BPS Structure of Argyres–Douglas Superconformal Theories

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We study geometric engineering of Argyres–Douglas superconformal theories realized by type IIB strings propagating in singular Calabi–Yau threefolds. We use this construction to count the degeneracy of light BPS states under small perturbations away from the conformal point, by computing the degeneracy of D3–branes wrapped around supersymmetric 3–cycles in the Calabi–Yau. We find finitely many BPS states, the number of which depends on how this deformation is done, similarly to the degeneracy of kink solutions for the deformation of $N = 2$ Landau–Ginzburg superconformal theories in two dimensions. Also, some aspects of worldsheet theories near general Calabi–Yau singularities are discussed.

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1. Introduction

In the past few years we have learned how to generate superconformal theories in various dimensions. Nevertheless we do not understand the properties of conformal theories in dimensions greater than two with the same precision as we do in the case of two dimensions.

In two dimensions, one powerful method to study conformal theories was pioneered by Zamolodchikov [1], who studied their properties under special integrable massive deformations. Zamolodchikov’s idea was to use the properties of the deformed theories, such as the spectrum of kinks and their scattering matrix amplitudes, to reconstruct the full data of the conformal theory. As a particular example of this philosophy, it was subsequently shown in the case of massive deformations of (2, 2) superconformal theories in two dimensions [2] that just the degeneracy of the (BPS saturated) kinks of the massive theory characterizes the dimensions of chiral operators at the conformal point.

It is thus natural, in the context of conformal theories in dimensions bigger than 2, to also ask the same question, namely, what is the number of BPS states for slight deformations of these theories away from the conformal point. For certain cases, this has been answered. For example we know the degeneracy for $N = 4$ theories in 4 dimensions [3] and some superconformal $N = 2$ theories (for example $SU(2)$ with four doublets) [4]. The aim of this paper is to widen the class of theories for which we know the BPS spectrum.

The case of $N = 2$ field theories in 4 dimensions is already very interesting from this point of view. In a limited set of cases, including certain $N = 2$ SU(2) gauge theories, the BPS spectrum can be computed exactly using field theory techniques [4][5]. However, there are many interesting 4D $N = 2$ theories for which we know very little about the structure of their BPS spectra, including the superconformal theories of Argyres and Douglas [6]. In this paper, our main result will be a determination of the degeneracy of the BPS states under slight deformations away from Argyres–Douglas points. We realize Argyres–Douglas superconformal theories and their deformations in terms of type IIB strings propagating in a (nearly) singular Calabi–Yau threefold. The BPS states are realized in this setup in terms of D3–branes wrapped around supersymmetric 3–cycles on the threefold. The structure of BPS states of Argyres–Douglas superconformal theories are among the simplest to study from this point of view because the counting of the supersymmetric 3-cycles gets

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3 A previous study of the spectrum of an Argyres–Douglas point using M–theory has been made by Gustavsson and Henningson [7].
related to counting certain special 1-cycles on a Riemann surface. We find a structure very reminiscent of BPS degeneracies of the deformations of minimal $N = 2$ superconformal theories in 2D. In particular we find that the spectrum of light BPS states is finite and depends on how we deform away from the conformal point.

The structure of this paper is as follows: In section 2 we review certain aspects of Calabi–Yau $d$-folds which have isolated singularities. We point out a connection between normalizable deformations of a Calabi–Yau singularity and unitarity bounds of the corresponding quantum field theory in the remaining $\mathbb{R}^n$. In section 3 we make some remarks about the string worldsheet description near a Calabi–Yau singularity. In section 4 we specialize to the case of Calabi–Yau threefold, which gives rise to $N = 2$ superconformal theories in 4 dimensions. We set up the computation for counting the number of supersymmetric 3-cycles for this case. In section 5 we apply these techniques to the case of Argyres–Douglas conformal theories and compute their spectra.

2. Calabi–Yau Singularities

In this section we consider Calabi–Yau manifolds with an isolated singularity. More precisely we consider a local model for a $d$-fold given by a hypersurface $W(x_i) = 0$ in $\mathbb{C}^{d+1}$. Moreover we assume $W(x_i)$ is a quasihomogeneous function of $x_i$:

$$W(\lambda^{q_i} x_i) = \lambda W(x_i). \quad (2.1)$$

We assume the singularity is isolated, namely

$$dW = 0 \quad \text{iff} \quad x_i = 0.$$ 

If $W$ is viewed as a LG superpotential of an $N = 2$ theory in 2 dimensions it would flow to a superconformal theory with central charge $\hat{c}$ given by

$$\hat{c} = \sum_{i=1}^{d+1} (1 - 2q_i) = (d + 1) - 2(\sum_i q_i). \quad (2.2)$$

The condition that the Calabi–Yau threefold with this singularity appear at finite distance in moduli space was recently studied in [10] with the conclusion that if

$$\hat{c} < d - 1 \quad (2.3)$$
the distance is finite and thus the study of singularity would be of physical interest only in this case.

An interesting question involves deformations of the singularity. In this context, as in the 2 dimensional case [8] [9], it is natural to consider the singularity ring of \( W \)

\[
\mathcal{R} = \mathbb{C}[x_i]/dW
\]
generated by the monomials

\[
x^\alpha = x_1^{\alpha_1}x_2^{\alpha_2} \cdots x^{\alpha_{d+1}}
\]
modulo setting to zero those polynomials which are in the ideal generated by \( \partial_i W \). One then considers deformations of the form

\[
W \rightarrow W + \sum_{\alpha \in \mathcal{R}} g_{\alpha} x^\alpha.
\]

For each element \( x^\alpha \) in the ring, consider the charge

\[
Q_\alpha = \sum_i q_i \alpha_i.
\]
The \( Q_\alpha \) lie in the range

\[
0 \leq Q_\alpha \leq \hat{c}.
\]

Moreover, the elements in the ring are paired so that for every element of the ring with charge \( Q_\alpha \) there is one with charge \( \hat{c} - Q_\alpha \). The degeneracy of the ring elements is captured by the function

\[
\sum_{\alpha \in \mathcal{R}} t^{Q_\alpha} = \prod_i \frac{(1 - t^{1-q_i})}{(1 - t^{q_i})}
\]
and the dimension of the ring \( \mathcal{R} \) is given by

\[
N = \text{dim} \mathcal{R} = \prod_{i=1}^{d+1} \frac{(1 - q_i)}{q_i}.
\]

This dimension is also the dimension of compact part of mid-dimension homology \( H_d(W = \mu) \). The compact homology can be realized by \( N \) spheres of dimension \( d \) with an intersection structure that can be obtained from the structure of \( W \) [11].

In questions in string theory, it is important to know how many cohomology elements of dimension \( d \) are supported on the singularity. These would correspond to normalizable
$d$-forms in the internal dimensions localized near the singularity. These forms can be obtained from the holomorphic $d$-form represented as

$$\Omega = \prod_i \frac{dx_i}{dW}.$$.

What this means is that we solve the equation $W = 0$ for one of the $x_i$ in terms of the others and view $dx_i/dW$ as $1/\partial_i W$ in the above expression. The other $d$-forms correspond to deformations of $\Omega$:

$$\Omega_\alpha = \partial \Omega/\partial g_\alpha.$$.

The condition that $\Omega_\alpha$ lead to a normalizable form localized at the singularity was studied in [10]. The condition for this is that $\int |\Omega_\alpha|^2$ diverge in the vicinity of $x_i = 0$. It was found that the deformation by monomial $x_\alpha$ corresponds to a normalizable cohomology element if

$$Q_\alpha < \frac{\hat{c} - d + 3}{2}. \quad (2.4)$$

Note that combined with (2.3) this implies that a normalizable cohomology deformation must have $Q_\alpha < 1$.

The condition (2.4) can also be understood physically by unitarity bounds of physical modes as follows. The coefficients $g_\alpha$ in the deformation of the singularity should correspond in the field theory setup, either to expectation values of scalar fields, or to coupling constants of field theory. We may assign dimensions to the coefficients $g_\alpha$ as follows: If we consider a $Dd$-brane (or a d-dimensional part of a higher dimensional brane) wrapped around a $d$-dimensional supersymmetric cycle $C$ in the local geometry, it gives a particle with BPS mass

$$M = \int_C \Omega$$

which implies that the mass dimension of $[\Omega] = 1$. By quasihomogeneity, we can assign a dimension to each variable $x_i$ proportional to its charge. The condition that $[\Omega] = 1$ implies that the dimension of a monomial $x_\alpha$ is given by

$$[x^\alpha] = \frac{2}{d - 1 - \hat{c}} \cdot Q_\alpha$$

and that the dimension of $g_\alpha$ is given by

$$[g_\alpha] = \frac{2(1 - Q_\alpha)}{(d - 1 - \hat{c})}. \quad (2.5)$$
The deformation parameter $g_a$ is the expectation value of a canonically normalized scalar field weighted with the Yang–Mills coupling; that is, $g_a \equiv g_{YM} \langle \phi_a \rangle$. Since $|g_{YM}| = (4 - D)/2$ in $D$ spacetime dimensions, the unitarity bound that $|\phi_\alpha| \geq (D - 2)/2$ in a superconformal theory requires that

$$[g_\alpha] = \frac{2(1 - Q_\alpha)}{(d - 1 - \hat{c})} > 1$$

and this condition is identical to (2.4). Thus, the mode is a physical field if and only if it satisfies the unitarity bound. This is quite a satisfactory result (which has also been obtained independently in [12]).

Let us now consider Calabi–Yau manifolds of various dimensions and see what this condition translates into in each case.

2.1. $d=2$

In the case of complex dimension 2, the finite distance condition (2.3) $\hat{c} < 1$ implies that $W$ is given by the LG superpotential of $N = 2$ minimal models, which gives the usual ADE classification of $K3$ singularities. Moreover the condition of physical fields (2.4) is automatically satisfied for all elements in the ring because all the elements have

$$Q_\alpha \leq \hat{c} < \frac{\hat{c} + 1}{2}.$$ 

This is in accordance (in the type IIA case) with the fact that all these deformations can be viewed as giving vev to scalars in commuting directions of the adjoint representation of the corresponding group.

2.2. $d=3$

For Calabi–Yau 3-fold singularities, the condition of being at finite distance implies that $\hat{c} < 2$. If we consider type IIB strings in the presence of such singularities we obtain an $N = 2$ superconformal theory in 4 dimensions. Examples of such $W$’s are provided by

$$W = F(x, y) + z^2 + w^2$$  

where $F(x, y)$ is a quasihomogeneous function of $x, y$. In this case the function $F(x, y)$ can be identified [13] with a fivebrane with worldvolume $R^4 \times \Sigma$, where $\Sigma$ is the Seiberg–Witten Riemann surface $F(x, y) = 0$. In this case the singular Calabi–Yau gets mapped to a singular Riemann surface. A special case of this

$$F(x, y) = x^n + y^2$$
corresponds to the Argyres–Douglas points that we will study in more detail later in this paper.

There are other singularities which are also at finite distance which are not of the form \((2.6)\). For example consider

\[ W = x^3 + y^3 + z^3 + w^{3N} \]

Type IIB string in the presence of such a singularity describes a superconformal theory, which does not have a description in terms of a Seiberg-Witten Riemann surface. Such singularities do arise in certain gauge theories. For example the above singularity appears in a gauge theory with gauge group \(SU(3N) \times SU(2N)^2 \times SU(N)^3\) with certain bi-fundamental matter dictated by the affine \(E_6\) Dynkin diagram \([14]\).

In the case of the threefold, the condition that deformations \(x^\alpha\) correspond to normalizable forms is \((2.4)\)

\[ Q_\alpha < \hat{c}/2. \]

Given the pairing of the ring elements and the fact that the sum of the charges of the pairs gives \(\hat{c}\), this means that the deformations that correspond to expectation values of dynamical fields are in one to one correspondence with the deformations that correspond to parameters in the theory. A natural interpretation of this pairing is to note that for each chiral operator in the \(N = 2\) superconformal theory \(\Phi_\alpha\) (whose lowest component is identified with \(\phi_\alpha\) where \(\langle \phi_\alpha \rangle = g_\alpha\)), we can consider the \(N = 2\) superspace integral

\[ S \rightarrow S + \int d^4x d^4\theta \ t_\alpha \Phi_\alpha \]

with \(t_\alpha\) being identified with the dual deformations of \(W\). The form of these deformations implies that the mass dimensions obey \([t_\alpha] + [g_\alpha] = 2\). In fact this is consistent with what we have found for the dimension of the parameters. Namely, since the charges of dual deformations add up to \(\hat{c}\) we deduce that the charge corresponding to the \(t_\alpha\) deformation is \(\hat{c} - Q_\alpha\) and using the dimension of the fields given by \((2.3)\) we see that indeed \([t_\alpha] + [g_\alpha] = 2\):

\[ [g_\alpha] = \frac{2(1 - Q_\alpha)}{(2 - \hat{c})} \]

\[ [t_\alpha] = \frac{2(1 - \hat{c} + Q_\alpha)}{(2 - \hat{c})} \]

\([g_\alpha] + [t_\alpha] = 2\).
2.3. $d=4$

The case of local singularities of Calabi–Yau 4-folds was studied in [10]. In particular, the case of theories with $\hat{c} < 1$ was studied in detail and it was shown that type IIA string in the presence of these singularities gives rise to certain $N = 2$ superconformal Kazama–Suzuki models in 2 dimensions. In these cases none of the deformations correspond to dynamical fields in the 2d theory.

3. Perturbative String Description

It is natural to consider the perturbative string theory in the presence of such singularities. Of course, if we are at the singularity the string theory is singular and there is no perturbative expansion. However, if we deform the quasihomogeneous function $W$ we can resolve the singularity. In such a case (if the singularity is resolved enough so that the wrapped branes are heavy enough) we can expect to have a perturbative string description. In this case one can use ideas developed in [15] [16] in the context of compact Calabi–Yau manifolds and for the non-compact case in [17] [18] [19] to find a Landau–Ginzburg description of the theory. Namely consider $d+2$ chiral fields $x_1, \ldots, x_{d+1}, y$ with $N=2$ $U(1)$ charge given by $q_1, \ldots, q_{d+1}, -1/\gamma$, and with a quasihomogeneous LG superpotential

$$\hat{W} = W(x_i) + \sum_{\alpha} g_{\alpha} x^\alpha y^{\gamma(Q_{\alpha} - 1)}. \quad (3.1)$$

$\gamma$ is fixed by the requirement that the total $\hat{c} = d$, which implies that

$$\gamma = \frac{2}{d - \hat{c} - 1}.$$ 

Note that the condition that the singularity be at finite distance (2.3) is that $\gamma > 0$ and the condition that the deformation given by $g_{\alpha}$ corresponds to a normalizable field in uncompactified theory is translated to the condition that the power of $y$ accompanying that deformation be less than $-1$. To obtain the full supersymmetric theory we also need to do the Gepner projection, which is a discrete orbifold of the above LG theory that keeps only the integral $U(1)$ charged fields.

A particular case of (3.1) is the deformation of $W$ by a constant:

$$\hat{W} = W + g_0 y^{-\gamma}$$

In this case it can be shown [17] [18] [19] that the resulting theory corresponds to the tensor product of the LG theory given by $W$ and a Kazama-Suzuki coset model $SL(2)/U(1)$ with level $k = 2 + \gamma$. This construction has also been noted independently in [12]. Certain aspects of conformal theories near singularities have also been discussed recently in [20].
4. Solitons in the $d = 3$ case

For the rest of this paper we will be interested in studying some aspects of Calabi–Yau threefold singularities corresponding to Argyres–Douglas points. As already noted these points correspond to local singularities given by

$$W = P(x) + y^2 + z^2 + w^2 = 0$$

(4.1)

with $P(x)$ a polynomial of degree $n$ in $x$. The superconformal point itself corresponds to $P(x) = x^n$, and there are $n - 1$ deformations given by

$$P(x) = x^n + \sum_{i=0}^{n-2} g_i x^i$$

where the $g_i$ correspond to expectation value of fields for $i < (n - 2)/2$ and to dual mass parameters for $i > (n - 2)/2$ \[21\]. Eq.(2.2) gives the central charge

$$\hat{c} = 1 - \frac{2}{n}$$

and (2.3) gives the dimensions

$$[g_i] = \frac{2(1 - i/n)}{1 + 2/n} = \frac{2(n - i)}{n + 2} \quad (i = 0..n-2)$$

in agreement with the result of \[22\].

We are interested in studying type IIB strings in the presence of such singularities. In particular, we would like to study the BPS states in this theory near the superconformal point and how their spectrum jumps as we change the parameters $g_i$ in the polynomial $P(x)$.

The BPS states in this case correspond to D3–branes wrapped around supersymmetric 3-cycles. A basis of vanishing 3-cycles (not necessarily supersymmetric) can be chosen to be $n - 1$ 3-spheres intersecting one another according to the Dynkin diagram of $A_{n-1}$. The intersection of two cycles can be interpreted as the skew symmetric form in the product of electric and magnetic charges. In particular, two $S^3$'s corresponding to adjacent nodes of $A_{n-1}$ carry, in some basis, electric versus magnetic charge.

To find the supersymmetric 3-cycles, one follows the strategy in \[13\] and considers the 3-cycle as a 3-sphere consisting of an $S^2$ fibered over a real curve in the $x$-plane. The $S^2$ above a given $x$ is $y^2 + z^2 + w^2 = -P(x)$ (in an appropriate real subspace). The projection
of the 3-cycle onto the $x$-plane is a curve in the $x$-plane which begins and ends at zeroes of $P(x)$. At the endpoints of the curve, which correspond to the ‘poles’ of $S^3$, the radius of the $S^2$ in the fiber goes to zero. A basis of 3-cycles (not necessarily supersymmetric) can be chosen to correspond to $n - 1$ intervals connecting the $n$ zeroes of $P(x)$ in a sequence. These would correspond to a set of 3-cycles whose intersection gives the Dynkin diagram of $A_{n-1}$.

As discussed in [15] the condition of having a supersymmetric cycle gets translated in the $x$-plane into the existence paths beginning and ending at the zeroes of $P(x)$ such that the phase of

$$
\int_{S^2} \Omega = \sqrt{P(x)} \, dx
$$

(4.2)

is constant along the path. This would guarantee in particular that the BPS inequality

$$
\int_{S^3} |\Omega| \geq |\int_{S^3} \Omega|
$$

is saturated. Alternatively, the required condition is that the image of the path under the Jacobian map with respect to the reduced one–form should be a straight line in the flat $W$-plane, that is,

$$
W(x(t)) = \int_{x_0}^{x(t)} \sqrt{P(x)} \, dx = \alpha t
$$

(4.3)

where $t$ is real parameterizing the path and $\alpha$ is a phase. In this case, the mass of a D3–brane wrapped around the supersymmetric cycle corresponding to such a path between zeros at $x_0$ and $x_1$ automatically saturates the BPS inequality

$$
M = \int_{x_0}^{x_1} \sqrt{P} \, dx \geq \left| \int_{x_0}^{x_1} \sqrt{P} \, dx \right|
$$

An efficient technique to find such paths is to solve the first–order differential equation

$$
\sqrt{P(x)} \frac{dx}{dt} = \alpha.
$$

for solutions $x(t)$ beginning and ending at roots of $P(x)$, for all possible values of the phase of $\alpha$. This is equivalent to finding integral curves of the vector field

$$
\frac{\alpha}{\sqrt{P(x)}} \frac{\partial}{\partial x}
$$

(4.4)

that start and end at roots of $P$. As we have explained, the existence or nonexistence of such a curve then implies the existence or nonexistence of a particular BPS state. This procedure is easily implemented.

In the next section we analyze the solutions to the condition (4.3) in various regimes of parameters.
5. Finding the BPS states

As we have discussed, we need to find solutions to

\[ \frac{dx}{dt} = \frac{\alpha}{\sqrt{P(x)}} \]  

(5.1)

where \( t \) is a real parameter and \( \alpha \) is a phase. Moreover the solution should begin and end at one of the \( n \) roots of \( P(x) \). We need to establish a few facts:

For each topology of path between two roots of \( P(x) \), \( \alpha \) is uniquely fixed (up to an overall sign). By topology of path we mean a homology class of paths with fixed endpoints on the complex \( x \)-plane with all the roots of \( P(x) \) deleted. To see the uniqueness of \( \alpha \) note that for each path \( \gamma \) in a particular homology class the integral

\[ \int_{\gamma} dx \sqrt{P(x)} = a_{\gamma}. \]  

(5.2)

which is well defined up to an overall sign, is independent of the precise choice of \( \gamma \) in that class. This is because two different paths in the same class will differ by integral of an analytic expression in the region bounded by the two curves. Physically, this is simply the reflection of the fact that each choice of path topology fixes the electric and magnetic charges of the BPS state and the integral (5.2) is just the central charge of the \( N = 2 \) algebra in that sector. It follows that \( \alpha \) is uniquely fixed for each choice of class of path to be

\[ \alpha = a_{\gamma}/|a_{\gamma}|. \]

The next fact we need to establish is that there is at most one solution connecting two roots. We first show that there cannot be two different solutions in the same path class. Let us first analyze the structure of the solution to (5.1) near each root. Near a root of \( P(x) \), which with no loss of generality we take to be at \( x = 0 \), one is solving an equation which can be approximated as

\[ \frac{dx}{dt} = \frac{\alpha}{\sqrt{x}} \]

whose solutions are given by

\[ x = \left( \frac{3}{2} \alpha t \right)^{\frac{2}{3}}. \]

In particular, for a given \( \alpha \) there are three solutions near \( x = 0 \) which make 120° angles relative to one another, corresponding to the 3 possible choices of cube roots of \( \alpha^2 \) in this solution. A typical set of integral curves near to a root is depicted in Figure 1.
Fig.1: A typical set of integral curves near to a root.

Now suppose there are two solutions in a given class (and thus with the same choice of $\alpha$) connecting two roots of $P$. In other words, let us assume that two of the three integral curves from one root join with two of the three integral curves from another root (see Figure 2a). The geometry should be as depicted in Figure 2a: first, the paths cannot intersect each other except at the two ends, as the vector field has a well defined direction at each point and that would not be the case at an intersection point. Secondly, the paths must make an angle of 120° on each side, otherwise the third integral curve emanating from either of the roots would have nowhere to go. (We can also rule out the case where all three curves are joined, by applying this argument to two of the curves making 120° angles relative to each other.) From the geometry of Figure 2a it is clear that there will necessarily be closed integral curves trapped inside. If we consider the winding of the phase

$$\Phi = |\sqrt{P(x)}|/\sqrt{P(x)}$$

defined by

$$w = \frac{1}{2\pi i} \int d \log \Phi$$

we see that $w = +1$ along these curves. Since $P(x)$ has no zeroes or poles inside this curve, this is impossible. We could also have chosen, instead of the trapped integral curve, an arbitrary interior curve to use for counting the winding of the phase $\Phi$. For example we can make the choice shown in Figure 2b. In this case we again get winding number $+1$, with the two small arcs near the roots each contributing $+1/6$ to the winding number and the remaining pieces along the integral curves giving a net winding of $+2/3$. 

11
Figs.2a and 2b: Impossibility of more than one integral curve of a given topology between two roots. The winding of the phase is calculated by integrating along the dashed line.

Now let us also establish that there can not be two solutions between two roots, even if the paths are in different topological classes. In such a case the two paths correspond to solutions to \((5.1)\) with two different phases \(\alpha_1\) and \(\alpha_2\). Let \(\epsilon = \alpha_1 \alpha_2^{-1}\) and define

\[
\delta \equiv \frac{1}{2\pi i} \log \epsilon.
\]

With no loss of generality we can assume \(\epsilon\) to be some generic phase (in other words for any special values of \(\epsilon\) we can change the coefficients of \(P(x)\) so that the two central charges have infinitesimally different phase ratios). Let us consider the winding of the phase \(\Phi(x)\) along the closed path shown in Figure 3. We will first assume, as in Figure 3, that the two curves do not intersect except at the endpoints. This assumption will be justified shortly.

Fig.3: Two integral curves between roots, with \(\alpha_1 \neq \alpha_2\).
This closed path consists of the two integral curves and the small paths connecting them near the roots. Even though the two integral curves are for different choices of $\alpha$, in either case $\alpha$ is a constant and does not vary over the path. In particular the integral $\int d \log \Phi$ along the two integral curves is the same as winding of the velocity vector, and gives a total contribution to the winding of $+2/3$. Along the arcs near each of the roots, the contribution to the winding will be given by $1/6$ up to a correction by $\delta/6$ at one endpoint and $-\delta/6$ at the other endpoint, which cancel out. We thus still obtain a net winding of $+1$ unit. However this is impossible as the winding associated with $1/\sqrt{P}$ is in general negative and given by $-m/2$ where $m$ denotes the number of roots of $P$ inside the closed curve. The reason for this is that $1/\sqrt{x-a}$ has a winding number $-1/2$ around $x=a$. (The other possible geometries for these two integral curves curves — in which their relative angle of approach to one of the endpoints is shifted by $\pm 2\pi/3$ — lead to similar contradictions.)

![Diagram](image)

**Fig.4:** Impossibility of intersection of two integral curves.

We also need to justify our assumption that the curves do not intersect each other except at the endpoints. Suppose they did. Pick an endpoint and consider the closed curve that forms between it and the first point where the two integral curves intersect (Figure 4). The winding number around that closed curve is again the sum of contributions of the two integral curves and the small arcs near the sharp ends. The integral curves contribute the sum of the two angles divided by $2\pi$, or $\frac{1}{3}(1+\delta) + \delta$, the left endpoint adds $\frac{1}{6}(1+\delta)$, and the right endpoint adds nothing. So the net winding number $\frac{1}{2} + \frac{3}{2}\delta$ is not even an integer, as $\epsilon$ can be assumed to be a generic phase. This is clearly a contradiction with the fact that the winding should be $-m/2$ for some integer $m$.

We have thus established that there is at most one solution connecting two distinct roots.

**Minimum Number of BPS states:** We can also argue that any pair of roots are connected by a sequence of BPS solutions. We can find such a sequence by minimizing
\[ \int |\sqrt{P(x)}dx| \] over arbitrary paths beginning at one root and ending on the other. Clearly there will be a solution to the minimization condition (paths going to infinity will give an infinite contribution to the above integral). Consider the path which minimizes it. Then for a generic point on this path, away from the roots of \( P \), the curve should satisfy (5.1). If not, fix two nearby points on the curve and take a solution of (5.1) between these two points. This will have a lower value for the integral, violating the assumption that we had found the curve which minimizes the integral. In general, this minimal curve may pass through some roots of \( P \). The above argument still shows that piecewise, between the roots, it should satisfy (5.1) for some \( \alpha \). This implies that any two roots are connected by a sequence of BPS solutions. If there are \( n \) roots in all, the minimum number of BPS solutions is thus \( n-1 \).

\[ \text{Fig.5a: Integral curve from 1 to 3 when } \theta < 120^\circ. \text{ Fig.5b: When } \theta > 120^\circ \text{ no 1–3 curve exists.} \]

\textit{Jumping Phenomena:} Consider three roots of \( P \) labeled by 1, 2, and 3. Suppose for some choice of parameters there are BPS solutions from 1 to 2, from 2 to 3, and from 1 to 3. \textit{A priori,} the phase of the central charge of the \( N = 2 \) algebra, which determines the phase \( \alpha \), will be different for the three different solitons. However, let us assume that as we change the parameters the phases become the same, \textit{i.e.,} the BPS charges from 1 to 2 and 2 to 3 get aligned and sum up to the charge from 1 to 3. This means that for the \textit{same} choice of \( \alpha \), \textit{i.e.} \( \alpha_{12} = \alpha_{23} = \alpha_{13} \), we have three integral solutions. In such a case the curve from 1 to 3 must coincide with the concatenation of the curves from 1 to 2 and 2 to 3.
3. If not, as in the previous arguments, by considering the trapped integral curves inside we would get a contradiction with the winding number. At the alignment point the angle

\[ \theta = \widehat{123} \]

which gives the angle between the solitons 1–2 and 2–3 is 120 degrees. However as we change parameters from one side to the other side the angle changes from less than to greater than 120°. We will argue below that the case where the 1–3 soliton does exist corresponds to when \( \theta < 120^\circ \). First, we argue that the 1–3 soliton should disappear when \( \theta > 120^\circ \). If the 1–3 solution continued to exist, the case before (when \( \theta < 120^\circ \)) and after (\( \theta > 120^\circ \)) the transition should look like that depicted in Figures 5a and 5b. This is because the phase of \( \alpha_{12}/\alpha_{13} \) should go from one sign to another, which implies, by looking at integral curves near the point 1, that the 1–3 curve is to one side or the other of the 1–2 curve. But the curve 1–3 depicted in Figure 5b is forbidden: The third integral curve emanating from point 2 would intersect it, which is not allowed. We thus see that the soliton represented by the 1–3 curve decays to 1–2 and 2–3 soliton as we pass through alignment.

The reverse of this argument can also be made. Suppose we originally have no soliton from 1 to 3. Let us choose parameters such that we are close to an aligned configuration, which as argued should correspond to \( \theta = 120^\circ \). Then we argue that as we decrease \( \theta \) a soliton should appear from 1 to 3. To see this note, as we discussed already, that there

![Fig.6: Proof of existence of integral curve from 1 to 3 when \( \theta < 120^\circ \).](image-url)
is a solution which minimizes the integral $\int |\sqrt{P} dx|$ between any pairs of point. Apply this to points 1 and 3. At the alignment point the minimal solution is given by the sum of the curves 1–2 and 2–3. If $\theta$ is decreased to less than 120° the class minimizing it is no longer sum of the 1–2 and 2–3 curves. The integral can be decreased, for example by considering a path such as the one shown in Figure 6: we pick two points $A$ and $B$, on 1–2 and 2–3 curves respectively, near root 2. We consider a solution of the integral curves $\gamma_{AB}$ of the vector field which interpolates between $A$ and $B$ for some $\alpha$. That this is possible is guaranteed if $\theta < 120^\circ$. Then the integral $\int |\sqrt{P} dx|$ is decreased if for the part near point 2, the curve $\gamma_{AB}$ is used, instead of the radial lines connecting $A$ and $B$ to point 2. Thus the minimum does not coincide with the sum of the 1–2 and 2–3 curves, and there appears a distinct 1–3 soliton as in Figure 5. The local analysis here is similar to what Joyce has considered in the context of 3-cycles on Calabi–Yau \cite{Joyce}.  

To summarize, whenever the phases of 3 such BPS charges are equal, we are in a situation of marginal stability. Under a perturbation away from alignment in one direction, all 3 states will be stable, but in the other direction, one of the states will become unstable to decay into the other two states.

5.1. Examples

In this part we show that with suitable choices of $P(x)$ we can have any number of solutions between $n - 1$ and $n(n - 1)/2$. That $n - 1$ is the minimum possible number has already been noted. Also that $n(n - 1)/2$ is the maximum number follows from the fact that there is at most one BPS state for each pair of roots of $P$. We will show that, for example, with $P(x) = x^n - 1$ we get $n(n - 1)/2$ solitons, one for each pair of roots of $P$. On the other hand with $P(x)$ having only real roots, we show that $P$ has exactly $n - 1$ solitons, corresponding to the solitons connecting adjacent roots. By our discussion about the jumping phenomenon, it follows that as we continuously change the polynomial $P(x)$ we get arbitrary number of solitons anywhere between these two bounds. In this way the story is very similar to that of the $A_n$ series for $N = 2$ LG theories in 2 dimensions \cite{Joyce}.

Let us first focus on the simplest nontrivial $A_n$ singularity, with $P(x) = x^3$, corresponding to the original SU(3) Argyres–Douglas point \cite{Argyres}. By the above arguments, the

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4 We would like to thank M. Douglas for pointing this reference out to us. See also \cite{Douglas1} \cite{Douglas2}.

5 In fact if we consider a degenerate choice of polynomial, where $P = (dW/dx)^2$ the problem of solving (5.1) is identical to the problem of finding solutions to 2d solitons in an $N = 2$ theory with LG potential given by $W$.  

16
minimum possible number of BPS states is \( n - 1 = 2 \), and the maximum is 3. A general deformation of \( P(x) \)

\[
P(x) = x^3 + ux + v
\]

involves two parameters \( u \) and \( v \), with \( v \) an operator expectation value and \( u \) a mass parameter. As we shall describe, the stable BPS spectrum near the singularity depends on the direction of approach, and the marginal stability surfaces (MSS’s) extend all the way down to the critical point. By tuning the dimensionless ratio \( u^3/v^2 \) appropriately as the scaling is performed, we may end up with either two or three light stable BPS states. (This modifies the result of [7], which found three states.) We will now exhibit examples in which each of these possibilities is realized.

First consider a situation where all three roots are arranged symmetrically, at cube roots of unity times a common scale factor, so that \( P(x) = x^3 - 1 \). Given any integral curve linking two roots at some value of \( \alpha \), we can construct two others, linking the other two pairs of roots, by multiplying \( \alpha \) and \( x \) by an appropriate cube root of unity (which preserves \( P(x) \) and the form of Eq.(4.3)). Hence, the total number of integral curves must be at least 3. Since the total number of such curves must also be either 2 or 3, this maximally symmetric configuration of roots leads to exactly 3 stable BPS states.

As another example, suppose the roots are all colinear; without loss of generality, we may suppose that

\[
P(x) = x(x - 1)(x - \lambda)
\]

where \( \lambda \) is real. We claim that there are exactly two integral curves for all \( \lambda > 1 \). By symmetry, all integral curves must either be at \( \alpha = \pm 1 \) or \( \alpha = \pm i \), and must lie along the real axis. Otherwise, given an integral curve not satisfying these conditions, we could construct another inequivalent geodesic between the same two roots, at the conjugate value of \( \alpha \) and along a conjugate path, in contradiction to the general uniqueness argument above. (Note that solutions at \( \alpha = i \) and \( \alpha = -i \) should be considered equivalent; the change in sign of \( \alpha \) simply means that the path is traversed in the opposite direction.) Now it is clear that there can be no integral curve from \( x = 0 \) to \( x = \lambda \), because it would have to pass either over or under \( x = 1 \), and thus could not lie entirely along the real axis. Thus, there must be exactly two integral curves (since this is the minimum possible), from 0 to 1 (which occurs at \( \alpha = 1 \)) and from 1 to \( \lambda \) (at \( \alpha = i \)). These correspond to real and imaginary 3-spheres in the Calabi–Yau manifold, intersecting transversely at a point.
Next, consider interpolating between these configurations by holding two of the roots fixed and moving the third root along a line midway between them. We do this by varying \( \lambda \) in Eq.(5.3) from \( \frac{1}{2} + \frac{i\sqrt{3}}{2} \) to \( \frac{1}{2} \), along the line \( \text{Re}(x) = \frac{1}{2} \). We find numerically that at approximately \( \lambda = \frac{1}{2} + .231i \), the number of integral curves jumps from 3 to 2. The curve of marginal stability in the \( \lambda \)-plane has roughly the appearance shown in Figure 7, and is preserved by the modular transformations

\[
\lambda \rightarrow \frac{1}{\lambda}, \quad \lambda \rightarrow 1 - \lambda, \quad \lambda \rightarrow \bar{\lambda}.
\]

In each of the regions containing the real \( \lambda \)-axis, the number of stable BPS states is 2; elsewhere in the \( \lambda \)-plane, there are 3 such states. By a local analysis, it can be shown that the components of the curve intersect the real axis at angles of \( \pm 60^\circ \), and approach asymptotes also making angles of \( \pm 60^\circ \).

![Fig.7: Curves of marginal stability for \( n = 3 \).](image)

The general \( A_n \) singularity with \( P(x) \) an \( n \)th order polynomial may be analyzed by the same methods. Again, when the roots are arranged with \( \mathbb{Z}_n \) symmetry, we will show that the maximum possible number \( n(n-1)/2 \) of stable BPS states is realized. And when the roots are colinear, the number of stable BPS states is minimized. Indeed, in the latter case, the same argument we gave for \( n = 3 \) extends to show that there are precisely \( n - 1 \) integral curves, connecting the roots in sequence, with \( \alpha \) alternating between 1 and \( i \).

On the other hand, suppose that the roots are located at the \( n \)th roots of unity, corresponding to the polynomial

\[
P(x) = x^n - 1,
\]
We will now show that there is a unique integral curve connecting any two roots, for a total of \( n(n - 1)/2 \) stable hypermultiplet states.

Assume for simplicity that \( n \) is even. In general, all integral curves of Eq.(5.1) for an \( n \)th–order \( P(x) \) that extend out to infinity in the \( x \)-plane must asymptote to one of \( n + 2 \) lines of constant phase. This is because, for large \( x \), Eq.(5.1) has the asymptotic solution

\[
x(t) = ((n + 2)\alpha t)^{1/(n+2)}.
\]

As we have discussed, there are exactly 3 integral curves emanating from each root; each of these must end either at another root or at one of these \( n + 2 \) asymptotic infinities. We consider the graph formed by the set of all integral curves starting or ending at a root, such as the graph shown in Figure 8 for the case \( n = 6 \) and \( \alpha = 1 \).

![Graph of integral curves ending on roots for \( n = 6 \) and \( \alpha = 1 \).](image)

**Fig.8:** Graph of integral curves ending on roots for \( n = 6 \) and \( \alpha = 1 \).

An important fact, which severely constrains the topologies of allowed graphs of this type, is that no two integral curves from a single root may approach the same asymptotic infinity. The proof is almost identical to the above argument that no two roots can be joined
by two distinct integral curves. If this did happen, we could consider the concatenation of two integral curves between a given root and a given infinity. A small deformation would produce a curve beginning and ending at the same infinity, with net index of +1/2. But as we have seen, closed curves always lead to indices of 0 or less, a contradiction.

We now argue that, for $\alpha=1$, there are integral curves connecting complex-conjugate $n$th roots of 1, and all other curves in the graph are unbounded. To see that only conjugate points can be connected, suppose that two non-conjugate roots $x_j = \exp(2\pi ij/(n+2))$ and $x_{j+k} = \exp(2\pi i(j+k)/(n+2))$, were joined by an integral curve. Then complex conjugation would map this curve into a different integral curve, also with $\alpha=1$, between a pair of roots conjugate to the original pair. By $\mathbb{Z}_n$ symmetry, at other values $\alpha = \exp(2\pi ik/n)$, we obtain $n$ rotated pairs of integral curves. Generically, this procedure leads to multiple geodesics between $x_j$ and $x_{j+k}$, at distinct values of $\alpha$. (A special case, when the original two roots are related by reflection about the imaginary axis, can be ruled out by showing, as we do below, that integral curves joining such points already exist for $\alpha = i$.)

Next, we need to show that, for $\alpha = 1$, all conjugate roots are indeed connected by integral curves. We first assume that $n$ is even. The roots $x = 1$ and $x = -1$ are special; they are not connected to any other roots, and each of them has 3 integral curves which must terminate at 3 different asymptotic infinities, as seen in Figure 8. By complex conjugation symmetry, one of the integral curves from $x = 1$ lies entirely along the real axis; the other two are asymptotically parallel to the lines $x(t) = \exp(\pm 2\pi i/(n+2))t$. In traversing a very large circle clockwise in the $x$-plane, the winding index of any vector field defined by (5.1) will be $(n+2)/2$. As we traverse the circle clockwise from the point where the first integral curve from $x = 1$ crosses this circle to the point where the third curve crosses, the index shifts by at least +1. Now consider any root connected to its conjugate; it must also have two unbounded integral curves, which similarly contribute at least +1/2 winding to the total index. If all conjugate roots are connected, the total index will be at least $(n+2)/2$; if any are not connected, their individual contributions to the index will be at least +1, and the lower bound on the index will be greater than its actual value. Therefore, for $\alpha = 1$, all conjugate roots are connected, and we have thus shown the existence of $\frac{n}{2} - 1$ integral curves of finite length. The graph of integral curves for the case $n = 8$ is shown in Figure 8.

The same sort of argument holds for each $\alpha$ which is an $n$th root of 1. In each of the $n/2$ cases (i.e., not distinguishing between $\alpha$ and $-\alpha$), we can exploit the symmetry about the line $\alpha t$ to obtain $\frac{n}{2} - 1$ integral curves, for a total of $\frac{n}{2}(\frac{n}{2} - 1)$ distinct cases. We can
also make use of the symmetry about the line $\alpha t$ when $\alpha$ is a $(2n)\text{th}$ root of 1; this gives an additional $(\frac{n}{2})^2$ integral curves, for a total of $n(n−1)/2$. Finally, if $n$ is odd, a similar counting scheme works: each $n\text{th}$ root of 1 corresponds to a distinct symmetry axis, and each such value of $\alpha$ leads to $(n−1)/2$ integral curves, for a grand total of $n(n−1)/2$.

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