‘Bureaucratic’ set systems, and their role in phylogenetics

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ABSTRACT

We say that a collection $C$ of subsets of $X$ is *bureaucratic* if every maximal hierarchy on $X$ contained in $C$ is also maximum. We characterize bureaucratic set systems and show how they arise in phylogenetics. This framework has several useful algorithmic consequences: we generalize some earlier results and derive a polynomial-time algorithm for a parsimony problem arising in phylogenetic networks.

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1. Bureaucratic sets and their characterization

In this work we introduce and study a class of set systems that arise in various ways from trees, graphs and intervals. We are interested in this class because it can provide a setting in which certain hard optimization problems can be solved efficiently, and we provide a particular example of this for a parsimony problem on phylogenetic networks.

We first recall some standard phylogenetic terminology (for more details, the reader can consult [1]). Recall that a hierarchy $H$ on a finite set $X$ is a collection of sets with the property that the intersection of any two sets is either empty or equal to one of the two sets.

A hierarchy is maximum if $|H| = 2|X| - 1$, which is the largest possible cardinality. In this case $H$ corresponds to the set of clusters $c(T)$ of some rooted binary tree $T$ with leaf set $X$ (a cluster of $T$ is the set of leaves that are separated from the root of the tree by any vertex). A maximum hierarchy necessarily contains $\{x\}$ for each $x \in X$, as well as $X$ itself; we will refer to these $|X| + 1$ sets as the *trivial clusters* of $X$. More generally, any hierarchy containing all the trivial clusters corresponds to the clusters $c(T)$ of a rooted tree $T$ with leaf set $X$ (examples of these concepts are illustrated in Fig. 1(a),(b)). Note that a hierarchy $H$ is maximum if and only if (i) $H$ contains all the trivial clusters, and (ii) each set $C \in H$ of size greater than 1 can be written as a disjoint union $C = A \sqcup B$, for two (disjoint) sets $A, B \in H$.

We now introduce a new notion.

**Definition.** We say that a collection $C$ of subsets of a finite set $X$ is a *bureaucracy* if (i) $\emptyset \neq C$ and $\emptyset \notin C$, and (ii) every hierarchy $H \subseteq C$ can be extended to a maximum hierarchy $H'$ such that $H \subseteq H' \subseteq C$. In this case, we also say that $C$ is bureaucratic.

Simple examples of bureaucracies include two extreme cases: the set of clusters of a binary tree, and the set $\mathcal{P}(X)$ of all non-empty subsets of $X$. Notice that $\{\{a\}, \{b\}, \{c\}, \{a, b\}, \{a, b, c\}\}$ and $\{\{b\}, \{c\}, \{b, c\}, \{a, b, c\}\}$ are both bureaucratic subsets of $\mathcal{P}(X)$ for $X = \{a, b, c\}$ but their intersection, $\{\{a\}, \{b\}, \{c\}, \{a, b, c\}\}$, is not. In particular, for an arbitrary subset $Y$ of $\mathcal{P}(X)$ (e.g. $Y = \{\{a\}, \{b\}, \{c\}, \{a, b, c\}\}$), there may not be a unique minimal bureaucratic subset of $\mathcal{P}(X)$ containing $Y$.

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Theorem 2. A collection $C$ of subsets of $X$ is bureaucratic if and only if it satisfies the following two properties:

1. (P1) $C$ contains all trivial clusters of $X$.
2. (P2) If $C_1, C_2, \ldots, C_k \subseteq C$ are disjoint and have union $\cup_i C_i$ in $C$ then there are distinct $i, j$ such that $C_i \cup C_j \in C$.

Proof. First suppose that $C$ is bureaucratic. Then $C$ contains a maximum hierarchy; in particular, it contains all the trivial clusters, and so (P1) holds. For (P2), suppose that $C'$ is a collection of $k \geq 3$ disjoint subsets of $X$, each element of $C$, and $\bigcup C' \in C$. Then $H = C \cup \{ \bigcup C' \}$ is a hierarchy. Let $H' \subseteq C$ be a maximum hierarchy on $X$ that contains $H$ (this exists, since $C$ is bureaucratic) and let $C$ be a minimal subset of $X$ in $H'$ that contains the union of at least two elements of $C'$. Since $H'$ is a maximum hierarchy, and $\bigcup C' \in H'$, $C$ is precisely the union of exactly two elements of $C'$; since $C \in H' \subseteq C$, this establishes (P2).

Conversely, suppose that a collection $C$ of subsets of $X$ satisfies (P1) and (P2), and that $H \subseteq C$ is a maximal hierarchy which is contained within $C$. Suppose that $H$ is not maximum (we will derive a contradiction). Then $H$ contains a set $C$ that is the disjoint union of $k \geq 3$ maximal proper subsets $A_1, \ldots, A_k$, each belonging to $H$ (and thereby $C$). Applying (P2) to $C' = \{A_1, \ldots, A_k\}$, there exist two sets, say $A_i, A_j$ for which $A_i \cup A_j \in C$. So, if we let $H' = H \cup \{A_i \cup A_j\}$, then we obtain a larger hierarchy containing $H$ that is still contained within $C$, which is a contradiction. This completes the proof. □

2. Examples of bureaucracies

We have mentioned two extreme cases of bureaucracies, namely the set of clusters of a rooted binary tree having leaf set $X$, and the full power set $P(X)$. Here are some further examples.

1. The set of intervals of $[n] = \{1, 2, \ldots, n\}$ is a bureaucracy where an interval is a set $[i, j] = \{k : i \leq k \leq j\}$, $1 \leq i \leq j \leq n$.

Proof. Let $C$ be the set of intervals of $[n]$. Then $C$ contains the trivial clusters. Also, a disjoint collection $I_1, \ldots, I_k$, $k > 2$, of intervals has union an interval if and only if every element of $[n]$ between $\min \bigcup I_i$ and $\max \bigcup I_i$ lies in (exactly) one interval, in which case the union of any pair of consecutive intervals is an interval, so (P2) holds. By Theorem 2, $C$ is bureaucratic. □

Similarly, if we order the elements of $X$ in any fashion, we can define the set of intervals on $X$ for that ordering by this construction (associating $x_i$ with $i$), and can thus obtain a bureaucracy.
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