Abstract

From a pseudo-triangulation with \( n \) tetrahedra \( T \) of an arbitrary closed orientable connected 3-manifold (for short, a 3D-space) \( M^3 \), we present a gem \( J' \), inducing \( S^3 \), with the following characteristics: (a) its number of vertices is \( O(n) \); (b) it has a set of \( p \) pairwise disjoint couples of vertices \( \{u_i, v_i\} \), each named a twistor; (c) in the dual \( (J')^* \) of \( J' \) a twistor becomes a pair of tetrahedra with an opposite pair of edges in common, and it is named a hinge; (d) in any embedding of \( (J')^* \subset S^3 \), the \( \epsilon \)-neighborhood of each hinge is a solid torus; (e) these \( p \) solid tori are pairwise disjoint; (f) each twistor contains the precise description on how to perform a specific surgery based in a Denh-Lickorish twist on the solid torus corresponding to it; (g) performing all these \( p \) surgeries (at the level of the dual gems) we produce a gem \( G' \) with \( |G'| = M^3 \); (h) in \( G' \) each such surgery is accomplished by the interchange of a pair of neighbors in each pair of vertices: in particular, \( |V(G')| = |V(J')| \).

This is a new proof, based on a linear polynomial algorithm, of the classical Theorem of Wallace (1960) and Lickorish (1962) that every 3D-space has a framed link presentation in \( S^3 \) and opens the way for an algorithmic method to actually obtaining the link by an \( O(n^2) \)-algorithm. This is the subject of a companion paper soon to be released.

1 Motivation

There exists a rather simple algorithm to go from a framed link inducing a space to a triangulation of the same space. This was first done in chapter 11 of [3] via graph encoded 3-manifolds or gems. This algorithm was improved and incorporated to the computational system BLINK, in [5]. Thus to get a blackboard framed link from a gem is a direct task. However, the contrary, given a gem to find by a polynomial algorithm a blackboard framed link inducing the same 3D-space is, as far as we know, an untouched problem in the literature. The reason why it is desirable to have such an algorithm stems from the fact that the quantum invariants are not computable from a triangulation based presentation of 3D-spaces. The two languages, triangulations and blackboard framed links have at present only a one way translation. This paper starts to fix the situation by providing a linear algorithm to prove the Lickorish-Wallace Theorem.
2 An overview of the Algorithm

Let $T$ be a pseudo triangulation inducing a 3D-space $M^3$. From it we construct the sequence

$$T \xrightarrow{\text{barycentric subdivision}} T' \xrightarrow{\text{dual gem}} G_0 \xrightarrow{\text{dipole cancelations}} G_1 \xrightarrow{\text{2-dipole creations}} G_2 \xrightarrow{\text{twisters}} G \xrightarrow{\text{localizing hinges in } X} G'. $$

The first passage is just the barycentric subdivision of $T$ producing $T'$. The barycentric subdivision of a 3-pseudo-complex has its 0-simplexes naturally colored by the dimensions it represents, namely, 0, 1, 2, 3, so that each tetrahedron receives the four colors. Define the color of a face of a tetrahedron to be the color of its opposite vertex. The second passage is given by dualization. The 1-skeleton of the dual cell complex $(T')^*$ of $T'$ is a graph $G_0$ so that its vertices and edges are 0-cells and 1-cells of $(T')^*$. The edges inherits the color of its dual triangular 2-face. This coloring of edges makes $G_0$ into a gem and from this colored 1-skeleton we can recover $T'$, whence $M^3$, see [6]. In the remaining passages we get gems $G_1$, $G_2$, $G$ and $G'$, all inducing $M^3$. From gem $G'$ we will get the gem $J'$ inducing $S^3$ by very simple local moves named $ji$-twists, which maintains the number of vertices.

There is an operation on gems named $k$-dipole ($k = 1, 2$) cancelation which together with its inverse the $k$-dipole creation is capable of linking any two gems inducing the same 3-manifold [2]. A gem without $1$-dipoles is called a crystallization. The following sequence $G_1, G_2, G$ are crystallizations. A $2$-dipole cancelation or creation in a crystallization yields a crystallization.

There is an operation on gems named $k$-dipole ($k = 1, 2$) cancelation which together with its inverse the $k$-dipole creation is capable of linking any two gems inducing the same 3-manifold [2], [9]. The cancelation of a dipole does not change the induced 3-manifold and decreases by two the number of vertices of a gem. Therefore in the passage $G_0 \rightarrow G_1$ we simplify the gem so that it has no $k$-dipoles, $k = 1, 2$. A gem without 1-dipoles is called a crystallization. In the above sequence $G_1, G_2, G$ are crystallizations. A 2-dipole cancelation or creation in a crystallization yields a crystallization.

The objective to be achieved in $G_2$ and the passage $G_1 \rightarrow G_2$ become transparents in the language of thin gems [11], here rebaptized knits. So, in this passage we start by switching from the gem language to the knit language, obtaining a knit $N_1$ from $G_1$. From $N_1$ we produce a spiked cactus knit $N_2$ having enough properties to meet our purposes. Then we switch back to the gem language by obtaining a gem $G_2$ from $N_2$. The passage $G_1 \rightarrow G_2$ is entirely obtained by 2-dipole creations. It increases the number of vertices, but as we shall see, $|V(G_2)| < 6 |V(G_1)|$. The important aspect of the spiked cactus gem $G_2$ (corresponding to a spiked cactus knit) is that a certain graph $G_2^{jk}$, defined from it, is connected. At this point we fix a spanning tree $X$ of $G_2^{jk}$, $|X| = p + 1$. Each one of the $p$ edges of $X$ corresponds in $G_2$ to a $ji$-pre-twistor. Again, all pertinent definitions are given in the next section.

To get the passage $G_2 \rightarrow G$ we start by replacing each edge of $X$ which corresponds to a $ji$-pre-twistor by an adequate number of parallel edges, each of which corresponding to a $j$-twistor. This can be done simply by creating special 2-tisters, the so called double-8-moves. After these moves all the edges of $X$ correspond in the dual $G^*$ of $G$ to pairs of tetrahedra with an opposite pair of vertices in common, named a hinge. The interior of these pairs of tetrahedra are pairwise disjoint. The final passage $G \rightarrow G'$ is the localization of the hinges and produce truly pairwise disjoint solid tori. It is accomplished by a local complexifying move that replaces a pair of vertices by a fixed configuration of 34 vertices. See Figure [3] in this work, as a consequence of the above passages, we prove
the following Theorem:

\[(2.1) \text{Theorem.}\] There is an algorithm which obtains, given a pseudo-triangulation \(T\) with \(n\) tetrahedra, \(|T| = M^3\), a gem \(J\), having \(O(n)\) vertices \(|J'| = S^3\), and a \(p\)-set of pairs of vertices \(H_\ell = \{u_\ell, v_\ell\}\), \(\ell = 1, \ldots, p\), \(\ell \neq \ell' \Rightarrow H_\ell \cap H_{\ell'} = \emptyset\) of \(J\) such that the \(\epsilon\)-neighborhood of \(H_\ell\) is a \(p\)-set of pairwise disjoint solid tori embedded in \(S^3\). Moreover, the \(ji\)-twists of the \(p\) \(k\)-twisters \(H_\ell = \{u_\ell, v_\ell\}\) produce a gem \(G'\), \(|V(G')| = |V(J')|\) with \(|G'| = M^3\). The framing of each component of the link induced by the collection of solid tori is the linking number of the two boundary component of a specific cylinder, the strip \(s_{uvij}\), obtained from the \(ji\)-twist of the \(\ell\)-th \(j\)-twistor.

This is a strengthening of the classical result of Wallace \[11\] and Lickorish \[4\] and its proof relies on a linear algorithm. It opens the way for an algorithmic method for actually obtaining the link by an \(O(n^2)\)-algorithm. This is the subject of a companion paper \[8\] currently under preparation: it awaits a proper computer implementation.

3  
Twistors, antipoles and their weaker versions

Let \((i, j, k)\) be a permutation of the non-null of colors of a bipartite gem \((1, 2, 3)\). An \(i\)-twistor in \(G\) is a pair of vertices of the same class of the bipartition \(\{u, v\}\) which are in the same 0i- and 0k-gon, in distinct 0j-, ik-, 0k- and \(ij\)-gons. In the dual \(G^*\) of the gem, an \(i\)-twistor becomes a pair of tetrahedra \(t_u \cup t_v\) with an opposite pair of edges in common, namely the pair of edges corresponding to the 0i- and 0k-gons containing both \(u\) and \(v\). Such an structure is named an \(i\)-hinge. If \(u\) and \(v\) satisfy all the connectivity conditions but are in distinct class of the bipartition, then \(\{u, v\}\) is called an \(i\)-antipole. The \(ij\)-twist of an \(i\)-twistor is the operation of exchanging the \(i\)- and \(j\)-colored neighbors of \(u\) and \(v\). A 3-residue in a gem is a connected component of a subgraph of \(G\) induced by three of the four colors. An \(ij\)-twist is an internal operation in the class of gems which does not change its number of 3-residues. We also have the following proposition.

\[(3.1) \text{Proposition.}\] For \(i, j \in \{1, 2, 3\}, i \neq j\), in dual terms of hinges an \(ij\)-twist of an \(i\)-twistor in a gem inducing \(M^3\) is a Dehn-Lickorish surgery in \(M^3\) \[11\].

\textbf{Proof.} We refer to Figure \[1\]. For a small \(\epsilon\) denote \(S_{uvij}\) and \(S'_{uvij}\) the solid tori which are \(\epsilon\)-neighborhoods of the hinges \(t_u \cup t_j\) and \(t'_u \cup t'_j\). The two \(i\)-colored and two \(j\)-colored faces of the hinge \(t_u \cup t_j\) is topologically a cylinder formed by four triangles which we call \textit{strip} denoting it by \(s_{uvij} = t_{ui} \cup t_{vi} \cup t_{uj} \cup t_{vj}\). Let \(\alpha\) be the closed curve in the boundary of \(S_{uvij}\) which goes “just above” at an \(\epsilon\)-distance of the medial curve of \(s_{uvij}\). Let \(\beta\) be the boundary of a meridian curve in \(S'_{uvij}\). The Dehn-Lickorish surgery is defined by attaching \(S'_{uvij}\) to the toroidal hole formed by the removal of \(S_{uvij}\) in such a way as to identify the curves \(\alpha\) and \(\beta\). Indeed, the only data needed to perform the surgery is a projection of the curve \(\alpha\) from \(M^3\) to a plane with the information of under and over passes. This projection becomes a blackboard framed knot and its framing is given by the linking number of the two components of the the strip \(s_{uvij}\). This number can be computed from a general position projection of the strip in a plane (again keeping the information of under and over passes). \(\square\)
Figure 1: $ij$-surgery on a $i$-hinge: replacement of $t_u \cup t_v$, by $t'_u \cup t'_v$.

For the proof of Theorem 2.1 we need a weakening of the concepts of twistors and antipoles. A $ji$-pre-twistor is a pair of vertices in the same class which are in the same $0j$- and $ik$-gon and in distinct $0i$- and $jk$-gons. If $u$ and $v$ satisfy these connectivity conditions but are in distinct classes, then the pair $\{u, v\}$ is called a $ji$-pre-antipole. An $i$-pair in a bipartite gem $G$ is a pair of vertices $\{u, v\}$ which are either a $ji$-pre-twistor, a $ki$-pre-twistor, a $ji$-pre-antipole or a $ki$-pre-antipole. Given a gem $G$ denote by $G^{jk}$ the graph whose vertices are the $jk$-gons of $G$ and the edges are the $i$-pairs of $G$. The ends of the edge corresponding to an $i$-pair $\{u, v\}$ are the vertices corresponding to the $jk$-gons which contain $u$ and $v$. In particular, $G^{jk}$ may have parallel edges but not loops.

The only property that we need in the fourth passage, $G_1 \rightarrow G_2$ is to get by 2-dipole creations a crystallization $G_2$ so that $G_2^{jk}$ is connected. We state the following Conjecture. In it 94-clusters are the 4-clusters of in [5], where is also defined a rigid gem: a crystallization whose 3-residues are 1-skeletons of polytopes. Loosely speaking a 94-cluster consist of a configuration of 9-vertices which has a central vertex incident to four square bigons. A 94-cluster implies a simplification: the 9 vertices become 7 without changing on the induced 3D-space. In a rigid gem each pre-twistor (resp. pre-antipole) is indeed a twistor (resp. an antipole).

(3.2) **Conjecture.** If $G$ is a rigid 3-gem without 94-clusters then $G^{jk}$ is connected for every choice of distinct $j$ and $k$ in $\{1, 2, 3\}$.

Unfortunately we have been unable so far to prove this deep structural property of gems and so we were led to use the spiked cactus construction, which follows. This construction
is needed only if $G_{1}^{jk}$ is not connected, otherwise we take $G_{2} = G_{1}$.

Figure 2: A knit inducing $S^1 \times S^1 \times S^1$ and its “cactification”

4 Knits and spiked cactus knits

We rename knit to be the object thin gem introduced for dimension 3 in [7] and its generalization for dimension 4 given in [1]. A knit is a plane bipartite graph with a perfect match of the sides of the edges. A $j$-knit $N$ is a knit obtained from a gem $G$ with edge colors $0, i, j, k$ as follows. Consider the 0-missing 3-residues of $G$ embedded in the plane so that the exterior faces is an $ik$-gon. Choose a representative white vertex to be an interior point of each $ij$-gon and a representative black vertex to be an interior point of each $jk$-gon. The black and white vertices are the vertices of a bipartite graph $N$. Link a black vertex to a white vertex by a curve so as to cross $G$ only once transversally in the interior of the $k$-colored edge of $G$ which separates the $ij$-gon and the $jk$-gon corresponding to the vertices. These curves are the edges of $N$. Make the cyclic order of these linking edges emanating from any vertex of $N$ to coincide with the cyclic edge of the dual $j$-colored edges in the corresponding $ij$-gon or $jk$-gon. Thus $N$ becomes a graph embedded in the plane. The vertices of $G$ are in $1 - 1$ correspondence with the sides of the edges $N$. These sides are matched by the 0-colored edges of $G$ and thus, $N$ is a knit. A gem $G$ is recoverable from its knit $N$: the angles of the knit are the $i$- and $k$-colored edges of the gem; the two sides of the edges of $N$ become the $j$-colored edges of the gem. If $G$ is a crystallization, $N$ is connected.

A cactus $j$-knit is one formed by a tree-like arrangement of polygons and single edges. In a cactus $j$-knit every edge is incident to the external face (the only white face — all the others in light gray). It is always possible to go from any knit $N_{1}$ to a cactus knit $N'_{2}$ by means of a sequence of trivial angles creations. See the right part of Figure 2. Each such operation is the creation of a 2-dipole at the gem level. We need our knit $N_{2}$ to have an extra property, namely, there should not be trivial angles at the black vertices (which correspond to the $jk$-gons of the associated gem.) This is not the case of the cactus knit of Figure 2. By extra trivial angle creations we get easily, in general, a cactus knit $N''_{2}$ which meets the extra property.

To go from $N''_{2}$ to our desired spiked cactus knit $N_{2}$ we create two spikes (two pendant edges) “trisecting” each 0-edge. There are three cases on how this must be done and
they are depicted on Figure 3. The creation of each spike is a 2-dipole creation in the gem language. In Figure 3 the black vertices with a white spot correspond to black monovalent vertices corresponding, in the gem, to $jk$-gons with two vertices. In cases 1A and 1B we might without loss of generality suppose that by going around the boundary of the external face and in the clockwise direction the directed edge $wx$ is at the left of the directed edge $yz$ and that the vertices $x$ and $y$ are white (they might coincide). Otherwise, they are black and $yz$ is at the left of $wx$ and $a$ and $w$ are white (in the complementary circular path). Interchanging the labels of $(w, y)$ and $(x, z)$ we get the assumption holding. For case 2 there is no changing in the argument if left and right are interchanged. Here is the crucial property of spiked cactus knits.

\[
\text{case 1A}
\]

\[
\text{case 1B}
\]

\[
\text{case 2}
\]

Figure 3: Spikes: \{a, c\}, \{b, d\}, \{c, e\}, \{d, f\} and \{a, b\}, \{b, c\}, \{c, d\} are $ji$-pre-twistors

(4.1) Lemma. Let $N_2$ be a spiked cactus knit of a crystallization $G_2$. Then every pair $(u, v)$ of edge sides in $N_2$ corresponding to a 0-colored edge $(u, v)$ in $G_2$ implies an edge of $G_2^{jk}$ linking the vertices corresponding to the $jk$-gons containing the vertices $u$ and $v$ of $G$. As a consequence, since $G_2$ is a crystallization, $G_2^{jk}$ is connected.

Proof. The proof follows from the facts that \{a, c\}, \{b, d\}, \{c, e\}, \{d, f\} are $ji$-pre-twistors in the first two cases of Figure 3 and the same is true for \{a, b\}, \{b, c\}, \{c, d\} in the third case. The connectivity conditions making these pairs pre-twistors are easily checked in the knit language.

In consequence of Lemma, $G_2$ induces a connected $G_2^{jk}$. Suppose that $|V(G_2^{jk})|=p+1$ and let $X$ be a subset of $p$ edges of $V(G_2^{jk})$ forming a spanning tree.
5 Pairwise disjoint solid tori corresponding to $X$

We turn back to the language of gems. The initial step in the passage $G_2 \rightarrow G$ is very simple. For each $ji$-pre-twistor in $X$ which is not a $j$-twistor we fix the situation by creating two 2-dipoles near $v$ (a double-8-move), as depicted in Figure 4. Note that after this move the vertices $u$ and $v$ are in distinct $0k$-gons and $ij$-gons. Thus $\{u, v\}$ becomes a $j$-twistor. A double-8-move replaces a twistor edge in $G$ by three twistor edges in parallel. Recall that each edge in $G^{jk}$ is labeled by a pair of vertices forming a pre-twistor or a pre-antipole. Actually the edges in $X$ correspond to pre-twistors. By repeating enough double-8-moves at appropriate vertices we might suppose that in gem $G$ the $p$ edges of $X$ have distinct $2p$ labels and that each pair of labels is a $j$-twistor.

Figure 4: Getting a $j$-twistor from a $ji$-pre-twistor by a $k$-double-8-move at $v$

Figure 5: Trisecting a bi-colored polygon in a gem

From now on we refer to Figure 6. In the passage $G \rightarrow G'$ we effect the process of localizing the hinges. This means that in $(G')^*$ the set of $p$ pairs of tetrahedra corresponding to $X$ are pairwise (entirely) disjoint. The passage $G \rightarrow G'$ is effected “at the spanning tree $X$ of twistors” replacing each pair of vertices of each twistor by the configuration of 34 vertices depicted in the upper right part of the figure. The localization moves transform each twistor in $X$ into a local object in the sense that any order to perform the twists in $X$ produce only gems and arrive at gem $J$: the $ji$-twistor $\{u_\ell, v_\ell\}$ does not disturb the other $j$-twistors $\{u_{\ell'}, v_{\ell'}\}$, $\ell' \neq \ell$. This “2 by 34 replacement” is essential for our work. The detailed proof that $G'$ simplifies to $G$ by dipole cancelations is given in the Appendix.
Gem $G'$ induces $M^3$ and we prove that performing the $ji$-twists at the whole set $X$ we get a gem $J'$ which induces $S^3$. We observe that to go from $J'$ to $G'$ we perform the inverse operation, namely, the $ij$-twist at $X$. It is rather easy to prove that gems $J''$ and $J$ (see Figure 6) induce the same space: from $J$ to $J''$ we have four 2-dipole
creations. From $J''$ to $J'$ we trisect (see Figure 5) the 0i-gon and the $jk$-gon incident to $u$ and $v$. These moves are factorable as 1-dipoles and 2-dipole creations. Finally, it is straightforward to prove that $J$ induces $S^3$ because it is a crystallization having a unique $jk$-gon: from $G$ we can arrive to $J$ directly by $p ji$-twists and each one of these decreases by one the number of $jk$-gons. In any crystallization the number of 0i-gons and of $jk$-gons coincide.

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6 Appendix: proof that $G' \longrightarrow G$ by dipole moves
Figure 7: Getting $G$ from $G'$: (1/12) canceling a \{k\}-dipole

Figure 8: Getting $G$ from $G'$: (2/12) canceling a \{0\}-dipole
Figure 9: Getting $G$ from $G'$: (3/12) canceling a $\{j\}$-dipole

Figure 10: Getting $G$ from $G'$: (4/12) canceling an $\{i\}$-dipole
Figure 11: Getting $G$ from $G'$: (5/12) canceling two $\{0,k\}$-dipoles

Figure 12: Getting $G$ from $G'$: (6/12) canceling $\{0,i\}$- and $\{j,k\}$-dipoles. No more dipoles
Figure 13: Getting \( G \) from \( G' \): (7/12) aligning move \( T S_4 \) (which factors by dipoles, see pag. 135 of [6])

{e,x} is \{j,k\}-dipole
{f,y} is \{0,i\}-dipole

Figure 14: Getting \( G \) from \( G' \): (8/12) canceling \{0,i\}-dipole and \{j,k\}-dipole
Figure 15: Getting $G$ from $G'$: (9/12) aligning the drawing

Figure 16: Getting $G$ from $G'$: (10/12) canceling $\{0, j\}$-dipole
Figure 17: Getting $G$ from $G'$: (11/12) canceling \{0, k\}-dipole

Figure 18: Getting $G$ from $G'$: (12/12) 2-dipole cancelations and $u, v$ label interchange