Remark on the relation between passive scalars and diffusion backward in time

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Abstract

The theory of stochastic differential equations is exploited to derive the Hopf’s identities for a passive scalar advected by a Gaussian drift field delta-correlated in time. The result holds true both for compressible and incompressible velocity field.

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1 Introduction

In the last years much attention (see for example \[1, 2, 3, 4, 5\] and references therein) has been devoted to the study of passive scalar advected by an external field \( \mathbf{v} = \mathbf{v}(x, t) \) and described by the equation

\[
\partial_t T + (\mathbf{v} \cdot \nabla) T - \nu \Delta T = f
\]

where \( \nu \) denotes the molecular diffusivity of the scalar \( T = T(x, t) \) and \( f = f(x, t) \) is an external source driving the system. The solution of (1) in a time interval \([0, t]\) is easily computed once it is known the kernel of the semigroup operator defined by the backward Fokker-Planck equation

\[
\begin{align*}
\partial_t P + \nabla \cdot (\mathbf{v} P) + \nu \Delta P &= 0 \\
\lim_{s \uparrow t} P(y, s | x, t) &= \delta(x - y)
\end{align*}
\]

describing the conditional probability density for a particle to be in position \( y \) at time \( s \leq t \) when it arrives in \( x \) at time \( t \).

The connection between the two equations is provided by the trajectories describing the realizations of a stochastic differential equation backward in time (BSDE), which represents the Lagrangian picture of the passive scalar equation (section 2). This is the generalisation of the fact that in order to solve equation (1) for zero viscosity one has to invert the solutions of the ODE generated by the flow \( \mathbf{v}(x, t) \).

The general solution of (2) can be written in the form of a path integral. This observation is very useful in the case of a random velocity (drift) field Gaussian in space and delta-correlated in time (section 3). There it is shown that for each N-points function of the theory i.e. for each average over the drift of the products of N fields at equal times, the Gaussian average defines a system of BSDEs of N variables. The equations satisfied by the N-points functions are the backward Kolmogorov equations corresponding to each of the BSDEs systems (section 4).

2 The Lagrangian picture

Let us consider the stochastic differential equation

\[
\begin{align*}
dx_s &= -\mathbf{v}(x_s, t - s)ds + \sqrt{2\nu}dw_s \quad s \leq t \\
x_0 &= x
\end{align*}
\]

In general the solution of (3) exists and is unique if the drift field is Lipschitz and does not grow at infinity faster than \(|x|^2\) (\([6, 7]\)). It describes the motion of a particle localised at time \( t \) in \( x \) and diffusing for increasing \( s \) i.e. backward in the global time \( s' = t - s \).

From the rules of Ito calculus the total differential of a scalar functional of (3) is given by:

\[
\begin{align*}
d_s T(x_s, t - s) &= -[\partial_{t-s} + \mathbf{v}(x_s, t - s) \cdot \nabla - \nu \Delta]T(x_s, t - s)ds + \sqrt{2\nu}dw_s \cdot \nabla T(x_s, t - s)
\end{align*}
\]
The average of an Ito stochastic integral is zero therefore if $T$ satisfies (1) with a given initial condition $T_0(x)$ at time $s = 0$ its expression at time $t$ is provided by (5):

$$T(x, t) = \int_{\mathbb{R}} d^D y T_0(y) P(y, 0 \mid x, t) + \int_0^t ds \int_{\mathbb{R}} d^D y f(y, s) P(y, s \mid x, t)$$

where $P$ is the transition probability density specified by (3):

$$P(y, s \mid x, t) = \int \delta(y - x_{t-s}) ds$$

If one uses again Ito calculus

$$\partial_s P(y, s \mid x, t) = \int \delta(y - x_{t-s}) ds$$

by exploiting the properties of the delta function one arrives to (6) as mentioned in the introduction. Therefore the solution to the problem of the passive scalar is equivalent to the investigation of the problem of backward diffusion.

The transition probability (6) can be expressed in the form of a path integral:

$$P(y, s \mid x, t) = \int_{x_{t-s}}^{x_{t}} D x_u \exp\left[ - \int_{t-s}^{t} \left( \frac{\|\dot{x}_u + v(x_u, u)\|^2}{4 \nu} - \frac{1}{2} \nabla \cdot v(x_u, u) \right) du \right]$$

The stochastic integral (8) is evaluated according to the Stratonovich’s mid-point prescription as stressed by the symbol ($S$). In the case of additive noise the Stratonovich prescription has the advantage to preserve under path integral sign the rules of ordinary calculus.

As it will become clear in the following, it is more useful for the scopes of the present paper to resort to the Ito pre-point prescription which allows to reabsorb in the measure the divergence term in the short time Lagrangian of (9):

$$P(y, s \mid x, t) = \int_{x_{t-s}}^{x_{t}} D x_u \exp\left[ - \int_{t-s}^{t} \left( \frac{\|\dot{x}_u + v(x_u, u)\|^2}{4 \nu} \right) du \right]$$

The Gaussian model

Let us now assume that the velocity field in (1) is a Gaussian random field with zero average and correlation:

$$\langle v_\alpha(x, t) v_\beta(y, s) \rangle = \delta(t - s) D_{\alpha\beta}(|x - y|)$$

Similar hypothesis are done for the external forcing:

$$\langle f(x, t) f(y, s) \rangle = \delta(t - s) C(x - y)$$

The assumption of time decorrelation is crucial in order to preserve under Gaussian average the locality in time of the short time action in (9). On the other hand no assumption is done on the incompressibility of the velocity field in contrast with (3).
In order to compute the general N-points function one needs to evaluate the path integral:

$$< \Pi_{i=1}^N P(y_i, 0 | x_i, t_i) >_v = \int \mathcal{D}v(x, u) \exp[-S(v)] \Pi_{i=1}^N \int_{x_i(t_i)=x_i}^{x_i(0)=y_i} \mathcal{D}x_i(u) \mathcal{D}\lambda_i(u) \exp[-S(x_i, \lambda_i)]$$

(12)

where

$$S(v) = \int_0^{t_{\text{Max}}} du ds \int \mathbb{R} dP \, d^P y \, v_\alpha(x, u) \, \frac{D_{\alpha \beta}^{-1}(|x - y|) \delta(t - s)}{2} \, v_\beta(y, s)$$

(13)

and, after the Hubbard-Stratonovich transform:

$$S(x_t, \lambda_i) = \int_0^{t_{\text{Max}}} du \theta(t_i - u) \left[ \nu |\lambda_i(u)|^2 + i \lambda_i(u) \cdot \dot{x}_i(u) + i \lambda_i(u) \cdot v(x_i(u), u) \right]$$

(14)

In (14) \( t_{\text{Max}} \) coincides with the largest \( t_i \) and \( \theta(t_i - u) \) is the usual step function. The drift field appears linearly in (14): the Gaussian average can be easily performed. The averaged action is given by

$$S_{\text{av}}(\{x_i\}_{i=1}^N, \{\lambda_i\}_{i=1}^N) = \sum_{i=1}^N \int_0^{t_i} du \{ \lambda_{t,\alpha}(u) \frac{2 \nu \delta_{\alpha \beta} + D_{\alpha \beta}(0)}{2} \lambda_{t,\beta}(u) + i \lambda_{t,\alpha}(u) \dot{x}_{t,\alpha}(u) \} + \sum_{i=1, k > l}^N \int_0^{\min(t_i, t_k)} du \lambda_{t,\alpha}(u) \frac{D_{\alpha \beta}(|x_i(u) - x_k(u)|)}{2} \lambda_{k,\beta}(u)$$

(15)

where the Einstein convention and Greek letters have been used for vector indices (\( \alpha, \beta = 1, ... D \)). A further simplification is introduced by setting \( t_i = t \forall i \):

$$S_{\text{av}}(\{x_i\}_{i=1}^N, \{\lambda_i\}_{i=1}^N) = \int_0^t du \{ \lambda_{t,\alpha}(u) \frac{G_{\alpha \beta}(x_i(u) - x_k(u))}{2} \lambda_{t,\beta}(u) + i \lambda_{t,\alpha}(u) \dot{x}_{t,\alpha}(u) \}$$

(16)

with the Einstein convention now extended to all indices and

$$G_{\alpha \beta}(x_i - x_k) = G_{\alpha \beta}(0) = \left[ 2 \nu \delta_{\alpha \beta} + D_{\alpha \beta}(0) \right] \quad l = k$$

$$G_{\alpha \beta}(x_i - x_k) = D_{\alpha \beta}(|x_l - x_k|) \quad l \neq k$$

(17)

From (16), (17) is immediately clear that the one point function is Gaussian with constant diffusion matrix given by \( G_{\alpha \beta}(0) \). On the other hand non-trivial behaviour is expected for \( N \geq 2 \). By integrating over the ghost trajectories \( \{\lambda_i\}_{i=1}^N \) one obtains

$$S_{\text{av}}(\{x_i\}_{i=1}^N) = \int_0^t du \dot{x}_{t,\alpha}(u) \frac{G_{\alpha \beta}^{-1}(x_i(u) - x_k(u))}{2} \dot{x}_{t,\beta}(u)$$

(18)

The averaged action (18) is exactly the one that we would have found starting from the system of Ito stochastic differential equations (SDE):

$$dx_{t,\alpha}(t) = G_{\alpha \beta}^+(x_i(t) - x_k(t)) \, dw_{k,\beta}(t) \quad l, k = 1, N$$

(19)
where \( dw(t) \) is the Wiener differential. For \( N=1 \) it is again evident that the diffusion is normal with diffusion matrix \( G_{\alpha,\beta}(0) \). The effect of the Gaussian average is therefore to associate to each of the \( N \) points function an effective diffusion.

Two remarks are needed. First, the system \( (19) \) is autonomous. This implies that the relation between the transition probability density for the forward process and that one of the backward process is particularly simple:

\[
P^{(N)}_{\{x_i\}_{i=1}^{N}}(\{x_i\}_{i=1}^{N}, t | \{y_i\}_{i=1}^{N}, s) = \mathcal{P}^{(N)}_{b}(\{x_i\}_{i=1}^{N}, s | \{y_i\}_{i=1}^{N}, t) = \mathcal{F}^{(N)}(\{x_i\}_{i=1}^{N}, \{y_i\}_{i=1}^{N}, t - s)
\]

(20)

for any \( t \geq s \) since that

\[
\partial_t \mathcal{F}^{(N)} = -\partial_s \mathcal{F}^{(N)} = \frac{1}{2} \partial_{l,\alpha} \partial_{k,\alpha} [G^{l,k}_{\alpha,\beta} \mathcal{F}^{(N)}]
\]

\[
\mathcal{F}^{(N)}(\{x_i\}_{i=1}^{N}, \{y_i\}_{i=1}^{N}, 0) = \prod_{i=1}^{N} \delta(x_i - y_i)
\]

(21)

Second, had we used the Stratonovich prescription we would have got to the averaged action:

\[
S_{av}(\{x_i\}_{i=1}^{N}) = \int_0^t du \left[ \dot{x}_{l,\alpha}(u) \frac{G^{-1}_{\alpha,\beta}(x_i(u) - x_k(u))}{2} \dot{x}_{k,\beta}(u) - \frac{1}{4} \partial_{l,\alpha} \partial_{k,\beta} D_{\alpha,\beta}(|x_i(u) - x_k(u)|) \right]
\]

(22)

Whilst its probabilistic meaning is a posteriori clear, (22) has the disadvantage of a less intuitive form.

**4 The Hopf’s identities**

Let us define the average of a given function \( F_0(x, ..., x) \) over the realizations of \( (19) \) interpreted as BSDEs:

\[
< F(x_1(t), ..., x_N(t)) >= \Pi_{i=1}^{N} \int_{-\infty}^{\infty} d^D y_t F_0(x, ..., x) \mathcal{P}^{(N)}_{b}(y_1, ..., y_N, s | x_1, ..., x_N, t) \]

(23)

Then \( [7] \)

\[
\partial_t < F(x_1(t), ..., x_N(t)) >= \frac{1}{2} G_{\alpha,\beta}(x_i - x_k) \partial_{l,\alpha} \partial_{k,\beta} < F(x_1^I, ..., x_N^I) >
\]

(24)

where \( \frac{1}{2} G_{\alpha,\beta}(x_i - x_k) \partial_{l,\alpha} \partial_{k,\beta} \) is equal to the operator \(-\mathcal{M}^F_{\alpha,\beta}\) of \( [8, 9, 10, 11, 12] \) Equation (24) is the backward Kolmogorov equation for the system \( (19) \) interpreted as BSDEs so it is perfectly consistent with initial conditions in time.

The Hopf’s identities are a straightforward consequence of this observation. Namely starting from (13), let us consider the case of the two point functions.

\[
\partial_t << T(x_1, t) T(x_2, t) >> = \int_0^t ds \int_{-\infty}^{\infty} d^D y_1 d^D y_2 [T_0(y_1) T_0(y_2) \delta(s) + C(y_1 - y_2)] \partial_t \mathcal{P}^{(2)}_b(y_1, y_2, s | x_1, x_2, t) + \int_{-\infty}^{\infty} d^D y_1 d^D y_2 C(y_1 - y_2) \mathcal{P}^{(2)}_b(y_1, y_2, t | x_1, x_2, t)
\]

(25)
where \(< ... >_{f}\) denotes the average over the external force. By means of (24), we recognise that (25) can be rewritten as:

\[
\partial_t \langle \langle T(x_1, t) T(x_2, t) \rangle_{f} \rangle_{\nu} = -\mathcal{M}_2 \langle \langle T(x_1, t) T(x_2, t) \rangle_{f} \rangle_{\nu} + C(x_1 - x_2) \tag{26}
\]

In the limit \(t \uparrow \infty\) (stationary case) we obtain the Hopf’s identity for the two point function.

Let us turn to the four point function.

\[
\partial_t \langle \langle \Pi_{i=1}^4 T(x_i, t) \rangle_{f} \rangle_{\nu} = \int_0^t ds \, du \int_\infty^{-\infty} \Pi_{i=1}^4 d^D y_i \, [\Pi_{i=1}^4 T_0(y_i) \delta(s) \delta(s - u) + T_0(y_1) T_0(y_2) \delta(s) C(y_3 - y_4) + C(y_1 - y_2) C(y_3 - y_4)] \partial_i \mathcal{P}_b^{(4)}(\{y_1, y_2, s\} \{y_3, y_4, u\} | x_1, ..., x_4, t) + \text{perm.} + C(x_1 - x_2) \int_0^t ds \int_\infty^{-\infty} d^D y_3 d^D y_4 \, [T_0(y_3) T_0(y_4) \delta(s) + C(y_3 - y_4)] \mathcal{P}_b^{(2)}(y_3, y_4, s | x_3, x_4, t) + \text{perm.} \tag{27}
\]

here perm. means permutation among the \(l - \text{indexes}\).

The expression \(\mathcal{P}_b^{(4)}(\{y_1, y_2, s\} \{y_3, y_4, u\} | x_1, ..., x_4, t)\) denotes the probability density conditioned upon an event at later time \(t\) that the components for \(l = 1, 2\) of the system (14) take the value \(y_1, y_2\) at a prior time \(s\) whilst those for \(l = 3, 4\) the value \(y_3, y_4\) at time \(u \leq t\). Since we are interpreting (19) as BSDEs with final conditions this event is perfectly well defined. Therefore we conclude:

\[
\partial_t \langle \langle \Pi_{i=1}^4 T(x_i, t) \rangle_{f} \rangle_{\nu} = -\mathcal{M}_4 \langle \langle \Pi_{i=1}^4 T(x_i, t) \rangle_{f} \rangle_{\nu} + C(x_1 - x_2) \langle \langle T(x_3, t) T(x_4, t) \rangle_{f} \rangle_{\nu} + \text{perm.} \tag{28}
\]

In the limit \(t \uparrow \infty\) we obtain:

\[
\mathcal{M}_4 \langle \langle T(x_1) T(x_2) T(x_3) T(x_4) \rangle_{f} \rangle_{\nu} = C(x_1 - x_2) \langle \langle T(x_3) T(x_4) \rangle_{f} \rangle_{\nu} + \text{permutations} \tag{29}
\]

The Hopf’s identity satisfied by the general \(N\)-points correlation function can be derived in the same way.

## 5 Conclusion

By means of the theory of stochastic differential equations the Hopf’s identities have been proven for a passive scalar advected by a Gaussian delta-correlated in time velocity field for both the compressible and incompressible case.
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