Determination of time-dependent coefficients for a hyperbolic inverse problem

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Abstract
For the time-dependent vector and scalar potentials \((A_0, \ldots, A_n)\) and \(V(t, x)\) respectively, the inverse boundary value problem for the hyperbolic partial differential equation
\[
(-i\partial_t + A_0(t, x))^2 u(t, x) - \sum_{j=1}^n (-i\partial_{x_j} + A_j(t, x))^2 u(t, x) + V(t, x)u(t, x) = 0
\]
is studied on a bounded and smooth cylindric domain \((\mathbb{R} \times \Omega)\). Using a geometric optics construction, it is shown that the boundary data allow for the recovery of integrals of the potentials along 'light rays'. The uniqueness of these potentials modulo a gauge transform is also established.

(Some figures may appear in colour only in the online journal)

1. Introduction
Let \(\Omega\) be a bounded simply connected domain in \(\mathbb{R}^n\) with \(n \geq 2\) and consider the hyperbolic equation with time-dependent coefficients
\[
(-i\partial_t + A_0(t, x))^2 u - \sum_{j=1}^n (-i\partial_{x_j} + A_j(t, x))^2 u + V(t, x)u = 0 \quad \text{in } \mathbb{R} \times \Omega,
\]
where \(V(t, x), A_j(t, x), 0 \leq j \leq n,\) are smooth functions vanishing when \(|x| > R\) for some \(R > 0.\) The smooth vector field \(A(t, x) = (A_0(t, x), \ldots, A_n(t, x))\) is called the vector potential and the function \(V(t, x)\) is called the scalar potential. Equation (1) is often referred to as the relativistic Schrödinger equation (see [16]).

For the above differential equation, the following initial and boundary conditions are imposed:
\[
u(t, x) = \partial_t u(t, x) = 0 \quad \text{for } t \ll 0
\]
\[
u(t, x) = f(t, x) \quad \text{on } \mathbb{R} \times \partial \Omega,
\]
where \(f\) is a compactly supported smooth function on \(\mathbb{R} \times \partial \Omega.\) Solutions to (1) satisfying (2) and (3) are unique and the Dirichlet to Neumann operator \(\Lambda\) is defined by
\[
\Lambda(f) := (\partial_v + iA(t, x) \cdot v)u(t, x)|_{\mathbb{R} \times \partial \Omega}
\]
where \( u \) is the solution of (1)–(3), \( v \) is the exterior unit normal to \( \partial \Omega \) and \( A(t, x) = (A_1(t, x), \ldots, A_n(t, x)) \). The inverse boundary value problem is the recovery of \( A(t, x) \) and \( V(t, x) \) knowing \( \Lambda(f) \) for all \( f \in C_0^\infty(\mathbb{R} \times \partial \Omega) \).

Inverse problems is a topic in mathematics that has been attracting a growing interest over the past decades, in part, due to its wide range of applications, from medicine to acoustics and electromagnetism, just to mention a few (see for instance [10] for some of the latest tools and techniques employed in the solutions of these problems). In the case of the hyperbolic inverse boundary value problem (1)–(4) with time-independent coefficients, a powerful tool called the boundary control method, or BC method for short, was discovered by Belishev (see [1]). It was later developed by Belishev, Kurylev, Lassas and others [11, 12], and more recently, a new approach to this problem based on the BC method was developed by Eskin for vector potentials in [2, 3]. On a similar note, Stefanov and Uhlmann established uniqueness and stability results for the wave equation in anisotropic media (see [18, 21] for a survey of these results).

Nevertheless, the case of time-dependent coefficients has seen very little progress in recent years. In the case of the vector potential being identically equal to zero \((A \equiv 0 \text{ in } (1))\), Stefanov [17] and Ramm–Sjöstrand [14] have shown that the Dirichlet to Neumann map completely determines the scalar potentials. More recently, Eskin [4] considered the case of time-dependent potentials that are analytic in time. The analyticity of the time variable is related to the use of a unique continuation theorem established by Tataru in [20]. In this paper, the restriction on the analyticity in the time variable is eliminated.

This work is structured as follows. In section 2, geometric optics solutions (GO for short) for equation (1) satisfying the set of initial conditions (2) are constructed, and a Green formula is also established. It is then shown that the light ray transforms of gauge-equivalent potentials agree. In section 3, the uniqueness of these potentials modulo a gauge transform is proven.

### 2. Construction of geometric optics solutions

**Definition 2.1.** The vector and scalar potentials \((A(t, x), V(t, x))\) and \((A'(t, x), V'(t, x))\) are said to be gauge equivalent if there exists \( g(t, x) \in C^\infty(\mathbb{R} \times \overline{\Omega}) \) such that \( g(t, x) \neq 0 \) on \( \mathbb{R} \times \partial \Omega \), \( g = 1 \) on \( \mathbb{R} \times \partial \Omega \) and

\[
A'(t, x) = A(t, x) - \frac{i}{g(t, x)} \nabla_{t,x} g(t, x)
\]

\[
V'(t, x) = V(t, x),
\]

where \( \nabla_{t,x} := (\partial_t, \partial_x) = (\partial_t, \partial_{x_1}, \ldots, \partial_{x_n}) \) is the \((n + 1)\)-dimensional gradient. The mapping \((A, V) \mapsto (A', V')\) is called a gauge transform.

When \( \Omega \) is simply connected, the gauge \( g \) has the particular form \( g(t, x) = e^{i\varphi(t, x)} \) where \( \varphi(t, x) \in C^\infty(\mathbb{R} \times \overline{\Omega}) \). Then \( -\frac{1}{g(t, x)} \nabla_{t,x} g(t, x) = \nabla_{t,x} \varphi(t, x) \) and two vector potentials are gauge equivalent if their difference is the gradient of a smooth function. The following is a well-known result stating that the recovery of these potentials is only possible up to a gauge transform.

**Proposition 2.1.** If \( u(t, x) \) is a solution of (1)–(3) and \( g(t, x) \) is as in definition (2.1), then \( v(t, x) = g(t, x)u(t, x) \) satisfies

\[
(-i\partial_t + A_0(t, x))^2 v - \sum_{j=1}^n (-i\partial_{x_j} + A_j(t, x))^2 v + V'(t, x)v = 0 \quad \text{in } \mathbb{R} \times \Omega
\]

\[
v = \partial_t v = 0 \quad \text{for } t \ll 0
\]

\[
v = fg|_{\mathbb{R} \times \partial \Omega} \quad \text{on } \mathbb{R} \times \partial \Omega
\]

with \((A', V')\) and \((A, V)\) gauge equivalent.
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In addition if $\Lambda'$ is the Dirichlet to Neumann operator associated with (5), then

$$\Lambda'(v|_{\mathbb{R} \times \partial \Omega}) = g|_{\mathbb{R} \times \partial \Omega} \Lambda(u|_{\mathbb{R} \times \partial \Omega})$$

i.e., $\Lambda' = \Lambda$ since $g|_{\mathbb{R} \times \partial \Omega} = 1$.

Proof. Let $x_0 = t$. For $0 \leq j \leq n$,

$$\left(-i\partial_{x_j} + A_j'(t, x)v(t, x) = g(t, x) \left(-i\partial_{x_j} + A_j'(t, x) - \frac{i}{g(t, x)} \partial_{x_j} g(t, x)\right) u(t, x).$$

If $A_j' = A_j + \frac{i}{g} \partial_{x_j} g$ for $0 \leq j \leq n$, then

$$\left(-i\partial_{x_j} + A_j'(t, x)\right)^2 v(t, x) = \left(-i\partial_{x_j} + A_j'(t, x)\right)(g(t, x)\left(-i\partial_{x_j} + A_j(t, x)\right)u(t, x))$$

$$= g(t, x)(-i\partial_{x_j} + A_j(t, x))^2 u(t, x);$$

thus,

$$\left(-i\partial_{x_j} + A_j'(t, x)\right)^2 v - \sum_{j=1}^{n} \left(-i\partial_{x_j} + A_j'(t, x)\right)^2 u + V'(t, x)v$$

$$= g(t, x)\left(-i\partial_{x_j} + A_j(t, x)\right)^2 u - \sum_{j=1}^{n} \left(-i\partial_{x_j} + A_j(t, x)\right)^2 u + V(t, x)u = 0$$

as $u$ is a solution of (1). Since $g$ is smooth and $u$ satisfies (2) and (3), it follows that for $t \ll 0$

$$v(t, x) = u(t, x)g(t, x) = 0$$

$$\partial_t v(t, x) = u(t, x)\partial_t g(t, x) + \partial_t u(t, x)g(t, x) = 0;$$

similarly $v(t, x)|_{\mathbb{R} \times \partial \Omega} = (g(t, x)u(t, x))|_{\mathbb{R} \times \partial \Omega} = f g|_{\mathbb{R} \times \partial \Omega}$. To conclude, note that

$$\Lambda'(v|_{\mathbb{R} \times \partial \Omega}) = (\partial_t (gu) + iA' \cdot v(gu))|_{\mathbb{R} \times \partial \Omega}$$

$$= ((\partial_t g)u + g(\partial_t u) + (\partial_t g) \cdot (\partial_t g) \cdot v(gu))|_{\mathbb{R} \times \partial \Omega}$$

$$= g(\partial_t u + iA \cdot v)u|_{\mathbb{R} \times \partial \Omega} + ((\partial_t g)u - (\partial_t g)u)|_{\mathbb{R} \times \partial \Omega}$$

$$= g|\mathbb{R} \times \partial \Omega \Lambda(u|_{\mathbb{R} \times \partial \Omega}).$$

If equation (6) holds, it is said that the Dirichlet to Neumann maps $\Lambda$ and $\Lambda'$ are gauge equivalent. Summarizing, proposition 2.1 shows that if the vector and scalar potentials are gauge equivalent, then the Dirichlet to Neumann maps are equal. In the following pages, the converse statement is proven; roughly speaking, if for a pair of vector and scalar potentials, the Dirichlet to Neumann operators associated with the hyperbolic equations (1)–(3) are equal, then so are the vector and scalar potentials.

The precise statement is the following.

Main Theorem. Let $\Omega$ be a bounded simply connected domain in $\mathbb{R}^n$ with $n \geq 2$. On $\mathbb{R} \times \Omega$, consider the relativistic Schrödinger equations

$$L_k u = \left(-i\partial_t + A_{0}^{(k)}(t, x)\right)^2 - \sum_{j=1}^{n} \left(-i\partial_{x_j} + A_j^{(k)}(t, x)\right)^2 + V^{(k)}(t, x) = 0, \quad k = 1, 2.$$

Here, $V^{(k)}, A_{0}^{(k)}, \ldots, A_n^{(k)}$ are locally integrable smooth functions, satisfying the growth condition $|A_{i}^{(k)}(t, x)| \leq C(1 + |x|)^{M}$ with $C, M > 0$ and $|t| \geq t_0$, as well as the support condition $A_{i}^{(k)}(t, x) = 0$ for $|x| \geq R > 0$. If the Dirichlet to Neumann operators associated with $L_1$ and $L_2$ are equal on the $\mathbb{R} \times \partial \Omega$, then the scalar potentials $V_1$ and $V_2$ are equal on $\mathbb{R} \times \Omega$, and there exists $\phi(t, x) \in C^{\infty}(t, x)$ such that
(ii) \( \phi_j^{(2)} - A^{(1)}_j = \partial_{\phi_j} \phi_j^{(2)} - A_j^{(1)} = \partial_{\phi_j} \phi(t, x), 1 \leq j \leq n, \) and

(ii) Supp \( \phi \subseteq \mathbb{R} \times \{ |x| \leq R \} \).

For the hyperbolic problem (1)–(3), GO solutions supported near light rays have the form

\[ u(t, x) = e^{ik(t-\omega x)} \sum_{p=0}^{N} \frac{v_p(t, x)}{(2ik)^p} + v^{(N+1)}(t, x), \quad \omega \in S^{n-1}, k \in \mathbb{R}. \] (7)

For \( u \), as above,

\[ (-i\partial_t + A_0)^2 u = e^{ik(t-\omega x)} \left[ (-i\partial_t + A_0)^2 + 2ik(-i\partial_t + A_0) - k^2 \right] \]

\[ \times \left( \sum_{p=0}^{N} \frac{v_p}{(2ik)^p} + e^{-ik(t-\omega x)} v^{(N+1)} \right), \]

and for \( 1 \leq j \leq n \)

\[ (-i\partial_{x_j} + A_j)^2 u = e^{ik(t-\omega x)} \left[ (-i\partial_{x_j} + A_j)^2 + 2ik\omega_j (-i\partial_{x_j} + A_j) - k^2 \omega_j^2 \right] \]

\[ \times \left( \sum_{p=0}^{N} \frac{v_p}{(2ik)^p} + e^{-ik(t-\omega x)} v^{(N+1)} \right). \]

Equation (1) becomes

\[ 0 = Lu = e^{ik(t-\omega x)} \left( (-i\partial_t + A_0)^2 - \sum_{j=1}^{n} (-i\partial_{x_j} + A_j)^2 + V \right) v \]

\[ + 2ik e^{ik(t-\omega x)} \left( (-i\partial_t + A_0) + \sum_{j=1}^{n} \omega_j (-i\partial_{x_j} + A_j) \right) v \]

\[ + e^{ik(t-\omega x)} \left( -k^2 + \sum_{j=1}^{n} (\omega_j k)^2 \right) v \]

\[ 0 = Lu = e^{ik(t-\omega x)} \left( L + 2ikL \right) v, \] (8)

where

\[ v(t, x) = \sum_{p=0}^{N} \frac{v_p(t, x)}{(2ik)^p} + e^{-ik(t-\omega x)} v^{(N+1)}(t, x) \] (9)

\[ L = (-i\partial_t + A_0(t, x))^2 - \sum_{j=1}^{n} (-i\partial_{x_j} + A_j(t, x))^2 + V(t, x) \] (10)

\[ \mathcal{L} = (-i\partial_t + A_0(t, x)) + \sum_{j=1}^{n} \omega_j (-i\partial_{x_j} + A_j(t, x)). \] (11)

Plugging in \( v \) into (8) leads to

\[ 0 = (2ik\mathcal{L} + L) \left( v_0 + \frac{1}{(2ik)^2} v_1 + \cdots + \frac{1}{(2ik)^N} v_N + e^{-ik(t-\omega x)} v^{(N+1)} \right), \] (12)

which can be written as

\[ (2ik)\mathcal{L}v_0 + (\mathcal{L}v_1 + L v_0) + \frac{1}{(2ik)^2} \left( \mathcal{L}v_2 + L v_1 \right) + \cdots + \frac{1}{(2ik)^N-1} \left( \mathcal{L}v_N + L v_{N-1} \right) \]

\[ + \frac{1}{(2ik)^N} L v_N + e^{-ik(t-\omega x)} v^{(N+1)} = 0, \] (13)

where the identity \((2ik\mathcal{L} + L)(e^{-ik(t-\omega x)} v^{(N+1)}) = e^{-ik(t-\omega x)} L v^{(N+1)}\) has been used.
A solution of (1)–(2) can be found via a two-step process, first by solving the \( N + 1 \) transport equations
\[
\mathcal{L}v_0 = 0, \quad \mathcal{L}v_j = -L v_{j-1}, \quad 1 \leq j \leq N
\] (14)
with initial conditions supported near a neighborhood of the light ray \( \gamma = \{(t', x') + s(1, \omega) : (t', x') \perp (1, \omega), s \in \mathbb{R}\} \) (where it is assumed that \( \gamma \) intersects the plane \( t = T_1 \) outside of the cylinder \( \mathbb{R} \times \Omega \)), and then by solving the second-order equation
\[
L v^{(N+1)} = -\frac{\sqrt{i k (t-\omega \cdot x)}}{(2 i k)^{N+2}} L v_N
\] (15)
with the initial and boundary conditions
\[
v^{(N+1)}(t, x) = 0 \quad \text{for} \quad t = T_1
\]
\[
\partial_t v^{(N+1)}(t, x) = 0 \quad \text{for} \quad t = T_1
\]
\[
v^{(N+1)}(t, x) = 0 \quad \text{for} \quad t \geq T_1, \quad x \in \partial \Omega.
\]

The above differential equation has a unique solution; moreover, if \( h \) represents the right-hand side of (15), then for \( T_1 < t < T \) and \( k > 1 \) (see for instance [10], p 185),
\[
||\partial_t v^{(N+1)}(t, \cdot)||_{L^2(\Omega)} + ||v^{(N+1)}(t, \cdot)||_{H^1(\Omega)} \leq C ||h||_{L^2(t, T) \times \Omega} \leq C\frac{1}{k^N}.
\] (16)

Therefore, if \( v_0 \) is a solution of the transport equation
\[
0 = \mathcal{L}v_0(t, x) = \sum_{j=0}^{n} \omega_j \partial_j v_0(t, x) + i \sum_{j=0}^{n} \omega_j A_j(t, x)v_0(t, x)
\] (17)
with \( \omega_0 = 1 \) and \( \partial_0 = \partial_t \), then \( u = e^{i k (t-\omega \cdot x)}(v_0 + \mathcal{O}(k^{-1})) \) solves (1) and satisfies the set of initial conditions (2).

Solutions of (17) have the form
\[
v_0(t, x) = \chi(t', x') \exp \left[ -i \int_{-\infty}^{(t'+\omega \cdot x)/2} \sum_{j=0}^{n} \omega_j A_j(t' + s, x' + s \omega) \, ds \right]
\] (18)
where \( (t', x') = (t, x) - \frac{1}{2} (t + \omega \cdot x)(1, \omega) \) is the projection of \( (t, x) \) into \( \Pi_{(1, \omega)} \), the \( n \)-dimensional linear subspace perpendicular to \( (1, \omega) \) (see figure 1), and \( \chi \) is any real-valued function that is constant along the direction given by \( (1, \omega) \), and whose support is contained in a neighborhood of the light ray \( \gamma = \{(t', x') + s(1, w) : s \in \mathbb{R}\} \).

Summarizing, a GO solution of (1)–(2) of the form
\[
u(t, x) = \exp[i k (t-\omega \cdot x) - i R_1(t, x; \omega)](\chi(t', x') + \mathcal{O}(k^{-1}))
\] (19)
can be constructed, where
\[
R_1(t, x; \omega) = \int_{-\infty}^{(t'+\omega \cdot x)/2} \sum_{j=0}^{n} \omega_j A_j(t' + s, x' + s \omega) \, ds.
\] (20)
Similarly, a GO solution for the backward hyperbolic problem
\[
Lv = 0 \quad \text{in} \quad (-\infty, T_2) \times \Omega
\]
\[
v = \partial_t v = 0 \quad \text{for} \quad t = T_2
\]
can be obtained in the same fashion.
Figure 1. Light rays \(\ell_1\) and \(\ell_2\) through an arbitrary point \((t, x)\). \(\ell_1\) does not intersect \(\mathbb{R} \times \Omega\) whereas \(\ell_2\) does. The point \((t', x')\) is the intersection of the ray through \((t, x)\) with direction \((1, \omega)\), with the \(n\)-dimensional linear subspace perpendicular to \((1, \omega)\) (denoted by \(\Pi_{(1, \omega)}\) in the figure).

A Green formula for these types of hyperbolic operators is needed in order to show that the Dirichlet to Neumann data determine the vectorial and scalar ray transforms of the potentials along 'light rays', that is, rays that make a 45° angle with the hyperplane \(t = 0\) (e.g. \(\ell_1\) and \(\ell_2\) in figure 1). This technique has had a lot of success in the context of inverse problems, in particular for the case of elliptic equations, the groundbreaking paper of Sylvester and Uhlmann [19] has been a source of inspiration for several other uniqueness results (see also Isakov’s review paper [9] for more information on this subject). For \(T_1\) and \(T_2\), two real numbers with \(T_1 < T_2\), consider the forward and backward hyperbolic equations

\[
\begin{align*}
L_1 u &= 0 \quad \text{in } [T_1, T_2] \times \Omega \\
u &= \mathcal{D}_t u = 0 \quad \text{for } t = T_1 \\
u &= f \quad \text{on } [T_1, T_2] \times \partial\Omega \\

L_2^* v &= 0 \quad \text{in } [T_1, T_2] \times \Omega \\
v &= \mathcal{D}_t v = 0 \quad \text{for } t = T_2 \\
v &= g \quad \text{on } [T_1, T_2] \times \partial\Omega,
\end{align*}
\]

where

\[
\begin{align*}
L_1 &= L(A^{(1)}, V^{(1)}) = (-i\partial_t + A_0^{(1)}(t, x))^2 - \sum_{j=1}^{n} (-i\partial_{x_j} + A_j^{(1)}(t, x))^2 + V^{(1)}(t, x) \\
L_2^* &= L(A^{(2)}, V^{(2)}) = (-i\partial_t + A_0^{(2)}(t, x))^2 - \sum_{j=1}^{n} (-i\partial_{x_j} + A_j^{(2)}(t, x))^2 + V^{(2)}(t, x),
\end{align*}
\]
and assume that the Dirichlet to Neumann operators

\[ \Lambda_1(f) = (\partial_x + i v \cdot A^{(1)}(t, x)) u(t, x)|_{(T_1, T_2) \times \partial \Omega} \]  
\[ \Lambda_2(g) = (\partial_x + i v \cdot A^{(2)}(t, x)) v(t, x)|_{(T_1, T_2) \times \partial \Omega} \]  

are equal on \((T_1, T_2) \times \partial \Omega\), i.e., \(\Lambda_1 f = \Lambda_2 f\) for all \(f\) smooth and supported on the set \((T_1, T_2) \times \partial \Omega\).

**Remark.** Note that for the operator \(L_2^*\), the associated Dirichlet to Neumann map is

\[ \Lambda_2^*(g) = (\partial_x + i v \cdot A^{(2)}(t, x)) v(t, x)|_{\mathbb{R} \times \partial \Omega}, \]

and that the main assumption is \(\Lambda_1 = \Lambda_2\) on \(\mathbb{R} \times \partial \Omega\). This is no mistake as later on it is shown that the notation is justified, as the \(L^2\) adjoint of \(\Lambda_2\) is indeed \(\Lambda_2^*\).

Denoting by \(\langle \cdot, \cdot \rangle_{(T_1, T_2) \times \Omega}\) and \(\langle \cdot, \cdot \rangle_{(T_1, T_2) \times \partial \Omega}\) the \(L^2\) inner products in \([T_1, T_2] \times \Omega\), \(\Omega\) and \([T_1, T_2] \times \partial \Omega\), respectively, integration by parts (using the initial conditions for \(A^{(1)}_0\)) leads to

\[ \langle -i \partial_t + A^{(1)}_0 \rangle^2 u |_{(T_1, T_2) \times \Omega} = \sum_{i=1}^n \langle -i \partial_t + A^{(1)} \rangle u |_{(T_1, T_2) \times \Omega} \]

where \(v = (v^{(1)}, \ldots, v^{(n)})\) is the exterior unit normal to \(\partial \Omega\) and \(u, v\) are solutions of the forward and backward hyperbolic equations, respectively. Also for \(A^{(1)}_j, j = 1, \ldots, n\),

\[ \sum_{j=1}^n \langle -i \partial_t + A^{(1)}_j \rangle^2 u |_{(T_1, T_2) \times \Omega} = \sum_{j=1}^n \langle -i \partial_t + A^{(1)}_j \rangle u |_{(T_1, T_2) \times \Omega} + \langle A_1(f), g\rangle |_{(T_1, T_2) \times \partial \Omega} \]

where \(f = u |_{(T_1, T_2) \times \Omega}\) and \(g = v |_{(T_1, T_2) \times \partial \Omega}\). Similarly for \(L_2^*\),

\[ \sum_{j=1}^n \langle -i \partial_t + A^{(2)}_j \rangle^2 v |_{(T_1, T_2) \times \Omega} = \sum_{j=1}^n \langle -i \partial_t + A^{(2)}_j \rangle v |_{(T_1, T_2) \times \Omega} \]

and

\[ 0 = \langle L_1 u, v \rangle |_{(T_1, T_2) \times \Omega} - \langle u, L_2^* v \rangle |_{(T_1, T_2) \times \Omega} \]

Combining expressions (23)–(26) and recalling that \(u, v\) are solutions of the forward and backward hyperbolic problem, respectively, the following expression is obtained:

\[ 0 = \langle L_1 u, v \rangle |_{(T_1, T_2) \times \Omega} - \langle u, L_2^* v \rangle |_{(T_1, T_2) \times \Omega} \]

\[ = \langle (\partial_t + (A^{(1)}_0)^T), (\partial_t + (A^{(1)}_0)^T) \rangle |_{(T_1, T_2) \times \Omega} - \langle (\partial_t + (A^{(2)}_0)^T), (\partial_t + (A^{(2)}_0)^T) \rangle |_{(T_1, T_2) \times \Omega} \]

\[ + \sum_{i=1}^n \langle -i \partial_t + A^{(1)}_i \rangle u |_{(T_1, T_2) \times \Omega} \]

\[ + \sum_{j=1}^n \langle -i \partial_t + A^{(2)}_j \rangle v |_{(T_1, T_2) \times \Omega} \]

\[ + \langle V^{(1)} u, v \rangle |_{(T_1, T_2) \times \Omega} - \langle V^{(2)} u, v \rangle |_{(T_1, T_2) \times \Omega} \]

\[ + \langle f, \Lambda_2^*(g) \rangle |_{(T_1, T_2) \times \partial \Omega} - \langle A_1(f), g\rangle |_{(T_1, T_2) \times \partial \Omega}. \]  

(27)
The sum $I_1 + I_2$ can be rewritten as

\[
I_1 + I_2 = \left( ((-i\partial_t + A_0^{(1)})u, (-i\partial_t + A_0^{(2)})u) \right)_{\Omega} + \left( ((-i\partial_t + A_0^{(2)})u, (-i\partial_t + A_0^{(1)})u) \right)_{\Omega} \\
- \left( ((-i\partial_t + A_0^{(1)})u, (-i\partial_t + A_0^{(2)})u) \right)_{\Omega} - \left( ((-i\partial_t + A_0^{(2)})u, (-i\partial_t + A_0^{(1)})u) \right)_{\Omega}
\]

That is

\[
I_1 + I_2 = \left( (A_0^{(1)} - A_0^{(2)})u, (-i\partial_t)v \right)_{\Omega} + \left( (A_0^{(1)} - A_0^{(2)})u, v \right)_{\Omega} + \left( (A_0^{(1)} - A_0^{(2)})u, v \right)_{\Omega} - \left( (A_0^{(1)} - A_0^{(2)})u, v \right)_{\Omega}
\]

A similar computation shows that

\[
I_2 + I_3 = \sum_{j=1}^{n} \left( (A_j^{(1)} - A_j^{(2)})u, (-i\partial_j)v \right)_{\Omega} + \sum_{j=1}^{n} \left( (A_j^{(1)} - A_j^{(2)})u, v \right)_{\Omega} + \sum_{j=1}^{n} \left( (A_j^{(1)} - A_j^{(2)})u, v \right)_{\Omega} - \sum_{j=1}^{n} \left( (A_j^{(1)} - A_j^{(2)})u, v \right)_{\Omega}
\]

Equations (27)–(29) then lead to the following Green formula:

\[
\langle \Lambda(f), g \rangle_{\Omega} - \langle \Lambda(f), \Lambda^*(g) \rangle_{\Omega} = \sum_{j=0}^{n} r_j \left( \left( A_j u, (-i\partial_j v) \right)_{\Omega} + \left( A_j (-i\partial_j u), v \right)_{\Omega} \right) + \sum_{j=0}^{n} r_j \left( \left( (A_j^{(2)})^2 - (A_j^{(1)})^2 \right)u, v \right)_{\Omega} - \left( V u, v \right)_{\Omega}. \tag{30}
\]

where $x_0 = t$, $A_j = A_j^{(2)} - A_j^{(1)}$ for $0 \leq j \leq n$, $V = V^{(2)} - V^{(1)}$, $r_0 = -1$ and $r_j = 1$ for $1 \leq j \leq n$.

Note that if the vector and scalar potentials in the forward and backward hyperbolic equation are taken to be the same (i.e., $\langle A_j^{(1)}, V_j^{(1)} \rangle = \langle A_j^{(2)}, V_j^{(2)} \rangle$), then

\[\langle \Lambda(f), g \rangle_{\Omega} - \langle \Lambda(f), \Lambda^*(g) \rangle_{\Omega} = 0,\]

proving that $\Lambda^* = \partial_x + iv \cdot A(t, x)$ is the $L^2$ adjoint of $\Lambda = \partial_x + iv \cdot A(t, x)$.

Since by assumption the Dirichlet to Neumann maps for the forward and backward hyperbolic equations agree (i.e., $A_1^* = A_2^*$ on $\mathbb{R} \times \Omega$), equation (30) leads to

\[0 = \langle \Lambda_1(f), g \rangle_{\Omega} - \langle \Lambda_2(f), g \rangle_{\Omega} = \sum_{j=0}^{n} r_j \left( \left( A_j u, (-i\partial_j v) \right)_{\Omega} + \left( A_j (-i\partial_j u), v \right)_{\Omega} \right) + \sum_{j=0}^{n} r_j \left( \left( (A_j^{(2)})^2 - (A_j^{(1)})^2 \right)u, v \right)_{\Omega} - \left( V u, v \right)_{\Omega}, \tag{31}\]

where, as before, $x_0 = t$, $A_j = A_j^{(2)} - A_j^{(1)}$ for $0 \leq j \leq n$, $V = V^{(2)} - V^{(1)}$, $r_0 = -1$ and $r_j = 1$ for $1 \leq j \leq n$. 

8
2.1. The x-ray transform

In the following paragraphs, the GO solutions of the forward and backward hyperbolic equations are combined with the Green formula developed before in preparation for the next section where the main result is proven.

Owing to (19) and (20), the GO solutions for \( u \) and \( v \) are given by

\[
\begin{align*}
  u(t, x) &= \exp[i k (t - \omega \cdot x) - i R_1(t, x, \omega)] \chi(t', x') + O(k^{-1}), \\
  v(t, x) &= \exp[-i k (t - \omega \cdot x) + i R_2(t, x, \omega)] \chi(t', x') + O(k^{-1}),
\end{align*}
\]

where

\[
\begin{align*}
  R_1(t, x, \omega) &= \int_{-\infty}^{(t+\omega \cdot x)/2} \sum_{j=0}^{n} \omega_j A_j^{(1)}(t' + s, x' + s\omega) ds, \\
  R_2(t, x, \omega) &= \int_{-\infty}^{(t+\omega \cdot x)/2} \sum_{j=0}^{n} \omega_j A_j^{(2)}(t' + s, x' + s\omega) ds.
\end{align*}
\]

For \( 0 \leq j \leq n \), the differentiation of (7) with respect to \( x_j \) combined with estimate (16) leads to the following formulas similar to (32)–(33):

\[
\begin{align*}
  \partial_{x_j} u &= k \exp[i k (t - \omega \cdot x) - i R_1(t, x, \omega)] (-ir_j \omega_j \chi + O(k^{-1})), \\
  (-i \partial_{x_j} u)(t, x) &= -k \exp[i \int_{-\infty}^{(t+\omega \cdot x)/2} (r_j \omega_j \chi(t', x')^2 + O(k^{-1})) ds,
\end{align*}
\]

and similarly

\[
\begin{align*}
  u(t, x) (-i \partial_{x_j} v(t, x)) &= -k \exp[i \int_{-\infty}^{(t+\omega \cdot x)/2} (r_j \omega_j \chi(t', x')^2 + O(k^{-1})) ds.
\end{align*}
\]

Thus, the Green formula reads

\[
0 = C k \int_{T_1} \int_{\Omega} \sum_{j=0}^{n} (A_j^{(2)}(t, x) - A_j^{(1)}(t, x)) r_j^2 \omega_j \chi^2(t', x')
\times \exp[i \int_{R_2(t, x, \omega)} - R_1(t, x, \omega)] dx \, dr + \cdots,
\]

where \( C \) is a (negative) constant and '…' represents terms of order \( O(1) \). Dividing the above expression by \( Ck \) and taking the limit as \( k \to +\infty \), all terms but the first disappear, and the previous expression leads to

\[
0 = \int_{T_1} \int_{\Omega} \sum_{j=0}^{n} \omega_j (A_j^{(2)}(t, x) - A_j^{(1)}(t, x)) \chi^2(t', x')
\times \exp[i \int_{R_2(t, x, \omega)} - R_1(t, x, \omega)] dx \, dr.
\]

Without loss of generality (cf remark 3.1 in [5]) it can be assumed that \( \operatorname{supp} A^{(j)} \subset \mathbb{R} \times \Omega \), \( j = 1, 2 \). Writing \( X' = (t', x') \) and letting \( A = (A_0, \ldots, A_n) = A^{(2)} - A^{(1)} \) be the difference of the vector potentials, the change of variables \( (t, x) = \sigma(1, \omega) + X' \) leads to

\[
0 = \int_{\Pi_{1(1), \omega}} \int_{-\infty}^{\infty} \sum_{j=0}^{n} \omega_j A_j(X' + \sigma(1, \omega)) \chi^2(X')
\times \exp[i \int_{-\infty}^{\sigma} \sum_{j=0}^{n} \omega_j A_j(X' + s(1, \omega)) ds] \, d\sigma \, dX'.
\]
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Since $\chi$ is an arbitrary function of $X'$, it can be concluded that

$$0 = \int_{-\infty}^{\infty} \sum_{j=0}^{n} \omega_j A_j(X' + \sigma(1, \omega)) \exp \left[ \int_{-\infty}^{\sigma} \sum_{j=0}^{n} \omega_j A_j(X' + s(1, \omega)) \, ds \right] \, d\sigma$$

$$= -i \int_{-\infty}^{\infty} \left( \frac{\partial}{\partial \sigma} \right) \exp \left[ \int_{-\infty}^{\sigma} \sum_{j=0}^{n} \omega_j A_j(X' + s(1, \omega)) \, ds \right] \, d\sigma$$

$$= -i \left( \exp \left[ \int_{-\infty}^{\sigma} \sum_{j=0}^{n} \omega_j A_j(X' + s(1, \omega)) \, ds \right] - 1 \right). \quad (36)$$

In summary, we have almost proved the following lemma.

**Lemma 2.1.** If the Dirichlet to Neumann operators $\Lambda_1$ and $\Lambda_2$ for the hyperbolic equations

$$L_k u = \left( (-i \partial_t + A_0^{(k)}(t, x))^2 - \sum_{j=1}^{n} (-i \partial_{\gamma_j} + A_j^{(k)}(t, x))^2 + V^{(k)}(t, x) \right) u = 0, \quad k = 1, 2$$

agree on $[T_1, T_2] \times \partial \Omega$, then for any light ray

$$\gamma(t, x; \omega) = \{(t, x) + s(1, \omega) : s \in \mathbb{R}\}, \quad (t, x) \in \mathbb{R}^{n+1}, \quad \omega \in S^{m-1},$$

the vectorial ray transform of $A = (A_0^{(2)} - A_0^{(1)}, \ldots, A_n^{(2)} - A_n^{(1)})$ along $\gamma(t, x; \omega)$ is equal to zero. In symbols, for all $(t, x) \in \mathbb{R}^{n+1}$, $\omega \in S^{m-1},$

$$(\mathcal{P} A)(t, x; \omega) := \int_{-\infty}^{\infty} \sum_{j=0}^{n} \omega_j A_j(t + s, x + s \omega) \, ds = 0 \quad (37)$$

where $A_j = A_j^{(2)} - A_j^{(1)}$ and $\omega_0 = 1$.

**Proof.** Equation $$(36)$$ can be rewritten as

$$\exp(i \mathcal{P} A)(t, x; \omega) = 1,$$

where $(t, x) \cdot (1, \omega) = 0$. In turn, this shows that if $(t, x)$ and $(1, \omega)$ are perpendicular, then $(\mathcal{P} A)(t, x; \omega) = 2 \pi r$. Incorporating the hypothesis of $A^{(1)}$ and $A^{(2)}$ being compactly supported in $x$, the exact value of $r$ can be determined. Since $$(37)$$ holds for any $(t, x, \omega) \in \mathbb{R}^{n+1} \times S^{m-1}$, in particular when $t = 0$ and $|x|$ is big enough and perpendicular to a fixed $\omega$, the light ray $(0, x) + s(1, \omega), s \in \mathbb{R}$, does not intersect the support of $\mathcal{A}$ (see $\ell_1$ in figure 1); hence

$$\int_{-\infty}^{\infty} \sum_{j=0}^{n} \omega_j A_j(t + s, x + s \omega) \, ds = 0 \quad (38)$$

holds when $(t, x) \perp (1, \omega)$. This orthogonality condition can now be removed via the following argument. If $(t, x)$ is an arbitrary point in $\mathbb{R}^{n+1}$, the change of variables $s = \sigma - \frac{1}{2}(t + \omega \cdot x)$ leads to

$$\int_{-\infty}^{\infty} \sum_{j=0}^{n} \omega_j A_j(t + \sigma, x + \sigma \omega) \, d\sigma = \int_{-\infty}^{\infty} \sum_{j=0}^{n} \omega_j A_j(t' + s, x' + s \omega) \, ds,$$

where $(t', x') = (t, x) - \frac{1}{2}(t + \omega \cdot x)(1, \omega)$ and $(t', x') \cdot (1, \omega) = 0$. \qed
3. Uniqueness of the time-dependent potentials

In this section, several uniqueness results for the integral of the vector potential along light rays are established. They will play an important role in the proof of the uniqueness of the inverse boundary value problem.

**Theorem 3.1.** Suppose that \( \mathcal{A}(t, x) = (A_0(t, x), \ldots, A_n(t, x)) \) with \( \mathcal{A} \in C^\infty \) in \( t \) and \( x \) is such that for any non-negative integers \( \alpha, \beta \) and for any \( 0 \leq j \leq n \), there exist positive constants \( c, C_{\alpha, \beta} \) such that for \( |t| \geq t_0 \), \( |\partial^\alpha \partial^\beta A_j(t, x)| \leq C_{\alpha, \beta} e^{-c|t|} \). If in addition, \( A_j(t, x) = 0 \) for \( |x| \geq R > 0 \) and

\[
\int \left( A_0 + \sum_{j=1}^n \omega_j A_j \right) (t + s, x + s\omega) \, ds = 0
\]

holds for all \((t, x) \in \mathbb{R} \times \mathbb{R}^n \) and all \( \omega \in S^{n-1} \), then there exist \( \psi(t, x) \in C^\infty (t, x) \) and positive constants \( \tilde{C}_{\alpha, \beta}, \tilde{c} \) such that

(i) \( A_0(t, x) = \partial_t \psi(t, x) \), \( A_j(t, x) = \partial_j \psi(t, x), \quad 1 \leq j \leq n \), and

(ii) \( \text{Supp } \psi \subseteq \mathbb{R} \times \{ |x| \leq R \} \), \( |\partial^\alpha \partial^\beta \psi(t, x)| \leq \tilde{C}_{\alpha, \beta} e^{-\tilde{c}|t|} \).

**Proof.** Let \( \chi \) be a non-negative smooth real-valued function that is identically equal to 1 when \([-1, 1]\) and vanishes outside \([-2, 2]\). For \( \epsilon > 0 \), let

\[
I_\epsilon(t, \xi; \omega) = \iint \chi(\epsilon t)(\epsilon s) e^{-\epsilon t - i\epsilon \xi} \left( A_0 + \sum_{j=1}^n \omega_j A_j \right) (t + s, x + s\omega) \, ds \, dt \, dx.
\]

On the one hand, as \( \epsilon \to 0 \)

\[
I_\epsilon(t, \xi; \omega) \to \iint e^{-\epsilon t - i\epsilon \xi} \left( \int_{-\infty}^{\infty} \left( A_0 + \sum_{j=1}^n \omega_j A_j \right) (t + s, x + s\omega) \, ds \right) \, dt \, dx = 0.
\]

On the other hand, Fubini’s theorem and the change of variables \( \tilde{x} = x + s\omega \) followed by \( \tilde{t} = t + s \) lead to

\[
I_\epsilon(t, \xi; \omega) = \iint \chi(\epsilon t)(\epsilon s) e^{-\epsilon t + i\epsilon (\omega - \xi)} \left( \int e^{-i\tilde{t} \tilde{\xi}} \left( \tilde{A}_0 + \sum_{j=1}^n \omega_j \tilde{A}_j \right) (\tilde{t} + \tilde{x}, \tilde{\xi}) \, d\tilde{t} \right) \, d\tilde{x} \, dx,
\]

where \( \tilde{A}_j(\tilde{t}, \tilde{x}) \) is the Fourier transform of \( A_j(\tilde{t}, x) \), \( 0 \leq j \leq n \), in the variables \( x_1, \ldots, x_n \). Since pointwise convergence implies convergence in the space of tempered distributions and since the Fourier transform preserves this type of convergence, the inner integral in (40) tends to \( \left( \tilde{A}_0 + \sum_{j=1}^n \omega_j \tilde{A}_j \right)(\tilde{t}, \tilde{x}) \) and \( I_\epsilon(t, \xi; \omega) \) tends to \( \delta(t + \omega \cdot \xi)(\tilde{A}_0 + \sum_{j=1}^n \omega_j \tilde{A}_j)(\tilde{t}, \tilde{x}) \). Therefore,

\[
0 = \delta(t + \omega \cdot \xi) \left( A_0 + \sum_{j=1}^n \omega_j A_j \right) (t, \xi)
\]

holds for all \((t, \xi) \), showing that the Fourier transform of \( A_0 + \sum_{j=1}^n \omega_j A_j \) vanishes on \( \Pi_{(1, \omega)} \), the \( n \)-dimensional linear subspace with normal \((1, \omega) \). This in turn implies that for an
appropriate choice of \(\omega\), this Fourier transform vanishes in the complement of the light cone \(C = \{(\tau, \xi) : |\tau| \geq |\xi|\}\). To see this, it suffices to note that if \((\tau, \xi) \notin C\), then \(
abla\frac{|\xi|}{|\xi|} \omega(\tau, \xi) = -\frac{\tau}{|\xi|}\). With this choice of \(\omega\), \(\tau + \omega(\tau, \xi) \cdot \xi = 0\) and the function \((A_0 + \sum_{j=1}^n \omega_j(\tau, \xi) A_j)\wedge (\tau, \xi)\) vanishes when \(|\tau| < |\xi|\). Moreover, this shows that the Fourier transform of the vector potential \(\hat{A}(\tau, \xi)\) is perpendicular to the \((n+1)\)-dimensional vector \((1, \omega(\tau, \xi))\) as
\[
(A_0 + \sum_{j=1}^n \omega_j(\tau, \xi) A_j)\wedge (\tau, \xi) = (1, \omega(\tau, \xi)) \cdot \hat{A}(\tau, \xi).
\]
Equation (43) also shows that \(\hat{A} = (\hat{A}_0, \ldots, \hat{A}_n)\) is orthogonal to all elements of \(E = \{(1, \omega(\tau, \xi)) : \tau + \omega(\tau, \xi) \cdot \xi = 0\}\). In the appendix, it is shown that the orthogonal complement \(E^\perp\) is one-dimensional, and since \((\tau, \xi)\) is perpendicular to any vector of the form \((1, \omega(\tau, \xi))\), this complement has to agree with the line \(\{c(\tau, \xi) : c \in \mathbb{R}\}\). Since the previous argument works for an arbitrary \(\tau\) and since the set \(\{\xi : |\tau| < |\xi|\}\) is an open subset in \(\mathbb{R}^n\), it follows that \(\hat{A}(\tau, \xi) = (\hat{A}_0(\tau, \xi), \ldots, \hat{A}_n(\tau, \xi))\) is proportional to the vector \((\tau, \xi)\) in the complement of the light cone. In other words, there exists a function \(\Phi\) such that whenever \(|\tau| < |\xi|\),
\[
(\hat{A}_0(\tau, \xi), \ldots, \hat{A}_n(\tau, \xi)) = i\Phi(\tau, \xi)(\tau, \xi).
\]
Since for any \(j\) the function \(A_j\) decays exponentially in \(t\) and is compactly supported in \(x\), then its Fourier transform \(\hat{A}_j\) is analytic in the strip \(|\text{Im} \tau| < c\). On the other hand, equation (44) gives
\[
\Phi(\tau, \xi) = -i\frac{\hat{A}_j(\tau, \xi)}{\xi^{(j)}}, \quad 1 \leq j \leq n,
\]
which shows that \(\Phi\) is analytic in the set \(\{(\tau, \xi) : |\text{Im} \tau| < c, (\tau, \xi) \neq (0, 0)\}\). Hartog’s theorem (see [7]) states that the concepts of removable singularities and isolated singularities agree in functions of several complex variables and it follows that \(\Phi\) is analytic in the strip \(|\text{Im} \tau| < c\). Moreover, if \(\varphi\) denotes the inverse Fourier transform of \(\Phi\), then \(\varphi\) and all of its derivatives are exponentially decaying in \(t\), and to finish the proof, it suffices to show that \(\varphi\) has the correct support properties.

Because of the assumptions on the support of the functions \(A_j\) it follows from the Paley–Wiener theorem that for \(0 \leq j \leq n,\)
\[
|\hat{A}_j(\tau, \xi)| \leq C_{N(j)} \exp[R|\text{Im} \xi|]\frac{1}{(1 + |\xi|)^N},
\]
and if \(|\xi^{(j)}| > 1\), then
\[
|\Phi(\tau, \xi)| = \left|\frac{\hat{A}_j(\tau, \xi)}{\xi^{(j)}}\right| \leq C_{N(j)} \exp[R|\text{Im} \xi|]\frac{1}{(1 + |\xi|)^N}
\]
for some \(C_N > 0\). Since \(h(\tau, \xi) = \Phi(\tau, \xi)(1 + |\xi|)^N \exp[-R|\text{Im} \xi|]\) is continuous when \(|\xi^{(j)}| \leq 1\), it is also bounded; hence \(|h(\tau, \xi)| \leq C_N\) for some positive \(C_N\) and the estimate
\[
|\Phi(\tau, \xi)| \leq C \frac{\exp(R|\text{Im} \xi|)}{(1 + |\xi|)^N}
\]
holds for any $\xi \in \mathbb{R}^n$. One more application of the Paley–Wiener theorem shows that the inverse Fourier transform of $\Phi(\tau, \xi)$ is supported in the set $[x : |x| \leq R]$. □

The conditions imposed on the potentials can be relaxed. The exchange of exponentially decaying by Schwartz functions produces the following similar result.

**Theorem 3.2.** Suppose that $A(t, x) = (A_0(t, x), \ldots, A_n(t, x))$ with $A \in C^\infty$ in $x$ and $t$ is such that for any $M > 0$ and non-negative integers $\alpha, \beta$, there exist constants $C_{M, \alpha, \beta} > 0$ such that $(1 + |\tau|)^M |\hat{\partial}^\beta \hat{A}_j(\tau, \xi)| \leq C_{M, \alpha, \beta}$ for $0 \leq j \leq n$. If in addition $A_j(t, x) = 0$ for $|x| \geq R > 0$ and (39) holds for all $(t, x) \in \mathbb{R} \times \mathbb{R}^n$ and all $\omega \in S^{n^*}$, then there exists $\varphi(t, x) \in C^\infty(t, x)$ such that

(i) $A_0(t, x) = \partial_i \varphi(t, x)$, \quad $A_j(t, x) = \partial_i \varphi(t, x)$, \quad $1 \leq j \leq n$, and

(ii) $\text{Supp } \varphi \subseteq \mathbb{R} \times [|x| \leq R]$, \quad $(1 + |\tau|)^M |\hat{\partial}^\beta \hat{\varphi}(\tau, \xi)| \leq C_{M, \alpha, \beta}$.

**Proof.** The proof goes along the same lines as the previous theorem except that in equation (44) it is only known that the left-hand side is entire in $\xi$. For $\tau_0 \neq 0$ fixed, Hartog’s theorem shows that $\Phi(\tau_0, \xi)$ is entire, and when $\tau_0 = 0$, equation (44) gives $\Phi(0, \xi) = -iA_j(0, \xi)/\xi^{(j)}$ for $1 \leq j \leq n$, proving that $\Phi$ has no singularities.

The part of the proof that deals with the support of $\varphi$ remains unchanged and it is only needed to show that $\Phi$ is a Schwartz function. If $\tilde{M} > 0$, $\beta$ is a non-negative integer, and if $|\xi| \leq R$,

$$
(1 + |\tau|)^{\tilde{M}} |\hat{\partial}^\beta \hat{\Phi}(\tau, \xi)| = (1 + |\tau|)^{\tilde{M}} \left| \hat{\partial}^\beta \left( i \frac{\hat{A}_0(\tau, \xi)}{\tau} \right) \right|
$$

$$
= (1 + |\tau|)^{\tilde{M}} \sum_{j=0}^{\beta} \frac{\partial^\beta}{\partial^\beta_j} \left| \hat{A}_0(\tau, \xi) \left( \frac{1}{\tau} \right)^j \right|
$$

$$
\leq C(1 + |\tau|)^{\tilde{M}} \sum_{j=0}^{\beta} \partial^\beta_j \left| \hat{A}_0(\tau, \xi) \right| \leq \tilde{C}(\tilde{M}, \beta, R),
$$

where the last inequality follows from the fact that the exponentially decaying $C^\infty$ functions transform into Schwartz functions. Since $\Phi$ is itself Schwartz, the desired function $\varphi$ is, like in the previous theorem, the inverse Fourier transform of $\Phi$. □

The conditions imposed so far on the potentials are such that they allow for the computation of the Fourier transform of equation (39). This transform can be computed under weaker assumptions and the following theorem shows that a similar result is still valid.

**Theorem 3.3.** Suppose that $A(t, x) = (A_0(t, x), \ldots, A_n(t, x))$ with $A \in C^\infty$ in $x$ and $t$ is such that for $0 \leq j \leq n$, $|A_j(t, x)| \leq C(1 + |t|)^M$ with $C, M > 0$ and $|t| \geq t_0$. If in addition the functions $|A_j(t, x)|$ are locally integrable in $\mathbb{R}^{n+1}$, satisfy the support condition $A_j(t, x) = 0$ for $|x| \geq R > 0$, and equation (39) holds for all $(t, x) \in \mathbb{R} \times \mathbb{R}^n$ and all $\omega \in S^{n^*}$, then there exists $\varphi(t, x) \in C^\infty(t, x)$ such that

(i) $A_0(t, x) = \partial_i \varphi(t, x)$, \quad $A_j(t, x) = \partial_i \varphi(t, x)$, \quad $1 \leq j \leq n$, and

(ii) $\text{Supp } \varphi \subseteq \mathbb{R} \times [|x| \leq R]$. 

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Proof. By the hypothesis on the growth of $A_j$ for $1 \leq j \leq n$, the Fourier transform of (39) can be computed to obtain $\delta(\tau + \omega \cdot \xi)(A_0 + \sum_{j=1}^{n} \omega_j A_j)^{(\omega_j, \alpha_j)}(\tau, \xi) = 0$, where $A_j(\tau, \xi)$ is an analytic function in $\xi$ and a distribution in $\tau$. In addition, since the wavefront set of $\delta(\tau + \omega \cdot \xi)$ and $A_0 + \sum_{j=1}^{n} \omega_j \cdot A_j$ do not intersect, the restriction of $A_0 + \sum_{j=1}^{n} \omega_j \cdot A_j$ on the set $\{(\tau, \xi) : \tau + \omega \cdot \xi = 0\}$ is well defined (cf [8]). Proceeding as before, when $|\tau| < |\xi|$, equation (42) can be solved (the set of solutions can be parametrized by $S^{n-2}$). Moreover, the change $(\tau, \xi) \to (\alpha \tau, \alpha \xi)$ in (42) leads to
\[
\frac{\alpha \xi}{|\alpha||\xi|} \cdot \omega(\alpha \tau, \alpha \xi) = -\frac{\alpha \tau}{|\alpha||\tau|}
\]
\[
\frac{\xi}{|\xi|} \cdot \omega(\alpha \tau, \alpha \xi) = -\frac{\tau}{|\tau|},
\]
showing that the solutions $\omega(\tau, \xi)$ of (42) are homogeneous of degree 0 in $(\tau, \xi)$ and that
\[
\hat{A}_0(\tau, \xi) + \sum_{j=1}^{n} \omega_j \hat{A}_j(\tau, \xi) = 0
\]
on the plane $\tau + \omega \cdot \xi = 0$. Replacing $(\tau, \xi)$ by $(\alpha \tau, \alpha \xi)$ leads to the equation
\[
\hat{A}_0(\alpha \tau, \alpha \xi) + \sum_{j=1}^{n} \omega_j \hat{A}_j(\alpha \tau, \alpha \xi) = 0.
\]
If $\chi(\alpha)$ is an arbitrary $C_0^\infty(\mathbb{R})$ function with support contained in the set $|\alpha - 1| < \epsilon$, multiplication of (47) by $\chi(\alpha)$ and integration with respect to $\alpha$ lead to
\[
a_0(\tau, \xi) + \sum_{j=1}^{n} \omega_j a_j(\tau, \xi) = 0,
\]
where $\tau + \omega \cdot \xi = 0$ and
\[
a_j(\tau, \xi) = \int_{-\infty}^{\infty} \hat{A}_j(\alpha \tau, \alpha \xi) \chi(\alpha) \, d\alpha.
\]
Noting that the expressions $a_j(\tau, \xi)$ are no longer distributions and that (48) holds with $\omega = \omega(\tau, \xi)$, it follows that $(a_0(\tau, \xi), \ldots, a_n(\tau, \xi)) = ib(\tau, \xi)(\tau, \xi)$ for some $b(\tau, \xi)$. That is,
\[
\frac{a_0(\tau, \xi)}{\tau} = \frac{a_1(\tau, \xi)}{\xi_1} = \cdots = \frac{a_n(\tau, \xi)}{\xi_n} = ib(\tau, \xi).
\]
Since $\chi(\alpha)$ is arbitrary,
\[
\frac{\hat{A}_0(\alpha \tau, \alpha \xi)}{\alpha \tau} = \frac{\hat{A}_1(\alpha \tau, \alpha \xi)}{\alpha \xi_1} = \cdots = \frac{\hat{A}_n(\alpha \tau, \alpha \xi)}{\alpha \xi_n} = i\hat{\Psi}(\alpha \tau, \alpha \xi),
\]
where $\hat{\Psi}(\alpha \tau, \alpha \xi)$ is a distribution in $\alpha \tau$ for all $\alpha \in (1 - \epsilon, 1 + \epsilon)$. Finally, when $\alpha = 1$
\[
\hat{A}_0(\tau, \xi) = i\tau \hat{\Psi}(\tau, \xi),
\]
\[
\hat{A}_1(\tau, \xi) = i\xi_1 \hat{\Psi}(\tau, \xi), \ldots, \hat{A}_n(\tau, \xi) = i\xi_n \hat{\Psi}(\tau, \xi)
\]
for $|\tau| < |\xi|$. Arguing as before, $\hat{\Psi}$ is entire in $\xi$ and $\hat{\Psi} \in S'$ in $\tau$ (since $A_j \in S'$ in $\tau$). Therefore $\varphi = \mathcal{F}^{-1}_{\tau,\xi} \hat{\Psi} \in S'$. Moreover, the identities $\partial_x \varphi = A_0, \partial_{\xi_1} \varphi = A_1, \ldots, \partial_{\xi_n} \varphi = A_n$ imply that $\varphi(t, x) \in C^\infty$ in $(t, x)$ and that $\varphi = 0$ for $|x| > R$. □

The main theorem is now restated and proved.

**Theorem 3.4.** Let $\Omega$ be a bounded simply connected domain in $\mathbb{R}^n$ with $n \geq 2$. On $\mathbb{R} \times \Omega$, consider the relativistic Schrödinger equations
\[
L_k u = \left(-i\partial_t + A_0^{(k)}(t, x)\right)^2 - \sum_{j=1}^{n} \left(-i\partial_j + A_j^{(k)}(t, x)\right)^2 + V^{(k)}(t, x) u = 0, \quad k = 1, 2.
\]
Here, \( V^{(k)}, A_{0}^{(k)}, \ldots, A_{n}^{(k)} \) are locally integrable smooth functions, satisfying the growth condition \( |A_{k}^{(j)}(t, x)| \leq C(1 + |t|)^M \) with \( C, M > 0 \) and \( |t| \geq t_{0} \), as well as the support condition \( A_{0}^{(k)}(t, x) = 0 \) for \( |x| \geq R > 0 \). If the Dirichlet to Neumann operators associated with \( L_1 \) and \( L_2 \) are equal on \( \mathbb{R} \times \partial \Omega \), then the scalar potentials \( V_1 \) and \( V_2 \) are equal on \( \mathbb{R} \times \Omega \), and there exists \( \varphi(t, x) \in C^{\infty}(t, x) \) such that

(i) \( A_{0}^{(j)} - A_{0}^{(1)} = \partial_{j} \varphi, A_{j}^{(2)} - A_{j}^{(1)} = \partial_{j} \varphi(t, x), 1 \leq j \leq n \), and

(ii) \( \text{Supp} \varphi \subseteq \mathbb{R} \times \{|x| < R\} \).

In other words, if \( A_{0}^{(k)} \) denotes the vector potential \( \{A_{0}^{(k)}, \ldots, A_{n}^{(k)}\} \), then the pair of vector and scalar potentials \( (A^{(1)}, V^{(1)}) \) and \( (A^{(2)}, V^{(2)}) \) are gauge equivalent.

**Proof.** By lemma 2.1, equation (39) holds for the components of the vector potential \( A = A^{(2)} - A^{(1)} \), and by theorem 3.3, \( A^{(2)} - A^{(1)} = \nabla_{t,x} \varphi \), where the support of \( \varphi \) is contained in the desired set. Replacing the pair of potentials \( (A^{(1)}, V^{(1)}) \) by \( (A^{(3)}, V^{(3)}) \) where \( A^{(3)} = A^{(1)} + \nabla_{t,x} \varphi \) and \( V^{(1)} = V^{(3)} \), proposition 2.1 shows that \( A^{(3)} = A^{(2)} \). The Green formula (31) applied to the pair of potentials \( (A^{(2)}, V^{(2)}) \) and \( (A^{(3)}, V^{(3)}) \) leads to

\[
0 = \langle (V^{(3)} - V^{(2)}) u, v \rangle_{[T_{1}, T_{2}] \times \Omega} = \int_{T_{1}}^{T_{2}} \int_{\Omega} (V^{(3)} - V^{(2)}) u \overline{v} \, dx \, dr.
\]

The GO representations (32)–(35) turn the previous integral into

\[
0 = \int_{T_{1}}^{T_{2}} \int_{\Omega} (V^{(3)}(t, x) - V^{(2)}(t, x))
\]

\[
\times \exp[i(\overline{R_{2}(t, x; \omega)} - R_{1}(t, x; \omega))]\chi^{2}(t', x') \, dx \, dr + \cdots,
\]

where, as before, ‘\( \cdots \)’ denotes terms of order \( \mathcal{O}(k^{-1}) \). Taking the limit as \( k \to +\infty \), all terms but one vanish, and the equality of the vector potentials implies that \( \overline{R_{2}} - R_{1} = 0 \). After a change of variables, (49) can be rewritten as

\[
0 = \int_{\Omega_{\gamma=0}} \left( \int_{-\infty}^{\infty} V^{(3)}(X' + s(1, \omega)) - V^{(2)}(X' + s(1, \omega)) \, ds \right) \chi^{2}(X') \, dS_{X'},
\]

and since \( \chi \) is arbitrary, the inner integral in the expression above vanishes, that is

\[
\int_{-\infty}^{\infty} (V^{(3)}(t' + s, x' + s\omega) - V^{(2)}(t' + s, x' + s\omega)) \, ds = 0,
\]

which shows that the light ray transforms of the scalar potentials \( V^{(2)} \) and \( V^{(3)} \) agree. Since \( V^{(3)} = V^{(1)} \), it follows that \( V^{(1)} = V^{(2)} \) and that \( (A^{(1)}, V^{(1)}) \) and \( (A^{(2)}, V^{(2)}) \) are gauge equivalent (see also [14, 17] for another proof of the equality of the scalar potentials).

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**Appendix**

**Lemma A.1.** The orthogonal complement of the set

\[
E = \{(1, \omega) : \omega \in S^{m-1}, \tau + \omega \cdot \xi = 0, |\tau| < |\xi|\}
\]

is a one-dimensional subspace of \( \mathbb{R}^{n+1} \).
Let $m \geq 2$. If $A \subseteq B \subseteq \mathbb{R}^m$ with

$$A \subseteq B \subseteq \text{Span}(A) = \left\{ \sum_{p=1}^{r} a_p \alpha_p : \alpha_p \in \mathbb{R}, \alpha_p \in A, \, r \in \mathbb{N} \right\};$$

then

$$A^\perp = B^\perp = \text{Span}(A)^\perp.$$  

Indeed, since orthogonal complements reverse inclusions, $\text{Span}(A)^\perp \subseteq B^\perp \subseteq A^\perp$. Also, if $x \cdot a = 0$ for all $a \in A$, then $x \cdot \sum_{p=0}^{r} a_p \alpha_p = \sum_{p=0}^{r} a_p (x \cdot \alpha_p) = 0$, showing that $A^\perp \subseteq \text{Span}(A)^\perp$. Therefore, $A^\perp = B^\perp = \text{Span}(A)^\perp$.

Denoting by $\text{CH}(A)$ the convex hull of $A$ and by $\mathcal{C}(A)$ the cone spanned by $A$, then

$$\text{CH}(A) = \left\{ \sum_{p=0}^{r} \alpha_p a_p : \alpha_p = 1, \, 0 \leq \alpha_p \leq 1, \, a_p \in A, \, r \in \mathbb{N} \right\};$$

$$\mathcal{C}(A) = \left\{ ta : t \in \mathbb{R}^+, \, a \in A \right\}.$$  

Clearly, $A \subseteq \text{CH}(A) \subseteq \mathcal{C}(\text{CH}(A))$, and since both sets contain particular linear combinations of elements of $A$, it follows that $\text{Span}(\mathcal{C}(\text{CH}(A))) = \text{Span}(A)$; therefore,

$$\text{Span}(\mathcal{C}(\text{CH}(A)))^\perp = \text{Span}(A)^\perp.$$  

The goal is now to apply this lemma to $E = \{(1, \omega(\tau, \xi)) : \tau + \omega(\omega, \xi) \cdot \xi = 0\}$. Before doing so, it is worth recalling that for $|\tau| < |\xi|$, the vectors $\omega$ satisfying $|\omega| = 1, \tau + \omega \cdot \xi = 0$ can be parametrized by $S^{n-2}$. Since rotations are non-singular transformations, the orthogonal complement of $E$ is the same as the orthogonal complement of the set

$$\tilde{E} = \left\{ (1, \omega_1, \ldots, \omega_{n-1}, a) : \omega_1^2 + \cdots + \omega_{n-1}^2 = 1 - a^2 \right\},$$

where $0 \leq a < 1$ is a fixed number. Then by the previous lemma,

$$\tilde{E}^\perp = \text{Span}(\mathcal{C}(\text{CH}(\tilde{E})))^\perp,$$

$$= \text{Span}(\mathcal{C}(\left\{ (1, \omega_1, \ldots, \omega_{n-1}, a) : \omega_1^2 + \cdots + \omega_{n-1}^2 \leq 1 - a^2 \right\}))^\perp,$$

$$= \text{Span}(\left\{ (t, \theta_1, \ldots, \theta_{n-1}, ta) : t, \theta_1, \ldots, \theta_{n-1} \in \mathbb{R}, \theta_1^2 + \cdots + \theta_{n-1}^2 \leq 1 - a^2 \right\})^\perp,$$

where it can be seen that $\tilde{E}^\perp$ is a one-dimensional subspace of $\mathbb{R}^{n+1}$ since clearly $\text{Span}(\mathcal{C}(\text{CH}(\tilde{E})))$ is $n$-dimensional.

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