Can God find a place in physics?  
St. Augustine’s philosophy meets general relativity  

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Abstract

In this speculative work I investigate whether God may find a place in theoretical physics. The comparison between some aspects of the nature of God, as deduced by the philosopher St. Augustine, and general relativity, suggests to identify God with a chronology violating region of spacetime. From this conclusion some physical suggestions can be drawn. Among them novel solutions to the homogeneity and entropy problems of cosmology.

1 Introduction

While discussing the possible limits of science we are often told that science cannot enter the realm of faith; that God will certainly remain outside the reach of any physical investigation. This seems to be a widespread and rather modern belief. Centuries of conflicting relations between scientists and religious institutions have pushed towards this peaceful compromise although, especially from the religious side, there are every now and then attempts to influence the teaching and the development of science.

In this work I will try to analyze if God can find a place in modern theoretical physics, and if so, to identify the mathematical object that represents it. The hope is that even a tentative answer may impulse the research into some unexplored directions. Of course mine will be a rather speculative exercise and I would have not tried to put my considerations in an organized work, were not for the opportunity given by this contest. It was important to me that the contest was indirectly founded, through FQXi, by the John Templeton Foundation, the latter being established to catalyze research on big questions not excluding those involving an interaction between science and religion.

2 St. Augustine’s conclusions on the nature of God

Since our main aim is to find if there could be a place for God in physics, we need first to have some facts stated concerning the nature of God. Here I shall consider some

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conclusions reached by the philosopher St. Augustine that are largely independent of
the sacred texts (the Bible) over which his thought had developed. St. Augustine
of Hippo (354 - 430) believed that men have the duty, by the help of reason, to
investigate as far as possible the message left by God to men through the sacred
texts. According to St. Augustine, a better understanding of God would lead to a
better understanding of God’s plans for men, and hence to a better behavior of men
in life. Here I will not focus on peculiarities of the Bible and consider instead as
starting point some aspects of the nature of God shared by different religions. They
are:

1. There is an entity which we call God that satisfies the following points.

2. God has created the world.

3. God cannot be wrong.

Our analysis will involve only these assumptions on the God side, while for those on
the scientific side we shall take from our present knowledge of physics.

At the time of St. Augustine the Manichæans asked the following question “What
was God doing before creating the world?”. Any answer to the question seems to
involve a paradox. If God created the world at one time and not at a previous time
then God changed his mind concerning the possibility of creating the world, thus he
was wrong in not creating it in the first place. The only conclusion is that God cannot
have created the world, because whatever decision God takes he has already taken it.
This is a clear conflict with point 2 that states that God has created the world, and
thus that the world has not always existed. The conclusion of the argument is in fact
more general: the will of God is eternal as there can be no discontinuity in it, and so
should be all the creations that follow from that will.

St. Augustine’s famous reply can be found in the XI book of the Confessions
[24, 23]. This book contains one of the most fortunate studies of the concept of time
especially in the chapters starting from 14 where one can find the famous sentence
“What, then, is time?” If no one ask me, I know; if I wish to explain to him who
asks, I know not.” The reply that interests us is contained in chapters 10-13 where
he considers some issues relating time, creation and God.

First he states that he will not reply (Chap. 12, Par. 14)

“like the man that, they say, answered avoiding with a joke the pressure
of the question: ‘God was preparing the hell for those who pry into such
deep mysteries’. A thing is to understand, and another thing is to jeer. I
will not answer this way. I would more likely answer: ‘I know not what I
know not’ [ ].”

He clarifies that with world one must understand all the creations of God. He accepts
the conclusion that the will of God is eternal, but denies that from that it follows
the eternity of the world. According to St. Augustine all the times are created by
God itself so that God comes “before” every time although this “before” must be
understood in a causal but not in a temporal way. In fact St. Augustine says (Chap.
13, Par. 16)
“It is not in time that you precedes the times. Otherwise you would not precede them all. You are always the same, your years never end. Your years nor come nor go; ours instead come and go, for all of them will come. Yours are all together because they are stable; they don’t go because of those coming, as they do not pass. Instead these, ours, will be when all shall cease to be. Your years are one day, and your day is not daily, but today; because your today yields no tomorrow, nor it follows yesterday. Your day is the eternity. You created all the times and before the times you were, and without a time there wouldn’t be any time”.

Note that St. Augustine deduces, as the Manichæans did, that the will of God cannot change, but he does not find in that any contradiction. For him, God does not perceive time as we do; not only God’s will is in a kind of permanent state but it is its very perception of time which shares this same permanence, this same eternal state.

I regard St. Augustine reply to the Manichæan question as logical given the premises. Of course although I claim that St. Augustine reply is logical I do not claim that with these considerations we are making science. Indeed, the main difficulty relies in the quite unclear subjects and verbs entering points 1, 2 and 3. However, this problem cannot be avoided from the start. The purpose of this work is to convert in a more rigorous language the sentences so far appeared in this essay. For the moment let me summarize what St. Augustine deduced from 1, 2 and 3 in the following additional points.

4. The will of God is eternal.

5. God created all the times, in particular God precedes all the times in a causal way. Nevertheless, God does not precede the times in a temporal way as the times did not exist before their creation.

6. Although God is not in our time, there is a kind of God’s perception of time radically different from that of humans. For God time is still, eternal, it is not perceived as a flow.

It is somewhat puzzling that St. Augustine used repeatedly the word ‘times’ in the plural form. Perhaps this is due to the fact that, although we often regard the Newtonian absolute time as the most intuitive and widespread notion of time, it wasn’t so at St. Augustine’s time. Another reason could be related with the concept of psychological and hence subjective time that St. Augustine had certainly elaborated (“Is in you, my mind, that I measure time” Chap. 28, Par. 36). We shall return on the relevance of this maybe accidental plurality later.

3 God and the chronology protection conjecture

Starting from assumptions 1, 2 and 3 we have been able to derive further facts on God’s nature given by points 4, 5 and 6. Despite their somewhat vague formulations these conclusions will prove quite stringent. Indeed, as we shall see, points 3,4 and 6
will suggest the mathematical object through which we should represent God, while points 1, 2 and 5 will allow us to put further constraints on a Universe admitting a God. In particular these constraints will offer new solutions to some old cosmological problems, a fact that some readers may regard as accidental while others, provided these solutions prove correct, may regard as evidence for the existence of God.

We now need to assume some familiarity of the reader with general relativity. In short the spacetime \((M, g)\), is a time oriented 4-dimensional manifold endowed with a Lorentzian metric of signature \((-,-,+,-)\).

It is believed that any reasonable spacetime should satisfy, along with Einstein’s equations, some additional causality requirement \([8]\). One of the weakest requirements that can be imposed on spacetime is that of chronology: there are no closed timelike curves (sometimes called CTC).

The fundamental problem of justifying chronology has received less attention than deserved. It is quite easy to construct solutions of the Einstein equations that violate chronology, consider for instance Minkowski spacetime with the slices \(t = 0\) and \(t = 1\) identified, or think of Gödel or Kerr’s spacetimes. Thus the problem is not if spacetime solutions of the Einstein equations can admit CTCs but rather if reasonable spacetimes not presenting CTCs may develop them.

S. Hawking argued that the laws of physics will always prevent a spacetime to form closed timelike curves, in fact he raised this expectation to the status of conjecture, now called chronology protection conjecture \([25]\). According to it the effects preventing the formation of CTCs may also be quantistic in nature, in fact Hawking claims that the divergence of the stress energy tensor at the chronology horizon (i.e. the boundary of the chronology violating set, the latter being the region over which CTCs pass) would be the principal candidate for a mechanism preventing the formation of CTCs.

Despite some work aimed at proving the chronology protection conjecture its present status remains quite unclear with some papers supporting it and other papers suggesting its failure \([26, 27, 13, 14, 10]\).

Apart from the technical motivations, the principal reason behind the rejection of spacetimes presenting chronology violations remains mostly a philosophical one. A closed timelike curve represents an observer which is forced to live an infinite number of times the same history (the grandfather paradox).

It is simply unacceptable that a human being, or any other entity presenting some form of free will, be stuck into a cycle in which always the same decisions are taken. Whatever a closed timelike curve might represent there seems to be consensus that it cannot represent the concept of “observer” to which we are used in physics.

Nevertheless, whereas the usual notion of “observer” cannot be represented by a CTC worldline, a God may indeed be represented by such worldline. Indeed, we have seen that God has an eternal will (point 4) thus faced with the same conditions he would pass through the same decisions. It cannot change direction because he confirms the correctness of the previous decision each time he is facing it.

Now, we have to expand some more on the consideration that God may be modeled by a CTC. First recall that the chronology violating set \(\mathcal{C}\) is made of all the points \(p \in M\) such that \(p \ll p\). This set splits into equivalence classes \([p]\) by means of the equivalence relation \(p \sim q\) if \(p \ll q \ll p\). In other words \(p\) and \(q\) belong to the same equivalence class if there is a closed timelike curve passing through both \(p\) and \(q\).
Moreover, in this case the timelike curve is entirely contained in \([p]\). It is possible to prove that the sets \([p]\) are all open in the manifold topology.

If \(p\) and \(q\) belong to the same chronology violating class then they have the same chronological role, in fact as \(p \ll q\) and \(q \ll p\) it is not possible to say which one comes before. They are in a sense ‘simultaneous’. Indeed, \(p\) can be connected to \(q\) also by a lightlike causal curve and the same holds in the other direction, thus it is indeed possible to move from \(p\) to \(q\) and then from \(q\) to \(p\) in zero proper time. In particular any timelike curve passing through \([p]\) would not cross events that follow ‘one after the other’ but rather almost equivalent events, actually chronologically undistinguishable. This picture fits well with point 6, that is with St. Augustine conclusion that “Your years are one day, and your day is not daily, but today; because your today yields no tomorrow, nor it follows yesterday. Your day is the eternity [ ].” All that suggests to regard God not as a single CTC, in fact given one one would get an infinite number of them in the same chronology violating class, but rather as a chronology violating class \([p]\) itself. This class \([p]\) has also to satisfy point 1, which we convert into the mathematical statement \(M = I^+([p])\), namely any point of \(M\) is chronologically preceded by a point of God.

Thus we are led to the following definition

**Definition 3.1.** On a spacetime \((M, g)\), \textit{God} is a chronology violating class \([p]\) such that \(M = I^+([p])\).

I will write this concept of God in italics in order to distinguish this technical notion from the God of the previous sections that has inspired it.

Note that given a \(\textit{God}\), then any point of \(\textit{God}\) generates \(M\) in the sense that \(p \in \textit{God} \Rightarrow I^+(p) = M\), and thus generates itself \(p \ll p\). In suggestive terms, any portion of \(\textit{God}\) creates itself and the whole world.

Provided \(\textit{God}\) exists it is unique, as the following theorem proves

**Theorem 3.2.** (Uniqueness of God) There is at most one chronology violating class \([p]\) such that \(M = I^+([p])\).

**Proof.** Indeed, suppose \(M = I^+([q]) = I^+([p])\) then \(q \in M = I^+([p]) = I^+(p)\), and with the roles of \(p\) and \(q\) interchanged we get \(p \in I^+(q)\), thus \(p \sim q\) and hence \([p] = [q]\). \(\square\)

Since to any chronology violating class \([r]\) not satisfying \(M = I^+([r])\) we can still apply the arguments relating it to points 4 and 6, we give the following definition

**Definition 3.3.** A \textit{minor God} is a chronology violating class which is not a \(\textit{God}\).

Now, the chronology protection conjecture in its original formulation may be rephrased as follows “there are no \textit{minor Gods}”, in fact the chronology protection conjecture, roughly speaking, states that chronology violating regions cannot form but does not state that they cannot exist since the beginning of the Universe. I must say, however, that any mechanism accomplishing the chronology protection would probably exclude, once applied to the backward direction, also any chronology violating region. Probably the issue as to whether there could be a mechanism that removes
while keeping a God could be answered only by showing the details of the chronology protection mechanism.

Let us assume for simplicity that there are no minor Gods and let us show in which way the definition of God satisfies point 5. Recall that a time function is a continuous function \( t : M \to \mathbb{R} \) such that \( x < y \Rightarrow t(x) < t(y) \), namely a function that increases over every causal curve. For instance any observer in Minkowski spacetime has its own time function.

Clearly, no time function can exist in the presence of a CTC, because if \( p \ll q \ll p \) then \( t(p) < t(q) < t(p) \), which is impossible. Indeed, the presence of a time function is equivalent to stable causality (i.e. causality is stable under sufficiently small perturbations of the metric) which is a much stronger causality property than chronology. Given one time function one has that a multitude of time functions exist.

Nevertheless, although \( M \) does not admit a time function, the spacetime \( M \setminus \mathcal{C} \) with the induced metric may indeed admit a time function and hence many of them. In other words, the part of spacetime not containing God (or better its closure) may admit time functions. In this sense God precedes the region of the Universe were time makes sense, but in a causal rather than a temporal way as those time functions are not defined in the region of God. This is exactly St. Augustine’s conclusion summarized by point 5.

The nice fact is that not only \( M \setminus \mathcal{C} \) may admit a time function, but that it must admit a time function, provided null geodesic completeness and other reasonable physical conditions are satisfied. For more details on these conditions see [18, 8].

**Theorem 3.4.** Let \( (M, g) \) be a spacetime that admits no minor Gods but possibly a God \([r]\). Assume that the spacetime is null geodesically complete and satisfies the null convergence condition and the null genericity condition on the null geodesics intersecting \( M \setminus [r] \). Then the spacetime \( M \setminus [r] \) is stably causal and hence admits a time function.

(For the proof see the appendix.)

The fact that the assumption of null geodesic completeness may be actually compatible with the singularity theorems has been discussed in [13].

In conclusions we have given a definition of God that satisfies some technical properties which represent pretty well points 2-6.

The figure summarizes the picture of a spacetime admitting a God. There are in fact solutions of the Einstein equations admitting a similar causal structure. The most important is the Taub-NUT metric, which so far has not been considered as a serious candidate for a cosmological solution. Here I would like to suggest that if not the metric structure, at least the causal structure of the Taub-NUT solution could indeed be similar to that of our Universe. In fact sometimes causal structures like Taub-NUT are dismissed on the ground that they have no ‘Big Bag’, no initial singularity, a fact which would contradict Hawking’s singularity theorem and observations.

This belief is incorrect: Hawking’s (1967) singularity theorem states that, given an expanding cosmological flow and some other conditions, there should be some past incomplete timelike geodesic. However, this timelike geodesic may well be totally imprisoned in a compact. In this case it may spiral towards the boundary of the chronology violating set without reaching it. In this picture the ‘Big Bang’ is replaced
by the boundary of the chronology violating region, exactly that slice that separates God from the rest of the Universe. Finally, its hot nature seems to fit well with the said divergence of the stress energy tensor that is expected according to the chronology protection conjecture. In fact there is also the possibility that the matching between the chronology violating set and the rest of the universe be accomplished up to a singular scale transformation. In this case the causal structure would be perfectly meaningful as a whole but the metric would not as it could not be continued through the boundary. For more details on these extension techniques see [15].

I conclude that it is possible to conceive a reasonable Universe whose causal structure has features analogous to Taub-NUT (Misner) and that then, after an initial phase, has the light cones tilted to match an expanding FLRW Universe. Spacetimes presenting some of these elements are for instance the $\lambda$-Taub-NUT spacetimes.

Similar models have already appeared in the literature. An important article that anticipated some ideas considered in this work is [7]. However, while in that article the authors focused on the problem of quantum field theory in spacetime with CTCs, here I shall consider mainly the problem of homogeneity and entropy and its relation with causality. In particular, I will introduce the idea of the rigidity at the boundary of the chronology violating region (see next section).

![Diagram of a Universe with S^1 section](image)

Figure 1: A Universe with $S^1$ section which gives an idea of the cosmological picture presented in this work. The region that admits time functions is causally preceded by the chronology violating set (God) as in St. Augustine’s conclusions.
4 The homogeneity and entropy problems

The cosmic microwave background (CMB) formed when, after a sufficient expansion of the Universe, the density of matter decreased to a level that light decoupled from it (the mean free path of light became infinite). The set of events of departure of those photons form an ideal last scattering hypersurface. Today we observe just a portion of that hypersurface, namely the intersection of it with our past light cone. Since we observe that the CMB radiation has the same spectra (temperature) independently of the direction of observation, there is the problem of justifying such homogeneity (or isotropy) given the fact that the regions that emitted that radiation were so far apart that, according to the FLRW scenario, they didn’t have any past point in common.

It is often claimed that inflation solves this problem. The idea is that if a patch of space expands so much, in the initial phase of the Universe, to include the whole surface we see today, then it should be natural to observe homogeneity. This argument works only if homogeneity is assumed at a different scale, actually at a much smaller scale, prior to inflation, namely if the initial patch is considered homogeneous.

This criticism has been moved to inflation by several authors, as rather that solving the problem of homogeneity, inflation seems to replace a type homogeneity assumption with another [2, 3, 22]. R. Penrose argues that inflation may well prove to be correct but not for the initial arguments moved in its favor [22].

Instead, the assumption that there is a chronology violating region generating the whole Universe, that is a God, explains rather easily the homogeneity of the CMB radiation. Indeed, the explanation has nothing to do with the expansion of the Universe (namely to the conformal scale factor) but rather to its causal structure. In our model any point \( r \) in the last scattering hypersurface contains, in its own past, the chronology violating region of God, namely \([p] \subset I^- (r)\) and in fact its boundary. Thus the chronological pasts of the points in the last scattering hypersurface share many points on spacetime, and thus it is reasonable that they have similar temperatures.

However, it would be incorrect to think that the points of the last scattering hypersurface have the same temperature because of thermalization. Instead, the Universe is already at a very homogeneous state at the boundary of the chronology violating region. Indeed, some mathematical results, connected with the concept of compactly generated Cauchy horizon and imprisoned curves [12, 17], suggest that this boundary must be generated by lightlike geodesics whose closure is exactly the boundary (as it happens in figure 1). Given any two points on the boundary \( p, q \), one would have \( q \in I^+ (p) \) and \( p \in I^+ (q) \), thus in practice they could be considered as causally related. As they can communicate through the boundary, this boundary is expected to attain an homogeneous temperature prior to any subsequent expansion.

This mechanism is welcome, indeed R. Penrose [21, 22] has pointed out that the thermalization mechanism cannot be considered as a satisfactory explanation for homogeneity as it conflicts with the so called entropy problem to which we shall return in a moment.

Of course this mechanism may be followed by that of inflation, but we point out that it does not seem to be necessary. Indeed, the main accomplishment of inflation seems to be its ability to predict the correct density inhomogeneities over the homogeneous background. Hollands and Wald [9] have recently argued that not
only inflation does not satisfactorily solve the homogeneity problem but also that the desired scale free spectra of the perturbations can be obtained even in absence of inflation. They therefore claim that the main problem is that of homogeneity/isotropy as they could not find any dynamical mechanism for it. We argued that such a mechanism exists, the solution lies in assuming the existence of a chronology violating region from which the Universe develops: a God in our terminology.

Let us come to the entropy problem. This difficulty of standard cosmology arises when considering the huge difference between the entropy of the Universe today with that at the time of the Big Bang. R. Penrose by taking into account also the gravitational entropy, has argued that the Universe at its beginning had probably to be thermalized, to account for the homogeneity problem, but nevertheless it had to be special as the calculation of the entropy shows that it was much smaller that today. In his view the Universe could increase in entropy despite its initial thermalization because in the beginning the gravitational degrees of freedom were almost frozen.

He also notes that when matter is left to the action of gravity it tends to clump, passing from an homogeneous state to an inhomogeneous one. The Weyl tensor increases because of this clumping, and therefore this tensor may quantify in some sense the amount of entropy contained in the gravitational degrees of freedoms. Thus Penrose ends suggesting that in the beginning of the universe the Weyl tensor had to be very small, and possibly zero. This is Penrose’s Weyl tensor hypothesis [21, 6].

The picture of the beginning of the Universe presented in this work is likely to satisfy the Penrose’s Weyl tensor hypothesis. Indeed, as I mentioned, the boundary of the chronology violating region would be generated by lightlike geodesics (which are moreover achronal). Now, there is a rigidity result [1] which states that an asymptotically simple vacuum spacetimes is isometric to Minkowski spacetime in a neighborhood of every achronal lightlike geodesic (Galloway’s null splitting theorem [4]). I expect that analogous results should hold for the case considered in this work, that is, I expect the spacetime near the boundary of the chronology violating region to be isometric to some highly symmetric spacetime. This rigidity would clearly fix the Weyl tensor and thus send to zero the degrees of freedom contained in it.

Note that in this case the null genericity condition would not hold for geodesics lying on the boundary of the chronology violating region. Fortunately we do not need it in theorem 3.4 hence its consequences are consistent with the rigidity of the boundary.

5 Conclusions

In this work I presented a picture for the beginning of the Universe which seems to be able to solve the homogeneity and entropy problems. The main idea is that the Big Bang has to be replaced with a null hypersurface such that all the points on it have the same chronological future (i.e. future distinction is violated). As a consequence they (almost) causally communicate between them and so homogeneity and thermalization hold there. Moreover, the points in the last scattering (spacelike) hypersurface have chronological pasts that contain one and hence all points of this null hypersurface, a fact which clarifies the observed temperature homogeneity.
If proved necessary, the just mentioned beginning of the Universe may be followed by a period of inflation, so that it is indeed possible to join the good accomplishments of inflation with the solution of the homogeneity and entropy problems given by the above idea.

By a stability argument, the spacetime once continued through the null hypersurface must develop closed timelike curves. Indeed, a spacetime in which the cones tilt in the opposite sense as in figure 37 of [8] would have a null hypersurface (and hence a failure of distinction) that disappears under a small perturbation of the metric.

With the aim of solving the homogeneity and entropy problem one is therefore naturally led to the idea of a chronology violating region from which the whole Universe has developed [7].

I showed that this picture for a Universe fits pretty well with some conclusions reached by St. Augustine while he was answering some questions raised by the Manichæans. To appreciate the correspondence it is necessary to identify God with the chronology violating set that precedes the whole Universe.

I must say that I was developing the physical content of this work before discovering St. Augustine thought in the Confessions. Nevertheless, I was so puzzled by the correspondence that decided to present them in conjunction so as to stress the similarities. While doing so I discovered some unexpected results like theorem 3.4 which I missed in previous analysis of similar problems, a fact which to my mind confirmed the correspondence.

One may ask how it happened that St. Augustine went so close to the model of Universe presented in these pages, given that he certainly ignored general relativity. My own opinion is that while thinking about a subject there are many ways of coming to trivial or incorrect conclusions, whereas only a few paths can lead to correct or at least interestingly structured thoughts. It is therefore not an accident that St. Augustine deep reflections on time, creation and God can find today a correspondence in general relativity. It should suffice to consider that the latter is the most advanced theory we have ever had on the dynamics of time.

Acknowledgments

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Appendix A: The boundary of the chronology violating region

For clarity I include some unpublished results on the boundary of the chronology violating region. The boundary of a set is denoted with a dot. In some cases in which this notation could be ambiguous the dot is replaced by the symbol $\partial$. The subset symbol $\subset$ is reflexive, i.e. $X \subset X$. 
Recall that a future lightlike ray is a future inextendible achronal causal curve, in particular it is a lightlike geodesic. Past lightlike rays are defined analogously. A lightlike line is an achronal inextendible causal curve. In particular a lightlike line is a lightlike geodesic without conjugate points.

For the proof of the next lemma see [11, Prop. 2], or the proof of [19, Theorem 12].

**Lemma 5.1.** Let \([r]\) be a chronology violating class. If \(p \in \hat{r}\) then through \(p\) passes a future lightlike ray contained in \([r]\) or a past lightlike ray contained in \([r]\) (and possibly both).

**Definition 5.2.** Let \([r]\) be a chronology violating class. The set \(R_f([r])\) is that subset of \([r]\) made of the points \(p\) through which passes a future lightlike ray contained in \([r]\). The set \(R_p([r])\) is defined analogously.

**Lemma 5.3.** The sets \(R_p([r])\) and \(R_f([r])\) are closed and \(\hat{r} = R_p([r]) \cup R_f([r])\).

**Proof.** It is a consequence of the fact that a sequence of future lightlike rays \(\sigma_n\) of starting points \(x_n \to x\) has as limit curve a future lightlike ray of starting point \(x\) [16], and analogously in the past case. Clearly, by lemma 5.1 \([r] = R_p([r]) \cup R_f([r])\). \(\square\)

Note that it can be \(R_p([r]) \cap R_f([r]) \neq \emptyset\).

**Lemma 5.4.** Let \([r]\) be a chronology violating class then \(I^-([r]) = I^-(\hat{r})\) and the following sets coincide:

(i) \(\overline{[r]} \setminus I^-([r])\),

(ii) \(\hat{r} \setminus I^-([r])\),

(iii) \(R_f([r]) \setminus I^-([r])\).

**Proof.** The inclusion \(I^-([r]) \subset I^-(\hat{r})\) is obvious. The other direction follows immediately from the fact that \(I^+\) is open.

\(R_f([r]) \setminus I^-([r]) \subset \overline{[r]} \setminus I^-([r])\) is trivial, \(\overline{[r]} \setminus I^-([r]) \subset \hat{r} \setminus I^-([r])\) follows from \([r] \subset I^-([r])\), and it remains to prove \(\hat{r} \setminus I^-([r]) \subset R_f([r]) \setminus I^-([r])\). Let \(p \in \hat{r} \setminus I^-([r])\), there is a sequence \(p_n \in [r]\), \(p_n \to p\). Since \(p_n \in [r]\) there are timelike curves \(\sigma_n\) entirely contained in \([r]\) which connect \(p_n\) to \(r\). By the limit curve theorem there is either (a) a limit continuous causal curve connecting \(p\) to \(r\), in which case as \([r]\) is open, \(p \in I^-([r])\), a contradiction, or (b) a limit future inextendible continuous causal curve \(\sigma\) starting from \(p\) and contained in \([r]\). Actually \(\sigma\) is contained in \([r]\) otherwise \(p \in I^-([r])\), a contradiction. Moreover, \(\sigma\) is a future lightlike ray, otherwise there would be \(q \in \hat{r} \cap \sigma\), \(p \ll q\) and as \(I^+\) is open \(p \in I^-([r])\), a contradiction. \(\square\)

Let us define the sets \(B_f([r]) = \overline{[r]} \setminus I^-([r])\) and \(B_p([r]) = \overline{[r]} \setminus I^+([r])\).

**Lemma 5.5.** The set \(B_f([r])\) is closed, achronal, and generated by future lightlike rays. Analogously, the set \(B_p([r])\) is closed, achronal, and generated by past lightlike rays.
Proof. Let us give the proof for $B_f([r])$, the proof for the other case being analogous. The closure of $B_f([r])$ is immediate from the definition. Let $p \in B_f([r])$, as $p \in R_f([r])$ there is a future lightlike ray starting from $p$ entirely contained in $[r]$ and hence in $R_f([r])$. Moreover, no point of this ray can belong to $I^-([r])$ otherwise $p$ would belong to $I^-([r])$. We conclude that the whole ray is contained in $B_f([r])$.

Let us come to the proof of achronality. Assume by contradiction that there is a timelike curve $\sigma : [0, 1] \to M$ whose endpoints $p = \sigma(0)$ and $q = \sigma(1)$ belong to $B_f([r])$. There cannot be a value of $t \in (0, 1)$ such that $\sigma(t) \in [r]$ otherwise as $I^+$ is open, and $p, q \in [r]$, we would have $r \ll \sigma(t) \ll r$, that is $\sigma(t) \in [r]$, in contradiction with $\sigma(t) \in [r]$. Thus either $\sigma((0, 1))$ is contained in $[r]$ or it is contained in $M \setminus [r]$. The former case would imply $p \in I^-([r])$, a contradiction. In the latter case it is possible to find $z \in \sigma((0, 1)) \cap M \setminus [r]$, and as $p \ll z \ll q$ and $I^+$ is open, $r \ll z \ll r$, a contradiction.

**Proposition 5.6.** Let $[r]$ be a chronology violating class, then $I^+([r]) \cap [r] \subset B_f([r])$ and $I^-(|r]) \cap [r] \subset B_p([r])$. Moreover, if $p \in I^+([r]) \cap R_p([r])$ or $I^-([r]) \cap R_f([r])$ then through $p$ passes a lightlike geodesic contained in $[r]$.

**Proof.** Let us prove the former inclusion, the latter being analogous.

Let $q \in I^+([r]) \cap [r]$, we have only to prove that $q \notin I^-(|r])$. If it were $q \in I^-(|r])$ then $r \ll q \ll r$, a contradiction.

Let us come to the last statement. As $p \in R_p([r])$ there is a past lightlike ray $\eta$ contained in $[r]$ ending at $p$. As $p \in I^+([r]) \cap [r] \subset R_f([r])$, there is a future lightlike ray $\sigma$ passing through $p$ and contained in $[r]$. This ray is the continuation of the past lightlike ray $\eta$. Indeed, assume that they do not join smoothly at $p$. Take a point $x \in I^+(r) \cap \eta \setminus \{p\}$, so that, because of the corner at $p$, $\sigma \setminus \{p\} \subset I^+(x)$. Again, since $I^+(x)$ is open and $\sigma \subset [r]$ we have $x \ll r$, thus since $r \ll x$, we conclude $x \ll z$ which is impossible as $x \in \eta \subset [r]$. We have therefore obtained a lightlike geodesic $\gamma = \sigma \circ \eta$ passing through $p$ entirely contained in $[r]$.

**Corollary 5.7.** The following identity holds: $[r] = B_p([r]) \cup B_f([r])$.

**Proof.** In a direction the inclusion is obvious, thus since $B_p([r]) = [r] \setminus I^+([r])$ and $B_f([r]) = [r] \setminus I^-([r])$ we have only to prove that if $p \in [r]$ then $p \notin I^+([r])$ or $p \notin I^-([r])$. Indeed, if $p$ belongs to both sets $r \ll p \ll r$, a contradiction.

Note that it can be $B_p([r]) \cap B_f([r]) \neq \emptyset$.

The previous results justify the following definition

**Definition 5.8.** The sets $B_f([r])$ and $B_p([r])$ are respectively the future and the past boundary of the chronology violating class $[r]$.

**Proposition 5.9.** Let $[r]$ be a chronology violating class then $I^+ (B_f([r])) \cap [r] = \emptyset$. Moreover, if $B_f([r]) \neq \emptyset$ then $I^-(B_f([r])) = I^-([r])$. Analogous statements hold in the past case.
Proof. If there were a \( p \in B_f([r]) \) such that \( I^+(p) \cap [r] \neq \emptyset \) then \( p \in I^-([r]) \), a contradiction.

In a direction, \( I^- (B_f([r])) \subset I^-(\overline{r}) = I^-(r) \). In the other direction, assume \( I^+(r) \cap B_f([r]) \neq \emptyset \), then there is \( q \in I^- (B_f([r])) \cap [r] \), hence \( I^-(r) = I^-(q) \subset I^- (B_f([r])) \).

The alternative \( I^+(r) \cap B_f([r]) = \emptyset \) cannot hold, indeed no point of \( I^+(r) \) would stay outside \([r]\) as this would imply that \( I^+(r) \cap \bar{r} \neq \emptyset \) and hence because of \( I^+(r) \cap [r] \subset B_f([r]) \), \( I^+(r) \cap B_f([r]) \neq \emptyset \). Thus the case \( I^+(r) \cap B_f([r]) = \emptyset \) leads to \( I^+([r]) \subset [r] \) and hence \( I^+(r) = [r] \), i.e. \([r]\) is a future set. As \( B_f([r]) \subset [r] \), and \( B_f([r]) \neq \emptyset \) taken \( x \in B_f([r]) \) by the property of future sets, \( I^+(x) \subset [r] \) hence \( x \in I^-([r]) \) in contradiction with the definition of \( B_f([r]) \).

\[ \Box \]

**Proposition 5.10.** Let \([r]\) be a chronology violating class then \( B_f([r]) = [r] \) if and only if \( B_p([r]) = \emptyset \). Analogously, \( B_p([r]) = [r] \) if and only if \( B_f([r]) = \emptyset \).

Proof. The direction \( B_p([r]) = \emptyset \Rightarrow B_f([r]) = [r] \) follows from \([r] = B_p([r]) \cup B_f([r]) \). For the converse, assume \( B_f([r]) = [r] \) and that, by contradiction, \( p \in B_p([r]) \) (hence \( p \in B_p([r]) \cap B_f([r]) \)), then \( I^-(p) \) has no point in \([r]\) otherwise \( p \in I^+(r) \) and hence \( p \notin B_f([r]) \), a contradiction. Thus if \( p \in B_p([r]) \) then \( I^-(p) \cap [r] = \emptyset \). Take \( q \preceq p \), as \( I^+ \) is open and \( p \in [r] \) there is a timelike curve joining \( q \) to \( r \). This curve intersects \([r]\) at some point \( x \), thus \( x \in [r] \cap I^-([r]) \), and \( x \notin B_f([r]) \), a contradiction. We conclude that \( B_p([r]) = \emptyset \). The proof of the time reversed case is analogous.

For the definition of the edge of an achronal set see [8] Sect. 6.5 or [1] Def. 14.27.

**Proposition 5.11.** \( \text{edge}(B_f([r])) = \text{edge}(B_p([r])) \).

Proof. Let \( q \in \text{edge}(B_f([r])) \) then for every neighborhood \( U \ni q \) there are \( x, y \in U \), \( x \preceq q \preceq y \) and a timelike curve \( \sigma \) not intersecting \( B_f([r]) \) connecting \( x \) to \( y \) entirely contained in \( U \). The point \( y \) cannot belong to \( \overline{r} \) for otherwise \( q \in I^-([r]) \) and hence \( q \notin B_f([r]) \) (recall that the edge of an achronal closed set belongs to the same set), a contradiction. Every intersection point of \( \sigma \) with \([r]\) does not belong to \( B_f([r]) \), and hence belongs to \( B_p([r]) \). There cannot be more than one intersection point otherwise if \( z_1 \preceq z_2 \) are any two intersection points, \( z_2 \in I^+(z_1) \subset I^+(r) \) thus \( z_2 \) cannot belong to \( B_p([r]) \), a contradiction. Moreover, \( \sigma \) cannot enter \([r]\) otherwise, by the same argument, the intersection point with \([r]\) would not belong to \( B_p([r]) \), a contradiction. We conclude that \( \sigma \setminus \{ x \} \subset M \setminus \overline{r} \) with possibly \( x \in B_p([r]) \). However, we can redefine \( x \) by slightly shortening \( \sigma \) so that we can assume \( \sigma \in M \setminus \overline{r} \). It remains to prove that \( q \in B_p([r]) \), from which it follows, as \( \sigma \) does not intersect \( B_p([r]) \), \( q \in \text{edge}(B_p([r])) \). Assume by contradiction, \( q \notin B_p([r]) \), so that \( q \in I^+(r) = I^+(r) \). Since the previous analysis can be repeated for every \( U \ni q \), we can find a sequence \( x_n \notin [r] \), \( x_n \to q \), \( x_n \preceq q \). As \( I^+(r) \) is open we can assume \( x_n \gg r \), but since \( x_n \preceq q \) and \( q \notin [r] \), we have also \( x_n \ll r \), thus \( x_n \in [r] \), a contradiction. We conclude that \( \text{edge}(B_f([r])) \subset \text{edge}(B_p([r])) \) and the other inclusion is proved similarly. 

\[ \Box \]
From the previous proposition it follows that edge($B_f([r])$) ⊂ $B_f([r]) \cap B_p([r])$, however, the reverse inclusion does not hold in general.

**Proposition 5.12.** Let [r] be a chronology violating class such that $I^+( [r] ) = M$ then $[r] = B_f([r])$, $J^- ([r]) = [r]$ and $I^- ([r]) = [r]$.

**Proof.** Since $I^+( [r] ) \cap [r] \subset B_f([r])$ we have $[r] \subset B_f([r])$ and hence the first equality. For the second equality the inclusion $[r] \subset J^- ([r])$ is obvious. For the other direction assume by contradiction, $p \in J^- ([r]) \setminus [r]$. Since $p \in M = I^+(r)$ there is a timelike curve joining $r$ to $p$ and a causal curve joining $p$ to $[r]$. By making a small variation starting near $p$ we get a timelike curve from $r$ to $[r]$, and hence equivalently, from $r$ to $r$ passing arbitrarily close to $p$, thus $p \in [r]$, a contradiction.

For the last equality it suffices to take the interior of the second one. □

The next result shows that, provided the chronal region is globally hyperbolic, the past Cauchy horizon of a suitable hypersurface is the future boundary of the chronology violating set. Thus this result relates our definition of boundary with the more restrictive one given in some other papers [25].

**Proposition 5.13.** Let [r] be a chronology violating class and assume that the manifold $N = M \setminus \overline{I^- ([r])}$ with the induced metric is globally hyperbolic, then for every Cauchy hypersurface $S$ of $N$, $S$ is edgeless in $M$ and $H^- (S) = \partial I^- ([r])$. In particular if $M = I^+( [r] )$ then $H^- (S) = [r] = B_f([r])$.

**Proof.** Since $S$ is a (acausal) Cauchy hypersurface for $N$, $\text{Int}D(S) = N$, thus $\partial D(S) \subset \overline{N} = \partial I^- ([r])$. The set $S$ has no edge in $N$, however, it has no edge also in $M$. Indeed, let $q \in \text{edge}(S)$, then as $S$ is closed in $N$, $q \in \partial I^- ([r])$. But $I^+(q)$ is an open set that cannot intersect the past set $I^- ([r])$, thus $I^+(q) \subset N$, moreover no inextendible timelike curve starting from $q$ (e.g. a geodesic) can intersect $S$ for otherwise $S$ would not be achronal. But since such curve would be inextendible in $N$ this would contradict the fact that $S$ is a Cauchy hypersurface. Thus $\text{edge}(S) = \emptyset$.

Note that $\partial D(S) = H^+(S) \cup H^- (S)$, thus $H^- (S) \subset I^- ([r])$. For the converse note that if $p \in \partial I^- ([r])$, $I^+(p)$ is an open set that cannot intersect $I^-(r)$, thus $I^+(p) \subset N$. Note that $p \in I^- (S)$ for otherwise a future inextendible timelike curve issued from $p$ would not intersect $S$, still when regarded as an inextendible curve in $N$ this empty intersection would contradict the fact that $S$ is a Cauchy hypersurface. Since $p \in I^- (S)$ the points in $I^+(p) \cap I^+(S)$ necessarily belong to $D^- (S)$ and moreover $p$ does not belong to $\text{Int}D^-(S)$ because the points in $I^- ([r])$ clearly do not belong to $D^- (S)$, as the future inextendible timelike curves issued from there may enter the chronology violating set $[r]$ and remain there confined. Thus $p \in H^- (S)$.

By the previous result if $M = I^+([r])$ then $I^- ([r]) = [r]$ and $B_f([r]) = [r]$. □

**Appendix B: The proof of theorem 3.4.**

Here I give the proof of theorem 3.4. It is a non-trivial generalization over that given in [19]. Note that null geodesic completeness is required only on those geodesics intersecting $M \setminus [r]$. These geodesics cannot be tangent to some geodesic generating the
boundary \([\mathcal{r}] = B_f(\mathcal{r})\), because since this boundary is generated by future lightlike rays contained in \([\mathcal{r}]\) the geodesic would have to be contained in \(\overline{\mathcal{r}}\), a contradiction.

**Proof.** Consider the spacetime \(N = M \setminus \mathcal{r}\) with the induced metric, and denote by \(J^+_N\) its causal relation. This spacetime is clearly chronological and in fact strongly causal. Indeed, if strong causality would fail at \(p \in N\) then there would be sequences \(p_n, q_n \to p\), and causal curves \(\sigma_n\) of endpoints \(p_n, q_n\), entirely contained in \(N\), but all escaping and reentering some neighborhood of \(p\). By an application of the limit curve theorem \([16, 1]\) there would be an inextendible lightlike geodesic \(\sigma\) passing through \(p\) and contained in \(\overline{N}\). With respect to \(N\) this geodesic could be incomplete but in \(M\) it is complete by assumption, and using the other hypothesis it also has two conjugate points. As a consequence it is not hard to show, by taking a timelike variation of \(\sigma\), and building up closed timelike curves as close as one wishes to \(p\), that \(p \in \mathcal{C}\), a contradiction. The next step is to prove that \(\overline{J^+_N}\) is transitive. In this case \(N\) would be causally easy \([20]\) and hence stably causal. The transitivity of \(\overline{J^+_N}\) is proved as done in \([19, \text{Theorem 5}]\), the only difference is that the argument allows us only to prove that if, \(x, y, z \in N\), \((x, y) \in \overline{J^+_N}\) and \((y, z) \in \overline{J^+_N}\), then \((y, z) \in \overline{J^+}(= \overline{I^+})\) as the limit causal curve passing through \(y\) may intersect \(N\). However, there are neighborhoods \(U\) and \(V\) such that any timelike curve connecting \(U \ni x, U \subset N\) to \(V \ni z, V \subset N\) must stay in \(N\), because otherwise there would be some \(w \in [\mathcal{r}]\) such that \(x' \leq w\), with \(x' \in U\). This is impossible because by proposition \([5, 12]\) \(J^-(\mathcal{r}) \subset [\mathcal{r}]\).

\[\square\]

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