COMPUTING ALL AFFINE SOLUTION SETS OF BINOMIAL SYSTEMS
(EXTENDED ABSTRACT)

DANKO ADROVIC AND JAN VERSCHELDE

Abstract. To compute solutions of sparse polynomial systems efficiently we have to exploit the structure of their Newton polytopes. While the application of polyhedral methods naturally excludes solutions with zero components, an irreducible decomposition of a variety is typically understood in affine space, including also those components with zero coordinates. For the problem of computing solution sets in the intersection of some coordinate planes, the direct application of a polyhedral method fails, because the original facial structure of the Newton polytopes may alter completely when selected variables become zero. Our new proposed method enumerates all factors contributing to a generalized permanent and toric solutions as a special case of this enumeration. For benchmark problems such as the adjacent 2-by-2 minors of a general matrix, our methods scale much better than the witness set representations of numerical algebraic geometry.

1. Introduction

Our investigation in [3] starts with the sparsest kind of polynomial systems: those with exactly two monomials with nonzero coefficients in every equation. This sparsest type of systems is called binomial. Software implementations of primary decompositions of binomial ideals [9] are described in [6] and [19]. Recent algebraic algorithms are developed in [15] and [18]. The complexity of counting the total number of affine solutions of a system of \( n \) binomials in \( n \) variables was shown as #P-complete [7]. In [13] combinatorial conditions for the existence of positive dimensional solution sets are given, for use in a geometric resolution. Symbolic polyhedral algorithms for computing isolated roots of sparse systems are in [12].

2. Monomial Maps representing Affine Solution Sets

Solution sets of binomial systems can be described as monomial maps, obtained via unimodular coordinate transformations [1], see also [10] and [14]. Note that some sparse polynomial systems such as the cyclic \( n \)-roots problems have monomial maps as solution sets [2].

Definition 2.1. A monomial map of a \( d \)-dimensional solution set in \( \mathbb{C}^n \) is

\[
x_k = c_k t_1^{v_{1,k}} t_2^{v_{2,k}} \cdots t_d^{v_{d,k}}, \quad c_k \in \mathbb{C}, v_{i,k} \in \mathbb{Z},
\]

for \( i = 1, 2, \ldots, d \) and \( k = 1, 2, \ldots, n \).

For a toric solution, all coefficients \( c_k \) in the monomial map (1) are nonzero. For an affine solution set, several coordinates may be zero. When setting variables to zero, it may happen that all constraints on some other variables vanish, then we say that those variables are free, while others are still linked to a toric solution of a subset of the original equations.

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3. A Generalized Permanent

To enumerate all choices of variables to be set to zero, we use the matrix of exponents of the monomials to define a bipartite graph between monomials and variables.

**Definition 3.1.** Let \( f(x) = 0 \) be a system. We collect all monomials \( x^a \) that occur in \( f \) along the rows of the matrix, yielding the *incidence matrix*

\[
M_f[x^a, x_k] = \begin{cases} 
1 & \text{if } a_k > 0 \\
0 & \text{if } a_k = 0.
\end{cases}
\]

Variables which occur anywhere with a negative exponent are dropped.

**Example 3.2.** For all adjacent minors of a 2-by-3 matrix, the incidence matrix is

\[
M_f = \begin{bmatrix}
 x_{11} & x_{12} & x_{13} & x_{21} & x_{22} & x_{23} \\
1 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 & 0 \\
\end{bmatrix}
\]

for the system defined by \( f = (f_1, f_2) \) with \( f_1 = x_{11}x_{22} - x_{21}x_{12} \) and \( f_2 = x_{12}x_{23} - x_{22}x_{13} \).

For this example, the rows of \( M_f \) equal the exponents of the monomials. We select \( x_{12} \) and \( x_{22} \) as variables to be set to zero, as overlapping columns \( x_{12} \) with \( x_{22} \) gives all ones.

**Proposition 3.3.** Let \( S \) be a subset of variables such that for all \( x^a \) occurring in \( f(x) = 0 \): \( M[x^a, x_k] = 1 \), for \( x_k \in S \), then setting all \( x_k \in S \) to zero makes all polynomials of \( f \) vanish.

**Proof.** \( M[x^a, x_k] = 1 \) means: \( x_k = 0 \Rightarrow x^a = 0 \). If the selection of the variables in \( S \) is such that all monomials in the system have at least one variable appearing with positive power, then setting all variables in \( S \) to zero makes all monomials in the system vanish. \( \square \)

Enumerating all subsets of variables so that \( f \) vanishes when all variables in a subset are set to zero is similar to a row expansion algorithm on \( M_f \) for a permanent:

**Algorithm 3.4** (recursive subset enumeration via row expansion of permanent).

Input: \( M_f \) is the incidence matrix of \( f(x) = 0 \); index of the current row in \( M_f \); and \( S \) is the current selection of variables.

Output: all \( S \) that make the entire \( f \) vanish.

if \( M[x^a, x_k] = 1 \) for some \( x_k \in S \)
then print \( S \) if \( x^a \) is at the last row of \( M_f \) or else go to the next row
else for all \( k: M[x^a, x_k] = 1 \) do
\( S := S \cup \{ x_k \} \)
if \( x^a \) is at the last row of \( M_f \)
then print \( S \)
else go to the next row
\( S := S \setminus \{ x_k \} \)

Greedy enumeration strategies can be applied in the algorithm above. The enumeration may generate subsets of variables that lead to affine monomial maps that are contained in other solution maps. For detailed membership tests we refer to [3].
4. Computational Experiments

The polynomial equations of adjacent minors are defined in [11, page 631]:

\[ x_{i,j} x_{i+1,j+1} - x_{i+1,j} x_{i,j+1} = 0, \quad i = 1, 2, \ldots, m - 1, \quad j = 1, 2, \ldots, n - 1. \]

For \( m = 2 \), the solution set is pure dimensional of degree \( 2^n \) and of dimension \( 2^n - (n - 1) = n + 1 \), the number of irreducible components of \( X \) equals the \( n \)th Fibonacci number [20, Theorem 5.9].

For a pure dimensional set, we restrict the enumeration: for every variable we set to zero, one equation has to vanish as well. Table 1 shows the comparison with a witness set construction, computed with version 2.3.70 of PHCpack [21]. Note that our method returns the irreducible decomposition, which is more than just a witness set. This system is one of the benchmarks in [4], but neither Bertini [5] nor Singular [8] can get as far as our method.

| \( n \) | \( 2^{n-1} \) | \#maps | search | witness |
|-------|----------------|--------|--------|---------|
| 3     | 4              | 2      | 0.00   | 0.03    |
| 4     | 8              | 3      | 0.00   | 0.16    |
| 5     | 16             | 5      | 0.00   | 0.68    |
| 6     | 32             | 8      | 0.00   | 2.07    |
| 7     | 64             | 13     | 0.01   | 7.68    |
| 8     | 128            | 21     | 0.01   | 28.10   |
| 9     | 256            | 34     | 0.02   | 71.80   |
| 10    | 512            | 55     | 0.05   | 206.01  |
| 11    | 1024           | 89     | 0.10   | 525.46  |
| 12    | 2048           | 144    | 0.24   | —       |
| 13    | 4096           | 233    | 0.57   | —       |
| 14    | 8192           | 377    | 1.39   | —       |
| 15    | 16384          | 610    | 3.33   | —       |
| 16    | 32768          | 987    | 8.57   | —       |
| 17    | 65536          | 1597   | 21.36  | —       |
| 18    | 131072         | 2584   | 55.95  | —       |
| 19    | 262144         | 4181   | 140.84 | —       |
| 20    | 524288         | 6765   | 372.62 | —       |
| 21    | 1048576        | 10946  | 994.11 | —       |

Table 1. The construction of a witness set for all adjacent minors of a general 2-by-\( n \) matrix requires the tracking of \( 2^{n-1} \) paths and is much more expensive than the combinatorial search. For \( n \) from 3 to 21 column 3 lists times in seconds on one core at 3.49GHz for the combinatorial search and times (\(< 1,000 \) seconds) for the witness construction are in the last columns.

Table 2 shows timings of the binomialCellularDecomposition in the Binomials [16] package of Macaulay2 [17] applied to the ideal defined by the adjacent minors.

| \( n \) | 3    | 4    | 5    | 6    | 7    | 8    | 9    | 10   | 11   | 12   | 13   | 14   | 15   |
|--------|------|------|------|------|------|------|------|------|------|------|------|------|------|
| time   | 0.01 | 0.03 | 0.06 | 0.11 | 0.24 | 0.49 | 0.98 | 1.97 | 4.11 | 8.96 | 22.3 | 54.7 | 106.8 |

Table 2. CPU time in seconds on one 3.49GHz core on the adjacent minors.
As for the adjacent minors of a general 2-by-$n$ matrix the number of components returned by the cellular decomposition equals the number of components in an irreducible decomposition, the comparison seems fair.

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