MATCHING COMPLEXES OF $3 \times n$ GRID GRAPHS

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Abstract. The matching complex of a graph $G$ is a simplicial complex whose simplices are matchings in $G$. In the last few years the matching complexes of grid graphs have gained much attention among the topological combinatorists. In 2017, Braun and Hough obtained homological results related to the matching complexes of $2 \times n$ grid graphs. Further in 2019, Matsushita showed that the matching complexes of $2 \times n$ grid graphs are homotopy equivalent to a wedge of spheres. In this article we prove that the matching complexes of $3 \times n$ grid graphs are homotopy equivalent to a wedge of spheres. We also give the comprehensive list of the dimensions of spheres appearing in the wedge.

1. Introduction

A matching in a (simple) graph $G$ is a collection of pairwise disjoint edges of $G$. The matching complex of $G$, denoted $M(G)$, is a simplicial complex whose vertex set is the edge set of $G$ and simplices are all the matchings in $G$. The matching complexes first appeared in the 1979 work of Garst [4], where the matching complexes of complete bipartite graphs (also known as the chessboard complexes) were studied while dealing with the Tits coset complexes. In 1992, Bouc [1] studied the matching complexes of complete graphs in connection with the Brown complexes and Quillen complexes. Thereafter, these complexes arose in connection with several areas of mathematics. For a broader perspective, see the 2003 survey article of Wachs [10].

Let $[n]$ denotes the set $\{1, 2, \ldots, n\}$. For two positive integers $m, n$, the $m \times n$ (rectangular) grid graph $\Gamma_{m,n}$ is a graph with vertex set $V(\Gamma_{m,n})$ and edge set $E(\Gamma_{m,n})$ defined as follows:

$$V(\Gamma_{m,n}) = \{(i, j) \in \mathbb{N}^2 : i \in [m], j \in [n]\}, \text{ and}$$

$$E(\Gamma_{m,n}) = \{((i, j), (i', j')) : |i - i'| + |j - j'| = 1\}.$$

The matching complex of $\Gamma_{1,n}$ was computed by Kozlov in [8]. In the same article, he also computed the matching complexes of cycle graphs. In 2005, Jonsson [6] studied the homotopical depth and topological connectivity of matching complexes of grid graphs and stated that “it is probably very hard to determine the homotopy type of” these complexes. In 2017, Braun and Hough [2] obtained homological results related to matching complexes of $2 \times n$ grid graphs. Matsushita [9], in 2019, extended their results by showing that the matching complexes of $2 \times n$ grid graphs are homotopy equivalent to a wedge of spheres. In this article, we compute the homotopy type of matching complexes of $3 \times n$ grid graphs. The main results of this article are summarised below.

Theorem 1.1. For $n \geq 1$, the matching complex of $\Gamma_{3,n}$ is homotopy equivalent to a wedge of spheres. Moreover, if $n \in \{9k, 9k + 1, \ldots, 9k + 8\}$ for some $k \geq 0$, then

$$M(\Gamma_{3,n}) \simeq \bigvee_{i=n-1}^{n+k-1} (b_i S^i),$$

where $b_i$’s are some positive integers and $\simeq$ denotes the homotopy equivalence of spaces.

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For a graph $G$, a subset $I \subseteq V(G)$ is said to be independent if there are no edges in the induced subgraph $G[I]$, i.e., $E(G[I]) = \emptyset$. The independence complex of $G$, denoted $\text{Ind}(G)$, is a simplicial complex whose vertex set is $V(G)$ and simplices are all the independent subsets of $G$. The line graph of a graph $G$, denoted $L(G)$, is a graph with $V(L(G)) = E(G)$ and two distinct vertices $(a_1, b_1), (a_2, b_2) \in V(L(G))$ are adjacent if and only if $\{a_1, b_1\} \cap \{a_2, b_2\} \neq \emptyset$. Note that the matching complex of $G$ is same as the independence complex of its line graph, i.e., $M(G) = \text{Ind}(L(G))$.

Let $G_n$ denotes the line graph of the grid graph $\Gamma_{3,n}$. To compute the homotopy type of $M(\Gamma_{3,n})$, we determine the homotopy type of $\text{Ind}(G_n)$. The main idea used in this article for the computation of $\text{Ind}(G_n)$ is to make a step by step careful choice to reduce the graph $G_n$ and arrive at different classes of graphs (a total of nine). All these new nine classes of graphs have been defined in Section 3.1. To obtain the Theorem 1.1, we use simultaneous inductive arguments on the independence complexes of these ten classes of graphs. For a quick overview of the relations between all these ten classes of graphs, we refer the reader to see Figure 5.1.

**Flow of the article:** In the following section, we list out various definitions and results that are used in this article. Section 3 is subdivided into three major subsections. The first two subsections deal with the base cases for the graph $G_n$ along with nine more associated classes of graphs. In the next subsection, Section 3.3, we provide and prove recursive formulae to compute the homotopy type of the independence complexes of these ten classes of graphs. The main result of Section 4 is Theorem 4.1, which gives the exact dimensions of the spheres occurring in the homotopy type of the independence complexes of the above mentioned ten classes of graphs.

2. **Preliminaries**

An (abstract) simplicial complex $\mathcal{K}$ is a collection of finite sets such that if $\tau \in \mathcal{K}$ and $\sigma \subset \tau$, then $\sigma \in \mathcal{K}$. The elements of $\mathcal{K}$ are called the simplices (or faces) of $\mathcal{K}$. If $\sigma \in \mathcal{K}$ and $|\sigma| = k + 1$, then $\sigma$ is said to be $k$-dimensional. The set of 0-dimensional simplices of $\mathcal{K}$ is denoted by $V(\mathcal{K})$, and its elements are called vertices of $\mathcal{K}$. A subcomplex of a simplicial complex $\mathcal{K}$ is a simplicial complex whose simplices are contained in $\mathcal{K}$. In this article, we always assume empty set as a simplex of any simplicial complex and we consider any simplicial complex as a topological space, namely its geometric realization. For the definition of geometric realization, we refer to Kozlov’s book [7].

For a simplex $\sigma \in \mathcal{K}$, define
\[
\text{lk}(\sigma, \mathcal{K}) := \{\tau \in \mathcal{K} : \sigma \cap \tau = \emptyset, \sigma \cup \tau \in \mathcal{K}\},
\]
\[
\text{del}(\sigma, \mathcal{K}) := \{\tau \in \mathcal{K} : \sigma \not\subseteq \tau\}.
\]

The simplicial complexes $\text{lk}(\sigma, \mathcal{K})$ and $\text{del}(\sigma, \mathcal{K})$ are called link of $\sigma$ in $\mathcal{K}$ and (face) deletion of $\sigma$ in $\mathcal{K}$ respectively. The join of two simplicial complexes $\mathcal{K}_1$ and $\mathcal{K}_2$, denoted as $\mathcal{K}_1 \ast \mathcal{K}_2$, is a simplicial complex whose simplices are disjoint union of simplices of $\mathcal{K}_1$ and $\mathcal{K}_2$. Let $\Delta^S$ denotes a $(|S| - 1)$-dimensional simplex with vertex set $S$. The cone on $\mathcal{K}$ with apex $a$, denoted as $C_a(\mathcal{K})$, is defined as
\[
C_a(\mathcal{K}) := \mathcal{K} \ast \Delta^{|a|}.
\]

For $a, b \notin V(\mathcal{K})$, the suspension of $\mathcal{K}$, denoted as $\Sigma(\mathcal{K})$, is defined as
\[
\Sigma(\mathcal{K}) := \mathcal{K} \ast \{a\} \cup \mathcal{K} \ast \{b\}.
\]
Observe that for any vertex $v \in V(K)$, we have
\[ K = C_v(\text{lk}(v, K)) \cup \text{del}(v, K) \text{ and } C_v(\text{lk}(v, K)) \cap \text{del}(v, K) = \text{lk}(v, K). \]
Clearly, $C_v(\text{lk}(v, K))$ is contractible. Therefore, from [5, Example 0.14], we have the following.

**Lemma 2.2.** Let $K$ be a simplicial complex and $v$ be a vertex of $K$. If $\text{lk}(v, K)$ is contractible in $\text{del}(v, K)$ then
\[ K \simeq \text{del}(v, K) \lor \Sigma(\text{lk}(v, K)), \]
where $\lor$ denotes the wedge of spaces.

A (simple) graph is an ordered pair $G = (V(G), E(G))$, where $V(G)$ is called the set of vertices and $E(G) \subseteq \binom{V(G)}{2}$, the set of (unordered) edges of $G$. The vertices $v_1, v_2 \in V(G)$ are said to be adjacent, if $(v_1, v_2) \in E(G)$.

The following observation directly follows from the definition of independence complexes of graphs.

**Lemma 2.3.** Let $G_1 \sqcup G_2$ denotes the disjoint union of two graphs $G_1$ and $G_2$. Then
\[ \text{Ind}(G_1 \sqcup G_2) \simeq \text{Ind}(G_1) * \text{Ind}(G_2). \]

Let $G$ and $H$ be two graphs. A map $f : V(G) \to V(H)$ is said to be a graph homomorphism if $(f(v), f(w)) \in E(H)$ for all $(v, w) \in E(G)$. A graph homomorphism is called an isomorphism if it is bijective and its inverse map is also a graph homomorphism. Two graphs $G$ and $H$ are said to be isomorphic if there is an isomorphism between them and we denote it by $G \simeq H$.

For a subset $A \subseteq V(G)$, the set of neighbours of $A$ is $N_G(A) = \{x \in V(G) : (x, a) \in E(G) \text{ for some } a \in A\}$. The closed neighbourhood set of $A \subseteq V(G)$, is $N_G[A] = N_G(A) \cup A$. If $A = \{v\}$ is a singleton set, then we write $N_G(v)$ (resp. $N_G[v]$) for $N_G(\{v\})$ (resp. $N_G(\{v\})$). A graph $H$ with $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$ is called a subgraph of the graph $G$. For a nonempty subset $U \subseteq V(G)$, the induced subgraph $G[U]$, is the subgraph of $G$ with $V(G[U]) = U$ and $E(G[U]) = \{(a, b) \in E(G) : a, b \in U\}$. The graph $G[V(G) \setminus A]$ is denoted by $G - A$, for $A \subseteq V(G)$. For a subset $B \subseteq E(G)$, we let $G - B$ to be the graph with the vertex set $V(G - B) = V(G)$ and the edge set $E(G - B) = E(G) \setminus B$.

**Lemma 2.4.** [3, Lemma 2.4] Let $G$ be a graph and $\{a, b\}$ be a 1-simplex in $\text{Ind}(G)$. If $\text{Ind}(G - N_G[\{a, b\}])$ is contractible, then $\text{Ind}(G) \simeq \text{Ind}(\tilde{G})$, where $V(\tilde{G}) = V(G)$ and $E(\tilde{G}) = E(G) \cup \{(a, b)\}$.

**Proof.** Let $x = \{a, b\}$. Observe that $\text{del}(x, \text{Ind}(G)) = \text{Ind}(\tilde{G})$ and $\text{lk}(x, \text{Ind}(G)) = \text{Ind}(\tilde{G}) = \text{Ind}(G - N_G[\{a, b\}])$. Since $\text{Ind}(G - N_G[\{a, b\}])$ is contractible, the result follows from Lemma 2.1. \qed

**Lemma 2.5.** [3, Lemma 2.5] Let $G$ be graph and $v$ be a simplicial vertex of $G$. Let $N_G(v) = \{w_1, w_2, \ldots, w_k\}$. Then
\[ \text{Ind}(G) \simeq \bigvee_{i=1}^{k} \text{Ind}(G - N_G[w_i]). \]

\[1\]A vertex $v$ of $G$ is called simplicial if $G[N_G(v)]$ is a complete graph, i.e. any two distinct vertices are adjacent.
For $r \geq 1$, the path graph $P_r$ is a graph with $V(P_r) = \{1, \ldots, r\}$ and $E(P_r) = \{(i, i+1) : i \in [r-1]\}$.

**Lemma 2.6.** [8, Proposition 4.6] For $r \geq 1$,

$$\text{Ind}(P_r) \simeq \begin{cases} S^{k-1} & \text{if } r = 3k, \\ \text{pt} & \text{if } r = 3k + 1, \\ S^k & \text{if } r = 3k + 2. \end{cases}$$

For $r \geq 3$, the cycle graph $C_r$ is the graph with $V(C_r) = \{1, \ldots, r\}$ and $E(C_r) = \{(i, i+1) : i \in [r-1]\} \cup \{(1, r)\}$.

**Lemma 2.7.** [8, Proposition 5.2] For $r \geq 3$,

$$\text{Ind}(C_r) \simeq \begin{cases} S^{k-1} \vee S^{k-1} & \text{if } r = 3k, \\ S^{k-1} & \text{if } r = 3k + 1. \end{cases}$$

We now proceed towards the main graph of this article. Recall that for $m, n \in \mathbb{N}$, the $m \times n$ grid graph is denoted by $\Gamma_{m,n}$ and $G_n = L(\Gamma_{3,n})$ denotes the line graph of $\Gamma_{3,n}$ (see Figure 2.1 for example). Formally we define $G_n$ with,

- $V(G_n) = \{u_i, v_j, w_i, x_j, y_i : i \in [n-1], j \in [n]\}$,
- $E(G_1) = \{(v_1, x_1)\}$, $E(G_2) = \{(v_1, x_1), (w_1, v_1), (w_1, x_1), (w_1, v_2), (u_1, v_1), (u_1, v_2), (v_2, x_2), (x_1, y_1), (y_1, x_2)\}$, and for $n \geq 3$,
- $E(G_n) = \{(u_i, v_i), (v_i, y_i), (w_i, x_i), (x_i, y_i), (y_i, x_{i+1}), (x_{i+1}, y_i), (w_i, x_{i+1}), (x_{i+1}, v_{i+1}), (v_{i+1}, v_i) : i \in [n-1]\} \cup \{(u_i, u_{i+1}), (w_i, w_{i+1}), (y_i, y_{i+1}) : i \in [n-2]\} \cup \{(v_i, v_i) : i \in [n]\}$.

![Figure 2.1](image)

**Figure 2.1**

3. Homotopy type of independence complexes of $G_n$ and associated classes of graphs

In this section, we define nine new graph classes viz. $\{B_n\}_{n \in \mathbb{N}}$, $\{A_n\}_{n \in \mathbb{N}}$, $\{D_n\}_{n \in \mathbb{N}}$, $\{J_n\}_{n \in \mathbb{N}}$, $\{O_n\}_{n \in \mathbb{N}}$, $\{M_n\}_{n \in \mathbb{N}}$, $\{Q_n\}_{n \in \mathbb{N}}$, $\{F_n\}_{n \in \mathbb{N}}$, $\{H_n\}_{n \in \mathbb{N}}$ and compute the homotopy type of their independence complexes along with that of $G_n$. The $n$-th member of each of these graph classes contains a copy of $G_n$ as an induced subgraph but not of $G_{n+1}$.

We divide this section into three subsections. In the first subsection, we define the above said nine classes of graphs and compute the homotopy type of independence complexes of these graphs along with $G_n$ for $n = 1$. In the next subsection, we compute the homotopy type of independence complexes of all these graphs for $n = 2$. The final subsection is devoted towards proving recursive formulae for the independence complexes of all ten graph classes, thereby computing their homotopy types. In particular, we show that the independence complex of each of the graphs from these graph classes is a wedge of spheres up to homotopy.
3.1. Graph definitions and \( n = 1 \) computations.

3.1.1. \( G_n \). Since \( G_1 \) is an edge (see Figure 2.1a), \( \text{Ind}(G_1) \simeq S^0 \).

3.1.2. \( B_n \). For \( n \geq 1 \), we define the graph \( B_n \) as follows:
\[
V(B_n) = V(G_n) \sqcup \{b_1, b_2, b_3, b_4\},
E(B_1) = E(G_1) \sqcup \{(b_1, v_1), (b_1, b_2), (b_2, b_3), (b_3, b_1), (b_4, x_1)\} \text{ and for } n \geq 2,
E(B_n) = E(G_n) \sqcup \{(b_1, u_1), (b_1, v_1), (b_1, b_2), (b_2, b_3), (b_3, b_4), (b_4, x_1), (b_4, y_1)\}.
\]

Since \( B_1 \cong C_6 \), Lemma 2.7 implies that \( \text{Ind}(B_1) \simeq S^1 \lor S^1 \).

3.1.3. \( A_n \). For \( n \geq 1 \), we define the graph \( A_n \) as follows: \( V(A_n) = V(G_n) \sqcup \{a\}, \)
\[
E(A_1) = E(G_1) \sqcup \{(a, x_1), (a, v_1)\} \text{ and for } n \geq 2,
E(A_n) = E(G_n) \sqcup \{(a, x_1), (a, v_1), (a, w_1)\}.
\]

Since \( A_1 \simeq C_3 \), Lemma 2.7 implies that \( \text{Ind}(A_1) \simeq S^0 \lor S^0 \).

3.1.4. \( D_n \). For \( n \geq 1 \), we define the graph \( D_n \) as follows:
\[
V(D_n) = V(G_n) \sqcup \{d\},
E(D_1) = E(G_1) \sqcup \{(v_1, d)\} \text{ and for } n \geq 2,
E(D_n) = E(G_n) \sqcup \{(d, v_1), (d, u_1)\}.
\]

Clearly, \( \text{Ind}(D_1) \simeq S^0 \).
3.1.5. $J_n$. For $n \geq 1$, we define the graph $J_n$ as follows:

$V(J_n) = V(G_n) \cup \{j_1, j_2, j_3, j_4, j_5, j_6\}$,
$E(J_1) = E(G_1) \cup \{(j_1, v_1), (j_1, x_1), (j_1, j_3), (j_2, j_3), (j_2, x_1), (j_2, j_5), (j_3, j_4), (j_3, j_5), (j_4, j_6), (j_5, j_6)\}$ and for $n \geq 2$,

$E(J_n) = E(G_n) \cup \{(j_1, v_1), (j_1, x_1), (j_1, j_3), (j_2, j_3), (j_2, j_5), (j_3, j_4), (j_3, j_5), (j_4, j_6), (j_5, j_6), (j_2, y_1), (j_1, w_1)\}$.

(a) $J_1$  
(b) $J_n, \ n \geq 2$

Since $N_{J_1}(j_6) \subseteq N_{J_1}(j_3)$, Lemma 2.4 implies that $\text{Ind}(J_1) \simeq \text{Ind}(J_1 - \{j_3\})$. Let $J'_1$ be the graph $J_1 - \{j_3\}$. Using the fact that $v_1$ is a simplicial vertex in $J'_1$ and $N_{J_1}(v_1) = \{j_1, x_1\}$, from Lemma 2.5 we get that $\text{Ind}(J'_1) \simeq \Sigma(\text{Ind}(J'_1 - N_{J_1}[v_1])) \vee \Sigma(\text{Ind}(J'_1 - N_{J_1}[x_1]))$. Observe that $J'_1 - N_{J_1}[j_1] \cong P_5 \cong J'_1 - N_{J_1}[x_1]$. Therefore using Lemma 2.6, we conclude that $\text{Ind}(J_1) \simeq S^1 \lor S^1$.

3.1.6. $O_n$. For $n \geq 1$, we define the graph $O_n$ as follows:

$V(O_n) = V(G_n) \cup \{o_1, o_2, o_3, o_4, o_5, o_6, o_7, o_8, o_9\}$,
$E(O_1) = E(G_1) \cup \{(o_1, v_1), (o_1, o_4), (o_2, v_1), (o_2, x_1), (o_2, o_4), (o_3, x_1), (o_3, o_6), (o_4, o_5), (o_5, o_7), (o_6, o_8), (o_7, o_9), (o_8, o_9)\}$ and for $n \geq 2$,

$E(O_n) = E(G_n) \cup \{(o_1, u_1), (o_1, v_1), (o_1, v_4), (o_2, v_1), (o_2, x_1), (o_2, o_4), (o_2, o_5), (o_2, w_1), (o_3, x_1), (o_3, y_1), (o_3, o_5), (o_4, o_5), (o_5 o_8), (o_7, o_9), (o_8, o_9)\}$.

(a) $O_1$  
(b) $O_n, \ n \geq 2$

Since $N_{O_1}(o_1) \subseteq N_{O_1}(o_2)$, Lemma 2.4 implies that $\text{Ind}(O_1) \simeq \text{Ind}(O_1 - \{o_2\})$. Observe that $O_1 - \{o_2\} \cong C_{10}$, thus by Lemma 2.7 we get that $\text{Ind}(O_1) \simeq S^2$.

3.1.7. $M_n$. For $n \geq 1$, we define the graph $M_n$ as follows:

(a) $M_1$  
(b) $M_n, \ n \geq 2$
Observe that
\[
\text{Ind}(M_1) = \text{Ind}(G_1) \cup \{(m_1, v_1), (m_1, x_1), (m_1, w_1), (m_1, m_2), (m_2, m_3), (m_3, x_1)\}
\] and for \( n \geq 2, \)
\[
\text{Ind}(M_n) = \text{Ind}(G_n) \cup \{(m_1, v_1), (m_1, x_1), (m_1, w_1), (m_1, m_2), (m_2, m_3), (m_3, x_1), (m_3, y_1)\}.
\]
Since \( N_{M_1}(m_2) \subseteq N_{M_1}(x_1) \) and \( M_1 - \{x_1\} \cong \text{pt} \) by Lemma 2.4 and Lemma 2.6 imply that \( \text{Ind}(M_1) \cong \text{Ind}(M_1 - \{x_1\}) \cong \text{Ind}(P_4) \cong \text{pt} \).

3.1.8. \( Q_n \). For \( n \geq 1 \), we define the graph \( Q_n \) as follows:
\[
V(Q_n) = V(G_n) \cup \{q_1, q_2, q_3, q_4, q_5, q_6, q_7\},
\]
\[
E(Q_1) = E(G_1) \cup \{(q_1, v_1), (q_1, q_3), (q_1, q_4), (q_2, v_1), (q_2, q_3), (q_2, q_5), (q_3, q_4), (q_4, q_6), (q_5, q_6), (q_5, q_7), (q_6, q_7)\} \text{ and for } n \geq 2,
\]
\[
E(Q_n) = E(G_n) \cup \{(q_1, u_1), (q_1, v_1), (q_1, q_3), (q_1, q_4), (q_2, v_1), (q_2, q_3), (q_2, q_5), (q_3, q_4), (q_3, q_5), (q_4, q_6), (q_5, q_6), (q_5, q_7), (q_6, q_7)\}.
\]

Since \( q_7 \) is a simplicial vertex in \( Q_1 \) and \( N_{Q_1}(q_7) = \{q_5, q_6\} \), Lemma 2.5 implies that
\[
\text{Ind}(Q_1) \cong \Sigma(\text{Ind}(Q_1 - N_{Q_1}[q_5])) \lor \Sigma(\text{Ind}(Q_1 - N_{Q_1}[q_6])).
\]
Observe that \( Q_1 - N_{Q_1}[q_5] \cong P_4 \), therefore \( \text{Ind}(Q_1 - N_{Q_1}[q_5]) \) is contractible by Lemma 2.6. Since \( N_{Q_1 - N_{Q_1}[q_5]}(q_7) \subseteq N_{Q_1 - N_{Q_1}[q_5]}(v_1) \), Lemma 2.4 implies that \( \text{Ind}(Q_1 - N_{Q_1}[q_6]) \cong \text{Ind}(Q_1 - N_{Q_1}[q_6] - \{v_1\}) \). Since \( Q_1 - N_{Q_1}[q_6] - \{v_1\} \cong P_4 \), we conclude that \( \text{Ind}(Q_1) \) is contractible.

3.1.9. \( F_n \). For \( n \geq 1 \), we define the graph \( F_n \) as follows:
\[
V(F_n) = V(G_n) \cup \{f_1, f_2, f_3, f_4\},
\]
\[
E(F_1) = E(G_1) \cup \{(f_1, v_1), (f_1, x_1), (f_1, f_2), (f_2, f_3), (f_2, f_4), (f_3, f_4), (f_4, x_1)\} \text{ and for } n \geq 2,
\]
\[
E(F_n) = E(G_n) \cup \{(f_1, v_1), (f_1, x_1), (f_1, f_2), (f_1, w_1), (f_1, f_3), (f_2, f_4), (f_3, f_4), (f_4, x_1), (f_4, y_1)\}.
\]

Observe that \( f_3 \) is a simplicial vertex in \( F_1 \) and \( N_{F_1}(f_3) = \{f_2, f_4\} \). Using Lemma 2.5, we get that
\[
\text{Ind}(F_1) \cong \Sigma(\text{Ind}(F_1 - N_{F_1}[f_2])) \lor \Sigma(\text{Ind}(F_1 - N_{F_1}[f_4])).
\]
Since \( F_1 - N_{F_1}[f_2] = F_1 - \{f_1, f_2, f_3, f_4\} \cong P_2 \cong F_1 - \{x_1, f_2, f_3, f_4\} = F_1 - N_{F_1}[f_4], \)
\[
\text{Ind}(F_1) \cong \Sigma(\text{Ind}(P_2)) \lor \Sigma(\text{Ind}(P_2)) \cong S^1 \lor S^1.
\]
3.1.10. \( H_n \). For \( n \geq 1 \), we define the graph \( H_n \) as follows:
\[
V(H_n) = V(G_n) \cup \{h_1, h_2, h_3, h_4\},
\]
\[
E(H_1) = E(G_1) \cup \{(h_1, v_1), (h_1, h_2), (h_2, h_3), (h_2, h_4), (h_3, h_4), (h_4, v_1), (h_4, x_1)\}
\] and for \( n \geq 2 \),
\[
E(H_n) = E(G_n) \cup \{(h_1, v_1), (h_1, u_1), (h_1, h_2), (h_2, h_3), (h_2, h_4), (h_3, h_4), (h_4, v_1), (h_4, x_1), (h_4, w_1)\}.
\]

![Graphs](image)

(a) \( H_1 \) (b) \( H_n, n \geq 2 \)

Since \( h_3 \) is a simplicial vertex in \( H_1 \) and \( N_{H_1}(h_3) = \{h_2, h_4\} \), Lemma 2.5 implies that

\[
\text{Ind}(H_1) \simeq \Sigma(\text{Ind}(H_1 - N_{H_1}[h_2])) \lor \Sigma(\text{Ind}(H_1 - N_{H_1}[h_4])).
\]

Note that \( H_1 - N_{H_1}[h_2] = H_1 - \{h_1, h_2, h_3, h_4\} \cong P_3 \) and \( H_1 - N_{H_1}[h_4] = H_1 - \{v_1, x_1, h_2, h_3, h_4\} \cong P_3 \), we get that \( \text{Ind}(H_1) \simeq \Sigma(\text{Ind}(P_3)) \lor \Sigma(\text{Ind}(P_1)) \simeq S^1 \lor \text{pt} \simeq S^1 \).

3.2. Case \( n = 2 \) computation.

3.2.1. \( G_2 \). In \( G_2 \), \( N_{G_2}(u_1) \subseteq N_{G_2}(w_1) \) (see figure on the right), thus by Lemma 2.4, \( \text{Ind}(G_2) \simeq \text{Ind}(G_2 - \{w_1\}) \). Since \( G_2 - \{w_1\} \cong C_6 \), Lemma 2.7 implies that \( \text{Ind}(G_2 - \{w_1\}) \simeq S^1 \lor S^1 \). Therefore \( \text{Ind}(G_2) \simeq S^1 \lor S^1 \).

3.2.2. \( B_2 \). We consider the vertex \( b_4 \) in \( B_2 \) (see figure on the right) and analyse \( \text{del}(b_4, \text{Ind}(B_2)) \) and \( \text{lk}(b_4, \text{Ind}(B_2)) \).

First note that \( \text{lk}(b_4, \text{Ind}(B_2)) = \text{Ind}(B_2 - N_{B_2}[b_4]) \). Let \( B'_2 \) denote the graph \( B_2 - N_{B_2}[b_4] \) (see Figure 3.10c). Since \( N_{B'_2}(b_2) \subseteq N_{B'_2}(v_1) \), Lemma 2.4 gives that \( \text{lk}(b_4, \text{Ind}(B_2)) \simeq \text{Ind}(B'_2 - \{v_1\}) \). Let \( B''_2 \) denote the graph \( B'_2 - \{v_1\} \setminus \{(u_1, v_2)\} \).

Claim: \( \text{Ind}(B''_2) \simeq \text{Ind}(B'_2 - \{v_1\}) \).

Since \((u_1, v_2) \notin E(B''_2), \{u_1, v_2\} \in \text{Ind}(B''_2) \). Observe that \( b_2 \) is an isolated vertex in \( B''_2 - N_{B''_2}[u_1, v_2] \) and hence Lemma 2.2 implies that \( \text{Ind}(B''_2 - N_{B''_2}[u_1, v_2]) \) is contractible. Therefore by Lemma 2.3, \( \text{Ind}(B''_2) \simeq \text{Ind}(B''_2 \cup \{(u_1, v_2)\}) = \text{Ind}(B'_2 - \{v_1\}) \).

Observe that \( N_{B'_2}(u_1) = N_{B'_2}(b_2) \), so Lemma 2.4 implies that \( \text{Ind}(B''_2) \simeq \text{Ind}(B''_2 - \{b_2\}) \). Since \( V(B''_2 - \{b_2\}) \cap N_{B''_2 - \{b_2\}}(b_4) = \emptyset \), \( \text{Ind}(B''_2 - \{b_2\}) \cap \{b_3\} \subseteq \text{Ind}(B_2 - \{b_4\}) = \text{del}(b_4, \text{Ind}(B_2)) \). Hence the inclusion map \( \text{Ind}(B''_2 - \{b_2\}) \rightarrow \text{del}(b_4, \text{Ind}(B_2)) \) is null homotopic. Therefore the following composition of maps is null homotopic

\[
\text{lk}(b_4, \text{Ind}(B_2)) \simeq \text{Ind}(B'_2 - \{v_1\}) \simeq \text{Ind}(B''_2) \simeq \text{Ind}(B''_2 - \{b_2\}) \rightarrow \text{del}(b_4, \text{Ind}(B_2)).
\]

Hence \( \text{lk}(b_4, \text{Ind}(B_2)) \) is contractible in \( \text{del}(b_4, \text{Ind}(B_2)) \) and therefore by Lemma 2.1,

\[
\text{Ind}(B_2) \simeq \text{del}(b_4, \text{Ind}(B_2)) \lor \Sigma(\text{lk}(b_4, \text{Ind}(B_2))).
\]
Note that $B'_2 \subseteq \{b_2\} \cong P_2 \sqcup A_1$, hence $\text{lk}(b_1, \text{Ind}(B_2)) \simeq \text{Ind}(B'_2 \subseteq \{b_2\}) \simeq \text{Ind}(P_2 \sqcup A_1)$. Also, $\text{del}(b_4, \text{Ind}(B_2)) = \text{Ind}(B_2 - \{b_1\})$ and $N_{B_2 - \{b_4\}}(b_3) \subseteq N_{B_2 - \{b_4\}}(b_1)$. Lemma 2.4 implies that $\text{del}(b_4, \text{Ind}(B_2)) \simeq \text{Ind}(B_2 - \{b_4, b_1\})$. However, $B_2 - \{b_4, b_1\}$ is isomorphic to $P_2 \sqcup G_2$ (see Figure 3.10b), therefore $\text{del}(b_4, \text{Ind}(B_2)) \simeq \Sigma(\text{Ind}(G_2))$. Hence from Equation (1), we get the following homotopy equivalence.

$$\text{Ind}(B_2) \simeq \Sigma(\text{Ind}(G_2)) \lor \Sigma^2(\text{Ind}(A_1)).$$

Since $\text{Ind}(A_1) \simeq S^0 \lor S^0$ (see Section 3.1.3) and $\text{Ind}(G_2) \simeq S^1 \lor S^1$ (see Section 3.2.1), we get that $\text{Ind}(B_2) \simeq (S^2 \lor S^2) \lor \Sigma^2(S^0 \lor S^0) \simeq \lor \lor S^2$.

3.2.3. $A_2$. Note that $N_{A_2}(u_1) \subseteq N_{A_2}(w_1)$ (see figure on the right), therefore by Lemma 2.4, $\text{Ind}(A_2) \simeq \text{Ind}(A_2 - \{w_1\})$. Let $A'_2 = A_2 - \{w_1\}$. Since $a$ is a simplicial vertex in $A'_2$ and $N_{A'_2}(a) = \{x_1, v_1\}$, Lemma 2.5 implies that $\text{Ind}(A'_2) \simeq \Sigma(\text{Ind}(A'_2 - N_{A'_2}[x_1])) \lor \Sigma(\text{Ind}(A'_2 - N_{A'_2}[v_1]))$. The graph $A'_2 - N_{A'_2}[x_1] = A_2 - \{w_1, a, x_1, v_1, y_1\} \cong P_3$. Therefore $\text{Ind}(A'_2 - N[x_1]) \simeq S^0$. Also, $A'_2 - N_{A'_2}[x_1] \cong A'_2 - N_{A'_2}[v_1]$. Hence $\text{Ind}(A'_2) \simeq \text{Ind}(A'_2) \simeq \Sigma(S^0) \lor \Sigma(S^0) = S^1 \lor S^1$.

3.2.4. $D_2$. Note that $N_{D_2}(y_1) \subseteq N_{D_2}(w_1)$ (see figure on the right), therefore by Lemma 2.4, $\text{Ind}(D_2) \simeq \text{Ind}(D_2 - \{w_1\})$. Let $D'_2 = D_2 - \{w_1\}$. Since $d$ is a simplicial vertex in $D'_2$ and $N_{D'_2}(d) = \{u_1, v_1\}$, Lemma 2.5 implies that $\text{Ind}(D'_2) \simeq \Sigma(\text{Ind}(D'_2 - N_{D'_2}[u_1])) \lor \Sigma(\text{Ind}(D'_2 - N_{D'_2}[v_1]))$. The graph $D'_2 - N_{D'_2}[u_1] = D_2 - \{w_1, d, u_1, v_1, v_2\} \cong P_3$. Therefore $\text{Ind}(D'_2 - N_{D'_2}[u_1]) \simeq S^0$. Also, $D'_2 - N_{D'_2}[u_1] \cong D'_2 - N_{D'_2}[v_1]$, therefore $\text{Ind}(D'_2 - N_{D'_2}[u_1]) \simeq S^0$. Hence $\text{Ind}(D_2) \simeq \text{Ind}(D'_2) \simeq \Sigma(S^0) \lor \Sigma(S^0) = S^1 \lor S^1$.

3.2.5. $J_2$. Since $N_{J_2}(j_6) \subseteq N_{J_2}(j_3)$ (see figure on the right), from Lemma 2.4, $\text{Ind}(J_2) \simeq \text{Ind}(J_2 - \{j_3\})$. Observe that $J_2 - \{j_3, x_1\} \cong O_1$ and therefore from Section 3.1.6 we see that $\text{del}(x_1, \text{Ind}(J_2 - \{j_3\})) \simeq \text{Ind}(O_1) \simeq S^2$.

Note that $\text{lk}(x_1, \text{Ind}(J_2 - \{j_3\})) \simeq \text{Ind}(J_2 - \{x_1, j_3, j_1, j_2, v_1, w_1, y_1\}) \simeq \text{Ind}(P_2 \sqcup P_3) \simeq S^1$. 

![Figure 3.10](image-url)
Since the fundamental group of $\mathbb{S}^2$ is trivial, $\text{lk}(x_1, \text{Ind}(J_2 - \{j_3\}))$ is contractible in $\text{del}(x_1, \text{Ind}(J_2 - \{j_3\}))$. Therefore Lemma 2.1 implies that $\text{Ind}(J_2) \simeq \mathbb{S}^2 \vee \Sigma(\mathbb{S}^1) \simeq \mathbb{S}^2 \vee \mathbb{S}^2$.

3.2.6. $O_2$. Since $N_{O_2 - \{o_9\}}(o_9) = \{o_9\} \subseteq N_{O_2 - \{o_9\}}(o_2) \cap N_{O_2 - \{o_9\}}(o_4)$ and $N_{O_2 - \{o_9\}}(o_8) = \{o_8\} \subseteq N_{O_2 - \{o_9\}}(o_3)$, from Lemma 2.4 we get that $\text{del}(o_9, \text{Ind}(O_2)) = \text{Ind}(O_2 - \{o_9\}) \simeq \text{Ind}(O_2 - \{o_9, o_2, o_4, o_3\})$. Observe that $O_2 - \{o_9, o_2, o_4, o_3\} \cong D_2 \sqcup P_2 \sqcup P_2$ (see Figure 3.11b), and therefore $\text{del}(o_9, \text{Ind}(O_2)) \simeq \Sigma^2(\text{Ind}(D_2))$.

We see that $\text{lk}(o_9, \text{Ind}(O_2)) = \text{Ind}(O_2 - N_{O_2 - \{o_9\}}) = \text{Ind}(O_2 - \{o_7, o_8, o_9\})$. Let $O'_2$ be the graph $O_2 - \{o_7, o_8, o_9\}$ (see Figure 3.11c). Since $N_{O_2}^{(2)}(o_6) \subseteq N_{O_2}(x_1)$, from Lemma 2.4, $\text{Ind}(O'_2) \simeq \text{Ind}(O'_2 - \{x_1\})$. Denote the graph $O'_2 - \{x_1\}$ by $O''_2$ (see Figure 3.11d). Since $O''_2 - N_{O''_2}^{(2)}\{y_1, x_2\}$ contains an isolated vertex $o_6$, we see that $\text{Ind}(O''_2 - N_{O''_2}^{(2)}\{y_1, x_2\})$ is a cone over $o_6$ and therefore contractible. Hence from Lemma 2.3, $\text{Ind}(O''_2) \simeq \text{Ind}(O''_2 - \{(y_1, x_2)\})$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure3.11.png}
\caption{Figure 3.11}
\end{figure}

Denote the graph $O''_2 - \{(y_1, x_2)\}$ by $O'''_2$. Since $N_{O''_2}^{(2)}(o_6) = \{o_3\} = N_{O''_2}^{(2)}(y_1)$, from Lemma 2.4 we get that $\text{Ind}(O'''_2) \simeq \text{Ind}(O''_2 - \{o_6\})$. Clearly $O''_2 - \{o_6\} \cong Q_1 \sqcup P_2$, and therefore $\text{lk}(o_9, \text{Ind}(O_2)) \simeq \Sigma(\text{Ind}(Q_1))$. Note that $V(O''_2 - \{o_6\}) \cap N_{O_2 - \{o_9\}}^{(o_8)} = \emptyset$ and therefore $\text{Ind}(O''_2 - \{o_6\}) \ast \{o_8\} \subseteq \text{Ind}(O_2 - \{o_9\}) = \text{del}(o_9, \text{Ind}(O_2))$. Hence the inclusion map $\text{Ind}(O''_2 - \{o_6\}) \hookrightarrow \text{del}(o_9, \text{Ind}(O_2))$ is null homotopic. Thus the composite map $\text{lk}(o_9, \text{Ind}(O_2)) = \text{Ind}(O'_2) \xrightarrow{\sim} \text{Ind}(O''_2) \xrightarrow{\sim} \text{Ind}(O'''_2) \xrightarrow{\sim} \text{Ind}(O''_2 - \{o_6\}) \hookrightarrow \text{del}(o_9, \text{Ind}(O_2))$ is null homotopic.

Therefore Lemma 2.1 implies that $\text{Ind}(O_2) \simeq \text{del}(o_9, \text{Ind}(O_2)) \vee \Sigma(\text{lk}(o_9, \text{Ind}(O_2))) \simeq \Sigma^2(\text{Ind}(D_2)) \vee \Sigma^2(\text{Ind}(Q_1))$. Since $\text{Ind}(D_2) \simeq \mathbb{S}^1 \vee \mathbb{S}^1$ (cf. Section 3.2.4) and $\text{Ind}(Q_1)$ is contractible (cf. Section 3.1.8), $\text{Ind}(O_2) \simeq \mathbb{S}^2 \vee \mathbb{S}^2$.

3.2.7. $M_2$. Since $N_{M_2}(m_2) \subseteq N_{M_2}(x_1)$ (see figure on the right), Lemma 2.4 implies that $\text{Ind}(M_2) \simeq \text{Ind}(M_2 - \{x_1\})$. Since $N_{M_2 - \{x_1\}}(w_1) \subseteq N_{M_2 - \{x_1\}}(w_1)$ and $M_2 - \{x_1\} \cong C_8$, Lemma 2.4 and Lemma 2.7 implies that $\text{Ind}(M_2) \simeq \mathbb{S}^2$.
3.2.8. **$Q_2$.** Since $q_7$ is a simplicial vertex in $Q_2$ (see Figure 3.12a), Lemma 2.5 implies that $\text{Ind}(Q_2) \simeq \Sigma(\text{Ind}(Q_2 - N_{Q_2[q_5]})) \lor \Sigma(\text{Ind}(Q_2 - N_{Q_2[q_6]}))$.

![Figure 3.12](image)

Note that $N_{Q_2 - N_{Q_2[q_5]}(q_4) = \{q_4\} \subseteq N_{Q_2 - N_{Q_2[q_5]}(u_1) \cap N_{Q_2 - N_{Q_2[q_5]}(u_2)$, therefore Lemma 2.4 gives $\text{Ind}(Q_2 - N_{Q_2[q_5]} \simeq \text{Ind}(Q_2 - N_{Q_2[q_5]} - \{u_1, v_1\}$). See that $Q_2 - N_{Q_2[q_5]} - \{u_1, v_1\} \simeq M_1 \sqcup P_2$ (see Figure 3.12b). Hence $\text{Ind}(Q_2 - N_{Q_2[q_5]} \simeq \Sigma(\text{Ind}(M_1))$. Also $Q_2 - N_{Q_2[q_6]} \simeq M_2$ (see Figure 3.12c), therefore $\text{Ind}(Q_2) \simeq \Sigma^2(\Sigma(\text{Ind}(M_1)) \lor \Sigma(\text{Ind}(M_2))$. Since $\text{Ind}(M_1)$ is contractible (cf. Section 3.1.7) and $\text{Ind}(M_2) \simeq S^2$ (cf. Section 3.2.7), we get that $\text{Ind}(Q_2) \simeq S^3$.

3.2.9. **$F_2$.** Since $f_3$ is a simplicial vertex in $F_2$, from Lemma 2.5 we have

$$\text{Ind}(F_2) \simeq \Sigma(\text{Ind}(F_2 - N_{F_2[f_2]})) \lor \Sigma(\text{Ind}(F_2 - N_{F_2[f_4]}))$$

Observe that $F_2 - N_{F_2[f_2]} = F_2 - \{f_1, f_2, f_3, f_4\} \simeq S_2$ and $F_2 - N_{F_2[f_4]} = F_2 - \{x_1, y_1, f_2, f_3, f_4\} \simeq H_1$ (see Figure 3.13a). Since $\text{Ind}(G_2) \simeq S^1 \lor S^1$ (cf. Section 3.2.1) and $\text{Ind}(H_1) \simeq S^1$ (cf. Section 3.1.10), $\text{Ind}(F_2) \simeq \Sigma(S^1 \lor S^1) \lor \Sigma(S^1) \simeq \lor_3 S^2$.

![Figure 3.13](image)

3.2.10. **$H_2$.** Observe that $h_3$ is a simplicial vertex in $H_2$ (see Figure 3.13b). Thus using Lemma 2.5, we get

$$\text{Ind}(H_2) \simeq \Sigma(\text{Ind}(H_2 - N_{H_2[h_2]})) \lor \Sigma(\text{Ind}(H_2 - N_{H_2[h_4]}))$$

Observe that $H_2 - N_{H_2[h_2]} \simeq G_2$ and $H_2 - N_{H_2[h_4]} \simeq P_5$. Therefore, $\text{Ind}(H_2) \simeq \Sigma(\text{Ind}(G_2)) \lor \Sigma(\text{Ind}(P_5)) \simeq \Sigma(S^1 \lor S^1) \lor \Sigma(S^1) \simeq \lor_3 S^2$.

3.3. **General case computation.** The main outcome of this subsection is that the independence complex of any graph among the ten classes of graphs (defined in Section 3.1) is a wedge of spheres up to homotopy. We prove this by induction on the subscript of the graphs, i.e., $n$. The cases $n = 1, 2$ follow from the Section 3.1 and Section 3.2. Fix $n \geq 3$, and inductively assume that for any $k < n$, the independence complex of any graph, among the ten classes of graphs, with subscript $k$ is a wedge of spheres up to homotopy.
3.3.1. $G_n$. For $n \geq 3$, we show that

$$\text{Ind}(G_n) \simeq \begin{cases} \vee_5 S^2 & \text{if } n = 3, \\ \text{Ind}(B_{n-1}) \vee \Sigma^3(\text{Ind}(A_{n-3})) & \text{if } n \geq 4. \end{cases} \quad (3)$$

We do this by analysing $\text{del}(w_1, \text{Ind}(G_n))$ and $\text{lk}(w_1, \text{Ind}(G_n))$. Since $\text{del}(w_1, \text{Ind}(G_n)) = \text{Ind}(G_n - \{w_1\})$ and $G_n - \{w_1\} \cong B_{n-1}$ (cf. Figure 3.14a), $\text{del}(w_1, \text{Ind}(G_n)) \simeq \text{Ind}(B_{n-1})$, for $n \geq 3$. Further note that, $\text{lk}(w_1, \text{Ind}(G_n)) = \text{Ind}(G_n - N_{G_n}[w_1])$.

For $n = 3$, $\text{del}(w_1, \text{Ind}(G_3)) \simeq \text{Ind}(B_2)$ thus by Section 3.2.2, $\text{Ind}(B_2) \simeq \vee_4 S^2$. Also, $G_3 - N_{G_3}[w_1]) \cong P_5$, therefore Lemma 2.6 implies that $\text{lk}(w_1, \text{Ind}(G_3)) \simeq S^1$. Since the fundamental group of $\vee_4 S^2$ is trivial, $\text{lk}(w_1, \text{Ind}(G_3))$ is contractible in $\text{del}(w_1, \text{Ind}(G_3))$. Hence, from Lemma 2.1, $\text{Ind}(G_3) \simeq \text{del}(w_1, \text{Ind}(G_3)) \vee \Sigma(\text{lk}(w_1, \text{Ind}(G_3))) \simeq \vee_4 S^2 \vee \Sigma(S^1) \simeq \vee_5 S^2$.

![Figure 3.14](image-url)

We now analyse $\text{lk}(w_1, \text{Ind}(G_n))$ for $n \geq 4$. Let $G_n'$ be the graph $G_n - N_{G_n}[w_1]$ (cf. Figure 3.14b). Since $N_{G_n}(y_1) \subseteq N_{G_n}(x_3)$, Lemma 2.4 implies that $\text{Ind}(G_n') \simeq \text{Ind}(G_n - \{x_3\})$. Observe that both the graphs $G_n' - \{x_3\} - N_{G_n - \{x_3\}}[\{y_3, x_4\}]$ and $G_n' - \{x_3\} - N_{G_n - \{x_3\}}[\{y_3, y_4\}]$ have $y_1$ as an isolated vertex (see Figure 3.14c), therefore the independence complexes of these graphs are contractible. Hence, Lemma 2.3 implies that $\text{Ind}(G_n' - \{x_3\}) \simeq \text{Ind}(G_n' - \{x_3\} - \{(y_3, x_4), (y_3, y_4)\})$.

Let $G_n'' = G_n' - \{x_3\} - \{(y_3, x_4), (y_3, y_4)\}$. Since $N_{G_n''}(y_3) \subseteq N_{G_n''}(y_1)$ (see Figure 3.14d), Lemma 2.4 implies $\text{Ind}(G_n'') \simeq \text{Ind}(G_n'' - \{y_1\})$. Note that $V(G_n'' - \{y_1\}) \cap N_{G_n - \{w_1\}}[x_1] = \emptyset$, therefore $\text{Ind}(G_n'' - \{y_1\}) \simeq \text{Ind}(G_n - \{w_1\}) = \text{del}(w_1, \text{Ind}(G_n))$. Hence the inclusion map $\text{Ind}(G_n'' - \{y_1\}) \hookrightarrow \text{del}(w_1, \text{Ind}(G_n))$ is null homotopic. Thus the following composition of maps is null homotopic:

$$\text{lk}(w_1, \text{Ind}(G_n)) \xrightarrow{\simeq} \text{Ind}(G_n' - \{x_3\}) \xrightarrow{\simeq} \text{Ind}(G_n'') \xrightarrow{\simeq} \text{Ind}(G_n'' - \{y_1\}) \hookrightarrow \text{del}(w_1, \text{Ind}(G_n)).$$

Therefore by Lemma 2.1,

$$\text{Ind}(G_n) \simeq \text{del}(w_1, \text{Ind}(G_n)) \vee \Sigma(\text{lk}(w_1, \text{Ind}(G_n))).$$

As shown earlier, $\text{del}(w_1, \text{Ind}(G_n)) \simeq \text{Ind}(B_{n-1})$, therefore to prove Equation (3), it suffices to show that $\text{lk}(w_1, \text{Ind}(G_n)) \simeq \Sigma^3(\text{Ind}(A_{n-3}))$. From the above discussion, we
know that \( \text{lk}(w_1, \text{Ind}(G_n)) \simeq \text{Ind}(G_{n}'' - \{y_1\}) \). Since \( N_{G_n'' - \{y_1\}}(u_1) \subseteq N_{G_n'' - \{y_1\}}(u_3) \cap N_{G_n'' - \{y_1\}}(v_3) \), Lemma 2.4 implies \( \text{Ind}(G_{n}'' - \{y_1\}) \simeq \text{Ind}(G_{n}'' - \{y_1\} - \{u_3, v_3\}) \). Moreover, \( G_n'' - \{y_1, u_3, v_3\} \) is isomorphic to \( P_2 \sqcup P_2 \sqcup A_{n-3} \) (see Figure 3.14f). Thus, by Lemma 2.2, \( \text{lk}(w_1, \text{Ind}(G_n)) \simeq \text{Ind}(G_{n}'' - \{y_1, u_3, v_3\}) \simeq \Sigma^2(\text{Ind}(A_{n-3})) \).

**Corollary 3.1.** For \( n \geq 1 \), \( \text{Ind}(G_n) \) is homotopy equivalent to a wedge of spheres.

**Proof.** The result follows from Section 3.1.1, Section 3.2.1, induction hypothesis, and Equation (3). \( \Box \)

3.3.2. \( B_n \). Let \( n \geq 3 \) and consider the vertex \( b_1 \) in \( B_n \). Let \( B'_n = B_n - N_{B_n}[b_1] \). Since both the graphs \( B'_n - \{v_1\} - N_{B'_n - \{v_1\}}[\{u_1, u_2\}] \) and \( B'_n - \{v_1\} - N_{B'_n - \{v_1\}}[\{u_1, v_2\}] \) have \( b_2 \) as an isolated vertex, their independence complexes are contractible. Let \( B''_n = B'_n - \{v_1\} - \{(u_1, u_2), (u_1, v_2)\} \). Now using the similar arguments as in the case of \( B_2 \), we get that \( \text{Ind}(B_n) \simeq \text{Ind}(B''_n) \). Moreover, the similar arguments imply that

\[
\text{Ind}(B_n) \simeq \Sigma(\text{Ind}(G_n)) \vee \Sigma^2(\text{Ind}(A_{n-1})). \tag{4}
\]

**Corollary 3.2.** For \( n \geq 1 \), \( \text{Ind}(B_n) \) is homotopy equivalent to a wedge of spheres.

**Proof.** The result follows from Section 3.1.2, Section 3.2.2, Corollary 3.1, induction hypothesis, and Equation (4). \( \Box \)

3.3.3. \( A_n \). For \( n \geq 3 \), we show that

\[
\text{Ind}(A_n) \simeq \begin{cases} \vee_5 S^2 & \text{if } n = 3, \\ \vee_2 \Sigma(\text{Ind}(D_{n-1})) \vee \Sigma^3(\text{Ind}(A_{n-3})) & \text{if } n \geq 4. \end{cases} \tag{5}
\]

In \( A_n \), a is a simplicial vertex with \( N_{A_n}(a) = \{v_1, w, x_1\} \). Therefore by Lemma 2.5,

\[
\text{Ind}(A_n) \simeq \Sigma(\text{Ind}(A_n - N_{A_n}[v_1])) \vee \Sigma(\text{Ind}(A_n - N_{A_n}[w])) \vee \Sigma(\text{Ind}(A_n - N_{A_n}[x_1])).
\]

Clearly \( A_n - N_{A_n}[x_1] \cong D_{n-1} \cong A_n - N_{A_n}[v_1] \) (see Figure 3.15a for \( n = 3 \) case). Therefore \( \text{Ind}(A_n) \simeq \vee_2 \text{Ind}(D_{n-1}) \vee \text{Ind}(A_n - N_{A_n}[w]). \) Since \( A_3 - N_{A_3}[w] \cong P_6 \), Lemma 2.6 and Section 3.2.4 implies \( \text{Ind}(A_3) \simeq \Sigma(S^1 \vee S^1) \vee \Sigma(S^1 \vee S^1) \vee \Sigma(S^1) = \vee_5 S^2.\)

![Figure 3.15](image)

It now suffices to show that \( \text{Ind}(A_n - N_{A_n}[w]) \simeq \Sigma^2(\text{Ind}(A_{n-3})) \), for \( n \geq 4 \). Let \( A'_n = A_n - N_{A_n}[w] \). Since \( N_{A'_n}(u_2) = \{u_2\} \subseteq N_{A'_n}(u_3) \cap N_{A'_n}(v_3) \) and \( N_{A'_n}(y_1) = \{y_1\} \subseteq N_{A'_n}(x_3) \cap N_{A'_n}(y_3) \) (see Figure 3.15b), Lemma 2.4 implies that \( \text{Ind}(A'_n) \simeq \text{Ind}(A'_n - \{v_3, u_3, x_3, y_3\}) \). Moreover, \( A'_n - \{v_3, u_3, x_3, y_3\} \cong P_2 \sqcup P_2 \sqcup A_{n-3} \) (see Figure 3.15c). Therefore, \( \text{Ind}(A_n - N_{A_n}[w]) \simeq \Sigma^2(\text{Ind}(A_{n-3})). \)

**Corollary 3.3.** For \( n \geq 1 \), \( \text{Ind}(A_n) \) is homotopy equivalent to a wedge of spheres.

**Proof.** The result follows from Section 3.1.3, Section 3.2.3, induction hypothesis, and Equation (5). \( \Box \)
3.3.4. \( D_n \). Let \( n \geq 3 \). Clearly \( d \) is a simplicial vertex of \( D_n \) and \( N_{D_n}(d) = \{u_1, v_1\} \). Therefore by Lemma 2.5, \( \text{Ind}(D_n) \simeq \Sigma(\text{Ind}(D_n - \text{N}_{D_n}[u_1])) \lor \Sigma(\text{Ind}(D_n - \text{N}_{D_n}[v_1])) \). Since \( D_n - \text{N}_{D_n}[v_1] \cong D_{n-1} \) and \( D_n - \text{N}_{D_n}[u_1] \cong J_{n-2} \) (see Figure 3.16), we have the following homotopy equivalence

\[
\text{Ind}(D_n) \simeq \Sigma(\text{Ind}(D_{n-1})) \lor \Sigma(\text{Ind}(J_{n-2})).
\] (6)

![Figure 3.16](image)

**Corollary 3.4.** For \( n \geq 1 \), \( \text{Ind}(D_n) \) is homotopy equivalent to a wedge of spheres.

**Proof.** The result follows from Section 3.1.4, Section 3.2.4, induction hypothesis, and Equation (6).

\( \Box \)

3.3.5. \( J_n \). For \( n \geq 3 \), we show that

\[
\text{Ind}(J_n) \simeq \text{Ind}(O_{n-1}) \lor \Sigma^2(\text{Ind}(D_{n-1})).
\] (7)

Since \( N_{J_n}(j_6) \subseteq N_{J_n}(j_3) \), Lemma 2.4 implies that \( \text{Ind}(J_n) \simeq \text{Ind}(J_n - \{j_3\}) \). Observe that \( J_n - \{j_3, x_1\} \cong O_{n-1} \) (see Figure 3.17a) and therefore \( \text{del}(x_1, \text{Ind}(J_n - \{j_3\})) = \text{Ind}(J_n - \{j_3, x_1\}) \cong \text{Ind}(O_{n-1}) \).

Let \( J'_n \) be the graph \( J_n - \{j_3\} \). We now analyse \( \text{lk}(x_1, \text{Ind}(J'_n)) \). Clearly \( \text{lk}(x_1, \text{Ind}(J'_n)) = \text{Ind}(J'_n - \text{N}_{J'_n}[x_1]) \). Let \( J''_n = J'_n - \text{N}_{J'_n}[x_1] \). Observe that the graph \( J''_n \cong D_{n-1} \lor P_3 \) (see Figure 3.17b). Since \( N_{J''_n}(j_5) = \{j_6\} = N_{J''_n}(j_5) \), Lemma 2.4 we get that \( \text{Ind}(J''_n) \simeq \text{Ind}(J''_n - \{j_5\}) \). Clearly \( J''_n - \{j_5\} \) is isomorphic to \( D_{n-1} \lor P_2 \). Hence \( \text{lk}(x_1, \text{Ind}(J''_n)) \simeq \text{Ind}(D_{n-1} \lor P_2) \cong \Sigma(\text{Ind}(D_{n-1})) \).

![Figure 3.17](image)

Note that \( V(J''_n - \{j_5\}) \cap N_{J''_n - \{j_3, x_1\}}(j_2) = \emptyset \) and therefore \( \text{Ind}(J''_n - \{j_5\}) \ast \{j_2\} \subseteq \text{Ind}(J''_n - \{j_3, x_1\}) = \text{del}(x_1, \text{Ind}(J''_n)). \) Hence the inclusion map \( \text{Ind}(J''_n - \{j_5\}) \hookrightarrow \text{del}(x_1, \text{Ind}(J''_n)) \) is null homotopic. Thus the composite map

\[
\text{lk}(x_1, \text{Ind}(J''_n)) = \text{Ind}(J''_n) \xrightarrow{\simeq} \text{Ind}(J''_n - \{j_5\}) \hookrightarrow \text{del}(x_1, \text{Ind}(J''_n))
\]

is null homotopic. Thus from Lemma 2.1 we get that \( \text{Ind}(J''_n) \simeq \text{del}(x_1, \text{Ind}(J''_n)) \lor \Sigma(\text{lk}(x_1, \text{Ind}(J''_n))) \simeq \text{Ind}(O_{n-1}) \lor \Sigma^2(\text{Ind}(D_{n-1})). \)
Corollary 3.5. For \( n \geq 1 \), \( \text{Ind}(J_n) \) is homotopy equivalent to a wedge of spheres.

**Proof.** The result follows from Section 3.1.5, Section 3.2.5, induction hypothesis, and Equation (7).

\[ 3.3.6. \] \( O_n \). For \( n \geq 3 \), we show that

\[
\text{Ind}(O_n) \simeq \Sigma^2(\text{Ind}(D_n)) \vee \Sigma^2(\text{Ind}(Q_{n-1})). \tag{8}
\]

Let \( O'_n = O_n - \{o_7, o_8, o_9\} \) (see Figure 3.18a). Using similar arguments as in the case of \( O_2 \), we get that \( \text{del}(o_9, \text{Ind}(O_n)) = \Sigma^2(\text{Ind}(D_n)) \) and \( \text{lk}(o_9, \text{Ind}(O_n)) \simeq \text{Ind}(O''_n) \), where \( O''_n = O'_n - \{x_1\} \) (see Figure 3.18b). Since \( O''_n - N_{O''_n}[\{y_1, x_2\}] \) and \( O''_n - N_{O''_n}[\{y_1, y_2\}] \) both contain \( o_6 \) as an isolated vertex, \( \text{Ind}(O''_n - N_{O''_n}[\{y_1, x_2\}]) \) and \( \text{Ind}(O''_n - N_{O''_n}[\{y_1, y_2\}]) \) both are cones with apex \( o_6 \) and therefore contractible. Hence from Lemma 2.3, \( \text{Ind}(O''_n) \simeq \text{Ind}(O''_n - \{(y_1, x_2), (y_1, y_2)\}). \) Denote the graph \( O''_n - \{(y_1, x_2), (y_1, y_2)\} \) by \( O'''_n \). Observe that the graph \( O'''_n \) is isomorphic to \( Q_{n-1} \sqcup P_3 \) (see Figure 3.18c) implying that \( \text{lk}(o_9, \text{Ind}(O_n)) \simeq \Sigma(\text{Ind}(Q_{n-1})). \)

Now similar arguments as in the case of \( O_2 \) imply that \( \text{lk}(o_9, \text{Ind}(O_n)) \) is contractible in \( \text{del}(o_9, \text{Ind}(O_n)) \). The Equation (8) then follows from Lemma 2.1.

![Figure 3.18](image_url)

**Corollary 3.6.** For \( n \geq 1 \), \( \text{Ind}(O_n) \) is homotopy equivalent to a wedge of spheres.

**Proof.** The result follows from Section 3.1.6, Section 3.2.6, Corollary 3.4, induction hypothesis, and Equation (8).

\[ 3.3.7. \] \( M_n \). For \( n \geq 3 \), we show that

\[
\text{Ind}(M_n) \simeq \Sigma(\text{Ind}(M_{n-1})) \vee \Sigma^2(\text{Ind}(F_{n-2})). \tag{9}
\]

Since \( N_{M_n}(m_2) \subseteq N_{M_n}(x_1) \), Lemma 2.4 implies that \( \text{Ind}(M_n) \simeq \text{Ind}(M_n - \{x_1\}) \). Let \( M'_n \) be the graph \( M_n - \{x_1\} \). We compute the link and deletion of the vertex \( m_1 \) in \( \text{Ind}(M'_n) \).

Note that \( \text{lk}(m_1, \text{Ind}(M'_n)) = \text{Ind}(M'_n - N_{M'_n}[m_1]) \). Let \( M''_n \) be the graph \( M'_n - N_{M'_n}[m_1] \) (see Figure 3.19a). Since \( N_{M''_n}(m_3) \subseteq N_{M''_n}(x_1) \), from Lemma 2.4, we get that \( \text{Ind}(M''_n) \simeq \text{Ind}(M''_n - \{x_1\}) \). Observe that both the graphs \( M''_n - \{x_1\} - N_{M''_n}[\{y_2, x_3\}] \) and \( M''_n - \{x_1\} - N_{M''_n}[\{y_2, y_3\}] \) contain an isolated vertex \( m_3 \) and therefore their independence complexes are contractible. Thus, using Lemma 2.3 we get that \( \text{Ind}(M''_n - \{x_2\}) \simeq \text{Ind}(M''_n - \{x_2\} - \{(y_2, y_3), (y_2, y_3)\}) \) (see Figure 3.19a). Denote the graph \( M''_n - \{x_2\} - \{(y_2, x_3), (y_2, x_3)\} \) by \( M''_n \). Since \( N_{M''_n}(y_2) \subseteq N_{M''_n}(m_3) \), \( \text{Ind}(M''_n) \simeq \text{Ind}(M''_n - \{m_3\}) \).

Observe that \( M''_n - \{m_3\} \simeq P_3 \sqcup F_{n-2} \) (see Figure 3.19a), therefore \( \text{lk}(m_1, \text{Ind}(M'_n)) \simeq \text{Ind}(M''_n) \simeq \text{Ind}(P_3 \sqcup F_{n-2}) \simeq \Sigma(\text{Ind}(F_{n-2})). \)
We now compute the homotopy type of $\text{del}(m_1, \text{Ind}(M'_n)) = \text{Ind}(M'_n - \{m_1\})$ (see Figure 3.19d). Since $N_{M'_n - \{m_1\}}(m_2) \subseteq N_{M'_n - \{m_1\}}(y_1)$ and $M'_n - \{m_1, y_1\} \cong F_2 \sqcup M_{n-1}$, we get that $\text{del}(m_1, \text{Ind}(M'_n)) \cong \text{Ind}(F_2 \sqcup M_{n-1}) \cong \Sigma(\text{Ind}(M_{n-1}))$.

From Lemma 2.1, it is now enough to show that the inclusion map $\text{lk}(m_1, \text{Ind}(M'_n)) \hookrightarrow \text{del}(m_1, \text{Ind}(M'_n))$ is null homotopic. We know that $\text{lk}(m_1, \text{Ind}(M'_n)) = \text{Ind}(M''_n) \cong \text{Ind}(M'_n - \{m_3\})$ and $\text{del}(m_1, \text{Ind}(M'_n)) = \text{Ind}(M'_n - \{m_1\})$. Note that $V(M''_n - \{m_3\}) \cap N_{M'_n - \{m_1\}}(m_2) = \emptyset$. Thus $\text{Ind}(M''_n - \{m_3\}) \ast \{m_2\} \subseteq \text{Ind}(M'_n - \{m_1\})$ implying that the map $\text{Ind}(M''_n - \{m_3\}) \hookrightarrow \text{Ind}(M'_n - \{m_1\})$ is null homotopic. Hence the composition map

$$\text{lk}(m_1, \text{Ind}(M'_n)) \hookrightarrow \text{Ind}(M''_n - \{m_3\}) \hookrightarrow \text{del}(m_1, \text{Ind}(M'_n))$$

is null homotopic.

**Corollary 3.7.** For $n \geq 1$, $\text{Ind}(M_n)$ is homotopy equivalent to a wedge of spheres.

**Proof.** The result follows from Section 3.1.7, Section 3.2.7, induction hypothesis, and Equation (9). □

3.3.8. $Q_n$. For $n \geq 3$, using the same arguments along the lines as in the case of $Q_2$ we get that,

$$\text{Ind}(Q_n) \cong \Sigma(\text{Ind}(M_n)) \vee \Sigma^2(\text{Ind}(M_{n-1})). \quad (10)$$

**Corollary 3.8.** For $n \geq 1$, $\text{Ind}(Q_n)$ is homotopy equivalent to a wedge of spheres.

**Proof.** The result follows from Section 3.1.8, Section 3.2.8, Corollary 3.7, induction hypothesis, and Equation (10). □

3.3.9. $F_n$. Observe that $f_3$ is a simplicial vertex in $F_n$ with $N_{F_n}(f_3) = \{f_2, f_4\}$, and therefore, $\text{Ind}(F_n) \cong \Sigma(\text{Ind}(F_n - N_{F_n}[f_2])) \vee \Sigma(\text{Ind}(F_n - N_{F_n}[f_4]))$ by Lemma 2.5. Since $F_n - N_{F_n}[f_2] \cong G_n$ and $F_n - N_{F_n}[f_4] \cong H_{n-1}$ (see Figure 3.20), we get that

$$\text{Ind}(F_n) \cong \Sigma(\text{Ind}(G_n)) \vee \Sigma(\text{Ind}(H_{n-1})). \quad (11)$$
Corollary 3.9. For \( n \geq 1 \), \( \text{Ind}(F_n) \) is homotopy equivalent to a wedge of spheres.

Proof. The result follows from Section 3.1.9, Section 3.2.9, Corollary 3.1, induction hypothesis, and Equation (11).

3.3.10. \( H_n \). Observe that \( h_3 \) is a simplicial vertex in \( H_n \) with \( N_{H_n}(h_3) = \{h_2, h_4\} \) (see Figure 3.21a). Therefore from Lemma 2.5, \( \text{Ind}(H_n) \cong \Sigma(\text{Ind}(H_n - N_{H_n}[h_2])) \vee \Sigma(\text{Ind}(H_n - N_{H_n}[h_4])) \).

![Figure 3.20](image)

\begin{align*}
\text{(a) } F_3 & \quad \text{(b) } F_3 - N_{F_3}[f_2] & \quad \text{(c) } F_3 - N_{F_3}[f_4]
\end{align*}

Figure 3.20

We see that \( N_{H_n - N_{H_n}[h_4]}(h_1) \subseteq N_{H_n - N_{H_n}[h_4]}(u_2) \cap N_{H_n - N_{H_n}[h_4]}(v_2) \), therefore Lemma 2.4 implies that \( \text{Ind}(H_n - N_{H_n}[h_4]) \cong \text{Ind}(H_n - N_{H_n}[h_4] - \{u_2, v_2\}) \). Since \( H_n - N_{H_n}[h_4] - \{u_2, v_2\} \cong P_2 \sqcup F_{n-2} \) (see Figure 3.21c) and \( H_n - N_{H_n}[h_2] \cong G_n \), we get that

\[ \text{Ind}(H_n) \cong \Sigma(\text{Ind}(G_n)) \vee \Sigma^2(\text{Ind}(F_{n-2})). \]  \hfill (12)

Corollary 3.10. For \( n \geq 1 \), \( \text{Ind}(H_n) \) is homotopy equivalent to a wedge of spheres.

Proof. The result follows from Section 3.1.10, Section 3.2.10, Corollary 3.1, induction hypothesis, and Equation (12).

4. Dimension of the spheres occurring in the homotopy type

In this section we determine the dimensions of all the spheres occurring in the homotopy type of the independence complexes of all ten classes of graphs defined in Section 3.1. For any \( m \geq n \geq 1 \), let \( [n, m] = \{a \in \mathbb{Z} : n \leq a \leq m\} \) and

\[ S^{[n,m]} = \{X : X \cong \bigvee_{d_1} S^n \vee \ldots \vee \bigvee_{d_{m+1}} S^{n+m} \text{ for some } d_1, \ldots, d_{m+1} > 0\}. \]

Theorem 4.1. Let \( n \geq 1 \).

1. If \( n \in [9k; 9k + 8] \), then \( \text{Ind}(G_n) \in S^{[n-1,n+k-1]} \).

2. If \( n \in [9k + 8, 9k + 16] \), then \( \text{Ind}(B_n) \in S^{[n,n+k+1]} \).
(3) If \( n \in [9k + 7, 9k + 15] \), then \( \text{Ind}(A_n) \in S^{[n-1,n+k]} \).

(4) If \( n \in [9k + 6, 9k + 14] \), then \( \text{Ind}(D_n) \in S^{[n-1,n+k]} \).

(5) If \( n \in [9k + 4, 9k + 12] \), then \( \text{Ind}(J_n) \in S^{[n,n+k+1]} \).

(6) If \( n \in [9k + 3, 9k + 11] \), then \( \text{Ind}(O_n) \in S^{[n+1,n+k+2]} \).

(7) If \( 1 \neq n \in [9k + 2, 9k + 10] \), then \( \text{Ind}(M_n) \in S^{[n,n+k]} \), and \( \text{Ind}(M_1) \simeq pt. \)

(8) If \( 1 \neq n \in [9k + 2, 9k + 10] \), then \( \text{Ind}(Q_n) \in S^{[n+1,n+k+1]} \), and \( \text{Ind}(Q_1) \simeq pt. \)

(9) If \( n \in [9k, 9k + 8] \), then \( \text{Ind}(F_n) \in S^{[n,n+k]} \).

(10) If \( n \in [9k, 9k + 8] \), then \( \text{Ind}(H_n) \in S^{[n,n+k]} \).

**Proof.** The proof is by induction on \( n \). For \( n \leq 8 \), the explicit homotopy types can be computed using the results from Section 3 and are listed in the Table 1.

| \( n \) | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  |
|-------|----|----|----|----|----|----|----|----|
| \( \text{Ind}(G_n) \) | \( S^0 \) | \( S^1 \) | \( S^1 \lor S^1 \) | \( V_2S^1 \) | \( V_3S^1 \) | \( V_4S^1 \) | \( V_5S^1 \) | \( V_6S^1 \) |
| \( \text{Ind}(B_n) \) | \( V_2S^1 \) | \( V_2S^1 \) | \( V_3S^1 \) | \( V_4S^1 \) | \( V_5S^1 \) | \( V_6S^1 \) | \( V_7S^1 \) | \( V_8S^1 \) |
| \( \text{Ind}(A_n) \) | \( V_2S^1 \) | \( V_2S^1 \) | \( V_3S^1 \) | \( V_4S^1 \) | \( V_5S^1 \) | \( V_6S^1 \) | \( V_7S^1 \) | \( V_8S^1 \) |
| \( \text{Ind}(D_n) \) | \( V_2S^1 \) | \( V_2S^1 \) | \( V_3S^1 \) | \( V_4S^1 \) | \( V_5S^1 \) | \( V_6S^1 \) | \( V_7S^1 \) | \( V_8S^1 \) |
| \( \text{Ind}(O_n) \) | \( V_2S^1 \) | \( V_2S^1 \) | \( V_3S^1 \) | \( V_4S^1 \) | \( V_5S^1 \) | \( V_6S^1 \) | \( V_7S^1 \) | \( V_8S^1 \) |
| \( \text{Ind}(M_n) \) | \( pt \) | \( S^0 \) | \( S^1 \) | \( S^1 \lor S^1 \) | \( V_2S^1 \) | \( V_3S^1 \) | \( V_4S^1 \) | \( V_5S^1 \) |
| \( \text{Ind}(Q_n) \) | \( pt \) | \( S^0 \) | \( S^1 \) | \( S^1 \lor S^1 \) | \( V_2S^1 \) | \( V_3S^1 \) | \( V_4S^1 \) | \( V_5S^1 \) |
| \( \text{Ind}(F_n) \) | \( V_2S^1 \) | \( V_2S^1 \) | \( V_3S^1 \) | \( V_4S^1 \) | \( V_5S^1 \) | \( V_6S^1 \) | \( V_7S^1 \) | \( V_8S^1 \) |
| \( \text{Ind}(H_n) \) | \( S^1 \) | \( V_2S^1 \) | \( V_3S^1 \) | \( V_4S^1 \) | \( V_5S^1 \) | \( V_6S^1 \) | \( V_7S^1 \) | \( V_8S^1 \) |

**Table 1.** Independence complexes of all ten classes of graphs for \( n \leq 8 \)

For \( n \geq 9 \), let us assume that the result holds for all \( m < n \).

(1) From Equation (3), we have the following:

\[
\text{Ind}(G_n) \simeq \text{Ind}(B_{n-1}) \lor \Sigma^3(\text{Ind}(A_{n-3})).
\]  

(13)

Let \( n \in [9k, 9k + 8] \), then \( n - 1 \in [9(k - 1) + 8, 9(k - 1) + 16] \). By induction, \( \text{Ind}(B_{n-1}) \in S^{[n-1,n+k-1]} \). Clearly \( n - 3 \in [9(k - 1) + 6, 9(k - 1) + 14] \). If \( n - 3 = 9(k - 1) + 6 = 9(k - 2) + 15 \), then by induction \( \text{Ind}(A_{n-3}) \in S^{[n-4,n-3+k-2]} \). and for \( n - 3 \in [9(k - 1) + 7, 9(k - 1) + 14] \), \( \text{Ind}(A_{n-3}) \in S^{[n-4,n-3+k-1]} \). Hence from Equation (13), \( \text{Ind}(G_n) \in S^{[n-1,n+k-1]} \).

(2) From Equation (4), we have the following:

\[
\text{Ind}(B_n) \simeq \Sigma(\text{Ind}(G_n)) \lor \Sigma^2(\text{Ind}(A_{n-1})).
\]  

(14)

Let \( n \in [9k + 8, 9k + 16] \). From part (1), we know that for \( n = 9k + 8 \), \( \Sigma(\text{Ind}(G_n)) \in S^{[n,n+k]} \) and for \( n \in [9k + 9, 9k + 16] = [9(k + 1), 9(k + 1) + 7] \), \( \Sigma(\text{Ind}(G_n)) \in S^{[n,n+k+1]} \). Clearly \( n \in [9k + 8, 9k + 16] \) implies \( n - 1 \in [9k + 7, 9k + 15] \). By induction, \( \text{Ind}(A_{n-1}) \in S^{[n-2,n+k-1]} \) and therefore \( \Sigma^2(\text{Ind}(A_{n-1})) \in S^{[n-2,n+k-1]} \). Thus Equation (14) implies that \( \text{Ind}(B_n) \in S^{[n,n+k+1]} \).

(3) From Equation (5), we have the following:

\[
\text{Ind}(A_n) \simeq \lor_2 \Sigma(\text{Ind}(D_{n-1})) \lor \Sigma^3(\text{Ind}(A_{n-3})).
\]  

(15)

Let \( n \in [9k + 7, 9k + 15] \). If \( n - 1 \in [9k + 6, 9k + 14] \), then by induction \( \text{Ind}(D_{n-1}) \in S^{[n-2,n+k-1]} \) and therefore \( \Sigma(\text{Ind}(D_{n-1})) \in S^{[n-1,n+k]} \). Clearly \( n \in
From Equation (6) we have the following:

\[
\text{Ind}(D_n) \simeq \Sigma(\text{Ind}(D_{n-1})) \lor \Sigma(\text{Ind}(J_{n-2})).
\]  

Let \( n \in [9k+6, 9k+14] \), then \( n-1 \in [9k+5, 9k+13] \). If \( n-1 = 9k+5 = 9(k-1) + 14 \), then by induction \( \text{Ind}(D_{n-1}) \in S^{[n-2,n+k-2]} \) and therefore \( \Sigma(\text{Ind}(D_{n-1})) \in S^{[n-1,n+k-1]} \). Otherwise \( n-1 \in [9k+7, 9k+14] \) and by induction \( \text{Ind}(D_{n-1}) \in S^{[n-2,n+k-1]} \) and thus \( \Sigma(\text{Ind}(D_{n-1})) \in S^{[n-1,n+k]} \). Also \( n-1 = 9k+6, 9k+14 \) implies \( n \in [9k, 9k+12] \). By induction \( \text{Ind}(J_{n-2}) \in S^{[n-2,n-2+k+1]} \), thereby implying that \( \Sigma(\text{Ind}(J_{n-2})) \in S^{[n-1,n+k]} \). Result then follows from Equation (16).

From Equation (7), we have the following:

\[
\text{Ind}(J_n) \simeq \text{Ind}(O_{n-1}) \lor \Sigma^2(\text{Ind}(D_{n-1})).
\]  

Let \( n \in [9k+4, 9k+12] \), then \( n-1 \in [9k+3, 9k+11] \). By induction, we have \( \text{Ind}(O_{n-1}) \in S^{[n,n+k+1]} \). Moreover, if \( n-1 \in [9k+3, 9k+5] = [9(k-1) + 12, 9(k-1) + 14] \), then by induction \( \Sigma^2(\text{Ind}(D_{n-1})) \in S^{[n,n+k]} \). For \( n-1 \in [9k+6, 9k+11] \), \( \Sigma^2(\text{Ind}(D_{n-1})) \in S^{[n,n+k+1]} \). Therefore the result follows from Equation (17).

From Equation (8), we have the following:

\[
\text{Ind}(O_n) \simeq \Sigma^2(\text{Ind}(D_n)) \lor \Sigma^2(\text{Ind}(Q_{n-1})).
\]  

Let \( n \in [9k+3, 9k+11] \). If \( n \in [9k+3, 9k+5] = [9(k-1) + 12, 9(k-1) + 14] \), then by induction \( \Sigma^2(\text{Ind}(D_n)) \in S^{[n+1,n+k+1]} \). Further, if \( n \in [9k+6, 9k+11] \), then again by induction \( \Sigma^2(\text{Ind}(D_n)) \in S^{[n+1,n+k+2]} \). Observe that \( n \in [9k+3, 9k+11] \) implies \( n-1 \in [9k+2, 9k+10] \) and therefore by induction we have \( \Sigma^2(\text{Ind}(Q_{n-1})) \in S^{[n+2,n+k+2]} \). The result then follows from Equation (18).

From Equation (9), we have the following:

\[
\text{Ind}(M_n) \simeq \Sigma(\text{Ind}(M_{n-1})) \lor \Sigma^2(\text{Ind}(F_{n-2})).
\]  

Let \( n \in [9k+2, 9k+10] \), then \( n-1 \in [9k+1, 9k+9] \). If \( n-1 = 9k+1 = 9(k-1) + 10 \), then \( \Sigma(\text{Ind}(M_{n-1})) \in S^{[n,n+k+1]} \). For \( n-1 \in [9k+2, 9k+9] \), \( \Sigma(\text{Ind}(M_{n-1})) \in S^{[n,n+k]} \) by induction. Clearly \( n \in [9k+2, 9k+10] \) implies \( n \in [9k, 9k+8] \). Therefore by induction, \( \text{Ind}(F_{n-2}) \in S^{[n-2,n+k-2]} \) and hence \( \Sigma^2(\text{Ind}(F_{n-2})) \in S^{[n,n+k]} \). Equation (19) then implies that for \( n \in [9k+2, 9k+10] \), \( \text{Ind}(M_n) \in S^{[n,n+k]} \).

From Equation (10), we have the following:

\[
\text{Ind}(Q_n) \simeq \Sigma(\text{Ind}(M_n)) \lor \Sigma^2(\text{Ind}(M_{n-1})).
\]  

Let \( n \in [9k+2, 9k+10] \), then \( \Sigma(\text{Ind}(M_n)) \in S^{[n+1,n+k+1]} \). Clearly \( n-1 \in [9k+1, 9k+9] \). If \( n-1 = 9k+1 = 9(k-1) + 10 \), then \( \text{Ind}(M_{n-1}) \in S^{[n-1,n+k-2]} \) thereby implying that \( \Sigma^2(\text{Ind}(M_{n-1})) \in S^{[n-1,n+k]} \). For \( n \in [9k+2, 9k+9] \), \( \Sigma^2(\text{Ind}(M_{n-1})) \in S^{[n+1,n+k+1]} \) by induction. The result now follows from Equation (20).

From Equation (11), we have the following:

\[
\text{Ind}(F_n) \simeq \Sigma(\text{Ind}(G_n)) \lor \Sigma(\text{Ind}(H_{n-1})).
\]
Let $n \in [9k, 9k + 8]$. From part (1), $\text{Ind}(G_n) \in S^{[n−1,n+k−1]}$ and therefore $\Sigma(\text{Ind}(G_n)) \in S^{[n,n+k]}$. Clearly $n - 1 \in [9k - 1, 9k + 7]$. If $n - 1 = 9k - 1 = 9(k - 1) + 8$, then by induction $\text{Ind}(H_{n-1}) \in S^{[n−1,n+k−2]}$ and therefore $\Sigma(\text{Ind}(H_{n-1})) \in S^{[n,n+k−1]}$. For $n - 1 \in [9k, 9k + 7]$, $\text{Ind}(H_{n-1}) \in S^{[n−1,n+k−1]}$ and hence $\Sigma(\text{Ind}(H_{n-1})) \in S^{[n,n+k]}$ by induction. The result now follows from Equation (21).

(10) From Equation (12), we have the following:

$$\text{Ind}(H_n) \simeq \Sigma(\text{Ind}(G_n)) \lor \Sigma^2(\text{Ind}(F_{n-2})). \quad (22)$$

Let $n \in [9k, 9k + 8]$. From part (1), $\text{Ind}(G_n) \in S^{[n−1,n+k−1]}$ and therefore $\Sigma(\text{Ind}(G_n)) \in S^{[n,n+k]}$. Clearly $n - 2 \in [9k - 2, 9k + 6]$. If $n - 2 \in [9k - 2, 9k - 1] = [9(k - 1) + 7, 9(k - 1) + 8]$, then by induction $\Sigma^2(\text{Ind}(F_{n-2})) \in S^{[n,n+k−1]}$. Further, if $n - 2 \in [9k, 9k + 6]$, then again from induction $\Sigma^2(\text{Ind}(F_{n-2})) \in S^{[n,n+k]}$. The result then follows from Equation (22). □

5. Summary

Here we summarize the homotopy equivalences that were obtained in Section 3.3. We depict this information via the following diagram. Each node in Figure 5.1 denotes a graph class defined in Section 3.1 and an edge $X \rightarrow Y$ indicates that $\lor, \Sigma^m(\text{Ind}(Y_{n-k}))$ appears in the homotopy type formula of $\text{Ind}(X_n)$ obtained in Section 3.3. For simplicity of notations, $1 \times (\Sigma^m, -k)$ is denoted by $(\Sigma^m, -k)$.

![Diagram](image_url)

**Figure 5.1.** Summary of homotopy type formulae obtained in Section 3.3

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From Theorem 4.1, we see that the number 9 plays an important role in determining the dimensions of spheres that occur in the homotopy type of \( M(\Gamma_{3,n}) \). It would be interesting to see if there is any relation between the number or the dimension of spheres in the homotopy type of \( M(\Gamma_{3,n}) \) and the combinatorial description of \( \Gamma_{3,n} \). Another interesting enumerative problem is to calculate the Betti numbers of \( M(\Gamma_{3,n}) \). More precisely,

**Question 5.1.** Can we determine the closed form formula for the homotopy type of \( M(\Gamma_{3,n}) \)?

Based on the main result of this article, our computer-based computations for various general grid graphs, and [9], we propose the following.

**Conjecture 1.** The complex \( M(\Gamma_{m,n}) \) is homotopy equivalent to a wedge of spheres for any grid graph \( \Gamma_{m,n} \).

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