Joint Detection and Tracking for Multipath Targets: A Variational Bayesian Approach

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Abstract—Different from traditional point target tracking systems assuming that a target generates at most one single measurement per scan, there exists a class of multipath target tracking systems where each measurement may originate from the interested target via one of multiple propagation paths or from clutter, while the correspondence among targets, measurements, and propagation paths is unknown. The performance of multipath target tracking systems can be improved if multiple measurements from the same target are effectively utilized, but suffers from two major challenges. The first is multipath detection that detects appearing and disappearing targets automatically, while one target may produce $s$ tracks for $s$ propagation paths. The second is multipath tracking that calculates the target-to-measurement-to-path assignment matrices to estimate target states, which is computationally intractable due to the combinatorial explosion. Based on variational Bayesian framework, this paper introduces a novel probabilistic joint detection and tracking algorithm (JDT-VB) that incorporates data association, path association, state estimation and automatic track management. The posterior probabilities of these latent variables are derived in a closed-form iterative manner, which is effective for dealing with the coupling issue of multipath data association identification risk and state estimation error. Loopy belief propagation (LBP) is exploited to approximate the multipath data association, which significantly reduces the computational cost. The proposed JDT-VB algorithm can simultaneously deal with track initiation, maintenance, and termination for multiple multipath target tracking with time-varying number of targets, and its performance is verified by a numerical simulation of over-the-horizon radar.

Index Terms—Joint detection and tracking, multipath data association, variational Bayesian, belief propagation

I. INTRODUCTION

Based on sensor data, multitarget detection and tracking is the problem of simultaneously detecting targets being present or not, and if present providing their trajectories. It is essential for many applications in the areas of defense, medical science, traffic control and navigation [1]. Traditional probabilistic algorithms [2,3] for target tracking address the point target, which assume that one target generates at most a single measurement per time step, and one measurement is originated from at most one target. In many practical scenarios, however, one target can result in more than one measurement per time step, including the extended target where multiple measurements are spatially distributed and multipath target where multiple measurements from different propagation paths are not spatially structured [4]. Typical multipath target tracking systems include the skywave over-the-horizon radar (OTHR) [5] in which high-frequency signals from the same one target are scattered through different ionospheric layers, giving rise to multiple unsolved detections at each dwell, the passive coherent location (PCL) system [6] with multistatic configurations where one target may generate multiple detections from different transmitters of opportunity, and sensors work in the urban environment where dense terrain can cause multiple delayed returns [7]-[8]. Most applications of extended target tracking are in computer vision [9].

In multipath target tracking systems, if a tracking algorithm is able to properly extract all available information contained in the multiple measurements, the accuracy of estimation for targets can be improved and the number of false tracks can be reduced due to the increase of signal-to-noise ratio [10]. However, because of the additional uncertainty of measurement-to-path association, its computational complexity increases significantly, as illustrated in the following two aspects.

One is multipath tracking with the following challenges:

- data association: measurements yielded from a sensor need to be associated with one of the tracked targets;
- path association: the target-originated measurements need to be associated with one of the propagation paths;
- high-dimensional estimation: the tracker has to handle high-dimensional latent-state parameters including target detection, estimation, and multipath data association;
- combinatorial explosion: the multipath data association problem consisting of data association and path association is combinatorial;
- coupling: the problem of target detection and tracking is coupled with the multipath data association.

The other is multipath detection: the number of targets to track is unknown and possibly time-varying. In the absence of prior information on the sensing environment, targets may appear and disappear at any time in any place. Moreover, one target may produce as many as $s$ tracks from $s$ propagation paths. Both multipath target detection and tracking are challenging on their own. Furthermore, they are tightly coupled. Many of previous researches follow the popular tracking-by-detection paradigm concentrated on the multipath tracking problem, which can be summarized as three categories.

Methods in the first category reformulate the multipath target tracking as two sequential sub-problems: single-path target tracking [11] and multipath track fusion [12]. They are computationally effective and easy to be implemented in practice, but the fused tracks are not always reliable by the
fact that multipath tracks are not accurate and even missed in the low detection probability case. Another challenge for this category is to account for the dependence of multipath tracks, which may arise due to common process noise for target dynamics, or if the same measurements are used to update two or more multipath track state estimates [13].

The second category algorithms address the multipath data association by extending the classical single-path data association algorithm to the multipath case, such as multipath probabilistic data association (MPDA) [14], [15], multihypothesis Viterbi data association [16], multi-detection joint probabilistic data association [17], multi-detection multiple hypothesis tracking [18], multi-detection probability hypothesis density [10], and multi-detection Bernoulli [19]. They exploit the available multipath measurements to improve tracking performance but suffer from greatly increased computational complexity. Another drawback of this category is that they cannot deal with the coupling of multipath data association and state estimation, which means an error (e.g., a track is by chance associated with clutter or/and choose a wrong path) can not be corrected once it has been made [20].

The key idea in the third category of algorithms is to perform the target state estimation and multipath data association jointly based on approximate Bayesian inference. The stochastic approximate approaches such as Markov Chain Monte Carlo (MCMC) which uses sampling strategy in the joint state space of target states and data association variables [21], [9], are computationally demanding, and therefore limited to small-scale problems [22]. The deterministic approximate approaches such as variational Bayesian (VB) [23], [24] and expectation-maximization (EM) [25] which turn the inference problem into an optimization problem and obtain the analytical approximations to the posterior distribution of the latent variables, tends to be faster and easier to scale to large data [26]. The existing EM-based algorithms [27], [28], [29], [30] carry out the state estimation and multipath data association identification iteratively in E-step and M-step, which are theoretically attractive and desirable to deal with the coupling of identification risk and estimation errors [31]. However, the EM-based algorithms are limited to the scenarios when the number of targets is known, and cannot detect the target automatically. The VB algorithm overcomes this drawback of the EM by introducing a prior information, and thus open the possibility of determining the optimal number of targets [32]. Moreover, VB is superior to EM in the case of high-dimensional latent variables [33].

Most recently, the variational Bayesian (VB) approaches have drawn attention in data association of point target tracking since they have advantages of closed-form analytical solution and convergence guarantee. VB was first applied to data association in [34], addressing the problem of distributed tracking using wireless sensor networks. Kanazaki et al. [35] derived a recursive variational approximation data association filter for approximating the marginalized likelihood of multiple targets state. Lázaro-Gredilla et al. [36] introduced a mixture of Gaussian processes to model every trajectory, and learn the hyperparameters of mixture Gaussian based on the VB algorithm. Turner et al. [20] proposed a complete variational tracker based on VB framework to retain the framing constraints and model tracks that are dropped in [30]. It can simultaneously perform inference for track management, data association, and state estimation with linear computation cost in batch window length as opposed to the exponential scaling of previous MAP-based approaches. Lau et al. [37] proposed a complete multitarget tracking algorithm based on structured mean field approach with existence-based model.

Inspired by the work of [20], we develop a VB-based method (JDT-VB) with polynomial computational complexity to solve the joint detection and tracking for multiple multipath targets in this paper. By constructing the conjugate prior distribution function, the posterior probability density or mass function of multipath data association, state estimation, and target detection are derived in the VB iterative loop. The Forney Style factor graph for modeling the data association assignment matrices is described in detail, and the loopy belief propagation (LBP) for approximating the expectation of the assignment matrices is proposed. The simulation of OTHR multipath multitarget tracking with the unknown number of targets is carried out to verify the effectiveness of the JDT-VB algorithm. In summary, we make the following contributions:

- we incorporate data association, path association, target detection and trajectory estimation in a unified Bayesian framework for multipath multitarget joint detection and tracking with time-varying number of targets.
- we derive the closed-form analytical expression for each posterior distribution of latent variables based on the variational Bayesian iteration.
- we employ a Forney-Style factor graph for modeling the multipath data association and apply the LBP algorithm for approximating the multipath assignment matrices.
- we present a multipath measurement clustering method for initialization of JDT-VB. The computational complexity of JDT-VB is analysed to be polynomial;
- we extract the multipath information both at the level of detection and at the level of tracking to improve the performance of our proposed algorithm.

A preliminary version of the results presented here appeared in a conference paper [38], [39]. Here, we present for the first time the entire formulation including important implementation details, the initialization, computational complexity analysis, and the additional simulation work.

The rest of the paper is organized as follows. The full Bayesian inference of joint multipath target detection and tracking and background of VB approximation are described in Section II. The closed-form analytical solutions of multipath data association, target detection, and state estimation are derived in Section III. The simulation analysis and the conclusion are given in Section IV and Section V, respectively.

Notation: Throughout this paper, \( \mathcal{N}(x|\mu, P) \) represents the Gaussian distribution of \( x \) with mean \( \mu \) and covariance \( P \), \( \mathbb{E}_x[f] \) represents the expectation of \( f \) w.r.t \( x \), \( H \) denotes the Shannon entropy, and \( I \) represents the indicator function. The variable-to-factor message \( \mu_{y\rightarrow f}(y) \) represents the message incoming factor \( f \) from edge \( y \), and factor-to-variable message \( \mu_{f\rightarrow y}(y) \) means the message outgoing factor \( f \) along edge \( y \).
II. BACKGROUND AND PRELIMINARIES

A. Joint Detection and Tracking for Multipath Target

1) Applications: Multipath targets occur in scenarios that are susceptible to the multipath propagation phenomenon, existing in OTHR, PCL, etc. OTHR exploits signal reflection from the ionosphere to detect and track airborne/surface targets at ranges an order of magnitude greater than conventional line-of-sight radars [5]. The target tracking and localization in OTHR depends on the Earth’s ionosphere, which is classified into several subregions according to the ionospheric virtual heights including the common E layer, and F layer. In such a case, all feasible ionospheric propagation paths would have to be modeled. Taking two ionospheric layers E layer and F layer for instance (shown in Fig. 1), the ionospheric heights on transmission $h^e_i$ and reception $h^r_i$ can be assigned to model each of these paths. For each path, it is assumed to produce a target-originated measurement independently with a known detection probability. However, which propagation path is a measurement originated from is unknown. Since it is usually impossible to select a radar operating frequency that results in single-path propagation for the region of interest, the multipath propagation is unavoidable in OTHR [14].

![Fig. 1: Illustration of OTHR multipath signal propagation.](image)

Remark 2.1: There are some differences between extended target and multipath target despite both of them result in multiple measurements per time step. The extended target occurs due to high sensor resolution but multipath target occurs due to multipath propagation phenomenon. The multiple measurements for an extended target are always spatially structured while that of multipath target are not. The focus of extended target tracking lies on target shape estimation, while for multipath target, it is multipath data association problem.

2) Problem Formulation: Consider the following discrete-time dynamic system of joint detection and tracking

$$x_{i,k+1} = f_k(x_{i,k}) + v_{i,k+1}, \quad i = 1, 2, \ldots, N_T,$$

where $x_{i,k} \in \mathcal{T}$ is the $i$th target state to be estimated in state space $\mathcal{T}$ at time $k$, $f_k$ is the known state transition function, $v_{i,k}$ is a zero-mean Gaussian white noise with known covariance $Q_{i,k}$. Here, $N_T$ is the maximum number of targets that simultaneously be present in the region of interest. This parameter is necessary in order to cast the problem at hand into a finite-dimensional state space [40]. The initial target states $x_{i,0}, i = 1, 2, \ldots, N_T$ are assumed to be Gaussian distributed.

In multipath target tracking, a received measurement might be originated from the underlying target through a particular propagation path or clutter. In other words, any a measurement belongs to the set containing the possible target-originated and clutter-originated measurements, denoted by $Y_k = \bigcup_{\tau=1}^{N_M} y_{r,k} \cup C_k$, where $N_M$ is the number of propagation paths. Here, $C_k$ is the set of clutter, which is independent and uniformly distributed. The total number of clutter is Poisson distributed [15]. The measurement $y_{r,k} \in \mathcal{O}$ in observation space $\mathcal{O}$, originated from the $i$th target via the $\tau$th path, is randomly available with probability $p^\tau_{i,k}$, and its model is given by

$$y_{r,k} = h^r_i(x_{i,k}) + w_{r,k}, \quad \tau = 1, 2, \ldots, N_M$$

where $h^r_i$ is the measurement function related with path $\tau$, and $w_{r,k}$ is zero-mean Gaussian white noise with known covariance $R_{r,k}$. Here, $v_{i,k}, w_{r,k}$ and $x_{i,0}$ are assumed to be mutually independent.

Coordinate registration (CR), which converts a target’s location from observation space $\mathcal{O}$ to state space $\mathcal{T}$, requires accurate information of association latent space $S$. The relationship among targets, measurements and paths is described via the following latent variables (see Fig. 2):

1) For each target $i \in \{1, \ldots, N_T\}$ and each measurement $j \in \{1, \ldots, N^E_F\}$, the data association variable $\phi^i_{j,k}$ represents the $i$th target is associated with the $j$th measurement. In particular, $\phi^0_{k,i}$ means the $i$th measurement is originated from clutter. Here, $N^E_F$ denotes the number of measurements at time $k$.

2) For each measurement $j \in \{1, \ldots, N^E_F\}$ and each path $\tau \in \{1, \ldots, N_M\}$, the path association variable $\tau^j_{k}$ represents the $j$th measurement is associated with the $\tau$th path; One measurement can originate from at most one target via a specific propagation path or from clutter.

3) The multipath data association variable $A_k \triangleq \{\phi_k, \tau_k\}$ with $\phi^k_{i,0,\tau} \in A_k$ represents the $j$th measurement is originated from the $i$th target via the $\tau$th propagation path. In particular, $a^0_{k,i,\tau}$ means the $i$th target is not detected by the $\tau$th propagation path, and $\phi^k_{0,j,\tau}$ means the $j$th measurement is originated from clutter.

![Fig. 2: Multipath data association problem.](image)

Definition 2.1: Define the binary meta-state $s_{i,k} \in \{0, 1\}$ to deal with the detection problem. For a sequence $S_k = \{s_{i,k}\}_{i=1}^{N_T}$, the element $s_{i,k} = 0$ means the $i$th target is dormant and $s_{i,k} = 1$ means the $i$th target is active. The goal of detection is thus to determine $S_k$ at each time $k$.

Definition 2.2: At time instant $k$, we denote all the target states $X_k = \{x_{i,k}\}_{i=1}^{N_T}$, measurements $Y_k = \{y_{j,k}\}_{j=1}^{N^E_F}$, meta-state $S_k = \{s_{i,k}\}_{i=1}^{N_T}$, and association event $A_k = \{a^k_{i,j,\tau}\}_{i=1,j=1,\tau=1}^{N_T,N^E_F,N_M}$, respectively. For a batch sequence over $K$
time steps, the corresponding variables sequence are denoted by \(X_1^K, Y_1^K, S_1^K\) and \(A_1^K\), respectively.

**Problem Statement:** The problem of joint detection and tracking for multipath target is to estimate the target state \(X_1^K\) (tracking) together with the meta-state \(S_1^K\) (detection) simultaneously, given measurements \(Y_1^K\) in the presence of unknown multipath data association \(A_1^K\).

In the sense of optimal Bayesian inference, the joint multitarget detection and tracking problem can be turned into solving the joint posteriori probability density function (PDF) \(L_1^K \triangleq p(X_1^K, S_1^K, A_1^K|Y_1^K)\), which can be decomposed according to the graphical model (see Fig. 3):

\[
L_1^K \propto p(Y_1^K|X_1^K, A_1^K)p(X_1^K)p(A_1^K|S_1^K)p(S_1^K),
\]

where the observation model \(p(Y_1^K|X_1^K, A_1^K)\), state prediction model \(p(X_1^K)\), meta-state model \(p(S_1^K)\), and multipath data association model \(p(A_1^K|S_1^K)\) will be given later.

![Fig. 3: Graphical model of joint detection and tracking.](image)

If \(A_1^K\) is known, it is tractable to infer the posterior PDF \(p(X_1^K|A_1^K, Y_1^K)\), using a Kalman or nonlinear smoother. With \(A_1^K\) unknown, however, the problem of tracking is formulated to calculate the posterior PDF, by considering all possible multipath data associations:

\[
p(X_1^K|Y_1^K) = \sum_{A_1^K} p(X_1^K|Y_1^K, A_1^K)p(A_1^K). \tag{4}
\]

Meanwhile, the detection problem, which determines meta-state \(S_1^K\) to be dormant or active, also requires the enumeration of all values of \(A_1^K\) due to the posterior PDF given by

\[
p(S_1^K|Y_1^K) = \sum_{A_1^K} p(S_1^K|Y_1^K, A_1^K)p(A_1^K). \tag{5}
\]

The optimal solution of joint multitarget detection and tracking, in the sense of Bayesian inference, is to solve the intractable summation problem of Eq. (4) and Eq. (5). In such situations, we have to resort to approximation schemes [41]. The sampling-based stochastic techniques such as MCMC are computationally demanding, often limiting their use to small-scale problems. The analytical-based deterministic approximations such as variational Bayesian (VB), scaling well to large-scale applications [26], are desirable for the joint detection and tracking of multipath target.

**B. Background to the Variational Bayesian Inference**

Variational Bayesian methods are a family of techniques for approximating intractable integrals or summation arising in Bayesian inference. They are typically used in complex statistical models consisting of observed variables (measurement \(Y\)), unknown parameters (target state \(X\) and meta-state \(S\)) and latent variables (multipath data association \(A\)), where the unknown parameters \(\{X, S\}\) and latent variables \(A\) are grouped together as unobserved variables \(Z\). The relationships among these types of random variables might be described by a graphical model (see Fig. 3). VB method provides a local-optimal, exact analytical solution to an approximation of the posterior PDF, which makes it more applicable in the case of high-dimensional unobserved variables.

The idea of variational inference is to approximate the posterior distribution \(p(Z|Y)\) with a simpler family of probability distributions and seek the distribution \(q(Z)\) from this family that is closest to the true posterior, i.e., \(p(Z|Y) \approx q(Z)\). The most common type of VB, known as mean-field VB [26], uses the Kullack-Leibler (KL) divergence as the choice of dissimilarity function. The KL-divergence is written as

\[
KL(q||p) = \sum_Z q(Z) \log \frac{q(Z)}{p(Z,Y)} + \log p(Y),
\]

\[
\Rightarrow \log p(Y) = KL(q||p) + B(Z).
\]

where \(B(Z) = \sum_Z q(Z) \log \frac{p(Z,Y)}{q(Z)}\) is known as the variational free energy, which is also the lower bound for the evidence \(\log p(Y)\) since the KL(\(q||p\)) is nonnegative.

The mean-field VB allows the elements of \(Z\) to be partitioned into \(M\) disjoint groups \(\{Z_i\}_{i=1}^M\) and each factor \(q_i(Z_i)\) is a probability distribution with a free functional form. By using an iterative mechanism which maximizes the joint posterior PDF \(p(Y, Z)\) with respect to each \(q_i(Z_i)\) in turn while holding other factors fixed, the unnormalized distribution \(q_i^* (Z_i)\) can be obtained as follows:

\[
q_i^* (Z_i) \propto \exp \left( \mathbb{E}_{z_{j\neq i}} \log p(Y, Z) \right)
\]

**Remark 2.2:** There are some characteristics of VB methods:

- much like the EM algorithm, the VB is also an iterative procedure that successively converges on (local) optimum parameter values. However, the VB algorithm is more suitable in high-dimensional latent variables case [26].
- the VB performs estimation in hyper-parameter space while the EM in parameter space. By introducing the prior distribution of the unobservable variables, the VB algorithm has better estimation accuracy and adaptive performance than the EM algorithm, such as the ability to determine the number of components adaptively in the Gaussian mixture model applications.
- the choice of conjugate priors is of key importance in the design of VB recursive algorithms, since they confine the shaping parameters and prevent a linear increase in the number of degrees-of-freedom with observed data. If the observation model has a unconjugate distribution on parameters, the computational complexity of full Bayesian inference is condemned to grow with time [42].
III. VB FOR JOINT DETECTION AND TRACKING

A. General Framework

The proposed JDT-VB algorithm is a joint multipath detection and tracking algorithm based on the VB framework, which integrates the target state estimation, multipath data association and automatic track management in an iterative loop. The information flow of the proposed JDT-VB algorithm is depicted in Fig. 4. By using measurements and constructing the prior probability of each unobserved variables, the posterior probability of target state, meta-state, and multipath data association are updated via the nonlinear smoother, forward-backward algorithm and loopy belief propagation, respectively. The details are further explained below:

- **prior probability model:** one very convenient and analytically favorable class of priors is conjugate priors in the exponential family. That is, the complete data likelihood function belongs to the exponential family while the prior distribution is conjugate to the likelihood function.
- **posterior state update (Module 1):** given the rth iterative multipath data association $\hat{A}^K_i$ and measurements $Y^K_i$, the path-conditional track $\hat{X}_{i,k}^K$ for each path is estimated via the nonlinear smoother, and then those path-conditional tracks are fused to obtain the target state $\hat{X}_{i,k}^K$.
- **posterior meta-state update (Module 2):** given the rth iterative multipath data association $\hat{A}^K_i$, the meta-state $\hat{S}^K_i$ is identified via forward-backward algorithm, which excavates the environmental information such as the probability of detection, and false-alarm rate.
- **posterior multipath data association update (Module 3):** given the rth iterative state estimate $\hat{X}^K_i$, meta-state $\hat{S}^K_i$ and measurements $Y^K_i$, the multipath data association is obtained via a LBP approximation. Then, data association and propagation paths are identified by computing marginal distribution density, and further the posterior probability of each path and its corresponding pseudo-marginal density are obtained.
- **iterative loop:** the above modules (1-3) are repeated until the values of the estimated variables at two consecutive iterations are close enough or the maximum number of iterations has been reached.

B. The Conjugate Exponential Family Prior

1) **State Model:** Although more general models are possible (i.e., formation target tracking where the target motion is under some constraint condition), within this paper we assume each track independently follows a first-order Markov process:

$$ p(X^K_{1,i}) = \prod_{i=1}^{N_T} p(x_{i,1}) \prod_{k=2}^{K} p(x_{i,k}|x_{i,k-1}) $$  

(9)

with $p(x_{i,k}|x_{i,k-1})$ being Gaussian distribution below

$$ p(x_{i,k}|x_{i,k-1}) = N(x_{i,k}|f_{k}(x_{i,k-1}), Q_{i,k}) $$  

(10)

2) **Meta-State Model:** The detection problem is addressed by augmenting track states with a two-state Markov model with an active/dormant meta-state $S_k$ in a 1-of-$N_T$ encoding:

$$ p(S^K_{1,i}) = \prod_{i=1}^{N_T} p(s_{i,1}) \prod_{k=2}^{K} p(s_{i,k}|s_{i,k-1}), \quad s_{i,k} \in \{0,1\} $$  

(11)

with the initial state distribution $\pi$ and state transition probability distribution $T_A(e,c) = p(s_{i,k} = c|s_{i,k-1} = e)$ for all $i$ and $k$. The existence variable $s_{i,k} = 1$ if the rth target is present at time $k$ and $s_{i,k} = 0$ otherwise. The number of targets being tracked at $k$, is thus denoted by $N^E_k = \sum_{i=1}^{N_T} s_{i,k}$.

3) **Multipath Data Association Model:** Different from the traditional data association algorithms that assume the assignment matrix $A_k$ is unknown constant, the VB algorithm treats the latent variable $A_k$ as a stochastic variable with its prior distribution $p(A_k)$ given by

$$ p(A_k|p^E_0) = \phi_k(\phi_k,p^E_0)p(\phi_k) \propto (\lambda V_G(k))^{N^E_k} \exp(-\lambda V_G(k)) / N^E_k! $$  

(12)

$$ \times \prod_{i=1}^{N_T} \prod_{\tau=1}^{N_M} p(\phi_k(i)\phi_k^{T})p^E_0(i)\phi_k^{T} (1-p^E_0(i))^{1-d_k^{i,\tau}} $$

where $\lambda$ is the clutter density, $V_G(k)$ is the volume of validation gate $[14]$, and $N^E_k$ is the number of clutter. $d_k^{i,\tau} = 1$ represents the $i$th target is detected via the $\tau$th path at time $k$, otherwise $d_k^{i,\tau} = 0$.

Deriving approximate inference procedures is often greatly simplified if the prior distribution on parameters is conjugate to the complete data likelihood $\mathcal{L}_k^E$, which can be expressed by the following exponential family form:

$$ \mathcal{L}_k^E = \prod_{j=1}^{N^E_k} \prod_{\tau=1}^{N_M} p_c(y_{j,k}|a_{i,j,\tau}^{E}) \prod_{i=1}^{N_T} p(y_{j,k}|x_{i,k}, a_{i,j,\tau}^{E} = 1)a_{i,j,\tau}^{E} $$  

(13)

$$ \times \prod_{i=1}^{N_T} p(x_{i,k}) $$

$$ = \prod_{i=1}^{N_T} p(x_{i,k}) \exp \left\{ 1^T (A_k \odot L_k)1 \right\} $$

where

$$ \mathcal{L}_k^E = \prod_{j=1}^{N^E_k} \prod_{\tau=1}^{N_M} p_c(y_{j,k}|a_{i,j,\tau}^{E}) \prod_{i=1}^{N_T} p(y_{j,k}|x_{i,k}, a_{i,j,\tau}^{E} = 1)a_{i,j,\tau}^{E} $$  

(13)

$$ \times \prod_{i=1}^{N_T} p(x_{i,k}) $$

$$ = \prod_{i=1}^{N_T} p(x_{i,k}) \exp \left\{ 1^T (A_k \odot L_k)1 \right\} $$

where $\mathcal{L}_k^E$ is the complete data likelihood $\mathcal{L}_k^E$, which can be expressed by the following exponential family form:
with
\[
\mathbf{1}^T(A_k \odot L_k) \mathbf{1} = \sum_{\tau=1}^{N_T} \sum_{i=0}^{N_T} \sum_{j=0}^{N_T} a_k^{i,j,\tau} \times L_k^{i,j,\tau},
\]
\[
L_k^{i,j,\tau} = \log p(y_{j,k} | x_{i,k}, a_k^{i,j,\tau}),
\]
\[
L_k^{0,j,\tau} = \log p_c(y_{j,k}), L_k^{1,0,\tau} = 0,
\]
where \(L_k\) represents log likelihood contributions from various assignments, and \(p_c\) denotes the PDF of the clutter with uniform distribution. Therefore, we have the exponential family, which implies the conjugate assignments prior \(p_a(A_k | \chi_k)\) can be given by
\[
p_a(A_k | \chi_k) = \mathcal{Z}(\chi_k) \mathbb{I}(A_k \in \mathcal{A}) \exp \{\mathbf{1}^T(\chi_k \odot A_k) \mathbf{1}\},
\]
where \(\mathcal{Z}(\chi_k)\) is the normalization constant, and \(\mathcal{A}\) is the set of all assignment hypothesis that obey the path-conditioned framing constraints. That is, each track is associated with at most one measurement via one path per frame, and vice versa.

 Recovering the prior distribution \(p(A_k)\) of Eq.(12) in the form of the conjugate assignments prior distribution, the hyperparameter \(\chi_k\) has the following settings,
\[
\chi_k^{i,j,\tau} = \log \left( \frac{p_g(i)}{(1 - p_g(i)) \lambda} \right), \chi_k^{0,j,\tau} = \chi_k^{1,0,\tau} = 0.
\]

4) Observation Model: Based on the assumption that the measurements are independently distributed conditioned on the state and assignment event, the observation likelihood function \(p(Y^1_k | X^1_k, A^1_k)\) can be factorized as:
\[
p(Y^1_k | X^1_k, A^1_k) = \prod_{k=1}^{K} \prod_{i=1}^{N_T} \prod_{j=1}^{N_T} \prod_{\tau=1}^{N_T} p(y_{j,k} | x_{i,k}, a_k^{i,j,\tau}).
\]

 The observation model depends upon whether the measurement comes from clutter or from an interested target:
\[
p(y_{j,k} | x_{i,k}, a_k^{i,j,\tau} = 1) = \mathcal{N}(y_{j,k} | h_k^c(x_{i,k}), R_k^c),
\]
\[
p(y_{j,k} | a_k^{i,j,\tau} = 1) = \mathcal{V}_{G}(k)^{-1}.
\]

Remark 3.1: Although the joint PDF \(L^k\) of Eq.(3) can be expressed by Eqs. (13), (21), a direct optimization of distribution \(p(X_k | Y^1_k)\) of Eq.(4) and \(p(S_k^1 | Y^1_k)\) of Eq.(5) with respect to the hidden variables is intractable, due to the combinatorial nature of the multipath data association problem. The VB-based approach for approximate inference in an iterative form is given by the following subsection.

C. VB for Posterior Probability Update

The key of VB algorithm for posterior probability update is to approximate the intractable \(p(X^1_k, S^1_k, A^1_k | Y^1_k)\) by a tractable distribution \(q(X^1_k, S^1_k, A^1_k)\), i.e.,
\[
p(X^1_k, S^1_k, A^1_k | Y^1_k) \approx q(X^1_k, S^1_k, A^1_k).
\]

According to the mean-field approximation [26], Eq.(22) is further decomposed as
\[
q(X^1_k, S^1_k, A^1_k) = q(X^1_k, S^1_k)q(A^1_k).
\]

In the VB framework, the multipath data association \(A^1_k\) is regarded as “latent variables” and \(\{X^1_k, S^1_k\}\) are regarded as “parameters”, and common variational practice of factorizing these two groups of variables is used. This gives the variational lower bound \(B(q)\) as follows
\[
B(q) = \mathbb{E}_q \left[ \log p(X^1_k, S^1_k, A^1_k, Y^1_k) \right] + \mathbb{H} \left[ q(A^1_k) \right].
\]

By observing the variational Bayesian lower bound Eq.(24) and Eq.(3), we arrive at the following induced factorizations without forcing further factorization upon Eq.(23):
\[
q(X^1_k, S^1_k) = \prod_{i=1}^{N_T} q(x_{i,1:k})q(s_{i,1:k}),
\]
\[
q(A^1_k) = \prod_{k=1}^{K} q(A^1_k).
\]

In other words, the approximate posterior on state and metastate factorizes across tracks, and multipath data association factorizes across time. Using the methods from calculus of variations for minimizing the KL-divergence, the VB update for \(q(X^1_k), q(S^1_k)\) and \(q(A^1_k)\) is given by (see Eq.(8)):
\[
\log q(X^1_k) = \mathbb{E}_{S^1_k,A^1_k} \left[ \log p(X^1_k, S^1_k, A^1_k | Y^1_k) \right],
\]
\[
\log q(S^1_k) = \mathbb{E}_{X^1_k,A^1_k} \left[ \log p(X^1_k, S^1_k, A^1_k | Y^1_k) \right],
\]
\[
\log q(A^1_k) = \mathbb{E}_{X^1_k,S^1_k} \left[ \log p(X^1_k, S^1_k, A^1_k | Y^1_k) \right].
\]

1) State Posterior Update \(q(x_{i,1:k})\): Based on the induced factorizations in Eq.(25), the updates of posterior PDF for each track are derived separately. Computing the expectation in Eq.(27) gives the following, denoting equality to an additive constant with \(c\):
\[
\log q(x_{i,1:k}) \propto \sum_{k=1}^{K} \log p(x_{i,k} | x_{i,k-1})
\]
\[
+ \sum_{k=1}^{K} \sum_{j=1}^{N_T} \sum_{\tau=1}^{N_T} \mathbb{E} \left[ a_k^{i,j,\tau} \right] \log \mathcal{N}(y_{j,k} | h_k^c(x_{i,k}), R_k^c).
\]

Using the product of Gaussian formula, yields,
\[
q(x_{i,1:k}) \propto p(x_{i,k}) \prod_{k=1}^{K} \prod_{\tau=1}^{N_T} \mathbb{E} \left[ q_{k}^{i,\tau} \right] \mathcal{N}(\tilde{y}_{i,\tau,k} | h_k^c(x_{i,k}), R_{i,\tau,k})
\]

with
\[
\tilde{y}_{i,\tau,k} = \frac{\sum_{j=1}^{N_T} \mathbb{E} \left[ q_{k}^{i,j,\tau} \right] y_{j,\tau,k}}{1 - \mathbb{E} \left[ q_{k}^{i,0,\tau} \right]}, \quad R_{i,\tau,k} = \frac{R_k^c}{1 - \mathbb{E} \left[ q_{k}^{i,0,\tau} \right]},
\]
where \(p(x_{i,k})\) is the prior distribution of the \(i\)th state, and \(\tilde{y}_{i,\tau,k}\) is the valid measurement that fall into the gate of the \(i\)th prediction \(P\) via the \(\tau\)th path. The form of the posterior \(q(x_{i,k})\) is a Gaussian mixture with weight \(\mathbb{E} \left[ q_{k}^{i,\tau} \right]\) and components of path-dependent local state \(q(x_{i,\tau,k})\). Here, \(q(x_{i,\tau,k})\) is equivalent to a dynamical system with pseudo-measurements \(\tilde{y}_{i,\tau,k}\) and non-stationary measurement covariance \(R_{i,\tau,k}\), which is simply implemented using Kalman smoother for linear case or Unscented Rauch-Tung-Striebel smoother (URTS) [43] for
nonlinear case, i.e.,
local state estimate and its covariance of the \( \tau \)th path:
\[
\hat{x}_{i,\tau,k} = \mathbb{E}[\hat{x}_{i,\tau,k}|\hat{y}_{i,\tau,1:k}]
\]
\[
P_{i,\tau,k} = \text{cov} [\hat{x}_{i,\tau,k}, \hat{x}_{i,\tau,k}|\hat{y}_{i,\tau,1:k}]
\]
global fused state and its covariance:
\[
\hat{x}_{i,k} = \sum_{\tau=1}^{N_M} \mathbb{E} [\varphi_{i,k}^T] \hat{x}_{i,\tau,k}, \quad P_{i,k} = \sum_{\tau=1}^{N_M} \mathbb{E} [\varphi_{i,k}^T] P_{i,\tau,k} \hat{x}_{i,\tau,k}
\]
The estimation of state \( X_1^K \) is summarized in Module 1.

**Module 1: State estimation (Tracking): URTS**

**Input:** Measurements \( Y_i^K \), and expectation of multipath data association \( \mathbb{E}[A_i^K] \) from Module 3.

**Output:** target state estimation \( \{\hat{X}_i^K, P_i^K\} \);
1: for each target \( i = 1 : N_T \) do
2: for each path \( \tau = 1 : N_M \) do
3: for each time instant \( k = 1 : K \) do
4: Select valid measurements subset \( \hat{Y}_{i,\tau,k} \) for target \( i \) and path \( \tau \) via gate technique:
5: \( \hat{y}_{i,\tau,k} \triangleq \{ y_{j,k} : D(\hat{y}_{i,\tau,k} - y_{j,k}, S_{i,\tau,k}) \leq \gamma_{i,\tau} \} \),
where \( y_{i,\tau,k} \) and \( S_{i,\tau,k} \) are the measurement prediction and innovation covariance of the \( i \)th target via the \( \tau \)th path, and scalar constant \( \gamma_{i,\tau} \) is chosen to make gate probability equal to \( p_g^\tau \) for path \( \tau \);
6: Calculate pseudo-measurements \( \{ \hat{y}_{i,\tau,k}, \hat{R}_{i,\tau,k} \} \) by using the local data association \( \mathbb{E}[\phi_{i,k}^T] \) via Eq. (32);
7: end for
8: Calculate the path-conditional local state estimate \( \{ \hat{x}_{i,\tau,k}, P_{i,\tau,k} \} \) of Eqs. (33-34) via the URTS;
9: end for
10: Calculate the global target state estimate \( \{ \hat{x}_{i,k}, P_{i,k} \} \) by using the path association \( \mathbb{E}[\phi_{i,k}^T] \) via Eq. (35).
11: end for

**2) Meta-State Posterior Update \( q(S_i^K) \):** The expectation on the meta-state in Eq. (28) can be computed as follows:
\[
\log q(s_{i,1:K}) \propto \mathbb{E}_q(A) \sum_{k=1}^{K} \log p(A_k|s_{i,k}) p(s_{i,k}|s_{i,k-1})
\]
Recall that the prior distribution of \( A_k \) is related with the \( i \)th state from Eq. (12). Substituting Eq. (12) into Eq. (36), yields
\[
q(s_{i,1:K}) \propto p(s_{i,1:K}) \prod_{k=1}^{K} \exp(s_{i,k} \xi_{i,k})
\]
with
\[
\xi_{i,k} = \sum_{\tau=1}^{N_M} \mathbb{E}[\varphi_{i,k}^T(1 - \mathbb{E}[\phi_{i,k}^{0,\tau}])] p_d^\tau(s_{i,k})
\]
Since \( p(s_{i,k}) \) follows a Markov chain of Eq. (11), the form of \( q(s_{i,k}) \) is the same as a hidden Markov model with the hidden variable \( s_{i,1:K} \) and model parameters \( \lambda = \{ p(A), b_k \} \), where observation symbol probability distribution \( b_k(s_{i,k}) = \exp(s_{i,k} \xi_{i,k}) \). Then, the meta-state posterior \( q(s_{i,1:k}) \) is updated via a forward-backward algorithm [34]. Note that the \( \xi_{i,k} \) integrates the detection of probability \( p_d^\tau \) from all propagation paths, which will improve the performance of detection by fusing multipath information.

The estimation of \( S_i^K \) is summarized in Module 2.

**Module 2: Meta-state identification (Detection): forward-backward algorithm**

**Input:** expectation of multipath data association \( \mathbb{E}[A_i^K] \) from Module 3;

**Output:** meta-state identification \( \{ \hat{S}_i^K, N_i^K \} \);
1: for each target \( i = 1 : N_T \) do
2: Define a forward variable \( \hat{\alpha}_{i,k}(e) \) as the probability of the observation sequence until time \( k \), with state \( e \in \{0,1\} \) at time \( k \). Execute the forward step as follows;
3: Initialize \( \hat{\alpha}_{i,1}(e) = \pi(e) b_{1}(e) \);
4: for each time instant \( k = 2 : K \) do
5: Update the forward variable \( \hat{\alpha}_{i,k}(e) \) by
6: end for
7: Define a backward variable \( \hat{\beta}_{i,k}(e) \) as the probability of the observation sequence from time \( k+1 \) to \( K \), given state \( e \) at time \( k \). Execute the backward step as follows;
8: Initialize \( \hat{\beta}_{i,k}(e) = 1 ; \)
9: for each time \( k = K-1 : 1 \) do
10: Update the backward variable \( \hat{\beta}_{i,k}(e) \) by
11: end for
12: Calculate the posterior probability in terms of forward and backward variables
13: Compare the \( q(s_{i,k}) \) with different thresholds \( \delta_b \) and \( \delta_m \) to do detection.
14: end for
15: Calculate the number of targets by \( N_i^K = \sum_{i=1}^{N_T} N_i^K \).

The detection is related with the meta-state \( S_i^K \) as follows:
- **initiation:** a new track for target \( i \) at time \( k \) is added by setting \( s_{i,k} = 1 \) if \( q(s_{i,k}) \geq \delta_b \).
- **maintenance:** the track of target \( i \) at time \( k \) is updated by setting \( s_{i,k} = 1 \) if \( q(s_{i,k}) = \delta_m \).
- **termination:** the track of target \( i \) at time \( k \) is deleted by setting \( s_{i,k} = 0 \) if \( q(s_{i,k}) < \delta_m \).
where $\delta_t$ and $\delta_m$ are birth target and survival target thresholds.

3) \textbf{Multipath Data Association Update} $q(A_k^K)$: The expectation on the multipath data association in Eq.(29) can be computed as follows,

$$\log q(A_k) = \mathbb{E}_{q(S_k)} \log p(A_k | S_k) + \sum_{j=1}^{N_E} \phi_{k,0} \log p_c(y_{j,k}) \quad (42)$$

$$+ \sum_{j=1}^{N_E} \sum_{i=1}^{N_T} \sum_{\tau=1}^{N_M} a_{k,j,\tau}^i \mathbb{E}_{q(x_{i,k}, \chi_k)} \log p(y_{j,k} | x_{i,k}; \phi_{k,\tau})$$

$$= \mathbb{E}_{q(S_k)} (1^T (\chi_k \odot A_k) 1) + \mathbb{E}_{q(S_k)} (1^T (L_k \odot A_k) 1).$$

Take the exponential of both sides, yields

$$q(A_k) = p_{a_k} (A_k | \mathbb{E}_{q(S_k)} [\chi_k] + \mathbb{E}_{q(S_k)} [L_k]). \quad (43)$$

Inspired by Eq.(43) and Eq.(17), the posterior distribution $q(A_k)$ has the same functional form as the prior distribution $p(A_k)$, and thus the update of multipath data association is equivalent to update conjugate prior parameter $\chi_k$ with $\chi_{p,k} = \mathbb{E}_{q(X_k)} [L_k] + \mathbb{E}_{q(S_k)} [\chi_k]$, which is given by the Appendix.

This imposes a challenging problem, as the state posterior $q(A_k)$ of path $\tau$ has a challenging problem, as the state posterior $q(A_k)$ of path $\tau$ is conditioned on the state parameter $\chi_k$ and the measurement parameter $\chi_{p,k}$.

The multipath data association $A_k^K = \{\phi_k, \varphi_k\}_{k=1}^{K}$ factorizes across time, and then can be further decomposed into path association and path-conditional data association, i.e.,

$$q(A_k^K) = \prod_{k=1}^{K} q(\phi_k | \varphi_k) q(\varphi_k). \quad (44)$$

For a given path $\tau$, the distribution of assignment matrix for data association conditioned on the $\tau$th path $q(A_k^\tau)$ can be naturally represented as a factor graph:

$$q(A_k^\tau) = q(\phi_k | \varphi_k = \tau) \propto \prod_{i=1}^{N_T} f_{i,R} (A_k^\tau_{i,.}) \prod_{j=1}^{N_E} f_{j,C} (A_k^\tau_{,.j}) \prod_{i=0}^{N_T} \prod_{j=0}^{N_E} f_{i,j}^S (A_k^\tau_{i,j}) \quad (45)$$

with

$$f_{i,R} (A_k^\tau_{i,.}) = \prod_{j=1}^{N_E} a_{k,ij,\tau}^i = 1, \quad (46)$$

$$f_{j,C} (A_k^\tau_{,.j}) = \prod_{i=1}^{N_T} a_{k,ij,\tau}^i = 1, \quad (47)$$

$$f_{i,j}^S (A_k^\tau_{i,j}) = \exp \{ \chi_{p,k} \times a_{k,ij,\tau}^i \}. \quad (48)$$

A factor graph can be constructed based on the assignment matrix and then $\chi_{p,k}$ is utilized to provide evidence for the corresponding evidence factor in order to initialize essential messages. Fig. 5 provides an example of constructing the corresponding Forney Style factor graph (FFG) \cite{51} of assignment matrix. There are four kinds of factors in our problem:

- evidence factor $f_{i,j}^E$: originated from the exponential part of the distribution for $A^\tau$;
- row factor $f_{j}^R$: obey the assumption that each target either is associated with a measurement or missed detection;
- column factor $f_{i}^C$: obey the assumption that each measurement is generated by either a target or clutter;
- equality factor $f_{i,j}^E$: clone some variables which play a role in more than two factors, which is only used in FFG.

![Factor Graph](image-url)

**Fig. 5:** Illustration of data association based on factor graph (Two targets to be associated with two measurements).

The LBP algorithm consists of message initialization step and message update step. The evidence factor $f_{i,j}^E$ is utilized to initialize messages, and row constraint factor $f_{j}^R$, column constraint factor $f_{i}^C$, and equality factor $f_{i,j}^E$ are used to update messages. The detailed illustration of data association based on factor graph (Fig. 5) and the message passing procedure refers to our previous work \cite{38, 39}.

Once the messages are updated and converge to a fixed point, marginal distribution of a certain variable can be computed by multiplying the corresponding two messages flowing in the opposite direction along its edge,

$$p(\alpha_{ij}) = \mu_{f_{i} \rightarrow \alpha_{ij}} (\alpha_{ij}) \times \mu_{f_{j} \rightarrow \alpha_{ij}} (\alpha_{ij}). \quad (49)$$

As all the variables in an assignment matrix are binary variables, the marginal distribution of any variable along its edge can be explicitly expressed as

$$p(\alpha_{ij} = 0) = \mu_{f_{i} \rightarrow \alpha_{ij}} (\alpha_{ij} = 0) \times \mu_{f_{j} \rightarrow \alpha_{ij}} (\alpha_{ij} = 0), \quad (50)$$

$$p(\alpha_{ij} = 1) = \mu_{f_{i} \rightarrow \alpha_{ij}} (\alpha_{ij} = 1) \times \mu_{f_{j} \rightarrow \alpha_{ij}} (\alpha_{ij} = 1). \quad (51)$$
The expectation of data association $E(\alpha_{ij})$ is given by

$$E(\alpha_{ij}) = \sum_{\alpha_{ij}=0}^{1} p(\alpha_{ij}) \times \alpha_{ij} = p(\alpha_{ij} = 1).$$

The Mahalanobis distance between the global state estimate and local path-dependent state estimate of path $\tau$ and target $i$ is used to calculate the path association probability $q(\varphi_{i,\tau})$.

$$q(\varphi_{i,\tau}) = \frac{C}{\sqrt{D(\hat{x}_{i,\tau,k} - \mu_{i,k}, P_{i,\tau,k} + \Sigma_{i,k})}},$$

where $C$ is the normalization constant, $\mu_{i,k}$ and $\Sigma_{i,k}$ are the initial state mean and covariance, given by

$$\mu_{i,k} = \sum_{\tau=1}^{N_{M}} P_{i,\tau,k}^{-1} \hat{x}_{i,\tau,k}, \quad \Sigma_{i,k} = \sum_{\tau=1}^{N_{M}} P_{i,\tau,k}^{-1}.$$

The computation of multipath data association $A_{1}^{K}$ is summarized in Module 3.

**Module 3 Multipath data association: LBP**

**Input:** measurements $Y_{1}^{K}$, target state estimate $\{\hat{X}_{1}^{K}, P_{1}^{K}\}$ from Module 1, meta-state estimate $S_{1}^{K}$ from Module 2;

**Output:** multipath data association $q(A_{1}^{K})$;

1. for each time instant $k = 1 : K$ do
2. Calculate $\chi_{p,k}$ according to the Appendix;
3. for each path $\tau = 1 : N_{M}$ do
4. Simplify assignment matrix $A_{1}^{\tau}$ via gate technique;
5. Construct the corresponding factor graph for $A_{1}^{\tau}$;
6. Initialize messages for each variable via Eq. (48);
7. while no convergence do
8. Start the message passing procedure and update the message iteratively;
9. end while
10. Calculate marginal distributions via Eqs. (50)-(51);
11. end for
12. Calculate path probability $q(\varphi_{k})$ via Eqs. (53)-(54);
13. end for
14. Output the multipath data association $q(A_{1}^{K})$ via Eq. (44);

**Remark 3.2:** The LBP approximation gives rise to a new variational lower bound $B_{\beta}(q)$, which merely replaces the entropy $H[q(A_{1}^{K})]$ in Eq. (44) with the Bethe entropy $H_{\beta}[q(A_{1}^{K})]$ given by

$$H_{\beta}[q(A_{1}^{K})] = H_{\beta}[q(A_{1}^{K})] + H_{\beta}[q(\varphi_{k})]$$

$$= \sum_{i=1}^{N_{T}} H_{\beta}[q(\alpha_{k}^{i})] + \sum_{j=1}^{N_{E}} H_{\beta}[q(\alpha_{c}^{j})]$$

$$- \sum_{i=1}^{N_{T}} \sum_{j=1}^{N_{E}} H_{\beta}[q(\alpha_{k}^{i}, \alpha_{c}^{j})].$$

Note that the inequality $H_{\beta}[q(A_{1}^{K})] \leq H[q(A_{1}^{K})]$ holds in most models [20], therefore $B_{\beta}(q)$ is a lower bound of $B(q)$.

**D. Performance Analysis of JDT-VB Algorithm**

1) **Summary:** The JDT-VB algorithm consisting of Module 1, 2 and 3 processing iteratively, is summarized in Table I.

**TABLE I: The summary of JDT-VB algorithm**

| Step | Description |
|------|-------------|
| 1: Initialization. | Given measurements $Y_{1}^{K}$, choose an initial value of $\{\hat{X}_{1}^{K(0)}, P_{1}^{K(0)}, S_{1}^{K(0)}, N_{T}\}$. |
| 2: Joint Detection and Tracking (the $r$th iteration) | 
\begin{enumerate}
\item Association: Calculate $A_{1}^{r}$ via Module 3.
\item Detection: Estimate $S_{1}^{r}$ via Module 2.
\item Tracking: Estimate $\hat{x}_{1}^{r}$ via Module 1.
\end{enumerate} |
| 3: Iterative Termination. | If iteration terminates, goto step (4); otherwise reset $r \leftarrow r + 1$ and return to (2.1). |
| 4: Outputs. | Output the joint detection and tracking results. |

2) **Initialization:** Although distinct methods may exist, our initialization procedure is given as follows:

- multiple path measurements cluster method is used to form track head. In scan $k$, consider any arbitrary collection of measurements $\{y_{j}(k)\}_{j=1}^{J}$ with $1 < J \leq N_{M}$. For any arbitrary two measurements $y_{i}(k)$ and $y_{j}(k)$, $i \neq j$, the difference $|y_{i}(k) - y_{j}(k)|$ should be less than a thick threshold $\rho$. For each candidate measurement $y_{j}(k)$, form multiple path state estimates $x_{\delta,m}(0|0)$ according to the active propagation paths $m = 1, \cdots, N_{M}$. Construct the state-to-path association hypotheses, and the probability of each hypothesis is evaluated to choose the most likely hypotheses. The initial state estimate $\hat{x}(0|0)$ is obtained by fusing those multiple-path states deemed to be associated with the same target. The diagonal components of the initial covariance $P(0|0)$ can be assigned based on the known measurement noise variances, as well as the initial meta-state $p(s_{0})$.
- track head is recursively updated by using JPDA and UKF algorithm. In scan $k + 1$, for each track head initiated in scan $k$, construct validation gates for each propagation path and perform path-dependent state estimation using the JPDA and UKF algorithm, and further to obtain the ultimate state estimate $\hat{x}(k+1|k+1)$ by fusing them. The measurements that do not fall into the validation gates are used to initial new track head. Meanwhile, the probability of meta-state $p(s_{k})$ is recursively updated by using the forward-backward algorithm according to the Modules 2.

- track deletion and confirmation by comparing the probability of meta-state $p(s_{k})$ with the threshold $\delta_{s}$, and $N_{T}$ is the number of confirmed temporary tracks.

3) **Computational Complexity:** The JDT-VB algorithm consists of an iteration loop among Modules 1, 2, and 3. Thus, the computational complexity of JDT-VB algorithm is proportional to the sum of the cost of these modules, i.e.,

$$c_{\text{tot}} = c_{\text{d}} \times (c_{\text{arts}} + c_{fb} + c_{\text{LBP}}).$$
where \( r_d \) is the number of iterations, and \( c_{urts}, c_{fb}, c_{lbp} \) are the computational cost of Modules 1, 2, and 3, respectively.

- \( c_{urts} \) (URTS and multipath fusion) is \( O(KN_TN_MD_z^2) \), being the most expensive operations the computation of the square root and the inverse of state covariance matrices, where \( D_z \) is the dimension of target state \( x' \);
- \( c_{fb} \) (forward-backward algorithm) is \( O(4KN_T) \);
- \( c_{lbp} \) (LBP algorithm) consists of an iteration loop among the procedure of message update. The complexity per iteration of this procedure is \( O(KN_M(N_E^3N_E^2 + N_E^2N_E^2)) \) \([47]\), since each iteration involves sending a message from each of the \( N_T \) target association variables to each of \( N_E \) measurement association variables (and vice versa).

**Remark 3.3:** There are several excellent characteristics of the proposed JDT-VB algorithm:

- it provides an integrated solution for joint multipath multitarget detection and tracking in the VB framework. The performance of detection and tracking will be improved by the fact that the multipath measurements are integrated for estimating the target state and meta-state.
- it has an iterative closed-loop among multipath data association, state estimation, and meta-state estimation, which is effective in dealing with the coupling relationship between estimation error and identification risk in the view of feedback control.
- it has polynomial computational complexity, and the combinatorial explosion problem of multipath data association is eliminated by LBP algorithm.
- it has convergence guarantee by the fact that the VB algorithm provides a variational lower bound \( \mathcal{B}(q) \) in Eq. (24) which should not decrease during the iterative procedure. The lower bound \( \mathcal{B}(q) \) is further approximated in the LBP algorithm to get a new variational lower bound \( \mathcal{B}_3(q) \leq \mathcal{B}(q) \leq \log p(Y) \). Moreover, the convergence of the LBP for data association is guaranteed \([47]\).

**IV. SIMULATION**

**A. Joint Detection and Tracking Using Simulated OTHR Data**

In this paper, we consider an idealized ionospheric model with two layers (E-layer and F-layer) and known ionospheric virtual heights (\( h_E \) and \( h_F \)). There are four one hop propagation paths: \( EE, EF, FE \) and \( FF \). The look-up path table with two layers \( E \) and \( F \) refers to Table II.

**TABLE II:** Indexing propagation paths (\( E \) and \( F \))

| Index | Path | \( h_E \) | \( h_F \) | \( p_d \) | Explanation |
|-------|------|---------|---------|---------|-------------|
| \( \tau = 1 \) | \( EE \) | \( h_E \) | \( h_E \) | \( p_{d1} \) | transmit on \( E \) and receive on \( E \) |
| \( \tau = 2 \) | \( EF \) | \( h_E \) | \( h_F \) | \( p_{d2} \) | transmit on \( E \) and receive on \( F \) |
| \( \tau = 3 \) | \( FE \) | \( h_F \) | \( h_E \) | \( p_{d3} \) | transmit on \( F \) and receive on \( E \) |
| \( \tau = 4 \) | \( FF \) | \( h_F \) | \( h_F \) | \( p_{d4} \) | transmit on \( F \) and receive on \( F \) |

The OTHR measurement \( y_k = [r_k, \dot{r}_k, \dot{\varphi}_k]^T \) in slant coordinates \( \mathcal{R} \) consists of slant range \( r_k \), range rate \( \dot{r}_k \) and azimuth \( \dot{\varphi}_k \). The target state \( x_k = [g_k, \dot{g}_k, \dot{\varphi}_k, \dot{\varphi}_k]^T \) in ground coordinates \( \mathcal{G} \) consists of ground range \( g \), range rate \( \dot{g} \), bearing \( \varphi \) and bearing rate \( \dot{\varphi} \). The measurement function, which is related with the path association since different path association leads to different ionospheric virtual heights (refer to Table II), is given by \([14]\)

\[
\begin{align*}
    r_k &= r_{\alpha,k} + r_{\beta,k} \\
    \dot{r}_k &= \frac{\dot{g}_k}{r_{\alpha,k}} + \frac{g_k - d \sin(\dot{\varphi}_k)}{r_{\alpha,k}} \\
    \dot{\varphi}_k &= \arcsin\left(\frac{2g_k \sin(\dot{\varphi}_k)}{r_{\alpha,k}}\right)
\end{align*}
\]

with

\[
\begin{align*}
    r_{\alpha,k} &= \sqrt{\frac{g_k^2}{4} + h_{r,k}^2} \\
    r_{\beta,k} &= \sqrt{\frac{g_k^2 - 2dg_k \sin(\dot{\varphi}_k) + d^2}{4} + h_{r,k}^2}
\end{align*}
\]

where \( r_\alpha \) and \( r_\beta \) are the ray path of transmitter and receiver, and \( d \) is the distance between receiver and transmitter.

1) **Scenario parameters:** The surveillance region is assumed to be \([1500, 2000]\) km in range, \([0.428, 0.608]\) rad in azimuth, and \([-0.524, 0.524]\) km/s in range rate. Consider six targets moving in the surveillance region with constant velocity model \( F_k = I_2 \otimes \begin{bmatrix} 1 & T_s \\ 0 & 1 \end{bmatrix} \). Targets initial states and duration time are given in Table III. The sampling period \( T_s = 16 \) s, and the number of dwells \( K = 30 \). The measurement noise covariance \( R_k \) for each path \( \tau = diag\{25 \text{ km}^2, 1e - 6 \text{ km}^2/s^2, 9e - 6 \text{ rad}^2\} \). The process noise covariance \( Q_k = 10^{-6} \times \text{blkdiag}\{\begin{bmatrix} 8 & 4 \\ 4 & 1 \end{bmatrix}, \begin{bmatrix} 0.01 & 0.1 \\ 0.01 & 0.1 \end{bmatrix}\} \). The detection probability of each path is \( p_{d}^\tau = 0.5 \), and the expected clutter density is 100 points per scan (\( \approx 2.2e - 3 \) per 3-D resolution cell). The signal-to-noise (SNR) is about 5.6 dB for independent path detection \( p_{d} = 0.5 \), and 12.7 dB for multipath detection \( p_{d} = 0.9375 \) if multipath measurement can be correctly identified. The number of paths \( N_M = 4 \), ionospheric heights \( [h_E = 100, h_F = 260] \) km, and the distance between receiver and transmitter \( d = 100 \) km.

**TABLE III:** Initial setting of targets

| Target | Initial state | Duration time |
|--------|---------------|---------------|
| \( x_1 \) | \([1700, 0.1, 0.48, 8.7e - 5]^T\) | \([1, 20]\) |
| \( x_2 \) | \([1800, 0.1, 0.48, 8.7e - 5]^T\) | \([1, 20]\) |
| \( x_3 \) | \([1950, -0.25, 0.55, 8.7e - 5]^T\) | \([8, 20]\) |
| \( x_4 \) | \([1880, 0.2, 0.54, 8.7e - 5]^T\) | \([20, 30]\) |
| \( x_5 \) | \([1650, -0.2, 0.56, 8.72e - 5]^T\) | \([20, 30]\) |
| \( x_6 \) | \([1600, 0.28, 0.55, 8.7e - 5]^T\) | \([12, 30]\) |

2) **JDT-VB algorithm parameters:** The gate probability of each path \( p_{g}^\tau = 0.971 \). The iterative terminated threshold \( \delta_T = 1e - 5 \) and the maximum number of iteration \( r_{max} = 20 \). The threshold of birth target \( \delta_b = 0.6 \) and survival target \( \delta_s = 0.85 \). The threshold \( \rho_T = [85 \text{ km}, 0.007 \text{ km/s}, 0.04 \text{ rad}]^T \). The initial meta-state probability \( p(s_0 = 1) = 0.3 \).
transition probability matrix $T_a(e,c) = \begin{bmatrix} 0.9, 0.1 \\ 0.1, 0.9 \end{bmatrix}$.

B. Performance Evaluation

1) The Performance Comparison with MPTF algorithm (50 Monte Carlo Runs): We compare our proposed JDT-VB algorithm with the well-known multi-hypothesis multipath track fusion algorithm (MPTF) \cite{12} for various levels of detection probability $p_d$, expected number of clutter $N_c$, and measurement noise covariance $R$. The MPTF algorithm belongs to the first category method aforementioned. Firstly, multiple path-dependent tracks are obtained in slant coordinate by using PDA and unscented Kalman filter (UKF). Then, MPTF algorithm fuses these path-dependent tracks in the ground coordinate. More specifically, propagation path-dependent track-to-target association hypotheses are constructed, and the probability of each hypothesis is evaluated before fusing those path-dependent tracks deemed to be associated with the same target within likely hypotheses. The JDT-VB and MPTF are implemented in MATLAB R2013a on a laptop computer with an Intel CORE i5 CPU with 4GB of RAM.

The multi-target trajectories and corresponding OTHR multipath detections and clutter are shown in Fig. 6. Two typical multiple targets tracking situation are designed in our scenario (Fig. 6(a)), including parallel-moving targets (target 1 and target 2) and cross-moving targets (target 3 and target 4; target 5 and target 6). Therefore, as shown in Fig. 6(b), the multipath detections of those targets are closely-spaced with 100 expected number of clutter in the tracking scene for each scan, which makes the multipath data association a very challenging problem.

Fig. 7 demonstrates the performance comparison of MPTF and JDT-VB algorithm in the aspect of both target detection and tracking. Both of the MPTF and JDT-VB algorithm initialize all of the target tracks successfully and create false tracks due to clutter or multipath propagation. However, the MPTF has more false tracks than JDT-VB since there exist multipath tracks in MPTF when the association hypotheses judgment is not correct. The tracking performance of JDT-VB is superior to the MPTF in the aspects of estimation precision, smoothness and track maintenance.

The performance comparison for various levels of the detection probability $p_d$ and the expected number of clutter $N_c$ is shown in Table VI and for various measurement noise covariance $R$ is shown in Table VII respectively. As expected, the performance of both the MPTF and JDT-VB is generally improved under the circumstances with higher SNR (higher probability of detection and less number of clutter), and higher measurement accuracy.

- Both the JDT-VB algorithm and MPTF algorithm have high target detection success ratio (TDSR). Even in the most challenging case of $p_d = 0.5$ and $N_c = 200$, the target detection success ratio of JDT-VB is larger than 0.75. The MPTF algorithm has higher detection ratio since once any path-dependent track has been successful started, then the target is detected.
- The JDT-VB algorithm has higher average track length ratio (ATLR) than that of MPTF algorithm. This is mainly because the MPTF algorithm deals with the multipath detection and tracking separately, and the fused tracks are not reliable by the fact that multipath tracks are not accurate and even missed in the low detection probability. The JDT-VB is a joint solution where multipath information are integrated into state estimation and target detection, and therefore has more steady tracking performance.
- The JDT-VB algorithm has less average number of false tracks (ANFT) but more average false track length (AFTL) than the MPTF algorithm. Meanwhile, as the SNR decreases or the measurement accuracy decreases, the number of false tracks increase for both JDT-VB and MPTF, whereas the increasing rate for JDT-VB is slower than that of MPTF. The short false track is easy to be initialized in the slant coordinate than in the ground coordinate. In MPTF algorithm, the fused tracks are not reliable by the fact that multipath tracks are not accurate and even missed in the low detection probability, which
TABLE IV: Performance Comparison in different SNR

| SNR   | $p_d = 0.75$ | $p_d = 0.5$ | $p_d = 0.5$ |
|-------|--------------|--------------|--------------|
|       | $N_c = 100$  | $N_c = 100$  | $N_c = 200$  |
| Metrics | JDT-VB | MPTF | JDT-VB | MPTF | JDT-VB | MPTF |
| TDSR | 0.93 | 1.00 | 0.86 | 0.95 | 0.82 | 0.96 |
| ATLR | 0.94 | 0.93 | 0.88 | 0.76 | 0.87 | 0.77 |
| ANFT | 0.85 | 2.15 | 0.95 | 3.55 | 1.75 | 11.60 |
| AFTL | 10.5 | 6.12 | 9.76 | 6.89 | 10.36 | 5.72 |
| RMSE | 0.82 | 3.39 | 1.11 | 12.59 | 3.40 | 11.32 |
| RMSEB | 1.40 | 4.30 | 2.40 | 6.80 | 2.7 | 7.20 |
| ACC(s) | 120.57 | 9.05 | 128.38 | 8.38 | 638.93 | 718.04 |

TABLE V: Performance Comparison in different $R$

| $R$ | $\delta_r = 5$km | $\delta_r = 10$km | $\delta_r = 5$km |
|-----|------------------|------------------|------------------|
| Metrics | JDT-VB | MPTF | JDT-VB | MPTF | JDT-VB | MPTF |
| TDSR | 0.86 | 0.95 | 0.86 | 0.93 | 0.83 | 0.83 |
| ATLR | 0.88 | 0.76 | 0.83 | 0.73 | 0.81 | 0.70 |
| ANFT | 0.95 | 3.55 | 1.65 | 5.50 | 2.45 | 8.60 |
| AFTL | 9.76 | 6.89 | 10.83 | 7.01 | 12.61 | 6.83 |
| RMSE | 1.11 | 12.59 | 3.80 | 16.12 | 4.95 | 17.36 |
| RMSEB | 2.4 | 6.80 | 2.70 | 8.80 | 4.70 | 11.60 |
| ACC(s) | 128.38 | 8.38 | 208.69 | 8.78 | 256.13 | 8.79 |

gives rise to the multipath tracks. Perhaps more precisely, part of the false tracks in MPTF algorithm are multipath tracks of targets of interest.

- The JDT-VB algorithm has higher estimation precision in both ground range (RMSE in ground range, RMSEB (mrad)) and bearing (RMSE in bearing, RMSEB (mrad)) than MPTF algorithm. As the decrease of SNR or and measurement accuracy, the estimation performance of JDT-VB algorithm decreases slightly. This is mainly because the proposed JDT-VB algorithm is an iterative batch processing consists of the URTS smoother while the MPTF algorithm is a recursion algorithm with a filter. Meanwhile, the multipath measurements can be effectively fused to improve the estimation performance of the JDT-VB algorithm.

- The average computational cost (ACC) of JDT-VB algorithm is larger than that of MPTF algorithm when the expected number of clutter $N_c = 100$. However, as the number of measurements increase (from $N_c = 100$ to $N_c = 200$), the increasing computational speed of MPTF (718.04/8.38 = 85.7) is much faster than that of JDT-VB algorithm (639.93/128.38 = 4.98). As stated in [12], the computational complexity of MPTF grows exponentially.

2) The Detailed Performance Analysis of JDT-VB Algorithm (Single Simulation Run): Figure. 8 shows the number of possible temporary tracks in the initialization step, which is 44 at time instant $k = 30$, i.e., the number of possible tracks $N_T = 44$. The probability of meta-state $p(s)$ (referred as active probability) is shown in Fig. 9 and the active probabilities of all six targets pass the threshold of steady track with high value. The local path-dependent and global fused state estimation and detection results are shown in Fig. 10 and Fig. 11 with its corresponding path association probability shown in Fig. 12 respectively. The fused state estimation and detection results are more stable than the path-dependent result. The path-dependent and global fused detection results are show in Fig. 13 most of the temporary false tracks (36 / 38 = 94.7 %) generated from the initialized step have been deleted during the VB loop, two false tracks have been appeared. The estimation and detection performance with different iteration number $r$ are demonstrated in Fig. 14 and Fig. 15. With the increasing number of iteration, the estimation error decreases and the active probability increases, which shows the convergence of the proposed JDT-VB algorithm and its improvement for both estimation and detection accuracy. In fact, the JDT-VB with $r = 1$ is just an open-loop smoother. The iteration result shows that the closed-loop processing has the advantages of dealing with the coupling of identification and estimation. The computational cost with respect to different number of targets $N_T$, number of clutter $N_c$, and window length $K$ are shown in Table VI. As expected, the JDT-VB algorithm has polynomial computational cost.

![Fig. 8: Number of tracks initialization.](Image)

![Fig. 9: Active probability estimation of tracks.](Image)
Fig. 7: The performance comparison of JDT-VB and MPTF (x-axis: Bearing (rad), y-axis: Ground range (km))
| $N_T$ | Time (s) | $N_c$ | Time (s) | $K$ | Time (s) |
|-------|----------|-------|----------|-----|----------|
| 5     | 132.35   | 50    | 93.37    | 10  | 28.03    |
| 10    | 199.24   | 100   | 123.51   | 15  | 57.76    |
| 15    | 259.48   | 150   | 281.83   | 20  | 94.51    |
| 20    | 393.71   | 200   | 638.82   | 25  | 130.38   |
| 25    | 519.64   | 250   | 1182.14  | 30  | 182.35   |

V. Conclusion

In this paper, the joint detection and tracking problem of multiple multipath targets where one target generates multiple detections was addressed under the VB framework. The proposed JDT-VB algorithm establishes the closed-loop solution among the multipath data association, state estimation, and target detection. The relevant analytical solutions are calculated iteratively via the LBP, URTS, and forward-backward algorithm, respectively. This paper compared the performance of the proposed JDT-VB algorithm with the MPTF algorithm by applying the OTHR multitarget tracking simulated data. The proposed algorithm outperformed the MPTF in the aspect of both state estimation and target detection, and it has less computational time than MPTF in the case of high false alarm.

Along the result of this paper, further work will focus on

- the extension of more challenging scenario with unknown or time-varying number of paths, and the prior information of propagation path (path rank information for OTHR) can be explored to improve the performance of the JDT-VB algorithm;
- the distributed parallel processing version of the proposed JDT-VB for multiple multipath target joint detection and tracking applications, and the distributed variational Bayesian based on consensus filter would be considered;
- the online recursion or adaptive length selection of processing window for real-time applications.

Appendix

Details of the derivations for updating the variational parameter $\chi_{p,k}$ are given as follows.

- For all $\tau = 1, \cdots, N_M$,
  \[
  \chi^{0,j,\tau}_{p,k} = p^{0,j,\tau}_{k} + \chi^{0,j,\tau} = \log p_c(y_{j,k}),
  \]
  \[
  \chi^{i,0,\tau}_{p,k} = p^{i,0,\tau}_{k} + \chi^{i,0,\tau} = 0.
  \]

- For all $i = 1, \cdots, N_T$ and $j = 1, \cdots, N^E_k$,
  \[
  \chi^{i,j,\tau}_{p,k} = \left\{
  \begin{array}{ll}
  q(s_{i,k}) \log(p(y_{j,k}|x_{i,k}, a^{i,j,\tau}) dx_{i,k}) \\
  \sum_{s_{i,k}} q(s_{i,k}) \chi^{i,j,\tau}_{k} \\
  E_s
  \end{array}
  \right.
  \]
  \[E_s = \sum_{s_{i,k}} q(s_{i,k}) \log \left( \frac{P^T_{d}(s_{i,k})}{1 - P^T_{d}(s_{i,k})} \right) \]
  \[= q(0) \log \left( \frac{P^T_{d}(0)}{1 - P^T_{d}(0)} \right) + q(1) \log \left( \frac{P^T_{d}(1)}{1 - P^T_{d}(1)} \right) \]

  \[E_x = -\frac{1}{2} \mathbb{E}_{q(x_{i,k})} \left[D(y_{j,k} - h^T(x_{i,k}), S_{i,k}) + C_k \right]
  \]
  \[= -\frac{1}{2} \mathbb{E} \left\{ \text{Tr} \left\{ S_{i,j,k}^{-1}(y_{j,k} - h^T(x_{i,k}, \cdot))(\cdot)^T \right\} \right\} + C_k
  \approx -\frac{1}{2} \text{Tr} \left\{ S_{i,j,k}^{-1}(y_{j,k} y_{j,k}^T) \right\} + C_k
  \]
  \[+ \frac{1}{2} \text{Tr} \left\{ S_{i,j,k}^{-1}(y_{j,k} H_k x_{i,k}^T + H_k^T x_{i,k} y_{j,k}^T) \right\}
  \]
  \[+ \frac{1}{2} \text{Tr} \left\{ S_{i,j,k}^{-1}(H_k^T x_{i,k}^T + P_{i,k})(H_k^T) \right\}
  \]
  \[= -\frac{1}{2} \text{Tr} \left\{ S_{i,j,k} H_k^T P_{i,k} \right\}
  \]
  \[= -\frac{1}{2} D(y_{j,k}, x_{i,k} - h^*(x_{i,k}), S_{i,k} + C_k
  \]

where $C_k = \log(2\pi|S_k|^{1/2})$, $H_k^*$ is the Jacobian matrix of function $h^*$, and the notation $\text{Tr}\{A\}$ is the trace of matrix $A$.

References

[1] Y. Bar-Shalom, X.R. Li, and T. Kirubarajan. Estimation with Application to Tracking and Navigation: Theory Algorithms and Software. New York: Wiley, 2001.
[2] I.J. Cox. Review of statistical data association techniques for motion correspondence. International Journal of Computer Vision, 10(1):53 – 66, 1993.
[3] G. W. Pulford. Taxonomy of multiple target tracking methods. IEEE Proceedings - Radar, Sonar and Navigation, 152(5):291–304, 2005.
[4] K. Granström and M. Baum. Extended object tracking: Introduction, overview and applications. arXiv preprint arXiv:1604.00970 2016.
[5] G.A. Fabrizio. High frequency over-the-horizon radar. McGraw-Hill, 2013.
[6] R. Tharmarasa, M. Subramaniam, N. Nadarajah, T. Kirubarajan, and M. McDonald. Multitarget passive coherent location with transmitter-origin and target-altitude uncertainties. IEEE Transactions on Aerospace and Electronic Systems, 48(3):2530–2550, 2012.
[7] M. Zhou, J. J. Zhang, and A. Papandreou-Suppappola. Multiple target tracking in urban environments. IEEE Transactions on Signal Processing, 64(5):1270–1279, 2016.
[8] L. Li and J. L. Krolik. Simultaneous target and multipath positioning. IEEE Journal of Selected Topics in Signal Processing, 8(1):153–165, 2014.
[9] Q. Yu and G. Medioni. Multiple-target tracking by spatiotemporal Monte Carlo Markov Chain data association. IEEE Transactions on Pattern Analysis and Machine Intelligence, 31(12):2196–2210, 2009.
[10] X. Tang, X. Chen, M. McDonald, R. Maher, R. Tharmarasa, and T. Kirubarajan. A multiple-detection probability hypothesis density filter. IEEE Transactions on Signal Processing, 63(8):2007 – 2019, 2015.
[11] G.W. Pulford and R.J. Evans. Probabilistic data association for systems with multiple simultaneous measurements. Automatica, 32(9):1311 – 1316, 1996.
[12] J.D. Percival and K.A.B. White. Multihypothesis fusion of multipath over-the-horizon radar tracks. In SPIE Conference on Signal and Data Processing of Small Targets, volume 3373, pages 440 – 451, Orlando, Florida, 1998.
[13] M.G. Rutten, D.J. Percival, and M. Rutten. Comparison of track fusion with measurement fusion for multipath OTHR surveillance. In Proceedings of the 4th International Conference on Information Fusion, pages 7 – 10, Montreal, Canada, 2001.
[14] G.W. Pulford and R.J. Evans. A multipath data association tracker for over-the-horizon radar. IEEE Transactions on Aerospace and Electronic Systems, 34(4):1165 – 1183, 1998.
Fig. 10: Path-dependent estimate error and fused estimate error for targets
(x-axis: time step, y-axis: [upper: Ground range (km), lower: Bearing (rad)])
(a) Target 1
(b) Target 2
(c) Target 3
(d) Target 4
(e) Target 5
(f) Target 6

Fig. 11: Path-dependent active probability and fused active probability for targets
(x-axis: time step, y-axis: active probability $p(s = 1)$)
Fig. 12: Path probability
(x-axis: time step, y-axis: path probability)
Fig. 13: Path-dependent active probability and fused active probability for false tracks (x-axis: time step, y-axis: active probability $p(s = 1)$)
Fig. 14: Estimate error with respect to different iterations $r = 1, 5, 20$
(x-axis: time step, y-axis: [upper: Ground range (km), lower: Bearing (rad)])
Fig. 15: Active probability with respect to different iterations $r = 1, 5, 20$
(x-axis: time step, y-axis: active probability $p(s = 1)$)
