ON THE EXISTENCE OF A $v_2^{32}$-SELF MAP ON $M(1,4)$ AT THE PRIME 2

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Abstract. Let $M(1)$ be the mod 2 Moore spectrum. J.F. Adams proved that $M(1)$ admits a minimal $v_1$-self map $v_1^1 : \Sigma^8 M(1) \rightarrow M(1)$. Let $M(1,4)$ be the cofiber of this self-map. The purpose of this paper is to prove that $M(1,4)$ admits a minimal $v_2$-self map of the form $v_2^{32} : \Sigma^{192} M(1,4) \rightarrow M(1,4)$. The existence of this map implies the existence of many 192-periodic families of elements in the stable homotopy groups of spheres.

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1. Introduction

Fix a prime $p$. The $p$-component of the stable homotopy groups of spheres admits a filtration called the chromatic filtration. Elements in the $n$th layer of this filtration fit into infinite $v_n$-periodic families. Theoretically, this process is well understood, thanks to the Nilpotence and Periodicity Theorems of Devinatz, Hopkins, and Smith [HS98], [DHS88].

It is difficult in practice, however, to explicitly identify $v_n$-periodic elements, and to determine their periods. One useful technique is to inductively form cofiber...
sequences:

\[ S \xrightarrow{p^0} S \rightarrow M(i_0), \]
\[ \Sigma^{2i_1(p-1)} M(i_0) \xrightarrow{v_{i_1}^1} M(i_0) \rightarrow M(i_0, i_1), \]
\[ \vdots \]
\[ \Sigma^{2i_n(p^n-1)} M(i_0, \ldots, i_{n-1}) \xrightarrow{v_{i_n}^n} M(i_0, \ldots, i_{n-1}) \rightarrow M(i_0, \ldots, i_n). \]

The maps \( v_k^i \) are \( v_k^i \)-self maps. The Periodicity Theorem guarantees their existence for large \( i \). The reader is warned that there are potentially many non-homotopic \( v_k^i \)-self maps, so the homotopy types of the spectra \( M(i_0, \ldots, i_n) \) are not determined merely from the sequence \((i_0, \ldots, i_n)\).

It is challenging to determine the minimal sequence \((i_0, i_1, \ldots, i_n)\). This minimal sequence determines the periods of the primary constituents of the \( v_n \)-periodic families in the stable homotopy groups of spheres. We refer the reader to [Rav86, Ch. 5.5], [Rav92], and [Beh07] for a more detailed discussion.

We give a brief synopsis of what is known concerning the minimal sequence of integers \((i_0, \ldots, i_n)\) so that the spectrum \( M(i_0, \ldots, i_n) \) exists at a given prime \( p \).

For \( p \geq 3 \), it is known that the complex \( M(1, 1) \) is minimal [Ada66], for \( p \geq 5 \), the complex \( M(1, 1, 1) \) is minimal [Smi70], and for \( p \geq 7 \), the complex \( M(1, 1, 1, 1) \) is minimal [Tod71]. For \( p = 2 \), the complex \( M(1, 4) \) is minimal [Ada66], and for \( p = 3 \), the complex \( M(1, 1, 9) \) is minimal [BP04].

In [DM81], it was argued that the complex \( M(1, 4, 8) \) is minimal at the prime 2, i.e. that there is a \( v_2 \)-self map:

\[ \Sigma^{48} M(1, 4) \xrightarrow{v_2^8} M(1, 4). \]

The result is incorrect: the image of \( v_2^8 \) in the Adams-Novikov spectral sequence for \( \text{tmf} \) is not a permanent cycle [HM], [Bau08]. In fact the first multiple of \( v_2 \) which is a permanent cycle in this spectral sequence is \( v_2^{32} \). The purpose of this paper is to prove the following theorem.

**Theorem 1.1.** There is a \( v_2^{32} \)-self map

\[ v : \Sigma^{192} M(1, 4) \rightarrow M(1, 4). \]

**Corollary 1.2.** At the prime 2, the complex \( M(1, 4, 32) \) is minimal.

**Remark 1.3.** A \( v_2^{32} \)-self map is, by definition, a map \( v \) whose induced map

\[ v_* : K(2)_* M(1, 4) \rightarrow K(2)_* M(1, 4) \]

is given by multiplication by \( v_2^{32} \). In particular, the map \( v \), and all of its iterates, must be essential. Since there is a map of ring spectra

\[ \text{tmf} \rightarrow K(2) \]

under which the periodicity generator \( v_2^{32} \in \pi_{192}(\text{tmf}_2) \) maps to \( v_2^{32} \in \pi_{192} K(2) \), to prove Theorem 1.1 it suffices to prove that there exists a self-map \( v \) such that

\[ v_* : \text{tmf}_* M(1, 4) \rightarrow \text{tmf}_* M(1, 4) \]

is given by multiplication by \( v_2^{32} \).
Remark 1.4. The fourth author reports that methods similar to those described in this paper show that the spectra $A_1$ and $M(2, 4)$ also admit $v_3^{32}$-self maps. Here, $A_1$ is a spectrum whose cohomology is a free module of rank 1 over the subalgebra $A(1)$ of the Steenrod algebra (see [DM81]).

The self-map of Theorem 1.1 produces many $v_3^{32}$-periodic infinite families of elements in the stable homotopy groups of spheres. These families are discussed in detail in [HM]. In fact, all of the results of [DM81] and [Mah81] concerning $v_2$-periodic families are valid with $v_3^8$ replaced by $v_3^{32}$.

Organization of the paper. In Section 2, we reduce Theorem 1.1 to showing that there exists a homotopy element

$$v \in \pi_{192}(M(1, 4) \wedge DM(1))$$

with Hurewitz image $v_3^{32} \in tmf_{192}(M(1, 4) \wedge DM(1))$. Here, $DM(1)$ is the Spanier-Whitehead dual of the spectrum $M(1)$.

In Section 3, we construct modified Adams spectral sequences (MASSs) of the form

$$\text{Ext}_{A_i}^{s,t}(F_2, H(1, 4) \otimes DH(1, 4)) \Rightarrow \pi_{t-s}(M(1, 4) \wedge DM(1, 4)),$$

$$\text{Ext}_{A_i}^{s,t}(F_2, H(1, 4) \otimes H_r(X)) \Rightarrow \pi_{t-s}(M(1, 4) \wedge X)$$

where $A_i$ is the dual Steenrod algebra, $H(1, 4)$ and $DH(1, 4)$ are objects in the derived category of $A_i$-comodules, and $\text{Ext}_{A_i}$ is a group of homomorphisms in the derived category. We show that (1.1) is a spectral sequence of algebras, and that (1.2) is a spectral sequence of modules over (1.1).

In Section 4, we prove that there exists an element

$$v_2^8 \in \text{Ext}_{A_1}^{i, j}(F_2, H(1, 4) \otimes DH(1, 4)).$$

In Section 5, we give a general overview of the theory of generalized Brown-Gitler $A_1$-comodules $M_i(j)$. We describe a spectral sequence which computes $\text{Ext}_{A_1}$ in terms of $\text{Ext}_{A_1(i)}$ of tensor products of these comodules. The case of interest is where $i = 2$, and the spectral sequence is an algebraic version of the tmf-resolution.

In Section 6, we compute

$$\text{Ext}_{A(2)_*}^{s,t}(H(1, 4) \otimes M_2(1)^{\otimes k})$$

for $k \leq 3$.

In Section 7, we establish vanishing lines for the Ext groups appearing in the algebraic tmf-resolution. These vanishing lines imply that the only targets of a potential differential supported by $v_3^{32}$ are detected in the algebraic tmf-resolution by the Ext groups computed in Section 6.

In Section 8, we completely compute the MASS for $tmf \wedge M(1, 4)$.

In Section 9, we show that in the MASS for $M(1, 4) \wedge DM(1, 4)$, the differential $d_2(v_3^8)$ is central. This allows us to deduce that $d_2(v_3^{16}) = 0$. We then argue that the differential $d_3(v_3^{24})$ is central, which implies that $d_3(v_3^{32}) = 0$. We just need to show that $v_3^{32}$ is a permanent cycle.

In Section 10, we show that $\bar{\kappa}^6$ is killed in the $E_3$-term of the MASS for $M(1, 4) \wedge DM(1, 4)$.

In Section 11, we prove the main theorem. We identify possible targets of $d_4(v_3^{32})$ in the MASS for $M(1, 4) \wedge DM(1)$ using the results of Sections 6 and 7, and then eliminate these possibilities using the differentials computed in Section 8 and 10.
Conventions. In this paper we shall always be implicitly working in the stable homotopy category localized at the prime 2. All homology and cohomology groups in this paper are implicitly taken with $\mathbb{F}_2$ coefficients.

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2. Generalized Moore spectra

Let $M(1)$ be the mod 2 Moore spectrum. There are many $v_1$-self-maps

$$\Sigma^8 M(1) \to M(1),$$

however, low dimensional calculations indicate that there is precisely one with Adams filtration 4. We shall call this map $v_1^4$, and its cofiber will be denoted $M(1, 4)$.

It is useful to regard the desired self-map $v$ of Theorem 1.1 as an element of the homotopy group $\pi_{192}(M(1, 4) \wedge DM(1, 4))$. The proof of the theorem is simplified by the following splitting result.

Proposition 2.1 (Davis-Mahowald [DM81, Lem 3.2]). The projection $$M(1, 4) \wedge DM(1, 4) \to M(1, 4) \wedge DM(1)$$ is a split surjection.

Corollary 2.2. An element $x \in \pi_k(M(1, 4))$ extends to a self-map

$$\bar{x} : \Sigma^k M(1, 4) \to M(1, 4)$$

if and only if $2x = 0$.

To prove Theorem 1.1 it therefore suffices to construct an appropriate element $v' \in \pi_{192}(M(1, 4) \wedge DM(1))$.

3. Modified Adams spectral sequences

For a graded Hopf algebra $\Gamma$ over a field $k$, let $\mathcal{D}_\Gamma$ denote the derived category of $\Gamma$-comodules. For objects $M$ and $N$ of $\mathcal{D}_\Gamma$, we define groups

$$\text{Ext}_{\Gamma}^{s,t}(M, N) = \mathcal{D}_\Gamma(\Sigma^s M, N[s])$$

as a group of maps in the derived category. Here $\Sigma^s M$ denotes the $t$-fold shift with respect to the internal grading of $M$, and $N[s]$ denotes the $s$-fold shift with respect to the triangulated structure of $\mathcal{D}_\Gamma$. This reduces to the usual definition of $\text{Ext}_\Gamma$ when $M$ and $N$ are $\Gamma$-comodules. We shall frequently use the abbreviation

$$\text{Ext}_{\Gamma}^{s,*}(M) := \text{Ext}_{\Gamma}^{s,*}(k, M).$$

For a left $\Gamma$-comodule $M$ and a right $\Gamma$-comodule $N$, let $C^*(N, \Gamma, M)$ denote the reduced cobar complex with

$$C^*(N, \Gamma, M) = N \otimes \mathbb{T}^{\otimes s} \otimes M.$$
Here, $\overline{\Gamma}$ is the cokernel of the unit $k \to \Gamma$. Then $C^*(\Gamma, \Gamma, M)$ is an injective resolution for $M$ in the category of $\Gamma$-comodules, and

$$\text{Ext}^s_\Gamma(M) = H^s(C^*(k, \Gamma, M)).$$

We refer the reader to [Rav86, Appendix 1] for details.

Let $A_*$ denote the dual Steenrod algebra. Let $H(1) = H_*(M(1))$ be the homology of the mod 2 Moore spectrum. There is a triangle in $D_{A_*}$:

$$\Sigma F_2[-1] \xrightarrow{h_\ast} F_2 \to H(1) \to \Sigma F_2.$$

Let $v_1^4 : \Sigma^{12} H(1)[-4] \to H(1)$ be the unique non-zero element of $\text{Ext}^{4,12}_A(H(1), H(1))$, which detects the $v_1$-self map of $M(1)$ in Adams filtration 4. Let $H(1,4)$ denote the cofiber

$$\Sigma^{12} H(1)[-4] \xrightarrow{v_1^4} H(1) \to H(1,4) \to \Sigma^{12} H(1)[-3].$$

Let

$$DM(1,4) = F(M(1,4), S) \simeq \Sigma^{-10} M(1,4)$$

denote the Spanier-Whitehead dual of $M(1,4)$, and let

$$DH(1,4) = \text{Hom}_{F_2}(H(1,4), F_2) \cong \Sigma^{-13} H(1,4)[3]$$

denote the corresponding object in $D_{A_*}$.

**Proposition 3.1.** For a finite complex $X$, there are modified Adams spectral sequences (MASSs) of the form:

$$E^s_{2,t}(M(1,4) \wedge X) = \text{Ext}^{s,t}_{A_*}(H(1,4) \otimes H_*(X)) \Rightarrow \pi_{t-s}(M(1,4) \wedge X),$$

$$E^s_{2,t}(M(1,4) \wedge DM(1,4)) = \text{Ext}^{s,t}_{A_*}(H(1,4) \otimes DH(1,4)) \Rightarrow \pi_{t-s}(M(1,4) \wedge DM(1,4)).$$

**Proof.** Consider the canonical Adams resolution of $M(1)$:

$$\begin{array}{c}
M(1) & \xrightarrow{} & M(1)_0 & \xrightarrow{} & M(1)_1 & \xrightarrow{} & M(1)_2 & \xrightarrow{} & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
K(1)_0 & & K(1)_1 & & K(1)_2 & & & & \\
\end{array}$$

where

$$M(1)_t = \overline{\mathcal{T}}_{\lambda^t} \wedge M(1),$$

$$K(1)_t = H \wedge \overline{\mathcal{T}}_{\lambda^t} \wedge M(1).$$

Here $H$ denotes the Eilenberg-MacLane spectrum $H F_2$, and $\overline{\mathcal{T}}$ denotes the fiber of the unit $S \to H$. Since the self-map $v_4^1 : \Sigma^8 M(1) \to M(1)$ has Adams filtration 4, there exists a lift:

$$\begin{array}{c}
\Sigma^8 M(1) & \xrightarrow{v_4^1} & M(1) \\
\downarrow \overline{v_4^1} & & \downarrow \overline{v_4^1} \\
M(1) & & M(1) \\
\end{array}$$
The lift \( \tilde{v}^4 \) induces a map of Adams resolutions:

\[
\begin{array}{ccccccc}
\Sigma^8 M(1)_0 & \cdots & \Sigma^8 M(1)_0 & \Sigma^8 M(1)_1 & \cdots \\
\downarrow \tilde{v}^4 & & \downarrow (\tilde{v}^4)_0 & \downarrow (\tilde{v}^4)_1 & \\
M(1)_0 & \cdots & M(1)_4 & M(1)_5 & \cdots \\
\end{array}
\]

where the maps \((\tilde{v}^4)_i\) are given by

\[
(\tilde{v}^4)_i : \Sigma^8 M(1)_i = \Sigma^8 H^A \wedge M(1) \xrightarrow{1 \wedge \tilde{v}^4} \Sigma^8 H^A \wedge H^4 \wedge M(1) = M(1)_{i+4}.
\]

The mapping cones of the vertical maps of (3.2)

\[
\Sigma^8 M(1)_{i-4} \xrightarrow{(\tilde{v}^4)_i} M(1)_i \to M(1)_i
\]

form a resolution:

\[
\begin{array}{ccccccc}
M(1,4) & \cdots & M(1,4)_0 & M(1,4)_1 & M(1,4)_2 & \cdots \\
\downarrow & & \downarrow & \downarrow & \downarrow & \\
K(1,4)_0 & K(1,4)_1 & K(1,4)_2 & & & \\
\end{array}
\]

Smashing this resolution with \( X \), we obtain a spectral sequence

\[
E^{s,t}_1(M(1,4) \wedge X) = \pi_{t-s}(K(1,4)_s \wedge X) \Rightarrow \pi_{t-s}(M(1,4) \wedge X).
\]

By the 3 \times 3 Lemma, the cofibers \( K(1,4)_i \) fit into cofiber sequences

\[
\Sigma^8 K(1)_{i-4} \xrightarrow{(\tilde{v}^4)_i} K(1)_i \to K(1,4)_i.
\]

Here we take \( K(1)_{i-4} = * \) if \( i < 4 \), and \((\tilde{v}^4)_i\) is the map induced by smashing \((\tilde{v}^4)_i\) with \( H \).

Using the \( A_* \)-comodule structure of \( H(1) \) together with the fact that the composite

\[
S^8 \to \Sigma^8 M(1) \xrightarrow{\tilde{v}^4} M(1)
\]

has Adams filtration 4, one may easily check that the map

\[
\Sigma^{12} H(1) = \Sigma^{12} \pi_* K(1)_0 \xrightarrow{(\tilde{v}^4)_0} \Sigma^4 \pi_* K(1)_4 = C^4(F_2, A_*, H(1))
\]

is injective. It follows that the maps

\[
\Sigma^{12} C^{i-4}(F_2, A_*, H(1)) = \Sigma^{8+i} \pi_* K(1)_{i-4} \xrightarrow{(\tilde{v}^4)_i} \Sigma^i \pi_* K(1)_i = C^i(F_2, A_*, H(1))
\]

are injective for all \( i \). We conclude that the cofiber sequences (3.4) give rise to short exact sequences

\[
0 \to \Sigma^{12} C^{i-4}(F_2, A_*, H(1) \otimes H_* X) \xrightarrow{(\tilde{v}^4)_i} C^i(F_2, A_*, H(1) \otimes H_* X) \to \Sigma^i \pi_* (K(1,4)_i \wedge X) \to 0.
\]
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In the derived category $D_{A_*}$ we have a map of triangles:

\[
\begin{array}{c}
\Sigma^{12}C^{*,-4}(A_*, A_*, H(1)) \xrightarrow{(v_1^4)_*} C^*(A_*, A_*, H(1)) \rightarrow Q(1, 4)_* \\
\Sigma^{12}H(1)[-4] \xrightarrow{v_1^4} H(1) \rightarrow H(1, 4)
\end{array}
\]

where $Q(1, 4)_i$ is the cokernel of the inclusion

\[
\Sigma^{12}C^{*-4}_i(A_*, A_*, H(1)) \xrightarrow{(v_1^4)_*} C^i(A_*, A_*, H(1)).
\]

Since we have isomorphisms of cochain complexes

\[
\pi_* (K(1, 4)_* \wedge X) \cong \text{Hom}_{A_*}(F_2, \text{Q}(1, 4)_* \otimes H_* X),
\]

we deduce that the $E_2$-term of the spectral sequence (3.3) is given by

\[
E_2^{s,t}(M(1, 4) \wedge X) = \text{Ext}^{s,t}_{A_*}(H(1, 4) \otimes H_* X).
\]

Consider the Adams resolution for the Spanier-Whitehead dual $DM(1)$:

\[
\begin{array}{c}
DM(1) \longrightarrow DM(1)_0 \longrightarrow DM(1)_1 \longrightarrow DM(1)_2 \longrightarrow \cdots \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
KD(1)_0 \quad KD(1)_1 \quad KD(1)_2
\end{array}
\]

where

\[
DM(1)_i = F(M(1), H^{\wedge i}),
\]

\[
KD(1)_i = F(M(1), H \wedge H^{\wedge i}).
\]

Define maps $(Dv_1^2)_i$ to be the composites

\[
(Dv_1^2)_i : DM(1)_i = F(M(1), H^{\wedge i}) \xrightarrow{u} F(H^{\wedge 4} \wedge M(1), H^{\wedge i+4}) \xrightarrow{(v_1^4)_*} F(\Sigma^8 M(1), H^{\wedge i+4}) = \Sigma^{-8} DM(1)_{i+4}
\]

where $u$ is the unit of the adjunction. These maps assemble to give a map of Adams resolutions:

\[
\begin{array}{c}
DM(1)_0 \longrightarrow \cdots \longrightarrow DM(1)_0 \longrightarrow M(1)_1 \longrightarrow \cdots \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
\Sigma^{-8} DM(1)_0 \longrightarrow \cdots \longrightarrow \Sigma^{-8} DM(1)_0 \longrightarrow \Sigma^{-8} DM(1)_4 \longrightarrow \Sigma^{-8} DM(1)_5 \longrightarrow \cdots
\end{array}
\]

Letting $DM(1, 4)_i$ denote the homotopy fibers of the vertical maps of $[3]$: $DM(1, 4)_i = \Sigma^{-8} DM(1)_{i+4}$.
we obtain a modified Adams resolution of \( DM(1,4) \):

\[
\begin{array}{ccccccccc}
\text{DM}(1,4) & \longrightarrow & \text{DM}(1,4)_{-4} & \longleftarrow & \text{DM}(1,4)_{-3} & \longleftarrow & \text{DM}(1,4)_{-2} & \longleftarrow & \cdots \\
\downarrow & & & & & & & & \\
\text{KD}(1,4)_{-4} & & & & & & & & \text{KD}(1,4)_{-3} & & & & & & & & \text{KD}(1,4)_{-2}
\end{array}
\]

and a corresponding modified Adams spectral sequence

\[
E_2^{s,t}(DM(1,4)) = \text{Ext}^{s,t}_{A}(DH(1,4)) \Rightarrow \pi_{t-s}(DM(1,4)).
\]

By taking iterated mapping cylinders, we may assume that the maps

\[
M(1,4)_{i+1} \rightarrow M(1,4)_i,
\]

\[
DM(1,4)_{i+1} \rightarrow DM(1,4)_i
\]

are inclusions of subcomplexes. Taking the smash product of resolutions \([BMMS86, \text{Ch. IV, Def. 4.2}]\)

\[
\{(M(1,4) \wedge DM(1,4))_i\} = \{M(1,4)_i\} \wedge \{DM(1,4)_i\}
\]

gives the spectral sequence

\[
\text{Ext}^{s,t}_{A}(H(1,4) \otimes DH(1,4)) \Rightarrow \pi_{t-s}(M(1,4) \wedge DM(1,4)).
\]

\(\square\)

**Proposition 3.2.** The spectral sequence \(\{E_r(M(1,4) \wedge DM(1,4))\}\) is a spectral sequence of algebras, and the spectral sequence \(\{E_r(M(1,4) \wedge X)\}\) is a spectral sequence of modules over \(\{E_r(M(1,4) \wedge DM(1,4))\}\).

**Proof.** The canonical Adams resolution for the sphere spectrum is given by \(\{\mathcal{H}^{\wedge i}\}\). The canonical evaluation maps

\[
DM(1,4)_i \wedge M(1,4)_j = (DM(1)_i \times_{(D^7_5)} (\Sigma^{-8}DM(1))^{j} \wedge (M(1)_j \cup_{(S^7_1)} C\Sigma^{8}M(1))_{j-4} \rightarrow \mathcal{H}^{\wedge i+j}
\]

induce maps of modified Adams resolutions

\[
\{(M(1,4) \wedge DM(1,4))_i\} \wedge \{(M(1,4) \wedge DM(1,4))_i\}
\]

\[
= \{M(1,4)_i\} \wedge \{(DM(1,4) \wedge M(1,4))_i\} \wedge \{DM(1,4)_i\}
\]

\[
\rightarrow \{M(1,4)_i\} \wedge \{\mathcal{H}^{\wedge i}\} \wedge \{DM(1,4)_i\}
\]

\[
= \{(M(1,4) \wedge DM(1,4))_i\},
\]

\[
\{(M(1,4) \wedge DM(1,4))_i\} \wedge \{M(1,4)_i \wedge X\}
\]

\[
= \{M(1,4)_i\} \wedge \{(DM(1,4) \wedge M(1,4))_i\} \wedge \{\mathcal{H}^{\wedge i} \wedge X\}
\]

\[
\rightarrow \{M(1,4)_i\} \wedge \{\mathcal{H}^{\wedge i}\} \wedge \{\mathcal{H}^{\wedge i} \wedge X\}
\]

\[
= \{M(1,4)_i \wedge X\}.
\]

These maps induce the desired pairings on the corresponding MASSs. \(\square\)
4. \(v_2^8\)-periodicity in \(\text{Ext}_{A_*}\)

A similar (but easier) argument to Proposition 2.1 proves the following lemma.

**Lemma 4.1.** The morphism

\[ H(1, 4) \wedge DH(1, 4) \to H(1, 4) \wedge DH(1) \]

is a split surjection.

**Corollary 4.2.** An element \(x \in \text{Ext}_{A_*}^*(H(1, 4))\) lifts to an element \(\bar{x}\) in \(\text{Ext}_{A_*}^*(H(1, 4) \otimes DH(1, 4))\) if and only if \(h_0x = 0\).

A computation of \(\text{Ext}_{A(2)_*}(H(1, 4))\) appears in Figure 8.1. Note that it is \(v_2^8\)-periodic.

**Proposition 4.3.** There exists an element

\[ \tilde{v}_2^8 \in \text{Ext}_{A_*}^{8,56}(H(1, 4) \otimes DH(1, 4)) \]

which maps to the element \(v_2^8 \in \text{Ext}_{A_*}^{8,56}(H(1, 4))\) under the composite

\[ \text{Ext}_{A_*}^*(H(1, 4) \otimes DH(1, 4)) \to \text{Ext}_{A_*}^*(H(1, 4)) \to \text{Ext}_{A(2)_*}^*(H(1, 4)). \]

**Proof.** In the May spectral sequence for \(\text{Ext}_{A(2)_*}(\mathbb{F}_2)\), the element \(v_3^8\) is detected by \(b_{2,0}^4\). Using Nakamura’s formula [Nak72], and the calculations of [Tan70], we see that in the May spectral sequence for \(\text{Ext}_{A_*}(\mathbb{F}_2)\), there are differentials:

\[ d_2(b_{3,0}^4) = b_{2,0}^4h_5, \]

\[ d_2(b_{2,0}^5) = h_0^4h_3h_5. \]

In the May spectral sequence, \(v_1^4\) multiplication corresponds to multiplication by \(b_{2,0}^2\). It follows that an element of \(\text{Ext}_{A_*}^{8,56}(H(1, 4))\) which maps to \(v_3^8 \in \text{Ext}_{A_*}^{8,56}(H(1, 4))\)

must have image \(h_0^3h_3h_5\) under the composite

\[ \text{Ext}_{A_*}^{8,56}(H(1, 4)) \xrightarrow{\delta^4} \text{Ext}_{A_*}^{5,44}(H(1)) \xrightarrow{\delta^3} \text{Ext}_{A_*}^{5,43}(\mathbb{F}_2). \]

Since the element \(h_0^3h_3h_5 \in \text{Ext}_{A_*}^{5,43}(\mathbb{F}_2)\) is killed by \(h_0\) multiplication, it lifts to an element \(h_0^3h_3h_5[1] \in \text{Ext}_{A_*}^{5,44}(H(1))\). Consider the exact sequence

\[ \text{Ext}_{A_*}^{8,56}(H(1)) \to \text{Ext}_{A_*}^{8,56}(H(1, 4)) \to \text{Ext}_{A_*}^{5,44}(H(1)) \xrightarrow{v_1^4} \text{Ext}_{A_*}^{9,56}(H(1)). \]

A computer calculation of \(\text{Ext}_{A_*}^{9,56}(H(1))\) using Bruner’s programs [Bru93] reveals that:

1. \(\text{Ext}_{A_*}^{9,56}(H(1)) = 0\),
2. Every element \(x \in \text{Ext}_{A_*}^{5,44}(H(1))\) satisfies \(h_0x = 0\),
3. Every element \(y \in \text{Ext}_{A_*}^{9,57}(H(1))\) satisfies \(y = h_0z\) for some \(z \in \text{Ext}_{A_*}^{8,56}(H(1))\).

These three facts allow us to deduce that there exists an element \(w \in \text{Ext}_{A_*}^{8,56}(H(1, 4))\) which maps to \(h_0^3h_3h_5[1]\), and for which we have \(h_0w = 0\). By Corollary 4.2, the element \(w\) lifts to the desired element \(v_2^8\) in \(\text{Ext}_{A_*}^{8,56}(H(1, 4) \otimes DH(1, 4))\).

We shall abusively refer to the element \(\tilde{v}_2^8 \in \text{Ext}_{A_*}^{8,56}(H(1, 4) \otimes DH(1, 4))\) as \(v_2^8\).
5. **Brown-Gitler comodules**

**Definitions.** Let $A(i)_*$ denote the quotient of the dual Steenrod algebra dual to the subalgebra $A(i)$ of the Steenrod algebra. There is an isomorphism

$$A(i)_* \cong \mathbb{F}[\bar{\xi}_1, \bar{\xi}_2, \bar{\xi}_3, \ldots, \bar{\xi}_{i+1}] / (\bar{\xi}_1^{2^{i+1}}, \bar{\xi}_2^{2^i}, \ldots, \bar{\xi}_{i+1}^{2^i}).$$

Here, $\bar{\xi}_i$ denotes the conjugate of $\xi_i$. We define a filtration on $A_*$ which induces a filtration on the $A_i$-subalgebra

$$(A//A(i))_* = A_i \square_{A(i)} \mathbb{F}_2 \cong \mathbb{F}_2[\bar{\xi}_1^{2^{i+1}}, \bar{\xi}_2^{2^i}, \ldots, \bar{\xi}_{i+1}^{2^i}, \bar{\xi}_i].$$

Our filtration is an increasing filtration of algebras given on generators by $|\bar{\xi}_j| = 2^{j-1}$. In particular, every element of $(A//A(i))_*$ has filtration divisible by $2^{i+1}$. The Brown-Gitler comodule $N_i(j)$ is the subspace of $(A//A(i))_*$ spanned by all elements of filtration less than or equal to $2^{i+1}j$. Using the coproduct formula

$$\psi(\bar{\xi}_k) = \sum_{k_1+k_2=k} \bar{\xi}_{k_1} \otimes \bar{\xi}_{k_2}^{2^{k_1}},$$

the submodule $N_i(j)$ is easily seen to be an $A_i$-subcomodule. Thus we have an increasing sequence of $A_i$-comodules:

$$\mathbb{F}_2 \cong N_i(0) \subset N_i(1) \subset N_i(2) \subset \cdots \subset (A//A(i))_*.$$

Define a map of ungraded rings

$$\phi_i : (A//A(i))_* \to (A//A(i-1))_*$$

whose effect on generators is given by:

$$\phi_i(\bar{\xi}_k^{2^l}) = \begin{cases} \bar{\xi}^{2^l}_{k-1}, & k > 1, \\ 1, & k = 1. \end{cases}$$

**Lemma 5.1.** The map $\phi_i$ is a map of ungraded $A(i)_*$-comodules.

**Proof.** As an $A(i)_*$-comodule algebra, $(A//A(i))_*$ is generated by the elements $\{\bar{\xi}_1^{2^{i+1}}, \bar{\xi}_2^{2^i}, \ldots\}$. It therefore suffices to check that $\phi_i$ commutes with the coaction on these generators. This is easily checked using the coproduct formula (5.1) and the relations in $A(i)_*$.

Let $M_i(j)$ denote the subspace of $(A//A(i))_*$ spanned by the monomials of filtration exactly $2^{i+1}j$.

**Lemma 5.2.** The map $\phi_i$ maps the subspace $M_i(j)$ isomorphically onto the $A_i$-subcomodule $N_i-1(j) \subset (A//A(i-1))_*$. 

**Proof.** The subspace of $M_i(j)$ spanned by monomials of the form $\bar{\xi}_1^{2^{i+1}}x$ where $x$ is a monomial involving $\bar{\xi}_k$ for $k > 1$ is mapped isomorphically onto the subspace $M_{i-1}(j-1) \subset N_{i-1}(j)$.

Using Lemma 5.1 we have the following corollaries.

**Corollary 5.3.** The subspace $M_i(j) \subset (A//A(i))_*$ is an $A(i)_*$-subcomodule.

**Corollary 5.4.** There is an isomorphism of (graded) $A(i)_*$-comodules

$$M_i(j) \cong \Sigma^{2^{i+1}} N_{i-1}(j).$$
**Corollary 5.5.** There is a splitting of $A(i)_*$-comodules

$$(A/A(i))_* \cong \bigoplus_{j \geq 0} M_i(j).$$

**Remark 5.6.** The comodule $N_{-1}(j)$ (respectively $N_0(j)$, $N_1(j)$) is isomorphic as an $A_*$-comodule to the homology of the $j$th $\mathbb{Z}/2$ (respectively integral, bo) Brown-Gitler spectrum. It is not known in general if the comodules $N_i(j)$ are realizable for $i > 1$.

**Algebraic resolutions.** We now describe an algebraic analog of an Adams resolution. For $i = -1$ (respectively $i = 0, 1, 2$) this algebraic resolution will correspond to the $H\mathbb{F}_2$ (respectively $HZ$, bo, tmf) Adams resolution.

Let $X$ be an object of the derived category $\mathcal{D}_{A_*}$. We define $T_i(X)^\bullet$ to be the following cosimplicial object.

$$(A/A(i))^* \otimes X \xrightarrow{u \otimes 1} (A/A(i))^{\otimes 2} \otimes X \xrightarrow{u \otimes 1} (A/A(i))^{\otimes 3} \otimes X$$

Here, $u$ is the unit

$$\mathbb{F}_2 \to (A/A(i))^*.$$

Since $(A/A(i))^*$ is an algebra, the canonical map

$$X \to \text{Tot}(T^i(X)^\bullet)$$

is a quasi-isomorphism (see, for instance, [Wei94, Prop. 8.6.8]). We therefore have a Bousfield-Kan spectral sequence

$$E_1^{s,t,n} = \text{Ext}_{A_*}^s((A/A(i))^* \otimes (A/A(i)))^n \otimes X[-n]) \Rightarrow \text{Ext}_{A_*}^{s,t}(X).$$

where

$$(A/A(i))^* = \text{coker} \left( \mathbb{F}_2 \xrightarrow{u} (A/A(i))^* \right).$$

The $E_1$-term can be simplified using a change of rings isomorphism, together with the splitting of Corollary 5.5

$$E_1^{s,t,n} = \text{Ext}_{A_*}^s((A/A(i))^* \otimes (A/A(i)))^n \otimes X[-n])$$

$$\cong \text{Ext}_{A_*}^s((A/A(i))^n \otimes X[-n])$$

$$\cong \bigoplus_{j_1, \ldots, j_n \geq 1} \text{Ext}_{A(i)}^{s,t}(M_i(j_1) \otimes \cdots \otimes M_i(j_n) \otimes X[-n]).$$

We shall call this spectral sequence (5.2) the $A/A(i)$-resolution for $X$. In this paper we are only be interested in the case where $i = 2$. In this case, we shall refer to the $A/A(2)$-resolution as the algebraic tmf-resolution.

**Lemma 5.7.** If $R$ is a monoid in the derived category $\mathcal{D}_{A_*}$, then the $A/A(i)$-resolution for $R$ is a spectral sequence of algebras. If $M$ is an $R$-module, then the $A/A(i)$-resolution for $M$ is a spectral sequence of modules over the $A/A(i)$-resolution for $R$. 
6. Ext computations

In this section we describe Ext$_{A(2)_*}^*(M)$ for various objects $M \in D_{A(2)_*}$. We first explain the computations, and then describe the methodology used to produce these computations. Charts displaying these Ext groups can be found in the following figures:

- Figure 6.1: Ext$_{A(2)_*}^*(\mathbb{F}_2)$ and Ext$_{A(2)_*}^*(M_2(1))$.
- Figure 6.2: Ext$_{A(2)_*}^*(M_2(1)\otimes^2)$ and Ext$_{A(2)_*}^*(M_2(1)\otimes^3)$.
- Figure 6.3: Ext$_{A(2)_*}^*(M_2(1)\otimes H(1))$ and Ext$_{A(2)_*}^*(M_2(1)\otimes^2 \otimes H(1))$.
- Figure 6.4: Ext$_{A(2)_*}^*(M_2(1)\otimes^3 \otimes H(1))$ and Ext$_{A(2)_*}^*(M_2(1)\otimes H(1,4))$.
- Figure 6.5: Ext$_{A(2)_*}^*(M_2(1)\otimes^2 \otimes H(1,4))$ and Ext$_{A(2)_*}^*(M_2(1)\otimes^3 \otimes H(1,4))$.

In each of these charts, the indexing has been modified to put the bottom generator of $M_2(1)\otimes^k$ in internal degree 0. The meaning of the notation in each of these charts is explained below.

**Ext$_{A(2)_*}^*(\mathbb{F}_2)$**. All of the elements are $c_4 = v_1^4$-periodic, and $v_2^5$-periodic. Exactly one $v_1^4$ multiple of each element is displayed with the • replaced by a o. Observe the wedge pattern beginning in $t - s = 35$. This pattern is infinite, propagated horizontally by $h_{2,1}$-multiplication and vertically by $v_1$-multiplication. Here, $h_{2,1}$ is the name of the generator in the May spectral sequence of bidegree $(t - s, s) = (5, 1)$, and $h_{2,1}^4 = g$.

**Ext$_{A(2)_*}^*(M_2(1)\otimes^k)$, for $k = 1, 2, 3$**. Every element is $v_2^8$-periodic. However, unlike Ext$_{A(2)_*}^*(\mathbb{F}_2)$, not every element of these Ext groups is $v_1^4$-periodic. Rather, it is the case that either an element $x \in$ Ext$_{A(2)_*}^*(M_2(1)\otimes^k)$ satisfies $v_1^4 x = 0$, or it is $v_1^4$-periodic. Each of the $v_1^4$-periodic elements fit into families which look like shifted and truncated copies of Ext$_{A(1)_*}^*(\mathbb{F}_2)$, and are labeled with a o. We have only included the beginning of these $v_1^4$-periodic patterns in the chart. The other generators are labeled with a •. A □ indicates a polynomial algebra $\mathbb{F}_2[h_{2,1}]$.

**Ext$_{A(2)_*}^*(M_2(1)\otimes^k \otimes H(1))$, for $k = 1, 2, 3$**. The notation in these charts is identical to that in the charts for Ext$_{A(2)_*}^*(M_2(1)\otimes^k)$, with the exception that the $v_1^4$-periodic patterns are truncated shifted copies of Ext$_{A(1)_*}^*(H(1))$.

**Ext$_{A(2)_*}^*(M_2(1)\otimes^k \otimes H(1,4))$, for $k = 1, 2, 3$**. Because we have taken the cofiber of $v_1^4$, none of the elements are $v_1^4$ periodic in these charts. The generators of the first $v_2^8$-periodic pattern are denoted with a • or a □, where again a □ denotes a polynomial algebra on $h_{2,1}$. In these charts, however, it is not the case that every element is $v_2^8$-periodic: some elements in the first lightening flash in the 0-stem fail to be $v_2^8$-periodic. We have conveyed this information by displaying the elements in the next $v_2^8$-pattern with o. With the exception of these first few generators, all of the other generators are $v_2^8$-periodic.
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$\Ext_{A(2)}(M_2(1))$
\[ \text{Ext}_{A(2)},(M_2(1)^{\otimes 2}) \]

Figure 6.2.

\[ \text{Ext}_{A(2)},(M_2(1)^{\otimes 3}) \]
Figure 6.3.

\[ \text{Ext}_{A(2)}(M_2(1) \otimes H(1)) \]

\[ \text{Ext}_{A(2)}(M_2(1)^{\otimes 2} \otimes H(1)) \]
$\text{Ext}_{A(2)}(M_2(1)^{\otimes 3} \otimes H(1))$

$\text{Ext}_{A(2)}(M_2(1) \otimes H(1, 4))$
ON THE EXISTENCE OF A $v^3_2$-SELF MAP ON $M(1,4)$ AT THE PRIME 2

\[ \text{Ext}_A(2) \ast (M_2(1) \otimes^H 1, 4)) \]

Figure 6.5.
Methodology. We explain how these charts were produced. The computation of \( \text{Ext}_{A(2)}(F_2) \) is well-known (see, for instance, [DM82]). The \( A(2)_* \) comodule \( M_2(1) \) can be described by the following diagram of generators.

Here, the dual action of the Steenrod algebra is encoded with a straight line denoting \( Sq^1 \), a curved line denoting \( Sq^2 \), and the bracket denoting \( Sq^4 \). A computation of \( \text{Ext}_{A(2)}(M_2(1)) \) can be found in [DM82]. The computation of \( \text{Ext}_{A(2)}(M_2(1) \otimes 2) \) was obtained from \( \text{Ext}_{A(2)}(M_2(1)) \) by inductively working up the skeletal filtration of the second factor of \( M_2(1) \). The computation of \( \text{Ext}_{A(2)}(M_2(1) \otimes 3) \) was then obtained from \( \text{Ext}_{A(2)}(M_2(1) \otimes 2) \) by inductively working up the skeletal filtration of the third factor of \( M_2(1) \). Along the way, because \( H(1) \) occurs as a subcomodule of \( M_2(1) \), we have computed \( \text{Ext}_{A(2)}(M_2(1) \otimes k \otimes H(1)) \) for \( k = 1, 2 \). We then use the long exact sequence induced by the triangle (3.1) to obtain \( \text{Ext}_{A(2)}(M_2(1) \otimes 3 \otimes H(1)) \).

Each of these manual computations was independently verified by R.R. Bruner’s computer program for computing \( \text{Ext} \) [Bru93]. This computer program constructs minimal resolutions of modules over the subalgebra \( A(2) \). We also used the computer program to gain complete understanding of \( v_4^1 \)-periodicity in these \( \text{Ext} \) groups, as we now explain. Note that there is an element

\[ v_1 \in \text{Ext}_{A(2)}^{1,3}(H(1) \otimes H_s C \eta) \]

where \( C \eta \) is the cofiber of \( \eta \in \pi_4 \). We used Bruner’s programs to compute minimal resolutions for

\[ \text{Ext}_{A(2)}(M_2(1) \otimes k \otimes H(1) \otimes H_s C \eta), \quad k = 1, 2, 3, \]

and read off all of the \( v_1 \)-multiplicative structure in these \( \text{Ext} \) groups from the minimal resolutions. We then used an \( \eta \)-Bockstein spectral sequence to recover the \( v_1^4 \)-multiplicative structure on

\[ \text{Ext}_{A(2)}(M_2(1) \otimes k \otimes H(1)), \quad k = 1, 2, 3. \]

From this, the computation of

\[ \text{Ext}_{A(2)}(M_2(1) \otimes H(1, 4)), \quad k = 1, 2, 3 \]

was easily determined by the long exact sequence arising from the triangle (3).

7. Reducing the Computation to \( M_2(1) \otimes k \) for \( k \leq 3 \)

Inductive Short Exact Sequences. We will construct some short exact sequences that relate the various Brown-Gitler comodules \( N_1(j) \). We have an isomorphism

\[ (A(2) \# A(1))_* \cong \Lambda[\xi_1, \xi_2, \xi_4]. \]
Observe that there is an isomorphism of $\mathbb{F}_2$-vector spaces
\[
\tau : (A/\langle A(1) \rangle)_* \xrightarrow{\cong} (A/\langle A(2) \rangle)_* \otimes (A(2)/\langle A(1) \rangle)_*
\]
given on the monomial basis by
\[
\tau(\xi_1^{i_1} \xi_2^{i_2} \cdots) = \xi_1^{i_1} \xi_2^{i_2} \xi_4^{i_4} \cdots \otimes \xi_1^{i_1} \xi_2^{i_2} \xi_3^{i_3}
\]
for $i_j \geq 0$ and $c_j = 0, 1$. The map $\tau$ is not an isomorphism of $A(2)_*$-comodules.

For instance, in $(A/\langle A(1) \rangle)_*$ we have the coaction
\[
\psi(\xi_1^4 \xi_2^2) = \xi_1^4 \xi_2^2 \otimes 1 + \xi_1^4 \otimes \xi_2^2 + \xi_1^2 \otimes \xi_4^1 + 1 \otimes \xi_1^4 \xi_2^2 + \xi_1^4 \otimes \xi_2^4 + \xi_1^2 \otimes \xi_4^1
\]
whereas in $(A/\langle A(2) \rangle)_* \otimes (A/\langle A(1) \rangle)_*$ we have
\[
\psi(1 \otimes \xi_1^4 \xi_2^2) = \xi_1^4 \xi_2^2 \otimes 1 + 1 \otimes \xi_1^4 \otimes \xi_2^2 + \xi_2^4 \otimes \xi_1^4 + 1 \otimes 1 \otimes \xi_1^4 \xi_2^2 + \xi_1^4 \otimes 1 \otimes \xi_1^2.
\]
However, there is a decreasing filtration
\[
(A/\langle A(1) \rangle)_* = F^0(A/\langle A(1) \rangle)_* \supset F^1(A/\langle A(1) \rangle)_* \supset \cdots
\]
of $A(2)_*$-comodules such that $\tau$ induces an isomorphism of the associated graded $A(2)_*$-comodules
\[
\tau : E^0(A/\langle A(1) \rangle)_* \xrightarrow{\cong} (A/\langle A(2) \rangle)_* \otimes (A(2)/\langle A(1) \rangle)_*.
\]
The decreasing filtration is given as follows: under the isomorphism
\[
(A/\langle A(2) \rangle)_* \cong \bigoplus_k M_2(k)
\]
of $A(2)_*$-comodules given by Corollary 5.5, we define
\[
F^j(A/\langle A(1) \rangle)_* := \tau^{-1}\left(\bigoplus_{k=j}^{\infty} M_2(k) \otimes (A(2)/\langle A(1) \rangle)_*\right).
\]
Using the coproduct formula 5.1, this easily verified to be a decreasing filtration by $A(2)_*$-comodules — the coaction preserves or raises the filtration.

Consider the quotients
\[
Q^j(A/\langle A(1) \rangle)_* := (A/\langle A(1) \rangle)_*/F^{j+1}(A/\langle A(1) \rangle)_*.
\]
The map $\tau$ induces isomorphisms of $\mathbb{F}_2$-vector spaces
\[
\tau : Q^j(A/\langle A(1) \rangle)_* \xrightarrow{\cong} N_2(j) \otimes (A(2)/\langle A(1) \rangle)_*.
\]
Furthermore, the filtration $\{F^k(A/\langle A(1) \rangle)_*\}$ projects to a finite decreasing filtration of $Q^j(A/\langle A(1) \rangle)_*$ by $A(2)_*$-comodules, such that $\tau$ induces an isomorphism of associated graded $A(2)_*$-comodules
\[
\tau : E^0Q^j(A/\langle A(1) \rangle)_* \xrightarrow{\cong} N_2(j) \otimes (A(2)/\langle A(1) \rangle)_*.
\]

\textbf{Lemma 7.1.} There is a short exact sequence of $A(2)_*$-comodules:
\[
0 \rightarrow \Sigma^{8j} N_1(j) \otimes N_1(1) \rightarrow N_1(2j + 1) \rightarrow Q^{j-1}(A/\langle A(1) \rangle)_* \rightarrow 0.
\]

\textbf{Lemma 7.2.} There is an exact sequence of $A(2)_*$-comodules:
\[
0 \rightarrow \Sigma^{8j} N_1(j) \rightarrow N_1(2j) \rightarrow Q^{j-1}(A/\langle A(1) \rangle)_* \rightarrow \Sigma^{8j+9} N_1(j - 1) \rightarrow 0.
\]
Proof of Lemma 7.2. Since the elements of \((A(2)/A(1))_*\) have Brown-Gitler filtration at most 12, the image of the composite

\[ N_2(j - 1) \otimes (A(2)/A(1))_* \leftarrow (A/A(2))_* \otimes (A(2)/A(1))_* \xrightarrow{\tau^{-1}} (A/A(1))_* \]

lies in \(N_1(2j + 1)\), giving a surjection of \(A(2)_*\)-comodules

\[ \rho : N_1(2j + 1) \twoheadrightarrow Q^{j-1}(A/A(1))_* \]

As \(\mathbb{F}_2\)-vector spaces, we have

\[ \tau(N_1(2j + 1)) = N_2(j - 1) \otimes (A(2)/A(1))_* \bigoplus M_2(j) \otimes N_1(1) \]

where the Brown-Gitler comodule \(N_1(1)\) is identified as the \(A(2)_*\)-subcomodule

\[ N_1(1) = \mathbb{F}_2\{1, \xi^4, \xi^2, \xi^3\} \subset (A(2)/A(1))_* \]

We deduce that the kernel of \(\rho\) is

\[ M_2(j) \otimes N_1(1) \cong \Sigma^g N_1(j) \otimes N_1(1) \]

\[ \square \]

Proof of Lemma 7.3. As an \(\mathbb{F}_2\)-vector space, the image of \(N_1(2j)\) in \((A/A(2))_* \otimes (A(2)/A(1))_*\) under the isomorphism \(\tau\) is given by

\[ \tau(N_1(2j)) \cong \begin{pmatrix} N_2(j - 2) \otimes (A(2)/A(1))_* \\
\oplus \\
M_2(j - 1) \otimes \mathbb{F}_2\{1, \xi^4, \xi^2, \xi^3, \xi^4 \xi^2, \xi^4 \xi^3, \xi^2 \xi^3\} \\
\oplus \\
M_2(j) \otimes \mathbb{F}_2\{1\} \end{pmatrix} \]

Thus, at least on the level of \(\mathbb{F}_2\)-vector spaces, we have an exact sequence

\[ 0 \rightarrow M_2(j) \otimes \mathbb{F}_2\{1\} \xrightarrow{\alpha} N_1(2j) \xrightarrow{\beta} Q^{j-1}(A/A(1))_* \]

\[ \xrightarrow{\gamma} M_2(j - 1) \otimes \mathbb{F}_2\{\xi^4 \xi^2, \xi^3\} \rightarrow 0 \]

We just need to prove that these are maps of \(A(2)_*\)-comodules. The map \(\gamma\) is clearly a map of \(A(2)_*\)-comodules. We have the following diagram of inclusions of \(A(2)_*\)-comodules.

\[ (7.2) \]

\[ \begin{array}{ccc}
M_2(j) \otimes \mathbb{F}_2\{1\} & \xrightarrow{\alpha} & N_1(2j) \\
\downarrow & & \downarrow \\
(A/A(2))_* & \xrightarrow{\subseteq} & (A/A(1))_* \\
\end{array} \]

\[ \begin{array}{ccc}
& & \delta \\
& \searrow & \\
& & Q^j(A/A(1))_* \\
\end{array} \]

In particular, the map \(\alpha\) is a map of \(A(2)_*\)-comodules. Let \(K\) be the cokernel of \(\alpha\). Then we get an induced map of short exact sequences of \(A(2)_*\)-comodules:

\[ \begin{array}{ccc}
M_2(j) \otimes \mathbb{F}_2\{1\} & \xrightarrow{\alpha} & N_1(2j) \\
\downarrow & & \downarrow \\
M_2(j) \otimes (A(2)/A(1))_* & \xrightarrow{\beta_1} & K \\
\downarrow & & \downarrow \\
M_2(j) \otimes (A(2)/A(1))_* & \xrightarrow{\beta_2} & Q^{j-1}(A/A(1))_* \\
\end{array} \]

We deduce that the map \(\beta\) is a map of \(A(2)_*\)-comodules, because it is given by the composite \(\beta_2 \circ \beta_1\) of \(A(2)_*\)-comodule maps. \(\square\)
Vanishing lines. We reduce the computations needed to those of $M_2(1)^\otimes k$ for $k \leq 3$ using vanishing lines for modified Adams $E_2$ terms. Note that after a finite range, $\text{Ext}_{A(2)}(H(1,4))$ has a vanishing line of slope $1/5$.

**Lemma 7.3.** We have

\[ \text{Ext}_{A(2)}^{s,t}(N_1(j) \otimes H(1,4)) = 0 \]

for

\[ s > \max \left\{ \frac{(t-s)+17}{7}, \frac{(t-s)+a_j}{6}, \frac{(t-s)+b_j}{5} \right\} \]

and the constants $a_j$ and $b_j$ are inductively defined by

\begin{align*}
    a_0 &= 21, \\
    b_0 &= 9, \\
    a_1 &= 15, \\
    b_1 &= 2, \\
    a_{2j} &= \max\{a_{j-1} - 8j - 2, a_j - 8j\}, \\
    b_{2j} &= \max\{b_{j-1} - 8j - 3, b_j - 8j\}, \\
    a_{2j+1} &= a_j - 8j, \\
    b_{2j+1} &= b_j - 8j.
\end{align*}

**Proof.** The case of $j = 0, 1$ is obtained by examining Figures 6.4 and 8.1. The case of $j \geq 2$ is established by induction using Lemmas 7.1 and 7.2. The terms involving $Q_j(A/sslash A(1))$ are handled using the spectral sequence

\[ \text{Ext}_{A(2)}^{s,t}(N_2(j) \otimes (A(2)/A(1)) \otimes H(1,4)) \Rightarrow \text{Ext}_{A(2)}^{s,t}(Q^j(A/sslash A(1)) \otimes H(1,4)) \]

induced from (7.1), and the change-of-rings isomorphism

\[ \text{Ext}_{A(2)}^{s,t}(N_2(j) \otimes (A(2)/A(1)) \otimes H(1,4)) \cong \text{Ext}_{A(1)}^{s,t}(N_2(j) \otimes H(1,4)). \]

The only non-zero values of $\text{Ext}_{A(1)}^{s,t}(H(1,4))$ are displayed below.

```
  2 0 2 4
```

In particular, we see that $\text{Ext}_{A(1)}^{s,t}(H(1,4))$ is zero for $s > \frac{(t-s)+17}{7}$. \hfill \Box

We extract the following estimate.

**Lemma 7.4.** Suppose that $j_1, \ldots, j_n$ is a sequence of positive integers such that for some $i$, $j_i \geq 2$. Then we have

\[ \text{Ext}_{A(2)}^{s,t}(M_2(j_1) \otimes \cdots \otimes M_2(j_n)[-n] \otimes H(1,4)) = 0 \]

for

\[ s > \max \left\{ \frac{(t-s)+17}{7}, \frac{(t-s)+2}{6}, \frac{(t-s)-12}{5} \right\}. \]
Proof. Assume that $n = 1$, and set $j$ equal to $j_1 \geq 2$. By Lemma 7.3, we have
\[ \text{Ext}_{A(2)}(M_2(j)[-1] \otimes H(1,4)) = 0 \]
if
\[ s > \max \left\{ \frac{(t-s) - 8j + 8 + 17}{7}, \frac{(t-s) - 8j + 7 + a_j}{6}, \frac{(t-s) - 8j + 6 + b_j}{5} \right\}. \]
It therefore suffices to prove that the following inequalities are satisfied:
\[ 17 \geq 17 - 8j + 8, \quad 2 \geq a_j - 8j + 7, \quad -12 \geq b_j - 8j + 6. \]
The inequalities are true for $j = 2, 3$. By induction, these inequalities hold for all $j$.

We now induct on $n$. We may as well assume that $j_1 \geq 2$. Assume that
\[ \text{Ext}_{A(2)}^s(M_2(j_1) \otimes \cdots \otimes M_2(j_{n-1})[-n+1] \otimes H(1,4)) = 0 \]
for
\[ s > \max \left\{ \frac{(t-s) + 17}{7}, \frac{(t-s) + 2}{6}, \frac{(t-s) - 12}{5} \right\}. \]
By filtering the $A_\ast$-comodule $M_2(j_n)$ by degree, we obtain an Atiyah-Hirzebruch type spectral sequence which converges to
\[ \text{Ext}_{A(2)}^s(M_2(j_1) \otimes \cdots \otimes M_2(j_n)[-n] \otimes H(1,4)) \]
and whose $E_1$-page is given by
\[ \bigoplus_x \text{Ext}_{A(2)}^{s,t}(\Sigma^x M_2(j_1) \otimes \cdots \otimes M_2(j_{n-1})[-n] \otimes H(1,4)). \]
where $x$ ranges over an $F_2$-basis of $M_2(j_n)$. The smallest value $|x|$ can take is 8, in the case $j_1 = 1$. By our inductive hypothesis, we have
\[ \text{Ext}_{A(2)}^s(\Sigma^8 M_2(j_1) \otimes \cdots \otimes M_2(j_{n-1})[-n] \otimes H(1,4)) = 0 \]
for
\[ s > \max \left\{ \frac{(t-s) + 17}{7}, \frac{(t-s) + 1}{6}, \frac{(t-s) - 14}{5} \right\}. \]
This verifies the inductive step. \qed

Lemma 7.5. Suppose that $n$ is greater than 3. Then we have
\[ \text{Ext}_{A(2)}^s(M_2(1)^\otimes n[-n] \otimes H(1,4)) = 0 \]
for
\[ s > \max \left\{ \frac{(t-s) + 17}{7}, \frac{(t-s) + 4}{6}, \frac{(t-s) - 17}{5} \right\}. \]
Proof. Examining Figure 6.5, we see that
\[ \text{Ext}_{A(2)}^s(N_1(1)^\otimes 3 \otimes H(1,4)) = 0 \]
for
\[ s > \max \left\{ \frac{(t-s) + 17}{7}, \frac{(t-s) + 8}{6}, \frac{(t-s) - 9}{5} \right\}. \]
The lemma follows from induction on $n$, using Atiyah-Hirzebruch type spectral sequences as in the proof of Lemma 7.3. \qed
In this section we describe a complete computation of the MAPS
\[ \text{Ext}_{A(2)}^s(H(1, 4)) \Rightarrow \pi_{t-s}(\text{tmf} \wedge M(1, 4)). \]

The spectral sequence is displayed in four pages in Figures 8.1 and 8.2. The entire spectral sequence is \( v_2^{32} \)-periodic.

We first explain what is happening in these charts. Then we explain the methodology used to produce these differentials.

**Page 1: dimensions 0–48.** The truncated wedge beginning in \( t-s = 35 \) is infinite, and propagated by \( g \)-multiplication. The entire chart is periodic under \( v_8^2 \)-multiplication. Classes born on the 0-cell of \( M(1) \) are denoted with a •, and classes born on the 1-cell of \( M(1) \) are denoted with a ◦. Although multiplication by \( c_4 = v_4^1 \) is faithful in \( \text{Ext}_{A(2)}^*(F_2) \), it is not faithful in \( \text{Ext}_{A(2)}^*(M(1)) \). We therefore get some classes coming from the 9-cell of \( M(1, 4) \), which we denote with a ◦.

There are only two possible Adams differentials through \( t-s = 47 \), and only one of them actually occurs. This differential is indicated on the chart.

**Page 2: dimensions 48–96.** We move to the region between the occurrence of \( v_2^8 \) and \( v_2^{16} \). There are numerous \( d_2 \) differentials in this range, displayed in the chart. In this chart, the classes propagated by \( v_2^1 \) are denoted with a •, and the classes coming from the truncated wedge starting in \( t-s = 35 \) are denoted with a ◦. Note that beginning in \( t-s = 91 \), we just have the following pattern.

```
89 91 93 95 97 99 101
18 20 • ◦• ◦• ◦• ◦• ◦•
```

**Page 3: dimensions 96–144.** We now move up to the region between \( v_2^{16} \) and \( v_2^{24} \). We propagate only the \( h_2^{1,1} \)-periodic pattern from the previous page (denoted with ◦); everything else is either the source or target of a \( d_2 \). We denote the elements propagated by \( v_2^{16} \) multiplication with a •.

**Page 4: dimensions 144–192.** We now introduce the differentials supported by \( v_2^{24} \) and its multiples. We see that eventually we get a small gap in homotopy between the 180 stem and the 192 stem. Then the pattern repeats with \( v_2^{32} \)-periodicity.

**Methodology.** In [HM], the structure of the Adams spectral sequence
\[ \text{Ext}_{A(2)}^{s,t}(F_2) \Rightarrow \pi_t \text{tmf}_2 \]
is completely determined. The Adams spectral sequence for \( \text{tmf}_* M(1, 4) \) is a module over the Adams spectral sequence for \( \text{tmf}_* \), and all of the differentials for \( \text{tmf}_* M(1, 4) \) were deduced from this structure. These computations were double-checked against the Atiyah-Hirzebruch spectral sequence
\[ H^*(M(1, 4), \text{tmf}_*) \Rightarrow \text{tmf}_* M(1, 4) \]
using the known values of \( \text{tmf}_* \). As a further consistency check, a combination of Gross-Hopkins duality [HG94] and Mahowald-Rezk [MR99] duality shows that
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MASS for $tmf, M(1, 4)$, $p3$:

Figure 8.2.

MASS for $tmf, M(1, 4)$, $p4$:

Figure 8.2.
We wish to lift the differential

and hence, using the fact that the MASS for \(\pi_4(M(1,4)\wedge DM(1,4))\) is a spectral sequence of algebras, we will deduce that 
\(d_r(v_2^{32})\) is zero for \(r < 4\).

**Lemma 9.1.** In the MASS for \(\pi_4(M(1,4)\wedge DM(1,4))\), there is a differential

\[d_2(v_2^8) = \tilde{e}_0 r,\]

where \(\tilde{e}_0 r\) is the image of the element \(e_0 r\) under the map

\[\text{Ext}_A^{10,57}(\mathbb{F}_2) \to \text{Ext}_A^{10,57}(H(1,4)\otimes DH(1,4)).\]

**Proof.** By Proposition 9.1, it suffices to establish that \(d_2(v_2^8) = \tilde{e}_0 r\) in the MASS

\[\text{Ext}_A^{14}(H(1,4)\otimes DH(1)) \to \pi_{t-s}(M(1,4)\wedge DM(1)).\]

The differential \(d_2(v_2^8)\) in the Adams spectral sequence for \(tmf\) maps to a differential \(d_2(v_2^8) = \tilde{e}_0 r\) under the map of (M)ASSs

\[\text{Ext}_A^{s,t}(\mathbb{F}_2) \to \pi_{t-s} tmf \to \text{Ext}_A^{s,t}(H(1,4)\otimes DH(1)) \to \text{Ext}_A^{s,t}(H(1,4)\otimes DH(1)) \to \pi_{t-s}(M(1,4)\wedge DM(1)).\]

where \(\tilde{e}_0 r\) is the image of \(e_0 r\) under the composite

\[\text{Ext}_A^{10,57}(\mathbb{F}_2) \to \text{Ext}_A^{10,57}(H(1,4)\otimes DH(1)) \to \text{Ext}_A^{10,57}(H(1,4)\otimes DH(1)).\]

We wish to lift the differential \(d_2(v_2^8) = \tilde{e}_0 r\) to \(d_2(v_2^8) = \tilde{e}_0 r\) using the map of MASSs:

\[\text{Ext}_A^{s,t}(H(1,4)\otimes DH(1)) \to \pi_{t-s}(M(1,4)\wedge DM(1)) \to \text{Ext}_A^{s,t}(H(1,4)\otimes DH(1)) \to \pi_{t-s}(M(1,4)\wedge DM(1)).\]

However, using

\[\text{Ext}_A^{s,t}(\mathbb{F}_2) = \begin{cases} e_0 r & (t-s,s) = (47,10), \\ e_0 r & (t-s,s) = (48,10) \end{cases}\]

and

\[\text{Ext}_A^{s,t}(H(1,4)) = \begin{cases} e_0 r & (t-s,s) = (47,10), \\ 0 & (t-s,s) = (46,10), (48,10), (55,5), (56,5), (57,5) \end{cases}\]

we may deduce that the map

\[\text{Ext}_A^{10,57}(H(1,4)\otimes DH(1)) \to \text{Ext}_A^{10,57}(H(1,4)\otimes DH(1))\]
is an isomorphism. This suffices to show that the differential \( d_2(v_2^8) \) lifts as desired. \( \square \)

Since \( d_2(v_2^8) \) is central, Proposition \[ \ref{cen} \] gives the following corollary.

**Corollary 9.2.** In the MASS for \( \pi_*(M(1,4) \wedge DM(1,4)) \), we have \( d_2(v_2^{16}) = 0 \).

We now investigate \( d_3(v_2^{16}) \).

**Lemma 9.3.** In the MASS for \( \pi_*(M(1,4) \wedge DM(1,4)) \), the element \( d_3(v_2^{16}) \) is in the image of

\[
\text{Ext}_{A_2}^{19, 114}(\mathbb{F}_2) \to \text{Ext}_{A_2}^{19, 114}(H(1,4) \otimes DH(1,4)).
\]

In particular, \( d_3(v_2^{16}) \) is central.

**Proof.** By Proposition 2.1 it suffices to establish that in the MASS for \( \pi_*(M(1,4) \wedge DM(1)) \), the element \( y = d_3(v_2^{16}) \) is in the image of the map

\[
\text{Ext}_{A_2}^{19, 114}(\mathbb{F}_2) \to \text{Ext}_{A_2}^{19, 114}(H(1,4) \otimes DH(1)).
\]

The differential \( d_3(v_2^{16}) \) in the ASS for \( tmf \) maps to a differential \( d_3(v_2^{16}) = z \) under the map of (M)ASSs

\[
\begin{align*}
\text{Ext}_{A(2)_*}^{s,t}(\mathbb{F}_2) & \xrightarrow{\pi_{t-s}tmf} \\
\text{Ext}_{A(2)_*}^{s,t}(H(1,4) \otimes DH(1)) & \xrightarrow{tmf_{t-s}(M(1,4) \wedge DM(1))} \\
\end{align*}
\]

where \( z \) is in the image of

\[
\text{Ext}_{A(2)_*}^{19, 114}(\mathbb{F}_2) \to \text{Ext}_{A(2)_*}^{19, 114}(H(1,4) \otimes DH(1)).
\]

Using the map of spectral sequences

\[
\begin{align*}
\text{Ext}_{A_2}^{s,t}(H(1,4) \otimes DH(1)) & \xrightarrow{\pi_{t-s}(M(1,4) \wedge DM(1))} \\
\text{Ext}_{A(2)_*}^{s,t}(H(1,4) \otimes DH(1)) & \xrightarrow{tmf_{t-s}(M(1,4) \wedge DM(1))} \\
\end{align*}
\]

we see that \( y \) maps to \( z \). Therefore \( z \) detects \( y \) in the algebraic \( tmf \)-resolution for \( \text{Ext}_{A_2}^{s,t}(H(1,4) \otimes DH(1)) \). Since the algebraic \( tmf \)-resolution is functorial, we deduce that \( y \) is in the image of the map

\[
\text{Ext}_{A_2}^{19, 114}(\mathbb{F}_2) \xrightarrow{\text{Ext}_{A_2}^{19, 114}(H(1,4) \otimes DH(1))} \\
\]

modulo higher terms of the algebraic \( tmf \)-resolution: that is to say, there exists an element

\[
x \in \text{Ext}_{A_2}^{19, 114}(\mathbb{F}_2)
\]

such that \( y - i_{\ast}(x) \) is detected in a higher filtration of the algebraic \( tmf \)-resolution.

We are left with showing that \( w = y - i_{\ast}(x) = 0 \). Suppose not. Using our vanishing lines from Section \[ \ref{van} \] and our \( \text{Ext}_{A(2)_*} \) computations from Section \[ \ref{com} \] we deduce that \( w \) is detected in the algebraic \( tmf \)-resolution by an element

\[
w \in \text{Ext}_{A(2)_*}^{19, 114}(M_2(1) \otimes H(1,4) \otimes DH(1)[-1])
\]
and the image of \( \overline{w} \) under the map

\[
\text{Ext}^{19,114}_{A(2),*}(M_2(1) \otimes H(1,4) \otimes DH(1)[-1]) \xrightarrow{1 \otimes p_* \otimes 1} \text{Ext}^{19,114}_{A(2),*}(M_2(1) \otimes \Sigma^{12} H(1) \otimes DH(1)[-4])
\]

is non-trivial, where \( p_* \) is the projection

\[
H(1,4) \to \Sigma^{12} H(1)[-3]
\]
in the derived category of \( A_* \)-comodules induced by the projection

\[ p : M(1,4) \to \Sigma^9 M(1). \]

We deduce that in the MASS for \( M(1) \land DM(1) \) there is a differential

\[
d_3((p_1 \otimes 1)(v_2^{16})) = (p_1 \otimes 1)(w).
\]

We will verify the following claim:

**Claim 9.4.** The element \((p_1 \otimes 1)(v_2^{16})\) is non-trivial in the \( E_3 \)-page of the MASS for \( M(1) \land DM(1) \).

Assuming Claim 9.4 we deduce that \( d_3((p_1 \otimes 1)(v_2^{16})) \) is non-trivial. However, the image of \( v_2^{16} \) under the map

\[
\text{Ext}^{16,112}_{A_*}(H(1,4) \otimes DH(1)) \xrightarrow{1 \otimes p_* \otimes 1} \text{Ext}^{16,112}_{A_*}(\Sigma^{12} H(1) \otimes DH(1)[-3])
\]

may be computed using the May spectral sequence. In the May spectral sequence, the element \( v_2^{16} \) is detected by \( b_{3,0}^6 \). Applying Nakamura’s formula \cite{Nak72} to the May spectral sequence differential \( d_8(b_{3,0}^4) = h_b b_{4,0}^2 \) in the proof of Proposition 4.3 gives

\[
d_{16}(b_{3,0}^6) = h_b b_{2,0}^6
\]

from which it follows that

\[
(p_1 \otimes 1)(v_2^{16}) = h_b b_{2,0}^6.
\]

The element \( b_{2,0}^6 \) detects the cube of the Adams map:

\[
v_1^{12} = (v_1^4)^3 \in \pi_{24}(M(1) \land DM(1)).
\]

Since this homotopy element has order 2, the Adams differential

\[
d_2(h_b) = h_b h_2^2
\]

implies that the element \( h_b b_{2,0}^6 \) detects the Toda bracket of the composite

\[
S^{86} \xrightarrow{h_b} S^{24} \xrightarrow{v_1^{12}} S^{24} \xrightarrow{v_1^{12}} M(1) \land DM(1).
\]

In particular, \( h_b b_{2,0}^6 \) is a permanent cycle in the MASS for \( M(1) \land DM(1) \), which contradicts the existence of a non-trivial differential \( d_3((p_1 \otimes 1)(v_2^{16})) \). Thus the assumption that \( w \neq 0 \) gives rise to a contradiction, and we conclude that \( w = 0 \), as desired.

We are left with verifying Claim 9.4. We will verify this claim by establishing:

1. The element

\[
(1 \otimes p_1 \otimes 1)(\overline{w}) \in \text{Ext}^{19,114}_{A(2),*}(M_2(1) \otimes \Sigma^{12} H(1) \otimes DH(1)[-4])
\]

is not the target of a differential in the algebraic \( tmf \)-resolution for \( \text{Ext}^{*}_{A_*}(H(1) \otimes DH(1)) \).
(2) The element 
\[(p_\ast \otimes 1)(w) \in \text{Ext}^{19,114}_{A_\ast}(\Sigma^{12}H(1) \otimes DH(1)[-3])\]
is not the target of a $d_2$ differential in the MASS for $M(1) \wedge DM(1)$.

Item (1) above is verified by observing that 
\[\text{Ext}^{s,t}_{A(2)_\ast}(F_2) = 0 \quad (t-s,s) = (86,15), (87,15), (88,15)\]
and so there are no possible contributions to 
\[\text{Ext}^{87,15}_{A(2)_\ast}(H(1) \otimes DH(1)),\]
and this is the only possible source for a differential in the algebraic $tmf$-resolution.

We now verify (2). The $A_\ast$-comodule $H(1) \otimes DH(1)$ has the following diagram of generators.

(9.1) 
\[
\begin{array}{ccc}
1 & \circ & \circ \\
0 & \bullet & \triangle \\
-1 & 1 & \square \\
\end{array}
\]

Here the straight lines encode the action of $Sq_1^\ast$ and the curved line denotes a $Sq_2^\ast$. Using Bruner’s computer generated $\text{Ext}_{A_\ast}(F_2)$ charts [Bru93], we compute the vicinity of $(p_\ast \otimes 1)(w)$ in $\text{Ext}_{A_\ast}^{s,t}(H(1) \otimes DH(1))$ in Table 9.1.

| $s$ | $t-s$ | 86 | 87 |
|-----|-------|----|----|
| 16  |      | $\circ$ | * |
| 15  |      | $(p_\ast \otimes 1)(w)$ | * |
| 14  |      | * | $\circ a_{87}$ |

Table 9.1. $\text{Ext}_{A_\ast}^{s,t}(H(1) \otimes DH(1))$ near $(p_\ast \otimes 1)(w)$

In this table, entries marked with * are not computed, otherwise, elements are denoted by the generator (as in (9.1)) that supports it. The only possible sources for a non-trivial $d_2$ are $a_{87}$ and $b_{87}$.

The element $a_{87}$ is the image of an element 
\[a_{88} \in \text{Ext}_{A_\ast}^{14,102}(F_2)\]
under the inclusion of the bottom generator 
\[\Sigma^{-1}F_2 \to H(1) \otimes DH(1).\]

Since $\text{Ext}_{A_\ast}^{16,103}(F_2) = 0$, we deduce that $d_2(a_{88}) = 0$ in the ASS for $\pi_\ast S$. The map of MASSs induced from the inclusion of the bottom cell of $M(1) \wedge DM(1)$ gives $d_2(a_{87}) = 0$.

We now turn our attention to $b_{87}$. Table 9.2 shows the portion of $\text{Ext}_{A_\ast}(H(1) \otimes DH(1))$ mapped to the vicinity of Table 9.1 under $h_2$-multiplication.
Using the $h_2$ multiplicative structure in Bruner’s tables [Bru93], we deduce that

\[
\begin{align*}
    h_2b_{84} &= b_{87}, \\
    h_2c_{83} &= 0, \\
    h_2c'_{83} &= 0, \\
    h_2c''_{83} &= 0.
\end{align*}
\]

Since $h_2$ is a permanent cycle in the ASS for the sphere, we have

\[
d_2(b_{87}) = d_2(h_2b_{84}) = h_2d_2(b_{84}) = 0.
\]

This completes our proof of Claim 9.4. \qed

Proposition 3.2 gives the following corollary.

**Corollary 9.5.** In the MASS for $\pi_\ast(M(1, 4) \wedge DM(1, 4))$, we have $d_3(v_2^{32}) = 0$.

## 10. Calculation of an Adams differential

The image of the element $\bar{\kappa} \in \pi_{20}(S)_2$ in $\pi_{20}(M(1, 4) \wedge DM(1, 4))$ gives rise to a self-map

\[
\bar{\kappa} : M(1, 4) \to M(1, 4).
\]

The element $g \in \Ext^{4,24}_{\mathbb{A}_*}(\mathbb{F}_2)$ which detects $\bar{\kappa}$ maps to a permanent cycle $\tilde{g} \in \Ext_{\mathbb{A}_*}(H(1, 4) \otimes DH(1, 4))$ which detects $\bar{\kappa} \in \pi_{20}(M(1, 4) \wedge DM(1, 4))$ in the MASS. The purpose of this section is to prove the following theorem.

**Theorem 10.1.**

1. The element $v_2^{32}h_1 \in \Ext_{\mathbb{A}_{(2)}_*}^{21,142}(H(1, 4))$ lifts to an element

\[
\widetilde{v_2^{32}h_1} \in \Ext_{\mathbb{A}_*}^{21,142}(H(1, 4) \otimes DH(1, 4)).
\]

2. There is a differential

\[
d_3(\widetilde{v_2^{32}h_1}) = \tilde{g}^6 + R
\]

in the MASS for $M(1, 4) \wedge DM(1, 4)$, where $R$ is an element of filtration greater than 0 in the algebraic tmf-resolution.

**Proof.** Table 10.1 displays a small portion of the $E_1$-page of the algebraic tmf-resolution for $\Ext_{\mathbb{A}_*}(H(1, 4) \wedge DH(1, 4))$. 

\[
\begin{array}{c|c|c}
  s \setminus t & 83 & 84 \\
  \hline
  15 & c_{83} & * \\
  14 & c'_{83} & * \\
  13 & * & \circ b_{84} \\
\end{array}
\]

**Table 9.2.** $\Ext_{\mathbb{A}_*}^{s,t}(H(1) \otimes DH(1))$ near $h_2^{-1}b_{87}$.
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Table 10.1. The algebraic $tmf$-resolution for $\text{Ext}_{A_*}(H(1,4) \otimes DH(1))$ near $v_2^{20}h_1$

| $s \backslash t - s$ | 120 | 121 |
|-------------------|-----|-----|
| 24                | $\bullet\bullet\bullet g^0$ | $\bullet$ |
|                   | $\circ$ | $\circ$ |
| 23                | $d_{120} \bullet b_{120}$ | $\bullet$ |
|                   | $\circ x_{120}$ | $\circ$ |
|                   | $\circ y_{120}$ | $\circ$ |
| 22                | $\bullet$ | $\circ$ |
|                   | $\circ z_{120}$ | * |
| 21                | $\bullet$ | $v_2^{20}h_1 \bullet$ |
|                   | $\circ \circ \circ \circ \circ$ | $\circ \circ \circ \circ$ |
|                   | $\circ \circ \circ \circ \circ$ | * |

We shall refer to all differentials in the algebraic $tmf$-resolution as $d_1$ differentials. Differentials in the MASS

$$E_2^{s,t} = \text{Ext}_{A_*}^{s,t}(H(1,4) \otimes DH(1)) \Rightarrow \pi_{t-s}(M(1,4) \wedge DM(1))$$

will be referred to by $d_r$ for $r \geq 2$.

In order to prove (1), we must show that the element $v_2^{20}h_1$ in Table 10.1 does not support a non-trivial $d_1$. There is one possible target $z_{120}$ in $(t - s, s) = (120, 22)$, but we will argue shortly that this possibility cannot occur. Assuming for the moment that $d_1(v_2^{20}h_1) = 0$, we would conclude that $v_2^{20}h_1$ lifts to an element

$$\overline{v_2^{20}h_1} \in \text{Ext}_{A_*}^{21,142}(H(1,4) \otimes DH(1,4)).$$

The composite

$$H(1,4) \wedge DH(1,4) \rightarrow H(1,4) \rightarrow tmf \wedge H(1,4)$$
induces a map of MASSES:

\[
\begin{align*}
\text{Ext}^s_t(\mathbb{A}(H(1,4) \otimes DH(1,4)) & \longrightarrow \pi_{t-s}(M(1,4) \wedge DM(1,4)) \\
\text{Ext}^s_t(\mathbb{A}(H(1,4)) & \longrightarrow \pi_{t-s}(\text{tmf} \wedge M(1,4))
\end{align*}
\]

In the MASS for \(\text{tmf} \wedge M(1,4)\), there is a differential

\[d_3(v_2^{20}h_1) = g^6.\]

In order to prove (2), we need to lift this differential to the MASS for \(M(1,4) \wedge DM(1)\). By Proposition 2.1, it suffices to lift this differential to the MASS for \(M(1,4) \wedge DM(1)\):

\[\text{Ext}^s_t(\mathbb{A}(H(1,4) \otimes DH(1)) \Rightarrow \pi_{t-s}(M(1,4) \wedge DM(1))).\]

The obstruction to lifting this differential is that \(\tilde{v}_2^{20}h_1\) could support a \(d_2\) in the MASS for \(M(1,4) \wedge DM(1)\). In fact, Table 10.1 demonstrates that there are four possible targets for such a \(d_2\) in \((t-s,s) = (120,23)\): these are labeled \(a_{120}, b_{120}, x_{120}, y_{120}\).

We now argue (1) and (2) by showing that the element \(v_2^{20}h_1\) in Table 10.1 cannot support a non-trivial \(d_1\) or \(d_2\). We will need Tables 10.2 and 10.3, which depict the \(\text{tmf}\)-resolution in the vicinities of \(gv_2^{20}h_1\) and \(gv_2^4h_1\), respectively.

| \(s\) \(\backslash\) \(t-s\) | 140 | 141 |
|----------------|-----|-----|
| 27             | \(ga_{120} \bullet \bullet g_{120}\) | \(\bullet\) |
|                | \(gx_{120}\) | \(\circ\) |
|                | \(gy_{120}\) | \(\circ\) |
| 26             | \(\bullet\) | \(x_{141}\) |
|                | \(gz_{120}\) | \(\circ\) |
| 25             | \(\bullet\) | \(gv_2^{20}h_1 \bullet \bullet\) |
|                | \(\circ \circ \circ\) | \(\circ \circ\) |
|                | \(\circ \circ \circ \circ \circ\) | \(\circ \circ \circ \circ\) |
|                | \(\ast\) | \(\ast\) |

Table 10.2. The algebraic \(\text{tmf}\)-resolution for \(\text{Ext}_{\mathbb{A}}(H(1,4) \otimes DH(1))\) near \(gv_2^{20}h_1\)

Write \(d_1(v_2^{20}h_1) = c \cdot z_{120}\) for \(c \in \mathbb{F}_2\). Then have

\[d_1(gv_2^{20}h_1) = c \cdot gz_{120}.\]

Table 10.3 shows that \(d_1(gv_2^4h_1) = 0\). Multiplying by the \(d_2\)-cycle \(v_2^{16}\) of Corollary 9.2 we deduce that we must have \(d_1(gv_2^{20}h_1) = 0\). Thus \(c\) equals 0, and we have proven (1).

Write

\[d_2(v_2^{20}h_1) = c_1a_{120} + c_2b_{120} + c_3x_{120} + c_4y_{120}\]
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| \( s \backslash t - s \) | 44 | 45 |
|---------------------|----|----|
| 11                  | ●  | ●  |
| 10                  | ●  | ●  |
| 9                   | ●  | ●  |

Table 10.3. The algebraic \( tmf \)-resolution for \( \text{Ext}_{A_2}(H(1, 4) \otimes DH(1)) \) near \( gv_2^h_1 \)

for \( c_i \in F_2 \). The image of \( v_2^{20}h_1 \) in \( \text{Ext}_{A_2}(H(1, 4)) \) is a \( d_2 \)-cycle in the MASS

\[
\text{Ext}^{s,t}_{A_2}(H(1, 4)) \to \pi_{t-s}(tmf \land M(1, 4)).
\]

We therefore deduce that \( c_1 = c_2 = 0 \). We wish to show that \( d_2(v_2^{20}h_1) = 0 \), i.e.
that it is contained in the image of \( d_1 \). We have

\[
d_2(gv_2^{20}h_1) = c_3gx_{120} + c_4gy_{120}.
\]

Examining Table 10.3 we see that \( d_2(gv_2^h_1) = 0 \). Since \( v_2^{16} \) is a \( d_2 \)-cycle, we deduce that \( d_2(gv_2^{20}h_1) = 0 \). This means that

\[
c_3gx_{120} + c_4gy_{120}
\]

is in the target of a \( d_1 \). With the exception of the element \( x_{141} \), all of the generators in \( (t - s, s) = (141, 26) \) are \( g \)-periodic. Thus we have

\[
d_1(E_1^{26, 167}) = g \cdot d_1(E_1^{22, 143}) + F_2\{d_1(x_{141})\}
\]

where \( E_1^{s,t} \) is the \( E_1 \)-term of the algebraic \( tmf \)-resolution for \( \text{Ext}^{s,t}_{A_2}(H(1, 4) \otimes DH(1)) \). However, we see from Table 10.3 that \( d_1(v_2^{-16}x_{141}) = 0 \), so it follows that \( d_1(x_{141}) = 0 \). We may therefore deduce the vanishing of \( d_2(gv_2^{20}h_1) \) from the vanishing of \( d_2(gv_2^{20}h_1) \). We have proven (2).

11. Proof of the Main Theorem

By Proposition 4.3 and Lemma 5.7 the element

\[
v_2^{32} \in \text{Ext}_{A_2}^{32, 224}(H(1, 4) \otimes DH(1))
\]

is a permanent cycle in the algebraic \( tmf \)-resolution, and it detects an element

\[
v_2^{32} \in \text{Ext}_{A_2}^{32, 224}(H(1, 4) \otimes DH(1)).
\]

By Corollary 9.5 the element \( v_2^{32} \) persists to the \( E_4 \)-page of the MASS for \( M(1, 4) \land DM(1) \). By Proposition 2.1 our main theorem (Theorem 1.1) is a consequence of the following lemma.

**Lemma 11.1.** The element

\[
v_2^{32} \in \text{Ext}_{A_2}^{32, 224}(H(1, 4) \otimes DH(1))
\]

cannot support a non-trivial \( d_r \) in the MASS for \( M(1, 4) \land DM(1) \) for \( r \geq 4 \).
Proof. We shall make use of the following tables. Table 11.1 depicts the algebraic \( \text{tmf} \)-resolution for \( \text{Ext}^{s,t}_{A^*}(H(1,4) \otimes DH(1)) \) in the region where all possible targets of \( d_r(v^{22}_3) \) can lie, for \( r \geq 4 \). Note that there are no non-zero elements in the algebraic \( \text{tmf} \)-resolution that can contribute to \( \text{Ext}^{s}_{A^*}(H(1,4) \otimes DH(1)) \) for \( s > 40 \). Table 11.2 depicts a region of the algebraic \( \text{tmf} \)-resolution which maps to the region of Table 11.1 under \( g^6 \)-multiplication. The notation in these tables is explained in Section 10. The subgroups labeled \( G_{191} \) and \( G_{71} \) are the subgroups generated by the contributions in the algebraic \( \text{tmf} \)-resolution labeled with a *.

| \( s \setminus t-s \) | 190 | 191 |
|------------------|-----|-----|
| 40               | ●   | ●   |
| 39               | ●   | ●   |
| 38               | \( g^6b_{70} \circ g^6c_{70} \) | \( g^6f_{71} \circ \) |
| 37               | \( v^2_{143}k_{143} \circ \) | \( v^2_{143}l_{143} \circ \) |
|                  | \( g^6a_{70} \circ \) | \( g^6a_{71} \circ \) |
|                  | \( G_{191} * \) | \( G_{71} * \) |

Table 11.1. The algebraic \( \text{tmf} \)-resolution for \( \text{Ext}^{s,t}_{A^*}(H(1,4) \otimes DH(1)) \) in the vicinity of \( (t-s,s) = (191,36) \)

| \( s \setminus t-s \) | 70 | 71 |
|------------------|----|----|
| 15               | ●  | ●  |
| 14               | \( b_{70} \circ c_{70} \) | \( f_{71} \circ \) |
| 13               | ●  | \( d_{71} \circ e_{71} \) |
|                  | \( a_{70} \circ \) | \( a_{71} \circ c_{71} \) |
| 12               | ●  | \( G_{71} * \) |

Table 11.2. The algebraic \( \text{tmf} \)-resolution for \( \text{Ext}^{s,t}_{A^*}(H(1,4) \otimes DH(1)) \) in the vicinity of \( (t-s,s) = (71,12) \)

The element \( v^{32}_2 \in \text{Ext}^{32,224}_{A(2)_*}(H(1,4)) \) detects a non-trivial permanent cycle of order 2 in \( \text{tmf}_{192} M(1,4) \). We deduce that the element \( v^{32}_2 \in \text{Ext}^{32,224}_{A(2)_*}(H(1,4) \otimes DH(1)) \) is a permanent cycle in the MASS for \( \text{tmf} \wedge M(1,4) \wedge DM(1) \). Consider the map of MASSs

\[
(11.1) \quad \text{Ext}^{s,t}_{A^*}(H(1,4) \otimes DH(1)) \longrightarrow \pi_{t-s}(M(1,4) \wedge DM(1))
\]

\[
\text{Ext}^{s,t}_{A^{(2)_*}}(H(1,4) \otimes DH(1)) \longrightarrow \text{tmf}_{t-s}(M(1,4) \wedge DM(1))
\]
induced by the map 

\[ M(1, 4) \land DM(1) \rightarrow tmf \land M(1, 4) \land DM(1). \]

Because \( v_2^{32} \) is a permanent cycle in the MASS for \( tmf \land M(1, 4) \land DM(1) \), we deduce that in the MASS for \( M(1, 4) \land DM(1) \), the differential \( d_r(v_2^{32}) \) cannot hit an element coming from \( Ext_A(2) \), in the algebraic \( tmf \)-resolution (these elements are represented by a \( \bullet \) in Table 11.1). Thus the only possible targets for \( d_r(v_2^{32}) \) in Table 11.1 are

\[ g^6 a_{71}, g^6 b_{71}, g^6 c_{71}, g^6 d_{71}, g^6 e_{71}, g^6 f_{71} \]

or an element of the group \( G_{191} \). We claim that none of these elements persist to detect a non-trivial element of the \( E_4 \)-page of the MASS.

Each of the elements in \( 11.2 \) is in the image of multiplication by \( g^6 \). Because the groups \( G_{71} \) and \( G_{191} \) lie on the edge of the slope 1/5 vanishing line of Lemmas 7.3 and 7.5, each of the elements in \( G_{191} \) are of the form \( g^6 y \) for \( y \in G_{71} \).

Suppose that \( x \) is a linear combination of the elements

\[ a_{71}, b_{71}, c_{71}, d_{71}, e_{71}, f_{71} \]

and the elements in \( G_{71} \). We must show that \( g^6 x \) cannot be the non-trivial image of \( d_r(v_2^{32}) \) for \( r \geq 4 \).

If \( x \) is a \( d_r \)-cycle for \( r \leq 3 \), then \( x \) persists to \( E_4 \). Using the multiplicative structure of the MASS (Proposition 3.2) together with the fact that \( g^6 = 0 \) in the \( E_4 \)-page of the MASS for \( M(1, 4) \land DM(1, 4) \) (Theorem 10.1), we deduce that \( g^6 x \) is zero in \( E_4 \). It therefore cannot be a non-trivial target for \( d_r(v_2^{32}) \).

Suppose, however, that \( d_r(x) \) is non-trivial for some \( r \leq 3 \). Since differentials in the algebraic \( tmf \) resolution must increase filtration, we deduce that the only possible targets for \( d_r(x) \) are linear combinations of

\[ a_{70}, b_{70}, c_{70} \]

and \( \bullet \)'s in Table 11.2 for which \( t - s = 70 \) and \( s \geq 14 \). However, each of these \( \bullet \)'s map to non-trivial permanent cycles under the map of spectral sequences (11.1), and therefore cannot be the target of MASS differentials. The only remaining possibilities are

Case (1): \( d_1(x) = a_{70}, \)

Case (2): \( d_2(x) = t_1 b_{70} + t_2 c_{70}, \)

for \( (0, 0) \neq (t_1, t_2) \in \mathbb{F}_2 \oplus \mathbb{F}_2 \). Using Theorem 10.1 we see that in these cases we would respectively have:

Case (1): \( d_1(g^6 x) = g^6 a_{70}, \)

Case (2): \( d_2(g^6 x) = t_1 g^6 b_{70} + t_2 g^6 c_{70}. \)

If we are in Case (1), we are done: the differential \( d_r(v_2^{32}) \) cannot be detected by \( g^6 x \) because \( g^6 x \) does not persist to \( E_2. \) If we are in Case (2), however, we must verify that \( t_1 g^6 b_{70} + t_2 g^6 c_{70} \) is not in the image of a \( d_1 \)-differential. The only possibility is

\[ (11.4) \quad d_1(s_1 v_2^8 k_{143} + s_2 v_2^8 l_{143}) = t_1 g^6 b_{70} + t_2 g^6 c_{70}. \]

\[ \text{This statement must be interpreted with care — Theorem 10.1 asserts that there is an element in } E_2 \text{ of the MASS for } M(1, 4) \land DM(1, 4) \text{ which is detected by } g^6 \text{ in the algebraic } tmf \text{-resolution, and which is the target of a } d_3 \text{ in the MASS.} \]
The algebraic tmf-resolution for Ext$_A^*(H(1,4) \otimes DH(1))$ in the vicinity of the elements $k_{143}$ and $l_{143}$ is displayed below.

|   | $s-t-s$ |
|---|---------|
| 30 | $\bullet$ |
| 29 | $\bullet \bullet$ $k_{143}$ $l_{143}$ |

We see that $k_{143}$ and $l_{143}$ must be $d_1$-cycles. By Proposition 4.3 and Lemma 5.7, we deduce that $v_9^2 k_{143}$ and $v_9^2 l_{143}$ must be $d_1$-cycles. Thus Possibility [11.4] cannot occur, and we deduce that in Case (2), $d_2(g^6 x)$ does not vanish. We conclude that in Case (2), $g^6 x$ cannot persist to $E_4$ and therefore it cannot be the target of $d_r(v_3^{32})$ for $r \geq 4$. □

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