On an Application of Jack’s Lemma, Starlikeness and \( k \)-Fold Symmetry in \( \mathbb{C}^n \)

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Abstract
It is known that the starlikeness plays a central role in complex analysis, similarly as the convexity in functional analysis. However, if we consider the biholomorphisms between domains in \( \mathbb{C}^n \), apart from starlikeness of domains, various symmetries are also important. This follows from the Poincaré theorem showing that the Euclidean unit ball is not biholomorphically equivalent to a polydisc in \( \mathbb{C}^n \), \( n > 1 \). From this reason the second author in 2003 considered some families of locally biholomorphic mappings defined in the Euclidean open unit ball using starlikeness factorization and a notion of \( k \)-fold symmetry. The 2017 paper of both authors contains some results on the absorption by a family \( S(k) \), \( k \geq 2 \), of the above kind, the families of mappings biholomorphic starlike (convex) and vice versa. In the present paper there is given a new sufficient criterion for a locally biholomorphic mapping \( f \), from the Euclidean ball \( \mathbb{B}^n \) into \( \mathbb{C}^n \), to belong to the family \( S(k) \), \( k \geq 2 \). The result is obtained using an \( n \)-dimensional version of Jack’s Lemma.

Keywords
Locally biholomorphic mappings in \( \mathbb{C}^n \) · \((j,k)\)-symmetrical mappings · Starlike mappings · \( n \)-dimensional version of Jack’s lemma

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1 Introduction and Auxiliary Results

In the present paper the symbol $\mathbb{C}^n$, $n \geq 1$, means the $n$ dimensional complex vector space with the Euclidean inner product

$$\langle z, w \rangle = \sum_{j=1}^{n} z_j \overline{w_j}, \quad z = (z_1, \ldots, z_n), \quad w = (w_1, \ldots, w_n) \in \mathbb{C}^n,$$

with the norm $||z||^2 = \langle z, z \rangle$ and with the open unit Euclidean ball

$$\mathbb{B}^n = \{ z \in \mathbb{C}^n : ||z|| < 1 \}.$$

By $\mathcal{L}(\mathbb{C}^n, \mathbb{C}^n)$ we denote the space of bounded linear operators $L : \mathbb{C}^n \to \mathbb{C}^n$, with the standard operator norm and the identity $I$. For a holomorphic mapping $f = (f_1, \ldots, f_n) : \mathbb{B}^n \to \mathbb{C}^n$, we denote by $Df(z)$ the Fréchet derivative of $f$ at $z \in \mathbb{B}^n$, i.e., an element of $\mathcal{L}(\mathbb{C}^n, \mathbb{C}^n)$ generated by the square matrix $[\frac{\partial f_j}{\partial z_l}(z)]_{1 \leq j, l \leq n}$ of the partial derivatives of the components $f_j$ of $f$ at the point $z$.

The paper concerns biholomorphic mappings $f : \mathbb{B}^n \to \mathbb{C}^n$, that is, mappings which are holomorphic and have the holomorphic inverse $f^{-1}$ which transforms the domain $f(\mathbb{B}^n) \subset \mathbb{C}^n$ onto the ball $\mathbb{B}^n$.

Let $S_t$ be the family of mappings $f : \mathbb{B}^n \to \mathbb{C}^n$, $f(0) = 0$, $Df(0) = I$, which transform biholomorphically the open unit Euclidean ball $\mathbb{B}^n$ onto starlike domain $f(\mathbb{B}^n) \subset \mathbb{C}^n$. A well known characterization of this family is due to Kikuchi [8], Matsuno [18] and Suffridge [23]. Many authors continued their considerations in this direction (see e.g. the monographs: [5] by Graham and Kohr [9] by Kohr and the second author). The above characterization of the family $S_t$ starts with locally biholomorphic mappings $f : \mathbb{B}^n \to \mathbb{C}^n$, that is holomorphic mappings $f : \mathbb{B}^n \to \mathbb{C}^n$ having the non singular Fréchet derivative $Df(z)$ at every $z \in \mathbb{B}^n$. The mentioned characterization of the family $S_t$ is presented in the lemma.

**Lemma 1** (Starlikeness factorization) A locally biholomorphic mapping $f : \mathbb{B}^n \to \mathbb{C}^n$, $f(0) = 0$, $Df(0) = I$, belongs to the family $S_t$, iff it satisfies the factorization

$$f(z) = Df(z)h(z), \quad z \in \mathbb{B}^n,$$

where $h : \mathbb{B}^n \to \mathbb{C}^n$, $h(0) = 0$, $Dh(0) = I$, is a holomorphic mapping such that

$$\text{Re} \langle h(z), z \rangle > 0, \quad z \in \mathbb{B}^n \setminus \{0\}.$$

The collection of all above mappings $h$ will be denoted by $\mathcal{P}$.

A very useful in research of biholomorphic starlike mappings is the following $n$-dimensional version [10, 13] of planar Jack’s lemma [7]. Some facts on its history can be found in the papers of Boas [2], Fournier [4] and in the monograph of Miller and Mocanu [19].
Lemma 2  \((C^n\text{-Jack’s Lemma})\)

Let \(q : B^n \to \mathbb{C}^n\) be a non-zero holomorphic mapping such that \(q(0) = 0\) and let \(a \in B^n\) be a point for which

\[
\|q(a)\| = \max_{\|z\| \leq \|a\|} \|q(z)\|.
\]

Then there exist numbers \(s \geq m \geq 1\) such that

\[
\langle Dq(a)a, q(a) \rangle = m \|q(a)\|^2, \|Dq(a)a\| = s \|q(a)\|.
\]

Moreover, if \(Dq(0) = 0\), then \(s \geq m \geq 2\). The equality \(s = m\) holds, iff \(n = 1\) or \(Dq(a)a = mq(a)\) for \(n > 1\).

Continuing the first section we give a very useful functional symmetry. In the papers \([3, 6, 11]\) there are considered the consequences of a modification of the above starlikeness factorization (1), using a unique decomposition of mappings \(f : B^n \to \mathbb{C}^n\) with respect to the cyclic group of \(k\)-th. roots of unity, \(k \geq 2\). Below we present such partition for mappings \(f : \Omega \supset \Omega_1 \to Y\), where \(\Omega, \Omega_1\) are normed complex vector spaces and \(\Omega_1\) is a \(k\)-symmetric nonempty subset of \(\Omega\) (\(\varepsilon \Omega = \Omega\) for the generator \(\varepsilon = \exp \frac{2\pi i}{k}\) of the above group) \([14]\).

By \(F_{j,k}(\Omega, Y)\), \(j = 0, ..., k - 1\), let us denote the collection of \((j, k)\)-symmetrical maps \(f : \Omega \to Y\), i.e., maps \(f\) satisfying the condition

\[
f(\varepsilon z) = \varepsilon^j f(z), z \in \Omega.
\]

Lemma 3  \((\text{partition of mappings})\)

For every mapping \(f : \Omega \to Y\), there exists exactly one sequence \(f_{0,k}, ..., f_{k-1,k}\) of mappings \(f_{j,k} \in F_{j,k}(\Omega, Y)\), \(j = 0, 1, ..., k - 1\), such that

\[
f = \sum_{j=0}^{k-1} f_{j,k}.
\]

Moreover,

\[
f_{j,k}(z) = \frac{1}{k} \sum_{l=0}^{k-1} \varepsilon^{-jl} f(\varepsilon^l z), z \in \Omega.
\]

Since the partition is unique, the maps \(f_{j,k}\) are called \((j, k)\)-symmetric components of \(f\). Let us observe that decomposition (3) generalizes the well known fact that every function \(f : \Omega \to Y\) is a sum of even function

\[
f_{0,2}(z) = \frac{1}{2} [f(z) + f(-z)], z \in \Omega
\]

and odd function

\[
f_{1,2}(z) = \frac{1}{2} [f(z) - f(-z)], z \in \Omega.
\]
Here $\varepsilon = -1$ and $\Omega \subset \mathbb{X}$ is a 2-symmetric set. Note that in the complex analysis, the above decomposition can be applied to solve some functional equations [15] to obtain Fourier series in an elementary way [16] and to get a Cartan uniqueness theorem in $\mathbb{C}^n$ [17].

Since the unit Euclidean ball $\mathbb{B}^n$ is $k$-symmetric and the function

$$\mathbb{B}^n \ni z \to Df(z) \in \mathcal{L}(\mathbb{C}^n, \mathbb{C}^n)$$

belongs to $\mathcal{F}_{0,k}(\mathbb{B}^n, \mathcal{L}(\mathbb{C}^n, \mathbb{C}^n))$ for $f \in \mathcal{F}_{1,k} = \mathcal{F}_{1,k}(\mathbb{B}^n, \mathbb{C}^n)$, we obtain that in the starlikeness factorization (1) the $(1,k)$-symmetry of $f$ implies the same for the mapping $h \in \mathcal{P}$. A reverse statement to this observation is more important. An open problem in this direction can be found in the paper [11] and its solution is given in [6].

In the paper [11] there was considered a case where the map $f$ on the right or left side of the starlikeness factorization (1) is replaced by its $(1,k)$-symmetrical part $f_{1,k}$. We recall only one of the possible cases. Let $f : \mathbb{B}^n \to \mathbb{C}^n$, $f(0) = 0$, $Df(0) = I$, be a holomorphic mapping with locally biholomorphic its $(1,k)$-symmetrical part $f_{1,k}$. By [11] $f \in \mathcal{S}_t \cap \mathcal{F}_{1,k}$, iff there exists a mapping $h \in \mathcal{P} \cap \mathcal{F}_{1,k}$ such that

$$f_{1,k}(z) = Df(z) h(z), z \in \mathbb{B}^n. \quad (4)$$

The above facts and an application of the mappings $f \in \mathcal{S}_t \cap \mathcal{F}_{1,k}$ in a proof of the Poincaré theorem, due to Barnard, FitzGerald and Gong [1], is a good motivation to consider the following family of maps introduced in [3].

**Definition** By $\mathcal{S}(k)$, $k \geq 2$, let us denote the family of all locally biholomorphic maps $f : \mathbb{B}^n \to \mathbb{C}^n$, $f(0) = 0$, $Df(0) = I$, satisfying the factorization (4) with $h \in \mathcal{P}$ (without the assumption that $h \in \mathcal{F}_{1,k}$).

Before we give the main theorem, we present some properties of the family $\mathcal{S}(k)$. First let us observe that the family $\mathcal{S}(2)$ is identical with the family $\mathcal{S}_{tS}$ of biholomorphic starlike mappings with respect to symmetric points in $\mathbb{B}^n$ [21], i.e., biholomorphic mappings $f : \mathbb{B}^n \to \mathbb{C}^n$, $f(0) = 0$, $Df(0) = I$, satisfying the condition

$$[f(z), f(-z)] \subset f(\mathbb{B}^n), z \in \mathbb{B}^n.$$  

Here $[f(z), f(-z)]$ means the segment with the ends $f(z), f(-z)$. This follows from the fact [21] that a locally biholomorphic mapping $f : \mathbb{B}^n \to \mathbb{C}^n$, $f(0) = 0$, $Df(0) = I$, belongs to $\mathcal{S}_{tS}$, iff there exists a mapping $h \in \mathcal{P}$ such that

$$f_{1,2}(z) = Df(z) h(z), z \in \mathbb{B}^n.$$  

The equality $\mathcal{S}(2) = \mathcal{S}_{tS}$ is a good geometrical interpretation of the family $\mathcal{S}(2)$. A geometrical interpretation of the family $\mathcal{S}(k)$ for every $k \geq 2$, implies that every $f \in \mathcal{S}(k)$ with $f_{1,k} \in \mathcal{S}_t$, is close-to-starlike (relative to $f_{1,k}$), in the sense of Pfaltzgraff-Suffridge definition [20]. More precisely, such $f$ is biholomorphic and for each $r \in (0, 1)$, the complement of $f(r \mathbb{B}^n)$ is the union of non intersecting rays [20].
Another interesting property of the families $S(k)$ is the fact [3] that the families $S(k)$ and the family $S_c$ of biholomorphic mappings $f : B^n \to \mathbb{C}^n$, $f(0) = 0$, $Df(0) = I$, with convex domains $f(B^n)$, are in some relations. It is known that $S_c$ is an essential subfamily of the family $S(2)$, i.e., there holds the inclusion $S_c \subsetneq S(2)$. However, $S_c \not\subset S(k)$ for $k \geq 3$. Usually, the elements $f$ of the family $S_c$ are called the biholomorphic convex mappings. Note that the family $S_c$ was investigated by many authors; a wide list of papers in this area includes the references in the monographs [5] and [9]. Let us recall also that in the paper [3] there is given the following relationship between the families $S(k)$, $k \geq 2$, and $S_t$:

$$S(k) \cap F_{1,k} = S_t \cap F_{1,k},$$

but

$$S(k) \nsubseteq S_t \nsubseteq S(k).$$

Now we are going to present the main result. The part (i) of the main theorem gives a sufficient condition for the relation $f \in S(k)$, $k \geq 2$. We construct this criterion using the following bounded linear operator $L \in \mathcal{L}(\mathbb{C}^n, \mathbb{C}^n)$:

$$L = D^2 f(a)(b, \cdot), \quad a \in B^n, \quad b \in \mathbb{C}^n,$$

obtained by restricting to $\{b\} \times \mathbb{C}^n$ a symmetric bounded bilinear operator $D^2 f(a)(\cdot, \cdot) : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}^n$, i.e., the second order Fréchet derivative of $f$ at the point $a$. Part (ii) gives an additional property of mappings $f \in \widehat{S(k)}$ defined in part (i).

### 2 The Main Result

**Theorem 1** For a locally biholomorphic map $f : B^n \to \mathbb{C}^n$, $f(0) = 0$, $Df(0) = I$ and $k = 2, 3, \ldots$, the following statements hold.

(i) If $f$ satisfies for $z \in B^n$ the inequality

$$\left\| I - Df(z)^{-1}Df_{1,k}(z) \right\| + (1 + \|z\|) \left\| Df(z)^{-1}D^2 f(z)(z, \cdot) \right\| < 2, \quad (5)$$

then $f$ belongs to the family $S(k)$.

(ii) The collection $\widehat{S(k)}$ of all mappings satisfying the condition (5), is a proper subfamily of the family $S(k)$.

**Proof** At first we prove the statement (i). Denoting

$$h(z) = Df(z)^{-1}f_{1,k}(z), \quad z \in B^n,$$
we see that $h(0) = 0$ and $h$ is holomorphic, because it is defined as a composition of the continuous bilinear operator

$$\mathbb{C}^n \times \mathcal{L}(\mathbb{C}^n, \mathbb{C}^n) \ni (u, L) \mapsto L(u) \in \mathbb{C}^n$$

and two holomorphic mappings

$$\mathbb{B}^n \ni z \mapsto f_{1,k}(z) \in \mathbb{C}^n,$$

$$\mathbb{B}^n \ni z \mapsto Df(z)^{-1} \in \mathcal{L}(\mathbb{C}^n, \mathbb{C}^n).$$

Moreover, the above definition of $h$ guarantees that our $f$ satisfies the Eq. (4). Of course, the right hand side of (4) is a composition of the above bilinear operator and the holomorphic mappings

$$\mathbb{B}^n \ni z \mapsto h(z) \in \mathbb{C}^n,$$

$$\mathbb{B}^n \ni z \mapsto Df(z) \in \mathcal{L}(\mathbb{C}^n, \mathbb{C}^n).$$

Therefore, using also the differentiation chain rule and a generalization of the differentiation product rule, we have

$$Df_{1,k}(z) = Df(z)Dh(z) + D^2f(z)(h(z), \cdot), z \in \mathbb{B}^n,$$

where $D^2f(z)(w, \cdot)$ for $z \in \mathbb{B}^n$ and $w \in \mathbb{C}^n$, means the linear operator defined as above. Thus, by the equalities $Df_{1,k}(0) = I = Df(0)$, we obtain the necessary normalization $Dh(0) = I$.

Therefore, it remains to prove that the mapping $h$ satisfies the condition (2). To this purpose it is sufficient to show that the mapping

$$q(z) = z - h(z), z \in \mathbb{B}^n$$

satisfies the inequality

$$\|q(z)\| \leq \|z\|^2, z \in \mathbb{B}^n. (7)$$

Indeed, for $z \in \mathbb{B}^n \setminus \{0\}$, we obtain step by step

$$\text{Re} \langle h(z), z \rangle = \text{Re} \langle z - q(z), z \rangle \geq \|z\|^2 - \|q(z)\| \|z\| \geq \|z\|^2 - \|z\|^3 > 0.$$

Now we show the inequality (7). To do that, in view of a strong version of the Schwarz lemma in $\mathbb{C}^n$ (see the monograph of Graham and Kohr [5, Chapt. 6]) and the condition $Dq(0) = 0$, it suffices to prove that

$$\|q(z)\| < 1, z \in \mathbb{B}^n. (8)$$
The supposition that (8) does not hold and the normalization $q(0) = 0$, imply that there exists a point $a \in \mathbb{B}^n$ such that 

$$1 = \|q(a)\| = \max_{\|z\| \leq \|a\|} \|q(z)\|.$$ 

Then, the lemma with the normalization $Dq(0) = 0$, gives that 

$$\|Dq(a)a\| \geq 2. \quad (9)$$

To complete the proof, we show the opposite inequality. We start with the equality 

$$f_{1,k}(z) = Df(z)(z - q(z)), z \in \mathbb{B}^n,$$

which follows from (4) and (6). Now, using the same differentiation rules as above, we obtain at $z \in \mathbb{B}^n$ and $w \in \mathbb{C}^n$

$$Df_{1,k}(z)w = Df(z)(I - Dq(z))w + D^2 f(z)(w, z - q(z)).$$

Hence we get

$$Dq(z)w = \left( I - Df(z)^{-1}Df_{1,k}(z) \right) w + Df(z)^{-1}D^2 f(z)(w, z - q(z)).$$

Therefore,

$$\|Dq(z)w\| \leq \left\| \left( I - Df(z)^{-1}Df_{1,k}(z) \right) w \right\| + \| Df(z)^{-1}D^2 f(z)(w, z - q(z)) \|.$$ 

Using the properties of the norm of linear operators, we obtain

$$\|Dq(z)w\| \leq \left\| I - Df(z)^{-1}Df_{1,k}(z) \right\| \|w\| + \| Df(z)^{-1}D^2 f(z)(w, \cdot) \| \|z - q(z)\|.$$ 

Choosing $z = a = w$ yields

$$\|Dq(a)a\| \leq \left\| I - Df(a)^{-1}Df_{1,k}(a) \right\| \|a\|$$

$$+ \| Df(a)^{-1}D^2 f(a)(a, \cdot) \| (\|a\| + \|q(a)\|)$$

and finally,

$$\|Dq(a)a\| \leq \left\| I - Df(a)^{-1}Df_{1,k}(a) \right\| \|a\| + \| Df(a)^{-1}D^2 f(a)(a, \cdot) \| (\|a\| + 1).$$

This and the assumption (5) imply the inequality

$$\|Dq(a)a\| < 2.$$
Since this is opposite to (9), the proof of the statement (i) is complete.

Now we prove the part (ii) of the theorem. To this aim, we will show that the mapping

\[ f(z) = \frac{z}{(1 - z_1^k)^{\frac{1}{k}}}, \quad z = (z_1, ..., z_n) \in \mathbb{B}^n, \]

with the branch of the power function \( x^{\frac{1}{k}} \) such that \( 1^{\frac{1}{2}} = 1 \), belongs to \( S(k) \setminus \widehat{S}(k) \) for every \( k \geq 2 \). Firstly, we prove that \( f \in S(k) \) for every \( k \geq 2 \). To do that let us observe that \( f \) is holomorphic, normalized and \((1, k)\)-symmetrical \((f_1 = f)\). Moreover, for \( z \in \mathbb{B}^n \), the Fréchet derivative of \( f \) at \( z \) has the form

\[
Df(z) = \frac{1}{(1 - z_1^k)^{1 + \frac{1}{k}}} \begin{bmatrix}
1 + z_1^k & 0 & \ldots & 0 \\
2z_1^{k-1}z_2 & 1 - z_1^k & \ldots & 0 \\
\ldots & \ldots & \ldots & \ldots \\
2z_1^{k-1}z_n & 0 & \ldots & 1 - z_1^k \\
\end{bmatrix}, \quad z \in \mathbb{B}^n
\]

Consequently, \( f \) is locally biholomorphic,

\[
Df(z)^{-1} = \frac{(1 - z_1^k)^{\frac{1}{k}}}{1 + z_1^k} \begin{bmatrix}
1 - z_1^k & 0 & \ldots & 0 \\
-2z_1^{k-1}z_2 & 1 + z_1^k & \ldots & 0 \\
\ldots & \ldots & \ldots & \ldots \\
-2z_1^{k-1}z_n & 0 & \ldots & 1 + z_1^k \\
\end{bmatrix}, \quad z \in \mathbb{B}^n
\]

and \( f \) satisfies equality (4) with

\[
h(z) = Df(z)^{-1}f_{1k}(z) = \frac{1 - z_1^k}{1 + z_1^k}, \quad z = (z_1, ..., z_n) \in \mathbb{B}^n.
\]

This shows that \( f \in S(k) \), because \( h \in \mathcal{P} \).

To confirm that \( f \notin \widehat{S}(k), k \geq 2 \), we will show that \( f \) does not satisfy the condition (5).

Using the fact that for \( a \in \mathbb{B}^n, b = (b_1, ..., b_n) \in \mathbb{C}^n \), the linear operator \( D^2 f(a)(b, \cdot) \) has the form

\[
D^2 f(a)(b, \cdot) = \left( \sum_{v=1}^{n} \frac{\partial^2 f_m}{\partial z_\mu \partial z_v} (a) b_v \right)_{1 \leq m, \mu \leq n},
\]

(cf., e.g., the monograph of Kohr and Liczberski [9]) and the fact that \((1, k)\)-symmetry of \( f \) reduces the condition (5) to the inequality

\[
(1 + \|z\|) \left\| Df(z)^{-1}D^2 f(z)(z, \cdot) \right\| < 2.
\]
we prove that for $z \in \mathbb{B}^n$, the norm of the linear operator $Df(z)^{-1} D^2 f(z)(z, \cdot)$ is infinite. To do that we compute firstly that

$$D^2 f(z)(z, \cdot) = \frac{2z^{k-1}}{(1-z^k)^{2+\frac{1}{2}}} \begin{bmatrix} z_1 (k+1+z_1^k) & 0 & \cdots & 0 \\ z_2 (k+2z_1^k) & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ z_n (k+2z_1^k) & 0 & \cdots & z_1 (1-z_1^k) \end{bmatrix}.$$ 

Therefore, for the points $z = (r, 0, \ldots, 0) \in \mathbb{B}^n$, we get

$$Df(z)^{-1} D^2 f(z)(z, \cdot) = \frac{2r^k}{(1+r^k)(1-r^k)} \begin{bmatrix} (k+1+r^k) & 0 & \cdots & 0 \\ 0 & 1+r^k & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1+r^k \end{bmatrix}.$$ 

Since

$$\|Df(z)^{-1} D^2 f(z)(z, \cdot)\| = \max_{\|u\|=1} \frac{\|Df(z)^{-1} D^2 f(z)(z, u)\|}{\|u\|}, u = (u_1, \ldots, u_n) \in \mathbb{C}^n,$$

we obtain for $z = (r, 0, \ldots, 0) \in \mathbb{B}^n$,

$$\|Df(z)^{-1} D^2 f(z)(z, \cdot)\| = \frac{2r^k}{1-r^k} \max_{\|u\|=1} \left( \frac{k+1+r^k}{1+r^k} \right)^2 |u_1|^2 + \sum_{\nu=2}^{n} |u_{\nu}|^2.$$ 

Thus, we conclude that for $\mathbb{B}^n \ni z = (r, 0, \ldots, 0) \to (1, 0, \ldots, 0)$,

$$\|Df(z)^{-1} D^2 f(z)(z, \cdot)\| \geq \frac{2r^k}{1-r^k} \max_{\|u\|=1} \sum_{\nu=2}^{n} |u_{\nu}|^2 = \frac{2r^k}{1-r^k} \to \infty.$$ 

Consequently, the inequality (5) is false for the above mapping $f$.

This completes the proof of the thesis $(ii)$.

### 3 Final Remarks

1. It is possible to consider the case $k = 1$ in the Theorem 1. Then we use the convention that $\varepsilon = 1$, hence $S(1) = St$, by (4). Therefore, the main Theorem 1 will take the following form.

**Theorem 2** For a locally biholomorphic map $f : \mathbb{B}^n \to \mathbb{C}^n$ such that $f(0) = 0$ and $Df(0) = I$, the following statements hold:
(i) If $f$ satisfies the condition
\[(1 + \|z\|) \left\| Df(z)^{-1} D^2 f(z)(z, \cdot) \right\| < 2, z \in \mathbb{B}^n, \tag{5*} \]
then $f$ belongs to $S(1) = St$.

(ii) The collection $\hat{S}(1)$ of all mappings satisfying the condition (5*), is a proper subfamily of the family $S(1) = St$.

2. The part (i) of Theorem 2 improves the result from the paper [12], where the stronger assumption
\[(1 + \|z\|) \left\| Df(z)^{-1} D^2 f(z)(z, \cdot) \right\| < 1, z \in \mathbb{B}^n, \]
gave the same thesis $f \in S(1) = St$ as in the main theorem from the article [12]. That was possible thanks to the use in the present paper a newer $n$-dimensional version of Jack’s lemma from [13], instead of its older version from the paper [10] used in [12].

3. The part (ii) of Theorem 2 was not included in the main theorem of the paper [12].

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References

1. Barnard, R.W., FitzGerald, C.H., Gong, S.: The growth and 1/4 theorem for starlike mappings in $\mathbb{C}^n$. Pacific J. Math. 150, 13–22 (1991)
2. Boas, H.P.: Julius and Julia: mastering the art of the Schwarz lemma. Am. Math. Monthly 127, 770–785 (2010)
3. Długosz, R., Liczberski, P.: Relations among starlikeness, convexity and k-fold symmetry of locally biholomorphic mappings in $\mathbb{C}^n$. J. Math. Anal. Appl. 450, 169–179 (2017)
4. Fournier, F.: On a new proof and an extension of Jack’s Lemma. J. Appl. Anal. 23, 21–24 (2017)
5. Graham, I., Kohr, G.: Geometric Function Theory in One and Higher Dimensions. Marcel Dekker INC, New York (2003)
6. Hamada, H., Kohr, G.: k-fold symmetrical mappings and Loewner chains. Demonstratio Math. 40, 85–94 (2007)
7. Jack, I.S.: Functions starlike and convex of order $\alpha$. J. Lond. Math. Soc. 3, 469–474 (1971)
8. Kikuchi, K.: Starlike and convex mappings in several complex variables. Pacific J. Math. 44, 569–580 (1973)
9. Kohr, G., Liczberski, P.: Univalent Mappings of Several Complex Variables. Cluj Univ. Press, Cluj-Napoca (1998)
10. Liczberski, P.: Jack’s Lemma for holomorphic mappings in $\mathbb{C}^n$. Ann. Univ. Mariae Curie-Skłodowska Sect. A 15, 131–139 (1986)
11. Liczberski, P.: Applications of a decomposition of holomorphic mappings in $\mathbb{C}^n$ with respect to a cyclic group. J. Math. Anal. Appl. 281, 276–286 (2003)
12. Liczberski, P.: A starlikeness criterion for holomorphic mappings in $\mathbb{C}^n$. Complex Variables 25, 193–195 (1994)
13. Liczberski, P.: Geometric properties of some classes of holomorphic mappings in $\mathbb{C}^n$. Sci. Bull. Łódź Tech. Univ. 826 (in Polish). (1999)
14. Liczberski, P., Polubiński, J.: On $(j, k)$–symmetrical functions. Math. Bohemica 120, 13–28 (1995)
15. Liczberski, P., Polubiński, J.: Functions $(j, k)$–symmetrical and functional equations with iterates of the unknown function. Publ. Math. Debrecen 60, 291–305 (2002)
16. Liczberski, P., Polubiński, J.: Symmetrical series expansion of complex valued functions. N. Z. J. Math. 27, 245–253 (1998)
17. Liczberski, P., Polubiński, J.: A uniqueness theorem of Cartan–Gutzmer type for holomorphic mappings in $\mathbb{C}^n$. Ann. Polon. Math. 79, 121–127 (2002)
18. Matsuno, T.: Star-like and convex-like theorems in the complex space. Sci. Rep. Tokyo Kyoiku Daigaku Sect. A 5, 88–95 (1955)
19. Miller, S.S., Mocanu, P.T.: Differential Subordinations. Theory and Applications. Marcell Dekker INC, New York (2000)
20. Pfaltzgraff, J.A., Suffridge, T.J.: Close-to-starlike holomorphic functions of several variables. Pacific J. Math. 57, 271–279 (1975)
21. Parvatham, R., Srinivasan, S.: Starlike functions with respect to symmetric points in $\mathbb{C}^n$. Sochoow J. Math. 20, 257–263 (1994)
22. Sakaguchi, K.: On a certain univalent mapping. J. Math. Soc. Japan 11, 72–75 (1959)
23. Suffridge, T.J.: The principle of subordination applied to functions of several variables. Pacific J. Math. 33, 241–248 (1970)

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