Reed-Muller Codes for Peak Power Control in Multicarrier CDMA

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Abstract

Reed-Muller codes are studied for peak power control in multicarrier code-division multiple access (MC-CDMA) communication systems. In a coded MC-CDMA system, the information data multiplexed from users is encoded by a Reed-Muller subcode and the codeword is fully-loaded to Walsh-Hadamard spreading sequences. The polynomial representation of a coded MC-CDMA signal is established for theoretical analysis of the peak-to-average power ratio (PAPR). The Reed-Muller subcodes are defined in a recursive way by the Boolean functions providing the transmitted MC-CDMA signals with the bounded PAPR as well as the error correction capability. A connection between the code rates and the maximum PAPR is theoretically investigated in the coded MC-CDMA. Simulation results present the statistical evidence that the PAPR of the coded MC-CDMA signal is not only theoretically bounded, but also statistically reduced. In particular, the coded MC-CDMA solves the major PAPR problem of uncoded MC-CDMA by dramatically reducing its PAPR for the small number of users. Finally, the theoretical and statistical studies show that the Reed-Muller subcodes are effective coding schemes for peak power control in MC-CDMA with small and moderate numbers of users, subcarriers, and spreading factors.

Index Terms

Boolean functions, Multicarrier code-division multiple access (MC-CDMA), Orthogonal frequency-division multiplexing (OFDM), Peak-to-average power ratio (PAPR), Reed-Muller codes, Spreading sequences, Walsh-Hadamard sequences, Walsh-Hadamard transform.

I. INTRODUCTION

Multicarrier communications have recently attracted much attention in wireless and mobile applications. The orthogonal frequency division multiplexing (OFDM) has been employed as a multiplexing and a
multiple access technique in a variety of wireless communication standards such as IEEE802.11 wireless LAN [1], IEEE802.16 mobile WiMAX [2], and 3GPP-LTE [3]. Also, the multicarrier code-division multiple access (MC-CDMA), a combined scheme of OFDM and CDMA [4]−[6], has been proposed to enjoy the benefits of OFDM and CDMA by allocating the spread data symbols to subcarriers. The popularity of multicarrier communications is mainly due to the robustness to multipath fading channels and the efficient hardware implementation employing fast Fourier transform (FFT) techniques. However, multicarrier communications have the major drawback of the high peak-to-average power ratio (PAPR) of transmitted signals, which may nullify all the potential benefits [7].

A number of techniques have been developed for PAPR reduction of OFDM signals. In particular, a constructive and theoretical approach is to employ a coding scheme [8]−[9] that provides low PAPR and good error correction capability for transmitted OFDM signals. The Golay complementary sequences [10], which belong to a coset of the first-order Reed-Muller code [11], are a good example of the coding scheme. Paterson [12] also discussed several coding schemes for PAPR reduction of multicode CDMA. In [13], he summarized the algebraic coding approaches for peak power control in OFDM and multicode CDMA. For a summary of the other PAPR reduction techniques for OFDM, see [14].

To reduce the PAPR of multicarrier CDMA (MC-CDMA) signals, on the other hand, numerous studies have been focused on the power characteristics of spreading sequences. Ochiai and Imai [15] presented statistical results of the PAPR in downlink MC-CDMA, where multiple users are supported by Walsh-Hadamard or Golay complementary spreading sequences. Considering a single user MC-CDMA, Popović [16] presented the basic criteria for the selection of spreading sequences by studying the crest factors (CF) − \( \sqrt{\text{PAPR}} \) of various binary and polyphase sequences. Similar studies can be found in [17] with multiple access interference (MAI) minimization. In MC-CDMA supporting multiple users or code channels, the crest factors of various spreading sequences have been compared in [18] and [19], where the Walsh-Hadamard spreading sequences showed the best PAPR properties, provided that a large number of spreading sequences are combined for the transmitted MC-CDMA signals. More studies can be found in [20]−[23] on the PAPR of various spreading sequences in MC-CDMA.

If MC-CDMA assigns multiple spreading sequences to a single user, the multicode MC-CDMA can be equivalently treated as the spread OFDM (S-OFDM) [24]−[26], where a data symbol of the user is spread across a set of subcarriers to enjoy frequency diversity. If the number of used spreading sequences is large, the Walsh-Hadamard spread OFDM can be viewed as a PAPR reducing scheme [18][27], compared to a conventional OFDM. Also, an error correction code may be applied prior to Walsh-Hadamard spreading for improving the error rate performance [26][28] or controlling the peak power [18] of S-OFDM.
To the best of our knowledge, most of the efforts on PAPR reduction of MC-CDMA and S-OFDM have been verified mainly by statistical experiments, not by thorough theoretical analysis. Through the experiments, the PAPR of the multicarrier signals has been statistically observed, but it has never been addressed whether it is theoretically bounded. In this paper, we propose a binary Reed-Muller coded MC-CDMA system and study its PAPR properties. In the coded MC-CDMA, the information data multiplexed from users is encoded by a Reed-Muller subcode and the codeword is then fully-loaded to Walsh-Hadamard spreading sequences. The coding scheme plays a role of reducing the PAPR of transmitted MC-CDMA signals as well as providing the error correction capability. We first establish the polynomial representation of a coded MC-CDMA signal for theoretical analysis of the PAPR. A recursive construction of Boolean functions is then presented for the Reed-Muller subcodes, where the PAPR of the MC-CDMA signal encoded by the subcode is proven to be theoretically bounded. The author of [29] pointed out that the construction is equivalent to Type-III sequences in [29] where he made a general and mathematical study for Boolean functions with bounded PAPR, not considering the application to MC-CDMA. We also discuss a connection between the code rate of the subcode and the maximum PAPR. Simulation results show that the PAPR of the coded MC-CDMA signal is not only theoretically bounded, but also statistically reduced. In particular, the coded MC-CDMA solves the major PAPR problem of uncoded MC-CDMA by dramatically reducing its PAPR for the small number of users. In conclusion, the Reed-Muller codes can be effectively utilized for peak power control in MC-CDMA with small and moderate numbers of users, subcarriers, and spreading factors. The PAPR properties of the coded MC-CDMA equivalently address those of the Reed-Muller coded S-OFDM which supports multiple data from a single user.

II. System Description

Throughout this paper, MC-CDMA abbreviates multicarrier CDMA — not multicode CDMA. This paper discusses a coded MC-CDMA system employing binary codewords, binary spreading sequences, and BPSK modulation. Hence, we focus our description on binary cases. The following notations will be used throughout this paper.

- \( L \) is a spreading factor or spreading sequence length.
- \( w \) and \( W \) are the actual and the maximum numbers of users supported by a coded MC-CDMA system, respectively, where \( w \leq W \). In the rest of this paper, we will use the context of \( w \) users, where the \( w \) users can be treated as \( w \) data bits of a single user in S-OFDM or multicode MC-CDMA.
- \( K \) is the codeword length of a \((K, W)\) code, where \( K \geq W \).
- \( N \) is the number of information bits that each user transmits in an OFDM symbol.
A. Coded MC-CDMA transmitter

Figure 1 illustrates a coded MC-CDMA transmitter proposed in this paper. Assume that \(w\) users access to the coded MC-CDMA system, where the \(i\)th user, \(0 \leq i \leq w - 1\), is actively transmitting the \(N\)-bit information \(a^{(i)}\) over an OFDM symbol. The \(n\)th information bit of \(a^{(i)}\) is multiplexed across all \(i\)'s and zero-tailed at the \(n\)th spreading process, \(0 \leq n \leq N - 1\). Note \(a^{(i)}_n \in \{0, 1\}\).

- \(a^{(i)} = (a^{(i)}_0, a^{(i)}_1, \ldots, a^{(i)}_{N-1})\) denotes the \(N\)-bit information of the \(i\)th user, \(0 \leq i \leq w - 1\), while \(a_n = (a^{(0)}_n, a^{(1)}_n, \ldots, a^{(w-1)}_n, 0, \ldots, 0)\) denotes the \(W\)-bit uncoded data multiplexed from \(w\) users and zero-tailed at the \(n\)th spreading process, \(0 \leq n \leq N - 1\). Note \(a^{(i)}_n \in \{0, 1\}\).
- \(b_n = (b^{(0)}_n, b^{(1)}_n, \ldots, b^{(K-1)}_n)\) denotes the coded output of \(a_n\) by a \(K\) code at the \(n\)th spreading process. Note \(b^{(k)}_n \in \{0, 1\}\) for \(0 \leq k \leq K - 1\).
- \(d_n = (d^{(0)}_n, d^{(1)}_n, \ldots, d^{(K-1)}_n)\) denotes the BPSK modulation output of \(b_n\) that experiences the \(n\)th spreading process. Hence, \(d^{(k)}_n = (-1)^{b^{(k)}_n} \in \{-1, +1\}\).
- \(c^{(k)} = (c^{(k)}_0, c^{(k)}_1, \ldots, c^{(k)}_{L-1})\) denotes the \(L\)-chip spreading sequence assigned for the \(k\)th coded bit of \(d_n\), while \(c_l = (c^{(0)}_l, c^{(1)}_l, \ldots, c^{(K-1)}_l)^T\) is a set of the \(l\)th spreading chips across all \(K\) spreading sequences, where \(0 \leq l \leq L - 1\). \(C\) is a \(K \times L\) orthogonal spreading matrix with \(L \geq K\), where \(c^{(k)}\) is the \(k\)th row vector and \(c_l\) is the \(l\)th column vector.
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- \(u_n = (u^{(0)}_n, u^{(1)}_n, \ldots, u^{(L-1)}_n)\) denotes the output data of length \(L\) of the \(n\)th spreading process.
then if \( w \leq W \), zeros are attached to form the \( W \)-bit uncoded data \( \mathbf{a}_n \), \( 0 \leq n \leq N - 1 \). Then, \( \mathbf{a}_n \) is encoded by a \((K, W)\) code to generate the \( K \)-bit codeword \( \mathbf{b}_n \) and its BPSK modulation \( \mathbf{d}_n \). The \( k \)th coded bit of \( \mathbf{d}_n \) is then spread by the \( L \)-bit spreading sequence \( \mathbf{c}^{(k)} \), \( 0 \leq k \leq K - 1 \), where a pair of the spreading sequences is mutually orthogonal. At the \( n \)th spreading process, the spread bits of length \( L \) are linearly combined over \( K \) spreading sequences to produce \( \mathbf{u}_n \), where each element of \( \mathbf{u}_n \) can take an arbitrary value. Obviously, the spreading process is equivalent to a transform by \( \mathbf{d}_n \) by the orthogonal spreading matrix \( \mathbf{C} \), i.e., \( \mathbf{u}_n = \mathbf{d}_n \cdot \mathbf{C} \), where \( \mathbf{c}^{(k)} \) is the \( k \)th row vector of \( \mathbf{C} \). The \( N \) blocks of the spread data \( \mathbf{u}_n \) of length \( L \) experience an \( N \times L \) block interleaver for frequency diversity, and total \( N \cdot L \) bits are allocated to \( N \cdot L \) subcarriers by inverse FFT (IFFT).

The MC-CDMA receiver accomplishes the reverse operation to recover the original information \( \mathbf{a}^{(i)} \) for the \( i \)th user, where the despreading process is equivalent to a transform by \( \mathbf{C}^T \), the transpose of \( \mathbf{C} \).

From Figure 1, the baseband transmission signal over an OFDM symbol duration \( T_s \) is given by

\[
s(t) = \sqrt{\frac{W}{K}} \cdot \sum_{n=0}^{N-1} \sum_{l=0}^{L-1} \sum_{k=0}^{K-1} d_n^{(k)} c_l^{(k)} e^{j2\pi(Nl+n)t/T_s}, \quad 0 \leq t < T_s
\]

where \( j = \sqrt{-1} \). Note that if the zero-tail processes and the \((K, W)\) encoders are removed from Figure 1 then \( s(t) \) with \( K = w \) is equivalent to a conventional uncoded MC-CDMA signal in [15]. The normalization factor \( \sqrt{\frac{W}{K}} \) is used in (1) to ensure that the average power of \( s(t) \) is equal to that of the uncoded MC-CDMA signal for \( w \) users, which will be shown in Section II-D.

### B. Boolean functions and Reed-Muller codes

Let \( \mathbf{x} = (x_0, \cdots, x_{m-1}) \) be a binary vector where \( x_i \in \{0, 1\} \), \( 0 \leq i \leq m - 1 \). A **Boolean function** \( f(\mathbf{x}) \) is defined by

\[
f(\mathbf{x}) = f(x_0, \cdots, x_{m-1}) = \sum_{i=0}^{2^m-1} v_i \prod_{l=0}^{m-1} x_l^{i_l}
\]

where \( v_i \in \{0, 1\} \) and \( i_l \) is obtained by a binary representation of \( i = \sum_{l=0}^{m-1} i_l 2^l \), \( i_l \in \{0, 1\} \). Note that the addition in a Boolean function is computed modulo-2. In (2), the order of the \( i \)th monomial with nonzero \( v_i \) is given by \( \sum_{l=0}^{m-1} i_l \), and the highest order of the monomials with nonzero \( v_i \)'s is called the (algebraic) **degree** of the Boolean function \( f \).

Associated with a Boolean function \( f \), a binary codeword of length \( 2^m \) is defined by

\[
f = (f_0, f_1, \cdots, f_{2^m-1}) \text{ where } f_j = f(j_0, j_1, \cdots, j_{m-1}), \quad j = \sum_{l=0}^{m-1} j_l 2^l
\]
where \( j_l \in \{0, 1\} \). In other words, the associated codeword \( f \) of length \( 2^m \) is obtained by the Boolean function \( f_j \) while \( j \) runs through 0 to \( 2^m - 1 \) in the increasing order.

The \( r \)th-order Reed-Muller code \( R(r, m) \) is defined by a set of binary codewords of length \( 2^m \) where each codeword is generated by a Boolean function of degree at most \( r \). In other words, each codeword in \( R(r, m) \) is the associated codeword \( f \) of length \( 2^m \) in (3) where the Boolean function \( f \) has the degree of at most \( r \). The \( r \)th-order Reed-Muller code \( R(r, m) \) has the dimension of \( 1 + \binom{m}{1} + \cdots + \binom{m}{r} \) and the minimum Hamming distance of \( 2^{m-r} \).

For more details on Boolean functions and Reed-Muller codes, see [30].

C. Walsh-Hadamard spreading sequences

The Walsh-Hadamard matrix is recursively constructed by 

\[
H_{2^m} = \frac{1}{\sqrt{2}} \begin{bmatrix} H_{2^{m-1}} & H_{2^{m-1}} \\ H_{2^{m-1}} & -H_{2^{m-1}} \end{bmatrix}.
\]  

(4)

Then, it is easy to see that 

\[
H_{2^m}^T = H_{2^m}, \quad H_{2^m} \cdot H_{2^m} = I_{2^m}
\]  

(5)

where \( T \) denotes a transpose and \( I_{2^m} \) the \( 2^m \times 2^m \) identity matrix. (5) shows that the Walsh-Hadamard matrix is symmetric and orthogonal, where the rows (or columns) are orthogonal vectors of length \( 2^m \), called Walsh-Hadamard sequences. A theoretically defined Walsh-Hadamard matrix [51] has no normalization factor \( \frac{1}{\sqrt{2}} \) in (4). However, we introduce it so that each Walsh-Hadamard sequence has the unit energy.

The Walsh-Hadamard sequences are described by the algebraic structure of Boolean functions and the first-order Reed-Muller code. In the Walsh-Hadamard matrix \( H_{2^m} \), let \( h_l \) be the \( l \)th column vector of length \( 2^m \), i.e., \( h_l = (h_{0,l}, h_{1,l}, \cdots, h_{2^m-1,l})^T \). Let \( l = \sum_{i=0}^{m-1} l_i 2^i \) and \( k = \sum_{i=0}^{m-1} k_i 2^i \), where \( l_i, k_i \in \{0, 1\} \). Then, \( h_{k,l} \) is given by 

\[
h_{k,l} = \frac{1}{\sqrt{2^m}} \cdot (-1)^{f_l(k_0,k_1,\cdots,k_{m-1})} = \frac{1}{\sqrt{2^m}} \cdot (-1)^{\sum_{i=0}^{m-1} l_i k_i}
\]  

(6)

where the addition in the exponent is the modulo-2 addition. Without the normalization factor \( \frac{1}{\sqrt{2^m}} \), the \( l \)th column vector of length \( 2^m \) is a ‘\( \pm 1 \)’-codeword of length \( 2^m \) associated with the Boolean function \( f_l \) of \( m \) variables. Precisely, \( f_l \) generates \( h_l \) through (6) while the row index \( k \) runs through 0 to \( 2^m - 1 \), where \( h_l \) is equivalent to a ‘\( \pm 1 \)’-codeword in the first-order Reed-Muller code \( R(1, m) \). Since each
column vector corresponds to a codeword of $R(1, m)$ of the minimum Hamming weight $2^{m-1}$, it is straightforward that the sum of the column elements is either $\sqrt{2^m}$ or 0, i.e.,

$$\sum_{k=0}^{2^m-1} h_{k, l} = \frac{1}{\sqrt{2^m}} \sum_{k=0}^{2^m-1} (-1)^{\sum_{i=0}^{m-1} l_i k_i} = \begin{cases} \sqrt{2^m}, & \text{if } l = 0 \\ 0, & \text{if } l \neq 0 \end{cases}$$

(7)

**Example 1:** From (4), a $4 \times 4$ Walsh-Hadamard matrix is given by

$$H_4 = \frac{1}{\sqrt{4}} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} = [h_0 \ h_1 \ h_2 \ h_3].$$

We have $f_0(k_0, k_1) = 0$, $f_1(k_0, k_1) = k_0$, $f_2(k_0, k_1) = k_1$, and $f_3(k_0, k_1) = k_0 + k_1$. Then, it is easily checked that each Boolean function $f_l$ generates the $l$th column vector $h_l$, $0 \leq l \leq 3$, through (6) while the row index $k = \sum_{i=0}^{1} k_i 2^i$ runs through 0 to 3. Also, (7) is true for each column vector.

The *Walsh-Hadamard transform* \(^{[30]}\) of a vector $g = (g_0, \ldots, g_{2^m-1})$ is defined by

$$\hat{g}_l = \frac{1}{\sqrt{2^m}} \sum_{k=0}^{2^m-1} g_k \cdot (-1)^{\sum_{i=0}^{m-1} l_i k_i}, \quad 0 \leq l \leq 2^m - 1$$

where $l = \sum_{i=0}^{m-1} l_i 2^i$ and $k = \sum_{i=0}^{m-1} k_i 2^i$. From the algebraic structure of the Walsh-Hadamard matrix described above, it is straightforward that the Walsh-Hadamard transform of $g$ is given as

$$\hat{g} = (\hat{g}_0, \ldots, \hat{g}_{2^m-1}) = g \cdot H_2^m.$$

**D. Peak-to-Average Power Ratio (PAPR)**

The peak-to-average power ratio (PAPR) \(^{[32]}\) of a multicarrier signal $s(t)$ is defined by

$$\text{PAPR} (s(t)) = \max_{0 \leq t < T_s} \frac{|s(t)|^2}{E[|s(t)|^2]}$$

where $T_s$ is an OFDM symbol duration and $E[\cdot]$ denotes the ensemble average. Using the orthogonality of spreading sequences, the approach made in \(^{[27]}\) implies that the average power of the MC-CDMA signal $s(t)$ in (1) is determined by

$$E[|s(t)|^2] = \frac{w}{K} \cdot \sum_{n=0}^{N-1} \sum_{l=0}^{L-1} \sum_{k=0}^{K-1} |d_n^{(k)}|^2 |c_l^{(k)}|^2.$$  

(8)

In particular, if $d_n^{(k)} \in \{-1, +1\}$ and $c^{(k)}$ has the unit energy, i.e., $\sum_{l=0}^{L-1} |c_l^{(k)}|^2 = 1$, then (8) becomes

$$E[|s(t)|^2] = \frac{w}{K} \cdot N \cdot K = Nw$$

(9)
which is equal to the average power of an uncoded MC-CDMA signal where each of \( w \) users transmits the \( N \)-bit information over an OFDM symbol duration.

In the following, we define a polynomial \( S(z) \) associated with \( s(t) \), similar to [33].

**Definition 1:** In general, the coded MC-CDMA signal \( s(t) \) in (1) has a form of
\[
s(t) = \sum_{i=0}^{NL-1} s_i e^{j2\pi it/T_s},
\]
where \( N \cdot L \) is the number of subcarriers over an OFDM symbol and \( s_i \) takes an arbitrary value. With \( z = e^{j2\pi t/T_s} \), the associated polynomial \( S(z) \) is defined by
\[
S(z) = \sum_{i=0}^{NL-1} s_i z^i.
\]

From (9) and (10), the PAPR of \( s(t) \) is translated into
\[
PAPR(s(t)) = \frac{\max_{|z|=1} |S(z)|^2}{Nw}.
\]

**III. POLYNOMIAL REPRESENTATION OF A CODED MC-CDMA SIGNAL**

We establish the polynomial representation of a coded MC-CDMA signal by presenting the associated polynomial introduced in Definition [1]. For simplicity, we first study the polynomial representation for \( N = 1 \), where each user transmits a single information bit with a single spreading process in an OFDM symbol. The general representation with \( N > 1 \) is then discussed.

**A. \( N = 1 \)**

With \( N = 1 \), a coded MC-CDMA signal is denoted by
\[
s_0(t) = \sqrt{\frac{w}{K}} \cdot \sum_{i=0}^{L-1} \sum_{k=0}^{K-1} d_0^{(k)} c_i^{(k)} e^{j2\pi it/T_s}, \quad 0 \leq t < T_s.
\]

Then, the polynomial representation of \( s_0(t) \) is established by the following theorem.

**Theorem 1:** The polynomial \( S_0(z) \) associated with \( s_0(t) \) in (12) is given by
\[
S_0(z) = \sqrt{\frac{w}{K}} \cdot d_0 \cdot C \cdot z
\]

where \( z = (1, z, z^2, \cdots, z^{L-1})^T \). In particular, if \( K = L = 2^m \) and \( C \) is a \( 2^m \times 2^m \) Walsh-Hadamard matrix, i.e., \( C = H_{2^m} \), then
\[
S_0(z) = \sqrt{\frac{w}{2^m}} \cdot d_0 \cdot H_{2^m} \cdot z = \sqrt{\frac{w}{2^m}} \cdot \hat{d}_0 \cdot z
\]

where \( \hat{d}_0 \) is the Walsh-Hadamard transform of \( d_0 \).
Proof. In [12], let \( s_0(t) = \sum_{l=0}^{L-1} u_0^{(l)} e^{j2\pi lt/T} \) where \( u_0^{(l)} = \sqrt{\frac{w}{K}} \sum_{k=0}^{K-1} d_0^{(k)} c_t^{(k)} \). Then,
\[
u_0 = (u_0^{(0)}, \ldots, u_0^{(L-1)}) = \sqrt{\frac{w}{K}} \cdot d_0 \cdot C. \tag{15}\]

With \( z = e^{j2\pi t/T} \), the associated polynomial is then given by \( S_0(z) = \sum_{l=0}^{L-1} u_0^{(l)} z^l = u_0 \cdot z \), which derives (13). If \( C = H_{2^m} \), then (14) is immediate.

**Corollary 1:** With \( N = 1 \), if a coded MC-CDMA signal \( s_0(t) \) has the PAPR of at most \( P \), then
\[
|S_0(z)|^2 \leq wP
\]
from (11). Also, we have from (13)
\[
|d_0 \cdot C \cdot z|^2 = \frac{K}{w} |S_0(z)|^2 \leq KP.
\]
where \( z = (1, z, z^2, \ldots, z^{L-1})^T \) with \( |z| = 1 \).

In (15), if \( C = H_{2^m} \), the spread output \( u_0 \) is the Walsh-Hadamard transform of \( d_0 \) with a scaling factor \( \sqrt{\frac{w}{K}} \) in a Walsh-Hadamard spread MC-CDMA, where \( K = L = 2^m \). A similar property has been noticed in a spreading process of multicode CDMA [12]. In addition, \( H_{2^m} \cdot z \) has the following structure.

**Lemma 1:** Let \( H_{2^m} \) be a \( 2^m \times 2^m \) Walsh-Hadamard matrix and \( z = (1, z, z^2, \ldots, z^{2^m-1})^T \). Then,
\[
T = H_{2^m} \cdot z = \begin{bmatrix}
\phi_0 \phi_1 \cdots \phi_{m-2} \phi_{m-1} \\
\theta_0 \phi_1 \cdots \phi_{m-2} \phi_{m-1} \\
\phi_0 \theta_1 \cdots \phi_{m-2} \phi_{m-1} \\
\vdots \\
\theta_0 \theta_1 \cdots \theta_{m-2} \theta_{m-1} \\
\phi_0 \phi_1 \cdots \phi_{m-2} \theta_{m-1} \\
\phi_0 \theta_1 \cdots \phi_{m-2} \theta_{m-1} \\
\vdots \\
\theta_0 \theta_1 \cdots \theta_{m-2} \theta_{m-1}
\end{bmatrix} = \begin{bmatrix}
G_0(\phi_0, \cdots, \phi_{m-1}, \theta_0, \cdots, \theta_{m-1}) \\
G_1(\phi_0, \cdots, \phi_{m-1}, \theta_0, \cdots, \theta_{m-1}) \\
\vdots \\
G_{2^m-1}(\phi_0, \cdots, \phi_{m-1}, \theta_0, \cdots, \theta_{m-1})
\end{bmatrix} \tag{16}\]

where \( G_i(\phi_0, \cdots, \phi_{m-1}, \theta_0, \cdots, \theta_{m-1}) = \prod_{t=0}^{m-1} \phi_t^{\overline{i_t} \phi_i^t} \theta_t^{\overline{i_t} \phi_i^t} \cdot i = \sum_{t=0}^{m-1} i_t \phi_i^t \), where \( \phi_i = \frac{1 + z^t}{\sqrt{2}} \) and \( \theta_i = \frac{1 - z^t}{\sqrt{2}} \). Note that \( \overline{i_t} = 0 \) if \( i_t = 1 \), or \( \overline{i_t} = 1 \) if \( i_t = 0 \).

**Proof.** If \( m = 1 \), then
\[
H_2 \cdot z = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ z \end{bmatrix} = \begin{bmatrix} \phi_0 \\ \theta_0 \end{bmatrix}. \tag{17}\]
Thus, (16) is true for \( m = 1 \). Assume (16) also holds for \( m = k - 1 \), i.e.,

\[
H_{2^{k-1}} \cdot z_1 = \begin{bmatrix}
G_0(\phi_0, \cdots, \phi_{k-2}, \theta_0, \cdots, \theta_{k-2}) \\
G_1(\phi_0, \cdots, \phi_{k-2}, \theta_0, \cdots, \theta_{k-2}) \\
\vdots \\
G_{2^{k-1}-1}(\phi_0, \cdots, \phi_{k-2}, \theta_0, \cdots, \theta_{k-2})
\end{bmatrix}
\]  \hspace{1cm} (18)

where \( z_1 = (1, z, z^2, \cdots, z^{2^{k-1}-1})^T \). From the recursive construction of \( H_{2^k} \), we have

\[
H_{2^k} \cdot z = \frac{1}{\sqrt{2}} \begin{bmatrix} H_{2^{k-1}} & H_{2^{k-1}} \\ H_{2^{k-1}} & -H_{2^{k-1}} \end{bmatrix} \begin{bmatrix}
z_1 \\
z_2^{2^{k-1}} \cdot z_1
\end{bmatrix} = \left[ \begin{array}{c}
(1 + z^{2^{k-1}}) \\
(1 - z^{2^{k-1}})
\end{array} \right] \cdot H_{2^{k-1}} \cdot z_1 = \left[ \begin{array}{c}
\phi_{k-1} \cdot H_{2^{k-1}} \cdot z_1 \\
\theta_{k-1} \cdot H_{2^{k-1}} \cdot z_1
\end{array} \right]
\]  \hspace{1cm} (19)

where \( z = (1, z, z^2, \cdots, z^{2^{k-1}})^T = (z_1^T, z_2^{2^{k-1}} \cdot z_1^T)^T \). Thus, from (18) and (19),

\[
H_{2^k} \cdot z = \begin{bmatrix}
\phi_{k-1} \cdot G_0(\phi_0, \cdots, \phi_{k-2}, \theta_0, \cdots, \theta_{k-2}) \\
\phi_{k-1} \cdot G_1(\phi_0, \cdots, \phi_{k-2}, \theta_0, \cdots, \theta_{k-2}) \\
\vdots \\
\phi_{k-1} \cdot G_{2^{k-1}-1}(\phi_0, \cdots, \phi_{k-2}, \theta_0, \cdots, \theta_{k-2}) \\
\phi_{k-1} \cdot G_{2^{k-1}-1}(\phi_0, \cdots, \phi_{k-2}, \theta_0, \cdots, \theta_{k-2}) \\
\vdots \\
\phi_{k-1} \cdot G_{2^{k-1}-1}(\phi_0, \cdots, \phi_{k-2}, \theta_0, \cdots, \theta_{k-2})
\end{bmatrix}
\]  \hspace{1cm} (20)

From (17) and (20), (16) is true by induction.

\( \square \)

**Remark 1:** In Definition 6 of [34], Parker and Tellambura defined \( G_m \), a set of normalized complex sequences of length \( 2^m \) by a tensor product, where \( |\phi|_2 + |\theta|_2 = 2 \). In fact, \( H_{2^m} \cdot z \) in Lemma \( \ref{lem:induction} \) is a special case of \( G_m \) with \( \phi_t = \frac{1 + z^t}{\sqrt{2}} \) and \( \theta_t = \frac{1 - z^t}{\sqrt{2}} \), \( 0 \leq t \leq m - 1 \).

**B. \( N > 1 \)**

In (12), replacing \( d_0^{(k)} \) by \( d_n^{(k)} \) leads us to \( s_n(t) \) and its associated polynomial \( S_n(z) \), i.e.,

\[
s_n(t) = \sqrt{\frac{w}{K} \cdot \sum_{l=0}^{L-1} \sum_{k=0}^{K-1} d_n^{(k)} c_l^{(k)} e^{j2\pi lt/2},} \quad S_n(z) = \sqrt{\frac{w}{K}} \cdot d_n \cdot C \cdot z, \quad 0 \leq n \leq N - 1 \]  \hspace{1cm} (21)

where \( z = (1, z, z^2, \cdots, z^{L-1})^T \). Obviously, \( s_n(t) \) is also a coded MC-CDMA signal where \( d_n \) is loaded with a single spreading process over an OFDM symbol. In particular, if \( C = H_{2^m} \), then the
spreading process is equivalent to the Walsh-Hadamard transform (WHT), which enables the efficient implementation of spreading and despreading processes.

**Theorem 2:** With \( S_n(z) \) in (21), the associated polynomial of a coded MC-CDMA signal \( s(t) \) in (1) is determined by

\[
S(z) = \sum_{n=0}^{N-1} S_n(z^N) \cdot z^n.
\]  

(22)

In other words, \( S(z) \) is a polynomial obtained by interleaving \( S_n(z) \)'s for \( 0 \leq n \leq N - 1 \).

**Proof.** In (1),

\[
s(t) = \sqrt{\frac{w}{K}} \cdot \sum_{n=0}^{N-1} \sum_{l=0}^{L-1} \sum_{k=0}^{K-1} d_n^{(k)} c_l^{(k)} e^{j2\pi(Nl+n)t/T} = \sum_{n=0}^{N-1} \tilde{s}_n(t)
\]

where \( \tilde{s}_n(t) = \sqrt{\frac{w}{K}} \cdot \sum_{l=0}^{L-1} \sum_{k=0}^{K-1} d_n^{(k)} c_l^{(k)} e^{j2\pi(Nl+n)t/T}. \)

For a given \( n \), \( \tilde{s}_n(t) \) is a signal assigned to the \((Nl+n)\)th subcarriers while \( l \) runs through 0 to \( L-1 \). From (23), it is straightforward that the associated polynomial of \( \tilde{s}_n(t) \) is given by

\[
\tilde{S}_n(z) = \sqrt{\frac{w}{K}} \cdot d_n \cdot C \cdot z^{(N)} \cdot z^n
\]

where \( z^{(N)} = (1, z^N, z^{2N}, \cdots, z^{(L-1)N})^T \). Compared to (21), \( \tilde{S}_n(z) = S_n(z^N) \cdot z^n \), and thus the associated polynomial \( S(z) \) is

\[
S(z) = \sum_{n=0}^{N-1} \tilde{S}_n(z) = \sum_{n=0}^{N-1} S_n(z^N) \cdot z^n.
\]

\( \square \)

**Remark 2:** Since \( s_n(t) \) is a coded MC-CDMA signal with a single spreading process over an OFDM symbol, Corollary 1 is also valid for \( S_n(z) \). Precisely, if \( s_n(t) \) has the PAPR of at most \( P \), then \( |S_n(z)|^2 \leq wP \) and \( |d_n \cdot C \cdot z|^2 \leq KP \) for \( 0 \leq n \leq N - 1 \), where \( z = (1, z, z^2, \cdots, z^{L-1})^T \) with \(|z| = 1\).

Using its associated polynomial \( S(z) \) in Theorem 2 we determine the PAPR bound of a coded MC-CDMA signal \( s(t) \) with \( N > 1 \).

**Theorem 3:** In (21), assume the maximum PAPR of \( s_n(t) \) is \( P \), i.e., \( \max_{0 \leq n \leq N-1} \text{PAPR}(s_n(t)) = P \). Then, the coded MC-CDMA signal \( s(t) \) in (1) has the PAPR of at most \( NP \), i.e.,

\[
\text{PAPR}(s(t)) \leq NP.
\]
Proof. From the associated polynomial $S(z)$ in (22),

$$|S(z)|^2 = |S_0(z^N) + S_1(z^N) \cdot z + \cdots + S_{N-1}(z^N) \cdot z^{N-1}|^2$$

$$\leq \sum_{n=0}^{N-1} |z^n|^2 \cdot \sum_{n=0}^{N-1} |S_n(z^N)|^2$$

$$\leq N^2 \cdot \max_{0 \leq n \leq N-1} |S_n(z^N)|^2$$

where $|z| = 1$. From Remark 2, $\max_{0 \leq n \leq N-1} \text{PAPR}(s_n(t)) = P$ implies $|S_n(z^N)|^2 \leq wP$ for every $n$. Thus, $|S(z)|^2 \leq N^2 \cdot wP$. Therefore, the PAPR of $s(t)$ is bounded by

$$\text{PAPR}(s(t)) = \frac{\max_{|z|=1} |S(z)|^2}{Nw} \leq \frac{N^2 \cdot wP}{Nw} = NP.$$

\[ \square \]

Although the proof is straightforward and the bound seems not so tight, Theorem 3 gives us an insight that the maximum PAPR of coded MC-CDMA signals increases as each user transmits more data bits ($N$) in an OFDM symbol. Therefore, $N$ should be as small as possible to remove the probability that the MC-CDMA signal has the high PAPR.

IV. REED-MULLER CODES FOR MC-CDMA

In this section, we develop a variety of subcodes of $R(r, m)$ for a $(K, W)$ coding scheme of a coded MC-CDMA in Figure 1 where the codeword of length $K = 2^m$ is associated with a Boolean function of degree $r$. We assume that the $K$-bit codeword is fully-loaded to all the available Walsh-Hadamard spreading sequences of length $2^m$, so $K = L = 2^m$. We analyze the PAPR properties of the fully-loaded, Reed-Muller coded, and Walsh-Hadamard spread MC-CDMA signals. First of all, we study the PAPR for $N = 1$. Then, the PAPR for $N > 1$ is investigated.

In the MC-CDMA system with $N = 1$, $d_0 = ((-1)^{b_0(0)}, (-1)^{b_0(1)}, \cdots, (-1)^{b_0(2^m-1)})$ where $b_0 = (b_0(0), b_0(1), \cdots, b_0(2^m-1))$ is a codeword of a $(2^m, W)$ Reed-Muller subcode. In this section, we denote $b_0 = (b_0, b_1, \cdots, b_{2^m-1})$ for simplicity.

A. The first-order Reed-Muller code

Let $b_0 = (b_0, b_1, \cdots, b_{2^m-1})$ be a codeword of the first-order Reed-Muller code $B_1^{(m)} = R(1, m)$. When it is employed as a $(K, W)$ coding scheme in a coded MC-CDMA, the dimension is $W = m + 1$ and the codeword length is $K = 2^m$. Each codeword is associated with a Boolean function of

$$b_1(x_0, \cdots, x_{m-1}) = \sum_{i=0}^{m-1} v_i x_i + e, \quad v_i, e \in \{0, 1\} \quad (24)$$
where the addition is computed modulo-2. The PAPR of the MC-CDMA signal encoded by a codeword in $B_1^{(m)}$ is determined in the following.

**Theorem 4:** With $K = L = 2^m$ and $N = 1$, let $s_0(t)$ be a Walsh-Hadamard spread MC-CDMA signal encoded by $b_0 = (b_0, b_1, \cdots, b_{2^m-1}) \in B_1^{(m)} = R(1, m)$. Then, the PAPR of $s_0(t)$ is

$$\text{PAPR}(s_0(t)) = 1.$$

**Proof.** From Theorem 1 the associated polynomial of $s_0(t)$ is given by $S_0(z) = \sqrt{\frac{w}{2^m}} \cdot \hat{d}_0 \cdot z$ where $\hat{d}_0$ is the Walsh-Hadamard transform of $d_0$, and $z = (1, z, z^2, \cdots, z^{2^m-1})^T$. By definition, $\hat{d}_0 = (\hat{d}_0, \hat{d}_1, \cdots, \hat{d}_{2^m-1})$ where

$$\hat{d}_l = \frac{1}{\sqrt{2^m}} \sum_{k=0}^{2^m-1} (-1)^{b_k + \sum_{i=0}^{m-1} l_i k_i} \text{ where } l = \sum_{i=0}^{m-1} l_i 2^i, \ k = \sum_{i=0}^{m-1} k_i 2^i.$$

In (24), $b_k = b_1(k_0, \cdots, k_{m-1})$, $0 \leq k \leq 2^m - 1$. From (7),

$$\hat{d}_l = \frac{1}{\sqrt{2^m}} \sum_{k=0}^{2^m-1} (-1)^{\sum_{i=0}^{m-1} v_i k_i + e} l_i k_i = \frac{1}{\sqrt{2^m}} \sum_{k=0}^{2^m-1} (-1)^{\sum_{i=0}^{m-1} (v_i + e) k_i} l_i k_i + e = \left\{ \begin{array}{ll} \pm \sqrt{2^m}, & \text{if } l = v \\ 0, & \text{otherwise} \end{array} \right. $$

where $v = \sum_{i=0}^{m-1} v_i 2^i$ for given $v_i$’s. Therefore, $S_0(z) = \sqrt{\frac{w}{2^m}} \cdot \hat{d}_0 \cdot z = \pm \sqrt{\frac{w}{2^m}} \cdot \sqrt{2^m} \cdot z^v = \pm \sqrt{w} \cdot z^v$. For any $v$, the PAPR of $s_0(t)$ is therefore

$$\text{PAPR}(s_0(t)) = \frac{\max_{|z|=1} |S_0(z)|^2}{w} = \frac{w \cdot |z^v|^2}{w} = 1.$$

□

From Theorem 4 we see that the first-order Reed-Muller code is a simple and effective coding scheme that provides the uniform power for the coded MC-CDMA signals. However, it has a relatively low code rate $R_1 = \frac{m+1}{2^m}$, which vanishes as the code length increases. Therefore, we need to develop high-rate coding schemes at the expense of the PAPR increases.

**B. Recursive construction**

From a seed pair of codes, we present how to recursively construct a new code using the associated Boolean functions. We also analyze the PAPR of the coded MC-CDMA signals.

**Theorem 5:** Let $f$ and $g$ be Boolean functions of $(m-1)$ variables, where $b_f \in F$ and $b_g \in G$ are the codewords of length $2^{m-1}$ associated with $f$ and $g$, respectively. Assume that the code rate of each
Then, the associated polynomial
\[ z \sqrt{z_m} \cdot \cdots \cdot z, z \] and \[ b \]
and \( b_g \), respectively, each of which has a form of (12) where \( K = L = 2^{m-1} \) and \( N = 1 \). Assume that each signal has the PAPR of at most \( P \).

Consider a Boolean function \( b \) of \( m \) variables defined by
\[
\begin{align*}
  b(x_0, \ldots, x_{m-1}) &= (1 + x_{m-1}) \cdot f(x_0, \ldots, x_{m-2}) + x_{m-1} \cdot g(x_0, \ldots, x_{m-2}).
\end{align*}
\] (25)

Then, a codeword \( b_0 \) of length \( 2^m \) associated with \( b \) has the code rate \( R = \frac{R_f + R_g}{2} \). Let \( s_0(t) \) be a coded MC-CDMA signal of (12) encoded by \( b_0 \), where \( K = L = 2^m \) and \( N = 1 \). Then, the PAPR of \( s_0(t) \) is
\[
PAPR(s_0(t)) \leq 2P.
\]

**Proof.** Obviously, \(|F| = 2^{2m-1} \cdot R_f\) and \(|G| = 2^{2m-1} \cdot R_g\), respectively. Thus, the number of codewords \( b_0 \) is \(|F| \cdot |G| = 2^{2m-1}(R_f + R_g)\) and the code rate is \( R = \frac{2^{m-1} \cdot (R_f + R_g)}{2^m} = \frac{R_f + R_g}{2^m} \). In particular, if \( F = G \), then we keep the code rate \( R = R_f = R_g \) while the codeword length doubles.

Let \( d_f \) and \( d_g \) be the BPSK modulation outputs of length \( 2^{m-1} \) from \( b_f \) and \( b_g \), respectively. From (25), it is straightforward that \( b_0 = (b_f \mid b_g) \) and \( d_0 = (d_f \mid d_g) \), where `|` denotes a concatenation. Then, the associated polynomial \( S_0(z) \) is determined by
\[
\begin{align*}
  S_0(z) &= \sqrt{\frac{w}{2^m}} \cdot d_0 \cdot H_{2^m} \cdot z \\
  &= \sqrt{\frac{w}{2^m}} \cdot (d_f \mid d_g) \cdot \frac{1}{\sqrt{2}} \left[ H_{2^m} \cdot z \right] \\
  &= \sqrt{\frac{w}{2^m}} \cdot (d_f \mid d_g) \cdot \left[ H_{2^m} \cdot z \right]
\end{align*}
\] (26)

where \( z = (1, z, z^2, \ldots, z^{2^{m-1}})^T \), \( z_1 = (1, z, z^2, \ldots, z^{2^{m-1}})^T \), \( \phi_{m-1} = \frac{1 + z^{2^{m-1}}}{\sqrt{2}} \), and \( \theta_{m-1} = \frac{1 - z^{2^{m-1}}}{\sqrt{2}} \). Let \( B_f(z) = \sqrt{\frac{w}{2^m}} \cdot d_f \cdot H_{2^m} \cdot z_1 \) and \( B_g(z) = \sqrt{\frac{w}{2^m}} \cdot d_g \cdot H_{2^m} \cdot z_1 \). Then, (26) becomes
\[
S_0(z) = B_f(z) \cdot \phi_{m-1} + B_g(z) \cdot \theta_{m-1}.
\]

Replacing the Boolean function \( g \) by \( g + 1 \) leads us to the change of the above associated polynomial to \( S'_0(z) \), i.e.,
\[
S'_0(z) = B_f(z) \cdot \phi_{m-1} - B_g(z) \cdot \theta_{m-1}.
\]

Then,
\[
|S_0(z)|^2 + |S'_0(z)|^2 = 2 \cdot \left( |B_f(z)|^2 \cdot |\phi_{m-1}|^2 + |B_g(z)|^2 \cdot |\theta_{m-1}|^2 \right).
\] (27)
If \( s_f(t) \) and \( s_g(t) \) have the PAPR of at most \( P \), then Corollary 1 implies \(|d_f \cdot H_{2^{m-1}} \cdot z_1| \leq 2^{m-1} P \) and \(|d_g \cdot H_{2^{m-1}} \cdot z_1| \leq 2^{m-1} P \), respectively. Thus,

\[
|B_f(z)|^2 \leq \frac{w}{2^m} \cdot 2^{m-1} \cdot P = \frac{w}{2} \cdot P, \quad |B_g(z)|^2 \leq \frac{w}{2} \cdot P.
\] (28)

Therefore, from (27) and (28),

\[
|S_0(z)|^2 + |S_0'(z)|^2 \leq 2 \cdot \max \left( |B_f(z)|^2, |B_g(z)|^2 \right) \cdot (|\phi_m-1|^2 + |\theta_m-1|^2)
\]
\[
= 2 \cdot \frac{w}{2} \cdot P \cdot 2 = w \cdot 2P
\]

where \(|\phi_m-1|^2 + |\theta_m-1|^2 = 2 \) from the definition of \( \phi_m-1 \) and \( \theta_m-1 \). Thus, the PAPR of \( s_0(t) \) is

\[
PAPR(s_0(t)) = \frac{\max_{|z|=1} |S_0(z)|^2}{w} \leq 2P.
\]

(\( \square \))

The recursive construction of a Boolean function has been originally discussed in [12] for the PAPR of multicode CDMA. In Theorem 5, we showed that the construction of (25) also provides the bounded PAPR for multicarrier CDMA. In general, if there is a seed code \( B_{r-1}^{(m-1)} \) of length \( 2^{m-1} \) and size \( |B_{r-1}^{(m-1)}| \), then we can construct a new code \( B_r^{(m)} \) of length \( 2^m \) and size \( |B_r^{(m)}| = |B_{r-1}^{(m-1)}|^2 \) by concatenating a pair of codewords from the seed. If the PAPR of each coded MC-CDMA signal for the seed \( B_{r-1}^{(m-1)} \) is at most \( P \), then each coded MC-CDMA signal encoded by the new code \( B_r^{(m)} \) provides the PAPR of at most \( 2P \). If each codeword in \( B_{r-1}^{(m-1)} \) is associated with a Boolean function of degree at most \( r-1 \), then \( B_r^{(m)} \) is a subcode of \( R(r,m) \) defined by a Boolean function of degree \( r \), where the minimum Hamming distance of \( B_r^{(m)} \) is at least \( 2^{m-r} \).

Construction 1 summarizes a recursive code construction for the application to MC-CDMA.

**Construction 1:** For positive integers \( r \) and \( m \), \( 2 \leq r \leq m \), let \( b_1(x_0, \cdots, x_{m-r}) = \sum_{i=0}^{m-r} v_i x_i + e \) and \( b_1'(x_0, \cdots, x_{m-r}) = \sum_{i=0}^{m-r} v_i' x_i + e' \), where \( v_i, v_i', e, e' \in \{0,1\} \). Starting with \( b_1 \) and \( b_1' \), the Boolean function \( b_r(x_0, \cdots, x_{m-1}) \) of degree \( r \) is constructed by the \( (r-1) \) successive recursions of

\[
b\hat{r}(x_0, \cdots, x_{m-r+\hat{r}-1}) = (1+x_{m-r+\hat{r}-1}) \cdot b_{\hat{r}-1}(x_0, \cdots, x_{m-r+\hat{r}-1}) + x_{m-r+\hat{r}-1} \cdot b_{\hat{r}-1}'(x_0, \cdots, x_{m-r+\hat{r}-1})
\] (29)

while \( \hat{r} \) runs through 2 to \( r \). In (29), the Boolean function \( b_{\hat{r}-1}' \) has the same form as \( b_{\hat{r}-1} \), but may have different coefficients. Let \( b_0 = (b_0, b_1, \cdots, b_{2^{m-1}}) \) be a codeword of \( B_r^{(m)} \subset R(r,m) \), associated with a Boolean function of \( b_r \). Then, \( B_r^{(m)} \) has total \( 2^{(m-r+2)-2^{r-1}} \) codewords through the \( (r-1) \) recursions.

In a Walsh-Hadamard spread MC-CDMA with \( K = L = 2^m \) and \( N = 1 \), the PAPR of \( s_0(t) \) encoded by \( b_0 \) is at most \( 2^{r-1} \) from Theorems 4 and 5.
Finally, the code parameters of $B^m_r$ are summarized as follows.

- Dimension $W = 2^r - 1 \cdot (m - r + 2)$ and code length $K = 2^m$,
- Code rate $R_r = \frac{2^{r-1} \cdot (m - r + 2)}{2^m} = 2^{r-1} - (m - r + 2)$,
- Minimum Hamming distance $\geq 2^{m-r}$,
- $\text{PAPR}(s_0(t)) \leq 2^{r-1}$.

First of all, we present a specific code example $B^m_2$ of length $2^m$ through a single recursion, where a pair of codewords in $R(1, m - 1)$ is employed as the seed.

**Construction 2:** Let $b_0 = (b_0, b_1, \ldots, b_{2^m-1})$ be a codeword of $B^m_2 \subset R(2^m)$ that is associated with a Boolean function $b_2$ defined by

$$b_2(x_0, \ldots, x_{m-1}) = \sum_{i=0}^{m-1} v_i x_i + x_{m-1} \cdot \sum_{i=0}^{m-2} v'_i x_i + e, \quad v_i, v'_i, e \in \{0, 1\}. \quad (30)$$

In a Walsh-Hadamard spread MC-CDMA with $K = L = 2^m$ and $N = 1$, the code parameters including the PAPR of a coded MC-CDMA signal $s_0(t)$ encoded by $b_0$ are summarized as follows.

- Dimension $W = 2m$ and code length $K = 2^m$,
- Code rate $R_2 = \frac{m-1}{2^m-1}$,
- Minimum Hamming distance $\geq 2^{m-2}$,
- $\text{PAPR}(s_0(t)) \leq 2$.

With a single recursion ($r = \tilde{r} = 2$) in Construction 1 (30) is straightforward by

$$b_2(x_0, \ldots, x_{m-1}) = (1 + x_{m-1}) \cdot \left( \sum_{i=0}^{m-2} v_i x_i + e \right) + x_{m-1} \cdot \left( \sum_{i=0}^{m-2} v'_i x_i + e' \right)$$

$$= \sum_{i=0}^{m-2} v_i x_i + (e + e') \cdot x_{m-1} + x_{m-1} \cdot \sum_{i=0}^{m-2} (v_i + v'_i) \cdot x_i + e$$

$$= \sum_{i=0}^{m-1} v_i x_i + x_{m-1} \cdot \sum_{i=0}^{m-2} v'_i x_i + e$$

where $v_{m-1} = e + e'$ and $v'_i = v_i + v'_i$.

**Remark 3:** By generalizing (30) to

$$g_2(x_0, \ldots, x_{m-1}) = \sum_{i=0}^{m-1} v_i x_i + x_{m-1} \cdot \sum_{i=0}^{\gamma} v'_i x_i + e, \quad 1 \leq \gamma \leq m - 1,$$

we obtain a code $GB^m_2$ associated with $g_2$. Obviously, the coding scheme $B^m_2$ in Construction 2 is a special case of $GB^m_2$ for $\gamma = m - 1$. It is not so hard to prove that a coded MC-CDMA signal $s_0(t)$
encoded by a codeword associated with $g_2$ has the PAPR of at most 2 for any $\gamma$. While $\gamma$ runs through 1 to $m - 1$, we have $2^{m+1} \cdot (2^{m-1} + 2^{m-2} - 1 + 2^{m-3} - 1 + \cdots + 2 - 1) = 2^{m+1} \cdot (2^m - m)$ distinct codewords in $GB_2^{(m)}$, more than the number of codewords in $B_2^{(m)}$. If we compare the code rates of $GB_2^{(m)}$ and $B_2^{(m)}$, however, the code rate difference of 
\[
\log_2 \left( \frac{2^{m+1} \cdot (2^m - m)}{2^m} \right) - 2m = \log_2 (2^m - m) + 1 - m < \frac{1}{2^m}
\]
is very small and approaches to 0 as $m$ increases. Meanwhile, the encoding and the decoding complexities of the generalized coding scheme $GB_2^{(m)}$ are obviously larger than those of $B_2^{(m)}$. For the little contribution to the code rate and the increase of the complexities from $GB_2^{(m)}$, we therefore consider $B_2^{(m)}$ as our coding scheme for low PAPR.

By employing the coding scheme $B_2^{(m)}$, the Walsh-Hadamard spread MC-CDMA system with $K = L = 2^m$ and $N = 1$ is able to support maximum $2m$ users or $2m$ information bits from a single user in an OFDM symbol, providing the PAPR of at most 2.

**Construction 3:** Let a Boolean function $b_3$ be defined by
\[
b_3(x_0, \cdots, x_{m-1}) = \sum_{i=0}^{m-1} v_i x_i + x_m x_{m-2}, \sum_{i=0}^{m-2} v'_i x_i + x_m x_{m-2}, \sum_{i=0}^{m-3} v''_i x_i + x_m x_{m-2}, e (31)
\]
where $v_i, v'_i, v''_i, e \in \{0, 1\}$. Let $b_0 = (b_0, b_1, \cdots, b_{2^m-1})$ be a codeword of $B_3^{(m)} \subset R(3, m)$ that is associated with $b_3$. In a Walsh-Hadamard spread MC-CDMA with $K = L = 2^m$ and $N = 1$, the code parameters including the PAPR of a coded MC-CDMA signal $s_0(t)$ encoded by $b_0$ are summarized as

- Dimension $W = 4(m - 1)$ and code length $K = 2^m$,
- Code rate $R_3 = \frac{m-1}{2^m-2}$,
- Minimum Hamming distance $\geq 2^{m-3}$,
- PAPR($s_0(t)$) $\leq 4$.

Similar to Construction 2, the Boolean function $b_3$ is immediate from the twice recursions of (25) for $\tilde{r} = 2$ and 3 with $r = 3$. With the coding scheme $B_3^{(m)}$, the Walsh-Hadamard spread MC-CDMA system with $K = L = 2^m$ and $N = 1$ is able to support maximum $(4m-4)$ users or $(4m-4)$ information bits from a single user in an OFDM symbol, providing the PAPR of at most 4.

Table II lists the parameters of $B_r^{(m)}$ of length $2^m$ from Construction 1 where $1 \leq r \leq 5$ and $m \geq r$. It elucidates a connection between the code rates of $B_r^{(m)}$ and the maximum PAPR of MC-CDMA signals encoded by $B_r^{(m)}$. Obviously, we obtain a high rate Reed-Muller subcode at the cost of high PAPR for the coded MC-CDMA signals.
TABLE I
THE PARAMETERS OF $B_r^{(m)}$ OF LENGTH $2^m$ FOR SOME $r$’S

| Code  | Dimension $W$ | Degree $r$ of a Boolean function | Minimum Hamming distance (for $K = L = 2^m$ and $N = 1$) | Maximum PAPR |
|-------|---------------|---------------------------------|--------------------------------------------------------|--------------|
| $B_1$ | $m + 1$       | 1                               | $\geq 2^{m-1}$                                         | 1            |
| $B_2$ | $2m$          | 2                               | $\geq 2^{m-2}$                                         | 2            |
| $B_3$ | $4(m-1)$      | 3                               | $\geq 2^{m-3}$                                         | 4            |
| $B_4$ | $8(m-2)$      | 4                               | $\geq 2^{m-4}$                                         | 8            |
| $B_5$ | $16(m-3)$     | 5                               | $\geq 2^{m-5}$                                         | 16           |

Remark 4: In Construction 1 if $r = m$, then $R_r = 1$ and $\text{PAPR}(s_0(t)) \leq 2^{m-1}$ in $B_r^{(m)}$. In other words, if we apply a trivial $m$th-order Reed-Muller code of length $2^m$ and code rate 1.0, or equivalently apply a Reed-Muller mapping to the information bits of length $2^m$, then the PAPR of the corresponding MC-CDMA signals is bounded by $2^{m-1}$. The Reed-Muller mapping may be useful for the applications of small spreading factors, e.g., $m = 2, 3, 4$, requiring low PAPR and no rate loss. However, if the spreading factor is large, we may have the problems of the high PAPR and the large complexity of demapping at the receiver.

C. Golay complementary sequences

Golay complementary sequences [10] provide the bounded PAPR $\leq 2$ for transmitted OFDM signals when they are employed as a coding scheme in an OFDM system. From [11], it is well known that each binary Golay complementary sequence of length $2^m$ is equivalent to the second-order coset of the first-order Reed-Muller code, where the associated Boolean function is defined by

$$b_c(x_0, \cdots, x_{m-1}) = \sum_{i=0}^{m-2} x_{\pi(i)}^2 x_{\pi(i+1)} + \sum_{i=0}^{m-1} v_i x_i + e$$

(32)

where $\pi$ is a permutation in $\{0, 1, \cdots, m-1\}$ and $v_i, e \in \{0, 1\}$. We now apply a binary code $B_c^{(m)}$ for coded MC-CDMA signals, where each codeword is a Golay complementary sequence defined by $b_c$.

Construction 4: Let $b_0 = (b_0, b_1, \cdots, b_{2^m-1})$ be a codeword of $B_c^{(m)} \subset \mathbb{R}(2, m)$ that is associated with a Boolean function $b_c$ in (32). Then, its code parameters are summarized as [11]

- Dimension $W = m + \log_2(m!)$ and code length $= 2^m$,
- Code rate $R_c = \frac{m + \log_2(m!)}{2^m}$,
- Minimum Hamming distance $\geq 2^{m-2}$. 
TABLE II

The maximum PAPR of Walsh-Hadamard spread MC-CDMA signals encoded by Golay complementary sequences of length $2^m$ ($K = L = 2^m$, $N = 1$)

| $m$ | Code length | Theoretical maximum (Theorem 6) | Actual maximum (Numerical experiments) |
|-----|-------------|---------------------------------|----------------------------------------|
| 3   | 8           | 4                               | 1.9654 ($\leq 2$)                      |
| 4   | 16          | 4                               | 1.9998 ($\leq 2$)                      |
| 5   | 32          | 8                               | 3.2184 ($\leq 4$)                      |
| 6   | 64          | 8                               | 3.8826 ($\leq 4$)                      |
| 7   | 128         | 16                              | 3.9930 ($\leq 4$)                      |
| 8   | 256         | 16                              | 5.8964 ($\leq 8$)                      |

Through a linear unitary transform (LUT), Parker and Tellambura \[34\] have implicitly investigated the PAPR of MC-CDMA signals encoded by the Golay complementary sequences. Therefore, Theorem 6 is immediate from the application of their work in a Walsh-Hadamard spread MC-CDMA.

**Theorem 6:** \[34\] In a Walsh-Hadamard spread MC-CDMA with $K = L = 2^m$ and $N = 1$, let $s_0(t)$ be the coded MC-CDMA signal encoded by a codeword $b_0 \in B_c^{(m)}$ in Construction 4. Then,

$$\text{PAPR}(s_0(t)) \leq 2^{m-\left\lfloor \frac{m}{2} \right\rfloor}. \quad (33)$$

**Proof.** From Lemma 1, $T = H_{2^m} \cdot z$ is a special case of a linear unitary transform (LUT) described in Theorem 6 of \[34\], where $\phi_t = \frac{1+z^2t}{\sqrt{2}}$ and $\theta_t = \frac{1-z^2t}{\sqrt{2}}$. Theorem 1 implies that $s_0(t)$ is equivalent to a unitary transform of a modulated Golay complementary sequence $d_0$ through $T$. Therefore, the bounded PAPR of (33) is obvious from Corollary 6 in \[34\]. \qed

**Remark 5:** Numerical experiments revealed that the actual maximum PAPR of MC-CDMA signals encoded by Golay complementary sequences is smaller than the upper bound predicted by Theorem 6. This may be the case because the PAPR bound of Theorem 6 has been established by general $\phi_t$ and $\theta_t$ with the requirement of $|\phi_t|^2 + |\theta_t|^2 = 2$ \[34\]. However, the Walsh-Hadamard spread MC-CDMA has the special values of $\phi_t = \frac{1+z^2t}{\sqrt{2}}$ and $\theta_t = \frac{1-z^2t}{\sqrt{2}}$, respectively, which may require the tighter bound on the PAPR than (33). Table II shows the PAPR comparison between the theoretical bound and the numerical results. From the numerical results, we conjecture $\text{PAPR}(s_0(t)) \leq 2^{\left\lfloor \frac{m+1}{2} \right\rfloor}$, where the proof is left open.

Table III compares the code rates of $B_2^{(m)}$, $B_3^{(m)}$, and $B_c^{(m)}$ for several $m$'s. We see from the table that $B_3^{(m)}$ is a proper coding scheme for a Walsh-Hadamard spread MC-CDMA with a small spreading factor, providing the bounded PAPR $\leq 4$ and the acceptable code rates.
### TABLE III

**THE CODE RATES OF VARIOUS CODES OF LENGTH $2^m$**

| $m$ | Code length | $B_2$ (Construction 2) | $B_3$ (Construction 3) | $B_c$ (Construction 4) |
|-----|-------------|------------------------|------------------------|------------------------|
| 3   | 8           | 0.75                   | 1.0                    | 0.6981                 |
| 4   | 16          | 0.5                    | 0.75                   | 0.5366                 |
| 5   | 32          | 0.3125                 | 0.5                    | 0.3721                 |
| 6   | 64          | 0.1875                 | 0.3125                 | 0.2421                 |
| 7   | 128         | 0.1094                 | 0.1875                 | 0.1508                 |

### D. Encoding and decoding

In Figure 1, the Reed-Muller subcode introduced in Construction 1 is applied at each spreading process for encoding the information from a single or multiple users in the Reed-Muller coded and Walsh-Hadamard spread MC-CDMA transmitter ($K = L = 2^m$). Precisely, a $(2^m, W)$ Reed-Muller subcode $B_r(m)$ encodes a $W$-bit input data $a_n = (a_n^{(0)}, \ldots, a_n^{(w-1)}, 0, \ldots, 0)$ at the $n$th spreading process, $0 \leq n \leq N - 1$, to produce a codeword $b_n$ of length $2^m$, which goes through Walsh-Hadamard spreading, interleaving, and IFFT in the sequel.

In the encoding process, the codeword $b_n$ is obtained by

$$b_n = a_n \cdot G_r^{(m)}$$

where $G_r^{(m)}$ is the $W \times 2^m$ generator matrix of $B_r(m)$, where $W = 2^{r-1}(m - r + 2)$. The recursion of Boolean functions in (29) equivalently derives the recursion of generator matrices of

$$G_r^{(m-r+\tilde{r})} = \begin{bmatrix} G^{(m-r+\tilde{r}-1)}_{\tilde{r}-1} & 0 \\ 0 & G^{(m-r+\tilde{r}-1)}_{\tilde{r}-1} \end{bmatrix}, \quad 2 \leq \tilde{r} \leq r \quad (34)$$

where $0 = (0, \ldots, 0)$ of length $2^{m-r+\tilde{r}-1}$ and $G_r^{(m-r+\tilde{r})}$ is a $2^{\tilde{r}-1}(m - r + 2) \times 2^{m-r+\tilde{r}}$ matrix. By elementary row operations, it is equivalent to

$$G_r^{(m-r+\tilde{r})} = \begin{bmatrix} G^{(m-r+\tilde{r}-1)}_{\tilde{r}-1} & G^{(m-r+\tilde{r}-1)}_{\tilde{r}-1} \\ 0 & G^{(m-r+\tilde{r}-1)}_{\tilde{r}-1} \end{bmatrix}, \quad 2 \leq \tilde{r} \leq r \quad (35)$$

While $\tilde{r}$ runs through 2 to $r$, the generator matrix $G_r^{(m)}$ is constructed by the $(r-1)$ recursions of (34) or (35), where the initial matrix $G_1^{(m-r+1)}$ is the $(m - r + 2) \times 2^{m-r+1}$ generator matrix of $R(1, m - r + 1)$.
given by

\[ G_{1}^{(m-r+1)} = \begin{bmatrix} 1111 & 1111 & \cdots & 1111 & 1111 \\ 0101 & 0101 & \cdots & 0101 & 0101 \\ 0011 & 0011 & \cdots & 0011 & 0011 \\ 0000 & 1111 & \cdots & 0000 & 1111 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0000 & 0000 & \cdots & 1111 & 1111 \end{bmatrix} = \begin{bmatrix} 1 \\ x_0 \\ x_1 \\ x_2 \\ \vdots \\ x_{m-r} \end{bmatrix}. \]

For the notations \( x_i \)'s of the generator matrix of the first-order Reed-Muller codes, see [30].

In particular, we are able to determine \( G_2^{(m)} \) and \( G_3^{(m)} \) directly from the Boolean expressions in (30) and (31), respectively. The generator matrices of \( B_2 \) and \( B_3 \) are \( 2m \times 2^m \) and \( (4m - 4) \times 2^m \) matrices, respectively. With each \( x_i \) of length \( 2^m \), we have

\[ G_2^{(m)} = \begin{bmatrix} 1 \\ x_0 \\ \vdots \\ x_{m-1} \\ x_{m-1}x_0 \\ \vdots \\ x_{m-1}x_{m-2} \end{bmatrix} \quad \text{and} \quad G_3^{(m)} = \begin{bmatrix} 1 \\ x_0 \\ \vdots \\ x_{m-1} \\ x_{m-1}x_0 \\ \vdots \\ x_{m-1}x_{m-2} \\ x_{m-1}x_{m-3} \\ \vdots \\ x_{m-1}x_{m-2}x_0 \\ \vdots \\ x_{m-1}x_{m-2}x_{m-3} \end{bmatrix}. \quad (36) \]

Note that \( G_2^{(m)} \) and \( G_3^{(m)} \) in (36) have the different orders of rows with those generated by the recursions of (34) or (35). In this paper, we use \( G_2^{(m)} \) and \( G_3^{(m)} \) in (36).

Example 2: Let \( m = 3 \). Then, the generator matrices of \( B_2 \) and \( B_3 \) are \( 6 \times 8 \) and \( 8 \times 8 \) matrices,
restrict our attention to a Walsh-Hadamard spread MC-CDMA employing

Similarly, if the system employs

the acceptable code rate as well as the low PAPR for the coded MC-CDMA signals. We show that the
determined by

For the encoding and decoding of Golay complementary sequences, see [11].

We briefly introduce some decoding techniques for Reed-Muller subcodes. The first-order Reed-Muller code can be decoded by the Fast Hadamard Transform (FHT) technique described in [35]. In general, the \( r \)-th-order Reed-Muller code is decoded by the Reed decoding algorithm [36]. In particular, if we consider \( B_2^{(m)} \) or \( B_3^{(m)} \) as a supercode of the union of cosets of \( R(1,m) \), then we can accomplish the soft decision decoding by removing each possible coset representative from the received codeword and then applying the FHT [37][11].

For the encoding and decoding of Golay complementary sequences, see [11].

E. PAPR of coded MC-CDMA signals with \( N > 1 \)

In what follows, we discuss the PAPR of coded MC-CDMA signals in a general case of \( N > 1 \). We restrict our attention to a Walsh-Hadamard spread MC-CDMA employing \( B_2^{(m)} \) or \( B_3^{(m)} \) that provides the acceptable code rate as well as the low PAPR for the coded MC-CDMA signals. We show that the maximum PAPR depends on the actual number of users supported by the MC-CDMA.

**Theorem 7:** Assume that \( B_2^{(m)} \) is employed in a Walsh-Hadamard spread MC-CDMA system in Figure 1 where \( K = L = 2^m \). The maximum PAPR of the coded MC-CDMA signal \( s(t) \) is then determined by

\[
PAPR(s(t)) \leq \begin{cases} 
N, & \text{if } 1 \leq w \leq m + 1, \\
2N, & \text{if } m + 2 \leq w \leq 2m. 
\end{cases} 
\]  

(37)

Similarly, if the system employs \( B_3^{(m)} \), then the maximum PAPR of \( s(t) \) is

\[
PAPR(s(t)) \leq \begin{cases} 
N, & \text{if } 1 \leq w \leq m + 1, \\
2N, & \text{if } m + 2 \leq w \leq 2m, \\
4N, & \text{if } 2m + 1 \leq w \leq 4m - 4. 
\end{cases} 
\]  

(38)
Proof. In Theorem 3, it is easy to see that $P = \max_{0 \leq n \leq N-1} \text{PAPR}(s_n(t)) = \max_{d_0} \text{PAPR}(s_0(t))$, where $s_0(t)$ and $s_n(t)$ are given in (12) and (21), respectively. Therefore, $P = 1, 2,$ and $4$ when $B_1^{(m)}$, $B_2^{(m)}$, and $B_3^{(m)}$ are employed as the coding scheme, respectively. In $G_2^{(m)}$ of (36), if $w \leq m + 1$, the first $(m + 1)$ rows participate in the encoding process, while the other rows are ignored by zero tailing. Since a linear combination of the first $(m + 1)$ rows generates a codeword of $B_1^{(m)}$, it is obvious that if $w \leq m + 1$, then $\text{PAPR}(s(t)) \leq N$ from Theorems 3 and 4. If $m + 2 \leq w \leq 2m$, on the other hand, $\text{PAPR}(s(t)) \leq 2N$ from Construction 2 and Theorem 3. Therefore, (37) is true for $B_2^{(m)}$. Similar to this approach, (38) is also true for $B_3^{(m)}$ from the generator matrix $G_3^{(m)}$ of (36) and Theorem 3.

In general, if $B_1^{(m)}$ is employed as the coding scheme, the Walsh-Hadamard spread MC-CDMA signals have the PAPR of at most $N \cdot 2^{r-1}$ from Construction 1 and Theorem 3. However, Theorem 7 is not true for the MC-CDMA signals if the generator matrix $G_r^{(m)}$ is recursively constructed by (34) or (35). We need to reorder the rows of $G_r^{(m)}$ to achieve the maximum PAPR depending on the number of actual users as in Theorem 7.

Remark 6: In Section IV-B, we developed various Reed-Muller subcodes of length $2^m$ to control the peak power of MC-CDMA signals with $2^m$ subcarriers in a systematic way. In fact, we may employ the coding scheme for a codeword of length $2^m N$, where $N = 2^h$, which encodes $a = (a_0 | a_1 | \cdots | a_{N-1})$, a concatenation of $N$ uncoded data block. Then, the codeword covers the entire $2^m N$ subcarriers to control the peak power and to ultimately reduce the maximum PAPR of the coded MC-CDMA signal. In this case, however, the code rate may be dramatically reduced for such a long codeword because the coding scheme is a subcode of the Reed-Muller code. This also enlightens a connection between the code rates and the maximum PAPR of coded MC-CDMA signals.

Remark 7: Virtually treating a single user’s data as multiple users’ one, a coded MC-CDMA system can be considered as an equivalent spread OFDM [25], where the single user’s data is spread across a set of subcarriers to enjoy frequency diversity. By applying the coding schemes introduced in this section, the spread OFDM additionally has the benefits of low PAPR and good error correction capability.

V. SIMULATION RESULTS AND DISCUSSIONS

This section provides simulation results to confirm our theoretical analysis and presents some discussions on statistical results of PAPR of MC-CDMA signals. The PAPR properties of a Reed-Muller coded and Walsh-Hadamard spread MC-CDMA system are compared to those of a pair of uncoded systems. In the uncoded systems, the one employs Walsh-Hadamard (WH) spreading sequences, while the other uses
Golay complementary (GC) spreading sequences each of which forms a row of a recursively constructed Golay complementary spreading matrix \cite{15}. In our coded MC-CDMA, we employ $B_3^{(m)}$ as the coding scheme which we believe is a good coding solution providing the acceptable code rates, the moderate complexity, and the low PAPR for the coded MC-CDMA signals.

For a fair comparison, we assume that all the MC-CDMA systems transmit the same number of information bits in an OFDM symbol from $w$ active users. If the uncoded systems transmit $Nw$ information bits in an OFDM symbol, our coded system of code rate $R$ then needs to transmit $\frac{Nw}{R}$ coded bits for the transmission of $Nw$ information bits. Therefore, while the uncoded ones have $NL$ subcarriers, the coded system needs to use $\frac{NL}{R}$ subcarriers in an OFDM symbol, where $L$ is a spreading factor used in the uncoded systems. In the following, our simulations employ $B_3^{(m)}$ of code rate $R = 0.5$ or $1$, where the coded MC-CDMA uses $2NL$ or $NL$ subcarriers in an OFDM symbol by employing the spreading sequences of length $2L$ or $L$.

In our simulations, we measure the discrete-time PAPR \cite{38} of each MC-CDMA signal from the IDFT (Inverse discrete Fourier transform) of the oversampling factor 8. Also, we statistically measure the PAPR over $N_s = 5 \times 10^5$ OFDM symbols for $NwN_s$ randomly generated information bits.

A. Code rate $R = 0.5$

In Figures 2 and 3, $w$ users access to each MC-CDMA system to transmit $N = 4$ information bits per each user in an OFDM symbol. The uncoded MC-CDMA systems use the spreading factor $L = 16$ and $NL = 64$ subcarriers, where maximum 16 users are supported, i.e., $1 \leq w \leq 16$. As the coded MC-CDMA system also needs to support up to 16 users, we choose a $(32, 16)$ code $B_3^{(5)}$ as its coding scheme, where $W = 16$, $K = 32$, and $R = 0.5$. Thus, our coded MC-CDMA uses the spreading factor $2L = 32$ and $2NL = 128$ subcarriers to transmit $Nw$ information bits in an OFDM symbol from the $w$ active users. In the coded MC-CDMA system, each 32-bit codeword is fully-loaded to all the available 32 Walsh-Hadamard spreading sequences regardless of $w$. On the other hand, the uncoded MC-CDMA systems assign the $w$ spreading sequences to $w$ users on demand, so they are fully-loaded only if $w = 16$.

Figure 2 shows the complementary cumulative distribution functions (CCDF) of $\Pr(\text{PAPR} > \lambda_0)$ of each MC-CDMA signal for $w = 8$ and 16. It reveals that the coded MC-CDMA is superior to the others when the number of active users is small. Precisely, if $w = 8$, it reduces the PAPR $\lambda_0$ achieving $\Pr(\text{PAPR} > \lambda_0) = 10^{-3}$ by more than 2 dB, compared to the uncoded systems. Moreover, Theorem 7 ensures that there exists no coded MC-CDMA signal with PAPR $> 9$ dB for $w = 8$, which implies that the coded MC-CDMA also outperforms the uncoded ones in theoretical aspects. If $w = 16$, the
Theoretical maximum of coded MC-CDMA ($w=8$)

Theoretical maximum of coded MC-CDMA ($w=16$)

Fig. 2. PAPR performance of MC-CDMA systems. All the MC-CDMA systems transmit $Nw = 32$ or 64 information bits in an OFDM symbol. The code rate of the coded MC-CDMA is 0.5.

coded MC-CDMA has almost the same PAPR $\lambda_0$ as the uncoded Walsh-Hadamard spread MC-CDMA for achieving $\Pr(\text{PAPR} > \lambda_0) = 10^{-3}$. Even in this case, it is theoretically guaranteed that no coded MC-CDMA signal has PAPR $> 12$ dB, which may not be true in the uncoded systems. Figure 2 also shows that most of the coded MC-CDMA signals in the statistical experiments have much smaller PAPR than the theoretical maximum predicted by Theorem 7.

Figure 3 displays the PAPR $\lambda_0$ of each MC-CDMA achieving $\Pr(\text{PAPR} > \lambda_0) = 10^{-3}$ according to the number of active users $w$, $1 \leq w \leq 16$. It is well known [15] that the uncoded Walsh-Hadamard (WH) spread MC-CDMA shows the high PAPR when the number of active users is small. The PAPR then decreases as the number of users increases. On the other hand, the uncoded Golay complementary (GC)
spread MC-CDMA has the low PAPR for the small number of users. However, the PAPR gets higher than that of the uncoded Walsh-Hadamard spread MC-CDMA as the number of users increases. Figure 3 shows that the coded MC-CDMA is a good alternative to the two uncoded systems by providing the smallest PAPR $\lambda_0$ for almost all user numbers. Moreover, Theorem 7 assures that the maximum PAPR of the coded system is theoretically limited to 6 dB for $1 \leq w \leq 6$, 9 dB for $7 \leq w \leq 10$, and 12 dB for $11 \leq w \leq 16$, respectively, where Figure 3 provides the numerical evidences. Therefore, it is theoretically guaranteed that there exists no coded MC-CDMA signal with the PAPR higher than the maximum values for each user, which is not generally true in the uncoded systems. The theoretical and statistical results show that the coded MC-CDMA dramatically reduces its PAPR for the small number of users, which

Fig. 3. Relationship between PAPR $\lambda_0$ and active number of users for MC-CDMA systems where $\Pr(PAPR > \lambda_0) = 10^{-3}$. The code rate of the coded MC-CDMA is 0.5.
effectively solves the high PAPR problem in the uncoded MC-CDMA. Figure 3 also shows that if the number of active users is large \((w \geq 11)\), the statistical PAPR \(\lambda_0\) is much smaller than the theoretical maximum predicted by Theorem 7. As a result, we claim that the coded MC-CDMA provides the best statistical and theoretical solution for PAPR reduction for any number of users.

**B. Code rate \(R = 1.0\)**

In Figures 4 and 5, the MC-CDMA systems support \(w\) users where each user transmits \(N = 8\) information bits in an OFDM symbol. The uncoded MC-CDMA systems use the spreading factor \(L = 8\) and \(NL = 64\) subcarriers to transmit \(Nw\) information bits in an OFDM symbol. To support up to 8 users, the coded MC-CDMA system employs \(B_3^{(3)}\), a \((8, 8)\) code with \(K = W = 8\) and \(R = 1\). In this case, \(B_3^{(3)}\) is used as a mapping scheme mentioned in Remark 4. In our MC-CDMA, each 8-bit uncoded data \(a_n, 0 \leq n \leq 7\), is transformed by the Reed-Muller mapping scheme for PAPR reduction. Thus, it uses the spreading factor 8 and 64 subcarriers to transmit \(Nw\) data bits in an OFDM symbol, which is the same as the uncoded systems. However, note that each 8-bit codeword of our MC-CDMA is fully-loaded to all the available spreading sequences of length 8 regardless of \(w\).

Figure 4 shows the results of \(\Pr(\text{PAPR} > \lambda_0)\) of each MC-CDMA for \(w = 4\) and 8. We observed that if \(w = 4\), our Reed-Muller mapped MC-CDMA system reduces the PAPR \(\lambda_0\) by about 3 dB to achieve \(\Pr(\text{PAPR} > \lambda_0) = 10^{-3}\), compared to the uncoded Walsh-Hadamard (WH) spread MC-CDMA employing the same spreading factor and the same number of subcarriers. Moreover, Theorem 7 ensures that our system has no signal with PAPR > 9 dB for \(w = 4\). Note that Theorem 7 determines the maximum PAPR of 9 dB (if \(w \leq 4\)), 12 dB (if \(5 \leq w \leq 6\)), and 15 dB (if \(7 \leq w \leq 8\)), respectively. Thus, even if the statistical PAPR property of our MC-CDMA is almost identical to that of the uncoded Walsh-Hadamard spread MC-CDMA for \(w = 8\), our system has no probability of signals with the PAPR higher than 15 dB, which is however unclear in the uncoded systems.

Similar to Figure 3, Figure 5 displays the PAPR \(\lambda_0\) of each MC-CDMA achieving \(\Pr(\text{PAPR} > \lambda_0) = 10^{-3}\) according to the number of users \(w, 1 \leq w \leq 8\). It shows that our MC-CDMA provides the smallest PAPR \(\lambda_0\) for almost all user numbers. Also, it numerically confirms that the theoretical maximums of PAPR in Theorem 7 hold for each user number. Figure 5 shows that if the number of active users is large, the statistical PAPR \(\lambda_0\) is much smaller than the theoretical maximum predicted by Theorem 7. For the small number of users, on the other hand, the coded MC-CDMA solves the high PAPR problem of the uncoded MC-CDMA by dramatically reducing its PAPR. Along with Figure 3, the coded MC-CDMA can be the best statistical and theoretical solution for PAPR reduction for any number of active users.
Fig. 4. PAPR performance of MC-CDMA systems. All the MC-CDMA systems transmit $N_w = 32$ or 64 information bits in an OFDM symbol employing the spreading factor 8 and 64 subcarriers. The code rate of the coded MC-CDMA is 1.0.

users regardless of its code rate. A drawback of the Reed-Muller mapped MC-CDMA is that it could be employed only for a small spreading factor due to the demapping complexity at the receiver.

VI. CONCLUSION

This paper has presented a coded MC-CDMA system where the information data is encoded by a Reed-Muller subcode for the sake of PAPR reduction. In the system, the codeword is then fully-loaded to Walsh-Hadamard spreading sequences, where the spreading and the despreading processes are efficiently implemented by the Walsh-Hadamard transform (WHT). We have established the polynomial representation of a coded MC-CDMA signal for theoretical analysis of the PAPR. We have then developed a recursive construction of the Reed-Muller subcodes which provide the transmitted MC-CDMA signals
with the bounded PAPR as well as the error correction capability. We have also investigated a theoretical connection between the code rates and the maximum PAPR in the coded MC-CDMA. Simulation results showed that the PAPR of the coded MC-CDMA signal is not only theoretically bounded, but also statistically reduced by the Reed-Muller coding schemes. In particular, it turned out that the coded MC-CDMA could solve the PAPR problem of uncoded MC-CDMA by dramatically reducing its PAPR for the small number of users. Finally, the theoretical and statistical studies exhibited that the Reed-Muller subcodes are effective coding schemes for peak power control in MC-CDMA with small and moderate numbers of users, subcarriers, and spreading factors. We believe this work gives us theoretical insights for PAPR reduction of MC-CDMA and S-OFDM by means of an error correction coding.
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