Damped quantum harmonic oscillator: density operator and related quantities

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Abstract

A closed expression for the density operator of the damped harmonic oscillator is extracted from the master equation based on the Lindblad theory for open quantum systems. The entropy and effective temperature of the system are subsequently calculated and their temporal behaviour is surveyed by showing how these quantities relax to their equilibrium values. The entropy for a state characterized by a Wigner distribution function which is Gaussian in form is found to depend only on the variance of the distribution function.

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1 Introduction

In the last two decades, the problem of dissipation in quantum mechanics, i.e. the consistent description of open quantum systems, was investigated by various authors \([1-5]\) (for a recent review on open quantum systems see ref. \([6]\)). It is commonly understood \([4, 7]\) that dissipation in an open system results from microscopic reversible interactions between the observable system and the environment. Because dissipative processes imply irreversibility and, therefore, a preferred direction in time, it is generally thought that quantum dynamical semigroups are the basic tools to introduce dissipation in quantum mechanics. In the Markov approximation the most general form of the generators of such semigroups was given by Lindblad \([8]\). This formalism has been studied for the case of damped harmonic oscillators \([3, 10]\) and applied to various physical phenomena, for instance, the damping of collective modes in deep inelastic collisions in nuclear physics \([11]\). In \([12]\) the Lindblad master equation for the harmonic oscillator was transformed into Fokker-Planck equations for quasiprobability distributions and a comparative study was made for the Glauber \(P\), antinormal ordering \(Q\) and Wigner \(W\) representations. In \([13]\) the density matrix for the coherent state representation and the Wigner distribution function subject to different types of initial conditions were obtained for the damped harmonic oscillator. A remarkable feature resulting from the preceding analysis \([14]\) is the agreement in form of the Lindblad master and Fokker-Planck equations with the corresponding equations familiar from quantum optics. Quantum optics is generally formulated with the help of the quantum mechanics of the damped harmonic oscillator \([4, 7]\) and the corresponding Brownian motion. Generally, the equation of motion treated in the theory of Brownian motion is the master equation satisfied by the density operator or the Fokker-Planck equation for the distribution function. The master and Fokker-Planck equations related to the problem of Brownian motion have been fully reviewed in the past \([4, 7, 15, 16]\).

In the present study we are also concerned with the observable system of a harmonic oscillator which interacts with an environment. The aim of this work is to explore further the physical aspects of the Fokker-Planck equation which is the \(c\)-number equivalent equation to the master equation. Generally the master equation for the density operator gains considerably in clarity if it is represented in terms of the Wigner distribution function which satisfies the Fokker-Planck equation. We shall describe, within the Lindblad theory for open quantum systems, how the system under consideration evolves to a final state of equilibrium by calculating first an explicit form of the density operator satisfying the operator master equation based on the Lindblad dynamics. We subsequently derive the entropy and the effective temperature of the quantum-mechanical system in a state characterized by a Wigner
distribution function which is Gaussian in form and show how these quantities relax to their equilibrium values.

Entropy is a quantity which may be interpreted physically as a measure of the lack of knowledge of the system. The idea of effective temperature associated with the Bose occupation distribution has already been introduced [17] in relation to the entropy which has been derived in the framework of quantum theory of harmonic oscillator relaxation. The derivation of the entropy associated with an infinite coupled harmonic oscillator chain has similarly been elaborated for classical [18] and quantum mechanical systems [19] represented by a phase space distribution function. In the present paper we derive first the density operator of the damped harmonic oscillator in the Lindblad theory by applying a procedure similar to the techniques developed in the description of quantum relaxation [17, 19, 20, 21].

When we denote by \( \rho(t) \) the density operator of the damped harmonic oscillator in the Schrödinger picture, the entropy \( S(t) \) is given by

\[
S(t) = -k \text{Tr}(\rho \ln \rho) = -k \sum_{m,n} <m|\rho|n> \ln <m|\rho|n>,
\]

where \( k \) is Boltzmann’s constant. While the explicit form of the general density matrix \( <m|\rho|n> \) is now available [14], the evaluation of the matrix element of the logarithmic operator \( \ln \rho \) is not an easy task. An alternative way of calculating the entropy is to compute straightway the expectation value of the logarithmic operator \( \langle \ln \rho \rangle = \text{Tr}(\rho \ln \rho) \). Accordingly, the problem amounts to derive the explicit form of the density operator.

The content of this paper is arranged as follows. In Sec. 2 we write the master equation for the density operator of the harmonic oscillator. Sec. 3 derives an explicit form of the density operator involved in the Lindblad master equation. Sec. 4 formulates the entropy and time-dependent temperature using the explicit form of the density operator and discusses their temporal behaviour. Finally, concluding remarks are given in Sec. 5.

## 2 Master equation for the damped quantum harmonic oscillator

The rigorous formulation for introducing the dissipation into a quantum mechanical system is that of quantum dynamical semigroups [4, 5, 6]. According to the axiomatic theory of Lindblad [8], the usual von Neumann-Liouville equation ruling the time evolution of closed quantum systems is replaced in the case of open systems by the following equation for the density operator \( \rho \):

\[
\frac{d\Phi_t(\rho)}{dt} = L(\Phi_t(\rho)).
\]
Here, $\Phi_t$ denotes the dynamical semigroup describing the irreversible time evolution of the open system in the Schrödinger representation and $L$ the infinitesimal generator of the dynamical semigroup $\Phi_t$. Using the structural theorem of Lindblad [8] which gives the most general form of the bounded, completely dissipative Liouville operator $L$, we obtain the explicit form of the most general time-homogeneous quantum mechanical Markovian master equation:

$$\frac{d\rho(t)}{dt} = L(\rho(t)), \quad (3)$$

where

$$L(\rho(t)) = -\frac{i}{\hbar} [H, \rho(t)] + \frac{1}{2\hbar} \sum_j (\{V_j \rho(t), V_j^\dagger\} + [V_j, \rho(t)V_j^\dagger]). \quad (4)$$

Here $H$ is the Hamiltonian of the system. The operators $V_j$ and $V_j^\dagger$ are bounded operators on the Hilbert space $\mathcal{H}$ of the Hamiltonian.

We should like to mention that the Markovian master equations found in the literature are of this form after some rearrangement of terms, even for unbounded Liouville operators. In this connection we assume that the general form of the master equation given by (3), (4) is also valid for unbounded Liouville operators.

In this paper we impose a simple condition to the operators $H, V_j, V_j^\dagger$ that they are functions of the basic observables $\hat{q}$ and $\hat{p}$ of the one-dimensional quantum mechanical system (with $[\hat{q}, \hat{p}] = i\hbar I$, where $I$ is the identity operator on $\mathcal{H}$) of such kind that the obtained model is exactly solvable. A precise version for this last condition is that linear spaces spanned by first degree (respectively second degree) noncommutative polynomials in $\hat{q}$ and $\hat{p}$ are invariant to the action of the completely dissipative mapping $L$. This condition implies [9] that $V_j$ are at most first degree polynomials in $\hat{q}$ and $\hat{p}$ and $H$ is at most a second degree polynomial in $\hat{q}$ and $\hat{p}$. Then the harmonic oscillator Hamiltonian $H$ is chosen of the form

$$H = H_0 + \frac{\mu}{2}(\hat{q}\hat{p} + \hat{p}\hat{q}), \quad H_0 = \frac{1}{2m}\hat{p}^2 + \frac{m\omega^2}{2}\hat{q}^2. \quad (5)$$

With these choices the Markovian master equation can be written [10]:

$$\frac{d\rho}{dt} = -\frac{i}{\hbar} [H_0, \rho] - \frac{i}{2\hbar}(\lambda + \mu)[\hat{q}, \rho\hat{p} + \hat{p}\rho] + \frac{i}{2\hbar}(\lambda - \mu)[\hat{p}, \rho\hat{q} + \hat{q}\rho]$$

$$- \frac{D_{pp}}{\hbar^2} [\hat{q}, [\hat{q}, \rho]] - \frac{D_{qq}}{\hbar^2} [\hat{p}, [\hat{p}, \rho]] + \frac{D_{pq}}{\hbar^2} ([\hat{q}, [\hat{p}, \rho]] + [\hat{p}, [\hat{q}, \rho]]), \quad (6)$$

where $D_{pp}, D_{qq}$ and $D_{pq}$ are the diffusion coefficients and $\lambda$ the friction constant. They satisfy the following fundamental constraints [10]:

$$i) \ D_{pp} > 0, \quad ii) \ D_{qq} > 0, \quad iii) \ D_{pp}D_{qq} - D_{pq}^2 \geq \lambda^2 \hbar^2/4. \quad (7)$$
In the particular case when the asymptotic state is a Gibbs state
\[
\rho_G(\infty) = e^{-\frac{\mathcal{H}_0}{kT}}/\text{Tr}e^{-\frac{\mathcal{H}_0}{kT}},
\]
these coefficients reduce to
\[
D_{pp} = \frac{\lambda + \mu}{2} \frac{\hbar}{m\omega} \coth \frac{\hbar\omega}{2kT}, \quad D_{qq} = \frac{\lambda - \mu}{2} \frac{\hbar}{m\omega} \coth \frac{\hbar\omega}{2kT}, \quad D_{pq} = 0,
\]
where \(T\) is the temperature of the thermal bath.

## 3 Evaluation of the density operator

There are several techniques available to solve equations such as (6). By introducing the real variables \(x_1, x_2\) corresponding to the operators \(\hat{q}, \hat{p}\):
\[
x_1 = \sqrt{\frac{m\omega}{2\hbar}}q, \quad x_2 = \frac{1}{\sqrt{2\hbar m\omega}}p,
\]
in [12, 13] we have transformed the operator form of the master equation into the following Fokker-Planck equation satisfied by the Wigner distribution function \(W(x_1, x_2, t)\):
\[
\frac{\partial W}{\partial t} = \sum_{i,j=1,2} A_{ij} \frac{\partial}{\partial x_i}(x_j W) + \frac{1}{2} \sum_{i,j=1,2} Q^W_{ij} \frac{\partial^2}{\partial x_i \partial x_j} W,
\]
where
\[
A = \begin{pmatrix} \lambda - \mu & -\omega \\ \omega & \lambda + \mu \end{pmatrix}, \quad Q^W = \frac{1}{\hbar} \begin{pmatrix} m\omega D_{qq} & D_{pq} \\ D_{pq} & D_{pp}/m\omega \end{pmatrix}.
\]
Since the drift coefficients are linear in the variables \(x_1\) and \(x_2\) and the diffusion coefficients are constant with respect to \(x_1\) and \(x_2\), Eq. (11) describes an Ornstein-Uhlenbeck process [22]. Following the method developed by Wang and Uhlenbeck [22], we solved [13] this Fokker-Planck equation, subject to either the wave-packet type or the \(\delta\)-function type of initial conditions.

When the Fokker-Planck equation is subject to a Gaussian (wave-packet) type of the initial condition (\(x_{10}\) and \(x_{20}\) are the initial values of \(x_1\) and \(x_2\) at \(t = 0\), respectively)
\[
W_w(x_1, x_2, 0) = \frac{1}{\pi \hbar} \exp\{-2[(x_1 - x_{10})^2 + (x_2 - x_{20})^2]\},
\]
the solution is found to be [13]:
\[
W_w(x_1, x_2) = \frac{\Omega}{\pi \hbar \omega \sqrt{-B_w}} \exp\{-\frac{1}{B_w} [\phi_w(x_1 - \bar{x}_1)^2 + \psi_w(x_2 - \bar{x}_2)^2 + \chi_w(x_1 - \bar{x}_1)(x_2 - \bar{x}_2)]\},
\]
where

\[ B_w = g_1 g_2 - \frac{1}{4} g_3^2, \quad g_1 = g_2^* = \frac{\mu a}{\omega} e^{2\Lambda t} + \frac{d_1}{\Lambda} (e^{2\Lambda t} - 1), \quad g_3 = 2[e^{-2\Lambda t} + \frac{d_2}{\Lambda}(1 - e^{-2\Lambda t})], \]  

(15)

\[ \phi_w = g_1 a^* + g_2 a^2 - g_3, \quad \psi_w = g_1 + g_2 - g_3, \quad \chi_w = 2(g_1 a^* + g_2 a) - g_3(a + a^*). \]  

(16)

We have put \( a = (\mu - i\Omega)/\omega, \quad \Lambda = -\lambda - i\Omega, \quad d_1 = (a^2 m\omega D_{qq} + 2a D_{pq} + D_{pp}/m\omega)/\hbar, \quad d_2 = (m\omega D_{qq} + 2\mu D_{pq}/\omega + D_{pp}/m\omega)/\hbar \) and \( \Omega^2 = \omega^2 - \mu^2 \). The functions \( \bar{x}_1 \) and \( \bar{x}_2 \), which are also oscillating functions, are given by \[ \bar{x}_1 = e^{-\Lambda t}[(x_{10}(\cos \Omega t + \frac{\mu}{\Omega} \sin \Omega t) + x_{10} \frac{\omega}{\Omega} \sin \Omega t)], \]  

(17)

\[ \bar{x}_2 = e^{-\Lambda t}[(x_{20}(\cos \Omega t - \frac{\mu}{\Omega} \sin \Omega t) - x_{10} \frac{\omega}{\Omega} \sin \Omega t)]. \]  

(18)

The solution (14) of the Fokker-Planck equation (11), subject to the wave-packet type of initial condition (13) can be written in terms of the coordinate and momentum (\( < \hat{A} > = \text{Tr}(\rho \hat{A}) \) denotes the expectation value of an operator \( \hat{A} \)) as \[ W(q, p) = \frac{1}{2\pi \sqrt{\delta}} \exp\{-\frac{1}{2\delta}[\phi(q - < \hat{q} >)^2 + \psi(p - < \hat{p} >)^2 - 2\chi(q - < \hat{q} >)(p - < \hat{p} >)]\}, \]  

(19)

where

\[ < \hat{q} >= e^{-\Lambda t}[(\cos \Omega t + \frac{\mu}{\Omega} \sin \Omega t) < \hat{q}(0) > + \frac{1}{m\Omega} \sin \Omega t < \hat{p}(0) >], \]  

(20)

\[ < \hat{p} >= e^{-\Lambda t}[-\frac{m\omega^2}{\Omega} \sin \Omega t < \hat{q}(0) > + (\cos \Omega t - \frac{\mu}{\Omega} \sin \Omega t) < \hat{p}(0) >], \]  

(21)

\[ \phi \equiv \sigma_{pp} = < \hat{p}^2 > - < \hat{p} >^2 = -\frac{\hbar m\omega^3}{4\Omega^2} \phi_w, \]  

(22)

\[ \psi \equiv \sigma_{qq} = < \hat{q}^2 > - < \hat{q} >^2 = -\frac{\hbar \omega}{4m\Omega^2} \psi_w, \]  

(23)

\[ \chi \equiv \sigma_{pq}(t) = \frac{1}{2} < \hat{q}\hat{p} + \hat{p}\hat{q} > - < \hat{q} > < \hat{p} > = \frac{\hbar \omega^2}{8\Omega^2} \chi_w, \quad \delta = \phi \psi - \chi^2. \]  

(24)

To get the explicit expression of the density operator, we use the relation \( \rho = 2\pi \hbar N \{ W_s(q, p) \} \), where \( W_s \) is the Wigner distribution function in the form of standard rule of association and \( N \) is the normal ordering operator \[ 23, 24 \] which acting on the function \( W_s(q, p) \) moves
all \( p \) to the right of the \( q \). By the standard rule of association is meant the correspondence \( p^m q^n \rightarrow \hat{q}^n \hat{p}^m \) between functions of two classical variables \((q,p)\) and functions of two quantum mechanical canonical operators \((\hat{q},\hat{p})\). The calculation of the density operator is then reduced to a problem of transformation of the Wigner distribution function by the \( N \) operator, provided that \( W_s \) is known. A special care is necessary for the \( N \) operation when the Wigner function is in the exponential form of a second order polynomial of \( q \) and \( p \). The Wigner distribution function (19) corresponds however to the form of the Weyl rule of association [25]. This function can be transformed into the form of standard rule of association [26] by

\[
W_s(q,p) = \exp\left(\frac{i\hbar}{2} \frac{\partial^2}{\partial p \partial q}\right) W(q,p),
\]

(25)

Upon performing the operation on the right-hand side, we get the Wigner distribution function:

\[
W_s(q,p) = \frac{1}{2\pi\sqrt{\xi}} \exp\left\{-\frac{1}{2\xi} \left[\phi(q - \langle \hat{q} \rangle)^2 + \psi(p - \langle \hat{p} \rangle)^2 - 2 \chi'(q - \langle \hat{q} \rangle)(p - \langle \hat{p} \rangle)\right]\right\},
\]

(26)

where

\[
\xi = \phi \psi - \chi'^2, \quad \chi' = \chi - i\frac{\hbar}{2}.
\]

(27)

The normal ordering operation of the Wigner function \( W_s \) in Gaussian form can be carried out by applying McCoy’s theorem [23, 24], which states that

\[
[J \exp(-i\hbar\gamma)]^{1/2} \exp(\alpha q^2 + \beta \hat{p}^2 + \gamma \hat{q}\hat{p}) = N[\exp(Aq^2 + Bp^2 + Gqp)],
\]

(28)

where \( \alpha = A/C, \beta = B/C, \ C = \sinh \Gamma/\Gamma J, \ \Gamma = -i\hbar(\gamma^2 - 4\alpha\beta)^{1/2} \), with \( J = \cosh \Gamma + i\hbar\gamma \sinh \Gamma/\Gamma = 1/(1 - i\hbar G) \). Having performed a straightforward calculation, we get the explicit form of the density operator:

\[
\rho = \frac{\hbar}{\sqrt{\xi}} \exp\left\{\frac{1}{2} \ln \frac{\xi - i\hbar\chi'}{2\hbar\sqrt{\xi - i\hbar\chi'} + \frac{1}{4}\hbar^2} \right\} \cosh^{-1}(1 + \frac{\hbar^2}{2(\xi - i\hbar\chi')})
\]

\[
\times \left[\phi(q - \langle \hat{q} \rangle)^2 + \psi(p - \langle \hat{p} \rangle)^2 - (\chi' + i\frac{\hbar}{2})[2(q - \langle \hat{q} \rangle)(p - \langle \hat{p} \rangle) - i\hbar]]\right],
\]

(29)

The density operator (29) is in a Gaussian form, as was expected from the initial form of the Wigner distribution function. While the density operator is expressed in terms of operators \( \hat{q} \) and \( \hat{p} \), the Wigner distribution is a function of real variables \( q \) and \( p \). When time \( t \) goes to infinity, the density operator approaches to
\[ \rho(\infty) = \frac{\hbar}{\sqrt{\sigma - \frac{\hbar^2}{4}}} \exp\left\{ -\frac{1}{2\hbar \sqrt{\sigma}} \ln \frac{2\sqrt{\sigma} + \hbar}{2\sqrt{\sigma} - \hbar} \right\} [\sigma_{pp}(\infty) \hat{q}^2 + \sigma_{qq}(\infty) \hat{p}^2 - \sigma_{pq}(\infty)(\hat{q}\hat{p} + \hat{p}\hat{q})], \quad (30) \]

where \( \sigma = \sigma_{pp}(\infty) \sigma_{qq}(\infty) - \sigma_{pq}^2(\infty) \) and [10]:

\[ \sigma_{qq}(\infty) = \frac{1}{2(m\omega)^2 \lambda(\lambda^2 + \Omega^2)} [(m\omega)^2 (2\lambda(\lambda + \mu) + \omega^2) D_{qq} + \omega^2 D_{pp} + 2m\omega^2(\lambda + \mu) D_{pq}], \quad (31) \]

\[ \sigma_{pp}(\infty) = \frac{1}{2\lambda(\lambda^2 + \Omega^2)} [(m\omega)^2 \omega^2 D_{qq} + (2\lambda(\lambda - \mu) + \omega^2) D_{pp} - 2m\omega^2(\lambda - \mu) D_{pq}], \quad (32) \]

\[ \sigma_{pq}(\infty) = \frac{1}{2m\lambda(\lambda^2 + \Omega^2)} [-(\lambda + \mu)(m\omega)^2 D_{qq} + (\lambda - \mu) D_{pp} + 2m(\lambda^2 - \mu^2) D_{pq}]. \quad (33) \]

In the particular case (9),

\[ \sigma_{qq}(\infty) = \frac{\hbar}{2m\omega} \coth \frac{h\omega}{2kT}, \quad \sigma_{pp}(\infty) = \frac{\hbar m\omega}{2} \coth \frac{h\omega}{2kT}, \quad \sigma_{pq}(\infty) = 0 \quad (34) \]

and the asymptotic state is a Gibbs state (8):

\[ \rho_G(\infty) = 2 \sinh \frac{h\omega}{2kT} \exp\left\{ -\frac{1}{kT} (\frac{1}{2m} \hat{p}^2 + \frac{m\omega^2}{2} \hat{q}^2) \right\}. \quad (35) \]

### 4 Entropy and effective temperature

Because of the presence of the exponential form in the density operator, the construction of the logarithmic density is straightforward. In view of the relations (22)-(24), the expectation value of the logarithmic density becomes

\[ <\ln \rho> = \ln \hbar - \frac{1}{2} \ln(\delta - \frac{\hbar^2}{4}) - \frac{\sqrt{\delta}}{\hbar} \ln \frac{2\sqrt{\sigma} + \hbar}{2\sqrt{\sigma} - \hbar}. \quad (36) \]

By putting \( \hbar \nu = \sqrt{\delta} - \hbar/2 \), we finally get the entropy in a closed form,

\[ S(t) = k[(\nu + 1) \ln(\nu + 1) - \nu \ln \nu]. \quad (37) \]

Because of the identity \( \delta = -\frac{h\omega^2}{4\Omega^2} B_w \), the function \( \nu \) takes explicitly the form

\[ \nu = \frac{\omega}{2\Omega} \sqrt{-B_w} - \frac{1}{2}, \quad (38) \]
where

\[ B_w = \exp(-4\lambda t)(2\mu \Re \frac{d_1 a^*}{\Lambda} - \frac{\Omega^2}{\omega^2} + \frac{|d_1|^2}{|\Lambda|^2} - \frac{d_2^2}{\lambda^2} + 2\frac{d_2}{\lambda}) \]

\[ -2 \exp(-2\lambda t)\Re \left( \frac{\mu d_1 a^*}{\Lambda} + \frac{|d_1|^2}{|\Lambda|^2} \exp 2i\Omega t \right) - \frac{d_2^2}{\lambda^2} + 2\frac{d_2}{\lambda} \right] + \frac{|d_1|^2}{|\Lambda|^2} - \frac{d_2^2}{\lambda^2} \]  

(39)

It is worth noting that the entropy depends only upon the variance of the Wigner distribution. When time \( t \to \infty \), the function \( \nu \) goes to \( s = \omega(d_2^2/\lambda^2 - |d_1|^2/(\lambda^2 + \Omega^2))^{1/2}/2\Omega - 1/2 \) and the entropy relaxes to its equilibrium value \( S(\infty) = k[(s + 1) \ln(s + 1) - s \ln s] \).

A similar expression to (37) has been formerly obtained [17, 18] in the framework of the quantum theory of oscillator relaxation. It should also be noted that the expression (37) has the same form as the entropy of a system of harmonic oscillators in thermal equilibrium. In the later case \( \nu \) represents, of course, the average of the number operator [19]. Eq. (37) together with the function \( \nu \) defined by (38), (39) is the desired entropy for the system. While the formal expression (37) for the entropy has a well-known appearance, the explicit form of the function \( \nu \) displays clearly a specific feature of the present entropy. We see that the time dependence of the entropy is represented by the damping factors \( \exp(-4\lambda t) \), \( \exp(-2\lambda t) \) and also by the oscillating function \( \exp 2i\Omega t \). The complex oscillating factor \( \exp 2i\Omega t \) occurs also in the second moments, as shown in [13]. This factor reduces, however, to a function of the frequency \( \omega \), expressly \( \exp 2i\omega t \) for \( \mu \to 0 \) or if \( \mu/\Omega \ll 1 \) (i.e. the frequency \( \omega \) is very large as compared to \( \mu \)).

In the case of a thermal bath (8), (9), we may define a time-dependent effective temperature \( T_e \), by remarking that at infinity of time the quantity \( \nu \) goes, according to (2.22) in ref. [13], to the average thermal phonon number \( < n > = (\exp(\hbar\omega/kT_e) - 1)^{-1} \). Thus \( \nu \) may be considered as the time variation of the thermal phonon number. Accordingly we may put in this case

\[ (\exp \frac{\hbar\omega}{kT_e} - 1)^{-1} = \nu. \]  

(40)

The function \( \nu \), which goes to \( < n > \) as \( t \) tends to infinity, vanishes at \( t = 0 \). From (40) the effective temperature \( T_e \) can be extracted as

\[ T_e(t) = \frac{\hbar\omega}{k[\ln(\nu + 1) - \ln \nu]} \]  

(41)

In terms of effective temperature we may say that the system at time \( t \) is found in thermal equilibrium at temperature \( T_e \). Then, in terms of the temperature, the entropy takes the form

\[ S = \frac{\hbar\omega}{T_e(\exp \frac{\hbar\omega}{kT_e} - 1)} - k \ln[1 - \exp(-\frac{\hbar\omega}{kT_e})]. \]  

(42)
This form of the entropy for the thermally excited oscillator state is well-known. The effective temperature approaches thermal equilibrium with the bath, $T_e \to T$, as the value of $t$ increases.

5 Concluding remarks

Recently we assist to a revival of interest in quantum brownian motion as a paradigm of quantum open systems. There are many motivations. The possibility of preparing systems in macroscopic quantum states led to the problems of dissipation in tunneling and of loss of quantum coherence (decoherence). These problems are intimately related to the issue of quantum-to-classical transition. All of them point the necessity of a better understanding of open quantum systems and all requires the extension of the model of quantum brownian motion. Our results allow such extensions. The Lindblad theory provides a selfconsistent treatment of damping as a possible extension of quantum mechanics to open systems. In the present paper we have studied the one-dimensional harmonic oscillator with dissipation within the framework of this theory. We have first obtained the density operator from the master and Fokker-Planck equations. The density operator in a Gaussian form is a function of the position and momentum operators in addition to several time dependent factors. The explicit form of the density operator has been subsequently used to calculate the entropy and the effective temperature. The temporal behaviour of these quantities displays how they relax to the equilibrium value. In a future work we plan to use the entropy of the system in order to quantify the extent of decoherence in the course of the system-environment interaction.

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