Topological Dirac variables in Abelian $U(1)$ theory.

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Abstract

In this study we, remembering the experience with topological Dirac variables in the non-Abelian Yang-Mills-Higgs (YMH) model with vacuum BPS monopole solutions, attempt to construct similar for the Abelian $U(1)$ model. We show that QED, as one understands it commonly, is only the topologically trivial sector ($n = 0$) of this Abelian $U(1)$ model. For $n \neq 0$ one gets Dirac monopole modes. In both the cases, $n = 0$ and $n \neq 0$, the theory can be quantized via the Hamiltonian reduction in terms of Dirac variables.

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1 Introduction

Of late, a good deal of efforts was expended to construct the constraint-shell formalism for the various Minkowskian non-Abelian models. The essence of this method consists in the reduction of the appropriate Hamiltonians in terms of physical, i.e. always transverse and gauge invariant, variables, called Dirac variables.

These Dirac variables can be got manifestly as the solutions to the Gauss law constraint

$$\partial W/\partial A_0 = 0$$  \hspace{1cm} (1.1)

(with $W$ being the action functional of the considered gauge theory and $A_0$ being the temporal components of the appropriate gauge field Dirac variables are just such variables. There are physical fields which are solutions to the Gauss law constraint 1).

In detail, the origin appearing Dirac variables in a particular gauge theory is following. In order to eliminate temporal components $A_0$ of gauge fields, which are undesirable therein, Dirac and, after him, other authors of the first classical studies in quantization of gauge fields, for instance, eliminated temporal components of gauge fields by gauge transformations. The typical look of such gauge transformations is

$$v^T(x,t)(A_0 + \partial_0)(v^T)^{-1}(x,t) = 0.$$  \hspace{1cm} (1.2)

This equation may be treated as that specifying the gauge matrices $v^T(x,t)$. This, in turn, allows to write down the gauge transformations for spatial components of gauge fields

$$\hat{A}_i^D(x,t) := v^T(x,t)(\hat{A}_i + \partial_i)(v^T)^{-1}(x,t); \quad \hat{A}_i = g\frac{\tau^a}{2i}A_{ai}.$$  \hspace{1cm} (1.3)

It is easy to check that the functionals $\hat{A}_i^D(x,t)$ specified in such a way are gauge invariant and transverse fields:

$$\partial_0 \partial_i \hat{A}_i^D(x,t) = 0; \quad u(x,t)\hat{A}_i^D(x,t)u(x,t)^{-1} = \hat{A}_i^D(x,t)$$  \hspace{1cm} (1.4)

1 Generally speaking, a constraint equation in the Hamiltonian formalism can be defined as following [1]. Constraint equations relate initial data for spatial components of the fields involved in a (gauge) model to initial data of their temporal components.

The Gauss law constraint has an additional specific that it is simultaneously the one of equations of motion (to solve these, it is necessary a measurement of initial data [1]). This is correctly for QED as well for non-Abelian theories (in particular, for QCD), i.e. for the so-called particular theories involving the singular Hessian matrix

$$M_{ab} = \frac{\partial^2 L}{\partial \dot{q}^a \partial \dot{q}^b}$$

(with $L$ being the Lagrangian of the studied theory, $q^i$ being the appropriate degrees of freedom and $\dot{q}^i$ being their time derivatives).

The Hessian matrix $M$ becomes singular (i.e. $\det M = 0$), since the identity

$$\partial L/\partial \dot{A}_0 = 0$$

which 'temporal' components $A_0$ of gauge fields always satisfy in particular theories (in other words, particular theories involve zero canonical momenta $\partial L/\partial \dot{A}_0$ for temporal components $A_0$ of gauge fields).

Thus temporal components $A_0$ of gauge fields are, indeed, non-dynamical degrees of freedom in particular theories, the quantization of which contradicts the Heisenberg uncertainty principle.
Following Dirac [3], we shall refer to the functionals $\hat{A}_i^D(x,t)$ as to the Dirac variables. The Dirac variables $\hat{A}_i^D$ may be derived by resolving the Gauss law constraint (1.1). Solving Eq. (1.1), one expresses temporal components $A_0$ of gauge fields $A$ through their spatial components; by that the nondynamical components $A_0$ are indeed ruled out from the apropriate Hamiltonians. Thus the reduction of particular gauge theories occurs over the surfaces of the apropriate Gauss law constraints. Only upon expressing temporal components $A_0$ of gauge fields $A$ through their spatial components one can perform gauge transformations (1.3) in order to turn spatial components $\hat{A}_i^D$ of gauge fields into gauge-invariant and transverse Dirac variables $\hat{A}_i^D$. Thus, formally, temporal components $A_0$ of these fields become zero. By that the Gauss law constraint (1.1) acquires the form

$$\partial_0 \left( \partial^i \hat{A}_i^D(x,t) \right) \equiv 0.$$ (1.5)

Such quantization method is suitable for non-Abelian as well as Abelian particualr gauge theories. A good patern, how to quantize in the above wise a non-Abelian (Minkowskian) gauge model is that procedure applied to the Minkowskian theory with the $SU(2) \rightarrow U(1)$ violated gauge symmetry involving Yang-Mills (YM) and Higgs vacuum modes in the shape of BPS monopoles [7].

Here we should like to enumerate gains which the above described Dirac (or fundamental) quantization method [3] gives for the Minkowskian gauge theory with vacuum YM and Higgs BPS monopole modes. Among them (the author holds it is the immediate of these gains) is the appearance of so-called zero modes, $\tilde{A}_0$, solutions to the Gauss law constraint (1.1) which can be recast to the shape of the homogeneous equation

$$(D^2)^{ab}\Phi_b(x) = 0,$$ (1.6)

involving the covariant derivative $D$ of the (topologically trivial) Higgs BPS monopole mode $\Phi_a$.

Herewith any zero mode $\tilde{A}_0$ becomes directly proportional to $\Phi_a$ with some only time depended coefficient $\dot{c}(t)$:

$$\tilde{A}_0^a(t,x) = \dot{c}(t)\Phi_a(x) \equiv Z^a.$$  

The variable $c(t)$ is related closely to the vacuum Chern-Simons functional:

$$\nu[A_0, \Phi] = \frac{g^2}{16\pi^2} \int_{t_{in}}^{t_{out}} dt \int d^3x F^a_{\mu\nu} \tilde{F}^{a\mu\nu} = \frac{\alpha_s}{2\pi} \int d^3x F^a_{\mu\nu} B^a_{\mu\nu}(\Phi) [c(t_{out}) - c(t_{in})]$$

$$= c(t_{out}) - c(t_{in}) = \int_{t_{in}}^{t_{out}} dt \dot{c}(t); \quad \alpha_s \equiv g^2/4\pi.$$ (1.7)
Here $F$ is the YM strength tensor with its dual $\tilde{F}^{\mu\nu} = 1/2 \varepsilon^{\mu\nu\lambda\rho} F_{\lambda\rho}$, $g$ is the YM coupling constant; $B_i^a(\Phi) = 1/2 \varepsilon^{aik} F_{ik} (F_{ik} = (\Phi^a/|\Phi|) F_{ika})$ is the "magnetic" field generated vacuum BPS monopole modes $\Phi$ via the Bogomol'nyi equation \[7, 8, 9\]
\[ B = \pm D\Phi. \] (1.8)

Zero modes $\tilde{A}_0^a(t, x)$ generate vacuum "electric" strength
\[ F_{i0}^a = \dot{c}(t) D_i^a(\Phi)\Phi_c(x) \] (1.9)
referred to as the so-called "electric monopole" (for instance, in \[1, 10\]).

In turn, "electric monopole" modes $F_{i0}^a \equiv E^a$ enter, in a natural wise, the action functional
\[ W_N = \int d^4x \frac{1}{2} (F_{ik}^c)^2 = \int dt \frac{e^2 I}{2}, \] (1.10)
involving the "rotary momentum"
\[ I = \int_V d^3x (D_i^{ac}(\Phi_k)\Phi_c)^2. \] (1.11)

Thus, as it was argued in \[1, 10, 11\], this action functional (1.10) describes correctly collective solid rotations inside the YM-Higgs (further, YMH) vacuum involving BPS monopole modes and quantized by Dirac in the above wise. The remarkable property of the action functional (1.10) is also the purely real, i.e. physical spectrum
\[ P_c = \dot{c}I = 2\pi k + \theta; \quad \theta \in [-\pi, \pi]; \quad k \in \mathbb{Z} \] (1.12)
of the topological momentum $P_c$.

In this purely real spectrum is the principal distinction of the Minkowskian YMH model (with vacuum BPS monopole modes) considered in the Dirac quantization scheme \[3\] from the Euclidian YM theory involving instantons \[12\]. In the latter theory, as it was discussed in the papers \[13, 14, 15\], the $\theta$-angle, playing the role of a quasi-momentum therein, takes indeed a complex value $\theta = \theta_1 + i\theta_2$. This implies the indefinite norm for quantum objects corresponding instantons \[12\], and thus this model encounters lot of problems.

The manifest rotary effect (1.10) inherent in the Minkowskian YMH model with vacuum BPS monopoles quantized by Dirac distinguishes that model, to a marked degree, from the well-known 't Hooft-Polyakov model (also associated with the Minkowski space) involving of the same name vacuum monopoles \[16, 17\]. The latter model does not contain rotary (vacuum) modes, but only stationary solutions, monopoles \[16, 17\].

The said, indeed, is a specific trace of the heuristical quantization scheme by Faddev and Popov (further, FP) \[18\] involving the relativistic (Poincare) invariant S-matrix and only mass-shell quantum fields. This quantization scheme \[18\] is not attached to a definite reference frame. On the contrary, the Dirac quantization scheme \[3\] is always associated
with a definite reference frame, for instance with the rest reference frame. Just in such rest reference frame the rotary effect (1.10) can be observed in the Minkowskian YMH model with vacuum BPS monopoles quantized by Dirac, while the S-matrix approach [18] to the ’t Hooft-Polyakov model [16, 17] gives only statical (stationary) solutions.

The next important distinction between the ’t Hooft-Polyakov model [16, 17] and the Minkowskian YMH model with vacuum BPS monopoles quantized by Dirac lies in thermodynamics. As it is well known, the second order phase transition occurs in the former theory and is reduced to the spontaneous and instantaneous breakdown of the initial SU(2) gauge symmetry group up to its U(1) subgroup. In the Minkowskian YMH model with vacuum BPS monopoles quantized by Dirac the first order phase transition takes place coming to coexisting (at the absolute zero temperature $T = 0$) two thermodynamic phases inside the BPS monopole vacuum.

The first of these two thermodynamic phases is the phase of collective vacuum rotations described by the action functional (1.10). The second one is the thermodynamic phase of superfluid potential motions set by the Bogomol’nyi equation (1.8) and the Gribov

\footnote{Side by side with the above second order phase transition involving $SU(2) \rightarrow U(1)$.}
ambiguity equation}

\[ [D^2_k(\Phi^i)_{ab}\Phi^a_k = 0. \]  

\[ (1.13) \]

The origin of the Gribov ambiguity equation (Gribov equation) \((1.13)\) is following. In the Minkowskian YMH model with vacuum BPS monopoles quantized by Dirac the general expression \((1.3)\) for the Dirac variables acquire the concrete look \([10][11]\)

\[ A^D_k = v^{(n)}(x)T \exp \left\{ \int_0^t d\tilde{\tau}_0(\tilde{\tau}, \mathbf{x}) \right\} \left( A^{(0)}_k + \partial_k \right) v^{(n)}(x)T \exp \left\{ \int_0^t d\tilde{\tau}_0(\tilde{\tau}, \mathbf{x}) \right\} \right]^{-1}, \]

with the symbol \(T\) standing for time ordering the matrices under the exponent sign.

Thus in the initial time instant \(t_0\), the topological degeneration of initial (YM) data comes thus to "large" stationary matrices \(v^{(n)}(x) \neq 0\) in the terminology \([19]\) depending on topological numbers \(n \neq 0\) and called the factors of the Gribov topological degeneration or simply the Gribov multipliers.

One attempts \([1][10][20]\) to find Gribov multipliers \(v^{(n)}(x)\), belonging to the \(U(1) \subset SU(2)\) embedding in the Minkowskian Higgs model, as

\[ \exp[n\Phi_0(x)], \]

implicating the Gribov phase \(\Phi_0(x)\), taking the shape \([6]\) of a scalar constructed by contracting the Pauli matrices \(\tau^a\) and Higgs vacuum BPS monopole modes:

\[ \Phi_0(r) = -i\pi \tau^a x_a f^{BPS}_{01}(r), \quad f^{BPS}_{01}(r) = \left[ \frac{1}{\tanh(r/\epsilon)} - \frac{\epsilon}{r} \right]. \]

In the initial time instant \(t_0\) the topological Dirac variables \(A^D_k\) acquire the look

\[ \hat{A}^{(n)}_k = v^{(n)}(x)(\hat{A}^{(0)}_k + \partial_k)v^{(n)}(x)^{-1}, \quad v^{(n)}(x) = \exp[n\Phi_0(x)]. \]

The important property of these topological is their "transverse" character: namely, that \([20]\)

\[ D^a_{ib}(\Phi^i_k)\hat{A}^{(n)}_b = 0 \]

with the covariant derivative \(D\) depending on the (topologically degenerated) vacuum YM BPS monopole modes \(\Phi^{(n)}_k\) (indeed, this is controlled by the Bogomol’nyi equation \((1.8)\)).

The above transverse gauge for the topological Dirac variables \(\hat{A}^D_k\) is not specified in a unique wise in each topological sector \(n\) of the considered YMH model. This phenomenon (correct for any transverse gauge in any non-Abelian gauge theory \([9]\)) is referred to as the Gribov ambiguity. The general analysis of this effect in the terminology of the trivial principal fibre bundle (involving the gauge group \(SU(2)\)) was given in the §T26 in the monograph \([9]\), which we recommend to our readers.

In our concrete Minkowskian YMH model with vacuum BPS monopoles quantized by Dirac the Gribov ambiguity phenomenon for (topologically degenerated) Dirac variables \(\hat{A}^D_k\) just comes to the Gribov ambiguity equation \((1.13)\). To ground this, it is necessary to write down explicitly the YM "magnetic" field

\[ B^a_i = \epsilon_{ijk}(\partial^j A^{sk} + \frac{g}{2}c^{abc}A^s_b A^c_k). \]

Then it is easy to see that the values \(D_iA^a\) (in particular, \(D_iA^{1D}\) if topological Dirac variables \(A^D\) are in question) have the same dimension that a "magnetic" YM field \(B^a_i\). Then it is easy to see that the Gribov ambiguity equation \((1.13)\) is the consequence of the Bogomol’nyi equation \((1.8)\), implicating (topologically trivial) Higgs vacuum BPS monopole modes \(\Phi_{(0)}\). Precisely, the connection between the Bogomol’nyi and Gribov ambiguity equations is set through the Bianchi identity

\[ \epsilon^{ijk} \nabla_i F^b_{jk} = 0, \]
This coexisting at the temperature \( T = 0 \) of two thermodynamic phases can continue infinitely long time until \( T = 0 \). Such first order phase transition was discussed in the recent paper [21] (see Conclusion therein), where it was referred to as the frozen supercooling situation. Herewith collective solid rotations inside the BPS monopole vacuum proceed without "friction forces". In this is the essence of the Josephson effect [15, 22, 23]: at \( T = 0 \), any "quantum train" cannot stop, moving permanently along closed trajectories.

The first order phase transition occurring in the Minkowskian YMH model with BPS monopole vacuum quantized by Dirac finds its reflection in the vacuum (Bose condensation) Hamiltonian \( H_{\text{cond}} \) [1, 10]

\[
H_{\text{cond}} = \frac{2\pi}{g^2\epsilon} \left[ P_c^2 \left( \frac{g^2}{8\pi^2} \right)^2 + 1 \right],
\]

written down over the YM Gauss law constraint (1.1) and containing the "electric" and "magnetic" contributions, given via Eqs. (1.10) and

\[
\frac{1}{2} \int d^3 x [B^a_i (\Phi_k)]^2 \equiv \frac{1}{2} V < B^2 > = \frac{1}{2\alpha_s} \int d^3 r \sim \frac{1}{2\alpha_s\epsilon} = \frac{2\pi g^2 m}{\sqrt{\lambda}} = \frac{2\pi}{g^2\epsilon},
\]

respectively. The latter, "magnetic", contribution is associated with the Bogomol'nyi equation (1.8).

The remarkable property of the vacuum (Bose condensation) Hamiltonian \( H_{\text{cond}} \) is its Poincare invariance (and thus also the CP-invariance) as that squared by the topological momentum \( P_c \). This solves the CP-problem (taking place in the Euclidian instanton model [12] involving the Poincare covariant \( \theta \)-item in its Lagrangian [4] in the Minkowskian YMH theory with vacuum BPS monopoles quantized by Dirac.

To explain the above (frozen) first order phase transition taking place in the Minkowskian YMH model with BPS monopole vacuum quantized by Dirac, the special assumption about the \( SU(2) \rightarrow U(1) \) gauge group space inherent in that model was made in the recent papers [21, 24].

The essence of this assumption is in the so-called "discrete" factorization

\[
SU(2) \equiv G \simeq G_0 \otimes \mathbb{Z}; \quad U(1) \equiv H \simeq U_0 \otimes \mathbb{Z}
\]

of the initial, \( SU(2) \), and residual, \( U(1) \), gauge symmetries groups.

In this case it can be shown that the apropriate YM degeneration space (vacuum manifold) \( R_{\text{YM}} \equiv SU(2)/U(1) \) acquires the look

\[
R_{\text{YM}} = \mathbb{Z} \otimes G_0/U_0.
\]

that is equivalent to Eq.

\[
DB = 0
\]

in terms of the (vacuum) "magnetic" field \( B \).

4We recomend our readers the paper [15] where the instanton model [12] was stated enough briefly but informatively.
$R_{YM}$ is a manifestly multiconnected (discrete) space: $\pi_0(R_{YM}) = \mathbb{Z}$. This means that different topological sectors of $R_{YM}$ are separated by \textit{domain walls}\footnote{As it is well known (see, e.g. Ref. \cite{25}), the width of a domain (or \textit{Bloch}, in the terminology \cite{25}) wall is roughly proportional to the inverse of the lowest mass of all the physical particles in the (gauge) model considered.}

And moreover, the YM degeneration space $R_{YM}$, (1.17), contains \textit{thread} and \textit{point} topological defects.

Thread topological defects (vortices) inside $R_{YM}$ are induced by the manifest isomorphism \cite{9}

$$\pi_1(R_{YM}) = \pi_0(H) \neq 0.$$  \hspace{1cm} (1.18)

It is easy to see that these are just responsible for all the (vacuum) rotary effects inherent in the Minkowskian Higgs model with vacuum BPS monopoles quantized by Dirac (in

In Minkowskian Higgs models (without quarks) the typical such scale is the (effective) Higgs mass $m/\sqrt{\lambda}$ (with $m$ being the Higgs mass and $\lambda$ being its selfinteraction constant). In particular, in the Minkowskian Higgs model with vacuum BPS monopoles quantized by Dirac (we discuss now) $m/\sqrt{\lambda}$ is the only mass scale different from zero in the Bogomol'nyi-Prasad-Sommerfeld (BPS) limit \cite{7}

$$m \to 0 \quad \lambda \to 0.$$  

If quarks are incorporated nevertheless in this model, one thinks that any “bare” flavour mass $m_0$ is by far less than the “effective” Higgs mass $m/\sqrt{\lambda}$:

$$m_0 \ll m/\sqrt{\lambda}.$$  

The typical value of the length dimension inversely proportional to $m/\sqrt{\lambda}$ is the (typical) size $\epsilon$ of BPS monopoles.

It can be given as \cite{1, 10, 11}

$$\frac{1}{\epsilon} = \frac{gm}{\sqrt{\lambda}} \sim \frac{g^2 <B^2> V}{4\pi},$$  

with $V$ being the volume occupied by the YMH vacuum configuration.

The said allows to assert that $\epsilon$ disappears in the infinite spatial volume limit $V \to \infty$, while it is maximal at the origin of coordinates (herewith it can be set $\epsilon(0) \to \infty$). This means, due to the above reasoning \cite{25}, that walls between topological domains inside $R_{YM}$ become infinitely wide, $O(\epsilon(0)) \to \infty$, at the origin of coordinates.

The fact $\epsilon(\infty) \to 0$ is also meaningful. This implies actual merging of topological domains inside the vacuum manifold $R_{YM}$, (1.17), at the spatial infinity. The said allows \cite{21} to interpret the discrete space $R_{YM}$ as the Riemann surface for the function $\lim_{n \to \infty} (1/(\sqrt{z})^n$ of the complex variable $z$ (with natural setting $V = \text{Re } z$). $z \to \infty$ serves as the branching point for this Riemann surface on which the above limit turns into zero, while $z \to 0$, the pole point for $1/(\sqrt{z})^n$ at any $n$, can be considered as another branching point.
particular for the action functional $\mathcal{I}_{\text{YM}}(\Phi)$.

Point topological defects inside the YM degeneration space $R_{\text{YM}}$ are generated by the isomorphism

$$\pi_2(R_{\text{YM}}) = \pi_1(H) = \mathbb{Z},$$

(1.19)
grounded in Ref. [21]. These topological defects come inside $R_{\text{YM}}$ to point hedgehog topological defects in the shape of vacuum Higgs and YM BPS monopoles [7]. It is obvious that these topological defects are responsible for all the superfluidity effects controlled by

6Indeed, vortices arising inside the vacuum manifold $R_{\text{YM}}$ are concentrated, in the main, in the spatial region along the axis $z$ of the chosen rest reference frame, infinitely close to this axis. In other words, it is just the above discussed limit $V \to 0 (\epsilon \to \infty)$.

In this spatial region vacuum vortices can be represented [9, 21] by the YM fields

$$A_\theta(\rho, \theta, z) \equiv A_\mu \partial x^\mu / \partial \theta = \exp(iM\theta)A_\theta(\rho)\exp(-iM\theta),$$

with $M$ being the generator of the group $G_1$ of rigid rotations compensating changes in the vacuum (Higgs-YM) “thread” configuration $(\Phi^a, A_\mu^a)$ at rotations around the axis $z$ of the chosen (rest) reference frame.

Herewith

$$A_\theta(\rho) = M + \beta(\rho),$$

where the function $\beta(\rho)$ approaches zero as $\rho \to \infty$.

The elements of $G_1$ can be set as [9]

$$g_\theta = \exp(iM\theta).$$

YM fields $A_\theta$ are manifestly invariant with respect to shifts along the axis $z$.

In turn, the Higgs rotary (vortex) modes $\Phi^a$ can be represented as [9]

$$\Phi^{(n)}(\rho, \theta, z) = \exp(M\theta)\phi(\rho) \ (n \in \mathbb{Z}), \ \nabla_\mu \phi(\rho) \leq \text{const} \rho^{-1-\delta}; \ \delta > 0.$$

These solutions are singular at $\rho \to 0$ but disappear as $\rho \to \infty$. This property of the Higgses $\Phi^{(n)}(\rho, \theta, z)$ allows to join them (in a smooth wise) with vacuum Higgs BPS monopoles belonging to the same topology $n$ and disappearing [7, 9, 6] at the origin of coordinates. Herewith, speaking ”in a smooth wise”, we imply that the covariant derivative $D\Phi$ of any vacuum Higgs field $\Phi_a^{(n)}$ merges with the covariant derivative of such a vacuum Higgs BPS monopole solution.

Following [9], the vacuum $z$-invariant, i.e. axially symmetric, (Higgs-YM) configuration $(\Phi^a, A_\mu^a)$, possessing, as it can be demonstrated [9], a finite linear energy density and obeying the apropriate equations of motions) can be treated as a rectilinear thread solution (called also the rectilinear thread vortex or the rectilinear thread).

Obvious locating of (topologically nontrivial) threads $A_\theta$ at the origin of coordinates (actually, in the spatial region $\rho \to 0$; the same is correctly also for Higgs thread modes $\Phi^a$) permits the natural geometrical interpretation of (topologically nontrivial) threads as infinitely narrow tubes around the axis $z$ over which the family of vortex solutions $(\Phi^a, A_\mu^a)$ is specified actually (disappearing rapidly outside this spatial region).

All the said provides that the action functional $\mathcal{I}_{\text{YM}}(\Phi)$, involving $D_\mu^a(\Phi_k)\Phi_c$, is described correctly by Higgs vacuum BPS monopole modes [7] as well as by ”rotary” Higgs modes $\Phi^{(n)}(\rho, \theta, z)$ [9]. And moreover, the topological momentum $P_c(n)$, (1.12), of the Minkowskian BPS monopole vacuum, running over the set $\mathbb{Z}$ of integers, takes the unique value $P_c(n) = \theta + 2\pi k$ in each topological sector $k$ of the Minkovskian degeneration space $R_{\text{YM}}$, corresponding to the family of vortex solutions with the topological number $k$. 

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the Bogomol’nyi, (1.8), and Gribov ambiguity, (1.13), equations in the Minkowskian YMH model with BPS monopole vacuum quantized by Dirac.

To finish our survey about the Minkowskian YMH model with BPS monopole vacuum quantized by Dirac, we should like to point of the peculiarities of QCD based on such model.

1. The above discussed [21] ”discrete vacuum geometry” (1.17) of the vacuum manifold \( R_{YM} \) (reduced to the \( \lim_{n \to \infty} (1/\sqrt{n}) \) Riemann surface), with merging topological domains at the spatial infinity, promotes the specific effect, the so-called topological confinement, coming [20] to decoupling from the real momentum spectrum \( P_c \), (1.12), of the free rotator (1.10) the series of values \( p \neq 2\pi k + \theta \ (k \in \mathbb{Z}) \). The physical sense of the topological confinement comes to surviving, in the vacuum Hamiltonian (1.14), the set \( \mathbb{Z} \) of integers, entering this Hamiltonian via the topological momentum \( P_c \), (1.12) inspite the manifest gauge invariance of this vacuum Hamiltonian due to the absorption of the Gribov topological multipliers \( v^{(n)}(x) \) therein.

2. The topological confinement, in the spirit of the complete destructive interference [1, 10, 20] of the topologically nontrivial Gribov multipliers \( v^{(n)}(x) \) \( (n \neq 0) \), implies the quark confinement as it is understood customary: one cannot observe colored quarks, i.e. those ”dressed” in Gribov topological multipliers \( v^{(n)}(x) \) \( (n \neq 0) \):

\[ q^Iv^{(n)}(x) \]

\( (q^I \) are topologically trivial quarks, which are gauge invariant, that means they are colorless).

At the QCD Hamiltonian level this implies its gauge invariance [20]:

\[ H[A^{(n)},q^{(n)}] = H[A^{(0)},q^I] \]  

This becomes transparent if one considers the wave function

\[ \Psi(c) = e^{ipc}, \]

corresponding to the free rotator (1.10). If one averages this function over all the values \( n \in \mathbb{Z} \) of the topological degeneration with the \( \theta \)-angle measure \( \exp(i\theta n) \), we get [20]

\[ \Psi(c)_{\text{observable}} = \lim_{L \to \infty} \frac{1}{2L} \sum_{n=-L}^{n=L} e^{i\theta n} \Psi(c + n) = \exp\{i(2\pi k + \theta)c\}. \]

It is so since an observer does not know where is the rotator (1.10). It can be at points \( N_{in}, N_{in} \pm 1, N_{in} \pm 2, N_{in} \pm 3, \ldots \)

At deriving \( \Psi(c)_{\text{observable}} \) the relation [20]

\[ \frac{1}{L} \sum_{n=-L/2}^{n=L/2} = 1 \]

was utilized.

Thus we see that if \( p \neq 2\pi k + \theta \), \( \Psi(c)_{\text{observable}} = 0 \). Just this phenomenon was referred to as the complete destructive interference in Refs. [1, 10, 20].
\(A^{(n)}\) are gluonic fields contained the Gribov topologically nontrivial multipliers \(v^{(n)}(x)\), \(n \neq 0\).

And moreover, as it was shown in Ref. [26], in the lowest order of the perturbation theory, averaging (quark) Green functions over all topologically nontrivial field configurations (including vacuum monopole ones) prove to be \([26, 27]\)

\[
G(x, y) = \frac{\delta}{\delta s^{*}(x)} \frac{\delta}{\delta s^{*}(y)} Z_{\text{conf}}(s^{*}, \bar{s}^{*}, J^{*}) \big|_{s^{*} = \bar{s}^{*} = 0} = G_{0}(x - y) f(x, y),
\]

(1.21)

with \(G_{0}(x - y)\) being the (one-particle) quark propagator in the perturbation theory and

\[
f(x, y) = \lim_{|x| \to \infty} \lim_{|y| \to \infty} \frac{1}{L} \sum_{n=-L/2}^{n=L/2} v^{(n)}(x) v^{(n)}(-y).
\]

(1.22)

Further, \(s^{*}, \bar{s}^{*}, J^{*}\) are the sources of the quark (\(q\)), antiquark (\(\bar{q}\)) and gluonic (\(A\)) fields, respectively, while \(Z_{\text{conf}}(s^{*}, \bar{s}^{*}, J^{*}) \big|_{s^{*} = \bar{s}^{*} = 0}\) is the generating functional given in the transverse gauge\(^{8}\)

\[D_{i}(A) \partial_{0} A^{i} = 0,\]

(1.23)

\(^{8}\)Fixing the gauge \([1.23]\) implies the Faddeev-Popov (FP) integral in the shape \([20]\)

\[
Z_{R,T}(s^{*}, \bar{s}^{*}, J^{*}) = \int DA^{*} Dq^{*} D\bar{q}^{*} \det \Delta \delta \left( \int d^{3}x \left[ L^{i}(A^{*}, q^{*}) + \bar{s}^{*} q^{*} + \bar{q}^{*} s^{*} + J^{*}_{a} A^{i}_{\bar{a}} \right] \right),
\]

involving the FP operator \([28]\)

\[
\hat{\Delta} = -\left( \partial_{i} D^{i}(A) \right) = -\left( \partial_{i}^{2} + \partial_{i} \text{ad}(A^{i}) \right)
\]

with

\[
\text{ad}(A)X \equiv [A, X]
\]

for an element \(X\) of the appropriate Lie algebra.

The very important feature of the FP integral \(Z_{R,T}(s^{*}, \bar{s}^{*}, J^{*})\) is its expressing in terms of Dirac variables \(A^{*}_{i}, q^{*}, \bar{q}^{*}\), i.e. its actual dependence on Gribov topological multipliers \(v^{(n)}(x)\).

With loss of generality, one can set \(T \to \infty\). The FP integral \(Z_{R,T}(s^{*}, \bar{s}^{*}, J^{*})\) includes the Lagrangian density \(L^{i}\) corresponding to the constraint-shall action of the Minkowskian non-Abelian theory (Minkowskian QCD) taking on the surface of the Gauss law constraint \([1.1]\); also \(R\) is a large real number, and one can assume that \(R \to \infty\).

Then the generating functional \(Z_{\text{conf}}(s^{*}, \bar{s}^{*}, J^{*})\) in Eq. (1.21) may be derived from the FP integral \(Z_{R,T}(s^{*}, \bar{s}^{*}, J^{*})\) by its averaging over the Gribov topological degeneration of initial data, i.e. over the set \(Z\) of integers

\[
Z_{\text{conf}}(s^{*}, \bar{s}^{*}, J^{*}) = \lim_{|x| \to \infty} \lim_{T \to \infty} \frac{1}{L} \sum_{n=-L/2}^{n=L/2} Z_{R,T}^{L}(s^{*}_{n, \phi_{i}}, \bar{s}^{*}_{n, \phi_{i}}, J^{*}_{n, \phi_{i}}),
\]

(1.21)
Indeed, it turns out that \( f(x, y) = 1 \) in (1.21). It is so due to the spatial asymptotic of the Gribov topological multipliers \( v^{(n)}(x) \) (at deriving the relation \( f(x, y) = 1 \) the same arguments as at deriving \( \Psi(c)_{\text{observable}} \) above were utilized).

Thus we see that in the lowest order of the perturbation theory any Green function \( G(x, y) \) becomes topologically trivial. And this means simultaneously the topological confinement and the quark in its generally accepted sense.

3. New interesting properties acquire fermionic (quark) degrees of freedom \( q^*, \bar{q}^* \) in Minkowskian constraint-shell QCD involving the spontaneous breakdown of the initial \( SU(3)_{\text{col}} \) gauge symmetry in the way.

The only specific of Minkowskian constraint-shell QCD (in comparison with the constraint-shell Minkowskian (YM-Higgs) theory) is the presence of three Gell-Mann matrices \( \lambda^a \), generators of \( SU(2)_{\text{col}} \) (just these matrices would enter G-invariant quark currents \( j^a_{\mu} \) in of Minkowskian constraint-shell QCD). In the constraint-shell Minkowskian (YM-Higgs) theory, involving the initial \( SU(2) \) gauge symmetry, the Pauli matrices \( \tau^a \) \((a = 1, 2, 3)\) would replace the Gell-Mann \( \lambda^a \) ones.

The very interesting situation, implying lot of important consequences, takes place to be in Minkowskian constraint-shell QCD involving the spontaneous breakdown (1.25) of the initial \( SU(3)_{\text{col}} \) gauge symmetry when the antisymmetric Gell-Mann matrices

\[
\lambda_2, \lambda_5, \lambda_7
\]

are chosen to be the generators of the \( SU(2)_{\text{col}} \) subgroup in (1.25), as it was done in Refs. 6 [20, 30].

As it was demonstrated in [6], the ”magnetic” vacuum field \( B^{ia}(\Phi_i) \) corresponding to Wu-Yang monopoles \( \Phi_i \) [31] acquires the form

\[
b_i^a = \frac{1}{g} \varepsilon_{iak} \frac{n_k(\Omega)}{r}; \quad n_k(\Omega) = \frac{x^l \Omega_{lk}}{r}; \quad n_k(\Omega)n^k(\Omega) = 1; \quad (1.27)
\]

in terms of the antisymmetric Gell-Mann matrices \( \lambda_2, \lambda_5, \lambda_7 \), (1.26), with \( \Omega_{lk} \) being an orthogonal matrix in the colour space.

\[\text{An of no small importance circumstance promoting the spatial asymptotic (1.24) for the Gribov multipliers } v^{(n)}(x), \text{ equal for different topologies } n \text{ inside the vacuum manifold } R_{YM} \text{ is decreasing (in effect down to zero), in this limit, the widths of domain walls between different topological sectors of this manifold.}\]
For the "antisymmetric" choice (1.26), we have

\[ b_i \equiv g \frac{b_1 \lambda^2 + b_2 \lambda^5 + b_3 \lambda^7}{2i}; \quad b_i = \frac{e^{aikn_k}}{gr^k} \quad (\tau_1 \equiv \lambda_2, \tau_2 \equiv \lambda_5, \tau_3 \equiv \lambda_7). \tag{1.28} \]

For the spontaneous breakdown of the initial SU(3)_{\text{col}} gauge symmetry in the (1.25) way, involving herewith antisymmetric Gell-Mann matrices \( \lambda_2, \lambda_5, \lambda_7 \) as generators of the "intermediate" SU(2)_{\text{col}} gauge symmetry, this BPS (Wu-Yang) monopole background takes the look (1.27) \[6\].

In another aspects such Minkowskian constraint-shell QCD possesses the in principle same "physics" that the Minkowskian YMH model with vacuum BPS monopoles quantized by Dirac, us discussed above.

In particular, if we factorize the vacuum manifold of the Minkowskian constraint-shell QCD,

\[ R_{\text{QCD}} = SU(2)_{\text{col}}/U(1) \]

in the (1.17) \[21\] wise, this involves the Gribov "discrete" factorisation of the (1.16) type for the "intermediate", SU(2)_{\text{col}}, and residual, U(1), gauge symmetries groups.

As regards the initial, SU(3)_{\text{col}}, gauge group, it is not important for us, generally speaking, what a geometrical structure ("continuous" or "discrete") has this group. The only important things are the geometries (topologies) of the "intermediate", SU(2)_{\text{col}}, and residual, U(1), gauge symmetries groups.

Supposing the Gribov "discrete" factorisation (1.16) for SU(2)_{\text{col}} and U(1) (implying the factorisation (1.17) for the QCD vacuum manifold \( R_{\text{QCD}} \)), we get once again topological rotations (1.10) (explained as a specific Josephson effect \[11, 13, 23\]) for the gluonic Bose condensate, involving vacuum "electric" monopoles (1.9) and the Poincare invariant Bose condensation Hamiltonian \( H_{\text{cond}} \), (1.14).

To write down the Dirac equation for a quark in the BPS (Wu-Yang) monopole background (1.27), (1.28)\[10\], note that each fermionic (quark) field may be decomposed by the complete set of the generators of the Lee group SU(2)_{\text{col}} (i.e. \( \lambda_2, \lambda_5, \lambda_7 \) in the considered case) completed by the unit matrix 1. This involves the following decomposition \[6\] of a quark field by the antisymmetric Gell-Mann matrices \( \lambda_2, \lambda_5, \lambda_7 \)

\[ \psi^{\alpha,\beta}_\pm = s_\pm \delta^{\alpha,\beta} + v^j_\pm \tau^j_{\alpha,\beta}, \tag{1.29} \]

involving some SU(2)_{\text{col}} isoscalar, \( s_\pm \), and isovector, \( v_\pm \), amplitudes. \( +, - \) are spinor indices, \( \alpha, \beta \) are SU(2)_{\text{col}} group space indices and

\( (\lambda_2, \lambda_5, \lambda_7) \equiv (\tau_1, \tau_2, \tau_3) \).

\[10\] This is the first step in getting spectra of mesonic and baryonic states. in QCD. For the detailed description of the algorithm how to do this in Minkowskian constraint-shell QCD we refer our readers to the survey \[20\] with the numerous references therein and also to the recent work \[35\].
The mix of group and spinor indices generated by Eqs. (1.27), (1.28) for the BPS (Wu-Yang) monopole background allows then to derive, utilising the decomposition (1.29), the system of differential equations in partial derivatives [6]

\[(\mp q_0 + m)s_\pm \mp i(\partial_a + \frac{n_a}{r})v^a_\pm = 0; \quad (1.30)\]
\[(\mp q_0 + m)v^a_\pm \mp i(\partial^a - \frac{n^a}{r})s_\pm - i\epsilon^{ab}j^a v^b_\pm = 0 \quad (1.31)\]

(implicating the mass \(m\) of a quark and its complete energy \(q_0\)), mathematically equivalent to the Dirac equation

\[i\gamma_0 \partial_0 \psi + \gamma_j [i\partial_j \psi + \frac{1}{2r} \tau^a e^{a|l} n_l \psi] - m\psi = 0 \quad (1.32)\]

for a quark in the BPS (Wu-Yang) monopole background.

The decomposition (1.29) [6] of a quark field implies that \(v_j^{\pm \tau_{\alpha,\beta}}\) is a three-dimensional axial vector in the colour space. Thus the spinor (quark) field \(\psi^{\alpha,\beta}_\pm\) is transformed, with the "antisymmetric" choice \(\lambda_2, \lambda_5, \lambda_7\), by the reducible representation of the \(SU(2)_{col}\) group that is the direct sum of the identical representation \(1\) and three-dimensional axial vector representation, we denote as \(3_{ax}\):

\[3_{ax} \oplus 1.\]

A new situation, in comparison with the usual \(SU(3)_{col}\) theory in the Euclidean space \(E_4\), appears in this case. That theory was worked out by Greenberg [32], Han and Nambu [33, 34]; its goal was getting hadronic wave functions (describing bound quark states) with the correct spin-statistic connection. To achieve this, the irreducible colour triplet (i.e. three additional degrees of freedom of quark colours, forming the polar vector in the \(SU(3)_{col}\) group space), was introduced. There was postulated that only colour singlets are physical observable states. So the task of the colours confinement was outlined.

Going over to the Minkowski space in Minkowskian constraint-shell QCD quantized by Dirac and involving the (1.25) breakdown of the \(SU(3)_{col}\) gauge symmetry, the antisymmetric Gell-Mann matrices \(\lambda_2, \lambda_5, \lambda_7\) and BPS (Wu-Yang) physical background, allows to introduce the new, reducible, representation of the \(SU(2)_{col}\) group with axial colour vector and colour scalar.

In this situation the question about the physical sense of the axial colour vector \(v_j^{\pm \tau_{\alpha,\beta}}\) is posed.

For instance, it may be assumed that the axial colour vector \(v_j^{\pm \tau_{\alpha,\beta}}\) has the form \(v_1 = r \times K\), with \(K\) being the polar colour vector (\(SU(2)_{col}\) triplet). These quark rotary degrees of freedom corresponds to rotations of fermions together with the gluonic BPS monopole vacuum describing by the free rotator action (1.10). The latter one is induced by vacuum "electric" monopoles (1.9). These vacuum "electric" fields are, apparently, the cause of above fermionic rotary degrees of freedom (similar to rotary singlet terms in two-atomic molecules; see e.g. §82 in [36]).
More exactly, repeating the arguments of Ref. [23], one can "nominate" the candidature of the "interference item"

\[ Z^a j_{Ia0} \tag{1.33} \]

involving the \( Z^a \) and the fermionic (quark) topologically trivial (i.e. gauge-invariant) current \( j^\mu_{Ia0} = e\bar{q}^I \gamma^\mu q^I \), in the constraint-shell Lagrangian density of Minkowskian QCD quantized by Dirac.

The appearance of fermionic rotary degrees of freedom \( \nu_1 \) in Minkowskian constraint-shell QCD quantized by Dirac confirms indirectly the existence of the BPS monopole background in that model (coming to the Wu-Yang one [31] at the spatial infinity). These fermionic rotary degrees of freedom testify in favour of nontrivial topological collective vacuum dynamics proper to the Dirac fundamental quantization [3] of Minkowskian constraint-shell QCD (this vacuum dynamics was described above).

4. This is the possibility to solve the \( U(1) \) problem basing upon the Minkowskian non-Abelian YMH model with vacuum BPS monopoles quantized by Dirac. In other words, one can find the \( \eta' \)-meson mass near to modern experimental data.

As it was demonstrated in the recent papers [1, 6, 10, 20, 30], the way to solve the \( U(1) \)-problem in the Minkowskian non-Abelian Higgs model quantized by Dirac is associated with the manifest rotary properties of the appropriate physical vacuum involving YM and Higgs BPS monopole solutions. The principal result obtained in the mentioned works regarding solving of the \( U(1) \)-problem in the Minkowskian non-Abelian Higgs model quantized by Dirac is the following.

The \( \eta' \)-meson mass \( m_{\eta'} \) proves to be inversely proportional to \( \sqrt{I} \), where the rotary momentum \( I \) of the physical Minkowskian (YM-Higgs) vacuum is given by Eq. [6]

\[ m_{\eta'} \sim 1/\sqrt{I}, \]

with the rotary momentum \( I \) given in (1.11).

More precisely,

\[ m_{\eta'}^2 \sim \frac{C^2_{\eta}}{IV} = \frac{N_f^2 \alpha_s^2 \langle B^2 \rangle}{2\pi^3}, \tag{1.34} \]

involving a constant \( C_{\eta} = (N_f/F_\pi) \sqrt{2/\pi} \), where \( F_\pi \) is the pionic decay constant and \( N_f \) the number of flavours in the considered Minkowskian non-Abelian Higgs model.

The explicit value (1.11) of the rotary momentum \( I \) of the physical Minkowskian (YM-Higgs) vacuum was substituted in this equation for the \( \eta' \)-meson mass \( m_{\eta'} \). The result (1.34) for the \( \eta' \)-meson mass \( m_{\eta'} \) is given in Refs. [1, 6, 10, 20, 30] for the Minkowskian non-Abelian Higgs model quantized by Dirac and implemented vacuum BPS monopole solutions allows to estimate the vacuum expectation value of the appropriate "magnetic" field \( B \) (specified in that case via the Bogomol’nyi equation (1.8))

\[ \langle B^2 \rangle = \frac{2\pi^3 F_\pi^2 m_{\eta'}^2}{N_f^2 \alpha_s^2} = \frac{0.06 GeV^4}{\alpha_s^2} \tag{1.35} \]
by using the estimation \( \alpha_s(q^2 \sim 0) \sim 0.24 \) [6, 37].

The constraint-shell Abelian model (the objective of the present study) is by far simpler than the constraint-shell non-Abelian model. But there is a common point by the both these models, this is the Gauss law constraint (1.1) resolved in terms of the Dirac variables (1.3) (where the gauge matrices \( \tau^a \) turn out to the trivial unit matrix in the \( U(1) \) case), always gauge invariant and Poincare covariant.

The goal of the present study is just to demonstrate this with the example of constraint-shell QED and to generalize this constraint-shell QED (which is the topologically trivial theory) on the case of nontrivial topologies inherent in the \( U(1) \) gauge group due to the natural isomorphism

\[
U(1) \simeq S^1
\]

(1.36)

with \( \pi_1S^1 = \mathbb{Z} \).

As it is well known (see, for instance, the monographs [38, 39]), these nontrivial topologies induce the Dirac monopole (Dirac string) [40], the purely gauge solution singular along the negative direction of the axis \( z \) in the chosen reference frame and the magnetic charge \( m \) satisfying the Dirac quantization condition [38, 39, 40]

\[
\frac{q m}{4\pi} = \frac{1}{2} n, \quad n \in \mathbb{Z},
\]

(1.37)

(with \( q \) being the electric charge inherent in the Abelian \( U(1) \) model with the unbroken gauge symmetry).

In these circumstances, we shall attempt to write down the topological Dirac variables in this Abelian \( U(1) \) constraint-shell model, similar to those \( \hat{A}_k \) [1, 10, 11] appearing in the Minkowskian non-Abelian YMH model with vacuum BPS monopoles quantized by Dirac. Such topological Dirac variables should take account of the Dirac quantization condition (1.37) and the Dirac monopoles being presented.

The article is organized as follows. Section 2 is devoted to the analysis of constraint-shell QED and contains two subsections: in the first one we construct Dirac variables and performe the reduction of the QED Lagrangian in terms of Dirac variables, removing the longitudinal degrees of freedom, which are unphysical. In the second subsection we study the Poincare covariance of the Dirac variables in constraint-shell QED. In Section 2 we utilize the results of the papers [20, 26, 27] and also get some new results.

Section 3 we devote to constructing Abelian \( U(1) \) constraint-shell model involving unbroken gauge symmetry.
2 Four-dimensional constraint-shall QED.

2.1 Constructing Dirac variables in four-dimensional constraint-shall QED.

Let us consider the standard QED action \[ W[A, \psi, \bar{\psi}] = \int dx \left[ -\frac{1}{4}(F_{\mu\nu})^2 + \bar{\psi}(i \gamma^\mu \nabla(A) - m^0)\psi \right], \] (2.1)

with

\[ \nabla_\mu(A) = \partial_\mu - ieA_\mu, \quad \nabla = \nabla_\mu \cdot \gamma^\mu; \]
\[ F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \] (2.2)

This action contains gauge fields more than the number of independent degrees of freedom.

The action (2.1) is invariant under the gauge transformations

\[ A_\mu = A_\mu + \partial_\mu \Lambda, \quad \psi^\Lambda = \exp[ie\Lambda]\psi, \quad \partial_\mu \partial^\mu \Lambda = 0. \] (2.3)

There may be shown (see §19.5 in [41]) that the gauge transformations (2.3) can be represented in the canonical form

\[ A^\Lambda_\mu = UA_\mu U^{-1}, \quad \psi^\Lambda = U\psi U^{-1}, \quad \bar{\psi}^\Lambda = U\bar{\psi} U^{-1} \] (2.4)

with a unitary operator \( U \):

\[ UU^{-1} = I. \]

Issuing from the gauge transformations (2.3), one can represent the operator \( U \) in the form

\[ U = e^{iF}, \] (2.5)

with \( F \) being a Hermitian operator, \( F = F^\dagger \), having the look

\[ F = \int \{\Lambda(x) \frac{\partial \chi}{\partial t} - \frac{\partial \Lambda(x)}{\partial t} \chi\} d^3x, \] (2.6)

with

\[ \chi = \frac{\partial A_\mu(x)}{\partial x_\mu}. \]

The gauge transformations (2.4) form, obviously, the Abelian group \( U(1) \).

For infinitesimal gauge transformations (2.3), (2.4) the function \( \Lambda(x) \) also will be infinitesimal. On the other hand, in this case

\[ U \approx I + iF, \]
and Eqs. (2.4) acquire the look
\[
A'_\mu \approx A_\mu + i[F, A_\mu], \\
\psi' \approx \psi + i[F, \psi], \\
\bar{\psi}' \approx \bar{\psi} + i[F, \bar{\psi}].
\] (2.7)

The comparison of the gauge transformations (2.3) and (2.7) results the relations
\[
i[F, A_\mu] = \frac{\partial \Lambda(x)}{\partial x_\mu}, \\
i[F, \psi] = ie\Lambda(x)\psi, \\
i[F, \bar{\psi}] = -ie\Lambda(x)\bar{\psi}.
\] (2.8)

Because of the infinitesimal nature of the considered gauge transformations, one can then identify the functions \(\Lambda(x)\) and \(F(x)\).

Note that the QCD Lagrangian (2.1) remains invariant under the transformations (2.4) combined with (2.3) [37]:
\[
L(\hat{A}^g) = L(\hat{A}) \quad \text{at} \quad \hat{A}^g = g(\hat{A} + \partial)g^{-1} \equiv \hat{A}_\mu \rightarrow \hat{A}_\mu - ie\partial_\mu \Lambda(x). 
\] (2.9)

Herewith
\[
g \equiv \exp[ie\Lambda(x)]. \tag{2.10}
\]

In Eq. (2.9) the record \(\hat{A}\) stands for denoting
\[
\hat{A}_\mu = \frac{e}{\hbar c} A_\mu, 
\] (2.11)
where the correct account of the elementary charge \(e\) and Planck constant \(\hbar\) in QED is taken.

In this case the ”coupling constant” \(e/(\hbar c)\) would also enter gauge transformations (2.10). Herewith the function \(\Lambda(x)\) is chosen in such a wise that exponential multipliers in (2.10) become dimensionless.

As one acts usually in QFT, we shall apply the Planck system of units, where \(\hbar = c = 1\) is set, in the majority of formulas in the present study. Simultaneously, sometimes we shall write down explicitly these constants in the cases when their role is important for understanding properties of physical models we represent in our work.

Let us now suppose that the invariance of QED with respect to the gauge transformations (2.3), (2.7) [20] [41] allows to remove a one field degree of freedom with the aid of an arbitrary gauge
\[
F(A_\mu) = 0, \quad F(A^u_\mu) = M_F u \neq 0, \tag{2.12}
\]
where the second equation means that the given gauge unambiguously fixes the field \(A\).
In general, to construct QED as a quantum-field theory obeyed the usual Feynman rules, one would always fix a certain gauge: say, $f_i = 0$; and this fact has several consequences [37].

1. The explicit solution to the constraint $f_i = 0$ gives the definite class of physical (gauge invariant) variables, functionals on initial gauge fields $A_i$ [23].

The most important patterns of such functionals are the transverse and longitudinal physical fields. Maxwell electrodynamics gives us an example of transverse physical fields. There are electric and magnetic tensions associated with the plane transverse physical fields. To get in this case the D’alembert equation (see e.g. §46 in [42])

$$\Delta A - \frac{\partial^2 A}{\partial t^2} = 0,$$

one would utilize two constraints in the QED Hamiltonian formalism.

The first of those constraints is the of the secondary constraint

$$A_0 \equiv \phi = 0,$$

referred to as the Weyl gauge in modern physical literature. This removal of the temporal component of a four-potential $A$ in QED is quite justified due to the trivial canonical momentum $\partial L/\partial \dot{A}_0 = 0$ inherent in this particular model [2] (see our in Introduction).

The second constraint one utilizes in Maxwell electrodynamics at deriving plane wave solutions is the combination of the secondary constraint (2.14) and the other secondary constraint, us familiar already the Gauss law constraint (1.1). Note here the remarkable property of the Gauss law constraint (1.1): it is simultaneously the motion equation and the secondary constraint [2] (that is correctly for any particular theory).

The more detailed look of the Gauss law constraint in QED will be us cited below; now we only note that the combination of the both constraints, (2.14) and (1.1) comes, indeed, to the radiation (Coulomb) gauge [11]

$$\text{div } A = 0,$$

that is the spatial part of the Lorentz gauge

$$\partial_{\mu} A^\mu = 0.$$

The latter one, in turns, comes, for the electric tension $E$, to the Maxwell equation

$$\text{div } E = -\text{div } \frac{\partial A}{\partial t} = -\frac{\partial}{\partial t} \text{ div } A = 0$$

when electromagnetic currents are absent (i.e. in the lowest order of the perturbation theory) and the Weyl gauge (2.14) is taken.

---

11 In 4-dimensional QED, if fermions are absent, the Gauss law constraint (1.1) is expressed as [2]

$$\Delta A_0 + \partial_0 (\partial_i A^i) = 0.$$
Note that the latter formula is the specific expression for the "Maxwell" Gauss law constraint when electromagnetic currents are absent and the Weyl gauge (2.14) is taken (cf. (15.12) in [2]).

\[ E = - \frac{\partial A}{\partial t} \]  

(2.17)

is an example of gauge invariant physical local functionals of gauge fields.

The Coulomb gauge

\[ \text{div} \ A = \text{div} \ E = 0 \]

implies that the four-potential \( A \) and electric tension \( E \) are orthogonal to the momentum \( p_i = -i \partial_i \), i.e. transverse.

Indeed there may be shown [43] that the secondary constraint (2.14) follows directly from the Lorentz gauge (2.16) since the four-vector of momentum, \( p \), is a null vector (i.e. \( |p| = 0 \)) for an electromagnetic field.

Really, one may rewrite Eq. (2.16) as

\[ p_\mu A^\mu = 0; \]  

(2.18)

therefore

\[ A_0 = p \cdot A / p_0 \]  

(2.19)

in the Minkowskian signature (+, −, −, −).

Thus the temporal component of \( A^\mu \) is eliminated by cancelling the longitudinal space-like component of the four-potential, that is

\[ p \cdot A = 0. \]  

(2.20)

At the quantum level, the Lorentz gauge (2.16), (2.18) come to the (weak) condition

\[ p \cdot A |\phi> = 0 \]  

(2.21)

imposed onto the physical state vectors of the Hilbert-Fock space \( \mathcal{H} \) of the second quantization.

Thus eliminating temporal component \( A^0 \) with cancelling the longitudinal space-like component \( p \cdot A \) of a four-potential \( A \) implies that the of negative norm states (ghosts) are absent in the Lorentz gauge (2.16), (2.18) in Gauss-shell electrodynamics (accompanied by the null energy-momentum four-vector \( p^2 = 0 \)).

Another example of transverse physical fields are Dirac variables [20], the topic of our discussion in the present study.

12In the case of massive vector bosons with the spin 1, described in the Proca model, \( p^2 = M \), with \( M \) being the mass of the spin 1 vector boson, and also in this case [39] \( p \cdot A \neq 0 \): massive vector bosons with the spin 1 always possess longitudinal space-like components.

Thus [43] for massive vector bosons with the spin 1 eliminating their temporal components \( A^0 \) is possible in the only case when \( |p| = 0 \), i.e. in a c.m. reference frame in the Minkowski space-time.

21
The change of the gauge for field functionals \((A^f_1 \rightarrow A^f_2)\) is fulfilled by substituting
\[
A^f_2[A^f_1] = V[A^f_1](A^f_1 + \partial) V^{-1}[A^f_1]; \quad \psi^f_2 = V[A^f_1] \psi^f_1.
\]
(2.22)

Herewith there may be shown that all the Green functions are invariant with respect to changes of gauges (2.22); for example:
\[
< \psi_{2}\cdots \bar{\psi}_{2} > \equiv < V[A^f_1] \psi_{1}\cdots \bar{\psi}_{1} V^{-1}[A^f_1] > \quad (2.23)
\]
(in theories without anomalies).

To proceed further, note that classical equations of a gauge model are split into the constraints, which relate initial data for spatial components of the fields involved in a (gauge) model to initial data of their temporal components, and the equations of motion: to solve these, it is necessary to measure initial data [10].

The both classes can intersect, for instance, in the case of the Gauss law constraint (1.1): as it was discussed above, it is simultaneously the motion equation and the secondary constraint [2].

This fact plays a very important role in the quantization of particular theories [2], in particular, in the quantization of constraint-shell QCD, the topic of the present study.

Above we have demonstrated that the Gauss law constraint (1.1) is reduced to Eq. (2.15) in pure “Maxwell” electrodynamics, i.e. when fermionic electromagnetic currents \(j\) are absent. Now let us assume that latter are “switched on”. In this case the Gauss law constraint (1.1) acquires the look
\[
\frac{\delta W}{\delta A_0} = 0 \Rightarrow \Delta A_0 = \partial_i \partial_0 A_i + j_0; \quad \Delta = \partial_i \partial_i, \quad j_\mu = e \bar{\psi} \gamma_\mu \psi
\]
(2.24)

(we refer, in the present work, the Latin indices to the spatial field components).

On the other hand, the set of equations of motion in such QED looks as [20]
\[
\frac{\delta W}{\delta A_k} = 0 \Rightarrow \partial_0^2 A_k - \partial_k \partial_0 A_0 - (\delta_{ki} \Delta - \partial_k \partial_i) A_i = j_k,
\]
(2.25)
\[
\frac{\delta W}{\delta \psi} = 0 \Rightarrow \bar{\psi}(i \nabla (A) + m^0) = 0,
\]
(2.26)
\[
\frac{\delta W}{\delta \bar{\psi}} = 0 \Rightarrow (i \nabla (A) - m^0) \psi = 0
\]
(here \(\nabla = \nabla \cdot \gamma\)).

The problem of the canonical quantization encounters the nondynamical status of temporal fields components \(A_0 = A_\mu \eta_\mu\) (with \(\eta_\mu\) being the chosen reference frame) [13]

The non-dynamic status of \(A_0\) is not compatible with the quantization of this component.
of an electromagnetic field as fixing $A_0$ (via the Gauss law constraint (2.24)) and its zero momentum

$$E_0 = \partial \mathcal{L}/\partial (\partial_0 A_0) = 0$$

(2.27)

contradict the commutation relations and Heisenberg uncertainty principle.

Besides that, temporal components $A_0$ of gauge fields result the of negative norm states (ghosts) at the second quantization of any physical theory $^{14}$.

To keep the quantum principles, Dirac excluded temporal components of gauge fields using the Gauss law constraint (2.24): herewith the explicit solution to the Gauss law constraint is explicit solution to the Gauss law constraint is

$$A_0(t, x) = a_0[A] + \frac{1}{\Delta} j_0(t, x),$$

(2.28)

where

$$a_0[A] = \frac{1}{\Delta} \partial_i \partial_0 A_i(t, x)$$

(2.29)

associates the initial data of $A_0(t_0, x)$ to the set of initial data of the longitudinal component $\partial_i \partial_0 A_i(t, y)$ and the (fermionic) current $j_0(t, y)$ in the whole space.

Here

$$\frac{1}{\Delta} f(x) = -\frac{1}{4\pi} \int d^3y \frac{f(y)}{|x-y|}$$

(2.30)

In this context one speak about inertial reference frames in special relativity (SR). This means that the given coordinate basis is connected with a heavy physical body moving without influences of any external forces $^{15}$.

Customary, inertial reference frames in the Minkowskian space-time are associated with the unit time axis

$$\eta_\mu = \left( \frac{1}{\sqrt{1-v^2}}, \frac{\vec{v}}{\sqrt{1-v^2}} \right),$$

with $\vec{v}$ being the velocity of a physical body.

The frame of reference $\eta_\mu = (1, 0, 0, 0)$ with $\vec{v} = 0$ is called the comoving frame (in the present study we shall apply often the term "rest reference frame" to such reference frames). In the present study we shall utilize very often comoving frames, in which a body is immovable.

In this terminology, one can define relativistic transformations as those that change initial data (these can be written down $^{15}$ as $L_{\mu\nu} \eta_\mu = \eta_\mu$) and gauge transformations as those that do not affect the readings of instruments and are associated with gauges of physical fields.

$^{14}$The Gauss law constraint (2.24) acquires in QED the look $^{12}$ $\text{div} \ E = j_0$ in the fermions present.

The Gauss law constraint (2.24) is the secondary constrain obtained $^{16}$ at the commutation relation between the QED Hamiltonian $\hat{H}$ and the canonical momentum $E_0$:

$$[E_0, \hat{H}] = i(\text{div} \ E - j_0) \approx 0;$$

at the commutation relation between the QED Hamiltonian $\hat{H}$ and the canonical momentum $E_0$:

$$[E_0, \hat{H}] = i(\text{div} \ E - j_0) \approx 0;$$

while the equal to zero canonical momentum $E_0$ is the the primary constraint in the QED Hamiltonian formalism. On the other hand, since $[E_0, \text{div} \ E - j_0] = 0$, both the mentioned constraints belong to the first class of constraints.
is the Coulomb kernel of the appropriate nonlocal distribution.

As we remember from mathematical physics (see e.g. p. 203 in [47]), the fundamental solution to the Laplace equation

$$\Delta \mathcal{E}_3 = \delta(x)$$

is

$$\mathcal{E}_3 = -\frac{1}{4\pi x}. \quad (2.32)$$

Just this specifies the action of the operator $\Delta^{-1}$, (2.30), on a continuous function $f(x)$.

Taking into account (2.30), Eq. (2.28) may be rewritten in the integral form as [26]

$$A_0(t, x) = -\frac{1}{4\pi} \int \frac{d^3y}{|x - y|} (\partial_i \partial_0 A_i(t, y) + j_0(t, y)). \quad (2.33)$$

One can substitute the solution (2.28) into the equation (2.25) for spatial components:

$$\frac{\delta W}{\delta A_i} \bigg|_{\delta W = 0} \Rightarrow [\delta_{ik} - \partial_i \frac{1}{\Delta} \partial_k] (\partial_0^2 - \Delta) A_k = j_i - \partial_i \frac{1}{\Delta} \partial_0 j_0. \quad (2.34)$$

We see that the constraint-shell equations of motion (2.34) contain only two transverse physical variables that are, indeed, gauge invariant functionals:

$$A^*_i(t, x) = [\delta_{ik} - \partial_i \frac{1}{\Delta} \partial_k] A_k. \quad (2.35)$$

(one can make sure directly that variables (2.35) are gauge invariant functionals by comparing the latter formula with the gauge transformations (2.3)).

Dirac rewrote these gauge invariant variables with the aid of the gauge transformations [20, 26]

$$\sum_{a=1,2} e^a_k A^D_a \equiv A^D_k[A] = v[A](A_k + i\frac{1}{e} \partial_k)v[A]^{-1},$$

$$\psi^D[A, \psi] = v[A] \psi, \quad (2.36)$$

where the gauge factor $v[A]$ [3] was defined as

$$v[A] = \exp\{-ie \int_{t_0}^t dt' a_0(t')\}. \quad (2.37)$$

It is obvious [27] that gauge invariant variables $A^D_k$, $\psi^D$ belong to the Heisenberg representation for quantum-field operators.

Let us now use the gauge transformation (2.3) for temporal components $a_0$ of electromagnetic fields:

$$a^\Lambda_0 = a_0 + \partial_0 \Lambda \Rightarrow v[A^\Lambda] = \exp[i e \Lambda(t_0, x)] v[A] \exp[-i e \Lambda(t, x)]. \quad (2.38)$$

But it is the same transformations law that (2.4) [41] upon identifying the functions $\Lambda$ and $F$ for infinitesimal gauge transformations.
Comparing then Eqs. (2.38) and (2.9), we draw the conclusion that should functionals (2.36) be, indeed, gauge invariant, it is sufficient that Dirac gauge factors transformed, (2.38), cancel the transformation law (2.9) \[15\].

Thus \[26\] one would claim
\[
v[A^\Lambda] = v[A]g^{-1}. \tag{2.39}
\]
In this case the simple computation \[23, 26\]
\[
A^D_k[A^\Lambda] = v[A] g^{-1}g(A_k + \frac{1}{e}\partial_k)g^{-1}g v[A]^{-1} = A^D_k \tag{2.40}
\]
gives the way to verify exactly the gauge invariance of nonlocal functionals (2.36).

We shall call functionals (2.36) (or, equivalent, (2.35)) the Dirac variables (for instance, in the terminology \[20\]).

The fact of gauge invariance of Dirac variables (2.36) is very remarkable. Indeed, the conception of gauge invariant Dirac variables as representative for four potentials in a gauge model is a new way therein in comparison with gauge fixing method (applied usually in QED, Yang-Mills theory, QCD and so on). This is the main advantage of this concept, on the author opinion.

We learn from (2.38) that the initial data of the gauge invariant Dirac variables (2.36) are degenerated with respect to a stationary phase
\[
\exp[ie\Lambda(t_0, x)] \equiv \exp[ie\hat{\Phi}_0(x)]. \tag{2.41}
\]
The Dirac variables (2.36), as functionals of initial data, satisfy the identity
\[
\partial_0 (\partial_i A^D_i(t, x)) \equiv 0 \tag{2.42}
\]
in the purely electromagnetic theory without electronic currents (i.e. if \(j_0 \equiv 0\)). This identity also may be checked directly (issuing from Eqs. (2.36), (2.37), which result Eq. (2.35) if decompose the exponent in (2.37); in turn, the functionals (2.35) satisfy (2.42)). Eq. (2.42) implies that Dirac variables (2.36) are transverse functionals of gauge fields.

The identity (2.42) is obtained formally from the Gauss law constraint
\[
\frac{\delta W}{\delta A_0} = 0 \Rightarrow \Delta A_0 = \partial_i \partial_0 A_i \tag{2.43}
\]
in the purely electromagnetic theory without electronic currents \[16\].

The Lagrangian density of the purely electromagnetic theory without electronic currents has the look \[21, 27\]
\[
\mathcal{L}_{ed} = \frac{1}{2}F_{0i}^2 - \frac{1}{2}B_i^2; \tag{2.44}
\]
\[15\]Herewith the stationary matrices \(\exp[ie\Lambda(t_0, x)]\) would be included in the appropriate gauge transformations (2.39) to cancel entirely this transformation law for multipliers \(v[A]\).

\[16\]It is correctly in the lowest order of the perturbation theory by \(e^2/(\hbar c)\) at substituting \(A^D_0 = 0\) in the latter equation, i.e. at the Dirac removal \[3\] of nondynamical temporal field components from the Gauss law constraint and the appropriate Lagrangian density.
\[ F_{0i} = \partial_0 A_i - \partial_i A_0; \quad B_i = \epsilon_{ijk} \partial^j A^k = \frac{1}{2} \epsilon_{ijk} F^{jk}. \]

The Gauss law constraint (2.43) permits in this case the particular solution

\[ A_0 = \frac{1}{\Delta} \partial_j \partial_0 A_j \equiv a_0. \] (2.45)

Substituting this solution for \( A_0 \) in Eq. for the electric field \( F_{0i} \), we get

\[ F_{0i} = \dot{A}_i - \partial_i A_0 = \dot{A}_i - \partial_i (\frac{1}{\Delta} \partial_j \dot{A}^j) = \delta_{ij}^D \dot{A}^j = \dot{A}_i^D \] (2.46)
due to (2.35), (2.36). Herewith the operator

\[ \delta_{ij}^D \equiv (\delta_{ij} - \partial_i (\frac{1}{\Delta} \partial_j)); \quad (\delta_{ij}^D)^2 = \delta_{ij}^D; \] (2.47)

projects out the vector \( \mathbf{A} = (A_1, A_2, A_3) \) into the plane perpendicular to the direction of propagation of the given electromagnetic wave.

As a result, the Lagrangian density of the purely electromagnetic theory becomes a gauge invariant expression involving only two nonlocal transverse field components \( A_i^D \) \((i = 1, 2)\): (2.35), (2.36), that are, in turn, gauge invariant under the transformations (2.9):

\[ \mathcal{L}_{ed}(x) = \frac{1}{2} (\dot{A}_i^D)^2 - \frac{1}{4} F_{ij}^2. \] (2.48)

Mathematically, all the said is equivalent to setting in zero temporal field components \( A_0 \); in the Gauss law constraint (2.43) as well as in the Lagrangian density (2.48).

In general, when one removes temporal components of Abelian gauge fields (this is also true for non-Abelian fields), the appropriate Dirac variables becomes equal to zero [11, 12, 26]:

\[ v[A](\hat{a}_0(t,x) + \partial_0)v^{-1}[A] = 0; \quad \hat{a}_0 = ieA_0. \] (2.49)

One can treat latter Eq. as the equation for specifying Dirac matrices \( v[A] \) and alone these matrices, (2.37), as solutions to this equation.

As one performs the gauge transformations (2.49) for temporal components of Abelian fields, he automatically turns spatial components of these fields into transverse and gauge invariant Dirac variables (2.36) satisfying the condition (2.42) [20].

Now let us again return to the QED model involving fermionic currents. In this case, as we have already ascertained, the Gauss law constraint (2.24) permits the particular solution (2.28).

The Lagrangian density of QED has the standard look (2.1) supplemented by the currents interaction item (see e.g. §17.2 in [11])

\[ \mathcal{L}_I = j_\mu(x) \cdot A_\mu(x). \] (2.50)
The QED Lagrangian density (2.1) supplemented by the currents interaction item (2.50) can be rewritten in terms of transverse and gauge invariant fields, Dirac variables (2.35), (2.36), upon substituting \( A_0 \), (2.28), in this Lagrangian density. This results [26, 27]

\[
\mathcal{L}(x) = \frac{1}{2} F_{0i}^{2}(A^{D}) - \frac{1}{4} F_{ij}^{2} - j_{i} A_{i}^{D} + j_{0} \frac{1}{\Delta} j_{0} + j_{0} \frac{1}{\Delta} \partial_{0} \partial_{i} A_{i} + \bar{\psi} \{ i \gamma_{\mu} \partial_{\mu} + ie \partial_{\mu} \left( \frac{1}{\Delta} \partial_{i} A_{i} \right) - m \} \psi;
\]

\[
(2.51)
\]

\[
F_{0i}(A^{D}) = \dot{A}_{i}^{D} - \partial_{i} A_{0}^{T}, \quad A_{0}^{T} = \frac{1}{\Delta} j_{0}(x).
\]

The electric field \( \partial_{i} A_{0}^{T} \), entering the Lagrangian density (2.51), satisfies the Poisson equation

\[
\Delta A_{0}^{T} = j_{0}(x).
\]

Due to the Dirac removal (2.49) [1, 3, 11] of temporal components of gauge fields, the fifth item in (2.51) would, indeed, vanish upon total going over to Dirac variables (involving the gauge (2.42)).

In this case, upon ruling out the surface items

\[
\bar{\psi} \{ -e \gamma_{\mu} \partial_{\mu} \left( \frac{1}{\Delta} \partial_{i} A_{i} \right) \} \psi,
\]

\[
\sim \dot{A}_{i}^{D} \partial_{i} A_{0}^{T} \quad \text{and} \quad (\partial_{i} A_{0}^{T})^2 \quad \text{from the Lagrangian density (2.51) and replacing fermionic fields by Dirac variables \( \psi^{D}, \bar{\psi}^{D} \): (2.36) (just as this was done for gauge fields \( A \) in (2.51)), it effectively acquires the gauge invariant look [27]
\]

\[
\mathcal{L}^{D}(x) = \frac{1}{2} (\dot{A}_{i}^{D})^{2} - \frac{1}{4} F_{ij}^{2} - j_{i} A_{i}^{D} + \frac{1}{2} j_{0} \frac{1}{\Delta} j_{0} + \bar{\psi}^{D} \{ i \gamma_{\mu} \partial_{\mu} - m \} \psi^{D},
\]

where the fermionic current \( j^{D} \) is written down in terms of Dirac variables \( \psi^{D}, \bar{\psi}^{D} \).

The gauge invariant Lagrangian density (2.53), written down in terms of Dirac variables \( A^{D}, \psi^{D}, \bar{\psi}^{D}, \) (2.36), describes correctly the equivalent unconstrained system (EUS) [20] for QED on the surface of the Gauss law constraint (2.24).

The appropriate Gauss law constraint-shell action [20], describing this EUS, can be written down as

\[
W^{*} = W|_{\delta W/\delta A_{0} = 0} = \int d^{4}x \mathcal{L}^{D}(x).
\]

(2.54)

To combine the nonlocal physical variables \( A^{D} \) and variation principle formulated for these nonlocal fields, one would consider the effective action [2, 20]

\[
W_{\text{eff}} = W^{*} + \int d^{4}x \lambda_{L}(x) \partial_{i} A_{i}^{D} \quad (i = 1, 2).
\]

(2.55)

with \( \lambda_{L}(x) \) being a Lagrange multiplier.

From the gauge invariant Lagrangian density (2.53) one reads the equations of motions [27] [17]

\[
\frac{\delta W_{\text{eff}}}{\delta A_{i}^{D}} = 0 \implies \square A_{i}^{D}(x) = \delta_{ki} j_{j}^{D}(x) \quad (i, j, k = 1, 2);
\]

(2.56)

\[\text{[17] Following Dirac [3], we change herewith the order constraining and varying at analysis of Gauss law constraint-shell QCD. The constraint-shell action (2.53) is got in the rest reference frame } \eta_{\mu} = (1, 0, 0, 0).\]
\[
\frac{\delta W^*}{\delta \psi^D} = 0 \implies (i\gamma_{\mu}\partial_{\mu} - m)\psi^D(x) = -e\gamma_{\mu}\psi^D(x)A_i^D(x) + \frac{1}{2}\gamma_0\{\psi^D(x), \frac{1}{\Delta}j_0^D(x)\}, \quad (2.57)
\]

with the projection operator \(\delta^D_{ki}\) given in \(2.47\).

It projects effectively QED into the time-like surface of three-vectors \(A^D \equiv (A^D_0, A^D_1, A^D_2)\) swept by the transverse components \(A^D_1, A^D_2\) of an "four-potential" in the Minkowski space (that are, indeed, the gauge invariant Dirac variables \(2.35, 2.36\)) and the temporal field component \(A^D_0\) specified in \(2.51, 2.52\) and then rewritten in terms of gauge invariant fermionic currents \(j^D\).

As a result, the temporal field component \(A^D_0\) becomes indeed "transverse" since the fermionic currents \(j^D_{\mu}\) are such (because of their explicit look \(e\bar{\psi}^D\gamma_{\mu}\psi^D\)).

The D’alembert equation \(2.56\) has the standard look \([42]\) describing plane electromagnetic waves involving transverse polarizations of electric and magnetic tensions.

By analogy with ordinary Maxwell electrodynamics (see e.g., §62 in \([42]\)), the solution to the D’alembert equation \(2.56\) can be represented in the shape of an "four-potential" in the Minkowski space

\[
A^D(x) = \frac{1}{c}\int \frac{j^D_{R/c}}{R} d^3x + A^D_{(0)}(x), \quad (2.58)
\]

with \(A^D_{(0)}\) being the solution to the homogeneous equation

\[
\Box A^D_k = 0 \quad (2.59)
\]

and \(R\) being the distance between the origin of coordinates and the observation point \(\text{[18]}\).

At the “Feynman” level, the solution \(2.58\) to the D’alembert equation \(2.56\) describes correctly the interaction of two fermionic currents (the four-fermionic interaction), that is the of the second order process in the perturbation theory.

Formally (see §32.1 in \([41]\)), such interaction of two fermionic currents has the look

\[
S^{(2)} = -\frac{1}{4} \int T[j_{\mu}(x)j_{\nu}(x')]T[A_{\mu}(x)A_{\nu}(x')]d^4xd^4x' = -\frac{1}{2} \int j_{\mu}(x)D_{\nu}(x-x')j_{\nu}(x')d^4xd^4x',
\]

with

\[
< 0|T[A_{\mu}(x)A_{\nu}(x')]|0> = \delta_{\mu\nu}D_{\nu}(x-x')
\]

being the photonic cause function (if physical photons are absent in the initial and final states; it is now just the case).

At these circumstances, the matrix element \(\langle f|S^{(2)}|i\rangle\) for the four-fermionic interaction can be represented in the shape of retarding potentials (where now we write down our equations in terms of Dirac variables, cf. §32.2 in \([41]\)):

\[
\langle f|S^{(2)}|i\rangle = i \int dt \int j^D_{\mu}(r_2, t)A^D_{\mu}(r_2, t)d^3x_2 = -e \int \bar{\psi}^D(x)\hat{A}(x)\psi^D(x)d^4x
\]

with

\[
A_{\mu}(r, t) = \frac{1}{4\pi} \int j_{\mu}(r')e^{-i\omega t + i\omega|\mathbf{r} - \mathbf{r}'|} \frac{d^3x'}{\mathbf{r} - \mathbf{r}'}
\]

and \(\omega\) being the energy loss along two fermionic lines: incoming and outcoming, of the vertex in the appropriate second order Feynman diagram (see Fig 32.1 in \([41]\)).
Thus the D'alembert equation (2.56) is on-shell of physical (transverse) photons, resulting the massless photon propagator [27]

\[ D_{ij}^D(q) = \frac{1}{q^2 + i\epsilon} (\delta_{ij} - q_i q_j \frac{1}{q^2} q_j) \quad (i, j = 1, 2); \quad D_{il} q^l = 0. \]  

(2.60)

The second equation of motion, (2.57), in the system of motion equations (2.56), (2.57) differs from the ordinary Dirac equation in QED by the second item on the right hand side of (2.57). It describes the Coulomb instantaneous interaction of fermionic currents remaining upon eliminating \( a_0 \) via (2.49) [10, 11].

Thus the solution to the system of equations of motion (2.56), (2.57) is a time-like three-vector \( A^D = (A^T_0, A^D_i) \quad (i = 1, 2) \) with the temporal component \( A^T_0 \) being the solution to the Poisson equation

\[ A^T_0 = \frac{1}{\Delta} j^D_0(x), \]  

(2.61)

coinciding mathematically with (2.52), and the transverse spatial components \( A^D_i \) given in (2.58) [42]. The latter ones are gauge invariant retarding potentials.

In turn [45], Eq. (2.61) is nothing else but the Gauss law constraint written down in terms of the fermionic Dirac variable \( j^D_0(x) \). Note that it is the effect of the manifest presence of fermionic currents and charges in (constraint-shell) QCD. In the “pure” electrodynamics (2.44) (with its constraint-shell reduction (2.48) [27], the right-hand side of Eq. (2.61) should vanish, and it becomes homogeneous Laplace equation [19].

From the physical standpoint [3, 45], all this is equivalent to the removal of the \( \partial_0 \partial_k A^k \) item in the Gauss law constraint (2.24); the latter one cannot be considered as a physical source of the Coulomb potential.

At least, it is possible the situation when the transverse (gauge invariant) potential \( A^D \) is a stationary field: \( \dot{A}^D = 0 \). In this case the first item in the gauge invariant QED Lagrangian density (2.53) becomes zero, the D’alembert equation (2.56) comes to the Poisson equation

\[ \Delta A^D_k(x) = \delta^D_{ki} j^D_i(x), \]  

(2.62)

permitting the \( A^D_k(x) \sim O(1/r) \quad (k = 1, 2) \) stationary solutions [42] (i.e. that having the look of Coulomb potentials [20] while the temporal component of a gauge invariant four-potential \( A^D \) is also a Coulomb field that is, indeed, the solution to the Poisson equation

\[ A^T_0 = \frac{1}{\Delta} j^D_0(x), \]

---

19 In Section 3 we shall once again give arguments in favour of impossibility to remove the temporal component of a four-potential and, consequently, the appropriate electric tension \( E \) when an electric charge/current is present in the considered theory.

20 In this case the appropriate electromagnetic Green function written down in the momentum representation possesses the asymptotic (infrared) behaviour (cf. §110 in [48])

\[ D_c \rightarrow 4\pi/q^2 \quad \text{as} \quad q^2 \rightarrow 0 \quad \text{and} \quad \omega = 0. \]
also possessing the $O(1/r)$ behaviour.

As a result, we come to the "pure" electrostatic (if $j_0^D(x) = j_0^D(x)$), now given in terms of gauge invariant transverse four-vectors $A^D = (A_0^D, A^D)$:

$$ p \cdot A^D = 0. \quad (2.63) $$

Indeed, this "pure" electrostatic should be supplemented by a stationary (electrostatic) solution $A_0(x)$ to the homogeneous Laplace equation [27]

$$ \Delta A_0(x) = 0, \quad (2.64) $$

that is the Gauss law constraint (2.24) resolving in terms of transverse Dirac variables $A_k^D$: (2.35), (2.36), in the purely electromagnetic theory (2.48) [27] (herewith temporal components of these Dirac variables vanish due to their removal (1.2) [11]).

The homogeneous Laplace equation (2.64), as it is well known (see e.g. §36 in [42]), describes electrostatic fields $E$ in emptiness:

$$ E = \frac{e}{R^3}, \quad A_0(R) \equiv \phi(R) = \frac{e}{R}, \quad (2.65) $$

obeyed the Coulomb law.

Just these electrostatic fields $E$ and electrostatic Coulomb potentials $\phi(r)$ [42] supplement the above described picture of electrostatic in terms of transverse Dirac variables $A^D$, (2.63).

Thus the solution to the Gauss law constraint (2.24) can be represented as the sum of the (general) solution (2.65) to the homogeneous Laplace equation (2.64) and the (particular) solution [20, 26] (2.28), (2.33) to the inhomogeneous equation (2.24).

Resuming the said above, one can speak that constructing transverse and physical Dirac variables in QED has the following common feature with constructing transverse and physical fields in usual electrodynamics on-shell of photon [41, 42, 43]. In the both cases one removes temporal components of Abelian fields resolving the Gauss law constraint (2.24) with the transverse Coulomb gauge. But in the Dirac fundamental quantization method [3, 20, 26, 27] one utilizes the nonlocal functionals (2.35), (2.36) [20] of gauge fields. These functionals are gauge invariant and transverse Dirac variables.

As we have noticed above, there is a definite ambiguity in specifying Dirac variables (2.36). They are specified indeed to within the stationary phase (2.41), extracting the subgroup of stationary gauge transformations in the general $U(1)$ gauge group inherent in QCD.

The stationary phase (2.41) is fixed via the additional constraint in the form of the time integral of the Gauss law constraint (2.42) getting from (2.24) upon setting $a_0$ in zero by the gauge transformations (1.2) [11] [11]:

$$ \partial_t A_i^D = 0. \quad (2.66) $$

We shall refer to this equation as to the constraint-shell or to the Coulomb gauge.
This gauge restricts initial data to within a phase specified by the equation

\[ \Delta \Phi_0(x) = 0, \quad (2.67) \]

that is the spatial part of the condition \( \partial_\mu \partial^\mu \Lambda = 0 \). \[23\] \[11\] \[21\]

Nontrivial solutions to this second-order differential equation in partial derivatives we shall call the degeneration of initial data or Gribov copying \[49\] the constraint-shell gauge. Mathematically, solutions to Eq. (2.67) always can be represented \[27\] as

\[ \frac{c_1}{r} + c_2, \quad (2.68) \]

with \( c_1 \) and \( c_2 \) being constants.

Finishing our discussion about constructing Dirac variables in constraint-shell four-dimensional QED \[20\] \[26\] \[27\] we should like to make the following important remark.

Constraint-shell QED model \[2.53\] \[26\] \[27\] has only three subtle differences from the initial gauge theory \[2.1\].

First of them is the origin of the current conservation law. In the initial constrained system \[2.1\] the current conservation law

\[ \partial_0 j_0 = \partial_i j_i, \]

may be derived from the equations for the gauge fields \[2.25\] \[42\], whereas the similar law

\[ \partial_0 j^D_0 = \partial_i j^D_i \]

in EUS \[2.53\] follows only from the "Dirac" equation \[2.57\] \[27\] for fermionic fields (cf. §7.5 in \[41\]) and from the explicit look \( \bar{\psi} \gamma^\mu \psi^D \) of gauge invariant fermionic currents, \[21\] Mathematically, Eq. (2.67) also coincides with the Laplace equation \[2.64\] and with the equation \[20\]

\[ \Delta \lambda_L(t, x) = 0 \]

imposed onto the Lagrange multiplier \( \lambda_L(t, x) \) entering the effective action \[2.55\].

\[22\] As there was demonstrated in \[42\], the Maxwell equations

\[ \frac{\partial F_{ik}}{\partial x^k} = -4\pi j^i \]

is mathematically equivalent to

\[ \frac{\partial^2 F_{ik}}{\partial x^i \partial x^k} = -4\pi \frac{\partial j^i}{\partial x^i}. \]

But the symmetric in the indices \( i, k \) operator \( \frac{\partial^2}{\partial x^i \partial x^k} \), being applied to the antisymmetric Maxwell strength tensor \( F^{ik} \), sets its identically in zero; thus one comes to the continuity equation (current

\[ \partial x^i = 0. \]

31
written down in terms of Dirac variables $\bar{\psi}^D, \psi^D$: (2.36). This difference becomes essential in quantum theory.

In the case of constraint-shell four-dimensional QED [20, 26, 27], involving EUS (2.53), generating the "Dirac" equation (2.57), we cannot use the current conservation law when \textit{quantum fermions are off-shell}: in particular, in an atom.

What one may observe in an atom? The bare fermions, or \textit{dressed} ones, (2.36)?

Dirac supposed [3] that we may observe only \textit{gauge invariant} quantities of the type of \textit{dressed} fields.

Really, we may convince ourselves (and this was done above) that the \textit{dressed fields} (2.36), as nonlocal functionals of initial gauge fields, are invariant with respect to the time-dependent gauge transformations of these initial fields: (2.3), (2.4), (2.9).

The gauge invariance with respect to the time dependent gauge transformations is the second difference of nonlocal Dirac variables (2.36) and EUS (2.53) [27] from the constrained system (2.1) involving ordinary transformational properties with respect to gauge and Lorentz transformations (see below).

The gauge constraint $\partial_i A_i = 0$ in the gauge-fixing method is associated (see the next subsection) with the relativistic non-covariance. In turn, the observable nonlocal variables (2.36) depend on the time axis in the \textit{relativistic covariant} manner. This is the third difference of the \textit{constraint-shell dynamic variables} (2.36) in EUS (2.53) from those in the gauge-fixing method.

The gauge-fixing method and its terminology "the Coulomb gauge" indeed do not reflect these three properties of Dirac observables in constraint-shell QED [20, 26, 27]: the off-shell non-conservation of the current, gauge invariance with respect to the transformations (2.3), (2.4), (2.9) and relativistic covariance.

In fact, the term \textit{gauge} (for example, the Coulomb gauge (2.66)) means a \textit{choice of nonlocal variables}, or more exactly, a \textit{choice of a gauge of physical sources} associated with these variables.

### 2.2 Relativistic covariance of Dirac variables in QED and minimal quantization scheme.

Dirac variables prove to be manifestly relativistically covariant. Relativistic properties of Dirac variables in gauge theories were investigated in the papers [4] (with the reference to the unpublished note by von Neumann), and then this job was continued by I. V. Polubarinov in his review [50]. These investigations displayed that there exist such relativistic transformations of Dirac variables that maintain transverse gauges of fields.

Let us demonstrate this now with the example of constraint-shell QED, us discussed in the previous subsection. If one makes therein the usual relativistic transformations of
initial fields $A_i, A_0, \psi$ with the parameter $\epsilon_i$ \cite{20}:

$$
\delta^0_L A_k = \epsilon_i (x'_i \partial_\nu - x'_0 \partial_\nu) A_k(x') + \epsilon_k A_0,
$$

$$
\delta^0_L \psi = \epsilon_i (x'_i \partial_\nu - x'_0 \partial_\nu) \psi(x') + \frac{1}{4} \epsilon_k [\gamma_i, \gamma_j] \psi(x'),
$$

then the physical Dirac variables \eqref{2.36} suffer the Heisenberg-Pauli transformations \cite{4}

$$
A^D_k[A_i + \delta^0_L A] - A^D_k[A] = \delta^0_L A^D_k + \partial_k \Lambda,
$$

$$
\psi^D[A + \delta^0_L A, \psi + \delta^0_L \psi] - \psi^D[A, \psi] = \delta^0_L \psi^D + i \epsilon \Lambda(x') \psi^D,
$$

with

$$
\Lambda[A^D, j^D_0] = \epsilon_k \frac{1}{\Delta} (\partial_0 A^D_k + \partial_k \frac{1}{\Delta} j^D_0).
$$

These transformations were interpreted as the transition from the Coulomb gauge with respect to the time axis in the fixed rest reference frame $\eta^0_\mu = (1, 0, 0, 0)$ to the Coulomb gauge with respect to the time axis in a moving reference frame (see Fig.1) \cite{20}

$$
\eta'_\mu = \eta^0_\mu + \delta_L \eta^0_\mu = (L \eta^0)^\_\mu.
$$
These transformations correspond to the "change of variables"
\[
\psi^D(\eta), \hat{A}^D(\eta) \rightarrow \psi^D(\eta'), \hat{A}^D(\eta'),
\]
so that these variables become transverse with respect to the new time axis \(\eta'\) (or, from the point of view of the "gauge-fixing" method of reduction, with respect to the transformations (2.69), (2.70), (2.71)) corresponding to the "change of a gauge":
\[
\partial_i \hat{A}^D_i = 0; \quad i = 1, 2; \quad A_T^D = 0.
\]

In the recent paper [51] it was argued that it is possible to extract the so-called 'small' subgroup \(G_0\) of Poincare invariant gauge transformations from the 'large' group \(G\) of all the gauge transformations in constraint-shell QED acting in the space of Dirac variables \(A^D\). Such 'small' subgroup \(G_0\) can be set by the condition
\[
\delta_0^L A^D_k = -\partial_k \Lambda,
\]
following directly from Eq. (2.70) for the appropriate Dirac variables \(A^D\).

As an important result, one gets (in the rest reference frame \(\eta^0\)) the relativistic covariant separation of the interaction with the Coulomb potential (instantaneous with respect to the time axis \(\eta_0^0\)) and retardations \(^{23}\) (say, (2.58)).

The Coulomb interaction herewith takes the manifest covariant look
\[
W_C = \int d^4x d^4y \frac{1}{2} j^D_\eta(x)V_C(z^+)j^D_\eta(y)\delta(\eta \cdot z).
\]
Here
\[
j^D_\eta = e\bar{\psi}^D_\eta \psi^D_\eta, \quad z^+ = z - \eta^0(z \cdot \eta), \quad z_\mu = (x - y)_\mu, \quad \eta^0 \equiv \eta^\mu \gamma^\mu,
\]
\(^{23}\)One can neglect retarded interactions if
\[
|x^0_{\text{out}} - x^0_{\text{in}}| \gg E^{-1}_{I,\text{min}}, \quad V_0^{1/3} \gg E^{-1}_{I,\text{min}}.
\]

This condition means that all stationary solutions with zero energy \(E_{I,\text{min}} \rightarrow 0\) cannot be considered as perturbations.

---

**Figure 1.**

| Lorentz Frame | Gauge | New Lorentz Frame | New Gauge |
|---------------|-------|-----------------|-----------|
| \(\eta^0 = (1, 0, 0, 0)\) | \(A_T^0 = 0\) | \(\eta' = \eta^0 + \delta_0^L \eta^0\) | \(A_T^{\prime 0} = (\eta' \cdot A) = 0\) |
| \(\eta = (1, 0, 0, 0)\) | \(\eta' = \eta + \delta_0^L \eta\) | | |

---

23 One can neglect retarded interactions if
\[
|x^0_{\text{out}} - x^0_{\text{in}}| \gg E^{-1}_{I,\text{min}}, \quad V_0^{1/3} \gg E^{-1}_{I,\text{min}}.
\]

This condition means that all stationary solutions with zero energy \(E_{I,\text{min}} \rightarrow 0\) cannot be considered as perturbations.
Knowing an instantaneous bound state (2.76) in constraint-shell QED, one can give the definition of relativistic (Lorentz) invariant states.

As it was noted in Ref. [45], the relativistic invariance means that a complete set of states \( \{|\Phi_I\rangle\}_{n=1}^\infty \) obtained by all Lorentz transformations of a state \(|\Phi_0\rangle\) in a rest frame \( \eta_\mu^0 = (1, 0, 0, 0) \) coincides with a complete set of states \( \{|\Phi_I\rangle\}_n \) obtained by all Lorentz transformations of this state \(|\Phi_0\rangle\) in another reference frame \( \eta_\mu = (\frac{1}{\sqrt{1-v^2}}, \frac{v}{\sqrt{1-v^2}}, 0, 0) \).

Issuing from this assertion, it is enough to find all the Lorentz transformations of the rest frame \( \eta_\mu^0 \), i.e. all the Lorentz transformations of the instantaneous interaction (2.76) obtained in this rest frame.

The finite Lorentz transformations from the time axis \( \eta^{(1)} \) to the time axis \( \eta^{(2)} \) were constructed in the paper [50] using the gauge transformations

\[
\begin{align*}
&ieA^{(2)} = U_{(2,1)}[ieA^{(1)} + \partial_0 U_{(2,1)}^{-1}], \quad \psi^{(2)} = U_{(2,1)}\psi^{(1)}, \\
&U_{2,1} = v^{(2)}v^{(1)},
\end{align*}
\]

where \( v^{(2)}, v^{(1)} \) are the Dirac gauge factors (2.37) associated with the time axes \( \eta^{(2)} \) and \( \eta^{(1)} \), respectively.

Returning to the initial QED action (2.1), again note that it is not compatible with quantum principles, as it contains the zero canonical momentum by temporal component of the electromagnetic field.

The Dirac formulation [3] of EUS (2.53) in four-dimensional QED [20, 26, 27] keeps the quantum principles by values that exclude the unphysical components.

One quantizes then EUS (2.53) involving gauge invariant physical Dirac variables (2.36). The appropriate commutation relations [20, 27]

\[
i[\partial_0 A^D_j(x,t), A^D_j(y,t)] = (\delta_{ij} - \partial_i \frac{1}{\Delta} \partial_j)\delta^3(x - y) \equiv \delta^D_{ij}\delta^3(x - y),
\]

\[
\{\hat{\psi}^D+(x,t), \hat{\psi}^D(y,t)\} = \delta^3(x - y)
\]

lead to the generating functional for Green’s function of the obtained unconstrained system in the form of the Feynman path integral in the fixed reference frame \( \eta \) [20, 52]:

\[
Z[s^*, \bar{s}^*, J^*] = \int D\psi^* D\bar{\psi}^* e^{i W^*[A^*, \psi^*, \bar{\psi}^*] + iS^*},
\]

including the external sources term:

\[
S^* = \int d^4x \left( \bar{s}^* \psi^s + \bar{\psi}^* s^s + J^*_i A^*_i \right)
\]
(here $A^*$ and $\psi^*$ are Dirac variables for gauge and fermionic fields, respectively; $J^*$ is the source of gauge fields, $s^*$ and $s^*$ are the sources of fermionic fields $\bar{\psi}^*$ and $\psi^*$, respectively; $W^*$ is the constraint-shell action of the considered theory.)

By constructing the unconstrained system, this generating functional is manifestly gauge invariant and relativistic covariant (due to the theory (2.70), (2.71)).

Relativistic transformation properties of quantum fields would repeat the ones of the Dirac variables (2.36) as nonlocal functionals of the initial fields.

As it was shown in the papers [20, 37, 50, 53, 54, 55, 56, 57], a quantum theory (say, four-dimensional QED) involving the gauge invariant Belinfante energy-momentum tensor

$$T_{\mu\nu} = F_{\mu\lambda}F_{\nu}^{\lambda} + \bar{\psi}\gamma_{\nu}[i\partial_{\nu} + eA_{\nu}]\psi - g_{\mu\nu}L + \frac{i}{4}\partial_{\lambda}[\bar{\psi}\Gamma_{\mu\nu}^\lambda\psi],$$

(2.82)

$$\Gamma_{\mu\nu}^\lambda = \frac{1}{2}[\gamma^\lambda\gamma_{\mu}]\gamma_{\nu} - g_{\mu\nu}\gamma^\lambda - g_{\nu}\gamma_{\mu},$$

on the surface of the Gauss law constraint (2.24), i.e., being rewritten in terms of the Dirac variables [37]:

$$T_{\mu\nu}\left[A_i, A_0 = \left(\frac{1}{\Delta}\partial_i\partial_0A^i + j_0\right)\right] = T_{\mu\nu}[A^D, \psi^D, \bar{\psi}^D];$$

(2.83)

completely reproduces the symmetry properties of the ”classical” theory (2.69)- (2.72):

$$i\epsilon_k[M_{0k}, \psi^D] = \delta_0^D\psi^D + i\epsilon[A^D, j^D_0]\psi^D;$$

$$i\epsilon_k[M_{0k}, A^D_\mu(x)] = \delta_0^D A^D_\mu(x) + \partial_{\mu}\Lambda;$$

$$M_{0k} = \int d^3x[k x T_{00} - t T_{0k}].$$

(2.84)

For us there will be very useful to cite now the explicit look of $T_{00}$ and $T_{0k}$, entering the expression (2.84) for the Lorentz boost $M_{0k}$ in constraint-shell QED [20, 26, 27].

---

24 Besides the above commutation relations, also it is worth to cite here the following ones [27], taking place in constraint-shell QED [20, 26, 27]. So,

$$i[\partial_0 A^D_i(x, t), A^D_j(y, t)] = \delta_{ij}^D\delta^3(x - y) \iff i[F^D_0(x, t), A^D_j(y, t)] = \delta_{ij}^D\delta^3(x - y),$$

with

$$F^D_0 = \hat{A}^D_i - \partial_i A^T_0.$$

The latter commutation relation follows from manifest commutations of bosonic fields $A^D_i$ ($i = 1, 2$) and fermionic charges $j^D_0$ due to

$$[\psi^D(x, t), A^D_i(y, t)] = 0.$$

On the other hand,

$$[A^T_0(x, t), \psi^D(x, t)] \sim -\frac{1}{4\pi|x - y|}\psi^D(y),$$

in the equal time instant $x_0 = y_0 = t_0.$
So, $T_{00}$ is determined by the explicit look of the constraint-shell gauge invariant Lagrangian density (2.53). Substituting the latter one in the expression (2.82) for the Belinfante energy-momentum tensor, we get [27]

$$T_{00} = \frac{1}{2} (F_{0i}^D)^2 + \frac{1}{4} F_{ij}^2 + \bar{\psi}^D (i\gamma_i \partial_i - m) \psi^D.$$  

(2.85)

Furthermore [27],

$$T_{0k} = F_{0i}^D F_{ki} + \psi^+ \gamma_0 \partial_k \psi^D + \frac{i}{4} \partial_i (\psi^+ \gamma_0 [\gamma_i, \gamma_k]) \psi^D.$$  

(2.86)

Herewith there is implemented the commutation relation [27]

$$i [T_{00}(x), T_{00}(y)] = -(T_{00}(x) + T_{00}(y)) \partial_k \delta^3 (x - y).$$  

(2.87)

Generally speaking, as there was explained in the paper [55], implementing the commutation relation of the (2.87) type in a gauge theory is the necessary condition for this theory to be manifestly relativistic (Lorentz) invariant 25.

However, in four-dimensional constraint-shell QED [20, 26, 27] the commutation relation (2.87) follows directly from explicit solving the Gauss law constraint (2.24) in terms of the Dirac variables (2.36).

Thus the commutation relation (2.87) acquires in constraint-shell QED [20, 26, 27] the rather another sense than in the Schwinger model [55] for describing gauge theories.

Moreover, there may be directly checked [27] that if the operators

$$H = \int d^3 x \, T_{00},$$  

(2.88)

$$P_k = \int d^3 x \, T_{0k}$$  

(2.89)

and also the momentum operators $M_{0k}, M_{ij}$ satisfy the ordinary algebra [27] of the Poincare group generators in the physical sector, (2.36), of gauge fields in constraint-shell QED [20, 26, 27] (we omit these commutation relations, since they are, perhaps, well-known to our readers), then the commutation relation (2.87) is automatically fulfilled.

It will be also useful to write down explicitly the commutation relations between the momentum operator $P$, given in (2.89), and Dirac variables $A_{\mu}^D$ and $\psi^D$, given in (2.36).

These are [27]

$$i [P_{\mu}, A_{\nu}^D] = \partial_\mu A_{\nu}^D, \quad i [P_{\mu}, \psi^D] = \partial_\mu \psi^D.$$  

(2.90)

25 In this case the commutation relation (2.87) is equivalent to the appearance of the Schwinger surface terms

$$\sim S_{ijk}^a(x) \int \frac{\partial}{\partial y_i} \delta^3(x - y) d^3 y = 0$$

in action functionals of gauge theories. This, in turn, implies their manifest Lorentz invariance.
Finally \[27\], the Poincare algebra of the operators \( H \), \( P_k \), \( M_{0k} \) and \( M_{ij} \), supplemented by the commutation relation \( (2.87) \), just results the Heisenberg-Pauli transformations \[20\] (2.70)- (2.72), accompanied, in turn, by the Lorentz transformation (2.73) of the chosen (rest) reference frame.

Thus the commutation relation \( (2.87) \) in four-dimensional QED \[20\] [26, 27] (unlike the analogous one in the Schwinger model \[55\] for describing gauge theories) cannot serve as the sufficient condition for the \textit{relativistic invariance} of the given theory. Rather the opposite effect takes place, \textit{the manifest relativistic covariance} of four-dimensional QED \[20\] [26, 27].

We should like make the following concluding remark \[27\] concerning the commutation relations in four-dimensional constraint-shell QED \[20\] [26, 27].

At going over from a classical to quantum theory, apart from defining simultaneous commutation relations, one must also eliminate pseudophysical quantities like a zero energy, a zero charge, etc.

To perform this elimination, one uses, as a rule, a normal product \[58\] for dynamical variables, depending quadratically on operators with identical arguments.

However \[55\] [59], utilizing \( N \)-product becomes an unnecessary although harmless thing in the spinor electrodynamics (generally speaking, it contradicts the gauge invariance in scalar electrodynamics and in the YM theory).

To achieve the manifest gauge invariance of a theory, it is sufficient to symmetrize the apropriate Hamiltonian in Bose operators and antisymmetrize it by Fermi operators, i.e. to use the Weyl quantization.

It is well known that this type of presentation of current densities \( j_\mu \) of spinor and scalar particles implies that particles and antiparticles enter \( j_\mu \) symmetrically, and the vacuum expectation value of \( j_\mu \) is equal to zero. It is just the recipe we shall follow in the present study.

For instance, in the case \[20\] [26, 27] of four-dimensional constraint-shell QED (that is the spinor electrodynamics), the symmetrization procedure of the operators \( A^D_i \) (\( i = 1, 2 \)), \( F_{0i}(A^D) \) (given in \( (2.51) \) \[27\]) and \( \psi^D \) does not influence the apropriate (constraint-shell) Hamiltonian \( H \) \( (2.88) \), momentum tensor \( M_{i,j} \) and boost tensor \( M_{0i} \) (given in \( (2.84) \) \[20\]).

The Lorentz transformation \( (2.73) \) of the chosen (rest) reference frame, at the level of the operator quantization, including the apropriate Feynman path integral, means the change of the time axis:

\[
Z^*_{\eta_\eta}[s^*, \bar{s}^*, J^*] = Z^*_\eta[Ls^*, L\bar{s}^*, LJ^*].
\]  

\[ (2.91) \]

This scheme of quantization, called \textit{the minimal quantization scheme} \[26\] [27, 37, 60, 61, 62], \[26\] explicitly depends on a choice of the time axis.

\[26\] As the additional claim to the minimal quantization scheme, one would diagonalize the Belinfante Hamiltonian \( T_{00} \). Then, on the operator level, the Dirac variables \[236\] coincides with the quantum ones.
If one chooses a definite reference frame with the initial time axis, any Lorentz transformation turns this time axis in a relativistic covariant manner. This implies that constraint dynamics is manifestly relativistic covariant.

Another problem is to find conditions at which measurable physical quantities and results of theoretical calculations do not depend on the time axis (identified with a physical device [10]).

This independence exists only for scattering amplitudes of particles on-shell [24, 63]. In this case one may speak about the relativistic invariance of scattering amplitudes squared as functionals of local degrees of freedom.

But it is well known that Green functions (in particular, one-particle Green functions) and instantaneous bound states depend on the choice of the time axis.

In general case, measurable quantities in electrodynamics depend on the time axis and other parameters of a physical device, including its size and energy resolution [23].

If a nonlocal process depends on the time axis, one should establish a principle how to choose this time axis.

In particular, this choice and the nonlocal relativistic transformations (2.84) remove all the infrared divergences from the one-particle Green function (for instance, in four-dimensional constraint-shell QED [20, 26, 27] written in terms of the radiation variables in the rest reference frame of an electron $p_\mu = (p_0, 0)$ for the time axis $l_\mu = (1, 0, 0, 0)$ [20, 27, 61, 64]:

$$i(2\pi)^4\delta^4(p - q)G(p - q) = \int d^4d^4y \exp(ipx - iqy) < 0|\bar{\psi}^D(x)\psi^D(y)|0 >, \tag{2.92}$$

$$G(p) = G_0(p) + G_0(p)\Sigma(p)G_0(p) + O(\alpha^4), \quad G_0(p) = [\not\!p - m]^{-1}$$

$$\Sigma(p) = \frac{\alpha}{8\pi^3i} \int \frac{d^4q}{q^2 + i\epsilon} \left[ \delta_{ij} - q_i \frac{1}{q^2} q_j \right] \gamma_i G_0(p - q)\gamma_j + \gamma_0 G_0(p - q)\gamma_0 \frac{1}{q^2} = \frac{\alpha}{4\pi} \Pi(p),$$

where $\Pi(p)$ is

$$m(3D + 4) - D(\not\!p - m) + \frac{1}{2}(\not\!p - m)^2 \left[ \frac{(\not\!p + m)}{p^2} \left( \ln \frac{m^2 - p^2}{m^2} \right) \left( 1 + \frac{\not\!p(\not\!p - m)}{2p^2} \right) - \frac{\not\!p}{2p^2} \right]$$

and $D$ is the ultraviolet dimensional-regularization parameter [27]:

$$D = 1/\epsilon - \gamma_\epsilon + \ln 4\pi.$$
As it is customary in QED, two items in the mass operator $\Sigma(p)$ in (2.92) describe, respectively, contributions from transverse Dirac fields $A_D^i$ ($i = 1, 2$) and from temporal Dirac components $A^T$ to the appropriate Green function (2.92).

Indeed, there may be demonstrated the manifest relativistic invariance of the mass operator $\Sigma(p)$ in constraint-shell QED [20, 26, 27] with respect to the Lorentz (Heisenberg-Pauli) transformations (2.70)–(2.72) of gauge fields $A_D^i$ ($i = 1, 2$) and fermionic ones $\psi^D$, $\bar{\psi}^D$ [4, 20, 27, 50]:

$$\delta_L^{\text{tot}} \Sigma(p) = (\delta_L^0 + \delta_\Lambda) \Sigma(p) = 0.$$ (2.93)

To prove the statement about the relativistic invariance of the mass operator $\Sigma(p)$ in constraint-shell QED [20, 26, 27] note firstly that the fermionic Green function $G_0(p-q) \equiv \tilde{G}_0$ entering the mass operator $\Sigma(p)$, (2.92), is just manifestly invariant with respect to the Lorentz transformations $\delta_L^0$ since $\tilde{p}$ and $\tilde{q}$ possess such property. This statement can be written down as $\delta_L^0 G_0(p) = 0$ [27].

Then we rewrite $\Sigma(p)$ as the sum

$$\Sigma(p) = \Sigma_F(p) + \Delta \Sigma(p),$$ (2.94)

where $\Sigma_F(p)$ is the mass operator (electron self-energy) in the Feynman gauge (see §76 in [48])

$$\Sigma_F(p) = -\frac{i e^2}{32\pi^4} \int \frac{d^4q}{q^2} \gamma_\mu \tilde{G}_0 \gamma_\mu,$$ (2.95)

invariant with respect to the Lorentz transformations $\delta_L^0$ ($\delta_L^0 \Sigma_F(p) = 0$), while the noninvariant addition $\Delta \Sigma(p)$ has the look

$$\Delta \Sigma(p) = \frac{i e^2}{32\pi^4} \int \frac{d^4q}{q^2} \left[ \bar{\psi} \gamma_\mu \tilde{G}_0 \gamma_\mu \psi + \bar{\psi} \gamma_0 \tilde{G}_0 \gamma_\mu \psi + \bar{\psi} \gamma_\mu \gamma_\mu \gamma_0 \tilde{G}_0 \psi \right]$$ (2.96)

$$= \frac{i e^2}{32\pi^4} \int \frac{d^4q}{q^2} \left[ B_k \tilde{G}_0 \psi + \bar{\psi} \tilde{G}_0 B_k \psi \right],$$ (2.98)

with $\epsilon_i$ being an infinitesimal Lorentz parameter involving Lorentz boosts

$$x_k' = x_k + \epsilon_k t, \quad t' = t + \epsilon_k x_k, \quad |\epsilon_k| \ll 1.$$
with
\[ B_k = q_k \gamma_0 + \gamma_k q_0 - \frac{2q_0 q_k}{q^2}q - \frac{q_0 q_k}{q^2} \lambda. \]  
(2.99)

However, the total Lorentz transformations for the Green function (2.92) also contain the gauge transformations \( \delta \Lambda \), (2.72):
\[
\delta \Lambda \Sigma(p) = \epsilon \int d^4x d^4y e^{ipx - iqv} \left[ <0|T(\psi^D(x)\bar{\psi}^D(y)\Lambda_k(y))|0> - <0|T(\Lambda_k(x)\psi^D(x)\bar{\psi}^D(y))|0> \right].
\]  
(2.100)

Using the explicit look (2.72) for \( \Lambda_k(x) = \Lambda^T_k(x) + \Lambda^\xi_k(x) \) with
\[
\Lambda^T_k(x) = \frac{1}{4\pi} \int d^3y \frac{\dot{A}_k(y, t)}{|x - y|}, \quad \Lambda^\xi_k(x) = \frac{1}{4\pi} \int d^3y \frac{\partial_k A_0^T(y, t)}{|x - y|},
\]  
(2.101)

we get
\[
\delta \Lambda \Sigma(p) = -\epsilon \int d^4q \frac{ie^2}{q^2 q^2} \left[ B_k \tilde{G}_0(p - m) + (p - m) \tilde{G}_0 B_k \right],
\]  
(2.102)

\[
B_k = (\delta_{ki} - \frac{q_k}{q^2}q_i)\gamma_i q_0 - \frac{q_0 q_k^2 \gamma_0}{q^2} + \gamma_0 q_k.
\]  
(2.103)

It is easy to check that \( B_k \) given by Eq. (2.103) coincide with those given by Eq. (2.99).

As
\[
(p - m)G_0(p - q) = 1 + \beta \tilde{G}_0(p - q)
\]  
(2.104)

and
\[
\int \frac{d^4q}{q^2 q^2} B_k = 0,
\]  
(2.105)

we get the final expression
\[
\delta \Lambda \Sigma(p) = \epsilon \int d^4q \frac{ie^2}{q^2 q^2} \left[ B_k \tilde{G}_0 \beta + \beta \tilde{G}_0 B_k \right] = -\delta^0 \Sigma(p).
\]  
(2.106)

The total response of \( \Sigma(p) \) to the Lorentz transformations (2.70)- (2.72) is thus equal to zero:
\[
\delta^{\text{tot}}_L \Sigma(p) = (\delta^0_L + \delta \Lambda) \Sigma(p) = 0.
\]  
(2.107)

Thus it is sufficient to calculate the electronic Green function in constraint-shell QED [20, 26, 27] in the rest reference frame of the observable electron, as it was done in (2.92).

Since [27]
\[
\partial_\mu \partial_\nu - \lambda_\nu (\partial \lambda) \leftrightarrow A^D = A^D - l(A^D \cdot l),
\]  
(2.108)

for the given time-like vector \( A^D = (A^T_0, A^D_i) (i = 1, 2) \) at a Lorentz transformation (2.73),
\[
l'_\mu = l^0_\mu + \delta^0 L_\mu,
\]  
41
of the rest reference frame \( l^0_\mu \), then appropriate passing
\[
p_\mu = (p_0, 0) \rightarrow p'_\mu = (p'_0, p'_\parallel \neq 0)
\]
in the momentum space, associated with the local Lorentz transformations (2.100), involves the change of gauge for the Dirac variables \( A^D \) (cf. Fig. 1):
\[
q_i A^D_i(q) = 0 \implies [q - l(ql)] A^D = 0,
\]
where the moving reference frame \( l' \) may be expressed through the momentum \( p'_\mu \) as
\[
l'_\mu = p'_\mu / \sqrt{p'^2}.
\]
It is enough transparent that the local Lorentz transformations (2.100) for the electronic Green function involve new Feynman diagrams, referred to as spurious ones (e.g. in [20]).
As a result, in another reference frame \( l' \) we get the same relativistic covariant expressions depending on the new momentum \( p' \) [27, 64].

It is also worth to note that the electronic self-energy \( \Sigma(p) \), given in (2.92) [20, 27], has no infrared divergences and allows the renormalization with subtracting at physical values of the momentum, \( \not{p}' = m \).

The next important property of the electronic Green function \( G(p) \), (2.92), written down in terms of Dirac variables \( A^D \), is that the probability to find an electron with the mass \( m \) given by the formula [27]
\[
R(p) = \lim_{\not{p}' \to m} (\not{p}' - m) G_{R}(p) = |\psi|^2
\]
is equal to unity (\( |\psi|^2 = 1 \)).

It is the consequence of the relativistic invariant look \((\not{p}' - m)^{-1}\) of the electronic Green function \( G(\not{p}') \) in (2.92).

The result (2.111) represents a solution to the renormalization problem on the mass-shell for transverse variables \( A^D \).

A mistake of the popular papers [66, 67] was not only ignoring correct transformation properties (2.70)- (2.72) of Dirac variables \( A^D \) in constructing \( \Sigma(p) \) in four-dimensional constraint-shell QED [20, 26, 27] but also in a unphysical choice of the time axis, that, in turn, incorrectly fixes temporal components of gauge fields, i.e. Coulomb (electrostatic) fields.

For instance, in Eq. (2.92), when \( p_\mu = (p_0, \not{p} \neq 0) \), the vector \( l^0_\mu = (1, 0, 0, 0) \) may be chosen so that an electron has the velocity different from that of its Coulomb field. As a result, there arise definite difficulties with the manifest Lorentz invariance and infrared

\[
\Sigma(\not{p}' = m) = \delta m = \frac{ma}{4\pi}(3D + 4)
\]
in this case.

\[27\] It can be proven that [65]
divergences. On the other hand, the correct transition to the rest reference frame $p_{\mu} = (p_0, 0)$ in (2.32) doesn’t remove these difficulties as one simultaneously rotate the initial rest reference frame $l_\mu^0 = (1, 0, 0, 0)$, thus leaving velocities of an electron and its proper field to be different from those in the Coulomb case.

So, the choice of $l_\mu^0$ must be specified by physical formulating the problem; in this case $l_\mu^0$ is, indeed, the unit vector along the momentum, $p_\mu \sim l_\mu^0$.

Comparing the formulas (2.92) and (2.23) [37], it is easy to see how “work” the theory (2.23) (say, in four-dimensional constraint-shell QED [20, 26, 27]) $A^D, \psi^D, \bar{\psi}^D$.

But such a modification does not affect the relativistic invariant $S$-matrix squared $|S|^2$ (that is on-shell of fields), invariant with respect to the Heisenberg-Pauli transformations (2.70)- (2.72) [4, 20, 27, 50] of the Dirac variables $A^D, \psi^D, \bar{\psi}^D$.

But off-shell various spurious diagrams appear induced by “gauge constituents” $\Lambda$ in Heisenberg-Pauli transformations (2.70)- (2.72).

The appearance of such spurious diagrams in constraint-shell QED [20, 26, 27] is associated with the $\Lambda$-transformations (2.100) [27] for the electronic Green function $G(p)$.

### 3 Expanding the Dirac quantization scheme from QED to the Abelian $U(1)$ theory.

The (Minkowskian) Abelian gauge model contains the Abelian group $U(1)$, and this determines its nontrivial topological content:

$$\pi_1(U(1)) = \pi_1S^1 = \mathbb{Z},$$

in turn specified by the radius $|x| < \infty$ of the circle $S^1$.

How this nontrivial topology (3.1) is embodied in the Dirac fundamental quantization [3] of the Abelian $U(1)$ model (with the exact $U(1)$ symmetry) we just attempt to elucidate in the present section.

The plan of this section is following. First, we shall demonstrate (although this is evident even on the face of it) that constraint-shell QED [20, 26, 27] is the topologically trivial sector ($n = 0$) of the constraint-shell Abelian $U(1)$ model we construct now.

Secondly, we implement the Gauss-shell reduction of the Abelian $U(1)$ model with the exact $U(1)$ symmetry (further AM) for the nontrivial topologies ($n \neq 0$) involving the Dirac monopole modes [39, 40] and construct the Dirac variables for this case.

Beginning with the point one of our programm, note that the three-dimensional configuration space $A^D = (A_0^T, A^D)$ ($i = 1, 2$) of Dirac variables in constraint-shell QED [20, 26, 27] is topologically equivalent to the flat space $\mathbb{R}^3$ with the deleted origin of coordinates:

$$A^D \sim \mathbb{R}^3 \setminus \{0\}.$$
origin of coordinates (one can trace this behaviour of Dirac variables, for example, in Eq. (2.30) for retarding potentials \[42\] in the case of "plane waves”, depending explicitly on the time \(t\), as well as for "electrostatic" solutions \(A_D^0(x)\) to the Poisson equation (2.62) and \(A^0_T(x)\) to the Poisson equation in (2.51)).

It is well known (see e.g. §1T1 in [9]) that \(\mathbb{R}^3 \setminus \{0\} \simeq S^2\).

Whence
\[
\pi_2(\mathbb{R}^3 \setminus \{0\}) = \pi_2 S^2 = \mathbb{Z} \neq 0.
\] (3.3)

Latter Eq. testifies in favour of the point hedgehog topological defect inside the manifold \(\mathbb{R}^3 \setminus \{0\}\) in an infinitesimal neighbourhood of the origin of coordinates (see §Φ1 in [9]).

On the other hand, the fundamental homotopical group \(\pi_1(\mathbb{R}^3 \setminus \{0\})\) is
\[
\pi_1(\mathbb{R}^3 \setminus \{0\}) = \pi_1 S^2 = 0.
\] (3.4)

Eqs. (3.3), (3.4) imply, respectively, the point hedgehog topological defect inside the configuration space \(A_D^0\) of QED Dirac variables, in an infinitesimal neighbourhood of the origin of coordinates, and the trivial fundamental homotopical group of this configuration space.

In effect, the topological equality (3.4), compared with (3.1), implies that the QED Dirac variables (2.36) belong to the trivial topological sector of the \(U(1)\) group space, in spite of the bad definition of gauge fields \(A_D^0\) at the origin of coordinates.

To perform the Gauss-shell reduction of AM, we should recall that in the basic of this model lies the \textit{duality} \[38\] between the set of Maxwell equations
\[
\begin{align*}
\nabla \cdot E &= \rho; \quad \nabla \times B - \partial_0 E = j; \\
\nabla \cdot B &= 0; \quad \nabla \times E + \partial_0 B = 0,
\end{align*}
\] (3.5)

written down as \(\partial_\mu F^{\mu\nu} = -j^\mu (j^\mu = (\rho, j))\) in terms of the Maxwell electromagnetic tensor \(F^{\mu\nu}\), and the set of equations
\[
\partial_\nu \tilde{F}^{\mu\nu} = -k^\mu; \quad \tilde{F}^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}, \quad k^\mu = (\sigma, k)
\] (3.6)

for its dual \(\tilde{F}\).

Otherwise (in 'classical' electrodynamics), it should be \(\partial_\nu \tilde{F}^{\mu\nu} = 0\) while \(\partial_\mu F^{\mu\nu} = -j^\mu\) ('classical' electrodynamics is not symmetrical with respect to the interchange of the electrical, \(E\), and magnetical, \(B\), tensities \[38\]: \(E \rightarrow B, B \rightarrow -E\)).

It is appropriate, in this point, to introduce (as it was done already, see eg. [68]) the tensor
\[
F^{\mu\nu} = F^{\mu\nu} + \tilde{F}^{\mu\nu}
\] (3.7)

for which the \textit{Cabibbo-Ferrari-Shanmugadhasan relation}
\[
F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu + \epsilon^{\mu\nu\rho\sigma} \partial_\rho \tilde{A}_\sigma
\] (3.8)
The *magnetical current* \( k^\mu = (\sigma, k) \) introduced in \((3.6)\) saves the situation, i.e. it restores the above \( E \rightarrow B; B \rightarrow -E \) (or \( F^{\mu\nu} \rightarrow \tilde{F}^{\mu\nu}; \tilde{F}^{\mu\nu} \rightarrow -F^{\mu\nu} \)) symmetry.

Herewith Eqs. \((3.5)\) and \((3.6)\) imply also the duality transformations

\[
j^\mu \rightarrow k^\mu; \quad k^\mu \rightarrow -j^\mu
\]

for the currents.

Since the electromagnetic current \( j^\mu \) is present in AM, we now consider, the Gauss law constraint \((1.1)\), with its solution of the shape \((2.28)\), remains valid in this model. Once again, \( A_0 \) proves to be the nondynamical degree of freedom, and now we attempt to remove it with the aid of some gauge transformations, the shape of which we shall elucidate below.

Following \([38]\), let us consider the Schrödinger equation for a fermion in the electromagnetic background \((A_0, A)\). This is

\[
\left[ \frac{1}{2m} (\mathbf{p} - eA)^2 + eA_0 \right] \psi = i \frac{\partial \psi}{\partial t},
\]

\((3.10)\)

invariant with respect to the gauge transformations

\[
A(x) \rightarrow A(x) + \frac{1}{e} \nabla \alpha(x);
\]

\[
\psi(x) \rightarrow e^{i\alpha(x)} \psi(x)
\]

\((3.11)\)

The situation with the Schrödinger equation \((3.10)\) is "good", i.e. this *does not have singular solutions*, when the magnetic current \( k^\mu \) is "switched off". But if the magnetic current \( k^\mu \) is "switched on", the vector potential \( A \) cannot exist everywhere, proving to be singular at definite values of \( x \). Now we attempt to find out why it is so, what these values of \( x \) are and how to avoid this difficulty.

Note firstly that the solution to the duality equation \((3.6)\) is the potential magnetic field \([38, 39]\)

\[
B = \frac{g}{4\pi r^2} \mathbf{n}; \quad \mathbf{n} = \frac{\mathbf{r}}{r}; \quad \text{div} \mathbf{B} = 4\pi g \delta^3(\mathbf{r}).
\]

\((3.12)\)

Here \( g \) is the *magnetic charge* \([38, 39, 40]\), connected with the \( \sigma \equiv k^0 \) component of \( k^\mu \) by the relation \([38]\)

\[
k^0 = \sum_i g_i \int dx_i^0 \delta^4(x - x_i).
\]

\((3.13)\)

Whence

\[
4\pi g = \oint_S \mathbf{B} \cdot d\mathbf{S}.
\]

\((3.14)\)

\(^{28}\)In this context the potential \( \tilde{A}^\mu \) concerns the dual tensor \( \tilde{F}^{\mu\nu} \) in the same wise as the potential \( A^\mu \) concerns the Maxwell tensor \( F^{\mu\nu} \). But below, when I will talk about magnetic monopoles, distracted somehow from electromagnetic field \( F^{\mu\nu} \), I will utilize merely the symbol \( A^\mu \) for such potential, associated with magnetic monopoles.

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It is obvious that $B$ cannot be written down as $\nabla \times A$, then

$$\text{div}(\nabla \times A) = \text{div}B = 0$$

and the integral (3.14) becomes zero: in this case no magnetic charges $g$ exist (we are not interested in this trivial case now). However, one can define $A$ in such a wise that $B$ is given as $\nabla \times A$ everywhere except on a line joining the origin of coordinates to infinity. To see that it is possible, let us consider the magnetic field created by the infinitely long and thin solenoid placed along the negative $z$-axis with its positive pole (with the strength $g$) at the origin:

$$B_{\text{sol}} = \frac{g}{4\pi r^2}n + g\theta(-z)\delta(x)\delta(y)\hat{z}, \quad (3.15)$$

where $\hat{z}$ is the unit vector in the $z$-direction. This magnetic field differs from the magnetic monopole field (3.12) by the singular magnetic flux along the solenoid, set by the second item in (3.15). Since the magnetic field given in (3.15) is source-free ($\nabla \cdot B_{\text{sol}} = 0$), one can write

$$B_{\text{sol}} = \nabla \times A. \quad (3.16)$$

Then from (3.12), (3.15), (3.16) one derives the monopole field given as

$$B = \frac{g}{4\pi r^2}n = \nabla \times A - g\theta(-z)\delta(x)\delta(y)\hat{z}. \quad (3.17)$$

The line occupied by the solenoid is called the Dirac string. The potential $A$, entering Eq. (3.17), singular along the negative $z$-axis, can be set as

$$A_x = \frac{-y}{r(r+z)}; \quad A_y = \frac{x}{r(r+z)}; \quad A_z = 0; \quad (3.18)$$

or

$$A_r = A_\theta = 0; \quad A_\phi = \frac{g}{r} \frac{1 - \cos \theta}{\sin \theta}. \quad (3.19)$$

in the spherical coordinates.

The Dirac string is a purely gauge artefact. So, for instance, if the Dirac string is located along the line $r = z$, instead of Eq. (3.19) one should write

$$A_r = A_\theta = 0; \quad A_\phi = -\frac{g}{r} \frac{1 + \cos \theta}{\sin \theta}. \quad (3.20)$$

The only physical singularity of the potential $A$ is its singularity at the origin of coordinates $r = 0$. This shows clearly that the Dirac monopole (3.17) is an example of point (hedgehog) topological defects (see e.g. §Φ1 in [9] and the discussion in [21]).

Now we get down directly to constructing Dirac variables in AM.

To do this, let us study the quantum behaviour of a charged particle (with the elementary charge $e$) in the magnetic monopole field. Its wave function is

$$\psi = |\psi| \exp[i\frac{\hbar}{\mu}(p \cdot r - Et)]. \quad (3.21)$$
When the magnetic monopole field is "switched on", we have $\mathbf{p} \to \mathbf{p} - (e/c)\mathbf{A}$ (where $\mathbf{A}$ is given in (3.18) or in (3.19)), and

$$\psi \to \psi \exp\left(-\frac{ie}{\hbar c}\mathbf{A} \cdot \mathbf{r}\right),$$

i.e. the change in the phase $\alpha$ of the wave function (3.21),

$$\alpha \to \alpha - \frac{e}{\hbar c}\mathbf{A} \cdot \mathbf{r} \quad (3.22)$$

occurs.

Let us consider now a closed contour at the fixed $r, \theta, \phi \in [0, 2\pi]$. Then the complete change in the phase $\alpha$ will be

$$\Delta \alpha = \frac{e}{\hbar c} \oint \mathbf{A} \cdot d\mathbf{l} = \frac{e}{\hbar c} \int \text{rot} \mathbf{A} \cdot d\mathbf{S} = \frac{e}{\hbar c} \int \mathbf{B} \cdot d\mathbf{S} = \frac{e}{\hbar c} \Phi(r, \theta), \quad (3.23)$$

where $\Phi(r, \theta)$ is the flux across the part of the sphere specified by some values of $r$ and $\theta$. As $\theta$ is changed, the flux across this part of the sphere is also changed. So, as $\theta \to 0$, the contour is shrunk to a point, and the flux passing across this part of the sphere goes to zero, $\Phi(r,0) = 0$.

As the contour is increased, the flux is also increased, and at last as $\theta \to \pi$,

$$\Phi(r, \pi) = \int \mathbf{B} \cdot d\mathbf{S} = 4\pi r^2 B = 4\pi g \quad (3.24)$$

due to (3.12).

But since at $\theta \to \pi$ the contour again is shrunk to a point, the potential $\mathbf{A}$ should be singular at $\theta = \pi$ in order for $\Phi(r, \theta)$ to be finite. And moreover, this conclusion is correct at any value of $r$, i.e. for a sphere of any radius; thus the potential $\mathbf{A}$ is singular along the negative half-axis $z$. This is an alternative deriving the above results (3.18), (3.19). It is obvious herewith that the Dirac string can be located along any direction (and, generally speaking, it is not definitely that it is a straight line, but it should be continuous; in other words, it should be a Jordan curve).

Note that the discussed singularity of the potential $\mathbf{A}$ implies the so-called "Dirac veto" [39]: the wave function $\psi$ goes to zero on the negative half-axis $z$. Therefore, its phase along this line is not specified and it follows from Eq. (3.23) that there are no necessity in the condition $\Delta \alpha \to 0$ at $\theta \to \pi$. But $\psi$ should be an uniquely defined function; thus the equality

$$\Delta \alpha = 2\pi n, \quad n \in \mathbb{Z},$$

should be satisfied.

Then from (3.23), (3.24) one derives

$$2\pi n = \frac{e}{\hbar c} 4\pi g$$
or
\[ eg = \frac{1}{2} nhc \]  
(3.25)

(note that it is precisely Eq. (1.37) in the \( h = c = 1 \) system of units; here is \( g \) instead of \( m \) stands for the magnetic charge).

The evident dualism between the radial magnetic field \( (3.12) \) and the tension \( E \) in usual electrostatics suggests the "Coulomb-like" behaviour of the force between two magnetic charges \( g_1 \) and \( g_2 \), just
\[ F = \frac{g_1 g_2}{r^2}. \]  
(3.26)

It can be also concluded that two magnetic charges of different signs are attracted while those with same sign repel. It is easy to see herewith that the "algebra" of magnetic charges coincides with that of integer numbers \( \mathbb{Z} \).

Now the look for the Dirac variables in AM can be derived by analogy with (2.36) \[20\], involving the gauge factor \( v[A] \), (2.37). Herewith the finiteness of the phase \( \Phi(r, \theta) \), (3.23), should be taken into account. Also it is important that now the gauge potential \( A \) is stationary \[29\], generating the Dirac monopole; thus instead of (2.37) it should be a stationary multiplier, as it was in the non-Abelian YMH model (quantized by Dirac), involving vacuum BPS monopole modes (us discussed briefly in Introduction).

Meanwhile, if the gauge potential \( A \) is given by Eqs. (3.19) or (3.20) \[39\], the Gauss law constraint (2.24) remains formally valid, but now its r.h.s. becomes
\[ \Delta A_0 = j_0 \]  
(3.27)

if the fermionic current \( j_0 \) exists. But if the latter one is "turned off", we encounter the simple homogeneous Laplace equation
\[ \Delta A_0 = 0, \]  
(3.28)

permitting the above citing solution (2.68) \[27\].

On the other hand, now (when the duality transformations (3.9) are performed, implying the Dirac conjecture (3.25) \[40\] is correct) there is no any sense to set \( A_0 = 0 \) for nontrivial topologies \( n \neq 0 \) (identically or by performing some transformations). Now we shall attempt to ground this.

If \( A_0 = 0 \), the electric tensity
\[ E = \partial A / \partial t + \nabla A_0 \]
turns into a zero field over the stationary monopole configuration (3.18) (when another vector potentials: say, electromagnetic, are absent). This means the absence of (stationary) electric charges $Qe \equiv Q$ (at setting $e = 1$) \[30\] (which serve, if exist, as sources of [electrostatic] fields) in the model. According Eq. \[3.25\] this implies that only the ”especial” case of the trivial topology $n = 0$ has right to exist without of essential problems; otherwise one encounter infinite magnetic charges $g$, and then the magnetic field $B$ can be finite/infinite (instead of to be zero) at the spatial infinity (this is according to \[3.12\]). This, in turn, can lead to the infinite energy integral
\[\sim \int d^4x (\tilde{F}^{\mu
u})^2\]
as a consequence.

And also all this implies the bad renormgroup properties of the theory in question and divergences in appropriate Feynman diagrams involving different degrees of $g$. And vice versa, the chance to get a good perturbation theory appears apparently at including (at nontrivial topologies) Feynman diagrams with $g$ as well as with $e$. Likely, in this case, a fine interplay between these diagrams implies a cancelation of divergences.

Thus if we desire to preserve nontrivial topologies $n \neq 0$ in our Abelian $U(1)$ model, we should rule out the gauges in which $A_0 = 0$ (the ”especial” case $n = 0$ in which only magnetic charges can exist is, indeed, an interesting case, we shall discuss below).

This means that ”turning off” the charge $j_0$ in the Gauss law constraint \[3.27\] has no physical sense in the us discussed theory, generally speaking (except the trivial topology case $n = 0$).

In the said is the specific of stationary monopole solutions \[3.18\] appearing in the ”complete” (topologically nontrivial) $U(1)$ gauge model in which the Dirac conjecture \[3.25\] is assumed. In this is the essential difference of this model from ”pure” constraint-shell QED \[2.48\] \[27\] permitting solutions (Dirac variables) depending manifestly on time.

Now we can begin directly with writing down Dirac variables in the above topologically nontrivial $U(1)$ gauge model involving the monopole configuration \[3.18\]. The experience of constraint-shell QED \[26\] \[27\] and Minkowskian YMH model with BPS monopoles quantized by Dirac \[24\] allows us to do this.

First of all, it is easy to see that the analogue of the Gribov phase $\hat{\Phi}_0(x)$ in the Minkowskian YMH model with BPS monopoles quantized by Dirac, in the $U(1)$ gauge model with Dirac monopoles will be the value $\Phi(r, \theta)$ \[39\]. This can be interpreted as a stationary phase for (topological) Dirac variables in the investigated $U(1)$ gauge model involving Dirac monopole modes.

As a result, by analogy with $\hat{A}_k^{(n)}$ in the non-Abelian gauge model, in our case the appropriate (topological) Dirac variables can be written down as

\[^{30}\text{Instead of the Dirac quantization conjecture \[40\] given in shape \[3.29\], one utilize very often its shape } \frac{Qg}{4\pi} = \frac{1}{2}n.\]

Grounding the latter equation is given, for instance, in the monograph \[38\] in §15.1).
\[ A^\mu_{Dm(n)} = v^{Dm(n)}(x)(A^{(0)}_\mu + \partial_\mu)v^{Dm(n)}(x)^{-1}, \quad v^{Dm(n)}(x) = \exp[n\Phi(r, \theta)]; \quad \mu = 0, 1, 2, 3. \]  

(3.29)

Here \( A^{(0)}_\mu \) is the topologically trivial field configuration (in which the electric charge takes arbitrary values while the magnetic charge of such a configuration is zero). From Eqs. (3.18) - (3.20) it can be concluded that the spatial components \( A^\mu_{Dm(0)} (i = 1, 2, 3) \) are exactly zero since now \( g = 0 \) is fixed, even in the singularity point \( r \to 0 \). But their topological copies \( A^\mu_{Dm(n)} \) in another sectors can be different from zero due to the presence of the value \( \Phi(r, \theta) \) in (3.29). \(^{31}\)

On the other hand, since Dirac variables in any model are gauge invariant inherently, the transformation (3.29) can be treated as a map from the zero to the \( n^{th} \), nontrivial, topological sector of the model discussed. \textit{But it is not a gauge transformation!} The same applies to Dirac variables in any model with nontrivial topologies, for instance, the non-Abelian YMH model \([1, 6, 10, 20, 30]\). In that case the ”large” Dirac variables \( A^D_{Dm(n)} \) is the image of ”small” \( A^{(0)}_k \) at the topological map of such kind.

As it was noted in \([69]\), there exists a simple mathematic model (see Lecture 2 in \([70]\)) which describes correctly such a topological map. In the studied case, the base of the covering consists of all the topologically trivial gauge fields \( A^{(0)}_\mu \), while its discrete infinitely-valent fibre is the set of all fields \( A^D_{Dm(n)} \).

As regards the temporal components \( A^D_{Dm(n)}(0) \) of the Dirac variables (3.29), these \( \text{different from zero} \) due to the above reasoning, as it is easy to understand, satisfy the Gauss law constraint (3.27), that is the Poisson equation, permitting purely stationary \( \text{O}(1/r) \) solutions. Thus they are associated with electric charges inherent in model we study now. And moreover, as it can be seen from Eq. (3.29) for a potential \( A^D_0 \), the topological copies with \( n \neq 0 \) for the topologically trivial temporal potential \( A^{(0)}_0 \) \text{coincide} with this \( A^{(0)}_0 \) since we deal in the considered model with the topological multipliers \( v^{Dm(n)}(x) \) which are commute and are independent on \( x_0 \).

The proof that these variables (3.29) are gauge invariant is similar to that \([26]\) in constraint-shell QED (see the calculations (2.40)), with only constructive addition that now there is no necessity in the nonstationary gauge matrices (such as \( g \), (2.10) in constraint-shell QED). Now the gauge matrices can be chosen to be purely stationary, say

\[ \tilde{g}(x) \equiv \exp(i\Lambda(x)) \]  

(3.30)

Also, formally, the condition \( \partial_\mu \partial^\mu \Lambda = 0 \) can be imposed onto \( \Lambda(x) \).

In contrast to the Dirac variables in constraint-shell QED or in the Minkowskian YMH model with BPS monopoles quantized by Dirac, the Dirac variables (3.29) in the \( U(1) \)

\(^{31}\)Such look for the topological multipliers \( v^{Dm(n)}(x) \) as in (3.29) is not a quite mandatory because \( \Phi(r, \theta) \) is a function of the topological number \( n \): this is according to Eqs. (3.24), (3.25). So, indeed, \( \Phi(r, \theta) \equiv \Phi_1(r, \theta, n) \) and the look \( v^{Dm(n)}(x) = \exp[\Phi_1(r, \theta, n)] \) for the above topological multipliers \( v^{Dm(n)}(x) \) is quite acceptable.
gauge model with Dirac monopoles are not transverse automatically but such condition can be imposed onto these variables. This involves some cumbersome mathematics affecting the monopole configuration \((3.18)-(3.20)\) [39] and the ”Gribov phase” \(\Phi(r, \theta)\). We omit them in the present study.

Note that \((3.29)\) is the purely stationary field configuration, in contrast to \(\hat{A}_k^D\) in the Minkowskian YMH model with BPS monopoles quantized by Dirac.

Now (upon grounding the gauge invariance of the Dirac variables \(A_{\mu}^{Dm(n)}\)), it becomes obvious that the gauge \(A_0^{Dm(0)} = 0\) (can exist in the unique case of arbitrary magnetic charges \(g\), zero electric charges and the zero topology \(n = 0\)) gives the gauge \(A_0^{Dm(n)} = 0\) in all the nontrivial topological sectors: it is according to Eq. (3.29). But such topologically nontrivial field configurations should be ruled out as those can lead to the infinite Hamiltonian (Lagrangian) density because of the now becoming infinite magnetic charges \(g \sim n/\infty\).

Due to the gauge invariance principle for the Dirac variables \(A_{\mu}^{Dm(n)}\), (3.29), we can write down for the model Hamiltonian \(H\):

\[
H(A_{\mu}^{Dm(0)}) = H(A_{\mu}^{Dm(n)}) \tag{3.31}
\]

The latter equation implies ruling out also the topologically trivial field configuration \(A_0^{Dm(0)}\). Thus one can suppose that purely magnetic and electrically neutral particles, creating the ”radial” magnetic field \(B\) according to Eq. (3.12) (such particles can be referred as ”magnons”), can not exist, probably, in the Nature.

Indeed, as we’ll argue now, the manifest gauge invariance (3.31) of the model Hamiltonian \(H\) prohibits, in general, the existence of magnetic charges in the Nature if the look (3.29) for the topological Dirac variables in the \(U(1)\) gauge theory is assumed.

Really, from Eq. (3.18) it follows that in the zero topological sector at an arbitrary electric charge \(Q \neq 0\) (then \(g = 0\) due to (3.25)), the vector potential \(A\) [38, 39], (3.18), generating the magnetic monopole solution (3.17), becomes zero (even in the physical singularity point \(r \to 0\) since \(g = 0\) is fixed).

Then the gauge invariance of the model Hamiltonian, \(H(A_{\mu}^{Dm(0)}) = H(A_{\mu}^{Dm(n)})\), implies, as it is easy to see, the impossibility to observe magnetic charges (but only electric charges!) if the shape (3.29) for the topological Dirac variables is assumed. The situation remind us that in the non-Abelian YMH gauge model (with vacuum BPS monopoles) quantized by Dirac. As it was argued in Ref. [20], due to the gauge invariance, \(H[A^{(n)}, q^{(n)}] = H[A^{(0)}, q^{(0)}]\), the QCD Hamiltonian \(H\) does not depend on the Gribov phase factors \(v^{(n)}(x)\) and ”it contains the perturbation series in terms of only the zero map fields (i.e., in terms of constituent color particles) that can be identified with the Feynman partons” [20] (in other words, only topologically trivial quarks \(q^{(0)}\) and gluons \(A^{(0)}\) can be observed).

In order to maintain in these circumstances the Dirac conjecture (3.25) [40] about the quantization of the magnetic charge, the following way out seems to be quite reasonable.
We propose to construct the topological (Gribov) copies of the potentials (3.18) containing manifestly the magnetic charge $g$ and generating the Dirac monopole given in Eq. (3.17). Note herewith that alone the gauge covariant potentials (3.18) are topologically nontrivial since they contain $g$ manifestly and then depend on the apropriate topological number $n$ via Eq. (3.25) [40].

By analogy with (3.29), we write now

$$A_i^{Dm(n)} = v^{Dm(n)}(x)(A_i^{(n)} + \partial_i) v^{Dm(n)}(x)^{-1}; \quad i = 1, 2, 3,$$

(3.32)

where now $A_i^{(n)}$ are the spatial potentials given in Eq. (3.18).

The same “transformation law” can be written down also for the temporal components of the potentials $A$: separately in each topological sector of the considered model:

$$A_0^{Dm(n)} = v^{Dm(n)}(x)(A_0^{(n)} + \partial_0) v^{Dm(n)}(x)^{-1} = A_0^{(n)}$$

(3.33)

since the topological multipliers $v^{Dm(n)}(x)$, (3.29), are the Abelian multipliers which commute with $A_0^{(n)}$ and due to their explicit look (they are manifestly stationary).

Note that setting $A_0^{(n)} = 0$ for any $n \neq 0$ in (3.33) implies $Q(n) = 0$ for this topology $n$ and then the infinite magnetic charges $g(n) \to n/\infty$ in this topological sector due to Eq. (3.25) [40]. As it was explained above, this can lead to the apropriate infinite energy integral and another undesirable consequences. Thus setting $A_0^{(n)} = 0$ is not profitable for us.

This has a very important consequence. At assuming $A_0^{(n)} = 0 \ (n \neq 0)$, one concludes easily that plane transverse waves (photons on the quantum level) are impossible for nontrivial topologies ($n \neq 0$) in the Abelian gauge $U(1)$ model involving magnetic charges. Such solutions, as it is well known (for example, §36 in [42]), must presume just the Coulomb gauge $A_0^{(n)} = 0 \ (n \neq 0)$. Thus, topologically nontrivial photons do not exist in the Abelian gauge $U(1)$ model involving magnetic charges.

Now it is possible to write down explicitly the Hamiltonian for the $U(1)$ gauge model involving magnetic monopoles now taking into account the transformation laws (3.32), (3.33). Such gauge invariant Hamiltonian is, indeed, a sum, running through all the topologies $n \in \mathbb{Z}$, of the Hamiltonians $H^{(n)}(A_i^{Dm(n)}, A_0^{(n)})$:

$$\mathcal{H} = \sum_n H^{(n)}(A_i^{Dm(n)}, A_0^{(n)}).$$

(3.34)

On the other hand, the gauge invariance of the Hamiltonian $\mathcal{H}$ implies

$$\mathcal{H} = \sum_n H^{(n)}(A_i^{(n)}, A_0^{(n)}),$$

(3.35)

where $A_i^{(n)}$ are the spatial potentials given in Eq. (3.18) and generating the Dirac monopoles [32].

---

[32] In this summing by topologies in the Hamiltonian $\mathcal{H}$ and also in fixing topological variables $A_i^{Dm(n)}$. 

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The notable feature of the quantization scheme \((3.32), (3.33)\) is that it \textit{permits} (unlike the quantization scheme \((3.29)\)) the existence of magnons (purely magnetically charged particles) in the zero topological sector of the considered model when \(A_{0}^{(0)} = 0\) and the magnetic charge \(g\) takes arbitrary, different from zero, values.

And vice versa, \(g = 0\) corresponds to the trivial topology \(n = 0\) with only arbitrary electric charges \(Q\) existing. This is just the electrostatics case when these electric charges \(Q\) induce the nonzero (electrostatic) potentials \(A_{0}^{(0)} \neq 0\).

It is easy to see now that "pure" constraint-shell QED \((2.48) [60]\), in which the electric charges are absent and which is the direct result of the removal \((2.49)\) of the temporal potentials of electromagnetic potentials, supplements the topologically trivial "magnon sector" of the generalized \(U(1)\) gauge model (involving Dirac monopoles). The same is correctly also for transverse electromagnetic waves (photons).

And at last, note that constraint-shell QED \((2.53)\), involving fermionic currents, can be considered as the topologically trivial part of the mentioned generalized \(U(1)\) gauge model in which \(g = 0\) is set. Herewith the "temporal" potentials \(A_{0}^{(0)} \neq 0\) are set by means of Eq. \((2.61) [60]\), while the "spatial" (retarding) potentials \(A^{D}(x), (2.58)\), again supplement the former.

The said is in a good agreement with our "topological" survey \((3.2)-(3.4)\) grounding that the Dirac variables \((2.36) [20]\) belong to the zero topological sector of the \(U(1)\) group space. [33]

We finish our investigation of the Hamiltonian \((3.35)\) by writing it explicitly. Note firstly that the Lagrangian density for the bosonic sector of the us investigated model has the look \([68]\)

\[
\mathcal{L}_{\text{bos}} = -\frac{1}{16\pi} F_{\mu\nu} F_{\mu\nu}
\]

according to \((3.3)\). Herewith in the product of two tensors \(F_{\mu\nu}\) we hold only the terms \(\tilde{F}_{\mu\nu} \tilde{F}_{\mu\nu}\) and \(F_{\mu\nu} F_{\mu\nu}\). The reasoning here is in neglecting (on the classical level) the interaction between the Maxwell electromagnetic field \(F_{\mu\nu}\) and its dual, \(B \equiv \tilde{F}_{\mu\nu}\) (that is the magnetic monopoles configuration). Such interaction refers to the quantum corrections of the fourth order (more exactly \(O(g\varepsilon^{4})\)), by analogy with the photon-photon scattering process in QED, involving four (virtual) fermions (see §41.1 in [41]).

\((3.32)\), in each topological sector of the \(U(1)\) Abelian model is the principled difference of this model from the quantized by Dirac YMH non-Abelian model \([1,6,10,20,30]\) with vacuum BPS monopole solutions. Here we are interested in fact only in the topologically trivial and gauge invariant Hamiltonian \(H[A^{(0)}, q^{(0)}]\), describing correctly the quark confinement \([20]\).

\(33\)Indeed, the topology \((3.2)-(3.4)\) relates equally to the Dirac potentials \(A^{D} = (A_{0}^{T}, A_{D}^{i})\) as well as to the conventional Coulomb-like electrostatic potentials \(\phi \equiv A_{0}\), obeying the Coulomb law in the shape \([9]\)

\[
\text{div grad } \phi = \text{div } B \neq 0
\]

and also to the magnetic monopole field \(B = \tilde{F}_{\mu\nu}\) (generating the differential 2-form \(\Omega = \tilde{F}_{\mu\nu} dx_{\mu} \wedge dx_{\nu}\)).

The important point in the both cases is the singularity of the physical fields at the origin of coordinates generating the topology \((3.3)\) (for details see § T7 (p. 278) in the monograph \([9]\) or p. 653 in the monograph \([71]\)).
Since $A^{(n)} = 0$ $(i = 1, 2, 3)$ for the stationary configuration (3.18) (or (3.19)) and since the canonical momenta
\[
\tilde{p}_n = \frac{\partial L_{\text{bos}}}{\partial \dot{A}^{(n)}_0} = 0; \quad n \neq 0
\]
(and the same is correct for the canonical momentum $p_0$ conjugate to $A_0$; as a result, the Gessian matrix $M$ of the investigated $U(1)$ Abelian gauge model is again degenerate, as in ordinary QED, for instance) [2], one can write down for the Hamiltonian $H$:
\[
H = p_0 \dot{A}_0 + \sum_{n \neq 0} \tilde{p}_n A^{(n)}_0 + (F^{i0})^2 + \tilde{p}_i A^{(n)}_i - L_{\text{bos}}. \quad (3.37)
\]
Here, obviously, [2]
\[
F^{i0} = E^i = p^{i0}.
\]
All this results
\[
H^{(n)}(\tilde{A}^{(n)}_i, \tilde{A}^{(n)}_0) = \frac{1}{16\pi} \tilde{F}_{ij(n)} \tilde{F}^{ij(n)}; \quad n \neq 0,
\]
where $\tilde{F}_{ij(n)}$ is the part of the tension tensor $F$ corresponding to the topology $n$.

If $n = 0$,
\[
H^{(0)}(A^{(0)}_i, A^{(0)}_0, \tilde{A}^{(0)}_i) = \frac{1}{16\pi} F_{ij} F^{ij} + \frac{1}{8\pi} F_{i0} F^{i0} + \frac{1}{16\pi} \tilde{F}_{ij(0)} \tilde{F}^{ij(0)}. \quad (3.39)
\]
The latter term in (3.39) takes account of the ”magnon” contribution (arbitrary $g$ at zero $Q$) into the zero topological sector of the discussed model, while first two describe correctly the ”purely Maxwell theory” (in particular, QED). As for the mixed items $\sim F_{ij} \tilde{F}^{i(0)}$ or $F_{i0} \tilde{F}^{i(0)}$ these give a disapearing contribution into the Hamiltonian $H^{(0)}(A^{(0)}_i, A^{(0)}_0, \tilde{A}^{(0)}_i)$ due to the reasoning given above.

Now let us analyse the fermionic sector of the Abelian $U(1)$ gauge model involving Dirac monopoles and how to incorporate the Dirac variables into this sector.

The question about the Dirac variables in the fermionic sector of the ”complete” $U(1)$ gauge model is very interesting and important. In the light of the ”dyon/magnon picture”, we assume the shape
\[
k_\mu = g \gamma_\mu \bar{\psi} \psi, \quad (3.40)
\]
with $\gamma_\mu$ being the usual Dirac matrices and $\psi, \bar{\psi} = \gamma_0 \psi$ being the fermionic field possessing the 1/2 spin for the magnetic current $k_\mu$ (which describe correctly magnons as well as dyons depending on the topological number $n$).

This assumption is quite natural and obvious since the Dirac conjecture [40] does not affect the Lorentz and spinoral properties of values involved in the $U(1)$ gauge model with Dirac monopoles.

\[\text{Now we write down the symbol ”tilde” over $A$ in order to distinguish the ”Maxwell” and its dual tensors.}\]
The "technology" writing down Dirac variables in the fermionic sector of the Abelian $U(1)$ gauge model involving Dirac monopoles is the same as in the gauge fields case discussed above (and also we have the pattern how to write down the Dirac variables in constraint-shell QED, see Section 2).

So we write down

$$
\psi'^{(n)} = v^{Dm(n)}\psi_0^{(n)},
$$

(3.41)

with $\psi_0^{(n)}$ being the "initial", topologically nontrivial, data for the fermionic field $\psi$, in the $n$th sector of the $U(1)$ gauge model. Herewith it is important to note that the field $\psi_0^{(n)}$ is gauge covariant while the Dirac variable $\psi'^{(n)}$ is gauge invariant.

Since the both electric and magnetic charges should always be present in the us discussed "complete" $U(1)$ gauge model, this, by analogy with QED, allows us now to write down the model Hamiltonian. This should have the shape

$$
H = \left( \sum_n H^{(n)} \right) - i\bar{\psi}_0^{(0)}(\partial_i - ieA_i^{(0)} - ig\tilde{A}_i^{(0)})\psi_0^{(0)} - i \sum_{n \neq 0} \bar{\psi}_0^{(n)}\gamma^i(\partial_i - ig\tilde{A}_i^{(n)})\psi_0^{(n)} - i \sum_{n \neq 0} \bar{\psi}_0^{(n)}\gamma^0(\partial_0 - ieA_0^{(n)})\psi_0^{(n)} - i \sum_{n \neq 0} \bar{\psi}_0^{(n)}\gamma^0(\partial_0 - igA_0^{(n)})\psi_0^{(n)} + M(\psi, \bar{\psi}) + \text{mixed items}
$$

(3.42)

Here $\sum_n H^{(n)}$ stands for the complete "bosonic" Hamiltonian including the items (3.38), (3.39) and $M(\psi, \bar{\psi})$ is the fermionic mass item. The remarkable point of Eq. (3.42) is also that $A_i^{(0)}$ are the electrostatic potentials, as it was argued above. Also we note that only "initial" values $\psi_0^{(n)}$ ($n \in \mathbb{Z}$) of fermionic fields enter the Hamiltonian (3.42) due to its manifest gauge invariance.

The "mixed items" in the last equation are very important and have the following meanings. There are the items describing the interactions between the fermions (including the coulombic one) via gauge fields belonging to different topological sectors of the considered model.

Concluding this section, the author should like express his opinion about the following objection against existing magnetic charges (and magnetic monopoles, as a consequence) in Nature.

So, for instance, in the paper [72], it was argued that the transformations

$$
\begin{align*}
  j_\mu &\rightarrow j_\mu \cos \theta + k_\mu \sin \theta; \\
  k_\mu &\rightarrow -j_\mu \sin \theta + k_\mu \cos \theta
\end{align*}
$$

(3.43)
for the electric/magnetic currents leave invariant the equations of motion (3.6), the Lorentz force
\[ f^\nu = j_\mu F^{\mu\nu} + k_\mu \tilde{F}^{\mu\nu} \] (3.44)
and also the appropriate energy-momentum tensor for the electromagnetic field if a dyon is involved.

Thus the question about the parameter \( \theta \), entering (3.43), with fixing its concrete value is rather a matter of convention but not of an experimental choice.

If now one considers a totality of dyons for which the \( g/Qe \) ratio has the same arbitrary chosen value, then the above parameter \( \theta \) can be connected with this ratio, for instance, as
\[ \theta = \arctan(g/Qe). \] (3.45)
Then with the aid of the dual rotations
\[ F_{\mu\nu} = (QeF_{\mu\nu} + g\tilde{F}_{\mu\nu})/q = F_{\mu\nu} \cos \theta + \tilde{F}_{\mu\nu} \sin \theta; \]
\[ \tilde{F}_{\mu\nu} = (Qe\tilde{F}_{\mu\nu} - gF_{\mu\nu})/q = -F_{\mu\nu} \sin \theta + \tilde{F}_{\mu\nu} \cos \theta, \] (3.46)
where \( q = \sqrt{Q^2e^2 + g^2} \), we come to the Maxwell equations with the one kind of sources:
\[ \partial^\nu F_{\mu\nu} = qj_\mu; \quad \partial^\nu \tilde{F}_{\mu\nu} = 0 \] (3.47)
and to the “usual” Lorentz force acting onto the trial charge \( q \),
\[ f^\nu = qj_\mu F^{\mu\nu}. \] (3.48)
Thus, formally, we have gone over (with the aid of the dual rotations (3.46)) to the usual Maxwell electrodynamics involving a one effective charge \( q \) from the electrodynamics involving dually charged particles with the universal ratio \( g/Qe \). This means that these both forms are equivalent.

The formal possibility of such going over has a profound physical justification. Since the presence of a field can be discovered only by its impact on a charged body, while the charge of a particle can be identified, in turn, only with the aid of the field, then only the interaction effects between charges and fields can be treated as immediately observable (measurable) phenomena, but not charges and fields taken separately. Therefore, it is impossible, in principle, to establish a difference between the effects (3.43)- (3.44) and (3.47)- (3.48) if one identifies the effective charge \( q \), (3.46), with the observable electric charge.

But, as it was stressed in [72], the dual rotation (3.46) leads to the electrodynamics with only a one charge for all the sources involving, only when the ratio \( g/Qe \) is the same for all the particles. Otherwise, going over to the system with an effective charge is possible only for the particles of the one kind: say, \( q_1 = \sqrt{(Q_1e_1)^2 + g_1^2} \). The particles with another ratio \( g/Qe \) \( (g_2/(Q_2e_2) \neq g_1/(Q_1e_1)) \) will possess both the electric, \( e' \), and magnetic, \( g' \), charges [73].
\[ e' = q_2 \cos(\theta_2 - \theta_1); \]
\[ g' = q_2 \sin(\theta_2 - \theta_1), \]
(3.49)

where \( q_2 = \sqrt{(Q_2 e_2)^2 + g_2^2} \), \( \theta_i = \arctan(g_i/Q_i e_i) \) \((i = 1, 2)\).

Thus it becomes evident that the universality of the ratio \( g/Qe \) purchases a crucial importance in electrodynamics of dually charged particles: if this ratio is the same for all the particles, the observable magnetic charge is absent.

The situation changes drastically when nontrivial topologies are involved: when one considers the abelian \( U(1) \) gauge theory.

This becomes obvious at examining the Dirac quantization condition

\[ \frac{Q_n g_n}{4\pi} = \frac{1}{2} n; \quad n \in \mathbb{Z}, \]
(3.50)

which is the generalization of the standard Dirac quantization condition \[3.25\] \[38, 39, 40\] onto the case when the electric charge \( Q_n \) and the magnetic one, \( g_n \), are involved in the \( n^{th} \) topological sector of the abelian \( U(1) \) gauge model (for instance, these charges are relevant to some dyon with the topological charge \( n \)).

As a consequence of Eq. (3.50),

\[ Q_n = n \frac{2\pi}{g_n} \iff \frac{Q_n}{g_n} = \frac{2\pi n}{g_n^2} \]
(3.51)

and these relations are different in each topological sector. This allows for observable magnetic charges to appear in the abelian \( U(1) \) gauge theory due to the above arguments [72].

Existence of nontrivial topologies is the distinctive feature of the world in which we live. In particular, such nontrivial topologies are inherent in the abelian \( U(1) \simeq S^1 \) gauge theory since \( \pi_1 S^1 = \mathbb{Z} \). So, to throw these nontrivial topologies (involving magnetic monopole configurations) and to leave only the trivial one, \( n = 0 \), which describe correctly Maxwell/quantum electrodynamics, it would be irrational.

4 Discussion.

In this last section of our study we should like stop on the topic of Section 3, on the author’s opinion, the most important in this article [35].

35 As to constraint-shell QED, we study in Section 2, it was rather the review of that made in the earlier papers [20, 60, 65]. The only point deserves here the especial our attention. It is the possibility to represent the Dirac variables \( \mathbf{A}_D(x) \) in the shape (2.57) of retarding potentials. This bridges constraint-shell QED [20, 60, 65] with the Feynman theory (more specifically, with the \( < f| \Sigma^{(2)}| i > \) matrix elements involving two fermionic currents)
In Section 3 the "complete" $U(1)$ gauge theory involving Dirac monopole configurations \((3.18)\) - \((3.20)\) \cite{38, 39} was discussed and its Dirac fundamental quantization was performed, leading us to the "reduced" Hamiltonian \((3.42)\).

The principal conclusion which can be drawn from this is that constraint-shell QED \cite{20, 60, 65} involving Dirac variables is, in fact, only some topologically trivial part of this "complete" theory. In this some analogy can be observed with the "complete" liquid He\(^4\) model. There, as it is well known, superfluidity in He\(^4\) \cite{64, 65} corresponds to the trivial $n = 0$ topology while at $n \neq 0$ vortices arise in a liquid He\(^4\) specimen (see the monograph \cite{76} and the discussion in the recent paper \cite{21}).

More in detail, the appearance of rectilinear vortices in a liquid helium II specimen is set by Eq.

$$n = \frac{m}{2\pi \hbar} \int_{\Gamma} \mathbf{v}^{(n)} d\mathbf{l}; \quad n \in \mathbb{Z}.$$  

This Eq. implicates the helium mass $m$ and the tangential velocity $\mathbf{v}^{(n)}$ of a rectilinear vortex. At $n = 0$, as it can be seen transparently from this Eq., the integral in its r.h.s. disappears. This means that the trivial topology $n = 0$ corresponds to irrotational, superfluid motions, $\text{rot} \mathbf{v}^{(n)} = 0$, inside the liquid helium II specimen.

Further, two kinds of (gauge) field configurations were us found with respect to the electric/magnetic charges ascribed to these configurations. For nontrivial topologies $n \neq 0$ there are dyon configurations, involving electric as well as magnetic charges (such configurations are, actually, well known in theoretical physics), while the trivial topology $n = 0$ permits purely magnetic states (we have called them "magnons" in the present study).

The interesting point of the model we discussed is the possibility of the topological expansion for the Hamiltonian \((3.42)\). In other words, any energy integral can be expanded by subintegrals equipped by topological numbers $n$.

The next interesting example permitting a “topological” expansion is the Bogomol’nyi bound \cite{9}

$$E_{\text{min}} = 4\pi m \frac{a}{g}; \quad a \equiv \frac{m}{\sqrt{\lambda}};$$

for the BPS monopole vacuum configuration (with $m$ and $\lambda$ being the Higgs mass and self-interaction constant, respectively, taken in the BPS limit \cite{7} $m \to \infty$, $\lambda \to \infty$).

In the latter equation the dependence of $E_{\text{min}}$ on the topological number $n$ originates from the dependence on $n$ of the magnetic charge $m$. Following \cite{9}, the latter one can be given (indeed, upon some fitting) as

$$m(\Phi, A) = C \zeta(\Phi, A), \quad \zeta(\Phi, A) \in \mathbb{Z}$$

for a magnetic monopole vacuum (Higgs-YM) configuration $\Phi, A$.

Here $C = \nu/4\pi$, where $\nu$ can be found from the conditions

$$\exp(\nu \hbar) = 1; \quad \exp(\lambda \hbar) \neq 1$$
\((h \equiv h(\Phi) \equiv \Phi/a)\) as \(0 \leq \lambda \leq \nu\). From the geometrical point of view, \(\nu\) is characterized as the length of the circle \(U(1) \simeq S^1\) (of the unit radius).

If \(H\) is the survived symmetry group in the considered model (in the present study we consider \(H = U(1)\)) and if \(t(h)\) is a representation of its Lee algebra, then the operator \(t(h)\) has the system of eigenvalues \(\{\lambda_k\}\) (since the operator \(h\) is anti-Hermitian, its eigenvalues \(\{\lambda_k\}\) are imaginary).

It follows from the relation \(\exp(\nu h) = 1\) that

\[
T(\exp(\nu h)) = \exp(\nu \ T(h)) = 1.
\]

Therefore for all the eigenvalues \(\lambda_k\) of the operator \(t(h)\) the equality \(\exp(\nu h) = 1\) is satisfied.

Whence

\[
\nu \lambda_k = 2\pi n i, \quad n \in \mathbb{Z};
\]

On the other hand, if \(\Phi = \sum_k \Phi^k f_k\), where \(f_1, \ldots, f_n\) are eigenVectors of the operator \(t(h)\) and if only the "electromagnetic" part of the gauge field \(A_\mu: a_\mu\) is different from zero (i.e. \(A_\mu = a_\mu h\)), then

\[
D_\mu \Phi \equiv \partial_\mu \Phi + t(a_\mu h) \Phi = \sum_k (\partial_\mu \Phi^k + \lambda_k a_\mu \Phi^k) f_k.
\]

Thus the electric charge entering this covariant derivative is inferred to be \(2\pi n/\nu\).

Since magnetic charges are integer multiples of the number \(\nu(4\pi)^{-1}\) \([9]\), the product of the electric charge \(e\) of a particle onto the magnetic charge \(m\) of (another) particle is a half-integer:

\[
e m = \frac{1}{2} n, \quad n \in \mathbb{Z},
\]

in agreement with the Dirac hypothesis \([40]\).

The author recognizes that his study of the Dirac "fundamental" quantization of the Abelian \(U(1)\) gauge model is only the first little step in this direction and that a rather large job awaits here.

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