DISCRETE ANALOGUES IN HARMONIC ANALYSIS: DIRECTIONAL MAXIMAL FUNCTIONS IN $\mathbb{Z}^2$

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Abstract. Let $V = \{v_1, \ldots, v_N\}$ be a collection of $N$ vectors that live near a discrete sphere. We consider discrete directional maximal functions on $\mathbb{Z}^2$ where the set of directions lies in $V$, given by

$$\sup_{v \in V, k \geq C \log N} \left| \sum_{n \in \mathbb{Z}} f(x - v \cdot n) \cdot \phi_k(n) \right|, \quad f : \mathbb{Z}^2 \to \mathbb{C},$$

where and $\phi_k(t) := 2^{-k} \phi(2^{-k} t)$ for some bump function $\phi$. Interestingly, the study of these operators leads one to consider an “arithmetic version” of a Kakeya-type problem in the plane, which we approach using a combination of geometric and number-theoretic methods. Motivated by the Furstenberg problem from geometric measure theory, we also consider a discrete directional maximal operator along polynomial orbits,

$$\sup_{v \in V} \left| \sum_{n \in \mathbb{Z}} f(x - v \cdot P(n)) \cdot \phi_k(n) \right|, \quad P \in \mathbb{Z}[\cdot]$$

for $k \geq C_d \log N$ sufficiently large.

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1. Introduction

Discrete analogues of (continuous) operators in harmonic analysis has been an active area of research since Bourgain initiated their study in the course of his work on pointwise ergodic theorems in the late 80s and early 90s, [2–4]. In particular, motivated by pointwise ergodic theoretic concerns (cf. Calderón’s transference principle
the initial focus of discrete harmonic analysis was in understanding maximal averaging operators along polynomial orbits, say

\[
\sup_k \left| \frac{1}{2^k} \sum_{n \leq 2^k} f(x - n^2) \right|, \ f : \mathbb{Z} \to \mathbb{C}.
\]

Although the analogous Euclidean operator,

\[
\sup_k \left| \frac{1}{2^k} \int_0^{2^k} f(x - t^2) \, dt \right|, \ f : \mathbb{R} \to \mathbb{C}
\]

is governed by the (continuous) Hardy-Littlewood maximal operator by a change of variables, there is no such connection between (1.1) and the discrete Hardy-Littlewood maximal function: averaging over the set of the squares

\[\{n^2 : n \geq 1\}\]

is more akin – from a density perspective – to a continuous averaging operator over a set of lower dimension. Although developing an appropriate \(\ell^p\) theory for these averages, or for the related singular integral formulation, required delicate analysis, necessitated by arithmetic complications unique to the discrete setting, the theory of polynomial radon transforms is essentially complete: aside from Bourgain’s work, major contributions to the field were made by Magyar, Stein, and Wainger [19], by Ionescu and Wainger [10], and more recently by Mirek, Stein, and Trojan [21, 22], and by Mirek, Stein, and Zorin-Kranich [23].

In light of the efforts of the above authors and others, the field has developed sufficiently robust tools that discrete analogues of more complicated questions can be meaningfully posed: in [18] and [17], the second author, partially in collaboration with Lacey, initiated a study into discrete analogues of maximally modulated oscillatory singular integrals, of the type considered by Stein [25] and Stein-Wainger [26]; in [15], [14], and [16], Kesler-Lacey, Kesler, and Kesler-Lacey-Menas establish \(\ell^p\)-improving (and sparse) estimates for spherical averages and maximal functions are established, in analogy with the work of Schlag [24].

The purpose of this paper is to begin an investigation into discrete analogues of directional maximal functions in the plane; we provide a brief summary of the \((L^2)\) planar theory, and refer the reader to the recent preprint of Di Plinio and Parissis [8] for a more comprehensive discussion.

The initial interest interest in the directional maximal function in the plane,

\[M_V f(x) := \sup_{v \in V \subset \mathbb{S}^1} \left| \int_0^1 f(x - t \cdot v) \, dt \right|, \ f : \mathbb{R}^2 \to \mathbb{C},\]

is due to its connection with the Kakeya maximal function over \(1/N \times 1\) tubes when \(|V| = N\); in particular, when \(V\) is uniformly distributed, the two maximal functions
both have operator norm on $L^2$ given by a universal constant times $\log^{1/2} N$, due to [6] (sharpness follows from the existence of Kakeya sets in the plane). For general sets of directions, $V \subset S^1$, the bound of $\log^{1/2} N$ was proven by Katz in [11], and re-proven by Demeter in [7]; we begin our study of the discrete directional maximal functions with the aim of understanding the (single-scale) maximal average in the plane,

$$A_{V,\phi_k} f(x) := \sup_{v \in V} \left| \sum_{n \in \mathbb{Z}} f(x - v \cdot n) \cdot \phi_k(n) \right|$$

where $\phi$ is some bump function (say) on the real line, $\phi_k(x) := 2^{-k} \phi(2^{-k} x)$ are the usual $L^1$-normalized dilations, and

$$V \subset \{ |x| \approx A, x \in \mathbb{Z}^2 \},$$

is a collection of $|V| = N$ vectors.\(^1\) (Because our estimates will be uniform over appropriately normalized bump functions, $\phi$, for notational ease we will often suppress the dependence of our maximal operators on these functions, writing, for instance, $A_{V,k} = A_{V,\phi_k}$.) In particular, we are after the operator norm of (1.2). Upon first consideration, this question is not very interesting: by considering bumps adapted to unit scales, we see that a sharp estimate is given by $N^{1/2}$. However, unlike the Euclidean setting, this problem is not scale-invariant, which leads to the natural question:

**Problem 1.3.** Can one find a scale $k = k(|V|)$ beyond which

$$\|A_{V,k}\|_{\ell^2(\mathbb{Z}^2) \to \ell^2(\mathbb{Z}^2)}$$

is essentially independent of $|V| = N$?

Although the continuous analogue of this problem could be (essentially) answered via a $TT^*$ argument, see [1], neither this approach, nor the more delicate of analysis of Katz, is appropriate in the discrete setting, where the geometry of the problem becomes radically different; for instance, in general, non-parallel lines

$$\{m + nv : n \in \mathbb{Z}\}, \{m' + nv' : n \in \mathbb{Z}\} \subset \mathbb{Z}^2, \ v \neq v'$$

will not intersect. Passing to Fourier space connects these geometric issues with arithmetic ones, and our answer to Problem 1.3 requires an interplay between these two perspectives: although a universal answer to Problem 1.3 seems out of reach of current techniques, we are able to construct sets of vectors for which we can answer this question quite thoroughly. In particular, we will show that for each $\epsilon > 0$ and $N \geq 1$, there exist collections of vectors $V = V_{N,\epsilon}$ so that $V$ lives near a discrete

\(^1\)In stricter analogy with the Euclidean case, one might expect to be able to select vectors on a (discrete) circle; the irregular distribution of lattice points introduces significant technical complications.
sphere\(^2\), has \(|V| = N\), and so that \(A_{V,k}\) has \(\ell^2(\mathbb{Z}^2)\) operator norm bounded by a constant multiple of \(|V| = N\):

**Theorem 1.4.** For any \(\epsilon > 0\), there exists an absolute constant \(C_\epsilon\) so that the following holds:

For \(V = V_{N,\epsilon}\) as above, we have the estimate

\[
\|A_{V,k}\|_{\ell^2(\mathbb{Z}^2) \to \ell^2(\mathbb{Z}^2)} \leq C_\epsilon \cdot N^\epsilon.
\]

provided that \(2^k \geq N^{C_0}\) for some absolute \(C_0\).

In fact, this construction is sufficiently robust to allow for a supremum over scales to be introduced. To this end, define

\[
A^*_V f(x) := \sup_{2^k \geq N^{C_0}, v \in V} \sum_{n} f(x - n \cdot v) \cdot \phi_k(n)
\]

**Corollary 1.6.** In the setting of Theorem 1.4, (1.5) is satisfied by \(A^*_V\) as well.

1.1. **Discrete Maximal Functions along Sparse Sequences.** We will also study discrete analogues where the geometry of the corresponding Euclidean problem in the plane is much less understood. While the Kakeya conjecture in the plane is fully resolved, very little progress has been made toward understanding the related Furstenberg conjecture in the plane.

Recall that a **Furstenberg set with parameter** \(\beta\) in the plane is a compact set \(K \subset \mathbb{R}^2\) such that for every direction \(w \in S^1\), there exists a line segment, \(l_w\), pointed in direction \(w\), so that

\[
\dim_H(K \cap l_w) \geq \beta
\]

for all \(w \in S^1\); here \(\dim_H\) refers to Hausdorff dimension. More generally, one may consider two-parameter \((\alpha, \beta)\) Furstenberg sets to be subsets \(K\) of the plane so that (1.7) holds for \(w \in V \subset S^1\), for some subset \(V \subset S^1\) of Hausdorff dimension \(\geq \alpha\). Thus, in analogy to the relationship between the Kakeya-Nikodym maximal operator and Kakeya sets, one would be motivated to introduce a Furstenberg-type maximal operator

\[
M^\beta_{V}(f)(x) = \sup_{v \in V} |\mu_v \ast f(x)|,
\]

where \(V\) is a subset of \(S^1\) of Hausdorff measure \(\alpha\) and \(\mu_v\) is the Hausdorff measure of a compact \(\beta\)-dimensional subset of the real line, embedded into the plane and rotated to be parallel with \(v\). The study of Furstenberg sets has a rich history in

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\(^2\)As the argument will reveal, for any \(\delta > 0\) one may find \(A\) sufficiently large with appropriate \(V \subset \{(1 - \delta) \cdot A \leq |x| \leq (1 + \delta) \cdot A : x \in \mathbb{Z}^2\}\)
geometric measure theory, and is deeply connected to the Falconer and sum-product conjectures, as explored, for instance, in [13].

In the discrete setting, we consider the maximal function,

$$A_{V,k}^2 f(x) := \sup_{v \in V} \left| \sum_n f(x - v \cdot n^2) \cdot \phi_k(n) \right|.$$ 

By density considerations, restricting integer sequences to squares may be viewed as the introduction of Radon-type behavior. Here, the counterpart to

$$\sum_n e^{-2\pi i n (v \cdot \beta)} \cdot \phi_k(n)^{\|v\|} = \sum_\substack{n \leq C \epsilon^{2k}, (a,q) = 1} 1 \cdot \|v - a/q\| \leq 2^{-2k},$$

(see §4.1 below for a precise statement); in particular, multi-frequency considerations are paramount.

Nevertheless, provided our sets of directions are given by $V_{N,\epsilon}$, we are able to bound this maximal function essentially independently of $|V|$. 

**Theorem 1.8.** For any $\epsilon > 0$, there exists an absolute constant $C_\epsilon$ so that the following holds:

For any $N$, there exists a set of vectors $|V| = |V_{N,\epsilon}| = N$ as above so that, if $k \geq C_2 \log N$ for some sufficiently large constant $C_2$,

$$\|A_{V,k}^2 \|_{\ell^2(\mathbb{Z}^2) \to \ell^2(\mathbb{Z}^2)} \leq C_\epsilon \cdot N^\epsilon.$$ 

In fact, one may replace the monomial sequence $\{n^2 : n \geq 1\}$ with the image of any polynomial with integer coefficients: with

$$A_{V,k}^P f(x) := \sup_{v \in V} \left| \sum_n f(x - v \cdot P(n)) \cdot \phi_k(n) \right|$$

we have the following theorem.

**Theorem 1.9.** For any $\epsilon > 0$, there exists an absolute constant $C_\epsilon$ so that the following holds:

For any $N$, there exists a set of vectors $|V| = |V_{N,\epsilon}| = N$ as above so that, if $k \geq C_d \log N$ for some sufficiently large constant $C_d$, and any polynomial $P$ of degree $d$,

$$\|A_{V,k}^P \|_{\ell^2(\mathbb{Z}^2) \to \ell^2(\mathbb{Z}^2)} \leq C_\epsilon \cdot N^\epsilon.$$
The key issue is the distribution of the sequence

\[ \{ P(n) \mod q : n \} \]

each integer \( q \geq 1 \); by a classical result of Hua, this uniform distribution can be quite precisely quantified in terms of decay of various exponential sums, cf. [9, §7, Theorem 10.1], and we are able to conclude our result. With this heuristic in mind, that we are able to conclude analogous results for (polynomial images, or even “thin” images [20], of) the primes, or for randomly generated sequences of prescribed density of the type introduced by Bourgain in [2, §8], should be somewhat expected.

Finally, we remark that in the foregoing discussion, all results are remain valid upon replacing our bump functions, \( \phi \), with appropriately dilated (one-dimensional) Calderón-Zygmund kernels, \( K \), see [7] for the continuous theory. As motivation for this perspective, we note that the discrete analogue of Carleson’s theorem \((d = 1)\), and the main result of [17] \((d \geq 2)\) exhibit the \( \ell^2 \) boundedness of

\[ \sum_m f(x - m, y - v(x) \cdot m^d) \]

for any \( v : \mathbb{Z} \to \mathbb{Z} \); this can be seen by taking a partial Fourier transform in the \( y \) variable. The content of our current theorem is the following.

**Theorem 1.10.** For any \( \epsilon > 0 \) and any polynomial of degree \( d \), there exists an absolute constant \( C_{\epsilon,d} \) so that one may find \( v = (v_1, v_2) : \mathbb{Z}^2 \to V \) with \( |V| = N \) arbitrary large, so that

\[ \| \sum_m f(x - v_1(x, y) \cdot m, y - v_2(x, y) \cdot P(m)) \|_{\ell^2(\mathbb{Z}^2)} \leq C_{\epsilon,d} \cdot N^\epsilon \cdot \| f \|_{\ell^2(\mathbb{Z}^2)}. \]

**Remark 1.11.** An inspection of the argument allows for \( v_1 \equiv 1 \) for certain choices of \( V \).

The structure of the paper is as follows:

In §2 we prove our multi-frequency incidence estimates;
In §3, we apply these incidence estimates to quickly prove Theorem 1.4 (and Corollary 1.6);
In §4 we combine the above analysis with further number-theoretic techniques from discrete harmonic analysis to prove Theorem 1.8 (and Theorem 1.9).

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1.3. **Notation.** Here and throughout, \( e(t) := e^{2\pi it} \) if \( x \equiv y \) will denote equivalence mod 1. Throughout, \( C \) will be a large number which may change from line to line.

For finitely supported functions on \( \mathbb{Z}^d \), \( d = 1, 2, \) we define the Fourier transform

\[
\mathcal{F}_{\mathbb{Z}} f(\beta) := \hat{f}(\beta) := \sum_n f(n) e(-\beta n),
\]

with inverse

\[
\mathcal{F}_{\mathbb{Z}}^{-1} g(n) := g^\vee(n) := \int g(\beta) e(\beta n) \, d\beta,
\]

where integration occurs over \( \mathbb{T} \) or \( \mathbb{T}^2 \) depending on dimension.

We will let \( \chi \) be an even non-negative compactly supported Schwartz function which approximates the indicator function of an interval centered at the origin.

\[
|\xi| \leq c \leq \chi \leq 1 \leq |\xi| \leq 2c.
\]

We will use

\[
\|f\|_{\ell^p} := \left( \sum_{x \in \mathbb{Z}^2} |f(x)|^p \right)^{1/p}
\]

with the obvious modification at \( p = \infty \). When we sum over \( \mathbb{Z} \) we will specify the domain explicitly.

We will make use of the modified Vinogradov notation. We use \( X \lesssim Y \), or \( Y \gtrsim X \), to denote the estimate \( X \leq CY \) for an absolute constant \( C \). We use \( X \approx Y \) as shorthand for \( Y \lesssim X \lesssim Y \). We also make use of big-O notation: we let \( O(Y) \) denote a quantity that is \( \lesssim Y \). If we need \( C \) to depend on a parameter, we shall indicate this by subscripts, thus for instance \( X \lesssim_p Y \) denotes the estimate \( X \leq C_p Y \) for some \( C_p \) depending on \( p \). We analogously define \( O_p(Y) \).

2. **A Multi-Frequency Incidence Problem**

Before turning to the main result of this section, we briefly pause to explain the nature of our multi-frequency incidence problem.

The departure point is that an upper bound for the “arithmetic” directional maximal function

\[
\|A_{V,k}\|_{\ell^2(\mathbb{Z}^2) \to \ell^2(\mathbb{Z}^2)}
\]

is – up to standard transference arguments – essentially given by

\[
(2.1) \quad \sup_{|\beta| \geq A^{-1} 2^{-k}} \left( \sum_{v \in V} \sum_{m \in \mathbb{Z}} 1_{|v, \beta - m| \leq 2^{-k} \beta} \right)^{1/2};
\]

here \( V \subset \{|x| \approx A, \ x \in \mathbb{Z}^2\} \), and \( 2^k \geq N^{C_0} \) is sufficiently large.
Note that the quantity (2.1) is the $L^\infty$ norm outside of a ball of radius $2^{-k}A^{-1}$ of a sum of characteristic functions of tubes of thickness $\ll 2^{-k}A^{-1}$ that are perpendicular to $v$ for some $v \in V$, where the tubes pointing in a given direction are spaced apart in an arithmetic progression with spacing $\approx A^{-1}$. Interestingly, a heuristic that emerges both in this linear setting, and then again later in the polynomial setting, where multi-frequency complications arise, is that those collections of vectors that are amenable to obtaining $\ell^2(\mathbb{Z}^2)$ bounds are those whose (normalized) coordinates exhibit an intermediate amount of arithmetic independence in terms of prime factorization. With these remarks in mind, we turn to our incidence problem.

In this section, $s \geq 1$ will be an integer, and $C_0$ will be a parameter which will assumed to be sufficiently large.

For $2^{s-1} \leq r < 2^s$, and $v \in V$, consider the union of tubes,

\[ K_{r,v} := \{ \beta \in \mathbb{T}^2, \ |eta| \gtrsim A^{-1}2^{-C_0 s} : |v \cdot \beta - m - b/r| \lesssim 2^{-C_0 s} \text{ for some } b \leq r, m \in \mathbb{Z} \}. \]

We will be interested in estimating

\[ C(s, V) := \left\| \sum_{2^{s-1} \leq r < 2^s, \ v \in V} 1_{K_{r,v}} \right\|_{L^\infty(\mathbb{T}^2)}. \]

By scaling considerations, with

\[ K^A(r, v) := \left\{ \beta \in \mathbb{T}^2 : |\beta| \gtrsim A^{-1}2^{-C_0 s}, |v \cdot \beta - m - b/rA| \lesssim A^{-1}2^{-C_0 s} \text{ for some } b \leq r, m \in \mathbb{Z} \right\}, \]

it suffices to estimate

\[ \left\| \sum_{2^{s-1} \leq r < 2^s, \ v \in A^{-1}V} 1_{K^A_{r,v}} \right\|_{L^\infty(\mathbb{T}^2)}. \]

Our main result in this section is the following proposition.

**Proposition 2.3.** Let $\epsilon > 0$ be arbitrary. Then, for each $N$, there exist collections of vectors $|V| = N$ so that for each $N^\epsilon \leq 2^s \leq N^\frac{7}{4}$, one may take

\[ C(s, V) \lesssim \epsilon N^\epsilon. \]

In particular, one may estimate

\[ C(s, V) \lesssim \epsilon \min\{2^{2s}, N, N^\epsilon, 1_{s=\log N}\}. \]

We may relate the quantity $C(s, V)$ to mixed norm estimates for the $X$-ray estimate in the plane. Such estimates are equivalent to certain $L^p$ norm estimates for sums.
of characteristic functions of tubes, and the endpoint $L^2$ X-ray estimate in the plane says the following.

**Proposition 2.4** *(Endpoint X-ray estimate in the plane)*. Let $1 \gg \delta > 0$ be arbitrary, and let $T := \{T\}$ be a set of $\delta$–tubes in $\mathbb{R}^2$ so that no tube is contained in the two-fold dilate of any other tube. Suppose furthermore that for each direction $v \in S^1$, at most $m$ tubes point in a direction that is $\delta$–close to $v$. Then for every $\epsilon > 0$,

$$\left\| \sum_{T \in T} 1_T \right\|_2 \lesssim \delta^{-\epsilon} \cdot m^{1/2} \cdot (\delta |T|)^{1/2}.$$  

This is similar to a dualized Kakeya maximal estimate, except the X-ray estimate involves multiple tubes in the same direction.

An estimate for $C(s,V)$ may thus be viewed as an $L^\infty$ arithmetic X-ray estimate in prescribed directions, since it involves sums of characteristic functions of tubes lying in a restricted set of directions such that centers of parallel tubes lie in arithmetic progression.

Before turning to the details, we describe the objective. The key object of consideration will be the intersections of the (thin) combs

$$I^A_{r,v} := \left\{ A^{-1}2^{-Cs} \lesssim |\beta| \lesssim 1 : v \cdot \beta = \frac{m}{A} + \frac{b}{rA} \text{ for some } b \leq r, m \in \mathbb{Z} \right\}, \ |v| \approx 1, \ Av \in \mathbb{Z}^2.$$  

The idea is that points living in – or near – many $I^A_{r,v}$ must have a great deal of arithmetic structure relative to each $v$; by choosing $\{v \in V\}$ to have a certain degree of arithmetic independence in terms of prime factorization, we can force $\beta$ living in – or near – the intersection

$$\bigcap_{2^{s-1} \leq r < 2^s, \ v \in S} I^A_{r,v}$$

to have an unacceptably large number of prime factors in its denominator, whenever $|S| \gtrsim |V|^\epsilon$.

Before turning to the proof, we pause to introduce some notation:

- Set throughout $\epsilon_0 := \epsilon^{A_0}$ for some large but absolute $A_0$;
- Each set of integers considered, which will be denoted $I$ or $J$ up to subscripts, will have length $\epsilon_0^{-1} \in \mathbb{N}$;
- Each integer involving subscripts of $q$ will be bounded above by $N^{C_0/3}$, where $C_0$ will be assumed to be sufficiently large, but independent of $N$;
- For each $I$, suppose that we are given a collection of real coefficients, $\{\alpha_j(I)\}$, a set of integers, $\{r_j(I)\}$, and a set of primes, $\{p_j(I)\}$; we will set $q_I$ to be the (largest) integer $q \leq N^{C_0/3}$ furnished by Dirichlet’s approximation theorem.
(Lemma 2.8 below), so that
\[ |\alpha_j(I) - a_j| \leq \frac{1}{q \cdot N^{\epsilon/2}} \] for each \( j \);
we also set \( R_I := \prod_j r_j(I) \) and similarly \( P_I := \prod_j p_j(I) \).

We now turn to the proof.

2.1. The Proof of Proposition 2.3. Recall our set-up: for some fixed \( A \), we have
\[ V \subset \{ x \in \mathbb{Z}^2 : |x| \approx A \} \]
we will work with \( V_A := \frac{1}{A} \cdot V \). As we travel down the argument, we will impose
further restrictions on our set. In particular, we will assume that \( A \) is sufficiently
composite that we may prescribe rational \( x \)-coordinates for \( v \in \frac{1}{A} \cdot V \); by scaling
considerations, we may in fact assume that \( A = 1 \), and that we are free to work with
rational vectors.

It suffices to choose a collection of vectors \( V = \{ v_1, v_2, \ldots, v_N \} \) with \( |V| = N \) so
that for any subcollection \( S \subset V \) with \( |S| = N^{\epsilon/2} \) and any choice of integers \( \{ r_i : 1 \leq i \leq N^{\epsilon/2} \} \), no point of \( \mathbb{R}^2 \) is simultaneously contained in
\[ \bigcap_{r_v: v \in S} K_{r_v, v}. \]

The Construction. Choose a large integer \( M \gg 1 \), to be determined later, and set \( P_N \)
to be the set of primes which do not divide \( N \), and collect the smallest \( N^{\epsilon/2} \) primes in
\[ P_N \cap [N^{\epsilon/2}, N^{M/\epsilon}]; \]
call this set \( P_M \). By Stirling’s approximation it is possible to define an integer
\[ \kappa \approx \epsilon^{-1} \]
so that
\[ \binom{N^{\epsilon/2}}{\kappa} \approx N. \]
Now choose our set \( V = \{ v_1, \ldots, v_N \} \) so that each \( |v_i| = 1 \) and the \( x \)-coordinate of
each \( v_i^\perp \), or the \( y \) coordinate of \( v_i \), which we denote \( (v_i)_y = (v_i^\perp)_x \), is of the form
\[ (v_i)_y = Q_i \cdot N^{-2M\kappa/\epsilon - 2\kappa} \cdot 2^{-10\kappa} \cdot p_{i_1} p_{i_2} \cdots p_{i_\kappa}, \]
where \( \{ p_{i_j} : 1 \leq j \leq \kappa \} \subset P_M \) are distinct, and so that no two \( v_i \) have the exact same
collection of corresponding primes \( \{ p_{i_1}, p_{i_2}, \ldots, p_{i_\kappa} \} \). Here, \( Q_i \) is a positive integer,
not divisible by any \( p_i \), so that \( (v_i)_y \) is as close to \( i/N \) as possible. (Note that since
\( N^{-2M\kappa/\epsilon - 2\kappa} \cdot 2^{-10\kappa} \cdot p_{i_1} \cdots p_{i_\kappa} \ll N^{-1} \), this forces the angle between any two distinct
\( v_i \) to be at least \( 1/(2N) \).)
**A Geometric Reduction.** As we observed that the difference in angle between any two distinct $v_i$’s is at least $\geq 1/2N$, so taking $C_0$ sufficiently large ($\geq 1$), the intersection of any two combs $K_{r_1,v_1} \cap K_{r_2,v_2}$ belongs to a small (say, $N^{-C_0/2}$) neighborhood of a grid, where the $x$-coordinates of all points in the grid belong to the rank 2 arithmetic progression

\[
(2.5) \quad \left\{ \frac{a}{r_2(v_1 \cdot v_2)} \cdot (v_1)_y + \frac{b}{r_1(v_1 \cdot v_2)} \cdot (v_2)_y : a, b \in \mathbb{Z} \right\}.
\]

Indeed, to see this, note that all points on the grid belong to a lattice with generators parallel to $v_1^\perp$ and $v_2^\perp$, which is a set of the form

\[
(c_1v_1^\perp) \cdot \mathbb{Z} + (c_2v_2^\perp) \cdot \mathbb{Z}
\]

for some constants $c_1, c_2$; by algebra, $(c_1, c_2)$ may be taken to be $(\frac{1}{r_2}, \frac{1}{r_1})$. We thus obtain that $(K_{r_1,v_1} \cap K_{r_2,v_2})_x$ is contained in a neighborhood of (2.5).

$N^\epsilon$-subcollections of $V$ have subsets that exhibit a certain degree of arithmetic independence. Suppose now that $S \subset V$ has $|S| = N^\epsilon$, and choose an integer $K$ so that

\[
\binom{K}{\kappa} \approx N^{\epsilon/2}.
\]

Then by Stirling’s approximation,

\[
K \approx \epsilon N^{\epsilon/2}.
\]

The following claim then follows from a counting argument.

**Claim 2.6.** For any choice of $S$ with $|S| = N^\epsilon$, one may choose at least $K$ different primes $\{p_{i_1}, p_{i_2}, \ldots, p_{i_K}\} \subset P_M$ so that we can choose disjoint sub-collections $\{S_1, \ldots, S_K\}$ of $S$, each of cardinality 2, so that if $S_j = \{v_j, w_j\}$, both

\[
(2.7) \quad (v_j)_y \cdot N^{M_k/\delta + \kappa} \cdot 2^{10\kappa}, \quad (w_j)_y \cdot N^{M_k/\delta + \kappa} \cdot 2^{10\kappa}
\]

are divisible by $p_{i_j}$, but $p_{i_l} \in P_m$ does not divide both factors in (2.7) for any $l \neq j$.

The proof of this claim is essentially a counting argument. In particular, we inductively choose $S_j$ as follows. If $j \leq \kappa$, take $S_j$ to be any two elements not contained in $\bigcup_{j' < j} S_{j'}$ such that the two elements are divisible by some single $p_{i_j} \notin \{p_{i_1}, \ldots, p_{i_{j-1}}\}$. If $j > \kappa'$, it is easy to see that this is possible, since the number of elements $v$ of $V$ (and hence $S$) satisfying that $(v)_y$ is divisible only by $p_{i_1}, \ldots, p_{i_{j-1}}$ is $\ll N^\epsilon$. Thus there remain that are divisible by some prime $p_i$ with $i \neq i_1, \ldots, i_{j-1}$, and since there are $N^{\epsilon/2}$ many primes to choose from and $\approx N^\epsilon$ different elements of $S$ that have not yet been chosen, by pigeonholing there must be some $p_i =: p_{i_j}$ for which at least two remaining elements of $S$ are divisible by $p_i$. This completes the proof of the claim.
Suppose that \( S \subset V \) with \( |S| = N^\epsilon \) is fixed. Recall that \( K \approx N^{\epsilon^2/2} \). Suppose that we have, as in the hypotheses of the previous claim, a collection of \( K \) many distinct primes \( \{p_{i_1}, p_{i_2}, \ldots, p_{i_K}\} \subset P_M \) and disjoint subcollections \( \{S_1, \ldots, S_K\} \) where each \( S_j = \{v_j, w_j\} \) has cardinality 2, and the \( S_j \) satisfy the conditions of the previous claim. Now we will use Dirichlet’s simultaneous approximation theorem, stated below.

**Lemma 2.8** (Dirichlet’s simultaneous approximation theorem). *Given real numbers \( \alpha_1, \ldots, \alpha_d \) and a natural number \( L \), there are integers \( r_1, \ldots, r_d \) and an integer \( q \in [1, L] \) such that \( |\alpha_i - \frac{r_i}{q}| \leq \frac{1}{qL^{1/d}} \).*

The above lemma, which we will apply with \( L = N^{C_0/3} \), is a classical result in Diophantine approximation theory and can be proven by an appropriate application of the pigeonhole principle. Noting that if we examine the expression (2.5) for vectors \( v, w \in S_i \) for some fixed \( i \), then the only irrational numbers appearing in the expression are the occurrences of \( (v^\perp \cdot w)^{-1} \). In particular, we will have \( \alpha_j(I) := (v_j^\perp \cdot w_j)^{-1} \) for \( \{v_j, w_j\} = S_j, j \in I \).

We would ideally like to apply Dirichlet’s theorem to the entire collection of real numbers

\[
\{(v^\perp \cdot w)^{-1}, v, w \in S_i \text{ for some } i\}.
\]

However, this would force us to take a very large value of \( d \) in Dirichlet’s theorem, which would be inadequate for our purposes. Instead, we opt to iteratively apply Dirichlet’s theorem to smaller subcollections of the set (2.9), each of size \( \epsilon_0^{-1} \).

An iterative setup. As a first step, we would first like to organize the set \( \{1, 2, \ldots, K\} \) into disjoint subcollections \( \{I_j : j\} \). We choose disjoint subsets \( \{I_j^1 : 1 \leq j \leq N_1\} \) as follows. Let \( I_1^1 \) be a subset of \( \{1, \ldots, K\} \) such that \( q_{I_1^1} \) is maximal, when the maximum is taken over all subcollections of (2.9) of size \( \epsilon_0^{-1} \). Inductively choose

\[
I_n^1 \subset \{1, \ldots, K\} \setminus \bigcup_{j<n} I_j^1
\]

such that \( q_{I_n^1} \) is maximal. Stop at \( n = N_1 \) when it is no longer possible to choose \( I_{N_1}^1 \). We now pigeon-hole to extract a subcollection,

\[
\{J_j^1 : 1 \leq j \leq M_1\}, \quad M_1 \gtrsim \frac{N_1}{\log N},
\]

so that \( q_{J_j^1} \) all lie in a single dyadic interval. Note that since \( K \approx N^{\epsilon^2/2} \), for \( N \) sufficiently large we have \( M_1 \gtrsim \frac{N^{\epsilon^2/2}}{\log N} \).

Now let us describe the inductive step:
Suppose that for $1 \leq l \leq k$ we have chosen disjoint subsets $\{J^k_j : 1 \leq j \leq M_k\}$ of $\{1, \ldots, K\}$. Then for each $J^k_j$ choose a representative $i^k_j \in J^k_j$. Organize the set of indices $\{i^k_j : i^k_j \in J^k_j \text{ for some } j\}$ into disjoint subsets $\{I^k_{n+1} : 1 \leq j \leq N_{k+1}\}$ as follows. Let $I^k_{n+1}$ be a subset of $\{i^k_j : i^k_j \in J^k_j \text{ for some } j\}$ such that the value of $q_{I^k_{n+1}}$ is maximal. Inductively choose $I^k_{n+1}$ to be a subset of

$$\{i^k_j : i^k_j \in J^k_j \text{ for some } j\} \setminus \bigcup_{j<n} I^k_{n+1}$$

such that $q_{I^k_{n+1}}$ is maximal. Stop at $n = N_{k+1}$ when it is no longer possible to choose $I^k_{N_{k+1}+1}$. Now choose a subcollection

$$\{J^{k+1}_j : 1 \leq j \leq M_{k+1}\}, \quad M_{k+1} \gtrsim \frac{N_{k+1}}{\log N}$$

where each $J^{k+1}_j$ is a distinct element of $\{I^{k+1}_n\}_n$ such that each $q_{J^{k+1}_j}$ lies in a single dyadic interval. Note that since $K \approx N^{c^2/2}$, for $N$ sufficiently large we have $M_{k+1} \gg N^{c^2/4}$ for all $k \leq k_0$ for some large $k_0$ – independent of $N$.

To summarize, for $0 \leq k \leq k_0$ we have defined disjoint sets of indices

(2.10) \quad \{J^k_j : 1 \leq j \leq M_k, \ 1 \leq k \leq k_0\}

such that for each $1 \leq k \leq k_0$, $\{q_{J^k_j} : j\}$ belongs to a single dyadic interval and $M_k \gg N^{c^2/4}$. Moreover, we have further imposed a tree structure on the set (2.10): each $J^{k+1}_j$ contains no more than one element in each $J^k_l$ for any $l$. The final – and crucial – observation is that for any $j, l$, whenever $m \leq k$ we have

$$q_{J^k_j} \leq 2 \cdot q_{J^k_l}.$$

**Arithmetic independence implies few overlaps.** Now we iteratively apply Dirichlet’s approximation theorem as follows. First, for vectors $v$ and $w$ and integers $r, r'$, consider the set

$$E_{r, r', v, w} := \left\{ \frac{a}{r'(v \cdot w)} \cdot (v)_y + \frac{b}{r(v \cdot w)} \cdot (w)_y : a, b \in \mathbb{Z} \right\} \subset \mathbb{R}.$$ 

As argued previously, for each $i \in \{1, \ldots, K\}$ we have

$$\left( K_{r_{i, v_i}} \cap K_{r'_{i, w_i}} \right)_x \subset E_{r_{i, r'_{i}}, v_i, w_i} + O(N^{-C_0/2}).$$

We now begin with the sets of indices $\{J^l_l : 1 \leq l \leq M_{l}\}$ and apply Dirichlet’s theorem to obtain that for every $j \in J^1_l$;

$$q_{J^l_j} \cdot T \cdot R_{J^l_j} \left( K_{r_{j, v_j}} \cap K_{r'_{j, w_j}} \right)_x \subset p_{ij} Z + O \left( N^{-C_{\text{gap}}} \cdot T \cdot R_{J^l_j} \right).$$
where
\[
T := T(N, M, \kappa, \delta) := N^{2M\kappa/\delta - 2\kappa} \cdot 2^{10\kappa} = N^{o(C_0)}
\]
for \(C_0\) sufficiently large. In particular, we may estimate
\[
N^{-\frac{C_0\kappa}{4}} \cdot T \ll N^{-\frac{C_0\kappa}{4}} \ll 1,
\]
which forces a near “exact incidence condition” among the above sets: if
\[
y \in q_{J_1} \cdot T \cdot R_{J_1} \cdot \bigcap_{j \in J_1} \left( K_{r_j, v_j} \cap K_{r_j', w_j} \right)
\]
\[
\Rightarrow y \in P_{J_1} \cdot \mathbb{Z} + O \left( N^{-\frac{C_0\kappa}{4}} \cdot R_{J_1} \right),
\]
for \(J \in \{J^k\}\) we have set \(P_J := \prod_{j \in J} P_{j_i}\). Rescaling, we have that
\[
y \in \bigcap_{j \in J_1} \left( K_{r_j, v_j} \cap K_{r_j', w_j} \right)
\]
\[
\Rightarrow y \in \left( q_{J_1} \cdot T \cdot R_{J_1} \right)^{-1} \cdot P_{J_1} \cdot \mathbb{Z} + O \left( q_{J_1}^{-1} \cdot N^{-\frac{C_0\kappa}{4}} \right).
\]
Now we move to the second stage of the inductive process and consider all sets of the form \(J^2_m\). Now if \(j \in J^1_l \cap J^2_m\) for some \(m, l\), then
\[
(2.11) \quad \left| (v_j^+ \cdot w_j)^{-1} - \frac{p}{q_{J_1}} \right| \leq \frac{1}{q_{J_1} \cdot N^{C_0\epsilon_0/4}}, \text{ for some } p
\]
where \(q_{J_1}^2 = b\) for \(a/b\) the reduced rational obtained from applying Dirichlet’s theorem to \(J^1_m\) such that
\[
\left| (v_j^+ \cdot w_j)^{-1} - a/b \right| \leq \frac{1}{q_{J_1}^2 N^{1/10}},
\]
note how we used the inequality \(q_{J_1}^2 \leq 2q_{J_1}^1\) in (2.11).

Now, if we apply Dirichlet’s theorem to the set of indices \(J^2_m\), then a similar argument shows that for every \(j \in J^2_m\), if \(l\) is the (unique) integer such that \(j \in J^1_l\), then
\[
q_{J_1} \cdot T \cdot R_{J_1} \cdot \bigcap_{k \in J_1} \left( K_{r_k, v_k} \cap K_{r_k', w_k} \right) \subset P_{J_1} \cdot \mathbb{Z} + O \left( N^{-\frac{C_0\kappa}{4}} \cdot R_{J_1} \right).
\]
But this forces a near exact incidence condition among the above sets in the sense that we have 

\[ y \in q_{J_m^2} \cdot T \cdot \prod_{l: J_l^1 \cap J_m^2 \neq \emptyset} R_{J_l^1} \cdot \bigcap_{l: J_l^1 \cap J_m^2 \neq \emptyset} \left( \bigcap_{k \in J_l^1} \left( K_{r_k, v_k} \cap K_{r'_k, w_k} \right) \right) \]

\[ \implies y \in \left( \prod_{l: J_l^1 \cap J_m^2 \neq \emptyset} P_{J_l^1} \right) \cdot \mathbb{Z} + O \left( N^{-\frac{C\alpha}{4}} \cdot \prod_{l: J_l^1 \cap J_m^2 \neq \emptyset} R_{J_l^1} \right) \]

Consequently,

\[ y \in \bigcap_{l: J_l^1 \cap J_m^2 \neq \emptyset} \left( \bigcap_{k \in J_l^1} \left( K_{r_k, v_k} \cap K_{r'_k, w_k} \right) \right) \]

\[ \implies y \in \left( q_{J_m^2} \cdot T \cdot \prod_{l: J_l^1 \cap J_m^2 \neq \emptyset} R_{J_l^1} \right)^{-1} \cdot \prod_{l: J_l^1 \cap J_m^2 \neq \emptyset} P_{J_l^1} \cdot \mathbb{Z} + O \left( q_{J_m^2}^{-1} N^{-\frac{C\alpha}{4}} \right). \]

We thus see that after the second stage of the inductive process, we have obtained that the intersection of the combs \( \bigcap_i \left( K_{r_i, v_i} \cap K_{r'_i, w_i} \right) \) is in a small \( o(N^{-\frac{C\alpha}{4}}) \) neighborhood of a number of size at least

\[ \left( q_{J_m^2} \cdot T \cdot \prod_{l: J_l^1 \cap J_m^2 \neq \emptyset} R_{J_l^1} \right)^{-1} \cdot \left( \prod_{l: J_l^1 \cap J_m^2 \neq \emptyset} P_{J_l^1} \right) \]

\[ \geq \left( q_{J_l^1} \cdot T \cdot \prod_{l: J_l^1 \cap J_m^2 \neq \emptyset} R_{J_l^1} \right)^{-1} \cdot \left( \prod_{l: J_l^1 \cap J_m^2 \neq \emptyset} P_{J_l^1} \right) \]

Continuing this inductive process, after stage \( k_0 \), we obtain that the intersection of the combs \( \bigcap_i \left( K_{r_i, v_i} \cap K_{r'_i, w_i} \right) \) is contained in a small \( o(N^{-\frac{C\alpha}{4}}) \) neighborhood of a number of size at least

\[ (2.12) \quad \left( q_{J_l^1} \cdot T \cdot \prod_{j \in E} r_j r_j' \right)^{-1} \cdot \prod_{j \in E} p_{ij}, \]

where \( E \subset \{1, \ldots, K\} \) has cardinality \( \geq \epsilon_0^{-k_0/2} \) (say). Now, we claim that the above quantity is \( \gg 1 \), which would necessarily lead to a contradiction, since clearly \( \bigcap_i \left( K_{r_i, v_i} \cap K_{r'_i, w_i} \right) \) is contained in a ball centered at the origin of radius 10. Indeed,
recalling that \( r_k \leq 2^s \leq N^{1/\epsilon} \) for all \( k \), we have
\[
q_{j_1} \cdot T \cdot \prod_{j \in E} r_{j'} \lesssim N^{C_0/3} \cdot T \cdot N^{2|E|} \lesssim N^{2(C_0/5 + |E|/\epsilon)}.
\]

On the other hand
\[
\prod_{k \in E} p_k \gtrsim N^{M|E|/\epsilon}.
\]

Choosing first \( C_0 \) sufficiently large, then \( k_0 \) large enough that 
\[
M \epsilon_0^{-k_0/2} \gg C_0 \text{ exhibits (2.12) } \gg 1 \text{ for the desired contradiction.}
\]

3. The Proof of Theorem 1.4 and Corollary 1.6

We now note that Theorem 1.4 follows by the previous estimate for \( C(s, N) \). Indeed, as previously alluded to, an upper bound for
\[
\|A_{V,k}\|_{\ell^2(\mathbb{Z}^2) \to \ell^2(\mathbb{Z}^2)}
\]
is given by the sum of the following two terms:

(3.1) \[
\| \sup_{v \in V} |A_{V,k}(\varphi * f)(x)|\|_{\ell^2 \to \ell^2}
\]

and

(3.2) \[
\sup_{|\beta| \gtrsim A^{-12^{-k}}} \left( \sum_{v \in V} \sum_{0 \leq m \leq A} 1_{|v \cdot \beta - m| \lesssim 2^{-k}}(\beta) \right)^{1/2}.
\]

In (3.1), \( \varphi \) is an appropriate Schwartz function with Fourier transform supported in \( \{ |\xi| \lesssim A^{-12^{-k}} \} \). We begin with (3.1):

Consider the following decomposition:
\[
A_{V,k}(\varphi * f) = A_{V,k}^R f + \mathcal{E}_v f,
\]
where \( A_{V,k}^R \) is a convolution operator with multiplier given by
\[
m_v(\beta) := m_v,k(\beta) := \hat{\varphi}(\beta) \cdot \sum_{m \in \mathbb{Z}} \phi_k(v \cdot \beta - m),
\]
and \( \{ \mathcal{E}_v : v \} \) are error terms, with uniformly small Fourier coefficients:
\[
\sup_{v,\beta} |\hat{\mathcal{E}}_v(\beta)| \lesssim 2^{-k}.
\]
(Here, we are conflating the convolution operator and its multiplier.) In particular, we may estimate
\[
\| \sup_v |\mathcal{E}_v f|\|_{\ell^2} \lesssim N^{1/2} \cdot 2^{-k} \cdot \|f\|_{\ell^2}.
\]

To handle the contribution of \( A_{V,k}^R \) we will need the following special case of a beautiful transference argument of Magyar, Stein, and Wainger [19, Lemma 2.1].
Lemma 3.3. Let $B_1, B_2$ be finite-dimensional Banach spaces, and

$$m : \mathbb{R}^2 \to L(B_1, B_2)$$

be a bounded function supported on a cube with side length one containing the origin that acts as a Fourier multiplier from

$$L^p(\mathbb{R}^2, B_1) \to L^p(\mathbb{R}^2, B_2),$$

for some $1 \leq p \leq \infty$. Here, $L^p(\mathbb{R}^2, B) := \{ f : \mathbb{R}^2 \to B : \| f \|_{L^p(\mathbb{R}^2)} < \infty \}$. Define

$$m_{\text{per}}(\beta) := \sum_{l \in \mathbb{Z}^2} m(\beta - l) \text{ for } \beta \in \mathbb{T}^2.$$

Then the multiplier operator

$$\| m_{\text{per}} \|_{\ell^p(\mathbb{Z}^2 ; B_1) \to \ell^p(\mathbb{Z}^2 ; B_2)} \lesssim \| m \|_{L^p(\mathbb{R}^2, B_1) \to L^p(\mathbb{R}^2, B_2)}.$$

The implied constant is independent of $p, B_1,$ and $B_2$.

In particular, we will apply Lemma 3.3 for

$$m(\beta) := \{ m_v(\beta) : v \in V \}$$

which takes values in

$$\ell^2(\mathbb{Z}^2 ; \ell^\infty(V)) := \{ f = \{ f_v : v \in V \} : \| \sup_{v \in V} | f_v | \|_{\ell^\infty(\mathbb{Z}^2)} < \infty \};$$

this accrues a norm loss of $\lesssim N^\epsilon$ by the continuous theory.

As for (3.2), we may bound this above by $N^\epsilon$ using the above construction (the argument slightly simplifies as no rational shifts are needed); note the necessity of the condition that $k \gg \log N$ (this has the effect of ensure that the quantity $C_0$ used in §2 above is sufficiently large).

To extend this result to Corollary 1.6, we simply observe that

$$\left( \sum_v \sup_{j \geq k} \left| \sum_n f(x - n \cdot v) \cdot \phi_j(n) \right|^2 \right)^{1/2}$$

has $\ell^2$ norm dominated by

$$\left( \sum_v \left| \sum_n f(x - n \cdot v) \cdot \tilde{\phi}_k(n) \right|^2 \right)^{1/2}$$

for an appropriate choice of $\phi, \tilde{\phi}$; this follows from the uniform-in-$v \in \mathbb{Z}^2 \ell^2$ boundedness of the maximal function

$$\sup_N \left| \frac{1}{N} \sum_{n \leq N} f(x - n \cdot v) \right|.$$
This boundedness may be seen by first establishing the weak-type $1 - 1$ estimate via the Hopf-Dunford-Schwartz maximal inequality from ergodic theory; the method of $TT^*$ also suffices.

4. The Proof of Theorems 1.8 and 1.9

The goal of this section is to prove Theorem 1.8, reproduced below for the reader’s convenience; Theorem 1.9 will follow by only minor modifications to the method.

**Theorem 4.1.** Let $\epsilon > 0$ be arbitrary. Then, for any $N$, there exists a set of vectors $|V| = |V_{N,\epsilon}| = N$ so that, if $k \gg_{\epsilon} \log N$,

$$
\|A_{V,k}^2 f\|_{\ell^2} \lesssim_{\epsilon} \epsilon N^\epsilon \cdot \|f\|_{\ell^2}.
$$

Our analysis combines discrete-harmonic analytic techniques, the incidence estimates from §2, and the estimate for the linear (non-Radon) maximal function, (1.2).

We turn to the details. The initial reductions we make are number-theoretic.

4.1. Approximations. In what follows, we will assume that $k \gg_{\epsilon} \log N$.

Set

$$
K_k := \sum_{n \in \mathbb{Z}} \phi_k(n) \cdot \delta_{n^2},
$$

where $\delta_m$ is the point-mass at the point $m$; with this in mind, we may view $A_{V,k}^2 f$ as a maximal multiplier operator, where the multipliers are given by

$$
\{\hat{K}_k(v \cdot \beta) : v\}.
$$

In particular, it is the (one-dimensional) Fourier transform of $K_k$ that we need to consider.

We will construct our analytic family of approximating multipliers by analyzing the behavior of $\hat{K}_k$ on the **major arcs**, which we now proceed to define. A reference for the material appearing below is [2, §5-6].

4.1.1. **Major Arcs.** Throughout this section, $\epsilon_0, \epsilon_1 \ll_{\epsilon} 1$ are sufficiently small constants (as one sends $\epsilon_0, \epsilon_1$ down to zero, the size of $k$ relative to $\log N$ will increase.)

We begin by collecting the rationals in the torus according to the size of their denominators; roughly speaking, these sets form approximate level sets for appropriate complete exponential sums (below).

For each $s \geq 1$, set

$$
\mathcal{R}_s := \{a/q \in \mathbb{T} \text{ reduced} : 2^{s-1} \leq q < 2^s\}.
$$
Definition 4.2. For $a/q \in \mathcal{R}_s$, where $s \leq \epsilon_0 k$, define the $k$th major arc at $a/q$ to be given by

\[(4.3) \quad \mathcal{M}_k(a/q) := \{ \beta \in \mathbb{T} : |\alpha - a/q| \leq 2^{(\epsilon_1 - 2)k} \} . \]

On each $\mathcal{M}_k(a/q)$, we may extremely accurately approximate the multiplier $\widehat{K}_k(\alpha)$. To do so, we introduce the complete exponential sums,

\[ S(a/q) := \frac{1}{q} \sum_{r \leq q} e(-a/q \cdot r^2), \quad (a, q) = 1. \]

By “completing the square” (and using periodicity of the phases $r \mapsto e(-a/q \cdot r^2)$, it is straightforward to check that

\[ |S(a/q)| \lesssim q^{-1/2}. \]

If one wishes to replace the monomial $r \mapsto r^2$ with a more general integer-valued polynomial $P$ of degree $d$, the estimate we use is due to Hua, [9, §7, Theorem 10.1], where the savings are on the order of $q^{d-1/d}$. For our purposes here, the key point is that, for each $d$, there exists some $\delta_d > 0$ so that $|S(a/q)| \lesssim q^{-\delta_d}$.

We will also need the continuous analogue of our discrete Fourier transform,

\[ V_k(\xi) := \int e(-2^{2k}t^2 \xi) \phi(t) \, dt. \]

Our main “local” characterization of $\widehat{K}_k(\alpha)$ is contained in the following proposition.

Proposition 4.4. On $\mathcal{M}_k(a/q)$, we may express

\[ \widehat{K}_k(\alpha) = S(a/q)V_k(\alpha - a/q) + O(2^{(2\epsilon_1 - 1)k}). \]

The proof of this proposition is entirely elementary.

Proof. By a summation by parts argument, taking into account the smoothness of the function $\phi$, it suffices to assume that

\[ \phi_k := \frac{1}{2^k}1_{[1,2^k]}. \]

Now, set $\alpha = a/q + \eta$, where $|\eta| \lesssim 2^{(\epsilon_1 - 2)k}$. Working mod 1, and expressing $r = lq + p$, we have

\[ \alpha \cdot r^2 = (a/q + \eta) \cdot (lq + p)^2 \equiv a/q \cdot p^2 + \eta \cdot (lq)^2 + O(2^{(2\epsilon_1 - 1)k}). \]
Consequently, we may express
\[
\hat{K}_k(\alpha) = \frac{1}{2^k} \sum_{n \leq 2^k} e(-\alpha n^2) = \sum_{p \leq q} e(-a/q \cdot p^2) \sum_{l \leq 2^k/q} e(-\eta \cdot (lq)^2) + O(2^{(2\epsilon_1-1)k}),
\]
where we have inserted and removed some terms, all absorbed into the big-O notation. Since the phase \( l \mapsto \eta q^2 \cdot l^2 \) has a tiny derivative (on the order of \( 2^{(2\epsilon_1-1)k} \)), we may use a Riemann-sum approximation to replace
\[
\frac{q}{2^k} \sum_{l \leq 2^k/q} e(-\eta \cdot (lq)^2) = V_k(\eta) + O(2^{(2\epsilon_1-1)k}),
\]
from which the result follows. \(\square\)

In light of this proposition, we know how to locally approximate \( \hat{K}_k \). To approximate it globally, we need to “patch together” our local approximates. Doing so takes a little more notation.

First, we collect the major boxes
\[
\mathcal{M}_k := \bigcup_{s \leq \epsilon_0 k} \bigcup_{a/q \in \mathcal{R}_s} \mathcal{M}_k(a/q).
\]

For \( s \leq \epsilon_0 k \), we define the multipliers
\[
L_{k,s}(\alpha) := \sum_{a/q \in \mathcal{R}_s} S(a/q) V_k(\alpha - a/q) \chi_k(\alpha - a/q)
\]
where \( \chi_k(\alpha) := \chi(2^{(2-\epsilon_1)k} \alpha) \), where \( \epsilon_1 \) is as in the definition of the major arcs, \( 4.3 \). Note that the sum over \( a/q \in \mathcal{R}_s \) in the definition of \( 4.5 \) is over disjointly supported terms.

We define
\[
L_k := \sum_{s \leq \epsilon_0 k} L_{k,s}.
\]

Then, we have the following important proposition.

**Proposition 4.6.** There exists some \( \kappa > 0 \) so that
\[
\| \hat{K}_k - L_k \|_{L^\infty(\mathbb{T})} \lesssim 2^{-\kappa k}.
\]

Before turning to the proof of this theorem we first recall Weyl’s Lemma.
Lemma 4.7. [Weyl’s Lemma] Suppose $|a_d - a/q| \leq \frac{1}{q^d}$. Then
\[
\left| \sum_{n \leq N} e(a_d \cdot n^d + \ldots + a_1 n) \right| \lesssim N^{1+\epsilon} \left( \frac{1}{N} + \frac{1}{q} + \frac{q}{N^d} \right)^{\frac{1}{2d-1}}.
\]

Remark 4.8. The power $\frac{1}{2d-1}$ is classical, but not sharp; see [27] for an improvement.

With this in mind, we turn to the proof of Theorem 4.6.

Proof of Theorem 4.6. We will establish this in cases. First we assume that $a/q \in R$ for some $s_0 \leq \epsilon_0 k$, and consider the behavior of $\hat{K}_k - L_k$ on $\mathcal{M}_k(a/q)$.

First observe that $\hat{K}_k - L_{k,s}$ vanishes on $\mathcal{M}_k(a/q)$. Suppose now that $s \neq s_0 \leq \epsilon k$. Then
\[
\sup_{\alpha \in \mathcal{M}_k(a/q)} |L_{k,s} (\beta)| \lesssim 2^{-s/2} \min_{b/r \in \mathcal{R}_s} |V_k (\alpha - b/r)| \lesssim 2^{-s/2} \cdot 2^{(\epsilon_0 - 1)k},
\]
which sums nicely over $s$; note our use of the lower bound
\[
|\alpha - b/r| \geq |a/q - b/r| - 2^{(\epsilon_1 - 2)k} \geq \frac{1}{qr} - 2^{(\epsilon_1 - 2)k} \gtrsim 2^{-2\epsilon_0 k},
\]
for $\alpha \in \mathcal{M}_k(a/q)$.

On the minor arcs, $\alpha \notin \mathcal{M}_k$, a similar calculation to (4.9) shows that $L_k$ has a power savings; as for $\hat{K}_k$, by Dirichlet’s principle one may choose a reduced rational, $b/r$, with $r \lesssim 2^{(2-\epsilon_1)k}$, so that
\[
|\alpha - b/r| \lesssim \frac{1}{r^{2(2-\epsilon_1)k}} \leq \frac{1}{r^{2}}.
\]
We are done unless $r \lesssim 2^{\epsilon_0 k}$ by Weyl’s Lemma 4.7. But in this case, we have shown that $\alpha \in \mathcal{M}_k(b/r)$, which is a contradiction. \(\square\)

The upshot from this discussion is that we may pass from the maximal function
\[
\sup_v |(\hat{K}_k(v \cdot) f)^\vee|,
\]
to the analytic approximate
\[
(4.10) \quad \sup_v |(L_k(v \cdot) f)^\vee|,
\]
via a square function argument, since
\[
\left( \sum_v |((L_k - \hat{K}_k)(v \cdot) f)^\vee|^2 \right)^{1/2}
\]
is bounded on $\ell^2$ for $k \gg \log N$ sufficiently large.

In the next subsection, we will turn our attention to the maximal function, (4.10).

4.2. **Estimating The Maximal function.** In light of the previous section, henceforth we need only consider the sum in $1 \leq s \leq \epsilon_0 k$ of the following operators,

$$\mathcal{L}_{V,k,s}f := \sup_v \left| \left( L_{k,s}(v \cdot \beta) \hat{f}(\beta) \right)^\vee \right|$$

With this in mind, we are ready for the proof of Theorem 1.8.

4.2.1. **The Proof of Theorem 1.8.** It suffices to prove the following proposition.

**Proposition 4.11.** Let $\epsilon > 0$ be arbitrary. Then there exists a set $|V| = N$ so that for each $1 \leq s \leq \epsilon_0 k$, with $2^{\epsilon s} \leq N \leq 2^{s/\epsilon}$, we may bound

$$\|\mathcal{L}_{V,k,s}f\|_2 \lesssim N^\epsilon \|f\|_2.$$ 

Indeed, with this proposition in hand, we can quickly deduce Theorem 1.8.

**Proof of Theorem 1.8 Assuming Proposition 4.11.** For $s \ll \log N$, we use the triangle inequality to simply bound,

$$|L_{k,s}f| \lesssim 2^{-s/2} \cdot \sum_{a/q \in \mathcal{R}_s} \sup_v \left| \sum_m \left( V_k(v \cdot \beta - a/q - m) \chi_k(v \cdot \beta - a/q - m) \hat{f}(\beta) \right)^\vee \right|$$

and apply the linear theory, Theorem 1.4, to control

$$\left\| \sup_v \left| \sum_m \left( V_k(v \cdot \beta - a/q - m) \chi_k(v \cdot \beta - a/q - m) \hat{f}(\beta) \right)^\vee \right| \right\|_{L^2} \lesssim N^\epsilon \|f\|_{L^2}$$

by convexity. In particular, an upper bound for the contribution of $s \ll \log N$ is

$$N^{C \cdot \epsilon} \cdot \|f\|_{L^2}.$$ 

For $s \gg \log N$, we replace the supremum in $v \in V$ with a square-sum,

$$|L_{k,s}f|^2 \leq \sum_v \sum_{a/q \in \mathcal{R}_s} S(a/q) \left( V_k(v \cdot \beta - a/q - m) \chi_s(v \cdot \beta - a/q - m) \hat{f}(\beta) \right)^\vee,$$

and then take into account the geometric decay of the Weyl sums,

$$\max_{a/q \in \mathcal{R}_s} |S(a/q)| \lesssim 2^{-s/2}.$$ 

\[ \Box \]

It remains to prove Proposition 4.11.
The Proof of Proposition 4.11. For each $2^{s-1} \leq q < 2^s$, let

$$\hat{f}_q := \hat{f} \cdot 1_{\bigcup_v K'_{q,v}};$$

where $K_{q,v}$ is defined in (2.2), and $K'_{q,v}$ is defined similarly, but only reduced rationals $a/q$, $(a,q) = 1$ are considered; this is done so that for each $v \in V$,

$$\sum_{q=2^{s-1}}^{2^s-1} 1_{K'_{q,v}},$$

is bounded by 1 on $\mathbb{T}^2$.

We now have

$$\sup_v \left| \left( L_{k,s}(v \cdot \beta) \hat{f}(\beta) \right) \right|$$

$$= \sup_v \left| \left( \sum_m \sum_{2^{s-1} \leq q < 2^s} \sum_{a \leq q, (a,q) = 1} S(a/q)V_k(v \cdot \beta - a/q - m)\chi_k(v \cdot \beta - a/q - m)\hat{f}_q(\beta) \right) \right|. $$

We now simply replace the supremum in $v$ with a square sum; it suffices to turn our attention to

$$\sup_{\beta} 2^{-s/2} \cdot \left( \sum_v \sum_q 1_{K'_{q,v}} \right)^{1/2} (\beta);$$

but this final expression is bounded by

$$2^{-s/2} \cdot N^c$$

using the incidence estimate from Proposition 2.3; replacing the sequence $n \mapsto n^2$ sees minor changes in the number theoretic approximation, see [4, §5-6], and replacing the gain of $2^{-s/2}$ from the Gauss sums by the corresponding more modest gain, of (say)

$$2^{-s/2} \cdot 2^{-\deg(P')},$$

The proof is complete. □

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