Effect of multiple degrees of ambivalence on the Naming Game

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Abstract—We examine a modified Naming Game in the mean field where there are multiple degrees of ambivalence. Once an agent in one state hears an opinion one way or another, he or she moves one step in the appropriate direction. In the absence of zealots, the two consensus states are stable steady states and the uniform distribution is an unstable steady state. With zealots for one opinion only, there is a critical value below which there are three steady states and above which there is only one. Consensus in favor of the zealots’ opinion is the steady state that always exists, and is stable. The second steady state is the uniform distribution in the absence of zealots, and moves away from the zealots’ opinion as the number of zealots increases. This state is unstable. The last steady state starts at the uniform distribution in the absence of zealots, and moves toward the zealots’ opinion as the number of zealots increases. This state is stable. When zealots are added on both sides, the “break” pattern observed for the Naming Game remains, with the region of multiple steady states growing with the addition of more intermediate states.

I. INTRODUCTION

The science of social interaction is a very alluring field. To understand how humans influence each other provides the opportunity to control how public opinion shapes itself. The difficulty is that the human brain is too complex to allow for a simple and clearly accurate model of human opinion formation. As such, a number of approximations have been proposed, each with its own shortcomings. Of interest is the rather simple voter model. Here, each person, or agent, has one of two conflicting opinions, A or B. As the agents talk to each other, they try to convince each other of their own opinions. One agent is chosen as the speaker and the other is chosen as the listener. If the speaker and listener agree, then there is no change in the opinion state, but if they initially disagree, then the listener is converted to the speaker’s opinion. [2, 4, 5, 9, 11, 12, 17, 18] Of particular interest is the parameter \( m \), the magnetization of the system. The magnetization is the expected poll result and is calculated by \( m = \frac{\rho_A - \rho_B}{\rho_A + \rho_B} \), where \( \rho_A \) represents the density of people in favor of opinion A and \( \rho_B \) represents the density of people in favor of opinion B. Note that, for the voter model, \( \rho_A + \rho_B = 1 \). In this case, the variables \( \rho_A \) and \( \rho_B \) are martingales, and given enough time, the system will almost surely go to one of two absorbing states, that of full consensus at A and that of full consensus at B. [2, 4, 5, 9, 11, 12, 18]

A more interesting model is the Naming Game model, or Binary Agreement Model. Here, the possibility of ambivalence is accounted for. In addition to the two extremist states A and B, there is the state AB, which represents ambivalence. In this model, agents in AB are no more or less likely to speak or be spoken to than anyone else. If the chosen speaker is in the ambivalent state, then he or she will subconsciously choose one of the two opinions. If the listener is in the ambivalent state, then he or she will change to whichever opinion he or she hears. [1, 3, 4, 5, 6, 7, 8, 13, 14, 15, 16, 17, 18, 19] While there are various psychological studies on the effect of voicing an opinion on an ambivalent speaker, this is ignored in the Listener-Only Naming Game model. [4, 18] In this model, the values \( \rho_A, \rho_B, \) and \( \rho_{AB} \) are no longer martingales. In fact, once one side begins to dominate, (this dominance can be determined by the magnetization, which is now calculated by \( m = \frac{\rho_{AB} - \rho_A}{\rho_{AB} + \rho_A} \)) this dominance will tend to grow until consensus is reached. The consensus points are stable steady states while perfect balance between \( \rho_A, \rho_B, \) and \( \rho_{AB} \) is an unstable steady state. [1, 3, 4, 5, 6, 7, 8, 13, 14, 15, 16, 17, 18, 19]

A further generalization allows for the agents to be leaning in one direction or another to varying degrees without full commitment to one idea or the other. If a listener hears one opinion or the other, he or she moves one step in the direction of the opinion he or she hears. [18] The parameter \( K \) represents the number of times that an agent convinced of opinion B needs to hear opinion A in order to be convinced of opinion A, and vice versa. The population densities in each state are now denoted \( \rho_0, \rho_1, \ldots, \rho_K \), where \( \rho_0 \) represents commitment to opinion B, \( \rho_K \) represents commitment to opinion A, and the other states are the various ambivalent states. The states themselves (as opposed to the populations of the states,) are denoted \( N_0, N_1, \ldots N_K \). An agent in state \( N_i \) will speak in favor of opinion A with probability \( \frac{K-i}{K} \) and opinion B with probability \( \frac{i}{K} \). Note that the Voter Model corresponds to \( K = 1 \) and the Naming Game model corresponds to \( K = 2 \). [18]

One other factor involved is the presence of zealots in favor of one opinion of the other. A zealot operates with a motive that ignores all logic and thus can never be convinced to change his or her opinion. No amount of convincing can turn a non-zealot, or normal agent, into a zealot. The presence of these zealots affects the dynamics of the system as well as the possible long-term outcomes. [1, 3, 4, 5, 9, 11, 12, 13, 14, 17, 18]

II. THE NO-ZEALOT CASE

The behavior of a network obeying one of these models can be characterized by its long term behavior in the mean field. That is to say, what possible steady-states are there and which of them are stable. [20] It has already been shown that for the standard Naming Game model, the two consensus points are stable steady states and that the uniform distribution between \( \rho_0, \rho_1, \) and \( \rho_2 \) is an unstable steady state. [1, 3, 4, 5, 6, 7, 8, 13, 14, 15, 16, 17, 18, 19] As it turns out, the analogous result is true for any value of \( K \geq 2 \). Namely, the state \( \rho_0 = 1, \rho_i = 0 \) for \( i \neq 0 \) is a stable steady state, as is the state \( \rho_K = 1, \rho_i = 0 \) for \( i \neq K \). The only other steady state is at \( \rho_i = \frac{1}{K+i} \) for all \( i \), and this state is unstable.

The outline of the proof is as follows: It has been previously shown that a necessary condition for a steady state is that
\[ \frac{p_i}{\rho_{i-1}} \] is constant (if finite) for \( i = 1, 2 \ldots K \). Furthermore, this quotient is equal to \( \frac{m}{1 - m} \), where \( m \) is the magnetization, or the expected poll result. The magnetization is calculated with the formula \( m = \sum_{i=1}^{K} \left( \frac{i}{K} \times \rho_i \right) \). It can be shown that this is a necessary and sufficient condition for a distribution to be a steady state. Thus, selecting a value of \( m \) between 0 and 1 fully determines the distribution by fixing the ratios of population densities of different states. However, the distribution has a magnetization of its own, and this may or may not be the same as the magnetization used to generate the distribution. If it is the same, then it is a steady state, otherwise it is not. It can be shown that for \( m = 0, m = 0.5, \) and \( m = 1, \) the resulting distribution has the same magnetization as the one used to generate it. For \( 0 < m < 0.5, \) the resulting magnetization is less than the value of \( m \) used to generate the distribution, and for \( 0.5 < m < 1 \) the resulting magnetization is greater than the one used to generate the distribution.

To address stability, we first look at the states corresponding to \( m = 0 \) and \( m = 1 \). Because the setup exhibits symmetry, we only need to show stability for one point, and the other point should have the same level of stability. It should be noted that the only distribution with \( m = 0 \) is \( \rho_0 = 1, \rho_i = 0 \) for \( i \neq 0 \). For distributions near this one, \( m \) tends to decrease over time. Thus, the distributions tend toward the minimum value of \( m \), which is \( m = 0 \), and there is only one distribution with that magnetization value. The point at \( m = 0.5 \) has a different property. Assuming a geometric distribution with \( m \) slightly greater than 0.5, the value of \( m \) will tend to increase, and if \( m \) is slightly less than 0.5, \( m \) will tend to decrease. Thus, if the distribution is slightly perturbed from the steady state, the perturbation will grow larger and larger, resulting in instability. A more detailed proof appears in the appendix.

III. Steady States for the Unilateral Zealot Case

We now examine the case where, in addition to the normal agents, there are agents with unshakable support for one of the two opinions. We perform calculations assuming the zealots support opinion B, and use symmetry to infer the corresponding results for zealots in favor of opinion A. We denote the population density of zealots in favor of opinion B as \( \rho_B \). It is worth mentioning that the notion of long-term behavior takes on multiple meanings in this case. If there are no zealots in favor of opinion A, but there is at least one in favor of opinion B, then there is only one absorbing state, namely, that of full consensus at opinion B. Furthermore, it can be shown that this consensus state will be reached with probability 1. [1, 3, 4, 9, 13, 14, 17, 18, 19, 20, 21, 22] So in a sense, the only truly stable steady state is that of consensus at opinion B. However, it is realistically possible for the system to hover in the neighborhood of a particular distribution for a relatively long period of time before drifting to the consensus state. [21, 22] It is these particular distributions which we are looking for. We will call these opinion states "metastable" states. Additionally, we look for the conditions under which these metastable states can exist.

In the case of the Voter Model, it is rather simple. The magnetization is a supermartingale, with martingality holding only for the consensus state. [3, 4, 9, 13, 17, 18] In the Naming Game model, things get more interesting. Consensus at B is always a stable steady state. It has been shown that in the case of \( \rho_B = 0 \), consensus at A is also a stable steady state, and that this stable state drifts toward lower values of \( m \) as \( \rho_B \) increases. This is also true for the case \( K = 3 \). Additionally, when \( \rho_B = 0 \), the uniform distribution is an unstable steady state. This unstable state moves to one of higher magnetization as \( \rho_B \) increases. [3, 4, 13, 14, 17, 18, 19, 21] This has also been found to be true in the case \( K = 3 \). To understand why this is, one needs to understand the relationship between the magnetization \( m \) and the magnetization of the normal agents, \( \rho_{\text{normal}} \). We calculate the magnetization of the normal agents by the formula \( \rho_{\text{normal}} = \frac{k}{K} \rho_B \). If we fix a value of \( m \) and force the normal agents into a geometric distribution obeying \( \frac{\rho_i}{\rho_{i-1}} = \frac{m}{1-m} \), then the value of \( \rho_{\text{normal}} \) is determined. It can be shown that the number of zealots committed to opinion B can be determined from the formula \( \rho_B = 1 - \frac{m}{\rho_{\text{normal}}} \). As it turns out, the results regarding the existence and metastability of steady states are true as long as \( d \rho_B dm < 0 \) over the interval (0, 1). Figures 2-5 examine the stability of steady states in the cases \( K = 3 \) and \( K = 10 \). The outline of the proof is as follows:

Similar to the no-zealot case, a necessary condition for a steady state is the relationship \( \frac{\rho_i}{\rho_{i-1}} = \frac{m}{1-m} \). A subtle difference between this case and the no-zealot case is that \( \rho_B \) is now a parameter, and while it does not directly affect any of the \( \rho_i \) values, it does directly affect \( m \). In particular, as \( \rho_B \) increases, \( m \) decreases. Thus, in cases where \( \rho_{\text{normal}} > m \), there is a suitable \( \rho_B \) that will enable the equation \( \frac{\rho_{i}}{\rho_{i-1}} = \frac{m}{1-m} \) to hold. Recall the result that if a geometric distribution is generated from a magnetization \( m \) then the resulting magnetization will exceed the generating magnetization for 0.5 < \( m < 1 \). Thus, for 0.5 < \( m < 1 \), there is a positive value of \( \rho_B \) that will make the distribution steady. By symmetry, for 0 < \( m < 0.5 \), there is a value of \( \rho_B \) that will make the distribution steady if \( \rho_B = 0 \).
Fig. 2. Setting $K = 3$, we pick initial conditions with random small perturbations off the intermediate steady state, and track the 2-norm of $d$, the difference between the perturbed state and the true steady state. Over the course of time, the difference tends to grow, showing that the steady state is unstable.

It can be shown that this special value of $\rho_B$ is unique and is a continuous function of $m$. This means that, as long as the special value of $\rho_B$ is of one concavity as $m$ varies, (in this case, concave down,) the steady states will each move in one direction as $m$ increases. Because $\rho_B$ is bounded by 0 and 1, there must be some maximum value of $\rho_B$ that yields a steady state other than consensus at B. The continuity of $\rho_B$ shows that multiple steady states must meet at this maximum. Empirical evidence suggests that the stability of these steady states does not change until they meet at this critical value of $\rho_B$. A more detailed proof appears in the appendix.

IV. TIPPING POINTS FOR THE CASE OF BILATERAL ZEALOTS

As we have already mentioned, stability, and thus, long-term sustainability of non-consensus states depends on the number of zealots. This effect is even further pronounced when there are not only zealots in favor of opinion B, but also in favor of opinion A. [10, 11, 12, 17, 19] Examining the data in figures 6-9, which examine the conditions under which multiple steady states exist, several phenomena are noticed. First, if there are sufficiently few committed agents total, there will be multiple steady states. Second, if there are significantly more zealots in favor of one opinion than the other, there is only a single steady-state. Finally, if there are enough zealots, there will only be one steady state. [10, 11, 12, 17, 19] This holds even when the zealots are perfectly evenly divided between opinions A and B. [11, 17, 19] This case is of particular interest, as it implies that, given enough zealots, the equilibrium uniform
distribution becomes a stable steady state. [10, 11, 12, 17, 19] More interestingly, it becomes the only stable steady state, and thus, given enough time, the system should always be near this state.

We find the critical value of $\rho_A$ and $\rho_B$ above which the uniform distribution of normal agents is stable. Let $\alpha$ represent the fraction of agents who are not zealots. We also assume that the system is near the uniform distribution. Because there is little drift here, we can assume that the system is near a geometric distribution, as the value of $m$ would have been approximately constant for a long time. Because the system is nearly uniform, we assume $r = 1 + \epsilon$. This gives us the following equations:

$$\rho_i = \rho_0 (1 + \epsilon)^i$$  \hspace{1cm} (1)$$

$$\rho_i = \rho_0 (1 + i \epsilon)$$  \hspace{1cm} (3)

for $i = 0, 1, 2, \ldots K$

Note that, because $\epsilon$ is small, $(1 + \epsilon)^i \approx 1 + i \epsilon$, giving us

$$\rho_i = \rho_0 (1 + i \epsilon)$$  \hspace{1cm} (4)

for $i = 0, 1, 2, \ldots K$

To calculate $m$ for this distribution, we can first calculate $m$ for the zealots, then calculate $m$ for the normal agents, and take a weighted average. Because the zealots are evenly distributed between $\rho_A$ and $\rho_B$, we know that, for this group of zealots, $m_{\text{zealots}} = 0.5$. For the normal agents, the calculation is more involved.
m_{normal} = \frac{\sum_{i=0}^{K} \frac{i + 1}{K} }{1 + \frac{K^2 + K}{2} \epsilon} \tag{5}

Evaluating the sum, and using the formula for the sum of squares, we get

m_{normal} = \frac{K + 1 + K(K+1)(2K+1)}{6K} \frac{\epsilon}{1 + \frac{K^2 + K}{2} \epsilon} \tag{6}

Factoring out a K+1 from the numerator and denominator, expanding in powers of epsilon, and dropping everything after the linear term, we get

m_{normal} = \frac{0.5 + (2K+1)\epsilon}{1 + \frac{K^2 + K}{2} \epsilon} \tag{7}

m_{normal} = (0.5 + \frac{2K+1}{6}\epsilon)(1 - \frac{K}{2} \epsilon) \tag{8}

m_{normal} = (0.5 + \epsilon(\frac{K + 2}{12})) \tag{9}

To find the overall value of $m$, we take a weighted average of $m_{normal}$ and $m_{zealots}$.

m = \alpha m_{normal} + (1 - \alpha)m_{zealots} \tag{10}

m = \alpha(0.5 + \epsilon(\frac{2K + 1}{12}))(1 - \alpha)(0.5) \tag{11}

m = 0.5 + \alpha \frac{K + 2}{12} \tag{12}

Next, to check for stability, we calculate $E[\frac{dm}{dt}]$. When this figure has the same sign as $\epsilon$, the equilibrium state is unstable, and when it has the opposite sign, the uniform distribution is stable.

\begin{align*}
E[\frac{dm}{dt}] &= m(1 - \rho_K) - (1 - m)(1 - \rho_0) \\
&= (0.5 + \alpha \frac{K + 2}{12})(1 - \frac{1 + K \epsilon}{K + 1 + \frac{K^2 + K}{2} \epsilon}) - (0.5 - \alpha \frac{K + 2}{12})(1 - \frac{1 - K \epsilon}{K + 1 + \frac{K^2 + K}{2} \epsilon}) \tag{13}
\end{align*}

Expanding in powers of $\epsilon$ and dropping everything after the linear terms, we get

\begin{align*}
E[\frac{dm}{dt}] &= (0.5 + \alpha \frac{K + 2}{12})(1 - \frac{1}{K + 1} - \frac{K \epsilon}{2}) - (0.5 - \alpha \frac{K + 2}{12})(1 - \frac{1}{K + 1} + \frac{K \epsilon}{2}) \\
&= \epsilon(-\frac{K}{4} + \alpha \frac{K^2 + 2K}{12K + 12}) - \epsilon(-\frac{K}{4} - \alpha \frac{K^2 + 2K}{12K + 12}) \tag{14}
\end{align*}

Note that the uniform distribution is stable whenever $\frac{K}{4} - \alpha \frac{K^2 + 2K}{12K + 12} > 0$. As a result, we get

\begin{align*}
\alpha < \frac{K}{4} - \frac{K^2 + 2K}{12K + 12} \tag{15}
\alpha < \frac{3}{K + 2} \tag{16}
\end{align*}

Figure 10. The graph of the difference between the highest and lowest values of $m$ for which a steady state exists in the case where $\rho_A = \rho_B$. For values of $\rho_A$ and $\rho_B$ slightly below the critical value, the difference is still fairly high.

APPENDIX A
Detailed Mathematical Proofs

In the case without zealots, $\rho_A = \rho_B = 0$. Let the distribution of agents in the various opinion states be represented by the K+1 dimensional vector $[\rho_0, \rho_1, \ldots, \rho_K]$. Let $m$ represent the magnetization of this distribution. Assume the distribution to be a steady state. Let $\phi_{i+}$ be the expected flux over one time step from state $N_i$ to state $N_{i+1}$. This is, by definition, the expected number of listeners in state $i$ who hear opinion $A$. We define a time step such that one conversation occurs per unit time. The probability that a listener is in state $i$ is simply $\rho_i$. The probability that a listener hears opinion $A$ is roughly the same for all potential listeners, (assuming a sufficiently large complete network,) and is equal to the magnetization. Thus, we have

$\phi_{i+} = m \times \rho_i \tag{20}$

for $i = 0, 1, 2, \ldots K-1$. $\phi_{K+} = 0$ because there is no state $N_{K+1}$ for the agents to move into. Similarly, we can define $\phi_{i-}$ as the expected flux from state $N_i$ to state $N_{i-1}$. We have

$\phi_{i-} = (1 - m) \times \rho_i \tag{21}$

If a distribution is a steady state, the flux from state $N_i$ to state $N_{i+1}$ must be matched by the flux from state $N_{i+1}$ to state $N_i$. Thus, we have the equations...
for $m \neq 1$. The special cases of $m = 0$ and $m = 1$ are the absorbing states, and are clearly steady states. The case $m = 0.5$ produces a steady state. In this case

$$m = \frac{1}{2}$$

$$\frac{\rho_{i+1}}{\rho_i} = \frac{0.5}{0.5} = 1$$

$$\rho_{i+1} = \rho_i$$

(25)

Because all $K+1$ population densities are equal and they sum to 1, we have $\rho_i = \frac{1}{K+1}$ for all $i$. The only question is whether or not the magnetization of the resulting distribution is, in fact, 0.5.

$$m = \frac{\sum_{i=1}^{K+1} i \times \rho_i}{K}$$

(26)

$$m = \frac{\sum_{i=1}^{K+1} i \times \frac{1}{K+1}}{K}$$

(27)

$$m = \frac{K \times (K+1)}{2} \times \frac{1}{K+1}$$

(28)

$$m = \frac{m}{m} = 0.5$$

(29)

(30)

Because the distribution is geometric and the resulting magnetization is equal to the magnetization used to generate the distribution, it is a steady state. Thus, steady states are achieved for the geometric distributions generated by magnetizations of 0, 0.5, and 1.

In the case where $0 < m < 0.5$, when $m$ is used to generate a distribution and $K \geq 2$, the resulting magnetization is strictly less than the one used to generate the distribution, and the when $0.5 < m < 1$, the resulting magnetization is strictly greater than the one used to generate the distribution. This means that, in either case, the distribution is not a steady state as the ratio $\frac{\rho_{i+1}}{\rho_i} \neq \frac{m}{m}$. The proof is shown for the case $0.5 < m < 1$ and an analogous proof can be used for the case $0 < m < 0.5$.

Suppose $K = 1$. Let $\rho_0 = 1 - x$ and $\rho_1 = x$. The magnetization of this distribution is

$$m = 1 \times x = x$$

(31)

and the ratio of adjacent states is

$$\frac{\rho_1}{\rho_0} = \frac{x}{1-x} = \frac{m}{1-m}$$

(32)

Thus, for $K = 1$, any geometric distribution is a steady state. If $K$ is increased by 1 and the same ratio $\frac{m}{1-m}$ between adjacent states and $0.5 < x < 1$, then the value of $m$ for the new distribution is strictly greater than the value of $m$ for the distribution with a lower value of $K$.

Consider a geometric distribution on $K+1$ states with $\rho_0 = \frac{m}{1-m}$. Suppose this distribution has magnetization $m_{old}$. Suppose that this distribution is extended to $K+2$ states by keeping the values of $\rho_0, \rho_1, \ldots, \rho_K$ and setting $\rho_{K+1} = \rho_K \times \frac{m}{1-m}$. We calculate the magnetization of the new distribution, $m_{new}$.

$$m_{new} = \frac{\sum_{i=1}^{K+1} (i \rho_i)}{\sum_{i=0}^{K+1} \rho_i}$$

(33)

$$m_{new} = \frac{K}{K+1} \sum_{i=0}^{K+1} \rho_i + \rho_{K+1}$$

(34)

$$m_{new} = m_{old} + \frac{1}{K+1} m_{old} \sum_{i=0}^{K+1} \rho_i + (1 - m_{old}) \rho_{K+1}$$

(35)

We examine the sign of $\frac{m_{new}}{m_{old}} > m_{old}$. Note that, because of the ratio $\frac{\rho_{i+1}}{\rho_i} = \frac{m}{1-m}$ and the restrictions taken on $x$, $\rho_K$ must be greater than any of the other values of $\rho_i$. Thus, $\rho_K > \sum_{i=0}^{K-1} \rho_i$, so $m_{old} > \frac{\sum_{i=0}^{K+1} \rho_i}{\sum_{i=0}^{K-1} \rho_i}$.

In the case $K = 1$, we have $m_{old} = x$ and we will show, by induction, that for $K \geq 2$, $m_{old} \geq x$, which gives us $m_{new} = m_{old} + \frac{1}{K+1} m_{old} \sum_{i=0}^{K+1} \rho_i + (1 - m_{old}) \rho_{K+1}$.

$$m_{new} = m_{old} + \frac{1}{K+1} m_{old} \sum_{i=0}^{K+1} \rho_i + (1 - m_{old}) \rho_{K+1}$$

(36)

$$m_{new} > m_{old} + \frac{1}{K+1} m_{old} \sum_{i=0}^{K+1} \rho_i + \frac{m_{old} \rho_{K+1}}{\sum_{i=0}^{K-1} \rho_i}$$

(37)

$$m_{new} > m_{old} + \sum_{i=0}^{K-1} \rho_i + \frac{m_{old}}{\sum_{i=0}^{K-1} \rho_i} \rho_{K+1}$$

(38)

$$m_{new} > m_{old}$$

(39)

Because $m_{new} > m_{old}$ and $m_{old} \geq x$, we have $m_{new} > \frac{x}{1-x}$ and the magnetization of the new distribution is not equal to $\frac{x}{1-x}$, so the distribution is not a steady state. Note that this inequality also validates the induction used earlier.

Having found the steady states, we examine their stability. Consider the state of consensus at B. The vector of population densities is $[1, 0, 0, \ldots, 0]$. For consensus at B, this is $[0, 0, \ldots, 0]$. Assume that the initial condition is a slight perturbation from consensus, and $[\rho_1, \rho_2, \ldots, \rho_K] = [\epsilon_1, \epsilon_2, \ldots, \epsilon_K]$. Consider the “magnetization norm” $||x||_m = (1/K) ||x||_K + 2 ||x||_K$.

$$||x||_m = ||c \times x||_m = \frac{|c \times x_1| + 2 |c \times x_2| + \ldots + K|c \times x_K|}{K}$$

(40)

$$= \frac{|c| ||x||_m}{K}$$

(41)

$$||x + y||_m = \frac{|x_1 + y_1| + 2 |x_2 + y_2| + \ldots + K|x_K + y_K|}{K}$$

(43)

$$\leq \frac{x_1 + |y_1| + \ldots + K|x_K| + K|y_K|}{K}$$

(44)

$$= ||x||_m + ||y||_m$$

(45)

Next, note that for realizable distributions, $x_i \geq 0$, so the absolute value bars can be ignored. Thus, $||x||_m = (1/K) (x_1 + 2 x_2 + \ldots + K x_K) = m$. At consensus, $||x||_m = 0$. We examine what happens in a neighborhood of consensus such that $||x||_m \leq \frac{m}{2K}$.

Note that this means that $\rho_i = 0$ for $i > K/2$ and that $\rho_0 \leq 1 - 2m$. We examine $\Delta m$, the change in $m$ over a single time step.
\[ \Delta m = \frac{m \rho_0 + (2m - 1)(1 - \rho_0)}{N K} \]
\[ \leq \frac{m(1 - 2m) + (2m - 1)(1 - (1 - 2m))}{N K} \]
\[ = \frac{m - 2m^2 + (4m^2 - 2m)}{N K} \]
\[ = \frac{-m + 2m^2 + (4m^2 - 2m)}{N K} \]
\[ < 0 \]  
(46)  
(47)  
(48)  
(49)  
(50)

for \(0 < m < 0.5\). Thus, the value of \(\|x\|_m\) will decrease, and the point is stable. An analogous argument holds for the point of consensus at \(A\).

Next, we examine the stability of the uniform distribution. In a sufficiently small neighborhood of this distribution, none of the population densities are 0. This allows us to describe the distribution with the vector of ratios of densities of adjacent states, \([\epsilon_1, \epsilon_2, \epsilon_3, \ldots, \epsilon_{K-1}]\). For the uniform distribution, this vector is \([1, 1, \ldots, 1]\). Assume a small perturbation of the form \([\epsilon, \epsilon, \ldots, \epsilon]\), so that the vector is \([1 + \epsilon, 1 + \epsilon, \ldots, 1 + \epsilon]\), where \(\epsilon > 0\). We claim that if the minimum ratio of densities of adjacent states is some \(r_{\text{min}}(t) > 1\), then after one time step, the minimum ratio of densities is some \(r_{\text{min}}(t + 1) > r_{\text{min}}(t)\). Let’s first examine the ratio \(\frac{\rho_{i+1}}{\rho_i}\) for \(i = 1, 2, \ldots, K - 2\). We fix \(\rho_i\) and attempt to minimize \(\frac{\rho_{i+1}}{\rho_i}(t + 1)\). Suppose that \(\rho_{i+1} > r_{\text{min}}(t)\rho_i\). Then, for sufficiently large \(N\), \(\rho_{i+1}\) and \(\rho_i\) would change sufficiently little to keep \(\frac{\rho_{i+1}}{\rho_i}(t + 1) > r_{\text{min}}(t)\). If \(\rho_{i+1} = r_{\text{min}}(t)\rho_i\), we attempt to minimize \(\frac{\rho_{i+1}}{\rho_i}(t + 1)\). To do this, we minimize \(\rho_{i+1}(t + 1)\) and maximize \(\rho_i(t + 1)\).

Obeying the constraint that \(\frac{\rho_{i+1}}{\rho_i} \geq r_{\text{min}}(t)\), we get
\[ \rho_{i-1}(t) \leq \frac{\rho_{i}(t)}{r_{\text{min}}(t)} \]
\[ \rho_i(t) = \rho_{i}(t) \]
\[ \rho_{i+1}(t) = \rho_i(t) \times r_{\text{min}}(t) \]
\[ \rho_{i+2}(t) \geq \rho_i(t) \times (r_{\text{min}}(t))^2 \]  
(51)  
(52)  
(53)  
(54)

The new values of the relevant densities are
\[ \rho_i(t + 1) = \rho_i(t) * \left(\frac{N - 1}{N}\right) + \rho_i(t) \times m(1/N) \]
\[ + \rho_{i+1}(t) \times (1 - m)(1/N) \]
\[ \rho_{i+1}(t + 1) = \rho_{i+1}(t) * \left(\frac{N - 1}{N}\right) + \rho_i(t) \times m(1/N) \]
\[ + \rho_{i+2}(t) \times (1 - m)(1/N) \]  
(55)  
(56)

The quotient of these terms is
\[ \frac{\rho_{i+1}(t + 1)}{\rho_i(t + 1)} = \frac{\rho_{i+1}(t)(\frac{N - 1}{N}) + \rho_i(t)m(1/N) + \rho_{i+2}(t)(1 - m)(1/N)}{\rho_i(t)(\frac{N - 1}{N}) + \rho_{i-1}(t)m(1/N) + \rho_{i+1}(t)(1 - m)(1/N)} \]
\[ \geq \frac{\rho_i(t)(r_{\text{min}}(t)(\frac{N - 1}{N}) + m(1/N) + r_{\text{min}}(t)(2 - m)(1/N))}{\rho_i(t)((\frac{N - 1}{N}) + m/r_{\text{min}}(t) + r_{\text{min}}(t)(1 - m)(1/N))} \]
\[ = r_{\text{min}}(t) \]  
(57)  
(58)  
(59)

The new values of the relevant densities are
\[ \rho_{i+1}(t + 1) = \rho_i(t) \times r_{\text{min}}(t) \]
\[ \rho_i(t + 1) = \rho_i(t) \times m(1/N) + \rho_{i+1}(t)(1 - m)(1/N) \]
\[ \rho_{i+1}(t + 1) = \rho_{i+1}(t) \times (1 - m)(1/N) \]
\[ \rho_{i+2}(t) \geq \rho_i(t) \times (r_{\text{min}}(t))^2 \]  
(60)  
(61)  
(62)  
(63)

The new values of the relevant densities are
\[ \rho_i(t + 1) = \rho_i(t) \times \left(\frac{N - m}{N}\right) \]
\[ + \rho_i(t) \times (1 - m)(1/N) \]
\[ \rho_{i+1}(t + 1) = \rho_{i+1}(t) \times \left(\frac{N - 1}{N}\right) \]
\[ + \rho_{i+1}(t) \times (1 - m)(1/N) \]  
(64)  
(65)  
(66)

The quotient of these terms is
\[ \frac{\rho_{i+1}(t + 1)}{\rho_i(t + 1)} = \frac{\rho_{i+1}(t)(\frac{N - 1}{N}) + \rho_i(t)m(1/N) + \rho_{i+2}(t)(1 - m)(1/N)}{\rho_i(t)(\frac{N - 1}{N}) + \rho_{i-1}(t)m(1/N) + \rho_{i+1}(t)(1 - m)(1/N)} \]
\[ \geq \frac{\rho_i(t)(r_{\text{min}}(t)(\frac{N - 1}{N}) + m(1/N) + r_{\text{min}}(t)(2 - m)(1/N))}{\rho_i(t)((\frac{N - 1}{N}) + m/r_{\text{min}}(t) + r_{\text{min}}(t)(1 - m)(1/N))} \]  
(67)  
(68)

The numerator of this fraction is a special case of the numerator of equation 58. The denominator differs by an additive term of \((\rho_i(t)) \times (N/m \times r_{\text{min}}(t) + 1/m)\). Because \(r_{\text{min}}(t) > 1\), we have
\[ m > r_{\text{min}}(t) \]
\[ \frac{m}{r_{\text{min}}(t)} > 1 - m \]
\[ \frac{m}{r_{\text{min}}(t)} > 1 - m \]
\[ \frac{m}{r_{\text{min}}(t)} > 1 - m \]  
(69)  
(70)  
(71)  
(72)

Thus, the denominator is smaller, so the quotient is larger, and we have
\[ \frac{\rho_{i+1}(t + 1)}{\rho_i(t + 1)} > r_{\text{min}}(t) \]  
(73)

Finally, we examine the case where \(i = K - 1\).
\[ \rho_{K-2}(t) \leq \frac{\rho_K - 1(t)}{r_{\text{min}}(t)} \]
\[ \rho_{K-1}(t) = \rho_K - 1(t) \]
\[ \rho_K(t) = \rho_K - 1(t) \times r_{\text{min}}(t) \]  
(74)  
(75)  
(76)

The new values of the relevant densities are
\[ \rho_{K-1}(t + 1) = \rho_{K-1}(t) \times (1 - m)(1/N) \]
\[ \rho_{K-2}(t) \times m(1/N) + \rho_K(t) \times (1 - m)(1/N) \]
\[ \rho_K(t + 1) = \rho_K(t) \times (\frac{N - 1}{N} + m) \]
\[ \rho_{K-1}(t) \times m(1/N) \]  
(77)  
(78)

Comparing this quotient to equation 64, we find that the denominator can be factored into the same form, but the numerator has an additive constant of \(\rho_K \times (\frac{m}{N} - \frac{1}{N} \times r_{\text{min}}(t))\).
Again, we have

\[ m \frac{1}{1 - m} > r_{\text{min}}(t) \]  \
\[ m > (1 - m) \times r_{\text{min}}(t) \]  \
\[ \frac{m}{N} > \frac{(1 - m) \times r_{\text{min}}(t)}{N} \]  \
\[ \frac{m}{N} - \frac{(1 - m) \times r_{\text{min}}(t)}{N} > 0 \]

Thus, the numerator is greater and the denominator is the same as in equation 58, so the quotient is greater. Again, we have

\[ \frac{\rho_K(t + 1)}{\rho_{K-1}(t + 1)} > r_{\text{min}}(t) \] (83)

Thus, \( r_{\text{min}}(t) \) in this case is a non-decreasing function of \( t \), so it will never drop back down to 1. The uniform distribution is unstable.

In the case of unilateral zealots, either \( \rho_A = 0 \) or \( \rho_B = 0 \). For simplicity, we assume \( \rho_A = 0 \) and infer the corresponding results for \( \rho_B = 0 \) by symmetry. The distribution of agents in each non-zealot opinion state must still be geometric, so for fixed \( \rho_B \) we can determine the distribution by the parameter \( m \). Note that, given a distribution of normal agents and zealots, we have the relation

\[ m = m_{\text{normal}}(1 - \rho_B - \rho_A) + m_{\text{zealots}}(\rho_A + \rho_B) \] (84)

Note that, in the case where \( \rho_A = 0 \), \( m_{\text{zealots}} = 0 \), so the equation reduces to

\[ m = m_{\text{normal}}(1 - \rho_B) \] (85)

In the case without zealots, we know that the only values of \( m \) which yield steady states are 0, 0.5, and 1. The question now arises, for other values of \( m \) is there a value of \( \rho_B \) such that there is a steady state with that value of \( m \). We solve for any value of \( \rho_B \) that would make this possible.

\[ m = m_{\text{normal}}(1 - \rho_B) \]  \
\[ m = m_{\text{normal}} - m_{\text{normal}}\rho_B \]  \
\[ \rho_B = \frac{m_{\text{normal}} - m}{m_{\text{normal}}} \] (88)

We have the added restrictions that \( 0 \leq \rho_B \leq 1 \). This can be achieved whenever \( m_{\text{normal}} - m \geq 0 \). As we have previously shown, this occurs whenever \( 0 < m < 0.5 \). Thus, for those values of \( m \) there exists a unique \( \rho_B \) such that equilibrium exists at that particular \( m \). This is consistent with what has been shown in the case \( K = 2 \). In that particular case, it was found that for sufficiently small \( \rho_B \) there were three values of \( m \) where equilibrium was achieved. One was consensus at B. The others started at \( m = 0.5 \) and \( m = 1 \) and as \( \rho_B \) increased, the corresponding equilibrium values of \( m \) approached each other. At some critical value of \( \rho_B \), the equilibrium states meet, and above that critical value, they vanish, and only the one equilibrium point, (consensus at B), remains. This will occur if \( \frac{d \rho_B}{dm} < 0 \) over the interval (0.5, 1).

The state of consensus at B can be shown to be stable using the aforementioned magnetization norm. Again, examine the neighborhood of consensus such that \( ||x||_m < \frac{1}{2K N} \).

\[ \Delta m = \frac{m(\rho_0 + \rho_B) + (2m - 1)(1 - \rho_0 - \rho_B) - m(1 - 2m) + (2m - 1)(1 - 1 - 2m)}{NK} \] (89)

\[ \leq \frac{m(1 - 2m) + (2m - 1)(1 - (1 - 2m))}{NK} \] (90)

\[ = \frac{m - 2m^2 + (4m^2 - 2m)}{NK} \] (91)

\[ = \frac{-m + 2m^2 + (4m^2 - 2m)}{NK} \] (92)

\[ < 0 \] (93)

over the interval (0, 0.5), showing that the point is stable.

Assuming that the pattern of three equilibrium points for sufficiently small \( \rho_B \) holds, it can be shown that the middle one is unstable. Note that, under these assumptions, the value of \( \rho_B \) needed to maintain equilibrium increases as \( m \) increases. Thus, as \( m \) increases and is used to generate a geometric distribution, the resulting value of \( m \) will be greater than the one used to generate the distribution. (It should be noted that in the case of the last equilibrium point, the opposite is true.) Because the quantities \( \rho_0, \rho_1, \rho_2, \ldots \) are nonzero, and because the value of \( \rho_B \) is fixed, we can determine the distribution (and any others near it) with the vector \( \{r_1, r_2, \ldots, r_K\} \), where \( r_1 = \rho_0 \). At equilibrium, there is a magnetization value \( m_{eq} \), and the vector of density ratios is \( \{m_{eq}, m_{eq}, \ldots, m_{eq}\} \). We assume that the initial state is perturbed slightly from the equilibrium state, and the vector of density ratios is now \( \{m_{eq} - \frac{m}{m_{eq}}, m_{eq} - \frac{m}{m_{eq}}, \ldots, m_{eq} - \frac{m}{m_{eq}}\} \), where \( m = m_{eq} + \epsilon \). We assume that if the minimum value of the ratio of densities of adjacent opinion states is no less than \( \frac{m}{m_{eq}} \), then after one time step the minimum ratio will still be no less than \( \frac{m}{m_{eq}} \), near enough the equilibrium point. Consider the ratio between two ambivalent states. Just like the case without zealots, we get

\[ \rho_i(t + 1) = \rho_i(t) \frac{(N - 1)}{N} + \rho_{i-1}(t) \times (1 - m)(1/N) \] (94)

\[ \rho_{i+1}(t + 1) = \rho_{i+1}(t) \frac{(N - 1)}{N} + \rho_{i}(t) \times (1 - m)(1/N) \] (95)

And the quotient is still bound from below by \( \frac{m}{m_{eq}} \).

Let’s consider the ratio between \( \rho_1 \) and \( \rho_0 \). Because of the assumption on the distribution, \( \rho_1 \geq \frac{m}{m_{eq}} \). As before, the only way that the ratio could be smaller than \( \frac{m}{m_{eq}} \) after one time step for sufficiently large \( N \) is if \( \rho_1 = \frac{m_{eq}}{m_{eq}} \). Furthermore, recall that \( \rho_2 \geq \frac{m_{eq}}{m_{eq}} \). The ratio of \( \rho_1 \) to \( \rho_0 \) after one time step is

\[ \rho_1(t + 1) = \frac{\rho_1(t)(\frac{N - 1}{N} + \rho_0(t)(m/N) + \rho_2(t)(1 - m)(1/N))}{\rho_0(t)(\frac{N - m}{N}) + \rho_1(t)(1 - m)(1/N)} \] (96)

\[ \geq \frac{\rho_0(t)(\frac{m}{m_{eq}} + \frac{m}{m_{eq}}) \times (1 - m)(1/N))}{\rho_0(t)(\frac{N - m}{N} + \frac{m_{eq}}{m_{eq}} \times (1 - m)(1/N))} \] (97)

Note that, for the greater value of \( m \), a greater value of \( \rho_B \) is needed to bring equilibrium. Thus, if \( m^* \) is above the equilibrium value, than the value of \( m \) of the distribution generated by \( m^* \) is greater than \( m^* \). With this, and logic similar to the no-zealot case, it can be shown that the ratio of \( \rho_1 \) to \( \rho_0 \) will remain above \( \frac{m}{m_{eq}} \). An analogous argument holds for the ratio of \( \rho_K \) to \( \rho_{K-1} \). Thus, the point is unstable.
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