Asymptotic Properties of Hilbert Geometry

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Abstract

We show that the spheres in Hilbert geometry have the same volume growth entropy as those in the Lobachevsky space. We give the asymptotic estimates for the ratio of the volume of metric ball to the area of the metric sphere in Hilbert geometry. Derived estimates agree with the well-known fact in the Lobachevsky space.

Key words: Hilbert geometry, Finsler geometry, balls, spheres, volume, area, entropy.

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1 Introduction

Hilbert geometry is the generalization of the Klein model of the Lobachevsky space. The absolute there is an arbitrary convex hypersurface unlike an ellipsoid in the Lobachevsky space. Hilbert geometries are simply connected, projectively flat, complete reversible Finsler spaces of constant negative flag curvature $-1$.

B. Colbois and P. Verovic proved in [12] that the balls in an $(n+1)$-dimensional Hilbert geometry have the same volume growth entropy as those in $\mathbb{H}^{n+1}$, namely $n$. We obtain the analogous result for the spheres in Hilbert geometry.

Theorem 1. Consider an $(n+1)$-dimensional Hilbert geometry associated with a bounded open convex domain $U \subset \mathbb{R}^{n+1}$ whose boundary is a $C^3$ hypersurface with positive normal curvatures. Then we have

$$\lim_{t \to \infty} \frac{\ln(Vol(S^n_t))}{t} = n$$

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It is known [4, 5, 6, 7] that in the Lobachevsky space $\mathbb{H}^{n+1}$ of constant curvature $-1$ for a family of metric balls $\{B_t^{n+1}\}_{t \in \mathbb{R}^+}$ the following equality holds
\[
\lim_{\rho \to \infty} \frac{\text{Vol}(B_{\rho}^{n+1})}{\text{Vol}(S_{\rho}^{n})} = \frac{1}{n}
\]

Such a ratio in a more general case for $\lambda$- and $h$-convex hypersurfaces in Hadamard manifolds was considered in [4, 6, 7] by A. A. Borisenko, V. Miquel, A. Reventos and E. Gallego.

Similar estimates in Finsler spaces were derived in [5] (see also [16]).

**Theorem** [5]. Let $(M^{n+1}, F)$ be an $(n+1)$-dimensional Finsler-Hadamard manifold that satisfies the following conditions:

1. Flag curvature satisfies the inequalities $-k_2^2 \leq K \leq -k_1^2$, $k_1, k_2 > 0$.
2. $S$-curvature satisfies the inequalities $n\delta_1 \leq S \leq n\delta_2$ such that $\delta_i < k_i$.

Then for a family $\{B_{\rho}^{n+1}(p)\}_{\rho \geq 0}$ we have
\[
\frac{1}{n(k_2 - \delta_2)} \leq \lim_{r \to \infty} \inf \frac{\text{Vol}(B_{\rho}^{n+1}(p))}{\text{Area}(S_{\rho}^{n}(p))} \leq \lim_{r \to \infty} \sup \frac{\text{Vol}(B_{\rho}^{n+1}(p))}{\text{Area}(S_{\rho}^{n}(p))} \leq \frac{1}{n(k_1 - \delta_1)}.
\]

Our goal is to prove analogous result in Hilbert geometry for a family $\{B_{t}^{n+1}\}_{t \in \mathbb{R}^+}$. Applying the theorem from [5] is the rather difficult task because the $S$-curvature in Hilbert geometry is difficult to calculate.

As the result the following theorem is obtained.

**Theorem 2.** Consider an $(n+1)$-dimensional Hilbert geometry associated with a bounded open convex domain $U \subset \mathbb{R}^{n+1}$ whose boundary is a $C^3$ hypersurface with positive normal curvatures. Fix a point $o \in U$, we will consider this point as the origin and the center of all the considered balls. Denote by $\omega(u) : S^n \to \mathbb{R}_+$ the radial function for $\partial U$, i.e. the mapping $\omega(u)u, u \in S^n$ is a parametrization of $\partial U$, and by $\iota : \mathbb{R}^{n+1} \to S^n$ the mapping such that $\iota(p) = \frac{u_p}{||u_p||}, u_p$ is the radius-vector of a point $p$.

Denote by $K$ and $k$ the maximum and minimum normal curvature of $\partial U$, $c = \max_{u \in S^n} \frac{\omega(u)}{\omega(-u)}, \omega_0 = \min_{u \in S^n} \omega(u), \omega_1 = \max_{u \in S^n} \omega(u)$. Then we have
\[
\lim_{\rho \to \infty} \sup \frac{\text{Vol}(B_{\rho}^{n+1})}{\text{Vol}(S_{\rho}^{n})} \leq \frac{1}{n} \frac{1}{c_{\rho}^{\frac{n}{2}}} \left( \frac{K}{k} \right)^{\frac{n}{2}} \frac{1}{(k\omega_0)^{n+1}} \frac{\int_{S^n} \omega(u) \, du}{\int_{\partial U} \omega(\iota(p)) \, dp} \]
\[
\lim_{\rho \to \infty} \inf \frac{\text{Vol}(B_{\rho}^{n+1})}{\text{Vol}(S_{\rho}^{n})} \geq \frac{1}{n} \frac{1}{c_{\rho}^{\frac{n}{2}}} \left( \frac{k}{K} \right)^{\frac{n}{2}} \frac{k\omega_0}{(k\omega_0)^{n+1}} \frac{\int_{S^n} \omega(u) \, du}{\int_{\partial U} \omega(\iota(p)) \, dp} \]

or, more simple expression
\[
\lim_{\rho \to \infty} \sup \frac{\text{Vol}(B_{\rho}^{n+1})}{\text{Vol}(S_{\rho}^n)} \leq \frac{1}{n} \left( \frac{K}{k} \right)^{\frac{n}{2}} \left( \frac{\omega_1}{\omega_0} \right)^{n+1} \left( \frac{\omega_1}{k} \right)^{\frac{n}{2}} \frac{1}{k \omega_1} \frac{\text{Vol}_E(S^n)}{\text{Vol}_E(\partial U)}
\]

\[
\lim_{\rho \to \infty} \inf \frac{\text{Vol}(B_{\rho}^{n+1})}{\text{Vol}(S_{\rho}^n)} \geq \frac{1}{n} \left( \frac{k}{K} \right)^{\frac{n}{2}} \left( \frac{\omega_0}{\omega_1} \right)^n \frac{\omega_1^n (k \omega_0)^{\frac{n}{2} + 1}}{\omega_0^n} \frac{\text{Vol}_E(S^n)}{\text{Vol}_E(\partial U)}
\]

If \( U \) is a symmetric domain with respect to \( o \) then we have

\[
\lim_{\rho \to \infty} \sup \frac{\text{Vol}(B_{\rho}^{n+1})}{\text{Vol}(S_{\rho}^n)} \leq \frac{1}{c} \left( \frac{k}{K} \right)^{\frac{n}{2}} \frac{\omega_1^n}{(k \omega_0)^{\frac{n}{2} + 1}} \frac{\text{Vol}_E(S^n)}{\text{Vol}_E(\partial U)}
\]

\[
\lim_{\rho \to \infty} \inf \frac{\text{Vol}(B_{\rho}^{n+1})}{\text{Vol}(S_{\rho}^n)} \geq \frac{1}{c} \left( \frac{k}{K} \right)^{\frac{n}{2}} (k \omega_0)^{\frac{n}{2}} \omega_0^n \frac{\text{Vol}_E(S^n)}{\text{Vol}_E(\partial U)}
\]

Notice that in this theorem the ratio of the volume of the ball to the internal volume of the sphere is considered, unlike theorem [5], where the induced volume is used.

## 2 Preliminaries

### 2.1 Finsler geometry

In this section we recall some basic facts and theorems from Finsler geometry that we need. See [16] for details.

Let \( M^n \) be an \( n \)-dimensional connected \( C^\infty \)-manifold. Denote by \( TM^n = \bigcup_{x \in M^n} T_x M^n \) the tangent bundle of \( M^n \), where \( T_x M^n \) is the tangent space at \( x \). A **Finsler metric** on \( M^n \) is a function \( F : TM^n \to [0, \infty) \) with the following properties:

1. \( F \in C^\infty(TM^n \setminus \{0\}) \);
2. \( F \) is positively homogeneous of degree one, i. e. for any pair \( (x, y) \in TM^n \) and any \( \lambda > 0 \), \( F(x, \lambda y) = \lambda F(x, y) \);
3. For any pair \( (x, y) \in TM^n \) the following bilinear symmetric form \( g_y : T_x M^n \times T_x M^n \to \mathbb{R} \) is positively definite,

\[
g_y(u, v) := \left. \frac{\partial^2}{\partial t \partial s} [F^2(x, y + su + tv)] \right|_{s=t=0}
\]

The pair \( (M^n, F) \) is called a **Finsler manifold**.

If we denote by

\[
g_{ij}(x, y) = \left. \frac{\partial^2}{\partial y^i \partial y^j} [F^2(x, y)] \right|
\]
then one can rewrite the form $g_g(u, v)$ as

$$
g_g(u, v) = g_{ij}(x, y)u^i v^j
$$

For any fixed vector field $Y$ defined on the subset $U \subset M^n$, $g_Y(u, v)$ is a Riemannian metric on $U$.

Given a Finsler metric $F$ on a manifold $M^n$. For a smooth curve $c : [a, b] \to M^n$ the length is defined by the integral

$$
L_F(c) = \int_a^b F(c(t), \dot{c}(t))dt = \int_a^b \sqrt{g_{ij}(\dot{c}(t)) \dot{c}^i(t) \dot{c}^j(t))dt}.
$$

Let $\{e_i\}_{i=1}^n$ be an arbitrary basis for $T_xM^n$ and $\{\theta^i\}_{i=1}^n$ the dual basis for $T^*_xM^n$. Consider the set $B^n_F(x) = \{ (y^i) \in \mathbb{R}^n : F(x, y^i e_i) < 1 \} \subset T_xM^n$. Denote by $\text{Vol}_E(A)$ the Euclidean volume of $A$. Then define the form

$$
dV_F = \sigma_F(x) \theta^1 \wedge \ldots \wedge \theta^n,
$$

here

$$
\sigma_F(x) := \frac{\text{Vol}_E(\mathbb{B}^n)}{\text{Vol}_E(B^n_F(x))}. \quad (1)
$$

and $\mathbb{B}^n$ is the unit ball in $\mathbb{R}^n$.

The volume form $dV_F$ determines a regular measure $\text{Vol}_F = \int dV_F$ and is called the Busemann-Hausdorff volume form.

For any Riemannian metric $g(u, v) = g_{ij}(x)u^i v^j$ the Busemann-Hausdorff volume form is the standard Riemannian volume form

$$
dV_g = \sqrt{\det(g_{ij})} \theta^1 \wedge \ldots \wedge \theta^n.
$$

It was proved in [9] that the Busemann-Hausdorff measure for reversible metric coincides with the $n$-dimensional outer Hausdorff measure. Recall that the $n$-dimensional outer Hausdorff measure of a set $A$ is defined by

$$
\nu_n = \lim_{r \to 0} \nu_{n,r},
$$

$$
\nu_{n,r} = \text{Vol}_E(\mathbb{B}^n) \inf \left( \sum_i \rho_i^n : 2\rho_i < r, A \subseteq \bigcup_i B[x_i, \rho_i], x_i \in A \right)
$$

It should be noticed here that if we calculate the Hausdorff measure for the submanifold in a Finsler manifold with the symmetric metric then we will obtain the internal volume on submanifold in the metric induced from the ambient space. But unfortunately using of this volume implies certain difficulties. In our case when we consider a sphere as the submanifold the following claim does not hold

$$
\text{Vol}(B^n_r) = \int_0^r \text{Vol}(S^{n-1}_t)dt,
$$

if we use the internal volume. For details, see [16].
2.2 Hilbert geometry

Consider a bounded open convex domain $U \subset \mathbb{R}^{n+1}$ whose boundary is a $C^3$ hypersurface with positive normal curvatures in $\mathbb{R}^n$ equipped with a Euclidean norm $\| \cdot \|$. For given two distinct points $p$ and $q$ in $U$, let $p_1$ and $q_1$ be the corresponding intersection point of the half line $p + \mathbb{R}_-(q-p)$ and $p + \mathbb{R}_+(q-p)$ with $\partial U$ (Fig. 1).

![Figure 1: Hilbert metric](image)

Then consider the following distance function.

$$
\begin{align*}
d_U(p, q) &= \frac{1}{2} \ln \frac{\|q - q_1\|}{\|q - p_1\|} \times \frac{\|p - p_1\|}{\|p - q_1\|}
\end{align*}
$$

(2)

$$
d_U(p, p) = 0
$$

The obtained metric space $(U, d_U)$ is called Hilbert geometry and is a complete noncompact geodesic metric space with the $\mathbb{R}^n$-topology and in which the affine open segments joining two points are geodesics [10].

The distance function is associated in a natural way with the Finsler metric $F_U$ on $U$. For a point $p \in U$ and a tangent vector $v \in T_p U = \mathbb{R}^n$

$$
F_U(p, v) = \frac{1}{2} \|v\| \left(\frac{1}{\|p - p_\|} + \frac{1}{\|p - p_+\|}\right)
$$

(3)

where $p_-$ and $p_+$ is the intersection point of the half-lines $p + \mathbb{R}_-v$ and $p + \mathbb{R}_+v$ with $\partial U$.

Then $d_U(p, q) = \inf \int_I F_U(c(t), \dot{c}(t))dt$ when $c(t)$ ranges over all smooth curves joining $p$ to $q$.

It is known (see for example [16]) that Hilbert metrics are the metrics of constant flag curvature $-1$. 

5
When \( U = B^n_r \) then we obtain the Klein model of the \( n \)-dimensional Lobachevsky space \( \mathbb{H}^n \) and the Finsler metric has the explicit expression

\[
F_{B^n_r}(p, v) = \sqrt{\frac{\|v\|^2}{r - \|p\|^2} + \frac{< v, p >^2}{(r^2 - \|p\|^2)^2}}
\] (4)

It is proved in [10] that the balls of arbitrary radii are convex sets in Hilbert geometry.

The asymptotic properties of Hilbert geometry have been obtained lately. All these properties mean that Hilbert geometry is ”almost” Riemannian at infinity. It is proved in [12] that Hilbert metric ”tends” to Riemannian metric as follows.

**Theorem [12].** Let \( C \in \mathbb{R}^n \) be a bounded open convex domain whose boundary \( \partial C \) is a hypersurface of class \( C^3 \) that is strictly convex. For any \( p \in C \) let \( \delta(p) > 0 \) be the Euclidean distance from \( p \) to \( \partial C \). Then there exists a family \((\tilde{l}_p)_{p \in C}\) of linear transformations in \( \mathbb{R}^n \) such that

\[
\lim_{\delta(p) \to 0} \frac{F_C(p, v)}{\|\tilde{l}_p(v)\|} = 1
\]

uniformly in \( v \in \mathbb{R}^n \setminus \{0\} \)

This means that the unit sphere in the tangent space of given Hilbert metric tends to ellipsoid in continuous topology as the tangent point goes to the absolute.

3 Calculating the volume growth entropy of spheres.

In this section we will prove that for an \((n+1)\)-dimensional Hilbert geometry

\[
\lim_{t \to \infty} \frac{\ln(\text{Vol}(S^n_r))}{t} = n,
\]

as it is in \( \mathbb{H}^{n+1} \).

Consider a bounded open convex domain \( U \subset \mathbb{R}^{n+1} \) whose boundary is a \( C^3 \) hypersurface with positive normal curvatures in \( \mathbb{R}^n \).

Fix a point \( o \in U \), we will consider this point as the origin and the center of all the considered balls. Denote by \( \omega(u) : S^n \to \mathbb{R}_+ \) the radial function for \( \partial U \), i.e. the mapping \( \omega(u)u, u \in S^n \) is a parametrization of \( \partial U \). Let \( B^{n+1}_r(o) \) be the metric ball of radius \( r \) centered at a point \( o \), \( S^n_r(o) = \partial B^{n+1}_r(o) \) be the metric sphere.

We will use the following lemma that shows the order of growth of the Hilbert distance from the sphere to \( \partial U \) in terms of the Euclidean distance. We also estimate the deviation of the tangent and normal vectors to sphere from those to \( \partial U \).
Lemma 1. Let \( \omega(u)u : S^n \rightarrow \mathbb{R}_+ \) be the parametrization of \( \partial U \), \( \rho_t(u) : S^n \rightarrow \mathbb{R}_+ \) the parametrization of the sphere of radius \( t \).

Then, as \( t \rightarrow \infty \):

1. \( \omega(u) - \rho_t(u) = \Delta(u)e^{-2t} + o(e^{-2t}) \);

   \[
   \Delta(u) = \omega(u) \left( \frac{\omega(u)}{\omega(-u)} + 1 \right)
   \]

2. \( \omega_i'(u) - \rho_{t,i}'(u) = \Delta_i(u)e^{-2t} + o(e^{-2t}) \);

   \[
   \Delta_i(u) = \left[ \omega_i'(u) \left( 2\frac{\omega(u)}{\omega(-u)} + 1 \right) + \left( \frac{\omega(u)}{\omega(-u)} \right)^2 \omega_i'(-u) \right]
   \]

3. \( \omega_{ij}''(u) - \rho_{t,ij}''(u) = \Delta_{ij}(u)e^{-2t} + o(e^{-2t}) \);

   \[
   \omega(u)^2 \Delta_{ij}(u) = \omega(u)[2\omega_i'(-u)\omega_j'(-u) - \omega(-u)\omega_{ij}''(-u)] + \omega(-u)^2[2\omega_i'(u)\omega_j'(u) + \omega(-u)\omega_{ij}'(u)] + \\
   + 2\omega(-u)\omega_i'(u)\omega_j'(u) + \omega_i'(-u)\omega_j'(u) + \omega_i'(-u)\omega_j'(u)
   \]

Proof of lemma 1. We are going to obtain the explicit expression for \( \rho_t(u) \). Let \( q = 0 \) be the center of the sphere, \( p \) be a point on the sphere. Using formula (3), we obtain the equation on the function \( \rho_t(u) \)

\[
\frac{1}{2} \ln \left[ \frac{\omega(u)}{\omega(-u)} \times \frac{\omega(-u) + \rho_t(u)}{\omega(u) - \rho_t(u)} \right] = t
\]

By the direct computation we have

\[
\rho_t(u) = \frac{\omega(-u)\omega(u)(e^{2t} - 1)}{\omega(u) + \omega(-u)e^{2t}}
\]

1. Consider the difference

\[
\omega(u) - \rho_t(u) = \omega(u) - \frac{\omega(-u)\omega(u)(e^{2t} - 1)}{\omega(u) + \omega(-u)e^{2t}} = \\
= \frac{\omega^2(u) + \omega(-u)\omega(u)}{\omega(u) + \omega(-u)e^{2t}} = \omega(u) \left( \frac{\omega(u)}{\omega(-u)} + 1 \right) e^{-2t} + o(e^{-2t}), t \rightarrow \infty
\]

2. We obtain analogously

\[
\omega'(u) - \rho_{t,i}'(u) = \\
= \frac{\omega_i'(u)\omega(-u)^2e^{2t} + 2e^{2t}\omega(u)\omega(-u)\omega_i'(u) + \omega(u)^2(\omega_i'(u) + \omega_i'(-u)(e^{2t} - 1))}{(\omega(u) + \omega(-u)e^{2t})^2} = \\
= \left[ \omega_i'(u) \left( 2\frac{\omega(u)}{\omega(-u)} + 1 \right) + \left( \frac{\omega(u)}{\omega(-u)} \right)^2 \omega_i'(-u) \right] e^{-2t} + o(e^{-2t}), t \rightarrow \infty
\]
3. It can be proved in the same manner. □

Denote by $k$ and $K$ the minimum and maximum Euclidean normal curvatures of $\partial U$.

We also use the notations $\omega_0 = \min_{u \in \mathbb{S}^n} \omega(u)$, $\omega_1 = \max_{u \in \mathbb{S}^n} \omega(u)$.

The following lemma gives the estimates on the angle between the radial and normal directions at the points from $\partial U$.

**Lemma 2.** For a given point $m = \omega(u_m)u_m \in \partial U$ denote by $N(m)$ the normal vector at $m$. Then

$$\cos \angle(u_m, N(m)) \geq \frac{\omega_0}{R}$$

**Proof of lemma 2.** This lemma follows from the more general theorem.

**Theorem [4, 6, 7].** Let $N$ be a hypersurface in a Riemannian manifold $M$. Consider $N$ as defined by the the equation $t = \rho(\theta)$ of class $C^2$, where $\rho(\theta)$ is the distance to a point $o$. $N$ can be seen as the 0-level set of the function $F = t - \rho$. For given point $P \in N$ we consider all the vectors to be attached at $P$. Denote by $Y = \frac{\text{grad}_N \rho}{\|\text{grad}_N \rho\|}$. Let $x$ be a unit vector in the plane spanned on $y$ and the radial direction that is orthogonal to the radial direction. Let $\varphi$ be the angle between the normal direction and the radial direction at the point $P \in N$.

If $k_n$ is the normal curvature at $P$ in the direction given by $Y$, $\mu_n$ is the normal curvature in the direction of $x$ of the sphere centered at $o$ of radius $\rho$ and $\frac{d\varphi}{ds}$ is the derivative of $\varphi$ with respect to the arc parameter of the integral curve of $Y$ by $P$ then

$$k_n = \mu_n \cos \varphi + \frac{d\varphi}{ds}$$

Now we can prove lemma 2.

Consider any integral curve $\gamma$ of $\frac{y}{\|y\|}$. Since the angle $\varphi$ takes its value in the interval $[0, \pi/2]$ then there is a supremum $\varphi_0$ of it. If at some point $\gamma(s_0)$ the value $\varphi_0$ is achieved then we have at this point $\varphi' = 0$ and

$$\cos \varphi = \frac{k_n}{\mu_n}$$

The minimum possible value of $k_n$ is equal to $k = \frac{1}{R}$, and the maximum possible value of $\mu_n$ is equal to $\frac{1}{\omega_0}$. Hence we have

$$\cos \varphi = \frac{k_n}{\mu_n} \geq \frac{\omega_0}{R}$$

And lemma 2 follows. □

**Proof of theorem 1.** Now we are going to estimate the volume of a sphere $S^n_\rho$ in Hilbert geometry. The idea of proof is to obtain the Hausdorff measure of this sphere. It follows from the reversibility of
Hilbert metrics that the Hausdorff measure coincides with the Finslerian Busemann-Hausdorff volume [9].

Fix the point \( p \) on the sphere \( S^n \). Since the spheres are convex we can choose the vector \( u \in S^n \) such that \( p = \rho_t(u) \). More generally, for a given origin \( o \in \mathbb{R}^{n+1} \) denote by \( u_p \) the corresponding radius vector and consider the function \( \iota : \mathbb{R}^{n+1} \to S^n \) such that \( \iota(p) = \frac{u_p}{||u_p||} \). Then we can write that \( p = \rho_t(\iota(u)) \).

Denote by \( m \) the point \( \omega(\iota(p))\iota(p) \in \partial U \). Consider the vector \( v_m \) which is tangent to \( \partial U \) at \( m \), the vector \( n_m \) which is orthogonal to \( v_m \) with respect to the Euclidean inner product such that the point \( o \) belongs to the plane \( \mathcal{P} \) spanned on \( v_m \) and \( n_m \). Let \( k_m \) be the curvature of the section of \( \partial U \) by \( \mathcal{P} \) at \( m \). Consider the special coordinate system in the plane \( \mathcal{P} \): let the axe \( z \) be directed as \( n_m \), and the axe \( x \) be directed as \( v_m \). Then in this special coordinate system the section of \( \partial U \) can be locally expressed as

\[
z(x) = \frac{1}{2}k_m x^2 + \bar{o}(x^2), \quad x \to 0
\]

Later on we will work with this section.

Draw the secant of the sphere that is parallel to the tangent vector at \( p \) (Fig. 2).

![Figure 2](image-url)

Put \( d = ||a_1 - a_2|| \), \( \delta(p) = ||m - p|| \), \( \delta_1 = ||b_1 - f|| \), \( \delta_2 = ||f - b_2|| \), \( h = ||p - f|| \).
Let us estimate the function $\delta(p)$. From lemma 1 we have

$$
\delta(p) \leq \omega(u) \left( \frac{\omega(u)}{\omega(-u)} + 1 \right) e^{-2t} + \bar{o}(e^{-2t}), t \to \infty
$$

(5)

From the triangle $pm_{1}m$ we have that $\delta(p) \approx \cos \angle(u_{m}, v_{m})(\omega(u) - \rho_{t}(u))$. Finally, using lemma 2 we obtain $\delta(p) \geq \frac{\omega_{0}}{R}(\omega(u) - \rho_{t}(u))$.

Consequently,

$$
\delta(p) \geq \frac{\omega_{0}}{R} \omega(u) \left( \frac{\omega(u)}{\omega(-u)} + 1 \right) e^{-2t} + \bar{o}(e^{-2t}), t \to \infty
$$

(6)

Then we estimate $\delta_1$ and $\delta_2$. Let

$$
z = a(p)x + \delta(p) + h
$$

be the equation of the secant in the special coordinates system. We will think at once that $h$ decreases faster than $\delta(p)$. Thus in the further computations we will neglect $h$.

We find the intersection points of this line with the boundary $\partial U$. From the expression for the boundary we have

$$
a(p)x + \delta(p) = \frac{1}{2}k_{m}x^{2}.
$$

Thus

$$
x_{1,2} = \frac{a(p) \pm \sqrt{a(p)^{2} + 2k_{m}\delta(p)}}{k_{m}}
$$

It follows from lemma 1 that $a(p) = a_{0}(u)e^{-2t}, t \to \infty$, for some function $a_{0}(u)$ and consequently $a(p) = O(\delta(p)), \delta(p) \to 0$. Therefore we have

$$
x_{1,2} = \pm \sqrt{\frac{2\delta(p)}{k_{m}}} + \bar{o}(\sqrt{\delta(p)}), \delta(p) \to 0
$$

$$
z_{1,2} = \frac{1}{2}k_{m}x^{2} + \bar{o}(x^{2})|_{x=x_{1,2}} = \frac{1}{2}\delta(p) + \bar{o}(\delta(p)), \delta(p) \to 0
$$

and

$$
\delta_1 = \sqrt{x_{1}^{2} + (z_{1} - \delta(p))^{2}} = \sqrt{\frac{2\delta(p)}{k_{m}}} + \bar{o}(\delta(p)) = \delta_2, \delta(p) \to 0
$$

Therefore the turning of the tangent as the point goes to the boundary does not influence on the asymptotic behavior of $\delta_i$. 





Compute the Hilbert length of the segment $a_1a_2$. Denote it by $d_U$. Then as $h \to 0$:

$$
\tilde{d}_U \approx \frac{1}{2} \ln \left[ \frac{d + \delta_1}{\delta_2} \times \frac{d + \delta_2}{\delta_1} \right] = \frac{1}{2} \ln \left[ \left( \frac{d}{\delta_2} + \frac{\delta_1}{\delta_2} \right) \times \left( \frac{d}{\delta_1} + \frac{\delta_2}{\delta_1} \right) \right] \approx \frac{1}{2} \left( \frac{d}{\delta_1} + \frac{d}{\delta_2} \right) \approx \frac{d}{\sqrt{\delta_1}} + \frac{d}{\sqrt{\delta_2}} + \tilde{o}(\sqrt{\delta_0(t)}), \delta_0(t) \to 0
$$

We are showing now that the limit of the ratio of $d_U$ to the Finslerian length $\tilde{d}_U$ of the geodesic arc $a_1a_2$ is equal to 1 as the arc is subtended to a point. Specialize the coordinate system on $\mathbb{R}^{n+1}$ so as $a_1 = 0$. Let $w(t) : [0, T] \to U$ be a parametrization of the arc. Then the segment from the point $a_1 = 0$ to the point $a_2 = w(t)$ can be parameterized by $v(s) = \frac{s}{t} w(t) : [0, t] \to U$. Calculate lengths of $v$ and $w$.

$$
\tilde{d}_U = \int_0^t F_U(w(s), \dot{w}(s)) \, ds
$$

$$
d_U = \int_0^t F_U(v(s), \dot{v}(s)) \, ds = \int_0^t F_U \left( \frac{s}{t} w(t), \frac{1}{t} \dot{w}(t) \right) \, ds.
$$

From the intermediate-value theorem for integrals we have

$$
\tilde{d}_U = \int_0^t F_U(w(s), \dot{w}(s)) \, ds = t F_U(w(s_0), \dot{w}(s_0)), s_0 \in [0, t]
$$

$$
d_U = \int_0^t F_U \left( \frac{s}{t} w(t), \frac{1}{t} \dot{w}(t) \right) \, ds = t F_U \left( \frac{s_1}{t} w(t), \frac{1}{t} \dot{w}(t) \right), s_1 \in [0, t]
$$

Now we are subtending the arc to a point, i.e. let $t \to 0$. Then $s_0, s_1 \to 0$, and we obtain:

$$
\frac{\tilde{d}_U}{d_U} = \frac{t F_U(w(s_0), \dot{w}(s_0))}{t F_U \left( \frac{s_1}{t} w(t), \frac{1}{t} \dot{w}(t) \right)} \to \frac{F_U(0, \dot{w}(0))}{F_U(0, \dot{w}(0))} = 1
$$

And the statement is proved.

Now our goal is to calculate the Hausdorff measure of the sphere $S^n_r$. Denote by $\delta_0(r)$ the Hausdorff distance from the points of the sphere to the absolute $\partial U$. Consider a covering $\{B_i\}$ of the sphere $S^n_r$ by balls of diameters $\tilde{d}_i$ centered at points $p_i \in S^n_r$. Denote by $k_i$ the normal curvature of $\partial U$ that corresponds to the $i$-th sphere from the covering (as above). As we saw, we can replace $\tilde{d}_i$ by the lengths of the corresponding chords $d_i$ of the sphere $S^n_r$.

Then the Hausdorff measure, and, consequently, the Finslerian Busemann-Hausdorff measure is given by

$$
\text{Vol}(S^n_r) = \text{Vol}_E(B^n) \inf_{d_{B_i}} \sum_i \left( \frac{d_i \sqrt{E_i}}{\sqrt{2(\delta_0(p_i) + h)}} \right)^n + \tilde{o}(\sqrt{1/\delta_0(t)}), \delta_0(t) \to 0,
$$

11
where infimum is calculated over all coverings of the sphere $S^n_r$.

Our metric sphere $S^n_r$ is sufficiently smooth, so we can proceed to the integral over $S^n_r$.

$$\text{Vol}(S^n_r) = \text{Vol}_E(\mathbb{B}^n) \inf_{d_{Bi}} \sum_i \left( \frac{d_i \sqrt{k_i}}{\sqrt{2(\delta(p_i) + h)}} \right)^n + \tilde{o}(\sqrt{1/\delta_0(t)^n}) =$$

$$\inf_{d_{Bi}} \sum_i \left( \frac{\sqrt{k_i}}{\sqrt{2(\delta(p_i) + h)}} \right)^n \text{Vol}_E(B_i) + \tilde{o}(\sqrt{1/\delta_0(t)^n}), \delta_0(t) \to 0,$$

Denote by $dp$ the area element of $S^n_t$. Proceeding to integral and estimating leads to

$$\text{Vol}(S^n_t) \geq k \frac{2}{n} \int_{S^n_t} \left( \frac{1}{2\delta(p)} \right)^{\frac{n}{2}} dp + \tilde{o}(\sqrt{1/\delta_0(t)^n}), \delta_0(t) \to 0$$

$$\text{Vol}(S^n_t) \leq K \frac{2}{n} \int_{S^n_t} \left( \frac{1}{2\delta(p)} \right)^{\frac{n}{2}} dp + \tilde{o}(\sqrt{1/\delta_0(t)^n}), \delta_0(t) \to 0$$

Using the explicit estimates (5), (6) for $\delta(p)$ as the result we have

$$\text{Vol}(S^n_t) \geq k \frac{2}{n} \int_{S^n_t} \left( 2\omega(\iota(p)) \left( \frac{\omega(\iota(p))}{\omega(-\iota(p))} + 1 \right) \right)^{-\frac{n}{2}} dp \cdot e^{nt} + \tilde{o}(e^{nt}), t \to \infty$$

$$\text{Vol}(S^n_t) \leq K \frac{2}{n} \int_{S^n_t} \left( 2\frac{\omega_0}{R} \omega(\iota(p)) \left( \frac{\omega(\iota(p))}{\omega(-\iota(p))} + 1 \right) \right)^{-\frac{n}{2}} dp \cdot e^{nt} + \tilde{o}(e^{nt}), t \to \infty$$

And theorem 1 follows. ■

4 Estimation of the ratio of the volume of the ball to the volume of the sphere.

Here we will find the asymptotic behavior of the volume of the metric ball $B^{n+1}_t$ in Hilbert geometry. We will use the method introduced in [13] in which some necessary estimates were improved.

The volume of a metric ball is given by the integral

$$\text{Vol}(B^{n+1}_t) = \int_{B^{n+1}_t} \sigma(p) dp$$

Here $\sigma(p)$ is the Busemann-Hausdorff volume form. And the volume estimating problem is reduced to the estimating of the volume form. Recall (1) that

$$\sigma(p) := \sigma_{F_U}(p) = \frac{\text{Vol}_E(\mathbb{B}^n)}{\text{Vol}_E(B^{n+1}_{F_U(p)})}.$$
Thus we have to estimate the volume of the unit sphere in the tangent space at the point \( p \in U \).

We will use the following simple lemma.

**Lemma 3.** There exists a value \( \rho_0 \) such that for any points \( p \in U \) in the neighborhood \( d(p, \partial U) \leq \rho_0 \) there exist a unique point \( \pi(p) \in \partial U \) : \( d(p, \pi(p)) = d(p, \partial U) \)

Put \( m = \pi(p) \in \partial U \). Denote by \( k \) and \( K \) the minimum and maximum Euclidean normal curvatures of \( \partial U \). Then at any point \( m \in \partial U \) the tangent sphere of radius \( R := \frac{1}{k} \) contains \( U \), the tangent sphere of radius \( r := \frac{1}{K} \) is contained in \( U \) [2]. On two tangent spheres of the radii \( r \) and \( R \) at this point we construct corresponding Klein metrics \( F_r \) and \( F_R \). We can give the explicit expressions (4) for them.

Then the following inequalities hold

\[
\text{Vol}_E \left( B_{F_r(p)}^{n+1} \right) \leq \text{Vol}_E \left( B_{F_U(p)}^{n+1} \right) \leq \text{Vol}_E \left( B_{F_R(p)}^{n+1} \right) \tag{9}
\]

As it was shown in [13]:

\[
\text{Vol}_E \left( B_{F_R(p)}^{n+1} \right) = \text{Vol}_E \left( \mathbb{B}^{n+1} \right) R^{n+1} \left\{ 1 - \left( 1 - \frac{d(p, m)}{R} \right)^2 \right\}^{\frac{n+2}{2}}
\]

\[
\text{Vol}_E \left( B_{F_r(p)}^{n+1} \right) = \text{Vol}_E \left( \mathbb{B}^{n+1} \right) r^{n+1} \left\{ 1 - \left( 1 - \frac{d(p, m)}{r} \right)^2 \right\}^{\frac{n+2}{2}}
\]

Thus, we have

\[
\frac{1}{R^{n+1} \left\{ 1 - \left( 1 - \frac{d(p, m)}{R} \right)^2 \right\}^{\frac{n+2}{2}}} \leq \sigma(p) \leq \frac{1}{r^{n+1} \left\{ 1 - \left( 1 - \frac{d(p, m)}{r} \right)^2 \right\}^{\frac{n+2}{2}}} \tag{10}
\]

Consider the mapping

\[
\Phi(u, s) = \tanh(s) \omega(u) u : \mathbb{S}^n \times \mathbb{R} \rightarrow U
\]

It was shown in [13] that the mapping \( \Phi(u, s) \) satisfies the following properties

1. \( \Phi(\mathbb{S}^n, [0, \rho - c]) \subseteq B_{\rho}^{n+1} \subseteq \Phi(\mathbb{S}^n, [0, \rho + 1]) \) where \( c = \sup_{u \in \mathbb{S}^n} \frac{\omega(u)}{\omega(-u)} \)

Hence,

\[
\text{Vol}(\Phi(\mathbb{S}^n, [0, \rho - c])) \leq \text{Vol}(B_{\rho}^{n+1}) \leq \text{Vol}(\Phi(\mathbb{S}^n, [0, \rho + 1]))
\]

2. \( |\text{Jac}(\Phi(u, s))| = \omega(u)^{n+1} \tanh^n(s)(1 - \tanh^2(s)) \)
We improve the first property.
Fix $d > 0$. Consider the difference

$$
\rho_t(u) - \omega(u) \tanh(t + d) = \omega(u) \left( 1 - \frac{\omega(u) + \omega(-u)}{e^{2t}\omega(-u) + \omega(u)} - \tanh(t + d) \right) =
$$

$$
= \omega(u) \left( \frac{2}{e^{2(t+d)} + 1} - \frac{\omega(u) + \omega(-u)}{e^{2t}\omega(-u) + \omega(u)} \right) =
$$

$$
= \rho_t(u) - \omega(u) \tanh(t + d) = \omega(u)e^{-2t} \left( 2e^{-2d} - 1 - \frac{\omega(u)}{\omega(-u)} + o(e^{-2t}), t \to \infty \right)
$$

Thus $B_{\rho}^{n+1} \subseteq \Phi(S^n, [0, \rho + d])$ for sufficiently large $\rho$ if

$$
2e^{-2d} - 1 - \frac{\omega(u)}{\omega(-u)} \leq 0
$$

$$
d \geq -\frac{1}{2} \ln \left[ \frac{1}{2} \left( 1 + \frac{1}{c} \right) \right] := d_1
$$

and $B_{\rho}^{n+1} \supseteq \Phi(S^n, [0, \rho + d])$ for sufficiently large finite $\rho$ if

$$
d \leq -\frac{1}{2} \ln \left[ \frac{1}{2} \left( 1 + c \right) \right] := d_2
$$

Fix the values $d_2$ and $d_1$ and choose sufficiently large $\rho_0$. Then

$$
\textbf{Vol}(\Phi(S^n, [\rho_0, \rho + d_2])) \leq \textbf{Vol}(B_{\rho}^{n+1}) \leq \textbf{Vol}(\Phi(S^n, [0, \rho + d_1]))
$$

(11)

Notice that if the domain $U$ is centrally-symmetric then $d_1 = d_2 = 0$. In the worst case when $c \to \infty$ we have $d_1 \to \ln \sqrt{2} \approx 0.347 < 1$. Inclusion (11) is more precise than it was obtained in [13]. It will be essentially used in the proof of theorem 2.

The volume of the set $\Phi(S^n, [\rho_0, \rho])$ is given by.

$$
\textbf{Vol}(\Phi(S^n, [\rho_0, \rho])) = \textbf{Vol}_E(B^n) \int_{S^n} \int_{\rho_0}^\rho \sigma(\Phi(u, s)) |\text{Jac}(\Phi(u, s))| ds du
$$

It is known [13] that

$$
|\text{Jac}(\Phi(u, s))| = \omega(u)^{n+1} \tanh^n(s)(1 - \tanh^2(s)) = \omega(u)^{n+1} \frac{4e^{2s}(e^{2s} - 1)^{n+1}}{e^{4s} - 1}
$$

And, using the estimates (10) we obtain

$$
\int_{S^n} \int_{\rho_0}^\rho \frac{4\omega(u)^{n+1} e^{2s} (e^{2s} - 1)^{n+1}}{e^{4s} - 1} ds du \leq \textbf{Vol}(\Phi(S^n, [\rho_0, \rho]))
$$
\[
\text{Vol}(\Phi(S^n, [\rho_0, \rho])) \leq \int_{S^n} \int_{\rho_0}^{\rho} 4\omega(u)^{n+1} e^{2s} \left( \frac{e^{2s+1} \theta}{e^{2s-1}} \right)^{n+1} dsdu
\]

Out next task is to find the asymptotic behavior of the integral

\[
\int_0^r \frac{4e^{2s} \left( \frac{e^{2s+1}}{e^{2s-1}} \right)^{n+1}}{(1 - (1 - C e^{-2s})^2)^{n+2}} ds
\]

After the changing of the variable \( y = e^{-2s} \), we obtain the integral

\[
\int_{e^{-2r}}^{1} -8 \frac{y^{-2} \left( \frac{e^{-2s+1}}{e^{2s-1}} \right)^{n+1}}{(1 - (1 - Cy)^2)^{n+2}} dy = \int_{e^{-2r}}^{1} \frac{8(1 - y)^{n+1}}{(1 + y)^{n+1}(y^2 - 1)(Cy(2 - Cy))^{n+2}} dy =
\]

\[
= \int_{e^{-2r}}^{1} \frac{1}{C^{n+1} y^{n+1/2} 2^{n+1/2}} \cdot \frac{1}{(1 + y)^{n+1}(y^2 - 1)(2 - Cy)^{n+2}} dy
\]

Notice that

\[
\lim_{y \to 0} \left[ \frac{(1 - y)^{n+1/2} 2^{n+1/2}}{(1 + y)^{n+1}(y^2 - 1)(2 - Cy)^{n+2}} \right] = -1
\]

Taking this into account and making the inverse change of variable we get

\[
\int_0^r \frac{4e^{2s} \left( \frac{e^{2s+1}}{e^{2s-1}} \right)^{n+1}}{(1 - (1 - C e^{-2s})^2)^{n+2}} ds = \frac{1}{nC^{n+1/2} 2^{n+1/2}} e^{nr} + o(e^{nr}), r \to \infty \quad (12)
\]

The expression for \( \text{Vol}_E(B_{F_p(x)}) \) includes the quantity \( d(p, m) = d(p, \partial U) \).

Thus we need the estimates of \( d(p, m) \) for the point \( p = \Phi(u, s) \). So,

\[
d(\Phi(u, s), \omega(u) u) = \omega(u) - \tanh(s) \omega(u) = \omega(u) - \frac{e^{2s} - 1}{e^{2s} + 1} \omega(u) = \frac{2\omega(u)}{1 + e^{2s}}
\]

Finally,

\[
d(\Phi(u, s), \partial U) \leq 2\omega(u) e^{-2s} + o(e^{-2s}) \quad (13)
\]

On the other hand analogously as formula (6) we get

\[
d(\Phi(u, s), \partial U) \geq 2\frac{\omega_0}{R} \omega(u) e^{-2s} + o(e^{-2s}) \quad (14)
\]

Using (12), (13), (14), one can compute that

\[
\frac{1}{n} C_1 e^n + o(e^n) \leq \text{Vol}(\Phi(S^n, [\rho_0, \rho])) \leq \frac{1}{n} C_2 e^n + o(e^n), \rho \to \infty \quad (15)
\]
\[
\begin{align*}
C_1 &= \frac{1}{2^n} \int_{S^n} \left( \frac{\omega(u)}{R} \right)^{\frac{n}{2}} du \\
C_2 &= \frac{1}{2^n} \frac{R^{n+2}}{\omega_0^{n+2}} \int_{S^n} \left( \frac{\omega(u)}{r} \right)^{\frac{n}{2}} du
\end{align*}
\]

And, taking into account (11), (15), we have

\[
\frac{1}{n} C_1 e^{nd_2} e^{n_\rho} + o(e^{n_\rho}) \leq \text{Vol}(B_{\rho}^{n+1}) \leq \frac{1}{n} C_2 e^{n_\rho} e^{nd_1} + o(e^{n_\rho}), \rho \to \infty \quad (16)
\]

\textbf{Proof of theorem 2.} It follows from (6), (7), (16) that:

\[
\lim_{\rho \to \infty} \sup \frac{\text{Vol}(B_{\rho}^{n+1})}{\text{Vol}(S^n_{\rho})} \leq \frac{1}{n} \frac{1}{2^{n/2}} e^{nd_2} \frac{R^{n+2}}{\omega_0^{n+2}} \frac{\int_{S^n} \left( \frac{\omega(u)}{r} \right)^{\frac{n}{2}} du}{k^{\frac{n}{2}} \int_{\partial U} \left( \omega(i(p)) \left( \frac{\omega(i(p))}{\omega(i(p))} + 1 \right) \right)^{-\frac{n}{2}} dp} \leq \frac{1}{n} \frac{1}{2^{n/2}} \frac{K}{k} \frac{1}{(k \omega_0)^{\frac{n}{2}+1}} \frac{\int_{S^n} \omega(u)^{\frac{n}{2}} du}{\int_{\partial U} \left( \omega(i(p)) \left( \frac{\omega(i(p))}{\omega(i(p))} + 1 \right) \right)^{-\frac{n}{2}} dp}
\]

\[
\leq \frac{1}{n} \frac{1}{2^{n/2}} \frac{K}{k} \frac{\omega_1^1}{(k \omega_0)^{\frac{n}{2}+1}} \frac{\text{Vol}_E(S^n)}{\text{Vol}_E(\partial U)}
\]

Note that \( c \leq \frac{\omega_1}{\omega_0}. \) Hence

\[
\lim_{\rho \to \infty} \sup \frac{\text{Vol}(B_{\rho}^{n+1})}{\text{Vol}(S^n_{\rho})} \leq \frac{1}{n} \frac{1}{2^{n/2}} e^{nd_2} \frac{1}{(k \omega_0)^{\frac{n}{2}+1}} \frac{\int_{S^n} \left( \frac{\omega(u)}{R} \right)^{\frac{n}{2}} du}{k^{\frac{n}{2}} \int_{\partial U} \left( \omega(i(p)) \left( \frac{\omega(i(p))}{\omega(i(p))} + 1 \right) \right)^{-\frac{n}{2}} dp}
\]

\[
\geq \frac{1}{n} \frac{1}{2^{n/2}} \frac{k}{K} \frac{\omega_1^1}{(k \omega_0)^{\frac{n}{2}}} \frac{\int_{S^n} \omega(u)^{\frac{n}{2}} du}{\int_{\partial U} \left( \omega(i(p)) \left( \frac{\omega(i(p))}{\omega(i(p))} + 1 \right) \right)^{-\frac{n}{2}} dp}
\]

\[
\geq \frac{1}{n} \frac{1}{2^{n/2}} \frac{k}{K} \frac{\omega_1^1}{(k \omega_0)^{\frac{n}{2}}} \frac{\text{Vol}_E(S^n)}{\text{Vol}_E(\partial U)} \geq \frac{1}{n} \frac{1}{2^{n/2}} \frac{k}{K} \frac{(\omega_1)}{\omega_1} \frac{\omega_1^1}{(k \omega_0)^{\frac{n}{2}}} \frac{\text{Vol}_E(S^n)}{\text{Vol}_E(\partial U)}
\]

And the theorem follows. \( \blacksquare \)

\textbf{Example 1.} Let \( U = B_{\rho}^{n+1}. \) Then we get the Klein model of the Lobachevsky space. Applying theorem 2 to this space implies

\[
\omega(u) = \frac{1}{k} = \frac{1}{K} = r = R = \omega_0 = \rho
\]
\[ c = 1 \]
\[ \int_{\partial U} du = \rho^n \text{Vol}_E(S^n) \]

Therefore we have obtained the well-known result
\[ \lim_{\rho \to \infty} \frac{\text{Vol}(B_{\rho}^{n+1})}{\text{Vol}(S^n_\rho)} = \frac{1}{n} \]

Example 2. One should not hope that for all metrics of negative curvature such result holds.

Let \( U \) be a open bounded strongly convex domain in \( \mathbb{R}^n \), \( o = 0 \in \mathbb{R}^n \). Given a point \( x \in U \) and a direction \( y \in T_x U \backslash \{0\} \simeq U \backslash \{0\} \). The Funk metric \( F(x, y) \) is a Finsler metric that satisfies the following condition
\[ x + \frac{y}{F(x, y)} \in \partial U. \]

Then Hilbert metric is a symmetrized Funk metric
\[ F_U(x, y) = \frac{1}{2} [F(x, y) + F(x, -y)] \]

Funk metrics are of constant negative curvature \(-1/4\), but for such metrics [5]:
\[ \lim_{r \to \infty} \frac{\text{Vol}(B_r^{n+1})}{\text{Vol}(S^n_r)} = \infty. \]

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