The Isotropic Cosserat Shell Model Including Terms up to $O(h^5)$. Part II: Existence of Minimizers

Ionel-Dumitrel Ghiba$^{1,2}$ · Mircea Bîrsan$^{3,1}$ · Peter Lewintan$^3$ · Patrizio Neff$^3$

Received: 19 March 2020 / Accepted: 17 September 2020 / Published online: 7 October 2020
© Springer Nature B.V. 2020

Abstract We show the existence of global minimizers for a geometrically nonlinear isotropic elastic Cosserat 6-parameter shell model. The proof of the main theorem is based on the direct methods of the calculus of variations using essentially the convexity of the energy in the nonlinear strain and curvature measures. We first show the existence of the solution for the theory including $O(h^5)$ terms. The form of the energy allows us to show the coercivity for terms up to order $O(h^5)$ and the convexity of the energy. Secondly, we consider only that part of the energy including $O(h^3)$ terms. In this case the obtained minimization problem is not the same as those previously considered in the literature, since the influence of the curved initial shell configuration appears explicitly in the expression of the coefficients of the energies for the reduced two-dimensional variational problem and additional mixed bending-curvature and curvature terms are present. While in the theory including $O(h^5)$ the conditions on the thickness $h$ are those considered in the modelling process and they are independent of the constitutive parameter, in the $O(h^3)$-case the coercivity is proven under some more restrictive conditions on the thickness $h$.

Keywords Geometrically nonlinear Cosserat shell · 6-parameter resultant shell · In-plane drill rotations · Thin structures · Dimensional reduction · Wryness tensor ·

I.-D. Ghiba
dumitrel.ghiba@uaic.ro

M. Bîrsan
mircea.birsan@uni-due.de

P. Lewintan
peter.lewintan@uni-due.de

P. Neff
patrizio.neff@uni-due.de

1 Department of Mathematics, Alexandru Ioan Cuza University of Iași, Blvd. Carol I, no. 11, 700506 Iași, Romania
2 Iași Branch, Octav Mayer Institute of Mathematics of the Romanian Academy, 700505 Iași, Romania
3 Head of Lehrstuhl für Nichtlineare Analysis und Modellierung, Fakultät für Mathematik, Universität Duisburg-Essen, Thea-Leymann Str. 9, 45127 Essen, Germany
1 Introduction

Shell and plate theories are intended for the study of thin bodies, i.e., bodies in which the thickness in one direction is much smaller than the dimensions in the other two orthogonal directions. In this follow up paper we investigate the existence of minimizers to a recently developed isotropic Cosserat shell model [6, 21], including higher order terms. The Cosserat shell model naturally includes an independent triad of rigid directors, which are coupled to the shell-deformation. From an engineering point of view, such models are preferred, since the independent rotation field allows for transparent coupling between shell and beam parts. It is interesting that the kinematical structure of 6-parameter shells [5, 19, 41] (involving the translation vector and rotation tensor) is identical to the kinematical structure of Cosserat shells (defined as material surfaces endowed with a triad of rigid directors describing the orientation of points). Using the derivation approach, Neff [28, 30, 31, 33, 51] has modelled and analysed the so-called nonlinear planar-Cosserat shell models, in which a full triad of orthogonal directors, independent of the normal of the shell, is taken into account. The results have been obtained by an 8-parameter ansatz of the deformation through the thickness and consistent analytic integration over the thickness in the case of a flat undeformed shell reference configuration. In previous papers, we have extended the modelling from flat shells to the most general case of initially curved shells [6, 21]. Our ansatz allows for a consistent shell model up to order $O(h^5)$ in the shell thickness. Interestingly, all $O(h^5)$-terms in the shell energy depend on the initial curvature of the shell and vanish for a flat shell. However, all occurring material coefficients of the shell model are uniquely determined in terms of the underlying isotropic three-dimensional Cosserat bulk-model and the given initial geometry of the shell. Thus, we fill a certain gap in the general 6-parameter shell theory, since all hitherto known models leave the precise structure of the constitutive equations wide open. In the present paper, we will show that our model is mathematically well-posed in the sense that global minimizers exist.

The topic of existence of solutions for the 2D equations of linear and nonlinear elastic shells has been treated in many works. The results that can be found in the literature refer to various types of shell models and they employ different techniques, see, e.g., [2, 4, 20, 23–25, 44–48]. The existence theory for linear or nonlinear shells is presented in details in the books of Ciarlet [12–14], together with many historical remarks and bibliographic references. A fruitful approach to the existence theory of 2D plate and shell models (obtained as limit cases of 3D models) is the $\Gamma$-convergence analysis of thin structures, see, e.g., [32, 35, 36, 40]. By ignoring the Cosserat effects, in order to start with a well-posed three-dimensional model, it is preferable to consider a polyconvex energy [3] in the three-dimensional formulation of the initial problem. In this direction, an example is the article [17], see also [9, 16], where the Ciarlet-Geymonat energy [15] is used. In these articles, no through the thickness integration is performed analytically and no reduced completely two-dimensional minimization problem is presented. The obtained problems are “two-dimensional” only in the sense that the final problem is to find three vector fields on a bounded open subset of $\mathbb{R}^2$. 
but all three-dimensional coordinates remain present in the minimization problem. By contrast, when a nonlinear three-dimensional problem in the Cosserat theory is considered, the three-dimensional problem is well-posed [34, 49, 50] and permits a complete dimensional reduction.

The classical geometrically nonlinear Kirchhoff-Love model (the Koiter model for short), is given by the minimization problem with respect to the midsurface deformation $m : \omega \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3$ of the type

$$
\int_{\omega} \left\{ h \left( \mu \| (I_m - I_{y_0}) \|^2 + \frac{\lambda \mu}{\lambda + 2 \mu} \text{tr} \left[ (I_m - I_{y_0})^2 \right] \right) + \frac{h^3}{12} \left( \mu \| (II_m - II_{y_0}) \|^2 + \frac{\lambda \mu}{\lambda + 2 \mu} \text{tr} \left[ (II_m - II_{y_0})^2 \right] \right) \right\} \text{d}a,
$$

(1.1)

where $I_m := [\nabla m]^T \nabla m \in \mathbb{R}^{2 \times 2}$ and $II_m := -[\nabla m]^T \nabla m \in \mathbb{R}^{2 \times 2}$ are the matrix representations of the first fundamental form (metric) and the second fundamental form on $m(\omega)$, respectively. However, this problem is notoriously ill-posed, since the first membrane term is non-convex in $\nabla m$ and is indeed a non-rank-one elliptic energy. Even the inclusion of the bending terms is not sufficient to regularize the problem [28, 33]. The very same problem arises in geometrically nonlinear Reissner-Mindlin (Naghdi) type shell models, which already include an independent director-vector-field that does not coincide with the normal to the surface, as in the Kirchhoff-Love model.

Let us explain the typical situation by looking at representative energy terms for the different models. Assume that $m : \omega \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3$ is the deformation of the midsurface of a flat shell, $n_m$ is the unit normal to the shell midsurface, the unit vector $d : \omega \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3$ is an independent director vector-field, and $\mathbf{R} : \omega \subset \mathbb{R}^2 \rightarrow \text{SO}(3)$ is an independent rotation field. Then the essence of a Kirchhoff-Love planar shell model is represented by the minimization problem with respect to $m : \omega \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3$ of the type

“Kirchhoff-Love type”

$$
\int_{\omega} \left( h \left( \frac{\| (\nabla m \mid n_m)^T (\nabla m \mid n_m) - 1_3 \|^2}{2} + \frac{h^3}{12} \| \nabla n_m \|^2 \right) \right) \text{d}a \quad (1.2)
$$

The essence of the corresponding Reissner-Mindlin problem is represented by the minimization problem with respect to $(m, d)$ of the type

“Reissner-Mindlin type”

$$
\int_{\omega} \left( h \left( \| (\nabla m \mid d)^T (\nabla m \mid d) - 1_3 \|^2 + \frac{h^3}{12} \| \nabla d \|^2 \right) \right) \text{d}a. \quad (1.3)
$$

And finally, the Cosserat flat shell model has the structure given by the minimization problem with respect to $(m, \mathbf{R})$ of the type

“Cosserat-shell”

$$
\int_{\omega} \left( h \left( \| (\nabla m \mid \mathbf{R} e_3) - 1_3 \|^2 + \frac{h^3}{12} \| \nabla \mathbf{R} \|^2 \right) \right) \text{d}a. \quad (1.4)
$$

\(^1\)For the sake of simplicity, in this overview in the introduction, we have ignored the dependence of the minimization problems on $\nabla \Theta$, where $\Theta$ is the diffeomorphism which maps the midsurface $\omega$ of the fictitious Cartesian parameter space onto the midsurface of the curved reference configuration, the focus being to present only the essential energy terms appearing in the variational problems.
Problems (1.2) and (1.3) are non-elliptic with respect to \( m \) at given \( d \), while problem (1.4) is even linear with respect to \( m \) at given rotation field \( \overline{R} \), which is itself controlled by the curvature term \( \| \nabla \overline{R} \|^2 \). Therefore, in principle, (1.4) admits minimizers, while (1.2) and (1.3) in general do not.

In view of these mathematical deficiencies, in the literature we find many types of existence theorems, which treat certain approximations of (1.1). The above mentioned approach by Ciarlet and his co-authors [9, 16, 17] falls into this category. It has already been noted by Neff [28], that an independent control of the continuum rotations in quadratic, non-rank-one convex energies like the membrane-term in (1.1) is sufficient to resolve the non-rank-one convexity issue. This is precisely, what the Cosserat shell model is incorporating from the outset by considering not a single director as additional independent field, but a triad of rigid directors - the rotation field \( R \in SO(3) \).

Concerning the geometrically nonlinear theory of elastic Cosserat shells with drilling rotations including \( O(h^3) \)-terms, there is no existence theorem published in the literature, except [7], as far as we are aware of. Existence results for the related Cosserat model of initially planar shells have been obtained earlier by Neff [28, 33]. For our new model, we search for the minimizing solution pair of class \( H^1(\omega, R^3) \) for the translation vector and \( H^1(\omega, SO(3)) \) for the rotation tensor. For the proof of existence, we employ the direct methods of the calculus of variations, extensions of the techniques presented in [7, 8, 28, 33], coercivity and uniform convexity of the energy in the appropriate geometrically nonlinear strain and curvature measure. A first task is to show the existence of the solution for the theory including \( O(h^5) \)-terms. In this case the expression of the energy allows us to have a decent control on each term of the energy density, in order to show the coercivity and the convexity of the energy. A second task is to consider that part of the energy which contains only \( O(h^3) \)-terms. In this case the obtained minimization problem is not the same as that considered in [7, 8, 10, 11, 18, 19], since additional mixed bending-curvature and curvature energy-terms are included and the influence of the curved initial shell configuration appears explicitly in the expression of the coefficients of the energies for the reduced two-dimensional variational problem. For the \( O(h^3) \)-model, the problem of coercivity turns out to be more delicate, since some steps used to prove the coercivity for the \( O(h^5) \)-model cannot be done in the same manner. As a preparation for the existence proofs we will rewrite the energy in an equivalent form that allows us to prove the coercivity and convexity of the energy. Moreover, for the \( O(h^3) \)-model, we need to impose either a stronger assumption on the constitutive parameters or a relation between the thickness and the characteristic length. This behaviour highlights the importance and interest for including \( O(h^3) \)-terms.

2 The New Geometrically Nonlinear Cosserat Shell Model

2.1 Notation

In this paper, for \( a, b \in \mathbb{R}^n \) we let \( \langle a, b \rangle_{\mathbb{R}^n} \) denote the scalar product on \( \mathbb{R}^n \) with associated (squared) vector norm \( \|a\|_{\mathbb{R}^n}^2 = \langle a, a \rangle_{\mathbb{R}^n} \). The standard Euclidean scalar product on the set of real \( n \times m \) second order tensors \( \mathbb{R}^{n \times m} \) is given by \( \langle X, Y \rangle_{\mathbb{R}^{n \times m}} = \text{tr}(X Y^T) \), and thus the (squared) Frobenius tensor norm is \( \|X\|_{\mathbb{R}^{n \times m}}^2 = \langle X, X \rangle_{\mathbb{R}^{n \times m}} \). In the following we omit the subscripts \( \mathbb{R}^n, \mathbb{R}^{n \times m} \). The identity tensor on \( \mathbb{R}^{n \times n} \) will be denoted by \( \mathbb{1}_n \), so that \( \text{tr}(X) = \langle X, \mathbb{1}_n \rangle \). We let \( \text{Sym}(n) \) and \( \text{Sym}^+(n) \) denote the symmetric and positive definite symmetric tensors, respectively. We adopt the usual abbreviations of Lie-group theory, e.g., \( \text{GL}(n) = \{ X \in \mathbb{R}^{n \times n} \mid \det(X) \neq 0 \} \) the general linear group, \( \text{SO}(n) = \{ X \in \text{GL}(n) \mid X^T X = \mathbb{1}_n \} \) the special orthogonal group.
Fig. 1 The shell in its initial configuration \( \Omega_\xi \), the shell in the deformed configuration \( \Omega_c \), and the fictitious planar Cartesian reference configuration \( \Omega_h \). Here, \( \vec{R}_\xi \) is the elastic rotation field, \( Q_0 \) is the initial rotation from the fictitious planar Cartesian reference configuration to the initial configuration \( \Omega_\xi \), and \( \vec{R} \) is the total rotation field from the fictitious planar Cartesian reference configuration to the deformed configuration \( \Omega_c \).

Let \( \Omega \) be an open domain of \( \mathbb{R}^3 \). The usual Lebesgue spaces of square integrable functions, vector or tensor fields on \( \Omega \) with values in \( \mathbb{R}^3, \mathbb{R}^{3 \times 3} \) or \( \text{SO}(3) \), respectively will be denoted by \( L^2(\Omega; \mathbb{R}) \), \( L^2(\Omega; \mathbb{R}^3) \), \( L^2(\Omega; \mathbb{R}^{3 \times 3}) \) and \( L^2(\Omega; \text{SO}(3)) \), respectively. Moreover, we use the standard Sobolev spaces \( H^1(\Omega; \mathbb{R}) \) [1, 22, 26] of functions \( u \). For vector fields \( u = (u_1, u_2, u_3)^T \) with \( u_i \in H^1(\Omega) \), \( i = 1, 2, 3 \), we define \( \nabla u := (\nabla u_1 | \nabla u_2 | \nabla u_3)^T \). The corresponding Sobolev-space will be denoted by \( H^1(\Omega; \mathbb{R}^3) \). If a tensor \( Q : \Omega \to \text{SO}(3) \) has the components in \( H^1(\Omega; \mathbb{R}) \), then we mark this by writing \( Q \in H^1(\Omega; \text{SO}(3)) \). When writing the norm in the corresponding Sobolev-space we will specify the space in subscript. The space will be omitted only when the Frobenius norm or scalar product is considered.

2.2 The Deformation of Cosserat Shells

Let \( \Omega_\xi \subset \mathbb{R}^3 \) be a three-dimensional shell-like thin domain. In a fixed standard base \( e_1, e_2, e_3 \) of \( \mathbb{R}^3 \), a generic point of \( \Omega_\xi \) will be denoted by \( (\xi_1, \xi_2, \xi_3) \). The elastic material constituting the shell is assumed to be homogeneous and isotropic and the reference configuration \( \Omega_\xi \) is assumed to be a natural state. The deformation of the body occupying the domain \( \Omega_\xi \) is described by a vector map \( \varphi_\xi : \Omega_\xi \subset \mathbb{R}^3 \to \mathbb{R}^3 \) (called deformation) and by a microrotation tensor \( \vec{R}_\xi : \Omega_\xi \subset \mathbb{R}^3 \to \text{SO}(3) \). We denote the current configuration (deformed configuration) by \( \Omega_c := \varphi_\xi(\Omega_\xi) \subset \mathbb{R}^3 \), see Fig. 1.

In what follows, we consider the fictitious Cartesian (planar) configuration \( \Omega_h \) of the body. This parameter domain \( \Omega_h \subset \mathbb{R}^3 \) is a right cylinder of the form

\[
\Omega_h = \left\{(x_1, x_2, x_3) \mid (x_1, x_2) \in \omega, \quad -\frac{h}{2} < x_3 < \frac{h}{2}\right\} = \omega \times \left(-\frac{h}{2}, \frac{h}{2}\right),
\]
where $\omega \subset \mathbb{R}^2$ is a bounded domain with Lipschitz boundary $\partial \omega$ and the constant length $h>0$ is the thickness of the shell. For shell–like bodies we consider the domain $\Omega_h$ to be thin, i.e., the thickness $h$ is small. We assume furthermore that there exists a $C^1$-diffeomorphism $\Theta : \mathbb{R}^3 \to \mathbb{R}^3$ in the specific form
\[
\Theta(x_1, x_2, x_3) = y_0(x_1, x_2) + x_3 n_0(x_1, x_2), \quad n_0 = \frac{\partial_{x_1} y_0 \times \partial_{x_2} y_0}{\|\partial_{x_1} y_0 \times \partial_{x_2} y_0\|},
\] (2.1)
where $y_0 : \omega \to \mathbb{R}^3$ is a function of class $C^2(\omega)$, so that $\Theta$ maps the fictitious planar Cartesian parameter space $\Omega_\omega$ onto the initially curved reference configuration of the shell $\Theta(\Omega_\omega) = \Omega_\xi$, $\Theta(x_1, x_2, x_3) = (\xi_1, \xi_2, \xi_3)$. The diffeomorphism $\Theta$ maps the mid-surface $\omega_\xi = y_0(\omega)$ of $\Omega_\xi$ and $n_0$ is the unit normal vector to $\omega_\xi$. For simplicity and where no confusions may arise, we will omit subsequently to write explicitly the arguments $(x_1, x_2, x_3)$ of the diffeomorphism $\Theta$ or we will specify only its dependence on $x_3$. We use the polar decomposition [37] of $\nabla_\xi \Theta(x_3)$ and write $\nabla_\xi \Theta(x_3) = Q_0(x_3) U_0(x_3)$, $Q_0(x_3) = \text{pol}(\nabla_\xi \Theta(x_3)) \in SO(3)$, $U_0(x_3) \in \text{Sym}^+(3)$.

Let us remark that
\[
\nabla_\xi \Theta(x_3) = (\nabla y_0|n_0) + x_3 (\nabla n_0|0) \forall x_3 \in \left(-\frac{h}{2}, \frac{h}{2}\right), \quad \nabla_\xi \Theta(0) = (\nabla y_0|n_0),
\]
and that $\det(\nabla y_0|n_0) = \sqrt{\det((\nabla y_0)^T \nabla y_0)}$ represents the surface element.

In the following, we consider the Weingarten map\(^2\) (or shape operator) on $y_0(\omega)$ defined by its associated matrix $L_{y_0} = I_{y_0}^T \Pi_{y_0} \in \mathbb{R}^{2 \times 2}$, where $I_{y_0} := [\nabla y_0]^T \nabla y_0 \in \mathbb{R}^{2 \times 2}$ and $\Pi_{y_0} := -[\nabla y_0]^T \nabla n_0 \in \mathbb{R}^{2 \times 2}$ are the matrix representations of the first fundamental form (metric) and the second fundamental form, respectively. Then, the Gauss curvature $K$ of the surface $y_0(\omega)$ is determined by $K := \det(L_{y_0})$ and the mean curvature $H$ through $2H := \text{tr}(L_{y_0})$.

We also need the tensors defined by:
\[
A_{y_0} := (\nabla y_0|0) \left[\nabla_\xi \Theta(0)\right]^{-1} \in \mathbb{R}^{3 \times 3}, \quad B_{y_0} := -(\nabla n_0|0) \left[\nabla_\xi \Theta(0)\right]^{-1} \in \mathbb{R}^{3 \times 3},
\] (2.3)
and the so-called alternator tensor $C_{y_0}$ of the surface [52]
\[
C_{y_0} := \det(\nabla_\xi \Theta(0)) \left[\nabla_\xi \Theta(0)\right]^{-T} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \left[\nabla_\xi \Theta(0)\right]^{-1}.
\] (2.4)

Now, let us define the map $\varphi : \Omega_h \to \Omega_e$, $\varphi(x_1, x_2, x_3) = \varphi_\xi(\Theta(x_1, x_2, x_3))$. We view $\varphi$ as a function which maps the fictitious planar reference configuration $\Omega_\omega$ into the deformed configuration $\Omega_e$. We also consider the elastic microrotation $Q_{e,s} : \Omega_h \to SO(3)$, $Q_{e,s}(x_1, x_2, x_3) := R_\xi(\Theta(x_1, x_2, x_3))$.

In [21], by assuming that $Q_{e,s}(x_1, x_2, x_3) = \overline{Q}_{e,s}(x_1, x_2)$ and considering an 8-parameter quadratic ansatz in the thickness direction for the reconstructed total deformation $\varphi_\xi : \Omega_h \subset \mathbb{R}^3 \to \mathbb{R}^3$ of the shell-like body, we have obtained a two-dimensional minimization problem in which the energy density is expressed in terms of the following tensor fields on the

\(^2\) We identify the Weingarten map, the first fundamental form and the second fundamental form with their associated matrices in the fixed base vector $e_1, e_2, e_3$. 
The Isotropic Cosserat Shell Model Including Terms up to $O(h^5)\ldots$
\[ W_{\text{shell}}(S, T) = \mu \langle \text{sym } S, \text{sym } T \rangle + \mu_c \langle \text{skew } S, \text{skew } T \rangle + \frac{\lambda \mu}{\lambda + 2\mu} \text{tr}(S) \text{tr}(T), \]

\[ W_{\text{mp}}(S) = \mu \| \text{sym } S \|^2 + \mu_c \| \text{skew } S \|^2 + \frac{\lambda^2}{2(\lambda + 2\mu)} \| \text{tr}(S) \|^2, \]

\[ W_{\text{curv}}(S) = \mu L_c^2 \left( b_1 \| \text{dev sym } S \|^2 + b_2 \| \text{skew } S \|^2 + b_3 \| \text{tr}(S) \|^2 \right). \]

The parameters \( \mu \) and \( \lambda \) are the Lamé constants of classical isotropic elasticity, \( \kappa = \frac{2\mu + 3\lambda}{3} \) is the infinitesimal bulk modulus, \( b_1, b_2, b_3 \) are non-dimensional constitutive curvature coefficients (weights), \( \mu_c \geq 0 \) is called the Cosserat couple modulus and \( L_c \) introduces an internal length which is characteristic for the material, e.g., related to the grain size in a polycrystal. The internal length \( L_c > 0 \) is responsible for size effects in the sense that smaller samples are relatively stiffer than larger samples. If not stated otherwise, we assume that \( \mu > 0, \kappa > 0, \mu_c > 0, b_1 > 0, b_2 > 0, b_3 > 0 \). Here, \( \mu L_c^2 \) plays the role of a dimensional agreement factor. Without loss of generality, we may assume that \( 0 < b_1 < 1, 0 < b_2 < 1, 0 < b_3 < 1 \). All constitutive coefficients are deduced from the three-dimensional formulation, without using any a posteriori fitting of some two-dimensional constitutive coefficients.

The potential of applied external loads \( \Pi(m, \overline{Q}_{e,s}) \) appearing in (2.6) is expressed by

\[ \Pi(m, \overline{Q}_{e,s}) = \Pi_\omega(m, \overline{Q}_{e,s}) + \Pi_{\gamma_t}(m, \overline{Q}_{e,s}), \quad \text{with} \]

\[ \Pi_\omega(m, \overline{Q}_{e,s}) = \int_\omega \langle f, u \rangle da + \Lambda_\omega(\overline{Q}_{e,s}) \quad \text{and} \quad \Pi_{\gamma_t}(m, \overline{Q}_{e,s}) = \int_{\gamma_t} \langle t, u \rangle ds + \Lambda_{\gamma_t}(\overline{Q}_{e,s}), \]

where \( u(x_1, x_2) = m(x_1, x_2) - y_0(x_1, x_2) \) is the displacement vector of the midsurface, \( \Pi_\omega(m, \overline{Q}_{e,s}) \) is the total potential of the external surface loads \( f \) and of the external applied body couples \( \Lambda_\omega(\overline{Q}_{e,s}) \), while \( \Pi_{\gamma_t}(m, \overline{Q}_{e,s}) \) is the total potential of the external boundary loads \( t \) and of the external boundary couples \( \Lambda_{\gamma_t}(\overline{Q}_{e,s}) \). Here, \( \gamma_t \) and \( \gamma_d \) are nonempty subsets of the boundary of \( \omega \) such that \( \gamma_t \cup \gamma_d = \partial \omega \) and \( \gamma_t \cap \gamma_d = \emptyset \). On \( \gamma_t \) we have considered traction boundary conditions, while on \( \gamma_d \) we have the Dirichlet-type boundary conditions:\(^3\)

\[ m|_{\gamma_d} = m^*, \quad \text{simply supported (fixed, welded)}, \]

\[ \overline{Q}_{e,s}|_{\gamma_d} = \overline{Q}_{e,s}^*, \quad \text{clamped}, \]

where the boundary conditions are to be understood in the sense of traces.

The functions \( \Lambda_\omega, \Lambda_{\gamma_t} : L^2(\omega, \text{SO}(3)) \to \mathbb{R} \) are expressed in terms of the loads from the three-dimensional parental variational problem, see [21], and they are assumed to be continuous and bounded operators.

**Remark 2.1** Our model [21] is constructed under the following assumptions upon the thickness

\[ h |\kappa_1| < \frac{1}{2} \quad \text{and} \quad h |\kappa_2| < \frac{1}{2}, \]

where \( \kappa_1 \) and \( \kappa_2 \) denote the principal curvatures of the initial underformed surface.

\(^3\)The existence theory works also for free microrotations at the boundary since \( \text{SO}(3) \) is a compact manifold.
We will consider materials for which the Poisson ratio \( \nu = \frac{\mu(3\lambda + 2\mu)}{2(\lambda + \mu)} \) and Young’s modulus \( E = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu} \) are such that \(-\frac{1}{2} < \nu < \frac{1}{2}\) and \( E > 0\). This assumption implies that \( 2\lambda + \mu > 0\). Under these assumptions on the constitutive coefficients, together with the positivity of \(\mu, \mu_c, b_1, b_2\) and \(b_3\), and the orthogonal Cartan-decomposition of the Lie-algebra \( \mathfrak{gl}(3) \), with

\[
W_{\text{shell}}(S) = \mu \|\text{dev sym } S\|^2 + \mu_c \|\text{skew } S\|^2 + \frac{2\mu (2\lambda + \mu)}{3(\lambda + 2\mu)} [\text{tr}(S)]^2, \tag{2.11}
\]

it follows that there exists the positive constants \( c_1^+, c_2^+, \lambda_1^+ \) such that

\[
\begin{align*}
C_1^+ &\|S\|^2 \geq W_{\text{shell}}(S) \geq c_1^+ \|S\|^2, \\
C_2^+ &\|S\|^2 \geq W_{\text{curv}}(S) \geq c_2^+ \|S\|^2 \quad \forall S \in \mathbb{R}^{3\times 3}.
\end{align*}
\tag{2.12}
\]

Hence, we note

\[
W_{\text{mp}}(S) = W_{\text{shell}}(S) + \frac{\lambda^2}{2(\lambda + 2\mu)} (\text{tr}(S))^2 \geq W_{\text{shell}}(S) \geq c_1^+ \|S\|^2. \tag{2.13}
\]

### 3 Existence of Minimizers for the Cosserat Shell Model of Order \( O(h^5) \)

In order to establish an existence result by the direct methods of the calculus of variations, we need to show the coercivity of the elastically stored shell energy density.

#### 3.1 Coercivity and Uniform Convexity in the Theory of Order \( O(h^5) \)

**Proposition 3.1** [Coercivity in the theory including terms up to order \( O(h^5) \)] For sufficiently small values of the thickness \( h \) such that \( h |\kappa_1| < \frac{1}{2} \) and \( h |\kappa_2| < \frac{1}{2} \) and for constitutive coefficients satisfying \( \mu > 0, \mu_c > 0, 2\lambda + \mu > 0, b_1 > 0, b_2 > 0 \) and \( b_3 > 0 \), the energy density

\[
W(\mathcal{E}_{m,s}, \mathcal{K}_{e,s}) = W_{\text{memb}}(\mathcal{E}_{m,s}) + W_{\text{memb,bend}}(\mathcal{E}_{m,s}, \mathcal{K}_{e,s}) + W_{\text{bend,curv}}(\mathcal{K}_{e,s}) \tag{3.1}
\]

is coercive in the sense that there exists a constant \( a_1^+ > 0 \) such that

\[
W(\mathcal{E}_{m,s}, \mathcal{K}_{e,s}) \geq a_1^+ (\|\mathcal{E}_{m,s}\|^2 + \|\mathcal{K}_{e,s}\|^2), \tag{3.2}
\]

where \( a_1^+ \) depends on the constitutive coefficients.

**Proof** In order to prove the coercivity note that the principal curvatures \( \kappa_1, \kappa_2 \) are the solutions of the characteristic equation of \( L_{y_0} \), i.e., \( \kappa^2 - \text{tr}(L_{y_0}) \kappa + \det(L_{y_0}) = \kappa^2 - 2H\kappa + K = 0 \). Therefore, from the assumptions \( h |\kappa_1| < \frac{1}{2}, h |\kappa_2| < \frac{1}{2} \), it follows that

\[
h^2 |K| = h^2 |\kappa_1| |\kappa_2| < \frac{1}{4} \quad \text{and} \quad 2h |H| = h |\kappa_1 + \kappa_2| < 1. \tag{3.3}
\]

Therefore, \( h - K \frac{h^3}{12} > 0 \) and \( \frac{h^3}{12} - K \frac{h^5}{80} > 0 \) and

\[
W(\mathcal{E}_{m,s}, \mathcal{K}_{e,s}) \geq \left( h + K \frac{h^3}{12} \right) W_{\text{shell}}(\mathcal{E}_{m,s}) + \left( \frac{h^3}{12} - K \frac{h^5}{80} \right) W_{\text{shell}}(\mathcal{E}_{m,s} B_{y_0} + C_{y_0} \mathcal{K}_{e,s})
\]
\[- \frac{h^3}{3} |H| |W_{\text{shell}}(E_{m,s}, E_{m,s}B_{y_0} + C_{y_0} K_{e,s})| \]
\[- \frac{h^3}{12} 2 |W_{\text{shell}}(E_{m,s}, (E_{m,s}B_{y_0} + C_{y_0} K_{e,s})B_{y_0})| \]
\[+ \frac{h^5}{80} W_{\text{mp}}((E_{m,s}B_{y_0} + C_{y_0} K_{e,s})B_{y_0}) + \left( h - \frac{h^3}{12} \right) W_{\text{curv}}(K_{e,s}). \quad (3.4)\]

Using the Cauchy–Schwarz inequality we deduce
\[W(E_{m,s}, K_{e,s}) \geq \left( h + K \frac{h^3}{12} \right) W_{\text{shell}}(E_{m,s}) \]
\[- \frac{1}{3} |H| \left[ h^2 W_{\text{shell}}(E_{m,s}) \right]^\frac{1}{2} \left[ h^4 W_{\text{shell}}(E_{m,s}B_{y_0} + C_{y_0} K_{e,s}) \right]^\frac{1}{2} \]
\[+ \left( \frac{h^3}{12} - K \frac{h^5}{80} \right) W_{\text{shell}}(E_{m,s}B_{y_0} + C_{y_0} K_{e,s}) \]
\[- \frac{1}{6} \left[ h W_{\text{shell}}(E_{m,s}) \right]^\frac{1}{2} \left[ h^5 W_{\text{shell}}((E_{m,s}B_{y_0} + C_{y_0} K_{e,s})B_{y_0}) \right]^\frac{1}{2} \]
\[+ \frac{h^5}{80} W_{\text{mp}}((E_{m,s}B_{y_0} + C_{y_0} K_{e,s})B_{y_0}) + \left( h - \frac{h^3}{12} \right) W_{\text{curv}}(K_{e,s}). \quad (3.5)\]

The arithmetic-geometric mean inequality leads to the estimate
\[W(E_{m,s}, K_{e,s}) \geq \left( h + K \frac{h^3}{12} - \frac{h^2}{6} \varepsilon |H| \right) W_{\text{shell}}(E_{m,s}) \]
\[+ \left( \frac{h^3}{12} - K \frac{h^5}{80} - \frac{h^4}{6} \varepsilon |H| \right) W_{\text{shell}}(E_{m,s}B_{y_0} + C_{y_0} K_{e,s}) \]
\[- \frac{h}{12} \delta W_{\text{shell}}(E_{m,s}) - \frac{h^5}{12 \delta} W_{\text{shell}}((E_{m,s}B_{y_0} + C_{y_0} K_{e,s})B_{y_0}) \]
\[+ \frac{h^5}{80} W_{\text{mp}}((E_{m,s}B_{y_0} + C_{y_0} K_{e,s})B_{y_0}) + \left( h - \frac{h^3}{12} \right) W_{\text{curv}}(K_{e,s}) \]
\[\forall \varepsilon > 0 \text{ and } \delta > 0. \quad (3.6)\]

Using (2.13), we obtain
\[W(E_{m,s}, K_{e,s}) \geq \left( h - \frac{h}{12} \delta + K \frac{h^3}{12} - \frac{h^2}{6} \varepsilon |H| \right) W_{\text{shell}}(E_{m,s}) \]
\[+ \left( \frac{h^3}{12} - K \frac{h^5}{80} - \frac{h^4}{6} \varepsilon |H| \right) W_{\text{shell}}(E_{m,s}B_{y_0} + C_{y_0} K_{e,s}) \]
\[+ \left( \frac{h^4}{80} - \frac{h^5}{12 \delta} \right) W_{\text{shell}}((E_{m,s}B_{y_0} + C_{y_0} K_{e,s})B_{y_0}) \]
\[+ \left( h - K \frac{h^3}{12} \right) W_{\text{curv}}(K_{e,s}) \quad \forall \varepsilon > 0 \text{ and } \delta > 0. \quad (3.7)\]
Taking δ = 8 and and ε = 2 we get\(^4\) that

\[
W(\mathcal{E}_{m,s}, K_{e,s}) \geq h \left[ \frac{1}{3} - \frac{\kappa}{12} - \frac{h}{3} |H| \right] W_{\text{shell}}(\mathcal{E}_{m,s}) \\
+ \frac{h^3}{12} \left( 1 - |K| \frac{12h^2}{80} - h |H| \right) W_{\text{shell}}(\mathcal{E}_{m,s} B_{y0} + C_{y0} K_{e,s}) \\
+ \left( h - |K| \frac{h^3}{12} \right) W_{\text{curv}}(K_{e,s}).
\]

(3.8)

In view of (3.3) and (2.12), we deduce

\[
W(\mathcal{E}_{m,s}, K_{e,s}) \geq h \left[ \frac{7}{48} W_{\text{shell}}(\mathcal{E}_{m,s}) + \frac{h^3}{12} \frac{37}{80} W_{\text{shell}}(\mathcal{E}_{m,s} B_{y0} + C_{y0} K_{e,s}) + h \frac{47}{48} W_{\text{curv}}(K_{e,s}) \right] \\
\geq h \left[ \frac{7}{48} c_1^+ \|\mathcal{E}_{m,s}\|^2 + \frac{h^3}{12} \frac{37}{80} c_1^+ \|\mathcal{E}_{m,s} B_{y0} + C_{y0} K_{e,s}\|^2 + h \frac{47}{48} c_2^+ \|K_{e,s}\|^2 \right].
\]

(3.9)

The desired constant \(a_1^+\) from the conclusion can be chosen as \(a_1^+ = \min \left\{ h \left[ \frac{7}{48} c_1^+, h \frac{47}{48} c_2^+ \right] \right\}.\)

\[\Box\]

**Corollary 3.2** [Uniform convexity in the theory including terms up to order \(O(h^5)\)] For sufficiently small values of the thickness \(h\) such that \(h |\kappa_1| < \frac{1}{2}\) and \(h |\kappa_2| < \frac{1}{2}\) and for constitutive coefficients such that \(\mu > 0, \mu_c > 0, 2 \lambda + \mu > 0, b_1 > 0, b_2 > 0, b_3 > 0\), the energy density

\[
W(\mathcal{E}_{m,s}, K_{e,s}) = W_{\text{memb}}(\mathcal{E}_{m,s}) + W_{\text{memb,bend}}(\mathcal{E}_{m,s}, K_{e,s}) + W_{\text{bend,curv}}(K_{e,s})
\]

(3.10)

is uniformly convex in \((\mathcal{E}_{m,s}, K_{e,s})\), i.e., there exists a constant \(a_1^+ > 0\) such that

\[
D^2 W(\mathcal{E}_{m,s}, K_{e,s}).[(H_1, H_2), (H_1, H_2)] \geq a_1^+ (\|H_1\|^2 + \|H_2\|^2) \quad \forall H_1, H_2 \in \mathbb{R}^{3 \times 3}.\]

(3.11)

**Proof** For a bilinear expression \(W(\mathcal{E}_{m,s}, K_{e,s})\) in terms of \(\mathcal{E}_{m,s}\) and \(K_{e,s}\), the second derivative with respect to these argument variables coincides with the function itself, modulo a scalar multiplication. We will prove this known fact only for two terms of the energy and we show that

\[
D^2(\|\text{sym} \mathcal{E}_{m,s}\|^2).[(H_1, H_2), (H_1, H_2)] = 2 \|\text{sym} H_1\|^2 \quad \text{and} \quad (3.12)
\]

\[
D^2(\langle \text{sym} \mathcal{E}_{m,s}, \text{sym}(\mathcal{E}_{m,s} B_{y0} + C_{y0} K_{e,s}) \rangle).[(H_1, H_2), (H_1, H_2)] \\
= 2 \langle \text{sym} H_1, \text{sym}(H_1 B_{y0} + C_{y0} H_2) \rangle.
\]

Indeed, on the one hand, we have \(D_F(\|F\|^2).H = 2 \langle F, H \rangle, \langle D^2_F(\|F\|^2).H, H \rangle = 2 \|H\|^2\). Useful in our calculation is that

\(^4\)This step cannot be repeated in the proof of the coercivity up to order \(O(h^3)\), since \(W_{\text{shell}}((\mathcal{E}_{m,s} B_{y0} + C_{y0} K_{e,s}) B_{y0})\) cannot be skipped. This is the reason why we have to choose another strategy to obtain the desired estimates.
D^2 W(\mathcal{E}_{m,s}, \mathcal{K}_{e,s}). [(H_1, H_2), (H_1, H_2)] = D^2_{\mathcal{E}_{m,s}, \mathcal{E}_{m,s}} W(\mathcal{E}_{m,s}, \mathcal{K}_{e,s}). (H_1, H_1) \\
+ 2D_{\mathcal{K}_{e,s}}[D_{\mathcal{E}_{m,s}} W(\mathcal{E}_{m,s}, \mathcal{K}_{e,s}). (H_1, H_2)]. (H_2, H_2) \\
+ D^2_{\mathcal{K}_{e,s}, \mathcal{K}_{e,s}} W(\mathcal{E}_{m,s}, \mathcal{K}_{e,s}). (H_2, H_2).

Since \text{sym} : \mathbb{R}^{3 \times 3} \rightarrow \text{Sym}(3) is a linear operator, we obtain

\[ D^2(\|\text{sym} \mathcal{E}_{m,s}\|^2). [(H_1, H_2), (H_1, H_2)] = D^2_{\mathcal{E}_{m,s}, \mathcal{E}_{m,s}} (\|\text{sym} \mathcal{E}_{m,s}\|^2). (H_1, H_1) = 2 \|\text{sym} H_1\|^2, \] (3.13)

which proves (3.12)_1. On the other hand, it holds

\[ D^2_{\mathcal{E}_{m,s}, \mathcal{E}_{m,s}} ([\text{sym} \mathcal{E}_{m,s}, \text{sym}(\mathcal{E}_{m,s}B_{y_0} + C_{y_0} \mathcal{K}_{e,s})]). (H_1, H_1) = 2(\text{sym} H_1, H_1B_{y_0}), \]

\[ D^2_{\mathcal{K}_{e,s}, \mathcal{K}_{e,s}} ([\text{sym} \mathcal{E}_{m,s}, \text{sym}(\mathcal{E}_{m,s}B_{y_0} + C_{y_0} \mathcal{K}_{e,s})]). (H_2, H_2) = 0, \] (3.14)

\[ D_{\mathcal{K}_{e,s}} [D_{\mathcal{E}_{m,s}} ([\text{sym} \mathcal{E}_{m,s}, \text{sym}(\mathcal{E}_{m,s}B_{y_0} + C_{y_0} \mathcal{K}_{e,s})]).(H_1, H_1)].(H_2, H_2) = [\text{sym} H_1, C_{y_0} H_2]. \]

Therefore

\[ D^2(\|\text{sym} \mathcal{E}_{m,s}\|^2). [(H_1, H_2), (H_1, H_2)] = 2(\text{sym} H_1, \text{sym}(H_1B_{y_0} + C_{y_0} H_2)), \] (3.15)

which proves (3.12)_2. In conclusion, after making similar calculations as above for the other terms appearing in the expression of \( W(\mathcal{E}_{m,s}, \mathcal{K}_{e,s}) \), we obtain

\[ D^2 W(\mathcal{E}_{m,s}, \mathcal{K}_{e,s}). [(H_1, H_2), (H_1, H_2)] = 2 W(H_1, H_2). \] (3.16)

Bounding the function \( W(\mathcal{E}_{m,s}, \mathcal{K}_{e,s}) \) for all \( \mathcal{E}_{m,s}, \mathcal{K}_{e,s} \in \mathbb{R}^{3 \times 3} \) away from zero amounts therefore to showing that \( D^2 W(\mathcal{E}_{m,s}, \mathcal{K}_{e,s}) \) is positive definite. Hence, the coercivity of \( W(\mathcal{E}_{m,s}, \mathcal{K}_{e,s}) \) expressed by Proposition 3.1 implies uniform convexity in the chosen variables.

3.2 The Existence Result in the Theory of Order \( O(h^5) \)

In this section, we prove the first main result of our paper. The admissible set \( \mathcal{A} \) of solutions is defined by

\[ \mathcal{A} = \{(m, \overline{Q}_{e,s}) \in H^1(\omega, \mathbb{R}^3) \times H^1(\omega, \text{SO}(3)) \mid m|_{\gamma_0} = m^*, \overline{Q}_{e,s}|_{\gamma_0} = \overline{Q}_{e,s}^*\}. \] (3.17)

where the boundary conditions are to be understood in the sense of traces.

**Theorem 3.3** [Existence result for the theory including terms up to order \( O(h^5) \)] Assume that the external loads satisfy the conditions

\[ f \in L^2(\omega, \mathbb{R}^3), \quad t \in L^2(\gamma_1, \mathbb{R}^3), \] (3.18)

and the boundary data satisfy the conditions

\[ m^* \in H^1(\omega, \mathbb{R}^3), \quad \overline{Q}_{e,s}^* \in H^1(\omega, \text{SO}(3)). \] (3.19)
Assume that the following conditions concerning the initial configuration are satisfied: \( y_0 : \omega \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3 \) is a continuous injective mapping and
\[
y_0 \in H^1(\omega, \mathbb{R}^3), \quad Q_0(0) \in H^1(\omega, SO(3)), \quad \nabla_x \Theta(0) \in L^\infty(\omega, \mathbb{R}^{3 \times 3}),
\]

where \( a_0 \) is a constant. Then, for sufficiently small values of the thickness \( h \) such that \( h|k_1| < \frac{1}{2} \) and \( h|k_2| < \frac{1}{2} \) and for constitutive coefficients such that \( \mu > 0, \mu_c > 0, 2\lambda + \mu > 0, b_1 > 0, b_2 > 0 \) and \( b_3 > 0 \), the minimization problem (2.6)–(2.10) admits at least one minimizing solution pair \((m, Q_{e,s}) \in A\).

**Proof** We employ the direct methods of the calculus of variations, similar to [7, 29, 34]. However, in comparison to [7], due to the fact that we use only matrix notation, some steps are shortened. In Proposition 3.1 and Corollary 3.2, we have shown that the strain energy density \( W(m,s) \) is a quadratic convex and coercive function of \((m,s)\).

The hypothesis (3.18) and the boundedness of \( \Pi_{S^0} \) and \( \Pi_{\beta_i} \) imply that there exists a constant \( C > 0 \) such that\(^5\)
\[
|\Pi(m, Q_{e,s})| \leq C \left( \|m - y_0\|_{L^2(\omega)} + \|m - y_0\|_{L^2(\gamma)} + \|Q_{e,s}\|_{L^2(\omega)} \right) \quad \forall (m, Q_{e,s}) \in H^1(\omega, \mathbb{R}^3) \times H^1(\omega, SO(3)).
\]

We have \( \|Q_{e,s}\|^2 = tr(Q_{e,s}^T Q_{e,s}) = tr(\mathbb{I}) = 3, \forall Q_{e,s} \in SO(3) \). Hence, there exists a constant \( C > 0 \) such that
\[
|\Pi(m, Q_{e,s})| \leq C \left( \|m\|_{H^1(\omega)} + 1 \right) \quad \forall (m, Q_{e,s}) \in H^1(\omega, \mathbb{R}^3) \times H^1(\omega, SO(3)). \tag{3.21}
\]

Considering
\[
R_s(x_1, x_2) = Q_{e,s}(x_1, x_2) Q_0(x_1, x_2, 0) \in SO(3), \tag{3.22}
\]
we observe that
\[
E_{m,s} = Q_0(R_s^T (\nabla m |Q_{e,s} \nabla_x \Theta(0) e_3) - Q_0^T (\nabla y_0 |n_0) [\nabla_x \Theta(0)]^{-1})
\]
\[
= Q_0(R_s^T \nabla m - Q_0^T \nabla y_0 |[\nabla_x \Theta(0)]^{-1}). \tag{3.23}
\]

The lifted quantity \( \tilde{y}_0 = (\nabla y_0 |n_0) \in \text{Sym}(3) \) is positive definite and also it’s inverse is positive definite. Using the above relation we obtain
\[
\|E_{m,s}\|^2 = \left\{ Q_0^T Q_0 (R_s^T \nabla m - Q_0^T \nabla y_0 |0, (R_s^T \nabla m - Q_0^T \nabla y_0 |0) \tilde{y}_0^{-1}\right\}
\]
\[
= \left\{ \tilde{y}_0^{-1} (R_s^T \nabla m - Q_0^T \nabla y_0 |0) \right\}^T \left\{ (R_s^T \nabla m - Q_0^T \nabla y_0 |0) \right\} \geq \lambda_0^2 \|R_s^T \nabla m - Q_0^T \nabla y_0 |0\|^2. \tag{3.24}
\]

\(^5\)By \( C \) and \( C_i, i \in \mathbb{N} \), we will denote (positive) constants that may vary from estimate to estimate but will remain independent of \( m, \nabla m \) and \( Q_{e,s} \).
where $\lambda_0$ is the smallest eigenvalue of the positive definite matrix $\hat{\Gamma}_{y_0}^{-1}$. Similarly, we deduce that
\[
\|k_{e,s}\|^2 = \|\langle axl(\hat{Q}_{e,s}^T \partial s_1 \hat{Q}_{e,s}) | axl(\hat{Q}_{e,s}^T \partial s_2 \hat{Q}_{e,s}) \rangle | 0 \|
\]
\[
\geq \lambda_0^2 \|\langle axl(\hat{Q}_{e,s}^T \partial s_1 \hat{Q}_{e,s}) | axl(\hat{Q}_{e,s}^T \partial s_2 \hat{Q}_{e,s}) \rangle | 0 \|^T\]
\[
\geq \lambda_0^2 \|\langle axl(\hat{Q}_{e,s}^T \partial s_1 \hat{Q}_{e,s}) | axl(\hat{Q}_{e,s}^T \partial s_2 \hat{Q}_{e,s}) \rangle | 0 \|^T\). \tag{3.25}
\]

From (3.24) we have
\[
\|e_{m,s}\|^2 \geq \lambda_0^2 \|\hat{R}_s^T \nabla m\|^2 - 2 \langle \hat{R}_s^T \nabla m, Q_{y_0}^T \nabla y_0 \rangle + \|Q_{y_0}^T \nabla y_0\|^2, \tag{3.26}
\]
Since $\|\hat{R}_s^T \nabla m\|^2 = \|\nabla m\|^2$ and $\|Q_{y_0}^T \nabla y_0\|^2 = \|\nabla y_0\|^2$, after integrating over $\omega$, using (3.2), the Cauchy–Schwarz inequality and the hypothesis upon $y_0$, gives us the estimate
\[
\|e_{m,s}\|^2_{L^2(\omega)} \geq \lambda_0^2 \|\nabla m\|^2_{L^2(\omega)} - 2 \|\nabla m\|_{L^2(\omega)} \|\nabla y_0\|_{L^2(\omega)} + \|\nabla y_0\|^2_{L^2(\omega)}
\]
\[
\geq \lambda_0^2 \|\nabla m\|^2_{L^2(\omega)} - C_1 \|\nabla m\|_{L^2(\omega)} + C_2, \tag{3.27}
\]
for some positive constants $C_1 > 0$, $C_2 > 0$.

By virtue of the coercivity of the internal energy and (3.20), (3.21) and (3.24), the functional $I(m, \hat{Q}_{e,s})$ is bounded from below
\[
I(m, \hat{Q}_{e,s}) \geq C_1 \int_\omega \|e_{m,s}\|^2 \det[\nabla \Theta(0)] da - \Pi(m, \hat{Q}_{e,s})
\]
\[
\geq C_2 a_0 \|e_{m,s}\|^2_{L^2(\omega)} - C_3 (\|m\|_{H^1(\omega)} + 1)
\]
\[
\geq C_4 \|\nabla m\|^2_{L^2(\omega)} - C_5 \|m\|_{H^1(\omega)} + C_6
\]
\[
\forall (m, \hat{Q}_{e,s}) \in H^1(\omega, \mathbb{R}^3) \times H^1(\omega, SO(3)), \tag{3.28}
\]
with $C_i > 0$, $i = 1, 2, \ldots, 6$. We also obtain, applying the Poincaré–inequality, that there exists a constant $C > 0$ such that
\[
\|\nabla m\|^2_{L^2(\omega)} \geq (\|\nabla (m - m^*)\|_{L^2(\omega)} - \|\nabla m^*\|_{L^2(\omega)})^2
\]
\[
\geq C \|m - m^*\|^2_{H^1(\omega)} - 2 \|m - m^*\|_{H^1(\omega)} \|\nabla m^*\|_{L^2(\omega)} + \|\nabla m^*\|^2_{L^2(\omega)}
\]
\[
\geq C \|m - m^*\|^2_{H^1(\omega)} - \frac{1}{\varepsilon} \|m - m^*\|^2_{H^1(\omega)} - \varepsilon \|\nabla m^*\|^2_{L^2(\omega)} + \|\nabla m^*\|^2_{L^2(\omega)}
\]
\[
\forall \varepsilon > 0. \tag{3.29}
\]
Therefore, by choosing $\varepsilon > 0$ small enough, (3.28) ensures the existence of constants $C_1 > 0$ and $C_2 \in \mathbb{R}$ such that
\[
I(m, \hat{Q}_{e,s}) \geq C_1 \|m - m^*\|^2_{H^1(\omega)} + C_2 \quad \forall (m, \hat{Q}_{e,s}) \in H^1(\omega, \mathbb{R}^3) \times H^1(\omega, SO(3)), \tag{3.30}
\]
i.e., the functional $I(m, \overline{Q}_{e,s})$ is bounded from below on $\mathcal{A}$.

Hence, there exists an infimizing sequence $\{m_k, \overline{Q}_{e,s}\}_{k=1}^{\infty}$ in $\mathcal{A}$, such that

$$
\lim_{k \to \infty} I(m_k, \overline{Q}_{e,s}) = \inf \{ I(m, \overline{Q}_{e,s}) \mid (m, \overline{Q}_{e,s}) \in \mathcal{A} \}. \tag{3.31}
$$

Since we have $I(m^*, \overline{Q}_{e,s}^*) < \infty$, in view of the conditions (3.19), the infimizing sequence $\{m_k, \overline{Q}_{e,s}\}_{k=1}^{\infty}$ can be chosen such that

$$
I(m_k, \overline{Q}_{e,s}) \leq I(m^*, \overline{Q}_{e,s}^*) < \infty, \quad \forall k \geq 1. \tag{3.32}
$$

Taking into account (3.30) and (3.32) we see that the sequence $\{m_k\}_{k=1}^{\infty}$ is bounded in $H^1(\omega, \mathbb{R}^3)$. Then, we can extract a subsequence of $\{m_k\}_{k=1}^{\infty}$ (not relabeled) which converges weakly in $H^1(\omega, \mathbb{R}^3)$ and moreover, according to Rellich’s selection principle, it converges strongly in $L^2(\omega, \mathbb{R}^3)$, i.e., there exists an element $\tilde{m} \in H^1(\omega, \mathbb{R}^3)$ such that

$$
m_k \rightharpoonup \tilde{m} \quad \text{in} \quad H^1(\omega, \mathbb{R}^3), \quad \text{and} \quad m_k \to \tilde{m} \quad \text{in} \quad L^2(\omega, \mathbb{R}^3). \tag{3.33}
$$

Corresponding to the fields $\{m_k, \overline{Q}_{e,s}\}$ we consider the strain measures $E^{(k)}_{m,s}, K^{(k)}_{e,s} \in L^2(\omega, \mathbb{R}^{3 \times 3})$. From the coercivity of the internal energy, (3.21) and (3.32) we get

$$
C_1 \|K^{(k)}_{e,s}\|_{L^2(\omega)}^2 \leq \int_{\omega} W(E^{(k)}_{m,s}, K^{(k)}_{e,s}) \det[\nabla_x \Theta(0)] \, da \leq I(m^*, \overline{Q}_{e,s}^*) + C_2 \left( \|m_k\|_{H^1(\omega)} + 1 \right),
$$

where $C_1, C_2$ are positive constants.

Since $\{m_k\}_{k=1}^{\infty}$ is bounded in $H^1(\omega, \mathbb{R}^3)$, it follows from the last inequalities that $\{K^{(k)}_{e,s}\}_{k=1}^{\infty}$ is bounded in $L^2(\omega, \mathbb{R}^{3 \times 3})$.

For tensor fields $P$ with rows in $H(curl; \Omega)$, i.e., $P = (P^T e_1 | P^T e_2 | P^T e_3)^T$ with $(P^T e_i)^T \in H(curl; \Omega), i = 1, 2, 3$, we define

$$
\text{Curl } P := \left( \text{curl } (P^T e_1)^T \mid \text{curl } (P^T e_2)^T \mid \text{curl } (P^T e_3)^T \right)^T.
$$

Since $\{K^{(k)}_{e,s}\}_{k=1}^{\infty}$ is bounded, so is $\{axl(\overline{Q}_{e,s}^T \partial_{\alpha} \overline{Q}_{e,s})\}_{k=1}^{\infty}, \alpha = 1, 2, 3$, in $L^2(\omega, \mathbb{R}^3)$ and it follows that $\overline{Q}_{k} \text{ Curl } \overline{Q}_{k}$ is bounded. Indeed, using the so-called wryness tensor (second order tensor) [18, 38]

$$
\Gamma_k := \left( axl(\overline{Q}_{k}^T \partial_{x_1} \overline{Q}_{k}) \mid axl(\overline{Q}_{k}^T \partial_{x_2} \overline{Q}_{k}) \mid 0 \right) \in \mathbb{R}^{3 \times 3}, \tag{3.34}
$$

we have (see [38]) the following close relationship (Nye’s formula) between the wryness tensor and the dislocation density tensor

$$
\alpha_k := \overline{Q}_{k}^T \text{ Curl } \overline{Q}_{k} = -\Gamma_k^T + \text{tr}(\Gamma_k) \mathbb{1}_3, \quad \text{or equivalently,}
$$

$$
\Gamma_k = -\alpha_k^T + \frac{1}{2} \text{tr}(\alpha_k) \mathbb{1}_3, \tag{3.35}
$$

because $\overline{Q}_{k} = \overline{Q}_{k}(x_1, x_2)$. Hence, $\{axl(\overline{Q}_{k}^T \partial_{\alpha} \overline{Q}_{k})\}_{k=1}^{\infty}$ is bounded if and only if $\overline{Q}_{k}^T \text{ Curl } \overline{Q}_{k}$ is bounded. Writing $\text{Curl } \overline{Q}_{k} = \overline{Q}_{k}^T \text{ Curl } \overline{Q}_{k}$ and using $\|\overline{Q}_{k}\|^2 = 3$, we deduce that the boundedness of $\overline{Q}_{k}^T \text{ Curl } \overline{Q}_{k}$ implies that $\text{Curl } \overline{Q}_{k}$ is bounded. Since the Curl-operator
bounds the gradient operator in SO(3), see \cite{38}, it follows that \( \{ \partial_{\alpha a} \hat{Q}_k \}_{k=1}^\infty \) is bounded in \( L^2(\omega, \mathbb{R}^{3\times3}) \), for \( \alpha = 1, 2 \). Since \( \hat{Q}_k \in SO(3) \) we have \( \| \hat{Q}_k \|^2 = 3 \) and thus we can infer that the sequence \( \{ \hat{Q}_k \}_{k=1}^\infty \) is bounded in \( H^1(\omega, \mathbb{R}^{3\times3}) \). Hence, there exists a subsequence of \( \{ \hat{Q}_k \}_{k=1}^\infty \) (not relabeled) and an element \( \hat{Q}_{e,s} \in H^1(\omega, \mathbb{R}^{3\times3}) \) with

\[
\hat{Q}_k \rightharpoonup \hat{Q}_{e,s} \quad \text{in} \quad H^1(\omega, \mathbb{R}^{3\times3}), \quad \text{and} \quad \hat{Q}_k \rightarrow \hat{Q}_{e,s} \quad \text{in} \quad L^2(\omega, \mathbb{R}^{3\times3}). \tag{3.36}
\]

Since \( \hat{Q}_k \in SO(3) \) we have

\[
\| \hat{Q}_k \hat{Q}_{e,s}^T - \mathbbm{1}_3 \|_{L^2(\omega)} = \| \hat{Q}_k (\hat{Q}_{e,s}^T - \hat{Q}_k^T) \|_{L^2(\omega)} = \| \hat{Q}_{e,s} - \hat{Q}_k \|_{L^2(\omega)} \rightarrow 0,
\]

i.e., \( \hat{Q}_k \hat{Q}_{e,s}^T \rightarrow \mathbbm{1}_3 \) in \( L^2(\omega, \mathbb{R}^{3\times3}) \). On the other hand, we can write

\[
\| \hat{Q}_k \hat{Q}_{e,s}^T - \hat{Q}_{e,s} \hat{Q}_{e,s}^T \|_{L^1(\omega)} = \| (\hat{Q}_k - \hat{Q}_{e,s}) \hat{Q}_{e,s}^T \|_{L^1(\omega)} \leq 3 \| \hat{Q}_k - \hat{Q}_{e,s} \|_{L^2(\omega)} \| \hat{Q}_{e,s} \|_{L^2(\omega)} \rightarrow 0,
\]

which means that \( \hat{Q}_k \hat{Q}_{e,s}^T \rightarrow \hat{Q}_{e,s} \hat{Q}_{e,s}^T \) in \( L^1(\omega, \mathbb{R}^{3\times3}) \). Consequently, we find \( \hat{Q}_{e,s} \hat{Q}_{e,s}^T = \mathbbm{1}_3 \) so that \( \hat{Q}_{e,s} \) belongs to \( H^1(\omega, SO(3)) \).

By virtue of the relations \( (m_k, \hat{Q}_k) \in A \) and (3.33), (3.36), we derive that \( \hat{m} = m^* \) on \( \gamma_d \) and \( \hat{Q}_{e,s} = \hat{Q}_{e,s}^* \) on \( \gamma^d \) in the sense of traces. Hence, we obtain that the limit pair satisfies \( (\hat{m}, \hat{Q}_{e,s}) \in A \).

Let us next construct the limit strain and curvature measures

\[
\hat{\epsilon}_{m,s} := \hat{Q}_{e,s}^T (\nabla \hat{m} | \hat{Q}_{e,s} \nabla \Theta(0) e_3) (\nabla \Theta(0))^{-1} - \mathbbm{1}_3 = Q_0 (\hat{R}_s \nabla \hat{m} - Q_0^T \nabla \gamma_0 | 0) (\nabla \Theta(0))^{-1},
\]

\[
\hat{k}_{e,s} := \text{axl}(\hat{Q}_{e,s}^T \partial_{x_1} \hat{Q}_{e,s}) - \text{axl}(Q_0^T \partial_{x_1} Q_0) (\nabla \Theta(0))^{-1} = Q_0 (\text{axl}(\hat{R}_s^T \partial_{x_1} \hat{R}_s) - \text{axl}(Q_0^T \partial_{x_1} Q_0)),
\]

where

\[
\hat{R}_s(x_1, x_2) := \hat{Q}_{e,s}(x_1, x_2) Q_0(x_1, x_2, 0) \in SO(3). \tag{3.38}
\]

As shown above, the sequence \( \{ m_k \}_{k=1}^\infty \) is bounded in \( H^1(\omega, \mathbb{R}^3) \). It follows that \( \{ \nabla m_k(0) \}_{k=1}^\infty \) is bounded in \( L^2(\omega, \mathbb{R}^{3\times3}) \). We define

\[
\bar{R}_k := \hat{Q}_k Q_0 \in SO(3). \tag{3.39}
\]

Then, the sequence \( \{ \bar{R}_k \nabla m_k(0) \}_{k=1}^\infty \) is bounded in \( L^2(\omega, \mathbb{R}^{3\times3}) \), since \( \bar{R}_k \in SO(3) \). Consequently, there exists a subsequence (not relabeled) and an element \( \xi \in L^2(\omega, \mathbb{R}^{3\times3}) \) such that

\[
\bar{R}_k \nabla m_k(0) \rightharpoonup \xi \quad \text{in} \quad L^2(\omega, \mathbb{R}^{3\times3}). \tag{3.40}
\]

On the other hand, let \( \Phi \in C_0^\infty(\omega, \mathbb{R}^{3\times3}) \) be an arbitrary test function. Then, using the properties of the scalar product we deduce
\[ \int_\omega \left( \overline{R}_k^T \nabla m_k | 0 \right) - \overline{R}_s^T (\nabla \hat{m} | 0), \Phi \right) \, da \]

\[ = \int_\omega \left( (\nabla m_k | 0) - (\nabla \hat{m} | 0) \right), \Phi \right) \, da + \int_\omega \left( (\overline{R}_k^T - \overline{R}_s^T) \nabla m_k | 0 \right), \Phi \right) \, da \]

\[ = \int_\omega \left( (\nabla m_k | 0) - (\nabla \hat{m} | 0), \overline{R}_s, (\nabla m_k | 0), \Phi \right) \, da + \int_\omega \left( (\nabla m_k | 0) - (\nabla \hat{m} | 0), \overline{R}_s, (\nabla m_k | 0), \Phi \right) \, da \]

\[ \leq \| \overline{R}_k - \overline{R}_s \|_{L^2(\omega)} \| (\nabla m_k | 0), \Phi \|_{L^2(\omega)} + \int_\omega \left( (\nabla m_k | 0) - (\nabla \hat{m} | 0), \overline{R}_s, (\nabla m_k | 0), \Phi \right) \, da . \]  

(3.41)

Since the relations (3.33), (3.36) and \( \overline{R}_s \Phi \in L^2(\omega, \mathbb{R}^{3 \times 3}) \) hold, and \( \| (\nabla m_k | 0), \Phi \| \) is bounded, we get

\[ \int_\omega \left( \overline{R}_k^T \nabla m_k | 0 \right), \Phi \right) \, da \rightarrow \int_\omega \left( \overline{R}_s^T \nabla \hat{m} | 0 \right), \Phi \right) \, da, \quad \forall \Phi \in C_0^\infty(\omega, \mathbb{R}^{3 \times 3}). \]  

(3.42)

By comparison of (3.40) and (3.42) we find \( \xi = \overline{R}_s^T (\nabla \hat{m} | 0) \), which means that \( \overline{R}_k^T \nabla m_k | 0 \rightarrow \overline{R}_s^T (\nabla \hat{m} | 0) \) in \( L^2(\omega, \mathbb{R}^{3 \times 3}) \), or equivalently

\[ \overline{R}_k^T \nabla m_k | 0 - Q_0^T (\nabla y_0 | 0) \rightarrow \overline{R}_k^T (\nabla \hat{m} | 0 - Q_0^T (\nabla y_0 | 0) \in L^2(\omega, \mathbb{R}^{3 \times 3}). \]  

(3.43)

Taking into account the hypotheses, we obtain from (3.43) that

\[ \mathcal{E}_{m,s}^{(k)} := Q_0^T \nabla m_k - Q_0^T \nabla y_0 | 0 [\nabla \xi \Theta(0)]^{-1} \rightarrow \mathcal{E}_{m,s} \]  

(3.44)

in \( L^2(\omega, \mathbb{R}^{3}) \).

We use now the fact that the sequence \( \{ \text{axl}(\overline{R}_k^T \partial_{x_a} \overline{R}_k) \}_{k=1}^\infty, \alpha = 1, 2, \) is bounded in \( L^2(\omega, \mathbb{R}^{3}) \), since we proved previously that \( \overline{R}_k^T \partial_{x_a} \overline{R}_k \) is bounded in \( L^2(\omega, \mathbb{R}^{3 \times 3}) \). Then, there exists a subsequence (not relabeled) and an element \( \zeta_a \in L^2(\omega, \mathbb{R}^{3}), \alpha = 1, 2, \) such that

\[ \text{axl}(\overline{R}_k^T \partial_{x_a} \overline{R}_k) \rightarrow \zeta_a \quad \text{in} \quad L^2(\omega, \mathbb{R}^{3}). \]  

(3.45)

On the other hand, for any test function \( \phi \in C_0^\infty(\omega, \mathbb{R}^{3}) \) we can write

\[ \int_\omega \left\{ \text{axl}(\overline{R}_k^T \partial_{x_a} \overline{R}_k - \overline{R}_s \partial_{x_a} \overline{R}_s) \right\}, \phi \right\}_{\mathbb{R}^{3}} \, da = \frac{1}{2} \int_\omega \left\{ \overline{R}_k^T \partial_{x_a} \overline{R}_k - \overline{R}_s \partial_{x_a} \overline{R}_s \right\}, \text{anti}(\phi) \right\}_{\mathbb{R}^{3 \times 3}} \, da \]

\[ = \frac{1}{2} \int_\omega \left\{ \overline{R}_s \partial_{x_a} \overline{R}_s - \partial_{x_a} \overline{R}_s \right\}, \text{anti}(\phi) \right\}_{\mathbb{R}^{3 \times 3}} \, da + \frac{1}{2} \int_\omega \left\{ (\overline{R}_k^T - \overline{R}_s^T) \partial_{x_a} \overline{R}_k \right\}, \text{anti}(\phi) \right\}_{\mathbb{R}^{3 \times 3}} \, da \]

\[ \leq \frac{1}{2} \int_\omega \left\{ \partial_{x_a} \overline{R}_k - \partial_{x_a} \overline{R}_s \right\}, \text{anti}(\phi) \right\}_{\mathbb{R}^{3 \times 3}} \, da + \frac{1}{2} \| \overline{R}_k - \overline{R}_s \|_{L^2(\omega)} \| \partial_{x_a} \overline{R}_k \| \text{anti}(\phi) \right\}_{\mathbb{R}^{3 \times 3}} \right\}_{\mathbb{R}^{3 \times 3}} \, da \]

\[ \rightarrow 0, \]

since \( \overline{R}_s \text{anti}(\phi) \in L^2(\omega, \mathbb{R}^{3 \times 3}), \| \partial_{x_a} \overline{R}_k \| \text{anti}(\phi) \right\}_{\mathbb{R}^{3 \times 3}} \) is bounded, and relations (3.36) hold. Consequently, we have

\[ \int_\omega \{ \text{axl}(\overline{R}_k^T \partial_{x_a} \overline{R}_k) \}, \phi \right\}_{\mathbb{R}^{3}} \, da \rightarrow \int_\omega \{ \text{axl}(\overline{R}_s^T \partial_{x_a} \overline{R}_s) \}, \phi \right\}_{\mathbb{R}^{3}} \, da, \quad \forall \phi \in C_0^\infty(\omega, \mathbb{R}^{3}). \]  

(3.47)
and by comparison with (3.45) we deduce that $\zeta_\alpha = axl(R_s^T \partial_{\alpha} \tilde{R}_s)$, i.e.,

\[
axl(R_k^T \partial_{\alpha} R_k) - axl(Q_0^T \partial_{\alpha} Q_0) = axl(R_s^T \partial_{\alpha} \tilde{R}_s) - axl(Q_0^T \partial_{\alpha} Q_0) \quad \text{in} \: L^2(\omega, \mathbb{R}^3), \quad (3.48)
\]

Hence, from (3.20) we derive the convergence

\[
K^{(k)}_{e,s} := Q_0(axl(R_k^T \partial_{\alpha} R_k) - axl(Q^T \partial_{\alpha} Q)) \rightharpoonup axl(R_s^T \partial_{\alpha} \tilde{R}_s) - axl(Q_0^T \partial_{\alpha} Q_0) \quad \text{in} \: L^2(\omega, \mathbb{R}^3),
\]

(3.49)

In the last step of the proof we use the convexity of the strain energy density $W$. In view of (3.44) and (3.49), we have

\[
\int_\omega W(\tilde{E}_{m,s}, \tilde{K}_{e,s}) \det(\nabla y_0|n_0) \, da \leq \liminf_{n \to \infty} \int_\omega W(\tilde{E}_{m,s}^{(k)}, \tilde{K}_{e,s}^{(k)}) \det(\nabla y_0|n_0) \, da.
\]

(3.50)

since $W$ is convex in $(\tilde{E}_{m,s}, \tilde{K}_{e,s})$. Taking into account the hypotheses (3.18), the continuity of the load potential functions, and the convergence relations (3.33)2 and (3.36)2, we deduce

\[
I(\tilde{m}, \tilde{Q}_{e,s}) \leq \liminf_{n \to \infty} I(m_k, Q_k).
\]

(3.51)

From (3.50) and (3.51) we get

\[
I(\tilde{m}, \tilde{Q}_{e,s}) \leq \liminf_{n \to \infty} I(m_k, Q_k).
\]

(3.52)

Finally, the relations (3.31) and (3.52) show that

\[
I(\tilde{m}, \tilde{Q}_{e,s}) = \inf \left\{ I(m, Q_{e,s}) \mid (m, Q_{e,s}) \in A \right\}.
\]

Since $(\tilde{m}, \tilde{Q}_{e,s}) \in A$, we conclude that $(\tilde{m}, \tilde{Q}_{e,s})$ is a minimizing solution pair of our minimization problem.

The boundary condition on $\tilde{Q}_{e,s}$ is not essential in the proof of the above theorem, and that one can prove the existence of minimizers for the minimization problem over a larger admissible set:

**Corollary 3.4** [Existence result for the theory including terms up to order $O(h^5)$ without boundary condition on the microrotation field] Under the hypotheses of Theorem 3.3, the minimization problem (2.6)–(2.10) admits at least one minimizing solution pair

\[
(m, \tilde{Q}_{e,s}) \in A = \left\{ (m, \tilde{Q}_{e,s}) \in H^1(\omega, \mathbb{R}^3) \times H^1(\omega, \text{SO}(3)) \mid m|_{\partial \omega} = m^* \right\}.
\]

(3.53)

4 Existence of Minimizers for the Cosserat Shell Model of Order $O(h^3)$

In this section we consider only terms up to order $O(h^3)$ in the expression of the energy density. Therefore, we obtain the following two-dimensional minimization problem for the deformation of the midsurface $m : \omega \to \mathbb{R}^3$ and the microrotation of the shell $\tilde{Q}_{e,s} : \omega \to \text{SO}(3)$ solving on $\omega \subset \mathbb{R}^2$: minimize with respect to $(m, \tilde{Q}_{e,s})$ the following functional

\[
I(m, \tilde{Q}_{e,s}) = \int_\omega W^{(h^3)}(\tilde{E}_{m,s}, \tilde{K}_{e,s}) \det(\nabla y_0|n_0) \, da - \Pi(m, \tilde{Q}_{e,s}),
\]

(4.1)
where the shell energy density $W^{(h^3)}(E_{m,s}, K_{e,s})$ is given by

$$W^{(h^3)}(E_{m,s}, K_{e,s}) = \left( h + K \frac{h^3}{12} \right) W_{\text{shell}}(E_{m,s}) + \frac{h^3}{12} W_{\text{shell}}(E_{m,s} B_{y_0} + C_{y_0} K_{e,s})$$

$$- \frac{h^3}{3} H W_{\text{shell}}(E_{m,s}, (E_{m,s} B_{y_0} + C_{y_0} K_{e,s})) + \frac{h^3}{6} W_{\text{shell}}((E_{m,s} B_{y_0} + C_{y_0} K_{e,s}) B_{y_0})$$

$$+ \left( h - K \frac{h^3}{12} \right) W_{\text{curv}}(K_{e,s}) + \frac{h^3}{12} W_{\text{curv}}(K_{e,s} B_{y_0}),$$

with all the other quantities having the same expressions and interpretations as in the theory up to order $O(h^5)$.

In [21] we have presented a comparison with the the general 6-parameter shell model [19]. While in the previous approaches [7, 10, 11, 19] the dependence of the coefficients upon the curved initial shell configuration is not specified, in our shell model, the constitutive coefficients are deduced from the three-dimensional formulation, while the influence of the curved initial shell configuration appears explicitly in the expression of the coefficients of the energies for the reduced two-dimensional variational problem. Another major difference between our model and the previously considered general 6-parameter shell model is that, even in the case of a simplified theory of order $O(h^3)$, additional mixed terms like the membrane–bending part $- \frac{h^3}{3} H W_{\text{shell}}(E_{m,s}, (E_{m,s} B_{y_0} + C_{y_0} K_{e,s}))$ and $\frac{h^3}{6} W_{\text{shell}}((E_{m,s} B_{y_0} + C_{y_0} K_{e,s}) B_{y_0})$, as well as $W_{\text{curv}}(K_{e,s} B_{y_0})$, are included, which are otherwise difficult to guess. Therefore, an existence proof for the new $O(h^3)$-model is of independent interest. First, we will show

**Proposition 4.1** [Coercivity in the theory including terms up to order $O(h^3)$] Assume that the constitutive coefficients are such that $\mu > 0$, $\mu_c > 0$, $2 \lambda + \mu > 0$, $b_1 > 0$, $b_2 > 0$ and $b_3 > 0$ and let $c_1^+$ denotes the smallest eigenvalue of $W_{\text{curv}}(S)$, and $c_1^+$ and $c_1^+ > 0$ denote the smallest and the largest eigenvalues of the quadratic form $W_{\text{shell}}(S)$. If the thickness $h$ satisfies one of the following conditions:

i) $h |\kappa_1| < \frac{1}{2}$, $h |\kappa_2| < \frac{1}{2}$ and $h^2 < \left( \frac{47}{4} \right)^2 \left( 5 - 2\sqrt{6} \frac{c_1^+}{C_1^+} \right)$;

ii) $h |\kappa_1| < \frac{1}{a}$, $h |\kappa_2| < \frac{1}{a}$ with $a > \max \left\{ 1 + \frac{\sqrt{2}}{2}, \frac{\sqrt{1 + 3 c_1^+}}{2}, \frac{1 + 3 c_1^+}{2}, \frac{1}{2} \right\}$,

then $W^{(h^3)}(E_{m,s}, K_{e,s})$ is coercive, in the sense that there exists a constant $a_1^+ > 0$ such that

$$W^{(h^3)}(E_{m,s}, K_{e,s}) \geq a_1^+ \left( \|E_{m,s}\|^2 + \|K_{e,s}\|^2 \right),$$

where $a_1^+$ depends on the constitutive coefficients.

**Proof** Using the properties presented in the Appendix, since $B_{y_0}^2 = 2HB_{y_0} + KA_{y_0} = 0$ and $E_{m,s} A_{y_0} = E_{m,s}$, it follows

$$(E_{m,s} B_{y_0} + C_{y_0} K_{e,s}) B_{y_0} = 2H(E_{m,s} B_{y_0} - K E_{m,s} + C_{y_0} K_{e,s} B_{y_0}).$$
Hence, we have

\[
(h + K \frac{h^3}{12}) W_{\text{shell}}(\varepsilon_{m,s}) - \frac{h^3}{3} K W_{\text{shell}}(\varepsilon_{m,s}, \varepsilon_{m,s} B_{y0} + C_{y0} \mathcal{K}_{e,s})
+ \frac{h^3}{6} W_{\text{shell}}(\varepsilon_{m,s}, \varepsilon_{m,s} B_{y0} + C_{y0} \mathcal{K}_{e,s}) B_{y0})
= \left( h + K \frac{h^3}{12} \right) W_{\text{shell}}(\varepsilon_{m,s}) - \frac{h^3}{3} H W_{\text{shell}}(\varepsilon_{m,s}, \varepsilon_{m,s} B_{y0}) - \frac{h^3}{3} H W_{\text{shell}}(\varepsilon_{m,s}, C_{y0} \mathcal{K}_{e,s})
+ \left( h - K \frac{h^3}{12} \right) W_{\text{shell}}(\varepsilon_{m,s}) - \frac{h^3}{3} H W_{\text{shell}}(\varepsilon_{m,s}, C_{y0} \mathcal{K}_{e,s}) + \frac{h^3}{6} W_{\text{shell}}(\varepsilon_{m,s}, C_{y0} \mathcal{K}_{e,s} B_{y0})
\]

(4.5)

Using (4.5) and the positive definiteness of the quadratic forms (2.8) and the Cauchy–Schwarz inequality we obtain

\[
W(\varepsilon_{m,s}, \mathcal{K}_{e,s}) \geq \left( h - K \frac{h^3}{12} \right) W_{\text{shell}}(\varepsilon_{m,s}) + \frac{h^3}{12} W_{\text{shell}}(\varepsilon_{m,s} B_{y0} + C_{y0} \mathcal{K}_{e,s})
- \frac{h^3}{6} \lvert W_{\text{shell}}(\varepsilon_{m,s}, C_{y0} \mathcal{K}_{e,s} B_{y0}) \rvert - \frac{h^3}{3} \lvert H \rvert \lvert W_{\text{shell}}(\varepsilon_{m,s}, C_{y0} \mathcal{K}_{e,s}) \rvert
+ \left( h - K \frac{h^3}{12} \right) W_{\text{curv}}(\mathcal{K}_{e,s}) + \frac{h^3}{12} W_{\text{curv}}(\mathcal{K}_{e,s} B_{y0})
\geq \left( h - K \frac{h^3}{12} \right) W_{\text{shell}}(\varepsilon_{m,s})
- \frac{1}{6} \left[ h W_{\text{shell}}(\varepsilon_{m,s}) \right]^\frac{1}{2} \left[ h^5 W_{\text{shell}}(C_{y0} \mathcal{K}_{e,s} B_{y0}) \right]^\frac{1}{2}
- \frac{1}{3} \lvert H \rvert \left[ h^2 W_{\text{shell}}(\varepsilon_{m,s}) \right]^\frac{1}{2} \left[ h^4 W_{\text{shell}}(C_{y0} \mathcal{K}_{e,s}) \right]^\frac{1}{2}
+ \frac{h^3}{12} W_{\text{shell}}(\varepsilon_{m,s} B_{y0} + C_{y0} \mathcal{K}_{e,s})
+ \left( h - K \frac{h^3}{12} \right) W_{\text{curv}}(\mathcal{K}_{e,s}) + \frac{h^3}{12} W_{\text{curv}}(\mathcal{K}_{e,s} B_{y0})
\]

(4.6)

In view of the arithmetic-geometric mean inequality it follows

\[
W(\varepsilon_{m,s}, \mathcal{K}_{e,s}) \geq \left( h - K \frac{h^3}{12} \right) W_{\text{shell}}(\varepsilon_{m,s}) - \frac{1}{12} \varepsilon \left( h W_{\text{shell}}(\varepsilon_{m,s})
- \frac{1}{12} h^5 W_{\text{shell}}(C_{y0} \mathcal{K}_{e,s} B_{y0}) - \frac{1}{6} \delta \lvert H \rvert h^2 W_{\text{shell}}(\varepsilon_{m,s})
+ \frac{h^3}{12} W_{\text{shell}}(\varepsilon_{m,s} B_{y0} + C_{y0} \mathcal{K}_{e,s})
+ \left( h - K \frac{h^3}{12} \right) W_{\text{curv}}(\mathcal{K}_{e,s}) + \frac{h^3}{12} W_{\text{curv}}(\mathcal{K}_{e,s} B_{y0}) \right) \forall \varepsilon > 0 \text{ and } \delta > 0.
\]
Using the inequalities $-h^2 |K| > -\frac{1}{4}$ and $-h |H| > -\frac{1}{2}$, we obtain

\[
W^{(h)}(E_{m,s}, K_{e,s}) \geq \frac{h}{12} \left( \frac{47}{4} - \delta - \epsilon \right) W_{\text{shell}}(E_{m,s}) + \frac{h^3}{12} W_{\text{shell}}(E_{m,s} B_{y_0} + C_{y_0} K_{e,s}) \\
- \frac{1}{12} h^3 W_{\text{shell}}(C_{y_0} K_{e,s}) - \frac{1}{12} h^3 W_{\text{shell}}(C_{y_0} K_{e,s} B_{y_0}) \tag{4.8}
\]

\[
+ \frac{47 h}{48} W_{\text{curv}}(K_{e,s}) + \frac{h^3}{12} W_{\text{curv}}(K_{e,s} B_{y_0}) \quad \forall \epsilon > 0 \text{ and } \delta > 0.
\]

In view of (3.3) and (2.12) and since the Frobenius norm is sub-multiplicative, we deduce

\[
W^{(h)}(E_{m,s}, K_{e,s}) \geq \frac{h}{12} \left( \frac{47}{4} - \delta - \epsilon \right) c_1^+ \|E_{m,s}\|^2 + \frac{h^3}{12} c_1^+ \|E_{m,s} B_{y_0} + C_{y_0} K_{e,s}\|^2 \\
- \frac{1}{12} h^3 c_1^+ \|C_{y_0}\|^2 \|K_{e,s}\|^2 - \frac{1}{12} h^3 c_1^+ \|C_{y_0} B_{y_0}\|^2 \tag{4.9}
\]

\[
+ \frac{47 h}{48} c_2^+ \|K_{e,s}\|^2 + \frac{h^3}{12} c_2^+ \|K_{e,s} B_{y_0}\|^2
\]

\forall \epsilon > 0 \text{ and } \delta > 0 \text{ such that } \frac{47}{4} > \delta + \epsilon.

Since $\|C_{y_0}\|^2 = 2$, the estimate (4.9) becomes

\[
W^{(h)}(E_{m,s}, K_{e,s}) \geq \frac{h}{12} \left( \frac{47}{4} - \delta - \epsilon \right) c_1^+ \|E_{m,s}\|^2 + \frac{h^3}{12} c_1^+ \|E_{m,s} B_{y_0} + C_{y_0} K_{e,s}\|^2 \tag{4.10}
\]

\[
+ c_1^+ h^3 \left( \frac{47}{8} h^2 c_2^+ - \frac{1}{\delta} \right) \|K_{e,s}\|^2 + c_1^+ h^3 \left( \frac{1}{2} h^2 c_2^+ - \frac{1}{\delta} \right) \|K_{e,s} B_{y_0}\|^2,
\]

for all $\epsilon > 0$ and $\delta > 0$ such that $\frac{47}{4} > \delta + \epsilon$. According to the properties presented in the Appendix, we have

\[
\|B_{y_0}\|^2 = \|B_{y_0}\| = 2 \text{H}(B_{y_0}, \mathbb{I}) - \text{K}(A_{y_0}, \mathbb{I}) = 4H^2 - 2K. \tag{4.11}
\]

Therefore, from (4.10) it follows

\[
W^{(h)}(E_{m,s}, K_{e,s}) \geq \frac{h}{12} \left( \frac{47}{4} - \delta - \epsilon \right) c_1^+ \|E_{m,s}\|^2 + \frac{h^3}{12} c_1^+ \|E_{m,s} B_{y_0} + C_{y_0} K_{e,s}\|^2 \tag{4.12}
\]

\[
+ h \left[ \frac{47}{48} c_2^+ - \frac{1}{6\delta} h^2 c_1^+ - \frac{1}{6\epsilon} h^4 c_1^+ \left( 4H^2 - 2K \right) \right] \|K_{e,s}\|^2 \\
+ h^3 \frac{1}{12} c_2^+ \|K_{e,s} B_{y_0}\|^2.
\]

Using again that $h$ is small, we obtain $-h^2 (4H^2 - 2K) > -\frac{3}{2}$ and
\[ W^{(h^3)}(\mathcal{E}_{m,s}, \mathcal{K}_{e,s}) \geq h \left( \frac{47}{4} - \delta - \varepsilon \right) c_1^+ \| \mathcal{E}_{m,s} \|^2 + \frac{h^3}{12} c_2^+ \| \mathcal{E}_{m,s} B_{y_0} + C_{y_0} \mathcal{K}_{e,s} \|^2 \] (4.13)

\[ + \frac{h^3}{12} C_1^+ \left[ \frac{47}{4} c_2^+ - \frac{2}{\delta} - \frac{3}{\varepsilon} \right] \| \mathcal{K}_{e,s} \|^2 + h^3 \frac{1}{12} c_2^+ \| \mathcal{E}_{m,s} B_{y_0} \|^2 \]

\[ \geq h \left( \frac{47}{4} - \delta - \varepsilon \right) c_1^+ \| \mathcal{E}_{m,s} \|^2 + \frac{h^3}{12} C_1^+ \left[ \frac{47}{4} c_2^+ - \frac{2}{\delta} - \frac{3}{\varepsilon} \right] \| \mathcal{K}_{e,s} \|^2. \]

We consider \( \delta = \gamma \varepsilon \) and we choose \( \delta > 0 \) and \( \gamma > 0 \) such that \( \frac{47}{4(1+\gamma)} > \varepsilon > \frac{2+3\varepsilon}{47} \frac{h^2 c_1^+}{c_2^+} \).

This choice of the variable \( \varepsilon \) is possible if and only if \( \frac{(47/4)^2}{(2+3\gamma)(1+\gamma)} > \frac{h^2 c_1^+}{c_2^+} \). At this point we use that \( \max_{\gamma>0} \frac{\gamma}{(2+3\gamma)(1+\gamma)} = 5 - 2\sqrt{6} \), and we take \( \gamma = \sqrt{\frac{2}{3}} \). Note that the considered values for \( \gamma \) and \( \varepsilon \) assure that the condition \( \frac{47}{4} > \delta + \varepsilon \) is automatically satisfied. Hence, we arrive at the following condition on the thickness \( h \):

\[ h^2 < \left( \frac{47}{4} \right)^2 \left( 5 - 2\sqrt{6} \right) \frac{c_2^+}{C_1^+} \approx 13.94 \frac{c_2^+}{C_1^+}, \] (4.14)

which proves the coercivity if the condition i) is satisfied.

Next, we consider coercivity for condition ii). Under the hypotheses of the theorem, using also the positive definiteness of the quadratic forms (2.8) and the Cauchy–Schwarz inequality, we have

\[ W^{(h^3)}(\mathcal{E}_{m,s}, \mathcal{K}_{e,s}) \geq \left( h + K \frac{h^3}{12} \right) W_{\text{shell}}(\mathcal{E}_{m,s}) \]

\[ - \frac{1}{3} |H| \left[ h^2 W_{\text{shell}}(\mathcal{E}_{m,s}) \right]^{\frac{1}{2}} \left[ h^4 W_{\text{shell}}(\mathcal{E}_{m,s} B_{y_0} + C_{y_0} \mathcal{K}_{e,s}) \right]^{\frac{1}{2}} \]

\[ + \frac{h^3}{12} W_{\text{shell}}(\mathcal{E}_{m,s} B_{y_0} + C_{y_0} \mathcal{K}_{e,s}) \]

\[ - \frac{1}{6} \left[ h W_{\text{shell}}(\mathcal{E}_{m,s}) \right]^{\frac{1}{2}} \left[ h^5 W_{\text{shell}}((\mathcal{E}_{m,s} B_{y_0} + C_{y_0} \mathcal{K}_{e,s}) B_{y_0}) \right]^{\frac{1}{2}} \]

\[ + \left( h - K \frac{h^3}{12} \right) W_{\text{curv}}(\mathcal{K}_{e,s}). \] (4.15)

Using the arithmetic-geometric mean inequality in the previous estimate, it follows

\[ W^{(h^3)}(\mathcal{E}_{m,s}, \mathcal{K}_{e,s}) \geq \left( h + K \frac{h^3}{12} - \frac{h^2}{6} \varepsilon |H| \right) W_{\text{shell}}(\mathcal{E}_{m,s}) \]

\[ + \left( \frac{h^3}{12} - \frac{h^4}{6} \varepsilon |H| \right) W_{\text{shell}}(\mathcal{E}_{m,s} B_{y_0} + C_{y_0} \mathcal{K}_{e,s}) \]

\[ - \frac{h}{12} \delta W_{\text{shell}}(\mathcal{E}_{m,s}) - \frac{h^5}{12 \delta} W_{\text{shell}}(\mathcal{E}_{m,s} B_{y_0} + C_{y_0} \mathcal{K}_{e,s}) B_{y_0} \] (4.16)

\[ + \left( h - K \frac{h^3}{12} \right) W_{\text{curv}}(\mathcal{K}_{e,s}). \]
We choose $\delta = 8$ and $\varepsilon = 2$ to obtain that

$$W^{(3)}(\mathcal{E}_{m,s}, K_{e,s}) \geq \frac{1}{3} - K \frac{h^2}{12} - \frac{h}{3} |H| W_{\text{shell}}(\mathcal{E}_{m,s})$$

$$+ \frac{h^3}{12} (1 - h |H|) W_{\text{shell}}(\mathcal{E}_{m,s} B_{y_0} + C_{y_0} K_{e,s})$$

$$- \frac{h^5}{96} W_{\text{shell}}((\mathcal{E}_{m,s} B_{y_0} + C_{y_0} K_{e,s}) B_{y_0}) + \left( h - |K| \frac{h^3}{12} \right) W_{\text{curv}}(K_{e,s}).$$

(4.17)

Let us consider $a > 0$ and impose $h |H| < \frac{1}{a}$, $h^2 |K| < \frac{1}{a^2}$. Therefore, using (2.12) and since the Frobenius norm is sub-multiplicative we deduce

$$W^{(3)}(\mathcal{E}_{m,s}, K_{e,s}) \geq h \frac{4a^2 - 4a - 1}{12 a^2} c_1^+ \|\mathcal{E}_{m,s}\|^2 + h \frac{12a^2 - 1}{12 a^2} c_2^+ \|K_{e,s}\|^2$$

$$+ \frac{h^3}{12} \frac{a - 1}{a} c_1^+ \|\mathcal{E}_{m,s} B_{y_0} + C_{y_0} K_{e,s}\|^2$$

$$- \frac{h^5}{96} C_1^+ \|\mathcal{E}_{m,s} B_{y_0} + C_{y_0} K_{e,s}\|^2 \|B_{y_0}\|^2.$$  

Moreover, using (4.11) we deduce $-h^2 \|B_{y_0}\|^2 > -\frac{6}{a^2}$ and the inequality

$$W^{(3)}(\mathcal{E}_{m,s}, K_{e,s}) \geq h \frac{4a^2 - 4a - 1}{12 a^2} c_1^+ \|\mathcal{E}_{m,s}\|^2 + h \frac{12a^2 - 1}{12 a^2} c_2^+ \|K_{e,s}\|^2$$

$$+ \frac{h^3}{12} \frac{a - 1}{a} \left[ \frac{c_1^+}{C_1^+} - \frac{3}{4a(a - 1)} \right] \|\mathcal{E}_{m,s} B_{y_0} + C_{y_0} K_{e,s}\|^2.$$  

(4.19)

Hence, choosing $a > 1 + \frac{\sqrt{3}}{2}$ we assure that $\frac{4a^2 - 4a - 1}{12 a^2} > 0$, $\frac{a - 1}{a} > 0$ and $\frac{12a^2 - 1}{12 a^2} > 0$. A suitable $a > 1 + \frac{\sqrt{3}}{2}$ should satisfy $\frac{c_1^+}{C_1^+} - \frac{3}{4a(a - 1)} > 0$, which is true if $a$ is such that $a > \frac{1 + \sqrt{3}}{2} > 1$. Therefore, if $a > \max \left\{ 1 + \frac{\sqrt{3}}{2}, \frac{1 + \sqrt{3}}{2} \right\}$, then inequality (4.19) yields

$$W^{(3)}(\mathcal{E}_{m,s}, K_{e,s}) \geq \frac{4a^2 - 4a - 1}{12 a^2} c_1^+ \|\mathcal{E}_{m,s}\|^2 + h \frac{12a^2 - 1}{12 a^2} c_2^+ \|K_{e,s}\|^2. \quad \square$$

**Corollary 4.2** [Uniform convexity for the theory including terms up to order $O(h^3)$] Under the hypotheses of Proposition 4.1, the energy density $W^{(3)}(\mathcal{E}_{m,s}, K_{e,s})$ is uniformly convex in $(\mathcal{E}_{m,s}, K_{e,s})$, i.e., there exists a constant $a_1^+ > 0$ such that

$$D^2 W^{(3)}(\mathcal{E}_{m,s}, K_{e,s}) \geq a_1^+ (\|H_1\|^2 + \|H_2\|^2) \quad \forall H_1, H_2 \in \mathbb{R}^{3 \times 3}. \quad (4.20)$$

**Proof** See Corollary 3.2. \square

Therefore, an existence result similar to Theorem 3.3 holds true for the theory including terms up to order $O(h^3)$:
Theorem 4.3 [Existence result for the theory including terms up to order \( O(h^3) \)] Assume that the external loads satisfy the conditions
\[
f \in L^2(\omega, \mathbb{R}^3), \quad t \in L^2(\gamma_1, \mathbb{R}^3), \tag{4.21}
\]
the boundary data satisfy the conditions
\[
m^* \in H^1(\omega, \mathbb{R}^3), \quad \overline{Q}_{e,s}^* \in H^1(\omega, \text{SO}(3)), \tag{4.22}
\]
and that the following conditions concerning the initial configuration are fulfilled:
\[
y_0: \omega \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3 \text{ is a continuous injective mapping and } \quad y_0 \in H^1(\omega, \mathbb{R}^3), \quad Q_0(0) \in H^1(\omega, \text{SO}(3)), \quad \nabla_x \Theta(0) \in L^\infty(\omega, \mathbb{R}^{3 \times 3}),
\]
\[
det[\nabla_x \Theta(0)] \geq a_0 > 0, \tag{4.23}
\]
where \( a_0 \) is a constant. Assume that the constitutive coefficients are such that \( \mu > 0, \mu_c > 0, 2\lambda + \mu > 0, b_1 > 0, b_2 > 0 \) and \( b_3 > 0 \). Then, if the thickness \( h \) satisfies at least one of the following conditions:

i) \( h|\kappa_1| < \frac{1}{2}, \quad h|\kappa_2| < \frac{1}{2} \) and \( h^2 < \left(\frac{47}{4}\right)^2 \frac{2}{(5 - 2\sqrt{6}) c_1^+} \);

ii) \( h|\kappa_1| < \frac{1}{a}, \quad h|\kappa_2| < \frac{1}{a} \) with \( a = \max\left\{1 + \frac{x_2^2}{2}, 1 + \frac{\frac{1 + x_2^2}{2}}{3} c_1^+\right\} \),

where \( c_1^+ \) denotes the smallest eigenvalue of \( W_{\text{curv}}(S) \), and \( c_1^+ \) and \( C_1^+ > 0 \) denote the smallest and the biggest eigenvalues of the quadratic form \( W_{\text{shell}}(S) \), the minimization problem corresponding to the energy density defined by (4.1) and (4.2) admits at least one minimizing solution pair \((m, \overline{Q}_{e,s}) \) \( \in \mathcal{A} \).

Corollary 4.4 [Existence result for the theory including terms up to order \( O(h^3) \) without boundary condition on the microrotation field] Under the hypotheses of Theorem 4.3, the minimization problem corresponding to the energy density defined by (4.1) and (4.2) admits at least one minimizing solution pair
\[
(m, \overline{Q}_{e,s}) \in \mathcal{A} = \left\{(m, \overline{Q}_{e,s}) \in H^1(\omega, \mathbb{R}^3) \times H^1(\omega, \text{SO}(3)) \mid m|_{\gamma_d} = m^*\right\}. \tag{4.24}
\]

5 Final Comments

Having the deformation of the midsurface \( m : \omega \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3 \) and the microrotation of the shell \( \overline{Q}_{e,s} : \omega \subset \mathbb{R}^2 \rightarrow \text{SO}(3) \) solving on \( \omega \) the minimization (two-dimensional) problem, we get the approximation of the deformation of the initial three-dimensional body using the following 6-parameter quadratic ansatz in the thickness direction for the reconstructed total deformation \( \varphi_s : \Omega_b \subset \mathbb{R}^3 \rightarrow \mathbb{R}^3 \) of the shell-like structure [21]
\[
\varphi_s(x_1, x_2, x_3) = m(x_1, x_2) + \left(x_3 \varphi_m(x_1, x_2) + \frac{x_3^2}{2} \varphi_b(x_1, x_2)\right) \overline{Q}_{e,s}(x_1, x_2) \nabla_x \Theta(x_1, x_2, 0) e_3, \tag{5.1}
\]
where \( \varrho_m, \varrho_b : \omega \subset \mathbb{R}^2 \rightarrow \mathbb{R} \) allow in principal for symmetric thickness stretch \( (\varrho_m \neq 1) \) and asymmetric thickness stretch \( (\varrho_b \neq 0) \) about the midsurface and they are given by

\[
\varrho_m = 1 - \frac{\lambda}{\lambda + 2\mu} \left[ (Q_{\varepsilon,s}^T(\nabla m|0)[\nabla \Theta(0)]^{-1}, \mathbb{I}_3) - 2 \right] = 1 - \frac{\lambda}{\lambda + 2\mu} \text{tr}(\mathcal{E}_{m,s}),
\]

\[
\varrho_b = -\frac{\lambda}{\lambda + 2\mu} \left[ (Q_{\varepsilon,s}^T(\nabla \Theta(0)e_3)|0)[\nabla \Theta(0)]^{-1}, \mathbb{I}_3 \right] + \frac{\lambda}{\lambda + 2\mu} \langle Q_{\varepsilon,s}^T(\nabla m|0)[\nabla \Theta(0)]^{-1}(\nabla n_0|0)[\nabla \Theta(0)]^{-1}, \mathbb{I}_3 \rangle
\]

\[
= -\frac{\lambda}{\lambda + 2\mu} \text{tr}\{C_{30}K_{\varepsilon,s} + \mathcal{E}_{m,s}B_{30}\}. \tag{5.2}
\]

Obviously, if we know the total microrotation \( \bar{R}_s(x_1, x_2) = Q_{\varepsilon,s}(x_1, x_2)Q_0(x_1, x_2, 0) \in \text{SO}(3) \), then we know the microrotation \( \bar{R}_\xi \) of the parental three-dimensional problem, since we assume it is independent of \( x_3 \).

It is noteworthy that the existence result in the \( O(h^3) \)-model is not simply the truncated version of the existence result for the \( O(h^5) \)-model. Both existence results require uniformly positive constitutive parameters, in particular \( \mu_c > 0 \). However, the existence result for the \( O(h^5) \)-model needs less stringent assumptions on the plate thickness versus the initial curvature. This may indicate that the \( O(h^5) \)-model is intrinsically more stable than the \( O(h^3) \)-model. We will illuminate this possible feature in future computational researches. Since classical shell models do not have any drill contribution, it is interesting to consider the no drill case generated by \( \mu_c = 0 \) (free drill, but no energy contribution associated to this deformation mode). In this interesting limit case, we would need new generalized Korn’s inequalities [27, 39, 42, 43], which couple the smoothness of the rotation field \( \bar{R}_s \) with the coercivity with respect to the deformation \( m \), in the sense that

\[
\| \bar{R}_s^T(\nabla m|0) + (\nabla m|0)^T \bar{R}_s \|^2_{L^2(\omega)} \geq c^+\|m\|^2_{H^1(\omega)}. \tag{5.3}
\]

However, such an estimate is currently only known for \( \bar{R}_s \in C(\bar{\omega}, \text{SO}(3)) \), but the elastic shell energy only assures \( \bar{R}_s \in H^1(\omega, \text{SO}(3)) \). More research is needed in this direction.

**Acknowledgements** This research has been funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) – Project no. 415894848 (M. Bîrsan, P. Lewintan and P. Neff). The work of I.D. Ghiba was supported by a grant of the Romanian Ministry of Research and Innovation, CNCS–UEFISCDI, project number PN-III-P1-1.1-TE-2019-0397, within PNCDI III.

**Publisher’s Note** Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

### Appendix: Properties of the Considered Tensors

In this paper we use some properties of the tensors involved in the variational formulation of the shell model [21].

**Proposition A.1** The following identities are satisfied:

i) \( \text{tr}[A_{30}] = 2, \quad \text{det}[A_{30}] = 0; \quad \text{tr}[B_{30}] = 2H, \quad \text{det}[B_{30}] = 0, \)
ii) $B_{y_0}$ satisfies the equation of Cayley-Hamilton type

$$B_{y_0}^2 - 2HB_{y_0} + KA_{y_0} = 0;$$

iii) $A_{y_0}B_{y_0} = B_{y_0}A_{y_0} = B_{y_0}, A_{y_0}^2 = A_{y_0};$

iv) $C_{y_0} \in \mathfrak{so}(3), \ C_{y_0}^2 = -A_{y_0}$ and it has the simplified form

$$C_{y_0} := Q_0(0) \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} Q_0^T(0) \in \mathfrak{so}(3).$$

v) \(Q^T_{e,s} \left( \nabla \left[ \Theta(x_3) \left[ \left[ \nabla \Theta(x_0) e_3 \right] \right] 0 \right] \left[ \left[ \nabla \Theta(x_0) \right] \right]^{-1} = C_{y_0} \kappa_{e,s} - B_{y_0};\)

vi) $C_{y_0} \kappa_{e,s} A_{y_0} = C_{y_0} \kappa_{e,s};$

vii) $E_{m,s} A_{y_0} = E_{m,s}.$

Proof For the proof of this proposition we refer to [21]. Here, we prove only the third identity of iv).

We have $[\nabla \Theta(x_3)] e_3 = n_0$. Let us recall that $X \in \text{GL}^+(3)$ satisfies the Generalized Kirchhoff Constraint (GKC) [29] if $X \in \text{GKC} := \{ X \in \text{GL}^+(3) | X^T X e_3 = \varrho^2 e_3, \varrho \in \mathbb{R}^+ \}.$ For all $X \in \text{GKC}$ with the polar decomposition $X = R U_0$, if follows that $U_0 \in \text{GKC}$. In view of this property and $\nabla \Theta(x_3) = Q_0(x_3) U_0(x_3)$, it follows\(^6\) $U_0(x_3) = \begin{pmatrix} * & * & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \text{Sym}^+(3).$ Since $Q_0 = 1$, we deduce

$$C_{y_0} = \text{Cof}(\nabla \Theta(x_0)) \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \left[ \nabla \Theta(x_0) \right]^{-1}$$

$$= Q_0(0) (\det U_0(0)) U_0^{-1}(0) \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} U_0^{-1}(0) Q_0^T(0).$$

(A.1)

Direct computations give us

$$\begin{pmatrix} a & x & 0 \\ x & b & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \frac{1}{ab - x^2};$$

$$\det \begin{pmatrix} a & x & 0 \\ x & b & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \frac{1}{ab - x^2}.$$ 

Using these calculation in (A.1), we obtain

$$\left( \det U_0(0) \right) U_0^{-1}(0) \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} U_0^{-1}(0) = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$ 

(A.2)

Hence, the alternator tensor has the representation given in iv). \(\square\)

\(^6\)Here, * denotes quantities having expressions which are not relevant for our calculations.
The Isotropic Cosserat Shell Model Including Terms up to $O(h^5)$... 289

References

1. Adams, R.A.: Sobolev Spaces, 1st edn. Pure and Applied Mathematics, vol. 65. Academic Press, London (1975)

2. Badur, J., Pietraszkiewicz, W.: On geometrically non-linear theory of elastic shells derived from pseudo-Cosserat continuum with constrained micro-rotations. In: Pietraszkiewicz, W. (ed.) Finite Rotations in Structural Mechanics, pp. 19–32. Springer, Berlin (1986)

3. Ball, J.M.: Convexity conditions and existence theorems in nonlinear elasticity. Arch. Ration. Mech. Anal. 63, 337–403 (1977)

4. Bîrsan, M.: Inequalities of Korn’s type and existence results in the theory of Cosserat elastic shells. J. Elast. 90, 227–239 (2008)

5. Bîrsan, M.: Derivation of a refined 6-parameter shell model: descent from the three-dimensional Cosserat elasticity using a method of classical shell theory. Math. Mech. Solids 25(6), 1318–1339 (2020)

6. Bîrsan, M., Ghika, I.D., Martin, R.J., Neff, P.: Refined dimensional reduction for isotropic elastic Cosserat shells with initial curvature. Math. Mech. Solids 24(12), 4000–4019 (2019)

7. Bîrsan, M., Neff, P.: Existence of minimizers in the geometrically non-linear 6-parameter resultant shell theory with drilling rotations. Math. Mech. Solids 19(4), 376–397 (2014)

8. Bîrsan, M., Neff, P.: Shells without drilling rotations: a representation theorem in the framework of the geometrically nonlinear 6-parameter resultant shell theory. Int. J. Eng. Sci. 80, 32–42 (2014)

9. Buniou, R., Ciarlet, Ph.G., Mardare, C.: Existence theorem for a nonlinear elliptic shell model. J. Elliptic Parabolic Equ. 1(1), 31–48 (2015)

10. Chróscielewski, J., Makowski, J., Pietraszkiewicz, W.: Statics and Dynamics of Multifold Shells: Non-linear Theory and Finite Element Method (in Polish). Wydawnictwo IPPT PAN, Warsaw (2004)

11. Chróscielewski, J., Pietraszkiewicz, W., Witkowski, W.: On shear correction factors in the non-linear theory of elastic shells. Int. J. Solids Struct. 47, 3537–3545 (2010)

12. Ciarlet, Ph.G.: Mathematical Elasticity, Vol. II: Theory of Plates, 1st edn. North-Holland, Amsterdam (1997)

13. Ciarlet, Ph.G.: Introduction to Linear Shell Theory. Gauthier-Villars, Paris (1998)

14. Ciarlet, Ph.G.: Mathematical Elasticity, Vol. III: Theory of Shells, 1st edn. North-Holland, Amsterdam (2000)

15. Ciarlet, Ph.G., Geymonat, G.: Sur les lois de comportement en élasticité non linéaire compressible. C. R. Acad. Sci., Ser. II 295, 423–426 (1982)

16. Ciarlet, Ph.G., Gogu, R., Mardare, C.: Orientation-preserving condition and polyconvexity on a surface: application to nonlinear shell theory. J. Math. Pures Appl. 99, 704–725 (2013)

17. Ciarlet, Ph.G., Mardare, C.: An existence theorem for a two-dimensional nonlinear shell model of Koiter’s type. Math. Models Methods Appl. Sci. 28(14), 2833–2861 (2018)

18. Eremeyev, V.A., Pietraszkiewicz, W.: The nonlinear theory of elastic shells with phase transitions. J. Elast. 74, 67–86 (2004)

19. Eremeyev, V.A., Pietraszkiewicz, W.: Local symmetry group in the general theory of elastic shells. J. Elast. 85, 125–152 (2006)

20. Fox, D.D., Simo, J.C.: A drill rotation formulation for geometrically exact shells. Comput. Methods Appl. Mech. Eng. 98, 329–343 (1992)

21. Ghika, I.D., Bîrsan, M., Lewintan, P., Neff, P.: The isotropic Cosserat shell model including terms up to $O(h^5)$. Part I: Derivation in matrix notation. J. Elast. (2020). https://doi.org/10.1007/s10659-020-09796-3. arXiv:2003.00549

22. Girault, V., Raviart, P.A.: Finite Element Approximation of the Navier-Stokes Equations. Lect. Notes Math., vol. 749. Springer, Heidelberg (1979)

23. Ibrahimbegović, A.: Stress resultant geometrically nonlinear shell theory with drilling rotations - Part I: A consistent formulation. Comput. Methods Appl. Mech. Eng. 118, 265–284 (1994)

24. Koiter, W.T.: Foundations and basic equations of shell theory. A survey of recent progress. In: Niordson, F.I. (ed.) Theory of Thin Shells, IUTAM Symposium Copenhagen 1967, pp. 93–105. Springer, Heidelberg (1969)

25. Koiter, W.T.: A consistent first approximation in the general theory of thin elastic shells. In: Koiter, W.T. (ed.) The Theory of Thin Elastic Shells, IUTAM Symposium Delft 1960, pp. 12–33. North-Holland, Amsterdam (1960)

26. Leis, R.: Initial Boundary Value Problems in Mathematical Physics. Teubner, Stuttgart (1986)

27. Neff, P.: On Korn’s first inequality with nonconstant coefficients. Proc. R. Soc. Edinb. A 132, 221–243 (2002)

28. Neff, P.: A geometrically exact Cosserat-shell model including size effects, avoiding degeneracy in the thin shell limit. Part I: Formal dimensional reduction for elastic plates and existence of minimizers for positive Cosserat couple modulus. Contin. Mech. Thermodyn. 16, 577–628 (2004)
29. Neff, P.: Geometrically exact Cosserat theory for bulk behaviour and thin structures. Modelling and mathematical analysis. Signatur HS 7/0973. Habilitationsschrift, Universitäts- und Landesbibliothek, Technische Universität Darmstadt, Darmstadt (2004)
30. Neff, P.: A geometrically exact viscoelastic membrane-shell with viscoelastic transverse shear resistance avoiding degeneracy in the thin-shell limit. Part I: The viscoelastic membrane-plate. Z. Angew. Math. Phys. 56(1), 148–182 (2005)
31. Neff, P.: Local existence and uniqueness for a geometrically exact membrane-plate with viscoelastic transverse shear resistance. Math. Methods Appl. Sci. 28, 1031–1060 (2005)
32. Neff, P.: The $\Gamma$-limit of a finite strain Cosserat model for asymptotically thin domains versus a formal dimensional reduction. In: Pietraszkiewicz, W., Szymczak, C. (eds.) Shell-Structures: Theory and Applications, pp. 149–152. Taylor and Francis Group, London (2006)
33. Neff, P.: A geometrically exact planar Cosserat shell-model with microstructure: existence of minimizers for zero Cosserat couple modulus. Math. Models Methods Appl. Sci. 17, 363–392 (2007)
34. Neff, P., Bîrsan, M., Osterbrink, F.: Existence theorem for the classical nonlinear Cosserat elastic model. J. Elast. 121(1), 119–141 (2015)
35. Neff, P., Chelmiński, K.: A geometrically exact Cosserat shell-model for defective elastic crystals. Justification via $\Gamma$-convergence. Interfaces Free Bound. 9, 455–492 (2007)
36. Neff, P., Hong, K.-I., Jeong, J.: The Reissner-Mindlin plate is the $\Gamma$-limit of Cosserat elasticity. Math. Models Methods Appl. Sci. 20, 1553–1590 (2010)
37. Neff, P., Lankeit, J., Madeo, A.: On Gorioli’s minimum property and its relation to Cauchy’s polar decomposition. Int. J. Eng. Sci. 80, 207–217 (2014)
38. Neff, P., Münch, I.: Curl bounds Grad on $SO(3)$. ESAIM Control Optim. Calc. Var. 14(1), 148–159 (2008)
39. Neff, P., Pompe, W.: Counterexamples in the theory of coerciveness for linear elliptic systems related to generalizations of Korn’s second inequality. Z. Angew. Math. Mech. 94, 784–790 (2014)
40. Paroni, R., Podio-Guidugli, P., Tomassetti, G.: The Reissner-Mindlin plate theory via $\Gamma$-convergence. C. R. Acad. Sci. Paris, Ser. I 343, 437–440 (2006)
41. Pietraszkiewicz, W., Konopińska, W.: Drilling couples and refined constitutive equations in the resultant geometrically non-linear theory of elastic shells. Int. J. Solids Struct. 51, 2133–2143 (2014)
42. Pompe, W.: Korn’s first inequality with variable coefficients and its generalizations. Comment. Math. Univ. Carol. 44(1), 57–70 (2003)
43. Pompe, W.: Counterexamples to Korn’s inequality with non-constant rotation coefficients. Math. Mech. Solids 16, 172–176 (2011). https://doi.org/10.1177/1081286510367554
44. Sansour, C., Bufler, H.: An exact finite rotation shell theory, its mixed variational formulation and its finite element implementation. Int. J. Numer. Methods Eng. 34, 73–115 (1992)
45. Simo, J.C., Fox, D.D.: On a stress resultant geometrically exact shell model. Part I: Formulation and optimal parametrization. Comput. Methods Appl. Mech. Eng. 72, 267–304 (1989)
46. Sprekels, J., Tiba, D.: An analytic approach to a generalized Naghdi shell model. Adv. Math. Sci. Appl. 12, 175–190 (2002)
47. Steigmann, D.J.: Two-dimensional models for the combined bending and stretching of plates and shells based on three-dimensional linear elasticity. Int. J. Eng. Sci. 46, 654–676 (2008)
48. Steigmann, D.J.: Extension of Koiter’s linear shell theory to materials exhibiting arbitrary symmetry. Int. J. Eng. Sci. 51, 216–232 (2012)
49. Tambaca, J., Velčić, I.: Semicontinuity theorem in the micropolar elasticity. ESAIM Control Optim. Calc. Var. 16(2), 337–355 (2010)
50. Tambaca, J., Velčić, I.: Existence theorem for nonlinear micropolar elasticity. ESAIM Control Optim. Calc. Var. 16, 92–110 (2010)
51. Weinberg, K., Neff, P.: A geometrically exact thin membrane model-investigation of large deformations and wrinkling. Int. J. Numer. Methods Eng. 74(6), 871–893 (2008)
52. Zhilin, P.A.: Applied Mechanics – Foundations of Shell Theory. State Polytechnical University Publisher, Sankt Petersburg (2006) (in Russian)