Riemannian Adaptive Optimization Algorithm and Its Application to Natural Language Processing

Hiroyuki Sakai, Hideaki Iiduka *

Abstract

This paper proposes a Riemannian adaptive optimization algorithm to optimize the parameters of deep neural networks. The algorithm is an extension of both AMSGrad in Euclidean space and RAMSGrad on a Riemannian manifold. The algorithm helps to resolve two issues affecting RAMSGrad. The first is that it can solve the Riemannian stochastic optimization problem directly, in contrast to RAMSGrad which only achieves a low regret. The other is that it can use constant learning rates, which makes it implementable in practice. Additionally, we apply the proposed algorithm to Poincaré embeddings, which embed the transitive closure of the WordNet nouns into the Poincaré ball model of hyperbolic space. Numerical experiments show that regardless of the initial value of the learning rate, our algorithm stably converges to the optimal solution and converges faster than RSGD, the most basic Riemannian stochastic optimization algorithm.

1 Introduction

Riemannian optimization has attracted a great deal of attention in light of developments in machine learning and deep learning. This paper focuses on Riemannian adaptive optimization algorithms for solving an optimization problem on a Riemannian manifold. In the field of machine learning, there is an important example of the Riemannian optimization problem. Nickel and Kiela [4] proposed Poincaré embeddings, which embed hierarchical representations of symbolic data (e.g., text, graph) into the Poincaré ball model of hyperbolic space. In fact, experiments on transitive closure of the WordNet noun hierarchy showed that embeddings into a 5-dimensional Poincaré ball are better than embeddings into a 200-dimensional Euclidean space. Since the Poincaré ball has a Riemannian manifold structure, the problem of finding Poincaré embeddings should be considered to be a Riemannian optimization problem.

*This work was supported by JSPS KAKENHI Grant Number JP18K11184.
Bonnabel [5] proposed Riemannian stochastic gradient descent (RSGD), the most basic Riemannian stochastic optimization algorithm. RSGD is a simple algorithm, but its slow convergence is problematic. In [6], Sato, Kasai, and Mishra proposed the Riemannian stochastic variance reduced gradient (RSVRG) algorithm and gave a convergence analysis under some natural assumptions. RSVRG converges to an optimal solution faster than RSGD; however, RSVRG needs to calculate the full gradient every few iterations. In Euclidean space, adaptive optimization algorithms, such as AdaGrad [7], Adam [8, Algorithm 1], Adadelta [9], and AMSGrad [10, Algorithm 2], are widely used for training deep neural networks. However, these adaptive algorithms cannot be naturally extended to general Riemannian manifolds, due to the absence of a canonical coordinate system. Therefore, special measures are required to extend the adaptive algorithms to Riemannian manifolds. For instance, Kasai, Jawanpuria, and Mishra [12] proposed adaptive stochastic gradient algorithms on Riemannian matrix manifolds by adapting the row, and column subspaces of gradients.

In the particular case of a product of Riemannian manifolds, Bécigneul and Ganea [13] proposed Riemannian AMSGrad (RAMSGrad) by regarding each component of the product Riemannian manifold as a coordinate component in Euclidean space. However, their convergence analysis had two points requiring improvement. First, they only performed a regret minimization (Theorem 3.1) and did not solve the Riemannian optimization problem. Second, they did a convergence analysis with only a diminishing learning rate; i.e., they did not perform a convergence analysis with a constant learning rate. Since diminishing learning rates are approximately zero after a large number of iterations, algorithms that use them are not implementable in practice. In contrast, a constant learning rate does not cause this problem. Therefore, the goal of this paper is to propose an algorithm that overcomes these issues and give its convergence analysis.

In particular, we propose modified RAMSGrad (Algorithm 1), which is an extension of RAMSGrad, to solve the Riemannian optimization problem (Problem 2.1). In addition, we give a convergence analysis (Theorem 3.2) valid for both a constant learning rate (Corollary 3.1) and diminishing learning rate (Corollary 3.2). In numerical experiments, we apply the proposed algorithm to Poincaré embeddings and compare it with RSGD (Section 4). We show that it converges to the optimal solution faster than RSGD and that it minimizes the objective function regardless of the initial learning rate. In particular, we show that the proposed algorithm with a constant learning rate is a good way of embedding the WordNet mammals subtree into a Poincaré ball.

This paper is organized as follows. Section 2 gives the mathematical preliminaries and states the main problem. Section 3 describes the modified RAMSGrad and gives its convergence analysis. Section 4 numerically compares the performance of the proposed learning algorithms with RSGD. Section 5 concludes the paper with a brief summary.
2 Mathematical Preliminaries

Let $M$ be a Riemannian manifold. An exponential map at $x \in M$, written as $\exp_x : T_x M \to M$, is a mapping from the tangent space $T_x M$ to $M$ with the requirement that a vector $\xi \in T_x M$ is mapped to the point $y := \exp_x(\xi) \in M$ such that there exists a geodesic $\gamma : [0, 1] \to M$, which satisfies $\gamma(0) = x$, $\gamma(1) = y$, and $\dot{\gamma}(0) = \xi$, where $\dot{\gamma}$ is the derivative of $\gamma$ (see [14]). Moreover, $\log_y : M \to T_x M$ denotes a logarithmic map at a point $x \in M$, which is defined as the inverse mapping of the exponential map at $x \in M$.

Next, we give the definitions of a geodesically convex set and function (see [15, Section 2]) that generalize the concepts of a convex set and function in Euclidean space.

Definition 2.1 (Geodesically convex set). Let $X$ be a subset of a Riemannian manifold $M$. $X$ is said to be geodesically convex if, for any two points in $X$, there is a unique minimizing geodesic within $X$ which joins those two points.

Definition 2.2 (Geodesically convex function). A smooth function $f : M \to \mathbb{R}$ is said to be geodesically convex if, for any $x, y \in M$, it holds that

$$f(y) \leq f(x) + \langle \text{grad } f(x), \log_x(y) \rangle_x,$$

where $\langle \cdot, \cdot \rangle_x$ is the Riemannian metric on $M$, and $\text{grad } f(x)$ is the Riemannian gradient of $f$ at a point $x \in M$ (see [14]).

For $i \in \{1, 2, \ldots, N\}$, let $M_i$ be a Riemannian manifold and $M$ be the Cartesian product of $n$ Riemannian manifolds $M_i$ (i.e., $M := M_1 \times \cdots \times M_N$).

For a geodesically convex set $X_i \subset M_i$, we define the projection operator as $\Pi_{X_i} : M_i \to X_i$, i.e., $\Pi_{X_i}(x^i)$ is the unique point $y^i \in X_i$ minimizing $d^i(x^i, \cdot)$, where $d^i(\cdot, \cdot) : M_i \times M_i \to \mathbb{R}$ denotes the distance function of $M_i$. The tangent space at a point $x = (x^1, x^2, \ldots, x^N) \in M$ is given by $T_x M = T_{x^1} M_1 \oplus \cdots \oplus T_{x^N} M_N$, by considering $T_{x^i} M_i$ to be a subspace of $T_x M$, where $\oplus$ is the direct sum of vector spaces. Then, for a point $x = (x^1, x^2, \ldots, x^N) \in M$ and a tangent vector $\xi \in T_x M$, we write $\xi = (\xi^1) = (\xi^1, \xi^2, \ldots, \xi^N)$, where $i \in \{1, 2, \ldots, N\}$, and $\xi^i \in T_{x^i} M_i$. Finally, for $x^i, y^i \in M_i$, $\varphi^i_{x^i \to y^i}$ denotes an isometry from $T_{x^i} M_i$ to $T_{y^i} M_i$ (e.g., $\varphi^i_{x^i \to y^i}$ stands for parallel transport from $T_{x^i} M_i$ to $T_{y^i} M_i$).

$E[X]$ denotes the expectation of a random variable $X$, and $t_{[n]}$ denotes the history of the process up to time $n$ (i.e., $t_{[n]} := (t_1, t_2, \ldots, t_n)$). $E[X|t_{[n]}]$ denotes the conditional expectation of $X$ given $t_{[n]}$.

Assumption 2.1. For $i \in \{1, 2, \ldots, N\}$, let $M_i$ be a complete Riemannian manifold with sectional curvature lower bounded by $\kappa_i \leq 0$. We define $M := M_1 \times \cdots \times M_N$. Then, we assume

(A1) For all $i \in \{1, 2, \ldots, N\}$, let $X_i \subset M_i$ be a compact geodesically convex set and $X := X_1 \times \cdots \times X_N$. In addition, $X_i \subset M_i$ has a diameter

---

1The geodesical convexity of $X_i$ implies the existence of a logarithmic map on $X_i$. 
bounded by \( D \); i.e., there exists a positive real number \( D \) such that

\[
\max_{i \in \{1, 2, \ldots, N\}} \sup \{d^i(x^i, y^i) : x^i, y^i \in X_i\} \leq D,
\]

where \( d^i(\cdot, \cdot) \) denotes the distance function of \( M_i \);

(A2) A smooth function \( f_t : M \to \mathbb{R} \) is geodesically convex, where \( t \) is a random variable whose probability distribution is a uniform distribution and supported on a set \( T := \{1, 2, \ldots, T\} \). The function \( f \) is defined for all \( x \in M \), by \( f(x) := \mathbb{E} \{ f_t(x) \} = (1/T) \sum_{t=1}^{T} f_t(x) \).

Note that, when we define a positive number \( G \) as

\[
G := \sup_{t \in T, x \in X} \| \nabla f_t(x) \|_x,
\]

we find that \( G < \infty \) from Assumption 2.1 (A1). The following is the main problem considered here [13, Section 4]:

**Problem 2.1.** Suppose that Assumption 2.1 holds. Then, we have

\[
x_* \in X_* := \left\{ x_* \in X : f(x_*) = \inf_{x \in X} f(x) \right\}.
\]

### 3 Riemannian Adaptive Optimization Algorithm

We propose the following algorithm (Algorithm 1). Algorithm 1 with \( n = t \in T \) coincides with RAMSGrad [13, Figure 1(a)]. A small constant \( \epsilon > 0 \) in the definition of \( \hat{v}_n^i \) guarantees that \( \sqrt{\hat{v}_n^i} > 0 \). (Adam [8, Algorithm 1] and AMSGrad [10, Algorithm 2], [11, Algorithm 1] use such a constant in practice).

In [13], Bécigneul and Ganea used “regret” to guarantee the convergence of RAMSGrad. The regret at the end of \( T \) iterations is defined as

\[
R_T := \sum_{t \in T} f_t(x_t) - \min_{x \in X} \sum_{j \in T} f_j(x),
\]

where \((f_t)_{t \in T}\) is a family of differentiable, geodesically convex functions from \( M \) to \( \mathbb{R} \) and \((x_t)_{t \in T}\) is the sequence generated by RAMSGrad. They proved the following theorem [13, Theorem 1]:

**Theorem 3.1** (Convergence of RAMSGrad). Suppose that Assumption 2.1 (A1) holds and that \( f_t \) is smooth and geodesically convex for all \( t \in T \). Let \((x_t)_{t \in T}\) and \((\hat{v}_t)_{t \in T}\) be the sequences obtained from RAMSGrad, \( \alpha_t = \alpha/\sqrt{T} \), \( \beta_1 = \beta_{11} \), \( \beta_{1k} \leq \beta_1 \) for all \( t \in T \), \( \alpha > 0 \), and \( \gamma := \beta_1/\sqrt{\beta_2} < 1 \). We then have:

\[
R_T \leq \frac{\sqrt{T}D^2}{2\alpha(1 - \beta_1)} \sum_{i=1}^{N} \sqrt{\hat{v}_T^i} + \frac{D^2}{2(1 - \beta_1)} \sum_{i=1}^{N} \sum_{t=1}^{T} \beta_{1i} \sqrt{\hat{v}_t} + \frac{\alpha\sqrt{1 + \log T}}{(1 - \beta_1)^2(1 - \gamma)\sqrt{1 - \beta_2}} \sum_{i=1}^{N} \zeta(\kappa_i, D) + 1 + \frac{1}{2} \sqrt{\sum_{i=1}^{T} \|g_i\|^2_{x_i}},
\]

where \( \zeta(\kappa_i, D) \) is defined as in Lemma A.7.
Algorithm 1 Modified RAMSGrad for solving Problem 2.1

Require: \((\alpha_n)_{n \in \mathbb{N}} \subset [0, 1), (\beta_{1n})_{n \in \mathbb{N}} \subset [0, 1), \beta_2 \in [0, 1), \epsilon > 0\)

1: \(n \leftarrow 1, x_1 \in X, \tau_0 = m_0 = 0 \in T_{x_0} M, v_0, \hat{v}_0 = 0 \in \mathbb{R}\)

2: loop
3:   \(g_{tn} = (g_{tn}) = \text{grad } f_{tn}(x_n)\)
4:     for \(i = 1, 2, \ldots, N\) do
5:       \(m_n^i = \beta_{1n} \tau_{n-1}^i + (1 - \beta_{1n}) g_n^i\)
6:       \(v_n^i = \beta_2 v_{n-1}^i + (1 - \beta_2) \|g_n^i\|_x^2\)
7:       \(\hat{v}_n^i = \max\{\hat{v}_{n-1}^i, v_n^i\} + \epsilon\)
8:       \(x_{n+1} = \Pi_{X_i} \left[ \exp_{x_n^i}^i \left( -\alpha_n \frac{m_n^i}{\sqrt{v_n^i}} \right) \right]\)
9:   end for
10:   \(n \leftarrow n + 1\)
11: end loop

Proof. See [13] Appendix A.

Here, we note that Theorem 3.1 asserts that the regret generated by RAMSGrad has an upper bound; however, RAMSGrad does not solve Problem 2.1. Additionally, Theorem 3.1 assumes a diminishing learning rate \(\alpha_t\) and does not assert anything about a constant learning rate. Our convergence analysis (Theorem 3.2) allows Algorithm 1 to use both constant and diminishing learning rates. Corollaries 3.1 and 3.2 are convergence analyses of Algorithm 1 with constant and diminishing learning rates, respectively.

Theorem 3.2. Suppose that Assumption 2.1 holds. Let \((x_n)_{n \in \mathbb{N}}\) and \((\hat{v}_n)_{n \in \mathbb{N}}\) be the sequences generated by Algorithm 1. We assume \(\beta_{1n} \leq \beta_{1,n-1}\) for all \(n \in \mathbb{N}\), and \((\alpha_n)_{n \in \mathbb{N}}\) is a sequence of positive learning rates, which satisfies \(\alpha_n (1 - \beta_{11}) \leq \alpha_{n-1} (1 - \beta_{1,n-1})\) for all \(n \in \mathbb{N}\). We define \(G := \max_{t \in T, x \in X} \|\text{grad } f_t(x)\|_x\). Then, for all \(x_* \in X_*\),

\[
E \left[ \frac{1}{n} \sum_{k=1}^{n} f(x_k) - f(x_*) \right] \leq \frac{NGD^2}{2(1 - \beta_{11})} \frac{1}{n \alpha_n} + \frac{G^2}{2 \sqrt{\epsilon} (1 - \beta_{11})} \sum_{i=1}^{N} \zeta(\kappa_i, D) \frac{1}{n} \sum_{k=1}^{n} \alpha_k \tag{1}
\]

\[
+ \frac{NGD}{1 - \beta_{11}} \frac{1}{n} \sum_{k=1}^{n} \beta_{1k}.
\]

Proof. See the Appendix [3]
for all \( x^* \in X^* \),
\[
\mathbb{E} \left[ \frac{1}{n} \sum_{k=1}^{n} f(x_k) - f(x^*) \right] \leq O \left( \frac{1}{n} \right) + C_1 \alpha + C_2 \beta,
\]
where \( C_1, C_2 > 0 \) are constants.

**Proof.** See the Appendix C.

Corollary 3.1 implies that, if we use sufficiently small constant learning rates \( \alpha \) and \( \beta \), then Algorithm 1 approximates the solution of Problem 2.1. Although RAMSGrad [13, Figure 1(a)] can only use diminishing learning rates such that \( \alpha_t := \alpha / \sqrt{t} \), Corollary 3.1 guarantees that Algorithm 1 with a constant learning rate can solve Problem 2.1.

**Corollary 3.2 (Diminishing learning rate).** Suppose that the assumptions in Theorem 3.2 hold, \( \alpha_n := 1/n^\eta \), where \( \eta \in [1/2, 1) \), and \( \sum_{k=1}^{\infty} \beta_{1k} < \infty \). Then, Algorithm 1 satisfies, for all \( x^* \in X^* \),
\[
\mathbb{E} \left[ \frac{1}{n} \sum_{k=1}^{n} f(x_k) - f(x^*) \right] =
\begin{cases}
O \left( \sqrt{\frac{1 + \log n}{n}} \right) & \text{if } \eta = \frac{1}{2} \\
O \left( \frac{1}{n^{1-\eta}} \right) & \text{if } \eta \in \left( \frac{1}{2}, 1 \right)
\end{cases}.
\]

**Proof.** See the Appendix C.

Corollary 3.2 implies that Algorithm 1 with a diminishing learning rate can solve Problem 2.1 while RAMSGrad [13, Figure 1(a)] only minimizes the regret (see Theorem 3.1) in the sense of the existence of a positive real number \( C \) such that
\[
\frac{R_T}{T} \leq C \sqrt{\frac{1 + \log T}{T}}.
\]
Additionally, RAMSGrad only works in the case where \( \eta = 1/2 \), but Algorithm 1 works for a wider range of \( \eta \).

### 4 Numerical Experiments

In [4], Nickel and Kiela developed Poincaré embeddings. Before describing the numerical experiments, we will review the fundamentals of hyperbolic geometry (see [4][15][16][17]). \( \mathcal{B}^d := \{ x \in \mathbb{R}^d : \|x\| < 1 \} \) denotes the open \( d \)-dimensional
unit ball, where $\|\cdot\|$ denotes the Euclidean norm. The Poincaré ball model of hyperbolic space $(B^d, \rho)$ is defined by a manifold $B^d$ equipped with the following Riemannian metric:

$$
\rho_x := \frac{4}{(1 - \|x\|^2)^2} \rho^E_x,
$$

where $x \in B^d$, and $\rho^E_x$ denotes the Euclidean metric tensor. The Riemannian manifold $(B^d, \rho)$ has a constant sectional curvature, $-1$. We define Möbius addition [16, Definition 1.10] of $x$ and $y$ in $B^d$ as

$$
x \oplus_M y := \frac{(1 + 2 \langle x, y \rangle + \|y\|^2) x + (1 - \|x\|^2) y}{1 + 2 \langle x, y \rangle + \|x\|^2 \|y\|^2},
$$

where $\langle \cdot, \cdot \rangle := \rho^E(\cdot, \cdot)$. Moreover, $\ominus_M x$ denotes the left inverse [16, Definition 1.7] of $x \in B^d$, and the Möbius gyrations [16, Definition 1.11] of $B^d$ are defined as

$$
gyr[x, y]z := \ominus_M (x \oplus_M y) \oplus_M \{x \oplus_M (y \oplus_M z)\},
$$

for all $x, y, z \in B^d$.

In accordance with the above statements, the induced distance function on $(B^d, \rho)$ (see [17, Eq. (6)]) is defined for all $x, y \in B^d$, by

$$
d(x, y) = 2 \tanh^{-1} (\|(-x) \oplus_M y\|).
$$

The exponential map on $(B^d, \rho)$ (see [17, Lemma 2]) is expressed as follows: for $x \in B^d$ and $\xi \neq 0 \in T_x B^d$,

$$
\exp_x(\xi) = x \oplus_M \left\{ \tanh \left( \frac{\|\xi\|}{1 - \|x\|^2} \right) \right\} \frac{\xi}{\|\xi\|},
$$

and, for $x \in B^d$ and $0 \in T_x B^d$,

$$
\exp_x(0) = x.
$$

Parallel transport of $(B^d, \rho)$ (see [13, Section 5]) along the unique geodesic from $x$ to $y$ is given by

$$
\varphi_{x \to y}(\xi) = \frac{1 - \|y\|^2}{1 - \|x\|^2} \text{gyr}[y, -x] \xi.
$$

The Riemannian gradient on $(B^d, \rho)$ (see [13, Section 5]) is expressed in terms of rescaled Euclidean gradients, i.e., for $x \in B^d$, and the smooth function $f : B^d \to \mathbb{R}$,

$$
\nabla f(x) = \frac{(1 - \|x\|^2)^2}{4} \nabla^E f(x),
$$

where $\nabla^E f(x)$ is the Euclidean gradient of $f$ at $x$. 

where $\nabla_E f(x)$ denotes the Euclidean gradient of $f$.

To compute the Poincaré embeddings for a set of $N$ symbols by finding the embeddings $\Theta = \{u_i\}_{i=1}^N$, where $u_i \in B^d$, we solve the following optimization problem: given $\mathcal{L} : B^d \times \cdots \times B^d \rightarrow \mathbb{R}$,

$$\text{minimize } \mathcal{L}(\Theta) \quad \text{subject to } u_i \in B^d.$$  \hfill (3)

The transitive closure of the WordNet mammals subtree consists of 1,180 nouns and 6,450 hypernymy Is-A relations. Let $\mathcal{D} = \{(u, v)\}$ be the set of observed hypernymy relations between noun pairs. We minimize a loss function defined by

$$\mathcal{L}(\Theta) = \sum_{(u,v) \in \mathcal{D}} \log \frac{e^{-d(u,v)}}{\sum_{v' \in \mathcal{N}(u)} e^{-d(u,v')}}$$  \hfill (4)

where $d(u,v)$ defined by (2) is the corresponding distance of the relation $(u,v) \in \mathcal{D}$, and $\mathcal{N}(u) = \{v' : (u,v') \notin \mathcal{D}\} \cup \{v\}$ is the set of negative examples for $u$ including $v$ (see [4, 13]). We embed the transitive closure of the WordNet mammals subtree into a 5-dimensional Poincaré ball $(B^5, \rho)$.

Let us define $M_i := B^5$ and $X_i := \{x \in B^5 : \|x\| \leq 1 - 10^{-5}\}$, whose projection operator $\Pi_{X_i} : B^5 \rightarrow X_i$ is computed as

$$\Pi_{X_i}(x) := \begin{cases} x & \text{if } \|x\| \leq 1 - 10^{-5} \\ (1 - 10^{-5}) \frac{x}{\|x\|} & \text{otherwise} \end{cases}.$$  \hfill (5)

Moreover, the geodesically convex set $X_i$ has a bounded diameter; in fact, let $D$ be the length of the diameter of $X_i$ as a closed disk, measured by the Riemann metric of $\rho$.

As in [4], we will introduce an index for evaluating the embedding. For each observed relation $(u,v) \in \mathcal{D}$, we compute the corresponding distance $d(u,v)$ in the embedding and rank it among the set of negative relations for $u$, i.e., among the set $\{d(u,v') : (u,v') \notin \mathcal{D}\}$. In addition, we assume the reconstruction setting (see [4]): i.e., we evaluate the ranking of all nouns in the dataset. Then, we record the mean rank of $v$ as well as the mean average precision (MAP) of the ranking. Thus, we evaluate the embedding in terms of the loss function values and the MAP rank.

We compared RSGD [5, Section 2] and Algorithm 1 (modified RAMSGrad) numerically. We experimented with a special iteration called the “burn-in phase” (see [4] Section 3) for the first 20 epochs. During the burn-in phase, the algorithm runs at a reduced learning rate of 1/100. When we minimize the loss function (4), we randomly sample 10 negative relations per positive relation. We set $\epsilon = 10^{-8}$ in Algorithm 1.

Our experiments were conducted on a fast scalar computation server at Meiji University. The environment has two Intel(R) Xeon(R) Gold 6148 (2.4

\url{https://www.meiji.ac.jp/isys/hpc/ia.html}
GHz, 20 cores) CPUs, an NVIDIA Tesla V100 (16GB, 900Gbps) GPU and a Red Hat Enterprise Linux 7.6 operating system. The experiment used the code of Facebook Research and we used the NumPy 1.17.3 package and PyTorch 1.3.0 package.

4.1 Constant learning rate

First, we compare algorithms with the following six constant learning rates:

(CS1) RSGD: $\alpha_n = 0.3$.

(CS2) RSGD: $\alpha_n = 0.1$.

(CA1) Algorithm 1: $\alpha_n = 0.3$, $\beta_{1n} = 0.9$, $\beta_2 = 0.999$.

(CA2) Algorithm 1: $\alpha_n = 0.3$, $\beta_{1n} = 0.001$, $\beta_2 = 0.999$.

(CA3) Algorithm 1: $\alpha_n = 0.1$, $\beta_{1n} = 0.9$, $\beta_2 = 0.999$.

(CA4) Algorithm 1: $\alpha_n = 0.1$, $\beta_{1n} = 0.001$, $\beta_2 = 0.999$.

The parameter $\alpha_n$ in (CS1) and (CS2) represents the learning rate of RSGD [5, Section 2]. The learning rates of (CA1)–(CA4) satisfy the assumptions of Corollary 3.1. The parameters $\beta_2 = 0.999$ and $\beta_{1n} = 0.9$ in (CA1) and (CA3) are used in [13, Section 5]. We used $\beta_{1n} = 0.001$ in (CA2) and (CA4) to compare (CA1) and (CA3) with Algorithm 1 with a small learning rate. Figs. 1–4 show the numerical results. Fig. 1 shows the performances of the algorithms for loss function values defined by (4) with respect to the number of epochs, while Fig. 2 presents those with respect to the elapsed time. Fig. 3 shows the MAP ranks of the embeddings with respect to the number of epochs, while Fig. 4 presents the MAP ranks with respect to the elapsed time. We can see that Algorithm 1 outperforms RSGD in every setting. In particular, Figs. 1–2 show that the learning outcomes of RSGD fluctuate greatly depending on the learning rate. In contrast, Algorithm 1 eventually reduces the loss function the most for any learning rate. Moreover, these figures show that the performance of (CA1) (resp. (CA3)) is comparable to that of (CA2) (resp. (CA4)).

4.2 Diminishing learning rate

Next, we compare algorithms with the following six diminishing learning rates:

(DS1) RSGD: $\alpha_n = 30/\sqrt{n}$.

(DS2) RSGD: $\alpha_n = 10/\sqrt{n}$.

(DA1) Algorithm 1: $\alpha_n = 30/\sqrt{n}$, $\beta_{1n} = 0.5^n$, $\beta_2 = 0.999$.

(DA2) Algorithm 1: $\alpha_n = 30/\sqrt{n}$, $\beta_{1n} = 0.9^n$, $\beta_2 = 0.999$.

https://github.com/facebookresearch/poincare-embeddings
Figure 1: Loss function value versus number of epochs in the case of constant learning rates.

Figure 2: Loss function value versus elapsed time in the case of constant learning rates.
Figure 3: MAP rank versus number of epochs in the case of constant learning rates.

Figure 4: MAP rank versus elapsed time in the case of constant learning rates.
Figure 5: Loss function value versus number of epochs in the case of diminishing learning rates.

\((\text{DA3})\) Algorithm 1: \(\alpha_n = 10/\sqrt{n}, \beta_{1n} = 0.5^n, \beta_2 = 0.999.\)

\((\text{DA4})\) Algorithm 1: \(\alpha_n = 10/\sqrt{n}, \beta_{1n} = 0.9^n, \beta_2 = 0.999.\)

The learning rates of (DA1)–(DA4) satisfy the assumptions of Corollary[3,2]. We implemented (DA2) and (DA4) to compare them with (CA1) and (CA3). We implemented (DA1) and (DA3) to check how well Algorithm 1 works depending on the choice of \(\beta_{1n}.\) Figs. 5–8 show the numerical results. Fig. 5 shows the behaviors of the algorithms for loss function values defined by (4) with respect to the number of epochs, whereas Fig. 6 shows those with respect to the elapsed time. Fig. 7 presents the MAP ranks of the embeddings with respect to the number of epochs, while Fig. 8 shows MAP ranks with respect to the elapsed time. Even in the case of diminishing learning rates, Algorithm 1 outperforms RSGD in every setting. The learning results of RSGD fluctuate greatly depending on the initial learning rate. In particular, (DS2) reduces the loss function more slowly than the other algorithms do. On the other hand, Algorithm 1 stably reduces the loss function, regardless of the initial learning rate. Moreover, these figures indicate that (DA2) outperforms (DA1) and that (DA3) performs comparably to (DA4).

From Figs. 2 and 6, we can see that (CA1) (resp. (CA3)) outperforms (DA2) (resp. (DA4)). The above discussion shows that Algorithm 1 with a constant learning rate is superior to the other algorithms at embedding the WordNet mammals subtree into a Poincaé ball.
Figure 6: Loss function value versus elapsed time in the case of diminishing learning rates.

Figure 7: MAP rank versus number of epochs in the case of diminishing learning rates.
5 Conclusion

This paper proposed modified RAMSGrad, a Riemannian adaptive optimization method, and presented its convergence analysis. The proposed algorithm solves the Riemannian optimization problem directly, and it can use both constant and diminishing learning rates. We applied it to Poincaré embeddings. The numerical experiments showed that it converges to the optimal solution faster than RSGD and minimizes the objective function regardless of the initial learning rate. In particular, an experiment showed that the proposed algorithm with a constant learning rate is a good way of embedding the WordNet mammals subtree into a Poincaré subtree.

A Lemmas

Zhang and Sra developed the following lemma in [15, Lemma 5].

**Lemma A.1** (Cosine inequality in Alexandrov spaces). Let \( a, b, c \) be the sides (i.e., side lengths) of a geodesic triangle in an Alexandrov space whose curvature is bounded by \( \kappa < 0 \) and \( A \) be the angle between sides \( b \) and \( c \). Then,

\[
a^2 \leq \zeta(\kappa, c)b^2 + c^2 - 2bc\cos(A),
\]

where

\[
\zeta(\kappa, c) = \frac{\sqrt{|\kappa|c}}{\tanh(\sqrt{|\kappa|c})}
\]
We prove the following lemma. All relations between random variables are supported to hold almost surely.

**Lemma A.2.** Suppose that Assumption 2.1 (A2) holds. We define \( G := \max_{t \in T, x \in X} \| \text{grad} f_t(x) \|_x \). Let \((x_n)_{n \in \mathbb{N}}\) and \((\tilde{v}_n)_{n \in \mathbb{N}}\) be the sequences generated by Algorithm 1. Then, for all \( i \in \{1, 2, \ldots, N\} \), and \( k \in \mathbb{N} \),

\[
\| m^i_k \|^2_{x^i_k} \leq G^2,
\]

and

\[
\sqrt{\tilde{v}_k} \leq G.
\]

**Proof.** First, we consider (5). The proof is by induction. For \( k = 1 \), from the convexity of \( \| \cdot \|_x \), we have

\[
\| m^i_1 \|^2_{x^i_1} \leq \beta_{11} \| \varphi^i_{x^i_0 \rightarrow x^i_1}(m^i_0) \|^2_{x^i_1} + (1 - \beta_{11}) \| g^i_1 \|^2_{x^i_1}
\]

\[
= (1 - \beta_{11}) \| g^i_1 \|^2_{x^i_1}
\]

\[
\leq \| g^i_1 \|^2_{x^i_1}
\]

\[
\leq G^2,
\]

where we have used \( 0 \leq \beta_{11} < 1 \) and \( \| g^i_1 \|_{x^i_1} \leq G \). Suppose that \( \| m^i_{k-1} \|^2_{x^i_{k-1}} \leq G^2 \). The convexity of \( \| \cdot \|^2_{x^i_k} \), together with the definition of \( m^i_k \), and \( \| g^i_k \|_{x^i_k} \leq G \), guarantees that,

\[
\| m^i_k \|^2_{x^i_k} \leq \beta_{1k} \| \varphi^i_{x^i_{k-1} \rightarrow x^i_k}(m^i_{k-1}) \|^2_{x^i_k} + (1 - \beta_{1k}) \| g^i_k \|^2_{x^i_k}
\]

\[
\leq \beta_{1k} \| m^i_{k-1} \|^2_{x^i_{k-1}} + (1 - \beta_{1k})G^2
\]

\[
\leq \beta_{1k}G^2 + (1 - \beta_{1k})G^2 = G^2.
\]

Induction thus ensures that, for all \( k \in \mathbb{N} \),

\[
\| m^i_k \|^2_{x^i_k} \leq G^2.
\]

(6) can be shown by the same way as (5). 

\[\Box\]

**B Proof of Theorem 3.2**
Proof of Theorem 3.2. We note that
\[ y_{k+1}^i := \exp_{x_k^i} \left( -\alpha_k \frac{m_k^i}{\sqrt{v_k^i}} \right). \]

Thus, we consider a geodesic triangle consisting of three points \( x_k^i, x_{k+1}^i \), and \( y_{k+1}^i \). Let the length of each side be \( a, b, \) and \( c \), respectively, such that
\[
\begin{align*}
  a &:= d(y_{k+1}^i, x_{k+1}^i) \\
b &:= d(y_{k+1}^i, x_k^i) \\
c &:= d(x_k^i, x_{k+1}^i)
\end{align*}
\]
(7)

It follows that
\[
\cos \left( \angle_{y_{k+1}^i x_k^i x_{k+1}^i} \right) := \frac{\left\langle \log_{x_k^i} (y_{k+1}^i), \log_{x_k^i} (x_{k+1}^i) \right\rangle_{x_k^i}}{|\log_{x_k^i} (y_{k+1}^i)_{x_k^i}| |\log_{x_k^i} (x_{k+1}^i)_{x_k^i}|} = \frac{\left\langle \log_{x_k^i} (y_{k+1}^i), \log_{x_k^i} (x_{k+1}^i) \right\rangle_{x_k^i}}{d(y_{k+1}^i, x_{k+1}^i) d(x_k^i, x_{k+1}^i)}.
\]

Using Lemma A.3 with (7) and the definition of \( \Pi_{X_i} \), we have
\[
\begin{align*}
d^i (x_{k+1}^i, x_k^i)^2 &\leq d^i (y_{k+1}^i, x_{k+1}^i)^2 \\
&\leq \zeta(\kappa^i, d^i (x_{k+1}^i, x_{k+1}^i)) d^i (y_{k+1}^i, x_k^i)^2 + d^i (x_k^i, x_{k+1}^i)^2 \\
&\quad - 2d^i (y_{k+1}^i, x_k^i) d^i (x_k^i, x_{k+1}^i) \left\langle \log_{x_k^i} (y_{k+1}^i), \log_{x_k^i} (x_{k+1}^i) \right\rangle_{x_k^i} d(y_{k+1}^i, x_{k+1}^i)^2 d(x_k^i, x_{k+1}^i)^2,
\end{align*}
\]
which, together with the definition of \( y_{k+1}^i \), implies that
\[
\left\langle \log_{x_k^i} (x_{k+1}^i) \right\rangle_{x_k^i} \leq \frac{\sqrt{v_k^i}}{2\alpha_k} (d^i (x_k^i, x_{k+1}^i)^2 - d^i (x_{k+1}^i, x_{k+1}^i)^2) \\
+ \zeta(\kappa^i, d^i (x_k^i, x_{k+1}^i)) \frac{\alpha_k}{2\sqrt{v_k^i}} \|m_k^i\|^2_{x_k^i}.
\]

Plugging \( m_k^i = \beta_{1k} \phi_k x_{k-1}^i \rightarrow x_k^i (m_{k-1}^i) + (1 - \beta_{1k}) g_k^i \) into the above inequality and
using (A1), we obtain
\[
\left\langle -g^i_{tk}, \log_{x^*_k} (x^*_i) \right\rangle_{x^*_k} \\
\leq \frac{\sqrt{v^i_k}}{2\alpha_k (1 - \beta_{1k})} (d^i(x^i_k, x^i_k)^2 - d^i(x^i_{k+1}, x^i_k)^2) \\
+ \frac{\zeta(i', D)}{2(1 - \beta_{1k})} \frac{\alpha_k}{\sqrt{v^i_k}} \| m^i_k \|_{x^*_k}^2 \\
+ \frac{\beta_{1k}}{1 - \beta_{1k}} \left\langle \varphi^i_{x^*_k} \ast x^i_k \rightarrow x^i_k, \ log_{x^*_k} (x^i_k) \right\rangle_{x^*_k}.
\]
(8)

Since (A2) implies that \( f \) is geodesically convex with \( g(x) = (g^i(x^i)) := \text{grad} f(x) \), we have
\[
f(x_k) - f(x_*) \leq \left\langle -g(x_k), \log_{x_k} (x_*) \right\rangle_{x_k} \\
= \sum_{i=1}^N \left\langle -g^i(x_k), \log^i_{x_k} (x_*) \right\rangle_{x_k}.
\]

Summing the above equality from \( k = 1 \) to \( n \), we obtain
\[
\frac{1}{n} \sum_{k=1}^n f(x_k) - f(x_*) \leq \frac{1}{n} \sum_{k=1}^n \sum_{i=1}^N \left\langle -g^i(x_k), \log^i_{x_k} (x_*) \right\rangle_{x_k}.
\]
(9)

Furthermore, the linearity of the Riemannian gradient ensures that
\[
\mathbb{E} \left[ \left\langle -g^i_{tk}, \log^i_{x_k} (x^*_i) \right\rangle_{x^*_k} \right] \\
= \mathbb{E} \left[ \mathbb{E} \left[ \left\langle -g^i_{tk}, \log^i_{x_k} (x^*_i) \right\rangle_{x^*_k} \left| t_{[k-1]} \right. \right] \right] \\
= \mathbb{E} \left[ \left\langle -\mathbb{E} \left[ g^i_{tk} \left| t_{[k-1]} \right. \right], \log^i_{x_k} (x^*_i) \right\rangle_{x_k} \right] \\
= \mathbb{E} \left[ \left\langle -g^i(x_k), \log^i_{x_k} (x^*_i) \right\rangle_{x_k} \right],
\]
which, together with (9), implies that
\[
\mathbb{E} \left[ \frac{1}{n} \sum_{k=1}^n f(x_k) - f(x_*) \right] \\
\leq \frac{1}{n} \mathbb{E} \left[ \sum_{k=1}^N \sum_{i=1}^N \left\langle -g^i(x_k), \log^i_{x_k} (x^*_i) \right\rangle_{x_k} \right] \\
= \frac{1}{n} \mathbb{E} \left[ \sum_{k=1}^N \sum_{i=1}^N \left\langle -g^i_{tk}, \log^i_{x_k} (x^*_i) \right\rangle_{x_k} \right].
\]
From (8) and the above inequality, we have

$$\begin{align*}
&\mathbb{E} \left[ \frac{1}{n} \sum_{k=1}^{n} f(x_k) - f(x_*) \right] \\
&\leq \frac{1}{n} \mathbb{E} \left[ \sum_{k=1}^{n} \sum_{i=1}^{N} \frac{\sqrt{\tilde{v}^i_k}}{2\alpha_k(1 - \beta^i_k)} \left( d^i(x^i_k, x^i_*)^2 - d^i(x^i_{k+1}, x^i_*)^2 \right) \right] \\
&+ \frac{1}{n} \mathbb{E} \left[ \sum_{k=1}^{n} \sum_{i=1}^{N} \frac{\zeta(i, D)}{2(1 - \beta^i_k)} \frac{\alpha_k}{\sqrt{\tilde{v}^i_k}} \| m_k^i \|^2 \right] \\
&+ \frac{1}{n} \mathbb{E} \left[ \sum_{k=1}^{n} \sum_{i=1}^{N} \frac{\beta_k}{1 - \beta^i_k} \langle \varphi_{x^i_{k-1}, x^i_k}^i(m_{k-1}^i), \log_{x^i_k}(x^i_*) \rangle \right].
\end{align*}$$ (10)

Here, let us consider the first term of the left-hand side of (10). We note that from the assumption for all \( k \in \mathbb{N}, \alpha_k(1 - \beta^i_k) \leq \alpha_{k-1}(1 - \beta_{1,k-1}) \), and \( \beta_k \leq \beta_{1,k-1} \),

$$\alpha_k(1 - \beta^i_k) \leq \alpha_{k-1}(1 - \beta_{1,k-1}) \leq \alpha_{k-1}(1 - \beta^i_k),$$

which implies \( \alpha_k \leq \alpha_{k-1} \). Using \( \beta_k \leq \beta_{11} \), \( \alpha_k \leq \alpha_{k-1} \), \( \sqrt{\tilde{v}^i_k} \geq \sqrt{\tilde{v}^i_{k-1}} \), and \( \alpha_k(1 - \beta^i_k) \leq \alpha_{k-1}(1 - \beta_{1,k-1}) \) for all \( k \in \mathbb{N} \), together with (A1), we have that

$$\begin{align*}
&\sum_{k=1}^{n} \sum_{i=1}^{N} \frac{\sqrt{\tilde{v}^i_k}}{2\alpha_k(1 - \beta^i_k)} \left( d^i(x^i_k, x^i_*)^2 - d^i(x^i_{k+1}, x^i_*)^2 \right) \\
&\leq \frac{1}{2(1 - \beta_{11})} \sum_{i=1}^{n} \left[ \sum_{k=2}^{n} \left( \frac{\sqrt{\tilde{v}^i_k}}{\alpha_k} - \frac{\sqrt{\tilde{v}^i_{k-1}}}{\alpha_{k-1}} \right) d^i(x^i_k, x^i_*)^2 \right] \\
&+ \frac{\sqrt{\tilde{v}^i_1}}{\alpha_1} d^i(x^i_1, x^i_*)^2 \\
&\leq \frac{1}{2(1 - \beta_{11})} \sum_{i=1}^{n} \left[ \sum_{k=2}^{n} \left( \frac{\sqrt{\tilde{v}^i_k}}{\alpha_k} - \frac{\sqrt{\tilde{v}^i_{k-1}}}{\alpha_{k-1}} \right) D^2 + \frac{\sqrt{\tilde{v}^i_1}}{\alpha_1} D^2 \right] \\
&= \frac{D^2}{2(1 - \beta_{11})} \sum_{i=1}^{n} \frac{\sqrt{\tilde{v}^i_1}}{\alpha_1} \\
&\leq \frac{NGD^2}{2\alpha_n(1 - \beta_{11})},
\end{align*}$$

where the last inequality is guaranteed by Lemma A.2. Namely,

$$\begin{align*}
&\mathbb{E} \left[ \sum_{k=1}^{n} \sum_{i=1}^{N} \frac{\sqrt{\tilde{v}^i_k}}{2\alpha_k(1 - \beta^i_k)} \left( d^i(x^i_k, x^i_*)^2 - d^i(x^i_{k+1}, x^i_*)^2 \right) \right] \\
&\leq \frac{NGD^2}{2\alpha_n(1 - \beta_{11})}.
\end{align*}$$ (11)
Next, let us consider the second term of the left-hand side of (10). From $\sqrt{\epsilon} \leq \sqrt{\hat{v}_k}$ and Lemma A.2, we have

\[
\sum_{k=1}^{n} \sum_{i=1}^{N} \zeta(\kappa^i, D) \frac{\alpha_k}{2(1 - \beta_{1k})} \|m_k^i\|_{x_k}^2 \leq \frac{G^2}{2\sqrt{\epsilon}(1 - \beta_{11})} \sum_{i=1}^{N} \zeta(\kappa_i, D) \sum_{k=1}^{n} \alpha_k.
\]

Namely,

\[
E \left[ \sum_{k=1}^{n} \sum_{i=1}^{N} \zeta(\kappa^i, D) \frac{\alpha_k}{2(1 - \beta_{1k})} \|m_k^i\|_{x_k}^2 \right] \leq \frac{G^2}{2\sqrt{\epsilon}(1 - \beta_{11})} \sum_{i=1}^{N} \zeta(\kappa_i, D) \sum_{k=1}^{n} \alpha_k.
\]

(12)

Now, let us consider the third term of the left-hand side of (10). Applying the Cauchy-Schwarz inequality to the term and using (A1) and Lemma A.2, it follows that

\[
\sum_{k=1}^{n} \sum_{i=1}^{N} \frac{\beta_{1k}}{1 - \beta_{1k}} \langle \varphi_{x^{i}_{k-1} \rightarrow x^i_k} \rangle \frac{\alpha_k}{2(1 - \beta_{1k})} \|m_k^i\|_{x_k} \|x_k\|_{x^i_k} \leq \frac{NGD}{1 - \beta_{11}} \sum_{k=1}^{n} \beta_{1k}.
\]

Namely,

\[
E \left[ \sum_{k=1}^{n} \sum_{i=1}^{N} \frac{\beta_{1k}}{1 - \beta_{1k}} \langle \varphi_{x^{i}_{k-1} \rightarrow x^i_k} \rangle \frac{\alpha_k}{2(1 - \beta_{1k})} \|m_k^i\|_{x_k} \|x_k\|_{x^i_k} \right] \leq \frac{NGD}{1 - \beta_{11}} \sum_{k=1}^{n} \beta_{1k}.
\]

(13)

Finally, together with (10), (11), (12), and (13), we have

\[
E \left[ \frac{1}{n} \sum_{k=1}^{n} f(x_k) - f(x_*) \right] \leq \frac{NGD^2}{2(1 - \beta_{11})} \frac{1}{n\alpha_n} + \frac{G^2}{2\sqrt{\epsilon}(1 - \beta_{11})} \sum_{i=1}^{N} \zeta(\kappa_i, D) \frac{1}{n} \sum_{k=1}^{n} \alpha_k + \frac{NGD}{1 - \beta_{11}} \frac{1}{n} \sum_{k=1}^{n} \beta_{1k}.
\]

This complete the proof. \qed
Proof of Corollary 3.1
The learning rates $\alpha_n := \alpha$ and $\beta_1^n := \beta$ satisfy that, for all $n \in \mathbb{N}$, $\beta_1^n \leq \beta_{1,n-1}$ and $\alpha_n(1 - \beta_1^n) \leq \alpha_{n-1}(1 - \beta_{1,n-1})$. Let us define

$$C_1 := \frac{G^2}{\sqrt{\epsilon(1 - \beta_{11})}} \sum_{i=1}^{N} \zeta(\kappa_i, D) > 0,$$

and

$$C_2 := \frac{NGD}{1 - \beta_{11}}.$$

Using the definitions of $C_1$, and $C_2$, (1) can be written as

$$\mathbb{E} \left[ \frac{1}{n} \sum_{k=1}^{n} f(x_k) - f(x^*) \right] \leq \frac{NGD^2}{2\alpha(1 - \beta_{11})} \frac{1}{n} + C_1 \alpha + C_2 \beta.$$

This complete the proof.

Proof of Corollary 3.2
Let $\alpha_n = 1/n^{\eta}$ ($\eta \in [1/2, 1)$) and $(\beta_1^n)_n \in \mathbb{N}$ satisfies $\beta_1^n \leq \beta_{1,n-1}$ and $\alpha_n(1 - \beta_1^n) \leq \alpha_{n-1}(1 - \beta_{1,n-1})$ for all $n \in \mathbb{N}$, and $\sum_{k=1}^{\infty} \beta_{1k} < \infty$. First, we obviously have

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \beta_{1k} \leq \lim_{n \to \infty} \frac{B_1}{n} = 0,$$

where $B_1 := \sum_{k=1}^{\infty} \beta_{1k} < \infty$. When $\eta = 1/2$, we obviously have

$$\lim_{n \to \infty} \frac{1}{n^{\alpha_n}} = \lim_{n \to \infty} \frac{1}{\sqrt{n}} = 0.$$

Furthermore, we have

$$\frac{1}{n} \sum_{k=1}^{n} \alpha_k \leq \frac{1}{n} \sqrt{\sum_{k=1}^{n} 1^2} \sqrt{\sum_{k=1}^{n} (\frac{1}{\sqrt{k}})^2} \leq \sqrt{\frac{1 + \log n}{n}},$$

where the first inequality comes from the Cauchy-Schwarz inequality and the second inequality comes from $\sum_{k=1}^{n} (1/k) \leq 1 + \log n$. On the other hand, if $\eta \in (1/2, 1)$, we obtain

$$\lim_{n \to \infty} \frac{1}{n^{\alpha_n}} = \lim_{n \to \infty} \frac{1}{n^{1-\eta}} = 0.$$
Moreover,

\[ \frac{1}{n} \sum_{k=1}^{n} \alpha_k \leq \frac{1}{n} \left( \sum_{k=1}^{n} 1^2 \right) \sqrt{\sum_{k=1}^{n} \left( \frac{1}{k^\eta} \right)^2} \leq \frac{B_2}{\sqrt{n}}, \]

where \( B_2 := \sum_{k=1}^{\infty} \left( \frac{1}{k^{2\eta}} \right) < \infty \). Therefore, it follows that for all \( \eta \in \left[ \frac{1}{2}, 1 \right) \),

\[ \lim_{n \to \infty} \frac{1}{n} \alpha_n = 0, \quad \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \alpha_k = 0. \]

Together with (1), (14), (15), and (16), the assertion of Corollary 3.2 is shown. \( \square \)

References

[1] A. Iosifidis, A. Tefas, and I. Pitas, “Graph embedded extreme learning machine,” IEEE Transactions on Cybernetics, vol. 46, no. 1, pp. 311–324, 2015.

[2] S. Mao, L. Xiong, L. Jiao, T. Feng, and S.-K. Yeung, “A novel Riemannian metric based on Riemannian structure and scaling information for fixed low-rank matrix completion,” IEEE Transactions on Cybernetics, vol. 47, no. 5, pp. 1299–1312, 2016.

[3] X. Shen and F.-L. Chung, “Deep network embedding for graph representation learning in signed networks,” IEEE Transactions on Cybernetics, 2018.

[4] M. Nickel and D. Kiela, “Poincaré embeddings for learning hierarchical representations,” in Advances in neural information processing systems, 2017, pp. 6338–6347.

[5] S. Bonnabel, “Stochastic gradient descent on Riemannian manifolds,” IEEE Transactions on Automatic Control, vol. 58, no. 9, pp. 2217–2229, 2013.

[6] H. Sato, H. Kasai, and B. Mishra, “Riemannian stochastic variance reduced gradient algorithm with retraction and vector transport,” SIAM Journal on Optimization, vol. 29, no. 2, pp. 1444–1472, 2019.

[7] J. Duchi, E. Hazan, and Y. Singer, “Adaptive subgradient methods for online learning and stochastic optimization,” Journal of machine learning research, pp. 2121–2159, 2011.

[8] D. P. Kingma and J. Ba, “Adam: A method for stochastic optimization,” Proceedings of The International Conference on Learning Representations, pp. 1–15, 2015.
[9] M. D. Zeiler, “Adadelta: an adaptive learning rate method,” arXiv preprint arXiv:1212.5701, 2012.

[10] S. J. Reddi, S. Kale, and S. Kumar, “On the convergence of Adam and beyond,” Proceedings of The International Conference on Learning Representations, pp. 1–23, 2018.

[11] H. Iiduka, “Appropriate learning rates of adaptive learning rate optimization algorithms for training deep neural networks,” arXiv preprint arXiv:2002.09647, 2020.

[12] H. Kasai, P. Jawanpuria, and B. Mishra, “Riemannian adaptive stochastic gradient algorithms on matrix manifolds,” in International Conference on Machine Learning, 2019, pp. 3262–3271.

[13] G. Bécigneul and O.-E. Ganea, “Riemannian adaptive optimization methods,” Proceedings of The International Conference on Learning Representations, 2019.

[14] P.-A. Absil, R. Mahony, and R. Sepulchre, Optimization algorithms on matrix manifolds. Princeton University Press, 2008.

[15] H. Zhang and S. Sra, “First-order methods for geodesically convex optimization,” in Conference on Learning Theory, 2016, pp. 1617–1638.

[16] A. A. Ungar, “A gyrovector space approach to hyperbolic geometry,” Synthesis Lectures on Mathematics and Statistics, vol. 1, no. 1, pp. 1–194, 2008.

[17] O.-E. Ganea, G. Bécigneul, and T. Hofmann, “Hyperbolic neural networks,” in Advances in neural information processing systems, 2018, pp. 5345–5355.