Byzantine Consensus in Directed Hypergraphs

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Abstract

Byzantine consensus is a classical problem in distributed computing. Each node in a synchronous system starts with a binary input. The goal is to reach agreement in the presence of Byzantine faulty nodes. We consider the setting where communication between nodes is modelled via a directed hypergraph. In the classical point-to-point communication model, the communication between nodes is modelled as a simple graph where all messages sent on an edge are private between the two endpoints of the edge. This allows a faulty node to equivocate, i.e., lie differently to its different neighbors. Different models have been proposed in the literature that weaken equivocation. In the local broadcast model, every message transmitted by a node is received identically and correctly by all of its neighbors. In the hypergraph model, every message transmitted by a node on a hyperedge is received identically and correctly by all nodes on the hyperedge. Tight network conditions are known for each of the three cases for undirected (hyper)graphs. For the directed models, tight conditions are known for the point-to-point and local broadcast models.

In this paper, we consider the directed hypergraph model that encompasses all the models above. Each directed hyperedge consists of a single head (sender) and at least one tail (receiver),

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This models a local multicast channel where messages sent by the head node (sender) are received identically by all the tail nodes (receivers) in the hyperedge. For this model, we identify tight network conditions for consensus. We observe how the directed hypergraph model (which we will also refer to as the local multicast model) reduces to each of the three models above under specific conditions. In each of the three cases, we relate our network condition to the corresponding known tight conditions. The local multicast model also encompasses other practical network models of interest that have not been explored previously, as elaborated in the paper.

1 Introduction

Byzantine consensus is a classical problem in distributed computing introduced by Lamport et al. [11, 14]. There are \( n \) nodes in a synchronous system. Each node starts with a binary input. At most \( f < n \) of these nodes can be Byzantine faulty, i.e., exhibit arbitrary behavior. The goal of a consensus protocol is for the non-faulty nodes to reach agreement on a single output value in finite time. To exclude trivial protocols, we require that the output must be an input of some non-faulty node.

In this paper, we study consensus under local multicast channels, which can be modelled as directed hypergraphs. A hypergraph is a generalization of graphs consisting of nodes and hyperedges. Unlike an edge in a graph, a hyperedge can connect any number of nodes. In the local multicast model, nodes are connected via a directed hypergraph \( G \). A local multicast channel is a directed hyperedge defined by a single sender and a non-empty set of receivers. Each node \( u \) may potentially serve as the sender on multiple local multicast channels/hyperedges. When node \( u \) sends a message on one of its local multicast channels/hyperedges, This model generalizes the following models that have been considered before in the literature.

1. **Point-to-point communication model:** In the classical point-to-point communication model, each edge \((u, v)\) in the communication graph represents a private link from node \( u \) to node \( v \). This model is well-studied [1, 3, 11, 12, 14, 16, 17]. It is well-known that, for undirected graphs \( n \geq 3f + 1 \) and node connectivity at least \( 2f + 1 \) are both necessary and sufficient in this model.

2. **Local broadcast model:** Recently, in [8, 9], we studied consensus under the local broadcast model [2, 10], where a message sent by any node is received identically by all of its neighboring nodes in the communication graph. For undirected graphs, minimum node degree at least \( 2f \) and node connectivity at least \( \lceil 3f/2 \rceil + 1 \) are both necessary and sufficient for Byzantine consensus [8] under the local broadcast model.

3. **Undirected hypergraph model:** Communication networks modelled as undirected hypergraphs have been studied in the literature [5, 6, 15]. A message sent by a node \( u \) on an undirected
hyperedge $e \ni u$ is received identically by all nodes in $e$. For this model, Ravikant et al. [15] gave tight conditions for Byzantine consensus on $(2,3)$-hypergraphs.\footnote{i.e., each hyperedge consists of either 2 or 3 nodes.} As we discuss in Section 4, these conditions extend to general undirected hypergraphs as well.

The classical point-to-point communication model allows a faulty node to equivocate, i.e., send conflicting messages to its neighbors without this inconsistency being observed by the neighbors. For example, a faulty node $z$ may tell its neighbor $u$ that it has input 0, but tell another neighbor $v$ that it has input 1. Since messages on each edge are private between the two endpoints, node $u$ does not overhear the message sent to node $v$ and vice versa. The local broadcast model and the hypergraph model restrict a faulty node’s ability to equivocate by detecting such attempts. In the local broadcast model, a faulty node’s attempt to equivocate is detected by its neighboring nodes in the communication graph. In the undirected hypergraph model, a faulty node’s attempt to equivocate on an (undirected) hyperedge is detected by the nodes in that hyperedge. In our local multicast model, a faulty node’s attempt to equivocate on a single multicast channel, i.e., on a single directed hyperedge, is detected by the receivers in that channel.

In this work, we introduce the local multicast model, that unifies the models identified above, and make the following main contributions:

1. **Necessary and sufficient condition for local multicast model:** In Section 3, we present a network condition, and show that it is both necessary and sufficient for Byzantine consensus under the local multicast model. The identified condition is inspired by the network conditions for directed graphs [9, 17], where node connectivity does not adequately capture the network requirements for consensus. We present a simple algorithm, inspired by [8, 9, 17].

2. **Reductions to the existing models:** The two extremes of the local multicast model are 1) each channel consists of exactly one receiver, and 2) each node has exactly one multicast channel. These correspond to the point-to-point communication model and the local broadcast model, respectively. In Section 4, we show how the network condition for the local multicast model reduces to the network requirements for the point-to-point model and the local broadcast model at the two extremes. On the other hand, if the hypergraph is undirected, then we show that the network condition reduces to the network requirements of the undirected hypergraph model given by Ravikant et al. [15]. Moreover, our algorithm for the local multicast model works for all the three models identified here as well.

3. **Extensions to other models:** The local multicast model also captures some other models of practical interest (see Section 5). For instance, consider the scenario where nodes are connected via a WiFi network. This can be modelled as local multicast over a graph $G_1$. Separately, the nodes are also connected via a bluetooth network, modelled using local multicast
over a graph $G_2$ (with the same node set as $G_1$). Then the union of these networks $G_1 \cup G_2$ can be captured using the local multicast model as well. As another example, consider the scenario where nodes are connected via point-to-point channels, in addition to a wireless network with local broadcast guarantees. As before, this can also be captured using the local multicast model. Our algorithm works for these cases as well.

In our recent work [7], we obtained an analogous tight condition for the “bidirectional” case when the underlying simple graph is undirected, i.e., if a node $u$ can send messages to a node $v$, then $v$ can also send messages to node $u$. The tight condition obtained here is a natural extension of the tight condition obtained in [7]. However, the results and proofs in this work are more general and encompass the results in [7].

2 System Model and Problem Formulation

We consider a synchronous system where nodes are connected via a directed hypergraph $G = (V, E)$. $V$ is the set of $n$ nodes. Each directed hyperedge $e \in E$ is of the form $e = (u, S)$, where $u \in V$ and $S \subseteq V - u$, representing a local multicast channel with sender $u$ and receivers $S$. By convention used here, $u$ is not included in $S$. However, trivially, each node receives its own message transmissions as well. $u$ is the head of $e$, denoted by $H(e) = u$, and each node in $S$ is a tail of $e$, denoted by $T(e) = S$. Observe that each hyperedge has a single head and at least one tail. For example, $(u, \{v, w\})$ is a hyperedge with head $u$ and two tail nodes $v$ and $w$. A message $m$ sent by a node $u$ on a hyperedge $e$ (such that $H(e) = u$) is received identically and correctly by all tail nodes $T(e)$ of $e$. Moreover, each recipient $v \in T(e)$ knows that $m$ was sent by $u$ on the hyperedge $e$. We assume that each hyperedge represents a FIFO multicast channel.

We use $\delta_G(u)$ to denote the set of hyperedges in $G$ that have $u$ as the head node, i.e.,

$$\delta_G(u) = \{ e \in E(G) \mid H(e) = u \} .$$

The hypergraph $G$ has an underlying directed simple graph, denoted by $\overline{G} = (\overline{V}, \overline{E})$, such that

$$\overline{V} := V ,$$

$$\overline{E} := \{ (u, v) \mid \exists e \in E : u = H(e), v \in T(e) \} .$$

**Neighbors:** A node $u$ is an in-neighbor of node $v$ in $G$ if there exists a hyperedge $e$ with $u = H(e)$ as the head and $v \in T(e)$ as one of the tails. We call $v$ an out-neighbor of $u$. Note that $u$ is an in-neighbor of node $v$ in $G$ if and only if $u$ is an in-neighbor of $v$ in $\overline{G}$. 

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• **In-neighborhood:** More generally, for two disjoint sets \(A, B \subseteq V(G)\), \(\Gamma_G(A, B)\) defined below is the set of in-neighbors of \(B\) in \(A\).

\[
\Gamma_G(A, B) = \{u \in A \mid \exists v \in B : u \text{ is an in-neighbor of } v \text{ in } G\}
\]

• **Adjacent:** We use \(A \rightarrow_G B\) (read as \(A\) is “adjacent” to \(B\) in \(G\)) to denote that either

(i) \(B = \emptyset\), or

(ii) nodes in \(B\) have at least \(f + 1\) in-neighbors in \(A\) in \(G\), i.e.,

\[
\left|\Gamma_G(A, B)\right| \geq f + 1.
\]

**Node split:** We now introduce the notion of a *node split* that is used to specify the necessary and sufficient condition under the local multicast model. As seen later, we will use the notion of node split to simulate possible equivocation by a faulty node. Intuitively, by splitting a node \(v\), we are creating two copies of \(v\) and dividing up the hyperedges amongst the two copies. Figure 1 shows two examples of node split. Formally, splitting a set of nodes \(X\) in \(G\) creates a new hypergraph \(G' = (V', E')\) as follows. Each node \(v \in X\) is replaced by two nodes \(v^0\) and \(v^1\), so that

\[
V' = (V - X) \cup \left\{v^0, v^1 \mid v \in X\right\}.
\]

Consider each node \(u \in V\) and hyperedge \(e = (u, S) \in \delta_G(u)\).

• If \(u \notin X\), then add a hyperedge \((u, S')\) in \(G'\), where

\[
S' = (S - X) \cup \left\{v^0, v^1 \mid v \in S \cap X\right\},
\]

i.e., each node \(v \in T(e) \cap X\) is replaced by the two nodes \(v^0\) and \(v^1\).

• If \(u \in X\), then choose a node \(u_e \in \{v^0, v^1\}\), and add a hyperedge \((u_e, S')\) in \(G'\), where

\[
S' = (S - X) \cup \left\{v^0, v^1 \mid v \in S \cap X\right\}.
\]

Recall that \(u \notin S\) by convention. Note that the choice of \(u_e \in \{v^0, v^1\}\) affects the set of hyperedges of hypergraph \(G'\). For simplicity, we say that the hyperedge \(e\) has been *assigned* to \(u_e\).

Observe that, for every node \(u \in V'\) in the hypergraph \(G'\), each hyperedge in \(\delta_G'(u)\) corresponds to a single hyperedge in \(G\). Similarly, for every node \(u \in V\) in the original hypergraph \(G\), each hyperedge in \(\delta_G(u)\) corresponds to a single hyperedge in \(G'\).
(a) Splitting a single node $v$. Only the hyperedges in $\delta_G(v)$ are drawn here. There are two hyperedges in $\delta_G(v)$: $(v, \{u, w\})$ and $(v, \{w, z\})$, drawn with blue and red colors, respectively. The three possible hypergraphs in $\Lambda_{\{v\}}(G)$, other than $G$, corresponding to the assignment of hyperedges when $v$ is split into $v^0$ and $v^1$. These are depicted as hypergraphs $G'_1$, $G'_2$, and $G'_3$.

(b) Splitting two nodes $u, v$ in a 4-node hypergraph $G$. Edges of the same color, which have the same head node, represent a single hyperedge. $G'$ is obtained by splitting nodes $u$ and $v$ into $u^0, u^1$ and $v^0, v^1$, respectively. The cyan hyperedge is assigned to $v^1$, the violet hyperedge is assigned to $v^0$, the red hyperedge is assigned to $u^0$, and the blue hyperedge is assigned to $u^1$.

Figure 1: Examples of the node split operation.
For a set $F \subseteq V(G)$, let $\Lambda_F(G)$ be the set of all hypergraphs that can be obtained from $G$ by splitting some subset of nodes in the set $F$. For a graph $G' \in \Lambda_F(G)$, we use $F'$ to denote the set of nodes in $G'$ that correspond to nodes in $F$ in $G$, i.e.,

$$F' := (V' \cap F) \cup (V' - V).$$

Note that there are two choices in the node split operation above which give rise to all the hypergraphs in $\Lambda_F(G)$:

1) choice of which nodes in $F$ to split, and

2) assignment of hyperedges for each split node.

As needed, we will occasionally clarify these choices to specify how a hypergraph $G' \in \Lambda_F(G)$ was constructed by splitting some nodes in $F$.

### 3 Main Result

The main result of this paper is a tight characterization of network requirements for Byzantine consensus under the directed hypergraph model. Recall that for a hypergraph $G' \in \Lambda_F(G)$ obtained by splitting some nodes in $F$, we use $F'$ to denote the set of nodes in $G'$ that correspond to nodes in $F$ in $G$. With a slight abuse of terminology, we allow a partition of a set to have empty parts.

**Theorem 3.1.** Byzantine consensus tolerating at most $f$ faulty nodes is achievable on a directed hypergraph $G$ if and only if for every $F \subseteq V$ of size at most $f$, every $G' \in \Lambda_F(G)$ satisfies the following: for every partition $(L, C, R)$ of $V'$, either

1) $L \cup C \rightarrow_{G'} R - F'$, or

2) $R \cup C \rightarrow_{G'} L - F'$.

Note that we allow a partition to have empty parts. However, the interesting partitions are those where both $L$ and $R$ are non-empty, but $C$ can be possibly empty. In Section 4, we show that when the directed hypergraph corresponds to the point-to-point, local broadcast, or undirected hypergraph model, the above condition reduces to the corresponding known tight network conditions in each of the three cases.

We prove the necessity portion of Theorem 3.1 in Section 6. In Section 7 we give an algorithm to constructively show the sufficiency. The above condition is similar to the network condition for directed graphs in the point-to-point communication model [16, 17] and in the local broadcast model [9]. For convenience, we give a name to the condition in Theorem 3.1.
**Definition 3.2** (Condition LCR-hyper). A graph $G$ satisfies condition LCR-hyper with parameter $F$ if for every $G' \in \Lambda_F(G)$ and every partition $(L, C, R)$ of $V'$, we have that either

1) $L \cup C \rightarrow_{G'} R - F'$, or

2) $R \cup C \rightarrow_{G'} L - F'$.

We say that $G$ satisfies condition LCR-hyper, if $G$ satisfies condition LCR-hyper with parameter $F$ for every set $F \subseteq V(G)$ of cardinality at most $f$.

### 4 Reductions to Other Models

In this section, we discuss how condition LCR-hyper relates to the tight conditions for the classical point-to-point communication model, the local broadcast model, and the undirected hypergraph model. In an undirected hypergraph, any node on the undirected hyperedge can act as the sender. Formally, we say that the hypergraph $G$ is **undirected** if

$$\exists (u, S) \in E \iff \forall v \in S, \exists (v, S \cup \{u\} - v) \in E.$$  

We say that the hypergraph $G$ is **bidirectional** if the underlying simple graph $\overline{G}$ is undirected. This corresponds to the case where for every pair of nodes $u, v \in V$,

$$\exists e \in E : u = H(e), v \in T(e) \iff \exists e' \in E : v = H(e'), u \in T(e').$$

Note that if a hypergraph $G$ is undirected, then it is bidirectional. However, the converse is not true. For example, the local broadcast model on undirected graphs is a special case of bidirectional hypergraphs but not undirected hypergraphs.

The classical point-to-point communication model corresponds to the case where each directed hyperedge has a single tail node. This means that the hypergraph $G$ is essentially the same as the underlying simple graph $\overline{G}$. So each edge $(u, v)$ in the graph $\overline{G}$ represents a point-to-point channel where the messages sent by node $u$ to node $v$ are private between $u$ and $v$. Under the point-to-point communication model, it is well known that $n \geq 3f + 1$ [4, 11, 14] and node connectivity at least $2f + 1$ [3, 4] are both necessary and sufficient for consensus in arbitrary undirected graphs. The following theorem states that if $G$ is bidirectional and has only point-to-point links, i.e., each hyperedge has a single tail node, then condition LCR-hyper reduces to $n \geq 3f + 1$ and node connectivity $\geq 2f + 1$.

**Theorem 4.1.** A bidirectional hypergraph $G$, such that each hyperedge has exactly one tail node, satisfies condition LCR-hyper if and only if

1) $n \geq 3f + 1$, and
2) the underlying undirected graph $\bar{G}$ has node connectivity at least $2f + 1$.

In Section A, we show a more general result (Theorem A.2) when $G$ is not necessarily bidirectional, but each hyperedge has exactly one tail node. This corresponds to the point-to-point communication model on arbitrary directed graphs [16, 17]. Theorem 4.1 follows as a corollary.

The local broadcast model corresponds to the other extreme where each node $u$ in $G$ has exactly one hyperedge in $\delta_G(u)$, so that the messages transmitted by $u$ are received identically and correctly by all out-neighbors of $u$. Under the local broadcast model, our earlier work [8] shows that node degree at least $2f$ and connectivity at least $\lceil 3f/2 \rceil + 1$ are both necessary and sufficient for consensus in arbitrary undirected graphs. The following theorem states that if $G$ is bidirectional and has only local broadcast channels, i.e., each node is a head node of a single hyperedge, then condition LCR-hyper reduces to minimum node degree $\geq 2f$ and node connectivity $\geq \lceil 3f/2 \rceil + 1$.

**Theorem 4.2.** A bidirectional hypergraph $G$, such that each node is a head node of exactly one hyperedge, satisfies condition LCR-hyper if and only if for the underlying undirected graph $\bar{G}$

1) each node in $\bar{G}$ has degree at least $2f$, and

2) $\bar{G}$ has node connectivity at least $\lceil 3f/2 \rceil + 1$.

In Section B, we show a more general result (Theorem B.2) when $G$ is not necessarily bidirectional, but each node is a head node of exactly one hyperedge. This corresponds to the local broadcast model on arbitrary directed graphs [9]. Theorem 4.2 follows as a corollary.

The last model we consider in this section is the undirected hypergraph model. Ravikant et al. [15] obtained tight conditions for this model. Recall that a hypergraph $G = (V, E)$ is undirected if, for every hyperedge $(u, S) \in E$ and tail node $v \in S$, there exists a hyperedge $(v, (S - v) \cup \{u\})$. For simplicity, an undirected hyperedge $e$ can be viewed as a subset of nodes $e \subseteq V$, representing $|e|$ directed hyperedges. $e$ is called an $|e|$-hyperedge. Each hyperedge is effectively a local multicast channel where any node $u \in e$ can send a message, which will be received identically and correctly by all nodes in $e - u$.

The tight characterization of undirected hypergraphs for consensus was given by Ravikant et al. [15]. We state this in Theorem 4.3 below. Observe that this is different from Theorem 1 in [15]. This is because we found a bug in the proof of Lemma 3 in [15] which is, in fact, not true, as documented in Appendix F. However, Theorem 4.3 still follows from the work in [15]. We observe that while this was presented as a tight characterization for $(2, 3)$-hypergraphs, it also holds for general undirected hypergraphs.

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2 An undirected hypergraph $G$ is a $(2, 3)$-hypergraph if each hyperedge is either a 2-hyperedge or a 3-hyperedge.
**Theorem 4.3** (Fixed version of Theorem 1 in [15]). Byzantine consensus tolerating at most $f$ faulty nodes is achievable on an undirected hypergraph $G = (V, E)$ if and only if $G$ satisfies each of the following:

1) $n \geq 2f + 1$,
2) the underlying simple graph $\overline{G}$ is either a complete graph or is $(2f + 1)$-connected,
3) for every $V_1, V_2, V_3 \subseteq V$ such that $V_1 \cup V_2 \cup V_3 = V$ and $|V_1| = |V_2| = |V_3| = f$, there exist three nodes
   
   (i) $u \in V_1 - (V_2 \cup V_3)$,
   (ii) $v \in V_2 - (V_1 \cup V_3)$, and
   (iii) $w \in V_3 - (V_1 \cup V_2)$,

   such that there is an undirected hyperedge in $G$ that contains $u$, $v$, and $w$.

The following theorem states that if $G$ is an undirected hypergraph, then condition LCR-hyper reduces to the conditions in Theorem 4.3.

**Theorem 4.4.** An undirected hypergraph $G$ satisfies condition LCR-hyper if and only if $G$ satisfies each of the following:

1) $n \geq 2f + 1$,
2) the underlying simple graph $\overline{G}$ is either a complete graph or is $(2f + 1)$-connected,
3) for every $V_1, V_2, V_3 \subseteq V$ such that $V_1 \cup V_2 \cup V_3 = V$ and $|V_1| = |V_2| = |V_3| = f$, there exist three nodes
   
   (i) $u \in V_1 - (V_2 \cup V_3)$,
   (ii) $v \in V_2 - (V_1 \cup V_3)$, and
   (iii) $w \in V_3 - (V_1 \cup V_2)$,

   such that there is an undirected hyperedge in $G$ that contains $u$, $v$, and $w$.

The formal proof of the theorem is given in Section C.

5 Application to New Models

As mentioned in Section 1, the local multicast model also encompasses some other network models of practical interest that, to the best of our knowledge, have not been considered before in the
literature. Suppose the \( n \) nodes are connected via a local multicast network represented as a directed hypergraph \( G_1 \). For example, network connectivity in \( G_1 \) can be via point-to-point links or via wireless channels modelled as local broadcast. Additionally, the \( n \) nodes are connected via another local multicast network represented as a directed hypergraph \( G_2 \). For example, \( G_2 \) may correspond to a wireless network with different frequencies and/or technologies. The complete system, where nodes can communicate on channels in \( G_1 \) as well as on channels in \( G_2 \), can also be characterized by the local multicast model. We omit details for brevity, but this corresponds to the natural union of \( G_1 \) and \( G_2 \), with each node now having access to its multicast channels in \( G_1 \) as well as its multicast channels in \( G_2 \).

\section{Necessity of Condition LCR-hyper}

Intuitively, consider a set \( F \subseteq V \) of size at most \( f \), such that \( G \) violates condition LCR-hyper with parameter \( F \). With \( F \) as a candidate faulty set, the splitting of nodes in \( F \) captures possible equivocation by nodes in \( F \): a faulty node can behave as if it has input 0 on some of its hyperedges and behave as if it has input 1 on the other hyperedges. Now consider the execution where non-faulty nodes in \( L \) have input 0. Since \( R \cup C \not\models_{G'} L-F' \), nodes in \( L-F' \) can not distinguish between \( F \) and its neighbors in \( R \cup C \), i.e., \( \Gamma_{G'}(R \cup C, L-F') \) as the set of faulty nodes. So non-faulty nodes in \( L \) are stuck with outputting 0 in this case. Similarly, if non-faulty nodes in \( R \) have input 1, then they have no choice but to output 1, creating the desired contradiction.

A formal necessity proof is given in Section D. It follows the standard state machine based approach \[1, 3, 4\], similar to \[9, 17\]. Suppose there exists a set \( F \subseteq V \), of size at most \( f \), such that \( G \) does not satisfy condition LCR-hyper with parameter \( F \), but there exists an algorithm \( A \) that solves consensus on \( G \). Algorithm \( A \) outlines a procedure \( A_u \) for each node \( u \) that describes \( u \)'s state transitions, as well as messages transmitted on each channel of \( u \) in each round. Now there exists a hypergraph \( G' \in \Lambda_F(G) \) and a partition of \( V' \) that does not satisfy the requirements of condition LCR-hyper. To create the required contradiction, we work with an algorithm for \( G' \) instead of \( A \). To see why this works, observe that an algorithm \( A \) on hypergraph \( G \) can be adapted to create an algorithm \( A' \) for a hypergraph \( G' \in \Lambda_F(G) \) as follows. Consider a round \( i \) in the algorithm \( A \). We specify the steps for each node in \( G' \) in round \( i \) for the algorithm \( A' \). Each node \( v \in V' \cap V \) that was not split runs \( A_v \) as specified for round \( i \). For a node \( v \in V'-V \) that was split into \( v^0, v^1 \in V' \), both \( v^0 \) and \( v^1 \) run \( A_v \) for round \( i \) with the following modification. Consider a hyperedge \( e \in \delta_G(v) \). Let \( e' \in \delta_{G'}(v^0) \) (resp. \( e' \in \delta_{G'}(v^1) \)) be the corresponding hyperedge in \( G' \). If the algorithm \( A_v \) wants to transmit a message on \( e \), then \( v^0 \) (resp. \( v^1 \)) sends the message on \( e' \), while \( v^1 \) (resp. \( v^0 \)) ignores this message transmission. Observe that, for any node \( u \in T(e') \), \( u \) receives messages on the hyperedge from exactly one of \( v^0 \) and \( v^1 \). Furthermore, by construction of \( G' \), each node \( v \in V' \) receives all messages needed to run the corresponding next steps in the
algorithm $A'$. 

Now, $A'$ might not solve consensus on $G'$, or may not even terminate. In the following lemma, we show that as long as care is taken with regards to which nodes are allowed to be faulty in $G'$ and the input of the split nodes, $A'$ indeed solves consensus in $G'$. So for necessity, it is enough to show that no algorithm exists for a hypergraph $G' \in \Lambda_F(G)$, under the two identified conditions. We use this in the formal necessity proof in Section D.

**Lemma 6.1.** For a directed hypergraph $G = (V, E)$, a set $F \subseteq V$ of size at most $f$, and a hypergraph $G' \in \Lambda_F(G)$, if there exists a Byzantine consensus algorithm $A$ on $G$ tolerating at most $f$ faulty nodes, then there exists an algorithm $A'$ on $G' = (V', E')$ that solves the Byzantine consensus problem under the following conditions.

1) The faulty nodes in $G'$ correspond to at most $f$ nodes in $G$.

2) For each node $v \in F - V'$ that was split into $v^0, v^1 \in V'$, either
   
   (i) both $v^0$ and $v^1$ have the same input, or
   
   (ii) at least one of $v^0$ and $v^1$ is faulty.

Proof. Suppose there exists an arbitrary directed hypergraph $G = (V, E)$ such that there is a consensus algorithm $A$ for $G$ tolerating $\leq f$ Byzantine faults. Consider any set $F \subseteq V$ of size at most $f$ and a hypergraph $G' \in \Lambda_F(G)$. Construct an algorithm $A'$ from $A$ as described in the text preceding the lemma. We show that $A'$ solves the Byzantine consensus problem under the conditions in the lemma statement.

Consider an execution $E'$ of $A'$ on $G'$ under the two conditions in the lemma statement. Without loss of generality, we assume that for every node $v \in F - V'$ that was split into $v^0, v^1 \in V'$, either both $v^0, v^1$ are non-faulty in $E'$ or both are faulty. Observe that the faulty nodes in $E'$ still correspond to at most $f$ nodes in $G$. Now for each node $v \in F - V'$ that was split into $v^0, v^1 \in V'$, either

i) both $v^0$ and $v^1$ have the same input, or

ii) both $v^0$ and $v^1$ are faulty.

It follows,\(^3\) by construction of $A'$, that the behavior of each node $v' \in V'$ on a hyperedge $e' \in \delta_{G'}(v')$ in any round of $E'$ is modelled by the behavior of the corresponding node $v \in V$ on the corresponding hyperedge $e \in \delta_G(v)$ in the corresponding round of $E$.

\(^3\)Recall from the split operation (Section 2) that for every node $v \in V'$ in the hypergraph $G'$, each hyperedge in $\delta_G(v)$ corresponds to a single hyperedge in $G$. 

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Since $A$ solves consensus on $G$ while tolerating $\leq f$ faulty nodes, so all non-faulty nodes in $E$ terminate in finite time, agreeing on an input of some non-faulty node. Recall that the behavior of each node $v' \in V'$ in execution $E'$ is modelled by the behavior of the corresponding node $v \in V$ in execution $E$. Therefore, as required, all non-faulty nodes in $E'$ also terminate in finite time, agreeing on an input of some non-faulty node.

7 Algorithm for Directed Hypergraphs

To prove the sufficiency portion of Theorem 3.1, we work with a different network condition, which we will show to be equivalent to condition LCR-hyper. We first introduce some notation that is used in the algorithm. For a set of nodes $U \subseteq V$, we use $G[U] = (V_U, E_U)$ to denote the sub-hypergraph induced by the nodes in $U$, i.e.,

$$V_U := U,$$
$$E_U := \{(u, S) \mid \exists e \in E : H(e) = u \in U, T(e) \cap U = S, S \neq \emptyset\}.$$ 

We use the shorthand $G - U$ to denote the sub-hypergraph $G[V - U]$. For an additional set of hyperedges $D \subseteq E$, we use $G[U, D] = (V_{U,D}, E_{U,D})$ to denote the sub-hypergraph induced by the nodes in $U$ and the hyperedges in $D$, i.e.,

$$V_{U,D} := U \cup \{u \mid \exists e \in D : u = H(e) \text{ or } u \in T(e)\},$$
$$E_{U,D} := D \cup E_U.$$ 

Observe that if $u \in V_{U,D} - U$, then $\delta_{G[U,D]}(u) \subseteq D$.

Paths in Hypergraph $G$: We use the following notations for paths. A path in a hypergraph $G$ is an alternating sequence of distinct nodes and hyperedges, starting and ending at two distinct nodes, such that

- if node $u$ immediately precedes a hyperedge $e$ in the sequence, then $u = H(e)$ is the head of $e$, and
- if a hyperedge $e$ immediately precedes a node $u$ in the sequence, then $u \in T(e)$ is a tail of $e$.

For a path $P = u_1, e_1, u_2, e_2, \ldots, e_{k-1}, u_k$, we say that $P$ passes through $u_1, \ldots, u_k$.

Observation 7.1. For a hypergraph $G$ with the underlying graph $\overline{G}$, there exists a path in $G$ that passes through some nodes $u_1, \ldots, u_k$ if and only if there exists a path in $\overline{G}$ that passes through $u_1, \ldots, u_k$. A path in $G$ corresponds to a unique path in $\overline{G}$ that passes through the same nodes. But a path in $\overline{G}$ can possibly correspond to multiple paths in $G$ that pass through the same nodes.
• **uv-paths:** For two nodes $u, v \in V$, a $uv$-path $P_{uv}$ is a path from $u$ to $v$. $u$ is called the *source* and $v$ the *terminal* of $P_{uv}$. Any other node in $P_{uv}$ is called an *internal* node of $P_{uv}$. Two $uv$-paths are *node-disjoint* if they do not share a common internal node.

• **Uv-paths:** For a set $U \subset V$ and a node $v \notin U$, a $Uv$-path is a $uv$-path for some node $u \in U$. All $Uv$-paths have $v$ as terminal. Two $Uv$-paths are node-disjoint if they do not have any nodes in common except the terminal node $v$. In particular, two node-disjoint $Uv$-paths have different source nodes. By definition, the number of disjoint $Uv$-paths is upper bounded by the size of the set $U$. Note the difference in definition between node-disjoint $uv$-paths and node-disjoint $Uv$-paths when $U = \{u\}$ is a singleton set. The former requires only internal nodes to be different, while the latter needs to have different source nodes as well. For the former, there can be more than one such node-disjoint path, while for the latter, there is at most one.

• **Propagate:** For two node sets $A, B \subseteq V$, we use $A \Rightarrow_G B$ (read as $A$ “propagates” to $B$ in $G$) to denote that either

1. $B = \emptyset$, or
2. for every $v \in B$, there exist at least $f + 1$ node-disjoint $Av$-paths in the hypergraph $G$.

Note that the subscript is important. For example, if $B \neq \emptyset$, then for a set $X \subseteq V$ that is disjoint from both $A$ and $B$, $A \Rightarrow_{G-X} B$ requires that for every $v \in B$, there exist at least $f + 1$ node-disjoint $Av$-paths in $G$ that do not contain any nodes from $X$.

### Directed Decomposition in Hypergraphs:

A directed hypergraph $G$ is *strongly connected* if, for every pair of nodes $u, v$, there is a $uv$-path as well as a $vu$-path in $G$. A strongly connected sub-hypergraph of $G$ is called a *component* of $G$. A directed decomposition of $G$ partitions $G$ into $H_1, \ldots, H_k$, with $k > 0$, such that each $H_i$ is a maximal component of $G$. A maximal component $H_i$ that has no in-neighbors, i.e., $\Gamma_G(V - H_i, H_i) = \emptyset$, is called a *source component* of the decomposition. In any directed decomposition of a hypergraph $G$, there always exists at least one source component.

We now give a different network condition which is equivalent to condition LCR-hyper, but will be useful for specifying an algorithm for the local multicast model and proving its correctness. Recall that we use $F'$ to denote the set of nodes in $G'$ corresponding to nodes in $F$ in $G$.

**Definition 7.2** (Condition AB-hyper). A hypergraph $G$ satisfies condition AB-hyper with parameter $F$ if for every $G' \in \Lambda_F(G)$ and every partition $(A, B)$ of $V'$, we have that either

1. $A \Rightarrow_{G'-B \cap F'} B - F'$, or
2. $B \Rightarrow_{G'-A \cap F'} A - F'$.
We say that $G$ satisfies condition AB-hyper, if $G$ satisfies condition AB-hyper with parameter $F$ for every set $F \subseteq V$ of cardinality at most $f$.

The following theorem states that condition LCR-hyper is equivalent to condition AB-hyper. It was shown for simple graphs in [9], but based on Observation 7.1, can be extended to directed hypergraphs as well.

**Theorem 7.3** ([9]). A hypergraph $G$ satisfies condition LCR-hyper if and only if $G$ satisfies condition AB-hyper.

We show the sufficiency of condition AB-hyper (and hence condition LCR-hyper) constructively. For the rest of this section, we assume that $G$ satisfies condition AB-hyper. We defer all proofs to Section E. The proposed algorithm is given in Algorithm 1. It draws inspiration from algorithms in [8, 9, 17]. Each node $v$ maintains a binary state variable $\gamma_v$, which we call $v$’s $\gamma$ value. Each node $v$ initializes $\gamma_v$ to be its input value.

The nodes use “flooding” to communicate with the rest of the nodes. We refer the reader to [8, 9] for details about the flooding primitive. Briefly, when a node $u$ wants to flood a binary value $b \in \{0, 1\}$, it transmits $b$ to all of its neighbors, who forward it to their neighbors, and so forth. If a node $u$ receives a message on a hyperedge $e$, then $u$ appends the channel id of $e$ when forwarding the message to its neighbors. This way a node $v$, on receiving a message, can trace the path that the message has travelled to reach $v$. By adding some simple sanity checks, one can assume that even a faulty node $v$ does indeed transmit some value, when it is $v$’s turn to forward a message. In at most $n$ synchronous rounds, the value $b$ will be “flooded” in $G$. However, faulty nodes may tamper messages when forwarding, so some nodes may receive a value $\bar{b} \neq b$ along paths that contain faulty nodes.

The algorithm proceeds in phases. Every iteration of the main for loop (starting at line 2) is a phase numbered $1, \ldots, 2^f$. Let $F^*$ denote the actual set of faulty nodes. Each iteration of the for loop, i.e. phase $> 0$, considers a candidate faulty set $F$. In this iteration, nodes attempt to reach consensus, by updating their $\gamma$ state variables, assuming the candidate set $F$ is indeed faulty. Each iteration has six steps.

- In step (a), each node $v$ performs a directed decomposition of $G - F$. This decomposition must have a unique source component:

**Lemma 7.4** (Similar to Lemma 6 in [9]). If a hypergraph $G$ satisfies condition AB-hyper, then for any set $F$ of size $\leq f$, the directed decomposition of $G - F$ has a unique source component.

We remind the reader that all proofs in this section are deferred to Section E. Each node $v$ identifies this unique source component $S$ of $G - F$. In steps (b)-(d), nodes in $S$ will attempt
Algorithm 1: Proposed algorithm for Byzantine consensus under the local multicast model: Steps performed by node $v$ are shown here.

*Initialization:* $\gamma_v := \text{input value of node } v$

*For each* $F \subseteq V$ *such that* $|F| \leq f$ *do*

**Step (a):** Perform directed decomposition of $G - F$. Let $S$ be the unique source component of the decomposition (Lemma 7.4).

**Step (b):** if $v \in S \cup \Gamma_G(F, S)$ then flood value $\gamma_v$.

**Step (c):** if $v \in S$ then

Create a hypergraph $G'_v$ by splitting all nodes in $F$ as follows. Set

$$F' := \left\{u^0 \mid u \in F\right\} \cup \left\{u^1 \mid u \in F\right\} \quad \text{and} \quad V(G'_v) := (V - F) \cup F'.$$

The edges of $G'_v$ are as determined by the split operation, with the following choices: for each node $u \in F$ and hyperedge $e \in \delta_G(u)$, identify a single $uv$-path $P_{uv}$ (if it exists) in $G[S, \{e\}]$. If $P_{uv}$ exists and $v$ received value 0 from $u$ along $P_{uv}$ in step (b), then assign $e$ to $u^0$. Else, assign $e$ to $u^1$.

For each node $u \in S$, identify a single $uv$-path $P_{uv}$ in $G - F$ (Lemma E.1). Note that path $P_{vv}$ trivially exists ($P_{vv}$ contains only $v$). Initialize $Z_v$ and $N_v$ as follows,

$$Z_v := \left\{u^0 \mid u \in \Gamma_G(F, S)\right\} \cup \left\{u \in S \mid v \text{ received 0 along } P_{uv} \text{ in step (b)}\right\},$$

$$N_v := \left\{u^1 \mid u \in \Gamma_G(F, S)\right\} \cup (S - Z_v).$$

**Step (d):**

if $Z_v \sim_{G'_v - (N_v \cap F')} N_v - F'$ then set $A_v := Z_v$ and $B_v := N_v$

else set $A_v := N_v$ and $B_v := Z_v$

if $v \in B_v - F'$ and, in step (b), $v$ received a value $b \in \{0, 1\}$ identically along any $f + 1$ node-disjoint $A_vv$-paths in $G'_v - (B_v \cap F')$ then

$\gamma_v := b$

**Step (e):** if $v \in S$ then flood value $\gamma_v$.

**Step (f):** if $v \in V - S - F$ and, in step (e), $v$ received a value $b \in \{0, 1\}$ identically along any $f + 1$ node-disjoint $Sv$-paths in $G - F$ then

$\gamma_v := b$

Output $\gamma_v$
to reach consensus on a single value, and then propagate that to the remaining nodes in steps (e) and (f).

• In step (b), each node \( v \in S \cup \Gamma_G(F, S) \) floods its \( \gamma_v \) value. Nodes in \( S \) may not be able to reach consensus by themselves, but they can pull in the nodes in \( \Gamma_G(F, S) \) to help:

**Lemma 7.5** (Similar to Lemma 7 in [9]). For a hypergraph \( G = (V, E) \), that satisfies condition AB-hyper, and a set \( F \subseteq V \) of size \( \leq f \), let \( S \) be the unique source component in the directed decomposition of \( G - F \). Then \( G[S \cup \Gamma_G(F, S)] \) satisfies condition AB-hyper with parameter \( \Gamma_G(F, S) \).

• In step (c), each node \( v \in S \) splits all nodes in the candidate faulty set \( F \) to construct a hypergraph \( G'_u \in \Lambda_F(G) \). For the assignment of hyperedges in \( G'_u \), consider a node \( u \in F \) and a hyperedge \( e \in \delta_G(u) \). \( v \) assigns \( e \) to \( v^0 \) if there exists a \( uv \)-path \( P_{uv} \) such that

1) \( e \) is the first hyperedge on \( P_{uv} \),

2) the rest of \( P_{uv} \) is contained entirely in \( G[S] \), and

3) \( v \) received value 0 from \( u \) along \( P_{uv} \) in step (b).

Otherwise, \( v \) assigns \( e \) to \( v^1 \).

Next, \( v \) partitions nodes in \( G'_u \) that correspond to nodes in \( S \cup \Gamma_G(F, S) \) in the original hypergraph \( G \), into sets \( Z_v \) and \( N_v \), as follows. If \( u \in \Gamma_G(F, S) \), then \( v \) places \( u^0 \) in \( Z_v \) and \( u^1 \) in \( N_v \). If \( u \in S \), then \( v \) identifies a single \( uv \)-path \( P_{uv} \) in \( G[S] \). Such a path always exists since \( S \) is strongly connected by construction. If \( v \) received 0 along \( P_{uv} \) in step (b), then \( v \) places \( u \) in \( Z_v \). Otherwise \( v \) places \( u \) in \( N_v \). For the purpose of step (c), node \( v \) is deemed to have received its own \( \gamma_v \) value along path \( P_{vv} \), containing only node \( v \), in step (b).

The hypergraph \( G'_u \), and sets \( Z_v \) and \( N_v \), are created in a manner so that

1) when \( F \neq F^* \), nodes in \( S \) may disagree on these constructions, i.e., it is possible that in this iteration, for two non-faulty nodes \( u, w \in S \), we have either

\[ G'_u \neq G'_w, \quad Z'_u \neq Z'_w, \quad \text{or} \quad N'_u \neq N'_w, \]

but

2) when \( F = F^* \), all nodes in \( S \) agree on these constructions, i.e., in this iteration, for any two non-faulty nodes \( u, w \in S \), we have

\[ G'_u = G'_w, \quad Z'_u = Z'_w, \quad \text{and} \quad N'_u = N'_w. \]

• In step (d), based on the estimates created in step (c), a node \( v \in S \) may update its \( \gamma_v \) value. The update rules ensure that
1) in each iteration, for each non-faulty node \( v \in S \), its \( \gamma_v \) value at the end of this equals the \( \gamma \) state value of some non-faulty node at the beginning of the iteration (Lemma 7.7).

2) when \( F = F^* \), all non-faulty nodes in \( S \) have identical \( \gamma \) values at the end of this step (Lemma 7.8).

- In the iteration where \( F = F^* \), by the end of step (d), nodes in \( S \) have reached consensus by adopting a single value in each of their \( \gamma \) states. In steps (e) and (f), nodes in \( S \) propagate the consensus value to the rest of the nodes, using the following property:

**Lemma 7.6** (Similar to Lemma 10 in [9]). For a hypergraph \( G = (V, E) \), that satisfies condition AB-hyper, and a set \( F \subseteq V \) of size \( \leq f \), let \( S \) be the unique source component in the directed decomposition of \( G - F \). Then \( S \xrightarrow{} G - F V - S - F \).

At the end, after all iterations of the main for loop, each output node \( v \) outputs its \( \gamma_v \) value.

The correctness of Algorithm 1 relies on the following two key lemmas, which are proven in Section E along with the 3 lemmas stated above. Recall that \( G \) satisfies condition AB-hyper and we use \( F^* \) to denote the actual set of faulty nodes.

**Lemma 7.7.** For a non-faulty node \( v \in V - F^* \), its state \( \gamma_v \) at the end of any given phase of Algorithm 1 equals the state of some non-faulty node at the start of that phase.

**Lemma 7.8.** Consider a phase \( > 0 \) of Algorithm 1 wherein \( F = F^* \). At the end of this phase, every pair of non-faulty nodes \( u, v \in V - F^* \) have identical state, i.e., \( \gamma_u = \gamma_v \).

**Lemma 7.7** ensures validity, i.e., that the output of each non-faulty node is an input of some non-faulty node. It also ensures that agreement among non-faulty nodes, once achieved, is not lost. **Lemma 7.8** ensures that agreement is reached in at least one phase of the algorithm. These two lemmas imply correctness of Algorithm 1 as shown in Section E.

### 8 Conclusion

In this paper, we introduced the local multicast model which, to the best of our knowledge, has not been studied before in the literature. The local multicast model corresponds to directed hypergraphs and encompasses the point-to-point, local broadcast, and undirected hypergraph communication models, as well as some new models which have not been considered before. We identified a tight network condition for Byzantine consensus under the local multicast model, along the lines of \([9, 17]\), and proved its necessity and sufficiency. When the local multicast model represents one of point-to-point, local broadcast, or undirected hypergraph communication models, we showed how the identified network condition reduces to the known tight requirements for the corresponding case.
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In this section, we consider the case where each hyperedge in the hypergraph $G$ has exactly one tail node. This corresponds to the classical point-to-point communication model on arbitrary directed graphs. In this case, Tseng and Vaidya [16, 17] showed that the following network condition, which is similar to condition LCR-hyper, is both sufficient and necessary.

A Reduction to Point-to-Point Channels

In this section, we consider the case where each hyperedge in the hypergraph $G$ has exactly one tail node. This corresponds to the classical point-to-point communication model on arbitrary directed graphs. In this case, Tseng and Vaidya [16, 17] showed that the following network condition, which is similar to condition LCR-hyper, is both sufficient and necessary.
Definition A.1 (Condition LCR-p2p [16, 17]). A directed graph $G$ satisfies condition LCR-p2p with parameter $F$ if for every partition $(L, C, R)$ of $V - F$, we have that either

1) $R \cup C \rightarrow_{G} L$, or

2) $L \cup C \rightarrow_{G} R$.

We say that $G$ satisfies condition LCR-p2p, if $G$ satisfies condition LCR-p2p with parameter $F$ for every set $F \subseteq V$ of cardinality at most $f$.

When $G$ is undirected, condition LCR-p2p reduces to $n \geq 3f + 1$ and node connectivity at least $2f + 1$ [16, 17]. Here, we show that condition LCR-hyper reduces to condition LCR-p2p when each hyperedge in the hypergraph $G$ has exactly one tail node. Theorem 4.1 follows as a corollary.

Theorem A.2. A directed hypergraph $G$, such that each hyperedge has exactly one tail node, satisfies condition LCR-hyper if and only if the underlying directed graph $\overline{G}$ satisfies condition LCR-p2p.

Proof. Since $G$ and $\overline{G}$ are essentially the same in this case, so we simply refer to $G$ for both in this proof. We show the contrapositive in both directions. First, consider a set $F \subseteq V$ of size at most $f$ such that $G$ does not satisfy condition LCR-hyper with parameter $F$. We will show that $G$ does not satisfy condition LCR-p2p with parameter $F$ either. Now, there exists a graph $G' \in \Lambda_F(G)$ and a partition $(L', C', R')$ of $V'$ such that, using $F'$ to denote the set of nodes in $G'$ corresponding to nodes in $F$ in $G$,

1) $L' \cup C' \not\rightarrow_{G'} R' - F'$, and

2) $R' \cup C' \not\rightarrow_{G'} L' - F'$.

We create a partition $(L, C, R)$ of $V - F$ as follows:

$L := L' - F'$,

$R := R' - F'$,

$C := C' - F'$.

Observe that $(L, C, R)$ is a partition of $V - F$. Furthermore, both $L$ and $R$ are non-empty. Now, we have

$$|\Gamma_G(L \cup C, R)| = |\Gamma_G((L' \cup C') - F', R' - F')|$$

$$= |\Gamma_{G'}((L' \cup C') - F', R' - F')|$$

$$\leq |\Gamma_{G'}(L' \cup C', R' - F')|$$

$$\leq f.$$
1) $L \cup C \not\rightarrow_{G} R$, and 
2) $R \cup C \not\rightarrow_{G} L$.

So $G$ does not satisfy condition LCR-p2p with parameter $F$, as required.

For the other direction, consider a set $F \subseteq V$ of size at most $f$ such that $G$ does not satisfy condition LCR-p2p with parameter $F$. We will show that $G$ does not satisfy condition LCR-hyper with parameter $F$ either. Now, there exists a partition $(L, C, R)$ of $V - F$ such that

1) $L \cup C \not\rightarrow_{G} R$, and 
2) $R \cup C \not\rightarrow_{G} L$.

We create a graph $G' \in \Lambda_F(G)$ by splitting all nodes in $F$, with the following choices: for each node $u \in F$ and an edge $(u, v)$, if $v \in L$, then add a hyperedge $(u^0, v)$ in $G'$; otherwise add $(u^1, v)$ in $G'$. We create a partition $(L', C', R')$ of $V'$ as follows:

$L' := L \cup \{u^0 \mid u \in F\}$,
$R' := R \cup \{u^1 \mid u \in F\}$,
$C' := C$.

Observe that, by construction, nodes in $R' - F' = R$ have no in-neighbors in $L' \cap F' = \{u^0 \mid u \in F\}$ and nodes in $L' - F' = L$ have no in-neighbors in $R' \cap F' = \{u^1 \mid u \in F\}$. So we have

$|\Gamma_{G'}(L' \cup C', R' - F')| = |\Gamma_{G}(L \cup C, R)| \leq f$,
$|\Gamma_{G'}(R' \cup C', L' - F')| = |\Gamma_{G}(R \cup C, L)| \leq f$.

Therefore,

1) $L' \cup C' \not\rightarrow_{G'} R' - F'$, and 
2) $R' \cup C' \not\rightarrow_{G'} L' - F'$.

So $G$ does not satisfy condition LCR-hyper with parameter $F$, as required.

\[ \square \]

## B Reduction to Local Broadcast Model

In this section, we consider the case where each node is a head node of exactly one hyperedge. This corresponds to the local broadcast model on arbitrary directed graphs. In this case, Khan et. al. [9] showed that the following network condition, which is similar to condition LCR-hyper, is both sufficient and necessary.
Definition B.1 (Condition LCR-local [9, 13]). A directed graph \( G \) satisfies condition LCR-local with parameter \( F \) if for every partition \((L, C, R)\) of \( V \), we have that either

1) \( R \cup C \rightarrow_G L - F \), or
2) \( L \cup C \rightarrow_G R - F \).

We say that \( G \) satisfies condition LCR-local, if \( G \) satisfies condition LCR-local with parameter \( F \) for every set \( F \subseteq V \) of cardinality at most \( f \).

When \( G \) is undirected, condition LCR-local reduces to minimum node degree at least \( 2f \) and node connectivity at least \( \lfloor 3f/2 \rfloor + 1 \). Here, we show that condition LCR-hyper reduces to condition LCR-local when each node in the hypergraph \( G \) is a head node of exactly one hyperedge. Theorem 4.2 follows as a corollary.

Theorem B.2. A directed hypergraph \( G \), such that each node is a head node of exactly one hyperedge, satisfies condition LCR-hyper if and only if the underlying directed graph \( \overline{G} \) satisfies condition LCR-local.

Proof. We show the contrapositive in both directions. First, consider a set \( F \subseteq V \) of size at most \( f \) such that \( G \) does not satisfy condition LCR-hyper with parameter \( F \). We will show that \( \overline{G} \) does not satisfy condition LCR-local with parameter \( F \). Now, there exists a hypergraph \( G' \in \Lambda_F(G) \) and a partition \((L, C, R)\) of \( V' \) such that, using \( F' \) to denote the set of nodes in \( G' \) corresponding to nodes in \( F \) in \( G \),

1) \( L \cup C \not\rightarrow_{G'} R - F' \), and
2) \( R \cup C \not\rightarrow_{G'} L - F' \).

Consider a node \( v \in F \) that was split into \( v^0, v^1 \) in \( G' \). Since there is exactly one hyperedge in \( \delta_G(v) \), it follows that either \( v^0 \) or \( v^1 \) has degree 0 in \( G' \). So at least one of \( v^0 \) and \( v^1 \) is neither in \( \Gamma_{G'}(L \cup C, R - F') \) nor in \( \Gamma_{G'}(R \cup C, L - F') \). Therefore, WLOG, we can assume that \( v \) was not split in \( G' \). Thus \( F' = F \), \( G' = G \), and \((L, C, R)\) is a partition of \( V \) such that

1) \( L \cup C \not\rightarrow_{G} R - F \), and
2) \( R \cup C \not\rightarrow_{G} L - F \).

Observe that

\[ \Gamma_G(L \cup C, R - F) = \Gamma_{\overline{G}}(L \cup C, R - F), \]
\[ \Gamma_G(R \cup C, L - F) = \Gamma_{\overline{G}}(R \cup C, L - F). \]
So $G$ does not satisfy condition LCR-local with parameter $F$, as required.

For the other direction, consider a set $F \subseteq V$ of size at most $f$ such that $\overline{G}$ does not satisfy condition LCR-local with parameter $F$. We will show that $G$ does not satisfy condition LCR-hyper with parameter $F$ either. Now, there exists a partition $(L, C, R)$ of $V$ such that

1) $L \cup C \not\rightarrow_{\overline{G}} R - F$ and so $L \cup C \not\rightarrow_{G} R - F$, and
2) $R \cup C \not\rightarrow_{\overline{G}} L - F$ and so $R \cup C \not\rightarrow_{G} L - F$.

Since $G \in \Lambda_{F}(G)$ and $(L, C, R)$ is a partition of $V$, so $G$ does not satisfy condition LCR-hyper with parameter $F$, as required. □

C Reduction to Undirected Hypergraphs

Proof of Theorem 4.4. Directly from Lemmas C.1, C.2, C.3, and C.4 below. □

We first show that if a hypergraph $G$ is undirected and satisfies condition LCR-hyper, then $G$ satisfies each of the conditions in Theorem 4.4.

Lemma C.1. If an undirected hypergraph $G$ satisfies condition LCR-hyper, then $n \geq 2f + 1$.

Proof. Consider an undirected hypergraph $G$. We show the contrapositive that if $n \leq 2f$, then there exists $F \subseteq V$ of size at most $f$ such that $G$ does not satisfy condition LCR-hyper with parameter $F$. Let $F = \emptyset$. Observe that $\Lambda_{F}(G) = \{G\}$. Partition $V$ into $(L, R)$ such that $0 < |L| \leq f$ and $0 < |R| \leq f$. With $C = \emptyset$, $(L, C, R)$ is a partition of $V$. But, since $0 < |L|, |R| \leq f$, we have that

1) $L \cup C = L \not\rightarrow_{G} R = R - F$, and
2) $R \cup C = R \not\rightarrow_{G} L = L - F$,

as required. □

Lemma C.2. If an undirected hypergraph $G$ satisfies condition LCR-hyper, then the underlying simple graph $\overline{G}$ is either a complete graph or is $(2f + 1)$-connected.

Proof. We show the contrapositive that, for an undirected hypergraph $G$, if the underlying simple graph $\overline{G}$ is neither a complete graph nor is $(2f + 1)$-connected, then $G$ does not satisfy condition LCR-hyper. If $n \leq 2f$, then by Lemma C.1, we have that $G$ does not satisfy condition LCR-hyper. So suppose that $n \geq 2f + 1$.
First, in each of the following two cases, we show that there exists a set $X$ of size at most $2f$ that partitions $V - X$ into $(A, B)$ such that $|A|, |B| > 0$ and there is no undirected hyperedge in $G$ that contains a node from both $A$ and $B$.

Case 1: $n = 2f + 1$.
Since the underlying simple graph $\overline{G}$ is not complete, there exist two nodes $u$ and $v$ such that there is no undirected hyperedge containing both $u$ and $v$. Then, choosing $A := \{u\}$, $B := \{v\}$, and $X = V - A - B$ satisfies the requirements above.

Case 2: $n > 2f + 1$.
Since the underlying simple graph $\overline{G}$ is not $(2f + 1)$-connected and $n > 2f + 1$, there exists a set $X$ of size at most $2f$ that partitions $V - X$ into $(A, B)$ such that $|A|, |B| > 0$ and there is no undirected hyperedge in $G$ that contains a node from both $A$ and $B$, as required.

Partition $X$ into $(F, C)$ such that $|F| \leq f$ and $|C| \leq f$. Recall that there is no undirected hyperedge that contains a node from both $A$ and $B$. It follows that, for each node $z \in F$ and directed hyperedge $e \in \delta_G(z)$, either $A \cap T(e) = \emptyset$ or $B \cap T(e) = \emptyset$.

Now, we create a graph $G' \in \Lambda_F(G)$ by splitting all nodes in $F$, with the following choices: for each node $z \in F$ and a directed hyperedge $e \in \delta_G(z)$, if $A \cap T(e) \neq \emptyset$, then assign $e$ to $z^0$; otherwise assign $e$ to $z^1$. Observe that $z^0$ does not have any hyperedge with tail nodes in $B$ and $z^1$ does not have any hyperedge with tail nodes in $A$. Let

$$L := A \cup \left\{ z^0 \mid z \in F \right\},$$
$$R := B \cup \left\{ z^1 \mid z \in F \right\}.$$  

Then $(L, C, R)$ is a partition of $V'$. We use $F'$ to denote the nodes in $G'$ corresponding to nodes in $F$ in $G$. Recall that $A = L - F'$ and $B = R - F'$ are both non-empty. Note that nodes in $L - F' = A$ (resp. $R - F' = B$) do not have any in-neighbors in $R$ (resp. $L$) and $|C| \leq f$. It follows that

1) $L \cup C \not\rightarrow_{G'} A = R - F'$, and
2) $R \cup C \not\rightarrow_{G'} B = L - F'$.

Thus, $G$ does not satisfy condition LCR-hyper, as required.

\[\square\]

**Lemma C.3.** If an undirected hypergraph $G$ satisfies condition LCR-hyper, then, for every $V_1, V_2, V_3 \subseteq V$ such that $V_1 \cup V_2 \cup V_3 = V$ and $|V_1| = |V_2| = |V_3| = f$, there exist three nodes

(i) $u \in V_1 - (V_2 \cup V_3)$,
such that there is an undirected hyperedge in $G$ that contains $u$, $v$, and $w$.

Proof. Consider an undirected hypergraph $G$. We show the contrapositive that if $G$ does not satisfy the condition in the lemma, then $G$ does not satisfy condition LCR-hyper. Suppose that there exist $V_1, V_2, V_3 \subseteq V$ such that

(i) $V_1 \cup V_2 \cup V_3 = V$,

(ii) $|V_1| = |V_2| = |V_3| = f$, and

(iii) no undirected hyperedge crosses all the three sets $V_1 - (V_2 \cup V_3)$, $V_2 - (V_1 \cup V_3)$, and $V_3 - (V_1 \cup V_2)$.

By (i) and (ii) above, we have $n \leq 3f$. Furthermore, if $n \leq 2f$, then we are done by Lemma C.1. So for the rest of the proof, we assume that $2f < n \leq 3f$. Let

$$V'_1 := V_1 - (V_2 \cup V_3),$$

$$V'_2 := V_2 - (V_1 \cup V_3),$$

$$V'_3 := V_3 - (V_1 \cup V_2).$$

Observe that if either of the three sets $V'_1$, $V'_2$, and $V'_3$ is empty, then $|V| \leq 2f$, a contradiction. So each of $V'_1$, $V'_2$, and $V'_3$ is non-empty. Recall, from (iii) above, that there is no undirected hyperedge crossing all the three sets $V'_1$, $V'_2$, and $V'_3$. It follows that, for each node $z \in V'_1$ and directed hyperedge $e \in \delta_G(z)$, either $V'_2 \cap T(e) = \emptyset$ or $V'_3 \cap T(e) = \emptyset$.

Let $F := V_1$. We create a graph $G' \in \Lambda_F(G)$ by splitting all nodes in $V'_1 = F - (V_2 \cup V_3)$, with the following choices in the node split operation: for each node $z \in V'_1$ and a directed hyperedge $e \in \delta_G(z)$, if $V'_2 \cap T(e) \neq \emptyset$, then assign $e$ to $z^0$; otherwise assign $e$ to $z^1$. Observe that $z^0$ does not have any edge with tail nodes in $V'_3$ and $z^1$ does not have any edge with tail nodes in $V'_2$. Let

$$L := (V_2 - V_3) \cup \{ z^0 \mid z \in V'_1 \},$$

$$R := (V_3 - V_2) \cup \{ z^1 \mid z \in V'_1 \},$$

$$C := V_2 \cap V_3.$$
Note that $L$, $C$, and $R$ are disjoint. Furthermore, we have

\[
L \cup C \cup R = \left\{ z^0, z^1 \mid z \in V'_1 \right\} \cup (V_2 - V_3) \cup (V_3 - V_2) \cup (V_2 \cap V_3) \\
= \left\{ z^0, z^1 \mid z \in V'_1 \right\} \cup (V_2 \cup V_3) \\
= (V - V'_1) \cup \left\{ z^0, z^1 \mid z \in V'_1 \right\} \\
= V'.
\]

Therefore, $(L, C, R)$ is a partition of $V'$. We use $F'$ to denote the nodes in $G'$ corresponding to nodes in $F$ in $G$. Now $R - F' = V'_3$ and $L - F' = V'_2$ are both non-empty. Therefore,

\[
\Gamma_{G'}(L \cup C, R - F') = \Gamma_{G'}(V_2 \cup \left\{ z^0 \mid z \in F \right\}, V'_3) \\
= \Gamma_{G'}(V_2, V'_3) \\
\subseteq V_2.
\]

Since $|V_2| = f$, so $L \cup C \not\Gamma_{G'} R - F'$. Similarly, $R \cup C \not\Gamma_{G'} L - F'$. Thus, $G$ does not satisfy condition LCR-hyper, as required.

We now show that if $G$ satisfies each of the three conditions in Theorem 4.4, then $G$ satisfies condition LCR-hyper.

**Lemma C.4.** An undirected hypergraph $G$ satisfies condition LCR-hyper if $G$ satisfies each of the following:

1) $n \geq 2f + 1$,

2) the underlying simple graph $\overline{G}$ is either a complete graph or is $(2f + 1)$-connected,

3) for every $V_1, V_2, V_3 \subseteq V$ such that $V_1 \cup V_2 \cup V_3 = V$ and $|V_1| = |V_2| = |V_3| = f$, there exist three nodes

   (i) $u \in V_1 - (V_2 \cup V_3)$,

   (ii) $v \in V_2 - (V_1 \cup V_3)$, and

   (iii) $w \in V_3 - (V_1 \cup V_2)$,

   such that there is an undirected hyperedge in $G$ that contains $u$, $v$, and $w$.

**Proof.** Consider an undirected hypergraph $G$. We show the contrapositive that if there exists a set $F \subseteq V$, of size at most $f$, such that $G$ does not satisfy condition LCR-hyper with parameter $F$, then one of the conditions in the lemma statement is violated. Now, there exists a graph $G' \in \Lambda_F(G)$ and a partition $(L, C, R)$ of $V'$ such that, using $F'$ to denote the set of nodes in $G'$ corresponding to nodes in $F$ in $G$,
1) \( L \cup C \not\rightarrow_{G'} R - F' \), and
2) \( R \cup C \not\rightarrow_{G'} L - F' \).

Note that this implies that both \( R - F' \) and \( L - F' \) are non-empty. There are the following cases to consider.

Case 1: Either \((L \cup C) \not\subseteq F' \cup \Gamma_G'(L \cup C, R - F')\) or \((R \cup C) \not\subseteq F' \cup \Gamma_G'(R \cup C, L - F')\).

Suppose that \((L \cup C) - F' - \Gamma_G'(L \cup C, R - F')\) is non-empty (the other case follows similarly). Let \( B := R - F' \) (recall that \( R - F' \) is non-empty). Let

\[
A := (L \cup C) - F' - \Gamma_G'(L \cup C, R - F'),
B := R - F',
X' := F' \cup \Gamma_G'(L \cup C, R - F').
\]

Observe that both \( A \) and \( B \) are non-empty. Then, removing \( X' \) partitions \( V' - X' \) into \((A, B)\) such that there is no undirected hyperedge between \( A \) and \( B \) in \( G' \). Note that \( A \cup B \subseteq V \cap V' \). Let \( X \) be the set of nodes in \( G \) that correspond to nodes in \( X' \) in \( G' \). Then \( X \) is a cut in the underlying simple graph \( \overline{G} \) that partitions \( V - X \) into \((A, B)\). We bound the size of \( X \) as follows. By definition of \( X \) and \( X' \),

\[
|X| = |F \cup (\Gamma_G'(L \cup C, R - F') - F')|
= |F \cup \Gamma_G'((L \cup C) - F', R - F')|
\leq |F| + |\Gamma_G'((L \cup C) - F', R - F')|
\leq f + f'
= 2f.
\]

Therefore, since \( A \) and \( B \) are both non-empty, \( X \) is a cut of size at most \( 2f \) in \( \overline{G} \). It follows that \( \overline{G} \) is neither a complete graph nor \((2f + 1)\)-connected. This violates the second condition in the lemma statement.

Case 2: \((L \cup C) \subseteq F' \cup \Gamma_G'(L \cup C, R - F')\), \((R \cup C) \subseteq F' \cup \Gamma_G'(R \cup C, L - F')\), and \( \Gamma_G'(C - F', R - F') \neq \Gamma_G'(C - F', L - F') \).

Without loss of generality assume that \( \Gamma_G'(C - F', R - F') - \Gamma_G'(C - F', L - F') \) is non-empty. Note that, by assumption of Case 2,

\[
V' = L \cup R \cup C
= F' \cup \Gamma_G'(L \cup C, R - F') \cup \Gamma_G'(R \cup C, L - F').
\]
Let
\[ A := L - F', \]
\[ B := \Gamma_{G'}(C - F', R - F') - \Gamma_{G'}(C - F', L - F'), \]
\[ X' := F' \cup \Gamma_{G'}(R \cup C, L - F'). \]

Observe that \( A \) is non-empty since \( L - F' \) is non-empty and \( B \) is non-empty by assumption of Case 2. Then, removing \( X' \) partitions \( V' - X' \) into \((A, B)\). By construction of \( B \), there is no undirected hyperedge between \( A \) and \( B \). Note that \( A \cup B \subseteq V \cap V' \). Let \( X \) be the set of nodes in \( G \) that correspond to nodes in \( X' \) in \( G' \). Then \( X \) is a cut in the underlying simple graph \( \overline{G} \) that partitions \( V - X \) into \((A, B)\). We bound the size of \( X \) as follows. By definition of \( X \) and \( X' \),
\[
|X| = |F \cup (\Gamma_{G'}(R \cup C, L - F') - F')|
= |F \cup \Gamma_{G'}((R \cup C) - F', L - F')|
\leq |F| + |\Gamma_{G'}((R \cup C) - F', L - F')|
\leq f + f
= 2f.
\]

Therefore, since \( A \) and \( B \) are both non-empty, \( X \) is a cut of size at most \( 2f \) in \( \overline{G} \). It follows that \( \overline{G} \) is neither a complete graph nor \((2f + 1)\)-connected. This violates the second condition in the lemma statement.

Case 3: \((L \cup C) \subseteq F' \cup \Gamma_{G'}(L \cup C, R - F'), \ (R \cup C) \subseteq F' \cup \Gamma_{G'}(R \cup C, L - F'), \) and \( \Gamma_{G'}(C - F', R - F') = \Gamma_{G'}(C - F', L - F') \).
First observe that, by assumption of Case 2,
\[
V' = L \cup R \cup C
= F' \cup \Gamma_{G'}(L \cup C, R - F') \cup \Gamma_{G'}(R \cup C, L - F')
\implies V = F \cup \Gamma_{G'}((L \cup C) - F', R - F') \cup \Gamma_{G'}((R \cup C) - F', L - F').
\]
This implies that \( n = |V| \leq 3f \) as follows.
\[
n = |V|
= |F \cup \Gamma_{G'}(L \cup C - F', R - F') \cup \Gamma_{G'}(R \cup C - F', L - F')|
\leq |F| + |\Gamma_{G'}(L \cup C - F', R - F')| + |\Gamma_{G'}(R \cup C - F', L - F')|
\leq 3f.
\]
Let
\[
A' := \Gamma_{G'}(L \cup C, R - F'),
B' := \Gamma_{G'}(R \cup C, L - F').
\]
Let $A$ and $B$ denote the set of nodes in $G$ corresponding to nodes in $A'$ and $B'$, respectively, in $G'$. Observe that $A \cup B \cup F = V$. Observe also that $|A| \leq |A'| \leq f$, $|B| \leq |B'| \leq f$, and $|F| \leq f$. If one of $A - (B \cup F)$, $B - (A \cup F)$, and $F - (A \cup B)$ is empty, then $n \leq 2f$, which violates the first condition in the lemma statement, and we are done. So assume that all three of the above sets is non-empty. By assumption of Case 3,

$$L - F' \subseteq \Gamma_{G'}(L - F', R - F') \subseteq L - F'.$$

Therefore, $L - F' = \Gamma_{G'}(L - F', R - F')$, and we have

$$A - (B \cup F) = (A - F) - (B - F)$$

$$= (A' - F') - (B' - F')$$

$$= \Gamma_{G'}((L \cup C) - F', R - F') - \Gamma_{G'}((R \cup C) - F', L - F')$$

$$= \Gamma_{G'}(L - F', R - F') - \Gamma_{G'}(R - F', L - F')$$

since $\Gamma_{G'}(C - F', R - F') = \Gamma_{G'}(C - F', L - F')$, by assumption of Case 3

$$= \Gamma_{G'}(L - F', R - F')$$

since $L - F' = \Gamma_{G'}(L - F', R - F')$.

Similarly, $B - (A \cup F) = R - F'$.

Now, we show that there is no undirected hyperedge in $G$ that crosses each of the 3 non-empty sets above. Consider any three nodes

(i) $u \in A - (B \cup F) = L - F'$,
(ii) $v \in B - (A \cup F) = R - F'$, and
(iii) $z \in F - (A \cup B)$.

If there is an undirected hyperedge that contains all three of $u, v, z$, then there is a directed hyperedge $e \in \delta_G(z)$ such that $u, v \in T(e)$. We will create a contradiction with (iii) above, by showing that $z \in A \cup B$. Observe that $u, v \in V \cap V'$, i.e., $u$ and $v$ were not split in $G'$. Let

$$z_e = \begin{cases} 
  z & \text{if } z \text{ was not split in } G', \\
  z^0 & \text{if } z \text{ was split into } z^0, z^1 \text{ in } G' \text{ and } e \text{ was assigned to } z^0, \\
  z^1 & \text{if } z \text{ was split into } z^0, z^1 \text{ in } G' \text{ and } e \text{ was assigned to } z^1.
\end{cases}$$

In each case, there is a directed hyperedge $e'$ in $G'$, corresponding to $e$ in $G$, such that $z_e = H(e')$ and $u, v \in T(e')$. Note that $z_e \in V' = L \cup C \cup R$ and there are two cases to consider.
$z_e \in L \cup C$: then $z_e \in \Gamma_{G'}(L \cup C, R - F') = A'$ since $v \in T(e')$ and $v \in R - F'$.

$z_e \in R$: then $z_e \in \Gamma_{G'}(R \cup C, L - F') = B'$ since $u \in T(e')$ and $u \in L - F'$.

In either case, $z_e \in A' \cup B'$. It follows that $z \in A \cup B$, a contradiction. Therefore, there is no undirected hyperedge that contains all three of $u, v,$ and $z$.

Finally, we show that the third condition in the lemma statement is violated. Since $n > f$, we can find three sets $V_1 \supseteq A, V_2 \supseteq B, V_3 \supseteq F$ such that $|V_1| = |V_2| = |V_3| = f$. Observe that $V_1 \cup V_2 \cup V_3 = A \cup B \cup F = V$, and so

$$V_1 - (V_2 \cup V_3) \subseteq A - (B \cup F),$$
$$V_2 - (V_1 \cup V_3) \subseteq B - (A \cup F),$$
$$V_3 - (V_1 \cup V_2) \subseteq F - (A \cup B).$$

Therefore, there is no undirected hyperedge in $G$ across the three (possibly empty) sets

(i) $V_1 - (V_2 \cup V_3),$
(ii) $V_2 - (V_1 \cup V_3),$ and
(iii) $V_3 - (V_1 \cup V_2)$.

This violates the third condition in the lemma statement.

In all cases, we have that one of the conditions in the lemma statement is violated. \qed

D Proof of Necessity of Condition LCR-hyper

In this section, we show the necessity portion of Theorem 3.1, following the discussion in Section 6.

Proof of Theorem 3.1 ($\Rightarrow$ direction). Suppose for the sake of contradiction that there exists a set $F$, of cardinality at most $f$, such that $G$ does not satisfy condition LCR-hyper with parameter $F$, but there exists an algorithm $A$ that solves Byzantine consensus on $G$. Then there is a hypergraph $G' \in \Lambda_F(G)$ and a partition $(L, C, R)$ of $V'$ such that, using $F'$ to denote the set of nodes in $G'$ corresponding to nodes in $F$ in $G$,

1) $L \cup C \not\leftrightarrow_{G'} R - F'$, and
2) $R \cup C \not\leftrightarrow_{G'} L - F'$.

Note that this implies that both $R - F'$ and $L - F'$ are non-empty. Consider the nodes in $C \cap F'$. By moving them from $C$ to $L$, the required condition stays violated. Therefore, without loss of
generality, we assume that $C \cap F' = \emptyset$ for the rest of the proof. As described in Section 6, we work with the algorithm $A'$ on $G'$ that corresponds to $A$, with appropriate inputs and faulty nodes to create the desired contradiction.

We first create a directed hypergraph $G = (V, E)$ to model the behavior of nodes in three different executions $E_1, E_2, \text{ and } E_3$ of algorithm $A'$ on $G' = (V', E')$. We will describe these executions later. Figure 2 depicts the underlying simple graph $\overline{G} = (\overline{V}, \overline{E})$. Recall that for the set $F \subseteq V$, we use $F'$ to denote the corresponding nodes in $V'$. Let

$$L' := L - \Gamma_{G'}(L, R - F'),$$
$$R' := R - \Gamma_{G'}(R, L - F'),$$
$$C' := C - \left(\Gamma_{G'}(C, L - F') \cup \Gamma_{G'}(C, R - F')\right).$$

A node $u$ in $G'$ may have up to 3 copies in $G$, denoted by $u_0, u_1, u_2$. If a node has a single copy in $G$, then we omit the subscript. This notation extends to sets as well so that $C_1', C_2', C_3'$ denote the three copies of the nodes in $C'$. The nodes have the following number of copies in $G$, as depicted in Figure 2.

- Nodes in $C'$ have three copies.
- Nodes in $\Gamma_{G'}(L, R - F'), \Gamma_{G'}(R, L - F')$, and $\Gamma_{G'}(C, L - F') \cap \Gamma_{G'}(C, R - F')$ have a single copy.
- All other nodes have two copies.

We describe the hyperedges of $G$ based on the hyperedges of $G'$ and the simple edges of $\overline{G}$ depicted in Figure 2. We use $v' \in V$ to denote a copy of a node $v \in V'$. Consider a copy $u' \in V$ of a node $u \in V'$. For a hyperedge $e = (u, S) \in E'$ of $G'$, let

$$S' := \left\{ v' \mid v \in S \text{ and } (u', v') \in E' \right\}.$$

If $S' \neq \emptyset$, then $(u', S')$ is a hyperedge in $G$. $\overline{G}$ has been constructed to ensure that for each edge $(u, v) \in E'$ of $G'$, each copy of $v$ has an edge from exactly one copy of $u$ in $\overline{G}$. Hence, for each hyperedge $e = (u, S) \in E'$ of $G'$ such that $v \in S$, each copy of $v$ receives messages on exactly one hyperedge corresponding to $e$ in $G$. However, there can be multiple copies of $v$ that receive messages from a copy of $u$.

The algorithm $A'$ outlines a procedure $A'_u$ for each node $u \in V'$ that describes $u$’s state transitions, as well as messages transmitted to each neighbor $v$ of $u$ in each round. We create an algorithm for $G$, corresponding to $A'$, as follows. Consider a hyperedge $(u, S) \in E'$ in $G'$. Let $u'$ be a copy of $u$ in $G$ and let $(u', S')$ be the hyperedge in $G$ corresponding to the hyperedge $(u, S)$ (using $S' = \emptyset$ for the case there is no such hyperedge). Then $u'$ runs the procedure $A'_u$, with the following modification.
When $\mathcal{A}_u'$ requires a message $m$ to be sent on the hyperedge $(u, S)$, $u'$ sends the message $m$ on the hyperedge $(u', S')$ in $\mathcal{G}$. Recall that, by construction of $\mathcal{G}$, for any in-neighbor $v$ of node $u$ in $G'$, each copy of $u$ receives messages from exactly one copy of $v$ in $\mathcal{G}$. So each copy of $u$ in $\mathcal{G}$ can correctly run the procedure $\mathcal{A}_u'$. Observe that it is not guaranteed that the nodes will agree on the same value, or even if the algorithm will terminate.

Consider an execution $\Sigma$ of the above algorithm on $\mathcal{G}$ with the following inputs. All (copies of) nodes denoted with subscript 0 have input 0. All (copies of) nodes denoted with subscript 1 have input 1. $C_2'$ is the only set with subscript 2, and has input 1. For the single copy nodes, $\Gamma_{G'}(L, R - F')$ has input 0, while all others have input 1. We show that with these inputs, the algorithm above does terminate, but the output of the nodes will help us in deriving the desired contradiction. We use the execution $\Sigma$ to model three executions $E_1$, $E_2$, and $E_3$ of $\mathcal{A}'$ on the hypergraph $G'$. In each of the three executions, we ensure that the conditions of Lemma 6.1 are met so that $\mathcal{A}'$ solves consensus in finite time. $E_1$, $E_2$, and $E_3$ are as follows.

$E_1 : \Gamma_{G'}(R \cup C, L - F')$ is the set of faulty nodes in this execution. Recall that $|\Gamma_{G'}(R \cup C, L - F')| \leq f$. All non-faulty nodes have input 0. Observe that this satisfies the conditions of Lemma 6.1 so that $\mathcal{A}'$ solves consensus in finite time in this execution. Figure 3 depicts the execution $E_1$. Consider any arbitrary round in $E_1$. We describe the messages transmitted by faulty nodes in this round. If a faulty node $u$ has a single copy in $\mathcal{G}$, then, in $E_1$, $u$ transmits the same messages as the copy in execution $\Sigma$. If a faulty node $u \in V'$ has two copies $u_0$ and $u_1$ in $\mathcal{G}$, then, in $E_1$, $u$ transmits the same messages as the copy $u_0$ in execution $\Sigma$. Figure 3 depicts how the behavior of each node, faulty or non-faulty, in $E_1$ is modelled by the corresponding copy in $\Sigma$. Observe that each node in $G'$ is being modelled by exactly one copy in $\mathcal{G}$. Since $\mathcal{A}'$ solves Byzantine consensus on $G'$, so all non-faulty nodes decide on output 0 (by validity) in finite time. In particular, all nodes in $L - F'$ in $E_1$ decide on output 0. In $\Sigma$, these are modelled by copies in either $(L' - F')_0$ or $\Gamma_{G'}(L - F', R - F')$. Therefore, all nodes in $(L' - F')_0$ and $\Gamma_{G'}(L - F', R - F')$ decide on output 0 in $\Sigma$.

$E_2 : \Gamma_{G'}(L \cup C, R - F')$ is the set of faulty nodes in this execution. Recall that $|\Gamma_{G'}(L \cup C, R - F')| \leq f$. All non-faulty nodes have input 1. Observe that this satisfies the conditions of Lemma 6.1 so that $\mathcal{A}'$ solves consensus in finite time in this execution. Figure 4 depicts the execution $E_2$. Consider any arbitrary round in $E_2$. We describe the messages transmitted by faulty nodes in this round. If a faulty node $u$ has a single copy in $\mathcal{G}$, then, in $E_2$, $u$ transmits the same messages as the copy in execution $\Sigma$. If a faulty node $u \in V'$ has two copies $u_0$ and $u_1$ in $\mathcal{G}$, then, in $E_2$, $u$ transmits the same messages as the copy $u_1$ in execution $\Sigma$. Figure 4 depicts how the behavior of each node, faulty or non-faulty, in $E_2$ is modelled by the corresponding copy in $\Sigma$. Observe that each node in $G'$ is being modelled by exactly one copy in $\mathcal{G}$. Since $\mathcal{A}'$ solves Byzantine consensus on $G'$, so all non-faulty nodes decide on output 1 (by validity)
in finite time. In particular, all nodes in $R - F'$ in $E_2$ decide on output 1. In $\Sigma$, these are modelled by copies in either $(R' - F')_1$ or $\Gamma_{G'}(R - F', L - F')$. Therefore, all nodes in $(R' - F')_1$ and $\Gamma_{G'}(R - F', L - F')$ decide on output 1 in $\Sigma$.

$E_3$: $F'$ is the set of faulty nodes. Recall that $C \cap F' = \emptyset$ and that some nodes in $F'$ in $G'$ might have been split from original nodes in $F$ in $G$. However, $|F| \leq f$, i.e. the total number of corresponding faulty nodes in $G$ is at most $f$. Figure 5 depicts the execution $E_3$. There are no split nodes outside of $F'$ in $G'$. Therefore, the conditions of Lemma 6.1 are satisfied and $A'$ solves consensus in finite time in this execution. All non-faulty nodes in the set $L - F'$ have input 0. All non-faulty nodes in the set $\Gamma_{G'}(C, L - F') - \Gamma_{G'}(C, R - F')$ also have input 0. All the other non-faulty nodes have input 1. Consider any arbitrary round in $E_3$. We describe the messages transmitted by faulty nodes in this round. If a faulty node $u \in F'$ has a single copy in $G$, then, in $E_3$, $u$ transmits the same messages as the copy in execution $\Sigma$. If a faulty node $u \in F'$ has two copies $u_0$ and $u_1$ in $G$, then, $u \in L' \cap F'$ (resp. $u \in R' \cap F'$) in $G'$. $u$ transmits the same messages as the copy $u_0$ (resp. $u_1$) in execution $\Sigma$. Figure 5 depicts how the behavior of each node, faulty or non-faulty, in execution $E_3$ is modelled by the corresponding copy in $\Sigma$. Observe that each node in $G'$ is being modelled by exactly one copy in $G$, even if it comes from an original node in $F - F'$ in $G$ that was split. We show that the output of nodes in execution $E_3$ is not the same, thus deriving the contradiction.

In execution $\Sigma$, nodes in $(L' - F')_0$ and $\Gamma_{G'}(L - F', R - F')$ output 0 while nodes in $(R' - F')_1$ and $\Gamma_{G'}(R - F', L - F')$ output 1. Observe that these copies model the nodes in $L - F'$ and $R - F'$, respectively, in $G'$. Therefore, in execution $E_3$, nodes in $L - F'$ output 0 while nodes in $R - F'$ output 1. Recall that both these sets are non-empty by construction. Thus algorithm $A'$ in execution $E_3$ on hypergraph $G'$ terminates without agreement between these two sets of nodes, a contradiction.

E Proof of Correctness of Algorithm 1

In this section, we show correctness of Algorithm 1 when the hypergraph $G$ satisfies condition AB-hyper. For the rest of this section, we assume that $G$ satisfies both condition AB-hyper and condition LCR-hyper (recall that, by Theorem 7.3, the two conditions are equivalent). Throughout this section, we use $F^*$ to denote the actual set of faulty nodes. We prove Lemma 7.7 first.

Proof of Lemma 7.7. Fix a phase $> 0$. We use $\gamma_u^{\text{start}}$ and $\gamma_u^{\text{end}}$ to denote the state $\gamma_u$ of node $u$ at the beginning and end of the phase, respectively. Consider an arbitrary non-faulty node $v$. If $v$ does not update its state in this phase, then the claim is trivially true since $\gamma_v^{\text{end}} = \gamma_v^{\text{start}}$. So
Figure 2: $\overline{\mathcal{G}}$ to model $E_1$, $E_2$, and $E_3$. The numbers adjacent to the sets are the corresponding inputs in execution $\mathcal{E}$; if there is no number adjacent to the set, then the input is the same as the subscript. An undirected edge denotes that edges can exist in both directions. A directed edge with a hollow arrow denotes that edges could only exist in one direction between the original nodes in $\overline{\mathcal{G}}$. A directed edge with a solid arrow denotes that edges could have existed in both directions between the original nodes in $\overline{\mathcal{G}}$, but in $\overline{\mathcal{G}}$, they only exist in one direction between the two copies.
Figure 3: Execution $E_1$ as modeled by execution $\Sigma$ on hypergraph $\mathcal{G}$ (the underlying simple graph $\mathcal{G}$ is shown here). The red nodes are faulty in $E_1$. The gray network is simulated by the faulty nodes (this includes edges between faulty nodes). Edges between faulty and non-faulty nodes are depicted in red.
Figure 4: Execution $E_2$ as modeled by execution $\Sigma$ on hypergraph $\mathcal{G}$ (the underlying simple graph $\overline{\mathcal{G}}$ is shown here). The red nodes are faulty in $E_2$. The gray network is simulated by the faulty nodes. Edges between faulty and non-faulty nodes are depicted in red.
Figure 5: Execution $E_3$ as modeled by execution $\Sigma$ on hypergraph $\mathcal{G}$ (the underlying simple graph $\overline{\mathcal{G}}$ is shown here). The red nodes are faulty in $E_3$. The gray network is simulated by the faulty nodes. Edges between faulty and non-faulty nodes are depicted in red.
suppose that $v$ did update its state in this phase. Then it must have done so in either step (d) or step (e) (but not both). We consider each case separately.

Case 1: $v \in S$ updated its state in step (d).

Suppose $v$ updated its state $\gamma_v$ to $\tau \in \{0, 1\}$ in step (d). Then, as per the update rules in step (d), $v$ must have received the value $\tau$ identically along $f + 1$ node-disjoint $A_v$-paths in step (b). Since there are at most $f$ faulty nodes, at least one of the $A_v$-paths, say $P$, must neither have any faulty internal node nor a faulty source node. Now $\tau$ was received along $P$, which has exclusively non-faulty internal nodes. So the source node of $P$, say $u$, flooded $\tau$ in step (b) of this phase. Furthermore, $u$ is non-faulty. Thus, $\gamma_u^{\text{start}} = \tau$ at the start of this phase. Therefore, the state of node $v$ at the end of this phase equals the state of a non-faulty node $u$ at the start of this phase.

Case 2: $v \in V - S - F$ updated its state in step (f).

Suppose $v$ updated its state $\gamma_v$ to $\tau \in \{0, 1\}$ in step (f). Then, as per the update rules in step (f), $v$ must have received the value $\tau$ identically along $f + 1$ node-disjoint $S_v$-paths in step (e). Since there are at most $f$ faulty nodes, at least one of the $S_v$-paths, say $P$, must neither have any faulty internal node nor a faulty source node. Now $\tau$ was received along $P$, which has exclusively non-faulty internal nodes. So the source node of $P$, say $u$, flooded $\tau$ in step (e) of this phase. Note that $u \in S$. If $u$ did not update its state in step (d), then $\tau = \gamma_u^{\text{end}} = \gamma_u^{\text{start}}$. Otherwise, by Case 1 above, $\tau = \gamma_u^{\text{end}} = \gamma_w^{\text{start}}$ for some non-faulty node $w$. In both cases, $\tau$ is a $\gamma$ value of some non-faulty node at the start of this phase. Therefore, the state of node $v$ at the end of this phase equals the state of some non-faulty node at the start of this phase.

In both cases, we have that $\gamma_v^{\text{end}} = \gamma_u^{\text{start}}$ for some non-faulty node $u$. \hfill $\square$

Before proving Lemma 7.8, we need some intermediate results. First, we show the proofs of Lemmas 7.4, 7.5, and 7.6, which are similar to Lemmas 6, 7, and 10 in [9].

**Proof of Lemma 7.4.** Fix an arbitrary set $F$. Suppose, for the sake of contradiction, that the directed decomposition of $G - F$ has two source components $S_1$ and $S_2$. To derive the contradiction, we show that $G$ does not satisfy condition LCR-hyper. Let

$$L := S_1,$$

$$R := S_2 \cup F,$$

$$C := V - S_1 - S_2 - F;$$

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so that \((L, R, C)\) is a partition of \(V\). Observe that both \(L - F = S_1\) and \(R - F = S_2\) are non-empty. Since \(S_1\) is a source component of the directed decomposition of \(G - F\),

\[
\Gamma_G(R \cup C, L - F) = \Gamma_G(V - S_1, S_1) \quad \text{since } R \cup C = V - S_1 \text{ and } L - F = S_1
\]

\[
\subseteq F \quad \text{since } S_1 \text{ is a source component of } G - F.
\]

Similarly, since \(S_2\) is also a source component of the directed decomposition of \(G - F\),

\[
\Gamma_G(L \cup C, R - F) = \Gamma_G(V - S_2 - F, S_2) = \emptyset \quad \text{since } L \cup C = V - S_2 - F \text{ and } R - F = S_2
\]

\[
\text{since } S_2 \text{ is a source component of } G - F.
\]

Therefore,

1) \(|\Gamma_G(L \cup C, R - F)| = 0 \leq f \implies L \cup C \not\rightarrow_G R - F, \text{ and}

2) \(|\Gamma_G(R \cup C, L - F)| \leq |F| \leq f \implies R \cup C \not\rightarrow_G L - F.

Note that \(F' = F\) for \(G' = G\). Since \(G \in \Lambda_F(G)\), this violates condition LCR-hyper, a contradiction. \(\square\)

**Proof of Lemma 7.5.** Fix an arbitrary set \(F\). Let \(S\) be the unique source component in the directed decomposition of \(G - F\), and let \(\Phi = \Gamma_G(F, S)\). Suppose, for the sake of contradiction, that \(G[S \cup \Phi]\) does not satisfy condition AB-hyper with parameter \(\Phi\). Then, by Theorem 7.3, \(G[S \cup \Phi]\) does not satisfy condition LCR-hyper with parameter \(\Phi\) either. So there exists a hypergraph \(G_{\Phi} \in \Lambda_{\Phi}(G[S \cup \Phi])\) and, using \(\Phi'\) to denote the set of nodes in \(G_{\Phi}\) corresponding to nodes in \(\Phi\) in \(G\), a partition \((L, C, R)\) of \(S \cup \Phi'\) such that

1) \(L \cup C \not\rightarrow_{G_{\Phi}} R - \Phi', \text{ and}

2) \(R \cup C \not\rightarrow_{G_{\Phi}} L - \Phi'.

Observe that this implies that both \(R - \Phi'\) and \(L - \Phi'\) are non-empty, by definition of \(\not\rightarrow\).

Since \(\Phi \subseteq F\), so there exists a hypergraph \(G' \in \Lambda_F(G)\) that is obtained by splitting \emph{exactly} the same nodes as were split to obtain \(G_{\Phi}\), and making the same assignments in the split operations in both graphs. So \(G'\) has the node set \(V' = \Phi' \cup (V - \Phi)\), and \(G_{\Phi} = G'[S \cup \Phi']\). Let \(F'\) denote the set of nodes in \(G'\) corresponding to nodes in \(F\) in \(G\) (i.e., \(F' = \Phi' \cup (F - \Phi)\)), and let

\[
C' := C \cup (V - S - \Phi).
\]

Then \((L, C', R)\) is a partition of \(V'\). To complete the contradiction, we show that \(L \cup C' \not\rightarrow_{G'} R - F'\) and \(R \cup C' \not\rightarrow_{G'} L - F'\), which violates condition LCR-hyper.
We first show that in $G'$, nodes in $(L \cup R) - F'$ have no in-neighbors in $C' - C$, as follows.

$$\Gamma_{G'}(C' - C, (L \cup R) - F') = \Gamma_{G'}(V - S - \Phi, (L \cup R) - F')$$

$$\subseteq \Gamma_{G'}(V - S - \Phi, S)$$

$$= \Gamma_G(V - S - \Phi, S)$$

$$= \emptyset,$$

where the last equality follows from the fact that $S$ is the source component in the directed decomposition of $G - F$. Now, we have

$$\left| \Gamma_{G'}(L \cup C', R - F') \right| = \left| \Gamma_{G'}(L \cup C, R - F') \right|$$

$$= \left| \Gamma_{G'}(L \cup C, R - \Phi') \right|$$

$$= \left| \Gamma_{G'}(L \cup C, R - \Phi') \right|$$

$$\leq f$$

$$\implies L \cup C' \not\rightarrow_{G'} R - F'$$

Similarly,

$$\left| \Gamma_{G'}(R \cup C', L - F') \right| = \left| \Gamma_{G'}(R \cup C, L - F') \right|$$

$$= \left| \Gamma_{G'}(R \cup C, L - \Phi') \right|$$

$$= \left| \Gamma_{G'}(R \cup C, L - \Phi') \right|$$

$$\leq f$$

$$\implies R \cup C' \not\rightarrow_{G'} L - F'$$

This violates condition LCR-hyper, a contradiction.

\[ \square \]

**Proof of Lemma 7.6.** Fix an arbitrary set $F$. Let $S$ be the unique source component in the directed decomposition of $G - F$. Let

$$A := S,$$

$$B := V - S,$$

so that $(A, B)$ is a partition of $V$. Now, since $A = S$ is the source component in the directed decomposition of $G - F$, we have

$$\Gamma_G(B, A) = \Gamma_G(V - S, S)$$

$$= \Gamma_G(F, S)$$

$$\subseteq F.$$

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That is, all the in-neighbors of $A$ in $B$ are contained entirely in $F$. So, by Menger’s Theorem, for any node $v \in A$, there can be at most $|F| \leq f$ node-disjoint $Bv$-paths in $G$. Thus, $B \nRightarrow_{G-(A \cap F)} A - F$. Since $G \in \Lambda_F(G)$, by condition AB-hyper, we have

$$S = A \twoheadrightarrow_{G-(B \cap F)} B - F = V - S - F.$$ 

The result follows from the fact that $B \cap F = F$. 

Now, we show that in every iteration of the main for loop of Algorithm 1, the paths in step (c) do exist.

**Lemma E.1.** In any phase $> 0$ of Algorithm 1 with a candidate faulty set $F$, for any two nodes $u, v \in S$, there exists a $uv$-path in $G[S]$.

*Proof.* Immediately since $S$ is strongly connected. 

In the flooding procedure ([8, 9]), when a non-faulty node wants to flood a value $b \in \{0, 1\}$, it sends a single value $b$ on all of its hyperedges. But a faulty node might send different messages on different hyperedges. Note, however, that even a faulty node must send the exact same value on a single hyperedge: if it sends two different values on the same hyperedge, then the receiving nodes can choose the first value and ignore the later one.

**Lemma E.2.** Consider a phase $> 0$ of Algorithm 1 wherein $F = F^*$. For any two non-faulty nodes $u, v \in S$, we have $G'_u = G'_v$ in step (c) of this phase. Furthermore, if in step (b) of this phase a faulty node $z \in \Gamma_G(F^*, S)$ transmitted $0$ (resp. $1$) on a hyperedge $e \in \delta_G(z)$, such that $T(e) \cap S$ is non-empty, then in step (c) of this phase $e$ is assigned to $z^0$ (resp. $z^1$) in $G'_u = G'_v$.

*Proof.* Consider the phase where $F = F^*$ and any two non-faulty nodes $u, v \in S$. Observe that the node set of the two hypergraphs $G'_u$ and $G'_v$ are the same. For the hyperedges, by construction, it is sufficient to show that, for any $z \in F^*$, the assignment of multicast channels to $z^0$ and $z^1$ in the split operation is the same in $G'_u$ as in $G'_v$. Consider an arbitrary node $z \in F^*$ and a hyperedge $e \in \delta_G(z)$. There are two cases to consider:

**Case 1:** There exists a node $w \in T(e)$ such that $w \in S$.

So $z \in \Gamma_G(F^*, S)$. By Lemma E.1, there exists a $wu$-path and a $wv$-path in $G[S]$. Therefore, there exists a $zu$-path and a $zw$-path, respectively, identified by nodes $u$ and $v$ in step (c). Observe that, for both these paths, the first hyperedge on the path is $e$ and $z$ is the only faulty node. Since $z$ is the source node in $P_{zu}$ and $P_{zu}$, both these paths do not have any faulty internal node. Therefore, in step (b), if $z$ transmitted $0$ on hyperedge $e$, then $u$ (resp. $v$) received value $0$.
along $P_{zu}$ (resp. $P_{zw}$). So both $u$ and $v$ assign $e$ to $z^0$ in $G_u'$ and $G_v'$, respectively. Similarly, if $z$ transmitted 1 on hyperedge $e$ in step (b), then both $u$ and $v$ assign $e$ to $z^1$ in $G_u'$ and $G_v'$, respectively.

Case 2: There does not exist any node $w \in T(e)$ such that $w \in S$.
Then there is no $zu$-path or $zw$-path in $G[S, \{e\}]$. Therefore, both $u$ and $v$ assign $e$ to $z^1$ in $G_u'$ and $G_v'$, respectively.

In both cases, we have that the hyperedge $e$ was assigned identically by both $u$ and $v$. Observe that if $z \in \Gamma_G(F^*, S)$ and $z$ transmitted 0 (resp. 1) on a hyperedge $e \in \delta_G(z)$, such that $T(e) \cap S$ is non-empty, then $e$ is assigned to $z^0$ (resp. $z^1$) by both $u$ and $v$, as required.  

**Lemma E.3.** Consider a phase $> 0$ of Algorithm 1 wherein $F = F^*$. Let

$$Z := \left\{ u^0 \mid u \in \Gamma_G(F^*, S) \right\} \cup \left\{ w \in S \mid w \text{ flooded value } 0 \text{ in step (b) of this phase} \right\},$$

$$N := \left\{ u^1 \mid u \in \Gamma_G(F^*, S) \right\} \cup \left\{ w \in S \mid w \text{ flooded value } 1 \text{ in step (b) of this phase} \right\}.$$

For any two non-faulty nodes $u, v \in S$, we have $Z = Z_u = Z_v$ and $N = N_u = N_v$ in step (c) of this phase.

**Proof.** Consider the phase where $F = F^*$ and $S \subseteq V - F^*$ is the unique source component in the directed decomposition of $G - F^*$. For any two non-faulty nodes $u, v \in S$, we show that $Z \subseteq Z_v$ and $N \subseteq N_v$ (resp. $Z \subseteq Z_u$ and $N \subseteq N_u$). Since $Z \cup N = Z_u \cup N_u = Z_v \cup N_v$, it follows that $Z = Z_u = Z_v$ and $N = N_u = N_v$. For a node $w \in \Gamma_G(F^*, S)$, the two split nodes $w^0$ and $w^1$ are assigned identically by both $u$ and $v$. So consider an arbitrary node $w \in S$. Recall that we are considering the phase $> 0$ of the algorithm where $F = F^*$ is the actual set of faulty nodes. So $w$ is non-faulty. There are two cases to consider:

Case 1: $w \in Z - F^*$, i.e., $w \in S$ flooded 0 in step (b) of this phase.

Let $P_{wv}$ be the $wv$-path identified by $v$ in step (c). Note that $P_{wv}$ is contained entirely in $G - F^*$ so that $P_{wv}$ does not have any faulty nodes. It follows that, in step (b), $w$ flooded the value 0. So $v$ received value 0 along $P_{wv}$. Therefore, in step (c) $v$ puts $w$ in the set $Z_v$.

Case 2: $w \in N - F^*$, i.e., $w \in S$ flooded 1 in step (b) of this phase.

Let $P_{wv}$ be the $wv$-path identified by $v$ in step (c). Note that $P_{wv}$ is contained entirely in $G - F^*$ so that $P_{wv}$ does not have any faulty nodes. It follows that, in step (b), $w$ flooded the value 1. So $v$ received value 1 along $P_{wv}$. Therefore, in step (c) $v$ puts $w$ in the set $N_v$. 

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So we have $Z \subseteq Z_u$ and $N \subseteq N_v$, as required. A symmetric argument gives $Z \subseteq Z_u$ and $N \subseteq N_v$. As argued before, this implies $Z = Z_u = Z_v$ and $N = N_u = N_v$.  

We are now ready to prove Lemma 7.8.

**Proof of Lemma 7.8.** Consider the phase where $F = F^*$ and $S \subseteq V - F^*$ is the unique source component in the directed decomposition of $G - F^*$. Suppose $u, v \in V - F^*$ are any two non-faulty nodes. Then,

1) by Lemma E.3, $Z = Z_u = Z_v$ and $N = N_u = N_v$, where $Z$ and $N$ are as in the statement of Lemma E.3, and

2) by Lemma E.2, $G'_u = G'_v$.

Let $G' = G'_u = G'_v$. We use $F'$ to denote the set of nodes in $G'$ corresponding to nodes in $F^*$ in $G$.

We first show that all non-faulty nodes in $S$ have identical state at the end of this phase. Observe that non-faulty nodes in $S$ update their states exclusively in step (d). So consider step (d) of this phase. At the start of step (d), by construction of $Z$ and $N$, all non-faulty nodes in $Z$ have identical state of 0, while all non-faulty nodes in $N$ have identical state of 1. We show that, in step (d), either all non-faulty nodes in $Z$ update their state to 1, or all non-faulty nodes in $N$ update their state to 0. Note that $G' \in \Lambda_{F^*}(G)$ and by condition AB-hyper, either $Z \leadsto_{G' - N \cap F'} N - F'$ or $N \leadsto_{G' - Z \cap F'} Z - F'$. We consider each case as follows.

**Case 1:** $Z \leadsto_{G' - (N \cap F')} N - F'$.

In this case, we show that all non-faulty nodes in $S$ have state 0 at the end of step (d). There are a further two cases to consider.

**Case (i):** $N - F'$ is empty.

Then all non-faulty nodes in $S = (Z \cup N) - F' = Z - F'$ have state 0 at the start of the phase. Each node $v \in S$ sets $B_v = N$ in step (d). We have $B_v - F' = N - F' = \emptyset$. So $v$ does not update its state in step (d). Therefore, all non-faulty nodes in $S$ have identical state 0 at the end of step (d).

**Case (ii):** $N - F'$ is non-empty.

Consider an arbitrary node $v \in S = (Z \cup N) - F'$. In step (d), $v$ sets $A_v = Z$ and $B_v = N$. If $v \in A_v - F' = Z - F'$, then $v$ has state 0 at the start of this phase and does not update it in step (d). So suppose that $v \in B_v - F' = N - F'$. Now, if in step (b) $v$ received the value 0 identically along some $f + 1$ node-disjoint $Zv$-paths in $G' - (N \cap F')$, then $v$ sets $\gamma_v = 0$. We show that such $f + 1$ node-disjoint $Zv$-paths do indeed exist. Since $Z \leadsto_{G' - (N \cap F')} N - F'$, there exist $f + 1$ node-disjoint $Zv$-paths
in $G' - (N \cap F')$. Without loss of generality only the source nodes on these paths are from $Z$. For each such path, observe that only the source node, say $z \in Z$, can be faulty. If the source node $z$ is faulty, then by Lemma E.2, and construction of $G'$ and $Z$, $z$ sent the value 0 on the first edge on this path in step (b). If $z$ is non-faulty, then by construction of $Z$, $z$ flooded value 0 in step (b). Now all other nodes on the path are non-faulty, so $v$ received value 0 identically along the $f + 1$ node-disjoint $Zv$-paths in step (b), as required.

Case 2: $Z \not\supseteq G' - (N \cap F') N - F'$ so that $N \sim_{G' - (Z \cap F')} Z - F'$ by condition AB-hyper.

In this case, we show that all non-faulty nodes in $S$ have state 1 at the end of step (d). There are a further two cases to consider.

Case (i): $Z - F'$ is empty.

Then, similar to Case 1(i), all non-faulty nodes in $S$ have state 1 at the start of the phase and they do not update their state in step (d). So all non-faulty nodes in $S$ have state identical state 1 at the end of step (d).

Case (ii): $Z - F'$ is non-empty.

Consider an arbitrary node $v \in S = (Z \cup N) - F'$. In step (d), $v$ sets $A_v = N$ and $B_v = Z$. If $v \in A_v - F' = N - F'$, then $v$ has state 1 at the start of this phase and does not update it in step (d). So suppose that $v \in B_v - F' = Z - F'$. As in Case 1(ii), since $N \sim_{G' - (Z \cap F')} Z - F'$, there exist $f + 1$ node-disjoint $Nv$-paths in $G' - (Z \cap F')$ such that $v$ received the value 1 identically along these paths in step (b). Therefore, $v$ sets $\gamma_v = 1$, as required.

In both cases, all non-faulty nodes in $S$ have identical state, say $\tau$, at the end of step (d). Since nodes in $S$ do not update their state after this step, nodes in $S$ have state $\tau$ at the end of this phase.

We now consider step (f) and an arbitrary non-faulty node $v \in V - S - F^*$. All nodes in $S$ are non-faulty, so each of them floods the value $\tau$ in step (e). By Lemma 7.6, $S \sim_{G - F^*} V - S - F^*$ and so there exist $f + 1$ node-disjoint $Sv$-paths in $G - F^*$. All the source nodes on these paths are non-faulty nodes in $S$. All the internal nodes on these paths are non-faulty as well. So $v$ receives the value $\tau$ identically along these $f + 1$ node-disjoint paths in step (e). It follows that $v$ updates $\gamma_v$ to the value $\tau$ in step (f). Therefore, all nodes in $V - S - F^*$ have state $\tau$ at the end of this phase, as required.

Using Lemmas 7.7 and 7.8, we can now prove the sufficiency of condition AB-hyper. Recall that condition AB-hyper is equivalent to condition LCR-hyper by Theorem 7.3. Thus, this shows the reverse direction of Theorem 3.1.
Proof of Theorem 3.1 ($\Leftarrow$ direction). Algorithm 1 satisfies the termination condition because it terminates in finite time.

In one of the iterations of the main for loop, we have $F = F^*$, i.e., $F$ is the actual set of faulty nodes. By Lemma 7.8, all non-faulty nodes have the same state at the end of this phase. By Lemma 7.7, these states remain unchanged in any subsequent phases. Therefore, all nodes output an identical state. So the algorithm satisfies the agreement condition.

At the start of phase 1, the state of each non-faulty node equals its own input. By inductively applying Lemma 7.7, we have that the state of a non-faulty node always equals the input of some non-faulty node, including in the last phase of the algorithm. So the output of each non-faulty node is an input of some non-faulty node, satisfying the validity condition.  

On Lemma 3 of [15]

The bug in proof of Lemma 3 in [15] is on the first line of page 457: sets $C_1, C_2, C_3$ may have negative size. Here, we present a counter example to the claim in Lemma 3 of [15]. We first need the following definition of hypergraph connectivity.

Definition F.1 (Definition 4 in [15]). For $\ell, t, k > 0$, an undirected hypergraph $G$ is $(\ell, t)$-hyper-$k$-connected, if, for any set $C$ of exactly $k-1$ nodes and any partition of $V-C$ into $\ell$ non-empty sets, each of size at most $t$, there exists an undirected hyperedge in $G$ that has a non-empty intersection with every set of the partition.

Recall that an undirected hyperedge $e \in E$ is a subset of nodes $e \subseteq V$ and is called an $|e|$-hyperedge. Recall also that an undirected hypergraph $G = (V, E)$ is a $(2, 3)$-hypergraph if each hyperedge is either a 2-hyperedge or a 3-hyperedge. The claim in Lemma 3 of [15] is as follows.

Claim F.2 (Lemma 3 in [15]). An undirected $(2, 3)$-hypergraph $G = (V, E)$ with $2f < n \leq 3f$ is $(3, f)$-hyper-$(3f - n + 1)$-connected if and only if, for every $V_1, V_2, V_3 \subseteq V$ such that $V_1 \cup V_2 \cup V_3 = V$ and $|V_1| = |V_2| = |V_3| = f$, there exist three nodes

(i) $u \in V_1 - (V_2 \cup V_3)$,

(ii) $v \in V_2 - (V_1 \cup V_3)$, and

(iii) $w \in V_3 - (V_1 \cup V_2)$,

such that $u, v, w \in E$.

Counter example. We show a counter example to the reverse direction. That is, we create an undirected $(2, 3)$-hypergraph $G = (V, E)$ with $2f < n \leq 3f$ that satisfies both of the following:
1) for every $V_1, V_2, V_3 \subseteq V$ such that $V_1 \cup V_2 \cup V_3 = V$ and $|V_1| = |V_2| = |V_3| = f$, there exist three nodes

(i) $u \in V_1 - (V_2 \cup V_3)$,
(ii) $v \in V_2 - (V_1 \cup V_3)$, and
(iii) $w \in V_3 - (V_1 \cup V_2)$,

such that $u, v, w \in E$.

2) $G$ is not $(3, f)$-hyper-$(3f - n + 1)$-connected.

Pick $f > 2$ and $n = 3f - 1 > 2f$. $G$ has all 2-hyperedges and has two parts $X$ and $Y$. $X$ consists of $2f + 1$ nodes and $Y$ consists of $f - 2$ nodes. Every 3 nodes in $X$ form a 3-hyperedge but no node in $Y$ is part of any 3-hyperedge. We show that $G$ satisfies each of the two condition above, as follows.

1) Consider any three sets $V_1, V_2, V_3 \subseteq V$ such that $V_1 \cup V_2 \cup V_3 = V$ and $|V_1| = |V_2| = |V_3| = f$.

By choice of $n$ and $f$ ($2f < n \leq 3f$), such sets do exist. Now,

\[ V_1 \cap X - (V_2 \cup V_3) = X - (V_2 \cup V_3) \]

since $V_1 \cup V_2 \cup V_3 = V \supseteq X$

\[ \neq \emptyset \]

since $|X| = 2f + 1 > 2f + |V_3| \geq |V_2 \cup V_3|$. 

Similarly, $V_2 \cap X - (V_1 \cup V_3) \neq \emptyset$ and $V_3 \cap X - (V_1 \cup V_2) \neq \emptyset$. It follows that there exist three nodes

(i) $u \in V_1 \cap X - (V_2 \cup V_3) \subseteq V_1 - (V_2 \cup V_3)$,
(ii) $v \in V_2 \cap X - (V_1 \cup V_3) \subseteq V_2 - (V_1 \cup V_3)$, and
(iii) $w \in V_3 \cap X - (V_1 \cup V_2) \subseteq V_3 - (V_1 \cup V_2)$.

By construction of $G$ and $X$, since $u, v, w \in X$, so $\{u, v, w\} \in E$, as required.

2) Pick any node $x \in X$. Let $C := \{x\}$. We create a partition $(S_1, S_2, S_3)$ of $V - C$ as follows. $S_1$ contains exactly $f$ nodes from $X - C$. $S_2$ contains the remaining $f$ nodes in $X - C - S_1$. $S_3 := Y$. Then,

\[ 0 < |S_1| = f \]
\[ 0 < |S_2| = f \]
\[ 0 < |S_3| = f - 2 \leq f. \]

Since no 3-hyperedge crosses $Y = S_3$ in $G$, so there is no undirected hyperedge in $G$ that has a non-empty intersection with each of $S_1, S_2$, and $S_3$. By Definition F.1, $G$ is not $(3, f)$-hyper-$(3f - n + 1)$-connected.
This completes the counter example to Claim F.2.