Some Identities on $\lambda$-Analogues of $r$-Stirling Numbers of the First Kind

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Abstract. In this paper, we study $\lambda$-analogues of the $r$-Stirling numbers of the first kind which have close connections with the $r$-Stirling numbers of the first kind and $\lambda$-Stirling numbers of the first kind. Specifically, we give the recurrence relations for these numbers and show their connections with the $\lambda$-Stirling numbers of the first kind and higher-order Daehee polynomials.

1. Introduction

It is known that the Stirling numbers of the first kind are defined as

$$(x)_n = \sum_{l=0}^{n} S_1(n, l)x^l, \quad \text{(see [1, 2, 6 - 9, 14])},$$  

(1)

where $(x)_0 = 1$, $(x)_n = x(x - 1) \cdots (x - n + 1)$, $(n \geq 1)$.

For $\lambda \in \mathbb{R}$, the $\lambda$-analogue of falling factorial sequence is defined by

$$(x)_n,\lambda = \sum_{l=0}^{n} S_1(n, l)x^l(\lambda), \quad \text{(see [2, 10, 14, 15, 17])},$$

(2)

In view of (1), we define $\lambda$-analogues of the Stirling numbers of the first kind as

$$(x)_n,\lambda = \sum_{l=0}^{n} S_{1,\lambda}(n, l)x^l, \quad \text{(see [2, 11 - 13, 16, 17])},$$

(3)

It is not difficult to show that

$$(1 + \lambda t)^x = \sum_{l=0}^{\infty} \binom{x}{l}\lambda (l) t^l = \sum_{l=0}^{\infty} \frac{(x)_l,\lambda}{l!} t^l, \quad \text{(see [4, 7 - 17])},$$

(4)

where $\binom{x}{l}\lambda$ are the $\lambda$-analogues of binomial coefficients $\binom{x}{l}$ given by $\binom{x}{l}\lambda = \frac{(x)_l,\lambda}{l!}$.
The \(r\)-Stirling numbers of the first kind are defined by the generating function
\[
\frac{1}{k!} \left( \log(1+t) \right)^{k} (1+t)^r = \sum_{n=0}^{\infty} S^{(r)}_{1}(n,k) \frac{t^n}{n!}, \quad \text{(see [3, 20 – 23])}.
\]
where \(k \in \mathbb{N} \cup \{0\}\) and \(r \in \mathbb{R}\).

The unsigned \(r\)-Stirling numbers of the first kind are defined as
\[
(x+r)(x+r+1) \cdots (x+r+n-1) = \sum_{k=0}^{n} \binom{n+r}{k+r} x^k, \quad \text{(see [1, 17, 22]).}
\]
Thus, by (5), we get
\[
(x+r)_n = (x+r)(x+r-1) \cdots (x+r-n+1) = \sum_{k=0}^{n} S^{(r)}_{1}(n,k) x^k, \quad \text{(see [1]).}
\]
From (5) and (7), we note that
\[
S^{(r)}_{1}(n,k) = (-1)^{n-k} \binom{n+r}{k+r}. \quad \text{(8)}
\]

The higher-order Daehee polynomials are defined by
\[
\frac{\left( \log(1+t) \right)^{k}}{k!} (1+t)^r = \sum_{n=0}^{\infty} D^{(k)}_{n}(x) \frac{t^n}{n!}, \quad \text{(see [5, 18, 19, 24]).}
\]
When \(x = 0\), \(D^{(k)}_{n}(0) = D^{(k)}_{n}(0)\) are called the higher-order Daehee numbers. In particular, for \(k = 1\), \(D_{n}(x) = D^{(1)}_{n}(x)\), \((n \geq 0)\), are called the ordinary Daehee polynomials.

In this paper, we consider \(\lambda\)-analogues of \(r\)-Stirling numbers of the first kind which are derived from the \(\lambda\)-analogues of the falling factorial sequence and investigate some properties for these numbers. Specifically, we give some identities and recurrence relations for the \(\lambda\)-analogues of \(r\)-Stirling numbers of the first kind and show their connections with the \(\lambda\)-Stirling numbers of the first kind and higher-order Daehee polynomials.

2. \(\lambda\)-analogues of \(r\)-Stirling numbers of the first kind

From (3) and (4), we have
\[
(1 + \lambda t)^{2} = \sum_{k=0}^{\infty} (x)_{k,\lambda} \frac{t^k}{k!} = \sum_{k=0}^{\infty} \left( \sum_{n=0}^{k} \binom{k}{n} S_{1,\lambda}(k,n) \frac{t^n}{n!} \right) \frac{t^k}{k!}
\]
\[
= \sum_{n=0}^{\infty} \left( \frac{n!}{n!} \sum_{k=n}^{\infty} S_{1,\lambda}(k,n) \frac{t^k}{k!} \right) \frac{t^n}{n!}. \quad \text{(10)}
\]
On the other hand, we also have
\[
(1 + \lambda t)^{2} = e^{\frac{t}{2} \log(1+\lambda t)} = \sum_{n=0}^{\infty} \left( \frac{\log(1+\lambda t)}{\lambda} \right)^n \frac{x^n}{n!}. \quad \text{(11)}
\]
Therefore, by (10) and (11), we get the generating function for \(S_{1,\lambda}(n,k)\), \((n,k \geq 0)\), which is given by
\[
\frac{1}{n!} \left( \frac{\log(1+\lambda t)}{\lambda} \right)^n = \sum_{k=n}^{\infty} S_{1,\lambda}(k,n) \frac{t^k}{k!}. \quad \text{(12)}
\]
Thus, by (15) and (16), we get
\[
\frac{1}{k!} \left( \frac{\log(1 + \lambda t)}{\lambda} \right)^k (1 + \lambda t)^\frac{k}{\lambda} = \sum_{n=k}^{\infty} S_{1,\lambda}^{(r)}(n, k) \frac{t^n}{n!},
\]
where \( k \in \mathbb{N} \cup \{0\} \), and \( r \in \mathbb{R} \). From (12) and (13), we note that \( S_{1,\lambda}^{(0)}(n, k) = S_{1,\lambda}(n, k), (n \geq k \geq 0) \). Also, it is easy to show that
\[
(1 + \lambda t)^\frac{k}{\lambda} (1 + \lambda t)^\frac{k}{\lambda} = \sum_{n=0}^{\infty} (x + r)_{n,\lambda} \frac{t^n}{n!}.
\]
By (14), we get
\[
\sum_{n=0}^{\infty} (x + r)_{n,\lambda} \frac{t^n}{n!} = \sum_{n=0}^{\infty} \binom{x + r}{n} \frac{t^n}{n!} = (1 + \lambda t)^\frac{k}{\lambda} e^{\lambda t} \sum_{r=0}^{\infty} \lambda r^r \frac{t^r}{r!} = \sum_{r=0}^{\infty} \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{m=k}^{\infty} \binom{m}{k} \frac{t^m}{m!} \left( \sum_{l=0}^{\infty} \binom{n}{l} \frac{t^l}{l!} \right) \frac{t^n}{n!}.
\]
Therefore, by comparing the coefficients on both sides of (15), we obtain the following theorem.

**Theorem 2.1.** For \( n \geq 0 \), we have
\[
(x + r)_{n,\lambda} = \sum_{k=0}^{n} S_{1,\lambda}^{(r)}(n, k) x^k.
\]
Now, we observe that
\[
\sum_{k=0}^{\infty} x^k \frac{1}{k!} \left( \frac{\log(1 + \lambda t)}{\lambda} \right)^k (1 + \lambda t)^\frac{k}{\lambda} = \sum_{m=0}^{\infty} \sum_{k=0}^{m} \binom{m}{k} \frac{1}{k!} S_{1,\lambda}^{(r)}(m, k) \frac{t^m}{m!} \left( \sum_{l=0}^{\infty} \binom{n}{l} \frac{t^l}{l!} \right) \frac{t^n}{n!},
\]
\[
= \sum_{m=0}^{\infty} \sum_{k=0}^{m} \binom{m}{k} \frac{1}{k!} S_{1,\lambda}^{(r)}(m, k) \frac{t^m}{m!} \left( \sum_{l=0}^{\infty} \binom{n}{l} \frac{t^l}{l!} \right) \frac{t^n}{n!}.
\]
Thus, by (15) and (16), we get
\[
\sum_{k=0}^{n} S_{1,\lambda}^{(r)}(n, k) x^k = \sum_{k=0}^{n} \sum_{m=k}^{n} \binom{n}{m} S_{1,\lambda}^{(r)}(m, k) \frac{t^m}{m!} \left( \sum_{l=0}^{\infty} \binom{n}{l} \frac{t^l}{l!} \right) x^k.
\]
Therefore, by comparing the coefficients on both sides of (17), we obtain the following theorem.
Theorem 2.2. For \( n \geq 0 \), we have
\[
S^{(r)}_{1,\lambda}(n, k) = \sum_{m=k}^{n} \binom{n}{m} S_{1,\lambda}(m, k)(r)_{n-m,\lambda}.
\]

Now, we define \( \lambda \)-analogues of the unsigned \( r \)-Stirling numbers of the first kind as follows:
\[
(x + r)(x + r + \lambda)(x + r + 2\lambda) + \cdots (x + r + (n-1)\lambda) = \sum_{k=0}^{n} \left[ \frac{n+r}{k+r} \right]_{\lambda} x^k.
\]

Note that \( \lim_{\lambda \to 1} \left[ \frac{n+r}{k+r} \right]_{\lambda} = \left[ \frac{n+r}{k+r} \right] \) \((n \geq k \geq 0)\).

By Theorem 2.1 and (18), we get
\[
(x - r)_{n,\lambda} = \sum_{k=0}^{n} S^{(-r)}_{1,\lambda}(n, k)x^k,
\]  
and
\[
(x - r)_{n,\lambda} = \sum_{k=0}^{n} (-1)^{n-k} \left[ \frac{n+r}{k+r} \right]_{\lambda} x^k.
\]

From (19) and (20), we can easily derive the following equation (21).
\[
S^{(-r)}_{1,\lambda}(n, k) = (-1)^{n-k} \left[ \frac{n+r}{k+r} \right] \) \((n \geq k \geq 0)\).
\]

For \( n \geq 1 \), by Theorem 2.1, we get
\[
(x + r)_{n+1,\lambda} = \sum_{k=0}^{n+1} S^{(r)}_{1,\lambda}(n+1, k)x^k = \sum_{k=1}^{n+1} S^{(r)}_{1,\lambda}(n+1, k)x^k + (r)_{n+1,\lambda}.
\]

On the other hand, by (2), we get
\[
(x + r)_{n+1,\lambda} = (x + r)_{n,\lambda}(x + r - n\lambda)
= x \sum_{k=0}^{n} S^{(r)}_{1,\lambda}(n, k)x^k - (n\lambda - r) \sum_{k=0}^{n} S^{(r)}_{1,\lambda}(n, k)x^k
= \sum_{k=1}^{n} S^{(r)}_{1,\lambda}(n, k - 1)x^k - (n\lambda - r)S^{(r)}_{1,\lambda}(n, k)x^k + (r - n\lambda)(r)_{n,\lambda} + x^{n+1}
= \sum_{k=1}^{n} \left( S^{(r)}_{1,\lambda}(n, k - 1) - (n\lambda - r)S^{(r)}_{1,\lambda}(n, k) \right) x^k + (r)_{n+1,\lambda} + x^{n+1}.
\]

Therefore, by Theorem 2.1 and (23), we obtain the following theorem.

Theorem 2.3. For \( 1 \leq k \leq n \), we have
\[
S^{(r)}_{1,\lambda}(n + 1, k) = S^{(r)}_{1,\lambda}(n, k - 1) - (n\lambda - r)S^{(r)}_{1,\lambda}(n, k).
\]
From (13), we note that
\[
\frac{1}{k!} \left( \frac{\log(1 + \lambda t)}{\lambda} \right)^k (1 + \lambda t)^\frac{k}{\lambda} = \frac{1}{k!} \left( \frac{\log(1 + \lambda t)}{\lambda} \right)^k \sum_{l=0}^{\infty} \frac{r^l}{l!} \left( \frac{\log(1 + \lambda t)}{\lambda} \right)^l
\]
\[
= \sum_{l=0}^{\infty} \sum_{i=0}^{k+l} \binom{k+l}{l} \frac{1}{(k+l)!} \left( \frac{\log(1 + \lambda t)}{\lambda} \right)^{k+l}
\]
\[
= \sum_{l=0}^{\infty} \sum_{i=0}^{k+l} \binom{k+l}{l} \sum_{m=n+k+l}^{\infty} S_{1,\lambda}(n, k + l) \frac{m^n}{m!}
\]
\[
= \sum_{m=0}^{\infty} \sum_{l=0}^{k} \binom{n}{m} D_i^{(k)} \lambda^i (r_{n+m-l}, \frac{m^n}{m!}) \frac{k^i}{i!}
\]
\[
(24)
\]
On the other hand, we have
\[
\frac{1}{k!} \left( \frac{\log(1 + \lambda t)}{\lambda} \right)^k (1 + \lambda t)^\frac{k}{\lambda} = \frac{1}{k!} \left( \frac{\log(1 + \lambda t)}{\lambda} \right)^k \sum_{l=0}^{\infty} \frac{r^l}{l!} \left( \frac{\log(1 + \lambda t)}{\lambda} \right)^l
\]
\[
= \sum_{l=0}^{\infty} D_i^{(k)} \frac{\lambda^i}{l!} \left( \sum_{m=0}^{\infty} (r_{m}, \frac{m^n}{m!}) \right) \frac{k^i}{i!}
\]
\[
= \sum_{m=0}^{\infty} \sum_{l=0}^{k} \binom{n}{m} D_i^{(k)} \lambda^i (r_{n+m-l}, \frac{m^n}{m!}) \frac{k^i}{i!}
\]
\[
(25)
\]
Thus, by (24) and (25), we get
\[
\sum_{l=0}^{n} r^l \left( \frac{k^i}{m+k} \right) S_{1,\lambda}(n+k+l) = \sum_{l=0}^{n} \binom{n}{l} D_i^{(k)} \lambda^i (r_{n+m-l}, \frac{m^n}{m!}) \frac{k^i}{i!}
\]
\[
(26)
\]
Therefore, by (26), we obtain the following theorem.

**Theorem 2.4.** For \( n \geq 0 \), we have
\[
\sum_{l=0}^{n} \binom{n}{l} D_i^{(k)} \lambda^i (r_{n+m-l}) = \sum_{l=0}^{n} \binom{n}{l} D_i^{(k)} \lambda^i (r_{n+m-l}) S_{1,\lambda}(n+k+l).
\]

Now, we observe that
\[
\frac{1}{k!} \left( \frac{\log(1 + \lambda t)}{\lambda} \right)^k (1 + \lambda t)^\frac{k}{\lambda} = \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} D_i^{(k)} \lambda^i (r_{n+m}, \frac{m^n}{m!}) \frac{k^i}{i!}
\]
\[
= \sum_{m=0}^{\infty} \sum_{l=0}^{k} \binom{n}{m} S_{1,\lambda}(m,k) (r_{n-m}, \frac{m^n}{m!}) \frac{k^i}{i!}
\]
\[
(27)
\]
Therefore, by (13) and (27), we obtain the following theorem.

**Theorem 2.5.** For \( n, k \geq 0 \), with \( n \geq k \), we have
\[
S_{1,\lambda}^{(n)}(m,k) = \sum_{m=k}^{n} \binom{n}{m} (r_{m-k}) S_{1,\lambda}(m,k).
\]
From (13), we note that
\[
\frac{1}{m!} \left( \frac{\log(1 + \lambda t)}{\lambda \lambda} \right)^m \frac{1}{k!} \left( \frac{\log(1 + \lambda t)}{\lambda \lambda} \right)^k (1 + \lambda t)^{\frac{k}{2}}
\]
\[
= \frac{(m + k)!}{m!k!} \frac{1}{(m + k)!} \left( \frac{\log(1 + \lambda t)}{\lambda \lambda} \right)^{m+k} (1 + \lambda t)^{\frac{k}{2}}
\]
\[
= \left( \frac{m + k}{m} \right) \sum_{n=m+k}^{\infty} \sum_{l=0}^{n} \binom{n}{l} S_{1,\lambda}^{(r)}(l, m + k) \frac{t^n}{n!}
\]

On the other hand,
\[
\frac{1}{m!} \left( \frac{\log(1 + \lambda t)}{\lambda \lambda} \right)^m \frac{1}{k!} \left( \frac{\log(1 + \lambda t)}{\lambda \lambda} \right)^k (1 + \lambda t)^{\frac{k}{2}}
\]
\[
= \left( \sum_{l=0}^{\infty} S_{1,\lambda}(l, m) \frac{t^l}{l!} \right) \left( \sum_{j=0}^{\infty} S_{1,\lambda}^{(r)}(j, k) \frac{t^j}{j!} \right)
\]
\[
= \sum_{n=m+k}^{\infty} \left( \sum_{l=0}^{n} \binom{n}{l} S_{1,\lambda}^{(r)}(l, k) S_{1,\lambda}(n-l, m) \right) \frac{t^n}{n!}
\]

Therefore, by (28) and (29), we obtain the following theorem.

**Theorem 2.6.** For \(m, n, k \geq 0\) with \(n \geq m + k\), we have
\[
\left( \frac{m + k}{m} \right) S_{1,\lambda}^{(r)}(n, m + k) = \sum_{l=0}^{n-m} \binom{n}{l} S_{1,\lambda}(l, k) S_{1,\lambda}(n-l, m)
\]

By (12), we get
\[
\sum_{n=k}^{\infty} S_{1,\lambda}(n, k) \frac{t^n}{n!} = \frac{1}{k!} \left( \frac{\log(1 + \lambda t)}{\lambda \lambda} \right)^k (1 + \lambda t)^{\frac{k}{2}}
\]
\[
= \left( \sum_{l=0}^{\infty} S_{1,\lambda}(l, k) \frac{t^l}{l!} \right) \left( \sum_{m=0}^{\infty} \binom{-\frac{k}{2}}{m} \lambda^m t^m \right)
\]
\[
= \left( \sum_{l=0}^{\infty} S_{1,\lambda}(l, k) \frac{t^l}{l!} \right) \left( \sum_{m=0}^{\infty} (-1)^m (r + (m - 1)\lambda)_{m, l} \frac{t^m}{m!} \right)
\]
\[
= \sum_{n=k}^{\infty} \left( \sum_{l=0}^{\infty} \binom{n}{l} S_{1,\lambda}^{(r)}(l, k)(-1)^{n-l}(r + (n - 1)\lambda)_{n-l, l} \right) \frac{t^n}{n!}
\]

Comparing the coefficients on both sides of (30), we have the following theorem.

**Theorem 2.7.** For \(n, k \geq 0\), with \(n \geq k\), we have
\[
S_{1,\lambda}(n, k) = \sum_{l=k}^{n} \binom{n}{l} S_{1,\lambda}^{(r)}(l, k)(-1)^{n-l}(r + \lambda(n - l - 1))_{n-l, l}
\]
From (9), we have
\[
\frac{1}{k!} \left( \frac{\log(1 + \lambda t)}{\lambda} \right)^k (1 + \lambda t) \overset{\lambda}{=} \frac{1}{k!} \left( \frac{\log(1 + \lambda t)}{\lambda t} \right)^k (1 + \lambda t) \overset{\lambda}{=}
\]
\[
= \frac{t^k}{k!} \sum_{m=0}^{\infty} D_m^{(r)} \frac{\mu^m}{m!} \left( \sum_{l=0}^{\infty} (r)_l \frac{\mu^l}{l!} \right) = \frac{t^k}{k!} \sum_{m=0}^{\infty} \sum_{l=0}^{n} \binom{n}{m} D_m^{(r)} \lambda^m (r)_{n-m, \lambda} \frac{\mu^l}{l!}.
\]
On the other hand, by (13), we get
\[
= \frac{t^k}{k!} \sum_{m=0}^{\infty} \sum_{l=0}^{n} \binom{n}{m} D_m^{(r)} \lambda^m (r)_{n-m, \lambda} \frac{\mu^l}{l!}
\]
Thus, by comparing the coefficients on both sides of (31) and (32), we get
\[
\sum_{m=0}^{\infty} \binom{n}{m} D_m^{(r)} \lambda^m (r)_{n-m, \lambda} = \frac{1}{\binom{n+r}{n}} S_n^{(r)} (n, k).
\]
From (9), we note that
\[
= \frac{t^k}{k!} \sum_{m=0}^{\infty} \lambda^m D_m^{(r)} (\frac{\lambda}{\lambda}) \frac{\mu^m}{m!}
\]
By (32) and (34), we get
\[
S_n^{(r)} (n, k) = \lambda^k \frac{(n+k)!}{n!k!} D_n^{(r)} (\frac{\lambda}{\lambda}) = \lambda^k \binom{n+k}{n} D_n^{(r)} (\frac{\lambda}{\lambda}), (n \geq 0).
\]
In particular, for \( r = 0 \), from (30) and (35) we have
\[
\lambda^k \binom{n+k}{k} D_n^{(r)} (\frac{\lambda}{\lambda}) = S_n^{(r)} (n, k, k)
\]
where \( n, k \geq 0 \).
Therefore, by (36), we obtain the following theorem.
Theorem 2.9. For \( n, k \geq 0 \), we have

\[
\lambda^n \binom{n + k}{k} D_n^{(\lambda)} = \sum_{l=0}^{n+k} \binom{n+k}{l} S_{1,\lambda}(l, k)(-1)^{n+k-l}(r + (n + k - l)\lambda)_{n+k-l,\lambda}.
\]

In addition,

\[
D_n^{(\lambda)} = \frac{1}{\lambda^{n+k}} \sum_{l=0}^{n+k} \binom{n+k}{l} l^k \left( \frac{1}{\lambda} \right)^{l+k-1} \times (r + (n + k - l)\lambda)_{n+k-l,\lambda}(-1)^{n+k-l} D_l^{(\lambda)}(\frac{z}{\lambda}).
\]

Now, we observe that

\[
\sum_{n=k}^{\infty} S_{1,\lambda}(n, k) \frac{t^n}{n!} = \frac{1}{k!} \left( \log(1 + \lambda t) \right)^k (1 + \lambda t)^{1/\lambda} e^{-t/\lambda} \log(1 + \lambda t)
\]

\[
= \left( \sum_{n=k}^{\infty} S_{1,\lambda}^{(\lambda)}(l, k) \frac{t^l}{l!} \right) \sum_{m=0}^{\infty} (-1)^m \frac{\lambda^m}{m!} \left( \log(1 + \lambda t) \right)^m
\]

\[
= \frac{1}{k!} \left( \sum_{n=k}^{\infty} S_{1,\lambda}^{(\lambda)}(l, k) \frac{t^l}{l!} \right) \sum_{m=0}^{\infty} (-1)^m \frac{\lambda^m}{m!} \sum_{j=m}^{\infty} S_{1,\lambda}(j, m) \frac{t^j}{j!}
\]

\[
= \sum_{n=k}^{\infty} \sum_{j=0}^{n-k} \binom{n}{j} (-1)^m \frac{\lambda^m}{m!} S_{1,\lambda}(j, m) S_{1,\lambda}(n-j, k) \frac{t^n}{n!}.
\]

Therefore, by comparing the coefficients on both sides of (37), we obtain the following theorem.

Theorem 2.10. For \( n, k \geq 0 \), with \( n \geq k \), we have

\[
S_{1,\lambda}(n, k) = \sum_{j=0}^{n-k} \sum_{m=0}^{\infty} \binom{n}{j} (-1)^m r^m S_{1,\lambda}(j, m) S_{1,\lambda}(n-j, k).
\]

For \( m, n \geq 0 \), we define \( \lambda \)-analogues of the Whitney’s type \( r \)-Stirling numbers of the first kind as

\[
(mx + r)_{n,\lambda} = (mx + r)(mx + r - \lambda)(mx + r - 2\lambda) \cdots (mx + r - (n-1)\lambda)
\]

\[
= \sum_{k=0}^{n} T_{1,\lambda}^{(\lambda)}(n, k|m)x^k.
\]

By (38), we get

\[
\sum_{n=0}^{\infty} (mx + r)_{n,\lambda} \frac{t^n}{n!} = \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} T_{1,\lambda}^{(\lambda)}(n, k|m)x^k \right) \frac{t^n}{n!}
\]

\[
= \sum_{k=0}^{\infty} \left( \sum_{n=k}^{\infty} T_{1,\lambda}^{(\lambda)}(n, k|m) \frac{t^n}{n!} \right) x^k.
\]

On the other hand, by binomial expansion, we get

\[
\sum_{n=0}^{\infty} (mx + r)_{n,\lambda} \frac{t^n}{n!} = \sum_{n=0}^{\infty} \binom{mx + r}{n} \frac{t^n}{n!}
\]

\[
= (1 + \lambda t)^{mx} = (1 + \lambda t)^{1/\lambda} e^{\lambda t(1 - 1/\lambda)}
\]

\[
= \sum_{k=0}^{\infty} \frac{m^k}{k!} \left( \frac{\log(1 + \lambda t)}{\lambda} \right)^k (1 + \lambda t)^{1/\lambda} x^k.
\]
Comparing the coefficients on both sides of (39) and (40), the generating function for $T_{1,\lambda}^{(r)}(n,k|m)$, $(n,k \geq 0)$, is given by

$$\frac{m^k}{k!} \left( \log(1 + \lambda t) \right)^k \left( 1 + \lambda t \right)^\lambda = \sum_{n=k}^{\infty} T_{1,\lambda}^{(r)}(n,k|m) \frac{t^n}{n!}. \quad (41)$$

From (13) and (41), we note that

$$S_{1,\lambda}^{(r)}(n,k) = \frac{1}{m^k} T_{1,\lambda}^{(r)}(n,k|m), \quad (n \geq k \geq 0). \quad (42)$$

It is known that the $r$-Whitney numbers are defined as

$$(mx + r)^x = \sum_{k=0}^{n} m^k W_{m,r}(n,k)(x)_k, \quad \text{(see [3]).} \quad (43)$$

By (3), we get

$$(mx + r)^{n,\lambda} = \sum_{j=0}^{n} S_{1,\lambda}(n,l) (mx + r)^l$$

$$= \sum_{j=0}^{n} S_{1,\lambda}(n,l) \sum_{j=0}^{l} m^i W_{m,r}(l,i)(x)_j$$

$$= \sum_{j=0}^{n} \sum_{l=j}^{n} S_{1,\lambda}(n,l) m^i W_{m,r}(l,i)(x)_j$$

$$= \sum_{j=0}^{n} \sum_{l=j}^{n} S_{1,\lambda}(n,l) m^i W_{m,r}(l,i) \sum_{k=0}^{j} S_1(j,k)x^k$$

$$= \sum_{k=0}^{n} \left( \sum_{j=k}^{n} \sum_{l=j}^{n} S_{1,\lambda}(n,l) S_1(j,k) m^i W_{m,r}(l,i) \right) x^k. \quad (44)$$

Therefore, by (38) and (44), we obtain the following theorem.

Theorem 2.11. For $n,k \geq 0$, with $n \geq k$, we have

$$T_{1,\lambda}^{(r)}(n,k|m) = \sum_{j=0}^{n} \sum_{l=j}^{n} S_{1,\lambda}(n,l) S_1(j,k) m^i W_{m,r}(l,i).$$

Acknowledgements. This paper is dedicated to 70th birthday of Professor Gradimir V. Milovanovic. Also, we would like to express our sincere condolences on the death of Professor Simsek’s mother.

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