Minimal annuli with constant contact angle
along the planar boundaries

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Abstract

We show that an immersed minimal annulus, with two planar boundary curves along
which the surface meets these planes with constant contact angle, is part of the catenoid.

1 Introduction

The catenoid is the simplest minimal surface in $\mathbb{R}^3$ except the plane. It is obtained by
revolving the catenary about an axis. The catenoid has been characterized by many authors.
For instance:

1. The catenoid is the only nonplanar minimal surface which is a surface of revolution (Bon-
   net [9]).

2. The catenoid is the only complete embedded minimal surface of total curvature $-4\pi$
   (Osserman [10]).

3. A complete minimal surface with two annular ends and of finite total curvature is the
catenoid (Schoen [11]).

4. A complete embedded nonplanar minimal surface of finite total curvature and genus zero
   is the catenoid (López and Ros [7]).

For more interesting characterizations see [1], [3] and [8].

On the other hand, one can also characterize a proper subset of the catenoid.
In 1869, Enneper [9] proved that a compact nonplanar minimal surface which is generated by
one-parameter family of circles is part of the catenoid or Riemann’s example. In 1956, Shiffman
[13] proved that a minimal annulus bounded by two horizontal circles is foliated by horizontal
circles.

In this paper we will also characterize a proper subset of the catenoid. Our characterizations
involves the hypothesis of a constant contact angle along the boundary of the minimal surface
as follows:

**Theorem 1.1.** Let $\Sigma$ be an immersed minimal annulus such that $\partial \Sigma$ consists of two $C^{2,\alpha}$
planar Jordan curves $\Gamma_1$ and $\Gamma_2$. If $\Sigma$ makes a constant contact angle with a plane $\Pi_i$ along $\Gamma_i$,
i = 1, 2, $\Pi_1 \neq \Pi_2$, then $\Sigma$ is part of the catenoid.

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Theorem 1.2. Let $\Sigma$ be an immersed minimal surface with boundary and let $\Gamma$ be one component of $\partial \Sigma$. If $\Gamma$ is a circle and $\Sigma$ meets a plane along $\Gamma$ at a constant angle, then $\Sigma$ is part of the catenoid.

Note that in Theorem 1.1 it is not necessary to assume that the planes $\Pi_1$ and $\Pi_2$ are parallel. Also it should be mentioned that Wente [16] proved every embedded annular capillary surface in a slab is a surface of revolution. But he also constructed many examples of immersed non-zero constant mean curvature (henceforth abbreviated as CMC) annular capillary surfaces lying in a slab which are not surfaces of revolution [17].

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2 Preliminaries

First, we review the Hopf differential. Let $\Sigma$ be a planar annulus $A = \{ (u, v) \in \mathbb{R}^2 : 1/R \leq u^2 + v^2 \leq R, R > 1 \}$. Suppose $u$ and $v$ are the isothermal coordinates on $A$ determined by $X$. We can write the first fundamental form and the second fundamental form of $\Sigma$ as follows

$$I_X = E(du^2 + dv^2),$$
$$II_X = Ldu^2 + 2M dudv + N dv^2.$$

The Hopf differential is the quadratic differential defined by $\Phi dw^2$, $\Phi = (L - N)/2 - iM$, $w = u + iv$. Then the Codazzi equation implies the following lemma.

Lemma 2.1. (See [2] or [6].) $\Phi$ is holomorphic on a CMC surface.

Second, we review some properties of umbilic points of a CMC surface. Umbilic points of $\Sigma$ are the zeros of $\Phi$. Lines of curvature of $\Sigma$ flow smoothly except at umbilic points. They rotate sharply around at an umbilic point. So we can define the rotation index of the lines of curvature at interior umbilic points.

Now we extend the rotation index to a boundary point. Let $p \in \partial \Sigma$ be a boundary point. We choose $X : D^+ \to \Sigma$ which is a conformal immersion of a half disk $D^+ = \{ (u, v) \in D : u^2 + v^2 \leq 1, v \geq 0 \}$ into the regular surface $\Sigma$ mapping the diameter $l$ of $D^+$ into $\partial \Sigma$. The lines of curvature of $\Sigma$ can be pulled back by $X$ to a line field on $D^+$. If $X(l)$ is a line of curvature of $\Sigma$, then this line field can be extended smoothly to a line field $F$ on $D$ by reflection about the diameter $l$. It is clear that $F$ has the well-defined rotation index at $X^{-1}(p)$ and furthermore, the rotation index does not depend on the choice of immersion $X$. So we can naturally define the rotation index of the lines of curvature at the umbilic point $p \in \partial \Sigma$ to be half the rotation index of $F$ at $X^{-1}(p)$.

Lemma 2.2. ( [2], Lemma 2) Let $\Sigma$ be a non-totally umbilic immersed CMC surface which is of class $C^{2,\alpha}$ up to and including the boundary $\partial \Sigma$. If the $\partial \Sigma$ are lines of curvature, then we have the following.

(a) The boundary umbilic points of $\Sigma$ are isolated.
(b) At an interior umbilic point the rotation index of lines of curvature is not bigger than $-1/2$.
(c) At a boundary umbilic point the rotation index of lines of curvature is not bigger than $-1/4$.

We now recall Björling’s theorem (see [4]). Let $c : [a, b] \to \mathbb{R}^3$ be any real analytic curve and $n : [a, b] \to S^2$ be any real analytic vector field perpendicular to the tangent vector of the curve $c(t)$. By the analyticity of $c$, there are unique analytic extensions $c : [a, b] \times (-\varepsilon, \varepsilon) \to \mathbb{C}^2$, and
n : [a, b] × (−ε, ε) → C³, where ε is a small enough positive number. Using these extensions, we define the unique immersion of surface as follows
\[ X(z) = \text{Re} \left( c(z) - i \int_0^z n(w) \times c'(w)dw \right), \]

where \( z \in [a, b] \times (−ε, ε) \). By a straightforward computation, this is a minimal immersion that extends \( c \) and \( n \) in the sense that for \( t \in [a, b] \times \{0\} \), \( X(t) = c(t) \) and \( n(t) \) is the surface normal.

3 Proof of the theorems

Proof of theorem 1.1. Step 1. We claim that \( \Pi_1 \) and \( \Pi_2 \) are parallel.

Let \( X : A = \{(u, v) \in \mathbb{R}^2 : 1/R \leq u^2 + v^2 \leq R \} \rightarrow \mathbb{R}^3 \) be a conformal immersion of \( \Sigma \). Because \( \Sigma \) is a minimal surface, \( \Delta X = 0 \) on \( A \), where \( \Delta \) is the Laplace-Beltrami operator on \( \Sigma \). So we have
\[ 0 = \int_\Sigma \Delta X dA = \int_{\Gamma_1} \nu_1 ds + \int_{\Gamma_2} \nu_2 ds, \]

where \( \nu_i \) denotes the outward pointing unit conormal vector along \( \Gamma_i \), \( i = 1, 2 \). By the boundary maximum principle [6], \( \theta_1 \neq 0, \pi \) (see, Figure 1). Let \( \Omega_1 \subset \Pi_1 \) be the domain bounded by \( \Gamma_1 \). The projection of \( \nu_1 \) to \( \Pi_1 \) is a normal vector field of \( \Gamma_1 \), and it is constant length. By the divergence theorem on \( \Omega_1 \), we see that the non-zero vector \( \int_{\Gamma_1} \nu_1 ds \) is perpendicular to \( \Pi_1 \).

Similarly, the non-zero vector \( \int_{\Gamma_2} \nu_2 ds \) is also perpendicular to \( \Pi_2 \). This implies that the two vectors are linearly dependent. So \( \Pi_1 \) and \( \Pi_2 \) are parallel.

Step 2. We claim that both \( \Gamma_1 \) and \( \Gamma_2 \) are convex.

The Terquem-Joachimsthal theorem [15] says that if \( \Gamma = \Sigma_1 \cap \Sigma_2 \) is a line of curvature in \( \Sigma_1 \), then \( \Gamma \) is also a line of curvature in \( \Sigma_2 \) if and only if \( \Sigma_1 \) and \( \Sigma_2 \) intersect at a constant angle along \( \Gamma \). Since \( \Gamma_1 \) is a line of curvature of \( \Pi_1 \) and \( \Sigma \) meets \( \Pi_1 \) in a constant contact angle along \( \Gamma_1 \), \( \Gamma_1 \) is also a line of curvature of \( \Sigma \). Similarly, \( \Gamma_2 \) is a line of curvature of \( \Sigma \).

Let \( \kappa_1, \kappa_2 \) be the principal curvatures of \( \Sigma \) along \( \Gamma_1, \Gamma_2 \) respectively. We want to show that neither \( \kappa_1 \) nor \( \kappa_2 \) has zeros.

First, let us suppose that \( \kappa_1 \) has zeros at finite points \( p_{ij}, j = 1, ..., m_i \) on \( \Gamma_i \), \( i = 1, 2 \). Then \( p_{ij}, j = 1, ..., m_i \) are the boundary umbilic points. Let \( q_k, k = 1, ..., n \) be the interior umbilic points. By the Poincaré-Hopf theorem and Lemma 2.2, we have
\[ \chi(\Sigma) = 0 = \sum_{p=p_{ij}, q_k} I(p) \leq \sum_i \sum_j \left( -\frac{1}{4} \right) + \sum_k \left( -\frac{1}{2} \right) < 0, \]

where \( \chi(\Sigma) \) is the Euler characteristic of \( \Sigma \) and \( I(p) \) is the rotation index at \( p \). Therefore neither \( \kappa_1 \) nor \( \kappa_2 \) has zeros.

Second, suppose either \( \kappa_1 \) or \( \kappa_2 \) has zeros at an infinite number of points. By lemma 2.1, \( \Phi \) is a holomorphic function. Since \( \Gamma_1, \Gamma_2 \) are compact sets, the Hopf differential \( \Phi \) of \( \Sigma \) is identically zero on the one or both of \( \Gamma_1 \) and \( \Gamma_2 \). So \( \Phi \) is identically zero. This means that \( \Sigma \) is a planar annulus. Since \( \Pi_1 \neq \Pi_2 \), this case cannot happen.

Hence both \( \kappa_1 \) and \( \kappa_2 \) cannot have zeros.

Let \( \kappa_i \) be the curvature of \( \Gamma_i \) in \( \Pi_i \). Since \( \kappa_i = \kappa_i \sin \theta_i, \kappa_i \) cannot be zero. So both \( \Gamma_1 \) and \( \Gamma_2 \) are convex. In fact they are strictly convex.

Step 3. We claim that \( \Sigma \) is part of the catenoid.

By Step 2 the total curvature of \( \partial \Sigma \) is \( 4\pi \). So by [4], \( \Sigma \) is embedded (see, Figure 1).

Hence we know that \( \Sigma \) is a surface of revolution by [16]. But we give a sketch of the proof for the sake of completeness.

Let \( v \) be a unit vector in \( \Pi_1 \). Now we apply the Alexandrov reflection principle [6] with the
one-parameter family of plane $\Pi_{v,t}$ which is orthogonal to $v$. Increasing $t$ one gets a first plane $\Pi_{v,\mathbb{T}}$ that reach $\Sigma$: that is $\Pi_{v,\mathbb{T}} \cap \Sigma \neq \emptyset$, but if $t < \mathbb{T}$ then $\Pi_{v,t} \cap \Sigma = \emptyset$. Let $\Sigma_t$ be the part of $\Sigma$ lying in $\Pi_{v,s}$, $s < t$ and let $\Sigma_{\Pi_{v,t}}$ be the symmetry of $\Sigma_t$ about $\Pi_{v,t}$. First, let us assume the first touching between $\Sigma$ and the reflected surface $\Sigma_{\Pi_{v,t_0}}$ by $\Pi_{v,t_0}$ for some $t_0$ occurs at an interior point. By the Hopf maximum principle, $\Sigma$ is symmetric with respect to $\Pi_{v,t_0}$. Second, let us assume the first touching between $\Sigma$ and $\Sigma_{\Pi_{v,t_1}}$ for some $t_1$ occurs at a boundary point. Since the contact angle is constant, the normal vector of $\Sigma$ at the touching point coincides with that of the reflected surface $\Sigma_{\Pi_{v,t_1}}$ at the touching point. So we can use the Hopf boundary maximum principle, then $\Sigma$ is symmetric with respect to $\Pi_{v,t_1}$. Otherwise, let us assume the first touching between $\Sigma$ and $\Sigma_{\Pi_{v,t_2}}$ for some $t_2$ occurs at a corner. Similarly the second case, the normal vector of $\Sigma$ at the corner coincides with that of the reflected surface $\Sigma_{\Pi_{v,t_2}}$ at the corner. Because of constant of the contact angle we can apply the Serrin’s boundary point lemma at a corner (Lemma 2.6 of [16] or [12]). Then $\Sigma$ is symmetric with respect to $\Pi_{v,t_2}$.

Since the first touching must occur at an interior point, a boundary point or a corner, $\Sigma$ is symmetric with respect to $\Pi_{v,T}$. By using the reflection principle for another unit vector $w$ in $\Pi_1$, we get that $\Sigma$ is symmetric with respect to $\Pi_{v,T_1}$. Hence, we conclude that $\Sigma$ is a rotational surface, i.e., it is part of the catenoid. □

Proof of theorem 1.2. Without loss of generality denote the circle by $c(t) = (\cos t, \sin t, 0)$. Since the surface has constant contact angle $\theta$ along the $c$, the surface normal vector becomes $n(t) = (\sin \theta \cos t, \sin \theta \sin t, \cos \theta)$. Then Björling’s formula (11) yields the unique minimal surface

$$X(z) = (X_1(z), X_2(z), X_3(z)) = \text{Re} \left( (\cos z, \sin z, 0) - i \int_0^z \left( -\cos \theta \cos w, \cos \theta \sin w, \sin \theta \right) dw \right)$$

$$= (\cos u \cosh v - \cos \theta \sin u \sinh v, \sin u \cosh v + \cos \theta \cos u \sinh v, \sin \theta v).$$

For each $X_3$, $X_1^2 + X_2^2 = \text{constant}$. This means that the minimal surface is foliated by coaxial circles. So $\Sigma$ is part of the catenoid. □
4 Remarks

Lemma 2.1, Lemma 2.2 and the Poincaré-Hopf theorem hold for CMC surfaces. So we drive a characterization of a sphere after an additional assumption.

**Corollary 4.1.** Let $\Sigma$ be an immersed non-zero CMC annulus such that $\partial \Sigma$ consists of two $C^{2,\alpha}$ planar Jordan curves $\Gamma_1$ and $\Gamma_2$. If $\Sigma$ makes a constant contact angle with a plane $\Pi_i$ along $\Gamma_i$, $i = 1, 2$, $\Pi_1 \neq \Pi_2$. In addition, $\Sigma$ has at least one umbilic point. Then $\Sigma$ is part of a sphere.

**Proof.** If $\Sigma$ has only finite umbilic points then it is contradiction to the Poincaré-Hopf theorem. So $\Sigma$ has infinitely many umbilic points. Since the Hopf differential $\Phi$ is holomorphic, it is identically zero. Hence, $\Sigma$ is part of a sphere.

So far, we have considered minimal annuli with boundary curves lying in a pair of planes. In case of minimal annuli with boundary curves lying on a sphere, there is a well-known open problem.

**Problem 4.2.** Let $\Gamma_1$, $\Gamma_2$ be $C^{2,\alpha}$ Jordan curves on a sphere. Show that if $\Sigma$ is an immersed minimal annulus meeting constant contact angles with the sphere along the boundary $\Gamma_1$, $\Gamma_2$, then $\Sigma$ is part of the catenoid.

**Remark 4.3.** By Theorem 1.2, Problem 4.2 is true if one of the boundary curves is a circle.

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