Sharp large deviations for the fractional Ornstein-Uhlenbeck process

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Outline

1. Introduction
   - On the Cramer-Chernov theorem
   - On the Bahadur-Rao theorem

2. Main results
   - Gaussian quadratic forms
   - Large deviation principle
   - Sharp large deviation principle

3. Statistical applications
   - Autoregressive process
   - Ornstein-Uhlenbeck process
   - Fractional Ornstein-Uhlenbeck process
The Gaussian example

Let \( (X_n) \) be a sequence of iid \( \mathcal{N}(0, \sigma^2) \) random variables. If

\[
S_n = \sum_{k=1}^{n} X_k
\]

we clearly have \( S_n \sim \mathcal{N}(0, \sigma^2 n) \). Consequently, for all \( c > 0 \)

\[
\mathbb{P}(S_n \geq nc) = \frac{\sigma}{c \sqrt{2\pi n}} \exp\left(-\frac{c^2 n}{2\sigma^2}\right) \left[1 + o(1)\right].
\]

Therefore,

\[
\lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}(S_n \geq nc) = -\frac{c^2}{2\sigma^2}.
\]

**Question.** What about the non-Gaussian case?
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Let \((X_n)\) be a sequence of iid random variables with mean \(m\). The \textbf{Fenchel-Legendre} dual of the log-Laplace \(L\) of \((X_n)\) is

\[
I(c) = \sup_{t \in \mathbb{R}} \{ct - L(t)\}.
\]

\textbf{Theorem (Cramer-Chernov)}

The sequence \((S_n/n)\) satisfies an \textbf{LDP} with rate function \(I\)

- **Upper bound**: for any closed set \(F \subset \mathbb{R}\)

\[
\limsup_{n \to \infty} \frac{1}{n} \log P\left(\frac{S_n}{n} \in F\right) \leq -\inf_{F} I,
\]

- **Lower bound**: for any open set \(G \subset \mathbb{R}\)

\[
\liminf_{n \to \infty} \frac{1}{n} \log P\left(\frac{S_n}{n} \in G\right) \geq -\inf_{G} I.
\]
Let \((X_n)\) be a sequence of iid random variables with mean \(m\). The **Fenchel-Legendre** dual of the log-Laplace \(L\) of \((X_n)\) is

\[
l(c) = \sup_{t \in \mathbb{R}} \{ct - L(t)\}.
\]

**Theorem (Cramer-Chernov)**

The sequence \((S_n/n)\) satisfies an LDP with rate function \(I\)

- **Upper bound:** for any closed set \(F \subset \mathbb{R}\)
  \[
  \limsup_{n \to \infty} \frac{1}{n} \log \mathbb{P} \left( \frac{S_n}{n} \in F \right) \leq - \inf_F I,
  \]

- **Lower bound:** for any open set \(G \subset \mathbb{R}\)
  \[
  \liminf_{n \to \infty} \frac{1}{n} \log \mathbb{P} \left( \frac{S_n}{n} \in G \right) \geq - \inf_G I.
  \]
On the Cramer-Chernov theorem

The rate function $I$ is convex with $I(m) = 0$. For all $c > m$,

$$
\lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}(S_n \geq nc) = -I(c).
$$

- **Gaussian**: If $X_n \sim \mathcal{N}(0, \sigma^2)$ with $\sigma^2 > 0$,

$$
I(c) = \frac{c^2}{2\sigma^2}.
$$

- **Exponential**: If $X_n \sim \mathcal{E}(\lambda)$ with $\lambda > 0$,

$$
I(c) = \begin{cases} 
\lambda c - 1 - \log(\lambda c) & \text{if } c > 0, \\
+\infty & \text{otherwise.}
\end{cases}
$$
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Theorem (Bahadur-Rao)

Assume that \( L \) is finite on all \( \mathbb{R} \) and that the law of \( (X_n) \) is absolutely continuous. Then, for all \( c > m \),

\[
\mathbb{P}(S_n \geq nc) = \frac{\exp(-nl(c))}{\sigma_c t_c \sqrt{2\pi n}} \left[ 1 + o(1) \right]
\]

where \( t_c \) is given by \( L'(t_c) = c \) and \( \sigma_c^2 = L''(t_c) \).

Remark. The core of the proof is the Berry-Esséen theorem.
Theorem (Bahadur-Rao)

\((S_n/n)\) satisfies an SLDP associated with \(L\). For all \(c > m\), it exists \((d_{c,k})\) such that for any \(p \geq 1\) and \(n\) large enough

\[
\mathbb{P}(S_n \geq nc) = \frac{\exp(-nl(c))}{\sigma_c t_c \sqrt{2\pi n}} \left[ 1 + \sum_{k=1}^{p} \frac{d_{c,k}}{n^k} + \mathcal{O}\left(\frac{1}{n^{p+1}}\right) \right].
\]

**Remark.** The coefficients \((d_{c,k})\) may be explicitly calculated as functions of the derivatives \(l_k = L^{(k)}(t_c)\). For example,

\[
d_{c,1} = \frac{1}{\sigma_c^2} \left( \frac{l_4}{8\sigma_c^2} - \frac{5l_3^2}{24\sigma_c^4} - \frac{l_3}{2t_c\sigma_c^2} - \frac{1}{t_c^2} \right).
\]
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Let \((X_n)\) be a centered stationary real Gaussian process with spectral density \(g \in L^\infty(\mathbb{T})\)

\[ E[X_nX_k] = \frac{1}{2\pi} \int_{\mathbb{T}} \exp(i(n - k)x)g(x) \, dx. \]

We are interested in the behavior of

\[ W_n = \frac{1}{n} X^{(n)t} M_n X^{(n)} \]

where \((M_n)\) is a sequence of Hermitian matrices of order \(n\) and

\[ X^{(n)} = \begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix}. \]
Let $T_n(g)$ be the covariance matrix of $X^{(n)}$. Via Cochran,

$$W_n = \frac{1}{n} \sum_{k=1}^{n} \lambda_k^n Z_k^n$$

- $\lambda_1^n, \ldots, \lambda_n^n$ are the eigenvalues of $T_n(g)^{1/2} M_n T_n(g)^{1/2}$
- $Z_1^n, \ldots, Z_n^n$ are iid with $\chi^2(1)$ distribution

The normalized cumulant generating function of $W_n$ is given by

$$L_n(t) = \frac{1}{n} \log \mathbb{E} \left[ \exp(n t W_n) \right] = -\frac{1}{2n} \sum_{k=1}^{n} \log(1 - 2t \lambda_k^n)$$

as soon as $t$ belongs to $\Delta_n = \left\{ t \in \mathbb{R} \mid 2t \lambda_k^n < 1 \right\}$. 
Toeplitz and Cochran

Let $T_n(g)$ be the covariance matrix of $X^{(n)}$. Via Cochran,

$$W_n = \frac{1}{n} \sum_{k=1}^{n} \lambda_k^n Z_k^n$$

- $\lambda_1^n, \ldots, \lambda_n^n$ are the eigenvalues of $T_n(g)^{1/2} M_n T_n(g)^{1/2}$
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The normalized cumulant generating function of $W_n$ is given by

$$L_n(t) = \frac{1}{n} \log \mathbb{E} \left[ \exp (ntW_n) \right] = -\frac{1}{2n} \sum_{k=1}^{n} \log (1 - 2t\lambda_k^n)$$

as soon as $t$ belongs to $\Delta_n = \{ t \in \mathbb{R} / 2t\lambda_k^n < 1 \}$. 

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Fractional Ornstein-Uhlenbeck process
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**LDP assumption.** There exists \( \varphi \in L^\infty(\mathbb{T}) \) not identically zero such that, if \( m_\varphi = \text{essinf} \varphi \) and \( M_\varphi = \text{esssup} \varphi \),

\[
(H_1) \quad m_\varphi \leq \lambda_k^n \leq M_\varphi
\]

and, for all \( h \in C([m_\varphi, M_\varphi]) \),

\[
(H_1) \quad \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} h(\lambda_k^n) = \frac{1}{2\pi} \int_{\mathbb{T}} h(\varphi(x)) \, dx.
\]

Under \((H_1)\), the asymptotic cumulant generating function is

\[
L(t) = -\frac{1}{4\pi} \int_{\mathbb{T}} \log(1 - 2t \varphi(x)) \, dx
\]

where \( t \) belongs to \( \Delta = \{ t \in \mathbb{R} / 2 \max(m_\varphi t, M_\varphi t) < 1 \} \).
The Fenchel-Legendre dual of $L$ is given by

$$I(c) = \sup_{t \in \mathbb{R}} \left\{ ct + \frac{1}{4\pi} \int_T \log(1 - 2t\varphi(x)) \, dx \right\}.$$

Theorem (Bercu-Gamboa-Lavielle)

If $(H_1)$ holds, the sequence $(W_n)$ satisfies an LDP with rate function $I$. In particular, for all $c > \mu$

$$\lim_{n \to \infty} \frac{1}{n} \log P(W_n \geq c) = -I(c).$$

with $\mu = \frac{1}{2\pi} \int_T \varphi(x) \, dx$. 
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**Sharp large deviation results**

**SLDP assumption.** There exists $H$ such that, for all $t \in \Delta$

\[
(H_2) \quad L_n(t) = L(t) + \frac{1}{n} H(t) + o\left(\frac{1}{n}\right)
\]

where the remainder is uniform in $t$.

**Theorem (Bercu-Gamboa-Lavielle)**

Assume that $(H_1)$ and $(H_2)$ hold. Then, for all $c > \mu$

\[
\mathbb{P}(W_n \geq c) = \frac{\exp(-nl(c) + H(t_c))}{\sigma_c t_c \sqrt{2\pi n}} \left[1 + o(1)\right]
\]

where $t_c$ is given by $L'(t_c) = c$ and $\sigma_c^2 = L''(t_c)$. 
**SLDP assumption.** For $p \geq 1$, there exists $H \in C^{2p+3}(\mathbb{R})$ such that, for all $t \in \Delta$ and for any $0 \leq k \leq 2p + 3$

$$(H_2(p)) \quad L_n^{(k)}(t) = L^{(k)}(t) + \frac{1}{n}H^{(k)}(t) + O\left(\frac{1}{n^{p+2}}\right)$$

where the remainder is uniform in $t$.

**Remark.** Assumption $(H_2(p))$ is not really restrictive. It is fulfilled in many statistical applications.
Theorem (Bercu-Gamboa-Lavielle)

For $p \geq 1$, assume that $(H_1)$ and $(H_2(p))$ hold. Then, $(W_n)$ satisfies an SLDP of order $p$ associated with $L$ and $H$. For all $c > \mu$, it exists $(d_{c,k})$ such that for $n$ large enough

$$\mathbb{P}(W_n \geq c) = \frac{\exp(-nl(c) + H(t_c))}{\sigma_c t_c \sqrt{2\pi n}} \left[ 1 + \sum_{k=1}^{p} \frac{d_{c,k}}{n^k} + \mathcal{O} \left( \frac{1}{n^{p+1}} \right) \right].$$

Remark. The coefficients $(d_{c,k})$ may be given as functions of the derivatives $l_k = L^{(k)}(t_c)$ and $h_k = H^{(k)}(t_c)$. For example,

$$d_{c,1} = \frac{1}{\sigma_c^2} \left( -\frac{h_2}{2} - \frac{h_1^2}{2} + \frac{l_4}{8\sigma_c^2} + \frac{l_3 h_1}{2\sigma_c^2} - \frac{5l_3^2}{24\sigma_c^4} + \frac{h_1}{t_c} - \frac{l_3}{2t_c \sigma_c^2} - \frac{1}{t_c^2} \right).$$
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Consider the stable autoregressive process

\[ X_{n+1} = \theta X_n + \varepsilon_{n+1}, \quad |\theta| < 1 \]

where \((\varepsilon_n)\) is iid \(\mathcal{N}(0, \sigma^2)\), \(\sigma^2 > 0\). If \(X_0\) is independent of \((\varepsilon_n)\) with \(\mathcal{N}(0, \sigma^2/(1 - \theta^2))\) distribution, \((X_n)\) is a centered stationary Gaussian process with spectral density given, for all \(x \in \mathbb{T}\), by

\[ g(x) = \frac{\sigma^2}{1 + \theta^2 - 2\theta \cos x}. \]
Let $\hat{\theta}_n$ be the **least squares** estimator of the parameter $\theta$

$$
\hat{\theta}_n = \frac{\sum_{k=1}^{n} X_k X_{k-1}}{\sum_{k=1}^{n} X_{k-1}^2}.
$$

We have $\hat{\theta}_n \xrightarrow{a.s.} \theta$ and $\sqrt{n}(\hat{\theta}_n - \theta) \xrightarrow{L} \mathcal{N}(0, 1 - \theta^2)$. One can also estimate $\theta$ by the **Yule-Walker** estimator

$$
\tilde{\theta}_n = \frac{\sum_{k=1}^{n} X_k X_{k-1}}{\sum_{k=0}^{n} X_k^2}.
$$
\[ a = \frac{\theta - \sqrt{\theta^2 + 8}}{4} \quad \text{and} \quad b = \frac{\theta + \sqrt{\theta^2 + 8}}{4}. \]

**Theorem (Bercu-Gamboa-Rouault)**

- \((\hat{\theta}_n)\) satisfies an **LDP** with rate function
  \[
  J(c) = \begin{cases} 
  \frac{1}{2} \log \left( \frac{1 + \theta^2 - 2\theta c}{1 - c^2} \right) & \text{if } c \in [a, b], \\
  \log |\theta - 2c| & \text{otherwise}. 
  \end{cases}
  \]

- \((\tilde{\theta}_n)\) satisfies an **LDP** with rate function
  \[
  I(c) = \begin{cases} 
  \frac{1}{2} \log \left( \frac{1 + \theta^2 - 2\theta c}{1 - c^2} \right) & \text{if } c \in ]-1, 1[, \\
  +\infty & \text{otherwise}. 
  \end{cases}
  \]
\[ a = \frac{\theta - \sqrt{\theta^2 + 8}}{4} \quad \text{and} \quad b = \frac{\theta + \sqrt{\theta^2 + 8}}{4}. \]

**Theorem (Bercu-Gamboa-Rouault)**

- \((\hat{\theta}_n)\) satisfies an **LDP** with rate function
  \[
  J(c) = \begin{cases} 
  \frac{1}{2} \log \left( \frac{1 + \theta^2 - 2\theta c}{1 - c^2} \right) & \text{if } c \in [a, b], \\
  \log |\theta - 2c| & \text{otherwise}.
  \end{cases}
  \]

- \((\tilde{\theta}_n)\) satisfies an **LDP** with rate function
  \[
  I(c) = \begin{cases} 
  \frac{1}{2} \log \left( \frac{1 + \theta^2 - 2\theta c}{1 - c^2} \right) & \text{if } c \in ]-1, 1[, \\
  +\infty & \text{otherwise}.
  \end{cases}
  \]
Least squares and Yule-Walker
Theorem (Bercu-Gamboa-Lavielle)

The sequence \((\tilde{\theta}_n)\) satisfies an SLDP. For all \(c \in \mathbb{R}\) with \(c > \theta\) and \(|c| < 1\), it exists a sequence \((d_{c,k})\) such that for any \(p \geq 1\) and \(n\) large enough

\[
P(\tilde{\theta}_n \geq c) = \frac{\exp(-nl(c) + H(c))}{\sigma_c t_c \sqrt{2\pi n}} \left[ 1 + \sum_{k=1}^{p} \frac{d_{c,k}}{n^k} + O\left(\frac{1}{n^{p+1}}\right) \right]
\]

where

\[
t_c = \frac{c(1 + \theta^2) - \theta(1 - c^2)}{1 - c^2},
\]

\[
\sigma_c^2 = \frac{1 - c^2}{(1 + \theta^2 - 2\theta c)^2},
\]

\[
H(c) = -\frac{1}{2} \log \left( \frac{(1 - c\theta)^4}{(1 - \theta)^2(1 + \theta^2 - 2\theta c)(1 - c^2)^2} \right).
\]
Yule-Walker

![Graph showing Yule-Walker in the stable case](image)
Yule-Walker

```
YULE WALKER

0.55 0.6 0.65 0.7 0.75 0.8 0.85 0.9
10^-9
10^-8
10^-7
10^-6
10^-5
10^-4
10^-3
10^-2
10^-1

STABLE CASE
```

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Fractional Ornstein-Uhlenbeck process
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Consider the explosive autoregressive process

$$X_{n+1} = \theta X_n + \varepsilon_{n+1}, \quad |\theta| > 1$$

where $\varepsilon_n$ is iid $\mathcal{N}(0, \sigma^2)$, $\sigma^2 > 0$. The Yule-Walker estimator satisfies

$$\hat{\theta}_n \longrightarrow 1/\theta \text{ a.s. together with}$$

$$|\theta|^n (\hat{\theta}_n - \frac{1}{\theta}) \xrightarrow{\mathcal{L}} \frac{(\theta^2 - 1)}{\theta^2} \mathcal{C}$$

where $\mathcal{C}$ stands for the Cauchy distribution.
Explosive autoregressive process

Theorem (Bercu)

The sequence $(\tilde{\theta}_n)$ satisfies an LDP with rate function

$$I(c) = \begin{cases} 
\frac{1}{2} \log \left( \frac{1 + \theta^2 - 2\theta c}{1 - c^2} \right) & \text{if } c \in ]-1, 1[, \ c \neq 1/\theta, \\
0 & \text{if } c = 1/\theta, \\
+\infty & \text{otherwise}. 
\end{cases}$$
The sequence $(\tilde{\theta}_n)$ satisfies an **SLDP**. For all $c \in \mathbb{R}$ with $c > 1/\theta$ and $|c| < 1$, it exists a sequence $(d_{c,k})$ such that for any $p \geq 1$ and $n$ large enough

$$
P(\tilde{\theta}_n \geq c) = \frac{\exp(-nl(c) + H(c))}{\sigma_c t_c \sqrt{2\pi n}} \left[ 1 + \sum_{k=1}^{p} \frac{d_{c,k}}{n^k} + \mathcal{O}\left(\frac{1}{n^{p+1}}\right) \right]
$$

$$
t_c = \frac{(\theta c - 1)(\theta - c)}{1 - c^2}, \quad \sigma_c^2 = \frac{1 - c^2}{(1 + \theta^2 - 2\theta c)^2},
$$

$$
H(c) = -\frac{1}{2} \log \left( \frac{(\theta c - 1)^2}{(1 + \theta^2 - 2\theta c)(1 - c^2)} \right).
$$
Yule-Walker

EXPLOSIVE CASE

YULE WALKER
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Consider the stable Ornstein-Uhlenbeck process

\[ dX_t = \theta X_t \, dt + dB_t, \quad \theta < 0 \]

with initial state \( X_0 = 0 \), where \((B_t)\) is a standard Brownian motion. We are interested in \text{SLDP} for the energy

\[ S_T = \int_0^T X_t^2 \, dt \]

and the maximum likelihood estimator of \( \theta \)

\[ \hat{\theta}_T = \frac{\int_0^T X_t \, dX_t}{\int_0^T X_t^2 \, dt} = \frac{X_T^2 - T}{2 \int_0^T X_t^2 \, dt}. \]
Theorem

We have the SLLN $S_T / T \rightarrow -1/2\theta$ a.s. Moreover, we have the CLT

$$\frac{1}{\sqrt{T}} \left( S_T + \frac{T}{2\theta} \right) \xrightarrow{\mathcal{L}} \mathcal{N} \left( 0, -\frac{1}{2\theta^3} \right).$$

Theorem

We have the SLLN $\hat{\theta}_T \rightarrow \theta$ a.s. Moreover, we have the CLT

$$\sqrt{T} \left( \hat{\theta}_T - \theta \right) \xrightarrow{\mathcal{L}} \mathcal{N} \left( 0, -2\theta \right).$$
The sequence \( (S_T/T) \) satisfies an \textbf{LDP} with rate function

\[
I(c) = \begin{cases} 
\frac{(2\theta c + 1)^2}{8c} & \text{if } c > 0, \\
+\infty & \text{otherwise.}
\end{cases}
\]

The sequence \( (\hat{\theta}_T) \) satisfies an \textbf{LDP} with rate function

\[
I(c) = \begin{cases} 
-\frac{(c - \theta)^2}{4c} & \text{if } c < \theta/3, \\
2c - \theta & \text{otherwise.}
\end{cases}
\]
**Theorem (Bercu-Rouault)**

The sequence \((S_T / T)\) satisfies an SLDP. For all \(c > -1/2\theta\), it exists a sequence \((b_{c,k})\) such that, for any \(p \geq 1\) and \(T\) large enough

\[
P(S_T \geq cT) = \frac{\exp(-TI(c) + H(c))}{\sigma_c t_c \sqrt{2\pi T}} \left[ 1 + \sum_{k=1}^{p} \frac{b_{c,k}}{T^k} + O \left( \frac{1}{T^{p+1}} \right) \right]
\]

where

\[
t_c = \frac{4\theta^2 c^2 - 1}{8c^2}, \quad H(c) = -\frac{1}{2} \log \left( \frac{1}{2} (1 - 2\theta c) \right)
\]

\[
\sigma_c^2 = 4c^3.
\]
The sequence \( \hat{\theta}_T \) satisfies an SLDP. For all \( \theta < c < \theta/3 \), it exists a sequence \( (d_{c,k}) \) such that, for any \( p \geq 1 \) and \( T \) large enough

\[
P^p(\hat{\theta}_T \geq c) = \frac{\exp(-TI(c) + H(c))}{\sigma_c t_c \sqrt{2\pi T}} \left[ 1 + \sum_{k=1}^{p} \frac{d_{c,k}}{T^k} + O \left( \frac{1}{T^{p+1}} \right) \right]
\]

\[
t_c = \frac{c^2 - \theta^2}{2c}, \quad H(c) = -\frac{1}{2} \log \left( \frac{(c + \theta)(3c - \theta)}{4c^2} \right)
\]

\[
\sigma_c^2 = -1/2c. \text{ Similar expansion holds for } c > \theta/3 \text{ with } c \neq 0.
\]
Theorem (Bercu-Rouault)

- For $c = 0$, it exists a sequence $(b_k)$ such that, for any $p \geq 1$ and $T$ large enough

$$
\mathbb{P}(\hat{\theta}_T \geq 0) = \frac{\exp(\theta T)}{\sqrt{\pi T} \sqrt{-\theta}} \left[ 1 + \sum_{k=1}^{p} \frac{b_k}{T^k} + O \left( \frac{1}{T^{p+1}} \right) \right].
$$

- For $c = \theta/3$, it exists a sequence $(d_k)$ such that, for any $p \geq 1$ and $T$ large enough

$$
\mathbb{P}(\hat{\theta}_T \geq c) = \frac{\exp(-Tl(c))}{4\pi T^{1/4} \tau_\theta} \left[ 1 + \sum_{k=1}^{2p} \frac{d_k}{(\sqrt{T})^k} + O \left( \frac{1}{T^{p+1/2}} \right) \right]
$$

where $\tau_\theta = \left(-\theta/3\right)^{1/4}/\Gamma(1/4)$. 
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Consider the fractional Ornstein-Uhlenbeck process

\[ dX_t = \theta X_t \, dt + dB_t^H, \quad \theta < 0 \]

where \((B_t^H)\) is a fractional Brownian motion with Hurst parameter \(0 < H < 1\), \((B_t^H)\) is a Gaussian process with continuous paths such that \(B_0^H = 0\), \(\mathbb{E}[B_t^H] = 0\) and

\[ \mathbb{E}[B_t^H B_s^H] = \frac{1}{2} \left( t^{2H} + s^{2H} - |t - s|^{2H} \right). \]

The weighting function

\[ w(t, s) = w_H^{-1} s^{-H+1/2} (t - s)^{-H+1/2} \]

plays a crucial role for stochastic calculus associated with \((B_t^H)\).
A Gaussian martingale

For all $t > 0$ and $H > 1/2$, let

$$M_t = \int_0^t w(t, s) \, dB^H_s.$$ 

Then, $(M_t)$ is a Gaussian martingale with quadratic variation

$$\langle M \rangle_t = \frac{t^{2-2H}}{\lambda_H}$$

$$\lambda_H = \frac{8H(1 - H)\Gamma(1 - 2H)\Gamma(H + 1/2)}{\Gamma(1/2 - H)}$$

where $\Gamma$ stands for the classical gamma function.
For all $t > 0$, let

$$Y_t = \int_0^t w(t, s) \, dX_s$$

$$Q_t = \frac{\ell H}{2} \left( t^{2H-1} Y_t + \int_0^t s^{2H-1} \, dY_s \right)$$

where $\ell_H = \lambda_H / (2(1 - H))$. The energy is given by

$$S_T = \int_0^T Q_t^2 \, d< M >_t$$

while the maximum likelihood estimator of $\theta$ is

$$\hat{\theta}_T = \frac{\int_0^T Q_t \, dY_t}{\int_0^T Q_t^2 \, d< M >_t}.$$
Theorem

We have the **SLLN** $S_T/T \to -1/2\theta$ a.s. Moreover, we have the **CLT**

$$\frac{1}{\sqrt{T}} \left( S_T + \frac{T}{2\theta} \right) \xrightarrow{\mathcal{L}} \mathcal{N} \left( 0, -\frac{1}{2\theta^3} \right).$$

Theorem

We have the **SLLN** $\hat{\theta}_T \to \theta$ a.s. Moreover, we have the **CLT**

$$\sqrt{T} \left( \hat{\theta}_T - \theta \right) \xrightarrow{\mathcal{L}} \mathcal{N} \left( 0, -2\theta \right).$$
The energy

**Theorem (Bercu-Coutin-Savy)**

The sequence \( (S_T / T) \) satisfies an **LDP** with rate function

\[
I(c) = \begin{cases} 
\frac{(2\theta c + 1)^2}{8c} & \text{if } 0 < c \leq -\frac{1}{2\theta \delta_H}, \\
\frac{c\theta^2}{2}(1 - \delta_H^2) + \frac{\theta}{2}(1 - \delta_H) & \text{if } c \geq -\frac{1}{2\theta \delta_H}, \\
+\infty & \text{otherwise.}
\end{cases}
\]

where \( \delta_H = (1 - \sin(\pi H))/(1 + \sin(\pi H)) \).

**Remark.** In the particular case \( H = 1/2, \delta_H = 0 \) and the **LDP** for \( (S_T / T) \) is exactly the one established by Bryc and Dembo.
Theorem (Bercu-Coutin-Savy)

The sequence \( (S_T/T) \) satisfies an SLDP. For all \( c > -1/(2\theta) \) with \( c < -1/(2\theta\delta_H) \), it exists a sequence \( (b_{c,k}^H) \) such that, for any \( p > 0 \) and \( T \) large enough,

\[
\mathbb{P}(S_T \geq cT) = \frac{\exp(-TI(c) + J(c) + K_H(c))}{\sigma_c t_c \sqrt{2\pi T}} \left[ 1 + \sum_{k=1}^{p} \frac{b_{c,k}^H}{T^k} + O \left( \frac{1}{T^{p+1}} \right) \right]
\]

where \( \sigma_c^2 = 4c^3 \),

\[
t_c = \frac{4\theta^2 c^2 - 1}{8c^2}, \quad J(c) = -\frac{1}{2} \log \left( \frac{1 - 2\theta c}{2} \right),
\]

\[
K_H(c) = -\frac{1}{2} \log \left( \frac{(1 + \sin(\pi H))(1 + 2\theta c\delta_H)}{2 \sin(\pi H)} \right).
\]
Theorem (Bercu-Coutin-Savy)

For all \( c > -1/(2\theta \delta_H) \), it exists a sequence \( (d_{c,k}^H) \) such that, for any \( p > 0 \) and \( T \) large enough,

\[
P(S_T \geq cT) = \frac{\exp(-TL(c) + P_H(c) + Q_H(c))}{\sigma_H t_H \sqrt{2\pi T}} \left[ 1 + \sum_{k=1}^{p} \frac{d_{c,k}^H}{T^k} + O \left( \frac{1}{T^{p+1}} \right) \right]
\]

where \( \sigma_H^2 = -1/2\theta^3 \delta_H^3 \),

\[
t_H = \frac{\theta^2(1 - \delta_H^2)}{2}, \quad P_H(c) = -\frac{1}{2} \log \left( \frac{-(1 + 2\theta c \delta_H)}{4\delta_H \sin(\pi H)} \right),
\]

\[
Q_H(c) = \frac{(2H - 1)^2 \sin(\pi H)(1 + 2\theta c \delta_H)}{2(1 - (\sin(\pi H))^2)}.
\]
Theorem (Bercu-Coutin-Savy)

For \( c = -1/(2\theta \delta_H) \), it exists a sequence \((d^H_k)\) such that, for any \( p > 0 \) and \( T \) large enough

\[
P(S_T \geq cT) = \exp(-TL(c) + K_H) \Gamma(1/4) \frac{2\pi \sigma_H t_H T^{1/4}}{2 \pi \sigma_H t_H T^{1/4}}
\]

\[
1 + \sum_{k=1}^{2p} \frac{d^H_k}{(\sqrt{T})^k} + \mathcal{O}\left(\frac{1}{T^p \sqrt{T}}\right)
\]

where

\[
K_H = \frac{1}{2} \log(\delta_H \sin(\pi H)) + \frac{1}{4} \log(-\theta \delta_H).
\]
Theorem (Bercu-Coutin-Savy)

The sequence \( (\hat{\theta}_T) \) satisfies an LDP with rate function

\[
l(c) = \begin{cases} 
-\frac{(c - \theta)^2}{4c} & \text{if } c < \frac{\theta}{3}, \\
2c - \theta & \text{otherwise}.
\end{cases}
\]

**Remark.** One can observe that \( (\hat{\theta}_T) \) shares the same LDP than the one established by Florens-Landais and Pham for \( H = 1/2 \).
The sequence \((\hat{\theta}_T)\) satisfies an SLDP. For all \(\theta < c < \theta/3\), it exists a sequence \(b^H_{c,k}\) such that, for any \(p > 0\) and \(T\) large enough,

\[
\mathbb{P}(\hat{\theta}_T \geq c) = \frac{\exp(-TI(c) + J(c) + K_H(c))}{\sigma_c t_c \sqrt{2\pi T}} \\
\left[1 + \sum_{k=1}^{p} \frac{b^H_{c,k}}{T^k} + O \left(\frac{1}{T^{p+1}}\right)\right]
\]

where \(\sigma^2_c = -1/2c\), \(p_H = (1 - \sin(\pi H))/\sin(\pi H)\),

\[
t_c = \frac{c^2 - \theta^2}{2c}, \quad J(c) = -\frac{1}{2} \log \left(\frac{(c + \theta)(3c - \theta)}{4c^2}\right),
\]

\[
K_H(c) = -\frac{1}{2} \log \left(1 + p_H \frac{(c - \theta)^2}{4c^2}\right).
\]
Theorem (Bercu-Coutin-Savy)

For all $c > \theta/3$ with $c \neq 0$, it exists a sequence $(d_{c,k}^H)$ such that, for any $p > 0$ and $T$ large enough,

$$
\mathbb{P}(\hat{\theta}_T \geq c) = \frac{\exp(-TL(c) + P(c)) \sqrt{\sin(\pi H)}}{\sigma^c t^c \sqrt{2\pi T}} \left[ 1 + \sum_{k=1}^{p} \frac{d_{c,k}^H}{T^k} + O \left( \frac{1}{T^{p+1}} \right) \right]
$$

where

$$
t^c = 2(c - \theta), \quad (\sigma^c)^2 = \frac{c^2}{2(2c - \theta)^3},
$$

$$
P(c) = -\frac{1}{2} \log \left( \frac{(c - \theta)(3c - \theta)}{4c^2} \right)
$$
Theorem (Bercu-Coutin-Savy)

For \( c = 0 \), it exists a sequence \( (b^H_k) \) such that, for any \( p \geq 1 \) and \( T \) large enough,

\[
P(\hat{\theta}_T \geq 0) = \frac{\exp(\theta T) \sqrt{\sin(\pi H)}}{\sqrt{\pi} T \sqrt{-\theta}} \left[ 1 + \sum_{k=1}^{p} \frac{b^H_k}{T^k} + \mathcal{O}\left( \frac{1}{T^{p+1}} \right) \right].
\]

For \( c = \theta/3 \), it exists a sequence \( (d^H_k) \) such that, for any \( p \geq 1 \) and \( T \) large enough, and \( \tau_\theta = (-\theta/3)^{1/4}/\Gamma(1/4) \),

\[
P(\hat{\theta}_T \geq c) = \frac{\exp(-TL(c)) \sqrt{\sin(\pi H)}}{4\pi T^{1/4} \tau_\theta} \left[ 1 + \sum_{k=1}^{2p} \frac{d^H_k}{(\sqrt{T})^k} + \mathcal{O}\left( \frac{1}{T^{p} \sqrt{T}} \right) \right].
\]