LARGE DEVIATION PRINCIPLE OF OCCUPATION MEASURE
FOR STOCHASTIC REAL GINZBURG-LANDAU EQUATION
DRIVEN BY $\alpha$-STABLE NOISES

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Abstract

We shall establish a large deviation principle for some occupation measure of the stochastic real Ginzburg-Landau equation driven by $\alpha$-stable noises. As a consequence, we obtain the exact rate of exponential ergodicity of the stochastic system under $\tau$-topology.

Keywords: Stochastic real Ginzburg-Landau equation; $\alpha$-stable noises; Large deviation principle; Strong Feller property; Irreducibility; Occupation measure.

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1. Introduction

We are concerned with the Large Deviation Principle (LDP in short) of the occupation measure for the stochastic real Ginzburg-Landau equation driven by $\alpha$-stable noises on torus $\mathbb{T} := \mathbb{R}/\mathbb{Z}$ as follows:

\begin{equation}
    dX - \partial_\xi^2 X dt - (X - X^3)dt = dL_t,
\end{equation}

where $X : [0, +\infty) \times \mathbb{T} \times \Omega \to \mathbb{R}$ and $L_t$ is an $\alpha$-stable noise. It is known (\cite{27}) that Eq. (1.1) admits a unique mild solution $X$ in the càdlàg space almost surely. As $\alpha \in (3/2, 2)$, $X$ is a Markov process with a unique invariant measure $\mu$. See Section 2 below for details. By the uniqueness (see \cite{2}), $\mu$ is ergodic in the sense that

\[
    \lim_{T \to \infty} \frac{1}{T} \int_0^T \Psi(X_t)dt = \int \Psi d\mu \quad \mathbb{P}\text{-a.s.}
\]

for all initial state $x_0$ and all continuous and bounded functions $\Psi$.

There are not many literatures on the study of invariant measures and the long time behavior of stochastic systems driven by $\alpha$-stable type noises. \cite{19, 20} studied the exponential mixing for a family of semi-linear SPDEs with Lipschitz nonlinearity, while \cite{9} obtained the existence of invariant measures for 2D stochastic Navier-Stokes equations forced by $\alpha$-stable noises with $\alpha \in (1, 2)$. \cite{28} proved the exponential mixing for a family of 2D SDEs forced by degenerate $\alpha$-stable noises. For the long term behaviour about stochastic system drive by Lévy noises, we refer to \cite{11, 18} and the literatures therein.

The large deviations for stochastic partial differential equations (SPDEs) have been intensively studied in recent years. Most of them, however, are concentrated on the
small noise large deviation principles of Freidlin-Wentzell’s type, which provide estimates for the probability that stochastic systems converge to their deterministic part as noises tend to zero, see [1, 10, 16, 22, 24, 29] and the references therein. But there are only very few papers on the large deviations of Donsker-Varadhan’s type for large time, which estimate the probability of the occupation measures’ deviation from invariant measure. Gourcy [12, 13] established the LDP for occupation measures of stochastic Burgers and Navier-Stokes equations perturbed by a rough white-noise force by the means of the hyper-exponential recurrence due to Wu [26]. Jakšić et al. [14] established the LDP for occupation measures of SPDE with smooth random perturbations by Kifer’s large deviation criterion [17]. They [15] also gave the large deviations estimates for dissipative PDEs with rough noise by Wu’s hyper-exponential recurrence criterion. The irreducibility and strong Feller property play the crucial role in their proof.

Our aim is to establish the LDP for the occupation measure \( \mathcal{L}_t \) of the system (1.1) given by

\[
\mathcal{L}_t(A) := \frac{1}{t} \int_0^t \delta_{\xi_s}(A) ds \quad \text{for any measurable set } A,
\]

where \( \delta_a \) is the Dirac measure at \( a \). As a consequence, we also obtain the exact rate of exponential ergodicity obtained in [27]. The LDP for empirical measures is one of the strongest ergodicity results for the long time behavior of Markov processes. It has been one of the classical research topics in probability since the pioneering work of Donsker and Varadhan [8].

Our approach to the LDP for the occupation measure \( \mathcal{L}_t \) of the system (1.1) is based on [26] by Wu, from which we need to prove irreducibility and verify the hyper-exponential recurrence condition. Due to the loss of the second moment of \( \alpha \)-stable noises, many nice analysis tools such as the Burkholder-Davis-Gundy inequality in the Wiener noise case are not available. We shall solve a control problem and apply a maximal inequality in [27] to get irreducibility. To verify the hyper-exponential recurrence condition, we shall sample a Markov chain from the stochastic system and estimate its hitting time to a compact set.

The paper is organized as follows. In Section 2, we first give a brief review of some known results about the existence and uniqueness of solutions and invariant probability measures for stochastic Ginzburg-Landau equations. In Section 3, we recall the results in [26] about the LDP for Markov processes with strong Feller property and topological irreducibility. We will also present the main theorem in this section. In the last two sections, we prove the irreducibility and verify hyper-exponential recurrence property, which imply the large deviation result.

2. Stochastic real Ginzburg-Landau equations

Let \( T = \mathbb{R}/\mathbb{Z} \) be equipped with the usual Riemannian metric, and let \( d\xi \) denote the Lebesgue measure on \( T \). Then

\[
H := \left\{ x \in L^2(T; \mathbb{R}); \int_T x(\xi) d\xi = 0 \right\}
\]
is a separable real Hilbert space with inner product
\[ \langle x, y \rangle_H := \int_T x(\xi)y(\xi)d\xi, \quad \forall \, x, y \in H. \]

Denote \( Z_* = \mathbb{Z} \setminus \{0\} \), it is well known that
\[ \{ e_k : e_k = e^{i2\pi k\xi}, \, k \in \mathbb{Z}_* \} \]
is an orthonormal basis of \( H \). For each \( x \in H \), it can be represented by Fourier series
\[ x = \sum_{k \in \mathbb{Z}_*} x_k e_k \quad \text{with} \quad x_k \in \mathbb{C}, \quad x_{-k} = \overline{x_k}. \]

Let \( \Delta \) be the Laplace operator on \( H \). It is well known that \( D(\Delta) = H^{2,2}(\mathbb{T}) \cap H \). In our setting, \( \Delta \) can be determined by the following relations: for all \( k \in \mathbb{Z}_* \),
\[ \Delta e_k = -\gamma_k e_k \quad \text{with} \quad \gamma_k = 4\pi^2|k|^2, \]
with
\[ H^{2,2}(\mathbb{T}) \cap H = \left\{ x \in H; \, x = \sum_{k \in \mathbb{Z}_*} x_k e_k, \quad \sum_{k \in \mathbb{Z}_*} |\gamma_k|^4|x_k|^2 < \infty \right\}. \]

Denote \( A = -\Delta \), \( D(A) = H^{2,2}(\mathbb{T}) \cap H \).

Define the operator \( A^\sigma \) with \( \sigma \geq 0 \) by
\[ A^\sigma x = \sum_{k \in \mathbb{Z}_*} \gamma_k^\sigma x_k e_k, \quad x \in D(A^\sigma), \]
where \( \{x_k\}_{k \in \mathbb{Z}_*} \) are the Fourier coefficients of \( x \), and
\[ D(A^\sigma) := \left\{ x \in H : \, x = \sum_{k \in \mathbb{Z}_*} x_k e_k, \quad \sum_{k \in \mathbb{Z}_*} |\gamma_k|^{2\sigma}|x_k|^2 < \infty \right\}. \]

Given \( x \in D(A^\sigma) \), its norm is
\[ \|A^\sigma x\|_H := \left( \sum_{k \in \mathbb{Z}_*} |\gamma_k|^{2\sigma}|x_k|^2 \right)^{1/2}. \]

Moreover, let
\[ V := D(A^{1/2}), \]
which is densely and compactly embedded in \( H \).

We shall study 1D stochastic Ginzburg-Landau equation on \( \mathbb{T} \) as the following
\[ (2.1) \]
\[ \begin{cases} dX_t + AX_t dt = N(X_t) dt + dL_t, \\ X_0 = x_0, \end{cases} \]
where
(i) the nonlinear term \( N \) is defined by
\[ N(u) = u - u^3, \quad u \in H. \]
(ii) $L_t = \sum_{k \in \mathbb{Z}} \beta_k e_k(t)e_k$ is an $\alpha$-stable process on $H$ with $\{l_k(t)\}_{k \in \mathbb{Z}}$, being i.i.d. 1-dimensional symmetric $\alpha$-stable process sequence with $\alpha > 1$, see [23]. Moreover, we assume that there exist some $C_1, C_2 > 0$ so that $C_1 \gamma_k^{-\beta} \leq |\beta_k| \leq C_2 \gamma_k^{-\beta}$ with $\beta > \frac{1}{2} + \frac{1}{2\alpha}$.

Here we consider a general $E$-valued càdlàg Markov process,

$$(\Omega, \{\mathcal{F}_t^0\}_{t \geq 0}, \mathcal{F}, \{X_t\}_{t \geq 0}, (\mathbb{P}_x)_{x \in E})$$

whose transition probability is denoted by $\{P_t(x, dy)\}_{t \geq 0}$, where $\Omega := D([0, + \infty); E)$ is the space of the càdlàg functions from $[0, + \infty)$ to $E$ equipped with the Skorokhod topology, $\mathcal{F}_t^0 = \sigma\{X_s, 0 \leq s \leq t\}$ is the natural filtration.

For all $f \in b\mathcal{B}(E)$ (the space of all bound measurable functions), define

$$(P_t f)(x) = \int_E P_t(x, dy) f(y) \quad \text{for all } t \geq 0, x \in E.$$  

For all $t > 0$, $P_t$ is said to be strong Feller if $P_t \varphi \in C_b(E)$ for any $\varphi \in b\mathcal{B}(E)$; $P_t$ is irreducible in $E$ if $P_t 1_O(x) > 0$ for any $x \in E$ and any non-empty open subset $O$ of $E$; $\{P_t\}_{t \geq 0}$ is accessible to $x \in E$, if the resolvent $\{R_\lambda\}_{\lambda > 0}$ satisfies

$$R_\lambda(y, \mathcal{U}) := \int_0^\infty e^{-\lambda t} P_t(y, \mathcal{U}) dt > 0, \quad \forall \lambda > 0$$

for all $y \in E$ and all neighborhoods $\mathcal{U}$ of $x$. Notice that the accessibility of $\{P_t\}_{t \geq 0}$ to any $x \in E$ is the so called topological transitivity in Wu [26].

[27] gives the following existence and uniqueness results for the solutions and the invariant measure.

**Theorem 2.1** (27). If $\alpha \in (3/2, 2)$ and $\frac{1}{2} + \frac{1}{2\alpha} < \beta < \frac{3}{2} - \frac{1}{\alpha}$, the following statements hold:

1. For every $x_0 \in H$ and $\omega \in \Omega$ a.s., Eq. (2.1) admits a unique mild solution $X_\omega(\omega) \in D([0, \infty); H) \cap D((0, \infty); V)$. Moreover, $X_\omega(\omega)$ has the following form:

$$X_t(\omega) = e^{-At} x_0 + \int_0^t e^{-A(t-s)} N(X_s(\omega)) ds + \int_0^t e^{-A(t-s)} dL_s(\omega), \quad \forall t > 0.$$  

2. $X$ is a Markov process, and its transitive probability $P_t$ is strong Feller in $H$ and in $V$ for any $t > 0$.

3. $X$ admits a unique invariant measure, and the invariant measure is supported on $V$.

For $E = H$ or $E = V$, let $\mathcal{M}_1(E)$ be the space of probability measures on $E$ equipped with the Borel $\sigma$-field $\mathcal{B}(E)$. On $\mathcal{M}_1(E)$, we consider $\sigma(\mathcal{M}_1(E), b\mathcal{B}(E))$, the so called $\tau$-topology of convergence against measurable and bounded functions which is much stronger than the usual weak convergence topology $\sigma(\mathcal{M}_1(E), C_b(E))$, see [3, Section 6.2].

3. INTRODUCTION TO LARGE DEVIATIONS AND THE MAIN RESULT

In this section, we recall some general results on the Large Deviation Principle for strong Feller and topologically irreducible Markov processes. We follow [23, 26].
3.1. **Entropy of Donsker-Varadhan.**

Consider a general $E$-valued càdlàg Markov process

$$(\Omega, \{\mathcal{F}_t\}_{t \geq 0}, \mathcal{F}, \{X_t(\omega)\}_{t \geq 0}, \{\mathbb{P}_x\}_{x \in E}),$$

where

- $\Omega = D([0, +\infty); E)$ (the space of the càdlàg functions from $[0, +\infty)$ to $E$ equipped with the Skorokhod topology; for any $\omega \in \Omega$, $X_t(\omega) = \omega(t)$);
- $\mathcal{F}^0_t = \sigma\{X_s; 0 \leq s \leq t\}$ for any $t \geq 0$ (nature filtration);
- $\mathcal{F} = \sigma\{X_t; t \geq 0\}$ and $\mathbb{P}_x(X_0 = x) = 1$.

Hence, $\mathbb{P}_x$ is the law of the Markov process with initial state $x \in E$. For any initial measure $\nu$ on $E$, let $\mathbb{P}_\nu(d\omega) := \int_E \mathbb{P}_x(d\omega) \nu(dx)$.

The empirical measure of level-3 (or process level) is given by

$$R_t := \frac{1}{t} \int_0^t \delta_{\theta_s X} ds$$

where $(\theta_s X)_t = X_{s+t}$ for all $t, s \geq 0$ are the shifts on $\Omega$. Thus, $R_t$ is a random element of $\mathcal{M}_1(\Omega)$, the space of all probability measures on $\Omega$.

The level-3 entropy functional of Donsker-Varadhan $H : \mathcal{M}_1(\Omega) \to [0, +\infty]$ is defined by

$$H(Q) := \begin{cases} \mathbb{E}^Q \log h_{\mathcal{F}_t}^\mathcal{F}(\bar{Q}_{w(\cdot), 0}; \mathbb{P}_{w(0)}) & \text{if } Q \in \mathcal{M}_1^s(\Omega); \\ +\infty & \text{otherwise,} \end{cases}$$

where

- $\mathcal{M}_1^s(\Omega)$ is the subspace of $\mathcal{M}_1(\Omega)$, whose elements are moreover stationary;
- $\bar{Q}$ is the unique stationary extension of $Q \in \mathcal{M}_1^s(\Omega)$ to $\bar{\Omega} := D(\mathbb{R}; E)$; $\mathcal{F}_t = \sigma\{X(u); s \leq u \leq t\}$, $\forall s, t \in \mathbb{R}, s \leq t$;
- $\bar{Q}_{w(\cdot), t}$ is the regular conditional distribution of $\bar{Q}$ knowing $\mathcal{F}_t^{-\infty}$;
- $h_{\mathcal{G}}(\nu; \mu)$ is the usual relative entropy or Kullback information of $\nu$ with respect to $\mu$ restricted to the $\sigma$-field $\mathcal{G}$, given by

$$h_{\mathcal{G}}(\nu; \mu) := \begin{cases} \int \frac{d\nu}{d\mu} |_{\mathcal{G}} \log \left( \frac{d\nu}{d\mu} |_{\mathcal{G}} \right) d\mu & \text{if } \nu \ll \mu \text{ on } \mathcal{G}; \\ +\infty & \text{otherwise.} \end{cases}$$

The level-2 entropy functional $J : \mathcal{M}_1(E) \to [0, +\infty]$ which governs the LDP in our main result is

$$(3.1) \quad J(\beta) = \inf\{H(Q) | Q \in \mathcal{M}_1^s(\Omega) \text{ and } Q_0 = \beta\}, \quad \forall \beta \in \mathcal{M}_1(E),$$

where $Q_0(\cdot) = Q(X_0 = \cdot)$ is the marginal law at $t = 0$.

3.2. **The hyper-exponential recurrence criterion.** Recall the following criterion of hyper-exponential recurrence established by Wu [26, Theorem 2.1], also see Gourcy [13, Theorem 3.2].

For any measurable set $K \in E$, let

$$(3.2) \quad \tau_K = \inf\{t \geq 0 \ s.t. \ X_t \in K\}, \quad \tau_K^{(1)} = \inf\{t \geq 1 \ s.t. \ X_t \in K\}.$$
Theorem 3.1. [26] Let \( A \in M_1(E) \) and assume that 
\( P_t \) is strong Feller and topologically irreducible on \( E \).

If for any \( \lambda > 0 \), there exists some compact set \( K \subset E \), such that 
\[
\sup_{\nu \in A} \mathbb{E}^\nu e^{\lambda \tau_{K}} < \infty, \quad \text{and} \quad \sup_{x \in K} \mathbb{E}^x e^{\lambda \tau_{K}} < \infty.
\]

Then the family \( \mathbb{P}_\nu (\mathcal{L}_t \in \cdot) \) satisfies the LDP on \( M_1(E) \) w.r.t. the \( \tau \)-topology with the rate function \( J \) defined by (3.1), and uniformly for initial measures \( \nu \) in the subset \( A \). More precisely, the following three properties hold:

(a1) for any \( a \geq 0 \), \( \{ \mu \in M_1(E); J(\mu) \leq a \} \) is compact in \( (M_1(E), \tau) \);
(a2) (the lower bound) for any open set \( G \) in \( (M_1(E), \tau) \),
\[
\liminf_{T \to \infty} \frac{1}{T} \log \inf_{\nu \in A} \mathbb{P}_\nu (\mathcal{L}_T \in G) \geq - \inf_F J;
\]
(a3) (the upper bound) for any closed set \( F \) in \( (M_1(E), \tau) \),
\[
\limsup_{T \to \infty} \frac{1}{T} \log \sup_{\nu \in A} \mathbb{P}_\nu (\mathcal{L}_T \in F) \leq - \inf_F J.
\]

Now, we are ready to state our main result.

Theorem 3.2. Assume that \( \alpha \in (3/2, 2) \) and \( \frac{1}{2} + \frac{1}{2\alpha} < \beta < \frac{3}{2} - \frac{1}{\alpha} \). Let 
\[
\mathcal{M}_L := \left\{ \nu \in M_1(H) \middle| \int_H \| u \|_H \nu(du) \leq L \right\}
\]
for any \( L > 0 \).

Then the family \( \mathbb{P}_\nu (\mathcal{L}_T \in \cdot) \) as \( T \to +\infty \) satisfies the large deviation principle with respect to the \( \tau \)-topology, with speed \( T \) and rate function \( J \) defined by (3.1), uniformly for any initial measure \( \nu \) in \( \mathcal{M}_L \). More precisely, the following three properties hold:

(a1) for any \( a \geq 0 \), \( \{ \mu \in M_1(H); J(\mu) \leq a \} \) is compact in \( (M_1(H), \tau) \);
(a2) (the lower bound) for any open set \( G \) in \( (M_1(H), \tau) \),
\[
\liminf_{T \to \infty} \frac{1}{T} \log \inf_{\nu \in \mathcal{M}_L} \mathbb{P}_\nu (\mathcal{L}_T \in G) \geq - \inf_F J;
\]
(a3) (the upper bound) for any closed set \( F \) in \( (M_1(H), \tau) \),
\[
\limsup_{T \to \infty} \frac{1}{T} \log \sup_{\nu \in \mathcal{M}_L} \mathbb{P}_\nu (\mathcal{L}_T \in F) \leq - \inf_F J.
\]

Proof. To prove the LDP, we only need to verify the conditions of Theorem 3.1 for the Markov process \( X \). Since the strong Feller property has been established in [27], it remains to prove the irreducibility in \( H \) and the hyper-exponential recurrence property. Those will be done in Section 4 and Section 5, respectively.

□

The LDP w.r.t. the \( \tau \)-topology is much stronger than that w.r.t. the usual weak convergence topology as in Donsker-Varadhan [8]. Some interesting consequence of this LDP can be deduced for many physical quantities of the system such as \( \| x \|_V \) and the Sobolev norm, which are not continuous on \( H \), see [13, Corollary 1.2, Proposition 1.3] for useful applications.
4. Irreducibility in $H$

In this section, we shall prove that $\{X_t\}_{t \geq 0}$ in the system (2.1) is irreducible in $H$. Together with the strong Feller property established in [27] (see Theorem 2.1), this gives another proof to the existence and uniqueness of invariant measure by classical Doob’s Theorem.

Let $C > 0$ be a constant and let $C_p > 0$ be a constant depending on the parameter $p$. We shall often use the following inequalities, see [27] or by similar calculation.

\begin{align*}
(4.1) & \quad \|A^\sigma e^{-At}\| \leq C_\sigma t^{-\sigma}, \quad \forall \, \sigma > 0 \quad \forall \, t > 0; \\
(4.2) & \quad \|N(x)\|_V \leq C(\|x\|_V + \|x\|_V^3), \quad \forall \, x \in V; \\
(4.3) & \quad \|AN(x)\|_H \leq C(1 + \|x\|_V^2)(1 + \|Ax\|_H^2); \\
(4.4) & \quad \|x\|_L^4 \leq \|x\|_V^2 \|x\|_H^2, \quad \forall \, x \in V.
\end{align*}

4.1. Irreducibility of stochastic convolution. Let us first consider the following Ornstein-Uhlenbeck process:

\begin{equation}
(4.5) \quad dZ_t + AZ_t dt = dL_t, \quad Z_0 = 0,
\end{equation}

where $L_t = \sum_{k \in \mathbb{Z}_+} \beta_k l_k(t)e_k$ is an $\alpha$-stable process on $H$. It is well known that

\begin{equation*}
Z_t = \int_0^t e^{-(t-s)A}dL_s = \sum_{k \in \mathbb{Z}_+} z_k(t)e_k,
\end{equation*}

where

\begin{equation*}
z_k(t) = \int_0^t e^{-\gamma_k(t-s)}\beta_k dl_k(s).
\end{equation*}

The following maximal inequality can be found in [27, Lemma 3.1].

**Lemma 4.1.** For any $T > 0, 0 \leq \theta < \beta - \frac{1}{2\alpha}$ and all $0 < p < \alpha$, we have

\[ E \sup_{0 \leq t \leq T} \|A^\theta Z_t\|_H^p \leq CT^{p/\alpha}, \]

where $C$ depends on $\alpha, \theta, \beta, p$.

The following lemma is concerned with the support of distribution of $(\{Z_t\}_{0 \leq t \leq T}, Z_T)$.

**Lemma 4.2.** For any $T > 0, 0 < p < \infty$, the random variable $(\{Z_t\}_{0 \leq t \leq T}, Z_T)$ has a full support in $L^p([0, T]; V) \times V$. More precisely, for any $\phi \in L^p([0, T]; V), a \in V, \epsilon > 0$,

\[ P \left( \int_0^T \|Z_t - \phi_t\|_V^p dt + \|Z_T - a\|_V < \epsilon \right) > 0. \]

**Proof.** First, by Lemma 4.1 we have $Z \in L^\infty([0, T]; V)$, a.s. For any $N \in \mathbb{N}$, let $H_N$ be the Hilbert space spanned by $\{e_k\}_{1 \leq k \leq N}$, and let $\pi_N : H \to H_N$ be the orthogonal projection. Furthermore, define

\[ \pi^N = I - \pi_N, \quad H^N = \pi^N H. \]
By the independence of $\pi_N Z$ and $\pi^N Z$, for any $\phi_t \in L^p([0, T]; V)$, $a \in V$, we have
\[
\mathbb{P} \left( \int_0^T \| Z_t - \phi_t \|^p_V \, dt + \| Z_T - a \|_V < \varepsilon \right) \\
\geq \mathbb{P} \left( \int_0^T \| \pi_N (Z_t - \phi_t) \|^p_V \, dt + \| \pi_N (Z_T - a) \|_V < \frac{\varepsilon}{2^{p+1}} \right) \\
= \mathbb{P} \left( \int_0^T \| \pi_N (Z_t - \phi_t) \|^p_V \, dt + \| \pi_N (Z_T - a) \|_V < \frac{\varepsilon}{2^{p+1}} \right) \\
\times \mathbb{P} \left( \int_0^T \| \pi^N (Z_t - \phi_t) \|^p_V \, dt + \| \pi^N (Z_T - a) \|_V < \frac{\varepsilon}{2^{p+1}} \right).
\]

By the same argument in the section 4.2 of [21], we obtain
\[
\mathbb{P} \left( \int_0^T \| \pi_N (Z_t - \phi_t) \|^p_V \, dt + \| \pi_N (Z_T - a) \|_V < \frac{\varepsilon}{2^{p+1}} \right) > 0.
\]

To finish the proof, it suffices to show
\[
\mathbb{P} \left( \int_0^T \| \pi^N (Z_t - \phi_t) \|^p_V \, dt + \| \pi^N (Z_T - a) \|_V < \frac{\varepsilon}{2^{p+1}} \right) > 0.
\]

For any $\theta \in \left( \frac{1}{2}, \beta - \frac{1}{2\alpha} \right)$, by Lemma 4.1 (with $p = 1$ therein), spectral gap inequality and Chebyshev inequality, we have
\[
\mathbb{P} \left( \sup_{0 \leq t \leq T} \| \pi^N Z_t \|_V \leq \delta \right) = 1 - \mathbb{P} \left( \sup_{0 \leq t \leq T} \| \pi^N Z_t \|_V > \delta \right) \\
\geq 1 - \mathbb{P} \left( \sup_{0 \leq t \leq T} \| \pi^N A^\theta Z_t \|_H > \delta \gamma_N^{\theta - \frac{1}{2}} \right) \\
\geq 1 - \mathbb{P} \left( \sup_{0 \leq t \leq T} \| A^\theta Z_t \|_H > \delta \gamma_N^{\theta - \frac{1}{2}} \right) \\
\geq 1 - C_{\alpha, \beta, T} \delta^{-1} \gamma_N^{\frac{1}{2} - \theta},
\]
where $C_{\alpha, \beta, T}$ depends on $\alpha, \beta, T$. By the previous inequality, as long as $N$ (depending on $\varepsilon, p, \phi$) is sufficiently large, we have
\[
\mathbb{P} \left( \sup_{0 \leq t \leq T} \| \pi^N Z_t \|_V \leq \frac{\varepsilon}{2^{2p+2}} \right) > 0,
\]
and
\[
\int_0^T \| \pi^N \phi_t \|^p_V \, dt + \| \pi^N a \|_V < \frac{\varepsilon}{2^{2p+2}}.
\]
Hence,
\[
\mathbb{P} \left( \int_0^T \left\| \pi^N (Z_t - \phi_t) \right\|_V^p dt + \left\| \pi^N (Z_T - a) \right\|_V < \frac{\varepsilon}{2^{p+1}} \right) \\
\geq \mathbb{P} \left( \int_0^T \left\| \pi^N Z_t \right\|_V^p dt + \left\| \pi^N Z_T \right\|_V < \frac{\varepsilon}{2^{2p+2}}, \int_0^T \left\| \pi^N \phi_t \right\|_V^p dt + \left\| \pi^N a \right\|_V < \frac{\varepsilon}{2^{2p+2}} \right) \\
= \mathbb{P} \left( \int_0^T \left\| \pi^N Z_t \right\|_V^p dt + \left\| \pi^N Z_T \right\|_V < \frac{\varepsilon}{2^{2p+2}} \right) > 0.
\]

The proof is complete. \(\square\)

4.2. A control problem for the deterministic system. Consider the deterministic system in \(H\),
\[
\partial_t x(t) + Ax(t) = N(x(t)) + u(t), \quad x(0) = x_0,
\]
where \(u \in L^2([0, T]; V)\). By using the similar argument in the proof of Lemma 4.2 in Xu [27], for every \(x(0) = x_0 \in H, u \in L^2([0, T]; V)\), the system (4.6) admits a unique solution \(x(\cdot) \in C([0, T]; H) \cap C((0, T]; V)\). Moreover, \(\{x(t)\}_{t \in [0, T]}\) has the following form:
\[
x(t) = e^{-A t} x_0 + \int_0^t e^{-A(t-s)} N(x(s))ds + \int_0^t e^{-A(t-s)} u(s)ds, \quad \forall t \in [0, T].
\]

Next, we shall prove that the deterministic system is approximately controllable in time \(T > 0\).

**Lemma 4.3.** For any \(T > 0, \varepsilon > 0, a \in V\), there exists some \(u \in L^\infty([0, T]; V)\) such that the system (4.6) satisfies that
\[
\left\| x(T) - a \right\|_V < \varepsilon.
\]

**Proof.** We shall prove the lemma by the following three steps.

**Step 1. Regularization.** For any \(\delta \in (0, T]\), let \(u(t) = 0\) for all \(t \in [0, \delta]\). Then the system (4.6) admits a unique solution \(x(\cdot) \in C([0, \delta]; H) \cap C((0, \delta]; V)\) with the following form:
\[
x(t) = e^{-A t} x_0 + \int_0^t e^{-A(t-s)} N(x(s))ds, \quad \forall 0 < t \leq \delta.
\]

**Step 2. Approximation at time \(T\) and Linear interpolation.** For any \(a \in V, \varepsilon > 0\), there exists a constant \(\theta > 0\) such that
\[
\left\| e^{-\theta A} a - a \right\|_V \leq \varepsilon.
\]
Setting \(x(t) = \frac{t-\delta}{T-\delta} e^{-\theta A} + \frac{T-t}{T-\delta} x(\delta)\) for all \(t \in [\delta, T]\). Then \(x(\cdot) \in C((0, T]; V)\). By (4.6), we have
\[
u(t) = \frac{e^{-\theta A} a - x(\delta)}{T-\delta} - A x(t) - N(x(t)), \quad \forall t \in [\delta, T].\]
Step 3. It remains to show that \( u \in L^\infty([0, T]; V) \). By (4.1), (4.2) and the construction of \( \{ x(t) \}_{t \in [0, T]} \) and \( \{ u(t) \}_{t \in [0, T]} \) above, it is sufficient to show that \( Ax(\delta) \in V \). For any \( t \in [\delta/2, \delta] \),

\[
x(t) = e^{(t-\delta/2)A}x(\delta/2) + \int_{\delta/2}^{t} e^{(t-s)A}N(x(s))\,ds,
\]

By (4.1), we have

\[
\|Ax(t)\|_H = \|Ae^{(t-\delta/2)A}x(\delta/2)\|_H + \left\|A \int_{\delta/2}^{t} e^{(t-s)A}N(x(s))\,ds\right\|_H \\
\leq \|Ae^{(t-\delta/2)A}x(\delta/2)\|_H + \int_{\delta/2}^{t} \|A^{1/2}e^{(t-s)A}\| \cdot \|A^{1/2}N(x(s))\|_H \,ds \\
\leq C_1(t - \delta/2)^{-1/2}\|x(\delta/2)\|_H + \int_{\delta/2}^{t} C_{1/2}(t-s)^{-1/2}\|N(x(s))\|_V \,ds.
\]

Since \( x(\cdot) \in C((0, \delta]; V) \), \( \sup_{t \in [\delta/2, \delta]} \|x(s)\|_V < \infty \). Together with (4.2) and (4.8), we obtain that

\[
\sup_{t \in [\delta/2, \delta]} \|Ax(t)\|_H < \infty.
\]

By the previous inequality and (4.3), we have

\[
\|A^{3/2}x(\delta)\|_H = \left\|A^{3/2}e^{(\delta/2)A}x(\delta/2) + A^{3/2} \int_{\delta/2}^{\delta} e^{(t-s)A}N(x(s))\,ds\right\|_H \\
\leq \left\|A^{3/2}e^{(\delta/2)A}x(\delta/2)\right\|_H + \int_{\delta/2}^{\delta} \|A^{1/2}e^{(\delta-s)A}\| \cdot \|AN(x(s))\|_H \,ds \\
\leq C_{3/2}(\delta/2)^{-3/2}\|x(\delta/2)\|_H + \int_{\delta/2}^{\delta} C_{1/2}(\delta-s)^{-1/2}\|AN(x(s))\|_H \,ds \\
\leq C_{3/2}(\delta/2)^{-3/2}\|x(\delta/2)\|_H + C \sup_{s \in [\delta/2, \delta]} (1 + \|x(s)\|_V^2) \cdot (1 + \|Ax(s)\|_H^2) \\
< \infty,
\]

which means that \( Ax(\delta) \in V \). The proof is complete.

\[\square\]

4.3. Irreducibility. Now we establish the irreducibility of the solutions to (2.41) in \( H \) by following the idea in [21, Theorem 5.4].

**Theorem 4.4.** For any initial value \( x_0 \in H \), the solution \( X = \{X_t\}_{t \geq 0} \) to the equation (2.41) is irreducible in \( H \).

**Proof.** Since for any \( 0 < t < T, X_t \in V \) a.s., for any \( a \in H, T > 0, \varepsilon > 0, \)

\[
P(\|X_T - a\|_H < \varepsilon) = \int_V P(\|X_T - a\|_H < \varepsilon | X_t = v) P(X_t \in dv),
\]

to prove that

\[
P(\|X_T - a\|_H < \varepsilon) > 0,
\]
it is sufficient to prove that for $0 < t < T$,
\[ P \left( \|X_T - a\|_H < \varepsilon | X_t = v \right) > 0 \quad \forall v \in V. \]

Without loss of generality, we can assume that the initial value $x_0$ is in the space $V$. Now we prove the theorem under this assumption in the following two steps.

**Step 1.** For any $a \in H, \varepsilon > 0$, there exists some $\theta > 0$ such that $e^{-\theta A}a \in V$ and
\[ \|a - e^{-\theta A}a\|_H \leq \frac{\varepsilon}{4}. \]

For any $T > 0$, by Lemma 4.3 and the spectral gap inequality, there exists some $u \in L^\infty([0, T]; V)$ such that the system
\[ \dot{x} + Ax = N(x) + u, \quad x(0) = x_0, \]

satisfies that
\[ \|x(T) - e^{-\theta A}a\|_H \leq \|x(T) - e^{-\theta A}a\|_V < \frac{\varepsilon}{4}. \]

Putting (4.9) and (4.10) together, we have
\[ \|x(T) - a\|_H < \frac{\varepsilon}{2}. \]

**Step 2:** We shall consider the systems (4.12) and (4.13) as follows:
\[ \begin{cases} 
\dot{z} + Az = u, & z(0) = 0, \\
\dot{y} + Ay = N(y + z), & y(0) = x_0 \in V,
\end{cases} \]

and
\[ \begin{cases} 
\text{d}Z_t + AZ_t \text{d}t = \text{d}L_t, & Z_0 = 0; \\
\text{d}Y_t + AY_t \text{d}t = N(Y_t + Z_t) \text{d}t, & Y_0 = x_0 \in V.
\end{cases} \]

By the arguments in the proof of Lemma 4.2 in [27], for any $x_0 \in V, u \in L^2([0, T]; V)$, the systems (4.12) and (4.13) admit the unique solutions $(y(\cdot), z(\cdot)) \in C([0, T]; V^2)$ and $(Y, Z) \in C([0, T]; V)^2$, a.s. Furthermore, denote
\[ x(t) = y(t) + z(t), \quad X_t = Y_t + Z_t, \quad \forall t \geq 0. \]

For any $0 \leq t \leq T$,
\[ \begin{align*}
\|Y_t - y(t)\|_H^2 &= 2 \int_0^t \|Y_s - y(s)\|_V^2 \text{d}s \\
&= 2 \int_0^t \langle Y_s - y(s), N(X_s) - N(x(s)) \rangle \text{d}s \\
&= 2 \int_0^t \|Y_s - y(s)\|_H^2 \text{d}s + 2 \int_0^t \langle Y_s - y(s), Z_s - z(s) \rangle \text{d}s \\
&- 2 \int_0^t \langle Y_s - y(s), X_s^3 - x^3(s) \rangle \text{d}s.
\end{align*} \]
Let us estimate the third term of the right hand side. Denoting $\Delta Y_s = Y_s - y(s)$ and $\Delta Z_s = Z_s - z(s)$, we have

\[
\int_0^t \langle Y_s - y(s), X_s^3 - x^3(s) \rangle \, ds \\
= \int_0^t \langle \Delta Y_s, [\Delta Y_s + \Delta Z_s + x(s)]^3 - x^3(s) \rangle \, ds \\
= \int_0^t \langle \Delta Y_s, [\Delta Y_s + \Delta Z_s]^3 + 3[\Delta Y_s + \Delta Z_s]^2 x(s) + 3[\Delta Y_s + \Delta Z_s] x^2(s) \rangle \, ds \\
= \int_0^t \langle \Delta Y_s, (\Delta Y_s)^3 + 3(\Delta Y_s)^2 \Delta Z_s + 3\Delta Y_s (\Delta Z_s)^2 + (\Delta Z_s)^3 \rangle \, ds \\
+ 3 \int_0^t \langle \Delta Y_s, [(\Delta Y_s)^2 + 2\Delta Y_s \Delta Z_s + (\Delta Z_s)^2] x(s) \rangle \, ds + 3 \int_0^t \langle \Delta Y_s, [\Delta Y_s + \Delta Z_s] x^2(s) \rangle \, ds.
\]

Since $\frac{3}{4}(\Delta Y_s)^4 + 3(\Delta Y_s)^3 x(s) + 3(\Delta Y_s)^2 x^2(s) \geq 0$, from the above relation we have

\[
\int_0^t \langle Y_s - y(s), X_s^3 - x^3(s) \rangle \, ds \\
\geq \int_0^t \langle \Delta Y_s, 3(\Delta Y_s)^2 \Delta Z_s + 3\Delta Y_s (\Delta Z_s)^2 + (\Delta Z_s)^3 \rangle \, ds \\
+ 3 \int_0^t \langle \Delta Y_s, [2\Delta Y_s \Delta Z_s + (\Delta Z_s)^2] x(s) \rangle \, ds \\
+ 3 \int_0^t \langle \Delta Y_s, \Delta Z_s x^2(s) \rangle \, ds + \frac{1}{4} \int_0^t \|\Delta Y_s\|_{L^4}^4 \, ds.
\]

Using the following Young inequalities: for all $y, z \in L^4(\mathbb{T}; \mathbb{R})$,

\[
\langle y, z \rangle = \left| \int_\mathbb{T} y(\xi) z(\xi) \, d\xi \right| \leq \frac{\int_\mathbb{T} y^4(\xi) \, d\xi}{80} + C \int_\mathbb{T} z^2(\xi) \, d\xi, \\
\langle y^2, z \rangle = \left| \int_\mathbb{T} y^2(\xi) z(\xi) \, d\xi \right| \leq \frac{\int_\mathbb{T} y^4(\xi) \, d\xi}{80} + C \int_\mathbb{T} z^2(\xi) \, d\xi, \\
\langle y^3, z \rangle = \left| \int_\mathbb{T} y^3(\xi) z(\xi) \, d\xi \right| \leq \frac{\int_\mathbb{T} y^4(\xi) \, d\xi}{80} + C \int_\mathbb{T} z^2(\xi) \, d\xi.
\]
and the Hölder inequality, we further get
\[
\int_0^t \langle Y_s - y(s), X^3_s - x^3(s) \rangle \, ds \\
\geq \frac{1}{80} \int_0^t \| \Delta Y_s \|_{L^4}^4 \, ds - 7C \int_0^t \| \Delta Z_s \|_{L^4}^4 \, ds \\
- 6C \int_0^t \| \Delta Z_s x(s) \|_{L^2}^2 \, ds - 3C \int_0^t \| (\Delta Z_s)^2 x(s) \|_{L^4}^4 \, ds \\
- 3C \int_0^t \| \Delta Z_s x^2(s) \|_{L^4}^4 \, ds
\]
\[
\geq \frac{1}{80} \int_0^t \| \Delta Y_s \|_{L^4}^4 \, ds - 7C \int_0^t \| \Delta Z_s \|_{L^4}^4 \, ds \\
- 6C \int_0^t \| \Delta Z_s \|_{L^4}^4 \| x(s) \|_{L^4}^4 \, ds - 3C \int_0^t \| \Delta Z_s \|_{L^4}^4 \| x(s) \|_{L^4}^4 \, ds \\
- 3C \int_0^t \| \Delta Z_s \|_{L^4}^4 \| x(s) \|_{L^4}^4 \, ds.
\]
Since \( x(t) = y(t) + z(t) \in C([0, T]; V) \), by (4.4), there exists a constant \( C_T \) such that
\[
\sup_{s \in [0, T]} \| y(s) + z(s) \|_{L^4} \leq \sup_{s \in [0, T]} \| y(s) + z(s) \|_{H^4}^4 \| y(s) + z(s) \|_{V}^4 \leq C_T.
\]
Consequently, there is some constant \( C_T > 0 \) satisfying that
\[
\| Y_t - y(t) \|_{H^4}^4 + 2 \int_0^t \| Y_s - y(s) \|_{V}^4 \, ds \\
\leq 3 \int_0^t \| Y_s - y(s) \|_{H^4}^4 \, ds + \int_0^t \| Z_s - z(s) \|_{H^4}^4 \, ds \\
+ C_T \int_0^t \left( \| Z_s - z(s) \|_{L^4}^4 + \| Z_s - z(s) \|_{H^4}^4 + \| Z_s - z(s) \|_{L^4}^4 + \| Z_s - z(s) \|_{H^4}^4 \right) \, ds.
\]
Therefore, by the spectral gap inequality and Gronwall’s inequality, we have
\[
(4.14) \quad \| Y_T - y(T) \|_{H^4}^4 \leq C_T \sum_{i \in \Lambda} \int_0^T \| Z_s - z(s) \|_{V}^4 \, ds,
\]
where \( \Lambda := \{4/3, 2, 8/3, 4\} \). This inequalities, together with Lemma 4.2, implies
\[
P(\| X_T - a \|_{H} < \varepsilon) \\
= P(\| Y_T - y(T) + Z_T - z(T) + x(T) - a \|_{H} < \varepsilon) \\
\geq P(\| Y_T - y(T) \|_{H} \leq \varepsilon/4, \| Z_T - z(T) \|_{H} \leq \varepsilon/4, \| x(T) - a \|_{H} < \varepsilon/2) \\
= P(\| Y_T - y(T) \|_{H} \leq \varepsilon/4, \| Z_T - z(T) \|_{H} \leq \varepsilon/4) \\
\geq P\left( \sum_{i \in \Lambda} \int_0^T \| Z_s - z(s) \|_{V}^4 \, ds + \| Z_T - z(T) \|_{V} \leq C_{T, \varepsilon} \right) \\
> 0.
\]
The proof is complete. □
5. Hyper-exponential Recurrence

In this section, we will verify the hyper-exponential recurrence condition (3.3).

5.1. Construction of the Markov chain.

**Lemma 5.1.** Let \( \{X_t\}_{t \geq 0} \) be the solution to the equation (2.1) with initial value \( x_0 \in H \).

For any \( T > 0 \), there exists some constant \( C \) depending on \( \alpha \) and \( \beta \) such that

\[
\int_0^T \mathbb{E}\|X_s\|_V \, ds \leq 2\|x_0\|_H + C(1 + T).
\]

**Proof.** Its proof can be found in the proof of Theorem 2.3 in [27] with a very slight modification, which is omit here. \( \square \)

**Lemma 5.2.** Let \( \{X_t\}_{t \geq 0} \) be the solution to the equation (2.1) with initial value \( x_0 \in H \).

For any \( T > 0 \), there exists a Markov chain \( \{X_{t_k}\}_{k \in \mathbb{N}} \) with \( t_0 = 0 \) satisfying that

\[
t_k - t_{k-1} \in \left[ \frac{T}{2}, T \right]
\]

and

\[
\mathbb{E}[\|X_{t_n}\|_V | \mathcal{F}_{t_{n-1}}] \leq \frac{4}{T}\|X_{t_{n-1}}\|_H + \frac{2C(1 + T)}{T}
\]

for some constant \( C > 0 \).

**Proof.**

**Step 1.** By Lemma 5.1, there exists some constant \( t_1 \in \left[ \frac{T}{2}, T \right] \) such that

\[
\mathbb{E}\|X_{t_1}\|_V \leq \frac{4}{T}\|x\|_H + \frac{2C(1 + T)}{T}.
\]

Otherwise, if for all \( t \in \left[ \frac{T}{2}, T \right] \),

\[
\mathbb{E}\|X_{t_1}\|_V > \frac{4}{T}\|x\|_H + \frac{2C(1 + T)}{T},
\]

then

\[
\int_0^T \mathbb{E}\|X_s\|_V \, ds > 2\|x\|_H + C(1 + T),
\]

which is contradict with Lemma 5.1.

**Step 2.** By the Markov property of \( \{X_t\}_{t \geq 0} \) and Lemma 5.1 again, we have

\[
\int_{t_1}^{t_1 + T} \mathbb{E}[\|X_s\|_V | \mathcal{F}_{t_1}] \, ds \leq 2\|X_{t_1}\|_H + C(1 + T).
\]

The same as that in Step 1, there exists a constant \( t_2 \in [t_1 + T/2, t_1 + T] \) such that

\[
\mathbb{E}[\|X_{t_2}\|_V | \mathcal{F}_{t_1}] \leq \frac{4}{T}\|X_{t_1}\|_H + \frac{2C(1 + T)}{T}.
\]

By induction, we complete the proof. \( \square \)
5.2. Estimates of the hitting time.

For any $M > 0$, define the hitting times
\begin{equation}
\tau_M = \inf \{ t_k > 0; \| X_{t_k} \|_V \leq M \}.
\end{equation}

Let
\begin{equation*}
K = \{ x \in V; \| x \|_V \leq M \}.
\end{equation*}

$K$ is clearly compact in $H$. Recall the definitions of $\tau_K$ and $\tau^1_K$ in (3.2). By the construction of the Markov chain \{X_{t_n}\}_{n \in \mathbb{N}} (taking $T > 2$ in Lemma 5.2), it is easy to see that
\begin{equation}
\tau_K \leq \tau_M, \quad \tau^1_K \leq \tau_M.
\end{equation}

The main result of this section is the following theorem, which implies the hyper-exponential recurrence condition (3.3).

**Theorem 5.3.** For any $\lambda > 0$ and sufficiently large $T > 0$, there exist constants $C = C_{\alpha, \beta, \eta}$ and $M = M_{\alpha, \beta, \eta, T}$ such that for any $x \in H$,
\begin{equation*}
\mathbb{E}_x \left[ e^{\lambda \tau_M} \right] \leq C(1 + \|x\|_H).
\end{equation*}

**Proof.** It sufficient to prove that for any $\theta \in (0, 1)$, there exist some positive constants $C = C_{\alpha, \beta, \eta}$ and $M = M_{\alpha, \beta, \eta, T}$ such that
\begin{equation}
P_x(\tau_M > t_k) \leq C\theta^k(1 + \|x\|_H)
\end{equation}
for any $x \in H$. Its proof follows the idea in [19, Lemma 6.5]. By Lemma 5.2 for any $k \geq 1$,
\begin{equation}
\mathbb{E}[\|X_{t_{k+1}}\|_V | F_{t_k}] \leq \frac{4}{T} \|X_{t_k}\|_V + C.
\end{equation}

By Chebyshev inequality,
\begin{equation}
P(\|X_{t_{k+1}}\|_V > M | F_{t_k}) \leq \frac{4}{TM} \|X_{t_k}\|_V + \frac{C}{M}.
\end{equation}

Denote
\begin{equation*}
B_k = \{ \|X_j\|_V > M; j = 0, \ldots, k \},
\end{equation*}
and
\begin{equation*}
p_k = P(B_k), \quad q_k = \mathbb{E}(\|X_{t_k}\|_V 1_{B_k}).
\end{equation*}

Integrating (5.5) over $B_k$, one has
\begin{equation}
p_{k+1} \leq \frac{4}{TM} q_k + \frac{C}{M}.
\end{equation}

Moreover, by integrating (5.4) over $B_k$,
\begin{equation}
q_{k+1} \leq \mathbb{E}(\|X_{t_{k+1}} 1_{B_k}\|) \leq \frac{4}{T} q_k + C p_k.
\end{equation}

From (5.6) and (5.7), one has
\begin{equation*}
\left( \begin{array}{c}
q_{k+1} \\
 p_{k+1}
\end{array} \right) \leq \left( \begin{array}{cc}
\frac{4}{T} & C/M \\
\frac{4}{TM} & \frac{C}{M}
\end{array} \right) \left( \begin{array}{c}
q_k \\
 p_k
\end{array} \right),
\end{equation*}
which clearly implies
\begin{equation}
\frac{4}{T}q_{k+1} + Cp_{k+1} \leq \left( \frac{4}{T} + \frac{C}{M} \right) \left( \frac{4}{T}q_k + Cp_k \right).
\end{equation}

We can choose $T$ and $M = M(\alpha, \beta)$ sufficiently large so that
\[ \frac{4}{T} + \frac{C}{M} \leq \theta. \]

Thus, by (5.8), we clearly have
\[ \frac{4}{T}q_k + Cp_k \leq \theta^{k-1} \left( \frac{4}{T}q_1 + Cp_1 \right). \]

This inequality, together with the easy fact $p_k = P_x(\tau_M > t_k)$, and
\[ q_1 \leq E\|X(t_1)\|_V \leq \frac{4}{T}\|x\|_H + C, \]

immediately implies the required estimate (5.3). The proof is complete.

\[ \square \]

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