Local limit theorems for a directed random walk on the backbone of a supercritical oriented percolation cluster

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May 20, 2021

We consider a directed random walk on the backbone of the supercritical oriented percolation cluster in dimensions $d+1$ with $d \geq 3$ being the spatial dimension. For this random walk we prove an annealed local central limit theorem and a quenched local limit theorem. The latter shows that the quenched transition probabilities of the random walk converge to the annealed transition probabilities reweighted by a function of the medium centred at the target site. This function is the density of the unique measure which is invariant for the point of view of the particle, is absolutely continuous with respect to the annealed measure and satisfies certain concentration properties.

Keywords: Random walk in dynamical random environment; oriented percolation; supercritical cluster; quenched local limit theorem in random environment, environment viewed from the particle

AMS Subject Classification: 60K37; 60J10; 82B43; 60K35;

1. Introduction

Random walks in a static or dynamic random environment arise in different models from physical and biological sciences. The investigation of such random walks under different conditions on the environment has been an active research area with a lot of recent progress. In this paper, we analyse a directed random walk on the backbone of a supercritical oriented percolation cluster on $\mathbb{Z}^d \times \mathbb{Z}$. This random walk was introduced and studied in [BCDG13]. There it was shown that the random walk satisfies a law of large numbers and a quenched central limit theorem for all spatial dimensions $d \geq 1$. The main purpose of this work is to extend these results and derive a quenched local limit theorem. For this, we will have to restrict ourselves to spatial dimensions $d \geq 3$. Analogous results for a class of ballistic random walks in uniformly elliptic i.i.d. random environments were recently obtained in [BCR16]. This paper has been an inspiration and a guide for the present study.

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1.1. The model and background

Consider a discrete space-time field $\omega := \{\omega(x, n) : (x, n) \in \mathbb{Z}^d \times \mathbb{Z}\}$ of i.i.d. Bernoulli random variables with parameter $p \in [0, 1]$ taking values in $\Omega := \{0, 1\}^{\mathbb{Z}^d \times \mathbb{Z}}$, which is defined on some (large enough) probability space equipped with a probability measure $P$.

As common in percolation theory, a space-time site $(x, n) \in \mathbb{Z}^d \times \mathbb{Z}$ is said to be open if $\omega(x, n) = 1$ and closed if $\omega(x, n) = 0$. A directed open path (with respect to $\omega$) from $(x, m)$ to $(y, n)$ for $m \leq n$ is a space-time sequence $(x_m, m), \ldots, (x_n, n)$ such that $x_m = x$, $x_n = y$, $\|x_k - x_{k-1}\| \leq 1$ for $k = m + 1, \ldots, n$ and $\omega(x_k, k) = 1$ for all $k = m, \ldots, n$. Here, and in the following $\|\cdot\|$ denotes the sup-norm on $\mathbb{R}^d$. We will write $(x, m) \xrightarrow{\omega} (y, n)$ if such an open path exists and $(x, m) \xrightarrow{\omega} \infty$ if there exists at least one infinite directed open path starting at $(x, m)$, i.e. if for each $n > m$ there is $y \in \mathbb{Z}^d$ so that $(x, m) \xrightarrow{\omega} (y, n)$.

It is well known that there is $p_c = p_c(d) \in (0, 1)$ such that $\mathbb{P}((0, 0) \xrightarrow{\omega} \infty) > 0$ if and only if $p > p_c$; see e.g. Theorem 1 in [GH02]. We consider here only the case of a fixed $p \in (p_c, 1]$. We define by

$$C := \{(x, n) \in \mathbb{Z}^d \times \mathbb{Z} : (x, n) \xrightarrow{\omega} \infty\}$$

the backbone of the space-time cluster of the oriented percolation, i.e. the set of all space-time sites which are connected to “time $+\infty$” by a directed open path. Note that $C$ depends on $\omega$ and that we have $\mathbb{P}(|C| = \infty) = 1$ for $p > p_c$. For future reference we define the process $\xi := (\xi_n)_{n \in \mathbb{Z}}$ on $(0, 1)^{\mathbb{Z}^d}$ by

$$\xi_n(x) = \mathbb{1}_C((x, n)).$$

The process $\xi$ can be interpreted as the time reversal of the stationary discrete time contact process. In particular, for any $n \in \mathbb{Z}$ the random field $\xi_n(\cdot)$ is distributed according to the upper invariant measure of the discrete time contact process, which is non-trivial in the case $p > p_c$. For more details we refer the reader to Section 1 (around equation (1.2)) in [BCDG13], see also [BG19].

Our goal is to study the directed random walk on the cluster $C$. This random walk was studied in [BCDG13] in the case that the initial point of the random walk belongs to the cluster. Here we want to compare the annealed and quenched laws for starting points without checking whether they are on the cluster or not. Thus, we define the random walk slightly differently: It behaves as a simple random walk (which jumps uniformly to one of the sites in the unit ball around the present site) as long as it is not on the cluster and once it hits the cluster it will behave as the random walk from [BCDG13]. For a site $(x, n) \in \mathbb{Z}^d \times \mathbb{Z}$ we define its neighbourhood at time $(n + 1)$ by

$$U(x, n) := \{(y, n + 1) : \|x - y\| \leq 1\}.$$  

Given $\omega$ and therefore the random cluster $C$ and $(y, m) \in \mathbb{Z}^d \times \mathbb{Z}$ we consider random walks $(X_n)_{n=m,m+1,\ldots}$ with initial position $X_m = y$ and transition probabilities for $n \geq m$ given by

$$\mathbb{P}(X_{n+1} = z | X_n = x, \omega) = \begin{cases} 
|U(x, n) \cap C|^{-1} & \text{if } (x, n) \in C \text{ and } (z, n + 1) \in U(x, n) \cap C, \\
|U(x, n)|^{-1} & \text{if } (x, n) \notin C \text{ and } (z, n + 1) \in U(x, n), \\
0 & \text{otherwise.}
\end{cases}$$
We write $P_\omega$ for the conditional law of $\mathbb{P}$ given $\omega$ and $E_\omega$ for the corresponding expectation. In particular, for the transition probabilities we have
\begin{equation}
P_\omega(X_{n+1} = z \mid X_n = x) = \mathbb{P}(X_{n+1} = z \mid X_n = x, \omega).
\end{equation}
For the above random walk starting from position $X_m = y \in \mathbb{Z}^d$ at time $m \in \mathbb{Z}$ we denote by $P^{(y,m)}_\omega$ its quenched law and by $E^{(y,m)}_\omega$ the corresponding expectation. The annealed (or averaged) law of that random walk is denoted by $\mathbb{P}^{(y,m)}$ and its expectation by $E^{(y,m)}$. Note that for any $A \in \sigma(X_n : n = m, m+1, \ldots)$ we have
\begin{equation}
\mathbb{P}^{(y,m)}(A) = \int P^{(y,m)}_\omega(A) \, d\mathbb{P}(d\omega).
\end{equation}

### 1.2. Main results: annealed and quenched local limit theorems

In Theorem 1.1 in [BCDG13] it is shown that the random walk $(X_n)$ starting in $0 \in \mathbb{Z}^d$ at time 0 satisfies an annealed central limit theorem and the limiting law is a non-trivial centred isotropic $d$-dimensional normal law. In particular its covariance matrix $\Sigma$ is of the form $\sigma^2 I_d$ for a positive constant $\sigma^2$ and the $d$-dimensional identity matrix $I_d$. Recall that in [BCDG13] it is assumed that the space-time origin is contained in $C$ so that the random walk starts and stays on $C$. This is not a big constraint because the time a random walk needs to hit the cluster $C$ has exponentially decaying tails; see Lemma B.1 in the appendix.

The annealed CLT from [BCDG13] can be strengthened to an annealed local CLT. For a proof of the following theorem we refer to Section 3.

**Theorem 1.1** (Annealed local CLT). For $d \geq 1$ and $\Sigma$ as above we have
\begin{equation}
\lim_{n \to \infty} \sum_{x \in \mathbb{Z}^d} \left| \mathbb{P}^{(0,0)}(X_n = x) - \frac{1}{(2\pi n)^{d/2} \sqrt{\det \Sigma}} \exp \left( -\frac{1}{2n} x^T \Sigma^{-1} x \right) \right| = 0.
\end{equation}

Theorem 3.1 in [BCDG13] extends the annealed CLT therein to a quenched version with the same limiting law. This result was later refined in [Ste17, Chapter 3], where a comparison result between the quenched and annealed laws after $N$ steps on the level of boxes of side length $N^{1/2}$ for $\theta \in (0, 1)$ was obtained. We recall this result in Theorem 8.1 below.

The main goal of this paper is to strengthen this further and prove a quenched local limit theorem which is an analogue of Theorem 1.1. In order to state the precise result, we need to introduce some notation. First, for $(y, m) \in \mathbb{Z}^d \times \mathbb{Z}$, we define the space-time shift operator $\sigma$ on $\Omega$ by
\begin{equation}
\sigma_{(y,m)} \omega(x, n) := \omega(x + y, n + m)
\end{equation}
and we write $\xi_m(y; \omega)$ for $\xi_m(y)$ read off from a given realization $\omega$ as in (1.1) and (1.2). We define the transition kernel for the environment seen from the particle (compare this with (1.4)) by
\begin{equation}
\mathcal{R} f(\omega) := \sum_{\|y\| \leq 1} g(y; \omega) f(\sigma_{(y,1)} \omega)
\end{equation}
acting on bounded measurable functions $f : \Omega \to \mathbb{R}$, where
\begin{equation}
g(y; \omega) := \mathbbm{1}_{\{\sum_{\|z\| \leq 1} \xi_1(z; \omega) > 0, \omega(0,0) = 1\}} \sum_{\|z\| \leq 1} \frac{\xi_1(y; \omega)}{\xi_1(z; \omega)} + \mathbbm{1}_{\{\sum_{\|z\| \leq 1} \xi_1(z; \omega) = 0 \text{ or } \omega(0,0) = 0\}} \frac{1}{3^d}.
\end{equation}
Definition 1.2. A measure $Q$ on $\Omega$ is called invariant with respect to the point of view of the particle if for every bounded continuous function $f : \Omega \rightarrow \mathbb{R}$

$$\int_{\Omega} f(\omega) \, dQ(\omega) = \int_{\Omega} f(\omega) \, dQ(\omega).$$  \hfill (1.11)

Theorem 1.3. Let $d \geq 3$ and $p \in (p_c, 1]$. Then there exists a unique measure $Q$ on $\Omega$ which is invariant with respect to the point of view of the particle satisfying $Q \ll \mathbb{P}$ and the concentration property (2.9) below.

The main result of this paper is a quenched local limit theorem which is an analogue of Theorem 1.11 in [BCR16] in the case of our model.

Theorem 1.4 (Quenched local limit theorem). Let $d \geq 3$ and $p \in (p_c, 1]$, let $Q$ be the measure from Theorem 1.3 and denote by $\varphi = dQ/d\mathbb{P} \in L_1(\mathbb{P})$ the Radon–Nikodym derivative of $Q$ with respect to $\mathbb{P}$. Then for $\mathbb{P}$ almost every $\omega$ we have

$$\lim_{n \to \infty} \sum_{x \in \mathbb{Z}^d} \left| P_{\omega}^{(0,0)}(X_n = x) - \mathbb{P}^{(0,0)}(X_n = x) \varphi(\sigma_{(x,n)} \omega) \right| = 0. \hfill (1.12)$$

Remark 1.5 (Uniqueness of $Q$). It will be proven in Lemma 10.1 that the function $\varphi$ in (1.12) is $\mathbb{P}$ almost surely unique. Furthermore, it will follow from the arguments in the proofs (cf. also Remark 2.6) that if a measure $Q'$ on $\Omega$ is invariant with respect to the point of view of the particle and satisfies $Q' \ll \mathbb{P}$ and (1.12) with $\varphi' = dQ'/d\mathbb{P}$ then this measure $Q'$ satisfies the concentration property (2.9) as well and thus in particular agrees with $Q$.

Related literature Random walks in static and dynamic random environments is a very active research area. For a review of random walks in random environments and basic concepts and objects we refer the reader to [Szn04]; for a more recent review see [DR14].

The random walk that we consider here can be seen as a random walk in a dynamic random environment. Its relation to random walks in dynamic random environments in the literature is discussed in some detail in [BCDG13, Remark 1.7]. The main differences are that the random environment is not uniformly elliptic and is not i.i.d. In fact the environment that we consider here has even infinite range dependencies, due to the fact that the steps of the random walk are restricted to the backbone of the oriented percolation cluster once it hits the cluster. The environment also does not satisfy mixing conditions such as (conditional) cone-mixing in contrast to the model considered in [HSS13]. In [BHT20] a much weaker mixing assumption than cone-mixing is introduced (literally for a continuous time model) and our model does satisfy their assumption. However, they only prove a LLN for a nearest neighbour random walk in $d = 1$. A comprehensive overview of the recent literature on random walks in dynamic random environments can be found in the introduction of [BHT20]. See also [BGS19, Remark 1.1].

Results on quenched local limit theorems for random walks in random environments are very recent. Our research is inspired by [BCR16] where a quenched local limit theorem was shown (in dimension $d \geq 4$) for the case of an i.i.d. random environment and where the random walk satisfies a ballisticity criterion and has uniformly elliptic transition probabilities. (Note that ballisticity is trivial in our model. Uniform ellipticity and the i.i.d. property are not satisfied.)

Other results on local limit theorems in random environments that we are aware of are concerned with specific models. In [DG19] the quenched local CLT is proven for random walks in a time-dependent balanced random environment. In [DG20b] and [DG20a] quenched local limit theorems are obtained for random walks in random environments on a strip. A different
class of random walks in random media for which quenched local CLTs have been obtained are the so called random conductance models. For a recent work in this direction and an overview of the literature see [ACS21] and references therein.

**Outlook and open questions** While we do exhibit a measure $Q$ which is invariant with respect to the point of view of the particle and absolutely continuous with respect to $\mathbb{P}$, we can currently establish uniqueness only in the class of such measures satisfying the additional property (2.9), see Remark 1.5. Furthermore, because of non-ellipticity, $Q$ is not equivalent to $\mathbb{P}$, see the discussion in Remark 2.6 below. We leave open the questions whether property (2.9) is necessary for uniqueness and whether $Q$ is equivalent to $\mathbb{P}$ when restricted to the set $\tilde{\Omega}$ from (2.11) in Remark 2.6.

We restrict our analysis to the case $d \geq 3$. This is essentially owed to the fact that Theorem 8.1, which we quote from [Ste17, Thm. 3.24], is presently only available under this assumption. It was proved there using an environment exposure technique from [BS02], which was also used by [BCR16], and the proof exploited the fact that in dimension at least 3, two independent random walks will almost surely meet only finitely often, irrespective of the number $N$ of steps they take. In spatial dimension $d = 2$, two independent walks will meet infinitely often, but the number of intersections up to time $N$ grows quite slowly (of order log $N$). It is conceivable that with substantial technical effort, the proof of [Ste17, Thm. 3.24] and also the results of the present investigation could be adapted to cover the case $d = 2$. We leave this question for future research. For our model, simulations suggest that Theorem 1.4 should hold even in spatial dimension $d = 1$. However, it seems that a rigorous analysis of the case $d = 1$ would require a completely new approach.

We prove in Theorem 1.4 a quenched local limit theorem for a very specific model of a non-elliptic random walk in a non-trivial dynamic random environment, and our proofs do exploit specific properties of this environment, namely the oriented percolation cluster. However, we think that this environment is prototypical for a large class of dynamic environments which can be “mapped” to it by suitable coarse-graining procedures, see [BˇCD16], Section 3 and the concrete example in Section 4 there. It seems quite possible that given substantial technical effort, our approach to Theorem 1.4 could be extended to the class of environments from [BˇCD16]. We leave this for future work.

**Outline of the paper** The proofs of the main results are long and quite technical. Let us describe the main ideas of the proofs and explain how the paper is organised: In Section 2 we first give several auxiliary results which we then use for the proofs of Theorem 1.3 and of Theorem 1.4.

**Annealed estimates:** In Section 3 we prove several annealed derivative estimates which build on, and extend somewhat, previous work by [Ste17]. These estimates will be used for the proof of the annealed local CLT, Theorem 1.1, also presented in Section 3. Starting with Section 4 the paper is devoted to the proofs of the auxiliary results from Section 2.

**Comparison of the quenched and annealed laws:** Lemma 2.1, proven in Section 4, provides a comparison between the quenched and annealed laws on the level of large (but finite) boxes. In particular it shows that the total variation distance between $\mathbb{P}(X_N \in \cdot)$ and $P_\omega(X_N \in \cdot)$ on the level of boxes of side length $M \gg 1$ is small with very high probability as $N \to \infty$ in a suitably quantified way; see equation (2.1). The starting point of the proof of Lemma 2.1 is [Ste17, Theorem 3.24], recalled in Theorem 8.1 below, which gives an analogous result for boxes whose size grows like $N^{\theta/2}$ with $0 < \theta < 1$ as $N \to \infty$, and therefore much slower than the
diffusive scale $N^{1/2}$. We augment this with an iteration scheme that is guided by the proof of Theorem 5.1 in [BCR16]. The main argument towards the proof of Lemma 2.1 is formulated as Proposition 4.1 which provides the crucial estimate for the iteration step. The proof of that proposition is long and relies to a large extent on ideas from [BCR16] and is postponed to Section 8. It requires a suitable control of the density of “good” boxes on which an estimate as in equation (2.1) from Lemma 2.1 holds locally uniformly, see Definition 8.2. This deviates from the set-up in [BCR16] because our environment is not i.i.d. and in fact here the boxes are in principle correlated over arbitrary lengths, albeit weakly.

**Measure for the point of view of the particle:** The function $\varphi = dQ/dP$ from (1.12) is the density of a measure $Q$ which is invariant with respect to the point of view of the particle and absolutely continuous with respect to $P$. For the existence of such a measure $Q$ we consider the quenched laws $Q_N$ of the environment seen from the particle after $N$ steps of the walk; see (2.4). The measure $Q$ is constructed as a weak limit of the Cesàro average of the measures $Q_N$ along a subsequence; see (2.6) and (2.8). In Proposition 2.2 and Corollary 2.4 we show that averages of $dQ_N/dP$ and $dQ/dP$ over large boxes are close to one with high probability depending on the size of the boxes. It will turn out that the measure $Q$ which we obtain as described above is unique, i.e. it does not depend on the particular subsequence; see Remark 2.6.

Proposition 2.2 and Corollary 2.4 are proven in Section 5. To this end we construct a coupling of $Q_N$ and $P_N$, the law of the environment viewed relative to the annealed walk (note that $P_N = P$ for all $N$). Lemma 2.1 allows for a coupling which puts both walks in the same $M$-box with very high probability. We strengthen this to a coupling which puts both walks at exactly the same spatial position with uniformly non-vanishing probability; see the proof of Lemma 5.3.

Since we average over the environment in the definition of the annealed law of the random walk in equation (1.6) it is clear that the annealed random walk does not see any specific environment. In contrast to that the quenched random walk knows the exact environment it walks in. So, to compare the annealed and quenched laws of the random walk, the annealed walk needs to see the environment of the quenched random walk. This is done through reweighting by $\varphi$. In particular, a consequence of multiplying the annealed law with $\varphi$ is that this product will be zero for all space-time points $(x, n) \in \mathbb{Z}^d \times \mathbb{Z}$ in which the contact process $\xi$ is 0 in the environment $\omega$.

In Proposition 2.8 we show that the annealed law of the random walk at time $n$ reweighted with the function $\varphi$ converges for almost all $\omega$ to a probability law on $\mathbb{Z}^d$. It is proven in Section 6.

In Lemma 10.1 we will see that a prefactor $\varphi$ satisfying (1.12) is unique. A quite general proof of that result is given in Section 10.

**Hybrid measures:** For the proof of Theorem 1.4, instead of comparing the quenched and annealed laws directly, we use the triangle inequality, some “hybrid” measures and space-time convolutions of quenched-annealed measures; see Definition 2.7. In Proposition 2.9, proven in Section 7, we show that the total variation distance of some of these measures converges to 0 as $n$, the number of steps of the random walk, goes to infinity. An essential tool of the proof of Proposition 2.9 is Lemma 7.1 in which we study the total variation distance of quenched laws of two random walks starting at different positions. The idea is to use couplings with the annealed measures on the level of large (growing) boxes combined with annealed derivative estimates in order to first ensure that the two walks are in the same box with probability bounded away from 0. Using connectivity properties of the oriented percolation clusters (see below) the above described procedure can be iterated to produce a literal coupling where the two walks coincide with high probability after sufficiently many steps. Lemma 7.1 is proven in Section 9.
2. Proofs of the main results

In this section we collect several important auxiliary results and present towards the end of this section how to utilise them to prove Theorem 1.3 and Theorem 1.4. The proofs of the auxiliary results are postponed to the subsequent sections.

Our starting point is a lemma which can be seen as an adaptation of Theorem 5.1 in [BCR16] to our setting. Recall between (1.5) and (1.6) the definitions of the quenched measure \( P^{(x,m)}_\omega \) and the annealed measure \( \mathbb{P}^{(x,m)} \) for the random walk \( (X_n)_{n=m,m+1,...} \) with \( X_m = x \). For any positive real number \( L \) we denote by \( \Pi_L \) a partition of \( \mathbb{Z}^d \) into boxes of side length \( [L] \).

**Lemma 2.1.** Let \( d \geq 3 \). For \( N, M \in \mathbb{N}, c, C > 0 \) denote by \( K(N) := K(N,M,c,C) \) the set of environments \( \omega \in \Omega \) such that for every \( x \in \mathbb{Z}^d \) satisfying \( \|x\| \leq N \)

\[
\sum_{\Delta \in \Pi_M} \left| P^{(x,0)}_\omega(X_N \in \Delta) - \mathbb{P}^{(x,0)}(X_N \in \Delta) \right| \leq \frac{C}{M^c} + \frac{C}{N^c}.
\]

(2.1)

If \( c > 0 \) is small enough and \( C < \infty \) large enough, there are universal positive constants \( \tilde{c}, \tilde{C} \), for which we have

\[
\mathbb{P}(K(N)) \geq 1 - \tilde{C}N^{-\tilde{c}\log N} \quad \text{for all } N.
\]

(2.2)

In words, Lemma 2.1 shows that the total variation distance between the annealed measure \( \mathbb{P}^{(x,0)}(X_N \in \cdot) \) and the quenched measure \( P^{(x,0)}_\omega(X_N \in \cdot) \) on the level of boxes of side length \( M \gg 1 \) is small with very high probability as \( N \to \infty \). The proof of Lemma 2.1 is given in Section 4. It builds on a preliminary result by Steiber [Ste17, Theorem 3.24] which we recall in Theorem 8.1 below. The latter gives an analogous result to Lemma 2.1 for boxes of side length \( N^{\theta/2} \) with \( 0 < \theta < 1 \) for large \( N \). In particular, for \( N \to \infty \) the side length of these boxes grows much more slowly than the diffusive scale \( N^{1/2} \).

Lemma 2.1 allows to construct a coupling of the quenched walk under \( P^{(x,0)}_\omega \) and the annealed walk under \( \mathbb{P}^{(x,0)} \) which puts both walks in the same \( M \)-box with very high probability. We strengthen this coupling to a coupling which puts both walks at exactly the same spatial position with uniformly non-vanishing probability; see Lemma 5.3 below. This, in turn, is essential for the next statement which concerns the difference between the annealed and quenched law of the environment viewed relative to the walk after \( N \) steps, which we denote by \( P_N \) and \( Q_N \) respectively. More precisely, for \( N \in \mathbb{N} \), we define \( Q_N \) and \( P_N \) by

\[
P_N(A) := \mathbb{E}\left[ \sum_{x \in \mathbb{Z}^d} \mathbb{P}^{(0,0)}(X_N = x) 1_{\{\sigma_{(x,N)} \omega \in A\}} \right]
\]

(2.3)

and

\[
Q_N(A) := \mathbb{E}\left[ \sum_{x \in \mathbb{Z}^d} P^{(0,0)}_\omega(X_N = x) 1_{\{\sigma_{(x,N)} \omega \in A\}} \right].
\]

(2.4)
Note that, in fact we have $P_N = P$ for all $N$; see (5.9).

The following proposition is proven in Section 5.

**Proposition 2.2.** For $M \in \mathbb{N}$ let $\Delta_0(M)$ denote a $d$-dimensional cube of side length $M$ in $\mathbb{Z}^d$ centred at the origin. There exists a universal constant $c > 0$ so that for every $\varepsilon > 0$ there is $M_0 = M_0(\varepsilon) \in \mathbb{N}$ so that for $M \geq M_0$ and all $N \in \mathbb{N}$

$$
\mathbb{P}\left( \left| \sum_{x \in \Delta_0(M)} \frac{dQ_N}{dP}(\sigma_{(x,0)}\omega) - 1 \right| > \varepsilon \right) \leq M^{-c \log M}.
$$

(2.5)

**Corollary 2.3.** Let $d \geq 3$ and $p > p_c$. Then, for every $k \in \mathbb{N}$, $\sup_N \mathbb{E}[\left( \frac{dQ_N}{dP} \right)^k] < \infty$.

**Proof.** For $M \in \mathbb{N}$ large enough, Proposition 2.2 implies

$$
\mathbb{P}\left( \frac{dQ_N}{dP}(\omega) > 2(2M + 1)^d \right) \leq \mathbb{P}\left( \sum_{x \in \{-M,\ldots,M\}^d} \frac{dQ_N}{dP}(\sigma_{(x,n)}\omega) > 2 \right) \leq M^{-c \log M},
$$

which implies the assertion.

We equip $\Omega$ with the product topology and consider the Cesàro sequence

$$
\tilde{Q}_n := \frac{1}{n} \sum_{N=0}^{n-1} Q_N, \quad n = 1, 2, \ldots.
$$

(2.6)

Using Corollary 2.3 and the Cauchy-Schwarz inequality for some finite positive constant $\tilde{c}$ we have

$$
\mathbb{E}\left[ \left( \frac{1}{n} \sum_{N=0}^{n-1} \frac{dQ_N}{dP} \right)^2 \right] = \frac{1}{n^2} \sum_{N,N'} \mathbb{P}\left[ \frac{dQ_N}{dP} \frac{dQ_{N'}}{dP} \right] \leq \tilde{c}.
$$

(2.7)

For $\varepsilon > 0$ let $K \subset \Omega$ be a compact subset such that $\mathbb{P}(K^c) < \varepsilon$. Then by the Cauchy-Schwarz inequality we obtain

$$
\tilde{Q}_n(K^c) = \int_{\Omega} 1_K \frac{dQ_n}{dP} \leq \sqrt{\varepsilon} \mathbb{P}(K^c)^{1/2} = \sqrt{\varepsilon}.
$$

Thus, the sequence $(\tilde{Q}_n)_n$ is tight. In particular, there is a weakly converging subsequence, say $(\tilde{Q}_{n_k})_k$, and we set

$$
Q := \lim_{k \to \infty} \tilde{Q}_{n_k}.
$$

(2.8)

A standard argument shows that $Q$ is invariant with respect to the point of view of the particle; see Proposition 1.8 in [Lig85] for an abstract argument or the proof of Lemma 1 in [DR14] for the argument in the case of random walks in random environments.

The proof of the following analogue of Proposition 2.2 for $Q$ instead of $Q_n$ is given in Section 5.

**Corollary 2.4.** Recall the notation of Proposition 2.2 and let $Q$ be the measure obtained as a limit in (2.8). There exists a universal constant $c > 0$ so that for every $\varepsilon > 0$ there is $M_0 = M_0(\varepsilon) \in \mathbb{N}$ and for every $M \geq M_0$ we have

$$
\mathbb{P}\left( \left| \sum_{x \in \Delta_0(M)} \frac{dQ}{dP}(\sigma_{(x,0)}\omega) - 1 \right| > \varepsilon \right) \leq M^{-c \log M}.
$$

(2.9)
Proof of Theorem 1.3. By construction and shift invariance of \( \mathbb{P} \) we have \( Q_N \ll \mathbb{P} \) for every \( N \) and therefore \( Q_n \ll \mathbb{P} \) for every \( n \). Furthermore, by (2.7) the family of Radon-Nikodym derivatives \( \{dQ_n/d\mathbb{P}\}_{n=1,2,...} \) is uniformly integrable. These facts together imply that we also have \( Q \ll \mathbb{P} \) for any \( Q \) obtained as in (2.8). The concentration property is the assertion of Corollary 2.4. For the question of uniqueness of \( Q \) see Remark 2.6 below.

Remark 2.5. Using shift-invariance of \( \mathbb{P} \), it is easy to see that for \( Q_N \) from (2.4) a version of \( dQ_N/d\mathbb{P} \) is given by
\[
\varphi_N(\omega) = \sum_{x \in \mathbb{Z}^d} P_\omega(-N,x)(X_0 = 0) \tag{2.10}
\]
(we have \( P_\omega^{(0,0)}(\omega)(X_N = x) = P_\omega(-N,x)(X_0 = 0) \), recall the notation introduced below (1.5)). This formula is the analogue of [BS02, Proposition 1.2] in our context. In particular, \( \varphi_N \) is a local function of the space-time values of \( \xi \) which themselves can be obtained as limits of local functions of \( \omega \). Thus, \( dQ/d\mathbb{P} \) can be considered as an almost sure limit of local functions of \( \omega \).

Remark 2.6 (Uniqueness of invariant \( Q \ll \mathbb{P} \) with concentration properties of the density). A measure \( Q \) obtained as in (2.8) may in principle depend on a particular subsequence. In the proof of Theorem 1.4 we will show that the density \( \varphi = dQ/d\mathbb{P} \) of any measure \( Q \) satisfying the concentration property (2.9) also satisfies (1.12). By Lemma 10.1 below, such a measure is unique. In particular, in (2.8) we have weak convergence towards the unique \( Q \) along any subsequence and therefore we have weak convergence of the Cesàro sequence \((\check{Q}_n)_{n \in \mathbb{N}}\) from (2.6) towards \( Q \). However, we currently do not know whether the sequence \((Q_N)_{N \in \mathbb{N}}\) from (2.4) converges itself.

Using Lemma B.1 and (2.10) from Remark 2.5 one can show that \( Q \) is concentrated on
\[
\check{\Omega} = \{ \omega \in \Omega : \omega \text{ contains a doubly infinite directed open path through } (0,0) \} \tag{2.11}
\]
and thus \( Q \) is not equivalent to \( \mathbb{P} \) because \( 0 < \mathbb{P}(\check{\Omega}) < 1 \). Note that Kozlov’s classical argument concerning equivalence, see e.g. [DR14, Thm. 2.12], does not apply because our walks are not elliptic. We do not know whether \( Q \) is equivalent to \( \mathbb{P}(\cdot | \check{\Omega}) \).

To prove Theorem 1.4 we want to make use of the good control of the difference between the quenched and annealed law on the level of boxes and various properties of the prefactor \( \varphi \) that we have formulated above in Lemma 2.1 and Corollary 2.4. Furthermore, instead of comparing \( \mathbb{P}^{(0,0)}(X_N \in \cdot) \) and \( P_\omega^{(0,0)}(X_N \in \cdot) \) directly, we compare both of these two measures with auxiliary “hybrid” measures which are introduced in the following definition.

Definition 2.7. Let \( Q \) be the measure on \( \Omega \) defined in (2.8), which by Theorem 1.3 and its proof is invariant with respect to the point of view of the particle with \( Q \ll \mathbb{P} \). Let \( \varphi = dQ/d\mathbb{P} \) be the corresponding Radon-Nikodym derivative. For \( \omega \in \Omega \) and a given partition \( \Pi \) of \( \mathbb{Z}^d \) into boxes of a fixed side length we define the following measures on \( \mathbb{Z}^{d+1} \):
\[
\nu^{\text{ann}\times \text{pre}}(x,n) := \nu^{\text{ann}\times \text{pre}}(\{(x,n)\}) := \frac{1}{Z_{\omega,n}} \mathbb{P}^{(0,0)}(X_n = x) \varphi(\sigma(x,n)\omega), \tag{2.12}
\]
\[
\nu^{\text{que}}(x,n) := \nu^{\text{que}}(\{(x,n)\}) := P_\omega^{(0,0)}(X_n = x), \tag{2.13}
\]
\[
\nu^{\text{box-que}\times \text{pre}}(x,n) := \nu^{\text{box-que}\times \text{pre}}(\{(x,n)\}) := P_\omega^{(0,0)}(X_n \in \Delta_x) \sum_{y \in \Delta_x} \varphi(\sigma(y,n)\omega). \tag{2.14}
\]
Here, \( Z_{\omega,n} = \sum_{x \in \mathbb{Z}^d} \mathbb{P}^{(0,0)}(X_n = x) \varphi(\sigma(x,n)\omega) \) is the normalizing constant in (2.12) and \( \Delta_x \) in (2.14) is the unique \( d \)-dimensional box that contains \( x \) in the partition \( \Pi \).
All of the measures introduced in the above definition are different measures of the random walk after \( n \) steps: \( \nu_{\omega}^{\text{ann} \times \text{pre}}(\cdot, n) \) is the annealed measure with a prefactor, \( \nu_{\omega}^{\text{que}}(\cdot, n) \) is the quenched measure and \( \nu_{\omega}^{\text{box-que} \times \text{pre}}(\cdot, n) \) is a “hybrid” measure, where the box is chosen according to the quenched measure but then the point inside the box is chosen according to the (annealed) normalised prefactor. Of course the measure \( \nu_{\omega}^{\text{box-que} \times \text{pre}}(\cdot, n) \) does depend on the particular partition \( \Pi \) but it will be clear from the context which partition is used.

First we study the behaviour of the normalizing constant in (2.12); see Section 6 for a proof of the following result.

**Proposition 2.8.** For \( \mathbb{P} \)-almost all \( \omega \in \Omega \) the normalizing constant \( Z_{\omega,n} \) satisfies

\[
\lim_{n \to \infty} Z_{\omega,n} = 1. \quad (2.15)
\]

The following proposition is the key result for the proof of Theorem 1.4. It states that for large \( n \) the above introduced measures are close to each other in a suitable norm. To state this precisely, for \( \omega \in \Omega \) and any two probability measures \( \nu_{\omega}^1 \) and \( \nu_{\omega}^2 \) on \( \mathbb{Z}^d \times \mathbb{Z} \) (more precisely these are transition kernels from \( \Omega \) to \( \mathbb{Z}^d \times \mathbb{Z} \)) let the \( L^1 \) distance of \( \nu_{\omega}^1 \) and \( \nu_{\omega}^2 \) at time \( n \in \mathbb{Z} \) be defined by

\[
\| \nu_{\omega}^1 - \nu_{\omega}^2 \|_{1,n} := \sum_{x \in \mathbb{Z}^d} |\nu_{\omega}^1(x, n) - \nu_{\omega}^2(x, n)|. \quad (2.16)
\]

Furthermore, for \( k \leq n \) the space-time convolution of \( \nu_{\omega}^1 \) and \( \nu_{\omega}^2 \) is defined by

\[
(\nu^1 \ast \nu^2)_{\omega,k}(x, n) := \sum_{y \in \mathbb{Z}^d} \nu^1_{\omega}(y, n-k)\nu^2_{\omega}(x-y, k). \quad (2.17)
\]

We can interpret (2.17) as follows: A random walk takes \( n-k \) steps in the random medium \( \omega \) according to \( \nu_{\omega}^1 \), then re-centers the medium at its current position in space-time and takes the remaining \( k \) steps according to \( \nu_{\omega}^2 \).

**Proposition 2.9.** Fix \( 0 < 2\delta < \varepsilon < \frac{1}{3} \), and for \( n \in \mathbb{N} \) set \( k = \lceil n^\varepsilon \rceil \) and \( \ell = \lfloor n^\delta \rfloor \). Let \( \Pi = \Pi(\ell) \) be a partition of \( \mathbb{Z}^d \) into boxes of side length \( \ell \). For \( \mathbb{P} \)-almost every \( \omega \in \Omega \) the measures from Definition 2.7 satisfy

\[
\lim_{n \to \infty} \| \nu_{\omega}^{\text{ann} \times \text{pre}} - (\nu_{\omega}^{\text{ann} \times \text{pre}} \ast \nu_{\omega}^{\text{que}})_{\omega,k} \|_{1,n} = 0, \quad (L1)
\]

\[
\lim_{n \to \infty} \| (\nu_{\omega}^{\text{ann} \times \text{pre}} \ast \nu_{\omega}^{\text{que}})_{\omega,k} - (\nu_{\omega}^{\text{box-que} \times \text{pre}} \ast \nu_{\omega}^{\text{que}})_{\omega,k} \|_{1,n} = 0, \quad (L2)
\]

\[
\lim_{n \to \infty} \| (\nu_{\omega}^{\text{box-que} \times \text{pre}} \ast \nu_{\omega}^{\text{que}})_{\omega,k} - (\nu_{\omega}^{\text{que}} \ast \nu_{\omega}^{\text{que}})_{\omega,k} \|_{1,n} = 0. \quad (L3)
\]

The proof of the above proposition is given in Section 7. With the results stated in the present section we can give a proof of the quenched local limit theorem.

**Proof of Theorem 1.4.** Using the triangle inequality we have

\[
\sum_{x \in \mathbb{Z}^d} |P_{\omega}(0)(X_n = x) - \mathbb{P}(0)(X_n = x)\varphi(\sigma(x,n)\omega)| \\
\leq \sum_{x \in \mathbb{Z}^d} |P_{\omega}(0)(X_n = x) - (\nu_{\omega}^{\text{box-que} \times \text{pre}} \ast \nu_{\omega}^{\text{que}})_{\omega,k}(x, n)| \quad (2.18)
\]
\begin{align*}
+ \sum_{x \in \mathbb{Z}^d} |(\nu_{\text{box}, n} \ast \nu_{\text{pre}, n})_{\omega, k}(x, n) - (\nu_{\text{ann}} \ast \nu_{\text{que}})_{\omega, n}(x, n)| & \quad (2.19) \\
+ \sum_{x \in \mathbb{Z}^d} |(\nu_{\text{ann}} \ast \nu_{\text{pre}, n})_{\omega, k}(x, n) - \nu_{\text{ann}} \ast \nu_{\text{pre}, n}(x, n)| & \quad (2.20) \\
+ \sum_{x \in \mathbb{Z}^d} |\nu_{\text{ann}} \ast \nu_{\text{pre}, n}(x, n) - \mathbb{P}^{(0, 0)}(X_n = x) \varphi(\sigma_{(x, n)} \omega)|. & \quad (2.21)
\end{align*}

By Proposition 2.9 the terms in (2.18), (2.19) and (2.20) tend to 0 as \( n \) goes to infinity. In order to compare (2.18) with (L3) literally note that we have \( \mathcal{P}^{(0, 0)}(X_n = x) = \nu_{\text{que}} \ast \nu_{\text{que}})_{\omega, n}(x, n) \) by construction. Finally, by definition of \( \nu_{\text{ann}} \ast \nu_{\text{pre}, n}(x, n) \) the term in (2.21) can be written as

\[ \left| \frac{1}{\mathcal{Z}_{\omega, n}} - 1 \right| \sum_{x \in \mathbb{Z}^d} \mathbb{P}^{(0, 0)}(X_n = x) \varphi(\sigma_{(x, n)} \omega) = \left| \frac{1}{\mathcal{Z}_{\omega, n}} - 1 \right| \mathcal{Z}_{\omega, n}. \quad (2.22) \]

By Proposition 2.8 it follows that the expression in (2.22) converges to 0 as \( n \) tends to infinity. \( \square \)

3. Annealed estimates and the proof of Theorem 1.1

In this section we collect estimates for the annealed walk that will be needed later in the proofs, and present a proof of Theorem 1.1.

**Lemma 3.1** (Annealed derivative estimates). For \( d \geq 3, j = 1, \ldots, d, x, y \in \mathbb{Z}^d, m, n \in \mathbb{Z}, m \in \mathbb{Z}, n \in \mathbb{N} \) denoting by \( e_j \) the \( j \)-th (canonical) unit vector we have

\[ \left| \mathbb{P}^{(y, m, n)}(X_{n+m} = x) - \mathbb{P}^{(y, m,e_j, n)}(X_{n+m} = x) \right| \leq Cn^{-(d+1)/2}, \quad (3.1) \]

\[ \left| \mathbb{P}^{(y, m, n)}(X_{n+m} = x) - \mathbb{P}^{(y, m+1, n)}(X_{n+m} = x) \right| \leq Cn^{-(d+1)/2}, \quad (3.2) \]

\[ \left| \mathbb{P}^{(y, m, n)}(X_{n+m} = x) - \mathbb{P}^{(y, m, x+e_j)}(X_{n+m} = x+e_j) \right| \leq Cn^{-(d+1)/2}, \quad (3.3) \]

\[ \left| \mathbb{P}^{(y, m, n)}(X_{n+m} = x) - \mathbb{P}^{(y, m, n)}(X_{n-1+m} = x) \right| \leq Cn^{-(d+1)/2}. \quad (3.4) \]

**Proof.** The estimates (3.1) and (3.2) are from [Ste17]; see Lemma 3.9 and its proof in Appendix A.2 there. By translation invariance we have

\[ \mathbb{P}^{(y+e_j, m, n)}(X_{n+m} = x) = \mathbb{P}^{(y, m)}(X_{n+m} = x - e_j) \]

and

\[ \mathbb{P}^{(y, m+1, n)}(X_{n+m} = x) = \mathbb{P}^{(y, m)}(X_{n-1+m} = x). \]

Thus, the estimates (3.3) and (3.4) follow from (3.1) and (3.2). \( \square \)

We will also need the following generalization of the annealed derivative estimates in the previous lemma.

**Lemma 3.2.** Let \( \varepsilon > 0 \). For \( n \in \mathbb{N} \) large enough and every partition \( \Pi_n^{(\varepsilon)} \) of \( \mathbb{Z}^d \) into boxes of side length \( \lfloor n^{\varepsilon} \rfloor \), we have

\[ \sum_{\Delta \in \Pi_n^{(\varepsilon)}} \sum_{x \in \Delta} \max_{y \in \Delta} \left| \mathbb{P}^{(0, 0)}(X_n = y) - \mathbb{P}^{(0, 0)}(X_n = x) \right| \leq Cn^{-\frac{d}{2} + 3d\varepsilon}. \quad (3.5) \]
Proof. We consider the following set of boxes around the origin of $\mathbb{Z}^d$ 
\[ \Pi^\varepsilon_n := \{ \Delta \in \Pi_n : \Delta \cap [-\sqrt{n} \log^3 n, \sqrt{n} \log^3 n]^d \neq \emptyset \}. \tag{3.6} \]
With this notation we can write the sum on the left hand side of (3.5) as 
\[ \sum_{\Delta \in \Pi^\varepsilon_n} \sum_{x \in \Delta} \max_{y \in \Delta} \left[ \mathbb{P}(X_n = y) - \mathbb{P}(X_n = x) \right] \]
\[ + \sum_{\Delta \in \Pi^\varepsilon_n \backslash \Pi^\delta_n} \sum_{x \in \Delta} \max_{y \in \Delta} \left[ \mathbb{P}(X_n = y) - \mathbb{P}(X_n = x) \right]. \tag{3.7} \]

So, it is enough to prove suitable upper bounds for these two sums. By Lemma 3.6 from [Ste17] we have 
\[ \sum_{\Delta \in \Pi^\varepsilon_n \backslash \Pi^\delta_n} \mathbb{P}(X_n = \Delta) \leq C n^{-c \log n} \tag{3.9} \]
for some positive constants $C$ and $c$. Thus, the double sum (3.8) is bounded from above by 
\[ \sum_{\Delta \in \Pi^\varepsilon_n \backslash \Pi^\delta_n} \sum_{x \in \Delta} \left[ \mathbb{P}(X_n = \Delta) - \mathbb{P}(X_n = x) \right] = \sum_{\Delta \in \Pi^\varepsilon_n \backslash \Pi^\delta_n} (|\Delta| - 1) \mathbb{P}(X_n = \Delta) \leq C n^{d_x} n^{-c \log n} \leq \tilde{C} n^{-c \log n} \]
for suitably chosen constants $\tilde{c}$ and $\tilde{C}$. Using annealed derivative estimates from Lemma 3.1 the double sum (3.7) is bounded above by 
\[ \sum_{\Delta \in \Pi^\varepsilon_n \backslash \Pi^\delta_n} \sum_{x \in \Delta} C n^{d_x} n^{-\frac{d+1}{2}} \leq C (n^{d_x} + \sqrt{n} \log^3 n)^d n^{c \log n} \leq C n^{3d_x} n^{-1/2}. \]
Combination of the last two displays completes the proof. \hfill \Box

Proof of Theorem 1.1. Let $\varepsilon, \delta > 0$ be small (they will later be tuned appropriately). Let $\Pi^\varepsilon_n$ be a partition of $\mathbb{Z}^d$ in boxes of side length $[\varepsilon \sqrt{n}]$. Let $C_\delta > 0$ be a constant such that $\mathbb{P}^{(0,0)}(\|X_n\| > C_\delta \sqrt{n}) < \delta$; such a constant exists by Lemma 3.6 from [Ste17]. Furthermore denote by $\Pi^\varepsilon_n \cap \Pi^\delta_n$ the subset of boxes in $\Pi^\varepsilon_n$ intersecting $\{ x \in \mathbb{Z}^d : \|x\| \leq C_\delta \sqrt{n} \}$. Then 
\[ \sum_{x \in \mathbb{Z}^d} \left| \mathbb{P}(X_n = x) - \frac{1}{(2\pi n)^d \sqrt{\det \Sigma}} \exp \left( -\frac{1}{2n} x^T \Sigma^{-1} x \right) \right| = \sum_{\Delta \in \Pi^\varepsilon_n \cap \Pi^\delta_n} \sum_{x \in \Delta} \left| \mathbb{P}(X_n = x) - \frac{1}{(2\pi n)^d \sqrt{\det \Sigma}} \exp \left( -\frac{1}{2n} x^T \Sigma^{-1} x \right) \right| \]
\[ + \sum_{\Delta \in \Pi^\varepsilon_n \cap \Pi^\delta_n} \sum_{x \in \Delta} \left| \mathbb{P}(X_n = x) - \frac{1}{(2\pi n)^d \sqrt{\det \Sigma}} \exp \left( -\frac{1}{2n} x^T \Sigma^{-1} x \right) \right|. \tag{3.10} \]

We will show that $\varepsilon$ can be chosen so small that the above sum is bounded by $4\delta$ for large enough $n$. We first find an upper bound for (3.10). By definition of $\Pi^\varepsilon_n \cap \Pi^\delta_n$ if $\Delta \in \Pi^\varepsilon_n \cap \Pi^\delta_n$ then we have $\|x\| > C_\delta \sqrt{n}$ for all $x \in \Delta$. Thus, (3.10) is bounded from above by 
\[ \sum_{x \in \mathbb{Z}^d \cap \|x\| > C_\delta \sqrt{n}} \left( \mathbb{P}(X_n = x) + \frac{1}{(2\pi n)^d \sqrt{\det \Sigma}} \exp \left( -\frac{1}{2n} x^T \Sigma^{-1} x \right) \right) \leq \delta + C \exp \left( -\frac{c}{2} C_\delta^2 \right). \]

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By choosing $C_\delta$ large enough we can ensure that (3.10) is bounded by $2\delta$.

Turning to (3.11) we first compare the two terms in $| \cdot |$ with the averages over appropriate boxes. First, let $x \in \mathbb{Z}^d$ be fixed and let $\Delta \in \Pi_n^{(\varepsilon, \delta)}$ be the box containing $x$. Using annealed derivative estimates from Lemma 3.9 in [Ste17] we obtain

$$
|\mathbb{P}^{(0,0)}(X_n = x) - \frac{1}{|\varepsilon\sqrt{n}|^d} \mathbb{P}^{(0,0)}(X_n \in \Delta)|
= \frac{1}{|\varepsilon\sqrt{n}|^d} \sum_{y \in \Delta} |\mathbb{P}^{(0,0)}(X_n = x) - \mathbb{P}^{(0,0)}(X_n = y)|
\leq \frac{1}{|\varepsilon\sqrt{n}|^d} |x - y| n^{-(d+1)/2} \leq |\varepsilon\sqrt{n}| \cdot n^{-(d+1)/2} \leq \frac{\varepsilon}{n^{d/2}}.
$$

Now consider $\Delta \in \Pi_n^{(\varepsilon, \delta)}$. For every $x \in \Delta$ we have

$$
\left| \exp\left(-\frac{1}{2n}x^T \Sigma^{-1} x\right) - \frac{1}{|\varepsilon\sqrt{n}|^d} \int_\Delta \exp\left(-\frac{1}{2n}y^T \Sigma^{-1} y\right) dy \right|
= \exp\left(-\frac{1}{2n}x^T \Sigma^{-1} x\right) \left| 1 - \frac{1}{|\varepsilon\sqrt{n}|^d} \int_\Delta \exp\left(-\frac{1}{2n}(y^T \Sigma^{-1} y - x^T \Sigma^{-1} x)\right) dy \right|
\leq \exp\left(-\frac{1}{2n}x^T \Sigma^{-1} x\right) \frac{1}{|\varepsilon\sqrt{n}|^d} \int_\Delta \left| 1 - \exp\left(-\frac{1}{2n}(y^T \Sigma^{-1} y - x^T \Sigma^{-1} x)\right) \right| dy
\leq \exp\left(-\frac{1}{2n}x^T \Sigma^{-1} x\right) \cdot C \varepsilon \leq C \varepsilon,
$$

where we have used $|x - y| \leq \varepsilon\sqrt{n}$ and $|x| \leq C_\delta\sqrt{n}$. Using first the triangle inequality and then combining the last two estimates we see that each summand in (3.11) is bounded from above by

$$
|\mathbb{P}^{(0,0)}(X_n = x) - \frac{1}{|\varepsilon\sqrt{n}|^d} \mathbb{P}^{(0,0)}(X_n \in \Delta)|
+ \left(\frac{1}{(2\pi n)^{d/2} \sqrt{\det \Sigma}} \right) \left| \exp\left(-\frac{1}{2n}x^T \Sigma^{-1} x\right) - \frac{1}{|\varepsilon\sqrt{n}|^d} \int_\Delta \exp\left(-\frac{1}{2n}y^T \Sigma^{-1} y\right) dy \right|
+ \left(\frac{1}{(2\pi n)^{d/2} \sqrt{\det \Sigma}} \right) \frac{1}{|\varepsilon\sqrt{n}|^d} \mathbb{P}^{(0,0)}(X_n \in \Delta) - \left(\frac{1}{(2\pi n)^{d/2} \sqrt{\det \Sigma}} \right) \int_\Delta \exp\left(-\frac{1}{2n}y^T \Sigma^{-1} y\right) dy
\leq \frac{C \varepsilon}{n^{d/2}} + \frac{C \varepsilon}{(2\pi n)^{d/2} \sqrt{\det \Sigma}} + \frac{1}{|\varepsilon\sqrt{n}|^d} \mathbb{P}^{(0,0)}(X_n \in \Delta) - \left(\frac{1}{(2\pi n)^{d/2} \sqrt{\det \Sigma}} \right) \int_\Delta \exp\left(-\frac{1}{2n}y^T \Sigma^{-1} y\right) dy.
$$

The number of vertices summed over all $\Delta \in \Pi_n^{(\varepsilon, \delta)}$ is bounded by $((C_\delta + \varepsilon)\sqrt{n})^d \leq C(C_\delta \sqrt{n})^d$. Thus,

$$
\sum_{\Delta \in \Pi_n^{(\varepsilon, \delta)}} \sum_{x \in \Delta} \left(\frac{C \varepsilon}{n^{d/2}} + \frac{C \varepsilon}{(2\pi n)^{d/2} \sqrt{\det \Sigma}} \right) \leq C \cdot C_\delta^d \varepsilon.
$$

(3.13)
3.12 We follow closely the proof of Theorem 5.1 in [BCR16] below along a sequence of more and more slowly growing box scales.

By applying the annealed CLT from [BCDG13] (and approximating the indicator \(1_\Delta\) appropriately by continuous and bounded functions) and noting that for fixed \(\varepsilon, \delta\) the set \(\Pi_n^{(\varepsilon,\delta)}\) is finite implies that (3.14) goes to zero as \(n\) tends to infinity. In particular it is smaller than \(\delta\) for large enough \(n\).

Combining the estimates above we obtain

\[
\sum_{x \in \mathbb{Z}^d} \left| \mathbb{P}^{(0,0)}(X_n = x) - \frac{1}{(2\pi n)^{d/2}\sqrt{\det \Sigma}} \exp\left( -\frac{1}{2n}y^T\Sigma^{-1}y \right) \right| \leq 2\delta + C \cdot C^d \varepsilon^2 + \delta < 4\delta
\]

for large enough \(n\) and choosing \(\varepsilon > 0\) so that \(C \cdot C^d \varepsilon < \delta\). This concludes the proof. \(\square\)

4. Proof of Lemma 2.1

For the proof of Lemma 2.1 we follow closely the proof of Theorem 5.1 in [BCR16] and adapt their arguments to our model. The general idea is to implement an iteration scheme that carries the annealed-quenched comparison from Theorem 8.1 below along a sequence of more and more slowly growing box scales.

Let us introduce some notation first. Let \(\theta > 0\) be a (small) constant to be determined in the proof. For \(j \in \mathbb{N}\), we set \(n_j := \lfloor N^{\frac{j}{M}} \rfloor\) and \(r(N) := \lfloor \log_2 \left( \frac{\log N}{\log M} \right) \rfloor\). Note that \(r(N)\) is the smallest integer satisfying \(n^\theta_{r(N)} \leq M\). Furthermore we set

\[
N_0 := N - \sum_{j=1}^{r(N)} n_j \quad \text{and} \quad N_k := \sum_{j=1}^{k} n_j + N_0 = N_{k-1} + n_k, \quad \text{for all } 1 \leq k \leq r(N). \quad (4.1)
\]

Finally, for \(0 \leq k \leq r(N)\), abusing the notation and suppressing the dependence on \(\theta\) and \(n\) we write for the rest of this section \(\Pi_k := \Pi_{n_k^\theta}\) and define

\[
\lambda_k(\omega) := \sum_{\Delta \in \Pi_k} \left| \mathbb{P}^{(0,0)}_\omega(X_{N_k} \in \Delta) - \mathbb{P}^{(0,0)}(X_{N_k} \in \Delta) \right|. \quad (4.2)
\]

Note in particular that \(\lambda_r(N)\) is twice the total variation distance between the quenched and the annealed measures on boxes of side length \(\leq M\), which is the term we wish to bound from above to show (2.1). If one wishes to be slightly more precise, then one should replace \(N_{r(N)}\) by \(M\), thus obtaining the total variation for boxes of side length \(M\) exactly. This, however, does not influence the estimates to follow.

The proof of the following proposition is long and technical and will be given in Section 8.

**Proposition 4.1.** There exists constants \(C, \varepsilon, \alpha > 0\) and events \(G(N), N \in \mathbb{N}\), with \(\mathbb{P}(G(N)) \geq 1 - CN^{-\alpha}\log N\) such that for all \(\omega \in G(N)\) we have

\[
\lambda_k \leq \lambda_{k-1} + C n_k^{-\alpha}, \quad \forall 1 \leq k \leq r(N). \quad (4.3)
\]

In particular, \(\lambda_r(N) \leq \lambda_1 + C \sum_{k=1}^{r(N)} n_k^{-\alpha}\) for \(\omega \in G(N)\).

**Proof of Lemma 2.1.** The assertion is a consequence of Proposition 4.1 and can be proven analogously to the argument in the last part of the proof of Theorem 5.1 in [BCR16], page 35. \(\square\)
5. Concentration from coupling: Proofs of Proposition 2.2 and Corollary 2.4

In this section we prove some analogues of the results of Section 6 in [BCR16] and present proofs of Proposition 2.2 and Corollary 2.4.

Lemma 5.1. There exists a constant \( c > 0 \) and set of environments \( K(N, c) \) satisfying
\[
\mathbb{P}(K(N, c)) \geq 1 - N^{-c \log N} \tag{5.1}
\]
such that for every \( \omega \) there exists a coupling \( \Theta_{\omega,N} \) of \( \mathbb{P}^{(0,0)}(X_N = \cdot) \) and \( P^{(0,0)}_{\omega}(X_N = \cdot) \) with the property
\[
\Theta_{\omega,N}(\Lambda) > c \text{ for every } \omega \in K(N, c), \tag{5.2}
\]
where \( \Lambda := \{(x, x) : x \in \mathbb{Z}^d\} \).

Proof. For \( \varepsilon > 0 \) and \( M \in \mathbb{N} \) denote by \( K(N) = K(N, M, \varepsilon) \) the set of environments \( \omega \in \Omega \) satisfying
\[
\sum_{\Delta \in \Pi_M} |P^{(0,0)}_{\omega}(X_N \in \Delta) - \mathbb{P}^{(0,0)}(X_N \in \Delta)| < \varepsilon, \tag{5.3}
\]
where \( \Pi_M \) is a partition of \( \mathbb{Z}^d \) into \( d \)-dimensional boxes of side length \( M \). By Lemma 2.1, for every \( \varepsilon \in (0, 1) \) there exists a \( M \in \mathbb{N} \) such that \( \mathbb{P}(K(N)) \geq 1 - N^{-c \log N} \). On the event \( K(N) \), the inequality (5.3) tells us that twice the total variation distance between \( \mathbb{P}^{(0,0)}(X_N \in \cdot) \) and \( P^{(0,0)}_{\omega}(X_N \in \cdot) \) on \( \Pi_M \) is less than \( \varepsilon \) and therefore there exists a coupling \( \Theta_{\omega,N,M} \) of both measures on \( \Pi_M \times \Pi_M \) such that \( \Theta_{\omega,N,M}(\Lambda_{\Pi_M}) > 1 - \varepsilon \), where \( \Lambda_{\Pi_M} = \{(\Delta, \Delta) : \Delta \in \Pi_M\} \).

Using the coupling \( \Theta \) we construct a new coupling of \( \mathbb{P}^{(0,0)}(X_N = \cdot) \) and \( P^{(0,0)}_{\omega}(X_N = \cdot) \) on \( \mathbb{Z}^d \times \mathbb{Z}^d \) which puts positive probability on the diagonal \( \Lambda = \{(x, x) : x \in \mathbb{Z}^d\} \). We define \( \Theta_{\omega,N} \) on \( \mathbb{Z}^d \times \mathbb{Z}^d \) by
\[
\Theta_{\omega,N}(x, y) := \sum_{\Delta, \Delta' \in \Pi_M} \tilde{\Theta}_{\omega,N-M,M}(\Delta, \Delta') \cdot \mathbb{P}^{(0,0)}(X_N = x|X_{N-M} \in \Delta) \cdot P^{(0,0)}_{\omega}(X_N = y|X_{N-M} \in \Delta'). \tag{5.4}
\]
Since \( \tilde{\Theta}_{\omega,N-M,M} \) is a coupling of \( \mathbb{P}^{(0,0)} \) and \( P^{(0,0)}_{\omega} \) on \( \Pi_M \times \Pi_M \) one can easily see that by the formula of total probability \( \Theta_{\omega,N} \) is indeed a coupling of \( \mathbb{P}^{(0,0)}(X_N = \cdot) \) and \( P^{(0,0)}_{\omega}(X_N = \cdot) \).

For \( x \in \mathbb{Z}^d \), let \( \Delta_x \) be the unique cube which contains \( x \) in the partition \( \Pi_M \). Since the side length of each box in the partition \( \Pi_M \) is \( M \) it follows that the annealed random walk can reach \( x \) from each point in the box \( \Delta_x \) in less than \( M \) steps.

Next we want to show that the coupling gives us a positive chance for the two walks to end up at the same position. In [BCR16] this is done by showing that \( \Theta_{\omega,N}(x, x) \) is bounded away from zero for all \( x \in \mathbb{Z}^d \). This is not true in our model because we do not have uniform ellipticity for the quenched measure. The idea here is to show that for “typical” \( \omega \) the measure \( \Theta_{\omega,N}(x, x) \) is bounded away from zero for “many” \( x \in \mathbb{Z}^d \). To this end for given \( \omega \) we define the set \( \Pi'_\omega \subset \Pi_M \) as the set of boxes \( \Delta \in \Pi_M \) satisfying
\[
P^{(0,0)}_{\omega}(X_N = x|X_{N-M} \in \Delta) > 0. \tag{5.5}
\]
Note that if \( \Pi^x_\omega = \emptyset \) for \( x \) and \( \omega \) then we have \( \Theta_{\omega,N}(x,x) = 0 \). Furthermore, by definition of \( P_{\omega}^{(0,0)}(X_N = x|X_{N-1} = y) \) we have
\[
P_{\omega}^{(0,0)}(X_N = x|X_{N-M} \in \Delta) \geq \left( \frac{1}{3^d} \right)^M
\]
for all \( \Delta \in \Pi^x_\omega \). Now using (5.4), (5.6) and uniform ellipticity of the annealed measure we obtain
\[
\Theta_{\omega,N}(x,x) = \sum_{\Delta \in \Pi^x_\omega} \tilde{\Theta}_{\omega,N-M,M}(\Delta, \Delta) \\
\cdot \mathbb{P}^{(0,0)}(X_N = x|X_{N-M} \in \Delta) \cdot P_{\omega}^{(0,0)}(X_N = x|X_{N-M} \in \Delta)
\geq \sum_{\Delta \in \Pi^x_\omega} \tilde{\Theta}_{\omega,N-M,M}(\Delta, \Delta) \eta^M \left( \frac{1}{3^d} \right)^M,
\]
where \( \eta \in (0,1) \) is the “uniform ellipticity bound” of the annealed random walk. Now it suffices to show
\[
\sum_{x \in \mathbb{Z}^d} \sum_{\Delta \in \Pi^x_\omega} \tilde{\Theta}_{\omega,N-M,M}(\Delta, \Delta) \geq \sum_{\Delta \in \Pi^x_\omega} \tilde{\Theta}_{\omega,N-M,M}(\Delta, \Delta).
\]
This follows immediately if we can show that for all \( \Delta \in \Pi_M \setminus \cup_{x \in \mathbb{Z}^d} \Pi^x_\omega \) we have
\[
\tilde{\Theta}_{\omega,N-M,M}(\Delta, \Delta) = 0.
\]
For that consider a box \( \Delta \in \Pi_M \setminus \cup_{x \in \mathbb{Z}^d} \Pi^x_\omega \), i.e. there is no \( x \in \mathbb{Z}^d \) with \( \Delta \in \Pi^x_\omega \) for the fixed \( \omega \). Thus, we have \( P_{\omega}^{(0,0)}(X_N = x|X_{N-M} \in \Delta) = 0 \) for all \( x \in \mathbb{Z}^d \). It follows that \( P_{\omega}^{(0,0)}(X_{N-M} \in \Delta) = 0 \), because there can be no infinitely long open path starting from \( \Delta \). We obtain
\[
\Theta_{\omega,N}(\Lambda) = \sum_{x \in \mathbb{Z}^d} \Theta_{\omega,N}(x,x) \geq \sum_{x \in \mathbb{Z}^d} \sum_{\Delta \in \Pi^x_\omega} \tilde{\Theta}_{\omega,N-M,M}(\Delta, \Delta) \eta^M \left( \frac{1}{3^d} \right)^M
\geq \sum_{\Delta \in \Pi^x_\omega} \tilde{\Theta}_{\omega,N-M,M}(\Delta, \Delta) \eta^M \left( \frac{1}{3^d} \right)^M \geq (1 - \varepsilon) \eta^M \left( \frac{1}{3^d} \right)^M
\]
for every \( \omega \in K(N) \).

Recall the definitions of \( P_N \) and \( Q_N \) from (2.3) respectively (2.4). Note that for every \( N \in \mathbb{N} \) the measure \( P_N \) is in fact the measure \( \mathbb{P} \) since for every measurable event \( A \in \Omega \) we have by translation invariance
\[
P_N(A) = \mathbb{E} \left[ \sum_{x \in \mathbb{Z}^d} \mathbb{P}^{(0,0)}(X_N = x) \mathbb{1}_{\{\sigma_{(x,N)} \omega \in A\}} \right] = \sum_{x \in \mathbb{Z}^d} \mathbb{P}^{(0,0)}(X_N = x) \mathbb{E} \left[ \mathbb{1}_{\{\sigma_{(x,N)} \omega \in A\}} \right]
= \sum_{x \in \mathbb{Z}^d} \mathbb{P}^{(0,0)}(X_N = x) \mathbb{P}(\sigma_{(x,-N)} A) = \sum_{x \in \mathbb{Z}^d} \mathbb{P}^{(0,0)}(X_N = x) \mathbb{P}(A) = \mathbb{P}(A).
\]

**Definition 5.2.** Given two environments \( \omega, \omega' \in \Omega \) we define their distance by
\[
\text{dist}(\omega, \omega') = \inf \{ \| (x,n) \| : \omega' = \sigma_{(x,n)} \omega \},
\]
where the infimum over an empty set is defined to be infinity.
We denote by $\Psi_N$ the coupling of $P_N$ and $Q_N$ from Lemma 5.1 extended to $\Omega \times \Omega$, that is,

$$\Psi_N(A) = \mathbb{E}\left[ \sum_{x,y \in \mathbb{Z}^d} \Theta_{\omega,N}(x,y) 1\{ (\sigma(y,N),\sigma(x,N)\omega,\sigma(y,N)\omega) \in A \} \right]. \tag{5.10}$$

The following result is an analogue to Lemma 6.6 in [BCR16].

**Lemma 5.3.** For $M, N \in \mathbb{N}$ let $D_{M,N}^{(1)} : \Omega \to [0, \infty]$ and $D_{M,N}^{(2)} : \Omega \to [0, \infty]$ be defined by

$$D_{M,N}^{(i)}(\omega_i) := \mathbb{E}_{\Psi_N}\left[ 1\{ \text{dist}(\omega_1,\omega_2) > M \} | \mathcal{F}_{\omega_i} \right](\omega_i), \quad i = 1, 2,$$

where $\mathcal{F}_{\omega_1}, \mathcal{F}_{\omega_2}$ are the $\sigma$-algebras generated by the first, respectively, second coordinate in $\Omega \times \Omega$ and $\Psi_N$ is defined in (5.10). For $M \in \mathbb{N}$, there exists an event $F_M$ with the following properties:

1. $\mathbb{P}(F_M) \geq 1 - M^{-c \log M}$.

2. For every $\varepsilon > 0$ one can choose $M = M(\varepsilon)$ large enough

$$\max\left\{ D_{M,N}^{(1)}(\omega), \frac{dQ_N}{dP}(\omega) D_{M,N}^{(2)}(\omega) \right\} \leq \varepsilon 1_{F_M}(\omega) + 1_{F_M^c}(\omega). \tag{5.11}$$

**Proof.** Let

$$F_M = \bigcap_{k > M/2} \left\{ \omega \in \Omega : \forall x \in [-k,k]^d \cap \mathbb{Z}^d, \sum_{\Delta \in \Pi_M} \left| \mathbb{P}(x,0)(X_k \in \Delta) - P(0,\omega)(X_k \in \Delta) \right| \leq \frac{C_2}{M^{c_1}} + \frac{C_1}{k^{c_1}} \right\}$$

where $\Pi_M$ is a partition of $\mathbb{Z}^d$ into boxes of side length $M$ and $C_2, c_1$ are the (renamed) constants from Lemma 2.1. Thus, $\mathbb{P}(F_M) \geq 1 - M^{-c \log M}$. Fix $\varepsilon > 0$. Then, by the definition of $F_M$ and the coupling $\tilde{\Theta}_{\omega,k,M}$ constructed in the proof of Lemma 5.1, for every $\omega \in F_M$, every $k > M/2$ and every $x \in [-k,k]^d \cap \mathbb{Z}^d$ we have

$$\tilde{\Theta}_{\sigma(x,k)\omega,k,M}(\Lambda_{\Pi_M}) > 1 - \frac{2C_2}{M^{c_1}} > 1 - \varepsilon \tag{5.12}$$

for large enough $M$, where $\Lambda_{\Pi_M} = \{ (\Delta, \Delta) : \Delta \in \Pi_M \}$. Note that for $k \leq M/2$ the left hand side of (5.12) is 1 and therefore (5.11) is trivially true for $N \leq M/2$.

Let us now verify the estimates (5.11) for $D_{M,N}^{(1)}$ and $\frac{dQ_N}{dP} D_{M,N}^{(2)}$ and $N > M/2$. Note that for $P$-almost every environment $\omega \in \Omega$ we have

$$D_{M,N}^{(1)}(\omega) = \sum_{x,y \in \mathbb{Z}^d} \Theta_{\sigma(y,N)\omega,N}(x,y) 1\{|x-y| > M\} \tag{5.13}$$

and for $Q_N$-almost every $\omega$ we have

$$D_{M,N}^{(2)}(\omega) = \left( \frac{dQ_N}{dP}(\omega) \right)^{-1} \sum_{x,y \in \mathbb{Z}^d} \Theta_{\sigma(x,N)\omega,N}(x,y) 1\{|x-y| > M\}. \tag{5.14}$$

Using (5.10) we have for every measurable event $A \subset \Omega$

$$E_{\Psi_N}\left[ 1\{|\omega_1,\omega_2| \in A \times \Omega \} 1\{ \text{dist}(\omega_1,\omega_2) > M \} \right]$$
which shows 

\[
Q_N \equiv \mathbb{E} \left[ \sum_{x,y \in \mathbb{Z}^d} \Theta_{\omega,N}(x,y) \mathbbm{1}_{\{ \Theta_{\omega,N}(x,y) \leq M, x \neq y \}} \right]
\]

where the last equality follows by translation invariance of \( \mathbb{P} \). Since \( \Psi_N \) is a coupling of \( P_N = \mathbb{P} \) and \( Q_N \) the last term equals

\[
E_{\Psi_N} \left[ \mathbbm{1}_{\{ (\omega, \omega') \in A \times \Omega \}} \sum_{x,y \in \mathbb{Z}^d} \Theta_{\sigma_{-N}(x,y)} \mathbbm{1}_{\{ \| x - y \| > M \}} \right],
\]

which implies (5.13).

For \( B_N := \{ \omega : \frac{dQ_N}{d\mathbb{P}}(\omega) \neq 0 \} \) we have \( Q_N(B^C_N) = \Psi_N(\Omega \times B^C_N) = 0 \), and we get similarly

\[
E_{\Psi_N} \left[ \mathbbm{1}_{\{ \omega \in A \times \Omega \}} \mathbbm{1}_{\{ \text{dist}(\omega_1, \omega_2) > M \}} \right]
\]

\[
= E_{\Psi_N} \left[ \mathbbm{1}_{\{ \omega \in A \cap B_N \}} \mathbbm{1}_{\{ \text{dist}(\omega_1, \omega_2) > M \}} \right]
\]

\[
= \Psi_N(\Omega \times (A \cap B_N) \cap \{ (\omega_1, \omega_2) : \text{dist}(\omega_1, \omega_2) > M \})
\]

\[
= \mathbb{E} \left[ \sum_{x,y \in \mathbb{Z}^d} \Theta_{\omega,N}(x,y) \mathbbm{1}_{\{ \Gamma_{(\omega_1,\omega_2)}(x,y) \in A \cap B_N \}} \mathbbm{1}_{\{ \text{dist}(\omega_1, \omega_2) > M \}} \right]
\]

\[
= \mathbb{E} \left[ \sum_{x,y \in \mathbb{Z}^d} \Theta_{\omega,N}(x,y) \mathbbm{1}_{\{ \Gamma_{(\omega_1,\omega_2)}(x,y) \in A \cap B_N \}} \mathbbm{1}_{\{ \| x - y \| > M \}} \right]
\]

\[
= E_{Q_N} \left[ \left( \frac{dQ_N}{d\mathbb{P}} \right)^{-1}(\omega) \sum_{x,y \in \mathbb{Z}^d} \Theta_{\sigma_{-N}(x,y)} \mathbbm{1}_{\{ \omega \in A \cap B_N \}} \mathbbm{1}_{\{ \| x - y \| > M \}} \right]
\]

\[
= E_{\Psi_N} \left[ \left( \frac{dQ_N}{d\mathbb{P}} \right)^{-1}(\omega_2) \sum_{x,y \in \mathbb{Z}^d} \Theta_{\sigma_{-N}(x,y)} \mathbbm{1}_{\{ (\omega_1, \omega_2) \in A \cap B_N \}} \mathbbm{1}_{\{ \| x - y \| > M \}} \right]
\]

which shows (5.14)

If \( \Theta_{\sigma_{-N}(x,y)}(x,y) > 0 \) then necessarily \( x \in [-N, N]^d \cap \mathbb{Z}^d \) because in \( N \) steps the annealed walk can only reach points in this box. It follows that for large enough \( M \), every \( \omega \in F_M \) and every \( N \geq M \) we have

\[
\sum_{x,y \in \mathbb{Z}^d} \Theta_{\sigma_{-N}(x,y)}(x,y) \mathbbm{1}_{\{ \| x - y \| > M \}}
\]

\[
= 1 - \sum_{x,y \in \mathbb{Z}^d} \Theta_{\sigma_{-N}(x,y)}(x,y) \mathbbm{1}_{\{ \| x - y \| \leq M \}}
\]

\[
\leq 1 - \min_{x \in [-N, N]^d} \sum_{x,y \in \mathbb{Z}^d} \Theta_{\sigma_{-N}(x,y)}(x,y) \mathbbm{1}_{\{ \| x - y \| \leq M \}}
\]
First we consider $B$ at the (spatial) origin, set $\omega$ whereas for $M$ we consider the events $\tilde{\Theta}$ where $\tilde{\Theta}$ is a positive constant. Therefore it is enough to show that $\tilde{\Theta}$ is the event from Lemma 6.5 in [BCR16]. We follow the ideas of the proof of Lemma 6.5 in [BCR16]. To this end, we consider the events

$$B^\varepsilon = \{ \omega : \frac{1}{|\Delta_0|} \sum_{x \in \Delta_0} \frac{dQ_N}{d\mathbb{P}}(\sigma(x,0)\omega) < 1 - \varepsilon \}$$

$$B^\varepsilon_0 = \{ \omega : \frac{1}{|\Delta_0|} \sum_{x \in \Delta_0} \frac{dQ_N}{d\mathbb{P}}(\sigma(x,0)\omega) > 1 + \varepsilon \}.$$

First we consider $B^-\varepsilon$. We decompose this event into two events, first of which has probability $M^{-c \log M}$ and the second is a $\mathbb{P}$ null set. We assume without loss of generality that $\Delta_0$ is centred at the (spatial) origin, set $M_\varepsilon = \frac{\varepsilon}{6d}M$, define $\Delta^-_0 = \{ x \in \mathbb{Z}^d : \|x\| < M - M_\varepsilon \}$ and $S^\varepsilon_0 = \{ \omega \in B^-\varepsilon_0 : \sigma(x,0)\omega \in F_{M_\varepsilon}, \forall x \in \Delta_0 \}$, where $F_{M_\varepsilon}$ is the event from Lemma 5.3. Due to property (1) of $F_{M_\varepsilon}$ from Lemma 5.3

$$\mathbb{P}(S^-_\varepsilon) \geq \mathbb{P}(B^-_\varepsilon) - |\Delta^-_0|\mathbb{P}(F^C_{M_\varepsilon}) \geq \mathbb{P}(B^-_\varepsilon) - M^d(M_\varepsilon)^{-c \log M_\varepsilon} \geq \mathbb{P}(B^-_\varepsilon) - M^{-\tilde{c} \log M},$$

where $\tilde{c}$ is a positive constant. Therefore it is enough to show that $\mathbb{P}(S^-_\varepsilon) = 0$.

We claim that there exists an event $K^\varepsilon \subset S^-_\varepsilon$ such that

$$\mathbb{P}(K^-) \geq \mathbb{P}(S^-_\varepsilon) \cdot ((4d)^d |\Delta_0|)^{-1} \tag{5.15}$$
and

\[ \text{if } \omega, \omega' \in K^{-}, \omega \neq \omega', \text{ then } \text{dist}(\omega, \omega') > 4M. \]  \hspace{1cm} (5.16)

For every \((x, n) \in \mathbb{Z}^d \times \mathbb{Z}\) let \(U_{(x,n)}\) be an independent (of everything else defined so far) random variable uniformly distributed on \([0, 1]\), and define

\[ K^{-} := \{ \omega \in S_{\varepsilon}^{-} : \forall (x, n) \in 4\Delta_{0} \text{ if } \sigma_{(x,n)} \omega \in B_{\varepsilon}^{-} \text{ then } U_{(x,n)} < U_{(0,0)} \}. \]

This means informally, that from each family of environments whose distance is smaller than \(4dM\) we choose one uniformly. This implies that property (5.16) for \(K\) holds. Property (5.15) holds because due to translation invariance of \(\mathbb{P}\) we have

\[ \mathbb{P}(S_{\varepsilon}^{-}) \leq \mathbb{P} \left( \bigcup_{x \in 4d\Delta_{0}} \sigma_{(x,0)} K^{-} \right) \leq \sum_{x \in 4d\Delta_{0}} \mathbb{P}(\sigma_{(x,0)} K^{-}) = (4d)^d |\Delta_{0}| \mathbb{P}(K^{-}). \]

Now, let

\[ G = \bigcup_{x \in \Delta_{0}} \sigma_{(x,0)} K^{-} \text{ and } G^{-} = \bigcup_{x \in \Delta_{0}^{-}} \sigma_{(x,0)} K^{-}. \]

By property (5.16) of \(K^{-}\) these are in both cases disjoint unions and therefore we have

\[ \mathbb{P}(G) = \sum_{x \in \Delta_{0}} \mathbb{P}(\sigma_{(x,0)} K^{-}) = |\Delta_{0}| \mathbb{P}(K^{-}) \quad \text{and} \]

\[ \mathbb{P}(G^{-}) = |\Delta_{0}^{-}| \mathbb{P}(K^{-}) = |\Delta_{0}| \left(1 - \frac{\varepsilon}{6d\varepsilon}\right)^d \mathbb{P}(K^{-}) > \left(1 - \frac{\varepsilon}{6}\right) \mathbb{P}(G). \] \hspace{1cm} (5.17)

Going back to the definition of the event \(B_{\varepsilon}^{-}\) and recalling that \(K^{-} \subset S_{\varepsilon}^{-} \subset B_{\varepsilon}^{-}\) we obtain

\[
Q_{N}(G) = \int_{G} \frac{dQ_{N}}{d\mathbb{P}}(\omega) d\mathbb{P}(\omega) = \sum_{x \in \Delta_{0}} \int_{\sigma_{(x,0)} K^{-}} \frac{dQ_{N}}{d\mathbb{P}}(\omega) d\mathbb{P}(\omega) = \int_{K^{-}} \sum_{x \in \Delta_{0}} \frac{dQ_{N}}{d\mathbb{P}}(\sigma_{(x,0)} \omega) d\mathbb{P}(\omega) \leq \int_{K^{-}} (1 - \varepsilon)|\Delta_{0}| d\mathbb{P}(\omega) = (1 - \varepsilon)|\Delta_{0}| \mathbb{P}(K^{-}) = (1 - \varepsilon) \mathbb{P}(G). 
\]

Combining this with (5.17), for small enough \(\varepsilon > 0\) we obtain

\[ Q_{N}(G) \leq (1 - \varepsilon) \mathbb{P}(G) = \frac{1 - \varepsilon}{1 - \varepsilon/6} \left(1 - \frac{\varepsilon}{6}\right) \mathbb{P}(G) < \frac{1 - \varepsilon}{1 - \varepsilon/6} \mathbb{P}(G^{-}) < \left(1 - \frac{\varepsilon}{3}\right) \mathbb{P}(G^{-}). \] \hspace{1cm} (5.18)

Let \(A^{-} = \{ (\omega, \omega') : \omega \in G^{-}, \omega' \notin G \}\). Then by (5.17) and (5.18)

\[ \Psi_{N}(A^{-}) \geq \mathbb{P}(G^{-}) - Q_{N}(G) \geq \mathbb{P}(G^{-}) - \left(1 - \frac{\varepsilon}{3}\right) \mathbb{P}(G^{-}) \]

\[ \geq \frac{\varepsilon}{3} \mathbb{P}(G^{-}) > \frac{\varepsilon}{3} \left(1 - \frac{\varepsilon}{6}\right) \mathbb{P}(G) > \frac{\varepsilon}{4} \mathbb{P}(G). \] \hspace{1cm} (5.19)
By construction of $K^-$, for every $(\omega, \omega') \in A^-$ we have $\text{dist}(\omega, \omega') > M_\varepsilon$ and, therefore,

$$
\int_G D^{(1)}_{M_\varepsilon,N} dP(\omega) = \int_{G \times \Omega} D^{(1)}_{M_\varepsilon,N} d\Psi_N(\omega, \omega') \geq \int_{G \times \Omega} D^{(1)}_{M_\varepsilon,N} d\Psi_N(\omega, \omega') = \int_{G \times \Omega} E\Psi_N(1_{\{\text{dist}(\omega, \omega') > M_\varepsilon\}} | G \times \Omega)(\omega, \omega') d\Psi_N(\omega, \omega')
$$

$$= \int_{G \times \Omega} E\Psi_N(1_{\{\text{dist}(\omega, \omega') > M_\varepsilon\}} | G \times \Omega)(\omega, \omega') d\Psi_N(\omega, \omega')
$$

$$= \int_{G \times \Omega} 1_{\{\text{dist}(\omega, \omega') > M_\varepsilon\}} 1_{\{G \times \Omega\}}(\omega, \omega') d\Psi_N(\omega, \omega') \geq \int_{G \times \Omega} 1_{\{\text{dist}(\omega, \omega') > M_\varepsilon\}} 1_{A^-}(\omega, \omega') d\Psi_N(\omega, \omega') = \Psi_N(A^-) \geq \frac{\varepsilon}{4} \mathbb{P}(G).
$$

Since $G \subset F_{M_\varepsilon}$ by definition, using Lemma 5.3 with $M_\varepsilon$ and $\varepsilon/5$ instead of $M$ and $\varepsilon$ we obtain

$$
\int_G D^{(1)}_{M_\varepsilon,N}(\omega) dP(\omega) \leq \int_G \frac{\varepsilon}{5} 1_{F_{M_\varepsilon}}(\omega) + 1_{F_{M_\varepsilon}}(\omega) dP(\omega) = \int_G \frac{\varepsilon}{5} dP(\omega) = \frac{\varepsilon}{5} \mathbb{P}(G).
$$

Combining (5.20) and (5.21) we conclude that $\mathbb{P}(G) = 0$ and, therefore $\mathbb{P}(K^-) = 0$. By property (5.15) of $K^-$ this implies that $\mathbb{P}(S^-_\varepsilon) = 0$ and finally $\mathbb{P}(B^-_\varepsilon) \leq M^{-c \log M}$.

Next we turn to the event $B^+_\varepsilon$. As before we set $M_\varepsilon = \frac{\varepsilon}{10} M$ and assume that $\Delta_0$ is centred at the origin. Define $\Delta_0^+ := \{x \in \mathbb{Z}^d : \|x\| < M + M_\varepsilon\}$ and let

$$
S^+_\varepsilon = \{\omega \in B^+_\varepsilon : \sigma(x,0) \in F_{M_\varepsilon}, \forall x \in \Delta_0^+\},
$$

where $F_{M_\varepsilon}$ is, as before, the event from Lemma 5.3. Due to property (1) of $F_{M_\varepsilon}$

$$
\mathbb{P}(S^+_\varepsilon) \geq \mathbb{P}(B^+_\varepsilon) - |\Delta_0^+| \mathbb{P}(F_{M_\varepsilon}) \geq \mathbb{P}(B^+_\varepsilon) - (1 + \frac{\varepsilon}{6d^2})^d M^d (M_\varepsilon)^{-c \log M_\varepsilon}
$$

and again it is enough to show that $\mathbb{P}(S^+_\varepsilon) = 0$. As for $S^-_\varepsilon$ we claim that there exists an event $K^+ \subset S^+_\varepsilon$ such that

$$
\mathbb{P}(K^+) \geq \mathbb{P}(S^+_\varepsilon) \cdot ((4d)^d|\Delta_0^+|)^{-1}
$$

and

$$
\text{if } \omega, \omega' \in K^+ \text{ with } \omega \neq \omega', \text{ then dist}(\omega, \omega') > 4(M + M_\varepsilon).
$$

Let

$$
H = \bigcup_{x \in \Delta_0^+} \sigma(x,0) K^+ \quad \text{and} \quad H^+ = \bigcup_{x \in \Delta_0^+} \sigma(x,0) K^+.
$$

Both are, by property (5.23) of $K^+$ disjoint unions. Therefore we have for $\varepsilon > 0$ small enough

$$
\mathbb{P}(H) = |\Delta_0^+| \mathbb{P}(K^+) \quad \text{and} \quad \mathbb{P}(H^+) = |\Delta_0^+| \mathbb{P}(K^+) = \left(1 + \frac{\varepsilon}{6d^2}\right)^d |\Delta_0^+| \mathbb{P}(K^+) < \left(1 + \frac{\varepsilon}{5}\right) \mathbb{P}(H).
$$
From $K^+ \subset S^+_\varepsilon \subset B^+_\varepsilon$ we obtain

$$Q_N(H) = \int_H \frac{dQ_N}{d\mathbb{P}}(\omega) d\mathbb{P}(\omega) = \sum_{x \in \Delta_0} \int_{\sigma(x,0)} K^+ \frac{dQ_N}{d\mathbb{P}}(\omega) d\mathbb{P}(\omega)$$

$$= \int_{K^+} \sum_{x \in \Delta_0} \frac{dQ_N}{d\mathbb{P}}(\sigma(x,0)\omega) d\mathbb{P}(\omega)$$

$$> \int_{K^+} |\Delta_0|(1 + \varepsilon) d\mathbb{P}(\omega) = (1 + \varepsilon)|\Delta_0|\mathbb{P}(K^+) = (1 + \varepsilon)\mathbb{P}(H).$$

Combination of this with (5.24), for small enough $\varepsilon > 0$ then yields

$$Q_N(H) > (1 + \varepsilon)\mathbb{P}(H) = \frac{1 + \varepsilon}{1 + \varepsilon/5} \mathbb{P}(H) > \frac{1 + \varepsilon}{1 + \varepsilon/5} \mathbb{P}(H^+) > \left(1 + \frac{\varepsilon}{3}\right) \mathbb{P}(H^+).$$

Let $A^+ := \{ (\omega, \omega') : \omega \notin H^+, \omega' \in H \}$. Then by (5.26)

$$\Psi_N(A^+) \geq Q_N(H) - \mathbb{P}(H^+) > Q_N(H) - \frac{1}{1 + \varepsilon/3}Q_N(H) = \frac{\varepsilon/3}{1 + \varepsilon/3}Q_N(H)$$

By the construction of $K^+$, for every $(\omega, \omega') \in A^+$ we have $\text{dist}(\omega, \omega') > M_\varepsilon$ and, therefore,

$$\int_H D^{(2)}_{M_\varepsilon,N}(\omega) dQ_N(\omega) = \int_{\Omega \times H} D^{(2)}_{M_\varepsilon,N}(\omega') d\Psi_N(\omega', \omega')$$

$$= \int_{\Omega \times \Omega} D^{(2)}_{M_\varepsilon,N}(\omega') \mathbb{1}_{(\Omega \times H)}(\omega, \omega') d\Psi_N(\omega', \omega')$$

$$= \int_{\Omega \times \Omega} E\Psi_N \mathbb{1}_{(\text{dist}(\omega, \omega') > M_\varepsilon)}(\omega') \mathbb{1}_{(\Omega \times H)}(\omega, \omega') d\Psi_N(\omega', \omega')$$

$$= \int_{\Omega \times \Omega} E\Psi_N \mathbb{1}_{(\text{dist}(\omega, \omega') > M_\varepsilon)}(\omega') \mathbb{1}_{(\Omega \times H)}(\omega, \omega') d\Psi_N(\omega', \omega')$$

$$= \int_{\Omega \times \Omega} \mathbb{1}_{(\text{dist}(\omega, \omega') > M_\varepsilon)}(\omega') \mathbb{1}_{A^+}(\omega, \omega') d\Psi_N(\omega', \omega')$$

$$= \frac{\varepsilon}{4}Q_N(H).$$

Since $H \subset F_{M_\varepsilon}$ by definition, $\mathbb{P}(H) \leq Q_N(H)$ by (5.25), and using Lemma 5.3 with $M_\varepsilon$ and $\frac{\varepsilon}{5}$
instead of $M$ and $\varepsilon$ we obtain
\[
\int_H D_{M,N}^{(2)} dQ_N(\omega) \leq \int_{H \cap B_N} \left( \frac{dQ_N}{d\mathbb{P}} \right)^{-1} \left[ \frac{\varepsilon}{5} \mathbb{1}_{F_M \cap B_N} + \mathbb{1}_{(F_M \cap B_N)^c} \right] dQ_N(\omega)
\]
\[
= \int_{H \cap B_N} \left[ \frac{\varepsilon}{5} \mathbb{1}_{F_M \cap B_N} + \mathbb{1}_{(F_M \cap B_N)^c} \right] d\mathbb{P}(\omega)
\]
\[
= \int_{H \cap B_N} \frac{\varepsilon}{5} d\mathbb{P}(\omega)
\]
\[
= \frac{\varepsilon}{5} \mathbb{P}(H \cap B_N) \leq \frac{\varepsilon}{5} \mathbb{P}(H) \leq \frac{\varepsilon}{5} Q_N(H),
\]
where we recall from Lemma 5.3 that $B_N = \{ \omega : \frac{dQ_N}{d\mathbb{P}}(\omega) \neq 0 \}$ and note that $B_N^C$ is a $Q_N$ null set. Combining (5.28) and (5.29), we conclude that $Q_N(H) = 0$ and, therefore, by (5.25) we have $\mathbb{P}(H) = 0$. It follows that $\mathbb{P}(K^+) = 0$, which by property (5.22) of $K^+$ implies that $\mathbb{P}(S_+^\ell) = 0$ and finally that (2.5) holds.

Proof of Corollary 2.4. To show that Proposition 2.2 holds for $Q$ as well we define $\Psi$ as the weak limit of $\{\frac{1}{n} \sum_{N=0}^{n-1} \Psi_N\}_{n=1}^\infty$ along any converging sub-sequence $\{n_k\}_{k \geq 1}$ (tightness of $\Psi_N$ follows similarly to the discussion below Corollary 2.3). Note that $\Psi$ is a coupling of $\mathbb{P}$ and $Q$ on $\Omega \times \Omega$. Furthermore let
\[
D_M^{(i)}(\omega) := E_\Psi[\mathbb{1}_{\text{dist}(\omega_1,\omega_2) > d_M} | F_\omega_i](\omega), \quad i = 1, 2.
\]

Now we want to prove inequality (5.11) from Lemma 5.3 for $D_M^{(1)}$ and $D_M^{(2)}$. It is enough to show that along some sub-sequence $\{n_\ell\}_{\ell \geq 1}$ of $\{n_k\}_{k \geq 1}$
\[
D_M^{(1)}(\omega) = \lim_{\ell \to \infty} \frac{1}{n_\ell} \sum_{N=0}^{n_\ell - 1} D_{M,N}^{(1)}(\omega) \quad \mathbb{P}\text{-a.s.}
\]
and
\[
D_M^{(2)}(\omega) = \left( \frac{dQ}{d\mathbb{P}}(\omega) \right)^{-1} \lim_{\ell \to \infty} \frac{1}{n_\ell} \sum_{N=0}^{n_\ell - 1} \frac{dQ}{d\mathbb{P}}(\omega) D_{M,N}^{(2)}(\omega) \quad Q\text{-a.s.}
\]
In fact, if the above equalities hold, then for $\mathbb{P}$-almost every $\omega$ we have
\[
D_M^{(1)}(\omega) = \lim_{\ell \to \infty} \frac{1}{n_\ell} \sum_{N=0}^{n_\ell - 1} D_{M,N}^{(1)}(\omega)
\]
\[
= \lim_{\ell \to \infty} \frac{1}{n_\ell} \left[ \sum_{N=0}^{M-1} D_{M,N}^{(1)}(\omega) + \sum_{N=M}^{n_\ell - 1} D_{M,N}^{(1)}(\omega) \right]
\]
\[
\leq \lim_{\ell \to \infty} \frac{1}{n_\ell} \left[ M + \sum_{N=M}^{n_\ell - 1} D_{M,N}^{(1)}(\omega) \right]
\]
\[
\leq \lim_{\ell \to \infty} \frac{1}{n_\ell} \left[ M + \sum_{N=M}^{n_\ell - 1} \left( \varepsilon \mathbb{1}_{F_M}(\omega) + \mathbb{1}_{F_M^C}(\omega) \right) \right]
\]
We have a subsequence that converges $\Psi$ where we used the definitions of $\Psi$ and of $D$

In addition for $D_M^{(2)}$ we have for $Q$ almost all $\omega$

$$\frac{dQ}{dP}(\omega)D_M^{(2)}(\omega) = \lim_{\ell \to \infty} \frac{1}{n_\ell} \sum_{N=0}^{n_\ell-1} \frac{dQ_N}{dP}(\omega)D_{M,N}^{(2)}(\omega)$$

$$\leq \lim_{\ell \to \infty} \frac{1}{n_\ell} \sum_{N=0}^{M-1} \frac{dQ_N}{dP}(\omega) + \sum_{N=M}^{n_\ell-1} \frac{dQ_N}{dP}(\omega)D_{M,N}^{(2)}(\omega)$$

$$\leq \lim_{\ell \to \infty} \frac{1}{n_\ell} \sum_{N=0}^{M-1} \frac{dQ_N}{dP}(\omega) + \sum_{N=M}^{n_\ell-1} (\varepsilon 1_{F_M}(\omega) + 1_{F_m^c}(\omega))$$

$$\leq \varepsilon 1_{F_M}(\omega) + 1_{F_m^c}(\omega).$$

Let us now prove (5.30) and (5.31). Starting with (5.30) let $A \subset \Omega$ be a measurable event. We have

$$E[D_M^{(1)}(\omega)1_A(\omega)]$$

$$= E[\Psi[1_{\{\text{dist}(\omega_1,\omega_2)>dM\}} 1_A \times \Omega(\omega_1,\omega_2)]$$

$$= \Psi(\{(\omega_1,\omega_2) \in \Omega \times \Omega : \text{dist}(\omega_1,\omega_2) > dM\} \cap A \times \Omega)$$

$$= \lim_{\ell \to \infty} \frac{1}{n_\ell} \sum_{N=0}^{n_\ell-1} \Psi_N(\{(\omega_1,\omega_2) \in \Omega \times \Omega : \text{dist}(\omega_1,\omega_2) > dM\} \cap A \times \Omega)$$

$$= \lim_{\ell \to \infty} \frac{1}{n_\ell} E[\Psi] \left[ \frac{1}{n_\ell} \sum_{N=0}^{n_\ell-1} D_M^{(1)}(\omega_1)1_A(\omega_1) \right]$$

$$= \lim_{\ell \to \infty} \left[ \frac{1}{n_\ell} \sum_{N=0}^{n_\ell-1} D_M^{(1)}(\omega_1)1_A(\omega_1) \right]$$

where we used the definitions of $\Psi$ and of $D_M^{(1)}$ as the conditional expectation. This implies convergence of $\frac{1}{n_\ell} \sum_{N=0}^{n_\ell-1} D_M^{(1)}(\omega_1)$ to $D_M^{(1)}$ in $L^1(P)$. Thus, by standard arguments we can choose a subsequence that converges $P$-almost surely. For $D_M^{(2)}$ we get in a similar way

$$E[D_M^{(2)}(\omega)1_A(\omega_2)]$$

$$= E[\Psi[1_{\{\text{dist}(\omega_1,\omega_2)>dM\}} 1_{\Omega \times A}(\omega_1,\omega_2)]$$

$$= \Psi(\{(\omega_1,\omega_2) \in \Omega \times \Omega : \text{dist}(\omega_1,\omega_2) > dM\} \cap \Omega \times A)$$

$$= \lim_{\ell \to \infty} \frac{1}{n_\ell} \sum_{N=0}^{n_\ell-1} \Psi_N(\{(\omega_1,\omega_2) > dM\} \cap \Omega \times A)$$

$$= \lim_{\ell \to \infty} \frac{1}{n_\ell} \sum_{N=0}^{n_\ell-1} E[\Psi] \left[ 1_{\{\text{dist}(\omega_1,\omega_2)>dM\}} 1_{\Omega \times A}(\omega_1,\omega_2) \right]$$

$$= \lim_{\ell \to \infty} \frac{1}{n_\ell} \sum_{N=0}^{n_\ell-1} E[\Psi] \left[ D_M^{(2)}(\omega_2)1_{\Omega \times A}(\omega_1,\omega_2) \right]$$
Thus, Lemma 5.3 holds for $D_{M, N}^{(1)}$ instead of $D_{M, N}^{(1)}$ and $D_{M, N}^{(2)}$, respectively.

Since the only tools we need for the proof of Proposition 2.2 are Lemma 2.1 and Lemma 5.3, we can walk through the proof of Proposition 2.2 and repeat the same steps for $\frac{dQ}{d\mathbb{P}}$ to show Corollary 2.4.

The following proposition is an analogue to Proposition 7.1 from [BCR16]. Note that the assertion expresses a general property of the density of a measure which is invariant for the point of view of the particle in the setting of a random walk in random environment. It is not model-specific.

**Proposition 5.4.** For $\mathbb{P}$-almost every $\omega$, every $n \in \mathbb{N}_0$, every $x \in \mathbb{Z}^d$ and all $k \leq n$

$$\varphi(\sigma(x, n)\omega) = \sum_{y \in \mathbb{Z}^d} P_{\omega}^{(x+y-n-k)}(X_n = x) \varphi(\sigma(x+y,n-k)\omega).$$

**Proof.** Let $n \in \mathbb{N}$. First we consider the case $k = 1$. For every bounded measurable function $h : \Omega \to \mathbb{R}$ we have (recall the notation in (1.9) and (1.10))

$$\int_{\Omega} h(\omega) \varphi(\sigma(x, n)\omega) \, d\mathbb{P}(\omega) = \int_{\Omega} h(\sigma(-x, -n)\omega) \varphi(\omega) \, d\mathbb{P}(\omega)$$

$$= \int_{\Omega} h(\sigma(-x, -n)\omega) \, dQ(\omega)$$

$$= \int_{\Omega} Q h(\sigma(-x, -n)\omega) \, dQ(\omega)$$

$$= \int_{\Omega} (Q h(\sigma(-x, -n)\omega)) \varphi(\omega) \, d\mathbb{P}(\omega)$$

$$= \int_{\Omega} \sum_{\|y\| \leq 1} g(\omega, y) h(\sigma(-x+y, n-1)\omega) \varphi(\omega) \, d\mathbb{P}(\omega)$$

$$= \int_{\Omega} \sum_{\|y\| \leq 1} g(\sigma(x-y, n-1)\omega, y) h(\omega) \varphi(\sigma(x-y, n-1)\omega) \, d\mathbb{P}(\omega).$$

Thus

$$\varphi(\sigma(x, n)\omega) = \sum_{\|y\| \leq 1} g(\sigma(x-y, n-1)\omega) \varphi(\sigma(x-y, n-1)\omega)$$

$$= \sum_{\|y\| \leq 1} P_{\omega}^{(x-y, n-1)}(X_1 = y) \varphi(\sigma(x-y, n-1)\omega)$$

$$= \sum_{\|y\| \leq 1} P_{\omega}^{(x-y, n-1)}(X_1 = x) \varphi(\sigma(x-y, n-1)\omega).$$
By applying the operator $R$ a second time we see that
\[
\int_{\Omega} h(\omega) \varphi(\sigma(x,n)\omega) \, d\mathbb{P} = \int_{\Omega} h(\omega) \sum_{\|y_1\| \leq 1} P_{\sigma(-x-y_1,-n+1)\omega}(X_1 = x) \varphi(\sigma(x+y_1,n-1)\omega) \, d\mathbb{P}(\omega)
\]
\[
= \int_{\Omega} h(\sigma(-x-y_1,-n+1)\omega) \sum_{\|y_1\| \leq 1} P_{\sigma(-x-y_1,-n+1)\omega}(X_1 = x) \varphi(\omega) \, d\mathbb{P}(\omega)
\]
\[
= \int_{\Omega} \left[ \left( R(h(\sigma(-x-y_1,-n+1)\omega) \sum_{\|y_1\| \leq 1} P_{\sigma(-x-y_1,-n+1)\omega}(X_1 = x)) \right) \varphi(\omega) \right] \, d\mathbb{P}(\omega)
\]
\[
= \int_{\Omega} \sum_{\|y_2\| \leq 1} g(\omega, y_2) h(\sigma(-x-y_1+y_2,-n+2)\omega) \sum_{\|y_1\| \leq 1} P_{\sigma(-x-y_1+y_2,-n+2)\omega}(X_1 = x) \varphi(\omega) \, d\mathbb{P}(\omega)
\]
\[
= \int_{\Omega} \sum_{\|y_2\| \leq 1} g(\sigma(x+y_1-y_2,n-2)\omega, y_2) h(\omega) \sum_{\|y_1\| \leq 1} P_{\omega}(x+y_1,y_2,n-2)(X_1 = x+y_1) \varphi(\sigma(x+y_1+y_2,n-2)\omega) \, d\mathbb{P}(\omega)
\]
\[
= \int_{\Omega} \sum_{\|y_2\| \leq 1} P_{\omega}(x+y_1,y_2,n-2)(X_1 = x+y_1) \varphi(\sigma(x+y_1+y_2,n-2)\omega) \, d\mathbb{P}(\omega).
\]

Thus,
\[
\varphi(\sigma(x,n)\omega) = \sum_{\|y_1\| \leq 1} \sum_{\|y_2\| \leq 1} P_{\omega}(x+y_1+y_2,n-2)(X_1 = x+y_1) P_{\omega}(x+y_1,n-1)(X_1 = x) \varphi(\sigma(x+y_1+y_2,n-2)\omega)
\]
\[
= \sum_{y \in \mathbb{Z}^d} P_{\omega}(x+y,n-2)(X_2 = x) \varphi(\sigma(x+y,n-2)\omega).
\]

Inductively we obtain
\[
\varphi(\sigma(x,n)\omega) = \sum_{y \in \mathbb{Z}^d} P_{\omega}(x+y,n-k)(X_k = x) \varphi(\sigma(x+y,n-k)\omega)
\]
for all $k \leq n$. \hfill \Box

6. Proof of Proposition 2.8

Let $\Pi$ be a partition of $\mathbb{Z}^d$ into boxes of side length $|n^\delta|$ with $0 < \delta < \frac{1}{d}$. Since $\mathbb{P}^{(0,0)}(X_n = x) = 0$ for $\|x\| > n$ only boxes in $\Pi_n := \{ \Delta \in \Pi : \Delta \cap [-n,n]^d \neq \emptyset \}$ have to be considered. We have
\[
|Z_{\omega,n} - 1| = \left| \sum_{x \in \mathbb{Z}^d} \mathbb{P}^{(0,0)}(X_n = x)[\varphi(\sigma(x,n)\omega) - 1] \right|
\]
\[
= \left| \sum_{\Delta \in \Pi_n} \sum_{x \in \Delta} \mathbb{P}^{(0,0)}(X_n = x)[\varphi(\sigma(x,n)\omega) - 1] \right|. \tag{6.1}
\]
By the annealed CLT from [BCDG13] for any \( \varepsilon > 0 \) there exists a constant \( C_\varepsilon > 0 \) such that

\[
P^{(0,0)}(\|X_n\| \geq C_\varepsilon \sqrt{n}) < \varepsilon
\]

We want to use this fact below and separate the sum in the last line of (6.1) into boxes in \( \hat{\Pi}_n = \{ \Delta \in \Pi_n : \Delta \cap \{ x \in \mathbb{Z}^d : \|x\| \leq C_\varepsilon \sqrt{n} \} \neq \emptyset \} \) and in \( \Pi_n \setminus \hat{\Pi}_n \). Using triangle inequality we obtain

\[
|Z_{\omega,n} - 1| \leq \left| \sum_{\Delta \in \Pi_n \setminus \hat{\Pi}_n} \sum_{x \in \Delta} P^{(0,0)}(X_n = x)[\varphi(\sigma(x,n)\omega) - 1] \right| + \left| \sum_{\Delta \in \hat{\Pi}_n} \sum_{x \in \Delta} \frac{1}{|\Delta|} \sum_{y \in \Delta} [P^{(0,0)}(X_n = y) - P^{(0,0)}(X_n = x)] [\varphi(\sigma(x,n)\omega) - 1] \right| + \left| \sum_{\Delta \in \Pi_n \setminus \hat{\Pi}_n} \sum_{x \in \Delta} \frac{1}{|\Delta|} \sum_{y \in \Delta} P^{(0,0)}(X_n = y)[\varphi(\sigma(x,n)\omega) - 1] \right|. \tag{6.2}
\]

We start with an upper bound of (6.2). By Corollary 2.4 there exists a constant \( C \), such that, due to translation invariance of \( P \), with \( P \) probability of a least \( 1 - C n^{-c \log n} \) for every \( \Delta \in \Pi_n \) we have \( \sum_{y \in \Delta} [\varphi(\sigma(y,n)\omega) + 1] \leq C |\Delta| \). Under this event we can bound (6.2) from above by

\[
\sum_{\Delta \in \Pi_n \setminus \hat{\Pi}_n} \sum_{x \in \Delta} P^{(0,0)}(X_n = x)[\varphi(\sigma(x,n)\omega) + 1] \leq C \sum_{\Delta \in \Pi_n \setminus \hat{\Pi}_n} \max_{x \in \Delta} P^{(0,0)}(X_n = x) |\Delta|.
\]

Using Lemma 3.2 with \( \delta > 0 \) replacing \( \varepsilon \) there we see that (6.2) is bounded from above by

\[
C \sum_{\Delta \in \Pi_n \setminus \hat{\Pi}_n} \sum_{y \in \Delta} \left[ \max_{x \in \Delta} P^{(0,0)}(X_n = x) - P^{(0,0)}(X_n = y) \right] + C \sum_{\Delta \in \Pi_n \setminus \hat{\Pi}_n} \sum_{y \in \Delta} P^{(0,0)}(X_n = y) \leq C \varepsilon + C n^{-1/2 + 3\delta}.
\]

Since \( \delta < \frac{1}{6d} \) it follows by the Borel-Cantelli lemma that

\[
\limsup_{n \to \infty} \left| \sum_{\Delta \in \Pi_n \setminus \hat{\Pi}_n} \sum_{x \in \Delta} P^{(0,0)}(X_n = x)[\varphi(\sigma(x,n)\omega) - 1] \right| \leq C \varepsilon, \quad P\text{-a.s.} \tag{6.5}
\]

Next we turn to (6.3). First note that by the annealed derivative estimates from Lemma 3.1 we have for \( x, y \in \Delta, \Delta \in \hat{\Pi}_n \)

\[
|P^{(0,0)}(X_n = x) - P^{(0,0)}(X_n = y)| \leq C \|x - y\| n^{-\frac{d+1}{2}} \leq C n^{-\frac{d+1}{2} + \delta}. \tag{6.6}
\]

By triangle inequality, (6.6) and again, as above, using Corollary 2.4 for the bound \( \sum_{y \in \Delta} [\varphi(\sigma(y,n)\omega) + 1] \leq C |\Delta| \) the expression (6.3) is bounded from above by

\[
\sum_{\Delta \in \hat{\Pi}_n} \sum_{y \in \Delta} \frac{1}{|\Delta|} \sum_{x \in \Delta} |P^{(0,0)}(X_n = y) - P^{(0,0)}(X_n = x)| [\varphi(\sigma(x,n)\omega) - 1] \leq C n^{-\frac{d+1}{2} + \delta} \sum_{\Delta \in \hat{\Pi}_n} \sum_{y \in \Delta} \frac{1}{|\Delta|} [\varphi(\sigma(y,n)\omega) + 1]
\]
\[ \leq Cn^{-d/2 + \delta} \sum_{\Delta \in \Pi_n} \sum_{y \in \Delta} C \]
\[ \leq C(C_{\varepsilon} \sqrt{n})^d n^{-d/2 + \delta} \leq \tilde{C}\varepsilon n^{-d/2 + \delta}. \]

with probability at least \( 1 - Cn^{-c\log n} \). Thus, as \( n \to \infty \), by the Borel-Cantelli lemma the expression (6.3) tends to 0 \( \mathbb{P} \)-almost surely.

Finally we consider (6.4). By triangle inequality and \( \mathbb{P}^{(0,0)}(X_n = y) \leq Cn^{-d/2} \) for all \( y \) we have
\[
\left| \sum_{\Delta \in \Pi_n} \sum_{x \in \Delta} \frac{1}{|\Delta|} \sum_{y \in \Delta} \mathbb{P}^{(0,0)}(X_n = y) |\varphi(\sigma(x,n)\omega) - 1| \right| 
\leq \sum_{\Delta \in \Pi_n} \frac{1}{|\Delta|} \sum_{y \in \Delta} \mathbb{P}^{(0,0)}(X_n = y) \sum_{x \in \Delta} |\varphi(\sigma(x,n)\omega) - 1| 
\leq Cn^{-d/2} \sum_{\Delta \in \Pi_n} \frac{1}{|\Delta|} \sum_{x \in \Delta} |\varphi(\sigma(x,n)\omega) - 1| 
= Cn^{-d(1/2 - \delta)} \sum_{\Delta \in \Pi_n} \frac{1}{|\Delta|} \sum_{x \in \Delta} |\varphi(\sigma(x,n)\omega) - 1|. 
\]

Using Corollary 2.4 we obtain
\[
\mathbb{P}\left( Cn^{d(1/2 - \delta)} \sum_{\Delta \in \Pi_n} \frac{1}{|\Delta|} \sum_{x \in \Delta} |\varphi(\sigma(x,n)\omega) - 1| > \varepsilon \right) 
\leq \mathbb{P}\left( \exists \Delta \in \Pi_n : \sum_{\Delta \in \Pi_n} \frac{1}{|\Delta|} \sum_{x \in \Delta} |\varphi(\sigma(x,n)\omega) - 1| > \frac{\varepsilon}{CC_{\varepsilon} d} \right) 
\leq n^{-d(1/2 - \delta)} \mathbb{P}\left( \frac{1}{|\Delta_0|} \sum_{x \in \Delta_0} |\varphi(\sigma(x,n)\omega) - 1| > \frac{\varepsilon}{CC_{\varepsilon} d} \right) 
\leq n^{-d(1/2 - \delta)} n^{-c\delta^2 \log n} \leq \tilde{C}n^{-c\log n},
\]
where \( \Delta_0 \in \Pi_n \) is an arbitrarily fixed box. Thus, for \( \varepsilon > 0 \) as \( n \to \infty \) the lim sup of (6.4) is bounded from above by \( \varepsilon \mathbb{P} \)-almost surely. Combining all three bounds of (6.2)–(6.4), we see that there is a constant \( \tilde{C} \) so that for all \( \varepsilon > 0 \)
\[
\limsup_{n \to \infty} |Z_{\omega,n} - 1| \leq \tilde{C}\varepsilon, \quad \mathbb{P} \text{-almost surely},
\]
which concludes the proof. \( \square \)

### 7. Proof of Proposition 2.9

The following result is an essential tool to prove Proposition 2.9 and will be proven in Section 9.

**Lemma 7.1.** Let \( 0 < \theta < 1/2 \) and \( b > 0 \). Define the set
\[
D(n) := \bigcap \left\{ \left\| P_{\omega}^{(x,0)}(X_n \in \cdot) - P_{\omega}^{(y,0)}(X_n \in \cdot) \right\|_{TV} \leq e^{-c\log n} \bigg\}, \quad (7.1)
\]

Then there are constants \( C, c > 0 \) so that \( \mathbb{P}(D(n)) \geq 1 - Cn^{-c\log n} \).
Note that the restriction $\|x\|, \|y\| \leq n^b$ in the definition of $D(n)$ in (7.1) is necessary because with probability 1 we have an environment where there exist (somewhere far out in space) two neighbouring points $x, y \in \mathbb{Z}^d$ so that the sites $(x, 0)$ and $(y, 0)$ are both connected to infinity but the respective clusters do not intersect for the first $n$ time steps.

Remark 7.2. The above lemma is the analogue of Lemma 7.7 from [BCR16] in our setting. Note that the bound stated in Lemma 7.7 from [BCR16] is too optimistic to hold in general. However, its assertion can be weakened and one obtains a bound which is still strong enough to prove Lemma 7.5 in [BCR16] by going a similar route as in the proof of Lemma 7.1 here.

Proof of Proposition 2.9, (L1). For this part we make use of the fact that, due to the annealed derivative estimates from Lemma 3.1 for $|x - y| \leq k$, $|\mathbb{P}^{(0,0)}(X_n = x) - \mathbb{P}^{(0,0)}(X_{n-k} = y)| \leq Ck/(n-k)^{(d+1)/2} \approx n^{-(d+1)/2 + \varepsilon}$, since $k = \lfloor n^\varepsilon \rfloor \ll n$. Furthermore we use the fact that by definition as a density of the invariant measure of the environment with respect to the point of view of the particle, the prefactor can be “transported” along the quenched transition probabilities; see Proposition 5.4. Finally we use the concentration property of Corollary 2.4; see equation (2.9).

We have to show

$$\lim_{n \to \infty} \sum_{x \in \mathbb{Z}^d} \frac{1}{Z_{\omega,n}} \mathbb{P}^{(0,0)}(X_n = x) \varphi(\sigma_{(x,n)} \omega)$$

$$- \frac{1}{Z_{\omega,n-k}} \sum_{y \in \mathbb{Z}^d} \mathbb{P}^{(0,0)}(X_{n-k} = y) \varphi(\sigma_{(y,n-k)} \omega) P_{\sigma_{(y,n-k)} \omega}^{(0,0)}(X_k = x - y) = 0. \quad (7.2)$$

Note that the by the triangle inequality the sum on the left hand side is bounded from above by

$$\sum_{x \in \mathbb{Z}^d} \frac{1}{Z_{\omega,n}} - \frac{1}{Z_{\omega,n-k}} \mathbb{P}^{(0,0)}(X_n = x) \varphi(\sigma_{(x,n)} \omega)$$

$$+ \frac{1}{Z_{\omega,n-k}} \sum_{x \in \mathbb{Z}^d} \mathbb{P}^{(0,0)}(X_n = x) \varphi(\sigma_{(x,n)} \omega)$$

$$- \sum_{y \in \mathbb{Z}^d} \mathbb{P}^{(0,0)}(X_{n-k} = y) \varphi(\sigma_{(y,n-k)} \omega) P_{\sigma_{(y,n-k)} \omega}^{(0,0)}(X_k = x - y).$$

By definition of $Z_{\omega,n}$, recall from Definition 2.7, the first sum in the above display equals to

$$\left| \frac{1}{Z_{\omega,n}} - \frac{1}{Z_{\omega,n-k}} \right| Z_{\omega,n},$$

which by Proposition 2.8 almost surely goes to 0 as $n$ and $n - k$ both tend to $\infty$. Thus, taking also into account the trivial deterministic bound on the speed of the random walk, for (7.2) it suffices to show

$$\lim_{n \to \infty} \sum_{x \in \mathbb{Z}^d \cap [-n,n]^d} \mathbb{P}^{(0,0)}(X_n = x) \varphi(\sigma_{(x,n)} \omega)$$

$$- \sum_{y \in \mathbb{Z}^d \cap [-n,n]^d} \mathbb{P}^{(0,0)}(X_{n-k} = y) \varphi(\sigma_{(y,n-k)} \omega) P_{\sigma_{(y,n-k)} \omega}^{(0,0)}(X_k = x - y) = 0. \quad (7.3)$$
Denoting by $B_n = \{ x \in \mathbb{Z}^d : \| x \| \leq \sqrt{n} \log^3 n \}$ and using the triangle inequality an upper bound of the sum in (7.3) is given by

$$
\sum_{x \in B_n} \sum_{y \in \mathbb{Z}^d \cap [-n,n]^d} \left| \mathbb{P}^{(0,0)}(X_n = x) - \mathbb{P}^{(0,0)}(X_{n-k} = y) \right| \times \varphi(\sigma_{(y,n-k)}^\omega) P^{(0,0)}_{\sigma_{(y,n-k)}^\omega}(X_k = x - y) 
+ \sum_{x \in B_n} \mathbb{P}^{(0,0)}(X_n = x) \times \left| \varphi(\sigma_{(x,n)}^\omega) - \sum_{y \in \mathbb{Z}^d \cap [-n,n]^d} \varphi(\sigma_{(y,n-k)}^\omega) P^{(0,0)}_{\sigma_{(y,n-k)}^\omega}(X_k = x - y) \right| 
+ \sum_{x \in \mathbb{Z}^d \cap [-n,n]^d \setminus B_n} \left| \mathbb{P}^{(0,0)}(X_n = x) \varphi(\sigma_{(x,n)}^\omega) - \sum_{y \in \mathbb{Z}^d \cap [-n,n]^d} \mathbb{P}^{(0,0)}(X_{n-k} = y) \varphi(\sigma_{(y,n-k)}^\omega) P^{(0,0)}_{\sigma_{(y,n-k)}^\omega}(X_k = x - y) \right|. 
$$

By the annealed derivative estimates (7.4) is bounded from above by

$$
\sum_{x \in B_n} \sum_{y \in \mathbb{Z}^d \cap [-n,n]^d} \left| \mathbb{P}^{(0,0)}(X_n = x) - \mathbb{P}^{(0,0)}(X_{n-k} = y) \right| \times \varphi(\sigma_{(y,n-k)}^\omega) P^{(0,0)}_{\sigma_{(y,n-k)}^\omega}(X_k = x - y) 
\leq \frac{2Ck}{(n-k)(d+1)/2} \sum_{x \in B_n} \sum_{y \in \mathbb{Z}^d \cap [-n,n]^d} \varphi(\sigma_{(y,n-k)}^\omega) P^{(0,0)}_{\sigma_{(y,n-k)}^\omega}(X_k = x - y) 
\leq \frac{2Ck(\sqrt{n} \log^3 n + k)^d}{(n-k)(d+1)/2} \frac{1}{(\sqrt{n} \log^3 n + k)^d} \sum_{y \in \mathbb{Z}^d \cap [-n,n]^d} \varphi(\sigma_{(y,n-k)}^\omega). 
$$

Now using Corollary 2.4 and the fact that $k = \lceil n^{1/4} \rceil < n^{1/4}$ for $\mathbb{P}$-almost every $\omega$ the last term tends to zero as $n$ tend to infinity.

Next we deal with (7.5). Recall that by Proposition 5.4 we have

$$
\varphi(\sigma_{(x,n)}^\omega) = \sum_{y \in \mathbb{Z}^d \cap [-n,n]^d} \varphi(\sigma_{(x,n-k)}^\omega) P^{(y,n-k)}_{\sigma_{(y,n-k)}^\omega}(X_k = x) 
$$
for every $x \in \mathbb{Z}^d$ such that $x + [-k,k]^d \cap \mathbb{Z}^d \subset [-n,n]^d \cap \mathbb{Z}^d$. This holds for every $x \in B_n$ and therefore the expression (7.5) equals 0.

Finally, for (7.6), using Lemma 3.6 from [Ste17], we have $\mathbb{P}^{(0,0)}(X_n \notin B_n) \leq C n^{-c \log n}$. Recall that $k = \lceil n^{1/4} \rceil$ and note that if $P^{(y,n-k)}_{\sigma_{(y,n-k)}^\omega}(X_k = x) > 0$ then $\| x - y \| \leq k$. Thus, for $x \in [-n,n]^d \cap \mathbb{Z}^d \setminus B_n$ and large enough $n$

$$
\| y \| \geq \| x \| - \| x - y \| \geq \sqrt{n} \log^3 n - k \geq \frac{1}{2} \sqrt{n} \log^3 n. 
$$
This implies, again due to Lemma 3.6 from [Stc17] that \( \mathbb{P}^{(0,0)}(X_{n-k} = y) \leq C n^{-c \log n} \). Therefore, the expression (7.6) is bounded from above by

\[
\sum_{x \in \mathbb{Z}^d \cap [0,n]^d \setminus B_n} \mathbb{P}^{(0,0)}(X_n = x) \varphi(\sigma(x,n)\omega) + \sum_{x \in \mathbb{Z}^d \cap [0,n]^d \setminus B_n} \sum_{y \in \mathbb{Z}^d \cap [0,n]^d} \mathbb{P}^{(0,0)}(X_{n-k} = y) \varphi(\sigma(y,n-k)\omega) P^{(y,n-k)}_{\omega}(X_k = x) + C n^{-c \log n} \sum_{y \in \mathbb{Z}^d \cap [0,n]^d} \varphi(\sigma(y,n-k)\omega) + C n^{-c \log n} \varphi(\sigma(x,n)\omega).
\]

By Corollary 2.4 we have

\[
\mathbb{P}\left( \sum_{x \in \mathbb{Z}^d \cap [0,n]^d} \varphi(\sigma(x,n)\omega) \leq 2n^d \right) > 1 - n^{-c \log n},
\]

as well as

\[
\mathbb{P}\left( \sum_{y \in \mathbb{Z}^d \cap [0,n]^d} \varphi(\sigma(y,n-k)\omega) \leq 2n^d \right) > 1 - C n^{-c \log n}.
\]

Thus, the probability of the event that (7.6) is bounded above by \( 4C n^{-c \log n} n^{-d} \) converges to 1 super-algebraically fast. Hence the expression (7.6) converges to 0 \( \mathbb{P} \)-almost surely.

**Proof of Proposition 2.9, (L2).** First note that, it is enough to show that

\[
\left\| \nu^{\text{ann} \times \text{pre}}_\omega - \nu^{\text{box} \times \text{pre}}_\omega \right\|_{1,n-k} \xrightarrow{n \to \infty} 0,
\]

since the last \( k \) steps are according to the quenched law for both hybrid measures. Then, as the measure \( \nu^{\text{box} \times \text{pre}}_\omega \) suggests, we make use of the comparison between the quenched and the annealed laws on the level of boxes we derived from Lemma 2.1. We also use the concentration properties of \( \varphi \) from Corollary 2.4.

Let \( k \in \{0, \ldots, n\} \) be fixed. Note that we have

\[
\left\| (\nu^{\text{ann} \times \text{pre}}_\omega * \nu^{\text{que}}_\omega)_{\omega,k} - (\nu^{\text{box} \times \text{que} \times \text{pre}}_\omega)_{\omega,k} \right\|_{1,n-k} \leq \left\| \nu^{\text{ann} \times \text{pre}}_\omega - \nu^{\text{box} \times \text{que} \times \text{pre}}_\omega \right\|_{1,n-k} = \sum_{x \in \mathbb{Z}^d} \varphi(\sigma(x,n-k)\omega) \left| \frac{\mathbb{P}^{(0,0)}(X_{n-k} = x)}{Z_{\omega,n-k}} - \frac{P^{(0,0)}_{\omega}(X_{n-k} \in \Delta_x)}{\sum_{y \in \Delta_x} \varphi(\sigma(y,n-k)\omega)} \right|.
\]

By Proposition 2.8 it is enough to show that \( \mathbb{P} \)-almost surely

\[
\lim_{n \to \infty} \sum_{x \in \mathbb{Z}^d} \varphi(\sigma(x,n-k)\omega) \left| \frac{\mathbb{P}^{(0,0)}(X_{n-k} = x)}{Z_{\omega,n-k}} - \frac{P^{(0,0)}_{\omega}(X_{n-k} \in \Delta_x)}{\sum_{y \in \Delta_x} \varphi(\sigma(y,n-k)\omega)} \right| = 0. \tag{7.7}
\]
Let $A_n = \{ x \in \mathbb{Z}^d : \|x\| \leq C_\varepsilon \sqrt{n} \}$, with $C_\varepsilon$ chosen so that $\mathbb{P}^{(0,0)}(\|X_{n-k}\| > C_\varepsilon \sqrt{n} - k) < \varepsilon$ for $n$ large enough. Using the triangle inequality the sum in (7.7) is bounded by

$$\sum_{x \in \mathbb{Z}^d \cap \{ -n, n \}^d \setminus A_n} \varphi(\sigma_{(x,n-k)}\omega) \left| \mathbb{P}^{(0,0)}(X_{n-k} = x) - \frac{\mathbb{P}^{(0,0)}(X_{n-k} \in \Delta_x)}{\|A_n\|} \right| \varphi(\sigma_{(y,n-k)}\omega) \right| \leq C \sum_{x \in A_n} \varphi(\sigma_{(x,n-k)}\omega) \frac{1}{\|A_n\|} \sum_{y \in \Delta_x} \frac{1}{(n-k)(d+1)/2} \|x - y\|$$

$$\leq Cdn^\delta \frac{1}{(n-k)(d+1)/2} \sum_{x \in A_n} \varphi(\sigma_{(x,n-k)}\omega)$$

$$= Cn^{\delta + d/2} \frac{1}{(n-k)(d+1)/2} \sum_{x \in A_n} \varphi(\sigma_{(x,n-k)}\omega) \rightarrow_0 0, \ \mathbb{P}\text{-a.s.},$$

where for the limit we use Proposition 2.2, the fact that $k = \lfloor n^{\varepsilon} \rfloor$ and $\delta < \varepsilon < \frac{1}{2}$.

Next we deal with (7.10). Writing $\hat{\Pi}_n = \{ \Delta \in \Pi : \Delta \cap A_n \neq \emptyset \}$, using annealed derivative estimates and Corollary 2.4 we see that (7.10) is bound by

$$\sum_{x \in A_n} \varphi(\sigma_{(x,n-k)}\omega) \frac{1}{\|A_n\|} \mathbb{P}^{(0,0)}(X_{n-k} \in \Delta_x) \right| 1 - \frac{1}{\|A_n\|} \sum_{y \in \Delta_x} \varphi(\sigma_{(y,n-k)}\omega) \right| \leq \varepsilon.$$
Using the same argument that was used for (6.4), we get that by the Borel-Cantelli lemma the last term goes to zero \( P \)-a.s.

Finally, we estimate (7.11). It is bounded from above by

\[
\sum_{x \in A_n} \varphi(\sigma_{(x,n-k)}\omega) \mathbb{P}(X_{n-k} \in \Delta_x) - P^{(0,0)}_{\omega}(X_{n-k} \in \Delta_x) = \sum_{\Delta \in \Pi_n} |\mathbb{P}(0,0)(X_{n-k} \in \Delta) - P^{(0,0)}_{\omega}(X_{n-k} \in \Delta)|.
\]

For the last term we can use Theorem 8.1 which implies that it is bounded by \( Cn^{-\frac{1}{4}} \) for \( P \)-almost every \( \omega \) and large enough \( n \). Therefore \( P \) almost surely converges to zero as \( n \) tends to infinity. \( \square \)

**Proof of Proposition 2.9.** (L3). Note that the first measure chooses, at time \( n-k \), a box according to the quenched law and a point in that box weighted by the prefactor, whereas the second measure chooses a box and a point in that box according to the quenched law at time \( n-k \). These points are then the starting points for the quenched random walks for the remaining \( k \) steps. We use the fact that, given enough time (much more than the square of the starting distance), the total variation distance for two quenched random walks starting from any pair of sites in a box with side length \( [n^\epsilon] \) is, given enough time, i.e. much more than the square of the side length of the box, is small with high probability, see Lemma 7.1.

The proof follows along the same lines as in [BCR16]. We will highlight the point in the proof where we deviate. We have

\[
\left\| (\nu^{\text{box-que} \times \text{pre}} * \nu^{\text{que}_{\omega}})_{\omega,k} - (\nu^{\text{que}} * L^{\text{que}})_{\omega,k} \right\|_{1,n} = \sum_{x \in \mathbb{Z}^d} \left| \left( (\nu^{\text{box-que} \times \text{pre}} * \nu^{\text{que}})_{\omega,k}(x,n) - (\nu^{\text{que}} * L^{\text{que}})_{\omega,k}(x,n) \right) \right|
\]

\[
= \sum_{x \in \mathbb{Z}^d} \left| \sum_{y \in \mathbb{Z}^d} P^{(0,0)}_{\omega}(X_{n-k} \in \Delta_y) \frac{\varphi(\sigma_{(y,n-k)}\omega)}{\varphi(\sigma_{(z,n-k)}\omega)} P^{(0,0)}_{\omega}(X_{k} = x - y) - \sum_{y \in \mathbb{Z}^d} P^{(0,0)}_{\omega}(X_{n-k} = y) P^{(0,0)}_{\sigma_{(y,n-k)}\omega}(X_{k} = x - y) \right|
\]

\[
= \sum_{x \in \mathbb{Z}^d} \sum_{\Delta \in \Pi_n} \sum_{y \in \Delta} P^{(y,n-k)}_{\omega}(X_{k} = x) P^{(0,0)}_{\omega}(X_{n-k} \in \Delta) \left( \frac{\varphi(\sigma_{(y,n-k)}\omega)}{\sum_{z \in \Delta} \varphi(\sigma_{(z,n-k)}\omega)} - P^{(0,0)}_{\omega}(X_{n-k} = y | X_{n-k} \in \Delta) \right)
\]
\begin{align*}\leq \sum_{x \in \mathbb{Z}^d} \sum_{\Delta \in \Pi} \left| \sum_{y \in \Delta} P^{(y,n-k)}_\omega(X_k = x)P^{(0,0)}_\omega(X_{n-k} \in \Delta) \right. & \cdot \left(\frac{\varphi(\sigma(y,n-k)\omega)}{\sum_{z \in \Delta} \varphi(\sigma(z,n-k)\omega)} - P^{(0,0)}_\omega(X_{n-k} = y \mid X_{n-k} \in \Delta)\right) \cdot (7.12) \\
& \text{Since for every } \Delta \in \Pi \text{ and } x \in \mathbb{Z}^d \text{ we have} \\
& \sum_{y \in \Delta} \frac{1}{|\Delta|} \sum_{v \in \Delta} P^{(y,n-k)}_\omega(X_k = x) \left[ \frac{\varphi(\sigma(y,n-k)\omega)}{\sum_{z \in \Delta} \varphi(\sigma(z,n-k)\omega)} - P^{(0,0)}_\omega(X_{n-k} = y \mid X_{n-k} \in \Delta) \right] = 0 \text{ it follows that (7.12) equals} \\
& \sum_{x \in \mathbb{Z}^d} \sum_{\Delta \in \Pi} P^{(0,0)}_\omega(X_{n-k} \in \Delta) \left| \sum_{y \in \Delta} \left[ P^{(y,n-k)}_\omega(X_k = x) - \left( \frac{1}{|\Delta|} \sum_{v \in \Delta} P^{(w,n-k)}_\omega(X_k = x) \right) \right] \right| \\
& = \sum_{x \in \mathbb{Z}^d} \sum_{\Delta \in \Pi} P^{(0,0)}_\omega(X_{n-k} \in \Delta) \left| \sum_{y \in \Delta} \frac{1}{|\Delta|} \sum_{w \in \Delta} \left[ P^{(y,n-k)}_\omega(X_k = x) - P^{(w,n-k)}_\omega(X_k = x) \right] \right| \\
& \leq \sum_{\Delta \in \Pi} \sum_{x \in \mathbb{Z}^d} P^{(0,0)}_\omega(X_{n-k} \in \Delta) \sum_{y \in \Delta} \frac{1}{|\Delta|} \sum_{w \in \Delta} \left[ P^{(y,n-k)}_\omega(X_k = x) - P^{(w,n-k)}_\omega(X_k = x) \right] \\
& \left| \frac{\varphi(\sigma(y,n-k)\omega)}{\sum_{z \in \Delta} \varphi(\sigma(z,n-k)\omega)} - P^{(0,0)}_\omega(X_{n-k} = y \mid X_{n-k} \in \Delta) \right| \right) = 0 \text{ if } \Delta \cap [-n+k-n-k]^d = \emptyset \text{ For } \Delta \cap [-n+k-n-k]^d \neq \emptyset \text{ we have } y,w \in \Delta \text{ implies that } \|y\|, \|w\| \leq n = k^{1/\epsilon} \text{ and } \|y - w\| \leq n^\epsilon = k^{1/\epsilon}. \\
& \text{Using Lemma 7.1 we see that (7.13) is bounded from above by} \\
& \sum_{\Delta \in \Pi} P^{(0,0)}_\omega(X_{n-k} \in \Delta) \sum_{y \in \Delta} \left| \frac{\varphi(\sigma(y,n-k)\omega)}{\sum_{z \in \Delta} \varphi(\sigma(z,n-k)\omega)} - P^{(0,0)}_\omega(X_{n-k} = y \mid X_{n-k} \in \Delta) \right| \\
& = e^{-\epsilon \log k \log \log \log n} \sum_{\Delta \in \Pi} \left| \sum_{y \in \Delta} \frac{\varphi(\sigma(y,n-k)\omega)}{\sum_{z \in \Delta} \varphi(\sigma(z,n-k)\omega)} - P^{(0,0)}_\omega(X_{n-k} = y \mid X_{n-k} \in \Delta) \right| \\
& \leq 2e^{-\epsilon \log k \log \log \log n} \sum_{\Delta \in \Pi} P^{(0,0)}_\omega(X_{n-k} \in \Delta) = 2e^{-\epsilon \log k \log \log \log n} \leq Ce^{-\epsilon \log n} \leq C \epsilon \log n \text{ since } k = \lceil n^\epsilon \rceil. \text{ The right hand side goes to 0 for } n \to \infty.$$

8. Proof of Proposition 4.1

The starting point is a result from [Ste17]. Define

$$P(N) := \left( \left[ -\frac{1}{24} \sqrt{N} \log^2 N, \frac{1}{24} \sqrt{N} \log^2 N \right]^d \times \left[ 0, \frac{1}{3} N \right] \right) \cap (\mathbb{Z}^d \times \mathbb{Z}). \quad (8.1)$$
For \( \theta \in (0,1) \) and \( (x, m) \in \mathcal{P}(N) \) let \( G'((x, m), N) \) denote the event that for every box \( \Delta \subset \mathbb{Z}^d \) of side length \( N^{\theta/2} \) we have
\[
\left| P_\omega^{(x, m)}(X_{m+N} \in \Delta) - \mathbb{P}^{(x, m)}(X_{m+N} \in \Delta) \right| \leq CN^{-d(1-\theta)/2-\frac{4}{5}}. \tag{8.2}
\]
Furthermore set
\[
G'(N) := \bigcap_{(x, m) \in \mathcal{P}(N)} \left( G'((x, m), N) \cup \{ \xi_m(x) = 0 \} \right). \tag{8.3}
\]

**Theorem 8.1** (Theorem 3.24 in [Ste17]). Let \( d \geq 3 \). There exist positive constants \( c \) and \( C \), such that for all \( (x, m) \in \mathcal{P}(N) \) we have
\[
\mathbb{P}^{(x, m)}(G'((x, m), N)) \geq 1 - CN^{-c \log N} \tag{8.4}
\]
and
\[
\mathbb{P}(G'(N)) \geq 1 - CN^{-c \log N}. \tag{8.5}
\]

The following notion of *good sites* and *good boxes* will be needed in the proof of Proposition 4.1. On such boxes the annealed and quenched laws are “close” to each other. Recall the process \( \xi = (\xi_n)_{n \in \mathbb{Z}} \) from (1.2) and the definition of \( n_k \) from the beginning of Section 4. Recall also that \( \Pi_k \) is a partition of \( \mathbb{Z}^d \) into the boxes of side length \( \lfloor n_k^\theta \rfloor \).

**Definition 8.2.** For a given realisation \( \omega \in \Omega \), we say that \( (x, m) \in \mathbb{Z}^d \times \mathbb{Z} \) is \((k-1, \theta, \varepsilon)-good\) if either \( \xi_m(x; \omega) = 0 \) or \( \xi_m(x; \omega) = 1 \) and the following two conditions are satisfied
\[
\sup_{\Delta' \in \Pi_k} \left| P_\omega^{(x, m)}(X_{m+n_k} \in \Delta') - \mathbb{P}^{(x, m)}(X_{m+n_k} \in \Delta') \right| \leq n_k^{\theta d - \frac{4}{5} - \varepsilon}, \tag{8.6}
\]
\[
P_\omega^{(x, m)}(\max_{s \leq n_k} \| X_{m+s} - x \| > \sqrt{n_k \log^3 n_k}) \leq C n_k^{-c \log n_k}. \tag{8.7}
\]

Otherwise the site is said to be \((k-1, \theta, \varepsilon)-bad\). We say that for \( \Delta \in \Pi_{k-1} \) and \( m \in \mathbb{Z} \) the box \( \Delta \times \{ m \} \) is \((k-1, \theta, \varepsilon)-good\) if each \( (x, m) \in \Delta \times \{ m \} \) is \((k-1, \theta, \varepsilon)-good\). Otherwise we say that \( \Delta \times \{ m \} \) is \((k-1, \theta, \varepsilon)-bad\).

The following lemma is a direct consequence of Theorem 8.1.

**Lemma 8.3.** For all \( \Delta \in \Pi_{k-1} \) there are positive constants \( C \) and \( c \) so that
\[
\mathbb{P}(\Delta \text{ is } (k-1, \theta, \varepsilon)-good) \geq 1 - C n_k^{-c \log n_k}. \tag{8.8}
\]

The assertion of Proposition 4.1 is the analogue of the inequality (5.1) in [BCR16]. The strategy of the proof there is as follows. First, using the triangle inequality and the Markov property an upper bound of \( \lambda_k \) is obtained which is given by a sum of four terms (5.2) – (5.5) in [BCR16]. Second, for each of these four terms an upper bound is shown. Three of these upper bounds, the ones for (5.2), (5.4) and (5.5), are not difficult and can be proven in the same way as in [BCR16]. For (5.3) Berger et. al use a notion of “good” boxes and the fact that they are independent at a large but finite distance. The definition of those good boxes translates to our Definition 8.2, where it is clear that the dependence on \( \xi \) prevents us from directly using any argument hinging on independence at a finite distance. We circumvent this problem by defining a new type of boxes for which we are able to work with independence, see the ideas below Proposition 8.4. Using those boxes as an approximation for the good boxes we prove a lower bound on the probability of hitting a good box in Proposition 8.4.
Proof of the analogue of an upper bound of (5.2) in [BCR16]. Consider
\[
\sum_{\Delta \in \Pi_k} \sum_{\Delta' \in \Pi_{k-1}} \left| \sum_{u \in \Delta'} P^{(u,N_{k-1})}_\omega (X_{N_k} \in \Delta) \times [P^{(\omega,0)}_\omega (X_{N_k-1} = u) - P^{(\omega,0)}_\omega (X_{N_k-1} \in \Delta') P^{(\omega,0)}_\omega (X_{N_k-1} = u|X_{N_k-1} \in \Delta')] \right|.
\] (8.9)

To get an upper bound for (8.9) the arguments in [BCR16] do not require any specific properties of the model and apply to our model as well. The steps are as follows: by the triangle inequality followed by elementary computations (8.9) is bounded from above by
\[
\sum_{\Delta \in \Pi_k} \sum_{\Delta' \in \Pi_{k-1}} \sum_{u \in \Delta'} P^{(u,N_{k-1})}_\omega (X_{N_k} \in \Delta) \times \left| P^{(\omega,0)}_\omega (X_{N_k-1} = u) - P^{(\omega,0)}_\omega (X_{N_k-1} \in \Delta') P^{(\omega,0)}_\omega (X_{N_k-1} = u|X_{N_k-1} \in \Delta') \right|
\] = \sum_{\Delta' \in \Pi_{k-1}} \sum_{u \in \Delta'} \left| P^{(\omega,0)}_\omega (X_{N_k-1} = u|X_{N_k-1} \in \Delta') [P^{(\omega,0)}_\omega (X_{N_k-1} \in \Delta') - P^{(\omega,0)}_\omega (X_{N_k-1} \in \Delta')] \right|
\] = \sum_{\Delta' \in \Pi_{k-1}} \left| P^{(\omega,0)}_\omega (X_{N_k-1} \in \Delta') - P^{(\omega,0)}_\omega (X_{N_k-1} \in \Delta') \right| = c_{\lambda k-1}.
\] (8.10)

Proof of the analogue of an upper bound of (5.3) in [BCR16]. Consider
\[
\sum_{\Delta \in \Pi_k} \sum_{\Delta' \in \Pi_{k-1}} \left| \sum_{u \in \Delta'} |P^{(0,0)}(X_{N_k-1} \in \Delta') P^{(0,0)}_\omega (X_{N_k-1} = u|X_{N_k-1} \in \Delta') \times [P^{(u,N_{k-1})}_\omega (X_{N_k} \in \Delta) - P^{(u,N_{k-1})}(X_{N_k} \in \Delta)] \right|.
\] (8.11)

First, by the triangle inequality (8.11) is bounded from above by
\[
\sum_{\Delta' \in \Pi_{k-1}} \sum_{u \in \Delta'} |P^{(0,0)}(X_{N_k-1} \in \Delta') P^{(0,0)}_\omega (X_{N_k-1} = u|X_{N_k-1} \in \Delta') \times \left| P^{(u,N_{k-1})}_\omega (X_{N_k} \in \Delta) - P^{(u,N_{k-1})}(X_{N_k} \in \Delta) \right|.
\] (8.12)

Next we define \(\Pi^1_{k-1}\) as the set of boxes \(\Delta' \in \Pi_{k-1}\) with the property
\[\Delta' \cap \{ x \in \mathbb{Z}^d : \| x \| \leq \sqrt{N_{k-1} \log^3 N_{k-1}} \} \neq \emptyset.\]
By Lemma 3.6 in [Ste17] it follows
\[
\sum_{\Delta' \in \Pi^1_{k-1}} |P^{(0,0)}(X_{N_k-1} \in \Delta') \leq C N_{k-1}^{-c \log N_{k-1}}
\] (8.13)
and consequently (8.11) is bounded from above by
\[ CN_k^{-c \log N_{k-1}} + \sum_{\Delta' \in \Pi_k} \sum_{u \in \Delta'} \P^{(0,0)}(X_{N_{k-1}} \in \Delta')P^{(0,0)}_\omega(X_{N_{k-1}} = u|X_{N_{k-1}} \in \Delta') \]
\[ \sum_{\Delta \in \Pi_k} |P^{(u,N_{k-1})}_\omega(X_{N_k} \in \Delta) - \P^{(u,N_{k-1})}(X_{N_k} \in \Delta)|. \] (8.14)

Recall Definition 8.2. We will write “good” for \((k-1, \theta, \varepsilon)\)-good to simplify the notation. By Lemma 8.3 we have \(\P(\Delta \text{ is } (k-1, \theta, \varepsilon)\)-good) \(\geq 1 - CN_k^{-c \log n_k}\). For \(u \in \Z^d\) define by \(\Pi_k^{1,u}\) the set of boxes \(\Delta \in \Pi_k\) satisfying (note that \(\P^{(u,0)}[X_{n_k}] = u\))
\[ \Delta \cap \{ x \in \Z^d : \|x - u\| \leq \sqrt{n_k \log^3 n_k} \} \neq \emptyset. \] (8.15)

If a box \(\Delta' \in \Pi_k^{1,u}\) is \((k-1, \theta, \varepsilon)\)-good, then for \(u \in \Delta'\)
\[ \sum_{\Delta \in \Pi_k^{1,u}} |P^{(u,N_{k-1})}_\omega(X_{N_k} \in \Delta) - \P^{(u,N_{k-1})}(X_{N_k} \in \Delta)| \]
\[ = \sum_{\Delta \in \Pi_k^{1,u}} |P^{(u,N_{k-1})}_\omega(X_{N_k} \in \Delta) - P^{(u,N_{k-1})}(X_{N_k} \in \Delta)| \]
\[ + \sum_{\Delta \in \Pi_k^{1,u} \setminus \Pi_k^{1,u}} |P^{(u,N_{k-1})}_\omega(X_{N_k} \in \Delta) - P^{(u,N_{k-1})}(X_{N_k} \in \Delta)| \]
\[ \leq \sum_{\Delta \in \Pi_k^{1,u}} |P^{(u,N_{k-1})}_\omega(X_{N_k} \in \Delta) - P^{(u,N_{k-1})}(X_{N_k} \in \Delta)| + Cn_k^{-c \log n_k} \] (8.16)

where we used in the first inequality that by Lemma 3.6 from [Ste17]
\[ \P^{(0,0)}(\|X_n\| > \sqrt{n \log^3 n}) \leq Cn^{-c \log n} \]
and that \(|\Pi_k^{1,u}| \leq Cn_k^{d/2 - \theta d} (\log n_k)^{3d} \).

It follows that (8.11) is bounded from above by
\[ CN_k^{-c \log N_{k-1}} + \sum_{\Delta' \in \Pi_k^{1,u} \text{ is good}} \sum_{u \in \Delta'} \P^{(0,0)}(X_{N_{k-1}} \in \Delta')P^{(0,0)}_\omega(X_{N_{k-1}} = u|X_{N_{k-1}} \in \Delta')Cn_k^{-\varepsilon/2} \]
\[ + \sum_{\Delta' \in \Pi_k^{1,u} \text{ is bad}} \sum_{u \in \Delta'} \P^{(0,0)}(X_{N_{k-1}} \in \Delta')P^{(0,0)}_\omega(X_{N_{k-1}} = u|X_{N_{k-1}} \in \Delta') \]
\[ \times \sum_{\Delta \in \Pi_k} |P^{(u,N_{k-1})}_\omega(X_{N_k} \in \Delta) - P^{(u,N_{k-1})}(X_{N_k} \in \Delta)| \]
\[ \leq CN_k^{-c \log N_{k-1}} + Cn_k^{-\varepsilon/2} + C \sum_{\Delta' \in \Pi_k^{1,u} \text{ is bad}} \P^{(0,0)}(X_{N_{k-1}} \in \Delta'). \] (8.17)
Now we want to find an estimate for the probability of hitting a bad box. For some \(\beta > 0\), to be chosen later, we consider the following event

\[
G_{N,n_k} := \left\{ \sum_{\Delta \in \Pi_{k-1}} \mathbb{1}_{\{\Delta \text{ is } (k-1,\theta,\varepsilon)\text{-good}\} \mathbb{P}^{(0,0)}(X_{N_{k-1}} \in \Delta) \geq 1 - C' n_k^{-\beta} \} \right. 
\]

(8.18)

and define

\[
G_N := \bigcap_{k=1}^{r(N)} G_{N,n_k}.
\]

(8.19)

We want to mimic the proof in [BCR16] and for that we need to define a new type of boxes to approximate the density of bad boxes. The problem with following the proof in [BCR16] arises from the fact that our environment is, due to the dependence on infinitely long open paths, not i.i.d. To overcome that problem the idea is to exchange the environment \(\xi\) with a process that only has finite range dependencies. We will use this idea to show in Proposition 8.4 below that

\[
\mathbb{P}(G_N) \geq 1 - C N^{-c\log(N)}.
\]

(8.20)

Note that \(n_{k-1} = n_k^2\). Thus, on \(G_N\) the expression (8.11) is bounded from above by

\[
CN_k^{-c\log N_{k-1}} + C n_k^{-\varepsilon/2} + C \sum_{\Delta' \in \Pi_{k-1}^{1}} \mathbb{P}^{(0,0)}(X_{N_{k-1}} \in \Delta') \\
\leq CN_k^{-c\log N_{k-1}} + C n_k^{-\varepsilon/2} + C' n_{k-1}^{-\beta} \leq C'' n_{k-1}^{-\beta}.
\]

(8.21)

As can be seen in the proof of Proposition 8.4 we can choose \(\beta \geq \varepsilon/4\) to obtain the last inequality in (8.21).

Proof of the analogue of an upper bound of (5.4) in [BCR16]. Consider

\[
\sum_{\Delta \in \Pi_k} \sum_{\Delta' \in \Pi_{k-1}} \left| \sum_{u \in \Delta'} \mathbb{P}^{(u,N_{k-1})}(X_{N_k} \in \Delta) \right. \times \left[ \mathbb{P}^{(0,0)}(X_{N_{k-1}} \in \Delta') P^{(0,0)}_{\omega}(X_{N_{k-1}} = u | X_{N_{k-1}} \in \Delta') - \mathbb{P}^{(0,0)}(X_{N_{k-1}} = u) \right]. 
\]

(8.22)

For any two probability measures \(\mu\) and \(\tilde{\mu}\) on \(\mathbb{Z}^d\) we have

\[
\sum_{u \in \Delta'} f(u)\mu(u) - \sum_{u \in \Delta'} f(u)\tilde{\mu}(u) \leq \max_{u \in \Delta'} f(u) - \min_{u \in \Delta'} f(u).
\]

Thus, the expression (8.22) can be bounded from above by

\[
\sum_{\Delta \in \Pi_k} \sum_{\Delta' \in \Pi_{k-1}} \mathbb{P}^{(0,0)}(X_{N_{k-1}} \in \Delta') \max_{u \in \Delta'} \mathbb{P}^{(u,N_{k-1})}(X_{N_k} \in \Delta) - \min_{u \in \Delta'} \mathbb{P}^{(u,N_{k-1})}(X_{N_k} \in \Delta) \\
\leq \sum_{\Delta' \in \Pi_{k-1}} \mathbb{P}^{(0,0)}(X_{N_{k-1}} \in \Delta') \\
\times \sum_{\Delta \in \Pi_k^{1(u)}} \left| \max_{u \in \Delta'} \mathbb{P}^{(u,N_{k-1})}(X_{N_k} \in \Delta) - \min_{u \in \Delta'} \mathbb{P}^{(u,N_{k-1})}(X_{N_k} \in \Delta) \right| + C n_k^{-c\log n_k},
\]

(8.23)
where \( \Pi_{1,u}^{(1,u)} \) is the set defined above.

Using \( \mathbb{P}^{(u,N_{k-1})}(X_{N_k} \in \Delta) = \sum_{v \in \Delta} \mathbb{P}^{(u,N_{k-1})}(X_{N_k} = v) \) we have

\[
\max_{u \in \Delta'} \mathbb{P}^{(u,N_{k-1})}(X_{N_k} \in \Delta) - \min_{u \in \Delta'} \mathbb{P}^{(u,N_{k-1})}(X_{N_k} = \Delta) \\
\leq \sum_{v \in \Delta} \max_{u \in \Delta'} \mathbb{P}^{(u,N_{k-1})}(X_{N_k} = v) - \min_{u \in \Delta'} \mathbb{P}^{(u,N_{k-1})}(X_{N_k} = v) \\
\leq \sum_{v \in \Delta} \text{diam}(\Delta') \frac{C}{n_k^{(d+1)/2}} \\
\leq (n_k^\theta)^d n_{k-1}^\theta \frac{C}{n_k^{(d+1)/2}},
\]

where the second to last inequality follows by the annealed derivative estimates from Lemma 3.9 in [Ste17]. Altogether the expression (8.22) is bounded from above by

\[
\sum_{\Delta' \in \Pi_{k-1}} \mathbb{P}^{(0,0)}(X_{N_{k-1}} \in \Delta') \sum_{\Delta \in \Pi_{1,u}^{(1,u)}} (n_k^\theta)^d n_{k-1}^\theta \frac{C}{n_k^{(d+1)/2}} + Cn_k^{-c \log n_k} \\
\leq C \sum_{\Delta' \in \Pi_{k-1}} \mathbb{P}^{(0,0)}(X_{N_{k-1}} \in \Delta') \left( \frac{dn_k^\theta}{n_k^\theta} \sqrt{\log n_k} \right)^3 \left( \frac{n_k^\theta}{n_{k-1}^\theta} \right)^d \frac{C}{n_k^{(d+1)/2}} + Cn_k^{-c \log n_k} \\
\leq C(\log n_k)^{3d} \frac{n_{k-1}^\theta}{n_k^{1/2}} + Cn_k^{-c \log n_k} \\
\leq C \left( \frac{\log n_k}{n_k^{1/2-\beta}} \right)^{3d} + Cn_k^{-c \log n_k}.
\]

(8.25)

\[\square\]

**Proof of the analogue of an upper bound of (5.5) in [BCR16].** Consider

\[
\sum_{\Delta \in \Pi_k} \sum_{\Delta' \in \Pi_{k-1}} \left| \sum_{u \in \Delta'} \mathbb{P}^{(0,0)}(X_{N_{k-1}} = u) \mathbb{P}^{(u,N_{k-1})}(X_{N_k} \in \Delta) - \mathbb{P}^{(0,0)}(X_{N_k} \in \Delta, X_{N_{k-1}} \in \Delta') \right|.
\]

(8.26)

Recall the regeneration times introduced in [BCGD13]. There they are defined for a random walk on the backbone of the oriented percolation cluster, whereas we allow the random walk to start outside the cluster. In Remark 2.3 Birkner et. al note that the local construction, which they use to obtain the regeneration times, can be extended to starting points outside the cluster. Let \( B_{m,n} \) be the event that the first regeneration time greater than \( m \) will happen before \( m + \bar{m}^\beta \), for some small constant \( \beta > 0 \) to be tuned appropriately later. By Lemma 2.5 from [BCGD13] the distribution of the regeneration increments has exponential tail bounds, and thus \( \mathbb{P}(B_{m,n}) \leq Ce^{-cm^\beta} \). First, note that by the theorem of total probability and the
triangle inequality \((8.26)\) is bounded from above by
\[
\sum_{\Delta' \in \Pi_{k-1}} \sum_{u \in \Delta'} \mathbb{P}^{(0,0)}(X_{N_{k-1}} = u) \sum_{\Delta \in \Pi_k} \left| \mathbb{P}^{(u,N_{k-1})}(X_{N_k} \in \Delta) - \mathbb{P}^{(0,0)}(X_{N_k} \in \Delta | X_{N_{k-1}} = u) \right|
\leq \sum_{\Delta' \in \Pi_{k-1}} \sum_{u \in \Delta'} \mathbb{P}^{(0,0)}(X_{N_{k-1}} = u) \\
\times \sum_{\Delta \in \Pi_k} \left( \left| \mathbb{P}^{(u,N_{k-1})}(X_{N_k} \in \Delta) - \mathbb{P}^{(0,0)}(X_{N_k} \in \Delta, B_{N_{k-1},n_k} | X_{N_{k-1}} = u) \right| \\
+ \mathbb{P}^{(0,0)}(X_{N_k} \in \Delta, B_{N_{k-1},n_k}^c | X_{N_{k-1}} = u) \right)
\tag{8.27}
\]

First note that
\[
\sum_{\Delta' \in \Pi_{k-1}} \sum_{u \in \Delta'} \mathbb{P}^{(0,0)}(X_{N_{k-1}} = u) \sum_{\Delta \in \Pi_k} \mathbb{P}^{(0,0)}(X_{N_k} \in \Delta, B_{N_{k-1},n_k}^c | X_{N_{k-1}} = u) \\
= \mathbb{P}(B_{N_{k-1},n_k}^c) \leq Ce^{-cn_k^\beta}.
\]

The remaining part of the right hand side of \((8.27)\) is bounded from above by
\[
\sum_{\Delta' \in \Pi_{k-1}} \sum_{u \in \Delta'} \mathbb{P}^{(0,0)}(X_{N_{k-1}} = u) \sum_{\Delta \in \Pi_k} \left( \mathbb{P}^{(u,N_{k-1})}(X_{N_k} \in \Delta, B_{0,n_k}^c) \\
+ \left| \mathbb{P}^{(u,N_{k-1})}(X_{N_k} \in \Delta, B_{0,n_k}) - \mathbb{P}^{(0,0)}(X_{N_k} \in \Delta, B_{N_{k-1},n_k} | X_{N_{k-1}} = u) \right| \right).
\]

Using the same arguments as above we obtain
\[
\sum_{\Delta' \in \Pi_{k-1}} \sum_{u \in \Delta'} \mathbb{P}^{(0,0)}(X_{N_{k-1}} = u) \sum_{\Delta \in \Pi_k} \mathbb{P}^{(u,N_{k-1})}(X_{N_k} \in \Delta, B_{0,n_k}^c) = \mathbb{P}(B_{N_{k-1},n_k}^c) \leq Ce^{-cn_k^\beta}
\]

and thus it remains to find a suitable upper bound for
\[
\sum_{\Delta' \in \Pi_{k-1}} \sum_{u \in \Delta'} \mathbb{P}^{(0,0)}(X_{N_{k-1}} = u) \sum_{\Delta \in \Pi_k} \left| \mathbb{P}^{(u,N_{k-1})}(X_{N_k} \in \Delta, B_{0,n_k}) - \mathbb{P}^{(0,0)}(X_{N_k} \in \Delta, B_{N_{k-1},n_k} | X_{N_{k-1}} = u) \right|
\]

Let \(\tilde{\tau}_{N_{k-1}}\) denote the first regeneration time greater than \(N_{k-1}\). By splitting the probabilities above into the sum over the possible sites at which the regeneration can occur and the possible times at which the random walk can be at the time of the regeneration we see that the term in the above display equals to
\[
\sum_{\Delta' \in \Pi_{k-1}} \sum_{u \in \Delta'} \mathbb{P}^{(0,0)}(X_{N_{k-1}} = u) \\
\times \sum_{\Delta \in \Pi_k} \left| \sum_{t \in [N_{k-1},N_{k-1}+n_k^\beta]} \sum_{v \in \mathbb{Z}^d; \|u-v\| \leq n_k^\beta} \mathbb{P}^{(v,t)}(X_{N_k} \in \Delta) \mathbb{P}^{(u,N_{k-1})}(\tilde{\tau}_{N_{k-1}} = t, X_{\tilde{\tau}_{N_{k-1}}} = v) \\
- \sum_{t \in [N_{k-1},N_{k-1}+n_k^\beta]} \sum_{v \in \mathbb{Z}^d; \|u-v\| \leq n_k^\beta} \mathbb{P}^{(v,t)}(X_{N_k} \in \Delta) \mathbb{P}^{(0,0)}(\tilde{\tau}_{N_{k-1}} = t, X_{\tilde{\tau}_{N_{k-1}}} = v | X_{N_{k-1}} = u) \right|.
\tag{8.28}
\]
The modulus in the last two lines of the above display is bounded from above by

\[
\left| \max_{t \in [N_{k-1}, N_{k-1} + n_k^\beta]} \mathbb{P}^{(v,t)}(X_{N_k} \in \Delta) - \min_{u \in Z^d, \|u-v\| \leq n_k^\beta} \mathbb{P}^{(v,t)}(X_{N_k} = u) \right|
\]

\[
= \max_{t \in [N_{k-1}, N_{k-1} + n_k^\beta]} \mathbb{P}^{(v,t)}(X_{N_k} \in \Delta) - \min_{u \in Z^d, \|u-v\| \leq n_k^\beta} \mathbb{P}^{(v,t)}(X_{N_k} = u) + \max_{t \in [N_{k-1}, N_{k-1} + n_k^\beta]} \mathbb{P}^{(u,N_{k-1})}(\tau_{N_{k-1}} > N_{k-1} + n_k^\beta) \]

\[
+ \min_{u \in Z^d, \|u-v\| \leq n_k^\beta} \mathbb{P}^{(v,t)}(X_{N_k} \in \Delta) \mathbb{P}^{(0,0)}(\tau_{N_{k-1}} > N_{k-1} + n_k^\beta, X_{N_{k-1}} = u)
\]

(8.29)

Plugging that into the sums in (8.28) we obtain that an upper bound of (8.26) is given by

\[
\sum_{\Delta' \in \Pi_{k-1}} \sum_{u \in \Delta'} \mathbb{P}^{(0,0)}(X_{N_{k-1}} = u) \cdot \mathbb{P}^{(v,t)}(X_{N_k} \in \Delta) - \min_{u \in Z^d, \|u-v\| \leq n_k^\beta} \mathbb{P}^{(v,t)}(X_{N_k} = u)
\]

\[
+ \sum_{\Delta' \in \Pi_{k-1}} \sum_{u \in \Delta'} \mathbb{P}^{(0,0)}(X_{N_{k-1}} = u) \cdot \mathbb{P}^{(v,t)}(X_{N_k} \in \Delta) \mathbb{P}^{(u,N_{k-1})}(\tau_{N_{k-1}} > N_{k-1} + n_k^\beta) \]

\[
+ \sum_{\Delta' \in \Pi_{k-1}} \sum_{u \in \Delta'} \sum_{\Delta \in \Pi_k} \mathbb{P}^{(v,t)}(X_{N_k} \in \Delta) \mathbb{P}^{(0,0)}(\tau_{N_{k-1}} > N_{k-1} + n_k^\beta, X_{N_{k-1}} = u) + C e^{-c n_k^\beta}
\]

(8.30)

Now define \(\Pi_k^{1,u,n}\) as the set boxes \(\Delta \in \Pi_k\) for which

\[
\Delta \cap \left( \bigcup_{v: \|v-u\| \leq n_k^\beta} \{ x \in Z^d : \|x - v\| \leq \sqrt{n_k^3 n_k} \} \right) \neq \emptyset.
\]

(8.31)

Using Lemma 3.6 from [Ste17] we obtain

\[
\sum_{\Delta \notin \Pi_k^{1,u,n}} \mathbb{P}^{(v,0)}(X_{N_{k-\ell}} \in \Delta) \leq \mathbb{P}^{(v,0)}(\|X_{N_{k-\ell}} - v\| > \sqrt{n_k^3 n_k} - t) \leq C n_k^{-c \log n_k}
\]
for all $v \in \mathbb{Z}^d$ with $\|v - u\| \leq n_k^\beta$ and all $t \in [N_{k-1}, N_{k-1} + n_k^\beta]$. Using this it follows

$$
\sum_{\Delta' \in \Pi_{k-1}} \sum_{u \in \Delta'} \mathbb{P}^{(0,0)}(X_{N_{k-1}} = u) \sum_{\Delta \in \Pi_k} \max_{t \in [N_{k-1}, N_{k-1} + n_k^\beta]} \mathbb{P}^{(v,t)}(X_N \in \Delta) \mathbb{P}^{(n,N_{k-1})}(\tau_{N_{k-1}} > N_{k-1} + n_k^\beta) \quad (8.32)
$$

$$
\leq |\Pi_k^{1,u,\beta}| \mathbb{P}^{(0,0)}(\tau_{N_{k-1}} > N_{k-1} + n_k^\beta) + C n_k^{-c \log n_k}
$$

$$
\leq n_k^\beta n_k^d/(2(1-2\theta) (\log n_k)^3 C e^{-cn_k^\beta} + C n_k^{-c \log n_k} \leq C n_k^{-c \log n_k},
$$

where we have used the fact that, by the definition of $\Pi_k^{(1,u)}$ in (8.15), $|\Pi_k^{1,u,\beta}| \leq n_k^{\beta d} |\Pi_k^{(1,u)}|$. Similarly

$$
\sum_{\Delta' \in \Pi_{k-1}} \sum_{u \in \Delta'} \mathbb{P}^{(0,0)}(X_{N_{k-1}} = u) \sum_{\Delta \in \Pi_k} \min_{t \in [N_{k-1}, N_{k-1} + n_k^\beta]} \mathbb{P}^{(v,t)}(X_N \in \Delta) \mathbb{P}^{(0,0)}(\tau_{N_{k-1}} > N_{k-1} + n_k^\beta, X_{N_{k-1}} = u) \quad (8.33)
$$

Altogether it follows that (8.26) is bounded from above by

$$
\sum_{\Delta' \in \Pi_{k-1}} \sum_{u \in \Delta'} \mathbb{P}^{(0,0)}(X_{N_{k-1}} = u) \times \sum_{\Delta \in \Pi_{k}^{1,u,\beta}} \left| \max_{t \in [N_{k-1}, N_{k-1} + n_k^\beta]} \mathbb{P}^{(v,t)}(X_N \in \Delta) - \min_{t \in [N_{k-1}, N_{k-1} + n_k^\beta]} \mathbb{P}^{(v,t)}(X_N \in \Delta) \right| + C n_k^{-c \log n_k} + C e^{-cn_k^\beta} \quad (8.34)
$$

Using the annealed derivative estimates from Lemma 3.1 we obtain

$$
\left| \max_{t \in [N_{k-1}, N_{k-1} + n_k^\beta]} \mathbb{P}^{(v,t)}(X_N \in \Delta) - \min_{t \in [N_{k-1}, N_{k-1} + n_k^\beta]} \mathbb{P}^{(v,t)}(X_N \in \Delta) \right| \leq |\Delta| \left| \max_{t \in [N_{k-1}, N_{k-1} + n_k^\beta]} \mathbb{P}^{(v,t)}(X_N = x) - \min_{t \in [N_{k-1}, N_{k-1} + n_k^\beta]} \mathbb{P}^{(v,t)}(X_N = y) \right| \leq |\Delta| C (4n_k^\beta + n_k^\theta) n_k^{-\frac{d+1}{2}}
$$

$$
\leq n_k^\beta C (4n_k^\beta + n_k^\theta) n_k^{-\frac{d+1}{2}}.
$$

(8.35)

Now if we choose $\beta = \theta$ and $\theta$ small enough, we get that the above expression is smaller than $C n_k^{-2d+1}$. Putting everything together we get the upper bound

$$
C e^{-cn_k^\beta} + C n_k^{-c \log n_k} + \sum_{\Delta' \in \Pi_{k-1}} \sum_{u \in \Delta'} \mathbb{P}^{(0,0)}(X_{N_{k-1}} = u) \sum_{\Delta \in \Pi_k} n_k^{-\frac{d+1}{2}}
$$

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\[ \leq C e^{-cn_k^\theta} + C n_k^{c \log n_k} + \sum_{\Delta' \in \Pi_{k-1}} \mathbb{P}^{(0,0)}(X_{N_{k-1}} \in \Delta') | \Pi_k^1 u | n_k^{-\frac{d-\theta}{2}} \]

\[ \leq C e^{-cn_k^\theta} + C n_k^{c \log n_k} + C n_k^{(1/2-\theta)d} (\log n_k)^{3d} n_k^{-\frac{d-\theta}{4}} \]

\[ = C e^{-cn_k^\theta} + C n_k^{c \log n_k} + C (\log n_k)^{3d} 2n_k^{\theta - 1/4} \]

Thus, recalling equation (8.26), we obtain

\[ \sum_{\Delta \in \Pi_k} \sum_{\Delta' \in \Pi_{k-1}} \left| \sum_{u \in \Delta'} \mathbb{P}^{(0,0)}(X_{N_{k-1}} = u) \mathbb{P}^{(u,N_{k-1})}(X_{N_k} \in \Delta) - \mathbb{P}^{(0,0)}(X_{N_k} \in \Delta, X_{N_{k-1}} \in \Delta') \right| \]

\[ \leq C n_k^{-c} \quad (8.36) \]

for some constants \( C, c > 0 \).

**Proof of Proposition 4.1.** To prove Proposition 4.1 we need to show inequality (4.3) which we recall here

\[ \lambda_k \leq \lambda_{k-1} + C n_k^{-\alpha}, \quad \forall 1 \leq k \leq r(N). \]

for some positive constants \( \alpha \) and \( C \) on the event \( G(N) \) from (8.19).

Fix \( \omega \in G(N) \). Recall the definition

\[ \lambda_k = \sum_{\Delta \in \Pi_k} \left| \sum_{u \in \Delta} P_{\omega}^{(0,0)}(X_{N_k} \in \Delta) - \mathbb{P}^{(0,0)}(X_{N_k} \in \Delta) \right| \]

from equation (4.2). Furthermore, we recall (8.9), (8.11), (8.22) and (8.26) for which we just estimated upper bounds.

\[ (8.9) = \sum_{\Delta \in \Pi_k} \sum_{\Delta' \in \Pi_{k-1}} \left| \sum_{u \in \Delta'} \mathbb{P}^{(0,0)}(X_{N_{k-1}} = u) \mathbb{P}^{(u,N_{k-1})}(X_{N_k} \in \Delta) \right| \times \left[ P_{\omega}^{(0,0)}(X_{N_{k-1}} = u) - \mathbb{P}^{(0,0)}(X_{N_{k-1}} \in \Delta') \mathbb{P}^{(0,0)}(X_{N_{k-1}} = u | X_{N_{k-1}} \in \Delta') \right], \]

\[ (8.11) = \sum_{\Delta \in \Pi_k} \sum_{\Delta' \in \Pi_{k-1}} \left| \sum_{u \in \Delta'} \mathbb{P}^{(0,0)}(X_{N_{k-1}} \in \Delta') \mathbb{P}^{(0,0)}(X_{N_{k-1}} = u | X_{N_{k-1}} \in \Delta') \right| \times \left[ \mathbb{P}^{(0,0)}(X_{N_{k-1}} \in \Delta') P_{\omega}^{(0,0)}(X_{N_{k-1}} = u | X_{N_{k-1}} \in \Delta') \right], \]

\[ (8.22) = \sum_{\Delta \in \Pi_k} \sum_{\Delta' \in \Pi_{k-1}} \left| \sum_{u \in \Delta'} \mathbb{P}^{(u,N_{k-1})}(X_{N_k} \in \Delta) \right| \times \left[ \mathbb{P}^{(0,0)}(X_{N_{k-1}} \in \Delta') \mathbb{P}^{(0,0)}(X_{N_{k-1}} = u | X_{N_{k-1}} \in \Delta') - \mathbb{P}^{(0,0)}(X_{N_{k-1}} = u) \right], \]

\[ (8.26) = \sum_{\Delta \in \Pi_k} \sum_{\Delta' \in \Pi_{k-1}} \left| \sum_{u \in \Delta'} \mathbb{P}^{(0,0)}(X_{N_{k-1}} = u) \mathbb{P}^{(u,N_{k-1})}(X_{N_k} \in \Delta) - \mathbb{P}^{(0,0)}(X_{N_k} \in \Delta, X_{N_{k-1}} \in \Delta') \right|. \]

Note that for \( \lambda_k \), by the triangle inequality, we obtain

\[ \lambda_k \leq (8.9) + (8.11) + (8.22) + (8.26). \]
Thus, using the proven estimates, (8.10), (8.21), (8.25) and (8.36), for each of the summands respectively we gain
\[
\lambda_k \leq \lambda_{k-1} + C' n_k^{-\epsilon/4} + C (\log n_k)^{3d} n_k^{1/2-2\theta} + C n_k^{-c\log n_k} + C n_k^{-c} \leq \lambda_{k-1} + \tilde{C} n_k^{-\alpha}
\]
for appropriate choices of $\alpha > 0$ and $\tilde{C} > 0$. The fact that $\mathbb{P}(G_N) \geq 1 - C N^{-c \log N}$ is proved in Proposition 8.4.

**Proposition 8.4.** For the events $G_N$ from (8.19) there exists $N_0 \in \mathbb{N}$ such that, for all $N \geq N_0$ we have that
\[
\mathbb{P}(G_N) \geq 1 - C N^{-c \log N}. \tag{8.37}
\]

Let $\beta > 0$ and put $f(n_k) = \log^2 n_k$. First we need another notion of good sites. Given a realization $\omega$ we define for all $(x, \ell) \in \mathbb{Z}^d \times \mathbb{Z}$ the set $C_m(x, \ell)$ as the set of sites at time $\ell + m \in \mathbb{Z}$ which can be reached from $(x, \ell)$ via an open path w.r.t. $\omega$. We start by defining for $k = 1, 2, \ldots$ a field $\xi^k := (\xi^k_t(x))_{t \in \mathbb{Z}^d}$ as follows
\[
\begin{align*}
(i) & \quad \xi^k_t(x) = \xi_t(x) \text{ for all } (x, t) \in \mathbb{Z}^d \times \{n_k + f(n_k), n_k + f(n_k) + 1, \ldots\} \\
(ii) & \quad \text{For all } (x, t) \in \mathbb{Z}^d \times \{n_k + f(n_k) - 2, n_k + f(n_k) - 1\} \text{ we set } \xi^k_t(x) = 1 \text{ if } C_{n_k + f(n_k) - t} \neq \emptyset. \text{ Otherwise we set } \xi^k_t(x) = 0.
\end{align*}
\]

Note that $\xi \leq \xi^k$ since for $(x, t)$ with $t < n_k + f(n_k)$ we set $\xi^k_t(x) = 1$ if $(x, t)$ has an open path of length at least $n_k + f(n_k) - t$ instead of requiring an infinite open path. For $\xi_t(x) \neq \xi^k_t(x)$ we necessarily must have $t < n_k + f(n_k)$ and there must exist an open path started at $(x, t)$ whose length is at least $n_k + f(n_k) - t$ but the contact process started at $(x, t)$ has to eventually die out, i.e. there is no infinite open path starting in $(x, t)$.

The following lemma gives us an upper bound on that probability. The result is well known in the oriented percolation and contact process world. For a proof see for instance in Lemma A.1. in [BCDG13].

**Lemma 8.5.** For $p > p_c$ there exist $C, c > 0$ such that for all $(x, t) \in \mathbb{Z}^d \times \mathbb{Z}$
\[
\mathbb{P}\left( (x, t) \rightarrow^\omega \mathbb{Z}^d \times \{t + 1\} \text{ and } (x, t) \rightarrow^\omega \mathbb{Z}^d \times \{\infty\} \right) \leq C e^{-cn}, \quad n \in \mathbb{N}.
\]

As a direct consequence we get the following corollary.

**Corollary 8.6.** For $x \in \mathbb{Z}^d$ define
\[
D_{n_k}(x) := (x + [-n^\theta_{k-1} - n_k, n^\theta_{k-1} + n_k] \times [0, n_k]) \cap (\mathbb{Z}^d \times \mathbb{Z}).
\]

For $p > p_c$ there exist constants $C, c > 0$ such that
\[
\mathbb{P}\left( \xi^k_t(y) = \xi_t(y) \text{ for all } (y, t) \in D_{n_k}(x) \right) \geq 1 - C e^{-c \log^2 n_k}. \tag{8.38}
\]

**Proof.** Note that $\theta > 0$ is a small constant and can be chosen such that we have $n^\theta_{k-1} = n^\theta_k \leq n_k$ and thus $|D_{n_k}(x)| \leq 2^d n_k^{d+1}$. By definition of $\xi^k \xi^k_t(y) \neq \xi_t(y)$ implies that there is at least one open but finite paths whose length is larger that $f(n_k)$. Using Lemma 8.5 the assertion (8.38) follows by the choice of $f(n_k) = \log^2 n_k$. (Here one can see that other choices of $f(n_k)$ are possible as well.)
Let $(\tilde{X})$ be a random walk in the environment $\hat{\xi}^k$ with transition probabilities given by

$$P_{\omega, \hat{\xi}^k}(\tilde{X}_{n+1} = x \mid \tilde{X}_n = y) = \begin{cases} |U(x, n) \cap \hat{\xi}^k|^{-1} & \text{if } (x, n) \in \hat{\xi}^k \text{ and } (y, n+1) \in U(x, n) \cap \hat{\xi}^k, \\ |U(x, n)|^{-1} & \text{if } (x, n) \notin \hat{\xi}^k \text{ and } (y, n+1) \in U(x, n), \\ 0 & \text{otherwise}, \end{cases}$$

(8.39)

where $\hat{\xi}^k := \{(x, n) \in \mathbb{Z}^d \times \mathbb{Z} : \hat{\xi}^k(x) = 1\}$.

Given a realisation $\omega$, we say that $(x, m)$ is $(k-1, \theta, \varepsilon, \hat{\xi}^k)$-good if it satisfies the conditions from Definition 8.2 with $\xi$ replaced by $\hat{\xi}^k$ and $X$ replaced by $\tilde{X}$ in the quenched probabilities.

**Lemma 8.7.** For all $(x, t) \in \mathbb{Z}^d \times \mathbb{Z}$ we have that

$$\mathbb{P}( (x, t) \text{ is } (k-1, \theta, \varepsilon, \hat{\xi}^k)-good) \geq 1 - \mathbb{C}n^{-c \log n_k}. \quad (8.40)$$

**Proof.** Due to Lemma 8.3 it suffices to show that with probability at least $1 - \mathbb{C}n^{-c \log n_k}$ we have $\hat{\xi}^k(x) = \xi_t(x)$ for all $(y, t) \in D_{n_k}(x)$. This exactly the assertion of Corollary 8.6. On that event $(x, t)$ is $(k-1, \theta, \varepsilon, \hat{\xi}^k)$-good. \qed

**Proof of Proposition 8.4.** Recall the definition of $G_{N,n_k-1}$ from (8.18). To estimate the probability of hitting a bad box we can now mimic the proof in [BCR16] since we get a lower bound by estimating the probability for the $(k-1, \theta, \varepsilon, \hat{\xi}^k)$-good boxes. By construction those boxes are independent of each other at distance $> 5n_k$. Define

$$\Pi_{k-1}^{(0)} = \{\Delta' \in \Pi_{k-1}^1 : \text{dist}(\Delta', 0) \leq \sqrt{n_{k-1}}\} \quad (8.41)$$

and for $r \geq 1$ let

$$\Pi_{k-1}^{(r)} = \{\Delta' \in \Pi_{k-1}^1 : 2^{r-1}\sqrt{n_{k-1}} < \text{dist}(\Delta', 0) \leq 2^r\sqrt{n_{k-1}}\}. \quad (8.42)$$

$(\Pi_{k-1}^{(r)})_{r \geq 0}$ is a partition of $\Pi_{k-1}^1$ into disjoint subsets according to the distance of the boxes from the origin which allows us to estimate the hitting probabilities of the bad boxes. Using the annealed local CLT (Theorem 1.1), we have

$$\sum_{\Delta' \in \Pi_{k-1}^1 \text{ is bad}} \tilde{\mathbb{P}}^{(0, 0)}(X_{N_{k-1}} \in \Delta') \leq \sum_{r=0}^{[\log_2(\log N_{k-1})^3]} \mathbb{P}_{k-1}^{(r)} \cap \{(k-1, \theta, \varepsilon, \hat{\xi}^k)-\text{bad boxes}\}|CN_{k-1}^{-d/2} e^{-cr^2} \quad (8.43)$$

holds for some constants $C, c > 0$ and $\tilde{\mathbb{P}}$ is the measure for the changed environments $\hat{\xi}^k$.

In order to estimate the number of bad boxes in each $\Pi_{k-1}^{(r)}$ we define the event $\tilde{G}_N = \tilde{G}_N(C)$ by

$$\tilde{G}_N := \bigcap_{k=1}^{r(N)[\log_2(\log N_{k-1})^3]} \bigcap_{r=0}^{[\log_2(\log N_{k-1})^3]} \{ \Pi_{k-1}^{(r)} \cap \{(k-1, \theta, \varepsilon, \hat{\xi}^k)-\text{bad boxes}\} \leq C|\Pi_{k-1}^{(r)}|n_{k-1}^{-\beta} \}, \quad (8.44)$$

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where \( \beta > 0 \) is a constant to be tuned later. Let \( \tilde{p}_{k-1} \) be the probability for a box \( \Delta' \in \Pi_{k-1} \) to be \((k-1, \theta, \varepsilon, \xi^k)\)-bad. Note that \( \tilde{p}_k \in \mathcal{O}(n_k^{-c \log n_k}) \) and on the event \( \tilde{G}_N \)

\[
\sum_{\Delta' \in \Pi_{k-1} \text{ is bad}} \overline{\mathbb{P}}(X_{N_{k-1}} \in \Delta') \leq \sum_{r=0}^{[\log_2(\log N_{k-1})]} C|\Pi_{k-1}^{(r)}|n_{k-1}^{-\beta}N_{k-1}^{d/2}e^{-cr^2}
\]

\[
\leq \sum_{r=0}^{[\log_2(\log N_{k-1})]} C2^{dr}(\sqrt{N_{k-1}/n_{k-1}})^dN_{k-1}^{d/2}e^{-cr^2}N_{k-1}^{-\beta} \leq Cn_{k-1}^{-(\beta+\theta d)}. \quad (8.45)
\]

Now it suffices to show that \( \mathbb{P}(\tilde{G}_N(C)) \geq 1 - C N^{-c \log(N)} \) for some constant \( C > 0 \). To do so, fix \( k \geq 1 \) and note that boxes \( \Delta' \in \Pi_{k-1} \) at distance \( 5n_k \) are, by construction of \( \xi^k \), good or bad independently of each other. To see this note that \( 2(n_{k-1}^\theta + n_k + f(n_k)) < 5n_k \) and recall that \( \xi^k(y) = 1 \) if there exists an open path connecting \((y, t) \) to \( \mathbb{Z}^d \times \{n_k + f(n_k)\} \) and \( \xi^k(y) = 0 \) otherwise. Let \( (\Pi_{k-1}^{(r)})_j \) be a partition of \( \Pi_{k-1}^{(r)} \) into at most \( (5n_k)^d \) subsets of boxes so that the distance between each pair of boxes in \( \Pi_{k-1}^{(r)} \) is bigger than \( 5n_k \), for every \( j \), and the number of boxes in \( \Pi_{k-1}^{(r)} \) is between \( |\Pi_{k-1}^{(r)}|/(2(5n_k)^d) \) and \( 2|\Pi_{k-1}^{(r)}|/(5n_k)^d). \)

If the number of \((k-1, \theta, \varepsilon, \xi^k)\)-bad boxes in \( \Pi_{k-1}^{r} \) is bigger than \( C|\Pi_{k-1}^{(r)}|n_{k-1}^{-\beta} \) then there exists at least one \( j \) so that the number of bad boxes in \( \Pi_{k-1}^{(r)} \) is larger than \( C|\Pi_{k-1}^{(r)}|n_{k-1}^{-\beta} \). Since the boxes in \( \Pi_{k-1}^{(r)} \) are good or bad independently of each other, their number is bounded and they are bad with probability \( \tilde{p}_{k-1} \), it follows by Hoeffding’s inequality that

\[
\overline{\mathbb{P}}(|\Pi_{k-1}^{(r)} \cap \{(k-1, \theta, \varepsilon, \xi^k)\}-bad boxes | > C|\Pi_{k-1}^{(r)}|n_{k-1}^{-\beta})
\]

\[
\leq (5n_k)^d\overline{\mathbb{P}}(|\Pi_{k-1}^{(r)} \cap \{(k-1, \theta, \varepsilon, \xi^k)\}-bad boxes | > [C|\Pi_{k-1}^{(r)}|n_{k-1}^{-\beta}/(5n_k)^d])
\]

\[
\leq (5n_k)^d \exp\left(-Cn_{k-1}^{-\beta} - \tilde{p} - \beta \right) \lvert \Pi_{k-1}^{(r)} \right\rvert \lvert 5n_k \right\rvert^d
\]

\[
\leq C(5n_k)^d \exp\left(-C2^{d}N^{\frac{d}{2}} + \frac{d}{2} - \frac{2\theta d}{n_k} - \frac{d}{2} \right)
\]

\[
= C(5n_k)^d \exp\left(-C2^{d}N^{\frac{d}{2}} - (4\beta + 2\theta d + d) \right),
\]

where the right hand side decays stretched exponentially in \( N \) for \( k \geq 4 \) if \( \beta \) is small enough, e.g. \( \beta = 1 \) (which is still sufficient for the proof of (8.11)). For \( 1 \leq k \leq 3 \) notice that

\[
\overline{\mathbb{P}}(|\Pi_{k-1}^{(r)} \cap \{(k-1, \theta, \varepsilon, \xi^k)\}-bad boxes | > C|\Pi_{k-1}^{(r)}|n_{k-1}^{-\beta})
\]

\[
\leq \overline{\mathbb{P}}(|\Pi_{k-1}^{(r)} \cap \{(k-1, \theta, \varepsilon, \xi^k)\}-bad boxes | \neq \emptyset)
\]

\[
\leq |\Pi_{k-1}^{(r)}|\tilde{p}_{k-1} \leq (\sqrt{N} \log^3(N))^d \tilde{p}_{k-1} \leq (\sqrt{N} \log^3(N))^d N^{-c \log(N)} \leq C N^{-c \log(N)}. \quad (8.47)
\]
Using the estimates above together with the definition of $\tilde{G}_N$ shows that

$$\mathbb{P}(\tilde{G}_N^C) = \mathbb{P} \left( \bigcup_{r=0}^{r(N)} \bigcup_{k=1}^{r(N) \left[ \log_2(\log N_{k-1}) \right]^3} \{ \Pi_{k-1}^{(r)} \cap \{ (k-1, \theta, \varepsilon, \tilde{\xi}^k) \text{-bad boxes} \} \mid > C|\Pi_{k-1}^{(r)}|n_{k-1}^{-\beta} \} \right)$$

\[
\leq \sum_{k=1}^{r(N) \left[ \log_2(\log N_{k-1}) \right]^3} \sum_{r=0}^{r(N)} \mathbb{P} \left( \{ \Pi_{k-1}^{(r)} \cap \{ (k-1, \theta, \varepsilon, \tilde{\xi}^k) \text{-bad boxes} \} \mid > C|\Pi_{k-1}^{(r)}|n_{k-1}^{-\beta} \} \right)
\]

\[
\leq r(N) \left[ \log_2(\log N_{k-1}) \right]^3] C N^{-c \log(N)} \leq C \log \log(N) \cdot \log(N)^{5/6} N^{-c \log(N)}
\]

\[
\leq N^{-c \log(N)}.
\]

(8.48)

Next we show that the number of $(k-1, \theta, \varepsilon)$-bad boxes in $\xi$ is on the same order as the number of $(k-1, \theta, \varepsilon, \tilde{\xi}^k)$-bad boxes in $\tilde{\xi}^k$ with high probability. First we define, in a slight abuse of notation, the sets

$$D_{n_k}(\Delta) := \{ (x, t) \in \mathbb{Z}^d \times \mathbb{Z} : \text{dist}(x, \Delta) \leq n_k, t \in [0, n_k] \},$$

$$A_{k, \Delta} := \{ \omega \in \Omega : \xi_i(x) = \tilde{\xi}^k(x) \text{ for all } (x, t) \in D_{n_k}(\Delta) \}$$

for all $\Delta \in \Pi_{k-1}^{(r)}$. Note that $D_{n_k}(\Delta)$ is the same box as $D_{n_k}(x)$ if $x$ is the center of $\Delta$. Using the above defined partitions $(\Pi_{k-1}^{(r)}, j)$ we see that for every choice of $\Delta, \Delta' \in \Pi_{k-1}^{(r)}$ the events $A_{k, \Delta}$ and $A_{k, \Delta'}$ are independent, since $\text{dist}(\Delta, \Delta') > 5n_k$. Since $\xi \leq \tilde{\xi}^k$ the number of $(k-1, \theta, \varepsilon)$-good boxes in $\xi$ is less or equal to the number of $(k-1, \theta, \varepsilon, \tilde{\xi}^k)$-bad boxes in $\tilde{\xi}^k$.

To shorten the notation we say for a box $\Delta \in \Pi_{k-1}^{(r)}$, that it is good in $\xi$ if it is $(k-1, \theta, \varepsilon)$-good and good in $\tilde{\xi}^k$ if it is $(k-1, \theta, \varepsilon, \tilde{\xi}^k)$-good. A box can only be bad in $\xi$ and good in $\tilde{\xi}^k$ for $\omega \in A_{k, \Delta}$. Using Corollary 8.6 we get $\mathbb{P}(A_{k, \Delta}^C) \leq C n_{k^{-c \log n_k}}$, and thus, again by Hoeffding’s inequality,

$$\mathbb{P} \left( |\Pi_{k-1}^{(r)} \cap \{ \text{bad in } \xi \} | - |\Pi_{k-1}^{(r)} \cap \{ \text{bad in } \tilde{\xi}^k \} | \geq C|\Pi_{k-1}^{(r)}|n_{k-1}^{-\beta} \right) \leq \mathbb{P} \left( \exists j \text{ s.t. } |\Pi_{k-1}^{(r)} \cap \{ \text{bad in } \xi \} | - |\Pi_{k-1}^{(r)} \cap \{ \text{bad in } \tilde{\xi}^k \} | \geq C|\Pi_{k-1}^{(r)}|n_{k-1}^{-\beta} \frac{1}{(5n_k)^d} \right)$$

\[
\leq (5n_k)^d \mathbb{P} \left( \sum_{\Delta \in \Pi_{k-1}^{(r)}} 1_{A_{k, \Delta}} \geq C|\Pi_{k-1}^{(r)}|n_{k-1}^{-\beta} \frac{1}{(5n_k)^d} \right)
\]

\[
\leq \tilde{C}(5n_k)^d \exp \left( -C2^{rd}N^2\left( \frac{4^{4d+2}d+4d+4}{2^d} \right) \right).
\]

Again the right hand side decays stretched exponentially in $N$ for $k \geq 4$ for $\beta > 0$ small enough. For $k \leq 3$ we can repeat the ideas of (8.47). The reason we have an upper bound in the same way as in (8.46) and (8.47) is that the probability for a box to be bad in $\tilde{\xi}^k$ is of the same order as $\mathbb{P}(A_{k, \Delta}^C)$, namely $n_{k^{-c \log n_k}}$. Define

$$A_N := \bigcap_{k=1}^{r(N) \left[ \log_2(\log N_{k-1}) \right]^3} \bigcap_{r=0}^{r(N)} \{ |\Pi_{k-1}^{(r)} \cap \{ \text{bad in } \xi \} | - |\Pi_{k-1}^{(r)} \cap \{ \text{bad in } \tilde{\xi}^k \} | \geq C|\Pi_{k-1}^{(r)}|n_{k-1}^{-\beta} \}$$

(8.50)
then by the same arguments as above we also get
\[ \mathbb{P}(A_C^N) \leq N^{-c \log N}. \] (8.51)
Since \( \tilde{G}_N \cap A_N \subset G_N \) the claim follows.

9. Mixing properties of the quenched law: proof of Lemma 7.1

Definition 9.1. Let \( \Pi_M \) be a partition of \( \mathbb{Z}^d \) into boxes of side lengths \( M \), let \( C > 0 \) and let \( \omega \) be a realisation of the environment. We call a box \( \Delta \in \Pi_M \) social with respect to \( \omega \) at time \( N \in \mathbb{N} \), if for any pair of points \( x, y \in \Delta \) there exists \( z \in \mathbb{Z}^d \) such that
\[ P^\omega_{(x,N)}(X_N + \lceil CM \rceil = z) > 0, \quad \text{and} \quad P^\omega_{(y,N)}(X_N + \lceil CM \rceil = z) > 0. \]

Note that if \( P^\omega_{(x,N)}(X_N + \lceil CM \rceil = z) > 0 \), then by construction \( P^\omega_{(x,N)}(X_N + \lceil CM \rceil = z) \geq (3^{-d})^{CM} \).

The next result shows that the density of social boxes is suitably high.

Lemma 9.2. For every \( \varepsilon > 0 \) there exists \( M_0 \in \mathbb{N} \) and constants \( c, C > 0 \) such that for all \( M \geq M_0 \) there exists a set of environments \( S_{M} \) satisfying
\[ \sum_{\Delta \in \Pi_M : \Delta \text{ is not social}} p_M^{(x,0)}(X_N \in \Delta) < \varepsilon \quad \text{for all } \omega \in S_M \]
and \( \mathbb{P}(S_M) \geq 1 - Ce^{-c \log n} \). (Recall that the property of \( \Delta \) being social depends on \( \omega \).)

Corollary 9.3. For every \( \varepsilon > 0 \) there exists \( M_0 \in \mathbb{N} \) so that for all \( M > M_0 \) there are environments \( S_M \) such that
\[ \sum_{\Delta \in \Pi_M : \Delta \text{ is not social}} p_M^{(x,0)}(X_N \in \Delta) < 2 \varepsilon \]
for all \( \omega \in S_M \) and \( \mathbb{P}(S_M) \geq 1 - Cn^{-c \log n} \).

Proof. Combine Lemma 9.2 and Lemma 2.1.

Proof of Lemma 9.2. The proof idea is similar to the one we have used to prove the high density of good boxes; see the proof of Proposition 8.4. We set
\[ p_M := \mathbb{P}(\Delta \text{ is not social}). \]
As a direct consequence of Lemma A.1 for every \( \Delta \in \Pi_M \) we have that \( p_M \leq Ce^{-cM} \) for some positive constants \( C, c \). We define
\[ S_M := \bigcap_{r=0}^{\log_2 \log^3 n} \left\{ |\Pi^{(r)}_M \cap \{ \text{not social boxes} \}| < C|\Pi^{(r)}_M|p_M \right\}, \] (9.1)
where
\[ \Pi^{(0)}_M = \{ \Delta \in \Pi_M : \text{dist}(\Delta, 0) \leq \sqrt{n} \}, \]
\[ \Pi^{(r)}_M = \{ \Delta \in \Pi_M : 2^{r-1} \sqrt{n} < \text{dist}(\Delta, 0) \leq 2^r \sqrt{n} \} \quad \text{for } r \geq 1. \]

By Lemma 3.6 from [Ste17] we have \( \mathbb{P}^{(0,0)}(\|X_n\| \geq \sqrt{n} \log^3 n) \leq C n^{-c \log n} \) and so for \( \omega \in S_M \) (note that being social depends on \( \omega \))

\[
\sum_{\Delta \in \Pi^{(r)}_M, \Delta \text{ is not social}} \mathbb{P}^{(0,0)}(X_n \in \Delta) \leq C n^{-c \log n} + \sum_{r=0}^{\log_2 \log^3 n} \sum_{\Delta \in \Pi^{(r)}_M, \Delta \text{ is not social}} \mathbb{P}^{(0,0)}(X_n \in \Delta)
\leq \sum_{r=0}^{\log_2 \log^3 n} C |\Pi^{(r)}_M|p_M \frac{1}{n^{d/2}} \exp\left(-\frac{1}{2n} (2^{r-1} \sqrt{n})^2 \right)
\leq C \sum_{r=0}^{\log_2 \log^3 n} \left( \frac{2^r \sqrt{n}}{M} \right)^d \frac{1}{n^{d/2}} \exp(-cr^2) p_M
\leq C p_M \sum_{r=0}^{\log_2 \log^3 n} \frac{1}{M^d} \exp(-cr^2 + rd \log 2)
\leq C' p_M
\]

where we used the annealed local CLT in the second inequality. It remains to show that \( \mathbb{P}^{(0,0)}(S_M) \geq 1 - Ce^{-c \log n} \). We have

\[
\mathbb{P}^{(0,0)}(S_M^n) = \mathbb{P}^{(0,0)}(\exists \omega \leq \log_2 \log^3 n : |\Pi^{(r)}_M| \cap \{ \text{not social boxes} \} > C |\Pi^{(r)}_M|p_M)
\leq \sum_{r=0}^{\log_2 \log^3 n} \mathbb{P}^{(0,0)}(|\Pi^{(r)}_M| \cap \{ \text{not social boxes} \} > C |\Pi^{(r)}_M|p_M).
\]

Next, let \((\Pi^{(r)}_M^j)_{j \in J}\) be a further partition of \(\Pi^{(r)}_M\) so that for each \(j \in J\) the distance between any pair of distinct boxes in \(\Pi^{(r)}_M^j\) is bigger than \(3CM\) and

\[
\frac{|\Pi^{(r)}_M^j|}{2(3CM)^d} \leq |\Pi^{(r)}_M^j| \leq \frac{2|\Pi^{(r)}_M|}{(3CM)^d}.
\]

Note that the index set \(J = J(M, r)\) is finite (in fact we have \(|J| \leq 2(3CM)^d\)) and that by construction the boxes in \(\Pi^{(r)}_M^j\) are social or not social independently of each other. If \(|\Pi^{(r)}_M \cap \{ \text{not social boxes} \}| > C |\Pi^{(r)}_M|p_M\) then there exists a \(j\) such that \(|\Pi^{(r)}_M^j \cap \{ \text{not social boxes} \}| > C |\Pi^{(r)}_M|p_M/(3CM)^d\). Using Hoeffding’s inequality for \(r \geq 1\) we obtain

\[
\mathbb{P}^{(0,0)}(|\Pi^{(r)}_M \cap \{ \text{not social boxes} \}| > C |\Pi^{(r)}_M|p_M)
\leq \sum_{j \in J} \mathbb{P}^{(0,0)}(|\Pi^{(r)}_M^j \cap \{ \text{not social boxes} \}| > C |\Pi^{(r)}_M|p_M/(3CM)^d)
\leq \sum_{j \in J} \mathbb{P}^{(0,0)}(|\Pi^{(r)}_M^j \cap \{ \text{not social boxes} \}| > |\Pi^{(r)}_M^j|p_M) > \left( \frac{C |\Pi^{(r)}_M|}{(3CM)^d} - |\Pi^{(r)}_M^j| \right)p_M
\leq \sum_{j \in J} \exp\left(-2p_M^2 \left( \frac{C |\Pi^{(r)}_M|}{(3CM)^d} - |\Pi^{(r)}_M^j| \right)^2 \right)
\]

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we have obtained integer part. Now choosing $M$ following we assume that the indices of the random walks are integers, otherwise we take the

Similarly for $r = 0$ we have

$$ \mathbb{P}^{(0,0)}(\|\Pi^{(0)}_M \cap \{\text{not social boxes}\} > C\|\Pi^{(0)}_M | p_M \) \leq 2(3CM)^d \exp\left(-Cp^3_M (\sqrt{n})^{2d}(3CM)^{2d}\right).$$

Using the above estimates we obtain

$$ \mathbb{P}^{(0,0)}(S^c_M) \leq 2(3CM)^d \exp\left(-Cp^3_M (\sqrt{n})^{2d}(3CM)^{2d}\right) + \sum_{r=1}^{\log_2 \log^3 n} 2(3CM)^d \exp\left(-Cp^3_M (\sqrt{n})^{2d}(3CM)^{2d}\right),$$

$$ \leq \log_2 \log^3 n \cdot \exp\left(-Cp^3_M (\sqrt{n})^{2d}(3CM)^{2d}\right) \leq Cn^{-c \log n}.$$

Proof of Lemma 7.1. The proof relies on a construction of a suitable coupling of $P^{(x,0)}(X_n \in \cdot)$ and $P^{(y,0)}(X_n \in \cdot)$. First we show that there is a coupling on the level of boxes with side length $M$, where $M$ is a constant. Let $\Pi_M$ be a partition of $\mathbb{Z}^d$ in boxes of side length $M$ and fix $x$ and $y$. Set

$$ F_n^\theta := \bigcap_{k \geq n^\theta} \left\{ \omega : \forall z \in [-k,k]^d \cap \mathbb{Z}^d, \sum_{\Delta \in \Pi_M} |\mathbb{P}(z,0)(X_k \in \Delta) - P^{(z,0)}(X_k \in \Delta)| \leq C_1(\sqrt{k} + C_1M^{c_2}) \right\}.$$

and

$$ F(x, y) := \bigcap_{(\tilde{x},m) \in \mathbb{Z}^d \times N_0} \sigma(x,m) F_n^\theta \bigcap \bigcap_{(\tilde{y},m) \in \mathbb{Z}^d \times N_0} \sigma(y,m) F_n^\theta.$$

By Lemma 2.1 we have $\mathbb{P}(F_n^\theta) \geq 1 - n^{-c \log n}$ and thus $\mathbb{P}(F(x, y)) \geq 1 - Cn^{-c \log n}$. In the following we assume that the indices of the random walks are integers, otherwise we take the integer part. Now choosing $M$ and $n$ large enough for $\|x - y\| \leq n^\theta$ on the event $F(x, y)$ we obtain

$$ \sum_{\Delta \in \Pi_M} |P^{(x,0)}(X_{n^{2\theta \log^8 n^\theta}} \in \Delta) - P^{(y,0)}(X_{n^{2\theta \log^8 n^\theta}} \in \Delta))|$$

$$ \leq \sum_{\Delta \in \Pi_M} |P^{(x,0)}(X_{n^{2\theta \log^8 n^\theta}} \in \Delta) - P^{(x,0)}(X_{n^{2\theta \log^8 n^\theta}} \in \Delta))|$$

$$ + \sum_{\Delta \in \Pi_M} |P^{(y,0)}(X_{n^{2\theta \log^8 n^\theta}} \in \Delta) - P^{(y,0)}(X_{n^{2\theta \log^8 n^\theta}} \in \Delta))|$$

50
we obtain

\[ \omega \]

the coupling attempt at the new positions. Successful, we let the random walks go along the same path until time

\[ x, y \]

coupling \( \Xi \). By Corollary

\[ P \]

boxes (Definition

\[ \omega \]

and we used Lemma 3.6 from [Ste17] and the annealed derivative estimates (Lemma 3.9 from [Ste17]). The number of steps we chose might seem a bit strange at first. The choice becomes more clear by looking at the last inequality above. There we see that, with the methods we use, we need a bit more steps than the square of the current distance. One can calculate that

\[ \begin{align*}
&= \frac{1}{2} + Cn^{-\log n} \left( \log(n^2 \log^d n^\theta) \right)^{\frac{3d}{4d-4}} - \frac{4d}{4d-4} \\
&= \frac{1}{4} + Cn^{-\log n} + C \left( \log(n^2 \log^d n^\theta) \right)^{\frac{3d}{4d-4}} - \frac{4d}{4d-4} \\
&< \frac{1}{2},
\end{align*} \]

for \( n \) large enough, where

\[ \Pi^{x,y}_M(m) := \{ \Delta \in \Pi_M : \Delta \cap \{ z \in \mathbb{Z}^d : \min(\|x-z\|, \|y-z\|) \leq \sqrt{m} \log^3 m \} \neq \emptyset \} \]

and we used Lemma 3.6 from [Ste17] and the annealed derivative estimates (Lemma 3.9 from [Ste17]). The number of steps we chose might seem a bit strange at first. The choice becomes more clear by looking at the last inequality above. There we see that, with the methods we use, we need a bit more steps than the square of the current distance. One can calculate that any additional factor \( \log^m(n^\theta) \) with \( m > 6d \) is enough to get the estimate. So there exists a coupling \( \Xi^{x,y}_{\omega,n^2 \log^d n^\theta} \) of \( P^{x,0}_\omega(X_{n^2 \log^d n^\theta} \in \cdot) \) and \( P^{y,0}_\omega(X_{n^2 \log^d n^\theta} \in \cdot) \) on \( \Pi_M \times \Pi_M \) such that for \( \omega \in F(x, y) \)

\[ \Xi^{x,y}_{\omega,n^2 \log^d n^\theta}(\{ (\Delta, \Delta) : \Delta \in \Pi_M \}) > \frac{1}{2}. \]

Recall \( \bar{S}_M \) from Corollary 9.3. We have for

\[ \omega \in H(x, y) := F(x, y) \cap \bigcap_{(\bar{x}, m) \mathbb{Z}^d \times \mathbb{Z}_0} \sigma(\bar{x}, m) \bar{S}_M \cap \bigcap_{(\bar{y}, m) \mathbb{Z}^d \times \mathbb{Z}_0} \sigma(\bar{y}, m) \bar{S}_M \]

that

\[ \sum_{\Delta \in \Pi_M} \Xi^{x,y}_{\omega,n^2 \log^d n^\theta}(\Delta, \Delta) > \frac{1}{2} - \varepsilon(M) > \frac{1}{4}. \]

By Corollary 9.3 we obtain \( \mathbb{P}(H(x, y)) \geq 1 - Cn^{-\log n} \). Thus, by the definition of social boxes (Definition 9.1), we can construct a coupling \( \tilde{\Xi}^{x,y}_{\omega,n^2 \log^d n^\theta} \) of \( P^{x,0}_\omega(X_{n^2 \log^d n^\theta + CM} \in \cdot) \) and \( P^{y,0}_\omega(X_{n^2 \log^d n^\theta + CM} \in \cdot) \) satisfying \( \tilde{\Xi}^{x,y}_{\omega,n^2 \log^d n^\theta}(\{ (z, z) : z \in \mathbb{Z}^d \}) > \frac{1}{4} \left( \frac{1}{2} \right)^{2CM} \). If this coupling is successful, we let the random walks go along the same path until time \( n \). In case it isn’t, we try to couple from their current position. Note that \( \omega \in H(x, y) \) ensures, that we can repeat the coupling attempt at the new positions.
For the rest of the proof let \( n_k := n^\theta \log^{k(4d+3)} n \), \( k \in \mathbb{N}_0 \) and \( s_k := n_k^2 \log^{8d} n_k + CM \). The \( n_k \) will represent the distance between the walkers at the start of an attempt at coupling and \( s_k \) will be the number of steps necessary for the attempt. Furthermore let \( S_k := \sum_{i=0}^k s_i \).

By Lemma 3.6 from [Ste17], we know that with probability of at least \( 1 - Cn^{-c \log n} \) the distance between the random walks will only be

\[
( n^{2\theta} \log^{8d} n^{-\theta} )^{1/2} \log^3 ( n^{2\theta} \log^{8d} n^{-\theta} ) \leq \log^{4d} ( n^{\theta} ) \log^3 ( n ) \leq \log^{4d+3} n = n_1,
\]
as long as \( 8d \leq (1 - 2\theta) \frac{\log n}{\log \log n} \). This condition is not a restriction, since we will let \( n \to \infty \).

Let us now iterate the coupling procedure. If the coupling in step \( k - 1 \) is not successful, i.e. if the walks are not at the same site, we try to couple again starting from the current positions. This leads to an iterative coupling \( \hat{\Xi} \) of the following form: \( \hat{\Xi}_{x,y}^\omega(0) = \hat{\Xi}_{x,y}^\omega(n_0) = \hat{\Xi}_{x,y}^\omega, n^\theta \) and for \( k \geq 1 \)

\[
\hat{\Xi}_{x,y}^\omega(z_1, z_2) = \sum_{a, b \in \mathbb{Z}^d} \hat{\Xi}_{x,y}^\omega, k-1(a, b) \cdot \left[ \mathbb{1}_{\{a=b\}} \mathbb{1}_{\{z_1 = z_2\}} P_{\omega}^{(a, S_{k-1})}(X_{S_k} = z_1) + \mathbb{1}_{\{0 < ||a-b|| \leq n_k\}} \Xi_{a,b}^{\omega, n_k}(z_1, z_2) + \mathbb{1}_{\{||a-b|| \geq n_k\}} P_{\omega}^{(a, S_{k-1})}(X_{S_k} = z_1) P_{\omega}^{(b, S_{k-1})}(X_{S_k} = z_2) \right],
\]

where \( \Xi_{a,b}^{\omega, n_k} \) is a coupling of \( P_{\omega}^{(a, S_{k-1})}(X_{S_k} \in \cdot) \) and \( P_{\omega}^{(b, S_{k-1})}(X_{S_k} \in \cdot) \). The idea is that the random walks will stay together once they are at the same site. We try to couple them via \( \hat{\Xi}_{a,b}^{\omega, n_k} \) if their distance is not too large and we let them evolve independently otherwise.

Since at distance \( n_k \) for the next coupling we walk \( s_k \) steps and with high probability have at most a distance of \( s_k^{1/2} \log^3 s_k \), the above coupling will work as long as \( k \leq \frac{(1 - 2\theta) \log n}{8d + 6} \) holds, which we show below. We obtain

\[
s_k^{1/2} \log^3 s_k = \left( n_k^2 \log^{8d} n_k + CM \right)^{1/2} \log^3 \left( n_k^2 \log^{8d} n_k + CM \right).
\]

Now for \( k \leq \frac{(1 - 2\theta) \log n}{(4d+3) \log \log n} \) and \( n \) large enough

\[
n_k^2 \log^{8d} n_k + CM \leq n_k^2 \log^{8d} n
\]

and

\[
\log^{4d} n_k = \log^{4d} \left( n^\theta \log^{k(4d+3)} (n) \right) \leq \log^{4d} n.
\]

Thus, we have

\[
s_k^{1/2} \log^3 s_k \leq n_k \log^{4d} (n) \log^3 \left( n_k^2 \log^{8d} n \right).
\]

Furthermore, if \( k \leq \frac{(1 - 2\theta) \log n}{8d + 6} \) then

\[
2 \log n_k + 8d \log \log n = 2 \log \left( n^\theta \log^{k(4d+3)} \right) + 8d \log n
\]

\[
= 2 \theta \log n + k (8d + 6) \log \log n + 8d \log \log n \leq \log n
\]

It follows that

\[
s_k^{1/2} \log^3 s_k \leq 2 n_k \log^{4d} n \log^3 n = 2 n^\theta \log^{(k+1)(4d+3)} (n) = n_{k+1}.
\]
So after we try the $k$-th coupling we are, with high probability, at distance $n_{k+1}$. The probability for each try to be successful is bounded from below by $\frac{1}{4} \left( \frac{1}{3^d} \right)^{2CM}$ and we have $\frac{(1-2\theta) \log n}{(8d+6) \log \log n - 1}$ attempts. So the time we need for those attempts is

$$\sum_{k=0}^{(1-2\theta) \log n / (8d+6) \log \log n - 1} s_k = \sum_{k=0}^{(1-2\theta) \log n / (8d+6) \log \log n - 1} n_k^2 \log^{8d} n_k + CM$$

$$\leq \sum_{k=0}^{(1-2\theta) \log n / (8d+6) \log \log n - 1} n^{2\theta} \log^{k(8d+6)}(n) \log^{8d}(n) + CM$$

$$= \frac{(1-2\theta) \log n}{(8d+6) \log \log n} CM + n^{2\theta} \log^{8d}(n) \sum_{k=0}^{(1-2\theta) \log n / (8d+6) \log \log n - 1} \left( \log^{(8d+6)}(n) \right)^k.$$

Note that

$$(\log n)^{(1-2\theta) \log n / (8d+6) \log \log n - (8d+6)} = \exp((1-2\theta) \log n) = n^{1-2\theta}$$

and therefore the right hand side of (9.2) is bounded from above by

$$\frac{(1-2\theta) \log n}{(8d+6) \log \log n} CM + n^{2\theta} \log^{8d}(n) \sum_{k=0}^{(1-2\theta) \log n / (8d+6) \log \log n - 1} \left( \log^{(8d+6)}(n) \right)^k < n,$$

for $n$ large enough. And the probability for the above coupling to fail is smaller than

$$(1-\frac{1}{4} \left( \frac{1}{3^d} \right)^{2CM})$$

where $p^* = \frac{1}{4} \left( \frac{1}{3^d} \right)^{2CM}$ and $c > 0$ is a constant. So for a fixed pair of points $x, y$ with $\|x - y\| \leq n^\theta$ we have

$$\left\| P^{(x,0)}_\omega(X_n \in \cdot) - P^{(y,0)}_\omega(X_n \in \cdot) \right\|_{TV} \leq e^{-c \log \log \log n}$$

with probability at least $1 - n^{-c \log n}$. Thus we get for every $b > 0$

$$\mathbb{P}(D(n)) = \mathbb{P}\left( \bigcap_{x, y \in \mathbb{Z}^d: \left\|x\right\|, \left\|y\right\| \leq n^b, \left\|x-y\right\| \leq n^\theta} \left\| P^{(x,0)}_\omega(X_n \in \cdot) - P^{(y,0)}_\omega(X_n \in \cdot) \right\|_{TV} \leq e^{-c \log \log \log n} \right)$$

$$\geq 1 - \sum_{x, y \in \mathbb{Z}^d: \left\|x\right\|, \left\|y\right\| \leq n^b, \left\|x-y\right\| \leq n^\theta} \mathbb{P}\left( \left\| P^{(x,0)}_\omega(X_n \in \cdot) - P^{(y,0)}_\omega(X_n \in \cdot) \right\|_{TV} > e^{-c \log \log \log n} \right)$$

$$\geq 1 - n^{d(b+\theta)} n^{-c \log n} \geq 1 - C n^{-c' \log n}.$$

Note that $b > 0$ can be chosen arbitrarily large, but the constants $C$ and $c'$ will have to adjusted accordingly. \qed
10. Uniqueness of the prefactor

With some minor adaptations of the ideas from [BCR16, Section 7.1] we can obtain the following result.

**Lemma 10.1.** Provided existence, the prefactor $\varphi$ in (1.12) is unique.

**Proof.** Assume that there are functions $f$ and $g$ which both satisfy (1.12) and denote $h = f - g$. We will check that $\mathbb{E}[|h|] = 0$ and hence that $h \equiv 0$ holds $\mathbb{P}$-a.s.

By the triangle inequality we have

$$
\sum_{x \in \mathbb{Z}^d} \mathbb{P}^{(0,0)}(X_n = x)|h(\sigma(x,n)\omega)| \leq \sum_{x \in \mathbb{Z}^d} |P^{(0,0)}(X_n = x) - \mathbb{P}^{(0,0)}(X_n = x)f(\sigma(x,n)\omega)|
+ \sum_{x \in \mathbb{Z}^d} |P^{(0,0)}(X_n = x) - \mathbb{P}^{(0,0)}(X_n = x)g(\sigma(x,n)\omega)|
$$

(10.1)

which by (1.12) implies

$$
\lim_{n \to \infty} \sum_{x \in \mathbb{Z}^d} \mathbb{P}^{(0,0)}(X_n = x)|h(\sigma(x,n)\omega)| = 0
$$

(10.2)

for $\mathbb{P}$ almost every $\omega$. That means $\lim_{n \to \infty} \mathbb{E}^{(0,0)}[|h(\sigma(x_n,n)\omega)|] = 0$ $\mathbb{P}$-a.s. Assume that $h \neq 0$, then there exists a measurable subset $A \subset \Omega$ and a constant $c > 0$ such that $\mathbb{P}(A) > 0$ and $|h| > c$ on $A$. Thus, for every $n \in \mathbb{N}$, an elementary computation shows

$$
\mathbb{E}\left[\mathbb{E}^{(0,0)}[|h(\sigma(x_n,n)\omega)|]\right] \geq \mathbb{E}\left[\mathbb{E}^{(0,0)}[|h(\sigma(x_n,n)\omega)|\mathbb{1}_{\{\sigma(x_n,n)\omega \in A\}}]\right]
$$

$$
\geq c\mathbb{E}\left[\mathbb{E}^{(0,0)}[\mathbb{1}_{\{\sigma(x_n,n)\omega \in A\}}]\right]
$$

$$
= c\mathbb{P}(A) > 0.
$$

(10.3)

Since

$$
\mathbb{E}\left[\mathbb{E}^{(0,0)}[|h(\sigma(x_n,n)\omega)|]\right] = \mathbb{E}\left[\sum_{y \in \mathbb{Z}^d} \mathbb{P}^{(0,0)}(X_n = y)|h(\sigma(y,n)\omega)|\right]
$$

$$
= \sum_{y \in \mathbb{Z}^d} \mathbb{P}^{(0,0)}(X_n = y)\mathbb{E}[|h(\sigma(y,n)\omega)|]
$$

$$
= \sum_{y \in \mathbb{Z}^d} \mathbb{P}^{(0,0)}(X_n = y)\mathbb{E}[|h(\omega)|] = \mathbb{E}[|h(\omega)|],
$$

(10.4)

the sequence $\{|h(\sigma(x_n,n)\omega)|\}_{n \in \mathbb{N}}$ is tight. Thus, (10.3) implies that for $\mathbb{P}$ almost all $\omega$ we have

$$
\lim_{n \to \infty} \mathbb{E}^{(0,0)}[|h(\sigma(x_n,n)\omega)|] > 0
$$

which is a contradiction to (10.2). \qed

A. Intersection of clusters of points connected to infinity

The following lemma is a quantification of Theorem 2 from [GH02] which was pointed out there without a proof. We give a proof using a key result from [GM14].

**Lemma A.1.** Let $d \geq 2$, $p > p_c$. Then there are positive constants $M$ and $C$ and $c$ such that for all $x,y \in \mathbb{Z}^d$ with $\|x - y\| \leq M$

$$
\mathbb{P}(B(x,y;M,C)|(x,0) \to \infty,(y,0) \to \infty) \geq 1 - \exp(-cM),
$$

(A.1)
where $B(x, y; M, C)$ is the set of all $\omega \in \Omega$ for which there is $z \in \mathbb{Z}^d$ satisfying

$$(x, 0) \xrightarrow{\omega} (z, CM), \quad (y, 0) \xrightarrow{\omega} (z, CM) \quad \text{and} \quad (z, CM) \xrightarrow{\omega} \infty.$$ 

**Proof.** For $A \subset \mathbb{Z}^d$ we put $\eta^A_t(x) = 1_{(y,0) \rightarrow (x,t)}$ for some $y \in A$ (this is the discrete time contact process starting from all sites in $A$ infected at time 0). Write

$$B(x, t) := \{ \exists z : \|x - z\| \leq c_1 t \text{ and } \eta^\{x\}_t(z) \neq \eta^{z,d}_t(z) \}$$

(A.2)

for the “bad” event that coupling in a ball around $x$ has not occurred at time $t$. We obtain from [GM14, Thm. 1, Formula (3)] that

$$\mathbb{P}(B(x, t) \cap \{(x, 0) \rightarrow \infty\}) \leq Ce^{-ct}$$

(A.3)

for certain constants $c_1, C, c \in (0, \infty)$ (which depend on $d$ and on $p > p_c$). Literally, the result in [GM14] is proved for the continuous time version of the contact process, but we believe that the same holds in discrete time.

Now consider $x, y \in \mathbb{Z}^d$ with $\|x - y\| \leq M$. Pick $C_2$ so large that

$$J := \{ z : \|z - x\| \leq C_2 M \text{ and } \|z - y\| \leq C_2 M \}$$

satisfies $\#J \geq M^d$. Applying (A.3) with $t = C_2 M$ for $x$ and for $y$ gives

$$\mathbb{P}\left( \left( B(x, C_2 M) \cup B(y, C_2 M) \right) \cap \{(x, 0) \rightarrow \infty, (y, 0) \rightarrow \infty\} \right) \leq \mathbb{P}(B(x, C_2 M) \cap \{(x, 0) \rightarrow \infty\}) + \mathbb{P}(B(y, C_2 M) \cap \{(y, 0) \rightarrow \infty\}) \leq 2Ce^{-cM_2}$$

(A.4)

hence

$$\mathbb{P}(\eta^{x}_{C_2 M}(z) = \eta^{z,d}_{C_2 M}(z) = \eta^{y}_{C_2 M}(z) \text{ for all } z \in J \mid (x, 0) \rightarrow \infty, (y, 0) \rightarrow \infty) \geq 1 - C'e^{-cC_2 M}. \quad (A.5)$$

Furthermore

$$\mathbb{P}\left( \exists z \in J : \eta^{z,d}_{C_2 M}(z) = 1 \text{ and } (z, C_2 M) \rightarrow \infty \mid (x, 0) \rightarrow \infty, (y, 0) \rightarrow \infty \right) \geq \mathbb{P}(\exists z \in J : \eta^{z,d}_{C_2 M}(z) = 1 \text{ and } (z, C_2 M) \rightarrow \infty) \geq 1 - Ce^{-cM^d} \quad (A.6)$$

where we used the FKG inequality in the first inequality. For the second inequality we use the fact that extinction starting from $A$ is exponentially unlikely in $\#A$ (see Theorem 2.30 (b) in [Lig99]) and the fact that $\eta^{z,d}_{C_2 M}$ dominates the upper invariant measure which itself dominates a product measure on $\{0, 1\}^{\mathbb{Z}^d}$ with some density $\rho > 0$ (see Corollary 4.1 in [LS06]).

Combining, we find

$$\mathbb{P}\left( \exists z \in \mathbb{Z}^d : (x, 0) \rightarrow (z, C_2 M), (y, 0) \rightarrow (z, C_2 M), (z, C_2 M) \rightarrow \infty \mid (x, 0) \rightarrow \infty, (y, 0) \rightarrow \infty \right) \geq 1 - C'e^{-cC_2 M} - Ce^{-cM^d}. \quad (A.7)$$

\qed
B. Quenched random walk finds the cluster fast

Since we allow the quenched random walk to start outside the cluster we need some kind of control on the time it needs to hit the cluster. The following lemma will yield exactly that.

**Lemma B.1.** Let \( d \geq 1 \) and define the set \( A_n = A_n(C', c') := \{ \omega \in \Omega : P_\omega^{(0,0)}(\xi_i(X_i) = 0, i = 1, \ldots, n) \leq C'e^{-c'n} \} \). There exist constants \( C, c > 0 \), so that for every \( p > p_c(d) \) and small enough \( C' \) and \( c' \) we have

\[
P(A_n^C) \leq Ce^{-cn} \quad \text{for all } n = 1, 2, \ldots.
\]

**Proof.** Note that by our definition of the quenched law, see equation (1.4), the quenched random walk performs a simple random walk until it hits the cluster \( C \). Thus, on the event that the random walk doesn’t hit the cluster, we can switch the random walk with a simple random walk \((Y_n)_n\) that is independent of the environment. Using Lemma 2.11 from [BCD16] it follows

\[
P^{(0,0)}(\xi_0(X_0) = \cdots = \xi_n(X_n) = 0)
= \sum_{x_1, \ldots, x_n} P^{(0,0)}((X_1, \ldots, X_n) = (x_1, \ldots, x_n), \xi_0(0) = \cdots = \xi_n(x_n) = 0)
= \sum_{x_1, \ldots, x_n} P^{(0,0)}((Y_1, \ldots, Y_n) = (x_1, \ldots, x_n), \xi_0(0) = \cdots = \xi_n(x_n) = 0)
\leq \tilde{C}e^{-\tilde{c}n},
\]

where \( \tilde{C} \) and \( \tilde{c} \) are certain constants depending only on \( p \) and \( d \).

Using the definition of the annealed law we get

\[
P^{(0,0)}(\xi_0(X_0) = \cdots = \xi_n(X_n) = 0)
\geq \int_{A_n^{C'}} P_\omega^{(0,0)}(\xi_i(X_i) = 0, i = 1, \ldots, n) dP(\omega)
\geq P(A_n^C)C' e^{-c'n}
\]

and since

\[
\int_{A_n^C} P_\omega^{(0,0)}(\xi_i(X_i) = 0, i = 1 \ldots, n) dP(\omega) \leq \tilde{C}e^{-\tilde{c}n}
\]

we obtain that \( P(A_n^C) \leq Ce^{-cn} \) with \( c = \tilde{c} - c' > 0 \) by choosing \( c' < \tilde{c} \).

**Acknowledgements**

We would like to thank Noam Berger for many useful discussions and for sharing various versions of [BCR16]. We also thank Nina Gantert for useful discussions and for her constant interest in this work.

MB and TS were supported by the DFG SPP Priority Programme 1590 “Probabilistic Structures in Evolution” through grant BI 1058/3-2.
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