The Dynamics of a Meandering River

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Abstract

We present a statistical model of a meandering river on an alluvial plane which is motivated by the physical non-linear dynamics of the river channel migration and by describing heterogeneity of the terrain by noise. We study the dynamics analytically and numerically. The motion of the river channel is unstable and we show that by inclusion of the formation of ox-bow lakes, the system may be stabilised. We then calculate the steady state and show that it is in agreement with simulations and measurements of field data.
Meandering rivers are ubiquitous in nature and can be found all over the world. It is a signature of their intellectual challenge that they have captured the imagination of scientists in quite differing fields [1]. They are important not only as examples of a system far from equilibrium undergoing an interesting dynamics but because abandoned channels of the main river become silted up and over thousands of years these silted channels become compressed to form shale beds between which oil is often found. Previous models that have tried to study river meanders fall into two broad classes. The first could be considered as *empiric* and are justified by the fact that Leopold and Wolman [2] showed by an analysis of field data from around the world, that there was a nearly constant ratio of radius of curvature to meander length and of radius of curvature to meander width. This ‘explains’ the visual ‘*scale invariance*’ when one observes rivers. They therefore concluded that “meander geometry is related in some unknown manner to a more general mechanical theory”. Consequently there have since been a number of papers [3–5] that modelled meanders as variants of random walks with essentially no justification. The second type is characterised by an analysis of the complex flow inside the channel [6–9] and as a result yields relatively little information about large scale or long time features of the river pattern. In this letter, a slightly different approach is taken; we attempt to find the simplest possible model of the behaviour of a meandering river which still retains the essential physics. The model is statistical in the sense that there is a spatio-temporal random aspect to the dynamics. Statistical models have been introduced in the study of river networks [10–12] but have not been used in the study of single channels. Also, to the authors knowledge, a stable state has not been constructed for any dynamic models of river meanders.

We consider the river on the length scale such that its channel may be considered to be a curve of length $L$ where $L \to \infty$, in two dimensions, $r(s, t) = [x(s, t), y(s, t)]$ where $s$ refers to the arc length position on the curve. We include all the major features of the dynamics of the channel and using them calculate the steady state. The derivation is outlined here and details will be reported elsewhere [13]. Our theory of meanders therefore acts as a connection between both previous approaches.
To calculate the equation of motion (EOM) of the river ‘curve’, we must begin by an analysis of the flow of water in a curved channel. The channel is considered to be narrow with width $b$ and we can describe the coordinates of the channel as $s$ and $n$ corresponding to moving along the centre-line of the channel and perpendicular to it, and note $n = n(s)$. We transform the Navier-Stokes equation for incompressible flow to the curved channel coordinates system. Using the similarly transformed continuity equation and assuming no stress at the top surface and bed stresses at the bottom of the channel, it can be shown using simple arguments that the major effect of this is that a velocity gradient develops across the channel with the velocity of water at the outer bank being greatest. That is if the outer bank corresponds to $n = b/2$ and the inner bank $n = -b/2$, then the velocity of fluid in the channel is given to leading order by $u(n) \approx u_0(1 + n \kappa(s))$ where $\kappa(s) = |\partial^2 r/\partial s^2|$ is the local curvature and $u_0$ is a constant. We do not include gravity so the plane is flat but we do have a pressure gradient driving the flow through the channel. In the analysis above the approximation is made that the channel is locally smooth, i.e there are no sharp changes in direction meaning the local curvature $\kappa(s)$ is small. Assuming erosion on the outer bank is balanced by deposition on the inner bank and that the rate of erosion on the outer bank is proportional to the velocity, the rate of normal, i.e. perpendicular to the curve, migration of the channel can be described by the relationship

$$R_m^0 \propto \kappa(s).$$

The condition that the curve is locally smooth is satisfied by adding another piece to the rate of migration; i.e.

$$R_m^1 \propto -\kappa^3 + \frac{\partial^2 \kappa}{\partial s^2}.$$  

This is obtained from the Rayleighian dynamics of a ‘global’ free energy which minimises curvature along the whole meandering river curve, $\mathcal{F}[r] = \int ds|\partial^2 r/\partial s^2|^2$. Evolution of curves undergoing similar dynamics as these first two bits of $R_m$ have been studied in the context of crystal growth and chemical front motion. Finally, we have a dynamic
noise term to model heterogeneities in space, such as different types of rock, and in time, for example, variation in flow of the river and re-eroding parts of the plain that have been eroded previously.

\[ R_m^2 \propto \eta(s,t). \]  

(3)

Dynamic noise is more appropriate than quenched noise because the river is constantly reworking its plain as it meanders and over long periods changing the nature of the terrain. For simplicity we take Gaussian uncorrelated noise.

To get the correct EOM we must appeal to reparametrisation invariance because the arc length, \( s \) is not constant in time. Physically, it is known that the normal velocity of a curve, in this case \( \hat{n} \cdot \partial_t r(s,t) = \sum_{i=0}^2 a_i R_m^i \) (\( \hat{n} \) is the unit normal) is independent of its parametrisation so for simplicity the EOM may be considered first in the 'normal gauge' where the tangential velocity of the curve is zero. Then using the Frenet formulae from differential geometry the EOM in the 'arc length gauge' may then be calculated to give,

\[
\frac{\partial}{\partial t}r(s,t) + \frac{\partial r}{\partial s} \int s ds' \kappa(s') V = \hat{n} V
\]  

(4)

where the normal velocity may be written \( V = A\kappa(s) - B (\kappa^3 - \kappa^2 \partial s^2) + \eta(s,t) = -\hat{n} \cdot (A\partial^2 r/\partial s^2 + B\partial^4 r/\partial s^4) + \eta(s,t) \). The smallest length-scale of the system is therefore \( \sqrt{B/A} \).

Recalling that \( \partial r/\partial s = \hat{s} \), the unit tangent vector to the curve, one sees that the second term on the lhs of equation (4) is a non-local tangential velocity corresponding to a 'stretching' or lengthening of the curve due the normal growth. To calculate the effective motion of the curve we replace the tangential velocity by its average, i.e. because we are dealing with a Langevin equation with the presence of a noise term, and we assume the system will always be near some steady state, we make the pre-averaging approximation where complicated parameters in the equation are replaced by their mean values, so we define \( \zeta(A,B) = \langle \int s \kappa V ds' \rangle = \int \mathcal{D}r \int s \kappa V ds' P([r],t) \), and consequently we may write the EOM of the river as

\[
\frac{\partial}{\partial t}r(s,t) + \zeta \frac{\partial r}{\partial s} = -A \frac{\partial^2 r}{\partial s^2} - B \frac{\partial^4 r}{\partial s^4} + \eta(s,t).
\]  

(5)
where

\[ \langle \eta(s, t) \rangle = 0 , \quad \langle \eta(s, t) \eta(s', t') \rangle = D \delta(s - s') \delta(t - t') \]  

(6)

We have thus calculated the first part of the problem, the local dynamics of the meandering river as a non-linear noisy equation. Noisy non-linear equations have been widely studied in the context of surface deposition [18] particularly when they lead to critical behaviour. Unlike these models meandering river systems do evolve into statistically invariant stable states. This equation also bears some superficial similarity to the Rouse equation of polymer dynamics [19], but its behaviour is actually quite different and is in fact extremely unstable and would lead to proliferation of length. This can be seen by a transformation to Fourier modes 

\[ r_k = \int ds r(s) \exp\{i k s\} \]  
gives 

\[ r_k(t) = r_k(0) \exp\{\Delta(k) t - i k \zeta t\} \]  

with \( \Delta(k) = A k^2 - B k^4 \) so there is always a band of unstable modes for which \( \Delta > 0 \).

In the form of equation(5) above, the essential dynamics is physically transparent and numerical interpretation very efficient. Similar equations (but without noise) such as those found in viscous fingering [20] could also be studied in this way.

We numerically integrated equation(5) above starting from an initial condition of a straight line and from Figure 1 it can be seen that meandering patterns are developed. Further evolution of the pattern leads to a plane filling, many time self-intersecting curve [13].

We now include the non-local part of the dynamics, the formation of ox-bow lakes; when the river intersects itself it loses that part of the curve between the two points of contact. This is the way the system is stabilised. We stress that locality here refers to s the position along the river and not \( r \) the position in space.

Before we proceed, it is useful to rewrite the EOM as a Fokker-Planck equation [21] so that the probability distribution \( P([r(s)], t) \) of the points on the river obey the equation

\[ \frac{\partial}{\partial t} P = \mathcal{O}_r P \]  

(7)

where

\[ \mathcal{O}_r = \int dL \int_0^L ds \delta \frac{\delta}{\delta r(s)} \cdot \left( \frac{\delta}{\delta r(s)} + \zeta \frac{\partial r}{\partial s} + A \frac{\partial^2 r}{\partial s^2} + B \frac{\partial^4 r}{\partial s^4} \right) \]
The integral over $L$ acknowledges the fact that the length does not remain constant. Since the river is taken as very long, the fact that the length $L$ varies is not so important as we assume that near the steady state $\langle L \rangle$ is a constant in time.

The oxbow interaction can be considered as a transition between states and we can define a transition probability in a Boltzmann type master equation [21]. The equation will be of the form $\partial_t P_a = O_r P_a + \sum_b [T_{ab}P_b - T_{ba}P_a]$ where $P_b$ refers to states scattering to $P_a$ or being scattered to from $P_a$ via oxbow creation and $T_{ab}$ is the ‘transition matrix’. For our purposes, it is sufficient to make the association $P([r(s)], t) = \lim_{N \to \infty} P([r_1, \ldots, r_N], t)$ so the functional can be taken as the infinite limit of a discrete function of many variables. In this notation the creation of an oxbow happens when the curve intersects itself at $i$ and $j$ where without loss of generality, $j > i$ corresponds to the transformation of an initial probability distribution $P_I$ for which $r_i = r_j$ to a final distribution $P_F$ for which the portion of the curve between $r_i$ and $r_j$ has vanished and so the curve must be re-labelled such that $\forall k > j, r_k \to r_k - (j - i)$. We can obtain an approximation of the final distribution in terms of the initial distribution by a generalisation of a Taylor expansion to functionals so that in the continuum notation, the initial and final distributions are given by, respectively, $P_I[r(s), L, t]$ and $P_F[r(s), L - \delta L, t + \delta t]$ where $\delta L$ is the length of the oxbow. Therefore the creation of an oxbow by intersection at $s_i$ and $s_j$ with $s_i < s_j$ is described by

$$P_F[\{r(s)\}, s_i, s_j] - P_I[\{r(s)\}, s_i, s_j] \approx \int dL \int_{s_i}^{L} ds' \frac{\delta P_I}{\delta r(s')} \left( s_{ij} \frac{\partial r}{\partial s'} + \frac{s_{ij}^2}{2} \frac{\partial^2 r}{\partial s'^2} + O(s_{ij}^3) \right)$$

(8)

where $s_{ij} = |s_i - s_j|$. Now this will only work if we can consider $|s_i - s_j|$ to be small and so by default this approximation will only be able to deal with small loops, i.e. $|s_i - s_j| << L$.

The transition probability $\tau$ will depend on the rate of approach of points $r(s_i)$ and $r(s_j)$ and so $\tau = \lambda \partial_t \{ \int ds_i ds_j \delta [r(s_i) - r(s_j)] \}$ where $\lambda$ is a constant. The sum over $b$ in the Fokker-Planck equation corresponds to a configurational sum over the state being scattered to or from.

To determine the effect of oxbows, we make the assumption that their effect added to our
prescription will give us, in the statistical sense, a steady state. In a sense, we construct such a steady state. The EOM becomes ∂tPs = OₚPs + IoₓPs where Ioₓ is an effective interaction due to the ox-bows. Because we are in the steady state, ⟨∂tPs⟩ = 0 so that we may write ⟨Ps⟩ ∝ exp{−F[r]} where F = F₀[r] + F'[r] and the operator acting on Ps is given by IoₓPs = ∫ ds δ/δr(δF'/δrPs) and the other part, δF₀/δr(s) = +ζ∂r/∂s + A∂²r/∂s² + B∂⁴r/∂s⁴. F₀ is the part of the free energy from the local dynamics and we want F' the part from the ox-bow to somehow cancel out the instabilities in the local EOM. It turns out that F' has two parts.

The first part of the interaction can be dealt by looking at the average value of the transition away from the present state by loss of an ox-bow, and we find an interaction term

$$F_1' \simeq \lambda \int ds ds' \delta[r(s) - r(s')]$$.

To obtain this we used the fact that ⟨∂t r⟩ = δF/δr(s) and δ⟨Ps⟩/δr = ⟨Ps⟩δF/δr and we rewrote the transition probability in the form τ = λ∂₁r(s₁) · δ/δr(s₁){∫ ∫ dsᵢ dsⱼ δ[r(sᵢ) - r(sⱼ)]}. This is the same form of term that would be obtained from the dynamics of the self-avoiding walk (SAW) [22] Hamiltonian.

The second part is obtained from the transition into the present state from another by loss of an ox-bow. Here we find that the effect of the transition is given by

$$\frac{δ}{δr} \cdot \frac{δF_2'}{δr} P \simeq \frac{δP}{δr} \left( g₁(λ) \frac{∂r}{∂s} + g₂(λ) \frac{∂²r}{∂s²} + \ldots \right)$$.

where $gₐ(λ) = \langle λ \int dsᵢ dsⱼ δ{[r(sᵢ) - r(sⱼ)]}|sᵢ - sⱼ|^{α} \rangle$ are a simple function of λ and measures the average rate of approach of points along the curve. The effect of this will be that if we consider the original equation we have a renormalisation of parameters due to the inclusion of the effective field of the ox-bow interaction, $A \rightarrow \tilde{A} = A - g₂(λ)$ and $ζ \rightarrow \tilde{ζ} = ζ - g₁(λ)$.

For the system to remain in a stable state, we must have $A - g₂(λ) = -2AD/B$ and $ζ = g₁(λ)$. We get the same results from independent scaling arguments. Combining both parts, we obtain a stable state that should correspond to a system always near that of a SAW confirming recent lattice simulations and measurements from field data [23].
We also included the creation of ox-bows to the numeric analysis of the meander dynamics by a relabelling procedure on contact, and we generated stable meander configurations in contrast to the situation when they are not present. In Figure 2 we have the configurations after 200,000 time-steps for the system with and without the inclusion of the ox-bow mechanism starting from an initial configuration of a straight line in both cases. The mean spread of the river path \( Y_g = \sqrt{\langle (Y - \bar{Y})^2 \rangle} \) as a function of length \( L \) was also calculated and we obtained the relationship \( Y_g \sim L \) for short distances and a cross-over to \( Y_g \sim L^{1/2} \) for long distance, agreeing with field data [23].

We now briefly consider some extensions to the model to make it more realistic; these will be discussed fully elsewhere [13]. In the expression for the fluid velocity for the channel, higher order terms are of the type \( \delta u \sim f^s \kappa \exp\{-\gamma (s - s')\} \) where \( \gamma \) is a constant depending on the parameters of the flow and infract it can be shown [13] that such corrections lead to an addition of a constant vector to the EOM. In other words it gives the river a well defined direction and does not change the dynamics described above. It is also instructive to consider higher order terms in the Taylor expansion undertaken to describe the ox-bow creation. These terms lead to a renormalisation [13] of the noise so that \( D \rightarrow \tilde{D} \). Finally we consider the effect of ’valley confinement’ and slope to the river channel. These will result in a potential \( U[\mathbf{r}] \) of the form \((g, 0) + (0, hy)\) added to the EOM.

In conclusion, we have developed a phenomenological model of river meanders which reproduces all the major features seen in nature motivated by an analysis of flow in a meandering channel. An essential feature of the model is the competition of ox-bow creation with a curvature instability of the river channel. It is particularly interesting that with the inclusion of ox-bows to the motion of the river, stable meanders form for all values of \( A, B, D \). The ratio \( B/A \) gives the lower length-scale of the river pattern or the size of the smallest ox-bows and the value of \( D \) gives the ’time scale’ over which the system retains memory of its previous behaviour. This explains the visual scale invariance of meandering river systems.

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FIGURES

FIG. 1. The evolution of a straight line at $t = 0.4 \times 10^4, 6 \times 10^4, 8 \times 10^4$ and $10^5$ (bold line) iterations with $A = 0.005$, $B = 0.000012$ and $D = 0.002$ via equation (5).

FIG. 2. Configuration after 200,000 iterations (a) with oxbows and (b) without oxbow creation.
