Characterising planar Cayley graphs and Cayley complexes in terms of group presentations

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ABSTRACT

We prove that a Cayley graph can be embedded in the Euclidean plane without accumulation points of vertices if and only if it is the 1-skeleton of a Cayley complex that can be embedded in the plane after removing redundant simplices. We also give a characterisation of these Cayley graphs in term of group presentations, and deduce that they can be effectively enumerated.

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1. Introduction

The study of groups that have Cayley graphs embeddable in the Euclidean plane $\mathbb{R}^2$, called planar groups, has a tradition starting in 1896 with Maschke's characterisation of the finite ones. Among the infinite planar groups, those that admit a flat Cayley complex, defined below, have received a lot of attention. They are important in complex analysis as they include the discontinuous groups of motions of the Euclidean and hyperbolic plane. Moreover, they are closely related to surface groups [19, Section 4.10]. These groups are now well understood due to the work of Macbeath [15], Wilkie [18], and others; see [19] for a survey. Planar groups that have no flat Cayley complex are harder to analyse, and they are the subject of on-going research [4–6,8,7].

All groups, Cayley graphs and Cayley complexes in this paper are finitely generated. Our first result is

**Theorem 1.1.** A planar Cayley graph of a group $\Gamma$ is accumulation-free if and only if it is the 1-skeleton of a flat Cayley complex of $\Gamma$.

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2 Partly supported by FWF grant P-19115-N18.
3 In [19] the term Cayley complex is not used but it is implicit in Theorems 4.5.6 and 6.4.7 that a group admits a flat Cayley complex if and only if it is a planar discontinuous group.
Here, a Cayley complex is flat if it can be embedded in $\mathbb{R}^2$ after removing redundant 2-simplices; see Section 2.1 for the precise definition. A planar graph is said to be accumulation-free, if it admits an embedding in $\mathbb{R}^2$ such that the images of its vertices have no accumulation point. The study of a planar graph is often simplified if one knows that the graph is accumulation-free; examples range from structural graph-theory [2] to percolation theory [14] and the study of spectral properties [13]. A further example is Thomassen’s Theorem 5.2 below, which becomes false in the non-accumulation-free case. Accumulation-free embeddings also appear with other names in the literature, most notably “locally finite”.

Theorem 1.1 implies that a group has a flat Cayley complex if and only if it has an accumulation-free Cayley graph, a fact that might be known to experts, and it should not be too hard to derive it from the results of [19]. Theorem 1.1 however strengthens this assertion into a theorem about all planar Cayley graphs, not just their groups. Since a single group can have a large variety of planar Cayley graphs (see Section 4 for some examples), it is in principle harder to prove results that hold for all planar Cayley graphs than proving the corresponding result for their groups. However, our proof is elementary and self-contained, avoiding the geometric machinery of [19].

We also prove that every accumulation-free Cayley graph admits an embedding the facial walks of which are preserved by the action of the group; see Corollary 3.6.

Finally, we derive a further characterisation of the accumulation-free Cayley graphs, and so by Theorem 1.1 also of the groups that admit a flat Cayley complex, by means of group presentations. We introduce a special kind of presentation, called a facial presentation, which is motivated by geometric intuition and can be easily recognised by an algorithm, and use it to obtain a further characterisation of the class of accumulation-free Cayley graphs:

**Corollary 1.2.** A Cayley graph admits an accumulation-free embedding if and only if it admits a facial presentation.

This implies that the accumulation-free Cayley graphs can be effectively enumerated (Corollary 5.4).

We prove Theorem 1.1 in Section 3. In Section 4 we examine accumulation-freeness as a group-theoretical invariant. Finally, in Section 5 we introduce facial presentations and prove Corollary 1.2.

2. Preliminaries

We will follow the terminology of [3] for graph-theoretical terms and that of [1,10] for group-theoretical ones.

Let us recall some standard definitions used in this paper. We say that a graph $G$ is $k$-connected if $G - X$ is connected for every set $X \subseteq V$ with $|X| < k$. A component of $G$ is a maximal connected subgraph of $G$.

A walk in $G$ is an alternating sequence $v_0 e_0 v_1 e_1 \cdots e_{k-1} v_k$ of vertices and edges in $G$ such that $e_i = \{v_i, v_{i+1}\}$ for all $i < k$. If $v_0 = v_k$, the walk is closed. If the vertices in a walk are all distinct, it is called a path (many authors use the word ‘path’ to denote a walk in our sense).

A 1-way infinite path is called a ray, a 2-way infinite path is a double ray. Two rays contained in a graph $G$ are equivalent if no finite set of edges separates them. The corresponding equivalence classes of rays are the ends of $G$.

By an embedding of a graph $G$ we mean a topological embedding of the corresponding 1-complex in the Euclidean plane $\mathbb{R}^2$; in simpler words, an embedding is a drawing of the graph in the plane with no two edges crossing. A graph is planar if it admits an embedding. A plane graph is a (planar) graph endowed with a fixed embedding.

A face of an embedding $\sigma : G \to \mathbb{R}^2$ is a component of $\mathbb{R}^2 \setminus \sigma(G)$. The boundary of a face $F$ is the set of vertices and edges of $G$ that are mapped by $\sigma$ to the closure of $F$. A path, or walk, in $G$ is called facial with respect to $\sigma$ if it is contained in the boundary of some face of $\sigma$.

One of our main tools will be the (finitary) cycle space $C_f(G)$ of a graph $G = (V, E)$, which is defined as the vector space over $\mathbb{Z}_2$ (the field of two elements) consisting of those subsets of $E$ such that can
be written as a sum (modulo 2) of a finite set of circuits, where a set of edges $D \subseteq E$ is called a circuit if it is the edge set of a cycle of $G$.

The cycle space is closely related to the first (simplicial) homology group $H_1(G)$ [10], and in fact the two objects coincide when the latter is defined over the field $\mathbb{Z}_2$. In this paper $H_1(G)$ will be defined over $\mathbb{Z}$ as usual, and so it should not be confused with $C_f(G)$.

2.1. (Flat) Cayley complexes

The Cayley complex $X$ of a group presentation $P = (S, R)$ is the universal cover of its presentation complex, which is a 2-dimensional cell complex with a single vertex, one loop at the vertex for each generator in $S$, and one 2-cell for each relation in $R$ bounded by the loops corresponding to the generators appearing in $R$, see [10]. The 1-skeleton of this Cayley complex is the Cayley graph corresponding to the group presentation $(S, R)$. From the Cayley complex we derive the simplified Cayley complex of $P$ as follows. Firstly, for every pair of parallel edges $e, e'$, resulting from an involution in $S$, we identify $e$ with $e'$, gluing them together according to a homeomorphism from $e$ to $e'$ that maps each endvertex to itself. We remove any 2-simplices of $X$ that were bounded by the circle $e \cup e'$; all other 2-simplices incident with $e$ or $e'$ are preserved. In the resulting 2-complex $X'$, we define two 2-simplices to be equivalent if they have the same boundary. Removing all but one of the elements of each equivalence class from $X'$ we obtain the simplified Cayley complex of $P$.

Equivalently, we can define the simplified Cayley complex of $P = (S, R)$ by building the corresponding Cayley graph, identifying each pair of parallel edges into a single edge, and then for every cycle $C$ of this graph induced by a relator in $R$, introducing a 2-simplex having $C$ as its boundary.

Definition 2.1. We say that a Cayley complex is flat, if the corresponding simplified Cayley complex is planar, that is, the latter admits an embedding into $\mathbb{R}^2$.

For example, the Cayley complex of the group presentation $\langle a \mid a^n \rangle$ has $n$ equivalent 2-simplices, while the corresponding simplified Cayley complex has only one, and is planar. As a further example, consider the presentation $\langle a, b \mid b^2, aba^{-1}b \rangle$. The corresponding simplified Cayley complex is a 2-way infinite ladder with each 4-gon bounding a 2-simplex (Fig. 1); note that this complex is planar, while the usual Cayley complex is not (to see this, note that for each $b$ edge, there are two distinct 4-cycles in the Cayley graph induced by the relation $aba^{-1}b$ and one 2-cycle induced by $b^2$). These two examples show that the above simplifications of the Cayley complex are necessary to make Theorem 1.1 true.

3. Proof of Theorem 1.1

In this section we will be assuming that our graphs have no parallel edges. In a Cayley graph this can be achieved by drawing, for every involution in the generating set, a single undirected edge rather than a pair of parallel edges with opposite directions. This convention affects neither planarity nor accumulation-freeness, and so our assumption comes without loss of generality for the proof of Theorem 1.1.

Our first lemma, a well-known fact which is easy to prove, relates $H_1(G)$ to group presentations. We will say that a closed walk $W$ in $G$ is induced by a relator $R$, if $W$ can be obtained by starting at some vertex $g$ and following the edges corresponding to the letters in $R$ in order; note that for a given $R$ there are several walks in $G$ induced by $R$, one for each starting vertex $g \in V(G)$. Note that every
closed walk in $G$ uniquely determines an element of $H_1(G)$, and we will, with a slight abuse, not make a distinction between the two.

**Lemma 3.1.** Let $G = \text{Cay}(\Gamma, S)$ be a Cayley graph of the group $\Gamma$, and let $\langle S \mid R \rangle$ be a presentation of $\Gamma$. Then the set of walks in $G$ induced by relators in $R$ generates $H_1(G)$.

Conversely, if $R'$ is a set of relations of $\Gamma'$ with letters in a generating set $S$ such that the set of closed walks of $\text{Cay}(\Gamma', S)$ induced by $R'$ generates $H_1(G)$, then $\langle S \mid R' \rangle$ is a presentation of $\Gamma'$.

Combined with the next easy fact, this allows one to deduce group presentations from accumulation-free embeddings of a Cayley graph.

**Lemma 3.2.** Let $G$ be an accumulation-free planar graph. Then the set of finite facial closed walks of $G$ generates $H_1(G)$.

**Proof.** It suffices to show that every cycle $C$ of $G$ is a sum of finite facial closed walks when seen as an element of $H_1(G)$. This is indeed the case, for as $G$ is accumulation-free there must be a side $A$ of $C$ containing only finitely many vertices, and so $E(C)$ is the sum of the facial closed walks corresponding to faces lying in $A$. □

We will also use the following basic characterisation of accumulation-free graphs.

**Lemma 3.3** ([16, Lemma 7.1]). A countable graph $G$ is accumulation-free if and only if some planar embedding of $G$ has the property that no cycle has both infinitely many vertices in its interior and infinitely many vertices in its exterior.

The following fact is probably known to experts in the study of infinite vertex transitive graphs. We include a proof sketch for the convenience of the non-expert. A double-ray is a 2-way infinite path (with no repetition of vertices).

**Lemma 3.4.** Let $G$ be an infinite, connected, vertex transitive graph which is not a double-ray. Then for every pair of vertices $x, y$ of $G$, no component of $G - \{x, y\}$ is finite.

**Proof.** To begin with, it is easy to prove that

$$\text{for every } x \in V(G), \text{ no component of } G - \{x\} \text{ is finite,} \quad (1)$$

by considering a minimal such component $C$ and mapping $x$ to some vertex of $C$.

Suppose that some component $C$ of $G - \{x, y\}$ is finite, and choose $x, y$ so as to minimise $|V(C)|$. We claim that the graph $C$ has no cut-vertex. Indeed, if $z \in V(C)$ separates $C$, then $G - \{x, z\}$ contains a component properly contained in $C$, contradicting the minimality of the latter. Moreover, each of $x, y$ has at least two neighbours in $C$; for if $y$ has a single neighbour $y'$ in $C$, then we could have replaced $y$ by $y'$ to obtain a separator $\{x, y'\}$ cutting off a smaller component, and if $y$ has no neighbour in $C$ then (1) is contradicted. These two observations, combined with Menger's theorem [3, Theorem 3.3.1], imply that there are two independent $x-y$ paths $P, Q$ through $C$. Moreover, (at least) one of $x, y$, say $x$, is contained in an infinite subgraph $X$ that does not meet $C \cup \{x, y\}$ except at $x$.

Let $z \in V(C)$, and consider an automorphism $g$ mapping $x$ to $z$. Then, there is a vertex $w = gy$ such that $\{z, w\}$ separates $G$. We consider three cases.

If $w$ lies in $C' := G \setminus (C \cup \{x, y\})$, then each of $gP, gQ, gX$ meets both $C'$ and $C$. But this is impossible since $C$ is separated from $C'$ by $x, y$ and the only vertices meeting more than one of $gP, gQ, gX$ are $z$ and $w$, none of which is $x$ or $y$.

If $w$ lies in $C$, then some component of $G - \{z, w\}$ is properly contained in $C$ contradicting its minimality.

Finally, if $w = y$, then as the component $gC$ of $G - \{z, w\}$ cannot be smaller than $C$, it must contain a vertex $x'$ in $G \setminus (C \cup \{y\})$. Note that there is a $x'-x$ path $P$ in $G \setminus (C \cup \{y\})$, because otherwise $x'$ lies in a component of $G - \{x, y\}$ sending no edges to $x$, contradicting (1). Recall that the infinite subgraph $X$ mentioned above does not meet $C \cup \{x, y\}$ except at $x$. Thus the infinite subgraph $X \cup P$ also does not meet $C \cup \{x, y\}$ except at $x$. Since this subgraph meets $gC$ (at the vertex $x'$), it is contained in $gC$ which is separated from the rest of the graph by $\{z, y\}$. This shows that $gC$ is infinite, contradicting the fact that it is a translate of the finite $C$.

Thus, in all three cases we obtained a contradiction. This proves Lemma 3.4. □
We can now prove Theorem 1.1, which we restate for the convenience of the reader.

**Theorem.** A planar Cayley graph of a group \( \Gamma \) is accumulation-free if and only if it is the 1-skeleton of a flat Cayley complex of \( \Gamma \).

**Proof.** For the forward implication, let \( G \) be a planar Cayley graph of the group \( \Gamma \), with respect to the generating set \( \delta \), admitting an accumulation-free embedding \( \sigma \). By Lemma 3.5 below, if \( F \) is a finite face boundary in \( \sigma \), then every translate of \( F \) is a face boundary. This means that if we let \( R' \) be the set of relations corresponding to the finite facial walks with respect to \( \sigma \) incident with the group identity \( e \), then every finite facial walk with respect to \( \sigma \) is induced by some element of \( R' \), and conversely any cycle induced by some element of \( R' \) bounds a face in \( \sigma \). By Lemmas 3.2 and 3.1, \( \{ \delta \ | \ R' \} \) is a presentation of \( \Gamma \). The corresponding simplified Cayley complex is planar and accumulation-free since we can embed its 1-skeleton \( G \) by \( \sigma \) and then every 2-simplex can be embedded into the face of \( \sigma \) bounded by the corresponding cycle.

For the backward implication, let \( X \) be a planar simplified Cayley complex and let \( G \) be its 1-skeleton. Let \( B \) be the set of closed walks in \( G \) bounding a 2-simplex of \( X \); in fact, each such closed walk is a cycle since \( X \) is planar, and it bounds a face of \( G \). Note that \( B \) generates \( H_1(G) \) by Lemma 3.1 and the definition of a Cayley complex. We will show that the condition in Lemma 3.3 is satisfied, i.e. no cycle of \( G \) has both infinitely many vertices in its interior and infinitely many vertices in its exterior. Indeed, every cycle \( K \) can be written as a finite sum of elements of \( B \) since the latter generates \( H_1(G) \). As each element of \( B \) bounds a face of \( G \), it is not hard to see that this sum comprises the face boundaries in the interior of \( K \). This implies that the interior of \( K \) contains only finitely many vertices. Thus by Lemma 3.3, \( G \) is accumulation-free. \( \square \)

A translate of a subgraph \( F \) of \( G \) is the image of \( F \) under an automorphism of \( G \). In the following lemma our assumption that \( G \) has no parallel edges becomes essential.

**Lemma 3.5.** Let \( G \) be a vertex transitive graph with an accumulation-free embedding \( \sigma \). If \( F \) is a finite face boundary in \( \sigma \) then every translate of \( F \) is a face boundary in \( \sigma \).

**Proof.** Suppose to the contrary that some image \( F' = gF \) of \( F \) under an automorphism \( g \) is not a face boundary. Then, as \( \sigma \) is accumulation-free, one of the sides of \( F' \) contains at least one finite bridge \( C \) of \( F' \), where by a bridge of \( F' \) we mean either a finite component of \( G - F' \) or an edge joining two vertices of \( F' \). Let \( N(C) \) be the set of vertices of \( F' \) incident with \( C \) (if \( C \) is an edge, then \( N(C) \) are its endvertices). Then \( F' - N(C) \) consists of a set of disjoint paths, which we call the intervals. Note that unless \( C \) is an edge, we have \( |N(C)| \geq 3 \) for otherwise Lemma 3.4 is contradicted as \( N(C) \) separates \( C \).

We claim that no bridge of \( F' \) is adjacent with more than one interval. \( \square \)

Indeed, if such a bridge \( C' \) existed, then, by a topological argument, it would be impossible to embed \( G \) is such a way that both \( C \) and \( C' \) lie in the same side of \( F' \) (Fig. 2), but such an embedding must be possible since \( F \) is a face boundary.

Next, we claim that at most one of the intervals sends an edge to an infinite component of \( G - F' \). For if there are intervals \( I \neq J \) adjacent with infinite components \( C_I, C_J \) of \( G - F' \), then replacing \( I \) in \( F' \) by a path through \( C_I \) we would obtain a cycle \( D \) that separates \( C_I \) from \( C_J \) by (2) (Fig. 2). But then \( g^{-1}I, g^{-1}J \) must lie in distinct sides of \( g^{-1}D \) since \( F = g^{-1}F' \) and \( F \) is a face boundary, contradicting the fact that \( \sigma \) is accumulation-free.

Thus our claim is proved, implying that there is a unique interval \( I \) adjacent with the infinite component of \( G - F' \). This fact, combined with (2), implies that deleting the vertices \( x, y \in N(C) \) bounding \( I \) leaves a finite component, namely the component \( K \) of \( G - \{x, y\} \) containing \( F' - I \); note that here we use the fact that \( G \) has no parallel edges to make sure that \( K \) contains at least one vertex. But this contradicts Lemma 3.4. \( \square \)

Lemma 3.5 implies that every accumulation-free Cayley graph admits an embedding that is topologically identical around any vertex. In order to make this more precise we will need a few definitions.
Given an embedding $\sigma$ of a Cayley graph $G$ with generating set $S$, we consider for every vertex $x$ of $G$ the embedding of the edges incident with $x$, and define the spin of $x$ to be the cyclic order of the set $E_x$ of edges incident with $x$ in which $e$ is a successor of $f$ whenever the edge $e$ comes immediately after the edge $f$ as we move clockwise around $x$. Note that the set $E_x$ depends only on $S$ and our convention on whether to draw one or two edges for involutions. This allows us to compare spins of different vertices, by identifying edges corresponding to the same generator in $S \cup S^{-1}$.

Call an edge of $G$ spin-preserving if its two endvertices have the same spin in $\sigma$, and call it spin-reversing if the spin of one endvertex can be obtained from the spin of the other by reversing the order. Call a colour in $S$ consistent if all edges bearing that colour are spin-preserving or all edges bearing that colour are spin-reversing in $\sigma$. Finally, call the embedding $\sigma$ consistent if every two vertices have the same spin up to reversing the order, and every colour is consistent in $\sigma$.

It is straightforward to check that $\sigma$ is consistent if and only if the action on the Cayley graph $G$ by its group preserves facial walks.

It is known that planar 3-connected Cayley graphs have a consistent embedding [8], while Cayley graphs of connectivity 2 do not always admit a consistent embedding [5]. Our next result shows that the latter cannot occur in the accumulation-free case.

**Corollary 3.6.** Every accumulation-free planar Cayley graph admits a consistent embedding.

Again, our convention that involutions are represented by a single edge rather than a pair of parallel edges is necessary here. For example, the Cayley graph of $\langle a, b, c | a^2, b^2, c^2, (ab)^3, (bc)^3, (ca)^3 \rangle$ is a hexagonal grid that does not admit a facial presentation if its generators are represented by pairs of parallel edges.

**Proof.** Let $G$ be a Cayley graph with an accumulation-free embedding $\sigma$, and let $S$ be its set of generators. We define an equivalence relation $\sim$ on $S \cup S^{-1}$ as follows. Declare two elements to be neighbours, if there is a finite face boundary incident with a fixed vertex $o \in V(G)$ containing the two edges corresponding to these elements, and let $\sim$ be the transitive closure of the neighbour relation. By Lemma 3.5, if two edges incident with some other vertex $x$ lie in a common finite face, then the corresponding edges incident with $o$ are also adjacent in the cyclic ordering. Thus neither the neighbour relation nor $\sim$ can depend on the choice of $o$.

Note that, by the definitions, equivalence classes of $\sim$ give rise to consecutive members of the spin of $o$, or any other vertex. This means that the spin of any vertex $x$ can be obtained from that of $o$ by changing the order in which the various $\sim$-classes appear or reversing the order in which the elements of a class appear.

Our next claim that the edges incident with any vertex $x$ corresponding to each $\sim$-class lie in distinct components of $G - x$; in particular, $G$ is not 2-connected unless $\sim$ only has one equivalence class. Indeed, between any two $\sim$-classes in the spin of $x$ there must be an infinite face by the definition of the neighbour relation. Now if there is a path $P$ in $G - x$ connecting the other endvertices of two edges $e, f$ incident with $x$ from distinct $\sim$-classes, then attaching $e$ and $f$ to $P$ we would obtain a cycle through $x$ that would separate two such infinite faces, contradicting accumulation-freeness.
Our last two observations combined show that we can modify our embedding of $G$ into an embedding in which $x$ and $o$ have the same spin up to reflection (i.e. reversing the order) by topological operations like reflecting the embedding of a single component of $G - x$ or changing the order in which two such components are embedded around $x$ (in the case where there are more than two of them). Note that such operations can only reverse the order of the spin of $o$ (if $o$ happens to lie in one of the components reflected), but they do not change adjacencies. Thus for every finite vertex set, we can make sure that all vertices in the set have the same spin up to reflection by finitely many such operations. By a standard compactness argument, we obtain an accumulation-free embedding of $G$ in which all vertices have the same spin as $o$ or its reflection.

It remains to show that each element of $S$ can be forced to be consistent as defined above. For this we distinguish two cases given an $s \in S$. The first case is when each edge of colour $s$ is a bridge, i.e. its removal separates $G$. In this case we can perform reflecting operations as above to make all such edges spin-preserving. In the other case, an argument similar to the one above shows that for each such edge $e$, at most one of the faces incident with $e$ is infinite in any accumulation-free embedding of $G$. Thus $e$ lies in a finite face boundary $C$, and by Lemma 3.5 all translates of $C$ are face boundaries as well. It is now easy to see that all translates of $e$ are spin-preserving or they are all spin-reversing, as $C$ forces one of the two behaviours.

In Section 5 we will prove a result that is, in a sense, the converse of Corollary 3.6; we will show how to use the ideas of spin and consistency to deduce accumulation-freeness from properties of a presentation.

### 4. Accumulation-freeness as a group-theoretical invariant

A planar group can admit both accumulation-free and non-accumulation-free Cayley graphs. For example, the Cayley graph corresponding to the presentation \( \langle a, b \mid b^2, abab \rangle \), of the infinite dihedral group, is accumulation-free planar, but adding the redundant generator \( c = ab \) keeps the Cayley graph planar and makes it non-accumulation-free as the reader can check. Thus accumulation-freeness is not group-theoretical invariant in general. However, it becomes an invariant if one only considers 3-connected Cayley graphs:

**Theorem 4.1.** If a group $\Gamma$ has a 3-connected accumulation-free planar Cayley graph and a group $\Delta$ has a 3-connected non-accumulation-free planar Cayley graph, then $\Gamma$ is not isomorphic to $\Delta$.

Before proving this let us see a further example showing that it is necessary that both graphs in the assertion be 3-connected. Consider the Cayley graph corresponding to the presentation \( \langle a, b \mid a^4, b^4 \rangle \). This is a free product of 4-cycles, and it is easy to see that it has an accumulation-free embedding and that its connectivity is 1. Now add the redundant generators $c = ab$ and $d = a^2b^2a^2$. Note that $d^2 = 1$. It is not hard to check that the corresponding Cayley graph is 3-connected, and that it is still planar: for every 4-cycle $C$ spanned by $a$, embed the four 4-cycles spanned by $b$ incident with $C$ alternatingly inside and outside $C$. Such an embedding is not accumulation-free, for $C$ separates two infinite subgraphs. It now follows easily from the following classical result, proved by Whitney [17, Theorem 11] for finite graphs and by Imrich [11] for infinite ones, that no embedding of this graph is accumulation-free.

**Theorem 4.2.** Let $G$ be a 3-connected graph embedded in the plane. Then every automorphism of $G$ maps each facial path to a facial path.

We will need a few lemmas for the proof of Theorem 4.1.

**Lemma 4.3.** Let $G$ be a 2-connected planar graph and let $\omega$, $\psi$ be distinct ends of $G$. Then there is a cycle $C$ in $G$ that separates $\omega$ from $\psi$, i.e. every double-ray with a tail in $\omega$ and a tail $\psi$ has a vertex in $C$.

**Proof.** Fix an embedding $\sigma$ of $G$. Consider a finite set of vertices $S = \{s_1, s_2, \ldots, s_t\}$ separating $\omega$ from $\psi$, and let $C_1$ be a cycle containing $s_1, s_2$; such a cycle exists since $G$ is 2-connected. If $C_1$ does not separate $\omega$ from $\psi$ then both ends lie in one of the sides of $C_1$, the outside say. Note that some vertex
Lemma 4.4. Let $G$ be a 2-connected graph with an accumulation-free embedding $\sigma$ and more than 1 end. Then at least one of the faces of $\sigma$ has infinite boundary.

Proof. By Lemma 4.3 there is a cycle $C$ separating two ends $\omega$, $\psi$ of $G$. Since $\sigma$ is accumulation-free, both these ends lie in the same side of $C$, the outside say. Let $K_{\omega}$ (respectively $K_{\psi}$) be the component of $G - C$ containing rays in $\omega$ (resp. $\psi$). Easily, it is possible to find independent subpaths $P_{\omega}$, $P_{\psi}$ of $C$ such that each vertex of $C$ adjacent with $K_{\omega}$ lies in $P_{\omega}$, and similarly for $K_{\psi}$ and $P_{\psi}$. Let $x$ be an endvertex of $P_{\omega}$; without loss of generality, $x$ is adjacent with $K_{\omega}$.

By the choice of $x$ we can choose an edge $e = xy$ with $y \in V(K_{\omega})$ and a further edge $f = xz$ incident with $x$ with $z \not\in K_{\omega}$ and $f \not\in P_{\omega}$, so that $e, f$ lie on a common face boundary $F$, bounding some face $F'$, say. Note that $f$ may or may not lie on $C$. Now if $F$ is infinite we are done, so suppose it is finite. Consider the subpath $F'$ of $F$ starting with the edge $xy$ and finishing at the first visit of $F$ to $C$. Thus one of the endvertices of $F'$ is $x$, and the other endvertex $x'$ must also lie on $P_{\omega}$ since, easily, $F' \setminus \{x, x'\}$ is contained in the component $K_{\omega}$ of $G \setminus C$. Now consider the cycle $P$ contained in $F' \cup P_{\omega}$. We claim that $K_{\omega}$, $K_{\psi}$ lie in distinct sides of $D$ which contradicts our assumption that $\sigma$ is accumulation-free.

To see this, note that as $P_{\omega} \cap D$ joins two vertices $x, x'$ on the face boundary $F$, it defines two regions in $\mathbb{R}^2 \setminus F$, one region $A$ bounded by $D$ and one region $B$ bounded by $(F \setminus F') \cup P_{\omega}$. By the definition of $K_{\omega}$, there is a ray in $\omega$ starting at $y$ and avoiding $C$, and so this ray is contained in $A$. By inspecting the cyclic ordering of the edges incident with $x$, it is easy to see that the edge of $C \setminus P_{\omega}$ incident with $x$ (which edge may coincide with $f$) lies in $B$ by the choice of $F$. Thus, any ray in $\psi$ starting with that edge and avoiding $P_{\omega}$, which exists by the definition of $K_{\omega}$, $K_{\psi}$, lies in $B$. This proves our claim that $D$ separates rays in $\omega$ from rays in $\psi$. □

Our last lemma is

Lemma 4.5. There is no 3-connected vertex-transitive accumulation-free planar graph with more than 1 end.

Proof. If such a graph $G$ exists, then by Lemma 4.4 it has an infinite face-boundary. By Theorem 4.2 this implies that every vertex of $G$ is incident with an infinite face-boundary.

Thus we can pick two vertices $x, y$ that lie in a common double ray $R$ of $G$ contained in a face-boundary. As $G$ is 3-connected, there are three independent $x$-$y$ paths $P_1$, $P_2$, $P_3$ by Menger’s theorem [3, Theorem 3.3.1]. By an easy topological argument, there must be a pair of those paths, say $P_1$, $P_2$, whose union is a cycle $C$ such that some side of $C$ contains a tail of $R$ and the other side of $C$ contains $P_3$. We may assume without loss of generality that $P_3$ is not a single edge, for we are allowed to choose $x$ and $y$ far apart. Thus the side of $C$ containing $P_3$ contains at least one vertex $z$. By our previous remarks, $z$ is incident with an infinite face-boundary. This means that both sides of $C$ contain infinitely many vertices, contradicting our assumption that $G$ is accumulation-free. □

We can now prove the main result of this section.

Proof of Theorem 4.1. If any of $\Gamma$, $\Delta$ is 1-ended then we are done since it is well-known, and not hard to prove, that all its planar Cayley graphs are accumulation-free in this case. The result now follows immediately from Lemma 4.5. □
5. Facial presentations

In this section we derive a further characterisation of the groups that admit a flat Cayley complex by means of group presentations. This characterisation is motivated by the concept of consistent embeddings introduced before Corollary 3.6.

Suppose we are given a group presentation \((S \mid R)\), with \(S, R\) finite, and a fixed spin \(\pi\) on \(S\), that is, a cyclic order of \(S \cup S^{-1}\) (note that \(|S \cup S^{-1}| = 2|S| - |B|\), where \(B \subseteq S\) is the set of \(b \in S\) with \(b = b^{-1}\), i.e. the set of involutions). Moreover, we fix an assignment \(f : S \rightarrow \{0, 1\}\), and say that \(s \in S\) is spin-preserving if \(f(s) = 0\) and spin-reversing if \(f(s) = 1\). Let \(T(S)\) be the Cayley graph corresponding to the presentation \((S \mid \{b^2 \mid b \in B\})\), with parallel edges corresponding to the elements of \(B\) replaced by single, undirected edges, and note that \(T(S)\) is a tree. Easily, \(T(S)\) has a consistent embedding \(\tau\) in which the spin of each vertex is either \(\pi\) or its reversal, and each \(s \in S\) is spin-preserving if and only if \(f(s) = 0\). Now call our presentation \((S \mid R)\) facial with respect to the data \(\pi, f\), if for every rotation \(w'\) of every word \(w \in R\), and any vertex \(t\) of \(T(S)\), the walk on \(T(S)\) that starts at \(t\) and is induced by \(w'\) is facial in \(\tau\).

It is a good exercise to try prove that each relator of a facial presentation contains an even number of occurrences of spin-reversing generators unless we are in the rather trivial case where \(|S| = 1\).

Note that every facial walk consisting of two edges of \(T(S)\), or every two elements of \(S \cup S^{-1}\) that are adjacent in \(\pi\), uniquely determine a 2-way infinite, periodic, ‘facial’ word. This easily implies that there is a canonical way to rewrite any facial presentation as \((S \mid E_1, \ldots, E_k)\), where \(E_i\) is aperiodic and each 2-way infinite facial walk in \(T(S)\) is obtained by repeatedly reading one of the \(E_i\). Moreover, we have \(k \leq |S \cup S^{-1}|\), but \(k\) can be as small as 1 even if \(S\) is large and all faces are finite; consider for example the presentation \((a, b, c \mid abcba)\) which is facial with respect to the spin \(a, c^{-1}, b, a^{-1}, c, b^{-1}\) and all edges spin-preserving.

We can now formulate the main result of this section, which complements Corollary 3.6:

**Theorem 5.1.** The Cayley graph corresponding to any facial presentation is planar and admits a consistent accumulation-free embedding.

As an example application, consider a Coxeter presentation
\[
\langle s_1, \ldots, s_k \mid s_i^2, (s_1 s_2)^{r_{12}}, (s_2 s_3)^{r_{23}}, \ldots, (s_k s_1)^{r_{1k}} \rangle
\]
with all exponents \(r_{ij}\) at least 2 and possibly infinite. It is straightforward to check that every such presentation is facial with respect to to the spin \(s_1, \ldots, s_k\) and all generators spin-reversing. Thus Theorem 5.1 tells us that the corresponding Cayley graph is planar and accumulation-free (this fact is probably well-known to experts in geometry).

For the proof of Theorem 5.1 we will use the following theorem of Thomassen, which generalises MacLane’s classical planarity criterion [3, Theorem 4.5.1] to infinite accumulation-free planar graphs. A 2-basis of \(G\) is a generating set \(B\) of the cycle space \(\mathcal{C}_{f}(G)\) such that no edge of \(G\) appears in more than two elements of \(B\).

**Theorem 5.2 (I6, Section 7).** A 2-connected graph has a 2-basis if and only if it is planar and has an accumulation-free embedding.

The requirement that \(G\) be 2-connected is essential in this assertion: consider for example the Cayley graph \(G\) corresponding to the presentation \((a_1, a_2, z \mid a_1 a_2 = a_2 a_1)\). Thus \(G\) is the free product of the square grid with the integer line. Note that the squares of the former factor form a 2-basis of \(G\), still \(G\) does not have an accumulation-free embedding.

In order to be able to still apply Theorem 5.2 in our setup, we will use the following fact. We say that a group presentation \(A\) contains a group presentation \(B\), if the Cayley graph corresponding to \(B\) is a subgraph of the Cayley graph corresponding to \(A\). Note that this means that the generating set of \(A\) contains that of \(B\), but the sets of relators can be quite different.

**Lemma 5.3.** Every facial presentation is contained in a facial presentation the Cayley graph of which is 2-connected.
Proof. Let \( \langle S \mid R \rangle \) be a facial presentation with respect to a spin \( \pi \) and an assignment \( f \) as in the above definition. If its Cayley graph \( G \) is 2-connected we are done, so suppose it is not. Then any vertex \( x \) separates \( S \) and we can find \( s, t \in S \cup S^{-1} \) that are consecutive in the spin \( \pi \) but \( xs \) and \( xt \) lie in distinct component of \( G - x \). We now construct a new presentation \( \langle S' \mid R' \rangle \) containing \( \langle S \mid R \rangle \) as follows. Firstly, we add a new generator \( z \) to \( S \) to obtain \( S' \). Secondly, add the relator \( z = t^{-1}s \) to \( R \). Finally, for every \( w \in R \) containing \( t^{-1}s \) (respectively \( s^{-1}t \)) as a subword – assume here that \( w \) is spelt without using exponents other than \( -1 \) – replace that subword by the letter \( z \) (resp. \( z^{-1} \)). Let \( R' \) be the set of relators obtained after all these changes.

It is easy to see that \( \langle S' \mid R' \rangle \) contains \( \langle S \mid R \rangle \), as it amounts to adding a redundant generator \( z \). It is also straightforward to see that the data \( \pi, f \) can be extended so as to make \( \langle S' \mid R' \rangle \) a facial presentation. Indeed, we can let \( f(z) = f(s) \XOR f(t) \). To extend \( \pi \) to \( S' \), let us assume without loss of generality that \( s \) immediately precedes \( t \) in \( \pi \). If \( s \) is spin-preserving, i.e. \( f(s) = 0 \), then we insert \( z^{-1} \) into \( \pi \) at the position just before \( s \). Otherwise, we insert \( z^{-1} \) into \( \pi \) at the position just after \( s \). Similarly, we insert \( z \) at the position just after (respectively, before) \( t \) if \( t \) is spin-preserving (resp. spin-reversing). It is now straightforward to check that every relator in \( R' \) is facial with respect to this data. \( \Box \)

We can now prove Theorem 5.1.

Proof. Let \( \langle S \mid R \rangle \) be a facial presentation. By our last lemma, we can find a facial presentation \( \langle S' \mid R' \rangle \) containing \( \langle S \mid R \rangle \) the Cayley graph \( G' \) of which is 2-connected. Let us show that \( G' \) admits an accumulation-free embedding.

Recall that \( G' \) admits a presentation of the form
\[
\langle S \mid E_1^{r_1}, \ldots, E_k^{r_k} \rangle,
\]
where each \( E_i \) is aperiodic facial word. We may assume without loss of generality that \( r_i \) is minimal with the property that \( E_i^{r_i} \) induces a closed walk in \( G' \), for otherwise we can replace \( r_i \) with some smaller value in the \( i \)th relator and obtain an equivalent presentation.

It would make our proof simpler if for every \( s \in S \cup S^{-1} \) and every \( E_i \), the letter \( s \) appears at most once in \( E_i \). This however is not always the case: the presentation \( \langle c, b \mid cbcb^{-1} \rangle \) for example, the Cayley graph of which is a square grid, is facial with respect to to the spin \( c, b, c^{-1}, b^{-1} \), with \( c \) being spin-preserving and \( b \) spin-reversing; but \( c \) appears twice in the word \( cb cb^{-1} \) (in this section we consider \( s \) and \( s^{-1} \) to be distinct letters). Still, we can easily modify our presentation whenever this situation occurs, to ensure that each letter \( s \) appears at most once in each \( E_i \). To begin with, note that \( s \) can appear at most twice in \( E_i \): to get \( E_i \) is uniquely determined (up to rotation and reversal) by any letter and ‘side’, if \( E_i \) contains three occurrences of \( s \), then two of them will correspond to the same side, implying that \( E_i \) is periodic contrary to our assumption. By the same arguments, \( s \) appears twice in \( E_i \), then this means that some rotation of the word \( E_i \) is read along both ‘sides’ of \( s \) in \( T(S) \). To avoid this situation, we can extend \( S \) by a new generator \( s' \), and add the relator \( s's^{-1} \) to our presentation. The new presentation yields the same Cayley graph with a parallel edge added to each \( s \) edge, and it is straightforward to amend the spin data to make sure that the presentation is still facial. Thus from now on we will assume that

for every \( s \in S \cup S^{-1} \) and every \( E_i \), the letter \( s \) appears at most once in \( E_i \). \hspace{1cm} (3)

It is still possible though that \( E_i \) contains both \( s \) and \( s^{-1} \).

Let \( W \) be the set of walks in \( G' \) induced by the above relators \( E_i^{r_i} \). Note that (3) and our choice of the \( r_i \) imply that

no walk in \( W \) traverses any edge of \( G' \) twice in the same direction. \hspace{1cm} (4)

For if this was the case, then the subwalk between to subsequent visits to the first endpoint of that edge would be closed, and by (3) it would be induced by (a rotation of) \( E_i^{m} \) with \( m < r_i \).

Recall that \( W \) generates \( H_1(G') \) (Lemma 3.1). The idea is to try apply Theorem 5.2 to \( W \), adapting the fact that every edge of a planar graph appears in just two facial walks to our situation.

Before we do that, we first simplify \( W \) as follows. For every walk \( w \in W \) traversing some edge \( e \) of \( G' \) in both directions, we split \( w \) into two closed walks \( w_1, w_2 \) that traverse \( e \) less often in total in
such a way that $w_1 + w_2$ corresponds to the same element of $H_1(G')$ as $w$. We repeat this recursively as often as needed until no walk traverses an edge in both directions. Finally, if two of the resulting closed walks can be obtained from one another by rotation or inversion, we delete one of them, and repeat until no such pairs exist. Let $W'$ denote the resulting set of walks.

By construction, $W'$ still generates $H_1(G')$. We claim that $W'$ has the desired property that every edge of $G'$ appears in at most two elements of $W'$, and at most once in each of them.

Indeed, since each element of $W$ traverses each edge at most once in each direction by (4), each element of $W$ traverses each edge at most once in total. Next, suppose that three distinct walks in $W'$ traverse some edge $e$. Again by (4), no two of them come from the same element of $W$. Then, as our presentation is planar, two of them, call them $w_1$, $w_2$, are induced by the same relator $E_{r_i}^{i}$. Thus each of $w_1$, $w_2$ contains the same number of edges of the colour of $e$ and, by (3), the same subword of $E_{r_i}^{i}$ (which must be a rotation of the word $E_i$) is read between any two subsequent traversals of such an edge. This easily implies that $w_1$ is a rotation of $w_2$, contradicting the construction of $W'$.

This proves that each edge appears at most twice in $W'$. Since $W'$ generates $H_1(G')$, the set of edge-sets of its elements generates $C_f(G')$. Splitting each such edge-set into edge-disjoint cycles – it is well-known that this is possible [3, Proposition 1.9.2.] – we obtain a 2-basis of $C_f(G')$. By Theorem 5.2, $G'$ admits an accumulation-free embedding, and so does its subgraph $G$. By Corollary 3.6, $G$ even admits a consistent accumulation-free embedding. □

Using the results of Section 3 we can prove now that the converse of Theorem 5.1 is also true, yielding Corollary 1.2; we repeat its statement here.

**Corollary.** A Cayley graph admits an accumulation-free embedding if and only if it admits a facial presentation.

**Proof.** Let $G$ be a Cayley graph with an accumulation-free embedding. Then $G$ admits a consistent accumulation-free embedding by Corollary 3.6 σ. As in the first part of the proof of Theorem 1.1, the set $R$ of relations corresponding to the finite facial closed walks of $σ$ incident with the group identity yields a presentation of $G$, and this presentation is, by construction, facial with respect to the spin data of $σ$. □

Note that this implies that every group admitting an accumulation-free planar Cayley graph is finitely presented. This fact extends to all planar groups [4]. In the accumulation-free case $|S ∪ S^{-1}|$ is an upper bound on the number of relators needed to present a group with generating set $S$, but in the general case this is not necessarily the case; see [8, Problem 10.2.].

**Remark.** One could modify the definition of a facial presentation by not giving involutions in $S$ any special treatment, that is, by letting $T(S)$ be the Cayley graph of the presentation $(S | ∅)$ (a tree of degree $2|S|$). Theorem 5.1 would then still be true by the same proof, but I suspect that its converse in Corollary 1.2 would fail; see Fig. 1.

It is known that the groups admitting an accumulation-free planar Cayley graph can be effectively enumerated [4]. Using Corollary 1.2 we can strengthen this as follows.

**Corollary 5.4.** The accumulation-free planar Cayley graphs can be effectively enumerated.

**Proof.** It is easy to construct an algorithm that given an abstract group presentation $(S | R)$, with both $S, R$ finite, decides whether this presentation is facial with respect to some spin data, since there are only finitely many possibilities for such data. The assertion thus follows from Corollary 1.2. □

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4 It is easy to see that the canonical projection of a generating set of $H_1(G)$ to $C_f(G')$ generates $C_f(G)$; the converse is not always true [12, Figure 9].
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