Hilbert’s Double Series Theorem’s Extensions via the Mathieu Series Approach†

Tibor K. Pogány 1,2

1 Institute of Applied Mathematics, Öbuda University, Bécsi út 96/b, 1034 Budapest, Hungary; pogany.tibor@m.uni-obuda.hu or tibor.poganji@uniri.com
2 Faculty of Maritime Studies, University of Rijeka, Studentska 2, 51000 Rijeka, Croatia
† Dedicated to Professor Jovan Mališić to his 85th birthday anniversary.

Abstract: The author’s research devoted to the Hilbert’s double series theorem and its various further extensions are the focus of a recent survey article. The sharp version of double series inequality result is extended in the case of a not exhaustively investigated non-homogeneous kernel, which mutually covers the homogeneous kernel cases as well. Particularly, novel Hilbert’s double series inequality results are presented, which include the upper bounds built exclusively with non-weighted \( \ell_p \)-norms. The main mathematical tools are the integral expression of Mathieu \((a, \lambda)\)-series, the Hölder inequality and a generalization of the double series theorem by Yang.

Keywords: Hilbert’s double series theorem; Hölder inequality; Dirichlet series; Jacobi theta function; Riemann zeta function; Mathieu \((a, \lambda)\)-series; non-homogeneous kernel; quasi-polynomial

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1. Introduction

Let \( \ell_p \) be the space of all complex sequences \( x = (x_n)_{n \geq 1}, \ x_n \in \mathbb{C} \), equipped with the finite norm \( \| x \|_p = \left( \sum_{n \geq 1} |x_n|^p \right)^{1/p} \). Hilbert’s double series theorem states that for all conjugated Hölder pairs of exponents \((p, q)\), \( p > 1 \), for which \( p^{-1} + q^{-1} = 1 \), there holds

\[
\sum_{m,n \geq 1} a_m b_n \frac{m+n}{\pi \sin(\pi / p)} \| a \|_p \| b \|_q < 1
\]

where the nonnegative \( a = (a_n)_{n \geq 1} \in \ell_p, b = (b_n)_{n \geq 1} \in \ell_q \) and the constant \( \pi \cotanh(\pi / p) \) is the best possible \cite{1} (p. 253) (also, see \cite{2} (pp. 357–358) for further information).

This inequality, induced also today and shortly after its emergence with Hardy’s \cite{3} and Mulholland’s \cite{4,5} results in the 1920s, has become one of the frequently revisited research topics in the theory of inequalities. Recent interest in this area is shown by Gao, Debnath, Pachpatte, Yang, Krnić and Pečarić, among others. The reference list of this paper contains mainly publications with discrete Hilbert’s inequalities with non-homegeneous kernels \cite{6–21}, since this version of a Hilbert’s-type inequality is in the focus of our interest. The usual way to obtain a Hilbert’s inequality is to apply the Hölder inequality to a conveniently rewritten bilinear form

\[
H^a_K = \sum_{m,n \geq 1} K(m, n) a_m b_n,
\]

in which \( a, b \geq 0 \) and \( K(\cdot, \cdot) \) we call the kernel function. To derive discrete Hilbert’s -type inequalities (or in other words, double series theorems) means to establish sharp upper bounds for \( H^a_K \) in terms of (not necessarily) weighted \( \ell_p \)-norms of the vectors \( a, b \).

The author’s approach was different in \cite{15,22–24}; he transforms the double series \( H^a_K \)
by means of the Gamma function formula $\Gamma(\mu)z^{\mu} = \int_0^\infty x^{\mu-1}e^{-zx} \, dx$, $\Re\{\mu\} > 0$ into a product of two Dirichlet series separating $a_n$ and $b_n$ and evaluates them using the Hölder inequality, which gives the (weighted) $\ell_p$-norms for both sequences $a_n, b_n$. Therefore, the main goal of this survey note is to present the author’s autonomously developed non-standard method to obtaining Hilbert-type inequalities [15,24], based on Cahen’s integral formula (see [25]) applied to the (auxiliary) Mathieu $(a, \lambda)$-series. Therefore, this article gives, for the first time, a complete overview of the author’s method for obtaining a whole class of Hilbert’s inequalities for the bilinear form (2) generated either by the non-homogeneous and/or homogeneous kernel function $K$. The exhaustive reference list covers all publication items in which the interested reader can clearly follow the derivation procedures and proofs. It has to be mentioned that, in [26,27], Hilbert’s-type inequalities are studied for multiple series, applying the same approach; in addition, consult [28] for another approach to multiple Hilbert’s inequalities.

A brief account of the problem considered is now presented. Consider the bilinear form (2), defined for $a, b \geq 0$, and $K(\cdot, \cdot)$ to be the kernel function of the double series (2). Hilbert’s double series theorems give sharp upper bounds for $K^{ab}$ in terms of (not necessarily) weighted $\ell_p$-norms of $a, b$. When $K(m, n) = (m^\kappa + n^\kappa)^{-\lambda}$, the kernel function is homogeneous of order $-\lambda \cdot \kappa$. Specifying $\kappa = \lambda = 1$, we obtain Hilbert’s expression in (1), and $\kappa = 1$ gives Yang’s result (3). Moreover, the homogeneous kernel function $K(\cdot, \cdot)$, in the case of non-conjugated Hölder exponents $(p, q)$, $p > 1, p^{-1} + q^{-1} \geq 1$ gives the classical Hilbert’s double series theorems established by Levin [29], Bonsall [30] and the recent ones by Krnić and Pečarić [11] (also, see the monograph [31] and the appropriate references therein). We point out that the best constant problem is longstanding for the non-conjugated couple of exponents $p, q$.

The bilinear form $H^{ab}_K$ in the case when the kernel is non–homogeneous, was mainly not considered and avoided. The rare Hilbert’s double series with non-homogeneous kernel theorem remains the ‘logarithmic’ inequality by Mulholland [32], in which $K(m, n) = \ln(mn)$, $m, n \geq 2$, and its extension by Yang, who takes

$$K(m, n) = (u(m) + u(n))^{-\lambda}, \quad m, n \geq n_0 \in \mathbb{N}; \lambda > 0,$$

assuming $u \in C^1(n_0 - 1, \infty), u((n_0 - 1)+) = 0, u(\infty) = \infty$ [33] (Chapters 3 and 4). In turn, the inequality by He et al. [34] (Theorem 3.4) was associated with $K(m, n) = (\Gamma(m) + \Gamma(n))^{-\lambda}$.

Finally, we need, in the exposition, the following result. Supposing that $p > 1, p^{-1} + q^{-1} = 1, 2 - \min\{p, q\} < \lambda \leq 2$, Bi Cheng Yang [33] (Equation (5)), [35] (Equation (1.8)) extended the double series theorem (1) to

$$\sum_{m,n \geq 1} \frac{a_m b_n}{(m+n)\lambda} < k_\lambda(p) \|u^{(1-\lambda)}/a\|_p \|u^{(1-\lambda)/q} b\|_q,$$

(3)

when, for non-negative sequences $a, b$, there hold $u^{1-\lambda} a = (u^{1-\lambda} a_n)_{n \geq 1} \in \ell_p, u^{1-\lambda} b \in \ell_q$, whilst $n := (n_k)_{k \geq 1}, n_k \in \mathbb{N}$ and $x^\mu = (x_n^\mu)_{n \geq 1}$. The constant

$$k_\lambda(p) := B(1 + (\lambda - 2)/p, 1 + (\lambda - 2)/q);$$

is the best possible;

$$B(u, v) = \int_0^1 t^{u-1}(1 - t)^{v-1} \, dt, \quad \min\{\Re(u), \Re(v)\} > 0,$$

stands for the Euler function of the first kind, in other words, the beta-function; while the integral

$$\Gamma(s) = \int_0^{\infty} e^{-t} t^{s-1} \, dt, \quad \Re(s) > 0,$$
defines the integral form of the Euler function of the second kind, which is the gamma function.

2. Preparation and Methodology

Consider monotone increasing functions \( \lambda, \rho : \mathbb{R}_+ \to \mathbb{R}_+ \), for which

\[
\lim_{x \to \infty} \lambda(x) = \lim_{x \to \infty} \rho(x) = \infty.
\]  

(4)

The restrictions \( \lambda|_{\mathbb{N}} = \lambda = (\lambda_n)_{n \geq 1}, \rho|_{\mathbb{N}} = \rho = (\rho_n)_{n \geq 1} \) occur in the infinite bilinear form presented as the double series

\[
H_{\lambda, \rho}^{a, b}(\mu) = \sum_{m \geq 1} \sum_{n \geq 1} \frac{a_m b_n}{(\lambda_m + \rho_n)^\mu}, \quad \mu > 0.
\]

(5)

Our main goal is to expose a sharp Hilbert’s bounding inequality for \( H_{\lambda, \rho}^{a, b}(\mu) \) built with the non–homogeneous kernel

\[
K(m, n) = (\lambda_m + \rho_n)^{-\mu}, \quad m, n \geq 1; \mu > 0.
\]

(6)

One of the main mathematical tools used is the so–called Mathieu \((a, \lambda)\)–series introduced by the author in [23] (Equation (2))

\[
S_\mu(\rho, a, \lambda) = \sum_{m \geq 0} \frac{a_m}{(\lambda_m + \rho)^\mu}, \quad \rho, \mu > 0,
\]

(7)

where \((\lambda)_m \geq 0\) is positive monotone increasing and \(\lambda_m \to \infty\) when \(m \to \infty\).

The author obtained the integral representation [23] (p. 686)

\[
S_\mu(\rho, a, \lambda) = \frac{a_0}{\rho^\mu} + \mu \int_{\lambda_1}^\infty \left( \sum_{n=1}^{\lfloor \lambda^{-1}(x) \rfloor} a_n \right) \frac{dx}{(\rho + x)^{\mu + 1}}.
\]

(8)

Here, the function \(a : \mathbb{R} \to \mathbb{R}_+\) and \(a \in C^1(\mathbb{R}_+)\); the sequence \(a_n = a|_{\mathbb{N}_0}\). By convention, \(f^{-1}(x)\) stands for the inverse of \(f(x)\) and \([w], \{w\}\) denotes here, and in what follows, the integer and the fractional part of \(w = [w] + \{w\}\), respectively. In (8), the counting function

\[
A_\lambda(x) := \sum_{n=1}^{\lfloor \lambda^{-1}(x) \rfloor} a_n
\]

can be summed up by the Euler–Maclaurin summation formula, for instance. However, for certain simple \(a_n\) we obtain \(A_\lambda(x)\) directly. The Euler–Maclaurin summation, in turn, results in the double integral representation established by the Cahen formula, see [36] (p. 8, Equation (1.15)) presented in our setting and the ancestor expression [25] (p. 97), which expresses the Dirichlet series in the form of a Laplace integral. The double integral expression for the Mathieu series reads [23] (p. 687, Theorem 1)

\[
S_\mu(\rho, a, \lambda) = \frac{a_0}{\rho^\mu} + \mu \int_{\lambda_1}^\infty \int_0^{\lfloor \lambda^{-1}(x) \rfloor} \frac{a(u) + a'(u) \{u\}}{(\rho + x)^{\mu + 1}} \, dx \, du.
\]

(9)

Remark 1. To simplify the exposition, it is of interest to assume that \(a_0 \equiv 0\). This results in the form of the double integral

\[
\sum_{m \geq 1} \frac{a_m}{(\lambda_m + \rho)^\mu} = \mu \int_{\lambda_1}^\infty \int_0^{\lfloor \lambda^{-1}(x) \rfloor} \frac{a(u) + a'(u) \{u\}}{(\rho + x)^{\mu + 1}} \, dx \, du,
\]

(10)
assumed the monotone behavior of \((\lambda_m)\), the differentiability of the function \(a \) on the positive real half axis and additional convergence conditions upon the both sides of (10).

The inequality’s derivation strategy is the following. Consider the Mathieu \((a, \lambda)\)-series (7) in which the parameter \(\rho\) becomes the member of a positive monotone sequence \((\rho_n)_{n \geq 0}\), which is the restriction \(\rho|_{\mathbb{N}} = \rho = (\rho_n)_{n \geq 1}\) of the function \(\rho \in C^1(\mathbb{R}+)\) such that \(\rho : \mathbb{R}_+ \to \mathbb{R}_+\), with divergence behavior (4), declared above. Then, the bilinear form (5) one splits into

\[
H_{\lambda, \rho}^{a, b}(\mu) = \sum_{m \geq 1} a_{m} \sum_{n \geq 1} \frac{b_{n}}{(\lambda_m + \rho_n)^{\mu}} = \sum_{n \geq 1} b_{n} \sum_{m \geq 1} S_{\mu}(\rho, a, \lambda_m).
\] (11)

Applying the Hölder inequality to the bilinear form with transformed summand as

\[
a_{m}b_{n} = \frac{a_{m}b_{n}}{(\lambda_m + \rho_n)^{\mu}} \leq \left( \sum_{m,n \geq 1} a_{m}^{p}b_{n}^{q} \right)^{1/p} \left( \sum_{m,n \geq 1} (\lambda_m + \rho_n)^{-qr} \right)^{1/q}
\]

where \(r > 1, s \) are suitable conjugated Hölder exponents, we infer the bound

\[
H_{\lambda, \rho}^{a, b}(\mu) \leq \left( \sum_{m,n \geq 1} a_{m}^{p}b_{n}^{q} \right)^{1/p} \left( \sum_{m,n \geq 1} (\lambda_m + \rho_n)^{-qr} \right)^{1/q}
\] (12)

The first sum on the right-hand-side is the \(r^{-1}\)th power of the Hilbert’s-type bilinear form, which we evaluate using Yang’s extension (3) of the Hilbert’s double series theorem with \(\lambda = r - 1 \in (2 - \min\{p, q\}, 2]\). The second term in (12) is nothing more than a sum of two Mathieu \((\mathbb{N}, \lambda)\)-series:

\[
\sum_{m,n \geq 1} \frac{m + n}{(\lambda_m + \rho_n)^{qr}} = \sum_{n \geq 1} S_{\mu}(\rho_n, \mathbb{N}, \lambda) + \sum_{m \geq 1} S_{\mu}(\rho, \mathbb{N}, \lambda_m),
\] (13)

both, we handle by virtue of the double integral representation (9). Accordingly, the extension of Hilbert’s double series theorem in terms of related \(\ell_p\)-norms follows.

**Theorem 1** ([15] (p. 1487, Theorem 1)). Assume that \(p > 1, p^{-1} + q^{-1} = 1, r \in (3 - \min\{p, q\}, 3]\), and let \(r, s\) be conjugated Hölder exponents: \(r^{-1} + s^{-1} = 1\). Let \(a, b\) be nonnegative sequences such that

\[
(n^{(2-r)/p}a_{n})_{n \geq 1} \in \ell_p, \quad (n^{(2-r)/q}b_{n})_{n \geq 1} \in \ell_q.
\]

Suppose that the positive monotone functions \(\lambda, \rho\) satisfy (4), whilst

\[
\int_{\lambda_1}^{\infty} \frac{(\lambda^{-1}(x))^{2}}{x^{3p/2+1}} \, dx < \infty, \quad \int_{\rho_1}^{\infty} \frac{(\rho^{-1}(x))^{2}}{x^{3q/2+1}} \, dx < \infty.
\] (14)

Then, we have

\[
H_{\lambda, \rho}^{a, b}(\mu) = \sum_{m,n \geq 1} \frac{a_{m}b_{n}}{(\lambda_m + \rho_n)^{\mu}} \leq C_{\lambda, \rho} \| n^{(2-r)/p}a^{r/p} \|_{p}^{1/r} \| n^{(2-r)/q}b^{r/q} \|_{q}^{1/r},
\] (15)

where the constant

\[
C_{\lambda, \rho} = \left( \frac{\mu s}{2} (\mu s + 1) \right)^{1/s} B^{1/r}(1 + (r - 3)/p, 1 + (r - 3)/q)
\]

\[
\cdot \left( \int_{\lambda_1}^{\infty} \int_{\rho_1}^{\infty} \frac{[\lambda^{-1}(x)][\rho^{-1}(y)] ([\lambda^{-1}(x)] + [\rho^{-1}(y)] + 2)}{(x + y)^{3p/2+1}} \, dx \, dy \right)^{1/s}
\]
is the best possible. The equality in (15) occurs for some absolute constant $C$, when

$$\frac{a_mb_n}{(\lambda m + \rho n)^\mu} = \frac{C (m + n)}{(\lambda m + \rho n)^\mu}, \quad m, n \geq 1.$$  

We point that out because Yang’s result (3) is sharp in the sense that $k_\lambda(p)$ is the best possible, and the transformation formula (13) is an equality; we have to discuss only the equality which follows by the use of the Hölder inequality, which confirms the validity of (16).

**Remark 2.** The derived Hilbert’s double series theorem (15) for the general non-homogeneous kernel

$$K(m,n) = \frac{1}{(\lambda m + \rho n)^\mu}, \quad \mu > 0,$$

requires only the monotonicity of sequences $\lambda, \rho \geq 0$. In particular, when $\lambda \equiv \rho$, assuming $\lambda \in C^1(\mathbb{R}_+)$ and $\lambda(0+) = 0$, we obtain the situation considered by Yang [33] (Equation (21)).

The range of the parameter $\lambda \in (2 - \min\{p, q\}, 2]$ in Yang’s result (3) was expanded to $\lambda \in (0, 14]$ by Krnić and Pečarić; see [11] (Theorems 1, 2) for the conjugate-, and [ibid., Theorem 4] for non-conjugate Hölder pair $p, q$. Hence, the range of $r$ in Theorem 1 can be mutually expanded. Obviously, we have to pay for further generalizations with related convergence conditions upon $a, b$ together with new parameter constraints.

3. Inequalities with Non-Weighted Norms

The finiteness of the constant $C_{\lambda, \rho}$ is ensured by the convergence of integrals (14):

$$\int_{\lambda_1}^{\infty} \frac{\lambda^{-1}(x)^2}{x^{\mu/2+1}} \, dx < \infty, \quad \int_{\rho_1}^{\infty} \frac{\rho^{-1}(x)^2}{x^{\mu/2+1}} \, dx < \infty.$$

In this section, the approach from [15] is changed to infer more general and, at the same time, as simple as possible discrete Hilbert’s-type inequalities. However, our main task is to find a sharp upper bound over $H_{a,b}^p$ in terms of $\|a\|_p$ and $\|b\|_q$, under the non-homogeneous kernel (6) assumption, in other words, we are looking for a sharp estimate of the form

$$\sum_{m,n \geq 1} a_mb_n (\lambda m + \rho n)^\mu \leq C^* \|a\|_p \|b\|_q, \quad C^* > 0.$$  

Moreover, we establish inequalities such as (17) by Hölder inequality with non-conjugate parameters, i.e., when $p, q > 1$ and $p^{-1} + q^{-1} \geq 1$, introducing a new incremental parameter

$$\Delta := \frac{1}{p} + \frac{1}{q} - 1 \geq 0;$$

consult, for $\Delta$, the classics such as Bonsall [30] and Levin [29], for instance.

Our reduction requirements should be balanced between a sharp, but at the same time quite complicated, constant $C^*$ occurring in (17). Accordingly, choosing suitable functions $\lambda$ and $\rho$ from one, and the non-conjugate Hölder exponents $p, q$ from the other hand, we simplify the Hilbert’s-type inequality into a set of corollaries, in which several special functions, such as the Jacobi theta function $\theta_3$ and Riemann zeta function $\zeta$ play important roles. We point out that all our derived upper bounds are new and sharp when $\Delta = 0$.

The following results are presented in [22], noting that, here and in the following, $I(x) = x$ denotes the identity.
Theorem 2 ([22] (pp. 89–90, Theorem 1)). Suppose \( p, q > 1, \mu > 0, a = (a_n)_{n \in \mathbb{N}} \in \ell_p, b = (b_n)_{n \in \mathbb{N}} \in \ell_q \) are non-negative sequences and \( \lambda, \rho \) are positive monotone increasing functions satisfying (4). Then,
\[
\sum_{m,n \geq 1} \frac{a_m b_n}{(Am + \rho_n)\mu} \leq C_{p,q}^\mu (\lambda, \rho) \|a\|_p \|b\|_q ,
\]
where the constant
\[
C_{p,q}^\mu (\lambda, \rho) = \frac{q^{1/q} p^{1/p}}{\Gamma(\mu)} \int_0^\infty x^{\mu+\Delta} \left( \int_0^\infty e^{-qtx} \left( f \right)^{1/q} df \right)^{1/q} \left( \int_0^\infty e^{-pxu} \left( \frac{u}{B} \right)^{1/p} du \right)^{1/p} dx .
\]
The equality in (18) appears for \( \lambda = \rho = I, \ p = q = 2 \) when
\[
\frac{a_m}{b_n} = C \delta_{mn}, \quad m, n \in \mathbb{N},
\]
where \( C \) is an absolute constant. Here, \( \delta_{mn} \) stands for the Kronecker delta.

Now, we specify step by step the parameters \( p, q \) and the functions \( \lambda, \rho \). Using this procedure, we will conclude various sharp corollaries of Theorem 2.

A. If we take \( \lambda(x) = Ax^\ell, \rho(x) = Bx^p \), their inverses are \( \lambda^{-1}(x) = (x/A)^{1/q}, \rho^{-1}(x) = (x/B)^{1/p} \) and the constant becomes
\[
C_{p,q}^\mu (A(\cdot)^\ell, B(\cdot)^p) = \frac{(Aq)^{1/q} (Bp)^{1/p}}{\Gamma(\mu)} \int_0^\infty x^{\mu+\Delta} \left( \int_A^\infty e^{-qtx} \left( \frac{t}{A} \right)^{1/q} dt \right)^{1/q} \left( \int_B^\infty e^{-pxu} \left( \frac{u}{B} \right)^{1/p} du \right)^{1/p} dx .
\]
The kernel \( K \) is obviously non-homogeneous for all \( p \neq q \).

Corollary 1. Assume \( p, q > 1, \mu > 0, \) and \( a = (a_n)_{n \in \mathbb{N}} \in \ell_p, b = (b_n)_{n \in \mathbb{N}} \in \ell_q \) to be non-negative sequences. Then,
\[
\sum_{m,n \geq 1} \frac{a_m b_n}{(Am + Bn\ell)^\mu} \leq C_{p,q}^\mu (A(\cdot)^\ell, Bx^p) \|a\|_p \|b\|_q ,
\]
where the constant
\[
C_{p,q}^\mu (A(\cdot)^\ell, B(\cdot)^p) = \left( \frac{(Aq)^{1/q} (Bp)^{1/p}}{\Gamma(\mu)} \right) \int_0^\infty x^{\mu+\Delta} \left( \int_A^\infty e^{-qtx} \left( \frac{t}{A} \right)^{1/q} dt \right)^{1/q} \left( \int_B^\infty e^{-pxu} \left( \frac{u}{B} \right)^{1/p} du \right)^{1/p} dx .
\]

B. For \( \lambda = \rho = I, \) the kernel is homogeneous. The inequality transforms into
\[
\sum_{m,n \geq 1} \frac{a_m b_n}{(m+n)^\mu} \leq \frac{\|a\|_p \|b\|_q}{\Gamma(\mu)} \int_0^\infty x^{\mu-1} \left( \sum_{m \geq 1} e^{-mqx} \right)^{1/q} \left( \sum_{n \geq 1} e^{-npnx} \right)^{1/p} dx ,
\]
\[
= \frac{\|a\|_p \|b\|_q}{\Gamma(\mu)} \int_0^\infty \frac{x^{\mu-1}}{(e^{qx} - 1)^{1/q} (e^{px} - 1)^{1/p}} dx .
\]
Corollary 2. Suppose $p, q > 1$, $\mu > \Delta + 1$ and let $a = (a_n)_{n \in \mathbb{N}} \in \ell_p$, $b = (b_n)_{n \in \mathbb{N}} \in \ell_q$ be non-negative sequences. Then,

$$\sum_{m,n \geq 1} \frac{a_mb_n}{(m+n)\mu} \leq C_{\mu,\Delta}^\mu(\mathcal{I},\mathcal{I}) \|a\|_p \|b\|_q,$$

where

$$C_{\mu,\Delta}^\mu(\mathcal{I},\mathcal{I}) = \frac{1}{\Gamma(\mu)} \int_0^\infty x^{\mu-1} \left( (\rho x - 1)^{1/q} (\rho x - 1)^{1/p} \right) dx.$$

C. In the case when $\lambda(x) = \rho(x) = x^2$, the kernel $K(m^2, n^2)$ is homogeneous.

Corollary 3. Let $p, q > 1$, $\mu > \Delta + 1$ and $a = (a_n)_{n \in \mathbb{N}} \in \ell_p$, $b = (b_n)_{n \in \mathbb{N}} \in \ell_q$ be non-negative sequences. Then,

$$\sum_{m,n \geq 1} \frac{a_mb_n}{(\mu m^2 + n^2)\mu} \leq C_{\mu,\Delta}^\mu(x^2, x^2) \|a\|_p \|b\|_q.$$

In this case

$$C_{\mu,\Delta}^\mu((\cdot)^2, (\cdot)^2) = \frac{1}{2^{\Delta+1} \Gamma(\mu)} \int_0^\infty x^{\mu-1} \left( \theta_3(0, e^{-px}) - 1 \right)^{1/p} \left( \theta_3(0, e^{-qx}) - 1 \right)^{1/q} dx,$$

where $\theta_3(\cdot, \cdot)$ stands for the Jacobi theta function

$$\theta_3(u, q) = 1 + 2 \sum_{n \geq 1} q^{n^2} \cos(2n\pi u), \quad |q| < 1.$$

The case $p = q = 2$ means a fortiori $\Delta = 0$ and also gives the constant in terms of the Riemann zeta function:

$$C_{2,2}^{\mu,0}(x^2, x^2) = \frac{1}{2\Gamma(\mu)} \int_0^\infty x^{\mu-1} \left( \theta_3(0, e^{-2x}) - 1 \right) dx = \frac{\zeta(2\mu)}{2^\mu},$$

which we obtain by the formula [37]:

$$\int_0^\infty t^{\mu-1} \left( \theta_3(0, e^{-At}) - 1 \right) dt = \frac{2\Gamma(\mu)}{A^{\mu+1}} \zeta(2\mu), \quad \Re\{A\} > 0, \Re\{\mu\} > 1/2.$$

D. Taking $\lambda \equiv \rho$, $p = q = 2$, the behavior of $\lambda(x)$ dictates the homogeneity of the kernel. Then, the constant becomes [22] (p. 93, Equation (14))

$$C_{2,2}^{\mu,0}(\lambda, \lambda) = \frac{\mu}{2^\mu} \int_1^\infty \frac{[\lambda^{-1}(t)]}{t^{\mu+1}} dt.$$

Corollary 4. Assume $\mu > 0$, $a = (a_n)_{n \in \mathbb{N}}$, $b = (b_n)_{n \in \mathbb{N}} \in \ell_2$ are non-negative sequences; and $\lambda$ is a positive monotone function which satisfies (4). Then,

$$\sum_{m,n \geq 1} \frac{a_mb_n}{(\lambda m + \lambda n)\mu} \leq C_{\mu,2}^{\mu,0}(\lambda, \lambda) \|a\|_2 \|b\|_2.$$

E. We close the specifications’ list with $\lambda = \rho = \mathcal{I}$, $p = q = 2$. With the aid of Corollary 4, we conclude.

Corollary 5. Suppose $\mu > 1$, $a = (a_n)_{n \in \mathbb{N}}$, and $b = (b_n)_{n \in \mathbb{N}} \in \ell_2$ are non-negative sequences. Then,

$$\sum_{m,n \geq 1} \frac{a_mb_n}{(m+n)^\mu} \leq 2^{-\mu} \zeta(\mu) \|a\|_2 \|b\|_2.$$
Remark 3. Except (1), no Hilbert’s and/or Hilbert-type inequalities are derived in terms of \( \|a\|_p, \|b\|_q \), until the appearance of Theorem 2. This was one of our main tasks in [22], namely, to express the sharp upper bound of \( H^{a,b}_K \) in terms of \( \|a\|_p, \|b\|_q \), when \( K \) is non-homogeneous.

Finally, let us remark that taking in (18) \( a_n \mapsto \phi(a_n) \), \( b_m \mapsto \psi(b_m) \); and \( \phi, \psi \) suitable functions, one can generalize Theorem 2 in another way.

4. Extensions of Theorem 2

Now, we generalize certain results of [22] presented and proved in the previous section to a new class of inequalities by introducing and studying an auxiliary function. To this end, we consider the Dirichlet-series

\[
H_{\lambda,\rho}^{a,b}(\mu; x) = \sum_{m,n \geq 1} \frac{a_m b_n}{(\lambda_m + \rho_n)^\mu} x^{\lambda_m + \rho_n}, \quad x > 0,
\]

(20)

associated with the Hilbert’s bilinear double series \( (2) \).

Since

\[
|H_{\lambda,\rho}^{a,b}(\mu; x)| \leq H_{\lambda,\rho}^{a,b}(\mu; |x|),
\]

it is sufficient to study the Dirichlet-series \( (20) \) for \( x > 0 \), bearing in mind that \( H_{\lambda,\rho}^{a,b}(\mu; 1) \equiv H_{\lambda,\rho}^{a,b} \) converges. Now, extension of certain bounding inequality for \( H_{\lambda,\rho}^{a,b}(\mu; \zeta) \) to the case \( \zeta \in \mathbb{C} \) is straightforward.

First, we transform the double series \( H_{\lambda,\rho}^{a,b}(\mu; x) \) with the aid of the familiar gamma function formula

\[
\Gamma(\mu)\zeta^{-\mu} = \int_0^\infty x^{\mu-1} e^{-x \zeta} \, dx, \quad \Re{\mu} > 0.
\]

Then, separating the kernel function into a product of two Dirichlet series, we evaluate them using the Hölder inequality with non-conjugated exponents \( p, p', \min\{p, p'\} > 1, p^{-1} + p'^{-1} \geq 1 \) and \( q, q' \), \( \min\{q, q'\} > 1, q^{-1} + q'^{-1} \geq 1 \), respectively. All these transformations give

\[
H_{\lambda,\rho}^{a,b}(\mu; x) = \frac{1}{\Gamma(\mu)} \int_0^\infty t^{\mu-1} \left( \sum_{m \geq 1} \frac{a_m}{(\lambda_m)^p} \right) \left( \sum_{n \geq 1} \frac{b_n}{(\rho_n)^q} \right) dt
\]

\[
\leq \frac{\|a\|_p \|b\|_q}{\Gamma(\mu)} \int_0^\infty t^{\mu-1} \left( \sum_{m \geq 1} (x^{-t})^{\lambda_m q} \right)^{1/p'} \left( \sum_{n \geq 1} (x^{-t})^{\rho_n q'} \right)^{1/q'} dt.
\]

(21)

By means of the Cahen’s Laplace integral expression of the Dirichlet series [38] (§5),

\[
D_{\lambda}(\zeta) = \sum_{m \geq 1} a_m e^{-\lambda_m \zeta} = \zeta \int_0^\infty e^{-\zeta t} \sum_{m=1}^{[\lambda^{-1}(t)]} a_m dt, \quad \Re{\zeta} > 0,
\]

(22)

for positive monotone increasing \( (\lambda_n)_{n \geq 1} \), which satisfies (4). By virtue of (22), the Dirichlet series in (21) becomes

\[
D_{\lambda}(p't) = \sum_{m \geq 1} x^{\lambda_m p'} e^{-\lambda_m (p't)} = p't \int_0^\infty e^{-p't u} \sum_{m : \lambda_m \leq a} x^{\lambda_m p'} du = p't \int_0^\infty e^{-p't u} \sum_{m=1}^{[\lambda^{-1}(u)]} x^{\lambda_m p'} du.
\]

Introducing the quasi-polynomial notation \( Q_a^\chi(z) = z^{p_1} + \cdots + z^{p_N} \), we can write

\[
D_{\lambda}(p't) = p't \int_{\lambda_1}^{\infty} e^{-p't u} Q_{[\lambda^{-1}(u)]}^{\lambda}(x^{p'}) \, du,
\]

(23)

\[
D_{\rho}(q't) = q't \int_{\rho_1}^{\infty} e^{-q't u} Q_{[\rho^{-1}(u)]}^{\rho}(x^{q'}) \, du.
\]

(24)
Replacing (23) and (24) into (21), we arrive at the new class of Hilbert-type inequalities with the scaling variable \( x > 0 \), which corresponds to the auxiliary Dirichlet-series (20).

**Theorem 3** ([24] (p. 49, Theorem 1)). Let \( p, q > 1; p', q' \) be non-conjugated Hölder exponents to \( p, q \), respectively; \( \mu > 0, a = (a_n)_{n \in \mathbb{N}} \in \ell_p, \ b = (b_n)_{n \in \mathbb{N}} \) and \( \ell_q \) are nonnegative sequences and \( \lambda, \rho \) are positive monotone increasing functions satisfying (4). Then,

\[
H_{\lambda, \rho}^{a, b}(\mu; x) \leq C_\mu^{a, b}(\lambda, \rho; x) \|a\|_p \|b\|_q,
\]

(25)

where the constant

\[
C_\mu^{a, b}(\lambda, \rho; x) = \frac{\mu^{1/P' \cdot q'/q}}{\Gamma(\mu)} \int_0^\infty \int_0^1 \int_0^\infty e^{-\mu'(u)} Q^\lambda(u) \bigg( x' \bigg) du \bigg( \int_{\nu_1}^{\infty} e^{-\nu q'} Q_{[\nu - 1]}^{\rho}(x') \, dv \bigg) \bigg( \int_{\rho_1}^{\infty} \nu \, dv \bigg) \, dt.
\]

(26)

The equality in (25) appears for \( \lambda(x) = \rho(x) = I(x) = 1 \) and conjugated Hölder exponents \( p = q = 2 \) when

\[
\frac{a_m}{b_n} = C \delta_{mn} \quad m, n \in \mathbb{N},
\]

(27)

Here, \( C \) is an absolute constant and \( \delta_{mn} \) stands for Kronecker’s delta.

**Remark 4.** The equality analysis in (25) we begin with applying \( a_m = C \delta_{mn} b_n \):

\[
H_{\lambda, \rho}^{a, b}(\mu; 1) = \sum_{m, n \geq 1} a_m / b_n \cdot b_n^2 = C \sum_{m, n \geq 1} \delta_{mn} b_n^2 = C \sum_{m, n \geq 1} \frac{b_n^2}{(m + n)^\mu} = C \frac{\zeta(\mu)}{2^\mu} \|b\|_2^2,
\]

which coincides with the right-hand-side expression in (25), when (27) holds. For \( p' = q' = 2 \) and \( \Delta = 0 \), constant (26) becomes

\[
C_\mu^{2, 2}(I, I; 1) = \frac{2}{\Gamma(\mu)} \int_0^\infty \int_1^\infty \int_0^\infty t^\mu e^{-2u} \left( \sum_{n=1}^{[u]} \frac{1}{n} \right) \, dt \, du = \frac{\mu}{2^\mu} \int_1^\infty \frac{[u]}{u^{\mu+1}} \, du.
\]

The integral expression of the Riemann zeta function [38] (p. 97, Corollary 6) is

\[
\zeta(s) = s \int_1^\infty \frac{x^s}{x^s + 1} \, dx, \quad s > 1.
\]

Thus, \( C_\mu^{2, 2}(I, I; 1) = \zeta(\mu) 2^{-\mu} \). On the other hand \( \|a\|_p \|b\|_q = C \|b\|_2^2 \).

5. Two Consequences of Theorem 3

F. An advantage of the inequality class covered by Theorem 3 is the fact that Hölder exponents \( p, q \) are independent and remain non-conjugated.

Letting \( p = q \) and \( p' = q' \), we obtain the next result.

**Corollary 6.** Assume the same situation as in preambula of Theorem 3. Then, we obtain the following sharp inequality:

\[
H_{\lambda, \rho}^{a, b}(\mu; x) \leq C_\mu^{a, b}(\lambda, \rho; x) \|a\|_p \|b\|_p,
\]

where the constant becomes

\[
C_\mu^{a, b}(\lambda, \rho; x) = \frac{\mu^{1/P' \cdot q'/q}}{\Gamma(\mu)} \int_0^\infty \int_0^1 \int_0^\infty e^{-\mu'(u)} Q^\lambda(u) \bigg( x' \bigg) Q_{[\nu - 1]}^{\rho}(x') \, du \, dv \, dt.
\]

(28)
G. Choosing \( p' = q' = 2 \), we deduce

**Corollary 7.** Let \( p' = q' = 2 \) be conjugated Hölder exponents and let other assumptions of Theorem 3 be fulfilled. For \( \lambda = \rho \), we have

\[
H_{\lambda, \lambda}^{a,b}(\mu; x) \leq C_{2,2}^{\mu}(\lambda; x) ||a||_{p} ||b||_{q},
\]

where

\[
C_{2,2}^{\mu}(\lambda; x) = \frac{\mu}{2\pi} \int_{\lambda}^{\infty} Q_{\lambda - (a)}^{1}(x^2) \frac{d\mu}{\mu^{\alpha+1}}.
\]

The inequality is sharp.

**Remark 5.** First, here, \( p, q \) remain independent; we can only say that their range is \( p, q \in (1, 2] \). Second, the equality of the kernel sequences \( \lambda = \rho \) does not mean that the kernel of the double Dirichlet series \( H_{\lambda, \lambda}^{a,b}(\mu; x) \) is homogeneous. Finally, let us note that Corollary 7 cannot be deduced from Corollary 6 by any specific application.

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