A Proximal Algorithm for Sampling from Non-convex Potentials

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Abstract

We study sampling problems associated with non-convex potentials that meanwhile lack smoothness. In particular, we consider target distributions that satisfy either logarithmic-Sobolev inequality or Poincaré inequality. Rather than smooth, the potentials are assumed to be semi-smooth or the summation of multiple semi-smooth functions. We develop a sampling algorithm that resembles proximal algorithms in optimization for this challenging sampling task. Our algorithm is based on a special case of Gibbs sampling known as the alternating sampling framework (ASF). The key contribution of this work is a practical realization of the ASF based on rejection sampling in the non-convex and semi-smooth setting. This work extends the recent algorithm in [24, 25] for non-smooth/semi-smooth log-concave distribution to the setting with non-convex potentials. In almost all the cases of sampling considered in this work, our proximal sampling algorithm achieves better complexity than all existing methods.

Key words. High-dimensional sampling, non-convex semi-smooth potential, complexity analysis, logarithmic-Sobolev inequality, Poincaré inequality

1 Introduction

The problem of drawing samples from an unnormalized probability distribution plays an essential role in data science and scientific computing [11] [6] [26]. It has been widely used in many areas such as Bayesian inference, Bayesian neural networks, probabilistic graphical models, biology, and machine learning [15] [29] [19] [21] [39] [11]. Compared with optimization oriented methods, sampling has the advantage of being able to quantify the uncertainty and confidence level of the solution, and often provides more reliable solutions to engineering problems. This advantage comes at the cost of higher computational cost. It is thus important to develop more efficient sampling algorithms.

In the classical setting of sampling, the potential function $f$ of an unnormalized target distribution $\exp(-f(x))$ is assumed to be smooth and (strongly) convex. Over the past decades, many sampling algorithms have been developed, including Langevin Monte Carlo (LMC), kinetic Langevin Monte Carlo (KLMC), Hamiltonian Monte Carlo (HMC), Metropolis-adjusted Langevin algorithm (MALA) etc [7] [17] [33] [36] [8] [2] [35] [36] [29]. Many of these algorithms are based on some type of discretization of the Langevin diffusion or the underdamped Langevin diffusion. They resemble the gradient-based algorithms in optimization. These algorithms work well for (strongly) convex and smooth potentials; many non-asymptotic complexity bounds have been proven. However, the cases where either the convexity or the smoothness is lacking are much less understood [5] [4] [8] [13] [24] [25] [27].

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In this paper, we consider the challenging task of sampling from potentials that are neither convex nor smooth. Many sampling problems in real applications fall into this setting. For instance, Bayesian neural networks are highly non-convex models corresponding to probability densities with multi-modality [18]. The lack of smoothness is due to the use of activation functions such as ReLU. The goal of this work is to develop an efficient algorithm with provable guarantees for a class of non-convex potentials that are not smooth. In particular, we consider probability distributions that satisfy either the log-Sobolev inequality (LSI) or the Poincaré inequality (PI) [41], and have potential functions $f$ that are semi-smooth, i.e., $\nabla f$ is Hölder continuous.

Inspired by the recent line of researches that lie in the interface of sampling and optimization, we examine this sampling task from an optimization perspective. We build on the intuition of proximal algorithms for non-smooth optimization problem and develop a proximal algorithm to sample from non-convex and semi-smooth potentials. Our algorithm is based on the alternating sampling framework (ASF) [22] developed recently to sample from strongly convex potentials. In a nutshell, the ASF is a Gibbs sampler over a carefully designed augmented distribution of the target one and can thus samples from the target distribution by sampling from the augmented distribution. The convergence results of ASF have been recently improved [4] to cover non-log-concave distributions that satisfy functional inequalities such as LSI.

The ASF is not directly implementable. In each iteration it needs to query the so-called restricted Gaussian oracle (RGO), which is itself a sampling task from a quadratically regularized distribution $\exp(-f(x) - \frac{1}{2\eta}\|x-y\|^2)$ for some given $\eta > 0$ and $y \in \mathbb{R}^d$. The RGO can be viewed as a sampling counterpart of the proximal map in optimization. The total complexity of ASF thus depends on that of the RGO. Except for a few special cases where $f$ has certain structures, the RGO is usually a challenging task. One key contribution of this work is a practical and efficient algorithm for RGO for potentials that are neither convex nor smooth. This algorithm extends the recent work [24] for convex and non-smooth potentials, and [25] for convex and semi-smooth potentials. Combining the ASF and our algorithm for RGO, we establish a proximal algorithm for sampling from non-convex and semi-smooth potentials. Our contributions are summarized as follows.

i) We develop an efficient sampling scheme of RGO for non-convex semi-smooth potentials and bound its complexity with a novel technique.

ii) We combine our RGO scheme and the ASF to develop a sampling algorithm that can sample from non-convex and semi-smooth potentials. Our algorithm has a better non-asymptotic complexity than all existing methods in the same setting.

iii) We further extend our algorithms for potentials that are the summation of multiple non-convex semi-smooth functions.

iv) Our results are the first high-accuracy guarantee for sampling problems with non-convex semi-smooth potentials.

**Related works:** MCMC sampling from non-convex potentials has been investigated in [34, 40, 42, 5, 12, 28]. There have also been some works on sampling without smoothness [23, 10, 3, 5, 27, 8, 11, 14, 37, 1, 31, 24, 25].

The literature for the case where the potential function lacks both convexity and smoothness is rather scarce. In [31, 12], the authors analyze the convergence of LMC for semi-smooth potentials that satisfy a dissipativity condition. The target distribution is assumed to satisfy some functional inequality. [12] also assumes an additional tail growth condition. The dissipativity condition is removed in [5]. The results in [31] are applicable to potentials that are the summation of multiple
semi-smooth functions, while those in [5, 12] are not. To compare our results with them, we make the simplification that the initial distance, either in KL or Rényi divergence, to the target distribution is $\tilde{O}(d)$. The results in cases with non-convex semi-smooth potentials are presented in Table 1 and that for composite semi-smooth potentials are in Table 2. It can be seen that our complexities have better dependence on all the parameters: LSI constant $C_{LSI}$, PI constant $C_{PI}$, semi-smooth coefficients $L_\alpha$, and dimension $d$. Moreover, our complexity bounds depend polylogarithmically on the accuracy $\varepsilon$ (thus $\varepsilon$ does not appear in the $\tilde{O}$ notation) while all the other results have polynomial dependence on $1/\varepsilon$. To the best of our knowledge, our results are the first high-accuracy guarantee for sampling with non-convex semi-smooth potentials.

| Source | Complexity | Assumption | Metric |
|--------|------------|------------|--------|
| [5]    | $\tilde{O}\left( \frac{C_{LSI}^{1 + 1/\alpha} L_\alpha^{2/\alpha} d^{2/\alpha}}{\varepsilon^{1/\alpha}} \right)$ | LSI, $\alpha > 0$, semi-smooth | Rényi |
| [5]    | $\tilde{O}\left( \frac{C_{PI}^{1 + 1/\alpha} L_\alpha^{2/\alpha} d^{1+1/\alpha}}{\varepsilon^{1/\alpha}} \right)$ | PI, $\alpha > 0$, semi-smooth | Rényi |
| [12]   | $\tilde{O}\left( \frac{C_{LSI}^{1 + 1/\alpha} L_\alpha^{2/\alpha} d^{1/\alpha}}{\varepsilon^{1/\alpha}} \right)$ | LSI, $\alpha > 0$, semi-smooth | KL |
| this paper (Thm. 3.1) | $\tilde{O}\left( C_{LSI} L_\alpha^{2/(1+\alpha)} d \right)$ | LSI, semi-smooth | KL |
| this paper (Thm. 3.2) | $\tilde{O}\left( C_{PI} L_\alpha^{2/(1+\alpha)} d^2 \right)$ | PI, semi-smooth | Rényi |

Table 1: Complexity bounds for sampling from non-convex semi-smooth potentials.

| Source | Complexity | Assumption | Metric |
|--------|------------|------------|--------|
| [31]   | $\tilde{O}\left( \frac{\left( C_{LSI} \max\left\{ \frac{1}{\alpha_i} \right\} \max\left\{ \frac{\varepsilon}{\alpha_i} \right\} \right)}{\varepsilon} \sum_{i=1}^n L_{\alpha_i} \right)$ | LSI, $\alpha_i > 0$, multiple semi-smooth | KL |
| this paper (Thm. 5.4) | $\tilde{O}\left( C_{LSI} \sum_{i=1}^n L_{\alpha_i}^{2/(\alpha_i+1)} d \right)$ | LSI, multiple semi-smooth | KL |
| this paper (Thm. 5.5) | $\tilde{O}\left( C_{PI} \sum_{i=1}^n L_{\alpha_i}^{2/(\alpha_i+1)} d^2 \right)$ | PI, multiple semi-smooth | Rényi |

Table 2: Complexity bounds for sampling from non-convex composite potentials.

### 2 Problem formulation and Background

We are interested in sampling from distributions with non-convex potentials that are not necessarily smooth. More specifically, we consider the sampling task with the target distribution

$$\nu \propto \exp(-f(x))$$

(1)

where the potential $f$ satisfies

$$\|f'(u) - f'(v)\| \leq \sum_{i=1}^n L_{\alpha_i} \| u - v \|^{\alpha_i}, \quad \forall u, v \in \mathbb{R}^d$$

(2)
for \( \alpha_1, \ldots, \alpha_n \in [0, 1] \) and \( L_{\alpha_1}, \ldots, L_{\alpha_n} > 0 \). Here \( f' \) denotes a subgradient of \( f \). When \( n = 1, f \) satisfying (2) is known as a semi-smooth or weakly-smooth function. Thus, the condition (2) can be viewed as a generalization of semi-smoothness. The target distribution \( \nu \) is assumed to satisfy LSI or PI, but \( f \) can be non-convex in general.

In this work, we develop a proximal algorithm for sampling from non-convex potentials that satisfy (2). Our method is built on the alternating sampling framework (ASF) introduced in [22]. Unlike most existing sampling algorithms that require the potential to be smooth, ASF is applicable to non-smooth problems. The ASF with target distribution \( \pi^X(x) \propto \exp(-f(x)) \) works as follows. Initialized at \( x_0 \sim \rho_0^X \), ASF with stepsize \( \eta > 0 \) performs the two alternating steps as follows.

\begin{algorithm}
\begin{enumerate}
\item Sample \( y_k \sim \pi^Y(y \mid x_k) \propto \exp\left[-\frac{1}{2\eta}\|x_k - y\|^2\right] \)
\item Sample \( x_{k+1} \sim \pi^X(x \mid y_k) \propto \exp\left[-f(x) - \frac{1}{2\eta}\|x - y_k\|^2\right] \)
\end{enumerate}
\end{algorithm}

The ASF is a special case of Gibbs sampling [16] of the joint distribution

\[
\pi(x, y) \propto \exp\left(-f(x) - \frac{1}{2\eta}\|x - y\|^2\right).
\]

In Algorithm 1 sampling \( y_k \) given \( x_k \) in step 1 is easy since \( \pi^Y(y \mid x_k) = \mathcal{N}(x_k, \eta I) \) is a standard Gaussian distribution. Sampling \( x_{k+1} \) given \( y_k \) in step 2 is however a nontrivial task; it corresponds to the so-called restricted Gaussian oracle for \( f \) introduced in [22], defined as follows.

**Definition 2.1.** Given a point \( y \in \mathbb{R}^d \) and stepsize \( \eta > 0 \), the restricted Gaussian oracle (RGO) for \( f : \mathbb{R}^d \rightarrow \mathbb{R} \) is a sampling oracle that returns a random sample from a distribution proportional to \( \exp(-f(-) - \| \cdot - y \|^2/(2\eta)) \).

The RGO is an analog of the proximal map in optimization, which is widely used in proximal algorithms for optimization [32]. To use the ASF in practice, one needs to efficiently implement the RGO. Some examples of \( f \) that admits a computationally efficient RGO have been presented in [27, 38]. These instances of \( f \) have simple coordinate-separable structures, \( \ell_1 \)-norm, and group Lasso. For general \( f \), especially non-convex and non-smooth ones considered in this work, it was not clear how to realize the RGO efficiently.

Assuming the RGO in the ASF can be realized, the ASF exhibits remarkable convergence properties. In [22] it was shown that Algorithm 1 converges linearly when \( f \) is strongly convex. This convergence result is recently improved in [4] under various weaker assumptions on the target distribution \( \pi^X \propto \exp(-f) \). Below we present several convergence results established in [4] that will be used in this paper, under the assumptions that \( \pi^X \) satisfies LSI or PI.

To this end, for two probability distributions \( \rho \ll \nu \), we denote by

\[
H_\nu(\rho) := \int \rho \log \frac{\rho}{\nu}, \quad \chi_\nu^2(\rho) := \int \frac{\rho^2}{\nu} - 1, \quad R_{q, \nu}(\rho) := \frac{1}{q-1} \log \int \frac{\rho^q}{\nu^{q-1}}
\]

the KL divergence, the Chi-squared divergence, and the Rényi divergence, respectively. Note that \( R_{2, \nu} = \log(1 + \chi_\nu^2) \), \( R_{1, \nu} = H_\nu \), and \( R_{q, \nu} \leq R_{q', \nu} \) for any \( 1 \leq q \leq q' < \infty \). We denote by \( W_2 \) the Wasserstein-2 distance defined by [11]

\[
W_2^2(\nu, \rho) := \min_{\gamma \in \Pi(\nu, \rho)} \int \|x - y\|^2 d\gamma,
\]
where \( \Pi(\nu, \rho) \) represents the set of all couplings between \( \nu \) and \( \rho \).

A probability distribution \( \nu \) satisfies LSI with constant \( C_{\text{LSI}} > 0 \) if for every \( \rho \), we have

\[
H_\nu(\rho) \leq \frac{C_{\text{LSI}}}{2} J_\nu(\rho),
\]

where the Fisher information \( J_\nu(\rho) \) is defined as

\[
J_\nu(\rho) = \mathbb{E}_\rho[\|\nabla \log \rho\|_\nu^2].
\]

**Theorem 2.2** ([4, Theorem 3]). Assume that \( \pi^X \propto \exp(-f) \) satisfies \( \lambda \)-LSI. For any initial distribution \( \rho_0^X \), the \( k \)-th iterate \( \rho_k^X \) of Algorithm 1 with step size \( \eta > 0 \) satisfies

\[
H_{\pi^X}(\rho_k^X) \leq \frac{H_{\pi^X}(\rho_0^X)}{(1 + \lambda \eta)^2k}.
\]

Moreover, for all \( q \geq 1 \),

\[
R_{q, \pi^X}(\rho_k^X) \leq \frac{R_{q, \pi^X}(\rho_0^X)}{(1 + \lambda \eta)^{2k/q}}.
\]

A probability distribution \( \nu \) satisfies PI with constant \( C_{\text{PI}} > 0 \) if for any smooth bounded function \( \psi : \mathbb{R}^d \to \mathbb{R} \), we have

\[
\mathbb{E}_\nu[(\psi - \mathbb{E}_\nu(\psi))^2] \leq C_{\text{PI}} \mathbb{E}_\nu[\|\nabla \psi\|^2].
\]

**Theorem 2.3** ([4, Theorem 4]). Assume \( \pi^X \propto \exp(-f) \) satisfies \( \lambda \)-PI. For any initial distribution \( \rho_0^X \), the \( k \)-th iterate \( \rho_k^X \) of Algorithm 1 with step size \( \eta > 0 \) satisfies

\[
\chi_{\pi^X}^2(\rho_k^X) \leq \frac{\chi_{\pi^X}^2(\rho_0^X)}{(1 + \lambda \eta)^2k}.
\]

Moreover, for all \( q \geq 2 \),

\[
R_{q, \pi^X}(\rho_k^X) \leq \begin{cases} 
R_{q, \pi^X}(\rho_0^X) - \frac{2k \log(1 + \lambda \eta)}{q}, & \text{if } k \leq \frac{q}{2 \log(1 + \lambda \eta)} (R_{q, \pi^X}(\rho_0^X) - 1), \\
1/(1 + \lambda \eta)^{2(k-k_0)/q}, & \text{if } k \geq k_0 := \left\lceil \frac{q}{2 \log(1 + \lambda \eta)} (R_{q, \pi^X}(\rho_0^X) - 1) \right\rceil.
\end{cases}
\]

To use the ASF for sampling problems, we need to realize the RGO with efficient implementations. In the rest of this paper, we develop an efficient algorithm for RGO associated with a potential satisfying (2), and then combine it with the ASF to establish a proximal algorithm for sampling. The complexity of the proximal algorithm can be obtained by combining the above convergence results for ASF and the complexity results we establish for RGO. The rest of the paper is organized as follows. In Section 3 we consider a special case of (2) with \( n = 1 \) (i.e., semi-smooth potentials) and establish complexity results for sampling from distributions with non-convex and semi-smooth potentials. In Section 4 we develop an efficient algorithm for RGO via rejection sampling, which is used in Section 3 combined with the ASF, to obtain the proximal sampling algorithm. In Section 5 we extend the results to the general cases (2). In Section 6 we present some concluding remarks. Finally, in Appendices A-C we present technical results, provide a self-contained discussion on solving an optimization subproblem in the proximal sampling algorithm, and comment on the proof techniques used in this paper compared with those of [24, 25].
3 Proximal sampling for non-convex and semi-smooth potentials

Our main goal in this section is to establish complexity results for sampling from distributions with non-convex and semi-smooth potentials, i.e., those satisfying (2) with \( n = 1 \). To better present our results, we begin with this simple case of (2), and the general setting of (2) is discussed in Section 5. In addition to the semi-smoothness of \( f \), we assume that \( \pi^X \propto \exp(-f) \) satisfies either LSI or PI, and discuss their corresponding complexity results in the rest of this section.

**Theorem 3.1.** Assume \( f \) is \( L_\alpha \)-semi-smooth with \( \alpha > 0 \) and \( \pi^X \propto \exp(-f) \) satisfies LSI with constant \( C_{\text{LSI}} \). With initial distribution \( \rho_0^X \) and stepsize \( \eta \propto 1/(L_\alpha^{\frac{2}{\alpha+1}} d) \), Algorithm 1 using Algorithm 2 as an RGO has the iteration-complexity bound

\[
\tilde{O} \left( C_{\text{LSI}} L_\alpha^{\frac{2}{\alpha+1}} d \right)
\]

(4)

to achieve \( \varepsilon \) error to the target \( \pi^X \) in terms of KL divergence. Each iteration queries \( \tilde{O}(1) \) subgradients of \( f \) and generates \( O(1) \) samples in expectation from Gaussian distribution.

**Proof:** The result follows directly from Theorem 2.2, Proposition 4.2 and Proposition 4.4 with the choice of stepsize \( \eta \propto 1/(L_\alpha^{\frac{2}{\alpha+1}} d) \).

**Theorem 3.2.** Assume \( f \) is \( L_\alpha \)-semi-smooth and \( \pi^X \propto \exp(-f) \) satisfies PI with constant \( C_{\text{PI}} \). With initial distribution \( \rho_0^X \) and stepsize \( \eta \propto 1/(L_\alpha^{\frac{2}{\alpha+1}} d) \), Algorithm 1 using Algorithm 2 as an RGO has the iteration-complexity bound

\[
\tilde{O} \left( C_{\text{PI}} L_\alpha^{\frac{2}{\alpha+1}} d \right)
\]

(5)

to achieve \( \varepsilon \) error to the target \( \pi^X \) in terms of Chi-squared divergence, and

\[
\tilde{O} \left( C_{\text{PI}} L_\alpha^{\frac{2}{\alpha+1}} qdR_{q,\nu}(\rho_0^X) \right)
\]

(6)

to achieve \( \varepsilon \) error in terms of Rényi divergence \( R_{q,\nu} \) (\( q \geq 2 \)). Each iteration queries \( \tilde{O}(1) \) subgradients of \( f \) and generates \( O(1) \) samples in expectation from Gaussian distribution.

**Proof:** The result is a direct consequence of Theorem 2.3, Proposition 4.2 and Proposition 4.4 with the choice of stepsize \( \eta \propto 1/(L_\alpha^{\frac{2}{\alpha+1}} d) \).

4 RGO for semi-smooth potentials

The bottleneck of the implementation of Algorithm 1 for sampling from a general distribution \( \exp(-f) \) is an efficient realization of the RGO, i.e., step 2 of Algorithm 1. We consider the setting where \( f \) is a non-convex and semi-smooth function, and develop an efficient algorithm for the corresponding RGO based on rejection sampling. With a sufficiently small \( \eta \) in RGO, we show that the expected number of rejection sampling steps to obtain one effective sample turns out to be bounded above by a dimension-free constant. The core to achieving such a constant bound on the expected rejection steps is a novel construction of proposal distribution that does not rely on the convexity of \( f \) and a refined analysis that captures the nature of semi-smooth functions. Relevant ideas have been explored in [9, 30] to design universal methods for convex semi-smooth optimization problems.
One of the most useful properties of semi-smooth functions is that they can be approximated by smooth functions to an arbitrary accuracy, at the cost of increasing their smoothness parameters. Intuitively, semi-smooth functions with order of growth $\alpha + 1 \in [1,2]$ are dominated by smooth functions with quadratic growth.

**Lemma 4.1.** Assume $f$ is an $L_\alpha$-semi-smooth function, then for $\delta > 0$ and every $u,v \in \mathbb{R}^d$, we have

$$|f(u) - f(v) - \langle f'(v), u - v \rangle| \leq \frac{M}{2} ||u - v||^2 + \frac{(1 - \alpha)\delta}{2},$$

where

$$M = \frac{L_{\alpha + 1}^2}{[(\alpha + 1)\delta]^{\frac{\alpha + 1}{2}}}. \quad (8)$$

**Proof:** It follows from the assumption that $f$ is $L_\alpha$-semi-smooth that for every $u,v \in \mathbb{R}^d$,

$$|f(u) - f(v) - \langle f'(v), u - v \rangle| \leq \frac{L_{\alpha}}{\alpha + 1} ||u - v||^{\alpha + 1}. \quad (9)$$

Using the Young's inequality $ab \leq a^p/p + b^q/q$ with

$$a = \frac{L_{\alpha}}{(\alpha + 1)\delta^{\frac{1}{2}}} ||u - v||^{\alpha + 1}, \quad b = \delta^{\frac{1}{2}}, \quad p = \frac{2}{\alpha + 1}, \quad q = \frac{2}{1 - \alpha},$$

we obtain

$$\frac{L_{\alpha}}{\alpha + 1} ||u - v||^{\alpha + 1} \leq \frac{L_{\alpha + 1}^2}{2[(\alpha + 1)\delta]^{\frac{\alpha + 1}{2}}} ||u - v||^2 + \frac{(1 - \alpha)\delta}{2}. \quad (7)$$

Plugging the above inequality into (9), we have

$$|f(u) - f(v) - \langle f'(v), u - v \rangle| \leq \frac{L_{\alpha + 1}^2}{2[(\alpha + 1)\delta]^{\frac{\alpha + 1}{2}}} ||u - v||^2 + \frac{(1 - \alpha)\delta}{2}.$$ 

This inequality and the definition of $M$ in (8) imply (7). □

Inspired by [25], rejection sampling for RGO heavily relies on the (approximate) minimizer of the potential function. We thus consider the regularized optimization problem

$$\min_{x \in \mathbb{R}^d} \left\{ f^\eta_y(x) := f(x) + \frac{1}{2\eta} ||x - y||^2 \right\}, \quad (10)$$

where $y \in \mathbb{R}^d$ is given. Even though $f$ is non-convex and semi-smooth, thanks to the strong regularization term $\frac{1}{2\eta}||x - y||^2$, $f^\eta_y$ is close to a strongly convex and smooth function with some approximation error. As a result, (10) can be solved by convex smooth optimization algorithms, such as Nesterov’s acceleration. For completeness, we describe a variant of the method in Algorithm 3 in Appendix B. The following proposition presents a complexity result of Algorithm 3 for finding an approximate stationary point of $f^\eta_y$ with a small $\eta$.

**Proposition 4.2.** Assume $\eta \leq \frac{1}{Md}$, and let $w \in \mathbb{R}^d$ be an approximate stationary point of $f^\eta_y$, i.e.,

$$||s|| \leq \sqrt{Md}, \quad s = f'(w) + \frac{1}{\eta}(w - y), \quad (11)$$

where $M$ is as in (8). Then, the iteration-complexity to find $w$ by using Algorithm 3 is $\tilde{O}(1)$. 

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Proof: It follows from Lemma A.2 that $f_y^\eta$ satisfies (32) with
\[
\mu = \frac{1}{\eta} - M, \quad L = \frac{1}{\eta} + M, \quad \theta = \frac{(1 - \alpha)\delta}{2}.
\] (12)
Since $\eta \leq 1/(Md)$, it is easy to verify that the assumption on $\rho$ (i.e., $\sqrt{Md}$ in our case) in Proposition B.4 is satisfied. Hence, it follows from Proposition B.4 and (12) that the proposition holds.

Our algorithm for RGO is based on rejection sampling. The next lemma gives two important functions that are useful for the analysis of rejection sampling, and shows that they bound $f_y^\eta$ in both directions.

**Lemma 4.3.** Let $w^* \in \mathbb{R}^d$ be a stationary point of $f_y^\eta$, i.e.,
\[
f'(w^*) + \frac{1}{\eta}(w^* - y) = 0.
\] (13)
Define
\[
h_1(x) := f(w) + \langle f'(w), x - w \rangle - \frac{M}{2}\|x - w\|^2 + \frac{1}{2\eta}\|x - y\|^2 - \frac{(1 - \alpha)\delta}{2},
\] (14)
\[
h_2(x) := f(w^*) + \langle f'(w^*), x - w^* \rangle + \frac{M}{2}\|x - w^*\|^2 + \frac{1}{2\eta}\|x - y\|^2 + \frac{(1 - \alpha)\delta}{2},
\] (15)
where $w$ is as in (11). Then, we have for every $x \in \mathbb{R}^d$,
\[
h_1(x) \leq f_y^\eta(x) \leq h_2(x).
\] (16)

**Proof:** Inequalities in (16) directly follow from (7) and the definitions of $h_1$ and $h_2$ in (14) and (15), respectively.

We are now ready to present the rejection sampling algorithm for RGO.

**Algorithm 2** RGO Rejection Sampling
1. Compute an approximate solution $w$ satisfying (11) by using Algorithm 3
2. Generate sample $X \sim \exp(-h_1(x))$
3. Generate sample $U \sim \mathcal{U}[0,1]$
4. If
\[
U \leq \frac{\exp(-f_y^\eta(X))}{\exp(-h_1(X))},
\]
then accept/return $X$; otherwise, reject $X$ and go to step 2.

By construction, the proposal $\propto \exp(-h_1(x))$ is close to the target $\pi_{X|Y}(x \mid y) \propto \exp(-f_y^\eta(x))$ when $\eta$ is sufficiently small, and hence the expected rejection steps is small. The following proposition rigorously justifies this intuition.

**Proposition 4.4.** Assume $f$ is $L_\alpha$-semi-smooth and let $f_y^\eta$ be as in (10). Then $X$ generated by Algorithm 2 follows the distribution $\pi_{X|Y}(x \mid y) \propto \exp(-f_y^\eta(x))$. Moreover, if
\[
\eta \leq \frac{1}{Md} = \left[\frac{(\alpha + 1)\delta}{\alpha + 1} \right]^{\frac{1}{\alpha + 1}},
\] (17)
then the expected number of rejection steps in Algorithm 2 is at most $\exp\left(\frac{3(1 - \alpha)\delta}{2} + 3\right)$. 

Proof: It is a well-known result for rejection sampling that \( X \sim \pi^Y(x|y) \) and the probability that \( X \) is accepted is
\[
\mathbb{P} \left( U \leq \frac{\exp(-f^y(u)(X))}{\exp(-h_1(X))} \right) = \frac{\int \exp(-f^y(x))dx}{\int \exp(-h_1(x))dx}.
\]

Using the above identity, (14), and Lemma A.3, we have
\[
\mathbb{P} \left( U \leq \frac{\exp(-f^y(u)(X))}{\exp(-h_1(X))} \right) \geq \frac{\int \exp(-h_2(x))dx}{\int \exp(-h_1(x))dx}
= \left( \frac{1 - \eta M}{1 + \eta M} \right)^{d/2} \frac{\exp \left( \frac{1}{2\eta} \|w^*\|^2 - f(w^*) + \langle f'(w^*), w^* \rangle - \frac{1}{2\eta} \|y\|^2 - \frac{1}{2} \alpha \delta \right)}{\exp \left( \frac{1}{2\eta} \|w\|^2 + \frac{\eta}{2(1 - \eta M)} \|s\|^2 - f(w) + \frac{1}{\eta} \langle w, y - w \rangle - \frac{1}{2\eta} \|y\|^2 + \frac{1}{2} \alpha \delta \right)}
= \left( \frac{1 - \eta M}{1 + \eta M} \right)^{d/2} \exp \left( -(1 - \alpha) \delta - \frac{\eta}{2(1 - \eta M)} \|s\|^2 \right)
\exp \left( \frac{1}{2\eta} \|w^*\|^2 - \frac{1}{2\eta} \|w\|^2 - f(w^*) + f(w) + \langle f'(w^*), w^* \rangle - \frac{1}{\eta} \langle w, y - w \rangle \right).
\]

It follows from (7) with \((u, v) = (w, w^*)\) that
\[
-f(w) + f(w^*) + \langle f'(w^*), w - w^* \rangle \leq \frac{M}{2} \|w - w^*\|^2 + \frac{(1 - \alpha) \delta}{2},
\]
which together with (13) implies that
\[
\frac{1}{2\eta} \|w^*\|^2 - \frac{1}{2\eta} \|w\|^2 - f(w^*) + f(w) + \langle f'(w^*), w^* \rangle - \frac{1}{\eta} \langle w, y - w \rangle
= \frac{1}{2\eta} \|w^*\|^2 - \frac{1}{2\eta} \|w\|^2 - f(w^*) + f(w) + \langle f'(w^*), w^* \rangle - \frac{1}{\eta} \langle w, \eta f'(w^*) + w^* - w \rangle
= \frac{1}{2\eta} \|w - w^*\|^2 - f(w^*) + f(w) + \langle f'(w^*), w^* - w \rangle
\geq \frac{1}{2\eta} \|w - w^*\|^2 - \frac{M}{2} \|w - w^*\|^2 - \frac{(1 - \alpha) \delta}{2}
\geq - \frac{(1 - \alpha) \delta}{2},
\]
where the last inequality is due to (17). Plugging the above inequality into (18), we have
\[
\mathbb{P} \left( U \leq \frac{\exp(-f^y(u)(X))}{\exp(-h_1(X))} \right) \geq \left( \frac{1 - \eta M}{1 + \eta M} \right)^{d/2} \exp \left( -\frac{3(1 - \alpha) \delta}{2} - \frac{\eta}{2(1 - \eta M)} \|s\|^2 \right).
\]

Hence, using the above bound, (17) and (11), we obtain the following bound on the expected number of rejection iterations
\[
\mathbb{P} \left( U \leq \frac{1}{\exp(-f^y(u)(X))} \right) \leq \left( \frac{1 + \eta M}{1 - \eta M} \right)^{d/2} \exp \left( \frac{3(1 - \alpha) \delta}{2} + \frac{\eta}{2(1 - \eta M)} \|s\|^2 \right)
\leq \left( 1 + \frac{2\eta M}{1 - \eta M} \right)^{d/2} \exp \left( \frac{3(1 - \alpha) \delta}{2} + \eta \|s\|^2 \right) \leq (1 + 4\eta M)^{d/2} \exp \left( \frac{3(1 - \alpha) \delta}{2} + \frac{\|s\|^2}{Md} \right)
\leq \left( 1 + \frac{4}{d} \right)^{d/2} \exp \left( \frac{3(1 - \alpha) \delta}{2} + 1 \right)
\leq \exp \left( \frac{3(1 - \alpha) \delta}{2} + 3 \right).
\]
5 Proximal sampling for non-convex composite potentials

This section is devoted to discussing the complexity of sampling from distributions with non-convex potential \( f \) satisfying (2). Results presented in this section generalize those given in Section 3, which are developed for the setting with \( f \) satisfying (2) with \( n = 1 \).

Note that although Section 4 is developed for the simple case where \( n = 1 \) in (2), the results and proof techniques apply to the general case of (2). Hence, to avoid duplication, we present the following results analogous to those in Section 4 without giving proofs.

The following lemma is a direct generalization of Lemma 4.1, which shows that functions satisfying (2) can be approximated by smooth functions up to some controllable approximation errors.

**Lemma 5.1.** Assume \( f \) satisfies (2), then for \( \delta > 0 \), we have

\[
|f(u) - f(v) - \langle f'(v), u - v \rangle| \leq \frac{M_n}{2} \|u - v\|^2 + \sum_{i=1}^{n} \frac{(1 - \alpha_i)\delta}{2}, \quad \forall u, v \in \mathbb{R}^d,
\]

where

\[
M_n = \sum_{i=1}^{n} \frac{L_{\alpha_i}^2}{\left[(\alpha_i + 1)\delta\right]^{1/\alpha_i+2}}.
\]

The next proposition is a counterpart of Proposition 4.2 and gives the complexity of solving optimization problem (10) in the context of (2).

**Proposition 5.2.** Assume \( \eta \leq 1/(M_n d) \), and let \( w_n \in \mathbb{R}^d \) be an approximate stationary point of \( f_\eta y \) such that \( \|f'(w_n) + \frac{1}{\eta}(w_n - y)\| \leq \sqrt{M_n d} \). Then, the iteration-complexity to find \( w_n \) by using Algorithm 3 is \( \tilde{O}(1) \).

The core to the proximal algorithm (Algorithm 1) is an efficient implementation of RGO. We use Algorithm 2 with slight modification as a realization of RGO in the context of (2). First, in step 1 of Algorithm 2, we use Algorithm 3 to compute \( w_n \) as in Proposition 5.2 instead of \( w \). Second, in steps 2 and 4 of Algorithm 2, instead of using \( h_1 \) as in (14), we define \( h_1 \) as follows,

\[
h_1(x) = f(w) + \langle f'(w), x - w \rangle - \frac{M_n}{2} \|x - w\|^2 + \frac{1}{2\eta} \|x - y\|^2 - \sum_{i=1}^{n} \frac{(1 - \alpha_i)\delta}{2},
\]

where \( M_n \) is as in (20).

The next proposition is a generalization of Proposition 4.4 in the generic setting where the potential \( f \) satisfies (2) with \( n \geq 1 \).

**Proposition 5.3.** Assume \( f \) satisfies (2) and let \( f_\eta y \) be as in (10). Then \( X \) generated by Algorithm 2 with modification follows the distribution \( \pi_X | Y(x | y) \propto \exp(-f_\eta y(x)) \). Moreover, if \( \eta \leq \frac{1}{M_n d} \), then the expected number of rejection steps in Algorithm 2 is at most \( \exp \left( \frac{3\sum_{i=1}^{n}(1 - \alpha_i)\delta}{2} + 3 \right) \).

In the rest of this section, we present two main results about the complexity of sampling for non-convex composite potentials. As discussed in Section 3, we also consider two cases where the target distribution \( \pi_X \propto \exp(-f) \) satisfies either LSI or PI.
Theorem 5.4. Assume \( f \) satisfies (2) and \( \pi^X \propto \exp(-f) \) satisfies LSI with constant \( C_{\text{LSI}} \). With initial distribution \( \rho_0^X \) and stepsize \( \eta \asymp 1/\left( \sum_{i=1}^n L_{\alpha_i}^{2+1} d \right) \), Algorithm 2 using the modified Algorithm \( \mathfrak{A} \) as an RGO has the iteration-complexity bound

\[
\tilde{O} \left( C_{\text{LSI}} \sum_{i=1}^n L_{\alpha_i}^{2+1} d \right)
\]

(21)

to achieve \( \varepsilon \) error to the target \( \pi^X \) in terms of KL divergence. Each iteration queries \( \tilde{O}(1) \) subgradients of \( f \) and generates \( \mathcal{O}(1) \) samples in expectation from Gaussian distribution.

Theorem 5.5. Assume \( f \) satisfies (2) and \( \pi^X \propto \exp(-f) \) satisfies PI with constant \( C_{\text{PI}} \). With initial distribution \( \rho_0^X \) and stepsize \( \eta \asymp 1/\left( \sum_{i=1}^n L_{\alpha_i}^{2+1} d \right) \), Algorithm 7 using the modified Algorithm \( \mathfrak{A} \) as an RGO has the iteration-complexity bound

\[
\tilde{O} \left( C_{\text{PI}} \sum_{i=1}^n L_{\alpha_i}^{2+1} d \right)
\]

(22)

to achieve \( \varepsilon \) error to the target \( \pi^X \) in terms of Chi-squared divergence, and

\[
\tilde{O} \left( C_{\text{PI}} \sum_{i=1}^n L_{\alpha_i}^{2+1} qdR_{q,\nu}(\rho_0^X) \right)
\]

(23)

to achieve \( \varepsilon \) error in terms of Rényi divergence \( R_{q,\nu} \) (\( q \geq 2 \)). Each iteration queries \( \tilde{O}(1) \) subgradients of \( f \) and generates \( \mathcal{O}(1) \) samples in expectation from Gaussian distribution.

6 Conclusions

In this paper, we develop a novel proximal sampling algorithm for distributions with non-convex and semi-smooth potentials, and establish complexity bound results for the proposed method under the assumption that distributions satisfy either LSI or PI. Our proximal algorithm is based on the ASF, which resembles the proximal point method in optimization. Each iteration of the ASF generates a sample from a regularized target distribution by querying the RGO, which is itself a challenging algorithmic task due to the lack of convexity and smoothness. The core to our development of the proximal sampling algorithm is an efficient realization of the RGO based on rejection sampling. More importantly, we develop a novel technique to bound the expected number of rejection steps in the RGO, which further leads to the complexity results of the proximal sampling algorithms in the context of LSI and PI. Finally, we extend our proximal algorithm and the novel analysis to the case where the potential has multiple semi-smooth components, and establish state-of-the-art complexity results for this setting.

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A Technical results

This section collects a few technical results that are useful for the analysis of RGO in Section 4.

The first result is the well-known Gaussian integral.

**Lemma A.1.** A useful Gaussian integral: for any \( \eta > 0 \),

\[
\int_{\mathbb{R}^d} \exp \left( -\frac{1}{2\eta} \|x\|^2 \right) dx = (2\pi\eta)^{d/2}.
\]

The following lemma shows that \( f_\eta^y \) as in (10) is close to a strongly convex and smooth function when \( \eta \) is small. This result is used in Proposition 4.2.

**Lemma A.2.** Let \( f_\eta^y := f + \frac{1}{2\eta} \| \cdot - y \|^2 \) and \( (f_\eta^y)' := f' + \frac{1}{\eta} (\cdot - y) \), then we have for every \( u, v \in \mathbb{R}^d \),

\[
\frac{1}{2} \left( \frac{1}{\eta} - M \right) \|u - v\|^2 - \frac{(1 - \alpha)\delta}{2} \leq f_\eta^y(u) - f_\eta^y(v) - \langle (f_\eta^y)'(v), u - v \rangle \leq \frac{1}{2} \left( \frac{1}{\eta} + M \right) \|u - v\|^2 + \frac{(1 - \alpha)\delta}{2}.
\]
Lemma A.3.

Proof: Using the definitions of \( f_y' \) and \((f_y')'\), we have

\[
f_y(u) - f_y(v) - \langle (f_y')'(v), u - v \rangle
= f(u) - f(v) - \langle f'(v), u - v \rangle + \frac{1}{2\eta} \| u - y \|^2 - \frac{1}{2\eta} \| v - y \|^2 - \frac{1}{\eta} (v - y, u - v)
= f(u) - f(v) - \langle f'(v), u - v \rangle + \frac{1}{2\eta} \| u - v \|^2.
\]

The lemma now follows from above identity \((\star)\).

The next lemma gives equivalent formulas of \( \int \exp(-h_1(x))\, dx \) and \( \int \exp(-h_2(x))\, dx \) that are useful in Proposition 4.3.

**Lemma A.3.** Recall \( w \) and \( w^* \) are defined in Proposition 4.2 and Lemma 4.3, respectively. Let \( h_1 \) and \( h_2 \) be as in \((\star)\) and \((\star)\), respectively. Then, we have the following integrals

\[
\int \exp(-h_1(x))\, dx = \left( \frac{2\pi \eta}{1 - \eta M} \right)^{d/2} \exp(H_1(w)) ,
\]

\[
\int \exp(-h_2(x))\, dx = \left( \frac{2\pi \eta}{1 + \eta M} \right)^{d/2} \exp(H_2(w^*)) ,
\]

where

\[
H_1(w) = \frac{1}{2\eta} \| w \|^2 + \frac{\eta}{2(1 - \eta M)} \| s \|^2 - f(w) + \frac{1}{\eta} \langle w, y - w \rangle - \frac{1}{2\eta} \| y \|^2 + \frac{1 - \alpha}{2} \delta,
\]

\[
H_2(w^*) = \frac{1}{2\eta} \| w^* \|^2 - f(w^*) + \langle f'(w^*), w^* \rangle - \frac{1}{2\eta} \| y \|^2 - \frac{1 - \alpha}{2} \delta.
\]

Proof: We first rewrite \( h_1 \) and \( h_2 \) as follows

\[
h_1(x) = f(w) + \langle f'(w), x - w \rangle - \frac{M}{2} \| x - w \|^2 + \frac{1}{2\eta} \| x - y \|^2 - \frac{(1 - \alpha)\delta}{2}
= \frac{1 - \eta M}{2\eta} \left\| x + \frac{\eta M}{1 - \eta M} w - \frac{1}{1 - \eta M} y + \frac{\eta}{1 - \eta M} f'(w) \right\|^2
- \frac{1}{2\eta(1 - \eta M)} \| \eta M w - y + \eta f'(w) \|^2
+ f(w) - \langle f'(w), w \rangle - \frac{M}{2} \| w \|^2 + \frac{1}{2\eta} \| y \|^2 - \frac{1 - \alpha}{2} \delta ,
\]

and

\[
h_2(x) = f(w^*) + \langle f'(w^*), x - w^* \rangle + \frac{M}{2} \| x - w^* \|^2 + \frac{1}{2\eta} \| x - y \|^2 + \frac{(1 - \alpha)\delta}{2}
= \frac{1 + \eta M}{2\eta} \left\| x - \frac{\eta M}{1 + \eta M} w^* - \frac{1}{1 + \eta M} y + \frac{\eta}{1 + \eta M} f'(w^*) \right\|^2
- \frac{1}{2\eta(1 + \eta M)} \| \eta M w^* + y - \eta f'(w^*) \|^2
+ f(w^*) - \langle f'(w^*), w^* \rangle + \frac{M}{2} \| w^* \|^2 + \frac{1}{2\eta} \| y \|^2 + \frac{1 - \alpha}{2} \delta .
\]
It follows from (28) and Lemma A.1 that
\begin{align}
\int \exp(-h_1(x))dx &= \left( \frac{2\pi\eta}{1-\eta M} \right)^{d/2} \exp(\hat{H}_1(w)), \\
\int \exp(-h_2(x))dx &= \left( \frac{2\pi\eta}{1+\eta M} \right)^{d/2} \exp(\hat{H}_2(w^*)),
\end{align}

where
\begin{align*}
\hat{H}_1(w) &= \frac{1}{2\eta(1-\eta M)}\|\eta Mw - y + \eta f'(w)\|^2 - f(w) + \langle f'(w), w \rangle \\
&\quad + \frac{M}{2} \|w\|^2 - \frac{1}{2\eta} \|y\|^2 + \frac{1}{2} - \frac{\alpha}{2} \delta, \\
\hat{H}_2(w^*) &= \frac{1}{2\eta(1+\eta M)}\|\eta Mw^* + y - \eta f'(w^*)\|^2 - f(w^*) + \langle f'(w^*), w^* \rangle \\
&\quad - \frac{M}{2} \|w^*\|^2 - \frac{1}{2\eta} \|y\|^2 - \frac{1}{2} - \frac{\alpha}{2} \delta.
\end{align*}

It suffices to show that $H_1(w) = \hat{H}_1(w)$ and $H_2(w^*) = \hat{H}_2(w^*)$ to complete the proof.

We first verify that $H_1(w) = H_1(w)$. Using the definition of $H_1(w)$ above and the definition of $s$ in (11), we have
\begin{align*}
H_1(w) &= \frac{1}{2\eta(1-\eta M)}\|\eta Mw - w + w - y + \eta f'(w)\|^2 - f(w) + \langle f'(w), w \rangle \\
&\quad + \frac{M}{2} \|w\|^2 - \frac{1}{2\eta} \|y\|^2 + \frac{1}{2} - \frac{\alpha}{2} \delta \\
&= \frac{1}{2\eta(1-\eta M)}\|\eta M - 1\|w - \eta s\|^2 - f(w) + \langle f'(w), w \rangle \\
&\quad + \frac{M}{2} \|w\|^2 - \frac{1}{2\eta} \|y\|^2 + \frac{1}{2} - \frac{\alpha}{2} \delta \\
&= \frac{1}{2\eta(1-\eta M)}\|w\|^2 + \frac{M}{2} \|w\|^2 - \langle w, s \rangle + \frac{\eta}{2(1-\eta M)}\|s\|^2 - f(w) + \langle f'(w), w \rangle \\
&\quad - \frac{1}{2\eta} \|y\|^2 + \frac{1}{2} - \frac{\alpha}{2} \delta \\
&= \frac{1}{2\eta} \|w\|^2 + \frac{\eta}{2(1-\eta M)}\|s\|^2 - f(w) + \frac{1}{\eta} \langle w, y - w \rangle - \frac{1}{2\eta} \|y\|^2 + \frac{1}{2} - \frac{\alpha}{2} \delta.
\end{align*}

In view of (26), we verify that $H_1(w) = \hat{H}_1(w)$ and hence (24) is proved.

We next verify that $H_2(w^*) = H_2(w^*)$. Using the definition of $H_2(w^*)$ and (23), we have
\begin{align*}
H_2(w^*) &= \frac{1}{2\eta(1+\eta M)}\|\eta Mw^* + w^* - w^* + y - \eta f'(w^*)\|^2 - f(w^*) + \langle f'(w^*), w^* \rangle \\
&\quad - \frac{M}{2} \|w^*\|^2 - \frac{1}{2\eta} \|y\|^2 - \frac{1}{2} - \frac{\alpha}{2} \delta \\
&= \frac{1}{2\eta(1+\eta M)}\|(\eta M + 1)w^*\|^2 - \frac{M}{2} \|w^*\|^2 - f(w^*) + \langle f'(w^*), w^* \rangle \\
&\quad - \frac{1}{2\eta} \|y\|^2 - \frac{1}{2} - \frac{\alpha}{2} \delta \\
&= \frac{1}{2\eta} \|w^*\|^2 - f(w^*) + \langle f'(w^*), w^* \rangle - \frac{1}{2\eta} \|y\|^2 + \frac{1}{2} - \frac{\alpha}{2} \delta.
\end{align*}
In view of (27), we verify that \( H_2(w^*) = \hat{H}_2(w^*) \) and hence (25) is proved.

## B Solving the optimization problem

In this section, we use Nesterov’s acceleration to establish the iteration-complexity for solving a general optimization problem that is nearly strongly convex and nearly smooth. This general result is then applied in Section 4 to find an approximate stationary point of \( f_\eta(y) \) (see Proposition 4.2).

We consider the optimization problem \( \min \{ g(x) : x \in \mathbb{R}^d \} \), where \( g \) satisfies

\[
\frac{\mu}{2} \| u - v \|^2 - \theta \leq g(u) - g(v) - \langle g'(v), u - v \rangle \leq \frac{L}{2} \| u - v \|^2 + \theta, \quad \forall u, v \in \mathbb{R}^d,
\]

for some given \( \theta > 0, \mu \geq 0, \) and \( L \geq 0 \). We use Nesterov’s accelerated gradient method to find a \( \rho \)-approximate stationary point \( w \) such that \( s \in \partial g(w) \) and \( \| s \| \leq \rho \). We also establish the iteration-complexity to obtain a \( \rho \)-approximate stationary point.

**Algorithm 3 Accelerated Gradient Method**

0. Let an initial point \( x_0 \), parameters \( L, \mu > 0 \) be given, and set \( x_0 = y_0 \), \( A_0 = 0 \), \( \tau_0 = 1 \), and \( k = 0 \);

1. Compute

\[
a_k = \frac{\tau_k + \sqrt{\tau_k^2 + 4 \tau_k LA_k}}{2L}, \quad A_{k+1} = A_k + a_k, \tag{33}
\]

\[
\tau_{k+1} = \tau_k + a_k \mu, \quad \bar{x}_k = \frac{A_k y_k + a_k x_k}{A_{k+1}} \tag{34}
\]

2. Compute

\[
y_{k+1} := \arg\min_{u \in \mathbb{R}^n} \left\{ \gamma_k(u) + \frac{L}{2} \| u - \bar{x}_k \|^2 \right\}, \tag{35}
\]

\[
x_{k+1} := \arg\min_{u \in \mathbb{R}^n} \left\{ a_k \gamma_k(u) + \frac{\tau_k}{2} \| u - x_k \|^2 \right\}, \tag{36}
\]

where

\[
\gamma_k(u) := g(\bar{x}_k) + \langle g'(\bar{x}_k), u - \bar{x}_k \rangle + \frac{\mu}{2} \| u - \bar{x}_k \|^2; \tag{37}
\]

3. Set \( k \leftarrow k + 1 \) and go to step 1.

**Lemma B.1.** For every \( k \geq 0 \) and \( u \in \mathbb{R}^n \), we have

\[
-\theta \leq g(u) - \gamma_k(u) \leq \frac{L - \mu}{2} \| u - \bar{x}_k \|^2 + \theta. \tag{38}
\]

**Proof:** This lemma directly follows from (32) and the definition of \( \gamma_k \) in (37).

**Lemma B.2.** For every \( k \geq 0 \), the following statements hold:

a) \( A_{k+1} = \frac{L a_k^2}{\tau_k} \); 

b) \( A_{k+1} \geq \frac{1}{L} \left( 1 + \frac{\sqrt{\tau}}{2 \sqrt{L}} \right)^{2k} \);
c) \( \sum_{i=0}^{k} A_{i+1} \geq \frac{\exp(2(k+1)(\beta-1)/\beta - 1)}{L(\beta^2 - 1)} \) where \( \beta = 1 + \frac{\sqrt{\tau}}{2\sqrt{L}} \).

**Proof:** a) This statement directly follows from (33).

b) It is easy to see from (33), \( A_0 = 0 \) and \( \tau_0 = 1 \) that \( A_1 = 1/L \). Using the definitions of \( A_k \) and \( \tau_k \) in (33) and (34), respectively, and the facts that \( A_0 = 0 \) and \( \tau_0 = 1 \), we easily derive that

\[
\tau_k = \tau_0 + A_k \mu = 1 + A_k \mu. \tag{39}
\]

It follows from (33) that

\[
A_{k+1} = A_k + a_k = A_k + \frac{\tau_k + \sqrt{\tau_k^2 + 4\tau_k L A_k}}{2L} \geq A_k + \frac{\tau_k}{2L} + \frac{\sqrt{\tau_k A_k}}{\sqrt{L}} \geq \left( \sqrt{A_k} + \frac{\sqrt{\tau_k}}{2\sqrt{L}} \right)^2.
\]

The above inequality and (39) imply

\[
\sqrt{A_{k+1}} \geq \sqrt{A_k} + \frac{\sqrt{\tau_k}}{2\sqrt{L}} = \sqrt{A_k} + \frac{\sqrt{1 + A_k \mu}}{2\sqrt{L}} \geq \left( 1 + \frac{\sqrt{\mu}}{2\sqrt{L}} \right) \sqrt{A_k}.
\]

This statement now follows from the above relation and the fact that \( A_1 = 1/L \).

c) Noting from b) that \( A_{k+1} \geq \beta^{2k}/L \), which together with the fact \( x \geq \exp((x - 1)/x) \) for \( x \geq 1 \), implies that

\[
\sum_{i=0}^{k} A_{i+1} \geq \frac{1}{L} \sum_{i=0}^{k} \beta^{2i} = \frac{\beta^{2(k+1)} - 1}{L(\beta^2 - 1)} \geq \frac{\exp(2(k+1)(\beta-1)/\beta - 1)}{L(\beta^2 - 1)}.
\]

**Lemma B.3.** For every \( k \geq 0 \), define

\[
t_k(u) = A_k \left[ g(y_k) - g(u) \right] + \frac{\tau_k}{2} \| u - x_k \|^2, \tag{40}
\]

then for every \( u \in \mathbb{R}^n \), we have

\[
\frac{\mu}{2} A_{k+1} \| y_{k+1} - \tilde{x}_k \|^2 \leq t_k(u) - t_{k+1}(u) + 2A_{k+1} \theta. \tag{41}
\]

**Proof:** Using the fact \( \gamma_k \) is convex and the definition of \( \tilde{x}_k \) in (34), we have

\[
A_k \gamma_k(y_k) + a_k \gamma_k(u) + \frac{\tau_k}{2} \| u - x_k \|^2 \geq A_{k+1} \gamma_k \left( \frac{A_k y_k + a_k u}{A_{k+1}} \right) + \frac{\tau_k A_{k+1}^2}{2a_k^2} \left\| \frac{A_k y_k + a_k u}{A_{k+1}} - \tilde{x}_k \right\|^2
\]

\[
= A_{k+1} \left[ \gamma_k \left( \frac{A_k y_k + a_k u}{A_{k+1}} \right) + \frac{L}{2} \left\| \frac{A_k y_k + a_k u}{A_{k+1}} - \tilde{x}_k \right\|^2 \right]
\]

\[
\geq A_{k+1} \min \left\{ \gamma_k(u) + \frac{L}{2} \| u - \tilde{x}_k \|^2 \right\}
\]

\[
= A_{k+1} \left[ \gamma_k(y_k) + \frac{L}{2} \| y_k - \tilde{x}_k \|^2 \right],
\]

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where the first identity is due to (33) and the second identity is due to the definition of \( y_{k+1} \) in (35). It follows from the second inequality of (38) with \( u = y_{k+1} \) and the above inequality that

\[
A_{k+1} \left[ g(y_{k+1}) - \theta + \frac{\mu}{2} \| y_{k+1} - \tilde{x}_k \|^2 \right] \\
\leq A_{k+1} \left[ \gamma_k(y_{k+1}) + \frac{L}{2} \| y_{k+1} - \tilde{x}_k \|^2 \right] \\
\leq A_k \gamma_k(y_k) + a_k \gamma_k(x_{k+1}) + \frac{\tau_k}{2} \| x_{k+1} - x_k \|^2 \\
\leq A_k \gamma_k(y_k) + a_k \gamma_k(u) + \frac{\tau_k}{2} \| u - x_k \|^2 - \frac{\tau_{k+1}}{2} \| u - x_{k+1} \|^2
\]

where the last inequality is due to (36) and the fact that \( a_k \gamma_k + \tau_k \cdot -x_k \|^2/2 \) is \( \tau_{k+1} \)-strongly convex. Rearranging the terms in the above inequality, we obtain

\[
\frac{\mu}{2} A_{k+1} \| y_{k+1} - \tilde{x}_k \|^2 \\
\leq A_k \gamma_k(y_k) + a_k \gamma_k(u) + \frac{\tau_k}{2} \| u - x_k \|^2 - \frac{\tau_{k+1}}{2} \| u - x_{k+1} \|^2 - A_{k+1} \left[ g(y_{k+1}) - \theta \right] \\
= A_k \left[ g(y_k) - g(u) \right] + \frac{\tau_k}{2} \| u - x_k \|^2 - A_{k+1} \left[ g(y_{k+1}) - g(u) \right] - \frac{\tau_{k+1}}{2} \| u - x_{k+1} \|^2 \\
+ A_k \left[ \gamma_k(y_k) - g(y_k) \right] + a_k \left[ \gamma_k(u) - g(u) \right] + A_{k+1} \theta \\
\leq t_k(u) - t_{k+1}(u) + 2A_{k+1} \theta
\]

where the identity is due to the fact that \( A_{k+1} = A_k + a_k \), and the last inequality is due to (40) and the first inequality of (38).

**Proposition B.4.** If \( \rho \geq 2\sqrt{2(\mu + L)\theta}/\sqrt{\mu} \), then the number of iterations \( k_0 \) to obtain a \( \rho \)-approximate stationary point of \( g \) is at most

\[
k_0 := \frac{2\sqrt{L + \sqrt{\mu}}}{\mu} \log \left( \frac{(\mu + L)^2 \rho^2}{2 \sqrt{\mu}} \frac{2\sqrt{L + \sqrt{\mu}}}{2\sqrt{\mu}} + 1 \right). \tag{42}
\]

**Proof:** It follows from the optimality condition of (35) that

\[g'(\tilde{x}_k) = (\mu + L)(\tilde{x}_k - y_{k+1}).\]

Using the above relation and summing (41) with \( u = x^* \) from \( k = 0 \) to \( k - 1 \), we have

\[
\frac{\mu}{2(\mu + L)^2} \sum_{i=0}^{k-1} A_{i+1} \| g'(\tilde{x}_i) \|^2 = \frac{\mu}{2} \sum_{i=0}^{k-1} A_{i+1} \| y_{i+1} - \tilde{x}_i \|^2 \\
\leq t_0(x^*) + 2 \sum_{i=0}^{k-1} A_{i+1} \theta = \frac{d_0^2}{2} + 2 \sum_{i=0}^{k-1} A_{i+1} \theta,
\]

where the last identity follows from the facts that \( A_0 = 0 \) and \( \tau_0 = 1 \). The above inequality and the assumption on \( \rho \) imply that

\[
\min_{0 \leq i \leq k-1} \| g'(\tilde{x}_i) \|^2 \leq \frac{(\mu + L)^2}{\mu} \left( \frac{d_0^2}{\sum_{i=0}^{k-1} A_{i+1}} + 4\theta \right) \leq \frac{(\mu + L)^2}{\mu} \frac{d_0^2}{\sum_{i=0}^{k-1} A_{i+1}} + \frac{\rho^2}{2}.
\]
In order to show \( \min_{0 \leq i \leq k-1} \| g'(\tilde{x}_i) \| \leq \rho \), it suffices to show
\[
\frac{(\mu + L)^2}{\mu} \frac{d_0^2}{\sum_{i=0}^{k-1} A_{i+1}} \leq \rho^2 \frac{2}{\rho^2}.
\]  
(43)

Using Lemma B.2 (c) and the fact that \( k \geq k_0 \) where \( k_0 \) is as in (42), we have
\[
\sum_{i=0}^{k-1} A_{i+1} \geq \exp(2k(\beta - 1)/\beta) - 1 \geq \frac{2(\mu + L)^2 d_0^2}{\rho^2},
\]
and hence (43) is proved.

\[\text{C Comparison with the proof techniques of [24, 25]}\]

In this section, we make some remarks on the proof techniques of this paper and [24, 25]. First, recall that the technical challenge in [24, 25] is to bound a modified Gaussian integral (see Proposition 6 of [25]),
\[
\int_{\mathbb{R}^d} \exp \left( -\frac{1}{2\eta} ||x||^2 - a ||x||^{\alpha+1} \right) \, dx \geq \frac{(2\pi \eta)^{d/2}}{\sqrt{e}},
\]  
(44)

where \( a > 0 \) and \( \alpha \in [0, 1] \). On the other hand, a key observation underlying the analysis of this paper is Lemma 5.1, i.e., the approximation of a function with multiple semi-smooth components by a smooth function. It would be very challenging to show complexity results of this paper, if one follows the proof techniques of [24, 25]. This is because it is hard to extend the bound (44) to the multiple semi-smooth components setting where \( ||x||^{\alpha+1} \) in (44) is replaced by \( \sum_{i=1}^n ||x||^{\alpha_i+1} \).

Moreover, Proposition C.1 below shows that the bound (44) can be tightened by using the key observation in this paper, i.e., Lemma 4.1.

\[\text{Proposition C.1. Let } \alpha \in [0, 1], \eta > 0, a \geq 0 \text{ and } d \geq 1. \text{ If}
\]
\[
2a(\eta d)^{(\alpha+1)/2} \leq 1,
\]  
(45)
then
\[
\int_{\mathbb{R}^d} \exp \left( -\frac{1}{2\eta} ||x||^2 - a ||x||^{\alpha+1} \right) \, dx \geq \frac{(2\pi \eta)^{d/2}}{\sqrt{e}}.
\]  
(46)

\[\text{Proof: Using the Young’s inequality } st \leq s^p/p + t^q/q \text{ with}
\]
\[
s = a^2 \frac{1-\alpha}{2} ||x||^{\alpha+1}, \quad t = \frac{1}{2^{1-\alpha}}, \quad p = \frac{2}{\alpha + 1}, \quad q = \frac{2}{1 - \alpha},
\]
we obtain
\[
a ||x||^{\alpha+1} \leq \frac{\alpha + \frac{1}{2} a^2 \frac{1-\alpha}{2} ||x||^2 + \frac{1 - \alpha}{4}}{2}. \]

This inequality and Lemma A.1 imply that
\[
\int_{\mathbb{R}^d} \exp \left( -\frac{1}{2\eta} ||x||^2 - a ||x||^{\alpha+1} \right) \, dx
\]
\[
\geq \int_{\mathbb{R}^d} \exp \left( -\frac{1}{2\eta} ||x||^2 - a^2 \frac{1-\alpha}{2} ||x||^{2\alpha+\alpha'} ||x||^{\alpha} - \frac{1 - \alpha}{4} \right) \, dx
\]
\[
= \exp \left( \frac{\alpha - 1}{4} \right) \int_{\mathbb{R}^d} \exp \left( -\frac{1}{2\eta} ||x||^2 \right) \, dx
\]
\[
= \exp \left( \frac{\alpha - 1}{4} \right) (2\pi \eta')^{d/2}
\]  
(47)
where
\[ 1 = \frac{1}{\eta'} = \frac{1}{\eta} + (\alpha + 1)a^{\frac{2}{\alpha + 1}}2^{\frac{1-\alpha}{\alpha + 1}}. \quad (48) \]

It follows from (45) that
\[ a^{2 \alpha} \leq \frac{1}{2^{1+\alpha} \eta d}, \]

which together with (48) implies that
\[ \frac{1}{\eta'} \leq \frac{1}{\eta} + \frac{(\alpha + 1)2^{\frac{1-\alpha}{\alpha + 1}}}{2^{1+\alpha} \eta d} = \left(1 + \frac{\alpha + 1}{2d}\right)^{-1} \frac{1}{\eta}, \]

and hence that
\[ \eta' \geq \left(1 + \frac{\alpha + 1}{2d}\right)^{-1} \eta. \]

Plugging the above inequality into (47), we have
\[
\int_{\mathbb{R}^d} \exp \left( -\frac{1}{2\eta} \|x\|^2 - a\|x\|^{\alpha+1} \right) dx \geq (2\pi \eta)^{\frac{d}{2}} \left(1 + \frac{\alpha + 1}{2d}\right)^{-\frac{d}{2}} \exp \left( \frac{\alpha - 1}{4} \right)
\geq (2\pi \eta)^{\frac{d}{2}} \exp \left( -\frac{\alpha + 1}{4} + \frac{\alpha - 1}{4} \right)
= (2\pi \eta)^{\frac{d}{2}} \exp \left( -\frac{1}{2} \right),
\]

where in the second inequality, we use the fact that
\[ \left(1 + \frac{\alpha + 1}{2d}\right)^{-\frac{d}{2}} \leq \exp \left( \frac{\alpha + 1}{4} \right). \]