Hodge and Prym tau functions, Jenkins-Strebel differentials and combinatorial model of $M_{g,n}$

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Abstract

The goal of the paper is to apply the approach inspired by the theory of integrable systems to construct explicit sections of line bundles over the combinatorial model of the moduli space of pointed Riemann surfaces based on Jenkins-Strebel differentials. The line bundles are tensor products of the determinants of the Hodge or Prym vector bundles with the standard tautological line bundles $L_j$ and the sections are constructed in terms of tau functions. The combinatorial model is interpreted as the real slice of a complex analytic moduli space of quadratic differentials where the phase of each tau-function provides a section of a circle bundle. The phase of the ratio of the Prym and Hodge tau functions gives a section of the $\kappa_1$-circle bundle.

By evaluating the increment of the phase around co-dimension 2 sub-complexes, we identify the Poincaré dual cycles to the Chern classes of the corresponding line bundles: they are expressed explicitly as combination of Witten’s cycle $W_5$ and Kontsevich’s boundary. This provides combinatorial analogues of Mumford’s relations on $M_{g,n}$ and Penner’s relations in the hyperbolic combinatorial model. The free homotopy classes of loops around $W_5$ are interpreted as pentagon moves while those of loops around Kontsevich’s boundary as combinatorial Dehn twists.

Throughout the paper we exploit the classical description of the combinatorial model in terms of Jenkins–Strebel differentials, parametrized in terms of period, or homological coordinates; we show that they provide Darboux coordinates for the symplectic structure introduced by Kontsevich. We also express the latter as the intersection pairing in the odd homology of the canonical double cover.

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1 Introduction

Tau-functions play an important role in the theory of moduli spaces starting from the works [52] and [39] where it was shown that the tau-function of a special solution of the Korteweg-de-Vries hierarchy generates intersection numbers of $\psi$-classes over $M_{g,n}$. This led to numerous further important results. For example intersection numbers of $\kappa$-classes with $\psi$-classes as well as Hodge integrals turn out to be generated by KdV and KP tau-functions [29, 11]; tau-functions of KP and Toda equation define also Hurwitz numbers of various kind [46]. We refer to [40, 47] and references therein for detailed discussion of the subject.

Tau-functions of another type were introduced in [38] while studying the Jimbo-Miwa isomonodromic deformations of quasi-permutation monodromy groups. They are special cases of general isomonodromic Jimbo-Miwa tau-functions [22, 42]; these tau-functions were named after Bergman due to their relation to the Bergman projective connection. The Bergman tau functions naturally appear in various contexts, including Dubrovin’s theory of Frobenius manifolds [10, 33].

In [32] it was shown that the Bergman tau-function on Hurwitz spaces is essentially a section of the determinant line bundle of the Hodge bundle. This observation was used to compute the Hodge class over spaces of admissible covers via boundary divisors, generalizing a result of Cornalba-Harris [9]. The theory of Bergman tau-functions was further applied to derive new relations in the Picard group of the spaces of abelian, quadratic and $n$–holomorphic differentials over Riemann surfaces [31, 35, 36, 34] and give an elementary proof of Farkas-Verra relations [15] in the Picard group of moduli spaces of spin curves [4].

In [30, 36] it was shown that there exist two natural tau-functions associated to spaces of quadratic differentials over Riemann surfaces. The first tau-function is a holomorphic section of the determinant line bundle of the Hodge vector bundle $\Lambda_H$; the second tau-function is a section of the determinant of the Prym vector bundle $\Lambda_P$. Therefore,
these two tau-functions (as well as their analogs in the case of Jenkins-Strebel differentials) are naturally called the Hodge ($\tau_+$) and the Prym ($\tau_-$) tau-functions, respectively (see Sec. 4.1 and Sec. 4.3). The study of the properties of $\tau_\pm$ gives in particular a direct analytical proof of Mumford’s formula $\lambda_2 - 13\lambda_1 = -\delta$ (which holds in $\text{Pic}(\mathcal{M}_g, \mathbb{Q})$), where $\delta$ is the class of Deligne-Mumford boundary [36]. The Hodge and Prym tau-functions turn out to be also relevant in the study of Lyapunov exponents of the Teichmuller flow [12].

The combinatorial model of $\mathcal{M}_{g,n}$, denoted by $\mathcal{M}_{g,n}[p]$, for fixed $p \in \mathbb{R}_{>0}^n$, is based on the theory of Jenkins-Strebel (JS) differentials. This model was developed starting from ideas of Harer, Mumford and Thurston (see [25]; a modern exposition is given in [45]). We call it the flat combinatorial model as opposed to Penner’s hyperbolic combinatorial model [48]. The flat combinatorial model was central in Kontsevich’s proof [39] of Witten’s conjecture [52].

Cells of $\mathcal{M}_{g,n}[p]$, with $p = (p_1, \ldots, p_n) \in \mathbb{R}_{>0}^n$, are labeled by ribbon graphs (or fatgraphs) of given topology on a Riemann surface $C$ of genus $g$ punctured at $n$ points. The vertices of the ribbon graph are zeros of the JS quadratic differential $Q$ while the faces correspond to poles of $Q$. The quadratic residues at the poles are given by $-p_i^2/4\pi^2$, where $p_1, \ldots, p_n$ are the perimeters of the faces. The lengths of the edges in the metric $|Q|$ coordinatize the cell. All cells with given multiplicities $k_1, \ldots, k_m$ of the zeros of $Q$ form a stratum of $\mathcal{M}_{g,n}[p]$ labeled by the vector $k$.

There exists a compactification $\overline{\mathcal{M}}_{g,n}[p]$ of the combinatorial model [45, 41] which provides a partial parametrization of the Deligne–Mumford compactification $\overline{\mathcal{M}}_{g,n}$: points of the combinatorial (Kontsevich’s) boundary correspond to quadratic differentials with simple poles, i.e., ribbon graphs with uni-valent vertices. The main stratum of $\mathcal{M}_{g,n}[p]$ corresponds to quadratic differentials with all simple zeros, so that the Strebel graph has only three-valent vertices; we denote this stratum by $W$. The boundaries of real co-dimension 1 of the cells of $W$ (the facets) correspond to JS differentials $Q$ with one double zero, while all other zeros remain simple; the corresponding ribbon graphs have one 4-valent vertex while all other vertices are three-valent. The union of cells of $W$ and their facets will be denoted by $\widetilde{W}$. The complement, $\overline{\mathcal{M}}_{g,n}[p] \setminus \widetilde{W}$, contains cells of (real) co-dimension 2 and higher.

In co-dimension 2 there exist two special sub-complexes of $\overline{\mathcal{M}}_{g,n}[p]$:

- The first, called “Witten’s cycle” $W_5$, corresponds to ribbon graphs with at least one 5-valent vertex. In the main stratum of $W_5$ the differential $Q$ has one zero of order 3 while all other zeros are simple.

- The second, $W_{1,1}$ (also a cycle) is the “Kontsevich’s boundary” of $\mathcal{M}_{g,n}[p]$. The main stratum corresponds to the collapse of two edges forming a homotopically non-trivial loop, and further resolution of the arising two-valent vertex into two 1-valent ones. Then the differential $Q$ acquires two simple poles at the nodal points in the normalization of the curve [45]. The corresponding ribbon graph either remains connected (and has two 1-valent vertices) or consists of two components with one 1-valent vertex each. The cycle $W_{1,1}$ is the sum of several sub-cycles that correspond to different topological types of the boundary of $\mathcal{M}_{g,n}$.

While $\mathcal{M}_{g,n}[p]$ is in one-to-one correspondence with $\mathcal{M}_{g,n}$ itself, this isomorphism does not extend to the boundary: some components of Deligne-Mumford boundary $\overline{\mathcal{M}}_{g,n} \setminus \mathcal{M}_{g,n}$ (the ones corresponding to homologically trivial vanishing cycles which leave all punctures in one of the connected components of the stable curve) are not represented by cells of highest dimension of $W_{1,1}$.

On the main stratum of each Witten-Kontsevich cycle (see Def. 2.8) of $\overline{\mathcal{M}}_{g,n}[p]$ there is an orientation that extends consistently from cell to cell. This orientation is defined by the top power of Kontsevich’s symplectic form

$$
\Omega = \sum_{i=1}^{n} p_i^2 \omega_i
$$

(1.1)

where $\omega_i$ is the representative of the first Chern class $\psi_i$ of the $i$-th tautological line bundle $\mathcal{L}_i$ [39], see (2.11). The integral of the top power of the form $\Omega$ over $W$ was used in [39] to generate the intersection indices of $\psi$-classes over $\mathcal{M}_{g,n}$.
Denote by $\lambda$ and $\lambda^{(n)}$ the first Chern classes of the determinant line bundles of the Hodge vector bundle $\Lambda_H$ and the vector bundle $\Lambda^{(n)}_2$ of quadratic differentials with at most simple poles at punctures over $\overline{M}_{g,n}$, respectively. Then Mumford formulas (Th. 7.6 of [3]) in $\text{Pic}(\overline{M}_{g,n}, \mathbb{Q})$ give the following relation between $\lambda$, $\lambda^{(n)}$, the classes $\psi_i$ and the Deligne–Mumford boundary $\delta_{DM}$:

$$\lambda^{(n)} - 13\lambda = \sum_{i=1}^{n} \psi_i - \delta_{DM}. \quad (1.2)$$

The formula (1.2) was established at the level of differential forms in [54] where the class $\sum_{i=1}^{n} \psi_i$ on $\overline{M}_{g,n}$ was expressed via Eisenstein series.

Another formula also due to Mumford expresses the kappa–class $\kappa_1$ as follows (see Th. 7.6 of [3])

$$\kappa_1 = 12\lambda + \sum_{i=1}^{n} \psi_i - \delta_{DM}. \quad (1.3)$$

The formulas (1.2) and (1.3) imply the expression of $\kappa_1$ via $\lambda$ and $\lambda^{(n)}$:

$$\kappa_1 = \lambda^{(n)} - \lambda, \quad (1.4)$$

therefore $\kappa_1$ coincides with the first Chern class of the following line bundle over $\overline{M}_{g,n}$:

$$\chi_{\kappa} = \frac{\det\Lambda^{(n)}_2}{\det\Lambda_H}. \quad (1.5)$$

In the hyperbolic combinatorial model of $\overline{M}_{g,n}$ the following relation was found by Penner [48]: $12\kappa_1 = W^{hyp}_5 + W^{hyp}_{1,1}$ where $W^{hyp}_5$ is the analog of Witten’s cycle and $W^{hyp}_{1,1}$ is the cycle corresponding to the boundary of $\overline{M}_{g,n}$ in the hyperbolic combinatorial model. An analog of Penner’s formula

$$12\kappa_1 = W_5 + W_{1,1} \quad (1.6)$$

in the flat combinatorial model was proved in [2]; see [44, 18] for alternative proofs (notice that the sum of boundary divisors in the second formula of Corollary A.1 of [44] coincides with $W_{1,1}$).

### 1.1 Summary of results

The principal goal of this paper is to express the Hodge class $\lambda$, the class $\lambda^{(n)}$ (and the closely related Prym class $\lambda_P = c_1(\det\Lambda_P)$, see details in Sec. 4.4) via $\psi$-classes and combinatorial cycles $W_5$ and $W_{1,1}$ using the Hodge and Prym tau functions $\tau_{\pm}$. As a corollary we provide a new proof of (1.6) and derive the analogs of Mumford’s relations (1.2) and (1.3) in the context of the flat combinatorial model.

Denote by $S[\chi_{\kappa}]$ the circle bundle associated to the line bundle $\chi_{\kappa}$ over the flat combinatorial model. We prove that a section of this “kappa circle bundle” is given by the argument of the ratio $\tau_+ / \tau_-$ (Corollary 5.13).

We show that the periods of the square root of the $JS$ differential on the two-sheeted canonical cover give Darboux coordinates for the Kontsevich symplectic form (1.1) which allows to effectively define the orientation of different cycles of the combinatorial model.

Let us now describe the content of the paper in more detail.

**Canonical cover defined by a meromorphic quadratic differential.** Let $Q$ be a meromorphic quadratic differential on a curve $C$ of genus $g$ such that the only even-order poles are of second order and denoted by $z_1, \ldots, z_n$. It is convenient to group the zeros of odd multiplicity together with the poles of odd order, denoting the total number by
We denote by $m_{\text{even}}$ the number of zeros of even multiplicity and write the divisor $(Q)$ (deg$(Q) = 4g - 4$) in form

$$
(Q) = \sum_{i=1}^{M} d_i q_i + \sum_{i=1}^{m_{\text{odd}}} (2k_i + 1) x_i + \sum_{i=m_{\text{odd}}+1}^{m} 2l_i x_i - \sum_{i=1}^{n} 2z_i .
$$

(1.7)

where $k_i \in \mathbb{Z}, l_i \in \mathbb{N}, m = m_{\text{odd}} + m_{\text{even}}$.

The moduli space of pairs $(C, Q)$ where divisor of $Q$ is of the form (1.7) will be denoted by $Q_{g,n}^{k,1}$.

The canonical covering $\hat{C}$ of $C$ is defined by the resolution of the nodes of the curve

$$
v^2 = Q,
$$

(1.8)

in $T^* C$. In the case of holomorphic quadratic differentials the canonical covering appeared first in Teichmüller theory (see [1]); it also gives a simplest example of the spectral curve of Hitchin’s systems [26] and plays a role in the theory of supersymmetric Yang-Mills equations [49] where the differential is allowed to be meromorphic.

The covering (1.8) is branched at poles and zeros of odd multiplicity $\{x_i\}_{i=1}^{m_{\text{odd}}}$. Note that $m_{\text{odd}} \in 2\mathbb{N}$ because deg$(Q) = 4g - 4 \in 2\mathbb{N}$. Therefore the genus of $\hat{C}$ equals

$$
\hat{g} = 2g + \frac{m_{\text{odd}}}{2} - 1.
$$

(1.9)

Denote by $\pi$ the canonical projection $\hat{C} \to C$ and by $\mu$ the involution of $\hat{C}$ interchanging the sheets.

Each of the zeros of even multiplicity $\{x_i\}_{i=1}^{m_{\text{odd}}+1}$ has two preimages on $\hat{C}$ which we denote by $\hat{x}_i$ and $\hat{x}_i^\mu$. Similarly, the preimages of a pole $z_i$ are denoted by $\hat{z}_i$ and $\hat{z}_i^\mu$. The pre-images on $\hat{C}$ of poles and zeros of odd multiplicity $\{x_i\}$ are branch points of the projection $\hat{C} \to C$; therefore, we shall continue to denote them by the same letters, omitting the hat.

The holomorphic involution $\mu : \hat{C} \to \hat{C}$ induces a linear map with eigenvalues $\pm 1$ on the cohomology of $\hat{C}$ and therefore the space $H^{(1,0)}(\hat{C})$ of holomorphic differentials on $\hat{C}$ can be decomposed into the direct sum of two eigenspaces $H^\pm$

$$
H^{(1,0)}(\hat{C}) = H^+ \oplus H^- , \quad \dim H^+ = g , \quad \dim H^- = g_-. \quad (1.10)
$$

where

$$
g_- = g + \frac{m_{\text{odd}}}{2} - 1 .
$$

(1.11)

The space $H^+$ can be identified with the fiber of the Hodge vector bundle $\Lambda_H$ over $Q_{g,n}^{k,1}$. The space $H^-$ is the space of holomorphic Prym differentials; it is the fiber of the Prym vector bundle $\Lambda_P$ over $Q_{g,n}^{k,1}$.

The differential $v$ satisfies $\mu^* v = -v$; it is a meromorphic Prym differential. The differential $v$ is holomorphic (i.e. $v \in H^-$) if $Q$ does not have poles of order higher than 1.

The homology group of $\hat{C} \setminus \{\hat{z}_j, \hat{z}_j^\mu\}_{j=1}^n$, relative to $\{\hat{x}_j, \hat{x}_j^\mu\}_{j=m_{\text{odd}}+1}$, which we denote by

$$
H_1(\hat{C} \setminus \{\hat{z}_j, \hat{z}_j^\mu\}_{j=1}^n; \{\hat{x}_j, \hat{x}_j^\mu\}_{j=m_{\text{odd}}+1})
$$

(1.12)

we consider it over $\mathbb{R}$) decomposes as $H_+ \oplus H_-$, where $\dim H_+ = 2g$, $\dim H_- = 2g_- + n + m_{\text{even}}$.

Notice that $\dim H_- = \dim Q_{g,n}^{k,1}$. Choosing some basis of cycles $\{s_j\}$ in $H_-$ we define the period, or homological coordinates on the moduli space $Q_{g,n}^{k,1}$ by

$$
P_{s_j} = \int_{s_j} v , \quad j = 1, \ldots, 2g_- + n + m_{\text{even}} .
$$

(1.13)

When $Q$ has at most double poles (therefore $k_i \geq -1$) the real slice $\{P_{s_j} \in \mathbb{R}, \forall j = 1 \ldots \hat{g}\}$ of $Q_{g,n}^{k,1}$ corresponds to Jenkins-Strebel differentials.
Hodge and Prym tau-functions. The Hodge ($\tau_+$) and Prym ($\tau_-$) tau-functions on spaces of quadratic differentials are the main analytical tools used in this paper; in their definition we follow [30, 36, 28]. They satisfy a system of differential equations with respect to the homological coordinates $P_j$ (1.13) which can be solved in terms of theta-functions and other canonical objects [37, 31]. Here we write the formula for $\tau_+$ on the space $Q_{g,n}^{k,1}$ referring to Section 4.3 for the formula for $\tau_-$. Denote by $E(x, y)$ the prime-form on $C$, by $A_x$ the Abel map with base-point $x$ and by $K^x$ the vector of Riemann constants. For $n \geq 1$ the fundamental polygon of $C$ can always be chosen so that (see Lemma 6 of [31]) $\frac{1}{2} A_x((Q)) + 2K^x = 0$.

Introduce the following multi-valued $g(1 - g)/2$ - differential $C(x)$ [17]:

$$C(x) = \frac{1}{W(x)} \left( \sum_{i=1}^{g} v_i(x) \frac{\partial}{\partial w_i} \right)^{g} \theta(w, \Omega) \bigg|_{w = K^x}, \quad W(x) := \det \left[ \frac{d^{k-1}}{d^{k-1}} v_j \right]_{1 \leq j, k \leq g} \quad (1.14)$$

where $\Omega$ is the period matrix of $C$, $\{v_j\}_{j=1}^{g}$ are holomorphic 1-forms on $C$ normalized by $\int_{a_i} v_j = \delta_{ij}$ and $\theta$ is the corresponding theta-function.

The Hodge tau function $\tau_+$ is then expressed in terms of the divisor $(Q) = \sum d_i q_i$ as follows:

$$\tau_+ = C^{2/3}(x) \left( \frac{Q(x)}{\prod_{i=1}^{M} E_{d_i}(x, q_i)} \right)^{(g-1)/6} \prod_{i<j} E(q_i, q_j)^{d_i d_j}. \quad (1.15)$$

Although the formula (1.15) seems to depend on $x$, in fact it is constant with respect to it. The prime-forms in (1.15) are evaluated at the points $q_i$ in the so-called distinguished local coordinates $\zeta_j$ as follows:

$$E(x, q_i) = \lim_{y \to q_i} E(x, y) \sqrt{d \zeta_i(y)}, \quad (1.16)$$

$$E(q_i, q_j) = \lim_{x \to q_i, y \to q_j} E(x, y) \sqrt{d \zeta_i(y)} \sqrt{d \zeta_j(x)}. \quad (1.17)$$

Near zeros $x_j$ of odd/even multiplicities (see (1.7)) the distinguished local coordinate are defined by, respectively,

$$\zeta_j(x) = \left[ \int_{x_j}^{x} v \right]^{2/(2k_j+3)}, \quad j = 1, \ldots, m_{\text{odd}}; \quad \zeta_j(x) = \left[ \int_{x_j}^{x} v \right]^{1/(l_j+1)}, \quad j = m_{\text{odd}} + 1, \ldots, m, \quad (1.18)$$

while, near the double poles $z_j$ they are given by

$$\xi_j(x) = \exp \left\{ \frac{2 \pi i}{p_j} \int_{x_j}^{x} v \right\} \quad (1.19)$$

where $x_1$ is a chosen zero of $Q$ and $p_k = 2 \pi i \text{res} |_{x_k} v$. The expression (1.15) depends on the choice of the "first" zero $x_1$ and on the integration paths in (1.19); different choices affect the coordinates $\{\xi_j\}$ by an $x$–independent factor. Under the change of Torelli marking on $C$ with an $Sp(2g, \mathbb{Z})$ matrix \[
\begin{pmatrix}
A & B \\
C & D
\end{pmatrix}
\] the function $\tau_+$ is multiplied by $\det(C \Omega + D)$ up to a root of unity.

The function $\tau_+$ (1.15) has degree of homogeneity equal to $-1/12$ with respect to this factor and hence a certain integer power of the expression

$$\tau_+^{12} \prod_{k=1}^{n} d \xi_k(z_k), \quad (1.20)$$

is invariant under the choice of local parameters $\xi_j$ near $z_j$ and also under the choice of signs in the definition of $v$, the prime-forms and local parameters $\zeta_j$.

The Prym tau-function $\tau_-$ is similarly constructed on any $Q_{g,n}^{k,1}$ in Section 4.3; under a change of canonical basis in $H_-$ by a symplectic matrix \[
\begin{pmatrix}
A_- & B_- \\
C_- & D_-
\end{pmatrix}
\] the function $\tau_-$ is multiplied by $\det(C_- \Omega_- + D_-)$ (up to a root of unity) where $\Omega_- = 2 \Pi$ with $\Pi$ being the Prym matrix of the canonical cover.
Homological symplectic structure on the combinatorial model. When all \( k_i \) and \( l_i \) in (1.7) are equal to 0 the space \( \Omega_{g,n}^{k,1} \) will be denoted by \( \Omega_{g,n}^0 \). In this case all zeros of \( Q \) are simple and all poles are of second order at \( z_1, \ldots, z_n \) i.e. \( m_{\text{even}} = 0 \) and \( m = m_{\text{odd}} = 4g - 4 + 2n \). Moreover \( g_- = \dim H^- = 3g - 3 + n \) and \( \dim H_- = \dim \Omega_{g,n}^0 = 6g - 6 + 3n = 2g_- + n \).

On the other hand, the intersection pairing on \( H_- \) has rank \( 6g - 6 + 2n = \dim H_- - n \). Given any two cycles \( s, \tilde{s} \in H_- \) and their corresponding homological coordinates \( P_s \) and \( P_{\tilde{s}} \) we define the homological Poisson bracket by the formula

\[
\{ P_s, P_{\tilde{s}} \} = s \circ \tilde{s} \quad \text{(1.21)}.
\]

For \( n = 0 \) (spaces of holomorphic quadratic differentials) the bracket (1.21) was introduced in [5].

The Casimir functions of (1.21) are \( P_{\gamma_j}, \ j = 1, \ldots, n \) where \( \gamma_j \in H_- \) is a half of the difference of two small positively oriented circles around the poles \( z_j, z_j' \) of \( v \) on the cover \( \tilde{C} \). Denote now by \( \Omega_{g,n}^0[p], \ p = (p_1, \ldots, p_n) \) the symplectic leaves of (1.21) in \( \Omega_{g,n}^0 \) where the quadratic residues at the poles \( z_j \)'s have the fixed values \(-\frac{p_j^2}{4\pi z_j^2}\). The symplectic form on the leaves \( \Omega_{g,n}^0[p] \) is given by

\[
\Omega_{\text{hom}} = \sum_{i=1}^{g_-} dA_i \wedge dB_i \quad \text{(1.22)}
\]

where \( A_j = P_{a_j}, B_j = P_{b_j}, j = 1, \ldots, g_- = 3g - 3 + n \) are the homological coordinates corresponding to a symplectic basis in \( (H_- \mod \mathbb{R}\{\gamma_j\})_j^{n=1} \). The coordinates \( A_j, B_j \) are thus defined up to addition of integer multiples of \( p_i \)'s, but the form (1.22) is well-defined on the symplectic leaves. Similarly, the form (1.22) remains well-defined on symplectic leaves \( \Omega_{g,n}^{k,0} \) when all zeros have odd multiplicity.

Real section of \( \Omega_{g,n}^0[p] \) and combinatorial model. The real slice of \( \Omega_{g,n}^0[p] \) (where all homological coordinates including the perimeters \( p_j \) are real) corresponds to the space of quadratic Jenkins–Strebel differentials with simple zeros, and hence it is in one-to-one correspondence with points of the largest stratum of the flat combinatorial model \( \mathcal{M}_{g,n}[p] \).

The complex symplectic form (1.22) induces a real symplectic form on each maximal cell of the flat combinatorial model. A parallel construction provides a symplectic structure on every cell where the Jenkins–Strebel differential \( Q \) has only zeros of odd multiplicity. The following theorem states the equality between Kontsevich’s and homological symplectic forms in the combinatorial setting.

**Theorem** [Thm. 2.11] The homological symplectic structure \( \Omega_{\text{hom}} \) (1.22) coincides with the Kontsevich symplectic form \( \Omega \) (1.1) on each cell of \( \mathcal{M}_{g,n}[p] \) labelled by ribbon graphs with only odd-valent vertices. Therefore the homological coordinates \( \{A_j, B_j\}_{j=1}^{g_-} \) are Darboux coordinates of \( \Omega \).

The symplectic form allows us to define an orientation within each of the cells mentioned in the theorem; analogously to [39, 45] we prove in Section 2.1 that this orientation propagates consistently between neighbouring cells of the Kontsevich-Witten cycles (see Def. 2.8 and Prop. 2.6). Moreover the orientation is also consistently propagating between Kontsevich-Witten cycles of different dimensions.

Hodge and Prym classes on \( \mathcal{M}_{g,n}[p] \) using \( \tau \). The Prym bundle \( \Lambda_P \) can be extended from \( \mathcal{M}_{g,n}[p] \) to \( \mathcal{M}_{g,n}[p] \) following [36, 34]. Over the largest stratum of \( \mathcal{M}_{g,n}[p] \) the fiber of \( \Lambda_P \) coincides with the space \( H^- \) of holomorphic Prym differentials. Over smaller strata the fibers of \( \Lambda_P \) contain also meromorphic Prym differentials.

In previous papers [32, 35, 36, 34] the study of the divisor of tau-functions allowed to express the first Chern class of the Hodge (and Prym in [36]) vector bundles via boundary divisors on various moduli spaces. In these cases
the tau-functions are complex-analytic on the corresponding moduli spaces. The straightforward application of this approach to the combinatorial model is not possible because \( \mathcal{M}_{g,n}[p] \) is not complex-analytic itself. On the other hand, since the combinatorial model is embedded into the space \( \mathcal{Q}_{g,n}[p] \) as its real slice, we can restrict \( \tau_\pm \) from strata of \( \mathcal{Q}_{g,n}[p] \) to strata of \( \mathcal{M}_{g,n}[p] \). This gives sections of (some integer powers of) the bundles \( \det \Lambda_{p,H}^{12} \prod_{i=1}^{n} L_i \), respectively. These sections are real-analytic within each maximal cell of the combinatorial model.

Therefore, similarly to [39] (see [55] for details) instead of working with line bundles it is more natural to consider the associated circle (or \( U(1) \)) bundles.

The computation of the Poincaré duals of \( c_1 \left( \det \Lambda_{p,H}^{12} \prod_{i=1}^{n} L_i \right) \) in the combinatorial model requires the evaluation of the increment of the arguments \( \Phi_\pm = \arg \tau_\pm \) around cycles of co-dimension 2, which are the Witten cycles \( W_5 \) and the Kontsevich boundary \( W_{1,1} \). Closed paths around \( W_5 \) are represented by pentagon moves, while closed paths around \( W_{1,1} \) are represented by combinatorial Dehn’s twists (see Section 2.3). The absolute values \( |\tau_\pm| \) vanish on all facets of co-dimension 1; on the other hand (Prop. 5.3) \( \Phi_\pm \) can be extended continuously across the facets of the complex allowing us to compute the increment around the co-dimension 2 sub-complex.

The computation of the increment of \( \Phi_\pm \) along closed paths can be performed by studying local models using results of [6] where special types of genus one Boutroux curves were analyzed. This leads to the following

**Theorem [Thm. 5.11]** The following relations hold:

\[
\lambda + \frac{1}{12} \sum \psi_i = \frac{1}{144} W_5 + \frac{13}{144} W_{1,1} \quad , \tag{1.23}
\]

\[
\lambda_P + \frac{1}{12} \sum \psi_i = \frac{13}{144} W_5 + \frac{25}{144} W_{1,1} \quad . \tag{1.24}
\]

We prove that the Prym vector bundle \( \Lambda_P \) over \( \overline{\mathcal{M}}_{g,n}[p] \) is isomorphic to the vector bundle \( \Lambda_{2}^{(n)} \) of meromorphic quadratic differentials on \( C \) with simple poles at punctures (an analog of this statement in the case of holomorphic quadratic differentials was established in [36, 34]). Therefore, taking into account that \( \kappa_1 = \lambda_{2}^{(n)} - \lambda \) (1.4) one can express the class \( \kappa_1 \) via \( W_5 \) and \( W_{1,1} \) reproducing the Arbarello-Cornalba formula (1.6).

Denote the circle bundle corresponding to the line bundle \( L_\kappa (1.5) \) by \( S[L_\kappa] \) (we recall that \( c_1(L_\kappa) = \kappa_1 \)). Then the relations (1.23), (1.24) imply the following

**Corollary [Cor. 5.13]** A section of the circle bundle \( S[(L_\kappa)^{48}] \) over \( \overline{\mathcal{M}}_{g,n}[p] \) is given by \( \Theta_\pm \), where \( \Theta_\pm = \left( \frac{\tau_\pm}{|\tau_\pm|} \right)^{48} \).

The elimination of the Witten cycle \( W_5 \) from (1.23), (1.24) leads to the following corollary

**Corollary [Cor. 5.13]** The following relation holds:

\[
\lambda_{2}^{(n)} - 13\lambda = -\sum_{i=1}^{n} \psi_i = -W_{1,1} . \tag{1.25}
\]
The advantage of explicit formulas for sections of line bundles is that, for example, questions about the Poincaré cycles representing a given line bundle can be immediately addressed by analyzing the divisor of the section. For example, a quite elementary question that remains unanswered is to identify the Poincaré dual of a \( \psi \)–class, and therefore also \( \sum_{j=1}^{n} \psi_j \) (Takhtajan–Zograf class [54]) in the combinatorial picture. In the \( \mathcal{M}_{1,1} \) case this question is elementary because of the identification between sections and modular forms but in general the answer is not known. Therefore the construction of explicit sections of line bundles has the twofold advantage of providing non-trivial special functions in the theory of integrable systems as well as providing tangible insight into the geometry of moduli spaces.

Organization of the paper. In Section 2 we discuss the flat combinatorial model of \( \mathcal{M}_{g,n} \) based on Jenkins-Strebel differentials. Here we interpret pentagon moves and combinatorial Dehn’s twist as paths around the Witten-Kontsevich cycles \( W_5 \) and \( W_{1,1} \), respectively. We also prove that homological, or period, coordinates are Darboux coordinates for the Kontsevich’s symplectic form used to orient the flat combinatorial model. In Section 3 we describe the geometry of canonical covering of a Riemann surface defined by a meromorphic quadratic differential and introduce homological coordinates on moduli spaces of quadratic differentials with given multiplicities of poles and zeros. In Section 4 we define and study the Hodge and Prym tau-functions on moduli spaces of quadratic differentials with given orders of poles and zeros. We treat in details two main examples in genus zero which play a key role in application to flat combinatorial model of \( \mathcal{M}_{g,n} \). In Section 5 we compute the increments of the argument of \( \tau \) with respect to pentagon move and combinatorial Dehn’s twist (i.e. the monodromy around Witten’s cycle \( W_5 \) and Kontsevich’s boundary \( W_{1,1} \)) which leads to the formulas (1.23) and (1.24) expressing Hodge and Prym classes via Kontsevich-Witten cycles in the combinatorial setting. As a corollary we prove relation (1.25) between classes \( \lambda_1 \) and \( \lambda_{2}^{(n)} \), the formula (1.6) and construct an explicit section of circle bundle \( S[L_{\kappa}] \) such that \( c_1(L_{\kappa}) \) is the first kappa-class \( \kappa_1 \).

2 Combinatorial model of \( \mathcal{M}_{g,n} \) via Jenkins-Strebel differentials

To introduce the flat combinatorial model of \( \mathcal{M}_{g,n} \) we start from the following definition (equivalent to the standard one [50]):

**Definition 2.1** A quadratic differential \( Q \) on an \( n \)–marked smooth curve \( C \) is called a Jenkins-Strebel differential if it has double poles at the marked points and all homological (or period) coordinates (1.13) are real:

\[
\mathcal{P}_s \in \mathbb{R}, \quad \forall s \in H_-,
\]

where \( H_- \) is the odd part of the homology of the canonical cover \( \hat{C} \) defined by (1.8).

The reality of the periods implies that the quadratic residues of \( Q \) at its double poles \( z_j \) are real and negative i.e. there exist \( p_j \in \mathbb{R}_+ \) such that in any local coordinate \( \zeta \) near \( z_j \) one has:

\[
Q(\zeta) = \frac{-(p_j/(2\pi))^2}{\zeta^2} \left(1 + O(\zeta)\right) (d\zeta)^2
\]

The Thm. 23.5 of Strebel’s book [50] states that for each Riemann surface \( C \) of genus \( g \) with \( n \) marked points \( z_1, \ldots, z_n \) and given \( p = (p_1, \ldots, p_n) \in \mathbb{R}_+^n \) there exists a unique Jenkins-Strebel differential \( Q \) with the expansion (2.2) near each \( z_j \) and no other poles.

The flat combinatorial model \( \mathcal{M}_{g,n}[p] \) of \( \mathcal{M}_{g,n} \) is constructed as follows (see [55] for references). Given a JS differential \( Q \) the corresponding ribbon graph \( \Gamma \) is the unoriented graph embedded in \( C \) defined as follows.

The vertices of \( \Gamma \) correspond to the zeroes and simple poles of \( Q \); zeros of multiplicity \( k \) are vertices of valence \( k + 2 \) and simple poles are uni-valent vertices. The edges of \( \Gamma \) correspond to arcs of horizontal trajectories (in the
Figure 1: Ribbon graph on a genus 1 Riemann surface representing a point in $\mathcal{M}_{1,1}[p]$. Two simple zeros $x_1$ and $x_2$ of $Q$ are connected by 3 edges of lengths $\ell_1$, $\ell_2$ and $\ell_3$. The ribbon graph has only one face of perimeter $p = 2(\ell_1 + \ell_2 + \ell_3)$. The lengths $\ell_1$ and $\ell_2$ can be used as coordinates on $\mathcal{M}_{1,1}[p]$.

The lengths $\ell_1$ and $\ell_2$ can be used as coordinates on $\mathcal{M}_{1,1}[p]$. The faces of $\Gamma$ are the connected components of $C \setminus \Gamma$, which are in one-to-one correspondence with the marked points.

Strebel’s result implies that there is a one-to-one correspondence between interior points of $\mathcal{M}_{g,n}$ (i.e. smooth curves) and metrized ribbon graphs $\Gamma$ with vertices of valence $\geq 3$, where the lengths $\ell_e$ of the edges $e$ of $\Gamma$ in the metric $|Q|$ provide local coordinates on $\mathcal{M}_{g,n}[p]$. The topological types of the graphs $\Gamma$ label cells of $\mathcal{M}_{g,n}[p]$. The strata of $\mathcal{M}_{g,n}[p]$ are labeled by the valences of the vertices. Namely, the stratum $\mathcal{M}^{k,1}_{g,n}[p]$ consists of punctured Riemann surfaces such that the Jenkins-Strebel differential $Q$ has zeros of odd multiplicities $2k_j + 1$, $j = 1, \ldots, m_{\text{odd}}$, $k_j \geq 1$ and even multiplicities $2l_j$ ($l_j \geq 1$), $j = 1, \ldots, m_{\text{even}}$ (1.7). The (real) dimension of the stratum $\mathcal{M}^{k,1}_{g,n}[p]$ is given by

$$\dim_{\mathbb{R}} \mathcal{M}^{k,1}_{g,n}[p] = 2g - 2 + m$$

where $m = m_{\text{even}} + m_{\text{odd}}$. The largest stratum, $W$, corresponds to Jenkins-Strebel differentials with simple zeros (all vertices of $\Gamma$ are then tri-valent) and has real dimension $6g - 6 + 2n$ (in this case $m = 4g - 4 + 2n$) which coincides with the real dimension of $\mathcal{M}_{g,n} \times \mathbb{R}^n$.

The length $\ell_e$ of an edge $e$ connecting two vertices $v_1, v_2$ is equal to the absolute value of the integral of $v$ along the horizontal trajectory connecting the two vertices. In turn, up to a factor of $\pm 1/2$, this length coincides with the period of $v$ over the integer cycles in $H_-$ consisting of the trajectory $e$ on one sheet of the projection $\hat{C} \to C$ and the same trajectory in the opposite direction on the other sheet. Fixing the vector $p \in \mathbb{R}^n$ one imposes $n$ linear constraints on the lengths of the edges: for the $j$-th face $f_j$, $j = 1, \ldots, n$ we have $\sum_{e \in \partial f_j} \ell_e = p_j$.

The faces of the graph $\Gamma$ are embedded disks in $C$. The face $f_j$ containing the pole $z_j$ is uniformized to the unit disk explicitly by the map

$$w_j(x) = \exp\left(\frac{2\pi i}{p_j} \int_{x_j^0}^x v\right);$$

with $z_j$ being mapped to the origin where one chooses the branch of $v$ which has residue $+\frac{p_j}{2\pi i}$ at $z_j$. Here $x_j^0$ is one of the zeros of $Q$ corresponding to a vertex on the boundary of $f_j$, arbitrarily chosen as basepoint of integration; $w_j(x_j^0) = 1$.

Within the face $f_j$ the flat metric $ds^2 = |Q|$ on $C$ is expressed as:

$$ds^2 = \frac{p_j^2}{4\pi^2} \left| \frac{dw_j}{w_j} \right|^2 \quad 0 < |w_j| \leq 1.$$  

For example, the ribbon graph corresponding to the stratum of highest dimension in the combinatorial model $\mathcal{M}_{1,1}[p]$ has only one face (Fig. 1). The constraint between the lengths of the edges in this case reads $2(\ell_1 + \ell_2 + \ell_3) = p$. 


Inverting this logic it is possible to construct a polyhedral Riemann surface (i.e. Riemann surface with flat metric with conical singularities) from a ribbon graph equipped with lengths of all edges by the procedure of “conformal welding” [50].

The union of all strata $\mathcal{M}_{g,n}^{k,l}[p]$ with $k_j \geq 0, l_j \geq 1$ for fixed $p \in \mathbb{R}_+^n$ forms the combinatorial model $\mathcal{M}_{g,n}[p]$ of $\mathcal{M}_{g,n}$, and it is set-theoretically isomorphic to $\mathcal{M}_{g,n}$.

**Compactification of $\mathcal{M}_{g,n}[p]$ and $\overline{\mathcal{M}}_{g,n}$.** We discuss now a compactification of $\mathcal{M}_{g,n}[p]$ (see [41, 55] for discussion and further references) and its relationship to the Deligne–Mumford compactification $\overline{\mathcal{M}}_{g,n}$. Denote by $\Delta_{DM} = \mathcal{M}_{g,n} \setminus \mathcal{M}_{g,n}$ the Deligne–Mumford boundary. Let $C^{(0)} \in \Delta_{DM}$ be a stable curve and let $C^{(0)} = \bigcup_j C_j^{(0)}$ be its decomposition into irreducible components. The normalization $\tilde{C}_j^{(0)}$ of each of these components is an element of $\mathcal{M}_{g,n}$, for some $g_j, n_j$, where $\sum n_j = n + 2\#\text{nodes}$. In other words, the set of marked points on $C_j^{(0)}$ consists of the original $n$ marked points and also marked points arising from resolution of nodes of $C^{(0)}$.

Let now $C^{(t)} \in \mathcal{M}_{g,n}$ be a smooth family of smooth curves for $t \neq 0$. Consider the corresponding family in the combinatorial model $\mathcal{M}_{g,n}[p]$; in general [55] the corresponding Jenkins–Strebel quadratic differentials $Q_j^{(t)}$ in the limit $|t| \to 0$ acquires at most a simple pole at the nodes of $C^{(0)}$.

Therefore each component $C_j^{(0)}$ gets equipped with the Jenkins–Strebel quadratic differentials $Q_j^{(0)}$ with (generically) simple poles at resolutions of nodal points. This differential is identically zero on components where all marked points are the result of the resolution of nodes, see [55]. On the remaining components the differential $Q_j^{(0)}$ defines the canonical covering $\tilde{C}_j^{(0)}$ of $C_j^{(0)}$ by the equation $v_j^2 = Q_j^{(0)}$. The resolutions of nodal points of $C^{(0)}$ are (generically) branch points of $\tilde{C}_j^{(0)}$.

Following [55] we extend Def. 2.1 to nodal stable curves.

**Definition 2.2** Let $C^{(0)} \in \Delta_{DM}$ be a stable nodal $n$–marked curve. A quadratic differential $Q$ on $C^{(0)}$ is called a Jenkins–Strebel differential if it has double poles at the marked points, at most simple poles at the nodes in the normalization of $C^{(0)}$ and in the normalization of each component all homological coordinates (1.13) are real.

The reality of homological coordinates immediately implies the vanishing of $Q$ on components of $C^{(0)}$ where no second order poles of $Q$ left: on those components the Abelian differential $\sqrt{Q}$ is holomorphic with real periods and therefore identically vanishing.

On those components $C_j^{(0)}$ where $Q_j^{(0)}$ has second order poles all nodal points are one-valent vertices of the corresponding ribbon graph (the valency 1 arises in generic case; if the length of the edge arriving to the one-valent vertex equals 0 then this vertex merges with another vertex of higher valency; this corresponds to node placed at zero of $Q_j^{(0)}$). Therefore, we assume $k_j \geq -1$ in the definition of $\mathcal{M}_{g,n}^{k,l}[p]$.

The union of all strata $\mathcal{M}_{g,n}^{k,l}[p]$ gives the compactification $\overline{\mathcal{M}}_{g,n}[p]$ of the combinatorial model. The corresponding Jenkins–Strebel differentials are now allowed to have simple poles and hence ribbon graphs may have univalent vertices [55, 41, 45].

Notice that the boundary $\overline{\mathcal{M}}_{g,n}[p] \setminus \mathcal{M}_{g,n}[p]$ has real co-dimension 2 because the degeneration of a single edge in the main stratum (where all vertices are tri-valent) cannot lead to a degenerate surface.

Rephrasing Strebel’s result (Thm. 23.5 [50]) for each $n$–pointed possibly singular, stable curve $C \in \overline{\mathcal{M}}_{g,n}$ and $n$–tuple of positive numbers $p = (p_1, \ldots, p_n)$ there is a unique JS quadratic differential (according to Def. 2.2) $Q$ on $C$ with the quadratic residues at the $n$–marked points given by $-p_j^2/(4\pi^2)$. In turns, $Q$ defines a ribbon graph for each component of the normalization, thus giving an element of $W_{1,1}$. This defines the Jenkins-Strebel map

$$J_p : \overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_{g,n}[p].$$

(2.6)
As we discussed above, the map $J_p$ is one-to-one on $M_{g,n}$. On the other hand, the part of Deligne-Mumford boundary $\Delta_{DM}$ which contains a stable component without marked points is mapped to a lower-dimensional component of $\overline{M}_{g,n}[p] \setminus M_{g,n}[p]$ since any component without marked points is blown–down to a point by $J_p$.

We are going to use the notation
\[ W_{1,1} = \overline{M}_{g,n}[p] \setminus M_{g,n}[p] = J_p[\Delta_{DM}] \] (2.7)
for the boundary of $M_{g,n}[p]$; the map $J_p : \Delta_{DM} \to W_{1,1}$ is surjective but not injective.

The set $W_{1,1}$ (which is also called the "Kontsevich’s boundary" of $M_{g,n}[p]$) is a subcomplex of $M_{g,n}[p]$, and, moreover, is known to be a cycle according to Prop.2.6 (see also [45]). The notation $W_{1,1}$ is motivated by valencies of non-standard (i.e. non-trivalent) vertices in its largest cells: these graphs have exactly 2 one-valent vertices.

**Co-dimension 2 subcomplexes.** In total there are only two subcomplexes of $\overline{M}_{g,n}[p]$ in co-dimension 2: $W_5$ and $W_{1,1}$. The largest cells of these two cycles are obtained by the contraction of two edges having at least one vertex in common: if the two edges have only one vertex in common, their contraction leads to the largest cells of $W_5$ (ribbon graphs with one vertex of valence $\geq 5$ and all other vertices of valence $\geq 3$).

On the other hand, if the two edges connect the same pair of vertices then they necessarily form a loop that is homotopically non-trivial on the Riemann surface (if these two edges had formed a homotopically trivial loop, then they would form the boundary of a face of the graph, but since the perimeters are kept fixed they cannot tend to zero simultaneously). Therefore, the simultaneous degeneration of two such edges pinches the Riemann surface and gives a point (of a cell of highest dimension of) the cycle $W_{1,1}$. These two types of degenerations are shown in Fig.2.

### 2.1 Orientation of $\overline{M}_{g,n}[p]$ in homological coordinates and Witten–Kontsevich cycles

**Symplectic structure.** Consider a point $(C, Q)$ in the stratum $\mathcal{M}_{g,n}^{k,1}[p]$ with $k_j \geq -1$. If $l = 0$ i.e. all multiplicities are odd (in which case we simply omit the superscript 1 and denote the stratum by $\mathcal{M}_{g,n}^k[p]$) then the canonical cover $\tilde{C}$ is branched at all the zeroes and at the simple poles. The odd homology $H_- = H_-(\tilde{C} \setminus \{\tilde{z}_j, \tilde{z}'_j\}_{i=1}^n)$ is generated by $2g_- + n$ cycles and the rank of intersection form equals $2g_-$. 

![Figure 2: A typical embedded Strebel graph. The two highlighted regions are where the contraction of edges leads either to the Witten’s cycle $W_5$ (left region) or to the Kontsevich’s boundary $W_{1,1}$ (right region).](image)
Definition 2.3 (Homological Poisson Structure) Given two arbitrary cycles \( \gamma, \tilde{\gamma} \in H_-(\hat{C} \setminus \{ z_j, z_j' \}_{j=1}^n) \) the Poisson bracket on \( \mathcal{M}_{g,n}^k[p] \) between the corresponding homological coordinates is defined by the formula

\[
\left\{ \int_{\gamma} v, \int_{\tilde{\gamma}} v \right\} = \gamma \circ \tilde{\gamma}, \quad \forall \gamma, \tilde{\gamma} \in H_-.
\]

(2.8)

where \( \gamma \circ \tilde{\gamma} \) denotes the intersection pairing.

The complex homological Poisson structure (2.8) in the complex case \( n = 0 \) was introduced in [5].

Let us now describe a suitable basis in \( H_- \) (recall that \( \dim H_- = 2g_- + n \)). For each point \( z_j, j = 1, \ldots, n \) let \( c_j \) be a small loop around it on the base curve \( C \) and denote \( \gamma_j = \frac{1}{2}(c_j - c_j') \in H_- \) so that the associated homological coordinates are \( \pm p_j \). Choose also \( 2g_- \) cycles \( \{ a^-_\ell, b^-_\ell \}_{\ell=1}^{g_-} \) with intersection index \( a^-_\ell \circ b^-_k = 1 \) and all other intersection indices vanishing (the choice of these cycles is ambiguous: they are defined up to a linear combination of cycles \( \gamma_j \)). Let \( \{ A_{\ell} = \int_{a^-_{\ell}} v, B_{\ell} = \int_{b^-_{\ell}} v \}_{\ell=1}^{g_-} \) be the corresponding real homological coordinates. The intersection form in \( H_- \) thus defines a Poisson structure if the perimeters \( \{ p_j \}_{j=1}^n \) are constants.

Proposition 2.4 On each cell of \( \mathcal{M}_{g,n}^k[p] \) the Poisson structure (2.8) is non-degenerate and admits an inverse given by the symplectic form

\[
\Omega_{\text{sym}} := \sum_{j=1}^{g_-} dA_j \wedge dB_j.
\]

(2.9)

Proof. First, it is obvious that the form (2.9) induces the Poisson structure (2.8). The expression (2.9) is invariant under the choice of cycles \( \{ a_{\ell}, b_{\ell} \}_{\ell=1}^{g_-} \) in \( H_- \) having the intersection indices \( a_{\ell} \circ b_{k} = \delta_{\ell k}, a_{\ell} \circ a_{k} = b_{\ell} \circ b_{k} = 0 \) since any two such sets of cycles are related by a symplectic transformation up to an addition of a linear combination of the cycles \( \gamma_j \). Moreover the periods over cycles \( \gamma_j \) (which are equal to perimeters \( p_j \)'s) are constant on the space \( \mathcal{M}_{g,n}^k[p] \).

\[ \blacksquare \]

The space \( \mathcal{M}_{g,n}^k[p] \) is the union of several cells of (real) dimension \( 2g_- \). Each of these cells is a polytope in \( \mathbb{R}^{2g_-} \) with the symplectic form (2.9) and with the induced volume form and orientation. The common boundaries of these cells are strata of real co-dimension 1 (in fact, pieces of hyperplanes).
Figure 4: Canonical bases of cycles in $H_-(\hat{C}_1)$ and $H_-(\hat{C}_2)$ in two cells of $W$ which agree on their common boundary points. This gives a natural way to propagate the symplectic basis in $H_-$ into the neighbouring cell.

Remark 2.5 The form (2.9) can also be defined on each cell of a stratum $M_{g,n}^{k,1}[p]$ with some $l_j \neq 0$ (i.e. with some zeros having even multiplicity). In that case the covering $\hat{C}$ has node at the corresponding zero $x_j$ of $Q$ of even multiplicity. However, the dimension of such stratum is greater than the number of periods $(A_j, B_j)$ since the full set of homological coordinates in this case contains also integrals between zeros of $v$ related by the involution $\mu$ (these integrals are Casimirs of the homological Poisson structure on such strata. Thus the form (2.9) can not be used to induce an orientation on $M_{g,n}^{k,1}[p]$ for $l \neq 0$.

Consider now the sub-complexes $W_r$ of $\overline{M}_{g,n}[p]$ whose cells of maximal dimension coincide with $M_{g,n}^{k,1}[p]$ (with $r_j = 2k_j + 1$ for $k_j \neq 1$ or $r_j = 2l_j$). By convention, the trivalent vertices are not listed in the valence vector $r$.

Proposition 2.6 The sub-complexes $W_r$ with all odd $r_j$'s are orientable cycles with the orientation induced by the $g$-th power of symplectic form (2.9).

Proof of Prop. 2.6. Using the definition 2.7 describing the propagation of the canonical basis on $\hat{C}$ under the Whitehead move we are going to verify that the orientation induced on each cell by the homological symplectic structure (2.9) defines a consistent orientation on $W_r$. This consistency is a corollary of two facts:

1. each boundary of co-dimension 1 is shared by an even number of cells (generically 2).
2. The orientation on each component of the co-dimension 1 boundary induced from a half of the cells is the opposite to the orientation induced from the other half.

The co-dimension 1 part of the boundary of a maximal cell of $W_r$ corresponds to one contraction of any of the edges of the corresponding ribbon graph. We start from analysis of the simplest case consisting of the degeneration of an edge between two tri-valent vertices and then analyze the general case.

Coalescence of two tri-valent vertices. Consider first the case where the vanishing edge connects two vertices of valence 3 each. We choose canonical symplectic bases in $H_-(\hat{C}_1)$ and $H_-(\hat{C}_2)$ as explained in Def. 2.7. Then the two
cells can be mapped locally to neighbourhoods in the half–spaces in \((A_1, B_1, \ldots, A_g, B_g) \in \mathbb{R}^{2g−}\) separated by the hyperplane \(A_1 = 0\) (on this hyperplane the cycle \(a_1^−\) is a vanishing cycle on \(\hat{C}_0\)). The orientation extends through the boundary if \(A_1\) has opposite signs in the two cells.

According to Fig. 4 the orientation of \(a_1^−\) before and after the Whitehead move is uniquely determined by the orientation of \(b_1^−\). The sign of \(A_1\) can be deduced from the following local considerations. In a distinguished parameter \(\zeta\) around the zero (denoted by \(x_1\) on both \(C_1\) and \(C_2\)) the differential \(v\) reads \(v = \frac{3}{2}\sqrt{\zeta}d\zeta\) and the local flat coordinate near \(x_1\) on \(\hat{C}_{1,2}\) is \(z = \zeta^{\frac{3}{2}}\); we choose the branch-cuts (shown in red) on both \(C_1\) and \(C_2\) along the positive \(\zeta\) axis.

The period \(A_1\) equals to \(2z(x_2)\) in both cases. To determine the sign of \(A_1\) in each case we notice that the flat coordinate \(z\) maps the upper rim of the cut to the positive \(z\)–axis and the lower rim to the negative \(z\)–axis. The remaining two horizontal trajectories (black) moving counterclockwise from the red trajectory, are mapped to the negative and positive \(z\)–axis, respectively.

Therefore, \(A_1\) is positive in the case of \(\hat{C}_1\) (left pane of Fig. 4) (\(x_2\) in that case lies on the trajectory corresponding to positive values of \(z\)). For analogous reasons, \(A_1\) is negative in the case of \(\hat{C}_2\) (right pane). Therefore, indeed, the two sides of the Whitehead move are locally described by \(A_1 > 0\) or \(A_1 < 0\) and we conclude that the orientation propagates consistently from cell to cell.

**General case.** Suppose now that the contracting edge connects two vertices of valences \(2k + 1, 2r + 1\), i.e. two zeroes of \(Q\) of odd multiplicities \(2k−1, 2r−1\), respectively. Then the merge produces a zero of even multiplicity \(2k + 2r − 2\) i.e. a vertex of valence \(2(k + r)\). This vertex can be resolved into two vertices of valences \(2k + 1, 2r + 1\) in \(2(k + r)\) ways. As in the tri-valent case, we choose the symplectic bases in \(H^−\) so that the vanishing cycle \(a_1^−\) goes around the contracting edge and the dual cycle goes around one of the adjacent edges (see Fig. 5).

In each resolution, the \(b_1^−\) cycle goes around one of the \(2(k + r)\) edges that are not contracted; as before we need to verify that the homological coordinate \(A_1\) is positive in half of the cases and negative in the other half.

We use again the flat coordinate centered at the vertex \(x_1\) incident to the \(b_1^−\) edge (which has valence either \(2k + 1\) or \(2r + 1\)). An elementary local consideration along the same lines as in the previous case (paying attention to the arguments of the distinguished and flat coordinates) shows that the sign of the product \(A_1B_1\) equals \((-1)^\ell\) where \(\ell\) is the number of edges between the \(b_1^−\) edge and the contracting edge, counterclockwise.
Since the sign of $B_1$ is the same in every resolution, there are half cases with positive and half cases with the negative sign of $A_1$ and the theorem is proved. ■

The above proposition justifies the following standard definition

**Definition 2.8 (Witten–Kontsevich cycles)** The sub-complexes $W_r$ with all $r_j$’s odd integers are called Witten–Kontsevich cycles.

If some of the $r_j$ is an even integer, the corresponding subcomplex is not orientable and therefore it is not a cycle [44].

### 2.1.1 Period coordinates as canonical coordinates for Kontsevich’s form $\sum p_f^2 \psi_f$

In [39] Kontsevich introduced the two form $\Omega = \sum_{f \in F(\Gamma)} p_f^2 \omega_f$ on each cell of $M_{g,n}[p]$ where $F(\Gamma)$ is the set of faces of $\Gamma$; $\omega_f$ is the two–form representing the Chern class $\psi_f$ of the tautological line bundle at the marked point corresponding to the face $f$ of the ribbon graph [39, 55].

In terms of lengths of the edges the form $\Omega$ can be written as follows:

$$\Omega = \sum_{f \in F(\Gamma)} \eta_f$$  \hspace{1cm} (2.10)

where

$$\eta_f = p_f^2 \omega_f = \sum_{e_j, e_k, \in E_f, i \leq j < k \leq n} d\ell_j \wedge d\ell_k,$$  \hspace{1cm} (2.11)

$e_1, \ldots, e_{n_f}$ are the edges bounding face $f$ ordered counterclockwise (the form is independent of the choice of the “first” edge on each face), and $\ell_j$ denotes the length of the edge $e_j$.

Consider now the following Poisson bivector on each cell of $M_{g,n}[p]$:

$$\mathcal{P} = \frac{1}{4} \sum_{x \in V(\Gamma)} \sum_{e_j, e_k, \in E_x, i \leq j < k \leq n_x} (-1)^{k-j-1} \frac{\partial}{\partial \ell_j} \wedge \frac{\partial}{\partial \ell_k}$$  \hspace{1cm} (2.12)

where $V(\Gamma)$ is the set of vertices of $\Gamma$ and $e_1, \ldots, e_{n_x}$ are edges incident to $x$ taken in counterclockwise order (the bivector $\mathcal{P}$ also does not depend on the choice of the “initial” edge).

In [39] it was shown that the two-form $\Omega$ is symplectic if all vertices of $\Gamma$ have odd valency.

**Theorem 2.9** [39] The form $\Omega$ (2.10) is symplectic on each top dimensional cell of any Witten–Kontsevich cycle $W_r$.

The proof of the next proposition (which in fact coincides with Lemma C.2 of [39] that was stated without proof there) is essentially contained in the proof of Lemma 5.4 of [45]. It shows that the bivector $\mathcal{P}$ (2.12) is the right inverse to the symplectic form $\Omega$ (2.10) on each cell of top dimension of any Witten–Kontsevich cycle $W_r$ (then it is also the left inverse since $\Omega$ is non-degenerate on those cells [39, 45]).

**Lemma 2.10** Let all vertices of $\Gamma$ have odd valence. Then for each edge $e$ of length $\ell$ the following relation holds:

$$\Omega[\mathcal{P}(d\ell)] = d\ell$$  \hspace{1cm} (2.13)

**Proof.** Let the edge $e$ connect vertices $x$ and $x'$ of valences $2k+1$ and $2k'+1$. Denote the remaining $2k$ edges emanating from $x$ by $e_1, \ldots, e_{2k}$ (enumerated counterclockwise starting from $e$). Similarly, the remaining $2k'$ edges emanating from $x'$ are denoted by $e_{1'}, \ldots, e_{2k'}$ (enumerated counterclockwise). Then

$$\mathcal{P}(d\ell) = \frac{1}{4} \left( \sum_{j=1}^{2k} (-1)^{j-1} \partial \ell_j + \sum_{j=1}^{2k'} (-1)^{j'-1} \partial \ell_{j'} \right)$$  \hspace{1cm} (2.14)
Figure 6:

The formula (2.14) implies that the bivector $\mathcal{P}$, up to factor $1/4$, coincides with the bivector $\beta$ from Sect.C of [39]. Let us label the faces involved as shown in Fig.6.

It is easy to verify that for any two consecutive edges $e$ and $e'$ (such that $e$ precedes $e'$ going counterclockwise) adjacent to the face $f$ we have

$$\eta_f \left[ \frac{\partial}{\partial \ell} - \frac{\partial}{\partial \ell'} \right] = d\ell + d\ell'.$$

(2.15)

We apply $\Omega$ in (2.10) to (2.14); using (2.15), the separate summands $\eta_f$ in $\Omega$ yield

$$\eta_f (\mathcal{P}(d\ell)) = \frac{1}{4} (d\ell_1 + 2d\ell + d\ell_{2k'}), \quad \eta_{f'} (\mathcal{P}(d\ell)) = \frac{1}{4} (d\ell_{2k} + 2d\ell + d\ell_{1'}) ,$$

(2.16)

$$\eta_f (\mathcal{P}(d\ell)) = \frac{(-1)^j}{4} (d\ell_{j+1} + d\ell_j), \quad \eta_{f'} (\mathcal{P}(d\ell)) = \frac{(-1)^{j'}}{4} (d\ell_{j'+1} + d\ell_{j'}) .$$

(2.17)

Summation of all terms in (2.16) and (2.17) gives $d\ell$.

The next theorem shows that the form $\Omega$ coincides with the homological symplectic form $\Omega_{\text{hom}}$ (2.9) when all vertices of $\Gamma$ have odd valences.

**Theorem 2.11** The Kontsevich’s two form $\Omega$ (2.10) coincides with the homological symplectic form $\Omega_{\text{hom}}$ (2.9) on each cell of maximal dimension of the Witten–Kontsevich cycle $W_r$ i.e.

$$\sum_{f \in P(\Gamma)} p_f^2 \omega_f = \sum_{j=1}^{g-1} dA_j \wedge dB_j$$

(2.18)

**Proof.** Denote by $\Gamma$ the ribbon graph of a cell of maximal dimension in $W_r$. We start from showing that the bi-vector representing the Poisson bracket (2.8) is given by (2.12) i.e. that

$$\{ \ell_e, \ell_{e'} \} = \mathcal{P}(d\ell_e, d\ell_{e'})$$

(2.19)

for any two edges $e, e'$ (where the Poisson bracket is defined in (2.8)).

We remind that for an edge $e$ the length is $\ell_e = \frac{1}{2} \oint_{\gamma_e} v > 0$ where $\gamma_e$ is the cycle in $H_+$ consisting of the edge $e$ on one sheet of $\hat{C}$ and the same edge on the other sheet in the opposite direction (with the overall orientation so that $\oint_{\gamma_e} v$ is positive). If two edges $e, e'$ have no common vertex, then clearly the intersection number $\gamma_e \circ \gamma_{e'} = 0$ and also $\mathcal{P}(d\ell_e, d\ell_{e'}) = 0.$
If $e, e'$ have one vertex $x_0$ of valence $2k+1$ in common then $P(d\ell_e, d\ell_{e'}) = \frac{1}{2} c(e, e')$ where $c(e, e') = \pm 1$ is the parity of the number of edges incident to $x_0$ between $e$ and $e'$ in counterclockwise direction (note that $c(e, e') = -c(e', e)$ since the valence of $x_0$ is odd). To show that (2.19) holds in this case we need to show that $\gamma_e \circ \gamma_{e'} = c(e, e')$ (since $\{\gamma_{e}, \gamma_{e'}\} = \gamma_{e'} \circ \gamma_{e}$ for any two cycles). Clearly $\gamma_e \circ \gamma_{e'} = \pm 1$. To decide on the sign let $\zeta$ be the distinguished coordinate at $x$ so that $v = \frac{2}{2k+1} \zeta \frac{2k+1}{2} d\zeta$. The flat coordinate at $x$ is $z = \zeta - \frac{2k+1}{2j}$ and the edges are arcs of the rays $\arg(\zeta) = j \frac{2k+1}{2}, j = 0, \ldots, 2k$. Placing the branch-cut along the $j = 0$-ray an elementary analysis shows that the orientations of the pair $(\gamma_e, \gamma'_{e'})$ that makes both integrals $\int_{\gamma_e} v$ and $\int_{\gamma_{e'}} v$ positive yields an intersection number equal to $c(e, e')$; the remaining factor of $4$ comes from the fact that $\ell_e = \frac{1}{2} \int_{\gamma_e} v$. This proves (2.19).

To prove the statement of the theorem it remains to use the relation between $P$ and $\Omega$ given by Lemma 2.10 since, as a corollary of (2.19), $P = \sum_{j=1}^{g} \frac{\partial}{\partial A_j} \wedge \frac{\partial}{\partial B_j}$ in terms of homological coordinates.

2.2 Pentagon moves and combinatorial Dehn twists

The largest stratum of the cycle $W_5$ corresponds to JS differentials with one zero of multiplicity 3 and all remaining zeros of multiplicity 1. Equivalently, the corresponding ribbon graphs have one vertex of valence 5 while all other vertices are tri-valent.

The largest stratum of the cycle $W_{1,1}$ corresponds to differentials with two simple poles and simple zeros on nodal curves. The corresponding ribbon graphs are either connected graphs with two uni-valent vertices (nodal irreducible curve) or union of two connected graphs each with one uni-valent vertex (two irreducible components). Therefore the cycle $W_{1,1}$ is the sum of a cycle formed by irreducible components $W_{irr}$ and several other cycles which correspond to degenerations where (in the largest stratum) the genus of one of the component is $g - j$ and the genus of the other is $j (0 \leq j \leq \lfloor \frac{g}{2} \rfloor)$, while the $n$ marked points are distributed between the components in all possible ways (with at least one marked point in each component).

2.2.1 Cycle $W_5$ and pentagon moves

For each point of Witten’s cycle $W_5$ the triple zero of $Q$ (i.e. the 5-valent vertex of the ribbon graph) can be split into 3 simple zeros (i.e. three-valent vertices) in 5 different ways shown in Figure 7. Therefore the star of the given cell $K$ of $W_5$ consists of five cells $K^{(0)}, \ldots, K^{(4)}$, cyclically ordered so that $K^{(j)}$ shares facets with $K^{(j+1)}$ and $K^{(j-1)}$ (indices taken modulo 5).

A loop in a transversal cross-section around a point of the cell $K$ consists of the union of five paths (one path in each $K^{(j)}$) that extend between different facets and form a continuous curve in $\hat{W}$ (recall that $\hat{W}$ denotes the union of cells of $W$ and their facets). The ordering of the cells $K^{(j)}$’s determines an orientation of the loop.

The positive orientation is, in turn, determined by the symplectic form (2.9) as we now explain, referring to Fig. 7. In each $K^{(j)}$ the homological coordinates are uniquely defined up to symplectic transformations. Let $A^{(j)}, B^{(j)}$ be the homological coordinates associated to the cycles $a^{(j)}_1, b^{(j)}_1$ in $\mathbb{R}^2_+$ and chosen so that they are both positive. Then the orientation in the cone $(A^{(j)}, B^{(j)}) \in \mathbb{R}^2_+$ is given by $dA^{(j)} \wedge dB^{(j)}$; the facets of $K^{(j)}$ correspond to the coordinate axes and the facet $A^{(j)} = 0$ (“vertical axis”) is the facet that follows the facet $B^{(j)} = 0$ (“horizontal axis”) in the positive orientation. This allows to uniquely determine the order of the $K^{(j)}$’s by propagating the homology basis as explained in Def. 2.7. Therefore, we get the following

**Proposition 2.12** The pentagon move, i.e. the sequence of five Whitehead moved depicted in Fig. 7 represents a path in $\hat{W}$ which goes around Witten’s cycle $W_5$ in the positive direction.
Figure 7: Resolution of a five-valent vertex and transformation of the pair of canonical cycles $a_1^-$ and $b_1^-$ on $\hat{C}$ under pentagon move

2.2.2 Kontsevich's boundary $W_{1,1}$ and combinatorial Dehn's twist

The Riemann surface $C$ corresponding to a point of the largest stratum of $W_{1,1}$ possesses a single nodal point $q_0$. The JS differential on such a curve has simple poles at the two points of the resolution of $q_0$ in the normalization of $C$ and $4g - 6 + 2n$ simple zeros.

If the degeneration of the two edges as shown in Fig. 2 (the region highlighted on the right) does not separate the Riemann surface (i.e. the shrinking loop is homologically non-trivial) then the corresponding ribbon graph is connected and has $4g - 6 + 2n$ tri-valent and 2 one-valent vertices. Such ribbon graphs correspond to the irreducible component $W^{irr}_{1,1}$ of $W_{1,1}$.

If the degeneration is separating, i.e., the loop is homologically trivial, and both components contain at least one double pole of $Q$, then the corresponding ribbon graph consists of two connected components $\Gamma_1$ and $\Gamma_2$ of genera $g_1$ and $g_2$ (with $g_1, g_2 \geq 0$ and $g = g_1 + g_2$). The numbers of faces of the ribbon graphs $\Gamma_{1,2}$ equal $n_{1,2}$ ($n_{1,2} \geq 1$). Such ribbon graphs correspond to the reducible component $W^r_{1,1}$ of $W_{1,1}$.

We now discuss a neighbourhood of $W_{1,1}$ in $\tilde{W}$. Starting from a point of $W_{1,1}$ (we denote the corresponding Riemann surface and quadratic differential by $C_0$ and $Q_0$) one replaces the two one-valent vertices of $\Gamma_0$ by two trivalent vertices by inserting two small edges of lengths $\alpha = tA$ and $\beta = tB$ with $A + B = 1$ connecting these new vertices, as shown in Fig. 8 (see [6] for details; the parameter $t \in R_+$ has the meaning of the length of the “vanishing cycle” going along horizontal trajectories which connect two arising simple zeros).

The resolution is done by inserting the annulus domain instead of two small circles around the simple poles of $Q_0$ via procedure of “conformal welding”, see Fig.8. In contrast to the resolution of a triple zero of $Q$ discussed above,
Figure 8: Resolution of a double point $x_{1,2}^0$ on $C$ by insertion of an annulus with two simple zeros $x_1$ and $x_2$

Figure 9: Resolution of a point of $W_{1,1}$. These two resolutions are related by a Whitehead move.

now there is only one cell of $W$ in a neighbourhood of a given point of $W_{1,1}$. The resolution can be done in countably many ways related by Whitehead moves; two of these possible resolutions are shown in Fig. 9.

The transversal coordinates to $W_{1,1}$ are the homological coordinates $(B, \tilde{B})$ along the cycles $b^{-}_1, \tilde{b}^{-}_1$ indicated in Figures 11, 12; these cycles correspond to the two vanishing edges.

Let us denote by $K$ the cell in $W$ obtained as a result of this resolution. This cell is “wrapped onto itself” in a neighbourhood of $W_{1,1}$ in the sense that the points on the facets of $K$ that correspond to the limits $B \to 0$ and $\tilde{B} \to 0$ are identified. However, the path between points $(0,1)$ and $(1,0)$ in the $(B, \tilde{B})$-plane for fixed $t$ is topologically non-trivial in $\tilde{W}$.

Let us explain how the orientation induced by the symplectic form $\Omega_{\text{Hom}}$ (2.9), together with the propagation of bases in Def. 2.7, translates to the $(B, \tilde{B})$ plane.

The intersection numbers of $\tilde{b}^{-}_1$ with the elements of the basis $\{a^{-}_1, b^{-}_1, \ldots\}$ are $\tilde{b}^{-}_1 \circ a^{-}_1 = -1, \tilde{b}^{-}_1 \circ b^{-}_1 = 2$ and all others are zero. In $H^-$ the intersection pairing has rank $2g^-$ and co-rank $n$, with the kernel being spanned by $\{\gamma_j\}_{j=1}^n$. Therefore, the intersection is well-defined only as a pairing in the quotient space $H^- \mod \{\gamma_j\}_{j=1}^n$. The computation of intersection numbers shows that

$$\tilde{b}^{-}_1 = b^{-}_1 - 2a^{-}_1 + \gamma$$

for some $\gamma \in \mathbb{Z}\{\gamma_j\}_{j=1}^n$.

**Lemma 2.13** The homological coordinates $(B, \tilde{B})$ are transversal coordinates to a component of $W_{1,1}$ within the adjacent cell of $W$ and
• the orientation induced by the symplectic structure $\Omega_{\text{hom}}$ (2.9) in the $(B, \bar{B})$-plane is given by $dB \wedge dB$.

• the homological coordinates $B, \bar{B}$ have opposite signs.

Proof. From the relation (2.20) it follows that the orientation induced by $\Omega_{\text{hom}}$ (2.9) in the transversal $(B, \bar{B})$-plane is $dB \wedge dB = 2dA \wedge dB$. The second item follows again from a local consideration (similar to the ones in the proof of Prop. 2.6) of the sign of $v = \sqrt{Q}$ near the zero $x_1$ that lies within all three $a_1^-, b_1^-, \bar{b}_1^-$ see Figures 11, 12. In the distinguished coordinate $\zeta$ at $x_1$ we have $\int_{x_1}^v \zeta = \zeta^2$; we choose the determination so that the cut is mapped to the positive $\zeta$-axis. Then $A_1 = \oint_{a_1^-} v$ is positive, $B > 0$ and $\bar{B} < 0$.

It is more convenient to use positive transversal coordinates; this can be done by defining $B' = -B/2$ and $A' = \bar{B}$. Then the orientation in the $(A', B')$-plane is $dA' \wedge dB'$ and we have:

**Corollary 2.14** The coordinates $B' = -\frac{1}{2} \oint_{a_1^-} v$ and $A' = \oint_{\bar{b}_1^-} v$ are transversal local coordinates in a neighborhood of $W_{1,1}$ on $\tilde{W}$. The local model of the transversal manifold is a cone $(\mathbb{R}^2_{\geq 0} \setminus \{(0,0)\})/\sim$ where the equivalence relation $\sim$ is the identification of the two axes $\{0\} \times \mathbb{R}_+$ and $\mathbb{R}_+ \times \{0\}$.

The restriction of $\Omega_{\text{hom}}$ (2.9) to the cone is given by $dA' \wedge dB'$, which defines the induced orientation.

Recall that the Dehn twist along a homotopically nontrivial loop $\ell$ of a Riemann surface corresponds to a closed path in $\mathcal{M}_{g,n}$ around a component of the Deligne–Mumford boundary $\Delta_{DM}$. Now Cor. 2.14 implies the following proposition.

**Proposition 2.15** Fix a cell of $W$ corresponding to a ribbon graph $\Gamma$ such that $x_1, x_2$ is a pair of vertices connected by two edges $e, \bar{e}$ which are homotopically distinct on $C$. Consider the associated coordinates $(A', B')$ defined in Cor. 2.14. Then the path in $\tilde{W}$ starting from an interior point $A' > 0 < B'$ to the “wall” $A' = 0$ followed by a path from the corresponding point on the other wall $B' = 0$ back to the original point represents the Dehn’s twist along the path formed by the edges $e, \bar{e}$. The orientation of this Dehn’s twist induced by the symplectic form (2.9) is positive.

On the transversal cone $(\mathbb{R}^2_{\geq 0} \setminus \{(0,0)\})/\sim$ (see Cor. 2.14), the Dehn twist is represented by a simple loop around the vertex of the cone. While traversing such path, the ribbon graph undergoes a single Whitehead move.

The next lemma describes the class of Dehn’s twists which can be obtained via this construction.

**Lemma 2.16** Let $\gamma$ be a loop on $C$ which is either non-separating or separates $C$ into 2 stable components each of which contains at least one marked point. Let, moreover, the curve $C$ be sufficiently close to the corresponding component $\Delta_{DM}$ of the Deligne–Mumford boundary. Then the Dehn’s twist based at the curve $C$ in $\mathcal{M}_{g,n}$ along the loop $\gamma$ can be realized as a path in $\tilde{W}$ defined in Proposition 2.15.

Proof. Let us start from an arbitrary curve $C_1$ with the contour $\gamma$ satisfying the condition of the lemma. The free homotopy class of the contour $\gamma$ identifies a component $\Delta_{\gamma,DM}$ of the Deligne–Mumford boundary $\Delta_{DM}$; the component $\Delta_{\gamma,DM}$ contains the stable nodal curves where $\gamma$ is shrunk to a point. Let $(C_t; \gamma_t)$ be a smooth family parametrized by $t \in [0,1]$ so that $C_0 \in \Delta_{DM}$ while $\gamma_0$ is collapsed to the node and $\gamma_1 = \gamma$. If $C_0$ is chosen generically in $\Delta_{DM}$ then the Jenkins–Strebel differential on the normalization of $C_0$ has two simple poles at the points obtained by resolving the node; the ribbon graph on the normalization of $C_0$ then has two uni-valent vertices as discussed in Section 2.2.2.

Thus for a sufficiently small $t$ the curve $C_t$ falls within a neighborhood which is parametrized in terms of the “plumbing” construction shown in Fig. 8. Then the Dehn’s twist along the loop $e \cup \bar{e}$ described in Proposition 2.15 (see Fig. 9) is homotopic to the Dehn’s twist along the loop $\gamma = \gamma_t$.■
2.3 Combinatorial Dehn’s twists in Chekhov-Fock modular groupoid

The fundamental group of the moduli space \( \pi_1(\mathcal{M}_{g,n}, C_0) \) i.e. the mapping class group \( MPG_{g,n} \) is known to be generated by a finite set of Dehn’s twists; an example of \( 2g + n \) Dehn’s twists \( \{D_i\}_{i=1}^{2g+n} \) generating the full \( MPG_{g,n} \) is shown in Fig. 4.10 of [14], see Fig. 10. Notice that all Dehn’s twists \( \{D_i\}_{i=1}^{2g+n} \) act along closed non-separating loops on the Riemann surface \( \mathcal{C} \) with \( n \) puncture. The Dehn’s twists represent paths around the components of the Deligne-Mumford boundary \( \Delta_{DM} \) obtained by pinching the loops. Any other Dehn twist (including those along separating loops) can therefore be represented as product of \( \{D_i\}_{i=1}^{2g+n} \).

This classical picture is naturally translated to the context of the combinatorial model \( \mathcal{M}_{g,n}[p] \). We start from the following

**Proposition 2.17** Let \( C_0 \) be mapped to a cell of maximal dimension by the Jenkins-Strebel map (2.6) \( \mathcal{J}_p : \mathcal{M}_{g,n} \to \mathcal{M}_{g,n}[p] \).
Then \( \mathcal{J}_p \) induces an isomorphism of the following fundamental groups:

\[
\pi_1(\mathcal{M}_{g,n}, C_0) \simeq \pi_1(\mathcal{M}_{g,n}[p], \mathcal{J}_p(C_0)) \simeq \pi_1(\tilde{W} \cup W_0^0 \cup W_4^0, \mathcal{J}_p(C_0)) .
\]

(2.21)

**Proof.** Since the Jenkins–Strebel map \( \mathcal{J}_p \) (2.6) from \( \mathcal{M}_{g,n} \) to \( \mathcal{M}_{g,n}[p] \) is a homeomorphism, the fundamental group \( \pi_1(\mathcal{M}_{g,n}, C_0) \) is isomorphic to \( \pi_1(\mathcal{M}_{g,n}[p], \mathcal{J}_p(C_0)) \). To prove the second isomorphism in (2.21) we observe that the fundamental group of the complex \( \mathcal{M}_{g,n}[p] \) is the same as the one of the sub-complex of all cells of codimension up to 2. The union of cells of highest dimension forms the stratum \( W \); the cells of codimension 1 form the stratum \( \mathcal{W}_4^0 \) whose cells correspond to ribbon graphs with exactly one vertex of valence 4 and all other vertices of valence 3; recall that we denoted \( \tilde{W} = W \cup \mathcal{W}_4^0 \). Finally, there are two strata of codimension 2: \( \mathcal{W}_5^0 \) (the stratum of highest dimension of the Witten’s cycle \( W_5 \)) and \( \mathcal{W}_{4,4}^0 \); the cells of \( \mathcal{W}_{4,4}^0 \) correspond to ribbon graphs which have two vertices of valence 4 and while other vertices have valence 3.

Now we define a graph \( G \) associated to the sub-complex \( \tilde{W} \), which naturally encodes its fundamental group.

Denote by \( K_0 \) the top-dimensional cell containing \( \mathcal{J}_p(C_0) \). We can represent an element of \( \pi_1(\mathcal{M}_{g,n}[p], \mathcal{J}_p(C_0)) \) by a loop based at \( \mathcal{J}_p(C_0) \) that remains within \( \tilde{W} \). Such a path traverses several cells of \( W \) crossing the facets between them. Transitions between cells of \( W \) correspond to the Whitehead moves on the corresponding edges.

Consider now the graph \( G \) whose vertices \( \{K_i\} \) are represented by cells of \( W \). Two vertices \( K_1 \) and \( K_2 \) are connected by an edge if they have a common facet i.e. if one can go from \( K_1 \) to \( K_2 \) via the Whitehead move. The edge then corresponds to a cell of \( \mathcal{W}_5^0 \). Since each cell in \( \tilde{W} \) is homeomorphic to a ball, the fundamental group \( \pi_1(\tilde{W}, \mathcal{J}_p(C_0)) \) is then isomorphic to \( \pi_1(G, K_0) \).

A path on the graph \( G \) can be thought of as a sequence of several Whitehead moves; the set of all Whitehead moves carries the name of modular groupoid introduced by Chekhov and Fock [8]. Two Whitehead moves \( M_1 \) and \( M_2 \) can be multiplied if the end-cell of the edge of \( G \) representing \( M_1 \) coincides with the initial cell of the edge representing \( M_2 \).

We now define a combinatorial complex \( \hat{G} \) obtained from \( G \) by adding “faces” to it; the faces of \( \hat{G} \) are defined to be in one-to-one correspondence with cells of co-dimension 2 in \( \mathcal{M}_{g,n}[p] \). For a face \( F \) of \( \hat{G} \) (corresponding to a cell
denoted by the same letter) the boundary is defined to be the union of vertices and edges of $G$ corresponding to cells of co-dimensions zero and one in the star of $F$ (the star of a cell $F$ is the union of all cells whose closure intersects $F$).

Since all cells of co-dimension 2 belong either to $W_0^0$ or $W_4.4$, there exist two types of faces in $\hat{G}$: quadrilaterals (for cells of $W_4.4^0$) and pentagons (for cells of $W_5^0$).

Within $\hat{G}$ there exist non-trivial homotopy deformations defined by the condition that all boundaries of faces are homotopically trivial (i.e. paths around $W_0^0$ and $W_4.4^0$ become trivial in $\pi_1(\hat{G})$).

Thus we have the following

**Proposition 2.18** The following fundamental groups are isomorphic

$$\pi_1(\hat{W}, \mathcal{F}_p(C_0)) \simeq \pi_1(G, K_0) \quad (2.22)$$

and

$$\pi_1(M_{g,n}, C_0) \simeq \pi_1(\hat{G}, K_0) \quad (2.23)$$

where $K_0$ is the cell of $W$ such that $\mathcal{F}_p(C_0) \in K_0$.

**Proof.** The isomorphism (2.22) is an immediate corollary of the definition of graph $G$ and the fact that all cells of $W$ and $W_4$ are homeomorphic to balls of corresponding dimensions.

To prove (2.23) we notice that the Prop. 2.17 establishes the isomorphism between $\pi_1(M_{g,n}, C_0)$ and $\pi_1(\hat{W} \cup W_0^0 \cup W_4.4^0, \mathcal{F}_p(C_0))$. The isomorphism between the latter fundamental group and $\pi_1(\hat{G}, K_0)$ is constructed as follows. For each class $[\gamma] \in \pi_1(\hat{W} \cup W_0^0 \cup W_4.4^0, \mathcal{F}_p(C_0))$ choose a representative $\gamma$ whose image $\mathcal{F}_p(\gamma)$ only traverses cells of $\hat{W}$ (avoiding strata of codimension greater than 1). This defines a loop in $G$ (and $\hat{G}$) along vertices and edges. Its representative in $\pi_1(\hat{G}, K_0)$ (with the homotopy induced by quadrilateral and pentagon moves) gives the image of the class $[\gamma]$. This image is independent of the choice of the loop within the class $[\gamma]$ since all cells in the combinatorial model are homeomorphic to balls. Conversely, let us fix a representative curve $C_j$ in each cell $K_j$ of $\hat{W}$. For the class $[\ell] \in \pi_1(\hat{G}, K_0)$ we select a representative that stays on edges of the graph $G$ and map it to the loop in $\hat{W}$ obtained by connecting the representatives $C_j$ of the cell traversed by $\ell$. Images of different representatives of the same class $[\ell]$ then also lie in the same equivalence class of loops in $\hat{W} \cup W_0^0 \cup W_4.4^0$, and therefore, represent the same element of $\pi_1(\hat{W} \cup W_0^0 \cup W_4.4^0, \mathcal{F}_p(C_0))$.

The fundamental group of $\hat{G}$ can be generated by the combinatorial Dehn’s twists in the same way as $\pi_1(M_{g,n})$ is generated by the ordinary Dehn’s twists.

**Definition 2.19** For each component $W_{1,1}^\gamma$ of $W_{1,1}$ (corresponding to the collapse of a non-separating loop $\gamma$) denote by $K_\gamma$ the adjacent cell of $W$. The (local) combinatorial Dehn twist $D_{\gamma,comb}^\gamma$ is defined to be the path in $G$ starting and ending at the vertex $K_\gamma$. This path follows the edge of $G$ which represents the facet of $K_\gamma$ obtained by the collapse of one of two edges of the corresponding ribbon graph. (see Cor. 2.14).
Figure 12: The same region as in Fig. 11 can be mapped to an annulus with the metric induced by the quadratic differential $Q$. The quadratic differential on this annulus is given by the restriction of a JS quadratic differential on $\mathbb{CP}^1$ with two poles of order 3 at $z = 0, z = \infty$ (more details are given in Section 4 of [6]).

Such local combinatorial Dehn’s twists can not be multiplied in the based fundamental group of $G$. To define the Dehn’s twist around $W_{1,1}$ based at any vertex $K_0$ of $G$ we choose a path $\ell_\gamma$ on $G$ which connects $K_0$ to the cell $K_\gamma$ and define

$$D_{\gamma}^{\text{comb}}[K_0] = \ell_\gamma D_{\gamma}^{\text{comb}} \ell_\gamma^{-1}. \tag{2.24}$$

The based combinatorial Dehn’s twist defined in this way, being considered as an element of $\pi_1(G, K_0)$ depends of course on the choice of the path $\ell_\gamma$. However, the equivalence class of $D_{\gamma}^{\text{comb}}[K_0]$ in $\pi_1(\hat{G}, K_0)$ does not depend on the choice of $\ell_\gamma$.

Combining Prop. 2.18 with the fact that $MPG_{g,n}$ is generated by Dehn twists along nonseparating loops $\gamma_1, \ldots, \gamma_{2g+n}$ shown in Fig.10 (Fig. 4.10 of [14]) we have proved

**Proposition 2.20** The fundamental group $\pi_1(\hat{G}, K_0)$ is generated by (equivalence classes of) combinatorial Dehn’s twists $D_{\gamma}^{\text{comb}}[K_0]$, $j = 1, \ldots, 2g+n$ given by (2.24).

This proposition implies in particular that the based Dehn’s twists around any part of the DM boundary can be expressed as products of based Dehn twists around only those components of $W_{1,1}$ which are isomorphic to the corresponding components of $\Delta_{DM}$.

In our framework, Theorem 1 of [8] can then be reformulated as follows:

**Proposition 2.21** The fundamental group $\pi_1(G, K_0) \simeq \pi_1(\hat{W}, J_p(C_0))$ is generated by combinatorial Dehn’s twists and paths around all pentagon and quadrilateral faces of $G$ which start and end at the same vertex $K_0$ of $G.$
3 Meromorphic quadratic differential and canonical covering

3.1 Canonical covering

Here we give more details about the geometry of the canonical covering $v^2 = Q$ (1.8) defined by a meromorphic quadratic differential $Q$ with divisor (1.7) (i.e. that the pair $(C, Q)$ is in the space $\mathcal{Q}^{k,1}_{g,n}$). The divisor of the Abelian differential $v$ on $\hat{C}$ takes the form

$$ (v) = \hat{M} \sum_{i=1}^{m_{\text{odd}}} \hat{d}_i \hat{q}_i \equiv \sum_{i=m_{\text{odd}}+1}^{m} l_i (\hat{x}_i + \hat{x}_i^\mu) - \sum_{i=1}^{n} (\hat{z}_i + \hat{z}_i^\mu). \quad (3.1) $$

To derive the expression (3.1) for the divisor of the abelian differential $v$ on $\hat{C}$ from the expression (1.7) for the divisor of quadratic differential $Q$ on $C$ one should use the fact that the local coordinates on $\hat{C}$ near poles and zeros of even multiplicity coincide with the local coordinates on $C$, while near the poles and zeros of odd multiplicity they are square roots of the local coordinate on $C$.

The genus of $\hat{C}$ equals $g = 2g + \frac{m_{\text{odd}}}{2} - 1$, or, equivalently, $\hat{g} = g + g_-$ with

$$ g_- = g + \frac{m_{\text{odd}}}{2} - 1. \quad (3.2) $$

For holomorphic $Q$ the canonical cover $\hat{C}$ is a classical ingredient of Teichmüller theory. In the case of meromorphic $Q$ the covering $\hat{C}$ appears in the theory of generalized $SL(2)$ Hitchin’s systems [7, 43] under the name of “spectral cover”.

Denote by $(a_i, b_i)$ a canonical basis in $H_1(C, \mathbb{Z})$. The dual basis in $H^{(1,0)}(C, \mathbb{C})$ will be denoted by $(v_1, \ldots, v_g)$ with the normalization $\oint_{a_i} v_j = \delta_{ij}$. Denote the period matrix of $C$ by $\Omega_{ij} = \oint_{b_i} v_j$.

A canonical basis of $H_1(\hat{C}, \mathbb{Z})$ can be chosen as shown in Fig.13 [16, 36, 5]:

$$ \{a_j, a_j^\mu, \tilde{a}_k, b_j, b_j^\mu, \tilde{b}_k\}, \quad j = 1, \ldots, g, \quad k = 1, \ldots, g_- - g, \quad (3.3) $$

where the subset $\{a_j, b_j, a_j^\mu, b_j^\mu\}_{j=1}^g$ is obtained by the lift of the canonical basis of cycles $\{a_j, b_j\}_{j=1}^g$ from $C$ to $\hat{C}$, such that

$$ \mu_* a_j = a_j^\mu, \quad \mu_* b_j = b_j^\mu, \quad \mu_* \tilde{a}_k + \tilde{a}_k = \mu_* \tilde{b}_k + \tilde{b}_k = 0. \quad (3.4) $$

We shall denote by

$$ \{\hat{v}_j, \hat{v}_j^\mu, \hat{w}_k\} \quad (3.5) $$

Figure 13: The canonical covering $\hat{C}$ and the symplectic basis of cycles (3.3).
the basis of normalized Abelian differentials on $\hat{C}$ dual to the basis of cycles (3.4). The differentials $v^+_j = \hat{v}_j + \hat{v}^\mu_j$, $j = 1, \ldots, g$, form a basis in the space $H^+$; these differentials are naturally identified with the normalized holomorphic differentials $v_j$ on $C$. Therefore, to simplify the notation, they shall often be denoted by $v_j$ instead of $v^+_j$. A basis in $H^-$ is given by the Prym differentials $v^-_l$, where

$$v^-_l = \begin{cases} \hat{v}_l - \hat{v}^\mu_l, & l = 1, \ldots, g, \\ \hat{w}_{l-g}, & l = g + 1, \ldots, g_-. \end{cases} \quad (3.6)$$

The following classes in $H_1(\hat{C}, \mathbb{R})$ form a symplectic basis in $H_+$

$$a^+_j = \frac{1}{2}(a_j + a^\mu_j), \quad b^+_j = b_j + b^\mu_j, \quad j = 1, \ldots, g, \quad a^+_j \circ b^+_k = \delta_{jk}, \quad (3.7)$$

whereas the classes

$$a^-_l = \frac{1}{2}(a_l - a^\mu_l), \quad b^-_l = b_l - b^\mu_l, \quad l = 1, \ldots, g, \quad (3.8)$$

$$a^-_l = a_{l-g}, \quad b^-_l = b_{l-g}, \quad l = g + 1, \ldots, g_- \quad (3.9)$$

form a symplectic basis in $H_-$. The basis $\{a^+_j, a^-_l, b^+_j, b^-_l\}$, $j = 1, \ldots, g$, $l = 1, \ldots, g_-$, is related to the canonical basis (3.3) as follows

$$\begin{pmatrix} b_+ \\ b_- \end{pmatrix} = T \begin{pmatrix} b \\ b^\mu \end{pmatrix}, \quad \begin{pmatrix} a_+ \\ a_- \end{pmatrix} = T^{-1} \begin{pmatrix} a \\ a^\mu \end{pmatrix} \quad (3.10)$$

with symmetric matrix

$$T = \begin{pmatrix} I_g & I_g & 0 \\ I_g & -I_g & 0 \\ 0 & 0 & I_{g-g_-} \end{pmatrix}, \quad T^{-1} = \begin{pmatrix} I_g/2 & I_g/2 & 0 \\ I_g/2 & -I_g/2 & 0 \\ 0 & 0 & I_{g-g_-} \end{pmatrix} \quad (3.11)$$

Integration of the differentials $v^+_j$ over the cycles $\{a^+_k\}$ gives $\int_{a^+_k} v^+_j = \delta_{jk}$. The integrals of $v^+_j$ over the cycles $\{b^+_k\}$ give twice the period matrix of $C$:

$$\Omega^+_jk \equiv \int_{b^+_k} v^+_j = 2\Omega_{jk} \quad (3.12)$$

Similarly, integration of the Prym differentials (3.6) over the cycles $\{a^-_l\}$ (3.8), (3.9) yields the $g_- \times g_-$ unit matrix, while their integrals over the cycles $\{b^-_l\}$ give a $g_- \times g_-$ symmetric matrix $\Omega^-$ which equals twice the Prym matrix $\Pi$ (as defined in [16], p.86):

$$\Omega^-jk \equiv \int_{b^-_k} v^-_j = 2\Pi_{jk}, \quad 1 \leq j, k \leq g_- \quad (3.13)$$

The Prym matrix can be written in block form as follows;

$$\Pi = \begin{pmatrix} \Pi_1 & \Pi_2 & \Pi_3 \\ \Pi^T_1 & \Pi^T_2 & \Pi^T_3 \end{pmatrix} \quad (3.14)$$

where $\Pi_1$ is $g \times g$ matrix; $\Pi_2$ is a $g \times (g_- - g)$ matrix and $\Pi_3$ is a $(g_- - g) \times (g_- - g)$ matrix. Then the period matrix $\Omega$ of the double cover $\hat{C}$ in the basis (3.3), (3.5) can be expressed in terms of $\Omega^+$ and $\Omega^-$ as follows:

$${\hat{\Omega}} = T^{-1} \begin{pmatrix} \Omega^+ & 0 \\ 0 & \Omega^- \end{pmatrix} T^{-1} = 2T^{-1} \begin{pmatrix} \Omega & 0 \\ 0 & \Pi \end{pmatrix} T^{-1} \quad (3.15)$$
which gives
\[ \tilde{\Omega} = \begin{pmatrix} \frac{\Omega + \Pi_1}{I} & \frac{\Omega - \Pi_1}{I} & \Pi_2 \\ \frac{2}{\Pi_1} & \frac{2}{\Pi_1} & -\Pi_2 \\ -\Pi_2 & -\Pi_2 & \Pi_3 \end{pmatrix} \]
(3.16)

The expression (3.16) coincides with (91) of [16] up to some signs and interchanged of blocks. This difference with [16] is due to a different choice of the canonical basis of cycles (3.3) made in this paper (our cycles \(a_i^0\) and \(b_j^0\) differ by sign from the ones used by Fay; also the ordering of the canonical cycles used by Fay is different from ours).

**Remark 3.1** The structure of the canonical cover is also extensively discussed in the recent papers [36, 5]. Conventions used in [5] differ from the ones used here due to a different normalization of the canonical basis (3.8) and (3.9) in \(H_-\) (and the dual basis (3.6) in \(H^-\)) used here and in [36]. In particular, the Prym matrix \(\tilde{\Pi}\) from [5] is related to the matrix (3.14) by the following transformation:
\[ \tilde{\Pi} = \begin{pmatrix} I_g & 0 \\ 0 & \sqrt{2} I_{g-} \end{pmatrix} \begin{pmatrix} I_0 & 0 \\ 0 & \sqrt{2} I_{g-} \end{pmatrix} \]
(3.17)

### 3.2 Distinguished local coordinates on \(C\) and \(\hat{C}\)

A given meromorphic quadratic differential \(Q\) with divisor as in (1.7) defines a set of distinguished local parameters on \(C\) and on the canonical covering \(\hat{C}\). The distinguished local parameters \(\zeta_i\) on \(C\) and \(\hat{\zeta}_i\) on \(\hat{C}\) coincide near all points except the branch points of the covering (the points \(\{x_i\}\) and \(\{y_i\}\) of (1.7)); near the branch points we have \(\zeta_i(x) = \hat{\zeta}_i^2(x)\).

- Near any point \(x_0 \in \hat{C}\) such that \(\pi(x_0) \notin \{Q\}\) the local coordinates on \(C\) and \(\hat{C}\) can be chosen to be \(\int_{x_0}^x v\).
- Near a point \(x_i, 1 = 1, \ldots, m_{odd}\) the distinguished local parameters \(\hat{\zeta}_i\) on \(\hat{C}\) and \(\zeta_i(x)\) on \(C\) are given by
\[ \hat{\zeta}_i(x) = \left[ \int_{x_i}^x v \right]^{1/(2i+3)}, \quad \zeta_i(x) = \hat{\zeta}_i^2(x). \]
(3.18)

- Near points \(x_i\) and \(x_i^\mu\) of \(\hat{C}\) for \(i = m_{odd} + 1, \ldots, m\) we define
\[ \hat{\zeta}_i(x) = \left[ \int_{x_i}^x v \right]^{1/(2i+1)}, \quad \hat{\zeta}_i^\mu(x) = \left[ \int_{x_i^\mu}^x v \right]^{1/(2i+1)} \]
(3.19)
which in fact coincide (up to a sign) with the local parameter
\[ \zeta_i(x) = \left[ \int_{x_i}^x v \right]^{1/(2i+1)} \]
(3.20)

- Near a second order pole \(z_i \in C\) and corresponding points \(\hat{z}_i, \hat{z}_i^\mu \in \hat{C}\) the distinguished local coordinates \(\zeta_i\) and \(\hat{\zeta}_i\) are defined as follows. Denote the quadratic residue of \(Q\) at \(z_i\) by \(-p_i^2/4\pi^2\). Then the residues of \(v\) at \(\hat{z}_i, \hat{z}_i^\mu\) are equal to \(p_i/2\pi i\) and \(-p_i/2\pi i\), respectively. The local parameters near \(\hat{z}_i\) and \(\hat{z}_i^\mu\) are given by
\[ \hat{\xi}_i^\pm(x) = \exp \left\{ \pm \frac{2\pi i}{p_i} \int_{x_1}^x v \right\} \]
(3.21)
where \(x_1\) is a chosen “first” zero of \(Q\). These parameters depend on the choice of the zero \(x_1\) and the paths of integration. To fix the integrals in (3.21) uniquely we consider the fundamental polygon of \(C_0\) of \(C\) and cut it along contours \(\gamma_1, \ldots, \gamma_n\) such that \(\gamma_i\) connects the zero \(x_1\) with the pole \(z_i\). The corresponding cuts \(\gamma_i\) and \(\hat{\gamma}_i^\mu\) on the fundamental polygon \(\hat{C}\) of \(\hat{C}\) are obtained by lifting \(\gamma_i\) from \(C\) to \(\hat{C}\) i.e. \(\pi^{-1}(\gamma_i) = \{\hat{\gamma}_i, \hat{\gamma}_i^\mu\}\). The paths of integration in (3.21) should then lie entirely in this fundamental polygon with these additional cuts.

The local parameters \(\zeta_i(x)\) on \(C\) near \(z_i\) can be chosen to coincide with any of \(\hat{\xi}_i^\pm\) and have the same ambiguity.
3.3 Homological coordinates on $Q^{k,1}_{g,n}$

As in the introduction, decompose the homology group of $\tilde{C} \setminus \{\pi^{-1}(z_j)\}_{j=1}^n$, relative to zeros of $v$ of even multiplicity, into even and odd parts under the involution $\mu$:

$$H_1(\tilde{C} \setminus \{\pi^{-1}(z_j)\}_{j=1}^n; \{\pi^{-1}(x_j)\}_{j=1}^m_{m_{od+1}}) = H_+ \oplus H_-.$$  

(3.22)

Then

$$\dim H_- = \dim Q^{k,1}_{g,n} = 2g - 2 + m + n$$  

(3.23)

which, since $m = m_{ev} + m_{od}$, can also be written as $\dim H_- = 2g_+ - n + m_{ev}$ with $g_+ = g + m_{od}/2 - 1$. For any basis of cycles $\{s_i\}$ in $H_-$ one defines homological coordinates on $Q^{k,1}_{g,n}$ as follows:

$$\mathcal{P}_{s_i} = \int_{s_i} v i = 1, \ldots, \dim H_-.$$  

(3.24)

We notice that $H_+$ can be identified with the relative homology group $H_1(C \setminus \{z_i\}_{i=1}^n, \{x_i\}_{i=1}^m_{m_{od+1}})$.

The homology group dual to (3.22) which can also be decomposed into corresponding even and odd parts is

$$H_1(\tilde{C} \setminus \{\pi^{-1}(x_j)\}_{j=1}^m; \{\pi^{-1}(z_j)\}_{j=1}^n) = H^+_\mp \oplus H^-_\mp.$$  

(3.25)

The subspaces $H^\pm_\mp$ are dual to $H^\pm_\mp$ respectively. The basis in $H^-_\mp$ dual to the basis $\{s_i\}$ in $H_-$ is denoted by $\{s^*_i\}$ (the intersection index is $s_i \circ s^*_j = \delta_{ij}$).

4 Hodge and Prym tau-functions on $Q^{k,1}_{g,n}$

The tau-functions $\tau_+$ and $\tau_-$ on the moduli spaces of holomorphic quadratic differentials were defined in [30, 36] by restriction of the Bergman tau-function on an appropriate stratum of moduli spaces of Abelian differentials [31]. For general theory of the Bergman tau-functions and its applications to the theory of isomonodromic deformations, spectral geometry and theory of matrix models and Frobenius manifolds we refer to [38, 33, 31, 13]. For applications of Bergman tau-functions to geometry of various moduli spaces and the theory of Teichmüller flow we refer to [32, 35, 36, 4, 34, 12].

Under a change of Torelli marking of the base Riemann surface the tau-function $\tau_+$ transforms as a section of determinant line bundle of the Hodge vector bundle; thus we call it Hodge tau-function. Similarly, $\tau_-$ transforms as a section of determinant line bundle of the Prym vector bundle over the space of quadratic differentials [36]; thus we call it Prym tau-function.

In our present setting of meromorphic quadratic differentials with second order poles we use the formalism of Bergman tau-functions on spaces of Abelian differentials of third kind [28]. We start from defining two tau-functions, $\tau_+$ and $\tilde{\tau}$ by explicit formulas using the framework of [28] and then define the Prym tau-function $\tau_-$ as the ratio $\tilde{\tau}/\tau_+$.

4.1 Hodge tau-function $\tau_+$

Introduce two vectors $r, s \in \frac{1}{2}\mathbb{Z}^g$ such that

$$\frac{1}{2} A_x((Q)) + 2K^x + \Omega r + s = 0$$  

(4.1)

and also the following notations [31, 36]:

$$E(x, q_i) = \lim_{y \to q_i} E(x, y) \sqrt{d\zeta_i(y)}$$  

(4.2)
Consider the multi-valued $g(1 - g)/2$-differential $C(x)$ (1.14) on $C$ where $K^x \in \mathbb{C}^g$ is the vector of Riemann constants corresponding to the basepoint $x$ [17].

Let us choose a system of cuts $\{\gamma_i\}_{i=1}^n$ on $C$ by connecting one zero (say, $x_1$) with $z_i$ by the cut $\gamma_i$, see Fig. 14. We assume that these cuts lie entirely in the fundamental polygon $C'$ of $C$. The integration contours in the definition of distinguished local parameters (3.21) near $z_i$ are assumed to not intersect $\{\gamma_i\}_{i=1}^n$.

**Definition 4.1** For a given choice of Torelli marking $t$ and cuts $\{\gamma_i\}$ the tau-function $\tau_+$ is given by the following expression which is independent of $x \in C$:

$$
\tau_+(C, Q, t, \{\gamma_i\}) = C^{2/3}(x) \left( \frac{Q(x)}{\prod_{i=1}^M E^d_i(x, q_i)} \right)^{(g-1)/6} \left( \prod_{i<j} E(q_i, q_j)^d_{i,j} \right)^{1/24} e^{-\frac{1}{24}(\langle r, r \rangle - 2\pi i \langle r, K^x \rangle)} \tag{4.4}
$$

where $\sum_{i=1}^M d_i q_i = (Q)$ is the divisor of quadratic differential $Q$ on $C$. 

Figure 14: Branch cuts and cuts $\{\gamma_i, \gamma_i'\}$ connecting $x_1$ with poles of $v = \sqrt{Q}$ on the canonical cover $\hat{C}$. 

$$
E(q_i, q_j) = \lim_{x \to q_i, y \to q_j} E(x, y) \sqrt{d\zeta_i(y)} \sqrt{d\zeta_j(x)} \tag{4.3}
$$

where $\zeta_i$ is the distinguished local parameter on $C$ near $q_i$.
In analogy to Sec.3 of [31] one can show that the expression (4.4) is independent of $x$. Namely, since the product of prime-forms in the denominator of (4.4) compensates all poles and zeros of $Q$, it is sufficient to verify that all holonomy factors of $\tau_+$ are powers of unity. This can be checked using (4.1) and transformation of $E(x,y)$ (see p.4 of [17]) and $C$ (see p.9 of [17]) between the opposite sides of the fundamental polygon:

$$E^2(x+a_j,y) = E^2(x,y)$$
$$C(x+a_j) = C(x)$$

Proposition 4.2 The tau-function $\tau_+$ has the quasi-homogeneity property $\tau_+(C,\epsilon Q) = \epsilon^{\kappa_+} \tau_+(C,Q)$ where

$$\kappa_+ = \frac{1}{48} \sum_{i=1}^{M} \frac{d_i(d_i + 4)}{d_i + 2}$$

Proof. The proof is based on the definition (4.4). The only terms which change under the transformation $Q \rightarrow \epsilon Q$ are the distinguished local parameters $\zeta_i$ with exception of the local parameters (3.21) near poles $z_j$ which are homogeneous in $\epsilon$. Collecting together all powers of $\epsilon$ we get the formula (4.7). An alternative proof of this proposition can be obtained using variational formulas for $\tau_+$ with respect to homological coordinates, following [32, 31, 36].

Next we discuss the dependence of $\tau_+$ on the choice of local parameters at poles $z_j$. Introduce the least common multiple of $d_i + 2$:

$$\alpha = LCM(d_1 + 2, \ldots, d_M + 2)$$

Proposition 4.3 The $(48\alpha)$th power of the expression

$$\tau_+(C,Q) \left( \prod_{i=1}^{n} d \xi_i(z_i) \right)^{1/12}$$

is independent of the choice of local parameters $\xi_i$ near $z_i$.

Proof. The degree of differential $d \xi_i(z_i)$ in (4.4) can be computed as follows. The contribution of multiplier containing $Q(x)$ equals $(g-1)/6$ multiplied with $-1/2$ (since $E(x,z_i)$ is in the denominator) and $-2$ (since $z_i$ enters $(Q)$ with coefficient $-2$), which gives $(g-1)/6$.

The contribution of remaining product of prime-forms can be computed as $(-2)(1/2)(1/24) \sum_{(Q) \backslash z_i} d_i$ which gives $(-1/24)(4g-2)$ (since $\deg(Q) = 4g - 4$). The sum of these two numbers equals $-1/12$ which proves the invariance of $(48\alpha)$th power of (4.9) under a change of local coordinates near $z_j$ (i.e. it is a $4\alpha$-differential with respect to $z_i$).

Proposition 4.4 Let $\sigma$ be a symplectic transformation of $H_1(C,\mathbb{Z})$ acting on canonical basis of cycles as follows:

$$\left( \begin{array}{c} b^\sigma \\ a^\sigma \end{array} \right) = \sigma \left( \begin{array}{c} b \\ a \end{array} \right), \quad \sigma = \left( \begin{array}{cc} A & B \\ C & D \end{array} \right)$$

Then the tau-function $\tau_+$ transforms under the action of $\sigma$ as follows:

$$\frac{\tau_+^\sigma}{\tau_+} = \gamma(\sigma) \det(C\Omega + D)$$

where $\gamma^{48\alpha}(\sigma) = 1$. 

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Proof. Recall the transformation of the vector \( v \) of normalized differentials, the period matrix, the vector of Riemann constants, the differential \( C(x) \) and the prime-form under symplectic transformation ([17], Lemma 1.5 and formulas (1.19), (1.20), (1.23)). Denote \( M = (C\Omega + D) \); then \( v^\sigma = M^{-1}v \) and

\[
\Omega^\sigma = (A\Omega + B)(C\Omega + D)^{-1},
\]

\[
E^{\sigma^2}(x, y) = E^2(x, y) \exp\{2\pi i((C\Omega + D)^{-1}CA_x, A_y)\}. \tag{4.12}
\]

Introduce two vectors whose entries equal to 0 or 1/2:

\[
\alpha_0 = \frac{1}{2}(C^tD)_0, \quad \beta_0 = \frac{1}{2}(A^tB)_0 \tag{4.13}
\]

where index 0 denotes the vector consisting of diagonal entries of the matrix module \( \mathbb{Z}^g \). Then

\[
(K^x)^\sigma = M^{-1}K^x + \Omega^\sigma \alpha_0 + \beta_0; \tag{4.14}
\]

\[
C^\sigma(x) = \gamma' \det^{3/2}(C\Omega + D)C(x) \times \exp\{-\pi i^4 \alpha_0 \Omega \alpha_0 + \pi i^4 K^x(C\Omega + D)^{-1}C K^x - 2\pi i^4 \alpha_0(C\Omega + D)^{-1}K^x\} \tag{4.15}
\]

where \((\gamma')^8 = 1\).

Substituting these transformations into (4.4) we get (4.11). The root of unity of degree \( 48\alpha \) appears due to ambiguity in the definition of distinguished local parameters at points of \( Q \) not coinciding with \( \{z_j\} \). In the process of computation it is convenient to assume that the fundamental polygon is chosen such that \( r = s = 0 \) (the proof that this choice is always possible for \( n \geq 1 \) is identical to the proof of Lemma 6 of [31]); then \( r^\sigma = -2\alpha_0 \) and \( s^\sigma = 2\beta_0 \).

Notice that the moduli-dependent term \( \det(C\Omega + D) \) in (4.11) can also be obtained using variational formulas for \( r_+ \), similarly to (4.23)-(4.25) of [28]. However, to prove that the constant multiplier \( \gamma \) is a root of unity of degree \( 48\alpha \) one needs to use the explicit formula (4.4).

The transformation (4.11) can be equivalently rewritten in terms of the basis \( \{a^+_j, b^+_j\} \) in \( H_+ \) given by (3.7). The period matrix \( \Omega^+ = 2\Omega \) according to (3.12) while the symplectic \( (\mathbb{Z}/2) \) matrix \( \sigma_+ \) acting in this basis is related to the matrix \( \sigma \) (4.10) by

\[
\sigma_+ = \begin{pmatrix} A_+ & B_+ \\ C_+ & D_+ \end{pmatrix} = \begin{pmatrix} A & 2B \\ C/2 & D \end{pmatrix} .
\]

Thus \( \det(C\Omega + D) = \det(C_+\Omega_+ + D_+) \) and transformation (4.11) can also be written as

\[
\tau^\sigma_+ = \gamma(\sigma) \det(C_+\Omega_+ + D_+) . \tag{4.16}
\]

### 4.2 Tau-function \( \hat{\tau} \)

Denote by \( \hat{C} \) the differential (1.14) corresponding to the Riemann surface \( \hat{C} \). Denote the Abel map on \( \hat{C} \) corresponding to initial point \( x \in \hat{C} \) by \( \hat{A} \), the prime-form on \( \hat{C} \) by \( \hat{E} \) and introduce two vectors \( \hat{r}, \hat{s} \in \mathbb{Z}^g \) via relation

\[
\hat{A}_x((v)) + 2\hat{K}^x + \hat{\Omega}\hat{r} + \hat{s} = 0 \tag{4.17}
\]

In analogy to (4.2), (4.3) we also define

\[
\hat{E}(x, \hat{q}_i) = \lim_{y \to \hat{q}_i} \hat{E}(x, y) \sqrt{d\hat{\zeta}_i(y)} \tag{4.18}
\]

\[
\hat{E}(\hat{q}_i, \hat{q}_j) = \lim_{x \to \hat{q}_i, y \to \hat{q}_j} E(x, y) \sqrt{d\zeta_i(y)} \sqrt{d\zeta_j(x)} \tag{4.19}
\]

where \( \hat{\zeta}_i \) is the distinguished local parameter on \( \hat{C} \) near \( \hat{q}_i \) given by (3.18).
The differential \( d\xi_i(z_i) \) enters the expression (4.4) with degree which can be computed as follows. The degree \( \frac{\hat{g}-1}{3} \) arises as contribution of the multiplier containing \( v(x) \): there are two prime-forms in the denominator which contribute \( d\xi_i(z_i) \) each with power \(-1)(1/2\) (since corresponding \( \hat{d}_i = -1\)).

The remaining sum is the contribution of other products of prime-forms where one of the arguments coincides with \( \hat{z}_i \) or \( z_i^\mu \). Since \( \text{deg}(v) = 2\hat{g} - 2\), this gives \( 1/6(1 - 2\hat{g}) \).

In total the degree of \( d\xi_i \) equals \(-1/6\) which shows that 48th power of (4.9) is invariant under the change of local coordinates near \( z_j \) (i.e. it is 8\( \alpha \)-differential with respect to each \( \xi_i \)).

Proposition 4.8 Let \( \hat{\sigma} \) be a symplectic transformation of \( H_1(C,\mathbb{Z}) \) acting on canonical basis of cycles as follows:

\[
\left( \frac{\hat{b}^\sigma}{\hat{a}^\sigma} \right) = \hat{\sigma} \left( \frac{b}{a} \right), \quad \hat{\sigma} = \begin{pmatrix} \hat{A} & \hat{B} \\ \hat{C} & \hat{D} \end{pmatrix}
\]

Then the tau-function \( \hat{\tau} \) transforms under the action of \( \hat{\sigma} \) as follows:

\[
\frac{\hat{\tau}^\hat{\sigma}}{\hat{\tau}} = \gamma(\hat{\sigma}) \det(\hat{C}\hat{\Omega} + \hat{D})
\]

where \( \gamma^{48\alpha}(\hat{\sigma}) = 1 \).

Proof. The transformation (4.11) is derived directly from (4.20) using transformation formulas (4.12), (4.14), (4.15) in the case of canonical covering \( \hat{C} \). The computation repeats the computation required to prove Prop.4.4.
4.3 Prym tau-function $\tau_-$

**Definition 4.9** Let $\{a_j, b_j\}_{j=1}^g$ be a canonical basis of cycles on $C$ and $\{\hat{a}_j, \hat{b}_j\}_{j=1}^g$ be an associate basis of cycles (3.3) on $\hat{C}$. Then the Prym tau-function $\tau_-$ on the space $Q^{k,1}_{g,n}$ is defined as ratio of $\hat{\tau}$ (4.20) and $\tau_+$ (4.4):

$$
\tau_- = \frac{\hat{\tau}(C, Q, \hat{t}, \{\gamma_i\}_{i=1}^n)}{\tau_+(C, Q, t, \{\gamma_i\}_{i=1}^n)}.
$$

(4.25)

The following properties of $\tau_-$ can be derived from the properties of tau-functions $\tau_+$ and $\hat{\tau}$.

**Proposition 4.10** Let $\hat{\sigma}$ be a symplectic transformation of $H_1(\hat{C}, \mathbb{Z})$ commuting with $\mu_+$, and let $\sigma_-$ be the matrix representing $\hat{\sigma}$ in the subspace $H_-$ such that

$$
\begin{pmatrix}
\hat{b}_- \\
\hat{a}_-
\end{pmatrix}^\sigma = \begin{pmatrix}
\sigma_-
\end{pmatrix} \begin{pmatrix}
b_- \\
a_-
\end{pmatrix},
\sigma_- = \begin{pmatrix}
A_- & B_- \\
C_- & D_-
\end{pmatrix}.
$$

(4.26)

Then the tau-function $\tau_-$ transforms under the action of $\sigma$ as follows:

$$
\frac{\tau_-^\sigma}{\tau_-} = \gamma_- \det(C_- \Omega + D_-)
$$

(4.27)

(\text{where according to (3.14)} \Omega_- = 2\Pi \text{ with } \Pi \text{ being the Prym matrix}) and $\gamma_-^{48\alpha} = 1$.

**Proof.** The transformation (4.27) of $\tau_-$ can be deduced from transformation (4.11) of $\tau_+$ and (4.24) of $\hat{\tau}$. Namely, any symplectic transformation $\hat{\sigma}$, commuting with $\mu_+$, has in the basis (3.3) the form

$$
\begin{pmatrix}
\hat{b} \\
\hat{a}
\end{pmatrix}^\hat{\sigma} = \sigma \begin{pmatrix}
b \\
a
\end{pmatrix}, \quad \hat{\sigma} = \begin{pmatrix}
T^{-1} & 0 & 0 & B_- \\
0 & A_- & 0 & B_-
\end{pmatrix} \begin{pmatrix}
A_+ & 0 & B_+ & 0 \\
0 & A_+ & 0 & B_-
\end{pmatrix} \begin{pmatrix}
T & 0 & 0 & 0 \\
0 & T & 0 & -1
\end{pmatrix}
$$

(4.28)

where

$$
\sigma_\pm = \begin{pmatrix}
A_\pm & B_\pm \\
C_\pm & D_\pm
\end{pmatrix}
$$

(4.29)

are symplectic matrices acting on bases $(b_\pm, a_\pm)$ given by (3.7), (3.8), (3.9). Then, since due to (3.15),

$$
\hat{\Omega} = T^{-1} \begin{pmatrix}
\Omega^+ & 0 \\
0 & \Omega^-
\end{pmatrix} T^{-1},
$$

we get

$$
\det(\hat{\Omega}) = \det(C_+ \Omega^+ + D_+) \det(C_- \Omega^- + D_-)
$$

(4.30)

which gives (4.27) as a corollary of (4.11) and (4.24).

Therefore, the Prym tau-function defined by (4.25) depends in fact on the following data: $(C, Q, t, \{\gamma_i\}_{i=1}^n)$ where $t_-$ is a choice of canonical basis $(a_-, b_-)$ in the space $H^-$ of holomorphic Prym differentials.

The following two properties of $\tau_-$ are parallel to the properties of $\tau_+$ and $\hat{\tau}$.

**Proposition 4.11** The tau-function $\tau_-$ has the quasi-homogeneity property

$$
\tau_-(C, eQ, \hat{t}, \{\gamma_i\}_{i=1}^n) = e^{\kappa_-} \tau_-(C, Q, \hat{t}, \{\gamma_i\}_{i=1}^n)
$$

where

$$
\kappa_- = \kappa_+ + \frac{1}{8} \sum_{i=1}^{\frac{m+n}{2}} \frac{1}{d_i + 2}
$$

(4.31)

and $\kappa_+$ is given by (4.7).
Proof. We have $\kappa_- = \hat{\kappa} - \kappa_+$. Using expressions (4.21) and (4.7) for $\kappa_+$ and $\hat{\kappa}$ we get (4.31).

Proposition 4.12 The 48th power of the expression

$$\tau_-(C, Q) \left( \prod_{i=1}^n d\xi_i(z_i) \right)^{1/12}$$

is independent of the choice of local parameters $\xi_i$ near $z_i$.

Proof. This proposition is an immediate corollary of Props. 4.3 and 4.4.

Finally, we discuss the dependence of $\tau_\pm$ on the choice of cuts $\{\gamma_i\}_{i=1}^n$ used to define the local parameters $\xi_i$ near $z_i$.

Proposition 4.13 Under a change of integration contours used to define distinguished local coordinates $\xi_i$ near $z_i$ (i.e. under a change of cuts defining the fundamental polygon $C'$ and cuts $\gamma_i$ between $x_1$ and $z_i$) the tau-functions $\tau_\pm$ transform as follows:

$$\tau_\pm(C, Q, t_\pm, \{\gamma_i\}_{i=1}^n) = \tau_\pm(C, Q, t_\pm, \{\gamma_i\}_{i=1}^n) \exp \left\{ \frac{-\pi i}{6} \sum_{j=1}^n \sum_{i=1}^{g-1} A_{ij} \frac{p_{i\gamma}}{p_j} \right\}$$

where the set $\{s_i\}$ contains cycles $\{a_i^-, b_i^+\}$, $l = 1, \ldots, g$ and $a_i^-, b_i^+, l = g + 1, \ldots, g_-$ (notations here follow (3.8),(3.9)) as well as $\frac{1}{2}(l_j - l_j^0)$ where $l_j$ is a small loop around $z_j$; $A_{ij}$ is a matrix with half-integer entries.

Proof. A change of integration contours $\gamma_i$ in the formulas (4.4), (4.25), (4.20) for $\tau_\pm$ obviously leads to appearance of an exponential factor of the form (4.33).

If the “first” zero is not changed then the corresponding matrix $A_{ij}$ has integer entries. If $x_1$ is changed to some other zero $x_2$ then the integrals get an additional contribution of the form $\int_{x_1}^{x_2} v$ with can be expressed as a half-integer combination of the cycles $s_i$. Therefore in general the matrix $A$ can have half-integer entries.

Remark 4.14 Relation (4.31) is analogous to the formula (2.4) of Theorem 2 of [12] (for the case of holomorphic $Q$) which relates sums of Lyapunov exponents corresponding to Hodge and Prym vector bundles. This indicates that the tau-function $\tau_-$ has a close relationship to the action of the Teichmüller flow on Prym vector bundle, similarly to the role of $\tau_+$ in the study of the Teichmüller flow on Hodge vector bundle, as described in Sect. 5.2 of [12].

Remark 4.15 In [36] it was considered the case when all $d_i = 1$ (i.e. $Q$ is a holomorphic quadratic differential with simple zeros). Then $M = 4g - 4$ and the formulas (4.7), (4.31) give $\kappa_+ = \frac{5}{36} (g - 1)$ and $\kappa_- = \frac{11}{36} (g - 1)$. These numbers differ from $p_+$ and $p_-$ from Theorem 2 of [36] by a factor of 48; this is due to the fact that the tau-functions $\tau_\pm$ from [36] equal to the 48th power of $\tau_\pm$ used in this paper. The function $\tilde{\tau}$ from [36] corresponds to 24th power of $\tilde{\tau}$ used here.

4.4 Hodge and Prym classes over $Q_{g,n}$ and $\tau_\pm$

The Hodge vector bundle $\Lambda_H$ over $Q_{g,n}$ is defined by the pullback of the Hodge vector bundle over $\mathcal{M}_{g,n}$ (the fiber of $\Lambda_H$ is the space of holomorphic abelian differentials on $C$).

To define the Prym vector bundle $\Lambda_P$ over $Q_{g,n}$ (following [36, 34]) we consider first the subspace $Q_{g,n}^0$ of $Q_{g,n}$ such that all zeros of $Q$ are simple; this is the largest stratum of $Q_{g,n}$. The fiber of $\Lambda_H$ over $Q_{g,n}^0$ coincides with $H_-$. At those boundary components of $Q_{g,n}^0$ where some zeros of $Q$ become multiple, the fiber of $\Lambda_P$ is obtained by decomposing the fiber of the Hodge vector bundle over $\mathcal{M}_{g}$ over the nodal curve $\tilde{C}$ into $H_+ \oplus H_-$. As a result $H_-$ consists of meromorphic differentials $u$ with poles of order $k_i$ at the branch point $x_i$ for odd $k_i$ and poles of order $k_i/2$ for even $k_i$. (In the next section we will call such differentials of $H_-$ “prymian local coordinates.”)
at points \((\hat{x}_i, \hat{\mu}_i \hat{n})\) for a zero \(x_i\) of even order \(k_i\). In this way one gets a well-defined vector bundle \(\Lambda_P\) over the whole space \(Q_{g,n}\) [34].

Let us denote by \(\Lambda_2^{(n)}\) the bundle over \(\mathcal{M}_{g,n}\), whose fibers consist of meromorphic quadratic differentials with at most simple poles at \(z_1, \ldots, z_n\) \((\Lambda_2^{(n)}\) is, in fact, the cotangent bundle over \(\mathcal{M}_{g,n}\)).

The following theorem is similar to the analogous statement from [36, 34]:

**Proposition 4.16** The fiber of the Prym vector bundle \(\Lambda_P\) over \(Q_{g,n}^0\) above the point \((C, Q)\) is isomorphic to the fiber of vector bundle \(\Lambda_2^{(n)}\) above the point \(C\). The isomorphism is given by

\[
\pi^* \hat{Q} = uv,
\]

where \(\pi: \hat{C} \to C\) is the canonical projection and \(\hat{Q}\) is a meromorphic quadratic differential on \(C\) with simple poles and \(u \in H^-\) is a Prym differential.

**Proof.** Both \(u, v\) are antisymmetric with respect to the involution \(\mu\) and therefore they have odd–multiplicity zeroes at the branch-points, with \(v\) having only simple zeros. The product has a zero of even multiplicity on \(\hat{C}\) at each branch-point. Denote by \(\hat{\zeta}\) a local parameter on \(\hat{C}\) near a branch-point, so that \(v \sim \hat{\zeta} d\hat{\zeta}\) and \(u \sim \hat{\zeta}^{2k+1} d\hat{\zeta}\).

Then \(uv \sim \hat{\zeta}^{2k+2} d\hat{\zeta}^2 \sim \hat{\zeta}^k d\hat{\zeta}^2\) with \(\zeta = \hat{\zeta}^2\). Thus \(uv\) is invariant under canonical involution on \(\hat{C}\) and hence corresponds to a quadratic differential on \(C\) with at most simple poles at punctures.

The isomorphism can be appropriately extended to all strata of \(Q_{g,n}\) as discussed in [34].

The following proposition is an immediate corollary of Props. 4.3, 4.4, 4.10 and 4.12:

**Proposition 4.17** \(\tau_{48}^k(C, Q)\) is a section of the line bundle \(\det^{48} \Lambda_H \otimes \prod_{i=1}^n L_i^4\) over the space \(Q_{g,n}^0\).

Following [32, 35, 36, 34] this proposition can be used to get relations between various classes in the Picard group of \(Q_{g,n}\). However, in this paper we don’t pursue this goal; instead we are going to use similar ideas in the context of combinatorial model of \(\mathcal{M}_{g,n}\).

### 4.5 Differential equations for \(\tau_{\pm}\) on \(Q_{g,n}^{k,1}\)

Differential equations for \(\tau_{\pm}\) can be derived in parallel to [31, 36, 28]. However, the presence of quadratic poles of \(Q\) introduces some new features.

Consider the system of cuts \(\{\gamma_i\}_{i=1}^n\) on \(C\) and the associate system of cuts \(\{\hat{\gamma}_i\}_{i=1}^n\) on \(\hat{C}\) used to define tau-functions \(\tau_{\pm}\). Choose a basis \(\{s_i\}_{i=1}^{g_+ + m_{odd}}\) in the odd subspace \(H'_-\) of \(H_1(\hat{C} \setminus \{\hat{\gamma}_i\}_{i=1}^n, \{\pi^{-1}(x_i)\}_{i=1}^m)\) and denote by \(\{\hat{s}_i\}_{i=1}^{g_+ + m_{even}}\) the dual basis in \(H_-^{\ast}\) which is the odd subspace in \(H_1(\hat{C} \setminus \{\hat{\gamma}_i\}_{i=1}^n \cup \{\pi^{-1}(x_i)\}_{i=1}^m)\).

Consider the meromorphic bidifferential \(\hat{B}(x, y) = d_x d_y \log \hat{E}(x, y)\), where \(\hat{E}\) corresponds to the same Torelli marking of \(\hat{C}\) as in the formula (4.20) for \(\hat{\tau}\).

Now put

\[
B_{\pm}(x, y) = \hat{B}(x, y) + \mu_* \hat{B}(y, x), \quad B_{\pm}(x, y) = \hat{B}(x, y) - \mu_* \hat{B}(y, x),
\]

(4.35)

(the subscript \(y\) at \(\mu_*\) means that we take the pullback with respect to the involution on the second factor in \(\hat{C} \times \hat{C}\)). The bidifferential \(B_{\pm}(x, y)\) is just the pullback of the canonical bidifferential \(B(x, y)\) on \(C \times C\). The bidifferential \(B_{\pm}\) was first introduced in [36]. It is called the Prym bidifferential.

Consider the regularization of the bidifferentials \(B_{\pm}\) near the diagonal:

\[
B_{\pm}^{reg}(x, x) = \left( B_{\pm}(x, y) - \frac{v(x) v(y)}{(\int_x^y v)^2} \right)\bigg|_{y=x}
\]

(4.36)
and introduce meromorphic Abelian differentials \( w^\pm \) on \( \hat{C} \) (anti-symmetric with respect to \( \mu \)) by

\[
w^\pm_v(x) = \frac{B'^g_v(x, x)}{v(x)}
\]  

(4.37)

The complete set of local coordinates on \( Q_{g,n}^{k,1} \), besides \( \{ P_{s_i}, \} \), contains the residues \( \{ p_i \} \). However, we are not going to consider derivatives of \( \tau^\pm \) with respect to \( p_i \), since they will be kept fixed in the context of the combinatorial model of \( M_{g,n} \). The derivatives with respect to \( P_{s_i} \) are given by the following proposition.

**Proposition 4.18** The tau-functions \( \tau^\pm \) satisfy the following equations:

\[
\frac{\partial \log \tau^\pm}{\partial P_{s_i}} = -\frac{1}{4\pi i} \int_{x_i^\pm} w^\pm_v, \quad i = 1, \ldots, g - m_{even}
\]  

(4.38)

**Proof.** The formula (4.38) is a corollary of two facts. The first is the variational formula for the Bergman tau-function on spaces of third kind Abelian differentials given by formula (4.4) of [28] (this formula is proved in parallel to the case of spaces of holomorphic Abelian differentials treated in detail in [31]). The second fact is the relationship between variational formulas for tau-functions on spaces of Abelian and quadratic differentials which was established in Section 4.1 of [36].

4.6 Examples.

The following two examples play an important role in the sequel since they allow to analyze the local behaviour of the tau-functions near different boundary components of the moduli spaces.

4.6.1 Space \( Q_{0,0}^{1,1,-7} \)

Here we consider the moduli space of quadratic differentials on Riemann sphere and \( n = 0 \) with the divisor

\[
(Q) = x_1 + x_2 + x_3 - 7x_4
\]

(this notation is motivated by application to the combinatorial model). Then \( \hat{g} = 1 \). The pole \( x_4 \) of multiplicity 7 will be put to infinity of complex plane with coordinate \( x \) while \( x_1, x_2 \) and \( x_3 \) are positions of zeros in \( x \)-plane. Using the rescaling and shift of coordinate \( x \) we can then write

\[
Q(x) = (x - x_1)(x - x_2)(x - x_3)(dx)^2
\]  

(4.39)

for \( x_1 + x_2 + x_3 = 0 \) and

\[
v(x) = [(x - x_1)(x - x_2)(x - x_3)]^{1/2} dx
\]  

(4.40)

such that on \( \hat{C} \) we have \( v = 2x_1 + 2x_2 + 2x_3 - 6\infty \).

**Hodge tau function** \( \tau_+ \). The distinguished local parameters on \( \mathbb{C}P^1 \) near points \( x_i \) are given by \( \zeta_i(x) = \left[ \int_{x_i}^x v \right]^{2/3} \) while the distinguished local parameter near \( y_1 = \infty \) equals \( \zeta_\infty(x) = \left[ \int_{x_1}^x v \right]^{-2/5} \). Then, for \( \zeta_i \) we have

\[
\frac{d\zeta_i(x)}{dx}\bigg|_{x=x_1} = (2/3)^{-1/3} \left[(x_1 - x_2)(x_1 - x_3)\right]^{1/3}
\]  

(4.41)

while \( \frac{d\zeta_\infty(x)}{d(1/x)}\bigg|_{x=\infty} = (2/5)^{2/5} \). In the sequel we shall omit the unessential multiplicative constants since all tau-functions are defined up to moduli-independent multiplicative factors. For the prime-form we have in genus zero: \( E(x, y) = \frac{y-x}{\sqrt{d(1/x)}} \). Then, say,

\[
E(x, x_1) = \text{const} \frac{(x - x_1)}{\sqrt{dx}} \left[(x_1 - x_2)(x_1 - x_3)\right]^{1/6} \quad E(x, \infty) = \text{const} \frac{1}{x \sqrt{d(1/x)}}
\]  

(4.42)
and similar expressions for $E(x_i, x_j)$ and $E(x_i, \infty)$. The formula (4.4) for the function $\tau_+$ takes the form:

$$
\tau_+ = \frac{Q^{-1/6}(x) \prod_{i=1}^{3} E^{1/6}(x_i, x) \prod_{i<j} E^{1/24}(x_i, x_j)}{E^{7/6}(\infty, x) \prod_{i=1}^{3} E^{7/24}(x_i, \infty)} .
$$

Substituting the expressions for the prime-forms computed in distinguished local parameters into this formula we get

$$
\tau_+ = |(x_1 - x_2)(x_2 - x_3)(x_3 - x_1)|^{1/36} .
$$

**Prym tau function $\tau_-$.** To get an expression for $\tau_-$ we shall first write down the formula for $\hat{\tau}$ and then find $\tau_-$ from relation $\hat{\tau} = \tau_+ \tau_-$. The function $\hat{\tau}$ is in this case coincides with the Bergman tau-function on the space of Abelian differentials of the form (4.40) i.e. on the stratum $H^{2,2,2,2,6}_1$ of meromorphic Abelian differentials of second kind [28]. The formula for $\hat{\tau}$ is given by (4.20).

Equivalently and more conveniently, we can use the formula (3.29) (and the next formula after that) of [31] which describe dependence of $\tau_+$ on the choice of Abelian differential while keeping the Riemann surface fixed. If we adjust this formula to the present situation we choose

$$
v_0 = \frac{dx}{\sqrt{(x-x_1)(x-x_2)(x-x_3)}}
$$

which is a differential without zeros or poles. In that case $\hat{\tau}(\hat{C}, v_0) = \eta^2(\omega_2/\omega_1)$ [31] where $\omega_1$ and $\omega_2$ are periods of the differential $v_0$:

$$
\omega_1 = \int_a v_0 , \quad \omega_2 = \int_b v_0
$$

and $\eta$ is the Dedekind’s eta-function. Then for an arbitrary differential $v$ of second kind with divisor $(v) = \sum_i m_i x_i$ an appropriate analog of the formula (3.29) of [31] looks as follows:

$$
\hat{\tau}(\hat{C}, v) = \eta^2(\omega_2/\omega_1) \prod_i \left( \frac{v_0}{d\zeta_i} \right)^{-m_i/12}
$$

where, $\zeta_i(x) = \left[ \int_{x_i}^x v \right]^{1/(d+1)}$ is the distinguished local parameter on $\hat{C}$ near $x_i$ defined by $v$.

If $v$ is given by (4.40) then

$$
\frac{v_0}{d\zeta_1} \big|_{x=x_1} = \text{const} \left[ (x_1 - x_2)(x_1 - x_3) \right]^{-2/3}
$$

and

$$
\frac{v_0}{d\zeta_\infty} \big|_{x=\infty} = \text{const} .
$$

Substituting (4.48), (4.49) (as well as similar formulas at $x_{2,3}$) into (4.47) we get

$$
\hat{\tau} = \eta^2(\omega_2/\omega_1) \left[ (x_1 - x_2)(x_1 - x_3)(x_2 - x_3) \right]^{2/9}
$$

and, dividing this expression by $\tau_+$ (4.44),

$$
\tau_- = \eta^2(\omega_2/\omega_1) \left[ (x_1 - x_2)(x_1 - x_3)(x_2 - x_3) \right]^{7/36} .
$$

Notice that the homogeneity coefficients $\hat{k} = 2/15$ and $k_- = 7/60$ are in agreement with the ones given by the general formulas (4.21), (4.31).

Finally, using the standard expression for the $\eta$-function via $w_1$ and the discriminant:

$$
\eta^{24}(\omega_2/\omega_1) = \frac{w_1^{12}}{(2\pi)^{12}} \left[ (x_1 - x_2)(x_1 - x_3)(x_2 - x_3) \right]^2
$$
we get, up to a non-essential constant
\[ \tau_- = \omega_1 [(x_1 - x_2)(x_1 - x_3)(x_2 - x_3)]^{13/36} \] (4.53)
or, equivalently,
\[ \tau_- = \omega_1 \tau_+^{13}. \]

4.6.2 Space \( Q_{0,0}^{1,1,-3,-3} \)
Here we put \( g = 0, \, n = 0 \) and choose the divisor \( (Q) = x_1 + x_2 - 3x_3 - 3x_4. \) Assuming \( x_3 = 0 \) and \( x_4 = \infty, \) we have
\[ Q = \frac{(x - x_1)(x - x_2)}{x^3} (dx)^2 \] (4.54)
and
\[ v(x) = x^{-3/2}[(x - x_1)(x - x_2)]^{1/2} dx. \] (4.55)
The poles \( y_1 \) and \( y_2 \) of order 3 are located at \( x = 0 \) and \( x = \infty. \)

**Hodge tau-function** \( \tau_+. \) The distinguished local parameters on \( \mathbb{C}P^1 \) at \( x_1 \) and \( x_2 \) are given by \( \xi_i(x) = \left[ \int_{x_i}^x v \right]^{2/3} \) and
\[ \frac{d\xi_1}{dx} \big|_{x=x_1} = \text{const} \mu_0^{1/3} (x_1 - x_2)^{1/3}, \quad \frac{d\xi_2}{dx} \big|_{x=x_2} = \text{const} \mu_0^{1/3} (x_1 - x_2)^{1/3}. \]
The distinguished local parameter at \( x = 0 \) is given by \( \zeta_0(x) = \left[ \int_{x_1}^x v \right]^{-2} \) such that \( \frac{d\zeta_0(x)}{dx} \big|_{x=0} = \text{const} (\mu_0 x_1 x_2)^{-1}. \)
Finally, the distinguished local parameter at \( x = \infty \) is \( \zeta_\infty(x) = \left[ \int_{x_1}^x v \right]^{-2} \) such that \( \frac{d\zeta_\infty(x)}{dx} \big|_{x=\infty} = \text{const} \mu_0^{-1}. \)
The expression (4.4) for \( \tau_+ \) takes the form:
\[ \tau_+ = Q^{-1/6} \left[ \frac{E(x_1, x) E(x_2, x)}{E(0, x) E(\infty, x)} \right]^{1/6} \left[ \frac{E(0, x) E(\infty, x)}{E(x_1, 0) E(x_1, \infty) E(x_2, 0) E(x_2, \infty)} \right]^{1/8} \] (4.56)
which gives
\[ \tau_+ = (x_1 x_2)^{1/12} (x_1 - x_2)^{1/36}. \] (4.57)

**Prym tau-function** \( \tau_- . \) Here we are going to use again the formula (4.47) with \( v_0 \) given by the standard holomorphic differential (4.45) to compute \( \check{\tau}(v, C), \) that is, the Bergman tau-function on the stratum \( \mathcal{H}^{[2,2,-2,-2]}_1 \) of the space of Abelian meromorphic differentials. The distinguished local coordinates near the points \( x = x_{1,2}, \, x = 0 \) and \( x = \infty \) of the divisor \( (v) \) are:
\[ \hat{\zeta}_i(x) = \left[ \int_{x_i}^x v \right]^{1/3}, \quad \hat{\zeta}_0(x) = \left[ \int_{x_1}^x v \right]^{-1}, \quad \hat{\zeta}_\infty(x) = \left[ \int_{x_1}^x v \right]^{-1} \] (4.58)
so that
\[ \frac{v_0}{d\hat{\zeta}_i} \big|_{x=x_i} = \text{const} (x_1 - x_2)^{-2/3}. \] (4.59)
Verification of formula (4.59) goes as follows: as \( x \to x_1 \) we have \( v_0 \sim [x_1 (x_1 - x_2)]^{-1/2} \left( \frac{dx}{x-x_1} \right)^{1/2} \) and
\[ d\hat{\zeta}_1(x) \sim \left( \frac{(x_1 - x_2)^{1/2}}{x_1^{3/2}} \right)^{3/2} \left( (x - x_1)^{2/3} \right)^{3/2} \left( x - x_1 \right)^{1/2} dx \] with the ratio leading to (4.59).
Similar computation gives
\[ \frac{v_0}{d\xi_0}_{x=0} = \text{const}; \quad \frac{v_0}{d\xi_\infty}_{x=\infty} = \text{const} \] (4.60)
such that the formula (4.47) with \( m_1 = m_2 = 2 \) and \( m_0 = m_\infty = -2 \) gives
\[ \tilde{\tau} = \eta^2(\omega_2/\omega_1)(x_1 - x_2)^{2/9} \] (4.61)
where, as before, \( \omega_1 \) and \( \omega_2 \) are \( a \)- and \( b \)- periods of the differential \( v_0 \), respectively, and
\[ \tau_- = \frac{\tilde{\tau}}{\tau_+} = \eta^2(\omega_2/\omega_1)(x_1 - x_2)^{7/36}(x_1x_2)^{-1/12} \] (4.62)
In this case the formula expressing the Dedekind’s eta-function in terms of branch points \( 0, x_1, x_2 \) and the Abelian integral \( \omega_1 \) looks as follows:
\[ \eta^{24}(\omega_2/\omega_1) = \frac{\omega_1^2}{(2\pi)^{12}}[x_1x_2(x_1 - x_2)]^2 \] (4.63)
and (4.62) becomes
\[ \tau_- = \omega_1(x_1x_2)^{1/12}(x_1 - x_2)^{13/36} \] (4.64)
or
\[ \tau_- = \frac{\omega_1}{x_1x_2}\tau_+^{13} \cdot \]

5 Hodge and Prym tau-functions in the combinatorial model of \( M_{g,n} \)

5.1 Monodromy of circle bundle and Poincaré dual of the first Chern class

We recall a few well-known facts (see for example [24] for details). An effective way to compute the Poincaré dual to the first Chern class of a complex line bundle \( L \) over a smooth (oriented) manifold \( M \) of dimension \( N \) goes as follows. Let \( f \) be a section of \( L \) which is smooth and non-vanishing outside of several oriented connected submanifolds \( M_1, \ldots, M_k \) of (real) co-dimension 2. Denote the circle (or \( U(1) \)) bundle over \( M \) associated to \( L \) by \( S[L] \). Then \( \Phi = f/f \) is a section of \( S[L] \) over \( M \setminus \{M_j\}_{j=1}^k \). Denote variation of \( \text{arg} \Phi \) along a small positively oriented loop around \( M_j \) by \( 2\pi k_j \) with \( k_j \in \mathbb{Z} \). Then the Poincaré dual to \( c_1(L) \) is given by a linear combination \( \sum_{j=1}^k k_jM_j \). The relation \( c_1(L) = \sum_{j=1}^k k_jM_j \) holds in the Picard group \( \text{Pic}(M, \mathbb{Z}) \).

Notice that the smooth section \( f \) may also vanish on a non-orientable submanifold \( M_0 \) of co-dimension 2. However, the variation of \( \text{arg}f \) around \( M_0 \) always vanishes since a small loop \( l_0 \) around \( M_0 \) can be smoothly deformed to \( l_0^{-1} \). (This observation is consistent with the standard fact that the Poincaré dual of a given form is always orientable).

The above construction needs to be adjusted if \( M \) is an orbifold (we assume that \( M \) is a “good” orbifold according to Thurston’s classification) i.e. it is a quotient of a smooth manifold \( \bar{M} \) over a discrete group. In this case \( M_j \) can also be suborbifolds of \( M \). The variation of \( \text{arg} \Phi \) around \( M_j \) is then equal to \( 2\pi i q_j \) where \( q_j \in \mathbb{Q} \). Then we have the relation \( c_1(L) = \sum_{j=1}^k q_jM_j \) which this time holds in \( \text{Pic}(M, \mathbb{Q}) \) (in the case of orbifold the group \( \text{Pic}(M, \mathbb{Q}) \) can have torsion while the group \( \text{Pic}(M, \mathbb{Q}) \) is torsion-free).

This description can be appropriately adjusted and applied to the combinatorial model \( M_{g,n}[p] \).

We consider the restrictions of the power \( \tau^\pm \) of the tau-functions \( \tau \) of Section 4 to each of the largest stratum \( W \) of \( M_{g,n}[p] \) (\( W \) is the “real locus” of the main stratum \( \mathcal{Q}_{g,n}^0[p] \) of \( Q_{g,n}^0[p] \)).

On each of those cells \( \tau^\pm \) are sections of the line bundles \( (\det \Lambda_M)^{48} \otimes \prod_{i=1}^n \mathcal{L}^i \) and \( (\det \Lambda_P)^{48} \otimes \prod_{i=1}^n \mathcal{L}^i \), respectively. Introduce the sections \( \Phi^\pm = \text{arg} \tau^\pm \) of these circle bundles on each cell of \( W \).

In Lemma 5.1 below it is shown that both sections \( \Phi^\pm \) admit continuous extension through the facets and provide continuous sections of the circle bundles over \( \hat{W} \) (we recall that \( \hat{W} \) denotes the union of \( W \) and facets between its
cells). Recall also that the complement of $\tilde W$ in $M_{g,n}[p]$ is the union of three co-dimension 2 subcomplexes: the Witten’s cycle $W_5$, Kontsevich’s boundary (also a cycle) $W_{1,1}$ and the non-orientable subcomplex $W_{2,2}$, which is not a cycle (we refer to [44, 45] for the proof of non-orientability of $W_{2,2}$). Due to its non-orientability, $W_{2,2}$ can not contribute to the Poincaré duals of $c_1((\det 4g\Lambda_H \otimes \prod_{i=1}^n L_i^\ast)$ and $c_1((\det 4g\Lambda_P \otimes \prod_{i=1}^n L_i^\ast)$. Therefore, in the Picard group $\text{Pic}(W, Q)$ (which coincides with $\text{Pic}(M_{g,n}[p], Q)$) these Chern classes are linear combinations of $W_5$ and $W_{1,1}$ with rational coefficients which are obtained by computing the monodromies of $\tau_{\pm}/|\tau_{\pm}|$ (i.e. the increments of $\Phi_{\pm} = \arg \tau_{\pm}$) around these cycles.

5.2 Tau-functions $\tau_{\pm}$ and sections of circle bundles over $\tilde W$

Since $M_{g,n}[p]$ is a slice of $Q_{g,n}[p]$ defined by the reality of all periods of the Abelian differential $v = \sqrt{Q}$ on $\hat C$, the Bergman tau-functions $\tau_{\pm}$ can be defined on each cell of $M_{g,n}[p]$ by the same formulas (4.4), (4.20), (4.25) as in the case of $Q_{g,n}[p]$.

Let us summarize the data which define $\tau_{\pm}$ on a given cell of $M_{g,n}[p]$ (we are mainly interested in the case when the cell is in the largest stratum $W$):

1. Torelli markings of $C$ and of $\hat C$: the tau-functions $\tau_{\pm}$ depend on the choice of the Lagrangian subspaces of $a$-cycles in $H_{\pm}$.

2. The choice of one of the zeros (say, $x_1$) of $Q$ which is used as an initial point of integration to define the distinguished local coordinates in neighbourhoods of poles $z_i$ via

$$\xi_i(x) = \exp \left\{ \frac{2\pi i}{p_i} \int_{x_1}^{x} v \right\}$$

(5.1)

3. The choice of cuts $\gamma_i$ between $x_1$ and $z_i$; the integration contours in (5.1) lie in $\hat C_0 \setminus \{\pi^{-1}\gamma_i\}_{i=1}^n$ where $\hat C_0$ is the fundamental polygon of $\hat C$ defined above.

The tau-functions $\tau_{\pm}$ are real-analytic within each cell of $M_{g,n}[p]$.

If the initial point of integration $x_1$ and/or the paths connecting $x_1$ with $z_1, \ldots, z_n$ are chosen differently then the tau-functions $\tau_{\pm}$ are multiplied by the factor of the form $(4.33)$, which on $M_{g,n}[p]$ is expressed as

$$\tau_{\pm} \to \tau_{\pm} \exp \left\{ -\frac{\pi i}{3} \sum_{j=1}^n \sum_{k=1}^{2g+2m-2} A_{kj} \frac{\ell_k}{p_j} \right\}.$$  

(5.2)

The matrix $A_{kj}$ is a matrix with integer or half-integer entries and $\ell_k$ are the Strebel lengths of the edges of the ribbon graph; notice that this multiplier is always unitary. The extra factor of 2 in the exponent of (5.2) comparing with (4.33) appears since the length of an edge is equal to 1/2 of an integer combination of periods $P_{\alpha}$ from (4.33).

Following Prop. 4.3, we introduce the following expressions on each stratum $M_{g,n}^d[p]$ of the combinatorial model $M_{g,n}[p]$: 

$$\tau_{\pm} \left( \prod_{i=1}^n d\xi_i(z_i) \right)^{1/12}$$

(5.3)

Similarly to Prop. 4.17, defining $\alpha$ via (4.8) on a given stratum, we conclude that the $48\alpha$th power of (5.3) is independent of the choice of the base-point $x_1$ and the cuts $\gamma_i$.

This allows to interpret $\tau_{\pm}^{48\alpha}$ as sections of the line bundles $(\det^{12} A_{H,P} \otimes \prod_{i=1}^n L_i)^{4\alpha}$ over the stratum of the combinatorial model $M_{g,n}^d[p]$.  

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For the main stratum $W$ $\alpha = 1$. Since $W$ consists of disconnected topologically trivial cells, we consider the topologically non-trivial space $W$ which contains also the facets of the cells of $W$ (as discussed above these facets correspond to ribbon graphs with one 4-valent vertex and the crossing from one cell of $W$ to another corresponds to a Whitehead move on one of edges of the ribbon graph).

The absolute values of $\tau_{\pm}$ vanish on the facets of cells of $W$ i.e. when two vertices of $\Gamma$ glue together. However, the following sections of the circle bundles over $W$:

$$\Theta_{\pm} = \left(\frac{\tau_{\pm}}{|\tau_{\pm}|}\right)^{48} = e^{48i\Phi_{\pm}}, \quad (5.4)$$

can be extended continuously through the facets.

The formalism of circle bundles (instead of line bundles) is adequate in the analysis of the intersection theory on moduli spaces using the JS combinatorial model; in [55] it was used to discuss various subtle points in Kontsevich’s proof of Witten’s conjecture [39]. To find the Poincaré dual to first Chern class of $\det^{48}\Lambda_{(H,P)} \otimes \prod_{i=1}^{n} L_{i}^{4}$ one needs to compute the contact of the argument $\Phi_{\pm}$ along closed paths in $W$ around cycles of co-dimension 2 in $\overline{\mathcal{M}}_{g,n}[p]$ (i.e. around $W_5$ and $W_{1,1}$). These increments (divided by $2\pi$) give coefficients in the linear combination of the cycles $W_5$ and $W_{1,1}$ (while $W_{1,1}$ is the sum of several cycles, all of them turn out to contribute with the same coefficient) when computing the Poincaré duals.

To continue $\arg \tau_{\pm}$ from one cell of $W$ to another via a Whitehead move one needs to study the asymptotics of $\tau_{\pm}$ as the length of one of the edges of the ribbon graph $\Gamma$ tends to zero while perimeters of all faces remain the same.

Denote the length of the “vanishing edge” by $t$ and the lengths of the edges of the limiting ribbon graph $\Gamma_0$ by $\ell_i$ (while keeping the perimeters constant). One can then assume that the limit $t \to 0$ of the graph $\Gamma$ is taken in such a way that all lengths $\ell_i$ of the edges of $\Gamma$ depend linearly on $t$ i.e. $\ell_i = \ell_i^0 + \alpha_i t$.

Denote the JS differential arising in the limit $t \to 0$ form by $Q_0$, multiplicity of its zeros are $(2,1,\ldots,1)$ the double zero of corresponding JS differential coincides with the four-valent vertex of $\Gamma$. The genus of the canonical cover given by $\nu^0_0 = Q_0$ (we denote it by $\tilde{C}_0$) equals $\hat{g} - 1$ since one of the branch cuts on $\tilde{C}$ degenerates as $t \to 0$. The set of homological coordinates on the facet includes the integral of $v^0_0 = \sqrt{Q_0}$ on $\tilde{C}_0$ between the double zeros of $Q_0$ (i.e. simple zeros of $v_0$) on the different sheets of $\tilde{C}_0$.

Tau-functions $\tau_{\pm}$ on each facet of $W$ i.e. on a cell of $W \setminus W$ are defined by the formulas $(4.4), (4.20), (4.25)$ with multiplicities of zeros of $Q_0$ having the above form. Such definition also depends on the choice of Torelli marking $t_0$ on $\tilde{C}_0$ and the choice of cuts $\{\lambda^0_j\}_{j=1}^4$ connecting a chosen zero (which we denote by $x_1$ and assume that it remains simple as $t \to 0$) of $Q_0$ with its poles.

Then the behaviour of $\tau_{\pm}$ in the limit $t \to 0$ is given by the following lemma:

**Lemma 5.1** Suppose that the Torelli marking $t_0$ of $C_0$ and contours $\gamma^0_\ell$ on $C_0$ are obtained from the Torelli marking $t$ on $C$ and the contours $\gamma_\ell$ on $C$ by a continuous deformation in the limit as $t \to 0$. Then the following asymptotics holds as $t \to 0$:

$$\tau_{\pm}(C, Q, t, \{\gamma_\ell\}) = \text{const} \times t^{1/72} (1 + o(1)) \tau_{\pm}(C, Q_0, t_0, \{\gamma^0_\ell\}) \quad (5.5)$$

and

$$\arg \tau_{\pm}(C, Q, t, \{\gamma_\ell\}) = \arg \tau_{\pm} + \text{const} + (C, Q_0, t_0, \{\gamma^0_\ell\}) + o(1) \quad (5.6)$$

where the constants are independent of a point of $(C_0, Q_0)$ of the facet.

**Proof.** The proof of asymptotic behaviour (5.5) is analogous to the proof of asymptotics of the tau-function on Hurwitz spaces given in Sec.3.2 of [32] and Sec.5.1 of [36]. Namely, one can use equations (4.38) for the tau-functions $\tau_+(C, Q)$ and $\tau_+(C_0, Q_0)$ and the homogeneity properties of these two tau-functions. The bidifferential $B(x, y)$ on $C$ tends in the limit $t \to 0$ to the bidifferential $B_0(x, y)$ on $C_0$ due to our assumption about Torelli markings on $C$ and
Moreover, $Q \to Q_0$; thus also the abelian differential $w^+_v$ on $\hat{C}$ tends to the corresponding Abelian differential $(w^+_v)^0$ on $\hat{C}_0$ (as long as one stays away from the neighbourhood of the degenerating edge). From (4.38) we see that partial derivatives of $\ln \tau_+(C, Q)$ with respect to all homological coordinates except $t$ tend to the defining equations for $\ln \tau_+(C_0, Q_0)$. It follows that as $t \to 0$ the function $\tau_+(C, Q)$ behaves as $f(t)\tau_+(C_0, Q_0)$. Since $\tau_+$ is algebraic in moduli, $f(t) = t^\alpha(1 + o(1))$ for some rational degree $\alpha$, i.e.

$$\tau_+(C, Q) = \text{const} \times t^\alpha (1 + o(1))\tau_+(C_0, Q_0) .$$

(5.7) 

Under rescaling $Q \to \epsilon Q$, $Q_0 \to \epsilon Q_0$ we have $t \to \epsilon^{1/2}t$. According to (4.7) the difference of homogeneity coefficients of $\tau_+(C, Q)$ and $\tau_+(C_0, Q_0)$ equals $1/48(10/3 - 3) = 1/144$ and, therefore, $\alpha = 1/72$. Notice that this asymptotics agrees with Lemma 9 of [36] (in [36] one uses the 48th power of $\tau_+$).

Taking the argument of (5.5) we get rid of the real-valued factor which diverges at facets of $W$ and get the formula (5.6) which allows to continue $\arg \tau_+$ from one cell of $W$ to another by appropriate choice of Torelli marking and contours $\gamma_i$ inside of the cells and at their facets. 

A similar statement holds for the Prym tau-function $\tau_-$, which depends on the choice of symplectic basis in the space $H_-$.

While the cycles from the symplectic basis in $H_+$ can always be chosen to avoid the degenerating edge of $\Gamma$, this is not the case for $H_-$. We have $\dim H^0_- = \dim H_- - 2$. Under the choice of canonical basis in $H_-$ shown in Fig.4, in the limit $t \to 0$ the cycle $a^-_i$ disappears while $b^-_i$ becomes non-closed. The remaining elements of the symplectic basis of $H_-$ naturally provide a symplectic basis in $H^0_-$.

**Lemma 5.2** Assume that the symplectic basis in $H^0_-$ is induced by symplectic basis in $H_-$ as described above; assume also that the integration contours $\gamma_i^0$ on $C_0$ are induced by these contours on $C$. Then the asymptotics of the Prym tau-function $\tau_-(C, Q, t, \{\gamma_i\})$ in the limit $t \to 0$ is:

$$\tau_-(C, Q, t, \{\gamma_i\}) = \text{const} \times t^{13/72}(1 + o(1))\tau_-(C_0, Q_0, t_0, \{\gamma_i^0\})$$

(5.8) 

and

$$\arg \tau_-(C, Q, t, \{\gamma_i\}) = \arg \tau_-(C_0, Q_0, t_0, \{\gamma_i^0\}) + \text{const} + o(1)$$

(5.9) 

where the constants are independent of a point of the facet.

**Proof.** The proof is completely parallel to the proof of the previous lemma. Under our assumption about Torelli marking of $\hat{C}$ the Bergman bidifferential $\hat{B}(x, y)$ on $\hat{C}$ tends to the Bergman bidifferential $\hat{B}_0(x, y)$ on $\hat{C}_0$. Therefore, the abelian differential $w^-_v$ on $\hat{C}$ also tends to the abelian differential $(w^-_v)^0$ on $\hat{C}_0$; the same statement holds for all of periods of $w^-_v$ over canonical cycles surviving in the limit. Therefore, since contours $\gamma_i^0$ on $\hat{C}_0$ are obtained by continuous deformation of contours $\gamma_i$ (we assume that the initial point of these contours does not coincide with any of zeros coalescing in the limit $t \to 0$), equations (4.38) imply Thus, again we have

$$\tau_-(C, Q, t, \{\gamma_i\}) = \text{const} \times t^\alpha (1 + o(1))\tau_-(C_0, Q_0, t^0, \{\gamma_i^0\})$$

(5.10) 

for some value of $\alpha$. Again, this value can be computed looking at rescaling $Q \to \epsilon Q$, $Q_0 \to \epsilon Q_0$ using homogeneity coefficients (4.31) of both tau-functions (recall that then $t \to \epsilon^{1/2}t$). This gives $\alpha = \frac{1}{12} + \frac{1}{72}$ implying (5.8).

The constant in (5.9) can be different when approaching $F$ from $K_1$ or $K_2$. Therefore, from Lemmas 5.1 and 5.2 we obtain the following

**Proposition 5.3** Let $F$ be the facet between two maximal cells $K_1, K_2$ of $\mathcal{M}_{g,n}[p]$. Then
1. The arguments of $\tau_\pm$ have well-defined limit at any point of $F$ while approaching from the interior of either $K_1$ or $K_2$.

2. Suppose that in $K_{1,2}$ the Torelli markings $t_{1,2}$ of $C$ and $t_{1,2}$ of $\hat{C}$ are chosen such that all pairs of canonical cycles except a pair shown in Fig.4 remain outside of a neighbourhood of the vanishing edge and continue smoothly from $K_1$ to $K_2$. The remaining pair is assumed to transform as shown in Fig. 4. (In particular the Torelli marking $t_0$ of $C_0$ and $t_0$ of $\hat{C}_0$ is the same whether it is induced from $K_1$ or $K_2$). Assume also that that the contours $\gamma_1^0$ on $C_0$ are obtained from the contours $\gamma_i^{1/2}$ on $C$ by continuity (both from $K_1$ and $K_2$). Then the limits of $\Phi_\pm = \arg(\tau_\pm)$ as $t \to 0$ taken from $K_1$ and $K_2$ differ by a constant $\phi_\pm$ independent of a point of $F$.

5.3 Monodromy of $\Phi_+$ around $W_5$ and $W_{1,1}$

5.3.1 Variation of $\Phi_+$ under a pentagon move

The canonical basis of cycles in $C$ can always be chosen so that a neighbourhood of the two edges involved in the pentagon move lies inside the fundamental polygon; thus the Torelli marking of $C$ remains invariant under the move.

Proposition 5.4 Under a positively oriented pentagon move (Prop. 2.12, Fig. 7), the increment of the continuous extension of $\Phi_+ = \arg \tau_+$ equals $\pi/72$.

Proof. First, notice that the variation of $\Phi_+ = \arg \tau_+$ along a closed path depends only on the free homotopy class of the path.

Therefore, to compute the variation of $\arg \tau_+$ under the pentagon move involving three selected three-valent vertices (denote these vertices by $x_1, x_2$ and $x_3$) we assume that the lengths of both edges connecting $x_2$ with $x_1$ and $x_3$ are small in comparison with lengths of the other edges, and, moreover, that these lengths remain small during the whole pentagon move.

Denote the lengths of the edges connecting the three selected zeros by $tA$ and $tB$ where $A + B = 1$, $A, B \in [0, 1]$ and $t$ is a small parameter which is kept fixed during the pentagon move. The three zeros $x_i$ are close to each other in the metric $|Q|$ and represent a small resolution of the triple zero. Let us rescale the flat coordinate $z(x) = t(A + B) + \int_{x_2}^{x} v$ in a neighbourhood of these three zeros by a factor of $1/t$ (here $x_2$ is the "central zero"; see Section 3.1 of [6] for justification of the choice of constant when defining $z(x)$). Then the surface $C$ can be represented via a plumbing construction as follows. Let $C_1$ be the surface with the non-resolved zero $\hat{x}$ of order 3 (five valent vertex of the corresponding ribbon graph) of the JS differential $Q_1$ such that the perimeters and all lengths of all edges except the edges connected to zeros $x_{1,2,3}$ are the same on $C$ as on $C_1$. Let $C_0$ be the Riemann sphere equipped with the quadratic differential of the form (4.39):

$$Q_0 = (x - x_1)(x - x_2)(x - x_3)(dx)^2.$$  

As in Section 4.6.1, we assume that $x_1 + x_2 + x_3 = 0$; the integrals of $\sqrt{Q_0}$ between $x_2$ and $x_1$ and $x_3$ are assumed now to be real and equal to $\pm A$ and $\pm B$. The flat coordinate on $C_0$ is given by the integral $w(x) = 1 + \int_{x_2}^{x} \sqrt{(x - x_1)(x - x_2)(x - x_3)} dx$. The plumbing of the Riemann surfaces $C_0$ and $C_1$ is defined by the equation (see Section 3.2 of [6])

$$\xi(x)\zeta(y) = t^{4/5}$$  

where $\xi(x)$ is the distinguished local coordinate near $x = \infty$ on $C_0$ given by $\xi(x) = \left[\frac{2}{5} w(x)\right]^{-2/5}$, $\zeta(y) = \left(\frac{5}{2} z(y)\right)^{2/5}$ is the distinguished local coordinate near the triple pole $\hat{x}$ of $C_1$; $z(x) = \int_{x}^{x_1} v_1$ is the flat coordinate on $C_1$ near $\hat{x}$.

The gluing of $C_0$ and $C_1$ is illustrated in Fig. 15. As $t \to 0$ one gets a union of Riemann sphere equipped with quadratic differential of the form (5.11), and the Riemann surface $C_1$ which belongs to the Witten’s stratum $W_5$.

The quadratic differentials of the form (5.11) on the Riemann sphere have three simple zeros and one pole of order 7; such differentials form the space $Q_0([1]^3, -7)$ discussed in Sec. 4.6.1.
Figure 15: Separating of the Riemann sphere with three simple zeros and a pole of order 7 of Q in a neighbourhood of $W_5$.

Denote a Torelli marking of $C$ by $t$; it naturally induces a Torelli marking $t_1$ on $C_1$ in the limit $t \to 0$ by continuous deformation of the canonical basis of cycles. Define a system of cuts $\{\gamma_i\}_{i=1}^n$ on $C$ choosing $x_1$ as an initial point. In the limit $t \to 0$ one gets a system of cuts $\{\gamma_i^1\}_{i=1}^n$ on $C_1$ which connect the triple zero $x_1$ with $n$ poles of $Q_1$ (one also gets cuts on $C_0$ connecting the point at infinity with $x_{1,2,3}$ but these cuts are inessential for our purposes).

Denote the resulting tau-function $\tau_+$ on the Witten’s stratum $W_5$ by $\tau_+(C_1, Q_1, t_1, \{\gamma_i^1\}_{i=1}^n)$ and on the space $Q_0([1]^3, -7)$ by $\tau_+(C_0, Q_0)$. The tau-functions $\tau_+(C_0, Q_0)$ is given by (4.44).

**Lemma 5.5** Assuming that the Torelli marking $t_1$ and the cuts $\{\gamma_i^1\}$ on $C_1$ are induced from $C$ in the limit $t \to 0$ as described above the following asymptotics holds:

$$\tau_+(C, Q, t, \{\gamma_i\}_{i=1}^n) = \text{const} \times (1 + o(1)) \tau_+(C_0, Q_0) \tau_+(C_1, Q_1, t_1, \{\gamma_i^1\}_{i=1}^n).$$

(5.13)

**Proof.** In the limit $t \to 0$ we have

$$\frac{\tau_+(C, Q)}{\tau_+(C_0, Q_0) \tau_+(C_1, Q_1)} = f(t)(1 + o(1))$$

(5.14)

where $f$ is some function independent of the moduli of $(C_0, Q_0)$ and $(C_1, Q_1)$. Similarly to the proof of Lemma 5.1 the asymptotics (5.14) follows from equations (4.38) for three tau-functions from the r.h.s. of (5.14) and the behaviour of $B(x, y)$ in the limit $t \to 0$. Namely, our assumption about the relationship between Torelli markings $t$ and $t_1$ implies that $B(x, y)$ tends to $B_0(x, y)$ or $B_1(x, y)$ if both points $x$ and $y$ remain in the limit $t \to 0$ on $C_0$ or $C_1$, respectively, see [16], Cor. 3.2. In turn, this implies that the meromorphic differential $w_0^+$ (4.37) on $C$ tends to the corresponding meromorphic differentials on $C_1$ and $C_0$. Using our assumptions about cuts $\gamma_i$ we conclude than that equations (4.38) for $\tau_+(C, Q)$ with respect to all homological coordinates except $t$ tend in the limit $t \to 0$ to the equations for the product $\tau_+(C_0, Q_0)\tau_+(C_1, Q_1)$ which implies (5.14).
To show that function $f$ is a constant up to terms vanishing in the limit $t \to 0$ (this fact will not be used here since we are interested only in the phase of tau-functions) one makes use of homogeneity property of the tau-functions $\tau_+$ on spaces $W$, $W_5$ and $Q_0([1]^3, -7)$. Namely, the homogeneity coefficient (4.7) of the function $\tau_+(C)$ equals $\frac{5}{3 \times 18} (4g - 4 + 2n)$ which exactly coincides with the sum of homogeneity coefficients of $\tau_+(C_0)$ and $\tau_+(C_1)$. Thus the ratio (5.14) tends to a constant as $t \to 0$.

Returning to the proof of Prop. 5.4, we observe that the tau-function $\tau_+(C_1, Q_1, t_1, \{\gamma_i\}_{i=1}^n)$ does not change under the pentagon move, and, therefore, the variation of its argument is only due to the variation of the argument of

$$\tau_+(C_0, Q_0) = [(x_1 - x_2)(x_2 - x_3)(x_3 - x_1)]^{1/36}$$

which equals $\pi/72$ according to Th. 2 of [6] (it is an elementary although technically non-trivial result).

5.3.2 Variation of $\Phi_+$ under combinatorial Dehn’s twist

To compute the variation of the argument of $\tau_+(C, Q, t, \{\gamma_i\}_{i=1}^n)$ under the combinatorial Dehn’s twist defined in Section 2.2.2 we assume that the Torelli marking $t$ is chosen such that the Lagrangian subspace of $a$-cycles remains invariant under the twist. Namely, if the twist goes around $W'_{1,1}$ (i.e. corresponding loop separates $C$ into two components) the Torelli marking $t$ of $C$ is obtained by taking the union of Torelli markings of these components. If the Dehn’s twist is performed around $W'_{1,1}$ i.e. along a non-separating loop then one of $a$-cycles should be chosen to follow this loop while all other $a$-cycles remain outside of the zone of the twist.

In the rest of this section we prove the following proposition.

Proposition 5.6 Let $e \cup \bar{e}$ be a closed loop on $C$ formed by the edges $e$ and $\bar{e}$ as in Prop. 2.15. Let the Torelli marking $t$ of $C$ be chosen as discussed above. Then the variation of the argument of the tau-function $\tau_+(C, Q, t, \{\gamma_i\}_{i=1}^n)$ under the combinatorial Dehn’s twist along $e \cup \bar{e}$ equals $13\pi/72$.

Proof. Under our assumption about the choice of Torelli marking on $C$ the Lagrangian subspace of $a$-cycles remains invariant after the combinatorial Dehn’s twist; thus the variation of $\arg \tau_+$ under the twist is independent of the lengths of edges within a given cell of $W$. Therefore, we can assume that the lengths of the edges $e$ and $\bar{e}$ are ”small” i.e. we denote them by $tA$ and $tB$ with $A + B = 1$ and $A, B, t \in \mathbb{R}_+$ and compute the variation of $\arg \tau_+$ in the limit $t \to 0$.

Consider the twists around reducible and irreducible components of Kontsevich’s boundary separately.

1. Combinatorial Dehn’s twist around $W'_{1,1}$.

In the limit $t \to 0$ when the lengths of both edges tend to $0$ while the lengths of other edges are adjusted accordingly to preserve the perimeters of all faces, $C$ splits into two Riemann surfaces, $C_1$ (of genus $g_1$) and $C_2$ (of genus $g_2$). Corresponding JS differentials We denote by $Q_{1,2}$ the corresponding JS differential, by $n_{1,2}$ the number of poles and by $\Gamma_{1,2}$ the ribbon graphs. The differentials $Q_{1,2}$ have simple poles at the points of resolution of the node and the ribbon graphs $\Gamma_{1,2}$ have one-valent vertices at these points.

According to our assumption the Torelli marking of $C$ is chosen in such a way that in the limit $t \to 0$ it gives rise to Torelli markings $t_{1,2}$ of $C_{1,2}$.

The ”first” zero $x_1$ and the cuts $\gamma_i$ are chosen as follows: the ”first” zero $x_1$ is chosen to be one of the zeros involved in the Dehn’s twist. Starting from this zero, we choose $n_1$ cuts $\{\gamma_i\}_{i=1}^{n_1}$ going towards the poles which remain on the $C_1$ side, and $n_2$ cuts $\{\gamma_i\}_{i=n_1+1}^{n_2}$ going towards the poles on $C_2$ side. Such a system of cuts naturally splits under degeneration into two systems: $\{\gamma_i\}_{i=1}^{n_1}$ on $C_1$ and $\{\gamma_i\}_{i=1}^{n_2}$ on $C_2$. The initial point of integration then becomes the one-valent vertex of the corresponding ribbon graphs $\Gamma_{1,2}$.

Introduce now the tau-functions $\tau_+(C, Q, t_1, \{\gamma_i\}_{i=1}^n)$, $\tau_+(C_1, Q_1, t_1, \{\gamma_i\}_{i=1}^n)$ and $\tau_+(C_2, Q_2, t_2, \{\gamma_i\}_{i=1}^n)$. 45
Consider also the following quadratic differential on the Riemann sphere $C_0$:

$$Q_0(x) = \frac{(x - x_1)(x - x_2)(dx)^2}{x^3}$$  \hfill (5.16)

The differential $Q_0$ has third order poles at the points $x = 0$ and $x = \infty$. The homological coordinates of $Q_0$ are equal to $\pm A$ and $\pm B$.

The differential (5.16) with real homological coordinates defines a point of the space $Q_0^{[1,1,-3,-3]}$ and the corresponding tau-function $\tau_+(C \mathbb{P}^1, Q_0)$ is given by (4.57).

The Riemann surface $C$ equipped with the differential $Q$ can be obtained via a “plumbing” construction by inserting the Riemann sphere equipped with the differential $Q_0$ between Riemann surfaces $(C_1, Q_1)$ and $(C_2, Q_2)$ (see for details [6]). A neighbourhood of the point $x = 0$ of the Riemann sphere is “plumbed” with a neighbourhood of the one-valent vertex (i.e. simple pole of $Q_1$) on $C_1$ while a neighbourhood of the point $x = \infty$ of the Riemann sphere is “plumbed” with a neighbourhood of the one-valent vertex (i.e. simple pole of $Q_2$) on $C_2$. This procedure is illustrated in Fig. 16.

The following analog of Lemma 5.5 holds in this case:

**Lemma 5.7** Assuming that the Torelli marking on $C_{1,2}$ and the cuts $\gamma_i^{(1,2)}$ are induced from $C$ in the limit $t \to 0$ the following asymptotics holds:

$$\tau_+(C, Q, t, \{\gamma_i\}) = \text{const} \times (1 + o(1)) \tau_+([bfCP^1, Q_0]) \tau_+(C_1, Q_1, t_1, \{\gamma_{i_1}^1\}) \tau_+(C_2, Q_2, t_2, \{\gamma_{i_2}^2\})$$  \hfill (5.17)

**Proof.** As in the proof of Lemma 5.5, consider the expression

$$\frac{\tau_+(C, Q, t, \{\gamma_i\})}{\tau_+([bfCP^1, Q_0]) \tau_+(C_1, Q_1, t_1, \{\gamma_{i_1}^1\}) \tau_+(C_2, Q_2, t_2, \{\gamma_{i_2}^2\})}.$$  \hfill (5.18)
Under our assumptions about Torelli markings and cuts \( \gamma_i \) on \( C \) and \( C_{1,2} \) the expression (5.18) behaves as \( f(t)(1 + o(1)) \) as \( t \to 0 \). This follows again from the asymptotics of the bidifferential \( B(x, y) \) as \( t \to 0 \) which in particular implies (under our choice of Torelli markings on \( C \) and \( C_{1,2} \)) that the Bergman projective connection \( S_B \) on \( C \) tends to the Bergman projective connections at the corresponding points of \( C_1, C_2 \) and \( C_0 \) (and also the fact that the quadratic differential \( Q \) tends to \( Q_{1,2} \) and \( Q_0 \) in the limit \( t \to 0 \) on corresponding components).

Using (4.7) we find that the expression (5.18) remains invariant under simultaneous rescaling of \( Q \) and \( Q_{0,1,2} \).

Therefore, \( f = \text{const} \).

In the limit \( t \to 0 \) the Whitehead move is performed entirely on the Riemann sphere equipped with \( Q_0 \); under this Whitehead move only the first term in the r.h.s. of (5.13) contributes to monodromy of \( \arg \tau_+ \). The monodromy of (4.57) under the Whitehead’s move was computed in Theorem 4 of [6]; it equals \( 13\pi/72 \) implying the statement of the proposition in the reducible case.

2. Combinatorial Dehn’s twist around the irreducible component \( W_{irr,1} \). In this case the edges, \( e_1 \) and \( e_2 \) connecting two vertices of the ribbon graph bound a loop which does not separate \( C \) into two components.

As stated before Prop.5.6, we choose the Torelli marking \( t \) such that the Lagrangian subspace of \( a \)-cycles remains invariant under the move. Namely, the “first” \( a \)-cycle \( a_1 \) goes along the loop formed by the edges \( (e_1, e_2) \). The cycle \( b_1 \) then goes “along” the corresponding cylinder (Fig. 17). The other cycles forming the canonical basis can be placed far from the zone of the Dehn’s twist. Then under the combinatorial Dehn’s twist shown in Fig. 17 the pair of cycles \( (a_1, b_1) \) transforms as

\[
(a_1, b_1) \rightarrow (a_1, b_1 - a_1)
\]

(5.19)

while all other cycles do not vary.

Denoting the lengths of the edges \( e_1 \) and \( e_2 \) by \( A \) and \( B \), we compute the variation of \( \arg \tau_+ \) as \( t \to 0 \). For that we assume that the basic cycles on \( C \) are chosen as discussed above and that the basic cycles on \( C \) (except \( (a_1, b_1) \)) are used to Torelli mark the Riemann surface \( C_1 \) of genus \( g - 1 \). Moreover, we assume that the cuts \( \gamma_i \) are chosen on \( C \) to start from, say, zero \( x_1 \) and all go from \( x_1 \) to the “right” inducing in the limit \( t \to 0 \) the system of cuts \( \{\gamma_i\}_{i=1}^n \) (then on \( C_1 \) they start at one of simple poles of \( Q_1 \)).

The pair \( (C, Q) \) can be then restored via plumbing construction by gluing the Riemann surface \( (C_1, Q_1) \) to the Riemann sphere equipped with the differential \( Q_0 \). The plumbing zones are placed on \( C_1 \) near the simple poles of \( Q_1 \) and on the Riemann sphere near the points \( x = 0 \) and \( x = \infty \) (see [6]).

The following lemma is and analog of lemmas 5.5, 5.7 and is proved in the same way.

**Lemma 5.8** Assuming that the Torelli marking and the cuts \( \{\gamma_i\}_{i=1}^n \) on \( C_1 \) are induced from cuts \( \{\gamma_i\}_{i=1}^n \) on \( C \) in the limit \( t \to 0 \) the following asymptotics holds:

\[
\tau_+(C, Q, t, \{\gamma_i\}) = \text{const} \times (1 + o(1)) \tau_+(Q_0) \tau_+(C_1, Q_1, t, 1, \{\gamma_i\})
\]

(5.20)

The variation of the argument of \( \tau_+(C, Q, t, \{\gamma_i\}) \) under the Dehn’s twist is again only due to variation of the argument of \( \tau_+(bfCP^1, Q_0) \) which gives \( 13\pi/72 \) as before. This ends the proof of Proposition 5.6.

\[\blacksquare\]
5.4 Monodromy of $\Phi_t$ around $W_5$ and $W_{1,1}$

When computing the monodromy of $\arg\tau_-$ around $W_5$ and $W_{1,1}$ a pair of cycles forming a canonical basis in $H_-(\hat{C})$ always passes through the zone of the corresponding move. Therefore, when computing the monodromy of $\arg\tau_-$ with respect to these moves one needs to take into account the contribution from the change of Torelli marking $t_-$. Let us call this pair of cycles $(a^-, b^-)$.

Under an elementary Whitehead move the cycle $a^-$ vanishes at the facet between two cells of $W$ and then reopens again. This requirement does not uniquely define the propagation of Torelli marking $t_-$ from one cell to another; Fig 4 gives just one of many possible ways of such propagation.

After a pentagon move or a combinatorial Dehn’s twist the Torelli marking in $t_-$ does not necessarily return back to the original one, which leads to an extra contribution to variation of $\arg\tau_-$ in comparison with $\tau_+$-case.

5.4.1 Variation of $\Phi_t$ under pentagon move

Assuming that the number of zeros of $Q$ is greater than 3 (the remaining case of $g = 1$ and $n = 1$ can be treated similarly) we choose the initial point $x_1$ of cuts $\{\gamma_i\}_{i=1}^n$ to not participate in the pentagon move; the cuts themselves are assumed to stay away from the zone of the move. Then the cuts $\{\gamma_i\}_{i=1}^n$ naturally propagate from cell to cell, and return to their initial positions after the pentagon move.

Indeed, if the evolution of cycles $(a_i^-, b_i^-)$ under five Whitehead moves is chosen as shown in Fig. 7 (in particular, we choose three branch cuts outgoing from the zone of the pentagon move as in Fig. 7; tilde marks the edge on which we perform the Whitehead move on each step) then the pair of cycles $(a_1^-, b_1^-)$ evolves to $(\tilde{a}_1^-, \tilde{b}_1^-)$ such that

$$\tilde{a}^- = a^- \quad \tilde{b}^- = b^- - a^-$$

(5.21)

Although the Torelli marking $t_-$ has monodromy under the pentagon move, this monodromy does not impact the $\arg\tau_-$ since the Lagrangian subspace of $a$-cycles in $H_-$ (and, therefore, the Prym bidifferential $B_-$) are invariant under such symplectic transformations.

**Proposition 5.9** Let the pentagon move induce a symplectic transformation in $H_-$ given by a matrix $\sigma_- = \begin{pmatrix} C_- & D_- \\ A_- & B_- \end{pmatrix} \in Sp(6g-6+2n, \mathbb{Z})$. Denote the Prym matrix of the canonical cover by $\Pi$. Then the variation of $\arg\tau_-$ under the pentagon move is given by:

$$\text{var} \arg\tau_- = \frac{13}{72} \pi + \arg \det (C_- \Pi + D_-) \ .$$

(5.22)

**Proof.** It is sufficient to prove (5.22) choosing the canonical basis $t_-$ in $H_-$ as described above; all canonical pairs $(a^-_i, b^-_i)$ except $(a^-_1, b^-_1)$ stay outside of the zone of the pentagon move, while the pair $(a^-_1, b^-_1)$ evolves as shown in Fig. 7 according to (5.21). Then the second term in the r.h.s. of (5.22) is absent and the variation of $\arg\tau_-(C, \{\gamma_i\})$ should be proven to equal $13\pi/72$.

To compute this variation we follow the logic of the proof of Prop.5.4. Namely, we represent the canonical cover $\hat{C}$ of the Riemann surface $C$ by plumbing of the canonical cover $\hat{C}_1$ of $C_1$ and the canonical cover $\hat{C}_0$ of the Riemann sphere equipped with quadratic differential $Q_0$ (5.11) ($Q_0$ is an element of the space $Q_0([1]^3, -7)$). In the limit as the plumbing parameter $t$ tends to 0 the cover $\hat{C}$ decomposes in the disjoint union of $\hat{C}_0$ and $\hat{C}_1$. The canonical bases $t^-_0$ in $H_- (\hat{C}_0)$ and $t^-_1$ in $H_- (\hat{C}_1)$ are naturally induced from the canonical base $t^-$ in $H_- (\hat{C})$ chosen according to the above convention.

In the limit $t \to 0$ the tau-function $\tau_-(C, Q, t^-)$ decomposes as

$$\tau_-(C, Q, t^-, \{\gamma_i\}) = \text{const} \times (1 + o(1)) \tau_-(C_0, Q_0, t^-_0) \tau_-(C_1, Q_1, t^-_1, \{\gamma_i\})$$

(5.23)
Similarly to the proof of Lemma 5.5, the proof of the asymptotics (5.23) is based on the fact that our choice of Torelli markings \( t^- \) and \( t_{0,1}^- \) implies that the Prym projective connection \( S_{\tilde{C}}(x) \) on \( \tilde{C} \) tends to the Prym projective connections \( S_{\tilde{C}_{0,1}} \) if the argument \( x \) remains in the limit on \( \tilde{C}_0 \) or \( \tilde{C}_1 \), respectively. Therefore, the same statement applies to the meromorphic differentials \( w^-_v \) (4.37) which enter the equations (4.38) for \( \tau_- \). The constant factor \( const \times (1 + o(1)) \) is obtained by comparing the homogeneity properties of the three tau-functions entering (5.23) (this coefficient equals \( \frac{5}{3 \times 48} (4g - 4 + 2n) \) on both sides).

The tau-function \( \tau_- (C, Q, t^-) \) is invariant under the pentagon move as \( t \to 0 \) i.e. in this limit the move is localized to \( C_0 \) part and the variation of \( arg \tau_- (C, Q, t^-, \gamma_i) \) is equal to the variation of the argument of \( \tau_- (C_0, Q_0, t_0^-) \).

Therefore, under the pentagon move around \( W_5 \) the variation of \( arg \tau_- (C, Q, t^-) \) is given by the sum of variation of the argument of \( [(x_1 - x_2)(x_1 - x_3)(x_2 - x_3)]^{13/36} \) and variation of the argument of the abelian integral \( \omega_1 \) over cycle \( a^-_1 \). According to Th.2 of [6] the variation of \( arg\left(\frac{1}{(x_1 - x_2)(x_1 - x_3)(x_2 - x_3)}\right) \) under the pentagon move equals \( 13/72 \pi \). On the other hand, the variation of \( arg \omega_1 \) is zero since the cycle \( a^-_1 \) transforms to itself.

The second term in (5.22) is non-trivial if \( t^- \) on the initial cell is chosen differently, or if Torelli’s marking suitable for description of the pentagon move on one set of three vertices is propagated along pentagon move on another set of three vertices (i.e. around another cell of \( W_5 \)).

### 5.4.2 Variation of \( \Phi_- \) under combinatorial Dehn’s twist

To study the variation of \( arg \tau_- (C, Q, t^-, \gamma_i) \) it is again convenient to choose a basis \( t^- \) in \( H_- (\tilde{C}) \) and the set of cuts \( \{\gamma_i\} \) which transform in the "best" way under the combinatorial Dehn’s twist both around reducible and irreducible components of \( W_{1,1} \). As before, we assume that all pairs of canonical cycles in \( H_- \) stay away from the twist zone. Two selected cycles, which we call \( a^-_1 \) and \( b^-_1 \) are chosen as shown in Fig.11.

The zone of the Dehn’s twist is also shown as an annulus in Fig.12. Under the action of the Dehn’s twist the cycles \((a^-, b^-)\) transform as follows:

\[
\tilde{a}^- = a^- - 2a^- \\
\tilde{b}^- = b^- + 2a^- \tag{5.25}
\]

therefore, under this choice of the basis in \( H_- (\tilde{C}) \) the Lagrangian subspace formed by \( a^- \)-cycles remains invariant under the Dehn’s twist.

**Proposition 5.10** Let the combinatorial Dehn twist induce a symplectic transformation given by a matrix \( \sigma_\pm = \left(\begin{array}{cc} C & D \\ A & B \end{array}\right) \in Sp(6g - 6 + 2n, \mathbb{Z}) \) in the space \( H_- \); denote the Prym matrix by \( \Pi \). Then the variation of \( arg \tau_- \) under the Dehn twist is given by:

\[
\text{var arg } \tau_- = \frac{25}{72} \pi + \text{arg det } (C_\pm II + D_-) \tag{5.26}
\]

**Proof.** The proof is parallel to the proof of the corresponding statement for \( \tau_+ \). Namely, in both reducible and irreducible case one can “localize” the Dehn’s twist.

In the irreducible case we represent the canonical cover \( \tilde{C} \) by plumbing of the canonical cover \( \tilde{C}_1 \) of the Riemann surface \( C_1 \) of genus \( g - 1 \) with the elliptic curve \( \tilde{C}_0 \) which is a canonical cover of the Riemann sphere with branch points at \( 0, \infty, x_1, x_2 \). Denoting the real plumbing parameter by \( t \), and the canonical bases in \( H_- (\tilde{C}_{0,1}) \) inherited from \( H_- (\tilde{C}) \) by \( t_{0,1} \), we have the asymptotics as \( t \to 0 \):

\[
\tau_- (C, Q, t, \gamma_i) = \text{const} \times (1 + o(1)) \tau_- (C_1, Q_1, t^-_1, \gamma_i) \tau_- (C_0, Q_0, t^-_0) \tag{5.27}
\]
It is assumed as before that the cuts \( \{ \gamma_i^1 \} \) are induced in \( C_1 \) from cuts \( \{ \gamma_i \} \) on \( C \) in the limit \( t \to 0 \). In the limit \( t \to 0 \) the function \( \tau_-(C_1, Q_1, t_1^1) \) does not change under the Dehn’s twist.

The function \( \tau_-(C_0, Q_0, t_0^0) \) is given explicitly by (4.64):

\[
\tau_- = \omega_1 (x_1 x_2)^{1/12} (x_1 - x_2)^{13/36}
\]  

(5.28)

where \( \omega_1 \) is the elliptic integral of the first kind over cycle \( \alpha^{-} \). According to our assumption about transformation of the \( a \)-cycle under the Dehn’s twist the integral \( \omega_1 \) remains invariant. The variation of the argument of the remaining explicit expression equals \( \frac{2\pi}{12} \) according to Theorem 4 of [6]. This gives (5.26) in the irreducible case.

The case of reducible component \( W_{1,1} \), when the Riemann surface \( C \) is separated into the union of two Riemann surfaces, \( C_1 \) and \( C_2 \), with the Riemann sphere equipped with a differential of the form (4.54) is glued between them, is treated similarly. Namely, he canonical cover \( \hat{C} \) can be obtained via plumbing of \( \hat{C}_0 \) with both \( \hat{C}_1 \) and \( \hat{C}_2 \). Then, under the usual assumption that the set of contours \( \{ \gamma_i^1 \} \) on \( \hat{C}_1 \) and \( \hat{C}_2 \) is inherited from \( \hat{C} \) and that the Torelli markings in \( H_-(\hat{C}_1), H_-(\hat{C}_2) \) and \( H_-(\hat{C}_0) \) are inherited from \( H_-(\hat{C}) \), we get the asymptotics as \( t \to 0 \):

\[
\tau_-(C, Q, t_-, \{ \gamma_i^1 \}) = \text{const} \times (1 + o(1)) \tau_-(C_1, Q_1, t_1^1, \{ \gamma_i^1 \}) \tau_-(C_1, Q_1, t_1^1, \{ \gamma_i^2 \}) \tau_-(C_0, Q_0, t_0^0)
\]  

(5.29)

where the last term is the tau-function (5.28) and the other two terms do not change under the Dehn’s twist. This leads to (5.26) in the irreducible case.

Again, the second term in (5.26) appears if we either start from a different Torelli marking in \( H_-(\hat{C}) \) or propagate the given Torelli marking along a different Dehn’s twist. 

\[\blacksquare\]

### 5.5 Expressing Hodge and Prym classes via \( W_5 \) and \( W_{1,1} \)

The Prym vector bundle \( \Lambda_P \) over \( \mathcal{M}_{g,n}[p] \) is defined by restriction of the Prym bundle over \( \mathcal{Q}_{g,n}[p] \) to \( \mathcal{M}_{g,n}[p] \), as in Section 4.4.

The theorem 4.16 is applicable also in the context of \( \mathcal{M}_{g,n}[p] \) and provides the isomorphism between the Prym vector bundle \( \Lambda_P \) with the vector bundle \( \Lambda_2^{(n)} \) of quadratic differentials with simple poles at the punctures, therefore the determinant line bundle \( \det \Lambda_P \) is the isomorphic to the determinant line bundle \( \det \Lambda_2^{(n)} \). We denote by \( \lambda_P \) the corresponding class: \( \lambda_P = c_1(\det \Lambda_P) \).

According to Proposition 4.17 the tau functions \( \tau^{48}_L \) are sections of the line bundles \( \lambda^{48} \otimes \prod_{i=1}^n \mathcal{L}_i^4 \) and \( \lambda_2^{48} \otimes \prod_{i=1}^n \mathcal{L}_i^4 \); furthermore, according to the discussion in Section 5.2 their phases provide sections of the corresponding circle bundles over the combinatorial model \( \mathcal{M}_{g,n}[p] \). Therefore, from the computations of the increment of \( \arg \tau_L \) around \( W_5 \) and \( W_{1,1} \) in Propositions 5.4, 5.6, 5.9, 5.10, we get the following relations between the first Chern classes of the determinants of the Hodge and Prym vector bundles (\( \lambda \) and \( \lambda_P \), respectively), \( \psi \)-classes, Witten’s cycle \( W_5 \) and Kontsevich’s boundary \( W_{1,1} \) of the combinatorial model:

**Theorem 5.11** The following relations hold in \( \text{Pic}(\mathcal{M}_{g,n}[p], Q) \):

\[
\lambda + \frac{1}{12} \sum \psi_i = \frac{1}{144} W_5 + \frac{13}{144} W_{1,1}
\]  

(5.30)

\[
\lambda_P + \frac{1}{12} \sum \psi_i = \frac{13}{144} W_5 + \frac{25}{144} W_{1,1}
\]  

(5.31)

The formula (5.30) is an analog of formulas for the \( \lambda \)-class derived in various complex-analytic frameworks: Hurwitz spaces [32], spaces of Abelian [35] and quadratic [36] differentials. The formula (5.31) is an analog of the expression of Prym class on spaces of quadratic differentials [36, 34].
5.6 The $\kappa_1$ circle bundle $S[\chi_\kappa]$ and $\tau_\pm$

Recall that the Mumford-Morita-Miller class $\kappa_1$ can be expressed as follows (1.4):

$$\kappa_1 = \lambda_2^{(n)} - \lambda_1$$  \hspace{1cm} (5.32)

Using (5.32) and taking into account that $\lambda_2^{(n)} = \lambda_P$ we have the following corollary of (5.30), (5.31):

**Corollary 5.12** The following formula holds in $\text{Pic}(\overline{M}_{g,n}[p], \mathbb{Q})$:

$$12\kappa_1 = W_5 + W_{1,1}$$  \hspace{1cm} (5.33)

The formula (5.33) was originally derived in [2] and then reproved in [44]. This paper gives an alternative proof of fact.

Observe now that $\kappa_1 = c_1(\chi_\kappa)$ where the line bundle $\chi_\kappa$ over $\mathcal{M}_{g,n}$ is defined by

$$\chi_\kappa = \frac{\det \Lambda_2^{(n)}}{\det \Lambda_H}$$  \hspace{1cm} (5.34)

As before, given any line bundle $\chi$ we denote the associated circle bundle by $S[\chi]$. As another corollary of (5.30), (5.31) we get

**Corollary 5.13** A section of the circle bundle $S[(\chi_\kappa)^{48}]$ over $\overline{M}_{g,n}[p]$ is given by $\frac{\Theta_-}{\Theta_+}$ where $\Theta_{\pm} = \left( \frac{\tau_{\pm}}{\tau_{\pm}} \right)^{48}$.

To get another corollary of formulas (5.30) and (5.31) one can eliminate $W_5$. Then we get the formula (using that $\lambda_P = \lambda_2^{(n)}$):

$$\lambda_2 - 13\lambda_1 = \sum_{i=1}^{n} \psi_i - W_{1,1}$$  \hspace{1cm} (5.35)

valid in $\overline{M}_{g,n}[p]$. This formula is the combinatorial analog of the Mumford’s formula (1.2) which is valid in the Deligne-Mumford compactification of $\mathcal{M}_{g,n}$.

5.7 Open problems

After the pioneering work [39] the approach to the study of of various tautological classes and their intersection numbers via the JS combinatorial model remained under-developed in spite of progress summarized in [45].

We mention only few natural questions which remain to be answered:

1. What is the combinatorial cycle which is Poincaré dual to the psi-class $\psi_i$? This seems to be a very elementary question whose answer we were not able to find in the literature.

2. How to complete the Kontsevich’s boundary $W_{1,1}$ of the JS combinatorial model to get the one-to-one correspondence with the Deligne-Mumford compactification $\overline{M}_{g,n}$ of the moduli space?

3. How to express all $\lambda$-classes directly via cycles of the JS combinatorial model without relying on relationship with $\kappa$-classes and expressions for $\kappa$-classes obtained in [2, 18, 44]?

4. How to get a self-contained proof, based on “flat” combinatorial model, of the theorem of [29] about generating function of linear Hodge integrals and KP hierarchy, without the use of inversion of the ELSV formula for Hurwitz numbers? Furthermore, intersection numbers of an arbitrary set of tautological classes ($\kappa$, $\lambda$- and $\psi$-) should admit a complete description based on the combinatorial model in the spirit of [39].
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References

[1] W. Abikoff, *The real analytic theory of Teichmüller space*, Lecture Notes in Math. 820 Springer, Berlin, 144 p. (1980)

[2] E. Arbarello and M. Cornalba, *Combinatorial and algebro-geometric cohomology classes on the moduli spaces of curves*, J. Algebraic Geom. 5 (1996), no. 4, 705-749.

[3] E. Arbarello, M. Cornalba, P. Griffiths, *Geometry of Algebraic Curves*, Grundlehren der mathematischen Wissenschaften, 268 vol.2, 963 pp., Springer (2011)

[4] M. Basok, *Tau function and moduli of spin curves*, Int. Math. Res. Not. 2015 no. 20, 10095-10117

[5] M. Bertola, D. Korotkin, C. Norton, *Symplectic geometry of the moduli space of projective structures in homological coordinates*, Invent.Math. 210, Issue 3, p. 759-814 (2017)

[6] M. Bertola, D. Korotkin, *Discriminant circle bundles over local models of Strebel graphs and Boutroux curves*, Theoretical and Mathematical Physics, to appear

[7] F. Bottacin, *Symplectic geometry on moduli spaces of stable pairs*, Ann. Sci. Ecole Norm. Sup. (4) 28 no. 4, 391-433 (1995)

[8] L. Chekhov, V. Fock, *Quantum Teichmüller spaces*, Theoret. and Math. Phys. 120 (1999), no. 3, 1245-1259, arXiv:math/9908165

[9] M. Cornalba, J. Harris, *Divisor classes associated to families of stable varieties, with applications to the moduli space of curves*, Ann. Sci. Ec. Norm. Super. (4) 21 455-475 (1988)

[10] B. Dubrovin, *Painlevé transcendentals in two-dimensional topological field theory, The Painlevé property*, 287-412, CRM Ser. Math. Phys., Springer, New York (1999)

[11] T. Ekedahl, S. Lando, M. Shapiro, A. Vainshtein, *Hurwitz numbers and intersections on moduli spaces of curves*, Invent.Math., 146 297-327 (2001)

[12] A. Eskin, M. Kontsevich, A. Zorich, *Sum of Lyapunov exponents of the Hodge bundle with respect to the Teichmüller geodesic flow*, Publ. Math. Inst. Hautes tudes Sci. 120 207-333 (2014)

[13] B. Eynard, A. Kokotov and D. Korotkin, “*Genus one contribution to free energy in Hermitian two-matrix model*”, Nucl. Phys. B694 443-472 (2004)

[14] B. Farb, D. Margalit, *A primer on mapping class groups*, Princeton University Press 472 pp (2002)
[15] G. Farkas, A. Verra. The geometry of the moduli space of odd spin curves, Ann. of Math. 180 no. 3, 927-970 (2014)
[16] John D. Fay. Theta functions on Riemann surfaces. Lecture Notes in Mathematics, Vol. 352. Springer-Verlag, Berlin-New York, 137 pp (1973)
[17] John D. Fay. Kernel functions, analytic torsion, and moduli spaces. Mem. Amer. Math. Soc. 96 (1992), no. 464, 123 pp.
[18] K. Igusa. Combinatorial Miller-Morita-Mumford classes and Witten cycles. Algebr. Geom. Topol. 4 (2004), 473-520
[19] K. Igusa. Graph cohomology and Kontsevich cycles. Topology 43 (2004), no. 6, 1469-1510
[20] K. Igusa, M. Kleber. Increasing trees and Kontsevich cycles. Geom. Topol. 8 (2004), 969-1012
[21] J. Jenkins. On the existence of certain general extremal metrics, Ann. of Math. (2) 66 (1957), 440-453
[22] M. Jimbo, T. Miwa, K. Ueno. Monodromy preserving deformation of linear ordinary differential equations with rational coefficients I, Physica 2D 306-352 (1981)
[23] G. van der Geer, A. Kouvidakis. The Hodge bundle on Hurwitz spaces, Pure Appl. Math. Q. 7, no. 4, Special Issue: In memory of Eckart Viehweg, 1297-1307, (2011)
[24] P. Griffiths, G. Harris. Principles of Algebraic Geometry, Wiley, NY, 1978 813 pp.
[25] J. Harer. The virtual cohomological dimension of the mapping class group of an orientable surface, Invent.Math. 84 (1986), no. 1, 157-176.
[26] N. Hitchin. The self-duality equations on a Riemann surface, Proc. LMS, 55 3 (1987) 59-126
[27] J. Hubbard, H. Masur. Quadratic differentials and foliations, Acta Math. 142 221-273 (1979)
[28] C. Kalla, D. Korotkin. Baker-Akhiezer spinor kernel and tau-functions on moduli spaces of meromorphic differentials, Commun.Math.Physics, 331, 1191-1235 (2014)
[29] M. Kazarian. KP hierarchy for Hodge integrals, Adv. Math. 221 1-21 (2009)
[30] A. Kokotov, D. Korotkin. Tau-functions on spaces of Abelian and quadratic differentials and determinants of Laplacians in Strebel metrics of finite volume, math.SP/0405042, preprint No. 46 of Max-Planck Institut for Mathematics in Science, Leipzig (2004)
[31] A. Kokotov, D. Korotkin. Tau-functions on spaces of Abelian differentials and higher genus generalization of Ray-Singer formula, J. Diff. Geom. 82 35-100 (2009)
[32] A. Kokotov, D. Korotkin, P. Zograf. Isomonodromic tau function on the space of admissible covers, Adv. Math., 227 1, 586-600 (2011)
[33] A. Kokotov, D. Korotkin. “On G-function of Frobenius manifolds related to Hurwitz spaces”, International Mathematics Research Notices 2004 no.7 343-360 (2004)
[34] D. Korotkin, A. Sauvaget, P. Zograf. Prym-Tyurin classes and tau-functions, arXiv:1710.01239
[35] D. Korotkin, P. Zograf, Tau function and moduli of differentials, Math. Res. Lett. 18 No.3 447-458 (2011)
[36] D. Korotkin, P. Zograf. Tau function and the Prym class, in “Algebraic and geometric aspects of integrable systems and random matrices”, 241–261, Contemp. Math., 593, Amer. Math. Soc., Providence, RI, 2013
[37] A.Kokotov, D.Korotkin, Isomonodromic tau function of Hurwitz Frobenius manifolds and its applications, IMRN, 2006 1-34 (2006)

[38] D.Korotkin, “Solution of an arbitrary matrix Riemann-Hilbert problem with quasi-permutation monodromy matrices”, Math. Ann. 329 No. 2, 335-364 (2004)

[39] M.Kontsevich, Intersection theory on the moduli space of curves and the matrix Airy function. Comm. Math. Phys. 147 (1992), no. 1, 1-23

[40] S.Lando, A.Zvonkin, Graphs on surfaces and their applications With an appendix by Don B. Zagier. Encyclopaedia of Mathematical Sciences, 141. Low-Dimensional Topology, II. Springer-Verlag, Berlin, 2004. xvi+455

[41] E.Looijenga, Cellular decompositions of compactified moduli spaces of pointed curves, The moduli space of curves, (Texel Island, 1994), Birkhäuser Boston, Boston, MA, 1995, pp. 369-400.

[42] B.Malgrange, Sur les déformations isomonodromiques. I: singularités régulières in Séminaire ENS, Progress in Math., Birkhäuser (1983)

[43] E. Markman, Spectral curves and integrable systems, Compositio Math. 93, 255-290 (1994)

[44] G.Mondello, Combinatorial classes on $M_{g,n}$ are tautological, Int. Math. Res. Not. 2004, no. 44, 2329-2390

[45] G.Mondello, Riemann surfaces, ribbon graphs and combinatorial classes, Handbook of Teichmüller theory. Vol. II, 151-215, IRMA Lect. Math. Theor. Phys., 13, Eur. Math. Soc., Zürich, 2009

[46] A.Okounkov, Toda equations for Hurwitz numbers, Math. Res. Lett. 7 447-453 (2000)

[47] R.Pandharipande, A calculus for the moduli space of curves, arXiv:1603.05151

[48] R.Penner, The Poincaré dual of the Weil-Petersson Kähler two-form, Comm. Anal. Geom. 1 (1993), no. 1, 43-69

[49] N.Seiberg, E.Witten, Monopole Condensation, And Confinement In N=2 Supersymmetric Yang- Mills Theory, Nucl. Phys. B426 (1994) 19-52

[50] K.Strebel, Quadratic differentials, Springer-Verlag, Berlin, 1984.

[51] O.Viro, Lectures on combinatorial presentations of manifolds, in: Differential Geometry and Topology (Alghero, 1992) (R. Caddeo and F. Tricerri, eds.), World Scientific, River Edge, NJ (1993), pp. 244-264.

[52] E.Witten, Two-dimensional gravity and intersection theory on moduli spaces, Surveys in Differential Geometry, 1, 243-310 (1991)

[53] S.Wolpert, On the homology of the moduli space of stable curves, Ann. of Math. (2) 118 no. 3, 491-523 (1983)

[54] L.Takhtajan, P.Zograf, A local index theorem for families of $\bar{\partial}$-operators on punctured Riemann surfaces and a new Kähler metric on their moduli spaces, Comm. Math. Phys. 137 (1991), no. 2, 399-426

[55] D.Zvonkine, An introduction to moduli spaces of curves and their intersection theory, Handbook of Teichmüller theory. Volume III, 667-716, IRMA Lect. Math. Theor. Phys., 17, Eur. Math. Soc., Zürich (2012)