REGULAR PROJECTIONS AND REGULAR COVERS IN O-MINIMAL STRUCTURES

M’HAMMED OUDRANE

Abstract. In this paper we prove that for any definable subset \( X \subset \mathbb{R}^n \) in a polynomially bounded o-minimal structure, with \( \dim(X) < n \), there is a finite set of regular projections (in the sense of Mostowski). We give also a weak version of this theorem in any o-minimal structure, and we give a counter example in o-minimal structures that are not polynomially bounded. As an application we show that in any o-minimal structure there exist a regular cover in the sense of Parusinski.

Introduction:
The regular projection theorem was first stated by Mostowski (see [11]) for complex analytic hypersurfaces, and then later a sub-analytic version was proved by Parusinski (see [13]). The theorem states that for any compact sub-analytic set \( X \subset \mathbb{R}^n \) there exists a finite set of regular projections (see Definition 2.1 below). In [13], Parusinski showed that a generic choice of \( n + 1 \) projections is sufficient. The theorem has many application, it leads to the proof of some important metric proprieties of sub-analytic sets, it has been used by Parusinski in [13] to prove the existence of Lipschitz stratification of sub-analytic sets, more precisely to show that every compact sub-analytic set can be decomposed in a finite union of L-regular sets in such a way that it is easy to glue Lipschitz stratifications of the pieces. Different ways to prove the existence of L-regular decomposition in any o-minimal structures were given by Kurdyka [5], Kurdyka and Parusinski [4], and Pawłucki [16]. The regular projection theorem has been also used recently by Parusński in [13] to prove the existence of regular cover for sub-analytic relatively compact open subsets which is used in [3] and [6] to construct Sobolev sheaves on the sub-analytic site.

In [14], Parusinski asked if the regular projection theorem and the existence of regular covers can be shown in any o-minimal structures. A first answer was given by Nguyen [12], he gives a Lipschitz version of this theorem, means that we can always find a finite set of regular projections after applying a bilipschitz homeomorphism of the ambient space. Nguyen’s proof is based on cell decomposition and Valette’s result on the existence of a good direction (see [17]). A direct consequence of this is the existence of regular covers in any o-minimal structure.

In this paper we give an answer to Parusński’s question, we will prove that the regular projection theorem works in any polynomially bounded o-minimal structure, and we will give an example showing that this is no longer true in non-polynomially bounded o-minimal structures. We will also give a weak version of the regular projection theorem in any o-minimal structure, which is a straightforward consequence of the techniques used in [12]. We will use this weak version to adopt the proof in [14] to show the existence of regular covers in any o-minimal structure. Our fundamental tools for the proof will be the cell decomposition and Miller’s result ([9]) to find a replacement for the Puiseux with parameter argument used in [13].

Key words and phrases. O-minimal geometry, metric proprieties of definable sets in o-minimal structures, regular projection, regular definable covers.
1. Definitions, terminology, notation:

1.1. Notations:
- \( \mathcal{P}(X) \): the set of subsets of \( X \).
- \( \text{grad}(f) \): is the gradient of a \( C^1 \) function \( f \).
- If \( f : A \times B \to C \) is a map, then for \( b \in B \) we denote by \( f(., b) \) (or also by \( f_b \)) the map,
  \[
  f(., b) : A \to C \quad x \mapsto f(x, b)
  \]
- \( B(v, r) \) is the open ball of radius \( r \) and center \( v \), and \( \overline{B}(v, r) \) is the closed ball of radius \( r \) and center \( v \).
- \( \text{Reg}^p(X) \): the set of points \( x \in X \) such that \( X \) is a \( C^p \) manifold near \( x \).
- For \( v \in \mathbb{R}^{n-1} \), \( \pi_v : \mathbb{R}^n \to \mathbb{R}^{n-1} \) is the linear projection parallel to \( \text{Vect}((v, 1)) \).
- For a set \( A \subset \mathbb{R}^n \times \mathbb{R}^m \), for \( x_0 \in \mathbb{R}^n \), we denote by \( A_{x_0} \) the set:
  \[
  A_{x_0} = \{ y \in \mathbb{R}^m : (x_0, y) \in A \}
  \]
- If \( F : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R} \) is \( C^1 \), we denote by \( D_1 F \) the differential of \( F \) with respect to the first variable, or we denote it also by \( D_y F \), and we denote by \( D_x F \) or \( D_2 F \) the differential of \( F \) with respect to the second variable.
- \( \mathcal{A} \) denotes the topological closure of \( A \).
- For a set \( U \subset \mathbb{R}^n \), \( \partial U \) denotes the frontier of \( U \), i.e. \( \partial U = \overline{U} \setminus U \).
- \( B \subset \mathbb{R}^k \) is called an open (closed) box if it can written as a product:
  \[
  B = I_1 \times \ldots \times I_k,
  \]
  where the \( I_i \) are open (closed) intervals in \( \mathbb{R} \).
- We denote by \( \mathbb{N} \) the set of nonnegative integers.
- If \( A \subset \mathbb{R} \), we denote by \( A_+ \), \( A_- \), and \( A^* \) as the following:
  \[
  A_+ = \{ x \in A : x \geq 0 \}, \\
  A_- = \{ x \in A : x \leq 0 \}, \\
  A^* = \{ x \in A : x \neq 0 \}
  \]
  And we denote by \( A^*_+ := A_+ \cap A^* \) and \( A^*_- := A_- \cap A^* \).
- A map \( f : \mathbb{R} \to \mathbb{R} \) is said to be ultimately equal to 0 if there is \( M \in \mathbb{R} \) such that \( f(x) = 0 \) for all \( x > M \).

1.2. O-minimal structures: An o-minimal structure on the field \((\mathbb{R}, +, .)\) is a family \( \mathcal{D} = (\mathcal{D}_n)_{n \in \mathbb{N}} \) such that for any \( n \), we have:
- (1) \( \mathcal{D}_n \subset \mathcal{P}(\mathbb{R}^n) \) is stable by complement and finite union.
- (2) For any \( P \in \mathbb{R}[X_1, \ldots, X_n] \), we have \( Z(P) \in \mathcal{D}_n \), where \( Z(P) = \{ x \in \mathbb{R}^n : P(x) = 0 \} \).
- (3) \( \pi(\mathcal{D}_n) \subset \mathcal{D}_{n-1} \), where \( \pi : \mathbb{R}^n \to \mathbb{R}^{n-1} \) is the standard projection.
- (4) For any \( A \in \mathcal{D}_1 \), \( A \) is a finite union of points and intervals.

For a fixed o-minimal structure \( \mathcal{D} \), we have the definitions:
- (a) Elements of \( \mathcal{D}_n \) are called definable sets.
- (b) If \( A \in \mathcal{D}_n \) and \( B \in \mathcal{D}_m \), then a map \( f : A \to B \) is called a definable map if its graph is a definable set.
- (c) The o-minimal structure \( \mathcal{D} \) is said to be polynomially bounded if for each definable function \( f : \mathbb{R} \to \mathbb{R} \), there is some \( n \in \mathbb{N} \) such that \( |f(x)| \leq x^n \) for \( x \) big enough.
- (d) For \( p \in \mathbb{N} \cup \{\infty\} \) a definable \( C^p \)-manifold is a manifold with finite atlas of definable transition maps and definable domains, see e.g. [2]. Most of the notions from the \( C^p \) manifolds can be extended to definable \( C^p \) manifolds in an obvious way.
Theorem 1.1. (Miller’s Dichotomy [10]). An o-minimal structure $\mathcal{D}$ is either polynomially bounded or contains the graph of the exponential function.

1.3. Cell decomposition. Take $p \in \mathbb{N}$, a definable set $C$ in $\mathbb{R}^n$ is said to be a $C^p$-cell if:

- case $n = 1$: $C$ is either a point or an open interval.
- case $n \geq 2$: $C$ is one of the following:
  - $C = \Gamma_\phi$ (the graph of $\phi$), where $\phi : B \to \mathbb{R}$ is a $C^p$ definable function, where $B$ is $C^p$-cell in $\mathbb{R}^{n-1}$.
  - $C = \Gamma(\phi, \varphi) = \{(x, y) \in B \times \mathbb{R} : \phi(x) < y < \varphi(x)\}$, where $\phi$ and $\varphi$ are two $C^p$ definable functions on a $C^p$-cell $B$, such that $\phi < \varphi$ with the possibility of $\phi = -\infty$ or $\varphi = +\infty$.

A $C^p$-cell decomposition of $\mathbb{R}^n$ is defined by induction as follows:

- A $C^p$-cell decomposition of $\mathbb{R}$ is finite partition by points and open intervals.
- A $C^p$-cell decomposition of $\mathbb{R}^n$ is a finite partition $A$ of $\mathbb{R}^n$ by $C^p$-cells, such that $\pi(A)$ is a $C^p$-cell decomposition of $\mathbb{R}^{n-1}$, where $\pi : \mathbb{R}^n \to \mathbb{R}^{n-1}$ is the standard projection and $\pi(A)$ is the family:
  $$\pi(A) = \{\pi(A) : A \in A\}.$$  

Theorem 1.2. Let $p \in \mathbb{N}$ and $\{X_1, \ldots, X_n\}$ be a finite family of definable sets of $\mathbb{R}^n$. Then there is a $C^p$-cell decomposition of $\mathbb{R}^n$ compatible with this family, i.e. each $X_i$ is a union of some cells.

Proof. See [1] or [13].

Now we can define the dimension of a definable set. Take $X$ a definable subset of $\mathbb{R}^n$ and $\mathcal{C}$ a cell decomposition of $\mathbb{R}^n$ compatible with $X$, then we define the dimension:

$$\dim_{\mathcal{C}}(X) = \max\{\dim(C) : C \subset X \text{ and } C \in \mathcal{C}\}.$$  

This number does not depend on $\mathcal{C}$, we denote it by $\dim(X)$.

2. Regular projection theorem

Let $X \subset \mathbb{R}^n$ be a definable subset of $\mathbb{R}^n$. For any $\lambda \in \mathbb{R}^{n-1}$, we denote by $\pi_\lambda : \mathbb{R}^n \to \mathbb{R}^{n-1}$, the projection parallel to the vector $(\lambda, 1) \in \mathbb{R}^n$. Fix $\varepsilon$ and $C$ positive real numbers and $p \in \mathbb{N}^*$. For $v \in \mathbb{R}^{n-1}$ and $x \in \mathbb{R}^n$ we define the cone $C_\varepsilon(x, v)$ by:

$$C_\varepsilon(x, v) = \{x + t(v', 1) : t \in \mathbb{R}^* \text{ and } v' \in B(v, \varepsilon)\}.$$  

Definition 2.1. We say that the projection $\pi_\lambda$ is $(\varepsilon, p)$-weak regular at a point $x \in \mathbb{R}^n$ (with respect to $X$) if:

1. $(\pi_\lambda)_X$ is finite.
2. the intersection of $X$ with the cone $C_\varepsilon(x, \lambda)$ is either empty or a disjoint finite union of sets of the form:
   $$A_{f_i} = \{x + f_i(\lambda')(\lambda', 1) : \lambda' \in B(\lambda, \varepsilon)\},$$  

   where $f_i$ are non-vanishing $C^p$ functions defined on $B(\lambda, \varepsilon)$.

   We say that the projection $\pi_\lambda$ is $(\varepsilon, C, p)$-regular at a point $x \in \mathbb{R}^n$ (with respect to $X$) if moreover we have:

3. $\|\text{grad}(f_i)\| \leq C \mid f_i \mid$ on $B(\lambda, \varepsilon)$ for all $i$.

Remark 2.2. The notion of regular projection introduced by Mostowski in [11] is not exactly the same as the notion of good direction (regular projection in the sense of Valette in [17], see also [12] for comparing both notions). One can show that a projection is regular in the sense of Mostowski if and only if it is weak regular and good.
2.1. Weak regular projection theorem: Let $A$ be an o-minimal structure on $\mathbb{R}$ (no condition on $A$).

**Theorem 2.3.** Let $X$ be a definable subset of $\mathbb{R}^n$ such that $\text{dim}(X) \leq n - 1$, and $p \in \mathbb{N}^*$. Then there is $\varepsilon > 0$ and $\{v_1, ..., v_k\} \subset \mathbb{R}^{n-1}$ such that for every $x \in \mathbb{R}^n$ there is $i$ such that $\pi_{v_i}$ is $(\varepsilon, p)$-weak regular at $x$ with respect to $X$.

**Proof.** For the proof we need a few lemmas.

**Lemma 2.4.** Take $C$ a definable subset of $\mathbb{R}^n$, and let $B$ to be a box in $\mathbb{R}^k$. Let $\Delta$ be a definable subset of $\mathbb{R}^n \times \mathbb{R}^k$ such that $\text{dim}(\Delta_x) \leq (k-1)$, for all $x \in C$. Then there exists a finite definable partition $C$ of $C$, such that for each $D \in C$ there is a box $B_D \subset B$ such that we have:

$$(D \times B_D) \cap \Delta = \emptyset.$$  

**Proof.** See [12].

**Remark 2.5.** We can easily extend Lemma 2.4 to the case of $C^p$-definable manifolds (we can replace $\mathbb{R}^k$ and $\mathbb{R}^n$ by definable manifolds) by choosing charts and reducing to the case of the last lemma.

**Lemma 2.6.** Let $M$, $J$, and $N$ be definable manifolds, and let $f : M \times J \rightarrow N$ be a $C^1$ definable submersion. Take $S$ a finite collection of $C^1$ definable submanifolds of $N$, then:

$$\rho(f, S) = \{ s \in J : f(., s) \text{ is transverse to all elements of } S \}$$

is a definable set and we have: $\text{dim}(J \setminus \rho(f, S)) < \text{dim}(J)$.

**Proof.** See [17, Lemma 3].

Take $m \in \mathbb{N}$. Let $P_X(x, v, \varepsilon)$ be a property on $(x, v, \varepsilon) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}_+^*$, i.e a map $P : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}_+^* \rightarrow \{0, 1\}$. Denote by $\pi : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \rightarrow \mathbb{R}^n \times \mathbb{R}^m$ the standard projection.

**Lemma 2.7.** If we have:

1. for all $(x,v) \in \mathbb{R}^n \times \mathbb{R}^m$, $\varepsilon$ and $\varepsilon'$ such that $\varepsilon < \varepsilon'$, we have:

$$P_X(x, v, \varepsilon') \text{ is true } \Rightarrow P_X(x, v, \varepsilon) \text{ is true}.$$  

2. $\mathcal{X} = \{(x, v, \varepsilon) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}_+^* : P_X(x, v, \varepsilon) \text{ is true}\}$ is definable and for all $x \in \mathbb{R}^n$ $\dim(B_x) < m$, where $B$ is defined by:

$$B = \{(x, v) \in \mathbb{R}^n \times \mathbb{R}^m : \pi^{-1}(x, v) \cap \mathcal{X} = \emptyset\}.$$  

Then there is $\varepsilon_0 > 0$ and $\mathcal{A} = \{v_1, ..., v_k\} \subset \mathbb{R}^m$ such that for all $x \in \mathbb{R}^n$ there is some $i$ such that $P_X(x, v_i, \varepsilon_0)$ is true.

**Proof.** By the assumptions (1) and (2), the function $F : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}_+$ given by:

$$F(x, v) = \begin{cases} \sup \{\varepsilon : P_X(x, v, \varepsilon) \text{ is true}\} & \text{if } \{\varepsilon : P_X(x, v, \varepsilon) \text{ is true}\} \neq \emptyset \\ 0 & \text{if } \{\varepsilon : P_X(x, v, \varepsilon) \text{ is true}\} = \emptyset \end{cases}$$

is well defined and definable.

Now take a cell decomposition $\mathcal{D}$ of $\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}$ compatible with $\mathbb{R}^n \times \mathbb{R}^m \times \{0\}$, $B \times \{0\}$ and $\Gamma_F$ (the graph of $F$).

If $\mathcal{D}_{n+m}$ is the cell decomposition of $\mathbb{R}^n \times \mathbb{R}^m$ determined by $\mathcal{D}$ (means that $\pi(D) = \mathcal{D}_{n+m}$), then for any $C \in \mathcal{D}_{n+m}$ of maximal dimension, some of the bands over $C$ (at least one) are in $\mathcal{X}$ and the others in $\mathcal{X}^c$. Hence by Lemma 2.4 we can find a cell decomposition $\mathcal{D}'_{n+m}$ finer than $\mathcal{D}_{n+m}$ such that for all $C \in \mathcal{D}'_{n+m}$ with $\dim(C) = n + m$ there is some
box $[a_C, b_C] \subset \mathbb{R}^+$ such that: $C \times [a_C, b_C] \subset \mathcal{X}$.

Now since $\dim(B_x) < m$ for all $x \in \mathbb{R}^n$, we can apply Lemma 2.4 again and find a partition $\mathcal{P}$ of $\mathbb{R}^n$ with the fact that for all $P \in \mathcal{P}$ there is a box $B_P = [v_P(1), v'_P(1)] \times ... \times [v_P(m), v'_P(m)] \subset \mathbb{R}^m$ such that $P \times B_P$ is included in some band $C_P$ of $\mathcal{D}'_{n+m}$, hence:

$$P \times B_P \times [a_{C_P}, b_{C_P}] \subset \mathcal{X}.$$ 

Finally we can take $\mathcal{A} = \{v_P = (v_P(1), ..., v_P(m))\}_{P \in \mathcal{P}}$ and $\varepsilon_0 = \min_{P \in \mathcal{P}}\{a_{C_P}\}$. \hfill $\square$

Now we define $P_X(x, v, \varepsilon)$ as follows:

$$P_X(x, v, \varepsilon) \text{ is true } \iff \pi_v \text{ is } (\varepsilon, p)\text{-weak regular at } x \text{ with respect to } X.$$ 

It’s obvious that $P_X$ satisfies (1) of the precedent lemma. Let’s prove that also (2) holds true for $P_X$. We have:

$$\mathcal{X} = \{(x, v, \varepsilon) : \pi_v \text{ is } \varepsilon - \text{weak regular at } x \text{ with respect to } X\}$$

$$= \mathcal{X}_1 \cup \mathcal{X}_2$$

where $\mathcal{X}_1$ and $\mathcal{X}_2$ are defined below.

$$\mathcal{X}_1 = \{(x, v, \varepsilon) : C_\varepsilon(x, v) \cap X = \emptyset\}$$

$$= \{(x, v, \varepsilon) : \forall t \in \mathbb{R}^n \text{ and } \forall v' \text{ such that } \|v - v'\| < \varepsilon : x + t(v', 1) \notin X\}$$

Hence $\mathcal{X}_1$ is defined by a first order formula, and therefore $\mathcal{X}_1$ is a definable set.

$$\mathcal{X}_2 = \{(x, v, \varepsilon) : \text{ such that } C_\varepsilon(x, v) \cap X \neq \emptyset, C_\varepsilon(x, v) \cap X \subset \text{Reg}^p(X), \text{ and } \forall y \in C_\varepsilon(x, v) \cap X, \text{ there is } t \in \mathbb{R}^n \text{ and } v' \in B(v, \varepsilon) \text{ with } y = x + t(v', 1) \text{ and the line through } x \text{ directed by } (v', 1) \text{ is transverse to } X \text{ at } y\}.$$ 

Hence $\mathcal{X}_2$ is also defined by a first order formula, thus it’s a definable set. Finally $\mathcal{X}$ is definable set. Let’s prove that $\dim(B_x) < n - 1$ for all $x \in \mathbb{R}^n$. Assume that this is not true. For $x \in \mathbb{R}^n$ and $v \in \mathbb{R}^{n-1}$ we denote by $L(x, v)$ the line that passes through $x$ and is directed by $(v, 1)$. For all $x \in \mathbb{R}^n$ we have:

$$B_x = B^1_x \cup B^2_x,$$

where:

$$B^1_x = \{v \in \mathbb{R}^{n-1} : \text{ such that } L(x, v) \text{ is not transverse to } \text{Reg}^p(X)\}$$

and:

$$B^2_x = \{v \in \mathbb{R}^{n-1} : L(x, v) \cap \text{Sing}^p(X) \neq \emptyset\}.$$ 

We define the definable map:

$$\phi : \mathbb{R}^{n-1} \times \mathbb{R}^* \to \mathbb{R}^n$$

$$(v, t) \mapsto \phi(v, t) = x_0 + t(v, 1).$$

A simple calculation shows that $\phi$ is a submersion, hence it is a local diffeomorphism, but since $\dim(\text{Sing}^p(X)) < (n - 1)$, we can deduce that:

$$\dim(\phi^{-1}(\text{Sing}^p(X))) < n - 1,$$

therefore this means that:
\[ \dim(\pi(\phi^{-1}(\text{Sing}^p(X)))) < n - 1, \]

hence:

\[ \dim(B^2_x) < n - 1. \]

But by our assumption there must be an \( x_0 \in \mathbb{R}^n \) such that: \( \dim(B^1_{x_0}) = n - 1 \), and

\[ B^1_{x_0} = \{ v \in \mathbb{R}^{n-1} : \text{such that } \phi(v,.) \text{ is not transverse to } \text{Reg}^p(X) \}. \]

Finally, by Lemma 2.6 we deduce that \( \dim(B^1_{x_0}) < n - 1 \), thus this is a contradiction with \( \dim(B^1_{x_0}) = n - 1 \). This complete the proof of Theorem 2.3.

\[ \square \]

**Remark 2.8.** (i) By the proof of Theorem 2.3 we can require \( \{v_1,...,v_k\} \) to be from an open subset of \((\mathbb{R}^{n-1})^k\).

(ii) If \( \dim(X) < (n - 1) \), then we can see that \( X_2 = \emptyset \), hence this means that we can find a finite set of projections \( \{v_1,...,v_k\} \subset \mathbb{R}^{n-1} \) and \( \varepsilon > 0 \) such that for any \( x \in \mathbb{R}^n \) there is an \( i \) such that:

\[ C_\varepsilon(x,v_i) \cap X = \emptyset. \]

**Question 1.** From the proof of Theorem 2.3 we can minimize the number of the projections by:

\[ k_m = \min\{k \in \mathbb{N} : k = |P| \text{ where } P \text{ is a partition of } \mathbb{R}^n \text{ such that for all } P \in \mathcal{P} \text{ there is a box } B_P = [v_P(1),v_P(1)] \times \ldots \times [v_P(m),v_P(m)] \subset \mathbb{R}^n \text{ such that } P \times B_P \text{ is included in some band } C_P \text{ of } D_{a+m} \}. \]

Clearly this number depends by \( n \) and \( X \), therefore we ask the question:

Is there a way to prove that the number of projections can be chosen independently of \( X \)?

**Question 2.** Can the set of projections be chosen in an open definable dense subset of \((\mathbb{R}^{n-1})^k\)?

2.2. Counter example in non-polynomially bounded o-minimal structures: Fix \( \mathcal{A} \) a non-polynomially bounded o-minimal structure on \((\mathbb{R},+.)\). Assume that the Regular projection theorem is true for this structure, and consider the set \( X \) defined by:

\[ X = X_1 \cup X_2, \]

where:

\[ X_1 = \{(x,axa+1,x^{a+1}) : x > 0 \text{ and } a \in \mathbb{R}\} \]

\[ X_2 = \{(x,-axa+1,x^{a+1}) : x > 0 \text{ and } a \in \mathbb{R}\}. \]

For all \( p \in \mathbb{N}^* \) we have: \( \text{Reg}^p(X) = X \setminus \{(x,0,x) : x > 0\} \), and the connected components of \( \text{Reg}^p(X) \) are \( C^p \)-manifolds.

By Miller’s Dichotomy Theorem [10], the graph of the exponential function \( x \rightarrow \exp(x) \) is a definable set, and therefore \( X \) is a definable set.

Take the germ of the curve \( x(s) = (s,0,0) \). By cell decomposition and the assumption there are \( \delta > 0 \), a vector \( v = (v_1,v_2) \in \mathbb{R}^2 \), \( \varepsilon > 0 \) and \( C > 0 \) such that for all \( s \in [0,\delta] \) we have one of the following two cases:

1. \( C_\varepsilon(x(s),v) \cap X = \emptyset \).
2. \( C_\varepsilon(x(s),v) \cap X = \sqcup A_{f^i_s} \), where \( f^i_s(v,\varepsilon) \rightarrow \mathbb{R}^* \) are \( C^p \)-regular definable functions such that for all \( s, i, \) and \( \lambda \in B(v,\varepsilon) \) we have:

\[ \frac{\|\text{grad } f^i_s(\lambda)\|}{|f^i_s(\lambda)|} \leq C. \]

We are interested in the second case, because for \( s \) small enough \( C_\varepsilon(x(s),v) \cap X \neq \emptyset \).

To obtain a contradiction, let’s prove the next two facts:
**Fact 1**: For $\lambda = (\lambda_1, \lambda_2) \in B(v, \varepsilon)$, the functions $s \mapsto f^i_s(\lambda)$ are characterized by the functional equation:

$$f^i_s(\lambda) = (s + \lambda_1 f^i_s(\lambda))^{\pm \lambda_2 + 1}.$$  

Indeed, take $s \in [0, \delta]$. Since $x(s) + f^i_s(\lambda)(\lambda, 1) \in C_x(x(s), v) \cap X$, there is some $x \in \mathbb{R}_+$ and $a \in \mathbb{R}$ such that:

$$x(s) + f^i_s(\lambda)(\lambda, 1) = (x, \pm ax^{\alpha+1}, xa^{\alpha+1}).$$

Hence:

$$x = s + f^i_s(\lambda)\lambda_1$$

$$f^i_s(\lambda) = x^{\alpha+1},$$

and this implies:

$$\lambda_2 = \pm a$$

and

$$f^i_s(\lambda) = (s + \lambda_1 f^i_s(\lambda))^{\pm \lambda_2 + 1}.$$  

But since the sets $\mathcal{A}_{f^i_s}$ are connected definable $C^p$-manifolds and the connected components of $\text{Reg}^p(X)$ are $C^p$-definable manifolds we deduce that the sign $\pm$ on $\lambda_2$ depends only on $i$. Reciprocally, take $\lambda \in B(v, \varepsilon)$ and take a definable continuous function $s \mapsto t_s \in \mathbb{R}_+$ such that:

$$t_s = (s + \lambda_1 t_s)^{\pm \lambda_2 + 1}.$$  

Fix $s \in [0, \delta]$, then we have:

$$x(s) + t_s(\lambda, 1) \in C_x(x(s), v).$$

Let’s take

$$x = s + \lambda_1 t_s$$

and

$$a = \pm \lambda_2.$$  

Then

$$x(s) + t_s(\lambda, 1) = (x, \pm ax^{\alpha+1}, xa^{\alpha+1}).$$

Hence

$$x(s) + t_s(\lambda, 1) \in C_x(x(s), v) \cap X.$$  

This implies that there is $i_s$ and $\lambda' \in B(v, \varepsilon)$ such that:

$$x(s) + t_s(\lambda, 1) = x(s) + f^i_s(\lambda')(\lambda', 1).$$

And this implies that $\lambda = \lambda'$ and $t_s = f^i_s(\lambda)$.

By the fact that $f^1_s < \ldots < f^k_s$, and the functions $s \mapsto t_s$ and $s \mapsto f^i_s(\lambda)$ are continuous and definable, we deduce that $i_s$ doesn’t depend on $s$, hence $s \mapsto t_s$ is one of the functions $s \mapsto f^i_s(\lambda)$.

This finishes the proof of **Fact 1**.

**Fact 2**:

(*) If $\lambda = (\lambda_1, \lambda_2) \in B(v, \varepsilon)$ such that $\lambda_2 > 0$, then there is a solution $s \mapsto t_s$ of the equation:

$$t_s = (s + \lambda_1 t_s)^{\lambda_2 + 1}$$

such that $\lim_{s \to 0} t_s = 0$.

For that, we define the function $F : \mathbb{R}^2 \rightarrow \mathbb{R}$ by:

$$F(s, t) = \begin{cases} t - (s + \lambda_1 t)^{\lambda_2 + 1} & \text{if } s + \lambda_1 t > 0 \\ t & \text{if } s + \lambda_1 t \leq 0. \end{cases}$$

$F$ is a $C^1$ function and we have:

$$F(0, 0) = 0 \text{ and } \frac{\partial F}{\partial t}(0, 0) = 1.$$
Hence by the Implicit Function theorem, there is a continuous function $s \mapsto t(s)$ such that $t(0) = 0$ and $F(s,t(s)) = 0$.

(●) If $\lambda \in B(v, \varepsilon)$ such that $\lambda_2 < 0$, then there is a solution $s \mapsto t_s$ for the equation:

$$t_s = (s + \lambda_1 t_s)^{-\lambda_2 + 1}$$

such that $\lim_{s \to 0} t_s = 0$. We use the same argument as in the first case by applying the Implicit Function theorem to the function:

$$F(s,t) = \begin{cases} t - (s + \lambda_1 t)^{-\lambda_2 + 1} & \text{if } s + \lambda_1 t > 0 \\ t & \text{if } s + \lambda_1 t \leq 0. \end{cases}$$

Now we will discuss the projection in two cases:

case 1: Assume that $v_2 \geq 0$, and take $\lambda = (\lambda_1, \lambda_2) \in B(v, \varepsilon)$ such that $\lambda_2 > 0$. From Facts 1 and 2 there is an $i_0$ such that $\lim_{s \to 0} f_s^{i_0}(\lambda) = 0$ and:

$$f_s^{i_0}(\lambda) = (s + \lambda_1 f_s^{i_0}(\lambda))^{\lambda_2 + 1}.$$

Therefore we have:

$$\frac{\partial f_s^{i_0}(\lambda)}{\partial \lambda_2}(\lambda) = (\ln(s + \lambda_1 f_s^{i_0}(\lambda)) + \frac{(\lambda_2 + 1)\lambda_1}{s + \lambda_1 f_s^{i_0}(\lambda)}) f_s^{i_0}(\lambda).$$

Then we have:

$$\frac{\partial f_s^{i_0}(\lambda)}{f_s^{i_0}(\lambda)}(1 - (\lambda_2 + 1)\lambda_1(s + \lambda_1 f_s^{i_0}(\lambda))^{-\lambda_2}) = \ln(s + \lambda_1 f_s^{i_0}(\lambda)).$$

Hence, since $\lambda_2 > 0$ and $\lim_{s \to 0} f_s^{i_0}(\lambda) = 0$, we deduce that:

$$\left| \frac{\partial f_s^{i_0}(\lambda)}{f_s^{i_0}(\lambda)} \right| \to \infty \text{ when } s \to 0.$$

Thus, this is a contradiction with the fact that: $\left| \frac{\partial f_s^{i_0}(\lambda)}{f_s^{i_0}(\lambda)} \right| \leq C \forall s$.

case 2: Assume that $v_2 < 0$, and take $\lambda = (\lambda_1, \lambda_2) \in B(v, \varepsilon)$ such that $\lambda_2 < 0$. From Fact 1 and 2 there is an $i_0$ such that $\lim_{s \to 0} f_s^{i_0}(\lambda) = 0$ and:

$$f_s^{i_0}(\lambda) = (s + \lambda_1 f_s^{i_0}(\lambda))^{-\lambda_2 + 1}$$

We have:

$$\frac{\partial f_s^{i_0}(\lambda)}{\partial \lambda_2}(\lambda) = (-\ln(s + \lambda_1 f_s^{i_0}(\lambda)) + (-\lambda_2 + 1)\lambda_1\frac{\partial f_s^{i_0}(\lambda)}{s + \lambda_1 f_s^{i_0}(\lambda)}) f_s^{i_0}(\lambda).$$

Then:

$$\frac{\partial f_s^{i_0}(\lambda)}{f_s^{i_0}(\lambda)}(1 - (-\lambda_2 + 1)\lambda_1(s + \lambda_1 f_s^{i_0}(\lambda))^{-\lambda_2}) = -\ln(s + \lambda_1 f_s^{i_0}(\lambda)).$$

Hence, since $-\lambda_2 > 0$ and $\lim_{s \to 0} f_s^{i_0}(\lambda) = 0$, we deduce that:

$$\left| \frac{\partial f_s^{i_0}(\lambda)}{f_s^{i_0}(\lambda)} \right| \to \infty \text{ when } s \to 0.$$

Hence this is a contradiction with the fact that: $\left| \frac{\partial f_s^{i_0}(\lambda)}{f_s^{i_0}(\lambda)} \right| \leq C \forall s$.

\[\square\]

2.3. Regular projection theorem in polynomially bounded structures. Let $\mathcal{A}$ be an o-minimal structure on $\mathbb{R}$ and let $\mathcal{K}$ be the field of exponents of $\mathcal{A}$, means that:

$$\mathcal{K} = \{ r \in \mathbb{R} : x \mapsto x^r \text{ is definable in } \mathcal{A} \}.$$ 

Before beginning the proof of the theorem we recall a few results:
Lemma 2.9. (Piecewise Uniform Asymptotics). Assume that \( A \) is polynomially bounded. Let \( f : \mathbb{R}^m \times \mathbb{R} \rightarrow \mathbb{R} \) be definable. Then there is a finite \( S \subset K \) such that for each \( y \in \mathbb{R}^m \) either the function \( t \mapsto f(y, t) \) is ultimately equal to 0 or there exists \( r \in S \) such that \( \lim_{t \to +\infty} \frac{f(y, t)}{t} \in \mathbb{R}^* \).

Proof. See [9], Proposition 5.2

Lemma 2.10. Let \( U \subset \mathbb{R}^k \) be a non-empty definable open set and \( M : U \times ]0, \alpha[ \rightarrow \mathbb{R} \) be a \( C^1 \) definable map. Suppose there exists \( K > 0 \) such that \( |M(y, t)| \leq K \), for \( (y, t) \). Then there is a closed definable subset \( F \) of \( U \) with \( \dim(F) < \dim(U) \) and continuous definable functions \( C, \tau : U \setminus F \rightarrow \mathbb{R}^*, \) such that for all \( y \in U \setminus F \) we have:

\[
\| D_y M(y, t) \| \leq C(t) \text{ for all } t \in ]0, \tau(y)[.
\]

Proof. See [8], Lemma 1.8

Lemma 2.11. (Monotonicity theorem). Let \( f : ]a, b[ \rightarrow \mathbb{R} \) be a definable map. Then there are points \( a = a_0 < a_1 < \ldots < a_k = b \) such that \( f \mid ]a, a_1 + 1[ \) is either constant or strictly monotonous on \( ]a_i, a_i + 1[ \) for each \( i = 0, \ldots, n - 1. \)

Proof. See [1] or [18].

Lemma 2.12. (The curve selection Lemma). Let \( A \) be a definable subset of \( \mathbb{R}^n \) and \( a \in \overline{A} \setminus A \). Let \( p \in \mathbb{N} \). Then there exists a \( C^p \) definable curve \( \gamma : [0, 1[ \rightarrow A \setminus \{a\}, \) such that \( \lim_{t \to 0^+} \gamma(t) = a. \)

Proof. See [1] or [18].

In all the rest of this section we assume that the o-minimal structure \( A \) is polynomially bounded.

Lemma 2.13. Take \( \Omega \) a definable open neighborhood of \( 0 \) in \( \mathbb{R} \times \mathbb{R}^m \), and a definable function:

\[
f : \Omega \rightarrow \mathbb{R} \quad (x, y) \mapsto f(x, y)
\]

such that: \( f^{-1}(0) \subset (\Omega \cap \mathbb{R}^m) \) and \( f \) is \( C^1 \) with respect to \( y \) on \( \Omega \setminus \mathbb{R}^m \). Then there exists a definable set \( W \subset (\Omega \cap \mathbb{R}^m) \) with \( \dim(W) < m \), such that for all \( (0, y) \in \Omega \setminus W \) there is some \( \varepsilon > 0 \) and \( C > 0 \) such that we have on \( B((0, y), \varepsilon) \cap (\Omega \setminus \mathbb{R}^m) \):

\[
\| D_y f \| \leq C \| f \|.
\]

Proof. We can assume that \( f \not\equiv 0 \). Let’s assume that the statement is not true, then we can find \( O \subset (\Omega \cap \mathbb{R}^m) \) with \( \dim(O) = m \) and for all \((0, y) \in O \) we have:

\[
(*) \quad \lim_{x \to 0^+} \frac{\| D_y f(x, y) \|}{\| f(x, y) \|} = +\infty.
\]

Now we can find an open definable set \( U \subset O \) and \( \alpha > 0 \) such that \( ]0, \alpha[ \times U \subset \Omega. \) Since \( f^{-1}(0) \subset (\Omega \cap \mathbb{R}^m) \), by Lemma 2.9 there exist an open set \( B \subset U \) and \( r \in \mathbb{R} \) such that:

\[
f(x, y) = c(y)x^r + \phi(x, y)x^r, \text{ for } y \in B \text{ and } \alpha' > x > 0 \text{ for some } \alpha' < \alpha,
\]

where \( c \) is a definable function on \( B \) with \( c(y) \neq 0 \) for all \( y \in B, \phi \) definable such that \( \lim_{x \to 0} \phi(x, y) = 0 \) for all \( y \in B \). Therefore for any \( y \in B \) and \( x \in ]0, \alpha' \) we have:

\[
\frac{\| D_y f(x, y) \|}{\| f(x, y) \|} = \frac{\| Dc(y) + D\phi(x, y) \|}{\| c(y) + \phi(x, y) \|} \leq \frac{\| Dc(y) \|}{\| c(y) + \phi(x, y) \|} + \frac{\| D\phi(x, y) \|}{\| c(y) + \phi(x, y) \|}.
\]

But by Lemma 2.10 and the fact that \( \lim_{x \to 0} \phi(x, y) = 0 \) for all \( y \in B \), we can shrink \( B \) and \( ]0, \alpha' \) and assume that there is a definable function \( y \mapsto M(y) > 0 \) such that for any \((x, y) \in ]0, \alpha'\times B \) we have:
case2: There is no cell with respect to $y$ for each $x$. Let Lemma 2.17.

It is obvious that if there is a finite definable cover (i.e. $\dim C < m$) such that:

\[ \forall x \in D \text{ and } \forall y \in B_D : \| D_y f(x, y) \| \leq c \| f(x, y) \| \]

and this contradicts $(\ast)$. 

\[ \square \]

Lemma 2.14. Let $f : [0, \varepsilon[ \times \Omega \to \mathbb{R}$ be a definable function, $C^1$ with respect to $y$ on $]0, \varepsilon[ \times \Omega$, where $\Omega$ is an open subset of $\mathbb{R}^m$. Then we can find a definable subset $W \subset \Omega$ with $\dim(W) < m$ and such that for every $y_0 \in \Omega \setminus W$ there are $r > 0$, $\varepsilon' > 0$, and $C > 0$ such that we have:

\[ \| D_y f(x, y) \| \leq C \| f(x, y) \| \text{ for all } (x, y) \in ]0, \varepsilon'[ \times B(y_0, r). \]

Proof. Take $Z = f^{-1}(0)$, and take $C$ a cell decomposition of $]0, \varepsilon[ \times \Omega$ compatible with $\{0\} \times \Omega$ and $Z \cap (]0, \varepsilon[ \times \Omega)$. Hence $\Omega$ is a finite disjoint union of cells of $C$. Take $O$ one of these cells with dimension equal to $m$, the cells of dimension smaller than $m$ will be chosen to be in the set $W$. We discuss two cases:

\begin{itemize}
  \item [case1:] There is a cell $B \subset Z$ of maximal dimension (i.e. $\dim(B) = m + 1$) such that $O \subset \overline{B}$. In this case at each point $(0, y) \in O$ we can find a neighborhood of $(0, y)$ in $]0, \varepsilon[ \times \Omega$, where $f \equiv 0$ on this neighborhood, hence in this neighborhood $\| D_y f \| \equiv 0$, then the result holds at $(0, y)$ for any $C > 0$.
  \item [case2:] There is no cell $B \subset Z$ of maximal dimension such that $O \subset \overline{B}$. In this case by Lemma 2.13 we can find a definable set $W_O \subset O$ with $\dim(W_O) < m$ and such that for all $(0, y) \in O \setminus W_O$ we can find $r_y > 0$, $\varepsilon_y > 0$, and $C_y > 0$, such that $\| D_y f \| \leq C_y \| f \|$ on $]0, \varepsilon[ \times B(y, r_y)$.
\end{itemize}

For cells like in case1 we define $W_O := \emptyset$. So finally the set:

\[ W := (\bigcup_{O, \dim(O) = m} W_O) \cup (\bigcup_{\dim(O) < m} O) \]

satisfies the required properties. 

\[ \square \]

Definition 2.15. Let $X$ be a definable subset of a definable manifold $N$, and $\Omega$ a definable open subset of $\mathbb{R}^m$. Take $f : X \times \Omega \to \mathbb{R}$, $(x, y) \mapsto f(x, y)$ a definable function and $C^1$ with respect to $y$. We say that $f$ is $X$-rectifiable with respect to $y$, if we can find a definable partition $\mathcal{P}$ of $X$, and $c > 0$ such that for every $D \in \mathcal{P}$ there is a box $B_D \subset \Omega$ such that:

\[ \forall x \in D \text{ and } \forall y \in B_D : \| D_y f(x, y) \| \leq c \| f(x, y) \|. \]

Remark 2.16. It is obvious that if there is a finite definable cover $(X_i)_i$ of $X$ such that for each $i$ $f_{X_i \times \Omega}$ is $X_i$-rectifiable with respect to $y$, then $f$ is $X$-rectifiable with respect to $y$.

Lemma 2.17. Let $X$ be a definable subset of $\mathbb{S}^n$, $\Omega$ be an open definable subset of $\mathbb{R}^m$, and $x_0 \in \partial X$. Let $f : X \times \Omega \to \mathbb{R}$, $(x, y) \mapsto f(x, y)$ be a definable function, $C^1$ with respect to $y$. Then there is a neighborhood $U$ of $x_0$ in $\mathbb{S}^n$ such that $f_{U \cap X \times \Omega}$ is $U \cap X$-rectifiable with respect to $y$, recall that it means we can find $\alpha > 0$, $C > 0$, a definable partition of $X$, and $\mathcal{B}$ a finite collection of boxes in $\Omega$ such that for every $C \in \mathcal{C}$ with $x_0 \in \overline{U}$, there is $B_C \in \mathcal{B}$ such that:

\[ \forall x \in C \text{ with } d(x, x_0) < \alpha, \forall y \in B_C : \| D_y f(x, y) \| \leq c \| f(x, y) \|. \]

Proof. Induction on $m$. 

The case of $m = 1$:

For the case of $n = 1$, i.e. $X \subset S^1$, by choosing a good coordinate (we can choose a stereographic projection) we can reduce this to the case of Lemma 2.14 and this finishes the proof of this case. Hence by contradiction we assume that $n > 1$, $\Omega = [a, b]$, and there is no neighborhood $U$ of $x_0$ in $S^n$ such that $f|_{U \cap X \times \Omega} = U \cap X$-rectifiable with respect to $y$.

We will discuss the next two cases:

Case1: Assume that $f^{-1}(0) = \emptyset$. We define the function:

$$g : X \times \Omega \to \mathbb{R}^+$$

$$(x, y) \mapsto \frac{\|D_x f(x,y)\|}{f(x,y)}.$$

Let’s define the set:

$$O = \{(x, y) \in X \times \Omega : \text{such that } g(x,.) \text{ is monotone near } y\}.$$

Since $O$ can be expressed by a first order formula, $O$ is a definable subset of $X \times \Omega$. Now take the set $\sum = (X \times \Omega) \setminus O$, by Lemma 2.11 we have for all $x \in X$:

$$\dim(\sum_x) = 0 < \dim(\Omega) = 1.$$

Hence by Lemma 2.14 we can find a definable partition $C_1$ of $X$ such that for all $C' \in C_1$

there is a box $B_{C'} = [a_{C'}, b_{C'}] \subset \Omega$ such that:

$$C' \times B_{C'} \cap \sum = \emptyset.$$

Then this means that for every $x \in C'$, $g(x,.)$ is monotone on $B_{C'}$, but we need that the monotonicity type does not depend on $x$ but only on $C'$. Let’s choose $C_0$ to be a partition of $X$ compatible with $C_1$ and a collection of sets $\{M^+, M^-, M_0\}_{C \in C_1}$, where $M^+, M^-$, and $M_0$ are defined by:

$$M^+ = \{x \in C' : g(x,.) \text{ is increasing}\}$$

$$M^- = \{x \in C' : g(x,.) \text{ is decreasing}\}$$

$$M_0 = \{x \in C' : g(x,.) \text{ is constant}\}$$

Hence the monotonicity type of $g(x,.)$ depends only on the elements of $C_0$, means that for any $C \in C_0$ there is a box $B_C = [a_C, b_C]$ in $\Omega$ such that for any $x \in C$, $g(x,.)$ is monotone on $B_C$, with the same monotonicity for all $x \in C$.

Now let’s apply our assumption to $C_0$, then we can replace $X$ by an element $C \in C_0$, with $x_0 \in C$ such that for every box $B$ in $\Omega$, $g$ is not bounded on $C \times B$. Take $D_C = [d_C, e_C] \subset B_C$ and let’s consider the graph of $g$ on $C \times D_C$:

$$\Gamma_g = \{(x, y, g(x, y)) : y \in D_C, x \in C\} \subset C \times D_C \times (\mathbb{R}^+ \cup \{+\infty\}).$$

Consider $\overline{\Gamma_g}$, the closure of $\Gamma_g$ in $C \times D_C \times (\mathbb{R}^+ \cup \{+\infty\})$. But since for any box $B$ in $D_C$, $g$ is not bounded on $C \times B$, we can find $y_0 \in D_C$ such that:

$$(x_0, y_0, +\infty) \in \overline{\Gamma_g} \setminus \Gamma_g.$$

By the curve selection Lemma there is a definable continuous curve $\gamma : [0, a] \to \overline{\Gamma_g}$ with $\gamma(t) = (x(t), y(t), g(x(t), y(t)))$ and such that:

$$\gamma(0) = (x_0, y_0, \infty) \text{ and } \gamma([0, a]) \subset \Gamma_g.$$

Assume that $g(x,.)$ is increasing for all $x \in C$ (the other case is similar). In this case by Lemma 2.14, we can shrink $[0, a]$ and we can find a box $B$ in $[e_C, b_C]$ (it is not empty because $e_C < b_C$) and $L > 0$ such that for all $y \in B$, and for all $t \in [0, a]$ we have:

$$g(x(t), y) \leq L.$$

But we have also for all $y \in B$, and for all $t \in [0, a]$: $g(x(t), y(t)) \leq g(x(t), y) < L$.

And this contradicts the fact that:

$$\lim_{t \to 0} g(x(t), y(t)) = +\infty.$$

Case2: We assume that $Z = f^{-1}(0) \neq \emptyset$. Take $\varepsilon > 0$, define the set $A = \{x \in B(x_0, \varepsilon) \cap X : \dim(Z_x) = 1\}$. We have the following cases:
case A: Assume that \( x_0 \in \overline{A} \), take \( D \) a definable partition of \( B(x_0, \varepsilon) \cap X \) compatible with \( A \). Since \( f \) is not \( B(x_0, \varepsilon) \cap X \)-rectifiable with respect to \( y \) near \( x_0 \), we can find \( C \in D \) with \( x_0 \in \overline{C} \) such that for any partition \( \mathcal{C}_C \) of \( C \) there is \( C' \in \mathcal{C}_C \) (with \( x_0 \in \overline{C'} \)) such that for any box \( B \subset \Omega \) there is no \( L > 0 \) such that:

\[
\| D_y f(x, y) \| \leq L \quad \text{for} \quad (x, y) \in C' \times B \quad \text{and} \quad x \text{ in some neighborhood of } x_0.
\]

Here we discuss two subcases:

1. case A.1: Suppose \( C \cap A = \emptyset \). Then by Lemma 2.4 we can find a partition \( \mathcal{C}_C \) of \( C \) such that for every \( C' \in \mathcal{C}_C \) there is a box \( B_{C'} \) in \( \Omega_2 \) such that \( (C' \times B_{C'}) \cap Z = \emptyset \), hence \( g \) is well-defined on \( C' \times B_{C'} \). Then we may apply the same argument as in the proof of case 1 by considering for every \( C' \in \mathcal{C} \) the function:

\[
g : C' \times B_{C'} \rightarrow \mathbb{R}
\]

\[
(x, y) \mapsto \frac{\| D_y f(x, y) \|}{\| f(x, y) \|}.
\]

2. case A.2: Suppose \( C \subset A \). Then again by Lemma 2.4 we can find a partition \( \mathcal{C}_C \) of \( C \) such that for every \( C' \in \mathcal{C}_C \) there is a box \( B_{C'} \) in \( \Omega \) such that \( (C' \times B_{C'}) \subset Z \) or \( (C' \times B_{C'}) \cap Z = \emptyset \). If \( (C' \times B_{C'}) \cap Z = \emptyset \), then we are again in the situation of case A.1. If \( (C' \times B_{C'}) \subset Z \), this gives that \( f = 0 \) and \( D_y f = 0 \) on \( C' \times B_{C'} \), and this is a contradiction because we can choose any \( L > 0 \) such that:

\[
\| D_y f(x, y) \| \leq L \quad \text{for} \quad (x, y) \in C' \times B_{C'}, \quad \text{and} \quad x \text{ in some neighborhood of } x_0.
\]

case B: Assume that \( x_0 \notin \overline{A} \). Then we can separate \( x_0 \) from \( A \) by some \( B(x_0, \varepsilon') \) and proceed as in the case A.2.

- **The case of** \( m > 1 \): Assume that the statement of the lemma is true for any positive integer smaller than \( m \). Fix \( I = I_1 \times \ldots \times I_m \) a box in \( \Omega \), with \( I_i = ]a_i, b_i[ \) and denote by \( I' = I_2 \times \ldots \times I_m \). Apply the induction hypothesis (the case of \( m - 1 \)) to the function:

\[
f : (X \times I_1) \times I' \rightarrow \mathbb{R}
\]

\[
((x, y_1), y') \mapsto f(x, (y_1, y')).
\]

Hence we can shrink \( I_1 \), and find a neighborhood \( U \) of \( x_0 \), such that \( f \) is \( (X \cap U) \times I_1 \)-rectifiable with respect to \( y' \), means that there is \( L > 0 \) and a partition \( \mathcal{P} \) of \( (X \cap U) \times I_1 \) (we can choose it to be compatible with \( X \cap U \)) such that for any \( C \in \mathcal{P} \) there is a box \( B \subset I' \) and we have:

\[
\forall(x, y_1) \in C, \forall y' \in B \quad | D_y f(x, y) | \leq L \quad | f(x, y) |.
\]

Since \( \mathcal{P} \) is compatible with \( X \cap U \), we can find a partition \( \mathcal{D} \) of \( X \cap U \) and a definable maps \( \phi_1, \ldots, \phi_k : D \rightarrow I_1 \) for every \( D \in \mathcal{D} \), such that every \( C \in \mathcal{P} \) is either the graph of one of these maps or a band between two graphs. Now take the set \( \sum_{x} \) defined by:

\[
\sum_{x} = \bigcup_{D \in \mathcal{D}} \bigcup_{i=1, \ldots, k} \Gamma_{\phi_i} \subset (X \cap U) \times I_1.
\]

Since for all \( x \in X \cap U \) we have \( \text{dim}(\sum_{x}) = 0 < 1 = \text{dim}(I_1) \), by Lemma 2.4 we can assume that the cells \( C \in \mathcal{P} \) are of the form:

\[
C = ]a_C, b_C[ \times I_1,
\]

where \( \{D_C\}_{C \in \mathcal{P}} \) is a partition of \( X \cap U \) and \( ]a_C, b_C[ \subset I_1 \). Hence to complete the proof we need to show that for such \( C \in \mathcal{P} \) with \( x_0 \in \overline{C} \), \( f \) is \( C \)-rectifiable with respect to \( y \) in some neighborhood of \( x_0 \). Now let's apply the case of \( m = 1 \) to the function:

\[
f : (D_C \times B_C) \times ]a_C, b_C[ \rightarrow \mathbb{R}
\]

\[
((x, y'), y_1) \mapsto f(x, y).
\]

We can find \( U_C \) a neighborhood of \( x_0 \) in \( X \) and we can shrink \( B_C \), so that \( f \) is \( (U_C \cap D_C) \times B_C \)-rectifiable with respect to \( y_1 \), that there is \( L' > 0 \) and a partition \( \mathcal{P}_C \) of \( (U_C \cap D_C) \times B_C \) compatible with \( U_C \times D_C \) such that for every \( P \in \mathcal{P}_C \) there is a box \( I_P \subset I_1 \) and:
\[\forall (x, y) \in P \text{ and } \forall y_1 \in I_P \text{ we have } \| D_{y_1} f(x, y) \| \leq L' | f(x, y) |\]

Now by Lemma 2.14 we can use the same argument as before and we can assume that every element \( P \in P_D \) can be written in the form \( P = M_P \times B_P \), where the collection \( \{ M_P \}_P \) is a partition of \( U_C \cap D_C \) and \( \{ B_P \}_P \) are boxes in \( B_C \). Therefore, finally, for any \( M_P \) there is a box \( I_P \times B_P \subset I \) such that on this box we have:

\[\| D_y f(x, y) \| \leq (L + L') | f(x, y) |\]

This finishes the proof of the lemma.

\[\square\]

**Theorem 2.18.** Let \( A \) be a definable subset of \( \mathbb{R}^n \) such that \( \text{dim}(A) \leq n - 1 \) and take \( p \in \mathbb{N}^* \). Then there is \( \varepsilon > 0, C > 0 \), and \( \{ v_1, ..., v_k \} \subset \mathbb{R}^{n-1} \) such that for every \( x \in \mathbb{R}^n \) there is \( i \in \{ 1, ..., k \} \) such that \( \pi_{v_i} \) is \((\varepsilon, C, p)\)-regular at \( x \) with respect to \( A \).

**Proof.** By Theorem 2.3 we can find a cell decomposition \( C = \{ C_1, ..., C_k \} \) of \( \mathbb{R}^n \), and \( \varepsilon > 0 \) such that for every \( i \in \{ 1, ..., k \} \) there is \( v_i \in \mathbb{R}^{n-1} \) such that \( \pi_{v_i} \) is \((\varepsilon, p)\)-regular at every point \( x \in C_i \) with respect to \( A \). Hence, for every \( i \in \{ 1, ..., k \} \) there are definable functions:

\[ f_{i,l} : C_i \times B(v_i, \varepsilon) \rightarrow \mathbb{R}^s \\
(x, v) \mapsto f_{i,l}(x, v) \]

such that each \( f_{i,l} \) is \( C^p \) with respect to \( v \) and we have:

\[ C_i(x, v_i) \cap X = \sqcup \{ x + f_{i,l}(x, v)(v, 1) : v \in B(v_i, \varepsilon) \}. \]

Now we will find a refinement of \( C \), constants \( \varepsilon' < \varepsilon \) and \( C > 0 \), and a \((C, \varepsilon', p)\)-regular projections of \( \mathbb{R}^n \) with respect to \( A \). For this, it is enough to prove that for every \( C_i \in C \) the maps \( f_{i,l} \) are \( C_i \)-rectifiable with respect to \( v \), means that there is \( c > 0 \) and a definable partition \( C_i \) of \( C_i \) such that for every \( D \in C_i \) there is a box \( B_D \subset B(v_i, \varepsilon) \) with:

\[ \frac{\| D_{x} f_{i,l} \|}{| f_{i,l} |} < c \text{ on } D \times B_D. \]

Take \( C_i \in C \), we have:

- If \( C_i \) is of dimension 0, means that \( C_i = \{ x \} \) is a point, then by continuity of \( \frac{\| D_{x} f_{i,l} \|}{| f_{i,l} |} \) we can find a closed ball \( \overline{B}(v_i, \varepsilon') \subset B(v_i, \varepsilon) \) such that \( \frac{\| D_{x} f_{i,l} \|}{| f_{i,l} |} \) is bounded by some \( c > 0 \) on this ball. Hence the maps \( f_{i,l} \) are \( C_i \)-rectifiable with respect to \( v \).

- Assume that \( C_i \) is of dimension \( \text{dim}(C_i) > 0 \). We know that we have a natural definable embedding (it is the inverse map of the stereographic projection, and it is semi-algebraic) of \( \mathbb{R}^n \) in \( \mathbb{S}^n \):

\[ E : \mathbb{R}^n \rightarrow \mathbb{S}^n. \]

We can replace \( C_i \) by \( E(C_i) \) and \( f_{i,l} \) by the maps \( f_{i,l} \circ (E^{-1}, Id_{B(v_i, \varepsilon)}) \). Indeed if the maps \( f_{i,l} \circ (E^{-1}, Id_{B(v_i, \varepsilon)}) \) are \( E(C_i) \)-rectifiable then the maps \( f_{i,l} \) are also \( C_i \)-rectifiable.

Take the closure \( \overline{C_i} \) of \( C_i \) in \( \mathbb{S}^n \) (it is a compact subset since \( \mathbb{S}^n \) is compact). Take \( (U_1, ..., U_k) \) an open definable cover of \( \overline{C_i} \) given by:

\[ U_j = \overline{C_i} \cap B(x_j, r_j) \text{ for } j = 1, ..., k, \text{ such that } x_j \in \overline{C_i} \text{ and } r_j > 0. \]

Now take \( x \in \partial C_i = \overline{C_i} \setminus C_i \), hence there is a \( j_x \in \{ 1, ..., k \} \) such that \( x \in U_{j_x} \). By applying Lemma 2.17 to the maps \( f_{i,l} : (U_{j_x} \setminus \partial C_i) \times B(v_i, \varepsilon) \rightarrow \mathbb{R} \) (here \( U_{j_x} \setminus \partial C_i = X \) and \( B(v_i, \varepsilon) = \Omega \)), we can find a neighborhood \( O_x \) of \( x \) in \( \overline{C_i} \) such that \( f_{i,l} \) are \( O_x \cap C_i \)-rectifiable.
with respect to \( v \). Then by compactness of \( \partial C_i \), we can choose a finite cover \((O_{x_1}, \ldots, O_{x_m})\) of an open neighborhood of \( \partial C_i \) in \( \overline{C} \) such that \( f_{i,l} \) are \((\bigcup_s O_{x_s}) \cap C_i\)-rectifiable with respect to \( v \). Now since \( d(\partial C_i, C_i \setminus (\bigcup_s O_{x_s})) > 0 \), we can find a compact subset \( K \subset C_i \) with \( C_i \setminus (\bigcup_s O_{x_s}) \subset K \), hence the functions \( \frac{d(p, v)}{d(p, f_{i,l}(v))} \) are bounded on \( C_i \setminus (\bigcup_s O_{x_s}) \), therefore \( f_{i,l} \) are \((\bigcup_s O_{x_s}) \cap C_i\)-rectifiable with respect to \( v \). Finally, since \((\bigcup_s O_{x_s}, C_i \setminus \bigcup_s O_{x_s})\) is a definable cover of \( C_i \), we deduce that the functions \( f_{i,l} \) are \( C_i\)-rectifiable with respect to \( v \).

\[ \square \]

3. Existence of Regular covers

Take \( \mathcal{D} \) an o-minimal structure on \((\mathbb{R}, +, .)\). Let \( U \) be an open definable relatively compact subset of \( \mathbb{R}^n \). By a regular cover of \( U \), we mean a finite cover \((U_i)\) by open definable sets, such that:

(1) each \( U_i \) is homeomorphic to the open unit ball in \( \mathbb{R}^n \) by a definable homeomorphism.

(2) there is a positive number \( C \) such that for all \( x \) in \( \mathbb{R}^n \) we have:

\[ d(x, \mathbb{R}^n \setminus U) \leq C \max d(x, M \setminus U_i). \]

**Example:** Take \( U = \{(x, y) \in \mathbb{R}^2 : 0 < x < 1 \text{ and } -1 < y < 1\} \), \( U \) is a definable set in any o-minimal structure on \( \mathbb{R} \). The definable cover \((U_1, U_2)\) of \( U \) defined by:

\[
\begin{align*}
U_1 &= \{(x, y) \in U : y > -\frac{1}{2}\} \\
U_2 &= \{(x, y) \in U : y < \frac{1}{2}\}
\end{align*}
\]

is a regular cover. Indeed, take \( p = (x, y) \in U \), then we have:

\[ d(p, \mathbb{R}^2 \setminus U) < \max (d(p, \mathbb{R}^2 \setminus U_1), d(p, \mathbb{R}^2 \setminus U_2)). \]

Now take the definable cover \((U_1', U_2')\) defined by:

\[
\begin{align*}
U_1' &= \{(x, y) \in U : y < x^2\} \\
U_2' &= \{(x, y) \in U : y > -x^2\}
\end{align*}
\]

Then \((U_1', U_2')\) is not a regular cover. Assume that this is a regular cover, take a point \( p_a = (a, 0) \in U \). Then we have:

\[ d(p_a, \mathbb{R}^2 \setminus U) = a \text{ for } a < \frac{1}{2}, \]

\[ d(p_a, \mathbb{R}^2 \setminus U_1') = d(p_a, \mathbb{R}^2 \setminus U_2') < a^2. \]

Hence we have for \( a < \frac{1}{2} \):

\[ \frac{d(p_a, \mathbb{R}^2 \setminus U)}{\max (d(p_a, \mathbb{R}^2 \setminus U_1'), d(p_a, \mathbb{R}^2 \setminus U_2'))} = \frac{d(p_a, \mathbb{R}^2 \setminus U)}{d(p_a, \mathbb{R}^2 \setminus U_1')} > \frac{a}{a^2} = \frac{1}{a} \xrightarrow{a \to 0} +\infty. \]

Hence this is a contradiction with the definition of a regular cover.

**Theorem 3.1.** For any open relatively compact definable subset \( U \) of \( \mathbb{R}^n \) there exists a regular cover.

**Proof.** Take \( X = \partial U = \overline{U} \setminus U \), \( X \) is compact and definable of dimension \( n - 1 \). Take \( Z = \text{Sing}(X) \) the set of the points where \( X \) is not \( n - 1 \) dimensional submanifold of \( \mathbb{R}^n \) near this points (i.e that \( Z = (\text{Reg}(X))^c \)). For any linear projection \( P : \mathbb{R}^n \to \mathbb{R}^{n-1} \) (i.e, a projection with respect to a vector in \( \mathbb{R}^n \)), we define the discriminant \( \Delta_P \) by:

\[ \Delta_P = P(Z) \cup CV(P_{\text{Reg}(X)}), \]
where $CV(\pi|_{Reg(X)})$ is the set of critical values of $P$ on $Reg(X)$. It’s obvious that $\Delta_P$ is definable subset of $\mathbb{R}^{n-1}$, because $P$ is a definable map and $CV(P|_{Reg(X)})$ can be described by a first order formula. And we have also:

$$P(U) = P(U) \cup \Delta_P.$$  

Take $\Lambda = \{\pi_1, ..., \pi_k\}$ a set of $\varepsilon$-weak regular projection with respect to $X$, and $v_1, ..., v_k \in \mathbb{R}^{n-1}$ such that $\pi_j = \pi_{v_j}$. For $x \in \mathbb{R}^n$ we denote by $C_j^\varepsilon(x)$ the cone:

$$C_j^\varepsilon(x) = C_\varepsilon(x, v_j) = \{x + t(v, 1) : t \in \mathbb{R}^*, v \in B(\varepsilon, v_j)\}.$$

**Lemma 3.2.** Take $\pi_j \in \Lambda$, and define:

$$R(\pi_j) = \{x \in U : \pi_j \text{ is } \varepsilon - \text{weak regular at } x \text{ with respect to } X\}.$$

Then we have $\pi_j(R(\pi_j)) \cap \Delta_{\pi_j} = \emptyset$ and there is some $C > 0$ such that for all $x \in R(\pi_j)$ we have:

$$d(x, X \setminus C_j^\varepsilon(x)) \leq C d(\pi_j(x), \pi_j(X \setminus C_j^\varepsilon(x))) \leq C d(\pi_j(x), \Delta_{\pi_j}).$$  

**Proof.** It’s enough to assume that $\pi_j$ is the standard projection $\pi_j = \pi_0 = \pi$. If $\pi(R(\pi)) \cap \Delta_{\pi} \neq \emptyset$, then there is $x' \in \pi(Z) \cup CV(\pi|_{Reg(X)})$ such that $x' = \pi(x)$ and $x \in R(\pi)$, but since $\pi$ is $\varepsilon$-weak regular a $x$ then $\pi^{-1}(x) \subset (Reg(X))$, hence the contradiction, so $\pi(R(\pi)) \cap \Delta_{\pi} = \emptyset$. Since $\Delta_{\pi} \subset \pi(X \setminus C_\varepsilon^0(x))$, we have $d(\pi(x), \pi(X \setminus C_\varepsilon^0(x))) \leq d(\pi(x), \Delta_{\pi})$, and to show the second claim it’s enough to find $C > 0$ such that:

$$d(x, X \setminus C_\varepsilon^0(x)) \leq C d(\pi(x), \pi(X \setminus C_\varepsilon^0(x))).$$

But for any $x' \notin C_\varepsilon^0(x)$ we have:

$$d(x, x') \leq 1 + \frac{1}{\varepsilon}.$$  

Indeed, take $x' = x + t(v, 1) \notin C_\varepsilon^0(x)$, with $t \in \mathbb{R}^*$ and $||v|| \geq \varepsilon$, hence $\pi(x') = \pi(x) + tv$, then we have:

$$d(x, x') = \frac{d(\pi(x), \pi(x'))}{d(\pi(x), \pi(x'))} = 1 + \frac{1}{\varepsilon}.$$  

Now for any $x' \in X \setminus C_\varepsilon(x)$ we have:

$$d(x, x') \leq (1 + \frac{1}{\varepsilon})d(\pi(x), \pi(x')).$$

But since this is for any $x' \in X \setminus C_\varepsilon(x)$, by the definition of the infimum we deduce that:

$$d(x, X \setminus C_\varepsilon(x)) \leq (1 + \frac{1}{\varepsilon})d(\pi(x), \pi(X \setminus C_\varepsilon(x))).$$

\[\square\]

**Remark 3.3.** For the left side inequality in Lemma 3.2 we don’t need $x$ to be in $R(\pi_j)$.  

For the proof of Theorem 3.1 we proceed by induction on $n$. Assume that Theorem 3.1 is true in $\mathbb{R}^{n-1}$. Fix $j \in \{1, ..., k\}$, then by the induction assumption and by Lemma 3.2 there is a $C_j > 1$ and a finite definable cover $(U_{j,i})_{i \in I_j}$ of $\pi_j(U) \setminus \Delta_{\pi_j}$ such that we have for all $x' \in \mathbb{R}^{n-1}$:

$$d(x', \mathbb{R}^{n-1} \setminus (\pi_j(U) \setminus \Delta_{\pi_j})) \leq C_j \max_{i \in I_j} d(x', \mathbb{R}^{n-1} \setminus (U_{j,i}))$$

and for all $x \in R(\pi_j)$ we have by Lemma 3.2:

$$d(x, X \setminus C_j^\varepsilon(x)) \leq C_j d(\pi_j(x), \pi_j(X \setminus C_j^\varepsilon(x))) \leq C_j d(\pi_j(x), \Delta_{\pi_j}).$$
Now take a cell decomposition of $\mathbb{R}^n$ compatible with $U$, $X$, $Z$, $CP((\pi_j)_x)$, and $\pi_j(U) \setminus \Delta_{\pi_j}$. Then for each $i \in I_j$ there are a definable functions:

$$\phi_1 < \phi_2 < \ldots < \phi_{l_i} : U_{j,i} \rightarrow \mathbb{R}$$

such that $X \cap \pi_j^{-1}(U_{j,i})$ is the disjoint union of graphs of these functions, and $U \cap \pi_j^{-1}(U_{j,i})$ is the disjoint union of the open sets bounded by the graphs of these functions. So for every $j \in \{1, \ldots, k\}$ and $i \in I_j$ we have that:

$$U \cap \pi_j^{-1}(U_{j,i}) = \bigcup_{m \in M_{j,i}} U_{j,i,m},$$

where $U_{j,i,m}$ are open definable subsets of $U$, given by:

$$U_{j,i,m} = \{(x', x_n) : x' \in U_{j,i} \text{ and } \phi_{m_1}(x') < x_n < \phi_{m_2}(x') \},$$

with $\phi_{m_1} \subset X$ and $\phi_{m_2} \subset X$.

Now take $x \in U$, hence by the weak projection theorem there is a projection $\pi_j \in \Delta$ such that $x \in R(\pi_j)$. Let's consider $i \in I_j$ such that $\pi_j(x) \in U_{j,i}$ and:

$$(3.2) \quad d(\pi_j(x), \mathbb{R}^{n-1} \setminus (\pi_j(U) \setminus \Delta_{\pi_j})) \leq C_j d(\pi_j(x), \mathbb{R}^{n-1} \setminus (U_{j,i})),$$

Since $\partial(\pi_j(U) \setminus \Delta_{\pi_j}) \subset \Delta_{\pi_j}$, we have:

$$(3.3) \quad d(\pi_j(x), \Delta_{\pi_j}) \leq C_j d(\pi_j(x), \mathbb{R}^{n-1} \setminus (U_{j,i})).$$

Now take $m \in M_{j,i}$ such that $x \in U_{j,i,m}$, then we claim that:

$$(3.4) \quad d(x, X) \leq (C_j)^2 d(x, \mathbb{R}^n \setminus U_{j,i,m}).$$

To prove (3.4) we discuss two cases (the first is obvious):

1. $d(x, \mathbb{R}^n \setminus U_{j,i,m}) \geq d(x, X)$.
2. $d(x, \mathbb{R}^n \setminus U_{j,i,m}) < d(x, X)$. In this case let $V = \partial U_{j,i,m} \cap \pi_j^{-1}(\partial U_{j,i})$ (the vertical part of $\partial U_{j,i,m}$). We have then:

$$d(x, \mathbb{R}^n \setminus U_{j,i,m}) = d(x, V),$$

because $d(x, \mathbb{R}^n \setminus U_{j,i,m}) = \min\{d(x, V), d(x, \partial U_{j,i,m} \cap X)\}$. Therefore by (3.1) and (3.3) we have:

$$d(x, X) \leq d(x, X \setminus C_j^1(x)) \leq C_j d(\pi_j(x), \Delta_{\pi_j}) \leq C_j^2 d(\pi_j(x), \mathbb{R}^{n-1} \setminus U_{j,i}) \leq C_j^2 d(x, V) \leq C_j^2 d(x, \mathbb{R}^n \setminus U_{j,i,m}).$$

And this proves (3.4).

Finally, we have a finite cover $(U_{j,i,m})_{m \in M_{j,i}, i \in I_j}$ of $U$. Take $C = \max_j C_j^2$. Hence for $x \in U$ we have:

$$d(x, \mathbb{R}^n \setminus U) = d(x, X) \leq (C_j)^2 d(x, \mathbb{R}^n \setminus U_{j,i,m}) \leq C \max_{j,i,m} d(x, \mathbb{R}^n \setminus U_{j,i,m}).$$

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Université Côte d’Azur, CNRS, LJAD, UMR 7351, 06108 Nice, France
Email address: oudrane@unice.fr