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Corrigendum to “Achievable Multiplicity partitions in the Inverse Eigenvalue Problem of a graph” [Spec. Matrices 2019; 7:276-290.]

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Abstract: We correct an error in the original Lemma 3.4 in our paper “Achievable Multiplicity partitions in the IEVP of a graph” [Spec. Matrices 2019; 7:276-290.]. We have re-written Section 3 accordingly.

Keywords: inverse eigenvalue problem, multiplicity partition, adjacency matrix, minimum rank, distinct eigenvalues, graphs

MSC: 05C50, 15A18

Corrigendum – Revising Section 3 : Titled “Complete Multipartite Graphs”

It has been brought to the attention of the authors that as originally stated from [1, Lemma 3.4] is not correct. Attempts have been made to adjust the hypotheses of this result to render a workable statement, but we have not been successful. As such, we use this note to highlight this error and have proposed the following re-write of Section 3 of our work on complete multipartite graphs.

3 Complete Multipartite Graphs

As is standard, we use the notation \( K_{p_1, p_2, \ldots, p_\ell} \) for the complete \( \ell \)-partite graph, where \( \ell \) is a positive integer and \( p_1 \geq p_2 \geq \cdots \geq p_\ell \). The set of vertices is partitioned into \( \ell \) disjoint parts \( V_1 \cup V_2 \cup \cdots \cup V_\ell \); part \( V_i \) has \( p_i \) vertices for \( i \in \{1, \ldots, \ell\} \); no two vertices from a part are adjacent, while any two vertices from different parts are adjacent.

In this section we verify that the value of \( q(K_{p_1, p_2, \ldots, p_\ell}) \) is either 2 or 3 assuming the graph is nonempty. One case remains unresolved namely when \( p_1 \leq p_2 + \cdots + p_\ell \). We also provide an upper bound for \( MB(K_{p_1, p_2, \ldots, p_\ell}) \) in the case of \( q(K_{p_1, p_2, \ldots, p_\ell}) = 2 \). In fact, when \( \ell = 2 \) or \( p_1 = p_2 = \cdots = p_\ell \) we show that \( MB(K_{p_1, p_2, \ldots, p_\ell}) = p_1 \). The question of which other multiplicity partitions can be achieved by a complete multipartite graph remains open.

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In an unpublished manuscript (see [6]), it is shown that any complete multipartite graph $K_{p_1, p_2, \ldots, p_\ell}$ satisfies $q(K_{p_1, p_2, \ldots, p_\ell}) \leq 3$. The basic idea employed in the proof of this inequality is to note that the matrix $B = [b_{u,v}]$ with entries defined as

$$b_{u,v} = \begin{cases} 0 & \text{if } u, v \in V_i, \\
\frac{1}{\sqrt{p_i(p_j)}} & \text{if } u \in V_i, v \in V_j, \text{ and } i \neq j, \\
\end{cases}$$

satisfies $B \in S(K_{p_1, p_2, \ldots, p_\ell})$ and $q(B) = 3$. Furthermore, it can be easily verified that the eigenvalues of $B$ are $\{-1, 0, \ell - 1\}$ with multiplicities $\ell - 1, 1, 1$, respectively. Thus if follows that $K_{p_1, p_2, \ldots, p_\ell} \in MP([- \sum_{i=1}^{\ell} (p_i - 1), \ell - 1, 1])$.

There are several known results for the partitions that are achievable along with the corresponding values of $q$ for complete multipartite graphs; we list them in the following lemma.

**Lemma 3.1.** For positive integers $\ell, p_1, q_i$ with $i = 1, 2, \ldots, \ell$

1. $q(K_{p_1, p_2, \ldots, p_\ell}) \in \{2, 3\}$.
2. $q(K_{p_1, q_1}) = \begin{cases} 2 & \text{if } p_1 = q_1, \\
3 & \text{otherwise}. \\
\end{cases}$
3. If $p_1 + p_2 + \cdots + p_\ell = q_1 + q_2 + \cdots + q_{\ell'}$ for $\ell, \ell' \geq 2$, then
$$q(K_{p_1, p_2, \ldots, p_\ell, q_1, q_2, \ldots, q_{\ell'}}) = 2.$$  
4. If $p_2 + \cdots + p_\ell < p_1$, then $q(K_{p_1, p_2, \ldots, p_\ell}) = 3$.

**Proof.** The first statement is from [3]. The second statement follows from [3, Cor. 6.5].

To observe that the third statement holds, note that $p_1 + p_2 + \cdots + p_\ell = q_1 + q_2 + \cdots + q_{\ell'}$ for $\ell, \ell' \geq 2$ implies that $K_{p_1, p_2, \ldots, p_\ell, q_1, q_2, \ldots, q_{\ell'}}$ is isomorphic to the join of $K_{p_1, p_2, \ldots, p_\ell}$ and $K_{q_1, q_2, \ldots, q_{\ell'}}$, and hence satisfies $q(K_{p_1, p_2, \ldots, p_\ell, q_1, q_2, \ldots, q_{\ell'}}) = 2$ by Theorem 2.1.

To see that the last statement holds, assume that $p_2 + \cdots + p_\ell < p_1$, and set $n = p_1 + p_2 + \cdots + p_\ell$; this implies $n - p_1 < p_1$. If $q(K_{p_1, p_2, \ldots, p_\ell}) = 2$, then $K_{p_1, p_2, \ldots, p_\ell} \in MP([n - k, k])$ with $p_1 \leq k \leq n - k$ (by Statement (3) of Lemma 2.3). Hence $p_1 \leq k \leq n - k \leq n - p_1$, implying $2p_1 \leq n$, which is a contradiction. Hence $q(K_{p_1, p_2, \ldots, p_\ell}) \geq 3$. Finally, the equality follows from the work in the unpublished manuscript [6].

Continuing with the complete multipartite graph, the only case that remains unresolved is when $p_1 \leq p_2 + \cdots + p_\ell$. We suspect that in this case $q(K_{p_1, p_2, \ldots, p_\ell}) = 2$.

To address the specific case when all $p_1 = k$, we begin by stating a technical result that is a special case of [7, Lemma 10] (in the notation of [7], we are setting $q = 0$ and $p = k \geq 2$).

**Lemma 3.2.** Let $k \geq 2$, and $M_1$ and $M_2$ be matrices that have $k$ rows and no zero columns. Then there exists a $k \times k$ matrix $R$ such that $R^T R = I_k$ and $M_1^T R M_2$ has no zero entries.

**Lemma 3.3.** For any graph $G$ with no isolated vertices and for any number $d$ such that $mr_+(G) \leq d \leq \lfloor V(G) \rfloor$, there exist vectors $q_1, \ldots, q_d$ such that $\sum_{i=1}^{d} q_i q_i^T \in \mathcal{S}(G)$ and each $q_i, i = 1, \ldots, d$, is entry-wise nonzero.

**Proof.** Since for the given parameter $d$ we have $mr_+(G) \leq d \leq \lfloor V(G) \rfloor$, it follows that there is an $A \in \mathcal{S}(G)$—the set of positive semidefinite matrices in $\mathcal{S}(G)$—with $\text{rank}(A) = d$. Since $G$ has no isolated vertices, $A$ has no zero rows or columns. Since $A$ is positive semidefinite, $A$ can be written as $A = U U^T$, for some $n \times d$ matrix $U$. If $U = [u_1, u_2, \ldots, u_d]$, where $u_1, u_2, \ldots, u_d$ are the columns of $U$, then we may assume that $u_1, u_2, \ldots, u_d$ are mutually orthogonal vectors in $\mathbb{R}^n$.

Set $M_1 = U^T$, and $M_2 = I_d$; each of these matrices have $d$ rows and do not have any zero columns (this follows since $A$ has no zero rows). Hence, by Lemma 3.2, there exists an orthogonal $d \times d$ matrix $R$ such that
$M_1^T R M_2 = UR$ has no zero entries. Let $Q = UR = [q_1, q_2, \ldots, q_d]$ then

$$QQ^T = URR^T U^T = UU^T = A \in S(G),$$

as needed.

The next result is a technical result concerning a bound on the minimum semidefinite rank of joins of graphs. This result is needed to study the case when all $p_i = k$ for determining the minimum number of distinct eigenvalues of the complete multipartite graph and establish a bound on the corresponding minimal multiplicity bipartition. In this proof $1_s$ is used to denote the vector in $\mathbb{R}^s$ with all entries equal to one. Similarly, $0_s$ is used to denote the vector in $\mathbb{R}^s$ with all entries equal to zero and $O$ is the all zeros matrix, the size will be clear from context.

**Lemma 3.4.** Consider the graph $G$ with no isolated vertices and positive integers $d, s_1, s_2, \ldots, s_d$. Assume $0 < \mr_r(G) \leq d \leq |V(G)|$. Suppose there exists $d$ mutually orthogonal vectors $z_1, z_2, \ldots, z_d$ such that if $Z = \{z_1, z_2, \ldots, z_d\}, then ZZ^T \in S(G)$ and $q(ZZ^T) = 2$. Then $q(G \vee (K_{s_1} \cup K_{s_2} \cup \cdots \cup K_{s_d})) = 2$; moreover, $MB(G \vee (K_{s_1} \cup K_{s_2} \cup \cdots \cup K_{s_d})) = d$.

**Proof.** Let $H = G \vee (K_{s_1} \cup K_{s_2} \cup \cdots \cup K_{s_d}), and s = \sum_{i=1}^d s_i$. We construct a matrix $B \in S(H)$ with $\sigma(B) = \{0, \beta\}$, where the eigenvalue $0$ has multiplicity $|V(H)| - d$ and the eigenvalue $\beta$ has multiplicity $d$. Up to translation, we may assume that the two distinct eigenvalues of $ZZ^T$ are $0$ and $\lambda$. Applying Lemma 3.3, we may replace $Z$ by $ZR$ (for some orthogonal matrix $R$) so that each column of $ZR$ is an entry-wise nonzero vector. If we set $Q = ZR$, then $QQ^T \in S(G)$, and $Q^T Q = R^T (Z^T Z) R = R^T (\lambda I_d) R = \lambda I_d$. Hence the columns of $Q$ are mutually orthogonal. Suppose the columns of $Q$ are the entry-wise nonzero vectors $q_1, \ldots, q_d$. Construct vectors $v_1, \ldots, v_d$ as follows

$$v_1 = \begin{bmatrix} \alpha_1 1_{s_1} \\ \alpha_2 1_{s_2} \\ \vdots \\ \alpha_d 1_{s_d} \end{bmatrix}, \quad v_2 = \begin{bmatrix} q_2 \\ \underline{0}_{s_2} \\ \underline{0}_{s_2-s_1} \end{bmatrix}, \quad \ldots \quad v_d = \begin{bmatrix} q_d \\ \underline{0}_{s_d} \\ \underline{0}_{s_d-s_1} \end{bmatrix}$$

where $\alpha_i$ will be determined. Let $q_i^T = [q_{1,i}, q_{2,i}, \ldots, q_{n,i}]$ for $i = 1, \ldots, d$ where $|V(G)| = n$.

For $j = 2, \ldots, d + 1$, let $B_{j-1} = [q_{j-1}^T, q_{j-1}^T, \ldots, q_{j-1}^T]$ be the $n \times s_{j-1}$ matrix with all columns equal to $q_{j-1}$. The matrix $B = \sum_{i=1}^d v_i v_i^T$ has the following form:

$$B = \begin{bmatrix} A & \alpha_1 B_{1,2} & \ldots & \alpha_d B_{1,d+1} \\ \alpha_1 B_{1,2}^T & \alpha_1^2 I_{s_1 \times s_1} & 0 & \ldots & 0 \\ 0 & \alpha_2^2 I_{s_2 \times s_2} & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \ldots & 0 & \alpha_d^2 I_{s_d \times s_d} \end{bmatrix},$$

Now, for a positive number $\beta > \max \{|q_i|, i = 1, \ldots, d\}$, set $\alpha_i = \sqrt{B - \|q_i\|^2 s_i}$. Then $B \in S(H)$ and rank($B$) = $d$, which implies the eigenvalue zero has multiplicity $|V(H)| - d$. On the other hand, $Bv_i = \beta v_i$ for each $i = 1, 2, \ldots, d$, and the vectors $v_1, \ldots, v_d$ are linearly independent. Thus $\sigma(B) = \{0, \beta\}$ with the desired multiplicities, and since $H$ has at least $d$ independent vertices, $d \leq MB(H)$.

Note that, in Lemma 3.4, if $G$ has $\ell$ isolated vertices, then these vertices form an independent set. By Lemma 2.4, in order for $q(G) = 2$, the union of the mutual common neighbours of an independent set cannot have more than $\ell$ elements; therefore, $\ell \leq \sum_{i=1}^d s_i$. Moreover, if $d = s_1 = 1$ and $G$ has isolated vertices, then
there is a unique path from a vertex of $G$ to an isolated vertex of $G$ using the vertex of $K_{1,1}$, which implies the graph cannot have only two distinct eigenvalues. It is unclear if the statement of Lemma 3.4 holds in other cases when $G$ has $\ell$ isolated vertices.

A specific case of Lemma 3.4 is when all the parts of a complete multipartite graph have the same size.

**Corollary 3.5.** Let $H = K_{k,k,...,k}$, $k \geq 2$ be the complete multipartite graph on $n$ vertices. Then $q(H) = 2$ and $MB(H) = k$.

**Proof.** Using Lemma 3.1, we know that $q(K_{k,k}) = 2$ and it is not difficult to determine that $MB(K_{k,k}) = k$ (since the minimum semidefinite rank of $K_{k,k}$ is equal to $k$). The graph $K_{k,k}$ represents the base case for an induction argument based on the number of parts. Assume for the complete multipartite $G = K_{k,k,...,k}$ on $l-1$ ($l \geq 3$) parts that $q(G) = 2$ with $MB(G) = k$. Let $H = K_{k,k,...,k}$ be the complete multipartite graph with $l$ parts. Then $H = G \lor (K_1 \cup K_1 \cup \cdots \cup K_1)$, where $G = K_{k,k,...,k}$ with one less part than $H$. The next step involves an application of Lemma 3.4. If we $d = k$ and using the fact that $q(G) = 2$ with $MB(G) = k$, then there exists a matrix $A$ in $S(G)$ such that $A = ZZ^T$, where $Z$ is an $n \times d$ matrix and with $Z^T Z = AI_d$. From Lemma 3.4 it follows that $q(H) = 2$.

For a vertex $v$ in a graph $G$, a new graph $G'$ can be constructed by cloning (or duplicating) $v$. The graph $G'$ has vertex set $V(G') = V(G) \cup \{v'\}$ and edge set $E(G') = E(G) \cup \{(v,u) \in N[v]\}$, where $N[v]$ is the closed neighbourhood of $v$ (that is, a neighbourhood of $v$ containing $v$). It turns out that cloning a vertex of a graph $G$ with $MB(G) = k$ results in a graph $G'$ with $MB(G') \leq k$. The following proposition is proved in Theorem 6.3 of [13], it is also implied by Corollary 4 of [2]. In [13], this is used to characterize graphs $G$ with $MB(G) = 2$ by constructing minimal such graphs (these are $K_1$, $K_1 \cup K_1$, $K_2, K_2, \ldots$, and $K_2, K_2, \ldots, K_2$) and constructing all the other such graphs by cloning vertices in the minimal graphs.

**Proposition 3.6.** Let $G$ be a graph with $q(G) = 2$. If $H$ is obtained from $G$ by cloning a vertex in $G$, then $H \in MP([n - k, k])$.

Suppose $G$ is a graph with $q(G) = 2$. If $H$ is obtained from $G$ by cloning a vertex, then $MB(H) \leq MB(G)$. It is not clear if this inequality is ever strict. By cloning vertices in $K_{k,k,...,k}$, where $k \geq 2$, we have the following consequence, reminiscent of Lemma 2.3.

**Corollary 3.7.** If $G = \bigvee_{i=1}^l (\cup_{j=1}^k A_{i,j})$ where $k \geq 2$, $i = 1, 2, \ldots, l$, $j = 1, 2, \ldots, k$, and $a_{i,j}$ are positive integers, then $q(G) = 2$ and $MB(G) = k$.

Unlike the case where $k = 2$, for general $k$ the previous corollary does not characterize all the graphs in $MP([n - k, k])$. In Section 4, several graphs in $MP([n - k, k])$ that are not included in Corollary 3.7 are given.

**Theorem 3.8.** Let $G$ and $H$ be two graphs with no isolated vertices. Further assume that $q(G) = q(H) = 2$ with $MB(G) = MB(H)$. Then $q(G \lor H) = 2$ and $MB(G \lor H) \leq MB(G) (= MB(H))$.

**Proof.** Assume $MB(G) = MB(H) = k$. If $k = 1$, there is nothing to prove, so assume $k \geq 2$. Let $n_1$ be the number of vertices in $G$ and $n_2$ the number of vertices in $H$. Let $A \in S(G)$ be such that $\sigma(A) = \{1^{(n_1-k)}, 1^{(k)}\}$ and let $B \in S(H)$ be such that $\sigma(B) = \{1^{(n_2-k)}, 1^{(k)}\}$. By Schur’s Theorem, there exists orthogonal matrices $Q_1$ and $Q_2$ such that

$$A = Q_1^T(I_k \otimes O_{n_1-k})Q_1 \quad \text{and} \quad B = Q_2^T(I_k \otimes O_{n_2-k})Q_2.$$

Let

$$M_1 = Q_1[1, \ldots, k | 1, \ldots, n_1] \quad \text{and} \quad M_2 = Q_2[1, \ldots, k | 1, \ldots, n_2],$$

so $A = M_1^TM_1$ and $B = M_2^TM_2$. This also implies that $M_1$ has no zero columns, since otherwise $A$ would have a row and column of zeros which would imply that $G$ would have an isolated vertex. Similarly, $M_2$ has no zero columns.
By Lemma 3.2, there exists a $k \times k$ matrix $R$ such that $R^T R = I_k$ and $M^T_1 R M_2$ has no zero entries. Define $C$ as follows:

$$C = \begin{bmatrix} M^T_1 \\ M^T_2 R^T \end{bmatrix} \begin{bmatrix} M_1 & R M_2 \end{bmatrix} = \begin{bmatrix} M^T_1 M_1 & M^T_1 R M_2 \\ M^T_2 R^T M_1 & M^T_2 R^T R M_2 \end{bmatrix} = \begin{bmatrix} A & M^T_1 R M_2 \\ M^T_2 R^T M_1 & B \end{bmatrix}.$$

Hence $C$ is positive semidefinite and $C \in S(G \lor H)$ since $M^T_1 R M_2$ is an entry-wise nonzero matrix. It is easy to note that $C$ has rank $k$ since $[M_1 \ R M_2]$ has a full-row rank. Therefore, $null(C) = n_1 + n_2 - k$. Moreover, $C^2 = 2C$ which implies $\sigma(C) = \{0^{(n_1 + n_2 - k)}, 2^{(k)}\}$. Hence $q(G \lor H) = 2$ since $C \in S(G \lor H)$ and $q(G \lor H) > 1$. □

In Theorem 3.8, where $k = MB(G) = MB(H)$, if $k = 1$, then the inequality $MB(G \lor H) \leq 1$ cannot be strict, because $MB(G \lor H) \geq 1$ for all graphs where it is defined. Similarly, if $k = 2$, then the inequality $MB(G \lor H) \leq 2$ cannot be strict, because $MB(G \lor H) = 1$ implies that $G \lor H$ is a complete graph with possibly isolated vertices, which is a contradiction. Moreover, if $k = 3$, then the inequality $MB(G \lor H) \leq 3$ cannot be strict, because $MB(G \lor H) = 1, 2$ implies that $G \lor H$ is either a complete graph with possibly isolated vertices, or a graph characterized in Lemma 2.3 (2), either case is a contradiction.

For cases of $k \geq 4$ in Theorem 3.8, it is still unclear if a strict inequality can hold in $MB(G \lor H) \leq k$. We suspect, that in fact, equality among $MB(G \lor H) = MB(G) = MB(H)$ holds under the hypothesis of Theorem 3.8. A related matter is to determine if a version of Theorem 3.8 still holds in the case when $MB(G) \neq MB(H)$. It turns out that the requirement of $MB(G) = MB(H)$ is essential in concluding that $q(G \lor H) = 2$ as in Theorem 3.8. Consider the following example. Let $G = Q_6$ (the 6-dimensional hypercube), and let $S = \{x, y, z\}$ be the independent set of vertices in $G$ consisting of $x = (000000), y = (010101)$, and $z = (111111)$. Also observe that there are no common neighbours among any pair of vertices from $S$ in $G$. Let $H = P_2$. Then we have $q(G) = q(H) = 2$, and $MB(G) \geq 3$ and $MB(H) = 1$. However, in the graph $G \lor H$, using the independent set $S$, it is easy to deduce that the condition of Lemma 2.4 fails. Hence $q(G \lor H) > 2$.

In fact, this idea can be easily generalized as follows: Suppose $G$ is a graph with $q(G) = 2$ that contains an independent set of vertices $S = \{v_1, v_2, \ldots, v_k\}$ in which $\bigcup_{v_i, v_j \in S} (N(v_i) \cap N(v_j)) = \emptyset$. Then for any graph $H$ with $q(H) = 2$ and $|H| < k$, we have $q(G \lor H) > 2$. To see this, it is enough to observe that in the graph $G \lor H$ we have

$$|\bigcup_{v_i, v_j \in S} (N(v_i) \cap N(v_j))| = |H| < |S|,$$

and hence the condition of Lemma 2.4 fails to hold.

We also note that the assumption of no isolated vertices in Theorem 3.8 is possibly a stronger condition than is in fact necessary; this assumption is used to ensure that the matrix $M_2$ in the proof has no zero columns. For instance, in the next result, which is a consequence of Lemma 3.4, all the vertices of the second graph are isolated vertices. The proof of Lemma 3.9 is the same as the proof of Theorem 3.8, except that the matrix $B$ is replaced with the identity matrix. We denote the graph on $k$ vertices with no edges by $K_k$.

**Lemma 3.9.** Let $G$ be a graph with no isolated vertices and $q(G) = 2$ with $G \in MP([n-k, k])$ for some $k \geq 2$. Then $q(G \lor K_k) = 2$ with $MB(G \lor K_k) \leq k$.

Note that the multiplicity bipartition $[n-k, k]$ for regular complete multipartite graphs $K_{k,k,\ldots,k}$ can also be obtained from the proof of Theorem 3.8 and induction.

It is also interesting to note that the minimum number of distinct eigenvalues of the join of two graphs can be large.

**Lemma 3.10.** For any graph $G$, $q(G \lor K_1) \geq \lceil \frac{q(G)+1}{2} \rceil$.

*Proof.* The eigenvalues for any matrix in $S(G)$ interlace the eigenvalues any matrix $S(G \lor K_1)$.

The next theorem is the main result of [15].
**Theorem 3.11** (Theorem 4.3 [15]). *Let G be a connected graph on n vertices and let \( \lambda_1, \lambda_2, \ldots, \lambda_n \) be distinct real numbers. Then there exists a real symmetric matrix \( A \in S(G) \) with eigenvalues \( \lambda_1, \lambda_2, \ldots, \lambda_n \) such that none of the eigenvectors of \( A \) has a zero entry.*

**Lemma 3.12.** *Let G be a connected graph on \( n \geq 2 \) vertices. Then \( q(G \vee \bar{K}_n) = 2 \) and \( MB(G \vee \bar{K}_n) = n \).*

**Proof.** Since \( G \) is a connected graph, by Theorem 3.11 there exists a matrix \( A \in S(G) \) with positive distinct eigenvalues

\[
\lambda_1 > \lambda_2 > \cdots > \lambda_n
\]

and corresponding entry-wise nonzero unit eigenvectors \( v_1, \ldots, v_n \) such that

\[
A = V^TAV = U^TU,
\]

where \( U = A^{\frac{1}{2}}V, \Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n), \) and \( V^T = [v_1, \ldots, v_n]. \) The rows of \( U \) are orthogonal since \( V \) is unitary. Let \( C \) be the \( n \times 2n \) matrix \( C = \begin{bmatrix} D & U \end{bmatrix} \), where

\[
D = \begin{bmatrix}
a_1 & 0 & \cdots & 0 \\
0 & a_2 & \cdots & 0 \\
& \vdots & \ddots & \vdots \\
0 & 0 & \cdots & a_n
\end{bmatrix},
\]

where the scalars \( a_i (i = 1, 2, \ldots, n) \) are to be determined. Since \( U \) is an entry-wise nonzero matrix, if each \( a_i \) is also nonzero, then

\[
C^T C = \begin{bmatrix} D^2 & DU \\ U^TD & U^TU \end{bmatrix} \in S(G \vee \bar{K}_n).
\]

Further, the rows of \( C \) are orthogonal and so

\[
CC^T = \begin{bmatrix}
a_1 & 0 & \cdots & 0 \\
0 & a_2 & \cdots & 0 \\
& \vdots & \ddots & \vdots \\
0 & 0 & \cdots & a_n
\end{bmatrix},
\]

where \( a_i = a_i^2 + \lambda_i^2, i = 1, \ldots, n. \) Therefore, the eigenvalues of \( CC^T \) are \( a_i, i = 1, \ldots, n. \) The values \( a_i \) can be set so that they are all strictly positive, and \( a_i \) for all \( i = 1, \ldots, n \) are equal to some \( \lambda_0 > \lambda_1^2. \) Then the spectrum of \( CC^T \) is 0 with multiplicity \( n, \) and \( \lambda_0 \) also with multiplicity \( n. \) This implies that \( q(G \vee \bar{K}_n) = 2 \) and \( MB(G \vee \bar{K}_n) \leq n. \) Finally, the vertices in \( \bar{K}_n \) form an independent set of size \( n, \) and so by Statement (3) of Lemma 2.3, it follows that \( MB(G \vee \bar{K}_n) = n. \)

The same proof can be used to prove the following result.

**Lemma 3.13.** *Let G be a connected graph on \( n \geq 2 \) vertices. Then \( q(G \vee \bar{K}_{n-1}) = 2 \) and \( MB(G \vee \bar{K}_{n-1}) = n - 1. \)*

**Proof.** As in the proof of the previous lemma, there exists a matrix \( A \in S(G) \) with eigenvalues

\[
\lambda_1 > \lambda_2 > \cdots > \lambda_{n-1} > \lambda_n = 0
\]

and corresponding entry-wise nonzero unit eigenvectors \( v_1, \ldots, v_n \) such that

\[
A = V^TAV = U^TU,
\]

where \( V^T = [v_1, \ldots, v_n], \Lambda = \text{diag}(\lambda_1, \ldots, \lambda_{n-1}, 0), U = DV[1, \ldots, n - 1 \mid 1, \ldots, n], \) and \( D = \text{diag}(\sqrt{\lambda_1}, \ldots, \sqrt{\lambda_{n-1}}). \) Hence the rows of \( U \) are orthogonal since \( V \) is unitary. Let \( C \) be the \((n - 1) \times (2n - 1)\) matrix \( C = \begin{bmatrix} D & U \end{bmatrix}. \) Since \( U \) is an entry-wise nonzero matrix, as long as each \( a_i \) is also nonzero, the matrix

\[
C^T C = \begin{bmatrix} D^2 & DU \\ U^TD & U^TU \end{bmatrix} \in S(G \vee \bar{K}_{n-1}).
\]
Further, the rows of $C$ are orthogonal and so

$$CC^T = \begin{bmatrix}
a_1 & 0 & \ldots & 0 \\
0 & a_2 & \ldots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & 0 & \ldots & a_{n-1}
\end{bmatrix},$$

where $a_i = a_i^2 + \lambda_i^2$, $i = 1, \ldots, n-1$. Similar to the proof of the previous lemma, the spectrum of $C^T C$ is 0 with multiplicity $n$, and $\lambda_0$ with multiplicity $n - 1$. This implies that $q(G \vee K_n) = 2$ and $MB(G \vee K_{n-1}) = n - 1$. □

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