The number of solutions to the boundary value problem with linear-quintic and linear-cubic-quintic nonlinearity

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Abstract: The nonlinear oscillators describing by differential equations of the form \( x'' = -ax + cx^5 \) and \( x'' = -ax + bx^3 + cx^5 \) are studied. Multiplicity results for both types of equations, given with the Neumann boundary conditions are obtained. It is shown that the number of solutions depend on the coefficient \( a \) only. The exact estimates of the number of solutions are obtained. Practical issues, such as the representation of solutions in terms of Jacobian elliptic functions and calculation of the initial values for solutions of boundary value problems, are considered also. The illustrative examples are provided. Outlines of future research conclude the article.

Key Words: Boundary value problem, quintic nonlinearity, cubic-quintic nonlinearity, phase trajectory, multiplicity of solutions, Jacobian elliptic function.

Received: June 4, 2020. Revised: October 21, 2020. Accepted: November 18, 2020. Published: December 8, 2020.

1 Introduction

Despite of the great number of papers devoted to non-linear oscillators, there are still unsolved and not well-researched topics. An overview of results and issues can be found in [15].

In this article we will consider the second order differential equations of two types. The first one is

\[ x'' = -ax + cx^5, \quad a > 0, \quad c > 0 \] (1)

with the linear-quintic nonlinearity. The second equation is of the form

\[ x'' = -ax + bx^3 + cx^5, \quad a > 0, \quad b > 0, \quad c > 0 \] (2)

and it contains the linear-cubic-quintic nonlinearity. Both equations appear in applications. The list of areas where equations (1) and (2) arise can be found in the works [1], [2], [3], [16], [7], [6]. Equations of the form (1) and (2) appear in the theory of Schrödinger equation, dealing with wave propagation in fluids plasmas, and nonlinear optical media [19, subsection 5.1]. Making the traveling wave transformation in nonlinear reaction-diffusion equations also can lead to cubic-quintic equations of the form (2) [19, subsection 5.2].

We are motivated, on one hand, by a series of papers dealing with equations (1) and (2). In the work [10], [11], [12], [13], [14] the results concerning the boundary value problems (BVP) for the equation

\[ x'' = -ax + bx^3, \quad a > 0, \quad b > 0. \] (3)

In articles [10] and [12] we have obtained also, convenient for use, formulas for solutions of BVP (3)

\[ x'(0) = 0, \quad x'(T) = 0. \] (4)

Moreover, the equations for the initial conditions of solutions to the problem (3), (4), were derived. The initial conditions are needed for effective computation of solutions of BVP. More on the problems arising when finding periodic solutions can be found in [4]. The estimates of the number of solutions were obtained. The proof was conducted showing that these estimates are precise. These results were based on the monotonicity of periods of some solutions on the initial conditions. More details are given below.

In this paper we extend our results concerning the BVP (3), (4) to the similar problems for equations (1) and (2). Our aim is to get the estimates of the number of solutions to the problem (1), (4) and to the problem (2), (4). Our results are:

- the monotonicity of periods for solutions of the equations (1) and (2) is proved;
- the exact estimates of the number of solutions for the BVP (1), (4) and (2), (4) are obtained;
Consider the Neumann boundary value problem (3), (4). In this section, we present the results about the exact number of solutions for this problem and using the theory of Jacobian elliptic functions, we recall expressions for solutions of the Cauchy problems (3),

\[ x(0) = x_\alpha, \quad x'(0) = 0, -\sqrt{\frac{a}{b}} < x_\alpha < \sqrt{\frac{a}{b}}, \quad x_\alpha \neq 0 \]

(5) obtained in the articles [10], [12], [13]. We show how to find the initial values \(x_\alpha\) of solutions of the problem (3), (4). The following statement is true for the problem (3), (4).

**Theorem 1** Let \(i\) be a positive integer such that

\[ \frac{i \pi}{\sqrt{a}} < T < \frac{(i + 1)\pi}{\sqrt{a}}, \]

where \(T\) is the end point of the interval \((0, T]\). The Neumann problem (3), (4) has exactly \(2i\) nontrivial solutions such that \(x(0) = x_\alpha \neq 0, x'(0) = 0, -\sqrt{\frac{a}{b}} < x_\alpha < \sqrt{\frac{a}{b}}\).

The proof can be found in the articles [12], [13]. Consider the Cauchy problem (3), (5). Let \(a\) and \(T\) be given. We wish to find \(x_\alpha\) such that the respective solutions \(x(t; x_\alpha)\) of the above problem satisfy the boundary condition \(x'(T) = 0\), i.e. \(x(t; x_\alpha)\) solve the Neumann problem (3), (4). The following assertion provides the explicit formula for the solution of (3), (4).

**Lemma 2** [10], [13] The function

\[ x(t, a, b, x_\alpha) = x_\alpha cd (\sqrt{a - \frac{1}{2}b x_\alpha^2 t}; k) \]

(7)

where \(k = \sqrt{\frac{b x_\alpha^2}{2a - b x_\alpha^2}}\) is a solution of the Cauchy problem (3), (5).

The proof of Lemma 2 is given in article [13].

Denote \(f(t, a, b, x_\alpha) = x'_\alpha(t, a, b, x_\alpha)\). This derivative can be computed and the following formula

\[ f(t, a, b, x_\alpha) = \frac{x_\alpha}{2a - b x_\alpha^2} (k^2 - 1) \times \]

\[ \times nd (\sqrt{a - \frac{1}{2}b x_\alpha^2 t}; k) \quad sd (\sqrt{a - \frac{1}{2}b x_\alpha^2 t}; k) \]

(8)

is valid.

**Lemma 3** [13] The boundary value problem (3), (4) for the coefficient \(a\) and \(T\) given can be solved now by solving the equation with respect to \(x_\alpha\)

\[ f(T, a, b, x_\alpha) = 0 \]

(9)

**Theorem 4** [13] A solution to the Neumann problem (3), (4) is given by (7), where \(x_\alpha\) is a solution of (9).
3 Main results for linear-quintic case

3.1 Multiplicity of solutions for linear-quintic case

Consider the equation (1). There are three critical points of equation (1): $x_{1,3} = \pm \sqrt[3]{\frac{a}{c}}$ both are saddle points, $x_2 = 0$ is a center. Two heteroclinic trajectories connect the two saddle points. $G_5$ is an open region bounded by the two heteroclinic trajectories connecting saddle points. The phase portrait of equation (1) is depicted in Fig. 2.

Consider trajectories (closed curves) that fill the region $G_5$. The heteroclinic solution at infinity satisfies

$$0 = x^2(\infty) = -a x^2(\infty) + \frac{1}{3} c x^6(\infty) + C = -a \sqrt[3]{\frac{a}{c}} + \frac{1}{3} c \left(\sqrt[3]{\frac{a}{c}}\right)^3 + C = -\frac{2a\sqrt[3]{a}}{3\sqrt{c}} + C,$$

where $x^2(\infty) = \sqrt[3]{\frac{a}{c}}$ and therefore the respective 

$$C = \frac{2a\sqrt[3]{a}}{3\sqrt{c}}.$$

Then any trajectory located in the region $G_5$ satisfies the relation (10), where $|C| < \frac{2a\sqrt[3]{a}}{3\sqrt{c}}$.

It was proved in the papers [11], [13] that the period of a solution to the cubic equation (3) is increasing function. In a similar manner, we can prove that a solution to the quintic equation (1) also is increasing function.

Since any trajectory in the region $G_5$ is closed it is convenient to consider the respective solutions in polar coordinates. Equation (1) written in polar coordinates

$$x(t) = \rho(t) \sin \phi(t), \quad x'(t) = \rho(t) \cos \phi(t)$$

turns to a system (12):

$$
\begin{align*}
\rho'(t) &= \rho^2(t) \phi'(t), \\
\phi'(t) &= \cos^2 \phi(t) + a \sin^2 \phi(t) - \rho^4(t) c \sin^6 \phi(t),
\end{align*}
$$

where $C$ is an arbitrary constant. It is clear from the analysis of the phase portrait that solutions (trajectories) of the Neumann problem can exist only in the region $G_5$ between the heteroclinic trajectories. Solutions $x(t; x_\alpha)$ of the Cauchy problem (1), (17)

$$x(0) = x_\alpha, \quad x'(0) = 0, \quad 0 < x_\alpha < \sqrt[3]{\frac{a}{c}}$$
satisfy the relation (16), where \(C = a x_\alpha^2 - \frac{1}{3} c x_\alpha^6\). It follows
\[
x^\prime 2(t; x_\alpha) = -a x^2(t; x_\alpha) + \frac{1}{3} c x^6(t; x_\alpha) + a x^2_\alpha - \frac{1}{3} c x_\alpha^6,
\]
\[
\frac{dx}{dt} = \pm \sqrt{-a x^2(t; x_\alpha) + \frac{1}{3} c x^6(t; x_\alpha) + a x^2_\alpha - \frac{1}{3} c x_\alpha^6},
\]
(18)

(notice that \(x'(t; x_\alpha) < 0\) and therefore
\[
-\frac{dx}{\sqrt{-a x^2(t; x_\alpha) + \frac{1}{3} c x^6(t; x_\alpha) + a x^2_\alpha - \frac{1}{3} c x_\alpha^6}} = dt.
\]
(19)
\[
-\frac{0}{x_\alpha} \frac{dx}{\sqrt{-a x^2 + \frac{1}{3} c x^6 + a x^2_\alpha - \frac{1}{3} c x_\alpha^6}} =
\]
\[
\frac{0}{x_\alpha} \frac{dx}{\sqrt{a x^2 + \frac{1}{3} c x^6 + a x^2_\alpha - \frac{1}{3} c x_\alpha^6}} = \int t dt = t.
\]
(20)

One has that
\[
\int_0^{x_\alpha} \frac{dx}{\sqrt{-a x^2 + \frac{1}{3} c x^6 + a x^2_\alpha - \frac{1}{3} c x_\alpha^6}} =
\]
\[
-\frac{\int_0^{x_\alpha} \frac{dx}{\sqrt{a (1-x^2) - \frac{1}{3} c x^6}}}{\int_0^{x_\alpha} \frac{dx}{\sqrt{a (1-x^2) - \frac{1}{3} c x^6}}} = T_{x_\alpha},
\]
(21)
where \(T_{x_\alpha}\) is the time needed to move on a phase plane from \((x_\alpha, 0)\) to the vertical axis \(x = 0\) (a quarter of a period). It follows then that \(T_{x_\alpha}\) is increasing function of \(x_\alpha\).

Compare \(T_{\alpha_1} = \int_0^{\sqrt{a (1-\xi^2) - \frac{1}{3} c x^6}} \frac{d\xi}{\sqrt{a (1-\xi^2) - \frac{1}{3} c x^6}}\) and
\[
T_{\alpha_2} = \int_0^{\sqrt{a (1-\xi^2) - \frac{1}{3} c x^6}} \frac{d\xi}{\sqrt{a (1-\xi^2) - \frac{1}{3} c x^6}}.\]
If \(x_\alpha_1 < x_\alpha_2\), then \(T_{\alpha_1} < T_{\alpha_2}\).

**Remark 8** The results on the monotonicity of periods of solution of equations similar to (1) and (2) were proved in [5], but they are not applicable to equations (1) and (2). Therefore we prove Lemma 7 and Lemma 14.

For the problem (1), (4) the following statement is true.

**Theorem 9** Let \(i\) be a positive integer such that
\[
\frac{i \pi}{\sqrt{a}} < T < \frac{(i+1) \pi}{\sqrt{a}},
\]
where \(T\) is as in (4). The Neumann problem (1), (4) has exactly \(2i\) nontrivial solutions such that \(x(0) = x_\alpha \neq 0, x'(0) = 0, -\sqrt{\frac{a}{c}} < x_\alpha < \sqrt{\frac{a}{c}}\).

**Proof:** Consider solutions of the Cauchy problem (1), \(x(0) = x_\alpha, x'(0) = 0, 0 < x_\alpha < \frac{\sqrt{a}}{\sqrt{c}}\). These solutions, denoted \(x(t; x_\alpha)\), are continuously dependent on the initial data, due to a simple form of the equation. Solutions for \(x_\alpha\) small enough are well approximated by solutions \(y(t)\) of the equation of variations \(y'' = -a y\) around the trivial solution. Solutions of the linearized equation which satisfy the initial conditions \(y(0) = x_0, y'(0) = 0\), are given by the formula
\[
y(t) = x_\alpha \cos \sqrt{a} t.
\]
Due to the assumption \(\frac{i \pi}{\sqrt{a}} < T < \frac{(i+1) \pi}{\sqrt{a}}\) solutions \(y(t)\) along with solutions \(x(t; x_\alpha)\) (for small enough \(x_\alpha\) have exactly \(i\) extrema in the interval \((0, T)\) and therefore \(T\) is an extremum point. Notice that zeros and extrema of solutions \(x(t; x_\alpha)\) alternate, and there is exactly one point of extremum between two consecutive zeros. These extrema due to Lemma 5 and Lemma 7 move monotonically to the right as \(x_\alpha\) increases. Solutions \(x(t; x_\alpha)\) with \(0 < x_\alpha < \frac{\sqrt{a}}{\sqrt{c}}\) and close enough to \(\frac{\sqrt{a}}{\sqrt{c}}\) have not extrema in \((0, T]\) since the respective trajectories are close to the upper heteroclinic (and the “period” of a heteroclinic solution is infinite) Therefore there are exactly \(i\) solutions of the problem (1), (4). The additional \(i\) solutions are obtained considering solutions with \(x_\alpha \in \left(-\frac{\sqrt{a}}{\sqrt{c}}, 0\right)\) due to symmetry arguments.

**3.2 Formulae of solutions for linear-quintic case**

Consider the problem
\[
x'' = -ax + cx^5,
\]
\[
x(0) = x_\alpha, \quad x'(0) = 0, \quad 0 < x_\alpha < \sqrt{\frac{a}{c}}.
\]
(24)

For \(a\) and \(T\) given we wish to find \(x_\alpha\) such that the respective solutions \(x(t; x_\alpha)\) of the above problem satisfy the boundary condition \(x'(T) = 0\), i.e. \(x(t; x_\alpha)\) solve the Neumann problem (1), (4). The following assertion provides the explicit formula for a solution of (24).

**Lemma 10** The function
\[
x(t; a, c, x_\alpha) = \frac{x_\alpha \cn \left( \frac{1}{3} c \xi_1 (x_\alpha^2 - \xi_2) t; k_1 \right)}{\sqrt{1 - \frac{x_\alpha^2}{\xi_1} \sn^2 \left( \frac{1}{3} c \xi_1 (x_\alpha^2 - \xi_2) t; k_1 \right)}}.
\]
(25)
where \( k_1 = \sqrt{\frac{x_α^2 (ξ_1 - ξ_2)}{ξ_t (x_α^2 - ξ_2)}} \).

\[ ξ_1 := ξ_1(x_α) = \frac{1}{2\sqrt{c}} \left(- x_α^2 \sqrt{c} + \sqrt{3(4a - x_α^4 c)}\right), \]

\[ ξ_2 := ξ_2(x_α) = \frac{1}{2\sqrt{c}} \left(- x_α^2 \sqrt{c} - \sqrt{3(4a - x_α^4 c)}\right), \tag{27} \]

is a solution of the Cauchy problem (24).

In equation (25) we introduce a new variable

\[ A := A(x_α) = \sqrt{\frac{1}{3} c \ ξ_1 (x_α^2 - ξ_2)} \tag{28} \]

and obtain the formula for a solution of the Cauchy problem (24)

\[ x(t, a, c, x_α) = \frac{x_α \ cn(A t; k_1)}{\sqrt{1 - x_α^2 \ sn^2(A t; k_1)}}, \tag{29} \]

where \( k_1 = \sqrt{\frac{x_α^2 (ξ_1 - ξ_2)}{ξ_t (x_α^2 - ξ_2)}} \), but \( ξ_1 \) and \( ξ_2 \) as in (26).

We can compute \( x'_1(t, a, c, x_α) \) and obtain

\[ x'_1(t, a, c, x_α) = x_α \ sn(A t; k_1) dn(A t; k_1) \times \]

\[ \frac{x_α^2 \ cn(A t; k_1)-ξ_t \sqrt{1 - x_α^2 \ sn^2(A t; k_1)}}{ξ_1 \left(\sqrt{1 - x_α^2 \ sn^2(A t; k_1)}\right)} \tag{30} \]

where \( ξ_1, ξ_2 \) and \( A \) are given by the formulas (26), (27). Denote \( g(T, a, c, x_α) = x'_1(t, a, c, x_α) \).

**Lemma 11** The problem (1), (4) for \( a \) and \( T \) given can be obtained as a solution of the equation

\[ g(T, a, c, x_α) = 0 \tag{31} \]

with respect to \( x_α \), where \( x_α \) is the initial value for solution.

**Theorem 12** A solution to the Neumann problem (1), (4) is given by (25), where \( x_α \) is a solution of (30).

## 4 Main results for linear-cubic-quintic case

### 4.1 Multiplicity of solutions for linear-cubic-quintic case

In this section we consider the linear-cubic-quintic equation (2).

There are three critical points of equation (2): \( x_2 = 0 \) is a center, \( x_{1,3} = \pm \sqrt{-\frac{b}{2c}} + \frac{1}{2c} \sqrt{b^2 + 4ac} \) both are saddle points. Two heteroclinic trajectories connect the two saddle points. \( G \) is an open region bounded by the two heteroclinic trajectories connecting saddle points. The phase portrait of equation (2) is depicted in Fig. 3.

![Figure 3: The phase portrait of equation (2), region G](image)

Consider trajectories (closed curves) that fill the region \( G \). The heteroclinic solution at infinity satisfies

\[ 0 = x^2(∞) = -ax^4(∞) + \frac{1}{2} b x^4(∞) + \frac{1}{3} c x^6(∞) + C, \tag{32} \]

where \( x^2(∞) = -b + \sqrt{b^2 + 4ac} \). Then any trajectory located in the region \( G \) satisfies the relation (31), where

\[ |C| < \frac{b + \sqrt{b^2 + 4ac}}{2c} \left(\frac{2a}{3} - \frac{b}{6} \left(\frac{b + \sqrt{b^2 + 4ac}}{2c}\right)\right). \]

Since any trajectory in the region \( G \) is closed it is convenient to consider the respective solutions in polar coordinates. Equation (2) written in polar coordinates (11) turns to a system (32):

\[ \begin{aligned}
  \phi'(t) &= \cos^2 \phi(t) + a \sin^2 \phi(t) - \rho^2(t) b \sin^4 \phi(t) - 2 \rho^2(t) c \sin^6 \phi(t), \\
  \rho'(t) &= \frac{1}{2} \rho(t) \sin 2\phi(t) \times (1 - a + \rho^2(t) b \sin^2 \phi(t) + \rho^4(t) c \sin^4 \phi(t)).
\end{aligned} \tag{33} \]

Consider any solution of equation (2) with the initial conditions \((x(t_0), x'(t_0)) \in G\). Let the initial conditions be written as

\[ \phi(t_0) = \phi_0, \rho(t_0) = \rho_0, (\phi_0, \rho_0) \in G, \rho_0 > 0. \tag{34} \]

**Lemma 13** The angular function of any solution of (32), (33) is monotonically increasing.

**Proof:** Consider the first equation of system (32) multiplied by \( \rho^2(t) \)

\[ \rho^2(t) \phi'(t) = \rho^2(t) \cos^2 \phi(t) + a \rho^2(t) \sin^2 \phi(t) - b \rho^4(t) \sin^4 \phi(t) - c \rho^6(t) \sin^6 \phi(t). \]
Returning to \((x', x'')\) coordinates

\[
\rho^2(t)\phi'(t) = x'^2(t) + a x^2(t) - b x^4(t) - c x^6(t) = x'^2(t) + x^2(t)(a - b x^2(t) - c x^4(t)) > 0.
\] (35)

Since \(-b\sqrt{b^2 - 4ac} < x^2(t) < -b\sqrt{b^2 + 4ac}\) for any solution of (32), (33) the angular function \(\phi(t)\) is increasing. \(\square\)

**Lemma 14** Suppose \(T_{x_\alpha}\) is the time needed to move on a phase plane from \((x_\alpha, 0)\) to the vertical axis \(x = 0\) (a quarter of a period). The function \(T_{x_\alpha}\) monotonically increases from \(\pi\) to \(+\infty\) as \(x_{\alpha}\) changes from zero to \(\sqrt{-\frac{b}{2c} + \frac{1}{2c}\sqrt{b^2 + 4ac}}\).

**Proof:** Equation (2) has an integral

\[
x'^2(t) = -a x^2(t) + \frac{1}{2} b x^4(t) + \frac{1}{3} c x^6(t) + C,
\] (36)

where \(C\) is an arbitrary constant. The solutions (trajectories) of the Neumann problem can exist only in the region \(G\) between the heteroclinic trajectories. Solutions \(x(t; x_{\alpha})\) of the Cauchy problem (2), (37)

\[
x(0) = x_{\alpha}, \quad x'(0) = 0,
\]

\[
0 < x_{\alpha} < \sqrt{-\frac{b}{2c} + \frac{1}{2c}\sqrt{b^2 + 4ac}}
\] (37)

satisfy the relation (36), \(C = a x_{\alpha}^2 - \frac{1}{2} b x_{\alpha}^4 - \frac{1}{3} c x_{\alpha}^6\).

Performing the transformations

\[
x'^2(t; x_{\alpha}) = -a x^2(t; x_{\alpha}) + \frac{1}{2} b x^4(t; x_{\alpha}) + \frac{1}{3} c x^6(t; x_{\alpha}) + a x_{\alpha}^2 - \frac{1}{2} b x_{\alpha}^4 - \frac{1}{3} c x_{\alpha}^6,
\]

\[
\frac{dx}{dt} = \pm \sqrt{-a x^2 + \frac{1}{2} b x^4 + \frac{1}{3} c x^6 + a x_{\alpha}^2 - \frac{1}{2} b x_{\alpha}^4 - \frac{1}{3} c x_{\alpha}^6},
\] (38)

(notice that \(x'(t; x_{\alpha}) < 0\) one obtains

\[
- \sqrt{-a x^2 + \frac{1}{2} b x^4 + \frac{1}{3} c x^6 + a x_{\alpha}^2 - \frac{1}{2} b x_{\alpha}^4 - \frac{1}{3} c x_{\alpha}^6} = dt.
\]

\[
- \int_{x_{\alpha}}^x \sqrt{-a x^2 + \frac{1}{2} b x^4 + \frac{1}{3} c x^6 + a x_{\alpha}^2 - \frac{1}{2} b x_{\alpha}^4 - \frac{1}{3} c x_{\alpha}^6} \, dx = \int_0^t dt = t.
\] (39)

One has that

\[
\int_0^{x_{\alpha}} \sqrt{-a x^2 + \frac{1}{2} b x^4 + \frac{1}{3} c x^6 + a x_{\alpha}^2 - \frac{1}{2} b x_{\alpha}^4 - \frac{1}{3} c x_{\alpha}^6} \, dx = T_{x_{\alpha}}
\]

\[
\left| \xi = \frac{x}{x_{\alpha}} \right| = \int_0^1 \sqrt{a(1-\xi^4) - \frac{1}{2} b x_{\alpha}^2(1-\xi^4) - \frac{1}{3} c x_{\alpha}^4(1-\xi^4)} \, d\xi = \int_0^{x_{\alpha}} \sqrt{-a x^2 + \frac{1}{2} b x^4 + \frac{1}{3} c x^6 + a x_{\alpha}^2 - \frac{1}{2} b x_{\alpha}^4 - \frac{1}{3} c x_{\alpha}^6} \, dx.
\]

\[
x'' = -a x + b x^3 + c x^5,
\]

\[
x(0) = x_{\alpha}, \quad x'(0) = 0,
\]

\[
0 < x_{\alpha} < \sqrt{-\frac{b}{2c} + \frac{1}{2c}\sqrt{b^2 + 4ac}}.
\] (40)

\[
4.2 \text{ Formulae of solutions for linear-cubic-quintic case}
\]

Consider the problem

\[
x'' = -a x + b x^3 + c x^5,
\]

\[
x(0) = x_{\alpha}, \quad x'(0) = 0,
\]

\[
0 < x_{\alpha} < \sqrt{-\frac{b}{2c} + \frac{1}{2c}\sqrt{b^2 + 4ac}}.
\] (41)
For $a$ and $T$ given we wish to find $x_\alpha$ such that the respective solutions $x(t; x_\alpha)$ of the above problem satisfy the boundary condition $x'(T) = 0$, i.e. $x(t; x_\alpha)$ solve the Neumann problem (2), (4). The following assertion provides the explicit formula for a solution of (43).

**Lemma 16** The function

$$x(t, a, b, c, x_\alpha) = \frac{x_\alpha \sqrt{\frac{1}{4} c \psi_1(x_\alpha^2 - \psi_2) t; k_2}}{\sqrt{1 - \frac{x_\alpha^2}{\psi_1} sn^2(B t; k_2)}},$$

$k_2 = \sqrt{\frac{x_\alpha^2}{\psi_1} (x_\alpha^2 - \psi_2)},$

$$\psi_1 := \psi_1(x_\alpha) = \frac{1}{4c}(-3b - 2cx_\alpha^2 - \sqrt{3(3b^2 + 4bcx_\alpha^2 - 4c^2x_\alpha^4 + 16ac)},
$$

$$\psi_2 := \psi_2(x_\alpha) = \frac{1}{4c}(-3b - 2cx_\alpha^2 + \sqrt{3(3b^2 - 4bcx_\alpha^2 - 4c^2x_\alpha^4 + 16ac)},$$

is a solution of the Cauchy problem (43).

In equation (44) we introduce a new variable

$$B := B(x_\alpha) = \sqrt{\frac{1}{3} c \psi_1(x_\alpha^2 - \psi_2)}$$

and obtain the formula for a solution of the Cauchy problem (43)

$$x(t, a, b, c, x_\alpha) = \frac{x_\alpha \sqrt{\frac{1}{4} c \psi_1(x_\alpha^2 - \psi_2) t; k_2}}{\sqrt{1 - \frac{x_\alpha^2}{\psi_1} sn^2(B t; k_2)}},$$

where $k_2 = \sqrt{\frac{x_\alpha^2}{\psi_1} (x_\alpha^2 - \psi_2)}$, but $\psi_1$ and $\psi_2$ as in (45).

We can compute $x'_i(t, a, b, c, x_\alpha)$ and obtain

$$x'_i(t, a, b, c, x_\alpha) = x_\alpha \frac{sn(B t; k_2)}{dn(B t; k_2)} dn(B t; k_2) \times
$$

$$\times \frac{x_\alpha^2 \sqrt{\frac{1}{4} c \psi_1(x_\alpha^2 - \psi_2) t; k_2} - \psi_1 \sqrt{1 - \frac{x_\alpha^2}{\psi_1} sn^2(B t; k_2)}}{\psi_1 \sqrt{1 - \frac{x_\alpha^2}{\psi_1} sn^2(B t; k_2)}},$$

where $\psi_1$, $\psi_2$ and $B$ are given by formulas (45), (46).

Denote $s(T, a, b, c, x_\alpha) = x'_i(t, a, b, c, x_\alpha)$.

We have arrived at the statement.

**Lemma 17** A solution to the problem (2), (4) for $a$ and $T$ given has the initial value $x(0) = x_\alpha$ that is a root of the equation

$$s(T, a, b, c, x_\alpha) = 0.$$

**Theorem 18** A solution to the Neumann problem (2), (4) is given by (44) where $x_\alpha$ is a solution of (49).

### 5 Example

**Example 1** Consider equation (1) with $a = 50$, $b = 25$:

$$x'' = -50x + 25x^5$$

with the initial conditions $x(0) = x_\alpha$, $x'(0) = 0$, $0 < x_\alpha < \sqrt{2}$. Then the number of solutions satisfying the boundary conditions (4), where $T = 1$, is two and, symmetrically, for initial conditions $x(0) = x_\alpha$, $x'(0) = 0$, $-\sqrt{2} < x_\alpha < 0$ there are two additional solutions to the problem (50), (4), totally four solutions. By Theorem 9, this is the case for $T = 1$ and $i = 2$ (namely $\frac{2\pi}{\sqrt{a}} < T = 1 < \frac{3\pi}{\sqrt{a}}$) in the inequality (22).

Consider equation (30) with $a = 50$, $b = 25$ and $T=1$. We have

$$g(1, 50, 25, x_\alpha) = x'_i(1, 50, 25, x_\alpha) = 0.$$

The next Figure shows that the equation (51) provides four nonzero initial values $x_\alpha$ for solutions of the problem (50), (4).

![Figure 4: The graph of $g(1, 50, 25, x_\alpha)$ for equation (50)](image)

In Fig. 5 and Fig. 6 the graphs of solutions $x(t)$ and the derivatives $x'(t)$ of the problem (50), (4) are depicted.

![Figure 5: Graphs $x(t)$ for solutions of the problem (50), (4), $x_\alpha \approx 0.8995$ (solid), $x_\alpha \approx 1.1831$ (dashed)](image)
Example 2
Equation (2) is of the form:

$$x'' = -50x + 25x^3 + 25x^5 \tag{52}$$

and the initial conditions are $x(0) = x_\alpha$, $x'(0) = 0$, $0 < x_\alpha < 1$, then the number of solutions satisfying the boundary conditions (4), where $T = 1$, is two and, symmetrically, for initial conditions $x(0) = x_\alpha$, $x'(0) = 0$, $-1 < x_\alpha < 0$ there are also two solutions to the problem (52), (4), totally four solutions.

By Theorem 15, this is the case for $T = 1$ and $i = 2$ (namely $\frac{2\pi}{\sqrt{50}} < T < \frac{3\pi}{\sqrt{50}}$) in the inequality (42).

Consider equation (49) with $a = 50$, $b = 25$, $c = 25$ and $T=1$, we have the equation

$$s(1,50,25,25,x_\alpha) = x'_1(1,50,25,25,x_\alpha) = 0. \tag{53}$$

In Fig. 7 four nonzero initial values $x_\alpha$ are indicated for solutions of the problem (50), (4).

In Fig. 8 and Fig. 9 the graphs of solutions $x(t)$ and the derivatives $x'(t)$ of the problem (52), (4) are depicted.

Concluding remarks

In this article, we have considered an undamped, unforced cubic-quintic oscillator. We have studied boundary value problems with the Neumann type boundary conditions. Our conclusion is that the number of solutions to the problem under consideration depends only on the coefficient $a$ in a linear part. Besides, the formulas for the number of solutions was obtained and, moreover, the proof was given that these estimates are exact. The complicated analytical expressions for solutions of the cubic-quintic equations were analyzed and as a result, equations for the initial conditions for solutions of the boundary value problems were obtained. This information makes it much easier to do the calculations.

In the work [12] some remarks were made concerning nonlinear oscillators with higher degree polynomial nonlinearities. In the work [11] cubic oscillators with non-constant (piece-wise linear) coefficient were considered, following the results and ideas in [18], [9]. Studies in both directions and combinations of both ones are possible as future research.

Acknowledgements

The author would like to sincerely thank Professor Felix Sadyrbaev for his useful comments.
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