Poisson-Lie structures on the external algebra of $SL(2)$ and their quantization

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Abstract

All possible graded Poisson-Lie structures on the external algebra of $SL(2)$ are described. We prove that differential Poisson-Lie structures prolonging the Sklyanin brackets do not exist on $SL(2)$. There are two and only two graded Poisson-Lie structures on $SL(2)$ and neither of them can be obtained by a reduction of graded Poisson-Lie structures on the external algebra of $GL(2)$. Both of them can be quantized and as a result we get new graded algebras of quantum right-invariant forms on $SL_q(2)$ with three generators.

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1 Introduction

Recently the problem of constructing differential calculi on the quantum special linear group $SL_q(N)$ \cite{1,2} has attracted considerable attention \cite{3-16}. A peculiar attention was paid to constructing bicovariant differential calculi on these quantum groups \cite{3-16}. Just the requirement of bicovariance, i.e. invariance under quantum version of right and left group transformations, aside from the obvious geometrical meaning has a physical interpretation \cite{17-20}. As it was recognized the proposed solution reveals some strange properties. Namely, the bicovariant differential complex involves an extra element which has no natural classical counterpart in $SL(N)$. In this paper we construct a quantum algebra of bicovariant forms associated with $SL_q(2)$ that is free from this defect.

Let us note that one can say that there are two main approaches to differential calculus on noncommutative spaces. The first is the Connes approach \cite{21} in which a complex of differential forms is constructed and the operator $d$ of exterior derivative plays a fundamental role. Such an approach was used for quantum groups by Woronowicz \cite{3}.

Another approach can be formulated following the Faddeev idea that all the objects in the theory of quantum groups should appear naturally as the result of quantization of appropriate Poisson brackets in the theory of Lie groups \cite{22}. This point of view pushes us to start a search for an algebra of quantum forms with finding natural Poisson-Lie structures on the external algebra of ordinary differential forms. These Poisson-Lie structures being quantized should give as a result algebras of quantum differential forms. The $d$ operator does not play a fundamental role in this approach, moreover, generally there are no reasons to be sure that quantum $d$ exists.

Just this point of view was advocated in the paper \cite{23}. It was found that the external algebra on $GL(N)$ can be equipped with the graded Poisson-Lie structures and just their quantization produces the bicovariant differential calculi on $GL_q(N)$ \cite{23}. This forces us to search directly for graded Poisson-Lie structures on $SL(N)$ whose subsequent quantization would give the bicovariant calculi on $SL_q(N)$ rather than to discuss possible reductions of $GL_q(N)$-calculi to the $SL_q(N)$ case.

We are going to describe all possible graded Poisson-Lie structures on the external algebra of $SL(2)$. We will prove that differential Poisson-Lie structures prolonging the Sklyanin brackets do not exist on $SL(2)$. This key observation explains why does not exist bicovariant differential calculi on $SL_q(2)$ which is in one to one correspondence with their classical counterpart. The absence of a unique preferred Poisson structure (differential Poisson-Lie structure) provokes the discussion of different Poisson brackets on the external algebra by relaxing some requirements on Poisson structures and their further quantization. The more natural possibility which we are going to deal with in this paper consists in consideration of a graded Poisson-Lie structure without requiring this structure to be the differential one.

We will find that there are two and only two graded Poisson-Lie structures on $SL(2)$. Neither of them can be obtained by a reduction of graded Poisson-Lie structures on the external algebra of $GL(2)$. Then we prove that both of them can
be quantized. As a result we get two graded algebras of quantum right-invariant forms on $SL_q(2)$. As we could have expected from classical treatment these algebras of quantum forms on $SL_q(2)$ do not admit the operator $d$ with standard properties.

The paper is organized as follows. In section 2 we fix our notations and introduce the notion of a graded Poisson-Lie algebra. The rest of the section is devoted to the description of all such structures on the external algebra of $SL(2)$. In section 3 the quantization of these graded Poisson-Lie structures is presented.

2 Graded Poisson-Lie structure associated with $SL(2)$

2.1 Definitions

The function algebra $\mathcal{A}$ on $SL(2, C)$ is defined as the free unital associative algebra generated by the entries of the matrix:

$$T = \begin{vmatrix} a & b \\ c & d \end{vmatrix},$$

modulo the additional relation $\det T = ad - bc = 1$.

Recall, that $\mathcal{A}$ has the natural Hopf algebra structure with comultiplication $\Delta$, the counit $\epsilon$ and the antipode $S$:

$$\Delta(T) = T \otimes T, \quad \epsilon(T) = I, \quad S(T) = T^{-1}.$$  

Specifying $T$ as the matrix (2.1) we have for the comultiplication $\Delta$:

$$\Delta \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) = \begin{array}{cc} a & b \\ c & d \end{array} \otimes \begin{array}{cc} a & b \\ c & d \end{array}. $$

According to the general theory of Poisson-Lie groups [2, 1, 24, 25] the function algebra on $SL(2)$ can be supplied with the Poisson structure $\{,\}$ compatible with the comultiplication $\Delta$, i.e. satisfying the condition

$$\Delta\{f, h\} = \{\Delta f, \Delta h\}, \quad f, h \in \mathcal{A}. $$

In terms of generators $t_i^j$ this Poisson structure is given by the Sklyanin bracket [24, 26]:

$$\{a, b\} = -ab, \quad \{b, d\} = -bd, \quad \{c, b\} = 0, \quad \{a, c\} = -ac, \quad \{c, d\} = -cd, \quad \{a, d\} = -2bc.$$  

Let

$$\theta = \begin{pmatrix} \theta^0 \\ \theta^1 \\ \theta^2 \end{pmatrix}$$

be the canonical right-invariant one-form on $SL(2)$ taking value in $sl(2)$. The components $\theta^i, \quad i = 0, 1, 2$ being the scalar right invariant one-forms can be viewed as
the generators of the external algebra on \( SL(2) \). In what follows we will refer to \( \theta \) as to the Maurer-Cartan form on \( SL(2) \). Note, that the choice of right-invariant forms is a matter of convention and they may be replaced by left-invariant ones.

Let us define now the basic object of our study. This is a \( \mathbb{Z}_2 \)-graded algebra \( \mathcal{M} \) generated by even \( t^n_m \in \mathcal{A} \) and odd \( \theta^i \) generators modulo the defining relations:

\[
\begin{align*}
  t^n_m t^l_k &= t^l_k t^n_m, \\
  t^n_m \theta^i &= \theta^i t^n_m, \\
  \theta^i \theta^j &= -\theta^j \theta^i,
\end{align*}
\]

\( \det T = ad - bc = 1. \)

Let \( \Omega \) (the algebra of external forms) is the subalgebra of \( \mathcal{M} \) generated by \( \theta^i \). Clearly, \( \mathcal{A} \) is also the subalgebra of \( \mathcal{M} \).

Now our task is to supply \( \mathcal{M} \) with a coalgebra structure, \textit{i.e.} one needs to define a homomorphism \( \Delta : \mathcal{M} \to \mathcal{M} \otimes \mathcal{M} \) which satisfies the axiom of coassociativity. Obviously, we require \( \Delta \) when being restricted on the subalgebra \( \mathcal{A} \) to coincide with the comultiplication (2.2). Since \( \Delta \) when acting on \( \Omega \) should encode the transformation law of the right-invariant form under the left and right group multiplications we define two different comultiplications which we call \( \Delta_R \) and \( \Delta_L \):

\[
\Delta_R : \Omega \to \mathcal{A} \otimes \Omega, \\
\Delta_L : \Omega \to \Omega \otimes \mathcal{A}.
\]

Indeed, the components \( \theta^i \) of the Maurer-Cartan form transforms under the left and right group transformations: \( g \to gg_1 \) and \( g \to g_1g \), \( g, g_1 \in SL(2), \) as

\[
\begin{align*}
  R_{g_1}^* \theta_{gg_1} &= \theta_g \\
  L_{g_1}^* \theta_{g_1g} &= Ad_{g_1} \theta_g
\end{align*}
\]

respectively. Therefore, in accordance with (2.7) we define

\[
\Delta_R \theta^i = \theta^i \otimes I \\
\Delta_L \theta^i = (Ad_g e_j)^i \otimes \theta^j
\]

where \( \{e_j\} \) is a basis in \( sl(2) \).

In addition to \( \Delta_{L,R} \) one can consider the comultiplication \( \Delta \) defined as the sum:

\[
\Delta_G = \Delta_L + \Delta_R
\]

Concerning this \( \Delta_G \) one can supply the \( \mathcal{M} \) by a counit \( \epsilon \) defined on \( \theta^i \) as \( \epsilon(\theta^i) = 0. \) Thus, there are three natural ways to prolong the comultiplication \( \Delta \) from \( \mathcal{A} \) to the hole algebra \( \mathcal{M} : \Delta_L, \Delta_R \) and \( \Delta_G \). Note, that just the comultiplication (2.10) was used for \( GL(N) \) [23].

Using the notations (2.1) for the generators of \( \mathcal{A} \) equation (2.4) can be rewritten as

\[
\Delta_L \theta^0 = (1 + 2bc) \otimes \theta^0 - ac \otimes \theta^1 + bd \otimes \theta^2
\]

\( ^1 \text{In the Woronowicz terminology} \) [3] \( \Omega \) is called a bicovariant-bimodule over \( \mathcal{A} \).
\[
\Delta_L \theta^1 = -2ab \otimes \theta^0 + a^2 \otimes \theta^1 - b^2 \otimes \theta^2, 
\]
\[
\Delta_L \theta^2 = 2cd \otimes \theta^0 - c^2 \otimes \theta^1 + d^2 \otimes \theta^2. 
\]

Let us recall that the graded algebra \( M \) becomes a graded Poisson algebra if we define a bilinear operation \( \{ , \} : M \otimes M \rightarrow M \) which satisfies

i) the super Jacobi identity:
\[
(-1)^{\deg x \deg z} \{ \{ x, y \}, z \} + (-1)^{\deg y \deg z} \{ \{ z, x \}, y \} + (-1)^{\deg x \deg y} \{ \{ y, z \}, x \} = 0, 
\]
(2.14)

ii) the graded Leibniz rule:
\[
\{ x \cdot y, z \} = x \{ y, z \} + (-1)^{\deg y \deg z} \{ x, z \} y, 
\]
(2.15)

iii) the graded symmetry property:
\[
\{ x, y \} = (-1)^{\deg x \deg y+1} \{ y, x \}, \quad \deg \{ x, y \} = (\deg x + \deg y) \mod 2. 
\]
(2.16)

The graded Poisson brackets on \( M \) can be extended to the brackets \( M \otimes M \) in a natural way:
\[
\{ x \otimes y, z \otimes w \}_{M \otimes M} = (-1)^{\deg y \deg z} \{ x, z \}_M \otimes yw + (-1)^{\deg y \deg z} xz \otimes \{ y, w \} 
\]
(2.17)
for any elements \( x, y, z, w \in M \).

**Definition 1** A graded Poisson algebra \( M \) is a graded Poisson-Lie algebra if it admits a Poisson-Lie structure, i.e. the following compatibility conditions are satisfied:

1) for any elements \( f, h \in A \)
\[
\Delta \{ f, h \}_M = \{ \Delta f, \Delta h \}_{M \otimes M}, 
\]
(2.18)

2) the brackets involving the odd generators \( \theta^i \) are covariant with respect to the both actions of \( \Delta_L \) and \( \Delta_R \), i.e.:
\[
\Delta_{L,R} \{ \theta^i, f \}_M = \{ \Delta_{L,R} \theta^i, \Delta f \}_{M \otimes M}, \quad i = 0, 1, 2, \quad f \in A 
\]
(2.19)
\[
\Delta_{L,R} \{ \theta^i, \theta^j \}_M = \{ \Delta_{L,R} \theta^i, \Delta_{L,R} \theta^j \}_{M \otimes M}, \quad i, j = 0, 1, 2. 
\]
(2.20)

We also say that a graded Poisson-Lie structure is of the first order if (2.18) and (2.19) are satisfied, and of the second order if (2.18) and (2.20) take place.

Now according to our general strategy we look for a graded Poisson structure (brackets) on \( M \) obeying the natural covariance conditions (2.18), (2.19) and (2.20).

One can also search for a graded Poisson structure covariant under the action of \( \Delta_G \) given by (2.10). Note that for \( GL(N) \) the graded Poisson-Lie structures are covariant with respect to \( \Delta_G \) [23], but in general a Poisson-Lie algebra can appear to be a \( \Delta_G \)-noncovariant one.
2.2 \{\theta, t\} brackets.

Due to the grading requirement (2.16) we can write down the general expression for the brackets as

\[
\{\theta^i, a\} = C_{a j}^i \theta^j, \quad \{\theta^i, b\} = C_{b j}^i \theta^j, \quad \{\theta^i, c\} = C_{c j}^i \theta^j, \quad \{\theta^i, d\} = C_{d j}^i \theta^j, \tag{2.21}
\]

where \(C\)-s are unknown structure functions of \(a, b, c\) and \(d\). We omit the cubic terms proportional to \(\theta^i \theta^j \theta^2\) from the very beginning since they are forbidden by the covariance condition (2.19). Note that one has to take into account that on \(SL(2)\) the matrix elements \(a, b, c, d\) are not independent: \(\det T = ad - bc = 1\). Therefore, one of the relations in eqs. (2.21) should follow from the three others. Considering the coordinate patch on \(SL(2)\) where \(d \neq 0\) we find:

\[
\{\theta^i, a\} = \{\theta^i, d^{-1}(1 + bc)\} = -d^{-1}a\{\theta^i, d\} + d^{-1}\{\theta^i, b\}c + d^{-1}\{\theta^i, c\}b. \tag{2.22}
\]

The covariance of eq.(2.21) with respect to the action of \(\Delta_R\) (eq.(2.19)) leads to the system of equations on the coefficients \(C\):

\[
\Delta C_{c j}^i = C_{c j}^i \otimes a + C_{d j}^i \otimes c,
\]

\[
\Delta C_{d j}^i = C_{d j}^i \otimes b + C_{d j}^i \otimes d,
\]

\[
\Delta C_{b j}^i = d^{-1}\left(-aC_{d j}^i + cC_{b j}^i + bC_{c j}^i\right) \otimes b + C_{b j}^i \otimes d \tag{2.23}
\]

for any \(i, j = 0, 1, 2\). Since the right tensor multipliers in the r.h.s. of eq(2.23) are linear in \(a, b, c, d\) we realize that \(C\)-s are also linear functions of matrix elements. Then comparing eqs.(2.23) with eqs.(2.24) one can easily find the following solutions:

\[
C_{c j}^i = \alpha_j^i a + \beta_j^i c, \quad C_{d j}^i = \alpha_j^i b + \gamma_j^i d, \quad C_{b j}^i = -\beta_j^i b + \gamma_j^i d, \tag{2.24}
\]

where \(\alpha_j^i, \beta_j^i, \gamma_j^i\) are arbitrary numerical coefficients. Substituting the coefficients (2.24) in eq.(2.22) we also get the coefficient \(C_{a j}^i\):

\[
C_{a j}^i = -\beta_j^i a + \gamma_j^i c. \tag{2.25}
\]

Having found the form of structure functions we can now use the covariance of the brackets with respect to the action of \(\Delta_L\). For example, the equation \(\{\Delta_L \theta^0, \Delta a\} = \Delta_L \left(C_{a j}^0 \theta^j\right)\) gives

\[
4abc \otimes a \partial^0 + b(1 + 3bc) \otimes a \partial^2 - a^2 c \otimes a \partial^1 + b^2 b \otimes c \partial^2 + abc \otimes c \partial^1 + \tag{2.26}
\]

\[
(1 + 2bc) a \otimes C_{a j}^0 \theta^j + b(1 + bc) \otimes C_{a j}^2 \theta^j - a^2 c \otimes C_{a j}^1 \theta^j + b(1 + 2bc) \otimes C_{a j}^0 \theta^j + b^2 b \otimes C_{a j}^2 \theta^j - abc \otimes C_{a j}^1 \theta^j =
\]

\[
\left[\Delta C_{a j}^0 (1 + 2bc \otimes I) - \Delta C_{a j}^0 (2ab \otimes I) + \Delta C_{a j}^0 (2cd \otimes I)\right] \left(I \otimes \theta^0\right) +
\]

\[
\left[-\Delta C_{a j}^0 (ac \otimes I) + \Delta C_{a j}^0 (a^2 \otimes I) - \Delta C_{a j}^0 (c^2 \otimes I)\right] \left(I \otimes \theta^1\right) +
\]

\[
\]
The result is the following:

\[
\eta
\]

Now we are in a position to determine the bracket containing two \( \theta \)-s. Having in mind the grading requirement (2.16) one can write for this bracket a following representation:

\[
\{ \theta^i, \theta^j \} = C^{ij}_{mn} \theta^m \theta^n.
\]  

(2.30)

where the coefficients \( C^{ij}_{mn} \in A \) are unknown functions that can be defined by requiring the bracket (2.30) to obey the bicovariance conditions (2.20). Equation for \( \Delta_R \) can be written as

\[
\{ \theta^i \otimes I, \theta^j \otimes I \} = \{ \theta^i, \theta^j \} \otimes I = \Delta C^{ij}_{mn} (\theta^m \theta^n \otimes I)
\]  

(2.30)

The brackets (2.28) and (2.29) have another interesting property, namely, they are also invariant with respect to the comultiplication \( \Delta_G = \Delta_L + \Delta_R \).

**Lemma 1** There exist two and only two graded Poisson-Lie structures of the first order on SL(2). They are given by:

**I-type**

\[
\{ \theta^0, a \} = -2c \theta^1, \quad \{ \theta^1, a \} = a \theta^1, \quad \{ \theta^2, a \} = 4c \theta^0 - a \theta^2,
\]

\[
\{ \theta^0, b \} = -2d \theta^1, \quad \{ \theta^1, b \} = b \theta^1, \quad \{ \theta^2, b \} = 4d \theta^0 - b \theta^2,
\]

\[
\{ \theta^0, c \} = 0, \quad \{ \theta^1, c \} = -c \theta^1, \quad \{ \theta^2, c \} = c \theta^2,
\]

\[
\{ \theta^0, d \} = 0, \quad \{ \theta^1, d \} = -d \theta^1, \quad \{ \theta^2, d \} = d \theta^2,
\]

**II-type**

\[
\{ \theta^0, a \} = 0, \quad \{ \theta^1, a \} = -a \theta^1, \quad \{ \theta^2, a \} = a \theta^2,
\]

\[
\{ \theta^0, b \} = 0, \quad \{ \theta^1, b \} = -b \theta^1, \quad \{ \theta^2, b \} = b \theta^2,
\]

\[
\{ \theta^0, c \} = -2a \theta^2, \quad \{ \theta^1, c \} = 4a \theta^0 + c \theta^1, \quad \{ \theta^2, c \} = -c \theta^2,
\]

\[
\{ \theta^0, d \} = -2b \theta^2, \quad \{ \theta^1, d \} = 4b \theta^0 + d \theta^1, \quad \{ \theta^2, d \} = -d \theta^2,
\]

The brackets (2.28) and (2.29) have another interesting property, namely, they are also invariant with respect to the comultiplication \( \Delta_G = \Delta_L + \Delta_R \).

**2.3 \( \{ \theta, \theta \} \) brackets**

Now we are in a position to determine the bracket containing two \( \theta \)-s. Having in mind the grading requirement (2.16) one can write for this bracket a following representation:

\[
\{ \theta^i, \theta^j \} = C^{ij}_{mn} \theta^m \theta^n,
\]  

(2.30)

where the coefficients \( C^{ij}_{mn} \in A \) are unknown functions that can be defined by requiring the bracket (2.30) to obey the bicovariance conditions (2.20). Equation for \( \Delta_R \) can be written as

\[
\{ \theta^i \otimes I, \theta^j \otimes I \} = \{ \theta^i, \theta^j \} \otimes I = \Delta C^{ij}_{mn} (\theta^m \theta^n \otimes I)
\]  

(2.30)
and it is obviously satisfied if all the coefficients \( C^i_{mn} \) are pure numbers.

To write down the equations for coefficients \( C \) that follows from the equation for \( \Delta_L \) we start with the bracket \( \{ \theta^0, \theta^1 \} \).

Applying \( \Delta_L \) to it’s both sides
\[
\Delta_L \{ \theta^0, \theta^1 \} = C_{01}^{00} \Delta_L(\theta^0 \theta^1) + C_{02}^{01} \Delta_L(\theta^0 \theta^2) + C_{12}^{00} \Delta_L(\theta^1 \theta^2),
\]
and using eq.(2.20) we arrive to
\[
-2ab(1 + 2bc) \otimes \{ \theta^0, \theta^0 \} - a^2 c \otimes \{ \theta^1, \theta^1 \} - b^2 d \otimes \{ \theta^2, \theta^2 \} +
8a^2 bc \otimes \theta^0 \theta^1 + 4b^2 (1 + 2bc) \otimes \theta^0 \theta^1 - 2ab(1 + 4bc) \otimes \theta^0 \theta^1 +
a^2 (1 + 4bc) \otimes \{ \theta^0, \theta^1 \} - b^2 (3 + 4bc) \otimes \{ \theta^0, \theta^2 \} + ab(1 + 2bc) \otimes \{ \theta^1, \theta^2 \} =
C_{01}^{01} (a^2 \otimes \theta^0 \theta^1 - ab \otimes \theta^1 \theta^2 + b^2 \otimes \theta^0 \theta^2) +
C_{02}^{01} (c^2 \otimes \theta^0 \theta^1 - cd \otimes \theta^1 \theta^2 + d^2 \otimes \theta^0 \theta^2) +
C_{12}^{01} (-2ac \otimes \theta^0 \theta^1 + (1 + 2bc) \otimes \theta^1 \theta^2 - 2bd \otimes \theta^0 \theta^2).
\]

Equating the terms containing the linear independent forms \( \theta^i \theta^j \) in the right multiples of tensor product we obtain the system of equations for coefficients \( C \). The solutions of this system are collected in the Table 1:

| \( C_{01}^{01} \) | \( C_{02}^{01} \) | \( C_{12}^{01} \) | \( C_{01}^{00} \) | \( C_{02}^{00} \) | \( C_{12}^{00} \) | \( C_{01}^{01} \) | \( C_{02}^{01} \) | \( C_{12}^{01} \) |
|---|---|---|---|---|---|---|---|---|
| -2 | 0 | 0 | \( \alpha \) | \( \beta \) | \( \gamma \) | 0 | 0 | 0 |
| 2\( \alpha \) | 2\( \beta \) | 4 + 2\( \gamma \) | 0 | 0 | 0 | 0 | 0 | 0 |

Table 1.

Here \( \alpha, \beta, \gamma \) are coefficients that are not fixed from the system. To find \( \alpha, \beta, \gamma \) let us consider, for instance, the l.h.s. of the following Jacobi identity:
\[
\{ \{ \theta^0, \theta^0 \}, a \} + \{ \{ a, \theta^0 \}, \theta^0 \} - \{ \theta^0, a \}, \theta^0 \} = 0.
\]

By using \( \{ \theta^0, \theta^0 \} = \alpha \theta^0 \theta^1 + \beta \theta^0 \theta^2 + \gamma \theta^1 \theta^2 \) eq.(2.33) is reduced to the form
\[
(-\alpha(1 + \eta)a + 2\eta \gamma + 4\eta c)\theta^0 \theta^1) + (\beta \eta c \theta^1 \theta^2 + \beta(1 + \eta) a \theta^0 \theta^2 = 0.
\]

Thus, we find \( \alpha = 0, \beta = 0 \) for both \( \eta = 0 \) and \( \eta = -2 \). Moreover, \( \eta(4 + 2\gamma) = 0 \), \( i.e. \gamma = -2 \) for \( \eta = -2 \). Let us show that when \( \eta = 0 \) we still have \( \gamma = -2 \). Clearly, taking the equation
\[
\{ \{ \theta^0, \theta^0 \}, c \} + \{ \{ c, \theta^0 \}, \theta^0 \} - \{ \theta^0, c \}, \theta^0 \} = 0
\]
when \( \eta = 0 \) and \( \alpha = \beta = 0 \) we obtain: \( (4 + 8) a \theta^0 \theta^2 = 0 \), \( i.e. \gamma = -2 \).

This means that using the bicovariance condition for the bracket \( \{ \theta^0, \theta^1 \} \) we determine all the brackets \( \{ \theta^i, \theta^j \} \) and, moreover, two systems of brackets (2.28) and (2.29) with \( \eta = 0 \) and \( \eta = -2 \), respectively define only one candidate for the bracket \( \{ \theta^i, \theta^j \} \). Thus, we arrive to the Lemma 2.
Lemma 2 There is a unique system of brackets of the second order invariant with respect to the both $\Delta_L$ and $\Delta_R$ and it is given by

$$\begin{align*}
\{\theta^0, \theta^0\} &= -2\theta^1\theta^2, & \{\theta^0, \theta^1\} &= -2\theta^0\theta^1, & \{\theta^0, \theta^2\} &= 2\theta^0\theta^2, \\
\{\theta^1, \theta^1\} &= 0, & \{\theta^1, \theta^2\} &= 0, & \{\theta^2, \theta^2\} &= 0.
\end{align*}$$

(2.34)

Proof. What we have to do is to show that the system of brackets given by eqs.(2.4), (2.28) or (2.29) and (2.34) satisfies the super Jacobi identity and the bicovariance condition (2.20). This is a matter of direct calculations.

From Lemma 1 and Lemma 2 follows the

Theorem 1 There exist two and only two graded Poisson-Lie structures on the external algebra of $\text{SL}(2)$ prolonging the Sklyanin brackets (2.4). They are given by (2.28) or (2.29) and (2.34).

2.4 Differential Poisson structures

In the theory of Lie groups the operator $d$ of exterior derivative plays an important role. There is a natural differential operator $d$ on $\mathcal{M}$, $d : \mathcal{M} \to \mathcal{M}$. $d$ is defined on generators of $\mathcal{M}$ by:

$$dT = \theta T, \quad d\theta = \theta \theta,$$

(2.35)

($\theta$ is the matrix (2.3) written in terms of $\theta^i$-generators) and extended to the hole algebra $\mathcal{M}$ by using $d^2 = 0$ and the Leibniz rule. In principle, graded Poisson structures are not connected in any way with the operator $d$. However, one can rise a question if there exist such Poisson structures with respect to which $d$ is a differentiation.

Definition 2 A graded Poisson structure on $\mathcal{M}$ is called a differential one if the operator $d$ satisfies the Leibniz-like rule:

$$d\{f, h\} = \{df, h\} + (-1)^{\deg f}\{f, dh\}$$

(2.36)

Now we are going to see whether the Poisson-Lie structures on $\text{SL}(2)$ given by (2.4), (2.28) or (2.29) and (2.34) are differential ones. Let us consider for example the bracket $\{a, b\} = -ab$. Then

$$\{da, b\} + \{a, db\} = -2ab\theta^0 - (3 + 2bc)\theta^1$$

(2.37)

and

$$d\{a, b\} = -d(ab) = -2ab\theta^0 - (1 + 2bc)\theta^1.$$

(2.38)

This shows that our Poisson-Lie structures on $\mathcal{M}$ are not the differential ones. Hence, we arrive to the

Theorem 2 Differential Poisson-Lie structures on the external algebra of $\text{SL}(2)$ prolonging the Sklyanin brackets do not exist.
3 Quantization

In this section we show that the Poisson-Lie structures on the external algebra of $SL(2)$ given by eqs. (2.4), (2.28) or (2.29) and (2.34) can be quantized. One defines the quantization of a commutative graded Poisson-Lie algebra $\mathcal{M}$ as a constructing of a non-commutative algebra $\mathcal{M}_q$ ($q = \exp h$) supplied with comultiplications $\Delta(h)$, $\Delta_{L,R}(h)$ that obeys the following conditions:

1) $\mathcal{M}_q$ is a free module over the ring $k[[h]]$, where $h$ is a parameter of quantization,
2) as a coalgebra $\mathcal{M}_q/h\mathcal{M}_q$ is isomorphic to $\mathcal{M}$,
3) one can define on $\mathcal{M}$ the graded Poisson bracket:

$$\{x, y\} = \lim_{h \to 0} \left( \frac{1}{h} [\hat{x}, \hat{y}] \right), \quad x, y \in \mathcal{M}, \quad \hat{x}, \hat{y} \in \mathcal{M}_q$$

(3.39)

which should coincide with the original bracket on $\mathcal{M}$ [4, 11].

First of all, note that the quantization of (2.4) gives the defining relations describing the function algebra on the quantum group $GL_q(2)$:

$$ab = qba \quad ac = qca \quad ad - da = \mu bc$$

$$bc = cb \quad cd = qdc \quad bd = qdb$$

(3.40)

where $\mu = q - 1/q$. The further factorization of $\text{Fun}(GL_q(2))$ modulo the quantum determinant $\det_q T = ad - qbc$ defines the function algebra $\mathcal{A}_q = \text{Fun}(SL_q(2))$ on the quantum group $SL_q(2)$. The quantization leaves the coalgebra structure intact. The quantization of the relations (2.28) and (2.29) and (2.34) is given by the following theorem.

Theorem 3 Noncommutative algebras generated by the symbols $\theta^i$, $i = 0, 1, 2$ and the generators of the quantum group $SL_q(2)$ modulo I-type relations:

$$\theta^0 a = a \theta^0 + q^2 \mu c \theta^1, \quad \theta^1 a = \frac{1}{q} a \theta^1, \quad \theta^2 a = q a \theta^2 - \mu c \theta^0,$$

$$\theta^0 b = b \theta^0 + q^2 \mu d \theta^1, \quad \theta^1 b = \frac{1}{q} b \theta^1, \quad \theta^2 b = q b \theta^2 - \mu d \theta^0,$$

$$\theta^0 c = c \theta^0, \quad \theta^1 c = q c \theta^1, \quad \theta^2 c = \frac{1}{q} c \theta^2,$$

$$\theta^0 d = d \theta^0, \quad \theta^1 d = q d \theta^1, \quad \theta^2 d = \frac{1}{q} d \theta^2$$

(3.41)

or II-type relations

$$\theta^0 a = a \theta^0, \quad \theta^1 a = q a \theta^1, \quad \theta^2 a = \frac{1}{q} a \theta^2,$$

$$\theta^0 b = b \theta^0, \quad \theta^1 b = q b \theta^1, \quad \theta^2 b = \frac{1}{q} b \theta^2,$$

$$\theta^0 c = c \theta^0 + \mu a \theta^2, \quad \theta^1 c = \frac{1}{q} c \theta^1 - \frac{\mu}{q^2} a \theta^0, \quad \theta^2 c = q c \theta^2,$$

$$\theta^0 d = d \theta^0 + \mu \theta^2, \quad \theta^1 d = \frac{1}{q} d \theta^1 - \frac{\mu}{q^2} b \theta^0, \quad \theta^2 d = q d \theta^2$$

(3.42)

with

$$\mu = q - 1/q, \quad \lambda = q + 1/q,$$

and the relations

$$(\theta^0)^2 = \frac{q^2 \mu}{\lambda} \theta^1 \theta^2, \quad (\theta^1)^2 = (\theta^2)^2 = 0, \quad \theta^1 \theta^2 = -\theta^2 \theta^1$$

(3.43)
\[
\theta^1 \theta^0 = -\frac{1}{q^2} \theta^0 \theta^1, \quad \theta^2 \theta^0 = -q^2 \theta^0 \theta^2.
\]

represent the quantization of the graded Poisson-Lie algebra (2.28) or (2.29) and (2.34), respectively. The quantum determinant is the central element of the both algebras introduced above.

**Remark** Since the quantization procedure implies that the comultiplications survive under quantization the Theorem 3 assumes the existence of the quantum version of left and right comultiplications \( \Delta_{L,R}(q) \).

**Proof** At first, one can easily show that in the quasiclassical limit \((h \to 0)\) the relations (3.41) and (3.42) produce the Poisson structures of the first and the second types, respectively. Next, one needs to check the consistency of the algebras introduced in the theorem with the defining relations of \( A_q \). For this purpose it is enough to consider the ordering of the cubic monomials. Let us, for example, bring the monomial \( ab\theta^0 \) into the form \( \theta^0 ab \) by two different ways:

\[
ab\theta^0 = a(\theta^0 b - q^2 \mu d\theta^1) = (\theta^0 a - q^2 \mu c \theta^1)b - q^2 \mu ad\theta^1 = \theta^0 ab - q\mu (qad + bc) \theta^1,
\]

\[
ab\theta^0 = qb(\theta^0 a - q^2 \mu c \theta^1) = q(\theta^0 b - q^2 \mu d\theta^1)a - q^3 \mu bc \theta^1 = \theta^0 ab - q\mu (qda + q^3 bc) \theta^1.
\]

Due to eqs.(3.41) the results of passing by two different ways are the same. Analogously, one can show the consistency of the defining relations (3.41) and (3.42) by considering all the other cubic monomials.

Covariance of (2.28), (2.29) and (2.34) with respect to \( \Delta_L \) and \( \Delta_R \) gives rise to the existence of comultiplications \( \Delta_{L,R}(q) \) being homomorphisms of the algebra described in the Theorem 3. On \( \theta \) generators the action of \( \Delta_L(q) \) reads

\[
\Delta_L(q) \theta^0 = (ad + \frac{1}{q} bc) \otimes \theta^0 - qac \otimes \theta^1 + bd \otimes \theta^2,
\]

(3.44)

\[
\Delta_L(q) \theta^1 = -\frac{\lambda}{q} ba \otimes \theta^0 + a^2 \otimes \theta^1 - \frac{1}{q} b^2 \otimes \theta^2,
\]

(3.45)

\[
\Delta_L(q) \theta^2 = \lambda dc \otimes \theta^0 - qc^2 \otimes \theta^1 + d^2 \otimes \theta^2.
\]

(3.46)

and

\[
\Delta_R(q) \theta^i = \theta^i \otimes I
\]

(3.47)

Such a coproduct is nothing but the adjoint coaction of \( A_h \) on \( \theta^i \) generators, representing the quantum analog of components of the right-invariant Maurer-Cartan form. To make it clear let us introduce a matrix \( \theta \)

\[
\theta = \| \theta_k^m \| = \left( \begin{array}{c} \theta^0 \\ \theta^2 \\ -\frac{1}{q^2} \theta^0 \end{array} \right)
\]

(3.48)

with the quantum trace \( \text{tr}_q \theta = \frac{1}{q} \theta_1^1 + q \theta_2^2 = 0 \). Then one can write

\[
\Delta_R(q) \theta^i = \theta^i \otimes I, \quad \Delta_L(q) \theta^j_k = t^k_i S(t^j_m) \otimes \theta^m_k
\]

(3.49)

Now one can show that \( \Delta_{L,R}(q) \) are homomorphisms. At last, using the defining relations (3.41) one can check that the quantum determinant \( \det_q T \) is the central element of the algebra under consideration.
4 Concluding Remarks and Discussion

A few remarks are now in order.

- To make the quantization of the brackets (2.28) and (2.34) more transparent it is suitable to rewrite them using the corresponding classical $r$-matrix. Introducing the standard tensor notations we have for the brackets of the first order eq.(3.15) from [23] with $\alpha = 0 = \beta$. In the case of $GL(N)$ group these brackets were forbidden by the total Jacobi identity. For the brackets (2.34) one can write:

$$\{\theta_1, \theta_2\} = -(\theta_1 \theta_1 + \theta_2 \theta_2) + r^1 \theta_1 \theta_2 + \theta_1 \theta_2 r^1 - \theta_1 r^1 \theta_2 - \theta_2 r^1 \theta_1,$$

(4.50)

where $r_+$ is the classical $r$-matrix for $sl(2)$ (see (C.1)) and $r_- = -r_{+}^{t}$.

Let us note that the corresponding quantum algebra (3.43) was found in [16] in the $R$-matrix form

$$R_{12}\theta_1 R_{21}\theta_2 + \theta_2 R_{12}\theta_1 R_{12}^{-1} = k_q(R_{12}\theta_1^2 R_{21}^{2} + \theta_1^2),$$

(4.51)

where $k_q = \frac{\mu^2}{q^{2}+1/q}$ and the quantum $R$-matrix for $SL_q(2)$ is given in Appendix B. It is clear that the brackets (4.50) are the quasiclassical limit of the relations (4.51). It seems natural that taken in the $r$-matrix form the graded Poisson-Lie structures can be generalized to $SL(N)$.

- Let us consider the $\ast$-involution on $SL_q(2)$ defined by

$$a^\ast = d, \quad d^\ast = a, \quad b^\ast = -qc, \quad c^\ast = -\frac{1}{q}b.$$

(4.52)

In addition to (4.52) we can introduce the operation $\ast$ on $\theta$-s:

$$(\theta^0)^\ast = -\theta^0, \quad (\theta^1)^\ast = -\theta^2, \quad (\theta^2)^\ast = -\theta^1.$$

(4.53)

One can easily check that the relations (3.43) are invariant with respect to the operation $\ast$. However, $\ast$ operation does not leave both (3.41) and (3.42) invariant but transforms the brackets (3.41) into (3.42) and vice versa.

- Let us briefly discuss the issue of existence of an operator $d_h$ of exterior derivative for the noncommutative associative algebras described in the Theorem 3. First of all we would like to stress that this question has essentially a classical origin. Obviously, an operator $d_h$ should be introduced in agreement with the defining relations (3.40) of $A_h$ that in the standard tensor notations read [1]:

$$RT_1 T_2 = T_2 T_1 R,$$

(4.54)

where $R$ is the quantum $R$-matrix for $SL_q(2)$ (see (B.1)). This means that

$$Rd_h T_1 T_2 + RT_1 d_h T_2 = d_h T_2 T_1 R + T_2 d_h T_1 R.$$

(4.55)

Now let us suppose that $R$-matrix and $d_h$ operator are quasiclassical, $i.e.$

$$R = 1 + hr + O(h^2), \quad d_h = d + h d^1 + O(h^2),$$

(4.56)
where $d$ is the ordinary operator of exterior derivative. Then by using eq. (3.39) the quasiclassical version of (2.30) takes the form

$$\{dT_1, T_2 \} + \{T_1, dT_2 \} = [r_{12}^+, d(T_1 T_2)] = d \left( [r_{12}^+, T_1 T_2] \right),$$

where $\{T_1, T_2 \} = [r_{12}^+, T_1 T_2]$ is the Sklyanin brackets (2.4) expressed via the classical $r_{12}^+$ matrix for $sl(2)$ (see (C.1)). Thus, we conjecture that any noncommutative algebra of quantum differential forms admits an exterior derivative if the corresponding graded Poisson-Lie structure is the differential one. As we have seen all graded Poisson-Lie structures on $SL(2)$ are not differential ones and, therefore, the corresponding algebras of quantum forms do not admit the exterior derivative.

This situation is quite opposite to what we have for $GL(N)$. In [23] it was shown that for all graded Poisson-Lie structures on $GL(N)$ the ordinary $d$ operator has the description in inner terms, namely

$$d = \{ \kappa \text{tr} \theta, \ldots \} ,$$

where $\kappa$ is a coefficient depending on the choice of a Poisson-Lie structure. Then the Leibniz rule eq. (2.36) reduces to the graded Jacobi identity

$$\{ \text{tr} \theta , \{ f, h \} \} = \{ \{ \text{tr} \theta , f \} , h \} + (-1)^{\deg f} \{ f , \{ \text{tr} \theta , h \} \}$$

(4.58)

and, therefore, all graded Poisson-Lie structures on $GL(N)$ appear to be the differential ones. Moreover, the possibility of expressing $d$ as in (4.57) allows one to define its quantum counterpart via the graded commutator $[,]$ as

$$d_h = \left[ \frac{1}{\mu} \text{tr} q \theta, \ldots \right].$$

In general, according to the Connes definition of differentiation of a noncommutative graded algebra the operator $d$ is expressed as

$$d = [\xi, \ldots]$$

where $\xi$ is some special element of degree one such that $[\xi, \xi] = 0$. For a given noncommutative graded algebra an appropriate element $\xi$ may or may not exist. In the last case one can try to extend the algebra in such a way as to include $\xi$ and, therefore to make the algebra a differential one. Just this situation takes place in the standard approach to differential calculi on quantum special groups [3,10,13]. In this case the graded algebras of quantum right(left)-invariant forms can be supplied with the exterior differentiation but the price we have to pay for doing this consists in extending these algebras by an extra element $\xi$ which has no natural classical counterpart in the corresponding classical group. Note, that many of the constructions of bicovariant differential calculi of such a type can be obtained by the reduction of bicovariant differential calculi on $GL(N)$ [10,13].

Let us compare now the graded Poisson-Lie structures on $GL(2)$ with the Poisson-Lie structures on $SL(2)$ in more detail. Among the graded $GL(2)$ Poisson-Lie brackets of the second order there is only one natural candidate for $SL(2)$ reduction. Its choice is dictated by the compatibility with the condition $\text{tr} \theta = 0$. 

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This selected Poisson-Lie bracket is given by the formula (4.50) where $\theta$ has four independent entries. The corresponding quantum algebra is compatible with the condition $\text{tr} \, q \theta = 0$ and also can be reduced to $SL_q(2)$. It is not surprising that it is given just by eq.(4.51).

The situation drastically changes when we search for a graded Poisson-Lie structure of the first order on $GL(N)$ that can admit the $SL(N)$-reduction. It turns out [16] that the reduction of any graded Poisson-Lie structure on $GL(N)$ involves inevitably the additional one-form generator $\xi = \text{tr} \, \theta$. Thus, the reduction yields the classical limit of the standard bicovariant calculi on quantum special groups [9], [10]. Note, that the relations (3.41) and (3.42) change the transformation property of (2.34) as compared with the $GL_q(2)$ case. Namely, despite the fact that

$$\Delta_G(q) = \Delta_L(q) + \Delta_R(q)$$

(4.59)

is the homomorphism of the algebras (3.41) and (3.42) the relations (2.34) in opposite to the $GL(N)$ case do not respect the action of (4.59).

Finally, one can address a question whether it is possible to find for the algebras (3.41) and (3.42) an $R$-matrix formulation. Note, that the existence of such a formulation can make the generalization of our $SL(2)$ constructions to $SL(N)$ case straightforward.

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APPENDIX

A  One-parameter family of candidates for graded Poisson-Lie structure on $SL(2)$

\[
\begin{align*}
\{\theta^0, a\} &= \eta c \theta^1, \\
\{\theta^0, b\} &= \eta d \theta^1, \\
\{\theta^0, c\} &= -(2 + \eta) a \theta^2, \\
\{\theta^0, d\} &= -(2 + \eta) b \theta^2, \\
\{\theta^1, a\} &= -(1 + \eta) a \theta^1, \\
\{\theta^1, b\} &= -(1 + \eta) b \theta^1, \\
\{\theta^1, c\} &= (4 + 2 \eta) a \theta^0 + (1 + \eta) c \theta^1, \\
\{\theta^1, d\} &= (4 + 2 \eta) b \theta^0 + (1 + \eta) d \theta^1, \\
\{\theta^2, a\} &= -2 \eta c \theta^0 + (1 + \eta) a \theta^2, \\
\{\theta^2, b\} &= -2 \eta d \theta^0 + (1 + \eta) b \theta^2, \\
\{\theta^2, c\} &= -(1 + \eta) c \theta^2, \\
\{\theta^2, d\} &= -(1 + \eta) d \theta^2.
\end{align*}
\]  

(B.1)

B  The quantum $R$-matrix for $SL_q(2)$

\[
R = \begin{pmatrix}
q & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & \mu & 1 & 0 \\
0 & 0 & 0 & q \\
\end{pmatrix}
\]  

(C.1)

C  The classical $r^{12}_+$ matrix for $sl(2)$

\[
r_+ = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 \\
0 & 0 & 0 & 1 \\
\end{pmatrix}
\]
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