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THE DUALITY BETWEEN SINGULAR POINTS AND INFLECTION POINTS ON WAVE FRONTS

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Abstract

In the previous paper, the authors gave criteria for $A_{k+1}$-type singularities on wave fronts. Using them, we show in this paper that there is a duality between singular points and inflection points on wave fronts in the projective space. As an application, we show that the algebraic sum of 2-inflection points (i.e. godron points) on an immersed surface in the real projective space is equal to the Euler number of $M/A_0$. Here $M$ is a compact orientable 2-manifold, and $M/A_0$ is the open subset of $M$ where the Hessian of $f$ takes negative values. This is a generalization of Bleecker and Wilson’s formula [3] for immersed surfaces in the affine 3-space.

1. Introduction

We denote by $K$ the real number field $R$ or the complex number field $C$. Let $n$ and $m$ be positive integers. A map $F: K^n \to K^m$ is called $K$-differentiable if it is a $C^\infty$-map when $K = R$, and is a holomorphic map when $K = C$. Throughout this paper, we denote by $P(V)$ the $K$-projective space associated to a vector space $V$ over $K$ and let $\pi: V \to P(V)$ be the canonical projection.

Let $M^n$ and $N^{n+1}$ be $K$-differentiable manifolds of dimension $n$ and of dimension $n + 1$, respectively. The projectified $K$-cotangent bundle

$$P(T^*N^{n+1}) = \bigcup_{p \in N^{n+1}} P(T^*_pN^{n+1})$$

has a canonical $K$-contact structure. A $K$-differentiable map $f: M^n \to N^{n+1}$ is called a frontal if $f$ lifts to a $K$-isotropic map $L_f$, i.e., a $K$-differentiable map $L_f: M^n \to P(T^*N^{n+1})$ such that the image $dL_f(TM^n)$ of the $K$-tangent bundle $TM^n$ lies in the contact hyperplane field on $P(T^*N^{n+1})$. Moreover, $f$ is called a wave front or a front if it lifts to a $K$-isotropic immersion $L_f$. (In this case, $L_f$ is called a Legendrian immersion.) Frontals (and therefore fronts) generalize immersions, as they allow for singular points. A frontal $f$ is said to be co-orientable if its $K$-isotropic lift $L_f$ can lift up to a $K$-differentiable map into the $K$-cotangent bundle $T^*N^{n+1}$, otherwise it is said to be non-co-orientable. It should be remarked that, when $N^{n+1}$ is a Riemannian mani-
fold, a front $f$ is co-orientable if and only if there is a globally defined unit normal vector field $\nu$ along $f$.

Now we set $N^{n+1} = K^{n+1}$. Suppose that a $K$-differentiable map $F: M^n \to K^{n+1}$ is a frontal. Then, for each $p \in M^n$, there exist a neighborhood $U$ of $p$ and a map

$$\nu: U \to (K^{n+1})^* \setminus \{0\}$$

into the dual vector space $(K^{n+1})^*$ of $K^{n+1}$ such that the canonical pairing $\nu \cdot dF(v)$ vanishes for any $v \in TU$. We call $\nu$ a local normal map of the frontal $F$. We set $G := \pi \circ \nu$, which is called a (local) Gauss map of $F$. In this setting, $F$ is a front if and only if

$$L := (F, \mathcal{G}): U \to K^{n+1} \times P((K^{n+1})^*)$$

is an immersion. When $F$ itself is an immersion, it is, of course, a front. If this is the case, for a fixed local $K$-differentiable coordinate system $(x^1, \ldots, x^n)$ on $U$, we set

$$v_p: K^{n+1} \ni v \mapsto \det(F_{x^i}(p), \ldots, F_{x^n}(p), v) \in K \quad (p \in U),$$

where $F_{x^j} := \partial F/\partial x^j$ ($j = 1, \ldots, n$) and ‘det’ is the determinant function on $K^{n+1}$. Then we get a $K$-differentiable map $\nu: U \ni p \mapsto v_p \in (K^{n+1})^*$, which gives a local normal map of $F$.

Now, we return to the case that $F$ is a front. Then it is well-known that the local Gauss map $\mathcal{G}$ induces a global map

$$(1.2) \quad \mathcal{G}: M^n \to P((K^{n+1})^*)$$

which is called the affine Gauss map of $F$. (In fact, the Gauss map $\mathcal{G}$ depends only on the affine structure of $K^{n+1}$.)

We set

$$h_{ij} := \nu \cdot F_{x^i/x^j} = -\nu_{x^i} \cdot F_{x^j} \quad (i, j = 1, \ldots, n),$$

where $\cdot$ is the canonical pairing between $K^{n+1}$ and $(K^{n+1})^*$, and

$$F_{x^i/x^j} = \frac{\partial^2 F}{\partial x^i \partial x^j}, \quad F_{x^i} = \frac{\partial F}{\partial x^i}, \quad v_{x^i} = \frac{\partial \nu}{\partial x^i}.$$ 

Then

$$H := \sum_{i, j=1}^n h_{ij} dx^i dx^j \quad (dx^i dx^j := (1/2)(dx^i \otimes dx^j + dx^j \otimes dx^i))$$
gives a $K$-valued symmetric tensor on $U$, which is called the **Hessian form** of $F$ associated to $v$. Here, the $K$-differentiable function

$$h := \det(h_{ij}): U \to K$$

is called the **Hessian** of $F$. A point $p \in M^n$ is called an **inflection point** of $F$ if it belongs to the zeros of $h$. An inflection point $p$ is called **nondegenerate** if the derivative $dh$ does not vanish at $p$. In this case, the set of inflection points $I(F)$ consists of an embedded $K$-differentiable hypersurface of $U$ near $p$ and there exists a non-vanishing $K$-differentiable vector field $\xi$ along $I(F)$ such that $H(\xi, v) = 0$ for all $v \in TU$. Such a vector field $\xi$ is called an **asymptotic vector field** along $I(F)$, and $[\xi] = \pi(\xi) \in P(K^{n+1})$ is called the **asymptotic direction**. It can be easily checked that the definition of inflection points and the nondegeneracy of inflection points are independent of choice of $v$ and a local coordinate system.

In Section 2, we shall define the terminology that

- a $K$-differentiable vector field $\eta$ along a $K$-differentiable hypersurface $S$ of $M^n$ is $k$-nondegenerate at $p \in S$, and
- $\eta$ meets $S$ at $p$ with multiplicity $k+1$.

Using this new terminology, $p \in I(F))$ is called an $A_{k+1}$-inflection point if $\xi$ is $k$-nondegenerate at $p$ but does not meet $I(F)$ with multiplicity $k+1$. In Section 2, we shall prove the following:

**Theorem A.** Let $F: M^n \to K^{n+1}$ be an immersed $K$-differentiable hypersurface. Then $p \in M^n$ is an $A_{k+1}$-inflection point (1 $\leq k \leq n$) if and only if the affine Gauss map $G$ has an $A_k$-Morin singularity at $p$. (See the appendix of [10] for the definition of $A_k$-Morin singularities, which corresponds to $A_{k+1}$-points under the intrinsic formulation of singularities as in the reference given in Added in Proof.)

Though our definition of $A_{k+1}$-inflection points are given in terms of the Hessian, this assertion allows us to define $A_{k+1}$-inflection points by the singularities of their affine Gauss map, which might be more familiar to readers than our definition. However, the new notion “$k$-multiplicity” introduced in the present paper is very useful for recognizing the duality between singular points and inflection points. Moreover, as mentioned above, our definition of $A_k$-inflection points works even when $F$ is a front. We have the following dual assertion for the previous theorem. Let $G: M^n \to P((K^{n+1})^*)$ be an immersion. Then $p \in M^n$ is an $A_{k+1}$-inflection point of $G$ if it is an $A_{k+1}$-inflection point of $v: M^n \to (K^{n+1})^*$ such that $\pi \circ v = G$. This property does not depend on a choice of $v$.

**Proposition A’.** Let $F: M^n \to K^{n+1}$ be a front. Suppose that the affine Gauss map $G: M^n \to P((K^{n+1})^*)$ is a $K$-immersion. Then $p \in M^n$ is an $A_{k+1}$-inflection point
of \( G \) (1 \( \leq \) \( k \) \( \leq \) \( n \)) if and only if \( F \) has an \( A_{k+1} \)-singularity at \( p \). (See (1.1) in [10] for the definition of \( A_{k+1} \)-singularities.)

In the case that \( K = \mathbb{R} \), \( n = 3 \) and \( F \) is an immersion, an \( A_3 \)-inflection point is known as a cusp of the Gauss map (cf. [2]).

It can be easily seen that inflection points and the asymptotic directions are invariant under projective transformations. So we can define \( A_{k+1} \)-inflection points of an immersion \( f: M^n \to \mathbb{P}(K^{n+2}) \). For each \( p \in M^n \), we take a local \( K \)-differentiable coordinate system \((U; x^1, \ldots, x^n) (\subset M^n)\). Then there exists a \( K \)-immersion \( F: U \to K^{n+2} \) such that \( f = [F] \) is the projection of \( F \). We set

\[
G: U \ni p \mapsto F_{x^1}(p) \land F_{x^2}(p) \land \cdots \land F_{x^n}(p) \land F(p) \in (K^{n+2})^*. 
\]

Here, we identify \((K^{n+2})^*\) with \( \bigwedge^{n+1} K^{n+2} \) by

\[
\bigwedge^{n+1} K^{n+2} \ni v_1 \land \cdots \land v_{n+1} \iff \det(v_1, \ldots, v_{n+1}, *) \in (K^{n+2})^*,
\]

where ‘det’ is the determinant function on \( K^{n+2} \). Then \( G \) satisfies

\[
(1.6) \quad G \cdot F = 0, \quad G \cdot dF = dG \cdot F = 0,
\]

where \( \cdot \) is the canonical pairing between \( K^{n+2} \) and \((K^{n+2})^*\). Since, \( g := \pi \circ G \) does not depend on the choice of a local coordinate system, the projection of \( G \) induces a globally defined \( K \)-differentiable map

\[
g = [G]: M^n \to \mathbb{P}((K^{n+2})^*),
\]

which is called the dual front of \( f \). We set

\[
h := \det(h_{ij}): U \to K \quad (h_{ij} := G \cdot F_{x^i x^j} = -G_{x^i} \cdot F_{x^j}),
\]

which is called the Hessian of \( F \). The inflection points of \( f \) correspond to the zeros of \( h \).

In Section 3, we prove the following

**Theorem B.** Let \( f: M^n \to \mathbb{P}(K^{n+2}) \) be an immersed \( K \)-differentiable hypersurface. Then \( p \in M^n \) is an \( A_{k+1} \)-inflection point (\( k \leq n \)) if and only if the dual front \( g \) has an \( A_k \)-singularity at \( p \).

Next, we consider the case of \( K = \mathbb{R} \). In [8], we defined the tail part of a swallow-tail, that is, an \( A_3 \)-singular point. An \( A_3 \)-inflection point \( p \) of \( f: M^2 \to \mathbb{P}(\mathbb{R}^4) \) is called positive (resp. negative), if the Hessian takes negative (resp. positive) values on the tail.
part of the dual of $f$ at $p$. Let $p \in M^2$ be an $A_3$-inflection point. Then there exists a neighborhood $U$ such that $f(U)$ is contained in an affine space $A^3$ in $P(\mathbb{R}^4)$. Then the affine Gauss map $\mathcal{G} : U \rightarrow P(A^3)$ has an elliptic cusp (resp. a hyperbolic cusp) if and only if it is positive (resp. negative) (see [2, p. 33]). In [13], Uribe-Vargas introduced a projective invariant $\rho$ and studied the projective geometry of swallowtails. He proved that an $A_3$-inflection point is positive (resp. negative) if and only if $\rho > 1$ (resp. $\rho < 1$). The property that $h$ as in (1.5) is negative is also independent of the choice of a local coordinate system. So we can define the set of negative points

$$M_- := \{ p \in M^2 ; h(p) < 0 \}.$$

In Section 3, we shall prove the following assertion as an application.

**Theorem C.** Let $M^2$ be a compact orientable $C^\infty$-manifold without boundary, and $f : M^2 \rightarrow P(\mathbb{R}^4)$ an immersion. We denote by $i^+_2(f)$ (resp. $i^-_2(f)$) the number of positive $A_3$-inflection points (resp. negative $A_3$-inflection points) on $M^2$ (see Section 3 for the precise definition of $i^+_2(f)$ and $i^-_2(f)$). Suppose that inflection points of $f$ consist only of $A_2$ and $A_3$-inflection points. Then the following identity holds

$$i^+_2(f) - i^-_2(f) = 2\chi(M_-).$$

(1.8)

The above formula is a generalization of that of Bleecker and Wilson [3] when $f(M^2)$ is contained in an affine 3-space.

**Corollary D** (Uribe-Vargas [13, Corollary 4]). Under the assumption of Theorem C, the total number $i^+_2(f) + i^-_2(f)$ of $A_3$-inflection points is even.

In [13], this corollary is proved by counting the parity of a loop consisting of flecnodal curves which bound two $A_3$-inflection points.

**Corollary E.** The same formula (1.8) holds for an immersed surface in the unit 3-sphere $S^3$ or in the hyperbolic 3-space $H^3$.

Proof. Let $\pi : S^3 \rightarrow P(\mathbb{R}^4)$ be the canonical projection. If $f : M^2 \rightarrow S^3$ is an immersion, we get the assertion applying Theorem C to $\pi \circ f$. On the other hand, if $f$ is an immersion into $H^3$, we consider the canonical projective embedding $\iota : H^3 \rightarrow S^3_+$ where $S^3_+$ is the open hemisphere of $S^3$. Then we get the assertion applying Theorem C to $\pi \circ \iota \circ f$. \[\square\]

Finally, in Section 4, we shall introduce a new invariant for $3/2$-cusps using the duality, which is a measure for acuteness using the classical cycloid.
This work is inspired by the result of Izumiya, Pei and Sano \cite{IzumiyaPeiSano} that characterizes $A_2$ and $A_3$-singular points on surfaces in $H^3$ via the singularity of certain height functions, and the result on the duality between space-like surfaces in hyperbolic 3-space (resp. in light-cone), and those in de Sitter space (resp. in light-cone) given by Izumiya \cite{Izumiya}. The authors would like to thank Shyuichi Izumiya for his impressive informal talk at Karatsu, 2005.

2. Preliminaries and a proof of Theorem A

In this section, we shall introduce a new notion "multiplicity" for a contact of a given vector field along an immersed hypersurface. Then our previous criterion for $A_k$-singularities (given in \cite{SajiUmeharaYamada2}) can be generalized to the criteria for $k$-multiple contactness of a given vector field (see Theorem 2.4).

Let $M^n$ be a $K$-differentiable manifold and $S \subset M^n$ an embedded $K$-differentiable hypersurface in $M^n$. We fix $p \in S$ and take a $K$-differentiable vector field $\eta: S \ni q \mapsto \eta_q \in T_q M^n$ along $S$ defined on a neighborhood $V \subset S$ of $p$. Then we can construct a $K$-differentiable vector field $\tilde{\eta}$ defined on a neighborhood $U \subset M^n$ of $p$ such that the restriction $\tilde{\eta}|_S$ coincides with $\eta$. Such an $\tilde{\eta}$ is called an extension of $\eta$. (The local existence of $\tilde{\eta}$ is mentioned in \cite[Remark 2.2]{SajiUmeharaYamada}.)

**Definition 2.1.** Let $p$ be an arbitrary point on $S$, and $U$ a neighborhood of $p$ in $M^n$. A $K$-differentiable function $\varphi: U \to K$ is called admissible near $p$ if it satisfies the following properties

1. $O := U \cap S$ is the zero level set of $\varphi$, and
2. $d\varphi$ never vanishes on $O$.

One can easily find an admissible function near $p$. We set $\varphi' := d\varphi(\tilde{\eta}): U \to K$ and define a subset $S_2 \subset O \subset S$ by

$$S_2 := \{q \in O : \varphi'(q) = 0\} = \{q \in O : \eta_q \in T_q S\}.$$

If $p \in S_2$, then $\eta$ is said to meet $S$ with multiplicity 2 at $p$ or equivalently, $\eta$ is said to contact $S$ with multiplicity 2 at $p$. Otherwise, $\eta$ is said to meet $S$ with multiplicity 1 at $p$. Moreover, if $d\varphi'(T_p O) \neq \{0\}$, $\eta$ is said to be 2-nondegenerate at $p$. The $k$-th multiple contactness and $k$-nondegeneracy are defined inductively. In fact, if the $j$-th multiple contactness and the submanifolds $S_j$ have been already defined for $j = 1, \ldots, k$ ($S_1 = S$), we set

$$\varphi^{(k)} := d\varphi^{(k-1)}(\tilde{\eta}): U \to K \quad (\varphi^{(1)} := \varphi')$$
and can define a subset of $S_k$ by
\[ S_{k+1} := \{ q \in S_k : \varphi^{(k)}(q) = 0 \} = \{ q \in S_k : \eta_q \in T_q S_k \}. \]

We say that $\eta$ meets $S$ with multiplicity $k + 1$ at $p$ if $\eta$ is $k$-nondegenerate at $p$ and $p \in S_{k+1}$. Moreover, if $d\varphi^{(k)}(T_p S_k) \neq \{0\}$, $\eta$ is called $(k + 1)$-nondegenerate at $p$. If $\eta$ is $(k + 1)$-nondegenerate at $p$, then $S_{k+1}$ is a hypersurface of $S_k$ near $p$.

**Remark 2.2.** Here we did not define ‘1-nondegeneracy’ of $\eta$. However, from now on, any $K$-differentiable vector field $\eta$ of $M^n$ along $S$ is always 1-nondegenerate by convention. In the previous paper [10], ‘1-nondegeneracy’ (i.e. nondegeneracy) is defined not for a vector field along the singular set but for a given singular point. If a singular point $p \in U$ of a front $f : U \to \mathbf{K}^{n+1}$ is nondegenerate in the sense of [10], then the function $\lambda : U \to \mathbf{K}$ defined in [10, (2.1)] is an admissible function, and the null vector field $\eta$ along $S(f)$ is given. When $k \geq 2$, by definition, $k$-nondegeneracy of the singular point $p$ is equivalent to the $k$-nondegeneracy of the null vector field $\eta$ at $p$ (cf. [10]).

**Proposition 2.3.** The $k$-th multiple contactness and $k$-nondegeneracy are both independent of the choice of an extension $\tilde{\eta}$ of $\eta$ and also of the choice of admissible functions as in Definition 2.1.

Proof. We can take a local coordinate system $(U; x^1, \ldots, x^n)$ of $M^n$ such that $x^n = \varphi$. Write
\[ \tilde{\eta} := \sum_{j=1}^n c^j \frac{\partial}{\partial x^j}, \]
where $c^j$ ($j = 1, \ldots, n$) are $K$-differentiable functions. Then we have that $\varphi' = \sum_{j=1}^n c^j \varphi_{x^j} = c^n$.

Let $\psi$ be another admissible function defined on $U$. Then
\[ \psi' = \sum_{j=1}^n c^j \frac{\partial \psi}{\partial x^j} = c^n \frac{\partial \psi}{\partial x^n} = \varphi' \frac{\partial \psi}{\partial x^n}. \]

Thus $\psi'$ is proportional to $\varphi'$. Then the assertion follows inductively. □

Corollary 2.5 in [10] is now generalized into the following assertion:

**Theorem 2.4.** Let $\tilde{\eta}$ be an extension of the vector field $\eta$. Let us assume $1 \leq k \leq n$. Then the vector field $\eta$ is $k$-nondegenerate at $p$, but $\eta$ does not meet $S$ with
multiplicity $k + 1$ at $p$ if and only if

$$
\varphi(p) = \varphi'(p) = \cdots = \varphi^{(k-1)}(p) = 0, \quad \varphi^{(k)}(p) \neq 0,
$$

and the Jacobi matrix of $K$-differentiable map

$$
\Lambda := (\varphi, \varphi', \ldots, \varphi^{(k-1)}): U \rightarrow K^k
$$
is of rank $k$ at $p$, where $\varphi$ is an admissible $K$-differentiable function and

$$
\varphi^{(0)} := \varphi, \varphi^{(1)} := \varphi', \ldots, \varphi^{(k)} := d\varphi^{(k-1)}(\eta).
$$

The proof of this theorem is completely parallel to that of Corollary 2.5 in [10]. To prove Theorem A by applying Theorem 2.4, we shall review the criterion for $A_k$-singularities in [10]. Let $U^n$ be a domain in $K^n$, and consider a map $\Phi: U^n \rightarrow K^m$ where $m \geq n$. A point $p \in U^n$ is called a singular point if the rank of the differential map $d\Phi$ is less than $n$. Suppose that the singular set $S(\Phi)$ of $\Phi$ consists of a $K$-differentiable hypersurface $U^n$. Then a vector field $\eta$ along $S$ is called a null vector field if $d\Phi(\eta)$ vanishes identically. In this paper, we consider the case $m = n$ or $m = n + 1$. If $m = n$, we define a $K$-differentiable function $\lambda: U^n \rightarrow K$ by

$$
(2.1) \quad \lambda := \det(\Phi_{x^1}, \ldots, \Phi_{x^n}).
$$

On the other hand, if $\Phi: U^n \rightarrow K^{n+1}$ ($m = n + 1$) and $v$ is a non-vanishing $K$-normal vector field (for a definition, see [10, Section 1]) we set

$$
(2.2) \quad \lambda := \det(\Phi_{x^1}, \ldots, \Phi_{x^n}, v).
$$

Then the singular set $S(\Phi)$ of the map $\Phi$ coincides with the zeros of $\lambda$. Recall that $p \in S(\Phi)$ is called nondegenerate if $d\lambda(p) \neq 0$ (see [10] and Remark 2.2). Both of two cases (2.1) and (2.2), the functions $\lambda$ are admissible near $p$ (cf. Definition 2.1), if $p$ is non-degenerate. When $S(\Phi)$ consists of nondegenerate singular points, then it is a hypersurface and there exists a non-vanishing null vector field $\eta$ on $S(\Phi)$. Such a vector field $\eta$ determined up to a multiplication of non-vanishing $K$-differentiable functions. The following assertion holds as seen in [10].

**Fact 2.5.** Suppose $m = n$ and $\Phi$ is a $C^\infty$-map (resp. $m = n + 1$ and $\Phi$ is a front). Then $\Phi$ has an $A_k$-Morin singularity (resp. $A_{k+1}$-singularity) at $p \in M^n$ if and only if $\eta$ is $k$-nondegenerate at $p$ but does not meet $S(\Phi)$ with multiplicity $k + 1$ at $p$. (Here multiplicity 1 means that $\eta$ meets $S(\Phi)$ at $p$ transversally, and 1-nondegeneracy is an empty condition.)

As an application of the fact for $m = n$, we now give a proof of Theorem A: Let $F: M^n \rightarrow K^{n+1}$ be an immersed $K$-differentiable hypersurface. Recall that a point $p \in$
$M^n$ is called a nondegenerate inflection point if the derivative $dh$ of the local Hessian function $h$ (cf. (1.5)) with respect to $F$ does not vanish at $p$. Then the set $I(F)$ of inflection points consists of a hypersurface, called the inflectional hypersurface, and the function $h$ is an admissible function on a neighborhood of $p$ in $M^n$. A nondegenerate inflection point $p$ is called an $A_{k+1}$-inflection point of $F$ if the asymptotic vector field $\xi$ is $k$-nondegenerate at $p$ but does not meet $I(F)$ with multiplicity $k+1$ at $p$.

Proof of Theorem A. Let $\nu$ be a map given by (1.1), and $G : M^n \to P((K^{n+1})^*)$ the affine Gauss map induced from $\nu$ by (1.2). We set

$$\mu := \det(\nu_{x^1}, \nu_{x^2}, \ldots, \nu_{x^n}, \nu),$$

where ‘det’ is the determinant function of $(K^{n+1})^*$ under the canonical identification $(K^{n+1})^* \cong K^{n+1}$, and $(x^1, \ldots, x^n)$ is a local coordinate system of $M^n$. Then the singular set $S(\mathcal{G})$ of $\mathcal{G}$ is just the zeros of $\mu$. By Theorem 2.4 and Fact 2.5, our criteria for $A_{k+1}$-inflection points (resp. $A_{k+1}$-singular points) are completely determined by the pair $(\xi, I(F))$ (resp. the pair $(\eta, S(\mathcal{G}))$). Hence it is sufficient to show the following three assertions (1)–(3).

(1) $I(F) = S(\mathcal{G})$.

(2) For each $p \in I(F)$, $p$ is a nondegenerate inflection point of $F$ if and only if it is a nondegenerate singular point of $\mathcal{G}$.

(3) The asymptotic direction of each nondegenerate inflection point $p$ of $F$ is equal to the null direction of $p$ as a singular point of $\mathcal{G}$.

Let $H = \sum_{i,j=1}^n h_{ij} \, dx^i \, dx^j$ be the Hessian form of $F$. Then we have that

$$h_{ij} \begin{pmatrix} h_{11} & \cdots & h_{1n} & * \\ \vdots & \ddots & \vdots & \vdots \\ h_{n1} & \cdots & h_{nn} & * \\ 0 & \cdots & 0 & v \cdot t_v \end{pmatrix} = \begin{pmatrix} \nu_{x^1} \\ \vdots \\ \nu_{x^n} \\ v \end{pmatrix} (F_{x^1}, \ldots, F_{x^n}, t_v),$$

where $v \cdot t_v = \sum_{j=1}^{n+1} (v^j)^2$ and $v = (v^1, \ldots, v^n)$ as a row vector. Here, we consider a vector in $K^n$ (resp. in $(K^n)^*$) as a column vector (resp. a row vector), and $t(\cdot)$ denotes the transposition. We may assume that $\nu(p) \cdot t_v(p) \neq 0$ by a suitable affine transformation of $K^{n+1}$, even when $K = \mathbb{C}$. Since the matrix $(F_{x^1}, \ldots, F_{x^n}, t_v)$ is regular, (1) and (2) follow by taking the determinant of (2.3). Also by (2.3), $\sum_{i=1}^n a_i h_{ij} = 0$ for all $j = 1, \ldots, n$ holds if and only if $\sum_{i=1}^n a_i \nu_{x^i} = 0$, which proves (3).

Proof of Proposition A’. Similar to the proof of Theorem A, it is sufficient to show the following properties, by virtue of Theorem 2.4.

(1’) $S(F) = I(\mathcal{G})$, that is, the set of singular points of $F$ coincides with the set of inflection points of the affine Gauss map.
(2') For each \( p \in I(\mathcal{G}) \), \( p \) is a nondegenerate inflection point if and only if it is a nondegenerate singular point of \( F \).

(3') The asymptotic direction of each nondegenerate inflection point coincides with the null direction of \( p \) as a singular point of \( F \).

Since \( \mathcal{G} \) is an immersion, (2.3) implies that
\[
I(\mathcal{G}) = \{ p; (F_{x^1}, \ldots, F_{x^s}, t^1v) \text{ are linearly dependent at } p \}
= \{ p; \lambda(p) = 0 \} \quad (\lambda := \det(F_{x^1}, \ldots, F_{x^s}, t^1v)).
\]

Hence we have (1'). Moreover, \( h = \det(h_{ij}) = \delta \lambda \) holds, where \( \delta \) is a function on \( U \) which never vanishes on a neighborhood of \( p \). Thus (2') holds. Finally, by (2.3), \( \sum_{j=1}^{n} b_j h_{ij} = 0 \) for \( i = 1, \ldots, n \) if and only if \( \sum_{j=1}^{n} b_j F_{t^j} = 0 \), which proves (3'). \( \square \)

**Example 2.6** (\( A_2 \)-inflection points on cubic curves). Let \( \gamma(t) := (x(t), y(t)) \) be a \( K \)-differentiable curve in \( K^2 \). Then \( v(t) := (\gamma'(t), \dot{x}(t)) \in (K^2)^s \) gives a normal vector, and
\[
h(t) = v(t) \cdot \ddot{y}(t) = \det(\gamma'(t), \dot{x}(t))
\]
is the Hessian function. Thus \( t = t_0 \) is an \( A_2 \)-inflection point if and only if
\[
\det(\gamma'(t_0), \dot{x}(t_0)) = 0, \quad \det(\gamma'(t_0), \ddot{y}(t_0)) \neq 0.
\]

Considering \( K^2 \subset P(K^3) \) as an affine subspace, this criterion is available for curves in \( P(K^3) \). When \( K = C \), it is well-known that non-singular cubic curves in \( P(C^3) \) have exactly nine inflection points which are all of \( A_2 \)-type. One special singular cubic curve is \( 2y^2 - 3x^3 = 0 \) in \( P(C^3) \) with homogeneous coordinates \([x, y, z] \), which can be parameterized as \( \gamma(t) = [\sqrt{2}t^2, \sqrt{3}t^3, 1] \). The image of the dual curve of \( \gamma \) in \( P(C^3) \) is the image of \( \gamma \) itself, and \( \gamma \) has an \( A_2 \)-type singular point \([0, 0, 1] \) and an \( A_2 \)-inflection point \([0, 1, 0] \).

These two points are interchanged by the duality. (The duality of fronts is explained in Section 3.)

**Example 2.7** (The affine Gauss map of an \( A_4 \)-inflection point). Let \( F: K^3 \rightarrow K^4 \) be a map defined by
\[
F(u, v, w) = \left( u, v, -u^2 - \frac{3v^2}{2} + uw^2 + vw^3 - \frac{w^4}{4} + \frac{w^5}{5} - \frac{w^6}{6} \right) \quad (u, v, w \in K).
\]

If we define \( \mathcal{G}: K^3 \rightarrow P(K^4) \cong P((K^4)^s) \) by
\[
\mathcal{G}(u, v, w) = [-2uw - 3vw^2 + w^3 - w^4 + w^5, 2u - w^2, 3v - w^3, 1]
\]
using the homogeneous coordinate system, \( G \) gives the affine Gauss map of \( F \). Then the Hessian \( h \) of \( F \) is

\[
\det \begin{pmatrix}
-2 & 0 & 2w \\
0 & -3 & 3w^2 \\
2w & 3w^2 & 2u + 6vw - w^2 + 4w^3 - 5w^4
\end{pmatrix} = 6(2u + 6vw - w^2 + 4w^3 - 2w^4).
\]

The asymptotic vector field is \( \xi = (w, w^2, 1) \). Hence we have

\[
h = 6(2u + 6vw - w^2 + 4w^3 - 2w^4),
\]

\[
h' = 12(3v + 6w^2 - w^3), \quad h'' = 144w, \quad h''' = 144,
\]

where \( h' = dh(\xi), \quad h'' = dh'(\xi) \) and \( h''' = dh''(\xi) \). The Jacobi matrix of \( (h, h', h'') \) at \( 0 \) is

\[
\begin{pmatrix}
2 & * & * \\
0 & 36 & * \\
0 & 0 & 144
\end{pmatrix}.
\]

This implies that \( \xi \) is 3-non-degenerate at \( 0 \) but does not meet \( I(F) = h^{-1}(0) \) at \( p \) with multiplicity 4, that is, \( F \) has an \( A_4 \)-inflection point at \( 0 \). On the other hand, \( G \) has the \( A_3 \)-Morin singularity at \( 0 \). In fact, by the coordinate change

\[
U = 2u - w^2, \quad V = 3v - w^3, \quad W = w,
\]

it follows that \( G \) is represented by a map germ

\[
(U, V, W) \mapsto -(UW + VW^2 + W^4, U, V).
\]

This coincides with the typical \( A_3 \)-Morin singularity given in (A.3) in [10].

### 3. Duality of wave fronts

Let \( P(K^{n+2}) \) be the \((n+1)\)-projective space over \( K \). We denote by \([x] \in P(K^{n+2})\) the projection of a vector \( x = (x^0, \ldots, x^{n+1}) \in K^{n+2} \setminus \{0\} \). Consider a \((2n + 3)\)-submanifold of \( K^{n+2} \times (K^{n+2})^* \) defined by

\[
\tilde{C} := \{(x, y) \in K^{n+2} \times (K^{n+2})^*; x \cdot y = 0\},
\]

and also a \((2n + 1)\)-submanifold of \( P(K^{n+2}) \times P((K^{n+2})^*) \)

\[
C := \{([x], [y]) \in P(K^{n+2}) \times P((K^{n+2})^*); x \cdot y = 0\}.
\]
As \( C \) can be canonically identified with the projective tangent bundle \( \text{PTP}(\mathbb{K}^{n+2}) \), it has a canonical contact structure: Let \( \pi: \tilde{C} \to C \) be the canonical projection, and define a 1-from

\[
\omega := \sum_{j=0}^{n+1} (x^j \, dy^j - y^j \, dx^j),
\]

which is considered as a 1-form of \( \tilde{C} \). The tangent vectors of the curves \( t \mapsto (tx, y) \) and \( t \mapsto (x, ty) \) at \((x, y) \in \tilde{C}\) generate the kernel of \( d\pi \). Since these two vectors also belong to the kernel of \( \omega \) and \( \dim(\ker \omega) = 2n + 2 \),

\[
\Pi := d\pi(\ker \omega)
\]
is a 2n-dimensional vector subspace of \( T_{\pi(x,y)}C \). We shall see that \( \Pi \) is the contact structure on \( C \). One can check that it coincides with the canonical contact structure of \( \text{PTP}(\mathbb{K}^{n+2}) \) (\( \cong C \)). Let \( U \) be an open subset of \( C \) and \( s: U \to K^{n+2} \times (K^{n+2})^* \) a section of the fibration \( \pi \). Since \( d\pi \circ ds \) is the identity map, it can be easily checked that \( \Pi \) is contained in the kernel of the 1-form \( s^*\omega \). Since \( \Pi \) and the kernel of the 1-form \( s^*\omega \) are the same dimension, they coincide. Moreover, suppose that \( p = \pi(x,y) \in C \) satisfies \( x^i \neq 0 \) and \( y^j \neq 0 \). We then consider a map of \( K^{n+1} \times (K^{n+1})^* \cong K^{n+1} \times K^{n+1} \) to \( K^{n+2} \times (K^{n+2})^* \cong K^{n+2} \times K^{n+2} \) defined by

\[
(a^0, \ldots, a^n, b^0, \ldots, b^n) \mapsto (a^0, \ldots, a^{i-1}, 1, a^{i+1}, \ldots, a^n, b^0, \ldots, b^{j-1}, 1, b^{j+1}, \ldots, b^n),
\]

and denote by \( s_{i,j} \) the restriction of the map to the neighborhood of \( p \) in \( C \). Then one can easily check that

\[
s_{i,j}^* \left[ \omega \wedge \left( \bigwedge_{u=0}^{n} dw \right) \right]
\]
does not vanish at \( p \). Thus \( s_{i,j}^*\omega \) is a contact form, and the hyperplane field \( \Pi \) defines a canonical contact structure on \( C \). Moreover, the two projections from \( C \) into \( P(\mathbb{K}^{n+2}) \) are both Legendrian fibrations, namely we get a double Legendrian fibration. Let \( f = [F]: M^n \to P(\mathbb{K}^{n+2}) \) be a front. Then there is a Legendrian immersion of the form \( L = ([F], [G]): M^n \to C \). Then \( g = [G]: M^n \to P((\mathbb{K}^{n+2})^*) \) satisfies (1.6) and (1.7). In particular, \( L := \pi(F, G): M^n \to C \) gives a Legendrian immersion, and \( f \) and \( g \) can be regarded as mutually dual wave fronts as projections of \( L \).

Proof of Theorem B. Since our contact structure on \( C \) can be identified with the contact structure on the projective tangent bundle on \( P(\mathbb{K}^{n+2}) \), we can apply the criteria of \( A_k \)-singularities as in Fact 2.5. Thus a nondegenerate singular point \( p \) is an \( A_k \)-singular point of \( f \) if and only if the null vector field \( \eta \) of \( f \) (as a wave front) is
(k − 1)-nondegenerate at p, but does not meet the hypersurface S(f) with multiplicity k at p. Like as in the proof of Theorem A, we may assume that \( F(p) \cdot F(p) \neq 0 \) and \( G(p) \cdot F(p) \neq 0 \) simultaneously by a suitable affine transformation of \( \mathbb{K}^{n+2} \), even when \( K = C \). Since \((F_x, \ldots, F_x, F, F', G)\) is a regular \((n + 2) \times (n + 2)\)-matrix if and only if \( f = [F] \) is an immersion, the assertion immediately follows from the identity

\[
\begin{pmatrix}
  h_{11} & \ldots & h_{1n} & 0 & * \\
  \vdots & \ddots & \vdots & \vdots & \vdots \\
  h_{n1} & \ldots & h_{nn} & 0 & * \\
  0 & \ldots & 0 & G \cdot G & 0 \\
  * & \ldots & * & iF \cdot F & 0
\end{pmatrix}
= \begin{pmatrix}
  G_{x_1} \\
  \vdots \\
  G_{x_n} \\
  G \\
  iF
\end{pmatrix}(F_x, \ldots, F_x, F, F', G).
\]

\[ \square \]

Proof of Theorem C. Let \( g : M^2 \to P((\mathbb{R}^4)^n) \) be the dual of \( f \). We fix \( p \in M^2 \) and take a simply connected and connected neighborhood \( U \) of \( p \).

Then there are lifts \( \hat{f}, \hat{g} : U \to S^3 \) into the unit sphere \( S^3 \) such that

\[
\hat{f} \cdot \hat{g} = 0, \quad d\hat{f}(v) \cdot \hat{g} = d\hat{g}(v) \cdot \hat{f} = 0 \quad (v \in TU),
\]

where \( \cdot \) is the canonical inner product on \( \mathbb{R}^4 \supset S^3 \). Since \( \hat{f} \cdot \hat{f} = 1 \), we have

\[
d\hat{f}(v) \cdot \hat{f}(p) = 0 \quad (v \in T_p M^2).\]

Thus

\[
d\hat{f}(T_p M^2) = \{ \xi \in S^3 : \xi \cdot \hat{f}(p) = \xi \cdot \hat{g}(p) = 0 \},
\]

which implies that \( df(TM^2) \) is equal to the limiting tangent bundle of the front \( g \). So we apply (2.5) in [9] for \( g \): Since the singular set \( S(g) \) of \( g \) consists only of cuspidal edges and swallowtails, the Euler number of \( S(g) \) vanishes. Then it holds that

\[
\chi(M_+) + \chi(M_-) = \chi(M^2) = \chi(M_+) - \chi(M_-) + i_2^+(f) - i_2^-(f),
\]

which proves the formula.  

\[ \square \]

When \( n = 2 \), the duality of fronts in the unit 2-sphere \( S^2 \) (as the double cover of \( P(\mathbb{R}^3) \)) plays a crucial role for obtaining the classification theorem in [6] for complete flat fronts with embedded ends in \( \mathbb{R}^3 \). Also, a relationship between the number of inflection points and the number of double tangents on certain class of simple closed regular curves in \( P(\mathbb{R}^3) \) is given in [11]. (For the geometry and a duality of fronts in \( S^2 \), see [1].) In [7], Porteous investigated the duality between \( A_k \)-singular points and \( A_k \)-inflection points when \( k = 2, 3 \) on a surface in \( S^3 \).
4. Cuspidal curvature on 3/2-cusps

Relating to the duality between singular points and inflection points, we introduce a curvature on 3/2-cusps of planar curves:

Suppose that \((M^2, g)\) is an oriented Riemannian manifold, \(\gamma : I \to M^2\) is a front, \(v(t)\) is a unit normal vector field, and \(I\) an open interval. Then \(t = t_0 \in I\) is a 3/2-cusp if and only if \(\dot{\gamma}(t_0) = 0\) and \(\Omega(\dot{\gamma}(t_0), \ddot{\gamma}(t_0)) \neq 0\), where \(\Omega\) is the unit 2-form on \(M^2\), that is, the Riemannian area element, and the dot means the covariant derivative. When \(t = t_0\) is a 3/2-cusp, \(\dot{v}(t)\) does not vanish (if \(M^2 = \mathbb{R}^2\), it follows from Proposition A'). Then we take the (arclength) parameter \(s\) near \(\gamma(t_0)\) so that \(|v'(s)| = \sqrt{g(v'(s), v'(s))} = 1\) \((s \in I)\), where \(v' = dv/ds\). Now we define the **cuspidal curvature** \(\mu\) by

\[
\mu := 2 \text{sgn}(\rho) \sqrt{\left. \frac{ds}{d\rho} \right|_{s=s_0}} \quad (\rho = 1/\kappa_g),
\]

where we choose the unit normal \(v(s)\) so that it is smooth around \(s = s_0\) \((s_0 = s(t_0))\).

If \(\mu > 0\) (resp. \(\mu < 0\)), the cusp is called positive (resp. negative). It is an interesting phenomenon that the left-turning cusps have negative cuspidal curvature, although the left-turning regular curves have positive geodesic curvature (see Fig. 4.1). Then it holds that

\[
\mu = \frac{\Omega(\dot{\gamma}(t), \ddot{\gamma}(t))}{|\dot{\gamma}(t)|^{5/2}} \left|_{t=t_0} \right. = 2 \frac{\Omega(v(t), \dot{v}(t))}{\sqrt{|\Omega(\dot{\gamma}(t), \ddot{\gamma}(t))|}} \left|_{t=t_0} \right.
\]

(4.1)

We now examine the case that \((M^2, g)\) is the Euclidean plane \(\mathbb{R}^2\), where \(\Omega(v, w)\) \((v, w \in \mathbb{R}^2)\) coincides with the determinant \(\det(v, w)\) of the \(2 \times 2\)-matrix \((v, w)\). A **cycloid** is a rigid motion of the curve given by \(c(t) := a(t - \sin t, 1 - \cos t)\) \((a > 0)\), and here \(a\) is called the **radius** of the cycloid. The cuspidal curvature of \(c(t)\) at \(t = 2\pi \mathbb{Z}\) is equal to \(-1/\sqrt{a}\). In [12], the second author proposed to consider the curvature as the inverse of radius of the cycloid which gives the best approximation of the given 3/2-cusp. As shown in the next proposition, \(\mu^2\) attains this property:

**Proposition 4.1.** Suppose that \(\gamma(t)\) has a 3/2-cusp at \(t = t_0\). Then by a suitable choice of the parameter \(t\), there exists a unique cycloid \(c(t)\) such that

\[
\gamma(t) - c(t) = o((t - t_0)^3),
\]

where \(o((t - t_0)^3)\) denotes a higher order term than \((t - t_0)^3\). Moreover, the square of the absolute value of cuspidal curvature of \(\gamma(t)\) at \(t = t_0\) is equal to the inverse of the radius of the cycloid \(c\).

**Proof.** Without loss of generality, we may set \(t_0 = 0\) and \(\gamma(0) = 0\). Since \(t = 0\) is a singular point, there exist smooth functions \(a(t)\) and \(b(t)\) such that \(\gamma(t) = t^2(a(t), b(t))\).
The duality on wave fronts

Fig. 4.1. A positive cusp and a negative cusp.

Since \( t = 0 \) is a 3/2-cusp, \((a(0), b(0)) \neq 0\). By a suitable rotation of \( \gamma \), we may assume that \( b(0) \neq 0 \) and \( a(0) = 0 \). Without loss of generality, we may assume that \( b(0) > 0 \). By setting \( s = t \sqrt{b(t)} \), \( \gamma(s) = \gamma(t(s)) \) has the expansion

\[
\gamma(s) = (\alpha s^3, s^2) + o(s^3) \quad (\alpha \neq 0).
\]

Since the cuspidal curvature changes sign by reflections on \( \mathbb{R}^2 \), it is sufficient to consider the case \( \alpha > 0 \). Then, the cycloid

\[
c(t) := \frac{2}{9\alpha^2} (t - \sin t, 1 - \cos t)
\]

is the desired one by setting \( s = t/(3\alpha) \). \( \Box \)

It is well-known that the cycloids are the solutions of the brachistochrone problem. We shall propose to call the number \( 1/|\mu|^2 \) the cuspidal curvature radius which corresponds the radius of the best approximating cycloid \( c \).

Remark 4.2. During the second author’s stay at Saitama University, Toshizumi Fukui pointed out the followings: Let \( \gamma(t) \) be a regular curve in \( \mathbb{R}^2 \) with non-vanishing curvature function \( \kappa(t) \). Suppose that \( t \) is the arclength parameter of \( \gamma \). For each \( t = t_0 \), there exists a unique cycloid \( c \) such that a point on \( c \) gives the best approximation of \( \gamma(t) \) at \( t = t_0 \) (namely \( c \) approximates \( \gamma \) up to the third jet at \( t_0 \)). The angle \( \theta(t_0) \) between the axis (i.e. the normal line of \( c \) at the singular points) of the cycloid and the normal line of \( \gamma \) at \( t_0 \) is given by

\[
\sin \theta = \frac{\kappa^2}{\sqrt{\kappa^4 + \kappa''^2}},
\]

and the radius \( a \) of the cycloid is given by

\[
a := \frac{\sqrt{\kappa^4 + \kappa''^2}}{|\kappa|^3}.
\]

One can prove (4.2) and (4.3) by straightforward calculations. The cuspidal curvature radius can be considered as the limit.
ADDED IN PROOF. In a recent authors’ preprint, “The intrinsic duality of wave fronts (arXiv:0910.3456), $A_{k+1}$-singularities are defined intrinsically. Moreover, the duality between fronts and their Gauss maps is also explained intrinsically.

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