Quiver Yangians and $\mathcal{W}$-algebras for generalized conifolds

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Received 5 September 2022; revised 13 February 2023
Accepted for publication 25 April 2023
Published 9 May 2023

Abstract

We focus on quiver Yangians for most generalized conifolds. We construct a coproduct of the quiver Yangian following the similar approach by Guay–Nakajima–Wendlandt. We also prove that the quiver Yangians related by Seiberg duality are indeed isomorphic. Then we discuss their connections to $\mathcal{W}$-algebras analogous to the study by Ueda. In particular, the universal enveloping algebras of the $\mathcal{W}$-algebras are truncations of the quiver Yangians, and therefore they naturally have truncated crystals as their representations.

Keywords: quiver, Yangians, $\mathcal{W}$-algebras

(Some figures may appear in colour only in the online journal)

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1. Introduction and summary

The quiver Yangians were first introduced in [1] as Bogomol’nyi–Prasad–Sommerfield (BPS) algebras [2] for Type IIA string theories on toric Calabi–Yau (CY) threefolds. Since then, the quiver Yangians and their cousins have been extensively studied in [3–9]. As a nice combinatoric way to encode the BPS spectrum, the crystal melting models [10–12] naturally provide representations for the algebras. It is also expected that the quiver Yangians have intimate relations with cohomological Hall algebras [13–17] and many other quantum algebras.

An important algebra structure of the quiver Yangian is its coproduct. As studied in [8, 9, 18–21], one can construct the \( R \)-matrices from the BPS algebras. This realizes the Bethe/gauge correspondence [22, 23] for supersymmetric gauge theories with (non-chiral) toric quiver descriptions. For instance, the rapidities in the Bethe ansatz equation are known to be identified with the supersymmetric vacua of the 2d \( \mathcal{N} = (2,2) \) gauge theory in this dictionary. Moreover, each 2-magnon \( S \)-matrix corresponds to the so-called bond factor (see (5.50)) stemmed from the action of the quiver Yangian generators on the 2d molten crystal [8]. By analyzing the coproduct structures of the algebra generators and the Lax operators, a remarkable no-go theorem for chiral quivers was discovered for the Bethe/gauge correspondence in [9]. In other words, it would require a more delicate and involved study if we want to extend such correspondence for shifted 2d crystals or the quivers associated to toric CY threefolds with compact divisors.

Here, we shall mainly focus on generalized conifolds \( xy = z^M w^N \) with \( M + N > 2, MN \neq 2 \) and \( M \neq N \). All but one toric CY threefolds without compact 4-cycles are generalized conifolds. Their quiver Yangians have salient features and have been systematically studied in [1]. In the construction of the coproduct, the restrictions on \( M, N \) are in fact due to the underlying untwisted affine Lie superalgebra \( \mathfrak{sl}_M|N \). To obtain the coproduct, we will discuss different presentations of the quiver Yangians. In fact, our approach is analogous to the case of Guay’s affine Yangian in [24]. It is also worth noting that a version of the affine super Yangian was introduced in [25]. However, as pointed out in [8], this super Yangian is different from the quiver Yangian (unless these two-parameter algebras degenerate to one-parameter families). This is because the Yangian in [25] does not respect the vertex constraints that will be mentioned later. Physically, such constraints can be viewed as removing the gauge redundancies [1]. Nevertheless, it is not surprising to find the expressions similar to the forms in [24, 25].

For affine Lie algebras and quiver Yangians with only bosonic generators, we only have even reflections, and there is one single toric phase for each quiver Yangian. For affine Lie superalgebras and quiver Yangians with fermionic generators, we would further have odd reflections [26–28] that do not respect the \( \mathbb{Z}_2 \)-grading of the algebras. However, it is exactly this property of the odd reflections that allows us to transform the quiver from one toric phase to another. Often in literature, only the distinguished case (which has two fermionic nodes)
would be discussed. It is always natural to expect the results obtained would still hold for those with more fermionic nodes due to the invariance of the underlying Kac–Moody algebra. Here, along with the help of a presentation obtained in studying the coproduct, we shall extend the odd reflections to the quiver Yangians and prove that the Seiberg dual quiver Yangians are indeed isomorphic algebras.

On the other hand, the $\mathcal{W}$-algebras [29–35] should play a crucial role in the tensionless limit of string theory in AdS$_3$ [36–39]. In particular, the rectangular $\mathcal{W}$-algebra can be realized as the symmetry algebra of the coset conformal field theory (CFT) whose holographic dual gives higher spin gravity [40]. Such vertex algebras have been well-studied in mathematics literature such as [41–43].

In this paper, we shall discuss the BPS/CFT (aka AGT, 2d/4d) correspondence [44, 45] at the level of algebras. The BPS algebras and the vertex operator algebras (VOAs) are expected to be contained in a broader picture under the BPS/CFT correspondence. Here, the BPS algebras are the quiver Yangians as they realize the BPS algebras in the Type IIA string theory setting on the toric CYs. The VOAs refer to the $\mathcal{W}$-algebras. They can arise from the $(p, q)$-webs of the corresponding CYs, which can also be interpreted as algebras from the junctions of the interfaces of 4d supersymmetric gauge theories [30, 46]. In the finite cases, the relations between Yangians and $\mathcal{W}$-algebras have been explored in [47–49]. For $\mathfrak{gl}_1$ whose associated CY is the simplest $\mathbb{C}^3$, it was shown in [50, 51] that the affine/quivier Yangian is isomorphic to the universal enveloping algebra of the $\mathcal{W}_{1+\infty}$-algebra. Moreover, in such case, the AGT conjecture was proven in [52] with a surjective homomorphism from the quiver Yangian for $\mathbb{C}^3$ to the universal enveloping algebra of the principal $\mathcal{W}$-algebra. Physically, the Nekrasov partition function of the 4d supersymmetric gauge theories can be identified with the conformal blocks of the corresponding VOAs. From a geometric perspective, the Verma module of the VOA results from its action on the equivariant intersection cohomology of the instanton moduli space [53].

Similar to the $\mathcal{W}_{1+\infty}$-algebra case whose truncation gives the algebra at the corner [46], the matrix-extented $\mathcal{W}$-algebra (called $\mathcal{W}_{M|N\times\infty}$ in [34]) for any generalized conifold has generators of spins $s = 1, 2, \ldots$ and is expected to truncate to VOAs describing certain interface of a 4d supersymmetric gauge theory. In [54], a surjective homomorphism from Ueda’s affine super Yangian to the rectangular $\mathcal{W}$-algebras was constructed in a way similar to the evaluation maps in Yangian algebras [25, 55]. However, as mentioned above, such super Yangian is not the BPS algebra physically. Therefore, due to this difference as well as the different conventions, we decide to give a careful construction of the surjective homomorphism from the quiver Yangian to the $\mathcal{W}$-algebras (under certain condition on the parameters of the quiver Yangian) although the approach is still similar to those in [54, 55]. This shows that the universal enveloping algebras of the VOAs are indeed the truncations of the BPS algebras, and the quiver Yangians can in this sense be viewed as some realization of $U(\mathcal{W}_{M|N\times\infty})$. As the (truncated) crystals are then naturally the representations of the universal enveloping algebras of the $\mathcal{W}$-algebras, we see that they are highest weight representations. We will also see that the parameters on the $\mathcal{W}$-algebra side can be related to the vacuum charge on the quiver Yangian side. Let us also mention here that similar to the quiver Yangian discussions, we shall also consider the $\mathcal{W}$-algebras not only restricted to the distinguished case. However, unlike the proof for the quiver Yangians, the isomorphisms for these $\mathcal{W}$-algebras are much more straightforward. This allows us to connect both the BPS algebras and the VOAs in different phases freely.

The paper is organized as follows. In section 2, we give a brief review on the concept of quiver Yangians for generalized conifolds. In section 3, we discuss different presentations of
the quiver Yangians which enable us to obtain the coproducts of them. We shall then prove that the quiver Yangians in different toric phases are isomorphic in section 4. In section 5, we shall study the relations between quiver Yangians and rectangular $W$-algebras. Some future directions will be mentioned in section 6. In appendix A, we shall also discuss the generators of the quiver Yangians for $\mathbb{C} \times \mathbb{C}^2 / \mathbb{Z}_2$ and the conifold, as well as $\mathbb{C}^3 / (\mathbb{Z}_2 \times \mathbb{Z}_2)$ which is the only toric CY$_3$ without compact four-cycles that is not a generalized conifold. We discuss the odd reflections of the Kac–Moody algebras in appendix B. In appendix C, we give an introduction to the rectangular $W$-algebras.

2. Quiver Yangians for generalized conifolds

Let us start with a recap on some fundamentals of the quiver Yangians. Here, we shall only give the defining relations for generalized conifolds $x y = z^M w^N$ ($M + N > 2$), but the quiver Yangians can be defined for any quivers as in [1]. We shall fix the convention that the set of natural numbers $\mathbb{N}$ is always counted from 0.

Given any generalized conifold with $M + N \geq 1$, its toric diagram is

![Toric Diagram](image)

Its quivers in different toric phases can be obtained from the triangulations of the lattice polygon [56, 57]. These triangulations can in turn be encoded by a sequence of signs $\varsigma = \{\varsigma_a | a \in \mathbb{Z}/(M+N)\mathbb{Z}\}$, one for each simplex in the toric diagram. There are $M$ minus ones and $N$ plus ones. When two simplices are glued side by side, they have the same sign. When they are glued in the alternative way, they have opposite signs. An illustration can be found in figure 1.

The quiver is constructed as follows. First, there is always a pair of opposite arrows connecting node $a$ and node $a + 1$ ($a \in \mathbb{Z}/(M+N)\mathbb{Z}$). Then the node $a$ is bosonic/even and has a self-loop if $\varsigma_a = \varsigma_{a+1}$. If $\varsigma_a = -\varsigma_{a+1}$, then it is fermionic/odd and has no self-loops. Hence, the resulting quiver is essentially the double of the untwisted affine $\mathfrak{sl}_M | \mathfrak{sl}_N$ Dynkin quiver with extra loops on the bosonic nodes. The superpotential $W$ can be fully determined in the toric quiver gauge theory and is composed of the terms

$$ W = \sum_{a} \varsigma_a \text{tr}(I_{a,a} I_{a,a+1} I_{a+1, a} - I_{a,a+1} I_{a+1, a}), \quad \varsigma_a = \varsigma_{a+1},$$

$$ W = \sum_{a} \varsigma_a \text{tr}(I_{a,a} I_{a,a+1} I_{a+1, a} - I_{a,a+1} I_{a+1, a}), \quad \varsigma_a = -\varsigma_{a+1},$$

where $I_{a,b}$ denotes the arrow/field from node $a$ to $b$.

To construct the quiver Yangian, we need to assign parameters $\tilde{\varsigma}_I$ to the arrows $I$ in the quiver. As $\tilde{\varsigma}_I$ can be viewed as charges under a global symmetry of the quiver quantum mechanics, they should be compatible with the superpotential. This yields the loop constraint

$$ \sum_{I \in L} \tilde{\varsigma}_I = 0,$$

for any closed loop $L$ in the periodic quiver. It turns out that the number of independent parameters is given by

$$ |Q_1| - |Q_2| - 1 = |Q_0| + 1,$$
where $Q_0, Q_1$ are the sets of nodes and arrows respectively. Moreover, $Q_2$ denotes the faces of the periodic quiver, or equivalently, the monomial terms in the superpotential.

Furthermore, as pointed out in [1], the mixing of global and gauge symmetries associated to each node would cause shifts of $\tilde{e}_I$. One can then introduce a gauge fixing condition to get rid of this shift. This is known as the vertex constraint:

$$\sum_{I \in a} \text{sgn}_a(I) \tilde{e}_I = 0,$$

(2.5)

where the sign function $\text{sgn}_a(I)$ is equal to +1 (resp. -1) when the arrow $I$ starts from (resp. ends at) the node $a$, and 0 otherwise. As an overall $U(1)$ symmetry decouples, the total number of the vertex constraints is $|Q_0|$. Together with the $|Q_0| + 1$ loop constraints, we are then left with two independent parameters denoted as $\epsilon_1, \epsilon_2$. Along with the R-symmetry, they parametrize the $U(1)^3$ isometry of the toric CY$_3$. Sometimes, it would also be convenient to introduce a third parameter $\epsilon_3$ such that $\epsilon_1 + \epsilon_2 + \epsilon_3 = 0$. This condition then specializes to the sub-torus $T^2$ that preserves the volume form.

Therefore, the quiver Yangian constructed therefrom is a two-parameter algebra\footnote{Later in section 5, we will discuss the connection of quiver Yangians to $\mathcal{W}$-algebras. These $\mathcal{W}$-algebras only depend on the ratio $\epsilon_1/\epsilon_2$ [34]. However, the two parameters are independent for generic quiver Yangians. This will also be reflected in the condition on $\epsilon_{1,2}$ in theorem 5.4.}. The general rule of the parameter assignment to the arrows is summarized in figure 2.

To write down the definition of the quiver Yangian, we need to set up some notations. In physics literature, the super bracket is often denoted as $[\cdot, \cdot]$. However, for convenience, we shall simply use $[x,y] = xy - (-1)^{|x||y|}yx$ here, where $|x| \in \{0, 1\}$ indicates the Bose–Fermi

Figure 1. In these examples, we have (a) $\varsigma = \{-1, +1\}$, (b) $\varsigma = \{-1, -1\}$ and (c) $\varsigma = \{-1, -1, +1, -1, +1, +1\}$.

Figure 2. The values of $\tilde{e}_I$ associated to bifundamentals and adjoints subject to the loop and vertex constraints for generalized conifolds. We have (a) $\omega = \omega_{a+1}$ and (b) $\omega = -\omega_{a+1}$, where $\epsilon_{1,2,3}$ are parameters of the quiver Yangian.
Recall that the superpotential is necessary in the definition of the algebra since it gives rise to the loop constraints (2.10) and (2.12). In the above relations, we also allow $\sigma_{ab}^0$ to be non-trivial when $M + N > 2$ if there is an arrow from $a$ to $b$. Otherwise, $\sigma_{ab}^1 = \sigma_{ab}^0 = 0$.

**Definition 2.1.** Given a quiver $Q = (Q_0, Q_1)$ and its superpotential $W$ (with $M + N > 2$), the *non-reduced* quiver Yangian is generated by the modes $e_n^{(a)}$, $f_m^{(a)}$, and $\psi_n^{(a)}$ ($a \in Q_0$, $n \in \mathbb{N}$) satisfying the relations

\[
\begin{align*}
[e_n^{(a)}, f_m^{(b)}] &= \delta_{ab} \psi_{m+n}^{(a)}, \\
[e_n^{(a)}, f_m^{(b)}] &= \sigma_{ab} \psi_{m+n}^{(a)}, \\
[\psi_{n+1}^{(a)}, e_m^{(b)}] &= \sigma_{1b}^{(a)} \psi_n^{(a)} e_m^{(b)} + \sigma_{1b}^{(a)} f_m^{(b)} \psi_n^{(a)}, \\
[\psi_{n+1}^{(a)}, f_m^{(b)}] &= -\sigma_{1b}^{(a)} \psi_n^{(a)} f_m^{(b)} - \sigma_{1b}^{(a)} f_m^{(b)} \psi_n^{(a)}, \\
[e_n^{(a)}, f_m^{(b)}] &= (\sigma_{ab}^1 = 0),
\end{align*}
\]

The generators $e_n^{(a)}$ and $f_m^{(a)}$ have the $\mathbb{Z}_2$-grading same as the corresponding node $a$ while $\psi_n^{(a)}$ is always bosonic. In the above relations, we also allow $\psi_{-1}^{(a)} := 1/(\epsilon_1 + \epsilon_2)$ so that

\[
\begin{align*}
\left[\psi_0^{(a)}, e_m^{(b)}\right] &= \frac{1}{\epsilon_1 + \epsilon_2} (\sigma_{ab}^1 + \sigma_{ab}^0) e_m^{(b)}, \\
\left[\psi_0^{(a)}, f_m^{(b)}\right] &= -\frac{1}{\epsilon_1 + \epsilon_2} (\sigma_{ab}^1 + \sigma_{ab}^0) f_m^{(b)}
\end{align*}
\]

can be deduced from the $\psi$ and $\psi_f$ relations.

Notice that for future convenience, we have rescaled the generators compared to the original convention in [1] with $e_n^{(a)}$, $f_n^{(a)}$, and $\psi_n^{(a)}$. The two sets of modes are related by $e_n^{(a)} = (\epsilon_1 + \epsilon_2)^{1/2} e_n^{(a)}$, $f_n^{(a)} = (\epsilon_1 + \epsilon_2)^{1/2} f_n^{(a)}$, and $\psi_n^{(a)} = (\epsilon_1 + \epsilon_2) \psi_n^{(a)}$ (including $\psi_0^{(a)}$).

To correctly recover the BPS degeneracies, we also need the Serre relations.

**Definition 2.2.** Given the above quiver data, the (reduced) quiver Yangian $Y_{Q,W}$ is the non-reduced quiver Yangian with the Serre relations given as follows. When $MN \neq 2$, we have

\[
\begin{align*}
\text{Sym}_{n_1, n_2} \left[ e_n^{(a)}, e_{n_1}^{(a)} \right] &= 0 \\
\text{Sym}_{m_1, m_2} \left[ e_m^{(a)}, e_m^{(a)} \right] &= 0
\end{align*}
\]

and the same relations with all $e$ replaced by $f$. When $(M,N) = (2,1)$ (or equivalently, $(M,N) = (1,2)$), namely for the suspended pinch point (SPP), we have

\[\text{Sym}_{n_1, n_2} \left[ e_n^{(a)}, e_{n_1}^{(a)} \right] = 0 \quad (|a| = 0),
\]

\[\text{Sym}_{m_1, m_2} \left[ e_m^{(a)}, e_{m_1}^{(a)} \right] = 0 \quad (|a| = 1),
\]

2 Recall that the superpotential is necessary in the definition of the algebra since it gives rise to the loop constraints which together with the vertex constraints restrict the parameters to be $\epsilon_{1,2}$. 

\[\text{Sym}_{n_1, n_2} \left[ e_n^{(a)}, e_{n_1}^{(a)} \right] = 0 \quad (|a| = 0),
\]

\[\text{Sym}_{m_1, m_2} \left[ e_m^{(a)}, e_{m_1}^{(a)} \right] = 0 \quad (|a| = 1),
\]
In this paper, the name quiver Yangian is always referred to as the reduced one. 

We will omit $\epsilon_j$ as well.

For toric CYs, as the superpotential can be unambiguously determined for a given quiver, we shall always choose $\psi_{0,1}$. Moreover, since the quiver Yangian\(^3\) is always a two-parameter Yangian algebra, we will omit $\epsilon_j$ as well.

3. Coproduct of quiver Yangians

Following the strategy in [24, 25], a coproduct of the quiver Yangian can be obtained based on the underlying Kac–Moody superalgebra $g = A_{M-1,N-1}^{(1)}$. In the Chevalley basis, we have the generators with $\left[ x_+^{(a)}, x_-^{(a)} \right] = h^{(a)}$ and $\left( x_+^{(a)}, x_-^{(a)} \right) = 1$, where $\langle -, - \rangle$ is an invariant inner product on the Kac–Moody superalgebra. Let $\Delta = \Delta_+ \cup \Delta_-$ be the set of roots composed of positive and negative roots. Denote the sets of real and imaginary roots as $\Delta^r$ and $\Delta^i$ respectively. Write $g_\alpha$ as the root space attached to the root $\alpha$, and the simple roots will be labeled as $\alpha^{(o)}$. In particular, $\Delta^r = \Delta_+ \cup \{ n\delta + \alpha | n \in \mathbb{Z}_+, \alpha \in \Delta \}$ and $\Delta^i = \{ n\delta | n \in \mathbb{Z}_+ \}$, where $\Delta$ is the set of roots of the underlying Lie superalgebra with the zeroth vertex removed in the Dynkin diagram of $g$ and $\delta = \sum\alpha^{(o)}$ is the minimal positive imaginary root of $g = A_{M-1,N-1}^{(1)}$. Notice that all the odd roots are isotropic (i.e. with vanishing inner product) in such a case.

Following the Cartan matrix (with the first non-zero diagonal element being 2), our convention would be taken as $\sigma^{\alpha \beta} = \delta^{\alpha \beta} = (\epsilon_1 + \epsilon_2) (\alpha^{(o)},\alpha^{(b)})$ for future convenience. In other words, we shall always choose $c_0 = -1$ for the corresponding simplex in the toric diagram. Therefore, $\sigma^{\alpha \beta} = -\frac{1}{2} \epsilon_3 (\alpha^{(o)},\alpha^{(b)})$. Then it is straightforward to see that there is an algebra homomorphism $\iota$ from $U(g)$ to $Y$ with $h^{(a)} \mapsto \psi^{(a)}_0$, $x_+^{(a)} \mapsto f^{(a)}_0$ and $x_-^{(a)} \mapsto f^{(a)}_0$. For each positive root $\alpha$, choose a basis $\{ x_+^{(\alpha,k)} \}$ of $g_\alpha$ with a dual basis $\{ x_-^{(\alpha,k)} \}$ of $g_{-\alpha}$ such that $\left( x_+^{(\alpha,k)}, x_-^{(\alpha,j)} \right) = \delta_{kj}$. We will also denote $e^{(\alpha,k)} = \iota \left( x_+^{(\alpha,k)} \right)$ and $f^{(\alpha,k)} = \iota \left( x_-^{(\alpha,k)} \right)$, where $k = 1, \ldots, \dim g_\alpha$. When $\alpha$ is a real root, $\dim g_\alpha = 1$ and we shall simply write $e^{(\alpha)} = e^{(\alpha,1)}$, $f^{(\alpha)} = f^{(\alpha,1)}$. In particular, given a simple root $\alpha^{(a)}$, we have $e^{(\alpha)} = e^{(\alpha^{(a)})}$ and $f^{(\alpha)} = f^{(\alpha^{(a)})}$.

3.1. A minimalistic presentation

The definition of the quiver Yangian in section 2 involves infinitely many generators. To write the coproduct, we first need to give a presentation with generators of a finite number.

As shown in [8], all the generators can in fact be inductively obtained from $e^{(a)}_0$, $f^{(a)}_0$ and $\psi^{(a)}_{0,1}$ by

\begin{equation}
\text{Sym}_{m_1,m_2} \text{Sym}_{m_1,m_2} \left[ e^{(0)}, e^{(2)}_1, e^{(0)}_{m_1}, e^{(2)}_2, e^{(2)}_1, e^{(1)}_k \right] 
= \text{Sym}_{m_1,m_2} \left[ e^{(2)}_1, e^{(0)}_{m_1}, e^{(2)}_2, e^{(2)}_1, e^{(1)}_k \right],
\end{equation}

and the same relation with all $e$ replaced by $f$, where the node (1) is taken to be the single bosonic node.

\(^3\) In this paper, the name quiver Yangian is always referred to as the reduced one.
We discuss how we can get all the generators from finitely many of them for the quiver Yangians associated to

Notice that the other toric phase for \((M,N)\) (though their possible minimalistic presentations with finitely many relations

\[(3.4)\]

\[(3.3)\]

exclude two special cases: (1) \((M,N)\) is bosonic (resp. fermionic). Therefore, it is natural to expect that the quiver Yangian can be

generated by the modes \(e^{(a)}\) and fermionic alternatively, though all cases with \(M = N\) will not be considered when we discuss coproducts later.

\[\psi_{m+1}^{(a)} = \frac{1}{\alpha_1^{(a)} \alpha_2^{(b)}} \left[ \gamma_1^{(b)} e_m^{(a)} + \frac{\sigma_1^{ab} - \sigma_2^{ab}}{2} \left[ \psi_0^{(b)} e_m^{(a)} \right] \right] \]

\[f_m^{(a)} = \frac{1}{\alpha_1^{(a)} \alpha_2^{(b)}} \left[ \gamma_1^{(b)} f_m^{(a)} + \frac{\sigma_1^{ab} - \sigma_2^{ab}}{2} \left[ \psi_0^{(b)} f_m^{(a)} \right] \right] \]

\[\psi_{m+1}^{(a)} = \left[ \epsilon_{m+1} + \epsilon_m \right], \quad (3.1)\]

where \(\gamma_1^{(b)} := \psi_1^{(b)} - \frac{\alpha_1 + \alpha_2}{\alpha_1^{(a)} \alpha_2^{(b)}} \left( \psi_0^{(b)} \right)^2\), and the node \(b\) can be taken as \(a\) (resp. \(a + 1\)) when \(a\) is bosonic (resp. fermionic). Therefore, it is natural to expect that the quiver Yangian can be

generated only by finitely many relations of the three sets of zero modes together with \(\psi_1^{(a)}\) (or equivalently, \(\omega_1^{(a)}\)). To confirm that this is the case, we need to show that they can recover all the defining relations of the quiver Yangian in section 2.

For this minimalistic presentation of the (reduced) quiver Yangians\(^4\), we also need to exclude two special cases: (1) \((M,N) = (2,1), (1,2)\); (2) \((M,N) = (2,2)\) with only fermionic nodes\(^5\). Their quivers are depicted in figure 3. We will further comment on this at the end of this subsection.

**Theorem 3.1.** For generalized conifolds with \(M + N > 2\), the non-reduced quiver Yangian is generated by the modes \(\epsilon_r^{(a)}, f_r^{(a)}\) and \(\psi_r^{(a)}\) \((a \in Q_0, r = 0, 1)\) satisfying the relations

\[\left[ \epsilon_r^{(a)}, \psi_s^{(b)} \right] = 0, \quad (3.2)\]

\[\left[ \epsilon_0^{(a)}, f_0^{(b)} \right] = \delta_{ab} \psi_0^{(a)}, \quad \left[ e_1^{(a)}, f_1^{(b)} \right] = \delta_{ab} \psi_1^{(a)}, \quad (3.3)\]

\[\left[ \psi_0^{(a)}, \psi_1^{(b)} \right] = \left( \alpha_1^{(a)} \alpha_2^{(b)} \right) \psi_r^{(b)}, \quad (3.4)\]

\(^4\) We discuss how we can get all the generators from finitely many of them for the quiver Yangians associated to \(\mathbb{C} \times \mathbb{C}^2/\mathbb{Z}_2\) and the conifold in appendix A (though their possible minimalistic presentations with finitely many relations are postponed to future study). We also mention the minimalistic presentation for \(\mathbb{C}^3/(\mathbb{Z}_2 \times \mathbb{Z}_2)\) therein although it is not a generalized conifold.

\(^5\) Notice that the other toric phase for \((M,N) = (2,2)\) is not excluded, where the quiver has four nodes being bosonic and fermionic alternatively, though all cases with \(M = N\) will not be considered when we discuss coproducts later.
The proof is similar to the ones in \((3.8)\) and \((3.9)\) can be deduced from \((R)\) for the non-reduced quiver Yangians below.

In terms of \(\psi_1^0\), the \(\psi_1 e_0\) and \(\psi_1 f_0\) relations can be written as

\[
\left[ \psi_1^0, e_0 \right] = \left( \alpha(a), \alpha(b) \right) e_1(b) + \frac{1}{2} \left( \sigma_1^{ba} - \sigma_1^{ab} \right) \left( \alpha(a), \alpha(b) \right) e_0(b),
\]

\[
\left[ \psi_1^0, f_0 \right] = - \left( \alpha(a), \alpha(b) \right) f_1(b) - \frac{1}{2} \left( \sigma_1^{ba} - \sigma_1^{ab} \right) \left( \alpha(a), \alpha(b) \right) f_0(b).
\]

It is worth noting that this resembles Drinfeld’s realization in [58]. For simplicity, we shall denote all the relations (3.2)–(3.10) as (R) and the relations (3.11) and (3.12) as (S).

**Proof.** The proof is similar to the ones in [24, 25]. Essentially, we need to show that the defining relations in section 2 can be deduced from \((R)\) for the non-reduced quiver Yangians and also \((S)\) for the reduced ones. This follows from lemmas 3.2–3.5 below.

**Lemma 3.2.** For \(m \in \mathbb{N}\), we have

\[
\left[ \psi_1^0, e_m \right] = \left( \alpha(a), \alpha(b) \right) e_m(b),
\]

\[
\left[ \psi_1^0, f_m \right] = \left( \alpha(a), \alpha(b) \right) e_m(b) + \sigma_1^{ba} \psi_0 e_m(b) + \sigma_1^{ab} \psi_0 e_m(b),
\]

and similar relations for \(f_m\) from \((R)\).

It would be helpful to also spell out these relations using \(\psi_1^0\):

\[
\left[ \psi_1^0, e_m \right] = \left( \alpha(a), \alpha(b) \right) e_{m+1}(b) + \frac{1}{2} \left( \sigma_1^{ba} - \sigma_1^{ab} \right) \left( \alpha(a), \alpha(b) \right) e_m(b),
\]

\[
\left[ \psi_1^0, f_m \right] = - \left( \alpha(a), \alpha(b) \right) f_{m+1}(b) - \frac{1}{2} \left( \sigma_1^{ba} - \sigma_1^{ab} \right) \left( \alpha(a), \alpha(b) \right) f_m(b).
\]
Proof. This can be proven by induction. When \( m = 0 \), this is the given (3.4). Now suppose both the \( \psi_0 e_m \) and \( \psi_1 e_m \) relations hold for \( m = k \). Then

\[
\left[ \psi_0^{(a)} , \epsilon_{k+1}^{(b)} \right] = \frac{1}{(\alpha^{(a)}, \alpha^{(b)})} \left[ \psi_1^{(c)} , \epsilon_k^{(b)} \right] - \frac{\alpha^{(b)} - \alpha^{(c)} - \frac{2}{2}}{2} \epsilon_k^{(b)} \\
= \frac{1}{(\alpha^{(a)}, \alpha^{(b)})} \left[ \psi^{(c)} , \left[ \psi_0^{(a)} , \epsilon_k^{(b)} \right] \right] - \frac{\alpha^{(b)} - \alpha^{(c)} - \frac{2}{2}}{2} \epsilon_k^{(b)} \\
= \left( \epsilon_k^{(b)} \right) \psi_1^{(a)} , \epsilon_{k+1}^{(b)},
\]

(3.17)

where the second equality follows from the fact that \( \psi_0^{(a)} \) commutes with \( \psi_1^{(c)} \), and we have used the \( \psi_0 e_m \) relation at \( m = k \) followed by expanding the only remaining bracket to obtain the third equality. The \( \psi_1 e_m \) relation, as well as the \( \psi_0 \psi_1 \) relations, can be shown in the same manner using \( \psi_1^{(c)} \).

With this useful lemma prepared, let us first show the \( \psi \psi \) and \( ef \) relations.

Lemma 3.3. For \( m, n \in \mathbb{N} \), we have (2.6) and (2.7) from (R).

Proof. Again, we prove this by induction. When \( n = m = 0 \), they are the given (3.2) and (3.3). Now suppose the relations hold when \( n + m = p + l = k + 1 \) for any such \( m, n \). Then we can consider the commutation relation of \( \psi^{(c)}_1 \) with the (super) commutators in question. For \( e_l^{(a)} \) and \( f_k^{(a)} \), we have

\[
0 = \left[ \psi_1^{(c)} , \psi_1^{(a)} \right] = \left[ \psi_1^{(c)} , \left[ e_l^{(a)} , f_k^{(a)} \right] \right],
\]

(3.18)

where \( c \) is taken to be \( a \) if \( |a| = 0 \) and \( a + 1 \) otherwise. Expanding the brackets and using lemma 3.2, this becomes

\[
(\alpha^{(a)}, \alpha^{(b)}) \left( \left[ e_l^{(a)} , f_k^{(a)} \right] - \left[ e_l^{(a)} , f_{k+1}^{(a)} \right] \right) = 0.
\]

(3.19)

Therefore,

\[
\left[ e_l^{(a)} , f_k^{(a)} \right] = \left[ e_l^{(a)} , f_{k+1}^{(a)} \right].
\]

(3.20)

We first take \( l = p \) and \( k = 0 \) and get

\[
\left[ e_p^{(a)} , f_0^{(a)} \right] = \left[ e_p^{(a)} , f_{0+1}^{(a)} \right] = \psi_0^{(a)},
\]

(3.21)

where the second equality follows from our definition of the higher modes as in (3.1). Next, we can take \( l = p - 1 \) and \( k = 1 \) to get the one for \( e_p^{(a)} \) and \( f_2^{(a)} \). Then all relations for the possible \((l, k)\) with \( l + k + 1 = p + 1 \) can be obtained in such way.

Likewise, for the \( ef \) relation with \( a \neq b \), we consider

\[
0 = \left[ \psi^{(c)}_1 , \left[ e_l^{(a)} , f_k^{(b)} \right] \right].
\]

(3.22)

Here, \( c \) can be chosen such that \( \sigma_1^{ac} \neq 0 \) and/or \( \sigma_1^{bc} \neq 0 \). Again, after using lemma 3.2, we can reach the relation for any \( l + k + 1 = p + 1 \). This proves the \( ef \) relation. The \( \psi \psi \) relation can be shown using \( \psi_1^{(c)} \) in the similar way.

Now, let us show the \( \psi e, \psi f, ee \) and \( ff \) relations.
Lemma 3.4. For \( m, n \in \mathbb{N} \), we have (2.8)–(2.12) from (R).

**Proof.** We again prove this by induction. When \( n = m = 0 \), they are given by (3.4)–(3.10). Suppose they hold for \( n = l, m = k \). We need to show that they are still true for \((l + 1, k)\) and \((l, k + 1)\). Consider

\[
\left[ \tilde{e}_i^{(a)}, e_{l+1}^{(a)} e_k^{(b)} - e_l^{(a)} e_{k+1}^{(b)} \right] = \left[ \tilde{e}_i^{(a)}, \sigma_f^{ba} e_l^{(a)} e_k^{(b)} + (-1)^{|a||b|} e_{l+1}^{(a)} e_i^{(a)} \right],
\]

which follows from our assumption when \( n = l, m = k \). Using lemma 3.2, a straightforward calculation yields

\[
2\sigma_1^{ab} \left[ \left( e_i^{(a)} e_{l+1}^{(a)} e_k^{(b)} - e_{l+1}^{(a)} e_i^{(a)} e_k^{(b)} \right) + \left( e_{l+1}^{(a)} e_i^{(a)} e_k^{(b)} - e_i^{(a)} e_{l+1}^{(a)} e_k^{(b)} \right) \right]
= 2\sigma_1^{ab} \left( \sigma_f^{ba} e_{l+1}^{(a)} e_k^{(b)} + (-1)^{|a||b|} e_i^{(a)} e_{l+1}^{(a)} e_k^{(b)} \right).
\]

Replacing \( \psi_1^{(a)} \) with \( \psi_1^{(b)} \), we obtain

\[
\left( \sigma_1^{ab} + \sigma_1^{ba} \right) \left( e_i^{(a)} e_{l+1}^{(a)} e_k^{(b)} - e_{l+1}^{(a)} e_i^{(a)} e_k^{(b)} \right) + 2\sigma_1^{ab} \left( \sigma_1^{ab} e_{l+1}^{(a)} e_k^{(b)} + (-1)^{|a||b|} e_i^{(a)} e_{l+1}^{(a)} e_k^{(b)} \right)
= \left( \sigma_1^{ab} + \sigma_1^{ba} \right) \left( \sigma_f^{ba} e_{l+1}^{(a)} e_k^{(b)} + (-1)^{|a||b|} e_i^{(a)} e_{l+1}^{(a)} e_k^{(b)} \right) + 2\sigma_1^{ab} \left( \sigma_1^{ab} e_{l+1}^{(a)} e_k^{(b)} + (-1)^{|a||b|} e_i^{(a)} e_{l+1}^{(a)} e_k^{(b)} \right).
\]

When \(|a||b| = 0\) or when \(|a||b| = 1\) with \(\sigma_1^{ab} \neq 0\), we essentially have two linearly independent equations with solution

\[
\left( \sigma_1^{ab} - \sigma_1^{ba} \right) \left( e_i^{(a)} e_{l+1}^{(a)} e_k^{(b)} - e_{l+1}^{(a)} e_i^{(a)} e_k^{(b)} \right) - \left( \sigma_1^{ab} e_1^{(a)} e_k^{(b)} + (-1)^{|a||b|} e_i^{(a)} e_{l+1}^{(a)} e_k^{(b)} \right) = 0,
\]

\[
\left( \sigma_1^{ab} - \sigma_1^{ba} \right) \left( e_i^{(a)} e_{l+1}^{(a)} e_k^{(b)} - e_{l+1}^{(a)} e_i^{(a)} e_k^{(b)} \right) - \left( \sigma_1^{ab} e_{l+1}^{(a)} e_k^{(b)} + (-1)^{|a||b|} e_i^{(a)} e_{l+1}^{(a)} e_k^{(b)} \right) = 0.
\]

When \(|a||b| = 1\) with \(\sigma_1^{ab} = 0\), the two linear equations are trivial, but we can instead use \(\psi_1^{(a\pm 1)}\) and \(\psi_1^{(b\pm 1)}\). This leads to the relation (2.11). The other relations can be shown in the similar way using \(\psi_1^{(a)}\).

Now, we have proven the non-reduced quiver Yangian part in theorem 3.1. To complete the proof for (reduced) quiver Yangians, we need to check the Serre relations.

Lemma 3.5. For \( m, n_{1,2}, n_{1,2} \in \mathbb{N} \), we have (2.14) and (2.15) from (S).

**Proof.** Again, we prove this by induction. When \(m, n_{1,2}, n_{1,2}\) are zero, they are given in (S). First, let us consider the relation (2.14) with \(|a| = 0\). For brevity, we shall write it as \(\text{Sym}_{1,1}(m) = 0\). Suppose this holds when \(n_1 + n_2 + m = p = l_1 + l_2 + k\) for any such \(n_{1,2}, m\).

We still consider the commutation relations with \(\psi_1^{(a)}\) and \(\psi_1^{(a\pm 1)}\), and we get the equations

\[
\text{Sym}_{l_1+1,l_2}(k) - 2\text{Sym}_{l_1,l_2+1}(k) + \text{Sym}_{l_1,l_2}(k) = 0,
\]

\[
\text{Sym}_{l_1+1,l_2}(k) + \text{Sym}_{l_1,l_2+1}(k) - 2\text{Sym}_{l_1,l_2}(k) = 0.
\]

Taking \(l_1 = l_2 = l\), we have two variables for the two linearly independent equations with solution

\[
\text{Sym}_{l+1,l}(k) = \text{Sym}_{l+1,l}(k+1) = 0.
\]
Next, taking \( l_1 = l - 1 \) and \( l_2 = l + 1 \), the two linear independent equations again have two variables solved by
\[
\text{Sym}_{l-1,l+2}(k) = \text{Sym}_{l-1,l+1}(k + 1) = 0. \tag{3.29}
\]
Keep this procedure, and we can prove this relation for any \( l_1 + l_2 + k + 1 = p + 1 \).

Now, let us consider the relation (2.15) with \(|a| = 1\). For brevity, we shall write it as \( \text{Sym}_{n_1,n_2}(m_1,m_2) = 0 \). Suppose this holds when \( n_1 + n_2 + m_1 + m_2 = p = l_1 + l_2 + k_1 + k_2 \) for any such \( n_1,n_2,m_1,m_2 \). We still consider the commutation relations with \( \psi_1^{(a)} \) and \( \psi_1^{(a\pm 1)} \), and we get the equations
\[
\begin{align*}
\left( \sigma^{a,a+1}_1 + \sigma^{a+1,a}_1 \right) \text{Sym}_{l_1,l_2}(k_1 + 1,k_2) &+ \left( \sigma^{a,a-1}_1 + \sigma^{a-1,a}_1 \right) \text{Sym}_{l_1,l_2}(k_1,k_2 + 1) = 0, \tag{3.30} \\
\left( \sigma^{a,a+1}_1 + \sigma^{a+1,a}_1 \right) \text{Sym}_{l_1,l_2}(k_1,k_2) &+ 2\sigma^{a+1,a+1}_1 \text{Sym}_{l_1,l_2}(k_1 + 1,k_2) \\
&+ \left( \sigma^{a,a+1}_1 + \sigma^{a+1,a}_1 \right) \text{Sym}_{l_1,l_2+1}(k_1,k_2) = 0, \tag{3.31} \\
\left( \sigma^{a,a-1}_1 + \sigma^{a-1,a}_1 \right) \text{Sym}_{l_1,l_2}(k_1,k_2) &+ 2\sigma^{a-1,a-1}_1 \text{Sym}_{l_1,l_2}(k_1,k_2 + 1) \\
&+ \left( \sigma^{a,a-1}_1 + \sigma^{a-1,a}_1 \right) \text{Sym}_{l_1,l_2+1}(k_1,k_2) = 0, \tag{3.32} \\
\end{align*}
\]
where we have used \( \sigma^{a-1,a+1}_1 = \sigma^{a+1,a-1}_1 = 0 \) since the quiver (with at least one fermionic node) would always have more than three nodes here. In particular, (3.30) can be simplified to
\[
\text{Sym}_{l_1,l_2}(k_1 + 1,k_2) - \text{Sym}_{l_1,l_2}(k_1,k_2 + 1) = 0 \tag{3.33}
\]
since \((a)\) is fermionic. Adding (3.31) and (3.32) together, we have
\[
2\sigma^{a+1,a+1}_1 \text{Sym}_{l_1,l_2}(k_1 + 1,k_2) + 2\sigma^{a-1,a-1}_1 \text{Sym}_{l_1,l_2}(k_1,k_2 + 1) = 0. \tag{3.34}
\]
To get more linearly independent equations, we can also consider the one using \( \text{ad}_{\psi_1^{(a)}} \), which yields
\[
\left( \sigma^{a+1,a+2}_1 + \sigma^{a+2,a+1}_1 \right) \text{Sym}_{l_1,l_2}(k_1 + 1,k_2) + \left( \sigma^{a+2,a-1}_1 + \sigma^{a-1,a+2}_1 \right) \text{Sym}_{l_1,l_2}(k_1,k_2 + 1) = 0. \tag{3.35}
\]
Since \( \sigma^{a+2,a-1}_1 = \sigma^{a-1,a+2}_1 = 0 \) here, this can be simplified to \( \text{Sym}_{l_1,l_2}(k_1 + 1,k_2) = 0 \).
Together with (3.34), we find \( \text{Sym}_{l_1,l_2}(k_1,k_2 + 1) = 0 \). Now, (3.31) becomes
\[
\text{Sym}_{l_1+1,l_2}(k_1,k_2) + \text{Sym}_{l_1,l_2+1}(k_1,k_2) = 0. \tag{3.36}
\]
Apply the same trick as before and take \( l_1 = l_2 = l \), we have \( \text{Sym}_{l,l+1}(k_1,k_2) = 0 \). Next, taking \( l_1 = l - 1, l_2 = l + 1 \), we get \( \text{Sym}_{l-1,l+1}(k_1,k_2) = 0 \). Keep this procedure, and we can find this holds for any \( l_1,l_2,k_1,k_2 \) satisfying \( l_1 + l_2 + k_1 + k_2 + 1 = p + 1 \). This completes the proof for the Serre relations.

Let us now make a comment on the cases in figure 3. We have to exclude them in theorem 3.1 since we do not have enough nodes to get sufficiently many linearly independent equations. For figure 3(a), we have five different modes in one relation, but there are only three nodes in the quiver. For figure 3(b), we have four modes in one relation while considering \( \text{ad}_{\psi_1^{(a\pm 1)}} \) is also not very useful since (3.35) would coincide with (3.34). We leave these cases to future work.
3.2. Another presentation and coproduct

From now on, besides the restrictions $M + N > 2$ and $MN \neq 2$, we will mainly focus on the cases with $M \neq N$ due to the subtleties from the underlying simple Lie superalgebra $\text{psl}(M|M)$ (when $M = N$). Analogous to [24], we can write an algebra homomorphism $\Delta_{V_1, V_2} : Y \rightarrow \text{End}_C(V_1 \otimes V_2)$ for any modules $V_{1, 2}$ in the category $O$. In particular, this can be promoted to a coproduct of the Yangian algebra by considering its completion $\bar{Y}$ following the argument in [24, section 5]. Then any $\Delta_{V_1, V_2}$ can be recovered from $\Delta : Y \rightarrow \bar{Y} \otimes \bar{Y}$, where $\bar{Y} \otimes \bar{Y}$ is the completion of $Y \otimes Y$ we are now going to discuss.

The quiver Yangian has the triangular decomposition $Y \cong Y^{+} \otimes Y^{0} \otimes Y^{-}$, where $Y^{+}$ (resp. $Y^{-}$, resp. $Y^{0}$) is generated by $e_n^{(a)} (f_n^{(a)}$, resp. $\psi^{(a)}_{n})$ for all $a \in Q_0$ and $n \in \mathbb{N}$ [1]. We shall also assume that the positive (resp. negative) part $Y^{+}$ (resp. $Y^{-}$) is isomorphic to the free algebra on $e_n^{(a)}$ (resp. $f_n^{(a)}$) quotiented out by the $ee$ (resp. $ff$) relations. We will denote the subalgebra generated by $e_n^{(a)}$ (resp. $f_n^{(a)}$) and $\psi^{(a)}_{n}$ as $Y^{\geq 0}$ (resp. $Y^{\leq 0}$).

We can set a degree as $deg e_n^{(a)} = 1$ whose grading is compatible with the algebra structure. With respect to this grading, $Y^{+} = \bigoplus_{k=0}^{\infty} Y_{k}^{+}$ with $Y_{k}^{+}$ spanned by monomials of degree $k$ in $Y^{+}$. We also write $Y_{k}^{\geq 0} := \bigoplus_{k \geq 0} Y_{k}^{+}$. Therefore, the quiver Yangian is a graded vector space as $Y = \bigoplus_{k=0}^{\infty} Y_{k}$, where $Y_{k} = Y^{\leq 0} \otimes Y_{k}^{+}$. Now consider the pair $(A_n, q_n)$ for $n \in \mathbb{N}$ with the left $Y$-module $A_n := Y / \left( Y \cdot Y_{\geq n} \right)$ and the natural quotient map $q_n$ from $Y$ to $A_n$. Then $q_{n-1}$ factors through $A_n$, that is, $p_n \circ q_n = q_{n-1}$ with the homomorphism $p_n : A_n \rightarrow A_{n-1}$. The pairs $(A_n, p_n)$ give rise to an inverse system of $Y$-modules, and we can define the completion of the quiver Yangian as the projective limit [59, section 10.1]:

$$\bar{Y} := \lim_{\longrightarrow} A_n.$$  

(3.37)

We will also write $Y \bar{\otimes} Y$ as the completion of $Y \otimes Y$.

To write down the coproduct of the quiver Yangian, we need another presentation of the algebra. Drinfeld’s $J$ presentation is used for finite dimensional cases in [60], but can be appropriately extended to affine cases following the recipe of [24]. In this presentation, the quiver Yangian is generated by $x$ and $J(x)$ for $x$ elements of the underlying Kac–Moody superalgebra $g$. Together with the Chevalley generators of $g$ mapped to the zero modes of $Y$ (recall the beginning of section 3), the isomorphism is given by

$$J\left( \psi^{(a)} \right) = \psi^{(a)}_1 + \psi^{(a)}_0, \quad J\left( e^{(a)}_0 \right) = e^{(a)}_1 + w^{(a)}_+, \quad J\left( f^{(a)}_0 \right) = f^{(a)}_1 + w^{(a)}_-,$$  

(3.38)

where\footnote{Notice that this is a well-defined operator when acting on modules in the category $O$ as $e^{(\alpha \cdot \lambda)}$ (i.e. $x^{(\alpha \cdot \lambda)}_+$) annihilates a vector for $\alpha$ with sufficiently large height.} [24]

$$\psi^{(a)} = \frac{1}{2} (\epsilon_1 + \epsilon_2) \sum_{\alpha \in \Delta_+} (\alpha, \alpha^{(a)}) \sum_{k=1}^{\dim g_0} \epsilon^{(\alpha \cdot \lambda)} e^{(\alpha \cdot \lambda)} = \frac{1}{2} (\epsilon_1 + \epsilon_2) \left( \psi^{(a)}_0 \right)^2.$$  

(3.39)

Then $w^{(a)}_\pm$ can be obtained by requiring

$$J\left( \left[ \psi^{(a)}_0, e^{(a)}_0 \right] \right) = J\left( J\left( \psi^{(a)}_0 \right), e^{(a)}_0 \right), \quad J\left( \left[ \psi^{(a)}_0, f^{(a)}_0 \right] \right) = J\left( J\left( \psi^{(a)}_0 \right), f^{(a)}_0 \right).$$  

(3.40)
In general, a direct computation shows that
\[
\left[J\left(\psi_0^{(a)}, e_0^{(b)}\right) - J\left(\psi_0^{(a)}, e_0^{(b)}\right)\right]
= \frac{c_1 + c_2}{2} \sum_{\alpha \in \Delta_+} \left(\alpha, \alpha \right) \sum_{k=1}^{\dim \mathfrak{g}_\alpha} f^{(\alpha,k)} e^{(\alpha,k)} e_0^{(a)} + \frac{\sigma_1^{ab} - \sigma_1^{ab}}{2} \left(\alpha^{(a)}, \alpha^{(b)}\right) e_0^{(b)} - \left(\alpha^{(a)}, \alpha^{(b)}\right) w_+^{(b)}.
\]  
(3.41)

To simplify this, we need a very useful lemma [61, lemma 18.4.1]:

**Lemma 3.6.** For \( z \in \mathfrak{g}_{\beta - \alpha} \), we have
\[
\sum_k \left[ x_+^{(\beta,k)}, z \right] \otimes x_+^{(\beta,k)} = \sum_k x_+^{(\alpha,k)} \otimes \left[ z, x_+^{(\alpha,k)} \right]
\]  
(3.42)
in \( \mathfrak{g} \otimes \mathfrak{g} \) and
\[
\sum_k \left[ x_+^{(\beta,k)}, z \right] x_+^{(\beta,k)} = \sum_k x_+^{(\alpha,k)} \left[ z, x_+^{(\alpha,k)} \right]
\]  
(3.43)
in \( \mathfrak{U}(\mathfrak{g}) \), where \( x_+^{(\alpha,k)} \) are the bases of the root spaces \( \mathfrak{g}_{\pm \alpha} \).

Then
\[
\sum_{\alpha \in \Delta_+} \left(\alpha, \alpha \right) \sum_{k=1}^{\dim \mathfrak{g}_\alpha} f^{(\alpha,k)} e^{(\alpha,k)} e_0^{(a)}
= \sum_{\alpha \in \Delta_+} \left(\alpha, \alpha \right) \sum_{k=1}^{\dim \mathfrak{g}_\alpha} f^{(\alpha,k)} \left[ e^{(\alpha,k)}, e_0^{(a)} \right] + \left( -1 \right)^{|a||b|} \left(\alpha^{(a)}, \alpha^{(b)}\right) \left[ f^{(b)} e_0^{(a)}, e_0^{(b)} \right] e_0^{(b)}
\]  
(3.44)
is equal to
\[
\sum_{\alpha \in \Delta_+ \setminus \{a^{(a)}\}} \left(\alpha, \alpha \right) \sum_{k=1}^{\dim \mathfrak{g}_\alpha} f^{(\alpha,k)} \left[ e^{(\alpha,k)}, e_0^{(a)} \right]
+ \left( -1 \right)^{|\beta||b|} \sum_{\beta \in \Delta_+ \setminus \{a^{(a)}\}} \left(\beta - \alpha^{(b)}, \alpha^{(a)}\right) \sum_{k=1}^{\dim \mathfrak{g}_\alpha} f^{(\beta,k)} \left[ e_0^{(b)}, e^{(\beta,k)} \right] + \left(\alpha^{(a)}, \alpha^{(b)}\right) \left[ f^{(b)} e_0^{(a)}, e_0^{(b)} \right] e_0^{(b)}.
\]  
(3.45)

Replacing the letter \( \beta \) in the second sum with \( \alpha \) and using the \( e f \) relation in the last term, this is simplified to
\[
\left(\alpha^{(a)}, \alpha^{(b)}\right) \sum_{\alpha \in \Delta_+} \sum_{k=1}^{\dim \mathfrak{g}_\alpha} f^{(\alpha,k)} \left[ e^{(\alpha,k)}, e_0^{(b)} \right] - \left(\alpha^{(a)}, \alpha^{(b)}\right) \psi_0^{(a)} e_0^{(b)}.
\]  
(3.46)

Therefore, we have
\[
\left(\alpha^{(a)}, \alpha^{(b)}\right) w_+^{(b)} = \left(\alpha^{(a)}, \alpha^{(b)}\right) \frac{c_1 + c_2}{2} \sum_{\alpha \in \Delta_+} \sum_{k=1}^{\dim \mathfrak{g}_\alpha} f^{(\alpha,k)} \left[ e^{(\alpha,k)}, e_0^{(b)} \right] - \psi_0^{(a)} e_0^{(b)}
- \frac{\sigma_1^{ab} - \sigma_1^{ab}}{2} \left(\alpha^{(a)}, \alpha^{(b)}\right) e_0^{(b)}.
\]  
(3.47)
Taking $b = a$, the requirement (3.40) on $w_+^{(a)}$ yields

$$w_+^{(a)} = \frac{\epsilon_1 + \epsilon_2}{2} \sum_{\alpha \in \Delta_+} \sum_{k=1}^{\dim g_0} f^{(\alpha, k)} (\epsilon_0^{(a)}, \epsilon_0^{(a)}) - \frac{\epsilon_1 + \epsilon_2}{2} \psi_0^{(a)} \psi_0^{(a)}.$$  \hspace{1cm} (3.48)

Likewise,

$$w_-^{(a)} = -\frac{\epsilon_1 + \epsilon_2}{2} \sum_{\alpha \in \Delta_+} \sum_{k=1}^{\dim g_0} f^{(\alpha, k)} (\epsilon_0^{(a)}, \epsilon_0^{(a)}) e^{(\alpha, k)} - \frac{\epsilon_1 + \epsilon_2}{2} \psi_0^{(a)} \psi_0^{(a)}.$$  \hspace{1cm} (3.49)

Notice that in (3.47), $(\alpha^{(a)}, \alpha^{(b)})$ is zero when taking $b = a$ for fermionic nodes. Nevertheless, we can check that the expressions for $w^{(a)}$ in (3.48) and (3.49) would still satisfy the requirement (3.40).

With the expressions for $v^{(a)}$ and $w^{(a)}$, we can write the commutation relations for the generators in the $J$ presentation. Similar relations can also be found in [24, 25] for similar Yangian algebras. For brevity, we shall also define

$$\overline{v}^{(a)} := v^{(a)} + \frac{\epsilon_1 + \epsilon_2}{2} (\psi_0^{(a)})^2 = \frac{1}{2} (\epsilon_1 + \epsilon_2) \sum_{\alpha \in \Delta_+} (\alpha, \alpha^{(a)}) \sum_{k=1}^{\dim g_0} f^{(\alpha, k)} e^{(\alpha, k)}.$$ \hspace{1cm} (3.50)

**Lemma 3.7.** We have

$$[\overline{v}_0^{(a)}, v^{(b)}] = 0,$$ \hspace{1cm} (3.51)

$$[\overline{v}^{(a)}, e_0^{(b)}] = \left( \alpha^{(a)}, \alpha^{(b)} \right) w^{(b)},$$ \hspace{1cm} (3.52)

$$[\overline{v}^{(a)}, f_0^{(b)}] = -\left( \alpha^{(a)}, \alpha^{(b)} \right) w^{(b)},$$ \hspace{1cm} (3.53)

$$[w_+^{(a)}, v^{(b)}] - \left[ e_0^{(a)}, w^{(b)} \right] = \frac{\epsilon_1 + \epsilon_2}{2} \left( \alpha^{(a)}, \alpha^{(b)} \right) \left[ e_0^{(a)}, e_0^{(b)} \right],$$ \hspace{1cm} (3.54)

$$[w_+^{(a)}, w^{(b)}] - \left[ e_0^{(a)}, w^{(b)} \right] = \frac{\epsilon_1 + \epsilon_2}{2} \left( \alpha^{(a)}, \alpha^{(b)} \right) \left[ f_0^{(a)}, f_0^{(b)} \right].$$ \hspace{1cm} (3.55)

**Proof.** All the proofs follow from direct calculations with the use of lemma 3.6. In particular, the second and third ones are essentially how we found $w^{(b)}$. Here, we would only explicitly write the calculation for the fourth one, which is the most lengthy, while the others should be much quicker.

Let us consider the $e_0 w_-$ relation, and the $w_+ f_0$ relation would follow from the same argument. For convenience, we write

$$-\frac{2}{\epsilon_1 + \epsilon_2} \left[ e_0^{(a)}, w^{(b)} \right] = \sum_{\alpha \in \Delta_+} \sum_{k=1}^{\dim g_0} \left( e_0^{(a)} f_0^{(b)} f^{(\alpha, k)} e^{(\alpha, k)} - (-1)^{[\alpha][b]} e_0^{(a)} f^{(\alpha, k)} f^{(\alpha, k)} e^{(\alpha, k)} - (-1)^{[\alpha][b]} f_0^{(b)} f^{(\alpha, k)} e^{(\alpha, k)} \right)$$

$$+ (-1)^{[\alpha][b]} (-1)^{[\alpha][b]} f^{(\alpha, k)} f^{(\alpha, k)} e^{(\alpha, k)} + \left[ e_0^{(a)} f_0^{(b)} \psi_0^{(b)} \right].$$ \hspace{1cm} (3.57)
where we have expanded all the brackets (except the last term) for clarity. By adding the terms
\[
\mp (-1)^{[\alpha \beta]} f_0^{(\alpha \beta)} e_0^{(\alpha \beta)} f^{(\alpha \beta)} e^{(\alpha \beta)}, \quad \mp (-1)^{[\alpha \beta]} f_0^{(\alpha \beta)} e_0^{(\alpha \beta)} f^{(\alpha \beta)} e^{(\alpha \beta)}
\]
inside the summations, we can group the terms into
\[
\sum_{\alpha \in \Delta_\alpha} \sum_{k=1}^{\dim \mathfrak{g}_\alpha} \left( \left[ \delta_{ab} f_0^{(\alpha \beta)} f^{(\alpha \beta)} e^{(\alpha \beta)} \right] + (-1)^{[\alpha \beta]} \left[ f_0^{(\alpha \beta)} e_0^{(\alpha \beta)} f^{(\alpha \beta)} e^{(\alpha \beta)} \right] \right)
\]
(3.58)

Picking out those with $\alpha = \alpha^{(a)}$ for the second and third terms in the summations, we have
\[
\sum_{\alpha \in \Delta_\alpha \setminus \{\alpha^{(a)}\}} \sum_{k=1}^{\dim \mathfrak{g}_\alpha} \left( (-1)^{[\alpha \beta]} \left[ f_0^{(\alpha \beta)} e_0^{(\alpha \beta)} f^{(\alpha \beta)} e^{(\alpha \beta)} \right] \right) + (-1)^{[\alpha \beta]} \left[ f_0^{(\alpha \beta)} e_0^{(\alpha \beta)} f^{(\alpha \beta)} e^{(\alpha \beta)} \right]
\]
(3.59)

\[
- \sum_{\alpha \in \Delta_\alpha} \sum_{k=1}^{\dim \mathfrak{g}_\alpha} \left( \delta_{ab} \left( \alpha^{(a)}, \alpha \right) f^{(\alpha \beta)} e^{(\alpha \beta)} \right) + (-1)^{[\alpha \beta]} \left[ f_0^{(\alpha \beta)} e_0^{(\alpha \beta)} f^{(\alpha \beta)} e^{(\alpha \beta)} \right] + \left[ f_0^{(\alpha \beta)} e_0^{(\alpha \beta)} f^{(\alpha \beta)} e^{(\alpha \beta)} \right].
\]
(3.60)

The first line vanishes using lemma 3.6 while the second line is simply $\frac{2}{\epsilon_1 + \epsilon_2} \delta_{ab} \nu^{(a)}$. This completes the proof.

From these relations, it is straightforward to get the following corollary by definitions of $J(\psi_0^{(a)}), J(e_0^{(a)})$ and $J(f_0^{(a)})$.

**Corollary 3.7.** We have
\[
J(\psi_0^{(a)}), J(e_0^{(a)}), J(f_0^{(a)})
\]
(3.61)

\[
J(e_0^{(a)}), e_0^{(b)} = (\alpha^{(a)}, \alpha^{(b)}) J(e_0^{(b)}) + \frac{\sigma_1^{ba} - \sigma_1^{ab}}{2} (\alpha^{(a)}, \alpha^{(b)}) e_0^{(b)},
\]
(3.62)

\[
J(e_0^{(a)}), f_0^{(b)} = (\alpha^{(a)}, \alpha^{(b)}) J(f_0^{(b)}) - \frac{\sigma_1^{ba} - \sigma_1^{ab}}{2} (\alpha^{(a)}, \alpha^{(b)}) f_0^{(b)},
\]
(3.63)

\[
J(f_0^{(a)}), f_0^{(b)} = \delta_{ab} J(\psi_0^{(a)})
\]
(3.64)

\[
J(e_0^{(a)}), e_0^{(b)} = \frac{1}{2} (\sigma_1^{ba} - \sigma_1^{ab}) e_0^{(a)} e_0^{(b)},
\]
(3.65)

\[
J(f_0^{(a)}), f_0^{(b)} = \frac{1}{2} (\sigma_1^{ba} - \sigma_1^{ab}) f_0^{(a)} f_0^{(b)},
\]
(3.66)

\[
J(e_0^{(a)}), e_0^{(b)} = J(f_0^{(a)}), f_0^{(b)} = 0 (\sigma_1 = 0).
\]
(3.67)

Notice that the last line follows from
\[
J(e_0^{(a)}), e_0^{(a)} = \frac{1}{(\alpha^{(c)}, \alpha^{(a)})} J(\psi_0^{(c)}), e_0^{(a)} - \frac{\sigma_1^{ac} - \sigma_1^{a0}}{2} e_0^{(a)}
\]
(3.68)

and likewise for $f_0^{(a)}$. When $a$ is bosonic, we can take $c = a$. When $a$ is fermionic, $c$ can be taken as one of $a \pm 1$ such that $\sigma_1^{ac} = 0$. It is straightforward to see that each relation in this corollary is equivalent to one of the relations (involving non-zero modes) in (R) in theorem 3.1.
It is also possible to write $J$ acting on any positive real roots besides the simple ones with the help of the Weyl group of the untwisted affine A-type superalgebra. Since $\dim_{\mathbb{R}} = 1$ for $\alpha \in \Delta_\text{aff}^+$, we shall omit the label $k$ in the corresponding elements. Due to the Serre relations, given an even simple root $\alpha^{(b)}$, the operator $\tau^{(b)} := \exp(\text{ad}_{\alpha^{(b)}}) \exp(-\text{ad}_{\bar{\alpha}^{(b)}}) \exp(\text{ad}_{\bar{\alpha}^{(b)}})$ is well-defined and is an automorphism of the quiver Yangian (see for example [28, 62]). Following the same argument as in [24, lemma 3.17], $\tau^{(b)}$ can be applied to $J(\psi_0^{(a)})$, $J(e_0^{(a)})$ and $J(f_0^{(a)})$ for any simple root $\alpha^{(a)}$. Moreover, we find that

$$\tau^{(b)} \left( J(\psi_0^{(a)}) \right) = J(\psi_0^{(a)}) - \frac{2(\alpha^{(b)} - \alpha^{(a)})}{(\alpha^{(b)}, \alpha^{(b)})} J(\psi_0^{(a)}) - \left( \frac{\alpha^{(b)} - \alpha^{(a)}}{\alpha^{(b)}, \alpha^{(b)}} \right) \psi_0^{(a)} \ . \ (3.69)$$

Suppose a root $\alpha$ can be obtained from a simple root $\alpha^{(a)}$ under the even reflections $s^{(b)}$ via $\alpha = s^{(b_1)} \ldots s^{(b_n)} (\alpha^{(a)})$. Then we may write $e^{(\alpha)} = \tau^{(b_1)} \ldots \tau^{(b_n)} (e^{(a)})$ and define $J(e^{(\alpha)}) := \tau^{(b_1)} \ldots \tau^{(b_n)} (J(e^{(a)}))$ (and likewise for $f$).

**Proposition 3.8.** For any positive real root $\alpha$ and $a \in Q_0$, we have

$$[J(\psi_0^{(a)}), e^{(a)}] = \left[ J(\psi_0^{(a)}), J(e^{(a)}) \right] + c^{\alpha \alpha} e^{(a)} = \left( \alpha^{(a)}, \alpha \right) J(e^{(a)}) + c^{\alpha \alpha} e^{(a)} ,$$

$$[J(\psi_0^{(a)}), f^{(a)}] = \left[ J(\psi_0^{(a)}), J(f^{(a)}) \right] - c^{\alpha \alpha} e^{(a)} = -\left( \alpha^{(a)}, \alpha \right) J(f^{(a)}) - c^{\alpha \alpha} f^{(a)} \ , \ (3.70)$$

where $c^{\alpha \alpha} \in \frac{\alpha - \alpha}{2} \mathbb{Z}$.

**Proof.** We will only show the first relation with $e^{(a)}$ as the proof of the second one with $f^{(a)}$ follows in the same way. We prove this by induction. When $p = 0$, this is automatically true since $\alpha$ is a simple root. Now suppose this holds for $\beta = s^{(b_2)} \ldots s^{(b_n)} (\alpha^{(a)})$. Then for $\alpha = s^{(b_1)} (\beta)$,

$$[J(\psi_0^{(a)}), e^{(a)}] = \tau^{(b_1)} \left[ \left( \tau^{(b_1)} \right)^{-1} \left[ J(\psi_0^{(a)}), e^{(b_1)} \right] \right]$$

$$= \tau^{(b_1)} \left[ J(\psi_0^{(a)}) - 2 \frac{\alpha^{(b_1)} - \alpha^{(a)}}{(\alpha^{(b_1)}, \alpha^{(b_1)})} J(\psi_0^{(b_1)}) - \frac{\alpha^{(b_1)} - \alpha^{(a)}}{\left( \alpha^{(b_1)}, \alpha^{(b_1)} \right)} \psi_0^{(b_1)}, e^{(\beta)} \right] . \ (3.71)$$

With the assumption of the proposition to hold for $\beta$, we have

$$[J(\psi_0^{(a)}), e^{(a)}] = \left( \alpha^{(a)}, \alpha \right) J(e^{(a)}) + c^{\alpha \alpha} e^{(a)} , \ (3.72)$$

where

$$c^{\alpha \alpha} = \frac{1}{2} - 2 \frac{(\alpha^{(b_1)}, \alpha^{(a)})}{(\alpha^{(b_1)}, \alpha^{(b_1)})} c^{\alpha \beta} + \left( \sigma^{ab} - \sigma^{ba} \right) \frac{(\alpha^{(b_1)}, \alpha^{(a)})}{(\alpha^{(b_1)}, \alpha^{(b_1)})} \left( \alpha^{(b_1)}, \beta \right) . \ (3.73)$$

Write $c^{\alpha \beta} = \frac{c_{1 - 2c}}{2} c'$, and hence $c' \in \mathbb{Z}$. Then

$$c^{\alpha \alpha} = \frac{c_1 - c_2}{2} \left( \left[ 1 - 2 \frac{(\alpha^{(b_1)}, \alpha^{(a)})}{(\alpha^{(b_1)}, \alpha^{(b_1)})} \right] c' + 2 \frac{(\alpha^{(b_1)}, \alpha^{(a)})}{(\alpha^{(b_1)}, \alpha^{(b_1)})} \left( \alpha^{(b_1)}, \beta \right) \right) . \ (3.74)$$
In particular, the term in the biggest bracket is an integer. The other equality involving the commutator of $\psi_0^{(a)}$ and $J(e^{(a)})$ in this relation can be proven in a similar way.

As a result, $J(e^{(a)})$ and $J(f^{(a)})$ are independent of the choice of the sequence of $\tau^{(b)}$ up to a constant multiple. From this proposition, it is also straightforward to obtain the following corollary.

**Corollary 3.8.1.** For any positive real root $\alpha$ and $a \in Q_0$, we have

$$
\left(\alpha^{(b)},\alpha\right) \left[ J\left(\psi_0^{(a)}\right), e^{(a)} \right] - \left(\alpha^{(a)},\alpha\right) \left[ J\left(\psi_0^{(b)}\right), e^{(a)} \right] = c_{\alpha}^b e^{(a)},
$$

where $c_{\alpha}^b = (\alpha^{(b)},\alpha) e^{\alpha} - (\alpha^{(a)},\alpha) e^{\alpha a}$.

Incidentally, following the similar proof as in [25], we also have

$$
\left[ J\left(\psi_0^{(a)}\right), \tilde{\gamma}^{(b)} \right] + \left[ J\left(\psi_0^{(b)}\right), \tilde{\gamma}^{(a)} \right] = \left[ J\left(\psi_0^{(a)}\right), J\left(\psi_0^{(b)}\right) \right] + \left[ \tilde{\gamma}^{(a)}, \tilde{\gamma}^{(b)} \right] = 0. \tag{3.76}
$$

Now, we are prepared to write our coproduct of the quiver Yangians. Recall that in general, $(x \otimes y)(z \otimes w) = (-1)^{|y||z|}[xz] \otimes [yw]$. For brevity, let us write a linear operator $\Box(x) := x \otimes 1 + 1 \otimes x$ and define a Casimir element

$$
\Omega_\pm := \sum_{\alpha \in \Delta_{\pm}} \sum_{k=1}^{\dim \mathfrak{g}_\alpha} f^{(\alpha,k)} \otimes e^{(\alpha,k)}. \tag{3.77}
$$

It is straightforward to get the following commutation relations:

**Lemma 3.9.** We have

$$
\left[ \Box\left(\psi_0^{(a)}\right), \Omega_- \right] = 0, \quad \left[ \Box\left(e^{(a)}\right), \Omega_- \right] = \psi_0^{(a)} \otimes e_0^{(a)}, \quad \left[ \Box\left(f_0^{(a)}\right), \Omega_- \right] = -f_0^{(a)} \otimes \psi_0^{(a)}. \tag{3.78}
$$

**Proof.** The relations directly follow the definitions of the operators. Here, we will only explicitly show the second one as an illustration. This can be seen by

$$
\left[ e^{(a)} \otimes 1, \Omega_- \right] = \sum_{\alpha \in \Delta_{+}} \sum_{k=1}^{\dim \mathfrak{g}_\alpha} f^{(\alpha,k)} \otimes e^{(\alpha,k)}
$$

$$
= \psi_0^{(a)} \otimes e_0^{(a)} + \sum_{\alpha \in \Delta_{+} \setminus \{\epsilon^{(a)}\}} \sum_{k=1}^{\dim \mathfrak{g}_{\alpha-\epsilon^{(a)}}} f^{(\alpha-\epsilon^{(a)},k)} \otimes \left[ e^{(\alpha-\epsilon^{(a)},k)}, e_0^{(a)} \right]
$$

$$
= \psi_0^{(a)} \otimes e_0^{(a)} - \left[ 1 \otimes e_0^{(a)}, \Omega_- \right], \tag{3.79}
$$

where we have used lemma 3.6 in the second equality.

Let us also introduce another map $\Delta$ defined by

$$
\Delta\left(\psi_0^{(a)}\right) = \Box\left(\psi_0^{(a)}\right), \quad \Delta\left(e_0^{(a)}\right) = \Box\left(e_0^{(a)}\right), \quad \Delta\left(f_0^{(a)}\right) = \Box\left(f_0^{(a)}\right),
$$

$$
\Delta\left(\psi_1^{(a)}\right) = \Box\left(\psi_1^{(a)}\right) + (\epsilon_1 + \epsilon_2) \psi_0^{(a)} \otimes \psi_0^{(a)} + (\epsilon_1 + \epsilon_2) \left[ \psi_0^{(a)} \otimes 1, \Omega_- \right]
$$

$$
= \Box\left(\psi_1^{(a)}\right) + (\epsilon_1 + \epsilon_2) \psi_0^{(a)} \otimes \psi_0^{(a)} - (\epsilon_1 + \epsilon_2) \sum_{\alpha \in \Delta_{+}^{\epsilon}} \left(\alpha^{(a)},\alpha\right) f^{(\alpha)} \otimes e^{(\alpha)}. \tag{3.80}
$$
Notice that this uniquely determines $\Delta$ as the actions on all modes can be obtained following the discussions in section 3.1. For instance,

**Proposition 3.10.** We have

\[
\Delta \left( \tilde{\psi}^{(a)}_1 \right) = \square \left( \tilde{\psi}^{(a)}_1 \right) + (\epsilon_1 + \epsilon_2) \left[ \psi^{(a)}_0 \otimes 1, \Omega_+ \right],
\]

(3.81)

\[
\Delta \left( e^{(a)}_1 \right) = \square \left( e^{(a)}_1 \right) - (\epsilon_1 + \epsilon_2) \left[ \Omega_-, e^{(a)}_0 \otimes 1 \right],
\]

(3.82)

\[
\Delta \left( f^{(a)}_1 \right) = \square \left( f^{(a)}_1 \right) + (\epsilon_1 + \epsilon_2) \left[ \Omega_-, 1 \otimes f^{(a)}_0 \right].
\]

(3.83)

**Proof.** The first one follows from a straightforward calculation using the linearity of $\Delta$ and (3.80). The second and third ones can be shown in the same way, and we will only explicitly verify the one for $e^{(a)}_1$ here. When $|a| = 0$, by (3.1) (with $b = a$), we have

\[
\Delta \left( e^{(a)}_1 \right) = \frac{1}{(\alpha^{(a)}), \alpha^{(a)}} \Delta \left( \left[ \psi_0^{(a)} \otimes 1, \Omega_+ \right] \right) = \square \left( e^{(a)}_1 \right) + \frac{\epsilon_1 + \epsilon_2}{(\alpha^{(a)}), \alpha^{(a)}} \left[ \psi_0^{(a)} \otimes 1, \Omega_+ \right],
\]

(3.84)

where we have applied the result of $\Delta \left( \tilde{\psi}^{(a)}_0 \right)$ in the second equality. Using the Jacobi identity $[x, [y, z]] = [[x, y], z] + (-1)^{|x||y|} [y, [x, z]]$, this is equal to

\[
\square \left( e^{(a)}_1 \right) - \frac{\epsilon_1 + \epsilon_2}{(\alpha^{(a)}), \alpha^{(a)}} \left( \left[ \psi_0^{(a)} \otimes 1, \left[ \Omega_-, \square \left( e^{(a)}_1 \right) \right] \right] - \left[ \Omega_-, \left[ \psi_0^{(a)} \otimes 1, \square \left( e^{(a)}_1 \right) \right] \right] \right)
\]

\[
= \square \left( e^{(a)}_1 \right) - \frac{\epsilon_1 + \epsilon_2}{(\alpha^{(a)}), \alpha^{(a)}} \left( \left[ \psi_0^{(a)} \otimes 1, -\psi_0^{(a)} \otimes e^{(a)}_0 \right] - \left[ \Omega_-, \left[ \psi_0^{(a)} \otimes 1, e^{(a)}_0 \otimes 1 \right] \right] \right)
\]

\[
= \square \left( e^{(a)}_1 \right) - (\epsilon_1 + \epsilon_2) \left[ \Omega_-, e^{(a)}_0 \otimes 1 \right].
\]

(3.85)

When $|a| = 1$, we take $b = a + 1$ in (3.1), and a similar calculation leads to the same result. \(\square\)

This operator in fact gives a coproduct of the quiver Yangian.

**Theorem 3.11.** For $M + N > 2$, $MN \neq 2$ and $M \neq N$, the map $\Delta : Y \rightarrow Y \otimes Y$ specified by (3.80) is a coassociative algebra homomorphism.

**Proof.** Let us first show that $\Delta$ is compatible with the relations (R) and (S) in theorem 3.1. This is immediate for those involving only the zero modes $\psi_0^{(a)}$, $e_0^{(a)}$ and $f_0^{(a)}$. For the $\psi_0 e_1$ relation (3.4) and the $\psi_0 f_1$ relation (3.6) (with $r = 1$), they can be checked following straightforward computations using proposition 3.10. For the $e_1 f_0$ relation, we have

\[
\left[ \Delta \left( e^{(a)}_1 \right), \Delta \left( f^{(a)}_0 \right) \right] = \delta_{ab} \square \left( \psi^{(a)}_1 \right) - (\epsilon_1 + \epsilon_2) \left[ \left[ \psi_0^{(a)} \otimes 1, \Omega_+ \right], \square \left( f^{(a)}_0 \right) \right],
\]

(3.86)

where we have used proposition 3.10 and (3.3). As before, using the Jacobi identity, this becomes

\[
\delta_{ab} \square \left( \psi^{(a)}_1 \right) + (\epsilon_1 + \epsilon_2) \left( \delta_{ab} \left[ \psi_0^{(a)} \otimes 1, f^{(b)} \otimes \psi_0^{(b)} \right] + \delta_{ab} \left[ \psi^{(a)} \otimes 1, \Omega_+ \right] \right)
\]

\[
= \delta_{ab} \square \left( \psi^{(a)}_1 \right) + (\epsilon_1 + \epsilon_2) \left( \delta_{ab} \psi_0^{(a)} \otimes \psi_0^{(b)} + \delta_{ab} \left[ \psi^{(a)} \otimes 1, \Omega_+ \right] \right)
\]

\[
= \delta_{ab} \Delta \left( \psi^{(a)}_1 \right). \]

(3.87)
The $e_0 f_1$ relation can be verified in the same way. Next, let us verify the $\psi_1 e_0$ relation, which will be similar for $\psi_1 f_0$. For convenience, let us consider the equivalent (3.13) with $\bar{\psi}_1^{\alpha(b)}$. Using proposition 3.10 and the Jacobi identity, we find

\[
\left[ \Delta \left( \bar{\psi}_1^{\alpha(a)} \right), \Delta \left( e_0^{(b)} \right) \right] = \left( \alpha^{(a)}, \alpha^{(b)} \right) \square \left( e_0^{(b)} \right) + \frac{\sigma_1^{ab} - \sigma_2^{ab}}{2} \left( \alpha^{(a)}, \alpha^{(b)} \right) \square \left( e_0^{(b)} \right)
\]

\[+ (\epsilon_1 + \epsilon_2) \left[ \left[ \psi_0^{(a)} \otimes 1, [\Omega_\alpha, \square \left( e_0^{(b)} \right)] \right] - \left[ \Omega_, \left[ \psi_0^{(a)} \otimes 1, \square \left( e_0^{(b)} \right) \right] \right] \right)
\]

\[= \left( \alpha^{(a)}, \alpha^{(b)} \right) \square \left( e_0^{(b)} \right) + \frac{\sigma_1^{ab} - \sigma_2^{ab}}{2} \left( \alpha^{(a)}, \alpha^{(b)} \right) \square \left( e_0^{(b)} \right) - (\epsilon_1 + \epsilon_2) \left( \alpha^{(a)}, \alpha^{(b)} \right) \left[ \Omega_\alpha, e^{(b)} \otimes 1 \right]
\]

\[= \left( \alpha^{(a)}, \alpha^{(b)} \right) \Delta \left( e_0^{(b)} \right) + \frac{\sigma_1^{ab} - \sigma_2^{ab}}{2} \left( \alpha^{(a)}, \alpha^{(b)} \right) \square \left( e_0^{(b)} \right).
\] (3.88)

The $ee$ and $ff$ relations can likewise be verified using the same method.

Now the only remaining relation yet to be checked is the $\bar{\psi}_1^{\alpha(a)}$ relation. For the $\psi_1^{\alpha(a)}$ case, the compatibility follows from the fact that $\square \left( \psi_0^{(a)} \right)$ commutes with $\Omega_\alpha$. For the $\psi_1^{\alpha(a)}$ case, it would again be more convenient to work with $\bar{\psi}_1^{\alpha(a)}$. Since

\[
\Delta \left( \bar{\psi}_1^{\alpha(a)} \right) = \square \left( \bar{\psi}_1^{\alpha(a)} \right) + (\epsilon_1 + \epsilon_2) \left[ \psi_0^{(a)} \otimes 1, \Omega_\alpha \right] \quad \text{and} \quad \square \left( \left[ \bar{\psi}_1^{\alpha(a)}, \bar{\psi}_1^{\alpha(b)} \right] \right) = 0,
\] (3.89)

it is equivalent to showing that

\[
\left[ \square \left( \bar{\psi}_1^{\alpha(a)} \right), \left[ \psi_0^{(a)} \otimes 1, \Omega_\alpha \right] \right] + \left[ \left[ \psi_0^{(a)} \otimes 1, \Omega_\alpha \right], \square \left( \bar{\psi}_1^{\alpha(b)} \right) \right]
\]

\[+ (\epsilon_1 + \epsilon_2) \left[ \left[ \psi_0^{(a)} \otimes 1, \Omega_\alpha \right], \left[ \psi_0^{(a)} \otimes 1, \Omega_\alpha \right] \right] \]

vanishes. In particular, after expanding the brackets and rearranging the terms, the second line in (3.90) becomes

\[
\frac{1}{2} (\epsilon_1 + \epsilon_2) \sum_{k > 0} \sum_{\alpha, \beta \in \Delta^+} \left( \alpha^{(a)}, \alpha \right) \left( \alpha^{(b)}, \beta \right) (-1)^{[\alpha][\beta]} \left( \left\{ f^{(\alpha)}, f^{(\beta)} \right\} \otimes \left\{ e^{(\alpha)}, e^{(\beta)} \right\} \right)
\]

\[+ \left( \left\{ f^{(\alpha)}, f^{(\beta)} \right\} \otimes \left\{ e^{(\alpha)}, e^{(\beta)} \right\} \right),
\] (3.91)

where $h_t$ denotes the height of a root, and we have defined $\{x, y\} = xy + (-1)^{|x||y|}yx$ for brevity. For the first line in (3.90), it equals

\[
\sum_{k > 0} \sum_{\alpha, \beta \in \Delta^+} \left( \alpha^{(a)}, \alpha \right) \left[ \square \left( \bar{\psi}_1^{\alpha(b)} \right) \otimes e^{(\alpha)} \right] - \sum_{k > 0} \sum_{\alpha, \beta \in \Delta^+} \left( \alpha^{(b)}, \alpha \right) \left[ \square \left( \bar{\psi}_1^{\alpha(a)} \right) \otimes e^{(\alpha)} \right].
\] (3.92)
Using proposition 3.8, this becomes

\[
\sum_{k>0} \sum_{\alpha \in \Delta_r^+} \left( \left( \alpha^{(a)} , \alpha \right) \tilde{v}_1^{(b)} - \left( \alpha^{(b)} , \alpha \right) \tilde{v}_1^{(a)} , f^{(a)} \right) \otimes e^{(a)} \\
+ \left[ \left( \alpha^{(a)} , \alpha \right) \tilde{v}_1^{(b)} - \left( \alpha^{(b)} , \alpha \right) \tilde{v}_1^{(a)} , e^{(a)} \right] \otimes f^{(a)} \right). \tag{3.93}
\]

Now, we claim that for each \( k \), we have

\[
\frac{1}{2} \left( \epsilon_1 + \epsilon_2 \right) \sum_{\alpha^{(a)} = k} \sum_{\alpha^{(b)} = k} \left( \alpha^{(a)} , \alpha \right) \left( \alpha^{(b)} , \beta \right) \left( -1 \right)^{\left| \alpha \right| \left| \beta \right|} \left( e^{(a)} , e^{(b)} \right) \otimes \left( f^{(a)} , f^{(b)} \right) \right)
\]

\[
= \sum_{\alpha^{(a)} = k} \sum_{\alpha^{(b)} = k} \left( \left( \alpha^{(a)} , \alpha \right) \tilde{v}_1^{(a)} - \left( \alpha^{(a)} , \alpha \right) \tilde{v}_1^{(b)} , e^{(a)} \right) \otimes f^{(a)} \right) \tag{3.94}
\]

and

\[
\frac{1}{2} \left( \epsilon_1 + \epsilon_2 \right) \sum_{\alpha^{(a)} = k} \sum_{\alpha^{(b)} = k} \left( \alpha^{(a)} , \alpha \right) \left( \alpha^{(b)} , \beta \right) \left( -1 \right)^{\left| \alpha \right| \left| \beta \right|} \left( e^{(a)} , e^{(b)} \right) \otimes \left( f^{(a)} , f^{(b)} \right) \right)
\]

\[
= \sum_{\alpha^{(a)} = k} \sum_{\alpha^{(b)} = k} \left( \left( \alpha^{(a)} , \alpha \right) \tilde{v}_1^{(a)} - \left( \alpha^{(a)} , \alpha \right) \tilde{v}_1^{(b)} , e^{(a)} \right) \otimes f^{(a)} \right) \tag{3.95}
\]

These two equalities would then imply (3.90) being zero. Here, we will only write the proof for (3.94) explicitly while (3.95) can be shown in a similar way.

We first notice that by definition of \( \tilde{v}^{(a)} \),

\[
\left[ \left( \alpha^{(b)} , \alpha \right) \tilde{v}_1^{(a)} - \left( \alpha^{(a)} , \alpha \right) \tilde{v}_1^{(b)} , f^{(a)} \right] = \\
\frac{1}{2} \left( \epsilon_1 + \epsilon_2 \right) \sum_{\beta \in \Delta_r^+} \left( \left( \alpha^{(a)} , \beta \right) \left( \alpha^{(b)} , \alpha \right) - \left( \alpha^{(b)} , \beta \right) \left( \alpha^{(a)} , \alpha \right) \right)
\]

\[
\times \left( f^{(a)} , f^{(b)} \right) e^{(a)} + f^{(b)} \left( e^{(a)} , f^{(a)} \right) \right). \tag{3.96}
\]

When \( \gamma := \beta - \alpha \in \Delta_r^+ \), we have

\[
f^{(b)} \left( e^{(b)} , f^{(a)} \right) = \left( f^{(a)} , e^{(b)} , f^{(a)} \right) f^{(b)} e^{(a)} \right) \tag{3.97}
\]

When \( \gamma := \alpha - \beta \in \Delta_r^+ \), we have

\[
f^{(b)} \left( e^{(b)} , f^{(a)} \right) = \left( f^{(a)} , e^{(b)} , f^{(a)} \right) f^{(b)} e^{(a)} = \left( f^{(a)} , e^{(b)} , f^{(a)} \right) f^{(b)} f^{(a)} \right) \tag{3.98}
\]
where we have used the invariance property \([x, y, z] = (x, [y, z])\) in the second equality. Therefore,

\[
\sum_{\beta \in \mathbb{A}_g^+} \left( (\alpha^{(a)}, \beta) (\alpha^{(b)}, \alpha) - (\alpha^{(b)}, \beta) \right) (\alpha^{(a)}, \alpha) \right) f^{(\beta)} \left[ e^{(\alpha)}, f^{(\alpha)} \right] \\
= \sum_{\beta, \gamma \in \mathbb{A}_g^+} \left( \left( \alpha^{(a), \delta} (\alpha^{(b)}, \beta - \gamma) - (\alpha^{(b)}, \beta) \right) \right) (\alpha^{(a)}, \beta - \gamma) \right) f^{(\gamma)} \left[ e^{(\beta)}, f^{(\alpha)} \right] f^{(\beta)} e^{(\gamma)} \\
- \sum_{\beta, \gamma \in \mathbb{A}_g^+} \left( \left( \alpha^{(a)}, \beta \right) (\alpha^{(b)}, \beta + \gamma) - \left( \alpha^{(a)}, \beta \right) \right) (\alpha^{(a)}, \beta + \gamma) \right) (-1)^{\beta|\gamma|} \\
\times (\left[ e^{(\beta)}, e^{(\gamma)} \right] f^{(\alpha)}) f^{(\beta)} e^{(\gamma)} \\
= \sum_{\beta, \gamma \in \mathbb{A}_g^+} \left( \left( \alpha^{(a), \gamma} (\alpha^{(b)}, \beta) - (\alpha^{(b)}, \beta) \right) \right) (\alpha^{(a), \gamma}) \right) (\alpha^{(a)}, \beta) \right) f^{(\gamma)} \left[ e^{(\beta)}, f^{(\alpha)} \right] f^{(\beta)} e^{(\gamma)} \\
- \sum_{\beta, \gamma \in \mathbb{A}_g^+} \left( \left( \alpha^{(a), \gamma} (\alpha^{(b)}, \beta) - (\alpha^{(b)}, \beta) \right) \right) (\alpha^{(a), \gamma}) \right) (-1)^{\beta|\gamma|} \left[ e^{(\beta)}, e^{(\gamma)} \right] f^{(\alpha)} f^{(\beta)} e^{(\gamma)},
\]

(3.99)

where the terms involving imaginary roots have vanishing coefficients. Likewise, the first part in the summation of (3.96) becomes

\[
- \sum_{\beta, \gamma \in \mathbb{A}_g^+} \left( \left( \alpha^{(a), \gamma} (\alpha^{(b)}, \beta) - (\alpha^{(b)}, \beta) \right) \right) (\alpha^{(a), \gamma}) \right) f^{(\gamma)} \left[ e^{(\beta)}, f^{(\alpha)} \right] f^{(\beta)} e^{(\gamma)}.
\]

(3.100)

Thus,

\[
\left[ (\alpha^{(b)}, \alpha) \gamma^{(a)} - (\alpha^{(a)}, \alpha) \gamma^{(b)} f^{(\alpha)} \right] e^{(a)} \\
= -\frac{1}{2} (\epsilon_1 + \epsilon_2) \sum_{\beta, \gamma \in \mathbb{A}_g^+} \left( \left( \alpha^{(a), \delta} (\alpha^{(b)}, \beta) - (\alpha^{(b)}, \beta) \right) \right) (\alpha^{(a)}, \beta) \right) (-1)^{\beta|\gamma|} \left[ e^{(\beta)}, e^{(\gamma)} \right] f^{(\alpha)} f^{(\gamma)} \gamma^{(\alpha)} e^{(\alpha)} \\
= -\frac{1}{2} (\epsilon_1 + \epsilon_2) \sum_{\beta, \gamma \in \mathbb{A}_g^+} \left( \left( \alpha^{(a), \delta} (\alpha^{(b)}, \beta) - (\alpha^{(b)}, \beta) \right) \right) (\alpha^{(a), \gamma}) \right) (-1)^{\beta|\gamma|} \left[ e^{(\beta)}, e^{(\gamma)} \right] f^{(\alpha)} f^{(\gamma)} \gamma^{(\alpha)} e^{(\alpha)}.
\]

(3.101)

This is also equal to

\[
-\frac{1}{2} (\epsilon_1 + \epsilon_2) \sum_{\beta, \gamma \in \mathbb{A}_g^+} \left( \left( \alpha^{(a), \delta} (\alpha^{(b)}, \beta) - (\alpha^{(b)}, \beta) \right) \right) (\alpha^{(a), \gamma}) \right) (-1)^{\beta|\gamma|} (-1)^{\beta|\gamma|} f^{(\gamma)} f^{(\gamma)} \gamma^{(\alpha)} e^{(\alpha)}.
\]

(3.102)
Adding (3.101) and (3.102) together and then dividing it by 2, we get

\[
\sum_{\alpha \in \Delta_{n+}^0} \left[ (\alpha^{(b)}, \alpha) \psi_1^{(a)} - (\alpha^{(a)}, \alpha) \psi_1^{(b)} \right] \otimes e^{(a)}
\]

\[
= \frac{1}{2} (\epsilon_1 + \epsilon_2) \sum_{\beta, \gamma \in \Delta_{n+}^0} \frac{1}{2} \left( (\alpha^{(a)}, \alpha) (\alpha^{(b)}, \beta) (\alpha^{(b)}, \gamma) - (\alpha^{(b)}, \beta) (\alpha^{(a)}, \gamma) \right) (-1)^{|\beta||\gamma|} \left\{ f^{(\beta)}, f^{(\gamma)} \right\} \otimes \left[ e^{(\gamma)}, e^{(\beta)} \right].
\]

(3.103)

To the first line in (3.94), we also add itself with \( \alpha \) and \( \beta \) exchanged and then divide the result by 2. This leads to

\[
\frac{1}{2} (\epsilon_1 + \epsilon_2) \sum_{\beta, \gamma \in \Delta_{n+}^0} \frac{1}{2} \left( (\alpha^{(a)}, \alpha) (\alpha^{(b)}, \beta) (\alpha^{(b)}, \gamma) - (\alpha^{(b)}, \beta) (\alpha^{(a)}, \gamma) \right) (-1)^{|\beta||\gamma|} \left\{ f^{(\beta)}, f^{(\gamma)} \right\} \otimes \left[ e^{(\gamma)}, e^{(\beta)} \right].
\]

(3.104)

where the terms with \( \alpha + \beta \in \Delta_{2m}^0 \) have vanishing coefficients. This proves the compatibility of \( \Delta \) with the \( \psi_1 \psi_1 \) relation, and hence with the all the relations in theorem 3.1. As a result, \( \Delta \) is an algebra homomorphism.

Now, let us check the coassociativity of \( \Delta \). It is immediate for the zero modes, and we only need to verify it for \( \psi_1^{(a)} \), or equivalently, \( \bar{\psi}_1^{(a)} \):

\[
(id \otimes \Delta) \left( \Delta \left( \bar{\psi}_1^{(a)} \right) \right) = \bar{\psi}_1^{(a)} \otimes (1 \otimes 1) + 1 \otimes \left( 1 \otimes \bar{\psi}_1^{(a)} \right) + 1 \otimes \left( \bar{\psi}_1^{(a)} \otimes 1 \right)
\]

\[
- 1 \otimes \left( (\epsilon_1 + \epsilon_2) \sum_{\alpha \in \Delta_{n+}^0} \left( \alpha^{(a)}, \alpha \right) \left( f^{(a)} \otimes e^{(a)} \right) \right)
\]

\[
- (\epsilon_1 + \epsilon_2) \sum_{\alpha \in \Delta_{n+}^0} \left( \alpha^{(a)}, \alpha \right) \left( f^{(a)} \otimes (1 \otimes e^{(a)}) + f^{(a)} \otimes (e^{(a)} \otimes 1) \right);
\]

(3.105)

\[
(\Delta \otimes id) \left( \Delta \left( \bar{\psi}_1^{(a)} \right) \right) = (1 \otimes 1) \otimes \bar{\psi}_1^{(a)} + (1 \otimes \bar{\psi}_1^{(a)}) \otimes 1 + \left( \bar{\psi}_1^{(a)} \otimes 1 \right) \otimes 1
\]

\[
- \left( (\epsilon_1 + \epsilon_2) \sum_{\alpha \in \Delta_{n+}^0} \left( \alpha^{(a)}, \alpha \right) \left( f^{(a)} \otimes e^{(a)} \right) \right) \otimes 1
\]

\[
- (\epsilon_1 + \epsilon_2) \sum_{\alpha \in \Delta_{n+}^0} \left( \alpha^{(a)}, \alpha \right) \left( (1 \otimes f^{(a)}) \otimes e^{(a)} + (f^{(a)} \otimes 1) \otimes e^{(a)} \right);
\]

(3.106)

Therefore, \( (id \otimes \Delta) \circ \Delta = (\Delta \otimes id) \circ \Delta \). This completes the proof of theorem 3.11.
Remark 1. In terms of the generators $ψ_0^{(a)} = (ε_1 + ε_2)ψ_0^{(a)}$, $θ_n^{(a)} = (ε_1 + ε_2)^{-1/2}e_n^{(a)}$, and likewise for $e^{(α, k)}$, $f^{(α, k)}$, the coproduct in (3.80) reads
\[
\Delta \psi_0^{(a)} = \square \psi_0^{(a)}, \quad \Delta θ_0^{(a)} = \square θ_0^{(a)}, \quad \Delta f_0^{(a)} = \square f_0^{(a)}.
\]
\[
\Delta (ψ_1^{(a)}) = \square (ψ_1^{(a)}) + θ_0^{(a)} ⊗ θ_0^{(a)} - (ε_1 + ε_2) \sum_{α, δ ∈ Δ^+} (α^{(a)}, δ^{(a)}) f_0^{(α)} ⊗ e^{(δ)}.
\] (3.107)

4. Isomorphism of quiver Yangians

Given any toric CY threefold, its quivers can be in different toric phases. Since these quivers are related by Seiberg/toric duality, it is natural to conjecture that their quiver Yangians are isomorphic. Here, we shall prove this for the generalized conifolds considered in this paper. As a result, different toric phases correspond to different triangulations of the toric diagram, i.e. different sequences $ς$.

A special feature for these generalized conifold is their underlying Kac–Moody superalgebras, which are of untwisted affine A-type. The zero modes of the quiver Yangians are actually different sets of Chevalley generators. For any two inequivalent sets of Chevalley generators, one can reach one from the other by odd reflections and Weyl groupoids [26–28]. We shall now extend this to isomorphisms of quiver Yangians. Recall that each quiver has an underlying Dynkin diagram associated to $sl(M|N)$. In such case, the odd reflection corresponding to an odd simple root $α^{(F)}$ acts on any simple root $α^{(a)}$ as

\[
α^{(F)} = r^{(F)} \left( α^{(a)} \right) = \begin{cases} 
-α^{(a)}, & a = F, \\
α^{(a)} + α^{(F)}, & a = F ± 1, \\
α^{(a)}, & \text{otherwise.}
\end{cases}
\] (4.1)

The Cartan matrix $A = (A_{ab})$ where $A_{ab} = (α^{(a)}, α^{(b)})$ is mapped to $A = RAR^T$ with $R = (R_{ab})$ given by

\[
R_{ab} = \begin{cases} 
-1, & a = b = F, \\
1, & a = b ≠ F, \\
1, & b = F, A_{af} ≠ 0, \\
0, & \text{otherwise.}
\end{cases}
\] (4.2)

In terms of Dynkin diagrams, this manipulation changes the $\mathbb{Z}_2$-grading of the nodes (and hence their $e, f$ generators) connected to the node $F$ (i.e. those with $A_{af} ≠ 0$) and leaves the remaining ones unchanged. The Chevalley generators are mapped to

7 More generally (especially when the Kac–Moody superalgebra has non-isotropic odd roots), the odd reflection associated to an isotropic simple odd root has the same action with the second condition as $α^{(a)} + α^{(F)}$ being a root.

8 It seems that there could often be typos when writing this transformation in literature. They would lead to inconsistency of signs in some of the relations. Therefore, we give an explicit proof of this in appendix B.
\[ \psi_0^{(a)} = \sum_{b=1}^{M+N} R_{ab} \psi_0^{(b)} = \begin{cases} -\psi_0^{(a)}, & a = F, \\ \psi_0^{(a)} + \psi_0^{(F)}, & a = F \pm 1, \\ \psi_0^{(a)}, & \text{otherwise}; \end{cases} \]

\[ e_0^{(a)} = \begin{cases} e_0^{(a)}, & a = F, \\ e_0^{(a)} + 2(e_0^{(a)}, e_0^{(a)}), & a = b = F \pm 1, \\ e_0^{(a)}, & \text{otherwise}; \end{cases} \]

\[ f_0^{(a)} = \begin{cases} f_0^{(a)}, & a = F, \\ f_0^{(a)} + 2(f_0^{(a)}, f_0^{(a)}), & a = b = F \pm 1, \\ f_0^{(a)}, & \text{otherwise}. \end{cases} \]  \tag{4.3}

Notice that \( A_{af} \) is simply \( \pm 1 \).

It would also be useful to spell out the following lemmas.

**Lemma 4.1.** We have

\[ \langle \alpha^{(a)} , \alpha^{(b)} \rangle = \begin{cases} -\langle \alpha^{(a)} , \alpha^{(b)} \rangle, & (a,b) = (F \pm 1, F), (F, F \pm 1), \\ \langle \alpha^{(a)} , \alpha^{(b)} \rangle + 2\langle \alpha^{(a)} , \alpha^{(b)} \rangle, & a = b = F \pm 1, \\ \langle \alpha^{(a)} , \alpha^{(b)} \rangle, & \text{otherwise}. \end{cases} \]  \tag{4.4}

**Proof.** This can be directly shown by considering how the Cartan matrix would change under the odd reflection. More specifically, \( A_{af} \) changes from \( \pm 2 \) to \( 0 \) (and vice versa) while \( A_{aF} = A_{Fa} \) changes from \( \mp 1 \) to \( \pm 1 \) for \( a = F \pm 1 \). The other elements in \( A \) are invariant. This can also be checked in the proof in appendix B. \( \square \)

**Lemma 4.2.** We have \( \sigma_1^{F \pm 1,F} = -\sigma_1^{F,F \pm 1} \) and \( \sigma_1^{F,F \pm 1} = -\sigma_1^{F \pm 1,F} \) while the other \( \sigma_1^{ab} \) are invariant. Therefore, \( \sigma_1^{ab} - \sigma_1^{ba} = \sigma_1^{ab} - \sigma_1^{ba} \).

**Proof.** This can be directly shown by considering how the charge assignment in figure 2(b) would change under the odd reflection. \( \square \)

Now, we are ready to show the algebra isomorphisms of the Seiberg/toric dual quiver Yangians.

**Theorem 4.3.** Given a generalized conifolds with \( M \neq N \), the quiver Yangians in different toric phases are isomorphic algebras.

**Proof.** When \( M + N > 2 \) and \( MN \neq 2 \), we shall show that the map induced by each odd reflection is an isomorphism. Then a quiver Yangian in any toric phase can be obtained from another under a sequence of such isomorphic maps. It suffices to check the relations involving non-zero modes. It would be more convenient to work with the \( J \) presentation. This is then given by (4.3) and
We have

\[ J(\psi'^{(a)}) = \begin{cases} -J(\psi'^{(a)}), & a = F, \\ J\left(\psi'^{(a)}\right) + J\left(\psi'^{(F)}\right) - \frac{1}{2} (\sigma_{1}^{a} F - \sigma_{1}^{a} F) \psi'^{(F)}, & a = F \pm 1, \\ J(\psi'^{(a)}), & \text{otherwise}; \end{cases} \]

\[ J(e'^{(a)}) = \begin{cases} J\left(e'^{(a)}\right), & a = F, \\ \left[e''_{0}, J\left(e'^{(a)}\right)\right], & a = F \pm 1, \\ J\left(e'^{(a)}\right), & \text{otherwise}; \end{cases} \]

\[ J(f'^{(a)}) = \begin{cases} -J\left(f'^{(a)}\right), & a = F, \\ -\frac{1}{\alpha F} \left[J\left(f'^{(a)}\right), J\left(f'^{(F)}\right)\right], & a = F \pm 1, \\ J\left(f'^{(a)}\right), & \text{otherwise}. \end{cases} \]

It is sufficient to show that this is consistent with the relations in corollary 3.7.1. There is no need to check the last equation therein as it is derived from the previous ones. The \( \psi' J(X') \) \( (X = \psi, e, f) \) relations are straightforward following linearity and the Jacobi identity when \( b = F \pm 1 \). The remaining relations are verified below in lemmas 4.4–4.6. This then also holds for its inverse.

Notice that we have also included \( C^3 (\mathbb{C} \times \mathbb{C}^2 / \mathbb{Z}_{2}, \text{conifold, resp. SPP}) \) with \( (M, N) = (1, 0) \) \((2, 0), (1, 1), \text{resp.} (2, 1)\) which do not belong to the family we mainly focus on in this paper, as well as \( C^3 / (\mathbb{Z}_{2} \times \mathbb{Z}_{2}) \) that is not a generalized conifold. Their isomorphisms are trivial since each of them only has one single toric phase.

**Lemma 4.4.** We have

\[ [J(\psi'^{(a)}), e'^{(b)}] = \left(\alpha'^{(a)}, \alpha'^{(b)}\right) J\left(e'^{(b)}\right) + \frac{\sigma_{1}^{b a} - \sigma_{1}^{a b}}{2} \left(\alpha'^{(a)}, \alpha'^{(b)}\right) e'^{(b)}, \tag{4.6} \]

\[ [J(\psi'^{(a)}), f'^{(b)}] = -\left(\alpha'^{(a)}, \alpha'^{(b)}\right) J\left(f'^{(b)}\right) - \frac{\sigma_{1}^{b a} - \sigma_{1}^{a b}}{2} \left(\alpha'^{(a)}, \alpha'^{(b)}\right) f'^{(b)}. \tag{4.7} \]

**Proof.** Here, we shall only explicitly write the proof for the \( J(\psi') e' \) relation, and the \( J(\psi') f' \) relation follows in the same manner. This can be divided into the following cases:

- \( a = b = F \pm 1 \): We have

\[ [J\left(\psi'^{(a)}\right), e'^{(b)}] = [J\left(\psi'^{(a)}\right), e'^{(F)}] + [J\left(\psi'^{(F)}\right), e'^{(b)}] + \frac{1}{2} (\sigma_{1}^{b} F - \sigma_{1}^{a} F) \left[\psi'^{(F)}, e'^{(b)}\right]. \tag{4.8} \]

Using the Jacobi identity, this is equal to

\[ \left[e'^{(b)}, [J\left(\psi'^{(a)}\right), e'^{(b)}] + [e'^{(F)}, [J\left(\psi'^{(F)}\right), e'^{(b)}] + e'^{(F)}, [J\left(\psi'^{(F)}\right), e'^{(b)}] \right] - 0 \frac{1}{2} (\sigma_{1}^{b} F - \sigma_{1}^{a} F) \left[\psi'^{(F)}, e'^{(b)}\right]. \tag{4.9} \]
Using the $J(\psi)e, J(e)e$ and $\psi e$ relations, the terms with $(\sigma_i^{bf} - \sigma_i^{fb})$ get cancelled and this becomes
\begin{equation}
\left(\alpha^{(b)}, \alpha^{(F)}\right) \left[ e_0^{(F)}, J \left( \psi_0^{(b)} \right) \right] + \left(\alpha^{(b)}, \alpha^{(b)}\right) \left[ e_0^{(F)}, J \left( \psi_0^{(b)} \right) \right] + \left(\alpha^{(b)}, \alpha^{(F)}\right) \left[ e_0^{(F)}, J \left( \psi_0^{(b)} \right) \right] = \left(\alpha^{(a)}, \alpha^{(b)}\right) e_0^{(b)}.
\end{equation}
(4.10)

- $a = F, b = F \pm 1$: We have
\begin{align*}
\left[ J \left( \psi_0^{(a)} \right), e_0^{(b)} \right] &= -\left[ J \left( \psi_0^{(F)} \right), e_0^{(b)} \right] = \left[ e_0^{(F)}, J \left( \psi_0^{(b)} \right) \right] = -\left(\alpha^{(a)}, \alpha^{(F)}\right) \left[ e_0^{(F)}, e_0^{(b)} \right] \\
&= \left(\alpha^{(a)}, \alpha^{(b)}\right) J \left( \psi_0^{(b)} \right) + \frac{1}{2} \left(\sigma_1^{ab} - \sigma_1^{ba}\right) \left(\alpha^{(F)}, \alpha^{(b)}\right) e_0^{(b)}.
\end{align*}
(4.11)

- $a = F \mp 1, b = F \pm 1$: This is similar to the case when $a = b = F \pm 1$. By using the Jacobi identity, as well as the $J(\psi)e, J(e)e$ and $\psi e$ relations, we have
\begin{align*}
\left[ J \left( \psi_0^{(a)} \right), e_0^{(b)} \right] &= \left(\alpha^{(a)}, \alpha^{(F)}\right) \left[ e_0^{(F)}, J \left( \psi_0^{(b)} \right) \right] + \frac{1}{2} \left(\sigma_1^{bf} - \sigma_1^{fb}\right) \left(\alpha^{(a)}, \alpha^{(F)}\right) e_0^{(b)} \\
&= \left(\alpha^{(a)}, \alpha^{(F)}\right) \left[ e_0^{(F)}, J \left( \psi_0^{(b)} \right) \right] - \frac{1}{2} \left(\sigma_1^{bf} - \sigma_1^{fb}\right) \left(\alpha^{(a)}, \alpha^{(F)}\right) e_0^{(b)}.
\end{align*}
(4.12)

This vanishes as $(\alpha^{(a)}, \alpha^{(F)}) = (\alpha^{(b)}, \alpha^{(F)})$ and $(\sigma_1^{bf} - \sigma_1^{fb}) = 0$.

- $a = b = F$: We have $\left[ J \left( \psi_0^{(a)} \right), e_0^{(b)} \right] = -\left[ J \left( \psi_0^{(F)} \right), e_0^{(F)} \right] = 0$.

- Otherwise: The remaining cases are immediate following the expressions of the primed generators and the similar arguments as above. \hfill \Box

**Lemma 4.5.** We have
\begin{equation}
\left[ J \left( e_0^{(a)} \right), J_0^{(b)} \right] = \left[ e_0^{(a)}, J \left( e_0^{(b)} \right) \right] = \delta_{ab} J \left( \psi_0^{(a)} \right).
\end{equation}
(4.13)

**Proof.** Here, we shall only explicitly write the proof for the $J(e)f^\prime$ relation, and the $e^\prime J(f^\prime)$ relation follows in the same manner. This can be divided into the following cases:

- $a = b = F$: We have $\left[ J \left( e_0^{(a)} \right), f_0^{(b)} \right] = -\left[ J \left( f_0^{(F)} \right), e_0^{(F)} \right] = -J \left( \psi_0^{(F)} \right) = J \left( \psi_0^{(a)} \right)$.

- $a = b = F \pm 1$: We have
\begin{equation}
\left[ J \left( e_0^{(a)} \right), J_0^{(b)} \right] = -\frac{1}{A_{af}} \left[ \left[ e_0^{(F)}, J \left( e_0^{(a)} \right) \right], \left[ J_0^{(F)}, J_0^{(a)} \right] \right].
\end{equation}
(4.14)

Using the Jacobi identity, this becomes
\begin{equation}
-\frac{1}{A_{af}} \left( \left[ \left[ e_0^{(F)}, J \left( e_0^{(a)} \right) \right], J_0^{(F)} \right] + \left[ \left[ e_0^{(F)}, J \left( e_0^{(a)} \right) \right], J_0^{(a)} \right] \right).
\end{equation}
(4.15)
In particular,
\[
\left[ e_0^{(F)} J \left( e_0^{(a)} \right) \right] J_0^{(a)} = \left[ e_0^{(F)} \left[ J \left( e_0^{(a)} \right) J_0^{(a)} \right] \right] - 0 = \left[ e_0^{(F)} J \left( \psi_0^{(a)} \right) \right]
\]
\[
= - \left( \alpha^{(a)}_0, \alpha^{(F)}_0 \right) J \left( e_0^{(F)} \right) - \frac{1}{2} \left( \sigma_1^{F} - \sigma_1^{F} \right) \left( \alpha^{(a)}_0, \alpha^{(F)}_0 \right) e_0^{(F)}.
\]
\[\text{(4.16)}\]

Likewise,
\[
\left[ e_0^{(F)} J \left( e_0^{(a)} \right) \right] J_0^{(a)} = \left( -1 \right)^{a_1} \left( \alpha^{(a)}_0, \alpha^{(F)}_0 \right) J \left( e_0^{(a)} \right).
\]
\[\text{(4.17)}\]

Therefore, (4.15) becomes
\[
\left[ J \left( e_0^{(a)} \right) J_0^{(a)} \right] + \left[ J \left( e_0^{(F)} \right) J_0^{(a)} \right] + \frac{1}{2} \left( \sigma_1^{F} - \sigma_1^{F} \right) \left( e_0^{(F)} J_0^{(a)} \right)
\]
\[
= J \left( \psi_0^{(a)} \right) + J \left( \psi_0^{(F)} \right) - \frac{1}{2} \left( \sigma_1^{F} - \sigma_1^{F} \right) \psi_0^{(F)}.
\]
\[\text{(4.18)}\]

\* \( a = F, b = F \pm 1 \): We have
\[
\left[ J \left( e_0^{(a)} \right) J_0^{(b)} \right] = \frac{1}{A_{bf}} \left[ \left[ J \left( e_0^{(F)} \right), J_0^{(b)} \right], \left[ J_0^{(a)} J_0^{(F)} \right] \right]
\]
\[
= \frac{1}{A_{bf}} \left( \left[ \left[ J \left( e_0^{(F)} \right), J_0^{(b)} \right] J_0^{(a)} \right] + \left( -1 \right)^{b_1 \left( 1 + a_1 \right)} \left[ J_0^{(b)} \left[ J_0^{(a)} J_0^{(F)} \right] \right] \right).
\]
\[\text{(4.19)}\]

The second term in the parenthesis is zero as \( J \left( e_0^{(F)} \right) J_0^{(a)} = 0 \) while using the \( J(f)J \) relation, the first term becomes
\[
\left[ \left[ J \left( e_0^{(F)} \right), J_0^{(b)} \right] J_0^{(a)} \right] - \frac{1}{2} \left( \sigma_1^{F} - \sigma_1^{F} \right) \left[ J_0^{(b)} \left[ J_0^{(a)} J_0^{(F)} \right] \right].
\]
\[\text{(4.20)}\]

Both of the terms vanish as \( \text{ad}_{e_0^{(F)}} = 0 \). Therefore, \( J \left( e_0^{(a)} \right) J_0^{(b)} \) = 0.

\* \( a = F \mp 1, b = F \pm 1 \): This is similar to the case when \( a = b = F \pm 1 \). Using the Jacobi identity, \( J \left( e_0^{(a)} \right) J_0^{(b)} \) becomes
\[
- \frac{1}{A_{bf}} \left( \left[ \left[ e_0^{(F)} J \left( e_0^{(a)} \right) J_0^{(b)} \right] J_0^{(F)} \right] + \left( -1 \right)^{b_1 \left( 1 + a_1 \right)} \left[ e_0^{(b)} \left[ J \left( e_0^{(a)} \right) J_0^{(F)} \right] \right] \right).
\]
\[\text{(4.21)}\]

In particular, \( \left[ e_0^{(F)} J \left( e_0^{(a)} \right) J_0^{(b)} \right] \) vanishes using the Jacobi identity. Moreover,
\[
\left[ \left[ e_0^{(F)} J \left( e_0^{(a)} \right) J_0^{(F)} \right] \right] = - \left( -1 \right)^{a_1} \left[ J \left( e_0^{(a)} \right) J_0^{(F)} \right].
\]
\[\text{(4.22)}\]

which consists of two terms proportional to \( J(e_0^{(a)}) \) and \( e_0^{(a)} \) respectively. Hence, \( \left[ J_0^{(b)}, \left[ J \left( e_0^{(F)} \right) J_0^{(a)} \right] J_0^{(F)} \right] \) vanishes. Therefore, \( J(e_0^{(a)}) J_0^{(b)} = 0 \).
Lemma 4.6. We have
\[ J\left(e_0^{(a)}, e_0^{(b)}\right) - e_0^{(a)} J\left(e_0^{(b)}\right) = \frac{1}{2} \left( \sigma_{1b}^{(a)} - \sigma_{1a}^{(b)} \right) \left[e_0^{(a)}, e_0^{(b)}\right], \]  
(4.23)
\[ J\left(f_0^{(a)}, f_0^{(b)}\right) - [f_0^{(a)}, J\left(f_0^{(b)}\right)] = -\frac{1}{2} \left( \sigma_{1b}^{(a)} - \sigma_{1a}^{(b)} \right) \left[f_0^{(a)}, f_0^{(b)}\right]. \]  
(4.24)

Proof. Here, we shall only explicitly write the proof for the \( J(e')e' \) relation, and the \( J(f')f' \) relation follows in the same manner. This can be divided into the following cases:

- \( a = b = F \pm 1 \): We have
  \[
  J\left(e_0^{(a)}, e_0^{(b)}\right) - e_0^{(a)} J\left(e_0^{(a)}\right) = \left[e_0^{(b)}, \left[e_0^{(a)}, J\left(e_0^{(b)}\right)\right]\right] - \left[e_0^{(b)}, \left[e_0^{(a)}, J\left(e_0^{(a)}\right)\right]\right].
  \]  
(4.25)

For \( |e^{(a)}| = 0 \), the two terms are the same and hence cancel each other. For \( |e^{(a)}| = 1 \), this is twice the first term. Using the Jacobi identity, we get
\[
\left[e_0^{(b)}, \left[e_0^{(a)}, J\left(e_0^{(b)}\right)\right]\right] - \left[e_0^{(b)}, \left[e_0^{(a)}, J\left(e_0^{(a)}\right)\right]\right] = \left[e_0^{(b)}, \left[e_0^{(a)}, \left[e_0^{(a)}, J\left(e_0^{(b)}\right)\right]\right]\right] - \left[e_0^{(b)}, \left[e_0^{(a)}, \left[e_0^{(a)}, J\left(e_0^{(a)}\right)\right]\right]\right].
\]  
(4.26)

The first term vanishes as \( \text{ad}_{e_0^{(a)}}^2 = 0 \). For the second term, using the \( eJ(e) \) relation, \( \left[e_0^{(a)}, J\left(e_0^{(a)}\right)\right] \) consists of two terms proportional to \( [J(e_0^{(a)}), e_0^{(a)}] \) and \( [e_0^{(a)}, e_0^{(a)}] \) respectively. Hence, the second term also vanishes as \( \text{ad}_{e_0^{(a)}}^2 = 0 \). Therefore, \( [J(e_0^{(a)}), e_0^{(a)}] - \left[e_0^{(b)}, J\left(e_0^{(b)}\right)\right] = 0 \).

- \( a = F, b = F \pm 1 \): We have
  \[
  J\left(e_0^{(a)}, e_0^{(b)}\right) - e_0^{(a)} J\left(e_0^{(a)}\right) = \frac{1}{2} \left( \sigma_{1b}^{(a)} - \sigma_{1a}^{(b)} \right) \left[e_0^{(a)}, e_0^{(b)}\right].
  \]  
(4.27)

using the Jacobi identity. On the other hand,
\[
\frac{1}{2} \left( \sigma_{1b}^{(a)} - \sigma_{1a}^{(b)} \right) \left[e_0^{(a)}, e_0^{(b)}\right] = \frac{1}{2} \left( \sigma_{1b}^{(a)} - \sigma_{1a}^{(b)} \right) \left[e_0^{(a)}, e_0^{(b)}\right] - \frac{1}{2} \left( \sigma_{1b}^{(a)} - \sigma_{1a}^{(b)} \right) \left[e_0^{(a)}, e_0^{(b)}\right].
\]  
(4.28)

- \( a = F \pm 1, b = F \pm 1 \): We have
  \[
  J\left(e_0^{(a)}, e_0^{(b)}\right) - e_0^{(a)} J\left(e_0^{(a)}\right) = (-1)^{1+|a|} e_0^{(a)}, \left[e_0^{(b)}, J\left(e_0^{(a)}\right)\right] - (-1)^{1+|a|} e_0^{(a)}, \left[e_0^{(b)}, J\left(e_0^{(a)}\right)\right].
  \]  
(4.29)
where we have used the Jacobi identity and \(\text{ad}^2 e^{(F)}_0 = 0\). Therefore, we need to show that the expression on the right hand side vanishes. In particular,

\[
\left[ e^{(F)}_0, J\left( e^{(a)}_0 \right), e^{(b)}_0 \right] = -(-1)^{|a|} J\left( e^{(a)}_0 \right), e^{(b)}_0 ]]. 
\]

(4.30)

Using

\[
J\left( e^{(a)}_0 \right) = \frac{1}{(a^{(c)}), a^{(a)}} \left[ J\left( e^{(c)}_0 \right), e^{(a)}_0 \right] - \frac{\sigma^{aa} - \sigma^{ca}}{2} e^{(a)}_0
\]

(4.31)

with \(c\) taken to be \(F\), the first term on the right hand side in (4.29) is equal to \((-1)^{1+|a|} [e^{(F)}_0, e^{(a)}_0, e^{(b)}_0] + \) plus a term proportional to \([e^{(F)}_0, e^{(a)}_0, e^{(b)}_0]\). In particular, \([e^{(F)}_0, e^{(a)}_0, e^{(b)}_0]\) vanishes due to the Serre relation for \(F\). Here,

\[
\mathcal{X}_1 = -(-1)^{|a|} \frac{1}{\Lambda_{df}} \left[ J\left( e^{(F)}_0 \right), e^{(a)}_0 \right], e^{(b)}_0 ]].
\]

(4.32)

Likewise, the second line in (4.29) is equal to \(-(-1)^{1+|a|} [e^{(F)}_0, e^{(a)}_0, e^{(b)}_0]\), where

\[
\mathcal{X}_2 = \frac{1}{\Lambda_{df}} \left[ J\left( e^{(F)}_0 \right), e^{(a)}_0 \right], e^{(b)}_0 ]].
\]

(4.33)

Therefore, showing that the right hand side in (4.29) vanishes is equivalent to showing \([e^{(F)}_0, \mathcal{X}_1] = [e^{(F)}_0, \mathcal{X}_2]\). Using the Jacobi identity and \(\text{ad}^2 e^{(F)}_0 = 0\), we have

\[
\mathcal{X}_1 = -(-1)^{|a|+|b|} \frac{1}{\Lambda_{df}} \left[ e^{(b)}_0, J\left( e^{(F)}_0 \right), e^{(a)}_0 ]] \right].
\]

(4.34)

Keep using the Jacobi identity, and we get

\[
\left[ e^{(F)}_0, \mathcal{X}_1 \right] = \left[ e^{(F)}_0, \mathcal{X}_3 \right] + \mathcal{X}_4,
\]

(4.35)

where

\[
\mathcal{X}_3 = -(-1)^{|a|+|b|} \frac{1}{\Lambda_{df}} \left[ e^{(b)}_0, J\left( e^{(F)}_0 \right), e^{(a)}_0 ]] \right].
\]

(4.36)

and \(\mathcal{X}_4\) is proportional to

\[
\left\{ e^{(F)}_0, J\left( e^{(F)}_0 \right), e^{(a)}_0 \right\}
\]

\[
= \left[ e^{(F)}_0, J\left( e^{(F)}_0 \right), e^{(a)}_0 \right] - \left[ J\left( e^{(F)}_0 \right), e^{(a)}_0 \right].
\]

(4.37)

Hence, \(\mathcal{X}_4\) vanishes due to \([e^{(F)}_0, J\left( e^{(F)}_0 \right)] = 0\) and the Serre relation for \(F\). Therefore, \([e^{(F)}_0, \mathcal{X}_1] = [e^{(F)}_0, \mathcal{X}_3]\). It is straightforward to see that \(\mathcal{X}_3 = \mathcal{X}_2\). This shows that (4.29) vanishes.
In terms of the notations and conventions see, the rectangular modes that implement the BPS/CFT correspondence. Indeed, as we are now going to use

\[ \psi_{9} \]

Here, we shall directly start with the commutation relations for the generators of rectangular \( W \)-algebras

\[ 5.1 \quad \text{From} \quad \mathcal{Y} \quad \text{to} \quad \mathcal{W} \]

Here, we shall directly start with the commutation relations for the generators of rectangular \( W \)-algebras for the generalized conifold \( \chi y = z^{M} w^{N} \). A mathematical definition of rectangular \( W \)-algebras \( \mathcal{W}^{k} (gl(M|N), (\mathcal{F}^{M|N})) \) is given in appendix C with the notations and conventions set up therein. For brevity, we shall abbreviate it as \( \mathcal{W}_{M|N} \).

The \( \mathcal{W} \)-algebras of interest in this paper can be generated by \( U_{ij}^{(s)} \) with spin \( s = 1, 2 \) and \( i, j \in \mathbb{Z}/(M + N)\mathbb{Z} \). Given a parity sequence \( \varsigma = \{ \varsigma_{a} \} \) as introduced in section 2, the generator \( U_{ij}^{(s)} \) has the \( \mathbb{Z}_{2} \)-grading given by \( (-1)^{p(i) + p(j)} \), where \( (-1)^{p(i)} = \varsigma_{i} \) (see also (C.1))\(^9\). The OPEs of the currents \( U_{ij}^{(s)}(z) \) were obtained in \([34, 35]\). The following commutation relations for their modes \( U_{ij}^{(s)}(m) \) can then be computed directly using (C.22).

\(^9\) As we will see shortly, \( |a| \) and \( p(i) \) are indeed consistent in the sense of \( \varsigma \) when relating \( \mathcal{Y} \) and \( \mathcal{W} \). In other words, \( |a| \) is bosonic when \( p(a) = p(a + 1) \) and fermionic otherwise.
Lemma 5.1. We have
\[
[U^{(1)}_{i_j}[m], U^{(1)}_{i_j}[n]] = \delta_{m-n}m \left( \delta_{i_j k} \delta_{i_j l} (-1)^{p(i_j)} \varkappa + \delta_{i_j k} \delta_{i_j l} \right)
\]
\[+ (-1)^{p(i_j) p(j_l) p(j_l) + p(i_j) p(j_l) + p(i_j) p(i_l) + p(j_l) p(i_l) + p(i_j) p(i_l) + p(j_l) p(i_l)} \delta_{i_j k} U^{(1)}_{i_j}[m+n] - (-1)^{p(i_j) p(j_l) p(i_l)} \delta_{i_j k} U^{(1)}_{i_j}[m+n],
\]
\[
(U^{(1)}_{i_j}[m], U^{(2)}_{i_j}[n]) = \frac{1}{2} l(l-1)m(m-1)z \delta_{m-n} \left( (-1)^{p(i_j)} \varkappa + \delta_{i_j k} \delta_{i_j l} \right)
\]
\[+ m(l-1) \left( (-1)^{p(i_j) p(j_l) + p(i_j) p(j_l) + p(i_j) p(i_l) + p(j_l) p(i_l) + p(i_j) p(i_l) + p(j_l) p(i_l)} \delta_{i_j k} U^{(1)}_{i_j}[m+n] + \delta_{i_j k} U^{(1)}_{i_j}[m+n] \right)
\]
\[+ (-1)^{p(i_j) p(i_l) + p(i_j) p(j_l) + p(i_j) p(i_l) + p(j_l) p(i_l) + p(i_j) p(i_l) + p(j_l) p(i_l)} \delta_{i_j k} U^{(2)}_{i_j}[m+n] - (-1)^{p(i_j) p(j_l) p(i_l)} \delta_{i_j k} U^{(2)}_{i_j}[m+n],
\]
\[
(U^{(2)}_{i_j}[m], U^{(2)}_{i_j}[n]) = \frac{1}{12} l(l-1)m(m-1)(m+1) \delta_{m-n} \left( 2 \varkappa ((1-2l) \alpha^2 + 1) (-1)^{p(i_j)} \varkappa -(4l-3) \alpha^2 + 1 \right)
\]
\[+ \frac{1}{2} m(m+1) (l-1)^2 \varkappa \left( U^{(1)}_{i_j}[m+n] - U^{(1)}_{i_j}[m+n] \right)
\]
\[- (m+1) \left( U^{(2)}_{i_j}[m+n] + U^{(2)}_{i_j}[m+n] + 2 \varkappa (-1)^{p(i_j) \delta_{i_j k} U^{(2)}_{i_j}[m+n]} \right)
\]
\[+ (m+1)(l-1) \left( \sum_{k \geq 0} U^{(1)}_{i_j}[m+n-k] + \sum_{k \geq 0} U^{(1)}_{i_j}[m+n-k] U^{(1)}_{i_j}[k] \right)
\]
\[+ (m+1)(l-1)(-1)^{p(i_j)} \varkappa \left( \sum_{k \geq 0} U^{(1)}_{i_j}[m+n-k] U^{(1)}_{i_j}[k] \right)
\]
\[- (m+1)(l-1) \varkappa (m+n+1) \left( 1 + (-1)^{p(i_j) \delta_{i_j k} U^{(1)}_{i_j}[m+n]} \right)
\]
\[- (m+n+2) \left( U^{(2)}_{i_j}[m+n] + (-1)^{p(i_j) \delta_{i_j k} U^{(2)}_{i_j}[m+n]} - 2 U^{(2)}_{i_j}[m+n] \right)
\]
\[+ (-1)^{p(i_j)} \left( \sum_{k \geq 1} U^{(2)}_{i_j}[k] U^{(1)}_{i_j}[m+n-k] + \sum_{k \geq 1} (-1)^{p(i_j) p(j_l) + p(i_j) p(j_l) + p(i_j) p(i_l) + p(j_l) p(i_l) + p(i_j) p(i_l) + p(j_l) p(i_l)} \delta_{i_j k} U^{(1)}_{i_j}[m+n-k] U^{(2)}_{i_j}[k] \right)
\]
\[- (-1)^{p(i_j)} \left( \sum_{k \geq 1} U^{(2)}_{i_j}[k] U^{(1)}_{i_j}[m+n-k] + \sum_{k \geq 1} (-1)^{p(i_j) p(j_l) + p(i_j) p(j_l) + p(i_j) p(i_l) + p(j_l) p(i_l) + p(i_j) p(i_l) + p(j_l) p(i_l)} \delta_{i_j k} U^{(1)}_{i_j}[m+n-k] U^{(2)}_{i_j}[k] \right)
\]
\[+ (l-1) \left( \sum_{k \geq 0} (-k-1) U^{(1)}_{i_j}[k] U^{(1)}_{i_j}[m+n-k] + \sum_{k \geq 0} (-k-1) U^{(1)}_{i_j}[m+n-k] U^{(1)}_{i_j}[k] \right)
\]
\[(l - 1)\zeta(-1)^{p(j)} \left( \sum_{k < 0} (-k - 1)U^{(1)}_{ij}[k]U^{(1)}_{j'i}[m + n - k] \right) + (-1)^{p(i) + \sigma(j)} \sum_{k > 0} (-k - 1)U^{(1)}_{ji}[m + n - k]U^{(1)}_{ij}[k] \right) \\
+ \frac{1}{2} (l - 1)(m + n + 1)(m + n + 2) \left( (l + 1)\zeta U^{(1)}_{ji}[m + n] - \zeta^2 U^{(1)}_{ii}[m + n] \right) \\
+ \frac{1}{2} (l - 1)(m + n + 1)(m + n + 2)(-1)^{p(i)} \zeta^2 \delta_{ij} U^{(1)}_{ii}[m + n]. \tag{5.3} \]

Notice that we only give the \(U^{(2)}_{ij}U^{(2)}_{ji}\) relation when \(i_1 = j_1\) and \(i_2 = j_2\) as this is sufficient for the use here. It is also straightforward to get the more general case from the OPE. In this paper, we shall always assume \(\zeta \neq 0\).

The \(W\)-algebra is often defined via the distinguished parity sequence, that is, only two fermionic \(p(i)\) (the non-super case \(M/0\) always has bosonic ones only). Here, we allow it to have different \(\zeta\). Analogous to the quiver Yangians related by Seiberg dualities, we would expect the \(W\)-algebras with different \(\zeta\) are essentially the same. In fact, the proof of this is much simpler than the quiver Yangian case in section 4 by virtue of the matrix presentation here.

**Proposition 5.2.** Given \(M, N\) and \(l\), the rectangular \(W\)-algebras \(W_{M, N, l}\) are isomorphic for different \(\zeta\).

**Proof.** The isomorphism can be constructed from a sequence of the following isomorphic maps. Suppose \(\zeta\) and \(\zeta'\) are related by \(\sigma \in \mathfrak{S}_{M+N}\) that permutes the \(i\)th and \((i + 1)\)th elements. Then the transformation is given by \(U^{(i)}_{ij} \to U^{(i)}_{\sigma(i)\sigma(j)}\). It is straightforward to see that this preserves the relations for the generators. \(\square\)

Therefore, when considering the map from the quiver Yangians to the (universal enveloping algebra of) \(\mathcal{Y}W\)-algebra below, we can simply take them to have the same \(\zeta\). The isomorphic ones are related by the transformations in theorem 4.3 and proposition 5.2 respectively.

Let \(\epsilon_{\pm} = \epsilon_1 \pm \epsilon_2\). Since
\[
\zeta_\sigma(\epsilon_1 - \epsilon_2) = \begin{cases} 
\zeta_{\sigma + 1}(\epsilon_1 - \epsilon_2), & \zeta_\sigma = \zeta_{\sigma + 1} \\
\zeta_{\sigma + 1}(\epsilon_2 - \epsilon_1), & \zeta_\sigma = -\zeta_{\sigma + 1}, 
\end{cases} \tag{5.4}
\]
we can take \(\epsilon_\pm = \epsilon^{a+1,a} - \epsilon^{a,a+1}\) for any \(a\) based on figure 2 without loss of generality. This allows us to consider another presentation of the quiver Yangian which would be convenient for our discussions. Let us prepare the generators defined as
\[
\begin{align*}
\mathcal{H}_0^{(a)} &= \psi_0^{(a)} \\
\mathcal{H}_1^{(a)} &= \psi_1^{(a)} + \frac{1}{2} \nu(a) \epsilon_\pm \psi_0^{(a)} \\
\mathcal{E}_0^{(a)} &= \epsilon_0^{(a)} \\
\mathcal{E}_1^{(a)} &= \epsilon_1^{(a)} + \frac{1}{2} \nu(a) \epsilon_\pm \epsilon_0^{(a)} \\
\mathcal{F}_0^{(a)} &= f_0^{(a)} \\
\mathcal{F}_1^{(a)} &= f_1^{(a)} + \frac{1}{2} \nu(a) \epsilon_\pm f_0^{(a)} 
\end{align*} \tag{5.5}
\]
where \(\nu(a)\) can be any function satisfying \(\nu(0) = 1\) for \(0 \leq a \leq M + N - 1\). In particular, this means that \(\nu(-1) \neq \nu(M + N - 1)\) and \(\nu(0) \neq \nu(M + N)\). For instance, the simplest example would be \(\nu(a) = a (-1 \leq a \leq M + N)\). We shall also pick a reference node labeled by \(a = 0\).
Proposition 5.3. The quiver Yangian is generated by $H_r^{(a)}$, $E_r^{(a)}$, $F_r^{(a)}$ ($a \in \mathbb{Q}_0, r = 0, 1$) subject to the relations
\begin{align*}
[H_r^{(a)}, H_r^{(b)}] &= 0, \\
[E_r^{(a)}, F_r^{(b)}] &= \delta_{ab} H_{r+}, \\
[H_0^{(a)}, E_r^{(a)}] &= A_{ab} E_r^{(a)}, \\
[H_1^{(a)}, E_0^{(b)}] &= \begin{cases} A_{ab} E_1^{(b)} + \frac{1}{2} A_{ab} \nu (M + N) \epsilon_0 E_0^{(b)}, & (a, b) = (M + N - 1, 0) \\
A_{ab} E_1^{(b)}, & (a, b) = (0, M + N - 1) \end{cases} \\
[F_0^{(a)}, F_0^{(b)}] &= -A_{ab} F_r^{(a)}, \\
[F_1^{(a)}, F_0^{(b)}] &= \begin{cases} -A_{ab} E_1^{(b)} - \frac{1}{2} A_{ab} \nu (M + N) \epsilon_0 E_0^{(b)}, & (a, b) = (M + N - 1, 0) \\
-A_{ab} E_1^{(b)} + \frac{1}{2} A_{ab} \nu (M + N) \epsilon_0 E_0^{(b)}, & (a, b) = (0, M + N - 1) \end{cases}
\end{align*}

Serre relations ($S$):
\begin{align*}
[E_0^{(a)}, E_0^{(b)}] &= [F_0^{(a)}, F_0^{(b)}] = 0 \quad (\sigma_1 = 0), \\
[E_1^{(a)}, E_0^{(b)}] &= \begin{cases} \frac{1}{2} A_{ab} \epsilon + \left( E_0^{(a)}, E_0^{(b)} \right) - \frac{1}{2} \nu (M + N) \epsilon_0 \left( E_0^{(a)}, E_0^{(b)} \right), & (a, b) = (0, M + N - 1) \end{cases} \\
[F_1^{(a)}, F_0^{(b)}] &= \begin{cases} \frac{1}{2} A_{ab} \epsilon + \left( F_0^{(a)}, F_0^{(b)} \right) - \frac{1}{2} \nu (M + N) \epsilon_0 \left( F_0^{(a)}, F_0^{(b)} \right), & (a, b) = (0, M + N - 1) \end{cases}
\end{align*}

where $H_1^{(a)} := H_1^{(a)} - \frac{1}{2} \epsilon + \left( H_0^{(a)} \right)^2$. Recall that $\{x, y\}$ is used to denote $xy + (-1)^{|x||y|}yx$ here. The higher modes with $r \geq 2$ can be obtained in a way similar to the presentation using $\psi, e, f$.
This can be verified by straightforward calculations. Hence, we omit the explicit proof here. When checking these relations, it is also worth noting that

\[ A_{ab} = \alpha^{(a)}(a), \alpha^{(b)} = \begin{cases} -(\xi_b + \xi_{b+1}), & a = b \\ \xi_{b+1}, & a = b + 1 \\ \xi_b, & b = a + 1 \\ 0, & \text{otherwise.} \end{cases} \] (5.16)

**Remark 3.** As pointed out in [8], the quiver Yangian is only isomorphic to Ueda’s affine super Yangian introduced in [25] when \( \epsilon = 0 \). By comparing the presentation above with the similar presentation for Ueda’s affine super Yangian in [54], it is straightforward to see that this difference is encoded by \( \nu(a) \) here and the coefficients in the presentation in [54].

Now, we are ready to bridge the quiver Yangians and \( W \)-algebras.

**Theorem 5.4.** Given a generalized conifold with \( M + N > 2 \), \( MN \neq 2 \) and \( M \neq N \), when \( \nu(M + N)\epsilon = (2\pi - M - N)\epsilon \), there is a surjective algebra homomorphism from the quiver Yangian to the universal enveloping algebra of \( W_{M,N} \). Fixing a parity sequence \( \varsigma \), such map \( \Phi : \mathcal{Y} \rightarrow U(W_{M,N}) \) can be uniquely determined by

\[
\Phi \left( h^{(a)}_0 \right) = \Phi \left( y^{(a)}_0 \right), \quad \Phi \left( h^{(a)}_1 \right) = \Phi \left( y^{(a)}_1 \right) - \frac{1}{2} \nu(a) \epsilon \Phi \left( y^{(a)}_0 \right)
\] (5.17)

for \( (X, Y) = (\psi, H), (\epsilon, E), (f, F) \), where

\[
\Phi \left( h^{(a)}_0 \right) = \begin{cases} U_{M+N, M+N}^{(1)}[0] - U_{11}^{(1)}[0] + l\epsilon, & a = 0 \\ U_{a+1, a+1}^{(1)}[0] - U_{a+1, a+1}^{(1)}[0], & a \neq 0, \end{cases}
\] (5.18)

\[
\Phi \left( e^{(a)}_0 \right) = \begin{cases} -(-1)^p U_{M+N, 1}^{(1)}[-1], & a = 0 \\ -(-1)^{p(a+1)} U_{a+1, a+1}^{(1)}[0], & a \neq 0, \end{cases}
\] (5.19)

\[
\Phi \left( f^{(a)}_0 \right) = \begin{cases} U_{1, M+N}^{(1)}[1], & a = 0 \\ U_{a+1, a}^{(1)}[0], & a \neq 0, \end{cases}
\] (5.20)

\[
\Phi \left( h^{(0)}_1 \right) = \epsilon_+ \left( U_{M+N, M+N}^{(2)}[0] - U_{11}^{(2)}[0] - U_{M+N, M+N}^{(1)}[0] U_{11}^{(1)}[0] \right) - \frac{1}{2} \sum_{c=1}^{M+N} \sum_{k=0}^{\infty} (-1)^{p(c)+p(M+N)} U_{c, M+N}^{(1)}[-k] U_{M+N, c}^{(1)}[k] + \frac{1}{2} \sum_{c=1}^{M+N} \sum_{k=0}^{\infty} (-1)^{p(c)+p(1)} U_{c, 1}^{(1)}[-k-1] U_{1, c}^{(1)}[k+1]
\] (5.21)
This is proven by directly applying lemma 5.1. The construction of this map is analogous to the one for Ueda’s affine Yangian in [54] as well as the evaluation maps for certain Yangians in [25, 55]. Nevertheless, we should still be careful about the slight differences, and
let us give a quick proof here. First, we shall check the relations involving only zero modes. This is essentially building the Chevalley generators from the matrices. Here, we would only explicitly write the proof for the $\mathcal{H}_0\mathcal{H}_0$, $\mathcal{H}_0\mathcal{E}_0$ and Serre relations. The other relations can be verified also following straightforward computations.

The $\mathcal{H}_0\mathcal{H}_0$ relation is immediate since $U_{\mathcal{H}_0}^{(1)}[0]$ and $U_{\mathcal{H}_0}^{(1)}[0]$ commute with each other for any $a, b$. Now, suppose $a, b \neq 0$, and the $\mathcal{H}_0\mathcal{E}_0$ relation is recovered by

$$
\left[ U_{\mathcal{H}_0}^{(1)}[0] - U_{\mathcal{H}_0}^{(1)}[0, a + 1, b + 1[0], -(-1)^{p(b+1)}U_{\mathcal{H}_0}^{(1)}[0, a + 1, b + 1[0] \right]
$$

\begin{align*}
&= -(-1)^{p(b+1)}\delta_{a,b+1}U_{\mathcal{H}_0}^{(1)}[0] + (-1)^{p(b+1)+p(a)}\delta_{b,a+1}U_{\mathcal{H}_0}^{(1)}[0] + (-1)^{p(b+1)+p(a)}\delta_{a,b+1}U_{\mathcal{H}_0}^{(1)}[0] + (-1)^{p(b+1)+p(a)}\delta_{b,a+1}U_{\mathcal{H}_0}^{(1)}[0] \\
&= \begin{cases} 
-(-1)^{p(b+1)}(-1)^{p(b+1)}U_{\mathcal{H}_0}^{(1)}[0] & a = b \\
-(-1)^{p(b+1)}(-1)^{p(b+1)}U_{\mathcal{H}_0}^{(1)}[0] & a = b + 1 \\
(-1)^{p(b)}(-1)^{p(b+1)}U_{\mathcal{H}_0}^{(1)}[0] & a + 1 = b \\
0 & \text{otherwise}
\end{cases}
\end{align*}

as required following (5.16). When $a$ and/or $b$ equal(s) zero, the relation can be checked similarly. The Serre relations also hold immediately as each term we get from the commutation relations has factor $\delta_{a,a+1}$, $\delta_{a,a+2}$ or $\delta_{a+1,a+2}$ that vanishes.

Now, let us check the relations involving higher modes. Here, we shall only verify the $\mathcal{H}_1\mathcal{H}_1$ and $\mathcal{H}_1\mathcal{F}_0$ relations explicitly, and the other relations can also be obtained following direct computations. For convenience, let us use the notation $\delta_{\text{cond}}$ which is equal to 1 when the condition cond is true and 0 otherwise.

Let us first consider the $\mathcal{H}_1\mathcal{H}_1$ relation for $a, b \neq 0$. In the total commutation relation, we have the piece

$$
\left[ U_{\mathcal{H}_1}^{(1)}[0] - U_{\mathcal{H}_1}^{(1)}[0, a + 1, b + 1[0], U_{\mathcal{F}_0}^{(2)}[0] - U_{\mathcal{F}_0}^{(2)}[0, a + 1, b + 1[0] \right]
$$

\begin{align*}
&= (l - 1) \left( \sum_{k > 0} kU_{\mathcal{H}_1}^{(1)}[k]U_{\mathcal{F}_0}^{(2)}[-k] - \sum_{k > 0} kU_{\mathcal{H}_1}^{(1)}[-k]U_{\mathcal{F}_0}^{(2)}[k] \right) \\
&+ (l - 1) \left( -(-1)^{p(b)}X \sum_{k > 0} kU_{\mathcal{H}_1}^{(1)}[k]U_{\mathcal{F}_0}^{(2)}[-k] - (-1)^{p(a)}X \sum_{k > 0} kU_{\mathcal{H}_1}^{(1)}[-k]U_{\mathcal{F}_0}^{(2)}[k] \right) \\
&+ (-1)^{p(a)}X \sum_{k > 1} U_{\mathcal{H}_1}^{(2)}[k]U_{\mathcal{F}_0}^{(1)}[-k] + (-1)^{p(b)}X \sum_{k > 1} U_{\mathcal{H}_1}^{(1)}[k]U_{\mathcal{F}_0}^{(2)}[-k] \\
&- (-1)^{p(a)}X \sum_{k > 1} U_{\mathcal{F}_0}^{(2)}[k]U_{\mathcal{H}_1}^{(1)}[-k] + (-1)^{p(b)}X \sum_{k > 1} U_{\mathcal{F}_0}^{(1)}[k]U_{\mathcal{H}_1}^{(2)}[-k] \\
&- \begin{bmatrix} a \rightarrow a \\ b \rightarrow b + 1 \end{bmatrix} - \begin{bmatrix} a \rightarrow a + 1 \\ b \rightarrow b \end{bmatrix} - \begin{bmatrix} a \rightarrow a + 1 \\ b \rightarrow b + 1 \end{bmatrix},
\end{align*}

\begin{equation}
(5.28)
\end{equation}
where all the single \( U_{ab}^{(i)}[0] \) and \( U_{bb}^{(i)}[0] \) get cancelled. We also have
\[
\begin{align*}
- \left[ \sum_{c=1}^{a} \sum_{k \geq 0} (-1)^{p(c)+p(a)} U_{ac}^{(1)}[-k] U_{ac}^{(1)}[k] + \sum_{c=a+1}^{M+N} (-1)^{p(c)+p(a)} U_{ac}^{(1)}[-k] U_{ac}^{(1)}[k] \right] \\
= - \sum_{k \geq 0} \left( k(l-1) U_{ac}^{(1)}[-k] U_{bb}^{(1)}[k] - k(l-1) \right) + \sum_{c=a+1}^{M+N} \ldots \\
+ \delta_{ab} \sum_{c=1}^{a} (-1)^{p(c)} U_{ac}^{(1)}[-k] U_{ac}^{(1)}[k] + \sum_{c=a+1}^{M+N} \ldots \\
- \delta_{ab} \sum_{c=1}^{a} (-1)^{p(c)} U_{ac}^{(2)}[-k] U_{ac}^{(2)}[k] + \sum_{c=a+1}^{M+N} \ldots \\
- k(l-1) U_{bb}^{(1)}[-k] U_{ac}^{(1)}[k] - [b \rightarrow b + 1] \right), \\
\end{align*}
\]
where each \( \sum_{c=a+1}^{M+N} \ldots \) indicates a term similar to its previous one (but with \( \pm k \) changed to \( \pm (k+1) \) in the modes only). Likewise,
\[
\begin{align*}
- \left[ U_{bb}^{(2)}[0] - U_{bb}^{(2)}[a+1,a+1][0], \sum_{c=1}^{b} \sum_{k \geq 0} (-1)^{p(c)+p(b)} U_{cb}^{(1)}[-k] U_{cb}^{(1)}[k] + \sum_{c=b+1}^{M+N} \ldots \right] \\
= - \sum_{k \geq 0} \left( k(l-1) U_{bb}^{(1)}[-k] U_{bb}^{(1)}[k] - k(l-1) \right) + \sum_{c=b+1}^{M+N} \ldots \\
+ \delta_{bc} \sum_{c=1}^{b} (-1)^{p(c)} U_{cb}^{(1)}[-k] U_{cb}^{(1)}[k] + \sum_{c=b+1}^{M+N} \ldots \\
- \delta_{bc} \sum_{c=1}^{b} (-1)^{p(c)} U_{cb}^{(2)}[-k] U_{cb}^{(2)}[k] + \sum_{c=b+1}^{M+N} \ldots \\
- [\text{blue terms}] + k(l-1) U_{bb}^{(1)}[-k] U_{bb}^{(1)}[k] - [a \rightarrow a + 1] \right), \\
\end{align*}
\]
where \( [\text{blue terms}] \) indicates a term similar to its previous one (but with \( \pm k \) changed to \( \pm (k+1) \) in the modes only). Notice that when cancelling the \( U^{(2)} U^{(1)} \) or \( U^{(1)} U^{(2)} \) terms (in pink and red), we need to apply the commutation relation for \( U^{(1)} \) and \( U^{(2)} \) for the modes 0 and 1. The resulted extra terms with single \( U^{(i)} \) are cancelled with those from a similar manipulation of the terms in a different color. The brown terms above are compensated by
\[
\begin{align*}
    &\sum_{c=1}^{a} \sum_{k \geq 0} (-1)^{p(c)+p(a)} U_{\alpha}^{(1)} [-k] U_{\alpha}^{(1)} [k] + \sum_{c=a+1}^{M+N} (\ldots), \quad \text{terms with } a \rightarrow b \bigg]
    \\
    &= \sum_{k \geq 0} \left( k(l-1) U_{\alpha}^{(1)} [-k] U_{\alpha}^{(1)} [k] - k(l-1) U_{\beta}^{(1)} [-k] U_{\alpha}^{(1)} [k] \right) .
\end{align*}
\]

(5.31)

One may check that \( U_{\alpha}^{(1)} [0] \) (and hence \( U_{\alpha}^{(1)} [0] U_{\alpha}^{(1)} [0] \)) commutes with other terms. This shows that \( \Phi \left( H_{1}^{(a)} \right) \) commutes with \( \Phi \left( H_{1}^{(b)} \right) \) when \( a, b \neq 0 \). Suppose \( a = 0 \) and \( b \neq 0 \), the commutation relations we need to compute are the same as the case for \( a, b \neq 0 \). When \( a = b = 0 \), this relation is trivial.

Let us next check the \( H_{1} F_{0} \) relation when \( a, b \neq 0 \). Similar to the computation for the zero modes, the commutation relation \( \epsilon_{+} \left[ U_{\alpha}^{(2)} [0] - U_{\alpha}^{(2)} [0], U_{b+1, b} U_{0, b} \right] \) is responsible for the \(-A_{ab} \epsilon_{+} U_{b+1, b} U_{0, b} \) term in \(-A_{ab} \Phi \left( F_{1} \right) \). Henceforth, we shall omit \(-A_{ab} \) and \( \epsilon_{+} \) when making similar statements for brevity (though the calculations would still carry \((-1)^{p(a)} \) factors that lead to the corresponding Cartan matrix element). Now, consider

\[
\begin{align*}
    &- \sum_{c=1}^{a} \sum_{k \geq 0} (-1)^{p(c)+p(a)} U_{\alpha}^{(1)} [-k] U_{\alpha}^{(1)} [k] + \sum_{c=a+1}^{M+N} (\ldots), U_{b+1, b} U_{0, b} [0] \\
    &= (-1)^{p(b)} \left( - \sum_{c=1}^{a} \sum_{k \geq 0} (-1)^{p(c)+p(b)+p(b+1)+p(c)+p(b)} \delta_{ab} U_{\alpha}^{(1)} [-k] U_{\alpha}^{(1)} [k] \right) \\
    &\quad + \sum_{k \geq 0} (-1)^{p(b)} \delta_{k+b+1} U_{\alpha}^{(1)} [-k] U_{\alpha}^{(1)} [k] - \sum_{k \geq 0} (-1)^{p(a)} \delta_{k+b+1} U_{\alpha}^{(1)} [-k] U_{\alpha}^{(1)} [k] \\
    &\quad - (-1)^{p(b+1)} \left( - \sum_{c=1}^{a} \sum_{k \geq 0} (-1)^{p(c)+p(b)+p(b+1)+p(c)+p(b)} \delta_{b+1, a} U_{\alpha}^{(1)} [-k] U_{\alpha}^{(1)} [k] \right) \\
    &\quad + (-1)^{p(b)} \left( - \sum_{c=a+1}^{M+N} \sum_{k \geq 0} (-1)^{p(c)+p(b)+p(b+1)+p(c)+p(b)} \delta_{b+1, c} U_{\alpha}^{(1)} [-k-1] U_{\alpha}^{(1)} [k+1] \right) \\
    &\quad + \sum_{k \geq 0} (-1)^{p(c)} \delta_{c+b+1} U_{\alpha}^{(1)} [-k-1] U_{\alpha}^{(1)} [k+1] - \sum_{k \geq 0} (-1)^{p(c)} \delta_{c+b+1, d} U_{\alpha}^{(1)} [-k-1] U_{\alpha}^{(1)} [k+1] \\
    &\quad - (-1)^{p(b)} \left( - \sum_{c=a+1}^{M+N} \sum_{k \geq 0} (-1)^{p(c)+p(b)+p(b+1)+p(c)+p(b)} \delta_{b+1, a} U_{\alpha}^{(1)} [-k-1] U_{\alpha}^{(1)} [k+1] \right) ,
\end{align*}
\]

(5.32)
and
\[
\sum_{c=1}^{d} \sum_{k \geq 0} (-1)^{p(c)+p(a+1)} T_{a+1}^{(1)} [-k] T_{a+1,a}^{(1)} [k] + \sum_{c=1}^{M+N} \cdots T_{b+1,b}^{(1)} [0]
\]
\[
= -(-1)^{p(b)} \left( \sum_{c=1}^{d} \sum_{k \geq 0} (-1)^{p(c)+p(b)p(c)+p(b)p(b+1)\delta_{a+1,b}} U_{ab}^{(1)} [-k] U_{b+1,c}^{(1)} [k] \right)
\]
\[
- \sum_{k \geq 0} (-1)^{p(a+1)} \delta_{a,b+1} T_{b+1,a+1}^{(1)} [-k] T_{a+1,a}^{(1)} [k] + \sum_{k \geq 0} (-1)^{p(a+1)} \delta_{a,b+1} T_{b+1,a+1}^{(1)} [-k] U_{a+1,b}^{(1)} [k+1]
\]
\[
+ (-1)^{p(b+1)} \left( \sum_{c=1}^{d} \sum_{k \geq 0} (-1)^{p(c)+p(b)p(c)+p(b)p(b+1)\delta_{a+1,b}} \delta_{a,b+1} U_{ab}^{(1)} [-k] T_{b+1,c}^{(1)} [k] \right)
\]
\[
- (-1)^{p(b)} \left( \sum_{c=1}^{M+N} \sum_{k \geq 0} (-1)^{p(c)+p(b)p(c)+p(b)p(b+1)\delta_{a+1,b}} \delta_{a,b+1} U_{cb}^{(1)} [-k - 1] U_{a+1,b}^{(1)} [k+1] \right)
\]
\[
- \sum_{k \geq 0} (-1)^{p(a+1)} \delta_{b+1,a} T_{b+1,a+1}^{(1)} [-k - 1] U_{a+1,b}^{(1)} [k+1]
\]
\[
+ (-1)^{p(b+1)} \left( \sum_{c=1}^{M+N} \sum_{k \geq 0} (-1)^{p(c)+p(b)p(c)+p(b)p(b+1)\delta_{a+1,b}} \delta_{a,b+1} U_{ab}^{(1)} [-k - 1] T_{b+1,c}^{(1)} [k+1] \right).
\]

(5.33)

The four green lines are equal to
\[
\begin{cases}
-\varsigma_{b+1} \text{ orange sums} & +(-1)^{p(b+1)} \delta_{b+1,a} U_{b+1,b}^{(1)} [0] U_{a+1,a}^{(1)} [0], \quad a = b + 1 \\
-\varsigma_{b} \text{ orange sums} & -(-1)^{p(b)} \delta_{a+1,b} U_{a+1,a+1}^{(1)} [0] U_{b+1,b}^{(1)} [0], \quad b = a + 1.
\end{cases}
\]

(5.34)

All the orange terms above then give the corresponding sum in \( \Phi \left( J_{k}^{(b)} \right) \). For the remaining red terms, altogether they become
\[
\begin{cases}
(-1)^{p(b+1)} \delta_{b+1,a} U_{b+1,b}^{(1)} [0] U_{a+1,a}^{(1)} [0], \quad a \geq b + 1 \\
-(-1)^{p(b)} U_{b+1,b}^{(1)} [0] U_{a+1,a}^{(1)} [0] + (-1)^{p(b+1)} U_{a+1,a+1}^{(1)} [0] U_{b+1,b}^{(1)} [0], \quad a = b \\
-(-1)^{p(b)} \delta_{a+1,b} U_{a+1,a+1}^{(1)} [0] U_{b+1,b}^{(1)} [0], \quad a < b - 1.
\end{cases}
\]

(5.35)

Since it would be more convenient to consider \( \widetilde{H}_{k}^{(a)} \), the \( U_{a+1,a+1}^{(1)} [0] U_{a}^{(1)} a[0] \) term gets cancelled and we are left with \( \frac{1}{2} \left( U_{a+1,a+1}^{(1)} [0] \right)^{2} + \left( U_{a+1,a+1}^{(1)} [0] \right)^{2} \). Then
Notice that this is not claiming that (5.36) shows the injectivity. However, this map does not seem to be well-defined when including some cases of BCD types) answers conjecture 2 in [the presentation in proposition] as we will see shortly, this is actually very natural on the quiver Yangian side (not just due to without choosing a reference

Therefore, we may view the quiver Yangian as some sort of ' algebra (see [34]). When a and/or b equal(s) zero, this can be verified in the same manner. Therefore, we would not write it explicitly here. Notice that the condition on $\epsilon_{\pm}$ would come from the cases when a and/or b equal(s) zero.

We have shown that $\Phi$ is a homomorphism, and we still need to check its surjectivity. This can be done by showing that any element $U_{ab}^{(1)}[m]$ ($s = 1, 2, a b = 1, \ldots, M + N, m \in \mathbb{Z}$) can be expressed in terms of $\Phi(H)$, $\Phi(E)$ and $\Phi(F)$. We shall omit the proof here as these generators will be constructed from $\Phi$ explicitly in the next subsection.

**Remark 4.** From theorem 5.4, we can see that the universal enveloping algebras of $\mathcal{W}_{M/N \times l}$ are essentially truncations of the quiver Yangians, that is,

$$Y/\ker(\Phi) \cong U(\mathcal{W}_{M/N \times l}).$$

(5.38)

Therefore, we may view the quiver Yangian as some sort of 'U($\mathcal{W}_{M/N \times l}$)' algebra (see [34])\(^{10}\). This allows us to apply our knowledge in BPS algebras to VOAs and vice versa.

At first glance, one might wonder whether we could write such surjective homomorphism without choosing a reference $a = 0$ so that the map would become more 'uniform'. However, as we will see shortly, this is actually very natural on the quiver Yangian side (not just due to the presentation in proposition 5.3), especially when discussing the crystal melting models.

### 5.1.1 Coproduct and parabolic induction.

As the coproduct for the quiver Yangians is obtained in section 3, we can then consider the parabolic induction for the $\mathcal{W}$-algebras. In other words, given representations $R_1, R_2$ of $U(\mathcal{W}_{M/N \times l})$, we have $R_1 \otimes R_2$ as a representation of $U(\mathcal{W}_{M/N \times (l_1 + l_2)})$. In particular, the study in [63] (see also [64] for non-super cases including some cases of BCD types) answers conjecture 2 in [34].

\(^{10}\) Notice that this is not claiming that $\Phi$ becomes an isomorphism when taking the limit $l \to \infty$. It still requires to show the injectivity. However, this map does not seem to be well-defined when $l$ diverges, and the factors $l$ cannot be fully absorbed under redefinitions of the generators in these expressions. One might think of taking $x \to 0$ as another possible way to bypass this divergence, but some properties of $\Phi$, such as surjectivity, rely on $x \neq 0$. 

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We are not adding extra labels. As studied in \[ (5.40) \]
truncations have generalized Miura/pseudo-differential operators of different types. In terms of \((p, q)\)-brane webs, certain stacks of D3s are stretched in different regions, indicating the multiplicities of smooth components of these divisors. See for example figure 6 in

\[ \eta_{\beta} (E_{ij}[m]) = E_{ij}[m] + \delta_{m,0} \delta_{ij,\beta} \] (5.39)
for some complex number \( \beta \). This yields an algebra automorphism

\[ \eta_{\beta}^{\otimes l} \eta_{\beta} \in \text{Aut} \left( U \left( \mathfrak{g}(M[N])_{x} \right) \right). \] (5.40)

Using (C.20), we have

\[ \eta_{\beta}^{\otimes l} \left( U^{(1)}_{ij}[m] \right) = U^{(1)}_{ij}[m] + \delta_{m,0} \delta_{ij,\beta}, \]
\[ \eta_{\beta}^{\otimes l} \left( U^{(2)}_{ij}[m] \right) = U^{(2)}_{ij}[m] + (l - 1) \alpha U^{(1)}_{ij}[m] + \frac{1}{2} \delta_{m,0} \delta_{ij}(l - 1)(\beta^2 - x \beta). \] (5.41)

To relate the parabolic induction with the coproduct of quiver Yangians, let us take \( l = l_1 + l_2 \) and \( k + l(M - N) = k_1 + l_1(M - N) = k_2 + l_2(M - N) \) such that \( x \) remains the same for \( \mathcal{W}_{M[N \times l]} \) and \( \mathcal{W}_{M[N \times l_1] \otimes \mathcal{W}_{M[N \times l_2]} \) that splits (C.18) into two pieces of sizes \( l_1 \) and \( l_2 \) (see also (5.5) in [34]). As computed in [34, 63], we have\(^{11}\)

\[ \Delta_{l_1, l_2} \left( U^{(1)}_{ij}[m] \right) = U^{(1)}_{ij}[m] \otimes 1 + 1 \otimes U^{(1)}_{ij}[m], \]
\[ \Delta_{l_1, l_2} \left( U^{(2)}_{ij}[m] \right) = U^{(2)}_{ij}[m] \otimes 1 + 1 \otimes U^{(2)}_{ij}[m] + \sum_{c=1}^{M+N} \sum_{k \in \mathbb{Z}} U^{(1)}_{ij}[k] \otimes U^{(2)}_{ij}[m - k] - (m + 1) l_1 \alpha 1 \otimes U^{(1)}_{ij}[m]. \] (5.42)

Let us also define the map \( \Delta_{l_1, l_2} = \left( \text{id}^{\otimes l_1} \otimes n_{-l_1, x}^{\otimes l_2} \right) \circ \Delta_{l_1, l_2} \). Then following the same proof as in [63], we have the commutative diagram

\[ \begin{array}{ccc}
\mathcal{Y} & \phi_1 & \to & U(\mathcal{W}_{M[N \times l]}) \\
\downarrow & & \downarrow \Delta & \\
\mathcal{Y} \otimes \mathcal{Y} & \phi_{l_1} \otimes \phi_{l_2} & \to & U(\mathcal{W}_{M[N \times l_1]}) \otimes U(\mathcal{W}_{M[N \times l_2]}),
\end{array} \] (5.43)

where we have labeled \( \Phi \) with subscripts \( l \) and \( l_{1,2} \) for clarity.

### 5.1.2. More general truncations

As studied in [30, 31, 46] for the \( C^3 \) case and [34] for any generalized conifold, there exist larger families of truncations of the \( \mathcal{W} \)-algebras. These truncations, which are dictated by the functions \( x^h y^j z^k w^l \in \mathbb{C}[x, y, z, w] / \langle xy = z^m w^n \rangle \), can be built from \( x \)-, \( y \)-, \( z \)- and \( w \)-algebras associated to different divisors in the CY3. In particular, the \( x \)-algebra just corresponds to the Miura operator of form (C.18). More generally, these truncations have generalized Miura/pseudo-differential operators of different types.

In terms of \((p, q)\)-brane webs, certain stacks of D3s are stretched in different regions, indicating the multiplicities of smooth components of these divisors. See for example figure 6 in

\(^{11}\) We are not adding extra labels \( l \) and \( l_{1,2} \) to these \( U^{(l)}_{ij} \) as it should be clear which elements belong to which parts.
[34] for an illustration of the patterns of these elementary truncated algebras. The web diagram encodes the loci in the base where the $T^2$ part of the fiber degenerates to a circle in the (resolved) CY threefold. The complex coordinates that are used in the moment maps parametrizing the base can be grouped into variables $x,y,z,w$. This gives rise to the perspective of a 4d $\mathcal{N} = 4$ gauge theory which is divided into a junction of four interfaces (or three for the $\mathbb{C}^3$ case).

The generators for the elementary building blocks of truncations can be found in [34]. Here, we would just like to mention that with

$$l = \frac{(N-M)\epsilon_1 - \epsilon_2)I_3 + \epsilon_2l_2 - \epsilon_1l_4 + \epsilon_1l_1}{(N-M)\epsilon_1 - \epsilon_2},$$

(5.44)

the same argument as in theorem 5.4 indicates that we have this surjective homomorphism from the quiver Yangian to the general $x^{(i)}y^{(j)}z^{(k)}w^{(l)}$-algebra, which reflects the feature of the VOAs as truncations. Notice that now the generators $U^{(1)}_{ij}$ and $U^{(2)}_{ab}$ are given in [34, (5.47)], satisfying the same OPEs as before.

5.2. Crystal melting

For quiver Yangians, the crystal melting representations encodes the information of BPS spectra with many salient features. Let us first give a quick recap on crystal melting. The crystal model is a 3d uplift of the periodic quiver. The atoms in the crystal are in one-to-one correspondence with the gauge nodes in the quiver while the bifundamental/adjoint arrows are the ‘chemical bonds’ connecting these atoms. Different atoms associated with different gauge nodes are then in different ‘colours’.

Let us pick an initial atom $o$ in the periodic quiver as the first atom being molten from the empty room configuration. From the above map $\Phi$, a natural choice would be an atom having color $a = 0$. All the other atoms are then arranged level by level following the arrows/chemical bonds, at the positions of the corresponding nodes in the periodic quiver. There could be more than one paths from one atom to another, but they should be equivalent due to the F-term relations. This gives rise to the path algebra $\mathbb{C}Q/(\partial W)$.

To recover the BPS counting, the crystal configurations should obey the melting rule. It states that an atom $a$ appears in the molten crystal $\mathcal{C}$ if there exists an arrow $l$ such that $l \cdot a \in \mathcal{C}$.

When considering the action of quiver Yangians on the crystal modules, it would be convenient to use the currents

$$\psi^{(a)}(z) = \sum_{n=-1}^{\infty} \frac{\psi^{(a)}_n}{z^{n+1}}, \quad e^{(a)}(z) = \sum_{n=0}^{\infty} \frac{e^{(a)}_n}{z^{n+1}}, \quad f^{(a)}(z) = \sum_{n=0}^{\infty} \frac{f^{(a)}_n}{z^{n+1}}. \quad (5.45)$$

We can write down the OPEs for these currents from the commutation relations that define the quiver Yangians. They can be found for instance in [1, 8]. In particular, $e^{(a)}$ should act like a creation operator that add atoms of color $a$ to the molten crystal while $f^{(a)}$ annihilates atoms in the crystal configuration. By analyzing how atoms can be added and removed following the melting rule, we can obtain the actions of these currents on any state $|\mathcal{C}\rangle$. Let us write $a \in \mathcal{C}_+$ (resp. $a \in \mathcal{C}_-$) such that $|\mathcal{C}\rangle$ would become $|\mathcal{C} + a\rangle$ (resp. $|\mathcal{C} - a\rangle$) after the atom $a$ with color $a$ is added to (resp. removed from) $\mathcal{C}$. From [1], we learn that the actions are

12 Recall that we have set $\psi^{(a)}_{-1} = 1/\epsilon_4$. Notice that this mode expansion here is only for symmetric quivers. For more general cases, there can be non-trivial $\psi^{(a)}_{n}$ with $n < 0$. 

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\[ \psi^{(a)}(z) | \mathcal{C} \rangle = \Psi^{(a)}_\epsilon(z) | \mathcal{C} \rangle, \]  
\[ \epsilon^{(a)}(z) | \mathcal{C} \rangle = \sum_{a \in \mathcal{E}_+} \frac{\pm \sqrt{(-1)^{b|\text{Res}_{\tilde{\epsilon}(a)} \Psi^{(a)}_\epsilon(u)}}}{z - \tilde{\epsilon}(a)} | \mathcal{C} + a \rangle, \]  
\[ f^{(a)}(z) | \mathcal{C} \rangle = \sum_{a \in \mathcal{E}_-} \frac{\pm \sqrt{\text{Res}_{\tilde{\epsilon}(a)} \Psi^{(a)}_\epsilon(u)}}{z - \tilde{\epsilon}(a)} | \mathcal{C} - a \rangle, \]  
where

\[ \Psi^{(a)}_\epsilon(z) := \left( \frac{z + C}{z} \right)^{\delta_{a,1}} \prod_{b \in \mathcal{Q}_0} \prod_{b \in \mathcal{E}} \phi^{b \rightarrow a}(z - \tilde{\epsilon}(b)), \]  
\[ \phi^{b \rightarrow a}(z) = \prod_{l \in a \rightarrow b} \frac{(z + \tilde{\epsilon}_l)}{(z - \tilde{\epsilon}_l)}, \]  
\[ \tilde{\epsilon}(a) = \sum_{l \in \text{path}[a \rightarrow a]} \tilde{\epsilon}_l. \]

Here, the numerical constant \( C \) is the vacuum charge. For the cases considered in this paper whose toric CY3 have no compact divisors, it can be identified as the central term \( \sum_{a \in \mathcal{Q}_0} \psi^{(a)}_\epsilon \). Later, we shall relate this to the parameters on the \( W \)-algebra side. The \( \pm \) signs in the actions depend on the statistics of the quiver Yangian (see [1, section 6] for more details). Recall that the charge assignment \( \tilde{\epsilon}_i \) should satisfy the constraints discussed in section 2 (and hence leads to the two parameters \( \epsilon_1, \epsilon_2 \) in the algebra). The action of each mode on the crystal can then be obtained via the corresponding contour integral around \( \infty \).

As argued in [1, 4], the truncations of quiver Yangians lead to truncated crystal configurations where the melting would stop at one or more atoms. Therefore, with the map \( \Phi \), we may consider the truncated crystals as modules of \( U(V_{\mathcal{M}[N \times l]}) \).

To get the actions of \( U^{(s)}_\psi \) on the truncated crystals, we need to know how they can be expressed using the map \( \Phi \). Here, let us find such expressions for all the elements at spin \( s = 1, 2 \), which essentially gives the proof of the surjectivity of \( \Phi \).

First, let us consider \( U^{(1)}_{ab}[m] \) for any \( a, b = 1, \ldots, M + N \) and \( m \in \mathbb{Z} \). The zero modes of \( \mathcal{H}, \mathcal{E}, \mathcal{F} \) already gives \( U^{(1)}_{a, a+1}[0], U^{(1)}_{a+1, a}[0] \) (with two exceptions) and \( U^{(1)}_{aa}[0] - U^{(1)}_{a+1,a+1}[0] \). It is immediate to obtain \( U^{(1)}_{aa}[0] - U^{(1)}_{bb}[0] \) for any \( a, b \). Using the commutation relations

\[ \begin{align*}
(-1)^p(a+1) & \begin{bmatrix}
U^{(1)}_{a,a+1} \quad U^{(1)}_{a+1,a} \\
U^{(1)}_{a+1,a+2} \quad U^{(1)}_{a+2,a+1}
\end{bmatrix} = U^{(1)}_{a+2,a+2} \\
(-1)^{p(a)}(p(a+1)+p(a+2)+p(a)+2) & \begin{bmatrix}
U^{(1)}_{a+1,a+1} \quad U^{(1)}_{a+1,a+2} \\
U^{(1)}_{a+2,a+1} \quad U^{(1)}_{a+2,a+2}
\end{bmatrix} = U^{(1)}_{a+1,a+2}
\end{align*} \]  

iteratively, we can get \( U^{(1)}_{ab}[0] \) for any \( a \neq b \) (including \( U^{(1)}_{1,M+N}[0] \) and \( U^{(1)}_{M+N,1}[0] \)). This is consistent with the fact that \( e^{(a)}_0 \) (and likewise for \( f^{(a)}_0 \)) is of form \([ \ldots, e^{(a)}_0, e^{(a)}_0, e^{(a)}_0, \ldots, e^{(a)}_0 ] \) when \( \alpha = \alpha^{(a)} + \ldots + \alpha^{(a)} \). Now, using \( \Phi(e^{(a)}_0) \) and \( \Phi(f^{(a)}_0) \), we can write
\[
\begin{align*}
\left\{ U^{(1)}_{a,M+N}, a \right\} & = \left\{ (-1)^{p(a)} U^0_{M+N,1}, U^{(1)}_{a,1} \right\}, \\
U^{(1)}_{a,M+N} & = (-1)^{p(M+N) + p(a)p(1)} \left[ U^{(1)}_{a,M+N}, U^{(1)}_{a,1} \right], \tag{5.53}
\end{align*}
\]

for \( a \neq M + N \). Hence, via
\[
\begin{align*}
U^{(1)}_{ab} & = (-1)^{p(M+N)p(a)+p(M+N)p(b)} \left[ U^{(1)}_{a,M+N}, U^{(1)}_{b,M+N} \right], \\
U^{(1)}_{ab} & = (-1)^{p(M+N)} \left[ U^{(1)}_{a,M+N}, U^{(1)}_{b,M+N} \right], \tag{5.54}
\end{align*}
\]

we can get \( U^{(1)}_{ab} \) for any \( a \neq b \). Keep this procedure, and we can obtain \( U^{(1)}_{ab} \) for any \( a \neq b \) and \( m \in \mathbb{Z} \).

For elements of spin 1, we are now left with \( U^{(1)}_{ab} \). Take
\[
X_{ab}[m] := U^{(1)}_{ab} [m], U^{(2)}_{bb} [0] - U^{(2)}_{b+1,b+1} [0] \nonumber
\]
\[
- \sum_{k \geq 0} U^{(1)}_{bb} [-k] U^{(1)}_{bb} [k] + \sum_{k \geq 0} U^{(1)}_{b+1,b+1} [-k] U^{(1)}_{b+1,b+1} [k+1] \tag{5.55}
\]

for \( a \neq b \). We are allowed to use this commutation relation because of \( \Phi \left( \hat{a}^{(b)} \right) \). A straightforward computation yields
\[
X_{ab}[m] = (-1)^{p(b)} U^{(1)}_{ab} [m] + (-1)^{p(b)} U^{(1)}_{ab} [m] U^{(1)}_{b+1,b+1} [0] + \sum_{k \geq 0} (-1)^{p(b)} U^{(1)}_{ab} [-k] U^{(1)}_{ab} [m+k]. \tag{5.56}
\]

Therefore,
\[
\begin{align*}
\left[ U^{(1)}_{ab} [1], (-1)^{p(b)} X_{ab}[m] \right] & = \left[ U^{(1)}_{ab} [1], (-1)^{p(b)} X_{ab}[m] \right] \\
& = (-1)^{p(b)} U^{(1)}_{ab} [m] - \delta_{m,0} (-1)^{p(b)} X_{ab}[m] \nonumber
\end{align*}
\]
\[
\begin{align*}
& \quad - \delta_{m,0} l(-1)^{p(a)} U^{(1)}_{ab} [m] \nonumber
\end{align*}
\]
\[
\begin{align*}
& \quad + \delta_{m,0} (r(-1)^{p(a)} U^{(1)}_{ab} [m] + (-1)^{p(b)} U^{(1)}_{ab} [m] U^{(1)}_{b+1,b+1} [0] - (-1)^{p(b)} U^{(1)}_{ab} [0] U^{(1)}_{ab} [m]). \tag{5.57}
\end{align*}
\]

Notice that most terms appeared in the calculation get cancelled since \( m = 1 \) is taken out. When \( m > 0 \), \( 5.57 \) is equal to
\[
-(-1)^{p(a)} U^{(1)}_{ab} [m] - l(-1)^{p(a)} U^{(1)}_{ab} [m] + (-1)^{p(b)} U^{(1)}_{ab} [m] U^{(1)}_{b+1,b+1} [0] - (-1)^{p(b)} U^{(1)}_{ab} [0] U^{(1)}_{ab} [m]. \tag{5.58}
\]

As a result, we obtain \( U^{(1)}_{ab} [m] \) for \( m > 0 \). This is likewise for \( m < 0 \). When \( m = 0 \), \( 5.57 \) is
\[
-(-1)^{p(a)} U^{(1)}_{ab} [0] - l(-1)^{p(a)} U^{(1)}_{ab} [0] + (-1)^{p(b)} U^{(1)}_{ab} [0] U^{(1)}_{b+1,b+1} [0] - (-1)^{p(b)} U^{(1)}_{ab} [0] U^{(1)}_{ab} [m]. \tag{5.59}
\]

This in particular gives
\[
-(-1)^{p(a)} U^{(1)}_{ab} [0] - l(-1)^{p(a)} U^{(1)}_{ab} [0] + (-1)^{p(b)} U^{(1)}_{ab} [0] U^{(1)}_{b+1,b+1} [0]. \tag{5.59}
\]

Together with \( U^{(1)}_{ab} [0] = U^{(1)}_{b+1,b+1} [0] \), we can get \( U^{(1)}_{ab} [0] \).

Now, we shall consider the elements of spin 2. From \( \Phi(\hat{a}^{(a)}) \), \( \Phi(\hat{c}^{(a)}) \), and \( \Phi(\hat{f}^{(a)}) \), we get \( U^{(2)}_{a,a+1} [0] \), \( U^{(2)}_{a+1,a+1} [0] \) (with two exceptions) and \( U^{(2)}_{a+1,a+1} [0] - U^{(2)}_{a,a+1} [0] \). Similar to the case of \( U^{(1)}_{ab} [m] \), we can then obtain \( U^{(2)}_{ab} [m] \), as well as \( U^{(2)}_{ab} [m] - U^{(2)}_{ab} [m] \), for any \( a \neq b \) and \( m \in \mathbb{Z} \) using the \( U^{(1)}(U^{(2)}) \) commutation relation.
To get the remaining elements $U^{(2)}_{ab}[m]$, let us compute
\[
(U^{(2)}_{ai}[1] - U^{(2)}_{bb}[1], U^{(2)}_{ai}[m] - U^{(2)}_{bb}[m]) - \left[ U^{(2)}_{ai}[1] - U^{(2)}_{bb}[1], U^{(2)}_{ai}[m + 1] - U^{(2)}_{bb}[m + 1] \right]
\]
(5.60)
for $a \neq b$. This could be tedious due to all the $U^{(2)}_{ab}[m]$ commutation relations, but we notice that most of the terms can be cancelled, and it becomes
\[
-2\varepsilon(-1)^{p(a)}U^{(2)}_{ab}[m + 1] - 2\varepsilon(-1)^{p(b)}U^{(2)}_{bb}[m + 1]
\]
\[
+(l - 1) \left( (1 + (-1)^{p(a)}) \left( \sum_{k > 0} U^{(1)}_{ba}[-k]U^{(1)}_{ab}[m + 1 + k] + \sum_{k > 0} U^{(1)}_{aa}[m + 1 - k]U^{(1)}_{aa}[k] \right) \right)
\]
\[
+(1 + (-1)^{p(b)} \left( \sum_{k > 0} U^{(1)}_{bb}[-k]U^{(1)}_{bb}[m + 1 + k] - \sum_{k > 0} U^{(1)}_{aa}[m + 1 - k]U^{(1)}_{bb}[k] \right)
\]
\[-(1)^{p(b)} \sum_{k > 0} U^{(1)}_{ab}[-k]U^{(1)}_{ab}[m + 1 - k] - (1)^{p(a)} \sum_{k > 0} U^{(1)}_{ba}[m + 1 - k]U^{(1)}_{ba}[k]
\]
\[-(1)^{p(a)} \sum_{k > 0} U^{(1)}_{ba}[-k]U^{(1)}_{ba}[m + 1 + k] - (1)^{p(b)} \sum_{k > 0} U^{(1)}_{ba}[m + 1 - k]U^{(1)}_{ba}[k]
\]
\[-(l - 1)(m + 2)\varepsilon^2 \left( (1)^{p(a)}U^{(1)}_{aa}[m + 1] + (1)^{p(b)}U^{(1)}_{bb}[m + 1] \right) \right].
\]
(5.61)
From this, we get $(-1)^{p(a)}U^{(2)}_{aa}[n] + (-1)^{p(b)}U^{(2)}_{bb}[n]$. Choose $a, b$ such that $p(a) = p(b)$ (which is always possible for the cases we focus on in this paper). Together with $U^{(2)}_{aa}[n] - U^{(2)}_{bb}[n]$, we can obtain $U^{(2)}_{ab}[n]$ for any $a = 1, \ldots, M + N$ and $n \in \mathbb{Z}$.

Now, we can in principle write the actions of any $U^{(2)}_{ab}[m]$ on the (truncated) crystals. Since the crystal configuration always starts from the empty state $|\emptyset\rangle$ on which only the $e_0^{(0)}$ and $\psi_0^{(0)}$ modes would have non-trivial action. It is natural to wonder whether the truncated crystal can be a highest weight representation of $U(\mathcal{W}_{M+N|L})$ with $|\emptyset\rangle$ being the highest weight vector. In particular, all the modes $U^{(2)}_{ab}[m]$ with $s \in \mathbb{Z}_+, a, b = 1, \ldots, M + N$ and $m > 0$ should annihilate the highest weight state. To make this state unique, we also need $U^{(1)}_{ab}[0]$ to act trivially on it for all $a > b$ in our convention here.

**Corollary 5.4.1.** The truncated crystal is a highest weight representation of $U(\mathcal{W}_{M+N|L})$ with the empty room configuration $|\emptyset\rangle$ as the highest weight state.

**Proof.** As pointed out in [35], following the OPEs, it suffices to show that $U^{(1)}_{a+1,a}[0]$, $U^{(1)}_{1,M+N}[1]$ and $U^{(1)}_{r+1,r}[1]$ ($r = 1, 2, 3$) would annihilate $|\emptyset\rangle$. This is obvious for $U^{(1)}_{a+1,a}[0]$ since it is mapped from $f_0^{(a)} = f_0^{(a)}$ ($a \neq 0$) which acts trivially on $|\emptyset\rangle$. Likewise, $U^{(1)}_{r+1,r}[1]$ comes from $f_0^{(0)} = f_0^{(0)}$.

To show that the remaining elements have zero eigenvalues, we first notice that $U^{(1)}_{a,b}[m]|\emptyset\rangle = 0$ for any $a > b$ and $m > 0$. This follows from the commutation relations we
used to construct these elements when showing the surjectivity of $\Phi$ above. On the other hand, $(U^{(l)}_{ba} |0) - U^{(l)}_{b+1,b+1} |0) = 0$ for $b \neq 0$. This is because $\psi^{(a)}_0 |\Omega\rangle = \delta_{a,0} C |\Omega\rangle$ from the action of $\psi^{(a)}(u)$. Besides, we have $\mathcal{H}_1^{(a)} |\Omega\rangle = \frac{1}{b} \nu(b) e_{-\delta_{a,b}} C |\Omega\rangle$ since $\psi^{(a)}_1 |\Omega\rangle = 0$. With these results, we find that $X_{ab}[m] |\Omega\rangle = 0$ for $a > b$ and $m = 0, 1$, where $X_{ab}[m]$ is defined in (5.55). Therefore,

$$
\left[ U^{(l)}_{ba} |0\rangle, (-1)^{p(b)} X_{ab}[m] \right] = \left[ U^{(l)}_{ba} |1\rangle, (-1)^{p(b)} X_{ab}[m-1] \right] |\Omega\rangle
$$

$$
= -(-1)^{p(a)} X_{ab}[m] U^{(l)}_{ba} |0\rangle |\Omega\rangle + (-1)^{p(a)} X_{ab}[m-1] U^{(l)}_{ba} |1\rangle |\Omega\rangle
$$

(5.62)

from (5.57) for $b < a$.

Let us take $a = 3$ and $b = 1$. Then $U^{(l)}_{13} |\Omega\rangle \propto \left[ U^{(l)}_{12} |0\rangle, U^{(l)}_{23} |0\rangle \right] |\Omega\rangle = 0$ since the only non-trivial $e^{(c)}_0 |\Omega\rangle$ is $c = 0$. This indicates that the first term in (5.62) vanishes. Repeat the similar procedure iteratively, and we find that $U^{(l)}_{1, M+N} |0\rangle |\Omega\rangle = 0$. Suppose $M + N \neq 3$, then $U^{(l)}_{13} |1\rangle |\Omega\rangle \propto U^{(l)}_{1, M+N} |1\rangle, U^{(l)}_{M+N, 3} |0\rangle |\Omega\rangle = 0$. If $M + N = 3$, then $U^{(l)}_{13} |1\rangle |\Omega\rangle = 0$ is automatic since this comes from $\mathcal{F}^{(0)}_0 |\Omega\rangle = 0$. As a result, the second term in (5.62), and hence the whole equation, would vanish.

Now, take $m = 1$. Recall that the action in (5.62) can be written using (5.57). As the latter two terms in (5.57) vanish when acting on $|\Omega\rangle$ as we have shown, the only two terms that survive under $m = 1$ would give a term proportional to $U^{(l)}_{11} |1\rangle$. Since (5.62) vanishes, we have $U^{(l)}_{11} |1\rangle |\Omega\rangle = 0$. Moreover, $U^{(l)}_{1,a} |\Omega\rangle \propto \left[ U^{(l)}_{11} |1\rangle, U^{(l)}_{1,0} |0\rangle \right] |\Omega\rangle = 0$ for $a > 1$. Keep this procedure, and we find that $U^{(l)}_{1d} |m\rangle |\Omega\rangle = 0$ for any $m > 0$. Together with the expression of $\Phi \left( \mathcal{F}^{(0)}_1 \right)$, we have $U^{(l)}_{1, M+N} |1\rangle |\Omega\rangle = 0$. Likewise, by considering $\mathcal{F}^{(a)}_2$ from $\mathcal{H}^{(a+1)}_1, \mathcal{F}^{(a)}_1$, we can get $U^{(l)}_{1, M+N} |1\rangle |\Omega\rangle = 0$ following the same steps. Notice that this uses the general $U^{(l)}_{ab} U^{(l)}_{cd}$ commutation relation which is not explicitly listed here, but it is straightforward from the OPE in [34].

**Remark 5.** Notice that the algebra only has generators $U^{(n)}$ up to level $l$. In other words, $U^{(a>b)}$ are automatically zero because the Miura operator reads

$$
\mathcal{L}^{(1)} |\Omega\rangle = \cdots \mathcal{L}^{(l)} |\Omega\rangle = (\mathcal{a} \partial + \mathcal{J}^{(1)}) \cdots (\mathcal{a} \partial + \mathcal{J}^{(l)}) = (\mathcal{a} \partial)^l + U^{(l)}(\mathcal{a} \partial)^{l-1} + \cdots + U^{(l)},
$$

(5.63)

where the highest $U$ terminates at $l$. See [34] for more details. Note that the superscripts here are not the superscripts $x, y, z, w$ used in the notations in [34]. In terms of the $x, y, z, w$ labels therein, we are only considering the $x$-algebra here. For the other types of truncations ($y, z, w$), since $U^{(n>l)}$ do not vanish due to the pseudo-differential operators, one further needs to show that $U^{(n>l)}$ annihilate all the states in the truncated crystal, and we will study this in our future work.

When considering $\mathcal{H}_0^{(a \neq 0)} |\Omega\rangle = \psi^{(a)}_0 |\Omega\rangle = 0$, we can see that $U^{(l)}_{11} |\Omega\rangle = U^{(l)}_{22} |\Omega\rangle = \cdots = U^{(l)}_{M+N,M+N} |\Omega\rangle$. On the other hand, $(U^{(l)}_{M+N,M+N} |0\rangle - U^{(l)}_{11} |0\rangle + 1) |\Omega\rangle = \mathcal{H}_0^{(0)} |\Omega\rangle = C |\Omega\rangle$. This then relates the parameter on the $\mathbf{W}$-algebra side with the vacuum charge in the quiver Yangian:
Corollary 5.4.2. We have \( C = l \sigma \).}

5.2.1. Example. Let us illustrate the above discussions with an example. The simplest case would be \( C \times C^2 / \mathbb{Z}_3 \) whose quiver and crystal model are depicted in figure 4. The possible configurations with the corresponding modes acting on \( | \varphi \rangle \) at low levels are listed in [1]. For instance, take \( | \mathcal{C} \rangle \) to be

\[
| \mathcal{C} \rangle = e^{(2)}_0 e^{(0)}_0 | \varphi \rangle = \quad \text{(image of a crystal configuration)}.
\] (5.64)

One possible way to add an atom is to act \( e^{(1)}_0 \) from (5.47) above, we have

\[
\psi^{(1)}_e = \frac{(z + \epsilon_{10})(z - \epsilon_{02} + \epsilon_{12})}{(z - \epsilon_{01})(z - \epsilon_{02} - \epsilon_{21})},
\] (5.65)

where we have kept the notation \( \epsilon_I \). Therefore,

\[
E^{(1)}_0 | \mathcal{C} \rangle = -\left( \frac{(\epsilon_{10} + \epsilon_{02} + \epsilon_{21})(\epsilon_{12} + \epsilon_{21})}{\epsilon_{01} - \epsilon_{02} - \epsilon_{21}} \right)^{1/2} | \mathcal{C} + 1 \rangle,
\] (5.66)

where we have taken \( + \) in \( \pm \) in (5.47). This gives

\[
U^{(1)}_{12} [0] | \mathcal{C} \rangle = \frac{3\epsilon_2(\epsilon_1 + \epsilon_2)}{\epsilon_1 - 2\epsilon_2} | \mathcal{C} + 1 \rangle,
\] (5.67)

where

\[
| \mathcal{C} + 1 \rangle = \quad \text{(image of a crystal configuration)}.
\] (5.68)

One may apply the procedure above used for constructing other elements and get their actions. Here, we list a few examples for \( U^{(1)}_{ij}[m] \):

\[
| \mathcal{C} \rangle \quad U^{(1)}_{22}[0] \quad U^{(1)}_{11}[0] \quad U^{(1)}_{12}[0] \quad U^{(1)}_{21}[0] \quad U^{(1)}_{12}[1] \quad U^{(1)}_{21}[1] \quad U^{(1)}_{21}[0] \quad U^{(1)}_{22}[1] \quad U^{(1)}_{22}[0] \quad U^{(1)}_{11}[1] \quad U^{(1)}_{22}[1]
\] (5.69)

where we omit the coefficients and only show the crystal configurations. By considering higher modes of the quiver Yangian generators, we can also get the actions of \( U^{(l)} \) with higher spins.

Let us take a brief look at how the truncations of the crystal could happen. Here, we shall only discuss the simplest example which truncates the algebra at \( l = 1 \). In such case, we just have the universal enveloping algebra of the Kac–Moody algebra with only zero modes for the quiver Yangian, or equivalently, only spin 1 elements for the \( \mathcal{W} \)-algebra. It is straightforward to see that the truncated crystal has the shape

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of semi-infinite length. We simply have $U^{(s>2)}_{ab}[m]|C⟩ = 0$ for any $a, b \in \{1, 2, 3\}$, $m \in \mathbb{Z}$ and any configuration $C$ as $U^{(1)}_{ab}[m]$ vanishes for $s > l$.

In general, for any quiver Yangian we focus on in this paper, the truncation at the very first level $l = 1$ can be described in this manner. For more general truncations of the crystal and larger $l$, this could be more involved, and we leave this to future work (see also section 6).

6. Outlook

We have discussed a few aspects of the quiver Yangians for most generalized conifolds and their connections to vertex algebras. Let us mention some possible directions for further research.

The construction of the coproduct we have benefits from the untwisted affine Lie superalgebra of A-type. A natural extension would be a more thorough study on quiver Yangians for the remaining generalized conifolds (i.e. $(M, N) = (2, 0), M = N$), as well as $\mathbb{C}^3/(\mathbb{Z}_2 \times \mathbb{Z}_2)$. All of them have underlying Kac–Moody algebras. It is worth noting that a method of computing the coproduct perturbatively is given in [9, Appendix C] for generalized conifolds. For the cases with $M = N$, due to the vanishing Killing forms, we would probably need to consider the algebra $gl(M|M)$. This could be similar to the Khoroshkin–Tolstoy approach [65, 66]. More generally, for toric CYs with compact 4-cycles, the quiver Yangians do not seem to have such underlying Kac–Moody algebras. It would be desirable to find the coproduct of the algebra associated to any quiver.

13 This is also the situation for $D(2, 1; \alpha)$ and $osp(2N+2|2N)$. The former is associated to the quiver Yangian for $\mathbb{C}^3/(\mathbb{Z}_2 \times \mathbb{Z}_2)$. For the latter case, we do not have associated quiver Yangians so far. It would also be interesting to see whether there can be similar (BPS) algebras for the orthosymplectic cases.
The coproduct plays an important role when studying the Bethe/gauge correspondence from the BPS quiver algebra [8, 9]. In particular, there are obstructions for the Bethe/gauge correspondence to work in the chiral quiver cases as pointed out in [9], and this requires further investigations. Similar to the map from the quiver Yangian to the \(W\)-algebra, it would be natural to wonder whether we can write some \(RTT\)-like presentation of the quiver Yangian, which might in turn shed light on the Bethe/gauge correspondence.

One may also consider the trigonometric and elliptic versions of the quiver Yangians [6]. They bear resemblance to the rational ones by means of the generators and their relations. However, the coproduct structures seem to behave very different.

We showed that the quiver Yangians discussed in this paper are isomorphic algebras under toric duality by virtue of the odd reflections of the Kac–Moody superalgebras. We expect similar isomorphic maps for quiver Yangians associated to the CYs without compact divisors that are not explored here. However, new methods are required when considering toric CYs with compact divisors. It is natural to conjecture that such quiver Yangians would still be isomorphic as their supersymmetric gauge theories are related by Seiberg duality. It could even be possible to consider the quivers outside the toric phases which can still be reached via Seiberg duality [67]. We can always define the corresponding quiver Yangians from the quiver data though whether/how they would implement the BPS algebras would need further checks. Mathematically, Seiberg dual quivers essentially transform under mutations. Therefore, cluster algebras might be useful in proving the isomorphisms.

Regarding the truncations and VOAs, we have only explicitly discussed the crystal configurations here for \(l = 1\). As analyzed in [1], when the crystal is truncated at some atom, we have the corresponding residue vanishing in the numerator of (5.47). This leads to some extra conditions that \(\epsilon_I\) should satisfy on the quiver Yangian side. On the other hand, recall that we also have certain condition on \(\epsilon_I\) for \(\Phi\) to be homomorphic, and the truncation comes from the parameter \(l\) on the \(W\)-algebra side. We expect that the cut at \(l\) would not provide all the possible truncations of the crystal. It is very likely that the coefficients in the actions of some \(U\)-modes become zero due to the truncation conditions on \(\epsilon_I\) from the quiver Yangians. Besides, there are more general truncations, namely the \(x^i y^j z^k w^l\)-algebras, as mentioned above. They might also give possible truncations on the crystal. Moreover, it would be interesting to see whether other crystal configurations, such as crystals in other chambers [68–70] and 2d crystals [8, 9, 71], could give similar relations.

It still remains an open question whether the quiver Yangians for more general geometry, especially those associated to toric CYs with compact divisors, could have some \(W\)-algebras as their truncations. It might be possible to construct the VOAs from the quiver Yangians in this setting and compare them with other constructions. This could provide more insights into the BPS/CFT correspondence.

**Data availability statement**

The data that support the findings of this study are available upon reasonable request from the authors. The data that support the findings of this study are available upon reasonable request from the authors.

**Acknowledgment**

The research is supported by a CSC scholarship.
Appendix A. Generators of quiver Yangians for $\mathbb{C} \times \mathbb{C}^2/\mathbb{Z}_2$ and conifold

Analogous to the cases discussed in the main context, the generators of the quiver Yangians with two gauge nodes can also be expressed using finitely many modes. The main difference is that there are more than one pairs of arrows connecting the two nodes in the quiver. See figure 3. Below we shall use the generators in the convention of $\psi$, $e$ and $f$.

A.1. Case 1: $\mathbb{C} \times \mathbb{C}^2/\mathbb{Z}_2$

We have

$$\sigma_1^{ab} = \sigma_1^{ba} = \begin{cases} \epsilon_1 + \epsilon_2 = -\epsilon_3, & a \neq b, \\ \epsilon_3, & a = b. \end{cases} \quad \sigma_2^{ab} = \sigma_2^{ba} = \begin{cases} \epsilon_1 \epsilon_2, & a \neq b \\ 0, & a = b. \end{cases} \quad (A.1)$$

The defining relations of the quiver Yangian are

$$[\psi_n^{(a)}, \psi_m^{(b)}] = 0, \quad (A.2)$$

$$e_n^{(a)} f_m^{(b)} = \delta_{ab} e_{n+m}^{(a)}, \quad (A.3)$$

$$[\psi_n^{(a)}, e_{m+1}^{(a)}] = [\psi_n^{(a)}, f_{m+1}^{(a)}] = \epsilon_3 \left\{ \psi_n^{(a)}, e_m^{(a)} \right\}, \quad (A.4)$$

$$[\psi_n^{(a)}, e_{m+1}^{(a)}] = -\epsilon_3 \left\{ \psi_n^{(a)}, f_m^{(a)} \right\}, \quad (A.5)$$

$$[e_n^{(a)}, e_{m+1}^{(a)}] = [e_n^{(a)}, f_{m+1}^{(a)}] = \epsilon_3 \left\{ e_n^{(a)}, e_m^{(a)} \right\}, \quad (A.6)$$

$$[e_n^{(a)}, f_{m+1}^{(a)}] = -\epsilon_3 \left\{ f_n^{(a)}, e_m^{(a)} \right\}, \quad (A.7)$$

$$[\psi_n^{(a)}, e_{m+1}^{(a+1)}] = -\epsilon_1 \epsilon_2 \left\{ \psi_n^{(a)}, e_{m+1}^{(a+1)} \right\} - \epsilon_3 \left\{ \psi_n^{(a)}, e_{m+1}^{(a+1)} \right\} + \epsilon_3 \left\{ \psi_n^{(a)}, e_{m+1}^{(a+1)} \right\}, \quad (A.8)$$

$$[\psi_n^{(a)}, f_{m+1}^{(a+1)}] = -\epsilon_1 \epsilon_2 \left\{ \psi_n^{(a)}, f_{m+1}^{(a+1)} \right\} - \epsilon_3 \left\{ \psi_n^{(a)}, f_{m+1}^{(a+1)} \right\} + \epsilon_3 \left\{ \psi_n^{(a)}, f_{m+1}^{(a+1)} \right\}, \quad (A.9)$$

$$[e_n^{(a)} e_{m+1}^{(a+1)}] = -2 \left\{ e_n^{(a)}, e_{m+1}^{(a+1)} \right\} + \left\{ e_n^{(a)}, e_{m+2}^{(a+1)} \right\}, \quad (A.10)$$

$$[e_n^{(a)} f_{m+1}^{(a+1)}] = -2 \left\{ e_n^{(a)}, f_{m+1}^{(a+1)} \right\} + \left\{ e_n^{(a)}, f_{m+2}^{(a+1)} \right\}, \quad (A.11)$$

$$[\psi_n^{(a)}, e_{m+1}^{(a+1)}] = -\epsilon_3 \left\{ \psi_n^{(a)}, e_{m+1}^{(a+1)} \right\} + \epsilon_3 \left\{ \psi_n^{(a)}, e_{m+1}^{(a+1)} \right\}, \quad (A.12)$$

Recall that $\psi_n^{(a)} = 1$. Take $n = -2$ in (A.8), and we have

$$[\psi_0^{(a)}, e_m^{(a+1)}] = -2 \epsilon_3 e_m^{(a+1)}, \quad \text{i.e.} \quad [\psi_0^{(a)}, e_m^{(a+1)}] = -2 \epsilon_3 e_m^{(a+1)}. \quad (A.13)$$

Then take $n = -1$ in (A.8), and we get

$$[\psi_1^{(a)}, e_m^{(a+1)}] = -2 \left\{ \psi_0^{(a)}, e_m^{(a+1)} \right\} = -\epsilon_3 \left\{ \psi_0^{(a)}, e_m^{(a+1)} \right\} + 2 \epsilon_3 e_m^{(a+1)} \quad (A.14)$$

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The defining relations of the quiver Yangian are

\[ e_{m+1}^{(a+1)} = -\frac{1}{6\epsilon_3} \left[ \psi_1^{(a)} \cdot e_m^{(a+1)} \right] + \frac{1}{6} \psi_0^{(a)} e_m^{(a+1)} + \frac{1}{6} e_m^{(a+1)} \psi_0^{(a)} \]

\[ = -\frac{1}{6\epsilon_3} \left[ \psi_1^{(a)} \cdot e_m^{(a+1)} \right] - \frac{1}{6} \psi_0^{(a)} e_m^{(a+1)} \left[ \psi_0^{(a)} + e_m^{(a+1)} \right] \psi_0^{(a)} \]

\[ = -\frac{1}{6\epsilon_3} \left[ \psi_1^{(a)} \cdot e_m^{(a+1)} \right] + \frac{1}{2} \left( \psi_0^{(a)} \right)^2 e_m^{(a+1)} \]  \hspace{1cm} (A.15)

and likewise for \( f_0^{(a)} \). Define \( \tilde{\psi}_1^{(a)} = \psi_1^{(a)} + \frac{1}{2} \left( \psi_0^{(a)} \right)^2 \) (notice the difference with the namesake for the cases in the main context). As a result, all the modes can be obtained from \( \psi_0^{(a)} \), \( \psi_1^{(a)} \), \( e_0^{(a)} \) and \( f_0^{(a)} \) (\( a \in Q_0 \)) inductively via

\[ e_{m+1}^{(a)} = \frac{1}{6(\epsilon_1 + \epsilon_2)} \left[ \tilde{\psi}_1^{(a)} \cdot e_m^{(a)} \right], \quad f_{m+1}^{(a)} = -\frac{1}{6(\epsilon_1 + \epsilon_2)} \left[ \tilde{\psi}_1^{(a)} \cdot f_m^{(a)} \right], \]

\[ \psi_{m+1}^{(a)} = \left[ e_{m+1}^{(a)} \cdot e_0^{(a)} \right]. \hspace{1cm} (A.16) \]

### A.2. Case 2: conifold

We have

\[ \sigma_1^{ab} = \sigma_1^{ba} = 0, \quad \sigma_2^{01} = -\epsilon_1, \quad \sigma_2^{10} = -\epsilon_2, \quad \sigma_2^{aa} = 0. \hspace{1cm} (A.17) \]

The defining relations of the quiver Yangian are

\[ \left[ \psi_n^{(a)} \cdot \psi_m^{(b)} \right] = 0, \hspace{1cm} (A.18) \]

\[ \left[ e_n^{(a)} \cdot f_m^{(b)} \right] = \delta_{ab} \psi_{m+n}^{(a)}, \hspace{1cm} (A.19) \]

\[ \left[ \psi_{n+2}^{(a+1)} \cdot e_m^{(a+1)} \right] - 2 \left[ \psi_{n+1}^{(a)} \cdot e_m^{(a+1)} \right] + \left[ \psi_n^{(a)} \cdot e_{m+2}^{(a+1)} \right] = -\sigma_2^{ba} \psi_n^{(a)} \cdot e_m^{(a+1)} + \sigma_2^{ab} \psi_n^{(a)} \cdot e_m^{(a+1)} \hspace{1cm} (A.20) \]

\[ \left[ \psi_{n+2}^{(a+1)} \cdot f_m^{(a+1)} \right] - 2 \left[ \psi_{n+1}^{(a)} \cdot f_m^{(a+1)} \right] + \left[ \psi_n^{(a)} \cdot f_{m+2}^{(a+1)} \right] = -\sigma_2^{ba} \psi_n^{(a)} \cdot f_m^{(a+1)} + \sigma_2^{ab} \psi_n^{(a)} \cdot f_m^{(a+1)} \hspace{1cm} (A.21) \]

\[ \left[ e_n^{(a)} \cdot e_{m+2}^{(a+1)} \right] - 2 \left[ e_{n+1}^{(a)} \cdot e_{m+1}^{(a+1)} \right] + \left[ e_n^{(a)} \cdot e_{m+2}^{(a+1)} \right] = -\sigma_2^{ba} \psi_n^{(a)} \cdot e_m^{(a+1)} + \sigma_2^{ab} \psi_n^{(a)} \cdot e_m^{(a+1)} \hspace{1cm} (A.22) \]
One can equivalently write the quiver Yangian using
\[ \sigma_2^{ab} f_n^{(a)} f_m^{(a+1)} = -\sigma_2^{ab} f_n^{(a)} f_m^{(a+1)} = -\sigma_2^{ab} f_n^{(a)} f_m^{(a+1)} f_n^{(a)}, \]
(A.23)
\[ \text{Sym}_{\alpha_1, \alpha_2} \text{Sym}_{\alpha_1, \alpha_2} \left[ f_{\alpha_1}, \left[ f_{\alpha_1}, f_{\alpha_1} \right] \right] \]
\[ = \text{Sym}_{\alpha_1, \alpha_2} \text{Sym}_{\alpha_1, \alpha_2} \left[ f_{\alpha_1}, \left[ f_{\alpha_1}, f_{\alpha_1} \right] \right] = 0. \]
(A.24)

Recall that we simply use \([\cdot, \cdot]\) to denote the supercommutator in our convention here. Take \( n = -2 \) in (A.20), and we have
\[ \left[ \psi_0^{(0)}, e_m^{(1)} \right] = 0. \]
(A.25)

Then take \( n = -1, 0 \) in (A.20), and we get
\[ \left[ \psi_1^{(0)}, e_m^{(1)} \right] = \left( \epsilon_1 - \epsilon_0 \right) e_m^{(1)}, \]
\[ \left[ \psi_2^{(0)}, e_m^{(1)} \right] = \left( \epsilon_2 - \epsilon_0 \right) \psi_1^{(0)} e_m^{(1)}. \]
(A.26)
\[ \left[ \psi_2^{(0)}, e_m^{(1)} \right] = 2 \psi_1^{(0)} e_{m+1}^{(1)} = \left( \epsilon_2 - \epsilon_0 \right) \psi_1^{(0)} e_m^{(1)} = \left[ \psi_0^{(0)}, e_m^{(1)} \right] \psi_1^{(0)} e_m^{(1)} = \left[ \psi_0^{(0)}, \psi_1^{(0)} e_m^{(1)} \right] = \left[ \psi_0^{(0)}, \psi_1^{(0)} e_m^{(1)} \right] = \left[ \psi_0^{(0)}, \psi_1^{(0)} e_m^{(1)} \right], \]
(A.27)

and likewise for \( a = 1 \), as well as \( f \). Define \( \overline{\psi}_2^{(a)} = \psi_2^{(a)} - \psi_1^{(a)} \psi_1^{(a)} \). As a result\(^{14} \), all the modes can be obtained from \( \psi_0^{(a)}, \psi_1^{(a)}, \psi_2^{(a)}, e_0^{(a)} \) and \( f_0^{(a)} (a \in \mathbb{Q}_2) \) inductively via
\[ e_0^{(1)} = \frac{1}{2(\epsilon_2 - \epsilon_0)} \left[ \psi_0^{(0)}, e_0^{(1)} \right], \quad e_1^{(1)} = \frac{1}{2(\epsilon_2 - \epsilon_0)} \left[ \psi_0^{(0)}, \psi_1^{(0)} e_m^{(1)} \right], \quad e_2^{(1)} = \frac{1}{2(\epsilon_2 - \epsilon_0)} \left[ \psi_0^{(0)}, \psi_2^{(0)} e_m^{(1)} \right], \quad e_3^{(1)} = \frac{1}{2(\epsilon_2 - \epsilon_0)} \left[ \psi_0^{(0)}, \psi_3^{(0)} e_m^{(1)} \right]. \]
\[ e_0^{(0)} = \frac{1}{2(\epsilon_2 - \epsilon_0)} \left[ \psi_0^{(0)}, e_0^{(0)} \right], \quad e_1^{(0)} = \frac{1}{2(\epsilon_2 - \epsilon_0)} \left[ \psi_0^{(0)}, \psi_1^{(0)} e_m^{(0)} \right], \quad e_2^{(0)} = \frac{1}{2(\epsilon_2 - \epsilon_0)} \left[ \psi_0^{(0)}, \psi_2^{(0)} e_m^{(0)} \right], \quad e_3^{(0)} = \frac{1}{2(\epsilon_2 - \epsilon_0)} \left[ \psi_0^{(0)}, \psi_3^{(0)} e_m^{(0)} \right]. \]
(A.28)

Unlike all the cases discussed above, we further need the extra \( e_2^{(a)} \) for the conifold case.

A.3. Comments on \( \mathbb{C}^3/(\mathbb{Z}_2 \times \mathbb{Z}_2) \)

Let us also briefly mention the quiver Yangian for \( \mathbb{C}^3/(\mathbb{Z}_2 \times \mathbb{Z}_2) \) here, which is the only toric CY3 without compact divisors that is not a generalized conifold. Its toric diagram and quiver are depicted in figure 6. The generators \( \psi_0^{(a)}, e_0^{(a)} \) and \( f_0^{(a)} \) of the quiver Yangian (whose underlying algebra is the affine \( D(2,1;\alpha) \)) satisfy the same defining relations as in definition 2.1. One can equivalently write the quiver Yangian using \( \psi_0^{(a)}, e_0^{(a)} \) and \( f_0^{(a)} \). Notice that the only vanishing \( e_0^{(ab)} \) in this case would have \( a = b \). It is straightforward to see that the quiver Yangian has a minimalistic presentation as given in theorem 3.1.

\(^{14} \) As \( \psi_0^{(a)} \) commutes with \( e_0^{(a+1)} \) and \( f_0^{(a+1)} \), it could be possible to add terms such as \( \frac{1}{4} \left( \psi_0^{(a)} \right)^3 \) to the definition of \( \overline{\psi}_2^{(a)} \), which might be more convenient when finding a minimalistic presentation.
Appendix B. Odd reflections and Chevalley generators

Let us verify that the primed Chevalley generators in (4.3) satisfy the corresponding relations of the Kac–Moody superalgebras, which are essentially the ones involving only zero modes in theorem 3.1. For convenience, (4.3) is reproduced here:

\[
\psi'^{(a)}_0 = \sum_{b=1}^{M+N} R_{ab} \psi'^{(b)}_0 = \begin{cases} 
-\psi_0^{(a)}, & a = F, \\
\psi_0^{(a)} + \psi_0^{(F)}, & a = F \pm 1, \\
\psi_0^{(a)}, & \text{otherwise}; 
\end{cases} \\
e'^{(a)}_0 = \begin{cases} 
\frac{1}{\sigma^{(a)}_0} f_0^{(a)}, & a = F, \\
\frac{f_0^{(a)} f_0^{(F)}}{\sigma^{(a)}_0} - \frac{1}{\Lambda_{ab}} \left[ f_0^{(a)} f_0^{(F)} \right], & a = F \pm 1, \\
f_0^{(a)}, & \text{otherwise}; 
\end{cases}
\]

We also recall that

\[
A'_{ab} = \begin{cases} 
-A_{ab}, & (a, b) = (F \pm 1, F), (F, F \pm 1), \\
A_{ab} + 2A_{abF}, & a = b = F \pm 1, \\
A_{ab}, & \text{otherwise}. 
\end{cases}
\]

The \(\psi'\psi'\) relation, the \(e'^{(a)}e'^{(b)}\), \(f'^{(a)}f'^{(b)}\) relations for \(\sigma^{ab}_1 = 0\) and the Serre relations whose right hand sides are all zero follow from direct computations. Now, let us check the \(e'f'\) relation:

- \(a = b = F\): We have

\[
\left[ e'^{(a)}_0, f'^{(b)}_0 \right] = - \left[ f_0^{(F)}, e_0^{(F)} \right] = -\psi_0^{(F)} = \psi_0^{(a)}. 
\]
\[ a = b = F \pm 1 : \text{We have} \]
\[
\begin{align*}
\left[ e_0^{(a)}, e_0^{(b)} \right] &= -\frac{1}{A_{af}} \left[ \left[ e_0^{(F)}, e_0^{(a)} \right], \left[ e_0^{(F)}, e_0^{(b)} \right] \right] \\
&= -\frac{1}{A_{af}} \left( \left[ e_0^{(F)}, e_0^{(a)} \right] - (-1)^{|a|} \left[ e_0^{(a)}, e_0^{(F)} \right] \right) - 0 \\
&= A_{ab} \left( e_0^{(F)}, e_0^{(a)} \right) - (-1)^{|a|} A_{af} \left( e_0^{(a)}, e_0^{(F)} \right) + A_{af} \left[ e_0^{(F)}, e_0^{(a)} \right] \\
&= (2A_{af} + A_{ab}) e_0^{(a)},
\end{align*}
\]  

(B.4)

where we have used the Jacobi identity in the second and third equalities.

- \( a = F, b = F \pm 1 \): We have \( e_0^{(a)} e_0^{(b)} = -\frac{1}{A_{af}} \left[ f_0^{(F)}, f_0^{(b)} \right] = 0 \) as \( A_{af}^2 = 0 \).

- \( a = F \pm 1, b = F \pm 1 \): This is similar to the case when \( a = b = F \pm 1 \). By using the Jacobi identity, we have \( e_0^{(a)} e_0^{(b)} = -\frac{1}{A_{af}} \left[ e_0^{(a)}, f_0^{(b)} \right] = 0 \) using the Jacobi identity.

- otherwise: The remaining cases are immediate following the expressions of the primed generators and the similar arguments as above.

Next, we shall check the \( \psi' e' \) relation:

- \( a = b = F \pm 1 \): We have
\[
\begin{align*}
\left[ \psi_0^{(a)}, e_0^{(b)} \right] &= \left[ \psi^{(a)}, \psi_0^{(b)} \right] \\
&= -\left[ e_0^{(F)}, \psi_0^{(b)} \right] + (-1)^{|a|} \left[ e_0^{(a)}, \psi_0^{(F)} \right] - 0 \\
&= A_{ab} \left[ e_0^{(F)}, e_0^{(a)} \right] - (-1)^{|a|} A_{af} \left[ e_0^{(a)}, e_0^{(F)} \right] + A_{af} \left[ e_0^{(F)}, e_0^{(a)} \right] \\
&= A_{ab} e_0^{(b)},
\end{align*}
\]  

(B.5)

where we have used the Jacobi identity in the second equality.

- \( a = F, b = F \pm 1 \): We have
\[
\begin{align*}
\left[ \psi_0^{(a)}, e_0^{(b)} \right] &= -\left[ \psi_0^{(a)}, e_0^{(F)}, e_0^{(b)} \right] \\
&= -\left[ e_0^{(F)}, A_{bf} e_0^{(b)} \right] = -A_{bf} e_0^{(b)} = A_{ab} e_0^{(b)}.
\end{align*}
\]  

(B.6)

- \( a = F \pm 1, b = F \pm 1 \): This is similar to the case when \( a = b = F \pm 1 \). By using the Jacobi identity, we have \( \psi_0^{(a)} e_0^{(b)} = -(-1)^{|a|} A_{bf} \left[ e_0^{(a)}, e_0^{(b)} \right] = 0 \).

- \( a \neq F, b = F \pm 1 \): We have \( \psi_0^{(a)}, e_0^{(b)} = \psi^{(a)}, e_0^{(F)}, e_0^{(b)} = 0 \) using the Jacobi identity.

- \( a = b = F \): We have \( \psi_0^{(a)} e_0^{(b)} = \psi^{(a)} e_0^{(F)} = 0 \).

- \( a = F \pm 1, b = F \): We have \( \psi_0^{(a)}, e_0^{(b)} = \psi^{(a)} e_0^{(F)} = 0 \).

- otherwise: The remaining cases are immediate following the expressions of the primed generators and the similar arguments as above.
The $\psi'f'$ relation can be verified in the same manner. Nevertheless, let us explicitly write the proof for three of the cases here:

- $a = F \pm 1, b = F$: We have
  \[
  \left[ \psi_0^{(a)}, f_0^{(b)} \right] = \left[ \psi_0^{(a)} + \psi_0^{(F)}, -e_0^{(F)} \right] = -A_{af} e_0^{(F)} = A_{ab} f_0^{(b)} = -A_{ab} f_0^{(b)}. \tag{B.7}
  \]

- $a = F, b = F \pm 1$: We have
  \[
  \left[ \psi_0^{(a)}, f_0^{(b)} \right] = \frac{1}{A_{af}} \left[ -\psi_0^{(F)}, -\left[ f_0^{(b)}, f_0^{(F)} \right] \right] = \frac{1}{A_{af}} \left[ -A_{bf} f_0^{(b)} f_0^{(F)} \right] = A_{af} f_0^{(b)} = -A_{ab} f_0^{(b)}. \tag{B.8}
  \]

- $a = b = F \pm 1$: We have
  \[
  \left[ \psi_0^{(a)}, f_0^{(b)} \right] = -\frac{1}{A_{af}} \left[ \psi_0^{(a)} + \psi_0^{(F)}, \left[ f_0^{(a)}, f_0^{(F)} \right] \right]
  = \frac{1}{A_{af}} \left( \left[ \left[ f_0^{(a)}, f_0^{(F)} \right], \psi_0^{(a)} \right] + \left[ f_0^{(a)}, f_0^{(F)} \right], \psi_0^{(a)} \right) \right)
  = \frac{1}{A_{af}} \left( -\left[ f_0^{(a)}, f_0^{(F)} \right], \psi_0^{(a)} \right) - (-1)^{|a|} \left[ f_0^{(F)}, \left[ f_0^{(a)}, \psi_0^{(a)} \right] \right]
  \times + 0 - (-1)^{|a|} \left[ f_0^{(F)}, \left[ f_0^{(a)}, \psi_0^{(a)} \right] \right]
  = \frac{1}{A_{af}} \left( A_{af} f_0^{(F)} f_0^{(a)} - (-1)^{|a|} A_{aa} f_0^{(F)} f_0^{(a)} - (-1)^{|a|} A_{af} f_0^{(F)} f_0^{(a)} \right)
  = -2A_{af} + A_{aa} \times \left( \frac{1}{A_{af}} \left[ f_0^{(F)}, f_0^{(a)} \right] \right)
  = -A_{ab} f_0^{(b)}. \tag{B.9}
  \]

**Appendix C. Rectangular $\mathcal{W}$-algebras**

In literature, the rectangular $\mathcal{W}$-algebra (of type A) is often defined based on the distinguished case for the Kac–Moody superalgebra where the number of fermionic nodes is minimized. Nevertheless, we can certainly consider $\mathcal{W}$-algebras with any underlying root systems/Dynkin diagrams as we are going to discuss now. Proposition 5.2 then ensures that they are isomorphic for a given generalized conifold.

In this appendix only, we will use $\mathfrak{g}$ to denote the algebra $\mathfrak{gl}(\mathcal{M}/\mathcal{N})$ for some positive integer $l$. We shall choose the convention such that the basis matrix $E_{ij}$ has entry $(-1)^{p(i,j)}$ at position $(i,j)$ with all other elements being zero\(^{15}\). Notice that we have used

\[
p(i) = \begin{cases} 
0, & i \text{ is bosonic} \\
1, & i \text{ is fermionic} 
\end{cases} \tag{C.1}
\]

\(^{15}\) Another convention often adopted in literature such as [54] would naturally be 1 at entry $(i,j)$. 

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so as to distinguish it from \(|a|\) in the quiver Yangians. Given a parity sequence \(\varsigma\) composed of \((-1)^{p(i)}\), the \(\mathbb{Z}_2\)-grading of \(E_{ij}\) is \(p(i) + p(j)\). In particular, \(E_{i,j} E_{i,j} = (-1)^{p(i) + p(j)} \delta_{i,j} E_{i,j}\) and \(\text{str}(E_{i,j} E_{i,j}) = (-1)^{p(i) + p(j)} \delta_{i,j} \delta_{i,j}\). Then
\[
\mathfrak{g} = \bigoplus_{1 \leq i,j \leq M+N} \mathbb{C} E_{(i-1)(M+N)+i,(i-1)(M+N)+j},
\]
(C.2)
and \(E_{(r-1)(M+N)+i,(r-1)(M+N)+j} = E_{ij} \otimes E_{rs}\) as \(\mathfrak{g}\) is isomorphic to \(\mathfrak{gl}(M|N) \otimes \mathfrak{gl}(l)\) as a vector space. We shall take the bosonic nilpotent matrix \(x^- = \sum_{i=1}^{M+N} E_{i(M+N)+1,(i-1)(M+N)+i}\) which can be written as \(\left(\sum_{i=1}^{M+N} E_{0} \otimes \left(\sum_{j=1}^{l-1} E_{s,j}-1\right)\right)\). This nilpotent matrix is of Jordan type with the rectangle Young tableau \((r^M|N)\) (and hence the name rectangular \(W\)-algebra). Given a complex number \(k\), there is an inner product of \(\mathfrak{g}\) given by
\[
(u|v) = \begin{cases}
\text{str}(uv), & u \in \mathfrak{sl}(M|N) \text{ or } v \in \mathfrak{sl}(M|N) \\
\text{str}(uv) + (-1)^{p(i)+p(j)}(1 - c), & u = E_{ij} \otimes E_{rs} \text{ and } v = E_{ij} \otimes E_{st}
\end{cases}
\]
(C.3)
for some \(c \in \mathbb{C}\).

Now, \(\mathfrak{g}\) has a good grading in the sense of [41] for the nilpotent element with
\[
\mathfrak{g}_r := \bigoplus_{0 \leq i \leq M+N, 0 \leq r \leq l-1} \mathbb{C} E_{(i-1)(M+N)+r(M+N)+j},
\]
(C.4)
We then have an \(\mathfrak{sl}_2\) triple \((h,x^+,x^-)\) such that \(\mathfrak{g}_r = \{y \in \mathfrak{g}| [h,y] = ry\}\). Define the subalgebras \(\mathfrak{b} = \mathfrak{g}_{<0} = \bigoplus_{r \geq 0} \mathfrak{g}_r\) and \(\mathfrak{g}_{=0} = \mathfrak{g}_{=0}\). We have an inner product on \(\mathfrak{b}\) which reads
\[
\kappa(u,v) = (u|v) + \frac{1}{2} (\kappa_{\mathfrak{g}}(u,v) - \kappa_{\mathfrak{b}_r}(\text{pr}(u),\text{pr}(v)))
\]
(C.5)
for any \(u,v \in \mathfrak{b}\), where \(\kappa_{\mathfrak{g}}\) (resp. \(\kappa_{\mathfrak{b}_r}\)) is the Killing form on \(\mathfrak{g}\) (resp. \(\mathfrak{g}_{=0}\)) and \(\text{pr} : \mathfrak{b} \rightarrow \mathfrak{g}_{=0}\) is the projection map. Recall that in general, the Killing form is \(\kappa_{\mathfrak{gl}(M|N)}(x,y) = 2(M-N) \text{str}(xy) - 2\text{str}(x)\text{str}(y)\). Then
\[
\kappa(E_{1(M+N)+1,1(M+N)+j}, E_{2(M+N)+1,2(M+N)+j}) = \delta_{r_1,r_2} \delta_{i_1,j_1} \delta_{i_2,j_2} (-1)^{p(i)+p(j)} \delta_{i_1,j_2} \delta_{i_2,j_1} (\delta_{r_1,r_2} - c),
\]
(C.6)
where \(c := k + (l - 1)(M-N)\). Consider the affinization \(\tilde{\mathfrak{b}} = \mathfrak{b} \rtimes \mathbb{C}\) (with \(1\) central). The commutation relation reads \([at^m, bt^n] = [a,b]t^{m+n} + \delta_{m,-n} m \kappa(a,b)1\). The associated (universal affine) vertex algebra \(V(\mathfrak{b})\) is defined to be \(U(\mathfrak{b})/U(\mathfrak{b})(\mathfrak{b}[t] \otimes \mathbb{C}(1-1)) \cong U(\tilde{\mathfrak{b}}) \otimes U(\mathfrak{b}[t] \otimes \mathbb{C})\). Here, \(\mathbb{C}\) denotes the one-dimensional representation of \(\mathfrak{b}[t] \otimes \mathbb{C}\) where \(\mathfrak{b}[t]\) acts trivially as 0 and \(1\) acts as 1. This is isomorphic to \(U(\mathfrak{b} \rtimes \mathbb{C})\) as a vector space by Poincaré–Birkhoff–Witt (PBW) theorem. Here, we shall take the mode expansion of a current \(a(z)\) in the vertex algebra depending on its spin \(s\):
\[
a(z) = \sum_{n \in \mathbb{Z}} a[n] z^{-s-1},
\]
(C.7)

Notice that the \(\mathfrak{gl}(l)\) part (with subscripts \(r,s\)) is always bosonic.
where we have denoted \( a^n \) as \( a[n] \) to avoid potential clutter of subscripts later on. In this paper, we use the normal ordered product with the convention\(^{17} \):

\[
:a(z)b(z): = a(z)\ll b(z) + (-1)^{p(a)+p(b)}b(z)a(z)\rr,
\]

where \( a(z)\ll = \sum_{n \leq -s} \frac{a[n]}{zn+s} \) and \( a(z)\rr = \sum_{n \geq -s} \frac{a[n]}{zn+s} \). (C.8)

In terms of modes, we have

\[
:a[n]b[m] := \begin{cases} a[n]b[m], & n \leq -s \\ (-1)^{p(a)+p(b)}b[m]a[n], & n > -s. \end{cases}
\] (C.9)

In the main context and below, we shall use \((ab)\) instead of \(:ab: \) to denote the normal ordering for convenience when it would not cause confusions. Of course, different conventions of the normal ordered product would not change our results in section 5. For instance, if we ‘split’ the normal ordering at the zero modes, one may check that the homomorphism \( \Phi \) from \( Y \) to \( WV \) would remain the same.

Let us also consider the Lie superalgebra \( a = \bigoplus_{\alpha \in \mathbb{C}} \mathfrak{A}(\alpha) \bigoplus \bigoplus_{\eta \in \mathbb{C}, \varrho < 0} \mathfrak{A}(\alpha) \) with \( p(A(\alpha)) = p(\alpha) \) and \( p(A(\alpha)) = p(\pi) + 1 \). The commutation relations are

\[
\begin{align*}
[A^{(\alpha)}, A^{(\beta)}] &= A^{(\alpha+\beta)}; \\
[A(\alpha), A(\beta)] &= 0,
\end{align*}
\] (C.10)

\[
\begin{align*}
[A^{(E_{\eta i_1})}, A^{(E_{\eta i_2})}] &= \delta_{i_1, i_2} A^{(E_{\eta i_1})} - \delta_{i_1, i_2} (-1)^{(p(\eta)+p(i_1))+(p(\eta)+p(i_2)+1)} A^{(E_{\eta i_1})}.
\end{align*}
\]

Suppose \( u = \sum_{i} a_i u_i \), then \( A(\alpha) \) is \( \sum_i a_i A^{(\alpha)} \) (and similarly for \( A(\alpha) \)). We can write the inner product determined by

\[
\kappa_{\alpha}(A^{(\alpha)}, A^{(\alpha)}) = \kappa_{\beta}(u, v), \quad \kappa_{\alpha}(A^{(\alpha)}, A^{(\alpha)}) = \kappa_{\alpha}(A^{(\alpha)}, A^{(\alpha)}) = 0, \quad (C.11)
\]

and likewise consider the affinization of \( a \). Then the associated vertex algebra \( V^h(a) \) contains \( V^h(b) \) as a subalgebra. Both of the vertex algebras can be regarded as non-associative algebras with respect to the normal ordered product.

The \( WV \)-algebra is then the collection of elements in \( V^h(b) \) that are annihilated by the Becchi-Rouet-Stora-Tyutin (BRST) charge. More specifically, the BRST cohomology has fermionic derivation \( Q : V^h(a) \rightarrow V^h(a) \) commuting with the translation operator \( \partial \) of the vertex algebra. The other commutation relations \( Q \) should satisfy can be found for example in [73, section 3] and in [54, section 3] (with the convention therein).

**Definition C.1.** Given the above data, the rectangular \( WV \)-algebra is \( WV \left( g\ell(M|N), \{i(M|N)\} \right) := \{ v \in V^h(b) \subset V^h(a) | Qv = 0 \} \).

---

\(^{17}\) One can also define the \( n \)-product given by

\[
(a_{\alpha}b)(z) = a(z)(a_{\alpha}b)(z) = \begin{cases} \text{Res}_{w}(w-z)^n[a(w), b(z)], & n \geq 0 \\ \frac{1}{n+1} \partial^n a(z)b(z), & n < 0, \end{cases}
\]

as well as the \( \lambda \)-bracket \( [a_{\lambda}b] = \sum_{n \in \mathbb{Z}} \frac{\lambda^n}{n!} a_{\alpha}b \) which enjoys certain properties such as the noncommutative Wick formula. See for example [72] for more details. The pair of fields is local if \( (a_{\alpha}b)(z) \) vanishes for sufficiently large (positive) \( n \). It is clear that the \((-1)\)-product coincides with the normal ordered product.
Notice that we have omitted the parity sequence \( \varsigma \) in the notation as different \( \varsigma \) give isomorphic \( \mathcal{V} \)-algebras by proposition 5.2. For our discussions, it would be of great help to obtain the generators of the \( \mathcal{V} \)-algebra. This can be constructed by considering the non-associative free algebra \( T(\mathfrak{gl}(l)_{\leq 0}[t^{-1}]^{-1}) \otimes \mathbb{C}[\tau] \) with the even element \( \tau \) commuting with \( 1 \) and \( [\tau, \mathbf{y} m] = -\mathbf{y} m - 1 \) for \( \mathbf{y} \in \mathfrak{gl}(l)_{\leq 0} \). We then have an algebra homomorphism \( \mathcal{T} : T(\mathfrak{gl}(l)_{\leq 0}[t^{-1}]^{-1}) \otimes \mathbb{C}[\tau] \to \mathfrak{gl}(M|N) \otimes \mathcal{V}^k(\mathfrak{b}) \otimes \mathbb{C}[\tau] \) such that
\[
\mathcal{T}(x) = \sum_{i,j=1}^{M+N} (-1)^{\mu(i)\mu(j)} E_{ij} \otimes \mathcal{T}_{ij}(x), \quad \mathcal{T}(\tau) = \tau, \tag{C.12}
\]
where
\[
\mathcal{T}_{ij}(x) = x \otimes E_{ij} \in \mathfrak{gl}(l)_{\leq 0}[t^{-1}]^{-1} \otimes \mathfrak{gl}(M|N) = \mathfrak{b} [t^{-1}]^{-1}.
\]
Since \( \mathcal{T}(xy) = \mathcal{T}(x)\mathcal{T}(y) \), we find that
\[
\mathcal{T}_{ij}(xy) = \sum_{r=1}^{M+N} \mathcal{T}_{ir}(x) \mathcal{T}_{rj}(y). \tag{C.14}
\]

Let us now consider the \( l \times l \) matrix
\[
B = \begin{pmatrix}
\sigma \tau + E_{11}[-1] & -1 & 0 & \cdots & 0 \\
E_{21}[-1] & \sigma \tau + E_{22}[-1] & -1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
E_{r-1,1}[-1] & E_{r-1,2}[-1] & \cdots & \sigma \tau + E_{r-1,r-1}[-1] & -1 \\
E_{rl}[-1] & E_{rl-1}[-1] & \cdots & E_{r,l-1}[-1] & \sigma \tau + E_{rl}[-1]
\end{pmatrix}, \tag{C.15}
\]
and compute its column determinant
\[
cdet(B) = \sum_{\sigma \in \mathcal{O}_l} \text{sgn} \sigma \ b_{\sigma(1)}(b_{\sigma(2)}(b_{\sigma(3)}\cdots(b_{\sigma(l-1)}l b_{\sigma(l),l})\cdots)). \tag{C.16}
\]
As the entries \( b_{\sigma} \) of \( B \) are in \( T(\mathfrak{gl}(l)_{\leq 0}[t^{-1}]^{-1}) \otimes \mathbb{C}[\tau] \), we can write
\[
\mathcal{T}_{ij}(cdet(B)) = \sum_{r=0}^{l} \hat{U}^{(r)}_{ij}(\sigma \tau)^{l-r}. \tag{C.17}
\]
We then have the remarkable results from [54, 73]:

**Theorem C.1.** The rectangular \( \mathcal{V} \)-algebra \( \mathcal{V}^k(\mathfrak{gl}(M|N), (\mathfrak{h}^{(M|N)}) \) is freely generated by \( \hat{U}^{(r)}_{ij} \) for \( 1 \leq r \leq l \) and \( 1 \leq i, j \leq M + N \). Moreover, when \( M \neq N \) and \( M + N \geq 2 \) (and \( \sigma \neq 0 \)), it is generated by \( \hat{U}^{(l)}_{ij} \) and \( \hat{U}^{(l)}_{ij} \).

Following [73, 74], the projection \( \mathfrak{b} \to \mathfrak{l} = (\mathfrak{gl}_{l_0} \otimes \mathfrak{gl}_{M+N}) \) induces an injective algebra homomorphism \( \mu : \mathcal{V}^k(\mathfrak{gl}(M|N), (\mathfrak{h}^{(M|N)}) \to \mathcal{V}^k(\mathfrak{b}) \) known as the (quantum) Miura transformation. Under the Miura transformation, we have
\[
\sum_{r=0}^{l} \mu \left( \hat{U}^{(r)}_{ij} \right) (\sigma \tau)^{l-r} = \mathcal{T}_{ij}((\sigma \tau + E_{11}[-1])(\sigma \tau + E_{22}[-1]) \cdots (\sigma \tau + E_{ll}[-1])). \tag{C.18}
\]
Let us write
\[
\mathcal{J}_{ij} = -E_{(s-1)(M+N)+i,(s-1)(M+N)+j}[{-1}], \quad \partial \mathcal{J}_{ij} = -E_{(s-1)(M+N)+i,(s-1)(M+N)+j}[{-2}]. \tag{C.19}
\]

\[\text{Notice that we could have also started with } -E_{ij} \text{ as our basis matrix from the very beginning.}\]
This gives the same convention as in [34]. The generators of the \( \mathcal{W} \)-algebra can be written as

\[
U^{(1)}_{ij} = \sum_{1 \leq s, \ell \leq l} J_{ij}^{s, \ell}, \quad U^{(2)}_{ij} = \sum_{1 \leq s, \ell \leq l} (s-1) \partial J_{ij}^{s, \ell} + \sum_{1 \leq s, \ell < 2 \leq l, 1 \leq n, M+N} \left( J_{ij}^{n, l} J_{ij}^{n, M+N} \right).
\]  

By definition of the vertex algebra, the OPE of \( J_{ij}^{s, \ell} \) reads

\[
J_{ij}^{s, \ell}(z) J_{ij}^{p, q}(w) \sim \frac{\kappa(E_{i,j} - 1)(M+N) + j_i, j_j, l_j, l_j(M+N) + j_i, j_j, l_j, l_j - 1]}{(z-w)^2} \times \left[ \frac{\kappa(M+N) + j_i, j_j, l_j, l_j(M+N) + j_i, j_j, l_j, l_j - 1]}{z-w} \right] \frac{(z-w)^2}{ \delta_{s, p} \delta_{\ell, q}} \left( -1 \right)^{\rho(j_i,j_j)} \frac{\delta_{\ell, q}}{z-w} J_{ij}^{s, \ell} J_{ij}^{p, q} - \left( -1 \right)^{\rho(j_i,j_j)} \frac{\delta_{\ell, q}}{z-w} J_{ij}^{s, \ell} J_{ij}^{p, q}.
\]  

The OPEs for \( U^{(r)}_{ij} \) can then be obtained from this, as well as the commutation relations for their modes via

\[
\left[ U^{(r)}_{ij} [m], U^{(r)}_{ij} [n] \right] = \frac{1}{(2\pi i)^2} \int_0^1 \int_0^1 \frac{dw}{w} \frac{dz}{z} z^{m+r-1} w^{n+r-1} U^{(r)}_{ij} (z) U^{(r)}_{ij} (w).
\]  

In this paper, we shall focus on the case when the parameter \( c = 0 \). The commutation relations used in this paper are listed in lemma 5.1.

To relate the non-associative \( \mathcal{W} \)- algebra with the quiver Yangian, we shall consider the universal enveloping algebra \( U(\mathcal{V}) \). In general, for any vertex algebra \( V \), its universal enveloping algebra \( U(V) \) is an associative algebra topologically generated by \( uv^m \) (or \( uv^{m+n-1} \) depending on the convention) for \( u \in V \) and \( m \in \mathbb{Z} \) which correspond to the modes \( u[m] \) in the vertex algebra. Therefore, we shall slightly abuse the notation and write \( u[m] \) as well for the elements in \( U(V) \). For more details on vertex algebras and their universal enveloping algebras, see for example [75].

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