Uniqueness of Some Weak Solutions for 2D Viscous Primitive Equations

Ning Ju

Communicated by M. Hieber

Abstract. First, a new sufficient condition for uniqueness of weak solutions is proved for the system of 2D viscous Primitive Equations. Second, global existence and uniqueness are established for several classes of weak solutions with partial initial regularity, including but not limited to those weak solutions with initial horizontal regularity, rather than vertical regularity. Moreover, new and improved regularity properties of weak solutions and strong solutions to the system of 2D viscous Primitive Equations are obtained. Our results and analyses for the problem with physical boundary conditions can be extended to those with other typical boundary conditions. Most of the results were not available before, even for the periodic case.

Mathematics Subject Classification. 35A01, 35A02, 35B40, 35Q10, 35Q35, 35Q86.

Keywords. Viscous primitive equations, Existence, Uniqueness.

Contents

1. Introduction 1
2. Preliminaries 5
3. Weak Solutions and Strong Solutions 8
4. A Sufficient Condition for Uniqueness 16
5. Global Existence 19
6. Uniqueness 26
References 28

1. Introduction

Consider, in the following two dimensional spatial domain

$$D := \{(x, z) \in \mathbb{R}^2 \mid 0 < x < 1, -h < z < 0\}$$,

with $h$ chosen as a positive constant\(^1\) in this paper, the following system of 2D viscous Primitive Equations (PE) for the three dimensional Geophysical Fluid motion:

*Horizontal momentum equations:*

$$\frac{\partial u}{\partial t} + (u, w) \cdot \nabla u = -\frac{\partial p}{\partial x} + v + \Delta u,$$

$$\frac{\partial v}{\partial t} + (u, w) \cdot \nabla v = -u + \Delta v.$$

\(^1\)This assumption is for simplicity of discussion only. The main results of this paper can be extended to the more general case for $h$ to be a positive function of $x \in [0, 1]$ with suitable assumptions on $h$ similar to those considered in some previous articles, e.g. [23]. The details in this aspect will be discussed elsewhere.
Hydrostatic balance:
\[ \frac{\partial p}{\partial z} + \theta = 0. \]

Continuity equation:
\[ \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0. \]

Heat equation:
\[ \frac{\partial \theta}{\partial t} + (u, w) \cdot \nabla \theta = \Delta \theta + Q. \]

In the above equations, the gradient \( \nabla \) and the Laplacian \( \Delta \) are defined as:
\[ \nabla := (\partial_x, \partial_z) = (\partial_1, \partial_2), \quad \Delta := \partial_x^2 + \partial_z^2 = \partial_1^2 + \partial_2^2. \]

The unknowns in the above system of 2D viscous PE are the fluid velocity \((u, v, w) \in \mathbb{R}^3\), the pressure \(p\) and the temperature \(\theta\). The heat source \(Q\) is given. For issues concerned in this article and for simplicity of presentation, \(Q\) is assumed to be independent of \(t\). Upon minor modifications, all the results obtained in this article can be extended to the case for time-dependent \(Q\) under suitable assumptions for \(Q\). Some of the coefficients in the above system are also simplified. In particular, viscosity constant, diffusivity constant and Coriolis rotational frequency from \(\beta\)-plane approximation are set as 1. The effect of salinity is omitted as well. All the above simplifications for conciseness of presentation sacrifice no mathematical generality for the main results of this paper.

The boundary \(\partial D\) of \(D\) is partitioned into three parts \(\Gamma_t \cup \Gamma_b \cup \Gamma_l\), where
\[ \Gamma_t := \{(x, 0) \in \overline{D}\}, \quad \Gamma_b := \{(x, -h) \in \overline{D}\}, \quad \Gamma_l := \{(x, z) \in \overline{D} | x = 0, 1\}. \]

The following boundary conditions of the PEs are used:
- on \(\Gamma_t\): \(u_z + \alpha_1 u = v_z + \alpha_2 v = w = \theta_z + \alpha_3 \theta = 0\),
- on \(\Gamma_b\): \(u = v = w = \theta_z = 0\),
- on \(\Gamma_l\): \(u = v = \theta_x = 0\),

where \(\alpha_i \geq 0\) for \(i = 1, 2, 3\) and \(u_z, v_z, \theta_x\) and \(\theta_z\) are the corresponding partial derivatives of \(u, v\) and \(\theta\).

For convenience of reference only, the above set of boundary conditions is called physical boundary conditions here, following [7]. \(\Gamma_t\) denotes the interface between the ocean and the air and is fixed by the “rigid lid” assumption. So, free surface problem is not considered. The boundary conditions for \(u, v\) and \(\theta\) on \(\Gamma_t\) reflect interaction between the ocean and the air. See [21, 23] and the references cited therein for geophysical background of 3D viscous PE along with these boundary conditions. See [1] for mathematical justification of the 3D viscous PE with this set of boundary conditions. See also [2] for physical background and mathematical analysis of the Robin type “friction” boundary condition for horizontal velocity at the bottom of the spatial domain.

Integrating the continuity equation and the hydrostatic balance equation and using the boundary condition \(w(x, 0, t) = 0\), one can express \(w\) and \(p\) as:
\[ w(x, z, t) = \int_z^0 u_x(x, \zeta, t) d\zeta, \quad (1.1) \]
\[ p(x, z, t) = q(x, t) + \int_z^0 \theta(x, \zeta, t) d\zeta, \quad (1.2) \]
where \( q(x,t) = p(x,0,t) \) is the pressure on \( \Gamma_i \). Eliminating \( w \) and \( p \) from the previous system of 2D viscous PE results in the following equivalent formulation:

\[
\begin{align*}
\begin{aligned}
u_t - \Delta u + uu_x + \left( \int_{-h}^0 u_x(x,\zeta,t) d\zeta \right) u_x + q_x + \int_{-h}^0 \theta_x(x,\zeta,t) d\zeta &= v, \\
v_t - \Delta v + uv_x + \left( \int_{-h}^0 u_x(x,\zeta,t) d\zeta \right) v_x &= -u, \\
\theta_t - \Delta \theta + u\theta_x + \left( \int_{-h}^0 u_x(x,\zeta,t) d\zeta \right) \theta_x &= Q;
\end{aligned}
\end{align*}
\] (1.3)

with the following boundary conditions:

\[
\begin{align*}
u_x + \alpha_1 u\bigg|_{z=0} &= u\bigg|_{z=-h} = u\bigg|_{x=0,1} = 0, \\
v_x + \alpha_2 v\bigg|_{z=0} &= v\bigg|_{z=-h} = v\bigg|_{x=0,1} = 0, \\
\int_{-h}^0 u_x(x,\zeta,t) d\zeta &= 0, \\
\theta_x + \alpha_3 \theta\bigg|_{z=0} &= \theta_x\bigg|_{z=-h} = \theta_x\bigg|_{x=0,1} = 0.
\end{align*}
\] (1.6) (1.7) (1.8) (1.9)

The above system of 2D viscous PE will be solved with suitable initial conditions:

\[
u(x,z) = \nu_0(x,z), \quad v(x,z) = \nu_0(x,z), \quad \theta(x,z) = \theta_0(x,z).
\] (1.10)

Notice that the boundary condition \( w(x,0,t) = 0 \) is already embedded in the expression (1.1). The other boundary condition for \( w \) is given in (1.8). It follows from (1.6) and (1.8) that

\[
\int_{-h}^0 u(x,z,t) dz = 0.
\] (1.11)

The mathematical framework of the viscous primitive equations for large scale ocean flow in three-dimensional spatial domain (3D viscous PE) was formulated in [21], where the notions of weak and strong solutions were defined and existence of weak solutions was proved. See also [1,2] for existence of weak solutions. Existence of strong solutions local in time and their uniqueness were proved in [9,26]. Existence of strong solutions global in time was proved in [5,15] for the case that \((u,v)\) satisfies Neumann boundary condition at the bottom and free boundary condition at the lateral side. Existence of strong solutions global in time was proved in [17] for the case that \((u,v)\) satisfies physical boundary conditions. See also the results in [10]. Uniform boundedness in \( H^1 \) of strong solutions global in time was proved in [11] and [18]. Uniform boundedness in \( H^2 \) of \( H^2 \) solutions global in time was proved in [12,14] for the case with \( u \) and \( v \) satisfying Neumann boundary condition at the bottom. Global uniform boundedness in \( H^m \) \((m \geq 2)\) of \( H^m \) solutions of the 3D PE with physical boundary conditions was recently proved in [13].

One of the outstanding unresolved mathematical problems for 3D viscous PE is about uniqueness of weak solutions. Global existence and uniqueness of \( z \)-weak solutions to 3D viscous PE, with the horizontal velocity satisfying Neumann boundary condition at the bottom, were proved in [24] for initial data \( (u_0,v_0,\theta_0) \) satisfying:

\[
(u_0,v_0,\theta_0) \in (L^6)^3 \quad \text{and} \quad \partial_z (u_0,v_0,\theta_0) \in (L^2)^3.
\]

A “\( z \)-weak” solution is a weak solution \((u,v,\theta)\) such that

\[
(u_x,v_x,\theta_z) \in L^\infty(0,T; (L^2)^3) \cap L^2(0,T; (H^1)^3).
\]

The extra condition \( (u_0,v_0,\theta_0) \in (L^6)^3 \) imposed in [24] was removed in [12] via an analysis different from the one used in [24]. Uniqueness of weak solutions was proved in [16] for continuous initial data and in [19] for a class of discontinuous initial data. Notice that Neumann boundary condition at the bottom is imposed for horizontal velocity in [16,19] as well. Furthermore, to avoid the boundary issue in horizontal directions, the whole space \( \mathbb{R}^2 \) is chosen as the horizontal spatial domain in [16] and periodicity
in horizontal directions is imposed in [19]. It is unclear if the results of [16,19] can be extended to the case with the domain bounded in horizontal directions without horizontal periodicity.

Indeed, the problem of uniqueness of weak solution is still open even for 2D viscous PE. One of the main reasons for the difficulty of the problem is that the nonlinearity in the weak formulation for the initial value problem of the 2D viscous PE seems as strong as that of the 3D NSE. See Remark 2.1. Existence and uniqueness of z-weak solutions, in the name of “weak vorticity solutions”, for 2D hydrostatic Navier-Stokes equations (2D hNSE) were proved in [3] for the case when $u$ satisfies a Robin type friction boundary condition at the bottom of spatial domain. The system of 2D hNSE is somewhat simplified from that of 2D PE (1.3)–(1.5) by neglecting $v$ and $\theta$. Existence and uniqueness of z-weak solution $u$ for 2D hNSE were also proved in [4] for the case with Dirichlet boundary condition at the bottom. For the case when the spatial domain is a rectangle, existence and uniqueness of the weak solution $u$ of the 2D hNSE were proved in [4] with even less demanding regularity:

$$u \in L^\infty(0,T; \mathbb{L}^2_x H^1_z \cap \mathbb{L}^2(0,T; H^1_x H^2_z),$$

for both the cases of Dirichlet and Neumann boundary conditions at the bottom. See [4] for notational details. Finally, we mention that existence and uniqueness of z-weak solution $(u,v,\theta)$ for 2D viscous PE were proved in [22] for the case with periodic boundary conditions on $(u,v,\theta)$.

This paper will focus on the problem of uniqueness of some classes of weak solutions of the 2D viscous PE and uniform boundedness of norms of partial regularity for these solutions with initial data having different types of partial regularity. Our analysis is for the system (1.3)–(1.5) with physical boundary conditions (1.6)–(1.9). However, it also applies to the cases with other typical boundary conditions.

To present our analysis in complete details, we first give the definition of a weak solution carefully and then prove several results about important properties of weak solutions and strong solutions of 2D viscous PEs. These results are included in Theorems 3.1–3.4. Closely related important discussions are also presented in Remarks 3.1–3.3. Part of the discussion in Sect. 3 shares some similarity with [19] in terms of strategy. However, the definition of a weak solution of the 2D viscous PE used in this paper is different from previously used definitions. The set of boundary conditions used in this paper is different from those used in [3,19]. Therefore, full proofs of all these results are presented since the ideas of the proofs along with detailed arguments are also different. These results provide fundamental technical support for our analysis in the rest sections of this paper. Moreover, some of these results seem also new and have independent interest.

The main results for existence and uniqueness of weak solutions with initial partial regularity are Theorems 4.1, 5.1, 5.2 and 6.1. Theorem 4.1 gives a new sufficient condition for uniqueness of weak solutions which improves the best existing result. Theorems 5.1 and 5.2 establish global existence of several classes of weak solutions with initial partial regularity. Finally, uniqueness of these classes of weak solutions with initial partial regularity follows as Theorem 6.1, giving five different cases of partial regularity for initial data which guarantee uniqueness of weak solutions. It should be mentioned that all the results of this paper are also valid for other typical boundary conditions for 2D viscous PEs, since the proofs presented can be directly extended to those cases.

Recall that uniqueness of z-weak solutions to 2D PE was proved in [22] for periodic case. However, its analysis does not apply to the case with physical boundary conditions (1.6)–(1.9). It is unclear either if other previous analyses for z-weak solutions can be extended to the case with boundary conditions (1.6)–(1.9). The only possible exception might be that of [4] for 2D hNSE, since Dirichlet boundary condition on $u$ at the bottom of the spatial domain was used in [4]. However, the definition of weak solutions as indicated in [4] is different from what is used in this paper. Moreover, explicit use of the definition of weak solutions was not presented in the analysis of [4]. In this paper, to deal with the boundary conditions (1.6)–(1.9), we give a fully detailed and explicit analysis of the z-weak solutions in an approach completely different from previous analyses. On the other hand, uniqueness of weak solutions as given by Theorem 6.1 for the other four cases of initial partial regularity was never discovered before for any boundary conditions. Especially, global existence and uniqueness are proved for weak solutions with initial partial regularity in horizontal direction, rather than vertical direction. The same is proved.
for three mixed cases as well. Notice that regularity of \((u_z, v_z, \theta_z)\) for weak solutions of 2D and 3D viscous PE played a prominent or even crucial role in almost all previous analytic works in dealing with solution regularity and uniqueness properties, due to the special feature of hydro-static balance. To the contrary of intuition, it is demonstrated in this paper that initial horizontal regularity seems to allow weak solutions to behave better for the 2D problem. This was not observed before.

The main new ideas of our analysis are some new manipulations of anisotropic Sobolev inequalities combined with new analysis of some important properties of weak solutions as presented in Sect. 3 and novel use of them in Sect. 5. It is especially worthy of pointing out that the main strategy of our proofs of Theorems 5.1 and 5.2 is quite non-traditional.

The rest of this article is organized as follows:

In Sect. 2, we give the notations and some definitions, briefly review the background and recall some facts and known results which are important for later analysis. Especially, we will prove Lemmas 2.1 and 2.2 which will be useful in later sections. In Sect. 3, we first give the definition of weak solutions and strong solutions of 2D viscous PE with physical boundary conditions. Then, we prove Theorem 3.1–3.4 about important properties of weak solutions and strong solutions of 2D viscous PE. These theorems will provide very important technical support for our analysis in the sequel. In Sect. 4, we prove Theorem 4.1. It gives a new sufficient condition for uniqueness of weak solutions and generalizes an already known one. It is a crucial observation which will be used later in proving our new uniqueness results. In Sect. 5, we prove Theorems 5.1 and 5.2. These global existence results, for weak solutions in specific partial regularity classes, not only give corresponding global-in-time uniform bounds and absorbing sets, but also prepare our proof of uniqueness in next section. In Sect. 6, we prove Theorem 6.1 for uniqueness of the weak solutions with the specific initial partial regularity.

2. Preliminaries

In this paper, \(C\) denotes a positive absolute constant, the value of which might vary from line to line. Similarly, \(C_\varepsilon\) denotes a positive constant depending on \(\varepsilon > 0\), the value of which may also vary at different occurrence. The following notations are used for real numbers \(A\) and \(B\):

\[
A \ll B \quad \text{if and only if} \quad A \leq C \cdot B,
\]

and

\[
A \approx B \quad \text{if and only if} \quad c \cdot A \leq B \leq C \cdot A,
\]

for some positive constants \(c\) and \(C\) independent of \(A\) and \(B\).

Denote by \(L^r(D), L^r((0, 1))\) and \(L^r((-h, 0)) (1 \leq r < +\infty)\) the classic Lebesgue \(L^r\) spaces with the norm:

\[
\|\varphi\|_r = \left\{ \begin{array}{ll}
(f_D |\varphi(x, z)|^r \, dx\, dz)^{\frac{1}{r}}, & \forall \varphi \in L^r(D); \\
(f_0^1 |\varphi(x)|^r \, dx)^{\frac{1}{r}}, & \forall \varphi \in L^r((0, 1)); \\
(f_h^0 |\varphi(z)|^r \, dz)^{\frac{1}{r}}, & \forall \varphi \in L^r((-h, 0)).
\end{array} \right.
\]

Standard modification is used when \(r = \infty\). When there is no confusion, index \(r = 2\) is omitted:

\[
\|\varphi\| := \|\varphi\|_2.
\]

Denote by \(H^m(D) (m \geq 1)\) the classic Sobolev spaces for square-integrable functions on \(D\) with square-integrable weak derivatives up to order \(m\). Domains of the functions spaces will be omitted from notations without confusion.

Some anisotropic Lebesgue spaces and Sobolev spaces will be used. For example, for \(r, s \in [1, \infty)\), \(L^r_x(L^s_z)\) denotes the standard function space of (classes of) Lebesgue measurable functions on \(D\) such
that
\[ \| \varphi \|_{L^r_s(L^r_t)} := \left( \int_0^1 \| \varphi(x,) \|_{L^r_t}^r dx \right)^{\frac{1}{r}} \]
\[ := \left[ \int_0^1 \left( \int_{-h}^0 |\varphi(x,z)|^s dz \right)^\frac{r}{s} dx \right]^{\frac{1}{r}} < \infty, \]
with stand modifications when \( r \) or \( s \) is \( \infty \).

We will also use \( C_B([\alpha, \infty)) \) to denote the set of uniformly bounded functions in \( C([\alpha, \infty)) \), for an interval \( [\alpha, \infty) \subset \mathbb{R} \). The following function spaces are defined:
\[ H := H_1 \times H_2 \times H_3, \quad V := V_1 \times V_2 \times V_3, \]
where
\[ H_1 := \left\{ \varphi \in L^2(D) \mid \int_{-h}^0 \varphi(x,z) dz = 0 \right\}, \]
\[ V_1 := \left\{ \varphi \in H^1(D) \mid \int_{-h}^0 \varphi(x,z) dz = 0, \ \varphi|_{\Gamma_1 \cup \Gamma_b} = 0 \right\}, \]
\[ H_2 := L^2(D), \quad V_2 := \left\{ \varphi \in H^1(D) \mid \varphi|_{\Gamma_1 \cup \Gamma_b} = 0 \right\}, \]
\[ H_3 := L^2(D), \quad V_3 := H^1(D), \quad \text{if } \alpha_3 > 0, \]
and
\[ H_3 := \left\{ \varphi \in L^2(D) \mid \int_D \varphi = 0 \right\}, \quad V_3 := H_3 \cap H^1(D), \quad \text{if } \alpha_3 = 0. \]
Define the bilinear form: \( a_i : V_i \times V_i \to \mathbb{R} \) for \( i = 1, 2, 3 \), such that
\[ a_i(\phi, \varphi) = \int_D \nabla \phi \cdot \nabla \varphi \ dx dz + \alpha_i \int_0^1 \phi(x,0)\varphi(x,0) \ dx, \]
and the corresponding linear operator \( A_i : V_i \to V_i' \), such that
\[ \langle A_i v, \varphi \rangle = a_i(v, \varphi), \quad \forall v, \varphi \in V_i, \]
where \( V_i' \) is the dual space of \( V_i \) and \( \langle \cdot, \cdot \rangle \) denotes scalar product between \( V_i' \) and \( V_i \) and the inner product in \( H_i \). Define:
\[ D(A_i) = \left\{ \phi \in V_i \mid A_i \phi \in H_i \right\}, \quad i = 1, 2, 3. \]
Define: \( A : V \to V' \) as \( A(u, v, \theta) := (A_1 u, A_2 v, A_3 \theta) \), for \( (u, v, \theta) \in V \). Then,
\[ D(A) = D(A_1) \times D(A_2) \times D(A_3). \]
It is easy to see that, for \( \phi \in V_i \),
\[ a_i(\phi, \phi) \leq \| \phi \|^2 + (1 + \alpha_i h)\| \phi \|^2. \]
Therefore, \( A_i^{-1} \) is a self-adjoint compact operator in \( H_i \). Hence, by the classic spectral theory, the power \( A_i^s \) can be defined for any \( s \in \mathbb{R} \). Then,
\[ D(A_i)' = D(A_i^{-1}) \]
is the dual space of \( D(A_i) \) and
\[ V_i = D(A_i^\frac{1}{2}), \quad V_i' = D(A_i^{-\frac{1}{2}}). \]
Moreover,
\[ D(A_i) \subset V_i \subset H_i \subset V_i' \subset D(A_i)', \]
and the embeddings above are all continuous and compact and each space above is dense in the one following it. Define the norm of $V_i$ as

$$\|\varphi\|^2_{V_i} = a_i(\varphi, \varphi) = \langle A_i \varphi, \varphi \rangle = \langle A_i^\frac{1}{2} \varphi, A_i^\frac{1}{2} \varphi \rangle, \quad i = 1, 2, 3.$$  

Then, for $\varphi \in V_i$ and $i = 1, 2, 3$,

$$\|\varphi\| \leq \|\varphi\|_{V_i} \approx \|\varphi\|_{H^1}.$$  

By the above discussion and elliptic regularity for linear 3D stationary PE (see e.g. [26]), we also have for $\varphi \in D(A_i)$ and $i = 1, 2, 3$,

$$\|\varphi\|_{V_i} \leq \|A_i \varphi\| \approx \|\varphi\|_{H^2}.$$

Next, we introduce the following anisotropic estimate which will be very useful for our later discussion:$^2$

**Lemma 2.1.** Let $\psi, \psi_x, \phi, \psi_z, \varphi \in L^2(D)$. Then,

$$\left| \int_D \psi \phi \varphi \right| \leq \|\psi\| \frac{3}{2} (\|\psi\| + \|\psi_x\|) \frac{1}{2} \|\phi\| \frac{1}{2} (\|\phi\| + \|\phi_z\|) \frac{1}{2} \|\varphi\|.$$  

**Proof.**

$$\left| \int_D \psi \phi \varphi \right| \leq \int_0^1 \|\psi\|_{L_x^\infty} \|\psi\|_{L_x^2} \|\phi\|_{L_x^2} \|\varphi\|_{L_x^2} \, dx$$

$$\leq \int_0^1 \|\psi\|_{L_x^\infty} \left( \|\phi\|_{L_x^2} + \|\phi_z\|_{L_x^2} \right)^{\frac{1}{2}} \|\psi\|_{L_x^2} \|\varphi\|_{L_x^2} \, dx$$

$$\leq \|\psi\|_{L_x^\infty} \left( \|\psi\|_{L_x^2} + \|\psi_x\|_{L_x^2} \right)^{\frac{1}{2}} \|\phi\| \frac{1}{2} (\|\phi\| + \|\phi_z\|)^{\frac{1}{2}} \|\varphi\|.$$  

Notice that, in the last step of the above derivations, we have used the following estimate:

$$\|\psi\|^2_{L_x^2(L_x^\infty)} = \int_{-h}^0 \|\psi\|^2_{L_x^2} \, dz$$

$$\leq \int_{-h}^0 \|\psi\|_{L_x^2} \left( \|\psi\|_{L_x^2} + \|\psi_x\|_{L_x^2} \right) \, dz$$

$$\leq \|\psi\| (\|\psi\| + \|\psi_x\|).$$

An immediate application of Lemma 2.1 is the following result:

**Lemma 2.2.** The following statements are valid:

(a) Suppose $(u, v, \theta) \in V$ and $w$ is given by (1.1). Then, for $\varphi \in V_1$,

$$|\langle uu_x, \varphi \rangle| \leq \|u\| \|u_x\| \|\varphi\| \|\varphi_x\|, \quad \langle uu_x, \varphi \rangle \leq \|u\| \|u_x\| \|\varphi\| \|\varphi_x\| + \|u\| \|u_x\| \|\varphi\| \|\varphi_x\|;$$  

(b) for $\varphi \in V_2$, with $i, j = 1, 2$, $i' = 3 - i$ and $j' = 3 - j$,

$$|\langle uv_x, \varphi \rangle| \leq \|u\| \|v\| \|\varphi\| \|\varphi_x\| \|\varphi_x\|, \quad \langle uv_x, \varphi \rangle \leq \|u\| \|v\| \|\varphi\| \|\varphi_x\| \|\varphi_x\| + \|u\| \|v\| \|\varphi\| \|\varphi_x\|;$$  

$$|\langle uv_z, \varphi \rangle| \leq \|u\| \|v\| \|\varphi\| \|\varphi_x\| \|\varphi_x\| + \|u\| \|v\| \|\varphi\| \|\varphi_x\|;$$  

$$|\langle uv_z, \varphi \rangle| \leq \|u\| \|v\| \|\varphi\| \|\varphi_x\| \|\varphi_x\| + \|u\| \|v\| \|\varphi\| \|\varphi_x\|;$$  

$^2$See also Lemma 2.2 of [6].
for \( \varphi \in V_3 \), with \( i, j = 1, 2 \), \( i' = 3 - i \) and \( j' = 3 - j \),

\[
|\langle u\theta_x, \varphi \rangle | \lesssim \|u_x\|\|\theta\|^{\frac{3}{2}}\|\partial_x \theta\|^{\frac{3}{2}}\|\partial_x \varphi\|^{\frac{1}{2}} + \|u\|\|\partial_t \varphi\|^{\frac{3}{2}} + \|\partial_t \theta\|^{\frac{1}{2}}\|\partial_x \varphi\|^{\frac{1}{2}},
\]

(2.5)

\[
|\langle w\theta_z, \varphi \rangle | \lesssim \|u_x\|\|\theta\|^{\frac{3}{2}}\|\partial_z \theta\|^{\frac{1}{2}}\|\partial_z \varphi\|^{\frac{3}{2}} + \|u_x\|\|\theta\|^{\frac{1}{2}}\|\partial_x \varphi\|^{\frac{3}{2}}.
\]

(2.6)

Therefore,

\[
uxz, uzx \in V_1', \quad vxz, wzv \in V_2', \quad \theta_xz, w\theta_z \in V_3'.
\]

(2.7)

(b) Suppose \((u, v, \theta) \in L^\infty(0, T; H) \cap L^2(0, T; V)\). Then,

\[
uxz \in L^2(0, T; V_1'), \quad wzv \in L^\frac{4}{3}(0, T; V_2'), \quad \theta_xz, w\theta_z \in L^\frac{4}{3}(0, T; V_3').
\]

Proof. For \(u, \varphi \in V_1\), Lemma 2.1 yields

\[
|\langle uxz, \varphi \rangle | \lesssim \|u\|\|\varphi\|\|\varphi\|^{\frac{3}{2}} \lesssim \|u\|\|u_x\|\|\varphi\|^{\frac{3}{2}}\|\varphi\|^{\frac{3}{2}}, \quad |\langle u^2, \varphi \rangle | \lesssim \|u\|\|u_x\|\|u_x\|\|\varphi\|\|\varphi\|^{\frac{3}{2}}.
\]

Then, a density argument using Lemma 2.1 again, along with the above two inequalities, proves

\[
\langle uxz, \varphi \rangle = -\frac{1}{2} \langle u^2, \varphi \rangle,
\]

from which (2.1) follows. Similarly, Lemma 2.1 yields

\[
|\langle wzv, \varphi \rangle | \lesssim \|w\|\|\varphi\|\|\varphi\|^{\frac{3}{2}} \lesssim \|w\|\|w_x\|\|\varphi\|\|\varphi\|^{\frac{3}{2}} \lesssim \|w\|\|u_x\|\|\varphi\|\|\varphi\|^{\frac{3}{2}}\|u_x\|\|\varphi\|^{\frac{3}{2}}.
\]

(2.8)

and

\[
|\langle wu, \varphi \rangle | \lesssim \|u\|\|u_x\|\|\varphi\|^{\frac{3}{2}} \lesssim \|u\|\|u_x\|\|\varphi\|^{\frac{3}{2}}\|\varphi\|^{\frac{3}{2}}.
\]

The above two inequalities plus (2.1), along with Lemma 2.1, then imply via a density argument that

\[
\langle wzv, \varphi \rangle = \langle u_x u, \varphi \rangle - \langle wu, \varphi \rangle,
\]

from which, we immediately prove (2.2) by (2.1) and (2.8). Similarly, we can prove (2.3)–(2.6). Then, it is easy to prove (2.7) and part (b) using (2.1)–(2.6).

Remark 2.1. Notice that Lemma 2.2 indicates that the nonlinearity of the 2D PE in the weak formulation (see Definition 3.1) seems as strong as that of the 3D Navier-Stokes Equations. This is one of the main reasons that the problem of uniqueness of the weak solutions of 2D PE is challenging.

\[\square\]

3. Weak Solutions and Strong Solutions

In this section, some important properties about weak solutions and strong solutions will be discussed. These will provide important technical support in the proofs of the main results of this paper to be presented in the next few sections. The following definitions of weak and strong solutions of the initial boundary value problem (1.3)–(1.10) for the 2D viscous PEs will be used in this paper:

Definition 3.1. Suppose \(Q \in L^2(D)\), \((u_0, v_0, \theta_0) \in H\) and \(T > 0\). The triple \((u, v, \theta)\) is called a weak solution of the viscous PEs (1.3)–(1.10) on the time interval \((0, T)\) if it satisfies (1.3)–(1.5) in weak sense, that is, if

\[
(u, v, \theta) \in L^\infty(0, T; H) \cap L^2(0, T; V),
\]

(3.1)
satisfies the follow equations in the sense of distribution on \((0, T)\):

\[
\begin{align*}
\langle u_t, \varphi \rangle + a_1(u, \varphi) + \left(\langle u, w \rangle \cdot \nabla u - v + \int_0^t \theta_x, \varphi \right) &= 0, \quad \forall \varphi \in V_1, \\
\langle v_t, \varphi \rangle + a_2(v, \varphi) + \left(\langle u, w \rangle \cdot \nabla v - u, \varphi \right) &= 0, \quad \forall \varphi \in V_2, \\
\langle \theta_t, \varphi \rangle + a_3(\theta, \varphi) + \left(\langle u, w \rangle \cdot \nabla \theta - Q, \varphi \right) &= 0, \quad \forall \varphi \in V_3, 
\end{align*}
\]  

(3.2)–(3.4)

where \(w\) is given by (1.1) in weak sense. Moreover,

\[
\lim_{t \to t_0^+} (u(t), v(t), \theta(t)) = (u_0, v_0, \theta_0),
\]

(3.5)
in weak topology of \(H\), and the following energy inequalities are satisfied for almost every \(t_0 \in [0, T)\) and almost every \(t \in (t_0, T)\):

\[
\begin{align*}
||u(t)||^2 + 2 \int_{t_0}^{t} \left( ||u(s)||^2_{V_1} + \left( -v + \int_0^t \theta_x, u \right) \right) ds &\leq ||u(t_0)||^2, \\
||v(t)||^2 + 2 \int_{t_0}^{t} \left( ||v(s)||^2_{V_2} + \langle u, v \rangle \right) ds &\leq ||v(t_0)||^2, \\
||\theta(t)||^2 + 2 \int_{t_0}^{t} \left( ||\theta(s)||^2_{V_3} - \langle Q, \theta(s) \rangle \right) ds &\leq ||\theta(t_0)||^2.
\end{align*}
\]  

(3.6)–(3.8)

Furthermore, the above energy inequalities (3.6)–(3.8) are also satisfied for \(t_0 = 0\) and for almost every \(t \in (0, T)\).

If \((u_0, v_0, \theta_0) \in V\), then \((u, v, \theta)\) is called a strong solution of (1.3)–(1.10) on the time interval \([0, T)\) if it satisfies (3.2)–(3.5) and

\[
(u, v, \theta) \in L^\infty(0, T; V) \cap L^2(0, T; D(A)).
\]

(3.9)

If \(T > 0\) in the above can be arbitrarily large, then the corresponding weak or strong solution is global.

\[\square\]

Remark 3.1. There are somewhat different ways to define weak solutions of the PE. For examples, see [1–3, 19, 21, 23, 26]. Especially, to define a weak solution of the 3D PE with physical boundary conditions, the domain of \(\varphi\) in (3.2)–(3.4) was chosen as \(D(A_i)\) in [21, 23, 26], for \(i = 1, 2, 3\) respectively instead of \(V_i\). Definition 3.1 is formally more restrictive than the one given in [21, 23, 26]. However, \(D(A_i)\) is dense in \(V_i\) and, by Lemma 2.2, the nonlinear terms of the 2D PE are in \(V_i^*\) for \((u, v, \theta) \in V\) and in \(L^2(0, T; V_i^*)\) for \((u, v, \theta) \in L^\infty(0, T; H) \cap L^2(0, T; V)\). Thus, for 2D case, a weak solution defined in [21, 23, 26], if satisfying (3.6)–(3.8), is also a weak solution in the sense of Definition 3.1.

\[\square\]

We first state and prove the following theorem on some basic properties satisfied by every weak solution.

Theorem 3.1. There exists at least one global weak solution of (1.3)–(1.10) in the sense of Definition 3.1. If \((u, v, \theta)\) is a weak solution on \((0, T)\), then\(^3\)

\[
\langle u, \varphi_1 \rangle, \langle v, \varphi_2 \rangle, \langle \theta, \varphi_3 \rangle \in C([0, T]), \quad \forall (\varphi_1, \varphi_2, \varphi_3) \in H.
\]

(3.10)

Moreover, there exists a zero measure set \(E \subset (0, \infty)\), such that

\[
\lim_{\epsilon \to 0^+} \| (u(t), v(t), \theta(t)) - (u_0, v_0, \theta_0) \|_H = 0,
\]

(3.11)

where \(E^c := (0, \infty) \setminus E\).

\(^3\)If \(T = \infty\), the space \(C([0, T])\) in (3.10) is replaced by \(C_B([0, \infty))\).
Proof. It is proved in [21] that there exists at least one global weak solution (in their sense) for 3D PE and it satisfies energy inequalities (3.6)–(3.8). Moreover, any weak solution as defined in [21] is weakly continuous from \([0, T] \rightarrow H\) if \(T\) is finite and weakly continuous from \([0, T] \rightarrow H\) if \(T = \infty\). By Remark 3.1, these results imply existence of at least one global weak solution \((u, v, \theta)\) of (1.3)–(1.10) in the sense of Definition 3.1 and that (3.10) is satisfied by any weak solution \((u, v, \theta)\) in the sense of Definition 3.1.

By Lemmas 2.1 and 2.2, we can also prove existence of at least one global weak solutions \((u, v, \theta)\) of the 2D problem (1.3)–(1.10) using Definition 3.1 directly, by following the standard approach of [25] in proving existence of weak solutions of Navier-Stokes equations for homogeneous and incompressible fluid in three dimensional spatial domain. Moreover, we can prove that any weak solution \((u, v, \theta)\) satisfies (3.10).

Finally, due to the fact that a weak solution satisfies the energy inequalities (3.6)–(3.8) for \(t = 0\) and for almost every \(t \in [0, T]\) by Definition 3.1, there exists a zero measure set \(E \subset (0, \infty)\) such that, for all \(t \in E^c\),

\[
\|u(t)\|^2 + 2 \int_0^t \left(\|u(s)\|^2_{V_2} + \left(-v + \int_0^s \theta_x, u\right)\right) ds \leq \|u_0\|^2, \tag{3.12}
\]

\[
\|v(t)\|^2 + 2 \int_0^t \left(\|v(s)\|^2_{V_2} + \langle u, v \rangle\right) ds \leq \|v_0\|^2, \tag{3.13}
\]

\[
\|	heta(t)\|^2 + 2 \int_0^t \left(\|	heta(s)\|^2_{V_3} - \langle Q, \theta(s) \rangle\right) ds \leq \|	heta_0\|^2. \tag{3.14}
\]

Notice that, by definition, \((u, v, \theta) \in L^2(0, T; V)\). Therefore, taking \(\limsup\) on both sides of (3.12) for \(t(\in E^c) \rightarrow 0^+\), we have

\[
\limsup_{E^c \ni t \rightarrow 0^+} \|u(t)\|^2 \leq \|u_0\|^2.
\]

By (3.10) and weak lower semi-continuity, we also have

\[
\|u_0\|^2 \leq \liminf_{E^c \ni t \rightarrow 0^+} \|u(t)\|^2.
\]

Thus,

\[
\lim_{E^c \ni t \rightarrow 0^+} \|u(t)\|^2 = \|u_0\|^2.
\]

Hence,

\[
\lim_{E^c \ni t \rightarrow 0^+} \|u(t) - u_0\|^2 = \lim_{E^c \ni t \rightarrow 0^+} (\|u(t)\|^2 - 2 \langle u(t), u_0 \rangle + \|u_0\|^2)
\]

\[
= \lim_{E^c \ni t \rightarrow 0^+} (\|u(t)\|^2 - \|u_0\|^2) = 0.
\]

The second equality above is due to weak continuity (3.10). This weak continuity argument was also used in [19] to prove (3.11).

Different proofs of existence of global weak solutions of 3D PE can also be found in [1, 2, 26]. Different sets of boundary conditions and different definitions of weak solution were used in [2, 3, 19].

Remark 3.2. Existence and uniqueness of global strong solutions for 3D PE with Neumann boundary condition for \((u, v)\) at bottom was proved in [5]. See also [15] for a different proof of existence of global strong solution with the same boundary conditions when initial data is in \(H^2\). Existence and uniqueness of global strong solution was proved in [17, 18] for 3D viscous PE with physical boundary condition. The strong solutions are uniformly bounded in \(V\) and a bounded absorbing set for the solutions exists in \(V\). These results apply to the 2D case as well. See also [23] for a direct proof of global existence of the strong solution of the 2D viscous PE (1.3)–(1.10). Moreover, following the argument of [11] for the case of 3D
PE with Neumann boundary conditions, we can prove (see [7]) for the 3D PE with physical boundary conditions that, if \((u_0, v_0, \theta_0) \in V\), then the strong solution \((u, v, \theta)\) satisfies
\[
(u_t, v_t, \theta_t) \in L^2(0, T; H), \quad \forall T > 0,
\]
and
\[
(u(t), v(t), \theta(t)) \in C_B([0, \infty); V).
\]

The next theorem shows that energy equalities are satisfied by every strong solution. Therefore, the strong solution is also a weak solution.

**Theorem 3.2.** Let \((u, v, \theta)\) be the unique global strong solution of (1.3)–(1.10) with \((u_0, v_0, \theta_0) \in V\). Then, for every \(t_0 \in [0, \infty)\) and \(t \in (t_0, \infty)\),
\[
\frac{1}{2} \frac{d}{dt} \|u\|^2 + \frac{d}{dt} \|u\|_{V_1}^2 + \left( \int_Z \theta_x \cdot u \right) = \|v\|^2 + \frac{d}{dt} \|v\|_{V_2}^2 + \langle u, v \rangle = 0,
\]
\[
\frac{1}{2} \frac{d}{dt} \|\theta\|^2 + \frac{d}{dt} \|\theta\|_{V_3}^2 = \|Q\|.
\]

Therefore, the strong solution is also a weak solution.

**Proof.** By (3.9), (3.15) and a lemma of Lions and Magenes (see [20, 25]), we have in the sense of distribution on \((0, \infty)\)
\[
\frac{d}{dt} \|u\|^2 = 2 \langle u_t, u \rangle, \quad \frac{d}{dt} \|v\|^2 = 2 \langle v_t, v \rangle, \quad \frac{d}{dt} \|\theta\|^2 = 2 \langle \theta_t, \theta \rangle.
\]
Therefore, by (3.16), (3.20) and Definition 3.1, we have in classic sense on \([0, \infty)\),
\[
\frac{1}{2} \frac{d}{dt} \|u\|^2 + \|u\|_{V_1}^2 + \left( \int_Z \theta_x \cdot u \right) = \|v\|^2 + \frac{d}{dt} \|v\|_{V_2}^2 + \langle u, v \rangle = 0,
\]
\[
\frac{1}{2} \frac{d}{dt} \|\theta\|^2 + \|\theta\|_{V_3}^2 = \|Q\|.
\]

In the above derivation, we have also used the cancellation property for \((u, v, \theta) \in V\):
\[
\langle (u, w) \cdot \nabla u, u \rangle = \langle (u, w) \cdot \nabla v, v \rangle = \langle (u, w) \cdot \nabla \theta, \theta \rangle = 0,
\]
which can be justified by Lemma 2.1, as shown in the proof of Lemma 2.2. Integrating (3.21)–(3.23) finishes the proof.

Weak–strong uniqueness was proved in [19] for 3D viscous PE with horizontal periodicity and with Neumann boundary condition for \((u, v)\) at the bottom of spatial domain. That is, a weak solution with initial data in the space of strong solutions \(V\) is the strong solution with the same initial data, and is thus the unique weak (and strong) solution. The definition of weak solution used in [19] is different from Definition 3.1 and the boundary conditions used in [19] are also different. In the following, we give a completely different proof of weak-strong uniqueness result. Our prove is a direct proof using Definition 3.1 and properties of weak and strong solutions. The argument of our proof is also different from that of [25] for a related uniqueness result.

**Theorem 3.3.** Let \((u, v, \theta)\) be a weak solution of (1.3)–(1.10) on \((0, T)\) in the sense of Definition 3.1 with \((u_0, v_0, \theta_0) \in V\). Then, \((u, v, \theta)\) is the strong solution of (1.3)–(1.10). Thus, \((u, v, \theta)\) is the unique weak (and strong) solution.
Proof. Let \((u_1, v_1, \theta_1)\) be the global strong solution and \((u_2, v_2, \theta_2)\) be a weak solution on \((0, T)\) for some \(T > 0\), with the same initial value \((u_0, v_0, \theta_0) \in V\). For convenience of presentation, assume \(T\) is finite. The case of \(T = \infty\) is then an easy consequence.

Use \(w_i\) and \(q_i\), for \(i = 1, 2\), to denote corresponding vertical velocity and surface pressure. Denote:
\[
(\tilde{u}, \tilde{v}, \tilde{\theta}, \tilde{w}, \tilde{q}) := (u_1 - u_2, v_1 - v_2, \theta_1 - \theta_2, w_1 - w_2, q_1 - q_2).
\]
Notice that, by (3.15) and (3.16), for the strong solution,
\[
u_1 \in C([0, T]; V_1) \cap L^2(0, T; D(A_1)), \quad u_{1,t} \in L^2(0, T; H_1).
\]
For a weak solution \(u_2\),
\[
u_2 \in L^\infty(0, T; H_1) \cap L^2(0, T; V_1), \quad u_{2,t} \in L^1(0, T; V'_1).
\]
Therefore, by standard regularization approximation on \((0, T)\), we can obtain sequences of functions \(\{u_{1,m}\}_{m=1}^\infty\) and \(\{u_{2,m}\}_{m=1}^\infty\) such that
\[
u_{1,m} \in C^\infty([0, T]; D(A_1)), \quad u_{2,m} \in C^\infty([0, T]; V_1), \quad \forall \: m \geq 1,
\]
and as \(m \to \infty\),
\[
u_{1,m} \to u_1 \quad \text{in} \quad L^2_{loc}(0, T; D(A_1)) \quad \text{and} \quad C([0, T]; V_1),
\]
\[
u_{1,m,t} \to u_{1,t} \quad \text{in} \quad L^2_{loc}(0, T; H_1),
\]
\[
u_{2,m} \to u_2 \quad \text{in} \quad L^2_{loc}(0, T; V_1)
\]
\[
u_{2,m,t} \to u_{2,t} \quad \text{in} \quad L^1_{loc}(0, T; V'_1)\]
\[
\text{(3.25)}
\]
It is obvious that, for any \(m \geq 1\),
\[
\frac{d}{dt} \langle u_{1,m}, u_{2,m} \rangle = \langle (u_{1,m})_t, u_{2,m} \rangle + \langle (u_{2,m})_t, u_{1,m} \rangle.
\]
\[
\text{(3.26)}
\]
As \(m \to \infty\), we have by (3.25) that, in \(L^1_{loc}(0, T)\),
\[
\langle u_{1,m}, u_{2,m} \rangle \to \langle u_1, u_2 \rangle,
\]
\[
\langle (u_{1,m})_t, u_{2,m} \rangle \to \langle u_{1,t}, u_2 \rangle,
\]
\[
\langle (u_{2,m})_t, u_{1,m} \rangle \to \langle u_{2,t}, u_1 \rangle.
\]
\[
\text{(3.27)}
\]
These convergences are also valid in the distribution sense. Therefore, we can take the limit \(m \to \infty\) in (3.26) in the sense of distribution to obtain
\[
\frac{d}{dt} \langle u_1, u_2 \rangle = \langle u_{1,t}, u_2 \rangle + \langle u_{2,t}, u_1 \rangle,
\]
\[
\text{(3.28)}
\]
in the sense of distribution. Notice that
\[
\langle u_{1,t}, u_2 \rangle \in L^2(0, T), \quad \langle u_{2,t}, u_1 \rangle \in L^1(0, T).
\]
Therefore, \(\langle u_1, u_2 \rangle \in W^{1,1}(0, T)\). Thus, it is absolutely continuous in \(t\) and for any \(t_0 \in [0, T]\) and \(t \in (t_0, T]\),
\[
\langle u_1(t), u_2(t) \rangle = \langle u_1(t_0), u_2(t_0) \rangle + \int_{t_0}^t (\langle u_{1,t}(s), u_2(s) \rangle + \langle u_{2,t}(s), u_1(s) \rangle) \, ds.
\]
\[
\text{(3.29)}
\]
By regularity of \((u_i, v_i, \theta_i)\), \(i = 1, 2\), we have as well, for \(j = 3 - i\),
\[
\langle u_{i,t}, u_j \rangle = -a_1(u_i, u_j) - \left( (u_i, w_i) \cdot \nabla u_i + v_i + \int_z \theta_{i,x}, u_j \right), \quad i = 1, 2.
\]
\[
\text{(3.30)}
\]
From (3.29)–(3.30), we obtain, for any $t_0 \in [0, T)$ and $t \in (t_0, T]$,
\begin{align*}
\langle u_1(t), u_2(t) \rangle + 2 \int_{t_0}^{t} a_1(u_1(s), u_2(s)) \, ds + \int_{t_0}^{t} \left( \langle (u_1, w_1) \cdot \nabla u_1 - v_1 + \int_{z}^{0} \theta_{1,z} \rangle, u_2 \right) \, ds \\
+ \int_{t_0}^{t} \left( \langle (u_2, w_2) \cdot \nabla u_2 - v_2 + \int_{z}^{0} \theta_{1,x} \rangle, u_1 \right) \, ds = \langle u_1(t_0), u_2(t_0) \rangle.
\end{align*}
(3.31)

Similarly, we can also obtain, for any $t_0 \in [0, T)$ and $t \in (t_0, T]$,
\begin{align*}
\langle v_1(t), v_2(t) \rangle + 2 \int_{t_0}^{t} a_2(v_1(s), v_2(s)) \, ds + \int_{t_0}^{t} \left( \langle (u_1, w_1) \cdot \nabla v_1 + u_2 \rangle, v_2 \right) \, ds \\
+ \int_{t_0}^{t} \left( \langle (u_2, w_2) \cdot \nabla v_2 + u_1 \rangle, v_1 \right) \, ds = \langle v_1(t_0), v_2(t_0) \rangle,
\end{align*}
(3.32)

and for any $t_0 \in [0, T)$ and $t \in (t_0, T]$,
\begin{align*}
\langle \theta_1(t), \theta_2(t) \rangle + 2 \int_{t_0}^{t} a_3(\theta_1(s), \theta_2(s)) \, ds + \int_{t_0}^{t} \langle (u_1, w_1) \cdot \nabla \theta_1, \theta_2 \rangle \, ds \\
+ \int_{t_0}^{t} \langle (u_2, w_2) \cdot \nabla \theta_2, \theta_1 \rangle \, ds = \langle \theta_1(t_0), \theta_2(t_0) \rangle + \int_{t_0}^{t} \langle Q, \theta_1 + \theta_2 \rangle \, ds.
\end{align*}
(3.33)

By Definition 3.1, $(u_2, v_2, \theta_2)$ satisfies the energy inequalities (3.6)–(3.8) for almost every $t \in (0, T]$. By Theorem 3.2, $(u_1, v_1, \theta_1)$ satisfies the energy equalities (3.17)–(3.19) for any $t_0 \in [0, T)$ and $t \in (t_0, T]$. Combining these with (3.31)–(3.33), we obtain, for almost every $t \in (0, T]$,
\begin{align*}
||\tilde{u}(t)||^2 + 2 \int_{0}^{t} ||\tilde{u}||_{V_1}^2 \, ds &\leq -2 \int_{0}^{t} \langle \tilde{u} u_{1,x} + \tilde{w} u_{1,z} + \int_{z}^{0} \theta_{1,z} \, ds + \tilde{v}, \tilde{u} \rangle \, ds, \\
||\tilde{v}(t)||^2 + 2 \int_{0}^{t} ||\tilde{v}||_{V_2}^2 \, ds &\leq -2 \int_{0}^{t} \langle \tilde{v} v_{1,x} + \tilde{w} v_{1,z} - \tilde{u}, \tilde{v} \rangle \, ds, \\
||\tilde{\theta}(t)||^2 + 2 \int_{0}^{t} ||\tilde{\theta}||_{V_3}^2 \, ds &\leq -2 \int_{0}^{t} \langle \tilde{\theta} \theta_{1,x} + \tilde{w} \theta_{1,z}, \tilde{\theta} \rangle \, ds.
\end{align*}
(3.34)–(3.36)

Notice that some cancellations are used in the derivation of the above inequalities, which can be justified using Lemma 2.1. We omit justification of these cancellations here, since we have done similar justifications before. Now, we estimate the terms on the right-hand side of equations (3.34)–(3.36). By Lemma 2.1, we have
\begin{align*}
\langle \tilde{u} u_{1,x}, \tilde{u} \rangle &\leq ||u_{1,x}|| ||\tilde{u}|| ||\tilde{u}||_{V_1}, \\
\langle \tilde{w} u_{1,z}, \tilde{u} \rangle &\leq ||u_{1,z}|| ||\tilde{u}||_{V_2}^2, \\
\langle \tilde{v} v_{1,x}, \tilde{v} \rangle &\leq ||v_{1,x}|| ||\tilde{u}||_{V_1} ||\tilde{v}||_{V_2}, \\
\langle \tilde{w} v_{1,z}, \tilde{v} \rangle &\leq ||v_{1,z}|| ||\tilde{u}||_{V_1} ||\tilde{v}||_{V_2}^2, \\
\langle \tilde{\theta} \theta_{1,x}, \tilde{\theta} \rangle &\leq ||\theta_{1,x}|| ||\tilde{u}||_{V_1} ||\tilde{\theta}||_{V_3}, \\
\langle \tilde{w} \theta_{1,z}, \tilde{\theta} \rangle &\leq ||\theta_{1,z}|| ||\tilde{u}||_{V_1} ||\tilde{\theta}||_{V_3}^2.
\end{align*}
(3.37)

Summing up (3.34)–(3.36), applying the estimates in (3.37) and using Cauchy–Schwartz inequality, we get for almost every $t \in (0, T]$,
\begin{align*}
||\tilde{u}(t), \tilde{v}(t), \tilde{\theta}(t)||_H^2 + \int_{0}^{t} ||(\tilde{u}(t), \tilde{v}(t), \tilde{\theta}(t))||_{V}^2 \, ds &\leq \int_{0}^{t} (1 + ||(u_1, v_1, \theta_1)||_{V}^2) ||(\tilde{u}, \tilde{v}, \tilde{\theta})||_H^2 \, ds.
\end{align*}

Applying a generalized version of Gronwall lemma to the above inequality yields
\begin{align*}
||\tilde{u}(t), \tilde{v}(t), \tilde{\theta}(t)||_H = 0, \quad \text{for a.e. } t \in [0, T].
\end{align*}
Therefore, \((u_1, v_1, \theta_1) = (u_2, v_2, \theta_2)\), for almost every \(t \in [0, T]\).

Remark 3.3. We have in fact proved, for general \((\tilde{u}(0), \tilde{v}(0), \tilde{\theta}(0)) \in H\) and \((u_1(0), v_1(0), \theta_1(0)) \in V\), the following Lipschitz continuity property for every\(^4\) \(t \in [0, T]\),

\[
\|\langle \tilde{u}(t), \tilde{v}(t), \tilde{\theta}(t) \rangle \|_H^2 \leq \|\langle \tilde{u}(0), \tilde{v}(0), \tilde{\theta}(0) \rangle \|_H^2 \exp \left\{ \int_0^t (1 + \|u_1(t), \theta_1(t)\|_V^4) ds \right\}.
\]

\(\Box\)

The following theorem gives a much deeper description of a weak solution than Definition 3.1 and Theorem 3.1 combined.

**Theorem 3.4.** Let \((0, T)\) be the largest interval of existence for a weak solution \((u, v, \theta)\) of the problem (1.3)–(1.10) with \((u_0, v_0, \theta_0) \in H\). Then, \(T = \infty\). Moreover,

\[
(u, v, \theta) \in C((0, \infty), V),
\]

\[
(u(t), v(t), \theta(t)) \in D(A), \quad \text{for a.e. } t > 0,
\]

and, for any \(t_0 \in [0, \infty)\), \(t \in (t_0, \infty)\), the energy equalities (3.17)–(3.19) are valid. Furthermore,

\[
\lim_{t \to 0^+} \|\langle u(t), v(t), \theta(t) \rangle - (u_0, v_0, \theta_0) \|_H = 0.
\]

Therefore,

\[
(u, v, \theta) \in C_B([0, \infty), H).
\]

**Proof.** Let \((0, T)\) be the largest interval of existence for a weak solution \((u, v, \theta)\) of (1.3)–(1.10). Since \((u, v, \theta) \in L^2(0, T; V)\), we have

\[
(u(t), v(t), \theta(t)) \in V, \quad \text{for a.e. } t \in (0, T).
\]

Choose \(\tau \in (0, T)\) such that \((u(\tau), v(\tau), \theta(\tau)) \in V\) and that (3.6)–(3.8) are satisfied with \(t_0 = \tau\). Then, by Remark 3.2, there is a strong solution \((u_1, v_1, \theta_1)\) of (1.3)–(1.10) on \([\tau, \infty)\) such that

\[
(u_1(\tau), v_1(\tau), \theta_1(\tau)) = (u(\tau), v(\tau), \theta(\tau)).
\]

By Definition 3.1, \((u, v, \theta)\) is a weak solution on \([\tau, T]\). Therefore, by Theorem 3.3, \((u_1, v_1, \theta_1) = (u, v, \theta)\) on \([\tau, T]\). By Theorem 3.2, \((u_1, v_1, \theta_1)\) is also a weak solution on \([\tau, \infty)\). By Theorem 3.3 again, \((u_1, v_1, \theta_1)\) is the unique weak solution on \([\tau, \infty)\). This proves \(T = \infty\).

Notice that the above \(\tau\) can be chosen arbitrarily small. Therefore, (3.39) follows from continuity property (3.16) for a strong solution, (3.40) follows from the definition of a strong solution, and by Theorem 3.2, for any \(t_0 > 0\) and all \(t \in (t_0, \infty)\), the energy equalities (3.17)–(3.19) are satisfied.

Next, we prove validity of (3.17)–(3.19) for \(t_0 = 0\) and all \(t > 0\). By Theorem 3.1, there exists a set \(E \subset (0, \infty)\) such that (3.11) is valid. So, we can choose a sequence

\[
\{t_n\}_{n=1}^\infty \subset (0, \infty) \setminus E,
\]

which is monotonically decreasing to 0 as \(n \to \infty\) and

\[
\lim_{n \to \infty} \|\langle u(t_n), v(t_n), \theta(t_n) \rangle - (u_0, v_0, \theta_0) \|_H = 0.
\]

\(^4\)See (3.42).
Since \( t_n > 0 \) for every \( n \geq 1 \), we have just proved, for any \( t > t_n \),
\[
\|u(t)\|^2 + 2 \int_{t_n}^{t} \left( \|u(s)\|^2_{V_1} + \left\langle -v(s) + \int_{x}^{0} \theta_x(s), u(s) \right\rangle \right) ds = \|u(t_n)\|^2, \tag{3.44}
\]
\[
\|v(t)\|^2 + 2 \int_{t_n}^{t} \left( \|v(s)\|^2_{V_2} + \left\langle u(s), v(s) \right\rangle \right) ds = \|v(t_n)\|^2, \tag{3.45}
\]
\[
\|	heta(t)\|^2 + \int_{t_n}^{t} \left( \|	heta(s)\|^2_{\dot{V}_3} - \left\langle Q, \theta(s) \right\rangle \right) ds = \|	heta(t_n)\|^2. \tag{3.46}
\]

Now, take the limit \( n \to \infty \) in (3.44)–(3.46) and using the continuity property (3.43) and the fact that \((u, v, \theta) \in L^2_0(0, \infty; V)\), we have, for any \( t > 0 \),
\[
\|u(t)\|^2 + 2 \int_{0}^{t} \left( \|u(s)\|^2_{V_1} + \left\langle -v(s) + \int_{x}^{0} \theta_x(s), u(s) \right\rangle \right) ds = \|u_0\|^2, \tag{3.47}
\]
\[
\|v(t)\|^2 + 2 \int_{0}^{t} \left( \|v(s)\|^2_{V_2} + \left\langle u(s), v(s) \right\rangle \right) ds = \|v_0\|^2, \tag{3.48}
\]
\[
\|	heta(t)\|^2 + \int_{0}^{t} \left( \|	heta(s)\|^2_{\dot{V}_3} - \left\langle Q, \theta(s) \right\rangle \right) ds = \|	heta_0\|^2. \tag{3.49}
\]

These are (3.17)–(3.19) for \( t_0 = 0 \) and all \( t > 0 \).

Moreover, by (3.47) and that \((u, v, \theta) \in L^2_0(0, \infty; V)\), we obtain
\[
\limsup_{t \to 0^+} \|u(t)\|^2 \leq \|u_0\|^2.
\]

By (3.10), we also have
\[
\|u_0\|^2 \leq \liminf_{t \to 0^+} \|u(t)\|^2.
\]

Thus,
\[
\lim_{t \to 0^+} \|u(t)\|^2 = \|u_0\|^2.
\]

Using (3.10) again, we obtain
\[
\lim_{t \to 0^+} \|u(t) - u_0\|^2 = 0.
\]

Similarly, we have
\[
\lim_{t \to 0^+} \|v(t) - v_0\|^2 = \lim_{t \to 0^+} \|	heta(t) - \theta_0\|^2 = 0.
\]

This proves (3.41).

Finally, by definition, as a weak solution,
\[
(u, v, \theta) \in L^\infty(0, T_0; H), \quad \forall T_0 > 0.
\]

As a strong solution, the uniform boundedness in \( V \) is valid:
\[
(u, v, \theta) \in L^\infty(T_0, \infty; V).
\]

Therefore,
\[
(u, v, \theta) \in L^\infty(0, \infty; H).
\]

Then, (3.42) follows form the above uniform boundedness in \( H \), (3.39) and (3.41). \( \square \)
4. A Sufficient Condition for Uniqueness

In this section, we present a new sufficient condition for uniqueness of weak solutions of 2D viscous PE (1.3)–(1.10). First, we mention the following result for a sufficient condition for uniqueness of weak solutions of (1.3)–(1.10):

**Proposition 4.1.** Let \((u_i, v_i, \theta_i)\), for \(i = 1, 2\), be weak solutions of (1.3)–(1.10). Suppose \((u_0, v_0, \theta_0) \in H\) and for some \(T > 0\),

\[
(u_{1,z}, v_{1,z}, \theta_{1,z}) \in L^4(0, T; [L^2(D)]^3).
\]

Then, \((u_1, v_1, \theta_1) \equiv (u_2, v_2, \theta_2)\) for \(t \in [0, T]\).

The condition \(u_{1,z} \in L^4(0, T; L^2)\) appeared first in [9] as a sufficient condition for uniqueness for the 2D hydrostatic Navier-Stokes equations, where \(v, \theta\) are neglected. This condition seems to be the best previous result for sufficiency of weak solutions of 2D hNSE. Especially, it was used in [4] in proving their uniqueness results. However, the definition of weak solutions used in [9] is somewhat different from Definition 3.1.

As the first main result of this section, the following Theorem 4.1 improves Proposition 4.1 and allows one to find new classes of weak solutions of the system of (1.3)–(1.10), within which the weak solutions are unique. Especially, it is crucial for proving our main uniqueness result in Sect. 6.

**Theorem 4.1.** Let \((u_i, v_i, \theta_i)\), for \(i = 1, 2\), be weak solutions of (1.3)–(1.10). Suppose \((u_0, v_0, \theta_0) \in H\) and for some \(T > 0\),

\[
(u_{1,z}, v_{1,z}, \theta_{1,z}) \in [L^4(0, T; L^2(D))] \cup L^2(0, T; L^\infty_x(L^2_z))\]

\(4.1\).

Then, \((u_1, v_1, \theta_1) \equiv (u_2, v_2, \theta_2)\) for all \(t \in [0, T]\).

**Proof.** We will prove Lipschitz continuity of the weak solutions with respect to initial data in \(L^2\), assuming \((u_1, v_1, \theta_1)\) satisfies the regularity condition (4.1). Denote:

\[
\tilde{u} = u_1 - u_2, \quad \tilde{v} = v_1 - v_2, \quad \tilde{\theta} = \theta_1 - \theta_2, \quad \tilde{w} = w_1 - w_2, \quad \tilde{q} = q_1 - q_2,
\]

\[
\tilde{u}_0 = u_{1,0} - u_{2,0}, \quad \tilde{v}_0 = v_{1,0} - v_{2,0}, \quad \tilde{\theta}_0 = \theta_{1,0} - \theta_{2,0}.
\]

Let \(t_0 \in (0, T)\). Then \((u_1, v_1, \theta_1)\) and \((u_2, v_2, \theta_2)\) are both strong solutions on \([t_0, T]\). Therefore, we can follow the proof of Theorem 3.3 and apply it to \((u_1, v_1, \theta_1)\) and \((u_2, v_2, \theta_2)\) on \([t_0, T]\) to obtain, for every \(t \in [t_0, T]\),

\[
\|\tilde{u}(t)\|^2 + 2\int_{t_0}^t \|\tilde{u}\|^2_1 ds = \|\tilde{u}(t_0)\|^2 - 2\int_{t_0}^t \left(\tilde{u}u_{1,x} + \tilde{w}u_{1,z} + \int_z^0 \tilde{\theta}_x d\zeta + \tilde{v}\right) ds,
\]

\(4.2\).

\[
\|\tilde{v}(t)\|^2 + 2\int_{t_0}^t \|\tilde{v}\|^2_1 ds = \|\tilde{v}(t_0)\|^2 - 2\int_{t_0}^t \left(\tilde{w}v_{1,x} + \tilde{u}v_{1,z} - \tilde{u}, \tilde{v}\right) ds,
\]

\(4.3\).

\[
\|\tilde{\theta}(t)\|^2 + 2\int_{t_0}^t \|\tilde{\theta}\|^2_1 ds = \|\tilde{\theta}(t_0)\|^2 - 2\int_{t_0}^t \left(\tilde{u}\theta_{1,x} + \tilde{w}\theta_{1,z}, \tilde{\theta}\right) ds.
\]

\(4.4\).

Notice that equalities (4.2)–(4.4) are obtained similarly to inequalities (3.34)–(3.36). Since \((u_1, v_1, \theta_1)\) and \((u_2, v_2, \theta_2)\) are both strong solutions on \([t_0, T]\), we indeed have above equalities and they are valid for every \(t \in [t_0, T]\).
By Theorem 3.4, \((u_i, v_i, \theta_i) \in C([t_0, T], V)\) for \(i = 1, 2\). Thus, (4.2)–(4.4) along with Lemma 2.1 imply \(\|\bar{u}\|, \|\bar{v}\|, \|\bar{\theta}\| \in C^1([t_0, T])\). Therefore, we have in classic sense for every \(t \in [t_0, T]\) that
\[
\frac{1}{2} \frac{d}{dt} \||\bar{u}\|^2 + \||\bar{v}\|^2 \rangle = -\langle \bar{u}u_{1,x} + \bar{v}u_{1,z} + \int_0^1 \bar{\theta}x_d\zeta + \bar{v}, \bar{u} \rangle, \tag{4.5}
\]
\[
\frac{1}{2} \frac{d}{dt} \||\bar{v}\|^2 + \||\bar{\theta}\|^2 \rangle = -\langle v_{1,x}\bar{u} + \bar{w}v_{1,z} - \bar{u}, \bar{v} \rangle, \tag{4.6}
\]
\[
\frac{1}{2} \frac{d}{dt} \||\bar{\theta}\|^2 \rangle = -\langle \bar{u}\theta_{1,x} + \bar{w}\theta_{1,z}, \bar{\theta} \rangle. \tag{4.7}
\]
Summing up (4.5)–(4.7) yields, for every \(t \in [t_0, T]\),
\[
\frac{1}{2} \frac{d}{dt} \left( \|\bar{u}\|^2 + \|\bar{v}\|^2 + \|\bar{\theta}\|^2 \right) + \|\bar{u}\|^2 \|\bar{v}\|^2 + \|\bar{\theta}\|^2 \leq -\langle \bar{u}u_{1,x} + \bar{v}u_{1,z} + \int_0^1 \bar{\theta}x_d\zeta, \bar{u} \rangle \tag{4.8}
\]
\[
- \langle v_{1,x}\bar{u} + \bar{w}v_{1,z}, \bar{v} \rangle - \langle \bar{u}\theta_{1,x} + \bar{w}\theta_{1,z}, \bar{\theta} \rangle.
\]
Now, we estimate all the terms on the right side of (4.8). The bilinear term is easily estimated by Cauchy–Schwartz inequality:
\[
\left| \int_z^0 \bar{\theta}x_d\zeta, \bar{u} \right| \leq \|\bar{\theta}x\| \|\bar{u}\| \leq C_{\varepsilon} \|\bar{u}\|^2 + \varepsilon \|\bar{\theta}x\|^2. \tag{4.9}
\]
Next, by Agmon’s inequality, we have
\[
|\langle \bar{u}u_{1,x}, \bar{u} \rangle| \leq \int_0^1 \|\bar{u}\|_{L^p}\left( \int_{-\varepsilon}^0 |u_{1,x}\bar{u}|dz \right) dx
\]
\[
\leq \int_0^1 \|\bar{u}\|_{L^p}\|\bar{u}_{1,z}\|_{L^2} \|u_{1,x}\|_{L^2} \|\bar{u}\|_{L^2} dx
\]
\[
\leq \|\bar{u}\|_{L^p(L^2)} \int_0^1 \|\bar{u}\|_{L^2} \|\bar{u}_{1,z}\|_{L^2} \|u_{1,x}\|_{L^2} dx
\]
\[
\leq \|\bar{u}\|_{L^p(L^2)} \|\bar{u}\|_{L^2} \|\bar{u}_{1,z}\|_{L^2} \|u_{1,x}\|,
\]
where Hölder’s inequality is applied in the last step. Then, by Minkowski’s inequality:
\[
\|\bar{u}\|_{L^p(L^2)} \leq \|\bar{u}\|_{L^2(L^p)}^rac{1}{\lambda} \|u_{1,x}\|_{L^2(L^p)}^rac{1}{\mu},
\]
we get
\[
|\langle \bar{u}u_{1,x}, \bar{u} \rangle| \leq \|\bar{u}\|_{L^2(L^p)} \|\bar{u}\|_{L^2(L^p)} \|\bar{u}_{1,z}\|_{L^2} \|u_{1,x}\|_{L^2(L^p)}^rac{1}{\mu}
\]
\[
\leq \|u_{1,x}\|_{L^2(L^p)} \|\bar{u}\|_{L^2(L^p)} \|\bar{u}_{1,z}\|_{L^2} \|u_{1,x}\|_{L^2(L^p)} \tag{4.10}
\]
\[
\leq \|u_{1,x}\|_{L^2(L^p)} \|\bar{u}\|_{L^2(L^p)} \|\bar{u}_{1,z}\|_{L^2} \|u_{1,x}\| \tag{4.10}
\]
\[
\leq \|u_{1,x}\|_{L^2(L^p)} \|\bar{u}\|_{L^2(L^p)} \|\bar{u}_{1,z}\|_{L^2} \|u_{1,x}\| \tag{4.10}
\]
\[
\leq C_{\varepsilon} \|u_{1,x}\|^2 \|\bar{u}\|^2 + \varepsilon \|\nabla \bar{u}\|^2.
\]
It follows similarly that
\[
|\langle \bar{v}v_{1,x}, \bar{v} \rangle| \leq \|v_{1,x}\| \|\bar{u}\|_{L^2(L^p)} \|\bar{u}_{1,z}\|_{L^2} \|v_{1,x}\|_{L^2(L^p)} \tag{4.11}
\]
\[
\leq C_{\varepsilon} \|v_{1,x}\|^2 (\|\bar{u}\|^2 + \|\bar{v}\|^2) + \varepsilon (\|\bar{u}_{1,x}\|^2 + \|\bar{v}_{1,z}\|^2),
\]
and
\[
|\langle \bar{u}\theta_{1,x}, \bar{\theta} \rangle| \leq \|\theta_{1,x}\| \|\bar{u}\|_{L^2(L^p)} \|\bar{u}_{1,z}\|_{L^2} \|\bar{\theta}\|_{L^2(L^p)} \tag{4.12}
\]
\[
\leq C_{\varepsilon} \|\theta_{1,x}\|^2 (\|\bar{u}\|^2 + \|\bar{\theta}\|^2) + \varepsilon (\|\bar{u}_{1,x}\|^2 + \|\bar{\theta}_{1,z}\|^2).
Finally, similar to above estimates, we have
\[ |\langle \tilde{w}u_{1,z}, \tilde{u} \rangle| \leq \int_0^1 \|\tilde{w}\|_{L^\infty_x(L^2)} \left( \int_{-h}^0 |u_{1,z}\tilde{u}|dz \right) dx \]
\[ \leq \int_0^1 \|\tilde{w}\|_{L^2} \|\tilde{w}_z\|_{L^2} \|u_{1,z}\|_{L^2} \|\tilde{u}\|_{L^2} dx \]
\[ \leq \int_0^1 \|\tilde{u}_x\|_{L^2} \|u_{1,z}\|_{L^2} \|\tilde{u}\|_{L^2} dx. \]

The right-hand side of the above inequality can be further estimated in two different ways:

\[ |\langle \tilde{w}u_{1,z}, \tilde{u} \rangle| \leq C \|\tilde{u}\|_{L^\infty_x(L^2)} \int_0^1 \|\tilde{u}_x\|_{L^2} \|u_{1,z}\|_{L^2} dx \]
\[ \leq C \|\tilde{u}\|_{L^2(L^\infty)} \|u_{1,z}\|_{L^2} \|\tilde{u}_x\| \]
\[ \leq C \|\tilde{u}\|_{L^2} \|u_{1,z}\|_{L^2} \|\tilde{u}_x\| \]
\[ \leq C \|\tilde{u}\|_{L^2} \|u_{1,z}\|_{L^2} \|\tilde{u}_x\| \] (4.13)

and

\[ |\langle \tilde{w}v_{1,z}, \tilde{v} \rangle| \leq C \|\tilde{v}\|_{L^\infty_x(L^2)} \int_0^1 \|\tilde{v}_x\|_{L^2} \|v_{1,z}\|_{L^2} dx \]
\[ \leq C \|\tilde{v}\|_{L^2(L^\infty)} \|v_{1,z}\|_{L^2} \|\tilde{v}_x\| \]
\[ \leq C \|\tilde{v}\|_{L^2} \|v_{1,z}\|_{L^2} \|\tilde{v}_x\| \]
\[ \leq C \|\tilde{v}\|_{L^2} \|v_{1,z}\|_{L^2} \|\tilde{v}_x\| \] (4.14)

In the above estimates (4.13) and (4.14), we have used ideas similar to those used in (4.10). It now follows, similar to (4.13) and (4.14), that

\[ |\langle \tilde{w}v_{1,z}, \tilde{v} \rangle| \leq C \|\tilde{v}\|_{L^\infty_x(L^2)} \int_0^1 \|\tilde{v}_x\|_{L^2} \|v_{1,z}\|_{L^2} dx \]
\[ \leq C \|\tilde{v}\|_{L^2(L^\infty)} \|v_{1,z}\|_{L^2} \|\tilde{v}_x\| \]
\[ \leq C \|\tilde{v}\|_{L^2} \|v_{1,z}\|_{L^2} \|\tilde{v}_x\| \]
\[ \leq C \|\tilde{v}\|_{L^2} \|v_{1,z}\|_{L^2} \|\tilde{v}_x\| \] (4.15)

\[ |\langle \tilde{w}v_{1,z}, \tilde{v} \rangle| \leq C \|\tilde{v}\|_{L^\infty_x(L^2)} \int_0^1 \|\tilde{v}_x\|_{L^2} \|v_{1,z}\|_{L^2} dx \]
\[ \leq C \|\tilde{v}\|_{L^2(L^\infty)} \|v_{1,z}\|_{L^2} \|\tilde{v}_x\| \]
\[ \leq C \|\tilde{v}\|_{L^2} \|v_{1,z}\|_{L^2} \|\tilde{v}_x\| \]
\[ \leq C \|\tilde{v}\|_{L^2} \|v_{1,z}\|_{L^2} \|\tilde{v}_x\| \] (4.16)

and

\[ |\langle \tilde{w}\theta_{1,z}, \tilde{\theta} \rangle| \leq C \|\tilde{\theta}_{1,z}\|^4 \|\tilde{\theta}\|^2 + \varepsilon (\|\tilde{u}_x\|^2 + \|\tilde{\theta}_x\|^2), \]
\[ |\langle \tilde{w}\theta_{1,z}, \tilde{\theta} \rangle| \leq C \|\tilde{\theta}_{1,z}\|^4 \|\tilde{\theta}\|^2 + \varepsilon \|\tilde{u}_x\|^2. \]

Plug the estimates (4.9)–(4.12), (4.13) or (4.14), (4.15) or (4.16), and (4.17) or (4.18) into (4.8) and choose sufficiently small $\varepsilon > 0$ to absorb the dissipation terms. Then, assume that $(u_1, v_1, \theta_1)$ satisfies (4.1) and apply Gronwall lemma to finish the proof. Notice that global regularity result of weak solutions of (1.3)–(1.10) is also used in justifying applicability of Gronwall lemma. As an example to demonstrate the details, we now finish the proof for a special case of (4.1) when $(u_1, v_1, \theta_1)$ satisfies:

\[(u_1, v_1, \theta_1) \in \left[L^2(0, T; L^\infty_x(L^2))\right]^3. \] (4.19)
Plug (4.9)–(4.12), (4.14), (4.16) and (4.18) into (4.8) and choose sufficiently small \( \varepsilon > 0 \), we obtain, for \( t \in [t_0, T] \),

\[
\frac{d}{dt} \| (\tilde{u}, \tilde{v}, \tilde{\theta}) \|_H^2 + \| (\tilde{u}, \tilde{v}, \tilde{\theta}) \|_V^2 \\
\leq \left[ 1 + \| (u_1, v_1, \theta_1)_x \|_H^2 + \| (u_1, v_1, \theta_1)_z \|_{(L^\infty_x(L^2_z))^3} \right] \| (\tilde{u}, \tilde{v}, \tilde{\theta}) \|_H^2.
\]

(4.20)

Noticing (4.19), we can use Gronwall inequality to obtain, for \( t \in [t_0, T] \),

\[
\| (\tilde{u}(t), \tilde{v}(t), \tilde{\theta}(t)) \|_H^2 \leq \| (\tilde{u}(t_0), \tilde{v}(t_0), \tilde{\theta}(t_0)) \|_H^2 \\
\times \exp \left\{ \int_{t_0}^t \left[ 1 + \| (u_1, v_1, \theta_1)_x \|_H^2 + \| (u_1, v_1, \theta_1)_z \|_{(L^\infty_x(L^2_z))^3} \right] ds \right\}.
\]

Now, take the limit \( t_0 \to 0^+ \) and use Theorem 3.4, we get

\[
\| (\tilde{u}(t), \tilde{v}(t), \tilde{\theta}(t)) \|_H^2 \leq \| (\tilde{u}_0, \tilde{v}_0, \tilde{\theta}_0) \|_H^2 \\
\times \exp \left\{ \int_{0}^{t} \left[ 1 + \| (u_1, v_1, \theta_1)_x \|_H^2 + \| (u_1, v_1, \theta_1)_z \|_{(L^\infty_x(L^2_z))^3} \right] ds \right\}.
\]

Since the above inequality is independent of \( t_0 \) and \( t_0 \) can be chosen arbitrarily small, it is valid for all \( t \in (0, T] \). This proves that \((u_1, v_1, \theta_1) \equiv (u_2, v_2, \theta_2)\), if \((\tilde{u}_0, \tilde{v}_0, \tilde{\theta}_0) = (0, 0, 0)\). The other cases covered by (4.1) can be similarly proved. \(\square\)

5. Global Existence

In this section, we prove global in time uniform boundedness of the norms of some partial derivatives of the weak solutions. These results are also important for proving our uniqueness result in Sect. 6.

Let us mention first that it is not immediately clear whether or not the global uniform \( L^2_t H^2_x \) boundedness for the solutions of the 2D hydrostatic Navier–Stokes equations as obtained in [4] can be extended to the problem of (1.3)–(1.10). This is due to the fact that the boundary conditions (1.6) and (1.7) for \((u, v)\) are different from the boundary condition (1.9) for \( \theta \) and the possibility that \( \alpha_1 \) and \( \alpha_2 \) may be different. This problem will be studied elsewhere.

We begin with a theorem for global in time uniform boundedness of \((u_z, v_z, \theta_z)\) in \([L^2(D)]^3\) for (1.3)–(1.10). Recall that global existence and uniqueness of \( z \)-weak solutions were proved in [22] for 2D viscous PE in the case of periodic boundary conditions; and in [12] for 3D viscous PE in case of Neumann boundary condition for horizontal velocity at bottom of the physical domain. However, these analyses do not apply to the system (1.3)–(1.10) due to different boundary conditions. A possible approach might be a proper modification of that of [4] in obtaining boundedness for \( z \)-weak solutions of the simplified 2D hydrostatic Navier-Stokes equations, where \( v \) and \( \theta \) were omitted. Nevertheless, new issues will come up again due to boundary conditions. Instead, we will take advantage of a result of [23] directly in our proof of the following Theorem 5.1. It is worthy of pointing out that the main idea of our proof of Theorem 5.1 (and of Theorem 5.2) is rather non-traditional.

**Theorem 5.1.** Suppose \( Q \in L^2(D), (u_0, v_0, \theta_0) \in H \) and \((u, v, \theta)\) is a weak solution of (1.3)–(1.10). The following statements are valid:

(a) If \( \partial_z u_0 \in L^2(D) \), then there exists a weak solution \((u, v, \theta)\) of (1.3)–(1.10), such that

\[ u_z \in L^\infty[0, \infty; L^2(D)] \cap L^2(0, \infty; H^1(D)) \]

Moreover, there exists a bounded absorbing set for \( u_z \) in \( L^2(D) \).

(b) If \((\partial_z u_0, \partial_z v_0) \in (L^2(D))^2 \), then there exists a weak solution \((u, v, \theta)\) of (1.3)–(1.10), such that

\[ (u_z, v_z) \in L^\infty[0, \infty; [L^2(D)]^2] \cap L^2(0, \infty; [H^1(D)]^2) \]

Moreover, there exists a bounded absorbing set for \((u_z, v_z)\) in \([L^2(D)]^2\).
Moreover, there exists an absorbing set for \((u, v, \theta)\) of (1.3)–(1.10), such that
\[ (u_t, \theta_t) \in L^\infty(0, \infty; [L^2(D)]^2) \cap L^2(0, \infty; [H^1(D)]^2). \]

Moreover, there exists a bounded absorbing set for \((u_t, \theta_t)\) in \([L^2(D)]^2\).

Proof. Step 1. Proof of part (a) of Theorem 5.1.
By Theorem 3.4, we can choose a monotonically decreasing sequence
\[ \{t_n\}_{n=1}^\infty \subset (0, \infty), \text{ such that } \lim_{n \to \infty} t_n = 0, \]
and
\[ (u, v, \theta) \in C([t_n, \infty), V) \cap L^2(t_n, \infty; D(A)), \forall n \geq 1. \]
Moreover, there exists an absorbing set for \((u, v, \theta)\) in \(V\), when the time interval \([t_1, \infty)\) is considered. Therefore, what is still needed to be proved is just the following:
\[ u_t \in L^\infty(0, t_1; [L^2(D)]^2) \cap L^2(0, t_1; [H^1(D)]^2). \]

By the estimate of \(\|u_t\|\) in §3.3 of [23] for a strong solution \((u, v, \theta)\) on \([t_n, \infty)\) with initial data \((u(t_n), v(t_n), \theta(t_n)) \in V\) and by Theorem 3.3, we have for almost every \(t \in [t_n, \infty)\),
\[
\frac{d}{dt} (\|u_t(t)\|^2 + \alpha_1 \|u(t)|_{z=0}\|^2 + \|\nabla u_z\|^2 + \alpha_1 \|u_{x}\|_{z=0}\|^2) \\
\leq \|\nabla u\|^2 + \|v\|^2 + \|\theta_{x}\|^2.
\]

Notice that (5.1) is used in deriving (5.3). Therefore, we have for \(t \in [t_n, t_1]\) with \(n > 1\),
\[
\|u_t(t)\|^2 + \alpha_1 \|u(t)|_{z=0}\|^2 + \int_{t_n}^{t} (\|\nabla u_z\|^2 + \alpha_1 \|u_{x}\|_{z=0}\|^2) \, ds \\
\leq \|u(t_n)|_{z=0}\|^2 + \alpha_1 \|u(t_n)|_{z=0}\|^2 + C \left[ \|\nabla u\|^2 + \|v\|^2 + \|\theta_{x}\|^2 \right] ds.
\]

Since \(u(t_n) \in V_1\), we have
\[
\|u(t_n)|_{z=0}\| = \left\| \int_{-\infty}^{0} u_z(t_n)dz \right\| \leq \|u_z(t_n)\|.
\]

Due to the fact that \((u, v, \theta)\) is a weak solution on \((0, \infty)\), we also have, for \(t \in [t_n, t_1]\),
\[
\int_{t_n}^{t} (\|\nabla u\|^2 + \|v\|^2 + \|\theta_{x}\|^2) ds \leq \int_{0}^{t_1} (\|\nabla u\|^2 + \|v\|^2 + \|\theta_{x}\|^2) ds,
\]
the upper bound of which depends only on \(\|(u_0, v_0, \theta_0)\|_{H}, \|Q\|\) and \(t_1\). Combining (5.4)–(5.6), we have, for \(t \in [t_n, t_1]\),
\[
\|u_t(t)\|^2 + \alpha_1 \|u(t)|_{z=0}\|^2 + \int_{t_n}^{t} (\|\nabla u_z\|^2 + \alpha_1 \|u_{x}\|_{z=0}\|^2) \, ds \\
\leq \|u(t_n)|_{z=0}\|^2 + \int_{t_n}^{t_1} \left[ \|\nabla u\|^2 + \|v\|^2 + \|\theta_{x}\|^2 \right] ds.
\]

Now, choose any \(\phi \in C_\infty^0(D)\). Then, \(\phi_z \in V_1\). Thus, by weak continuity of \((u, v, \theta)\) on \([0, \infty)\) (see Theorem 3.1),\(^5\) we have
\[
\lim_{n \to \infty} \langle u_z(t_n) - \partial_z u_0, \phi \rangle = - \lim_{n \to \infty} \langle u(t_n) - u_0, \phi \rangle = 0.
\]

Since \(C_\infty^0(D)\) is dense in \(L^2(D)\), we have weak convergence:
\[ u_z(t_n) \rightharpoonup \partial_z u_0, \text{ in } L^2(D). \]

\(^5\) Indeed, even the strong continuity (3.42) is also valid by Theorem 3.4.
Therefore, \( \{u_z(t_n)\}_{n=1}^{\infty} \) is bounded in \( L^2(D) \). Now, taking the limit \( t_n \to 0 \) in (5.7), we have, for all \( t \in (0, t_1) \),
\[
\|u_z(t)\|^2 + \alpha_1 \|u(t)\|_{z=0}\|^2 + \int_0^t \left( \|\nabla u_z\|^2 + \alpha_1 \|u_x\|_{z=0}\|^2 \right) ds \\
\leq \sup_{n \geq 1} \|u_z(t_n)\|^2 + \int_0^{t_1} \left[ \|\nabla u\|^2 + \|v\|^2 + \|\theta_x\|^2 \right] ds.
\] (5.8)

This proves (5.2) and thus finishes Step 1.

**Step 2.** Proof of Theorem 5.1 part (b).

For simplicity of presentation, in the proof of part (b) of Theorem 5.1, we will only provide the key estimate of \( \|v_z\| \) under the assumption that \((u, v, \theta)\) is a strong solution on \([0, \infty)\). The justification that this estimate is sufficient for a rigorous proof of part (b) of Theorem 5.1 is almost the same as the one we provided in Step 1 for our proof of part (a) of Theorem 5.1. Thus, it is omitted for conciseness.

Taking inner product of (1.4) with \(-v_{zz}\) yields:
\[
\frac{1}{2} \frac{d}{dt} \left( \|v_z\|^2 + \alpha_2 \|v(z = 0)\|^2 \right) + \|\nabla v_z\|^2 + \alpha_2 \|v_x(z = 0)\|^2 = \langle uv_x + vw_z - u, v_{zz} \rangle.
\] (5.9)

The following computations are used in deriving (5.9):
\[
- \int v_tv_{zz} = - \int_0^1 \left( v_tv_z \big|_{z=-h}^0 - \int_{-h}^0 v_x dv_z \right) dx \\
= \frac{1}{2} \frac{d}{dt} \|v_z\|^2 + \int_0^1 v_t \alpha_2 v_z \big|_{z=0} \partial_z \left( v^2_z \right) dx \\
= \frac{1}{2} \frac{d}{dt} \|v_z\|^2 + \frac{\alpha_2}{2} \frac{d}{dt} \|v(z = 0)\|^2, \\
\int_D v_{zz} v_{xx} = \int_{-h}^0 \left( v_{zz} v_x \big|_{x=0}^1 - \int_0^1 v_{zzz} v_x dx \right) dz \\
= - \int_D v_{zzz} v_x dx dz \\
= - \int_0^1 \left( v_{zz} v_x \big|_{z=-h}^0 - \int_{-h}^0 v_{zzz}^2 dz \right) dx \\
= \|v_{zz}\|^2 + \alpha_2 \|v_x(z = 0)\|^2.
\]

The two tri-linear terms on the right-hand side of (5.9) will be estimated in the following.
First, we have
\[
\int_D wv_z v_{zz} = \frac{1}{2} \int_D w \partial_z (v_z^2) \\
= \frac{1}{2} \int_0^1 \left( wv_z^2 \big|_{z=-h}^0 - \int_{-h}^0 w_z v_z^2 \right) dx \\
= \frac{1}{2} \int_D u_x v_z^2 \\
\leq C \|u_x\| \|v_z\| \|v_{zz}\|^{1/2} \|v_{zz}\|^{1/2} \\
\leq C \varepsilon \|u_x\|^2 \|v_z\|^2 + \varepsilon \|\nabla v_z\|^2.
\] (5.10)
Contrary to the common intuition from experience, the other tri-linear term is more complicated to deal with. Integrating by parts and applying boundary conditions, one has

\[
\int_D uv_xv_z = \int_0^1 \left[ uv_xv_z \right]_{z=-h}^0 - \int_0^0 (u_zv_xv_z + uv_xv_z)dz \right] dx = -\alpha_2 \int_0^1 uv_xv_z |_{z=0} dx - \int_D u_xv_xv_z dxz - \int_D uv_xv_z dxz =: I_0 + I_1 + I_2.
\]

In the following, we estimate \(I_0, I_1\) and \(I_2\) respectively. First, we have

\[
|I_1| \leq C \|u_z\| \|v_x\| \frac{1}{2} \|v_{xx}\| \frac{1}{2} \|v_x\| \frac{1}{2} \|v_{xx}\| \frac{1}{2}
\]

\[
= C \|u_z\| \|v_x\| \frac{1}{2} \|v_x\| \frac{1}{2} \|v_{xx}\|
\]

\[
\leq C \epsilon (\|u_z\|^4 + \|v_x\|^2 \|v_x\|^2) + \frac{\epsilon}{2} \|v_{xx}\|^2,
\]

and

\[
|I_2| \leq C \|v_{xx}\| \|u\| \frac{1}{2} \|u_z\| \frac{1}{2} \|v_x\| \frac{1}{2} \|v_{xx}\| \frac{1}{2}
\]

\[
= C \|u\| \frac{1}{2} \|u_z\| \frac{1}{2} \|v_x\| \frac{1}{2} \|v_{xx}\|
\]

\[
\leq C \epsilon \|u\|^2 \|u_z\|^2 \|v_x\|^2 + \frac{\epsilon}{2} \|v_{xx}\|^2.
\]

Noticing that \(u(0, z, t) = 0\), we have

\[
u^2(x, 0, t) = 2 \int_0^x u(\xi, 0, t)u_x(\xi, 0, t) d\xi.
\]

Thus,

\[
\|u(z = 0)\|^2_{L^\infty} \leq 2 \|u(z = 0)\| \|v_x(z = 0)\|.
\]

So, we can estimate \(I_0\) as following:

\[
|I_0| \leq \alpha_2 \|u(z = 0)\| \|v(z = 0)\| \|v_x(z = 0)\|
\]

\[
\leq \frac{\alpha_2}{2} \|u(z = 0)\|^2_{L^\infty} \|v(z = 0)\|^2 + \frac{\alpha_2}{2} \|v_x(z = 0)\|^2
\]

\[
\leq \alpha_2 \|u(z = 0)\| \|v_x(z = 0)\| \|v(z = 0)\| \leq \frac{\alpha_2}{2} \|v(x = 0)\|^2.
\]

Due to the boundary condition \(u(z = -h) = 0\), it holds that,

\[
u(z = 0) = \int_{-h}^0 u_z(x, z, t) dz,
\]

and thus, by Minkowski inequality,

\[
\|u(z = 0)\| = \left[ \int_0^1 \left( \int_{-h}^0 u_z d\zeta \right)^2 dx \right]^\frac{1}{2}
\]

\[
\leq \left[ \int_0^1 \left( \int_{-h}^0 |u_z| d\zeta \right)^2 dx \right]^\frac{1}{2}
\]

\[
\leq h^\frac{1}{2} \|u_z\|.
\]
Similar to (5.12), we also have \( \|u_x(z = 0)\| \leq h^{\frac{1}{2}} \|u_{xx}\| \). Therefore,
\[
I_0 \leq \alpha_2 h \|u_x\| \|u_{xx}\| \|v(z = 0)\|^2 + \frac{\alpha_2}{2} \|v_x(z = 0)\|^2.
\]

Combining the above estimates of \( I_1, I_2 \) and \( I_0 \), we have
\[
\int_D uv_x v_z dx dz \leq C_\varepsilon (\|u\|^2 \|u_x\|^2 + \|v_x\|^2) \|v_z\|^2 + \alpha_2 h \|u_x\| \|u_{xx}\| \|v(z = 0)\|^2 + C_\varepsilon \|u_x\|^4
\]
\[
+ \varepsilon \|v_{xx}\|^2 + \frac{\alpha_2}{2} \|v_x(z = 0)\|^2; \tag{5.13}
\]

Finally, it follows from (5.9), (5.10) and (5.13) that, for \( \varepsilon > 0 \) chosen sufficiently small,
\[
\frac{d}{dt} (\|v_z\|^2 + \alpha_2 \|v(z = 0)\|^2) + \|\nabla v_z\|^2 + \alpha_2 \|v_x(z = 0)\|^2
\]
\[
\leq (\|u_x\|^2 + \|v_x\|^2 + \|u\|^2 \|u_x\|^2) \|v_z\|^2 + \alpha_2 \|u_x\| \|u_{xx}\| \|v(z = 0)\|^2 + \|u_x\|^4 + \|u\|^2
\]
\[
\leq (\|u_x\|^2 + \|v_x\|^2 + \|u\|^2 \|u_x\|^2 + \|u_x\| \|u_{xx}\|)
\]
\[
\times (\|v_z\|^2 + \alpha_2 \|v(z = 0)\|^2) + \|u_x\|^4 + \|u\|^2. \tag{5.14}
\]

Notice that, by (5.12), for \( v_{0,z} \in L^2 \),
\[
\|v_0(z = 0)\| \leq h^{\frac{1}{2}} \|v_{0,z}\| < \infty.
\]

By, Theorem 5.1 (a), (5.14) and Gronwall lemma, we have local in time boundedness of \( \|v_z\| \) on some interval \([0, t_0]\), with \( t_0 > 0 \). This yields global uniform boundedness and an absorbing set for \( \|v_z\| \), since we have uniform boundedness and an absorbing set for \( (u, v, \theta) \) in \( V \) when considered in \([t_0, \infty)\) as a strong solution.

Finally, with (5.14) we can justify as in our proof of part (a), that Theorem 5.1 (b) is valid for a weak solution \( (u, v, \theta) \) when \( \partial_x u_0, \partial_x v_0 \in L^2(D) \).

Proof of Theorem 5.1 (c) is similar to that for (b). \( \square \)

**Remark 5.1.** Quite unexpectedly, it is unclear and also seems non-trivial whether similar global regularity property for \( (u_z, v_x) \), or \( (u_z, \theta_x) \) is valid when \( (u_{0,z}, v_{0,x}) \in [L^2(D)]^2 \), or \( (u_{0,z}, \theta_{0,x}) \in [L^2(D)]^2 \) respectively.

Notice that vertical regularity \( (u_z, v_x, \theta_x) \in L^2 \) of weak solutions of 2D and 3D viscous PE played a very prominent or even crucial role in almost all previous analytic works in dealing with solution regularity and uniqueness properties, for example, as shown in (4.1). This is mainly due to the special feature of the hydro-static balance. However, to the contrary of this intuitive impression, we see next that horizontal regularity might actually force weak solutions to behave somewhat better, at least for the 2D problem. This is manifested in the following Theorem 5.2, which is our second main result of this section.

**Theorem 5.2.** Suppose \( Q \in L^2(D) \), \( (u_0, v_0, \theta_0) \in H \) and \( (u, v, \theta) \) is a weak solution of (1.3)–(1.10). The following statements are valid:

(a) If \( \partial_x u_0 \in L^2(D) \), then
\[
\|u_x\| \in L^\infty(0, +\infty; L^2(D)) \cap L^2(0, \infty; H^1(D)).
\]

Moreover, there exists a bounded absorbing set for \( u_x \) in \( L^2(D) \).

(b) If \( (\partial_x u_0, \partial_x v_0) \in [L^2(D)]^2 \), then
\[
(u_x, v_x) \in L^\infty(0, +\infty; [L^2(D)]^2) \cap L^2(0, \infty; [H^1(D)]^2).
\]

Moreover, there exists a bounded absorbing set for \( (u_x, v_x) \) in \( [L^2(D)]^2 \).
(c) If \((\partial_x u_0, \partial_x \theta_0) \in [L^2(D)]^2\), then
\[
(u_x, \theta_x) \in L^\infty(0, +\infty; [L^2(D)]^2) \cap L^2(0, \infty; [H^1(D)]^2).
\]
Moreover, there exists a bounded absorbing set for \((u_x, \theta_x)\) in \([L^2(D)]^2\).

(d) If \((\partial_x u_0, \partial_x v_0) \in [L^2(D)]^2\), then
\[
(u_x, v_z) \in L^\infty(0, +\infty; [L^2(D)]^2) \cap L^2(0, \infty; [H^1(D)]^2).
\]
Moreover, there exists a bounded absorbing set for \((u_x, v_z)\) in \([L^2(D)]^2\).

(e) If \((\partial_x u_0, \partial_x \theta_0) \in [L^2(D)]^2\), then
\[
(u_x, \theta_z) \in L^\infty(0, +\infty; [L^2(D)]^2) \cap L^2(0, \infty; [H^1(D)]^2).
\]
Moreover, there exists a bounded absorbing set for \((u_x, \theta_z)\) in \([L^2(D)]^2\).

Proof. Again, for simplicity of presentation, in the proof of Theorem 5.2, we will only provide the key estimates of \(\|u_x\|, \|v_z\|, \|\theta_x\|, \|\theta_z\|\) and \(\|\theta_z\|\) under the assumption that \((u, v, \theta)\) is a strong solution on \([0, \infty)\). The justification that these estimates are sufficient for a rigorous proof of Theorem 5.2 is almost the same as the one we provided in our proof of Theorem 5.1 (a). Thus, it is omitted for conciseness.

We provide these key estimates in three steps.

Step 1. Estimate for \(\|u_x\|_2\).

Taking inner product of (1.3) with \(-u_{xx}\) yields
\[
\frac{1}{2} \frac{d}{dt} \|u_x\|^2 + \|\nabla u_x\|^2 + \alpha_1 \|u_x(z = 0)\|^2 = \left< uu_x + uw_z + v + q_x + \int_0^t \theta(x, \zeta, t) d\zeta, u_{xx} \right>. \tag{5.15}
\]

The following computations, along with the boundary conditions (1.6)–(1.9) and the constraint (1.11), are used in the derivation of (5.15):

\begin{align*}
- \int_D u_t u_{xx} dx dz &= - \int_{-h}^0 \left( u_t u_{x} \right)_{x=0}^1 - \int_0^1 u_t u_x \right) dx = \frac{1}{2} \frac{d}{dt} \|u_x\|^2, \\
\int_D u_x u_{xx} dx dz &= \int_{-h}^0 \left( u_x u_{xx} \right)_{x=0}^1 - \int_0^1 u_x u_{xx} dx \right) dz \\
&= - \int_D u_x u_{xx} dx dz \\
&= - \int_0^1 \left( u_x u_{xx} \right)_{z=-h}^0 - \int_{-h}^0 |u_{xx}|^2 \right) dx \\
&= \alpha_1 \|u_x(z = 0)\|^2 + \|u_{xx}\|^2, \\
\int_D q_x u_{xx} dx dz &= \int_0^1 q_x \left( \int_{-h}^0 u_{xx} dx \right) dx = \int_0^1 q_x \partial_x \left( \int_{-h}^0 u_x dx \right) dx = 0.
\end{align*}

Now, we estimate the two tri-linear terms on the right-hand side of (5.15) as following:

\[
\langle uu_x, u_{xx} \rangle = \int_{-h}^0 \int_0^1 u \partial_x \left( \frac{u_x^2}{2} \right) dx dz = - \frac{1}{2} \int_D u_x^3 \\
\leq C \|u_x\|^2 \|u_{xx}\|^\frac{3}{2} \|u_x\|^\frac{1}{2} \\
\leq C \|u_x\|^4 + \varepsilon \|\nabla u_x\|^2.
\]

\[
\langle uw_z, u_{xx} \rangle \leq C \|w\|^\frac{3}{2} \|u_x\|^\frac{3}{2} \|u_{xx}\|^\frac{3}{2} \|u_{xx}\|^\frac{1}{2} \|u_{xx}\| \\
\leq C \|u_x\|^2 \|u_{xx}\|^2 + \varepsilon \|\nabla u_x\|^2.
\]
Therefore, it follows from (5.15) that

\[
\frac{1}{2} \frac{d}{dt} \left( \|u_x\|^2 + \|\nabla u_x\|^2 + \alpha_1 \|u_x(z = 0)\|^2 \right) \\
\leq 2\varepsilon \|\nabla u_x\|^2 + C_\varepsilon (1 + \|u_x\|^2) \|u_x\|^4 + (\|v\| + \|\theta_x\|) \|u_{xx}\| \\
\leq 3\varepsilon \|\nabla u_x\|^2 + C_\varepsilon (1 + \|u_x\|^2) \|u_x\|^4 + C_\varepsilon (\|v\|^2 + \|\theta_x\|^2)
\]

Thus, if \( \varepsilon > 0 \) is chosen sufficiently small, then

\[
\frac{d}{dt} \|u_x\|^2 + \|\nabla u_x\|^2 + \alpha_1 \|u_x(z = 0)\|^2 \leq (1 + \|u_x\|^2) \|u_x\|^4 + \|v\|^2 + \|\theta_x\|^2. \tag{5.16}
\]

A local in time upper bound of \( \|u_x\| \) can be obtained from (5.16) as follows. By (5.16),

\[
\frac{d}{dt} \|u_x\|^2 \leq C(1 + \|u_x\|^2 + \|v\|^2 + \|\theta_x\|^2)(\|u_x\|^2 + 1)^2.
\]

Denote:

\[
y(t) := \|u_x\|^2 + 1, \quad g(t) := C(1 + \|u_x\|^2 + \|v\|^2 + \|\theta_x\|^2).
\]

Then

\[
y'(t) \leq g(t)g^2(t).
\]

Notice that \( g(t) \geq 1, g(t) \geq C(>0) \) and \( g \in L^1(0, +\infty) \). Therefore,

\[
y(t) \leq \frac{y(0)}{1 - y(0) \int_0^t g(s)ds},
\]

for \( t \in (0, t_*) \) where \( t_* \) is decided by the following equation:

\[
y(0) \int_0^{t_*} g(s)ds = 1.
\]

Thus, for \( t \in (0, t_*) \)

\[
\|u_x\|^2 + 1 \leq \frac{\|u_{0,x}\|^2 + 1}{1 - (\|u_{0,x}\|^2 + 1) \int_0^t g(s)ds}.
\]

This finishes the proof of local in time boundedness of \( \|u_x\| \).

One the other hand, there exists \( t_0 \in (0, t_*) \), such that

\[
(u(t_0), v(t_0), \theta(t_0)) \in V.
\]

Therefore, by the result of uniform boundedness of strong solutions (see §3.3 of [18,23]) and its uniqueness on \( [t_0, +\infty) \), there exists a bounded absorbing set for \( (u, v, \theta) \) in \( V \) for \( t \in [t_0, \infty) \), thus a bounded absorbing set for \( u_x \) in \( L^2 \) for \( t \in [0, \infty) \). It also proves the uniform boundedness:

\[
u_x \in L^\infty(0, +\infty; L^2).
\]

Then, integrating (5.16) for \( t \) from 0 to \( \infty \) proves

\[
\nabla u_x \in L^2(0, +\infty; L^2).
\]

This finishes proof of Theorem 5.2 (a).

**Step 2.** Estimate of \( \|v_x\|_2 \) and \( \|\theta_x\| \).

Similar to **Step 1**, taking inner product of (1.4) with \(-v_{xx}\) yields:

\[
\frac{1}{2} \frac{d}{dt} \|v_x\|^2 + \|\nabla v_x\|^2 + \alpha_2 \|v_x(z = 0)\|^2 = \langle uv_x + wv_z - u, v_{xx} \rangle. \tag{5.17}
\]
The tri-linear terms on the right-hand side of (5.17) can be estimated as following:
\[
\int_D uw_x v_{xx} = \frac{1}{2} \int_D u \partial_x (v_x^2) dx dz = \frac{1}{2} \int_D u_x v_x^2 dx dz \\
\leq C \|u_x\|_2 \|v_x\|_2 \|v_{xx}\|_2 \|v_x\|_2 \varepsilon \\
\leq C \epsilon \|u_x\|^2 \|v_x\|^2 + \varepsilon \|\nabla v_x\|^2.
\]
\[
\int_D uv_z v_{xz} \leq C \|u\|^2 \|v_z\|^\frac{1}{2} \|v_x\|^\frac{1}{2} \|v_{xz}\|^\frac{1}{2} \|v_{xx}\| \\
\leq C \epsilon \|u_x\|^4 \|v_x\|^2 + \varepsilon \|\nabla v_x\|^2.
\]
Thus, if \( \epsilon > 0 \) is chosen sufficiently small, then
\[
\frac{d}{dt} \|v_x\|^2 + \|\nabla v_x\|^2 + \alpha_2 \|v_x(z = 0)\|^2 \leq C (\|u_x\|^2 \|v_x\|^2 + \|u_x\|^4 \|v_x\|^2 + \|u\|^2).
\] (5.18)
Similarly, we have
\[
\frac{d}{dt} \|\theta_x\|^2 + \|\nabla \theta_x\|^2 + \alpha_0 \|\theta_x(z = 0)\|^2 \leq C (\|u_x\|^2 \|\theta_x\|^2 + \|u_x\|^4 \|\theta_x\|^2 + \|Q\|^2).
\] (5.19)
Notice the fact that \( v_z, \theta_z \in L^2(0, \infty; L^2) \). Thus, as argued in Step 1, Theorem 5.2 (b) follows from Theorem 5.2 (a) and (5.18); Theorem 5.2 (c) follows from Theorem 5.2 (a) and (5.19).

**Step 3.** Estimate of \( \|v_z\|_2 \) and \( \|\theta_z\|_2 \).

Recall (5.9). The two tri-linear terms on the right-hand side of (5.9) can be estimated in the following. First, we have
\[
\int_D uv_z v_{zz} \leq C \|u\|^2 \|v_z\|^\frac{1}{2} \|v_x\|^\frac{1}{2} \|v_{xz}\|^\frac{1}{2} \|v_{zz}\| \\
\leq C \epsilon \|u_x\|^2 \|u_x\|^2 \|v_x\|^2 + \varepsilon \|\nabla v_x\|^2
\] (5.20)
Next, the estimate (5.10) will be used again. Thus, if \( \epsilon > 0 \) is chosen sufficiently small, then
\[
\frac{d}{dt} (\|v_z\|^2 + \alpha_2 \|v(z = 0)\|^2) + \|\nabla v_z\|^2 + \alpha_2 \|v_z(z = 0)\|^2 \leq \|u\|^2 \|u_x\|^2 \|v_x\|^2 + \|u_x\|^2 \|v_z\|^2 + \|u\|^2.
\] (5.21)
Similarly, we have
\[
\frac{d}{dt} (\|\theta_z\|^2 + \alpha_0 \|\theta(z = 0)\|^2) + \|\nabla \theta_z\|^2 + \alpha_2 \|\theta_z(z = 0)\|^2 \leq \|u\|^2 \|u_x\|^2 \|\theta_x\|^2 + \|u_x\|^2 \|\theta_z\|^2 + \|Q\|^2.
\] (5.22)
Now, Theorem 5.2 (d) and (e) follow from (5.21) and (5.22) respectively as in Step 1 and Step 2. □

**6. Uniqueness**

In this section, we state and prove our last main result, Theorem 6.1, on uniqueness of weak solutions of 2D viscous PE (1.3)–(1.10) when some initial partial regularity is assumed.
Theorem 6.1. Suppose $Q \in L^2(D)$, $(u_0, v_0, \theta_0) \in H$ and $(u, v, \theta)$ is a weak solution of (1.3)–(1.10). Suppose further that one of the following initial regularity is valid:

$$(\partial_x u_0, \partial_x v_0, \partial_x \theta_0) \in (L^2(D))^3,$$

or

$$(\partial_x u_0, \partial_x v_0, \partial_x \theta_0) \in (L^2(D))^3,$$

or

$$(\partial_x u_0, \partial_x v_0, \partial_x \theta_0) \in (L^2(D))^3,$$

or

$$(\partial_x u_0, \partial_x v_0, \partial_x \theta_0) \in (L^2(D))^3.$$

Then, the following are valid:

(a) The norm for the corresponding solution regularity is uniformly bounded for all time $t \geq 0$ and an absorbing set exists for the norm of the corresponding solution regularity.

(b) The weak solution is unique.

Proof. The claim (a) for solution regularity is an immediate consequence of Theorems 5.1 and 5.2. Moreover, assuming any one of the above five initial conditions, we have

$$(u_{xz}, v_{xz}, \theta_{xz}) \in [L^2(0, \infty; L^2(D))]^3. \tag{6.1}$$

Notice that

$$\| u_z \|_{L^\infty_x (L^2_z)} = \left\| \left( \int_{-h}^{0} |u_z|^2 \right)^{\frac{1}{2}} \right\|_{L^\infty_x} \leq \left( \int_{-h}^{0} \| u_z \|_{L^\infty_x}^2 \right)^{\frac{1}{2}} \text{ (Minkowski inequality)} \leq C \left( \int_{-h}^{0} \| u_z \|_{L^2_z}^2 \| u_{xz} \|_{L^2_z} \right)^{\frac{1}{2}} \text{ (Agmon’s inequality)} \leq C \| u_z \|^{\frac{1}{2}} \| u_{xz} \|^{\frac{3}{2}}. \tag{6.2}$$

Therefore, for any $T > 0$,

$$\int_0^T \| u_z \|_{L^\infty_x (L^2_z)}^2 dt \leq C \int_0^T \| u_z \| \| u_{xz} \| dt \leq C \left( \int_0^T \| u_z \|^2 dt \right)^{\frac{1}{2}} \left( \int_0^T \| u_{xz} \|^2 dt \right)^{\frac{1}{2}}. \tag{6.2}$$

Notice that for any weak solution $(u, v, \theta)$,

$$(u_z, v_z, \theta_z) \in L^2(0, T; L^2(D)).$$

Thus, by (6.1) and (6.2), we have

$$u_z \in L^2(0, T; L^\infty_x (L^2_z)).$$

Similarly,

$$(u_z, v_z, \theta_z) \in [L^2(0, T; L^\infty_x (L^2_z))]^3.$$

Thus, Theorem 6.1 (b) is proved by Theorem 4.1. \hfill \Box

Remark 6.1. Due to the reason mentioned in Remark 5.1, it is unclear and also seems nontrivial if uniqueness is still valid for weak solutions with one of the following three in $(L^2(D))^3$:

$$(\partial_z u_0, \partial_z v_0, \partial_z \theta_0), \text{ or } (\partial_z u_0, \partial_z v_0, \partial_z \theta_0), \text{ or } (\partial_z u_0, \partial_z v_0, \partial_z \theta_0).$$
Acknowledgements. The author thanks the referee for careful reading and kind comments. He also thanks for the kind help of Professors Giovanni Galdi and Matthias Hieber taking care of this manuscript.

Declarations
Conflict of interest There is no conflict of interest.

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

References

[1] Azérad, P., Guillen, F.: Mathematical justification of the hydrostatic approximation in the primitive equations of geophysical fluid dynamics. SIAM J. Math. Anal. 33(4), 847–859 (2001)
[2] Bresch, D., Guillon-Gonzáez, F., Masmoudi, N., Rodríguez-Bellido, M.A.: Asymptotic derivation of a Navier condition for the primitive equations. Asympt. Anal. 33(1), 237–259 (2003)
[3] Bresch, D., Guillon-Gonzáez, F., Masmoudi, N., Rodríguez-Bellido, M.A.: On the uniqueness of weak solutions of the two-dimensional primitive equations. Differ. Integral Equ. 16(1), 77–94 (2003)
[4] Bresch, D., Kazhikhov, A., Lemoine, J.: On the two-dimensional hydrostatic Navier–Stokes equations. SIAM J. Math. Anal. 36(3), 796–814 (2005)
[5] Cao, C., Titi, E.S.: Global well-posedness of the three-dimensional primitive equations of large scale ocean and atmosphere dynamics. Ann. Math. (2) 166(1), 245–267 (2007)
[6] Cao, C., Wu, J.: Global regularity for the two-dimensional anisotropic Boussinesq equations with vertical dissipation. Arch. Ration. Mech. Anal. 208(3), 985–1004 (2013)
[7] Evans, L., Gastler, R.: Some results for the primitive equations with physical boundary conditions. Z. Angew. Math. Phys. 64(6), 1729–1744 (2013)
[8] Galdi, G.P.: An introduction to the Navier–Stokes initial-boundary value problem. In: Galdi, G.P., Heywood, J.G., Rannacher, R. (eds.) Fundamental Directions in Mathematical Fluid Mechanics. Advances in Mathematical Fluid Mechanics, pp. 1–70. Basel, Birkhäuser (2000)
[9] Guillon-Gonzáez, F., Masmoudi, N., Rodríguez-Bellido, M.A.: Anisotropic estimates and strong solutions of the Primitive Equations. Differ. Integral Equ. 14(1), 1381–1408 (2001)
[10] Hieber, M., Kashiwabara, T.: Global well-posedness of the three-dimensional primitive equations in Lp-space. Arch. Ration. Mech. Anal. 221, 1077–1115 (2016)
[11] Ju, N.: The global attractor for the solutions to the 3D viscous primitive equations. Discrete Contin. Dyn. Syst. 17(1), 159–179 (2007)
[12] Ju, N.: On $H^2$ solutions and $z$-weak solutions of the 3D primitive equations. Indiana Univ. Math. J. 66(3), 973–996 (2017)
[13] Ju, N.: Global uniform boundedness of solutions to viscous 3D Primitive Equations with physical boundary conditions. Indiana Univ. Math. J. 69(5), 1763–1784 (2020)
[14] Ju, N., Temam, R.: Finite dimensions of the global attractor for 3D primitive equations with viscosity. J. Nonlinear Sci. 25(1), 131–155 (2015)
[15] Kobelkov, G.: Existence of a solution “in the large” for ocean dynamics equations. J. Math. Fluid Mech. 9(4), 588–610 (2007)
[16] Kukavica, I., Pei, Y., Rusin, W., Ziane, M.: Primitive equations with continuous initial data. Nonlinearity 27(6), 1135–1155 (2014)
[17] Kukavica, I., Ziane, M.: On the regularity of the primitive equations of the ocean. Nonlinearity 20(12), 2739–2753 (2007)
[18] Kukavica, I., Ziane, M.: Uniform gradient bounds for the primitive equations of the ocean. Differ. Integral Equ. 21(9–10), 837–849 (2008)
[19] Li, J., Titi, E.: Existence and uniqueness of weak solutions to viscous primitive equations for a certain class of discontinuous initial data. SIAM J. Math. Anal. 49(1), 1–28 (2017)
[20] Lions, J., Magenes, E.: Non-Homogeneous Boundary Value Problems and Applications. Springer, Berlin (1972)
[21] Lions, J., Temam, R., Wang, S.: On the equations of the large scale Ocean. Nonlinearity 5, 1007–1053 (1992)
[22] Petcu, M.: On the backward uniqueness of primitive equations. J. Math. Pures Appl. 87, 275–289 (2007)
[23] Petcu, M., Temam, R., Ziane, M.: Some mathematical problems in geophysical fluid dynamics. In: Temam, R.M., Tribbia, J.J., Ciarlet, P.G. (eds.) Handbook of Numerical Analysis, Special Volume: Computational Methods for the Atmosphere and the Oceans, vol. XIV, pp. 577–750. Elsevier/North-Holland, Amsterdam (2009)
[24] Tachim Medjo, T.: On the uniqueness of z-weak solutions of the three-dimensional primitive equations of the ocean. Nonlinear Anal. Real World Appl. 11(3), 1413–1421 (2010)
[25] Temam, R.: Navier–Stokes Equations: Theorey and Numerical Analysis. Reprinted by American Mathematical Society (2001)
[26] Temam, R., Ziane, M.: Some mathematical problems in geophysical fluid dynamics. In: Friedlander, S., Serre, D. (eds.) Handbook of Mathematical Fluid Dynamics, vol. 3, pp. 535–658. Elsevier, Amsterdam (2004)

Ning Ju
Department of Mathematics
Oklahoma State University
401 Mathematical Sciences
Stillwater OK74078
USA
e-mail: ning.ju@okstate.edu

(accepted: August 10, 2021; published online: September 2, 2021)