MOMENT ASYMPTOTIC EXPANSIONS OF THE WAVELET TRANSFORMS

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Abstract. Using distribution theory we present the moment asymptotic expansion of continuous wavelet transform in different distributional spaces for large and small values of dilation parameter $a$. We also obtain asymptotic expansions for certain wavelet transform.

1. Introduction

In past few decades there were many mathematician who has done great work in the field of asymptotic expansion like Wong 1979 [10] using Mellin transform technique has obtained asymptotic expansion of classical integral transform and after that Pathak & Pathak 2009 [3] [4] [5] [6] has found the asymptotic expansion of continuous wavelet transform for large and small values of dilation and translation parameters. Estrada & Kanwal 1990 [7] has obtained the asymptotic expansion of generalized functions on different spaces of test functions. In present paper using Estrada & Kanwal technique we have obtained the asymptotic expansion of

2000 Mathematics Subject Classification. 42C40; 34E05.
Key words and phrases. Asymptotic expansion, Wavelet transform, Distribution.

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wavelet transform in different distributional spaces.

The continuous wavelet transform of $f$ with respect to wavelet $\psi$ is defined by

$$\left( W_\psi f \right)(a,b) = \frac{1}{\sqrt{a}} \int_{-\infty}^{\infty} f(x) \psi \left( \frac{x-b}{a} \right) dx, \quad b \in \mathbb{R}, a > 0,$$  \hspace{1cm} (1.1)

provided the integral exists \[3\]

Now, from (1.1) we get

$$\left( W_\psi f \right)(a,b) = \sqrt{a} \int_{-\infty}^{\infty} f(x) \psi \left( \frac{x-b}{a} \right) dx = \sqrt{a} \left\langle f(ax), \psi \left( x - \frac{b}{a} \right) \right\rangle$$  \hspace{1cm} (1.2)

This paper is arranged in following manner. In section second, third, fourth and fifth we drive the asymptotic expansion in the distributional spaces $\mathcal{E}'(\mathbb{R})$, $\mathcal{D}'(\mathbb{R}), \mathcal{O}'_\gamma(\mathbb{R}), \mathcal{O}'_\varepsilon(\mathbb{R})$ and $\mathcal{O}'_M(\mathbb{R})$ respectively, studied in \[7\]

2. THE MOMENT ASYMPTOTIC EXPANSION OF $\left( W_\psi f \right)(a,b)$ AS $a \to \infty$ IN THE SPACE $\mathcal{E}'(\mathbb{R})$ FOR GIVEN $b$

The space $\mathcal{E}(\mathbb{R})$ is the space of all smooth functions on $\mathbb{R}$ and it’s dual space $\mathcal{E}'(\mathbb{R})$, the space of distribution with compact support. If $\psi \in \mathcal{E}(\mathbb{R})$, then $\psi \left( x - \frac{b}{a} \right) \in \mathcal{E}(\mathbb{R})$. So consider the seminorms

Case 1 For $b \geq 0$

$$\left\| \psi \left( x - \frac{b}{a} \right) \right\|_{\alpha,M} = \text{Max} \left\{ D^\alpha \psi \left( x - \frac{b}{a} \right) : \frac{b}{a} - M < x < b + M \right\}$$  \hspace{1cm} (2.1)

Case 2 For $b < 0$

$$\left\| \psi \left( x - \frac{b}{a} \right) \right\|_{\alpha,M} = \text{Max} \left\{ D^\alpha \psi \left( x - \frac{b}{a} \right) : b - M < x < \frac{b}{a} + M \right\}$$  \hspace{1cm} (2.2)
for $\alpha \in \mathbb{N}$ and $M > 0$, these seminorm generate the topology of $\mathcal{E}(\mathbb{R})$. If $q = 0, 1, 2, 3, \ldots$, we set

$$X_q = \{ \psi \in \mathcal{E}(\mathbb{R}) : D^\alpha \psi(0) = 0 \text{ for } \alpha < q \} \quad (2.3)$$

**Lemma 2.1.** Let $\psi \in X_q$, then for every $\alpha \in \mathbb{N}$ and $M > 0$,

$$\left\| \psi \left( \frac{x - b}{a} \right) \right\|_{\alpha, M} = O \left( \frac{1}{a^q} \right) \text{ as } a \to \infty \quad (2.4)$$

**Proof 1.** For $b \geq 0$. For $\psi \in X_q$ we can find a constant $K$ such that

$$\left| \psi \left( x - \frac{b}{a} \right) \right| \leq K \left| x - \frac{b}{a} \right|^q, \quad b/\alpha - 1 < x < b + 1. \quad (2.5)$$

Therefore, if $a > M$ we obtain

$$\left\| \psi \left( \frac{x - b}{a} \right) \right\|_{0,M} = \text{Max} \left\{ \left| \psi \left( \frac{x - b}{a} \right) \right| : \frac{b}{\alpha} - M < x < b + M \right\} \leq O \left( \frac{M}{a^q} \right). \quad (2.6)$$

If $\alpha \leq q$ and $\psi \in X_q$ then $D^\alpha \psi \in X_{q-\alpha}$ and thus

$$\left\| \psi \left( \frac{x - b}{a} \right) \right\|_{\alpha, M} = \left\| a^\alpha D^\alpha \psi \left( \frac{x - b}{a} \right) \right\|_{0, M} = \frac{1}{a^\alpha} O \left( \frac{1}{a^{q-\alpha}} \right) = O \left( \frac{1}{a^q} \right)$$

Similarly by using (2.2) we can prove that

$$\left\| \psi \left( \frac{x - b}{a} \right) \right\|_{\alpha, M} = O \left( \frac{1}{a^q} \right) \text{ for } b < 0.$$
Theorem 2.2. Let wavelet \( \psi \in \mathcal{E}(\mathbb{R}) \), \( f \in \mathcal{E}'(\mathbb{R}) \) and \( \mu_\alpha = \langle f, x^\alpha \rangle \) be its moment sequence. Then for a fixed \( b \) the moment asymptotic expansion of wavelet transform is

\[
\sqrt{a}\left\langle f(ax), \psi \left( x - \frac{b}{a} \right) \right\rangle = \sum_{\alpha=0}^{N} \frac{\mu_\alpha D^\alpha \psi(-b/a)}{\alpha! a^{\alpha+1/2}} + O\left( \frac{1}{a^{N+1/2}} \right) \text{ as } a \to \infty. \tag{2.7}
\]

Proof 2. Let \( P_N(x, b/a) = \sum_{\alpha=0}^{N} \frac{D^\alpha \psi(-b/a)}{a^{\alpha}} x^\alpha \) be the polynomial of order \( N \) of the function \( \psi \left( x - \frac{b}{a} \right) \). Then we have

\[
\left\langle f(ax), \psi \left( x - \frac{b}{a} \right) \right\rangle = \left\langle f(ax), P_N(x, b/a) \right\rangle + \left\langle f(ax), \psi \left( x - \frac{b}{a} \right) - P_N(x, b/a) \right\rangle
\]

\[
= \sum_{\alpha=0}^{N} \frac{\mu_\alpha D^\alpha \psi(-b/a)}{\alpha! a^{\alpha+1}} + R_N(a)
\]

where the remainder \( R_N(a) \) is given as \( R_N(a) = \left\langle f(ax), \psi \left( x - \frac{b}{a} \right) - P_N(x, b/a) \right\rangle \).

Since \( \psi \left( x - \frac{b}{a} \right) - P_N(x, b/a) \) in \( X_{N+1} \) we obtain

\[
|R_N(a)| = \left| \left\langle f(ax), \psi \left( x - \frac{b}{a} \right) - P_N(x, b/a) \right\rangle \right|
\]

\[
= \frac{L}{a} \sum_{\alpha=0}^{q} \| \psi_N \left( \frac{x - b}{a} \right) \|_{\alpha, M}
\]

\[
= O\left( \frac{1}{a^{N+1}} \right)
\]

where the existence of \( L, q \) and \( M \) is guaranteed by the continuity of \( f \). Hence we get the required asymptotic expansion \( 2.7 \).

Example 2.3. In this example we choose \( \psi \) to be Mexican-Hat wavelet and derive the asymptotic expansion of Mexican-Hat wavelet transform by using Theorem 2.2.

The Mexican-Hat wavelet is given by \( 3 \)

\[
\psi(x) = (1 - x^2)e^{-x^2} \in \mathcal{E}(\mathbb{R}) \tag{2.8}
\]
Let
\[ P_2(x, b/a) = e^{-\frac{x^2}{2}} \left( (a^2 - b^2) + \frac{b(3a^2 - b^2)}{a} x + \frac{(6a^2b^2 - 3a^4 - b^4)}{2a^2} x^2 \right). \]

Now, using Theorem 2.2 we get the asymptotic expansion of Mexican-Hat wavelet transform
\[ \sqrt{a} \langle f(ax), \psi \left( x - \frac{b}{a} \right) \rangle = e^{-\frac{x^2}{2}} \left( \frac{(a^2 - b^2)}{\sqrt{a}} \mu_0 + \frac{b(3a^2 - b^2)}{a^{3/2}} \mu_1 + \frac{(6a^2b^2 - 3a^4 - b^4)}{2a^{5/2}} \mu_2 \right) + O\left( \frac{1}{a^{9/2}} \right) \text{ as } a \to \infty \]
where \( \mu_i = \langle f, x^i \rangle, i = 0, 1, 2. \)

3. The moment asymptotic expansion of \((W_\psi f)(a, b)\) for large and small values of \(a\) in the space \(\mathcal{P}'(\mathbb{R})\) for a given \(b\)

Case 1. Let \(\psi \in \mathcal{P}(\mathbb{R})\).

We now consider the moment asymptotic expansion in the space \(\mathcal{P}'(\mathbb{R})\) of distributions of "less than exponential growth". The space \(\mathcal{P}(\mathbb{R})\) consist of those smooth functions \(\phi(x)\) that satisfy
\[ \lim_{x \to \infty} e^{-\gamma|x|} D^\beta \phi(x) = 0 \text{ for } \gamma > 0 \text{ and each } \beta \in \mathbb{N}, \]
with seminorms
\[ \|\phi(x)\|_{\gamma, \beta} = \sup \left\{ | e^{-\gamma|x|} D^\beta \phi(x) | : x \in \mathbb{R} \right\}. \]
Let wavelet $\psi(x) \in \mathcal{P}(\mathbb{R})$. Then

$$\|\psi(x)\|_{\gamma, \beta, \frac{b}{a}} = \sup \left\{ \left| e^{-\gamma|x|} D^\beta (x - \frac{b}{a}) \right| : x \in \mathbb{R} \right\}$$

$$= \sup \left\{ \left| e^{-\gamma|x - \frac{b}{a}|} D^\beta (x - \frac{b}{a}) e^{-\gamma|x|} \right| : x \in \mathbb{R} \right\}$$

$$= \|\psi \left( x - \frac{b}{a} \right) \|_{\gamma, \beta} A(x, b/a),$$

where

$$A(x, b/a) = \sup \left\{ \left| \frac{e^{-\gamma|x|}}{e^{-\gamma|x - \frac{b}{a}|}} \right| : x \in \mathbb{R} \right\} \leq e^{\gamma \left| \frac{b}{a} \right|} < \infty,$$

for a given $\gamma > 0$ and $b \in \mathbb{R}$.

So $\|\psi(x)\|_{\gamma, \beta, \frac{b}{a}}$ is also seminorm on $\mathcal{P}(\mathbb{R})$ for $\gamma > 0$, $\beta \in \mathbb{N}$ and for a given $b \in \mathbb{R}$. Therefore these seminorm generate the topology of the space $\mathcal{P}(\mathbb{R})$. If

$$X_q = \{ \psi \in \mathcal{P}(\mathbb{R}) : D^\alpha \psi(0) = 0, \text{ for } \alpha < q \}.$$

Therefore for any $\gamma > 0$ we can find a constant $C$ such that

$$\left| \psi \left( x - \frac{b}{a} \right) \right| \leq C \left| x - \frac{b}{a} \right|^q e^{-\gamma \left| \frac{b}{a} \right|}$$

if $a > 1$

$$e^{-\gamma|x|} \left| \psi \left( x - \frac{b}{a} \right) \right| \leq C \left| x - \frac{b}{a} \right|^q e^{-\gamma \left| \frac{b}{a} \right|} \leq \frac{C_1}{a^q}$$

and thus

$$\left\| \psi \left( \frac{x}{a} \right) \right\|_{\gamma, 0, \frac{b}{a}} = O \left( \frac{1}{a^q} \right) \text{ as } a \to \infty, \psi \in X_q. \quad (3.1)$$

Hence using above equation we get

$$\left\| \psi \left( \frac{x}{a} \right) \right\|_{\gamma, \beta, \frac{b}{a}} = O \left( \frac{1}{a^q} \right) \text{ as } a \to \infty. \quad (3.2)$$
Using (3.2) we obtain the following theorem

**Theorem 3.1.** Let $\psi \in \mathcal{P} (\mathbb{R})$, $f \in \mathcal{P}' (\mathbb{R})$ and $\mu_\alpha = \langle f, x^\alpha \rangle$ be its moment sequence. Then for a fixed $b$ the asymptotic expansion of wavelet transform is

$$
\sqrt{a} \left\langle f(ax), \psi \left( x - \frac{b}{a} \right) \right\rangle \sim \sum_{\alpha = 0}^{\infty} \frac{\mu_\alpha D_\alpha \psi (\frac{-b}{a})}{\alpha! a^{\alpha+1/2}} \quad \text{as } a \to \infty. \quad (3.3)
$$

**Proof 3.** Similarly as Theorem 2.2

**Example 3.2.** Let $\psi(x) = (1 - x^2)e^{-\frac{x^2}{2}} \in \mathcal{P} (\mathbb{R})$ is Mexican-Hat wavelet and $f(x) \in \mathcal{P}' (\mathbb{R})$. Therefore by Theorem 3.3 moment asymptotic expansion of continuous Mexican-Hat wavelet transform for large $a$ in $\mathcal{P}' (\mathbb{R})$ is given by

$$
\sqrt{a} \left\langle f(ax), \psi \left( x - \frac{b}{a} \right) \right\rangle \sim \sum_{\alpha = 0}^{\infty} \frac{\mu_\alpha D_\alpha [(1 - x^2)e^{-\frac{x^2}{2}}]_{x = \frac{-b}{a}}}{\alpha! a^{\alpha+1/2}} \quad \text{as } a \to \infty.
$$

Case 2. In this case we consider wavelet $\psi(x) \in \mathcal{P}' (\mathbb{R})$ and $f(x) \in \mathcal{P} (\mathbb{R})$.

Then the wavelet transform (1.1) can we rewrite as

$$
(W_\psi f)(a,b) = \frac{1}{\sqrt{a}} \left\langle \psi \left( \frac{x}{a} \right), f\left( x + \frac{b}{a} \right) \right\rangle
$$

Similarly as Theorem 3.1 we can also obtain the following theorem

**Theorem 3.3.** Let $\psi \in \mathcal{P}' (\mathbb{R})$, $f \in \mathcal{P} (\mathbb{R})$ and $\mu_\alpha = \langle \psi, x^\alpha \rangle$ be its moment sequence. Then for a fixed $b$ the asymptotic expansion of wavelet transform is

$$
\frac{1}{\sqrt{a}} \left\langle \psi \left( \frac{x}{a} \right), f(x + b) \right\rangle \sim \sum_{\alpha = 0}^{\infty} \frac{\mu_\alpha D_\alpha f(b) a^{\alpha+1/2}}{\alpha!} \quad \text{as } a \to 0. \quad (3.4)
$$

**Example 3.4.** In this example again we consider the Mexican-Hat wavelet which is less then exponential growth , so by applying Theorem 3.3 and using formula [30,pp.320, 1], we get the asymptotic expansion of wavelet transform for small
values of $a$

$$\frac{1}{\sqrt{a}} \left\langle \psi \left( \frac{x}{a} \right), f(x+b) \right\rangle \sim \sum_{\alpha=0}^{\infty} -2^{\frac{1}{2}(2\alpha-1)} \Gamma \left( \frac{2\alpha+1}{2} \right) \frac{D^{2\alpha} f(b)}{(2\alpha)!} \frac{a^{2\alpha+1/2}}{2^{\alpha}} f(a)_{2\alpha+1} a^{1/2} \quad \text{as } a \to 0.$$

4. **The moment asymptotic expansion of** $$(W_{\psi} f)(a,b)$$ **as** $a \to \infty$ **in the space** $\mathcal{O}_{\gamma}'(\mathbb{R})$ **for given** $b$

A test function $\psi$ belongs to $\mathcal{O}_{\gamma}(\mathbb{R})$, if it is smooth and $D^\alpha \psi(x) = O(|x|^\gamma)$ as $x \to \infty$ for every $\alpha \in \mathbb{N}$ and $\gamma \in \mathbb{R}$. The family of seminorms

$$\|\psi(x)\|_{\alpha,\gamma} = \sup \{\rho_\gamma(|x|)|D^\alpha \psi(x)| : x \in \mathbb{R}\}$$

where

$$\rho_\gamma(|x|) = \begin{cases} 1, & 0 \leq |x| \leq 1 \\ |x|^{-\gamma}, & |x| > 1 \end{cases}$$

(4.1)

generates a topology for $\mathcal{O}_{\gamma}(\mathbb{R})$. Now with the help of the translation version of $\psi(x)$, we can define the seminorms on $\mathcal{O}_{\gamma}(\mathbb{R})$ as

$$\|\psi(x)\|_{\alpha,\gamma,b/a} = \sup \left\{ \rho_\gamma(|x|)|D^\alpha \psi \left( x - \frac{b}{a} \right)| : x \in \mathbb{R} \right\}$$

$$= \sup \left\{ \rho_\gamma \left( \left| x - \frac{b}{a} \right| \right) \left| D^\alpha \psi \left( x - \frac{b}{a} \right) \right| : x \in \mathbb{R} \right\} \nabla(x,b/a)$$

$$= \left\| \psi \left( x - \frac{b}{a} \right) \right\|_{\alpha,\gamma} \nabla(x,b/a)$$

where $\nabla(x,b/a) = \sup \left\{ \frac{\rho_\gamma(|x|)}{\rho_\gamma(|x-b/a|)} : x \in \mathbb{R} \right\}$. So, for $\gamma > 0$.

$$\nabla(x,b/a) \leq \begin{cases} 1, & \text{for } 0 \leq |x| \leq 1 \text{ and } 0 \leq |x-b/a| \leq 1 \\ \left( 1 + \frac{|b/a|}{1-|b/a|} \right)^\gamma, & \text{for } |x| > 1 \text{ and } |x-b/a| > 1 \\ (1 + \frac{1}{a})^\gamma, & \text{for } 0 \leq |x| \leq 1 \text{ and } |x-b/a| > 1 \\ 1, & \text{for } |x| > 1 \text{ and } |x-b/a| \leq 1 \end{cases}.$$
Similarly for $\gamma < 0$, we have
\[
\nabla(x, b/a) \leq \begin{cases} 
1, & \text{for } 0 \leq |x| \leq 1 \text{ and } 0 \leq |x - b/a| \leq 1 \\
\left(1 + \frac{|b/a|}{1 + |b/a|}\right)^{-\gamma}, & \text{otherwise}
\end{cases}
\]

Thus $\sup \left\{ \frac{\rho(|x|)}{\rho(|x - \frac{b}{a}|)} : x \in \mathbb{R} \right\} \leq \left(1 + \frac{|b/a|}{1 + |b/a|}\right)^{|\gamma|} = K < \infty$, $\forall \gamma \in \mathbb{R}$

Therefore $\|\psi(x)\|_{\alpha, \gamma, b/a}$ are also seminorm on $\mathcal{O}_{\gamma}(\mathbb{R})$. These seminorm generate the topology of the space $\psi(x) \in \mathcal{O}_{\gamma}(\mathbb{R})$. If

$$X_q = \{ \psi \in \mathcal{O}_{\gamma}(\mathbb{R}) : D^\alpha \psi(0) = 0, \text{ for } \alpha < q \}.$$ 

So for any $\gamma$ we can find a constant $C$ such that

$$\rho(|x|) \left| \psi \left( x - \frac{b}{a} \right) \right| \leq C \rho(|x|) \left| x - \frac{b}{a} \right|^q \nabla(x, b/a).$$

If $a > 1$

$$\rho(|x|) \left| \psi \left( x - \frac{b}{a} \right) \right| \leq \frac{M}{a^q}$$

Hence using above equation we get

\[
\left\| \psi \left( \frac{x}{a} \right) \right\|_{\alpha, \gamma, b/a} = O \left( \frac{1}{a^q} \right) \text{ as } a \to \infty. \tag{4.2}
\]

Similarly as Theorem 3.3 we can obtain the following theorem

**Theorem 4.1.** Let $\psi \in \mathcal{O}_{\gamma}(\mathbb{R})$, $f \in \mathcal{O}_{\gamma}'(\mathbb{R})$, $N = ||[\gamma]|| - 1$ and $\mu_\alpha = \langle f, x^\alpha \rangle$ be its moment sequence. Then for a fixed $b \in \mathbb{R}$ the asymptotic expansion of wavelet transform is

\[
\sqrt{a} \langle f(ax), \psi \left( x - \frac{b}{a} \right) \rangle = \sum_{\alpha=0}^{N} \frac{\mu_\alpha D^\alpha \psi(-b/a)}{\alpha!a^{\alpha+1/2}} + O \left( \frac{1}{a^{N+1/2}} \right) \text{ as } a \to \infty. \tag{4.3}
\]

Since $\mathcal{O}_{\gamma}'(\mathbb{R}) = \bigcap \mathcal{O}_{\gamma}'(\mathbb{R})$, we obtain the asymptotic expansion of wavelet transform in the space $\mathcal{O}_{\gamma}'(\mathbb{R})$. 
Theorem 4.2. Let $\psi \in \mathcal{O}_c(\mathbb{R})$, $f \in \mathcal{O}'_c(\mathbb{R})$ and $\mu_\alpha = \langle f, x^\alpha \rangle$ be its moment sequence. Then for a fixed $b \in \mathbb{R}$ the asymptotic expansion of wavelet transform is

$$\sqrt{a} \left\langle f(ax), \psi \left( x - \frac{b}{a} \right) \right\rangle \sim \sum_{\alpha=0}^{\infty} \frac{\mu_\alpha D^\alpha \psi \left( -\frac{b}{a} \right)}{\alpha! a^{\alpha+1/2}} + O \left( \frac{1}{a^{N+1/2}} \right) \text{ as } a \to \infty. \quad (4.4)$$

5. The moment asymptotic expansion of $(W_\psi f)(a,b)$ as $a \to \infty$ in the space $\mathcal{O}'_M(\mathbb{R})$ for given $b$

The space $\mathcal{O}_M(\mathbb{R})$ consist of all $c^\infty$-function whose derivatives are bounded by polynomials (of probably different degrees). Let $\psi \in \mathcal{O}_M(\mathbb{R})$ then its translation version is also in $\mathcal{O}_M(\mathbb{R})$. Then by using Theorem 9 \cite{7} we can also derive the asymptotic expansion of wavelet transform in $\mathcal{O}'_M(\mathbb{R})$

Theorem 5.1. Let $\psi \in \mathcal{O}_M(\mathbb{R})$, $f \in \mathcal{O}'_M(\mathbb{R})$ and $\mu_\alpha = \langle f, x^\alpha \rangle$ be its moment sequence. Then for a fixed $b \in \mathbb{R}$ the asymptotic expansion of wavelet transform is

$$\langle f(x), \psi_{a,b}(x) \rangle \sim \sum_{\alpha=0}^{\infty} \frac{\mu_\alpha(f) D^\alpha \psi \left( -\frac{b}{a} \right)}{\alpha! a^{\alpha+1/2}} \text{ as } a \to \infty. \quad (5.1)$$

Proof. By using (1.7.1) \cite{3} we can be write the wavelet transform

$$\sqrt{a} \left\langle f(ax), \psi \left( x - \frac{b}{a} \right) \right\rangle = \sqrt{a} \langle e^{i\omega \hat{f}(\omega)}, \hat{\psi}(a\omega) \rangle \quad (5.2)$$

where $\psi(x) \in \mathcal{O}_M(\mathbb{R})$ and $f(x) \in \mathcal{O}'_M(\mathbb{R})$ then its Fourier transforms $\hat{\psi}(\omega) \in \mathcal{O}'_c(\mathbb{R})$ and $\hat{f}(\omega) \in \mathcal{O}_c(\mathbb{R})$ respectively.

Now by using Theorem 4.2 we get

$$\langle f(x), \psi_{a,b}(x) \rangle \sim \sum_{\alpha=0}^{\infty} \frac{\mu_\alpha(e^{-i\omega \hat{\psi}(\omega)}) D^\alpha(\hat{f}(0))}{\alpha! a^{\alpha+1/2}} \text{ as } a \to \infty. \quad (5.3)$$
But by the properties of Fourier transform we have

$$\mu_\alpha(e^{-ib\omega}\hat{\psi}(\omega)) = \langle e^{-i\frac{b}{a}\omega}\hat{\psi}(\omega), \omega^\alpha \rangle = i^{-\alpha}D^\alpha \psi\left(-\frac{b}{a}\right)$$

and hence

$$\langle f(x), \psi_{a,b}(x) \rangle \sim \sum_{\alpha=0}^{\infty} \frac{\mu_\alpha(f)D^\alpha \psi\left(-\frac{b}{a}\right)}{\alpha! a^{\alpha+1/2}} \quad \text{as} \quad a \to \infty. \quad (5.4)$$

\[\square\]

\section*{Acknowledgment}

The work of second author is supported by U.G.C. start-up grant.

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