Abstract

We introduce a higher rank analog of the Pandharipande-Thomas theory of stable pairs [RR09] on a Calabi-Yau threefold $X$. More precisely, we develop a moduli theory for frozen triples given by the data $\mathcal{O}_X^{\text{gr}}(-n) \to F$ where $F$ is a sheaf of pure dimension 1. The moduli space of such objects does not naturally determine an enumerative theory: that is, it does not naturally possess a perfect symmetric obstruction theory. Instead, we build a zero-dimensional virtual fundamental class by hand, by truncating a deformation-obstruction theory coming from the moduli of objects in the derived category of $X$. This yields the first deformation-theoretic construction of a higher-rank enumerative theory for Calabi-Yau threefolds. We calculate this enumerative theory for local $\mathbb{P}^1$ using the Graber-Pandharipande [GP99] virtual localization technique. In a sequel to this article [She10], we show how to compute similar invariants associated to frozen triples using Kontsevich-Soibelman [MK08], Joyce-Song [JS09] wall-crossing techniques.

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1 Introduction

The work of algebraic geometers to understand the rigorous mathematical structure of Gromov-Witten invariants has led to introduction of new theories such as Donaldson-Thomas [Tho00] and Pandharipande-Thomas theories [RR09]. The numerical invariants computed in each theory are conjecturally related to each other but the complete understanding of the connection between these invariants and invariants in Gromov-Witten theory has not yet been achieved. During several past years there has been a growth of interest in computing invariants associated to higher rank analogue of these theories. Toda [Tod10] and Nagao [Nag10] have succeeded in computing a class of higher rank Donaldson-Thomas type invariants using the wall-crossing techniques developed by Kontsevich-Soibelman [MK08] and Joyce-Song [JS09]. In the present article we construct a machinery to compute the higher rank Donaldson-Thomas type invariants using intersection theory. The outcome is a deformation theoretic higher rank enumerative theory for Calabi-Yau threefolds. One of our main results is the construction of a zero-dimensional virtual fundamental class for objects given as higher rank analogue of stable pairs in [RR09]. We carry out calculations over toric Calabi-Yau threefolds such as local $\mathbb{P}^1$ to compute invariants associated to these objects using the method of virtual localization [GP99]. In what follows we explain some of the required background in more detail:

In [PT09] and [RR09] the authors introduce stable pairs given by a tuple $(F,s)$ where $s \in H^0(X,F)$ and $F$ is a pure sheaf with fixed Hilbert polynomial and fixed second Chern character which has one dimensional support. It is shown that there exists a virtual fundamental class of degree zero over the moduli space of stable pairs and the invariants are defined by integration against this class. In the present article we develop a higher rank analogue of the theory of stable pairs; Let $X$ be a nonsingular Calabi-Yau 3-fold over $\mathbb{C}$ with $H^1(O_X) = 0$ and with a fixed polarization $L$. A triple of type $(P_E, P_F)$ over $X$ is given by a tuple $(E,F,\phi)$ where $E$ and $F$ have fixed Hilbert polynomials $P_E$ and $P_F$ respectively, $F$ is a pure sheaf with one dimensional support over $X$ and $\phi : E \to F$ is a holomorphic morphism. We will introduce the notion of frozen triples of type $(P_F, r)$ which means that in a given triple $(E,F,\phi)$, $E \cong O_X^{\oplus r}(-n)$ and $F$ has fixed Hilbert polynomial $P_F$. In other words we “freeze” $E$ to be isomorphic to $O_X^{\oplus r}(-n)$ but the choice of this isomorphism is not fixed. We will also work with closely related objects
called highly frozen triples given as quadruples \((E,F,\phi,\psi)\) where \(E\), \(F\) and \(\phi\) have the same definition as before but this time we have “highly” frozen the triple by fixing a choice of isomorphism \(\psi : E \cong \mathcal{O}_X^{-n} \). The stability condition for frozen and highly frozen triples is compatible with PT stability of stable pairs \[\text{RR09}\] (Lemma 1.3). We call this stability condition \(\tau\)'-limit-stability or in short \(\tau\)'-stability. By definition (Lemma 4.1) a frozen (respectively highly frozen) triple \((E,F,\phi)\) of type \((P_F,r)\) is \(\tau\)'-limit-stable if and only if the map \(E \xrightarrow{\phi} F\) has zero dimensional cokernel. We give a construction of the moduli space of \(\tau\)'-stable frozen and highly frozen triples as stacks. We show that these moduli stacks are given as algebraic stacks. More precisely, the moduli stack of \(\tau\)'-stable highly frozen triples of type \((P_F,r)\) \(\mathcal{M}_{s,HFT}^{(P_F,r,n)}(\tau')\) is given as a Deligne-Mumford (DM) stack while the moduli stack of \(\tau\)'-stable frozen triples of type \((P_F,r)\) \(\mathcal{M}_{s,FT}^{(P_F,r,n)}(\tau')\) is given as an Artin stack. We also show that \(\mathcal{M}_{s,HFT}^{(P_F,r,n)}(\tau')\) is a \(\text{GL}_r(C)\)-torsor over \(\mathcal{M}_{s,FT}^{(P_F,r,n)}(\tau')\) (Proposition 5.5 Corollary 6.4 Theorems 6.2 6.5). The notion of \(\tau\)'-stability condition turns out to be a limiting GIT stability and thus one can apply the results of Wandel \[\text{Wan10}\] (Section 3) to prove that the DM stack \(\mathcal{M}_{s,HFT}^{(P_F,r,n)}(\tau')\) has the stronger property of being given as a quasi-projective scheme.

**Remark 1.1.** One can construct the higher rank theory of stable pairs over a nonsingular projective 3-fold or a noncompact 3-fold such as a toric variety. However in the instance that the base 3-fold is chosen to be non-compact, in order to get well behaved moduli spaces, we require the condition that the one dimensional support of the sheaf \(F\) appearing in a frozen or a highly frozen triple is compact. We will show later an example of this situation when \(X\) is chosen to be a toric variety given by the total space of \(\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \to \mathbb{P}^1\) (local \(\mathbb{P}^1\)).

**Remark 1.2.** As we will show later, at some instances we may need to work over non-compact moduli spaces. However, despite the fact that we prove our results over noncompact stacks, the torus fixed loci of these moduli stacks are compact which enable us to carry out localization computations over them.

For a 3-fold \(X\) the natural deformation obstruction theories of stable frozen and highly frozen triples fail to provide well behaved complexes of correct amplitude over \(\mathcal{M}_{s,FT}^{(P_F,r,n)}(\tau')\) and \(\mathcal{M}_{s,HFT}^{(P_F,r,n)}(\tau')\) and they do not admit virtual cycles. We show that viewing the frozen and highly frozen triples as more complicated objects in \(\mathcal{D}^b(X)\) given by \(P^* : E \to F\) and computing
the fixed-determinant obstruction theory of \( I^\bullet \) will be the starting step in finding a well behaved deformation obstruction theory for the moduli stacks of frozen and highly frozen triples. This method has been successfully used in [RR09] to obtain an alternative candidate for the obstruction theory of the moduli space of stable pairs. It is important to note that in the higher rank case, despite the fact that the object \( I^\bullet \) (with the fixed determinant) does not distinguish between a frozen or a highly frozen triple, its deformation space does. In other words, it can be shown that given a frozen triple \((E, F, \phi)\) and a highly frozen triple \((E, F, \phi, \psi)\), both associated to the same object \( I^\bullet \in D^b(X) \), the space of flat deformations of \((E, F, \phi)\) and \( I^\bullet \) are equally governed by the group \( \text{Ext}^1(I^\bullet, I^\bullet)_0 \) while the space of flat deformations of \((E, F, \phi, \psi)\) is not equal to that of \( I^\bullet \).

We summarize this remark as follows:

**Theorem.** ([Proposition 7.2, Theorem 7.3, Theorem 7.6]).

1. Fix a map \( f : S \to \mathcal{M}^{(P_F, r, n)}_{s, \text{HFT}}(\tau') \). Let \( S' \) be a square-zero extension of \( S \) with ideal \( \mathcal{I} \). Let \( \text{Def}_S(S', \mathcal{M}^{(P_F, r, n)}_{s, \text{HFT}}(\tau')) \) denote the deformation space of the map \( f \) obtained by the set of possible deformations, \( f' : S' \to \mathcal{M}^{(P_F, r, n)}_{s, \text{FT}}(\tau') \). The following statement is true:

\[
\text{Def}_S(S', \mathcal{M}^{(P_F, r, n)}_{s, \text{HFT}}(\tau')) \cong \text{Hom}(I^\bullet_S, F) \otimes \mathcal{I}
\]  

(1.1)

2. Similarly for frozen triples let \( f : S \to \mathcal{M}^{(P_F, r, n)}_{s, \text{FT}}(\tau') \). Let \( S' \) be a square-zero extension of \( S \) with ideal \( \mathcal{I} \). Let \( \text{Def}_S(S', \mathcal{M}^{(P_F, r, n)}_{s, \text{FT}}(\tau')) \) denote the deformation space of the map \( f \) obtained by the set of possible deformations, \( f' : S' \to \mathcal{M}^{(P_F, r, n)}_{s, \text{FT}}(\tau') \). The following statement is true:

\[
\text{Def}_S(S', \mathcal{M}^{(P_F, r, n)}_{s, \text{FT}}(\tau')) \cong \text{Ext}^1(I^\bullet_S, I^\bullet_S)_0 \otimes \mathcal{I}.
\]  

(1.2)

We show that over \( \mathcal{M}^{(P_F, r, n)}_{s, \text{FT}}(\tau') \) deforming objects in the derived category leads to a 4-term deformation-obstruction complex of perfect amplitude \([-2, 1]\):
Theorem. (Theorem 9.5). There exists a map in the derived category given by:

\[ R\pi_{\mathcal{M}}(R\mathcal{H}om(\mathcal{P}^\bullet, \mathcal{P}^\bullet)_0 \otimes \pi_X^*\omega_X)[2] \to \mathbb{L}_{\mathcal{M}_{(P,F)^{r,n}}}(\tau'). \]

After suitable truncations, there exists a 4 term complex \( E^\bullet \) of locally free sheaves, such that \( E^\bullet \) is self-symmetric of amplitude \([-2, 1]\) and there exists a map in the derived category:

\[ \mathbb{L}_{\mathcal{M}_{(P,F)^{r,n}}}(\tau') \to \mathbb{L}_{\mathcal{M}_{(P,F)^{r,n}}}(\tau') \]

such that \( h^{-1}(ob^t) \) is surjective, and \( h^0(ob^t) \) and \( h^1(ob^t) \) are isomorphisms. Here \( \mathbb{L}_{\mathcal{M}_{(P,F)^{r,n}}}(\tau') \) stands for the truncated cotangent complex of the Artin stack \( \mathcal{M}_{(P,F)^{r,n}}(\tau') \) which is of amplitude \([-1, 1]\).

The computation of invariants over \( \mathcal{M}_{(P,F)^{r,n}}(\tau') \) requires constructing a well-behaved virtual fundamental class. When \( X \) is given as a local toric variety (such as local \( \mathbb{P}^1 \)) we use torus-equivariant cohomology and Graber Pandharipande virtual localization technique [GP99] to compute the invariants. At the moment it is not clear to us how to classify the torus-fixed loci of \( \mathcal{M}_{(P,F)^{r,n}}(\tau') \). However this obstacle does not exist for the case of highly frozen triples; The existence of an additional "non-geometric" torus action (Section 12) over \( \mathcal{M}_{(P,F)^{r,n}}(\tau') \) makes it possible to classify the torus-fixed loci of \( \mathcal{M}_{(P,F)^{r,n}}(\tau') \) as a finite union of nonsingular compact components.

Hence instead of developing a virtual class over \( \mathcal{M}_{(P,F)^{r,n}}(\tau') \) we pull back the 4-term deformation obstruction complex in Theorem 9.5 via the forgetful map \( \pi \) in Proposition 5.5 (Diagram (5.1)) and try to construct a virtual fundamental class for \( \mathcal{M}_{(P,F)^{r,n}}(\tau') \). We summarize by saying that the construction of an enumerative theory over \( \mathcal{M}_{(P,F)^{r,n}}(\tau') \) has two advantages:

1. The construction of virtual fundamental classes and integration over DM stacks is in much more developed stage than over Artin stacks

2. There exists a perfect classification of the torus fixed loci of highly frozen triples which makes it possible to do the computations using the virtual localization technique [GP99].

Let \( \pi : \mathcal{M}_{(P,F)^{r,n}}(\tau') \to \mathcal{M}_{(P,F)^{r,n}}(\tau') \) denote the natural forgetful map in Proposition 5.5 (Diagram (5.1)). The complex \( \pi^*\mathbb{L}_{\mathcal{M}_{(P,F)^{r,n}}}(\tau') \) is perfect of amplitude \([-2, 1]\) and the main obstacle in constructing a well-behaved deformation
obstruction theory over the DM stack $\mathcal{H}^{(P_{F,r,n})}_{s,\text{HFT}}(\tau')$ is to truncate $\pi^*E^\bullet\lor$ in to a 2-term complex and define (globally) a well-behaved deformation-obstruction theory of perfect amplitude $[-1,0]$. The simplest solution to this problem is to apply a cohomological truncation operation. Doing so requires obtaining a certain lifting map from $g : \Omega_{\pi} \to \pi^*E^\bullet\lor$ (Proposition 9.12 and Lemma 9.14), taking the mapping cone of this lift (and shifting by $-1$) and proving that the resulting complex satisfies the conditions of being a perfect obstruction theory for $\mathcal{M}^{(P_{F,r,n})}_{s,\text{HFT}}(\tau')$ (Lemma 9.15). Here $\Omega_{\pi}$ is the relative cotangent sheaf of $\pi : \mathcal{M}^{(P_{F,r,n})}_{s,\text{HFT}}(\tau') \to \mathcal{M}^{(P_{F,r,n})}_{s,\text{FT}}(\tau')$. This procedure will remove the degree 1 term from the complex $\pi^*E^\bullet\lor$. We also require to remove the degree $-2$ term of $\pi^*E^\bullet\lor$. We use the self symmetry of $E^\bullet\lor$ and apply the same procedure to the dual map $g^\lor : \pi^*E^\bullet \to T_{\pi}$ (Lemma 9.16 and Lemma 9.17) obtained from dualizing the map $g$. We finally obtain a local truncation of $\pi^*E^\bullet\lor$ of perfect amplitude $[-1,0]$ which we denote by $G^\bullet$. Assuming that $\pi^*E^\bullet\lor$ is given by a 4 term complex of vector bundles:

$$
\pi^*E^{-2} \to \pi^*E^{-1} \to \pi^*E^0 \to \pi^*E^1
$$

(1.4)

it can be seen from our construction (Lemma 8.18) that locally the complex $G^\bullet$ is given by

$$
\pi^*E^{-2} \xrightarrow{d'} \pi^*E^{-1} \oplus T_{\pi} \to \pi^*E^0 \oplus \Omega_{\pi} \xrightarrow{d} \pi^*E^1
$$

(1.5)

which is quasi-isomorphic to a 2-term complex of vector bundles

$$
\text{Coker}(d') \to \text{Ker}(d)
$$

concentrated in degree $-1$ and 0. The existence of the lifting map $g$ is guaranteed Zariski locally over $\mathcal{M}^{(P_{F,r,n})}_{s,\text{HFT}}(\tau')$ but not globally. Hence our strategy is to locally truncate $\pi^*E^\bullet\lor$ as explained above, construct the corresponding local virtual cycles and glue the local cycles to define a globally-defined virtual fundamental class. Our main summarizing theorem of this part is as follows:

**Theorem.** (Theorem 9.11). Consider the 4-term deformation obstruction theory $E^\bullet\lor$ of perfect amplitude $[-2,1]$ over $\mathcal{M}^{(P_{F,r,n})}_{s,\text{FT}}(\tau')$.

1. Locally in the Zariski topology over $\mathcal{M}^{(P_{F,r,n})}_{s,\text{HFT}}(\tau')$ there exists a perfect two-term deformation obstruction theory of perfect amplitude $[-1,0]$ which is obtained from the suitable local truncation of the pullback $\pi^*E^\bullet\lor$.
2. This local theory defines a globally well-behaved virtual fundamental class over $\mathcal{M}^{(P_{F,r,n})}_{s,\text{HFT}}(\tau')$. 

7
Proving the second part of Theorem 9.11 requires a technical assumption which we explain in Subsection 10.1. As discussed earlier if $X$ has a torus action, then the moduli space of highly frozen triples inherits it (once we have chosen an equivariant structure of $\mathcal{O}_X(-n)$) (Section 11.1). Let $G$ denote the torus action induced on $\mathcal{M}_{s,\text{HFT}}^{(P_F,r,n)}(\tau')$. It can be shown that a torus fixed point in the moduli stack corresponds to a $G$-equivariant highly frozen triple of type $(P_F,r)$ (Proposition 12.2). The key observation is that a $G$-equivariant highly frozen triple of rank $r$ is always written as a direct sum of $r$-copies of $(\mathbb{C}^*)^3$-equivariant PT stable pairs (Remark 13.3 and Remark 13.4):

$$I^{\cdot G} \cong \bigoplus_{i=1}^{r} (\mathcal{O}_X(-n) \rightarrow F_i)^{(\mathbb{C}^*)^3}. \quad (1.6)$$

**Remark 1.3.** The consequence of identity $(1.6)$ is of significant importance since it enables one to immediately realize that the $G$-fixed loci of $\mathcal{M}_{s,\text{HFT}}^{(P_F,r,n)}(\tau')$ are given as $r$-fold product of $(\mathbb{C}^*)^3$-fixed loci of PT moduli space of stable pairs which are conjectured by Pandharipande and Thomas in [RR09] (Conjecture 2) to be nonsingular and compact. Hence, though our original moduli stack is constructed as a non-compact space, its torus-fixed locus is given as a finite union of compact and non-singular components.

Let $\iota_Q : Q \rightarrow \mathcal{M}_{s,\text{HFT}}^{(P_F,r,n)}(\tau')$ denote a non-singular compact component of the torus fixed locus of $\mathcal{M}_{s,\text{HFT}}^{(P_F,r,n)}(\tau')$. Let $G_{0,Q} : G_0 \rightarrow G_1,Q$ be the dual of the restriction of the deformation obstruction complex in $(1.5)$ to $Q$. Using the method mentioned earlier, we construct the virtual fundamental class over all such $Q$ and obtain the virtual localization formula (Equation 11.5):

$$\left[\mathcal{M}_{s,\text{HFT}}^{(r,n)}(\tau')\right]^{\text{vir}} = \sum_{Q \subset \mathcal{M}_{s,\text{HFT}}^{(r,n)}(\tau')} \iota_Q \left( \frac{e(G_1,Q)}{e(G_0,Q)} \cdot e(T_Q) \cap [Q] \right). \quad (1.7)$$

The invariants associated to the highly frozen triples can be obtained by

$$\text{HFT}(r, n, \beta) = \int \left[\mathcal{M}_{s,\text{HFT}}^{(r,n)}(\tau')\right]^{\text{vir}}.1.$$

We compute the partition function associated to the invariants of highly frozen triples in complete generality and show that this partition function is given as $r$-fold product of PT partition function:

$$Z_{\text{HFT}} = \left( \sum_{m} P_{m,\beta} q^{m \cdot m} \right)^r. \quad (1.8)$$
Here $n$ is our fixed large enough choice of twisting. Moreover using the computational techniques in equivariant obstruction theory, we compute the 1-legged equivariant Calabi-Yau vertex for $X$ given as the total space of $O_{\mathbb{P}^1}(-1) \oplus O_{\mathbb{P}^1}(-1) \to \mathbb{P}^1$ (Example [16.2]):

$$W_{1,0,0}^{\text{HFT}}|_{r>2} = \left(1 + q \frac{(n+1)(s+1)}{s_1} \right)^r$$  (1.9)

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3 Definition of triples

**Definition 3.1.** Let $X$ be a nonsingular projective Calabi-Yau 3-fold over $\mathbb{C}$ (i.e $K_X \cong O_X$ and $\pi_1(X) = 0$ which implies $H^1(O_X) = 0$) with a fixed polarization $L$. A holomorphic triple supported over $X$ is given by $(E, F, \phi)$ consisting of a torsion free coherent sheaf $E$ and a pure sheaf with one dimensional support $F$, together with a holomorphic morphism $\phi : E \to F$. A homomorphism of triples from $(E', F', \phi')$ to $(E, F, \phi)$ is a commutative diagram:
Now let $S$ be a $\mathbb{C}$ scheme of finite type and let $\pi_X : X \times S \to X$ and $\pi_S : X \times S \to S$ be the corresponding projections. An $S$-flat family of triples over $X$ is a triple $(\mathcal{E}, \mathcal{F}, \phi)$ consisting of a morphism of $O_{X \times S}$ modules $\mathcal{E} \xrightarrow{\phi} \mathcal{F}$ such that $\mathcal{E}$ and $\mathcal{F}$ are flat over $S$ and for every point $s \in S$ the fiber $(\mathcal{E}, \mathcal{F}, \phi) \mid_s$ is given by a holomorphic triple over $X$. Two $S$-flat families of triples $(\mathcal{E}, \mathcal{F}, \phi)$ and $(\mathcal{E}', \mathcal{F}', \phi')$ are isomorphic if there exists a commutative diagram of the form:

\[
\begin{array}{ccc}
E' & \xrightarrow{\phi'} & F' \\
\downarrow & & \downarrow \\
E & \xrightarrow{\phi} & F
\end{array}
\]

**Remark 3.2.** A triple $(E, F, \phi)$ of type $(P_E, P_F, \beta)$ is given by a triple such that $P(E(m)) = P_E$ and $P(F(m)) = P_F$ and $\beta = \chi_2(F)$. During the discussion, for simplicity, we omit $\beta$ and write a triple of type $(P_E, P_F)$. Since by assumption the sheaf $F$ has one dimensional support, the Hilbert polynomial of $F$ in variable $m$ satisfies:

\[ P_F := P(F(m)) = \chi(F(m)) = m \int_\beta c_1(L) + d. \quad (3.1) \]

Here $c_1(L)$ is the first Chern class of the fixed polarization $L$ over $X$ and $d \in \mathbb{Z}$ and $\beta$ as before is $\chi_2(F)$. Note that $P_F$ is a polynomial of degree $= \dim(\text{Supp}(F)) = 1$ and by rank of $F$ (denoted by $rk(F)$) we mean the leading coefficient of $P_F$.

**Definition 3.3.** Define a **frozen triple** of rank $r$ as a holomorphic triple in Definition 3.1 such that $E \cong O_X(-n)^{\oplus r}$ for some fixed large enough $n \in \mathbb{Z}$. We call the frozen triples of rank $r$ “of type $(P_F, r)$” when the sheaf $F$ has fixed Hilbert polynomial $P_F$. Moreover, an $S$-flat family of frozen-triples is a triple $(\mathcal{E}, \mathcal{F}, \phi)$ consisting of a morphism of $O_{X \times S}$ modules $\phi : \mathcal{E} \to \mathcal{F}$ such that $\mathcal{E}$ and $\mathcal{F}$ satisfy the condition of Definition 3.1 and moreover $\mathcal{E} \cong \pi_X^*O_X(-n) \otimes \pi_S^*\mathcal{M}_S$ where $\mathcal{M}_S$ is a vector bundle of rank $r$ on $S$. Two $S$-flat families of frozen-triples $(\mathcal{E}, \mathcal{F}, \phi)$ and $(\mathcal{E}', \mathcal{F}', \phi')$ are isomorphic if there exists a commutative diagram:
Now we define highly frozen triples.

**Definition 3.4.** A **highly frozen triple** is a quadruple \((E, F, \phi, \psi)\) where \((E, F, \phi)\) is a frozen triple as in Definition 3.3 and \(\psi : E \xrightarrow{\cong} O_X(-n)^{\oplus r}\) is a fixed choice of isomorphism. A morphism between highly frozen triples \((E', F', \phi', \psi')\) and \((E, F, \phi, \psi)\) is a morphism \(F' \xrightarrow{\rho} F\) such that the following diagram is commutative.

\[
\begin{array}{ccc}
\mathcal{E}' & \xrightarrow{\phi'} & \mathcal{E}_2 \\
\mathcal{E} & \xrightarrow{\phi} & \mathcal{F}
\end{array}
\]

An \(S\)-flat family of highly frozen-triples is a quadruple \((\mathcal{E}, \mathcal{F}, \phi, \psi)\) consisting of a morphism of \(O_{X \times S}\) modules \(\mathcal{E} \xrightarrow{\phi} \mathcal{F}\) such that \(\mathcal{E}\) and \(\mathcal{F}\) satisfy the condition of Definition 3.1 and moreover \(\psi : \mathcal{E} \xrightarrow{\cong} \pi_X^* O_X(-n) \otimes \pi_S^* O_S^{\oplus r}\) is a fixed choice of isomorphism. Two \(S\)-flat families of highly frozen-triples \((\mathcal{E}, \mathcal{F}, \phi, \psi)\) and \((\mathcal{E}', \mathcal{F}', \phi', \psi')\) are isomorphic if there exists a commutative diagram:

\[
\begin{array}{ccc}
\pi_X^* O_X(-n) \otimes \pi_S^* O_S^{\oplus r} & \xrightarrow{\psi'^{-1}} & \mathcal{E}' \\
\pi_X^* O_X(-n) \otimes \pi_S^* O_S^{\oplus r} & \xrightarrow{\psi^{-1}} & \mathcal{E}
\end{array}
\]

\[
\begin{array}{ccc}
E' & \xrightarrow{\phi'} & F' \\
E & \xrightarrow{\phi} & F
\end{array}
\]

4 Stability of frozen triples and its relationship to stability of PT pairs

**Definition 4.1.** Let \(q_1(m)\) and \(q_2(m)\) be positive rational polynomials of degree at most 2. A holomorphic triple \(T = (E, F, \phi)\) of type \((P_E, P_F)\) is called \(\tau\)-semistable (respectively, stable) if for any subsheaves \(E'\) of \(E\) and
of $F$ such that $0 \neq E' \oplus F' \neq E \oplus F$ and $\phi(E') \subset F'$:

\[
q_2(m) \left( P_{E'} - rk(E') \left( \frac{P_E}{rk(E)} - \frac{q_1(m)}{rk(E)} \right) \right) + q_1(m) \left( P_{E'} - rk(F') \left( \frac{P_F}{rk(F)} + \frac{q_2(m)}{rk(F)} \right) \right) \leq 0/\text{resp.} < 0.
\]  

(4.1)

\[\tau\text{-stability versus Le Potier’s stability}\]

Consider the special case in which the triples are given as coherent systems, i.e. when $E \cong \Gamma \otimes O_X$ such that $\Gamma \subset H^0(F)$. For simplicity we denote this coherent system by $(\Gamma, F)$. Recall that by a sub-coherent system we mean a pair $(\hat{\Gamma}, \hat{F}) \subset (\Gamma, F)$ which is given by $\hat{\Gamma} \otimes O_X \rightarrow F'$, i.e. a subsheaf $0 \rightarrow \hat{F} \rightarrow F$ and $\hat{\Gamma} \subset H^0(\hat{F})$ such that $i(\hat{\Gamma}) \subset \Gamma$. We intend to work with one stability parameter. We use the rational function $q(m)$ instead of $q_1(m)$ and $q_2(m)$ by setting $q_2/m/q_1/m := q(m)$. Assume $\Gamma \otimes O_X \rightarrow F$ is $\tau$-semistable, i.e for all $(\hat{\Gamma}, \hat{F}) \subset (\Gamma, F)$:

\[
q(m) \left( \dim(\Gamma') \cdot P_{O_X} - \dim(\Gamma') \cdot \left( \frac{\dim(\Gamma) \cdot P_{O_X}}{\dim(\Gamma)} - \frac{q_1(m)}{\dim(\Gamma)} \right) \right) + \left( P_{E'} - rk(F') \left( \frac{P_F}{rk(F)} + \frac{q_2(m)}{rk(F)} \right) \right) \leq 0,
\]  

(4.2)

Hence one obtains:

\[
q(m) \left( \dim(\Gamma') \cdot P_{O_X} - \dim(\Gamma') \cdot P_{O_X} + \frac{\dim(\Gamma') \cdot q_1(m)}{\dim(\Gamma)} \right) + \left( P_{E'} - rk(F') \left( \frac{P_F}{rk(F)} + \frac{q_2(m)}{rk(F)} \right) \right) \leq 0,
\]  

(4.3)

By carefully rewriting, one obtains:

\[
\frac{P_F}{rk(F)} + \frac{q_2(m)}{\dim(\Gamma)} \cdot \frac{\dim(\Gamma)}{rk(F')} \geq \frac{P_{E'}}{rk(F')} + \frac{q_2(m)}{\dim(\Gamma)} \cdot \frac{\dim(\Gamma)}{rk(F')}
\]  

(4.4)

Which is similar to Le Potier’s criteria for stability of $(\Gamma, F)$ if we require this inequality to hold for every choice of sub-coherent systems $(\hat{\Gamma}, \hat{F}')$. We explain this similarity in the remark below.
4.1 Statement of $\tau$-stability for frozen triples of type $(P_F, r)$

We study the stability of frozen triples of type $(P_F, r)$. Fix a frozen triple $(E, F, \phi)$ of type $(P_F, r)$. The subtriples of this frozen triple are given by triples of the form $(E', F', \psi)$ for which the following diagram commutes:

\[
\begin{array}{ccc}
E' & \xleftarrow{\psi} & F' \\
\downarrow & & \downarrow \\
E & \xrightarrow{\phi} & F \\
\end{array}
\]

The stability assumption means that for $(E, F, \phi)$ the following condition is satisfied:

\[
\forall E' \subset E \text{ and } \forall F' \subset F \text{ such that } 0 \neq E' \oplus F' \neq O_X(-n)^{\oplus r} \oplus F \neq 0 \text{ and } \phi(E') \subset F':
\]

\[
q_2(m) \left( P_{E'} - rk(E') \left( \frac{P_E}{r} - \frac{q_1(m)}{r} \right) \right) + q_1(m) \left( P_{F'} - rk(F') \left( \frac{P_F}{rk(F)} + \frac{q_2(m)}{rk(F)} \right) \right) \leq 0.
\]

(4.6)

Taking the sub-triple to be $O_X(-n)^{\oplus r} \xrightarrow{\psi} F'$ such that $F' \subset F$, then the stability condition is written as:

\[
q_2(m) \left( P_{O_X(-n)^{\oplus r}} - r \left( \frac{P_{O_X(-n)^{\oplus r}}}{r} \right) \right) + q_2(m) \cdot q_1(m) + q_1(m) \left( P_{F'} - rk(F') \left( \frac{P_F}{rk(F)} + \frac{q_2(m)}{rk(F)} \right) \right) < 0.
\]

(4.7)

Dividing by $q_1(m)$ and setting $q(m) = \frac{q_1(m)}{q_1(m)}$ we obtain:

\[
\frac{P_{F'}}{rk(F')} + \frac{q(m)}{rk(F')} \leq \frac{P_F}{rk(F)} + \frac{q(m)}{rk(F)}.
\]

(4.8)

Which is again somewhat similar to Le Potier’s condition for coherent systems [Pot93]. We are now ready to give a complete $\tau$-stability condition for frozen triples of type $(P_F, r)$:
**Definition 4.2.** Let \( q(m) \) be given by a polynomial with rational coefficients such that its leading coefficient is positive. A frozen triple \((E, F, \phi)\) of type \((P_F, r)\) is \( \tau' \)-stable with respect to \( q(m) \) if and only if:

1. for all proper nonzero subsheaves \( G \subset F \) for which \( \phi \) does not factor through \( G \) we have:

\[
\frac{P_G}{rk(G)} < \frac{P_F}{rk(F)} + \frac{q(m)}{rk(F)}.
\]

2. For all subsheaves, \( G \subset F \) which the map \( \phi \) factors through:

\[
q(m) + \left( P_G - rk(G) \left( \frac{P_F}{rk(F)} + \frac{q(m)}{rk(F)} \right) \right) < 0.
\]

(4.9)

It is trivially seen that equations (4.8) and (4.9) are exactly equivalent to each other.

### 4.2 \( q(m) \to \infty \) limit stability for frozen triples of type \( (P_F, r) \)

We show that the \( \tau' \)-stability condition for frozen and highly frozen triples is asymptotically similar to stability of PT pairs [RR09] (Lemma 1.3).

**Definition 4.3.** Fix \( q(m) \) to be given as a polynomial of degree at least 2 with rational coefficients such that its leading coefficient is positive. A frozen (respectively highly frozen) triple of type \((P_F, r)\) is called to be \( q(m) \to \infty \) \( \tau' \)-limit-stable if it is stable in the sense of Definition 4.2 with respect to this fixed choice of \( q(m) \).

**Lemma 4.4.** Let \( q(m) \) be a polynomial as in Definition 4.3. A frozen triple \((E, F, \phi)\) of type \((P_F, r)\) is \( \tau \)-limit-stable if and only if the map \( E \xrightarrow{\phi} F \) has zero dimensional cokernel.

**Proof.** For simplicity, we use \( \mathcal{O}_X^\oplus(-n) \) instead of \( E \). The exact sequence

\[
0 \to K \to \mathcal{O}_X(-n)^{\oplus r} \xrightarrow{\phi} F \to Q \to 0
\]

induces a short exact sequence \( 0 \to \text{Im}(\phi) \to F \to Q \to 0 \). Therefore one obtains the following commutative diagram of the triples:

\[
\begin{array}{ccc}
\mathcal{O}_X(-n)^{\oplus r} & \xrightarrow{\phi} & \text{Im}(\phi) \\
\downarrow & & \downarrow \\
\mathcal{O}_X(-n)^{\oplus r} & \xrightarrow{} & F
\end{array}
\]

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Now we assume that $O_X(-n)^{\oplus r} \to F$ is a $q(m) \to \infty$-limit-stable triple:

$$q(m) + \left( P_{\text{Im}(\phi)} - r k(\text{Im}(\phi)) \cdot \left( \frac{P_F}{r k(F)} + \frac{q(m)}{r k(F)} \right) \right) < 0.$$  \hspace{1cm} (4.10)

In other words by rearrangement:

$$q(m) \left( 1 - \frac{r k(\text{Im}(\phi))}{r k(F)} \right) < r k(\text{Im}(\phi)) \frac{P_F}{r k(F)} - P_{\text{Im}(\phi)}.$$  

Consider the polynomials on both sides of inequality (4.10) with respect to the variable $m$. One sees that the right hand side of (4.10) is a polynomial in $m$ of degree at most 1. However by the choice $q(m)$ as in Definition 4.3 one sees that the left hand side of the inequality is given by a polynomial of degree at least two with positive leading coefficient. Hence the left hand side becomes larger than the right hand side and the only way for the inequality to make sense is to have the left hand side to be equal to zero, i.e $r k(\text{Im}(\phi)) = r k(F)$ and therefore $Q$ must be a zero dimensional sheaf. For the other direction: Assume that $Q$ is not a zero dimensional sheaf and the triple is $\tau'$-limit-stable. Now by similar argument, since degree of $q(m)$ is chosen to be sufficiently large enough, $r k(\text{Im}(\phi)) = r k(F)$ which contradicts the assumption of $Q$ not being zero dimensional sheaf and this finishes the proof.

\begin{remark}
By Lemma 4.4 it is seen that $q(m) \to \infty \tau'$-limit stable pairs are given as the higher rank analog of PT stable pairs [RR09]. We proved in Lemma 4.4 that the notion of $q(m) \to \infty$ coincides with the notion of $q(m)$-stability for a suitable choice of $q(m)$ given in Definition 4.3. The important outcome of this conclusion is that for a suitable choice of $q(m)$ the notion of $q(m) \to \infty \tau'$-limit-stability comes from a GIT notion of stability.
\end{remark}

5 Construction of moduli stacks

Throughout the rest of this article by $\tau'$-stability we mean the $q(m) \to \infty$ $\tau'$-limit-stability.

5.1 Definition of moduli stacks as categories fibered in groupoids

Definition 5.1. Define $\mathcal{M}^{(P_F,r,n)}_{s,\text{HFT}}(\tau')$ to be the fibered category $p : \mathcal{M}^{(P_F,r,n)}_{s,\text{HFT}}(\tau') \to \text{Sch}/\mathbb{C}$ such that for all $S \in \text{Sch}/\mathbb{C}$ the objects in $\mathcal{M}^{(P_F,r,n)}_{s,\text{HFT}}(\tau')$ are $S$-flat
families of $\tau'$-stable highly frozen triples of type $(P_F, r)$ as in Definition 3.3.

Given a morphism of $\mathcal{C}$-schemes $g : S \to K$ and two families of highly frozen triples $T_S := (E, E', \phi, \psi)_S$ and $\hat{\mathcal{T}}_K := (E', E'', \phi', \psi')_K$ as in Definition 3.3 (sub-index indicates the base parameter scheme over which the family is constructed), a morphism $T_S \to \hat{\mathcal{T}}_K$ in $\mathcal{M}^{(P_F, r, n)}_{s, \text{HFT}}(\tau')$ is defined by an isomorphism:

$$\nu_S : T_S \xrightarrow{\cong} (g \times 1_X)^* \hat{\mathcal{T}}_K.$$ 

**Definition 5.2.** Define $\mathcal{M}^{(P_F, r, n)}_{s, \text{FT}}(\tau')$ to be the fibered category $p : \mathcal{M}^{(P_F, r, n)}_{s, \text{FT}}(\tau') \to \text{Sch}/\mathcal{C}$ such that for all $S \in \text{Sch}/\mathcal{C}$ the objects in $\mathcal{M}^{(P_F, r, n)}_{s, \text{FT}}(\tau')$ are $S$-flat families of frozen triples of type $(P_F, r)$ as in Definition 3.3. Given a morphism of $\mathcal{C}$-schemes $g : S \to K$ and two families of frozen triples $T_S := (E, E', \phi)_S$ and $\hat{\mathcal{T}}_K := (E', E'', \phi')_K$ as in Definition 3.3 (sub-index indicates the base parameter scheme over which the family is constructed), a morphism $T_S \to \hat{\mathcal{T}}_K$ in $\mathcal{M}^{(P_F, r, n)}_{s, \text{FT}}(\tau')$ is defined by an isomorphism:

$$\nu_S : T_S \xrightarrow{\cong} (g \times 1_X)^* \hat{\mathcal{T}}_K.$$ 

**Proposition 5.3.** Use definitions 5.1 and 5.2. The fibered categories $\mathcal{M}^{(P_F, r, n)}_{s, \text{HFT}}(\tau')$ and $\mathcal{M}^{(P_F, r, n)}_{s, \text{FT}}(\tau')$ are stacks.

**Proof.** This is immediate from faithfully flat descent of coherent sheaves and homomorphisms of coherent sheaves [vis04] (Theorem 4.23). \[\square\]

**Remark 5.4.** There exists a forgetful morphism $g' : \mathcal{M}^{(P_F, r, n)}_{s, \text{FT}}(\tau') \to \mathcal{B} \text{GL}_r(\mathbb{C})$ which is given by taking a frozen triple $\{(E, F, \phi)\} \in \mathcal{M}^{(P_F, r, n)}_{s, \text{FT}}(\tau')$ to $\{E\} \in \mathcal{B} \text{GL}_r(\mathbb{C})$ by forgetting $F$ and $\phi$.

**Proposition 5.5.** The natural diagram:

$$\begin{array}{ccc}
\mathcal{M}^{(P_F, r, n)}_{s, \text{HFT}}(\tau') & \xrightarrow{g} & \text{pt} = \text{Spec}(\mathbb{C}) \\
\pi \downarrow & & \downarrow i \\
\mathcal{M}^{(P_F, r, n)}_{s, \text{FT}}(\tau') & \xrightarrow{\hat{g}} & \mathcal{B} \text{GL}_r(\mathbb{C}) = \left[\text{Spec}(\mathbb{C}) \backslash \text{GL}_r(\mathbb{C})\right],
\end{array}
$$

is a fibered diagram in the category of stacks. In particular $\mathcal{M}^{(P_F, r, n)}_{s, \text{HFT}}(\tau')$ is a $\text{GL}_r(\mathbb{C})$-torsor over $\mathcal{M}^{(P_F, r, n)}_{s, \text{FT}}(\tau')$. It is true that locally in the flat topology $\mathcal{M}^{(P_F, r, n)}_{s, \text{FT}}(\tau') \cong \mathcal{M}^{(P_F, r, n)}_{s, \text{HFT}}(\tau') \times \left[\text{Spec}(\mathbb{C}) \backslash \text{GL}_r(\mathbb{C})\right]$. This isomorphism does not hold true globally unless $r = 1$. 

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Proof. We show that there exists a forgetful map \( \pi : \mathcal{M}_{s,\text{HFT}}^{(P_{F,r,n})}(\tau') \to \mathcal{M}_{s,\text{FT}}^{(P_{F,r,n})}(\tau') \) which induces a map from \( \mathcal{M}_{s,\text{HFT}}^{(P_{F,r,n})}(\tau') \times [\text{Spec}(\mathbb{C})/\text{GL}_r(\mathbb{C})] \to \mathcal{M}_{s,\text{FT}}^{(P_{F,r,n})}(\tau') \) and show that this map has an inverse locally but not globally unless \( r = 1 \). First we prove the claim for \( r = 1 \).

By Definition 5.6.

To proceed further we state the following definition.

To proceed further we state the following definition.

Note that tensoring \( O \) which is obtained by tensoring with \( \text{GL}_r \) which is obtained by tensoring with \( \text{GL}_1(\mathbb{C}) = \mathbb{G}_m \). For a \( \mathbb{C} \)-scheme \( S \), an \( S \)-point of \( \mathcal{M}_{s,\text{HFT}}^{(P_{F,1})}(\tau') \times [\text{Spec}(\mathbb{C})/\mathbb{G}_m] \) is identified with the data \( (\mathcal{O}_{X \times S}(-n) \to F, \mathcal{L}_S) \) where \( \mathcal{L}_S \) is a \( \mathbb{G}_m \) line bundle over \( S \). Let \( \pi_S : X \times S \to S \) be the natural projection onto the second factor.

There exists a map that sends this point to an \( S \)-point \( a \in \left( \mathcal{M}_{s,\text{FT}}^{(P_{F,1})}(\tau') \right)(S) \) which is obtained by tensoring with \( \mathcal{L}_S \), i.e \( \mathcal{O}_X(-n) \boxtimes \mathcal{L}_S \xrightarrow{\phi} F \boxtimes \mathcal{L}_S \).

Note that tensoring \( \mathcal{O}_{X \times S}(-n) \) with \( \pi_S^* \mathcal{L}_S \) does not change the fact that \( \mathcal{O}_{X \times S}(-n) \mid_{s \in S} \cong \mathcal{O}_X(-n) \boxtimes \mathcal{L}_S \mid_{s \in S} \) fiber by fiber.

Moreover, there exists a section map \( s : \mathcal{M}_{s,\text{FT}}^{(P_{F,1})}(\tau') \to \mathcal{M}_{s,\text{HFT}}^{(P_{F,1})}(\tau') \times \left[ \text{Spec}(\mathbb{C})/\mathbb{G}_m \right] \). Simply take an \( S \)-point \( [\mathcal{O}_X(-n) \boxtimes \mathcal{L}_S \to F] \in \left( \mathcal{M}_{s,\text{HFT}}^{(P_{F,1})}(\tau') \right)(S) \) and send to an \( S \)-point in \( \left( \mathcal{M}_{s,\text{HFT}}^{(P_{F,1})}(\tau') \times \left[ \text{Spec}(\mathbb{C})/\mathbb{G}_m \right] \right)(S) \) by the map

\[
[\mathcal{O}_X(-n) \boxtimes \mathcal{L}_S \to F] \mapsto ([\mathcal{O}_X(-n) \to F \otimes \pi_S^* \mathcal{L}_S^{-1}], \mathcal{L}_S).
\]

Note that since \( \mathcal{L}_S \) is a line bundle over \( S \) then it is invertible and hence a section map is always well defined and \( \mathcal{M}_{s,\text{HFT}}^{(P_{F,1})}(\tau') \) is a gerbe over \( \mathcal{M}_{s,\text{FT}}^{(P_{F,1})}(\tau') \).

To proceed further we state the following definition.

Definition 5.6. Consider a stack \( (\mathcal{Y}, p_0) : \mathcal{Y} \to \text{Sch}/\mathbb{C} \). Given Two morphism of stacks \( \pi_1 : \mathcal{X} \to \mathcal{Y} \) and \( \pi_2 : \mathcal{X}' \to \mathcal{Y} \), the fibered product of \( \mathcal{X} \) and \( \mathcal{X}' \) over \( \mathcal{Y} \) is defined by the category whose objects are defined by triples \((x, x', \alpha)\) where \( x \in \mathcal{X} \) and \( x' \in \mathcal{X}' \) respectively and \( \alpha : \pi_1(x) \to \pi_2(x') \) is an arrow in \( \mathcal{Y} \) such that \( p_0(\alpha) = \text{id} \). Moreover the morphisms \((x, x', \alpha) \to (y, y', \beta)\) are defined by the tuple \((\phi : x \to y, \psi : x' \to y')\) such that

\[
\pi_1(\psi) \circ \alpha = \beta \circ \pi_2(\phi) : P_E(x) \to P_E(y').
\]

Now let \( r > 1 \). There exists a forgetful map \( \pi : \mathcal{M}_{s,\text{HFT}}^{(P_{F,r,n})}(\tau') \to \mathcal{M}_{s,\text{FT}}^{(P_{F,r,n})}(\tau') \) which takes \((E, F, \phi, \psi)\) to \((E, F, \phi)\) by forgetting the choice of isomorphism, \( \psi \). Moreover, there exists a map \( g' : \mathcal{M}_{s,\text{FT}}^{(P_{F,r,n})}(\tau') \to \mathcal{B}\text{GL}_r(\mathbb{C}) \) by Remark 5.4. Finally there exists the natural projection \( i : \text{Spec}(\mathbb{C}) \to \left[ \text{Spec}(\mathbb{C})/\text{GL}_r(\mathbb{C}) \right] = \mathcal{B}\text{GL}_r(\mathbb{C}) \). It follows directly from Definition 5.6 that the dia-
is a fibered diagram and $\mathcal{M}_{s,HFT}^{(P_{\tau,n})}(\tau') = \mathcal{M}_{s,FT}^{(P_{\tau,n})}(\tau') \times B_{GL_r}(C) = \left[ Spec(C) \right]_{GL_r(C)}$. Here one cannot use the same argument used for frozen triples of rank 1 to conclude that there exists a section map $s : \mathcal{M}_{s,FT}^{(P_{\tau,n})}(\tau') \rightarrow \mathcal{M}_{s,HFT}^{(P_{\tau,n})}(\tau') \times \left[ Spec(C) \right]_{GL_r(C)}$, since as we showed, the $S$-point of $B_{GL_r}(C)$ is a $GL_r(C)$ bundle of rank $r$ over $S$ and this vector bundle is trivializable locally but not globally. Therefore locally in the flat topology one may think of $\mathcal{M}_{s,FT}^{(P_{\tau,n})}(\tau')$ as isomorphic to $\mathcal{M}_{s,HFT}^{(P_{\tau,n})}(\tau') \times \left[ Spec(C) \right]_{GL_r(C)}$ but not globally.

Next we show that the moduli stacks of frozen and highly frozen triples are given as algebraic stacks.

### 5.2 Boundedness

As we will show in this section, the moduli stacks of frozen and highly frozen triples are given as the stacky quotients of a base parametrizing scheme by group actions. In order to construct this parameter scheme we need a certain boundedness property for the family of triples of fixed given type. In [Wan10](Definition 1.1) Wandel studies the construction of the moduli space of objects $\phi : D \rightarrow E$ denoted as pairs. These objects are defined similar to triples in Definition 3.1. The author introduces the notion of Hilbert polynomial and reduced Hilbert polynomial for a pair [Wan10](Definition 1.3). Moreover, the author defines a semistability condition denoted as $\delta$-semistability [Wan10](Definition 1.4) where $\delta$ is given as a stability parameter. Replacing $\delta$ with $q(m)$, it is easily seen that Wandel’s notion of $\delta$-semistability is completely compatible with our notion of $\tau'$-semistability in Definition 4.2. Here we state a proposition in [Wan10](Proposition 2.1) without proof which ensures one to obtain the required boundedness condition for the family of triples. The following statement can be adapted to our case once one replaces the notion of pairs and $\delta$-semistability in [Wan10] with our notion of triples and $\tau'$-semistability respectively.

**Proposition 5.7.** [Wan10](Proposition 2.1) Given a pair $\phi : D \rightarrow E$, Let $P$ and $\delta$ be polynomials. Then there is a constant $C$ depending only on $P$ and $D$ such that for every $O_X$-module $E$ occurring in a $\delta$-semistable pair we
have $\mu_{\text{max}}(\mathcal{E}) \leq C$. In particular, the family of pairs which are semistable with respect to any stability parameter $\delta$ having the fixed Hilbert polynomial $P$ is bounded.

Following this proposition it is shown in [Wan10] (Proposition 2.4) that a family of $\delta$-semistable pairs with given fixed numerical data (such as fixed Hilbert polynomials) satisfy a regularity condition. Hence, it is shown that given a bounded family of $\delta$-semistable pairs, the sheaves $\mathcal{D}$ and $\mathcal{E}$ appearing in the family satisfy the condition that for some large enough integer $n'$ the sheaves $\mathcal{D}(n')$ and $\mathcal{E}(n')$ are globally generated [Wan10] (look following Definition 3.2).

6 Moduli stacks as algebraic stacks

6.1 The Parameter Scheme of $\tau'$-stable highly frozen triples of type $(P_F, r)$

Replacing the pairs and $\delta$-semistability in [Wan10] (Section 2) with triples and $\tau'$-semistability and adapting the results of propositions 2.1 and 2.4 in [Wan10] to our case one finds that there exists an integer $n'$ such that for all coherent sheaves $E$ and $F$ appearing in a family of $\tau$-(semi)stable triples $(E, F, \phi)$, $E(n')$ and (in particular) $F(n')$ are globally generated. Now use notation of Definition 3.1. One first constructs an $S$-flat family of coherent sheaves $F$ with fixed Hilbert polynomial $P_F$. By construction the family of coherent sheaves $F$ appearing in a $\tau'$-stable triple is bounded and moreover the large enough twist $F(n')$ is globally generated. Fix such $n'$ and let $V_F$ be a complex vector space of dimension $d_F = P_F(n')$ given as $V_F = H^0(F \otimes L^{n'})$. Let $Q_F$ denote Quot$_{P_F}(V_F \otimes \mathcal{O}_X(-n'))$. Now we fix a large enough integer $n$ (not necessarily equal to $n'$). We construct a scheme which parameterizes morphisms $\mathcal{O}_X^{\oplus r}(-n) \to F$. There exists a bundle $\mathcal{P}$ over $Q_F$ whose fibers parametrize $H^0(F(n))$. It is trivially seen that the fibers of the bundle $\mathcal{P}^{\oplus r}$ parametrize $H^0(F(n))^{\oplus r}$. In other words the fibers of $\mathcal{P}^{\oplus r}$ parametrize the maps $E \to F$ such that $E = \mathcal{O}_X^{\oplus r}(-n)$. Now let

$$\mathcal{E}^{(P_F, r, n)}_{\tau'}(\tau') \subset \mathcal{P}^{\oplus r} \quad (6.1)$$
be given as an open subscheme of $\mathcal{P}^{\oplus r}$ whose fibers parametrize $\tau'$-stable highly frozen triples $E \to F$.

\[
\begin{align*}
V_F \otimes \mathcal{O}_X(-n') \\
\mathcal{O}_X(-n)^{\oplus r} & \overset{\phi}{\longrightarrow} F
\end{align*}
\]

**Definition 6.1.** Given a quasi-projective $\mathbb{C}$-scheme $A$ and a complex group $G$, the quotient stack $[A/G]$ is given as a fibered category over $\text{Sch}/\mathbb{C}$ such that for any $\mathbb{C}$-scheme $S$, $[A/G](S)$ consists of pairs $(P, \pi)_S$, where $P$ is a principal $G$-bundle over $S$ and $\pi : P \to A$ is a $G$-equivariant morphism.

**Theorem 6.2.** Let $\mathcal{S}_s(P_F,r,n)(\tau')$ be the stable locus of the parametrizing scheme of highly frozen triples of type $(P_F,r)$ as in (6.1). Let $[\mathcal{S}_s(P_F,r,n)(\tau')_{\text{GL}(V_F)}]$ be the stack-theoretic quotient of $\mathcal{S}_s(P_F,r,n)(\tau')$ by $\text{GL}(V_F)$ where $V_F$ is defined as in Section 6.1. Then there exists an isomorphism of groupoids

\[
\mathcal{M}^{(P_F, r, n)}_{s, \text{HFT}}(\tau') \cong \left[ \mathcal{S}_s^{(P_F, r, n)}(\tau')_{\text{GL}(V_F)} \right].
\]

**Proof.** Consider the scheme $\mathcal{S}_s^{(P_F, r, n)}(\tau')$. First, one shows that there exists a functor $q : \left[ \mathcal{S}_s^{(P_F, r, n)}(\tau')_{\text{GL}(V_F)} \right] \to \mathcal{M}^{(P_F, r, n)}_{s, \text{HFT}}(\tau')$. Then one shows that there exists a functor in the opposite direction and finally one proves that the composition of the two functors is a natural isomorphism of categories fibered in groupoids. Look at [Gom99] for more general treatment. Diaconescu [Dia08] uses a similar proof to construct the moduli stack of ADHM sheaves supported over a curve as a quotient stack. Fix a parametrizing scheme $S$ over $\mathbb{C}$. The fiber of the quotient stack $\left[ \mathcal{S}_s^{(P_F, r, n)}(\tau')_{\text{GL}(V_F)} \right]_S$ over $S$ consists of pairs $(P, \pi_S)_S$ as in Remark 6.1 (where $A$ in Remark 6.1 is replaced by $\mathcal{S}_s^{(P_F, r, n)}(\tau')$). Let $T := E_1 \overset{\phi}{\longrightarrow} E_2$ be the universal $\tau'$-stable frozen triple of type $(P_F, r)$ over $X \times \mathcal{S}_s^{(P_F, r, n)}(\tau')$. Given:

\[
\begin{array}{ccc}
\mathcal{P} & \overset{\pi_S}{\longrightarrow} & \mathcal{S}_s^{(P_F, r, n)}(\tau') \\
\downarrow p & & \\
S
\end{array}
\]

(6.3)
one obtains a diagram:

\[
\begin{array}{ccc}
P \times X & \xrightarrow{(\pi_S)_X} & \mathcal{G}_s^{(P_F,r,n)}(\tau') \times X \\
(p \times 1_X) \downarrow & & \downarrow \\
S \times X & & \\
\end{array}
\]

Let \( K \) be a \( \mathbb{C} \)-scheme and let \( g : S \to K \) be a morphism of \( \mathbb{C} \)-schemes. A morphism in \( \mathcal{S}(P_F,r,n) \) between two objects \((P',\pi_S)_S\) and \((P,\pi'_S)_K\) is given by a commutative diagram:

\[
\begin{array}{ccc}
P & \xrightarrow{\nu} & \mathcal{G}_s^{(P_F,r,n)}(\tau') \\
p \downarrow & & \downarrow \\
S & \xrightarrow{\pi_S} & S \\
\end{array}
\]

such that \( \nu \) is an isomorphism of principal \( \text{GL}(V_F) \)-bundles over \( S \). Note that by construction in Section 6.1 the objects parametrized by \( \mathcal{G}_s^{(P_F,r,n)}(\tau') \) are given by a morphism \( \mathcal{O}_X(-n') \to F \) such that \( F \) (itself) is given as a flat quotient \( V_F \otimes \mathcal{O}_X(-n') \to F \). Now define a morphism

\[ q' : \mathcal{G}_s^{(P_F,r,n)}(\tau') \to \mathcal{M}_{s,\text{HFT}}^{(P_F,r,n)}(\tau') \]

by forgetting the surjection \( V_F \otimes \mathcal{O}_X(-n') \to F \). Note that by construction and since the map \( \pi_S \) in diagram (6.3) is \( \text{GL}(V_F) \)-equivariant then one obtains a map from \( S \) to \( \mathcal{M}_{s,\text{HFT}}^{(P_F,r,n)}(\tau') \) i.e one obtains an induced diagram:

\[
\begin{array}{ccc}
P & \xrightarrow{\pi_S} & \mathcal{G}_s^{(P_F,r,n)}(\tau') \\
p \downarrow & & \downarrow q' \\
S & \xrightarrow{\pi_S} & \mathcal{M}_{s,\text{HFT}}^{(P_F,r,n)}(\tau'). \\
\end{array}
\]

Since \( g'^* \pi'_S \circ \nu = \pi_S \), it is guaranteed that \((\pi_S \times 1_X)^* T \cong (g \times 1_X)^* ((\pi'_S) \times 1_X)^* T \) and this isomorphism descends to \( T_S \xrightarrow{\cong} (g \times 1_X)^* T_K \) where \( T_S \) and
Hence for every closed point $p \in X$. We use this fact to construct a family of stable highly cohomology one obtains an isomorphism:

$$\pi^* \mathcal{B} \otimes_{X \times \mathbb{C}^*} \pi_X^*(\mathcal{O}_X(-n')) \rightarrow \mathcal{F} \rightarrow 0.$$  \hspace{1cm} (6.6)

Now let the principal $GL(V_F)$-bundle $p : \mathcal{P} \rightarrow S$ be defined as: $\mathcal{P} = \text{Isom}(V_F \otimes \mathcal{O}_S \rightarrow \mathcal{B})$. Since $\mathcal{P}$ is given by a frame bundle over $S$ then one has $p^* \mathcal{B} \cong V_F \otimes \mathcal{O}_\mathcal{P}$. Now pull back by $(p \times 1_X) : \mathcal{P} \times X \rightarrow S \times X$ and obtain:

$$(p \times 1_X)^*(\pi^* \mathcal{B} \otimes_{X \times \mathbb{C}^*} \pi_X^*(\mathcal{O}_X(-n'))) \rightarrow (p \times 1_X)^* \mathcal{F} \rightarrow 0.$$  \hspace{1cm} (6.7)

On the other hand:

$$(p \times 1_X)^*(\pi^* \mathcal{B} \otimes_{X \times \mathbb{C}^*} \pi_X^*(\mathcal{O}_X(-n'))) \cong V_F \otimes (p \times 1_X)^*(\pi_X^*(\mathcal{O}_X(-n'))) \hspace{1cm} (6.8)$$

Now denote $\mathcal{F}^\mathcal{P} = (p \times 1_X)^* \mathcal{F}$. So one obtains an isomorphism

$$p^* \mathcal{B} \cong (\pi^*_\mathcal{P})_* \left[ \mathcal{F}^\mathcal{P} \otimes_{\mathcal{P} \times \mathcal{P}} (p \times 1_X)^*(\pi_X^*(\mathcal{O}_X(n'))) \right].$$

Let $\{b\} \in \mathcal{P}$ be a closed point. By evaluation at $\{b\}$ one obtains $p^* \mathcal{B} |_{X \times \{b\}} \cong V_F \otimes \mathcal{O}_{X \times \{b\}}$ and:

$$\left( \pi^*_\mathcal{P} \right)_* \left[ \mathcal{F}^\mathcal{P} \otimes_{\mathcal{P} \times \mathcal{P}} (p \times 1_X)^*(\pi_X^*(\mathcal{O}_X(n'))) \right] |_{X \times \{b\}} \cong H^0 \left[ \mathcal{F}^\mathcal{P} (n') |_{X \times \{b\}} \right].$$  \hspace{1cm} (6.9)

twisting the map $V_F \otimes \mathcal{O}_{X \times \{b\}}(-n') \rightarrow \mathcal{F}^\mathcal{P} |_{X \times \{b\}}$ by $n$ and taking the zero cohomology one obtains an isomorphism:

$$H^0 \left[ V_F \otimes \mathcal{O}_{X \times \{b\}} \right] \cong H^0 \left[ \mathcal{F}^\mathcal{P} (n') |_{X \times \{b\}} \right].$$

Hence for every closed point $\{b\} \in \mathcal{P}$ one gets an isomorphism in the level of zero cohomologies. We use this fact to construct a family of stable highly
frozen triples parametrized by $P$. This family is obtained by applying $(\pi_P)_*$ to the following morphism over $P$:

$$V_F \otimes_X \operatorname{p}(p \times 1_X)^*(\pi_X^* \mathcal{O}_X(-n')) \to \mathcal{F}^P.$$ (6.10)

which is naturally $\operatorname{GL}(V_F)$-equivariant by construction and it gives rise to a classifying $\operatorname{GL}(V_F)$-equivariant morphism $P \to \mathcal{S}^{(P_F, r, n)}(\tau')$. Now consider two objects in $\mathcal{M}^{(P_F, r, n)}_{s, \text{HFT}}(\tau')$ and a morphism between them. This data by Definition 5.2 is a pair $(g, \nu_S)$ such that $g : S \to K$ and $\nu_S : T_S \to (g \times 1_X)^* \hat{T}_K$. However since these two families determine principal $\operatorname{GL}(V_F)$-bundles over $S$ and $K$ respectively, we obtain a morphism of principal $\operatorname{GL}(V_F)$-bundles:

$$P \cong (g \times 1_X)^* \hat{P} \xrightarrow{\nu} \hat{P}$$

$$\begin{array}{c}
S \\
g \\
\downarrow \nu \\
K
\end{array}$$

(6.11)

Let $h : P \cong (g \times 1_X)^* \hat{P}$, it is verified that the family $h^* \nu^* \hat{T}_K$ and $T_S$ are isomorphic. Therefore there exists a functor $j : \mathcal{M}^{(P_F, r, n)}_{s, \text{HFT}}(\tau') \to \left[\mathcal{S}^{(P_F, r, n)}(\tau') / \operatorname{GL}(V_F)\right]$ and also it is verified that $q \circ j$ and $j \circ q$ are natural isomorphisms.

One may use the above results (i.e the natural isomorphism in Theorem 6.2) in order to obtain an alternative definition of the moduli stack of $\tau'$-stable highly frozen triples of type $(P_F, r)$ as the quotient stack $\left[\mathcal{S}^{(P_F, r, n)}(\tau') / \operatorname{GL}(V_F)\right]$.

**Remark 6.3.** By Definition 5.1 and construction of $\mathcal{S}^{(P_F, r, n)}(\tau')$ in Section 6.1, $\operatorname{GL}(\mathbb{C})$ acts compatibly on both sides of the isomorphism 6.2. The next corollary gives the algebraic structure of the moduli stack of frozen triples.

**Corollary 6.4.** Use Proposition 5.5, Theorem 6.2 and Remark 6.3. Let $\left[\mathcal{S}^{(P_F, r, n)}(\tau') / \operatorname{GL}(V_F)\right]$ be the stack-theoretic quotient of $\mathcal{S}^{(P_F, r, n)}(\tau')$ by $\operatorname{GL}(\mathbb{C}) \times \operatorname{GL}(V_F)$ where $V_F$ is defined as in Section 6.1. There exists an isomorphism of groupoids:

$$\mathcal{M}^{(P_F, r, n)}_{s, \text{FT}}(\tau') \cong \left[\mathcal{S}^{(P_F, r, n)}(\tau') / \operatorname{GL}(\mathbb{C}) \times \operatorname{GL}(V_F)\right].$$

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**Theorem 6.5.** Consider \( q(m) \to \infty \) \( \tau' \)-limit stability as in Lemma 4.4. The moduli stack \( \mathcal{M}^{(p_F,r,n)}_{s,HFT} (\tau') \) for such choice of stability parameter \( q(m) \) is a Deligne-Mumford (DM) stack.

**Proof.** It is enough to show that for every \( \mathbb{C} \)-point \( p \in \mathcal{M}^{(p_F,r,n)}_{s,HFT} (\tau')(\text{Spec}(\mathbb{C})) \) its’ stabilizer group \( \text{Stab}^{(p_F,r,n)}_{s,HFT} (\tau') (p) \) is finite. Since the point \( p \) is represented by a \( \tau' \)-stable highly frozen triple \( (E,F,\phi,\psi) \), then \( \text{Stab}^{(p_F,r,n)}_{s,HFT} (\tau') (p) \) is obtained by the automorphism group of \( (E,F,\phi,\psi) \). Hence it is enough to show that the automorphism group of any such \( (E,F,\phi,\psi) \) is a finite group. The following lemma shows that the automorphism group of a \( \tau' \)-limit-stable highly frozen triple has one element which is the identity.

**Lemma 6.6.** Given a \( \tau' \)-limit-stable highly frozen triple \( (E,F,\phi,\psi) \) as in Definition 3.4 and a commutative diagram

\[
\begin{array}{ccc}
\mathcal{O}_X(-n) \oplus r & \xrightarrow{\psi^{-1}} & E \\
\downarrow \text{id} & & \downarrow \phi \\
\mathcal{O}_X(-n) \oplus r & \xrightarrow{\psi^{-1}} & E \\
\end{array}
\]

the map \( \rho \) is given by \( \text{id}_F \).

**Proof.** Since \( \psi \) is a choice of isomorphism, for simplicity replace \( E \) by \( \mathcal{O}_X(-n) \oplus r \) and consider the diagram:

\[
\begin{array}{ccc}
\mathcal{O}_X(-n) \oplus r & \xrightarrow{\phi} & F \\
\downarrow \text{id} & & \downarrow \rho \\
\mathcal{O}_X(-n) \oplus r & \xrightarrow{\phi} & F \\
\end{array}
\]  

(6.12)

The diagram (6.12) induces:

\[
\begin{array}{ccc}
\mathcal{O}_X(-n) \oplus r & \xrightarrow{\phi} & \text{Im}(\phi) & \xrightarrow{\rho} & E \\
\downarrow \text{id} & & \downarrow \rho_{|\text{Im}(\phi)} & & \downarrow \rho \\
\mathcal{O}_X(-n) \oplus r & \xrightarrow{\phi} & \text{Im}(\phi) & \xrightarrow{\rho} & E \\
\end{array}
\]

By commutativity of (6.12) \( \rho \circ \phi = \phi \circ \text{id} = \phi \), then \( \rho(\text{Im}(\phi)) = \text{Im}(\phi) \). Hence \( \rho(\text{Im}(\phi)) \subset \text{Im}(\phi) \). It follows that \( \rho_{|\text{Im}(\phi)} = \text{id}_{\text{Im}(\phi)} \). Indeed if \( s \in \text{Im}(\phi)(\mathcal{U}) \)
where $\mathcal{U} \subset X$ is affine open with $\tilde{s} \in \mathcal{O}_X(-n)^{\oplus r}(\mathcal{U})$ satisfying $\phi(\tilde{s}) = s$, then $\rho(s) = \rho(\phi(\tilde{s})) = \phi(id(\tilde{s})) = \phi(\tilde{s}) = s$. Now apply $\text{Hom}(-, F)$ to the short exact sequence $0 \rightarrow \text{Im}(\phi) \rightarrow F \rightarrow Q \rightarrow 0$ where $Q$ denotes the corresponding cokernel. One obtains:

$$0 \rightarrow \text{Hom}(Q, F) \rightarrow \text{Hom}(F, F) \rightarrow \text{Hom}(\text{Im}(\phi), F).$$

Since $(E, F, \phi, \psi)$ is $q(m) \rightarrow \infty \tau'$-limit-stable then by Lemma 4.4 $Q$ is a sheaf with 0-dimensional support. Hence by purity of $F$, $\text{Hom}(Q, F) \cong 0$. Hence one obtains an injection $\text{Hom}(F, F) \hookrightarrow \text{Hom}(\text{Im}(\phi), F)$. Now

$$\rho \mid_{\text{Im}(\phi)} = \text{id}_{\text{Im}(\phi)} = (\text{id}_F) \mid_{\text{Im}(\phi)}.$$

So $\rho = \text{id}_F$. This finishes the proof of Lemma 6.6 as well as Theorem 6.5.

**Remark 6.7.** As was described in Remark 4.5, the notion of $\tau'$-stability condition turns out to be a limiting GIT stability and thus one can apply the results of Wandel [Wan10] (Section 3) to prove that the DM stack $\mathcal{M}^{(p_F, r, n)}_{s, \text{HFT}}(\tau')$ has the stronger property of being given as a quasi-projective scheme. We will use this fact later in discussing the construction of deformation obstruction theory over $\mathcal{M}^{(p_F, r, n)}_{s, \text{HFT}}(\tau')$.

### 7 Deformations of triples

In this section, we describe the deformation theory of frozen and highly frozen triples.

**7.1 Preliminaries**

As we showed, the construction of the moduli stack of stable frozen triples depends on a choice of two fixed large enough integers $n \gg 0$ and $n' \gg 0$. The first integer appears in the description of a stable highly frozen triple $\mathcal{O}_X(-n)^{\oplus r} \rightarrow F$ and the second integer is the one for which $F(n')$ becomes globally generated and hence there exists a surjective map $V_F \otimes \mathcal{O}_X(-n') \rightarrow F$. The existence of the integer $n'$ is according to Wandel [Wan10] (Proposition 2.4) where he shows that given a bounded family of stable triples $E \rightarrow F$ there exists such integer such that for every tuple $(E, F)$ appearing in the family $E(n')$ and $F(n')$ are globally generated over $X$. The fact that the sheaf $F(n')$ is globally generated for large enough values of $n'$ does not a priori imply that $H^i(F(n)) = 0$ for all $i > 0$ and our fixed choice of $n$. Hence we introduce the following definition:
Definition 7.1. Consider $\mathcal{M}_{s,\text{HFT}}^{(P,F,r,n)}(\tau')$ and $\mathcal{M}_{s,\text{FT}}^{(P,F,r,n)}(\tau')$ in definitions 5.1 and 5.2 respectively. Define the open substacks $\mathcal{H}_{s,\text{HFT}}^{(P,F,r,n)}(\tau') \subset \mathcal{M}_{s,\text{HFT}}^{(P,F,r,n)}(\tau')$ and $\mathcal{H}_{s,\text{FT}}^{(P,F,r,n)}(\tau') \subset \mathcal{M}_{s,\text{FT}}^{(P,F,r,n)}(\tau')$ as follows:

1. $\mathcal{H}_{s,\text{HFT}}^{(P,F,r,n)}(\tau') = \{(E,F,\phi,\psi) \in \mathcal{M}_{s,\text{HFT}}^{(P,F,r,n)}(\tau') \mid H^1(F(n)) = 0\}$.
2. $\mathcal{H}_{s,\text{FT}}^{(P,F,r,n)}(\tau') = \{(E,F,\phi) \in \mathcal{M}_{s,\text{FT}}^{(P,F,r,n)}(\tau') \mid H^1(F(n)) = 0\}$.

From now on all our calculations are carried out over $\mathcal{H}_{s,\text{HFT}}^{(P,F,r,n)}(\tau')$ and $\mathcal{H}_{s,\text{FT}}^{(P,F,r,n)}(\tau')$ and the results in the following sections hold true for $\mathcal{H}_{s,\text{HFT}}^{(P,F,r,n)}(\tau')$ and $\mathcal{H}_{s,\text{FT}}^{(P,F,r,n)}(\tau')$ only. Also we assume that it is implicitly understood that in the following sections by the “moduli stack of frozen or highly frozen triples” we mean the open substack of the corresponding moduli stacks as defined in Definition 7.1.

7.2 Deformations of $\mathcal{O}_X^{(−n)⊕r} \xrightarrow{\phi} F$

Proposition 7.2. Given a $\tau'$-stable highly frozen triple $(E,F,\phi,\psi)$ represented by the complex $I^\bullet : \mathcal{O}_X^{(−n)⊕r} \to F$ its space of infinitesimal deformations is given by $\text{Hom}(I^\bullet, F)$.

Proof. A square zero embedding $S \hookrightarrow S'$ is a closed immersion whose defining ideal $\mathcal{I}$ satisfies $\mathcal{I}^2 = 0$. Given a square zero embedding and a family of highly frozen triples over $S$, a flat deformation of this family over $\hat{S}$ is a completion of the following commutative diagram with the missing arrows (and exact rows).

\[
\begin{array}{cccccc}
0 & \to & \mathcal{O}_{X\times S}^{(−n)⊕r} \otimes \mathcal{I} & \to & \mathcal{O}_{X\times S}^{(−n)⊕r} & \to & \mathcal{O}_{X\times S}^{(−n)⊕r} & \to & 0 \\
0 & \to & \mathcal{F} \otimes \mathcal{I} & \to & \mathcal{F} & \to & \mathcal{F} & \to & 0.
\end{array}
\]

(7.1)

Following a method described by Illusie in [Ill71] (section IV) for deformation of graded modules and graded morphisms of graded modules, one needs to think of $\mathcal{O}_{X\times S}$ as a graded algebra in degree zero. Therefore one obtains from $\mathcal{F}$ the graded $\mathcal{O}_{X\times S}$-algebra, $\mathcal{F}_{gr} := \mathcal{O}_{X\times S} \oplus \mathcal{F}$ such that $\mathcal{O}_{X\times S}$ sits in degree zero and the second summand sits in degree one. We similarly define
\( F' \). Hence one obtains a commutative diagram of graded \( \mathcal{O}_{X \times S} \)-algebras.

\[
\begin{array}{c}
0 \\[-1ex] \\
\downarrow \\
0 \quad \downarrow \\
\mathcal{F} \otimes \mathcal{I} \quad \mathcal{F}'_{gr} \\
0 \quad \mathcal{O}_{X \times S} + \mathcal{F} \quad 0
\end{array}
\]

(7.2)

here the bottom row in degree zero is given by (Ill71), 3.1):

\[
0 \to 0 \to \mathcal{O}_{X \times S} \to \mathcal{O}_{X \times S} \to 0
\]

(7.3)

and in degree one it is given by the bottom row in (7.1). We know that the obstruction to complete this diagram is given by composition of morphisms:

\[
L_{\mathcal{O}_{X \times S} \oplus \mathcal{F}/\mathcal{O}_{X \times S}} \to L_{\mathcal{O}_{X \times S} \oplus \mathcal{O}_{X \times S}(-n)^{\oplus r}/\mathcal{O}_{X} \otimes \mathcal{F}[1]}
\]

\[
\to \mathcal{I} \otimes (\mathcal{O}_{X \times S}(-n) \oplus \mathcal{O}_{X \times S}(-n)^{\oplus r}) \otimes (\mathcal{O}_{X \times S} \oplus \mathcal{F}) \to \mathcal{I} \otimes \mathcal{F}[2]
\]

(7.4)

where \( L \) is the cotangent complex. Let \( k^1(-) \) of a graded module denote the degree one component of that module. Now we state Illusie’s result:

**Theorem 7.3.** [Ill71] (section IV 3.2.12). Given \( I^\bullet := \mathcal{O}_{X \times S}(-n)^{\oplus r} \to \mathcal{F} \), there exists an element

\[
\text{ob} \in \text{Ext}^2_{D^b(X \times S)}(\text{Cone}(\phi), \mathcal{I} \otimes \mathcal{F})
\]

whose vanishing is necessary and sufficient to complete Diagram (7.2). If \( \text{ob} = 0 \) then the set of isomorphism classes of completions forms a torsor under \( \text{Ext}^1_{D^b(X \times S)}(\text{Cone}(\phi), \mathcal{I} \otimes \mathcal{F}) \).

Here, \( \text{Cone}(\phi) = I^\bullet_S[1] \). Moreover, the obstructions \( \text{ob} : \text{Cone}(\phi) \to \mathcal{I} \otimes \mathcal{F} \) are given by the composite morphism [Ill71] (3.2.14.3):

\[
\text{Cone}(\phi) \to k^1(L_{\mathcal{O}_{X \times S} \oplus \mathcal{O}_{X \times S}(-n)^{\oplus r}/\mathcal{O}_{X} \otimes \mathcal{F}[1])
\]

\[
\to k^1(\mathcal{I} \otimes (\mathcal{O}_{X \times S}(-n) \oplus \mathcal{O}_{X \times S}(-n)^{\oplus r}) \otimes (\mathcal{O}_{X \times S} \oplus \mathcal{F})) \to \mathcal{I} \otimes \mathcal{F}[2].
\]

(7.5)

Another way of stating this theorem is to say that the obstructions are given by:

\[
\text{Cone}(\phi) \to L_{\mathcal{O}_{X \times S}/\mathcal{O}_{X} \otimes \mathcal{F}[1]} \to \mathcal{I} \otimes \mathcal{F}[2],
\]

(7.6)
the set of such composite homomorphisms is given by \( \text{Hom}(I^*_{S[1]}, \mathcal{I} \otimes \mathcal{F}[2]) \cong \text{Ext}^1(I^*_{S}, \mathcal{I} \otimes \mathcal{F}) \cong \text{Ext}^1(I^*_{S}, \mathcal{F}) \otimes \mathcal{I} \), similarly if \( ob = 0 \), then the set of isomorphism classes of deformations of highly frozen triples makes a torsor under \( \text{Ext}^1(I^*_{S[1]}, \mathcal{I} \otimes \mathcal{F}) \cong \text{Hom}(I^*_{S}, \mathcal{I} \otimes \mathcal{F}) \cong \text{Hom}(I^*_{S}, \mathcal{F}) \otimes \mathcal{I} \) and this finishes the proof of Proposition 7.2.

Now we use the result of Proposition 7.2 and Proposition 5.5 to study the space of infinitesimal deformations of a frozen triple.

**Proposition 7.4.** The tangent space of the moduli stack of \( \mathfrak{t} \)-limit-stable frozen triples at a point \( \{p\} : (E, F, \phi) \) represented by a complex \( I^* := [E \to F] \) (where \( E \cong \mathcal{O}_X(-n) \)) is given by:

\[
T_{\{p\}} \mathfrak{m}^{(P_F, r, n)}_{s, \mathcal{F}T}(\tau') \cong \text{Hom}(I^*, F) / \text{Im} (\mathfrak{gl}_r(\mathbb{C}) \to \text{Hom}(I^*, F)). \tag{7.7}
\]

Equivalently

\[
T_{\{p\}} \mathfrak{m}^{(P_F, r, n)}_{s, \mathcal{F}T}(\tau') \cong \text{Coker} \left[ \text{Hom}(I^*, \mathcal{O}_X(-n) \otimes \mathfrak{S}') \to \text{Hom}(I^*, F) \right]. \tag{7.8}
\]

**Proof.** Since our analysis is over a point in the moduli stack, we assume that \( S = \text{Spec}(\mathbb{C}) \) and \( S' \) is a square-zero extension over \( S \). Therefore via \( S \cong S' \) one writes \( \mathcal{O}_{S'} \cong \mathcal{O}_S \oplus \mathcal{O}_{S'} \otimes \mathcal{I} \) as an \( \mathcal{O}_S \)-module. Now use the result of Proposition 7.2. The tangent space of \( \mathfrak{m}^{(P_F, r, n)}_{s, \mathcal{F}T}(\tau') \) at a stable frozen triple of type \( (P_F, r) \) is given by the space of infinitesimal deformations of that triple. Use the notation in Definition 3.4. Suppose that \( \mathcal{O}_X(-n) \otimes \pi^*_S \mathcal{M}_{S'} \xrightarrow{\phi'} \mathcal{F} \) is a flat deformation of the family of frozen triples \( \mathcal{O}_X(-n) \otimes \pi^*_S \mathcal{M}_S \xrightarrow{\phi} \mathcal{F} \) over \( S' \). Similar to 7.1 to obtain the set of such flat deformations one needs to consider the commutative diagram below:

\[
\begin{array}{ccc}
0 & \to & (\mathcal{O}_X(-n) \otimes \pi^*_S \mathcal{M}_{S'}) \otimes \mathcal{I} \\
& \downarrow & \downarrow \\
0 & \to & \mathcal{F} \otimes \mathcal{I} \quad \to \quad \mathcal{F} \quad \xrightarrow{\phi'} \quad \mathcal{F} \quad \xrightarrow{\phi} \quad 0
\end{array}
\]

\[
\text{Ext}^1 \left( \mathcal{O}_X(-n) \otimes \pi^*_S \mathcal{M}_{S'}, \phi' \right) \to \mathcal{F}, (\mathcal{O}_X(-n) \otimes \pi^*_S \mathcal{M}_{S'}) \otimes \mathcal{I} \xrightarrow{\phi'} \mathcal{F} \otimes \mathcal{I}, \tag{7.10}
\]

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we use the isomorphisms \( (O_{X \times S}(-n)_{\mathbb{C}^r}) \otimes \mathcal{I} \cong (O_{X \times S}(-n)_{\mathbb{C}^r}) \otimes \mathcal{I} \) and the notation introduced earlier. Now fix a trivialization \( \psi_M : M_S \rightarrow O_{\mathbb{C}^r} \). This induces a fixed choice of isomorphism \( \psi : O_X(-n) \otimes \pi_S^*M_S \xrightarrow{\cong} O_{X \times S}(-n) \). Now use the fact that \( S \) is a point hence \( S' \) is split over \( S \). Therefore one obtains the following splitting of \( O_{X \times S} \)-modules:

\[
O_{X \times S}(-n)_{\mathbb{C}^r} \cong O_{X \times S}(-n)_{\mathbb{C}^r} \oplus \left( O_{X \times S}(-n)_{\mathbb{C}^r} \right) \otimes \mathcal{I}, \tag{7.11}
\]

Now replace \( O_X(-n) \otimes \pi_S^*M_S \) with the fixed choice of \( O_{X \times S}(-n) \) in the top row of (7.9). Moreover use the splitting property in (7.11). The commutative diagram in (7.9) induces:

\[
\begin{array}{ccc}
0 & \to & 0 \\
\downarrow & & \downarrow \\
0 & \to & F \otimes \mathcal{I} \\
\downarrow & & \downarrow \\
0 & \to & F \\
\downarrow & & \downarrow \\
0 & \to & 0
\end{array}
, \tag{7.12}
\]

where the \( O_{X \times S}(-n)_{\mathbb{C}^r} \) appearing in the upper row are given as a choice of trivialization of \( E \) appearing in the frozen triple \( (E,F,\phi) \). The set of extensions in (7.12) is given by:

\[
\text{Ext}^1(O_{X \times S}(-n) \to F, (F \otimes \mathcal{I})[-1]) \cong \text{Ext}^0(I_{\mathbb{C}^r}, F \otimes \mathcal{I}) \cong \text{Hom}(I_{\mathbb{C}^r}, F) \otimes \mathcal{I} \tag{7.13}
\]

where by Proposition (7.2) \( \text{Hom}(I_{\mathbb{C}^r}, F) \) is the space of infinitesimal deformations of the highly frozen triple represented by the complex \( I_{\mathbb{C}^r} := O_{X \times S}(-n)_{\mathbb{C}^r} \to F \). Hence it is seen that when \( S = \text{Spec}(\mathbb{C}) \) one obtains the deformation space of a \( \mathbb{C} \)-point in \( \mathcal{H}(P_{r,n}, \tau) \) from the deformation space of a \( \mathbb{C} \)-point in \( \mathcal{H}(P_{r,n}, \tau) \) by making a choice of isomorphism \( \psi : O_X(-n) \otimes \pi_S^*M_S \xrightarrow{\cong} O_{X \times S}(-n) \). It is also seen that there exists a map \( \text{Hom}(I_{\mathbb{C}^r}, F) \to T_{\{p\}} \mathcal{H}(P_{r,n}, \tau) \) and the kernel of this map corresponds to the choices of trivialization of \( O_X(-n) \otimes \pi_S^*M_S \) which were not fixed in obtaining the diagram in (7.12), i.e \( \mathfrak{g}r(\mathbb{C}) \). In other words over a \( \mathbb{C} \)-point in the moduli stack one obtains a short exact sequence of \( \mathbb{C} \)-vector spaces:

\[
\mathfrak{g}r(\mathbb{C}) \to \text{Hom}(I_{\mathbb{C}^r}, F) \to T_{\{p\}} \mathcal{H}(P_{r,n}, \tau) \to 0. \tag{7.14}
\]

Note that when \( S = \text{Spec}(\mathbb{C}) \) then \( I_{\mathbb{C}^r} \cong I^* \) canonically. Also it is true that for large \( n \) one has \( \text{Hom}(I^*, O_X(-n)^{\mathbb{C}^r}) \cong \text{End}(O_X(-n)^{\mathbb{C}^r}) \cong \mathfrak{g}r(\mathbb{C}) \).
Now replace $gl_r(C)$ with $\text{Hom}(I^*, \mathcal{O}_X(-n)^{\oplus r})$ and conclude that the space of infinitesimal deformations of a frozen triple in $\mathcal{S}_{s,FT}^{(F,r,n)}(\tau')$, i.e the tangent space of the moduli stack at a $C$-point, is obtained as

$$T_{\{p\}} \mathcal{S}_{s,FT}^{(F,r,n)}(\tau') \cong \text{Coker} \left[ \text{Hom}(I^*, \mathcal{O}_X(-n)^{\oplus r}) \to \text{Hom}(I^*, F) \right] \quad (7.15)$$

**Remark 7.5.** Another way of observing the result obtained in 7.4 is to compare the tangent spaces of the moduli stacks of $\tau$-stable highly frozen triples and frozen triples. Since $\mathcal{S}_{s,FT}^{(F,r,n)}(\tau')$ is a $GL_r(C)$ torsor over $\mathcal{S}_{s,FT}^{(F,r,n)}(\tau')$, therefore at every point $\{p\}$ one obtains the following exact sequence of the corresponding tangent spaces:

$$gl_r(C) \to T_{\{p\}} \mathcal{S}_{s,FT}^{(F,r,n)}(\tau') \to T_{\{p\}} \mathcal{S}_{s,FT}^{(F,r,n)}(\tau') \to 0, \quad (7.16)$$

hence it is immediately seen that $\mathcal{S}_{s,FT}^{(F,r,n)}(\tau') \cong \text{Coker} [gl_r(C) \to T_{\{p\}} \mathcal{S}_{s,FT}^{(F,r,n)}(\tau')]$.

However, $T_{\{p\}} \mathcal{S}_{s,FT}^{(F,r,n)}(\tau') \cong \text{Hom}(I^*, F)$ by Proposition 7.2 and this proves the result obtained in 7.4.

Now we analyze the infinitesimal deformations of frozen triples in more generality, i.e. we do not assume that $S$ is a point or $S'$ is an $S$-scheme. We assume that $S$ is an affine scheme of finite type over $C$ and $S \to S'$ is a square-zero embedding of $C$-schemes.

**Theorem 7.6.** Use notation in Definition 3.4. Fix a map $f : S \to \mathcal{S}_{s,HT}^{(F,r,n)}(\tau')$. Fixing $f$ corresponds to fixing an $S$-flat family of frozen triples given by $[\mathcal{O}_X(-n) \boxtimes \mathcal{M}_S \to \mathcal{F}]$ as in Definition 3.3. Let $S'$ be a square-zero extension of $S$ with ideal $I$. Let $\mathcal{D}e f_S(S', \mathcal{S}_{s,FT}^{(F,r,n)}(\tau'))$ denote the deformation space of the map $f$ obtained by the set of possible deformations, $f' : S' \to \mathcal{S}_{s,FT}^{(F,r,n)}(\tau')$. The following statement is true:

$$\mathcal{D}e f_S(S', \mathcal{S}_{s,FT}^{(F,r,n)}(\tau')) \cong \text{Hom}(I_S^*, F) \otimes I / \text{Im} \left( (gl_r(\mathcal{O}_S) \to \text{Hom}(I_S^*, \mathcal{F})) \otimes I \right) \quad (7.17)$$

**Proof.** Let $g : S \to \mathcal{S}_{s,HT}^{(F,r,n)}(\tau')$ denote the map of $C$-stacks. Given the square-zero extension $S'$ one may ask if the map $g$ is extendable to a map $g' : S' \to \mathcal{S}_{s,HT}^{(F,r,n)}(\tau')$. If $g$ is extendable, then by Proposition 7.2 we know that the set of such extensions is given by $\text{Hom}(I_S^*, \mathcal{F}) \otimes I$. Let $\pi : \mathcal{S}_{s,HT}^{(F,r,n)}(\tau') \to \mathcal{S}_{s,FT}^{(F,r,n)}(\tau')$ denote the forgetful map in Proposition 5.5.
Via composition, one obtains a map $\pi \circ g : S \to \mathcal{S}_{s,\text{FT}}^{(P_F,r,n)}(\tau')$. One may ask further if the map $\pi \circ g$ can be extended to a map $\pi \circ g' : S' \to \mathcal{S}_{s,\text{FT}}^{(P_F,r,n)}(\tau')$. We consider the following commutative diagram:

$$
\begin{array}{ccc}
\mathcal{S}_{s,\text{FT}}^{(P_F,r,n)}(\tau') & \xrightarrow{\pi \circ g} & S \\
\pi \downarrow & & \downarrow \pi \circ g \\
\mathcal{S}_{s,\text{FT}}^{(P_F,r,n)}(\tau') & \xrightarrow{\pi \circ g'} & S'
\end{array}
$$

(7.18)

by Theorem 8.4 we have shown that the following exact sequence exists over $X \times S$.

$$
\mathfrak{gl}_r(O_S) \otimes \mathcal{I} \to \text{Hom}(I_s^\bullet, \mathcal{F}) \otimes \mathcal{I} \to \text{Ext}^1(I_s^\bullet, I_s^\bullet) \otimes \mathcal{I} \to 0.
$$

Let $\mathcal{D}ef_S(S', \mathcal{S}_{s,\text{FT}}^{(P_F,r,n)}(\tau'))$ denote the deformation space of the map $\pi \circ g$ obtained by set of possible extensions $\pi \circ g'$. By Proposition 7.2 we have shown that

$$
\mathcal{D}ef_S(S', \mathcal{S}_{s,\text{FT}}^{(P_F,r,n)}(\tau')) \cong \text{Hom}(I_s^\bullet, \mathcal{F}) \otimes \mathcal{I}.
$$

Moreover, by Definition 3.4 and via evaluating the moduli functor associated to the moduli stack of frozen triples on $S'$, one obtains a family of frozen triples represented by the complex $O_X^{\oplus r}(-n) \boxtimes \mathcal{M}_S \to \mathcal{F}$ over $X \times S$. By assumption both $S$ and $S'$ are chosen to be affine schemes therefore it is not hard to see that the flat deformation of the locally free sheaf $M_S$ over $S'$ is trivial:

**Lemma 7.7.** Let $\mathcal{M}_S$ be a vector bundle of rank $r$ over $S$ such that $\mathcal{M}_S \cong O_S^{\oplus r}$. Given a square-zero extension $S' \hookrightarrow S'$ of affine $\mathbb{C}$-schemes, the flat deformations of $\mathcal{M}_S$ over $S'$ is trivial, i.e the flat extension of $\mathcal{M}_S$ over $S'$ is given by $\mathcal{M}_{S'}$ a vector bundle of rank $r$ over $S'$ such that $\mathcal{M}_{S'} \cong O_S^{\oplus r}$.

**Proof.** Replace $\mathcal{M}_S$ with $O_S^{\oplus r}$. There exists an exact sequence

$$
0 \to O_S^{\oplus r} \otimes \mathcal{I} \to M_S' \to O_S^{\oplus r} \to 0.
$$

Since $S'$ is affine, we get an exact sequence:

$$
0 \to \text{H}^0(O_S^{\oplus r} \otimes \mathcal{I}) \to \text{H}^0(M_S') \to \text{H}^0(O_S^{\oplus r}) \to 0.
$$

Let $E, F, \cdots, e_r$ be the canonical generators of $O_S^{\oplus r}$. Choose lifts $E', F', \cdots, e_r' \in \text{H}^0(M_S')$. These sections define a homomorphism $\phi : O_S^{\oplus r} \xrightarrow{E', F', \cdots, e_r'} M_S'$. Moreover the homomorphism $\phi$ becomes an isomorphism upon restriction to $S$. Since $S \subset S'$ is a nilpotent thickening, by Nakyama’s lemma, this
implies that $\phi$ is an isomorphism. 

No we can see that there exists a surjective map

$$\mathcal{D}ef_S(S', \mathcal{S}_{r,n}^{(P_F, r, n)}(\tau')) \to \mathcal{D}ef_S(S', \mathcal{S}_{s,FT}^{(P_F, r, n)}(\tau')) \to 0.$$ 

Moreover by construction, there exists a natural map $\mathcal{D}ef_S(S', \mathcal{S}_{r,n}^{(P_F, r, n)}(\tau')) \to \mathcal{D}ef_S(S', \mathcal{S}_{s,FT}^{(P_F, r, n)}(\tau')) \to 0$. 

by commutativity of the above diagram and surjectivity of the maps $k$ and $k'$, one concludes the following isomorphisms:

$$\text{Ext}^1(I_s^*, I_s^*) \otimes \mathcal{I} \cong \text{Hom}(I_s^*, \mathcal{F}) \otimes \mathcal{I} / \text{Im}(e)$$

$$\cong \mathcal{D}ef_S(S', \mathcal{S}_{s,FT}^{(P_F, r, n)}(\tau')) / \text{Im}(e') \cong \mathcal{D}ef_S(S', \mathcal{S}_{s,FT}^{(P_F, r, n)}(\tau')),$$

therefore

$$\mathcal{D}ef_S(S', \mathcal{S}_{s,FT}^{(P_F, r, n)}(\tau')) \cong \text{Hom}(I_s^*, \mathcal{F}) \otimes \mathcal{I} / \text{Im} \left( (\mathfrak{gl}_r(\mathcal{O}_S) \to \text{Hom}(I_s^*, \mathcal{F}) \otimes \mathcal{I}) \right).$$

This finishes the proof of Theorem 7.6.

8 Deformations of frozen triples versus deformations of objects in the derived category

In this section we give a comparison between deformation space of frozen triples as in Proposition 7.4 and deformation space of an object in the derived category of $X$ with fixed determinant. We need the following two lemmas in proving the main result of this section:

**Lemma 8.1.** Let $[\mathcal{O}_X(-n) \otimes_F \phi \to F]$ correspond to a point of $\mathcal{D}_{s,HT}^{(P_F, r, n)}(\tau')$ or $\mathcal{D}_{s,FT}^{(P_F, r, n)}(\tau')$. Then:

$$\text{Ext}^2(F, \mathcal{O}_X(-n)) \cong 0 \cong \text{Ext}^1(F, \mathcal{O}_X(-n)).$$ 

(8.1)
Proof. Use Serre duality and obtain:
\[
\text{Ext}^i(F, \mathcal{O}_X(-n)) \cong (\text{Ext}^{3-i}(\mathcal{O}_X(-n), F \otimes \omega_X)^\vee \cong \text{Ext}^{3-i}(\mathcal{O}_X(-n), F)^\vee \cong H^{3-i}(F(n))^\vee.
\]
(8.2)
The statement follows from the definitions of $\mathcal{S}^{(P_F,r,n)}_{s,HFT}(\tau')$ and $\mathcal{S}^{(P_F,r,n)}_{s,FT}(\tau')$.

Let $\mathcal{D}^b(X)$ be the bounded derived category of coherent sheaves on $X$. Let $I^\bullet$ be an object of the derived category given by the complex $I^\bullet := \mathcal{O}_X(-n)^{\oplus r} \xrightarrow{\phi} F$ with $\mathcal{O}_X(-n)^{\oplus r}$ in degree 0 and $F$ in degree 1. Let $K := \text{Ker}(\phi)$ and $Q := \text{Coker}(\phi)$. There exist the following exact triangles in the derived category:

\[
F[-1] \rightarrow I^\bullet \rightarrow \mathcal{O}_X(-n)^{\oplus r} \rightarrow F \rightarrow \cdots \quad (8.3)
K \rightarrow I^\bullet \rightarrow Q[-1] \rightarrow K[1] \rightarrow \cdots \quad (8.4)
\]

Lemma 8.2. Suppose that a frozen triple $(E,F,\phi)$ of type $(P_F,r)$ is $\tau'$-stable. Then $\text{Ext}^{\leq -1}(I^\bullet, I^\bullet) = 0$.

Proof. Note that $\text{Ext}^k(I^\bullet, I^\bullet) = 0$ for $k \leq -2$ for degree reasons. We now consider $k = -1$. Apply $\text{Hom}(I^\bullet, \cdot)$ to (8.3) and obtain:

\[
\cdots \rightarrow \text{Ext}^{-2}(I^\bullet, F) \rightarrow \text{Ext}^{-1}(I^\bullet, I^\bullet) \rightarrow \text{Ext}^{-1}(I^\bullet, \mathcal{O}_X^{\oplus r}(-n)) \rightarrow \text{Ext}^{-1}(I^\bullet, F) \rightarrow \cdots \quad (8.5)
\]

Now apply $\text{Hom}(\cdot, F)$ to (8.4) and obtain:

\[
\cdots \rightarrow \text{Ext}^i(Q[-1], F) \rightarrow \text{Ext}^i(I^\bullet, F) \rightarrow \text{Ext}^i(K, F) \rightarrow \text{Ext}^{i+1}(Q[-1], F) \cdots \quad (8.6)
\]

Combining the exact sequence (8.5) and exact sequences obtained from (8.6) for $i = -2$ and $i = -1$ we obtain the following commutative diagram:

\[
\begin{array}{cccccc}
& & & & & \\
& & & & & \\
& & & & & \\
& & & & & \\
\text{Ext}^{-2}(Q[-1], F) & \rightarrow & \text{Ext}^{-1}(Q[-1], F) & \rightarrow & \\
\downarrow & & \downarrow & & \\
\text{Ext}^{-2}(I^\bullet, F) & \rightarrow & \text{Ext}^{-1}(I^\bullet, I^\bullet) & \rightarrow & \text{Ext}^{-1}(I^\bullet, \mathcal{O}_X^{\oplus r}(-n)) & \rightarrow & \text{Ext}^{-1}(I^\bullet, F) & \rightarrow & \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\text{Ext}^{-2}(K, F) & \rightarrow & \text{Ext}^{-1}(K, F) & \rightarrow & \\
\downarrow & & \downarrow & & \\
\text{Ext}^{-1}(Q[-1], F) & \rightarrow & \text{Hom}(Q[-1], F) & \\
\end{array}
\]
(8.7)
It is easy to see that $\text{Ext}^{-2}(Q[-1], F) \cong 0$, $\text{Ext}^{-2}(K, F) \cong 0$ and $\text{Ext}^{-1}(K, F) \cong 0$ for degree reasons. Moreover, $\text{Ext}^{-1}(Q[-1], F) = \text{Hom}(Q, F) \cong 0$ since $Q$ is zero dimensional (by $\tau'$-stability) and $F$ is of pure dimension one. Hence the above commutative diagram takes the following form:

\[
\begin{array}{ccccccc}
& & & & & & \\
& & & & & & \\
& & & & & & \\
& & & & & & \\
& & & & & & \\
& & & & & & \\
& & & & & & \\
& & & & & & \\
0 & \rightarrow & \text{Ext}^{-2}(I^\bullet, F) & \rightarrow & \text{Ext}^{-1}(I^\bullet, I^\bullet) & \rightarrow & \text{Ext}^{-1}(I^\bullet, O_X^{\oplus r}(-n)) & \rightarrow & \text{Ext}^{-1}(I^\bullet, F) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0 & & 0 & & 0 \\
\end{array}
\]

hence

\[
\text{Ext}^{-2}(I^\bullet, F) \cong 0 \quad \text{and} \quad \text{Ext}^{-1}(I^\bullet, F) \cong 0,
\]

and therefore $\text{Ext}^{-1}(I^\bullet, I^\bullet) \cong \text{Ext}^{-1}(I^\bullet, O_X^{\oplus r}(-n))$. Now apply $\text{Hom}(\cdot, O_X^{\oplus r}(-n))$ to (8.3) and obtain:

\[
\begin{align*}
\text{Ext}^{-1}(F, O_X^{\oplus r}(-n)) & \rightarrow \text{Ext}^{-1}(O_X^{\oplus r}(-n), O_X^{\oplus r}(-n)) \rightarrow \text{Ext}^{-1}(I^\bullet, O_X^{\oplus r}(-n)) \\
& \rightarrow \text{Hom}(F, O_X^{\oplus r}(-n)).
\end{align*}
\]

(8.10)

Now $\text{Ext}^{-1}(O_X^{\oplus r}(-n), O_X^{\oplus r}(-n)) \cong 0$ by degree reasons and $\text{Hom}(F, O_X^{\oplus r}(-n)) \cong 0$ by purity of $O_X^{\oplus r}(-n)$. Hence $\text{Ext}^{-1}(I^\bullet, O_X^{\oplus r}(-n)) \cong 0$ and $\text{Ext}^{-1}(I^\bullet, I^\bullet) \cong \text{Ext}^{-1}(I^\bullet, O_X^{\oplus r}(-n)) \cong 0$.

**Lemma 8.3.** Let $I^\bullet$ be an object in the derived category representing a $\tau'$-limit-stable frozen triple $(E, F, \phi)$ of type $(P_F, r)$. Then there exists an injective map: $\text{Hom}(I^\bullet, I^\bullet) \hookrightarrow \text{End}(O_X(-n)^{\oplus r})$.

**Proof.** Apply $\text{Hom}(I^\bullet, \cdot)$ to $F[-1] \rightarrow I^\bullet \rightarrow O_X(-n)^{\oplus r}$ and obtain the following exact sequence:

\[
\begin{align*}
\text{Ext}^{-1}(I^\bullet, F) & \rightarrow \text{Hom}(I^\bullet, I^\bullet) \rightarrow \text{Hom}(I^\bullet, O_X(-n)^{\oplus r}) \rightarrow \text{Hom}(I^\bullet, F) \\
& \rightarrow \text{Ext}^1(I^\bullet, I^\bullet) \rightarrow \text{Ext}^1(I^\bullet, O_X(-n)^{\oplus r}) \rightarrow \text{Ext}^1(I^\bullet, F) \rightarrow \text{Ext}^2(I^\bullet, I^\bullet).
\end{align*}
\]

(8.11)
Observe that the leftmost term in (8.11) vanishes because of degree reasons:

\[
0 \to \text{Hom}(I^\bullet, I^\bullet) \to \text{Hom}(I^\bullet, \mathcal{O}_X(-n)^{\oplus r}) \to \text{Hom}(I^\bullet, F) \\
\to \text{Ext}^1(I^\bullet, I^\bullet) \to \text{Ext}^1(I^\bullet, \mathcal{O}_X(-n)^{\oplus r}) \to \text{Ext}^1(I^\bullet, F) \to \cdots
\]  

(8.12)

Now apply \(\text{Hom}(\cdot, \mathcal{O}_X(-n)^{\oplus r})\) to (8.3) and obtain:

\[
\text{Hom}(F, \mathcal{O}_X(-n)^{\oplus r}) \to \text{End}(\mathcal{O}_X(-n)^{\oplus r}) \\
\to \text{Hom}(I^\bullet, \mathcal{O}_X(-n)^{\oplus r}) \to \text{Ext}^1(F, \mathcal{O}_X(-n)^{\oplus r})
\]  

(8.13)

Using Lemma 8.1, it is immediately seen that the leftmost and the rightmost terms in (8.13) vanish. Hence:

\[
\text{End}(\mathcal{O}_X(-n)^{\oplus r}) \cong \text{Hom}(I^\bullet, \mathcal{O}_X(-n)^{\oplus r}).
\]  

(8.14)

Hence it is seen from (8.12) that \(\text{Hom}(I^\bullet, I^\bullet) \to \text{End}(\mathcal{O}_X(-n)^{\oplus r})\) is injective.

\[\square\]

**Theorem 8.4.** Let \(p \in \mathcal{S}_{s,FT}^{(P_F, r, n)}(\tau')\) be a point represented by a \(\tau'\)-limit-stable frozen triple \((E, F, \phi)\) of type \((P_F, r)\). Let \(I^\bullet := \mathcal{O}_X(-n)^{\oplus r} \xrightarrow{\phi} F\) be a complex with fixed determinant. The following is true:

\[
T_p \mathcal{S}_{s,FT}^{(P_F, r, n)}(\tau') \cong \text{Ext}^1(I^\bullet, I^\bullet)_0
\]

**Proof.** Consider the exact sequence in (8.12). Now use the result of Lemma 8.3 to obtain the following exact sequence:

\[
0 \to \text{Hom}(I^\bullet, I^\bullet) \to \mathfrak{gl}_r(\mathbb{C}) \to \text{Hom}(I^\bullet, F) \\
\to \text{Ext}^1(I^\bullet, I^\bullet) \to \text{Ext}^1(I^\bullet, \mathcal{O}_X(-n)^{\oplus r}) \to \text{Ext}^1(I^\bullet, F) \to \cdots
\]

(8.15)

where we have replaced \(\text{End}(\mathcal{O}_X(-n)^{\oplus r})\) with \(\mathfrak{gl}_r(\mathbb{C})\). Now recall that \(H^1(\mathcal{O}_X) \cong 0\) by assumption. In that case, assuming that the complex \(I^\bullet\) has fixed determinant, then we have \(\text{Ext}^1(I^\bullet, I^\bullet)_0 \cong \text{Ext}^1(I^\bullet, I^\bullet)\). Hence the exact sequence in (8.15) is rewritten as:

\[
0 \to \text{Hom}(I^\bullet, I^\bullet) \to \mathfrak{gl}_r(\mathbb{C}) \to \text{Hom}(I^\bullet, F) \to \text{Ext}^1(I^\bullet, I^\bullet)_0 \to 0 \to \text{Ext}^1(I^\bullet, F).
\]

(8.16)

Hence we obtain \(\text{Hom}(I^\bullet, F)/\text{Im}[\mathfrak{gl}_r(\mathbb{C}) \to \text{Hom}(I^\bullet, F)] \cong \text{Ext}^1(I^\bullet, I^\bullet)_0\). Now use Proposition 7.4 and obtain

\[
T_{\{p\}} \mathcal{S}_{s,FT}^{(P_F, r, n)}(\tau') \cong \text{Ext}^1(I^\bullet, I^\bullet)_0.
\]

\[\square\]
The result obtained in Theorem 8.4 implies that the $S$-points of the moduli stack of frozen triples (i.e. $S$-flat families of frozen triples) deform as objects in the derived category, in other words:

**Corollary 8.5.** Let $I_S^\bullet$ be defined as in Theorem 8.4. The higher order deformation $I_S^\bullet$ over $\mathcal{S}$ of $I_S^\bullet$ with trivial determinant is quasi-isomorphic to a complex:

$$[O_{X \times C}(−n) \oplus \phi'] \to F]$$

**Proof.** This is a direct consequence of Theorem 8.4. 

---

### 9 Deformation-obstruction theories

**Notation 1:** By a perfect complex $E^\bullet$ in $\mathcal{D}(X)$ of perfect amplitude $[a, b]$ we mean a complex satisfying the condition that for every point $p \in X$ there exists an open neighborhood $\mathcal{U}_p$ over which there exists a complex of vector bundles $\mathbb{R}^\bullet$ whose terms $R^i$ vanish for $i < a$ and $i > b$ and $E^\bullet | \mathcal{U}_p$ is quasi-isomorphic to $\mathbb{R}^\bullet$. Moreover, by a perfect complex of strongly perfect amplitude $[a, b]$ we mean a complex $E^\bullet$ in $\mathcal{D}(X)$ satisfying the condition that there exists globally a complex of vector bundles $\mathbb{R}^\bullet$ such that $R^i = 0$ for $i < a$ or $i > b$ and such that $E^\bullet \cong \mathbb{R}^\bullet$ in $\mathcal{D}(X)$.

By Theorem 6.2 the moduli stack of stable frozen triples $\mathcal{H}^{(PF, r, n)}_{s, FT}(\tau')$ is an Artin stack. Hence, we give the definition of a perfect deformation-obstruction theory for an Artin stack:

**Definition 9.1.** Following [LMB00] and [Ols], by definition, a perfect deformation-obstruction theory for $\mathcal{H}^{(PF, r, n)}_{s, FT}(\tau')$ is given by a perfect 3-term complex $E^\bullet \vee$ of strongly perfect amplitude $[-1, 1]$ and a map in the derived category $E^\bullet \phi \to L^\bullet \mathcal{H}^{(PF, r, n)}_{s, FT}(\tau')$ such that $h^1(\phi)$ and $h^0(\phi)$ are isomorphisms and $h^{-1}(\phi)$ is an epimorphism.

**Remark 9.2.** The reason for having superscript $\vee$ in $E^\bullet \vee$ appearing in statement of Definition 9.1 will be justified through our construction later. Here $L^\bullet \mathcal{H}^{(PF, r, n)}_{s, FT}(\tau')$ is the truncated cotangent complex of the Artin moduli stack of $\tau'$-stable frozen triples concentrated in degrees $−1, 0$ and $1$ whose pullback via the projection map $\pi : \mathcal{H}^{(PF, r, n)}_{s, FT}(\tau') \to \mathcal{H}^{(PF, r, n)}_{s, FT}(\tau')$ has the form:

$$\pi^*L^\bullet \mathcal{H}^{(PF, r, n)}_{s, FT}(\tau') : I/I^2 \to \Omega^1 [\mathcal{H}^{(PF, r, n)}_{s, FT}(\tau') \to (gl_\tau(\mathbb{C}))^\vee \otimes O_{\mathcal{H}^{(PF, r, n)}_{s, FT}(\tau')}.$$
Note that \((gl_r(\mathbb{C}))^\vee \otimes O_{\mathcal{H}^{(PF,r,n)}_s,\text{HFT}}(\tau') \cong \Omega_{\mathfrak{X}}\). Moreover, \(\mathfrak{X}\) denotes the smooth Artin stack that one needs to embed \(\mathcal{H}^{(PF,r,n)}_s,\text{HFT}(\tau')\) into, in order to obtain the truncated cotangent complex, \(L^{\bullet}_{\mathcal{H}^{(PF,r,n)}_s,\text{HFT}}(\tau')\). Finally \(I\) is the ideal corresponding to this embedding. From now on by a perfect complex of perfect amplitude \([a,b]\) we mean a perfect complex of strongly perfect amplitude \([a,b]\).

9.1 Discussion on perfect obstruction theory over \(\mathcal{H}^{(PF,r,n)}_{s,\text{FT}}(\tau')\)

Given \(X\) a smooth projective Calabi-Yau threefold over \(\mathbb{C}\) and \(S\) a parametrizing scheme of finite type over \(\mathbb{C}\), by Theorem 8.4, we showed that the tangent space at every point of the moduli stack of \(\tau\)-limit-stable frozen triples is isomorphic to the space of deformations of the complex with fixed determinant which represents the stable frozen triple. In this section we use this result to construct a deformation obstruction complex for \(\mathcal{H}^{(PF,r,n)}_{s,\text{FT}}(\tau')\). Throughout this section in order to save space when ever required, we use the following notation:

1. \(\mathcal{H} := \mathcal{H}^{(PF,r,n)}_{s,\text{FT}}(\tau')\).

2. \(A := O_X \otimes M \otimes O_{X \times \mathcal{H}}(-n)\) where \(M\) is the universal vector bundle of rank \(r\) on \(X \times \mathcal{H}\) whose fiber over every point of the moduli space is isomorphic to \(O^r_X\).

3. \(\pi_{\mathcal{H}} : X \times \mathcal{H} \to \mathcal{H}\) and \(\pi_X : X \times \mathcal{H} \to X\).

Consider the universal exact triangle determined by the universal complex representing a universal stable frozen triple over \(X \times \mathcal{H}\):

\[
I^\bullet \to M \otimes O_{X \times \mathcal{H}}(-n) \to F
\]  

(9.1)

Apply \(R \mathcal{H}om(\cdot, I^\bullet) \otimes \pi_X^* \omega_X[2]\) to (9.1) and obtain the composition of morphisms:

\[
R \mathcal{H}om(I^\bullet, I^\bullet) \otimes \omega_{\mathcal{H}}[2] \to R \mathcal{H}om(F, I^\bullet) \otimes \omega_{\mathcal{H}}[3] \\
\to R \mathcal{H}om(M \otimes O_{X \times \mathcal{H}}(-n), I^\bullet) \otimes \omega_{\mathcal{H}}[3]
\]  

(9.2)
There exists a map from the trace-free $\mathcal{H}om$ to $\mathcal{H}om$ so we get the following composition of morphisms in $\mathcal{D}^b(X \times \mathcal{H})$:

$$
\begin{align*}
\mathcal{H}om(\mathcal{I}^\bullet, \mathcal{I}^\bullet)_0 \otimes \omega_{\mathcal{H}}[2] & \to \mathcal{H}om(\mathcal{I}^\bullet, \mathcal{I}^\bullet) \otimes \omega_{\mathcal{H}}[2] \\
& \to \mathcal{H}om(\mathcal{F}, \mathcal{I}^\bullet) \otimes \omega_{\mathcal{H}}[3] \to \mathcal{H}om(M \otimes \mathcal{O}_{X \times \mathcal{H}}(-n), \mathcal{I}^\bullet) \otimes \omega_{\mathcal{H}}[3]
\end{align*}
$$

(9.3)

**Theorem 9.3.** There exists a map in $\mathcal{D}^b(\mathcal{H})$ given by $R\pi_{\mathcal{H}}^*(\mathcal{H}om(\mathcal{F}, \mathcal{I}^\bullet) \otimes \omega_{\mathcal{H}}[3]) \to \mathcal{L}_{\mathcal{H}}^\bullet$.

**Proof.** One needs to apply the result of Illusie [Ill71] (Section 4.2) in Theorem 7.3 to the universal complex $\mathcal{I}^\bullet : M \otimes \mathcal{O}_{X \times \mathcal{H}}(-n) \xrightarrow{\phi} \mathcal{F}$. Since we will not eventually use $R\pi_{\mathcal{H}}^*(\mathcal{H}om(\mathcal{F}, \mathcal{I}^\bullet) \otimes \omega_{\mathcal{H}}[3])$ as a suitable candidate for the deformation obstruction theory of $\mathcal{H} := \mathcal{H}(P_{F,s},r,n)$ and since the proof follows directly from the proof of Joyce and Song in [JS09] (Theorem 14.7) applied to $\mathcal{I}^\bullet : M \otimes \mathcal{O}_{X \times \mathcal{H}}(-n) \xrightarrow{\phi} \mathcal{F}$ we omit providing a detailed proof here and leave this to the reader. □

**Remark 9.4.** Note that the complex $\mathcal{H}om(\mathcal{F}, \mathcal{I}^\bullet) \otimes \omega_{\mathcal{H}}[3]$ is neither perfect of amplitude $[-1, 1]$ nor it defines a deformation theory for moduli stack of frozen triples. However by (9.3) one obtains:

$$
\begin{align*}
\mathcal{H}om(\mathcal{I}^\bullet, \mathcal{I}^\bullet)_0 \otimes \omega_{\mathcal{H}}[2] & \to \mathcal{H}om(\mathcal{I}^\bullet, \mathcal{I}^\bullet) \otimes \omega_{\mathcal{H}}[2] \\
& \to R\pi_{\mathcal{H}}^*(\mathcal{H}om(\mathcal{F}, \mathcal{I}^\bullet) \otimes \omega_{\mathcal{H}}[3]) \to \mathcal{L}_{\mathcal{H}}^\bullet
\end{align*}
$$

(9.4)

Now consider the composite morphism in the derived category:

$$
\mathcal{H}om(\mathcal{I}^\bullet, \mathcal{I}^\bullet)_0 \otimes \omega_{\mathcal{H}}[2] \to \mathcal{L}_{\mathcal{H}}^\bullet.
$$

(9.5)

Note that for every point $\{p\} \in \mathcal{H}(P_{F,s},r,n)$ the fiber of $\mathcal{H}om(\mathcal{I}^\bullet, \mathcal{I}^\bullet)_0 \otimes \omega_{\mathcal{H}}[2]$ over $\mathcal{I}^\bullet$ is a complex which has 4 non-vanishing cohomologies (by taking cohomologies in degrees $-2, \cdots, 1$) equal to $\text{Ext}^{2+i}(I^\bullet, I^\bullet)_0$.

### 9.2 Non-perfect deformation-obstruction theory of amplitude $[-2, 1]$ over $\mathcal{H}(P_{F,s},r,n)(\tau')$

**Theorem 9.5.** (a). There exists a map in the derived category given by:

$$
\begin{align*}
R\pi_{\mathcal{H}}^*(\mathcal{H}om(\mathcal{I}^\bullet, \mathcal{I}^\bullet)_0 \otimes \pi_X^*\omega_{\mathcal{H}})[2] \xrightarrow{\text{ob}} \mathcal{L}_{\mathcal{H}}^\bullet(\mathcal{H}(P_{F,s},r,n)(\tau')).
\end{align*}
$$
(b). After suitable truncations, there exists a 4 term complex $E^\bullet$ of locally free sheaves, such that $E^{\bullet\vee}$ is self-symmetric of amplitude $[-2,1]$ and there exists a map in the derived category:

$$E^{\bullet\vee} \xrightarrow{obf} \mathbb{L}^\bullet_{S_h}((P_F,r,n)_{\tau'}),$$

such that $h^{-1}(obf)$ is surjective, and $h^0(obf)$ and $h^1(obf)$ are isomorphisms.

Proof. Here we prove (a). Consider the universal complex:

$$\mathbb{P}^\bullet = [M \otimes \mathcal{O}_{X \times \mathcal{H}}(-n) \rightarrow \mathbb{E}] \in \mathcal{D}^b(X \times \mathcal{H}).$$

Since the composition of the maps $id : \mathcal{O}_{X \times \mathcal{H}} \rightarrow R\mathcal{H}om(\mathbb{I}^\bullet, \mathbb{I}^\bullet)$ and $tr : R\mathcal{H}om(\mathbb{I}^\bullet, \mathbb{I}^\bullet) \rightarrow \mathcal{O}_{X \times \mathcal{H}}$ is multiplication by $rk(\mathbb{I}^\bullet)$, one obtains a splitting $R\mathcal{H}om(\mathbb{I}^\bullet, \mathbb{I}^\bullet) \cong R\mathcal{H}om(\mathbb{I}^\bullet, \mathbb{I}^\bullet)_0 \oplus \mathcal{O}_{X \times \mathcal{H}}$. Recall that by discussions in Section 6.1 $\mathcal{H} = [\mathbb{G}^{(G,F,r,n)}_{\tau'}]$, where $G = GL_r(\mathbb{C}) \times GL(V_F)$. For simplicity denote $\mathbb{S} := \mathbb{G}^{(G,F,r,n)}_{\tau'}$. Let $\mathbb{I}_\mathbb{S}$ denote the pullback of $\mathbb{I}_\mathbb{S}$ to $X \times \mathbb{S}$. We write $L^\bullet$ to mean the full, untruncated cotangent complex, and write $L^\bullet_{\tau'} = \tau^{-1}L^\bullet$ for the truncated cotangent complex. Consider the Atiyah class $\mathbb{I}_\mathbb{S} \rightarrow L^\bullet_{X \times \mathbb{S}} \otimes \mathbb{I}_\mathbb{S}[1]$ defined by Illusie [III71] (Section IV2.3.6). The Atiyah class can be identified with a class in $\mathbb{E}xt^1(\mathbb{I}_\mathbb{S}, L^\bullet_{X \times \mathbb{S}} \otimes \mathbb{I}_\mathbb{S})$. The composite

$$\mathbb{I}_\mathbb{S} \rightarrow L^\bullet_{X \times \mathbb{S}} \otimes \mathbb{I}_\mathbb{S}[1] \rightarrow \tau^{-1}L^\bullet_{X \times \mathbb{S}} \otimes \mathbb{I}_\mathbb{S}[1] = L^\bullet_{X \times \mathbb{S}} \otimes \mathbb{I}_\mathbb{S}[1]$$

is the truncated Atiyah class of [HT10], Section 2.2. By [III97] (Proposition 2.1.10) the complex $\mathbb{I}^\bullet$ is perfect. It then follows from Corollaire IV.2.3.7.4 of [III71] that the composite $\mathbb{I}_\mathbb{S} \rightarrow L^\bullet_{X \times \mathbb{S}} \otimes \mathbb{I}_\mathbb{S}[1] \rightarrow \Omega^1_{X \times \mathbb{S}} \otimes \mathbb{I}_\mathbb{S}[1]$, when identified with a 1-extension, agrees with the canonical 1-extension

$$0 \rightarrow \Omega^1_{X \times \mathbb{S}} \otimes \mathbb{I}_\mathbb{S} \rightarrow \mathcal{P}^1_{X \times \mathbb{S}} \otimes \mathbb{I}_\mathbb{S} \rightarrow \mathbb{I}_\mathbb{S} \rightarrow 0$$

defined by tensoring with the first-order principal parts $\mathcal{P}^1_{X \times \mathbb{S}}$. We want to show that the Atiyah class descends to $X \times \mathcal{H} = X \times [\mathbb{S}]$ where $G = GL_r(\mathbb{C}) \times GL(V_F)$ (where this identification comes from discussions in Section 6). More precisely, this means the following. Let $q_\mathbb{S} : X \times \mathbb{S} \rightarrow X \times \mathcal{H}$ denote the projection. Then we want a morphism $\mathbb{I}^\bullet \rightarrow L^\bullet_{X \times \mathbb{S}} \otimes \mathbb{I}^\bullet_{\tau'}[1]$ on $\mathcal{H}$, such that the natural composite $q_\mathbb{S}^!\mathbb{I}^\bullet \rightarrow q_\mathbb{S}^!L^\bullet_{X \times \mathbb{S}} \otimes q_\mathbb{S}^!\mathbb{I}^\bullet_{\tau'}[1] \rightarrow L^\bullet_{X \times \mathbb{S}} \otimes \mathbb{I}^\bullet_{\tau'}[1]$ agrees with the Atiyah class of Illusie. The complex $\mathbb{I}_\mathbb{S}$ is $G$-equivariant by construction (it comes via pullback from $X \times \mathcal{H}$), and the construction of the Atiyah class shows that it too is naturally $G$-equivariant. The pulled
back cotangent complex $q_\delta^* L^\bullet_{X \times \delta}$ has the following description. There is a natural composite map $L^\bullet_{X \times \delta} \to \Omega^1_{X \times \delta} \to \mathfrak{g}^\vee \otimes \mathcal{O}_{X \times \delta}$, where the second map is dual to the infinitesimal $\mathfrak{g}$-action (and $\mathfrak{g} = \text{Lie}(G)$). Then $q_\delta^* L^\bullet_{X \times \delta} \simeq \text{Cone}[L^\bullet_{X \times \delta} \to \mathfrak{g}^\vee \otimes \mathcal{O}_{X \times \delta}][-1]$. Thus, to prove that the Atiyah class descends to $X \times \delta$ in the sense explained above, it suffices to show that the composite $\mathfrak{l}^\bullet \to L^\bullet_{X \times \delta} \otimes \mathfrak{l}^\bullet[1] \to \Omega^1_{X \times \delta} \otimes \mathfrak{l}^\bullet[1] \to \mathfrak{g}^\vee \otimes \mathfrak{l}^\bullet[1]$ represents an equivariant split extension. By the above discussion, this extension is obtained by pushing out the principal parts extension along the natural map $\Omega^1_{X \times \delta} \otimes \mathfrak{l}^\bullet \to \mathfrak{g}^\vee \otimes \mathfrak{l}^\bullet$. Just as a splitting of the principal parts extension corresponds to a choice of connection, however, a splitting of its pushout corresponds to a choice of an $L$-connection (Section 4) where $L = \mathfrak{g} \otimes \mathcal{O}_{X \times \delta}$ is the action Lie algebroid associated to the infinitesimal $G$-action. Since $\mathfrak{l}^\bullet$ is $G$-equivariant, it comes equipped with a $\mathfrak{g} \otimes \mathcal{O}_{X \times \delta}$-connection, hence a $G$-equivariant splitting of the required $1$-extension. It follows that the Atiyah class descends to $X \times \delta$. We now have the truncated Atiyah class of the universal complex, given by a class in

$$\text{Ext}^1_{X \times \delta}(\mathfrak{l}^\bullet \otimes \mathfrak{l}^\bullet, \mathfrak{l}^\bullet, \mathfrak{l}^\bullet) \cong \text{Ext}^1_{X \times \delta}(\mathcal{R}\mathcal{H}om(\mathfrak{l}^\bullet, \mathfrak{l}^\bullet), \mathfrak{l}^\bullet) \cong \text{Ext}^1_{X \times \delta}(\mathcal{R}\mathcal{H}om(\mathfrak{l}^\bullet, \mathfrak{l}^\bullet)_0 \oplus \mathcal{O}_{X \times \delta}, \mathfrak{l}^\bullet), \mathfrak{l}^\bullet),$$(9.8)

where $\mathfrak{l}^\bullet$ denotes the truncated cotangent complex of $X \times \delta$. Note that over $X \times \delta$, $\mathfrak{l}^\bullet = L\mathfrak{p}_X^+ L\mathfrak{p}_X \otimes L\mathfrak{p}_\delta^+ L\mathfrak{p}_\delta$. Since the projection maps are flat then the derived pullbacks are the same as the usual pullbacks. One obtains the following map between the Ext groups:

$$\text{Ext}^1_{X \times \delta}(\mathcal{R}\mathcal{H}om(\mathfrak{l}^\bullet, \mathfrak{l}^\bullet)_0 \oplus \mathcal{O}_{X \times \delta}, \mathfrak{l}^\bullet) \to \text{Ext}^1_{X \times \delta}(\mathcal{R}\mathcal{H}om(\mathfrak{l}^\bullet, \mathfrak{l}^\bullet)_0, \pi_\delta^* \mathfrak{l}^\bullet).$(9.9)

On the other hand:

$$\text{Ext}^1_{X \times \delta}(\mathcal{R}\mathcal{H}om(\mathfrak{l}^\bullet, \mathfrak{l}^\bullet)_0, \pi_\delta^* \mathfrak{l}^\bullet)
\cong \text{Ext}^1_{X \times \delta}(\mathcal{R}\mathcal{H}om(\mathfrak{l}^\bullet, \mathfrak{l}^\bullet)_0 \otimes \omega_{X \times \delta})
\cong \text{Ext}^1_{X \times \delta}(\mathfrak{l}^\bullet, \mathfrak{l}^\bullet) \otimes \omega_{X \times \delta})
\cong \text{Ext}^1_{\mathfrak{g}}(\mathfrak{l}^\bullet, \mathfrak{l}^\bullet) \otimes \omega_{X \times \delta})
\cong \text{Ext}^1_{\mathfrak{g}}(\mathfrak{l}^\bullet, \mathfrak{l}^\bullet) \otimes \omega_{X \times \delta}), (9.10)$$

where the first isomorphism is obtained by Serre duality, the second isomorphism is induced by the adjointness property of the left derived pullback.
and the right derived pushforward and the third isomorphism is obtained by Serre duality. By projection formula and the definition of the relative dualizing sheaf
\[ \omega_{\pi'_{\delta}} = \omega_{X \times \delta} \otimes \omega_{\delta}^{-1} = \pi^*_X \omega_X \]
the last term in (9.10) is rewritten as:
\[ \Ext^{-\dim(X)+1}_{\delta} (R\pi_{\delta}^*(R\Hom(I^\bullet, I^\bullet)_0 \otimes \pi^*_X \omega_X), \mathbb{L}_{\delta}^\bullet). \]  
(9.11)
Since \(X\) is a three-fold, (9.11) is rewritten as:
\[ R\pi_{\delta}^*(R\Hom(I^\bullet, I^\bullet)_0 \otimes \pi^*_X \omega_X) \rightarrow \mathbb{L}_{\delta}^\bullet. \]  
(9.12)
Therefore, it is seen that the truncated Atiyah class of the universal complex over the moduli stack of \(\tau'\)-stable frozen triples, induces a well defined map in the derived category as in (9.5).

Next, to prove (b), we show that (9.12) defines a relative deformation-obstruction theory for \(\mathcal{X}_{s,F_T}(\tau')\):

**Proposition 9.6.** The morphism given by (9.12) defines a (non-perfect) relative deformation-obstruction theory for \(\mathcal{X}_{s,F_T}(\tau')\).

**Proof.** We follow the same strategy as in [HT10], [RR09]. Given a morphism of \(\mathbb{C}\)-stacks \(\delta : S \rightarrow \mathcal{X}_{s,F_T}(\tau')\) and a square zero embedding \(S \hookrightarrow \hat{S}\), by the theory of cotangent complexes there exists a morphism in \(D^b(S)\):
\[ L_S^\bullet \rightarrow L_{S/S'}^\bullet \cong [\mathcal{I}_{S \subset S'} \rightarrow \Omega_{S'}^1 \mid s]. \]
There exists a morphism:
\[ [\mathcal{I}_{S \subset S'} \rightarrow \Omega_{S'}^1 \mid s] \rightarrow \mathcal{I}_{S \subset S'}^1[1], \]
hence we obtain a morphism in \(D^b(S)\) given by \(e : \Ext^{1}(g^*L^\bullet_{\mathcal{X}_{s,F_T}(\tau')}, \mathcal{I}_{S \subset S'}^1) \rightarrow L_S^\bullet \rightarrow \mathcal{I}_{S \subset \hat{S}}[1]\) where \(\mathcal{I}_{S \subset \hat{S}}\) is the ideal of \(S \subset \hat{S}\) and \(e \in \Ext^{1}(g^*L^\bullet_{\mathcal{X}_{s,F_T}(\tau')}, \mathcal{I}_{S \subset S'})\). Now \(e\) is equal to zero if and only if there exists a lift \(\hat{g} : \hat{S} \rightarrow \mathcal{X}_{s,F_T}(\tau')\) and moreover if such \(g'\) exists then the set of isomorphism extensions forms a torsor under \(\Hom(g^*L^\bullet_{\mathcal{X}_{s,F_T}(\tau')}, \mathcal{I}_{S \subset \hat{S}})\).

Consider the following commutative diagram:
\[
\begin{array}{ccc}
X \times \mathcal{X}_{s,F_T}(\tau') & \xrightarrow{\mathcal{H}} & X \times S \\
| \quad \pi_{\delta} & & \pi_S \\
\mathcal{X}_{s,F_T}(\tau') & \xleftarrow{\mathcal{H}} & S
\end{array}
\]  
(9.13)
Pullback the morphism in (9.12) by \( g \) and obtain:

\[
g^* R\pi_{\delta*} (R\mathcal{H}om(\mathbb{I}^*, \mathbb{I}^*_{0} \otimes \pi_X^* \omega_X))[2] \to g^* \mathbb{L}_L^{*}(P_{F,r,n})(\tau')
\]

(9.14)

This induces a natural composite morphism in \( D^b(S) \):

\[
o : g^* R\pi_{\delta*} (R\mathcal{H}om(\mathbb{I}^*, \mathbb{I}^*_{0} \otimes \pi_X^* \omega_X))[2] \to g^* \mathbb{L}_L^{*}(P_{F,r,n})(\tau') \to \mathbb{L}_S \to \mathcal{I}_{S \subset S}[1],
\]

(9.15)

where \( o \in \text{Ext}^{-2}(g^* R\pi_{\delta*} (R\mathcal{H}om(\mathbb{I}^*, \mathbb{I}^*_{0} \otimes \pi_X^* \omega_X)), \mathcal{I}_{S \subset S}) \). One shows that there exists an extension of \( g \) to \( \tilde{g} \) if and only if \( o \) vanishes and moreover the set of such extensions forms a torsor under

\[
\text{Ext}^{-2}(g^* R\pi_{\delta*} (R\mathcal{H}om(\mathbb{I}^*, \mathbb{I}^*_{0} \otimes \pi_X^* \omega_X)), \mathcal{I}_{S \subset S}).
\]

By (9.13) and the flatness of \( \pi_S \) one obtains the following isomorphism:

\[
g^* R\pi_{\delta*} (R\mathcal{H}om(\mathbb{I}^*, \mathbb{I}^*_{0} \otimes \pi_X^* \omega_X))[2] \cong R\pi_{\delta*} (R\mathcal{H}om(\mathcal{F}^* \mathbb{T}^*, \mathcal{F}^* \mathbb{T}^*_{0} \otimes \pi_X^* \omega_X))[2].
\]

(9.16)

Hence one obtains:

\[
R\pi_{\delta*} (R\mathcal{H}om(\mathcal{F}^* \mathbb{T}^*, \mathcal{F}^* \mathbb{T}^*_{0} \otimes \pi_X^* \omega_X))[2] \to g^* \mathbb{L}_L^{*}(P_{F,r,n}) \to \mathbb{L}_S \to \mathcal{I}_{S \subset S}[1]
\]

(9.17)

therefore:

\[
o \in \text{Ext}^{-1}(R\pi_{\delta*} (R\mathcal{H}om(\mathcal{F}^* \mathbb{T}^*, \mathcal{F}^* \mathbb{T}^*_{0} \otimes \pi_X^* \omega_X)), \mathcal{I}_{S \subset S})
\]

\[
\cong \text{Ext}^2(X \times S, (\mathcal{F}^* \mathbb{T}^*, \mathcal{F}^* \mathbb{T}^*_{0} \otimes \pi_X^* \mathcal{I}_{S \subset S}[0])
\]

(9.18)

by a similar argument to (9.10). By results of Thomas in [Tho00] the trace of the obstruction class is the obstruction to deform \( det(\mathcal{F}^* \mathbb{T}^*) \). So this is enough to conclude that \( o = 0 \) if and only if there exist deformations of \( \mathcal{F}^* \mathbb{T}^* \) from \( X \times S \) to \( X \times \hat{S} \). Moreover the set of such deformations forms a torsor under \( \text{Ext}^1(X \times S, (\mathcal{F}^* \mathbb{T}^*, \mathcal{F}^* \mathbb{T}^*_{0} \otimes \pi_X^* \mathcal{I}_{S \subset S}[0]) \). By definition of relative moduli stack, the deformations of \( \mathcal{F}^* \mathbb{T}^* \) are in one to one correspondence with deformations of \( g \) to \( \tilde{g} \).

\[\square\]

**Remark 9.7.** Recall that the truncated cotangent complex of \( \mathcal{Y}_{s_{,FT}}^{(P_{F,r,n})}(\tau') \) is concentrated in degrees \(-1, 0 \) and \( 1 \) whose pullback via the projection map \( \pi : \mathcal{Y}_{s_{,FT}}^{(P_{F,r,n})}(\tau') \to \mathcal{Y}_{s_{,FT}}^{(P_{F,r,n})}(\tau') \) has the form:

\[
\pi^* \mathbb{L}_L^{*}(P_{F,r,n})(\tau') : \mathcal{I}/\mathcal{I}^2 \to \Omega_A |_{\mathcal{Y}_{s_{,FT}}^{(P_{F,r,n})}(\tau')} \to (\mathcal{g}l_\tau(\mathbb{C}))^\vee \otimes \mathcal{O}_{\mathcal{Y}_{s_{,FT}}^{(P_{F,r,n})}(\tau')},
\]

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By Theorem 9.5 and Proposition 9.6 one obtains a morphism in the derived category $E^\bullet \cong \mathbf{L} \mathcal{A}_s^{(P,F,r,n)}(\tau')$. It is seen that by construction, for a point $b \in \mathcal{H}_s^{(P,F,r,n)}(\tau')$, $h^{-1}(ob^t |_b)$ is an epimorphism. Moreover, by Propositions 7.4 and 7.6 the cohomology map in degree zero

$$h^0(ob^t |_b) : \text{Ext}^1(I^\bullet, I^\bullet)_0 \to \text{Coker}(\text{gl}_r(\mathbb{C}) \to \text{Hom}(I^\bullet, F))$$

is an isomorphism. Finally, $h^1(ob^t |_b)$ is an isomorphism mapping the automorphisms of the object in the derived category to the automorphisms of the associated frozen triple.

To complete the proof of part (b) of Theorem 9.5 we show that the deformation-obstruction theory in Proposition 9.6 is globally quasi-isomorphic to a 4 term complex of vector bundles:

**Lemma 9.8.** Given $S$ a smooth scheme of finite type over $\mathbb{C}$ and $X \to S$ a smooth projective morphism of relative dimension $n$, If $F$ is a flat family of coherent sheaves on the fibers of $f : X \to \mathbb{C}$ then there exists a locally free resolution $0 \to F_n \to F_{n-1} \to \cdots \to F_0 \to F$. Such that $R^mf_*F_m$ is locally free for $m = 0, \cdots, n$, $R^if_*F_m = 0$ for $i \neq n$ and $m = 0, \cdots, n$.

**Proof.** Look at [HL97] (Proposition 2.1.10).

**Proposition 9.9.** The complex $R\pi_{\mathcal{H}} (R\mathcal{H} \text{om}(\mathfrak{I}^\bullet, \mathfrak{I}^\bullet)) = 0 \otimes \pi^\wedge_X \omega_X [2]$ in (9.12) is quasi-isomorphic to a 4 term complex of locally free sheaves.

**Proof.** Consider the universal complex $\mathfrak{I}^\bullet$ over $X \times \mathcal{H}_s^{(P,F,r,n)}(\tau')$. By Lemma 9.8 there exists a finite locally free resolution $A^\bullet$ of $\mathfrak{I}^\bullet$. There exists an isomorphism $(A^\bullet)^\vee \otimes A^\bullet \cong \mathcal{O}_{X \times \mathcal{H}_s^{(P,F,r,n)}}(\tau') \oplus ((A^\bullet)^\vee \otimes A^\bullet)_0$. Now define the quasi-isomorphism class of trace-free Homs by $R\mathcal{H} \text{om}(\mathfrak{I}^\bullet, \mathfrak{I}^\bullet)_0 = ((A^\bullet)^\vee \otimes A^\bullet)_0$. Each term in the complex $(A^\bullet)^\vee \otimes A^\bullet)_0$ is a coherent locally free sheaf over $X \times \mathcal{H}_s^{(P,F,r,n)}(\tau')$ flat over $\mathcal{H}_s^{(P,F,r,n)}(\tau')$. Since the projection map $\pi_{\mathcal{H}} : X \times \mathcal{H}_s^{(P,F,r,n)}(\tau') \to \mathcal{H}_s^{(P,F,r,n)}(\tau')$ has relative dimension 3, by Lemma 9.8 there exists a locally free resolution of length 4 associated to each term in $(A^\bullet)^\vee \otimes A^\bullet)_0$. From this point the proof follows Lemma 2.10 [RR09]. Let the complex $B^\bullet$ be a sufficiently negative locally free resolution of $(A^\bullet)^\vee \otimes A^\bullet)_0$ trimmed to start at least 4 places earlier than $(A^\bullet)^\vee \otimes A^\bullet)_0$, then $R^{m-2} \pi_{\mathcal{H}} (\mathcal{H}_s^{(P,F,r,n)})^\vee B^m = 0$ for all $m$. and $R^{m+3} \pi_{\mathcal{H}} (\mathcal{H}_s^{(P,F,r,n)}) B^m$ is locally free. Let $E^\bullet$ be defined as the complex with

$$E^j \cong R^{m+3} \pi_{\mathcal{H}} (\mathcal{H}_s^{(P,F,r,n)}) B^{j+3}.$$ (9.19)
The complex $E^\bullet$ is a complex of locally free sheaves, and quasi-isomorphic to $R\pi_{0,s}R\mathcal{H}om(I^\bullet, I^\bullet)_0$. Restricting this complex to a point $\{b\} \in \mathcal{H}_{s,FT}^{(P_F, r, n)}(\tau')$ (i.e base change) one obtains a complex whose cohomologies compute $\text{Ext}^i(I^\bullet, I^\bullet)_0$. By the property of $I^\bullet$ shown earlier, the negative Ext groups vanish. Hence, one obtains a complex whose nonvanishing cohomologies are given by

$$\text{Ext}^0(I^\bullet, I^\bullet)_0, \cdots, \text{Ext}^4(I^\bullet, I^\bullet)_0, \cdots$$

However since $X$ is Calabi-Yau, by Serre duality $\text{Ext}^i(I^\bullet, I^\bullet)_0 \cong \text{Ext}^{3-i}(I^\bullet, I^\bullet)$. Hence for $i > 3$, $\text{Ext}^i(I^\bullet, I^\bullet)_0 \cong 0$ and the only non-vanishing cohomologies are $\text{Ext}^3(I^\bullet, I^\bullet)_0 \cdots$. Now apply Lemma 2.10 in [RR09] to $E^\bullet$ in (9.19). The complex $E^\bullet$ is quasi-isomorphic to a 4 term complex of locally free sheaves. □

We proved in Proposition 9.6 that the map in (9.12) defines a relative perfect deformation-obstruction theory over $\mathcal{F}_s$. Moreover, in Proposition 9.9 we proved that this relative theory has global resolution by a 4-term complex of locally free sheaves over $\mathcal{F}_s$. This finishes the proof of part of part (b) of Theorem 9.5. □

As was explained in introduction, we aim at producing an enumerative theory for highly frozen triples since firstly, their moduli space is a DM stack and secondly there exists a perfect classification of torus-fixed loci of the moduli space of highly frozen triples. Next we show how to obtain a perfect deformation obstruction theory for HFT using the non-perfect deformation obstruction theory of FT.

### 9.3 A perfect Deformation-obstruction theory of amplitude $[-1, 0]$ over $\mathcal{F}_{s, HFT}^{(P_F, r, n)}(\tau')$

In this section we construct a suitable deformation-obstruction theory over the moduli stack of highly frozen triples. First we prove a statement about the self duality of the complex $E^\bullet$ obtained in Proposition 9.9.

**Lemma 9.10.** The complex $E^\bullet$ in Proposition 9.9 is self-dual in the sense of [Beh09]. In other words there exists a quasi-isomorphism of complexes $E^\bullet \cong E^\bullet[1]$.

**Proof.** Use the notation in Section 9.1. The derived dual of $E^\bullet$ over $\mathcal{F}_{s, FT}^{(P_F, r, n)}(\tau')$ is given by

$$E^\bullet := R\mathcal{H}om(E^\bullet, O_{\mathcal{S}}).$$
By Proposition [9.9] $\mathcal{E}^\bullet$ is quasi-isomorphic to $R\pi_{\mathcal{B},*}(R\mathcal{H}om(\mathcal{I}^\bullet, \mathcal{I}^\bullet)_0 \otimes \pi_X^* \omega_X)$ [2]. Now use Grothendieck duality and obtain the following isomorphisms:

$$R\mathcal{H}om(\mathcal{E}^\bullet, \mathcal{O}_\mathcal{S}) \cong R\pi_{\mathcal{B},*}(R\mathcal{H}om_{\mathcal{X} \times \mathcal{S}}(\mathcal{E}^\bullet, \mathcal{I}^\bullet)_0 \otimes \pi_X^* \omega_X) [2, \pi^1 \mathcal{O}_\mathcal{S})]$$

$$\cong R\pi_{\mathcal{B},*}(R\mathcal{H}om_{\mathcal{X} \times \mathcal{S}}(\mathcal{E}^\bullet, \mathcal{I}^\bullet)_0 \otimes \pi_X^* \omega_X) [2, \pi_X^* \omega_X [3]]$$

$$\cong R\pi_{\mathcal{B},*}R\mathcal{H}om(\mathcal{O}_{\mathcal{X} \times \mathcal{S}}, R\mathcal{H}om(\mathcal{I}^\bullet, \mathcal{I}^\bullet)_0 [1]) \cong \mathcal{E}^\bullet [-1]. \quad (9.20)$$

Hence we conclude that $\mathcal{E}^{\bullet\vee}[1] \cong \mathcal{E}^\bullet$. Note that the second isomorphism in (9.20) is obtained using the fact that $\mathcal{X}$ is a Calabi-Yau threefold and hence $\omega_X \cong \mathcal{O}_X$.

**An alternative obstruction bundle for HFT**

Consider the forgetful map $\pi : \mathcal{S}^{(P,F,r,n)}_{s,\text{HFT}}(\tau') \to \mathcal{S}^{(P,F,r,n)}_{s,\text{FT}}(\tau')$. We pullback the four-term deformation obstruction theory of perfect amplitude $[-2,1]$ over $\mathcal{S}^{(P,F,r,n)}_{s,\text{FT}}(\tau')$ via $\pi$. After suitably truncating the pulled-back complex we (locally) define a perfect two-term deformation obstruction theory of amplitude $[-1,0]$ over $\mathcal{S}^{(P,F,r,n)}_{s,\text{HFT}}(\tau')$:

**Theorem 9.11.** Consider the 4-term deformation obstruction theory $\mathcal{E}^{\bullet \vee}$ of perfect amplitude $[-2,1]$ over $\mathcal{S}^{(P,F,r,n)}_{s,\text{FT}}(\tau')$.

1. Locally in the Zariski topology over $\mathcal{S}^{(P,F,r,n)}_{s,\text{HFT}}(\tau')$ there exists a perfect two-term deformation obstruction theory of perfect amplitude $[-1,0]$ which is obtained from the suitable local truncation of the pullback $\pi^* \mathcal{E}^{\bullet \vee}$.

2. This local theory defines a globally well-behaved virtual fundamental class over $\mathcal{S}^{(P,F,r,n)}_{s,\text{HFT}}(\tau')$.

**Proof:** Here we prove the first part of the theorem. Proving the second part of Theorem 9.11 requires a technical assumption which we will explain in the next section. As we showed before, since $\mathcal{S}^{(P,F,r,n)}_{s,\text{FT}}(\tau')$ is an Artin stack, the cotangent complex of $\mathcal{S}^{(P,F,r,n)}_{s,\text{FT}}(\tau')$ has a term in degree 1. By the canonical exact triangle of relative cotangent complexes in the derived category, we have:

$$\pi^* \mathbb{L}_{\mathcal{S}^{(P,F,r,n)}_{s,\text{FT}}(\tau')} \to \mathbb{L}_{\mathcal{S}^{(P,F,r,n)}_{s,\text{HFT}}(\tau')} \to \Omega_\pi \to \pi^* \mathbb{L}_{\mathcal{S}^{(P,F,r,n)}_{s,\text{FT}}(\tau')} [1], \quad (9.21)$$

By Theorem 9.5 $\mathcal{E}^{\bullet \vee} \xrightarrow{ob} \mathbb{L}_{\mathcal{S}^{(P,F,r,n)}_{s,\text{FT}}(\tau')}$ is a perfect deformation obstruction theory of amplitude $[-2,1]$ for $\mathcal{S}^{(P,F,r,n)}_{s,\text{FT}}(\tau')$, such that $h^0(\text{ob}), h^1(\text{ob})$ are isomorphisms and $h^{-1}(\text{ob})$ is an epimorphism.
Proposition 9.12. Let \( \mathcal{U} = \prod_\alpha \mathcal{U}_\alpha \) be an atlas of affine schemes for \( \mathcal{D}^{(P,F,r,n)}(\tau') \). Fix one of the maps \( q : \mathcal{U}_\alpha \to \mathcal{D}^{(P,F,r,n)}(\tau') \). The following isomorphism holds true in \( \mathcal{D}^h(\mathcal{U}_\alpha) \):

\[
\text{Hom}(q^*\Omega_\pi, q^*(\pi^*E^\bullet_{(P,F,r,n)}(\tau')[1])) \cong \text{Hom}(q^*\Omega_\pi, q^*(\pi^*L^\bullet_{(P,F,r,n)}(\tau')[1])).
\]

(9.22)

Hence, in particular it is true that there exists a nontrivial lift \( g_\alpha : q^*\Omega_\pi \to q^*(\pi^*E^\bullet_{(P,F,r,n)}(\tau')[1]). \)

Proof. Consider the exact triangle

\[
q^*(\pi^*E^\bullet) \xrightarrow{ob'_\alpha} q^*(\pi^*\mathbb{L}^\bullet_{(P,F,r,n)}(\tau')) \to \text{Cone}(ob'_\alpha)
\]

(9.23)

induced by pulling back the deformation obstruction theory in Theorem 9.5 via \( \pi \circ q : \mathcal{U}_\alpha \to \mathcal{D}^{(P,F,r,n)}(\tau') \). By Proposition 5.5 and the exact triangle in (9.21):

\[
\Omega_\pi \cong (gl_\pi(\mathbb{C}))^\vee \otimes \mathcal{O}_{\mathcal{D}^{(P,F,r,n)}(\tau')}.
\]

Hence \( q^*\Omega_\pi \cong (gl_\pi(\mathbb{C}))^\vee \otimes \mathcal{O}_{\mathcal{U}_\alpha} \). Now apply \( \text{Hom}^0(q^*\Omega_\pi, \cdot) \) to the exact triangle (9.23) and obtain

\[
\text{Hom}^0(q^*\Omega_\pi, \text{Cone}(ob'_\alpha)) \to \text{Hom}^0(q^*\Omega_\pi, q^*(\pi^*E^\bullet)[1]) \to \text{Hom}^0(q^*\Omega_\pi, \text{Cone}(ob'_\alpha)[1]).
\]

(9.24)

We prove the statement of the theorem by showing that

\[
\text{Hom}^0(q^*\Omega_\pi, \text{Cone}(ob'_\alpha)) \cong 0 \cong \text{Hom}^0(q^*\Omega_\pi, \text{Cone}(ob'_\alpha)[1]).
\]

(9.25)

Now consider the long exact sequence of cohomology induced by the exact triangle in (9.23)

\[
0 \to h^{-3}(q^*(\pi^*E^\bullet)) \to h^{-3}(q^*(\pi^*\mathbb{L}^\bullet)) \to h^{-3}(\text{Cone}(ob'_\alpha)) \cong h^{-2}(q^*(\pi^*E^\bullet))
\]

\[
\to h^{-2}(q^*(\pi^*\mathbb{L}^\bullet)) \to h^{-2}(\text{Cone}(ob'_\alpha)) \to h^{-1}(q^*(\pi^*E^\bullet)) \to h^{-1}(q^*(\pi^*\mathbb{L}^\bullet))
\]

\[
\to h^{-1}(\text{Cone}(ob'_\alpha)) \to h^0(q^*(\pi^*E^\bullet)) \cong h^0(q^*(\pi^*\mathbb{L}^\bullet)) \to h^0(\text{Cone}(ob'_\alpha))
\]

\[
\to h^1(q^*(\pi^*E^\bullet)) \cong h^1(q^*(\pi^*\mathbb{L}^\bullet)) \to h^1(\text{Cone}(ob'_\alpha))
\]

\[
\to h^2(q^*(\pi^*E^\bullet)) \to h^2(q^*(\pi^*\mathbb{L}^\bullet)) \to 0
\]

(9.26)

where we have used the fact that \( q^*(\pi^*\mathbb{L}^\bullet) \) and \( q^*(\pi^*E^\bullet) \) are perfect complexes of amplitudes \([-1,1]\) and \([-2,1]\) respectively and \( h^i(ob'_\alpha) \) is an isomorphism for \( i = 0, 1 \) and a surjection for \( i = -1 \). Hence we conclude that
Cone(obₜ) has cohomologies on degrees -2 and -3 only. Now use the fact that one can replace the complex Cone(obₜ) with a representative complex A* such that Aₖ = 0 for k ≥ -1. Now use the following lemma:

**Lemma 9.13.** If U is an affine scheme and A* is a complex with Aₖ = 0 for k ≥ -1, then Hom⁰(Oₜ, A*[l])  is 0 for all l ≥ 0.

**Proof.** We use the general fact that given complexes G and F, in order to compute the Grothendieck hypercohomology Hom¹(G, F), one replaces F with its injective resolution F → K*. Moreover replacing F with K* is equivalent with replacing G with P* such that P* → G is a projective resolution. Now use the fact that locally over U, Oₜ is given as a free and in particular projective module hence its projective resolution consists of one term and one can use P* by construction.\(\square\)

Now use the fact that by construction \(q^*Ω_{\pi} \cong (gl_r(\mathbb{C}))^v \otimes O_{U}\) and apply the result of Lemma 9.13 by replacing \(O_{U}\) with \(q^*Ω_{\pi}\) and obtain the vanishings in (9.25). This finishes the proof of Proposition 9.12.\(\square\)

**Lemma 9.14.** Let \(q: U_\alpha \to \mathcal{H}_{s, HFT}^{(P, r, n)}(\tau')\) and \(q': U_\beta \to \mathcal{H}_{s, HFT}^{(P, r, n)}(\tau')\) be given as fixed affine charts over \(\mathcal{H}_{s, HFT}^{(P, r, n)}(\tau')\) such that the isomorphism in Proposition 9.12 holds true over \(U_\alpha\) and \(U_\beta\). Let \(f_\alpha : U_\alpha \times_{q \times q'} U_\beta \to U_\alpha\) and \(f_\beta : U_\alpha \times_{q \times q'} U_\beta \to U_\beta\) be the corresponding projection to \(U_\alpha\) and \(U_\beta\). Then

\[
\text{Hom}^0(f_\alpha(q^*Ω_{\pi}, q^*(π^*E^v))[1]) \cong \text{Hom}^0(f_\beta(q^*Ω_{\pi}, q^*(π^*E^v))[1]).
\]

Moreover the same statement is true for \(f_\alpha\) and \(q\) replaced by \(f_\beta\) and \(q'\).

**Proof.** Because \(\mathcal{H}_{s, HFT}^{(P, r, n)}(\tau')\) is a quasi-projective scheme (Remark 6.7) then an intersection of affine subschemes of \(\mathcal{H}_{s, HFT}^{(P, r, n)}(\tau')\) is affine. Now apply Proposition 9.12 to \(U_\alpha \times_{q \times q'} U_\beta\).\(\square\)

In what follows in order to save space we denote \(\mathcal{H}_{FT} := \mathcal{H}_{s, FT}^{(P, r, n)}(\tau')\) and \(\mathcal{H}_{HFT} := \mathcal{H}_{s, HFT}^{(P, r, n)}(\tau')\). Now fix \(U_\alpha\). By the local existence of the map \(g_\alpha\) in Proposition 9.12 there exists a commutative diagram over \(U_\alpha\):

\[
\begin{array}{ccccccccc}
π^*E^v |_{U_\alpha} & \longrightarrow & \text{Cone}(g_\alpha)[-1] & \longrightarrow & Ω_{\pi} |_{U_\alpha} & \longrightarrow & \pi^*E^v[1] |_{U_\alpha} & \longrightarrow & \text{Cone}(g_\alpha) \\
\mid & & \mid & & \mid & & \mid & & \\
π^*(ob) |_{U_\alpha} & \longrightarrow & ob' & \longrightarrow & \text{id} & \longrightarrow & π^*ob[1] |_{U_\alpha} & \longrightarrow & π^*ob[1] |_{U_\alpha} \\
\end{array}
\]

\[
\begin{array}{ccccccccc}
π^*\mathbb{L}_{\mathcal{H}_{FT}} |_{U_\alpha} & \longrightarrow & \mathbb{L}_{\mathcal{H}_{HFT}} |_{U_\alpha} & \longrightarrow & Ω_{\pi} |_{U_\alpha} & \longrightarrow & π^*\mathbb{L}_{\mathcal{H}_{FT}}[1] |_{U_\alpha} & \longrightarrow & \mathbb{L}_{\mathcal{H}_{HFT}}[1] |_{U_\alpha} \\
\end{array}
\]

(9.27)
Lemma 9.15. The map $ob' : \text{Cone}(g_a)[-1] \to \mathbb{L}^{\bullet}_{\Delta_{s,HFT}^{(P_F,t,r,n)}}(\tau') |_{U_a}$ defines a perfect 3-term deformation obstruction theory of amplitude $[-2,0]$ for $\Delta_{s,HFT}^{(P_F,t,r,n)}(\tau')$ over $U_a$.

Proof. We show that $\text{Cone}(g)[-1]$ is concentrated in degrees $-2,-1$ and 0, moreover $h^0(ob')$ is an isomorphism and $h^{-1}(ob')$ is an epimorphism. The proof uses the long exact sequence of cohomologies. For $h^{-1}(ob')$ one obtains:

$$
0 \longrightarrow h^{-1}(\pi^*E^\bullet |_{U_a}) \cong \pi^*h^{-1}(E^\bullet |_{U_a}) \longrightarrow h^{-1}(\text{Cone}(g_a)[-1]) \longrightarrow 0
$$

the top horizontal isomorphism is due to the fact that

$$
\text{Cone}(g_a)[-1] : \pi^*E^{-2} \to \pi^*E^{-1} \to \pi^*E^0 \oplus \Omega_\pi \to \pi^*E^1,
$$

where $E^i$ correspond to the terms of the complex $E^\bullet |_{U_a}$. The vanishing on the left and right of the top and bottom rows of (9.28) are due to the fact that $\Omega_\pi$ is a sheaf concentrated in degree zero. By Theorem 9.5 the second vertical map (from left) is a surjection and by commutativity of the diagram (9.28) the map $h^{-1}(ob')$ is surjective. In degrees 0 and 1 one obtains:

$$
0 \longrightarrow \pi^*h^0(E^\bullet |_{U_a}) \longrightarrow h^0(\text{Cone}(g_a)[-1]) \longrightarrow \Omega_\pi |_{U_a} \longrightarrow \pi^*h^1(E^\bullet |_{U_a}) \longrightarrow h^1(\text{Cone}(g)[{-1}]) \longrightarrow 0
$$

In this diagram, $h^1(\mathbb{L}^{\bullet}_{\Delta_{s,HFT}^{(P_F,t,r,n)}}(\tau') |_{U_a}) \cong 0$ since over $\Delta_{s,HFT}^{(P_F,t,r,n)}(\tau')$ the truncated cotangent complex does not have cohomology in degree 1. Moreover $\pi^*h^1(ob') |_{U_a}$ is an isomorphism by Theorem 9.5. Hence $h^1(ob') \cong 0$. Moreover by Theorem 9.5, $\pi^*h^0(ob')$ is an isomorphism, hence by the commutativity of the diagram (9.29), $h^0(ob')$ is an isomorphism. This finishes the proof of Lemma 9.15.

In order to obtain a perfect deformation obstruction theory of amplitude $[-1,0]$, one needs to truncate the complex $\text{Cone}(g_a)[-1]$ so that it does not
have any cohomology in degree $-2$. The self-duality of $E^\bullet$ gives a diagram of morphisms in the derived category:

\[
\begin{array}{ccc}
E^\bullet & \xrightarrow{\cong} & E^\bullet[1] \\
\downarrow & & \downarrow \\
T_\pi|_{u_\alpha}[1] & & \text{Cone}(g_\alpha)
\end{array}
\]

(9.30)

**Lemma 9.16.** The natural map

\[
\text{Hom}_{D(U_\alpha)}^0(Cone(g_\alpha), T_\pi|_{u_\alpha}[1]) \to \text{Hom}_{D(U_\alpha)}^0(E^\bullet, T_\pi|_{u_\alpha}[1]) \quad (9.31)
\]

is an isomorphism.

**Proof.** Note that $U_\alpha$ is affine and $T_\pi|_{u_\alpha}[1] \cong O_{U_\alpha}^{\text{dim}(U)}[1]$, so the statement reduces to knowing that $H^1(E^\bullet[1]) \to H^1(\text{Cone}(g_\alpha)^\vee)$ is an isomorphism. This follows since $E^\bullet[1] \to \text{Cone}(g_\alpha)$ is an isomorphism on $H^{-1}$ as shown in diagram (9.28).

By Lemma 9.16, $g_\alpha^\vee$ factors through a map $\text{Cone}(g_\alpha) \to T_\pi|_{u_\alpha}[1]$ which is unique up to homotopy. We make a choice of such lift and denote it again by $g_\alpha^\vee$. Now consider the exact triangle

\[
\text{Cone}(g_\alpha)[-1] \to \text{Cone}(g_\alpha)[-1] \xrightarrow{g_\alpha^\vee} T_\pi|_{u_\alpha} \to \text{Cone}(g_\alpha)^\vee.
\]

(9.32)

Denote $G^\bullet|_{U_\alpha} := \text{Cone}(g_\alpha)[-1]$. In order to finish the proof of Theorem 9.11, we need one more lemma.

**Lemma 9.17.** The complex $G^\bullet|_{U_\alpha}$ defines a perfect deformation obstruction theory of amplitude $[-1,0]$ for $U_\alpha$.

**Proof.** : By construction

\[
G^\bullet|_{U_\alpha} := \pi^*E^{-2} \to \pi^*E^{-1} \oplus T_\pi|_{U_\alpha} \to \pi^*E^0 \oplus \Omega_\pi|_{U_\alpha} \to \pi^*E^1.
\]

This complex has no cohomology in degree 1 and $-2$, i.e. in the following commutative diagram, the top row is quasi-isomorphic to the bottom row:

\[
\begin{array}{cccccccc}
\pi^*E^{-2} & \xrightarrow{d'} & \pi^*E^{-1} \oplus T_\pi|_{U_\alpha} & \xrightarrow{\pi^*E^0 \oplus \Omega_\pi|_{U_\alpha} \xrightarrow{d} \pi^*E^1} \\
\downarrow & & \downarrow & & \downarrow \\
0 & \xrightarrow{\text{Coker}(d')} & \text{Ker}(d) & \xrightarrow{0}
\end{array}
\]

(9.33)
moreover there exists a morphism \( G^\bullet \mid_{U_\alpha} \to \mathbb{L}_{\mathcal{S}_{s,HFT}^{(P_F,r,n)}(\tau')}^\bullet \mid_{U_\alpha} \) which is given by the composition of \( G^\bullet \mid_{U_\alpha} \to \text{Cone}(g)[-1] \) and \( \text{Cone}(g)[-1] \to \mathbb{L}_{\mathcal{S}_{s,HFT}^{(P_F,r,n)}(\tau')}^\bullet \mid_{U_\alpha} \). By Lemma 9.15 this map satisfies the condition of being a deformation obstruction theory. This finishes the first part of the proof of Theorem 9.11.

Remark 9.18. The difference between the construction in Theorem 9.11 and the construction in [RR09] (2.3) for rank 1 triples (stable pairs) is that, for theory of stable pairs, the terms \( \text{Hom}(I^\bullet,I^\bullet)_0 \) and \( \text{Ext}^3(I^\bullet,I^\bullet)_0 \) are equal to zero by stability, however in higher rank, the stability condition does not ensure that the stable objects, \( I^\bullet \), in the derived category are simple objects. Hence applying Theorem 9.11 is necessary to obtain a perfect deformation obstruction theory of amplitude \([-1,0]\) for \( U_\alpha \).

10 Virtual fundamental class for HFT

In Proposition 9.12 we proved the local existence of a map in the derived category \( \Omega_\pi \mid_{U_\alpha} \to \pi^*E^\bullet\mid[1] \mid_{U_\alpha} \), where \( U_\alpha \) were open subsets given as elements of a smooth cover \( \mathcal{U} = \bigsqcup U_\alpha \) of \( \mathcal{S}_{s,HFT}^{(P_F,r,n)}(\tau') \) (Noseda in [Nos07] refers to charts in Proposition 9.12 as charts with lifting property). We locally constructed the perfect deformation obstruction theory in the first part of Theorem 9.11. To prove the second part of Theorem 9.11 we need to prove that the local virtual cycles obtained from Theorem 9.11 glue to each other and define a well-behaved global virtual fundamental cycle.

10.1 Semi-perfect obstruction theories

Followed by constructions in [BF97] (Definition 3.7), one needs to choose a local embedding over \( \mathcal{S}_{s,HFT}^{(P_F,r,n)}(\tau') \). Then one constructs the normal cone associated to a perfect local deformation-obstruction theory of amplitude \([-1,0]\) over this local embedding and proves that this normal cone is independent of the local embedding, i.e. it remains invariant under the base change. The base-change invariance enables one to glue the local normal cones constructed over each local embedding and obtain a global cone stack. Essentially a global virtual fundamental class is constructible from a global normal cone stack [BF97]. The gluability of the so-called local normal cone stacks depends on whether the local deformation obstruction theories over each chart satisfy the condition of being given as a “semi-perfect obstruction theory” in the sense of Chang-Li [Hua11]. To continue we include some
background about the semi-perfect obstruction theories from [Hua11]. First we state the definition of the numerical equivalence.

**Definition 10.1.** [Hua11] (Definition 2.5) Let \( i: T \to T' \) be a closed subscheme with \( T' \) local Artinian. Let \( \mathcal{I} \) be the ideal sheaf of \( T \) in \( T' \) and let \( m \) be the ideal sheaf of the closed point of \( T' \). We call \( i \) a small extension if \( \mathcal{I} \cdot m = 0 \). Now let \( \mathcal{M} \) be an Artin stack and \( X \to \mathcal{M} \) a representable morphism of a DM stack to an Artin stack. Let \( \mathcal{U} = \coprod_{\alpha} \mathcal{U}_{\alpha} \in \Lambda \) be a DM cover of \( X \) by affine schemes. Consider a small extension \((T, T', \mathcal{I}, m)\) that fits into a commutative square

\[
\begin{array}{ccc}
T & \xrightarrow{g} & \mathcal{U}_{\alpha} \\
\downarrow i & & \downarrow \\
T' & \longrightarrow & \mathcal{M}
\end{array}
\]

(10.1)

so that the image of \( g \) contains a closed point \( p \in \mathcal{U}_{\alpha} \). Finding a morphism \( g': T' \to \mathcal{U}_{\alpha} \) that commutes with the arrows in (10.1) is called “infinitesimal lifting problem of \( \mathcal{U}_{\alpha}/\mathcal{M} \) at \( p \).”

**Notation:** Given a perfect relative deformation obstruction theory \( G^\bullet \to L^\bullet_{X/\mathcal{M}} \) denote by \( \phi: G^\bullet_{\mathcal{U}} \to L^\bullet_{\mathcal{U}/\mathcal{M}} \) the restriction of the deformation obstruction theory to \( \mathcal{U} \).

**Definition 10.2.** Given a \( \mathcal{U} \to \mathcal{M} \) let \( \phi: G^\bullet_{\mathcal{U}} \to L^\bullet_{\mathcal{U}/\mathcal{M}} \) be a perfect obstruction theory. For the infinitesimal lifting problem in Definition 10.1 we call the image

\[
ob(\phi, g, T, T') := H^1(\phi^\vee)(\omega(g, T, T')) \in \text{Ext}^1(g^*G^\bullet_{\mathcal{U}}, \mathcal{I}) = \text{Ob}(\phi, p) \otimes \mathcal{I}
\]

(10.2)

the obstruction class (of \( \phi \)) to the lifting problem.

**Definition 10.3.** [Hua11] (Definition 2.9) Given two (local) deformation obstruction theories \( \phi: G^\bullet_{\mathcal{U}} \to L^\bullet_{\mathcal{U}/\mathcal{M}} \) and \( \phi': G'^\bullet_{\mathcal{U}} \to L'^\bullet_{\mathcal{U}/\mathcal{M}} \) over \( \mathcal{U} \) as in Definition 10.2 we call them \( \nu \)-equivalent if there exists an isomorphism of sheaves:

\[
\psi: H^1(G^\bullet_{\mathcal{U}}) \to H^1(G'^\bullet_{\mathcal{U}})
\]

(10.3)

so that for every closed point \( p \in \mathcal{U}_{\alpha} \) and any infinitesimal lifting problem of \( \mathcal{U}_{\alpha}/\mathcal{M} \) at \( p \) (as in Definition 10.1) we have

\[
\psi \mid_p (ob(\phi, g, T, T')) = ob(\phi', g, T, T') \in \text{Ob}(\phi', p) \otimes_k \mathcal{I}.
\]
Now consider $\mathcal{H}_{\text{HFT}} := \mathcal{H}_{s,\text{HFT}}^{(p_F,r,n)}(\tau')$. Let $U_\alpha$ and $U_\beta$ be given as two charts over $\mathcal{H}_{s,\text{HFT}}^{(p_F,r,n)}(\tau')$ with the lifting property as in Theorem 9.11. Let $\phi_\alpha : \mathcal{G}_\alpha^* \to L\mathcal{U}_\mathcal{A}/\mathcal{M}$ and $\phi_\beta : \mathcal{G}_\beta^* \to L\mathcal{U}_\mathcal{B}/\mathcal{M}$. Moreover, let $U_{\alpha\beta} = U_\alpha \cap U_\beta$, $f_\alpha : U_{\alpha\beta} \hookrightarrow U_\alpha$ and $f_\beta : U_{\alpha\beta} \hookrightarrow U_\beta$.

**Proposition 10.4.** Let $f_\alpha^*\phi_\alpha$ and $f_\beta^*\phi_\beta$ denote the pullback of $\phi_\alpha$ and $\phi_\beta$ to $U_{\alpha\beta}$. Then $f_\alpha^*\phi_\alpha$ and $f_\beta^*\phi_\beta$ are $\nu$-equivalent over $U_{\alpha\beta}$.

**Proof.** We have to show that given a diagram

$$
\begin{array}{c}
T \\
\downarrow T' \\
\mathcal{H}_{\text{HFT}}
\end{array}
\xrightarrow{g_{\alpha\beta}} \begin{array}{c}
U_{\alpha\beta} \\
U_{\alpha\beta}
\end{array}
$$

there exists a map $\psi : H^1(f_\alpha^*\mathcal{G}_\alpha^*)^\vee \xrightarrow{\sim} H^1(f_\beta^*\mathcal{G}_\beta^*)^\vee$ such that given a class $\text{ob}(f_\alpha^*\phi_\alpha, g_{\alpha\beta}, T, T') \in H^1(f_\beta^*(L\mathcal{H}_{\text{HFT}}|_{U_\alpha})^\vee)$ (Look at diagram (10.4)) and for every point $p \in U_{\alpha\beta}$ we have

$$
\psi|_p \text{ob}(f_\alpha^*\phi_\alpha, g_{\alpha\beta}, T, T') = \text{ob}(f_\beta^*\phi_\beta, g_{\alpha\beta}, T, T').
$$

Apply the result of Proposition 9.12 over $U_\alpha$ and $U_\beta$ and obtain unique isomorphisms as in (9.22) over $U_\alpha$ and $U_\beta$. Now use the fact that $U_{\alpha\beta}$ is affine and pull back the obtained isomorphisms via $f_\alpha$ and $f_\beta$ to $U_{\alpha\beta}$ and obtain a unique isomorphism

$$
\text{Hom}(\Omega_\pi|_{U_{\alpha\beta}}, E_{\alpha\beta}^*) \cong \text{Hom}(\Omega_\pi|_{U_{\alpha\beta}}, L_{\alpha\beta}^*).
$$

Now by the uniqueness property there exists an isomorphism in $D^b(U_{\alpha\beta})$ given by

$$
\kappa_{\alpha\beta} : f_\alpha^* E_{\alpha}^* \to f_\beta^* E_{\beta}^*.
$$

By assumption $U_\alpha$ and $U_\beta$ are given as charts with lifting property (Theorem 9.11), hence there exists lifts $\text{Hom}(\Omega_\pi|_{U_\alpha}, E_{\alpha}^*[1])$ and $\text{Hom}(\Omega_\pi|_{U_\beta}, E_{\beta}^*[1])$ given by $g_\alpha : \Omega_\pi|_{U_\alpha} \to E_{\alpha}^*[1]$ and $g_\beta : \Omega_\pi|_{U_\beta} \to E_{\beta}^*[1]$ over $U_\alpha$ and $U_\beta$ respectively. Now consider the pullbacks $f_\alpha^*\Omega_\pi|_{U_\alpha}[-1] \to f_\alpha^* E_{\alpha}^*$ and $f_\beta^*\Omega_\pi|_{U_\beta}[-1] \to f_\beta^* E_{\beta}^*$ and note that by Proposition 9.12 $f_\alpha^* g_\alpha$ and $f_\beta^* g_\beta$ are homotopic to each other over $U_{\alpha\beta}$ and satisfy the equation:

$$
f_\alpha^* g_\alpha - f_\beta^* g_\beta = d \circ h_{\alpha\beta} + h_{\alpha\beta} \circ d.
$$

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where $h_{\alpha\beta}$ is given as a choice of homotopy. Now take the cone of $f_\alpha^*g_\alpha$ and $f_\beta^*g_\beta$ and obtain the following commutative diagram:

$$
\begin{array}{cccccc}
\text{Cone}(f_\alpha^*g_\alpha)[-1] & \to & f_\alpha^*\Omega_\pi|\mathcal{U}_\alpha & \xrightarrow{f_\alpha^*g_\alpha} & f_\alpha^*E^*_\alpha[1] & \to & \text{Cone}(f_\alpha^*g_\alpha) \\
\downarrow J_{\alpha\beta}[-1] & & \downarrow \text{id} & & \downarrow \text{id} & & \downarrow J_{\alpha\beta} \\
\text{Cone}(f_\beta^*g_\beta)[-1] & \to & f_\beta^*\Omega_\pi|\mathcal{U}_\beta & \xrightarrow{f_\beta^*g_\beta} & f_\beta^*E^*_\beta[1] & \to & \text{Cone}(f_\beta^*g_\beta),
\end{array}
$$

(10.5)

where $J_{\alpha\beta} := \begin{pmatrix} \text{id} & h_{\alpha\beta} \\ 0 & \text{id} \end{pmatrix}$. Since the first and the second rows in diagram (10.5) are given by exact triangles one computes the long exact sequence of cohomologies and obtains the following commutative diagram:

$$
\begin{array}{cccccccc}
\cdots & \to & H^i(f_\alpha^*\Omega_\pi|\mathcal{U}_\alpha[-1]) & \xrightarrow{\text{id}} & H^i(f_\alpha^*E^*_\alpha) & \xrightarrow{\text{id}} & H^i(\text{Cone}(f_\alpha^*g_\alpha))[-1] & \to & \cdots \\
& & \downarrow \text{id} & & \downarrow \text{id} & & \downarrow H^i(J_{\alpha\beta})[-1] & \\
\cdots & \to & H^i(f_\beta^*\Omega_\pi|\mathcal{U}_\beta[-1]) & \xrightarrow{\text{id}} & H^i(f_\beta^*E^*_\beta) & \xrightarrow{\text{id}} & H^i(\text{Cone}(f_\beta^*g_\beta))[-1] & \to & \cdots,
\end{array}
$$

(10.6)

Now use [ea87] (Proposition 4.10) and conclude that the left, middle and right squares in (10.6) are commutative square diagrams for all $i$. By computing the cohomologies in the level of $i = -1$ one obtains:

$$
\begin{array}{cccccc}
0 & \to & H^{-1}(f_\alpha^*E^*_\alpha) & \xrightarrow{\cong \rho_1} & H^{-1}(\text{Cone}(f_\alpha^*g_\alpha)[-1]) & \to & 0 \\
0 & \to & H^{-1}(f_\beta^*E^*_\beta) & \xrightarrow{\cong \rho_2} & H^{-1}(\text{Cone}(f_\beta^*g_\beta)) & \to & 0,
\end{array}
$$

(10.7)

where the vanishings on the ends are due to the fact that $H^i(f_\alpha^*\Omega_\pi|\mathcal{U}_\alpha[-1]) \cong 0$ and $H^i(f_\beta^*\Omega_\pi|\mathcal{U}_{\beta\alpha}[-1]) \cong 0$ for $i = -1,0$. Hence we conclude that by commutativity of the middle square $H^{-1}(J_{\alpha\beta}[-1])$ is an isomorphism of cohomologies and moreover, given any $\nu \in H^{-1}(\text{Cone}(f_\alpha^*g_\alpha)[-1])$:

$$
\text{id} \circ \rho_1^{-1}(\nu) = \rho_2^{-1} \circ H^{-1}(J_{\alpha\beta}[-1])(\nu).
$$

(10.8)

Note that given a choice of homotopy $h_{\alpha\beta}^\gamma$ satisfying

$$
f_\alpha^*g_\alpha - f_\beta^*g_\beta = d \circ h_{\alpha\beta}^\gamma = h_{\alpha\beta}^\gamma \circ d
$$

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and via restriction of the exact triangle in (9.32) to $U_{\alpha\beta}$ and similar to the above procedure we obtain a commutative diagram:

$$
\begin{array}{cccc}
\text{Cone}(f^*_\alpha g^\vee_\alpha)[-1] & \longrightarrow & \text{Cone}(f^*_\alpha g_\alpha)[-1] & \longrightarrow & \text{Cone}(f^*_\alpha g^\vee_\alpha) \\
\big\uparrow J^\vee_{\alpha\beta}[-1] & & \big\uparrow J_{\alpha\beta}[-1] & & \big\uparrow J^\vee_{\alpha\beta} \\
\text{Cone}(f^*_\beta g^\vee_\beta)[-1] & \longrightarrow & \text{Cone}(f^*_\beta g_\beta)[-1] & \longrightarrow & \text{Cone}(f^*_\beta g^\vee_\beta).
\end{array}
$$

Similarly obtain a commutative diagram induced by the long exact sequences of cohomologies:

$$
\begin{array}{cccc}
H^i(\text{Cone}(f^*_\alpha g^\vee_\alpha)[-1]) & \longrightarrow & H^i(\text{Cone}(f^*_\alpha g_\alpha)[-1]) & \longrightarrow & H^i(\text{Cone}(f^*_\alpha g^\vee_\alpha)) \\
\big\uparrow H^i(J^\vee_{\alpha\beta}[-1]) & & \big\uparrow H^i(J_{\alpha\beta}[-1]) & & \big\uparrow H^i(J^\vee_{\alpha\beta}) \\
H^i(\text{Cone}(f^*_\beta g^\vee_\beta)[-1]) & \longrightarrow & H^i(\text{Cone}(f^*_\beta g_\beta)[-1]) & \longrightarrow & H^i(\text{Cone}(f^*_\beta g^\vee_\beta)).
\end{array}
$$

Now use [ea87] (Proposition 4.10) and conclude that the left, middle and right squares in (10.6) are commutative square diagrams for all $i$ and in particular for $i = -1$:

$$
\begin{array}{ccc}
0 & \longrightarrow & H^{-1}(\text{Cone}(f^*_\alpha g^\vee_\alpha)[-1]) \\
& \big\uparrow q_1 & \big\uparrow H^{-1}(J^\vee_{\alpha\beta}[-1]) \\
0 & \longrightarrow & H^{-1}(\text{Cone}(f^*_\beta g^\vee_\beta)[-1]) \\
& \big\uparrow q_2 & \big\uparrow H^{-1}(J_{\alpha\beta}[-1]) \\
0 & \longrightarrow & H^{-1}(\text{Cone}(f^*_\beta g^\vee_\beta)[-1])
\end{array}
$$

Hence by commutativity of the left square and the fact that $H^{-1}(J_{\alpha\beta}[-1])$ is an isomorphism, then $H^{-1}(J^\vee_{\alpha\beta}[-1])$ is an isomorphism and moreover, for any $\mu \in H^{-1}(\text{Cone}(f^*_\alpha g^\vee_\alpha)[-1])$ we have:

$$
H^{-1}(J^\vee_{\alpha\beta}[-1]) \circ q_1(\mu) = q_2 \circ H^{-1}(J_{\alpha\beta}[-1])(\mu)
$$

Now take an element $\mu \in H^{-1}(\text{Cone}(f^*_\alpha g^\vee_\alpha)[-1])$ and note that by (10.8) and (10.12) we have:

$$
id \circ \rho_1^{-1} \circ H^{-1}(J_{\alpha\beta}[-1]) \circ q_1(\mu) = \rho_2^{-1} \circ H^{-1}(J_{\alpha\beta}[-1]) \circ q_2 \circ H^{-1}(J^\vee_{\alpha\beta}[-1])(\mu)
$$

Moreover $L_{9\text{HFT}}^\bullet$ and $L_{9\text{FT}}^\bullet$ satisfy the condition that $H^{-1}(L_{9\text{HFT}}^\bullet) \cong H^{-1}(L_{9\text{FT}}^\bullet)$ hence one easily observes that there exist maps $\lambda_1 : H^{-1}(f^*_\alpha g^\vee_\alpha) \rightarrow H^{-1}(f^*_\alpha L_{9\text{HFT}}^\bullet | u_\alpha$
and \( \lambda_2 : H^{-1}(f_\beta^* E^*) \to H^{-1}(f_\beta^*(\mathcal{L}_{HFT}^\bullet|_{\mathcal{U}_\beta})) \) such that the following diagram commutes:

\[
\begin{array}{ccc}
H^{-1}(f_\alpha^* E^*) & \xrightarrow{\lambda_1} & H^{-1}(f_\alpha^*(\mathcal{L}_{HFT}^\bullet|_{\mathcal{U}_\alpha})) \\
\text{id} & & \text{id} \\
H^{-1}(f_\beta^* E^*) & \xrightarrow{\lambda_2} & H^{-1}(f_\beta^*(\mathcal{L}_{HFT}^\bullet|_{\mathcal{U}_\beta})).
\end{array}
\]

(10.14)

Now by (10.14) and (10.13) it is seen that given \( \mu \in H^{-1}(\text{Cone}(f_\alpha^* g_\alpha^\vee)[-1]) \) we obtain an identity

\[
\text{id} \circ \lambda_1 \circ \text{id} \circ \rho_1^{-1} \circ H^{-1}(J_{\alpha\beta}[-1]) \circ q_1(\mu) = \lambda_2 \circ \text{id} \circ \rho_2^{-1} \circ H^{-1}(J_{\alpha\beta}[-1]) \circ q_2 \circ H^{-1}(J_{\alpha\beta}^\vee[-1])(\mu).
\]

(10.15)

Let \( \psi^\vee := \text{id} \circ \rho_2^{-1} \circ H^{-1}(J_{\alpha\beta}[-1]) \circ q_2 \circ H^{-1}(J_{\alpha\beta}^\vee[-1]) \). So far we have seen that in the level of \( H^{-1} \) cohomology there exists a map \( \psi^\vee : H^{-1}(\text{Cone}(f_\alpha^* g_\alpha^\vee)[-1]) \xrightarrow{\cong} H^{-1}(\text{Cone}(f_\beta^* g_\beta^\vee)[-1]) \) such that \( \lambda_2 \circ \text{Im}(\psi^\vee) = \text{Im}(\lambda_1) \). Recall that by our notation \( G_{\alpha}^\bullet := \text{Cone}(g_\alpha^\vee)[-1] \) and \( G_{\beta}^\bullet|_{\mathcal{U}_\beta} := \text{Cone}(g_\beta^\vee)[-1] \). Now dualize the construction and conclude that there exists a map \( \psi : H^1(f_\alpha^* G_{\alpha}^\bullet)^\vee \xrightarrow{\cong} H^1(f_\beta^* G_{\beta}^\bullet)^\vee \) such that given a class \( \text{ob}(f_\alpha^* \phi_\alpha, g_{\alpha\beta}, T, T') \in H^1(f_\alpha^*(\mathcal{L}_{HFT}^\bullet|_{\mathcal{U}_\alpha})) \) (Look at diagram (10.4)) and for every point \( p \in \mathcal{U}_{\alpha\beta} \) we have

\[
\psi|_p \text{ob}(f_\alpha^* \phi_\alpha, g_{\alpha\beta}, T, T') = \text{ob}(f_\beta^* \phi_\beta, g_{\alpha\beta}, T, T').
\]

This finishes the proof of Proposition 10.4.

\[
\square
\]

**Definition 10.5.** [Hua11](Definition 3.1). Let \( X \) be a DM stack of finite type over an Artin stack \( \mathcal{M} \). A semi perfect obstruction theory over \( X \to \mathcal{M} \) consists of an étale covering \( \mathcal{U} = \coprod_{\alpha \in \Lambda} \mathcal{U}_\alpha \) of \( X \) by schemes, and a truncated perfect relative obstruction theory

\[
\phi_\alpha : G_{\alpha}^\bullet \to \mathcal{L}_{\mathcal{U}_\alpha/\mathcal{M}} \n
\]

for each \( \alpha \in \Lambda \) such that

1. for each \( \alpha, \beta \) in \( \Lambda \) there is an isomorphism

\[
\psi_{\alpha\beta} : H^1(G_{\alpha}^\bullet|_{\mathcal{U}_{\alpha\beta}}) \xrightarrow{\cong} H^1(G_{\beta}^\bullet|_{\mathcal{U}_{\alpha\beta}})
\]

so that the collection \( (H^1(G_{\alpha}^\bullet), \psi_{\alpha\beta}) \) forms a descent datum of sheaves.

2. For any pair \( \alpha, \beta \in \Lambda \) the obstruction theories \( \phi_\alpha|_{\mathcal{U}_{\alpha\beta}} \) and \( \phi_\beta|_{\mathcal{U}_{\alpha\beta}} \) are \( \nu \)-equivalent.

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The condition (1) above, that the \( \nu \)-equivalences we have constructed induce a descent datum of sheaves on \( H^1 \) requires that we carefully choose homotopies \( h_{\alpha \beta} \) and \( h_{\alpha \beta}^\vee \) on \( U_{\alpha \beta} \) so that the induced composite quasi-isomorphisms \( \psi_{\gamma \alpha} \circ \psi_{\beta \gamma} \circ \psi_{\alpha \beta} \) induce the identity maps on \( H^1 \). We believe that when \( X \) is given as local \( \mathbb{P}^1 \) it is possible to give a rigorous construction of such homotopical maps [She11]. Thus for the purpose of calculations that we intend to carry out in this article, for now we assume:

**Assumption 10.6.** The homotopies \( h_{\alpha \beta} \) and \( h_{\alpha \beta}^\vee \) can be chosen so that the collection \( (H^1(G^\bullet_{\alpha}), \psi_{\alpha \beta}) \) forms a descent datum of sheaves.

**Theorem 10.7.** Suppose that Assumption [10.6] holds. Then the local deformation obstruction theory obtained in Proposition [9.17] defines a semi-perfect obstruction theory over \( \mathcal{F}^{(P_F, r, n)}_{s, \text{HFT}}(\tau') \).

**Proof.** For part (2) of Definition [10.5] apply Proposition [10.4] and conclude that \( \phi_\alpha |_{U_{\alpha \beta}} = f^*_{\alpha} \phi_\alpha \) and \( \phi_\beta |_{U_{\alpha \beta}} = f^*_{\beta} \phi_\beta \) are \( \nu \)-equivalent. To prove part (1), first apply Proposition [10.4] and obtain the map

\[
psi_{\alpha \beta} : H^1(G^{\bullet \vee}_{\alpha} |_{U_{\alpha \beta}}) \xrightarrow{\cong} H^1(G^{\bullet \vee}_{\beta} |_{U_{\alpha \beta}}).
\]

Now by Assumption [10.6] \( (H^1(G^{\bullet \vee}_{\alpha}), \psi_{\alpha \beta}) \) forms a descent datum. This completes the proof. \( \square \)

**Remark 10.8.** The assumption that the descent condition holds should, morally speaking, be unnecessary. The local models \( G^\bullet_{\alpha} \) can always be glued up to higher homotopies, and thus should always give an \( \infty \)-stack in which the virtual normal cone lives. We expect that in the future a good intersection theory for \( \infty \)-stacks would allow one to construct a virtual cycle using this \( \infty \)-stack. Such a construction is beyond the scope of the present article, however.

Since the deformation obstruction theory constructed in Section [9.3] satisfies the condition of being given as a semi-perfect obstruction theory in the sense of [Hua11], now we apply [Hua11] (Lemma 3.3, Proposition 3.4, Proposition 3.8) and conclude that the local virtual cycles obtained from each locally truncated deformation obstruction theory (Lemma [9.17]) glue together and define a globally well-behaved virtual fundamental class of degree zero over \( \mathcal{F}^{(P_F, r, n)}_{s, \text{HFT}}(\tau') \). This finishes the proof of the second part of Theorem [9.11].
Remark 10.9. It is important to note that essentially (as we will show later) the result of our calculations do not depend on the choice of the homotopy maps \( h_{\alpha\beta} \) and \( h_{\alpha\beta}' \) in Assumption 10.6. In other words, the existence of the so-called well-defined homotopy maps will guarantee the existence of a theory of highly frozen triples for Calabi-Yau threefolds but no matter what choice of homotopy maps we make, it does not have any effect on the value of the numerical invariants which we calculate in this theory using the equivariant computations.

11 Classification of torus-fixed HFT over a toric Calabi-Yau threefold

Let \( X \) be given as total space of \( \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \to \mathbb{P}^1 \). Consider the ample line bundle over \( X \) given by \( \mathcal{O}_X(1) \). Recall the earlier discussions where \( H(\mathbb{P}_F, r, n) \), \( H_{\text{FT}}(\tau') \) and \( H(\mathbb{P}_F, r, n) \), \( \text{FT}(\tau') \) were defined in Definition 7.1. It is easily seen that when \( X = \text{Tot}(\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \to \mathbb{P}^1) \) then \( H(\mathbb{P}_F, r, n) \), \( H_{\text{FT}}(\tau') = H_{\text{FT}}(\mathbb{P}_F, r, n) \), \( \text{FT}(\tau') = \text{FT}(\mathbb{P}_F, r, n) \), hence in this section we switch back to our earlier notation and use \( H_{\text{FT}}(\mathbb{P}_F, r, n) \) instead. Note that a point in \( M_{\text{FT}}(\mathbb{P}_F, r, n) \) or \( M_{\text{FT}}(\mathbb{P}_F, r, n) \) is represented by a stable highly frozen or frozen triple respectively such that the Hilbert polynomial of \( F \) is equal to \( P_F \). However in the setting of this section \( X \) is given as a toric non-compact variety and the Hilbert polynomial of \( F \) is not well-defined. Therefore in order to define stability, we use the geometric stability criteria for triples which is equivalent to \( \tau' \)-limit stability.

Definition 11.1. Given \( X \) as the total space of \( \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \to \mathbb{P}^1 \), the highly frozen triples \((E, F, \phi, \psi)\) and frozen triples \((E, F, \phi)\) are called stable if \( \text{Coker}(\phi) \) has zero-dimensional support.

Now redenote \( M_{s,\text{HFT}}(\tau') \) as the stack of \( \tau' \)-stable highly frozen triples over \( X \). The reason to change our notation is that from now on we use a geometric criterion for stability of triples and we omit the superscript \( P_F \) in the notation.

11.1 The geometric action of \( T = (\mathbb{C}^*)^3 \) on \( M_{s,\text{HFT}}(\tau') \) over local \( \mathbb{P}^1 \)

We study the natural induced action of \( T = (\mathbb{C}^*)^3 \) on the moduli stack of highly frozen triples supported over a local Calabi-Yau threefold \( X \) given
by the total space of \( N = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \to \mathcal{C} \) where \( \mathcal{C} \cong \mathbb{P}^1 \). It is true that algebraically \( X = \text{Spec}(\text{Sym}^\bullet(\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1))) \). Let \( L_1 \) and \( L_2 \) denote the first and second copy of \( \mathcal{O}_{\mathbb{P}^1}(-1) \) in \( N \). Let \( \mathcal{U}_\alpha \) be given as a local affine chart around \( 0 \in \mathcal{C} \). By fixing the equivariant structures on \( L_1 \) and \( L_2 \) we see that there exists an action of \( (\mathbb{C}^*)^3 \) on \( X \) which is given locally over \( \mathcal{U}_\alpha \) by \( (\lambda_1, \lambda_2, \lambda_3) \cdot (l_1, l_2, s) = (\lambda_1 l_1, \lambda_2 l_2, \lambda_3 s) \), where \( l_1 \) and \( l_2 \) denote any local non-vanishing sections of \( L_1 \) and \( L_2 \) respectively and \( s \) denotes the local coordinate along \( \mathcal{C} \). Later in Example 16.2 we carry out computations with a two dimensional sub-torus of \( T \) which fixes the Calabi-Yau form of \( X \), however for now we stick to this notation.

**Definition 11.2.** Define the divisor \( D_1 \subset X \) as the fiber of \( X \) over \( 0 \in \mathcal{C} \), \( D_2 \subset X \) as \( \text{Tot}(L_1 \to \mathcal{C}) \) and \( D_3 \subset X \) as \( \text{Tot}(L_2 \to \mathcal{C}) \) respectively. Moreover, let \( l_1^\vee \) and \( l_2^\vee \) denote any local sections of \( L_1^\vee \) and \( L_2^\vee \) over \( \mathcal{U}_\alpha \).

Here we give a local description of the modules associated to the structure sheaves of \( D_1, D_2, D_3 \) in \( \mathcal{U}_\alpha \). It follows by the usual arguments that there exists an equivalence of categories \( \mathcal{F} : \text{Coh}(X) \overset{\sim}{\to} \text{Mod}_{\text{Sym}^\bullet(N^\vee)} \). Locally over \( \mathcal{U}_\alpha \) the module over \( \mathbb{C}[l_1^\vee, l_2^\vee, s] \) associated to the structure sheaf of \( X \) is given by the polynomial ring \( \mathcal{F}(\mathcal{O}_{\mathcal{U}_\alpha}) = \mathbb{C}[l_1^\vee, l_2^\vee, s] \). Let \( t_1, t_2 \) and \( t_3 \) denote the weights of the action of \( (\lambda_1, \lambda_2, \lambda_3) \) on \( (0, l_1^\vee, 0) \), \( (0, 0, l_2^\vee) \) and \( (s, 0, 0) \) respectively. One observes that the action of \( T \) on the localized structure module obtained above provides it with a decomposition into torus weight spaces, i.e locally:

\[
\mathcal{F}(\mathcal{O}_{\mathcal{U}_\alpha}) = \bigoplus_{(m, n, l)} \mathbb{C}[l_1^\vee, l_2^\vee, s](m, n, l). \tag{11.1}
\]

**Remark 11.3.** Locally over \( \mathcal{U}_\alpha \) the divisor \( D_1 \) is understood by the vanishing locus of \( s \), therefore inorder to obtain the module associated to \( \mathcal{O}_{\mathcal{U}_\alpha}(kD_1) \) we consider the \( \mathbb{C}[l_1^\vee, l_2^\vee, s] \)-module generated by \( \frac{1}{s} \). i.e \( \mathcal{F}(\mathcal{O}_{\mathcal{U}_\alpha}(kD_1)) = \left( \frac{1}{s} \right)^k \mathbb{C}[s, l_1^\vee, l_2^\vee] \) and since this module is generated by \( \left( \frac{1}{s} \right)^k \), it lies in \( \frac{1}{k} \) weight space. One may consider divisors \( D_2 \) and \( D_3 \) in \( X \) and similarly, the module structure associated to \( \mathcal{O}_{\mathcal{U}_\alpha}(kD_i) \) for \( i = 1, 2 \) is given by \( \mathcal{F}(\mathcal{O}_{\mathcal{U}_\alpha}(kD_i)) = \left( \frac{1}{l_i} \right)^k \mathbb{C}[s, l_1^\vee, l_2^\vee] \).

**12 The induced action of \( T \) on \( \mathcal{M}_{s, \text{HFT}}^{(r, n)}(\tau') \)**

We show that the \( T \)-action on \( X \) induces an action on \( \mathcal{M}_{s, \text{HFT}}^{(r, n)}(\tau') \). Given a \( \tau' \)-stable highly frozen triple \( (E, F, \phi, \psi) \) represented by the complex \( \mathcal{O}_X(-n) \oplus \phi \to \)}
\[ F \text{ and } t \in T \text{ we obtain a new highly frozen triple as follows: Let } U \subset X \text{ be an open subset. Given } t \in T. \text{ Identify } O_X(-1) \text{ with } O_X(-D_1) \text{ where } D_1 \text{ is defined in Definition}\, 11.2. \text{ Hence locally over } U \text{ sections of } O_X(-n) \text{ are identified with sections of } O_X(-nD_1). \text{ Now given a section } s \in \Gamma(O_X(-1) \mid t^{-1}U), \text{ the composition } s \circ t^{-1} \text{ defines a map:}
\]
\[ t^*O_X(-n) \mid U \overset{\psi}{\longrightarrow} O_X(-n) \mid U,
\]

which is an isomorphism. In other words we have chosen an equivariant structure for \( O_X(-n) \). Therefore the induced inverse isomorphism \( \psi^{-1} \) defines a map \( O_X^{\oplus}(n) \overset{\psi^{-1}}{\longrightarrow} t^*O_X(-n) \). Now compose with sections of \( F \) and obtain a highly frozen triple:

\[
\begin{align*}
O_X^{\oplus}(n) & \xrightarrow{\psi^{-1}} t^*O_X(-n) \\
& \xrightarrow{t^*\phi} t^*F,
\end{align*}
\]

(12.1)

Hence we are able to obtain a new \( \tau' \)-stable highly frozen triple \( (E, t^*F, \phi', \psi) \) represented by the complex \( O_X^{\oplus}(n) \overset{\phi'}{\longrightarrow} t^*F \) such that \( \phi' = t^*\phi \circ \psi^{-1} \) in \( \text{[12.1]} \). One needs to show that the composite morphism in \( \text{[12.1]} \) induces a well-behaved pointwise action of \( T \) on \( \mathcal{M}^{(r, n)}_{s, \text{HFT}}(\tau') \). We prove this fact in several steps. First we show in more generality that there exists an action of \( T \) on the moduli stack of triples of type \((P_E, P_F)\) (i.e \( \mathcal{M}^{(P_E, P_F)}_{s}(\tau')\)). Then we specialize to frozen triples and show that there exists a well-behaved action of \( T \) on \( \mathcal{M}^{(r, n)}_{s, \text{FT}}(\tau') \) induced by the pullback. Then by Definition \( \text{[12.1]} \) and since the pointwise action of \( T \) on highly frozen triples is induced by the composition of the isomorphism \( \psi^{-1} \) and pulling back by the torus \( (t^*\phi) \), the existence of a well-behaved action of \( T \) on \( \mathcal{M}^{(r, n)}_{s, \text{HFT}}(\tau') \) follows as a corollary.

**Proposition 12.1.** Let \( X \) be given as the total space of \( O_{\mathbb{P}^1}(-1) \oplus O_{\mathbb{P}^1}(-1) \). Let \( T \) be the \( (\mathbb{C}^*)^3 \) action on \( X \). Having fixed an equivariant structure on \( O_X(1) \), there exists an induced action of \( T \) on moduli stack of stable highly frozen triples \( \overline{m}^T : T \times \mathcal{M}^{(r, n)}_{s, \text{HFT}}(\tau') \rightarrow \mathcal{M}^{(r, n)}_{s, \text{HFT}}(\tau') \) given by \( \overline{m}^T(t, (E,F,\phi,\psi)) = \overline{m}^t(E,F,\phi,\psi) \) where \( \overline{m}^t(E,F,\phi,\psi) \) is defined by the composite morphism in \( \text{[12.1]} \).

**Proof.** The sketch of the proof of this statement is completely borrowed from arguments in \[ \text{[Koo08]} \text{ (Proposition 4.1). Since Pic}(T) = 0 \text{ any line bundle on} \]
Note that $T$ is the projective second factor. The $T$-isomorphism of functors:

$$
\phi \circ \sigma_1 \phi = T_1 \sigma_1 T_2
$$

Let $\text{Quot}$ the quotient stack of a scheme $Q$ Consider the Quot schemes $E$ sheaves on $X$ and $F$ appearing in the family of triples are globally generated. Let $(H,\sigma)$ be a suitable integer as in Section 6.1. We know that for any $n \geq n_3$ the sheaves $E$ and $F$ of sheaves $V_1 \otimes O_X(-n')$ and $V_F \otimes O_X(-n')$ respectively with Hilbert polynomial $P_E$ and $P_F$. The moduli stack of triples $E \to F$ is obtained as the quotient stack of a scheme $\mathcal{S}$ by the action of the group $\text{GL}(V_1) \times \text{GL}(V_F)$. Note that $\mathcal{S}$ is defined as a closed subscheme of $A = Q_1 \times Q_2 \times P$ where $P$ is the projective Hom bundle given as $P = P(\text{Hom}(V_1, V_F))$. Let $H_1 = V_1 \otimes O_X(-n')$ and $H_2 = V_F \otimes O_X(-n')$. Let $\sigma : T \times X \to X$ denote the action of $T$ on $X$ and $P_F : T \times X \to X$ denote the projection onto the second factor. The $T$-equivariant structure on $X$ induces the isomorphisms $\phi_{H_i} : \sigma^* H_i \cong \sigma^* H_i$ for $i = 1, 2$ (where $T$ acts trivially on $V_1$ and $V_F$). It is easy to see that the action of $T$ on $X$ lifts to $Q_1 \times Q_2$.

Let $\text{Quot}(H_1, P_E) \times \text{Quot}(H_1, P_E)$ be the functor which is representable by the product of Quot schemes $Q_1 \times Q_2$. In other words there exists an isomorphism of functors:

$$
\Theta : \text{Quot}(H_1, P_E) \times \text{Quot}(H_1, P_E) \cong Q_1 \times Q_2,
$$

where $Q_1 \times Q_2 = \text{Hom}(-, Q_1 \times Q_2)$. Our goal is to define a regular action of $T$ on $Q_1 \times Q_2$ given by the map $\overline{m}^T : T \times Q_1 \times Q_2 \to Q_1 \times Q_2$. Let $p_{Q_1 \times Q_2} : T \times X \times Q_1 \times Q_2 \to T \times X$ be the projection onto the first two factors and $\overline{m}_{Q_1 \times Q_2} = \sigma \times 1_{Q_1 \times Q_2}$ be the lift of the action of $T$ on $X \times Q_1 \times Q_2$. Let $(H,Q_1 \overset{u_1}{\to} E, Q_2 \overset{u_2}{\to} F)$ be the universal family over $Q_1 \times Q_2$. It is seen that pre-composing $(\overline{m}_{Q_1 \times Q_2}^* (u_1, u_2))$ with $(p_{Q_1 \times Q_2}^* \sigma_{Q_1 \times Q_2}^*)^{-1}$ gives an element of $\text{Quot}(H_1, P_E) \times \text{Quot}(H_1, P_E)(T \times Q_1 \times Q_2) \cong \text{Hom}(T \times Q_1 \times Q_2, Q_1 \times Q_2)$, which defines the regular action of $T$ on $Q_1 \times Q_2$. Let $t = (\lambda_1, \lambda_2, \lambda_3) \in T$ be a closed point. Let $i_t : X \hookrightarrow T \times X$ denote the injection. Let

$$
\phi_{H_1 \times H_2, t} = i_t^* (\phi_{H_1}, \phi_{H_2}) = (i_t^* \phi_{H_1}, i_t^* \phi_{H_2}) : t^* (H_1, H_2) \cong (H_1, H_2).
$$

Let $q \in Q_1 \times Q_2 : ([H_1 \overset{u_1}{\to} E], [H_2 \overset{u_2}{\to} F])$ be a closed point. It is easy to see that there exists a lift of the action of $t \in T$ on $Q_1 \times Q_2$ which is obtained by $\overline{m}^T (t, q) = q \cdot t$ and it corresponds to

$$
([H_1 \overset{(i_t^* \phi_{H_1})^{-1}}{\to} t^* H_1 \overset{t^* u_1}{\to} t^* E], [H_2 \overset{(i_t^* \phi_{H_2})^{-1}}{\to} t^* H_2 \overset{t^* u_2}{\to} t^* F]).
$$

(12.3)
The composite morphisms in (12.3) define the lifted action of $T$ on $Q_1 \times Q_2$. Since the action of $T$ on points of $P$ is trivial one lifts the action of $T$ to $Q_1 \times Q_2 \times P$ where $T$ acts on $Q_1 \times Q_2$ as described above and it acts trivially on the points $p = \text{Hom}(V_1, V_F)^V \in P$. Let $S$ be the scheme parametrizing triples of type $(P_E, P_F)$. Let $U \subset S$ be the open subscheme of $\tau'$-limit stable triples of type $(P_E, P_F)$. The regular action of $T$ on $Q_1 \times Q_2 \times P$ restricts to the action of $T$ on $U$:

\[
\begin{array}{ccc}
T \times U & \overset{m^T}{\longrightarrow} & U \\
id_T \times \pi & | & \\
T \times M_{s}(P_E, P_F)(\tau') & \overset{\pi}{\longrightarrow} & M_{s}(P_E, P_F)(\tau'). \\
\end{array}
\]

Note that the action of $G = \text{GL}(V_1) \times \text{GL}(V_F)$ is trivial on $T$ and the maps $m^T$, $\pi$ and $\pi \circ m^T$ are $G$-equivariant. By the property of quotient stacks, this induces a $G$-equivariant map $T \times M_{s}(P_E, P_F)(\tau') \to M_{s}(P_E, P_F)(\tau')$ which defines the induced action of $T$ on $M_{s}(P_E, P_F)(\tau')$. This proof restricts easily to the case where $M_{s}(P_E, P_F)(\tau')$ is replaced by $M_{s}(P_{F, r, n})(\tau')$ and one obtains the action of $T$ in Proposition [12.1] over moduli stack of stable frozen triples. Now we use the fact that given any $t \in T$ the action of $T$ on highly frozen triples is obtained by pre-composition of the the map $t^*$ and the map defined by fixed choice of $\psi^{-1}$ which we denoted by the choice of equivariant structure on $O_X(-n)^{\oplus r}$. This by construction will automatically define an action of $T$ on $M_{s, \text{HFT}}(\tau')$ (and hence on $M_{s, \text{HFT}}(\tau')$).

For more detailed discussion look at [Koo08] (Proposition 4.1).

**Proposition 12.2.** Let $S$ be a parametrizing scheme of finite type over $\mathbb{C}$. Let $(\mathcal{E}, \mathcal{F}, \phi, \psi)_S$ denote a family of stable highly frozen triples over $S$. Suppose that for all $t = (\lambda_1, \lambda_2, \lambda_3) \in T$ we have $t^*((\mathcal{E}, \mathcal{F}, \phi, \psi)_S) \cong (\mathcal{E}, \mathcal{F}, \phi, \psi)_S$, then $(\mathcal{E}, \mathcal{F}, \phi, \psi)_S$ admits a $T$-equivariant structure.

**Proof.** We give an adaptation of the proof given in [Nev02] (Lemma 3.3) to our case. By assumption for any $t \in T$ one has $t^*((\mathcal{E}, \mathcal{F}, \phi, \psi)_S) \cong (\mathcal{E}, \mathcal{F}, \phi, \psi)_S$. Let $\sigma : T \times X \to X$ denote the torus action on $X$ and $\pi_2 : T \times X \to X$ be the projection onto the second factor. Let $q : X \times S \to S$ be the projection onto $S$. One needs to show that there exists a map:

\[
\rho : \text{Ext}^0_{\mathcal{O}_T \times q}((\pi_2 \times \text{id}_S)^*(\mathcal{E}, \mathcal{F}, \phi, \psi)_S, (\sigma \times \text{id}_S)^*(\mathcal{E}, \mathcal{F}, \phi, \psi)_S) \to \mathcal{O}_T \times S,
\]

(12.5)
which is an isomorphism of line bundles over \( T \times S \). Here

\[
\text{Ext}^0_{id_T \times q}(\pi_2 \times \text{id}_S)^*(\mathcal{E}, \mathcal{F}, \phi, \psi)_S, (\sigma \times \text{id}_S)^*(\mathcal{E}, \mathcal{F}, \phi, \psi)_S) \\
\cong R^0(q \times \text{id}_T)_*(\mathcal{H}om((\pi_2 \times \text{id}_S)^*(\mathcal{E}, \mathcal{F}, \phi, \psi)_S, (\sigma \times \text{id}_S)^*(\mathcal{E}, \mathcal{F}, \phi, \psi)_S)).
\]

(12.6)

By definition of \( \mathcal{M}^{(r,n)}_{s,\text{HFT}}(\tau') \), choosing a family of stable highly frozen triples over \( S \) is equivalent to choosing a unique map \( S \to \mathcal{M}^{(r,n)}_{s,\text{HFT}}(\tau') \). Since \((\sigma \times \text{id}_S)^* (\mathcal{E}, \mathcal{F}, \phi, \psi)_S\) and \((\pi_2 \times \text{id}_S)^*(\mathcal{E}, \mathcal{F}, \phi, \psi)_S\) are two families over \( \mathcal{M}^{(r,n)}_{s,\text{HFT}}(\tau') \), they both define maps \( f : T \times S \to \mathcal{M}^{(r,n)}_{s,\text{HFT}}(\tau') \) and \( g : T \times S \to \mathcal{M}^{(r,n)}_{s,\text{HFT}}(\tau') \) respectively. By the uniqueness property, both maps are uniquely isomorphic to each other. On the other hand by Lemma 6.6 the complexes representing \( \tau' \)-stable highly frozen triples are simple objects hence:

\[
\text{Ext}^0_{id_T \times q}(\pi_2 \times \text{id}_S)^*(\mathcal{E}, \mathcal{F}, \phi, \psi)_S, (\sigma \times \text{id}_S)^*(\mathcal{E}, \mathcal{F}, \phi, \psi)_S) \\
\cong \text{Ext}^0_{id_T \times q}(\mathcal{E}, \mathcal{F}, \phi, \psi)_{T \times S}, (\mathcal{E}, \mathcal{F}, \phi, \psi)_{T \times S}) \cong \mathcal{O}_{T \times S}.
\]

(12.7)

Now the inverse image of 1 \( \in \mathcal{O}_{T \times S} \) via the map \( \rho \) in (12.5) gives a section of

\[
\text{Ext}^0_{id_T \times q}(\mathcal{E}, \mathcal{F}, \phi, \psi)_{T \times S}, (\mathcal{E}, \mathcal{F}, \phi, \psi)_{T \times S})
\]

which induces a section of

\[
\text{Ext}^0_{id_T \times q}(\pi_2 \times \text{id}_S)^*(\mathcal{E}, \mathcal{F}, \phi, \psi)_S, (\sigma \times \text{id}_S)^*(\mathcal{E}, \mathcal{F}, \phi, \psi)_S)
\]

which induces a morphism \((\pi_2 \times \text{id}_S)^*(\mathcal{E}, \mathcal{F}, \phi, \psi)_S \to (\sigma \times \text{id}_S)^*(\mathcal{E}, \mathcal{F}, \phi, \psi)_S\). Moreover, it can be checked that this morphism is an isomorphism (point-wise) for every point in the moduli stack of stable highly frozen triples hence it is an isomorphism and this finishes the proof. \( \square \)

13 The non-geometric action of \( T_0 = (\mathbb{C}^*)^r \) on \( \mathcal{M}^{(r,n)}_{s,\text{HFT}}(\tau') \) and the splitting property of stable highly frozen triples

Definition 13.1. Define an action \( \sigma_0 : T_0 \times \mathcal{M}^{(r,n)}_{s,\text{HFT}}(\tau') \to \mathcal{M}^{(r,n)}_{s,\text{HFT}}(\tau') \) where \( T_0 = (\mathbb{C}^*)^r \) and \( \sigma_0 \) acts on \( \mathcal{M}^{(r,n)}_{s,\text{HFT}}(\tau') \) by rescaling in components of
$O_X(-n)^{\oplus r}$. In other words, given $[(E, F, \phi, \psi)] \in \mathcal{M}_{s, \text{HFT}}^{(r,n)}(\tau')$ and $(z_1, \ldots, z_r) \in T_0$ we have:

$$\sigma_0((z_1, z_2, \ldots, z_r), (E, F, \phi, \psi)) = (E, F, \phi \circ \nu, \psi), \quad (13.1)$$

where $\nu := E\frac{z_1^{-1} \cdots 0}{\vdots \ddots \vdots} \frac{0 \cdots z_r^{-1}}{E}$. 

**Proposition 13.2.** Let $S$ be a parametrizing scheme of finite type over $\mathbb{C}$. Let $(\mathcal{E}, \mathcal{F}, \phi, \psi)_S$ denote a family of stable highly frozen triples over $S$. Suppose that for all $t_0 = (z_1, \ldots, z_r) \in T_0$ $\sigma_0(t_0, (\mathcal{E}, \mathcal{F}, \phi, \psi)_S) \cong (\mathcal{E}, \mathcal{F}, \phi, \psi)_S$ then $(\mathcal{E}, \mathcal{F}, \phi, \psi)_S$ admits a $T_0$-equivariant structure:

$$\sigma_0^*(\mathcal{E}, \mathcal{F}, \phi, \psi)_S \cong \tilde{p}_2^*(\mathcal{E}, \mathcal{F}, \phi, \psi)_S,$$

where $\tilde{p}_2: T_0 \times \mathcal{M}_{s, \text{HFT}}^{(r,n)}(\tau') \to \mathcal{M}_{s, \text{HFT}}^{(r,n)}(\tau')$ is the projection onto the second factor.

**Proof.** The action of $T_0$ is directly defined over $\mathcal{M}_{s, \text{HFT}}^{(r,n)}(\tau')$ hence one may directly apply the proof of Proposition [12.2] to $T_0$ instead of $T$ and the universal family $(\mathcal{E}, \mathcal{F}, \phi, \psi)$ instead of $(\mathcal{E}, \mathcal{F}, \phi, \psi)_S$ and use the simpleness property of $\tau'$-limit stable highly frozen triples. $\square$

**Remark 13.3.** Since the stable highly frozen triples are $T_0$-equivariant, by Proposition [13.2] it is easily seen that the action of $T_0$ on a point $p \in \mathcal{M}_{s, \text{HFT}}^{(r,n)}(\tau')$ (represented by a stable highly frozen triple $(E, F, \phi, \psi)$) as in (13.1) induces a $T_0$-weight decomposition on $E \cong O_X^{\oplus r}(-n)$. Let $(w_1, \ldots, w_r)$ denote the $r$-tuple of weights. In fact the only nontrivial weights due to the action of $T_0$ are

$$(1, 0, \ldots, 0), (0, 1, \ldots, 0), \ldots, (0, \ldots, 0, 1).$$

Now consider the module $M$ associated to the sheaf $O_X^{\oplus r}(-n)$ and denote by $M^0$ the module associated to the sheaf $O_X(-n)$. The graded piece of $M^T$ which sits in $(0, \ldots, i, \ldots, 0)$ weight space is given by the module $0 \oplus \cdots \oplus M^0 \oplus \cdots \oplus 0$ with $M^0$ in $i$'th position which we denote by $M^0_i$. On the other hand the $T$-weight decomposition of $M^T$ is given by (11.1). Therefore the $T \times T_0$-weight decomposition of $M^{T \times T_0}$ is given by

$$M^{T \times T_0} \cong \bigoplus_{i=1}^{r} \left( \bigoplus_{(m_1, m_2, m_3)} M^0_i(m_1, m_2, m_3) \right) \quad (13.2)$$

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According to Propositions 12.2 and 13.2 the \( T \times T_0 \)-fixed points of \( \mathcal{M}_{s,\text{HFT}}(\tau') \) are represented by highly frozen triples which admit \( T \times T_0 \)-equivariant structure. Now let \( N \) denote the module associated to \( F \). Given a \( T \times T_0 \)-equivariant highly frozen triple \( M \to N \), by the property of morphism between two graded sheaves of modules, the sheaf \( N \) admits a weight decomposition compatible to that of \( M \). Hence it is seen that a torus equivariant highly frozen triple admits a \( T \times T_0 \)-weight decomposition of the following form:

\[
[M \to N] \sim \bigoplus_{i=1}^{r} \left[ \bigoplus_{(m_1,m_2,m_3)} M_i^0(m_1,m_2,m_3) \to \bigoplus_{(m'_1,m'_2,m'_3)} N_i(m'_1,m'_2,m'_3) \right]
\]

(13.3)

**Remark 13.4.** The weight decomposition in (13.3) clarifies the fact that a \( T \times T_0 \)-equivariant \( \tau' \)-limit stable highly frozen triple is decomposable into \( r \) copies of \( T \)-equivariant \( \tau' \)-limit stable highly frozen triples of the form \( \mathcal{O}_X(-n) \to F_i \):

\[
[\mathcal{O}_X(-n) \to F_i]^{T \times T_0} \cong \bigoplus_{i=1}^{r} \left[ \mathcal{O}_X(-n) \to F_i \right]^T.
\]

(13.4)

### 14 HFT partition function over local \( \mathbb{P}^1 \)

Let \( G = T \times T_0 \). The identification of \( G \)-fixed points in the moduli space of stable highly frozen triples with \( r \)-fold direct sum of \( T \)-equivariant PT stable pairs (Remark 13.4) enables one to easily calculate the partition function for the stable highly frozen triples. Recall from [RR09] that for nonzero \( \beta \in H^2(X,\mathbb{Z}) \) the partition function for stable pairs was given as:

\[
Z_{\text{PT},\beta} = \sum_m P_{m,\beta} q^m.
\]

(14.1)

Here \( m \) denotes the Euler characteristic of the sheaf \( F \) appearing in a stable pair given by \( \mathcal{O}_X \to F \). Now suppose \( X = \text{Tot}(\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)) \to \mathbb{P}^1 \). By [PT09] when \( X \) is given as a local Toric Calabi-Yau threefold the \( T \)-fixed points of the moduli space of stable pairs are given by isolated points and \( P_{m,\beta} = \#\{T \text{- fixed points in the moduli space}\} \). Now let \( Q_{m,\beta} \) denote the
the G-fixed locus of $\mathcal{M}_{s,HFT}^{(r,n)}(\tau')$ which parametrizes the G-equivariant highly frozen triples

$$[\mathcal{O}_X(-n) \to F]^G \cong \bigoplus_{i=1}^r [\mathcal{O}_X(-n) \to F_i]^T \cong \bigoplus_{i=1}^r [\mathcal{O}_X \to F_i(n)]^T.$$ 

with $\chi(F^G) = m$ and $\text{ch}_2(F^G) = \beta$. Note that if $\chi(F^G) = m$ and $\chi(F^T_i) = m_i$ then $\sum_i m_i = m$. By the result in Remark 13.4 it is easily seen that

$$Q_{m,\beta} = \prod_{i=1}^r Q_{m_i,\beta},$$

such that $Q_{m_i,\beta}$ for $i = 1, \cdots, r$ are given by isolated points (this fact is true when $X$ is fixed to be local $\mathbb{P}^1$). Hence, we obtain the partition function for the highly frozen triples:

$$Z_{HFT} = \sum_m \sum_{m_1 + \cdots + m_r = m} (P_{m_1,\beta} q^{n \cdot m_1} \cdots \cdots \cdot P_{m_r,\beta} q^{n \cdot m_r})$$

$$= \sum_m \sum_{m_1 + \cdots + m_r = m} \left( \prod_{i=1}^r P_{m_i,\beta} q^{n \cdot m_i} \right) = \left( \sum_m P_{m,\beta} q^{n \cdot m} \right)^r. \quad (14.2)$$

Here the occurrence of $n$ in the exponent is due to the effect of twisting the highly frozen triples by the large enough integer $n$.

15 The threefold vertex via HFT

In what follows we compute the Calabi-Yau equivariant vertex via the equivariant deformation-obstruction theory of the moduli space of stable highly frozen triples.

15.1 Equivariant obstruction theory on HFT

The identification of the highly frozen triples of rank $r$ as direct sum of $r$ copies of PT stable pairs makes it easy to see that the G-fixed components of the moduli stack of highly frozen triples are obtained as $r$-fold products of T-fixed components of the moduli stack of stable pairs which are conjectured by Pandharipande and Thomas in [RR09] (Conjecture 2) to be nonsingular and compact.

Let $Q$ denote the G-fixed locus of $\mathcal{M}_{s,HFT}^{(r,n)}(\tau')$. We assume that $Q$ is non-singular, connected and compact. Let $\iota_Q : Q \hookrightarrow \mathcal{M}_{s,HFT}^{(r,n)}(\tau')$ denote the
natural embedding. Let $G^\bullet_Q := (\iota_Q)^* G^\bullet$ where $G^\bullet$ is the deformation obstruction theory obtained in Theorem 9.11. Let $G^G_Q$ and $G^m_Q$ denote the sub-bundles of $G^\bullet_Q$ with zero and nonzero characters respectively. By Theorem 9.11, the $G$-fixed deformation obstruction theory restricted to $Q$ is given by a map of perfect complexes:

$$G^G_Q \xrightarrow{\phi} L^Q.$$ (15.1)

Here $G^G_Q$ is represented by a two term complex of vector bundles $G^{-1}_Q \xrightarrow{\phi} G^0_Q$. By the virtual localization formula:

$$\left[\mathcal{M}_{s,HFT}^{(r,n)}(\tau')\right]^{vir} = \sum_{Q \subset \mathcal{M}_{s,HFT}^{(r,n)}(\tau')} \iota_Q \left( e(G^m_{G,0,Q}) \cap [Q]^{vir} \right).$$ (15.2)

Where $G^m_{G,0,Q}$ and $G^m_{G,1,Q}$ denote the dual of $G^0_{G,0,Q}$ and $G^{-1}_{G,1,Q}$ respectively. Now we rewrite (15.2) with respect to the Euler classes $e(G^m_{G,1,Q})$ and $e(G^m_{G,0,Q})$ where $G^0_{G,0,Q}$ and $G^{-1}_{G,1,Q}$ denote the dual of $G^0_{G,0,Q}$ and $G^{-1}_{G,1,Q}$ respectively. In order to do so ones needs to have the description of the virtual tangent space with respect to the $G$-fixed deformation obstruction theory. If $Q$ is assumed to be nonsingular, then $L^Q := 0 \to \Omega_Q$. The $G$-fixed deformation obstruction theory (15.1) induces a composite morphism

$$G^{-1}_{G,0,Q} \to G^0_{G,0,Q} \xrightarrow{\phi} \Omega_Q.$$ (15.3)

The kernel of this composite morphism is the obstruction bundle $K$ and by definition $[Q]^{vir} = e(K) \cap [Q]$. One computes the $K$-theory class of $K^V$ as follows:

$$[K^V] = [G^G_{G,1,Q}] - [G^G_{G,0,Q}] + [T_Q],$$ (15.4)

where $G^G_{G,0,Q}$ and $G^G_{G,1,Q}$ denote the dual of $G^0_{G,0,Q}$ and $G^{-1}_{G,1,Q}$ respectively. Therefore one has:

$$e(K^V) = \frac{e(G^G_{G,1,Q})}{e(G^G_{G,0,Q})} \cdot e(T_Q).$$ (15.5)

By (15.2) and (15.4) the virtual fundamental class of $\mathcal{M}_{s,HFT}^{(r,n)}(\tau')$ is obtained as

$$\left[\mathcal{M}_{s,HFT}^{(r,n)}(\tau')\right]^{vir} = \sum_{Q \subset \mathcal{M}_{s,HFT}^{(r,n)}(\tau')} \iota_Q \left( \frac{e(G^G_{G,1,Q})}{e(G^G_{G,0,Q})} \cdot e(T_Q) \cap [Q] \right).$$ (15.5)
Now we compute the difference $[G_{0,Q}] - [G_{1,Q}]$ in the $G$-equivariant $K$-theory of $Q$. Consider a point $p \in Q$ represented by the complex $I^*G := [O_X^{\oplus}(-n) \to F]^G$. The difference $[G_{0,Q}] - [G_{1,Q}]$ over this point is the virtual tangent space at this point. We use the quasi isomorphism in diagram (9.33) to compute the virtual tangent space:

$$T^Q_p = [\text{Coker}(d')] - [\text{Ker}(d')] = ([\pi^*E^1] - [\pi^*E^0] + [\pi^*E^{-1}] - [\pi^*E^{-2}]) + ([T_p] - [\Omega_p]),$$

(15.6)

where $E^i$ for $i = -1, \cdots, 2$ are the corresponding terms of $E^{\text{**}}$ in Lemma 9.9 and the cancellation in the second row is due to isomorphism of $\Omega_p$ and $T_p$ which is seen from their triviality.

By the construction of $E^{\text{**}}$ in Proposition 9.9 and since the point $p \in Q$ is represented by $I^*G$ the following identities hold true:

$$T^Q_p = \sum_{i=0}^{3} (-1)^i \cdot \text{Ext}^i(I^*G, I^*G)_0 = [\chi(O_X, O_X)] - [\chi(I^*G, I^*G)].$$

(15.7)

### 15.2 Computation of $\chi(O_X, O_X) - \chi(I^*G, I^*G)$

By definition:

$$\chi(I^*G, I^*G) = \sum_{i,j=0}^{3} (-1)^{i+j} H^i(\text{Ext}^j(I^*G, I^*G))$$

and

$$\chi(O_X, O_X) = \sum_{i,j=0}^{3} (-1)^{i+j} H^i(\text{Ext}^j(O_X, O_X)).$$

By replacing the cohomology terms with the Čech complex obtained with respect to an affine open cover $\bigcup \alpha U_\alpha$ we obtain:

$$\chi(I^*G, I^*G) = \sum_{i,j=0}^{3} (-1)^{i+j} \mathcal{C}^i(\text{Ext}^j(I^*G, I^*G))$$

and

$$\chi(O_X, O_X) = \sum_{i,j=0}^{3} (-1)^{i+j} \mathcal{C}^i(\text{Ext}^j(O_X, O_X)).$$
By definition the sheaf $F$ appearing in the stable highly frozen triples is pure of dimension 1. Therefore, the restriction of $F$ over the triple and quadruple intersections of $U_{\alpha}$'s vanishes and over such intersections $I^* \cong \mathcal{O}_X^\mathbb{P}(-n)$.

**Definition 15.1.** Define:

\[
\mathcal{T}_{[\mathbf{i}^*]}^1 = \bigoplus_{\alpha} \left( H^0(U_{\alpha}, \mathcal{O}_X) - \sum_j (-1)^j H^0(U_{\alpha}, \mathcal{E}xt^j(I^*, I^*)) \right)
\]

\[
\mathcal{T}_{[\mathbf{i}^*]}^2 = \bigoplus_{\alpha, \beta} \left( H^0(U_{\alpha\beta}, \mathcal{O}_X) - \sum_j (-1)^j H^0(U_{\alpha\beta}, \mathcal{E}xt^j(I^*, I^*)) \right)
\]

\[
\mathcal{T}_{[\mathbf{i}^*]}^3 = \bigoplus_{\alpha, \beta, \gamma} ((1 - r^2) H^0(U_{\alpha\beta\gamma}, \mathcal{O}_X)) \text{ and } \mathcal{T}_{[\mathbf{i}^*]}^4 = \bigoplus_{\alpha, \beta, \gamma, \delta} (1 - r^2) H^0(U_{\alpha\beta\gamma\delta}, \mathcal{O}_X).
\]

(15.8)

By Definition 15.8 and (15.7) the virtual tangent space is obtained as:

\[
\mathcal{T}_{[\mathbf{i}^*]} = \mathcal{T}_{[\mathbf{i}^*]}^1 - \mathcal{T}_{[\mathbf{i}^*]}^2 + \mathcal{T}_{[\mathbf{i}^*]}^3 - \mathcal{T}_{[\mathbf{i}^*]}^4.
\]

(15.9)

Now let $T$ and $T_0$ be defined as before and $G = T \times T_0$. Let $(t_1, t_2, t_3)$ be defined as the weights of $T$. Moreover let $(w_1, \cdots, w_r)$ be defined as weight of the action of $T_0$. Here $w_i$ is given by tuples $(0, \cdots, 1, \cdots, 0)$ where 1 is positioned in the $i$'th position in the tuple. In this section we compute the $G$-character of $\mathcal{T}_{[\mathbf{i}^*]}$ for $i = 1, \cdots, 4$ in (15.9). We compute the vertex for $G$-equivariant stable highly frozen triples which are identified with

\[
\bigoplus_{i=1}^r \left( \mathcal{O}_X^T(-n) \to F_i^T \right).
\]

(15.10)

Recall that super-index $T$ indicates equivariance with respect to the action of $T$. Choose a Čech cover $\mathcal{U} = \bigcup_{\alpha} U_{\alpha}$ of $X$. The restriction of each copy of $\mathcal{O}_X^T(-n) \to F_i^T$ in (15.10) to the underlying supporting curve $\mathcal{C}_{\alpha}$ of $F_i^T$ induces an exact sequence of the form:

\[
0 \to \mathcal{O}_{\mathcal{C}_{\alpha}}^T(-n) \rightarrow (F_i^T)_{\alpha} \rightarrow (Q_i^T)_{\alpha} \rightarrow 0,
\]

(15.11)

By $\tau'$-stability the sheaf $(F_i^T)_{\alpha}$ may be zero and if it is nonzero then the cokernel $(Q_i^T)_{\alpha}$ has to be zero dimensional. Moreover by the splitting property of $G$-equivariant highly frozen triples it is easily seen that $Q_i^G := \bigoplus_{i=1}^r (Q_i^T)_{\alpha}$, such that each $(Q_i^T)_{\alpha}$ has zero dimensional support.
one has \( \text{Supp}(Q^G) := \bigcup_{i=1}^r \text{Supp}(Q_i^T) \) and if there exists \((Q_i^T)_\alpha\) for some \(i\) with one dimensional support then it contradicts with stability of the original highly frozen triple. Given \( F^G_\alpha = \bigoplus_{i=1}^r (F_i^T)_\alpha \), we use the procedure similar to \[\text{PT09}\] (Section 4.4) and \[\text{D.06}\] (Section 4.7) to compute the \( T \)-character of each summand, \((F_i^T)_\alpha\). Let \( F^T_\alpha \) denote the \( T \)-character of each summand. Let \((P_i)_\alpha(t_1, t_2, t_3)\) denote the associated Poincaré polynomial of

\[
(\mathbb{I}_i^t)_\alpha := \left( \mathcal{O}_X^T(-n) \rightarrow F_i^T \right) |_\alpha .
\]

The Poincaré polynomial of \((\mathbb{I}_i^t)_\alpha\) is related to the \( T \) character of \( F_i \) as:

\[
F^T_{i,\alpha} = \frac{C^\alpha \cdot (P_i)_\alpha}{(1-t_1)(1-t_2)(1-t_3)},
\]

where the correction term \( C^\alpha \) is the \( T \)-character of \( \mathcal{O}_X(-n) \) with the chosen equivariant structure. Now the \( G \)-character of \( F_i \) is given by:

\[
F^G_{i,\alpha} = w_i \cdot F^T_{i,\alpha} = \frac{C^\alpha \cdot w_i + w_i \cdot (P_i)_\alpha}{(1-t_1)(1-t_2)(1-t_3)},
\]

where \( w_i \) is the weight corresponding to the action of \( T_0 \) on the \( i \)'th copy of \( \mathcal{O}_X(-n) \) and on \( F_i^T \). The description of \( C^\alpha \) depends on one’s choice of equivariant structure. The \( T \)-character of each \( \text{tr}_\chi((\mathbb{I}_i^t)_\alpha, (\mathbb{I}_i^t)_\alpha) \) as computed in \[\text{D.06}\] (Section 4.7) is given as follows:

\[
\text{tr}_\chi((\mathbb{I}_i^t)_\alpha, (\mathbb{I}_i^t)_\alpha) = \frac{w_i \cdot w_i^{-1} \cdot (P_i)_\alpha (\overline{P}_i)_\alpha}{(1-t_1)(1-t_2)(1-t_3)} = \frac{(P_i)_\alpha (\overline{P}_i)_\alpha}{(1-t_1)(1-t_2)(1-t_3)}.
\]

The dual bar operation is negation on \( K(Q|\mathcal{U}_\alpha) \) and \( t_i \rightarrow \frac{1}{t_i} \) on the equivariant variables \( t_i \). Since \( \mathbb{I}_\alpha^G := \bigoplus_{i=1}^r (\mathbb{I}_i^t)^G \), the \( G \)-character of \( \chi(i_\alpha^G, i_\alpha^G) \) is obtained as:

\[
\text{tr}_\chi(\mathbb{I}_\alpha^G, \mathbb{I}_\alpha^G) = \sum_{1 \leq i \leq r} \frac{w_i \cdot w_i^{-1} \cdot (P_i)_\alpha (\overline{P}_j)_\alpha}{(1-t_1)(1-t_2)(1-t_3)}.
\]

Moreover the \( G \)-character of \( F_\alpha \) appearing in \( \mathbb{I}_\alpha^G \) is given by:

\[
F^G_\alpha = \frac{\sum_{i=1}^r w_i \cdot C^\alpha + \sum_{i=1}^r w_i \cdot (P_i)_\alpha}{(1-t_1)(1-t_2)(1-t_3)},
\]
since the $G$-character of the $α$-summand of $T^1_{[\bullet]}$ in (15.8) is given by:

$$1 - \sum_{1 \leq i \leq r} w_i w_j^{-1} \cdot (P_i)_{\alpha} (P_j)_{\alpha}$$

one computes the $α$-summand of the $G$-character of $T^1_{[\bullet]}$ as a function of

$$F^G_{α}:$$

$$\text{tr}_R - χ((\bullet), G)_{α, (\bullet), G}_{α} = F^G_{α} \cdot (\sum_{j=1}^{r} w_j^{-1}) \cdot C_{α}^{n} - \frac{F^G_{α} \cdot (\sum_{i=1}^{r} w_i) \cdot C_{α}^{n}}{t_1 t_2 t_3}$$

$$+ F^G_{α} F^G_{α} (1 - t_1)(1 - t_2)(1 - t_3) + \frac{1 - (\sum_{i,j=1}^{r} w_i w_j^{-1}) \cdot C_{α}^{n} C_{α}^{n}}{(1 - t_1)(1 - t_2)(1 - t_3)}$$

(15.18)

Now we compute the $G$-character of $T^2_{[\bullet]}, T^3_{[\bullet]}$ and $T^4_{[\bullet]}$. Assume that $U_{αβ}$ is the affine patch over which the equivariant parameter $t_1$ is invertible. Given $F = \bigoplus_{i=1}^{r} F_i$, Let $(F_i)_{αβ}$ denote the restriction of $F_i$ to $U_{αβ}$. Let

$$F^T_{i,αβ} = \sum_{k_2, k_3, k_2 \in \mu_{αβ}} t_2^{k_2} t_3^{k_3}$$

denote the $T$-character associated to this restriction (Look at [D. 06] (4.10)). The $G$-character of $F_{αβ}$ is obtained as

$$F^G_{αβ} = \sum_{i=1}^{r} F^T_{i,αβ} \cdot w_i.$$  

By the same argument as above and similar to computations in [D. 06] (4.10) one relates the $G$-character of $αβ$’th summand of the virtual tangent space $T^2_{[\bullet]}$ in (15.8) to $F^G_{αβ}$:

$$\text{tr}_R - χ((\bullet), G)_{αβ} = \left[ F^G_{αβ} \cdot (\sum_{j=1}^{r} w_j^{-1}) \cdot C_{αβ}^{n} - \frac{F^G_{αβ} \cdot (\sum_{i=1}^{r} w_i) \cdot C_{αβ}^{n}}{t_2 t_3} \right] \cdot \delta(t_1),$$

(15.19)

here $C_{αβ}^{n}$ is a function of $n$ and the correction term that needs to be inserted into description of the Poincaré polynomial of $O_X | U_{αβ}$ in order to obtain the
Poincaré polynomial of $\mathcal{O}_X(-n)\big|_{U_{\alpha,\beta}}$. Moreover, we have used the notation $\delta(t_1) = \sum_{k \in \mathbb{Z}} t_1^k$. Now assume $U_{\alpha,\beta,\gamma}$ is the affine patch over which the equivariant parameters $t_1$ and $t_2$ are invertible. The $\alpha,\beta,\gamma$'th summand of $T_3[I\cdot]$ in (15.8) is obtained as follows:

$$tr_{R - \chi(I\cdot)_{\alpha\beta\gamma},(I\cdot)_{\alpha\beta\gamma}} = (1 - \sum_{i,j=1}^r w_i w_j^{-1}) \delta(t_1) \delta(t_2).$$

(15.20)

Finally the $T$-character of $T_1[I\cdot]$ in (15.8) is obtained as:

$$tr_{R - \chi(I\cdot)_{\alpha\beta\gamma\delta},(I\cdot)_{\alpha\beta\gamma\delta}} = (1 - \sum_{i,j=1}^r w_i w_j^{-1}) \delta(t_1) \delta(t_2) \delta(t_3).$$

(15.21)

Based on above discussion the $G$-character of the virtual tangent space over a point is obtained as follows:

$$tr_{R - \chi(I\cdot),(I\cdot)_{\alpha}} = \sum_{\alpha} tr_{R - \chi(I\cdot)_{\alpha\beta},(I\cdot)_{\alpha\beta}} - \sum_{\alpha,\beta} tr_{R - \chi(I\cdot)_{\alpha\beta},(I\cdot)_{\alpha\beta}}$$

$$+ \sum_{\alpha,\beta,\gamma} tr_{R - \chi(I\cdot)_{\alpha\beta\gamma},(I\cdot)_{\alpha\beta\gamma}} - \sum_{\alpha,\beta,\gamma,\delta} tr_{R - \chi(I\cdot)_{\alpha\beta\gamma\delta},(I\cdot)_{\alpha\beta\gamma\delta}}$$

(15.22)

### 15.3 HFT Redistribution and calculation of equivariant vertex

The $G$-character of the virtual tangent space in (15.22) is equal to the addition of vertex contributions (the first summand on right hand side of (15.22)) and the remaining edge contributions. Similar to discussions in [PT09] (Section 4.6) one may redistribute the terms in (15.18), (15.19), (15.20) and (15.21) so that they become Laurent polynomials in the variables $t_1$.

Define

$$G_{\alpha,\beta} = F_{\alpha,\beta}^G \left( \sum_{j=1}^r w_j^{-1} \right) \cdot C_{\alpha,\beta}^m - \frac{F_{\alpha,\beta}^G \cdot (\sum_{i=1}^r w_i) \cdot C_{\alpha,\beta}^m}{t_2 t_3}$$

$$+ F_{\alpha,\beta}^G \cdot \frac{1 - t_2}{t_2 t_3} + \frac{1 - (\sum_{i,j=1}^r w_i w_j^{-1}) \cdot C_{\alpha,\beta}^m \cdot C_{\alpha,\beta}^m}{(1 - t_2)(1 - t_3)}.$$
In that case one can rewrite the edge character (15.19) similar to \textcolor{red}{PT09} (Equation 4.11). Similarly define

\[ G_{\alpha\beta\gamma} = \frac{(1 - \sum_{i,j=1}^{r} w_i w_j^{-1})}{(1 - t_3)}. \] (15.24)

Hence (15.20) is rewritten as

\[ \left( \frac{G_{\alpha\beta\gamma}(t_3)}{1 - t_1} + t_1^{-1} \frac{G_{\alpha\beta\gamma}(t_3)}{1 - t_1^{-1}} \right) \frac{1}{1 - t_2} + t_2^{-1} \left( \frac{G_{\alpha\beta\gamma}(t_3)}{1 - t_1} + t_1^{-1} \frac{G_{\alpha\beta\gamma}(t_3)}{1 - t_1^{-1}} \right) \frac{1}{1 - t_3}, \] (15.25)

Note that here we expand the first term of the edge character in ascending powers of \( t_1 \) and the second term in descending powers of \( t_1 \). We follow the same rule and expand the first term in (15.25) in ascending powers of \( t_2 \) and the second term in descending powers of \( t_2 \). Finally define

\[ G_{\alpha\beta\gamma\delta} = (1 - \sum_{i,j=1}^{r} w_i w_j^{-1}). \] (15.26)

Hence (15.21) is rewritten as

\[
\begin{aligned}
& \left( \frac{G_{\alpha\beta\gamma\delta}}{1 - t_1} + t_1^{-1} \frac{G_{\alpha\beta\gamma\delta}}{1 - t_1^{-1}} \right) \frac{1}{1 - t_2} + t_2^{-1} \left( \frac{G_{\alpha\beta\gamma\delta}}{1 - t_1} + t_1^{-1} \frac{G_{\alpha\beta\gamma\delta}}{1 - t_1^{-1}} \right) \frac{1}{1 - t_3} \\
& + t_3^{-1} \left( \frac{G_{\alpha\beta\gamma\delta}}{1 - t_1} + t_1^{-1} \frac{G_{\alpha\beta\gamma\delta}}{1 - t_1^{-1}} \right) \frac{1}{1 - t_2} + t_2^{-1} \left( \frac{G_{\alpha\beta\gamma\delta}}{1 - t_1} + t_1^{-1} \frac{G_{\alpha\beta\gamma\delta}}{1 - t_1^{-1}} \right) \frac{1}{1 - t_3},
\end{aligned}
\] (15.27)

where we expand the first term in (15.27) in ascending powers of \( t_3 \) and the second term in descending powers of \( t_3 \). Now for each \( U_{\alpha} \) define a new vertex character similar to \textcolor{red}{PT09} (Equation 4.12):

\[ V_{\alpha} = tr_{R^{-\chi(\bullet,G_{\alpha},(\bullet,G_{\infty}))}} + \sum_{i=1}^{3} \frac{G_{\alpha\beta_i}(t_i',t_i'' \cdot t_i)}{1 - t_i}. \] (15.28)

where \( \beta_1, \beta_2, \beta_3 \) are the three neighboring vertices and

\[ (t_i, t_i', t_i'') = (t_1, t_2, t_3). \]
Moreover redefine the edge character $E_{\alpha\beta}$ as in \cite{PT09} (Section 4.6):

$$E_{\alpha\beta} = t_1^{-1} \frac{G_{\alpha\beta}(t_2, t_3)}{1 - t_1^{-1}} - \frac{G_{\alpha\beta}(t_2 t_1^{-m_{\alpha\beta}}, t_3 t_1^{-m'_{\alpha\beta}})}{1 - t_1^{-1}}$$ (15.29)

Here the integers $m_{\alpha\beta}$ and $m'_{\alpha\beta}$ are determined by the normal bundle $\mathcal{N}_{C_{\alpha\beta}/X}$ to the supporting curve $C_{\alpha\beta} := \text{Supp}(F_{\alpha\beta})$:

$$\mathcal{N}_{C_{\alpha\beta}/X} = \mathcal{O}(m_{\alpha\beta}) \oplus \mathcal{O}(m'_{\alpha\beta}).$$

Similarly redefine $E_{\alpha\beta\gamma}$ and $E_{\alpha\beta\gamma\delta}$ respectively as:

$$E_{\alpha\beta\gamma} = t_2^{-1} \left( t_1^{-1} \frac{G_{\alpha\beta\gamma}(t_3)}{1 - t_1^{-1}} - \frac{G_{\alpha\beta\gamma}(t_3 t_1^{-m'_{\alpha\beta}})}{1 - t_1^{-1}} \right) \frac{1}{1 - t_2^{-1}}$$

and

$$E_{\alpha\beta\gamma\delta} = t_3^{-1} \left( t_2^{-1} \left( t_1^{-1} \frac{G_{\alpha\beta\gamma\delta}}{1 - t_1^{-1}} - \frac{G_{\alpha\beta\gamma\delta}}{1 - t_1^{-1}} \right) \frac{1}{1 - t_3^{-1}} \right)$$

and

$$- \left( t_1^{-1} \frac{G_{\alpha\beta\gamma\delta}}{1 - t_1^{-1}} - \frac{G_{\alpha\beta\gamma\delta}}{1 - t_1^{-1}} \right) \frac{1}{1 - t_2^{-1}}$$

$$- \left( t_2^{-1} \left( t_1^{-1} \frac{G_{\alpha\beta\gamma\delta}}{1 - t_1^{-1}} - \frac{G_{\alpha\beta\gamma\delta}}{1 - t_1^{-1}} \right) \frac{1}{1 - t_2^{-1}} \right) \frac{1}{1 - t_3^{-1}}$$ (15.30)

According to the above redistributions the $G$-character of the virtual tangent space in (15.22) can be rewritten as:

$$\text{tr}_{R-\chi([\bullet, \bullet])} = \sum_{\alpha} V_{\alpha} + \sum_{\alpha\beta} E_{\alpha\beta} + \sum_{\alpha\beta\gamma} E_{\alpha\beta\gamma\delta} + \sum_{\alpha\beta\gamma\delta} E_{\alpha\beta\gamma\delta}$$ (15.32)

**Remark 15.2.** Given a torus fixed component $Q^k$ of the moduli stack of highly frozen triples (here $k$ denotes the length of the zero dimensional cokernel sheaf associated to the highly frozen triples) denote $V_{Q^k} = \sum_{\alpha} V_{\alpha}$ where $V_{\alpha}$ are defined as in (15.28). By discussions in \cite{PT09} (Section 4.7)
one defines the integral of the evaluation of the contribution of $V_{Q^k}$ on $Q^k$, i.e:

$$w(Q) = \int_Q e(T_Q)e(-V_Q). \quad (15.33)$$

Hence by substituting $w(Q^k)$ in (15.33) in Equation 4.14 of [PT09] one obtains a definition for the equivariant Calabi-Yau vertex associated to the moduli stack of highly frozen triples:

$$W_{Q}^{\text{HFT}} = \sum_k w(Q^k).q^k \quad (15.34)$$

16 Equivariant vertex for local $\mathbb{P}^1$

**Proposition 16.1.** Use the result obtained in Lemma 4.4 and Remark 13.3. Given a $\tau'$-limit stable $G$-equivariant highly frozen triple $O_X^G(-(n)) \xrightarrow{\phi^G} F^G$ of type $(P_F,2)$ with supporting curve $C$ for $F$ consider the finite length $G$-equivariant cokernel $Q^G$ given by $\text{Coker}(\phi)^G$. Then $Q^G \cong Q^T_1 \oplus \cdots \oplus Q^T_r$ such that each $Q^T_i$ for $i = 1, \cdots, r$ is given as a subsheaf of

$$\mathcal{H} = \lim \rightarrow \left( \mathcal{H}om(m^l, \mathcal{O}_C) / \mathcal{O}_C \right). \quad (16.1)$$

In other words a $\tau'$-limit stable $G$-equivariant highly frozen triple of rank $r$ with support $C$ is equivalent to a subsheaf of $\mathcal{H}$ in (16.1) for $r \gg 0$.

**Proof.** Since

$$O_X^G(-(n)) \xrightarrow{\phi^G} F^G := \bigoplus_{i=1}^r (O_X^T(-n) \to F^T_i),$$

each $O_X^T(-n) \to F^T_i$ restricted to the supporting curve of $F_i$, is identified with $Q^T_i$ appearing in

$$0 \to O_C^T(-n) \to F^T_i \to Q^T_i \to 0,$$

and by Proposition 1.8 of [RR09] $Q_i$ is identified with a subsheaf of the quasi-coherent sheaf

$$\lim \rightarrow \mathcal{H}om(m^l, \mathcal{O}_C) / \mathcal{O}_C.$$ 

It is easily seen that the cokernel of the original $G$-equivariant highly frozen triple, restricted to $C$ and identified with $\bigoplus_{i=1}^r Q^T_i$, is a subsheaf of the direct sum of two copies of the same quasi-coherent sheaf. \qed

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Example 16.2. (One legged vertex over Local $\mathbb{P}^1$). Assume that $X$ is given as the total space of $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$ over $\mathbb{P}^1$ (local $\mathbb{P}^1$) and $r = 2$.

There exists two affine patches $U_\alpha$ and $U_\beta$ covering $X$. According to [PT09] (Example 4.9) there exists a combinatorial way of describing $X$ using 3 dimensional Young tableaux diagrams. The partitions associated to the Newton polyhedron of $X$ on each patch are given as three dimensional partitions with $\mu_1 = (1), \mu_2 = (0), \mu_3 = (0)$ (Look at [PT09] (Example 4.9) for more detail on terminology). We compute the 1-legged equivariant structure on $X$. We identify an action of $1$-legged vertex over $\mathbb{P}^1$. Consider $X$ as a quotient $X \cong (\mathbb{C}^4 \setminus Z)/\mathbb{C}^*$ where $Z \subset \mathbb{C}^4$ is obtained by setting $x_0 = x_1 = 0$. Let $([x_0 : x_1, x_2, x_3])$ denote the coordinates in $X$ where $[x_0 : x_1]$ denote the homogeneous coordinates along the base $\mathbb{P}^1$ and $x_2, x_3$ denote the fiber coordinates. Locally in the $U_\alpha$ and $U_\beta$ the defining coordinates are given as $(\frac{x_1}{x_0}, x_2x_0, x_3x_0)$ and $(\frac{x_1}{x_0}, x_2x_1, x_3x_1)$ respectively. Consider $U_\alpha$. Let us denote the local coordinates in this chart by $(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3)$ where $\tilde{x}_1 = \frac{x_1}{x_0}, \tilde{x}_2 = x_2x_0, \tilde{x}_3 = x_3x_0$. Let $H \subset X$ denote the hyperplane obtained as the fiber of $X$ over $0 \in \mathbb{P}^1$, i.e locally in $U_\alpha$ by setting $\tilde{x}_1 = 0$. Throughout this calculation we fix the hyperplane $H$ as a choice of equivariant structure on $\mathcal{O}_X(1)$. Now consider the action of $T' = \mathbb{C}^3$ on $X$ where locally over $U_\alpha$ is given by $(\lambda_1, \lambda_2, \lambda_3) \cdot \tilde{x}_i = \lambda_i \cdot \tilde{x}_i$. We identify an action of $(\mathbb{C}^*)^2$ on $X$ which preserves the Calabi-Yau form by considering a subtorus $T' \subset T$ such that

$$T' = \{ (\lambda_1, \lambda_2, \lambda_3) \in T \mid \lambda_1\lambda_2\lambda_3 = 1 \}. \quad (16.2)$$

Let $\hat{\ell}_1, \cdots, \hat{\ell}_3$ denote the characters corresponding to the action of $\lambda_i$. Identify $\mathcal{O}_X(-1) \cong \mathcal{O}_X(-H)$. Locally over $U_\alpha$ the Poincaré polynomial of $\mathcal{O}_X(-n) |_{U_\alpha}$ is obtained as,$$
\frac{\hat{P}_1}{(1 - \hat{\ell}_1)(1 - \hat{\ell}_2)(1 - \hat{\ell}_3)}.
$$
Restriction to the affine open patch $\beta$ is equivalent to the change of local variables,

$$\hat{\ell}_1 \mapsto \hat{\ell}_1^{-1} \hat{\ell}_2 \mapsto \hat{\ell}_2 \hat{\ell}_1 \hat{\ell}_3 \mapsto \hat{\ell}_3 \hat{\ell}_1. \quad (16.3)$$

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Hence the Poincaré polynomial of $\mathcal{O}_X(-n)\mid_{\mathcal{U}_\beta}$ is obtained by:

$$1 \over (1 - \tilde{t}_1^{-1})(1 - (t_2 \tilde{t}_1))(1 - (t_3 \tilde{t}_1)).$$

Note that in this case the terms $C^m_\alpha$ and $C^n_\beta$ in [15.18] are $\tilde{t}_1^n$ and 1 respectively. Finally the $T$-character of the Poincaré polynomial of $\mathcal{O}_X(-n)\mid_{\mathcal{U}_\alpha}\mid_{\mathcal{U}_\beta}$ is obtained as $(1/((1-t_2)(1-t_3))) \delta(\tilde{t}_1)$. Here the term $C^n_{\alpha,\beta}$ in [15.19] is equal to 1.

To compute the contributions in [15.28] we need to compute the trace character in [15.19] over the two patches $\alpha$ and $\beta$ and the edge redistribution in [15.23]. The former are obtained by:

$$tr R_{-\chi((\bullet)_{\alpha} \cdot (\bullet)_{\beta})} = F^G_\alpha \cdot \left( {w_1^{-1} + w_2^{-1}} \right) - F^G_\alpha \cdot \left( w_1 + w_2 \right) \cdot \tilde{t}_1^n \over t_1 \tilde{t}_2 \tilde{t}_3 + F^G_\alpha \over (1 - \tilde{t}_1)(1 - \tilde{t}_2)(1 - \tilde{t}_3)$$

$$+ F^G_\beta \over (1 - \tilde{t}_1)(1 - \tilde{t}_2)(1 - \tilde{t}_3)$$

$$tr R_{-\chi((\bullet)_{\beta} \cdot (\bullet)_{\alpha})} = F^G_\beta \cdot \left( w_1^{-1} + w_2^{-1} \right) - F^G_\beta \cdot \left( w_1 + w_2 \right) \over t_1 \tilde{t}_2 \tilde{t}_3 + F^G_\beta \over (1 - \tilde{t}_1)(1 - \tilde{t}_2)(1 - \tilde{t}_3)$$

$$+ F^G_\beta \over (1 - \tilde{t}_1^{-1})(1 - (t_2 \tilde{t}_1))(1 - (t_3 \tilde{t}_1))$$

$$\left( 1 - \tilde{t}_1^{-1} \right) \over (1 - \tilde{t}_1) \tilde{t}_2 \tilde{t}_3 \left( 1 - \tilde{t}_1 \right)$$

$$1 - \left( w_1 + w_2 \right) \over (1 - \tilde{t}_1)(1 - \tilde{t}_2)(1 - \tilde{t}_3)$$

(16.4)

Before computing the edge redistribution in [15.23] we compute $F^G_\alpha, F^G_\beta, F^G_{\alpha,\beta}$. Let us for the moment assume that the $G$-fixed locus of the moduli stack is composed of only one component which parametrizes the stable highly frozen triples satisfying the condition that the cokernel of the map $\mathcal{O}_X^{12}(-n)^G \rightarrow F^G$ is given by a skyscraper sheaf which is supported over $k_1$ torus fixed points in $\mathcal{U}_\alpha$ and $k_2$ tours fixed points in $\mathcal{U}_\beta$. Later we extend our computation to the more general case for all such $k_1, k_2$. Let $Q^G$ denote $\text{Coker}(\phi^G)$ associated to $\mathcal{O}_X(-n) \oplus^2 G \rightarrow F^G$. Let $l(Q^G) = k$ be the length of $Q^G$. Suppose $l(Q^G \mid_{\mathcal{U}_\alpha}) = k_1$ and $l(Q^G \mid_{\mathcal{U}_\beta}) = k_2$ hence $k_1 + k_2 = k$. Now use the fact that by construction $Q^G \cong Q^T_1 \oplus Q^T_2$. Let $l(Q^T_1 \mid_{\mathcal{U}_\alpha}) = d_1$ and $l(Q^T_2 \mid_{\mathcal{U}_\alpha}) = d_2$. Moreover assume $l(Q^T_1 \mid_{\mathcal{U}_\beta}) = c_1$ and $l(Q^T_2 \mid_{\mathcal{U}_\beta}) = c_2$. So this means that we have the constraint that $d_1 + d_2 = k_1$ and $c_1 + c_2 = k_2$. Hence:

$$F^G_\alpha = w_1 \cdot \tilde{1}^{-d_1} \over (1 - \tilde{t}_1) + w_2 \cdot \tilde{1}^{-d_2} \over (1 - \tilde{t}_1), F^G_\beta = w_1 \cdot \tilde{c}_1 \over (1 - \tilde{t}_1) + w_2 \cdot \tilde{c}_2 \over (1 - \tilde{t}_1)$$

(16.5)
Now we consider the case that the $G$-fixed locus of the moduli stack contains more than one component. In this case to compute the contribution of box configurations one needs to consider all possible tuples of six integers $(k_1, k_2, d_1, d_2, c_1, c_2)$ such that for a fixed value of $k$ the following three relations are satisfied: $d_1 + d_2 = k_1$, $c_1 + c_2 = k_2$ and $k_1 + k_2 = k$. Hence we obtain the following identities:

$$
P^G_\alpha = \sum_{d_1 + d_2 = k_1} \left( w_1 \cdot \frac{\tilde{t}_1^{d_1}}{1 - \tilde{t}_1} + w_2 \cdot \frac{\tilde{t}_2^{d_2}}{1 - \tilde{t}_2} \right)$$

$$
P^G_\beta = \sum_{c_1 + c_2 = k_2} \left( w_1^{-1} \cdot \frac{\tilde{t}_1^{c_1}}{1 - \tilde{t}_1^{-1}} + w_2^{-1} \cdot \frac{\tilde{t}_2^{c_2}}{1 - \tilde{t}_2^{-1}} \right)$$

for all $k_1, k_2$ such that $k_1 + k_2 = k$. 

(16.6)

The trace character in (15.18) is obtained as:

$$
tr_{R-\chi(\bullet, \bullet, \bullet)} = (w_1^{-1} + w_2^{-1}) \cdot \sum_{d_1 + d_2 = k_1} \left( w_1 \cdot \frac{\tilde{t}_1^{n-d_1}}{1 - \tilde{t}_1} + w_2 \cdot \frac{\tilde{t}_2^{n-d_2}}{1 - \tilde{t}_2} \right)
$$

$$
- (w_1 + w_2) \cdot \frac{1}{t_1 t_2 t_3} \sum_{d_1 + d_2 = k_1} \left( w_1^{-1} \cdot \frac{\tilde{t}_1^{n+d_1}}{1 - \tilde{t}_1^{-1}} + w_2^{-1} \cdot \frac{\tilde{t}_2^{n+d_2}}{1 - \tilde{t}_2^{-1}} \right)
$$

$$
- \sum_{d_1 + d_2 = k_1} \left[ \frac{\tilde{t}_1^{n_1-d_1}}{1 - \tilde{t}_1} + \frac{\tilde{t}_2^{n_2-d_2}}{1 - \tilde{t}_2} \right] 
$$

$$
\cdot \frac{(1 - \tilde{t}_2)(1 - \tilde{t}_3)}{t_2 t_3} + \frac{1 - (w_1 + w_2)^2}{w_1 w_2} \cdot \frac{1}{(1 - \tilde{t}_1)(1 - \tilde{t}_2)(1 - \tilde{t}_3)}. 
$$

(16.7)
The trace character over $U_\beta$ is obtained as:

$$
tr_{R-\chi^{(\ast,\ast)}_\beta} = (w_1^{-1} + w_2^{-1}) \cdot \sum_{c_1+c_2=k_2} \left( w_1 \cdot \frac{\tilde{t}_1^{c_1}}{(1-\tilde{t}_1)} + w_2 \cdot \frac{\tilde{t}_1^{c_2}}{(1-\tilde{t}_1)} \right) - (w_1 + w_2) \cdot \frac{1}{\tilde{t}_1^{-1}(\tilde{t}_2\tilde{t}_1)(\tilde{t}_3\tilde{t}_1)} \sum_{c_1+c_2=k_2} \left( w_1 \cdot \frac{\tilde{t}_1^{-c_1}}{(1-\tilde{t}_1)} + w_2 \cdot \frac{\tilde{t}_1^{-c_2}}{(1-\tilde{t}_1)} \right) + \sum_{c_1+c_2=k_2} \left[ \frac{\tilde{t}_1^{c_1-m_1+1}}{(1-\tilde{t}_1)} + \frac{w_1 \tilde{t}_1^{c_1-m_2+1}}{w_2 (1-\tilde{t}_1)} + \frac{w_2 \tilde{t}_1^{c_2-m_1+1}}{w_1 (1-\tilde{t}_1)} + \frac{\tilde{t}_1^{c_2-m_2+1}}{(1-\tilde{t}_1)} \right] \cdot \frac{(1-(\tilde{t}_2\tilde{t}_1))(1-(\tilde{t}_3\tilde{t}_1))}{(\tilde{t}_2\tilde{t}_1)(\tilde{t}_3\tilde{t}_1)} + \frac{1-\frac{(w_1+w_2)^2}{w_1w_2}}{(1-\tilde{t}_2)(1-\tilde{t}_3)}. \tag{16.8}
$$

By equation (15.23), $G_{\alpha\beta}$ is obtained as:

$$
G_{\alpha\beta}(\tilde{t}_2, \tilde{t}_3) = (w_1^{-1} + w_2^{-1}) \cdot \tilde{t}_1^{-n} \cdot \frac{1}{\tilde{t}_2\tilde{t}_3} + \frac{1-(w_1+w_2)^2}{(1-\tilde{t}_2)(1-\tilde{t}_3)}. \tag{16.9}
$$

Now by equation (15.28), the contributions $V_\alpha$ and $V_\beta$ are obtained by the following equations:

$$
V_\alpha = tr_{R-\chi^{(\ast,\ast)}_\beta} + \frac{G_{\alpha\beta}(\tilde{t}_2, \tilde{t}_3)}{(1-\tilde{t}_1)} \tag{16.10}
$$

$$
V_\beta = tr_{R-\chi^{(\ast,\ast)}_\beta} + \frac{G_{\alpha\beta}(\tilde{t}_2, \tilde{t}_3)}{(1-\tilde{t}_1)} \tag{16.10}
$$

Let $Q^k$ denote the $G$-fixed component of the moduli stack of rank 2 highly frozen triples over which the highly frozen triples $O_X^\infty(-n)G \to F^G$ satisfy the condition that $l(Coker(\phi)^G) = k$. By (15.32) and (16.10), the $G$-equivariant vertex character written as $V_{Q^k} = V_\alpha + V_\beta$ is obtained as:

$$
V_{Q^k} = \sum_{d_1+d_2=k} \left( (w_1^{-1} + w_2^{-1}) \cdot \left( w_1 \cdot \sum_{i=1}^{d_1} \tilde{t}_1^{i-n} + w_2 \cdot \sum_{i=1}^{d_2} \tilde{t}_1^{-i-n} \right) \right) - (w_1 + w_2) \cdot \left( w_1^{-1} \cdot \sum_{i=0}^{d_1} \frac{\tilde{t}_1^{i+n}}{\tilde{t}_2\tilde{t}_3} + w_2^{-1} \cdot \sum_{i=0}^{d_2} \frac{\tilde{t}_1^{-i+n}}{\tilde{t}_2\tilde{t}_3} \right). \tag{16.11}
$$

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Note the similarity between (16.11) and the vertex obtained in [PT09] (Lemma 5). Now let $s_i$ for $i = 1, 2, 3$ and $v_j$ for $j = 1, 2$ denote the equivariant parameters corresponding to characters $\tilde{t}_i$ and $w_j$ respectively. By the definition of the equivariant vertex in (15.2) the coefficient of the degree $k$ term in the equivariant vertex in (15.34) is obtained by the integral of the evaluation of the contribution of $V_{Q^k}$ on $Q^k$, i.e:

$$w(Q^k) = \int_{Q^k} e(T_{Q^k})e(-V_{Q^k}) =$$

$$\prod_{d_1 + d_2 = k} \left[ \frac{v_1(v_2 - 1) + \prod_{i=0}^{d_1-1} ((i + n)s_1) - (s_2 + s_3)}{v_1(1 - v_2) + \prod_{i=1}^{d_1} (-1)^i \cdot (i + n)s_1} \cdot \frac{v_2(v_1 - 1) + \prod_{i=0}^{d_2-1} ((i + n)s_1) - (s_2 + s_3)}{v_2(1 - v_1) + \prod_{i=1}^{d_2} (-1)^i (i + n)s_1} \right].$$

(16.12)

It is easy to see that following the same calculation the $k$'th coefficient of the equivariant vertex for $r > 2$ case is obtained as:

$$w(Q^k) = \prod_{d_1 + d_2 = k} \left[ \frac{\prod_{j=1}^{r} ((-v_j + \prod_{l=1}^{r} v_l) + \prod_{i=1}^{d_j-1} ((i + n)s_1) - (s_2 + s_3))}{\prod_{j=1}^{r} ((v_j + (-1)^{r-1} \prod_{l=1}^{r} v_l) + \prod_{i=1}^{d_j} (-1)^i (i + n)s_1)} \right].$$

(16.13)

**Remark 16.3.** Setting $v_1 = v_2 = 1$ in (16.12) would result in the following equation:

$$w(Q^k) = \prod_{d_1 + d_2 = k} \left[ \frac{\prod_{i=0}^{d_1-1} ((i + n)s_1) - (s_2 + s_3)) \cdot \prod_{i=0}^{d_2-1} ((i + n)s_1) - (s_2 + s_3))}{\prod_{i=1}^{d_1} (-1)^i (i + n)s_1) \cdot \prod_{i=1}^{d_2} (-1)^i (i + n)s_1)} \right].$$

(16.14)

Now use the condition on Calabi-Yau torus and set $s_1 + s_2 + s_3 = 0$. This is equivalent to $ns_1 - (s_2 + s_3) = -(n + 1)(s_2 + s_3)$. Now use this fact to
simplify (16.13) and obtain:

\[ w(Q^k) = \prod_{d_1 + d_2 = k} \left[ \left( \prod_{i=0}^{d_1-1} (i - (n + 1) \frac{s_2 + s_3}{s_1}) \right) \cdot \left( \prod_{i=0}^{d_2-1} (i - (n + 1) \frac{s_2 + s_3}{s_1}) \right) \right] \]

\[ = \prod_{d_1 + d_2 = k} \left[ \frac{(-1)^{d_2+d_1} (n + d_1)! (n + d_2)!}{(n + 1)!} \right] \]

(16.15)

which is (up to a shift by \( n + 1 \)) the \( q^k \)th coefficient of the generating series associated to the equivariant vertex given by:

\[ W_{\text{HFT}}^{1,0,0} |_{r=2} = \left( 1 + q \frac{(n+1)(s_2+s_3)}{s_1} \right)^2. \]

(16.16)

This result confirms the identity in Equation (14.2). Moreover, similar to above, applying the same simplification to (16.13) we obtain the 1-legged equivariant vertex over local \( \mathbb{P}^1 \) associated to highly frozen triples:

\[ W_{\text{HFT}}^{1,0,0} |_{r>2} = \left( 1 + q \frac{(n+1)(s_2+s_3)}{s_1} \right)^r. \]

(16.17)

**Remark 16.4.** The computation of equivariant vertex for more general local toric Calabi-Yau threefolds with outgoing partitions \( <\mu_1, \mu_2, \mu_3> \) requires more detailed combinatorial calculations. However, one can compute the associated partition functions in those general cases if one fully understands the equivariant PT vertex in rank 1. In other words it is seen from our calculations that if the PT vertex with respect to variable \( q \) is given by \( W_{<\mu_1,\mu_2,\mu_3>}^P = G(q) \) then the HFT partition function is obtained by

\[ W_{<\mu_1,\mu_2,\mu_3>}^{\text{HFT}} |_{r} = (G(q))^{(n+1)(r)}. \]

**Remark 16.5.** Note that in rank 2 case unless special choices are made for the values of \( w_1, w_2 \), the result of the theory depends on \( v_1, v_2 \) and the choice of twisting \( n \). However, it is interesting to point out that by looking at (15.23) and (15.18) it is observed that having the characters \( w_1, w_2 \) to satisfy the relation \( w_1 + w_2 = 0 \) would eliminate all terms in (15.23) and (15.18) with \( \tilde{t}_1^n \) or \( \tilde{t}_1^{-n} \) occurrences and hence results in a theory independent of the twisting \( n \). Moreover similar situation exists when carrying out calculations for rank> 2. This seems to be a very interesting feature of the theory of highly frozen triples which we will approach in more detail in our future endeavors.
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