Existence and Construction of Exact FRG Flows of a UV-Interacting Scalar Field Theory

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We prove the existence and give a construction procedure of Euclidean-invariant exact solutions to the Wetterich equation[1] in \( d > 2 \) dimensions satisfying the naive boundary condition of a massive and interacting real scalar \( \phi^4 \) theory in the ultraviolet limit as well as a generalised free theory in the infrared limit. The construction produces the momentum-dependent correlation functions to all orders through an iterative scheme, based on a self-consistent ansatz for the four-point function. The resulting correlators are bounded at all regulator scales and we determine explicit bounds capturing the asymptotics in the UV and IR limits. Furthermore, the given construction principle may be extended to other systems and might become useful in the study of general properties of exact solutions.

I. INTRODUCTION

In quantum field theory and related fields one rarely has access to exact expressions for quantities of interest. Instead, one generally resorts to approximation schemes such as truncations of power series or lattice discretisations. But the use of such approximations raises the question of their respective reliability. In terms of observables, one is interested in quantitative bounds on deviations from exact values. However, the necessity of renormalisation turns the analysis of such deviations into a complicated task. They are commonly studied by investigating artificial regulator dependencies, the apparent convergence of truncation schemes or by purely qualitative methods such as apparent stability of features like fixed points or phase transitions. Nonetheless, it usually remains very difficult and often practically impossible to provide quantitative bounds on absolute errors and hence to explicitly specify the region of applicability of any given approximation procedure.

There are some notable exceptional cases in which exact results have been obtained such as the Schwinger model[2], the Thirring model[3] and the lattice \( \phi^4 \) and \( \phi^4_{d>4} \) theories[4,5]. Further exact results in quantum field theoretical models[6], condensed matter physics[7] as well as in hydrodynamics[8] and statistical mechanics[9] have been obtained through the use of the "functional renormalisation group" which is also at the core of this paper. It constitutes a renormalisation scheme of the path integral quantisation and leads to well known expressions for the renormalisation group flow of the "functional renormalisation group flow" of the correlation functions of the quantum field theory at hand. As will be demonstrated, it is possible to bootstrap formally (in the sense of not necessarily analytic) exact solutions to these equations by providing a well-behaved, consistent set of low-order correlation functions and giving an explicit construction procedure for the higher-order ones.

In this paper the above method is employed to construct exact solutions to the Wetterich equation for quantum field theories on Euclidean spacetimes of dimensions \( d > 2 \) that satisfy the naive boundary conditions of massive and interacting real scalar \( \phi^4 \) theories in the classical limit. This boundary condition corresponds to strictly finite renormalisations of all coupling constants and hence does not agree with the rigorously known results for the \( \phi^4_3 \) theory. In particular, the constructed solutions are shown to correspond to generalised free quantum field theories.

Nonetheless, I believe that exact solutions may provide good grounds for further research on the functional renormalisation group and its applications. Through their constructive nature the solutions given in this paper may also be able to open the door to more rigorous error estimates because the knowledge of bounds on lower-order correlators may be employed to produce bounds on higher-order ones.

II. THE FUNCTIONAL RENORMALISATION GROUP

Let us start with the Euclidean path integral quantisation of a classical action \( S_\Lambda \) for a real scalar field at a UV regularisation scale \( \Lambda > 0 \). Then

\[
\exp \left[ -\Gamma (\phi) \right] = \int \mathcal{D}_\Lambda \psi \exp \left[ -S_\Lambda (\phi + \psi) + \left( \mathcal{D}_\phi \Gamma \right) (\psi) \right],
\]

where \( \mathcal{D}_\Lambda \) denotes the regularised path integral measure and \( \Gamma \) is the effective action. For clarity and brevity we shall use Fréchet derivatives instead of functional deriva-
where $\Gamma^{(2)}_{k,A}$ denotes the second derivative of $\Gamma_{k,A}$ at $\phi$ interpreted as an operator. For particularly well-behaved regulators one may now simply take the limit $\Lambda \to \infty$ in this equation, removing the necessity of a UV cutoff $\Lambda$ and leading to the “$\Lambda$-free” $\varphi$ action.

Let us refer to the resulting object of interest as $\partial_k \Gamma_{k,A}(\phi)$ which may be achieved inductively by noting that

$$ \left( D|_{\phi} A \right) (\psi) = -A(\phi) \circ \left( D|_{\phi} \Gamma^{(2)}_{k} \right) (\psi) \circ A(\phi) , $$

or for short

$$ D|_{\phi} A = -A \circ D \Gamma^{(2)}_{k} \circ A. $$

An educated guess produces the induction hypothesis

$$ D^n A = \sum_{c \in C(n)} (-1)^{\#c} \frac{n!}{c!} A \circ \prod_{l=1}^{\#c} \left[ D^{c_l} \Gamma^{(2)}_{k} \circ A \right] , $$

where $C(n)$ denotes the set of all multi-indices with positive entries that are combinations of the natural number $n$, e.g.

$$ C(3) = \{(1,1,1), (1,2), (2,1), (3)\} . $$

In equation $[9]$ $\#c$ is the length of such a multi-index and

$$ c! = \prod_{l=1}^{\#c} (c_l!) , \quad |c| = \sum_{l=1}^{\#c} c_l = n $$

for all $n \in \mathbb{N}$ and any $c \in C(n)$. The inductive proof of equation $[9]$ is given in appendix $[A]$. Inserting this result into equation $[6]$ then yields

$$ \partial_k D^n |_{0} \Gamma_{k}(\phi^{\otimes n}) = \sum_{c \in C(n)} (-1)^{\#c} \frac{n!}{c!} \text{Tr} \left\{ \left( D^{c_1} A \right)(0) \left( D^{c_2} A \right)(0) \right\} . $$

Equation $[12]$ expresses all possible one-loop diagrams generated by an arbitrary action $\Gamma_{k}$ contributing to the renormalisation group flow of a given correlation function. As is common practice, we shall work with them explicitly in the Fourier picture. Restricting ourselves to translation-invariant quantum field theories, for every $n \in \mathbb{N}$ there is a $(k$-dependent) function $\kappa_n$ such that

$$ (D^n |_{0} \Gamma_{k})(\phi_1 \otimes \ldots \otimes \phi_n) = \left( 2\pi \right)^{\frac{d}{2}} \int_{(\mathbb{R}^d)^{n-1}} \kappa_n (p_1, \ldots, p_{n-1}; k) \phi_1 (p_1) \ldots \phi_{n-1} (p_{n-1}) $$

$$ \hat{\phi}_n (- [p_1 + \ldots + p_{n-1}]) \, dp_1 \ldots dp_{n-1} $$

for all test functions $\phi_1, \ldots, \phi_n$. These $\kappa_n$ are precisely the commonly considered one-particle irreducible

\footnote{i.e. for all test functions $\psi_1, \psi_2$ on $\mathbb{R}^d$ we have

$$ \left\langle \psi_1^{(2)} \Gamma^{(2)}_{k,A} \psi_2 \right\rangle = \int_{\mathbb{R}^d} \psi_1^{(2)} \Gamma^{(2)}_{k,A} \psi_2 = \left\langle D^2 \Gamma_{k} \right\rangle (\psi_1, \psi_2) . $$

\footnote{Partitions including permutations

\footnote{We define the Fourier transform $\hat{f}$ of a measurable function $f : \mathbb{R}^d \to \mathbb{R}$ as

$$ \hat{f} (\mu) = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} \exp [-ipx] f (x) \, dx $$

whenever the integral converges.

\footnote{The prefactors $(2\pi)^{\frac{d}{2}}$ are chosen such that they vanish in position space.}}
n-point functions in Fourier space stripped of their delta-functions:
\[ \Gamma_k^{(n)}(p_1, ..., p_n) = \kappa_n(p_1, ..., p_{n-1}) \delta(p_1 + ... + p_n) \]  
Consequently, any such \( D^n \mid_0 \Gamma_k \) is translation invariant in the sense that
\[ (D^n \mid_0 \Gamma_k)(T \phi_1 \otimes ... \otimes T \phi_n) = (D^n \mid_0 \Gamma_k)(\phi_1 \otimes ... \otimes \phi_n) \]
for all translations \( T \) of \( \mathbb{R}^d \) by the properties of the Fourier transform. Furthermore, such a \( D^n \mid_0 \Gamma_k \) is obviously \( \mathcal{O}(d) \)-invariant whenever the corresponding \( \kappa_n \) is.\(^5\) To simplify equations from this point on, any \( k \)-dependence will be suppressed whenever it does not lead to ambiguities. Since Fréchet derivatives are invariant under permutations there are corresponding symmetries of the \( \kappa_n \): For all \( \sigma \in \Sym_{n-1} \)
\[ \kappa_n(p_{\sigma(1)}, ..., p_{\sigma(n-1)}) = \kappa_n(p_1, ..., p_{n-1}) \]
(16) and also
\[ \kappa_n(p_1, ..., p_{n-1}) = \kappa_n(−[p_1 + ... + p_{n-1}], p_2, ..., p_{n-1}) \]  
(17)
for all \( p_1, ..., p_{n-1} \in \mathbb{R}^d \). We shall refer to functions \( f \) satisfying these symmetries as \( \Sym_{n-1}^* \) symmetrical.\(^6\) It remains to phrase equation \([12] \) in terms of the correlation functions \( \kappa_n \). While the left-hand side is simple, let us look at the right-hand side first: If the expression within the trace is viewed as an integral operator the trace can be evaluated by integration along the diagonal. From the definition of the \( \kappa_n \) we already know the integral form of the derivatives of \( \Gamma_k \) and it only remains to express \( R_k \) appropriately. It is common practice to define \( R_k \) in momentum space as a family of multiplication operators parameterised by \( k \), i.e.
\[ \left[ \mathcal{F}R_k \mathcal{F}^{-1} \phi \right](p) = \tilde{r}(p; k) \phi(p) \]
(18)
for some \( \tilde{r} : \mathbb{R} \times \mathbb{R}_{>0} \to \mathbb{R} \). The role of \( \tilde{r} \) is to contribute a “momentum-dependent mass” that protects against IR singularities and at the same time screens UV divergences at any finite scale \( k > 0 \) by a rapid decay for large momenta. Thus, \( \tilde{r} \) and \( k_2 \) have to be treated on similar footings so that we have to demand
\[ \tilde{r}(q) = \tilde{r}(-q) \]
(19)
for all \( q \in \mathbb{R}^d \) in accordance with the \( \Sym_{n-1}^* \) symmetry of \( k_2 \). Choosing a \( \Sym_{n-1}^* \)-violating regulator would generate further symmetry-breaking terms leading to undesirable contributions that are not translation invariant.

The trace in equation \([12] \) then becomes
\[ \text{Tr} \{ ... \} = (2\pi)^{−|c|} \int_{(\mathbb{R}^d)^{−|c|}} \lambda_c(p_1, ..., p_{|c|−1}) \]
\[ \times \tilde{\phi}(p_1) ... \tilde{\phi}(p_{|c|−1}) \]  
(20)
\[ \times \tilde{\phi}(−[p_1 + ... + p_{|c|−1}]) \, dp_1 ... dp_{|c|−1} , \]
where
\[ \lambda_c(p_1, ..., p_{|c|−1}) = \int_{\mathbb{R}^d} \frac{(\partial_k \tilde{r})(q)}{\tilde{r}(q) + \tilde{r}(-q)}^{-2} \]
\[ \kappa_{|2+c|+c} (p_1^{#c+1}, ..., p_{|c|−1}, 1, \sum_{a=1}^{#c+1} \sum_{b=1}^{#c−1} p_1^a q^b) \]
\[ \Pi_{l=1}^{#c−1} \left[ \kappa_{2+cl} (p_1^{1,1} p_{c+1}^{l}, q - \sum_{a=1}^{#c−1} q^a b_1 p_b^a) \right] dq \]  
(21)
This represents an integral over an arbitrary one-loop diagram containing all possible vertices in closed form.

Let us now collect the above and rewrite equation \([12] \) as
\[ 0 = \int_{(\mathbb{R}^d)^{n−1}} \left[ (2\pi)^{d} (\partial_k \kappa_n)(p_1, ..., p_{n−1}) \right] \]
\[ \times \tilde{\phi}(−[p_1 + ... + p_{n−1}]) \tilde{\phi}(p_1) ... \tilde{\phi}(p_{n−1}) \]
\[ \times \sum_{c \in \mathcal{C}(n)} \left( −1 \right)^{#c} n_1 c ! \lambda_c(p_1, ..., p_{n−1}) \, dp_1 ... dp_{n−1} . \]
Before the fundamental lemma of the calculus of variations may be invoked here, we need to polarise this equation, allowing for arbitrary test functions of the form \( \phi_1 \otimes ... \otimes \phi_n \) instead of purely diagonal ones \( \phi^\otimes_n \). However, this polarisation will leave the \( \kappa_n \) part invariant (after proper substitutions of the integral variables) by equation \([13] \) due to its \( \Sym_{n−1}^* \) symmetry. Therefore, such a polarisation is exactly the same as a \( \Sym_{n−1}^* \) symmetrisation. Hence, simply defining
\[ \tilde{\lambda}_c(p_1, ..., p_{n−1}) = \frac{1}{n!} \sum_{\sigma \in \Sym_n} \lambda_c(p_{\sigma(1)}, ..., p_{\sigma(n−1)}) \]  
(23)
where we set \( p_n = −[p_1 + ... + p_{n−1}] \) sidesteps the explicit polarisation. Invoking the fundamental lemma of the calculus of variations then leads to
\[ \partial_k \kappa_n = \frac{1}{2 (2\pi)^d} \sum_{c \in \mathcal{C}(n)} (−1)^{#c} n_1 c ! \tilde{\lambda}_c . \]  
(24)
\(^5\) The action of \( \mathcal{O}(d) \) on a function \( g : (\mathbb{R}^d)^n \to \mathbb{R} \) is the standard one, defined as
\[ (Og)(p_1, ..., p_n) = g(O^{-1}p_1, ..., O^{-1}p_n) \]
for all \( O \in \mathcal{O}(d) \) and \( p_1, ..., p_n \in \mathbb{R}^d \).

\(^6\) \( \Sym \) standing for the symmetric (permutation) group and “s” for the involution given by
\[ (p_1, ..., p_{n−1}) \mapsto (−[p_1 + ... + p_{n−1}], p_2, ..., p_{n−1}) . \]
The full group \( \Sym_{n−1}^* \) of symmetries is isomorphic to \( \Sym_n \) but the underlying action is a non-standard one on \( (n−1)\)-tuples, hence the alternative naming.

\(^7\) We use \( \tilde{r} \) to avoid confusion with the commonly used shape function \( r \) defined as
\[ r(p) = p^2 r \left( \frac{p^2}{k^2} \right) . \]
This is an equivalent formulation of equation [12] and will be referred to as the flow equation of the correlation function $\kappa_n$. While these equations for arbitrary $n \in \mathbb{N}$ are certainly implied by the A-free form of equation [4] if

- $\Gamma_k$ is analytic,
- $\phi \mapsto \left( \Gamma_k^{(2)} \bigg| \phi \right)^{-1}$ is analytic,
- the sum in equation [5] may be pulled out of the trace

the converse is not necessarily true: A given solution might not correspond to an analytic $\Gamma_k$, that is the formal series

$$\sum_{n=1}^{\infty} \frac{(2\pi)^{\frac{d}{2}(2-n)}}{n!} \int_{[0,1]^{n-1}} \kappa_n (p_1, ..., p_{n-1}) \times \hat{\phi} (\{- [p_1 + ... + p_{n-1}]\}) \hat{\phi} (p_1) ... \hat{\phi} (p_{n-1}) \ dp_1 ... dp_{n-1}$$

might diverge for some non-zero test function $\phi$. Nonetheless, in the study of differential equations a lot of insight is often gained by an initial broadening of the space of admissible solutions and in this spirit, one might even expect such formal solutions to be very important for the general study of the Wetterich equation.

One further remark is in order at this point: Upon solving equation [4] it is not clear whether there always exists a corresponding $S_\lambda$ satisfying equation [3] amounting to the reconstruction problem [12]. Especially, a possible non-uniqueness of solutions to equation [4] casts doubts on a positive conjecture. The situation is made even less clear by studying solutions to the A-free version of the Wetterich equation due to the difficulty of non-regularised path integrals.

### III. A CONSTRUCTIVE SOLUTION FOR THE CORRELATION FUNCTIONS

A full solution to the flow equations [24] with $\kappa_m \neq 0$ for some $m \in \mathbb{N} \geq 3$ of course seems rather difficult to find due to the non-linear structure of the $\lambda_c$ terms. This is the reason why one in practise usually truncates the equations at a finite $n \in \mathbb{N}$. There are, however, precisely three terms on the right hand side of equation [24] for $n \in \mathbb{N} \geq 3$ revealing a somewhat linearish structure, namely

- $c = (n) \quad \Rightarrow \lambda_c$ depends linearly on $\kappa_{n+2}$
- $c = (n-1, 1) \quad \Rightarrow \lambda_c$ depends linearly on $\kappa_{n+1}$
- $c = (1, n-1) \quad \Rightarrow \lambda_c$ depends linearly on $\kappa_{n+1}$

Phrased differently, for all $n \in \mathbb{N} \geq 3$ there exist linear operators $I_n$ implicitly depending on $\{\kappa_2, r\}$ and $J_n$ implicitly depending on $\{\kappa_2, \kappa_3, F\}$ such that

$$I_n \kappa_{n+2} = -2 (2\pi)^d \partial_k \kappa_n + n J_n \kappa_{n+1}$$

$$+ \sum_{c \in C(n) \setminus \{(n), (n-1, 1), (1, n-1)\}} (-1)^{c^t} n! \frac{\partial c}{\partial \lambda_c}.$$  

The significance of this equation lies in the fact, that the right-hand side depends only on $\{\kappa_2, \kappa_{n+1}, r\}$. Suppose now, that all $I_n$ possess right inverses $\rho_n$, i.e mappings such that $I_n \circ \rho_n = \text{id}$. Then, setting

$$\kappa_{n+2} = \rho_n \left[ -2 (2\pi)^d \partial_k \kappa_n + n J_n \kappa_{n+1}$$

$$+ \sum_{c \in C(n) \setminus \{(n), (n-1, 1), (1, n-1)\}} (-1)^{c^t} n! \frac{\partial c}{\partial \lambda_c} \right]$$

will evidently solve equation [27]. This fact suggests the following approach for solving the flow equation for the correlators:

1. For some $N \in \mathbb{N}$ find $\kappa_1, ..., \kappa_{N+1}$ satisfying equation [24] for all $n \in \mathbb{N} \leq N$
2. Find a right inverse $\rho_N$ of $I_N$
3. Construct $\kappa_{N+2}$ as in equation [28]
4. Increase $N$ by 1 and go back to step 2

This iterative construction will produce $\kappa_n$ for all $n \in \mathbb{N}$ and they will satisfy their respective flow equation. Evidently, this construction depends crucially on the initial $\kappa_1, ..., \kappa_{N+1}$ which have to be given as input for all values of momenta and the scale $k$. Furthermore, in every iteration there may be several right inverses to choose from because the kernel of any $I_n$ might be non-empty. Hence, this procedure is quite different from the usual approach of giving specific boundary conditions at some scale $k$ (or at $k \to \infty$). In fact, it shall be demonstrated that imposing the naive boundary condition of a real scalar $\phi^4$ theory in the UV limit $k \to \infty$ does not guarantee the uniqueness of solutions to equation [24]. However, before diving into the specifics of $\phi^4$ theory, we shall give explicit expressions for the $I_n$ and particularly simple choices of linear right inverses $\rho_n$. For brevity, define

$$K(q) = \frac{\langle \partial_k \bar{r} \rangle (q)}{(\kappa_2(q) + \bar{r}(q))^2},$$

allowing to write

$$\lambda_{(n)} (p_1, ..., p_{n-1}) = \int_{\mathbb{R}^d} K(q) \times \kappa_{n+2} (p_1, ..., p_{n-1}, -[p_1 + ... + p_{n-1}], q) \ dq.$$  

By the $\text{Sym}^*_{n+1}$ symmetry of $\kappa_{n+2}$, this is $\text{Sym}^*_{n-1}$ symmetric such that $\bar{\lambda}_{(n)} = \lambda_{(n)}$ and

$$\bar{\lambda}_{(n)} (p_1, ..., p_{n-1})$$

$$= \int_{\mathbb{R}^d} K(q) \kappa_{n+2} (p_1, ..., p_{n-1}, -q, q) \ dq.$$  

(31)
Thus one may write

\[(I_n f)(p_1, \ldots, p_{n-1}) = \int_{\mathbb{R}^d} K(q) f(p_1, \ldots, p_{n-1}, -q, q) dq\] (32)

for all functions \(f: (\mathbb{R}^d)^{n+1} \rightarrow \mathbb{R}\) where the integral exists. The reason for allowing arbitrary functions \(f\) and not just \(\text{Sym}_{n+1}^*\) symmetric ones is to facilitate the proof given in appendix B that the yet to be defined \(\rho_n\) are indeed right inverses of the corresponding \(I_n\). An obvious choice of linear right inverse of the above \(I_n\) is given by

\[(\tilde{\rho}_n g)(p_1, \ldots, p_{n+1}) = \frac{g(p_1, \ldots, p_{n-1})}{\int_{\mathbb{R}^d} K} \] (33)

However, in general such \(\tilde{\rho}_n g\) will not be \(\text{Sym}_{n+1}^*\) symmetric whenever \(g\) is \(\text{Sym}_{n-1}^*\) symmetric which is unacceptable here, as it would generate terms that are not momentum conserving. Taking \(\tilde{\rho}_n\) as an ansatz and successively eliminating all \(\text{Sym}_{n+1}^*\)-violating terms generated by the action of \(\text{Sym}_{n+1}^*\) on functions of the form \(\rho_n g\) where \(g\) is taken to be \(\text{Sym}_{n-1}^*\) symmetric leads to the much better choice

\[(\rho_n g)(p_1, \ldots, p_{n+1}) = \sum_{J \subseteq \{0, \ldots, n+1\}} \sum_{l=0}^{n-l-\#J} \frac{\alpha_{a,b}^{n,l,J}}{(\int_{\mathbb{R}^d} K)^{n-\#J-1}} \int_{(\mathbb{R}^d)^{n-\#J-1}} g(p_1, \ldots, p_{n-1-J-1}) \times K(s_1) \cdots K(s_l) K(t_1) \cdots K(t_{n-1-J-1}) d_{s_1} \cdots d_{t_l} \] (34)

with

\[\alpha_{a,b}^{n,l,J} = \frac{(-1)^{n-1-a-b}}{n} 2^{n-1-a-2b} \binom{n-1}{b} \] (35)

In the above expression we have defined \(p_0 = -[p_1 + \ldots + p_{n+1}]\) and introduced the shorthand notation \(p_J := p_{j_1}, \ldots, p_{j_{\#J}}\). Note that the particular order of the corresponding momenta \(p\) in the above expression does not matter since \(g\) is presumed symmetric. Hence, we do not need another sum over all permutations of index sets \(J\). For a proof that \(\rho_n\) when restricted to \(\text{Sym}_{n-1}^*\)-symmetric functions is indeed a right inverse of \(I_n\) see appendix B.

It is obvious that \(\rho_n\) is a linear operator and thus a particular simple choice of right inverse of \(\kappa_n\). Furthermore, it preserves \(O(d)\)-invariance provided \(K\) itself is \(O(d)\)-invariant. In our naive approach to \(O^d\) theory, we shall consider a two-point function that does not scale with \(k\) and approximates the free propagator

\[\kappa_{2,\text{free}}(p) = m^2 + \|p\|^2\] (36)

for some mass \(m\). Hence, any \(k\) scaling of \(K\) comes from the choice of a regulator. Furthermore, common regulators scale like \(k^2\) at small momenta leading to an overall \(k\) scaling of \(K\) as \(k^{-3}\). A simple power counting in equation \(B4\) then reveals that \(\rho_n\) scales like \(k^{d-3}\). This fact is remarkable, as it indicates that in \(d > 3\) dimensions the correlators constructed through \(\rho_n\) are strongly suppressed for large \(k\). This simplifies the control of the “classical limit” \(k \rightarrow \infty\), as one usually considers only a finite set of nonzero correlation functions in this limit. The small \(k\) behaviour is precisely the opposite. Here \(\rho_n\) grows arbitrarily large, possibly leading to IR divergences.

IV. SOLVING THE FLOW EQUATIONS

We shall consider a real scalar quantum field theory in \(d\) Euclidean dimensions without spontaneous symmetry breaking with the “classical limit”

\[
\lim_{k \rightarrow \infty} \kappa_2(p) = \kappa_{2,\text{free}}(p) = m^2 + |p|^2 \\
\lim_{k \rightarrow \infty} \kappa_4(p, q, r) = \frac{\lambda}{|m|^{d-4}} \\
\forall n \in \mathbb{N} \setminus \{2, 4\} : \lim_{k \rightarrow \infty} \kappa_n = 0
\] (37)

for some \(m \in \mathbb{R}\), \(\lambda > 0\) where the limits should be understood in a distributional sense. In particular, for \(k \rightarrow \infty\) all odd correlation functions vanish. We shall now set \(N = 3\) and proceed as outlined in the preceding section. The reason for setting \(N = 3\) is of course to be able to satisfy the boundary condition for \(\kappa_{N+1} = \kappa_4\) for \(k \rightarrow \infty\). We thus choose the ansatz

\[
\kappa_4(p, q, r; k) = \frac{\lambda}{|m|^{d-4}} \exp \left[ - |p|^d + |q|^d + |r|^d + |p + q + r|^d + |m|^d \right] \] (38)

which is obviously \(\text{Sym}_3^*\) and \(O(d)\) invariant and satisfies equation \(B1\). The rationale for choosing this particular form for \(\kappa_4\) is to keep the upcoming integrals as simple as possible and to ensure a rapid decrease of \(\kappa_4\) and its \(k\) derivatives for \(k \rightarrow 0\). The latter is paramount for controlling the divergent \(k\) behaviour of \(\rho_n\) in this limit. At the same time, all higher correlators as generated by the \(\rho_n\) will vanish in the UV due to the very same \(k\)-scaling. The most natural choice for the lower odd correlators is

\[\kappa_3 = 0 \quad \text{and} \quad \kappa_1 = 0,\] (39)

which alongside the given construction procedure guarantees the vanishing of all odd correlators because

- for all odd \(n \in \mathbb{N}\) any \(c \in \mathbb{C}(n)\) contains an odd entry,

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8 The \(\kappa_4\) limit has been chosen such that \(\lambda\) is dimensionless.

9 Technically speaking, \(\kappa_n\) is a distribution on \(\mathbb{R}^{(n-1)d}\) and the \(k\) limits should be understood as pointwise convergence.
correlation functions is straightforward employing equation \[40\]. Their respective \((d)-\)invariance follows from that of \(K\). It remains to discuss the behaviour of the correlators in the IR limit \(k \to 0\) and the UV limit \(k \to \infty\) respectively: Obviously \(\kappa_4\) vanishes in the limit of \(k \to 0\). As is proved in appendix \([D]\) for all \(n \in \mathbb{N}_{>2}\) there are constants \(B^{0,1}_{2n} > 0\) such that
\[
\|\kappa_{2n}\|_{L^\infty} \leq B^{0,1}_{2n} \frac{|m|^{2+(2-d)(n-1)+(n-2)(1+\Delta)}}{(k + |m|)^{(n-2)(1+\Delta)+1}} k^n
\]
for
\[
\Delta = \left\{ \begin{array}{ll}
1 & d \geq 4 \\
 d - 3 & d < 4
\end{array} \right.
\]  

These equations establish the central result of this work: For \(d > 2\) all higher correlators vanish in both limits \(k \to 0\) and \(k \to \infty\). Thus, the IR limit is a non-interacting theory with a non-trivially momentum dependent propagator \(\kappa_2\) - a generalised free theory. It may also be possible that the given solutions generalise to \(d = 2\), since the proofs only make use of the property that the UV behaviour of \(|\partial_k \kappa_4|\) is bounded by \(\sim k^{-l}\). It is, however, even bounded by \(\sim k^{l-1}\) whenever \(l \in \mathbb{N}\) which should guarantee the correct UV limits, while equation \([40]\) still ensures trivial IR limits. A formal argument showing this has not yet been produced.

In the definition of \(\kappa_4\) in equation \([38]\), note that the argument in the exponential can be multiplied by any positive real number and still all estimates hold analogously with modified constants. Furthermore, the boundary conditions at \(k \to \infty\) remain satisfied and all higher correlators vanish at \(k = 0\) upon such a modification of \(\kappa_4\). At the same time, the IR limit of \(\kappa_2\) will in general be different. Such ansätze do not correspond to a rescaling of \(\kappa\) since the dependence of \(\bar{r}\) remains unaltered. Instead, they lead to different flows solving the flow equations for the correlators.

V. THE FLOW OF THE DIMENSIONLESS POTENTIAL

It is possible to extract the quantum potential from the correlators by examining their behaviour at zero momentum. Of particular interest is the flow of the dimensionless potential \(v\) given by
\[
v(s) := \sum_{n=1}^{\infty} \frac{\kappa_{2n}(0, \ldots, 0)}{k^{2d+(2-d)(n-1)}(2n)!} s^{2n}.
\]
It is appropriate to analyse its dimensionless flow, i.e. \(k \partial_k v\) which we shall examine in the limits \(k \to 0\) and \(k \to \infty\). The \(\kappa_2\) contribution is determined by equation \([45]\) where the second term on the right-hand side is non-negative for all \(p \in \mathbb{R}^d\). Hence,
\[
\lim_{k \to 0} \frac{\kappa_2(0)}{k^2} \geq \lim_{k \to 0} \frac{\kappa_{2,\text{free}}(0)}{k^2} = \infty
\]
so that the resulting two-point correlator contains a gap that is bounded from below by the bare gap. Furthermore,

\[ \lim_{k \to 0} k \partial_k \frac{\kappa_2}{k^2} \leq \lim_{k \to 0} \left[ \frac{\| \partial_k \kappa_2 \|_{L^\infty}}{k} - 2 \frac{\kappa_{\text{free}}(0)}{k^2} \right] \leq \lim_{k \to 0} \left[ \frac{(2\pi)^{-d}}{2} R_1 A_0^2 \frac{\| \partial_k \kappa_2 \|_{L^\infty}}{k} - 2 \frac{\kappa_{\text{free}}(0)}{k^2} \right] \]

\[ = -\infty \]

for constants \( R_1, A_0^2 \geq 0 \), where the \( \| \partial_k \kappa_2 \|_{L^\infty} \) estimate is taken from equation \( D26 \) in appendix \( D \). Thus, the contribution of the propagator to the dimensionless potential diverges in the limit of \( k \to 0 \) which may be expected, since \( m \) is taken to not scale with \( k \). The UV limits become

\[ \lim_{k \to \infty} \frac{\kappa_2}{k^2} = \lim_{k \to \infty} \frac{m^2}{k^2} = 0 \]

and

\[ \lim_{k \to \infty} k \partial_k \frac{\kappa_2(0)}{k^2} \leq \lim_{k \to \infty} \left[ \frac{\| \partial_k \kappa_2 \|_{L^\infty}}{k} - 2 \frac{m^2}{k^2} \right] \leq \lim_{k \to \infty} \left[ \frac{(2\pi)^{-d}}{2} R_1 A_0^2 \frac{\| \partial_k \kappa_2 \|_{L^\infty}}{k} - 2 \frac{m^2}{k^2} \right] \]

\[ = 0 . \]

Thus in the limit of \( k \to \infty \) the corresponding contribution to \( v \) vanishes and the solution lives in the deep Euclidean region. For the contributions from the higher correlators, we use theorem \( [1] \) from appendix \( D \) to produce the estimates

\[ \| \kappa_{2n} \|_{L^\infty} \leq B^{0,x}_{2n,1} \frac{|m|^{2+(2-d)(n-1)+(n-2)(1+\Delta)}}{k^{x}} \cdot (n-2)(1+\Delta) + x, \]

\[ \| \partial_k \kappa_{2n} \|_{L^\infty} \leq B^{1,x}_{2n,1} \frac{|m|^{2+(2-d)(n-1)+(n-2)(1+\Delta)}}{k^{x} + |m|^n} \cdot (n-2)(1+\Delta) + 1 + x \]

with constants \( B^{0,x}_{2n,1}, B^{1,x}_{2n,1} \geq 0 \) for all \( x \in N \) and \( n \in N \geq 2 \). Hence, for all such \( n \),

\[ |\partial_k \kappa_{2n}(0, \ldots, 0, \kappa_{2n}(0, \ldots, 0) | \leq \frac{1}{k^{1+(2-d)(n-1)}} \left( \frac{2+(2-d)(n-1)}{k^{2+(2-d)(n-1)}} \right) \kappa_{2n}(0, \ldots, 0) \]

With the previous inequalities, we then obtain

\[ \lim_{k \to 0} \frac{\kappa_{2n}(0, \ldots, 0, \kappa_{2n}(0, \ldots, 0) | \leq B^{0,x}_{2n,1} \frac{|m|^{2+(2-d)(n-1)+\max(1,3+(2-d)(n-1))}}{k^{x} + |m|^n} \cdot (n-2)(1+\Delta) + 1 + x \]

Likewise

\[ \lim_{k \to 0} \frac{\partial_k \kappa_{2n}(0, \ldots, 0) | \leq B^{1,x}_{2n,1} \frac{|m|^{2+(2-d)(n-1)+\max(1,3+(2-d)(n-1))}}{k^{x} + |m|^n} \cdot (n-2)(1+\Delta) + 1 + x \]

so that \( v \) and \( k \partial_k v \) in the limit of small \( k \) are fully determined by the \( \kappa_2 \) contributions. For large \( k \) the estimates

\[ \| \kappa_{2n}(0, \ldots, 0) \|_{L^\infty} \leq B^{0,1}_{2n,1} \frac{|m|^{(3-d+\Delta)(n-2)+4-d}}{k^{2+(2-d)(n-1)}} \]

\[ \| \partial_k \kappa_{2n}(0, \ldots, 0) \|_{L^\infty} \leq B^{1,1}_{2n,1} \frac{|m|^{(3-d+\Delta)(n-2)+4-d}}{k^{1+(2-d)(n-1)}} \]

produce meaningful bounds whenever \( d \leq 4 \):

\[ \lim_{k \to \infty} \frac{\kappa_{2n}(0, \ldots, 0) | \leq \begin{cases} 0 & d \leq 4 \\ B^{0,1}_{2n,1} \frac{|m|^{(3-d+\Delta)(n-2)+4-d}}{k^{2+(2-d)(n-1)}} & d = 4 \\ \infty & d > 4 \end{cases} \]

\[ \lim_{k \to \infty} \frac{|\partial_k \kappa_{2n}(0, \ldots, 0) | \leq \begin{cases} 0 & d \leq 4 \\ B^{1,1}_{2n,1} \frac{|m|^{(3-d+\Delta)(n-2)+4-d}}{k^{1+(2-d)(n-1)}} & d = 4 \\ \infty & d > 4 \end{cases} \]

Thus,

\[ \lim_{k \to \infty} |v(s)| \leq \begin{cases} 0 & d \leq 4 \\ \sum_{n=2} B^{0,1}_{2n,1} \frac{|m|^{2n}}{(2n)!} & d = 4 \\ \infty & d > 4 \end{cases} \]

and

\[ \lim_{k \to \infty} |k \partial_k v(s)| \leq \begin{cases} 0 & d \leq 4 \\ \sum_{n=2} \frac{Y_{2n}^{2n}}{(2n)!} \frac{|m|^{2n}}{(2n)!} & d = 4 \\ \infty & d > 4 \end{cases} \]

for \( Y_{2n} = B^{0,1}_{2n,1} + (2+(2-d)(n-1))B^{0,1}_{2n,1,1} \). In particular no definite statement is obtained by these methods for \( d > 4 \). However, the \( \kappa_4 \) contribution to \( v \) may be calculated explicitly:

\[ k \partial_k \kappa_4(0, 0, 0, 0) = \lambda \exp \left( \frac{|m|}{k} \right) \times \left( \frac{|m|}{k} \right)^{4-d} \times \left( \frac{|m|}{k} \right)^{5-d} \]

Hence,

\[ \lim_{k \to \infty} k \partial_k \kappa_4(0, 0, 0, 0) = \begin{cases} 0 & d \leq 4 \\ \infty & d > 4 \end{cases} \]

so that for \( d = 4 \) the beta function of the quartic term of the dimensionless potential vanishes in the limit of \( k \to \infty \).

In \( d < 4 \) we see that the dimensionless potential as well as its flow vanishes in the large \( k \) limit owing to the fact that \( \kappa_2 \) and \( \kappa_4 \) are bounded and have positive mass dimensions. In the case of \( d = 4 \) the dimensionless potential is obviously non-vanishing while the fate of its flow in the limit of \( k \to \infty \) is unclear. A numerical analysis
showed that the coefficients $Y_{2n}$ with $d = 4$ grow so fast that a zero radius of convergence is probable. Thus, we do not obtain a useful estimate of $\lim_{k \to \infty} |k \partial_k v(s)|$ in this case.

VI. CONCLUSIONS

It has been demonstrated that a Euclidean invariant exact solution to equation 24 satisfying the boundary conditions 37 exists and may be constructed as outlined in the last section. Furthermore, explicit bounds on the flow as given by equation 46 and more generally by the methods applied in appendices C and D may be utilized to approximate the flow of any given correlation function to arbitrary precision. By construction, the mass and the quartic coupling only undergo finite renormalisations during the flow from $k \to \infty$ to $k \to 0$. Thus, the theory in the latter limit does not correspond to the known $\phi_3^4$ result which requires infinite renormalisations. This raises the question of how to determine the physically correct boundary conditions in the large $k$ limit which is of course intimately connected with the physically appropriate choice of classical action $S_A$. Conversely, one may ask how a given renormalisation group flow determines $S_A$ which precisely amounts to the reconstruction problem 23. With $S_A$ being unknown in this case, it is unclear whether $\lim_{k \to 0} \Gamma_k$ is independent of the choice of renormalisation scheme. In particular it was demonstrated that the flow was not uniquely determined by $\lim_{k \to \infty} \Gamma_k$. Hence, it may be expected that there is a yet to be uncovered connection between exact solutions to the flow equations and a possibly unique physical one.

The given solution was obtained through a very straightforward construction procedure that essentially enables the extrapolation of higher-order correlation functions from a set of lower-order ones. Though these extrapolations should not be expected to be unique, one may hope that their asymptotic behaviour for small and large values of $k$ are strongly constrained. Such constraints can then reveal lots of structure of the higher correlators. In particular, the construction principle may be extended to models with multiple scalar fields as well as fermions without gauge symmetries. Applying similar choices of $\rho_n$ operators to systems truncated at finite $n \in \mathbb{N}$ may then give hints for or against the applicability of the truncations in use and possibly even enable the explicit calculation of uncertainties.

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Appendix A: Proof of the Derivative Identity for the Propagator

Before stepping into the induction proof, note that equation 9 corresponds to equation 8 for $n = 1$. In order to further shorten notation, let us write

$$\langle c \rangle = \langle c_1, \ldots, c_l \rangle = A \circ \prod_{l=1}^{#c} (D^{c_l} \Gamma_k^{(2)} \circ A) \quad (A1)$$

for all $l \in \mathbb{N}$ and any multi-index $c \in \mathbb{N}^l$. Furthermore, define two operations on such multi-indices. First, let

$$s_j : \mathbb{N}^l \to \mathbb{N}^l \quad (n_1, \ldots, n_l) \mapsto (n_1, \ldots, n_{j-1}, 1 + n_j, n_{j+1}, \ldots, n_l) \quad (A2)$$

for any $j \in \mathbb{N}$ with $j \leq l$ and second,

$$t_j : \mathbb{N}^l \to \mathbb{N}^{l+1} \quad (n_1, \ldots, n_l) \mapsto (n_1, \ldots, n_{j-1}, 1, n_j, \ldots, n_l) \quad (A3)$$

for any $j \in \mathbb{N}$ with $j \leq l + 1$. Assuming the validity of equation 9 for a fixed $n \in \mathbb{N}$, we obtain

$$D^{n+1} A = \sum_{c \in C(n)} (-1)^{1+\#c} \frac{n!}{c!} \sum_{j=1}^{1+\#c} \langle t_j (c) \rangle \quad (A4)$$

where it is now obvious that

$$D^{n+1} A = \sum_{c \in C(n+1)} (-1)^{\#c} \frac{n!}{c!} \sum_{j=1}^{\#c} \langle c_j (c) \rangle$$

where

$$D^{n+1} A = \sum_{c \in C(n+1)} (-1)^{\#c} \frac{(n+1)!}{c!} \sum_{j=1}^{\#c} \langle c_j (c) \rangle\quad (A5)$$

and

$$D^{n+1} A = \sum_{c \in C(n+1)} (-1)^{\#c} \frac{n!}{c!} \sum_{j=1}^{\#c} \langle c_j (c) \rangle\quad (A6)$$

This proves equation 9.
Appendix B: Proof that $I_n \circ \rho_n = \text{id}$

Let us fix a real $\text{Sym}^n_{n-1}$-symmetric function $g$ on $(\mathbb{R}^d)^{n-1}$ and compute $I_n \rho_n g$. In order to facilitate the proof, let us split $\rho_n g$ into the following parts defined by restricting the sum over $J$ in equation $[34]$:

- $\rho_n^1 g$ where $J$ contains no index $\geq n$
- $\rho_n^2 g$ where $J$ contains precisely one index $\geq n$
- $\rho_n^3 g$ where $J$ contains precisely two indices $\geq n$

Then, $I_n \rho_n g = I_n \rho_n^1 g + I_n \rho_n^2 g + I_n \rho_n^3 g$ by the linearity of $I_n$.$^{10}$ Hence, it suffices to analyse the three parts individually: The first part becomes

$$\left(I_n \rho_n^1 g\right)(p_1, ..., p_{n-1}) = \sum_{J \subseteq \{0, ..., n-1\}} \sum_{l=0}^{n-1-\#J-1} \frac{\alpha^n_{J,l}}{\left(\int_{\mathbb{R}^d} K\right)^{n-\#J-1}} \int_{\mathbb{R}^d} K(q) \ g(p_j, -s_1, s_1, ..., -s_l, s_l, t_1, ..., t_{n-1-\#J-2l}) \ K(s_1) ... K(s_l) K(t_1) ... K(t_{n-1-\#J-2l}) \ dq \ ds_1 ... dt_l, \tag{B1}$$

where $p_j$ contains neither $q$ nor $-q$ allowing the evaluation of the $q$ integral. Thus,

$$\left(I_n \rho_n^3 g\right)(p_1, ..., p_{n-1}) = \sum_{J \subseteq \{0, ..., n-1\}} \sum_{l=0}^{n-1-\#J-1} \frac{\alpha^n_{J,l}}{\left(\int_{\mathbb{R}^d} K\right)^{n-\#J-1}} \int_{\mathbb{R}^d} K(q) \ g(p_j, -s_1, s_1, ..., -s_l, s_l, t_1, ..., t_{n-1-\#J-2l}) \ K(s_1) ... K(s_l) K(t_1) ... K(t_{n-1-\#J-2l}) \ dq \ ds_1 ... dt_l. \tag{B2}$$

In the second part $p_j$ contains either $q$ or $-q$. But since $K(q) = K(-q)$ by equation $[19]$ both contributions are identical. Removing the index $n$ or $n+1$ respectively from $J$ and inserting $q$ explicitly then leads to

$$\left(I_n \rho_n^2 g\right)(p_1, ..., p_{n-1}) = 2 \sum_{J \subseteq \{0, ..., n-1\}} \sum_{l=0}^{n-2-\#J-1} \frac{\alpha^n_{J+1,l}}{\left(\int_{\mathbb{R}^d} K\right)^{n-\#J-1}} \int_{\mathbb{R}^d} K(q) \ g(p_j, q_j, -s_1, s_1, ..., -s_l, s_l, t_1, ..., t_{n-2-\#J-2l}) \ K(s_1) ... K(s_l) K(t_1) ... K(t_{n-2-\#J-2l}) \ dq \ ds_1 ... dt_l, \tag{B3}$$

where the factor of 2 comes from the two possibilities of picking either $n$ or $n+1$. Relabelling $q$ to $t_{n-1-\#J-2l}$ simplifies this part to

$$\left(I_n \rho_n^2 g\right)(p_1, ..., p_{n-1}) = \sum_{J \subseteq \{0, ..., n-1\}} \sum_{l=0}^{n-2-\#J-1} \frac{2\alpha^n_{J+1,l}}{\left(\int_{\mathbb{R}^d} K\right)^{n-\#J-1}} \int_{\mathbb{R}^d} K(q) \ g(p_j, -s_1, s_1, ..., -s_l, s_l, t_1, ..., t_{n-1-\#J-2l}) \ K(s_1) ... K(s_l) K(t_1) ... K(t_{n-1-\#J-2l}) \ ds_1 ... dt_l, \tag{B4}$$

Relabelling $q$ to $s_{l+1}$ and shifting the index $l$ by 1 leads to

$$\left(I_n \rho_n^3 g\right)(p_1, ..., p_{n-1}) = \sum_{J \subseteq \{0, ..., n-1\}} \sum_{l=1}^{n-2-\#J-1} \frac{\alpha^n_{J+2,l-1}}{\left(\int_{\mathbb{R}^d} K\right)^{n-\#J-1}} \int_{\mathbb{R}^d} K(q) \ g(p_j, -q_j, -s_1, s_1, ..., -s_l, s_l, t_1, ..., t_{n-3-\#J-2l}) \ K(s_1) ... K(s_l) K(t_1) ... K(t_{n-3-\#J-2l}) \ ds_1 ... dt_l. \tag{B5}$$

Furthermore, the coefficients $\alpha^n_{a,b}$ may be determined by demanding $I_n \rho_n g = g$ translating to

$$n \alpha^n_{n-1,0} = 1$$

$$\forall a \in \{0, ..., n-4\}, \ b \in \left\{1, ..., \left\lfloor \frac{n-2-a}{2} \right\rfloor \right\}: \quad \alpha^n_{a,b} + 2\alpha^n_{a+1,b} + \alpha^n_{a+2,b-1} = 0$$

$$\forall a \in \{0, ..., n-3\}, \ n - a \text{ odd}: \quad \alpha^n_{a,(n-1-a)/2} + \alpha^n_{a+2,(n-3-a)/2} = 0$$

$$\forall a \in \{0, ..., n-2\}: \quad \alpha^n_{a,0} + 2\alpha^n_{a+1,0} = 0.$$
sets of \(\{0, \ldots, n-1\}\) of length \(n-1\). All these subsets give the same contribution to \(I_{n, p, q}\) due to the \(\text{Sym}_{n-1}\) symmetry of \(g\). As may easily be verified, equation \(35\) solves these recursion relations. Furthermore, this solution is unique because all \(\alpha_n\) for \(n \in \mathbb{N}\) and \(a, b \in \mathbb{N}_0\) with \(a+2b \leq n-1\) are uniquely determined by the values of \(\alpha_n\).

**Appendix C: Existence Proof of \(\kappa_2\)**

Let \(\kappa_2^1(p; k) = m^2 + \|p\|^2\) and for any \(n \in \mathbb{N}\) define

\[
\kappa_2^{n+1}(p; k) = \kappa_2^n(p) + \frac{1}{2} \int_0^\infty \int_{\mathbb{R}^d} \frac{\partial_k r(q, k')}{[\kappa_2^n(q, k') + r(q, k')]^2} \kappa_4(p, -q, q; k') \, dq \, dk' \tag{C1}
\]

which obviously satisfies the boundary condition \(37\) if the integrals are finite. The first thing to note is that \(\kappa_2^n\) is finite for all \(n \in \mathbb{N}\), since \(\kappa_2^1 > 0\), \(\kappa_2^n \geq 0\) and by equation \(42\) the regulator contribution is positive. Thus, by equation \(44\)

\[
\frac{1}{\kappa_2^n(q) + \tilde{r}(q)} \leq \frac{1}{\kappa_2^n(q) + \tilde{r}(q)} \tag{C2}
\]

which together with

\[
\frac{\partial_k \tilde{r}(q)}{[\kappa_2^n(q) + \tilde{r}(q)]^2} \leq \frac{2k}{m^2 + k^2} \tag{C3}
\]

leads to

\[
\frac{\partial_k \tilde{r}(q)}{[\kappa_2^n(q) + \tilde{r}(q)]^2} \leq \frac{2k}{m^2 + k^2} \tag{C4}
\]

Inserting this into the recursion relation \(C1\) leads to

\[
\kappa_2^n(p) \leq \kappa_2^n(p) + \frac{(2\pi)^d \lambda}{m^d - 2m^2 + k^2} \int_0^\infty \int_{\mathbb{R}^d} \frac{2 \|p\|^2 + 2 \|q\|^2 + |m|^2}{k'} \exp \left[ - \frac{2 \|p\|^2 + |m|^2}{k' m^d - 1} \right] dq \, dk' \tag{C5}
\]

\[
= \kappa_2^n(p) + (2\pi)^d \frac{s_{d-1}}{2d} \lambda |m|^3 \int_0^\infty \frac{k'^2}{m^2 + k'^2} \exp \left[ - \frac{2 \|p\|^2 + |m|^2}{k' m^d - 1} \right] dk' \tag{C6}
\]

where \(s_n\) denotes the surface area of the unit \(n\)-sphere. Estimating the exponential by \(1\) and extending the integral to \([0, \infty)\) immediately gives the result

\[
\kappa_2^n(p) \leq \kappa_2^n(p) + (2\pi)^d \frac{s_{d-1}}{2d} \lambda |m|^3 \int_0^\infty \frac{k'^2}{m^2 + k'^2} \, dk' \leq \kappa_2^n(p) + (2\pi)^d \frac{s_{d-1}}{2d} \lambda m^2 \tag{C6}
\]

which in a slightly more compact form reads

\[
\|\kappa_2^n - \kappa_2^n\|_{L^\infty} \leq \frac{(2\pi)^d \pi s_{d-1}}{8d} \lambda m^2 := t_d \lambda m^2 \tag{C7}
\]

for all \(n \in \mathbb{N}\). Note that the numerical factor \(t_d\) in front of \(\lambda m^2\) is rather small: It is \(1/8\) for \(d = 1\) and goes to zero rather rapidly for larger values of \(d\).

We shall now show that the mapping \(\kappa_2^n \rightarrow \kappa_2^{n+1}\) given by equation \(C1\) actually is a contraction for values of \(\lambda\) not being too large. To this end, note that

\[
\frac{\kappa_2^n(q) + \tilde{r}(q)}{\kappa_2^n(q) + \tilde{r}(q)} = \frac{\kappa_2^n(q) + \tilde{r}(q)}{\kappa_2^n(q) + \tilde{r}(q)} + \frac{\kappa_2^n(q) - \kappa_2^n(q)}{\kappa_2^n(q) + \tilde{r}(q)} \tag{C8}
\]

and hence

\[
\left[\left[\kappa_2^{n+1}(q) + \tilde{r}(q)\right] - \left[\kappa_2^n(q) + \tilde{r}(q)\right]\right]^2 \leq \left[\frac{\kappa_2^n(q) + \tilde{r}(q)}{\kappa_2^n(q) + \tilde{r}(q)}\right] \left[\frac{\kappa_2^{n+1}(q) + \tilde{r}(q)}{\kappa_2^{n+1}(q) + \tilde{r}(q)}\right] \leq \left[\frac{\kappa_2^n(q) + \tilde{r}(q)}{\kappa_2^n(q) + \tilde{r}(q)}\right]^2 \leq 2 \left[\frac{\kappa_2^n(q) - \kappa_2^{n+1}(q)}{\kappa_2^n(q) + \tilde{r}(q)}\right]^3 \leq 2 \left(\frac{\kappa_2^n(q) - \kappa_2^{n+1}(q)}{\kappa_2^n(q) + \tilde{r}(q)}\right)^2 \tag{C9}
\]

Using this estimate to compare two successive iterates one finally arrives at

\[
\frac{\kappa_2^{n+2}(p) - \kappa_2^{n+1}(p)}{\kappa_2^{n+2}(q) - \kappa_2^{n+2}(q)} \leq 2 \left(\frac{\kappa_2^n(p) - \kappa_2^n(p)}{\kappa_2^n(q) + \tilde{r}(q)}\right)^3 \leq 2 \left(\frac{\kappa_2^n(p) - \kappa_2^n(p)}{\kappa_2^n(q) + \tilde{r}(q)}\right)^3 \tag{C10}
\]

or for short

\[
\left\|\kappa_2^{n+2} - \kappa_2^{n+1}\right\|_{L^\infty} \leq 2 \left(\frac{\kappa_2^n(p) - \kappa_2^n(p)}{\kappa_2^n(q) + \tilde{r}(q)}\right)^3 \tag{C11}
\]

The factor in front is smaller than one whenever

\[
0 \leq \lambda < \frac{\sqrt{3} - 1}{2d} \tag{C12}
\]

or equivalently equation \(E3\) is satisfied. The upper bound is a function that grows rather rapidly starting at a value of approximately \(2.93\) for \(d = 1\). From now on, we assume \(\lambda\) to satisfy inequality \(C12\). Thus, by the completeness of
$L^\infty(\mathbb{R}^d)$ we have proven the convergence of the sequence $(p \mapsto \kappa_p^k (p; k))_{k \in \mathbb{N}}$ to some $p \mapsto \kappa_p^k (p; k)$ in $L^\infty(\mathbb{R}^d)$ for all $k \in [0, \infty)$. Also, $\kappa_2$ has to be a fixed point of the iteration map such that equation (15) is satisfied where the right hand side is continuous with respect to $k$, since the integrand is non-singular for all $k' \geq 0$. Thus, $\kappa_2$ is also $k$-continuous on $[0, \infty)$ as well. But then the right-hand side is differentiable with respect to $k$ on all of $[0, \infty)$, so that

$$
\partial_k \kappa_2 (p) = -\frac{1}{2} (2\pi)^{-d} \int_{\mathbb{R}^d} \frac{\partial_k \tilde{r} (q)}{[\kappa_2 (q) + r (q)]^{2}} \kappa_2 (p, q, q) \, dq
$$

for all $k \in \mathbb{R}_{\geq 0}$. Hence, $\kappa_2$ satisfies the flow equation. Furthermore, the right-hand side is obviously $k$-differentiable so that $\partial^2_k \kappa_2$ may be expressed through $\kappa_2$ and $\partial_k \kappa_2$. Hence, $\partial^2_k \kappa_2$ is again $k$-differentiable. Iterating this argument then shows that $\kappa_2$ is smooth with respect to $k$. The $p$-smoothness of $\kappa_2$ is immediate from equation (15) by the regularity of $\kappa_4$. For the $O (d)$-invariance of $\kappa_2$, note that $\kappa_4$ and $\tilde{r}$ as well as $\kappa_2^2$ are $O (d)$-invariant. Thus, by equation (C1) each iterate $\kappa_2^2$ is also $O (d)$-invariant. Since the set of all $O (d)$-invariant functions in $L^\infty(\mathbb{R}^d)$ is closed, the limit point $\kappa_2$ has to lie in this set as well.

Appendix D: Bounding the Higher Correlators

Let us assume that all higher correlators have been constructed by virtue of equation (4). It then remains to find useful bounds ascertaining the correct UV limits as well as non-singular IR limits. The key to this, is a proper estimate for the $k$-derivatives of $\kappa_2$. Before we can produce such estimates, we shall need corresponding ones for $\kappa_4$ and $\tilde{r}$. Let us begin with the regulator for which we have the relation

$$
\partial_k \tilde{r} (q) = 2 k \tilde{r} (q) \left( \|q\|^2 + \tilde{r} (q) \right)
$$

that may easily be derived from equation (2) It hints at the following identity for all $l \in \mathbb{N}_0$ and some constants $\beta_{a,b}^l \in \mathbb{R}$:

$$
\partial_k^l \tilde{r} (q) = \sum_{a=1}^{l} \sum_{b=0}^{l-1} \beta_{a,b}^l k^{2-l-2a-2b} \tilde{r} (q)^a \left( \|q\|^2 + \tilde{r} (q) \right)^b
$$

which can straightforwardly be proved by induction. The constants $\beta_{a,b}^l$ are recursively defined by

$$
\begin{align*}
\beta_{a,b}^{l+1} &= (2-l-2a-2b) \beta_{a,b}^l + 2a \beta_{a,b}^{l-1} + 2b \beta_{a-1,b}^{l-1} , \\
\beta_{a,b}^0 &= \begin{cases} 1 & a = 1, b = 0 , \\
0 & \text{otherwise} \end{cases}
\end{align*}
$$

for all $l \in \mathbb{N}_0$ and $a, b \in \mathbb{Z}$. The next theorem will allow to find an estimate for such expressions.

**Theorem 1.** Let $a \in \mathbb{N}$ and $b \in \mathbb{N}_0$. Then,

$$
\sup_{q \in \mathbb{R}^d} \left| \tilde{r} (q)^a \left( \|q\|^2 + \tilde{r} (q) \right)^b \right| \leq k^{2(a+b)} \left( 1 + \frac{b}{a} \right)^b .
$$

(D4)

**Proof.** For $b = 0$ the statement is obvious since $\tilde{r} (q) \leq k^2$. Hence, let us assume that $b \in \mathbb{N}$. Since

$$
\tilde{r} (q)^a \left( \|q\|^2 + \tilde{r} (q) \right)^b
$$

is actually a smooth function of $\|q\|^2$, we may look for local extrema by differentiating with respect to $\|q\|^2$. Then, a necessary condition for $\|q\|^2$ at a maximum is

$$
\begin{align*}
a \left[ \|q\|^2 + \tilde{r} (q) \right] \partial_{\|q\|^2} \tilde{r} (q) \\
+ b \tilde{r} (q) \left[ 1 + \partial_{\|q\|^2} \tilde{r} (q) \right] = 0 .
\end{align*}
$$

We find the following identity for $q \neq 0$:

$$
\partial_{\|q\|^2} \tilde{r} (q) = \frac{r (q)}{\|q\|^2} \left[ 1 - \frac{1}{2 k} \left( \|q\|^2 + \tilde{r} (q) \right) \right] .
$$

(D6)

which inserted into the previous equation gives us the equivalent condition

$$
1 = \tilde{r} (q) + \frac{a}{a + b} \left( \frac{\|q\|^2}{k^2} \right) .
$$

(D7)

after some simple algebra. We perform a change of variables to $y = \frac{\|q\|^2}{k^2}$ and obtain

$$
\exp y = 1 + \frac{a+b}{a+b-ay}
$$

(D8)

as a further equivalent expression for the extremality even including the case $q = 0$. Note, that the apparently excluded case $ay = a + b$ is not relevant, since it does not solve the derivative test as is obvious from equation (D7). Furthermore,

$$
\frac{\partial}{\partial y} \left( \frac{a+b}{a+b-ay} \right) = \left( \frac{a+b}{a+b-ay} \right)^2 > 0 ,
$$

(D9)

such that for $ay > a + b$ we have the right-hand side of equation (D8) being monotonically increasing with $y$ and

$$
\lim_{y \rightarrow \infty} \left( 1 + \frac{a+b}{a+b-ay} \right) = 1 - \frac{a+b}{a} = - \frac{b}{a} < 0 ,
$$

(D10)

spoiling equation (D8). Thus, we conclude that all extrema lie in the interval $y \in \left[ \frac{a}{a+b}, \frac{a}{a+b} \right]$. But then, at a maximum
we have
\[
\bar{r}(q)^a \left[ \|q\|^2 + 2 \bar{r}(q) \right]^b
= \left( \frac{\|q\|^2}{\exp y - 1} \right)^a \left( \|q\|^2 + \frac{\|q\|^2}{\exp y - 1} \right)^b
= k^{2(a+b)} \left( \frac{y}{\exp y - 1} \right)^a \left( y + \frac{y}{\exp y - 1} \right)^b
= k^{2(a+b)} \left( \frac{y}{\exp y - 1} \right)^a \left( y + \frac{y}{\exp y - 1} \right)^b
\]
\[
= k^{2(a+b)} \left( \frac{1 - \frac{a}{a+b} y}{a+b} \right)^a \left( 1 + \frac{b}{a+b} y \right)^b
\leq k^{2(a+b)} \left( 1 + \frac{b}{a} \right)^b,
\]
where we have used \( y \in [0, \frac{a+b}{a}] \) in the last estimate. □

**Corollary 2.** Applying this estimate to the regulator derivatives, we obtain
\[
\|\partial_k^a \bar{r}\|_{L^\infty} \leq k^{2-n} \sum_{a=1}^{n} \sum_{b=0}^{n-a} \beta_{a,b}^n \left( 1 + \frac{b}{a} \right)^b.
\]
Hence, there is a constant \( R_n \geq 0 \) such that
\[
\|\partial_k^a \bar{r}\|_{L^\infty} \leq R_n k^{2-n}
\]
for all \( n \in \mathbb{N}_0 \).

**Corollary 3.** Applying the estimate to \( K \) and employing equation \[D12\] leads to
\[
\|K\|_{L^\infty} \leq R_1 \frac{k}{(k^2 + m^2)^2}.
\]

Having obtained the appropriate estimates for the regulator, the next step towards the \( \kappa_2 \) estimates is to study \( \kappa_4 \).

**Theorem 4.** For all \( l \in \mathbb{N}_0 \) there exist constants \( A_l^4 \geq 0 \) such that
\[
\sup_{p \in \mathbb{R}^d} \int_{\mathbb{R}^d} |\partial_k^l \kappa_4 (p, q, -q)| \, dq \leq A_l^4 \left| \frac{|m|}{k l!} \right|^3 k^{l+1}.
\]

*Proof.* As can easily be proved by induction, we have
\[
\partial_k^l \kappa_4 (p, q, r) = \kappa_4 (p, q, r) \sum_{a=0}^l \gamma_a^l \left| m \right|^{a-d} k^{a-l}
\times \left( |p|^d + \|q\|^d + |r|^d + \|p+q+r|^d + |m|^d \right)^a
\]
for all \( l \in \mathbb{N}_0, p, q, r \in \mathbb{R}^d \). The constants \( \gamma_a^l \in \mathbb{R} \) are determined by
\[
\gamma_a^{l+1} = - (a + l) \gamma_a^l + \gamma_{a-1}^l,
\gamma_a^0 = \begin{cases} 1 & \text{if } a = 0, \\ 0 & \text{otherwise} \end{cases}
\]
for all \( a \in \mathbb{Z} \). Expanding the above, we get
\[
|\partial_k^l \kappa_4 (p, q, -q)| \leq \lambda \sum_{a=0}^l \sum_{b=0}^a \left( \frac{a}{b} \right)^2 |\gamma_a^l| \left| \frac{|m|}{k} \right|^d k^{a-b} \left( 2 \|p\|^d + |m|^d \right)^{a-b}
\times \left| \frac{2 \|p\|^d + |m|^d}{k} \right| \left( \frac{2 \|q\|^d + |m|^d}{k} \right)^{d-1}
\]
which allows us to perform the \( q \) integral, such that
\[
\int_{\mathbb{R}^d} |\partial_k^l \kappa_4 (p, q, -q)| \, dq \leq \frac{8^d-1}{2^d} \lambda \sum_{a=0}^l \sum_{b=0}^a \left( \frac{a}{b} \right)^2 |\gamma_a^l| \left| \frac{|m|}{k} \right|^3 \left( 2 \|p\|^d + |m|^d \right)^{a-b}
\times \left| \frac{2 \|p\|^d + |m|^d}{k} \right| \left( \frac{2 \|q\|^d + |m|^d}{k} \right)^{d-1}
\]
Let us again expand this, leading to
\[
\int_{\mathbb{R}^d} |\partial_k^l \kappa_4 (p, q, -q)| \, dq \leq \frac{8^d-1}{2^d} \lambda \sum_{a=0}^l \sum_{b=0}^a \left( \frac{a}{b} \right)^2 |\gamma_a^l| \left| \frac{|m|}{k} \right|^3 \left( \frac{2 \|p\|^d + |m|^d}{k} \right)^{a-b}
\times k^{1+b-a-1} \left| |p|^d \right| \left( \frac{2 \|p\|^d + |m|^d}{k} \right)^{d-1}
\]
allowing us to produce the estimate
\[
\sup_{p \in \mathbb{R}^d} \int_{\mathbb{R}^d} |\partial_k^l \kappa_4 (p, q, -q)| \, dq \leq \frac{8^d-1}{2^d} \lambda \sum_{a=0}^l \sum_{b=0}^a \left( \frac{a}{b} \right)^2 |\gamma_a^l| \left| \frac{|m|}{k} \right|^3 \left( \frac{2 \|p\|^d + |m|^d}{k} \right)^{a-b}
\times k^{1+b-a-1} \left| |p|^d \right| \left( \frac{2 \|p\|^d + |m|^d}{k} \right)^{d-1} \exp \left[ - \frac{|m|}{k} \right]
\]
For \( l = 0 \), the above reduces to
\[
\sup_{p \in \mathbb{R}^d} \int_{\mathbb{R}^d} |\kappa_4 (p, q, -q)| \, dq
\leq \frac{8^d-1}{2^d} \lambda |\gamma_0^0| \left| \frac{|m|}{k} \right|^3 \left( \frac{2 \|p\|^d + |m|^d}{k} \right)^{d-1} \exp \left[ - \frac{|m|}{k} \right]
\]
which is precisely of the desired form. For \( l \in \mathbb{N} \) and all
a, b, c ∈ N₀ with a − b − c ≥ 0 the following is valid:

\[
\exp \left[ -\frac{|m|}{k} \right] \\
\leq \left[ \frac{(|m|)^{a-b-c}}{(a-b-c)!} + \frac{(|m|)^{l+a-b-c}}{(l+a-b-c)!} \right]^{-1} \\
= \frac{k^{l+a-b-c}|m|^{b+c-a}}{k^{l+b+c-a} + \frac{1}{(l+a-b-c)!}|m|^{l}} \\
\leq (l - a - b - c)! \frac{k^{l+a-b-c}|m|^{b+c-a}}{k^{l} + |m|^{l}}.
\]  

Inserting this into equation (D21) yields

\[
\sup_{p \in \mathbb{R}^d} \int_{\mathbb{R}^d} |\partial^l_k \kappa_4(p, q, -q)| \, dq \\
\leq \frac{8d-1}{2d} \lambda \sum_{a=0}^{d} \sum_{b=0}^{a} \frac{(a)}{(b)} b! |\gamma^l_1| (l + a - b) \frac{|m|^{3} k}{k^{l} + m^{l}} \\
+ \frac{8d-1}{2d} \lambda \sum_{a=0}^{d} \sum_{b=0}^{a} \sum_{c=1}^{d} \frac{(a)}{(b)} \left( \frac{a - b}{c} \right) b! \left( \frac{c}{e} \right) |\gamma^l_1| \\
\times (l + a - b - c)! \frac{|m|^{3} k}{k^{l} + m^{l}}.
\]

Then

\[
\|\partial^{n+1}_k \kappa_2\|_{L^\infty} \\
\leq \frac{(2\pi)^{-d}}{2} \sum_{l=0}^{n} \frac{1}{l!} \left( \frac{m}{k} \right)^{l} \|\partial^l_k \kappa_2\|_{L^\infty} \\
\times \sup_{p \in \mathbb{R}^d} \int_{\mathbb{R}^d} |\partial^{l-a}_k \kappa_4(p, q, -q)| \, dq \\
\leq \frac{(2\pi)^{-d}}{2} \sum_{l=0}^{n} \frac{1}{l!} \frac{A_{n-l} |m|^{3} k}{k^{n-l} + |m|^{n-l}} \|\partial^l_k \kappa_2\|_{L^\infty}.
\]  

But, also

\[
\|\partial^l_k \kappa_2\|_{L^\infty} \\
\leq \sum_{a=0}^{l} \sum_{b=0}^{a} \frac{(a)}{(b)} \|\partial^{l-a}_k \kappa_2\|_{L^\infty} \\
\times \left( \frac{\partial^l_k (\kappa_2 + \overline{r})}{(\kappa_2 + \overline{r})} \right)^{-1} \|\partial^l_k \kappa_2\|_{L^\infty} \\
\leq \sum_{a=0}^{l} \sum_{b=0}^{a} \frac{(a)}{(b)} R_{1+l-a} k^{1+a-l} \\
\times \left( \frac{\partial^l_k (\kappa_2 + \overline{r})}{(\kappa_2 + \overline{r})} \right)^{-1} \|\partial^l_k \kappa_2\|_{L^\infty}.
\]

While equation (9) has been derived in a non-commutative algebra of operators, it also holds in a similar form in the commutative algebra of functions. Thus, together with equation (14) and the induction hypothesis, we obtain

\[
\|\partial^l_k (\kappa_2 + \overline{r})\|_{L^\infty} \\
\leq \sum_{c \in \mathcal{C}(l)} \frac{l!}{c!} \frac{1}{m^{2} + k^{2}} \prod_{a=1}^{#c} \frac{\|\partial^l_k \kappa_2\|_{L^\infty} + \|\partial^l_k \overline{r}\|_{L^\infty}}{m^{2} + k^{2}}
\]

\[
\leq \sum_{c \in \mathcal{C}(l)} \frac{l!}{c!} \frac{1}{m^{2} + k^{2}} \prod_{a=1}^{#c} (B_{c}^{a} + R_{c,a}) k^{-c_{a}}
\]

\[
= \frac{k^{-l}}{m^{2} + k^{2}} \sum_{c \in \mathcal{C}(l)} \frac{l!}{c!} \prod_{a=1}^{#c} (B_{c}^{a} + R_{c,a})
\]

\[
=: B_{l,r}^{k-1} m^{2} + k^{2}
\]

for all \( l \in N_{\leq n} \cup \{0\} \), where we set \( B_{l,0}^{k} = 1 \). Inserted into the previous equation, this gives us

\[
\|\partial^l_k \kappa_2\|_{L^\infty} \\
\leq \sum_{a=0}^{l} \sum_{b=0}^{l-a} \frac{(a)}{(b)} R_{1+l-a} k^{1+a-l} \\
\times \frac{k^{-b}}{m^{2} + k^{2}} \frac{1}{B_{l,r}^{k-1} m^{2} + k^{2}}
\]

\[
=: B_{l,r}^{k-1} \frac{k^{-l}}{m^{2} + k^{2}}
\]

for all \( l \in N_{\leq n} \cup \{0\} \). Finally, inserting this into equation

\[
\text{Theorem 5. Let } n \in N. \text{ Then, there exists a constant } B_{l}^{k} \geq 0 \text{ such that}
\]

\[
\|\partial^l_k \kappa_2\|_{L^\infty} \leq B_{l}^{k} m^{2} k^{n}.
\]  

\[
\text{Proof. Let us first consider the case } n = 1:
\]

\[
\|\partial_k \kappa_2\|_{L^\infty} \\
\leq \frac{(2\pi)^{-d}}{2} \|K\|_{L^\infty} \sup_{p \in \mathbb{R}^d} \int_{\mathbb{R}^d} |\kappa_4(p, q, -q)| \, dq \\
\leq \frac{(2\pi)^{-d}}{2} R_1 A_{4}^{0} \frac{|m|^{3} k^{2}}{(k^{2} + m^{2})^{2}} \\
\leq \frac{(2\pi)^{-d}}{4} R_1 A_{4}^{0} m^{2} k^{-1},
\]

where the second inequality follows from corollary 3 and theorem 4. Let us now proceed by induction. Fix some \( n \in N \) and assume that the theorem holds for all \( l \in N_{\leq n} \).
Proof. We obviously have

\[
\| \partial_k^n K \|_{L^1} \leq \sum_{l=0}^{n} \sum_{a=0}^{l} \left( \binom{n}{a} \right) \left( \frac{k}{n} \right)^{l-n} \| \partial_k^{l-n} \varphi \|_{L^1} \times \left( \frac{k}{n} + \left| \varphi \right| \right)_{L^1} \leq \sum_{l=0}^{n} \sum_{a=0}^{l} B_{2,r}^{l-a} \bar{R}_{l+n-l} \left( m^2 + k^2 \right)^{\frac{d+1-n}{2}}.
\]

\[
\text{Theorem 10. For all natural numbers } n \in \mathbb{N}_0, \text{ there exist constants } C_K^n \geq 0 \text{ such that}
\]

\[
\left( \frac{\partial_k^n K}{m^2 + k^2} \right)^{\frac{1}{2}} \leq C_K^n \left( \frac{k^2 + m^2}{k^{d+1}} \right)^{\frac{1}{2}}.
\]

Proof. We have

\[
K \leq \int_{\mathbb{R}^d} \left( \frac{\partial_k \varphi}{m^2 + k^2} \right)^2 dq \geq \int_{\mathbb{R}^d} \left( \frac{\partial_k \varphi}{m^2 + k^2} \right)^2 dq
\]

\[
= s_{d-1} \int_{0}^{\infty} \lambda \left( m^2 + k^2 \right) \left( \varphi \left( \sqrt{\lambda} \right) \right)^2 \lambda^{\frac{d-1}{2}} \lambda^{\frac{d-3}{2}} dt
\]

\[
\geq \frac{s_{d-1}}{\max \left( 2, 1 + t_d \lambda \right)^{\frac{d-1}{2}}} X_d \left( \frac{k^2 + m^2}{k^{d+1}} \right)^{\frac{1}{2}}
\]

with \( X_d > 0 \) being the numerical value of the integral which is finite for \( d \geq 1 \). Thus, by inverting both sides the theorem is true for \( n = 0 \). For \( n \in \mathbb{N} \) note that

\[
\left( \frac{\partial_k^n K}{m^2 + k^2} \right)^{\frac{1}{2}} \leq C_K^n \left( \frac{k^2 + m^2}{k^{d+1}} \right)^{\frac{1}{2}}
\]

Proof. The use of equation (D22) yields

\[
\| \partial_k^n \varphi \|_{L^1} \leq \frac{1}{2} m^2 \lambda^{d-1} \sum_{a=0}^{l-n} \lambda^{d-1} \beta_a \left( k^2 + m^2 \right)^{d-1} \int_{\mathbb{R}^d} \left( \frac{\partial_k \varphi}{m^2 + k^2} \right)^2 dq
\]

\[
= s_{d-1} \int_{0}^{\infty} \lambda \left( m^2 + k^2 \right) \left( \varphi \left( \sqrt{\lambda} \right) \right)^2 \lambda^{\frac{d-1}{2}} \lambda^{\frac{d-3}{2}} dt
\]

\[
\geq \frac{s_{d-1}}{\max \left( 2, 1 + t_d \lambda \right)^{\frac{d-1}{2}}} X_d \left( \frac{k^2 + m^2}{k^{d+1}} \right)^{\frac{1}{2}}
\]

\[
\| \partial_k^n K \|_{L^1} \leq C_K^n \left( \frac{k^2 + m^2}{k^{d+1}} \right)^{\frac{1}{2}}.
\]

Now, we finally turn to our estimates of the higher correlation functions.

\[
\text{Theorem 11. Define } \Delta \text{ as in equation (D7). Then, for all } n \in \mathbb{N}_{\geq 2}, x \in \mathbb{N} \text{ and } l \in \mathbb{N}_0 \text{ there exist constants } B_{2,n}^x \geq 0
\]

\[
\text{such that}
\]

\[
\text{Corollary 6. For all } l \in \mathbb{N}_0, \text{ there is a constant } B_{2,r}^l \geq 0 \text{ such that}
\]

\[
\left( \frac{\partial_k^l \varphi}{m^2 + k^2} \right)^{\frac{1}{2}} \leq B_{2,r}^l \left( \frac{k^2 + m^2}{k^{d+1}} \right)^{\frac{1}{2}}.
\]

\[
\text{Corollary 7. For all } l \in \mathbb{N}_0, \text{ there is a constant } B_{l,K}^l \geq 0 \text{ such that}
\]

\[
\| \partial_k^l K \|_{L^1} \leq B_{l,K}^l \left( \frac{k^2 + m^2}{k^{d+1}} \right)^{\frac{1}{2}}.
\]

Having obtained these estimates concerning \( \kappa_2 \), only a few estimates regarding the exponential regulator are needed before turning to \( \rho_{2n} \). As a start, we have the following theorem.

\[
\| \partial_k^l \kappa_2 \|_{L^1} \leq \bar{R}_l k^{2d-l}.
\]

\[
\| \partial_k^l \varphi \|_{L^1} \leq \bar{R}_l k^{2d-l}.
\]

\[
\| \partial_k^l K \|_{L^1} \leq B_{l,K}^l \left( \frac{k^2 + m^2}{k^{d+1}} \right)^{\frac{1}{2}}.
\]

\[
\| \partial_k^l \varphi \|_{L^1} \leq \bar{R}_l k^{2d-l}.
\]
such that
$$\|\partial_k^\kappa 2n\|_{L^\infty} \leq B_{2n}^{l} \frac{|m|^{d-(2-d)n+(n-2)(1+\Delta)}}{(k + |m|)^{(n-2)(1+\Delta)+x+l}}, \quad (D41)$$

Proof. Let us begin with a proof of the statement for \(\kappa_4\) i.e. for \(n = 2\). We know from equation \(\text{D16}\) that
$$\|\partial_k^\kappa 4\|_{L^\infty} \leq \sum_{a=0}^{l} \left| \frac{|m|^{4-a}}{k^{a+l}} \right| \exp \left[ \frac{-|m|}{k} \right] \times \sup_{y \in \mathbb{R}} \left( e^{y^2} + \left| \frac{y}{k} \right| |m| \right) \exp \left[ -\frac{y^2}{k |m|} \right]. \quad (D42)$$

This expands to
$$\|\partial_k^\kappa 4\|_{L^\infty} \leq \lambda \sum_{a=0}^{l} \sum_{b=0}^{a} \left( \frac{a}{b} \right) \left| \frac{|m|^{4-d-a}}{k^{a+l}} \right| \exp \left[ -\frac{|m|}{k} \right] \times \left( e^{y^2} + \left| \frac{y}{k} \right| |m| \right) \exp \left[ -\frac{y^2}{k |m|} \right], \quad (D43)$$
such that
$$\|\partial_k^\kappa 4\|_{L^\infty} \leq \lambda \sum_{a=0}^{l} \left| \frac{|m|^{4-d-a}}{k^{a+l}} \right| \exp \left[ -\frac{|m|}{k} \right] \times \sum_{a=0}^{l} \left( \frac{a}{b} \right) \left| \frac{|m|^{4-d-a}}{k^{a+l}} \right| \exp \left[ -\frac{|m|}{k} \right] \times \left( e^{y^2} + \left| \frac{y}{k} \right| |m| \right) \exp \left[ -\frac{y^2}{k |m|} \right]. \quad (D44)$$

But from equation \(\text{D23}\) we have for all \(a, b \in \mathbb{N}_0\) with \(a \geq b\) and all \(x \in \mathbb{N}\)
$$\exp \left[ -\frac{|m|}{k} \right] \leq (x + l + a - b)^{-l} \frac{k^{x+l+a-b} |m|^{b-a}}{k^{x+l} + |m|^{x+l}}. \quad (D45)$$

Inserted into the previous equation, this yields
$$\|\partial_k^\kappa 4\|_{L^\infty} \leq \lambda \sum_{a=0}^{l} \left| \frac{|m|^{4-d-a}}{k^{a+l}} \right| \exp \left[ -\frac{|m|}{k} \right] \times \sum_{a=0}^{l} \left( \frac{a}{b} \right) \left| \frac{|m|^{4-d-a}}{k^{a+l}} \right| \exp \left[ -\frac{|m|}{k} \right] \times \left( e^{y^2} + \left| \frac{y}{k} \right| |m| \right) \exp \left[ -\frac{y^2}{k |m|} \right]. \quad (D46)$$
The result then follows from
$$\frac{1}{k^{x+l} + |m|^{x+l}} \leq 2^{2x+l-1} \frac{1}{(k + |m|)^{x+l}}. \quad (D47)$$

Let us now fix some \(n \in \mathbb{N}_{\geq 2}\) and assume the theorem to be true for all \(l \in \mathbb{N}_{\geq 2}\) with \(l \leq n\). It needs to be shown that the theorem also holds for \(\kappa_{2n+2}\) as given by equation \(\text{D40}\). By the linearity of \(\rho_{2n}\) it suffices to show this for \(\rho_{2n} \partial_k \kappa_{2n}\) and \(\rho_{2n} \lambda_{\kappa}\) separately for all \(c \in C(2n) \setminus \{(2n)\}\). In either case, for \(l \in \mathbb{N}_0\) and a sufficiently regular \(\text{Sym}_{2n-1}\)-symmetric function \(g\) we have
$$\|\partial_k^l \rho_{2n} g\|_{L^\infty} \leq \sum_{J \subseteq \{0, \ldots, 2n+1\}} \sum_{l=0}^{2n-1-l} \sum_{a=0}^{l} \sum_{b=0}^{a} \left( \frac{a}{b} \right) \left| \frac{|m|^{4-a}}{k^{a+l}} \right| \exp \left[ -\frac{|m|}{k} \right] \times \left( e^{y^2} + \left| \frac{y}{k} \right| |m| \right) \exp \left[ -\frac{y^2}{k |m|} \right]. \quad (D48)$$

where we have used that \(\int_\mathbb{R} K = \|K\|_{L^1}\) since \(K > 0\). Employing corollary \(\text{D9}\) we get
$$\|\partial_k^l \kappa \|_{L^\infty} \leq \sum_{\alpha \in \mathbb{N}_0^\alpha} \left( \frac{a}{b} \right) \left| \frac{|m|^{4-a}}{k^{a+l}} \right| \exp \left[ -\frac{|m|}{k} \right] \times \left( e^{y^2} + \left| \frac{y}{k} \right| |m| \right) \exp \left[ -\frac{y^2}{k |m|} \right]. \quad (D49)$$

for all \(a, b \in \mathbb{N}_0^{\alpha}\) Furthermore, from theorem \(\text{D10}\) one has
$$\|\partial_k^a \|K\|_{L^\infty} \leq \sum_{\alpha \in \mathbb{N}_0^{\alpha}} \left( \frac{a}{b} \right) \left| \frac{|m|^{4-a}}{k^{a+l}} \right| \exp \left[ -\frac{|m|}{k} \right] \times \left( e^{y^2} + \left| \frac{y}{k} \right| |m| \right) \exp \left[ -\frac{y^2}{k |m|} \right]. \quad (D50)$$

for all \(a \in \mathbb{N}_0\) and \(b \in \mathbb{N}\). The insertion of these two inequalities into equation \(\text{D48}\) reveals the important in-
termediate result
\[
\| \partial_k^l \rho_{2n} g \|_{L^\infty} \\
\leq \sum_{J \subseteq \{1, \ldots, 2n+2\}} \sum_{j=0}^{2n-1-l} \sum_{a=0}^{l} \sum_{b=0}^{l} \frac{(l)}{(a) (b)} \left| \partial_{2n-j, j} \right| x D_{K}^{a-b, 2n-1-J-j} D_{K}^{b, 2n-J-j} \times \frac{(m^2 + k^2)^2}{k^{d+1+a}} \| \partial_k^{l-a} g \|_{L^\infty}.
\]

(D51)

The divergent behaviour for \( k \to 0 \) elucidates the need for the extremely strong IR regularity of \( \kappa_4 \) as imposed in equation [33]. Now, consider the case \( g = \partial_k \kappa_{2n} \) and \( x \in \mathbb{N} \):
\[
\| \partial_k^l \rho_{2n} \partial_k \kappa_{2n} \|_{L^\infty} \\
\leq \sum_{a=0}^{l} E_{2n}^{a} (m^2 + k^2)^2 \left| \partial_k^{l-a} \kappa_{2n} \right|_{L^\infty} \\
\leq \sum_{a=0}^{l} E_{2n}^{a} B_{2n}^{l-a, x+1, a+1} \frac{(m^2 + k^2)^2}{k^{d+1+a}} \\
\times \left| m \right|^{d+(2-d)(n+1)+1+(1+\Delta) k_x+1} \\
\times \left| m \right|^{(n-1)(1+\Delta)+1} \\
= \sum_{a=0}^{l} E_{2n}^{a} B_{2n}^{l-a, x+1, a+1} \\
\times \frac{(m^2 + k^2)^2 \left| m \right|^{d-3-\Delta}}{k^{d+1+\Delta}} \\
\times \frac{(k + |m|)^{(n-1)(1+\Delta)+1} k_x}{k^{d+1+\Delta}} \\
\leq \sum_{a=0}^{l} E_{2n}^{a} B_{2n}^{l-a, x+1, a+1} \\
\times \frac{(m^2 + k^2)^2 \left| m \right|^{d-3-\Delta}}{(k + |m|)^{(n-1)(1+\Delta)+1} k_x}.
\]

(D52)

This is the expected result and also shows that the use of these methods requires \( d - 3 - \Delta \geq 0 \). Otherwise, the last inequality would not generally hold. It remains to estimate the \( \rho_{2n} \tilde{\lambda}_c \) terms. Let \( c \in \mathcal{C}(2n) \setminus \{(2n)\} \). Then, obviously
\[
\| \partial_k^l \rho_{2n} \tilde{\lambda}_c \|_{L^\infty} \leq \| \partial_k^l \lambda_c \|_{L^\infty},
\]

(D53)

so that we need not bother with symmetrisation. In total, for \( g = \tilde{\lambda}_c \) and \( l \in \mathbb{N}_0 \) we get
\[
\| \partial_k^l \rho_{2n} \tilde{\lambda}_c \|_{L^\infty} \\
\leq \sum_{a=0}^{l} E_{2n}^{a} \left( m^2 + k^2 \right)^2 \left| \partial_k^{l-a} \lambda_c \right|_{L^\infty}.
\]

(D54)

Estimating \( \| \partial_k^l \lambda_c \|_{L^\infty} \) is rather cumbersome with
\[
\| \partial_k^l \lambda_c \|_{L^\infty} \leq \sum_{a=0}^{l} \sum_{b=0}^{l} \left( \frac{l}{a} \right) \left( \frac{a}{b} \right) \left| \partial_k^{l-a} K \right|_{L^1} \\
\times \left| \partial_k^{a-b} \left( [k_2 + r]^{-1} \right) \right|_{L^\infty} \\
\times \left| \partial_k \kappa \right|_{L^\infty}
\]

(D55)

for all \( l \in \mathbb{N}_0 \). However, using corollary [9] and [6] one obtains
\[
\| \partial_k^l \lambda_c \|_{L^\infty} \\
\leq \sum_{a=0}^{l} \sum_{b=0}^{l} \left( \frac{l}{a} \right) \left( \frac{a}{b} \right) \left| \partial_k^{l-a} K \right|_{L^1} \\
\times \sum_{a=0}^{l} \sum_{b=0}^{l} \left( \frac{l}{a} \right) \left( \frac{a}{b} \right) \left| \partial_k^{l-a} K \right|_{L^1} \\
= \sum_{a=0}^{l} \sum_{b=0}^{l} \left( \frac{l}{a} \right) \left( \frac{a}{b} \right) \left| \partial_k^{l-a} K \right|_{L^1} \\
\times \left( \sum_{\beta \in \mathbb{N}_0^c} \frac{b_1}{\beta} \prod_{i=1}^{\#c} \left| \partial_k^{\beta_i} \kappa_{2+c_i} \right|_{L^\infty} \right)
\]

(D56)
Inserting this result into equation D54 yields
\[
\| \partial_k^\ell p_{2n+1} \tilde{\lambda}_c \|_{L^\infty} \\
\leq \sum_{a=0}^{l} \sum_{\beta \in \mathbb{N}^{|\beta|} \setminus \{0\}} \sum_{|\beta| \leq l-a} E_{2n+1} F_{l-a, \beta} \\
\quad \times \frac{k|\beta|-l}{(m^2 + k^2)^{\#c-1}} \prod_{j=1}^{\#c} \| \partial_k^\beta \kappa_{2+c_j} \|_{L^\infty} \\
= \sum_{\beta \in \mathbb{N}^{|\beta|} \setminus \{0\}} \frac{G_{c, \beta} \kappa_{|\beta| - l}}{(m^2 + k^2)^{\#c-1}} \prod_{j=1}^{\#c} \| \partial_k^\beta \kappa_{2+c_j} \|_{L^\infty}.
\]

so that we just need to use proper estimates for \( \| \partial_k^\beta \kappa_{2+c} \|_{L^\infty} \). To that end, let \( x \in \mathbb{N} \) be arbitrary and fix some multi-index \( X \in \mathbb{N}^{|\beta|}_0 \) with \( |X| = x+l \). Invoking the induction hypothesis leads to the conclusion
\[
\| \partial_k^\beta \kappa_{2+c_j} \|_{L^\infty} \\
\leq B_{2+c_j} \left( \frac{|m|^{|d+(2-d)\left(\frac{c_j}{2}+1\right) + \left(\frac{c_j}{2}-1\right)(1+\Delta)} k^x_j}{(k + |m|)^{(\frac{c_j}{2}-1)(1+\Delta)+X_j+\beta_j}} \right)
\]
for all even \( c_j \in \mathbb{N} \) and all \( \beta_j \in \mathbb{N}_0 \). In particular, this translates to
\[
\prod_{j=1}^{\#c} \| \partial_k^\beta \kappa_{2+c_j} \|_{L^\infty} \\
\leq \frac{|m|^{(1-\Delta)(n+\Delta-d)n \Delta + l} \prod_{j=1}^{\#c} B_{2+c_j}}{(k + |m|)^{(1-\Delta)(n-\#c-\Delta)+x+l}}
\]
\[
= \frac{|m|^{(1-\Delta)(n+\Delta-d)n + l}}{(k + |m|)^{(1-\Delta)(\#c-1)}}
\]
\[
\times \frac{|m|^{(3+\Delta-d)n+1-\Delta k^x_l}}{(k + |m|)^{n+\Delta-n-2\#c+x+b+l+1-\Delta}} \prod_{j=1}^{\#c} B_{2+c_j}
\]
\[
\leq \frac{|m|^{(3+\Delta-d)n+1-\Delta k^x_l}}{(k + |m|)^{n+\Delta-n-2\#c+x+b+l+1-\Delta}} \prod_{j=1}^{\#c} B_{2+c_j}
\]
\[
\leq \frac{|m|^{(3+\Delta-d)n+1-\Delta k^x_l \left(k^2 + m^2\right)^{\#c-1}}}{(k + |m|)^{n+\Delta+n+l-1-\Delta}} \times 2^{\#c-1} \prod_{j=1}^{\#c} B_{2+c_j}
\]

with \( |c| = 2n \) and \( |\beta| = b \). Here, it may be seen that it was important to choose \( \Delta \leq 1 \) Otherwise, the second inequality would in general not hold. Thus, the largest \( \Delta \) that is possible using these methods is max \{d - 3, 1\} which precisely corresponds to the choice made in equation 47. We may now insert this result into equation D57.

Due to our appropriate choice of \( X \) and our careful estimates the right-hand side precisely corresponds to the one of equation D41 with \( n \) replaced by \( n+1 \).
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