On the dispersionless Davey-Stewartson hierarchy: the tau function, the Riemann-Hilbert problem and the Hamilton-Jacobi theory

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Abstract

The dDS (dispersionless Davey-Stewartson) hierarchy is constructed by two eigenfunctions of a special Hamiltonian vector field. This hierarchy consists of the infinite symmetries of the dDS system. Further, this paper explores the tau function, the Riemann-Hilbert problem and Hamilton-Jacobi theory related to dDS hierarchy.

1. Introduction

The DS (Davey-Stewartson) system is one of the most notable (2+1)-dimensional integrable systems. In 1974, Davey and Stewartson employed a multi-scale analysis to derive a set of nonlinear partial differential equations that describe the evolution of a three-dimensional wave packet in water with finite depth [1]. The DS system (parameterized by $\varepsilon > 0$) can be expressed in a general form reads as follows:

$$i\varepsilon q_x + \frac{\varepsilon^2}{2}(q_{xx} + \sigma^2 q_{yy}) + \delta q\phi = 0,$$

$$\sigma^2 \phi_{yy} - \phi_{xx} + (|q|^2)_{xx} + \sigma^2 (|q|^2)_{yy} = 0,$$

where $x, y, t \in \mathbb{R}$ are the variables of the complex field $q(x, y, t)$ and the real field $\phi(x, y, t)$. We will denote (1) as the DS-I (Davey-Stewartson-I) system when $\sigma = 1$, and as the DS-II (Davey-Stewartson-II) system when $\sigma = 1$. We also denote the case with $\delta = 1$ as the focusing case and the case with $\delta = -1$ as the defocusing case, referring to (1) accordingly. The DS system has undergone extensive examination and has yielded a plenty of noteworthy research discoveries, establishing itself as an exemplary integrable model. For DS-II type equations, a productive high-precision numerical method has been proposed in [2]. It was suggested in [3] to use commutator identities on associative algebras as the basis for the method for deriving (2+1)-dimensional nonlinear integrable equations, and this method has now been extended to the standard hierarchy of integrable equations. The (1+1)-dimensional reduction of the DS system, the NLS (nonlinear Schrödinger) equation, which is one of the most important integrable models, is widely applied in the research of rogue waves recently. Wang and Tian studied the general higher-order rogue wave solutions of the nonlocal NLS equation with parity-time symmetric [4]. Konopelchenko and Taimanov investigated the numerous symmetries of the DS system and highlighted that every symmetry leads to an extensive range of distinct geometric deformations of tori in $\mathbb{R}^4$ while maintaining the preservation of the Willmore functional. They established the DS hierarchy by examining the compatibility of undetermined differential operators with respect to $\partial_2$ and $\partial_y$ [5–7] and provided illustrations of flows corresponding to $t_2$ and $t_y$. Recently, one of the authors gave a direct expression of the DS hierarchy by two scalar pseudo-differential operators [8].

The dispersionless integrable systems are a significant subclass of integrable systems in addition to the classical integrable systems. The dispersionless (semiclassical) limits of the classical integrable systems can sometimes be used to obtain these systems. They are extensively researched and frequently appear in a variety of mathematical physics problems. Jin, Levermore, and McLaughlin have examined the semi-classical limit of the nonlinear Schrödinger equation in arbitrary space dimension [9, 10]. The semi-classical limit of the nonlinear Schrödinger equation in both the defocusing and the focusing cases has been considered by Bronski and McLaughlin in [11]. Krichever took a deep look into the hierarchy of the dispersionless Lax equations [12, 13].
In section 6, we summarise the results of this paper. In section 4, we discuss the related Riemann-Hilbert problem. In section 5, we study the Hamilton-Jacobi theory.

Many compatible partial differential equations were introduced by one of the authors in [22, 23]. This paper is organized as follows. In section 2, we introduce the dDS hierarchy which includes in [22, 23]. This recent research from one of the authors [22, 23] presents that the following dDS system

\[ u_t + 2(uS_z)_z - 2(uS_z)_z = 0, \]  
\[ s_t + S_z^2 - S_z^2 + \phi = 0, \]  
\[ \phi_{zz} + 2(u_{zz} - u_{zz}) = 0 \]

arises from the commutation condition of the Hamiltonian vector fields Lax pair and associates with a hierarchy of infinite symmetrical nonlinear systems. Recently, the dDS system has been studied from the Lie symmetry algebra point of view and some of its exact solutions has been obtained [24].

The hierarchy of partial differential equations, which encompasses an infinite set of symmetries, is commonly associated with the integrable system of partial differential equations. Guha, Takasaki and Takebe have made a series of important and insightful studies for dispersionless integrable hierarchies [14–16, 25–27]. Sato theory is one of the important branches of research on the integrable systems and Soliton. It is widely used in the field of modern mathematics and physics. In particular, the tau function is the most central element of Sato theory, it not only facilitates the analysis of PDEs but also plays a pivotal role in various related fields. The study of the twistor construction for the self-dual vacuum Einstein equation and its hyper-Kähler version [28, 29] is important in geometry and mathematical physics, one of the key steps is to consider a nonlinear Riemann-Hilbert problem. This theory can also be extended to the dDS case. And it is well known that, in classical mechanics, the Hamilton-Jacobi theory is a way to integrate a Hamiltonian system through an appropriate canonical transformation. The generating function of this transformation, which provides an integration of the original system, is a partial differential equation. The Hamilton-Jacobi theory derives from the analytical mechanics and has been an important method in classical mechanics, theoretical physics, differential equations and differential geometry. We will study the tau function, the Riemann-Hilbert problem and Hamilton-Jacobi theory for the dHS hierarchy based on the previous research [22, 23].

This paper is organized as follows. In section 2, we introduce the dDS hierarchy which includes infinitely many compatible flows, derive the twistor structure and Lax-Sato formism for the dDS hierarchy, and show some meaningful examples. In section 3, we present the existence for the tau function of the dDS hierarchies. In section 4, we discuss the related Riemann-Hilbert problem. In section 5, we study the Hamilton-Jacobi theory. In section 6, we summarise the results of this paper.

2. The dispersionless Davey-Stewartson hierarchy

A new hierarchy of compatible partial differential equations was introduced by one of the authors in [23]. This hierarchy consists the infinite symmetries of the dDS system (2) is called as the dDS hierarchy. In this section, we will review some basic concepts and properties of this dispersionless hierarchy.

As an extension of the dDS system, the construction of the dDS hierarchy is based on two eigenfunctions of this following special Hamiltonian vector field

\[ P = \partial_z - \{ \hat{H}, \cdot \} \]  
\[ = \partial_z - \{ v + \frac{u}{p}, \cdot \} \]  
\[ = \partial_z + \frac{u}{p^2} \partial_z + (v_z + \frac{u_z}{p}) \partial_p, \]

where \( p \) is a complex parameter and \( u = u(t), v = v(t) \) depend on complex variables \( t = (t_m) \) \( (m, n \in \mathbb{N}, m + n \geq 1) \) and we denote \( t_0 \equiv z, t_{m1} \equiv \hat{z} \) in this paper).

Given an arbitrary simple closed curve \( \Gamma \) around the origin of the complex \( p \)-plane, the two eigenfunctions \( \mathcal{L} \) and \( \hat{\mathcal{L}} \) of \( P \) are described by the two formal Laurent series

\[ \mathcal{L} = p + \sum_{i<0} f_i(t)p^i, \]  
\[ \hat{\mathcal{L}} = p + \sum_{i<0} f_i(t)p^{-i}, \]  
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\[ \hat{\mathcal{L}} = p + \sum_{i<0} f_i(t)p^{-i}, \]  
\[ \mathcal{L} = p + \sum_{i<0} f_i(t)p^i, \]  
\[ \hat{\mathcal{L}} = p + \sum_{i<0} f_i(t)
\[ \mathcal{L} = \frac{u(t)}{p} + \sum_{i \geq 0} g_i(t)p^i. \] (3b)

Remark 1. Both \( f_i \) and \( g_i \) are dependent on the two real functions \( u \) and \( v \), as well as their derivatives or integrals with respect to the independent variables \( z \) and \( \dot{z} \). The coefficients \( f_i^{(m)} \) and \( g_i^{(n)} \) of the Laurent expansions

\[ \mathcal{L}^m = p^m + \sum_{i \leq m-1} f_i^{(m)} p^i, \]

\[ \mathcal{L}^n = \frac{u^n}{p^n} + \sum_{i \geq 1-n} g_i^{(n)} p^i, \]

are also influenced by the variables \( u \), \( v \) as well as their derivatives or integrals with respect to the independent variables \( z \), \( \dot{z} \).

The alternative formulation of the second Hamiltonian for the dDS system (2) can be represented as \((\mathcal{L}^0)^0 + (\mathcal{L}^0)\leq 0\). By introducing the following Hamiltonian, this concept can be extended to any non-negative integer \( m \), \( n \) as

\[ H_{mn} = (\mathcal{L}^m)^{>0} + (\mathcal{L}^n)\leq 0, \]

\[ H_{m0} = (\mathcal{L}^m)^{>0}, \]

\[ H_{0n} = (\mathcal{L}^n)^{\leq 0}. \]

Here and hereafter in this paper, the symbol \((\cdot)^{>0}\) stands for extracting the positive powers of \( p \) and similarly the symbol \((\cdot)\leq 0\) stands for extracting the non-positive part.

Subsequently, the Zakharov-Shabat equations

\[ \frac{\partial \hat{H}}{\partial t_{mn}} - \frac{\partial H_{mn}}{\partial \ddot{z}} + \{\hat{H}, H_{mn}\} = 0 \] (4)

can be transformed into a set of closed systems in \((2 + 1)\) dimensions, where the unknowns are denoted as \( u \) and \( v \),

\[ u_{mn} - (uf_i^{(m)})_z - g_i^{(n)}\dot{z} = 0, \] (5a)

\[ v_{mn} + f_i^{(m)} - g_i^{(n)}\ddot{z} = 0. \] (5b)

The concept of dDS hierarchy is a direct extension of the above special case, which includes a specific Hamiltonian \( \hat{H} \). Recently, in [23], one of the authors gave three equivalent definitions of the dDS hierarchy as follows, respectively in the forms of Zakharov-Shabat equation, quadratic differential form and Lax-Sato equation.

Definition 1. The dDS hierarchy is defined by the Zakharov-Shabat equations

\[ \frac{\partial H_{mn}}{\partial t_M} - \frac{\partial H_M}{\partial t_{mn}} + \{H_{mn}, H_M\} = 0. \] (6)

The equation (6) known as the Zakharov-Shabat equation is encountered in the analysis of self-dual vacuum Einstein and hyper-Kähler geometry [30]. In the investigation of the hyper-Kähler version of the vacuum Einstein equation, a Kähler-like 2-form and its associated 'Darboux coordinates' play a significant role [31, 32].

Definition 2. By introducing an exterior differential 2-form

\[ \omega = \sum_{m+n \geq 1} dH_{mn} \wedge dt_{mn} = dp \wedge dz + d\hat{H} \wedge d\dot{z} + \sum_{m+n \geq 2} dH_{mn} \wedge dt_{mn}, \] (7)

the dDS hierarchy can be expressed as

\[ \omega = d\mathcal{L} \wedge d\mathcal{M} = d\hat{\mathcal{L}} \wedge d\hat{\mathcal{M}}, \] (8)

in which

\[ \mathcal{M} = \sum_{m+n \geq 2} nt_{mn} \mathcal{L}^{m-1} + z + \sum_{i=1}^{\infty} \nu_i \mathcal{L}^{-i-1}, \] (9a)

\[ \hat{\mathcal{M}} = \sum_{m+n \geq 2} nt_{mn} \hat{\mathcal{L}}^{m-1} + \dot{z} + \sum_{i=1}^{\infty} \phi_i \hat{\mathcal{L}}^{-i-1}. \] (9b)

The symbol 'd' stands for total differentiation in \( p, z, \dot{z} \) and \( t_{mn} \) \((m + n \geq 2)\).
Definition 3. The dDS hierarchy consists of the following Lax-Sato equations

$$\frac{\partial K}{\partial t_{mn}} = [H_{mn}, K]$$

(10)

where eigenfunctions $K = \mathcal{L}, \mathcal{M}, \hat{\mathcal{L}}, \hat{\mathcal{M}}$ adhere to canonical Poisson relations

$$\{\mathcal{L}, \mathcal{M}\} = 1, \quad \{\hat{\mathcal{L}}, \hat{\mathcal{M}}\} = 1.$$

To the end of this section, we present some meaningful examples of the dDS hierarchy below.

Example 1. By taking $m = 0, n = 1; k = 2$, $l = 2$ and $t_{01} = \hat{z}, t_{22} = t$, Hamiltonians read as follows

$$H_{01} = v + \frac{u}{p},$$

$$H_{22} = (p^2 + 2f_0) + \left[(v^2 + 2ug_0) + \frac{2uv}{p} + \frac{u^2}{p^2}\right]$$

$$= (p^2 + 2fp) + \left[w + \frac{2uv}{p} + \frac{u^2}{p^2}\right].$$

(11b)

Then nonlinear system arising from the Zakharov-Shabat equation (6) read as

$$2u_z + u_{zz} - 2tv_z = 0,$$

$$u_t - 2(u^2)_z - 2(uv)_z = 0,$$

$$v_t - 2u_z - 2fv_z - w_z = 0,$$

$$f_z + v_z = 0.$$

(12d)

which can be precisely simplified to the dDS system (2) by selecting $v = S_z$.

Example 2. By selecting one of the two Hamiltonians as $\hat{H} = H_{01}$, equations (5) and (4) can be used to obtain the integrable flow equation. By choosing another Hamiltonian $H_{33}$ and setting $v = S_z$, we can determine the functions $f_0^{(3)}, f_1^{(3)}, g_0^{(3)},$ and $g_1^{(3)}$ in expression (5),

$$f_0^{(3)} = -S_z^3 + 6S_zV + \partial_z^{-1}(uS_z - S_zV),$$

$$f_1^{(3)} = 3(S_z^2 - V),$$

$$g_0^{(3)} = S_z^3 - 6S_z W + \partial_z^{-1}(S_zW - uS_zW),$$

$$g_1^{(3)} = 3u(S_z^2 - W),$$

in which

$$W_z = u_z, \quad V_z = u_z.$$

Then the following system is obtained,

$$u_{t_{01}} - 3[u(S_z^2 - V)]_z - 3[u(S_z^2 - W)]_z = 0,$$

$$S_{t_{01}} - (S_z^3 + S_z^2) + 3(S_zV + S_zW) + 3\phi = 0,$$

$$\phi_{z} = (uS_z)_zz + (uS_z)_{zz},$$

which is analogous to the dDS system (2).

3. The tau function

The tau function theory is the most central element of Sato theory, it is not only facilitates the analysis of PDEs but also plays a pivotal role in various related fields. The tau function encompasses all aspects of integrable systems that can be used to define solutions for the whole integrable hierarchy. By applying the definitions of dDS hierarchy introduced in the preceding section, we can discuss the existence of the tau function from the 2-form (7).

Theorem 1. There exists the tau function $\tau_{\text{dDS}}(t)$ satisfying

$$d \log \tau_{\text{dDS}}(t) = \sum_{m \geq n \geq 2} (v_{mn+1} + \hat{\nu}_{n+1})dt_{mn}$$

(13)

This is due to the basic fact that the right hand side of the equation (13) is a closed form and we will prove this below.
Firstly, let us introduce the notion of formal residue of 1-forms
\[ \text{res} \sum a_{n}p^{n}dp = a_{-1}. \]

For the eigenfunctions \( \mathcal{L} \) and \( \hat{\mathcal{L}} \), similar to the literature [15], the following formulas are also true
\[ \text{res} \mathcal{L}^{n}d\mathcal{L} = \delta_{n,-1}, \]
\[ \text{res} \hat{\mathcal{L}}^{n}d\hat{\mathcal{L}} = -\delta_{n,-1}, \quad n \in \mathbb{Z}. \]

**Proposition 1.** The coefficients \( \nu_{i} \) and \( \hat{\nu}_{i} \) from the Orlov eigenfunctions \( \mathcal{M} \) and \( \hat{\mathcal{M}} \), which are defined by (9), satisfy
\[ \frac{\partial \hat{\nu}_{i+1}}{\partial t_{mn}} = - \text{res} \hat{\mathcal{L}}^{i}dH_{mn}, \quad \text{(14a)} \]
\[ \frac{\partial \nu_{i+1}}{\partial t_{mn}} = \text{res} \mathcal{L}^{i}dH_{mn}, \quad \text{(14b)} \]

**Proof.** We only give the proof of the equations (14). By the chain rule of differentiation, one obtains
\[ \frac{\partial \hat{\mathcal{M}}}{\partial t_{mn}} = \frac{\partial \hat{\mathcal{M}}}{\partial \mathcal{L}} \bigg|_{\mathcal{L} \text{ fixed}} + \frac{\partial \hat{\mathcal{L}}}{\partial t_{mn}} + n\mathcal{L}^{n-1} + \sum_{i=1}^{\infty} \frac{\partial \hat{\nu}_{i+1}\mathcal{L}^{i-1}}{\partial t_{mn}}. \quad \text{(15)} \]

Therefore, using the equations of the eigenfunctions \( \mathcal{L} \) and \( \hat{\mathcal{L}} \) as well, one has
\[ \frac{\partial \hat{\nu}_{i+1}}{\partial t_{mn}} = - \text{res} \hat{\mathcal{L}}\left(\frac{\partial \hat{\mathcal{M}}}{\partial \mathcal{L}} \bigg|_{\mathcal{L} \text{ fixed}} \right) d\hat{\mathcal{L}} \]
\[ = - \text{res} \hat{\mathcal{L}}^{i} \left( \frac{\partial \hat{\mathcal{M}}}{\partial t_{mn}} \right) d\hat{\mathcal{L}} + \text{res} \hat{\mathcal{L}}^{i} \left( \frac{\partial \hat{\mathcal{M}}}{\partial t_{mn}} \right) d\hat{\mathcal{M}} \]
\[ = - \text{res} \hat{\mathcal{L}}^{i} \left( \frac{\partial H_{mn}}{\partial \mathcal{L}} \right) d\mathcal{L} + \text{res} \hat{\mathcal{L}}^{i} \left( \frac{\partial \mathcal{M}}{\partial \mathcal{L}} \right) d\mathcal{M} \]
\[ = - \text{res} \hat{\mathcal{L}}^{i} \left( \frac{\partial H_{mn}}{\partial \mathcal{L}} \right) d\mathcal{L} + \text{res} \hat{\mathcal{L}}^{i} \left( \frac{\partial \mathcal{M}}{\partial \mathcal{L}} \right) d\mathcal{M} \]
\[ = - \text{res} \hat{\mathcal{L}}^{i} dH_{mn}, \]

where we have used the Lax equations (10) and the canonical Possion relation \( \{ \hat{\mathcal{L}}, \mathcal{M} \} = 1 \). The equation (4) can be proved similarly.

**Proposition 2.** The coefficients \( u_{m+1} \) and \( \hat{u}_{m+1} \) from the Orlov eigenfunctions \( \mathcal{M} \) and \( \hat{\mathcal{M}} \) which are defined by (9), satisfy the condition
\[ \frac{\partial (u_{m+1} + \hat{u}_{m+1})}{\partial t_{ij}} = \frac{\partial (v_{k+1} + \hat{v}_{k+1})}{\partial t_{mn}}, \quad m,n \in \mathbb{Z}. \quad \text{(16)} \]

**Proof.** By applying the proposition 1, one obtains
\[ \frac{\partial (u_{m+1} + \hat{u}_{m+1})}{\partial t_{ij}} - \frac{\partial (v_{k+1} + \hat{v}_{k+1})}{\partial t_{mn}} = \text{res} (\mathcal{L}^{m} - \hat{\mathcal{L}}^{m})dH_{ij} - \text{res} (\mathcal{L}^{k} - \hat{\mathcal{L}}^{k})dH_{mn} \]
\[ = \text{res} (\mathcal{L}^{m} - \hat{\mathcal{L}}^{m})dH_{ij} - \text{res} (\mathcal{L}^{k} - \hat{\mathcal{L}}^{k})dH_{mn} \]
\[ = \text{res} (\mathcal{L}^{m} - \hat{\mathcal{L}}^{m})dH_{ij} - \text{res} (\mathcal{L}^{k} - \hat{\mathcal{L}}^{k})dH_{mn} \]
\[ = \text{res} \mathcal{L}^{m}d\mathcal{L} + \text{res} \mathcal{L}^{k}d\hat{\mathcal{L}}, \]

which vanishes because \( m \) and \( n \) are positive integers. \( \square \)

Based on the above facts, we arrive the conclusion that the right hand side of (13) is closed. The theorem 1 shows that the coefficients \( u_{i}, \hat{u}_{i}, \nu_{i} \) and \( \hat{\nu}_{i} \) from the two pairs of eigenfunctions (\( \mathcal{L}, \mathcal{M} \)) and (\( \hat{\mathcal{L}}, \hat{\mathcal{M}} \)) can be expressed by this tau function \( \tau_{\nabla_{DS}} \). For the case of \( n = 0 \), it reduces exactly to the tau function \( \tau_{\nabla_{KPI}} \) of dKP hierarchy [15].
4. The Riemann-Hilbert problem

The study of the twistor construction for the self-dual vacuum Einstein equation and its hyper-Kähler version [28, 29] is important in geometry and mathematical physics. This theory can be also extended to consider the dDS hierarchy and the key step is to solve the following

\[ f(\mathcal{L}, \mathcal{M}) = \tilde{f}(\mathcal{L}, \mathcal{M}), \quad (16a) \]
\[ g(\mathcal{L}, \mathcal{M}) = \tilde{g}(\mathcal{L}, \mathcal{M}), \quad (16b) \]

where \((f, g)\) and \((\tilde{f}, \tilde{g})\) are two pairs of holomorphic functions satisfying the canonical Poisson relation \(\{f, g\} = \{\tilde{f}, \tilde{g}\} = 1\). In this section, we will describe the details of the Riemann-Hilbert problem based on the two pairs of eigenfunctions \((\mathcal{L}, \mathcal{M}), (\mathcal{L}, \mathcal{M})\) defined in (3) (9).

**Theorem 2.** The solutions \((\mathcal{L}, \mathcal{M})\) and \((\mathcal{L}, \mathcal{M})\) of the Riemann-Hilbert problem (16) solve the dDS hierarchy. Namely, they satisfy the Lax-Sato equations (10) and the canonical Poisson relations \(\{\mathcal{L}, \mathcal{M}\} = \{\tilde{\mathcal{L}}, \tilde{\mathcal{M}}\} = 1\). The holomorphic functions \((f, g, \tilde{f}, \tilde{g})\) are defined as the twistor data of this solution.

**Proof.**

Firstly, the determinant of the both hand sides of this equation and using the relations

\[ \{f, g\} = \{\tilde{f}, \tilde{g}\} = 1, \]

one abstains

\[ \{\mathcal{L}, \mathcal{M}\} = \{\tilde{\mathcal{L}}, \tilde{\mathcal{M}}\}. \]

Then

\[ \{\mathcal{L}, \mathcal{M}\} = \frac{\partial \mathcal{L}}{\partial p} \frac{\partial \mathcal{M}}{\partial z} - \frac{\partial \mathcal{L}}{\partial z} \frac{\partial \mathcal{M}}{\partial p} = \frac{\partial \mathcal{L}}{\partial p} \frac{\partial \mathcal{M}}{\partial \mathcal{L}} \left( \frac{\partial \mathcal{L}}{\partial z} + 1 + \sum_{i=1}^{\infty} \frac{\partial \mathcal{L} \mathcal{M}^{i-1}}{\partial z} \right)^{-1} - \frac{\partial \mathcal{L}}{\partial z} \frac{\partial \mathcal{M}}{\partial \mathcal{L}} \frac{\partial \mathcal{L}}{\partial \mathcal{M}}. \]

The formal Laurent expansion of \(\{\tilde{\mathcal{L}}, \tilde{\mathcal{M}}\}\) contains only nonnegative power of \(p\), therefore

\[ \{\mathcal{L}, \mathcal{M}\} = \{\tilde{\mathcal{L}}, \tilde{\mathcal{M}}\} = 1. \]

This gives the canonical Poisson relations for the solutions of the equations (16). Further, differentiating (16) by \(t_{\text{non}}\) gives

\[ \left( \begin{array}{c} \frac{\partial \mathcal{L}}{\partial z} \frac{\partial \mathcal{M}}{\partial \mathcal{M}} \\ \frac{\partial \mathcal{M}}{\partial \mathcal{M}} \frac{\partial \mathcal{M}}{\partial \mathcal{M}} \\ \frac{\partial \mathcal{M}}{\partial \mathcal{M}} \frac{\partial \mathcal{M}}{\partial \mathcal{M}} \end{array} \right) \left( \begin{array}{c} \frac{\partial \mathcal{L}}{\partial \mathcal{M}} \\ \frac{\partial \mathcal{M}}{\partial \mathcal{M}} \\ \frac{\partial \mathcal{M}}{\partial \mathcal{M}} \end{array} \right) = \left( \begin{array}{c} \frac{\partial \mathcal{L}}{\partial \mathcal{L}} \frac{\partial \mathcal{M}}{\partial \mathcal{M}} \\ \frac{\partial \mathcal{M}}{\partial \mathcal{M}} \frac{\partial \mathcal{M}}{\partial \mathcal{M}} \\ \frac{\partial \mathcal{M}}{\partial \mathcal{M}} \frac{\partial \mathcal{M}}{\partial \mathcal{M}} \end{array} \right). \]

By combining (17), we can rewrite the formula (19) as follows

\[ \left( \begin{array}{c} \frac{\partial \mathcal{M}}{\partial \mathcal{M}} - \frac{\partial \mathcal{L}}{\partial \mathcal{M}} \\ \frac{\partial \mathcal{M}}{\partial \mathcal{M}} \frac{\partial \mathcal{M}}{\partial \mathcal{M}} \\ \frac{\partial \mathcal{M}}{\partial \mathcal{M}} \frac{\partial \mathcal{M}}{\partial \mathcal{M}} \end{array} \right) = \left( \begin{array}{c} \frac{\partial \mathcal{L}}{\partial \mathcal{L}} \frac{\partial \mathcal{M}}{\partial \mathcal{M}} \\ \frac{\partial \mathcal{M}}{\partial \mathcal{M}} \frac{\partial \mathcal{M}}{\partial \mathcal{M}} \\ \frac{\partial \mathcal{M}}{\partial \mathcal{M}} \frac{\partial \mathcal{M}}{\partial \mathcal{M}} \end{array} \right). \]
The first component of the left hand side for (20) is given by

$$\begin{align*}
\frac{\partial M}{\partial z} \frac{\partial L}{\partial t_{mn}} - \frac{\partial L}{\partial z} \frac{\partial M}{\partial t_{mn}} &= \left( \frac{\partial M}{\partial \dot{L}} \mid_{p,v \text{ fixed}} \frac{\partial L}{\partial \dot{z}} + 1 + \sum_{j=1}^{\infty} \frac{\partial v_{j+1}}{\partial \dot{z}} L^{-j-1} \right) \frac{\partial L}{\partial t_{mn}} \\
- \left( \frac{\partial M}{\partial \dot{L}} \mid_{p,v \text{ fixed}} \frac{\partial L}{\partial t_{mn}} + mL^{-1} + \sum_{j=1}^{\infty} \frac{\partial v_{j+1}}{\partial t_{mn}} L^{-j-1} \right) \frac{\partial L}{\partial z} \\
= -\frac{\partial L^m}{\partial z} + \frac{\partial L}{\partial t_{mn}} + \left( \sum_{j=1}^{\infty} \frac{\partial v_{j+1}}{\partial t_{mn}} L^{-j-1} \right) \frac{\partial L}{\partial t_{mn}} = \left( \sum_{j=1}^{\infty} \frac{\partial v_{j+1}}{\partial t_{mn}} L^{-j-1} \right) \frac{\partial L}{\partial z},
\end{align*}$$

which contains limited positive powers of $p$. Similarly, the second component of the right hand side of (20) is given by

$$\begin{align*}
\frac{\partial \hat{M}}{\partial z} \frac{\partial \hat{L}}{\partial t_{mn}} - \frac{\partial \hat{L}}{\partial z} \frac{\partial \hat{M}}{\partial t_{mn}} &= \left( \frac{\partial \hat{M}}{\partial \hat{\dot{L}}} \mid_{p,\hat{v} \text{ fixed}} \frac{\partial \hat{L}}{\partial \hat{\dot{z}}} + \sum_{j=1}^{\infty} \frac{\partial \hat{v}_{j+1}}{\partial \hat{\dot{z}}} (\hat{L})^{-j-1} \right) \frac{\partial \hat{L}}{\partial t_{mn}} \\
- \left( \frac{\partial \hat{M}}{\partial \hat{\dot{L}}} \mid_{p,\hat{v} \text{ fixed}} \frac{\partial \hat{L}}{\partial t_{mn}} + nL^{-1} + \sum_{j=1}^{\infty} \frac{\partial \hat{v}_{j+1}}{\partial t_{mn}} (\hat{L})^{-j-1} \right) \frac{\partial \hat{L}}{\partial z} \\
= -\frac{\partial \hat{L}^n}{\partial z} + \left( \sum_{j=1}^{\infty} \frac{\partial \hat{v}_{j+1}}{\partial t_{mn}} (\hat{L})^{-j-1} \right) \frac{\partial \hat{L}}{\partial t_{mn}} = \left( \sum_{j=1}^{\infty} \frac{\partial \hat{v}_{j+1}}{\partial t_{mn}} (\hat{L})^{-j-1} \right) \frac{\partial \hat{L}}{\partial z},
\end{align*}$$

which contains limited non-positive powers of $p$. By considering both sides of (20), one obtains

$$\begin{align*}
\frac{\partial M}{\partial z} \frac{\partial L}{\partial t_{mn}} - \frac{\partial L}{\partial z} \frac{\partial M}{\partial t_{mn}} &= \frac{\partial \hat{M}}{\partial \hat{\dot{L}}} \mid_{p,\hat{v} \text{ fixed}} \frac{\partial \hat{L}}{\partial \hat{\dot{z}}} + \sum_{j=1}^{\infty} \frac{\partial \hat{v}_{j+1}}{\partial \hat{\dot{z}}} (\hat{L})^{-j-1}, \\
\frac{\partial M}{\partial p} \frac{\partial L}{\partial t_{mn}} - \frac{\partial L}{\partial p} \frac{\partial M}{\partial t_{mn}} &= \frac{\partial \hat{M}}{\partial \hat{\dot{L}}} \mid_{p,\hat{v} \text{ fixed}} \frac{\partial \hat{L}}{\partial \hat{\dot{p}}} + \sum_{j=1}^{\infty} \frac{\partial \hat{v}_{j+1}}{\partial \hat{\dot{p}}} (\hat{L})^{-j-1}.
\end{align*}$$

Then these equations can be readily solved as

$$\begin{align*}
\frac{\partial \hat{L}}{\partial t_{mn}} &= \{H_{mn}, L\}, \\
\frac{\partial \hat{L}}{\partial t_{mn}} &= \{H_{mn}, M\}, \\
\frac{\partial \hat{M}}{\partial t_{mn}} &= \{H_{mn}, \hat{M}\}.
\end{align*}$$

\[\square\]

5. The Hamilton-Jacobi theory

It is well known that, in classical mechanics, the Hamilton-Jacobi theory is a way to integrate a Hamiltonian system through an appropriate canonical transformation. The generating function of this transformation, which provides an integration of the original system, is a partial differential equation. The Hamilton-Jacobi theory derives from the analytical mechanics and has been an important method in classical mechanics, theoretical physics, differential equations and differential geometry. In this section, we will continue to discuss the Hamilton-Jacobi theory related to the dDS hierarchy.

Firstly, we define the functions $p(\lambda, \mu, t) = p, z(\hat{\lambda}, \hat{\mu}, t) = z$ implicitly by

$$\begin{align*}
\mathcal{L}(p, z, t) &= \lambda, \\
\mathcal{M}(p, z, t) &= \mu,
\end{align*}$$

with the parameters $\lambda, \mu$.

Similarly, we can define functions $\hat{p}(\hat{\lambda}, \hat{\mu}, t) = \hat{p}, \hat{z}(\hat{\lambda}, \hat{\mu}, t) = \hat{z}$ implicitly by

$$\begin{align*}
\hat{\mathcal{L}}(p, z, t) &= \hat{\lambda}, \\
\hat{\mathcal{M}}(p, z, t) &= \hat{\mu},
\end{align*}$$

with the parameters $\hat{\lambda}, \hat{\mu}$.

Then we can define a couple of Hamiltonian systems as described in the following theorem.
Theorem 3. Both \((p(\lambda, \mu, t), z(\lambda, \mu, t))\) and \((\hat{p}(\hat{\lambda}, \hat{\mu}, t), \hat{z}(\hat{\lambda}, \hat{\mu}, t))\) satisfy the multi-time Hamiltonian system

\[
\begin{align*}
\frac{dp}{d\nu} &= \frac{\partial H_{mn}}{\partial z}, \quad \frac{\partial H_{mn}}{\partial \nu} = \frac{\partial \nu}{\partial p}, \\
\frac{dz}{d\nu} &= -\frac{\partial H_{mn}}{\partial p}, \quad \frac{d\nu}{dp} = \frac{\partial H_{mn}}{\partial \nu},
\end{align*}
\]

with time-dependent Hamiltonians \(H_{mn}\).

Proof. In this proof, we abbreviate \((p(\lambda, \mu, t), z(\lambda, \mu, t)), (\hat{p}(\hat{\lambda}, \hat{\mu}, t), \hat{z}(\hat{\lambda}, \hat{\mu}, t))\) as \((p(t), z(t))\), \((\hat{p}(t), \hat{z}(t))\) to avoid complicated symbols. According to the definition, the following identities are true

\[
\mathcal{L}(p(t), z(t), t) = \lambda, \quad \mathcal{M}(p(t), z(t), t) = \mu, \\
\hat{\mathcal{L}}(\hat{p}(t), \hat{z}(t), t) = \hat{\lambda}, \quad \hat{\mathcal{M}}(\hat{p}(t), \hat{z}(t), t) = \hat{\mu}.
\]

By considering the derivative of \(t_{mn}\) for these identities, one obtains

\[
\begin{align*}
\frac{\partial \mathcal{L}}{\partial p} \frac{dp}{dt_{mn}} + \frac{\partial \mathcal{L}}{\partial z} \frac{dz}{dt_{mn}} + \frac{\partial \mathcal{L}}{\partial \nu} = 0, \\
\frac{\partial \mathcal{M}}{\partial p} \frac{dp}{dt_{mn}} + \frac{\partial \mathcal{M}}{\partial z} \frac{dz}{dt_{mn}} + \frac{\partial \mathcal{M}}{\partial \nu} = 0, \\
\frac{\partial \hat{\mathcal{L}}}{\partial \hat{p}} \frac{d\hat{p}}{dt_{mn}} + \frac{\partial \hat{\mathcal{L}}}{\partial \hat{z}} \frac{d\hat{z}}{dt_{mn}} + \frac{\partial \hat{\mathcal{L}}}{\partial \hat{\nu}} = 0, \\
\frac{\partial \hat{\mathcal{M}}}{\partial \hat{p}} \frac{d\hat{p}}{dt_{mn}} + \frac{\partial \hat{\mathcal{M}}}{\partial \hat{z}} \frac{d\hat{z}}{dt_{mn}} + \frac{\partial \hat{\mathcal{M}}}{\partial \hat{\nu}} = 0.
\end{align*}
\]

Then by applying the Lax equations of the dDS hierarchy (10), we derive the following Hamilton-Jacobi systems

\[
\begin{align*}
\left(\begin{array}{c}
\frac{\partial \mathcal{L}}{\partial p} \\
\frac{\partial \mathcal{M}}{\partial p}
\end{array}\right) \frac{dp}{dt_{mn}} &= \left(\begin{array}{c}
\frac{\partial \mathcal{L}}{\partial z} \\
\frac{\partial \mathcal{M}}{\partial z}
\end{array}\right) \frac{dz}{dt_{mn}} - \left(\begin{array}{c}
\frac{\partial \mathcal{L}}{\partial \nu} \\
\frac{\partial \mathcal{M}}{\partial \nu}
\end{array}\right), \\
\left(\begin{array}{c}
\frac{\partial \hat{\mathcal{L}}}{\partial \hat{p}} \\
\frac{\partial \hat{\mathcal{M}}}{\partial \hat{p}}
\end{array}\right) \frac{d\hat{p}}{dt_{mn}} &= \left(\begin{array}{c}
\frac{\partial \hat{\mathcal{L}}}{\partial \hat{z}} \\
\frac{\partial \hat{\mathcal{M}}}{\partial \hat{z}}
\end{array}\right) \frac{d\hat{z}}{dt_{mn}} - \left(\begin{array}{c}
\frac{\partial \hat{\mathcal{L}}}{\partial \hat{\nu}} \\
\frac{\partial \hat{\mathcal{M}}}{\partial \hat{\nu}}
\end{array}\right).
\end{align*}
\]

The trajectories of this Hamiltonian system should form a two-dimensional family because it resides in a two-dimensional phase space. Both sets of parameters \((\lambda, \mu)\) and \((\hat{\lambda}, \hat{\mu})\) should have a functional relationship. Let us write the functional relations as

\[
\lambda = f(\hat{\lambda}, \hat{\mu}), \quad \mu = g(\hat{\lambda}, \hat{\mu}).
\]

This implies that four eigenfunctions \((\mathcal{L}, \mathcal{M}, \hat{\mathcal{L}}, \hat{\mathcal{M}})\) satisfy the following equations

\[
\mathcal{L}(p, z, t) = f(\hat{\mathcal{L}}, \hat{\mathcal{M}}), \quad \mathcal{M}(p, z, t) = g(\hat{\mathcal{L}}, \hat{\mathcal{M}}).
\]

Therefore, the Riemann-Hilbert problem can be reproduced from the multi-time Hamiltonian system.

Because there are two different multi-time Hamiltonian trajectories parameterizations, two different canonical transformations \((p, z)\) to the parameterizations \((\lambda, \mu)\) and \((\hat{\lambda}, \hat{\mu})\) can be obtained. These two different canonical transformations are defined by two generation functions. One of the canonical transformations is \((p, z) \mapsto (\lambda, \mu)\), which is defined by a generating function \(S(\lambda, z, \hat{z}, t)\) as

\[
\frac{\partial S}{\partial \lambda} = \mu, \quad \frac{\partial S}{\partial \hat{z}} = \hat{\mu}, \\
\frac{\partial S}{\partial z} = u, \quad \frac{\partial S}{\partial \hat{\nu}} = H_{mn}.
\]

The transformed Hamiltonian system with zero Hamiltonians is given by

\[
\frac{d\lambda}{dt_{mn}} = 0, \quad \frac{d\mu}{dt_{mn}} = 0.
\]

Similarly, the other canonical transformation \((p, z) \mapsto (\hat{\lambda}, \hat{\mu})\) is defined by a generating function

\[
\hat{S} = \hat{S}(\hat{\lambda}, z, \hat{z}, t)\]
The transformed Hamiltonian system with zero Hamiltonians is given by
\[
\frac{\partial \hat{S}}{\partial \hat{\lambda}} = \hat{\mu}, \quad \frac{\partial \hat{S}}{\partial \hat{z}} = p, \\
\frac{\partial \hat{S}}{\partial \hat{\varepsilon}} = \frac{u}{p}, \quad \frac{\partial \hat{S}}{\partial t_{mn}} = H_{mn}.
\]

Then the above canonical transformations can be rewritten as
\[
d\hat{S} = \mu d\hat{\lambda} + pdz + \frac{u}{p} d\hat{\varepsilon} + \sum_{m+n \geq 2} \infty H_{mn} dt_{mn}, \\
d\hat{S} = \hat{\mu} d\hat{\lambda} + pdz + \frac{u}{p} d\hat{\varepsilon} + \sum_{m+n \geq 2} \infty H_{mn} dt_{mn},
\]
which are actually the \(S\) and \(\hat{S}\) functions related to the tau functions introduced in [23].

6. Conclusion

In this paper, the existence of the tau function, the Riemann-Hilbert problem and Hamilton-Jacobi theory for the dDS hierarchy are studied. Based on these results, we will continue to consider some problems related to the twistor theory, exact solutions and algebraic structures and the results will present in the subsequent papers.

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Data availability statement

No new data were created or analysed in this study.

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