simply denote it as a convolution of an indefinite \( \bar{z} \)-integral and a contour \( z \)-integral, which are the actual operations we are going to carry out.

To begin with the integration, we find a primitive of \( f \) with respect to \( \bar{z} \), say \( F \), by keeping \( z \) as independent variable,
\[ F(z, \bar{z}) = \int d\bar{z} \; f(z, \bar{z}) \ , \] (15)
so that \( \Delta \) becomes,
\[ \Delta = \oint dz \; F(z, \bar{z}) = \oint dz \int d\bar{z} \; F_{\bar{z}} \ , \] (16)
where \( F_{\bar{z}} \) is a short-hand notation for \( \partial F/\partial \bar{z} \). Before proceeding with the final integration on the \( z \)-variable, let us analyse the structure of \( F \). Since \( F \) is the primitive of a rational function, its general form can only contain two types of terms: a rational term and a logarithmic one,
\[ F(z, \bar{z}) = F_{\text{rat}}(z, \bar{z}) + F_{\text{log}}(z, \bar{z}) \ . \] (17)
It is important to notice that the presence of the term \( F_{\text{rat}} \) depends on the powers of \( t \) in Eq. [11]: \( F_{\text{rat}} \) can be generated, after integrating \( f \) in \( \bar{z} \), only if \( \alpha_R + \alpha_L \geq 0 \). The \( z \)-integration will be performed by applying Cauchy’s Residue Theorem, therefore the final structure of the double-cut is determined by the nature of \( F \). Namely, the \( z \)-integration of \( F_{\text{rat}} \) \( [F_{\text{log}}] \) is responsible of the rational [logarithmic] term of \( \Delta \).

- Double-cut of the Scalar Function \( I_2 \). Let us evaluate the double-cut of the 2-point scalar function \( I_2 \), which also is a prototype example:
\[ \Delta_{I_2} = 2\pi i \left( \text{Res}_{z=0} F_{\text{rat}}(z, \bar{z}) + \text{Res}_{\bar{z}=0} F_{\text{rat}}(z, \bar{z}) \right) \ . \] (19)

where the \( z \)-integration is performed via Cauchy’s Residue Theorem. The integrand \( F_{\text{rat}} \) is rational in \( z \), and contains poles whose location in the complex plane is a unique signature of the Feynman integral they come from [17 18 24]. The choice of \( p \) and \( q \) specified in Eqs. [4] grants that there exists a pole at \( z = 0 \) associated to the 2-point function in the \( K^2 \)-channel, \( I_2(K^2) \); while the reduction of higher-point functions that have \( I_2(K^2) \) as subdiagram can generate poles at finite \( z \)-values. Because of the presence of \( \bar{z} \), through the term \( (1 + z\bar{z}) \), \( F_{\text{rat}} \) is non-analytic. The Residue Theorem has to be applied by reading the residues in \( z \), and substituting the corresponding complex-conjugate values where \( \bar{z} \) appears. Therefore, the result of \( \Delta_{I_2} \) can be implicitly written as,
\[ \Delta_{I_2} = (2\pi i) \left( \text{Res}_{z=0} F_{\text{rat}}(z, \bar{z}) + \text{Res}_{\bar{z}=0} F_{\text{rat}}(z, \bar{z}) \right) \ . \] (20)

For the last integration in \( z \), by applying the Residue Theorem, we take the residue of the unique simple pole at \( z = 0 \), since the term \( (1 + z\bar{z}) \equiv (1 + |z|^2) \) never vanishes, being always positive. The final result of the double-cut of the scalar 2-point function reads,
\[ \Delta_{I_2} = (2\pi i) \text{Res}_{z=0} F_{\text{rat}}(z, \bar{z}) = -2\pi i \ . \] (22)
- **Coefficient of the 2-point Function.** The expression of the 2-point coefficient can be finally obtained by taking the ratio of $\Delta^{\text{rat}}$ in (19) and the double-cut of $I_2$ in (22),
\[
c_2 = \frac{\Delta^{\text{rat}}}{\Delta I_2} = -\text{Res}_{z=0}F^{\text{rat}}(z, \bar{z}) - \text{Res}_{z\neq 0}F^{\text{rat}}(z, \bar{z}) . \tag{23}
\]

- **Hermite Polynomial Reduction.** To optimize the integration algorithm, one can use the so called Hermite Polynomial Reduction (HPR), a technique enabling the direct extraction of the rational term of the primitive of a rational function, without computing the integral as a whole. Based on the square-free factorization of the integrand, HPR can be used to write the result of any integral of a rational function as a pure rational term plus another integral that, if explicitly computed, would generate the logarithmic remainder.

As written in Eq. (23), the coefficient of the 2-point function comes only from the term $F^{\text{rat}}$, and not from $F^{\text{log}}$, see Eqs. (15, 17): $F^{\text{rat}}$ is the rational term in the result of the $\bar{z}$-integration of $f$, which is rational in $\bar{z}$, see Eq. (13). Therefore HPR is suitable for extracting $F^{\text{rat}}$ out of the $\bar{z}$-integration.

The integration algorithm of Sec. 1 can be implemented with SOM [23] together with the routine [20] for Hermite Polynomial Reduction.

### 2. Stokes’ Theorem

In this section we give a formal definition of the $z$-$\bar{z}$ integration used in Sec. 1 as an application of Stokes’ Theorem for differential forms. In what follows, we use the notation: $g_z = \partial g/\partial z$ and $g_{\bar{z}} = \partial g/\partial \bar{z}$.

Let us recall that the complex 1-form
\[
\chi = \frac{1}{z - z_0} dz , \tag{24}
\]
which is defined for all $z$ except $z_0$, is a closed form,
\[
d\chi = d \left( \frac{1}{z - z_0} \right) \wedge dz = \frac{(-1)}{(z - z_0)^2} dz \wedge dz = 0 . \tag{25}
\]

We consider any complex smooth function $F$ and differentiate the 1-form $\omega = F \chi$,
\[
\omega = (z - z_0)^{-1} F dz , \tag{26}
\]
\[
\text{obtaining the 2-form},
\]
\[
d\omega = dF \wedge \chi = (z - z_0)^{-1} F \bar{z} d\bar{z} \wedge dz . \tag{27}
\]

Now we take a domain $D$ in the complex plane and apply Stokes’ Theorem to $d\omega$. Due to the singularity of $\omega$ at $z_0$, we remove a tiny disk $D(z_0; r)$, centered at $z_0$ with radius $r$, from $D$. Then $\omega$ has no singularity in the regulated domain $D_r = D - D(z_0; r)$, and we may apply Stokes’ Theorem:
\[
\iint_{D_r} d\omega = \int_{\partial D_r} \omega = \int_{\partial D} \omega - \int_{\partial D(z_0; r)} \omega . \tag{28}
\]

Here $\partial D(z_0; r)$ is a circle $\gamma$ around the point $z_0$, which is described by the parametric equation $\gamma(t) = z_0 + re^{it}$. Since $F(z_0 + re^{it})$ converges to $F(z_0)$ as the radius $r$ shrinks to 0, the last integral in Eq. (28),
\[
\int_{\partial D(z_0; r)} \omega = \int_0^{2\pi} F(z_0 + re^{it}) dt , \tag{29}
\]
\[
\text{converges to } 2\pi i F(z_0) \text{ as } r \text{ goes to } 0. \quad \text{Letting } r \to 0 \text{ in Eq. (28), the disk } D(z_0; r) \text{ disappears and } D_r \text{ fills up } D. \quad \text{Consequently Stokes’ Theorem can be reformulated as,}
\]
\[
\iint_{D} d\omega = \int_{\partial D} \omega - 2\pi i F(z_0) . \tag{30}
\]

By using the explicit expression of $\omega$ and $d\omega$, in Eqs. (26, 27), and rearranging terms, we obtain the so called Generalised Cauchy Formula or Cauchy-Pompeiu Formula,
\[
2\pi i F(z_0) = \int_{\partial D} \frac{F(z)}{z - z_0} dz - \iint_{D} \frac{F(\bar{z})}{z - z_0} d\bar{z} \wedge dz . \tag{31}
\]

Let us discuss two special cases.

First, when $F$ is analytic, $\bar{F} = 0$, hence we obtain,
\[
F(z_0) = \frac{1}{2\pi i} \int_{\partial D} \frac{F(z)}{z - z_0} dz \tag{32}
\]
which is the well-known Cauchy Formula, where $\partial D$ is any closed curve surrounding $z_0$.

Secondly, when $F$ vanishes on the boundary of $D$, that is $F|_{\partial D} = 0$, Eq. (31) becomes,

$$F(z_0) = \frac{1}{2\pi i} \oint_D \frac{F_z}{z - z_0}dz \land d\bar{z}. \quad (33)$$

where we used $d\bar{z} \land dz = -dz \land d\bar{z}$.

The expression (33) is what needed to define properly the double-cut $\Delta$ given in Eqs. (12, 16), which we rewrite here as,

$$\Delta \equiv \int\int_D F(z, \bar{z}) dz \land d\bar{z} = \int\int_D \frac{F_z}{z - z_0}dz \land d\bar{z}, \quad (34)$$

by identifying $f = F_z = F_{\bar{z}}/(z - z_0)$, and $F = F/(z - z_0)$, where the functions $f$ and $F$ were defined in Eqs. (13, 15, 17). The integration domain, $D$, is the whole complex plane. The vanishing of $F$ on the boundary is granted by the structure of the rational integrand and relations (14) among the degrees of numerator and denominator.

To deal with the general case, where more than one pole might appear, the calculation of $\Delta$ trivially generalises, by the superimposition principle, to the sum of the residues at all the poles in $z$,

$$\Delta \equiv \int\int_D F_z dz \land d\bar{z} = \sum_j \int\int_D \frac{F_{z}^{(j)}}{z - z_j} dz \land d\bar{z} = 2\pi i \sum_{j \in \text{poles}} F^{(j)}(z_j), \quad (35)$$

due to the subtraction of a disk around each of the $z$-poles from the domain $D$.

Finally, Eq. (35) validates Eq. (19), hence the expression for the coefficient $c_2$ in Eq. (23). Notice that the role of $z$ and $\bar{z}$ in the application of Stokes’ Theorem can be interchanged, reflecting the symmetry of $c_2$ under the exchange $p \leftrightarrow q$ in (4).

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