A numerical rotating neutron star solver is used to study the temporal evolution of accreting neutron stars using a multi-polytrope model for the nuclear equation of state named ACB5. The solver is based on a quadrupole expansion of the metric, but confirms the results of previous works, revealing the possibility of an abrupt transition of a neutron star from a purely hadronic branch to a third-family branch of stable hybrid stars, passing through an unstable intermediate branch. The accretion is described through a sequence of stationary rotating stellar configurations which lose angular momentum through magnetic dipole emission while, at the same time, gaining angular momentum through mass accretion. The model has several free parameters which are inferred from observations. The mass accretion scenario is studied in dependence on the effectiveness of angular momentum transfer which determines at which spin frequency the neutron star will become unstable against gravitational collapse to the corresponding hybrid star on the stable third-family branch. It is conceivable that the neutrino burst which accompanies the deconfinement transition may trigger a pulsar kick which results in the eccentric orbit. A consequence of the present model is the prediction of a correlation between the spin frequency of the millisecond pulsar in the eccentric orbit and its mass at birth.

**KEYWORDS:**
neutron star, accretion, binary systems

1 INTRODUCTION

One of the most intriguing phenomena in compact star physics is the appearance of eccentric orbits in binaries of a millisecond pulsar (MSP) with a low-mass white dwarf in a rather narrow region of orbital periods, see Fig. [1]. In (J. Antoniadis, 2014) it has been demonstrated that the observed eccentricities of $e \approx 0.01-0.15$ can be generated from the dynamical interaction of the binary with a circumbinary disc for periods between 15 and 50 days. P. C. C. Freire & Tauris (2014), however, proposed that these binary MSPs may form from the rotationally...
In order to develop the scenario, we shall first consider a model for disc accretion that results in spin-up and mass increase of the compact star. We then suggest an equation of state (EoS) with a strong phase transition for the compact star matter fulfilling the criterion for the formation of a third-family branch of hybrid stars (Gerlach, 1968), separated from the one for ordinary neutron stars by a sequence of unstable configurations (Alford, Han, & Prakash, 2013). With such a setting one can suggest a prompt catastrophic rearrangement of the stellar structure as the origin for a compact star kick and thus an eccentric orbit. The EoS shall be chosen such that the modulus structure as the origin for a compact star kick and thus an angular branch which thus involves a decent spin-up and structural reconfiguration of the compact star interior that may result in energy release by violent neutrino emission. The latter may result in the neutrino rocket mechanism that produces a compact star kick (see, e.g., Berdermann, Blaschke, Grigorian, & Voskresensky, 2006). The equations of all perturbation functions are given by

\[ ds^2 = -e^\phi [1 + 2(h_0 + h_2 P_2) dt^2 + e^\phi \left[ 1 + \frac{2(m_0 + m_2 P_2)}{r - 2M} \right] dr^2 + r^2 \left( 1 + 2(v_2 - h_2 P_2) \right) d\theta^2 + \sin^2(\phi) d\phi^2 ] + O(\Omega^3), \]

where the functions \( M(r) \) and \( \lambda(r) \) are solutions of the Tolman-Oppenheimer-Volkoff (TOV) equation \( (e^\phi = (1 - 2M/r)^{-1}) \), and \( h_0(r), m_0(r), h_2(r), m_2(r) \), and \( v_2(r) \) denote monopole and quadrupole perturbation functions. The stellar deformation is described by the Legendre polynomial

\[ P_2 = P_2(\cos \theta) = (3 \cos^2 \theta - 1)/2, \]

and \( \omega(r) \) is the angular (frame-dragging) frequency relative to a stationary observer at spacelike infinity. The energy-momentum tensor is characterized by two more perturbation functions, \( p^0_2(r) \) and \( p^2_2(r) \), once again contributing to radial and \( \theta \)-dependent perturbations (through \( P_2 \)). The equations of all the perturbations can be readily derived from Einstein’s field equations (Hartle & Thorne, 1968). The perturbation equations can be integrated by starting from the TOV solution by choosing a guess value for the angular velocity and fixing it through multiple integrations with the shooting method. The angular velocity for an inertial local observer \( \omega \) is obtained in this way. It is related to the angular frequency \( \omega \) through \( \omega = \Omega - \omega \), where \( \Omega = \omega + i/\omega \) is the star’s angular velocity (defined by the 4-velocity components of a co-rotating observer). The asymptotics of the perturbation functions allow one to find them unambiguously for a given angular velocity \( \Omega \) or angular momentum boundary value \( J \).
The code is conceptually divided into a TOV integrator and a function which takes the TOV input and integrates all the perturbations radially outward with the corresponding boundary value settings (Hartle & Thorne 1968). A 4th order Runge-Kutta method has been used due to its relative simplicity and accuracy. Several versions of the code were developed, one of which allowing us to determine the properties of stars with predefined baryon numbers (listed further down in the paper). In addition to the standard procedure of Hartle and Thorne, care has been taken to compute the proper mass shedding limit by taking into account the dragging of inertial frames as described in (Weber 1999).

The model for matter accretion from a neutron star’s disc is based on (Bejger, Fortin, Haensel, & Zdunik 2011), considering the baryon mass $M_b$ and angular momentum $J$ to be the variables of such an evolution. A single star’s evolution is seen as a set of stationary rotating configurations of different $M_b$ and $J$ values. Angular momentum can be gained from infalling particles and lost through the interaction between the star’s magnetic field and the accretion disk. The relevant evolution equation is

$$\frac{dJ}{dM_b} = l_{\text{tot}} = \kappa l(r_0) - \omega,$$

where $l(r_0)$ is the specific angular momentum of the infalling particle at the innermost radius of the disc, $r_0$, while $\omega$ gives the magnetic torque divided by the accretion rate. The quantity $\kappa$ describes the amount of angular momentum of the infalling particles which the star can actually receive. The value of $\kappa$ is thus in the range of [0, 1]. The magnetic torque is given in terms of the dipole moment of the star, $\mu = BR^3$ as

$$l_m = \frac{\mu^2}{9r_0^3 M_b} \left(3 - 2 \sqrt{\frac{r_0^3}{r_c^3}}\right).$$

The inner disc radius is determined through a transcendental equation relating the magnetospheric radius $r_m$, the corotation radius $r_c$, and the relativistic marginally stable orbit $r_{ms}$ as follows

$$\frac{1}{\Omega} \frac{dl}{dr} = \frac{1}{2} l_m(r_0) = \left(\frac{r_m}{r_0}\right)^{7/2} \left(\sqrt{\frac{r_c^3}{r_0^3}} - 1\right),$$

where the corresponding radii are given by

$$r_m = \frac{\mu^2}{(GM)^{1/3} M_b^2},$$

$$r_c = \frac{(GM)^{1/3}}{\Omega},$$

$$0 = f_{ms}(r_{ms}).$$

Evolution through accretion is modelled by numerical integration of (3) which makes use of the boundary $J$ for a given $M_b$ integration of neutron star models to generate a sequence based on initial baryon mass and magnetic field values. The results further down were obtained by preparing an array of $J$ values and integrating $b$ with the corresponding $M_{bg}$ values and a given value of the angular momentum transfer coefficient $\kappa$. Last but not least, a decrease of the neutron star magnetic field has been considered through a simple phenomenological model in which the accreted mass $M_{\text{acc}}$ "buries" the initial magnetic field $B_0$, described by

$$B(M_{\text{acc}}) = \frac{B_0}{1 + M_{\text{acc}}/m_B}.$$  \hspace{1cm} (9)

This model was suggested by (Shibazaki, Murakami, Shaham, & Nomoto 1989). It has a free parameter $m_B$ which controls the speed at which the magnetic field decreases. Values of $m_B$ based on observations are in the range $10^{-5} - 10^{-4} M_\odot$.

A final estimate on the time during which accretion takes place depends on the value of $M_b$. It should be noted that the overall evolution is degenerate since the ratio $\mu^2/M_b$ determines the RHS of Eq. (6). Changing the values of $\mu$ and $M_b$ simultaneously can lead to the same track if properly scaled, but the time in which the overall evolution takes place depends only on $M_b$. For that reason, several different values of the initial magnetic field $B_0$ and $M_b$ have been considered in the listed results to show the dependence on these quantities. All of them, however, are of the order of $M_b \approx 10^{-3} M_\odot$, $B_0 \approx 10^8 G$.

3 | EQUATION OF STATE

For the present study we will exploit an equation of state in the form of a multi-polytrope parametrisation as it can be found, e.g., in (Alvarez-Castillo & Blaschke 2017), Hebeler, Lattimer, Pethick, & Schwenk 2013, Read, Lackey, Owen, & Friedman 2009. It consists of piecewise polytrope regions in density that are joined thermodynamically consistently. For the lowest density parts that correspond to the outer NS crust ($n < 0.5 n_0$) we implement the BPS EoS (Baym, Pethick, & Sutherland 1971), to be followed by an intermediate phase of homogeneous matter in $\beta$-equilibrium ($0.5 n_0 < n < 1.1 n_0$). Above $1.1 n_0$, we introduce four polytropic segments described by:

$$P(n) = \kappa_i (n/n_0)^{\Gamma_i},$$

where $n_0 < n < n_{i+1}$, $i = 1 \ldots 4$ define the regions with constant $\kappa_i$ and polytropic index $\Gamma_i$. Since $P(n) = n^2 (d(\epsilon(n)/n)/(dn))$, the remaining thermodynamical variables can be easily determined (Zdunik, Bejger, Haensel, & Gourgoulhon 2006):

$$\epsilon(n)/n = \frac{1}{n_0} \int_{n_0}^{n} d\nu \kappa \nu^{\Gamma_i-2} = \frac{1}{n_0^{\Gamma_i}} \kappa \nu^{\Gamma_i-1} + C,$$

$$\mu(n) = \frac{P(n) + \epsilon(n)}{n} = \frac{1}{n_0^{\Gamma_i}} \kappa \nu^{\Gamma_i-1} + m_0,$$
where $C$ is fixed by the condition $(n \to 0) = m_0 n$. The above equations can be recast into the following forms:

$$n(\mu) = \left[ n_0^i (\mu - m_0) \left( \frac{\Gamma_i - 1}{\kappa_i} \right) \right]^{1/(\Gamma_i - 1)}, \quad (13)$$

$$P(\mu) = \kappa \left[ n_0^i (\mu - m_0) \left( \frac{\Gamma_i - 1}{\kappa_i} \right) \right]^{\Gamma_i/(\Gamma_i - 1)}, \quad (14)$$

which are suitable for performing a Maxwell construction of a first-order phase transition between the hadron and quark phases. Each of the polytropic regions has physical meaning. The first one is defined as the result of a fit to the stiffest EoS version provided in (Hebeler et al., 2013). We are interested in an EoS with a strong first order phase transition that would produce a large jump in energy density $\Delta \varepsilon$ at the critical pressure $P_c$ in order to fulfill the Seidov criterion (Seidov, 1971) for a gravitational instability: $\Delta \varepsilon > (\varepsilon_c + 3 P_c)/2$. Therefore we define the second polytrope as a region of constant pressure $P_2 = \kappa_2$ ($\Gamma_2 = 0$), which can result from a Maxwell construction using (14) for the neighboring regions. The remaining polytropes, in the regions $i = 3, 4$, at the highest densities beyond the first order phase transition shall correspond to stiff quark matter. It shall be sufficiently stiff to result in stable hybrid star configurations with a maximum mass of at least $2 M_\odot$, thus fulfilling the constraint from the lower bound of the 2$\sigma$ region of PSR J0740+6620 (Cromartie, n.d.), and similar constraints stemming from PSR J0348+0432 (J, et. al. Antoniadis, 2013) and PSR J1614-2230 (Demorest, Pennucci, Ransom, Roberts, & Hessels, 2010).

### Table 1: EOS for the ACB5 (Paschalidis et al., 2018)

The functional form of the polytrope is defined by Eq. (10) in the main text. The first polytrope ($i = 1$) is fitted to the nuclear EoS at supersaturation densities, the second one ($i = 2$) describes a first-order phase transition with constant pressure $P_2 = \kappa_2$ and $\Gamma_2 = 0$ in the $n_2 < n < n_3$ region. The polytropes that lie in regions 3 and 4, i.e., above the phase transition represent high-density matter, e.g., quark matter. Maximum masses $M_{\text{max}}$ on the hadronic and hybrid branches correspond to region 1 and 4. The minimal mass $M_{\text{min}}$ on the hybrid branch lies in region 3 and marks the onset of the third family solution.

| i   | $\Gamma_i$ | $\kappa_i$ [MeV/fm$^3$] | $n_i$ [1/fm$^3$] | $m_{0,i}$ [MeV] | $M_{\text{max/min}}$ [M$_\odot$] |
|-----|------------|--------------------------|------------------|----------------|--------------------------|
| 1   | 4.777      | 2.1986                   | 0.1650           | 939.56         | 1.40                     |
| 2   | 0.0        | 33.969                   | 0.2838           | 939.56         | –                        |
| 3   | 4.000      | 0.4373                   | 0.4750           | 995.03         | 1.39                     |
| 4   | 2.800      | 2.7919                   | 0.7500           | 932.48         | 2.00                     |

As it has been noted in (Alvarez-Castillo & Blaschke, 2017), it is rather challenging to find a parametrisation with only one polytrope for the high-density region that would obey thermodynamic consistency and at the same time provide a high maximum mass without violating the causality constraint on the squared speed of sound $c_s^2 = dP/de < 1$. The multi-polytrope parametrisation has also proven successful for analyses of the compactness constraint that follows from the recent measurements of tidal deformability (Abbott, 2018) (see also (De et al., 2018; Zhao & Lattimer, 2018)) of compact stars in the mass range around 1.36 $M_\odot$ by analysing the gravitational wave signal from the inspiral phase of the binary merger GW170817 (Annala, Gorda, Kurkela, & Vuorinen, 2018; Paschalidis et al., 2018) and by placing constraints on the EoS (Miller, Chirenti, & Lamb, 2019).

The four-polytrope model which was introduced in (Paschalidis et al., 2018) and denoted there as "ACB5" fulfills the present constraints from multimessenger astronomy on maximum mass and compactness of compact stars (given at the end of the Introduction) and has recently also been studied concerning its behaviour under fast rotation (Blaschke et al., 2019; Bozola, Espino, Lewin, & Paschalidis, 2019). In Table 1 we provide the parameters of this EoS.

### 4 RESULTS, DISCUSSION AND PERSPECTIVES

We have utilized our rotating neutron star solver in order to obtain solutions for the axially symmetric solutions of the Einstein equations with the EoS ACB5. On this basis we studied the spin and mass evolution of the compact star within the accretion scenario defined in Section 2 in dependence on the effectiveness of angular momentum transfer, as described also in (Bejger, Blaschke, Haensel, Zdunik, & Fortin, 2017) for a similar case.

In Fig. 2 we show the baryon mass versus equatorial radius for given angular momenta $J = 0.05, 0.1, 0.15, \ldots, 0.45 J_0$, with $J_0 = GM_\odot^2/c$ (blue solid lines), together with the static case (black solid line) and the Keplerian limit (red solid line). In order to guide the eye we have shown also a few lines of constant baryon mass which allow to estimate the path of a collapsing star from the edge of the hadronic branch to the corresponding point on the stable hybrid star branch (third family) under simultaneous conservation of angular momentum and baryon mass. We want to note that for the present type of EoS such a catastrophic rearrangement of the stellar interior is possible, in principle, at the free-fall time scale. In previous works on quark deconfinement in accreting low-mass binaries (Glenedenning, Pei, & Weber, 1997) this was not the case and the rearrangement would have taken a secular time scale of about
10⁶ − 10⁸ yr in order to get rid of the angular momentum difference by, e.g., electromagnetic dipole radiation. However, such a direct transition scenario would also entail that the orbits would re-circularize quite fastly (on a scale of ∼ 10³ yrs) due to mass-accretion and/or tidal forces. The collapse to a hybrid star as an origin for the eccentric MSP orbit in the prompt collapse scenario should therefore have occurred rather recently.

Alternatively, in order to circumvent the problem with re-circularization in the prompt collapse scenario, one can discuss a delayed collapse scenario on the basis of the calculation presented in Fig. 2. If the binary star gains mass by accretion but fails to reach the onset mass for the transition when the accretion and spin-up process stops, it could then undergo a spin-down evolution and thus reach the instability line for transition to a hybrid star configuration by loss of angular momentum. There are several realisations of this process, depending on the period of the binaries: a) short period binaries which would always reach the maximum mass during accretion, resulting in circular orbits, b) long period binaries that would collapse only after spin-down, producing eccentric orbits, and c) very long period binaries that would never undergo the transition.

In Fig. 3 we show the gravitational mass versus equatorial radius for the same ACB5 EoS and for the same grid of fixed angular momenta (green solid lines). The light grey lines are curves of fixed spin frequency \( f = 1/P \), where \( P \) is the period. The blue lines with different styles describe in the text, for different efficiencies \( x \) of angular momentum transfer. The solid blue line corresponds to direct collapse while the magenta dash-double-dotted line shows the path for delayed collapse.
In order to put our scenario for the origin of the eccentric MSPs to the test, we look forward to precise mass measurements, in particular for highly spun-up pulsars like PSR J0955-6150 with a spin period of 2.0 ms (Octau et al. 2018).

For the present scenario it is not immediately obvious how more massive eMSPs could be explained like PSR J1946+3417 (Barr et al. 2017) with a precisely determined high mass of $1.828^{(22)} M_\odot$. It is, however, also possible to develop the present scenario further so that another major structural rearrangement contributing to a pulsar kick may take place, even leading from a third to a fourth family, see (Alford & Sedrakian 2017, Li, Sedrakian, & Alford 2019).

**Acknowledgements**

We acknowledge the Russian Science Foundation for supporting the part of the work on phenomenology of compact stars with strong first order phase transition under grant number 17-12-01427. The work of F.W. was supported by the National Science Foundation (USA) under Grant PHY-1714068.

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