The optimal free knot spline approximation of stochastic differential equations with additive noise

Mehdi Slassi*

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Abstract

In this paper we analyse the pathwise approximation of stochastic differential equations by polynomial splines with free knots. The pathwise distance between the solution and its approximation is measured globally on the unit interval in the $L_\infty$-norm, and we study the expectation of this distance. For equations with additive noise we obtain sharp lower and upper bounds for the minimal error in the class of arbitrary spline approximation methods, which use $k$ free knots. The optimal order is achieved by an approximation method $\hat{X}_k$, which combines an Euler scheme on a coarse grid with an optimal spline approximation of the Brownian motion $W$ with $k$ free knots.

Keywords: Stochastic differential equation; Pathwise uniform approximation; Spline approximation; Free knots

1 Introduction

Consider a scalar stochastic differential equation (SDE) with additive noise

$$dX(t) = a(t, X(t))dt + \sigma(t)dW(t), \quad t \in [0,1], \quad (1)$$

with initial value $X(0)$. Here $W = (W(t))_{t \geq 0}$ denotes a one-dimensional Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We study pathwise approximation of equation (1) on the unit interval by polynomial splines with free knots.

Let $X$ and $\hat{X}$ denote the strong solution and an approximate solution on $[0,1]$, respectively. For the pathwise error we consider the distance in $L_\infty$-norm

$$\|X - \hat{X}\|_{L_\infty[0,1]} = \sup_{0 \leq t \leq 1} |X(t) - \hat{X}(t)|,$$

*Fachbereich Mathematik, Technische Universität Darmstadt, Schloßgartenstraße 7, 64289 Darmstadt, Germany
and we define the error $e_q(\hat{X})$ of the approximation $\hat{X}$ by averaging over all trajectories, i.e.,

$$e_q(\hat{X}) = \left(E^*\|X - \hat{X}\|^q_{L_\infty[0,1]}\right)^{1/q}, \quad 1 \leq q < \infty. \quad (2)$$

Here we use the outer expectation value $E^*$ in order to avoid cumbersome measurability considerations. The reader is referred to [14] for a detailed study of the outer integral and expectation. In the sequel, for two sequences $(a_k)_{k\in\mathbb{N}}$ and $(b_k)_{k\in\mathbb{N}}$ of positive real numbers we write $a_k \approx b_k$ if $\lim_{k \to \infty} a_k / b_k = 1$ and $a_k \gtrsim b_k$ if $\liminf_{k \to \infty} a_k / b_k \geq 1$. Additionally $a_k \asymp b_k$ means $C_1 \leq a_k / b_k \leq C_2$ for all $k \in \mathbb{N}$ and some positive constants $C_i$.

Typically, piecewise linear functions with fixed knots or with sequential selection of knots are used to approximate the solution of SDEs globally on a time interval, and approximations of this kind are considered in the present paper as particular cases, too.

For $k \in \mathbb{N}$ we use $\hat{X}_k^e$ to denote the piecewise interpolated Euler scheme with constant step-size $1/k$. In [8] Hofmann et al. have determined the strong asymptotic behaviour of $e_q(\hat{X}_k^e)$ with an explicitly given constant, namely

$$e_q(\hat{X}_k^e) \approx \frac{C_e}{\sqrt{2}} \cdot (\ln k / k)^{1/2} \quad (3)$$

with

$$C_e = \|\sigma\|_{L_\infty[0,1]},$$

where $\|\sigma\|_{L_\infty[0,1]} = \sup_{t \in [0,1]} |\sigma(t)|$. Note that the upper bound in (3) has first been given in [6] with an unspecified constant.

Now, we recall known results concerning the approximations that are based on a sequential selection of knots to evaluate $W$, see [8][12] for a formal definition of such methods. This includes numerical methods with adaptive step size control. In [8] Hofmann et al. show that a step size proportional to the inverse of the current value of $\sigma^2$ leads to an asymptotically optimal method $\hat{X}_k^a$, more precisely

$$e_q(\hat{X}_k^a) \approx \frac{C_a}{\sqrt{2}} \cdot (\ln k / k)^{1/2} \quad (4)$$

and

$$C_a = \|\sigma\|_2,$$

where $\|\sigma\|_2 = \left(\int_0^1 (\sigma(t))^2 \, dt\right)^{1/2}$. Moreover, they establish strong asymptotic optimality of the sequence $\hat{X}_k^a$, i.e., for every sequence of methods $\hat{X}_k$ that use $k$ sequential observations of $W$

$$e_q(\hat{X}_k) \gtrsim \frac{C_a}{\sqrt{2}} \cdot (\ln k / k)^{1/2}. \quad (5)$$

Typically, $C_a < C_e$ and $C_a > 0$, so that the convergence order $(\ln k / k)^{1/2}$ cannot be improved by sequential observation of $W$. A generalization of the results [3].
and (5) to the case of systems of equations with multiplicative noise has
been achieved in [12]. In the present paper we do not impose any restriction on the selection of the knots.

For \( k \in \mathbb{N} \) and \( r \in \mathbb{N}_0 \) we let \( \Pi_r \) denote the set of polynomials of degree at most \( r \), and we consider the space \( \Phi_{k,r} \) of polynomial splines \( \varphi \) of degree at most \( r \) with \( k-1 \) free knots, i.e.,

\[
\varphi = \sum_{j=1}^{k} \mathbf{1}_{[t_{j-1}, t_j]} \cdot \pi_j,
\]

where \( 0 = t_0 < \cdots < t_k = 1 \) and \( \pi_1, \ldots, \pi_k \in \Pi_r \). Note that the spline \( \varphi \) uses \( k+1 \) knots, whereof \( k-1 \) can be chosen freely. Then, any approximation method \( \tilde{X}_k \) by splines with \( k-1 \) free knots can be thought of as a mapping

\[
\tilde{X}_k : \Omega \longrightarrow \Phi_{k,r},
\]

and we denote this class of mappings by \( \mathcal{N}_{k,r} \).

Furthermore, we define the minimal error

\[
e_{k,q}^{\min}(X) = \inf \left\{ e_q(\tilde{X}_k) : \tilde{X}_k \in \mathcal{N}_{k,r} \right\},
\]

i.e., the \( q \)-average \( L_\infty \)-distance of the solution \( X \) to the spline space \( \Phi_{k,r} \). We shall study the strong asymptotic behaviour of \( e_{k,q}^{\min}(X) \) as \( k \) tends to infinity.

Note that spline approximation with free knots is a nonlinear approximation problem in the sense that the approximants do not come from linear spaces but rather from nonlinear manifolds \( \Phi_{k,r} \). Nonlinear approximation for deterministic functions has been extensively studied in the literature, see [5] for a survey. In the context of stochastic processes much less is known, and we refer the reader to [1, 2, 4, 9, 13]. At first in [9] and thereafter in [4, 13] approximation by splines with free knots is studied, while wavelet methods are employed in [1, 2].

From Creutzig et al. [4] we know, that

\[
e_{k,q}^{\min}(X) \approx (1/k)^{1/2}.
\]

Hence free knot spline approximation yields a better rate of convergence than (3) and (4). We add, that the same order of convergence is achieved by the average Kolmogorov widths, see [3, 10, 11], but asymptotically optimal subspaces seem to be unknown.

In [13] we analyse an approximation method \( \tilde{X}^*_k \), which achieves the convergence order \( 1/\sqrt{k} \). The method \( \tilde{X}^*_k \) combines a Milstein scheme on a coarse grid with an optimal spline approximation of the Brownian motion \( W \). The approximation method \( \tilde{X}^*_k \) basically works in two steps. First, we take the Milstein scheme to estimate the drift and diffusion coefficients at equidistant discrete points \( t_\ell \). At the second stage we piecewise freeze the drift and diffusion coefficients and we consider on each subinterval \([t_{\ell-1}, t_\ell]\) the asymptotically optimal
spline approximation of the Brownian motion $W(t) - W(t_0)$ with equal number of free knots fixed a priori. For adaptive step size control a similar idea has been used in [7]. In the particular case of SDEs with additive noise, we show that the error of $\hat{X}^*_k$ satisfies
\[ e_q(\hat{X}^*_k) \approx (E(\tau_{1,1}))^{-1/2} \cdot C_e \cdot (1/k)^{1/2}, \]
where
\[ \tau_{1,1} = \inf \left\{ t > 0 \mid \inf_{\pi \in \Pi_r} \| W - \pi \|_{L_\infty(t_{j-1}, t_j]} > \varepsilon \right\}. \]
Hence the stopping time $\tau_{1,1}$ yields the maximal length of a subinterval $[0, t]$ that permits best approximation of $W$ by polynomials of degree at most $r$ with error at most one.

In order to improve the asymptotic constant in (8) we introduce in the present paper an approximation method $\hat{X}^\dagger_k$. The method $\hat{X}^\dagger_k$ is defined in the same way as $\hat{X}^*_k$, where the number of free knots used in each subinterval $[t_{j-1}, t_j]$ is roughly proportional to $(\sigma(t_{j-1}))^2$. For the error of $\hat{X}^\dagger_k$ we establish the strong asymptotic behaviour with an explicitly given constant, namely
\[ e_q(\hat{X}^\dagger_k) \approx (E(\tau_{1,1}))^{-1/2} \cdot C_a \cdot (1/k)^{1/2}. \] (9)
Note that the new approximation performs asymptotically better than the approximation $\hat{X}^*_k$ in many cases.

In [4] the lower and upper bound in (7) are proven non-constructively and the method of proof does not allow to control asymptotic constants. In this paper we wish to find sharp lower and upper bounds for the minimal error (6) for SDEs with additive noise. We show that the minimal errors satisfy
\[ e_{k,q}^{\min}(X) \approx (E(\tau_{1,1}))^{-1/2} \cdot C_a \cdot (1/k)^{1/2}. \] (10)
We note that the order of convergence in (9) and (10) does not depend on the degree $r$ of the approximation splines. The parameter $r$ has only an impact on the asymptotic constant $E(\tau_{1,1})$. We add that due to (9) and (10) the method $\hat{X}^\dagger_k$ is asymptotically optimal in the class $\mathcal{N}_{k,r}$ for every equation (1) with additive noise.

The structure of the paper is as follows. In Section 2 we specify our assumptions regarding the equation (1). The drift and diffusion coefficients must satisfy Lipschitz conditions, and the initial value must have a finite $q$-moment for all $q \geq 1$. Moreover, we briefly recall some definitions and results from [4] concerning the optimal approximation of $W$ by polynomial splines with free knots. We introduce the approximation method $\hat{X}^\dagger_k$ and state the main results. Proofs are given in Section 3.

2 Main result

Given $\varepsilon > 0$, we define a sequence of stopping times by $\tau_{0,\varepsilon} = 0$ and
\[ \tau_{j,\varepsilon} = \tau_{j,\varepsilon}(W) = \inf \left\{ t > \tau_{j-1,\varepsilon} \mid \inf_{\pi \in \Pi_r} \| W - \pi \|_{L_\infty(\tau_{j-1,\varepsilon}, t]} > \varepsilon \right\}, \quad j \geq 1. \]
For \( j \in \mathbb{N} \) we define
\[
\xi_{j, \varepsilon} = \tau_{j, \varepsilon} - \tau_{j-1, \varepsilon}.
\]
These random variables yield the lengths of consecutive maximal subintervals that permit best approximation from the space \( \Pi_r \) with error at most \( \varepsilon \). For every \( \varepsilon > 0 \) the random variables \( \xi_{j, \varepsilon} \) form an i.i.d. sequence with
\[
\xi_{j, \varepsilon} \overset{d}{=} \varepsilon^2 \cdot \tau_{1,1} \quad \text{and} \quad E \left( \tau_{1,1}^m \right) < \infty
\]
for every \( m \in \mathbb{N} \), see [4]. Furthermore, we consider the pathwise minimal approximation error by splines using \( k - 1 \) free knots
\[
\gamma_k = \gamma_k(W) = \inf \{ \varepsilon > 0 \mid \tau_{k, \varepsilon} \geq 1 \}.
\]
An optimal spline approximation of \( W \) on \([0, 1]\) with \( k - 1 \) free knots is given by
\[
\tilde{W}_k = \sum_{j=1}^{k} 1_{[\tau_{j-1, \gamma_k}, \tau_{j, \gamma_k}]} \cdot \arg\min_{\pi \in \Pi_r} \| W - \pi \|_{L_\infty[\tau_{j-1, \gamma_k}, \tau_{j, \gamma_k}]}.
\]

(11)

More precisely, from [4] we know that
\[
\| W - \tilde{W}_k \|_{L_\infty[0, 1]} = \gamma_k \approx (E (\tau_{1,1}) \cdot k)^{-1/2} \quad \text{a.s.} \] (12)
and
\[
\left( E^* \left( \| W - \tilde{W}_k \|_{L_\infty[0, 1]}^q \right) \right)^{1/q} \approx (E (\tau_{1,1}) \cdot k)^{-1/2} \] (13)

We assume that the drift coefficient \( a : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R} \) and the diffusion coefficient \( \sigma : [0, 1] \rightarrow \mathbb{R} \) and the initial value \( X(0) \) have the following properties.

\begin{itemize}
  \item (A) \( a \) is differentiable with respect to the state variable. Moreover, there exists a constant \( K > 0 \), such that
    \[
    |a(t, x) - a(t, y)| \leq K \cdot |x - y|,
    \]
    \[
    |a(s, x) - a(t, x)| \leq K \cdot (1 + |x|) \cdot |s - t|,
    \]
    \[
    |a^{(0,1)}(t, x) - a^{(0,1)}(t, y)| \leq K \cdot |x - y|
    \]
    for all \( s, t \in [0, 1] \) and \( x, y \in \mathbb{R} \).
  \item (B) There exists a constant \( K > 0 \), such that
    \[
    |\sigma(s) - \sigma(t)| \leq K \cdot |s - t|
    \]
    and
    \[
    |\sigma(t)| > 0
    \]
    for all \( s, t \in [0, 1] \).
\end{itemize}
• (C) The initial value \( X(0) \) is independent of \( W \) and

\[
E(\|X(0)\|_q^q) < \infty \quad \text{for all } q \geq 1.
\]

Note that (A) yields the linear growth condition, i.e., there exists a constant \( C > 0 \) such that

\[
|a(t,x)| \leq C \cdot (1 + |x|)
\]

for all \( t \in [0,1] \) and \( x \in \mathbb{R} \).

Conditions (A) and (C) are standard assumptions for analysing stochastic differential equations, while (B) is slightly stronger than the standard assumption for equations with additive noise. We conjecture, that the weaker condition \( \sigma \neq 0 \) would be sufficient to obtain the results in the paper. Given the above properties, a pathwise unique strong solution of equation (1) with initial value \( X(0) \) exists. In particular the conditions assure that

\[
E\left(\|X\|_{L_{\infty}[0,1]}^q\right) < \infty \quad \text{for all } q \geq 1.
\]

Next, we turn to the definition of the spline approximation scheme \( \hat{X}_k^\dagger \). Fix \( \delta \in (1/2,1) \) and for \( k \in \mathbb{N} \) take

\[
n_k = \left\lceil k^\delta \right\rceil.
\]

Note that

\[
\lim_{k \to \infty} \frac{n_k}{k} = 0 \quad \text{and} \quad \lim_{k \to \infty} \frac{\sqrt{k}}{n_k} = 0.
\]

We take the Euler scheme to compute an approximation to \( X \) at the discrete points

\[
t_\ell = \frac{\ell}{n_k}, \quad \ell = 0, \ldots, n_k.
\]

This scheme is defined by

\[
\hat{X}(t_0) = X(0)
\]

and

\[
\hat{X}(t_{\ell+1}) = \hat{X}(t_\ell) + a(t_\ell, \hat{X}(t_\ell)) \cdot (t_{\ell+1} - t_\ell) + \sigma(t_\ell) \cdot (W(t_{\ell+1}) - W(t_\ell)).
\]

For every \( \ell \in \{0, \ldots, n_k - 1\} \) we consider the Brownian motion \( W^\ell \), defined by

\[
W^\ell(t) = W(t) - W(t_\ell), \quad t \in [t_\ell, t_{\ell+1}].
\]

Put

\[
\sigma_\ell = \sigma(t_\ell)
\]

and let

\[
m_{\ell,k} = \left\lceil \left( \frac{\sigma_\ell^2}{\sum_{i=0}^{n_k-1} \sigma_i^2} \right) \cdot (k - n_k) \right\rceil + 1.
\]
Let \( \hat{W}^\ell_{m_{\ell,k}} \) denote the asymptotically optimal spline approximation of \( W^\ell \) on the interval \([t_\ell, t_{\ell+1}]\) with \( m_{\ell,k} - 1 \) free knots, cf. (11). Now, the approximation method \( \hat{X}^\dagger_k \) is given by

\[
\hat{X}^\dagger_k (t_0) = X (0)
\]

and for \( t \in [t_\ell, t_{\ell+1}] \)

\[
\hat{X}^\dagger_k (t) = \bar{X} (t_\ell) + a (t_\ell, \bar{X} (t_\ell)) \cdot (t - t_\ell) + \sigma_\ell \cdot \hat{W}^\ell_{m_{\ell,k}} (t).
\]

(20)

Note that the number of free knots on \([t_\ell, t_{\ell+1}]\) is given by \( m_{\ell,k} - 1 \). Since

\[
k - n_k \leq n_k + 1 + \sum_{\ell=0}^{n_k-1} (m_{\ell,k} - 1) \leq k + 1,
\]

the method \( \hat{X}^\dagger_k \) uses at most \( k + 1 \) knots for every trajectory. Due to (17) the upper bound \( k + 1 \) is sharply asymptotical. By formally introducing a few additional knots we get a method with \( k - 1 \) free knots, i.e., \( \hat{X}^\dagger_k \in \mathcal{N}_{k,r} \).

Now we can state the main results of the paper.

**Theorem 1.** Assume that (A), (B) and (C) hold for equation (1). Then we have

\[
\lim_{k \to \infty} \sqrt{k} \cdot e^q (\hat{X}^\dagger_k) = \left( E (\tau_{1,1}) \right)^{-1/2} \cdot \| \sigma \|_2
\]

for all \( q \geq 1 \).

**Theorem 2.** Assume that (A), (B) and (C) hold for equation (1). Then, the minimal errors satisfy

\[
\lim_{k \to \infty} \sqrt{k} \cdot e^\min_{k,q} (X) = \left( E (\tau_{1,1}) \right)^{-1/2} \cdot \| \sigma \|_2
\]

for all \( q \geq 1 \).

Due to (21) and (22) the method \( \hat{X}^\dagger_k \) is asymptotically optimal in the class \( \mathcal{N}_{k,r} \) for every equation (1) with additive noise.

### 3 Proof of main result

For the proof of Theorem 1 we need the following Lemma.

For every \( \ell = 0, \ldots, n_k - 1 \) we consider the pathwise minimal approximation error of \( W^\ell \)

\[
\gamma^\ell_{m_{\ell,k}} = \gamma^\ell_{m_{\ell,k}} (W^\ell) = \inf \{ \varepsilon > 0 \mid \tau^\ell_{m_{\ell,k},\varepsilon} \geq t_{\ell+1} \},
\]

where \( \{ \tau^\ell_{j,\varepsilon} \}_{j \in \mathbb{N}} \) denotes the sequence of stopping times on \([t_\ell, t_{\ell+1}]\), defined by

\[
\tau^\ell_{0,\varepsilon} = t_\ell
\]

\[
\tau^\ell_{j+1,\varepsilon} = \inf \{ t \in [t_j, t_{j+1}] \mid a (t, \bar{X} (t)) \cdot (t - t_\ell) + \sigma_\ell \cdot \hat{W}^\ell_{m_{\ell,k}} (t) > \varepsilon \},
\]

for \( j \geq 0 \).

Now consider the stopping times

\[
\tau^\ell_{m_{\ell,k},\varepsilon} = \inf \{ t \in [t_\ell, t_{\ell+1}] \mid \gamma^\ell_{m_{\ell,k}} (W^\ell) > \varepsilon \}
\]

and define the sequence of stopping times

\[
\tau^\ell_{n_k - 1,\varepsilon} = \inf \{ t \in [t_\ell, t_{\ell+1}] \mid \gamma^\ell_{m_{\ell,k}} (W^\ell) > \varepsilon/n_k \},
\]

for \( \ell = 0, \ldots, n_k - 1 \).
and

\[ \tau_{j,\varepsilon}^\ell = \tau_{j-1,\varepsilon}^\ell (W^\ell) = \inf \left\{ t > \tau_{j-1,\varepsilon}^\ell \mid \inf_{\pi \in \Pi_r} \| W^\ell - \pi \|_{L^\infty[\tau_{j-1,\varepsilon}, t]} > \varepsilon \right\}, \quad j \geq 1. \]

So, we have

\[ \| W^\ell - \hat{W}_{m, k}^\ell \|_{L^\infty[t, t+1]} = \gamma_{m, k}^\ell \quad \text{a.s.} \] (23)

Renormalizing each interval \([t, t+1]\) to \([0, 1]\) it can easily be shown that

\[ \gamma_{m, k}^\ell = \frac{1}{\sqrt{n_k}} \cdot \gamma_{m, k}^\ell \] (24)

and

\[ \gamma_{m, k}^\ell \approx (E_{\tau_{1,1}})^{-1/2} \cdot \frac{1}{\sqrt{m_k} \cdot \sqrt{n_k}} \quad \text{a.s.} \] (25)

for every \(\ell \in \mathbb{N}_0\), by Lemma 8 in [4]. Furthermore, due to (17) we have

\[ |\sigma_{\ell} \cdot \gamma_{m, k}^\ell | \approx (E_{\tau_{1,1}})^{-1/2} \cdot \| \sigma \|_2 \cdot (1/k)^{1/2} \quad \text{a.s.} \] (26)

for every \(\ell \in \mathbb{N}_0\).

From now on let \(C\) denote unspecified positive constants, which only depend on the constant \(K\) from condition (A), as well as on \(a(0,0), \sigma(0,0)\) and \(E|X(0)|^q\).

**Lemma 3.** For all \(q \geq 1\) we have

\[ \lim_{k \to \infty} \sqrt{k} \cdot \left( E_{0 \leq \ell \leq n_k-1} \left( |\sigma_{\ell} | \cdot \gamma_{m, k}^\ell \right)^q \right)^{1/q} = (E_{\tau_{1,1}})^{-1/2} \cdot \| \sigma \|_2. \] (27)

**Proof.** We have

\[ E_{0 \leq \ell \leq n_k-1} \left( |\sigma_{\ell} | \cdot \gamma_{m, k}^\ell \right)^q = \left( \sum_{i=0}^{n_k-1} \sigma_i^2 \right)^{q/2} \cdot E_{0 \leq \ell \leq n_k-1} \left( \left( \sum_{i=0}^{n_k-1} \sigma_i^2 \right)^{1/2} \cdot \gamma_{m, k}^\ell \right)^q. \]

Let \(\rho > 1\), and put \(\mu = E_{\tau_{1,1}}\) and \(a_k = \frac{1}{\sqrt{n_k (k-n_k)} \cdot \sqrt{n_k \rho}}\). Then,

\[ E_{0 \leq \ell \leq n_k-1} \left( |\sigma_{\ell} | / \left( \sum_{i=0}^{n_k-1} \sigma_i^2 \right)^{1/2} \right)^{q/2} \cdot \gamma_{m, k}^\ell \leq a_k + I(k), \]

where

\[ I(k) = \int_{a_k}^{\infty} \mathbb{P} \left( \max_{0 \leq \ell \leq n_k-1} \left( |\sigma_{\ell} | / \left( \sum_{i=0}^{n_k-1} \sigma_i^2 \right)^{1/2} \right)^{q/2} \cdot \gamma_{m, k}^\ell > t \right) dt. \]
Firstly, by (17) we have
\[
\lim_{k \to \infty} \left( \sqrt{k} \right)^q \cdot \left( \sum_{i=0}^{n_k-1} \sigma_i^2 \right)^{q/2} \cdot a_k = \frac{\|\sigma\|_q^2}{\sqrt{\mu/\rho}}.
\] (28)

Using (24) we get the estimate
\[
I (k) \leq \sum_{\ell=0}^{n_k-1} \int_{\frac{\mu}{\rho} \cdot n_k^2}^{\mu/\rho} \mathbb{P} \left( \gamma_{m_{\ell,k}} > t^{1/q} \cdot \sqrt{n_k} \cdot \left( \sum_{i=0}^{n_k-1} \sigma_i^2 \right)^{1/2} / |\sigma\ell| \right) dt.
\]

Then, we split the above right-hand side into terms \( I_1 (k) \) and \( I_2 (k) \), where
\[
I_1 (k) = \sum_{\ell=0}^{n_k-1} \int_{\frac{\mu}{\rho} \cdot n_k^2}^{\mu/\rho} \mathbb{P} \left( \frac{\gamma_{m_{\ell,k}} > t^{1/q} \cdot \sqrt{n_k} \cdot \left( \sum_{i=0}^{n_k-1} \sigma_i^2 \right)^{1/2} / |\sigma\ell| \right) dt
\]
and
\[
I_2 (k) = \sum_{\ell=0}^{n_k-1} \int_{\frac{\mu}{\rho} \cdot n_k^2}^{\infty} \mathbb{P} \left( \gamma_{m_{\ell,k}} > t^{1/q} \cdot \sqrt{n_k} \cdot \left( \sum_{i=0}^{n_k-1} \sigma_i^2 \right)^{1/2} / |\sigma\ell| \right) dt.
\]

We put
\[
S_n = \sum_{j=1}^{n} \xi_{j,1}.
\]

Using the fact that for all \( \varepsilon > 0 \)
\[
\mathbb{P} \left( \gamma_{m_{\ell,k}} \leq \varepsilon \right) = \mathbb{P} \left( S_{m_{\ell,k}} \geq 1/\varepsilon^2 \right)
\] (29)
and the random variables \( \xi_{j,1} \) form an i.i.d. sequence (see [4]), it follows by substitution on the one hand that
\[
I_1 (k) = \frac{q}{2} \sqrt{\frac{\mu/\rho}{(k-n_k) \cdot n_k^2}} \sum_{\ell=0}^{n_k-1} \int_{\frac{\mu}{\rho} \cdot n_k^2}^{\mu/\rho} t^{-\left( q/2+1 \right)} \cdot \mathbb{P} \left( \frac{S_{m_{\ell,k}}}{m_{\ell,k}} < t \right) dt.
\]

For \( \mu / (\rho \cdot n_k^2) \leq t \leq \mu / \rho \) we use Höfdding’s inequality to obtain
\[
t^{-\left( q/2+1 \right)} \cdot \mathbb{P} \left( \frac{S_{m_{\ell,k}}}{m_{\ell,k}} < t \right) \leq t^{-\left( q/2+1 \right)} \cdot \mathbb{P} \left( \frac{S_{m_{\ell,k}}}{m_{\ell,k}} - \mu \right) > \mu - \mu / \rho \)
\[
\leq \frac{(\rho \cdot n_k^2)^{q/2+1}}{\mu^{q/2+1}} \cdot 2 \exp \left( -2m_{\ell,k} \cdot (\mu - \mu / \rho)^2 \right)
\]
for every \( \ell = 0, \ldots, n_k - 1 \). This yields
\[
\lim_{k \to \infty} \left( \sqrt{k} \right)^q \cdot \left( \sum_{i=0}^{n_k-1} \sigma_i^2 \right)^{q/2} \cdot I_1 (k) = 0.
\] (30)
To verify this, it suffices to show
\[
\lim_{k \to \infty} n_k^{q+2} \sum_{\ell=0}^{n_k-1} \exp (-2m_{\ell,k} \cdot c) = 0
\]
with \( c > 0 \). In fact we have
\[
m_{\ell,k} \approx \frac{\sigma^2_\ell}{\|\sigma\|_2^2} \cdot \frac{k}{n_k}
\]
for every \( \ell \in \mathbb{N}_0 \). Let \( \alpha = \inf_{0 \leq t \leq 1} (\sigma(t))^2 \). Using the definition of \( n_k \) in (16) we get for \( k \) sufficiently large
\[
n_k^{q+2} \sum_{\ell=0}^{n_k-1} \exp (-2m_{\ell,k} \cdot c) \leq \frac{\sigma^2_\ell}{\|\sigma\|_2^2} \cdot \frac{k}{n_k} \cdot c \leq k^{\delta(q+3)} \cdot \exp \left( -\frac{\alpha}{\|\sigma\|_2^2} \cdot k^{1-\delta} \cdot c \right),
\]
which yields (31).

On the other hand, using (29) we obtain
\[
\begin{align*}
I_2(k) &= \sum_{\ell=0}^{n_k-1} \int_{\tau_1,1}^{\infty} P \left( m_{\ell,k} \leq \frac{\sigma^2_\ell}{\|\sigma\|_2^2} \cdot \frac{k}{n_k} \cdot c \right) dt \\
&\leq \sum_{\ell=0}^{n_k-1} \int_{\tau_1,1}^{\infty} P \left( \tau_1,1 \leq \frac{\sigma^2_\ell}{\|\sigma\|_2^2} \cdot \frac{k}{n_k} \cdot \sum_{i=0}^{n_k-1} \sigma_i^2 \right) m_{\ell,k} dt.
\end{align*}
\]
Note that for all \( \eta \leq 1 \)
\[
P (\tau_1,1 \leq \eta) \leq \exp (-C \cdot \eta^{-1})
\]
with some constant \( C > 0 \); see the proof of Lemma 8 in [4]. From (17) we have
\[
\frac{k - n_k}{n_k^2} \cdot (\mu/\rho) \leq 1
\]
for \( k \) sufficiently large. Then, for all \( t \geq n_k^q \cdot a_k \) we have
\[
\frac{\sigma^2_\ell}{t^{2/q} \cdot n_k \cdot \sum_{i=0}^{n_k-1} \sigma_i^2} \leq 1
\]
for every \( \ell = 0, \ldots, n_k - 1 \).
Hence, we get

\[ I_2(k) \leq n_k \cdot \int_{n_k}^{\infty} \exp \left(-C \cdot (k-n_k) \cdot t^{2/q} \right) dt \]

\[ = \frac{q}{2} \cdot n_k \cdot \frac{1}{\left(\sqrt{n_k} \cdot (k-n_k)\right)^q} \int_{n_k}^{\infty} t^{q/2-1} \cdot \exp \left(-C \cdot t \right) dt \]

\[ \leq \frac{q \cdot \mu}{2 \cdot \rho} \cdot \frac{1}{\left(\sqrt{n_k} \cdot (k-n_k)\right)^q} \int_{n_k}^{\infty} t^{q/2} \cdot \exp \left(-C \cdot t \right) dt, \]

which implies

\[ \lim_{k \to \infty} \left( \sqrt{k} \cdot \left( \sum_{i=0}^{n_k-1} \sigma_i^2 \right)^{q/2} \cdot I_2(k) \right) = 0. \quad (32) \]

Finally, combining (28)-(32), we obtain

\[ \limsup_{k \to \infty} \sqrt{k} \cdot \left( E \max_{0 \leq t \leq n_k} \left| \sigma_\ell \cdot \gamma^\ell_{m,t,k} \right| \right)_{q/2} \leq \frac{\|\sigma\|_2^q}{\sqrt{\mu} \cdot \rho^{q/2}}. \]

Letting \( \rho \) tend to 1 yields the upper bound in (27).

For establishing the lower bound in (27) it suffices to study the case \( q = 1 \). In fact we have

\[ E \left( \max_{0 \leq t \leq n_k} \left| \sigma_\ell \cdot \gamma^\ell_{m,t,k} \right| \right) \geq E \left( |\sigma_0| \cdot \gamma^0_{m,0,k} \right). \]

We use (20) and Fatou’s Lemma to obtain

\[ \liminf_{k \to \infty} \sqrt{k} \cdot E \left( \max_{0 \leq t \leq n_k} \left| \sigma_\ell \cdot \gamma^\ell_{m,t,k} \right| \right) \geq \frac{\|\sigma\|_2^q}{\sqrt{\mu}}, \]

which completes the proof. \( \square \)

In order to prove the main result given in Theorem 11 we introduce the process \( \overline{X}_{n_k} \) as follows. For \( k \in \mathbb{N} \) let

\[ 0 = t_0 < t_1 < \cdots < t_{n_k} = 1 \]

be the discretization (13) of \([0, 1]\). The process \( \overline{X}_{n_k} \) is given by \( \overline{X}_{n_k}(0) = X(0) \) and for \( t \in [t_\ell, t_{\ell+1}] \)

\[ \overline{X}_{n_k}(t) = \overline{X}_{n_k}(t_\ell) + a(t_\ell, \overline{X}_{n_k}(t_\ell)) \cdot (t - t_\ell) + \sigma_\ell \cdot (W(t) - W(t_\ell)) \]. \quad (33)

Note that \( \overline{X}_{n_k} \) coincides with the Euler scheme (19) at the discretization points \( t_\ell \). Instead of estimating \( X - \bar{X}_k \) directly, we consider \( X - \overline{X}_{n_k} \), as well as \( \overline{X}_{n_k} - \bar{X}_k \) separately. From Proposition 3 in [13] we know that

\[ \left( E \left| X - \overline{X}_{n_k} \right|_{L_{\infty}[0,1]}^q \right)^{1/q} \leq C \cdot \frac{1}{n_k}. \quad (34) \]
From this and (17) it follows that
\[
\lim_{k \to \infty} \sqrt{k} \cdot \left( E \left\| X - \overline{X}_{nk} \right\|_{L_\infty[0,1]}^q \right)^{1/q} = 0,
\] (35)
and so \( E \left\| \overline{X}_{nk} - \hat{X}_k \right\|_{L_\infty[0,1]}^q \right)^{1/q} \) is the asymptotically dominating term.

**Proof of Theorem 1** In view of the lower bound in Theorem 2 it suffices to show
\[
\limsup_{k \to \infty} \sqrt{k} \cdot \left( E \left\| X - \hat{X}_k \right\|_{L_\infty[0,1]}^q \right)^{1/q} \leq (E (\tau_{1,1}))^{-1/2} \cdot \| \sigma \|_2 .
\] (36)
For \( t \in [t_\ell, t_{\ell+1}] \) we have
\[
| \overline{X}_{nk}(t) - \hat{X}_k(t) | = \left| \sigma_{\ell} \cdot \left( W^\ell(t) - \hat{W}_{m_{\ell,k}}^\ell(t) \right) \right| .
\]
Thus
\[
\left\| \overline{X}_{nk} - \hat{X}_k \right\|_{L_\infty[0,1]} = \max_{0 \leq \ell \leq n_k - 1} \left( | \sigma_{\ell} | \cdot \sup_{t_\ell \leq t \leq t_{\ell+1}} \left| W^\ell(t) - \hat{W}_{m_{\ell,k}}^\ell(t) \right| \right) .
\] (37)
Then, the estimate (36) is a direct consequence of (35) together with the equation (37), (23) and Lemma 3.

**Proof of Theorem 2** The upper bound in (22) is a direct consequence from (21). For establishing the lower bound it suffices to study the case \( q = 1 \). For \( k \in \mathbb{N} \) take \( n_k \in \mathbb{N} \) such that
\[
\lim_{k \to \infty} \frac{n_k}{k} = 0 \quad \text{and} \quad \lim_{k \to \infty} \frac{\sqrt{k}}{n_k} = 0.
\] (38)
Let
\[
\bar{\ell}_k = \frac{\ell}{n_k}
\]
for \( \ell = 0, \ldots, n_k \), and consider the process \( \overline{X}_{nk} \) for this discretization; see (33). At first, by Minkowski’s inequality and (34) we have for every approximation \( \hat{X}_k \in \Phi_{k,r} \)
\[
E \left\| X - \hat{X}_k \right\|_{L_\infty[0,1]} \geq E \left\| \overline{X}_{nk} - \hat{X}_k \right\|_{L_\infty[0,1]} - C/n_k.
\] (39)
For a fixed \( \omega \in \Omega \) let \( \hat{X}_k(\omega) \in \Phi_{k,r} \) be given by
\[
\hat{X}_k(\omega) = \sum_{j=1}^k 1_{[t_{j-1}, t_j]} \cdot \pi_j.
\]
Let
\[
D \left( \hat{X}_k(\omega) \right) = \{ t_j : j = 0, \ldots, k \}
\]
to the set of knots used by \( \hat{X}_k(\omega) \), and put
\[
d_{\ell-1} = \sharp \left(D \left( \hat{X}_k(\omega) \right) \cap [\tilde{t}_{\ell-1}, \tilde{t}_{\ell}] \right), \quad \ell = 1, \ldots, n_k.
\]
We refine the corresponding partition to a partition
\[
0 = \tilde{t}_0 < \cdots < \tilde{t}_k = 1,
\]
that contains all the points \( \ell/n_k \), and we define the polynomials \( \tilde{\pi}_j \in \Pi_r \) by
\[
\hat{X}_k(\omega) = \sum_{j=1}^{k} 1_{[t_{j-1}, t_j]} \cdot \tilde{\pi}_j.
\]
Furthermore, for \( t \in [\tilde{t}_{j-1}, \tilde{t}_j] \subseteq [\tilde{t}_{\ell-1}, \tilde{t}_{\ell}] \) we define \( \tilde{\pi}_j \in \Pi_r \) by
\[
\tilde{\pi}_j(t) = \nabla_{n_k} (\tilde{t}_{\ell-1}, \omega) + a (\tilde{t}_{\ell-1}, \nabla_{n_k} (\tilde{t}_{\ell-1}, \omega)) \cdot (t - \tilde{t}_{\ell-1}) + \sigma_{\ell-1} \cdot (\tilde{\pi}_j(t) - W(\tilde{t}_{\ell-1}, \omega)).
\]
Put
\[
\tilde{f} = \sum_{j=1}^{k} 1_{[t_{j-1}, t_j]} \cdot \tilde{\pi}_j.
\]
Then, we have
\[
\left| \nabla_{n_k} (\omega) - \hat{X}_k(\omega) \right|_{L_{\infty}[0,1]} \geq \max_{1 \leq \ell \leq n_k} \left( |\sigma_{\ell-1}| \cdot \sup_{\tilde{t}_{\ell-1} < \ell \leq \ell} |W(t, \omega) - \tilde{f}(t)| \right).
\]
Note that there exists an \( \ell_0 = \ell_0(\omega) \in \{1, \ldots, n_k\} \), so that
\[
d_{\ell_0-1} \leq m_{\ell_0-1} \cdot k + 2.
\]
To see this, suppose that
\[
d_{\ell-1} > m_{\ell-1} \cdot k + 2 \quad \forall \ell \in \{1, \ldots, n_k\}.
\]
This implies
\[
k \geq \sum_{\ell=1}^{n_k} d_{\ell-1} > \sum_{\ell=1}^{n_k} (m_{\ell-1} \cdot k + 2) \geq k - n_k + 2n_k = k + n_k,
\]
which leads to a contradiction. Hence we a.s. have
\[
\sup_{\tilde{t}_{\ell_0-1} < \ell \leq \tilde{t}_{\ell_0}} |W(t, \omega) - \tilde{f}(t)| \geq \inf_{\varphi \in \Phi_{d_{\ell_0-1}}} \|W - \varphi\|_{L_{\infty}[\tilde{t}_{\ell_0-1}, \tilde{t}_{\ell_0}]} = \gamma_{d_{\ell_0-1}} \geq \gamma_{m_{\ell_0-1}}^{-1/2}
\]
by (38). Hence we use (25), (26), (39) and (40) to obtain
\[
\liminf_{k \to \infty} \sqrt{k} \cdot E \|X - \hat{X}_k\|_{L_{\infty}[0,1]} \geq \left( E_{r_1,1} \right)^{-1/2} \cdot \|\sigma\|_2
\]
by Fatou’s Lemma. This completes the proof of Theorem 2.

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