How efficient is transport of quantum cargo through multiple highways?

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Quantum states can be efficiently transferred over a long distance if the entire quantum channel can be divided into several small blocks. We consider a scenario in which each block consists of two copies of a multipartite state – one is used for distributing an arbitrary quantum state to multiple parties while the other channel is required to concentrate it back to a single party. Both in noiseless and local noisy scenarios, we find one-shot quantum capacities of these channels in terms of fidelity, when the initial shared states in each block are the generalized Greenberger-Horne-Zeilinger and the generalized W states. We also consider a situation where optimal local measurements transform multipartite states to bipartite ones which can then be used as single-path channels for quantum state transmission in each segment. We show that in some parameter ranges, the former protocol provides strictly better fidelities than that of the latter, thereby establishing the importance of distributing and concentrating arbitrary quantum states via multipartite entangled states. In long distance quantum communication, over the local measurement based protocol. Moreover, we show that in presence of bit flip or bit-phase flip noise, shared generalized Greenberger-Horne-Zeilinger states possess an inherent noise detection and correction mechanism, leading to the same fidelity as in the noiseless case. We consider further noise models also, which do not enjoy the same mechanism.

I. INTRODUCTION

From the dawn of civilization, communication between different persons and groups played a crucial role in shaping human society. With the emergence of modern science and technologies, a new era in the transmission of information with and without security has been developed. In both the cases, it has been realized, both theoretically \cite{1-5} and experimentally \cite{6, 7}, that quantum technologies can provide higher efficiencies \cite{1} than their classical counterparts. In most of the cases, the main ingredient that helps to achieve such advantage is the bipartite entanglement of a shared quantum state between the sender and the receiver \cite{1}, (cf. \cite{8-13} for other quantum information processing tasks).

One of the most interesting consequences of bipartite entanglement is the ability to transfer an arbitrary quantum state, with the additional help of a finite amount of classical communication – quantum teleportation \cite{5}. It is an indispensable feature of quantum mechanics, since to achieve this task by using unentangled states, one requires infinite amount of classical communication between the sender, Alice, and the receiver, Bob (cf. \cite{14, 15}). Significantly, within few years of its discovery, it has been realized experimentally \cite{5, 7}, first by using photons \cite{16-19} and then in other physical substrates \cite{7, 20-25}, thereby establishing a new epoch in communication, which can potentially lead to a quantum internet \cite{10}, parallel to the extremely useful internet already in use.

One of the avenues by which this breakthrough can become more prominent is by the involvement of multiple parties, which will be a step-forward towards building a quantum network. Over the years, several quantum information transmission protocols with multipartite states were proposed and some of them have also been demonstrated in experiments \cite{26, 27}. On the other hand, if Alice and Bob are located over a long distance, it has been noticed that the task of teleporting an unknown quantum state from Alice to Bob by using a single entangled state is not the best resort. For example, in case of photonic systems with polarization degrees of freedom, photon loss approximately becomes exponential with the length of the channel \cite{17, 19}, reducing the quantum capacity of a quantum channel. As a remedy, it was proposed that in a noisy environment, quantum state transfer over a long distance can be divided into several small segments, in which entanglement distillation \cite{28} followed by a modified version of quantum teleportation, known as entanglement swapping \cite{29} are performed to obtain quantum efficiency – the entire process being called a quantum repeater (QR) \cite{30}.

Motivated by recent experimental achievements which establish an entangled channel over a few thousand kilometers \cite{19} and developments on QRs \cite{31}, we consider a quantum state transmission protocol, which is divided into several small blocks, involving multipartite states. In particular, each block consists of two multipartite states, among which one is used to transfer an arbitrary quantum state among multiple parties while the other one is for accumulating it back to a single one. We call this process as the “multipath quantum repeater” which has two components, “teledistribution” and “teleconcentration” (TD-TC) (see Fig. 1). The port which possesses the initial state to be transferred can be referred as an input port while the receiver who finally receives the unknown state is the output port, and other parties involved in this protocol can be called the auxiliary nodes.

The multiple-path protocol can reduce to a bipartite linear-chain scenario if all the auxiliary parties except-
A multiparty state, \( \rho \) state to be teleported to a receiver, Bob, denoted by \( A \) and \( B \), while \( B \) is also connected with the same Claires via another multiparty state, \( \tilde{\rho} \). A sender, Alice, \( A \) and \( B \) are respectively called input and output ports while Claires are the auxiliary nodes.

The paper is arranged as follows. Sec. II is devoted to a more detailed discussion of TD-TC protocol involving multipartite entangled states, between the source and the receiver. In Sec. III, we discuss the entire protocol in details for two families of three-qubit pure states, namely the generalized GHZ and the generalized W states belonging to two inequivalent SLOCC classes of three-qubit states, as quantum channels in a multipath protocol while optimal fidelities in local measurement-based case are also derived for the similar families of multiparty quantum states in Sec. IV for comparison. Sec. V deals with the case when one of the parties are affected by different kinds of local noise. Before conclusion (Sec. VII), we discuss the capacities of quantum state transfer when the entire distance is divided into several small blocks (see Sec. VI).

II. MULTIPATH QUANTUM REPEATER: SETTING THE STAGE

In this section, we will discuss a communication protocol where a sender and a receiver are connected via multiple channels. Alice, \( A \), intends to send an unknown quantum state, \( |\psi\rangle \), given by

\[
|\psi\rangle = a |0\rangle + b |1\rangle,
\]

with \( a \) and \( b \) being complex numbers, satisfying \( |a|^2 + |b|^2 = 1 \), to Bob, \( B \), with the help of \( N - 1 \) Claires, \( C_1, C_2, \ldots, C_{N-1} \). Two multiparty quantum states, \( \rho_N \), and \( \tilde{\rho}_N \) are shared between Alice and Claires, and between Claires and the Bob respectively (See Fig. 1 for illustration with \( N = 3 \)). The protocol consists of the following steps:

Step 1. Alice initiates the process by performing a joint measurement, \( M_A \), on the unknown input state \( |\psi\rangle \), and her part of the quantum state \( \rho_N \).

Step 2. Alice communicates her measurement outcome classically to the Claires, \( C_1, C_2, \ldots, C_{N-1} \), post which each of them performs local unitary operations, \( \{U_{C_i}^j\}, i = 1, \ldots, N-1, j = 1, \ldots, d \), on their respective parts of the shared state \( \rho_N \), where \( d \) is the number of elements in the measurement basis.

Step 3. The \( i \)th Claire performs a measurement \( \{M_{C_i}\} \), jointly on her part of the rotated post-measured state \( \rho_N \) and her part of the shared \( \tilde{\rho}_N \), at some later time which is possibly predecided by Alice and Bob.

Step 4. Based on the measurement results communicated by the Claires, Bob rotates his part of the quantum state with unitary operators, \( \{U_B^k\} \). We refer to such a protocol as "multipath QR", where Steps 1 and 2 are parts of TD while Steps 3 and 4 together constitutes the TC (for the same, (cf. [36])).

\[\text{FIG. 1. A schematic diagram of multipath quantum repeater. It consists of teledistribution and teleconcentration parts. A sender, Alice, } A, \text{ who posse a unknown quantum state to be teleported to a receiver, Bob, denoted by } B \text{ shares a multiparty state, } \rho, \text{ with Claires, } C_1 \text{ and } C_2, \text{ while } B \text{ is also connected with the same Claires via another multiparty state, } \tilde{\rho}. \text{ A and } B \text{ are respectively called input and output ports while Claires are the auxiliary nodes.}\]
Let us suppose that at the end of the protocol, Bob obtains the state $\rho_\psi$. The fidelity under TD-TC is then defined as

$$\mathcal{F}^{DC} = \int \langle \psi | \rho_\psi | \psi \rangle d\psi,$$

which depends on the choice of measurements, $M_A, \{M_{C_i}\}_{i=1}^{N-1}$ at the Alice’s and $N - 1$ Claire’s nodes respectively and also on the unitary rotations $\{U_{C_i}^j\}$ and $\{U_B^j\}$. Hence, the optimal fidelity of multipath QR by shared multiparty quantum channels, $\rho_N$ and $\rho_N$, is obtained by maximizing Eq. (2), over all measurement strategies and unitary operators, given by

$$\mathcal{F}^{DC}(\rho_N, \tilde{\rho}_N) = \max_{M_A, \{M_{C_i}\}, \{U_{C_i}^j\}, \{U_B^j\}} \int \langle \psi | \rho_\psi | \psi \rangle d\psi.$$

Note that if a sender and a receiver share a quantum state having vanishing entanglement, the teleportation fidelity of sending a qubit cannot go beyond $\frac{2}{3}$, with the help of classical communication, while the fidelity reduces to $\frac{1}{2}$, when there is no classical communication allowed between them. In this paper, whenever we encounter a product state, shared between Alice and Claire or between Claire and Bob, we put the value of fidelity to be $\frac{2}{3}$. Note that a pure quantum state between two parties, if without entanglement, can only be product.

Before presenting the results, let us discuss the scenario where multipath QR can be useful. The advantages of QR is well known and hence we only concentrate on the benefit of TD-TC protocol. Apart from the advantages obtained in terms of fidelity, we discuss the importance of the auxiliary nodes in the protocol and then the benefit of several nodes instead of a single one. First of all, consider a scenario where Alice wants to send an unknown quantum state at some predecided later time, when Alice is not available and Claire is ready to help Alice. In particular, Claire’s presence is essential when Alice needs to leave the laboratory at some earlier time, or the location of her laboratory is in such a place that she could not be present at the time of transmission. For example, such a situation can arise if Alice is a part of secret agency and she has to perform her measurement as soon as she gets the unknown state, any delay on her part can cause unnecessary threat to her as well as to the protocol. Secondly, it can also be argued that if the protocol involve a single Claire, and if she is “uncooperative” or is compromised by any third party (enemy), there is a possibility of a measurement and a feed forward procedure, resulting in information leakage before the time. We will show that the introduction of many Claire’s, not only helps Alice-Bob to overcome Alice’s constraints, but also leads to a better fidelity compared to a bipartite scenario. In this paper, we also assume that the Claire’s are not allowed to communicate between each other even classically, and classical communication is only between (Alice, $C_i$) for $i = 1, 2, \ldots N$ and between ($C_i$, Bob) pairs.

Our aim here is to analyze the performance of the shared quantum channels, both noiseless and noisy, in terms of fidelity of the TD-TC protocol defined in Eq. (3). However, finding optimal fidelity after maximizing over measurements and unitaries in a multipartite scenario is not easy. As a way-out, looking at the symmetries of the states involved, we choose a particular kind of measurement and unitary operators, which provides a lower bound on $\mathcal{F}(\rho)$. In this paper, we deal with two information-theoretically important families of three-qubit pure states, the generalized Greenberger-Horne-Zeilinger (GHZ) [33], and the generalized W states [32], shared between (A, $C_1, C_2$) and ($C_1, C_2, B$)-trios.

### III. TWO-PATH QUANTUM REPEATER: NOISELESS SCENARIO

In this section, we assume that the shared three-qubit states used as quantum channels are noiseless, and we explicitly discuss the protocols as well as the evaluation of their fidelities. The effects of local noise on the protocol will be discussed in the succeeding sections.

#### A. TD-TC protocol via Generalized GHZ state

Let us first consider the situation, in which Alice shares a three-qubit generalized GHZ state $|gGHZ\rangle$, given by

$$|gGHZ(\alpha)\rangle = \sqrt{\alpha} |000\rangle + \sqrt{1 - \alpha} e^{i\phi} |111\rangle,$$

where $\alpha \in (0, 1)$ and $\phi \in [0, 2\pi)$, with $C_1$ and $C_2$. The receiver, Bob, shares another copy of the same state, $|gGHZ(\alpha)\rangle$, with the auxiliary nodes, $C_1$ and $C_2$. Note here that all the parties except Bob (to whom the unknown state has to be teleported) initially possess two qubits. Alice first performs a joint Bell measurement $\{|\phi^\pm\rangle, |\psi^\pm\rangle\}$ [37] on the input state $|\psi\rangle$, and the subsystem of $|gGHZ(\alpha)\rangle$ in her possession. The unnormalized post measured states (PMS) at $C_1$ and $C_2$ read as

$$|\xi_+\rangle_{C_1C_2} = A' |\phi^+\rangle_{\psi, \chi} \otimes |gGHZ\rangle_{AC_1C_2},$$

$$|\xi_+\rangle_{C_1C_2} = A' |\phi^+\rangle_{\psi, \chi} \otimes |gGHZ\rangle_{AC_1C_2},$$

$$|\xi_\psi\rangle_{C_1C_2} = (b\sqrt{\alpha}|00\rangle + a\sqrt{1 - \alpha} e^{i\phi}|11\rangle)/\sqrt{2}$$

$$(5)$$

$$|\xi_\psi\rangle_{C_1C_2} = (b\sqrt{\alpha}|00\rangle + a\sqrt{1 - \alpha} e^{i\phi}|11\rangle)/\sqrt{2}$$

$$(6)$$

Depending on the measurement outcomes obtained and communicated by Alice, the Claire’s perform local unitary operations, chosen from the Pauli operators, $\{I, \sigma_x, \sigma_y, \sigma_z\}$ on their respective parts of the shared $|gGHZ(\alpha)\rangle$ state. The set of local unitaries at the end of $C_1$ and $C_2$ can jointly be represented as $\{I \otimes I \otimes \sigma_x, \sigma_y, \sigma_z \}$, corresponding to $\{|\phi^\pm\rangle, |\psi^\pm\rangle\}$ clickings.

After performing these unitary operations, the PMS shared by $C_1$ and $C_2$ are $|\xi_+\rangle_{C_1C_2}$ and $|\xi_\psi\rangle_{C_1C_2}$, with normalization. Note that the normalization constants of these states are the probabilities of obtaining the
outcomes of the Bell measurement performed by Alice. The first two steps is a part of TD protocol. After this, $C_1$ and $C_2$ perform two independent Bell measurements on their auxiliary nodes, communicate their measurement results to Bob and finally Bob chooses unitary operators given in Ref. [38], depending on the eight measurement outcomes of gGHZ state guarantees that if $|\phi^\pm\rangle$ clicks at $C_1$’s port, then $|\psi^\pm\rangle$ can never click at $C_2$’s end and vice-versa. Thus we can have only eight possible outcomes of the measurements instead of sixteen, which would later turn out to be useful in a noisy scenario. In each of the cases, the (unnormalized) quantum state, teleported to Bob, is given in Table I.

$$\begin{array}{|c|c|c|}
\hline
\text{Outcomes by} & \text{Teleported state to} & \text{Bob (B)} \\
\hline
C_1 & C_2 & \text{When Alice obtains $|\phi^\pm\rangle$} \\
\hline
|\phi^\pm\rangle & |\phi^\pm\rangle & (a|0\rangle + b(1-\alpha)e^{2\text{i}\phi}|1\rangle)/2\sqrt{2} \\
|\psi^\pm\rangle & |\psi^\pm\rangle & \sqrt{\alpha(1-\alpha)}e^{\text{i}\phi}|a|0\rangle + b|1\rangle)/2\sqrt{2} \\
\hline
|\phi^\pm\rangle & |\phi^\pm\rangle & \sqrt{\alpha(1-\alpha)}|\psi^\pm\rangle \\
|\psi^\pm\rangle & |\psi^\pm\rangle & (a-\alpha)e^{2\text{i}\phi}|0\rangle + b|1\rangle)/2\sqrt{2} \\
\hline
\end{array}$$

TABLE I. Table of four possible teleported states at the output port, B, after different measurement outcomes by Alice and Claire, $C_1$ and $C_2$ and the unitary rotations by Claire and Bob.

Therefore, the fidelity in multipath QR by using gGHZ state is constrained by

$$\mathcal{F}^{DC}(gGHZ) \geq \int d^2\alpha d^2\beta \left| \frac{|a|^2 + |b|^2(1-\alpha)e^{2\text{i}\phi}|^2}{2} + 2\alpha(1-\alpha) + |a|^2(1-\alpha)e^{2\text{i}\phi} + |b|^2|a|^2(1-\alpha)e^{2\text{i}\phi} + |b|^2|a|^2(1-\alpha)e^{2\text{i}\phi} \right|^2 \right. \\
\left. = \frac{2}{3} + \frac{4}{3}\alpha(1-\alpha) \cos^2 \phi. \right. \quad (7)$$

The lower bound on $\mathcal{F}^{DC}(gGHZ)$ can be improved if one absorbs the phase factor of the gGHZ state to any one of the measurements by the Claire or in any one of the unitary operations by Claire or Bob. For example, in the Bell measurement, performed by Alice, she can choose the basis by redefining $|1\rangle \rightarrow |\tilde{1}\rangle = e^{\text{i}\phi}|1\rangle$. Immediately, Eq. (7) reduces to

$$\mathcal{F}^{DC}(gGHZ(\alpha)) \geq \frac{2}{3} + \frac{4}{3}\alpha(1-\alpha) = F(gGHZ). \quad (8)$$

Notice that it reaches to unity, when $\alpha = \frac{1}{2}$, implying the protocol to be optimal when the shared multiparty quantum state is the GHZ state. It is tempting to conjecture at this point that the TD-TC protocol described above is possibly the optimal one also for the gGHZ state.

![Map of fidelity of a generalized W state vs. state parameters, $\alpha$ and $\beta$, in the multipath QR process. Both axes are dimensionless.](image)

**FIG. 2.** Map of fidelity of a generalized W state vs. state parameters, $\alpha$ and $\beta$, in the multipath QR process. Both axes are dimensionless.

### B. Generalized W states as multipath quantum channels

We now move on to a scenario, where the distribution and concentration channels are from the two-parameter family of three-qubit generalized W states [32], given by

$$|gW(\alpha,\beta)\rangle = \sqrt{\alpha}|001\rangle + \sqrt{\beta}|010\rangle + \sqrt{1-\alpha-\beta}|100\rangle, \quad (9)$$

where $\alpha, \beta \in (0, 1)$ and $\alpha + \beta < 1$. Like the case of the gGHZ state, $(A, C_1, C_2)$ and $(C_1, C_2, B)$ share two copies of gW state, $|gW(\alpha,\beta)\rangle$. At the input port and at both the auxiliary nodes, Bell measurements are carried out as before. However, the choice of unitary operators are different than that in the preceding section. $C_1$ and $C_2$ perform identities if $|\phi^\pm\rangle$ or $|\psi^\pm\rangle$ clicks at Alice’s node while for the rest of the measurement outcomes, they operate $\sigma_z$ in their subsystems. The applications of these local unitaries reduce the PMS from four to two, shared by $C_1$ and $C_2$, and are given by

$$|\zeta_{\phi}^+\rangle_{C_1} = \frac{1}{\sqrt{2}}\left( a(\sqrt{\alpha}|01\rangle + \sqrt{\beta}|10\rangle) + \sqrt{1-\alpha-\beta}|00\rangle \right), \quad (10)$$

$$|\zeta_{\psi}^+\rangle_{C_1} = \frac{1}{\sqrt{2}}\left( a\sqrt{1-\alpha-\beta}|00\rangle + b(\sqrt{\alpha}|01\rangle + \sqrt{\beta}|10\rangle) \right). \quad (11)$$

In the TC part, local unitaries on the output port, $B$, are given in Table IX of Appendix C, depending on the results of two Bell measurements, executed by $C_1$ and $C_2$. Therefore, the fidelity of the TD-TC protocol, described above, for the shared generalized W states can then be estimated as
\[ \mathcal{F}^{DC}(gW(\alpha, \beta)) \geq F(gW) = 2 \int a^2 da^2 b \left[ \alpha \beta (|a|^4 + |b|^4) + \frac{1}{4} \left( (|a|^2(\alpha + \beta) + |b|^2(1 - \alpha - \beta))^2 + (|a|^2(1 - \alpha - \beta) + |b|^2(\alpha + \beta))^2 \right) + 2(\alpha + \beta)(1 - \alpha - \beta), \right] \quad (12) \]

\[ = \frac{2}{3} + \frac{2}{3}(2\alpha + \beta)(1 - \alpha - \beta). \quad (13) \]

**FIG. 3.** Local measurement-based single-path QR. Optimal local measurements are performed by \( C_2 \), reducing tripartite channels to bipartite ones. For example, executing local measurements by \( C_2 \) disentangles \( C_2 \) from \( A \) and \( C_1 \) and the entire TD-TC scheme reduces to the one consisting of \((A, C_1)\) and \((C_1, B)\) duos.

**Remark.** The behaviour of fidelity for the generalized W state in the \((\alpha, \beta)\)-plane is depicted in Fig. 2. From the figure, it is clear that the choice of measurements and unitaries discussed above leads to a maximal fidelity in a region where \( \alpha \approx \frac{1}{2} \) and \( \beta \approx 0 \). This situation arises when the shared gW state is close to a product state in the \( A(B)C_2 : C_1 \) bipartitions, having negligible genuine multipartite entanglement. A similar scenario is true when the state is product across \( A(B)C_1 : C_2 \) bipartition, although the choice of unitary operators in this case needs to be different.

**IV. LOCAL MEASUREMENT-BASED SINGLE-PATH QUANTUM REPEATER**

Instead of the TD-TC scenario, we now consider a scheme where one of the Claires initially performs optimal local measurements in the shared state, thereby reducing the protocol consisting of two linear quantum channels. Our aim is to compare fidelities obtained by multipath protocol with that of the local measurement-based single-path ones. Specifically, from three-party quantum states, \( \rho \) and \( \tilde{\rho} \), shared between \((A, C_1, C_2)\) and \((C_1, C_2, B)\) trios, certain two-party states, \( \sigma_\rho \) and \( \tilde{\sigma}_\rho \) are obtained by operating suitable rank-one projective measurements by one of the Claires, say, \( C_2 \) (see Fig. 3). In this case, \( C_1 \) which actively participates in the protocol can be called functioning port where as \( C_2 \) is the non-functioning port. As in the previous case, initially, either two copies of the generalized GHZ or the generalized W state as multiparty channels are shared among all the parties.

The motivation behind considering such a protocol is as follows. First of all, there can be a situation where one of the parties, say \( C_2 \), for some reason, has to leave the protocol, reducing the initial pure state to a mixed state. In case of the gGHZ state, such scenario leads to a separable state between \( A \) and \( C_1 \) as well as \( C_1 \) and \( B \), although for the gW state, the reduced state, \( P[\sqrt{3}|01\rangle + \sqrt{1-\alpha-\beta}|10\rangle + \alpha P[|00\rangle] \), where \( P[|\psi\rangle] = |\psi\rangle \langle \psi| \) is entangled and hence can be used for teleportation protocol with quantum fidelity. The example of the gGHZ state shows that such betrayal or absence can sometimes eliminate the quantum advantage from the communication protocol. Under such circumstances, a profitable situation can be regained if one considers a local measurement-based protocol, if the situation is not of betrayal and the absent party can pre-measure in some basis of his/her parts of the system. In particular, instead of leaving the protocol, \( C_2 \) can help to share highly entangled state to \((A, C_1)\) and \((C_1, B)\) pairs by performing suitable local measurements on her parts, transforming tripartite states to bipartite ones, which are useful for quantum teleportation. In the local measurement-based scenario, after optimization performed over local measurements, the optimal fidelity can be obtained by using the formula of teleportation fidelity in terms of singlet fraction \([14]\). Note that, similar motivation leads to the concept of entanglement assistance or localizable entanglement \([39]\).

**A. Generalized GHZ vs. generalized W states for local measurement-based scheme**

Let us start the discussion with a scenario where an arbitrary three-qubit gGHZ state is shared between the Alice (Bob) and Claires. As mentioned, complete ignorance in the non-functioning port leads to a two party
which reaches its maximum value for $\alpha = 1$. For the details, see Appendix A.

As it is a separable state in the $A(B) : C_1$ bipartition, $F_1 = \frac{2}{3}$. On the other hand, if $C_2$ chooses an arbitrary rank-one projective measurement, given by

$$|M\rangle = \sqrt{x}|0\rangle + e^{i\theta} \sqrt{1-x}|1\rangle,$$

$$|M^\perp\rangle = \sqrt{1-x}|0\rangle - e^{i\theta} \sqrt{x}|1\rangle,$$

the PMS between Alice (Bob) and $C_1$ can be represented as

$$|\xi_M\rangle = \frac{1}{\sqrt{p_M}}(\sqrt{x}\alpha|00\rangle + e^{i(\phi-\theta)} \sqrt{(1-x)(1-\alpha)}|11\rangle),$$

$$|\xi_{M^\perp}\rangle = \frac{1}{\sqrt{p_M}}(\sqrt{x(1-\alpha)}|00\rangle - e^{i(\phi-\theta)} \sqrt{x(1-\alpha)}|11\rangle).$$

where $p_M = x\alpha + (1-x)(1-\alpha)$ is the probability of obtaining $|M\rangle$ and $p_{M^\perp} = x(1-\alpha) + (1-x)\alpha$ for $|M^\perp\rangle$. The teleportation fidelity by using PMS as channels between Alice to $C_1$ is [14]

$$f_{max}^{\perp} = \frac{2}{3} + \frac{2}{3} \sqrt{x\alpha(1-x)(1-\alpha)},$$

which reaches its maximum value for $x = \frac{1}{2}$, representing the optimal measurement basis as $|\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$. We now assume that the teleportation fidelity, by using single-path QR, attains its maximum value when each segment can teleport at its maximum capacity. Starting from the gGHZ state, optimal measurements on both parts of the shared state by $C_2$ lead to the bipartite states shared between Alice (Bob) and $C_1$, given by

$$|\mu^{\perp}\rangle^{gGHZ}_{A(B)C_1} = \sqrt{\alpha}|00\rangle \pm \sqrt{1-\alpha e^{i\phi}}|11\rangle,$$

with probability $\frac{1}{2}$. The fidelity of single-path QR, via quantum channels, (A, $C_1$) and ($C_1$, B) pairs, finally reads

$$F_1^{gGHZ} = \frac{2}{3} + \frac{4}{3} \alpha(1-\alpha).$$

For the details, see Appendix A.

Comparing Eqs. (8) and (21), we find that the fidelities obtained by the specific TD-TC protocol with that of the local measurement-based ones exactly match and therefore, we have

$$F^{DC}(gGHZ) \geq F_2^{gGHZ} = F(gGHZ),$$

where the equality holds for the GHZ state with $\alpha = \frac{1}{2}$. If we now assume that with the shared gGHZ state, the protocol presented in Sec III is the optimal one, we have that multipath QR in terms of teleportation fidelity does not provide any advantage over local measurement-based single-path QR. We will show that when the shared state is the gW state, two scenarios no more remains the same. Let us now move to the protocol with the initial state being the generalized W state. Let us first state the following proposition.

**Proposition:** Consider two copies of a generalized W state shared between $A, C_1, C_2$ and $C_1, C_2, B$. Suppose that $C_2$ must remain non-functional during the actual implementation of the teleportation, but agrees to make measurements before. The optimal fidelity in the single-path quantum repeater can be obtained when the local measurements at $C_2$ are performed in the computational basis.

**Proof:** After performing measurements by $C_2$ in an arbitrary basis given in Eqs. (15) and (16), the PMS and their probabilities of occurrence can be computed (see Table II). The maximum teleportation fidelity, $f_{max}^{gW}$, via PMS from A to $C_1$ in terms of the maximal singlet fraction [14], $F_{max}$, is given by

$$f_{max}^{gW} = p_M f_{max}^{M} + p_{M^\perp} f_{max}^{M^\perp},$$

where

$$f_{max}^{M^\perp} = 2 F_{max}^{M^\perp} + 1, \quad \text{and}$$

$$F_{max}^{M^\perp} = \frac{1}{2}(1 + 2 \sqrt{A_{M^\perp}^+ A_{M^\perp}^-}).$$

Substituting values from Table. II, and maximizing over the measurement basis, we obtain

$$f_{max}^{gW} = \frac{2}{3} + \frac{2}{3} \sqrt{\beta(1-\alpha-\beta)}.$$
We now note that if $C_2$ makes the measurement in the \{0,1\} basis, the PMS reduces to

$$|\zeta_0\rangle = \frac{1}{\sqrt{p_0}}(\sqrt{\beta}|01\rangle + \sqrt{1-\alpha-\beta}|10\rangle), \quad (27)$$

$$|\zeta_1\rangle = |00\rangle. \quad (28)$$

with $p_0 = 1 - \alpha$ and $p_1 = \alpha$. Following the protocol described in Sec. III B, we can easily find that $F_{\text{max}}^W$ can be maximized. If we now assume that the fidelity of local measurement-based single-path QR is maximized when the fidelities at each segment (i.e., $A \rightarrow C_1$ and $C_1 \rightarrow B$) are maximized, the optimal measurement basis is the computational basis.

After proving the optimality of the measurement basis, let us now explicitly evaluate the optimal fidelity in this process. If $C_2$ performs measurements in the computational basis on her parts of the shared states, the PMS between Alice (Bob) and $C_1$ are given in Table III. From this table, it is clear that except the first outcome, a pure product state is shared between $A$ and $C_1$, or between $C_1$ and $B$, or between both in all the three cases, and hence for these situations, the fidelity reduces to $\frac{2}{3}$. In the first situation, i.e., when the outcome is \{0,0\}, the fidelity reads as $\frac{2}{3} + \frac{4}{3(p_0)}\beta(1-\alpha-\beta)$. For a detailed calculation see Appendix A. Therefore, the optimal fidelity in the local measurement-based single-path QR, when the initial shared state is the generalized W state, can be shown to be

$$F^g_{W}(gW) = (1 - (p_0)^2)\frac{2}{3} + (p_0)^2 \left(\frac{2}{3} + \frac{4}{3(p_0)}\beta(1-\alpha-\beta)\right)$$

$$= \frac{2}{3} + \frac{4}{3}\beta(1-\alpha-\beta). \quad (29)$$

Remark. For the shared gW state, instead of performing any measurement, if $C_2$ leaves her laboratory, or if $C_1$ tries to communicate the quantum state secretly to Bob ignoring $C_2$, one can show that sending unknown quantum state with fidelity better than the classical is still possible, and the fidelity is given by

$$F^{l}_{\text{mixed}}(gW) = \frac{2}{3} + \frac{2}{3}\left(2\beta(1-\alpha-\beta) - \alpha(1-\alpha)\right). \quad (30)$$

Since, $\alpha(1-\alpha) > 0$, we conclude that the local measurement-based protocol is always better than the scheme where $C_2$ just leaves the protocol without performing the measurement.

Let us now state one of the main results of the paper. In particular, we compare the quantum capacities of multipath QR with that of the local measurement-based single-path ones. Before discussing the results for the entire family of the gW state, let us state the following theorem which is true for the W state with $\alpha = \beta = \frac{1}{3}$:

**Theorem:** If two copies of the three-qubit W state are shared between $A, C_1, C_2$ and $C_1, C_2, B$, sending an unknown qubit in the teledistribution and teleconcentration protocol is always beneficial than using a single-path quantum repeater, consisting of two bipartite quantum states derived from optimal local measurements in the non-functioning port, and the corresponding advantage is $9.1\%$ or better.

**Proof:** For the W state, $|W\rangle = \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle)$, $F^{DC}(W) \geq \frac{\sqrt{6}}{3} > \frac{2}{3} = F^l_{W}(W)$, by using Eqs. (13) and (30), thereby establishing the advantage of multipath protocol for sending quantum information over the single-path ones.

The above Theorem holds even for the shared generalized W states, for some values of $\alpha$ and $\beta$. In particular, we show that the TD-TC scheme is better than the optimal local measurement-based ones when

$$\alpha \geq \frac{\beta}{2}. \quad (31)$$

In general, for gW states with $\alpha \geq \frac{2}{3}$, the TD-TC protocol performs better than the local measurement based scheme by

$$\frac{(2\alpha-\beta)(1-\alpha-\beta)}{1+2\beta(1-\alpha-\beta)} \times 100\%$$

or better. Note, however, that it is still possible that the region in which the TD-TC protocol can not give better fidelity, may show the benefit over the local measurement-based scheme if one can construct optimal measurements and unitaries involved in the two-path protocol.

### V. NOISY CHANNELS

Until now, the results obtained involve quantum channels which are not affected by any kind of noise. In practice, quantum states can never be kept completely isolated from the environment, and hence it is important to investigate the effects of environmental interaction on the quantum capacities of the protocol. In particular, we assume that local noise acts on both the qubits, possessed by $C_1$, as shown in Fig. 4 in both multipath and local measurement-based single-path protocols. In this section, we find out the robustness of fidelities, against noise in one of the subsystems of the shared channels. The initial shared state is again two copies of a three-qubit generalized GHZ or a generalized W state. We will
Consider five different kinds of noise [34] models – (1) Bit flip noise, (2) phase flip noise, (3) bit-phase flip noise, (4) amplitude damping, and (5) phase damping. The detailed actions of these noise models on the quantum state are given in Appendix B. We assume that irrespective of the noise acting on the subsystems, Alice, Claire’s (A, C1, and C2) and Bob continue with their protocol of the noiseless scenario, described in Secs. II and IV. This is probably a natural and important assumption since we believe that the senders and the receivers may not always be in a position to alter their actions, depending on the noise or they may not always be aware of the types of noise acting on the system.

A. Generalized GHZ state against noise: Inherent detection and rectification

As before, A (B), C1 and C2 share a $|gGHZ(\alpha)\rangle$ state, and local noise acts on both the subsystems of C1. Before comparing the fidelities obtained from the two-path and local measurement-based single-path QRs, let us first discuss that the multipath TD-TC protocol enables us to identify and rectify certain kinds of noise in the system when the gGHZ state is shared between them.

First note that in the TD-TC scheme, due to the symmetry of the gGHZ states, when $|\phi^+\rangle$ ($|\psi^\pm\rangle$) clicks in the measurement performed by C1, the outcome of the measurement at the node of C2 can not be $|\psi^\mp\rangle$ in a noiseless scenario or when phase flip error occurs. However, when bit flip or bit-phase flip noise happens on both the qubits of C1, such correlations in measurement results are broken and new correlations appear, leading to identification of noise models acting on C1. In particular, in these cases, when $|\phi^\pm\rangle$ clicks in C1’s port, $|\psi^\mp\rangle$ can only be the outcome at C2 and vice-versa. Therefore, this contrasting feature in the measurement outcomes enables us to conclusively distinguish the local bit flip and bit-phase flip noises acting on one of the Claire’s subsystem in the TD-TC protocol from the noiseless and phase flip noisy scenarios. Interestingly, by designing suitable local unitary operators at Bob’s port, the effects of noise, either bit or bit-phase flip errors, on fidelities can be corrected. Let us illustrate the unitary operators of B which are appropriate to correct the bit flip error at the node of C1. When the measurement outcomes at the end of C1 and C2 are $\{|\phi^\pm\rangle |\psi^\mp\rangle, |\phi^\mp\rangle |\psi^\pm\rangle\}$, the corresponding unitary operators at the output port can be set to $\{I, \sigma_z, \sigma_x, \sigma_y\}$ respectively. If Bob now employs the above set of unitaries, the lower bound on the fidelity for TD-TC under bit flip channel coincides with the noiseless case. Interestingly, the detection and subsequent rectification are not possible in the local measurement-based case, where the fidelity depends on the noise parameter, p, (see Table IV). This rectification procedure again shows another superior characteristic of multipath QR protocol over the single-path one.

Suppose now that the sender and the receiver a priori know the more probable error at the C1’s port to be the bit-phase flip one. Hence the circuit for implementing the multipath protocol can be designed in such a way that the fidelity remains unaltered in this scenario. Specifically, the bit-phase flip error can be corrected if...
Bob applies \( \{\sigma_i \otimes U_i\} \) \((i = 1, 2, \ldots, 8)\) where \( \{U_i\} \) are the set of unitaries used in the bit flip case. In presence of different kinds of noise models, the lower bounds on \( F_{DC} \), and the exact values of \( F_{l} \) with the measurement being performed in the \( \{|+, -\rangle\} \) basis are given in Table IV for the shared gGHZ state. As discussed before, for bit flip/bit-phase flip errors, the multipath setting turns out to be maximally robust compared to any other noise models while the opposite occurs for the amplitude damping ones, provided the set of measurements and unitaries discussed before in the TD-TC protocol for the shared gGHZ state is optimal. On the other hand, the lower bounds on the TD-TC, match with \( F_{l} \), when other noises like phase flip, bit-phase flip, phase damping act locally on \( C_{1} \), if we assume that the rectifying unitary operators are only applied for bit flip errors.

Remark. For an arbitrary noise model, for example for amplitude damping, when Claires perform measurements in the Bell basis, all possible (sixteen) outcomes can arise, and hence Bob applies all the sixteen unitary rotations mentioned in the noiseless and bit flip cases. In this sense, when the shared state is the gGHZ state, one can always distinguish amplitude damping or other noise models from the errors like bit, phase, bit-phase flips and phase damping channels, although rectification might not always be possible.

| Noise type           | Multipath QR                                                                 | Single-path QR                                                                 |
|----------------------|------------------------------------------------------------------------------|--------------------------------------------------------------------------------|
| Bit flip             | \( \frac{2}{3} + \frac{2}{3} \left(\frac{2\alpha + \beta(1 - \alpha - \beta)}{(1 - 2p + 2p^2) + p(p - 1)}\right) \) | \( \frac{2}{3} + \frac{4}{5} \beta (1 - \alpha - \beta) - \frac{2}{5} p(1 - p) \times (1 - \alpha)^2 - 4\beta(1 - \alpha - \beta) \) |
| Phase flip           | \( \frac{2}{3} + \frac{3}{5} \left(\frac{2\alpha(1 - \alpha) + \beta(1 - \beta)(1 - 2p^2) - 3\alpha\beta + 4\alpha\beta p(1 - p)}{(1 - 2p + 2p^2) + p(p - 1)}\right) \) | \( \frac{2}{3} + \frac{4}{5} \beta (1 - \alpha - \beta) (1 - 4p(1 - p)) \) |
| Bit phase flip       | \( \frac{2}{3} + \frac{3}{5} \left(\frac{2\alpha + \beta(1 - \alpha - \beta)}{(1 - 2p + 2p^2) + p(p - 1)}\right) \) | \( \frac{2}{3} + \frac{4}{5} \beta (1 - \alpha - \beta) - \frac{2}{5} p(1 - p) \times (1 - \alpha)^2 - 4\beta(1 - \alpha - \beta) \) |
| Amplitude damping    | \( \frac{2}{3} + \frac{2}{3} \left(\frac{2\alpha + \beta(1 - \alpha - \beta)}{1 + \beta^2(1 + p)}\right) + p(\alpha\beta - 2\beta + \beta^2(1 + p)) \) | \( \frac{2}{3} + \frac{4}{5} \beta (1 - \alpha - \beta) - 2\beta p(1 - \alpha) + \frac{4}{5} p\beta^2(2 + p) \) |
| Phase damping        | \( \frac{2}{3} + \frac{3}{5} \left(\frac{2\alpha(1 - \alpha) + \beta(1 - \beta)(1 - 2p^2) - 3\alpha\beta + 4\alpha\beta p(2 - p)}{(1 - 2p + 2p^2) + p(p - 1)}\right) \) | \( \frac{2}{3} + \frac{4}{5} \beta (1 - \alpha - \beta) (1 - p(2 - p)) \) |

**TABLE V.** Similar consideration as in Table IV when the shared state is the generalized W state instead of the gGHZ state.

Both the cases, the local measurement-based single-path protocol outperforms the two-path ones for most of the regions in system parameters of gW, i.e., in \((\alpha, \beta)\)-plane. On the other hand, there is a contrasting situation in presence of phase damping and phase flip errors – TD-TC yields a better fidelity than the single-path case for almost the entire parameter space of gW state. Moreover, there exists a finite region in the \((\alpha, \beta)\)-plane of the shared gW state with local amplitude damping noises at \( C_{1} \) where multipath performs better than the single ones.

To put things in a quantitative perspective, we calculate percentages of gW states for which two-path TD-TC protocol provides better quantum capacities than that of the local measurement-based single-path ones under the actions of various local noises. For such comparison, we fix the noise parameter at \( p = 0.3 \), for all the noise models. We generate \( 10^6 \) gW states according to the parameterisations, given in [32]. The observations are the following:

1. For bit flip and bit-phase flip noises, only 9.1% of the states are useful in TD-TC setting.

2. In case of amplitude damping channel, the percentage of the TD-TC with multipath goes up to 41.9%.

3. For phase damping and phase flip errors, \( F_{l}(gW) \leq F_{DC}(gW) \) holds for 80.3% and 92.6% respectively, thereby establishing the two-path protocol as a better ones than the local measurement-based ones.
Here $p = 0.3$. Both axes are dimensionless.

![Contour plots of fidelities for multipath TD-TC scheme with few blocks in $(\alpha, \beta)$-plane of the generalized W state. Number of blocks are chosen as (a) $m = 2$, (b) $m = 4$. Both the axes are dimensionless.](image)

![Multipath QR: a schematic diagram of QR with $m$ blocks, each consists of a multipath TD-TC scenario with shared tripartite entangled quantum states. (b) Local measurement-based QR: in each block, one of the parties performs an optimal local measurement on her/his qubits, which converts each block to two single-path quantum channels.](image)

**VI. MULTIPLE BLOCKS OF MULTIPATH VS. LOCAL MEASUREMENT-BASED PROTOCOLS**

We now illustrate the multiple blocks scheme with tripartite states which can be easily generalized to arbitrary number of parties. As shown in Fig. 6, the total length between the sender and the receiver is divided into arbitrary, say $m$, number of units (blocks) — each unit shares two copies of the same given state as in Fig. 1, which execute a multipath TD-TC scheme following the same steps as given in the beginning of Sec. II. The output state after implementing TD-TC protocol for $i - 1$ blocks, becomes the input state for the $i^{th}$ block. See Fig. 6(a). We are interested in evaluating quantum capacities of an entire quantum channel in terms of the fidelity, $F^D_{m}$, after $m$ blocks. We will compare the above scenario with the local measurement-based scheme. In the local measurement-based scheme, as before, one of the Claire’s of each block, perform optimal local measurements, reducing the entire block structure to a single-path, consisting of $2m$ bipartite states as depicted in Fig. 6(b).

When three-qubit gGHZ states are used as quantum channels and Bell measurements as well as the same unitary operations are performed as described before, the fidelity for the entire QR process reads as

$$F_m(gGHZ) = 2^{2m} \alpha^m (1 - \alpha)^m \int 2|a|^2|b|^2 d^2a d^2b$$

$$+ \sum_{k=0}^{2m} \binom{2m}{k} \alpha^k (1 - \alpha)^{2m-k} \int (|a|^4 + |b|^4) d^2a d^2b$$

$$= \frac{2}{3} + \frac{2^{2m}}{3} (\alpha(1 - \alpha))^m \leq F^D_{m}(gGHZ).$$

On the other hand, if one obtains $m$ blocks of single-path quantum channels (which therefore consists of $2m$ bipartite states) by optimal local measurements which in this case is $\{|+, -\}$, iterative methods leads to the fidelity given by

$$F_m^I(gGHZ) = \frac{2}{3} + \frac{2^{2m}}{3} (\alpha(1 - \alpha))^m,$$

which happens to coincide with $F_m(gGHZ)$, which in turn is upper bounded by $F^D_{m}(gGHZ)$. If the similar QR problem is considered for the shared generalized W states of $m$ blocks, the fidelity, after a rather tedious algebraic iterative calculation, is given by

$$F_m(gW) = \frac{2}{3} \left(1 + 2^{m-1} (2\alpha + \beta)^m (1 - \alpha - \beta)^m \right).$$

On the other hand, the local measurement-based protocol leads to the fidelity

$$F_m^I(gW) = \frac{2}{3} + \frac{2^{m}}{3} \beta^m (1 - \alpha - \beta)^m.$$

For comparison, $F_m(gW)$, for $m = 2$ and $m = 4$, is depicted in Fig. 7. We observe that the region in the $(\alpha, \beta)$-plane, in which the specific TD-TC protocol shows quantum advantage, shrinks with the increase of the number of
of blocks. Moreover, comparing Eqs. (33) and (34), we find that the benefit of the two-path scheme over the single-path ones by using gW states reduces with m.

VII. CONCLUSION

Quantum teleportation is a pioneering discovery which forms one of the pillars in the success story of quantum information science. Its exclusivity to the quantum domain puts it in sharp contrast with classical ideas. In this work, we address the question of sending a quantum state over a long distance in a multiparty framework. In particular, we investigate the performance of this multiparty protocol (in terms of its fidelity) for two important families of three-qubit states, the generalized Greenberger-Horne-Zeilinger (gGHZ) and the generalized W states. We then compare them with a protocol consisting of linear chains obtained from these multiparty states by performing optimal local measurements in one of the nodes. In both noiseless and noisy scenarios, we showed that for certain families of multiparty shared states, one shot capacities, in terms of average output to input fidelity, of multipath protocols are strictly higher than that of the corresponding single-path cases. We also observed that the protocol proposed in this paper inherently possesses a noise correcting mechanism, when local noise is either bit flip or bit-phase flip, acting on one of the parties of the shared gGHZ state. Moreover, we found the capacities of long distance quantum channels, consisting of arbitrary number of multipartite as well as bipartite units by using iterative methods. Advantages in quantum state transfer by using multiple path protocol show the importance of creating multipartite entangled states in quantum communication protocols over the bipartite ones.

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\[ |0\rangle = \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle), \quad |\psi^+\rangle = \frac{1}{\sqrt{2}} (|0\rangle + i|1\rangle), \quad |\psi^-\rangle = \frac{1}{\sqrt{2}} (|0\rangle - |1\rangle) \]

TABLE VI. Table of the unitary rotations performed by Bob corresponding to the possible Bell basis outcomes of the Claire.

| Measurement Outcomes | Units |
|----------------------|-------|
| \(|\psi^+\rangle\)   | \(|\psi^+\rangle\) | \(I\) |
| \(|\phi^+\rangle\)   | \(|\phi^+\rangle\) | \(I\) |
| \(|\phi^-\rangle\)   | \(|\phi^-\rangle\) | \(\sigma_z\) |
| \(|\phi^0\rangle\)    | \(|\phi^0\rangle\) | \(\sigma_z\) |
| \(|\psi^+\rangle\)   | \(|\psi^+\rangle\) | \(\sigma_z\) |
| \(|\psi^-\rangle\)   | \(|\psi^-\rangle\) | \(\sigma_y\) |
| \(|\psi^0\rangle\)    | \(|\psi^0\rangle\) | \(\sigma_y\) |

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Appendix A: Fidelity of a configuration of two different linear chains

Consider now a teleportation protocol with two different linear chains, $|\psi_1\rangle_{AC_1} = \sqrt{\alpha_1}|00\rangle + \sqrt{\alpha_2}|11\rangle$, connecting Alice and $C_1$ and $|\psi_2\rangle_{C_1B} = \sqrt{\alpha_2}|00\rangle + \sqrt{\alpha_1}|11\rangle$, connecting $C_1$ and Bob. For such a situation, the optimal teleportation fidelity from Alice to Claire (Claire to Bob) can be obtained by performing Bell measurements and unitary rotations as given in Ref. [5]. To compute the optimal fidelity for two linear chains, we again use Bell measurements and the respective unitaries.

Suppose, $|\psi\rangle$, is the arbitrary state that Alice wants to teleport to Bob. In the first step, Alice performs a joint Bell measurement on the unknown state and on her part of the shared state, followed by a classical communication of measurement results to Claire. Claire then applies local unitary operator on her part, depending on the measurement outcomes that Alice obtained. The teleported state (un-normalized) on Claire’s part in the first chain, is given in Table VII.

| Outcome (Alice’s) | Teleported state |
|-------------------|------------------|
| $|\phi^+\rangle$  | $\frac{1}{2}(a\sqrt{\alpha_1}|0\rangle + b\sqrt{1-\alpha_1}|1\rangle)$ |
| $|\psi^\pm\rangle$ | $\frac{1}{2}(a\sqrt{\alpha_1}|1\rangle - b\sqrt{1-\alpha_1}|0\rangle)$ |

TABLE VII. Un-normalized teleported state in the Claire’s port, after Alice performs Bell measurements and Claire rotates her part with proper unitary operators.

In the second step, Claire teleports each of the quantum states that she obtains (see table, VIII) to Bob by applying the similar protocol as above.

| Outcome (Claire’s) | Teleported state |
|-------------------|------------------|
| $|\phi^+\rangle$  | $\frac{1}{2}(a\sqrt{\alpha_1\alpha_2}|0\rangle + b\sqrt{1-\alpha_1}(1-\alpha_2)|1\rangle)$ |
| $|\psi^\pm\rangle$ | $\frac{1}{2}(a\sqrt{1-\alpha_1}\alpha_2|0\rangle + b\sqrt{\alpha_1}(1-\alpha_2)|1\rangle)$ |

TABLE VIII. Un-normalized teleported state in Bob’s port, depending on Claire’s measurement outcomes.

The repetitive iteration of the above protocol for the two linear chains, leads to the total fidelity, given by

$$F = 4 \int d^2a d^2b \left[ \frac{1}{4} \left( |a|^2 \sqrt{\alpha_1\alpha_2} + |b|^2 \sqrt{(1-\alpha_1)(1-\alpha_2)} \right)^2 + \left( |a|^2 \sqrt{\alpha_1}(1-\alpha_2) + |b|^2 \sqrt{(1-\alpha_1)\alpha_2} \right)^2 + \left( |a|^2 \sqrt{(1-\alpha_1)\alpha_2} + |b|^2 \sqrt{\alpha_1(1-\alpha_2)} \right)^2 \right]$$

$$= \frac{2}{3} + \frac{4}{3} \sqrt{\alpha_1\alpha_2}(1-\alpha_1)(1-\alpha_2). \quad (A1)$$

Note, when $\alpha_1 = \alpha_2 = \alpha$, Eq. (A1) reduces to

$$F = \frac{2}{3} + \frac{4}{3} \alpha(1-\alpha). \quad (A2)$$
Appendix B: Various noisy channels

Let us briefly discuss about various kinds of noisy channels [34, 35] required in the main text.

**Bit flip channel:** The bit-flip operation is achieved by applying the Pauli operator $\sigma_x$. As the name suggests, it flips the quantum state $|0\rangle$ to $|1\rangle$ and vice-versa with a probability $1 - p$ while it keeps the state unchanged with a probability $p$, $(0 < p < 1)$. Hence in the presence of bit flip channel, a quantum state $\rho$ is transformed as

$$\rho \xrightarrow{\text{bit flip}} p\rho + (1 - p)\sigma_x\rho\sigma_x. \quad (B1)$$

**Phase flip channel:** The phase flip operation transforms $|1\rangle \rightarrow -|1\rangle$ The transformation in this case reads as

$$\rho \xrightarrow{\text{phase flip}} p\rho + (1 - p)\sigma_z\rho\sigma_z. \quad (B2)$$

**Bit-phase flip channel:** Bit-phase flip channel changes a quantum state as follows:

$$\rho \xrightarrow{\text{bit-phase flip}} p\rho + (1 - p)\sigma_y\rho\sigma_y. \quad (B3)$$

**Amplitude damping channel:** A quantum state, $\rho$, under the action of amplitude damping channel, transforms as

$$\rho \xrightarrow{\text{amplitude damping}} M_0\rho M_0^\dagger + M_1\rho M_1^\dagger, \quad (B4)$$

where $M_i$, $i = 0, 1$, are the Krauss operators, given by

$$M_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1 - p} \end{pmatrix}, \quad M_1 = \begin{pmatrix} 0 & \sqrt{p} \\ 0 & 0 \end{pmatrix},$$

Satisfying the condition $M_0^\dagger M_0 + M_1^\dagger M_1 = I$.

**Phase damping channel:** The Kraus operator representation of the phase damping channel, when a quantum state $\rho$ is passing through it is given by

$$\rho \xrightarrow{\text{phase damping}} M_0\rho M_0^\dagger + M_1\rho M_1^\dagger + M_2\rho M_2^\dagger, \quad (B5)$$

where

$$M_0 = \sqrt{1 - p}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad M_1 = \sqrt{p}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad M_2 = \sqrt{p}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$
We notice from the table that like the gGHZ states, when $|\psi^+\rangle$ clicks in one of the Claire’s port, say, $C_1$, $C_2$ can never obtain $|\psi^-\rangle$ as her measurement outcome. Note, however that the correlations in the measurement outcomes for gGHZ is more prominent than that of the gW states.

| Alice’s measurement outcome $|\phi^\pm\rangle$ | Alice’s measurement outcome $|\psi^\pm\rangle$ |
|---|---|
| Outcomes | Unitary | Fidelity | Outcomes | Unitary | Fidelity |
| $C_1$ | $C_2$ | B | $|\langle \text{out} | \phi \rangle|^2 \times 8$ | $C_1$ | $C_2$ | B | $|\langle \text{out} | \phi \rangle|^2 \times 8$ |
| $|\phi^+\rangle$ | $|\phi^+\rangle$ | $I$ | $\left( |a|^2 (\alpha + \beta) + |b|^2 (1 - \alpha - \beta) \right)^2$ | $|\phi^+\rangle$ | $|\phi^+\rangle$ | $\sigma_x$ | $\left( |a|^2 (1 - \alpha - \beta) + |b|^2 (\alpha + \beta) \right)^2$ |
| $|\phi^+\rangle$ | $|\phi^-\rangle$ | $\sigma_z$ | $\left( |a|^2 (\alpha - \beta) + |b|^2 (1 - \alpha - \beta) \right)^2$ | $|\phi^+\rangle$ | $|\phi^-\rangle$ | $\sigma_y$ | $\left( |a|^2 (1 - \alpha - \beta) + |b|^2 (\alpha - \beta) \right)^2$ |
| $|\phi^-\rangle$ | $|\phi^-\rangle$ | $\sigma_x$ | $\alpha (1 - \alpha - \beta)$ | $|\phi^-\rangle$ | $|\phi^-\rangle$ | $I$ | $\alpha (1 - \alpha - \beta)$ |
| $|\phi^-\rangle$ | $|\phi^-\rangle$ | $\sigma_y$ | $\alpha (1 - \alpha - \beta)$ | $|\phi^-\rangle$ | $|\phi^-\rangle$ | $I$ | $\alpha (1 - \alpha - \beta)$ |
| $|\phi^-\rangle$ | $|\phi^-\rangle$ | $\sigma_x$ | $\alpha (1 - \alpha - \beta)$ | $|\phi^-\rangle$ | $|\phi^-\rangle$ | $I$ | $\alpha (1 - \alpha - \beta)$ |
| $|\psi^+\rangle$ | $|\psi^+\rangle$ | $\sigma_x$ | $\beta (1 - \alpha - \beta)$ | $|\psi^+\rangle$ | $|\psi^+\rangle$ | $I$ | $\beta (1 - \alpha - \beta)$ |
| $|\psi^+\rangle$ | $|\psi^+\rangle$ | $\sigma_x$ | $\beta (1 - \alpha - \beta)$ | $|\psi^+\rangle$ | $|\psi^+\rangle$ | $I$ | $\beta (1 - \alpha - \beta)$ |
| $|\psi^-\rangle$ | $|\psi^-\rangle$ | $I$ | $4 \alpha \beta |a|^4$ | $|\psi^-\rangle$ | $|\psi^-\rangle$ | $\sigma_x$ | $4 \alpha \beta |b|^4$ |
| $|\psi^-\rangle$ | $|\psi^-\rangle$ | $I$ | $0$ | $|\psi^-\rangle$ | $|\psi^-\rangle$ | $\sigma_x$ | $0$ |
| $|\psi^-\rangle$ | $|\psi^-\rangle$ | $\sigma_y$ | $\beta (1 - \alpha - \beta)$ | $|\psi^-\rangle$ | $|\psi^-\rangle$ | $\sigma_x$ | $\beta (1 - \alpha - \beta)$ |
| $|\psi^-\rangle$ | $|\psi^-\rangle$ | $\sigma_y$ | $\beta (1 - \alpha - \beta)$ | $|\psi^-\rangle$ | $|\psi^-\rangle$ | $\sigma_x$ | $\beta (1 - \alpha - \beta)$ |
| $|\psi^-\rangle$ | $|\psi^-\rangle$ | $I$ | $0$ | $|\psi^-\rangle$ | $|\psi^-\rangle$ | $\sigma_x$ | $0$ |
| $|\psi^-\rangle$ | $|\psi^-\rangle$ | $I$ | $4 \alpha \beta |a|^4$ | $|\psi^-\rangle$ | $|\psi^-\rangle$ | $\sigma_x$ | $4 \alpha \beta |b|^4$ |

**TABLE IX.** When the initial state is the generalized W states, the computation of fidelity for each measurement outcomes obtained in the ports of Alice, $C_1$ and $C_2$ are listed. The corresponding unitary operators at Bob’s node are also mentioned.