Prediction and suppression of chaotic instability in brake squeal

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Abstract Chaotic instability in a vibration phenomenon, known as brake squeal, is investigated including the combined effects of falling friction, sprag-slip and mode coupling. Brake squeal is a high-pitched noise that occurs sometimes when a vehicle is decelerated using disc brakes. The equations of motion for the two dominant coupled modes of the brake system reduce to two autonomous coupled nonlinear second-order systems. The mode coupling instability via friction causes limit cycle behaviour via a supercritical Hopf bifurcation. This limit cycle is shown to break up into chaotic motion characterised by a Poincare map with a less than one-dimensional attractor, similar to that found in a forced dry friction oscillator. For the first time, conservative analytical conditions for brake squeal chaos are developed and verified numerically over a range of sprag angles and brake pressures for fundamental and real brake systems. The predictive model is then used to identify and quantify means to suppress brake squeal chaos to unlikely, excessive friction levels. The results provide predictive insight into conditions under which brake squeal chaos occurs and its suppression.

Keywords Brake squeal · Chaotic instability · Hopf bifurcation

1 Introduction

Brake squeal is a high-pitched tonal noise that occurs during braking of a vehicle using disc brakes. For the automotive and railway industries, this causes undesirable noise and vibration resulting in excessive customer complaints and warranty costs [1]. In general, brake squeal occurs due to a friction-induced instability of one or more brake system modes of vibration that grows to a nonlinear limit cycle and/or bounded chaotic oscillations. An understanding of the occurrence of brake squeal chaos requires insight into the underlying mechanisms of its instability. There have been several specific mechanisms of brake squeal investigated, as reviewed in [2], including falling friction, spragging [3, 4] and mode coupling [5, 6].

Spurr [3] first suggested the “sprag-slip” mechanism for the instability of brake squeal. Sprag-slip can be considered a type of stick–slip vibrations (i.e. [7]) related to a beam- or strut-like structure sliding at an angle to a surface. He found a semi-rigid strut inclined to a moving surface could “sprag” or dig into the moving surface and then dynamically slip based on the static friction and the sprag angle [4]. This was verified using “beam on disc” testing [8] and beam-on-belt numerical simulations [9] including beam elasticity and finite thickness. Fieldhouse [1, 10] applied sprag-slip theory to a real brake system geometry to identify the sprag angle and determine how the position of the dynamic centre of pressure of a brake pad affects squeal [11]. North [5] identified the mechanism of
mode coupling in automotive brake squeal, analogous to “binary flutter”, and Hoffman [6] performed a reduced model numerical analysis to provide important mechanistic insight and the effects of damping [12, 13]. The relationship between spragging and mode coupling was solidified in [14] where spragging of the dynamic modes was identified as an important condition for mode coupled squeal [14]. Complex eigenvalue analysis can be used to numerically predict brake squeal mode coupling in complex geometries, although recently an analytical solution for a reduced two-mode system has been identified [14]. The simplified test rig results have confirmed that mode coupled squeal primarily occurs due to two distinct modes with closely spaced natural frequencies [15]. In full brake assemblies squeal frequencies were found to typically vary from 2 to 15 kHz due to a combinations of rotor circumferential in-plane modes, diametric out-of-plane modes and radial in-plane modes [16–22]. A comprehensive FEA and experimental investigation of a 3 kHz squeal in a real full brake system was performed by Park [20]. A comparison of these real full brake system measurements to the squeal model used in the present research has already been performed by the author’s research team recently in [14] which focused on the linear instability behaviour of mode coupling. In addition, a closed form prediction for the occurrence, growth and amplitude of brake squeal was identified [14], but chaos was not investigated. One of the purposes of the present paper is to extend this research to nonlinear behaviour.

Oberst and Lai [23] provided a comprehensive review of research into chaotic instabilities in brake squeal and related friction coupled systems. Although externally forced, experimental and numerical chaos due to instability was identified by Feeny and Moon [24] in a fundamental dry friction oscillator. The chaotic attractors were found to have common features despite the different friction laws used. In particular, the only minor differences found were that true sticking motion produced cusps in the funnel structure of the chaotic attractor as opposed to tiny loops from “almost sticking” motion for the continuous friction law. More specifically, in a simplified two-degree-of-freedom brake pad and disc system, Shin et al. [25] performed a numerical stability analysis and showed squeal limit cycle attractors in phase space, dependent on the closeness of the modes and stabilising or destabilising damping. Large nonlinear chaotic-like motion was also noticed following a period doubling route; however, the motion was not conclusively identified [26]. In [27], nonlinear time domain analysis of a brake model showed intermittent bursts of very high amplitude vibrations via an intermittency route to chaos. Chaos inherent to a stick–slip oscillator was quenched using large amplitude dither in Feeny and Moon [28] and Lin et al. [29] numerically identified and controlled brake disc squeal using state feedback control. Conversely chaotic instability in a full-scale brake system was confirmed numerically and experimentally and the transition from a limit cycle to an unstable torus attractor [23] was quantified. In addition, a novel method to estimate Lyapunov exponents from brake squeal measurements was demonstrated in [30] using noise in an Eckmann–Ruelle statistical procedure and showed quasi-periodic or slightly chaotic behaviour. Limit cycles, quasi-periodic motion, weakly chaotic attractors and different types of stick–slip vibrations were measured experimentally in friction excited vibrations with transitions characterised by beating phenomena, sudden energy exchange and intermittent dynamics [31]. Numerical work also confirmed observations of complex dynamical behaviour and chimera-alike dynamics in a self-excited frictional oscillator with a weak stiffness nonlinearity [32].

Although this previous research has numerically or experimentally identified chaotic instability in brake squeal, no closed form analytical predictive criteria for its onset have been developed. Due to nonlinearities, this means chaos can only be predicted numerically using a limited set of parameters under restricted computational requirements. Therefore, in addition, there has been no predictive insight to analytically quantify measures to avoid brake squeal chaos. In addition, the combined mechanisms of mode coupling, spragging and falling friction have not been specifically investigated. To address this, the present research is focused first on the predictive identification and quantification of chaotic instabilities in brake squeal under “falling friction”, mode coupled and spragging conditions. Subsequently the model is used to predict and quantify means of eliminating chaos in brake squeal. The major contributions include:

1. Closed form analytical predictions and quantification of chaotic instability in brake squeal, under falling friction, mode coupling and spragging
mechanisms, validated with numerical simulations and nonlinear dynamics tools.
2. Further insight into the conditions associated with brake squeal chaotic instability.
3. Analytical prediction to identify and quantify parametric control methods to suppress brake squeal chaos.

This paper will first describe the reduced two-mode brake squeal model consisting of two autonomous coupled nonlinear second-order systems with a nonlinear friction law. Closed form analyses to predict the local instability occurrence, growth and limit cycle amplitude of brake squeal are then reviewed and developed. Conservative analytical criteria for the onset of brake squeal chaos are then derived. Numerical simulations are then performed and quantitative nonlinear dynamics tools (including Poincare maps and Lyapunov exponents) are used to identify chaotic instability onset and compared with the analytical necessary criteria in simplified fundamental and real brake model cases. The criteria are used to efficiently predict a range of typical brake parameters under which chaotic squeal could occur. The criteria are then used to efficiently investigate a range of typical brake parameters under which brake squeal chaos can be eliminated or suppressed to unrealistically high, static friction levels.

2 Methodology

A reduced two-mode brake squeal model consisting of two autonomous coupled nonlinear second-order systems with a nonlinear friction law is first described [14]. Closed form analyses to predict the local instability occurrence, growth and limit cycle amplitude of brake squeal are then described in Sect. 2.2. These are then used with an invariant measure in nonlinear dynamics theory to develop conservative analytical criteria for the onset of brake squeal chaos in Sect. 2.3.

Figure 1 describes the fundamental mechanical interactions causing brake squeal limit cycle oscillations and noise.

Essentially the contact mechanics couples the brake pad mechanics and coupled modal dynamics of the brake pad and rotor system. The brake pad mechanics determines the centre of pressure of the contact at which point the sliding velocity and its component in the squealing mode direction are determined. The friction behaviour and parameters are governed by the contact mechanics which transfer friction forces and couple with the dynamics of the dominant modes of vibration of the brake system. If the interaction is unstable, the resultant brake system vibrations grow into a squeal limit cycle which in turn creates noise [14, 33]. The main purpose of this paper is to analytically predict when the squeal limit cycle breaks up into chaotic motion and use this insight to suppress the phenomenon. It is possible that chaotic motion

Fig. 1 Conceptual model of brake squeal noise [14]
could occur under mode coupling, causing very large instability levels resulting in averaged localised phase space expansion [34]. In the following, the full nonlinear numerical and simplified analytical models used for this purpose are described.

2.1 Full nonlinear brake squeal model

The full nonlinear interactions of the brake system of Fig. 1 can be encapsulated by the two-degree-of-freedom model of Fig. 2 [14]. The derivation of the equations of motion is provided in [14], so for convenience are just summarised in the following. Figure 2a shows the total sliding velocity, \( V \), at a brake pad angle of attack, \( \theta_A \), to the perpendicular components, \( V_o \) and lateral crabbing (or creep) velocity, \( V_c \), which are tangential and normal to the pad centre of mass relative motion. The contact between the rotor and pad under static and dynamic normal force, \( N \), \( F_N \), induces a friction force, \( F_f \), that couple the two dominant spragging and out-of-plane modes, \( a \), \( b \), at a sprag angle, \( \theta \), shown in Fig. 2b.

According to Fig. 2a, the quasi-static brake pad angle of attack, \( \theta_A \), is determined geometrically as,

\[
\sin(\theta_A) = V_c / V = x_{CoP} / r_{CoP},
\]  

(1)

where \( x_{CoP} \) and \( r_{CoP} \), define the position of the Centre of Pressure. Note that tangential or radial pad motion, which have both been associated with brake squeal [23], can be investigated using this model. The dynamic sliding ratio which includes the vibration component in the contact plane is defined as,

\[
\zeta = (V_{c,o} + (\delta_a \dot{y}_a(t) \cos(\theta))) / V,
\]  

(2)

where \( \delta_a \) is the sliding direction defined as +1 in Fig. 2b. The contact mechanics couples two dominant modes at a sprag angle according to Fig. 2b, as,

\[
\begin{align*}
m_a \ddot{y}_a(t) + c_a \dot{y}_a(t) + k_a y_a(t) &= -F_N \sin(\theta) - F_f \cos(\theta), \\
m_b \ddot{y}_b(t) + c_b \dot{y}_b(t) + k_b y_b(t) &= F_N
\end{align*}
\]  

(3)

where

\[
F_N = k_H (y_a \sin(\theta) - y_b), F_f = \delta_a \mu(\zeta)(N + F_N),
\]  

(4)

where \( y \), \( m \), \( k \) and \( c \) are the modal displacements, masses, stiffnesses and damping coefficients, respectively. The recently published research of the authors showed that one of the key mechanisms governing the linear instability occurrence under different braking

![Fig. 2 Two-degree-of-freedom model for brake squeal consisting of; a brake pad mechanics with sliding and friction forces acting through the centre of pressure and b coupled modal dynamics [14]. Insets: Abaqus & Free3D Car Brake System models](image)
pressure regimes is the pressure-dependent contact stiffness, \( kH \), between the brake pad and disc due to contact area and multi-scale effects. This was fully investigated and validated with full brake system measurements in [14]. The present paper focuses on the subsequent nonlinear squeal behaviour governed by the nonlinear sliding friction physics. A smooth nonlinear friction law is used from [24] to represent both large and small vibrations as a function of the dynamic sliding ratio,

\[
\mu(\zeta) = \mu_s \tanh(z_\mu \zeta) \left[ (1 - \mu_k \operatorname{sech}(\beta_\mu \zeta)) \right].
\]

(5)

where \( \mu_s \) and \( \mu_k \) represent the static and kinetic friction coefficients and \( z_\mu \) and \( \beta_\mu \) determine the shape of the friction curve. The friction curves of (5) are shown in the results as Fig. 3, for a fundamental and real brake system that has been tuned to measurements in [35]. The sound pressure level of brake squeal has already been predicted in [14]. Comparison of predictions with field and experimental sound pressure level measurements, under the falling friction instability causing wheel squeal, has also been provided in [36]. This showed that the sound pressure level may be predicted, assuming that the sound radiation is proportional to the squared vibration velocity normal to the squealing surfaces. Hence, in this paper we focus on predicting the vibration velocity amplitude.

Equations (1)–(5) represent the full nonlinear equations of motion of the system that can be solved numerically. The model has already been validated to existing brake squeal theory and full brake squeal measurements in [14]. Note that the two-degree-of-freedom model described by Fig. 2 is a modal variational model from nominal conditions. Therefore in practice, loss of contact between the two degrees of freedom can occur if the dynamic normal force amplitude exceeds the static normal force. This condition has been neglected in the present model and would introduce addition nonlinear effects. Analytical solutions for predicting the occurrence growth and amplitude of squeal limit cycles are described in subsequent Sect. 2.2.

### 2.2 Analytical solutions for brake squeal limit cycles

The local stability and growth of brake squeal can be determined using a complex eigenvalue analysis. This is typically performed numerically due to mathematical difficulties; however, an analytical solution was recently obtained in [37] under the assumptions of small non-proportional damping. In such a way, coupled Eqs. (3) and (4) can be decoupled to the form,

\[
\mathbf{p}^T \mathbf{M} \ddot{\mathbf{Y}} + \mathbf{p}^T \mathbf{C} \dot{\mathbf{Y}} + \mathbf{p}^T \mathbf{K} \mathbf{Y} = \mathbf{p}^T \mathbf{F},
\]

\[
\mathbf{Y} = \mathbf{p}^{-1} \begin{bmatrix} y_a & y_b \end{bmatrix}^T, \quad \mathbf{F} = \begin{bmatrix} -\delta_\mu(\zeta) N \cos(\theta) & 0 \end{bmatrix}^T
\]

(6)

where \( \mathbf{Y} \) is the decoupled modal displacement vector and \( \mathbf{M}, \mathbf{K}, \mathbf{C} \) and \( \mathbf{F} \) are the, mass, stiffness, damping matrices and forcing vector of Eqs. (3) and (4), respectively. The left and right undamped eigenvectors \( \mathbf{p}_L \) and \( \mathbf{p}_R \) are defined as,

\[
\mathbf{p}_L = \begin{bmatrix} 1 & P_{L2} \\ P_{L1} & 1 \end{bmatrix} \quad \text{where} \quad \mathbf{p}_L^T \mathbf{K} = \lambda^2 \mathbf{p}_L^T \mathbf{M},
\]

\[
\mathbf{p}_R = \begin{bmatrix} 1 & P_2 \\ P_1 & 1 \end{bmatrix} \quad \text{where} \quad \mathbf{K} \mathbf{p}_R = \lambda^2 \mathbf{p}_R \mathbf{M}.
\]

(7)

where \( p_i \) and \( p_{Li} \) are the eigenvector elements that determine the eigenvalues and therefore the system modal damping as a function of system parameters, as [37],

\[
p_1 = \left[ \omega_a^2 - \omega_b^2 + \sqrt{\Delta} \right] /[2f(\mu, 0)kH/(m_a \sin(\theta))], \quad P_2 = 1/p_1 \text{, } P_{L1} = -m_a/p_1 m_b, \quad P_{L2} = -m_b/p_2 m_a
\]
therefore,

\[ p_L p_i = 4f(\mu, 0)(k_H/m_i)^2 \left[ \omega_a^2 - \omega_b^2 - \sqrt{\Delta} \right]^2, \]

where \( f(\mu, 0) = \sin^2(\theta)(1 + \delta_a \mu/\tan(\theta)), \)

\[ \omega_a^2 = (k_a + k_H f(\mu, 0))/m_a, \omega_b^2 = (k_b + k_H)/m_b, \]

\[ \Delta = (\omega_a^2 - \omega_b^2)^2 + 4f(\mu, 0)k_H^2/(m_a m_b) \]

Equation (8)

Here \( i \) is the mode number, \( j = b, a \) is the conjugate, \( f(\mu, 0) \) represents Spurr’s [1, 3, 10] criteria for spragging \( \tan(\theta) < -\delta_a \mu \) and \( \Delta \) is the discriminant that determines whether the eigenmodes are real or complex and therefore the type of mode coupling (see [14] for more details). In particular, recent research [13, 14, 37] has more clearly identified and characterised two types of mode coupling: stiffness mode coupling where the closeness of the modal frequencies causes instability coupling via the stiffness matrix and viscous mode coupling where the modal damping causes instability coupling via the damping matrix. Stiffness mode coupling is the well-recognised form of mode coupling instability where the natural frequencies of the uncoupled system are close. Viscous mode coupling is less well recognised as it can occur when the uncoupled natural frequencies are not close. More details are provided in [14, 37]. Mathematically, if \( \Delta \) is positive, the eigenvalues are real and viscous mode coupling occurs (coupled determined by the damping matrix). Conversely, if \( \Delta \) is negative, the eigenvalues

\[
\mu(\zeta) = \begin{cases} 
    k_{\mu 1} \zeta' & \text{for } \zeta' \leq 1 \\
    k_{\mu 1} + k_{\mu 2}(\zeta' - 1) & \text{for } \zeta' > 1
\end{cases}
\]

where \( \zeta \geq -\zeta_c \) and \( \zeta_c = \zeta_c/\zeta_c \)

Equation (9) highlights that the system modal damping, \( c_{si} \), depends on the complexity of the eigenmodes and therefore the type of mode coupling (i.e. viscous or stiffness). Since the structural modal damping factors are usually well below critical, it may be deduced from (13) that stiffness mode coupling (second line) provides a much larger contribution to

\[
c_{si} = \begin{cases} 
    c_i & \text{if } p_i, p_{Li} = \text{Re}(p_i), \text{Re}(p_{Li}) \\
    \text{Re} \left( c_i/(2m_i) \pm \sqrt{c_i^2/(2m_i)^2 - k_i/m_i} \right) 2m_i & \text{if } p_i, p_{Li} \neq \text{Re}(p_i), \text{Re}(p_{Li})
\end{cases}
\]

Equation (13)
the system modal damping (and local eigenvalue magnitude) than viscous mode coupling (first line). The stability and hence occurrence of brake squeal can then be simply expressed according to when the effective system damping becomes negative and analytically determined using Eqs. (11) and (13) as,

\[
\text{Instability / squeal when: } c_{\text{eff}} = \left[ c_{\text{si}} + k_{F} k_{I2} N / (\zeta_{c} V) \right] < 0
\]

where \(c_{\text{eff}}\) is the effective system damping, \(c_{\text{si}}\) is structural damping, \(k_{F}\) is friction slope damping, \(k_{I2}\) is mode coupling, \(N\) is normalized load, and \(\zeta_{c}\) is damping ratio.

The brake squeal criterion (14) concisely combines and superposes the effects of mode coupling and falling friction via the effective system damping, \(c_{\text{eff}}\). Squeal instability will initially grow exponentially when the effective system damping, \(c_{\text{eff}}\), is negative but this growth will change with amplitude as \(c_{\text{eff}}\) changes. In particular, when the sliding oscillation amplitude encroaches on the friction stick region below critical friction, positive damping will be added. At a particular oscillation amplitude, \(A_{\text{o}}\), where the power from the negative damping exactly balances the positive damping, a steady-state limit cycle will occur according to [14, 24],

\[
A_{\text{o}} t_{c} = \left( V_{c,0} - \zeta_{c} V / \left( \cos(\theta) \cos(\omega_{i} t_{c}) \right) \right) \\
\text{where } \sin(2\omega_{i} t_{c}) - 2\omega_{i} t_{c} = 2\pi \left( \zeta_{c} V / \left( k_{F} N \right) + k_{I2} \right) / \left( k_{I1} - k_{I2} \right)
\]

where \(t_{c}\) is the time in the squeal cycle that the sliding ratio reaches the critical peak of the friction curve. The closed form solution of (15) assumes a linear sinusoidal limit cycle. This assumption may be broken under larger amplitude nonlinear phenomena such as chaotic motion [23, 24, 34]. Therefore, Eqs. (8), (11)–(15) provide closed form analytical solutions to brake squeal limit cycle occurrence, initial growth and amplitude under the instability mechanisms of falling friction, mode coupling and spragging.

2.3 Prediction of brake squeal chaos

What happens when the squeal limit cycle breaks down due to excessive instability levels causing phase space expansion? In the following, two analytical criteria for the prediction of chaotic vibrations in brake squeal are developed based on the Lyapunov exponent (expansion) test for chaos and the analytical solutions developed in 2.2. The Lyapunov exponent test measures the sensitivity of the system to changes in initial conditions. In particular, the Lyapunov exponent, \(\lambda\), defines the power law growth or expansion of the change in size of an initial error condition according to [38],

\[
d = d_{0} 2^{(t - t_{0})},
\]

where \(d\) is the maximum diameter of the ellipsoid in phase space that grows from the initial conditions at time \(t\) and \(d_{0}\) is the diameter of the sphere of initial conditions at time \(t_{0}\). A positive Lyapunov exponent implies chaos, i.e.

Chaos possible if \(\lambda > 0\).

Note this Lyapunov exponent analysis is merely a necessary condition for chaos. A positive maximal Lyapunov exponent characterises sensitivity to initial conditions, which on its own does not imply any sort of complicated behaviour in the lack of other conditions such as phase space compactness, topological transitivity and a dense set of periodic orbits. In a chaotic system the motion is bounded by nonlinearities so its divergence, \(\lambda\) is measured locally using small time increments \(t - t_{0}\) or linearly using partial derivatives (the Jacobian matrix) and continual resetting to obtain an average local divergence over a long period of time. Similarly, the eigenvalues of the Jacobian matrix determine the local stability such that there is a close relationship between the maximum Lyapunov exponent and system eigenvalue evaluated dynamically, i.e. in a general case, the maximum Lyapunov exponent can be shown to be an average of the real part of the maximum dynamic eigenvalue integrated over a sufficiently long time period [39]. Therefore, chaos is most likely to occur when the largest eigenvalue is most positive, indicating the highest
local divergence. In the present brake squeal case, the stability and eigenvalue analysis has been determined analytically in Sect. 2.2. It is found that the largest eigenvalue is most positive when the effective system damping $c_{\text{eff}}$ in the modal Eq. (11) is most negative. This occurs under stiffness mode coupling conditions because the modal damping factor is less than critical, causing the complex stiffness to have a larger instability effect than the modal damping (13). Therefore, a simple, conservative analytical solution for the critical friction coefficient at which brake squeal chaos occurs may be obtained, based on the occurrence of stiffness mode coupling, as:

Chaos possible if $\mu_s > \mu_{\text{crit}}$, $\Delta(\mu_{\text{crit}}) = 0$, \hspace{1cm} (18)

where a fully closed form analytical solution for the critical friction for chaos may be obtained using (8) and (18), as,

$$\mu_{\text{crit}} = \left\{ \frac{m_a}{k_H} \left\{ \frac{\sqrt{k_H (k_a - k_b) (k_H - k_a - k_b)}}{m_a m_b} - \frac{(k_H + k_a - k_b)}{m_a} \left( \frac{k_H}{m_a} - \frac{k_b}{m_b} \right) \right\} \right\} \sin^2(\theta) / (\delta_a \cos(\theta) \sin(\theta)).$$ \hspace{1cm} (19)

Equations (18) and (19) represent a conservative criterion for brake squeal chaos in that only the condition for high local phase space expansion due to stiffness mode coupling is considered. As it is based on the largest eigenvalue of the system, it has not taken into account the continuous averaging of the local expansion at many points around the phase space. Also bounded chaos requires sufficient nonlinearities for folding and contraction in the phase space. This may be taken into account by also defining an amplitude criterion at which substantial expansion and nonlinearities occur.

For this purpose, the closed form limit cycle solution for brake squeal (15) may be used to solve for when the bilinear friction law (9) is exceeded causing large nonlinearities associated with negative creep oscillations [34]. Therefore, the critical condition for the possible breakup of the limit cycle to nonintegrable behaviour may be defined by the necessary condition,

$$A_d \omega_i \geq V_{c,o} / \cos(\theta) \hspace{1cm} \text{for} \hspace{1cm} V_c > \zeta_c V_o.$$ \hspace{1cm} (20)

This conservative condition may be simplified by solving for the critical system modal damping using (20) and the solution (15), as,

Chaos if $\mu_s > \mu_{\text{crit}}$, $c_{\text{eff}}(\mu_{\text{crit}}) = k_{\text{eff}} N [(\sin(2\omega_i t_c) - 2\omega_i t_c) (k_{\mu 1} - k_{\mu 2}(\mu_{\text{crit}})) / (2\pi) - k_{\mu 2}(\mu_{\text{crit}})] / (\zeta_c V_c)$, $\omega_i t_c = a \cos(1 - \zeta_c V_c / V_c).$ \hspace{1cm} (21)

Equation (21) represents a less conservative criterion for brake squeal chaos than (18) and (19), i.e. in addition to large local expansion it assumes negative sliding is required to provide sufficient nonlinearities, to bound the brake squeal phase space behaviour. In this case, the analytical solution is not fully closed form, so a numerical root finding function is required to find the solution for $\mu_{\text{crit}}$ from the analytical equations for $c_{\text{eff}}$ in (8), (11)–(13). However, in practice, this computational time was found to be almost instantaneous. Note that the effect of parametric excitation due to changing friction levels [34] has been neglected due to the very steep positive slope of the friction curve and to avoid multiple lengthy integrations. This assumption is tested in the next section. In particular, comparisons of numerical solutions of the full nonlinear equations of motion are compared with these conservative predictive criteria for brake squeal chaotic instability (18, 19, 21) subsequently in Sect. 3. Finally, it is noted that the energy provision/generation for chaotic motion comes from the frictional dynamics of the squeal instability. In particular, any chaotic attractor that develops is powered by individual or combined frictional instability mechanisms identified in Sect. 2.2, i.e. falling friction, mode coupling and spragging.

3 Results

The occurrence of chaotic instability in brake squeal was investigated for two case studies [14] using the full nonlinear time domain model, described in Sect. 2.1. The numerical solution of the full nonlinear equations of motion in Sect. 2.1 was solved on an INTEL Core i7 using the fourth- and fifth-order Runge–Kutta routine using DYNAMICS by Nusse and Yorke, or the Radua method in MathCad 15.0 and an initial velocity at or smaller than 0.01 m/s. The time
step was set at least 100 times smaller than the sprag mode natural period of oscillation, $2\pi/\sqrt{(k_H/m_a)}$. The nonlinear phenomena was investigated using various phase space tools and the Lyapunov spectrum of exponents using algorithms described in [40] and iterated at least 1000 squeal oscillation periods. Although this Lyapunov exponent algorithm [40] can have difficulties with discontinuities, as identified in recent research, i.e. [41], the friction law described by Eq. (5) is a continuous approximation of the friction oscillator switching mechanism and no exponent convergence issues were found for the parameters investigated. Comparisons of the full numerical solutions and the conservative analytical criteria derived in Sect. 2.3 were performed and used to identify conditions under which chaotic instability occurs and its suppression over a range of friction coefficients, modal damping constants, sprag angles and brake pressures. The system nominal parameter conditions were chosen according to fundamental and real investigations of brake squeal, [13, 14, 20, 35, 42] as described in Table 1.

The nonlinear friction model of Eq. (5) was tuned experimentally to a typical automobile pad (Volvo 850/S70 X8 pad) [35]. Note that lower friction curve steepness parameters, $\mu_s$, to [14] were used to provide the same accuracy of fit while enhancing the numerical stability and speed of the solver. The approximate bilinear friction curve parameters $k_{\mu 1}$ and $k_{\mu 2}$ were obtained automatically from the experimentally tuned nonlinear friction model according to [24],

$$k_{\mu 1} \approx \mu_s(\zeta_c) \quad \text{and} \quad k_{\mu 2} \approx (\mu(2\theta_A) - \mu_s)\zeta_c/(2\theta_A).$$

These experimentally tuned nonlinear and approximate bilinear friction curves of Eq. (5) and (22) were plotted over a large sliding velocity range in Fig. 3.

The curves of Fig. 3 show the experimental friction curve including falling friction can be well represented by the full nonlinear curve (5) and the analytical bilinear approximation with less than 2% and 5%
error, respectively, for the fundamental model. The nominal higher static friction level of the real brake system model is also represented reasonably with the falling friction notably having a higher slope according to (5) and (22). These friction models are used to investigate the onset and control of chaotic instability in brake squeal in the following.

3.1 Numerical identification of brake squeal chaos in a fundamental system

A fundamental two degree-of-freedom brake squeal model [13, 14, 42] was investigated for nonlinear squeal instability as the friction coefficient was changed. The fundamental model parameters were chosen as described in [14] and are listed in the first column of Table 1. The local stability of the equations of motion using (13) were verified with full numerical complex eigenvalue solutions in [14] for varying friction coefficient. It showed two-mode solutions with distinct frequencies up to a critical value of \( \mu = 0.453 \). Beyond this, stiffness mode coupling occurs where the modes have the same natural frequency but opposing complex stiffness damping levels that are increased by the structural modal damping. The complex stiffness negative damping increases as the friction coefficient increases further until it is balanced by the positive structural damping at \( \mu = 0.47 \) at which point a squeal limit cycle occurs. This transition to a stable limit cycle as the friction increases represents a Hopf bifurcation. Since it is found, the previously stable equilibrium point representing no motion is now unstable, the bifurcation is supercritical. The amplitude of the limit cycle is determined by the nonlinear friction curve which may introduce nonlinear instability, i.e. chaotic motion.

To investigate the nonlinear behaviour, time histories and phase spaces of numerical solutions of the nonlinear differential equations of motion were first determined under the static friction levels, \( \mu_s \), of 0.65 and 0.85 as shown in Fig. 4.

The time history of Fig. 4a shows that brake squeal occurs after a small period (\( \sim 7 \) s) of exponential growth to reach a constant limit cycle amplitude for a static friction level, \( \mu_s = 0.65 \), exceeding the supercritical Hopf bifurcation value. The oscillations are approximately sinusoidal as predicted by the local stability analysis. The corresponding friction oscillations shown in Fig. 4b are mainly on the negatively sloped region of the friction curve, encroaching only a very small part of the steep positively sloped region to provide the necessary power dissipation to achieve the

![Fig. 4 Simulated squeal oscillations for fundamental brake system; a sprag mode vibration velocity (blue line) and b corresponding friction oscillation range (blue solid line) under static friction \( \mu_s = 0.65 \) and c sprag mode vibration velocity (blue line) and d corresponding friction oscillation range (blue solid line) under static friction \( \mu_s = 0.85 \). (Color figure online)](image-url)
limit cycle energy balance. In contrast, when the static friction is increased to $\mu_s = 0.85$, Fig. 4c shows much larger, irregular, non-sinusoidal, sprag velocity oscillations. Similarly, comparison of the cases of Fig. 4b, d highlights much larger friction oscillations with larger encroachment on both the positive and negative slopes. Note that in this case, although it is approaching negative (or reverse) friction as the static friction is increased, the large irregular squeal oscillations have not encroached into this negative sliding state which is in contrast with the predictions of chaotic motion for an analogous wheel squeal case [34]. This should be investigated further as it appears the nonlinear effects associated with the chaotic motion are due to the friction curve shape over the positive stick and sliding region alone (see Fig. 4d). This is found in the forced one-degree-of-freedom friction oscillator system of [24] but in contrast to the positive and negative sliding behaviour found in [34]. Based on the damping magnitudes from the system structural damping ($c_d = -0.11 \text{ Ns/m}$) and positive and negative sloped stick region ($k_F k_F (2 N / (\zeta_s V)) = 1532$ and $-0.99 \text{ Ns/m}$, respectively), it would appear that there is a sudden change in damping force at critical sliding that in this case is distorting the sliding velocity from a linear sinusoidal shape. This would tend to suggest that the chaotic behaviour is due to the bilinear nature (with one slope approaching infinity) of the friction curve (not trilinear) as consistent with [24, 43, 44]. In particular, the large non-sinusoidal oscillations for the case Fig. 4c, d indicate linear assumptions of the predictive limit cycle analysis (9) and (15) are invalidated. To investigate this possible chaotic instability in more detail, further numerical nonlinear analyses were performed in the following.

The onset of brake squeal chaos in the phase space was investigated under increasing static friction coefficient as shown in Fig. 5.

Figure 5 depicts brake squeal behaviour changing from a one-loop limit cycle and bifurcating to more complex nonlinear behaviour. In particular, Fig. 5a shows a small closed single loop limit cycle for $\mu_s = 0.65$ indicating approximately sinusoidal squeal behaviour. For initial conditions further away from this smaller loop, a different larger limit cycle was found similar to the inner loop of Fig. 5b. As the static friction coefficient is increased to $\mu_s = 0.75$ (Fig. 5b) the limit cycle bifurcates into two-loop motion that bifurcates again as the friction is increased to $\mu_s = 0.8$. Further increases in static friction to $\mu_s = 0.85$ caused further increasingly compressed bifurcations to a wandering orbit that appears to not be closed. Note this attractor in Fig. 5d) appears to have the same small loops from “almost sticking” behaviour found in [24], although this squeal is unforced. This could indicate chaotic instability. For confirmation, the corresponding Poincare maps and bifurcation diagram were calculated as all shown in Fig. 6. The Poincare plane of $y_b = 0.6$ was chosen for convenience.

The nonlinear behaviour of the phase spaces of Fig. 5 appears to be confirmed in Fig. 6. In particular, the bifurcation diagram of Fig. 6a shows a period 4 limit cycle, at low static friction coefficient, bifurcating via a well-known period doubling route to chaos as friction increases. The two-loop phase space behaviour of Fig. 5b at $\mu_s = 0.75$ is confirmed by the Poincare map of Fig. 6b with a closed set of 8 points indicating period 8 limit cycle behaviour. The bifurcation diagram of Fig. 6a shows a branching to period 16 limit cycle in Fig. 6c and further increasingly compressed bifurcations to the onset of chaos at $\mu_s > 0.82$. This is shown in the Poincare map of Fig. 6d of a fractal of less than one dimension (note the full phase space is higher), characterised by the nonperiodic random like behaviour in the phase space of Fig. 5d. The strange attractor was determined to have a Lyapunov dimension of 2.85 with the fractal nature confirmed by the non-integer value. This was calculated using the Lyapunov spectrum of [0.142 0.0 $-0.167 - 454$] with the positive maximal exponent confirming chaotic instability via exponential sensitivity to initial conditions. Note that for a static friction coefficient of $\mu_s = 0.883$ there occurs a window of periodic behaviour with a closed set of approximately 12 points (with evidence of some overlapping in Fig. 6a). This typical period doubling route to chaos in brake squeal is in contrast to the quasi-periodic route found for wheel squeal in [34] but is consistent with the forced friction oscillator chaos with the same friction model in [24]. Interestingly, it is noted that in this case, chaotic instability occurs without a nonlinear jump to brake squeal negative sliding amplitudes. This is inconsistent with [34] but encapsulated by the first conservative criteria developed in Sect. 2.3. This identification and onset of brake squeal chaos was investigated further in a real brake system.
3.2 Numerical identification of brake squeal chaos in a real brake system

A similar nonlinear investigation to Sect. 3.1 using the two-degree-of-freedom brake squeal model (2.1) was performed on a real brake system with coupled in- and out-of-plane modes that was experimentally verified in [20]. The analytical and numerical predictions of the occurrence and growth of limit cycle squeal are provided in [14] and are verified by a comprehensive set of brake squeal experiments detailed in [20]. In particular, the measured braking conditions under which brake squeal occurred on the floating calliper brake assembly were verified with the theoretical modelling but chaotic phenomena were not investigated. The squeal mode was measured to have a frequency of 3 kHz [20] and found to correspond to the finite element model results in Fig. 7a [14].

Dynamic transformations of Fig. 7a were used to determine the parameters listed in Table 1 for modes a

![Phase spaces of brake squeal oscillations under increasing static friction coefficient](image)

**Fig. 5** Phase spaces of brake squeal oscillations under increasing static friction coefficient; a $\mu_s = 0.65$, b $\mu_s = 0.75$, c $\mu_s = 0.8$ and d $\mu_s = 0.85$
and $b$ in Fig. 7b, based on the coupled rotor and brake calliper dynamics, as detailed in [14]. One of the key characteristics of real brake squeal is its dependence on brake pressure [20]. This is most likely due to contact stiffness variation from loading of rough elastic surfaces and non-elastic behaviour that may be modelled as [14, 45],

$$k_{H}(p) = k_{Ho}p^{1/2},$$

(23)

where $p$ and $k_{Ho}$ are the brake pressure and a proportional factor tuned to the nominal conditions, respectively. In this case, the brake pressure exponential coefficient of $\frac{1}{2}$ is chosen for a typical contact roughness fractal distribution according to [46]. Hence, the contact stiffness and modal stiffness and mass parameters, $b$, are functions of pressure $p$ as detailed in [14]. In the following cases, when the damping was changed from the nominal damping ratios of 3.6% and 4.2% for the $a$ and $b$ modes, respectively, the scaling $c_b = 3.613 c_a$ was preserved.
To investigate the nonlinear behaviour in this real brake system, numerical simulations of the time history and friction curve behaviour were first investigated under nominal and greatly reduced (10%) damping levels as shown in Fig. 8.

Fig. 7 Real brake system squeal model a coupled in-plane radial (IPR) and out-of-plane diametrical (OPD) modes, FEA performed in [14, 20], b coupled two-mode brake squeal model [14]

Fig. 8 Simulated squeal oscillations for real brake system; a sprag mode vibration velocity (blue line) and b corresponding friction oscillation range (blue solid line) under damping $c_{sp} = 3600$ Ns/m and c sprag mode vibration velocity (blue line) and d corresponding friction oscillation range (blue solid line) under damping $c_{sp} = 360$ Ns/m. In all cases $c_\mu = 3.613 c_a$. (Color figure online)

Similar to Fig. 4, Fig. 8 shows brake squeal due to: (a) a periodic limit cycle due to stiffness mode coupling and (b) a larger irregular amplitude time history for a much lower modal damping (10%). The corresponding friction oscillations shown in Fig. 8b, d
highlight the low damping case has very large, positive and negative sliding which is not sinusoidal. In particular, comparing the time histories of Fig. 8a, c, in the period up to 0.05 s, it may be deduced that the oscillations grew to a nonlinear irregular amplitude once it encroached into negative (or reverse) sliding, as consistent with [34], but in contrast to Fig. 4. The very large irregular oscillations for the low damping case in the zoom-up of Fig. 8c and the sliding oscillations in Fig. 8d indicate linear limit cycle assumptions of Eqs. (9) and (15) are invalidated. In the following, further phase space and Poincare map investigations were performed.

A more detailed investigation of the onset of chaotic brake squeal under progressively lower damping is shown in the phase spaces of Fig. 9.

Figure 9 shows that the brake squeal behaviour changes from a one-loop limit cycle and breaks into more complex possibly chaotic behaviour. In particular, Fig. 9a shows a simple closed limit cycle for $c_a = 3600$ Ns/m indicating sinusoidal-like squeal behaviour. As the damping was decreased to 50% nominal, shown in b), the limit cycle breaks into

Fig. 9 Simulated phase spaces of squeal oscillations in a real brake system under decreasing damping: a $c_a = 3600$ Ns/m, b $c_a = 1800$ Ns/m, c $c_a = 900$ Ns/m and d $c_a = 360$ Ns/m. In all cases $c_b = 3.613 c_a$. 
complex periodic motion that complicates further as
the damping is decreased to 25% nominal damping.
Further decreases in damping to 10% nominal damp-
ing leads to a wandering orbit that appears to not be
closed. For confirmation, the corresponding bifurca-
tion diagram and Poincare maps are shown in Fig. 10.
The Poincare plane of \( y_b = 0.0 \) was chosen for
convenience.

Figure 10 confirms the nonlinear behaviour shown
in the phase spaces of Fig. 9 is brake squeal chaos. The
bifurcation diagram (Fig. 9a) shows chaotic behaviour
at the lowest damping levels of 10% nominal that
gradually becomes more ordered as the damping is
increased. In particular, the sinusoidal squeal with a
single loop phase space at nominal damping in Fig. 9a
is confirmed to be period 2 motion in the Poincare map
of Fig. 10b. The breakdown to complex periodic
motion occurring at around 50% nominal damping in
Fig. 9b is shown to be approximately a period 16 limit
cycle in Fig. 10b. This breakdown appears to be
through a sudden crisis phenomenon rather than period
doubling bifurcations as shown in the bifurcation

![Bifurcation Diagram](image)

**Fig. 10** a Bifurcation diagram of Poincare maps for increasing damping in real brake system. Poincare maps for decreasing damping; b \( c_a = 3600 \text{ Ns/m} \), c \( c_a = 1800 \text{ Ns/m} \) and d \( c_a = 360 \text{ Ns/m} \). In all cases \( c_b = 3.613 c_a \).
diagram of Fig. 10a. The bifurcation diagram shows further slurring of the Poincare map periodic points as the damping reduces to 10% nominal to obtain a fractal in the Poincare map of Fig. 10d characterised by the nonperiodic random like behaviour in the phase space of Fig. 9d. Brake squeal chaos is confirmed by a positive maximal exponent of the Lyapunov spectrum $[71.0.0 - 1062 - 1.61 \times 10^5]$. The fractal nature of the strange attractor was confirmed with a non-integer of 2.07 for the Lyapunov dimension based on the Lyapunov spectrum. One of the key observations of the these fundamental and real brake squeal cases is that chaotic motion occurs with and without a nonlinear jump to negative sliding oscillations. It is interesting to see if these contrasting cases can both be predicted using the necessary analytical criteria developed in Sect. 2.3.

3.3 Comparison of predictions of brake squeal chaos

The onset of steady-state brake squeal chaos over a range of parameters was numerically and analytically investigated. The occurrence of a non-closed limit cycle was detected using the phase space and Poincare map analyses of Figs. 9 and 10 to determine the critical friction coefficient for a range of contact angles and brake pressures. These numerical solutions were then quantitatively compared with the conservative analytical predictions of chaotic instability using Eqs. (18)–(21). For the fundamental brake model conditions, two cases with different chaotic behaviour were compared, the first being parameters studied in Table 1 and the second where the damping and normal force was dropped to 30% and 0.8% of the nominal values, respectively. For the real brake model, the brake pressure was varied with the damping held at 10% of the nominal values of Table 1 under which chaotic vibrations occurred as shown in Figs. 9 and 10d. Figure 11 summarises these results and highlights the conservative nature of the analytical prediction for the onset of steady-state brake squeal chaos.

The fundamental model behaviour of Fig. 11a shows a good comparison between the analytically predicted onset of chaotic instability at critical static friction coefficient levels and that from the full nonlinear numerical simulations. In particular, the onset of chaos over the range of sprag angles is conservatively predicted by the negative sliding criterion to within 15% errors for the first case with lower damping and normal force (red circle). A very similar trend is also predicted by the more conservative, but simpler, stiffness mode coupling criterion with larger errors for this case. For the nominal case (blue square box), the negative sliding criterion does not exist as the normal load is too large to reach negative sliding under those parameters. This was confirmed by the friction oscillation ranges in Fig. 4d. In this case, it appears chaos is occurring due to different nonlinearities in the friction curve, and therefore, the negative sliding criterion does not apply. This could be investigated further. Conversely, the
stiffness mode coupling criteria is still relevant and provides a useful conservative prediction of the initial occurrence of chaos, with notably the trend of critical static friction coefficient vs sprag angle predicted well. Chaotic squeal is characterised by high static friction coefficient over 0.6 for a range of sprag angles causing stiffness mode coupling squeal.

The real model behaviour of Fig. 11b shows a well-matched comparison, of critical static friction levels, between the conservative analytical predictions and the numerical simulations for the onset of chaotic instability over a range of brake pressures with errors less than 25%. In particular, the onset of chaos over the range of brake pressures is also well predicted by both stiffness mode coupling and negative sliding criteria (18)–(21) which are almost overlapping each other. This closeness may be due to the very high sensitivity of brake squeal amplitude with static friction coefficient near to stiffness mode coupling conditions. This indicates that brake squeal chaos is first determined by the onset of stiffness mode coupling and second large nonlinearities in the friction curve due to negative sliding. Suppression of chaotic instability can therefore be achieved using the quantitative understanding of the parameters and their effect on verified criteria demonstrated in Fig. 11.

### 3.4 Suppression of chaotic instability in brake squeal

The suppression of the onset of chaotic instability in the real brake squeal model is explored further in Table 2 with the nominal damping at 10% of the values of Table 1. In particular, the % change in nominal parameter value required to avoid chaotic instability by ensuring the critical static friction coefficient exceeds realistic values of $\mu_{\text{crit}} > 1$, was determined, using the verified analytical criterion for negative sliding. Note, in this suppression investigation, the friction sliding slope, $k_{l2}$, is held constant as the friction coefficient changes to isolate its effects.

Table 2 highlights multiple means of eliminating chaotic instability in brake squeal by parameter control. In general, the results appear to reflect that chaotic instability is dependent upon the stiffness mode coupling occurrence that in turn is dependent upon how close the uncoupled mode frequencies are. In particular, the following observations from the results of Table 2 can be made:

| Parameter description | Change required for chaos elimination ($\mu_{\text{crit}} > 1$) | Example practical suppression method (control/change in) |
|-----------------------|-------------------------------------------------------------|--------------------------------------------------------|
| Structural vibration parameters | | |
| Modal mass (for sprag angle, $\theta$ constant) ($m_a$, $m_b$) | $-62\%$ or $+29\%$, $-23\%$ or $+166\%$ | Rotor/calliper structural design |
| Modal stiffness (for sprag angle, $\theta$ constant) ($k_a$, $k_b$) | $-24\%$ or $+88\%$, $+31\%$ | Rotor/calliper structural design |
| Modal damping (for sprag angle, $\theta$ constant) ($c_a$, $c_b$) | $+6000\%$, $+7160\%$ | Structural/material/shim damping |
| Contact stiffness (for sprag angle, $\theta$ constant) ($k_l$) | $-77\%$ or $+2300\%$ | Material/geometry/surface roughness |
| Brake pressure ($p$) | $-72\%$ or $+346\%$ | Braking functional requirement |
| Sprag angle ($\theta$) | $-98\%$ or $+158\%$ | Structural geometry/dynamic parameter |
| Friction parameters | | |
| Critical sliding ratio $\zeta_c$ | Not possible | – |
| Sliding slope ($k_{l2}$) | $-789,700\%$ | Pad material/friction modifier |
| Brake pad angle of attack ($\theta_A$) | $-95\%$ | Structural geometry/pad end condition |
| Nominal sliding velocity (for angle of attack, $\theta_A$ constant) ($V$) | $-99\%$ | Braking functional requirement (difficult) |
| Normal loading ($N$) | $+6100\%$ | – |
Brake squeal chaos can be suppressed by small changes in modal mass or stiffness (≥ ± 31%) by greatly reducing the squeal growth rate due to stiffness mode coupling. These results are similar to means of avoiding brake squeal entirely as found in [14].

Very large changes in modal damping are needed (≥ +73x) to override the negative damping due to the very high complex stiffness from stiffness mode coupling.

Large changes in contact stiffness are required (≥ − 77%) to eliminate chaos as it has a less direct effect of detuning the modes to lessen stiffness mode coupling.

Brake pressure has a stronger effect than contact stiffness as it affects this and the sprag modal properties. As the brake pressure necessarily changes as the brakes are applied over a full range, it is difficult to avoid chaos by controlling this parameter.

Brake squeal can be eliminated by avoiding the sprag condition for mode coupling [14] by large increases (≥ +158%) or decreases (< − 98%) in the sprag angle, θ. The sprag angle is determined by the mode shapes and brake structural geometry.

Brake squeal chaos suppression via friction parameter control is generally harder to achieve according to:

- Brake squeal chaos cannot be eliminated by controlling the critical sliding ratio (at maximum friction) because the falling friction mechanism is secondary in this case compared to stiffness mode coupling.

- An extremely large increase in magnitude (≥ 7898x) and change in sign of the sliding slope is required for positive friction to dampen the negative damping from stiffness mode coupling. In this case, this change is not realistic due to the very high complex stiffness.

- Chaotic instability can be avoided by large decreases in sliding velocity, V, or brake pad angle of attack, θ_A, to reduce squeal amplitudes by moving the initial sliding oscillations closer to the positive damped stick region [36].

- Very large increases (≥ 62x) in normal loading are required to avoid brake squeal chaos, in this case, although normal loading may affect the stability characteristics in a way not modelled by the criteria.

These results show the conservative analytical criteria can very efficiently evaluate and quantify several brake squeal chaos suppression techniques, without the need for lengthy and extensive multiple numerical integrations of the nonlinear equations of motion. Note that Table 2 summarises the results for directly eliminating chaotic instability, not necessarily the brake squeal limit cycles which can still occur under positive sliding conditions. By inspection of Fig. 9, it may be deduced that suppressing chaotic squeal to limit cycles can substantially reduce vibration velocity amplitudes by the order of 6 times leading to large reductions in sound pressure levels of approximately 15 dB, in that case (using squeal sound radiation model in [14]). The frequency content of the squeal limit cycle will be necessarily more tonal than the squeal chaos (as found in [34]).

4 Conclusion

Brake squeal chaos, under “falling friction”, mode coupling and spragging mechanisms, has been efficiently predicted and suppressed based on a two-degree-of-freedom coupled model. Numerical solutions of the full nonlinear equations of motion show brake squeal chaotic instability was characterised by a period doubling route in a fundamental brake squeal model as the static friction coefficient was increased. A similar analysis for a real brake system model also shows a less clear bifurcation process (via maybe crises) to chaotic instability as the modal damping is decreased. Mode coupling instability is shown to provide the necessary (phase space) expansion (or positive Lyapunov Exponent) for chaos via friction which first causes limit cycle behaviour via a super-critical Hopf bifurcation. This limit cycle is shown to break up into chaotic motion characterised by a Poincare map with an approximate one-dimensional attractor, similar to that found in a forced dry friction oscillator [24]. For the first time, conservative analytical criteria for brake squeal chaos are derived and numerically verified over a range of static friction coefficients, structural damping, sprag angles and brake pressures for both the fundamental and real brake models. The predictive criteria are based on high
local phase space expansion due to the occurrence of stiffness mode coupling and very large squeal instability causing negative sliding nonlinearities. The analytical criteria are found to conservatively predict the critical friction levels over a range of sprag angles and brake pressures to within 25% error and are noted to be calculated almost instantaneously. Interestingly a fundamental model case of chaotic instability without negative sliding squeal oscillations is identified under mode coupling. This case is also conservatively predicted by the first criterion but further research is required to understand and predict its occurrence more accurately. Further insight may be gained from the forced bilinear model of [43] which showed chaos when one slope tends to infinity (like the stick slope of the friction curve in the present model). Despite this, the results provide important predictive insight into conditions under which brake squeal chaos occurs and its suppression. In particular, the efficient criteria are used to perform a parametric control investigation to suppress brake squeal chaos by increasing the critical static friction level at which chaos occurs to above 1. The results show small changes in modal mass or stiffness ($>\pm 31\%$) can suppress chaos by greatly reducing the squeal growth rate due to stiffness mode coupling (due to closeness of the uncoupled modal frequencies). The brake pressure and sprag angle are shown to be important control parameters for avoiding chaos in brake squeal. Physically the sprag angle could be controlled via the structural dynamic parameters and the geometry of the brake system. Very large increases in modal damping ($> + 73x$) are needed to override the negative damping from the local phase space expansion due to stiffness mode coupling. The falling friction mechanism is shown to be secondary in this case (by approximately 7900x) compared to stiffness mode coupling. The results could be verified experimentally but provide predictive insight into conditions under which brake squeal chaos occurs and its suppression. It is hoped that the analytical results for this generalised model may provide predictive insight into other types of friction-induced chaotic instability.

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**Declarations**

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