A faithful analytical effective one body waveform model for spin-aligned, moderately eccentric, coalescing black hole binaries

Danilo Chiaramello\textsuperscript{1,2} and Alessandro Nagar\textsuperscript{2,3}
\textsuperscript{1} Dipartimento di Fisica, Università di Torino, via P. Giuria 1, 10125 Torino, Italy
\textsuperscript{2}INFN Sezione di Torino, Via P. Giuria 1, 10125 Torino, Italy and
\textsuperscript{3}Institut des Hautes Études Scientifiques, 91440 Bures-sur-Yvette, France

We present a new effective-one-body (EOB) model for eccentric binary coalescences. The model stems from the state-of-the-art model \texttt{TEOBiResumSM} for circularized coalescing black-hole binaries, that is modified to explicitly incorporate eccentricity effects both in the radiation reaction and in the waveform. Using Regge-Wheeler-Zerilli type calculations of the gravitational wave losses as benchmarks, we find that a rather accurate (\(~1\%) expression for the radiation reaction along mildly eccentric orbits (\(\epsilon \sim 0.3\)) is given by dressing the current, EOB-resummed, circularized angular momentum flux, with a leading-order (Newtonian-like) prefactor valid along general orbits. An analogous approach is implemented for the waveform multipoles. The model is then completed by the usual merger-ringdown part informed by circularized numerical relativity (NR) simulations. The model is validated against the few, publicly available, NR simulations calculated by the Simulating eXtreme Spacetime (SXS) collaboration, with mild eccentricities, mass ratios between 1 and 3 and up to rather large dimensionless spin values (\(\pm 0.7\)). We find that the maximum EOB/NR unfaithfulness, calculated with Advanced LIGO noise, is at most of order 3\% except for 1 single outlier, with eccentricity \(\sim 0.2\), that is just above 4\%. The analytical framework presented here should be seen as a promising starting point for developing highly-faithful waveform templates driven by eccentric dynamics for present, and possibly future, gravitational wave detectors.

Introduction. — Parameter estimates of all gravitational wave (GW) signals from coalescing binaries are done under the assumption that the inspiral is quasi-circular \cite{1}. This is motivated by the efficient circularization of the inspiral due to gravitational wave emission. In addition, no explicit evidence for eccentricity for some events was found \cite{2,4}. However, recent population synthesis studies \cite{5–8} suggest that active galactic nuclei and globular clusters may host a population of eccentric binaries. Currently, there are no ready-to-use waveform models that accurately combine both eccentricity and spin effects over the entire parameter space. Recently, numerical relativity (NR) started producing surveys of eccentric, spinning binary black hole (BBH) coalescence waveforms \cite{9,11}, and a NR-surrogate waveform model for nonspinning eccentric binaries up to mass ratio \(q = 10\) exists \cite{11}. On the analytical side, Ref. \cite{12,13} provided closed-form eccentric inspiral templates (based on the Quasi-Keplerian approximation). Similarly, a few exploratory effective-one-body (EOB)-based \cite{14,17} studies were recently performed \cite{18,20}. In particular Ref. \cite{18,20} introduced and tested \texttt{SEOBNRE}, a way to incorporate eccentricity within the \texttt{SEOBNRv1} \cite{21} circularized waveform model. However, the \texttt{SEOBNRv1} model is outdated now, since it does not accurately cover high-spins, nor mass ratios up to 10. This drawback is inherited by the \texttt{SEOBNRE} model \cite{20}.

In this Letter, we modify a highly NR-faithful EOB multipolar waveform model for circularized coalescing BBHs, \texttt{TEOBiResumSM} \cite{22,23}, to incorporate eccentricity-dependent effects. The EOB formalism relies on three building blocks: (i) a Hamiltonian, that describes the conservative part of the relative dynamics; (ii) a radiation reaction force, that accounts for the back-reaction onto the system due to the GW losses of energy and angular momentum; (iii) a prescription for computing the waveform. Including eccentricity requires modifications to blocks (ii) and (iii) with respect to the quasi-circular case.

Radiation reaction and waveform for eccentric inspirals. — Within the EOB formalism, we use phase-space variables \((r, \varphi, p_r, p_\varphi)\), related to the physical ones by \(r = R/(GM)\) (relative separation), \(p_r = P_{Rr}/\mu\) (radial momentum), \(p_\varphi = P_{\varphi}/(\mu GM)\) (angular momentum) and \(t = T/(GM)\) (time), where \(\mu \equiv m_1 m_2/M\) and \(M = m_1 + m_2\). The radial momentum is \(p_r \equiv (A/B)^{1/2} p_{r,\text{orb}}\), where \(A\) and \(B\) are the EOB potentials. The EOB Hamiltonian is

\[
\hat{H}_{\text{EOB}} \equiv \frac{H_{\text{EOB}}}{\mu} = \nu^{-1} \sqrt{1 + 2\nu (H_{\text{eff}} - 1)},
\]

with \(\nu \equiv \mu/M\) and \(H_{\text{eff}} = G\hat{p}_\varphi + H_{\text{orb}}\), where \(G\hat{p}_\varphi\) incorporates odd-in-spin (spin-orbit) effects while \(H_{\text{orb}}\) even-in-spin effects \cite{22}. We denote dimensionless spin variables as \(\chi_i \equiv S_i/m_i^2\). The \texttt{TEOBiResumSM} \cite{23,24} waveform model is currently the most NR faithful model versus the \texttt{zero_det_highP} Advanced LIGO design sensitivity \cite{26}. Reference \cite{23} found that the maximum value of the EOB/NR unfaithfulness is always below 0.5\% all\cite{27} over the current release of the SXS NR waveform catalog \cite{29,30}. This is achieved by NR informing a 4.5PN spin-orbit effective function \(c_3(\nu, \chi_1, \chi_2)\) and an effective 5PN function \(\hat{a}_0(\nu)\) entering the Padé resummed radial potential \(A(r)\). [see Eqs. (39) and (33) of Ref. \cite{11}] The two Hamilton’s equations...
that take account of GW losses are
\[ \dot{p}_\varphi = \dot{\tilde{F}}_\varphi, \quad \dot{p}_r = \sqrt{\frac{A}{B}} \left( -\partial_r \tilde{H}_{\text{EOB}} + \tilde{F}_r \right), \]

where \( \tilde{F}_\varphi, \tilde{F}_r \) are the two radiation reaction forces. In the quasi-circular case \[22, 12\] one sets \( \tilde{F}_r = 0 \). Here, we use \( \tilde{F}_r \neq 0 \) and \( \tilde{F}_\varphi \) explicitly includes noncircular terms. The main technical issue is to build (resummed) expressions of \( \tilde{F}_\varphi, \tilde{F}_r \) that are reliable and robust up to merger. Building upon Ref. \[33\], Ref. \[44\] derived the 2PN-accurate, generic expressions of \( \tilde{F}_\varphi, \tilde{F}_r \), that are unsuited to drive the transition from the EOB inspiral to plunge and merger: they are nonresummed and generally unreliable in the strong-field regime (see below). The forces are related to the instantaneous losses of energy and angular momentum through GWs. Following Ref. \[44\], there exists a gauge choice such that the balance equations read
\[
- \dot{J}^\infty = F_\varphi, \\
- \dot{E}^\infty = \dot{r} F_r + \dot{\varphi} F_\varphi + \dot{E}_{\text{Schott}},
\]

where \( E_{\text{Schott}} \) is the Schott energy, \( (\dot{E}, \dot{J})^\infty \) are the energy and angular momentum fluxes at infinity, while \( F_{\varphi,r} \) are the energy and angular momentum losses at infinity, while

in the circularized case \[23, 25, 45, 48\]: any analytical choice for \( (F_\varphi, F_r, E_{\text{Schott}}) \) is tested by comparisons with the energy and angular momentum fluxes emitted by a test particle orbiting a Schwarzschild black hole on eccentric orbits. We focus first on \( F_\varphi \). We start with the 2PN-accurate result of Ref. \[44\] [see Eq. (3.70) and Appendix D therein], \( F_{\varphi,2PN}(r, p_r, p_\varphi) \), reexpress it in terms of \( p_r, p_\varphi \), and factor it in a circular part (defined imposing \( p_r = \dot{p}_r = 0 \), \( F_{\varphi,2PN}^\circ \)), and a noncircular contribution, \( F_{\varphi,2PNnc}^\circ \). Following Ref. \[44\], there exists a gauge choice such that the balance equations read
\[
\dot{p}_\varphi = \dot{\tilde{F}}_\varphi, \quad \dot{p}_r = \sqrt{\frac{A}{B}} \left( -\partial_r \tilde{H}_{\text{EOB}} + \tilde{F}_r \right), \]

where \( \tilde{F}_\varphi, \tilde{F}_r \) are the two radiation reaction forces. In the quasi-circular case \[22, 12\] one sets \( \tilde{F}_r = 0 \). Here, we use \( \tilde{F}_r \neq 0 \) and \( \tilde{F}_\varphi \) explicitly includes noncircular terms. The main technical issue is to build (resummed) expressions of \( \tilde{F}_\varphi, \tilde{F}_r \) that are reliable and robust up to merger. Building upon Ref. \[33\], Ref. \[44\] derived the 2PN-accurate, generic expressions of \( \tilde{F}_\varphi, \tilde{F}_r \), that are unsuited to drive the transition from the EOB inspiral to plunge and merger: they are nonresummed and generally unreliable in the strong-field regime (see below). The forces are related to the instantaneous losses of energy and angular momentum through GWs. Following Ref. \[44\], there exists a gauge choice such that the balance equations read
\[
- \dot{J}^\infty = F_\varphi, \\
- \dot{E}^\infty = \dot{r} F_r + \dot{\varphi} F_\varphi + \dot{E}_{\text{Schott}},
\]

where \( E_{\text{Schott}} \) is the Schott energy, \( (\dot{E}, \dot{J})^\infty \) are the energy and angular momentum fluxes at infinity, while \( F_{\varphi,r} \) are the energy and angular momentum losses at infinity, while
Although this expression incorporates formally less noncircular PN information than Eq. (3), the time-derivatives (and \( f(\Omega) \) as well) are obtained from the full EOB (resummed) equations of motion rather than the 2PN ones used in \( F_{\text{2PN}}^2 \). For \( F_r \), we build on Ref. [44] and we use \( F_r = 32/3p_c/r^4P_0^2(F_{\text{2PN}}^2) \), where \( P_0^2 \) is the \((0,2)\) Padé approximant and \( F_{\text{2PN}}^2 = f_r^N + c^{-2}f_{\text{ipn}}^N + c^{-4}f_{\text{ipn}}^N \) is the 2PN accurate expression calculated from Eqs. (3.70) and (D9-D11) of Ref. [44]. We adopt an analogous approach to deal with the Schott energy, as given by Eqs. (3.57) and (C1-C4) of Ref. [44]. We factorize it in a circular and noncircular part that are both resummed with the \( P_0^2 \) Padé approximant, so to have \( E_{\text{Schott}} = 16/5p_c/r^4P_0^2[|E_{\text{ipn}}^2|E_{\text{ipn}}^2|E_{\text{ipn}}^2] \), where \( E_{\text{Schott}} = E_{\text{Schott}}^{\text{p,c,0}} + c^{-2}E_{\text{Schott}}^{\text{ipn,0}} + c^{-4}E_{\text{Schott}}^{\text{ipn,2PN}} \). Equations (3) and (4) are specialized to the test particle limit \((\nu = 0)\) and computed along the eccentric, conservative, dynamics of a particle orbiting a Schwarzschild black hole. The result is compared with the fluxes computed using Regge-Wheeler-Zerilli (RWZ) black hole perturbation theory [51, 52]. To accurately extract waves at future null infinity, we adopt the hyperboloidal layer method of [53] and compute the fluxes with the usual expressions of Ref. [53], including all multipoles up to \( \ell = 8 \). Figure 1 shows the illustrative case of an orbit with semilatus rectum \( p = 9 \) and eccentricity \( e = 0.3 \). The apastron is \( r_1 = p/(1 - e) \), and the periastron is \( r_2 = p/(1 + e) \) [50]. The figure indicates that \( F_{\text{EOB}}^{\text{PNuc}} \) delivers analytical energy and angular momentum fluxes (red lines) that are, on average, in better agreement with the RWZ ones than those obtained from \( F_{\text{EOB}}^{\text{PNuc}} \), that increase up to a 10% fractional difference at apastron. We adopt then \( F_{\text{EOB}}^{\text{Newtnc}} \) as analytical representation of the angular momentum flux along generic orbits. The maximal analytical/RWZ flux relative differences is \( \sim 10^{-2} \). The robustness of this result is checked by considering several orbits with \( p \) varying from just above the stability threshold \((p = 6 + 2e)\) up to \( p = 21 \), and for each \( p \) we consider \( 0 \leq e \leq 0.9 \). We then compute the relative flux differences \((\delta \hat{E}, \delta \hat{J})\) at periastron for each \((p,e)\). We find that, for each value of \( p \), \((\delta \hat{E}, \delta \hat{J})\) are at most of the order of 10% for \( e = 0.9 \). More interestingly if \( e \lesssim 0.3 \), the fractional differences do not exceed the 5% level. Let us consider now the waveform emitted from the transition from inspiral to plunge, merger and ringdown as driven by \((F_r, F_{\text{EOB}}^{\text{Newtnc}})\) focusing on a test-particle (of mass ratio \( \mu = M/\tilde{M} = 10^{-3} \) on a Schwarzschild background. To efficiently compute, along the relative dynamics, up to the third time-derivative of the phase-space variables entering Eq. (1), we suitably generalize the iterative analytical procedure used in Appendix A of Ref. [57] to calculate \( \hat{r} \). We checked that two itera-

**TABLE I.** SXS simulations with eccentricity analyzed in this work. From left to right: the ID of the simulation; the mass ratio \( q \equiv m_1/m_2 \geq 1 \); the individual spins \((\chi_1, \chi_2)\); the estimated NR eccentricity at first apastron \( e_{\text{NR}} \); the initial EOB eccentricity \( e_{\text{EOB}} \) and apastron frequency \( \omega_{\text{EOB}} \) that allow to get a good EOB/NR frequency agreement.

| SXS | \( q \) | \( \chi_1 \) | \( \chi_2 \) | \( e_{\text{NR}} \) | \( e_{\text{EOB}} \) | \( \omega_{\text{EOB}} \) | max(\( \hat{F} \)) \( \% \) |
|-----|-----|-----|-----|-----|-----|-----|-----|
| 1355 | 1   | 0   | 0   | 0.062 | 0.089 | 0.028025 | 1.55   |
| 1356 | 1   | 0   | 0   | 0.102 | 0.1503 | 0.019019 | 2.07   |
| 1359 | 1   | 0   | 0   | 0.112 | 0.193 | 0.020 | 1.06   |
| 1357 | 1   | 0   | 0   | 0.114 | 0.193 | 0.019465 | 1.92   |
| 1361 | 1   | 0   | 0   | 0.160 | 0.23329 | 0.020 | 1.52   |
| 1360 | 1   | 0   | 0   | 0.161 | 0.242 | 0.01948 | 2.12   |
| 1362 | 1   | 0   | 0   | 0.217 | 0.30 | 0.01903 | 1.52   |
| 1364 | 2   | 0   | 0   | 0.049 | 0.085 | 0.025155 | 1.04   |
| 1365 | 2   | 0   | 0   | 0.067 | 0.11 | 0.02395 | 1.15   |
| 1367 | 2   | 0   | 0   | 0.105 | 0.1494 | 0.026 | 1.32   |
| 1369 | 2   | 0   | 0   | 0.201 | 0.31 | 0.01731 | 1.42   |
| 1371 | 3   | 0   | 0   | 0.063 | 0.092 | 0.0290 | 0.74   |
| 1372 | 3   | 0   | 0   | 0.107 | 0.15 | 0.02598 | 1.61   |
| 1374 | 3   | 0   | 0   | 0.208 | 0.31523 | 0.0161 | 4.32   |
| 89   | 1   | -0.5 | 0   | 0.047 | 0.07729 | 0.01776 | 0.95   |
| 1136 | 1   | -0.75 | -0.75 | 0.078 | 0.124 | 0.0271 | 1.11   |
| 1321 | 1.22 | +0.33 | -0.44 | 0.048 | 0.078 | 0.02684 | 1.43   |
| 1322 | 1.22 | +0.33 | -0.44 | 0.063 | 0.099 | 0.02687 | 1.41   |
| 1323 | 1.22 | +0.33 | -0.44 | 0.104 | 0.14 | 0.02606 | 1.64   |
| 3241 | 1.22 | +0.33 | -0.44 | 0.205 | 0.30 | 0.0187 | 3.10   |
| 1149 | 3   | +0.70 | +0.60 | 0.037 | 0.062 | 0.026664 | 3.33   |
| 1169 | 3   | -0.70 | -0.60 | 0.036 | 0.0472 | 0.025 | 0.17   |
LLS:...LLS:...LLS:...

FIG. 3. Illustrative EOB/NR time-domain waveform comparison for the nonspinning configuration SXS:BBH:1369, with $e^\text{NR} = 0.201$, $q = 2$ and $\chi_1 = \chi_2 = 0$.

FIG. 4. EOB/NR unfaithfulness computed over the eccentric SXS simulations publicly available. The horizontal lines mark the 3% and 1% values.

sitions are sufficient to obtain an excellent approximation ($\approx 10^{-3}$) of the derivatives computed numerically. An illustrative waveform is displayed in Fig. 2 for $p = 8$ and $e = 0.3$ (initial values). The top three rows of the figure highlight the numerical consistency ($\approx 10^{-2}$) between the RWZ angular momentum and energy fluxes and their analytical counterparts. The corresponding waveform is shown (in black) in the fourth row of the plot. The gravitational waveform is decomposed in multipoles as $h_+ - i h_\times = D_L^{-1}\sum_s h_{\ell m} \sim Y_{\ell m}$, where $D_L$ is the luminosity distance and $Y_{\ell m}$ the $s = -2$ spin-weighted spherical harmonics. We use below the RWZ normalized variable $\Psi_{\ell m} = h_{\ell m}/\sqrt{(\ell + 2)(\ell + 1)!/(\ell - 1)!}$.

A detailed analysis of the properties of the RWZ waveform, such as the excitation of QNMs, etc., will be presented elsewhere. Here we employ it as a target, “exact” waveform to validate the EOB one. Within the EOB formalism, each multipole is factorized as $h_{\ell m} = h_{\ell m}^{(N)} h_{\ell m}^{(NC)}$.

where $h_{\ell m}^{(N,e)}$ is the Newtonian (leading-order) prefactor, $h_{\ell m}$ is the resummed relativistic correction [46] and $e$ the parity of $\ell + m$. The circularized prefactor $h_{\ell m}^{(N,e)}$ is replaced by its general expression obtained computing the time-derivatives of the Newtonian mass and current multipoles. We have $h_{\ell m}^{(N,0)} \propto e^{\text{imp}} I_{\ell m}^{(f)}$ and $h_{\ell m}^{(N,1)} \propto e^{\text{imp}} S_{\ell m}^{(f)}$, where $(f)$ indicates the $\ell$-th time-derivative and $I_{\ell m} \equiv r^3 e^{-\text{imp}}$ and $S_{\ell m} \equiv r^{\ell+1} \Omega e^{-\text{imp}}$ are the Newtonian mass and current multipoles. The $\ell = m = 2$ mode of the analytical waveform is superposed, as a red line, in the bottom row of Fig. 2 showing excellent agreement with the RWZ one essentially up to merger [55]. A similar agreement is found for subdominant modes.

Comparison with numerical relativity simulations.—The complete, $\nu$-dependent, radiation reaction of above replaces now the standard one used in EOBResumS, so to consistently drive an eccentric inspiral. Everything is analogous to the test-particle case, aside from (i) the initial conditions at the apastron, which are more involved because of the presence of spin, though they are a straightforward generalization of those of [19]; (ii) similar complications for the time-derivatives needed in $f^{\text{Newt,c}}$. The EOB waveform with the noncircular Newtonian prefactors is completed by next-to-quasi-circular (NQC) corrections and the NR-informed circularized ringdown [22] [23]. Differently from the circularized case, the NQC correction factor is smoothly activated in time just when getting very close to merger, to avoid spurious contaminations during the inspiral. Also, no iteration on the NQC amplitude parameters is performed [23]. We assess the quality of the analytic waveforms by comparing them with the sample of eccentric NR simulations publicly available in the SXS catalog [28] [40] that are listed in Table 1. We carry out both time-domain comparisons and compute the EOB/NR unfaithfulness. To do so correctly the EOB evolution should be started in such a way that the eccentricity-induced frequency oscillations are consistent with the corresponding ones in the NR simulations. Since the eccentricities are gauge dependent, their nominal values are meaningless for this purpose. EOB and NR waveforms are then aligned in the
time-domain \cite{57} during the early inspiral and then we progressively vary the initial GW frequency at apastron, \( \omega_a^{\text{EOB}} \), and eccentricity, \( e_a^{\text{EOB}} \), until we achieve minimal fractional differences (~10^{-2}) between the EOB and NR GW frequencies. To facilitate the parameter choice, we also estimate the initial (at first apastron) eccentricity of each NR simulation, \( e_a^{\text{NR}} \), using the method proposed in Eq. (2.8) of Ref. \cite{10}, where \( e_a \) is deduced from the frequency oscillations; we here employ, however, the frequency of the (2, 2) mode, as opposed to the orbital frequency as done in Ref. \cite{10}. The last two columns of Table \ref{tab:tab1} contain the values of \( \omega_a^{\text{EOB}}, e_a^{\text{EOB}} \) that lead to the best agreement between NR and EOB. The corresponding time-domain agreement, using SXS:BBH:1369 as an illustrative, typical, example, is presented in Fig. 3. The accumulated phase difference partly comes from the values of \( (a_0^c, c_3) \), that were obtained using the circularized radiation reaction of Ref. \cite{23}. It is possible, using the new expressions of \( (F_0^c, F_0^c) \) introduced here, and working with initial eccentricity \( \sim 5 \times 10^{-5} \) to retune \( a_0^c \) so to achieve a phase agreement up to merger well below 0.1 rad. An extensive discussion is postponed to future work. Figure 3 shows the EOB/NR unfaithfulness \( \bar{F} \equiv 1 - F \) (see Eq. (48) of Ref. \cite{23}) computed with the \textsc{zerodet\_highp} \cite{26} Advanced-LIGO power spectral density. Both NR and EOB waveforms (starting at approximately the same frequency) were suitably tapered in the early inspiral. We find that \( \max(\bar{F}) \) is at most \( \sim 3\% \) all over the NR-covered parameter space, except for a single outlier, SXS:BBH:1374 with \( \max(\bar{F}) \sim 4.32\% \) and \( e_a^{\text{NR}} = 0.208 \). The difference comes both from the EOB side, due to the lack of higher-order corrections and because of the nonoptimal values of \( (a_0^c, c_3) \), and from the NR simulations, that for such values of \( e_a^{\text{NR}} \) are more noisy (see e.g. bottom panel of Fig. 3). Subdominant multipoles are rather robust in the nonspinning case, see e.g. Fig. 3. For large spins, we find the same problems related to the correct determination of NQC corrections found for \textsc{TEObiResumS\_SM} \cite{23}. Highly-accurate NR simulations covering a larger portion of the parameter space (see e.g. Ref. \cite{10}) are thus needed to robustly validate the model when \( e_a^{\text{NR}} \gtrsim 0.2 \).

Conclusions.— We illustrated that minimal modifications to \textsc{TEObiResumS\_SM} \cite{23} enabled us to build a (mildly) eccentric waveform model that is reasonably NR-faithful over a nonnegligible portion of the parameter space. This model could provide new eccentricity measurements on LIGO-Virgo events. Our approach can be applied, straightforwardly, also in the presence of tidal effects. The current analytical framework can be improved in several ways. First, \( (a_0^c, c_3) \) should be NR-recalibrated, so to construct a more faithful model (ideally with \( \max(F) \lesssim 5 \times 10^{-5} \)) for circularized binaries. This would translate to a reduction of \( \max(F) \) of Table \ref{tab:tab1}. In addition, higher-order corrections in the waveform and flux (see Refs. \cite{18,19,59}) should be incorporated, so to better handle higher eccentricities. Provided high-order, gravitational-self-force informed, resummed expressions for the EOB potentials \cite{60,63}, as well as analytically improved fluxes to enhance the analytical/numerical agreement of Fig. 1 for larger eccentricities, we believe that our approach would eventually also be useful for the efficient construction of EOB-based waveform templates for eccentric extreme-mass-ratio inspirals, as interesting sources for LISA \cite{63}.

We are grateful to T. Damour, G. Pratten and I. Romero-Shaw for useful comments.

\begin{thebibliography}{100}
\bibitem{1} B. P. Abbott \textit{et al.} (LIGO Scientific, Virgo), (2018), arXiv:1811.12907 [astro-ph.HE].
\bibitem{2} B. P. Abbott \textit{et al.} (Virgo, LIGO Scientific), Class. Quant. Grav. \textbf{34}, 104002 (2017) arXiv:1611.07531 [gr-qc].
\bibitem{3} A. H. Nitz, A. Lenon, and D. A. Brown, (2019), arXiv:1912.05464 [astro-ph.HE].
\bibitem{4} T. Hinderer and S. Babak, Phys. Rev. D\textbf{99}, 044028 (2019), arXiv:1901.07038 [gr-qc].
\bibitem{5} K. Belczynski, D. E. Holz, T. Bulik, and R. O’Shaughnessy, Nature \textbf{534}, 512 (2016), arXiv:1602.04531 [astro-ph.HE].
\bibitem{6} J. Samsing, Phys. Rev. D\textbf{97}, 103014 (2018), arXiv:1711.07452 [astro-ph.HE].
\bibitem{7} I. Hinder, L. E. Kidder, and H. P. Pfeiffer, (2017), arXiv:1709.02007 [gr-qc].
\bibitem{8} A. Ramos-Buades, S. Husa, G. Pratten, H. Estells, C. Garcia-Quirs, M. Mateu, M. Collonzi, and R. Jaume, (2019), arXiv:1909.11011 [gr-qc].
\bibitem{9} A. Buonanno, Y. Pan, A. Buonanno, E. Barausse, et al., Phys. Rev. D\textbf{93}, 084029 (2016), arXiv:1602.02444 [astro-ph.HE].
\bibitem{10} K. Belczynski, D. E. Holz, T. Bulik, and R. O’Shaughnessy, Nature \textbf{534}, 512 (2016), arXiv:1602.04531 [astro-ph.HE].
\bibitem{11} J. Samsing, Phys. Rev. D\textbf{97}, 064003 (2019) arXiv:1901.07038 [gr-qc].
\bibitem{12} A. Klein, Y. Boetzel, A. Gopakumar, P. Jetzer, and L. de Vittori, Phys. Rev. D\textbf{98}, 104043 (2018) arXiv:1801.08542 [gr-qc].
\bibitem{13} S. Tiwari, G. Achanvendi, M. Haney, and P. Hemantakumar, Phys. Rev. D\textbf{99}, 124008 (2019), arXiv:1905.07956 [gr-qc].
\bibitem{14} A. Buonanno and T. Damour, Phys. Rev. D\textbf{59}, 084006 (1999) arXiv:gr-qc/9811091.
\bibitem{15} A. Buonanno and T. Damour, Phys. Rev. D\textbf{62}, 064015 (2000) arXiv:gr-qc/0001013.
\bibitem{16} A. Buonanno, Y. Chen, and T. Damour, Phys. Rev. D\textbf{74}, 104005 (2006) arXiv:gr-qc/0508067.
\bibitem{17} T. Damour, P. Jaranowski, and G. Sch"{a}fer, Phys. Rev. D\textbf{91}, 084024 (2015) arXiv:1502.07245 [gr-qc].
\bibitem{18} Z. Cao and W.-B. Han, Phys. Rev. D\textbf{96}, 044028 (2017), arXiv:1708.00166 [gr-qc].
\bibitem{19} T. Hinderer and S. Babak, Phys. Rev. D\textbf{96}, 104048 (2017) arXiv:1707.08426 [gr-qc].
\bibitem{20} X. Liu, Z. Cao, and L. Shao, (2019), arXiv:1910.00784 [gr-qc].
\bibitem{21} A. Taracchini, Y. Pan, A. Buonanno, E. Barausse,
M. Boyle, et al., Phys. Rev. D86, 024011 (2012), arXiv:1202.0790 [gr-qc].
[22] A. Nagar et al., Phys. Rev. D98, 104052 (2018), arXiv:1806.01772 [gr-qc].
[23] A. Nagar, G. Riemen Schneider, G. Pratten, P. Rettega, and F. Messina, (2020), arXiv:2001.09082 [gr-qc].
[24] A. Nagar and A. Shah, Phys. Rev. D94, 104017 (2016), arXiv:1606.00207 [gr-qc].
[25] F. Messina, A. Maldarella, and A. Nagar, Phys. Rev. D97, 084016 (2018) arXiv:1801.02366 [gr-qc].
[26] D. Shoemaker, https://dec.ligo.org/cgi-bin/DocDB/ShowDocument?docid=2974.
[27] Modulo a single outlier at 0.7%.
[28] T. Chu, H. P. Pfeiffer, and M. A. Scheel, Phys. Rev. D80, 124051 (2009) arXiv:0909.1313 [gr-qc].
[29] G. Lovelace, M. Scheel, and B. Szilagyi, Phys. Rev. D83, 024010 (2011) arXiv:1010.2777 [gr-qc].
[30] G. Lovelace, M. Boyle, M. A. Scheel, and B. Szilagyi, Class. Quant. Grav. 29, 045003 (2012) arXiv:1110.2229 [gr-qc].
[31] L. T. Buchman, H. P. Pfeiffer, M. A. Scheel, and B. Szilagyi, Phys. Rev. D86, 084033 (2012) arXiv:1206.3015 [gr-qc].
[32] D. A. Hemberger, G. Lovelace, T. J. Loredo, L. E. Kidder, M. A. Scheel, B. Szilagyi, N. W. Taylor, and S. A. Teukolsky, Phys. Rev. D88, 064014 (2013) arXiv:1305.5991 [gr-qc].
[33] M. A. Scheel, M. Giesler, D. A. Hemberger, G. Lovelace, K. Kuper, M. Boyle, B. Szilagyi, and L. E. Kidder, Class. Quant. Grav. 32, 105009 (2015) arXiv:1412.1803 [gr-qc].
[34] J. Blackman, S. E. Field, C. R. Galley, B. Szilagyi, M. A. Scheel, M. Tiglio, and D. A. Hemberger, Phys. Rev. Lett. 115, 121102 (2015) arXiv:1502.07788 [gr-qc].
[35] G. Lovelace et al., Class. Quant. Grav. 32, 065007 (2015) arXiv:1411.7297 [gr-qc].
[36] A. H. Mroue, M. A. Scheel, B. Szilagyi, H. P. Pfeiffer, M. Boyle, et al., Phys. Rev. Lett. 111, 241104 (2013) arXiv:1304.6077 [gr-qc].
[37] P. Kumar, K. Barkett, S. Bhagwat, N. Afshari, D. A. Brown, G. Lovelace, M. A. Scheel, and B. Szilagyi, Phys. Rev. D92, 102001 (2015) arXiv:1507.00103 [gr-qc].
[38] T. Chu, H. Feng, P. Kumar, H. P. Pfeiffer, M. Boyle, D. A. Hemberger, L. E. Kidder, M. A. Scheel, and B. Szilagyi, Class. Quant. Grav. 33, 165001 (2016), arXiv:1512.06800 [gr-qc].
[39] M. Boyle et al., Class. Quant. Grav. 36, 195006 (2019) arXiv:1904.04383 [gr-qc].
[40] “SXS Gravitational Waveform Database,” https://data.black-holes.org/waveforms/index.html.
[41] A. Nagar, G. Pratten, G. Riemen Schneider, and R. Gamba, (2019), arXiv:1904.09550 [gr-qc].
[42] S. Ossokine et al., in preparation, 2019.
[43] A. Gopakumar, B. R. Iyer, and S. Iyer, Phys. Rev. D55, 6060 (1997) [Erratum-Phys. Rev.D57,6562(1998)], arXiv:gr-qc/9703075 [gr-qc].