On the time-dependent extra spatial dimensions in six dimensional space-time

Phan Hong Lien¹ and Do Thi Hong Hai²
¹Le Quy Don Academy of Technology, 236 Hoang Quoc Viet, Hanoi, Vietnam
²Hanoi University of Mining and Geology, Duc Thang, Bac Tu Liem, Hanoi, Vietnam
E-mail: phhlien.mta@gmail.com, dothihonghai@humg.edu.vn

Abstract. In this paper, we analyze the time-dependent extra spatial dimensions in six dimensional (6D) space-time. The 4-brane is assumed to be a de Sitter space. Based on the form of the brane-world energy-momentum tensor proposed by Shiromizu et al. and the five dimensions by Peter K. F. Kuhfittic, we extended the theory to the 2-codimension embedded in higher dimensions. The inflation scenario in 6D is investigated in two cases of cosmological constant: \( \Lambda > 0 \) and \( \Lambda < 0 \). The energy of two extra dimensions is calculated too.

1. Introduction
Nowadays, the models of extra dimensions introduce completely new perspectives of theoretical physics. Extra dimensions were proposed to assure a true solution for the hierarchy problem [1], and to explain the embedding of the Standard Model in a Grand Unified Theory [2] and the affecting of particle physics and cosmology. Particularly, in the brane-world theories, the D-brane inflation in warped compactification has attracted much attention recently [3]. In that case the inflation is identified as the separation between a brane and an anti-brane [4], which describes by 2-codimension branes couple to bulk metric in six and ten dimensions [5].

In this paper, we analyze the time-dependent spatial dimensions in six dimensional space-time. The crossover distance \( R_c \) between the two regimes is [6]

\[
R_c = \frac{M_{PL}}{2M_6^2}.
\]

Here the 4-brane is assumed to be a de Sitter space. Based on the form of the brane-world energy momentum tensor proposed by Shiromizu et al. and the five dimensions by Peter K. F. Kuhfittic [7], the theory to the 2-codimension embedded in higher dimensions is extended. The inflation scenario in 6D is investigated in two cases of cosmological constant: \( \Lambda > 0 \) and \( \Lambda < 0 \).

This paper is organized as follows. In Sec. 2 we present the formulation of Einstein field equation and the metric in 6D, where all spatial dimensions are time-dependent. The general solutions of these equations when \( \Lambda > 0 \) are investigated in Sec. 3. It shows that the extra dimensions expand by the factor \( e^{Ht} \) in the branes. Sec. 4 is devoted to the case \( \Lambda < 0 \). The conclusion and discussion are given in Sec. 5.
2. Formulation

We start from the bulk metric in six-dimensional space-time

\[ ds^2 = -dt^2 + [R(t)]^2 \left[ dr^2 + r^2 \left( d\theta^2 + \sin^2 \theta d\varphi^2 \right) \right] + [\chi(t)]^2 \left[ dy_1^2 + dy_2^2 \right], \]

where \( y_1, y_2 \) are the coordinates in two extra dimensions. Note that the spatial dimensions are assumed to be time dependent by the scalar factors \( R(t) \) and \( \chi(t) \).

It is clear that the manifold (1) in six dimensions contains two five-dimensional space-time as its submanifolds at \( y_1 = 0 \) and \( y_2 = 0 \). The hypersurface \( y_1 = 0 \) and \( y_2 = 0 \) corresponds to 3-brane.

If we impose \( y_1 = \rho \sin \phi, y_2 = \rho \cos \phi \), then the metric (1) takes the form

\[ ds^2 = -dt^2 + [R(t)]^2 \left[ dr^2 + r^2 \left( d\theta^2 + \sin^2 \theta d\varphi^2 \right) \right] + [\chi(t)]^2 \left[ d\rho^2 + \rho^2 d\phi^2 \right], \]

where \( \rho, \phi \) are the central symmetric coordinates in two extra dimensions, which could be compactified on a 2-dimensional sphere \( (S^2/Z_2) \).

We assume that the energy-momentum tensor, which satisfies the six-dimensional Einstein field equations in the Early Universe

\[ G_{MN} = k_6^2 T_{MN}, \]

could be written in the form

\[ T_{MN} = -\Lambda g_{MN} + \delta(\rho) \delta(\phi) \left( -\lambda q_{MN} + \tau_{MN} \right). \]

Here \( G_{MN} \) is Einstein tensor, \( g_{MN} \) is the metric tensor in six dimensions (6D), \( \Lambda \) is the cosmological constant, the Latin letters denote the 6D indices \( M, N = 0, 1, 2, 3, 5, 6 \).

The \( \delta \)-function appears due to the boundary surface \( \rho = 0 \) and \( \phi = 0 \). \( \lambda \) and \( \tau_{MN} \) are the vacuum energy and the energy-momentum tensor of the 4-brane.

In this case the inflation is identified as the separation between a brane and an anti-brane like codimension-2 brane \([3]\).

\[ T_{\rho\rho} = -\Lambda g_{\rho\rho} + \delta(\rho) \left( -\lambda q_{\rho\rho} + \tau_{\rho\rho} \right), \]
\[ T_{\phi\phi} = -\Lambda g_{\phi\phi} + \delta(\phi) \left( -\lambda q_{\phi\phi} + \tau_{\phi\phi} \right). \]

The 4-brane is assumed to be a de Sitter space, meaning \( R(t) = e^{Ht} \) and \( H = \sqrt{\Lambda_5}/3 \), where \( \Lambda_5 \) is the cosmological constant of the 4-brane. The nonzero components of Einstein tensor in the orthonormal frame are determined from the Ricci tensor (see Appendix)

\[ G_{tt} = 3 \left( \frac{R'(t)}{R(t)} \right)^2 + 6 \frac{R'(t)\chi'(t)}{R(t)\chi(t)} + \left( \frac{\chi'(t)}{\chi(t)} \right)^2, \]
\[ G_{rr} = -2 \left( \frac{R'(t)}{R(t)} \right)^2 - 4 \frac{R'(t)\chi'(t)}{R(t)\chi(t)} - 2 \left( \frac{\chi''(t)}{\chi(t)} \right)^2 - \left( \frac{\chi'(t)}{\chi(t)} \right)^2, \]
\[ G_{\theta\theta} = G_{\varphi\varphi}, \]
\[ G_{\rho\rho} = -3 \left( \frac{R'(t)}{R(t)} \right)^2 - 3 \left( \frac{R'(t)\chi'(t)}{R(t)\chi(t)} \right)^2 - 3 \frac{\chi''(t)}{\chi(t)} \]
\[ G_{\phi\phi} = -3 \left( \frac{R'(t)}{R(t)} \right)^2 - 3 \left( \frac{R'(t)\chi'(t)}{R(t)\chi(t)} \right)^2 - \left( \frac{\chi''(t)}{\chi(t)} \right)^2. \]
The time-dependent extra spatial dimensions are derived from the system of Eqs. (7), (8), and (9)
\[ G_{tt} + G_{rr} - 2G_{pp} = k_6^2 (T_{tt} + T_{rr} - 2T_{pp}) , \]
\[ 8 \left( \frac{R'(t)}{R(t)} \right)^2 + 8 \frac{R'(t)\chi'(t)}{R(t)\chi(t)} + 4 \frac{R''(t)}{R(t)} = -2k_6^2 T_{pp} . \] (11)

Since \( \delta(\rho), \delta(\phi) = 0 \) in the bulk, i.e. \( T_{MN} = -\Lambda g_{MN} \) and \( R(t) = e^{Ht} \) with \( H = \sqrt{\Lambda_5/3} \), we have the following equation for \( \chi(t) \)
\[ 12H^2 + 8H \frac{\chi'(t)}{\chi(t)} - 2k_6^2 \Lambda = 0 , \]
\[ \frac{\chi'(t)}{\chi(t)} = -\frac{3}{2} H + \frac{1}{4H} k_6^2 \Lambda . \] (12)

Let \( A_{in} \) be the initial value of \( \chi(t) \) (at the onset of inflation), i.e. \( \chi(0) = A_{in} \), then the solutions of Eq. (12) are
\[ \chi(t) = A_{in} e^{-\frac{3}{2}Ht} e^{\frac{k_6^2 \Lambda}{4Ht}} . \] (13)

In the next two sections, we will analyze two cases of the cosmological constant \( \Lambda \).

3. The case \( \Lambda > 0 \)

From Eq. (9) we have
\[ G_{pp} = -\frac{\chi''(t)}{\chi(t)} - 3 \left( \frac{R'(t)}{R(t)} \right)^2 - 3 \frac{R'(t)\chi'(t)}{R(t)\chi(t)} = k_6^2 T_{pp} , \]
or, equivalently
\[ \chi''(t) + 3H \chi'(t) + \left( 6H^2 - k_6^2 \Lambda \right) \chi(t) = 0 . \] (14)

Its general solutions are given by
\[ \chi(t) = C_1 e^{-\frac{3}{2}Ht} e^{\frac{1}{2}\sqrt{4k_6^2 \Lambda - 15H^2 t}} + C_2 e^{-\frac{3}{2}Ht} e^{-\frac{1}{2}\sqrt{4k_6^2 \Lambda - 15H^2 t}} , \]
where \( C_1 \) and \( C_2 \) are arbitrary constants.
The solutions (13) and (15) must agree, particularly at \( t = \tau \), the end of inflation. It implies that
\[ C_1 = A_{in} \quad and \quad C_2 = 0 . \] (16)

In fact, the second term \( e^{-\frac{3}{2}Ht} e^{-\frac{1}{2}\sqrt{4k_6^2 \Lambda - 15H^2 t}} \) is many orders of magnitude below Planck scale, due to \( H\tau = 100 \) [9], therefore it does not make physical sense and could be eliminated. So
\[ \chi(t) = A_{in} e^{-\frac{3}{2}Ht} e^{\frac{1}{2}\sqrt{4k_6^2 \Lambda - 15H^2 t}} . \] (17)

Compare (13) with (15), we get
\[ \frac{k_6^2 \Lambda}{4H} = \frac{1}{2} \sqrt{4k_6^2 \Lambda - 15H^2} . \] (18)

Two solutions of this equation are \( \Lambda_1 = 2 \Lambda_5/k_6^2 \) and \( \Lambda_2 = 10 \Lambda_5/3k_6^2 \).

It is easy to show that these solutions are consistent with \( G_{pp} \) and \( G_{\phi\phi} \). From Eq. (12) we can derive
\[ \frac{\chi'(t)}{\chi(t)} = \frac{9}{4} H^2 + \frac{3}{16} k_6^2 \Lambda + \frac{k_6^4 \Lambda^2}{16H^2} . \] (19)
and Eq. (14) becomes

\[
\frac{5\Lambda_5}{4} + \frac{3(k_6^2 \Lambda)^2}{16\Lambda_6} = -k_6^2 \Lambda. \tag{20}
\]

This equation is satisfied, i.e. \(\delta(\rho)\tau_{\rho\rho} = 0\) and \(\delta(\phi)\tau_{\phi\phi} = 0\) in the brane and anti-brane if and only if \(\Lambda = \Lambda_1\) or \(\Lambda = \Lambda_2\).

We emphasize that due to the property of \(\delta\)-function for any continuos \(\tau_{MN}\)

\[
\delta(\rho)\delta(\phi)\tau_{MN} = \delta(\rho)\delta(\phi) (\tau_{MN} \mid _{\rho=0, \phi=0}) \tag{21}
\]

which describes the confinement to the brane and anti-brane, respectively.

For \(\Lambda = 2\Lambda_5/k_6^2\), Eq (13) yields

\[
\chi(t) = A_{in}e^{-\frac{3}{2}Ht}e^{\frac{3}{2}Ht} = A_{in}. \tag{22}
\]

For \(\Lambda = 10\Lambda_5/3k_6^2\), from both Eq. (13) and (15) we obtain

\[
\chi(t) = A_{in}e^{-\frac{3}{2}Ht}e^{\frac{5}{2}Ht} = A_{in}e^{Ht}. \tag{23}
\]

So, the extra dimensions expand by the factor \(e^{Ht}\) in the branes, which is similar to the case of the four dimensional space.

4. The case \(\Lambda < 0\)

If \(\Lambda < 0\), the solution (13) of Eq. (12)

\[
\chi(t) = A_{in}e^{-\frac{3}{2}Ht}e^{\frac{5}{2}Ht} \tag{14}
\]

is not acceptable, since at the end of inflation \((t = \tau, H\tau = 100)\), \(\chi(t)\) becomes many orders of magnitude below Planck scale.

In this case, the solution of (14) takes the form

\[
\chi(t) = e^{-\frac{3}{2}Ht} \left( C_1 \cos \frac{1}{2} \sqrt{15H^2 - 4k_6^2 \Lambda t} + C_2 \sin \frac{1}{2} \sqrt{15H^2 - 4k_6^2 \Lambda t} \right). \tag{24}
\]

It is plausible, but as in 5D [7], it also shows that \(\chi(t)\) has shrunk significantly at the end of inflation.

5. Conclusion and discussion

In the above sections, the brane-world scenarios matter field are analyzed in the frame of the theory with two extra dimensions, where the spatial dimensions are time-dependent. It shows that when the cosmological constant \(\Lambda > 0\), all spatial distance expands by the factor \(e^{Ht}\) in the brane and anti-brane. When \(\Lambda < 0\), the scalar factor \(\chi(t)\) of external dimensions shrinks significantly at the end of inflation.

However, if the extra spatial are tightly curled up, as in conventional assumptions, a huge amounts of potential energy should be stored. It could be estimated, for example

\[
R_{\mu\rho}^{\phi} = R_{\mu\phi}^{\rho} = \left| \frac{\chi'(t)}{\chi(t)} \right| = H^2 = \frac{\Lambda_5}{3} \tag{25}
\]

where the Hubble parameter is determined by [10]

\[
H^2 = \frac{k_6^4 T^4}{(10^{19} GeV)^2}. \tag{26}
\]

The source of this energy might just be the dark energy that accelerate the Universe’s expansion rate [7].
Appendix
The components of the Ricci tensor are

\[
\begin{align*}
R_{tt} &= -3 \frac{R''(t)}{R(t)} - 2 \frac{\chi''(t)}{\chi(t)}, \\
R_{rr} &= \frac{R''(t)}{R(t)} + 2 \left( \frac{R'(t)}{R(t)} \right)^2 + 2 \frac{R'(t) \chi'(t)}{R(t) \chi(t)}, \\
R_{\theta\theta} &= R_{\varphi\varphi} = R_{pp} = 3 \frac{R'(t) \chi'(t)}{R(t) \chi(t)} + \frac{\chi''(t)}{\chi(t)} + \left( \frac{\chi'(t)}{\chi(t)} \right)^2, \\
R_{\rho\rho} &= R_{\phi\phi}.
\end{align*}
\]

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