Convergence problems along curves for generalized Schrödinger operators with polynomial growth

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Abstract: In this paper we build the relationship between smoothness of the functions and convergence rate along curves for a class of generalized Schrödinger operators with polynomial growth. We show that the convergence rate depends only on the growth condition of the phase function and regularity of the curve. Our result can be applied to a wide class of operators. In particular, convergence results along curves for a class of generalized Schrödinger operators with non-homogeneous phase functions is built and then the convergence rate is established.

Keywords: Schrödinger operator; Convergence; Polynomial growth.

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1 Introduction

Consider the generalized Schrödinger equation

\[
\begin{cases}
\partial_t u(x,t) - iP(D)u(x,t) = 0 & x \in \mathbb{R}^n, t \in \mathbb{R}^+, \\
u(x,0) = f
\end{cases}
\]  

(1.1)

where \( D = \frac{1}{i}(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n}) \), \( P(\xi) \) is a real continuous function defined on \( \mathbb{R}^n \), \( P(D) \) is defined via its real symbol

\[
P(D)f(x) = \int_{\mathbb{R}^n} e^{ix \cdot \xi} P(\xi) \hat{f}(\xi) d\xi.
\]

The solution of (1.1) can be formally written as

\[
e^{itP(D)}f(x) := \int_{\mathbb{R}^n} e^{ix \cdot \xi + itP(\xi)} \hat{f}(\xi) d\xi,
\]

(1.2)

where \( \hat{f} \) denotes the Fourier transform of \( f \).

The Carleson’s problem, that is. to determine the optimal \( s \) for which

\[
\lim_{t \to 0^+} e^{itP(D)}f(x) = f(x)
\]

(1.3)

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almost everywhere whenever \( f \in H^s(\mathbb{R}^n) \), has been widely studied since the first work by Carleson ([2]), later works see [10], [18], [16], [17], [14] and references therein. Sharp results were derived in some cases, such as the elliptic case ([7, 8], when \( n \geq 1 \), \( P(\xi) = |\xi|^2 \)); the non-elliptic case ([9], when \( n \geq 1 \), \( P(\xi) = \xi_1^2 - \xi_2^2 \pm \cdots \pm \xi_n^2 \)) and the fractional case ([3], when \( n \geq 1 \) and \( P(\xi) = |\xi|^\alpha, \alpha > 1 \)).

A natural generalization of the convergence problem is to consider almost everywhere convergence along variable curves instead of vertical lines. Let \( \gamma \) be a continuous function such that

\[
\gamma(x,t) : \mathbb{R}^n \times [0,1] \rightarrow \mathbb{R}^n, \quad \gamma(x,0) = x.
\]

Consider the pointwise convergence problem along the curve \( (\gamma(x,t),t) \), that is, to determine the optimal \( s \) for which

\[
\lim_{t \to 0^+} e^{itP(D)}(f)(\gamma(x,t)) = f(x)
\]  \hspace{1cm} (1.4)

almost everywhere whenever \( f \in H^s(\mathbb{R}^n) \). This problem has been considered by [4–6, 11, 13]. When \( \gamma(x,t) \) is Hölder continuous with respect to \( t \) uniformly for each \( x \) and bilipschitz with respect to \( x \) for each \( t \in [0,1] \), sharp results were obtained by [4] for \( n = 1 \), \( P(\xi) = |\xi|^2 \). For smooth curve \( \gamma(x,t) \in C^1(\mathbb{R}^n \times \mathbb{R}) \), since the corresponding convergence result is equivalent to the pointwise convergence of solutions to the free Schrödinger equation with initial data, the convergence rate follows from [12].

However, how is the relationship between smoothness of the functions and the convergence rate along the curves with less smooth condition? Hence, it is interesting to seek the convergence rate of \( e^{itP(D)} f(x) \) as \( t \) tends to \( 0 \) along the curve \( (\gamma(x,t),t) \) if \( f \) has more regularity. The problem is, suppose that \( e^{itP(D)}(f)(\gamma(x,t)) \) converge pointwisely to \( f(x) \) for \( f \in H^s(\mathbb{R}^n) \) as \( t \) tends to \( 0 \), whether or not it is possible that, for \( f \in H^{s+\delta}(\mathbb{R}^n), \delta \geq 0 \),

\[
e^{itP(D)}(f)(\gamma(x,t)) - f(x) = o(e^{\theta(\delta)})
\]  \hspace{1cm} (1.5)

almost everywhere for some \( \theta(\delta) \geq 0 \)? Cao, Fan and Wang [1] proved this property in the vertical case \( \gamma(x,t) = x \) when \( n \geq 1 \), \( P(\xi) = |\xi|^2 \), and when \( n = 1 \), \( P(\xi) = |\xi|^\alpha, \alpha > 1 \). Authors of this paper improved the results in [1] to general \( P(\xi) \) with polynomial growth. It is proved in [12] that the convergence rate in the vertical case depends only on the growth condition of \( P(\xi) \).

In this paper, we first discuss the convergence rate for a class of Schrödinger operators with polynomial growth along curves. Denote by \( B(x_0,R) \) the ball with center \( x_0 \in \mathbb{R}^n \) and radius \( R \lesssim 1 \). Suppose that \( \gamma(x,t) \) satisfies

\[
|\gamma(x,t) - \gamma(x,t')| \leq C|t - t'|^\alpha, \quad 0 < \alpha \leq 1
\]  \hspace{1cm} (1.6)

uniformly for \( x \in B(x_0,R) \) and \( t, t' \in [0,1] \). Then our first main result is as follows:

**Theorem 1.1.** If there exist \( m \geq 1 \), \( s_0 > 0 \) such that

\[
|P(\xi)| \lesssim |\xi|^m, |\xi| \rightarrow +\infty,
\]  \hspace{1cm} (1.7)
and for each $s > s_0$,

$$
\left\| \sup_{0 < t < 1} |e^{itP(D)} f(\gamma(x,t))| \right\|_{L^p(B(x_0,R))} \lesssim \|f\|_{H^s(\mathbb{R}^n)}, \quad p \geq 1, \quad (1.8)
$$

then for all $f \in H^{s+\delta}(\mathbb{R}^n)$, $0 \leq \delta < m$,

$$
e^{itP(D)} f(\gamma(x,t)) - f(x) = o(t^{\delta/m}), \quad \text{a.e.} \quad x \in B(x_0,R) \quad \text{as} \quad t \to 0^+. \quad (1.9)
$$

Note that the convergence rate in Theorem 1.1 depends on the growth condition of the phase function and the regularity of the curve, but independent of the gradient of the phase function and the dimension of the spatial space. In particular, for non-zero Schwartz functions, the convergence rate seems no faster than $t^a$ as $t$ tends to 0 along the curve $(\gamma(x,t), t)$, see Theorem 2.2 below. Theorem 1.1 is quite general and can be applied to a wide class of operators, such as the non-elliptic Schrödinger operators $(\mathcal{P}(\xi) = \xi_1^2 - \xi_2^2 \pm \cdots \pm \xi_n^2)$, the fractional Schrödinger operators $(\mathcal{P}(\xi) = |\xi|^\alpha, \alpha > 1)$. Therefore, the convergence rate along curves can be deduced from Theorem 1.1 provided that the pointwise convergence along curves is proved.

Next, we consider a class of operators with non-homogeneous phase function and obtained the corresponding convergence result and convergence rate. For convenience, we concentrate ourselves on the case $n = 2$ and consider a class of operators with phase function

$$
\mathcal{P}_{m_1,m_2}(\xi) = \xi_1^{m_1} \pm \xi_2^{m_2},
$$

where $m_1, m_2 \in \mathbb{N}^+$, $2 \leq m_1 \leq m_2$, and a class of curves $\gamma(x,t) : \mathbb{R}^2 \times [0,1] \to \mathbb{R}^2$, $\gamma(x,0) = x$ satisfies

$$
|\gamma(x,t) - \gamma(y,t)| \sim |x-y|, \quad (1.10)
$$

$$
|\gamma(x,t) - \gamma(x,t')| \lesssim |t-t'|^{-1/m_2}, \quad (1.11)
$$

for each $x, y \in B(x_0,R)$ and $t, t' \in [0,1]$. We have the following result:

**Theorem 1.2.** For each $s > 1/2$,

$$
\left\| \sup_{0 < t < 1} |e^{it\mathcal{P}_{m_1,m_2}(D)} f(\gamma(x,t))| \right\|_{L^2(B(x_0,R))} \lesssim \|f\|_{H^s(\mathbb{R}^2)} \quad (1.12)
$$

whenever $f \in H^s(\mathbb{R}^2)$. Moreover, by Theorem 1.1, for all $f \in H^{s+\delta}(\mathbb{R}^2)$, $0 \leq \delta < m_2$,

$$
e^{it\mathcal{P}_{m_1,m_2}(D)} f(\gamma(x,t)) - f(x) = o(t^{\delta/(m_1-1)m_2}), \quad \text{a.e.} \quad x \in B(x_0,R) \quad \text{as} \quad t \to 0^+. \quad (1.13)
$$

In the rest of the introduction, we briefly sketch the proof of Theorem 1.2. We only need to prove (1.12), (1.13) then follows from Theorem 1.1. Recall that Theorem 4.1 in [9] by Kenig-Ponce-Vega shows that for each $s > 1/2$,

$$
\left\| \sup_{0 < t < 1} |e^{it\mathcal{P}_{m_1,m_2}(D)} f| \right\|_{L^2(B(0,1))} \lesssim \|f\|_{H^s(\mathbb{R}^2)} \quad (1.14)
$$

whenever $f \in H^s(\mathbb{R}^n)$. (1.12) follows from (1.14) and Theorem 1.3 below.
Theorem 1.3. If there exists $s_0 > 0$ such that for each $s > s_0$,
\[
\left\| \sup_{0 < t < 1} |e^{itDf}| \right\|_{L^p(B(0,1))} \lesssim \|f\|_{L^\infty(R^d)}
\] (1.15)
whenever $f \in H^s(R^d)$, then for each $s > s_0$,
\[
\left\| \sup_{0 < t < 1} |e^{itDf}(\gamma(t))| \right\|_{L^p(B(x_0, R))} \lesssim \|f\|_{H^s(R^d)}
\] (1.16)
whenever $f \in H^s(R^d)$.

When $m_1 = m_2$, result as in Theorem 1.3 was first obtained by Cho, Lee and Vargas ([4], Proposition 4.3). In order to prove Theorem 1.3, after Littlewood-Paley decomposition, we only need to consider $f$, supp $\hat{f} \subset \{\xi : |\xi| \sim \lambda\}$, $\lambda \gg 1$. We decompose $[0,1]$ into bounded overlap intervals $J = \bigcup_{J \in \mathcal{J}} J$, each $J$ is of length $\lambda^{1-m_1}$. For each $J$, it follows from (1.15) that
\[
\left\| \sup_{t \in J} |e^{itDf}| \right\|_{L^p(B(x_0, R))} \lesssim \lambda^{s_0+\epsilon} \|f\|_{L^2}, \ \forall \epsilon > 0
\] (1.17)
As Lemma 2.2 in [13], inequalities (1.17), (1.10), (1.11) imply
\[
\left\| \sup_{t \in J} |e^{itDf}(\gamma(t))| \right\|_{L^p(B(x_0, R))} \lesssim \lambda^{s_0+\epsilon} \|f\|_{L^2}, \ \forall \epsilon > 0.
\] (1.18)
Inequality (1.16) then follows from (1.18) and a time localizing lemma:

Theorem 1.4. Let $\mathcal{J} = \{J\}$ be a collection of intervals of length $\lambda^{1-m_1}$ with bounded overlap, $[0,1] = \bigcup_{J \in \mathcal{J}} J$. Suppose that for some $\alpha > 0$, $p \geq 2$,
\[
\left\| \sup_{t \in J} |e^{itDf}(\gamma(t))| \right\|_{L^p(B(x_0, R))} \lesssim \lambda^{\alpha} \|f\|_{L^2}
\] (1.19)
provided that supp $\hat{f} \subset \{\xi : |\xi| \sim \lambda\}$, then for any $\epsilon > 0$, we have
\[
\left\| \sup_{0 < t < 1} |e^{itDf}(\gamma(t))| \right\|_{L^p(B(x_0, R))} \lesssim \lambda^{\alpha+\epsilon} \|f\|_{L^2}
\] (1.20)
whenever supp $\hat{f} \subset \{\xi : |\xi| \sim \lambda\}$.

Based on our previous argument, we will omit some details and only prove Theorem 1.4 in Section 3.

2 Proof of Theorem 1.1

We first prove the following Lemma 2.1.

Lemma 2.1. Assume that $g$ is a Schwartz function whose Fourier transform is supported in the annulus $A(\lambda) = \{\xi \in \mathbb{R}^n : |\xi| \sim \lambda\}$. $\gamma(x,t)$ satisfies
\[
|\gamma(x,t) - x| \lesssim t^\alpha, \quad \gamma(x,0) = 0
\]
for all $x \in B(x_0, R)$ and $t \in (0, \lambda^{-\frac{1}{q}})$. Then for each $x \in B(x_0, R)$ and $t \in (0, \lambda^{-\frac{1}{q}})$,

$$|e^{itP(D)}g(\gamma(x, t))| \leq \sum_{l \in \mathbb{Z}^n} \frac{C_n}{(1 + |l|)^{n+1}} \left| \int_{\mathbb{R}^n} e^{it(x + \frac{1}{q}) \xi + itP(\xi)} \hat{g}(\xi) d\xi \right|. \quad (2.1)$$

**Proof.** As in [13], we introduce a cut-off function $\phi$ which is smooth and equal to 1 on $B(0, 2)$ and supported on $(-\pi, \pi)^n$. After scaling we have

$$e^{itP(D)}(g)(\gamma(x, t)) = \lambda^n \int_{\mathbb{R}^n} e^{i\lambda \gamma(x, t) \eta + itP(\lambda \eta)} \phi(\eta) \hat{g}(\lambda \eta) d\eta$$

$$= \lambda^n \int_{\mathbb{R}^n} e^{i\lambda \gamma(x, t) \eta - i\lambda x \eta + itP(\lambda \eta)} \phi(\eta) \hat{g}(\lambda \eta) d\eta. \quad (2.2)$$

Since

$$|\lambda \gamma(x, t) - \lambda x| \lesssim 1,$$

then by Fourier expansion,

$$\phi(\eta)e^{i\lambda|\gamma(x, t) - x| \eta} = \sum_{l \in \mathbb{Z}^n} c_l(x, t) e^{it \eta},$$

where

$$|c_l(x, t)| \lesssim \frac{C_n}{(1 + |l|)^{n+1}}$$

uniformly for each $l \in \mathbb{Z}^n$, $x \in B(x_0, R)$ and $t \in (0, \lambda^{-\frac{1}{q}})$. Then we have

$$|e^{itP(D)}g(\gamma(x, t))| \leq \sum_{l \in \mathbb{Z}^n} \frac{C_n \lambda^n}{(1 + |l|)^{n+1}} \left| \int_{\mathbb{R}^n} e^{it(\eta + i\lambda x \eta + itP(\lambda \eta))} \hat{g}(\lambda \eta) d\eta \right|$$

$$= \sum_{l \in \mathbb{Z}^n} \frac{C_n}{(1 + |l|)^{n+1}} \left| \int_{\mathbb{R}^n} e^{it(x + \frac{1}{q} \xi + itP(\xi)} \hat{g}(\xi) d\xi \right|,$$

then we arrive at (2.1). \hfill \Box

**Proof of Theorem 1.1.** It suffices to show that for some $q \geq 1$ and $\forall \epsilon > 0$, $s_1 = s_0 + \epsilon$,

$$\left\| \sup_{0 < t < 1} \frac{|e^{itP(D)}(f)\gamma(x, t) - f(x)|}{t^{\alpha/m}} \right\|_{L^q(B(x_0, R))} \lesssim \|f\|_{H^{s_1 + \epsilon}((\mathbb{R}^n))}. \quad (2.3)$$

In fact, if (2.3) holds, then fix $\lambda > 0$, choose $g \in C^\infty_0(\mathbb{R}^n)$ such that

$$\|f - g\|_{H^{s_1 + \epsilon}((\mathbb{R}^n))} \leq \lambda \epsilon^{1/q}.$$

it follows

$$\left\{ x \in B(x_0, R) : \sup_{0 < t < 1} \frac{|e^{itP(D)}(f - g)(\gamma(x, t)) - (f - g)(x)|}{t^{\alpha/m}} > \frac{\lambda}{2} \right\} \leq \frac{2q}{\lambda^q} \left\| \sup_{0 < t < 1} \frac{|e^{itP(D)}(f - g)(\gamma(x, t)) - (f - g)(x)|}{t^{\alpha/m}} \right\|_{L^q(B(x_0, R))}^q$$

$$\leq \frac{2q}{\lambda^q} \|f - g\|_{H^{s_1 + \epsilon}((\mathbb{R}^n))}^q$$

$$\leq \epsilon. \quad (2.4)$$
Moreover,

\[
\frac{|e^{itP(D)}(g)(\gamma(x,t)) - g(x)|}{t^{\alpha \delta/m}} \to 0, \quad \text{if} \quad t \to 0^+
\]

uniformly for \( x \in B(x_0, R) \). Indeed, for each \( x \in B(x_0, R) \),

\[
\lim_{t \to 0^+} \frac{|e^{itP(D)}(g)(\gamma(x,t)) - g(x)|}{t^{\alpha \delta/m}} \leq \lim_{t \to 0^+} \frac{|e^{itP(D)}(g)(\gamma(x,t)) - g(\gamma(x,t))|}{t^{\alpha \delta/m}} + \lim_{t \to 0^+} \frac{|g(\gamma(x,t)) - g(x)|}{t^{\alpha \delta/m}}.
\]  

(2.6)

By mean value theorem, we have

\[
\frac{|e^{itP(D)}(g)(\gamma(x,t)) - g(x)|}{t^{\alpha \delta/m}} \leq t^{1-\alpha \delta/m} \int_{\mathbb{R}^n} |P(\xi)||\dot{g}(\xi)|d\xi,
\]

(2.7)

and

\[
\frac{|g(\gamma(x,t)) - g(x)|}{t^{\alpha \delta/m}} \leq \frac{|\gamma(x,t) - x|}{t^{\alpha \delta/m}} \int_{\mathbb{R}^n} |\xi| |\dot{g}(\xi)|d\xi \leq t^{\alpha-\alpha \delta/m} \int_{\mathbb{R}^n} |\xi| |\dot{g}(\xi)|d\xi.
\]

(2.8)

Inequalities (2.6) - (2.8) imply (2.5).

By (2.4) and (2.5) we have

\[
\left| \left\{ x \in B(x_0, R) : \limsup_{t \to 0^+} \frac{|e^{itP(D)}(f)(\gamma(x,t)) - f(x)|}{t^{\alpha \delta/m}} > \lambda \right\} \right| \leq \epsilon,
\]

(2.9)

which implies (1.9) for \( f \in H^{s+\delta}(\mathbb{R}^n) \) and almost every \( x \in B(x_0, R) \). By the arbitrariness of \( \epsilon \), in fact we can get (1.9) for all \( f \in H^{s+\delta}(\mathbb{R}^n) \), \( s > s_0 \). Next we will prove (2.3) for \( q = \min\{p, 2\} \).

In order to prove (2.3), we decompose \( f \) as

\[ f = \sum_{k=0}^{\infty} f_k, \]

where \( \text{supp} f_0 \subset B(0,1) \), \( \text{supp} f_k \subset \{ \xi : |\xi| \sim 2^k \} \), \( k \geq 1 \). It follows that

\[
\left\| \sup_{0 < t < 1} \frac{|e^{itP(D)}(f)(\gamma(x,t)) - f(x)|}{t^{\alpha \delta/m}} \right\|_{L^q(B(x_0,R))} \leq \sum_{k=0}^{\infty} \left\| \sup_{0 < t < 1} \frac{|e^{itP(D)}(f_k)(\gamma(x,t)) - f_k(x)|}{t^{\alpha \delta/m}} \right\|_{L^q(B(x_0,R))}.
\]

(2.10)

For \( k \lesssim 1 \), because (2.7), (2.8) and \( P(\xi) \) is continuous,

\[
\left\| \sup_{0 < t < 1} \frac{|e^{itP(D)}(f_k)(\gamma(x,t)) - f_k(x)|}{t^{\alpha \delta/m}} \right\|_{L^q(B(x_0,R))} \leq \left\| \sup_{0 < t < 1} \frac{|e^{itP(D)}(f_k)(\gamma(x,t)) - f_k(\gamma(x,t))|}{t^{\alpha \delta/m}} \right\|_{L^q(B(x_0,R))} + \left\| \sup_{0 < t < 1} \frac{|f_k(\gamma(x,t)) - f_k(x)|}{t^{\alpha \delta/m}} \right\|_{L^q(B(x_0,R))}
\]

\[
\lesssim \|f\|_{H^{s+\delta}(\mathbb{R}^n)}.
\]

(2.11)
For $k \gg 1$,

$$
\left\| \sup_{0 < t < 1} \frac{|e^{itP(D)}(f_k)(\gamma(x,t)) - f_k(x)|}{t^{\alpha/m}} \right\|_{L^\gamma(B(x_0,R))} \\
\leq \left\| \sup_{2^{-m} \leq t < 1} \frac{|e^{itP(D)}(f_k)(\gamma(x,t)) - f_k(x)|}{t^{\alpha/m}} \right\|_{L^\gamma(B(x_0,R))} \\
+ \left\| \sup_{0 < t < 2^{-m} \frac{1}{2}} \frac{|e^{itP(D)}(f_k)(\gamma(x,t)) - f_k(x)|}{t^{\alpha/m}} \right\|_{L^\gamma(B(x_0,R))} \\
:= I + II. \tag{2.12}
$$

We first estimate $I$, from (1.8) we have

$$
\left\| \sup_{2^{-m} \leq t < 1} \frac{|e^{itP(D)}(f_k)(\gamma(x,t))|}{|\gamma(x,t)|} \right\|_{L^p(B(x_0,R))} \leq 2^{(\alpha_0 + \frac{\alpha}{m})k} \|f_k\|_{L^2(\mathbb{R}^n)}, \tag{2.13}
$$

hence,

$$
I \leq 2^{\delta k} \left\| \sup_{2^{-m} \leq t < 1} \frac{|e^{itP(D)}(f_k)(\gamma(x,t)) - f_k(x)|}{t^{\alpha/m}} \right\|_{L^\gamma(B(x_0,R))} \\
\leq 2^{\delta k} \left\{ \left\| \sup_{2^{-m} \leq t < 1} \frac{|e^{itP(D)}(f_k)(\gamma(x,t))|}{t^{\alpha/m}} \right\|_{L^\gamma(B(x_0,R))} + \|f_k\|_{L^\gamma(B(x_0,R))} \right\} \\
\leq 2^{\delta k} \left\{ \left\| \sup_{2^{-m} \leq t < 1} \frac{|e^{itP(D)}(f_k)(\gamma(x,t))|}{t^{\alpha/m}} \right\|_{L^\gamma(B(x_0,R))} + \|f_k\|_{L^2(B(x_0,R))} \right\} \\
\leq 2^{\delta k} 2^{(\alpha_0 + \frac{\alpha}{m})k} \|f\|_{L^2(\mathbb{R}^n)} \\
\leq 2^{-\frac{\delta}{1}} \|f\|_{H^{\alpha+\delta}(\mathbb{R}^n)}. \tag{2.14}
$$

For $II$, by triangle inequality,

$$
II \leq \left\| \sup_{0 < t < 2^{-m}} \frac{|e^{itP(D)}(f_k)(\gamma(x,t))|}{t^{\alpha/m}} \right\|_{L^\gamma(B(x_0,R))} \\
+ \left\| \sup_{0 < t < 2^{-m} \frac{1}{2}} \frac{|(f_k)(\gamma(x,t)) - f_k(x)|}{t^{\alpha/m}} \right\|_{L^\gamma(B(x_0,R))}. \tag{2.15}
$$

Mean value theorem and Lemma 2.1 imply

$$
\left\| \sup_{0 < t < 2^{-m} \frac{1}{2}} \frac{|f_k(\gamma(x,t)) - f_k(x)|}{t^{\alpha/m}} \right\|_{L^\gamma(B(x_0,R))} \\
\leq \left\| \sup_{0 < t < 2^{-m} \frac{1}{2}} \frac{|f_k(\gamma(x,t)) - f_k(x)|}{t^{\alpha/m}} \right\|_{L^\gamma(B(x_0,R))} \\
\leq 2^{-mk + \delta k} \sum_{h=1}^{n} \sum_{\xi \in \mathbb{Z}^n} C_n \frac{C_n}{(1 + ||\xi||)^{n+\delta}} \left\| \int_{\mathbb{R}^n} e^{i\xi \cdot (x + \frac{1}{2}t)} \xi_h \hat{f}_k(\xi) d\xi \right\|_{L^2(B(x_0,R))} \\
\leq 2^{-(m-1)k + \delta k} \|\hat{f}_k\|_{L^2(\mathbb{R}^n)} \\
\leq 2^{-s_1 k} \|f\|_{H^{\alpha+\delta}(\mathbb{R}^n)}, \tag{2.16}
$$
where $\theta(x,t) \in [0,1]$.

By Taylor’s formula and Lemma 2.1, we get

\[
\left\| \sup_{0 < t < 2^{-\frac{mk}{n}} - \frac{mk}{n} + \delta k} \frac{|e^{itP(D)}(f_k)(\gamma(x,t)) - f_k(x)|}{|f|} \right\|_{L^n(B(x_0,R))} \leq \sum_{j=1}^{\infty} \frac{2^{-\frac{mk}{n} + \delta k}}{j!} \sum_{\xi \in \mathbb{Z}^n} C_{n} \frac{e^{-|x|^{2} - \frac{1}{|\xi|^{2}}}}{1 + |\xi|^{2n+1}} \left\| \frac{e^{i(x+\frac{\xi}{|\xi|})} P(\xi)^j \hat{f}_k(\xi)}{|f|} \right\|_{L^n(B(x_0,R))}
\]

\[
\leq \sum_{j=1}^{\infty} \frac{2^{-\frac{mk}{n} + \delta k}}{j!} \sum_{\xi \in \mathbb{Z}^n} C_{n} \frac{e^{-|x|^{2} - \frac{1}{|\xi|^{2}}}}{1 + |\xi|^{2n+1}} \left\| P(\xi)^j \hat{f}_k(\xi) \right\|_{L^2(\mathbb{R}^n)}
\]

\[
\leq \sum_{j=1}^{\infty} \frac{2^{-\frac{mk}{n} + \delta k}}{j!} \left\| \hat{f}_k(\xi) \right\|_{L^2(\mathbb{R}^n)} \lesssim \frac{1}{n} \int \left\| \hat{f}(\xi) \right\|_{L^2(\mathbb{R}^n)} \lesssim 2^{-\frac{1}{n} \alpha k} \|f\|_{H^{1+\delta}(\mathbb{R}^n)}. \tag{2.17}
\]

Inequalities (2.14), (2.16) and (2.17) yield for $k \gg 1$,

\[
\left\| \sup_{0 < t < T} \frac{|e^{itP(D)}(f_k)(\gamma(x,t)) - f_k(x)|}{|f|} \right\|_{L^n(B(x_0,R))} \lesssim 2^{-\frac{1}{n} \alpha} \|f\|_{H^{1+\delta}(\mathbb{R}^n)}. \tag{2.18}
\]

It is clear that (2.3) follows from (2.10), (2.11) and (2.18).

**Theorem 2.2.** For each Schwartz function $f$, there exists

\[\gamma(x,t) = x - e_1 t^\alpha, \quad e_1 = (1,0,...,0), \quad 0 < \alpha \leq 1,\]

such that if

\[
\lim_{t \rightarrow 0^+} \frac{e^{itP(D)}(f)(\gamma(x,t)) - f(x)}{t^\alpha} = 0, \quad a.e. \quad x \in \mathbb{R}^n, \tag{2.19}
\]

then $f \equiv 0$.

**Proof.** when $0 < \alpha < 1$, by Taylor’s formula,

\[
e^{itP(D)} f(\gamma(x,t)) - f(x) = t^\alpha \int_{\mathbb{R}^n} e^{ix \xi_1} \hat{f}(\xi) d\xi + o(t^{2\alpha}) + o(t). \tag{2.20}
\]

Therefore,

\[
\lim_{t \rightarrow 0^+} \frac{|e^{itP(D)}(f)(\gamma(x,t)) - f(x)|}{t^\alpha} \geq \frac{1}{2} \int_{\mathbb{R}^n} e^{ix \xi_1} \hat{f}(\xi) d\xi. \tag{2.21}
\]

If $f$ is not zero, then there is a set $A$ with positive measure and a constant $c > 0$, such that

\[
\int_{\mathbb{R}^n} e^{ix \xi_1} \hat{f}(\xi) d\xi \geq c.
\]

this contradicts (2.19). The same method is valid for $\alpha = 1$. \qed
3 Proof of Theorem 1.4

Proof of Theorem 1.4. Set

\[ A_k =: \{ \xi : \frac{|\xi_1|}{2^{m_1/k/m_1}} + \frac{|\xi_2|}{2^k} \sim 1 \} \]

Consider \( k \) such that

\[ \{ \xi : |\xi| \sim \lambda \} \cap A_k \neq \emptyset, \]

then

\[ \lambda^{m_1/m_2} \leq 2^k \leq \lambda, \]

therefore

\[ k \sim \log \lambda. \]

Decompose

\[ f = \sum_k f_k, \]

where \( \text{supp } \hat{f}_k \subset \{ \xi : |\xi| \sim \lambda \} \cap A_k \). If for each \( k \),

\[ \left\| \sup_{0 < t < 1} |e^{itP_{m_1,m_2}(D)}(f_k)(\gamma(x,t))| \right\|_{L^p(B(x_0,R))} \lesssim \lambda^\alpha \|f\|_{L^2}, \tag{3.1} \]

then

\[ \left\| \sup_{0 < t < 1} |e^{itP_{m_1,m_2}(D)}(f)(\gamma(x,t))| \right\|_{L^p(B(x_0,R))} \]

\[ \lesssim \sum_k \left\| \sup_{0 < t < 1} |e^{itP_{m_1,m_2}(D)}(f_k)(\gamma(x,t))| \right\|_{L^p(B(x_0,R))} \]

\[ \lesssim \lambda^\alpha \sum_k \|f\|_{L^2} \]

\[ \lesssim \lambda^{\alpha+\varepsilon} \|f\|_{L^2}. \tag{3.2} \]

It is sufficient to show that if

\[ \left\| \sup_{t \in J} |e^{itP_{m_1,m_2}(D)}(f_k)(\gamma(x,t))| \right\|_{L^p(B(x_0,R))} \lesssim \lambda^\alpha \|f\|_{L^2} \tag{3.3} \]

for each \( J \in \mathcal{J} \), then (3.1) holds.

For each \( g \in L^2 \), \( G(x,t) \in L^{p'}(B(x_0,R), L^1_t(0,1)) \), \( 1/p + 1/p' = 1 \). Define the operator \( T \) by

\[ Tg := \int_{\mathbb{R}^2} e^{i\gamma(x,t) \cdot \xi + itP_{m_1,m_2}(\xi)} \Psi(\xi) \hat{g}(\xi) d\xi, \]

\[ G_f(x,t) = G(x,t) \chi_f(t), \]
where \( \Psi(\xi_1, \xi_2) = \psi\left(\frac{\xi_1 - 2k}{2m_1\lambda_1}, \frac{\xi_2}{2m_2\lambda_2}\right) \psi\left(\frac{\xi_1}{2m_2\lambda_2}, \frac{\xi_2 - 2k}{2m_1\lambda_1}\right) \), \( \psi \in C^\infty_c \) equals to 1 on \( \{\xi : |\xi| \sim 1\} \) and rapidly decay outside. \( \chi_J(t) \) is the characteristic function of \( J \). By duality, it is enough to show that

\[
\|T^*G_J\|_{L^2} \lesssim \lambda^n \|G_J\|_{L^n'_x(B(x_0, R), L^1_y(0, 1))}, \quad J \in \mathcal{J}
\]

implies

\[
\|T^*G\|_{L^2} \lesssim \lambda^n \|G\|_{L^n'_x(B(x_0, R), L^1_y(0, 1))}.
\]

Inequality (3.5) is equivalent to

\[
\left| \sum_{J, J' \in \mathcal{J}} \langle T^*G_J, T^*G_{J'} \rangle \right| \lesssim \lambda^{2n} \|G\|^2_{L^n'_x(B(x_0, R), L^1_y(0, 1))}.
\]

When \( \text{dist}(J, J') \geq 100\lambda_1^{-m_1} \),

\[
\|T^*G_J, T^*G_{J'}\| = \|G_J, \chi_J TT^*G_{J'}\|
\]

\[
\leq \|G_J\|_{L^n'_x(B(x_0, R), L^1_y(0, 1))} \|\chi_J TT^*G_{J'}\|_{L^n'_x(B(x_0, R), L^1_y(0, 1))}
\]

\[
\leq \|G_J\|_{L^n'_x(B(x_0, R), L^1_y(0, 1))} \|\chi_J TT^*G_{J'}\|_{L^n'_x(B(x_0, R), L^1_y(0, 1))}.
\]

For each \( x \in B(x_0, R), t \in J \),

\[
\left| \chi_J TT^*G_{J'}(x, t) \right| \leq \int_{B(x_0, R)} \int_0^1 |K(x, y, t, t')| |G_{J'}(y, t')| dt' dy,
\]

where

\[
|K(x, y, t, t')| = \left| \int_{\mathbb{R}^2} e^{i\gamma(x, t) - \gamma(y, t') - i(\xi_1, \xi_2) + i(t - t')P_{m_1, m_2}(\xi_1, \xi_2)} \psi^2(\xi_1, \xi_2) d\xi_1 d\xi_2 \right|
\]

\[
= 2^{\left(\frac{m_1}{2} + 1\right)k} \left| \int_{\mathbb{R}^2} e^{i\Phi_{x, y, t, t'}(\eta_1, \eta_2)} \psi^2(2^{\frac{m_2}{2}}\eta_1, 2^k \eta_2) d\eta_1 d\eta_2 \right|,
\]

in which

\[
\Phi_{x, y, t, t'}(\eta_1, \eta_2) = [\gamma(x, t) - \gamma(y, t')] \cdot (2^{\frac{m_2}{2}}\eta_1, 2^k \eta_2) + 2^{m_2} (t - t')P_{m_1, m_2}(\eta_1, \eta_2).
\]

Since

\[
|t - t'| \geq 100\lambda_1^{-m_1} \geq 1002^{\left(\frac{m_2}{2} - m_2\right)k}, \quad |\eta_1| + |\eta_2| \sim 1,
\]

it is easy to check that

\[
|\nabla_{\eta} \Phi_{x, y, t, t'}(\eta_1, \eta_2)| \gtrsim 2^{m_2}|t - t'|,
\]

\[
|D^\beta_{\eta} \Phi_{x, y, t, t'}(\eta_1, \eta_2)| \lesssim 2^{m_2}|t - t'|, \quad |\beta| \geq 2.
\]

Integration by parts implies that for positive integer \( N \gg \frac{m_2}{m_1} \),

\[
|K(x, y, t, t')| \leq 2^{\left(\frac{m_2}{2} + 1\right)k} \frac{C_N}{(1 + 2^{m_2}|t - t'|)^N} \left(\frac{2^{m_2/k}}{\lambda}\right)^N \lesssim \lambda^{-O(N)}.
\]

(3.10)
Inequalities (3.7), (3.8), (3.10) and Hölder’s inequality imply
\[
|\langle T^*G_J, T^*G_{J'} \rangle| \lesssim \lambda^{-O(N)} \|G_J\|_{L^p_x(B(x_0, R), L^1_t(0, 1))} \|G_{J'}\|_{L^p_x(B(x_0, R), L^1_t(0, 1))}
\] (3.11)
when \(\text{dist}(J, J') \geq 100\lambda^{1-m_1}\).

For \(J, J'\) such that \(\text{dist}(J, J') \leq 100\lambda^{1-m_1}\), by Hölder’s inequality and (3.4) we have
\[
|\langle T^*G_J, T^*G_{J'} \rangle| \lesssim \|T^*G_J\|_{L^2} \|T^*G_{J'}\|_{L^2} \\
\lesssim \lambda^{2\alpha} \|G_J\|_{L^p_x(B(x_0, R), L^1_t(0, 1))} \|G_{J'}\|_{L^p_x(B(x_0, R), L^1_t(0, 1))}.
\] (3.12)

It follows from (3.11), (3.12) and \(1 \leq p' \leq 2\) that
\[
\left| \sum_{J, J' \in J} \langle T^*G_J, T^*G_{J'} \rangle \right| \lesssim \sum_{J, J' \in J, \text{dist}(J, J') \leq 100\lambda^{1-m_1}} |\langle T^*G_J, T^*G_{J'} \rangle| \\
+ \sum_{J, J' \in J, \text{dist}(J, J') \geq 100\lambda^{1-m_1}} |\langle T^*G_J, T^*G_{J'} \rangle| \\
\lesssim \lambda^{2\alpha} \sum_{J \in J} \|G_J\|_{L^p_x(B(x_0, R), L^1_t(0, 1))}^2 \\
\lesssim \lambda^{2\alpha} \|G\|_{L^p_x(B(x_0, R), L^1_t(0, 1))}^2
\]
which implies (3.6).

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