An Introduction to
Hyperbolic Barycentric Coordinates
and their Applications

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ABSTRACT Barycentric coordinates are commonly used in Euclidean geometry. The adaptation of barycentric coordinates for use in hyperbolic geometry gives rise to hyperbolic barycentric coordinates, known as gyrobarycentric coordinates. The aim of this article is to present the path from Einstein’s velocity addition law of relativistically admissible velocities to hyperbolic barycentric coordinates, along with applications.

1. Introduction

A barycenter in astronomy is the point between two objects where they balance each other. It is the center of gravity where two or more celestial bodies orbit each other. In 1827 M"obius published a book whose title, Der Barycentrische Calcul, translates as The Barycentric Calculus. The word barycenter means center of gravity, but the book is entirely geometrical and, hence, called by Jeremy Gray [15], M"obius’s Geometrical Mechanics. The 1827 Möbius book is best remembered for introducing a new system of coordinates, the barycentric coordinates. The historical contribution of Möbius’ barycentric coordinates to vector analysis is described in [5 pp. 48–50].

The Möbius idea, for a triangle as an illustrative example, is to attach masses, \(m_1, m_2, m_3\), respectively, to three non-collinear points, \(A_1, A_2, A_3\), in the Euclidean plane \(\mathbb{R}^2\), and consider their center of mass, or momentum, \(P\), called barycenter, given by the equation

\[
P = \frac{m_1 A_1 + m_2 A_2 + m_3 A_3}{m_1 + m_2 + m_3}.
\]

The barycentric coordinates of the point \(P\) in \(\mathbb{R}^2\) in the plane of triangle \(A_1A_2A_3\), relative to this triangle may be considered as weights, \(m_1, m_2, m_3\), which if placed at vertices \(A_1, A_2, A_3\), cause \(P\) to become the balance point for the plane. The point \(P\) turns out to be the center of mass when the points of \(\mathbb{R}^2\) are viewed as position vectors, and the center of momentum when the points of \(\mathbb{R}^2\) are viewed as relative velocity vectors.
In the transition from Euclidean to hyperbolic barycentric coordinates we partially replace vector addition by Einstein addition of relativistically admissible velocities, and replace masses by relativistic masses. Barycentric coordinates are commonly used in Euclidean geometry [47], convex analysis [28], and non-relativistic quantum mechanics [2]. Evidently, Einstein addition is tailor made for the adaptation of barycentric coordinates for use in hyperbolic geometry [42, 43], hyperbolic convex analysis and, perhaps, relativistic quantum mechanics [3]. Our journey to hyperbolic barycentric coordinates thus begins with the presentation of Einstein addition, revealing its intrinsic beauty and harmony.

2. Einstein Addition

Let $c > 0$ be an arbitrarily fixed positive constant and let $\mathbb{R}^n = (\mathbb{R}^n, +, \cdot)$ be the Euclidean $n$-space, $n = 1, 2, 3, \ldots$, equipped with the common vector addition, $+$, and inner product, $\cdot$. The home of all $n$-dimensional Einsteiniann velocities is the $c$-ball

$$\mathbb{R}^n_c = \{ v \in \mathbb{R}^n : \|v\| < c \}.$$  

The $c$-ball $\mathbb{R}^n_c$ is the open ball of radius $c$, centered at the origin of $\mathbb{R}^n$, consisting of all vectors $v$ in $\mathbb{R}^n$ with magnitude $\|v\|$ smaller than $c$.

Einstein velocity addition is a binary operation, $\oplus$, in the $c$-ball $\mathbb{R}^n_c$ given by the equation [33], [29, Eq. 2.9.2], [25, p. 55], [12],

$$u \oplus v = \frac{1}{1 + \frac{u \cdot v}{c^2}} \left\{ u + \frac{1}{\gamma_u} v + \frac{1}{c^2} \gamma_u (u \cdot v) u \right\},$$

for all $u, v \in \mathbb{R}^n_c$, where $\gamma_u$ is the Lorentz gamma factor given by the equation

$$\gamma_v = \frac{1}{\sqrt{1 - \frac{\|v\|^2}{c^2}}}.$$

where $u \cdot v$ and $\|v\|$ are the inner product and the norm in the ball, which the ball $\mathbb{R}^n_c$ inherits from its space $\mathbb{R}^n$, $\|v\|^2 = v \cdot v$. A nonempty set with a binary operation is called a groupoid so that, accordingly, the pair $(\mathbb{R}^n_c, \oplus)$ is an Einstein groupoid.

In the Newtonian limit of large $c$, $c \to \infty$, the ball $\mathbb{R}^n_c$ expands to the whole of its space $\mathbb{R}^n$, as we see from (2), and Einstein addition $\oplus$ in $\mathbb{R}^n_c$ reduces to the ordinary vector addition $+$ in $\mathbb{R}^n$, as we see from (3) and (4).

When the nonzero vectors $u$ and $v$ in the ball $\mathbb{R}^n_c$ of $\mathbb{R}^n$ are parallel in $\mathbb{R}^n$, $u \parallel v$, that is, $u = \lambda v$ for some $\lambda \in \mathbb{R}$, Einstein addition (3) reduces to the Einstein addition of parallel velocities,

$$u \oplus v = \frac{u + v}{1 + \frac{u \cdot v}{c^2}}, \quad u \parallel v,$$
which was partially confirmed experimentally by the Fizeau’s 1851 experiment [24]. Following [3] we have, for instance,

\[ \|u\| \oplus \|v\| = \frac{\|u\| + \|v\|}{1 + \frac{1}{c^2} \|u\| \|v\|} \]

for all \( u, v \in \mathbb{R}^n \).

The \textit{restricted Einstein addition} in (5) and (6) is both commutative and associative. Accordingly, the restricted Einstein addition is a commutative group operation, as Einstein noted in [6]; see [7, p. 142]. In contrast, Einstein made no remark about group properties of his addition (3) of velocities that need not be parallel. Indeed, the general Einstein addition is not a group operation but, rather, a gyrocommutative gyrogroup operation, a structure discovered more than 80 years later, in 1988 [30, 31, 32], formally defined in Sect. 5.

In physical applications, \( \mathbb{R}^n = \mathbb{R}^3 \) is the Euclidean 3-space, which is the space of all classical, Newtonian velocities, and \( \mathbb{R}^n_c = \mathbb{R}^3_c \subset \mathbb{R}^3 \) is the \( c \)-ball of \( \mathbb{R}^3 \) of all relativistically admissible, Einsteinian velocities. The constant \( c \) represents in physical applications the vacuum speed of light. Since we are interested in both physics and geometry, we allow \( n \) to be any positive integer and, sometimes, replace \( c \) by \( s \).

Einstein addition (3) of relativistically admissible velocities, with \( n = 3 \), was introduced by Einstein in his 1905 paper [6, 7, p. 141] that founded the special theory of relativity, where the magnitudes of the two sides of Einstein addition (3) are presented. One has to remember here that the Euclidean 3-vector algebra was not so widely known in 1905 and, consequently, was not used by Einstein. Einstein calculated in [6] the behavior of the velocity components parallel and orthogonal to the relative velocity between inertial systems, which is as close as one can get without vectors to the vectorial version (3) of Einstein addition. Einstein was aware of the nonassociativity of his velocity addition law of relativistically admissible velocities that need not be collinear. He therefore emphasized in his 1905 paper that his velocity addition law of relativistically admissible collinear velocities forms a group operation [6, p. 907].

We naturally use the abbreviation \( u \ominus v = u \oplus (-v) \) for Einstein subtraction, so that, for instance, \( v \ominus v = 0 \) and

\[ \ominus v = 0 \ominus v = -v \]

Einstein addition and subtraction satisfy the equations

\[ \ominus (u \oplus v) = \ominus u \ominus v \]

and

\[ \ominus u \ominus (u \oplus v) = v \]

for all \( u, v \) in the ball \( \mathbb{R}^n_c \), in full analogy with vector addition and subtraction in \( \mathbb{R}^n \). Identity (8) is called the \textit{gyroautomorphic inverse property} of Einstein addition, and
Identity (9) is called the *left cancellation law* of Einstein addition. We may note that Einstein addition does not obey the naive right counterpart of the left cancellation law (9) since, in general,

\[(u\oplus v)\ominus v \neq u.\]

However, this seemingly lack of a *right cancellation law* of Einstein addition is repaired, for instance, in [43, Sect. 1.9].

Einstein addition and the gamma factor are related by the *gamma identity*,

\[
\gamma_{u\oplus v} = \gamma_u \gamma_v \left(1 + \frac{u \cdot v}{c^2}\right),
\]

which can be written, equivalently, as

\[
\gamma_{u\ominus v} = \gamma_u \gamma_v \left(1 - \frac{u \cdot v}{c^2}\right)
\]

for all \(u, v \in \mathbb{R}^n_c\). Here, (12) is obtained from (11) by replacing \(u\) by \(\ominus u = -u\) in (11).

A frequently used identity that follows immediately from (4) is

\[
\frac{v^2}{c^2} = \frac{\|v\|^2}{c^2} = \frac{\gamma_v^2 - 1}{\gamma_v^2}
\]

and useful identities that follow immediately from (11) – (12) are

\[
\frac{u \cdot v}{c^2} = -1 + \frac{\gamma_{u\oplus v}}{\gamma_{u\ominus v}} = 1 - \frac{\gamma_{u\ominus v}}{\gamma_{u\oplus v}}.
\]

Einstein addition is noncommutative. Indeed, while Einstein addition is commutative under the norm,

\[
\|u \oplus v\| = \|v \oplus u\|,
\]

in general,

\[
u \oplus v \neq v \oplus u,
\]

\(u, v \in \mathbb{R}^n_c\). Moreover, Einstein addition is also nonassociative since, in general,

\[
(u \oplus v) \oplus w \neq u \oplus (v \oplus w),
\]

\(u, v, w \in \mathbb{R}^n_c\).

As an application of the gamma identity (11), we prove the Einstein gyrotriangle inequality.

**Theorem 1. (The Gyrotriangle Inequality).**

\[
\|u \oplus v\| \leq \|u\| \oplus \|v\|
\]

for all \(u, v\) in an Einstein gyrogroup \((\mathbb{R}^n_c, \oplus)\).
Proof. By the gamma identity (11) and by the Cauchy-Schwarz inequality \[23\], we have
\[
\gamma_{\|u\|\oplus\|v\|} = \gamma_u \gamma_v \left(1 + \frac{\|u\|\|v\|}{s^2}\right)
\]
\[
\geq \gamma_u \gamma_v \left(1 + \frac{u \cdot v}{s^2}\right)
\]
\[
= \gamma_u \gamma_v
\]
\[
= \gamma_{\|u\|\oplus\|v\|}
\]
for all \(u, v\) in an Einstein gyrogroup \((\mathbb{R}^n_s, \oplus)\). But \(\gamma_x = \gamma_{\|x\|}\) is a monotonically increasing function of \(\|x\|\), \(0 \leq \|x\| < s\). Hence (19) implies
\[
\|u \oplus v\| \leq \|u\|\oplus\|v\|
\]
for all \(u, v \in \mathbb{R}^n_s\). \[\square\]

Remark 2. (Einstein Addition Domain Extension). Einstein addition \(u \oplus v\) in (3) involves the gamma factor \(\gamma_u\) of \(u\), while it is free of the gamma factor \(\gamma_v\) of \(v\). Hence, unlike \(u\), which must be restricted to the ball \(\mathbb{R}^n_c\) in order to insure the reality of a gamma factor, \(v\) need not be restricted to the ball. Hence, the domain of \(v\) can be extended from the ball \(\mathbb{R}^n_c\) to the whole of the space \(\mathbb{R}^n\). Moreover, also the gamma identity (11) remains valid for all \(u \in \mathbb{R}^n_c\) and \(v \in \mathbb{R}^n\) under appropriate choice of the square root of negative numbers. If \(1 + u \cdot v/c = 0\), then \(u \oplus v\) is undefined, and, by (11), \(\gamma_{u \oplus v} = 0\), so that \(\|u \oplus v\| = \infty\).

3. Einstein Addition Vs. Vector Addition

Vector addition, +, in \(\mathbb{R}^n\) is both commutative and associative, satisfying
\[
\begin{align*}
\mathbf{u} + \mathbf{v} &= \mathbf{v} + \mathbf{u} & \text{Commutative Law} \\
\mathbf{u} + (\mathbf{v} + \mathbf{w}) &= (\mathbf{u} + \mathbf{v}) + \mathbf{w} & \text{Associative Law}
\end{align*}
\]
(21)
for all \(u, v, w \in \mathbb{R}^n\). In contrast, Einstein addition, \(\oplus\), in \(\mathbb{R}^n_c\) is neither commutative nor associative.

In order to measure the extent to which Einstein addition deviates from associativity we introduce gyrations, which are self maps of \(\mathbb{R}^n\) that are trivial in the special cases when the application of \(\oplus\) is associative. For any \(u, v \in \mathbb{R}^n_c\) the gyration \(\text{gyr}[u, v]\) is a map of the Einstein groupoid \((\mathbb{R}^n_c, \oplus)\) onto itself. Gyrations \(\text{gyr}[u, v] \in \text{Aut}(\mathbb{R}^n_c, \oplus), u, v \in \mathbb{R}^n_c\), are defined in terms of Einstein addition by the equation
\[
\text{gyr}[u, v]w = \ominus(u \oplus v) \oplus (u \oplus (v \oplus w))
\]
(22)
for all \(u, v, w \in \mathbb{R}^n_c\), and they turn out to be automorphisms of the Einstein groupoid \((\mathbb{R}^n_c, \oplus), \text{gyr}[u, v] : \mathbb{R}^n_c \to \mathbb{R}^n_c\).
We recall that an automorphism of a groupoid \((S, \oplus)\) is a one-to-one map \(f\) of \(S\) onto itself that respects the binary operation, that is, \(f(a \oplus b) = f(a) \oplus f(b)\) for all \(a, b \in S\). The set of all automorphisms of a groupoid \((S, \oplus)\) forms a group, denoted \(\text{Aut}(S, \oplus)\). To emphasize that the gyrations of an Einstein gyrogroup \((\mathbb{R}^n_e, \oplus)\) are automorphisms of the gyrogroup, gyrations are also called gyroautomorphisms.

A gyration \(\text{gyr}[u, v], u, v \in \mathbb{R}^n_e\), is trivial if \(\text{gyr}[u, v]w = w\) for all \(w \in \mathbb{R}^n_e\). Thus, for instance, the gyrations \(\text{gyr}[0, v], \text{gyr}[v, v]\) and \(\text{gyr}[v, \ominus v]\) are trivial for all \(v \in \mathbb{R}^n_e\), as we see from (22).

Einstein gyrations, which possess their own rich structure, measure the extent to which Einstein addition deviates from commutativity and from associativity, as we see from the gyrocommutative and the gyroassociative laws of Einstein addition in the following identities [33, 35, 37]:

\[
\begin{align*}
  u \oplus v &= \text{gyr}[u, v](v \oplus u) & \text{Gyrocommutative Law} \\
  u \oplus (v \oplus w) &= (u \oplus v) \oplus \text{gyr}[u, v]w & \text{Left Gyroassociative Law} \\
  (u \oplus v) \oplus w &= u \oplus (v \oplus \text{gyr}[v, u]w) & \text{Right Gyroassociative Law} \\
  \text{gyr}[u \oplus v, v] &= \text{gyr}[u, v] & \text{Gyro Left Reduction Property} \\
  \text{gyr}[u, v \oplus u] &= \text{gyr}[u, v] & \text{Gyro Right Reduction Property} \\
  \text{gyr}[\ominus u, \ominus v] &= \text{gyr}[u, v] & \text{Gyro Even Property} \\
  (\text{gyr}[u, v])^{-1} &= \text{gyr}[v, u] & \text{Gyro Inversion Law}
\end{align*}
\]

(23)
for all \(u, v, w \in \mathbb{R}^n_e\).

Einstein addition is thus regulated by gyrations to which it gives rise owing to its nonassociativity, so that Einstein addition and its gyrations are inextricably linked. The resulting gyrocommutative gyrogroup structure of Einstein addition was discovered in 1988 [30]. Interestingly, gyrations are the mathematical abstraction of the relativistic effect known as Thomas precession [37, Sec. 10.3] [45]. Thomas precession, in turn, is related to the mixed state geometric phase, as Lévy discovered in his work [21] which, according to [21], was motivated by the author work in [34].

The left and right reduction properties in (23) present important gyration identities. These two gyration identities are, however, just the tip of a giant iceberg. The identities in (23) and many other useful gyration identities are studied in [33, 35, 37, 39, 42, 43].

4. Gyraions

An explicit presentation of the gyrations, \(\text{gyr}[u, v] : \mathbb{R}^n_e \to \mathbb{R}^n_e\), of Einstein groupoids \((\mathbb{R}^n_e, \oplus)\) in (22) in terms of vector addition rather than Einstein addition is given by the equation

\[
\text{gyr}[u, v]w = w + \frac{Au + Bv}{D},
\]

(24)
where

\[
A = -\frac{1}{c^2} \frac{\gamma_u^2}{(\gamma_u + 1)} (\gamma_v - 1)(u \cdot w) + \frac{1}{c^2} \gamma_u \gamma_v (v \cdot w) \\
+ \frac{2}{c^4} \frac{\gamma_u^2 \gamma_v^2}{(\gamma_u + 1)(\gamma_v + 1)} (u \cdot v)(v \cdot w)
\]

(25)

\[
B = -\frac{1}{c^2} \frac{\gamma_v}{\gamma_v + 1} \{\gamma_u (\gamma_v + 1)(u \cdot w) + (\gamma_u - 1)\gamma_v (v \cdot w)\}
\]

\[
D = \gamma_u \gamma_v (1 + \frac{u \cdot v}{c^2}) + 1 = \gamma_u \oplus v + 1 > 1
\]

for all \(u, v, w \in \mathbb{R}^n_c\).

**Remark 3. (Gyration Domain Extension).** The domain of \(u, v \in \mathbb{R}^n_c \subset \mathbb{R}^n\) in (24) – (25) is restricted to \(\mathbb{R}^n_c\) in order to insure the reality of the gamma factors of \(u\) and \(v\) in (25). However, while the expressions in (24) – (25) involve gamma factors of \(u\) and \(v\), they involve no gamma factors of \(w\). Hence, the domain of \(w\) in (24) – (25) can be extended from \(\mathbb{R}^n_c\) to \(\mathbb{R}^n\). Indeed, extending in (24) – (25) the domain of \(w\) from \(\mathbb{R}^n_c\) to \(\mathbb{R}^n\), gyrations \(\text{gyr}[u, v]\) are expanded from maps of \(\mathbb{R}^n_c\) to linear maps of \(\mathbb{R}^n\) for any \(u, v \in \mathbb{R}^n_c\), \(\text{gyr}[u, v] : \mathbb{R}^n \rightarrow \mathbb{R}^n\).

In each of the three special cases when (i) \(u = 0\), or (ii) \(v = 0\), or (iii) \(u\) and \(v\) are parallel in \(\mathbb{R}^n\), \(u \parallel v\), we have \(Au + Bv = 0\) so that \(\text{gyr}[u, v]\) is trivial. Thus, we have

\[
\text{gyr}[0, v]w = w \\
\text{gyr}[u, 0]w = w \\
\text{gyr}[u, v]w = w, \quad u \parallel v
\]

(26)

for all \(u, v \in \mathbb{R}^n_c\) such that \(u \parallel v\) in \(\mathbb{R}^n\), and all \(w \in \mathbb{R}^n\).

It follows from (24) that

\[
\text{gyr}[v, u](\text{gyr}[u, v]w) = w
\]

(27)

for all \(u, v \in \mathbb{R}^n_c\), \(w \in \mathbb{R}^n\), or equivalently,

\[
\text{gyr}[v, u] \circ \text{gyr}[u, v] = I
\]

(28)

for all \(u, v \in \mathbb{R}^n_c\), where \(I\) denotes the trivial map, also called the identity map.

Hence, gyrations are invertible linear maps of \(\mathbb{R}^n\), the inverse, \(\text{gyr}^{-1}[u, v]\), of \(\text{gyr}[u, v]\) being \(\text{gyr}[v, u]\). We thus have the gyration inversion property

\[
\text{gyr}^{-1}[u, v] = \text{gyr}[v, u]
\]

(29)

for all \(u, v \in \mathbb{R}^n_c\).

Gyrations keep the inner product of elements of the ball \(\mathbb{R}^n_c\) invariant, that is,

\[
\text{gyr}[u, v]a \cdot \text{gyr}[u, v]b = a \cdot b
\]

(30)
for all $a, b, u, v \in \mathbb{R}_c^n$. Hence, $\text{gyr}[u, v]$ is an isometry of $\mathbb{R}_c^n$, keeping the norm of elements of the ball $\mathbb{R}_c^n$ invariant,

$$\|\text{gyr}[u, v]w\| = \|w\|. \quad (31)$$

Accordingly, $\text{gyr}[u, v]$ represents a rotation of the ball $\mathbb{R}_c^n$ about its origin for any $u, v \in \mathbb{R}_c^n$.

The invertible map $\text{gyr}[u, v]$ of $\mathbb{R}_c^n$ respects Einstein addition in $\mathbb{R}_c^n$,

$$\text{gyr}[u, v](a \oplus b) = \text{gyr}[u, v]a \oplus \text{gyr}[u, v]b \quad (32)$$

for all $a, b, u, v \in \mathbb{R}_c^n$, so that $\text{gyr}[u, v]$ is an automorphism of the Einstein groupoid $(\mathbb{R}_c^n, \oplus)$.

**Example 4.** As an example that illustrates the use of the invariance of the norm under gyrations, we note that

$$\|\ominus u \oplus v\| = \|u \ominus v\| = \|\ominus v \oplus u\|. \quad (33)$$

Indeed, we have the following chain of equations, which are numbered for subsequent derivation,

$$\|\ominus u \oplus v\| \overset{(1)}{=} \|\ominus (\ominus u \oplus v)\| \overset{(2)}{=} \|\ominus u \oplus v\| \overset{(3)}{=} \|\text{gyr}[u, \ominus v](\ominus v \oplus u)\| \overset{(4)}{=} \|\ominus v \oplus u\| \quad (34)$$

for all $u, v \in \mathbb{R}_c^n$. Derivation of the numbered equalities in $(34)$ follows:

1. Follows from the result that $\ominus w = -w$, so that $\|\ominus w\| = \|-w\| = \|w\|$ for all $w \in \mathbb{R}_c^n$.
2. Follows from the automorphic inverse property $(8)$, p. 3, of Einstein addition.
3. Follows from the gyrocommutative law of Einstein addition.
4. Follows from the result that, by $(31)$, gyrations keep the norm invariant.

5. **From Einstein Addition to Gyrogroups**

Taking the key features of the Einstein groupoid $(\mathbb{R}_c^n, \oplus)$ as axioms, and guided by analogies with groups, we are led to the formal gyrogroup definition in which gyrogroups turn out to form a most natural generalization of groups.

**Definition 5. (Gyrogroups [37, p. 17]).** A groupoid $(G, \oplus)$ is a gyrogroup if its binary operation satisfies the following axioms. In $G$ there is at least one element, 0, called a left identity, satisfying

$$(G1) \quad 0 \oplus a = a$$
for all $a \in G$. There is an element $0 \in G$ satisfying axiom $(G1)$ such that for each $a \in G$ there is an element $\ominus a \in G$, called a left inverse of $a$, satisfying

\[(G2) \quad a \ominus a = 0.\]

Moreover, for any $a, b, c \in G$ there exists a unique element $\text{gyr}[a, b]c \in G$ such that the binary operation obeys the left gyroassociative law

\[(G3) \quad a \ominus (b \ominus c) = (a \ominus b) \ominus \text{gyr}[a, b]c.\]

The map $\text{gyr}[a, b] : G \to G$ given by $c \mapsto \text{gyr}[a, b]c$ is an automorphism of the groupoid $(G, \oplus)$, that is,

\[(G4) \quad \text{gyr}[a, b] \in \text{Aut}(G, \oplus),\]

and the automorphism $\text{gyr}[a, b]$ of $G$ is called the gyroautomorphism, or the gyration, of $G$ generated by $a, b \in G$. The operator $\text{gyr} : G \times G \to \text{Aut}(G, \oplus)$ is called the gyrator of $G$. Finally, the gyroautomorphism $\text{gyr}[a, b]$ generated by any $a, b \in G$ possesses the left reduction property

\[(G5) \quad \text{gyr}[a, b] = \text{gyr}[a \oplus b, b].\]

The gyrogroup axioms $(G1)-(G5)$ in Definition 5 are classified into three classes:

1. The first pair of axioms, $(G1)$ and $(G2)$, is reminiscent of the group axioms.
2. The last pair of axioms, $(G4)$ and $(G5)$, presents the gyrator axioms.
3. The middle axiom, $(G3)$, is a hybrid axiom linking the two pairs of axioms in (1) and (2).

As in group theory, we use the notation $a \ominus b = a \oplus (\ominus b)$ in gyrogroup theory as well. In full analogy with groups, gyrogroups are classified into gyrocommutative and non-gyrocommutative gyrogroups.

**Definition 6. (Gyrocommutative Gyrogroups).** A gyrogroup $(G, \oplus)$ is gyrocommutative if its binary operation obeys the gyrocommutative law

\[(G6) \quad a \oplus b = \text{gyr}[a, b](b \oplus a)\]

for all $a, b \in G$.

It was the study of Einstein’s velocity addition law and its associated Lorentz transformation group of special relativity theory that led to the discovery of the gyrogroup structure in 1988 [30]. However, gyrogroups are not peculiar to Einstein addition [38]. Rather, they are abound in the theory of groups [13, 14, 8, 9, 10], loops [17], quasigroup [18, 20], and Lie groups [19]. The path from Möbius to gyrogroups is described in [38].

### 6. Einstein Scalar Multiplication

The rich structure of Einstein addition is not limited to its gyrocommutative gyrogroup structure. Indeed, Einstein addition admits scalar multiplication, giving
rise to the Einstein gyrovector space. Remarkably, the resulting Einstein gyrovector spaces form the setting for the Cartesian-Beltrami-Klein ball model of hyperbolic geometry just as vector spaces form the setting for the standard Cartesian model of Euclidean geometry, as shown in [33, 35, 37, 39, 42, 43] and as indicated in the sequel.

Let \( k \otimes v \) be the Einstein addition of \( k \) copies of \( v \in \mathbb{R}^n_c \), that is \( k \otimes v = v \oplus v \ldots \oplus v \) (\( k \) terms). Then,

\[
k \otimes v = c \frac{\left(1 + \frac{\|v\|}{c}\right)^k - \left(1 - \frac{\|v\|}{c}\right)^k}{\left(1 + \frac{\|v\|}{c}\right) + \left(1 - \frac{\|v\|}{c}\right)^k} v.
\]

The definition of scalar multiplication in an Einstein gyrovector space requires analytically continuing \( k \) off the positive integers, thus obtaining the following definition.

**Definition 7. (Einstein Scalar Multiplication).** An Einstein gyrovector space \((\mathbb{R}^n_s, \oplus, \otimes)\) is an Einstein gyrogroup \((\mathbb{R}^n_s, \oplus)\) with scalar multiplication \( \otimes \) given by

\[
r \otimes v = s \frac{\left(1 + \frac{\|v\|}{s}\right)^r - \left(1 - \frac{\|v\|}{s}\right)^r}{\left(1 + \frac{\|v\|}{s}\right) + \left(1 - \frac{\|v\|}{s}\right)^r} v = s \tanh(r \tanh^{-1} \frac{\|v\|}{s}) \frac{v}{\|v\|}
\]

where \( r \) is any real number, \( r \in \mathbb{R} \), \( v \in \mathbb{R}^n_s \), \( v \neq 0 \), and \( r \otimes 0 = 0 \), and with which we use the notation \( v \otimes r = r \otimes v \).

As an example, it follows from Def. 7 that Einstein half is given by the equation

\[
\frac{1}{2} \otimes v = \frac{\gamma}{1 + \gamma} v,
\]

so that, as expected, \( \frac{\gamma}{1 + \gamma} v \oplus \frac{\gamma}{1 + \gamma} v = v \).

Einstein gyrovector spaces are studied in [33, 35, 37, 39, 42, 43]. Einstein scalar multiplication does not distribute over Einstein addition, but it possesses other properties of vector spaces. For any positive integer \( k \), and for all real numbers \( r, r_1, r_2 \in \mathbb{R} \) and \( v \in \mathbb{R}^n_s \), we have

\[
k \otimes v = v \oplus v \ldots \oplus v \quad k \text{ terms}
\]

\[
(r_1 + r_2) \otimes v = r_1 \otimes v \oplus r_2 \otimes v \quad \text{Scalar Distributive Law}
\]

\[
(r_1 r_2) \otimes v = r_1 \otimes (r_2 \otimes v) \quad \text{Scalar Associative Law}
\]

in any Einstein gyrovector space \((\mathbb{R}^n_s, \oplus, \otimes)\).

Additionally, Einstein gyrovector spaces possess the scaling property

\[
\frac{|r| \otimes a}{\|r \otimes a\|} = \frac{a}{\|a\|}
\]
\(\mathbf{a} \in \mathbb{R}^n, \mathbf{a} \neq \mathbf{0}, r \in \mathbb{R}, r \neq 0,\) the \textit{gyroautomorphism property}

\[ \text{gyr}[\mathbf{u}, \mathbf{v}](r \otimes \mathbf{a}) = r \otimes \text{gyr}[\mathbf{u}, \mathbf{v}](\mathbf{a}) \]

\(\mathbf{a}, \mathbf{u}, \mathbf{v} \in \mathbb{R}^n, r \in \mathbb{R},\) and the identity gyroautomorphism

\[ \text{gyr}[r_1 \otimes \mathbf{v}, r_2 \otimes \mathbf{v}] = I \]

for all \(r_1, r_2 \in \mathbb{R}, \mathbf{v} \in \mathbb{R}^n.\)

Any Einstein gyrovector space \((\mathbb{R}^n_s, \oplus, \otimes)\) inherits an inner product and a norm from its vector space \(\mathbb{R}^n.\) These turn out to be invariant under gyrations,

\[ \text{gyr}[\mathbf{a}, \mathbf{b}](\mathbf{u} \cdot \text{gyr}[\mathbf{a}, \mathbf{b}](\mathbf{v})) = \mathbf{u} \cdot \mathbf{v} \]

\[ \|\text{gyr}[\mathbf{a}, \mathbf{b}](\mathbf{v})\| = \|\mathbf{v}\| \]

for all \(\mathbf{a}, \mathbf{b}, \mathbf{u}, \mathbf{v} \in \mathbb{R}^n,\) as indicated in Sect. 4.

7. From Einstein Scalar Multiplication to Gyrovector Spaces

Taking the key features of Einstein scalar multiplication as axioms, and guided by analogies with vector spaces, we are led to the formal gyrovector space definition in which gyrovector spaces turn out to form a most natural generalization of vector spaces.

**Definition 8. (Real Inner Product Gyrovector Spaces)** \(^{[37, \text{p. 154}]}\). A real inner product gyrovector space \((G, \oplus, \otimes)\) (gyrovector space, in short) is a gyrocommutative gyrogroup \((G, \oplus)\) that obeys the following axioms:

1. \(G\) is a subset of a real inner product vector space \(V\) called the carrier of \(G, G \subset V,\) from which it inherits its inner product, \(\cdot,\) and norm, \(\|\cdot\|,\) which are invariant under gyroautomorphisms, that is,

\[ \text{gyr}[\mathbf{u}, \mathbf{v}](\mathbf{a} \cdot \text{gyr}[\mathbf{u}, \mathbf{v}](\mathbf{b})) = \mathbf{a} \cdot \mathbf{b} \] \hspace{1cm} \text{Inner Product Gyroinvariance}

for all points \(\mathbf{a}, \mathbf{b}, \mathbf{u}, \mathbf{v} \in G.\)

2. \(G\) admits a scalar multiplication, \(\otimes,\) possessing the following properties. For all real numbers \(r, r_1, r_2 \in \mathbb{R}\) and all points \(\mathbf{a} \in G:\)

\[ 1 \otimes \mathbf{a} = \mathbf{a} \] \hspace{1cm} \text{Identity Scalar Multiplication}

\[ (r_1 + r_2) \otimes \mathbf{a} = r_1 \otimes \mathbf{a} \oplus r_2 \otimes \mathbf{a} \] \hspace{1cm} \text{Scalar Distributive Law}

\[ (r_1 r_2) \otimes \mathbf{a} = r_1 \otimes (r_2 \otimes \mathbf{a}) \] \hspace{1cm} \text{Scalar Associative Law}

\[ \frac{|r| \otimes \mathbf{a}}{\|r \otimes \mathbf{a}\|} = \frac{\mathbf{a}}{\|\mathbf{a}\|}, \quad \mathbf{a} \neq \mathbf{0}, r \neq 0 \] \hspace{1cm} \text{Scaling Property}

\[ \text{gyr}[\mathbf{u}, \mathbf{v}](r \otimes \mathbf{a}) = r \otimes \text{gyr}[\mathbf{u}, \mathbf{v}](\mathbf{a}) \] \hspace{1cm} \text{Gyroautomorphism Property}

\[ \text{gyr}[r_1 \otimes \mathbf{v}, r_2 \otimes \mathbf{v}] = I \] \hspace{1cm} \text{Identity Gyroautomorphism.}

3. Real, one-dimensional vector space structure \((\|G\|, \oplus, \otimes)\) for the set \(\|G\|\) of one-dimensional “vectors”

\[ \|G\| = \{ \pm \|\mathbf{a}\| : \mathbf{a} \in G \} \subset \mathbb{R} \] \hspace{1cm} \text{Vector Space}
with vector addition \( \oplus \) and scalar multiplication \( \otimes \), such that for all \( r \in \mathbb{R} \) and \( a, b \in G \),

(V9) \( \| r \otimes a \| = |r| \| a \| \) \quad \text{Homogeneity Property}
(V10) \( \| a \oplus b \| \leq \| a \| \oplus \| b \| \) \quad \text{Gyrotriangle Inequality.}

Einstein addition and scalar multiplication in \( \mathbb{R}^n_a \) thus give rise to the Einstein gyrovector spaces \( (\mathbb{R}^n_a, \oplus, \otimes) \), \( n \geq 2 \).

8. **Gyrolines – The Hyperbolic Lines**

In applications to geometry it is convenient to replace the notation \( \mathbb{R}^n_c \) for the \( c \)-ball of an Einstein gyrovector space by the \( s \)-ball, \( \mathbb{R}^n_s \). Moreover, it is understood that \( n \geq 2 \), unless specified otherwise.

Let \( A, B \in \mathbb{R}^n_s \) be two distinct points of the Einstein gyrovector space \( (\mathbb{R}^n_s, \oplus, \otimes) \), and let \( t \in \mathbb{R} \) be a real parameter. Then, the graph of the set of all points

(43) \[ A \oplus (\ominus A \oplus B) \otimes t \]

\( t \in \mathbb{R} \), in the Einstein gyrovector space \( (\mathbb{R}^n_s, \oplus, \otimes) \) is a chord of the ball \( \mathbb{R}^n_s \). As such, it is a geodesic line of the Beltrami-Klein ball model of hyperbolic geometry, shown
in Fig. 1 for $n = 2$. The geodesic line (43) is the unique gyroline that passes through the points $A$ and $B$. It passes through the point $A$ when $t = 0$ and, owing to the left cancellation law, (9), it passes through the point $B$ when $t = 1$. Furthermore, it passes through the midpoint $m_{A,B}$ of $A$ and $B$ when $t = 1/2$. Accordingly, the gyrosegment $AB$ that joins the points $A$ and $B$ in Fig. 1 is obtained from gyroline (43) with $0 \leq t \leq 1$.

Gyrolines (43) are the geodesics of the Beltrami-Klein ball model of hyperbolic geometry. Similarly, gyrolines (43) with Einstein addition $\oplus$ replaced by Möbius addition $\oplus_M$ are the geodesics of the Poincaré ball model of hyperbolic geometry. These interesting results are established by methods of differential geometry in [36].

Each point of (43) with $0 < t < 1$ is said to lie between $A$ and $B$. Thus, for instance, the point $P$ in Fig. 1 lies between the points $A$ and $B$. As such, the points $A$, $P$ and $B$ obey the gyrotriangle equality according to which

\[ d(A, P) \oplus d(P, B) = d(A, B) \]

in full analogy with Euclidean geometry. Here

\[ d(A, B) = \| A \oplus_B B \| \]

$A, B \in \mathbb{R}^n$, is the Einstein gyrodistance function, also called the Einstein gyrometric. This gyrodistance function in Einstein gyrovector spaces corresponds bijectively to a standard hyperbolic distance function, as demonstrated in [37, Sect. 6.19], and it gives rise to the well-known Riemannian line element of the Beltrami-Klein ball model of hyperbolic geometry, as shown in [36].

9. EUCLIDEAN ISOMETRIES

In this section and in Sect. 10 we present well-known results about Euclidean isometries and Euclidean motions in order to set the stage for the introduction of hyperbolic isometries (gyroisometries) and motions (gyromotions) in Sects. 11 and 12.

The Euclidean distance function (distance, in short) in $\mathbb{R}^n$,

\[ d(A, B) = \| A - B \| , \]

$A, B \in \mathbb{R}^n$, gives the distance between any two points $A$ and $B$. It possesses the following properties:

1. $d(A, B) = d(B, A)$
2. $d(A, B) \geq 0$
3. $d(A, B) = 0$ if and only if $A = B$
4. $d(A, B) \leq d(A, C) + d(C, B)$ (the triangle inequality)
5. $d(A, B) = d(A, C) + d(C, B)$ (the triangle equality, for $A, B, C$ collinear, $C$ lies between $A$ and $B$)

for all $A, B, C \in \mathbb{R}^n$. 
Definition 9. (Isometries). A map $\phi : \mathbb{R}^n \to \mathbb{R}^n$ is a Euclidean isometry of $\mathbb{R}^n$ (isometry, in short) if it preserves the distance between any two points of $\mathbb{R}^n$, that is, if

$$d(\phi A, \phi B) = d(A, B)$$

for all $A, B \in \mathbb{R}^n$.

An isometry is injective (one-to-one into). Indeed, if $A, B \in \mathbb{R}^n$ are two distinct points, $A \neq B$, then

$$0 \neq \| - A + B \| = \| - \phi A + \phi B \| ,$$

so that $\phi A \neq \phi B$. We will now characterize the isometries of $\mathbb{R}^n$, following which we will find that isometries are surjective (onto).

For any $X \in \mathbb{R}^n$, a translation of $\mathbb{R}^n$ by $X$ is the map $\lambda_X : \mathbb{R}^n \to \mathbb{R}^n$ given by

$$\lambda_X A = X + A$$

for all $A \in \mathbb{R}^n$.

Theorem 10. (Translational Isometries). Translations of a Euclidean space $\mathbb{R}^n$ are isometries.

Proof. The proof is trivial, but we present it in order to set the stage for the gyro-counterpart Theorem 16, p. 18, of this theorem. Let $\lambda_X, X \in \mathbb{R}^n$, be a translation of a Euclidean space $\mathbb{R}^n$. Then $\lambda_X$ is an isometry of the space, as we see from the following obvious chain of equations,

$$\| - \lambda_X A + \lambda_X B \| = \| - (X + A) + (X + B) \| = \| - A + B \| .$$

\[ \square \]

Theorem 11. (Isometry Characterization [27, p. 19]). Let $\phi : \mathbb{R}^n \to \mathbb{R}^n$ be a map of $\mathbb{R}^n$. Then the following are equivalent:

1. The map $\phi$ is an isometry.
2. The map $\phi$ preserves the distance between points.
3. The map $\phi$ is of the form

$$\phi X = A + RX ,$$

where $R \in O(n)$ is an $n \times n$ orthogonal matrix (that is, $R^t R = RR^t = I$ is the identity matrix) and $A = \phi O \in \mathbb{R}^n$, $O = (0, \ldots, 0)$ being the origin of $\mathbb{R}^n$.

Proof. By definition, Item 1 implies Item 2 of the Theorem. Suppose that $\phi$ preserves the distance between any two points of $\mathbb{R}^n$, and let $R : \mathbb{R}^n \to \mathbb{R}^n$ be the map given by

$$RX = \phi X - \phi O .$$
Then \( RO = O \), and \( R \) also preserves the distance. Indeed, for all \( A, B \in \mathbb{R}^n \)

\[
(53) \quad \| - RA + RB \| = \| - (\phi A - \phi O) + (\phi B - \phi O) \| = \| - \phi A + \phi B \| = \| - A + B \| .
\]

Hence, \( R \) preserves the norm,

\[
(54) \quad \| RX \| = \| - RO + RX \| = \| - O + X \| = \| X \| .
\]

Consequently, \( R \) is orthogonal, \( R \in O(n) \). Indeed, for all \( X, Y \in \mathbb{R}^n \) we have

\[
(55) \quad \| X - Y \|^2 = (X - Y) \cdot (X - Y) = \| X \|^2 + \| Y \|^2 - 2X \cdot Y ,
\]

so that

\[
(56) \quad 2RX \cdot RY = \| RX \|^2 + \| RY \|^2 - \| RX - RY \|^2 = \| X \|^2 + \| Y \|^2 - \| X - Y \|^2 = 2X \cdot Y .
\]

Thus, following (52), there is an orthogonal \( n \times n \) matrix \( R \) such that

\[
(57) \quad \phi X = \phi O + RX ,
\]

and so (2) implies (3).

If \( \phi \) is of the form (51) then \( \phi \) is the composite of an orthogonal transformation followed by a translation, and so \( \phi \) is an isometry. Thus, (3) implies (1), and the proof is complete. \( \square \)

Following Theorem 11 it is now clear that isometries of \( \mathbb{R}^n \) are surjective (onto), the inverse of isometry \( A + RX \) being

\[
(58) \quad (A + RX)^{-1} = - R^t A + R^t X .
\]

**Theorem 12. (Isometry Unique Decomposition).** Let \( \phi \) be an isometry of \( \mathbb{R}^n \). Then it possesses the decomposition

\[
(59) \quad \phi X = A + RX ,
\]

where \( A \in \mathbb{R}^n \) and \( R \in O(n) \) are unique.

**Proof.** By Theorem 11 \( \phi X \) possesses a decomposition (52). Let

\[
(60) \quad \phi X = A_1 + R_1 X \quad \phi X = A_2 + R_2 X
\]

be two decompositions of \( \phi X, X \in \mathbb{R}^n \). For \( X = O \) we have \( R_1 O = R_2 O = O \), implying \( A_1 = A_2 \). The latter, in turn, implies \( R_1 = R_2 \), and the proof is complete. \( \square \)
Let $R$ be an orthogonal matrix. As $RR^t = I$, we have that $(\det R)^2 = 1$, so that $\det R = \pm 1$. If $\det R = 1$, then $R$ represents a rotation of $\mathbb{R}^n$ about its origin. The set of all rotations $R$ in $O(n)$ is a subgroup $SO(n) \subset O(n)$ called the special orthogonal group. Accordingly, $SO(n)$ is the group of all $n \times n$ orthogonal matrices with determinant 1.

The set of all isometries $\phi X = A + RX$ of $\mathbb{R}^n$, $A, X \in \mathbb{R}^n$, $R \in O(n)$, forms a group called the isometry group of $\mathbb{R}^n$. Following [1, p. 416],

(1) the isometries $\phi X = A + RX$ of $\mathbb{R}^n$ with $\det R = 1$ are called direct isometries, or motions, of $\mathbb{R}^n$; and
(2) the isometries $\phi X = A + RX$ of $\mathbb{R}^n$ with $\det R = -1$ are called opposite isometries.

The motions of $\mathbb{R}^n$, studied in Sect. 10, form a subgroup of the isometry group of $\mathbb{R}^n$.

10. The Group of Euclidean Motions

The Euclidean group of motions of $\mathbb{R}^n$ is the direct isometry group. It consists of the (i) commutative group of all translations of $\mathbb{R}^n$ and (ii) the group of all rotations of $\mathbb{R}^n$ about its origin.

A rotation $R$ of $\mathbb{R}^n$ about its origin is an element of the group $SO(n)$ of all $n \times n$ orthogonal matrices with determinant 1. The rotation of $A \in \mathbb{R}^n$ by $R \in SO(n)$ is $RA$. The map $R \in SO(n)$ is a linear map of $\mathbb{R}^n$ that keeps the inner product invariant, that is

$$R(A + B) = RA + RB$$
$$RA \cdot RB = A \cdot B$$

for all $A, B \in \mathbb{R}^n$ and all $R \in SO(n)$.

The *Euclidean group of motions* is the semidirect product group

$$\mathbb{R}^n \times SO(n)$$

of the Euclidean commutative group $\mathbb{R}^n = (\mathbb{R}^n, +)$ and the rotation group $SO(n)$. It is a group of pairs $(X, R)$, $X \in (\mathbb{R}^n, +)$, $R \in SO(n)$, acting isometrically on $\mathbb{R}^n$ according to the equation

$$X, R)A = X + RA$$

for all $A \in \mathbb{R}^n$. Each pair $(X, R) \in \mathbb{R}^n \times SO(n)$, accordingly, represents a rotation of $\mathbb{R}^n$ followed by a translation of $\mathbb{R}^n$.

The group operation of the semidirect product group (62) is given by action composition. Accordingly, let $(X_1, R_1)$ and $(X_2, R_2)$ be any two elements of the semidirect product group $\mathbb{R}^n \times SO(n)$. Their successive applications to $A \in \mathbb{R}^n$ is
equivalent to a single application to $A$, as shown in the following chain of equations (64), in which we employ the associative law of vector addition, $+$, in $\mathbb{R}^n$.

\[
(X_1, R_1)(X_2, R_2)A = (X_1, R_1)(X_2 + R_2A) \\
= X_1 + R_1(X_2 + R_2A) \\
= X_1 + (R_1X_2 + R_1R_2A) \\
= (X_1 + R_1X_2) + R_1R_2A \\
= (X_1 + R_1X_2, R_1R_2)A
\]

(64)

for all $A \in \mathbb{R}^n$.

It follows from (64) that the group operation of the semidirect product group (62) is given by the semidirect product

\[
(X_1, R_1)(X_2, R_2) = (X_1 + R_1X_2, R_1R_2)
\]

(65)

for any $(X_1, R_1), (X_2, R_2) \in \mathbb{R}^n \times SO(n)$.

**Definition 13.** (Covariance). A map

\[
T : (\mathbb{R}^n)^k \to \mathbb{R}^n
\]

from $k$ copies of $\mathbb{R}^n$ into $\mathbb{R}^n$ is covariant (with respect to the motions of $\mathbb{R}^n$) if its image $T(A_1, A_2, \ldots, A_k)$ co-varies (that is, varies together) with its preimage points $A_1, A_2, \ldots, A_k$ under the motions of $\mathbb{R}^n$, that is, if

\[
X + T(A_1, \ldots, A_k) = T(X + A_1, \ldots, X + A_k)
\]

(67)

\[
RT(A_1, \ldots, A_k) = T(RA_1, \ldots, RA_k)
\]

for all $X \in \mathbb{R}^n$ and all $R \in SO(n)$. In particular, the first equation in (67) represents covariance with respect to (or, under) translations, and the second equation in (67) represents covariance with respect to (or, under) rotations.

Following Theorem 31, p. 40, we will see that Euclidean barycentric coordinate representations of points of $\mathbb{R}^n$ are covariant.

The importance of covariance under the motions of a geometry was first recognized by Felix Klein (1849–1924) in his *Erlangen Program*, the traditional professor’s inaugural speech that he gave at the University of Erlangen in 1872. The thesis that Klein published in Erlangen in 1872 is that a geometry is a system of definitions and theorems that express properties invariant under a given group of transformations called motions. The Euclidean motions of Euclidean geometry are described in this section, and the hyperbolic motions of hyperbolic geometry are described in Sect. 12. It turns out that the Euclidean and the hyperbolic motions share remarkable analogies.

11. **Gyroisometries – The Hyperbolic Isometries**

Our study of hyperbolic isometries is guided by analogies with Euclidean isometries, studied in Sect. 9. The hyperbolic counterpart of the Euclidean distance
function $d(A, B)$ in $\mathbb{R}^n$, given by (46), is the gyrodistance function $d(A, B)$ in an Einstein gyrovector space $\mathbb{R}^n_s = (\mathbb{R}^n_s, \oplus, \otimes)$, given by

$$d(A, B) = \|\ominus A \oplus B\|,$$

$A, B \in \mathbb{R}^n_s$, giving the gyrodistance between any two points $A$ and $B$. It should always be clear from the context whether $d(A, B)$ is the distance function in $\mathbb{R}^n$ or the gyrodistance function in $\mathbb{R}^n_s$. Like the distance function, the gyrodistance function possesses the following properties for all $A, B, C \in \mathbb{R}^n_s$:

1. $d(A, B) = d(B, A)$
2. $d(A, B) \geq 0$
3. $d(A, B) = 0$ if and only if $A = B$.
4. $d(A, B) \leq d(A, C) \oplus d(C, B)$ (The gyrotriangle inequality).
5. $d(A, B) = d(A, C) \oplus d(C, B)$ (The gyrotriangle equality, for $A, B, C$ gyrocollinear, $C$ lies between $A$ and $B$).

The gyrotriangle inequality in Item (4) is presented, for instance, in [42, p. 94].

**Definition 14. (Gyroisometries).** A map $\phi : \mathbb{R}^n_s \rightarrow \mathbb{R}^n_s$ is a gyroisometry of $\mathbb{R}^n_s$ if it preserves the gyrodistance between any two points of $\mathbb{R}^n_s$, that is, if

$$d(\phi A, \phi B) = d(A, B)$$

for all $A, B \in \mathbb{R}^n_s$.

A gyroisometry is injective (one-to-one into). Indeed, if $A, B \in \mathbb{R}^n_s$ are two distinct points, $A \neq B$, then

$$0 \neq \|\ominus A \oplus B\| = \|\ominus \phi A \oplus \phi B\|,$$

so that $\phi A \neq \phi B$. We will now characterize the gyroisometries of $\mathbb{R}^n_s$, following which we will find that gyroisometries are surjective (onto).

For any $X \in \mathbb{R}^n_s$, a left gyrotranslation of $\mathbb{R}^n_s$ by $X$ is the map $\lambda_X : \mathbb{R}^n_s \rightarrow \mathbb{R}^n_s$ given by

$$\lambda_X A = X \oplus A$$

for all $A \in \mathbb{R}^n_s$.

**Theorem 15. (Left Gyrotranslation Theorem).** [35, p. 29] [37, p. 23] [42, p. 82] [43, p. 39]. Let $(G, \oplus)$ be a gyrogroup. Then,

$$\ominus (X \oplus A) \oplus (X \oplus B) = \text{gyr}[X, A](\ominus A \oplus B)$$

for all $A, B, X \in G$.

**Theorem 16. (Left Gyrotranslational Gyroisometries).** Left gyrotranslations of an Einstein gyrovector space are gyroisometries.
Proof. Let $\lambda_X, X \in \mathbb{R}_s^n$, be a left gyrotranslation of an Einstein gyrovector space $(\mathbb{R}_s^n, \oplus, \otimes)$. Then $\lambda_X$ is a gyroisometry of the space, as we see from the following chain of equations, which are numbered for subsequent derivation:

$$
\|\ominus \lambda_X A \oplus \lambda_X B\| \overset{(1)}{=} \|\ominus (X \oplus A) \oplus (X \oplus B)\|
$$

$$
\overset{(2)}{=} \|\text{gyr}[X, A](\ominus (A \oplus B))\|
$$

$$
\overset{(3)}{=} \|\ominus A \oplus B\|
$$

for all $A, B, X \in \mathbb{R}_s^n$. Derivation of the numbered equalities in (73) follows:

1. Follows from (71).
2. Follows from (1) by the Left Gyrotranslation Theorem 15.
3. Follows from (2) by the norm invariance (31) under gyrations.

□

Theorem 17. (Gyroisometry Characterization). Let $\phi : \mathbb{R}_s^n \rightarrow \mathbb{R}_s^n$ be a map of $\mathbb{R}_s^n$. Then the following are equivalent:

1. The map $\phi$ is a gyroisometry.
2. The map $\phi$ preserves the gyrodistance between points.
3. The map $\phi$ is of the form

$$
\phi X = A \oplus RX ,
$$

where $R \in O(n)$ is an $n \times n$ orthogonal matrix (that is, $R^t R = RR^t = I$ is the identity matrix) and $A = \phi O \in \mathbb{R}_s^n$, $O = (0, \ldots, 0)$ being the origin of $\mathbb{R}_s^n$.

Proof. By definition, Item 1 implies Item 2 of the Theorem.

Suppose that $\phi$ preserves the gyrodistance between any two points of $\mathbb{R}_s^n$,

$$
\|\ominus \phi A \oplus \phi B\| = \|\ominus A \oplus B\| ,
$$

and let $R : \mathbb{R}_s^n \rightarrow \mathbb{R}_s^n$ be the map given by

$$
RX = \ominus \phi O \oplus \phi X .
$$

Then $RO = O$ and, by the left cancellation law (9),

$$
\phi X = \phi O \oplus RX .
$$
Furthermore, $R$ also preserves the gyrodistance. Indeed, for all $X, Y \in \mathbb{R}^n_s$ we have the following chain of equations, which are numbered for subsequent explanation:

\[
\| \ominus RX \oplus RY \| = \| \ominus (\ominus \phi O \ominus \phi X) \oplus (\ominus \phi O \ominus \phi Y) \| \quad (1)
\]

\[
= \| \text{gyr}[\ominus \phi O, \phi X] [\ominus \phi X \oplus \phi Y] \| \quad (2)
\]

\[
= \| \ominus \phi X \oplus \phi Y \| \quad (3)
\]

\[
= \| \ominus X \oplus Y \|. \quad (4)
\]

Derivation of the numbered equalities in (78) follows:

1. Follows from (76).
2. Follows from (1) by the Left Gyrotranslation Theorem 15.
3. Follows from (2) by the invariance (42) of the norm under gyrations.
4. Follows from (3) by Assumption (75).

The map $R$ preserves the norm since, by (78),

\[
\| RX \| = \| \ominus RO \ominus RX \| = \| \ominus O \ominus X \| = \| X \|. \quad (79)
\]

Moreover, $R$ preserves the inner product as well. Indeed, by the gamma identity \[12\], p. 4 in $\mathbb{R}^n_s$ and by (78) – (79), and noting that $\gamma_A = \gamma_{\| A \|}$ for all $A \in \mathbb{R}^n_s$, we have the following chain of equations,

\[
\gamma_X \gamma_Y (1 - \frac{X \cdot Y}{s^2}) = \gamma_{\ominus X \oplus Y} = \gamma_{\ominus RX \ominus RY}
\]

\[
= \gamma_{RX} \gamma_{RY} (1 - \frac{RX \cdot RY}{s^2})
\]

\[
= \gamma_X \gamma_Y (1 - \frac{RX \cdot RY}{s^2}),
\]

implying

\[
RX \cdot RY = X \cdot Y, \quad (80)
\]

as desired, so that $R$ is orthogonal.

Thus, following (76), there is an orthogonal $n \times n$ matrix $R$ such that

\[
\phi X = \phi O \ominus RX, \quad (82)
\]

and so Item (2) implies Item (3) of the Theorem.

If $\phi$ is of the form (74) then $\phi$ is the composite of an orthogonal transformation followed by a left gyrotranslation, and so $\phi$ is a gyroisometry. Thus, Item (3) implies Item (1) of the Theorem, and the proof is complete. \hfill \square
Following Theorem 17 it is now clear that gyroisometries of $\mathbb{R}^n_s$ are surjective (onto), the inverse of gyroisometry $A \oplus RX$ being

$$(A \oplus RX)^{-1} = \ominus R' A \ominus R' X.$$  

**Theorem 18.** (Gyroisometry Unique Decomposition). Let $\phi$ be a gyroisometry of $\mathbb{R}^n_s$. Then it possesses the decomposition

$$(84) \quad \phi X = A \oplus RX,$$

where $A \in \mathbb{R}^n_s$ and $R \in O(n)$ are unique.

**Proof.** By Theorem 17, $\phi X$ possesses a decomposition (84). Let

$$(85) \quad \phi X = A_1 \oplus R_1 X$$

$$\phi X = A_2 \oplus R_2 X$$

be two decompositions of $\phi X$, for all $X \in \mathbb{R}^n_s$. For $X = O$ we have $R_1 O = R_2 O = O$, implying $A_1 = A_2$. The latter, in turn, implies $R_1 = R_2$, and the proof is complete. \qed

Let $R$ be an orthogonal matrix. As $RR^t = I$, we have that $(\det R)^2 = 1$, so that $\det R = \pm 1$. If $\det R = 1$, then $R$ represents a rotation of $\mathbb{R}^n_s$ about its origin. The set of all rotations $R$ in $O(n)$ is a subgroup $SO(n) \subset O(n)$ called the special orthogonal group. Accordingly, $SO(n)$ is the group of all $n \times n$ orthogonal matrices with determinant 1.

In full analogy with isometries, the set of all gyroisometries $\phi X = A \oplus RX$ of $\mathbb{R}^n_s$, $A, X \in \mathbb{R}^n_s$, $R \in O(n)$, forms a group called the gyroisometry group of $\mathbb{R}^n_s$. Accordingly, by analogy with isometries,

1. the gyroisometries $\phi X = A \oplus RX$ of $\mathbb{R}^n_s$ with $\det R = 1$ are called *direct gyroisometries, or motions, of $\mathbb{R}^n_s$*, and
2. the gyroisometries $\phi X = A \oplus RX$ of $\mathbb{R}^n_s$ with $\det R = -1$ are called *opposite gyroisometries*.

In gyrolanguage, the motions of $\mathbb{R}^n_s$ are called *gyromotions*. They form a subgroup of the gyroisometry group of $\mathbb{R}^n_s$, studied in Sect. 12 below.

**12. Gyromotions – The Motions of Hyperbolic Geometry**

The group of gyromotions of $\mathbb{R}^n_s = (\mathbb{R}^n_s, \oplus, \otimes)$ is the direct gyroisometry group of $\mathbb{R}^n_s$. It consists of the gyrocommutative gyrogroup of all left gyrotranslations of $\mathbb{R}^n_s$ and the group $SO(n)$ of all rotations of $\mathbb{R}^n_s$ about its origin.

A rotation $R$ of $\mathbb{R}^n_s$ about its origin is an element of the group $SO(n)$ of all $n \times n$ orthogonal matrices with determinant 1. The rotation of $A \in \mathbb{R}^n_s$ by $R \in SO(n)$ is
RA. The map $R \in SO(n)$ is a \textit{gyrolinear} map of $\mathbb{R}^n_s$ that respects Einstein addition and keeps the inner product invariant, that is

\begin{align}
R(A \oplus B) &= RA \oplus RB \\
RA \odot RB &= A \odot B
\end{align}

for all $A, B \in \mathbb{R}^n_s$ and all $R \in SO(n)$, in full analogy with (61), p. 14.

The group of gyromotions of $\mathbb{R}^n_s$ possesses the \textit{gyrosemidirect product group} structure. It is the gyrosemidirect product group

\begin{align}
\mathbb{R}^n_s \times SO(n)
\end{align}

of the Einstein gyrocommutative gyrogroup $\mathbb{R}^n_s = (\mathbb{R}^n_s, \oplus)$ and the rotation group $SO(n)$. More specifically, it is a group of pairs $(X, R), X \in (\mathbb{R}^n_s, \oplus), R \in SO(n)$, acting gyroisometrically on $\mathbb{R}^n_s$ according to the equation

\begin{align}
(X, R)A &= X \oplus RA
\end{align}

for all $A \in \mathbb{R}^n_s$. Each pair $(X, R) \in \mathbb{R}^n_s \times SO(n)$, accordingly, represents a rotation of $\mathbb{R}^n_s$ followed by a left gyrotranslation of $\mathbb{R}^n_s$.

The group operation of the gyrosemidirect product group $\mathbb{R}^n_s$ is given by action composition. Accordingly, let $(X_1, R_1)$ and $(X_2, R_2)$ be any two elements of the gyrosemidirect product group $\mathbb{R}^n_s \times SO(n)$. Their successive applications to $A \in \mathbb{R}^n_s$ is equivalent to a single application to $A$, as shown in the following chain of equations (89), in which we employ the left gyroassociative law of Einstein addition, $\oplus$, in $\mathbb{R}^n_s$.

\begin{align}
(X_1, R_1)(X_2, R_2)A &= (X_1, R_1)(X_2 \oplus R_2 A) \\
&= X_1 \oplus R_1(X_2 \oplus R_2 A) \\
&= X_1 \oplus (R_1 X_2 \oplus R_1 R_2 A) \\
&= (X_1 \oplus R_1 X_2, \text{gyr}[X_1, R_1 X_2] R_1 R_2 A) \\
&= (X_1 \oplus R_1 X_2, \text{gyr}[X_1, R_1 X_2] R_1 R_2 A)
\end{align}

for all $A \in \mathbb{R}^n_s$.

It follows from (89) that the group operation of the gyrosemidirect product group $\mathbb{R}^n_s$ is given by the \textit{gyrosemidirect product}

\begin{align}
(X_1, R_1)(X_2, R_2) &= (X_1 \oplus R_1 X_2, \text{gyr}[X_1, R_1 X_2] R_1 R_2)
\end{align}

for any $(X_1, R_1), (X_2, R_2) \in \mathbb{R}^n_s \times SO(n)$.

Gyrocovariance with respect to gyromotions is formalized in the following two definitions:

\textbf{Definition 19. (Gyrocovariance).} A map

\begin{align}
T : (\mathbb{R}^n_s)^k \rightarrow \mathbb{R}^n_s
\end{align}

from $k$ copies of $\mathbb{R}^n_s = (\mathbb{R}^n_s, \oplus, \odot)$ into $\mathbb{R}^n_s$ is \textit{gyrocovariant} (with respect to the gyromotions of $\mathbb{R}^n_s$) if its image $T(A_1, A_2, \ldots, A_k)$ co-varies (that is, varies together)
with its preimage points \( A_1, A_2, \ldots, A_k \) under the gyromotions of \( \mathbb{R}_n^s \), that is, if
\[
X \oplus T(A_1, \ldots, A_k) = T(X \oplus A_1, \ldots, X \oplus A_k)
\]
(92)
\[
RT(A_1, \ldots, A_k) = T(RA_1, \ldots, RA_k)
\]
for all \( X \in \mathbb{R}_n^s \) and all \( R \in SO(n) \). In particular, the first equation in (92) represents gyrocovariance with respect to (or, under) left gyrottranslations, and the second equation in (92) represents gyrocovariance with respect to (or, under) rotations.

**Definition 20. (Gyrocovariance in Form).** Let
\[
T_1(A_1, \ldots, A_k) = T_2(A_1, \ldots, A_k)
\]
be a gyrovector space identity in an Einstein gyrovector space \( \mathbb{R}_n^s = (\mathbb{R}_n^s, \oplus, \odot) \), where
\[
T_i : (\mathbb{R}_n^s)^k \to \mathbb{R}_n^s
\]
i = 1, 2, is a map from \( k \) copies of \( \mathbb{R}_n^s \) into \( \mathbb{R}_n^s \).

The identity is gyrocovariant in form (with respect to the gyromotions of \( \mathbb{R}_n^s \)) if
\[
T_1(X \oplus A_1, \ldots, X \oplus A_k) = T_2(X \oplus A_1, \ldots, X \oplus A_k)
\]
(95)
\[
T_1(RA_1, \ldots, RA_k) = T_2(RA_1, \ldots, RA_k)
\]
for all \( X \in \mathbb{R}_n^s \) and all \( R \in SO(n) \).

We will see from the Gyrobarycentric Representation Gyrocovariance Theorem 11, p. 18 that hyperbolic barycentric (gyrobarycentric, in gyrolanguage) coordinate representations of points of \( \mathbb{R}_n^s \) are gyrocovariant, Theorem 11 in turn, provides a powerful tool to determine analytically various properties of hyperbolic geometric objects.

The importance of hyperbolic covariance (gyrocovariance) under hyperbolic motions (gyromotions) of hyperbolic geometry (gyrogeometry) lies in Klein's Erlangen Program, as remarked below the Covariance Definition 13, p. 17.

13. **Lorentz Transformation and Einstein Addition**

The Newtonian, classical mass of a particle system suggests the introduction of barycentric coordinates into Euclidean geometry. In full analogy, the Einsteinian, relativistic mass of a particle system suggests the introduction of barycentric coordinates into hyperbolic geometry as well, where they are called gyrobarycentric coordinates. The relativistic mass, which is velocity dependent [44], thus meets hyperbolic geometry in the context of gyrobarycentric coordinates, just as the classical mass meets Euclidean geometry in the context of barycentric coordinates.

Interestingly, unlike classical mass, relativistic mass is velocity dependent. “Coincidentally”, the velocity dependence of relativistic mass has precisely the form
that gives rise to the requested analogies. Our mission to capture the requested analogies that lead to the adaptation of barycentric coordinates for use in hyperbolic geometry begins with the study of the Lorentz transformation as a coordinate transformation regulated by Einstein Addition.

The Lorentz transformation is a linear transformation of spacetime coordinates that fixes the spacetime origin. A Lorentz boost, \( L(\mathbf{v}) \), is a Lorentz transformation without rotation, possessing the matrix representation

\[
L(\mathbf{v}) = \begin{pmatrix}
\gamma_v & c^{-2}\gamma_v v_1 & c^{-2}\gamma_v v_2 & c^{-2}\gamma_v v_3 \\
\gamma_v v_1 & 1 + c^{-2}\gamma_v^2 v_1^2 & c^{-2}\gamma_v^2 v_1 v_2 & c^{-2}\gamma_v^2 v_1 v_3 \\
\gamma_v v_2 & c^{-2}\gamma_v^2 v_2 v_1 & 1 + c^{-2}\gamma_v^2 v_2^2 & c^{-2}\gamma_v^2 v_2 v_3 \\
\gamma_v v_3 & c^{-2}\gamma_v^2 v_3 v_1 & c^{-2}\gamma_v^2 v_3 v_2 & 1 + c^{-2}\gamma_v^2 v_3^2
\end{pmatrix}
\]

Employing the matrix representation (96) of the Lorentz transformation boost, the Lorentz boost application to spacetime coordinates takes the form

\[
L(\mathbf{v}) \begin{pmatrix} t \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} = L(\mathbf{v}) \begin{pmatrix} t' \\ x'_1 \\ x'_2 \\ x'_3 \end{pmatrix} = \begin{pmatrix} t' \\ x'_1 \\ x'_2 \\ x'_3 \end{pmatrix}
\]

where \( \mathbf{v} = (v_1, v_2, v_3)^t \in \mathbb{R}^3 \), \( \mathbf{x} = (x_1, x_2, x_3)^t \in \mathbb{R}^3 \), \( \mathbf{x'} = (x'_1, x'_2, x'_3)^t \in \mathbb{R}^3 \), and \( t, t' \in \mathbb{R} \), where exponent \( t \) denotes transposition.

In the Newtonian limit of large vacuum speed of light \( c \), \( c \to \infty \), the Lorentz boost \( L(\mathbf{v}) \), (96) – (97), reduces to the Galilei boost \( G(\mathbf{v}) \), \( \mathbf{v} = (v_1, v_2, v_3) \in \mathbb{R}^3 \),

\[
G(\mathbf{v}) \begin{pmatrix} t \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} = \lim_{c \to \infty} L(\mathbf{v}) \begin{pmatrix} t \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} t \\ x_1 + vt \\ x_2 + vt \\ x_3 + v^2 t \end{pmatrix}
\]

where \( \mathbf{x} = (x_1, x_2, x_3)^t \in \mathbb{R}^3 \) and \( t \in \mathbb{R} \).

The representation of spacetime coordinates as \( (t, \mathbf{x})^t \) in (97) is more advantageous than its representation as \( (ct, \mathbf{x})^t \). Indeed, unlike the latter representation, the former representation of spacetime coordinates allows one to recover the Galilei boost from the Lorentz boost by taking the Newtonian limit of large speed of light \( c \), as shown in the transition from (97) to (98).

As a result of adopting \( (t, \mathbf{x})^t \) rather than \( (ct, \mathbf{x})^t \) as our four-vector that represents four-position, our four-velocity is given by \( (\gamma_v, \gamma_v \mathbf{v}) \) rather than \( (\gamma_v c, \gamma_v \mathbf{v}) \),
\[ p_0 = \frac{E}{c^2} = m \gamma_v \gamma_v v \]

rather than the standard four-momentum, which is given by \((p_0, p) = (E/c, p) = (m \gamma_v c, m \gamma_v v)^t\), as found in most relativity physics books. According to (99) the relativistically invariant mass (that is, rest mass) \(m\) of a particle is the ratio of the particle’s four-momentum \((p_0, p)^t\) to its four-velocity \((\gamma_v, \gamma_v v)^t\).

For the sake of simplicity, and without loss of generality, some authors normalize the vacuum speed of light to \(c = 1\) as, for instance, in [11]. We, however, prefer to leave \(c\) as a free positive parameter, enabling related modern results to be reduced to classical ones under the limit of large \(c, c \to \infty\), as, for instance, in the transition from a Lorentz boost into a corresponding Galilei boost in (96) – (98), and the transition from Einstein addition (3) into a corresponding vector addition (21).

The Lorentz boost (96) – (97) can be written vectorially in the form

\[
L(u) \begin{pmatrix} t \\ x \end{pmatrix} = \begin{pmatrix} \gamma_u (t + \frac{1}{c^2} u \cdot x) \\ \gamma_u u t + x + \frac{1}{c^2} \frac{\gamma_u^2}{1 + \gamma_u} (u \cdot x) u \end{pmatrix}.
\]

Being written in a vector form, the Lorentz boost \(L(u)\) in (100) survives unimpaired in higher dimensions. Rewriting (100) in higher dimensional spaces, with \(x = vt, u, v \in \mathbb{R}^n\subset \mathbb{R}^n\), we have

\[
L(u) \begin{pmatrix} t \\ vt \end{pmatrix} = \begin{pmatrix} \gamma_u (t + \frac{1}{c^2} u \cdot vt) \\ \gamma_u u t + vt + \frac{1}{c^2} \frac{\gamma_u^2}{1 + \gamma_u} (u \cdot vt) u \end{pmatrix}
\]

\[
= \begin{pmatrix} \frac{u \otimes v}{\gamma_v} t \\ \frac{\gamma_u \otimes v}{\gamma_v} (u \oplus v)t \end{pmatrix}.
\]

Equation (101) reveals explicitly the way Einstein velocity addition underlies the Lorentz boost. The second equation in (101) follows from the first by Einstein addition formula (3) and the gamma identity [11], p. 4.

The special case of \(t = \gamma_v\) in (101) proves useful, giving rise to the elegant identity

\[
L(u) \begin{pmatrix} \gamma_v \\ \gamma_v v \end{pmatrix} = \begin{pmatrix} \gamma_u \otimes v \\ \gamma_u \otimes v (u \oplus v) \end{pmatrix}
\]

of the Lorentz boost of four-velocities, \(u, v \in \mathbb{R}^n\). Since in physical applications \(n = 3\), in the context of \(n\)-dimensional special relativity we call \(v\) a three-vector and \((\gamma_v, \gamma_v v)^t\) a four-vector, etc., even when \(n \neq 3\).

The four-vector \(m(\gamma_v, \gamma_v v)^t\) is the four-momentum of a particle with invariant mass (or, rest mass) \(m\) and velocity \(v\) relative to a given inertial rest frame \(\Sigma_0\). Let \(\Sigma_0\) be an inertial frame that moves with velocity \(\otimes u = -u\) relative to the rest
frame $\Sigma_0$, $u, v \in \mathbb{R}_{\Sigma_0}^n$. Then, a particle with velocity $v$ relative to $\Sigma_0$ has velocity $u \oplus v$ relative to the frame $\Sigma \oplus u$. Owing to the linearity of the Lorentz boost, it follows from (102) that the four-momentum of the particle relative to the frame $\Sigma \oplus u$ is

$$L(u)m \begin{pmatrix} \gamma_v \\ \gamma_v v \end{pmatrix} = mL(u) \begin{pmatrix} \gamma_v \\ \gamma_v v \end{pmatrix}$$

(103)

$$= m \begin{pmatrix} \gamma_{u \oplus v} \\ \gamma_{u \oplus v} (u \oplus v) \end{pmatrix}.$$  

Similarly, it follows from the linearity of the Lorentz boost and from (102) that

$$L(w) \sum_{k=1}^{N} m_k \begin{pmatrix} \gamma_{v_k} \\ \gamma_{v_k} v_k \end{pmatrix} = \sum_{k=1}^{N} m_k L(w) \begin{pmatrix} \gamma_{v_k} \\ \gamma_{v_k} v_k \end{pmatrix}$$

$$= \sum_{k=1}^{N} m_k \begin{pmatrix} \gamma_{w \oplus v_k} \\ \gamma_{w \oplus v_k} (w \oplus v_k) \end{pmatrix}$$

(104)

$$= \begin{pmatrix} \sum_{k=1}^{N} m_k \gamma_{w \oplus v_k} \\ \sum_{k=1}^{N} m_k \gamma_{w \oplus v_k} (w \oplus v_k) \end{pmatrix},$$

where $m_k \in \mathbb{R}$ and $w, v_k \in \mathbb{R}_{\Sigma_0}^n$, $k = 1, \ldots, N$.

The chain of equations (103) reveals the interplay of Einstein addition, $\oplus$, in $\mathbb{R}_{\Sigma_0}^n$ and vector addition, $+$, in $\mathbb{R}^n$ that appears implicitly in the $\Sigma$-notation for scalar and vector addition. This harmonious interplay between $\oplus$ and $+$, which will prove crucially important in our approach to hyperbolic barycentric coordinates, reveals itself in (104) where Einstein’s three-vector formalism of special relativity, embodied in Einstein addition $\oplus$, meets Minkowski’s four-vector formalism of special relativity.

The (Minkowski) norm of a four-vector is Lorentz transformation invariant. The norm of the four-position $(t, \mathbf{x})^t$ is

$$\left\| \begin{pmatrix} t \\ \mathbf{x} \end{pmatrix} \right\| = \sqrt{t^2 - \frac{||\mathbf{x}||^2}{c^2}}$$

(105)

and, accordingly, the norm of the four-velocity $(\gamma_v, \gamma_v v)^t$ is

$$\left\| \begin{pmatrix} \gamma_v \\ \gamma_v v \end{pmatrix} \right\| = \gamma_v \left\| \begin{pmatrix} 1 \\ \mathbf{v} \end{pmatrix} \right\| = \gamma_v \sqrt{1 - \frac{||v||^2}{c^2}} = 1.$$  

14. Invariant Mass of Particle Systems

The results in (103) – (104) follow from the linearity of Lorentz boosts. We will now further exploit that linearity to obtain the relativistically invariant mass of particle systems. Being observer’s invariant, the Newtonian, rest mass, $m$, is referred
to as the (relativistically) invariant mass. In contrast, the relativistic mass \( m\gamma \) is velocity dependent and, hence, observer’s dependent.

Let

\[
S = S(m_k, v_k, \Sigma_0, k = 1, \ldots, N)
\]

be an isolated system of \( N \) noninteracting material particles the \( k \)-th particle of which has invariant mass \( m_k > 0 \) and velocity \( v_k \in \mathbb{R}^n_c \) relative to an inertial frame \( \Sigma_0, k = 1, \ldots, N \).

Classically, the Newtonian mass \( m_{\text{newton}} \) of the system \( S \) is additive in the sense that it equals the sum of the Newtonian masses of its constituent particles, that is

\[
m_{\text{newton}} = \sum_{k=1}^{N} m_k.
\]

In full analogy, also the relativistic mass of a system is additive, as we will see in \( \text{[150]} \), p. \( \text{35} \) provided that the relativistically invariant mass of particle systems is appropriately determined by Theorem \( \text{26} \), p. \( \text{33} \).

In order to determine

1. the relativistically invariant mass \( m_0 \) of the system \( S \), and
2. the velocity \( v_0 \) relative to \( \Sigma_0 \) of a fictitious inertial frame, called the center of momentum frame, relative to which the three-momentum of \( S \) vanishes,

we make the natural assumption that the four-momentum is additive. Then, the sum of the four-momenta of the \( N \) particles of the system \( S \) gives the four-momentum \((m_0\gamma v_0, m_0\gamma v_0 v_0)\) of \( S \), where (i) \( m_0 \) is the invariant mass of \( S \), and (ii) \( v_0 \) is the velocity of the center of momentum of \( S \) relative to \( \Sigma_0 \). This assumption yields the equation

\[
\sum_{k=1}^{N} m_k \begin{pmatrix} \gamma v_k \\ \gamma_{v_k} v_k \end{pmatrix} = m_0 \begin{pmatrix} \gamma v_0 \\ \gamma_{v_0} v_0 \end{pmatrix}
\]

from which \( m_0 \) and \( v_0 \) are determined. In \( \text{[109]} \),

1. the invariant masses \( m_k > 0 \) and the velocities \( v_k \in \mathbb{R}^n_c \), \( k = 1, \ldots, N \), relative to \( \Sigma_0 \) of the constituent particles of \( S \) are given, while
2. the invariant mass \( m_0 \) of \( S \) and the velocity \( v_0 \) of the center of momentum frame of \( S \) relative to \( \Sigma_0 \) are to be determined uniquely by \( \text{[109]} \) in the Resultant Relativistically Invariant Mass Theorem, which is Theorem \( \text{26} \) in Sect. \( \text{15} \).

If \( m_0 > 0 \) and \( v_0 \in \mathbb{R}^n_c \) that satisfy \( \text{[109]} \) exist then, as anticipated, the three-momentum of the system \( S \) relative to its center of momentum frame vanishes since, by \( \text{[103]} \) and \( \text{[109]} \), the four-momentum of \( S \) relative to its center of momentum frame
frame is given by

\[ L(\oplus v_0) \sum_{k=1}^{N} m_k \left( \gamma v_k \right) = L(\oplus v_0)m_0 \left( \gamma v_0 \right) \]

\[ = m_0 \left( \gamma \oplus v_0 \oplus v_0 \right) \left( \gamma v_0(\oplus v_0 \oplus v_0) \right) \]

\[ = m_0 \left( \begin{array}{c} 1 \\ 0 \end{array} \right) \]

noting that \( \gamma \oplus v_0 \oplus v_0 = \gamma_0 = 1. \)

15. Resultant Relativistically Invariant Mass

The following five Lemmas 21–25 lead to the Resultant Relativistically Invariant Mass Theorem 26, p. 33.

**Lemma 21.** Let \( N \) be any positive integer, and let \( m_k \in \mathbb{R} \) and \( v_k \in \mathbb{R}^n_\circ \), \( k = 1, \ldots, N \), be \( N \) scalars and \( N \) points of an Einstein gyrogroup \( \mathbb{R}^n_\circ = (\mathbb{R}^n_\circ, \oplus) \). Then

\[
\left( \sum_{k=1}^{N} m_k \gamma v_k \frac{v_k}{c} \right)^2
\]

\[ = \left( \sum_{k=1}^{N} m_k \gamma v_k \right)^2 - \left\{ \sum_{k=1}^{N} m_k \right\}^2 + 2 \sum_{j,k=1}^{N} m_j m_k \left( \gamma \oplus v_j \oplus v_k - 1 \right) \]
**Proof.** The proof is given by the following chain of equations, which are numbered for subsequent explanation:

\[
\left( \sum_{k=1}^{N} m_k \gamma v_k \right)^2 \overset{(1)}{=} \sum_{k=1}^{N} m_k^2 \gamma v_k^2 + 2 \sum_{j,k=1, j<k}^{N} m_j m_k \gamma v_j \gamma v_k \frac{v_j \cdot v_k}{c^2}
\]

\[
\overset{(2)}{=} \sum_{k=1}^{N} m_k^2 (\gamma v_k^2 - 1) + 2 \sum_{j,k=1, j<k}^{N} m_j m_k (\gamma v_j \gamma v_k - \gamma v_j \oplus v_k)
\]

\[
\overset{(3)}{=} \sum_{k=1}^{N} m_k^2 \gamma v_k^2 - \sum_{k=1}^{N} m_k^2 + 2 \sum_{j,k=1, j<k}^{N} m_j m_k \gamma v_j \gamma v_k - 2 \sum_{j,k=1, j<k}^{N} m_j m_k \gamma v_j \oplus v_k
\]

\[
\overset{(4)}{=} \left( \sum_{k=1}^{N} m_k \gamma v_k \right)^2 - \left\{ \sum_{k=1}^{N} m_k^2 + 2 \sum_{j,k=1, j<k}^{N} m_j m_k \gamma v_j \oplus v_k \right\}
\]

\[
\overset{(5)}{=} \left( \sum_{k=1}^{N} m_k \gamma v_k \right)^2 - \left\{ \left( \sum_{k=1}^{N} m_k \right)^2 + 2 \sum_{j,k=1, j<k}^{N} m_j m_k (\gamma v_j \oplus v_k - 1) \right\}
\]

The assumption \( v_k \in \mathbb{R}^n \) implies that all gamma factors in (111) – (112) are real and greater than 1. Derivation of the numbered equalities in (112) follows:

1. This equation is obtained by an expansion of the square of a sum of vectors in \( \mathbb{R}^n \).
2. Follows from (1) by (13) – (14), p. 4.
3. Follows from (2) by an obvious expansion.
4. Follows from (3) by an expansion of the square of a sum of real numbers.
5. Follows from (4) by an expansion of another square of a sum of real numbers.

\[\square\]

**Lemma 22.** Let \((\mathbb{R}_c^n, \oplus)\) be an Einstein gyrogroup, and let \( m_k \in \mathbb{R} \) and \( v_k \in \mathbb{R}_c^n \), \( k = 1, 2, \ldots, N \), be \( N \) scalars and \( N \) elements of \( \mathbb{R}_c^n \), such that

\[
\sum_{k=1}^{N} m_k \gamma v_k \neq 0.
\]

If the \((n + 1)\)-vector equation

\[
\sum_{k=1}^{N} m_k \left( \gamma v_k \gamma v_k \right) = m_0 \left( \gamma v_0 \right)
\]
for the unknowns $m_0 \in \mathbb{R}$ and $v_0 \in \mathbb{R}^n$ possesses a solution, then $m_0$ is given by the equation

\[
(115) \quad m_0 = \sqrt{ \left( \sum_{k=1}^{N} m_k \right)^2 + 2 \sum_{j, k=1}^{N} m_j m_k (\gamma \circ v_j \oplus v_k - 1) }
\]

where the sign of $m_0$ equals the sign of the left-hand side of (113).

**Proof.** The norms of the two sides of (114) are equal while, by (106), the norm of the right-hand side of (114) is $m_0$. Hence, the norm of the left-hand side of (114) equals $m_0$ as well, obtaining the following chain of equations, which are numbered for subsequent explanation:

\[
(116) \quad m_0^2 \overset{(1)}{=} \left\| \sum_{k=1}^{N} m_k \left( \frac{\gamma v_k}{\gamma v_k \cdot v_k} \right) \right\|^2 \\
\quad \overset{(2)}{=} \left\| \left( \sum_{k=1}^{N} m_k \gamma v_k \right) \right\|^2 \\
\quad \overset{(3)}{=} \left( \sum_{k=1}^{N} m_k \gamma v_k \right)^2 - \left( \sum_{k=1}^{N} m_k \gamma v_k \cdot v_k \right)^2 \\
\quad \overset{(4)}{=} \left( \sum_{k=1}^{N} m_k \right)^2 + 2 \sum_{j, k=1}^{N} m_j m_k (\gamma \circ v_j \oplus v_k - 1)
\]

Derivation of the numbered equalities in (116) follows:

1. This equation follows from the result that the norm of the left-hand side of (114) equals the norm of the right-hand side of (114), the latter being $m_0$ by (106).
2. Follows from Item (1) by the common “four-vector” addition of $(n + 1)$-vectors (where $n = 3$ in physical applications).
3. Follows from Item (2) by (105).
4. Follows from Item (3) by Identity (111) of Lemma 21.

□

**Lemma 23.** Let $(\mathbb{R}^n_\gamma, \oplus)$ be an Einstein gyrogroup, let $v_k \in \mathbb{R}^n_\gamma$ be $N$ elements of the gyrogroup, and let $m_k \in \mathbb{R}$, $k = 1, 2, \ldots, N$, be $N$ scalars, such that

\[
(117) \quad \sum_{k=1}^{N} m_k \gamma v_k \neq 0
\]
Furthermore, let
\[ N \sum_{k=1}^{N} m_k \left( \gamma_{v_k} \right) = \left( \begin{array}{c} r \\ r v_0 \end{array} \right) \]
be an \((n + 1)\)-vector equation for the two unknowns \( r \in \mathbb{R} \) and \( v_0 \in \mathbb{R}^n \).

Then \((118)\) possesses a unique solution \((r, v_0)\), where
\[ r = \sum_{k=1}^{N} m_k \gamma_{v_k} \]
\[ v_0 = \frac{\sum_{k=1}^{N} m_k \gamma_{v_k} v_k}{\sum_{k=1}^{N} m_k \gamma_{v_k}} \]

**Proof.** The proof is immediate. \(\square\)

Taking \( r = m_0 \gamma_{v_0} \), Lemma \(23\) gives rise to the following Lemma \(24\).

**Lemma 24.** Let \((\mathbb{R}_c^n, \oplus)\) be an Einstein gyrogroup, let \( v_k \in \mathbb{R}_c^n \) be \( N \) elements of the gyrogroup, and let \( m_k \in \mathbb{R}, \ k = 1, 2, \ldots, N \), be \( N \) scalars such that
\[ \sum_{k=1}^{N} m_k \gamma_{v_k} \neq 0 \]
Furthermore, let
\[ \sum_{k=1}^{N} m_k \left( \gamma_{v_k} \right) = m_0 \left( \begin{array}{c} \gamma_{v_0} \\ \gamma_{v_0} v_0 \end{array} \right) \]
be an \((n + 1)\)-vector equation for the two unknowns \( m_0 \in \mathbb{R} \) and \( v_0 \in \mathbb{R}^n \).

Then \((121)\) possesses a unique solution \((m_0 \gamma_{v_0}, v_0)\), where
\[ \gamma_{v_0} = \frac{\sum_{k=1}^{N} m_k \gamma_{v_k}}{m_0} \]
\[ v_0 = \frac{\sum_{k=1}^{N} m_k \gamma_{v_k} v_k}{\sum_{k=1}^{N} m_k \gamma_{v_k}} \]
where
\[ m_0 = \sqrt{\left( \sum_{k=1}^{N} m_k \right)^2 + 2 \sum_{j,k=1}^{N} m_j m_k (\gamma_{v_j} \oplus v_k - 1)} \]

(1) If \( m_0^2 > 0 \) then \( m_0 \neq 0 \) is real and
\[ \gamma_{v_0} = \frac{\sum_{k=1}^{N} m_k \gamma_{v_k}}{m_0} \]

is real, so that \( v_0 \) lies inside the ball \( \mathbb{R}_c^n, v_0 \in \mathbb{R}_c^n \subset \mathbb{R}_c^n \).
(2) If \( m_0^2 < 0 \) then \( m_0 \) is purely imaginary. Hence,

\[
\gamma_{v_0} = \frac{\sum_{k=1}^{N} m_k \gamma_{v_k}}{m_0}
\]

is purely imaginary, so that \( v_0 \in \mathbb{R}^n \) lies outside the closure of the ball \( \mathbb{R}^n_c \).

(3) If \( m_0^2 = 0 \) then \( m_0 = 0 \), while \( m_0 \gamma_{v_0} \neq 0 \). Hence,

\[
v_0 = \frac{\sum_{k=1}^{N} m_k \gamma_{v_k} v_k}{\sum_{k=1}^{N} m_k \gamma_{v_k}}
\]

and \( \gamma_{v_0} = \infty \), so that \( v_0 \) lies on the boundary of the ball \( \mathbb{R}^n_c \), \( v_0 \in \partial \mathbb{R}^n_c \).

**Proof.** Equation (123) is established in Lemma 22, and (122) is established in Lemma 23 with \( r = m_0 \gamma_{v_0} \). The proof of (124) – (126) in Items (1) – (3) is immediate. \( \square \)

**Lemma 25.** Let \((\mathbb{R}^n_c, \oplus)\) be an Einstein gyrogroup, let \( v_k \in \mathbb{R}^n_c \) be \( N \) elements of the gyrogroup, and let \( m_k \in \mathbb{R}, k = 1, 2, \ldots, N \), be \( N \) scalars.

Let us assume that the vector \( v_0 \in \mathbb{R}^n \) satisfies the \((n+1)\)-vector equation

\[
\sum_{k=1}^{N} m_k \left( \gamma_{v_k} \gamma_{v_k} v_k \right) = m_0 \left( \gamma_{v_0} \gamma_{v_0} v_0 \right)
\]

together with the condition

\[
\sum_{k=1}^{N} m_k \gamma_{v_k} \neq 0.
\]

Then, for any \( w \in \mathbb{R}^n_c \), \( v_0 \) satisfies the \((n+1)\)-vector equation

\[
\sum_{k=1}^{N} m_k \left( \gamma_{w \oplus v_k} \gamma_{w \oplus v_k} (w \oplus v_k) \right) = m_0 \left( \gamma_{w \oplus v_0} \gamma_{w \oplus v_0} (w \oplus v_0) \right)
\]

together with the condition

\[
\sum_{k=1}^{N} m_k \gamma_{w \oplus v_k} \neq 0
\]

where

\[
m_0 = \sqrt{\left( \sum_{k=1}^{N} m_k \right)^2 + 2 \sum_{j,k=1}^{N} m_j m_k (\gamma_{\oplus v_j \oplus v_k} - 1)}
\]

\[
= \sqrt{\left( \sum_{k=1}^{N} m_k \right)^2 + 2 \sum_{j,k=1}^{N} m_j m_k (\gamma_{\oplus (w \oplus v_j) \oplus (w \oplus v_k)} - 1)}
\]
Proof. The vector $v_0 \in \mathbb{R}^n$ need not be an element of $\mathbb{R}_c^n$. Yet, the Einstein sum $w \oplus v_0$ is defined, as explained in Remark 2 p. 5.

The two representations of $m_0$ in (131) are equal since $m_0$ is invariant under left gyrotranslations, as we see from the following chain of equations,

\[
\| \oplus (w \oplus v_j) \oplus (w \oplus v_k) \| = \| \text{gyr}[w, v_j] \oplus v_k \| = \| \oplus v_j \oplus v_k \|
\]

which implies

\[
\gamma \oplus (w \oplus v_j) \oplus (w \oplus v_k) = \gamma \oplus v_j \oplus v_k
\]

The chain of equations (132), in turn, follows from the Left Gyrotranslation Theorem 15, p. 15, and from the norm invariance (42) under gyrations.

Applying the Lorentz boost $L(w)$, $w \in \mathbb{R}_c^n$, to each side of (127), we have

\[
L(w) \sum_{k=1}^{N} m_k \begin{pmatrix} \gamma v_k \\ \gamma v_k \gamma v_0 v_k \end{pmatrix} = L(w)m_0 \begin{pmatrix} \gamma v_0 \\ \gamma v_0 \gamma v_0 v_0 \end{pmatrix}
\]

Following the linearity of the Lorentz boost, illustrated in (103) and (104), and the invariance under left gyrotranslations of $m_k$ (these are constants) and $m_0$ (given by (131)), the $(n + 1)$-vector equation (134) can be written as

\[
\sum_{k=1}^{N} m_k \begin{pmatrix} \gamma w \oplus v_k \\ \gamma w \oplus v_k \gamma w \oplus v_k \end{pmatrix} = m_0 \begin{pmatrix} \gamma w \oplus v_0 \\ \gamma w \oplus v_0 \gamma w \oplus v_0 \end{pmatrix}
\]

In (134) – (135) we recover (129) as a Lorentz transformation of (127).

Moreover, being bijective and linear, the Lorentz transformation takes only the zero $(n + 1)$-vector, $\mathbf{0}$, into $\mathbf{0}$, implying that condition (130) is equivalent to condition (128).

Hence, $v_0$ satisfies (127) – (128) if and only if $v_0$ satisfies (129) – (130), as desired.

Theorem 26. (Resultant Relativistically Invariant Mass Theorem). Let $(\mathbb{R}_c^n, \oplus)$ be an Einstein gyrogroup, let $v_k \in \mathbb{R}_c^n$ be $N$ elements of the gyrogroup, and let $m_k \in \mathbb{R}$, $k = 1, 2, \ldots, N$, be $N$ scalars such that

\[
\sum_{k=1}^{N} m_k \gamma v_k \neq 0.
\]

Furthermore, let

\[
\sum_{k=1}^{N} m_k \begin{pmatrix} \gamma v_k \\ \gamma v_k \gamma v_0 v_k \end{pmatrix} = m_0 \begin{pmatrix} \gamma v_0 \\ \gamma v_0 \gamma v_0 v_0 \end{pmatrix}
\]

be an $(n + 1)$-vector equation for the two unknowns $m_0 \in \mathbb{R}$ and $v_0 \in \mathbb{R}^n$.

Then (137) possesses a unique solution $(m_0, v_0)$. Moreover, the solution $(m_0, v_0)$ satisfies the following three identities for all $w \in \mathbb{R}_c^n$ (including, in particular, the
interesting special case when \( w = 0 \):

\[
(138) \quad w \oplus v_0 = \frac{\sum_{k=1}^{N} m_k \gamma_{w \oplus v_k} (w \oplus v_k)}{\sum_{k=1}^{N} m_k \gamma_{w \oplus v_k}}
\]

\[
(139) \quad \gamma_{w \oplus v_0} = \frac{\sum_{k=1}^{N} m_k \gamma_{w \oplus v_k}}{m_0}
\]

\[
(140) \quad \gamma_{w \oplus v_0} (w \oplus v_0) = \frac{\sum_{k=1}^{N} m_k \gamma_{w \oplus v_k} (w \oplus v_k)}{m_0}
\]

where

\[
(141) \quad m_0 = \sqrt{\left( \sum_{k=1}^{N} m_k \right)^2 + 2 \sum_{j<k}^{N} m_j m_k \left( \gamma_{\oplus (w \oplus v_j) \oplus (w \oplus v_k)} - 1 \right)}
\]

and where the sign of \( m_0 \) equals the sign of the left-hand side of (136).

1. If \( m_0^2 > 0 \) then \( m_0 \neq 0 \) is real and

\[
(142) \quad \gamma_{w \oplus v_0} = \frac{\sum_{k=1}^{N} m_k \gamma_{w \oplus v_k}}{m_0}
\]

is real, so that \( v_0 \) lies inside the ball \( \mathbb{R}^n_c \), \( v_0 \in \mathbb{R}^n \subset \mathbb{R}^n_c \).

2. If \( m_0^2 < 0 \) then \( m_0 \) is purely imaginary. Hence,

\[
(143) \quad \gamma_{w \oplus v_0} = \frac{\sum_{k=1}^{N} m_k \gamma_{w \oplus v_k}}{m_0}
\]

is purely imaginary, so that \( v_0 \in \mathbb{R}^n \) lies outside the closure of the ball \( \mathbb{R}^n_c \).

3. If \( m_0^2 = 0 \) then \( m_0 = 0 \), while \( m_0 \gamma_{w \oplus v_0} \neq 0 \). Hence,

\[
(144) \quad w \oplus v_0 = \frac{\sum_{k=1}^{N} m_k \gamma_{w \oplus v_k} (w \oplus v_k)}{\sum_{k=1}^{N} m_k \gamma_{w \oplus v_k}}
\]

and \( \gamma_{w \oplus v_0} = \infty \), so that \( w \oplus v_0 \), and hence \( v_0 \), lies on the boundary of the ball \( \mathbb{R}^n_c \), \( v_0 \in \partial \mathbb{R}^n_c \).

**Proof.** Identity (140) is a trivial consequence of (138) – (139) (but, it is presented in the Theorem for later convenience). Hence, Theorem 26 reduces to Lemma 24 when \( w = 0 \).

By Lemma 25 the condition and the equation in (136) – (137) are equivalent to the equation and the condition in (127) – (128).

Replacing (136) – (137) by their equivalent counterparts (127) – (128), Theorem 26 coincides with Lemma 24 in which \( v_k \in \mathbb{R}^n_c \) and \( v_0 \in \mathbb{R}^n \) are renamed as \( w \oplus v_k \in \mathbb{R}^n_c \) and \( w \oplus v_0 \in \mathbb{R}^n \). Lemma 24 therefore, completes the proof. \( \square \)
In physical applications to particle systems the dimension of $\mathbb{R}^n_c$ is $n = 3$, and the scalars $m_k$ in Theorem 26 represent particle masses. As such, $m_k$ are positive so that assumption (136) is satisfied. However, anticipating applications of Theorem 26 to barycentric coordinates in hyperbolic geometry, in Sect. 18, we need the validity of Theorem 26 for any natural number $N$, and for scalars $m_k$ that need not be positive.

We have thus established in Theorem 26 the following four results concerning an isolated system $S$, (107),

$$S = S(m_k, v_k, \Sigma_0, k = 1, \ldots, N),$$

of $N$ noninteracting material particles the $k$-th particle of which has invariant mass $m_k > 0$ and velocity $v_k \in \mathbb{R}^n_c$ relative to an inertial frame $\Sigma_0$, $k = 1, \ldots, N$:

1. The relativistically invariant (or, rest) mass $m_0$ of the system $S$ is given by

$$m_0 = \sqrt{\left(\sum_{k=1}^N m_k\right)^2 + 2 \sum_{j<k}^N m_j m_k (\gamma_{v_j} \gamma_{v_k} - 1)}$$

according to (141) with $w = 0$.

2. The relativistic mass of the system $S$ is

$$m_0 \gamma_{v_0}$$

relative to the rest frame $\Sigma_0$, 

(a) where $v_0$ is the velocity of the center of momentum frame of $S$ relative to $\Sigma_0$, given by

$$v_0 = \frac{\sum_{k=1}^N m_k \gamma_{v_k} v_k}{\sum_{k=1}^N m_k \gamma_{v_k}}$$

according to (138) with $w = 0$;

(b) where

$$\gamma_{v_0} = \frac{\sum_{k=1}^N m_k \gamma_{v_k}}{m_0}$$

according to (139) with $w = 0$; and

(c) where $m_0$ is given by (146).

3. Like energy and momentum, the relativistic mass is additive, that is, in particular for the system $S$ relative to the rest frame $\Sigma_0$, by (139) with $w = 0$,

$$m_0 \gamma_{v_0} = \sum_{k=1}^N m_k \gamma_{v_k}$$

4. The relativistic mass $m_0 \gamma_{v_0}$ of a system meshes well with the Minkowskian four-vector formalism of special relativity. In particular, for the system $S$
relative to the rest frame $\Sigma_0$, we have, by (137),

$$
\sum_{k=1}^{N} \left( \frac{m_k \gamma v_k}{m_k \gamma v_k} \right) = \left( \frac{m_0 \gamma v_0}{m_0 \gamma v_0} \right)
$$

where $m_0$ and $v_0$ are given uniquely by (146) and (148).

Thus, the relativistically invariant mass $m_0$ of a particle system $S$ in (146) gives rise to its associated relativistic mass $m_0 \gamma v_0$ relative to the rest frame $\Sigma_0$. The latter, in turn, brings in (151) the concept of the relativistic mass into conformity with the Minkowskian four-vector formalism of special relativity.

To appreciate the power and elegance of Theorem 26 in relativistic mechanics in terms of analogies that it shares with familiar results in classical mechanics, we present in Theorem 27 below the classical counterpart of Theorem 26. Theorem 27 is obtained from Theorem 26 by approaching the Newtonian (or, equivalently, Euclidean) limit when $c$ tends to infinity. The resulting Theorem 27 is immediate, and its importance in classical mechanics is well-known.

**Theorem 27. (Resultant Newtonian Invariant Mass Theorem).** Let $(\mathbb{R}^n, +)$ be a Euclidean $n$-space, and let $m_k \in \mathbb{R}$ and $v_k \in \mathbb{R}^n$, $k = 1, 2, \ldots, N$, be $N$ scalars and $N$ elements of $\mathbb{R}^n$ satisfying

$$
\sum_{k=1}^{N} m_k \neq 0
$$

Furthermore, let

$$
\sum_{k=1}^{N} m_k \left( \begin{array}{c} 1 \\ v_k \end{array} \right) = m_0 \left( \begin{array}{c} 1 \\ v_0 \end{array} \right)
$$

be an $(n+1)$-vector equation for the two unknowns $m_0 \in \mathbb{R}$ and $v_0 \in \mathbb{R}^n$.

Then (153) possesses a unique solution $(m_0, v_0)$, $m_0 \neq 0$, satisfying the following equations for all $w \in \mathbb{R}^n$ (including, in particular, the interesting special case of $w = 0$):

$$
w + v_0 = \frac{\sum_{k=1}^{N} m_k (w + v_k)}{\sum_{k=1}^{N} m_k}
$$

and

$$
m_0 = \sum_{k=1}^{N} m_k
$$

**Proof.** While a straightforward proof of Theorem 27 is trivial, our point is to present a proof that emphasizes how Theorem 27 is derived from Theorem 26. Indeed, in the limit as $c \to \infty$, the results of Theorem 26 tend to corresponding results of Theorem 27 noting that in this limit gamma factors tend to 1. In this sense, Theorem 27 is a special case of Theorem 26 corresponding to $c = \infty$. □
In physical applications to particle systems the dimension of \( \mathbb{R}^n \) is \( n = 3 \), and the scalars \( m_k \) in Theorem 27 represent particle masses and, hence, they are positive. However, anticipating applications of Theorem 27 to barycentric coordinates in Euclidean geometry, in Sect. 16 we need the validity of Theorem 27 for any natural number \( N \), and for scalars \( m_k \) that need not be positive.

Identity (154) of Theorem 27 is immediate. Yet, it is geometrically important. The geometric importance of the validity of (154) for all \( w \in \mathbb{R}^n \) lies on its implication that the velocity \( v_0 \) of the center of momentum frame of a particle system relative to a given inertial rest frame in classical mechanics is independent of the choice of the origin of the classical velocity space \( \mathbb{R}^n \) with its underlying standard Cartesian model of Euclidean geometry.

Unlike Identity (154) of Theorem 27 which is immediate, its hyperbolic counterpart in Theorem 26 Identity (138), is not immediate. Yet, in full analogy with Theorem 27 the validity of Identity (138) in Theorem 26 for all \( w \in \mathbb{R}^c_n \) is geometrically important. This geometric importance of Identity (138) lies on its implication that the velocity \( v_0 \) of the center of momentum frame of a particle system relative to a given inertial rest frame in relativistic mechanics is independent of the choice of the origin of the relativistic velocity space \( \mathbb{R}^c_n \) with its underlying Cartesian-Beltrami-Klein ball model of hyperbolic geometry.

16. Barycentric Coordinates

The use of barycentric coordinates in Euclidean geometry, dates back to Möbius, is described, for instance, in [47, 16], and the historical contribution of Möbius’ barycentric coordinates to vector analysis is described in [5, pp. 48–50]. In this section we set the stage for the adaptation in Sect. 18 of barycentric coordinates for use in hyperbolic geometry by illustrating the way Theorem 27 suggests the introduction of barycentric coordinates as a mathematical tool in Euclidean geometry.

For any positive integer \( N \), let \( m_k \in \mathbb{R}, \ k = 1, \ldots, N \), be \( N \) given scalars such that

\[
(156) \quad \sum_{k=1}^{N} m_k \neq 0
\]

and let \( A_k \in \mathbb{R}^n \) be \( N \) given points in the Euclidean \( n \)-space \( \mathbb{R}^n \), \( k = 1, \ldots, N \). Theorem 27 states the trivial, but geometrically significant, result that the equation

\[
(157) \quad \sum_{k=1}^{N} m_k \begin{pmatrix} 1 \\ A_k \end{pmatrix} = m_0 \begin{pmatrix} 1 \\ P \end{pmatrix}
\]

for the unknowns \( m_0 \in \mathbb{R} \) and \( P \in \mathbb{R}^n \) possesses the unique solution given by

\[
(158) \quad m_0 = \sum_{k=1}^{N} m_k
\]
and

(159) \[ P = \sum_{k=1}^{N} \frac{m_k A_k}{\sum_{k=1}^{N} m_k}, \]
satisfying for all \( X \in \mathbb{R}^n \),

(160) \[ X + P = \sum_{k=1}^{N} \frac{m_k (X + A_k)}{\sum_{k=1}^{N} m_k}. \]

We view (159) as the representation of a point \( P \in \mathbb{R}^n \) in terms of its barycentric coordinates \( m_k, k = 1, \ldots, N \), with respect to the set of points \( S = \{A_1, \ldots, A_N\} \).

Identity (160), then, implies that the barycentric coordinate representation (159) of \( P \) with respect to the set \( S \) is covariant (or, invariant in form) in the following sense.

The point \( P \) and the points of the set \( S \) of its barycentric coordinate representation vary together under translations. Indeed, a translation \( X + A_k \) of each \( A_k, k = 1, \ldots, N \), in (160) results in the translation \( X + P \) of \( P \) by \( X \).

In order to insure that barycentric coordinate representations with respect to a set \( S \) are unique, we require \( S \) to be barycentrically independent, as defined below.

**Definition 28. (Barycentric Independence, Flats).** A set \( S \) of \( N \) points \( S = \{A_1, \ldots, A_N\} \) in \( \mathbb{R}^n, n \geq 2 \), is barycentrically independent if the \( N-1 \) vectors \(-A_1 + A_k, k = 2, \ldots, N\), are linearly independent. The \((N-1)\)-dimensional subspace \( L \) of \( \mathbb{R}^n \) spanned by the \( N-1 \) linearly independent vectors \(-A_1 + A_k\) is denoted by

(161) \[ L = \text{Span}\{-A_1 + A_2, \ldots, -A_1 + A_N\}. \]

A translate, \( A + L \), of \( L \) by \( A \in \mathbb{R}^n \) is the set of all points \( A + X \) where \( X \in L \), called an \((N-1)\)-dimensional flat, or simply \((N-1)\)-flat in \( \mathbb{R}^n, n \geq N \). Flats of dimension 1, 2, and \( n-1 \) are also called lines, planes, and hyperplanes, respectively.

The \((N-1)\)-flat \( A_{N,k}^{\text{euc}} \),

(162) \[ A_{N,k}^{\text{euc}} = A_k + \text{Span}\{-A_k + A_1, -A_k + A_2, \ldots, -A_k + A_N\} \subset \mathbb{R}^n, \]

for any \( 1 \leq k \leq N \), associated with a barycentrically independent set \( S = \{A_1, \ldots, A_N\} \) in \( \mathbb{R}^n \), proves useful in the study of barycentric coordinates. Note that one of the vectors \(-A_k + A_i, 1 \leq i \leq N \), in (162) vanishes.

We are now in the position to present the formal definition of Euclidean barycentric coordinates, as suggested by Theorem 27, p. 36.

**Definition 29. (Barycentric Coordinates).** Let

(163) \[ S = \{A_1, \ldots, A_N\} \]

be a barycentrically independent set of \( N \) points in a Euclidean \( n \)-space \( \mathbb{R}^n \). The scalars \( m_k, k = 1, \ldots, N \), satisfying

(164) \[ \sum_{k=1}^{N} m_k \neq 0, \]
are barycentric coordinates of a point \( P \in \mathbb{R}^n \) with respect to the set \( S \) if
\[
P = \frac{\sum_{k=1}^{N} m_k A_k}{\sum_{k=1}^{N} m_k}.
\]

Barycentric coordinates are homogeneous in the sense that the barycentric coordinates \((m_1, \ldots, m_N)\) of the point \( P \) in (165) are equivalent to the barycentric coordinates \((\lambda m_1, \ldots, \lambda m_N)\) for any nonzero scalar \( \lambda \in \mathbb{R}, \lambda \neq 0 \). Since in barycentric coordinates only ratios of coordinates are relevant, the barycentric coordinates \((m_1, \ldots, m_N)\) are also written as \((m_1 : \ldots : m_N)\).

Barycentric coordinates that are normalized by the condition
\[
\sum_{k=1}^{N} m_k = 1
\]
are called special barycentric coordinates.

The point \( P \) in (165) is said to be a barycentric combination of the points of the set \( S \), possessing the barycentric coordinate representation (barycentric representation, in short) (165) with respect to \( S \).

The barycentric combination (165) is positive (non-negative) if all the coefficients \( m_k, k = 1, \ldots, N \), are positive (non-negative). The set of all positive (non-negative) barycentric combinations of the points of the set \( S \) is called the convex span (convex hull) of \( S \).

The constant
\[
m_P = \sum_{k=1}^{N} m_k
\]
is called the constant of the barycentric representation of \( P \) with respect to the set \( S \).

The hyperbolic counterpart (205), p. 46, of the representation constant \( m_P \) in (167) proves crucially important in the adaptation of barycentric coordinates and convexity considerations for use in hyperbolic geometry. Convexity considerations are, for instance, important in quantum mechanics where mixed states are positive barycentric combinations of pure states [2].

**Definition 30. (Simplex).** The convex hull of the barycentrically independent set \( S = \{A_1, \ldots, A_N\} \) of \( N \geq 2 \) points in \( \mathbb{R}^n \) is an \((N-1)\)-dimensional simplex, called an \((N-1)\)-simplex and denoted \( A_1 \ldots A_N \). The points of \( S \) are the vertices of the simplex. The convex hull of \( N - 1 \) of the points of \( S \) is a face of the simplex, said to be the face opposite to the remaining vertex. The convex hull of each two of the vertices is an edge of the simplex.

For \( K < N \), a \((K-1)\)-subsimplex, or a \((K-1)\)-face of an \((N-1)\)-simplex, is a \((K-1)\)-simplex whose vertices form a subset of the vertices of the \((N-1)\)-simplex.
The convex span of the set $S = \{A_1, \ldots, A_N\}$ in Def. 30 is thus the interior of the $(N - 1)$-simplex $A_1 \ldots A_N$.

**Theorem 31. (Barycentric Representation Covariance).** Let

\[
P = \frac{\sum_{k=1}^{N} m_k A_k}{\sum_{k=1}^{N} m_k}
\]

be the barycentric representation of a point $P \in \mathbb{R}^n$ in a Euclidean $n$-space $\mathbb{R}^n$ with respect to a barycentrically independent set $S = \{A_1, \ldots, A_N\} \subset \mathbb{R}^n$. The barycentric representation \[168\] is covariant, that is,

\[
X + P = \frac{\sum_{k=1}^{N} m_k (X + A_k)}{\sum_{k=1}^{N} m_k}
\]

for all $X \in \mathbb{R}^n$, and

\[
RP = \frac{\sum_{k=1}^{N} m_k R A_k}{\sum_{k=1}^{N} m_k}
\]

for all $R \in SO(n)$.

**Proof.** The proof is immediate, noting that addition of vectors in $\mathbb{R}^n$ distributes over scalar multiplication, and that rotations $R \in SO(n)$ of $\mathbb{R}^n$ about its origin are linear maps of $\mathbb{R}^n$. \qed

**Theorem 32. (Barycentric Representation Existence).** Let $S = \{A_1, \ldots, A_N\}$ be a barycentrically independent set of $N$ points in a Euclidean $n$-space $\mathbb{R}^n$, $n \geq N - 1$. Then, $P \in \mathbb{R}^n$ possesses a barycentric representation

\[
P = \frac{\sum_{k=1}^{N} m_k A_k}{\sum_{k=1}^{N} m_k}
\]

with respect to $S$, with homogeneous barycentric coordinates $m_k \in \mathbb{R}$, $k = 1, 2, \ldots, N$, that satisfy the condition

\[
\sum_{k=1}^{N} m_k \neq 0
\]

if and only if

\[
P \in \mathbb{A}_{N,1}^{euc}.
\]

**Proof.** Assuming \[173\], we have

\[
-A_1 + P \in \text{Span}\{-A_1 + A_2, \ldots, -A_1 + A_N\},
\]

so that there exist scalars $m_k \in \mathbb{R}$, $k = 2, \ldots, N$, such that

\[
-A_1 + P = m_2(-A_1 + A_2) + \ldots m_N(-A_1 + A_N)
\]

\[
= m_1(-A_1 + A_1) + m_2(-A_1 + A_2) + \ldots m_N(-A_1 + A_N)
\]
for any scalar \( m_1 \in \mathbb{R} \), where \( m_k, k = 2, \ldots, N \), are determined uniquely by the vector \(-A_1 + P\).

We now select the special scalar \( m_1 \) that is uniquely determined by the normalization condition

\[
\sum_{k=1}^{N} m_k = 1,
\]

that is,

\[
m_1 = 1 - \sum_{k=2}^{N} m_k.
\]

Then, (175) can be written as

\[
-A_1 + P = \frac{m_1(-A_1 + A_1) + m_2(-A_1 + A_2) + \ldots + m_N(-A_1 + A_N)}{m_1 + m_2 + \ldots + m_N},
\]

implying, by the barycentric representation covariance (168) – (169),

\[
P = \frac{m_1A_1 + m_2A_2 + \ldots + m_NA_N}{m_1 + m_2 + \ldots + m_N}.
\]

Owing to the homogeneity of the coordinates \( m_k \) in (175), the representation of \( P \) in (179) remains valid if we replace the normalization condition (176) by the weaker condition (172), thus obtaining the desired barycentric representation (171) – (172) of \( P \).

Conversely, assuming (171) – (172), we have, by Result (169) of Theorem 31 with \( X = -A_1 \),

\[
-A_1 + P = \frac{\sum_{k=1}^{N} m_k(-A_1 + A_k)}{\sum_{k=1}^{N} m_k},
\]

implying (173), as desired. \( \square \)

**Lemma 33.** Let \( S = \{A_1, A_2, \ldots, A_N\} \), \( N \geq 2 \), be a barycentrically independent set of \( N \) points in a Euclidean space \( \mathbb{R}^n \), \( n \geq N - 1 \), and let \( \mathcal{A}_{N,k}^{\text{euc}} \) be the \((N-1)\)-flat (162) associated with \( S \), for each \( k, 1 \leq k \leq N \). Then, \( \mathcal{A}_{N,k}^{\text{euc}} \) is independent of \( k \).

**Proof.** Let \( k_1 \) and \( k_2 \) be two distinct integers, \( 1 \leq k_1, k_2 \leq N \), and let \( P \in \mathcal{A}_{N,k_1}^{\text{euc}} \). Then, by the Barycentric Representation Existence Theorem 32, \( P \) possesses a barycentric representation

\[
P = \frac{\sum_{k=1}^{N} m_k A_k}{\sum_{k=1}^{N} m_k},
\]

\( \sum_{k=1}^{N} m_k \neq 0 \).
Applying the Barycentric Representation Covariance Theorem \[31\] p. \[103\] with \(X = -A_{k_2}\) to \[(181)\], we obtain the equation

\[
(182) \\
- A_{k_2} + P = \sum_{k=1}^{N} m_k(-A_{k_2} + A_k) / \sum_{k=1}^{N} m_k.
\]

Hence,

\[
(183) \\
- A_{k_2} + P \in \text{Span}\{-A_{k_2} + A_1, \ldots, -A_{k_2} + A_N\} \subset \mathbb{R}^n,
\]

so that \(P \in \mathcal{A}_{euc}^{k_1, k_2}\). Hence, \(\mathcal{A}_{euc}^{k_1, k_2} \subset \mathcal{A}_{euc}^{k_1, k_2}\). The proof of the reverse inclusion is similar (just interchanging \(k_1\) and \(k_2\)), so that \(\mathcal{A}_{euc}^{k_1, k_2} = \mathcal{A}_{euc}^{k_1, k_2}\), as desired. \(\square\)

Following the vision of Felix Klein in his Erlangen Program \[26\], it is owing to the covariance with respect to translations and rotations that barycentric representations possess geometric significance. Indeed, translations and rotations in Euclidean geometry form the group of motions of the geometry, as explained in Sect. \[10\] and according to Felix Klein’s Erlangen Program, a geometric property is a property that remains invariant in form under the motions of the geometry.

### 17. Segments

A study of the Euclidean segment is presented here as an example that illustrates a simple, common use of barycentric coordinates. The purpose of this simple example is to set the stage for its hyperbolic counterpart in Sect. \[24\] which is far away from being simple.

Let \(A_1, A_2 \in \mathbb{R}^2\) be two distinct points of the Euclidean plane \(\mathbb{R}^2\), and let \(P \in \mathcal{A}_{euc}^{2,1}\), where \(\mathcal{A}_{euc}^{2,1}\) is the 1-flat (line)

\[
(184) \\
\mathcal{A}_{euc}^{2,1} = A_1 + \text{Span}\{-A_1 + A_1, -A_1 + A_2\} = A_1 + \text{Span}\{-A_1 + A_2\} \subset \mathbb{R}^2,
\]

so that \(P\) is a point on the line that passes through the points \(A_1\) and \(A_2\). Then, by Theorem \[32\] \(P\) possesses a barycentric representation

\[
(185) \\
P = \frac{m_1 A_1 + m_2 A_2}{m_1 + m_2}
\]

with respect to the barycentrically independent set \(S = \{A_1, A_2\}\), with barycentric coordinates \(m_1\) and \(m_2\) satisfying \(m_1 + m_2 \neq 0\). In particular:

1. If \(m_1 = 0\), then \(P = A_2\).
2. If \(m_2 = 0\), then \(P = A_1\).
3. If \(m_1, m_2 > 0\), or \(m_1, m_2 < 0\), then \(P\) lies on the interior of segment \(A_1 A_2\), that is, between \(A_1\) and \(A_2\).
4. If \(m_1\) and \(m_2\) are nonzero and have opposite signs, then \(P\) lies on the exterior of segment \(A_1 A_2\).
Owing to the homogeneity of barycentric coordinates, these can be normalized by the condition

\[(186) \quad m_1 + m_2 = 1,\]

so that, for instance, we can parametrize \(m_1\) and \(m_2\) by a parameter \(t\) according to the equations \(m_1 = t\) and \(m_2 = 1 - t, 0 \leq t \leq 1.\) Then, the point \(P\) possesses the special parametric barycentric representation

\[(187) \quad P = tA_1 + (1 - t)A_2.\]

Owing to the covariance of barycentric representations with respect to translations, the barycentric representation \((187)\) of \(P\) obeys the identity

\[(188) \quad X + P = t(X + A_1) + (1 - t)(X + A_2)\]

for all \(X \in \mathbb{R}^2.\) The derivation of Identity \((188)\) from \((187)\) is trivial. However, Identity \((188)\) serves as an illustration of its hyperbolic counterpart in \((239), p. 56,\) which is far away from being trivial.

18. Gyrobarycentric Coordinates

Guided by analogies with Sect. 16 in this section we introduce barycentric coordinates into hyperbolic geometry \([40, 42, 43]\), where they are called gyrobarycentric coordinates. For any positive integer \(N,\) let \(m_k \in \mathbb{R}\) be \(N\) given scalars, and let \(A_k \in \mathbb{R}_n^s\) be \(N\) given points in an Einstein gyrovector space \((\mathbb{R}_n^s, \oplus, \otimes), k = 1, \ldots, N,\) satisfying

\[(189) \quad \sum_{k=1}^{N} m_k \gamma A_k \neq 0.\]

According to Theorem 26 p. 33 the equation

\[(190) \quad \sum_{k=1}^{N} m_k \begin{pmatrix} \gamma A_k \\ \gamma A_k A_k \end{pmatrix} = m_0 \begin{pmatrix} \gamma P \\ \gamma P P \end{pmatrix}\]

for the unknowns \(m_0 \in \mathbb{R}\) and \(P \in \mathbb{R}^n\) possesses the unique solution \((m_0, P)\) given by

\[(191a) \quad m_0 = \sqrt{\left(\sum_{k=1}^{N} m_k \right)^2 + 2 \sum_{j,k=1 \atop j < k}^{N} m_j m_k (\gamma \otimes A_j \oplus A_k - 1)}\]

or, equivalently,

\[(191b) \quad m_0 = \sqrt{\sum_{k=1}^{N} m_k^2 + 2 \sum_{j,k=1 \atop j < k}^{N} m_j m_k \gamma \otimes A_j \oplus A_k},\]
\( m_0 \neq 0 \), obeying the left gyrotranslation invariance condition

\[
(192) \quad m_0 = \sqrt{\left(\sum_{k=1}^{N} m_k\right)^2 + 2 \sum_{j,k=1}^{N} m_j m_k \left(\gamma_{\oplus (X \oplus A_j) \oplus (X \oplus A_k)} - 1\right)}
\]

for all \( X \in \mathbb{R}_s^n \), and

\[
(193) \quad P = \frac{\sum_{k=1}^{N} m_k \gamma_{A_k} A_k}{\sum_{k=1}^{N} m_k \gamma_{A_k}},
\]

obeying the left gyrotranslation covariance condition

\[
(194) \quad X \oplus P = \frac{\sum_{k=1}^{N} m_k \gamma_{X \oplus A_k} (X \oplus A_k)}{\sum_{k=1}^{N} m_k \gamma_{X \oplus A_k}}
\]

for all \( X \in \mathbb{R}_s^n \).

**Remark 34.** We may remark that Equation (191a) for \( m_0 \) is preferable over (191b) when we wish to emphasize that we are guided by analogies that (i) relativistic mechanics and its regulating hyperbolic geometry share with (ii) classical mechanics and its regulating Euclidean geometry. It is clear from (191a) that in the Euclidean-Newtonian limit, \( s \to \infty \), gamma factors tend to 1, so that (191a) tends to (158). In applications, however, Equation (191b) for \( m_0 \) is preferable over (191a) for its simplicity.

Furthermore, Theorem 26, p. 33 states that \( P \) and \( m_0 \) satisfy the two identities

\[
(195) \quad \gamma_P = \frac{\sum_{k=1}^{N} m_k \gamma_{A_k}}{m_0}
\]

and

\[
(196) \quad \gamma_P P = \frac{\sum_{k=1}^{N} m_k \gamma_{A_k} A_k}{m_0},
\]

obeying the left gyrotranslation covariance condition

\[
(197) \quad \gamma_{X \oplus P} = \frac{\sum_{k=1}^{N} m_k \gamma_{X \oplus A_k}}{m_0}
\]

and

\[
(198) \quad \gamma_{X \oplus P}(X \oplus P) = \frac{\sum_{k=1}^{N} m_k \gamma_{X \oplus A_k} (X \oplus A_k)}{m_0}
\]

for all \( X \in \mathbb{R}_s^n \).

We view (193) as the representation of a point \( P \in \mathbb{R}_s^n \) in terms of its hyperbolic barycentric coordinates \( m_k, k = 1, \ldots, N \), with respect to the set of points \( S = \{A_1, \ldots, A_N\} \). Naturally in gyrolanguage, hyperbolic barycentric coordinates are called gyrobarcentric coordinates. Identity (194) implies that the gyrobarcentric coordinate representation (193) of \( P \) with respect to the set \( S \) is gyrocovariant with respect to left gyrotranslations in the sense of Def. 19, p. 22 as stated in
Theorem 41, p. 48. The point $P$ and the points of the set $S$ of its gyrobarycentric coordinate representation vary together under left gyrotranslations. Indeed, a left gyrotranslation $X \oplus A_k$ of each $A_k$ by $X$, $k = 1, \ldots, N$ in (194) results in the left gyrotranslation $X \oplus P$ of $P$ by $X$.

In order to insure that gyrobarycentric coordinate representations with respect to a set $S$ are unique, we require $S$ to be gyrobarycentrically independent, as defined below.

Definition 35. (Gyrobarycentric Independence, Gyroflats). A set $S$ of $N$ points $S = \{A_1, \ldots, A_N\}$ in $\mathbb{R}^n$, $n \geq 2$, is gyrobarycentrically independent if the $N-1$ gyrovectors in $\mathbb{R}^n_s$, $\odot A_1 \oplus A_k$, $k = 2, \ldots, N$, considered as vectors in $\mathbb{R}^n$, are linearly independent in $\mathbb{R}^n$. The $(N-1)$-dimensional subspace $L$ of $\mathbb{R}^n$ spanned by the $N-1$ gyrovectors $\odot A_1 \oplus A_k \in \mathbb{R}^n_s \subset \mathbb{R}^n$, considered as vectors in $\mathbb{R}^n$, is denoted by

\[ L = \text{Span}\{\odot A_1 \oplus A_2, \ldots, \odot A_1 \oplus A_N\}. \]

A left gyrotranslate, $A \oplus L$, of $L$ by $A \in \mathbb{R}^n_s$ is the set of all points $A \oplus X$ where $X \in L$, called an $(N-1)$-dimensional gyroflat, or simply $(N-1)$-gyroflat in $\mathbb{R}^n$, $n \geq N$. Gyroflats of dimension 1, 2, and $n-1$, restricted to $\mathbb{R}^n \cap \mathbb{R}^n_s$, are also called gyrolines, gyroplanes, and hypergyroplanes, respectively.

The $(N-1)$-gyroflat $A_{N,k}$,

\[ A_{N,k} = A_k \oplus \text{Span}\{\odot A_k \oplus A_1, \odot A_k \oplus A_2, \ldots, \odot A_k \oplus A_N\} \subset \mathbb{R}^n, \]

for any $1 \leq k \leq N$, associated with a gyrobarycentrically independent set $S = \{A_1, \ldots, A_N\}$ in $\mathbb{R}^n$, proves useful in the study of gyrobarycentric coordinates. Note that one of the gyrovectors $\odot A_k \oplus A_i$, $1 \leq i \leq N$, in (200) vanishes.

We are now in the position to present the formal definition of gyrobarycentric coordinates, that is, hyperbolic barycentric coordinates, as suggested by Theorem 26, p. 33.

Definition 36. (Gyrobarycentric Coordinates). Let

\[ S = \{A_1, \ldots, A_N\} \]

be a gyrobarycentrically independent set of $N$ points in an Einstein gyrovector space $\mathbb{R}^n_s = (\mathbb{R}^n_s, \odot, \oplus)$, $n \geq N-1$, The scalars $m_1, \ldots, m_N$, satisfying

\[ \sum_{k=1}^{N} m_k \gamma_{A_k} \neq 0, \]

are gyrobarycentric coordinates of a point $P \in \mathbb{R}^n$ with respect to the set $S$ if

\[ P = \frac{\sum_{k=1}^{N} m_k \gamma_{A_k} A_k}{\sum_{k=1}^{N} m_k \gamma_{A_k}} \in \mathbb{R}^n. \]
Gyrobarycentric coordinates are homogeneous in the sense that the gyrobarycentric coordinates \((m_1, \ldots, m_N)\) of the point \(P\) in (203) are equivalent to the gyrobarycentric coordinates \((\lambda m_1, \ldots, \lambda m_N)\) for any nonzero scalar \(\lambda \in \mathbb{R}, \lambda \neq 0\). Since in gyrobarycentric coordinates only ratios of coordinates are relevant, the gyrobarycentric coordinates \((m_1, \ldots, m_N)\) are also written as \((m_1 : \ldots : m_N)\).

Gyrobarycentric coordinates that are normalized by the condition

\[(204) \quad \sum_{k=1}^{N} m_k = 1\]

are called special gyrobarycentric coordinates.

The point \(P\) in (203) is said to be the gyrobarycentric combination of the points of the set \(S\), possessing the gyrobarycentric coordinate representation (gyrobarycentric representation, in short) (203) with respect to the set \(S\).

The gyrobarycentric combination (or, representation) (203) is positive (non-negative) if all the coefficients \(m_k, k = 1, \ldots, N\), are positive (non-negative). The set of all positive (non-negative) gyrobarycentric combinations of the points of the set \(S\) is called the gyroconvex span (gyroconvex hull) of \(S\).

The constant \(m_P\), given by

\[(205) \quad m_P = \left( \sum_{k=1}^{N} m_k \right)^2 + 2 \sum_{j,k=1 \atop j<k}^{N} m_j m_k (\gamma_\ominus A_j \oplus A_k - 1),\]

is called the constant of the gyrobarycentric representation (203) of \(P\) with respect to the set \(S\).

In the Euclidean-Newtonian limit \(s \to \infty\), gamma factors tend to 1 and the \(s\)-ball \(\mathbb{R}^n_s\) expands to the whole of its space, \(\mathbb{R}^n\). Hence, in that limit Def. 36 of gyrobarycentric coordinates reduces to Def. 29 of barycentric coordinates.

**Remark 37.** Gyrobarycentric representation constants will prove useful. Owing to the homogeneity of gyrobarycentric coordinates, the value of the constant \(m_P\) in (205) of the gyrobarycentric representation (203) of \(P\) has no significance. Significantly, however, is whether (i) \(m_P^2\) is positive (implying that \(m_P\) is a nonzero real number), (ii) \(m_P^2\) is zero (implying \(m_P = 0\)), and (iii) \(m_P^2\) is negative (implying that \(m_P\) is purely imaginary). Also significant are ratios like \(m_k/m_P\), etc.

**Remark 38.** It should be noted that while the point \(P\) is gyrobarycentrically represented in (203) with respect to a set \(S \subset \mathbb{R}^n_s\) of points in \(\mathbb{R}^n_s\), in general \(P\) lies in \(\mathbb{R}^n \supset \mathbb{R}^n_s\). Hence, it is important to associate a gyrobarycentric representation of a point \(P\) with respect to a set \(S \subset \mathbb{R}^n\) with the constant \(m_P\) of the gyrobarycentric representation. Indeed, as we see from Corollary 43, p. 49, it is the gyrobarycentric representation constant \(m_P\) that determines whether the point \(P\) lies inside the
s-ball $\mathbb{R}^n_s$, or on the boundary $\partial\mathbb{R}^n_s$ of the ball, or does not lie in the closure $\overline{\mathbb{R}^n_s}$ of the ball.

The concept of the gyroconvex hull in Def. 36 enables the concept of the Euclidean simplex in Def. 30, p. 39, to be translated into a corresponding concept of the Einsteinian gyrosimplex in the following definition.

**Definition 39. (Gyrosimplex).** The gyroconvex hull of a gyrobarycentrically independent set $S = \{A_1, \ldots, A_N\}$ of $N \geq 2$ points in $\mathbb{R}^n_s$ is an $(N - 1)$-dimensional gyrosimplex, called an $(N - 1)$-gyrosimplex and denoted by $A_1 \ldots A_N$. The points of $S$ are the vertices of the gyrosimplex. The gyroconvex hull of $N - 1$ of the points of $S$ is a gyroface of the gyrosimplex, said to be the gyroface opposite to the remaining vertex. The gyroconvex hull of each two of the vertices is a gyroedge of the gyrosimplex.

For $K < N$, a $(K - 1)$-subgyrosimplex, or a $(K - 1)$-gyroface of an $(N - 1)$-gyrosimplex, is a $(K - 1)$-gyrosimplex whose vertices form a subset of the vertices of the $(N - 1)$-gyrosimplex.

The gyroconvex span of the set $S = \{A_1, \ldots, A_N\}$ in Def. 39 is thus the interior of the $(N - 1)$-gyrosimplex $A_1 \ldots A_N$.

Any two distinct points $A_1, A_2$ of an Einstein gyrovector space $\mathbb{R}^n_s$ are gyrobarycentrically independent, and their gyroconvex span is the interior of the gyrosegment $A_1A_2$, which is a 1-gyrosimplex. Similarly, any three non-gyrocollinear points (that is, points that do not lie on the same gyroline; see [37, Remark 6.23] for this terminology) $A_1, A_2, A_3$ of $\mathbb{R}^n_s$, $n \geq 2$, are gyrobarycentrically independent, and their gyroconvex span is the interior of the gyrotriangle $A_1A_2A_3$, which is a 2-gyrosimplex. An illustrative example follows.

**Example 40.** Low $N$-dimensional gyrosimplices, $1 \leq N \leq 4$, are:

1. A 0-dimensional gyrosimplex is a point $A_1$ in an Einstein gyrovector space $(\mathbb{R}^n_s, \oplus, \otimes)$, $n \geq 1$.
2. A 1-dimensional gyrosimplex is a gyrosegment $A_1A_2$ the 2 vertices of which form the gyrobarycentrically independent set $S = \{A_1, A_2\}$ in an Einstein gyrovector space $(\mathbb{R}^n_s, \oplus, \otimes)$, $n \geq 1$.
3. A 2-dimensional gyrosimplex is a gyrotriangle $A_1A_2A_3$ the 3 vertices of which form the gyrobarycentrically independent set $S = \{A_1, A_2, A_3\}$ in an Einstein gyrovector space $(\mathbb{R}^n_s, \oplus, \otimes)$, $n \geq 2$.
4. A 3-dimensional gyrosimplex is a gyrotetrahedron $A_1A_2A_3A_4$ the 4 vertices of which form the gyrobarycentrically independent set $S = \{A_1, A_2, A_3, A_4\}$ in an Einstein gyrovector space $(\mathbb{R}^n_s, \oplus, \otimes)$, $n \geq 3$.
5. Generally, an $(N - 1)$-dimensional gyrosimplex, $N \geq 2$, is a geometric object denoted by $A_1 \ldots A_N$, the $N$ vertices of which form the gyrobarycentrically independent set $S = \{A_1, \ldots, A_N\}$ in an Einstein gyrovector space $(\mathbb{R}^n_s, \oplus, \otimes)$, $n \geq N - 1$. 
Theorem 41. (Gyrobarcentric Representation Gyrocovariance).

Let \( S = \{A_1, \ldots, A_N\} \) be a gyrobarcentrically independent set of \( N \) points in an Einstein gyrovector space \((\mathbb{R}^n_s, \oplus, \otimes)\), \( n \geq N - 1 \), and let \( P \in \mathbb{R}^n \) be a point that possesses the gyrobarcentric representation

\[
P = \frac{\sum_{k=1}^{N} m_k \gamma A_k A_k}{\sum_{k=1}^{N} m_k \gamma A_k}
\]

with respect to \( S \).

Then

\[
\gamma_P = \frac{\sum_{k=1}^{N} m_k \gamma A_k}{m_P}
\]

and

\[
\gamma_P P = \frac{\sum_{k=1}^{N} m_k \gamma A_k A_k}{m_P},
\]

where the constant \( m_P \) of the gyrobarcentric representation (206a) of \( P \) is given by

\[
m_P = \sqrt{\left( \sum_{k=1}^{N} m_k \right)^2 + 2 \sum_{j,k=1}^{N} m_j m_k (\gamma \otimes A_j \otimes A_k - 1)}.
\]

Furthermore, the gyrobarcentric representation (206a) and its associated identities in (206b) – (206d) are gyrocovariant, that is,

\[
X \oplus P = \frac{\sum_{k=1}^{N} m_k \gamma X \otimes A_k (X \oplus A_k)}{\sum_{k=1}^{N} m_k \gamma X \otimes A_k}
\]

(207a)

\[
\gamma_{X \oplus P} = \frac{\sum_{k=1}^{N} m_k \gamma X \otimes A_k}{m_P}
\]

(207b)

\[
\gamma_{X \oplus P} (X \oplus P) = \frac{\sum_{k=1}^{N} m_k \gamma X \otimes A_k (X \oplus A_k)}{m_P}
\]

(207c)

\[
m_P = \sqrt{\left( \sum_{k=1}^{N} m_k \right)^2 + 2 \sum_{j,k=1}^{N} m_j m_k (\gamma \otimes (X \otimes A_j) \otimes (X \otimes A_k) - 1)}.
\]

(207d)

for all \( X \in \mathbb{R}^n_s \), and

\[
RP = \frac{\sum_{k=1}^{N} m_k \gamma R A_k}{\sum_{k=1}^{N} m_k \gamma R A_k}
\]

(208a)
\( \gamma_{RP} = \frac{\sum_{k=1}^{N} m_k \gamma_{RA_k}}{m_P} \)  
\[ \text{(208b)} \]

\[ \gamma_{RP}(RP) = \frac{\sum_{k=1}^{N} m_k \gamma_{RA_k}(RA_k)}{m_P} \]  
\[ \text{(208c)} \]

\[ m_P = \sqrt{\left( \sum_{k=1}^{N} m_k \right)^2 + 2 \sum_{j,k=1}^{N} m_j m_k (\gamma_{\ominus(RA_j)\ominus(RA_k)} - 1)} \]  
\[ \text{(208d)} \]

for all \( R \in SO(n) \).

**Proof.** The pair \((m_P, P)\) is a solution of the \((n + 1)\)-vector equation

\[ \sum_{k=1}^{N} m_k \left( \begin{array}{c} \gamma_{A_k} \\ \gamma_{A_k} A_k \end{array} \right) = m_P \left( \begin{array}{c} \gamma_P \\ \gamma_P P \end{array} \right) \]

as we see from Theorem 26, p. 33.

Hence, by Theorem 26 with \( w = 0 \), \( \gamma_P \) and \( \gamma_P P \) are given by (206b) – (206c). Furthermore, by Theorem 26 the pair \((m_P, P)\), and \( \gamma_P \) and \( \gamma_P P \) are gyrocovariant under left gyrotranslations, thus proving (207a) – (207d).

Finally, the proof of (208a) – (208d) follows immediately from the linearity and gyrolinearity of \( R \in SO(n) \), noting that \( R \) preserves the norm; see (61), p. 16 and (86), p. 22.

\[ \square \]

**Remark 42.** Gyrocovariance of a real or a purely imaginary number means that the number is invariant under gyromotions, that is, under left gyrotranslations and rotations. Hence, in particular, the gyrocovariance of gamma factors, like \( \gamma_P \), and representation constants, like \( m_P \), in Theorem 41, as well as any gyrobarycentric coordinate \( m_k \), means that each of these is invariant under gyromotions.

The unique solution of (137) that Theorem 26, p. 33, provides, implies immediately the following corollary about gyrobarycentric representations.

**Corollary 43.** Let \( P \in \mathbb{R}^n \) be a point that possesses the gyrobarycentric representation

\[ P = \frac{\sum_{k=1}^{N} m_k \gamma_{A_k} A_k}{\sum_{k=1}^{N} m_k \gamma_{A_k}} \]

with respect to a gyrobarycentrically independent set \( S = \{A_1, \ldots, A_N\} \subset \mathbb{R}_s^n \subset \mathbb{R}^n \) in an Einstein gyrovector space \((\mathbb{R}_s^n, \oplus, \otimes)\). Then, either

1. \( P \) lies in \( \mathbb{R}_s^n \), or
2. \( P \) lies on the boundary \( \partial \mathbb{R}_s^n \) of \( \mathbb{R}_s^n \), or
3. \( P \) does not lie in the closure \( \overline{\mathbb{R}_s^n} \) of \( \mathbb{R}_s^n \) or, equivalently, \( P \) lies beyond \( \overline{\mathbb{R}_s^n} \),

if and only if, respectively, either
(1) $\gamma_P$ is real, or
(2) $\gamma_P = \infty$, or
(3) $\gamma_P$ is purely imaginary,
or, equivalently, if and only if, respectively, either

(1) $m_P^2 > 0$ (so that without loss of generality we can select $m_P > 0$), or
(2) $m_P^2 = 0$ (so that $m_P = 0$), or
(3) $m_P^2 < 0$ (so that $m_P$ is purely imaginary).

Proof. The proof of the Corollary follows immediately from the definition of gamma factors and from Theorem 41. □

Additionally, the point $P$ in Corollary 43 lies in the interior of gyrosimplex $A_1 \ldots A_N$ if and only if the gyrobarycentric coordinates of $P$ are all positive or all negative.

Theorem 44. (Gyrobarycentric Representation Existence). Let $S = \{A_1, \ldots, A_N\}$ be a gyrobarycentrically independent set of $N$ points in an Einstein gyrovector space $(\mathbb{R}_s^n, \oplus, \otimes)$, $n \geq N - 1$. Then, $P \in \mathbb{R}^n$ possesses a gyrobarycentric representation

$$P = \frac{\sum_{k=1}^{N} m_k \gamma_A A_k}{\sum_{k=1}^{N} m_k \gamma_A}$$

with respect to $S$, with homogeneous gyrobarycentric coordinates $m_k \in \mathbb{R}$, $k = 1, 2, \ldots, N$, that satisfy the condition

$$\sum_{k=1}^{N} m_k \gamma_A \neq 0$$

if and only if

$$P \in \mathbb{A}_{N,1},$$

where $\mathbb{A}_{N,1}$ is the $(N - 1)$-gyroflat

$$\mathbb{A}_{N,1} = A_1 \oplus \text{Span}\{\ominus A_1 \oplus A_2, \ldots, \ominus A_1 \oplus A_N\} \subset \mathbb{R}^n.$$

Proof. Assuming (213), we have by the left cancellation law [9], p. 3, of Einstein addition,

$$\ominus A_1 \oplus P \in \text{Span}\{\ominus A_1 \oplus A_2, \ldots, \ominus A_1 \oplus A_N\}.$$ 

Hence, there exist scalars $m_k \in \mathbb{R}$, $k = 2, \ldots, N$, such that

$$\ominus A_1 \oplus P = \sum_{k=2}^{N} m_k \gamma_A \ominus A_k (\ominus A_1 \oplus A_k) = \sum_{k=1}^{N} m_k \gamma_A (\ominus A_1 \oplus A_k) \subset \mathbb{R}^n$$

for any scalar $m_1 \in \mathbb{R}$. The arbitrariness of $m_1$ follows from $\ominus A_1 \oplus A_1 = \mathbf{0}$ for $k = 1$ in (216). Owing to the gyrobarycentrically independence of $S$, the coefficients $m_k$, $k = 2, \ldots, N$, in (216) are determined uniquely by the gyrovector $\ominus A_1 \oplus P$ and
by the gamma factors $\gamma_{\oplus A_1 \oplus A_k}$. Here, the gyrovectors $\oplus A_1 \oplus P$ and $\oplus A_1 \oplus A_k$ are considered as vectors in $\mathbb{R}^n \supset \mathbb{R}^n_s$.

We now select the special scalar $m_1$ that is uniquely determined by the normalization condition

\begin{equation}
\sum_{k=1}^{N} m_k \gamma_{\oplus A_1 \oplus A_k} = 1,
\end{equation}

that is,

\begin{equation}
m_1 = 1 - \sum_{k=2}^{N} m_k \gamma_{\oplus A_1 \oplus A_k}.
\end{equation}

Then, (216) can be written as

\begin{equation}
\oplus A_1 \oplus P = \frac{\sum_{k=1}^{N} m_k \gamma_{\oplus A_1 \oplus A_k} (\oplus A_1 \oplus A_k)}{\sum_{k=1}^{N} m_k \gamma_{\oplus A_1 \oplus A_k}}.
\end{equation}

Following Identity (207a), p. 13, of the Gyrobarycentric Representation Gyrovariance Theorem 41, with $X = \oplus A_1$, (219) yields

\begin{equation}
P = \frac{\sum_{k=1}^{N} m_k \gamma_{A_k} A_k}{\sum_{k=1}^{N} m_k \gamma_{A_k}}.
\end{equation}

Owing to the homogeneity of the coordinates $m_k$ in (220), the representation of $P$ in (220) remains valid if we replace the normalization condition (217) by the weaker condition (212), thus obtaining in (220) the desired gyrobarycentric representation (211) – (212) of $P$.

Conversely, assuming (211) – (212), we have by Result (207a) of Theorem 41 with $X = \oplus A_1$,

\begin{equation}
\oplus A_1 \oplus P = \frac{\sum_{k=1}^{N} m_k \gamma_{\oplus A_1 \oplus A_k} (\oplus A_1 \oplus A_k)}{\sum_{k=1}^{N} m_k \gamma_{\oplus A_1 \oplus A_k}},
\end{equation}

implying (213), as desired. □

**Lemma 45.** Let $S = \{A_1, A_2, \ldots, A_N\}$, $N \geq 2$, be a gyrobarycentrically independent set of $N$ points in an Einstein gyrovector space $(\mathbb{R}_s^n, \oplus, \otimes)$, $n \geq N - 1$, and let

\begin{equation}
A_{N,k} = \text{Span} \{ \oplus A_k \oplus A_1, \oplus A_k \oplus A_2, \ldots, \oplus A_k \oplus A_N \} \subset \mathbb{R}^n
\end{equation}

for each $k$, $1 \leq k \leq N$.

Then, $A_{N,k} := A_k$ is independent of $k$.

**Proof.** Let $k_1$ and $k_2$ be two distinct integers, $1 \leq k_1, k_2 \leq N$, and let $P \in A_{N,k_1}$. Then, by the Gyrobarycentric Representation Existence Theorem 44, $P$ possesses...
a gyrobarycentric representation

\[ P = \frac{\sum_{k=1}^{N} m_k \gamma_{A_k} A_k}{\sum_{k=1}^{N} m_k \gamma_{A_k}}. \]

\[ \sum_{k=1}^{N} m_k \gamma_{A_k} \neq 0. \]

Applying the Gyrobarycentric Representation Gyrocovariance Theorem \[ 41 \] p. 48, with \[ X = \oplus A_k \] to (223), we obtain the equation

\[ \ominus A_k \oplus P = \frac{\sum_{k=1}^{N} m_k \gamma_{\oplus A_k} (\ominus A_k \oplus A_k)}{\sum_{k=1}^{N} m_k \gamma_{A_k}}. \]

Hence,

\[ \ominus A_k \oplus P \in \text{Span} \{ \ominus A_k + A_1, \ldots, \ominus A_k + A_N \} \subset \mathbb{R}^n, \]

so that, by means of the left cancellation law \[ [4] \] p. 3, \( P \in \mathbb{A}_{N,k_2} \). Hence, \( \mathbb{A}_{N,k_1} \subset \mathbb{A}_{N,k_2}. \) The proof of the reverse inclusion is similar (just interchange \( k_1 \) and \( k_2 \)), so that \( \mathbb{A}_{N,k_1} = \mathbb{A}_{N,k_2}, \) as desired. \( \square \)

19. Uniqueness of Gyrobarycentric Representations

Remark 46. (The Index Notation). It will prove useful to use the index notation for indexed points \( A_k, k \in \mathbb{N} \), in Einstein gyrovector spaces \( (\mathbb{R}^n, \oplus, \otimes) \) as follows:

\[ a_{ij} = \ominus A_i \oplus A_j, \quad a_{ij} = \|a_{ij}\|, \quad \gamma_{ij} = \gamma_{a_{ij}}, \]

noting that \( a_{ij} = a_{ji}, \gamma_{ij} = \gamma_{ji}, a_{ii} = 0, a_{ii} = 0 \) and \( \gamma_{ii} = 1. \)

Theorem 47. (Gyrobarycentric Representation Uniqueness). A gyrobarycentric representation of a point in an Einstein gyrovector space \( (\mathbb{R}^n, \oplus, \otimes) \) with respect to a gyrobarycentrically independent set \( S = \{A_1, \ldots, A_N\} \) is unique.

Proof. Let

\[ P = \frac{\sum_{k=1}^{N} m_k \gamma_{A_k} A_k}{\sum_{k=1}^{N} m_k \gamma_{A_k}} = \frac{\sum_{k=1}^{N} m_k' \gamma_{A_k} A_k}{\sum_{k=1}^{N} m_k' \gamma_{A_k}} \in \mathbb{R}^n \]

be two gyrobarycentric representations of a point \( P \),

\[ P \in \mathbb{A}_N \subset \mathbb{R}^n \]

with respect to a gyrobarycentrically independent set \( S = \{A_1, \ldots, A_N\} \subset \mathbb{R}^n \) in an Einstein gyrovector space \( (\mathbb{R}^n, \oplus, \otimes) \).
Then, by Theorem 41 with $X = \ominus A_j$ in (207a), along with the convenient index notation (226), we have from (227)

$$\ominus A_j \oplus P = \frac{\sum_{k=1}^{N} m_k \gamma_{\ominus A_j \oplus A_k} (\ominus A_j \oplus A_k)}{\sum_{k=1}^{N} m_k \gamma_{\ominus A_j \oplus A_k}}$$

(229)

$$= \frac{\sum_{k=1}^{N} m_k \gamma_j a_{jk}}{\sum_{k=1}^{N} m_k \gamma_j}$$

$$= \frac{\sum_{k=1}^{N} m'_k \gamma_j a_{jk}}{\sum_{k=1}^{N} m'_k \gamma_j}$$

for any $A_j$, $1 \leq j \leq N$. Note that when $k = j$ in (229), $a_{jk} = \ominus A_j \oplus A_k = 0$ and $\gamma_{jk} = \gamma_{a_{jk}} = \gamma_0 = 1$.

The set $S = \{A_1, \ldots, A_N\} \subset \mathbb{R}^n$ is gyrobarically independent. Hence, by Def. 35 the set of gyrovectors $a_{jk} = \ominus A_j \oplus A_k$, $k = 1, \ldots, N$, $k \neq j$, considered as vectors in $\mathbb{R}^n$, forms a set of $N - 1$ linearly independent vectors for each $j$. Owing to this linear independence,

$$m'_k = cm_k$$

(230)

for all $k = 1, \ldots, N$, where $c$ is a nonzero constant. Since gyrobarionic coordinates are homogeneous, the nonzero common factor, $c$, of the gyrobarionic coordinates of a gyrobarionic representation is irrelevant. Hence, the two gyrobarionic representations of $P$ in (227) coincide, so that the gyrobarionic representation (227) of $P$ with respect to a given gyrobarically independent set is unique.

20. **Gyrovector Gyroconvex Span**

Let $P \in \mathbb{R}^n$ be a point in an Einstein gyrovector space $(\mathbb{R}_s^n, \oplus, \otimes)$ that possesses a gyrobarionic representation,

$$P = \frac{\sum_{k=1}^{N} m_k \gamma_{A_k} A_k}{\sum_{k=1}^{N} m_k \gamma_{A_k}}$$

(231)

with respect to a gyrobarically independent set $S = \{A_1, \ldots, A_N\} \subset \mathbb{R}^n$.

Then, by Identity (207a) of the Gyrobarionic Representation Gyrocovariance Theorem 41 with $X = \ominus A_0$, the gyrobarionic representation (231) gives rise to Identity (232) that we employ in the following Definition.

**Definition 48. (Gyrovector Gyroconvex Span).** The Identity

$$\ominus A_0 \oplus P = \frac{\sum_{k=1}^{N} m_k \gamma_{\ominus A_0 \oplus A_k} (\ominus A_0 \oplus A_k)}{\sum_{k=1}^{N} m_k \gamma_{\ominus A_0 \oplus A_k}}$$

(232)
in an Einstein gyrovector space \((\mathbb{R}_s^n, \oplus, \otimes)\) represents the gyrovector \(A_0 \otimes P\) as a gyrovector gyroconvex span of the \(N\) gyrovectors \(A_k, k = 1, \ldots, N\).

The geometric significance of gyrovector gyroconvex spans is established in the following theorem.

**Theorem 49. (Gyrovector Gyroconvex Span Gyrocovariance).** The representation (232) of a gyrovector as a gyrovector gyroconvex span in an Einstein gyrovector space \((\mathbb{R}_s^n, \oplus, \otimes)\) is gyrocovariant in form.

**Proof.** Rotations \(R, R \in SO(n)\), of \(\mathbb{R}_s^n\) are linear maps of \(\mathbb{R}_s^n\) onto itself expandable to linear maps of \(\mathbb{R}^n\) onto itself, which respect both Einstein addition in \(\mathbb{R}_s^n\) (see the first equation in (86), p. 22) and vector addition in \(\mathbb{R}^n\), and which keep the norm invariant. Hence, following (232) we have

\[
\oplus R A_0 \oplus R P = R (\oplus A_0 \oplus P)
\]

(233)

for all rotations \(R \in SO(n)\). Hence, (232) remains invariant in form under rotations.

In the following chain of equations, which are numbered for subsequent explanation, we complete the proof by demonstrating that (232) remains invariant in form under left gyrotranslations as well.

\[
\sum_{k=1}^{N} m_k \gamma_{\ominus A_0 \oplus A_k} \ominus (X \oplus A_0) \oplus (X \oplus A_k)
\]

(234)

Derivation of the numbered equalities in (234) follows:

1. The left-hand side of the first equation in (234) is recognized as the left gyrotranslation by \(X \in \mathbb{R}_s^n\) of the right-hand side of (232). The right-hand
side of the first equation in (234) follows from the left-hand side of the first equation in (234) by the Left Gyrotranslation Theorem 15, p. 18, noting (133), p. 33.

(2) Follows from (1) since gyrations of \( R^n_s = (R^n, \oplus, \otimes) \) form a subset of \( SO(n) \) so that, as such, gyrations are linear maps of \( R^n \) (see Remark 3, p. 7).

(3) Follows from (2) by (232).

(4) Follows from (3) by the Left Gyrotranslation Theorem 15, p. 18, thus obtaining the desired expression, which is recognized as the left gyrotranslation by \( X \in R^n_s \) of the left-hand side of (232).

\[ \Box \]

Being invariant in form under hyperbolic motions, that is, under both rotations and left gyrotranslations, Identity (232) is gyrocovariant in form according to Def. 20, p. 23.

21. Gyrosegments

A study of the gyrosegment is presented here as an example that illustrates a simple use of gyrobarycentric coordinates in a way analogous to the study of the segment in Sect. 17.

Let \( A_1, A_2 \in R^2_s \) be two distinct points of the Einstein gyrovector plane \( R^2_s = (R^2_s, \oplus, \otimes) \), and let \( P \in A_{N,1} \), where \( A_{N,1} \) is the 1-gyroflat (gyroline)

\[ A_{N,1} = A_1 \oplus \text{Span}\left\{ \ominus A_1 \oplus A_1, \ominus A_1 \oplus A_2 \right\} = A_1 \oplus \text{Span}\left\{ \ominus A_1 \oplus A_2 \right\} \subset R^2. \]

Then, by Theorem 44, \( P \) possesses a gyrobarycentric representation

\[ P = \frac{m_1 \gamma_{A_1} A_1 + m_2 \gamma_{A_2} A_2}{m_1 \gamma_{A_1} + m_2 \gamma_{A_2}} \]

with respect to the gyrobarycentrically independent set \( S = \{ A_1, A_2 \} \), with gyrobarycentric coordinates \( m_1 \) and \( m_2 \) satisfying \( m_1 \gamma_{A_1} + m_2 \gamma_{A_2} \neq 0 \). In particular:

(1) If \( m_1 = 0 \), then \( P = A_2 \).

(2) If \( m_2 = 0 \), then \( P = A_1 \).

(3) If \( m_1, m_2 > 0 \), or \( m_1, m_2 < 0 \), then \( P \) lies on the interior of gyrosegment \( A_1 A_2 \), that is, between \( A_1 \) and \( A_2 \).

(4) If \( m_1 \) and \( m_2 \) are nonzero and have opposite signs, then \( P \) lies on the exterior of gyrosegment \( A_1 A_2 \).

Owing to the homogeneity of gyrobarycentric coordinates, these can be normalized by the condition

\[ m_1 + m_2 = 1, \]

so that, for instance, we can parametrize \( m_1 \) and \( m_2 \) by a parameter \( t \) according to the equations \( m_1 = t \) and \( m_2 = 1 - t \), \( 0 \leq t \leq 1 \). Then, the point \( P \) possesses the
special gyrobarycentric representation

\[
P = \frac{t\gamma_{A_1}A_1 + (1-t)\gamma_{A_2}A_2}{t\gamma_{A_1} + (1-t)\gamma_{A_2}}.
\]

Following the Gyrobarycentric Representation Gyrocovariance Theorem \[41\] p. 48, the gyrobarycentric representation (238) of \(P\) obeys the identity

\[
X \oplus P = \frac{t\gamma_{X \oplus A_1}(X \oplus A_1) + (1-t)\gamma_{X \oplus A_2}(X \oplus A_2)}{t\gamma_{X \oplus A_1} + (1-t)\gamma_{X \oplus A_2}}
\]

for all \(X \in \mathbb{R}_s^2\). Unlike its Euclidean counterpart \[188\], which is trivial, Identity (239) is, indeed, far away from being trivial.

Gyrobarycentric coordinates form an incisive tool that has proved amenable to the extension of classical geometric concepts in Euclidean geometry to the hyperbolic geometry setting \[40, 41, 42, 43\].

22. Gyromidpoint

The use of gyrobarycentric coordinates is demonstrated here by determining the gyromidpoints of gyrosegments. Let \(A_1A_2\) be a gyrosegment in an Einstein gyrovector space \((\mathbb{R}_s^n, \oplus, \otimes)\), \(n \geq 1\), formed by two distinct points \(A_1, A_2 \in \mathbb{R}_s^n\). The gyromidpoint \(M_{12} \in A_1 \oplus \text{Span}\{ \otimes A_1 \oplus A_2 \}\) of gyrosegment \(A_1A_2\), shown in Fig. 2, is the point of the gyrosegment that is equigyrodistant from \(A_1\) and \(A_2\), that is,

\[
\| \otimes A_1 \oplus M_{12} \| = \| \otimes A_2 \oplus M_{12} \|.
\]

In order to determine the gyromidpoint \(M_{12}\) of gyrosegment \(A_1A_2\), let \(M_{12}\) be given by its gyrobarycentric representation (203) with respect to the set \(S = \{A_1, A_2\}\),

\[
M_{12} = \frac{m_1\gamma_{A_1}A_1 + m_2\gamma_{A_2}A_2}{m_1\gamma_{A_1} + m_2\gamma_{A_2}},
\]

where the gyrobarycentric coordinates \(m_1\) and \(m_2\) are to be determined in (249) below.

The constant \(m_{M_{12}}\) of the gyrobarycentric representation (241) of \(M_{12}\) is given by the equation

\[
m_{M_{12}} = \sqrt{(m_1 + m_2)^2 + 2m_1m_2(\gamma_{12} - 1)},
\]

according to (205).
\[ M_{12} = \frac{\gamma_{A_1} A_1 + \gamma_{A_2} A_2}{\gamma_{A_1} + \gamma_{A_2}} \]

\[ M_{12} = \frac{1}{2} \otimes (A_1 \boxplus A_2) \]

\[ m_1 \gamma_{A_1} a_{12} + m_2 \gamma_{A_2} a_{12} \]

\[ (243) \]

\[ \otimes A_1 \otimes M_{12} = \frac{m_1 \gamma_{A_1} \otimes A_1 (\otimes A_1) + m_2 \gamma_{A_1} \otimes A_2 (\otimes A_2)}{m_1 \gamma_{A_1} \otimes A_1 + m_2 \gamma_{A_1} \otimes A_2} \]

\[ = \frac{m_2 \gamma_{12} a_{12}}{m_1 + m_2 \gamma_{12}} \]

\[ \otimes A_2 \otimes M_{12} = \frac{m_1 \gamma_{A_2} \otimes A_2 (\otimes A_2) + m_2 \gamma_{A_2} \otimes A_2 (\otimes A_2)}{m_1 \gamma_{A_2} \otimes A_1 + m_2 \gamma_{A_2} \otimes A_2} \]

\[ = \frac{m_1 \gamma_{12} a_{21}}{m_1 \gamma_{21} + m_2} \]
where, as indicated in Fig. [2] we use the convenient index notation (226), noting that 
\( a_{12} = a_{21} \) and \( \gamma_{12} = \gamma_{21} \), while, in general, \( a_{12} \neq a_{21} \) since, by the gyrocommutative law, \( a_{21} = \text{gyr}[\ominus A_2, A_1]a_{12} \).

In each of the two equations in (243) we employ the frequently used trivial identities
\[
\ominus A \ominus A = 0 \\
\gamma \ominus A \ominus A = \gamma_0 = 1
\]
for all \( A \in \mathbb{R}_s^n \).

Taking magnitudes of the extreme sides of each of the two equations in (243), we have
\[
\|\ominus A_1 \ominus M_{12}\| = \frac{m_2}{m_1 + m_2 \gamma_{12}} \gamma_{12} a_{12} \\
\|\ominus A_2 \ominus M_{12}\| = \frac{m_1}{m_1 \gamma_{12} + m_2} \gamma_{12} a_{12},
\]
so that by (245) and (246) we have
\[
\frac{m_1}{m_1 \gamma_{12} + m_2} = \frac{m_2}{m_1 + m_2 \gamma_{12}},
\]
implying \( m_1 = \pm m_2 \neq 0 \).

For \( m_1 = m_2 = : m \), the constant \( m_{M_{12}} \) of the gyrobaricentric representation (241) of \( M_{12} \) is given by
\[
m_{M_{12}}^2 = (m_1 + m_2)^2 + 2m_1 m_2 (\gamma_{12} - 1) = 2m^2 (\gamma_{12} + 1) > 0 \quad (\text{Accepted}),
\]
so that, being positive, \( m_{M_{12}}^2 \) is acceptable since it implies, by Corollary 43, that \( M_{12} \in \mathbb{R}_s^n \).

In contrast, for \( m_1 = -m_2 = : m \), the constant \( m_{M_{12}} \) of the gyrobaricentric representation (241) of \( M_{12} \) is given by
\[
m_{M_{12}}^2 = (m_1 + m_2)^2 + 2m_1 m_2 (\gamma_{12} - 1) = -2m^2 (\gamma_{12} - 1) < 0 \quad (\text{Rejected}),
\]
so that, being negative, \( m_{M_{12}}^2 \) is rejected since it implies, by Corollary 43, that \( M_{12} \notin \mathbb{R}_s^n \). Hence, the solution \( m_1 = -m_2 \) of (246) is rejected, allowing the unique solution \( m_1 = m_2 \).

The unique solution for the gyrobaricentric coordinates of the midpoint \( M_{12} \) (modulo a nonzero multiplicative scalar) is, therefore, \( (m_1 : m_2) = (m : m) \) or, equivalently,
\[
(m_1 : m_2) = (1 : 1).
\]

Substituting the gyrobaricentric coordinates (249) into (241) we, finally, obtain the gyromidpoint \( M_{12} \) in terms of its vertices \( A_1 \) and \( A_2 \) by the gyromidpoint equation
\[
M_{12} = \frac{\gamma A_1 A_1 + \gamma A_2 A_2}{\gamma A_1 + \gamma A_2}.
\]
Following (247) and (249) – (250), the constant \( m_{M_{12}} \) of the gyrobarycentric representation of the gyromidpoint \( M_{12} \) in (250) is

\[
m_{M_{12}} = \sqrt{2\sqrt{\gamma_{12} + 1}}.
\]

Hence, by the Gyrobarycentric Representation Gyrocovariance Theorem 41, p. 48, the gyromidpoint \( M_{12} \) possesses the three identities

\[(252a)\]
\[X \oplus M_{12} = \frac{\gamma_X \oplus A_1 (X \oplus A_1) + \gamma_X \oplus A_2 (X \oplus A_2)}{\gamma_X \oplus A_1 + \gamma_X \oplus A_2},\]

\[(252b)\]
\[\gamma_{X \oplus M_{12}} = \frac{\gamma_X \oplus A_1 + \gamma_X \oplus A_2}{\sqrt{2\sqrt{\gamma_{12} + 1}}},\]

\[(252c)\]
\[\gamma_{X \oplus M_{12}} (X \oplus M_{12}) = \frac{\gamma_X \oplus A_1 (X \oplus A_1) + \gamma_X \oplus A_2 (X \oplus A_2)}{\sqrt{2\sqrt{\gamma_{12} + 1}}},\]

for all \( X \in \mathbb{R}_s^n \), where (252a) follows immediately from (252a) – (252b).

Following (252a) with \( X = \ominus A_1 \), by Einstein half (37), p. 10, we have

\[(253)\]
\[\ominus A_1 \oplus M_{12} = \frac{\gamma_{\ominus A_1 \oplus A_2} (\ominus A_1 \oplus A_2)}{1 + \gamma_{\ominus A_1 \oplus A_2}} = \frac{\gamma_{12}}{1 + \gamma_{12}} a_{12} = \frac{1}{2} \otimes a_{12},\]

so that by the scaling property \( V(5) \) of Einstein gyrovector spaces (Def. 8),

\[(254)\]
\[\| \ominus A_1 \oplus M_{12} \| = \frac{1}{2} \otimes a_{12} = \frac{1}{2} \| a_{12} \| = \frac{1}{2} \otimes a_{12}.\]

Similarly, following (252b) – (252c) with \( X = \ominus A_1 \) we have

\[(255)\]
\[\gamma_{\ominus A_1 \oplus M_{12}} = \frac{1 + \gamma_{\ominus A_1 \oplus A_2}}{\sqrt{2\sqrt{1 + \gamma_{12}}}} = \frac{1 + \gamma_{12}}{\sqrt{2\sqrt{1 + \gamma_{12}}}} = \frac{\sqrt{1 + \gamma_{12}}}{\sqrt{2}},\]

and

\[(256)\]
\[\gamma_{\ominus A_1 \oplus M_{12}} (\ominus A_1 \oplus M_{12}) = \frac{\gamma_{\ominus A_1 \oplus A_2} (\ominus A_1 \oplus A_2)}{\sqrt{2\sqrt{1 + \gamma_{12}}}} = \frac{\gamma_{12} a_{12}}{\sqrt{2\sqrt{1 + \gamma_{12}}}}.\]

Hence, by (253) and (255),

\[(257)\]
\[\frac{1}{2} \otimes a_{12} = \sqrt{\frac{1 + \gamma_{12}}{2}}\]

and, by (253) and (256),

\[(258)\]
\[\frac{1}{2} \otimes a_{12} (\frac{1}{2} \otimes a_{12}) = \frac{\gamma_{12} a_{12}}{\sqrt{2\sqrt{1 + \gamma_{12}}}}.\]

The squared gyrolength of \( \frac{1}{2} \otimes a_{12} \) is, by (257) and (13), p. 4

\[(259)\]
\[\| \frac{1}{2} \otimes a_{12} \|^2 = s^2 \frac{\gamma_{(1/2) \otimes a_{12}}^{-1}}{\gamma_{(1/2) \otimes a_{12}}} - 1 = s^2 \frac{\gamma_{12} - 1}{\gamma_{12} + 1}.\]
and the squared gyrolength of $a_{12}$ is, by (13), p. 4
\[(260) \quad \|a_{12}\|^2 = \|\oplus A_1 \oplus A_2\|^2 = s^2 \frac{\gamma_{12}^2 - 1}{\gamma_{12}^2}.
\]
Indeed, by (260),
\[(261) \quad \|a_{12}\| = \|\frac{1}{2} \otimes a_{12} \| \oplus \|\frac{1}{2} \otimes a_{12}\|
= s \sqrt{\frac{\gamma_{12}^2 - 1}{\gamma_{12}^2 + 1}} \oplus s \sqrt{\frac{\gamma_{12}^2 - 1}{\gamma_{12}^2 + 1}}
= s \sqrt{\frac{\gamma_{12}^2 - 1}{\gamma_{12}^2}},
\]
as expected from (260).

23. Gyroline Boundary Points

A gyroline in an Einstein gyrovector space $(\mathbb{R}_s^n, \oplus, \otimes)$ approaches the boundary of the ball $\mathbb{B}_s^n$ of its space at its two boundary points, as shown in Fig. 3, p. 62.

Let $A_1, A_2 \in \mathbb{R}_s^n$ be two distinct points of an Einstein gyrovector space $(\mathbb{R}_s^n, \oplus, \otimes)$, and let $P$ be a generic point on the gyroline, (43), p. 12,
\[(262) \quad L_{A_1A_2} = A_1 \oplus (\ominus A_1 \oplus A_2) \otimes t,
\]
$t \in \mathbb{R}$, that passes through these two points. Furthermore, let
\[(263) \quad P = \frac{m_1 \gamma_{A_1} A_1 + m_2 \gamma_{A_2} A_2}{m_1 \gamma_{A_1} + m_2 \gamma_{A_2}}
\]
be a gyrobarycentric representation of a generic point $P$ on the gyroline $L_{A_1A_2}$ with respect to the gyrobarycentrically independent set $S = \{A_1, A_2\}$. We wish to determine the gyrobarycentric coordinates $m_1$ and $m_2$ in (263) for which the point $P$ is a boundary point of the gyroline $L_{A_1A_2}$.

Owing to the homogeneity of gyrobarycentric coordinates, we can select $m_2 = -1$, obtaining from (263) the gyrobarycentric representation
\[(264) \quad P = \frac{m \gamma_{A_1} A_1 - \gamma_{A_2} A_2}{m \gamma_{A_1} - \gamma_{A_2}}.
\]

According to Def. 36 of the gyrobarycentric representation of $P$ in (263) and its constant $m_P$ in (265), the constant $m_P$ of the gyrobarycentric representation of $P$ in (263) - (264) satisfies the equation
\[(265) \quad m_P^2 = m_1^2 + m_2^2 + 2m_1 m_2 \gamma_{\ominus A_1 \oplus A_2}
= m_2^2 + 1 - 2m_P \gamma_{12},
\]
where we use the index notation (226).
By Corollary 43, p. 49, the point $P$ lies on the boundary of the ball $\mathbb{R}^n_s$ if and only if $m_P = 0$, that is by (265), if and only if

$$m^2 - 2m\gamma_{12} + 1 = 0.$$  

Indeed, the two solutions of (266), which are

$$m = \gamma_{12} + \sqrt{\gamma_{12}^2 - 1}$$

(267)

$$m = \gamma_{12} - \sqrt{\gamma_{12}^2 - 1},$$

correspond to the two boundary points of gyroline $L_{A_1A_2}$, as shown in Fig. 3.

The substitution into (264) of each of the two solutions (267) gives the two boundary points $E_{A_1}$ and $E_{A_2}$ of the gyroline $L_{A_1A_2}$ in (262),

$$E_{A_1} = \frac{(\gamma_{12} + \sqrt{\gamma_{12}^2 - 1})\gamma_{A_1} - \gamma_{A_2}A_1}{(\gamma_{12} + \sqrt{\gamma_{12}^2 - 1})\gamma_{A_1} - \gamma_{A_2}}$$

(268)

$$E_{A_2} = \frac{(\gamma_{12} - \sqrt{\gamma_{12}^2 - 1})\gamma_{A_1} - \gamma_{A_2}A_1}{(\gamma_{12} - \sqrt{\gamma_{12}^2 - 1})\gamma_{A_1} - \gamma_{A_2}}.$$  

Being points on the boundary of the $s$-ball $\mathbb{R}^n_s$, the points $E_{A_1}$ and $E_{A_2}$ are not in the Einstein gyrovector space $(\mathbb{R}^n_s, \oplus, \otimes)$ and their gamma factors are undefined,

$$\gamma_{E_{A_1}} = \gamma_{E_{A_2}} = \infty.$$  

(269)

Yet, their left gyrotranslation by any $X \in \mathbb{R}^n_s$, shown in Fig. 3, are well-defined. Thus, for instance, the left gyrotranslation $X \oplus E_{A_1}$ of the boundary point $E_{A_1}$ by any $X \in \mathbb{R}^n_s$, which involves the gamma factor of $X$, does not involve the undefined gamma factor of $E_{A_1}$, as we see from the definition of Einstein gyrosums in (3), p. 4.

The magnitude of a boundary point of $\mathbb{R}^n_s$ is $s$, and, conversely, a point of $\mathbb{R}^n$ with magnitude $s$ is a boundary point of $\mathbb{R}^n_s$. Furthermore, the magnitude of any left gyrotranslated boundary point remains $s$, as indicated in (271) below and in Fig. 3. Hence, a left gyrotranslated boundary point remains a boundary point. A left gyrotranslation of a boundary point thus results in the rotation of the boundary point about the origin of its $s$-ball $\mathbb{R}^n_s$, as shown in Fig. 3.

The left gyrotranslated boundary points $\ominus A_1 \oplus E_{A_1}$ and $\ominus A_1 \oplus E_{A_2}$ that follow from (268) by means of the gyrocovariance identity (207a) in Theorem 41 are particularly elegant. Indeed, by the gyrocovariance identity (207a) with $X = \ominus A_1$, etc.
applied to each of the two equations in (268), we have
\[ \ominus A_1 \oplus E_{A_1} = \frac{(\gamma_{12} + \sqrt{\gamma_{12}^2 - 1})\gamma_{\ominus A_1 \oplus A_1} (\ominus A_1 \oplus A_1) - \gamma_{\ominus A_1 \oplus A_2} (\ominus A_1 \oplus A_2)}{(\gamma_{12} + \sqrt{\gamma_{12}^2 - 1})\gamma_{\ominus A_1 \oplus A_1} - \gamma_{\ominus A_1 \oplus A_2}} \]
\[ = \frac{-\gamma_{12}a_{12}}{(\gamma_{12} + \sqrt{\gamma_{12}^2 - 1}) - \gamma_{12}} = \ominus \frac{\gamma_{12}a_{12}}{\sqrt{\gamma_{12}^2 - 1}} \]
\[ \ominus A_1 \oplus E_{A_2} = \frac{\gamma_{12}a_{12}}{\sqrt{\gamma_{12}^2 - 1}}, \]
where we use the index notation (226), noting (244), p. 58.

Note that by (270) and (13), p. 4
\[ \| \ominus A_1 \oplus E_{A_1} \|^2 = \| \ominus A_1 \oplus E_{A_2} \|^2 = \frac{\gamma_{12}a_{12}^2}{\gamma_{12}^2 - 1} = s^2. \]

Hence, the gyrodistance between \( E_k \) and \( A_1, k = 1, 2 \), is \( s \), as expected, since boundary points of gyrolines are located on the boundary of the \( s \)-ball of their Einstein gyrovector space.

The equations in (270) imply, by means of the left cancellation law (9), p. 3, that the boundary points \( E_{A_1} \) and \( E_{A_2} \) of the gyroline \( L_{A_1, A_2} \) that passes through the

Figure 3. Gyroline Boundary Points. Any gyroline in an Einstein gyrovector space \( \mathbb{R}^n_s \) approaches the boundary \( \partial \mathbb{R}^n_s \) of the \( s \)-ball \( \mathbb{R}^n_s \) at two points, called the boundary points of the gyroline. Here, a gyroline \( L_{A_1, A_2} \) and its two boundary points \( E_{A_1} \) and \( E_{A_2} \) in an Einstein gyrovector space \( (\mathbb{R}^n_s, \oplus, \otimes), n = 2 \), are shown along with their left gyrotranslation by some \( X \in \mathbb{R}^n_s \). It is indicated that a gyroline and its boundary points are gyrocovariant (vary together, in the hyperbolic geometric sense) with respect to left gyrotranslations.
points $A_1$ and $A_2$ are given by the equations

$$E_{A_1} = A_1 \oplus \frac{\gamma_{12}^2 a_{12}}{\sqrt{\gamma_{12}^2 - 1}}$$

(272)

$$E_{A_2} = A_1 \ominus \frac{\gamma_{12}^2 a_{12}}{\sqrt{\gamma_{12}^2 - 1}},$$

as shown graphically in Fig. 3.

Interesting applications of gyrobarycentric coordinates in hyperbolic geometry are found in [42, 43, 46].

It is well-known, as emphasized in [2], that Euclidean barycentric coordinates prove useful in the geometry of quantum states. Barycentric coordinate systems underlie the study of convex analysis [25], and Convexity considerations are important in non-relativistic quantum mechanics where mixed states are positive barycentric combinations of pure states, and where barycentric coordinates are interpreted as probabilities [28, p. 11]. The success in [2] and [4] of the study of the geometry of quantum states in terms of barycentric coordinates suggests that relativistic barycentric coordinates can prove useful in the geometry of relativistic quantum states as well.

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