HUSIMI’S Q-FUNCTION OF THE ISOTONIC OSCILLATOR IN A GENERALIZED NEGATIVE BINOMIAL STATES REPRESENTATION

ZOUHAİR MOUAYN

ABSTRACT. While considering a class of generalized negative binomial states, we verify that the basic minimum properties for these states to be considered as coherent states are satisfied. We particularize them for the case of the Hamiltonian of the isotonic oscillator and we determine the corresponding Husimi’s Q-function. This function may be used to determine a lower bound for the thermodynamical potential of the Hamiltonian by applying a Berezin-Lieb inequality.

1 INTRODUCTION

The negative binomial states (NBS) are the field states that are superposition of the number states with appropriately chosen coefficients [1]. Precisely, these labeling coefficients are such that the associated photon-counting distribution is a negative binomial probability distribution [2]. As matter of fact, these coefficients turn out to be an orthonormal basis of a weighted Bergman space of analytic functions on the complex unit disk satisfying a certain growth condition. Furthermore, the NBS are considered as intermediate states between pure coherent states and pure thermal states [3] and reduce to Susskind-Glogower phases states for a particular limit of the parameter [4].

Now, as in [5, 6], we replace the labeling coefficients by an orthonormal basis of a Hilbert space that generalize the weighted Bergman space of analytic functions on the unit disk we have mentioned to consider a class of generalized negative binomial states (GNBSs) in this sense. Here, we precisely verify that the basic minimum properties for the constructed states to be considered as coherent states are satisfied. Namely, the conditions which have been formulated by Klauder [7]: (a) the continuity of labeling, (b) the fact that these states are normalizable but not orthogonal and (c) these states fulfilled the resolution of the identity with a positive weight function.

Next, we particularize the GNBSs formalism for the case of the isotonic oscillator (IO) [8] whose importance consists in the fact that it admits exact analytic solutions an being in a certain sense an intermediate potential between the three dimensional harmonic oscillator potential and other anharmonic potentials such as Pöschl-Teller or Morse potentials [9]. Next, we
determine the Husimi’s Q-function \[10\] which turns out to be the expectation value in the GNBSs representation of the heat semigroup operator associated with the IO. The obtained Q-function can be considered as a lower symbol for this heat semigroup operator. Finally, a lower bound for the thermodynamical potential of the IO may be obtained by applying a Berezin-Lieb inequality \[11, 12\].

The paper is organized as follows. In Section 2, we review briefly the coherent states formalism we will be using. Section 3 deals with some needed facts on the generalized weighted Bergman spaces on the disk. In Section 4, we attach to each of these spaces a set of coherent states generalizing the negative binomial states and verify that they satisfy the basic minimum properties of coherent states. In Section 5, we recall briefly some needed spectral properties of the isotonic oscillator Hamiltonian. In Section 6, we obtain the Husimi’s Q-function associated with the IO in the coherent state representation and we deduce a lower bound for the thermodynamical potential of the IO.

2 Coherent states and Berezin-Lieb inequalities

Here, we review a coherent states formalism starting from a measure space “it as a set of data” as presented in \[13\]. Let \(X = \{x \mid x \in X\}\) be a set equipped with a measure \(d\mu\) and \(L^2(X,d\mu)\) the space of \(d\mu\)-square integrable functions on \(X\). Let \(A^2 \subset L^2(X,d\mu)\) be a subspace of infinite dimension with an orthonormal basis \(\{\Phi_j\}_{j=0}^{\infty}\). Let \(\mathcal{H}\) be another (functional) space with \(\text{dim} \mathcal{H} = \infty\) and \(\{\phi_j\}_{j=1}^{\infty}\) is a given orthonormal basis of \(\mathcal{H}\). Then consider the family of states \(\{|x>\}_{x \in X}\) in \(\mathcal{H}\), through the following linear superpositions:

\[
|x> := (\mathcal{N}(x))^{-\frac{1}{2}} \sum_{j=0}^{+\infty} \Phi_j(x) |\phi_j> \tag{2.1}
\]

where

\[
\mathcal{N}(x) = \sum_{j=0}^{+\infty} \Phi_j(x) \overline{\Phi_j(x)}. \tag{2.2}
\]

These coherent states obey the normalization condition

\[
\langle x | x \rangle_{\mathcal{H}} = 1 \tag{2.3}
\]

and the following resolution of the identity of \(\mathcal{H}\)

\[
1_{\mathcal{H}} = \int_X |x><x| \mathcal{N}(x) d\mu(x) \tag{2.4}
\]

which is expressed in terms of Dirac’s bra-ket notation \(|x><x|\) meaning the rank-one operator \(|\phi> \mapsto \langle \phi | x \rangle_{\mathcal{H}} |x>\). The choice of the Hilbert space \(\mathcal{H}\) define in fact a quantization of the space \(X\) by the coherent states in (2.1), via the inclusion map \(X \ni x \mapsto |x> \in \mathcal{H}\) and the property (2.4).
is crucial in setting the bridge between the classical and the quantum mechanics.

Now, given a set of coherent states \( \{|x\rangle\} \), the concept of upper and lower symbols of an operator \( A \) was separately introduced by Berezin [11] and Lieb [12] by

\[
A = \int_X d\mu(x) \hat{A} |x><x| \tag{2.5}
\]

to define the upper symbol \( \hat{A} \) of \( A \), and the expectation value

\[
\hat{A}(x) := \langle x | A | x \rangle \tag{2.6}
\]

for the definition of the lower symbol \( \hat{A} \) of \( A \). Note that given an operator \( A \) its upper symbol is not unique in general. It can be proved [12] that given any convex function \( \phi \), the following inequalities

\[
\int_X \phi(\hat{A}) d\mu(x) \leq \text{Tr}(\phi(A)) \leq \int_X \phi(\hat{A}) d\mu(x) \tag{2.7}
\]

hold and are called Berezin-Lieb inequalities.

3 Generalized Bergman spaces on \( \mathbb{D} \)

Let \( \mathbb{D} = \{z \in \mathbb{C}, |z| < 1\} \) be unit disk endowed with its usual Kähler metric \( ds^2 = -\partial\overline{\partial} \log (1-z\overline{z}) dz \otimes d\overline{z} \). The Bergman distance on \( \mathbb{D} \) is given by

\[
\cosh^2 d(z, w) = \frac{(1-z\overline{w})(1-z\overline{w})}{(1-z\overline{z})(1-w\overline{w})} \tag{3.1}
\]

and the volume element reads

\[
d\mu(z) = \frac{1}{(1-z\overline{z})^2} dv(z) \tag{3.2}
\]

with the Lebesgue measure \( dv(z) \). Let us consider the 1–form on \( \mathbb{D} \) defined by \( \theta = -i(\partial - \overline{\partial}) \log (1-z\overline{z}) \) to which the Schrödinger operator

\[
H_{\sigma} := \left(d + i\frac{\sigma}{2} \text{ext}(\theta)\right)^* \left(d + i\frac{\sigma}{2} \text{ext}(\theta)\right) \tag{3.3}
\]

can be associated. Here \( \sigma \geq 0 \) is a fixed number, \( d \) denotes the usual exterior derivative on differential forms on \( \mathbb{D} \) and \( \text{ext}(\theta) \) is the exterior multiplication by \( \theta \) while the symbol \( * \) stands for the adjoint operator with respect to the Hermitian scalar product induced by the Bergman metric \( ds^2 \) on differential forms. Actually, the operator \( H_{\sigma} \) is acting on the Hilbert space \( L^2(\mathbb{D}, d\mu(z)) \) and can be unitarily intertwined as

\[
(1-z\overline{z})^{\frac{\sigma}{2}} \Delta_{\sigma} (1-z\overline{z})^{-\frac{\sigma}{2}} = H_{\sigma} \tag{3.4}
\]

in terms of the second order differential operator

\[
\Delta_{\sigma} := -4(1-z\overline{z}) \left(1-z\overline{z} \frac{\partial^2}{\partial z \partial \overline{z}} - \sigma \frac{\partial}{\partial \overline{z}}\right). \tag{3.5}
\]
The latter one is acting on the Hilbert space

\[ L^{2,\sigma}(\mathbb{D}) = L^2(\mathbb{D}, (1 - z\overline{z})^{\sigma - 2} \, d\nu(z)) \] (3.6)

The spectral analysis of \( \Delta_\sigma \) have been studied by many authors, see [5] and references therein. Note that this operator is an elliptic densely defined operator on \( L^2(\mathbb{D}) \) and admits a unique self-adjoint realization that we denote also by \( \Delta_\sigma \). The part of its spectrum is not empty if and only if \( \sigma > 1 \). This discrete part consists of eigenvalues occurring with infinite multiplicities and having the expression

\[ \epsilon^m_\sigma := 4m(\sigma - m - 1) \] (3.7)

for varying \( m = 0, 1, \ldots, \lfloor (\sigma - 1)/2 \rfloor \). Here, \( \lfloor x \rfloor \) denotes the greatest integer not exceeding \( x \). Moreover, it is well known that the functions given in terms of Jacobi polynomials \([15, 17]\) by

\[ \Phi^{\sigma, m}_k(z) = \sqrt{\frac{(\sigma - 2m - 1)k!\Gamma(\sigma - m)}{\pi m!\Gamma(\sigma - 2m + k)}} \times (-1)^k (1 - z\overline{z})^{-m} z^{m - k} P_{m - k}^{(m - k, \sigma - 2m - 1)}(1 - 2z\overline{z}) \] (3.8)

constitute an orthonormal basis of the eigenspace

\[ \mathcal{A}^{2,\sigma}_m(\mathbb{D}) := \{ \phi \in L^{2,\sigma}(\mathbb{D}) , \Delta_\sigma \phi = \epsilon^m_\sigma \phi \} \] (3.9)

of \( \Delta_\sigma \) associated with the eigenvalue \( \epsilon^m_\sigma \) in (3.7). Finally, the \( L^2 \)-eigenspace \( \mathcal{A}^{2,\sigma}_0(\mathbb{D}) = \{ \phi \in L^{2,\sigma}(\mathbb{D}) , \Delta_\sigma \phi = 0 \} \) corresponding to \( m = 0 \) and associated to \( \epsilon^0_\sigma = 0 \) in (3.7) reduces further to the weighted Bergman space consisting of holomorphic functions \( \phi : \mathbb{D} \to \mathbb{C} \) with the growth condition

\[ \int_\mathbb{D} |\phi(z)|^2 (1 - z\overline{z})^{\sigma - 2} \, d\nu(z) < +\infty \] (3.10)

This is why the eigenspaces in (3.9) have been called generalized Bergman spaces on the complex unit disk.

**Remark 3.1.** In [5], we have used the basis in (3.8) to perform a class of coherent states belonging to the Hilbert space \( L^2(\mathbb{R}_+^*, x^{-1} dx) \). The associated coherent state transform have been considered as a generalization of the second Bargmann transform ([18, p.203]).

4 **GENERALIZED NEGATIVE BINOMIAL STATES**

The negative binomial states are labeled by points \( z \in \mathbb{D} \) and are of the form

\[ |z, \sigma, 0 > := (1 - z\overline{z})^{\frac{1}{2} \sigma} \sum_{k=0}^{+\infty} \sqrt{\frac{\Gamma(\sigma + k)}{\Gamma(\sigma) k!}} z^k | \psi_k >, \] (4.1)
where $\sigma > 1$ is a fixed parameter and $|\psi_k>$ are Fock states. Their photon probability distribution

$$|<\psi_k | z, \sigma >|^2 = (1 - z\bar{z})^\sigma (\bar{z}z)^k \frac{\Gamma(\sigma + k)}{\Gamma(\sigma) k!}$$

(4.2)

obeys the negative binomial probability distribution with parameters $\lambda = z\bar{z}$ and $\sigma$. Also, observe that the coefficients in the superposition (4.1): 

$$\Phi_k^{\sigma,0}(z) := \sqrt{\frac{\Gamma(\sigma + k)}{\pi \Gamma(\sigma) k!}} z^k, k = 0, 1, 2, ...$$

(4.3)

constitute an orthonormal basis of the eigenspace $A_0^2, \sigma(\mathbb{D})$ associated with the first eigenvalue $\epsilon_0^\sigma = 0$ and consisting of analytic functions on $\mathbb{D}$ with the growth condition (3.10).

For instance, let $\sigma > 1$ and $m = 0, 1, ...; (\sigma - 1)/2$ be fixed parameters and let $\{|\psi_k>\}_{k=0}^\infty$ be a set of Fock states in a Hilbert space $\mathcal{H}$. Then a class of generalized negative binomial states (GNBS) can be defined as in [5, 6] by

$$|z, \sigma, m> := (\mathcal{N}_{\sigma, m}(z))^\frac{1}{2} \sum_{k=0}^\infty \Phi_k^{\sigma,m}(z) |\psi_k>,$$

(4.4)

where $\mathcal{N}_{\sigma, m}(z)$ is a normalization factor and $\{\Phi_k^{\sigma,m}(z)\}_{k=0}^\infty$ is the orthonormal basis of the generalized Bergman space in (3.9). Now, one of the important tasks to do is to determine is the overlap relation between two GNBSs.

**Proposition 4.1.** Let $\sigma > 1$ and $m = 0, 1, ...; (\sigma - 1)/2$. Then, for every $z, w \in \mathbb{D}$, the overlap relation between two GNBSs is given through the scalar product

$$<w, \sigma, m | z, \sigma, m >_\mathcal{H} = \frac{(\sigma - 2m - 1) \Gamma(\sigma - m)}{\pi m!} \mathcal{N}(z) \mathcal{N}(w)^{-\frac{1}{2}}$$

$$\times \left(1 - z\bar{w}\right) \left(1 - \bar{z}w\right) \mathcal{S}_{z, w},$$

(4.5)

$$\times \frac{\left(1 - z\bar{w}\right) \left(1 - \bar{z}w\right)}{(1 - z\bar{z})(1 - w\bar{w})}^m \mathcal{S}_{z, w},$$

where $2F_1$ is a terminating Gauss hypergeometric sum.

**Proof.** In view of Eq. (4.4), the scalar product of two GNBS $|z, \sigma, m>$ and $|w, \sigma, m>$ in $\mathcal{H}$ reads

$$<w, \sigma, m | z, \sigma, m >_\mathcal{H} = \frac{(\sigma - 2m - 1) \Gamma(\sigma - m)}{\pi} \mathcal{N}(z) \mathcal{N}(w)^{-\frac{1}{2}} \sum_{k=0}^\infty \Phi_k^{\sigma,m}(z) \Phi_k^{\sigma,m}(w)$$

$$\mathcal{S}_{z, w},$$

(4.6)
where

\[ S^{\sigma,m}_{z,w} = \frac{\Gamma (\sigma - m) (zw)^m}{m! \Gamma (\sigma - 2m)} \sum_{k=0}^{+\infty} \frac{k!}{(\sigma - 2m)_k} \left( \frac{1}{zw} \right)^k \times P_k^{(m-k,\sigma-2m-1)} (1 - 2zw) P_k^{(m-k,\sigma-2m-1)} (1 - 2w\overline{w}). \]  

(4.7)

Making use of the following identity ([19, p.1329]):

\[ \sum_{n=0}^{+\infty} \frac{n! 1^n}{(1 + \alpha)_n} P_n^{(\gamma-n,\alpha)} (x) P_n^{(\gamma-n,\alpha)} (y) = \left( 1 - \frac{1}{4} (x - 1) (y - 1) t \right)^{1+\gamma+\alpha} \times (1 - t)^{\gamma/2} \frac{4}{}\text{2F1} \left( 1 + \gamma + \alpha, -\gamma, 1 + \alpha; \frac{-(x + 1) (y + 1) t}{(1 - t) (4 - (x - 1) (y - 1) t)} \right) \]

(4.8)

for \( n = k, t = 1/zw, \gamma = m, \alpha = \sigma - 2m - 1, x = 1 - 2zw \) and \( y = 1 - 2w\overline{w} \), we obtain, after calculations, the expression

\[ S^{\sigma,m}_{z,w} = \frac{\Gamma (\sigma - m) (-1)^m \left( (1 - z\overline{w}) (1 - w\overline{z}) \right)^m}{m! \Gamma (\sigma - 2m) (1 - z\overline{w})^\sigma} \times \text{2F1} \left( -m, \sigma - m, \sigma - 2m; \frac{(1 - z\overline{w}) (1 - w\overline{z})}{(1 - z\overline{w}) (1 - w\overline{z})} \right). \]

(4.9)

Returning back to Eq. (4.6) and inserting the expression (4.9) we arrive at the announced formula. 

**Corollary 4.2.** The normalization factor in (4.4) is given by

\[ \mathcal{N}_{\sigma,m} (z) = \frac{(\sigma - 2m - 1)}{\pi (1 - z\overline{z})^{\sigma}}, \]

(4.10)

for every \( z \in \mathbb{D}. \)

**Proof.** We first make appeal to the relation ([17, p.212]):

\[ \text{2F1} \left( -n, n + \kappa + q + 1, 1 + \kappa; \frac{1 - \tau}{2} \right) = \frac{n! \Gamma (1 + \kappa)}{\Gamma (1 + \kappa + n)} P_n^{(\kappa,\kappa)} (\tau) \]

(4.11)

connecting the \( \text{2F1} \)–sum with the Jacobi polynomial for the parameters \( n = m, \kappa = \sigma - 2m - 1, q = 0 \) and the variable

\[ \tau = 1 - 2 \left( \frac{(1 - z\overline{z}) (1 - w\overline{w})}{(1 - z\overline{w}) (1 - w\overline{z})} \right) \]

(4.12)

to rewrite Eq. (4.5) as

\[ \sqrt{\mathcal{N}_{\sigma,m} (z) \mathcal{N}_{\sigma,m} (w)} = \frac{(-1)^m (\sigma - 2m - 1) (1 - z\overline{z})^{-\sigma}}{\pi < w; \sigma, m | z, \sigma, m > \mathcal{H}} \times P_m^{(\sigma-2m-1,0)} \left( 1 - 2 \left( \frac{(1 - z\overline{z}) (1 - w\overline{w})}{(1 - z\overline{w}) (1 - w\overline{z})} \right) \right). \]

(4.13)

The factor \( \mathcal{N}_{\sigma,m} (z) \) should be such that

\[ < z, \sigma, m | z, \sigma, m > \mathcal{H} = 1. \]

(4.14)
So that we put \( z = w \) in (4.13) and we use the symmetry identity ([17] p.210):

\[
P_m^{(\gamma,\varphi)}(\xi) = (-1)^m P_m^{(\delta,\gamma)}(-\xi)
\]

(4.15)
to obtain the expression

\[
\mathcal{N}_{\sigma,m}(z) = \frac{(\sigma - 2m - 1)}{\pi (1 - z\bar{z})^\sigma} p_1^{(0,\sigma - 2m - 1)}(1)
\]

(4.16)
Finally, we apply the fact that ([17] p.1329):

\[
p_1^{(\alpha)}(1) = \frac{\Gamma(n + \alpha + 1)}{n! \Gamma(\alpha + 1)}
\]

(4.17)
in the case of \( \alpha = 0, n = m \) and \( \varphi = \sigma - 2m - 1 \). This ends the proof. \( \blacksquare \)

**Proposition 4.3.** Let \( \sigma > 1 \) and \( m = 0, 1, \ldots, (\lfloor \sigma - 1 \rfloor / 2) \). Then, the GNBS in (4.4) satisfy the following resolution of the identity

\[
\mathbf{1}_\mathcal{H} = \int_\mathcal{D} |z, \sigma, m > < z, \sigma, m | d\mu_{\sigma,m}(z),
\]

(4.18)
where \( \mathbf{1}_\mathcal{H} \) is the identity operator and \( d\mu_{\sigma,m}(z) \) is a measure which can be expressed through a Meijer’s G-function as

\[
d\mu_{\sigma,m}(z) := \pi^{-1}(\sigma - 2m - 1)G_{11}^{11} \left( -z\bar{z} \left| \begin{array}{c} \begin{array}{c} -1 \\ 0 \\ \end{array} \end{array} \right. \right) d\nu(z),
\]

(4.19)
and \( d\nu(z) \) being the Lebesgue measure on \( \mathcal{D} \).

**Proof.** Let us assume that the measure takes the form

\[
d\mu_{\sigma,m}(z) = \mathcal{N}_{\sigma,m}(z) \Omega(z) d\nu(z),
\]

(4.20)
where \( \Omega(z) \) is an auxiliary density to be determined. Let \( \varphi \in \mathcal{H} \) and let us start by writing the following action

\[
\mathcal{O} \left[ \varphi \right] := \left( \int_\mathcal{D} |z, \sigma, m > < z, \sigma, m | d\mu_{\sigma,m}(z) \right) [\varphi]
\]

(4.21)
\[
= \int_\mathcal{D} \varphi |z, \sigma, m > < z, \sigma, m | d\mu_{\sigma,m}(z).
\]

(4.22)
Making use Eq. (4.4), we obtain successively

\[
\mathcal{O} \left[ \varphi \right] = \int_\mathcal{D} \varphi |z, \sigma, m > < z, \sigma, m | d\mu_{\sigma,m}(z)
\]

\[
\quad \quad \quad \quad \quad + \sum_{k=0}^{\infty} \Phi_k^{(m)}(z) \psi_k |z, \sigma, m | d\mu_{\sigma,m}(z)
\]

(4.23)
\[
\quad \quad \quad \quad \quad = \int_\mathcal{D} \sum_{k=0}^{\infty} \Phi_k^{(m)}(z) < \psi_k |z, \sigma, m | d\mu_{\sigma,m}(z)
\]

(4.24)
\[
\quad \quad \quad \quad \quad = \left( \sum_{j,k=0}^{\infty} \int_\mathcal{D} \bar{\Phi}_j^{(m)}(z) \Phi_k^{(m)}(z) |z, \sigma, m | d\mu_{\sigma,m}(z) \right) \left[ \varphi \right].
\]

(4.25)
We replace the measure $d\mu_{\sigma,m}(z)$ by the expression in the right hand side of Eq. (4.20), then Eq. (4.25) can be written without $\varphi$ as follows

$$
O = \sum_{j,k=0}^{+\infty} \int_{\mathbb{D}} \Phi^{m}_j(z) \Phi^{m}_k(z) \Omega(z) d\nu(z) | \psi_j > < \psi_k | .
$$

(4.26)

Therefore, we need to have

$$
\int_{\mathbb{D}} \Phi^{m}_j(z) \Phi^{m}_k(z) \Omega(z) d\nu(z) = \delta_{jk}.
$$

(4.27)

For this we recall the orthogonality relation of the $\Phi^{m}_k(z)$ in the Hilbert space $L^2_{\sigma}(\mathbb{D})$, which reads

$$
\int_{\mathbb{D}} \Phi^{m}_j(z) \Phi^{m}_k(z) (1-z\bar{z})^{\sigma-2} d\nu(z) = \delta_{jk}.
$$

(4.28)

This suggests us to set $\Omega(z) := (1-z\bar{z})^{\sigma-2}$. Therefore, we get that

$$
d\mu_{\sigma,m}(z) = \frac{(\sigma-2m-1)}{\pi (1-z\bar{z})^{\sigma}} d\nu(z).
$$

(4.29)

By making use of the identity [20]:

$$
G_{11}^{11} \left( \varpi \bigg| \begin{array}{c} a \\ b \end{array} \right) = \Gamma(1-a+b) \varpi^b (1+\varpi)^{a-b-1}
$$

(4.30)

for $\varpi = -z\bar{z}$, $a = -1$ and $b = 0$, we arrive at the expression of the measure in (4.19). Therefore, Eq. (4.26) reduces to

$$
O = \sum_{j,k=0}^{+\infty} \delta_{jk} | \psi_j > < \psi_k | = 1\mathcal{H}.
$$

(4.31)

The proof is finished. ■

**Proposition 4.4.** Let $\sigma > 1$ and $m = 0, 1, ..., \lfloor (\sigma - 1) / 2 \rfloor$. Then, the states $|\ z, \sigma, m >$ satisfy the continuity property with respect to the label $z \in \mathbb{D}$. That is, the norm of the difference of two states

$$
d_{\sigma,m}(z,w) := || | z, \sigma, m > - | w, \sigma, m > ||_{\mathcal{H}}
$$

(4.32)

goes to zero whenever $z \to w$.

**Proof.** By using the fact that any GNBS is normalized by the factor given in (4.10), direct calculations enable us to write the square of the quantity in (4.32) as

$$
\sqrt{d_{\sigma,m}^2(z,w)} = 2 (1 - \Re < z, \sigma, m | w, \sigma, m >).
$$

(4.33)

Next, we use of the expression of the scalar product in (3.9) form which it is clear that the overlap takes the value 1 as $z \to w$ and consequently $d_{\sigma,m}(z,w) \to 0$. ■

We end this section by the following remarks.
Remark 4.5. By a general fact on reproducing kernels \[21\], the proof of proposition (2.1) also says that the knowledge of the explicit orthonormal basis in \(3.8\) leads directly to expression of the reproducing kernel of the generalized Bergman space in \(3.9\) via calculations using the formula \(4.8\) due to A. Srivastava and A. B. Rao \[19\].

Remark 4.6. In \[6\], we have used the same basis \(3.8\) under another form as labeling coefficients in order to consider the photon-counting probability distribution with the mass function

\[
P_k^{(\sigma,m,\lambda)} := \gamma_{\sigma,m,k} (1 - \lambda)^{\sigma-2m} \lambda^{m-k} \left( \frac{p_{(m-k,\sigma-2m-1)}}{2(m+k-m-k)}(1-2\lambda) \right)^2,
\]

where \(k = 0,1,2,\ldots, \gamma_{\sigma,m,k} > 0\) is a constant, \(m = 0,1,\ldots, [\sigma-1]/2\) and \(\lambda = z\bar{z}\). We have calculated the associated Mandel parameter \[22\] and we have discussed the classicality/nonclassicality of the GNBS with respect to the location of their labeling points \(z\) inside the hyperbolic disk \(D\). Similar results, in the Euclidean plane and the Riemann sphere settings have been obtained respectively in \[23\] and \[24\].

Remark 4.7. The fact that we have written the measure \(d\mu_{\sigma,m}(z)\) in \(4.19\) in terms of the Meijer’s G-function could be of help when tackling the "photon-added coherent states (PACS)" problem for the GNBS under consideration.

5 The isotonic oscillator \(L_\alpha\)

Not all quantum Hamiltonians are known to have exact solutions. An important model of a solvable class is the isotonic oscillator \[25\]

\[
L_\alpha := \frac{1}{2} \left( -\frac{d^2}{dx^2} + x^2 + \frac{(\alpha^2 - 1/4)}{x^2} \right), \quad \alpha \geq 1/2 \tag{5.1}
\]

acting in the Hilbert space \(\mathcal{H} := L^2(\mathbb{R}_+, dx)\) and the eigenfunctions \(\psi \in \mathcal{H}\) satisfy the Dirichlet boundary condition \(\psi(0) = 0\). This operator appears in the literature under many names such as Gol’dman-Krivchenkov Hamiltonian \[26\] or pseudoharmonic oscillator \[8\] or Laguerre operator \[27\]. It is the generalization of the harmonic oscillator in three dimensions where the generalization lies in the parameter \(\nu = \alpha^2 - 1/4\) ranging over \((0, +\infty)\) instead of the angular momentum quantum numbers \(l = 0,1,2,\ldots\). This operator may be factorized as follows

\[
L_\alpha = D^+_\alpha D^-_\alpha + \alpha + 1 \tag{5.2}
\]

in terms of the operators, having the form

\[
D^-_\alpha = \frac{1}{\sqrt{2}} \left( -\frac{\alpha + 1/2}{x} + x \pm \frac{d}{dx} \right). \tag{5.3}
\]

It is well known that the Hamiltonian \(L_\alpha\) admits exact solutions of the form

\[
\psi^\alpha_k(x) = \left( \frac{2\Gamma(k+1)}{\Gamma(k+\alpha+1)} \right)^{\frac{1}{2}} x^{\alpha+\frac{1}{2}} e^{-\frac{1}{2}x^2} L^k_\alpha \left( x^2 \right), \tag{5.4}
\]
where $L_k^{(\alpha)}(\cdot)$ denotes the Laguerre polynomial \[16\] and the corresponding eigenvalues are given by (28):

$$\lambda_k^{(\alpha)} = 2\alpha + k + 1, k = 0, 1, 2, \ldots$$  \hspace{1cm} (5.5)

Note also that the functions $\psi_k^{(\alpha)}$ can be obtained by $k$-fold application of a creation operator to the ground state wavefunction $\psi_0^{(\alpha)}$. The vectors $\{|\psi_k^{(\alpha)}\rangle\}_{k=0}^{\infty}$ satisfy the orthogonality relation

$$\langle \psi_k^{(\alpha)} | \psi_j^{(\alpha)} \rangle_{\mathcal{H}} = \int_0^{+\infty} \psi_k^{(\alpha)}(x) \psi_j^{(\alpha)}(x) \, dx = \delta_{kj}$$  \hspace{1cm} (5.6)

and constitute a complete orthonormal basis for the Hilbert space $\mathcal{H}$. Furthermore, they can be used together with the eigenvalues in (5.5) to define the heat semigroup associated with $L^{(\alpha)}$ as

$$e^{-tL^{(\alpha)}} [f] := \sum_{k=0}^{+\infty} e^{-t\lambda_k^{(\alpha)}} < f | \psi_k^{(\alpha)} >_{\mathcal{H}} \cdot \psi_k^{(\alpha)}$$  \hspace{1cm} (5.7)

for any function $f \in L^2(\mathbb{R}_+, dx)$. It is also well known that by using the Hille-Hardy formula ([17, p.242]), this semigroup has an integral representation, i.e.,

$$e^{-tL^{(\alpha)}} [f] (x) = \int_0^{+\infty} W_t(x, y) f(y) \, dy,$$  \hspace{1cm} (5.8)

where

$$W_t(x, y) = \frac{2 \sqrt{xy} e^{-t}}{(1 - e^{-2t})} I_{\alpha} \left( \frac{2xy e^{-t}}{1 - e^{-2t}} \right) \exp \left( -\frac{1}{2} \left( x^2 + y^2 \right) \frac{1 + e^{-2t}}{1 - e^{-2t}} \right).$$  \hspace{1cm} (5.9)

Here $I_{\alpha}(\cdot)$ denotes the modified Bessel function of the first kind and order $\alpha$ ([17, p.66]).

6 Husimi’s Q-function attached to $L^{(\alpha)}$

For $\sigma > 1$ and $m = 0, 1, \ldots, \lfloor (\sigma - 1) / 2 \rfloor$. A class of generalized negative binomial states (GNBS) attached to the isotonic oscillator $L^{(\alpha)}$ can be defined by setting

$$|z, \sigma, m, \alpha> = (N_{\sigma,m}(z))^{-\frac{1}{2}} \sum_{k=0}^{+\infty} \Phi_{\sigma,m}^{(\alpha)}(z) | \psi_k^{(\alpha)} >$$  \hspace{1cm} (6.1)

where $N_{\sigma,m}(z)$ is the factor in (4.10), $\{\Phi_{\sigma,m}^{(\alpha)}(z)\}$ are defined in (3.8) and $\{|\psi_k^{(\alpha)}\rangle\}$ are the Fock vectors given in (5.4). The diagonal representation of $e^{-tL^{(\alpha)}}$ in the GNBSs representation is now precised as follows.

**Definition 6.1.** The Husimi’s Q-function attached to the operator $L^{(\alpha)}$ is given through the mean value

$$Q_m^t (L^{(\alpha)}) (z) = \mathbb{E}_{\{|z, \sigma, m, \alpha>\}} \left( e^{-tL^{(\alpha)}} \right) = <z, \sigma, m, \alpha | e^{-tL^{(\alpha)}} | z, \sigma, m, \alpha>$$  \hspace{1cm} (6.2)

with respect to the set of GNBSs defined in (6.1).
Finally, we replace the factor \( N \) (6.1). We obtain successively the equation (6.2) in which we also replace the GNBS by their definition in with the operator \( L \) to write an inequality involving the thermodynamical potential associated \( \eta \) where

\[
\begin{align*}
\mathbb{E}_{\{z, \sigma, m, \alpha\}} (e^{-t L_\alpha}) & = \frac{\pi (1 - z \bar{z})^{\sigma} e^{-(2\alpha + 1)t}}{(\sigma - 2m - 1)} \left( \frac{(z \bar{z} - e^{-t}) (1 - z \bar{z} e^{-t})}{(1 - z \bar{z})^2} \right)^m \\
& \times \left( \frac{1 - z \bar{z}}{1 - z \bar{z} e^{-t}} \right)^\sigma \sum_{m=0}^{\infty} \frac{P_m^{(\sigma - 2m, 0)}}{(1 + \frac{2e^{-t} (1 - z \bar{z})^2}{(z \bar{z} - e^{-t})(1 - z \bar{z} e^{-t})})}. \quad (6.3)
\end{align*}
\]

**Proof.** We start by inserting the expression (5.7) of the operator \( e^{-t L_\alpha} \) into the equation (6.2) in which we also replace the GNBS by their definition in (6.1). We obtain successively

\[
Q_m^\ell (L_\alpha) (z) = \sum_{k=0}^{\infty} \exp (-t \lambda_k^m) < z, \sigma, m, \alpha | \psi_k^m > | z, \sigma, m, \alpha > \quad (6.4)
= \sum_{k=0}^{\infty} \exp (-t \lambda_k^m) < z, \sigma, m, \alpha | \psi_k^m >^2 \quad (6.5)
= \sum_{k=0}^{\infty} \exp (-t \lambda_k^m) | (\mathcal{N}_{\sigma, m} (z))^{-\frac{1}{2}} \Phi_k^{\sigma, m} (z) |^2 \quad (6.6)
= (\mathcal{N}_{\sigma, m} (z))^{-1} \sum_{k=0}^{\infty} \exp (-t (2\alpha + k + 1)) \Phi_k^{\sigma, m} (z) \Phi_k^{\sigma, m} (z) \quad (6.7)
= (\mathcal{N}_{\sigma, m} (z))^{-1} e^{-(2\alpha + 1)t} \sum_{k=0}^{\infty} (e^{-t})^k \Phi_k^{\sigma, m} (z) \Phi_k^{\sigma, m} (z). \quad (6.8)
\]

Now, to calculate the sum in (6.8) we make use of the expression of the functions \( \Phi_k^{\sigma, m} (z) \) in (3.8) involving Jacobi polynomials and we apply the identity (4.8). This allows us to obtain the expression

\[
Q_m^\ell (L_\alpha) (z) = (\mathcal{N}_{\sigma, m} (z))^{-1} e^{-(2\alpha + 1)t} \left( \frac{(z \bar{z} - e^{-t})(1 - z \bar{z} e^{-t})}{(1 - z \bar{z})^2} \right)^m \\
\times \left( \frac{1 - z \bar{z}}{1 - z \bar{z} e^{-t}} \right)^\sigma \sum_{m=0}^{\infty} \frac{P_m^{(\sigma - 2m, 0)}}{(1 + \frac{2e^{-t} (1 - z \bar{z})^2}{(z \bar{z} - e^{-t})(1 - z \bar{z} e^{-t})})}. \quad (6.9)
\]

Finally, we replace the factor \( \mathcal{N}_{\sigma, m} (z) \) by its expression in (4.10).}

In the following we will use the \( Q_m \)-function presented above in order to write an inequality involving the thermodynamical potential associated with the operator \( L_\alpha \). This potential reads

\[
\Omega_\alpha := -\frac{1}{\beta} \text{Tr} \left( \log \left( 1 + e^{-\beta (L_\alpha - \eta)} \right) \right) \quad (6.10)
\]

where \( \eta \) is the chemical potential and \( \beta = 1/k_B T \), \( k_B \) is the Boltzmann constant and \( T \) denotes the temperature. Let us put \( \epsilon = e^{\beta \eta} > 0 \) and state the following inequality.
Proposition 6.3. Let $\sigma > 1$. Then, the thermodynamical potential in (6.10) satisfy the inequality
\[
\max_{m \in \mathbb{Z}_+ \cap [0, (\sigma - 1)/2]} \left[ \frac{1}{\beta} \int_{D} \log \left( \frac{1}{1 + e^{Q^l_{m} (\mathcal{L}_a)} (z)} \right) d\mu_{\sigma,m} (z) \right] \leq \Omega_\alpha \tag{6.11}
\]
for every $\beta > 0$.

Proof. The form of the potential $\Omega_\alpha$ in (6.10) suggests us to consider the function
\[
\phi_\epsilon (u) = - \log (1 + \epsilon u). \tag{6.12}
\]
So that we can rewrite (6.10) as
\[
tf \Omega_\alpha = \text{Tr} \left( \phi_\epsilon \left( e^{-t L_\alpha} \right) \right). \tag{6.13}
\]
where $\beta = t \in \mathbb{R}_+$. We now apply the Berezin-Lieb inequality (2.7) for the lower symbol $Q_m$ of the operator $e^{-t L_\alpha}$ in the GNBS representation (6.2) to obtain the following inequality
\[
\int_{D} \left[ \phi_\epsilon \circ Q^l_{m} (\mathcal{L}_a) \right] (z) d\mu_{\sigma,m} (z) \leq \text{Tr} \left( \phi_\epsilon \left( e^{-t L_\alpha} \right) \right). \tag{6.14}
\]
Making use of (6.12) and replacing the right hand side of (6.14) by $t\Omega_\alpha$ as in (6.13), we get an inequality that holds for every $m = 0, 1, ..., \lfloor (\sigma - 1)/2 \rfloor$. Therefore, we consider the maximum with respect to the integer $m$ of the quantity in the left hand side of (6.14) in order to be close as possible to the value of $\Omega_\alpha$.

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FACULTY OF SCIENCES & TECHNICS (M’GHILA)
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