Derivatives of $p$-adic $L$-functions, Heegner cycles and monodromy modules attached to modular forms

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1 Introduction

Let \( f(z) \) be an elliptic newform of even weight \( k = n + 2 \geq 4 \) and level \( N \) and let \( p \) be a prime number. Let us assume that \( f(z) \) corresponds to a modular form on a Shimura curve via the theory of Jaquet-Langlands and let us fix a quadratic imaginary field \( K \). Throughout the introduction we will assume for simplicity that \( N \) is square free with an even number of prime factors, \( p|N \), all prime divisors of \( N \) are inert in \( K \) and finally that \( K \) has class number 1 (see section 5 and 8 for a more general set-up).

In such a situation in [BDIS] we have defined the anticyclotomic \( p \)-adic \( L \)-function of \( f(z) \) over \( K \), \( L_p(f/K,s) \) and have proved that it vanishes at the central critical point \( s = k/2 \). The main purpose of this paper is to prove that the derivative of \( L_p(f/K,s) \) at \( s = k/2 \) can be interpreted as the \( e_f \)-component (\( e_f \) is the idempotent in the Hecke algebra corresponding to \( f(z) \)) of the image of a Heegner cycle under a \( p \)-adic Abel-Jacobi map. The complex \( L \)-function of \( f(z) \) over \( K \) also vanishes at \( s = k/2 \) and its derivative can be expressed in terms of the height of a Heegner cycle. Therefore we consider our result as a \( p \)-adic Gross-Zagier type formula though the reader should be aware that no \( p \)-adic heights are involved (a short explanation of how our result fits into the broader picture of the Bloch-Beilinson conjectures is given in Remark 9.4).

With some slight modifications our argument would also give a similar result for the weight \( k = 2 \). If \( f(z) \) corresponds to an elliptic curve \( E/Q \) this is then a reformulation of the main result of [BD2]. We also remark that a \( p \)-adic Gross-Zagier type formula involving \( p \)-adic heights has been obtained by Nekovar [Ne3]. However the situation he deals with is different; he considers a two-variable \( p \)-adic \( L \)-function and the case where \( p \) does not devide the level and \( f(z) \) is ordinary at \( p \).

In establishing our formula we make extensive use of \( p \)-adic Hodge theory. Namely, if \( G_Q \to \text{GL}(V_p(f)) \) is the Galois representation attached to \( f(z) \) as in [De1], its restriction to a decomposition group at \( p \) is a semistable representation. Our proof relies on the fact that we are able to explicitly describe the semistable Dieudonné module attached to \( V_p(f) \) (via Fontaine’s theory) in terms of \( p \)-adic integration. We think that this description is interesting in itself and as a separate application we use it to prove that the \( L_p \)-invariants of Fontaine-Mazur and Teitelbaum attached to \( f(z) \) are equal.

Now we describe our results in more detail. Let \( X/Q \) be the Shimura
curve which classifies abelian surfaces with an action of a maximal order in the quaternion algebra over \( \mathbb{Q} \) of discriminant \( N \). The Jacquet-Langlands theory associates to \( f(z) \) a Hecke eigenform on \( X \) which by abuse of notation we also denote by \( f(z) \). Let \( F_f \) be the finite extension of \( \mathbb{Q} \) generated by the Hecke eigenvalues of \( f(z) \). The Galois representation \( V_p(f) \) attached to \( f(z) \) (two-dimensional as a \( F_f \otimes \mathbb{Q}_p \)-module) occurs in the \( p \)-adic cohomology of a certain Kuga-Sato variety. It can be also realized as a direct summand of the \( (n+1) \)-th \( p \)-adic cohomology of the \( m = \frac{n}{2} \)-self product \( \mathcal{A}^m \) of the universal abelian surface \( \mathcal{A} \) over \( X \) (in the introduction we neglect the fact that \( X \) is only a coarse moduli space; for precise statements we refer to section 5). In fact the Galois representation attached to the whole space of modular forms of weight \( k \) which we denote by \( H_p(\mathcal{M}_n) \) (since it is the \( p \)-adic realisation of a certain motive \( \mathcal{M}_n \)) can be identified with the first étale cohomology of \( X = X \otimes \mathbb{Q} \) with coefficients in a \( p \)-adic local system \( L \) which occurs in a certain tensor power of the first relative \( p \)-adic cohomology of \( \mathcal{A} \) over \( X \). As a representation of \( G_{\mathbb{Q}_p} \), \( H_p(\mathcal{M}_n) \) is semisimple. We can apply Fontaine’s theory which associates to local \( p \)-adic Galois representations simpler objects – filtered \((\phi,N)\)-modules – which still encode all the information. When restricting the Galois action to the inertia subgroup \( I_p \subseteq G_{\mathbb{Q}_p} \) and applying Fontaine’s functor \( D_{\text{st}} \) we obtain a filtered \((\phi,N)\)-module \( D_{\text{st}}(H_p(\mathcal{M}_n)) \) the semistable Dieudonné module of \( H_p(\mathcal{M}_n) \). That is \( D_{\text{st}}(H_p(\mathcal{M}_n)) \) is a finite dimensional vector space over the completion \( \mathbb{Q}_p^{ur} \) of the maximal unramified extension of \( \mathbb{Q}_p \) endowed with Frobenius \( \phi \) and monodromy operator \( N \). In section 3 we give a concrete description of \( D_{\text{st}}(H_p(\mathcal{M}_n)) \) by using the Čerednik-Drinfeld uniformisation of \( X_{\mathbb{Q}_p^{ur}} = X \otimes \mathbb{Q} \mathbb{Q}_p^{ur} \) by the \( p \)-adic upper half plane \( \mathcal{H}_p \). More precisely we can identify \( X_{\mathbb{Q}_p^{ur}} \) with the Mumford curve \( X_\Gamma = \Gamma \backslash \mathcal{H}_p \) (the group \( \Gamma \subseteq \text{GL}_2(\mathbb{Q}_p) \) is given in terms of the definite quaternion algebra over \( \mathbb{Q} \) of discriminant \( N/p \)). We show that \( D_{\text{st}}(H_p(\mathcal{M}_n)) \) is equal to \( H^1_{\text{DR}}(X_\Gamma, \mathcal{V}_n) \) where \( \mathcal{V}_n \) is a filtered \( F \)-isocrystal on \( X_\Gamma \) associated to the \( \text{GL}_2(\mathbb{Q}_p) \)-representation \( V_n : = n - \text{th symmetric power of } \mathbb{Q}_p^2 \). The Frobenius operator is given in terms of Coleman integration of 1-forms on \( \mathcal{H}_p \) and the monodromy operator in terms of Schneider’s residue map \( \omega \mapsto \text{Res}_e(\omega) \) which associates to a 1-form a harmonic cocyle on the Bruhat-Tits tree (see Theorem 5.9). We also have a result for open subschemes of \( X \) which is needed for the computation of the \( p \)-adic Abel-Jacobi image of Heegner cycles.

The main ingredients in the proof of Theorem 5.9 are a comparison theorem between \( p \)-adic étale cohomology of semistable curves and log-crystalline
cohomology (with coefficients) in \([Fa2]\) and the description of the log-crystalline cohomology groups in \([CI]\). These facts are reviewed in section 3. In section 4 we attach to a \(\mathbb{Q}_p\)-rational representation of \(\text{GL}_2\) a filtered \(F\)-isocrystal on \(X_\Gamma\). A key result is then that the local system corresponding to the filtered \(F\)-isocrystal attached to \(V_n\) is the sheaf \(\mathbb{L}_n\) (see Lemma 5.10).

The application to \(L\)-invariants is given in section 6. Recall that the \(L\)-invariant for a weight two modular form \(f(z)\) corresponding to an elliptic curve \(E/\mathbb{Q}\) with split multiplicative reduction at \(p\) is defined as the ratio \(L(f) = \log_p(q_E)/\text{ord}_p(q_E)\) where \(q_E\) is the Tate period of \(E\) over \(\mathbb{Q}_p\). In the higher weight case three possible definitions of \(L(f)\) have been given (by Teitelbaum \([Te]\), Coleman \([Co]\) and Fontaine-Mazur \([Ma]\)). The first and second are defined in terms of Coleman integration and residues on the Shimura curve and modular curve respectively whereas the last in terms of the semistable Dieudonné module \(D_{\text{st}}(V_p(f))\). As an application of Theorem 5.9 we will show that the first and last are equal. We remark that the work of \([CI]\) also establishes the equality of the Coleman and Fontaine-Mazur \(L\)-invariant (as explained in \([Co]\)) so that all three \(L\)-invariants are the same.

In section 9 we apply our results to Heegner cycles and \(p\)-adic \(L\)-functions. In \([BDIS]\) we proved the following formula describing the derivative of \(L_p(f/K,s)\) at \(s = k/2\) as a \(p\)-adic integral

\[
L_p'(f/K, k/2) = \int_{z_0}^{\bar{z}_0} f_p(z) P(z)^m \, dz.
\]

Here \(f_p(z)\) is the rigid analytic modular form on \(\mathcal{H}_p\) for \(\Gamma\) corresponding to \(f(z)\), \(P(z)\) a certain polynomial of degree 2 and \(z_0\) is a point on \(\mathcal{H}_p\) lying over a Heegner point on \(X\).

In section 9 we show that the right hand side of (1) can be interpreted as the \((e_f\)-component) of the image of a Heegner cycle on \(\mathcal{A}^m\) (defined in section 8) under a \(p\)-adic Abel-Jacobi map (the latter will be defined in section 7). We briefly explain the main steps of the proof. Firstly a Heegner cycle defines an extension of \(D_{\text{st}}(H_p(\mathcal{M}_n))[-(m+1)]\) by the trivial \((\phi, N)\)-module \(\mathbb{Q}_p^{ur}\). The group of extension classes \(\text{Ext}^1(\mathbb{Q}_p^{ur}, D_{\text{st}}(H_p(\mathcal{M}_n))[-(m+1)])\) can be identified with the dual of the space \(M_k(\Gamma)\) of weight \(k\) modular forms on \(\mathcal{H}_p\) for \(\Gamma\). It is then shown that under this isomorphism the extension class corresponding to the Heegner cycle is the functional

\[
f_p(z) \mapsto \int_{z_0}^{\bar{z}_0} f_p(z) P(z)^m \, dz.
\]
This last step uses a formula expressing the cup-product of 1-forms on $X_{\Gamma}$ in terms of the periods (this formula is a generalisation of a result of E. de Shalit in [4S2]).

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**Notation.**

For a field $K$ we let $\overline{K}$ be an algebraic closure and $G_K = \text{Gal}(\overline{K}/K)$ the absolute Galois group.

Let $L/K$ be a field extension. For a $K$-module $M$ we set $M_L = M \otimes_K L$. Also for a $K$-scheme $X$ we write $X_L = X \otimes_K L$.

For a $\mathbb{Q}$-module $M$ we write $M_{\mathbb{Q}_p} = M \otimes_{\mathbb{Q}} \mathbb{Q}_p$.

We denote by $\mathbb{C}_p$ the completion of the algebraic closure of $\mathbb{Q}_p$. We let $\mathbb{Q}_p^{ur} \subseteq \mathbb{C}_p$ be the closure of the maximal unramified extension of $\mathbb{Q}_p$ and $\mathbb{Z}_p^{ur}$ its valuation ring. For a positive integer $\mu$ we let $\mathbb{Q}_p^{\mu} \subseteq \mathbb{Q}_p^{ur}$ be the unramified extension of degree $\mu$ of $\mathbb{Q}_p$ and $\mathbb{Z}_p^{\mu}$ its ring of integers.

### 2 Filtered Frobenius monodromy modules

Let $K$ be a field of characteristic 0 which is complete with respect to a discrete valuation and has a perfect residue field $\kappa$ of characteristic $p > 0$. Let $K_0 \subseteq K$ be the maximal subfield of $K$ of absolute ramification index 1, i.e. $K_0$ is the quotient field of the ring of Witt vectors of $\kappa$.

Let $\sigma : K_0 \to K_0$ be the absolute Frobenius automorphism. In this section we deal with $\sigma$-linear algebra. Specifically we recall (form [4S2]) the relation between semistable $p$-adic representations of $G_K$ and filtered Frobenius monodromy modules (filtered $(\phi, N)$-modules for short) and then give a description of certain Ext-groups in the category of (filtered $(\phi, N)$-modules) which will be used in our discussion of the $p$-adic Abel-Jacobi map in section 7. We also introduce the notion of monodromy modules and their
\(L\)-invariants which is needed for the definition of the Fontaine-Mazur \(L\)-
invariant of modular forms.

A filtered \((\phi, N)\)-module consists of the following data:
1. A finite dimensional \(K_0\)-vector space \(D\) with an exhaustive and separated
decreasing filtration \(F^i D_K\) on \(D_K\) called Hodge filtration.
2. A \(\sigma\)-linear automorphism \(\phi = \phi_D : D \rightarrow D\) (the Frobenius of \(D\)).
3. A \(K\)-linear endomorphism \(N = N_D : D \rightarrow D\) (the monodromy operator)
such that \(N\phi = p\phi N\).

A filtered Frobenius module is a \((\phi, N)\)-module with trivial monodromy
operator. Morphisms of filtered \((\phi, N)\)-modules are \(K_0\)-linear maps which
respect the Frobenii, the filtrations and the monodromy operators. The
category of filtered \((\phi, N)\)-modules \(MF_K(\phi, N)\) is an additive tensor category
admitting kernels and cokernels. There is also the notion of a short exact
sequence of filtered \((\phi, N)\)-modules. We consider \(K\) as a filtered \((\phi, N)\)-
module by setting \(\phi_K = \sigma\), \(N = 0\) and \(F^i K = K\) (resp. \(= 0\)) for \(i \leq 0\) (resp.
\(i > 0\)). For \(D \in \text{Ob}(MF_K(\phi, N))\) its \(i\)-fold twist \(D[i]\) is defined as \(D[i] = D\)
as vector spaces, \(\phi_D[i] = p^i \phi_D\), \(F^j D[i]_K = F^{j - i} D_K\) and \(N_D[i] = N_D\).

The correspondence between filtered \((\phi, N)\)-modules and semistable Ga-
lois representations is given in terms of Fontaine’s ring \(B_{st}\) (defined in [Fo1]).
It is a topological \(K_0\)-algebra whose construction depends on choosing a branch
of the \(p\)-adic logarithm. It is equipped with the following structure:
1. A continuous action of \(G_K\) such that \(B_{st}^{G_K} = K_0\).
2. A \(G_K\)-equivariant embedding \(K_{0}^{ur} \rightarrow B_{st}\) of the maximal unramified
extension \(K_{0}^{ur}\) of \(K_0\).
3. A \(\sigma\)-linear continuous automorphism \(\phi : B_{st} \rightarrow B_{st}\) commuting with the
\(G_K\)-action.
4. An exhaustive and separated decreasing filtration \(F^i\) on \((B_{st})_K\) which is
stable under the \(G_K\)-action.
5. A \(K_0\)-linear operator \(N : B_{st} \rightarrow B_{st}\) such that \(N\phi = p\phi N\).

Let \(\text{Rep}(G_K)\) be the category of \(p\)-adic representations of \(G_K\), i.e. finite
dimensional \(\mathbb{Q}_p\)-vector spaces with a continuous linear \(G_K\)-action. It is an
abelian tensor category with twists given by tensoring with appropriate pow-
ers of the Tate representation \(\mathbb{Q}_p(1) = (\lim \mu_{p^n}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p\). For a \(p\)-adic representa-
tion \(V\) of \(G_K\) one defines \(D_{st}(V)\) : \((V \otimes B_{st})^{G_K}\). It inherits from \(B_{st}\) the
structure of a filtered \((\phi, N)\)-module. We have \(D_{st}(V(n)) = D_{st}(V)[-n]\). A
\(p\)-adic representation \(V\) of \(G_K\) is called semistable if the canonical (injective)
map
\[ \alpha : D_{st}(V) \otimes_{K_0} B_{st} \longrightarrow V \otimes_{Q_p} B_{st} \]
is bijective. The full subcategory \( \text{Rep}_{ss}(G_K) \) of semistable representations is an abelian tensor category.

Conversely given a filtered \((\phi, N)\)-module \( D \) the module \( D \otimes_{K_0} B_{st} \) has a natural structure as a filtered \((\phi, N)\)-module and one defines \( V_{st}(D) := \text{Hom}_{MF_K(\phi, N)}(K, D \otimes_{K_0} B_{st}) \). It is a \( p \)-adic representation of \( G_K \). The module \( D \) is called admissible if it is isomorphic to \( D_{st}(V) \) for some semistable representation \( V \) of \( G_K \).

The full subcategory \( \text{MF}_{ad}^K(\phi, N) \) of \( \text{MF}_K(\phi, N) \) of admissible filtered \((\phi, N)\)-modules is an abelian tensor category such that exact sequences remain exact in \( \text{MF}_K(\phi, N) \). Moreover the restriction of the functors \( D_{st} \) and \( V_{st} \) to \( \text{Rep}_{ss}(G_K) \) and \( \text{MF}_{ad}^K(\phi, N) \) are mutually quasi-inverse \( \otimes \)-equivalences.

Any finite dimensional \( K_0 \)-vector space \( D \) equipped with a bijective \( \sigma \)-linear endomorphism \( \phi : D \rightarrow D \) admits a canonical slope decomposition (see [Zi])
\[
D = \bigoplus_{\lambda \in \mathbb{Q}} D_{\lambda}.
\]

(2) For a rational number \( \lambda = \frac{r}{s}, r, s \in \mathbb{Z}, s > 0 \) the subspace \( D_{\lambda} \) of \( D \) (called the isotypical component of \( D \) of slope \( \lambda \)) is the largest subspace of \( D \) which has an \( O_{K_0} \)-stable lattice \( M \) with \( \phi^s(M) = p^r M \). The slopes of \( D \) are the rational numbers \( \lambda \) for which \( D_{\lambda} \neq 0 \). The pair \((D, \phi)\) is called isotypical of slope \( \lambda_0 \) if \( \lambda_0 \) is the only slope. If \( D \) is a filtered \((\phi, N)\)-module then \( N(D_{\lambda}) \subseteq D_{\lambda-1} \) for all \( \lambda \in \mathbb{Q} \).

**Lemma 2.1** Let \( D \) be a filtered \((\phi, N)\)-module, \( n \) be an integer and assume that \( N \) induces an isomorphism between the isotypical components \( D_n \) and \( D_{n-1} \). Then there is a canonical isomorphism
\[
\text{Ext}^1_{MF_K(\phi, N)}(K[n], D) \cong D/F^n.
\]

(3) **Proof.** Firstly we remark that though \( MF_K(\phi, N) \) is not an abelian category a group structure on \( \text{Ext}^1_{MF_K(\phi, N)}(K[n], D) \) can be defined in the usual way. For an extension
\[
eq 0 \rightarrow D \rightarrow E \rightarrow K[n] \rightarrow 0
\]

(4)
we obtain a diagram with exact rows

\[
\begin{array}{cccccc}
0 & \longrightarrow & D_n & \longrightarrow & E_n & \longrightarrow & K_0 & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & D_{n-1} & \longrightarrow & E_{n-1} & \longrightarrow & 0 & \longrightarrow & 0
\end{array}
\]

where the vertical arrows are induced by the respective monodromy operators. By assumption the first vertical map is an isomorphism. Hence the upper row admits a canonical splitting that is the map \( r : E \to K[n] \) has a canonical splitting \( s : K[n] \to E \) (compatible with Frobenius and monodromy operators but not necessarily with filtrations).

We define the map (3) by sending the class of (4) to \( s(1)+F^n \in E_K/F^n \cong D_K/F^n \). A simple computation shows that (3) is an isomorphism. \( \square \)

We finish this section with a discussion of monodromy modules and its \( \mathcal{L} \)-invariants. These notions were introduced in [Ma] (in fact we work here with a slightly different definition; see Remark 2.5 below). Let \( T \) be a finite dimensional commutative semisimple \( \mathbb{Q}_p \)-algebra. For simplicity we assume from now on that \( K = K_0 \).

**Definition 2.2** Let \( D \) be a \( T \)-object in \( MF_K(\phi, N) \), i.e. an object together with a \( \mathbb{Q}_p \)-algebra homomorphism of \( T \to \text{End}_{MF_K(\phi, N)}(D) \). \( D \) is called a (two-dimensional) monodromy \( T \)-module if the following conditions hold:

(i) \( D \) is a free \( T_K \)-module of rank 2.

(ii) The sequence \( D \xrightarrow{N} D \xrightarrow{N} D \) is exact.

(iii) There exists an integer \( j_0 \) such that \( F^{j_0}D \) is a free \( T_K \)-submodule of rank 1 and \( F^{j_0}D \cap \text{Ker}(N_D) = 0 \).

Note that condition (i) and (iii) imply that \( \text{Ker}(N_D) \) is a free \( T_K \)-submodule of rank 1. Hence (iii) implies that \( F^{j_0}D \oplus \text{Ker}(N_D) = D \).

**Lemma 2.3** Let \( D \) be a monodromy \( T \)-module. Then there exists a decomposition \( D = D^{(1)} \oplus D^{(2)} \) where \( D^{(1)}, D^{(2)} \) are \( \phi \)-stable free rank-1 \( T_K \)-submodules such that \( N : D \to D \) induces an isomorphism \( N|_{D^{(2)}} : D^{(2)} \to D^{(1)} \). Moreover the decomposition is uniquely determined by these properties.
Proof: We may assume that $T$ is a field. We claim that $D^{(1)} = \text{Ker}(N_D)$ is isotypical. For this it is enough to consider the case where $\kappa$ is algebraically closed. Let $\lambda \in \mathbb{Q}$ with $D^{(1)}(\lambda) \neq 0$. By a theorem of Dieudonné-Manin the algebra $\text{End}(D^{(1)}(\lambda), \phi)$ of the Frobenius isocrystal $(D^{(1)}(\lambda), \phi)$ is a central simple $\mathbb{Q}_p$-algebra of dimension $(\dim_K D^{(1)}(\lambda))^2$. Since $T \subseteq \text{End}(D^{(1)}(\lambda), \phi)$ we deduce that $\dim_K D^{(1)}(\lambda) \geq \dim_{\mathbb{Q}_p} T = \dim_K D^{(1)}$, hence $D^{(1)}(\lambda) = D^{(1)}$ as claimed.

Property (ii) implies now that $D = D_\lambda \oplus D_{\lambda+1}$, $D^{(1)} = D_\lambda$ and $N : D_{\lambda+1} \to D_\lambda$ is bijective. We define $D^{(2)} = D_{\lambda+1}$. $\square$

Definition 2.4 Let $D$ be a monodromy $T$-module, let $j_0$ be the integer appearing in part (iii) of definition 2.2 and $D = D^{(1)} \oplus D^{(2)}$ be the decomposition into $\phi$-stable free rank-1 $T_K$-submodules of $D$ as in lemma 2.3. The $L$-invariant $L(D)$ of $D$ is defined as the unique element in $T_K$ such that $x - L(D)N(x) \in F^{j_0}D_K$ for every $x \in D^{(2)}$.

Remark 2.5 (a) Let $\kappa'$ be a perfect extension field of $\kappa$ and let $K' \supseteq K$ be the quotient field of $W(k')$. Let $D$ be a $T$-object in $MF_K(\phi, N)$. Then $D$ is a monodromy $T$-module if and only if $D_L = D \otimes_K L$ is a monodromy $T$-module in $MF_L(\phi, N)$ and that in this case the $L$-invariants coincide $L(D) = L(D_L)$.

(b) The notion of a monodromy module as introduced in ([Ma], section 9) is not quite adequate for the application to modular forms. In fact with the notation as in ([Ma], section 12) the filtered $(\phi, N)$-module $D_p(f)$ considered there is a two-dimensional $F$-module but the $\phi$-action is $F$-linear (not $\sigma_F$-linear) and therefore $D_p(f)$ is neither a (two-dimensional) monodromy module in $MF_F(\phi, N)$ nor in $MF_{\mathbb{Q}_p}(\phi, N)$ if $F \neq \mathbb{Q}$ in the sense of ([Ma], sect. 9).

3 Covergent filtered $F$-isocrystals and Dieudonné modules

Let $K$ be as in section 3 and assume further that $K$ is of absolute ramification index 1, i.e. $K = K_0$. Let $\mathcal{O}_K$ be its ring of integers. Let $Z$ be a $p$-adic formal $\mathcal{O}_K$-scheme i.e. a formal $\mathcal{O}_K$-scheme locally of finite type in the sense of ([Ber], (0.2.1)). We assume that $Z$ is analytically smooth ([Og], p.772), so that the
rigid analytic $K$-variety $Z^{an}$ associated to $Z$ is smooth (for the construction of $Z^{an}$ we refer to [Ber], (0.2.1)).

We want to define the notion of a filtered $F$-isocrystal on $Z$. It is a convergent $F$-isocrystal $E$ together with a filtration on the coherent $\mathcal{O}_{Z^{an}}$-module $E^{an}$ on $Z^{an}$ associated to $E$ satisfying “Griffith transversality” with respect to the connection. To give a precise definition we first recall briefly the notion of an $F$-isocrystal from ([Og], 2.17).

An enlargement of $Z$ is a pair $(T, z_T)$ consisting of a flat formal $\mathcal{O}_K$-scheme $T$ and a morphism of formal $\mathcal{O}_K$-schemes $z_T : T_0 \rightarrow Z$, where the subscript 0 means the reduced closed subscheme of the closed subscheme of $T$ defined by the ideal $p\mathcal{O}_T$ ([Og], 2.1).

**Definition 3.1** A convergent isocrystal $E$ on $Z$ consists of the following data:

(a) For every enlargement $T = (T, z_T)$ of $Z$ a coherent $\mathcal{O}_T \otimes_{\mathcal{O}_K} K$-module $E_T$.

(b) For every morphism of enlargements $g : (T', z_{T'}) \rightarrow (T, z_T)$ an isomorphism of $\mathcal{O}_T \otimes_{\mathcal{O}_K} K$-modules

$$\theta_g : g^*(E_T) \rightarrow E_{T'}.$$  

The collection of isomorphisms $\{\theta_g\}$ is required to satisfy the cocycle condition.

If $T$ is an enlargement of $Z$ then $E_T$ may be interpreted as a coherent sheaf $E_T^{an}$ on the rigid space $T^{an}$ ([Og], 1.5). If we assume that $T$ is analytically smooth over $\mathcal{O}_K$ then there is a natural integrable connection

$$\nabla_T : E_T^{an} \rightarrow E_T^{an} \otimes \Omega^1_{T^{an}}$$  

(cf. [Og], 1.20, 2.81).

Note that the notion of an isocrystal on $Z$ depends only on the $\kappa$-scheme $Z_0$. More precisely there is an equivalence of categories between convergent isocrystals on $Z$ and on $Z_0$. We identify an isocrystal on $Z$ and $Z_0$ in the following. Let $F = F_{Z_0}$ denote the absolute Frobenius of $Z_0$.

**Definition 3.2** A convergent $F$-isocystal on $Z$ is a convergent isocrystal $E$ on $Z$ together with an isomorphism of isocrystals $\Phi : F^*E \rightarrow E$. 

\[10\]
We define filtered $F$-isocrystals only in the case where $Z$ is analytically smooth over $\mathcal{O}_K$ (which we assume from now on) so that $E^{an} := E^an_Z$ carries a natural connection $\nabla$.

**Definition 3.3** A filtered convergent $F$-isocrystal on $Z$ consists of an $F$-isocrystal $E$ together with an exhaustive and separated decreasing filtration $F^iE^{an}$ of coherent $\mathcal{O}^{an}_Z$-submodules such that $\nabla(F^i) \subseteq F^{i-1} \otimes \mathcal{O}^{an}_Z \Omega^1_X$ for all $i$.

The category of convergent filtered isocrystals on $Z$ is an additive tensor category.

**Examples 3.4** (a) The category of convergent filtered isocrystals on $Z$ is an additive rigid tensor category. The assignment $T \mapsto \mathcal{O}_T \otimes K$ – denoted by $\mathcal{O}Z$ – with the canonical Frobenius and filtration given by $F^i = \mathcal{O}_Z^{an}$ for $i \leq 0$ and $F^i = 0$ for $i > 0$ is the identity object.

(b) A filtered Frobenius module on $Z = \text{Spf}(\mathcal{O}_K)$ is a filtered Frobenius ($K$-)module in the sense of section 2.

(c) If $f : X \to Z$ is a smooth proper morphism of $p$-adic formal schemes then the $F$-isocrystal $R^df_*\mathcal{O}_{X/K}$ defined in ([Og], 3.1) – built using crystalline cohomology sheaves $\otimes K$ – is a convergent filtered $F$-isocrystal in a natural way. In fact the associated coherent $\mathcal{O}^{an}_Z$-module $(R^df_*\mathcal{O}_{X/K})^{an}$ is isomorphic to the relative de Rham cohomology $H^q_{\text{DR}}(X^{an}/Z^{an}) = R^qf_*\mathcal{O}^{an}_{X^{an}/Z^{an}}$ and the connection ([E]) coincides with the Gauss-Manin connection ([Og], 3.10). The filtration on $(R^df_*\mathcal{O}_{X/K})^{an} \cong H^q_{\text{DR}}(X^{an}/Z^{an})$ is the Hodge filtration.

Let $X \to \text{Spec}(\mathcal{O}_K)$ be a proper semistable curve with connected fibers. We assume that the generic fiber $X$ is smooth and projective and that the irreducible components $C_1, \ldots, C_r$ of the special fiber $C$ are all smooth and geometrically connected and that there is more than one of them. We assume moreover that the singular points of $C$ are all $\kappa$-rational ordinary double points. Let $E$ be a convergent filtered $F$-isocrystal on the formal completion $\hat{X}$ of $X$. The rigid analytic space $X^{an}$ associated to $X$ (see [Ber], 0.3) can be identified with $\hat{X}^{an}$ and the coherent $\mathcal{O}^{an}_X$-module $E^{an}$ with connection and filtration is defined algebraically i.e. there exists a coherent locally free $\mathcal{O}_X$-module $E$ with connection $\nabla : E \to E \otimes \mathcal{O}_X \Omega^1_X$ and filtration $F^iE$ by $\mathcal{O}_X$-submodules such that $\nabla(F^i) \subseteq F^{i-1} \otimes \mathcal{O}_X \Omega^1_X$ and the data yield the connection and filtration after passing to $E^{an}$.
In [Cq], R. Coleman has defined a structure of a filtered \((\phi,N)\)-module on the de Rham cohomology group \(H^{1}_{DR}(X,E)\) by using \(p\)-adic integration. We briefly recall his construction (for details we refer to [Cq] and [CI]).

The Frobenius and monodromy operator are defined using the admissible \(C\)-integral of elements \(\omega\) on the de Rham cohomology group \(X\). It is well defined up to a rigid horizontal section of \(E^{an}\) on the de Rham cohomology group \(X\). The left and right terms in the exact sequence (7) have natural Frobenius \(\Phi\) and monodromy \(\iota\) operators.

The Hodge filtration on \(H^{1}_{DR}(X,E)\) is defined as

\[
F^{i}H^{1}_{DR}(X,E) = \text{Im} \left( H^{1}(X,F^{i}E \overset{\nabla}{\rightarrow} F^{i-1}E \otimes \Omega_{X}^{1}) \rightarrow H^{1}(X,E \otimes \Omega^{*}) \right)
\]

The Frobenius and monodromy operator are defined using the admissible \(C\)-covering of elements \(\omega\) on \(X\). The vertices \(V(G)\) are the irreducible components of \(X\) and the oriented edges \(\overrightarrow{E}(G)\) are triples \((x,C_{i},C_{j})\) where \(x\) is a singular point of \(X\) and \(C_{i},C_{j}\) are the two components on which \(x\) lies. If \(e = (x,C_{i}C_{j})\) is an edge we set \(o(e) = C_{i}, t(e) = C_{j}\) and let \(\bar{e}\) be the opposite edge \((x,C_{j}C_{i})\). For a vertex \(v = C_{i}\) of \(G\) we let \(U_v = \text{red}^{-1}(C_{i})\) be the tube associated to it where \(\text{red} : X^{an} \rightarrow C(k)\) is the reduction map.

Similarly for an edge \(e = (x,C_{i}C_{j})\) we let \(A_{e}\) be the wide open annulus \(\text{red}^{-1}(x) = U_{o(e)} \cap U_{t(e)}\). The orientation of \(e\) induces an orientation on \(A_{e}\).

The open cover \(\{U_{v}\}_{v \in V(G)}\) of \(X^{an}\) is admissible. The corresponding Mayer-Vietoris sequence yields a short exact sequence

\[
0 \rightarrow \left( \bigoplus_{e \in \overrightarrow{E}(G)} H^{0}_{DR}(A_{e},E^{an}) \right)^{-}/\left( \bigoplus_{v \in V(G)} H^{0}_{DR}(U_{v},E^{an}) \right) \overset{i}{\rightarrow} H^{1}_{DR}(X,E) \rightarrow \text{Ker} \left( \bigoplus_{v \in V(G)} H^{1}_{DR}(U_{v},E^{an}) \rightarrow \bigoplus_{e \in \overrightarrow{E}(G)} H^{1}_{DR}(A_{e},E^{an}) \right) \rightarrow 0.
\]

The superscript \(-\) indicates the subspace of \(\bigoplus_{e \in \overrightarrow{E}(G)} H^{0}_{DR}(A_{e},E^{an})\) consisting of elements \(\{f_{e}\}_{e \in \overrightarrow{E}(G)}\) with \(f_{\bar{e}} = -f_{e}\) for all \(e \in \overrightarrow{E}(G)\).

The left and right terms in the exact sequence (7) have natural Frobenius \(\Phi\) and monodromy \(\iota\) operators. Moreover the map \(i\) admits a natural left inverse \(s\) defined as follows: by identifying \(H^{1}_{DR}(X,E)\) with the Cech hypercohomology of the covering \(\{U_{v}\}_{v \in V(G)}\) we can represent elements \(\omega \in H^{1}_{DR}(X,E)\) as pairs of collections \((\{\omega_{v}\}_{v \in V(G)}, \{f_{e}\}_{e \in \overrightarrow{E}(G)})\) where \(\omega_{v} \in (E^{an} \otimes \Omega_{X}^{1}(U_{v}))\) and \(f_{e} \in E^{an}(A_{e})\) are such that \(f_{\bar{e}} = -f_{e}\) and \(\omega_{o(e)}|_{A_{e}} - \omega_{t(e)}|_{A_{e}} = \nabla(f_{e})\) for all \(e \in \overrightarrow{E}(G)\). Then \(s(\omega)\) will be represented by the family \(\{g_{e}\}_{e \in \overrightarrow{E}(G)}\) where \(g_{e} = f_{e} - (\lambda_{o(e)}|_{A_{e}} - \lambda _{t(e)}|_{A_{e}})\). Here, if \(v \in V(G)\) then \(\lambda_{v}\) is a \(p\)-adic integral of \(\omega_{v}\). It is well defined up to a rigid horizontal section of \(E^{an}|_{U_{v}}\) once we have fixed a branch of the \(p\)-adic logarithm (which we assume from now on). The map \(s\) defines a splitting of the sequence (7) and the Frobenius \(\Phi\).
on $H_{DR}^1(X, E)$ is defined by requiring it to be compatible with the Frobenii on the right and left term and the splitting.

The monodromy operator is defined using residues. There is a natural residue map

$$\text{Res}_e : H_{DR}^1(A_e, E^{an}) \to H_{DR}^0(A_e, E^{an}) \simeq (E^{an}|_{A_e})_\nabla = 0.$$  

(to define it one has to assume that $E^{an}|_{A_e}$ has a basis of horizontal sections; however this is always the case as it is shown in [CI], section 3.2). The monodromy operator is defined as the composition

$$N = \iota \circ \left( \bigoplus_e \text{Res}_e : H_{DR}^1(X, E) \to H_{DR}^1(X, E) \right)$$

$$\omega \mapsto \iota \left( \{ \text{Res}_e(\omega|_{A_e}) \}_{e \in \mathcal{E}(\mathfrak{g})} \right) \quad (8)$$

It satisfies the relation $N\Phi = p\Phi N$.

We also need to consider the de Rham cohomology of an open subscheme $U$ of $X$ which is the complement of a finite number of points which specialise to smooth points on $C$. More precisely let $U = X - S$ where $S$ is a finite set of $K$-rational points of $X$ which – considered as points on $\mathfrak{X}$ – are all smooth and which specialise to pairwise different (smooth) points on $C$. Then one can define in a similar way a structure of a filtered $(\phi, N)$-module on $H_{DR}^1(U, E)$. In fact if we replace $X$ by $U$ and $U_v$ by $U_v - S$ in the sequence (7) and in (8) the sequence (7) is also exact and the monodromy operator is defined again by (8). Moreover $\iota$ has a left inverse defined as before. Note that the term on the left in (7) remains the same since $H_{DR}^0(U_v, E^{an})$ does not change if we remove a finite number of points from $U_v$. However to define the Frobenius on the right term (and thus on $H_{DR}^1(U, E)$) is more involved as one has to work with logarithmic isocrystals. Associated to the divisor $\sum_{P \in S} P$ on $\mathfrak{X}$ is a fine logarithmic structure $M_S$ on $\mathfrak{X}$ and $\mathfrak{X}$ (see [Ka]). By pulling back $\mathfrak{E}$ under the canonical morphism of formal log-schemes $j : (\mathfrak{X}, M) \to (\mathfrak{X}, \text{trivial log-structure})$ one gets a convergent log-$F$-isocrystal (a log-isocrystal is defined in terms of log-enlargements – a log-version of enlargements). Then the de Rham cohomology $H_{DR}^1(U_v - S, E^{an})$ can be described in terms of log-crystalline cohomology with coefficients in $j^*\mathfrak{E}$ of the component $C_i = v$ of $C$ and thus carries a Frobenius. For details we refer to [CI].

The Gysin sequence

$$0 \longrightarrow H_{DR}^1(X, E) \longrightarrow H_{DR}^1(U, E) \xrightarrow{\bigoplus_{x \in S} \text{Res}_x} \bigoplus_{x \in S} \mathcal{E}_x[1] \quad (9)$$

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becomes then an exact sequence of filtered \((\phi, N)\)-modules. Here \(\mathcal{E}_x[1]\) is the stalk of \(\mathcal{E}\) at \(x\); by Example 3.4(b) it is a filtered Frobenius module.

In \([CI]\) the following technical assumption on a convergent \(F\)-isocrystal \(\mathcal{E}\) on \(C\) is made in order to compare Coleman’s definition of the \((\phi, N)\)-module \(H^1_{\text{DR}}(X, \mathcal{E})\) with Faltings’ log-crystalline cohomology. Let \(i : x \to C\) be the embedding of a closed point in \(C\) and \(j : C^0_v \to C\) be the embedding of the complement of the singular points of \(C\) in a component. Then

**Assumption 3.5** The isocrystal \(H^0_{\text{crys}}(x, i^*(\mathcal{E}))\) is isotypical and its slope does not occur in the set of slopes of \(H^1_{\text{crys}}(C^0_v, j^*(\mathcal{E}))\).

Consider now the case where \(\mathcal{E}\) is a convergent filtered isocrystal of the type considered in Example 3.4(c). More precisely let \(f : Y \to X\) be a smooth proper morphism, \(Y = \mathfrak{Y}_K\) the generic fiber of \(\mathfrak{Y}\) and let \(H^q_{\text{DR}}(Y/X) = R^q f_* \mathcal{O}_\mathfrak{Y}/K\) be the convergent filtered \(F\)-isocrystal of 3.4(c). The results of \([Fa1, Fa2]\) and \([CI]\) imply the following.

**Theorem 3.6** (a) The representation \(H^1_{\text{ét}}(X, R^q f_* \mathbb{Q}_p)\) is semistable. If assumption 3.5 is satisfied for the isocrystal \(H^q_{\text{DR}}(Y/X)\) then \(D_{\text{st}}(H^1_{\text{ét}}(X, R^q f_* \mathbb{Q}_p))\) is canonically isomorphic as a filtered \((\phi, N)\)-module to \(H^1_{\text{DR}}(X, H^q_{\text{DR}}(Y/X))\).

(b) More generally let \(S\) be a finite set of smooth sections of \(f : X \to \text{Spec}(\mathcal{O}_K)\) which specialise to pairwise different (smooth) points on \(C\) and let \(U = X - S, \bar{U} = U \otimes_K \overline{K}\) and \(Y_x\) be the geometric fiber of \(f : Y \to X\) over \(x \in S\). Then we have an exact sequence of semistable Galois representations

\[
0 \to H^1_{\text{ét}}(X, R^q f_* \mathbb{Q}_p) \to H^q_{\text{ét}}(U, R^q f_* \mathbb{Q}_p) \to \bigoplus_{x \in S} H^q_{\text{ét}}(Y_x, \mathbb{Q}_p(-1))
\]

which is - after applying the functor \(D_{\text{st}}\) - isomorphic to the sequence (9) (for \(\mathcal{E} = H^q_{\text{DR}}(Y/X)\).

## 4 Convergent filtered \(F\)-isocrystals associated to representations of \(\text{GL}_2\)

Let \(M_2\) be the algebra scheme of \(2 \times 2\)-matrices and \(\text{GL}_2\) the group scheme of invertible elements in \(M_2\). By \(\mathcal{H}_p\) we denote the \(p\)-adic upper half plane over \(\mathbb{Q}_p\); it is a rigid analytic \(\mathbb{Q}_p\)-variety whose \(\mathbb{C}_p\)-valued points are \(\mathcal{H}_p(\mathbb{C}_p) = \mathbb{C}_p - \mathbb{Q}_p\). We denote by \(\hat{\mathcal{H}}\) the canonical formal model of \(\mathcal{H}_p\) over \(\mathbb{Z}_p^{ur}\). We
have a left $\text{GL}_2(\mathbb{Q}_p)$-action through linear transformations on $\mathcal{H}_p$. The set $\mathcal{H}_p(\mathbb{Q}_p^2)$ will be often identified with the set of $\mathbb{Q}_p$-algebra homomorphisms $\text{Hom}(\mathbb{Q}_p^2, M_2(\mathbb{Q}_p))$ as follows. Any $\psi \in \text{Hom}(\mathbb{Q}_p^2, M_2(\mathbb{Q}_p))$ defines an action of $\mathbb{Q}_p^2$ on $\mathcal{H}_p(\mathbb{Q}_p^2)$ and the point $z \in \mathbb{Q}_p^2 - \mathbb{Q}_p$ corresponding to $\psi$ is characterized by the property

$$\psi(a) \left( \begin{array}{c} z \\ 1 \end{array} \right) = a \left( \begin{array}{c} z \\ 1 \end{array} \right) \quad \forall a \in \mathbb{Q}_p^2.$$  

(10)

For an algebraic group $G$ over $\mathbb{Q}_p$ we let $\text{Rep}_{\mathbb{Q}_p}(G)$ be the category of finite-dimensional $\mathbb{Q}_p$-rational representations $\rho : G \to \text{GL}(V)$. In this section we construct for every representation $V$ in $\text{Rep}_{\mathbb{Q}_p}(\text{GL}_2 \times \text{GL}_2)$ a filtered $F$-isocrystal $\mathcal{E}(V)$ on $\hat{\mathcal{H}}$ and for every $\mathbb{Q}_p^2$-rational point $\psi \in \text{Hom}(\mathbb{Q}_p^2, M_2(\mathbb{Q}_p))$ a filtered Frobenius module $V_{\psi} \in \text{MF}_{\mathbb{Q}_p}(\phi, N)$ which turns out to be the fiber of $\mathcal{E}(V)$ at $\Psi$ (compare also [RZ], 1.31 for a related construction).

For an object $(V, \rho : \text{GL}_2 \times \text{GL}_2 \to \text{GL}(V))$ in $\text{Rep}_{\mathbb{Q}_p}(\text{GL}_2 \times \text{GL}_2)$ we also use the notation $(V, \rho_1, \rho_2)$ where $\rho_1 : \text{GL}_2 \to \text{GL}(V)$ (resp. $\rho_2$) is the restriction of $\rho$ to the first (resp. second) factor. If $(V_1, \rho_1), (V_2, \rho_2)$ are two representations of $\text{GL}_2$ we define $V_1 \circ V_2 = (V_1, \rho_1) \circ (V_2, \rho_2)$ to be the $\text{GL}_2 \times \text{GL}_2$-representation

$$V_1 \circ V_2 = (V_1 \otimes V_2, \rho_1 \otimes 1_{V_2}, 1_{V_1} \otimes \rho_2).$$

For $V \in \text{Rep}_{\mathbb{Q}_p}(\text{GL}_2)$ and $m \in \mathbb{Z}$ we write $V\{m\} : = V \otimes \text{det}^{\otimes m}$, $\{m\}V : = \text{det}^{\otimes m} \circ V$.

A representation $(V, \rho) \in \text{Rep}_{\mathbb{Q}_p}(\text{GL}_2)$ is said to be pure of weight $n$ if every element $a$ of the center $Z(\text{GL}_2) = \mathbb{G}_m$ acts by multiplication with $a^n$. In general $(V, \rho)$ can be written as

$$\rho(n) = \bigoplus_{n \in \mathbb{Z}} (V(n), \rho^{(n)})$$

where $(V^{(n)}, \rho^{(n)})$ is pure of weight $n$.

For $n \geq 0$ let $\mathcal{P}_n$ denote the vector space of polynomials of degree $\leq n$ over $\mathbb{Q}_p$ with right $\text{GL}_2$-action

$$P(X) \cdot A = (cX + d)^n P(\frac{aX + b}{cX + d}) \quad A = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \text{GL}_2, P(X) \in \mathcal{P}_n.$$
Therefore the dual $V_n^\ast = \mathcal{P}_n^\ast = \text{Hom}_{\mathbb{Q}_p}(\mathcal{P}_n, \mathbb{Q}_p)$ has a left $GL_2$-action given by

$$\forall \, P(X) \in \mathcal{P}_n : (A \cdot R)(P(X)) = R(P(X) \cdot A)$$

We need the following simple fact (see [De2], Proposition 3.1).

**Lemma 4.1** Any object of $\text{Rep}_{\mathbb{Q}_p}(GL_2)$ is a direct summand of a sum of representations of the form $V^{\otimes m} \otimes (V^\vee)^{\otimes n}, m, n \geq 0$.

For $(V, \rho_1, \rho_2)$ in $\text{Rep}_{\mathbb{Q}_p}(GL_2 \times GL_2)$ let $\phi_V = \rho_2\left(\begin{pmatrix} 0 & p \\ 1 & 0 \end{pmatrix}\right) : V \to V$.

Note that if $(V, \rho_1, \rho_2)$ is of the form $V\{m\}$ with $V \in \text{Ob}(\text{Rep}_{\mathbb{Q}_p}(GL_2))$ then $\phi_V\{m\} = p^m 1_V$.

For $\Psi \in \text{Hom}(\mathbb{Q}_p^2, M_2(\mathbb{Q}_p))$ the underlying vector space of the filtered Frobenius module $V_\Psi$ is $V_{Qur}$. The filtration is defined in terms of the first $GL_2$-action $\rho_1$ solely. Consider first the case where $(V, \rho_1)$ is pure of weight $n$. Let $V^{\otimes r}_{Qur, i}$ be the subspace of $v \in V^{\otimes r}_{Qur}$ such that $\rho_1(\Psi(a))(v) = a^i \sigma(a)^{n-i}v$ for all $a \in \mathbb{Q}_p^\times$ and $F^i_{\Psi}V^{\otimes r}_{Qur} = \bigoplus_{j \geq i} V^{\otimes r}_{Qur, j}$.

For arbitrary $V$ we use the decomposition (11) and set

$$F^i_{\Psi}V^{\otimes r}_{Qur} = \bigoplus_{n \in \mathbb{Z}} F^i_{\Psi}V^{(n)}_{Qur}$$

Then $F^\ast_\Psi$ is an exhaustive and separated filtration on $V^{\otimes r}_{Qur}$ and we define $V_\Psi$ to be $V^{\otimes r}_{Qur}$ together with Frobenius $\Phi = \phi_V \otimes \sigma$ and filtration $F^\ast_\Psi$. The functor $V \mapsto V_\Psi$ from $\text{Rep}_{\mathbb{Q}_p}(GL_2 \times GL_2)$ to $MF^{\otimes r}_{Qur}(\phi, N)$ is an exact tensor functor.

The convergent filtered $F$-isocrystal $E(V)$ associated to $(V, \rho_1, \rho_2)$ in $\text{Rep}_{\mathbb{Q}_p}(GL_2 \times GL_2)$ is defined as follows. As an isocrystal it is just $V \otimes_{\mathbb{Q}_p} \mathcal{O}_{\hat{H}_p}$ where $\mathcal{O}_{\hat{H}_p}$ is the isocrystal of example (3.4) (a). The Frobenius is given by $\phi_V \otimes \Phi_{\mathcal{O}_{\hat{H}_p}}$. The filtration is again defined in terms of $\rho_1 : GL_2 \to GL(V)$ only. Consider first the case $(V, \rho_1) = V_1$. We have a canonical map of sheaves on $\mathcal{H}_p$

$$V_1 \otimes \mathcal{O}_{\mathcal{H}_p} \to \mathcal{O}_{\mathcal{H}_p}, \quad R \otimes f \mapsto (z \mapsto R(X - z)f(z))$$

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\( R(X - z) = R(X) - zR(1) \). The filtration of \( V_1 \otimes_{\mathbb{Q}_p} \mathcal{O}_{\hat{\mathcal{H}}} \) is given by \( F^0 = V_1 \otimes_{\mathbb{Q}_p} \mathcal{O}_{\hat{\mathcal{H}}} \), \( F^2 = 0 \) and \( F^1 \) is the kernel of \([-I^3] \). We get an induced filtration on \( V_1^{\mathbb{Z}} \otimes (V_1')^{\mathbb{Z}} \otimes_{\mathbb{Q}_p} \mathcal{O}_{\hat{\mathcal{H}}} \) for all \( m, n \geq 0 \) and thus by Lemma 4.1 on \( \mathcal{E}(V)^{an} \) for an arbitrary representation \( V \) in \( \text{Rep}_{\mathbb{Q}_p}(\text{GL}_2 \times \text{GL}_2) \).

Again the assignment \( V \mapsto \mathcal{E}(V) \) is an exact tensor functor. For an even integer \( n = 2m \) we write \( V_n \) for \( \mathcal{E}(V,\{m\}) \).

**Lemma 4.2** For \( (V, \rho_1, \rho_2) \) in \( \text{Rep}_{\mathbb{Q}_p}(\text{GL}_2 \times \text{GL}_2) \) and \( \Psi \in \text{Hom}(\mathbb{Q}_p^2, M_2(\mathbb{Q}_p)) \) we have \( \mathcal{E}(V)_\Psi \cong V_\Psi \), where \( \mathcal{E}(V)_\Psi \) is the stalk of \( \mathcal{E}(V) \) at \( \Psi \).

**Proof.** That the Frobenii are the same is obvious. To see that the filtrations also agree it is enough, by Lemma 4.1, to consider the case \( (V, \rho_1) = V_1 \). Note that \((V_1 \otimes_{\mathbb{Q}_p} \mathcal{O}_{\hat{\mathcal{H}}})_\Psi \) is generated by \( ev_z \in (V_1')^{\mathbb{Z}} = \text{Hom}(\mathcal{P}_1, \mathbb{Q}_p^{ur}) \), \( ev_z(P(X)) = P(z) \) and that

\[
(\Psi(a) \cdot ev_z)(P(X)) = aP(z) = a(ev_z)(P(X)) \quad \forall a \in \mathbb{Q}_p^{ur}, P(X) \in \mathcal{P}_1.
\]

Here \( z \in \mathbb{Q}_p^2 - \mathbb{Q}_p \) is the point corresponding to \( \Psi \) via \([-I^3] \). \( \square \)

The filtered isocrystals \( \mathcal{E}(V) \) constructed above also descend to isocrystals on Mumford curves. Let \( \Gamma \subseteq \text{SL}_2(\mathbb{Q}_p) \) be a discrete cocompact subgroup and \( X = X_\Gamma \) the associated Mumford curve over \( \mathbb{Q}_p^{ur} \) so that \( X^{an}_\Gamma = \Gamma \backslash \mathcal{H}_p \). Assume first that \( \Gamma \) is torsionfree. The action of \( \text{GL}_2 \) on \( \mathcal{E}(V) \) is compatible with the action on \( \mathcal{H}_p \) and therefore \( \mathcal{E}(V) \) descends to a filtered isocrystal on \( X_\Gamma \) which will be also denote by \( \mathcal{E}(V) \). We let \( E(V) \) be the associated coherent locally free \( \mathcal{O}_{X_\Gamma} \)-module \( E \) with connection and filtration.

We will give now a concrete description of the \((\phi, N)\)-module structure on \( H^1_{\mathcal{D}R}(X_\Gamma, E(V)) \). For this we have to recall the description of the special fiber \( \mathcal{H}_s \) of \( \mathcal{H} \). Let \( \mathcal{T} \) be the Bruhat-Tits tree for \( \text{PGL}_2(\mathbb{Q}_p) \); the vertices are homothety classes of lattices in \( \mathbb{Q}_p^2 \). Two vertices \( v_1, v_2 \) are adjacent if one can choose representing lattices \( M_1, M_2 \) with \( M_1 \subset M_2 \subset pM_1 \). An oriented edge of \( \mathcal{T} \) is a pair \( e = (v_1, v_2) \) where \( v_1, v_2 \) are adjacent vertices. We set \( o(e) = v_1, t(e) = v_2 \) and \( \bar{e} = (v_2, v_1) \).

The set of components of \( \mathcal{H}_s \) is in one-to-one correspondence to the set vertices \( \mathcal{V}(\mathcal{T}) \) of \( \mathcal{T} \), each component being isomorphic to the projective line over \( \mathbb{F}_p \). We write \( \{\mathbb{P}^1_{\mathbb{F}_p}\}_{v \in \mathcal{V}(\mathcal{T})} \) for the set of components of \( \mathcal{H}_s \). The singular points of \( \mathcal{H}_s \) are ordinary \( \mathbb{F}_p \)-rational double points; they correspond to the
set of (unoriented) edges of $\mathcal{T}$; two components $P^1, P^2$ intersect if and only if $v_1, v_2$ are adjacent and in this case transversally in an $\mathbb{F}_p$-rational point. The curve $X_\Gamma$ has a canonical semistable model $X_\Gamma$ whose formal completion is $\Gamma \setminus \mathcal{H}$. Hence the special fiber of $X_\Gamma$ is $\Gamma \setminus \mathcal{H}$.

Let $C^0(V_{Q_{p^r}})$ (resp. $C^1(V_{Q_{p^r}})$) be the set of maps $f : \mathcal{V}(\mathcal{T}) \to V_{Q_{p^r}}$ (resp. maps $f : \mathcal{E}(\mathcal{T}) \to V_{Q_{p^r}}$ such that $f(\bar{e}) = -f(e)$ for all $e \in \mathcal{E}(\mathcal{T})$). The short exact sequence

$0 \to V_{Q_{p^r}} \to C^0(V_{Q_{p^r}}) \xrightarrow{\partial} C^1(V_{Q_{p^r}}) \to 0$

(where $\partial(f)(e) = f(o(e)) - f(t(e))$) yields an isomorphism

$\epsilon : C^1(V_{Q_{p^r}}) / C^0(V_{Q_{p^r}}) \cong H^1(\Gamma, V_{Q_{p^r}})$.

From the above description of $X_\Gamma$ we see that the left term in the sequence (7) is equal to $C^1(V_{Q_{p^r}})$ and therefore can be replaced by $H^1(\Gamma, V_{Q_{p^r}})$. In this situation we write $P$ instead of the splitting $s$ introduced in the last section. Explicitly it is given as follows. We identify $H^1_{DR}(X_\Gamma, E(V))$ with the space of $V$-valued $\Gamma$-invariant meromorphic differential forms of the second kind on $\mathcal{H}_p$ (modulo exact forms). Given such a form $\omega$ we let $F_\omega$ be a primitive of it (see [dS1]). Then $P$ is given by

$P : H^1_{DR}(X_\Gamma, E(V)) \to H^1(\Gamma, V_{Q_{p^r}}), \omega \mapsto (\gamma \mapsto \gamma(F_\omega) - F_\omega)$

The monodromy operator $N$ is the composite of $\iota \circ (-\epsilon) \circ I$ where $I$ is Schneider’s integration map (see [dS3])

$I : H^1_{DR}(X_\Gamma, E(V)) \to C^1(V_{Q_{p^r}}), \omega \mapsto (e \mapsto \text{Res}_e(\omega)).$

As for the Frobenius we first note that under the isomorphism (14) we obtain a Frobenius on $H^1(\Gamma, V_{Q_{p^r}})$. It is the one induced by the map $\phi_V \otimes \sigma$ on the coefficients $V_{Q_{p^r}}$.

We will assume from now on that $\Gamma$ is arithmetic (see [dS1]). We need the following lemma to characterise the Frobenius.

**Lemma 4.3** Assume that $\Gamma$ is arithmetic. Then sequence

$0 \to H^1(\Gamma, V_{Q_{p^r}}) \xrightarrow{\iota} H^1_{DR}(X_\Gamma, E(V)) \xrightarrow{I} C^1(V_{Q_{p^r}}) / C^0(V_{Q_{p^r}}) \to 0$

is exact.
Proof. We may assume that \( V \) is an irreducible representation of \( \text{GL}_2 \). Then \( V \cong V_n \otimes \text{det}^\otimes m \) for some \( m, n \in \mathbb{Z}, n \geq 0 \), hence \( V \cong V_n \) as a \( \Gamma \)-module. The assertion follows now from (\[\text{IS}1\] 3.9) and (\[\text{IS}3\] 1.6).

Hence the sequence

\[
H^1_{DR}(X_{\Gamma}, E(V)) \xrightarrow{N} H^1_{DR}(X_{\Gamma}, E(V)) \xrightarrow{N} H^1_{DR}(X_{\Gamma}, E(V))
\]

is exact and it follows that there is a unique Frobenius operator on \( H^1_{DR}(X_{\Gamma}, E(V)) \) satisfying \( N\phi = p\phi N \) and which is compatible (with respect to \( \iota \) and \( P \)) with the Frobenius on \( H^1(\Gamma, V_{\text{Qur}}^p) \).

It is easy to see that if \((V, \rho_2)\) is pure of weight \( n \) then the isocrystal \( E(V) \) on \( X_{\Gamma} \) satisfies assumption \(3.3\) (in fact with the notation there \( H^0_{\text{crys}}(x, j^*(E(V))) \) and \( H^1_{\text{crys}}(X_{\Gamma}, j^*(E(V))) \) are isotypical of slope \( n/2 \) and \( n/2 + 1 \) respectively).

If \( \Gamma \) is not torsionfree we define a filtered \((\phi, N)\)-module \( H^1_{DR}(X_{\Gamma}, E(V)) \) as follows. We choose a free normal subgroup of finite index \( \Gamma' \subseteq \Gamma \). The group \( \Gamma/\Gamma' \) acts then on \( H^1_{DR}(X_{\Gamma'}, E(V)) \) as automorphisms of a filtered \((\phi, N)\)-module and thus the filtered \((\phi, N)\)-module structure on \( H^1_{DR}(X_{\Gamma'}, E(V)) \) induces one on \( H^1_{DR}(X_{\Gamma}, E(V)) \). This construction is clearly independent of the choice of \( \Gamma' \).

In the case of an open subscheme \( U = X_{\Gamma} - S \) of the type considered in the last section the description of the \((\phi, N)\)-module structure on \( H^1_{DR}(U, E(V)) \) is very similar. The monodromy operator is defined as before. Let \( \pi : \mathcal{H}_p \rightarrow X_{\Gamma}^\text{an} \) be the canonical projection. We identify \( H^1_{DR}(U, E(V)) \) with the space of \( V \)-valued \( \Gamma \)-invariant meromorphic differential forms on \( \mathcal{H}_p \) which are of the second kind when restricted to \( \pi^{-1}(U) \). The left inverse \( P : H^1_{DR}(U, E(V)) \rightarrow H^1(\Gamma, V_{\text{Qur}}^p) \) of \( \iota \) is defined by the same formula (\[\text{IF}3\]). We choose for each \( x \in S \) a point \( \Psi_x \in \text{Hom}(\mathbb{Q}_p^2, M_2(\mathbb{Q}_p)) \) with \( \pi(\Psi_x) = x \). One can easily verify that there is a unique Frobenius on \( H^1_{DR}(U, E(V)) \) such that the Gysin sequence

\[
0 \longrightarrow H^1_{DR}(X, E(V)) \longrightarrow H^1_{DR}(U, E(V)) \xrightarrow{\oplus_{x \in S} \text{Res}_x} \oplus_{x \in S} V_{\Psi_x}[1]
\]

is a sequence of \((\phi, N)\)-modules and such that \( P \) is compatible with Frobenius.
5  Representations and Dieudonné module attached to modular forms

Let $f_\infty$ be a modular form of a fixed even weight $k \geq 4$ and level $N$ and assume that the prime number $p$ divides $N$ exactly once. Under certain conditions on the level one can associate to $f_\infty$ a modular form $f$ on a Shimura curve via the Jacquet-Langlands correspondence and the Galois representation associated to $f_\infty$ can be identified with a direct summand, denoted by $V_p(f)$, in the $p$-adic cohomology group of a Kuga-Sato variety over the Shimura curve. In this section we will define these representations and show – using the results of the previous sections and the Theorem of Cerednik-Drinfeld – that they are semistable as representations of the decomposition group at $p$ and describe explicitly their Dieudonné modules using $p$-adic integration.

To begin with we fix some notation. Put $n = k - 2$, $m = \frac{n}{2}$. For the rest of this paper, $N^-$ denotes a positive squarefree integer with an odd number of prime divisors none of which equals $p$, $N^+$ denotes a positive integer relatively prime to $pN^-$ and $N$ the product $pN^-N^+$. Let $B$ be the indefinite quaternion algebra over $\mathbb{Q}$ of discriminant $pN^-$. Let $\mathcal{B}$ be an Eichler order of level $N^+$ contained in $\mathcal{B}^\max$. Associated to these data is a Shimura curve $X = X_{N^+,pN^-}$ whose description as a coarse moduli scheme we briefly recall from [3C].

**Definition 5.1** Let $S$ be a $\mathbb{Q}$-scheme. An abelian surface with quaternionic multiplication (by $\mathcal{R}^\max$) and level $N^+$-structure over $S$ (abelian surface with $\text{QM}$ for short) is a triple $(A, \iota, C)$ where
1. $A$ is an abelian scheme over $S$ of relative dimension 2;
2. $\iota : \mathcal{R}^\max \to \text{End}_S(A)$ is an inclusion defining an action of $\mathcal{R}^\max$ on $A$;
3. $C$ is a subgroup scheme of $A$ which is locally isomorphic to $\mathbb{Z}/N^+\mathbb{Z}$ and is stable and locally cyclic under the action of $\mathcal{R}$.

**Definition 5.2** The Shimura curve $X = X_{N^+,pN^-}/\mathbb{Q}$ is the coarse moduli scheme of the moduli problem

$$S \mapsto \{ \text{isomorphism classes of abelian surfaces with QM over } S \}.$$ 

The scheme $X$ is a smooth projective geometrically connected curve over $\mathbb{Q}$. We recall its description as a Mumford curve over $\mathbb{Q}_p^2$. Let $B/\mathbb{Q}$ be
the definite quaternion algebra over $\mathbb{Q}$ of discriminant $N^-$ and let $R$ be an Eichler $\mathbb{Z}[\frac{1}{p}]$-order of level $N^+$ in $B$. By fixing an isomorphism $B_p \cong M_2(\mathbb{Q}_p)$ the group $\Gamma$ of elements of reduced norm 1 in $R$ can be viewed as a discrete cocompact subgroup of $\text{SL}_2(\mathbb{Q}_p)$.

**Theorem 5.3** (Čerednik-Drinfeld; see [BC], Chapitre III, 5.3.1, [Ce], [Dr])

We have $X_{\mathbb{Q}_p^2} \cong X_\Gamma$.

We recall now the Jacquet-Langlands correspondence between cusp forms of weight $k$ and level $N$ which are $pN^-$-new and modular forms of weight $k$ on $X$. Let us first clarify the latter notion.

**Definition 5.4** Let $K$ be a field of characteristic 0. A $K$-valued modular form of weight $k$ on $X$ is a global section of $\Omega_{X/K}^\otimes m+1$. We denote the space of these modular forms by $M_k(X, K)$.

We remark that over the field $K = \mathbb{Q}_{ur}$ (or any complete field $K \subseteq \mathbb{C}_p$ containing $\mathbb{Q}_p$) one can give a more concrete description of $M_k(X, K)$ due to theorem 5.3. A $p$-adic modular form of weight $k$ for $\Gamma$ is a rigid analytic function $f$ on $\mathcal{H}_p$ defined over $K$ such that

$$f(\gamma z) = (cz + d)^k f(z) \quad \text{for all } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma.$$ 

The space of these $p$-adic modular forms will be denoted by $M_k(\Gamma) = M_k(\Gamma, K)$. By identifying $M_k(X, \mathbb{Q}_{ur})$ with the space of global section of $\Omega_{X/K}^\otimes m+1$ we get an isomorphism

$$M_k(\Gamma) \longrightarrow M_k(X, \mathbb{Q}_{ur}), f \mapsto f(z)dz^\otimes m+1 \quad (19)$$

Returning to the case of arbitrary coefficients $K$ the Jacquet-Langlands correspondence is an isomorphism (uniquely determined up to scaling)

$$M_k(X, K) \cong S_k(\Gamma_0(N), K)^{pN^- - new} \quad (20)$$

which is compatible with the action of the Hecke operators and Atkin-Lehner involutions.

Let $f_\infty \in S_k(\Gamma_0(N), \mathbb{Q})$ be a newform and $f \in M_k(X, \mathbb{Q})$ the corresponding newform on $X$ and let $F = F_f$ be the finite extension of $\mathbb{Q}$ generated by
the eigenvalues of the Hecke operators acting on \( f \) (or \( f_\infty \)). Associated to \( f_\infty \) is a two-dimensional (as \( F_p \)-module) \( G_\mathbb{Q} \)-representation \( V_p(f_\infty) \). It is defined as a direct summand of the \((n + 1)\)-th \( p \)-adic cohomology of a suitable compactification of the \( n \)-fold fibre product of the universal elliptic curve (with full level \( N \)-structure) over the modular curve \( X(N) \) (see [De1]).

A similar construction for \( f \) can be done by using the universal abelian surface over the Shimura curve. For this we have to work with a more refined moduli problem then in definition 5.2.

**Definition 5.5** Let \( M \geq 3 \) be an integer relatively prime to \( N \) and \( S \) be a \( \mathbb{Q} \)-scheme. An abelian surface with quaternionic multiplication (by \( R_{\text{max}} \)), level \( N^\dagger \)-structure and full level \( M \)-structure is a quadruple \( (A, \iota, C, \bar{\nu}) \) where \( (A, \iota, C) \) is a triple as in 5.1 and \( \bar{\nu} : (R_{\text{max}}/MR_{\text{max}})_S \to A[M] \) is a \( R_{\text{max}} \)-equivariant isomorphism from the constant group scheme \( (R_{\text{max}}/MR_{\text{max}})_S \) to the group scheme of \( M \)-division points of \( A \).

The corresponding moduli problem admits a fine moduli scheme which we denote by \( X_M \) (see [BC]). Again it is a smooth projective curve over \( \mathbb{Q} \) (however it is not geometrically connected). We have a Galois covering \( q : X_M \to X \) with Galois group \( \cong G/\{\pm 1\} \) where

\[
G = G_M = (R_{\text{max}}/MR_{\text{max}})^* \cong (R/MR)^* \cong GL_2(\mathbb{Z}/MZ) \cong (R/MR)^*
\]

obtained by forgetting the level \( M \)-structure. Let \( \pi : \mathcal{A} \to X_M \) be the universal abelian surface over \( X_M \). The action of \( R_{\text{max}} \) on \( \mathcal{A} \) induces an action of \( B^s \) on \( R^d f_\ast Q_p \). We let

\[
L_2 = \bigcap_{b \in B} \text{Ker}(b - N(b) : R^2 \pi_{\ast} Q_p \to R^2 \pi_{\ast} Q_p).
\]

It is a 3-dimensional \( p \)-adic local system on \( X_M \). Let

\[
\Delta_m : \text{Sym}^m L_2 \longrightarrow (\text{Sym}^{m-2} L_2)(-2),
\]

be the Laplace operator associated to the non-degenerated pairing

\[
( , ) : L_2 \otimes L_2 \hookrightarrow R^2 \pi_{\ast} Q_p \otimes R^2 \pi_{\ast} Q_p \xrightarrow{\cup} R^4 \pi_{\ast} Q_p \xrightarrow{\text{tr}} Q_p (-2).
\]

Symbolically (21) is given by

\[
\Delta_m(x_1 \cdots x_m) = \sum_{1 \leq i < j \leq m} (x_i, x_j) x_1 \cdots \hat{x_i} \cdots \hat{x_j} \cdots x_m.
\]

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For $n > 2$ we define $L_n$ as the kernel of $\Delta_m$. Note that we have an action of $G$ on $L_n$ compatible with the action on $X_M$.

**Definition 5.6** The $p$-adic $G_\mathbb{Q}$-representation attached to the space $M_k(X, \mathbb{Q})$ is defined to be the representation

$$H_p(M_n) = H^1(X_M, \mathbb{L}_n)^G$$

where $\overline{X}_M = X_M \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}$.

To justify the notation $H_p(M_n)$ note that it is the $(n+1)$-th $p$-adic realization of the motive $M_n$ constructed in appendix 10.1 (see 10.1). As in [De1] or ([Sch1], section 4) one can define Hecke operators and Atkin-Lehner involutions acting on $H_p(M_n)$ (and on $M_n$). Let $T$ be the subalgebra of $\text{End}(S_k(\Gamma_0(N), \mathbb{Q}))_{\text{new}} \cong \text{End}(M_k(X, \mathbb{Q}))$ generated by the Hecke operators $T_\ell$ for $\ell \nmid N$ and $U_\ell$ for $\ell | N$. Then $M_k(X, \mathbb{Q})$ is a free $T$-module of rank one. By multiplicity one to every newform $f \in M_k(X, \mathbb{Q})$ corresponds a primitive idempotent $e_f \in T$ such that $e_f \cdot M_k(X, \mathbb{Q}) = F_f \cdot f$. The Betti realization $H_B(M_n)$ has a Hodge structure of type $(0,n+1)$, $(n+1,0)$, $H_B(M_n) \otimes \mathbb{C} = H^{(0,n+1)} \oplus H^{(n+1,0)}$ with $H^{(0,n+1)} \cong M_k(X, \mathbb{C})$. The comparison isomorphism $H_B(M_n) \otimes \mathbb{C} \cong H_p(M_n) \otimes_{\mathbb{Q}_p, \tau} \mathbb{C}$ where $\tau : \mathbb{Q}_p \hookrightarrow \mathbb{C}$ is any embedding shows that $H_p(M_n)$ is a free $T_{\mathbb{Q}_p}$-module of rank 2.

**Definition 5.7** The $p$-adic $G_\mathbb{Q}$-representation $V_p(f)$ attached to $f$ is given by $V_p(f) = e_f \cdot H_p(M_n)$.

**Lemma 5.8** We have $V_p(f) \cong V_p(f_\infty)$ as $G_\mathbb{Q}$-representations and $T$-modules.

**Proof:** The proof is similar to the one in the weight two case given by Ribet ([Ri2], lemme to théorème 2). The Eichler-Shimura relations imply that the traces of the Frobenii at primes $\ell \nmid N$ operating on $V_p(\phi)$ and $V_p(\phi')$ are given by the action of the $T_\ell$ and thus are the same. Cebotarev’s density theorem and the theorem of Brauer-Nesbitt allow us to deduce that the semisimplifications of the $G_\mathbb{Q}$-representations $V_p(f), V_p(f_\infty)$ are isomorphic. But in [Ri2] it is shown that $V_p(f_\infty)$ is already a simple $G_\mathbb{Q}$-representation. Thus $V_p(f) \cong V_p(f_\infty)$. □

The main result of this section is
Theorem 5.9 Considered as a $G_{\mathbb{Q}_p}$-representation $H_p(M_n)$ is semistable. We have an isomorphism of $(\phi, N)$-modules (canonical up to scaling)

$$D_{st,\mathbb{Q}_p}(H_p(M_n)) = H^1_{DR}(X_\Gamma, V_n).$$

Proof. For the first statement it is enough to see that $H_p(M_n)$ is semistable as a $G_{\mathbb{Q}_p}$-representation.

Over $\mathbb{Q}_p$ the curve $X_M$ admits a $p$-adic uniformization similar to 5.3. We have (see [BC], Chapitre III, 5.3.1)

$$\mathbb{Q}_p \cong (X_M)^{an} \cong \Gamma \backslash (\mathcal{H}_p \times (R/MR)^*)$$

where $\Gamma$ acts diagonally on $\mathcal{H}_p \times (R/MR)^*$ (on the factor $(R/MR)^* \times R$ an element $\gamma \in \Gamma$ acts by left multiplication with $\gamma \mod M$). Since the orbits of the $\Gamma$-action on $(R/MR)^*$ are the fibers of the reduced norm $Nrd : (R/MR)^* \to (\mathbb{Z}/M\mathbb{Z})^*$ we can write (23) also as

$$\mathbb{Q}_p \cong (X_M)^{an} \cong \bigoplus_{(\mathbb{Z}/M\mathbb{Z})^*} \Gamma_M \backslash \mathcal{H}_p$$

where $\Gamma_M := \{ \gamma \in \Gamma \mid ga \equiv 1 \mod M \}$, i.e. $(X_M)^{an}$ is the disjoint union of Mumford curves $X_{\Gamma_M}$ (indexed by the set $(\mathbb{Z}/M\mathbb{Z})^*$).

To deduce 5.9 from theorem 3.6 we need an explicit description of the filtered isocrystal $H^1_{DR}(A/X_M)$ (over $\mathbb{Q}_p$). Note that $H^1_{DR}(A/X_M)$ carries a natural $B_{\mathbb{Q}_p}$-action induced by the $R^{max}$-action on $A$.

On $M_2$ we have the following two commuting left $GL_2$-actions $\rho_1, \rho_2 : GL_2 \to GL(M_2)$ given by

$$\rho_1(A)(B) := AB \quad \rho_2(A)(B) := B A$$

for $A \in GL_2$ and $B \in M_2$. Here for $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2$, $\overline{A}$ denotes the matrix $\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$. Let $E(M_2)$ be the convergent filtered $F$-isocrystal attached to $(M_2, \rho_1, \rho_2)$. 

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Lemma 5.10 There exists an (up to scaling) canonical isomorphism of filtered isocrystals on $H_p$

$$H^1_{DR}(\mathcal{A}/X_M) \cong \mathcal{E}(M_2).$$

Proof. This is essentially proved in section 5 of [Fa2]. We will explain how the result shown there can be reformulated as above.

Let $D = B_p$ be the unique quaternion algebra over $\mathbb{Q}_p$ with maximal order $\mathcal{O}_D$. Explicitly it can be written as $D = \mathbb{Q}_p^2[\Pi]$ with $\Pi^2 = p, \Pi x = \sigma(x)\Pi$.

We recall the description of $\hat{\mathcal{H}}$ as the moduli space of special formal $\mathcal{O}_D$-modules. Let $R$ be a $\mathbb{Z}_p^ur$-algebra. A formal group $G$ over $R$ with $\mathcal{O}_D$-multiplication is called special if the tangent space of $G$ is a free $\mathbb{Z}_p^ur \otimes \mathbb{Z}_p$ $R$-module of rank 1 (where $\mathbb{Z}_p^ur$ acts on $G$ via $\mathbb{Z}_p^ur \hookrightarrow D$). Fix a special formal group $G_0$ over $\bar{\mathbb{F}}_p$. Then $\hat{\mathcal{H}}$ represents the functor which associates to every $\mathbb{Z}_p^ur$-algebra $R$ on which $p$ is nilpotent the set of isomorphism classes of pairs $(\mathcal{G}, \lambda)$ where $\mathcal{G}$ is a special formal group over $R$ and $\lambda : \mathcal{G} : = G \otimes_R R/pR \to G_0 \otimes \mathbb{F}_p R/pR$ is a $\mathbb{Q}_p$-isogeny of height 0 of special formal $\mathcal{O}_D$-modules, i.e. $\lambda = p^{-h}\lambda_0$ for some positive integer $h$ and isogeny $\lambda_0$ of height $h$. Let $(\mathcal{G}, \lambda)$ be the universal such pair over $\hat{\mathcal{H}}$. The algebra $\text{End}_{\mathcal{O}_D}(G_0)_p$ of $\mathbb{Q}_p$-homomorphisms of $G_0$ is isomorphic to $M_2(\mathbb{Q}_p)$. We fix such an isomorphism $\text{End}_{\mathcal{O}_D}(G_0)_p \cong M_2(\mathbb{Q}_p)$. It induces compatible $\text{GL}_2(\mathbb{Q}_p)$-actions on $\mathcal{G}$ and $\hat{\mathcal{H}}$.

Let $H^1_{DR}(\mathcal{G}/\hat{\mathcal{H}})$ be the dual of the Lie algebra of the universal vectorial extension of $\mathcal{G}$. It is a filtered convergent $F$-isocrystal on $\hat{\mathcal{H}}$. Let $\mathbb{D}(G_0)$ be the Dieudonné module of $G_0$ and define $H^1_{\text{cris}}(G_0)$ to be the dual of $\mathbb{D}(G_0)$. Then as a convergent $F$-isocrystal $H^1_{DR}(\mathcal{G}/\hat{\mathcal{H}})$ is constant isomorphic to $H^1_{\text{cris}}(G_0) \otimes \mathcal{O}_{\mathcal{H}_p}$ (the isomorphism is induced by the isogeny $\lambda : \mathcal{T} \to G_0$). The $\mathbb{Q}_p^ur$-vector space $H^1_{\text{cris}}(G_0)$ is endowed with the following structure:

- A $\sigma$-linear Frobenius $\Phi : H^1_{\text{cris}}(G_0) \to H^1_{\text{cris}}(G_0)$ such that $(H^1_{\text{cris}}(G_0), \Phi)$ is isotypical of slope $\frac{1}{2}$.
- A $D$-module structure given by an embedding $j : D \to \text{End}_{\mathbb{Q}_p^ur}(H^1_{\text{cris}}(G_0))$ which commutes with $\Phi$ and such that $H^1_{\text{cris}}(G_0)$ is a free $D \otimes_{\mathbb{Q}_p} \mathbb{Q}_p^ur$-module of rank 1.
- A $\mathbb{Q}_p$-algebra embedding $\iota : M_2(\mathbb{Q}_p) \longrightarrow \text{End}_{\mathbb{Q}_p^ur}(H^1_{\text{cris}}(G_0))$ (induced by the isomorphism $\text{End}_{\mathcal{O}_D}(G_0)_p \cong M_2(\mathbb{Q}_p)$) which commutes with the $D$-action and the Frobenius.

Set $\Phi' = j(\Pi)^{-1}\Phi$. Then $(H^1_{\text{cris}}(G_0), \Phi')$ is isotypical of slope 0, hence $V : = H^1_{\text{cris}}(G_0)^{\Phi'=id}$ is a four-dimensional $\mathbb{Q}_p$-vector space. Therefore $H^1_{\text{cris}}(G_0)$
\( \cong V_{Q_p} \) and under this isomorphism \( \Phi \) corresponds to \( \phi_V \otimes \sigma \) where \( \phi_V \) denotes the restriction of \( j(\Pi) \) to \( V \). Thus we obtain an isomorphism of convergent \( F \)-isocrystals
\[
(25) \quad H^1_{DR}(\mathcal{G}/\mathcal{H}) \cong V \otimes_{Q_p} \mathcal{O}_{\mathcal{H}}
\]
where the Frobenius on the right hand side is given by \( \phi_V \otimes \Phi_{\mathcal{O}_{\mathcal{H}}} \). By restriction, the embedding \( \iota \) induces an embedding \( \iota_1 : M_2(Q_p) \to \text{End}_{Q_p}(V) \) and it is shown in [Fa2] that the filtration on \( H^1_{DR}(\mathcal{G}/\mathcal{H}) \) corresponds under \( (25) \) to the filtration on \( V \otimes_{Q_p} \mathcal{O}_{\mathcal{H}} \) induced by the representation \( \eta_1 : = \iota_1 |_{\mathcal{GL}_2} : \mathcal{GL}_2 \to \mathcal{GL}(V) \) as in section [3].

Let \( Z \) be the centralizer of \( \iota_1(M_2(Q_p)) \) in \( \text{End}_{Q_p}(V) \). Note that \( \phi_V \in Z \) and \( \phi_V^2 = p \). Clearly \( Z \cong M_2(Q_p) \). Choose an embedding \( \iota_2 : M_2(Q_p) \to \text{End}_{Q_p}(V) \) with image \( Z \) and such that \( \iota_2\left( \begin{pmatrix} 0 & p \\ 1 & 0 \end{pmatrix} \right) = \phi_V \). Let \( \eta_2 : = \iota_2 \mid_{\mathcal{GL}_2} : \mathcal{GL}_2 \to \mathcal{GL}(V) \). Then \( (25) \) can be interpreted as an isomorphism \( H^1_{DR}(\mathcal{G}/\mathcal{H}) \cong \mathcal{E}((V, \eta_1, \eta_2)) \). On the other hand \( (V, \eta_1, \eta_2) \cong (M_2, \rho_1, \rho_2) \). Thus we finally obtain an isomorphism of filtered \( F \)-isocrystals
\[
(26) \quad H^1_{DR}(\mathcal{G}/\mathcal{H}) \cong \mathcal{E}(M_2).
\]
and the Lemma follows by descending \( (26) \) to \( X_F \).

It is apparent from the construction of \( (25) \) that there is an isomorphism \( D_{Q_p} = B_{Q_p} \cong M_2(Q_p^ur) \) such that the \( D_{Q_p}^ur \)-action on the left hand side of \( (26) \) is compatible with the \( \rho_2 \)-action of \( \mathcal{GL}_2(Q_p^ur) \)-on the right hand side.

Hence from Lemma 5.10 we obtain also an isomorphism of filtered isocrystals with \( B_{Q_p}^ur \)-action
\[
H^1_{DR}(\mathcal{A}/X_M) \cong \Lambda^2 H^1_{DR}(\mathcal{A}/X_M) \cong \mathcal{E}(\Lambda^2 M_2)
\]
Note that \( (M_2, \rho_1, \rho_2) \) is canonically isomorphic to \( V_1 \circ V_1 \). Hence
\[
(\Lambda^2 M_2, \Lambda^2 \rho_1, \Lambda^2 \rho_2) = \Lambda^2 V_1 \circ \text{Sym}^2 V_1 \oplus \text{Sym}^2 V_1 \circ \Lambda^2 V_1 = \{1\} V_2 \oplus V_2 \{1\}
\]
and therefore \( \bigcap_{b \in B_{Q_p}^ur} \ker(b - N(b) : \mathcal{E}(\Lambda^2 M_2) \to \mathcal{E}(\Lambda^2 M_2)) \cong V_2 \).

Applying theorem 4.6 (a) we obtain
\[
D_{st, Q_p^ur}(H^1_{et}(X_M, \mathbb{L}_2)) \cong H^1_{DR}((X_M)_{Q_p^ur}, V_2).
\]
Passing on both sides to $G$-invariants yields an isomorphism

$$D_{st,q_{p}^{ur}}(H_{p}(\mathcal{M}_{2})) = H^{1}_{DR}(X_{\Gamma}, V_{2}) \tag{27}$$

i.e. the assertion for $n = 2$.

Now let $n \geq 4$. There exists a canonical nondegenerated symmetric bilinearform in $\text{Rep}_{q_{p}}(\text{GL}_{2})$

$$< , >: V_{2} \otimes V_{2} \longrightarrow \det \otimes^{2} \tag{28}$$

whose definition we briefly recall from ([BDIS], 1.2; actually (28) is $-2 \times$ the pairing considered there). Let $\mathcal{U} = \{U \in M_{2} \mid \text{trace}(U) = 0\}$ with right $\text{GL}_{2}$-action

$$U \cdot A = \overline{A} U A$$

for $U \in \mathcal{U}$ and $A \in \text{GL}_{2}$. For $U \in \mathcal{U}$ we set

$$P_{U}(X) = \text{trace} \left( U \begin{pmatrix} X & -X^{2} \\ 1 & -X \end{pmatrix} \right) \tag{29}$$

The map $\mathcal{U} \rightarrow \mathcal{P}_{2}, U \mapsto P_{U}(X)$ is an isomorphism of right $\text{GL}_{2}$-modules. There is a pairing on $\mathcal{P}_{2}$ given by

$$< P_{U_{1}}(X), P_{U_{2}}(X) > : = -\text{trace}(U_{1} \overline{U_{2}}) \quad \forall U_{1}, U_{2} \in \mathcal{U}. \tag{29}$$

The isomorphism $\mathcal{P}_{2} \rightarrow V_{2}, R(X) \mapsto (P(X) \mapsto < R(X), P(X) >)$ yields the pairing (28) by transport of structure. Note that $< Av_{1}, v_{2} > = < v_{1}, \overline{A} v_{2} >$ for all $v_{1}, v_{2} \in V_{2}$ and $A \in \text{GL}_{2}$.

By ([BDIS], 1.2) the $\text{GL}_{2}$-subrepresentation $V_{2m}$ of $\text{Sym}^{m} V_{2}$ is the kernel of the Laplace operator associated to (28)

$$\Delta : \text{Sym}^{m} V_{2} \longrightarrow \text{Sym}^{m-2} V_{2} \otimes \det \otimes^{2} \tag{30}$$

By tensoring (28) with $\otimes \det \otimes^{2}$ we obtain a pairing

$$< , >: V_{2}\{1\} \otimes V_{2}\{1\} \longrightarrow \{2|2\}. \tag{31}$$

We need the following elementary lemma.
Lemma 5.11  The pairing (31) coincides with the pairing

$$V_2\{1\} \otimes V_2\{1\} \leftrightarrow \Lambda^2 M_2 \otimes \Lambda^2 M_2 \rightarrow \Lambda^4 M_2 \cong \{2|2\}$$

(as a map in $\text{Rep}_{Q_p}(\text{GL}_2 \times \text{GL}_2)$).

By applying theorem 1.5 (a) to $A^i \rightarrow X_M$ and using the Kuenneth formula for $H^{2i}_{\text{DR}}(A^i/X_M)$ one can easily deduce from Lemma 5.10 that

$$D_{st, Q_p}(H^1(X_M, \text{Sym}^i L_2)) \cong H^1_{\text{DR}}((X_M)\text{Q}^{ur}, \text{Sym}^i V_2),$$

for all $i \geq 1$. Lemma 5.11 implies that the diagram

$$\begin{array}{ccc}
D_{st, Q_p}(H^1(X_M, \text{Sym}^m L_2)) & \xrightarrow{\cong} & H^1_{\text{DR}}((X_M)\text{Q}^{ur}, \text{Sym}^m V_2) \\
\downarrow & & \downarrow \\
D_{st, Q_p}(H^1(X_M, \text{Sym}^{m-2} L_2)) & \xrightarrow{\cong} & H^1_{\text{DR}}((X_M)\text{Q}^{ur}, \text{Sym}^{m-2} V_2),
\end{array}$$

commutes where the vertical maps are induced by (21) and (30) respectively. Hence the kernels of the vertical arrows are isomorphic as well. Passing again to $G$-invariants yields Theorem 5.9 in the case $n \geq 4$.

Remark 5.12 The pairing (31) induces a symmetric bilinearform

$$< , >: V_n \otimes V_n \rightarrow \otimes^n$$

(compare [BDIS], 1.2). It follows from Lemma 5.11 that the (nondegenerate) pairing

$$< , >: H^1_{\text{DR}}(X_\Gamma, V_n) \otimes H^1_{\text{DR}}(X_\Gamma, V_n) \rightarrow H^2_{\text{DR}}(X_\Gamma, V_n \otimes V_n) \rightarrow$$

$$\rightarrow H^2_{\text{DR}}(X_\Gamma) \cong Q_p^{ur}$$

(where $*$ is induced by (32)) is given by the cup-product

$$H^{n+1}_{\text{DR}}(M_n) \otimes H^{n+1}_{\text{DR}}(M_n) \subseteq H^{n+1}_{\text{DR}}(A^m) \otimes H^{n+1}_{\text{DR}}(A^m) \rightarrow H^{2n+2}_{\text{DR}}(A^m) \cong Q_p^{ur},$$

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hence under the isomorphism (27) coincides with the map

\[ D_{st, Q_p'}(H_p(M_n)) \otimes D_{st, Q_p'}(H_p(M_n)) \longrightarrow Q_p'[2m + 1] \]

induced by the cup-product \( \mathbb{L}_n \otimes \mathbb{L}_n \rightarrow Q_p[-n] \). More precisely the isomorphism (27) can and will be chosen in such a way that it is compatible with to (33) and (34). This fixes it up to sign (and not only up to scaling).

Hence we can view (33) as a map

\[ < , > : H^1_{DR}(X, V_n) \otimes H^1_{DR}(X, V_n) \longrightarrow Q_p'[2m + 1] \]

in \( MF_{Q_p'}(\phi, N) \).

We finish this section by describing explicitly the sequence of Dieudonné modules of a certain Gysin sequence associated to a finite number of rational points on \( X \). Let \( x_1, \ldots, x_r \in X(\mathbb{Q}_{ur}) \) and fix points \( z_1, \ldots, z_r \in H_p(\mathbb{Q}_p^{ur}) \) such that \( \Gamma z_i = x_i \) (under the isomorphism \( X(\mathbb{Q}_p^{ur}) \cong \Gamma \backslash H_p(\mathbb{Q}_p^{ur}) \)). For simplicity we assume that the stabilizers of \( z_1, \ldots, z_r \) in \( \Gamma \) are all \( \{ \pm 1 \} \). Let \( X_0 \cong \Gamma \backslash H_p \) be one of the components of \( (X_M)_{Q_p} \) and let \( x'_i := \Gamma_M z_i \in X' \) and \( U' \) their complement in \( (X_M)_{Q_p} \). We have a short exact (Gysin) sequence of \( G_{Q_p'} \)-modules

\[ 0 \longrightarrow H^1(X_M, \mathbb{L}_n) \longrightarrow H^1(U \otimes_{Q_p^{ur}} \overline{Q}_{p}^{ur}, \mathbb{L}_n) \longrightarrow \bigoplus_{i=1}^{r} (\mathbb{L}_n)_{x'_i}(-1) \longrightarrow 0. \]

Let

\[ 0 \longrightarrow H_p(M_n) \longrightarrow E \longrightarrow \bigoplus_{i=1}^{r} (\mathbb{L}_n)_{x'_i}(-1) \longrightarrow 0 \]

be its push-out under the canonical projection \( H^1(X_M, \mathbb{L}_n) \rightarrow H_p(M_n) \).

Similarly as above we can deduce from Theorem 3.6 (b) the following

**Theorem 5.13** After applying \( D_{st, Q_p'} \) the sequence (36) becomes isomorphic to the sequence of filtered \( (\phi, N) \)-modules

\[ 0 \longrightarrow H^1_{DR}(X, V_n) \longrightarrow H^1_{DR}(U, V_n) \oplus \bigoplus_{i=1}^{r} \text{Res}_{z_i} \longrightarrow \bigoplus (V_{n})_{z_i}[1] \longrightarrow 0 \]

where \( U : = X_{Q_p'} - \{ x_1, \ldots, x_r \} \).
**Remark 5.14** In particular for $x \in X_M(\mathbb{Q}_p^{ur})$ and $z \in \mathcal{H}_p$ lying above $x$ there is a canonical isomorphism
\[ D_{st}(\mathbb{L}_n) \cong (\mathcal{V}_n)_z. \]

It is easy to see that it respects the pairings on both sides induced by the cup product $\mathbb{L}_n \otimes \mathbb{L}_n \to \mathbb{Q}_p[-n]$ and (32) respectively.

### 6 Comparison of $\mathcal{L}$-invariants

As in the last section let $f_\infty$ be a newform of weight $k \geq 4$ and level $N$ corresponding to a newform $f$ on $X$. The exceptional zero conjecture of [MTT] relates the derivative of a $p$-adic $L$-function to the value of the complex $L$-function of $f_\infty$ at the central critical point $k/2$. In this conjecture a certain local factor $\mathcal{L}(f_\infty)$ appears the so called $\mathcal{L}$-invariant of $f_\infty$. Three possible definitions for it have been given. The Fontaine-Mazur $\mathcal{L}$-invariant $\mathcal{L}_{FM}(f_\infty)$ is defined in terms of the semistable Dieudonné module $D_{st}(V_p(f_\infty))$. The $\mathcal{L}$-invariants of Teitelbaum and Coleman, $\mathcal{L}_T(f_\infty)$ and $\mathcal{L}_C(f_\infty)$ are defined in terms of $p$-adic integration on the Shimura curve $X$ and the modular curve $X_0(N)$ respectively. The exceptional zero conjecture for $f_\infty$ has been proved by Stevens (using $\mathcal{L}_C(f_\infty)$) and by Kato, Kurihara and Tsuji (using $\mathcal{L}_{FM}(f_\infty)$). In [BDIS] a version of the exceptional zero conjecture is proved involving the anticyclotomic $p$-adic $L$-function of $f_\infty$ and $\mathcal{L}_T(f_\infty)$.

In this section we show that $\mathcal{L}_{FM}(f_\infty) = \mathcal{L}_T(f_\infty)$. Our proof is based on the explicit description given in the previous section of the Dieudonné module of $V_p(f_\infty) \cong V_p(f)$ (or more precisely of $H_p(M_n)$) considered as a representation of the inertia group $I_p \cong G_{\mathbb{Q}_p^{ur}}$ which in turn is based on the comparison theorems of Faltings and Coleman-Iovita.

We begin by giving an explicite description of the Hodge filtration of $H^1(X_\Gamma, \mathcal{V}_n) = H^1_{DR}(X_\Gamma, \mathcal{V}_n)$.

**Proposition 6.1** We have,
\[ F^iH^1(X_\Gamma, \mathcal{V}_n) = \begin{cases} 
  H^1(X_\Gamma, \mathcal{V}_n) & \text{if } i \leq 0, \\
  M_k(\Gamma) & \text{if } 1 \leq i \leq k - 1, \\
  0 & \text{if } i \geq k
\end{cases} \]
Proof. For \( i \in \{1, \ldots, n\} \) we let \( \partial^i \in V_n \otimes \mathcal{O}(\mathcal{H}_p) \) be given by \( \partial^i(z)(P(X)) := (\frac{d^i}{dz^i}P(X))|_{x=z} \) for \( z \in \mathcal{H}_p, P(X) \in \mathcal{P}_n \). It is easy to verify (and will be left to the reader) that the filtration \( \mathcal{F}^iV_n \) on \( V_n \) is given by

\[
\mathcal{F}^iV_n = \begin{cases} 
  V_n & \text{if } j \leq 0, \\
  \sum_{i=0}^{n-j} \mathcal{O}_{\mathcal{H}_p} \partial^i & \text{if } 0 \leq j \leq n, \\
  0 & \text{if } i \geq n + 1
\end{cases}
\]

Consequently \( F^iH^1(X_\Gamma, V_n) = H^1(X_\Gamma, V_n) \) if \( i \leq 0 \), \( F^iH^1(X_\Gamma, V_n) = 0 \) if \( i \geq n + 2 \) and the image of the embedding

\[
M_k(\Gamma) \rightarrow H^1(X_\Gamma, V_n), f(z) \mapsto \omega_f = f(z)\partial^0 \otimes dz
\]

lies in \( F^{n+1}H^1(X_\Gamma, V_n) \). Hence,

\[
\dim F^{n+1}H^1(X_\Gamma, V_n) \geq \dim M_k(\Gamma) = \frac{1}{2} \dim H^1(X_\Gamma, V_n)
\]

(see [IS3]). Since \( F^i \) and \( F^{n+1} \) are orthogonal with respect to \( \langle \rangle \) we obtain

\[
\dim F^{n+1} \leq \dim F^1 \leq \dim H^1(X_\Gamma, V_n)/F^{n+1} \leq \frac{1}{2} \dim H^1(X_\Gamma, V_n)
\]

and therefore \( F^1 = F^2 = \ldots = F^{n+1} = M_k(\Gamma) \). \( \square \)

By Theorem [5.9], \( V_p(f) \) is considered as a \( G_{Q_p} \)-representation – is semisimple. The associated Dieudonné module \( D_{st,Q_p}(V_p(f)) \) is a two-dimensional \( F_{Q_p} \)-module where \( F = F_f \). The Fontaine-Mazur \( \mathcal{L} \)-invariant of \( f_\infty \) is defined as the \( \mathcal{L} \)-invariant of \( D_{st,Q_p}(V_p(f_\infty)) \cong D_{st,Q_p}(V_p(f)) \). The following result has been conjectured by Mazur ([Ma], section 12).

Lemma 6.2 \( D_{st,Q_p}(V_p(f)) \) is a monodromy \( F_{Q_p} \)-module.

Proof. By remark 2.3 (a) above it is enough to consider \( D_{st,Q_p'}(V_p(f)) \) instead. Clearly property (i) of 2.2 is satisfied and that (ii) holds is a consequence of the exactness of \( \langle \rangle \). Proposition 6.1 implies that \( F^{m+1}D_{st,Q_p'}(V_p(f)) = e_f(F^{m+1}H^1(X_\Gamma, V_n)) \cong e_fM_k(\Gamma) \), hence \( F^{m+1}D_{st,Q_p'}(V_p(f)) \) is free a \( F_{Q_p'} \)-module of rank 1. Finally the fact that the restriction of \( N : H^1(X_\Gamma, V_n) \rightarrow H^1(X_\Gamma, V_n) \) to \( M_k(\Gamma) \) is injective (see [IS1], 3.9) implies that

\[
F^{m+1}D_{st,Q_p'}(V_p(f)) \cap \text{Ker}(N) = 0.
\]

\( \square \)
Definition 6.3 The Fontaine-Mazur $\mathcal{L}$-invariant $\mathcal{L}_{FM}(f)$ of $f$ (or $f_\infty$) is defined as the $\mathcal{L}$-invariant of the monodromy module $D_{st,Q_p}(V_p(f))$.

We also recall the definition of the Teitelbaum $\mathcal{L}$-invariant $\mathcal{L}_T(f)$. Let $P_0: M_k(\Gamma) \to H^1(\Gamma, V_n)$ be the composition of the inclusion

$$M_k(\Gamma) \cong F^{m+1}H^1(X_\Gamma, V_n) \hookrightarrow H^1(X_\Gamma, V_n)$$

with the map $P$, i.e. $P(f)$ is represented by the cocycle $\gamma \mapsto \gamma(F_f) - F_f$ where $F_f$ is a primitive of $\omega_f$. Both maps $P_0$ and

$$(37) \quad M_k(\Gamma) \hookrightarrow H^1(X_\Gamma, V_n) \xrightarrow{\text{col}} H^1(\Gamma, V_n)$$

are homomorphisms of free $\mathbb{T}_{\mathbb{Q}_p}$-modules of rank one and (37) is an isomorphism. Hence there is an element $\mathcal{L}_T \in \mathbb{T}_{\mathbb{Q}_p}$ with $P_0(g) = \mathcal{L}_T\epsilon(I(\omega_g))$ for every $g \in M_k(\Gamma)$. Then $\mathcal{L}_T(f) = e_f \mathcal{L}_T \in e_f \mathbb{T}_{\mathbb{Q}_p} = F_{\mathbb{Q}_p}$.

Theorem 6.4 Let $f \in M_k(X, \mathbb{Q})$ be a newform. Then, $\mathcal{L}_{FM}(f) = \mathcal{L}_T(f)$.

Proof. To simplify the notation we work with $D_{st}(H_p(M_n)) \cong H^1(X_\Gamma, V_n)$ rather than $D_{st,Q_p}(V_p(f))$ (though strictly speaking $D_{st,Q_p}(V_p(f))$ is not a monodromy $\mathbb{T}_{\mathbb{Q}_p}$-module in the sense of definition 2.2 since $\mathbb{T}_{\mathbb{Q}_p}$ is in general not semisimple). The slope decomposition of $H^1(X_\Gamma, V_n)$ is of the form

$$(38) \quad H^1(X_\Gamma, V_n) = H^1(X_\Gamma, V_n)_m \oplus H^1(X_\Gamma, V_n)_{m+1},$$

where $H^1(X_\Gamma, V_n)_m = \iota(H^1(\Gamma, V_n)) = \text{Ker}(N)$ and $H^1(X_\Gamma, V_n)_{m+1}$ is the kernel of $P$. If we decompose an element $x \in H^1(X_\Gamma, V_n)$ as $x = x_m + x_{m+1}$ according to (38) then $\mathcal{L}_T$ is thus characterised by the property

$$\mathcal{L}_TN(x) = -x_m \quad \forall x \in F^{m+1}H^1(X_\Gamma, V_n),$$

whereas $\mathcal{L}_{FM} = \mathcal{L}_{FM}(H^1(X_\Gamma, V_n)) \in \mathbb{T}_{\mathbb{Q}_p}$ is characterised by

$$x - \mathcal{L}_{FM}N(x) \in F^{m+1}H^1(X_\Gamma, V_n) \quad \forall x \in H^1(X_\Gamma, V_n)_{m+1}.$$ 

By ([IS3], Theorem 1.6) any element $x$ of $H^1(X_\Gamma, V_n)$ can be uniquely written as $x = x' + x''$ with $x' \in F^{m+1}$ and $x'' \in H^1(X_\Gamma, V_n)_m$. Therefore if $x \in H^1(X_\Gamma, V_n)_{m+1}$ then $x'_m = -x''$ and thus

$$x - \mathcal{L}_TN(x) = x' + x'' - \mathcal{L}_TN(x') = x' + x'' + x'_m = x' \in F^{m+1}.$$

Since this holds for all $x \in H^1(X_\Gamma, V_n)_{m+1}$ we conclude $\mathcal{L}_T = \mathcal{L}_{FM}$. \qed
7 The \( p \)-adic Abel-Jacobi map

In this section we will define the \( p \)-adic Abel-Jacobi maps for the motives \( \mathcal{M}_n \) (see appendix 10.1) where \( n = 2m \) is a positive even integer. It is a map from the Chow group \( CH^{m+1}(\mathcal{M}_n) \) to the dual of the space of weight \( k \)-modular modular forms on \( X \).

We begin by reviewing briefly the definition of the cohomological \( \ell \)-adic Abel-Jacobi map (see e.g. [Ja]). Let \( K \) be a field of characteristic 0 and \( \ell \) a prime number. For a smooth projective variety \( X \) over \( K \) we denote by \( CH^i(X) = CH^i(X)_\mathbb{Q} \) the Chow group of codimension \( i \)-cycles (with rational coefficients) and by \( CH^i(X)_0 \) be the subgroup of cycle classes homologous to zero, i.e. the kernel of the cycle class map

\[
cl = cl^{X,i}: CH^i(X) \longrightarrow H^{2i}(\overline{X}, \mathbb{Q}_\ell(i))^G_K
\]

where \( \overline{X} : = X \otimes_K \overline{K} \). The map (39) factors through \( H^1_{\text{cont}}(K, CH^{2i-1}(\overline{X}, \mathbb{Q}_\ell(i))) \neq 0 \). The \( \ell \)-adic Abel-Jacobi map

\[
cl_0 = cl^{X,i}_0: CH^i(X)_0 \longrightarrow H^1_{\text{cont}}(K, H^{2i-1}(\overline{X}, \mathbb{Q}_\ell(i))) = \text{Ext}^1_{G_K}(\mathbb{Q}_\ell, H^{2i-1}(\overline{X}, \mathbb{Q}_\ell(i)))
\]

is defined as follows: Let \( z \in CH^i(X)_0 \) and \( Z \) be a cycle representing \( z \). Then \( cl_0(z) \) is the extension class

\[
0 \longrightarrow H^{2i-1}(\overline{X}, \mathbb{Q}_\ell(i)) \longrightarrow E \longrightarrow \mathbb{Q}_\ell \longrightarrow 0
\]

given by the pull-back of

\[
0 \longrightarrow H^{2i-1}(\overline{X}, \mathbb{Q}_\ell(i)) \longrightarrow H^{2i-1}(\overline{X} - |Z|, \mathbb{Q}_\ell(i)) \longrightarrow \relarg{\text{Ker}}{H^{2i}_|Z|(\overline{X}, \mathbb{Q}_\ell(i)) \rightarrow H^{2i}(\overline{X}, \mathbb{Q}_\ell(i))) \longrightarrow 0
\]

via \( \mathbb{Q}_\ell \cong \mathbb{Q}_\ell \cdot cl^{X,i}_0(Z) \mapsto H^{2i}_|Z|(\overline{X}, \mathbb{Q}_\ell(i)). \)

Let \( K \) be a finite extension of \( \mathbb{Q}_p \). For any \( p \)-adic Galois representation \( V \) of \( G_K \), Bloch and Kato and Nekovar have introduced subspaces \( H^1_f(K, V) \subseteq H^1_{\text{st}}(K, V) \subseteq H^1_g(K, V) \) of \( H^1(K, V) \) given by

\[
H^1_f(K, V) = \text{Ker}(H^1(K, V) \rightarrow H^1(K, V \otimes_{\mathbb{Q}_p} B_{\text{cris}}))
\]

\[
H^1_{\text{st}}(K, V) = \text{Ker}(H^1(K, V) \rightarrow H^1(K, V \otimes_{\mathbb{Q}_p} B_{\text{st}}))
\]

\[
H^1_g(K, V) = \text{Ker}(H^1(K, V) \rightarrow H^1(K, V \otimes_{\mathbb{Q}_p} B_{\text{DR}}))
\]

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(for the first and the last see [BK] and for the second see [Ne2]). It is known [Ne2] that if V is a semistable (resp. crystalline, resp. de Rham) representation of G_K then H^1_{st}(K,V) (resp. H^1_f(K,V), resp. H^1_g(K,V)) can be identified with the group of extension classes of V by Q_p in the category of semistable (resp. crystalline resp. de Rham) representations of G_K.

**Lemma 7.1** Let X be smooth projective variety over K. Then the image of (40) for \( \ell = p \) is contained in H^1_{st}(K, H^{2i-1}(\overline{X}, Q_p(i))).

Nekovar has announced a proof of this fact in (see [Ne4]). His argument is based on a spectral sequence relating Ext-groups in the category of semistable p-adic representations to log-syntomic cohomology. If X has good reduction a proof along these lines appears already in [Ne5]. Nekovar has indicated also a different approach for proving this lemma ([Ne4], 3.11): firstly by using de Jong’s theorem on alterations he shows that it is enough to consider the case where X has semistable reduction ([Ne4], page 6 above). Then H^{2i-1}(\overline{X}, Q_p(i))) is a semistable representation (see [Ts]) and therefore H^2_{st}(K, H^{2i-1}(\overline{X}, Q_p(i))) = H^2_{st}(K, H^{2i-1}(\overline{X}, Q_p(i))) by [Hy] or [Ne4], proposition 1.24). Consequently it is enough to show that in the extension (41) corresponding to an element \( z \in CH^i(X)_0 \), the middle term is a de Rham representation or – if Z be a cycle representing \( z \) – that H^{2i-1}(\overline{X} - |Z|, Q_p(i)) is de Rham. However this follows from [K1] where it is shown that H^{2i-1}(\overline{X} - |Z|, Q_p(i)) is potentially semistable hence de Rham.

Now assume that K is a number field and let v be a place of K above \( p \) which for simplicity we assume to be unramified over \( p \). Let X be a smooth projective variety over K and assume that H^i(\overline{X}, Q_p) is semistable as a representation of the local Galois group G_Kv for all j. Then the functor D_{st,Kv} yields isomorphisms

(42) \[ H^{2i}(\overline{X}, Q_p(i))^{G_{Kv}} \cong \Gamma(D_{st,Kv}(H^{2i}(\overline{X}, Q_p(i)))) \],

(43) \[ H^1_{st}(K_v, H^{2i-1}(\overline{X}, Q_p(i))) \cong \text{Ext}^1_{\text{Rep}_{ss}(G_{Kv})}(Q_p, H^{2i-1}(\overline{X}, Q_p(i))) \cong \text{Ext}^1_{MF_{Kv}^{\text{ad}}(\phi,N)}(K_v, D_{st}(H^{2i-1}(\overline{X}, Q_p(i)))) \],

where for a filtered \((\phi,N)\)-module D, \( \Gamma(D) \) is given by

\[ \Gamma(D) = \text{Hom}_{MF_{Kv}^{\text{ad}}(\phi,N)}(K_v, D) = F^0D \cap D^{\phi=0,N=0} \].

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By composing (39) and (40) for \( \ell = p \) with the restriction maps for \( K_v/K \) and with (42) and (43) respectively we obtain maps

\[
cl = cl^{X,i} : CH^i(X) \longrightarrow \Gamma(D_{st,K_v}(H^{2i}(\overline{X},\mathbb{Q}_p(i)))),
\]

(44)

\[
cl_0 = cl_0^{X,i} : CH^i(X)_0 \longrightarrow \text{Ext}^1_{MF^q_{K_v}(\phi,N)}(K_v[i],D_{st,K_v}(H^{2i-1}(\overline{X},\mathbb{Q}_p))).
\]

(45)

One can easily extend the definition of (39), (40), (44) and (45) to Chow motives. In the following we use the notation of the appendix 10.1 (for definitions and general facts about Chow motives we refer to [Sch2]). Note that \( \text{Corr}(X,X) \) acts on the source and the target of (39) and (40) and that both maps are homomorphisms of \( \text{Corr}(X,X) \)-modules. Thus if \( \mathcal{M} = (X, p) \) is a motive over \( K \) such that its \( p \)-adic realizations are semistable as \( G_{K_v} \)-representations we obtain maps

\[
cl = cl^{M,i} : CH^i(\mathcal{M}) \longrightarrow \Gamma(D_{st,K_v}(H^{2i}_p(\mathcal{M})(i))),
\]

(46)

\[
CH^i(\mathcal{M})_0 \longrightarrow \text{Ext}^1_{MF^q_{K_v}(\phi,N)}(K_v[i],D_{st,K_v}(H^{2i-1}_p(\mathcal{M}))).
\]

(47)

This applies in particular to the sequence of motives \( \mathcal{M}_n \). By Lemma [10.1] of the appendix we see that \( CH^{m+1}((\mathcal{M}_n)_K)_0 = CH^{m+1}((\mathcal{M}_n)_K) \). Lemma 2.4 together with ([Ne2], 1.27) (and appendix 10.1) shows that the target of (47) for \( \mathcal{M} = (\mathcal{M}_n)_K, i = m + 1 \) can be identified with

\[
\text{Ext}^1_{MF^q_{K_v}(\phi,N)}(K_v[m + 1],D_{st,K_v}(H_p(\mathcal{M}_n))) \cong H^{m+1}_{DR}(\mathcal{M}_n)_K/F^{m+1}
\]

\[
\cong F^{m+1}H^{m+1}_{DR}(\mathcal{M}_n)_K \sim M_k(X, K_v)^{\vee}.
\]

(48)

\[
\text{Definition 7.2} \quad \text{Let}
\]

\[
\rho_K = \rho_{K,v} : CH^{m+1}((\mathcal{M}_n)_K) \longrightarrow M_k(X, K_v)^{\vee}
\]

be the composite of (17) (for \( \mathcal{M} = \mathcal{M}_n, i = m + 1 \)) with (18). It will be called the \( v \)-adic Abel-Jacobi map for \( (\mathcal{M}_n)_K \).
Let $K \hookrightarrow \mathbb{Q}_p^{ur}$ be an inclusion which induces the place $v$ of $K$. By abuse of notation we will denote the composition of $\rho_{K,v}$ with the inclusion $M_k(X, K^\vee) \hookrightarrow M_k(X, \mathbb{Q}_p^{ur})^\vee \cong M_k(\Gamma)^\vee$ also by $\rho_K$

\begin{equation}
\rho_K = \rho_{K,v} : \text{CH}^{m+1}((\mathcal{M}_n)_K) \longrightarrow M_k(\Gamma)^\vee.
\end{equation}

Let $\tilde{P}$ be a closed point of $(X_M)_K$ and let $\mathcal{A}_P^m$ be the fiber of $(\mathcal{A}^m)_K \to (X_M)_K$ over $P$. We assume that the residue field $L:= K(\tilde{P})$ of $\tilde{P}$ is Galois over $\mathbb{Q}$ and that $p$ is unramified in $L$. We let $P$ be the image of $\tilde{P}$ under $p : (X_M)_K \to X_K$ with residue field $H$ (later $K$ will be an imaginary quadratic field and $P$ a Heegner point so that $H$ is the Hilbert class field of $K$ and $L = H(\mu_M)$). Our aim is to compute the image of a cycle class under $\rho_K$ which has support on $A_m \tilde{P}$. More precisely we fix an embedding $L \hookrightarrow \mathbb{Q}_p^{ur}$ which induces the place $v$ on $K$ (the induced place on $H$ and $L$ will be also denoted by $v$). It determines a point on $(X_M)_{\mathbb{Q}_p^{ur}}$ above $\tilde{P}$ which according to (24) can be written as $\Gamma_M z, z \in H_p$ (lying in one of the components $\Gamma_M \setminus H_p$). We also assume that the stabilizer of $z$ (in $\Gamma$) is $\{ \pm 1 \}$.

There is an exact sequence of $G_L$-modules

\begin{equation}
0 \longrightarrow H^1(X_M, \mathbb{L}_n)(m+1) \longrightarrow H^1(X_M - (\tilde{P} \otimes L \mathbb{Q}), \mathbb{L}_n)(m+1) \longrightarrow \\
\to (\mathbb{L}_n)_{\tilde{P} \otimes L \mathbb{Q}}(m) = H_p^{2m}((\mathcal{A}_P^m, \epsilon_n)/L)(m) \longrightarrow 0,
\end{equation}

where $H_p^{2m}((\mathcal{A}_P^m, \epsilon_n)/K)$ is the $p$-adic realisation of the motive $(\mathcal{A}_P^m, \epsilon_n)/L$.

Let

\begin{equation}
0 \longrightarrow H_p(\mathcal{M}_n)(m+1) \longrightarrow E \longrightarrow (\mathbb{L}_m)_{\tilde{P} \otimes L \mathbb{Q}}(m) \longrightarrow 0
\end{equation}

be the push-out of (50) under $H^1(X_M, \mathbb{L}_n) \to H_p(\mathcal{M}_n)$. By restricting the Galois action to $G_{Q^{ur}} \subseteq G_L$ and applying $D_{st, Q^{ur}}$ to (51) we obtain a short exact sequence in $MF_{Q^{ur}}^{ad}(\phi, N)$ which according to Theorem 5.13 is isomorphic to the sequence

\begin{equation}
0 \longrightarrow H^1(X_{Q^{ur}}, \mathcal{V}_n)[-m+1] \longrightarrow H^1(U, \mathcal{V}_n)[-m+1] \overline{\text{Res}}_{\mathcal{V}_n}(\mathcal{V}_n)_z[-m] \longrightarrow 0
\end{equation}

where $U := X_{\Gamma} - \{ \Gamma z \}$.
Proposition 7.3 The diagram

$$CH^m((\mathcal{A}_P^m, \epsilon_n)) \longrightarrow \Gamma((\mathcal{V}_n)\mathcal{z}[-m])$$

commutes. Here the upper horizontal map is the composition of (46) for $M = (A^m_P, \epsilon_n)/L$ with the restriction map to $\mathbb{Q}_p^w$. The right vertical map is induced by $A^m_P \hookrightarrow (A^m)_H$ and the left is a connecting homomorphism in the long exact Ext-groups sequence corresponding to (52).

Proof. This can be easily deduced from the commutativity of the diagram

$$\epsilon_n CH^m((A^m_P, \epsilon_n)) \longrightarrow \epsilon_n H^{2m}(A^m_P \otimes_L \overline{\mathbb{Q}}, \mathbb{Q}_p(m))^{GL}$$

where the lower arrow is given by the composition

$$\epsilon_n CH^{m+1}((A^m)_H) \longrightarrow H^1_{cont}(L, \epsilon_n H^{2m-1}((A^m)_{\overline{\mathbb{Q}}}, \mathbb{Q}_p(m))) \longrightarrow$$

$$\longrightarrow H^1_{cont}(L, \epsilon_n H^{2m-1}((A^m)_{\overline{\mathbb{Q}}}, \mathbb{Q}_p(m))).$$

\[\square\]

8 Heegner cycles

In this section we define Heegner cycles. They are not really algebraic cycles but elements in $CH^{m+1}(\mathcal{M}_n)$. We then compute their images under the $p$-adic Abel-Jacobi map.

To begin with we recall the definition of CM-points and Heegner points on $X = X_{N^+, pN^-}$. Let $F$ be a field of characteristic 0. Recall that any $F$-valued point of $X$ can be represented by an abelian surface with QM, i.e. by a triple $(A, \iota, C)$ where $A$ is an abelian surface, $\iota : \mathcal{R}^{max} \rightarrow \text{End}_F(A)$ a ring monomorphism and $C$ a cyclic subgroup of order $N^+$ in $A$ which is stable under the action of the Eichler order $\mathcal{R}$. The abelian surface $A$ is said to
have complex multiplication if $\text{End}_{R_{\text{max}}}(A) \neq \mathbb{Z}$. In this case it is an order $\mathcal{O}$ in an imaginary quadratic number field and $A$ is said to have complex multiplication by $\mathcal{O}$. An $F$-valued point on $X$ is called CM-point if it can be represented by a triple $(A, \iota, C)$ such that $A$ has complex multiplication. It is called Heegner point if $(A, \iota, C)$ can be chosen so that $A$ has complex multiplication by $\mathcal{O}$ and $C$ is $\mathcal{O}$-stable.

Before we proceed with the definition of Heegner cycles we need to determine the structure of the Neron-Severi group (tensored by $\mathbb{Q}$) of an abelian surface with QM. Let $A$ be an abelian surface over $F$, $\iota : R_{\text{max}} \to \text{End}_F(A)$ a monomorphism and set $\text{NS}(A)_{\mathbb{Q}} = \text{NS}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$. We have a natural right $\mathcal{B}^*$-action on $\text{NS}(A)_{\mathbb{Q}}$ given by $\mathcal{L} \cdot b = \iota(b)^*(\mathcal{L})$. By $\text{ad}^0(\mathcal{B})$ we denote the representation of $\mathcal{B}^*$ consisting of elements of trace $= 0$ on which $\mathcal{B}^*$ acts from the right by $u \cdot b = \bar{b}u$.

**Lemma 8.1** As a right $\mathcal{B}^*$-module $\text{NS}(A) \otimes \mathbb{Q}$ decomposes as follows into irreducible $\mathcal{B}^*$-representations.

(a) If $A$ has no complex multiplication then $\text{NS}(A)_{\mathbb{Q}} \cong \text{ad}^0(\mathcal{B})$.

(b) If $A$ has complex multiplication then $\text{NS}(A)_{\mathbb{Q}} \cong \text{ad}^0(\mathcal{B}) \oplus \text{Nrd}$ where $\text{Nrd}$ denotes the one-dimensional representation of $\mathcal{B}^*$ given by the reduced norm $\text{Nrd} : \mathcal{B}^* \to \mathbb{Q}$. In this case the two summands are orthogonal to each other under the intersection pairing on $\text{NS}(A)_{\mathbb{Q}}$.

**Proof.** See [Be].

We fix now (for the rest of this paper) an imaginary quadratic number field $K$. We denote the non-trivial automorphism of $K$ by $a \mapsto \bar{a}$. We assume that the following conditions hold.

- The discriminant $D_K > 0$ of $K$ is relatively prime to $N = N^+ N^- p$.
- All prime divisors of $N^- p$ are inert in $K$.
- All prime divisors of $N^+$ are split in $K$.

Note that the second condition implies that the prime ideal $p\mathcal{O}_K$ splits completely in the Hilbert class field $H/K$. For simplicity we also assume that $\mathcal{O}_K^* = \{ \pm 1 \}$, a condition which is satisfied as soon as $D_K > 4$.

In the case where $A$ has complex multiplication by $\mathcal{O}_K$ we can pick a canonical generator (up to sign) of the one-dimensional subspace of $\epsilon_2 \text{NS}(A)_{\mathbb{Q}} \subseteq \text{NS}(A)_{\mathbb{Q}}$ where $\mathcal{B}^*$ acts via the reduced norm.
Proposition 8.2 Assume that $A$ has complex multiplication by $\mathcal{O}_K$. Then there exists an element $y_H$ in $\text{NS}(A)$ such that 
(a) $\iota(b)^*(y_{CM}) = N(b)y_{CM}$ for all $b \in \mathcal{R}$ 
(b) The self-intersection number of $y_{CM}$ is $-2D_K$ (thus $y_{CM} \in \epsilon_2\text{NS}(A)_{\mathbb{Q}}$).

Up to sign, $y_{CM}$ is uniquely determined by these properties.

Proof. The uniqueness of $\pm y_{CM}$ follows immediately from (8.1). For the existence we note first that $\text{End}(A) \cong \mathcal{O}_K \otimes \mathcal{R}^{\text{max}} \cong M_2(\mathcal{O}_K)$.

Once we have fixed a bijection $\text{End}(A) \overset{\sim}{\rightarrow} M_2(\mathcal{O}_K)$ we get an isomorphism $A \cong E \times E$ where $E$ is an elliptic curve over $H$ with $\text{End}(E) \cong \mathcal{O}_K$. Denote by $\Gamma_{\sqrt{-D_K}}$ the graph of the endomorphism $\sqrt{-D_K} \in \text{End}(E)$. Then 

$$y_{CM} := [\Gamma_{\sqrt{-D_K}} - D_K[0 \times E] - [E \times 0] \in \text{NS}(A)$$

has the desired properties (see [Ne3], II.3.3(3)). \hfill \Box

A Heegner point on $X$ associated to $K$ is a Heegner point which is represented by a triple $(A, \iota, C)$ where $A$ has complex multiplication by the ring of integers $\mathcal{O}_K$ of $K$ (and $C$ is stable under the $\mathcal{O}_K$-action). Since from now on we are dealing only with Heegner points associated to $K$ we often drop the reference to $K$.

Let $t$ denote the number of prime factors of $N$ and $h$ the class number of $K$. There are precisely $2^t h$ different Heegner points on $X$ associated to $K$ which are all defined over the Hilbert class field $H$ of $K$ (cf. [BD1], 2.5). Let $\mathcal{W} \cong (\mathbb{Z}/2\mathbb{Z})^t$ denote the subgroup of $\text{Aut}(X)$ generated by the Atkin-Lehner involutions $w_\ell : X \rightarrow X$, for $\ell$ a prime factor of $N$ and let $\Delta = \text{Pic}(\mathcal{O}_K)$ which we identify with $\text{Gal}(H/K)$ via the Artin map. There is a natural action of $\mathcal{W} \times \Delta$ on the set of Heegner points which is free and transitive.

To define the Heegner cycle for $\mathcal{M}_n$ we have to work again with the finer moduli problem [5.3]. We fix an integer $M \geq 3$ relatively prime to $N$ and let $q : X_M \rightarrow X$ be the projection and $\pi : A \rightarrow X_M$ be the universal abelian surface. Let $P$ be a Heegner point on $X$ (viewed as a closed point on $X_H$) and let $\tilde{P}$ be any point of $(X_M)_H$ above $P$. The fiber $A_{\tilde{P}}$ is an abelian surface with $\text{End}_{\mathcal{R}^{\text{max}}}(A_{\tilde{P}}) = \mathcal{O}_K$. Hence by proposition 8.2 there exists an – up to sign – unique element $y_{CM,\tilde{P}} \in \text{NS}(A_{\tilde{P}})$ satisfying (a) and (b) of 8.2. We choose a representative $z_{\tilde{P}}$ of $y_{CM,\tilde{P}}$ in $\epsilon_2 \text{Pic}(A_{\tilde{P}}) = \epsilon_2 CH^1(A_{\tilde{P}})$. 

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One can choose the elements $z_{\tilde{P}}, \tilde{P} \in q^{-1}(P)$ in such a way that they are compatible with the $G = G_{M'}$-action. More precisely any $g \in G$ extends uniquely to an automorphism $A \to A$, also denoted by $\tilde{g}$, and thus induces an isomorphism

$$g : A_{\tilde{P}} \to A_{\tilde{g}(\tilde{P})}$$

for $\tilde{P} \in q^{-1}(P)$. We require that the elements $z_{\tilde{P}} \in \epsilon_2\text{CH}^1(A_{\tilde{P}})$ satisfy

$$g_*(z_{\tilde{P}}) = z_{\tilde{g}(\tilde{P})}$$

for all $\tilde{P} \in q^{-1}(P)$ and $g \in G$.

We define the cycle class $y_P = y_P^{(2)}$ in $CH^2((\mathcal{M}_2)_H)$ to be the image of $z_{\tilde{P}}$ under

$$\epsilon_2\text{CH}^1(A_{\tilde{P}}) \xrightarrow{(\iota_{\tilde{P}})_*} \epsilon_2\text{CH}^2(A_H) \xrightarrow{(\rho_G)_*} (\epsilon_2\text{CH}^2(A_H))^G = CH^2((\mathcal{M}_2)_H)$$

with $p_G = \frac{1}{|G|} \sum_{g \in G} g \in \mathbb{Q}[G]$. By (55) this is independent of the point $\tilde{P}$ we have chosen above $P$.

We also require that the elements $\{y_P\}$ are compatible with respect to the action of the Atkin-Lehner involutions and the $\Delta$-action. Any $w \in \mathcal{W}$ extends canonically to an involution $w : X_M \to X_M$ which commutes with the $G$-action. Also the action of $\Delta$ on $H$ induces an action on $(X_M)_H, A_H^m$ etc. which commutes with the $\mathcal{W}$- and $G$-operations. We can choose the elements $z_{\tilde{P}}$ for different Heegner points $P$ and $\tilde{P} \in q^{-1}(P)$ so that

$$w_*(z_{\tilde{P}}) = z_{w(\tilde{P})}, \quad \delta_*(z_{\tilde{P}}) = z_{\delta(\tilde{P})}$$

for all $w \in \mathcal{W}$ and $\delta \in \Delta$ and all $\tilde{P}$. With this normalisation we obtain

$$w_*(y_P) = y_{w(P)}, \quad \delta_*(y_P) = y_{\delta(P)} \quad \forall w \in \mathcal{W}, \delta \in \Delta.$$  

\textbf{Remark 8.3} Albeit the cycle class $y_P \in CH^2((\mathcal{M}_n)_H)$ is defined using $z_{\tilde{P}} \in CH^1(A_{\tilde{P}})$ conjectures of Bloch and Beilinson would imply that it depends only on the the class of $z_{\tilde{P}}$ modulo homological equivalence i.e. on $y_{CM,\tilde{P}}$.

For arbitrary (even) $n \geq 2$ we define cycles $y_P^{(n)} \in CH^{m+1}((\mathcal{M}_n)_H)$ as follows. Firstly for $\tilde{P} \in q^{-1}(P)$ let $z_{\tilde{P}}^{(n)}$ be the image of the $m$th exterior...
product $z_P \times \ldots \times z_P$ under the projector $\epsilon_n$ acting on $CH^m(A_P^n)$. Then $y_P^{(n)} \in CH^{m+1}(\mathcal{M}_n)_H$ is defined as the image of $z_P^{(n)}$ under

$$\epsilon_n CH^m(A_P^n) \xrightarrow{(\iota_P)_*} \epsilon_n CH^{m+1}(A_H^n) \xrightarrow{(\rho_G)_*} CH^{m+1}(\mathcal{M}_n)_H$$

As before $y_P^{(n)}$ does not depend on the point $\tilde{P}$ above $P$ and we have

$$w_*(y_P^{(n)}) = y_{w(P)}^{(n)}, \quad \delta_*(y_P^{(n)}) = y_{\delta(P)}^{(n)}$$

for all $w \in \mathcal{W}$, $\delta \in \Delta$ and all Heegner points $P$. The cycle class $y_P^{(n)}$ is called Heegner cycle on $\mathcal{M}_n$.

We want to compute the image of $y_P^{(n)}$ under the map $\rho_H$ (cf. (49)). For that we need to recall the $p$-adic analytic description of the set of Heegner points on $X$ given in ([BD2], section 5). Let $\mathcal{O}_K = \mathcal{O}_K[\frac{1}{p}]$. An embedding $\Psi : K \rightarrow B$ is called optimal if $\Psi(\mathcal{O}) \subseteq R$. The group $\Gamma$ acts naturally by conjugation on the set of optimal embeddings and we denote by $\text{emb}(\mathcal{O}, R)$ the set of conjugacy classes (note that contrary to [BD2] we do not consider orientations here). For an optimal embedding $\Psi$ we denote by $[\Psi] \in \text{emb}(\mathcal{O}, R)$ its class. There is a natural action of $\text{Pic}(\mathcal{O}) \cong \Delta$ on $\text{emb}(\mathcal{O}, R)$ given as follows (see [BDIS], 2.3). Let $\alpha \in \Delta$ and choose an ideal $a \in \mathcal{O}$ representing it. Let $\Psi : K \rightarrow B$ be an optimal embedding. The right order of the left $R$-ideal $R\Psi(a)$, denoted by $R_a$, is an Eichler $\mathbb{Z}[\frac{1}{p}]$-order of level $N^+$ in $B$. The right action of $\Psi(\mathcal{O})$ on $R\Psi(a)$ yields an embedding $\tilde{\Psi}_a : \mathcal{O} \rightarrow R_a$. Since all Eichler $\mathbb{Z}[\frac{1}{p}]$-orders are conjugated there is an element $a \in B^*$ such that $\text{ord}_p(Nrd(a)) = 0$ and $R = aR_a a^{-1}$. Then,

$$\alpha \cdot [\Psi] = [\tilde{\Psi}_a^\alpha].$$

By ([BD2], section 5) there is a bijection

$$\text{Heegner points on } X_H \xrightarrow{\cong} \text{emb}(\mathcal{O}, R)$$

which is compatible with the $\Delta$-action on both sides. It is given as follows. We fix an embedding $\iota : H(\mu_M) \hookrightarrow \mathbb{Q}_p^{ur}$ and denote by $v$ be the corresponding place of $H(\mu_M)$ (and also of $H$). The embedding $\iota$ allows us to identify
$K_p := K \otimes \mathbb{Q}_p$ and $H_v$ with $\mathbb{Q}_p^2 \subseteq \mathbb{Q}_p^{ur}$. Using the $p$-adic uniformisation of $X_{\mathbb{Q}_p^2}$ we view $\text{emb}(\mathcal{O}, R)$ as a subset of $X(K_p)$ via

$$\text{emb}(\mathcal{O}, R) \hookrightarrow \Gamma \setminus \text{Hom}(K_p, B_p) \cong \Gamma \setminus \text{Hom}(\mathbb{Q}_p^2, M_2(\mathbb{Q}_p)) = X_\Gamma(K_p)$$

The image of $\text{emb}(\mathcal{O}, R)$ under (59) is the set of Heegner points (considered as a subset of $X(K_p)$ via $X(H) \hookrightarrow X(H_v) = X(K_p)$).

Let $P$ be a Heegner point and let $\Psi : K \rightarrow B$ be an optimal embedding corresponding to $P$ which we view as an element of $\text{Hom}(\mathbb{Q}_p^2, M_2(\mathbb{Q}_p)) \subseteq \mathcal{H}_p$. We fix one of the components $\Gamma M_2 \setminus \mathcal{H}_p$ of $(X_M)_{\mathbb{Q}_p^{ur}}$ and let $\bar{P} \in q^{-1}(P)$ be the point $\Gamma M_2$. Let $P_\Psi(X) \in \mathcal{P}_2$ be the polynomial $P_\Psi(\sqrt{-D_K})(X)$ defined as in equation (29) in section 3 and define $d_{\Psi}^{(n)} \in V_n$ by

$$d_{\Psi}^{(n)}(P(X)) = \langle P(X), P_\Psi^m \rangle.$$  

**Lemma 8.4** Let

$$cl_{p, \Psi} : CH^m(\mathbb{A}_{\mathbb{Q}_p}, \epsilon_n) \longrightarrow \Gamma((V_n)_\Psi[-m])$$

denote the upper horizontal homomorphism in diagram (53). We have,

$$cl_p(Z_{\bar{P}}^{(n)}) = \pm d_{\Psi}^{(n)}.$$  

**Proof.** It is enough to consider the case $n = 2$ since the case of an arbitrary (even) $n$ follows formally from the first case. We have an isomorphism

$$U_{\mathbb{Q}_p^{ur}} = \{ U \in M_2(\mathbb{Q}_p^{ur}) | \text{trace}(U) = 0 \} \longrightarrow V_{\mathbb{Q}_p^{ur}}, U \mapsto <P_U(X), \ldots>.$$  

The Frobenius on $V_\Psi[-1]$ corresponds under (51) to the map $1_U \otimes \sigma$ on $U_{\mathbb{Q}_p^{ur}}$ and the subspace $V_{\mathbb{Q}_p^{ur}, i}$ to

$$U_i := \{ U \in U_{\mathbb{Q}_p^{ur}} | \Psi(\bar{a})U\Psi(a) = a^i \bar{a}^{2-i}U \}.$$  

A simple computation now shows that $\Gamma(V_\Psi[-1])$ is an one-dimensional $\mathbb{Q}_p$-vector space generated by $< P_\Psi, \ldots > = cl_{\Psi}^{(2)}$. Since $< P_\Psi, P_\Psi > = -\text{trace}(\sqrt{-D_K}(\sqrt{-D_K})) = -2D_K$ the assertion follows from (5.14) and (8.2).
Remark 8.5  Recall that the isomorphism (27) was so far well-defined only up to sign (Remark 5.12) and that changing the sign also changes the sign of the map (60). We fix the sign by requiring that $c_{p}(z_{n}^{(n)}) = cl_{p}^{(n)}$. This automatically implies that $c_{p}(z_{n}^{(n)}) = cl_{p}^{(n)}$ for any optimal embedding $\Psi : K \to B$ since $\mathcal{W} \times \Delta$-acts transitively on the set of Heegner points. For example consider the Atkin-Lehner involution $w_{p}$ at $p$ and choose $\gamma \in R^{*}$ such that $\text{ord}_{p}(Nrd(\gamma))$ is odd. We have a commutative diagram

$$
\begin{array}{ccc}
CH^{m}((\mathcal{A}_{\psi}^{m}, \epsilon_{n})) & \xrightarrow{cl_{p}, \Psi} & \Gamma((V_{n})_{\psi}[-m]) \\
\downarrow (w_{p})_{*} & & \downarrow \gamma_{*} \\
CH^{m}((\mathcal{A}_{w_{p}(\bar{P})}^{m}, \epsilon_{n})) & \xrightarrow{cl_{p}, \Psi, \gamma} & \Gamma((V_{n})_{\psi}[-m])
\end{array}
$$

where $\gamma_{*}(v) = Nrd(\gamma)^{-m} \gamma v$. Since $\gamma_{*}(cl_{\psi}^{(n)}) = cl_{\psi}^{(n)}$ we get

$$
cl_{p, \Psi, \gamma}(y_{w_{p}(\bar{P})}^{(n)}) = (w_{p})_{*}(y_{w_{p}(\bar{P})}^{(n)}) = \gamma_{*}(cl_{p, \Psi}^{(n)}) = cl_{\psi}^{(n)}.
$$

For $\alpha \in \Delta$ and $a \in B^{*}$ as in (57) we get a diagram as above with $\alpha$ and $a$ instead of $w_{p}$ and $\gamma$.

9 Derivatives of anticyclotomic $p$-adic $L$-functions

In this section we show that the derivative of the anticyclotomic $p$-adic $L$-function attached to a newform of even weight $k \geq 4$ on $X$ at the central critical point $s = k/2$ can be expressed in terms of the $p$-adic Abel-Jacobi image of Heegner cycles.

To begin with we briefly review the construction of the anticyclotomic $p$-adic $L$-function $L_{p}(f, K, s)$ from ([BDIS] section 2.5). We keep the notation and assumptions of the last section. Let $f(z) \in M_{k}(\Gamma)$ and let $\mathcal{A}_{n}$ denote the set of $\mathbb{C}_{p}$-valued functions on $\mathbb{P}^{1}(\mathbb{Q}_{p})$ which are locally analytic except for a pole of order at most $n$ at $\infty$. Associated to $f(z)$ there is a distribution

$$
\mu_{f} : \mathcal{A}_{n} \to \mathbb{C}_{p}, \vartheta \mapsto \mu_{f}(\vartheta) = \int_{\mathbb{P}^{1}(\mathbb{Q}_{p})} \vartheta(x) \mu_{f}(dx)
$$
which is characterised by the property
\[
\int_{U(e)} P(x) \mu_f(dx) = \text{Res}_e (P(z) f(z) dz)
\]
for all \( P(X) \in \mathcal{P}_n \) and \( e \in \mathcal{E}(\mathcal{T}) \). Here \( U(e) \) denotes as usual the compact subset of \( \mathbb{P}^1(\mathbb{Q}_p) \) corresponding to the ends of \( \mathcal{T} \) containing \( e \) (see e.g. [BDIS], section 1.3).

Let \( K_\infty/K \) be the maximal abelian extension of \( K \) which is unramified outside \( p \) and anticyclotomic, i.e. the involution in \( \text{Gal}(K/\mathbb{Q}) \) acts as \(-1\) on \( \text{Gal}(K_\infty/K) \). Let \( G = \text{Gal}(K_\infty/H) \) and \( \tilde{G} = \text{Gal}(K_\infty/K) \) so that \( \tilde{G}/G \cong \Delta \). Class field theory identifies \( G \) with \( K^*_p/\mathbb{Q}^*_p \). An optimal embedding \( \Psi : K \to B \) yields an embedding
\[
G \cong K^*_p/\mathbb{Q}^*_p \xrightarrow{\Psi} B^*_p/\mathbb{Q}^*_p \cong \text{PGL}_2(\mathbb{Q}_p)
\]
hence a (simply transitive) action of \( G \) on \( \mathbb{P}^1(\mathbb{Q}_p) \). For a fixed base point \( * \in \mathbb{P}^1(\mathbb{Q}_p) \) we let \( \eta_{\Psi,*} : G \to \mathbb{P}^1(\mathbb{Q}_p) \) be the bijection given by letting \( G \) act on \(*\). We obtain a distribution \( \mu_{f,\Psi,*} \) on \( G \), i.e. a functional on the set \( \mathcal{A}(G) \) of \( \mathbb{C}_p \)-valued locally analytic functions on \( G \) by setting
\[
\int_G \vartheta(\alpha) \mu_{f,\Psi,*}(d\alpha) : = \int_{\mathbb{P}^1(\mathbb{Q}_p)} \vartheta \circ \eta_{\Psi,*}^{-1}(x) P_\Psi(x)^m \mu_f(dx)
\]
for \( \vartheta \in \mathcal{A}(G) \). For \( \gamma \in \Gamma \) we have \( \mu_{f,\Psi,\gamma,*} = \mu_{f,\Psi,*} \). The distribution \( \mu_{f,\Psi,*} \) depends on the base point \(*\) and the representative \( \Psi \) of \([\Psi] \in \text{emb}(\mathcal{O},R)\) only up to translation by an element in \( G \).

Define \( \log : G \to \mathbb{Q}^ur_p \) to be the composite
\[
(62) \quad \log : G \cong K^*_p/\mathbb{Q}^*_p \cong K_{p,1} \xrightarrow{\log_p} \mathbb{Q}^2_p \subseteq \mathbb{Q}^ur_p
\]
where \( K_{p,1} = \{ x \in K^*_p \mid N_{K_p/\mathbb{Q}_p}(x) = 1 \} \) and where the second isomorphism is given by \( x \mapsto x/\bar{x} \). The map (62) extends uniquely to \( \tilde{G} \). For \( \alpha \) in \( G \) or \( \tilde{G} \) and \( s \in \mathbb{Z}_p \) we define \( \alpha^s : = \exp(s \log(\alpha)) \). The partial \( p \)-adic \( L \)-function attached to the datum \((f, \Psi, *)\) is defined as
\[
L_p(f, \Psi, *, s) : = \int_G \alpha^{s-k/2} \mu_{f,\Psi,*}(d\alpha)
\]
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If $\star' \in \mathbb{P}^1(\mathbb{Q}_p)$ is another base point and $\Psi' : K \to B$ another optimal embedding which is conjugate under $\Gamma$ to $\Psi$ then there exists an element $\alpha_0 \in G$ such that

$$L_p(f, \Psi', \star', s) = \alpha_0^{s-k/2} L_p(f, \Psi, \star, s).$$

In particular since $L_p(f, \Psi, \star, k/2) = 0$ we see that the first derivative $L'_p(f, \Psi, \star, k/2)$ is independent of the choice of $\Psi$ in $[\Psi]$ and $\star$.

The distribution $\mu_{f,\Psi,\star}$ extends canonically to a distribution on $\widetilde{G}$ denoted by $\mu_{f,K}$ for short. For that let $\Lambda$ denote the set of $\Gamma$-conjugacy classes of pairs $(\Psi, \star)$ where $\Psi : K \to B$ is an optimal embedding and $\star \in \mathbb{P}^1(\mathbb{Q}_p)$. The action of $\mathcal{W} \times \Delta$ on $\text{emb}(\mathcal{O}, R)$ lifts canonically to a simply transitive action of $\mathcal{W} \times \widetilde{G}$ on $\Lambda$ such that $\alpha(\Psi, \star) = (\Psi, \Psi(\alpha)\star)$ for $\alpha \in G$ and $(\Psi, \star) \in \Lambda$ (see [BDIS], Lemma 2.14). We denote by $\Xi$ the set of $\Delta$-orbits in $\text{emb}(\mathcal{O}, R)$ (or $G$-orbits in $\Lambda$). It is a principal homogeneous $\mathcal{W}$-space.

Fix $\xi$ in $\Xi$ and a pair $(\Psi, \star)$ representing it (depending on the circumstances we consider elements in $\Xi$ either as $\Delta$-orbits in $\text{emb}(\mathcal{O}, R)$ or $G$-orbits in $\Lambda$). For every $\delta \in \Delta$ we choose a lift $\alpha_\delta \in \widetilde{G}$ and write $(\Psi_\delta, \star_\delta) : = \alpha_\delta(\Psi, \star)$. The distribution $\mu_{f,K} = \mu_{f,K,\xi}$ is given by the formula

$$\int_{\widetilde{G}} \vartheta(\alpha) \mu_{f,K}(d\alpha) : = \sum_{\delta \in \Delta} \int_{\widetilde{G}} \vartheta(\alpha \alpha_\delta) \mu_{f,\Psi_\delta,\star_\delta}(d\alpha)$$

for $\vartheta \in A(\widetilde{G})$. It is independent of the choice of the $\alpha_\delta$ and depends on the pair $(\Psi, \star) \in \xi$ only up to translation by an element in $\widetilde{G}$. The anticyclotomic $p$-adic $L$-function attached to the datum $(f, K, \xi)$ is a function in the $p$-adic variable $s \in \mathbb{Z}_p$ defined as

$$L_p(f, K, \xi, s) : = \int_{\widetilde{G}} \alpha^{s-k/2} \mu_{f,\Psi,\star}(d\alpha).$$

Up to a multiple of the form $s \mapsto \alpha_0^{s-k/2}$ it is independent of the pair $(\Psi, \star) \in \xi$ (for the justification of the term anticyclotomic $p$-adic $L$-function we refer to [BDIS]). $L_p(f, K, \xi, s)$ can be written as a sum of partial $p$-adic $L$-functions

$$L_p(f, K, \xi, s) = \sum_{i=1}^{h} L_p(f, \Psi_i, \star_i, s)$$

for $h \in \mathbb{N}$.
where $\Psi_1, \ldots, \Psi_h$ are representatives of the $\Delta$-orbit $\xi$ and $\star_1, \ldots, \star_h$ are suitably chosen base points. In particular $L_p(f, K, \xi, k/2) = 0$ and $L'_p(f, K, \xi, k/2) = \sum_{i=1}^h L'_p(f, \Psi_i, \star_i, k/2)$ is independent of the pair $(\Psi, \star)$.

Remark 9.1 Assume that $f(z)$ is a newform, i.e. it corresponds to a newform $f_\infty \in S_k(\Gamma_0(N))$ under the isomorphism (20). In this case $L_p(f, K, \xi, s)$ is independent of $\xi$ up to sign. In fact if $w \in W$ and $\epsilon \in \{\pm 1\}$ such that $w(f) = \epsilon f$ then

$$L_p(f, K, w\xi, s) = \epsilon L_p(f, K, \xi, s).$$

In [BDIS] the dependence of $L_p(f, K, \xi, s)$ on $\xi$ was surpressed. We had considered there always a fixed $\Delta$-orbit $\xi$ in $\text{emb}(O, R)$. However the reader should notice that there is no canonical choice for it.

For $[\Psi] \in \text{emb}(O, R)$ we define

$$L'_p(\Psi, k/2) : M_k(\Gamma) \longrightarrow \mathbb{Q}_p^{ur}, f(z) \mapsto L'_p(f, \Psi, \star, k/2)$$

and for $\xi \in \Xi$ we let $L'_p(K, \xi, k/2)$ be the functional

$$M_k(\Gamma) \longrightarrow \mathbb{Q}_p^{ur}, f(z) \mapsto L'_p(f, K, \xi, k/2),$$

so that $L'_p(K, \xi, k/2) = L'_p(\Psi_1, k/2) + \ldots + L'_p(\Psi_h, k/2)$ where $\Psi_1, \ldots, \Psi_h$ are representatives of $\xi$. Let $P_1, \ldots, P_h$ be the Heegner points on $X_H$ corresponding to $\Psi_1, \ldots, \Psi_h$ via (58) and let $\overline{P_i} \in X(H)$ be the complex conjugate of $P_i$. We denote by $y^{(n)}_\xi$ the image of the Heegner cycle $y^{(n)}_{P_i}$ under the push-forward $CH^{m+1}((\mathcal{M}_n)_H) \rightarrow CH^{m+1}((\mathcal{M}_n)_K)$. It does not depend on the choice of $i \in \{1, \ldots, h\}$ and is mapped to $y^{(n)}_{P_1} + \ldots + y^{(n)}_{P_h}$ under pull-back to $H$. Complex conjugation also acts on the set of optimal embeddings (by $\overline{\Psi(a)} = \Psi(\overline{a})$) and likewise on $\text{emb}(O, R)$ and $\Xi$.

Our main result is:

**Theorem 9.2**

$$L'_p(K, \xi, k/2) = \rho_H(y^{(n)}_{P_1} + \ldots + y^{(n)}_{P_h} + (-1)^{m+1}(w_p)_*(y^{(n)}_{\overline{P_1}} + \ldots + y^{(n)}_{\overline{P_h}})) =$$

$$= \rho_K(y^{(n)}_\xi + (-1)^{m+1}(w_p)_*(y^{(n)}_\xi)).$$

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Proof. The second equality follows immediately from the definition of $y_z^{(n)}$. For the first equality it is enough to show that

$$L'_p(\Psi, k/2) = \rho_H(y_P^{(n)} + (-1)^{m+1}y_{wp(\overline{P})}^{(n)})$$

for any Heegner point $P$ and corresponding optimal embedding $\Psi : K \to B$. Fix $P, \Psi$ and let $z_0$ and $z_1$ be the points on $H_p$ which correspond to $\Psi$ and $\Psi$ respectively via (1). Note that the class $[\nabla] \in \text{emb}(O, R)$ belongs to $w_p(\overline{P})$ via (58). Let $U = X - \{P, w_p(\overline{P})\}$ and let

$$0 \to H^1(X, \mathcal{V}_n) \to E \to \mathbb{Q}^{ur}_p[m + 1] \to 0$$

be the pull-back of

$$0 \to H^1(X, \mathcal{V}_n) \xrightarrow{j^*} H^1(U_{\mathbb{Q}^{ur}_p}, \mathcal{V}_n) \to (V_n)_{\Psi}[1] \oplus (V_n)_{\overline{\Psi}}[1] \to 0$$

with respect to

$$\mathbb{Q}^{ur}_p[m] \to (V_n)_{\Psi} \oplus (V_n)_{\overline{\Psi}}, 1 \mapsto (c^{(n)}_{\Psi}, -c^{(n)}_{\overline{\Psi}}).$$

By Proposition 7.3, Lemma 60 and Remark 8.5 the extension class (63) corresponds to $\rho_H(y_P^{(n)} + (-1)^{m+1}y_{wp(\overline{P})}^{(n)})$ under the isomorphism

$$\text{Ext}^1_{\mathcal{M} \mathcal{F}^{ad}_{\mathbb{Q}^{ur}_p}(\phi, N)}(\mathbb{Q}^{ur}_p[m + 1], H^1(X, \mathcal{V}_n)) \cong H^1(X, \mathcal{V}_n)/F^{m+1} \cong M_k(\Gamma)^\vee.$$

Concretely, let $\alpha$ be the uniquely determined element in $H^1(U_{\mathbb{Q}^{ur}_p}, \mathcal{V}_n)_{m+1}$ with $N(\alpha) = 0$ and such that

$$\text{Res}_{z_0}(\alpha) = c^{(n)}_{\Psi} = -< P_{\Psi}^m, \ldots >, \quad \text{Res}_{z_1}(\alpha) = (-1)^{m+1}c^{(n)}_{\overline{\Psi}} = -< P_{\overline{\Psi}}^m, \ldots >.$$

Let $\beta \in H^1(X, \mathcal{V}_n)$ with $j_*(\beta) \equiv \alpha \mod F^{m+1}$. Then,

$$\rho(y_P^{(n)} + (-1)^{m+1}y_{wp(\overline{P})}^{(n)})(f(z)) = < [\omega_f], \beta >$$

for all $f(z) \in M_k(\Gamma)$.

Since $H^1(X, \mathcal{V}_n) = H^1(X, \mathcal{V}_n)_m \oplus F^{m+1}H^1(X, \mathcal{V}_n)$ (see the proof of Theorem 6.4) we may assume that $\beta \in H^1(X, \mathcal{V}_n)_m = c(H^1(\Gamma, (V_n)_{\mathbb{Q}^{ur}_p}))$, i.e. $\beta = c(c)$ for some $c \in H^1(\Gamma, (V_n)_{\mathbb{Q}^{ur}_p})$. In order to compute $< [\omega_f], \beta >$ we use the results and notation of appendix 10.2. By Theorem 10.3 we have

$$< [\omega_f], \beta > = -< I([\omega_f], c >_{\Gamma}.$$
Let \( \chi \) be a \( \Gamma \)-invariant \( V_n \)-valued meromorphic differential form on \( \mathcal{H}_p \) which is holomorphic outside of \( \pi^{-1}(U^{an}) \), which has simple poles at \( z_0, \bar{z}_0 \) and whose class in \( F_{m+1}^1(U^{ur}_{\mathbb{Q}_p}, V_n) \) represents \( \alpha - j_*(\beta) \). Hence \( P_U([\chi]) = -c, N([\chi]) = 0 \) and

\[
\text{Res}_{z_0}(\chi) = \text{Res}_{z_0}(\alpha) = <P_{\Psi}^m, \ldots > = -\text{Res}_{z_0}(\alpha) = -\text{Res}_{z_0}(\chi).
\]

We can apply Corollary 10.7 and obtain

\[
< [\omega_f], \beta > = -< I([\omega_f]), c > = < I([\omega_f]), P_U([\chi]) > = \int_{z_0}^{z_0} f(z) P_{\Psi}(z)^m dz.
\]

By ([BDIS], Theorem 3.5) the last expression is equal to \( L'_p(f, \Psi, *, k/2) \) hence together with (66) we get

\[
\rho(y_p^{(n)} + (-1)^{m+1}y_{w_p(\overline{f})}(f(z)) = L'_p(f, \Psi, *, k/2)
\]

as claimed.

Let \( f(z) \in M_k(\Gamma, \mathbb{C}_p) \) be a newform and \( w \) be the eigenvalue of the Atkin-Lehner involution \( w_p \) acting on \( f(z) \) (so \( w = -1 \) if \( f(z) \) is of split multiplicative type and \( w = 1 \) otherwise). Let

\[
\rho_{f,p} : CH^{m+1}((\mathcal{M}_n)_{K}) \rightarrow \mathbb{C}_p, z \mapsto \rho_K(z)(f(z))
\]

be the \( f(z) \)-component of \( \rho_K \).

**Corollary 9.3** We have the following equality

\[
L'_p(f, K, \xi, k/2) = \rho_{f,p}(y_{\xi}^{(n)} + w(-1)^{m+1}y_{\xi}^{(n)}).
\]

**Remark 9.4** Let \( f(z) \in M_k(\Gamma, \mathbb{C}_p) \) be a newform. In [BK] the authors have defined a subgroup \( H^1_f(K, V_p(f)) \) of \( H^1_{cont}(K, V_p(f)) \) which one might consider as a cohomological version of a Selmer group. Corollary 9.3 together with Kolyvagin’s method of Euler systems (compare [Ne1]) can be used to show that

\[
L'_p(f, K, \xi, k/2) \neq 0 \implies \text{dim}(\text{Im}(cl_{0,f})) = \text{dim}(H^1_f(K, V_p(f))) = 1.
\]

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Here \( cl_{0, f} : CH^{m+1}((\mathcal{M}_n)_K) \to H^1_{\mathrm{cont}}(K, V_p(f)) \) is the \( f \)-component of the cohomological Abel-Jacobi map \([10]\) (for the motive \( \mathcal{M}_n \)). This statement can be interpreted as an affirmative answer to a Bloch-Beilinson type conjecture relating the order of vanishing of \( L_p(f, K, \xi, s) \) at \( s = k/2 \) to the dimension of the Abel-Jacobi image of a cycle class group attached to the motive “\( M(f) \)”. The reader is invited to formulate a precise conjecture (generalising conjecture 4.1 of [BD1]).

10 Appendix

10.1 Chow motives attached to modular forms on Shimura curves

Attached to the space of modular forms of weight \( k = n + 2 \) on \( X \) is a certain motive \( \mathcal{M}_n \) which is the analogue in the context of Shimura curves of the motive attached to the space of cusp forms of a fixed weight and level constructed by Scholl \([Sch1]\). The motive \( \mathcal{M}_n \) has been considered first by Besser (see \([Be]\)). He constructed it as a motive for absolute Hodge cycles. It is possible to refine his construction and define \( \mathcal{M}_n \) in the category of Chow motives (see \([Wo]\)). For convenience we briefly recall here the main steps of the construction and some properties of \( \mathcal{M}_n \).

Let \( K \) be a field of characteristic 0 and \( S \) be a smooth quasiprojective connected variety over \( K \). We denote by \( \mathcal{M}^0(S) \) the category of relative Chow motives over \( S \) with respect to graded correspondences (see \([DM, K\ddot{u}]\)). Recall that the objects of \( \mathcal{M}^0(S) \) are triples \((X, p, i)\) where \( X \) is a smooth projective \( S \)-scheme, \( p \) is a projector (i.e. an idempotent in the ring \( \text{Corr}_S(X, X) : = \bigoplus \nu \ CH^{\dim(X_\nu/S)}(X_\nu \times_S X) \) of relative correspondences where \( X = \bigsqcup_\nu X_\nu \) is the decomposition of \( X \) into connected components) and \( i \) is an integer. Morphisms are given by

\[
\text{Hom}_{\mathcal{M}^0(S)}((X, p, i), (Y, q, j)) = q \circ CH^{\dim(X_\nu/S)+j-i}(X_\nu \times_S Y) \circ p.
\]

Composition is induced by composition of correspondences. The category \( \mathcal{M}^0(S) \) is an additive pseudo-abelian \( \mathbb{Q} \)-linear rigid tensor category. In particular the kernel of projector exists in \( \mathcal{M}^0(S) \). For a smooth projective \( S \)-scheme \( X \) we denote by \( h(X/S) : = (X, \Delta_X) \) its motive (we abbreviate \( (X, p) \) for \( (X, p, 0) \)).
Let $S = X_M/\mathbb{Q}$ be the fine moduli scheme of the moduli problem for some $M \geq 3$ and let $\pi : \mathcal{A} \to X_M$ be the universal abelian surface with quaternionic multiplication. The motive $h(\mathcal{A}) = h(\mathcal{A}/X_M)$ admits a canonical decomposition

$$h(\mathcal{A}) = h^0(\mathcal{A}) \oplus h^1(\mathcal{A}) \oplus h^2(\mathcal{A}) \oplus h^3(\mathcal{A}) \oplus h^4(\mathcal{A})$$

with $h^i(\mathcal{A}) \cong \Lambda^i h^1(\mathcal{A})$ and $h^i(\mathcal{A})^\vee \cong h^{4-i}(\mathcal{A})(2)$ (see [DM] or [Kü]). The $\mathcal{R}^{\text{max}}$-action induces an embedding $\mathcal{B} \to \text{End}(h^1(\mathcal{A}))$. There exists a unique idempotent $e \neq 0$ in the subring $\text{Sym}^2 \mathcal{B}$ of $\text{End}(\Lambda^2 h^1(\mathcal{A}))$ with $x^2 \cdot e = Nrd(x)e$ for all $x \in \mathcal{B}$. The projector $\epsilon_2 \in \text{Corr}_{X_M}(\mathcal{A}, \mathcal{A})$ is defined by $(\mathcal{A}, e) = h^2(\mathcal{A})_\cdot = \ker(e)$.

Let $\langle , \rangle : \text{Sym}^2 h^2(\mathcal{A})_\sim \to \mathbb{Q}(2)$ be the restriction of the symmetric pairing

$$h^2(\mathcal{A}) \otimes h^2(\mathcal{A}) \cong \Lambda^2 h^1(\mathcal{A}) \otimes \Lambda^2 h^1(\mathcal{A}) \overset{\wedge}{\to} \Lambda^4 h^1(\mathcal{A}) \cong \mathbb{Q}(-2)$$

to $h^2(\mathcal{A})_\cdot$. Let

$$\Delta_m : \text{Sym}^m h^2(\mathcal{A})_\sim \to (\text{Sym}^{m-2} h^2(\mathcal{A})_\cdot)(-2)$$

be the Laplace operator associated to $\langle , \rangle$ and let $\lambda_{m-2} : \text{Sym}^{m-2} h^2(\mathcal{A})_\sim \to (\text{Sym}^m h^2(\mathcal{A})_\cdot)(2)$ be the map given symbolically by $\lambda_{m-2}(x_1 x_2 \ldots x_{m-2}) = x_1 x_2 \ldots x_{m-2} \mu$ where $\mu : \mathbb{Q} \to (\text{Sym}^2 h^2(\mathcal{A})_\sim)(2)$ is the dual of $\langle , \rangle$ (twisted by 2). Then $\Delta_m \circ \lambda_{m-2}$ is an isomorphism. Put $p : = \lambda_{m-2} \circ (\Delta_m \circ \lambda_{m-2})^{-1} \circ \Delta_m$. Since $p^2 = p$, $\ker(p)$ exists (hence also the kernel of $\Delta_m$ exist and is equal to $\ker(p)$). The projector $\epsilon_n \in \text{Corr}_{X_M}(\mathcal{A}^m, \mathcal{A}^m)$ is defined by $(\mathcal{A}^m, \epsilon_n) = \ker(p)$.

The $p$-adic realisation of $(\mathcal{A}^m, \epsilon_n)$ is equal to the sheaf $\mathbb{L}_n$ (shifted by $-n$), i.e. we have

$$(67) \quad R_p(\mathcal{A}^m, \epsilon_n) = \mathbb{L}_n[-n]$$

where $R_p : \mathcal{M}^0(X_M) \to D^b(X_M, \mathbb{Q}_p)$ is the $p$-adic realization functor from $\mathcal{M}^0(X_M)$ to the bounded derived category of $\mathbb{Q}_p$-sheaves (see [DM], 1.8).

The group $G = (\mathcal{R}^{\text{max}}/M\mathcal{R}^{\text{max}})^* \cong \text{GL}_2(\mathbb{Z}/M\mathbb{Z})$ acts canonically as $\mathcal{X}$-automorphisms on $X_M$ and $\mathcal{A}^m$. We can therefore view the idempotent $p_G : = \frac{1}{|G|} \sum_{g \in G} g \in \mathbb{Q}[G]$ as an element in $\text{Corr}_X(\mathcal{A}^m, \mathcal{A}^m)$. Since the projectors $\epsilon_n$ and $p_G$ commute their product is a projector, too. The relative
Chow motive $\mathcal{M}_n$ over $X$ is defined as $(\mathcal{M}_n, p_G \circ \epsilon_n)$. It is independent of the integer $M$. When considered as a Chow motive over $\mathbb{Q}$ (by applying the canonical functor $\mathcal{M}^0(X) \to \mathcal{M}^0(\mathbb{Q})$) its $p$-adic realization is given by the following lemma.

**Lemma 10.1** We have,

$$H^i_p(\mathcal{M}_n) \cong \begin{cases} H^1(\overline{X}_M, \mathbb{L}_n)^G & \text{if } i = n + 1, \\ 0 & \text{otherwise.} \end{cases}$$

To see this note that by (67) we have

$$H^i_p(\mathcal{M}_n) = (p_G)_*(H^i(\overline{X}_M, R_p((\mathcal{A}^m, \epsilon_n)/X_M))) = H^{i-n}(\overline{X}_M, \mathbb{L}_n)^G$$

Since $H^i(\overline{X}_M, \mathbb{L}_n) = 0$ for all $i \neq 1$ the lemma follows.

Similarly for the de Rham realization we have $H^i_{DR}(\mathcal{M}_n) = 0$ for $i \neq n + 1$ and $F^{m+1}H^{n+1}_{DR}(\mathcal{M}_n) \cong M_k(X)$ (see also §1).

### 10.2 A formula for the cup product on open subschemes of Mumford curves

Let $k$ be a finite extension of $\mathbb{Q}_p$ and $\Gamma \subseteq \text{SL}_2(k)/\{\pm 1\}$ be a discrete cocompact subgroup. Let $X_\Gamma$ be the associated Mumford curve; as a rigid analytic space it is $\Gamma \backslash \mathcal{H}_p$ where $\mathcal{H}_p = \mathbb{P}_k^1 - \mathbb{P}(k)$ is the $p$-adic upper half plane defined over $k$. Let $V$ be a finite-dimensional $\Gamma$-representation (over $k$) which carries a $\Gamma$-invariant bilinearform

$$< , > : V \otimes V \rightarrow k \quad \text{(68)}$$

The locally free $\mathcal{O}_{\mathcal{H}_p}$-module $\mathcal{O}_{\mathcal{H}_p} \otimes V$ descends to a local system on $X_\Gamma$ denoted by $\mathcal{V}$. The pairing (68) induces a pairing

$$< , >_{X_\Gamma} : H^1_{DR}(X_\Gamma, \mathcal{V}) \otimes H^1_{DR}(X_\Gamma, \mathcal{V}) \xrightarrow{\cup} H^2_{DR}(X_\Gamma, \mathcal{V} \otimes \mathcal{V}) \rightarrow \quad \text{(69)}$$

$$\rightarrow H^2_{DR}(X_\Gamma) \cong k.$$

In [dS2] de Shalit proves a formula for (69) in terms of “Coleman” and “Schneider integration” (see Theorem 10.3 below). In this section we generalise this formula to certain open subschemes of $X_\Gamma$ in the case where $X_\Gamma$
is of arithmetic origin (i.e. a Shimura curve). Under this hypothesis we give also a new short proof of 10.3 based on Theorem 5.9.

To begin with we recall some facts from [dS1, dS3]. There are pairings

\[<\ ,\ >_{\Gamma}: H^1(\Gamma, V) \otimes \text{Char}(V)^\Gamma \longrightarrow k,\]

\[<\ ,\ >_{\Gamma}: \text{Char}(V)^\Gamma \otimes H^1(\Gamma, V) \longrightarrow k\]

defined as

\[H^1(\Gamma, V) \otimes \text{Char}(V)^\Gamma \xrightarrow{\cup} H^1(\Gamma, V \otimes \text{Char}(V)) \xrightarrow{\ast} H^1(\Gamma, \text{Char}(k)) \xrightarrow{\text{tr}} k,\]

\[\text{Char}(V)^\Gamma \otimes H^1(\Gamma, V) \xrightarrow{\cup} H^1(\Gamma, \text{Char}(V) \otimes V) \xrightarrow{\ast} H^1(\Gamma, \text{Char}(k)) \xrightarrow{\text{tr}} k,\]

where \((\ast)\) is induced by (68). The map \(\text{tr}\) can be defined as follows. We have short exact sequences

\[0 \longrightarrow H^i(\Gamma, V) \xrightarrow{\iota} H^i_{\text{DR}}(X_{\Gamma}, V) \longrightarrow H^{i-1}(\Gamma, \text{Char}(V)) \longrightarrow 0.\]

For \(i = 2\) the first group vanishes since \(\Gamma\) has a subgroup of finite index which is free. Hence for \(V = k\) we have an isomorphism

\[\text{tr} : H^1(\Gamma, \text{Char}(k)) \cong H^2_{\text{DR}}(X_{\Gamma}) \cong k.\]

Explicitly the pairing (70) is given by

\[<[\mathfrak{z}], f >_{\Gamma} = \frac{1}{[\Gamma : \Gamma']} \sum_{i=1}^{g} <\mathfrak{z}(\gamma_i), f(c_i) >\]

for \(f \in \text{Char}(V)^\Gamma, \mathfrak{z} \in Z^1(\Gamma, V)\). Here \(\Gamma'\) is a free subgroup of finite index in \(\Gamma\), \(b_1, \ldots, b_g, c_1, \ldots, c_g\) are the “free edges” of a good fundamental domain \(\mathfrak{F}\) for \(\Gamma' \backslash \mathcal{T}\) (in the sense of [dS1], 2.5) and \(\gamma_1, \ldots, \gamma_g\) are generators of \(\Gamma'\) with \(\gamma_i(b_i) = \bar{c}_i\).

**Lemma 10.2** Let \(I : H^1_{\text{DR}}(X_{\Gamma}, V) \to \text{Char}(V)^\Gamma\) be the map (14).

(a) \(<\iota(x), y >_{X_{\Gamma}} = < x, I(y) >_{\Gamma}\) for all \(x \in H^1(\Gamma, V)\) and \(y \in H^1_{\text{DR}}(X_{\Gamma}, V)\).

(b) \(< y, \iota(x) >_{X_{\Gamma}} = -< I(y), x >_{\Gamma}\) for all \(x \in H^1_{\text{DR}}(X_{\Gamma}, V)\) and \(y \in H^1(\Gamma, V)\).
Proof. We have $H^i_{DR}(X,\mathcal{V}) = H^i(\Gamma, \Omega^\bullet(\mathcal{H}_p) \otimes V)$ and there is a distinguished triangle

$$V \xrightarrow{i} \Omega^\bullet(\mathcal{H}_p) \otimes V \xrightarrow{I} C_{\text{har}}(V)[-1] \longrightarrow V[1]$$

which induces the sequences (72). Part (a) follows from the commutativity of the diagram

$$\begin{array}{ccc}
\Omega^\bullet(\mathcal{H}_p) \otimes V \otimes V & \longrightarrow & \Omega^\bullet(\mathcal{H}_p) \\
\downarrow I \otimes 1 & & \downarrow I \\
C_{\text{har}}(V)[-1] \otimes V & \longrightarrow & C_{\text{har}}(k)[-1].
\end{array}$$

and (b) by applying (a) to the pairing $< v_1, v_2 >' = < v_2, v_1 >$ instead of (68).

In [41], the following formula is proved through some complicated explicite computations.

**Theorem 10.3** For all $x, y \in H^1_{DR}(X,\mathcal{V})$ we have

$$(74) \quad < x, y >_{X} = < P(x), I(y) >_{\Gamma} - < I(x), P(y) >_{\Gamma}.$$  

Before we explain how this can be generalised to open subschemes we show how to deduce 10.3 from 5.9 in the case where $X$ is the Shimura curve $X$ (over $\mathbb{Q}_{ur}$), $V = V_n$ with $n = 2m \geq 2$ even and (68) is the pairing (32).

Under these assumptions the pairing (68) can be interpreted as a map in $MF_{Q_{ur}}(\phi, N)$ (see remark 5.12) and therefore the isotypical components of $H^1_{DR}(X,\mathcal{V}_n)$ in the slope decomposition (38) are isotropic. If we decompose elements $x, y \in H^1_{DR}(X,\mathcal{V}_n)$ as $x = x_m + x_{m+1}, y = y_m + y_{m+1}$ according to (38) then $x_m = \iota(P(x)), y_m = \iota(P(y))$ and $I(x) = I(x_{m+1}), I(y) = I(y_{m+1})$. Together with Lemma 10.2 we obtain

$$(x, y) = < x_m, y_{m+1} > + < x_{m+1}, y_m > = < P(x), I(y) > - < I(x), P(y) >.$$  

Let $K$ be an extension of $k$ contained in $\mathbb{C}_p$ which is complete with respect to the $p$-adic valuation and of ramification index 1 over $k$ (e.g $k = \mathbb{Q}_p$ and
We pass from $k$ to $K$ in order to have rational points on $X_\Gamma$ (i.e. we replace $X_\Gamma$ by $(X_\Gamma)_K$, $V$ by $V_K$ but we will suppress the subscript from now on). The condition on the ramification index implies that the image of every $K$-valued point on $H_p$ under the reduction map $\text{red} : H_p \to \vert T \vert$ is a vertex).

Let $j : U \to X_\Gamma$ be the complement of finitely many points $x_1, \ldots, x_r \in X_{\Gamma}(K)$ and choose preimages $z_1, \ldots, z_r \in \mathcal{F} \subseteq H_p(K)$ of $x_1, \ldots, x_r$ under the projection $\pi : H_p \to X^{an}_{\Gamma}$. To simplify the notation we assume that the stabilizer of each point $z_i$ under the action of $\Gamma$ is trivial.

We have a pairing

$$< , >_U : H^1\text{DR}_{c}(U, V) \otimes H^1\text{DR}(U, V) \xrightarrow{\cup} H^2\text{DR}(X_\Gamma, V \otimes V) \longrightarrow$$

$$\longrightarrow H^2\text{DR}_{c}(U) \cong H^2\text{DR}(X_\Gamma) \cong k.$$  

The inclusion $j$ induces maps $j_* : H^1\text{DR}_{c}(U, V) \to H^1\text{DR}(X_\Gamma, V), j^* : H^1\text{DR}(X_\Gamma, V) \to H^1\text{DR}(U, V)$ such that $< j_*(x), y >_{X_{\Gamma}} = < x, j^*(y) >_U$.

We are going to introduce now the analogues of the maps (15) and (16) for $H^1\text{DR}_{c}(U, V), H^1\text{DR}(U, V)$. We write $\text{Ind}^\Gamma(V)$ for the $\Gamma$-module $\text{Maps}(\Gamma, V)$ with $\Gamma$-action is given by $(\gamma f)(\tau) : = \gamma f(\gamma^{-1}\tau)$. Let $ad : V \to \text{Ind}^\Gamma(V), v \mapsto (\tau \mapsto v)$ and let $K^\bullet(V)$ denote the complex

$$K^\bullet(V) : = \text{Cone}(V \xrightarrow{ad, \ldots, ad} \bigoplus_{i=1}^r \text{Ind}^\Gamma(V))[-1].$$

Let $D$ be the divisor $x_1 + \ldots + x_r$. The de Rham cohomology with compact support $H^i\text{DR}_{c}(U, V)$ can be identified with the hypercohomology

$$H^i(X_\Gamma, [\mathcal{L}(-D) \otimes V \to \Omega^1_{X_{\Gamma}} \otimes V]) \cong H^i(\Gamma, \text{Cone}(\Omega^\bullet(\mathcal{H}_p) \otimes V \xrightarrow{\alpha} \bigoplus_{i=1}^r \text{Ind}^\Gamma(V))[\alpha])$$

where $\alpha$ is given (in degree 0) by $F \mapsto (F(z_1), \ldots, F(z_r)), F \in \mathcal{O}(\mathcal{H}_p) \otimes V$.

The distinguished triangle

$$K^\bullet(V) \longrightarrow \text{Cone}(\Omega^\bullet(\mathcal{H}_p) \otimes V \xrightarrow{\alpha} \bigoplus_{i=1}^r \text{Ind}^\Gamma(V))[-1] \longrightarrow C_{\text{har}}(V) \longrightarrow K^\bullet(V)[1]$$

yields short exact sequences

$$0 \longrightarrow H^i(\Gamma, K^\bullet(V)) \xrightarrow{\cup_U} H^i_{\text{DR}, c}(U, V) \longrightarrow H^{i-1}(\Gamma, C_{\text{har}}(V)) \longrightarrow 0.$$  

(76)
and for \( i = 1 \) we get a map \( H^{1}_{\text{DR}, c}(U, V) \rightarrow C_{\text{har}}(V)^{\Gamma} \) which we denote by \( I_{U,c} \). The injection \( i_{U,c} : H^{1}(\Gamma, K^{\bullet}(V)) \rightarrow H^{1}_{\text{DR}, c}(U, V) \) has again a left inverse \( P_{U,c} : H^{1}_{\text{DR}, c}(U, V) \rightarrow H^{1}(\Gamma, K^{\bullet}(V)) \) defined in terms of Coleman integration. For that let \( \mathcal{F}(V) \) be the subspace of \( V \)-valued locally analytic functions on \( \mathcal{H}_{p} \) which are primitives of elements of \( \Omega^{1}(\mathcal{H}_{p}) \otimes V \) (thus the definition of \( \mathcal{F}(V) \) depends on the choice of the branch of the \( p \)-adic logarithm). Since \( V = \text{Ker}(d : \mathcal{F}(V) \rightarrow \Omega^{1}(\mathcal{H}_{p}) \otimes V) \) the complex \( [\mathcal{F}(V) \xrightarrow{(d,\alpha)} \Omega^{1}(\mathcal{H}_{p}) \otimes V \oplus \bigoplus_{i=1}^{r} \text{Ind}^{\Gamma}(V)] \) (concentrated in degrees 0 and 1) is quasiisomorphic to \( K^{\bullet}(V) \). We defined \( P_{U,c} : H^{1}_{\text{DR}, c}(U, V) = H^{1}(\Gamma, \mathcal{O}(\mathcal{H}_{p}) \otimes \Omega^{1}(\mathcal{H}_{p}) \otimes V \oplus \bigoplus_{i=1}^{r} \text{Ind}^{\Gamma}(V)) \rightarrow H^{1}(\Gamma, K_{\mathcal{H}}^{\bullet}(V)) \). One easily checks that \( I_{U,c} = I \circ j_{*} \) and \( q_{*} \circ P_{U,c} = P \circ j_{*} \) (\( g : V \rightarrow K^{\bullet}(V) \) is the canonical map).

As in section 4 let \( C^{0}(V) \) (resp. \( C^{1}(V) \)) be the set of maps \( f : \mathcal{Y}(T) \rightarrow V \) (resp. maps \( f : \mathcal{E}(T) \rightarrow V \) such that \( f(\bar{e}) = -f(e) \) for all \( e \in \mathcal{E}(T) \)). We have an exact sequence

\[
0 \rightarrow C_{\text{har}}(V) \rightarrow C^{1}(V) \xrightarrow{\delta} C^{0}(V) \rightarrow 0
\]

where \( \delta(f)(v) = \sum_{\alpha(e)=v} f(e) \). For \( i \in \{1, \ldots, r\} \) we set \( v_{i} = \text{red}(z_{i}) \) and define \( \chi_{i} : \text{Ind}^{\Gamma}(V) \rightarrow C^{0}(V) \) by

\[
\chi_{i}(f)(v) = \begin{cases} f(\gamma) & \text{if } v = \gamma v_{i} \text{ for some } \gamma \in \Gamma, \\ 0 & \text{otherwise}. \end{cases}
\]

Let \( C_{U}(V) \) be the kernel of

\[
C^{1}(V) \oplus \bigoplus_{i=1}^{r} \text{Ind}^{\Gamma}(V) \rightarrow C^{0}(V), (f, f_{1}, \ldots, f_{r}) \mapsto \delta(f) - \sum_{i=1}^{r} \chi_{i}(f_{i}).
\]

We have a short exact sequence

\[
(77) \quad 0 \rightarrow C_{\text{har}}(V) \rightarrow C_{U}(V) \xrightarrow{\delta} \bigoplus_{i=1}^{r} \text{Ind}^{\Gamma}(V) \rightarrow 0
\]
For the de Rham cohomology $H^i_{DR}(U, \mathcal{V})$, we have

$$H^i_{DR}(U, \mathcal{V}) = H^i(X, \Omega^*_{X, \text{log}}(\log(|D|) \otimes \mathcal{V})) \cong H^i(\Gamma, \Omega^*_{\mathcal{H}}(\log(|\pi^{-1}(D)|) \otimes \mathcal{V}).$$

There is a distinguished triangle

$$V \rightarrow \Omega^*_{\mathcal{H}}(\log(|\pi^{-1}(D)|)) \otimes V \xrightarrow{J} C_U(V)[-1] \rightarrow V[1]$$

where $I(\omega)(e, \gamma_1, \ldots, \gamma_r) = (\text{Res}_e(\omega), \text{Res}_{\gamma_1(\varepsilon_1)}(\omega), \ldots, \text{Res}_{\gamma_r(\varepsilon_r)}(\omega))$. We get short exact sequences

$$(78) \quad 0 \rightarrow H^i(\Gamma, V) \xrightarrow{\iota} H^i_{DR}(U, \mathcal{V}) \rightarrow H^{i-1}(\Gamma, C_U(V)) \rightarrow 0.$$  

For $i = 1$ the map $H^1_{DR, c}(U, \mathcal{V}) \rightarrow C_U(V)^\Gamma$ will be denoted by $I_U$.

By identifying $H^1_{DR, c}(U, \mathcal{V})$ with the space of $\Gamma$-invariant $V$-valued meromorphic differential forms which are of the second kind on $\pi^{-1}(U_{\text{an}})$ modulo exact differentials we can define a left inverse $P_U : H^1_{DR, c}(U, \mathcal{V}) \rightarrow H^1(\Gamma, V)$ of $\iota_U$ by the same formula as (15). We have $P_U \circ j^* = P$ and $I_U \circ j^* = R$.

Lemma 10.4 There exists a pairing

$$(79) \quad < , >_{\Gamma, U} : H^1(\Gamma, \mathcal{K}^* (V)) \otimes C_U(V)^\Gamma \rightarrow K,$$

such that

$$(80) \quad < \iota_{U, c}(x), y >_{\Gamma, U} = < x, I_U(y) >_{\Gamma, U}$$

for all $x \in H^1(\Gamma, \mathcal{K}^* (V))$ and $y \in H^1_{DR, c}(U, \mathcal{V})$.

Proof. The exact sequences (70), (78) for $i = 1$ guarantee the existence of (79) with the property (80) once we have shown that $\iota_{U, c}(H^1(\Gamma, \mathcal{K}^* (V)))$ and $\iota_U(H^1(\Gamma, V))$ are orthogonal under (75). This follows by considering the diagram

$$
\begin{array}{ccc}
H^1(\Gamma, \mathcal{K}^* (V)) \otimes H^1(\Gamma, V) & \rightarrow & H^2(\Gamma, \mathcal{K}^* (K)) \\
\downarrow \iota \otimes \iota & & \downarrow \iota \\
H^1_{DR, c}(U, \mathcal{V}) \otimes H^1_{DR}(U, \mathcal{V}) & \rightarrow & H^2_{DR, c}(U)
\end{array}
$$

using the fact that $H^2(\Gamma, \mathcal{K}^* (K)) = 0$. The upper map is the composite of the cup-product with the map $H^2(\Gamma, \mathcal{K}^* (V) \otimes V) \rightarrow H^2(\Gamma, \mathcal{K}^* (K))$ induced by (68). \qed
We also need a concrete description of the pairing \((73)\) similar to \((79)\). We assume for simplicity that \(\Gamma\) is free and leave the formulation of the general case to the reader. Elements of \(H^1(\Gamma, K^\bullet(V))\) can be represented by \((r + 1)\)-tuples \((z, f_1, \ldots, f_r)\) such that \(ad \circ z = \partial(f_1) = \ldots = \partial(f_r)\) where \(z \in Z^1(\Gamma, V)\), \(f_1, \ldots, f_r \in \text{Ind}^\Gamma(V)\) and \(\partial(f_i)(\gamma) = \gamma f_i - f_i\). An element of \(C_U(V)^\Gamma\) is given by a tuple \((g, g_1, \ldots, g_r)\) such that \(\delta(g) = \sum_{i=1}^r \chi_i(g_i)\). With the notation as in \((73)\) we have
\[
<([z, f_1, \ldots, f_r]), (g, g_1, \ldots, g_r)>_{\Gamma, U} = \sum_{i=1}^g <z(\gamma_i), g(c_i)> + \sum_{j=1}^{r} <f_j(1), g_j(1)>.
\]
We leave the verification of this formula as an exercise to the reader.

**Conjecture 10.5** For all \(x \in H^1_{\text{DR}, c}(U, V), y \in H^1_{\text{DR}}(U, V)\) we have
\[
< x, y >_{U} = < P_U(x), I_U(y) >_{\Gamma, U} - < I_{U,c}(x), P_U(y) >_{\Gamma}.
\]
It is likely that this can be proved by the methods of [dS1, appendix]. Here we will prove it only in the case needed for the application.

**Theorem 10.6** Let \(X_\Gamma\) be the Shimura curve \(X\) (over \(\mathbb{Q}_{ur}\)), \(V = V_n\) for \(n = 2m \geq 2\) even and the pairing is pairing \((72)\). Then Conjecture 10.5 holds.

**Proof.** In this case \((73)\) is nondegenerated. Hence there is unique structure of a filtered \((\phi, N)\)-module on \(H^1_{\text{DR}, c}(U, V)\) such that \((73)\) is a map
\[
H^1_{\text{DR}, c}(U, V_n) \otimes H^1_{\text{DR}}(U, V_n) \rightarrow \mathbb{Q}_{ur}[2m + 1]
\]
in \(MF_{\mathbb{Q}_{ur}}(\phi, N)\). By Theorem 5.13, \(H^1_{\text{DR}}(U, V)\) has a slope decomposition of the form
\[
H^1_{\text{DR}}(U, V_n) = H^1_{\text{DR}}(U, V_n)_m \oplus H^1_{\text{DR}}(U, V_n)_{m+1}
\]
where \(H^1_{\text{DR}}(U, V_n)_m = H^1_{\text{DR}}(X_\Gamma, V_n)_m = \iota(H^1(\Gamma, V_n))\) and \(H^1_{\text{DR}}(X_\Gamma, V_n)_{m+1} = \text{Ker}(P_U)\). Thus the slope decomposition of \(H^1_{\text{DR}, c}(U, V_n)\) is also of the form
\[
H^1_{\text{DR}, c}(U, V_n) = H^1_{\text{DR}, c}(U, V_n)_m \oplus H^1_{\text{DR}, c}(U, V_n)_{m+1}
\]
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and $H^1_{\text{DR},c}(U, V_n)_{m+1} = H^1_{\text{DR}}(X, V_n)_{m+1}$. Since $i_U(H^1(\Gamma, K^\bullet(V)))$ and $i_U(H^1(\Gamma, V))$ are orthogonal we see that $i_U(H^1(\Gamma, K^\bullet(V))) \subseteq H^1_{\text{DR},c}(U, V_n)_{m}$. But both vector spaces have the same dimension, hence $i_U(H^1(\Gamma, K^\bullet(V))) = H^1_{\text{DR},c}(U, V_n)_{m}$. If we write $x \in H^1_{\text{DR},c}(U, V_n)$ as $x = x_m + x_{m+1}$ and $y \in H^1_{\text{DR}}(U, V_n)$ as $y = y_m + y_{m+1}$ according to (53) and (54), then $x_m = i(P_U(x)), y_m = i_U(P_U(y)) = j^*(i(U(y))$ and we obtain as before

$$< x, y >_U = < i(P_U(x)), y_m + x_{m+1} + j^*(i(U(y)) > x =$$

$$= < P_U(x), I_U(y) >_{\Gamma, U} - < I_U(x), P_U(y) >_{\Gamma}$$

(the last equality follows from Lemma 10.2 and Lemma 10.4).

Now assume that $r = 2$ and that $\text{red}(z_1) = v = \text{red}(z_2)$. Let $\chi$ be a $\Gamma$-invariant $V_n$-valued meromorphic differential form on $H_p$ which is holomorphic outside of $\pi^{-1}(U^m)$ and which has simple poles at $z_1, z_2$ with $\text{Res}_{z_1}(\chi) = -\text{Res}_{z_2}(\chi)$. The residue theorem implies then that $c_\chi(e) = \text{Res}_{e}(\chi)$ is a harmonic cocycle. We also assume that the class $[\chi]$ lies in the kernel of $N : H^1_{\text{DR}}(U, V_n) \rightarrow H^1_{\text{DR}}(U, V_n)$. Note that this implies that $c_\chi = 0$ (see [543]).

**Corollary 10.7** For every $f \in M_k(\Gamma)$ we have

$$< I([\omega_f]), P_U(\chi) >_{\Gamma} = < F_{\omega_f}(z_1), \text{Res}_{z_1}(\chi) > + < F_{\omega_f}(z_2), \text{Res}_{z_2}(\chi) > .$$

**Proof.** Since $F^{m+1}H^1_{\text{DR},c}(U, V_n)$ and $F^{m+1}H^1_{\text{DR}}(U, V_n)$ are orthogonal we have $< [\omega_f], [\chi] >_{U} = 0$. Hence by (52) we get $< I([\omega_f]), P_U(\chi) >_{\Gamma} = < P_U([\omega_f]), I_U([\chi]) >_{\Gamma, U}$. However using the formula (51) it is easy to see that

$$< P_U([\omega_f]), I_U([\chi]) >_{\Gamma, U} =$$

$$= < F_{\omega_f}(z_1), \text{Res}_{z_1}(\chi) > + < F_{\omega_f}(z_2), \text{Res}_{z_2}(\chi) > .$$

$\Box$

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