Sharp Steklov upper bound for revolution submanifolds

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Abstract

In this note, we find sharp upper bound for the Steklov spectrum on revolution manifolds of the Euclidean space with one boundary component.

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1 Introduction

The Steklov eigenvalues of a smooth, compact, connected Riemannian manifold $(M, g)$ of dimension $n \geq 2$ with boundary $\Sigma$ are the real numbers $\sigma$ for which there exists a nonzero harmonic function $f : M \to \mathbb{R}$ which satisfies $\partial_\nu f = \sigma f$ on the boundary $\Sigma$. Here and in what follows, $\partial_\nu$ is the outward-pointing normal derivative on $\Sigma$. The Steklov eigenvalues form a discrete sequence $0 = \sigma_0 < \sigma_1 \leq \sigma_2 \leq \cdots \to \infty$, where each eigenvalue is repeated according to its multiplicity.

Recently, relationships between the geometry of boundary $\Sigma$ and the spectrum are very much investigated. In [4], the authors show that fixing only the geometry of the boundary and letting the arbitrary Riemannian metric inside $M$ is not sufficient to control the Steklov eigenvalues: they can be as large or as small as one wishes. On the other side, it was shown in [5, 8] that fixing $g$ in a neighborhood of $\Sigma$ has a much stronger influence on the spectrum.

In [3], the authors consider $n$-dimensional revolution manifolds $M$ of the Euclidean space $\mathbb{R}^{n+1}$ with one boundary component $S^{n-1} \subset \mathbb{R}^n \times \{0\}$. They show the sharp lower bound

$$\sigma_k(M) \geq \sigma_k(\mathbb{B}^n), \quad \text{for each } k \in \mathbb{N} \cup \{0\},$$

where $\mathbb{B}^n$ denotes the revolution manifold given by the $n$ dimensional Euclidean ball. For $n \geq 3$, in case of equality for one of the eigenvalues $\sigma_k$, $k \geq 1$, $M$ has to be isometric to $\mathbb{B}^n$. They also find an upper bound for $\sigma_k$ which is not sharp. Note that, for $n = 2$, all revolution manifolds with boundary $S^1$ are all isospectral (see [3], Proposition 1.10).

The goal of this work is to investigate sharp upper bounds for revolution manifold $M$ of the Euclidean space $\mathbb{R}^{n+1}$ with one boundary component $S^{n-1} \subset \mathbb{R}^n \times \{0\}$. We denote by

$$0 = \sigma_{(0)}(M) < \sigma_{(1)}(M) < \sigma_{(2)}(M) < \ldots$$

the distinct (counted without multiplicity) eigenvalues of the revolution manifold $M$. Now we state our main result.
Theorem 1. Let $M \subset \mathbb{R}^{n+1}$ be a $n$-dimensional revolution manifold with one boundary component isometric to the round sphere $S^{n-1}$. Then for $n \geq 3$, we have for each $k \geq 1$,

$$\sigma_k(M) < k + n - 2.$$  

Moreover, these bounds are sharp. For each $\epsilon > 0$ and each $k \geq 1$, there exists a revolution manifold $M$, such that $\sigma_k(M_\epsilon) > k + n - 2 - \epsilon$.

However, the inequality is strict: for each $k$, there does not exist $M$ such that $\sigma_k(M) = k + n - 2$.

Note that such bounds exist for abstract revolution metrics on the ball $\mathbb{B}^n$ if we add bounds on the curvature of $(M, g)$ (see [6], [7]). Basically in [6], the author considers Steklov problem on a ball with rotationally invariant metric under the assumption that radial curvature is bounded below (or bounded above) by some real number and proves a two sided bound for the Steklov eigenvalues. For warped product manifold with only one boundary component, the author in [7] has given a lower bound (upper bound) for Steklov eigenvalues under the hypothesis that the manifold has nonnegative (nonpositive) Ricci curvature and strictly convex boundary.

Theorem 1 will be a consequence of the study of the mixed Steklov Dirichlet and Steklov Neumann spectrum on an annulus and Proposition 6 telling that given a revolution hypersurface $M_1$ with one boundary component, it is always possible to construct another revolution hypersurface $M_2$ with larger Steklov eigenvalue.

The rest of the paper is organized as follows. In Section 2 we present the Steklov and mixed Steklov problem. In Section 3 we consider the specific situation of revolution hypersurfaces of the Euclidean space with one boundary component. Finally, in Section 4 we give the proof of Theorem 1.

2 Some general facts about Steklov and mixed problems

Let $(M, g)$ be a compact Riemannian manifold with boundary $\Sigma$. The Steklov eigenvalues of $(M, g)$ can be characterized by the following variational formula

$$\sigma_j(M) = \min_{E \in \mathcal{H}_j} \max_{0 \neq f \in E} R_M(f), \quad j \geq 0,$$

where $\mathcal{H}_j$ is the set of all $(j+1)$-dimensional subspaces in the Sobolev space $H^1(M)$ and

$$R_M(f) = \frac{\int_M |\nabla f|^2 dV_M}{\int_{\Sigma} |f|^2 dV_{\Sigma}}$$

is the Rayleigh quotient.

In order to obtain bound for $\sigma_j(M)$, we will compare the Steklov spectrum with the spectra of mixed Steklov-Dirichlet or Steklov-Neumann problems on domains $A \subset M$ such that $\Sigma \subset A$. Let $\partial_{\text{int}} A$ denote the intersection of the boundary of $A$ with the interior of $M$. Also, we suppose that it is smooth.

The mixed Steklov-Neumann problem on $A$ is the eigenvalue problem

$$\Delta f = 0 \text{ in } A, \quad \partial_{\nu} f = \sigma f \text{ on } \Sigma, \quad \partial_{\nu} f = 0 \text{ on } \partial_{\text{int}} A,$$

where $\nu$ denotes the outward-pointing normal to $\partial A$. The eigenvalues of this mixed problem form a discrete sequence

$$0 = \sigma_0^N(A) \leq \sigma_1^N(A) \leq \sigma_2^N(A) \leq \cdots \nearrow \infty,$$
and for each \( j \geq 0 \), the \( j \)th eigenvalue is given by

\[
\sigma_j^N(A) = \min_{E \in \mathcal{H}_j(A)} \max_{\nu \neq f \in E} \frac{\int_A |\nabla f|^2 \, dV_A}{\int_\Sigma |f|^2 \, dV_\Sigma},
\]

where \( \mathcal{H}_j(A) \) is the set of all \((j + 1)\)-dimensional subspaces in the Sobolev space \( H^1(A) \).

The mixed Steklov-Dirichlet problem on \( A \) is the eigenvalue problem

\[
\Delta f = 0 \text{ in } A,
\]

\[
\partial_\nu f = \sigma f \text{ on } \Sigma, \quad f = 0 \text{ on } \partial_{\text{int}}A.
\]

The eigenvalues of this mixed problem form a discrete sequence

\[
0 < \sigma_0^D(A) \leq \sigma_1^D(A) \leq \cdots \leq \infty,
\]

and the \( j \)th eigenvalue is given by

\[
\sigma_j^D(A) = \min_{E \in \mathcal{H}_{j,0}(A)} \max_{\nu \neq f \in E} \frac{\int_A |\nabla f|^2 \, dV_A}{\int_\Sigma |f|^2 \, dV_\Sigma},
\]

where \( \mathcal{H}_{j,0}(M) \) is the set of all \((j + 1)\)-dimensional subspaces in the Sobolev space \( H^1_0(A) = \{ u \in H^1(A) : u = 0 \text{ on } \partial_{\text{int}}A \} \).

For each \( j \in \mathbb{N} \), comparisons between the variational formulae give the following bracketing:

\[
\sigma_j^N(A) \leq \sigma_j(M) \leq \sigma_j^D(A). \tag{2}
\]

Note in particular that for \( j = 0 \), we have

\[
0 = \sigma_0^N(A) = \sigma_0(M) < \sigma_0^D(A).
\]

3 Revolution hypersurfaces of the Euclidean space

A compact revolution manifold \( M^n \) with one boundary component is a revolution metric on the \( n \)-dimensional ball. It can be seen as the warped product \([0, L] \times S^{n-1}\) with the Riemannian metric

\[
g(r, p) = dr^2 + h^2(p)g_0(p),
\]

where \((r, p) \in [0, L] \times S^{n-1}\), \(g_0\) is the canonical metric on the sphere of radius one and \(h \in C^\infty([0, L])\) satisfies \(h > 0\) on \([0, L], h'(L) = 1\) and \(h^2(L) = 0\) for all integers \(l \geq 0\). If we suppose that the boundary is the round sphere of radius one, we also have \(h(0) = 1\). Moreover, the fact that \(M\) is an \(n\)-dimensional revolution submanifold of the Euclidean space implies

\[
1 - r \leq h(r) \leq 1 + r.
\]

For more details, see [3].
3.1 Steklov spectrum and the eigenfunctions of Revolution hypersurface

The Steklov spectrum and the eigenfunctions of a revolution manifold with one connected component are very well explained in Proposition 8 of [7]. Before proceeding further, we would like to mention that by Laplace-Beltrami operator, we mean \( \Delta = -\text{div} \text{ grad} \), which is positive, whereas in [7], the author considers \( \Delta = \text{div} \text{ grad} \). This explains the difference of the signs in the following.

**Proposition 2.** Each eigenfunction \( f \) of the Steklov problem on Revolution hypersurface \( M \) can be written as \( f(r, p) = u(r)v(p) \), where \( v \) is a spherical harmonic of the sphere \( S^{n-1} \) of degree \( k \geq 0 \), i.e.,

\[
\Delta v = \lambda_k v \text{ on } S^{n-1},
\]

where \( \lambda_{(k)} = k(n-2-k) \) and \( u \) is a solution of the equation

\[
\frac{1}{h^{n-1}} \frac{d}{dr} \left( h^{n-1} \frac{d}{dr} u \right) - \frac{1}{h^2} \lambda_{(k)} u = 0
\]

on \((0, L)\) and under the condition \( u(L) = 0 \). The Steklov eigenvalue \( \sigma_{(k)} \) has the multiplicity same as \( \lambda_{(k)} \), \( k \)th eigenvalue (counted without multiplicity) of the round sphere \( S^{n-1} \).

For a proof and more details, see [7].

Roughly speaking, this comes from the fact that

\[
\Delta(u(r)v(p)) = -u''(r)v(p) - \frac{(n-1)h'}{h}u'(r)v(p) + \frac{u(r)}{h^2} \Delta_{S^{n-1}} v(p).
\]

If \( v \) is the \( k \)th eigenfunction of the sphere \( S^{n-1} \) (counted without multiplicity), we obtain

\[
\Delta(u(r)v(p)) = -\frac{1}{h^{n-1}} \frac{d}{dr} \left( h^{n-1} \frac{d}{dr} u \right)v(p) + \frac{u(r)}{h^2} \lambda_k v(p),
\]

and, because \( f \) is harmonic, we have

\[
-\frac{1}{h^{n-1}} \frac{d}{dr} \left( h^{n-1} \frac{d}{dr} u \right) + \lambda_k \frac{u(r)}{h^2} = 0.
\]

The condition \( u(L) = 0 \) comes from the fact that \( f \) has to be smooth at the point where \( r = L \).

3.2 The mixed Steklov-Dirichlet eigenvalues on annular domains

**Proposition 3.** Let \( B_1 \) and \( B_L \) be the balls in \( \mathbb{R}^n \), \( n \geq 3 \), centered at the origin of radius one and \( L \), respectively. Consider the following eigenvalue problem on \( \Omega_0 = B_L \setminus B_1 \)

\[
\begin{align*}
\Delta f &= 0 & \text{in } B_L \setminus B_1, \\
\frac{\partial f}{\partial \nu} &= \sigma^D f & \text{on } \partial B_L,
\end{align*}
\]

(3)

Then for \( 0 \leq k < \infty \),

\[
\sigma_{(k)}^D(\Omega_0) = \frac{k}{L^{2k+n-2} - 1} + \frac{(k + n - 2) L^{2k+n-2}}{L^{2k+n-2} - 1}.
\]
The eigenfunctions of \( f_k(r, p) = u(r)v(p) \), where \( v \) is an eigenfunction for the \( k^{th} \) eigenvalue on the sphere \( S^{n-1} \) and \( u \) is a real valued function defined on the interval \([1, L]\). For \( f_k(r, p) \) to be an eigenfunction, corresponding to the \( k^{th} \) eigenvalue (counting without multiplicity) of the mixed Steklov Dirichlet problem on \( \Omega_0 \), \( u \) should satisfy the following

\[
\begin{align*}
\Delta f &= 0 \quad \text{in } \Omega_0, \\
\frac{\partial f}{\partial n} &= 0 \quad \text{on } \partial \Omega_0, \\
\sigma^N f &= -\sigma^D f \quad \text{on } \partial \Omega_0.
\end{align*}
\]

Then for \( 0 \leq k < \infty \),

\[
\sigma^N_{(k)}(\Omega_0) = k \left( L_k^{(n+2k-2)} \right) / \left( L_k^{(n+2k-2)} + (n + k - 2) \right).
\]

Proof. Note that the eigenfunctions \( f_k(r, p) \) of \([\nabla] \) can be expressed as \( f_k(r, p) = u(r)v(p) \), where \( v \) is an eigenfunction for the \( k^{th} \) eigenvalue on the sphere \( S^{n-1} \) and \( u \) is a real valued function defined on \([1, L]\). If the function \( u \) corresponds to the \( k^{th} \) eigenvalue (counting without multiplicity) of the mixed Steklov-Neumann problem on \( \Omega_0 \), then

\[
\begin{align*}
\Delta f &= 0 \quad \text{in } \Omega_0, \\
u(1) &= 0, u'(1) = -\sigma^D f(1).
\end{align*}
\]

These conditions give

\[
\begin{align*}
akL_k^{k-1} - b(n + k - 2)L_k^{(n+k-1)} &= 0, \\
ka + (-k + 2 - n)b &= -\sigma^D f(a + b).
\end{align*}
\]

By eliminating \( a \) and \( b \), we obtain

\[
-(k + \sigma^N_{(k)})(n + k - 2)L_k^{(n+k-1)} + kL_k^{k-1}(n + k - 2 - \sigma^N_{(k)}) = 0
\]
and
\[ \sigma_{(k)}^N(kL^{k-1} + (n + k - 2)L^{-(k+n-1)}) = k(n + k - 2)(L^{k-1} - L^{-(k+n-1)}). \]

Multiplying by \( L^{(n+k-1)} \) to get
\[ \sigma_{(k)}^N(kL^{n+2k-2} + (n + k - 2)) = k(n + k - 2)(L^{(n+2k-2)} - 1), \]

and
\[ \sigma_{(k)}^N = \frac{(n + k - 2)(L^{(n+2k-2)} - 1)}{kL^{(n+2k-2)} + (n + k - 2)}. \]

\[ \square \]

4 Proof of the main theorem

4.1 Comparison of revolution hypersurfaces

Recall that for an \( n \)-dimensional revolution submanifold \( M \) of the Euclidean space \( \mathbb{R}^{n+1} \) with one boundary component \( S^{n-1} \subset \mathbb{R}^n \times \{0\} \) the induced Riemannian metric may be written as
\[ g(r,p) = dr^2 + h^2(r)go(p), \]
where \( go \) is the canonical metric of \( S^{n-1} \), \( r \in [0,L] \) and \( h(0) = 1, h(L) = 0, h(r) > 0 \) if \( 0 < r < L \), \( h'(L) = 0 \) and \( 1 - r \leq h(r) \leq 1 + r \).

Proposition 5. Let \( M = [0, L] \times S^{n-1} \) be a Riemannian manifold with metric \( g_i = dr^2 + h_i^2(r)g_{S^{n-1}}, \)
\( i = 1,2. \) Moreover suppose that \( h_1(0) = h_2(0) = 1 \) and \( h_1(r) \leq h_2(r) \). Consider the mixed Steklov-Neumann problem on \( M \) (Steklov at \( r = 0 \) and Neumann at \( r = L \)). Then the Rayleigh quotient of a function \( f(r,p) \) defined on \( M \) is given by
\[ R_{g_i}(f) = \frac{\int_0^L \int_{S^{n-1}} \left( \frac{\partial f}{\partial r} \right)^2 + \frac{1}{h_i(r)} \| \nabla f \|^2 \right) h_i^n(r)dr \ dv_{g_{S^{n-1}}}}{\int_{S^{n-1} \times \{0\}} f^2(0,p) \ dv_{g_{S^{n-1}}}}, \]
where \( \nabla \) is the exterior derivative in the direction of \( S^{n-1} \). The condition \( h_1(r) \leq h_2(r) \) gives that \( R_{g_1}(f) \leq R_{g_2}(f) \). Hence, we have \( \sigma_{(k)}^N(M, g_1) \leq \sigma_{(k)}^N(M, g_2) \) for all \( k \in \mathbb{N} \cup \{0\} \).

Proposition 6. For any hypersurface of revolution \( (M_1, g_1) \subset \mathbb{R}^{n+1}, \) with boundary \( S^{n-1} \times \{0\} \), there exists a hypersurface of revolution \( (M_2, g_2) \subset \mathbb{R}^{n+1} \) with the same boundary such that, for all \( k \geq 1, \sigma_{(k)}(M_2) > \sigma_{(k)}(M_1) \).

Proof. Note that \( M_1 \) will be of the form \( [0, L_1] \times S^{n-1} \) with metric \( g_1 = dr^2 + h_1^2(r)g_{S^{n-1}}, \) where \( h_1 \) satisfies \( h_1(0) = 1, |h_1(r)| \leq 1 \) and \( h_1(L_1) = 0 \). The condition \( |h_1'(r)| \leq 1 \) gives \( 1 - r \leq h_1(r) \leq 1 + r \).

Consider a hypersurface of revolution \( M_2 = [0, L_2] \times S^{n-1} \) with metric \( g_2 = dr^2 + h_2^2(r)g_{S^{n-1}}, \) where \( L_2 = 2L_1 + 2 \) and
\[ h_2(r) = \begin{cases} 1 + r, & \text{if } r \leq L_1, \\ L_2 - r, & \text{if } L_1 + 1 \leq r \leq L_2. \end{cases} \]
For $L_1 \leq r \leq L_1 + 1$, we just ask $h$ to join smoothly $h(L_1)$ and $h(L_1 + 1)$. Note that $h_1(r) \leq h_2(r)$ for $r \leq L_1$.

Now consider the mixed Steklov-Neumann problem on $\tilde{M} \equiv [0, L_1 - \epsilon] \times S^{n-1}$ with two metrics $g_1$ and $g_2$. Then from Proposition 5 we get $\sigma_k^N(M, g_1) < \sigma_k^N(M, g_2)$ for all $k \geq 1$. The strict inequality follows from Proposition 5 applied to eigenfunctions of $(\tilde{M}, g_1)$ and from the fact that there exist points in $\tilde{M}$ such that $h_2(r) > h_1(r)$ at those points.

Recall that because of the bracketing

$$\sigma_k(M_2, g_2) \geq \sigma_k^N(\tilde{M}, g_2), \quad k \in \mathbb{N} \cup \{0\}.$$ 

Using the method of Anné (see [2], Theorem 2, and [1] for a less general but easiest version of the result), we have that as $\epsilon \to 0$, $\sigma_k^N(M, g_1) \to \sigma_k(M_1, g_1)$. As a consequence, we get $\sigma_k(M_2, g_2) > \sigma_k(M_1, g_1)$. Note that the multiplicity of $\sigma_k(M_2, g_2)$ and $\sigma_k(M_1, g_1)$ is same for all $k \in \mathbb{N} \cup \{0\}$, this proves the result. \hfill \Box

Next we prove Theorem 1 by using Proposition 5.

**Proof.** Note that $M$ will be of the form $[0, L] \times S^{n-1}$ with metric $g = dr^2 + h^2(r)g_{S^{n-1}}$, where $h$ satisfies $h(0) = 1$, $|h'(r)| \leq 1$ and $h(L) = 0$.

Proposition 5 already shows that it is always possible to increase strictly the spectrum of $M$. Moreover, Proposition 5 gives the existence of a sequence of hypersurfaces of revolution $M_i = [0, L_i] \times S^{n-1}$, $1 \leq i < \infty$, with boundary $S^{n-1} \times \{0\}$ and metric $g_i = dr^2 + h_i^2(r)g_{S^{n-1}}$ ($h_i$ and $L_i$ are constructed as in Proposition 5) such that

$$\sigma_{(k)}(M) < \sigma_{(k)}(M_1) < \sigma_{(k)}(M_2) < \cdots .$$

Also, for $i \geq 2$,

$$\sigma^N_{(k)}(A_i) \leq \sigma_{(k)}(M_i) \leq \sigma^D_{(k)}(A_i),$$

where $A_i$ is a neighborhood of the boundary of $M_i$, which is annular domain with inner radius one and outer radius $1 + L_{i-1}$.

Moreover, we have $L_i \to \infty$ as $i \to \infty$. Note that for $k > 0$,

$$\lim_{i \to \infty} \sigma^D_{(k)}(A_i) = \lim_{i \to \infty} \sigma^N_{(k)}(A_i) = k + n - 2.$$ 

This shows $\lim_{i \to \infty} \sigma_{(k)}(M_i) = k + n - 2$. Combining this with the fact that $\sigma_{(k)}(M_i)$ is an increasing sequence proves the theorem. \hfill \Box

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