ITERATED SPANS AND “CLASSICAL” TOPOLOGICAL FIELD THEORIES

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ABSTRACT. We construct higher categories of iterated spans, possibly equipped with extra structure in the form of “local systems”, and classify their fully dualizable objects. By the Cobordism Hypothesis, these give rise to framed topological quantum field theories, which are the framed versions of the “classical” TQFTs considered in the quantization programme of Freed-Hopkins-Lurie-Teleman.

More explicitly, we construct two families of $(\infty, n)$-categories: First, $(\infty, n)$-categories $\text{Span}_n(C)$, for $C$ an $\infty$-category with finite limits, whose objects are the objects of $C$ and whose $i$-morphisms are $i$-fold spans of morphisms in $C$. Second, $(\infty, n)$-categories $\text{Span}_n(\delta; D)$, where $D$ is an $(\infty, n)$-category, whose objects are spaces equipped with a morphism to the space of objects of $D$ and whose $i$-morphisms are $i$-fold spans in $\delta$ equipped with a map to the space of $i$-morphisms in $D$; more generally, we can replace $\delta$ by an arbitrary $\infty$-topos and $D$ by an internal $(\infty, n)$-category in this $\infty$-topos. We show that the $(\infty, n)$-categories of the first class have natural symmetric monoidal structures such that all their objects are fully dualizable, and that the same holds for those of the second second class when the internal $(\infty, n)$-category $D$ is symmetric monoidal and has duals.

Using this machinery, we also construct an $(\infty, 1)$-category of Lagrangian correspondences between symplectic derived algebraic stacks and show that all its objects are fully dualizable.

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1. INTRODUCTION

1.1. Motivation from TQFTs. Topological quantum field theories (or TQFTs) are a particularly simple class of quantum field theories, which we can reasonably expect to understand from the mathematical point of view. TQFTs were first formalized mathematically by Atiyah [Ati88] in the late 1980s, and have been the subject of much research over the past two decades. TQFTs in Atiyah’s sense are symmetric monoidal functors from a category of closed $(n - 1)$-dimensional manifolds, with morphisms given by $n$-dimensional cobordisms and the symmetric monoidal structure by taking disjoint unions, to some other symmetric monoidal category, typically that of complex vector spaces.

Recently, much work has focused on the related notion of extended topological quantum field theories; this was first formalized in terms of $n$-categories by Baez and Dolan [BD95], building on earlier work by a number of mathematicians, including Lawrence [Law93] and Freed [Fre94]. As reformulated in the language of $(\infty, n)$-categories by Lurie, extended TQFTs of dimension $n$
are symmetric monoidal functors from an \((\infty,n)\)-category whose \(k\)-morphisms are \(k\)-dimensional cobordisms, to some other symmetric monoidal \((\infty,n)\)-category. Baez and Dolan conjectured that framed \(n\)-dimensional extended TQFTs (where we consider cobordisms equipped with a framing of their tangent bundle) valued in any symmetric monoidal \((\infty,n)\)-category \(\mathcal{C}\) are classified by the fully dualizable objects in \(\mathcal{C}\) — this is the Cobordism Hypothesis, which has been proved by Lurie [Lur10c].

Typically, TQFTs take values in an \((\infty,n)\)-category of “linear” or algebraic objects, reflecting the linear nature of quantum mechanics. Work of, among others, Freed [Fre94] and Kapustin [Kap10] suggests that from the point of view of physics the natural target is an \((\infty,n)\)-category with objects linear \((n-1)\)-categories and iterated correspondences or bimodules as morphisms — this, however, has not yet been rigorously constructed. Moreover, even after its construction has been carried out we will still be left with the task of defining interesting examples of extended TQFTs.

Ideas from physics suggest that many interesting examples of TQFTs should arise as “quantizations” of “classical” topological field theories — in the context of topological field theories this proposal goes back at least to [Fre94]. Roughly speaking, for a non-extended theory the idea is that the “classical” field theory assigns to a manifold \(M\) the “space of fields” \(\mathcal{F}(M)\) on \(M\), which will typically be some form of stack, and to a cobordism \(K\) from \(M\) to \(M'\) the span

\[
\begin{tikzcd}
\mathcal{F}(K) \\
\mathcal{F}(M) \\
\mathcal{F}(M')
\end{tikzcd}
\]

obtained by restricting the fields on \(K\) to the boundary. The quantization, which is supposed to be analogous to the path integral of quantum field theory, would then assign some algebraic object to the stacks \(\mathcal{F}(M)\), for example the value of some cohomology theory \(E^*\mathcal{F}(M)\), and to the span a push-pull composite \(t_\ast s^\ast\): \(E^*\mathcal{F}(M) \to E^*\mathcal{F}(K) \to E^*\mathcal{F}(M')\), where the pushforward \(t_\ast\) is thought of as “integrating” over the fibres of \(t\). A more general version of such a construction, for fully extended TQFTs, has been proposed by Freed, Hopkins, Lurie, and Teleman [FHLT10]: They consider “classical” extended TQFTs valued in \((\infty,n)\)-categories whose higher morphisms are iterated spans in, for example, stacks, and propose that the “quantization” is obtained by composing these with a “linearization functor” from iterated spans to an algebraic \((\infty,n)\)-category.

Constructing such “linearizations” in general will require substantial progress in the theory of higher categories. They have, however, been constructed in some special cases — for example, Morton has studied extended TQFTs valued in 2-fold spans of groupoids [Mor11] and 2-fold spans of groupoids equipped with \(U(1)\)-valued cocycles [Mor10], and has constructed linearization functors to linear categories (or “2-vector spaces”) in both cases. Freed, Hopkins, Lurie, and Teleman observe that the existence of a quantization requires that certain left and right adjoints coincide — such ambidexterity has recently been studied by Hopkins and Lurie [HL13] in the case of \((\infty,1)\)-categories.

1.2. Summary of Results. The results of this paper can be viewed as a modest contribution towards this deep programme for constructing TQFTs: we do not consider the difficult problem of constructing linearization functors, but rather look at the starting point of “classical” topological field theories.

In the first part of the paper we construct two families of \((\infty,n)\)-categories. We first construct \((\infty,n)\)-categories \(\text{Span}_n(\mathcal{C})\) of iterated spans in an \(\infty\)-category \(\mathcal{C}\) with finite limits. More explicitly, the objects of \(\text{Span}_n(\mathcal{C})\) are the objects of \(\mathcal{C}\), a 1-morphism from \(A\) to \(B\) is a span

\[
\begin{tikzcd}
X \\
A \\
B
\end{tikzcd}
\]
in $\mathcal{C}$, with composition given by taking fibre products, a 2-morphism is a span of spans, and so forth. The construction we use generalizes that of Barwick [Bar13a] in the case $n = 1$.

We then use the $(\infty,n)$-category $\text{Span}_n(\mathcal{S})$ to construct, for any $(\infty,n)$-category $\mathcal{C}$, an $(\infty,n)$-category $\text{Span}_n(\mathcal{S};\mathcal{C})$ of $n$-fold iterated spans in spaces equipped with local systems in $\mathcal{C}$. More precisely, the objects of $\text{Span}_n(\mathcal{S};\mathcal{C})$ are spaces equipped with a map to the space of objects of $\mathcal{C}$, the morphisms are spans of spaces

\[
\begin{array}{ccc}
X & \xleftarrow{f} & \mathcal{S} \\
\downarrow & & \downarrow \\
A & \xrightarrow{g} & B
\end{array}
\]

where $X$ is equipped with a map to the space of morphisms in $\mathcal{C}$ compatible with the maps from $A$ and $B$ to the space of objects of $\mathcal{C}$ via the source and target projections, and in general the $i$-morphisms are $i$-fold spans where the top object is equipped with a map to the space of $i$-morphisms in $\mathcal{C}$. These are the $(\infty,n)$-categories considered as targets for classical topological field theories by Freed, Hopkins, Lurie, and Teleman, who propose that for good choices of $\mathcal{C}$ there should be a symmetric monoidal “linearization functor” from $\text{Span}_n(\mathcal{S};\mathcal{C})$, or at least from the subcategory of spans of $\pi$-finite spaces, to $\mathcal{C}$.

More generally, we also consider the case where we replace $\mathcal{S}$ by a general $\infty$-topos $\mathcal{X}$, and consider iterated spans in $\mathcal{X}$ equipped with local systems in an internal $(\infty,n)$-category $\mathcal{C}$ in $\mathcal{X}$.

In the second part of the paper we prove that, under reasonable hypotheses, all objects of these $(\infty,n)$-categories are fully dualizable. More precisely, we prove:

**Theorem 1.1.** Let $\mathcal{C}$ be an $\infty$-category with finite limits. Then the $(\infty,n)$-category $\text{Span}_n(\mathcal{C})$ has a natural symmetric monoidal structure, with respect to which $\text{Span}_n(\mathcal{C})$ has duals.

Recall that, in Lurie’s terminology, a symmetric monoidal $(\infty,n)$-category has duals if all of its objects have duals, and all of its $i$-morphisms have a left and a right adjoint for all $i = 1, \ldots, n - 1$. We generalize this notion to $(\infty,n)$-categories internal to an $\infty$-topos, and using results of Riehl and Verity [RV13] on adjunctions in $(\infty,2)$-categories we prove the following:

**Theorem 1.2.** Let $\mathcal{X}$ be an $\infty$-topos, and suppose $\mathcal{D}$ is a symmetric monoidal $(\infty,n)$-category internal to $\mathcal{X}$. Then $\text{Span}_n(\mathcal{X};\mathcal{D})$ has a natural symmetric monoidal structure. Moreover, if $\mathcal{D}$ has duals, then so does $\text{Span}_n(\mathcal{X};\mathcal{D})$ with respect to this symmetric monoidal structure.

Thus, in both cases all objects determine framed TQFTs via the cobordism hypothesis. An interesting topic for future research is to describe other flavours of TQFTs with these targets, e.g. oriented and unoriented ones. Lurie proves that these are classified by homotopy fixed points of various group actions induced from an action of $O(n)$ on the space of fully dualizable objects. In [Lur09c, §3.2], Lurie makes the following conjecture:

**Conjecture 1.3** (Lurie). The $O(k)$-action on the underlying $\infty$-groupoid of $\text{Span}_k(\mathcal{C})$ is trivial, for all $\infty$-categories $\mathcal{C}$ with finite limits.

We also state the obvious generalization of this to the case of spans with local systems:

**Conjecture 1.4.** Let $\mathcal{D}$ be a symmetric monoidal $(\infty,n)$-category with duals. Then the $O(n)$-action on underlying $\infty$-groupoid of $\text{Span}_n(\mathcal{S};\mathcal{D})$ is induced by composition with the $O(n)$-action on the space of objects of $\mathcal{D}$.

Finally, using the work of Pantev, Toën, Vaquié and Vezzosi [PTVV13] on symplectic structures on derived algebraic stacks, we construct an $\infty$-category $\text{Lag}^{cl}_{\sigma(\infty,1)}$ of Lagrangian correspondences between $n$-symplectic derived Artin stacks as a subcategory of $\text{Span}_{1}(\text{dSt}_{k,\mathcal{A}^{2}_{cl}[\mathcal{N}]})$, where $\text{dSt}_{k}$ is the $\infty$-topos of derived stacks over a base field $k$ and $\mathcal{A}^{2}_{cl}[\mathcal{N}]$ is the derived stack of $n$-shifted closed 2-forms. We then show:
Theorem 1.5. The ∞-category $\text{Lag}_\ast^{\infty}(\varnothing,1)$ inherits a symmetric monoidal structure from $\text{Span}_1(\text{dSt}_k; A_{cl}^n[n])$, with respect to which all $n$-symplectic derived Artin stacks are dualizable.

This partly generalizes results of Calaque [Cal13] from the level of 1-categories to ∞-categories. Calaque also proposes that an interesting class of “semi-classical” TQFTs takes values in an $(\infty,k)$-category of derived $n$-symplectic stacks with morphisms given by “higher Lagrangian correspondences”, which should form a sub-$(\infty,k)$-category of $\text{Span}_k(\text{dSt}_k; A_{cl}^n[n])$. Constructing these $(\infty,k)$-categories will require studying the appropriate notion of iterated Lagrangian correspondence, which I hope to do in a sequel to this paper.

1.3. Related Work. The “classical” TQFTs considered in this paper have previously been discussed by a number of authors; particularly inspirational were the accounts of Freed, Hopkins, Lurie, and Teleman [FHLT10] and of Calaque [Cal13].

I learned the construction of the ∞-category of spans in an ∞-category from Barwick, who has since made extensive use of this and variants of it [Bar13a, Bar14]. In unpublished work, Barwick has also given an alternative definition of higher categories of iterated spans, in the setting of Rezk’s $\Theta_n$-spaces.

The idea that the $(\infty,n)$-category of iterated spans could most easily be constructed as underlying an $n$-uple ∞-category I gained from the definition sketched by Schreiber in [Sch13, §3.9.14.2]. Schreiber and collaborators have also extensively studied quantization by linearizing iterated spans of stacks, for example in [Sch14a, Sch14b] and [Sch13, §3.9.14]; they consider not necessarily topological quantum field theories valued in iterated spans in a cohesive ∞-topos under the name local prequantum field theories. Nuiten [Nui13] has also recently studied the quantization of these.

In the 1-categorical setting, a construction of “weak $n$-fold categories” of iterated cospans in a category has been given by Grandis [Gra07]. Finally, the $(\infty,n)$-categories $\text{Span}_n(\mathcal{S}; \mathcal{C})$ also appear in Lurie’s work on the cobordism hypothesis [Lur09c], under the name $\text{Fam}_n(\mathcal{C})$, but only a sketch of a definition is given there. In unpublished work, Lurie has also given a construction of the $(\infty,2)$-category of 2-fold spans in the setting of scaled simplicial sets.

1.4. Overview. We begin by briefly reviewing the model of $(\infty,n)$-categories we will use, namely iterated Segal spaces, in §2. Then we construct the $(\infty,n)$-category $\text{Span}_n(\mathcal{C})$ of iterated spans in an ∞-category $\mathcal{C}$ as an $n$-fold Segal space in §3, and the $(\infty,n)$-category $\text{Span}_n(\mathcal{X}; \mathcal{D})$ of iterated spans in an ∞-topos $\mathcal{X}$ equipped with local systems in an $(\infty,n)$-category $\mathcal{D}$ internal to $\mathcal{X}$ in §4. In §5 we review the notions of duals and adjoints in $(\infty,n)$-categories, and generalize these to $(\infty,n)$-categories in an ∞-topos. In §6 we then prove that $\text{Span}_n(\mathcal{C})$ is symmetric monoidal and that all its objects are fully dualizable, and in §7 we show the same holds for $\text{Span}_n(\mathcal{X}; \mathcal{D})$. Finally, in §8 we construct an ∞-category of Lagrangian correspondences between derived algebraic stacks and prove that all of its objects are dualizable.

1.5. Notation. This article relies heavily on the theory of ∞-categories as developed in the guise of quasicategories in the work of Joyal and Lurie, and we generally reuse the notation and terminology used by Lurie in [Lur09a, Lur09c, Lur14]. We note the following conventions, some of which differ slightly from those of Lurie:

- $\Delta$ is the simplicial indexing category, with objects the non-empty finite totally ordered sets $[n] := \{0, 1, \ldots, n\}$ and morphisms order-preserving functions between them.
- If $\mathcal{C}$ is an ∞-category, we write $\text{ic}(\mathcal{C})$ for the interior or underlying space of $\mathcal{C}$, i.e. the largest subspace of $\mathcal{C}$ that is a Kan complex.
- If $f: \mathcal{C} \to \mathcal{D}$ is left adjoint to a functor $g: \mathcal{D} \to \mathcal{C}$, we will refer to the adjunction as $f \dashv g$. 
\begin{itemize}
  \item $\mathcal{S}$ is the $\infty$-category of spaces; this can be defined as the coherent nerve $\mathrm{N}(\mathbf{Set}_\Lambda^\infty)$ of the full simplicial subcategory $\mathbf{Set}_\Lambda^\infty$ spanned by the Kan complexes in the simplicial category $\mathbf{Set}_\Lambda$ of simplical sets.
  \item We say a functor $F: \mathcal{C} \to \mathcal{D}$ of $\infty$-categories is \textit{coinitial} if the opposite functor $F^\mathrm{op}: \mathcal{C}^\mathrm{op} \to \mathcal{D}^\mathrm{op}$ is cofinal, i.e. if composing with $F$ always takes $\mathcal{D}$-indexed limits to $\mathcal{C}$-indexed limits.
\end{itemize}

1.6. \textbf{Acknowledgements.} I first learned about $\infty$-categories of spans from conversations with Clark Barwick four years ago. The present work was inspired by a number of discussions during my visit to the MSRI programme on algebraic topology in the spring of 2014, in particular with Hiro Tanaka and Owen Gwilliam. I also thank Gregor Schaumann, Chris Schommer-Pries, and Peter Teichner for helpful comments.

2. \textbf{Review of Iterated Segal Spaces}

In this section we briefly review the model for $(\infty, n)$-categories we use in this paper: the iterated Segal spaces of Barwick [Bar05], as generalized to the context of arbitrary $\infty$-topoi by Lurie [Lur09b].

\textbf{Definition 2.1.} Suppose $\mathcal{C}$ is an $\infty$-category with finite limits. A \textit{category object} in $\mathcal{C}$ is a simplicial object $C_n: \Delta^\mathrm{op} \to \mathcal{C}$ such that the natural maps

$$C_n \to C_1 \times C_0 \cdots \times C_0 C_1$$

induced by the maps $\sigma_i: [0] \to [n]$ sending 0 to $i$ and $\rho_i: [1] \to [n]$ sending 0 to $i-1$ and 1 to $i$ are equivalences in $\mathcal{C}$ for all $n$. We write $\mathrm{Cat}(\mathcal{C})$ for the full subcategory of $\mathrm{Fun}(\Delta^\mathrm{op}, \mathcal{C})$ spanned by the category objects. We refer to a category object in the $\infty$-category $\mathcal{S}$ of spaces as a (1-uple) \textit{Segal space}.

\textbf{Definition 2.2.} An $n$-uple \textit{category object} in an $\infty$-category $\mathcal{C}$ is inductively defined to be a category object in the $\infty$-category of $(n-1)$-uple category objects. We write $\mathrm{Cat}^n(\mathcal{C}) := \mathrm{Cat}(\mathrm{Cat}^{n-1}(\mathcal{C}))$ for the $\infty$-category of $n$-uple category objects in $\mathcal{C}$. We refer to a category object in $\mathcal{S}$ as an $n$-uple \textit{Segal space}.

\textbf{Remark 2.3.} The term $n$-uple Segal space is motivated by the observation that 2-uple (or double) Segal spaces model double $\infty$-categories, i.e. category objects in $\mathrm{Cat}_\infty$. More generally, $n$-uple Segal spaces can be considered as a model for $n$-uple $\infty$-categories, i.e. internal $\infty$-categories in internal $\infty$-categories in $\ldots$ in $\infty$-categories. However, the $\infty$-category $\mathrm{Cat}^n(\mathcal{S})$ is not the correct $\infty$-category of $(n-1)$-uple $\infty$-categories, as we need to invert an appropriate class of “fully faithful and essentially surjective functors”. We will not consider this localization here, however.

\textbf{Remark 2.4.} Unwinding the definition, we see that an $n$-uple Segal space $\mathcal{C}: (\Delta^\mathrm{op})^n \to \mathcal{S}$ consists of the data of:

\begin{itemize}
  \item a space $\mathcal{C}_0 \ldots, 0$ of objects
  \item spaces $\mathcal{C}_{1,0,\ldots,0}, \ldots, \mathcal{C}_{0,\ldots,0,1}$ of $n$ different kinds of 1-morphism, each with a source and target in $\mathcal{C}_{0,\ldots,0}$,
  \item spaces $\mathcal{C}_{1,1,0,\ldots,0}$, etc., of “commutative squares” between any two kinds of 1-morphism,
  \item spaces $\mathcal{C}_{1,1,1,0,\ldots,0}$, etc., of “commutative cubes” between any three kinds of 1-morphism,
  \item $\ldots$
  \item a space $\mathcal{C}_{1,1,\ldots,1}$ of “commutative $n$-cubes”,
\end{itemize}

\begin{itemize}
  \item together with units and coherently homotopy-associative composition laws for all these different types of maps.
\end{itemize}

We can view $(\infty, n)$-categories as given by the same kind of data, except that there is only one type of 1-morphism, so we require certain spaces to be “trivial”. This leads to Barwick’s definition of an $n$-fold Segal object in an $\infty$-category:

\textbf{Definition 2.5.} Suppose $\mathcal{C}$ is an $\infty$-category with finite limits. A 1-fold \textit{Segal object} in $\mathcal{C}$ is just a category object in $\mathcal{C}$. For $n > 1$ we inductively define an $n$-fold \textit{Segal object} in $\mathcal{C}$ to be an $n$-uple category object $\mathcal{D}$ such that
(i) the \((n-1)\)-uple category object \(D_0, \ldots, \bullet\) is constant,
(ii) the \((n-1)\)-uple category object \(D_k, \ldots, \bullet\) is an \((n-1)\)-fold Segal object for all \(k\).

We write \(\text{Seg}_{n}(\mathcal{C})\) for the full subcategory of \(\text{Cat}^{n}(\mathcal{C})\) spanned by the \(n\)-fold Segal objects. When \(\mathcal{C}\) is the \(\infty\)-category \(\mathcal{S}\) of spaces, we refer to \(n\)-fold Segal objects in \(\mathcal{S}\) as \(n\)-fold Segal spaces.

**Remark 2.6.** Unwinding the definition, we see that an \(n\)-fold Segal space \(\mathcal{C}\) consists of
- a space \(\mathcal{C}_{0, \ldots, 0}\) of objects,
- a space \(\mathcal{C}_{1,0, \ldots, 0}\) of 1-morphisms,
- a space \(\mathcal{C}_{1,1,0, \ldots, 0}\) of 2-morphisms,
- \(\ldots\)
- a space \(\mathcal{C}_{1, \ldots, 1}\) of \(n\)-morphisms,

together with units and coherently homotopy-associative composition laws for these morphisms.

**Remark 2.7.** Although \(n\)-fold Segal spaces describe \((\infty, n)\)-categories, the \(\infty\)-category \(\text{Seg}_{n}(\mathcal{S})\) is not the correct homotopy theory of \((\infty, n)\)-categories, as we have not inverted the appropriate class of fully faithful and essentially surjective maps. This localization can be obtained by restricting to the full subcategory of \(n\)-fold Segal spaces, as proved by Rezk [Rez01] for \(n = 1\) and Barwick [Bar08] for \(n > 1\). This result was generalized to arbitrary \(\infty\)-topoi by Lurie [Lur09b], and we will now briefly review this generalization:

**Definition 2.8.** A distributor consists of an \(\infty\)-category \(\mathcal{Y}\) together with a full subcategory \(\mathcal{X}\) such that:
1. The \(\infty\)-categories \(\mathcal{X}\) and \(\mathcal{Y}\) are presentable.
2. The full subcategory \(\mathcal{X}\) is closed under small limits and colimits in \(\mathcal{Y}\).
3. If \(Y \to X\) is a morphism in \(\mathcal{Y}\) such that \(X \in \mathcal{X}\), then the pullback functor \(X/\mathcal{X} \to Y/\mathcal{Y}\) preserves colimits.
4. Let \(\mathcal{O}\) denote the full subcategory of \(\text{Fun}(\Delta^1, \mathcal{Y})\) spanned by those morphisms \(f : Y \to X\) such that \(X \in \mathcal{X}\), and let \(\pi : \mathcal{O} \to \mathcal{X}\) be the functor given by evaluation at 1 \(\in \Delta^1\). Since \(\mathcal{Y}\) admits pullbacks, the evaluation-at-1 functor \(\text{Fun}(\Delta^1, \mathcal{Y}) \to \mathcal{Y}\) is a Cartesian fibration, hence so is \(\pi\).

Let \(\chi : \mathcal{X} \to \text{Cat}_{\infty}^{\mathcal{O}}\) be a functor that classifies \(\pi\). Then \(\chi\) preserves small limits.

If \(\mathcal{X} \subseteq \mathcal{Y}\) is a distributor, an \(\mathcal{X}\)-Segal object in \(\mathcal{Y}\) is a Segal object \(\mathcal{C} : \Delta^{\mathcal{O}} \to \mathcal{Y}\) such that \(\mathcal{C}_{0} \in \mathcal{X}\). We write \(\text{Seg}_{\mathcal{X}}(\mathcal{Y})\) for the full subcategory of \(\text{Seg}(\mathcal{Y})\) spanned by the \(\mathcal{X}\)-Segal objects.

**Remark 2.9.** It follows from the definition that if \(\mathcal{X} \subseteq \mathcal{Y}\) is a distributor, then \(\mathcal{X}\) is an \(\infty\)-topos.

**Definition 2.10.** Write \(\text{Gpd}(\mathcal{X})\) for the full subcategory of \(\text{Seg}(\mathcal{X})\) spanned by the groupoid objects, i.e. the simplicial objects \(X\) such that for every partition \([n] = S \cup S'\) where \(S \cap S'\) consists of a single element, the diagram

\[
\begin{array}{ccc}
X([n]) & \longrightarrow & X(S) \\
\downarrow & & \downarrow \\
X(S') & \longrightarrow & X(S \cap S')
\end{array}
\]

is a pullback square. Let \(\mathcal{X} \subseteq \mathcal{Y}\) be a distributor, and let \(\Lambda : \mathcal{Y} \to \mathcal{X}\) denote the right adjoint to the inclusion \(\mathcal{X} \hookrightarrow \mathcal{Y}\). The inclusion \(\text{Gpd}(\mathcal{X}) \hookrightarrow \text{Seg}(\mathcal{X}) \hookrightarrow \text{Seg}(\mathcal{Y})\) admits a right adjoint \(\iota : \text{Seg}(\mathcal{Y}) \to \text{Gpd}(\mathcal{X})\), which is the composite of the functor \(\Lambda : \text{Seg}(\mathcal{Y}) \to \text{Seg}(\mathcal{X})\) induced by \(\Lambda\), and the functor \(\iota : \text{Seg}(\mathcal{X}) \to \text{Gpd}(\mathcal{X})\) right adjoint to the inclusion, which exists by [Lur09b, Proposition 1.1.14]. We say an \(\mathcal{X}\)-Segal object \(F : \Delta^{\mathcal{O}} \to \mathcal{Y}\) is complete if the groupoid object \(\iota F\) is constant, and write \(\text{CSS}_{\mathcal{X}}(\mathcal{Y})\) for the full subcategory of \(\text{Seg}_{\mathcal{X}}(\mathcal{Y})\) spanned by the complete \(\mathcal{X}\)-Segal objects.

**Definition 2.11.** Let \(\mathcal{Y} \subseteq \mathcal{X}\) be a distributor, and suppose \(f : \mathcal{C} \to \mathcal{D}\) is a morphism in \(\text{Seg}_{\mathcal{X}}(\mathcal{Y})\). We say that \(f\) is fully faithful and essentially surjective if:
1. The map \(|\text{Gpd}(\mathcal{C})| \to |\text{Gpd}(\mathcal{D})|\) is an equivalence in the \(\infty\)-topos \(\mathcal{X}\).
(2) The diagram

\[
\begin{align*}
\mathcal{E}_1 & \longrightarrow \mathcal{D}_1 \\
\mathcal{E}_0 \times \mathcal{E}_0 & \longrightarrow \mathcal{D}_0 \times \mathcal{D}_0
\end{align*}
\]

is a pullback square in \(\mathcal{Y}\).

**Theorem 2.12** ([Lur09b, Theorem 1.2.13]). Let \(\mathcal{X} \subseteq \mathcal{Y}\) be a distributor. Then the left adjoint

\[ L_{\mathcal{X} \subseteq \mathcal{Y}} : \text{Seg}_{\mathcal{X}}(\mathcal{Y}) \to \text{CSS}_{\mathcal{X}}(\mathcal{Y}) \]

exhibits \(\text{CSS}_{\mathcal{X}}(\mathcal{Y})\) as the localization of \(\text{Seg}_{\mathcal{X}}(\mathcal{Y})\) with respect to the fully faithful and essentially surjective morphisms.

**Theorem 2.13** ([Lur09b, Proposition 1.3.2]). Suppose \(\mathcal{X} \subseteq \mathcal{Y}\) is a distributor. Then so is \(\mathcal{X} \subseteq \text{CSS}_{\mathcal{X}}(\mathcal{Y})\), where we regard \(\mathcal{X}\) as a full subcategory of \(\text{CSS}_{\mathcal{X}}(\mathcal{Y})\) via the diagonal embedding \(\epsilon^* : \mathcal{X} \to \text{Fun}(\Delta^{op}, \mathcal{X})\).

We can therefore inductively define distributors \(\mathcal{X} \subseteq \text{CSS}^n_{\mathcal{X}}(\mathcal{Y}) := \text{CSS}_{\mathcal{X}}(\text{CSS}^{n-1}_{\mathcal{X}}(\mathcal{Y}))\); we refer to the objects of \(\text{CSS}^n_{\mathcal{X}}(\mathcal{Y})\) as complete \(n\)-fold \(\mathcal{X}\)-Segal objects in \(\mathcal{Y}\).

**Definition 2.14.** Let \(\mathcal{X}\) be an \(\infty\)-topos. We write \(\text{CSS}^n(\mathcal{X})\) for \(\text{CSS}^n_{\mathcal{X}}(\mathcal{X})\), which we may regard as a full subcategory of \(\text{Seg}_{\mathcal{X}}(\mathcal{X})\). It is clear that the inclusion \(\text{CSS}^n(\mathcal{X}) \hookrightarrow \text{Seg}_{\mathcal{X}}(\mathcal{X})\) has a left adjoint \(L_n, \mathcal{X} : \text{Seg}_{\mathcal{X}}(\mathcal{X}) \to \text{CSS}^n(\mathcal{X})\), obtained inductively as the composite

\[
\text{Seg}_{\mathcal{X}}(\mathcal{X}) \xrightarrow{\text{Seg}_{\mathcal{X}}(L_{n-1, \mathcal{X}})} \text{Seg}_{\mathcal{X}}(\text{CSS}^{n-1}(\mathcal{X})) \xrightarrow{L_{\mathcal{X} \subseteq \text{CSS}^{n-1}(\mathcal{X})}} \text{CSS}^n(\mathcal{X}).
\]

In the case of Segal spaces, we will also make use of Rezk’s characterizations of the complete objects. To state these we must first introduce some notation:

**Definition 2.15.** Let \(\mathcal{E}_\bullet\) be a Segal space. A morphism in \(\mathcal{E}\) is a point of \(\mathcal{E}_1\). Let \(E^1\) denote the contractible groupoid with two objects and a unique morphism between any pair of objects; we may regard this as a Segal space by taking its nerve and viewing this as a discrete simplicial space. An equivalence in \(\mathcal{E}\) is then a map of Segal spaces \(E_1 \to \mathcal{E}\). There are two obvious inclusions \([1] \to E^1\) which thus induce maps \(\text{Map}(E^1, \mathcal{E}) \to \mathcal{E}_1\). We write \(\mathcal{E}_{eq}\) for the subspace of \(\mathcal{E}_1\) consisting of the components in the image of this map.

**Theorem 2.16** (Rezk). Let \(\mathcal{E}\) be a Segal space. Then the map \(\text{Map}(E^1, \mathcal{E}) \to \mathcal{E}_{eq}\) is an equivalence, and the following are equivalent:

(i) \(\mathcal{E}\) is complete.

(ii) The map \(\mathcal{E}_0 \to \mathcal{E}_{eq}\) induced by the degeneracy map \(s_0\) is an equivalence.

(iii) The map \(\mathcal{E}_0 \to \text{Map}(E^1, \mathcal{E})\) induced by composition with either of the maps \([0] \to E^1\) is an equivalence.

**Proof.** This is Theorem 6.2 and Proposition 6.4 of [Rez01]. □

The usual notion of a map between \(\infty\)-topoi is that of a geometric morphism: an adjunction where the left adjoint preserves finite limits. The \(\infty\)-categories of complete Segal spaces are functorial with respect to geometric morphisms of \(\infty\)-topoi — in fact, it will be useful to observe that they are functorial for a slightly more general class of maps:

**Definition 2.17.** Let \(\mathcal{X}\) and \(\mathcal{Y}\) be \(\infty\)-topoi. A pseudo-geometric morphism from \(\mathcal{X}\) to \(\mathcal{Y}\) is a functor \(f_* : \mathcal{X} \to \mathcal{Y}\) such that \(f_*\) admits a left adjoint \(f^*\) which preserves pullbacks.

**Definition 2.18.** Let \(\mathcal{X} \subseteq \mathcal{Y}\) and \(\mathcal{X}' \subseteq \mathcal{Y}'\) be distributors. A pseudo-geometric morphism from \(\mathcal{Y}\) to \(\mathcal{Y}'\) is a functor \(G : \mathcal{Y} \to \mathcal{Y}'\) such that:

1. \(G\) takes \(\mathcal{X}\) to \(\mathcal{X}'\).
(2) $G$ has a left adjoint $F : \mathcal{Y}' \to \mathcal{Y}$
(3) $F$ takes $\mathcal{X}'$ to $\mathcal{X}$.
(4) If $\phi : \Delta^1 \times \Delta^1 \to \mathcal{Y}'$ is a pullback diagram such that $\phi(1, 1) \in \mathcal{X}'$, then $F(\phi)$ is a pullback diagram in $\mathcal{Y}$.

**Remark 2.19.** It is clear that a pseudo-geometric morphism of distributors as above determines a pseudo-geometric morphism of $\infty$-topoi

$$F|_{\mathcal{X}'} : \mathcal{X}' \rightleftharpoons \mathcal{X} : G|_{\mathcal{X}}.$$

**Proposition 2.20.**

(i) Let $\mathcal{X} \subseteq \mathcal{Y}$ and $\mathcal{X}' \subseteq \mathcal{Y}'$ be distributors. Suppose $G : \mathcal{Y} \to \mathcal{Y}'$ is a pseudo-geometric morphism of distributors with left adjoint $F$. Then composition with $F$ and $G$ induces an adjunction

$$LF_* : \text{CSS}_{\mathcal{X}'}(\mathcal{Y}') \rightleftharpoons \text{CSS}_{\mathcal{X}}(\mathcal{Y}) : G_*,$$

and this is also a geometric morphism.

(ii) Suppose $f^* : \mathcal{X}' \rightleftharpoons \mathcal{X} : f_*$ is a pseudo-geometric morphism of $\infty$-topoi. Then the functors given by composition with $f^*$ and $f_*$ induce an adjunction

$$L_{n,\mathcal{X}}(f^*)_* : \text{CSS}^n(\mathcal{X}') \rightleftharpoons \text{CSS}^n(\mathcal{X}) : (f_*)_*.$$

For the proof we need the following observation:

**Lemma 2.21.**

(i) Let $\mathcal{X} \subseteq \mathcal{Y}$ be a distributor. Then the localization $L_{\mathcal{X} \subseteq \mathcal{Y}} : \text{Seg}_{\mathcal{X}}(\mathcal{Y}) \to \text{CSS}_{\mathcal{X}}(\mathcal{Y})$ preserves fibre products over constant diagrams $c^*\mathcal{X}$ where $X \in \mathcal{X}$; in particular, $L_{\mathcal{X} \subseteq \mathcal{Y}}$ preserves products.

(ii) Let $\mathcal{X}$ be an $\infty$-topos. Then the localization $L_{n,\mathcal{X}} : \text{Seg}_n(\mathcal{X}) \to \text{CSS}^n(\mathcal{X})$ preserves products.

**Proof.** Since the inclusion $\text{CSS}_{\mathcal{X}}(\mathcal{Y}) \hookrightarrow \text{Seg}_{\mathcal{X}}(\mathcal{Y})$ is a right adjoint, it preserves limits. Thus we must show that if $c$ and $D$ are $\mathcal{X}$-Segal objects of $\mathcal{Y}$ over $c^*\mathcal{X}$, then the natural map $L_{\mathcal{X} \subseteq \mathcal{Y}}(c \times c^*\mathcal{X} D) \to L_{\mathcal{X} \subseteq \mathcal{Y}}(c) \times L_{\mathcal{X} \subseteq \mathcal{Y}}(D)$ in $\text{Seg}_{\mathcal{X}}(\mathcal{Y})$ is an equivalence. By Theorem 2.12, this is equivalent to proving that the map $c \times c^*\mathcal{X} D \to L_{\mathcal{X} \subseteq \mathcal{Y}}(c) \times L_{\mathcal{X} \subseteq \mathcal{Y}}(D)$ is fully faithful and essentially surjective. Condition (1) in the definition holds since pullbacks over $\mathcal{X}$ preserve colimits in the $\infty$-topos $\mathcal{X}$, and the colimit in question is sifted, and condition (2) holds since limits commute. This proves (i); then (ii) follows inductively as the functor $L_{n,\mathcal{X}}$ is a composite of functors constructed from the functors in (i). \qed

**Proof of Proposition 2.20.** We obviously have an adjunction

$$F_* : \text{Fun}(\Delta^{op}, \mathcal{Y}') \rightleftharpoons \text{Fun}(\Delta^{op}, \mathcal{Y}) : G_*$$

It is clear from the definition of a pseudo-geometric morphism that $F_*$ and $G_*$ preserve $\mathcal{X}'$- and $\mathcal{X}$-Segal objects, respectively, so there is an induced adjunction

$$F_* : \text{Seg}_{\mathcal{X}'}(\mathcal{Y}') \rightleftharpoons \text{Seg}_{\mathcal{X}}(\mathcal{Y}) : G_*.$$

We clearly have a commutative diagram of left adjoints

$$\begin{array}{ccc}
\text{Gpd}(\mathcal{X}') & \xrightarrow{F_*} & \text{Gpd}(\mathcal{X}) \\
\downarrow & & \downarrow \\
\text{Seg}_{\mathcal{X}'}(\mathcal{Y}') & \xrightarrow{F_*} & \text{Seg}_{\mathcal{X}}(\mathcal{Y}).
\end{array}$$

where the vertical morphisms denote the obvious inclusions, hence the corresponding diagram of right adjoints also commutes, giving an equivalence $G_*(\text{Gpd}(c)) \simeq \text{Gpd}(G_* c)$. It follows that $G_*$ preserves complete Segal objects, hence there is an induced adjunction

$$L_{\mathcal{X} \subseteq \mathcal{Y}}F_* : \text{CSS}_{\mathcal{X}'}(\mathcal{Y}') \rightleftharpoons \text{CSS}_{\mathcal{X}}(\mathcal{Y}) : G_*.$$
To complete the proof of (i), we must show that this is a pseudo-geometric morphism. It is clear that $L_{X \subseteq Y} F_*$ and $G_*$ preserve constant simplicial objects valued in $X$ and $X'$, so it remains to show that, given a pullback diagram

$$
\begin{array}{ccc}
\mathcal{E} \times_X \mathcal{D} & \longrightarrow & \mathcal{E} \\
\downarrow & & \downarrow \\
\mathcal{D} & \longrightarrow & c^*X
\end{array}
$$

in $CSS(X';\mathcal{Y}')$, where $c^*X$ is the constant simplicial object with value $X \in X'$, its image under $L_{X \subseteq Y} F_*$ is also a pullback. Since limits in $CSS(X';\mathcal{Y}')$ are computed in $\text{Seg}_{X'}(\mathcal{Y}')$, and these in turn are computed objectwise, it follows that $F_*$ takes this to a pullback diagram in $\text{Seg}_{X'}(\mathcal{Y}')$. Now applying Lemma 2.21 we conclude that the image of this under $L_{X \subseteq Y}$ is also a pullback. This completes the proof of (i), and (ii) is just a special case of (i) obtained by induction.\hfill $\square$

**Corollary 2.22.** Suppose $X$ is an $\infty$-topos and $C_\bullet$ is a Segal object in $X$. Then $C_\bullet$ is complete if and only if the Segal spaces $\text{Map}_X(X,C_\bullet)$ are complete for all $X \in X$.

**Proof.** Let $r^* : S \rightarrow X$ denote the unique colimit-preserving functor such that $r^*(\cdot)$ is a terminal object of $X$, and let $r_* := \text{Map}_X(\cdot,r)$ be its right adjoint. By [Lur09a, Proposition 6.3.4.1], the adjunction $r^* \dashv r_*$ is a geometric morphism. It is clear that for any $X \in X$ the functor $\text{Map}_X(X,-)$ has a left adjoint given by $X \times r^*(\cdot)$, and this preserves pullbacks since $r^*$ preserves finite limits. Thus the adjunction $X \times r^*(-) \dashv \text{Map}_X(X,-)$ is a pseudo-geometric morphism of $\infty$-topoi, and so by the proof of Proposition 2.20 we have an equivalence $\text{Map}_X(X,Gpd(C_\bullet)) \simeq Gpd(\text{Map}_X(X,C_\bullet))$. Clearly $Gpd(C_\bullet)$ is constant if and only if $\text{Map}_X(X,Gpd(C_\bullet))$ is constant for all $X \in X$, so $C_\bullet$ is complete if and only if $Gpd(\text{Map}_X(X,C_\bullet))$ is constant for all $X \in X$, i.e. if and only if $\text{Map}_X(X,C_\bullet)$ is complete for all $X \in X$.\hfill $\square$

Lemma 2.21 also implies the following, which lets us define internal hom objects between complete Segal spaces:

**Lemma 2.23.** The Cartesian product in $CSS_n(X)$ preserves colimits separately in each variable.

**Proof.** Colimits in $CSS_n(X)$ are computed by applying the localization $L$ to the colimit of the same diagram in $\text{Seg}_n(X)$. Thus the result follows by combining Lemma 2.21 with the observation that the product clearly preserves colimits in $\text{Seg}_n(X)$.\hfill $\square$

**Definition 2.24.** It follows that $CSS_n(X)$ has internal Hom objects. We denote the internal Hom of morphisms from $C$ to $D$ by $\text{Hom}^D_C$. If $X \in X$ we abbreviate $\text{Hom}^X_C$ by $\text{Hom}_X$. We also write $\text{MAP}(C,D)$ for the object of $X$ that represents the functor $\text{Map}_{CSS_n(X)}(C \times c^*(-),D) : X \rightarrow S$. Equivalently, this is just $(\text{Hom}^D)^{0,\ldots,0}_0$.

It will be useful to restate the definition of an $n$-fold Segal object in a more formal way. To do this we first introduce some notation:

**Definition 2.25.** A map $\phi : [n] \rightarrow [m]$ is inert if $\phi$ is the inclusion of a subinterval, i.e. we have $\phi(i) = \phi(0) + i$ for all $i$. We write $\Delta_{\text{int}}$ for the subcategory of $\Delta$ containing only the inert maps. Let $\Delta_{\text{int}}^\text{op}$ denote the full subcategory of $\Delta_{\text{int}}^\text{op}$ spanned by the objects $[0]$ and $[1]$, i.e. the category

$$
[1] \Rightarrow [0],
$$

and let $\Delta_{\text{int}}^\text{op}/[n] := \Delta_{\text{int}}^\text{op} \times \Delta_{\text{int}}^\text{op}/[n]$.

**Remark 2.26.** This is a special case of the general notion of an inert map defined by Barwick [Bar13b] in the context of operator categories, which can be viewed as settings for different kinds of algebraic structures.

Then we have the following simple restatement of the definition:
Lemma 2.27. A functor $\Phi: (\Delta^{op})^n \to \mathcal{C}$ is a k-uple Segal object if and only if the restriction $\Phi|_{(\Delta^{op})^n \times k}$ is the right Kan extension of $\Phi|_{(\Delta^{int})^n}$ along the inclusion $j^k: (\Delta^n)^k \to (\Delta^{op})^n$, i.e. the natural map $\Phi|_{(\Delta^n)^n} \to j^k \Phi|_{(\Delta^n)^n}$ is an equivalence.

We will now show that there is a canonical way to extract an $n$-fold Segal space from an $n$-uple Segal space; in the next section we will apply this to construct an $n$-fold Segal space of iterated spans from an $n$-uple Segal space.

Proposition 2.28. The inclusion $\text{Seg}_n(S) \hookrightarrow \text{Cat}^n(S)$ has a right adjoint $U_{\text{Seg}}: \text{Cat}^n(S) \to \text{Seg}_n(S)$.

For the proof we need the following observation:

Lemma 2.29. Suppose $\pi: \mathcal{E} \to \mathcal{C}$ is a Cartesian fibration and $j: \mathcal{C}_0 \hookrightarrow \mathcal{C}$ is a fully faithful functor with a right adjoint $r: \mathcal{C} \to \mathcal{C}_0$. Let

$$
\begin{array}{ccc}
\mathcal{E}_0 & \xrightarrow{j} & \mathcal{E} \\
\downarrow \pi_0 & & \downarrow \pi \\
\mathcal{C}_0 & \xrightarrow{j} & \mathcal{C}
\end{array}
$$

be a pullback square. Then the inclusion $J$ has a right adjoint $R: \mathcal{E} \to \mathcal{E}_0$ such that the counit map $J \pi(X) \to X$ is a $\pi$-Cartesian morphism over the counit map $jr \pi(X) \to \pi(X)$.

Proof. Let $\epsilon: \mathcal{E} \times \Delta^1 \to \mathcal{E}$ be the counit natural transformation $ir \to \text{id}_\mathcal{E}$. The functor $\text{Fun}(\mathcal{E}, \mathcal{E}) \to \text{Fun}(\mathcal{E}, \mathcal{C})$ given by composition with $\pi$ is a Cartesian fibration by [Lur09a, Corollary 3.2.2.12], hence we may choose a Cartesian lift of the identity functor of $\mathcal{E}$ along $\epsilon \circ (\pi \times \text{id}) : \mathcal{E} \times \Delta^1 \to \mathcal{E}$ to get a natural transformation $\epsilon: \mathcal{E} \times \Delta^1 \to \mathcal{E}$, where $\epsilon_X$ is a $\pi$-Cartesian morphism for every $X \in \mathcal{E}$. We let $R := \epsilon|_{\mathcal{E} \times \{0\}}$. Since by construction $\pi \circ R \simeq r \circ \pi$, it is clear that $R(X)$ lies in $\mathcal{E}_0$ for all $X \in \mathcal{E}$, so we may regard $R$ as a functor $\mathcal{E} \to \mathcal{E}_0$. By the dual of [Lur09a, Proposition 5.2.2.8], to show that $R$ is right adjoint to $J$ it suffices to show that for all $X \in \mathcal{E}_0$ and $Y \in \mathcal{E}$, the map

$$
\text{Map}_{\mathcal{E}_0}(X, RY) \simeq \text{Map}_{\mathcal{E}}(JX, JRY) \to \text{Map}_{\mathcal{E}}(JX, Y)
$$

arising from composition with $\epsilon_Y$ is an equivalence. By construction the map $\epsilon_Y$ is a $\pi$-Cartesian morphism, hence the commutative square

$$
\begin{array}{ccc}
\text{Map}_{\mathcal{E}_0}(X, RY) & \xrightarrow{\epsilon_Y} & \text{Map}_{\mathcal{E}}(JX, Y) \\
\downarrow & & \downarrow \\
\text{Map}_{\mathcal{E}_0}(\pi(X), r\pi(Y)) & \xrightarrow{\epsilon_Y} & \text{Map}_{\mathcal{E}}(j\pi(X), \pi(Y))
\end{array}
$$

is Cartesian. But the lower horizontal map is an equivalence since $\epsilon$ is the counit for the adjunction $j \dashv r$, hence so is the upper horizontal map. \hfill \Box

Proof of Proposition 2.28. We define $X_n$ to be the full subcategory of $\Delta^\times n$ spanned by the objects of the form $([k_1], \ldots, [k_n])$ where $k_i$ is 0 or 1 for all $i$, and non-zero for at most one $i$, and let $i_n: X_n \hookrightarrow \Delta^\times n$ denote the inclusion. Let $e_n: X_1 \to X_n$ be the (fully faithful) functor that sends $[0]$ to $([0], \ldots, [0])$ and $[1]$ to $([1], [0], \ldots, [0])$. The functor $e_n^*: \text{Fun}(X_n^{op}, S) \to \text{Fun}(X_1^{op}, S)$ given by composition with $e_n$ has a left adjoint $e_{n,!}$ (given by left Kan extension), and it is easy to see that for $\phi: X_1 \to S$ the functor $e_{n,!}\phi$ is given by $e_{n,!*}[\phi([i]), [0], \ldots, [0]) \simeq \phi([i])$ for $i = 0, 1$, and $e_{n,!*}[\phi([0], [1], \ldots, [0]) \simeq \phi([0])$ with face and degeneracy maps given by the identity of $\phi([0])$. It
is then clear that we have a Cartesian square

\[
\begin{array}{c}
\text{Seg}_{n}(S) \\
\downarrow \epsilon_{n,t}
\end{array}
\begin{array}{c}
\text{Cat}^{e}(S) \\
\downarrow i_{n,t}
\end{array}
\begin{array}{c}
\text{Fun}(X_{1}^{op}, S) \\
\downarrow \epsilon_{n,t}
\end{array}
\begin{array}{c}
\text{Fun}(X_{n}^{op}, S).
\end{array}
\]

It follows from [GH13, Lemma A.1.6] that $i_{n,t}$ is a Cartesian fibration, and the functor $\epsilon_{n,t}$ is fully faithful since the composite $\epsilon_{n,t}^{*}e_{n,t}$ is equivalent to the identity functor. The existence of the right adjoint therefore follows from Lemma 2.29.

**Definition 2.30.** Let $\mathcal{E}$ be an $n$-uple Segal space. We refer to $U_{\text{Seg}}\mathcal{E}$ as the **underlying $n$-fold Segal space of** $\mathcal{E}$.

**Remark 2.31.** Unwinding the definitions we see that if $\mathcal{E}$ is an $n$-uple Segal space, then the space $(U_{\text{Seg}}\mathcal{E})_{1,...,1,0,...,0}$ of $k$-morphisms in $U_{\text{Seg}}\mathcal{E}$ is given by the pullback of $(U_{\text{Seg}}\mathcal{E})_{1,...,1,0,...,0} \to (i_{n,t}^{*}i_{n,t}^{e}\mathcal{E})_{1,...,1,0,...,0}$ along $(i_{n,t}^{*}i_{n,t}^{e}\mathcal{E})_{1,...,1,0,...,0} \to (i_{n,t}^{*}i_{n,t}^{e}\mathcal{E})_{1,...,1,0,...,0}$, where $i_{n,t}$ denotes the right adjoint to $i_{n,t}^{*}$ given by right Kan extension.

**Remark 2.32.** In the definition of $n$-fold Segal space, we privileged one of the $n$ possible spaces of $1$-morphisms. By making a different choice (or permuting the coordinates in $(\Delta^{op})^{\times n}$) we get $n$ different $n$-fold Segal spaces from an $n$-uple Segal space.

**Definition 2.33.** Suppose $\mathcal{E}$ is an $n$-fold Segal space. The two face maps $[1] \to [0]$ induce a map of $(n - 1)$-fold Segal spaces from $\mathcal{E}_{1}$ to the constant $(n - 1)$-fold Segal space $\mathcal{E}_{0} \times \mathcal{E}_{0}$. Given two objects $X, Y$ of $\mathcal{E}$, i.e. a point of $\mathcal{E}_{0} \times \mathcal{E}_{0}$, we define the mapping $(\infty, n - 1)$-category $\mathcal{E}(X, Y)$ to be the pullback of $\mathcal{E}_{1} \to \mathcal{E}_{0} \times \mathcal{E}_{0}$ along the map $* \to \mathcal{E}_{0} \times \mathcal{E}_{0}$ determined by $(X, Y)$. It is clear that this is also an $(n - 1)$-fold Segal space.

**Definition 2.34.** Suppose $\mathcal{E}$ is an $n$-fold Segal object in $\mathcal{X}$. The **underlying $k$-fold Segal object** $u_{(\infty,k)}\mathcal{E}$ of $\mathcal{E}$ is the $k$-fold simplicial object obtained by restricting $\mathcal{E}$ along the inclusion $(\Delta^{op})^{\times k} \to (\Delta^{op})^{\times n}$ that is $[0]$ in the last $n - k$ components.

We will make use of the following alternative characterization of completeness for $n$-fold Segal spaces:

**Theorem 2.35.** Suppose $\mathcal{E}$ is an $n$-fold Segal space. Then the following are equivalent:

(i) $\mathcal{E}$ is complete.

(ii) The Segal space $\mathcal{E}_{*,0,...,0}$ is complete, and the $(n - 1)$-fold Segal spaces $\mathcal{E}(X,Y)$ are complete for all objects $X, Y$ in $\mathcal{E}$.

For convenience, we make the following inductive definition:

**Definition 2.36.** Let $\mathcal{E}$ be an $n$-fold Segal space. We say that $\mathcal{E}$ is **pseudo-complete** if

1. the Segal space $\mathcal{E}_{*,0,...,0}$ is complete,
2. the $(n - 1)$-fold Segal spaces $\mathcal{E}(X,Y)$ are pseudo-complete for all objects $X, Y$ in $\mathcal{E}$.

Our goal is then to show that an $n$-fold Segal space is complete if and only if it is pseudo-complete. Before we give the proof we need to make a number of observations:

**Lemma 2.37.** Let $\mathcal{X} \subseteq \mathcal{Y}$ be a distributor, and suppose $\mathcal{E} \in \text{Seg}_{\mathcal{X}}(\text{CSS}_{\mathcal{X}}^{n-1}(\mathcal{Y}))$. Then $\mathcal{E}$ is in $\text{CSS}_{\mathcal{Y}}^{n}(\mathcal{Y})$ if and only if the Segal object $\mathcal{E}_{*,0,...,0}$ in $\text{Seg}_{\mathcal{X}}(\mathcal{Y})$ is complete.

**Proof.** The inclusion $\text{Seg}_{\mathcal{X}} \hookrightarrow \text{Seg}_{\mathcal{Y}}$ clearly factors through the inclusion $\text{Seg}_{\mathcal{X}}(\mathcal{Y}) \hookrightarrow \text{Seg}_{\mathcal{X}}\text{CSS}_{\mathcal{X}}^{n-1}(\mathcal{Y})$ induced by the functor $\mathcal{Y} \to \text{CSS}_{\mathcal{X}}^{n-1}(\mathcal{Y})$ that sends an object of $\mathcal{Y}$ to the constant $(n - 1)$-simplicial object with that value. Thus the right adjoint $\text{Seg}_{\mathcal{X}}\text{CSS}_{\mathcal{X}}^{n-1}(\mathcal{Y}) \to \text{Seg}(\mathcal{X})$ is the
composite of the right adjoint \( \text{Seg}_X \rightarrow \text{CSS}_X \) \( n-1 \) \( \text{Seg}_X(Y) \rightarrow \text{Seg}_X \). In particular, the groupoid object \( \text{Gpd}(\mathcal{C}) \) is equivalent to \( \text{Gpd}(\mathcal{C}_{\bullet,0,\ldots,0}) \) and so \( \mathcal{C} \) is complete if and only if \( \mathcal{C}_{\bullet,0,\ldots,0} \) is.

**Lemma 2.38.** Let \( \mathcal{C} \) be an \( n \)-fold Segal space. Then the following are equivalent:

(i) \( \mathcal{C} \) is complete.

(ii) The Segal space \( \mathcal{C}_{\bullet,0,\ldots,0} \) is complete, and the \((n - 1)\)-fold Segal space \( \mathcal{C}_{\bullet,\ldots,\bullet} \) is complete.

**Proof.** By Lemma 2.37 we know that \( \mathcal{C} \) is complete if and only if \( \mathcal{C}_{\bullet,0,\ldots,0} \) is complete and the \((n - 1)\)-fold Segal spaces \( \mathcal{C}_{\bullet,\ldots,\bullet} \) are complete for each \( n \). But \( \mathcal{C}_{\bullet,0,\ldots,0} \) is constant and so obviously complete, and thus for \( n > 1 \) the Segal condition implies that \( \mathcal{C}_{\bullet,\ldots,\bullet} \) is complete if \( \mathcal{C}_{\bullet,\ldots,\bullet} \) is complete, since complete Segal objects are closed under limits. \( \square \)

**Lemma 2.39.** Suppose given an \( n \)-fold Segal space \( \mathcal{C} \) together with a map \( \pi: \mathcal{C} \rightarrow X \) where \( X \) is a constant Segal space. Then for any two objects \( c, d \in \mathcal{C} \), there is a map \( \mathcal{C}(c, d) \rightarrow \Omega_{\pi(c), \pi(d)} X \) whose fibres are of the form \( \mathcal{C}_{\pi(c)}(c, d') \), where \( d' \) is the image of \( d \) in \( \mathcal{C}_{\pi(c)} \) under the equivalence determined by the path from \( \pi(c) \) to \( \pi(d) \).

**Proof.** The map \( \pi \) gives a commutative square

\[
\begin{array}{ccc}
\mathcal{C}_1 & \rightarrow & X \\
\downarrow & & \downarrow \\
\text{Ob}(\mathcal{C}) \times^2 & \rightarrow & X \times^2
\end{array}
\]

so taking fibres over a point \((c, d) \in \text{Ob}(\mathcal{C}) \times^2 \) we get a map \( \mathcal{C}(c, d) \rightarrow \Omega_{\pi(c), \pi(d)} X \). The fibre \( \mathcal{C}(c, d)_p \) of this map at a point of \( p \in \Omega_{\pi(c), \pi(d)} X \) we can identify with the limit of the commutative cube

\[
\begin{array}{ccc}
\mathcal{C}(c, d)_p & \rightarrow & \mathcal{C}_1 \\
\downarrow & & \downarrow \\
* & \rightarrow & (\text{ObC}) \times^2 \\
\downarrow & & \downarrow \\
* & \rightarrow & X \times^2
\end{array}
\]

where the bottom square is determined by \( p \). Now taking fibres at the points determined in the bottom square, we get a pullback square

\[
\begin{array}{ccc}
\mathcal{C}(c, d)_p & \rightarrow & (\mathcal{C}_{\pi(c)})_1 \\
\downarrow & & \downarrow \\
* & \rightarrow & (\mathcal{C}_{\pi(c)})_0 \times^2,
\end{array}
\]

which identifies \( \mathcal{C}(c, d)_p \) with a mapping \((n - 1)\)-fold Segal space in \( \mathcal{C}_{\pi(c)} \). \( \square \)

**Lemma 2.40.** Let \( \pi: \mathcal{C} \rightarrow X \) be a map of \( n \)-fold Segal spaces, where \( X \) is constant. Suppose the fibres \( \mathcal{C}_x \) are pseudo-complete for each \( x \in X \). Then \( \mathcal{C} \) is also pseudo-complete.
Proof. We prove this by induction on $n$. The map $\pi$ induces a commutative diagram

$$
\begin{array}{ccc}
\text{Map}(E^1, \mathcal{E}_{\bullet,0,...,0}) & \longrightarrow & \mathcal{E}_{0,...,0} \\
\downarrow & & \downarrow \\
X & & X
\end{array}
$$

where the map on fibres at $x \in X$ is clearly $\text{Map}(E^1, (\mathcal{E}_x)_{\bullet,0,...,0}) \rightarrow (\mathcal{E}_x)_{0,...,0}$. This map is an equivalence for all $x$, since $\mathcal{E}_x$ is pseudo-complete, hence the horizontal map in the triangle is also an equivalence and thus $\mathcal{E}_{\bullet,0,...,0}$ is complete.

Now given objects $c,d \in \mathcal{E}$, by Lemma 2.39 there is a map $\mathcal{E}(c,d) \rightarrow \Omega_{\pi(c),\pi(d)}X$ whose fibres are mapping $(n-1)$-fold Segal spaces in $\mathcal{E}_{\pi(c)}$. By assumption these are pseudo-complete, hence by the inductive hypothesis $\mathcal{E}(c,d)$ is pseudo-complete for all $c,d$. This completes the proof. □

Proof of Theorem 2.35. We will show, by induction on $n$, that an $n$-fold Segal space is complete if and only if it is pseudo-complete. For $n = 1$ the two notions coincide, so there is nothing to prove. Suppose we have shown that they agree for $n < k$, and let $\mathcal{E}$ be a $k$-fold Segal space. By Lemma 2.38 $\mathcal{E}$ is complete if and only if $\mathcal{E}_{\bullet,0,...,0}$ is complete and the $(k-1)$-fold Segal space $\mathcal{E}_1$ is complete. Since by assumption the notions of complete and pseudo-complete $(k-1)$-fold Segal spaces coincide, it remains to show that $\mathcal{E}_1$ is complete if and only if the fibres $\mathcal{E}(c,d)$ at $(c,d) \in \text{Ob}(\mathcal{E}) \times 2$ are all complete. One direction follows from applying Lemma 2.40 to the map $\mathcal{E}_1 \rightarrow \text{Ob}(\mathcal{E}) \times 2$, and the other is obvious since complete $(k-1)$-fold Segal spaces are closed under limits in $(k-1)$-fold Segal spaces. □

3. The $(\infty,n)$-Category of Iterated Spans

In this section we construct the $(\infty,n)$-category of iterated spans in an $\infty$-category with finite limits, in the form of an $n$-fold Segal space. In the case $n = 1$, the construction we use is due to Barwick [Bar13a], and in general we consider a simple inductive generalization of Barwick's definition. We begin by defining an $n$-uple Segal space from which we will extract the desired $(\infty,n)$-category as its underlying $n$-fold Segal space. This requires introducing some notation:

Definition 3.1. Let $\Sigma^n$ be the partially ordered set with objects pairs $(i, j)$ with $0 \leq i \leq j \leq n$, where $(i, j) \leq (i', j')$ if $i \leq i'$ and $j' \leq j$. It is clear that a map of totally ordered sets $\phi: [n] \rightarrow [m]$ induces a functor $\Sigma^n \rightarrow \Sigma^m$ by sending $(i, j)$ to $(\phi(i), \phi(j))$; we thus get a functor $\Sigma^*: \Delta \rightarrow \text{Cat}$. We will also write $\Sigma^{m_0,\ldots,m_k}$ for the product $\Sigma^{m_0} \times \cdots \times \Sigma^{m_k}$, which defines a functor $\Sigma^{*\cdots*}: \Delta \rightarrow \text{Cat}$.

Definition 3.2. Let $\Lambda^n$ denote the full subcategory of $\Sigma^n$ spanned by those pairs $(i,j)$ such that $j - i \leq 1$. These subcategories are not in general preserved by the functors $\Sigma(\phi)$ for $\phi$ in $\Delta$, but they are preserved by the functors arising from inert maps. We thus get a functor $\Lambda^*: \Lambda_{\text{int}} \rightarrow \text{Cat}$ with a natural transformation $i: \Lambda^* \rightarrow \Sigma^*|_{\Lambda_{\text{int}}}$. We will also write $\Lambda^{m_0,\ldots,m_k}$ for the product $\Lambda^{m_0} \times \cdots \times \Lambda^{m_k}$, which defines a functor $\Lambda^{*\cdots*}: \Lambda_{\text{int}} \rightarrow \text{Cat}$, with a natural transformation $i: \Lambda^{*\cdots*} \rightarrow \Sigma^{*\cdots*}|_{\Lambda_{\text{int}}}$.\]

Examples 3.3.

(i) The category $\Sigma^0 = \Lambda^0$ is the trivial one-object category.

(ii) The category $\Sigma^1 = \Lambda^1$ can be depicted as

$$(0,0) \leftrightarrow (0,1) \rightarrow (1,1).$$
(iii) The category $\Sigma^2$ can be depicted as

```
(0,2)  (0,1)  (1,2)
|      |      |
(0,0)  (1,1)  (2,2),
```

and the subcategory $\Lambda^2$ as

```
(0,1)  (1,1)  (2,2).
|      |      |
(0,0)  (1,1)  (2,2).
```

**Remark 3.4.** There is an obvious identification between $\Sigma^n$ and the category $(\Delta_{[n]/n})^\op$. Similarly, the subcategory $\Lambda^n$ corresponds to the full subcategory $\Sigma^\Lambda_{[n]/n}$ as defined in Definition 2.25.

**Definition 3.5.** Suppose $\mathcal{C}$ is an $\infty$-category with finite limits. A functor $f: \Sigma^{n_1,\ldots,n_k} \to \mathcal{C}$ is Cartesian if $f$ is a right Kan extension of its restriction to $\Lambda^{n_1,\ldots,n_k}$ along the inclusion $i_{n_1,\ldots,n_k}$, i.e. the unit map

$$f \to (i_{n_1,\ldots,n_k})_* f$$

is an equivalence, where $i_{n_1,\ldots,n_k}: \text{Fun}(\Sigma^{n_1,\ldots,n_k}, \mathcal{C}) \to \text{Fun}(\Lambda^{n_1,\ldots,n_k}, \mathcal{C})$ is the functor given by composition with $i_{n_1,\ldots,n_k}$, and $(i_{n_1,\ldots,n_k})_*$ denotes its right adjoint, given by right Kan extension. We write $\text{Fun}^{\text{Cart}}(\Sigma^{n_1,\ldots,n_k}, \mathcal{C})$ for the full subcategory of $\text{Fun}(\Sigma^{n_1,\ldots,n_k}, \mathcal{C})$ spanned by the Cartesian functors, and $\text{Map}^{\text{Cart}}(\Sigma^{n_1,\ldots,n_k}, \mathcal{C})$ for its underlying space.

**Remark 3.6.** Expanding the definition, we see that a functor $f: \Sigma^n \to \mathcal{C}$ is Cartesian if and only if it is given by taking iterated pullbacks of the $n$ spans $f_1, \ldots, f_n$ given by restricting $f$ along the inclusions $\Sigma(\rho_i): \Sigma^1 \to \Sigma^n$ coming from the maps $\rho_i: [1] \to [n]$ in $\Delta$ that send $0$ to $i - 1$ and $1$ to $i$. In other words, $f$ is Cartesian if and only if it presents the $n$-fold composite of these spans as $1$-morphisms in our desired $\infty$-category of spans. Similarly, a functor $f: \Sigma^{n_1,\ldots,n_k} \to \mathcal{C}$ is Cartesian if it presents the appropriate composite of spans in $k$ different directions.

**Definition 3.7.** Suppose $\mathcal{C}$ is an $\infty$-category with finite limits. Let $\overline{\text{SPAN}}_k(\mathcal{C}) \to (\Delta^\op)^{\times k}$ be a left fibration associated to the functor $(\Delta^\op)^{\times k} \to \mathcal{S}$ given by $\text{Map}(\Sigma^{\bullet,\ldots,\bullet}, \mathcal{C})$. Then we define $\text{SPAN}_k(\mathcal{C})$ to be the full subcategory of $\overline{\text{SPAN}}_k(\mathcal{C})$ spanned by the Cartesian functors $\Sigma^{\bullet,\ldots,\bullet} \to \mathcal{C}$.

Our goal is now to prove that $\text{SPAN}_k(\mathcal{C}) \to (\Delta^\op)^{\times k}$ is a $k$-uple Segal space, or more precisely is the left fibration associated to a $k$-uple Segal space.

**Proposition 3.8.** The restricted projection $\text{SPAN}_k(\mathcal{C}) \to (\Delta^\op)^{\times k}$ is a left fibration.

**Proof.** It suffices to check that if $\alpha \to \beta$ is a coCartesian morphism in $\overline{\text{SPAN}}_k(\mathcal{C})$ such that $\alpha$ is a Cartesian functor, then so is $\beta$. In other words, given a Cartesian functor $\alpha: \Sigma^{n_1,\ldots,n_k} \to \mathcal{C}$ and a morphism $(\phi_1, \ldots, \phi_k): ([n_1], \ldots, [n_k]) \to ([m_1], \ldots, [m_k])$ in $\Delta^{\times k}$, we must show that the composite functor $(\phi_1, \ldots, \phi_k)^\ast \alpha: \Sigma^{m_1,\ldots,m_k} \to \mathcal{C}$ is also Cartesian.

Let $\Lambda^{n_1,\ldots,n_k} \subseteq \Sigma^{n_1,\ldots,n_k}$ be the full subcategory spanned by the objects of $\Lambda^{n_1,\ldots,n_k}$ together with the objects $X = (X_1, \ldots, X_k)$ such that $X_i \geq \phi_i(Y)$ for some $Y = (Y_1, \ldots, Y_k) \in \Lambda^{m_1,\ldots,m_k}$, and let $k: \Lambda^{n_1,\ldots,n_k} \to \overline{\Lambda}^{n_1,\ldots,n_k}$ and $j: \overline{\Lambda}^{n_1,\ldots,n_k} \to \Sigma^{n_1,\ldots,n_k}$ denote the inclusions. Then $\phi = (\phi_1, \ldots, \phi_k)$ induces a map

$$\lambda: \Lambda^{m_1,\ldots,m_k} \to \overline{\Lambda}^{n_1,\ldots,n_k}.$$
such that the diagram

\[
\begin{array}{ccc}
\Lambda^{m_1, \ldots, m_k} & \xrightarrow{\lambda} & \Lambda^{n_1, \ldots, n_k} \\
\downarrow i & & \downarrow j \\
\Sigma^{m_1, \ldots, m_k} & \xrightarrow{\phi} & \Sigma^{n_1, \ldots, n_k}
\end{array}
\]

commutes (and, in fact, is a pullback square). We wish to show that the natural transformation \( \phi^* \alpha \to i_* i^* \phi^* \alpha \) is an equivalence, where we abbreviate \( i \) := \( i_{m_1, \ldots, m_k} \) and \( \phi^* := (\phi_1, \ldots, \phi_k)^* \). We have a natural equivalence \( i_* i^* \phi^* \simeq i_* \lambda^* j^* \), and our natural transformation factors as

\[
\phi^* \alpha \to \phi^* j_* j^* \alpha \to i_* \lambda^* j^* \alpha,
\]

where the second map comes from the Beck-Chevalley transformation

\[
\phi^* j_* \to i_* i^* \phi^* j_* \simeq i_* \lambda^* j^* j_* \to i_* \lambda^* j^*
\]

from the commutative square above. Now, writing \( i = jk \) for the inclusion \( \Lambda^{n_1, \ldots, n_k} \hookrightarrow \Sigma^{n_1, \ldots, n_k} \), we know by assumption that the natural map \( \alpha \to i'_s(i'^s)^* \alpha \) is an equivalence. As \( j \) is fully faithful, we have that \( j^* j_* \simeq \text{id} \), and hence we get

\[
\begin{aligned}
j_* j^* \alpha & \simeq j_* i'^s(i'^s)^* \alpha \simeq j_* j^* j_* k_* (i'^s)^* \alpha \simeq i'_s(i'^s)^* \alpha \simeq \alpha,
\end{aligned}
\]

so we see that the natural map \( \alpha \to j_* j^* \alpha \) is also an equivalence. Thus \( \phi^* \alpha \to \phi^* j_* j^* \alpha \) is an equivalence, and it suffices to show that the natural map \( \phi^* j_* j^* \alpha \to i_* \lambda^* j^* \alpha \) is an equivalence. In other words, we must show that for all \( X \in \Sigma^{m_1, \ldots, m_k} \), the natural map

\[
\lim_{\overline{\Lambda}_{\phi(X)/}} j^* \alpha \to \lim_{\overline{\Lambda}_{\phi(X)/}^{\lambda}} \lambda^* j^* \alpha
\]

is an equivalence. To see this, it suffices to prove that the functor \( \lambda_X : \Lambda^{m_1, \ldots, m_k}_{X/} \to \Lambda^{n_1, \ldots, n_k}_{\phi(X)/} \) is coinitial. By [Lur09a, Theorem 4.1.3.1] this is equivalent to proving that for every \( Y \in \Lambda^{n_1, \ldots, n_k}_{\phi(X)/} \), the \( \infty \)-category \( \Lambda^{m_1, \ldots, m_k}_{X/Y} := \Lambda^{m_1, \ldots, m_k}_{X/} \times_{\Lambda^{n_1, \ldots, n_k}_{\phi(X)/}} \Lambda^{n_1, \ldots, n_k}_{\phi(X)/} \) is weakly contractible.

Since all the categories involved are partially ordered sets, we can identify this category with the full subcategory \( \Lambda^{m_1, \ldots, m_k}_{X,Y} \) of \( \Lambda^{m_1, \ldots, m_k} \) spanned by those objects \( Z \) such that \( Z \geq X \) and \( \phi(Z) \leq Y \). We thus want to show that this subcategory is weakly contractible.

It suffices to show this in the case \( k = 1 \). Since \( \Lambda^m \) is then just a wedge of \( \Delta^1 \)'s, it is clear that a full subcategory is weakly contractible if and only if it is connected. If \( \phi : [m] \to [n] \) and \( Y = (c,d) \), let \( \hat{a} \) be maximal in \([m]\) such that \( \phi(\hat{a}) \leq c \) and let \( \hat{b} \) be minimal such that \( \phi(\hat{b}) \geq d \). If \( c < \hat{a} \) then clearly \( \hat{a} < \hat{b} \), and it is clear that \( (\phi(a), \phi(b)) \leq Y \) if and only if \((a, b) \leq (\hat{a}, \hat{b}) \). Moreover, by definition of \( \overline{\Lambda} \) we know that \( \hat{b} - \hat{a} = 1 \), and so the subcategory \( \Lambda^m_{X,Y} \) consists of the single object \((\hat{a}, \hat{b})\). On the other hand, if \( c = d \), then \( \hat{a} = \hat{b} \) and \((\phi(a), \phi(b)) \leq Y \) if and only if \( \hat{b} \geq \hat{b} \) and \( a \leq \hat{a} \). This partially ordered set is clearly connected, which completes the proof. \( \square \)

**Proposition 3.9.** In the \( \infty \)-category of \( \infty \)-categories, the natural map \( \Lambda^1 \Pi_{\Lambda^n} \to \Lambda^n \) coming from the inert maps \([0], [1] \to [n] \) in \( \Delta \) is an equivalence of \( \infty \)-categories.

**Proof.** This colimit in \( \text{Cat}_\infty \) is equivalent to the corresponding iterated homotopy pushout in the Joyal model structure. The diagram of simplicial sets obtained by taking the nerve of the corresponding diagram in the category of categories contains only injective maps of simplicial sets. Since these are cofibrations and the Joyal model structure is left proper, we conclude that this homotopy colimit is simply given by (a fibrant replacement for) the iterated pushout of simplicial sets \( \text{NA}^1 \Pi_{\text{NA}^n} \to \text{NA}^n \). It is easy to see that this is isomorphic to the simplicial set \( \text{NA}^1 \)). \( \square \)

**Theorem 3.10.** The functor associated to the left fibration \( \text{SPAN}_{k}(E) \to (\Delta^0)^{\times k} \) is a \( k \)-uple Segal space.
Proof. Unwinding the definitions, we must show that for each \([n_1], \ldots, [n_k]\) in \(\Delta^k\), the natural map
\[
\text{Map}^{\text{Cart}}(\Sigma^{n_1,\ldots,n_k}, \mathcal{C}) \to \lim_{(i_1,\ldots,i_k) \in \Pi_n} \text{Map}(\Sigma^{i_1}, \mathcal{C}) \times \cdots \times \text{Map}(\Sigma^{i_k}, \mathcal{C})
\]
is an equivalence. Now using Proposition 3.9 and the fact that products in \(\text{Cat}_\infty\) commute with colimits, it follows that the target of this map is equivalent to \(\text{Map}(\Lambda^{n_1,\ldots,n_k}), \mathcal{C})\), and under this equivalence the Segal map corresponds to the map given by composing with the inclusion \(\Lambda^{n_1,\ldots,n_k} \hookrightarrow \Sigma^{n_1,\ldots,n_k}\). Since this is fully faithful, and \(\text{Map}^{\text{Cart}}(\Sigma^{n_1,\ldots,n_k}, \mathcal{C})\) is precisely the space of functors that are right Kan extensions along this inclusion, it follows that our map is an equivalence. \(\square\)

**Definition 3.11.** Let \(\mathcal{C}\) be an \((\infty, k)\)-category with finite limits. The \((\infty, k)\)-category \(\text{Span}_k(\mathcal{C})\) of iterated spans in \(\mathcal{C}\) is the \(k\)-fold Segal space \(U_{\text{Seg}} \text{SPAN}_k(\mathcal{C})\) associated to the \(k\)-uple Segal space \(\text{SPAN}_k(\mathcal{C})\).

**Remark 3.12.** We can similarly define \(\text{SPAN}_k^+(\mathcal{C}) \to (\Delta^\text{op})^k\) to be a coCartesian fibration associated to the multisimplicial \(\infty\)-category \(\text{Fun}(\Sigma^* \times \cdots \times \Sigma^*, \mathcal{C})\), and let \(\text{SPAN}_k^+(\mathcal{C})\) be the full subcategory spanned by the Cartesian functors. The proofs of Proposition 3.8 and Theorem 3.10 also imply that \(\text{SPAN}_k^+(\mathcal{C}) \to (\Delta^\text{op})^k\) is the coCartesian fibration corresponding to a \(k\)-uple category object in \(\text{Cat}_\infty\). We can think of these as \((k + 1)\)-uple \(\infty\)-categories, and again define a right adjoint to a full subcategory that corresponds to \((\infty, k + 1)\)-categories (cf. [Lur09b]). This gives an \((\infty, k + 1)\)-category \(\text{Span}_k^+(\mathcal{C})\) whose underlying \((\infty, k)\)-category is \(\text{Span}_k(\mathcal{C})\), with the \((k + 1)\)-morphisms given by morphisms of \(k\)-fold spans.

We will now prove that the \(n\)-fold Segal space \(\text{Span}_n(\mathcal{C})\) is always complete. We first consider the case \(n = 1\), which is due to Barwick:

**Proposition 3.13** ([Bar14, Proposition 3.4]). Let \(\mathcal{C}\) be an \((\infty, k)\)-category with finite limits. Then the Segal space \(\text{Span}_1(\mathcal{C})\) is complete.

To prove this, we must first recall the definition of twisted arrow \(\infty\)-categories:

**Definition 3.14.** Let \(e : \Delta \to \Delta\) be the functor given by \([n] \mapsto [n]^{\text{op}} \times [n] \cong [2n + 1]\). The edgewise subdivision of a simplicial set \(K\) is \(e^* K\). By [Lur11, Proposition 4.2.3] if \(\mathcal{C}\) is a quasicategory then so is \(e^* \mathcal{C}\). We will refer to this as the *twisted arrow \(\infty\)-category* \(\text{Tw}(\mathcal{C})\) of \(\mathcal{C}\); see [Bar13a, §2] for a more extensive discussion of these objects.

Observe that \(\Sigma^n\) is precisely \(\text{Tw}(\Delta^n)\), which gives us the following description of maps to \(\text{SPAN}_1^+(\mathcal{C})\):

**Lemma 3.15.** Let \(\mathcal{C}\) be an \((\infty, k)\)-category with finite limits, and \(K\) an arbitrary \((\infty, k)\)-category. Then there is a natural equivalence \(\text{Map}_{\text{Cat}_\infty}(K, \text{SPAN}_1^+(\mathcal{C})) \simeq \text{Map}_{\text{Cart}}(\text{Tw}(K)), \mathcal{C})\), where Map denotes the enrichment in simplicial sets on both sides.

**Proof.** Regard \(\text{SPAN}_1(\mathcal{C})\) as a Segal space, instantiated as a simplicial object in simplicial sets, and \(K\) as a quasicategory, which we may also regard \(K\) as a discrete simplicial space. Then the space \(\text{Map}_{\text{Cat}_\infty}(K, \text{SPAN}_1^+(\mathcal{C}))\) can be identified with the natural simplicial set of maps between bisimplicial sets, i.e. the end \(\text{Map}(K, \Sigma^*) \simeq \text{Map}(\text{colim}_{(\Delta^\text{op})} K \times \text{Tw}(\Delta^*) \mathcal{C})\), since \(\text{Tw}\) preserves colimits (as it is given by composition with an endofunctor of \(\Delta^\text{op}\)) this coend is equivalent to \(\text{Tw}(K)\), as required.

**Proof of Proposition 3.13.** We may clearly identify \(\text{Map}(E^1, \text{Span}_1(\mathcal{C}))\) with a subspace of components in \(\text{Map}(E^1, \text{SPAN}_1(\mathcal{C}))\). By Lemma 3.15 the latter is equivalent to \(\text{Map}(\text{Tw}(E^1)), \mathcal{C})\), and we may identify \(\text{Map}(E^1, \text{Span}_1(\mathcal{C}))\) with the maps \(\text{Tw}(E^1) \to \mathcal{C}\) such that for every \(n\)-simplex of \(E^1\), the composite \(\Sigma^n \to \text{Tw}(E^1) \to \mathcal{C}\) is Cartesian. But it is easy to see that \(\text{Tw}(E^1)\) is the contractible category with four objects, so not only are these composites all Cartesian, the space \(\text{Map}(\text{Tw}(E^1)), \mathcal{C})\) is equivalent to \(\mathcal{C}\). Thus \(\text{Span}_1(\mathcal{C})\) is local with respect to \(E^1 \to *\), i.e. it is a complete Segal space. \(\square\)
To extend this to iterated spans, we first identify the mapping \((\infty, n - 1)\)-categories in \(\text{Span}_n(\mathcal{C})\):

**Proposition 3.16.** Let \(\mathcal{C}\) be an \(\infty\)-category with finite limits. If \(X\) and \(Y\) are objects of \(\mathcal{C}\), then the \((\infty, k - 1)\)-category \(\text{Span}_k(\mathcal{C})(X, Y)\) of maps from \(X\) to \(Y\) in \(\text{Span}_k(\mathcal{C})\) is equivalent to \(\text{Span}_{k-1}(\mathcal{C}/X \times Y)\).

For the proof we need a simple observation:

**Lemma 3.17.** Suppose \(X\) and \(Y\) are objects of an \(\infty\)-category \(\mathcal{C}\) that have a product \(X \times Y\). Then for any \(\infty\)-category \(K\) there is a natural pullback square

\[
\begin{array}{ccc}
\text{Map}(K, \mathcal{C}/X \times Y) & \longrightarrow & \text{Map}(\Sigma^1 \times K, \mathcal{C}) \\
\downarrow & & \downarrow \\
\{(c_X, c_Y)\} & \longrightarrow & \text{Map}(\Sigma^0 \times K, \mathcal{C})^2,
\end{array}
\]

where \(c_X\) and \(c_Y\) denote the functors constant at \(X\) and \(Y\).

**Proof.** Since \(X \times Y\) is a product, the \(\infty\)-category \(\mathcal{C}/X \times Y\) is equivalent to \(\mathcal{C}/p\) where \(p\) is the diagram \(\{0, 1\} \to \mathcal{C}\) sending 0 to \(X\) and 1 to \(Y\). The \(\infty\)-category \(\mathcal{C}/p\) has the universal property that for all \(\infty\)-categories \(K\) there are natural pullback squares

\[
\begin{array}{ccc}
\text{Map}(K, \mathcal{C}/p) & \longrightarrow & \text{Map}(K \star \{0, 1\}, \mathcal{C}) \\
\downarrow & & \downarrow \\
\{(X, Y)\} & \longrightarrow & \text{Map}(\{0, 1\}, \mathcal{C}).
\end{array}
\]

Clearly \(\Sigma^1\) is equivalent to \(\{0, 1\}^\diamond\), i.e. \(\ast \amalg_{\{0, 1\} \times \{0\}} \{0, 1\} \times \Delta^1\), and since products in \(\text{Cat}_\infty\) preserve colimits, this gives an equivalence

\[
K \times \Sigma^1 \simeq K \amalg_{K \times \{0, 1\} \times \{0\}} K \times \{0, 1\} \times \Delta^1.
\]

Moreover, the \(\infty\)-category \(K \star \{0, 1\}\) is equivalent to the pushout (in \(\text{Cat}_\infty\))

\[
K \amalg_{K \times \{0, 1\} \times \{0\}} K \times \{0, 1\} \times \Delta^1 \amalg_{K \times \{0, 1\} \times \{1\}} \{0, 1\},
\]

thus we get a pullback square

\[
\begin{array}{ccc}
\text{Map}(K \star \{0, 1\}, \mathcal{C}) & \longrightarrow & \text{Map}(K \times \Sigma^1, \mathcal{C}) \\
\downarrow & & \downarrow \\
\text{Map}(\{0, 1\}, \mathcal{C}) & \longrightarrow & \text{Map}(K \times \{0, 1\}, \mathcal{C}).
\end{array}
\]

Putting these two pullbacks squares together then completes the proof. \(\square\)

**Proof of Proposition 3.16.** By Lemma 3.17 we have natural pullback squares

\[
\begin{array}{ccc}
\text{SPAN}_{k-1}(\mathcal{C}/X \times Y)_{[n_1], \ldots, [n_k-1]} & \longrightarrow & \text{SPAN}_k(\mathcal{C})_{[1], [n_1], \ldots, [n_k-1]} \\
\downarrow & & \downarrow \\
\{(c_X, c_Y)\} & \longrightarrow & \text{SPAN}_k(\mathcal{C})_{[0], [n_1], \ldots, [n_k-1]}^2.
\end{array}
\]
By the definition of Cartesian functors from $\Sigma^*\ldots\cdot$, it is easy to see that this restricts to a pullback square

$$\begin{array}{ccc}
\text{SPAN}_{k-1}(E/X \times Y)_{[n_1],\ldots,[n_{k-1}]} & \to & \text{SPAN}_k(E)_{[1],[n_1],\ldots,[n_{k-1}]}
\\
\{(c_X,c_Y)\} & \to & \text{SPAN}_k(E)_{[0],[n_1],\ldots,[n_{k-1}]}^2
\end{array}$$

The functor $U_{\text{Seg}}$ is a right adjoint, and it is clear from its definition that for any $n$-uple Segal space $E$, the $(n-1)$-fold Segal spaces $U_{\text{Seg}}(E)^{[i],\ldots,\cdot}$ can be identified with the $(n-1)$-fold Segal space $U_{\text{Seg}}(E)^{[i],\ldots,\cdot}$ extracted from the $(n-1)$-uple Segal space $E^{[i],\ldots,\cdot}$. Thus we have natural pullback squares

$$\begin{array}{ccc}
\text{Span}_{k-1}(E/X \times Y)_{[n_1],\ldots,[n_{k-1}]} & \to & \text{Span}_k(E)_{[1],[n_1],\ldots,[n_{k-1}]}
\\
\{X,Y\} & \to & \text{Span}_k(E)_{[0],[n_1],\ldots,[n_{k-1}]}^2
\end{array}$$

which completes the proof. \qed

**Corollary 3.18.** Let $\mathcal{C}$ be an $\infty$-category with finite limits. Then the $n$-fold Segal space $\text{Span}_n(\mathcal{C})$ is complete.

**Proof.** We prove this by induction on $n$. The case $n = 1$ is Proposition 3.13. Suppose the result holds for all $n < k$. By Theorem 2.35 to show that $\text{Span}_k(\mathcal{C})$ is complete it suffices to prove that the Segal space $\text{Span}_k(\mathcal{C})^{0,\ldots,0}$ is complete, and the $(n-1)$-fold Segal spaces $\text{Span}_n(\mathcal{C})(X,Y)$ are complete for all $X,Y$ in $\mathcal{C}$. But $\text{Span}_k(\mathcal{C})^{0,\ldots,0}$ is clearly equivalent to $\text{Span}_1(\mathcal{C})$, which we know is complete, and by Proposition 3.16 we can identify $\text{Span}_k(\mathcal{C})(X,Y)$ with $\text{Span}_{k-1}(E/X \times Y)$, which is complete by the inductive hypothesis. \qed

4. The $(\infty,n)$-Category of Iterated Spans with Local Systems

In this section we will use the $(\infty,k)$-category $\text{Span}_k(X)$ to construct for every $k$-fold Segal object $E : (\Delta^o)^{\times k} \to X$ in $\mathcal{C}$ an $(\infty,k)$-category $\text{Span}_k(X;E)$ of $k$-fold iterated spans in $X$ equipped with local systems in $\mathcal{C}$. More precisely, an object of $\text{Span}_k(X;E)$ is an object $X \in X$ equipped with a morphism $X \to E^{0,\ldots,0}$ to the objects of $E$. A 1-morphism is a span

$$\begin{array}{ccc}
X & \to & B \\
\downarrow & & \downarrow \\
A & \to & 0
\end{array}$$

in $X$ over the span

$$\begin{array}{ccc}
\mathcal{C}^{1,0,\ldots,0} & \to & \mathcal{C}^{0,\ldots,0}
\\
\downarrow & & \downarrow \\
\mathcal{C}^{0,\ldots,0} & \to & \mathcal{C}^{0,\ldots,0}
\end{array}$$

given by the source and target maps for 1-morphisms in $\mathcal{C}$. Similarly a 2-morphism is a 2-span equipped with a map to the 2-span given by the source and target maps for 2-morphisms, and so forth. To define this we will show that any $k$-fold Segal object in $X$ determines a section of the projection $\text{SPAN}_k^+(X) \to (\Delta^o)^{\times k}$, and then use results of Lurie to construct a fibrewise over-category for this section.
Definition 4.1. Let \( \mathcal{S} \to \Delta^{op} \) be the Grothendieck fibration associated to the functor \( \Sigma^* : \Delta \to \text{Cat} \). Explicitly, \( \mathcal{S} \) is the category with objects pairs \( ([n], (i, j)) \) with \( [n] \in \Delta \) and 0 \( \leq i \leq j \leq n \), and morphisms \( ([n], (i, j)) \to ([m], (i', j')) \) given by a morphism \( \phi : [m] \to [n] \) in \( \Delta \) and a morphism \( (i, j) \to (\phi(i'), \phi(j')) \) in \( \Sigma^n \). It is clear that \( \hat{\mathcal{S}} \to (\Delta^{op})^{xk} \) is the Grothendieck fibration associated to the functor \( \hat{\Sigma}^{xk} : (\Delta^{op})^{xk} \to \text{Cat} \).

By [GH13, Proposition A.1.5], the \( \infty \)-category \( \text{SPAN}^+_k(X) \) has a universal property: for every map of simplicial sets \( K \to (\Delta^{op})^{xk} \), the \( \infty \)-category \( \text{Fun}_{(\Delta^{op})^{xk}}(K, \text{SPAN}^+_k(X)) \) is naturally equivalent to \( \text{Fun}(K \times (\Delta^{op})^{xk} \hat{\mathcal{S}}^{xk}, X) \). In particular, giving a section \( (\Delta^{op})^{xk} \to \text{SPAN}^+_k(X) \) is equivalent to giving a functor \( \hat{\mathcal{S}}^{xk} \to X \).

Definition 4.2. Let \( \Pi : \hat{\mathcal{S}} \to \Delta^{op} \) denote the functor that sends \( ([n], (i, j)) \) to \( [j - i] \) and a map \( (\phi : [m] \to [n], (i, j) \to (\phi(i'), \phi(j'))) \) to the map \( [j - i] \to [j' - i'] \) in \( \Delta^{op} \) corresponding to the map of ordered sets taking \( s \in [j' - i'] \) to \( \phi(i' + s) - i \in [j - i] \).

Thus any map \( \Phi : (\Delta^{op})^{xk} \to X \) determines a section \( s_\Phi : (\Delta^{op})^{xk} \to \text{SPAN}^+_k(X) \) via the functor \( (\Pi^{xk}) : \hat{\mathcal{S}}^{xk} \to X \).

Lemma 4.3. Suppose \( \Phi : (\Delta^{op})^{xk} \to X \) is a k-uple category object in \( X \). Then the section \( s_\Phi : (\Delta^{op})^{xk} \to \text{SPAN}^+_k(X) \) factors through \( \text{SPAN}^+_k(X) \).

Proof. Using the description of the correspondence between sections and functors from \( \hat{\mathcal{S}}^{xk} \) from the proof of [GH13, Proposition A.1.5], we see that it suffices to check that for each \( ([n_1], \ldots, [n_k]) \) the restriction \( \Pi_{n_1, \ldots, n_k} : \hat{\mathcal{S}}^{n_1, \ldots, n_k} \to X \) is a Cartesian functor. But under the identification of \( \Sigma^{n_1, \ldots, n_k} \) with \( (\Delta^{op})^{xk} \) of Remark 3.4, this functor corresponds to the composite

\[
(\Delta^{op})^{xk}([n_1], \ldots, [n_k]) \to (\Delta^{op})^{xk} \to X,
\]

where the first map is the obvious forgetful functor. By Lemma 2.27, the restriction of \( \Phi \) to \( (\Delta^{op})^{xk} \) is the right Kan extension of its restriction to \( (\Delta^{op})^{xk} \), from which it clearly follows that \( \Pi_{n_1, \ldots, n_k} \) is the right Kan extension of its restriction to \( \Delta^{n_1, \ldots, n_k} \), since this corresponds to \( \prod_i \Delta^{[n_i]} \) under this identification.

Definition 4.4. Suppose \( \Phi : (\Delta^{op})^{xk} \to X \) is a k-uple category object in \( X \). Using [Lur09a, Definition 4.2.2.1], we define \( \text{SPAN}^+_k(X; \Phi) \to (\Delta^{op})^{xk} \) to be the \( \infty \)-category with the universal property that a map \( K \to \text{SPAN}^+_k(X; \Phi) \) over \( (\Delta^{op})^{xk} \) is equivalent to a map

\[
K \times \Delta^1 \Pi_{n_1, \ldots, n_k} (\Delta^{op})^{xk} \to \text{SPAN}^+_k(X),
\]

that restricts to \( s_\Phi \) on \( (\Delta^{op})^{xk} \).

Proposition 4.5. Suppose \( \Phi : (\Delta^{op})^{xk} \to X \) is a k-uple category object in \( X \). Then \( \pi : \text{SPAN}^+_k(X; \Phi) \to (\Delta^{op})^{xk} \) is a coCartesian fibration, and the associated functor \( (\Delta^{op})^{xk} \to \text{Cat}_\infty \) is a k-uple Segal object of \( \text{Cat}_\infty \).

Proof. It follows from [Lur09a, Proposition 4.2.2.4] that \( \pi \) is a coCartesian fibration. Using the universal property of \( \text{SPAN}^+_k(X; \Phi) \) we see that a map from \( K \) to the fibre \( \text{SPAN}^+_k(X; \Phi)|_{([n_1], \ldots, [n_k])} \) is naturally equivalent to a map \( K' \to \text{SPAN}^+_k(X)|_{([n_1], \ldots, [n_k])} \) that restricts to \( s_\Phi |_{([n_1], \ldots, [n_k])} \) at the cone point. In other words, we have a natural equivalence

\[
\text{SPAN}^+_k(X; \Phi)|_{([n_1], \ldots, [n_k])} \simeq \text{SPAN}^+_k(X)|_{([n_1], \ldots, [n_k])}|_{s_\Phi([n_1], \ldots, [n_k])} \simeq \text{Fun}^{\text{Cart}}(\Sigma^{n_1, \ldots, n_k}, X)/\Phi \circ \Pi_{n_1, \ldots, n_k}.
\]

Since \( \Phi \circ \Pi_{n_1, \ldots, n_k} \) clearly restricts to \( \Phi \circ \Pi_{n_1, \ldots, 0} \) and \( \Phi \circ \Pi_{0, \ldots, n_k} \) under the appropriate inclusions, it follows that this is a k-uple Segal object.
**Definition 4.6.** Suppose \( \mathcal{C} \) is a \( k \)-fold Segal object in \( \mathcal{X} \). We let \( \text{SPAN}_k(\mathcal{X}; \mathcal{C}) \rightarrow (\Delta^\text{op})^\times k \) denote the left fibration obtained from the coCartesian fibration \( \text{SPAN}_k^+(\mathcal{X}; \mathcal{C}) \rightarrow (\Delta^\text{op})^\times k \) by discarding the non-coCartesian morphisms; this left fibration classifies a \( k \)-uple Segal space. The \((\infty, k)\)-category \( \text{Span}_k(\mathcal{X}; \mathcal{C}) \) of iterated spans in \( \mathcal{X} \) with local systems valued in \( \mathcal{C} \) is the underlying \( k \)-fold Segal space \( U_{\text{seq}}\text{SPAN}_k(\mathcal{X}; \mathcal{C}) \) associated to the \( k \)-uple Segal space \( \text{SPAN}_k(\mathcal{X}; \mathcal{C}) \).

We will now show that \( \text{Span}_k(\mathcal{X}; \mathcal{C}) \) is a complete \( k \)-fold Segal space, provided \( \mathcal{C} \) is a complete \( k \)-fold Segal object in \( \mathcal{X} \). We first consider the case \( k = 1 \), which follows from the following observation:

**Lemma 4.7.** Let \( \mathcal{X} \) be an \( \infty \)-topos and \( \mathcal{C} \) a Segal object in \( \mathcal{X} \). Then there is an equivalence

\[
\text{Span}_1(\mathcal{X}; \mathcal{C})_{\text{eq}} \simeq \prod_{X \in \pi_0 \mathcal{X}} \text{Map}_\mathcal{X}(X, \mathcal{C})_{\text{eq}}.
\]

**Proof.** By definition, \( \text{Span}_1(\mathcal{X}; \mathcal{C})_{\text{eq}} \) is the subspace of \( \text{Span}_1(\mathcal{X}; \mathcal{C})_1 \simeq \text{Fun}(\Sigma^1, \mathcal{C})_{/\text{eq}(\Sigma(1))} \) consisting of those components that correspond to equivalences. The forgetful functor \( \text{Span}_1(\mathcal{X}; \mathcal{C}) \rightarrow \text{Span}_1(\mathcal{X}) \) induces a map \( \text{Span}_1(\mathcal{X}; \mathcal{C})_{\text{eq}} \rightarrow \text{Span}_1(\mathcal{X})_{\text{eq}} \), so by Proposition 3.13 the underlying span of an equivalence is trivial. We may thus identify \( \text{Span}_1(\mathcal{X}; \mathcal{C})_{\text{eq}} \) with a collection of components in \( \prod_{X \in \pi_0 \mathcal{X}} \text{Map}_\mathcal{X}(X, \mathcal{C})_1 \). But it is easy to see from the definition of an equivalence in a Segal space that a map \( X \rightarrow \mathcal{C}_1 \) is an equivalence in \( \text{Span}_1(\mathcal{X}; \mathcal{C}) \) if and only if it is an equivalence when considered as a morphism in \( \text{Map}_\mathcal{X}(X, \mathcal{C}_\bullet) \). \( \square \)

**Proposition 4.8.** Suppose \( \mathcal{X} \) is an \( \infty \)-topos and \( \mathcal{C} \) is a complete Segal object in \( \mathcal{X} \). Then \( \text{Span}_1(\mathcal{X}; \mathcal{C}) \) is a complete Segal space.

**Proof.** By Theorem 2.16 it suffices to show that the degeneracy map \( \text{Span}(\mathcal{X}; \mathcal{C})_0 \rightarrow \text{Span}(\mathcal{X}; \mathcal{C})_{\text{eq}} \) is an equivalence. By Lemma 4.7 we may identify this with the map

\[
\prod_{X} \text{Map}(X, \mathcal{C}_0) \rightarrow \prod_{X} \text{Map}(X, \mathcal{C})_{\text{eq}}.
\]

But by Corollary 2.22 the Segal spaces \( \text{Map}(X, \mathcal{C})_0 \) are complete since \( \mathcal{C} \) is complete, and by Theorem 2.16 it follows that for each \( X \) the map \( \text{Map}(X, \mathcal{C}_0) \rightarrow \text{Map}(X, \mathcal{C})_{\text{eq}} \) is an equivalence. \( \square \)

To extend this to iterated Segal spaces, we first identify the mapping \((\infty, k - 1)\)-categories of \( \text{Span}_k(\mathcal{X}; \mathcal{C})_0 \):

**Proposition 4.9.** Suppose \( \mathcal{C} \) is a \( k \)-fold Segal object in \( \mathcal{X} \), and that \( \xi: X \rightarrow \text{Ob}(\mathcal{C}) \) and \( \eta: Y \rightarrow \text{Ob}(\mathcal{C}) \) are objects of \( \text{Span}_k(\mathcal{X}; \mathcal{C})_0 \). Then the \((\infty, k - 1)\)-category \( \text{Span}_{k-1}(\mathcal{X}; \mathcal{C})_0(\xi, \eta) \) of maps from \( \xi \) to \( \eta \) in \( \text{Span}_k(\mathcal{X}; \mathcal{C}) \) is equivalent to \( \text{Span}_{k-1}(\mathcal{X}; \mathcal{C}_{\xi, \eta})_0 \), where \( \mathcal{C}_{\xi, \eta} \) is the \((k - 1)\)-fold Segal object given by the pullback square

\[
\begin{array}{ccc}
\mathcal{C}_{\xi, \eta} & \rightarrow & \mathcal{C}_1 \\
\downarrow & & \downarrow \\
X \times Y & \rightarrow & \mathcal{C}_0^2.
\end{array}
\]

To prove this, we first make the following observations:

**Lemma 4.10.** The natural map of simplicial sets

\[
(A^q \times B \amalg_{A \times B} A \times B^q)^{\bullet} \rightarrow A^q \times B^q
\]

is an isomorphism for all simplicial sets \( A \) and \( B \).

**Proof.** Since both sides preserve colimits in \( A \) and \( B \), it suffices to consider the case where both are standard simplices. Thus, as \( (\Delta^n)^\bullet \cong \Delta^{n+1} \), we must show that the map

\[
(\Delta^{n+1} \times \Delta^m \amalg_{\Delta^m \times \Delta^n} \Delta^n \times \Delta^m)^\bullet \rightarrow \Delta^{n+1} \times \Delta^{m+1}
\]

is an isomorphism for all simplicial sets \( A \) and \( B \).
is an isomorphism for all $n, m$. Now $\Delta^{n+1} \times \Delta^{m+1}$ is the nerve of the category $[n+1] \times [m+1]$, which is clearly the join (of categories) $[0] \star \mathbf{C}$, where $\mathbf{C}$ is the subcategory spanned by all objects except $(0,0)$. The nerve functor takes the join of categories to the join of simplicial sets, so it follows that $\Delta^{n+1} \times \Delta^{m+1} \simeq (\mathbf{N}C)^{\circ}$. Moreover, the simplicial set $\mathbf{N}C$ is clearly the subcomplex of $\Delta^{n+1} \times \Delta^{m+1}$ containing only the simplices that do not have $(0,0)$ as a vertex, which we can identify with $\Delta^{n+1} \times \Delta^m \sqcup \Delta^n \times \Delta^{m+1}$.

Lemma 4.11. Let $A$ be a simplicial set, and suppose given a functor $\mu: A \times \Delta^1 \sqcup A \rightarrow \mathcal{C}$ with limit $X \in \mathcal{C}$. Then there is a natural pullback diagram

$$
\begin{array}{ccc}
\mathcal{C}_X & \longrightarrow & \text{Fun}(A^q, \mathcal{C}) / \alpha \\
\downarrow & & \downarrow \\
\{v\} & \longrightarrow & \text{Fun}(A, \mathcal{C}) / \beta
\end{array}
$$

where $\alpha = \mu|_{A^q}$, $\beta = \mu|_A$, and $v$ is the object corresponding to $\mu|_{A \times \Delta^1}$.

Proof. Since $X$ is the limit of $\mu$, the $\infty$-category $\mathcal{C}_X$ is equivalent to $\mathcal{C}_/\mu$, which, since

$$(A \times \Delta^1 \sqcup A^q)^{\circ} \simeq A^q \times \Delta^1$$

by Lemma 4.10, fits in a natural pullback square

$$
\begin{array}{ccc}
\mathcal{C}_/\mu & \longrightarrow & \text{Fun}(A^q \times \Delta^1, \mathcal{C}) \\
\downarrow & & \downarrow \\
\{\mu\} & \longrightarrow & \text{Fun}(A \times \Delta^1 \sqcup A^q, \mathcal{C}).
\end{array}
$$

But then $\mathcal{C}_/\mu$ is equivalently the limit of the diagram

which, taking fibres, we can clearly identify with the pullback

$$
\begin{array}{ccc}
\mathcal{C}_/\mu & \longrightarrow & \text{Fun}(A^q, \mathcal{C}) / \alpha \\
\downarrow & & \downarrow \\
\{v\} & \longrightarrow & \text{Fun}(A, \mathcal{C}) / \beta
\end{array}
$$

as required. □
Proof of Proposition 4.9. Using Lemma 4.11, we have natural pullback diagrams

\[
\begin{array}{ccc}
\text{Fun}(\Sigma^{n_1,\ldots,n_{k-1}}, X) \times_{s\mathcal{E},\xi,\eta} (n_1,\ldots,n_{k-1}) & \longrightarrow & \text{Fun}(\Sigma^{1,n_1,\ldots,n_{k-1}}, X) \times_{s\mathcal{E},(1,n_1,\ldots,n_{k-1})} \\
\{ (c_\xi, c_\eta) \} & \longrightarrow & \text{Fun}(\Sigma^{0,n_1,\ldots,n_{k-1}}, X) \times_{s\mathcal{E},(0,n_1,\ldots,n_{k-1})} \\
\end{array}
\]

It is easy to see that these restrict to pullback diagrams

\[
\begin{array}{ccc}
\text{Fun}^\text{Cart}(\Sigma^{n_1,\ldots,n_{k-1}}, X) \times_{s\mathcal{E},\xi,\eta} (n_1,\ldots,n_{k-1}) & \longrightarrow & \text{Fun}^\text{Cart}(\Sigma^{1,n_1,\ldots,n_{k-1}}, X) \times_{s\mathcal{E},(1,n_1,\ldots,n_{k-1})} \\
\{ (c_\xi, c_\eta) \} & \longrightarrow & \text{Fun}^\text{Cart}(\Sigma^{0,n_1,\ldots,n_{k-1}}, X) \times_{s\mathcal{E},(0,n_1,\ldots,n_{k-1})} \\
\end{array}
\]

The functors $U_{\mathcal{E}_{\Sigma}}$ and $\iota$, which takes the underlying space of an $\infty$-category, are right adjoints, and so preserve limits, and it is clear from its definition that for any $n$-uple Segal space $\mathcal{E}$, the $(n - 1)$-fold Segal spaces $U_{\mathcal{E}_{\Sigma}}(\mathcal{E})_{[i]}$ can be identified with the $(n - 1)$-fold Segal space $U_{\mathcal{E}_{\Sigma}}(\mathcal{E})_{[i]}$ extracted from the $(n - 1)$-uple Segal space $\mathcal{E}_{[i]}$. Thus, we have natural pullback squares

\[
\begin{array}{ccc}
\text{Span}_{k-1}(X; \mathcal{E}, \xi, \eta) & \longrightarrow & \text{Span}_k(X; \mathcal{E})_1 \\
\{ (c_\xi, c_\eta) \} & \longrightarrow & \text{Span}_k(X; \mathcal{E})_0^{\times 2,} \\
\end{array}
\]

as required. □

Remark 4.12. In particular, if $X \simeq Y \simeq s$, so that $\xi$ and $\eta$ are determined by two objects $x$ and $y$ of $\mathcal{E}$, then $\text{Span}_k(X; \mathcal{E})(\xi, \eta) \simeq \text{Span}_{k-1}(X; \mathcal{E}(x, y))$.

Corollary 4.13. Suppose $X$ is an $\infty$-topos and $\mathcal{E}$ is a complete $k$-fold Segal object in $X$. Then $\text{Span}_k(X; \mathcal{E})$ is a complete $k$-fold Segal space.

Proof. The case $k = 1$ is Proposition 4.8; we will prove the general case by induction on $k$. Suppose we know the result for $k$-fold Segal objects for all $k < n$. By Theorem 2.35 to show that $\text{Span}_n(X; \mathcal{E})$ is complete it suffices to prove that the Segal space $\text{Span}_n(X; \mathcal{E})_{[0],\ldots,0}$ is complete, and the $(n - 1)$-fold Segal spaces $\text{Span}_n(X; \mathcal{E})(\xi, \eta)$ are complete for all $\xi, \eta$. But $\text{Span}_n(X; \mathcal{E})_{[0],\ldots,0}$ is clearly equivalent to $\text{Span}_n(X; \mathcal{E}_{[0],\ldots,0})$, which we know is complete, and by Proposition 4.9 we can identify $\text{Span}_n(X; \mathcal{E})(\xi, \eta)$ with $\text{Span}_{n-1}(X; \mathcal{E}_{\xi,\eta})$ — this is complete by the inductive hypothesis, since complete $(n - 1)$-fold Segal objects in $X$ are closed under pullback, so $\mathcal{E}_{\xi,\eta}$ is complete. □

5. Adjoints and Duals in Iterated Segal Spaces

In this section we first review the notions of ($\infty$, $k$)-categories with adjoints and (symmetric) monoidal ($\infty$, $k$)-categories with duals from [Lur09c], and then extend these notions to ($\infty$, $k$)-categories internal to an $\infty$-topos. We begin by recalling some key facts about adjunctions in ($\infty$, 2)-categories due to Riehl and Verity:

Definition 5.1. Let $\text{Adj}$ denote the generic adjunction, i.e. the universal 2-category containing an adjunction between two 1-morphisms. An explicit description of $\text{Adj}$ can be found in [RV13, §4]. We will think of $\text{Adj}$ as a 2-fold Segal space via the obvious nerve functor from 2-categories to 2-fold Segal spaces. An adjunction in a (complete) 2-fold Segal space $\mathcal{E}$ is then a map of 2-fold Segal spaces $\text{Adj} \rightarrow \mathcal{E}$. If $\mathcal{E}$ is a complete 2-fold Segal space, we write $\text{Adj}(\mathcal{E}) := \text{Map}(\text{Adj}, \mathcal{E})$ for the space of adjunctions in $\mathcal{E}$. 
Theorem 5.2 ([RV13, Theorem 5.3.9]). Every adjunction in the homotopy 2-category of an $(\infty, 2)$-category extends to an adjunction in the $(\infty, 2)$-category. In particular, a 1-morphism in an $(\infty, 2)$-category has a (left or right) adjoint if and only if it has one in the homotopy 2-category.

Definition 5.3. We write $C_k$ for the $k$-cell, i.e. the generic $k$-morphism, thought of as an $(\infty, n)$-category for any $n > k$. Concretely, it is the representable $n$-fold simplicial object represented by $([1], \ldots, [1], [0], \ldots, [0])$ where $[1]$ occurs $k$ times. If $\mathcal{C}$ is a complete $n$-fold Segal space, we write $\text{Mor}_k(\mathcal{C})$ for the space $\text{Map}(C_k, \mathcal{C})$ of $k$-morphisms in $\mathcal{C}$, i.e. $\mathcal{C}_{1, \ldots, 1, 0, \ldots, 0}$.

Definition 5.4. More or less keeping the notation of [RV13], among the data defining the $(\infty, 2)$-category $\text{Adj}$ we have:

- two objects $+$ and $-$,
- 1-morphisms $f: + \to -$ (the left adjoint) and $g: + \to -$ (the right adjoint),
- 2-morphisms $u: \text{id}_+ \to g f$ (the unit) and $c: f g \to \text{id}_-$ (the counit), satisfying the triangle identities.

Theorem 5.5 ([RV13, Theorem 5.4.22]). Suppose $\mathcal{C}$ is a complete 2-fold Segal space. The maps $f^*$ and $g^*$: $\text{Adj}(\mathcal{C}) \to \text{Mor}_1(\mathcal{C})$ sending an adjunction in $\mathcal{C}$ to the left and right adjoint, respectively, are $(-1)$-connected, i.e. their fibres are either empty or contractible.

Remark 5.6. The results of Riehl and Verity are proved in the context of categories strictly enriched in simplicial sets equipped with the Joyal model structure. Reformulating these theorems in terms of complete 2-fold Segal spaces is justified because these two models of $(\infty, 2)$-categories are equivalent by the unicity theorem of Barwick and Schommer-Pries [BSP11]. (An explicit equivalence can also be obtained by combining Theorem 5.9 and Corollary 7.21 of [Hau13].)

Now we recall what it means for an $(\infty, n)$-category to have adjoints:

Definition 5.7. Suppose $\mathcal{C}$ is a (complete) $n$-fold Segal space with $n > 1$. We say that $\mathcal{C}$ has adjoints for 1-morphisms if every 1-morphism in the homotopy 2-category of $\mathcal{C}$ has a left and a right adjoint. Equivalently, $\mathcal{C}$ has adjoints for 1-morphisms if the maps $f^*, g^*: \text{Adj}(u_{(\infty, 2)}\mathcal{C}) \to \text{Mor}_1(u_{(\infty, 2)}\mathcal{C})$ are both equivalences.

Definition 5.8. Suppose $\mathcal{C}$ is a (complete) $n$-fold Segal space with $n > 1$. For $n > k > 1$ we say that $\mathcal{C}$ has adjoints for $k$-morphisms if for all objects $X, Y$ of $\mathcal{C}$ the $(n - 1)$-fold Segal space $\mathcal{C}(X, Y)$ has adjoints for $(k - 1)$-morphisms. We say that $\mathcal{C}$ has adjoints if it has adjoints for $k$-morphisms for all $k = 1, \ldots, n - 1$.

Remark 5.9. To see that a not necessarily complete $n$-fold Segal space $\mathcal{C}$ has adjoints, it is not necessary to complete it: Whether $\mathcal{C}$ has adjoints for 1-morphisms only depends on the homotopy 2-category, which is easy to describe without completing $\mathcal{C}$. Moreover, the mapping $(n - 1)$-fold Segal spaces in the completion of $\mathcal{C}$ are the completions of the mapping $(n - 1)$-fold Segal spaces of $\mathcal{C}$, so by induction we do not need to complete to see that $\mathcal{C}$ has adjoints for $k$-morphisms also for $k > 1$.

Definition 5.10. A monoidal $n$-fold Segal space is an associative algebra object in $\text{Seg}_n(\mathcal{S})$, in the sense of [Lur14, §4.1.1]. By [Lur14, Proposition 4.1.2.6] these are equivalent to $\Delta^{\text{op}}$-monoids in $\text{Seg}_n(\mathcal{S})$, i.e. $(n + 1)$-fold Segal spaces $\mathcal{C}^{\otimes}: (\Delta^{\text{op}})^{\times(n + 1)} \to \mathcal{S}$ such that $\mathcal{C}^{\otimes}_{0, \ldots, 0, 1} \simeq \mathcal{S}$. We then say that a monoidal $n$-fold Segal space $\mathcal{C}^{\otimes}$ has duals if $\mathcal{C}$ has adjoints when regarded as an $(n + 1)$-fold Segal space.

Definition 5.11. A symmetric monoidal or $E_k$-monoidal $n$-fold Segal space is a commutative algebra or an $E_k$-algebra in $\text{Seg}_n(\mathcal{S})$, respectively. We say a symmetric monoidal or $E_k$-monoidal $n$-fold Segal space has duals if the underlying monoidal $n$-fold Segal space has duals.

Remark 5.12. Combining [Lur14, Theorem 5.1.2.2] and [Lur14, Proposition 4.1.2.6], we may identify $E_k$-monoidal $n$-fold Segal spaces with $E_{k - 1}$-algebras in $\Delta^{\text{op}}$-monoids in $n$-fold Segal spaces.
By induction, we may therefore identify $E_{k}$-monoidal $n$-fold Segal spaces with $(n + k)$-fold Segal spaces $C^\otimes : (\Delta^n)^\times (n + k)$ such that $\text{Ob}(C^\otimes) \cong C^\otimes_{0, 0, \ldots, 0}$, $\text{Mor}_r(C^\otimes) \cong C^\otimes_{1, 0, \ldots, 0}$, ..., $\text{Mor}_n(C^\otimes) \cong C^\otimes_{1, \ldots, 1, 0, \ldots, 0}$ are all points. Given a pointed $(n + k)$-fold Segal space, it is clear that we may extract an $(n + k)$-fold Segal space of this type, which thus gives an $E_k$-monoidal structure on the endomorphisms of the identity $(k - 1)$-morphism of the object picked out by the base point.

**Remark 5.13.** It follows from [Lur14, Corollary 5.1.1.5] that a symmetric monoidal $n$-fold Segal space is equivalent to a sequence $C_1, C_2, \ldots$ where $C_i$ is an $E_i$-monoidal $n$-fold Segal space whose underlying $E_{i-1}$-monoidal $n$-fold Segal space is $C_{i-1}$. Using Remark 5.12, it follows that to exhibit a symmetric monoidal structure on an $n$-fold Segal space $C$ it suffices to find a sequence $(C_1, C_1), (C_2, C_2), \ldots$ where $(C_i, C_i)$ is a pointed $(n + i)$-fold Segal space such that $C_{i-1}$ is equivalent to the mapping $(n + i - 1)$-fold Segal space $C_i((c_i, c_i))$.

**Lemma 5.14.** We may regard a space $X$ as an $n$-fold Segal space for any $n$ by taking the constant functor with value $X$. If $X$ is an associative algebra object in $S$ (or in other words an $A_\infty$-space), then $X$ has duals if and only if $X$ is group-like.

**Proof.** It suffices to check that the homotopy 1-category of $X$, equipped with the induced monoidal structure, has duals. But this is just the fundamental 1-groupoid of $X$, and it is obvious that an object of a monoidal groupoid has a dual if and only if it has an inverse. \hfill $\square$

We now extend these notions to the setting of $(\infty, k)$-categories internal to an $\infty$-topos.

**Definition 5.15.** Suppose $\mathcal{X}$ is an $\infty$-topos, and let

$$r^*_\mathcal{X} : X \Rightarrow r_s : \mathcal{X} : r_s$$

denote the unique geometric morphism from the $\infty$-category of spaces. By Proposition 2.20, this induces an adjunction

$$L_n(r^*_\mathcal{X}) : \text{CSS}_n(\mathcal{S}) \rightleftarrows \text{CSS}(\mathcal{X}) \colon (r_s)^*$$

If $C$ is a complete 2-fold Segal object in $\mathcal{X}$, then an adjunction in $C$ is a functor $(r^*_\mathcal{X}), \text{Adj} \to C$. We write $\text{Adj}(C) \in \mathcal{X}$ for the mapping object $\text{MAP}(r^*_\mathcal{X}, \text{Adj}, C)$ in $\mathcal{X}$, defined in Definition 2.24. Similarly, if $C$ is a complete $k$-fold Segal object in $\mathcal{X}$, we write $\text{Ob}(C) := \text{MAP}(r^*_\mathcal{X}, C_0, C)$ $\cong C_{0, \ldots, 0}$ and $\text{Mor}_n(C) := \text{MAP}(r^*_\mathcal{X}, C_n, C)$ for $n = 1, \ldots, k$.

**Lemma 5.16.** Let $C$ be a complete 2-fold Segal object in an $\infty$-topos $\mathcal{X}$. Then the morphisms $r^*_\mathcal{X}$ and $g^*_\mathcal{X} : \text{Adj}(\mathcal{X}) \to \text{Mor}_1(\mathcal{X})$ are $(-1)$-truncated.

**Proof.** We must show that for any $X \in \mathcal{X}$, the map $\text{Map}_\mathcal{X}(X, \text{Adj}(C)) \to \text{Map}_\mathcal{X}(X, \text{Mor}_1(\mathcal{X}))$ is $(-1)$-truncated. But there is a natural equivalence

$$\text{Map}_\mathcal{X}(X, \text{Adj}(C)) \cong \text{Map}_{\text{CSS}_2(\mathcal{X})}(X \times (r^*_\mathcal{X}), \text{Adj}, C) \cong \text{Map}_{\text{CSS}_2(\mathcal{X})}((r^*_\mathcal{X}), \text{Adj}, C^X) \cong \text{Map}_{\text{CSS}_2(\mathcal{S})}((r_s)^_\mathcal{S}, C^X) \cong \text{Adj}((r_s)^_\mathcal{S}, C^X),$$

and similarly $\text{Map}_\mathcal{X}(X, \text{Mor}_1(\mathcal{X})) \cong \text{Mor}_1((r_s)^_\mathcal{S}, C^X)$. Thus this follows by applying Theorem 5.5 to the complete 2-fold Segal spaces $(r_s)^_\mathcal{S}, C^X$ for all $X \in \mathcal{X}$. \hfill $\square$

**Definition 5.17.** Suppose $C$ is a complete $n$-fold Segal object in $\mathcal{X}$ with $n > 1$. We say that $C$ has adjoints for 1-morphisms if the maps $r^*_\mathcal{X}, g^*_\mathcal{X} : \text{Adj}(u_{(\infty, 2)}(C)) \to \text{Mor}_1(u_{(\infty, 2)}(C))$ are both equivalences.

**Definition 5.18.** Suppose $C$ is a complete $n$-fold Segal object in $\mathcal{X}$ with $n > 1$. For $n > k > 1$ we say that $C$ has adjoints for $k$-morphisms if for all maps $\phi : X \to \text{Ob}(C)^{\times 2}$ in $\mathcal{X}$, the complete $(n - 1)$-fold
Segal object \( \mathcal{C}_p \), defined by the pullback square

\[
\begin{array}{ccc}
\mathcal{C}_p & \longrightarrow & \mathcal{C}_1 \\
\downarrow & & \downarrow \\
X & \longrightarrow & \mathcal{C}_0 \\
\end{array}
\]

in \((k-1)\)-fold Segal objects, has adjoints for \((k-1)\)-morphisms. We say that \( \mathcal{C} \) has adjoints if it has adjoints for \( k \)-morphisms for all \( k = 1, \ldots, n-1 \).

**Definition 5.19.** If \( \mathcal{C} \) is a (not necessarily complete) \( n \)-fold Segal object in \( \mathcal{X} \), we say that \( \mathcal{C} \) has adjoints (for \( k \)-morphisms) if this is true of the completion \( L\mathcal{C} \).

**Definition 5.20.** A monoidal complete \( n \)-fold Segal object in \( \mathcal{X} \) is an associative algebra object in \( \text{CSS}^n(\mathcal{X}) \). As in Definition 5.10, we may identify these with certain \((n+1)\)-fold Segal objects in \( \mathcal{X} \) whose objects are a point. We then say that a monoidal complete \( n \)-fold Segal space \( \mathcal{C} \) has duals if \( \mathcal{C} \) has adjoints (after completion) when regarded as an \((n+1)\)-fold Segal space.

**Definition 5.21.** A symmetric monoidal or \( E_k \)-monoidal complete \( n \)-fold Segal object in \( \mathcal{X} \) is a commutative algebra or an \( E_k \)-algebra in \( \text{CSS}^n(\mathcal{X}) \), respectively. We say a symmetric monoidal or \( E_k \)-monoidal complete \( n \)-fold Segal object has duals if the underlying monoidal complete \( n \)-fold Segal object has duals.

**Remark 5.22.** By the same argument as in Remarks 5.12 and 5.13, we may identify \( E_k \)-monoidal complete \( n \)-fold Segal objects with certain \((n+k)\)-fold complete Segal objects, and symmetric monoidal complete Segal objects with a compatible sequence of these. In particular, after completing as necessary, we see that from a symmetric monoidal complete \( n \)-fold Segal object \( \mathcal{C} \) we obtain a sequence \((\mathcal{C}_i, c_i)\), \((\mathcal{C}_2, c_2), \ldots \) where \((\mathcal{C}_i, c_i)\) is a pointed complete \((n+i)\)-fold Segal object, and \( \mathcal{C}_{i-1} \) is equivalent to the mapping \((n+i-1)\)-fold Segal object \( \mathcal{C}_i(c_i, c_i) \).

We now check that our new definition reduces to the previous one when the \(\infty\)-topos \( \mathcal{X} \) is the \(\infty\)-category \( \mathcal{S} \) of spaces:

**Proposition 5.23.** Suppose \( \mathcal{C} \) is a complete \( n \)-fold Segal space. Then the following are equivalent:

(i) \( \mathcal{C} \) has adjoints for \( k \)-morphisms in the (inductive) sense of Definition 5.18.

(ii) \( \mathcal{C} \) has adjoints for \( k \)-morphisms in the (inductive) sense of Definition 5.8.

The proof depends on the following observation:

**Lemma 5.24.** Suppose given a morphism of \( n \)-fold Segal spaces \( \mathcal{C} \to X \), where \( X \) is constant. If all the fibres \( \mathcal{C}_x \) for \( x \in X \) have adjoints for \( k \)-morphisms in the sense of Definition 5.8, then so does \( \mathcal{C} \).

**Proof.** To prove this, we induct on \( n \). For \( n = 1 \), there is nothing to prove, so we may suppose that the statement is true for \((n-1)\)-fold Segal spaces for all \( k = 1, \ldots, n-1 \).

We first consider the case \( k = 1 \). Since \( \text{Adj}(X) \simeq \text{Mor}_1(X) \simeq X \), we have a commutative diagram

\[
\begin{array}{ccc}
\text{Adj}(\mathcal{C}) & \longrightarrow & \text{Mor}_1(\mathcal{C}) \\
\downarrow & & \downarrow \\
X & & \\
\end{array}
\]

Since the functors \( \text{Adj}(\cdot) \) and \( \text{Mor}_1(\cdot) \) clearly preserve limits, the induced map on fibres over \( p \in X \) can be identified with \( \text{Adj}(\mathcal{C}_p) \to \text{Mor}_1(\mathcal{C}_p) \). By assumption this is an equivalence for all \( p \in X \), and so \( \text{Adj}(\mathcal{C}) \to \text{Mor}_1(\mathcal{C}) \) is also an equivalence, i.e. \( \mathcal{C} \) has adjoints for 1-morphisms.
For $k > 2$, we must show that $\mathcal{C}(c,d)$ has adjoints for $(k - 1)$-morphisms for all $c, d \in \text{Ob}(\mathcal{C})$. But by Lemma 2.39 there is a map $\mathcal{C}(c,d) \to \Omega_{\pi(c),\pi(d)}X$ whose fibres are mapping $(n - 1)$-fold Segal spaces in the fibres of $\pi$, and so have adjoints for $(k - 1)$-morphisms. The result therefore holds by the inductive hypothesis.

Proof of Proposition 5.23. Clearly (ii) is a special case of (i), so we must show that (ii) implies (i). We prove this by induction on $n$. Suppose we know the definitions are equivalent for $i$-fold Segal spaces for all $i < n$, and all $k = 1, \ldots, n - 1$.

The case $k = 1$ is the same in both definitions. For $k > 1$, we must show that if $\mathcal{C}$ has adjoints for $k$-morphisms in the sense of Definition 5.8 then $\mathcal{C}_\phi$ has adjoints for $(k - 1)$-morphisms in the sense of Definition 5.18 for every map $\phi: X \to \text{Ob}(\mathcal{C})^{\times^2}$. By the inductive hypothesis, the two definitions coincide for $(n - 1)$-fold Segal spaces, so it suffices to prove that $\mathcal{C}_\phi$ has adjoints for $(k - 1)$-morphisms in the sense of Definition 5.8. By Lemma 5.24, to see this it suffices to show that the fibres of the map $\mathcal{C}_\phi \to X$ have adjoints for $(k - 1)$-morphisms. The fibre of this map at $p \in X$ is clearly $\mathcal{C}(a,b)$ where $\phi(p) \simeq (a,b)$, so this follows from the assumption that $\mathcal{C}$ has adjoints for $k$-morphisms.

Proposition 5.25. We may regard an object $X \in \mathcal{X}$ as a complete $n$-fold Segal object for any $n$ by taking the constant functor with value $X$. If $X$ is an associative algebra object in $\mathcal{X}$ then $X$ has duals if and only if $X$ is grouplike.

Proof. Write $\mathcal{C}$ for the associative monoid corresponding to $X$, regarded as an $(n + 1)$-fold Segal object in $\mathcal{X}$. Then by Lemma 5.16, the $(n + 1)$-fold Segal object $\mathcal{C}$ has adjoints for 1-morphisms if and only if for every $Y \in \mathcal{X}$ the $(n + 1)$-fold Segal space $(r_*)_* \mathcal{C}^Y$ has adjoints for 1-morphisms. Similarly, $\mathcal{C}$ is a groupoid object if and only if $(r_*)_* \mathcal{C}^Y$ is a groupoid object for all $Y \in \mathcal{X}$. The result therefore follows by Lemma 5.14.

6. Full Dualizability for Iterated Spans

In this section we will show that $\text{Span}_k(\mathcal{C})$ is symmetric monoidal, and that all its objects are fully dualizable — in fact, we will show that $\text{Span}_k(\mathcal{C})$ has duals.

Proposition 6.1. Suppose $\mathcal{C}$ is an $\infty$-category with finite limits. Then the $(\infty,k)$-category $\text{Span}_k(\mathcal{C})$ is symmetric monoidal.

Proof. By Proposition 3.16 we can identify $\text{Span}_k(\mathcal{C})$ with the $(\infty,k)$-category $\text{Span}_{k+1}(\mathcal{C})(*,*)$ of endomorphisms of $*$ in $\text{Span}_{k+1}(\mathcal{C})$. By induction, for every $n \geq 1$ we can thus identify $\text{Span}_k(\mathcal{C})$ with the endomorphism $(\infty,k)$-category in $\text{Span}_{k+n}(\mathcal{C})$ of the identity $(n - 1)$-morphism of the final object $*$ of $\mathcal{C}$. By Remark 5.13 it follows that $\text{Span}_k(\mathcal{C})$ has a symmetric monoidal structure.

Remark 6.2. In the case $k = 1$, an explicit construction of this symmetric monoidal structure has also been carried out in unpublished work of Barwick.

Lemma 6.3. Let $\mathcal{C}$ be an $\infty$-category with finite limits. For all $k \geq 2$, the 1-morphisms in $\text{Span}_k(\mathcal{C})$ have adjoints.

Proof. It suffices to check this in the homotopy 2-category of $\text{Span}_k(\mathcal{C})$. A 1-morphism $\phi: A \to B$ in $\text{Span}_k(\mathcal{C})$ is a span

$$
\begin{array}{ccc}
X & \xleftarrow{f} & B \\
\downarrow & & \downarrow \\
A & \rightarrow & B
\end{array}
$$
We will show that the reversed span $\bar{\phi}$ given by

\[ \begin{array}{c}
X \\
\downarrow \\
A
\end{array} \xleftarrow{\phi} \begin{array}{c}
B \\
\downarrow \\
X
\end{array} \xrightarrow{\eta} \begin{array}{c}
\Delta \\
\downarrow \\
X \times_B X
\end{array} \xrightarrow{\epsilon} \begin{array}{c}
\Delta \\
\downarrow \\
X \times_A X
\end{array} \xrightarrow{\phi} \begin{array}{c}
\Delta \\
\downarrow \\
B
\end{array} \]

is a right adjoint to this, with unit $\eta : \text{id}_A \to \bar{\phi}$ given by the span

\[ \begin{array}{c}
X \\
\downarrow \\
A
\end{array} \xleftarrow{\eta} \begin{array}{c}
X \\
\downarrow \\
X \times_B X
\end{array} \xrightarrow{\epsilon} \begin{array}{c}
\Delta \\
\downarrow \\
X \times_A X
\end{array} \xrightarrow{\epsilon} \begin{array}{c}
\Delta \\
\downarrow \\
B
\end{array} \]

over $A \times A$, and counit $\epsilon : \phi \bar{\phi} \to \text{id}_B$ given by

\[ \begin{array}{c}
\Delta \\
\downarrow \\
X \times_A X
\end{array} \xrightarrow{\epsilon} \begin{array}{c}
\Delta \\
\downarrow \\
B
\end{array} \]

over $B \times B$, where $\Delta$ denotes the relevant diagonal maps. To see this it suffices to check that the triangle equations hold up to homotopy. The 2-morphism $\phi \eta : \phi \to \phi \bar{\phi}$ is given by the span

\[ \begin{array}{c}
\pi_1 \\
\downarrow \\
X
\end{array} \xleftarrow{\phi \eta} \begin{array}{c}
X \times_B X \\
\downarrow \\
X \times_B X \times_A X
\end{array} \xrightarrow{\text{id} \times \Delta} \begin{array}{c}
\text{id} \\
\downarrow \\
X \times_B X \times_A X
\end{array} \]

and $\epsilon \phi$ is given by

\[ \begin{array}{c}
\Delta \times \text{id} \\
\downarrow \\
X \times_B X \times_A X
\end{array} \xrightarrow{\epsilon \phi} \begin{array}{c}
\text{id} \\
\downarrow \\
X
\end{array} \xrightarrow{\pi_2} \begin{array}{c}
X \times_B X \times_A X \\
\downarrow \\
X
\end{array} \]

The composite $\phi \to \phi$ of these two maps is therefore given by the pullback

\[ (X \times_B X) \times_{(X \times_B X \times_A X)} (X \times_A X) \]

But this pullback can also be identified with the limit of the diagram

\[ \begin{array}{c}
X \\
\downarrow \\
A
\end{array} \xleftarrow{\text{id}} \begin{array}{c}
X \\
\downarrow \\
X
\end{array} \xrightarrow{\text{id}} \begin{array}{c}
X \\
\downarrow \\
X
\end{array} \xrightarrow{\text{id}} \begin{array}{c}
X \\
\downarrow \\
B
\end{array} \]

which can clearly be identified with $X$. Thus $(\epsilon \phi) \circ (\phi \eta) \simeq \text{id}_\phi$, and the other triangle equivalence, $(\phi \epsilon) \circ (\eta \phi) \simeq \text{id}_\phi$, is proved similarly. □

**Theorem 6.4.** The $(\infty, k)$-category $\text{Span}_k(\mathcal{C})$ has adjoints.

**Proof.** We prove this by induction on $k$. For $k = 1$, there is nothing to prove. Suppose we have shown that for all $\mathcal{C}$ the $(\infty, k - 1)$-category $\text{Span}_{k-1}(\mathcal{C})$ has adjoints. We saw in Lemma 6.3 that $\text{Span}_k(\mathcal{C})$ has adjoints for 1-morphisms, and for every pair $X, Y$ of objects in $\mathcal{C}$ the $(\infty, k - 1)$-category $\text{Span}_{k}(\mathcal{C})(X, Y)$ can be identified with $\text{Span}_{k-1}(\mathcal{C}/_{X \times Y})$ by Proposition 3.16, and so has adjoints by the inductive hypothesis. Thus $\text{Span}_k(\mathcal{C})$ also has adjoints. □
Corollary 6.5. The (symmetric) monoidal \((\infty,k)\)-category \(\text{Span}_k(\mathcal{C})\) has duals.

Proof. As a monoidal \((\infty,k)\)-category, we may identify \(\text{Span}_k(\mathcal{C})\) with the endomorphism \((\infty,k)\)-category \(\text{Span}_{k+1}(\mathcal{C})(\ast,\ast)\). Since \(\text{Span}_{k+1}(\mathcal{C})\) has adjoints, it follows that \(\text{Span}_k(\mathcal{C})\) has duals. □

Invoking the cobordism hypothesis, we get:

Corollary 6.6. Suppose \(\mathcal{C}\) is an \(\infty\)-category with finite limits. Then every object \(C \in \mathcal{C}\) defines a framed \(k\)-dimensional TQFT \(Z^k_C : \text{Bord}^k \rightarrow \text{Span}_k(\mathcal{C})\), where \(\text{Bord}^k\) denotes the \((\infty,k)\)-category of framed cobordisms.

Remark 6.7. In fact, these framed TQFTs can all naturally be extended to unoriented TQFTs. To see this, we will sketch a description of the TQFT \(Z^k_C\) valued in \(\text{Span}_k(\mathcal{C})\) associated to an object \(C \in \mathcal{C}\):

1. If \(\mathcal{D}\) is an \(\infty\)-category with finite colimits, let \(\text{Cosp}_k(\mathcal{D})\) denote \(\text{Span}_k(\mathcal{D}^{\text{op}})\).

2. Let \(\text{Bord}^k_{\text{fin}}\) denote the \((\infty,k)\)-category of unoriented bordisms, and \(\mathcal{S}^{\text{fin}}\) the full subcategory of \(\mathcal{S}\) spanned by the finite complexes. There is a symmetric monoidal “forgetful functor” \(\text{Bord}^k_{\text{fin}} \rightarrow \text{Cosp}_k(\mathcal{S}^{\text{fin}})\) that sends a cobordism to its underlying \(\infty\)-groupoid together with the inclusions of the boundary, etc.

3. If \(\mathcal{C}\) is an \(\infty\)-category with finite limits, then \(\mathcal{C}\) is cotensored over \(\mathcal{S}^{\text{fin}}\). Thus given \(C \in \mathcal{C}\) there is a functor \(C(-) : (\mathcal{S}^{\text{fin}})^{\text{op}} \rightarrow \mathcal{C}\). Since \(\text{Span}_k(-)\) is natural in limit-preserving functors, this induces a functor \(C(-) : \text{Cosp}_k(\mathcal{S}^{\text{fin}}) \rightarrow \text{Span}_k(\mathcal{C})\) for all \(k\).

4. Identifying \(\text{Span}_k(\mathcal{C})\) as an endomorphism \((\infty,k)\)-category in \(\text{Span}_{k+n}(\mathcal{C})\) for all \(n\), we conclude that the functor \(C(-)\) is \(E_n\)-monoidal for all \(n\), hence symmetric monoidal.

5. Composing, we get a symmetric monoidal functor \(\hat{Z}^k_C : \text{Bord}^k_{\text{fin}} \rightarrow \text{Span}_k(\mathcal{C})\) that sends a cobordism \(X\) to \(C^X\) and the iterated span coming from the iterated boundary of \(X\).

6. By the cobordism hypothesis, the composite of \(\hat{Z}^k_C\) with the forgetful functor \(\text{Bord}^k_{\text{fin}} \rightarrow \text{Bord}^k\) is the unique symmetric monoidal functor \(\text{Bord}^k_{\text{fin}} \rightarrow \text{Span}_k(\mathcal{C})\) that sends the point to \(C\), hence it must be equivalent to \(Z^k_C\).

7. In particular, \(Z^k_C\) extends to the unoriented TQFT \(\hat{Z}^k_C\).

Note, however, that \(\hat{Z}^k_C\) is not the unique unoriented TQFT extending \(Z^k_C\).

7. Full Dualizability for Iterated Spans with Local Systems

We will now prove that if \(\mathcal{C}\) is a symmetric monoidal complete \(k\)-fold Segal object in a \(\infty\)-topos \(\mathcal{X}\), then the \((\infty,k)\)-category \(\text{Span}_k(\mathcal{X};\mathcal{C})\) is symmetric monoidal, and it has duals provided \(\mathcal{C}\) does.

Proposition 7.1. Suppose \(\mathcal{C}\) is a symmetric monoidal complete \(k\)-fold Segal object of \(\mathcal{X}\). Then the \((\infty,k)\)-category \(\text{Span}_k(\mathcal{X};\mathcal{C})\) is symmetric monoidal.

Proof. Since \(\mathcal{C}\) is symmetric monoidal, by Remark 5.22 we can choose a sequence of “deloopings”, i.e. pointed \((k+i)\)-fold complete Segal objects \((\mathcal{E}_i,\mathcal{C}_i)\) such that \(\mathcal{E}_0 = \mathcal{C}\) and \(\mathcal{E}_i \simeq \mathcal{E}_{i+1}(\mathcal{C}_{i+1},\mathcal{C}_{i+1})\). By Proposition 4.9, we can then identify \(\text{Span}_{k+i}(\mathcal{X};\mathcal{C}_i)\) with the mapping \((\infty,k+i)\)-category

\[
\text{Span}_{k+i+1}(\mathcal{X};\mathcal{E}_{i+1})(x_{i+1},x_{i+1})
\]

in \(\text{Span}_{k+i+1}(\mathcal{X};\mathcal{E}_{i+1})\), where the object \(x_{i+1}\) is the map \(* \rightarrow \text{Ob}(\mathcal{E}_{i+1})\) corresponding to the object \(\mathcal{C}_{i+1}\). Thus by Remark 5.13 it follows that \(\text{Span}_k(\mathcal{C})\) has a symmetric monoidal structure. □

Proposition 7.2. Suppose \(\mathcal{C}\) is a complete \(k\)-fold Segal object in \(\mathcal{X}\) that has adjoints for \(1\)-morphisms. Then \(\text{Span}_k(\mathcal{X};\mathcal{C})\) has adjoints for \(1\)-morphisms.

Proof. Suppose given a \(1\)-morphism in \(\text{Span}_k(\mathcal{X};\mathcal{C})\), i.e. a span \(A \leftarrow X \rightarrow B\) in \(\mathcal{X}\) equipped with a map to the span \(\text{Ob}(\mathcal{C}) \leftarrow \text{Mor}_1(\mathcal{C}) \rightarrow \text{Ob}(\mathcal{C})\). We will show that a right adjoint to this morphism is given by \(B \leftrightarrow X \rightarrow A\), now with \(X\) equipped with the map

\[
X \rightarrow \text{Mor}_1(\mathcal{C}) \xrightarrow{(f^*)^{-1}} \text{Adj}(\mathcal{C}) \xrightarrow{g^*} \text{Mor}_1(\mathcal{C}),
\]
The triangle identities for the adjunction then follow by combining the proof of Lemma 6.3 with the homotopies coming from the triangle identities for the generic adjunction. Thus all 1-morphisms have adjoints such that \( \xi \circ \eta = 1_{\mathcal{C}} \) and \( \xi \) has adjoints for \( \mathcal{C} \rightarrow \mathcal{C} \). To see that they also have left adjoints, we simply interchange the roles of the morphisms \( f \) and \( g \) above.

**Theorem 7.3.** Suppose \( \mathcal{C} \) is a complete k-fold Segal object in \( \mathcal{X} \) that has adjoints. Then \( \text{Span}_k(\mathcal{X}; \mathcal{C}) \) has adjoints.

**Proof.** We will show that if \( \mathcal{C} \) has adjoints for i-morphisms then \( \text{Span}_k(\mathcal{X}; \mathcal{C}) \) also has adjoints for i-morphisms. The case \( i = 1 \) was proved in Proposition 7.2. Suppose \( i > 1 \), then we must show that \( \text{Span}_k(\mathcal{X}; \mathcal{C})(\xi, \eta) \) has adjoints for \( (i - 1) \)-morphisms for all \( \xi, \eta \in \text{Span}_k(\mathcal{X}; \mathcal{C}) \). By Proposition 4.9, this \( (k - 1) \)-fold Segal space can be identified with \( \text{Span}_{k-1}(\mathcal{X}; \mathcal{C}^{\otimes}, \eta) \), and by definition \( \mathcal{C}^{\otimes} \) has adjoints for \( (i - 1) \)-morphisms if \( \mathcal{C} \) has adjoints for i-morphisms. Thus by induction we see that \( \text{Span}_k(\mathcal{X}; \mathcal{C}) \) has adjoints for i-morphisms.

**Corollary 7.4.** Suppose \( \mathcal{C} \) is a (symmetric) monoidal complete k-fold Segal object in \( \mathcal{X} \) that has duals. Then the (symmetric) monoidal \((\infty, k)\)-category \( \text{Span}_k(\mathcal{X}; \mathcal{C}) \) also has duals.

**Proof.** Since \( \mathcal{C} \) is monoidal, by Definition 5.21 there is a pointed complete \((n+1)\)-fold Segal object \( (\mathcal{C}^{\otimes}, \ast) \) with adjoints such that \( \mathcal{C} \) is the endomorphism \( n \)-fold Segal object \( \mathcal{C}^{\otimes}(\ast, 
\ast) \). Then \( \text{Span}_k(\mathcal{X}; \mathcal{C}) \) is the endomorphism \((\infty, n)\)-category \( \text{Span}_k(\mathcal{X}; \mathcal{C}^{\otimes})(x, x) \), where \( x \) is the object \( \ast \rightarrow \text{Ob}(\mathcal{C}^{\otimes}) \) corresponding to the base point. By Theorem 7.3 the \((\infty, n+1)\)-category \( \text{Span}_k(\mathcal{X}; \mathcal{C}^{\otimes}) \) has adjoints, hence so does the monoidal \((\infty, n)\)-category \( \text{Span}_k(\mathcal{X}; \mathcal{C}) \).

Invoking the cobordism hypothesis, we get:

**Corollary 7.5.** Suppose \( \mathcal{C} \) is a symmetric monoidal complete k-fold Segal object in \( \mathcal{X} \) that has duals. Every morphism \( \phi: X \rightarrow \text{Ob}(\mathcal{C}) \) in \( \mathcal{X} \) defines a framed k-dimensional TQFT \( \mathcal{Z}_k^{fr}: \text{Bord}_k^{fr} \rightarrow \text{Span}_k(\mathcal{X}; \mathcal{C}) \), where \( \text{Bord}_k^{fr} \) denotes the \((\infty, k)\)-category of framed bordisms.

**Example 7.6.** Suppose \( A \) is a grouplike \( E_\infty \)-algebra in an \( \infty \)-topos \( \mathcal{X} \). Then by Proposition 5.25 we may regard \( A \) as a symmetric monoidal internal \((\infty, n)\)-category with duals in \( \mathcal{X} \) for any \( n \), and so we get for every \( n \) a symmetric monoidal \((\infty, n)\)-category \( \text{Span}_n(\mathcal{X}; A) \). The underlying \((\infty, n)\)-category of this is just \( \text{Span}_n(\mathcal{X}/A) \), but the symmetric monoidal structure is not that coming from the Cartesian product in \( \mathcal{X}/A \) (i.e. the fibre product over \( A \)); instead, the tensor product of two maps \( X, Y \rightarrow A \) is the product \( X \times Y \) equipped with the composite map \( X \times Y \rightarrow A \times A \rightarrow A \) where the second map is the multiplication in \( A \). Similarly, the unit for the symmetric monoidal structure is the unit map \( \ast \rightarrow A \), and the dual of an object \( X \rightarrow A \) is the composite \( X \rightarrow A \rightarrow A \) where the second map is the inverse mapping for \( A \).

8. The \((\infty, 1)\)-Category of Lagrangian Correspondences

In this section we will use the theory of symplectic derived stacks and Lagrangian morphisms developed by Panetev, Toën, Vaquié and Vezzosi [PTVV13] to construct an \( \infty \)-category \( \text{Lag}^n \) of \( n \)-symplectic derived stacks, with morphisms given by Lagrangian correspondences between them. We begin by briefly recalling their setup for derived stacks and (closed) \( p \)-forms; for full details we refer to [TV08] and [PTVV13].
Definition 8.1. Let \(k\) be a field of characteristic 0. We write \(\text{cdga}_k^{\leq 0}\) for the category of non-positively graded commutative differential graded \(k\)-algebras, equipped with the usual model structure, and \(\text{dAff}_k^{\text{op}}\) for the associated \(\infty\)-category. We may equip this \(\infty\)-category with an étale topology, as described in [TV08], and we write \(\text{dSt}_k := \text{Sh}_{\text{ét}}(\text{dAff}_k)\) for the associated \(\infty\)-topos of étale sheaves of spaces.

Remark 8.2. It is not necessary to take \(k\) to be a field of characteristic zero. However, for more general rings commutative differential graded algebras are often not the most appropriate notion of “derived rings”, the more useful notions being simplicial commutative algebras and (connective) \(E_\infty\)-algebras. For a field of characteristic zero, however, all three notions coincide.

Remark 8.3. For many purposes we do not want to consider arbitrary objects of \(\text{dSt}_k\), but only some subclass of “geometric” objects. The notion of derived Artin stack provides a good notion of such geometric stacks; roughly speaking they are derived stacks obtained as iterated realizations of smooth groupoids — see [TV08] for details. We write \(\text{dSt}_k^{\text{Art}}\) for the full subcategory of \(\text{dSt}_k\) spanned by the derived Artin stacks — by [TV08, Corollary 1.3.3.5] this is closed under finite limits in \(\text{dSt}_k\).

Definition 8.4. In [PTV13], functors \(\Omega^p\) and \(\Omega^p_{\text{cl}}\) from \(\text{dAff}_k^{\text{op}}\) to the \(\infty\)-category of cochain complexes that take a derived affine scheme to its complex of \(p\)-forms and closed \(p\)-forms, respectively, are constructed and shown to be étale sheaves. Extending by taking colimits and regarding cochain complexes as modules over the Eilenberg-MacLane ring spectrum \(Hk\), we may regard these as sheaves of spectra on \(\text{dSt}_k\). We let \(\mathcal{A}^p[n] := \Omega^p \Sigma^n \Omega^p\) and \(\mathcal{A}^p_{\text{cl}}[n] := \Omega^p \Sigma^n \Omega^p_{\text{cl}}\) denote the derived stacks (i.e. sheaves of spaces on \(\text{dSt}_k\)) obtained by delooping shifts of these sheaves of spectra (for all \(n \in \mathbb{Z}\)). If \(X\) is a derived stack, we refer to \(\mathcal{A}^p[n](X)\) as the space of \(n\)-shifted \(p\)-forms on \(X\). There is a “forgetful” map \(\Omega^p_{\text{cl}} \to \Omega^p\), which induces natural transformations \(\mathcal{A}^p_{\text{cl}}[n] \to \mathcal{A}^p[n]\), but the components are not in general monomorphisms (i.e. inclusions of a subset of the connected components).

Remark 8.5. Since \(\mathcal{A}^p_{\text{cl}}[n]\) comes from a sheaf of spectra on \(\text{dSt}_k\), we may regard it as a sheaf of group-like \(E_\infty\)-spaces, i.e. a grouplike \(E_\infty\)-monoid in \(\text{dSt}_k\). Thus by Example 7.6 there is a symmetric monoidal (\(\infty, k\))-category \(\text{Span}_k(\text{dSt}_k; \mathcal{A}^p_{\text{cl}}[n])\) with duals for all \(p, n\).

Definition 8.6. If \(X\) is a derived Artin stack, an \(n\)-shifted 2-form \(\omega \in \mathcal{A}^2[n](X)\) corresponds to a morphism \(T_X \to \mathbb{L}_X[n]\) of quasi-coherent sheaves on \(X\), where \(\mathbb{L}_X\) is the cotangent complex and \(T_X\) is its dual, the tangent complex. We say that \(\omega\) is non-degenerate if this morphism is an equivalence, and write \(\mathcal{A}^2_{\text{nd}}[n](X)\) for the collection of components of \(\mathcal{A}^2[n](X)\) corresponding to the non-degenerate \(n\)-shifted 2-forms.

Definition 8.7. An \(n\)-shifted symplectic form on a derived Artin stack \(X\) is a non-degenerate closed 2-form, i.e. an element of the pullback

\[
\text{Sympl}_n(X) := \mathcal{A}^2_{\text{cl}}[n](X) \times_{\mathcal{A}^2[n](X)} \mathcal{A}^2_{\text{nd}}[n](X),
\]

which is a subset of the connected components of \(\mathcal{A}^2_{\text{cl}}[n](X)\). An \(n\)-symplectic derived Artin stack \((X, \omega)\) is a derived Artin stack \(X\) equipped with an \(n\)-shifted symplectic form \(\omega\).

Definition 8.8. Suppose \(X\) is a derived stack, and \(\omega\) is an \(n\)-shifted closed 2-form on \(X\). If \(f: L \to X\) is a morphism of derived stacks, then an \(\omega\)-isotropic structure on \(f\) is a commutative square

\[
\begin{array}{ccc}
L & \xrightarrow{f} & X \\
\downarrow & & \downarrow \omega \\
\ast & \xrightarrow{0} & \mathcal{A}^2_{\text{cl}}[n],
\end{array}
\]
Equivalently, it is a path from $0$ to the composite closed $n$-shifted $2$-form $f^*\omega$ in $\mathcal{A}_d^2[n](L)$.

**Definition 8.9.** Suppose $(X,\omega)$ is an $n$-symplectic derived Artin stack, and $f: L \to X$ is a morphism of derived Artin stacks. Then an isotropic structure on $f$ induces a commutative square (cf. [PTVV13, §2.2] for the details)

\[
\begin{array}{ccc}
\mathcal{T}_L & \xrightarrow{f^*} & \mathcal{T}_X \\
\downarrow & & \downarrow \\
0 & \xrightarrow{\mathbb{I}_L[n]} & \mathbb{I}_L[n]
\end{array}
\]

of quasi-coherent sheaves on $L$. We say that the isotropic structure is Lagrangian if this square is Cartesian.

**Definition 8.10.** Suppose $X$ and $Y$ are $n$-shifted symplectic derived Artin stacks. A span

\[
\begin{array}{ccc}
f & \xrightarrow{L} & X \\
\downarrow & & \downarrow \\
Y & \xrightarrow{g} & Y
\end{array}
\]

in $(\text{dSt}^\text{Art}_{k}/\mathcal{A}_d^2[n])$ induces a commutative square

\[
\begin{array}{ccc}
\mathcal{T}_L & \xrightarrow{f^*} & \mathcal{T}_X \\
\downarrow & & \downarrow \\
\mathcal{S}^*\mathcal{T}_Y & \xrightarrow{\mathbb{I}_L[n]} & \mathbb{I}_L[n]
\end{array}
\]

of quasi-coherent sheaves on $L$. We say the span is a Lagrangian correspondence if this square is Cartesian.

**Theorem 8.11** ([Cal13, Theorem 4.4]). Suppose $X$, $Y$, and $Z$ are $n$-symplectic derived Artin stacks, and $X \xleftarrow{f} K \xrightarrow{g} Y$ and $Y \xleftarrow{h} L \xrightarrow{k} Z$ are Lagrangian correspondences. Then the composite span

\[
\begin{array}{ccc}
K \times_Y L & \xrightarrow{k} & Z \\
\downarrow & & \downarrow \\
X & \xrightarrow{f} & Y
\end{array}
\]

is also a Lagrangian correspondence.

**Proof.** A Cartesian square

\[
\begin{array}{ccc}
N & \xrightarrow{\phi} & K \\
\downarrow \phi & & \downarrow \\
L & \xrightarrow{h} & Y
\end{array}
\]
induces a commutative diagram

\[
\begin{array}{ccc}
T_N & \to & \phi^* T_K & \to & \phi^* f^* T_X \\
\downarrow & & \downarrow & & \downarrow \\
\psi^* T_L & \to & \phi^* \gamma^* T_Y & \to & \phi^* L_K [n] \\
\downarrow & & \downarrow & & \downarrow \\
\psi^* k^* T_Z & \to & \psi^* L_L [n] & \to & L_N [n].
\end{array}
\]

Here the upper right square is Cartesian since \(X \leftarrow K \to Y\) is a Lagrangian correspondence, and the bottom left square is Cartesian since \(Y \leftarrow L \to Z\) is a Lagrangian correspondence. The top left square is Cartesian since \(N\) is the fibre product of \(K\) and \(L\) over \(Y\). Finally, since \(Y\) is symplectic, we have an equivalence \(\phi^* \gamma^* T_Y \simeq \phi^* \gamma^* L_Y [n]\), and so we can identify the bottom right square with a shift of the dual of the top left square. The bottom right square is therefore coCartesian, but we’re in a stable \(\infty\)-category so coCartesian and Cartesian squares coincide. Thus the boundary square in the diagram is also Cartesian, which by definition means that \(X \leftarrow N \to Z\) is a Lagrangian correspondence. \(\square\)

**Definition 8.12.** By Theorem 8.11, Lagrangian correspondences are closed under composition in the homotopy category of \(\text{Span}_1 (\text{dSt}_k; A^2_{G_1} [n])\). We can therefore define the \(\infty\)-category \(\text{Lag}_{\text{dSt}_k (\infty, 1)}\) to be the subcategory of \(\text{Span}_1 (\text{dSt}_k; A^2_{G_1} [n])\) whose objects are the \(n\)-symplectic derived Artin stacks and whose 1-morphisms are the Lagrangian correspondences between these.

**Remark 8.13.** The idea of considering symplectic derived stacks and Lagrangian correspondences as a subcategory of \(\text{Span}_1 (\text{dSt}_k; A^2_{G_1} [n])\) is taken from [Sch14b].

**Lemma 8.14.** The symmetric monoidal structure on \(\text{Span}_1 (\text{dSt}_k; A^2_{G_1} [n])\) induces a symmetric monoidal structure on \(\text{Lag}_{\text{dSt}_k (\infty, 1)}\).

**Proof.** To show that \(\text{Lag}_{\text{dSt}_k (\infty, 1)}\) inherits a symmetric monoidal structure, it suffices to prove that it contains the unit of the symmetric monoidal structure on \(\text{Span}_1 (\text{dSt}_k; A^2_{G_1} [n])\), and that its objects and morphisms are closed under this. The unit is the map \(* \to A^2_{G_1} [n]\) corresponding to 0, which is obviously symplectic. If \(X\) and \(Y\) are \(n\)-symplectic derived Artin stacks, then their tensor product is the Cartesian product \(X \times Y\) equipped with the sum of the symplectic forms on \(X\) and \(Y\), which is again symplectic. Finally, the tensor product of two Lagrangian correspondences is again their Cartesian product, which is Lagrangian with respect to the sum symplectic structures (since we just get the direct sum of the two Cartesian squares of quasi-coherent sheaves, which is again Cartesian). \(\square\)

**Proposition 8.15.** With respect to the induced symmetric monoidal structure, all \(n\)-symplectic derived Artin stacks are dualizable in \(\text{Lag}_{\text{dSt}_k (\infty, 1)}\).

**Proof.** Let \((X, \omega)\) be an \(n\)-symplectic derived Artin stack. We must show that the dual of \(X\) is also an \(n\)-symplectic derived Artin stack, and the evaluation and coevaluation maps, as described in the proof of Proposition 7.2, are Lagrangian correspondences. By Example 7.6 the dual of \(X\) is \(\overline{X}\), meaning \(X\) equipped with the negative \(-\omega\) of its symplectic form. This is again symplectic, as the morphism \(T_X \to L_X [n]\) induced by \(-\omega\) is simply the negative of that induced by \(\omega\), and so is also an equivalence. The coevaluation map is given by the span \(* \leftarrow X \xrightarrow{\Delta} X \times X\), where \(\Delta\) is the diagonal and \(X \times X\) is equipped with the sum symplectic structure \((-\omega, \omega)\). The induced diagram
of quasi-coherent sheaves on $X$ is

$$
\begin{array}{ccl}
T_X & \longrightarrow & T_X \oplus T_X \\
\downarrow & & \downarrow \\
0 & \longrightarrow & L_X[n],
\end{array}
$$

where the top horizontal map is $(-\operatorname{id}, \operatorname{id})$. This is Cartesian if and only if the square

$$
\begin{array}{ccl}
T_X & \longrightarrow & T_X \\
\downarrow & & \downarrow \\
T_X & \longrightarrow & L_X[n]
\end{array}
$$

is Cartesian, where the top horizontal and left vertical maps are both the identity, but this is true since $X$ is symplectic as this means the other two maps, which are also identical, are equivalences. Thus this span is a Lagrangian correspondence. Similarly, the evaluation map $X \times X \leftarrow \Delta X \rightarrow \ast$ is likewise a Lagrangian correspondence. □

**Corollary 8.16.** Every $n$-symplectic derived Artin stack $X$ determines a framed $1$-dimensional TQFT $Z_X : \text{Bord}^\text{fr}_1 \to \text{Lag}^n_{(\infty,1)}$.

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