Equivalence of Fokker-Planck approach and non-linear $\sigma$-model for disordered wires in the unitary symmetry class

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Abstract

The exact solution of the Dorokhov-Mello-Pereyra-Kumar-equation for quasi one-dimensional disordered conductors in the unitary symmetry class is employed to calculate all $m$-point correlation functions by a generalization of the method of orthogonal polynomials. We obtain closed expressions for the first two conductance moments which are valid for the whole range of length scales from the metallic regime ($L \ll Nl$) to the insulating regime ($L \gg Nl$) and for arbitrary channel number. In the limit $N \to \infty$ (with $L/(Nl) = \text{const.}$) our expressions agree exactly with those of the non-linear $\sigma$-model derived from microscopic Hamiltonians.

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The current understanding of transport and localization in quasi one-dimensional disordered wires is essentially based on two competing approaches. The first one is a random matrix approach in which the transfer matrix is written as a product of statistical independent building blocks modelling the quasi one-dimensional structure of the wire. The distribution of each building block is calculated by maximizing its entropy (local maximum entropy approach). This procedure relies on the assumption that all scattering channels are equivalent. The probability distribution \( p(\lambda_i, L) \) (the \( \lambda_i \) denote the usual radial coordinates which are related to the transmission eigenvalues via \( T_i = (1 + \lambda_i)^{-1} \)) is determined by a Fokker-Planck equation (henceforth called DMPK-equation) describing the evolution with the length \( L \) of the wire:

\[
\frac{\partial p(\{\lambda_i\}, L)}{\partial L} = \frac{2}{\xi} \sum_i \frac{\partial}{\partial \lambda_i} J(\{\lambda_i\}) \lambda_i (1 + \lambda_i) \times \frac{\partial}{\partial \lambda_i} \frac{p(\lambda_i, L)}{J(\{\lambda_i\})}, \tag{1}
\]

\[
J(\{\lambda_i\}) = \prod_{i > j} |\lambda_i - \lambda_j|^{\beta}. \tag{2}
\]

The three possible values of \( \beta = 1, 2, 4 \) correspond to the orthogonal, unitary or symplectic symmetry classes, respectively, and \( \xi = (\beta N + 2 - \beta) l \) is the localization length.

An alternative description of quasi one-dimensional wires starts from a more direct, microscopic approach. The metallic regime \( (l \ll L \ll Nl) \) is treated in terms of diagrammatic perturbation theory. To extend the results to all length scales it is necessary to consider the one-dimensional supersymmetric non-linear \( \sigma \)-model. Developing the Fourier analysis on the matrix space defining the \( \sigma \)-model, the average of the conductance and its variance have been calculated for all length scales from the metallic to the insulating regime \( (L \gg Nl) \). Since the one-dimensional \( \sigma \)-model can be derived from a Hamiltonian with white-noise potential, Wegner’s \( N \)-orbital model, and a random band matrix approach for the Hamiltonian, it provides a description of quasi one-dimensional disordered conductors that is rather independent of microscopic details.

The results for the universal conductance fluctuations and weak-localization corrections in the metallic regime obtained by the Fokker-Planck approach and by the \( \sigma \)-model (and also by the diagrammatic perturbation theory) are in agreement. In the insulating regime, the behavior of the probability distribution following from the DMPK-equation is rather well understood. The Lyapunov exponents are known analytically and the \( \lambda_i \) (and hence the conductance \( g = \sum_i (1 + \lambda_i)^{-1} \)) follow a log-normal distribution. However, a direct comparison with the results of the \( \sigma \)-model is rather difficult because only the averages \( \langle g \rangle \) and \( \langle g^2 \rangle \) (instead of \( \langle \ln g \rangle \)) are known. The issue is rather important due to the unusual results of the \( \sigma \)-model for the symplectic symmetry class (systems with time-reversal symmetry and rather strong spin-orbit coupling): Both the average conductance and its variance were found to have finite values as \( L/\xi \to \infty \). It is therefore of considerable interest to derive from the DMPK-equation closed expressions of the average conductance in the insulating regime, even though \( \langle g \rangle \) does not coincide with the typical conductance \( \exp(\langle \ln g \rangle) \).

The situation in the unitary symmetry class (systems with no time reversal symmetry) is particularly favorable because recently the DMPK-equation has been solved exactly.
by means of a Sutherland-transformation \[17\] leading to an independent Fermion problem. Based on the expression of \( p(\{\lambda_i\}, L) \) given in Ref. \[16\], this letter is concerned with the exact calculation of the \( m \)-point correlation functions

\[
R_m(\lambda_1, \ldots, \lambda_m, L) = \frac{N!}{(N-m)!} \int_0^\infty d\lambda_{m+1} \cdots d\lambda_N \ p(\{\lambda_i\}, L)
\]

that are usually considered in the theory of random matrices \[18\].

The exact result for the joint probability distribution given in Ref. \[16\] can be expressed in terms of the variables \( \lambda_i \) as

\[
p(\{\lambda_i\}, t) \propto \prod_{i>j}(\lambda_i - \lambda_j) \ \det(g_{m-1}(\lambda_j, t)) e^{C_N t}
\]

with

\[
g_m(\lambda, t) = \int_0^\infty dk \ (k^2)^m \ \frac{\theta(k)}{\sinh(\pi k/2)}
\]

\[
\times P_{\frac{1}{2}(ik-1)}(1 + 2\lambda)e^{-k^2 t}
\]

and a constant \( C_N \) to be found in \[16,19\]. The functions \( P_{\frac{1}{2}(ik-1)}(1 + 2\lambda) \) denote the generalized Legendre functions. We have introduced the abbreviation \( t = L/2\xi \), i.e. the system length measured in units of (twice) the localization length.

In random matrix theory, one typically applies the method of orthogonal polynomials \[18\] to calculate the \( m \)-point functions \( (3) \). This method cannot directly be carried over to the present case because the pairwise “interaction” between the \( \lambda_i \) eigenvalues is no longer purely logarithmic \[16\]. A first generalization \[20\] of the method is based on the application of biorthogonal polynomials. Extending the idea of Ref. \[20\], we write the probability distribution \( (4) \) as a product of two determinants

\[
p(\{\lambda_i\}, t) \propto \det(Q_{n-1}(\lambda_j, t)) \ \det(h_{m-1}(\lambda_j, t))
\]

with “dual” functions \( Q_n(\lambda, t) \) and \( h_m(\lambda, t) \) \((n, m = 0, 1, \ldots, N-1)\) that fulfill the biorthogonality relation

\[
\int_0^\infty d\lambda \ Q_n(\lambda, t) \ h_m(\lambda, t) = \delta_{nm}.
\]

Expressing the product of the \( \lambda_i \)-differences in \( (4) \) as a Vandermonde determinant with (arbitrary) polynomials \( Q_n(\lambda, t) \) of degree \( n \), we arrive at \( (3) \). The biorthogonality \( (7) \) can be achieved by a suitable linear transformation of the functions \( g_m(\lambda, t) \). In the following, we choose the \( Q_n(\lambda, t) \) as

\[
Q_n(\lambda, t) = P_n(1 + 2\lambda) e^{-\varepsilon_n t}, \quad \varepsilon_n = (1 + 2n)^2
\]

where \( P_n \) are the Legendre polynomials.

The next step is the evaluation of the integral of each combination of \( Q_n(\lambda, t) \) with \( g_m(\lambda, t) \). The Legendre functions \( P_{\frac{1}{2}(ik-1)}(1 + 2\lambda) \) (see Eq. \( (3) \)) are eigenfunctions of the differential operator
\[ D = - \left( 4 \frac{\partial}{\partial \lambda} \lambda (1 + \lambda) \frac{\partial}{\partial \lambda} + 1 \right) \]  

(9)

with eigenvalue \( k^2 \). The operator \( D \) is related to the one particle Hamiltonian \( \mathcal{H}_0 \) used in Ref. [16] by a suitable transformation. In the following, we need three properties of \( g_m(\lambda, t) \), namely

\[ g_m(\lambda, t) = D^m g_0(\lambda, t) \]  

(10)

\[ \frac{\partial}{\partial t} g_0(\lambda, t) = -D g_0(\lambda, t) \]  

(11)

\[ g_0(\lambda, 0) = \delta(\lambda) \]  

(12)

Eqs. (10), (11) follow directly from (5). Eq. (12) reflects the initial condition for the one particle Green’s function [16]. In order to proceed, we need the identity

\[ \int_0^\infty d\lambda \ Q_n(\lambda, t) g_m(\lambda, t) = (-\varepsilon_n)^m \]  

(13)

which follows from the fact that the Legendre polynomials \( P_n(1 + 2\lambda) \) are also eigenfunctions of the differential operator \( D \) (with eigenvalue \( -\varepsilon_n \)). One can verify for \( m = 0 \) via (11,12) that the integral in (13) does not depend on \( t \) and is equal to unity. The generalization to arbitrary \( m \) follows directly from Eq. (13).

We now construct the functions \( h_m(\lambda, t) \) in analogy to Eq. (5), i.e.

\[ h_m(\lambda, t) = \int_0^\infty dk \ L_m(k^2) \frac{k}{2} \tanh \left( \frac{\pi k}{2} \right) \times P_{\frac{1}{2}(ik-1)}(1 + 2\lambda)e^{-k^2t} \]  

(14)

Here, the \( L_m(\cdots) \) denote a set of linearly independent polynomials of (maximal) degree \( N - 1 \). The integral (13) with the \( g_m \) replaced by \( h_m \) takes the value \( L_m(-\varepsilon_n) \). The biorthogonality relation (7) is therefore fulfilled if we choose the polynomials \( L_m \) as the Lagrangian interpolation polynomials

\[ L_m(z) = \prod_{l=0,(l\neq m)}^{N-1} \frac{z - (-\varepsilon_l)}{(-\varepsilon_m) - (-\varepsilon_l)} \]  

(15)

As a key step [20] to apply standard methods from random matrix theory [18], we consider the function

\[ K_N(\lambda, \tilde{\lambda}; t) = \sum_{m=0}^{N-1} Q_m(\lambda, t) \ h_m(\tilde{\lambda}, t) \]  

(16)

with the two important properties

\[ \int_0^\infty d\lambda \ K_N(\lambda, \lambda; t) = N \]  

(17)

\[ \int_0^\infty d\mu \ K_N(\lambda, \mu; t) K_N(\mu, \tilde{\lambda}; t) = K_N(\lambda, \tilde{\lambda}; t) \]  

(18)
The application of a general theorem of the theory of random matrices \[21,22\] immediately yields the \(m\)-point correlation functions

\[
R_m(\lambda_1, \ldots, \lambda_m; t) = \det (K_N(\lambda_i, \lambda_j; t)_{1 \leq i,j \leq m}) .
\] (19)

This expression together with Eqs. (8, 14, 16) forms the key result of this letter. For \(m = 1\) (or \(m = 2\)) it represents the density (the two-point function)

\[
R_1(\lambda; t) = K_N(\lambda, \lambda; t) ,
\] (20)

\[
R_2(\lambda, \tilde{\lambda}; t) = R_1(\lambda; t) R_1(\tilde{\lambda}; t) - K_N(\lambda, \tilde{\lambda}; t) K_N(\tilde{\lambda}, \lambda; t) .
\] (21)

In the insulating regime \(t \gg 1\) the \(k\)-integration in (14) can be evaluated by a saddle-point approximation \[19\] which yields for the density of the variable

\[
x = \text{arsinh}(\sqrt{\lambda})
\]

a sum of Gaussian distributions with centers

\[
2t (1 + 2m) (m = 0, 1, \ldots, N - 1)
\]

and variances \(2t\). This result reflects the self-averaging behavior of \(x\) in the localized regime and is well understood by a direct analysis of the DMPK-equation \[13–15\] in terms of Lyapunov exponents. The results of a numerical evaluation of (14) for \(N = 5\) and three typical length scales \((L/\xi = 0.2, 1, 10)\) which correspond to the metallic, crossover, and insulating regime are given in Fig. 1 which shows the density \(\rho(x) = \text{sinh}(2x) R_1(\text{sinh}^2 x, L/(2\xi))\) versus \(x\). Both axes have been rescaled with the factor \(L/\xi\) for convenience. In all three cases, the effect of the level repulsion is easily recognized because it leads to distinct maxima and minima in the level density. This effect is most prominent in the localized regime (curve (a), \(L = 10 \xi\)) where each level has its own Gaussian peak well separated from the other levels. In the crossover regime (curve (b), \(L = \xi\)) the levels begin to mix but the amplitudes of the maxima are still rather large in comparison to the minima. Even in the so-called metallic regime (curve (c), \(L = 0.2 \xi\)) the oscillations are still visible and the density is not entirely constant. On the other hand, a constant density is expected for this regime \[16\]. This discrepancy is a finite \(N\) effect (note that \(N = 5\) in Fig. 1). The usual metallic regime, \(l \ll L \ll \xi = 2Nl\), exists in the large \(N\) limit only.

Now, we focus attention on the average conductance and its second moment. The conductance is expressed in terms of the density \[23\] as

\[
\langle g \rangle = \int_0^\infty d\lambda \frac{1}{1 + \lambda} \sum_{m=0}^{N-1} Q_m(\lambda, t) h_m(\lambda, t) .
\] (22)

Since \(Q_m(\lambda, t)\) is a polynomial of degree \(m\) in the variable \(1 + \lambda\), we can use the decomposition

\[
\frac{1}{1 + \lambda} Q_m(\lambda, t) = \frac{(-1)^m}{1 + \lambda} e^{-\varepsilon_m t} + r_{m-1}(\lambda)
\] (23)

where \(r_{m-1}(\lambda)\) is a polynomial of degree \(m - 1\) that does not contribute in the integral \[22\] due to the biorthogonality between the \(Q_n\) and the \(h_m\). The \(\lambda\)-integration can be done \[19\] resulting in

\[
\langle g \rangle = 2 \sum_{m=0}^{N-1} \int_0^\infty dk e^{-(1+2m)^2 k^2 t} k \tanh \left( \frac{2k}{2t} \right)
\times \frac{2m + 1}{k^2 + (1 + 2m)^2} a(N, m, k) .
\] (24)
where we have introduced the coefficient

\[ a(N, m, k) = \frac{\Gamma \left( N + \frac{1}{2} + i \frac{k}{2} \right) \Gamma \left( N + \frac{1}{2} - i \frac{k}{2} \right)}{\Gamma(N - m) \Gamma(N + m + 1)} . \]  

Using the expressions (20) and (21) for the density and the two-point function, it is also possible to get by a straightforward calculation the second moment of the conductance

\[ \langle g^2 \rangle = \frac{1}{2} \sum_{m=0}^{N-1} \int_0^\infty dk \, e^{-(1+2m)^2+k^2)t} \, k \tanh \left( \frac{\pi k}{2} \right) \times (2m+1) \, a(N, m, k) . \]

We note that this result can also be derived in a more direct way by the general identity

\[ \frac{\partial \langle g \rangle}{\partial t} = -4 \left( \langle g^2 \rangle + \frac{\beta}{2} \left( \langle g^2 \rangle - \langle g^2 \rangle \right) \right) \]

that follows directly from the original Fokker-Planck equation for arbitrary \( \beta \) (with \( g_2 = \sum_i (1 + \lambda_i)^{-2} \)). In the unitary case \( \beta = 2 \) the average \( \langle g_2 \rangle \) drops out and \( \langle g^2 \rangle \) is entirely determined by (24) and (27).

Eqs. (24) and (26) are exact for all values of \( N \) and \( t = L/(2\xi) \). In the insulating regime, they can be simplified to give

\[ \langle g \rangle \simeq 4 \langle g^2 \rangle \simeq \frac{\pi^{3/2}}{4} a(N, 0, 0) \left( \frac{L}{2\xi} \right)^{-3/2} e^{-L/2\xi} , \]

where the coefficient \( a(N, 0, 0) \) takes values between \( \pi/4 \) for \( N = 1 \) and 1 for \( N \to \infty \). Eq. (28) differs considerably from the behavior of the typical conductance: \( \exp(\langle \ln g \rangle) \sim e^{-2L/\xi} \).

We can also consider the limit \( N \to \infty \) for all regimes. Then, the coefficient \( a(N, m, k) \) tends to unity and the range of the \( m \)-sum is extended to infinity. In this case, the resulting expressions for the average conductance and the second moment become identical with those obtained from the one-dimensional non-linear \( \sigma \)-model (for the unitary case) by Zirnbauer et al. \[9,10\]. (There is still a factor 2 that accounts for a different consideration of the spin degeneracy in the Landauer formula.)

This exact agreement for the whole range of length scales from the metallic to insulating regime strongly suggests that the Fokker-Planck approach \[1,2\] which leads to the DMPK-equation is completely equivalent (in the large \( N \)-limit) to the quasi one-dimensional microscopic models from which the one-dimensional \( \sigma \)-model is derived \[8,11,12\]. We have proven this equivalence for the first two conductance moments.

This result sheds new light on the symplectic case at \( \beta = 4 \) where a similar agreement seems hard to believe. At the moment, exact expressions of \( \langle g \rangle \) and \( \langle g^2 \rangle \) for the DMPK-equation and \( \beta = 4 \) are not known but it is nevertheless clear that the typical conductance decreases exponentially with the length of the wire. In fact, the Lyapunov exponents are known for every value of \( \beta = 1, 2, 4 \) \[13,14\] and one expects for the density of the \( x \)-variable in the insulating regime a similar behavior as in Fig. \[4\] (a), with Gaussian maxima at the positions \( 2t(1 + \beta m) \). The typical conductance corresponds to the first maximum whereas the average conductance \( \langle g \rangle = \int dx \rho(x) \cosh^{-2}(x) \) is determined by the density.
at small $x$ where the deviations from the Gaussian approximation are rather strong. A precise quantitative estimation of $\langle g \rangle$ is thus not possible within this simple picture. In Ref. [23] it was found by an involved and approximate calculation that the average conductance decreases exponentially with the length of the wire for all values of $\beta$.

On the other hand, a simple and precise argument can be given by the exact expression (27). Let us assume that the average conductance approaches exponentially its “finite non-vanishing limit” for $t \to \infty$, as suggested by the $\sigma$-model result for $\beta = 4$. Then $\partial \langle g \rangle / \partial t$ vanishes and we get from Eq. (27) and the inequality $0 \leq g_2 \leq g^2$ immediately $\langle g_2 \rangle = \langle g^2 \rangle = 0$ thus contradicting the assumption.

Apparently, the Fokker-Planck approach and the supersymmetric description for quasi one-dimensional conductors seem to be equivalent for $\beta = 2$ and disagree in the insulating regime for $\beta = 4$. It must be emphasized that the unusual results of the $\sigma$-model approach [9,10] are due to a so-called “zero mode contribution” which is absolutely necessary [10] to reproduce the correct behavior in the metallic regime. At the moment, it is not clear whether there is a particular problem with the $\sigma$-model, probably related to the large $N$ limit, or whether in the Fokker-Planck approach the assumption of an isotropic diffusion in transfer matrix space needs revision. This assumption could in principle be wrong just for $\beta = 4$. A satisfactory answer to these questions remains an outstanding and important problem.

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FIGURES

FIG. 1. The density $\rho(x) L/\xi$ of the variable $x = \text{arsinh}(\sqrt{\lambda})$ versus $x \xi/L$ for the cases $L = 10 \xi$ (a), $L = \xi$ (b), and $L = 0.2 \xi$ (c).