Monte Carlo Simulations of the SU(2) Vacuum Structure

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Lattice Monte Carlo simulations are performed for the SU(2) Yang Mills gauge theory in the presence of an Abelian background with external sources to obtain information on the effective potential. The goal is to investigate the lowest Landau mode that, in the continuum one-loop effective potential, is the crucial mode for instability. It is shown that also in the lattice formulation this lowest Landau mode plays a very peculiar role, and it is important for the understanding of the vacuum properties.

1. INTRODUCTION

To understand the IR behavior of non-Abelian gauge theory a non-perturbative framework is necessary. Therefore, lattice formulation is particularly useful. However, the comparison between one-loop perturbative expansion and lattice regularization might give important information. From this comparison we can learn about the trustworthiness of perturbative expansion. Furthermore, the loop expansion can provide indicative information for the lattice quantities.

Despite the importance of Yang-Mills theories, the complete solution of non-Abelian gauge theories has yet to be found. In order to gain a better understanding of these theories, a necessary first step is the study of their vacuum structures. Nevertheless, the vacuum structure of such theories has yet to be understood even in the simplest case, i.e. SU(2) without matter. Moreover, the infrared properties of such theories must be studied systematically if we want to have some clue on confinement.

Many authors have studied one-loop effective potential for SU(2) and the possible consequences \cite{1,2}. In addition, several Monte Carlo simulations have already been done with a wide range of techniques in 3 and 4 Dimensions \cite{2,3}.

2. ONE-LOOP APPROACH

A powerful method to investigate the properties of Yang-Mills theories is to compute of the effective potential in the background gauge. This method has been discussed extensively in the literature \cite{4}. The technique is to split the gauge field into a background $A^b_\mu$ and a quantized field $\eta^b_\mu$, as

$$A^b_\mu = A^b_\mu + \eta^b_\mu ,$$

and subsequently performing a loop expansion. This manifestly gauge invariant scheme is based on the observation that the loop expansion corresponds to an expansion in the parameter $\bar{h}$ which multiplies the entire action. Hence, a shift of the fields or a redefinition of the division of the Lagrangian into free and interacting parts can be performed at any finite order of the loop expansion without violating the gauge invariance. Nevertheless, there are several subtleties on the exponentiation of the gauge constraint and in the ghosts contributions to the finite part of the effective action \cite{5}.

We want to investigate the situation where $A$ generates a static constant chromomagnetic field. A possible choice for $A^b_\mu$ is the so called Abelian background:

$$A^b_\mu = \frac{1}{2} H \delta^b \delta^3(x \delta_{\mu 2} - y \delta_{\mu 1}) .$$

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With this choice, the well known Savvidy result for the one-loop effective potential is obtained

\[ E(H) = \frac{H^2}{2} + \frac{11g^2H^2}{48\pi^2} \left( \ln \frac{gH}{\mu^2} - \frac{1}{2} - \frac{6\pi}{11} \right) + \ldots \]  

The remarkable feature of this expression is that it exhibits a minimum for \( H \) different from zero. However, as Nielsen and Olesen realized, due to the imaginary part of the effective potential, this minimum has unstable modes. Since the above mentioned preliminary studies were done, the property of this non-trivial vacuum has been thoroughly investigated. In particular a scenario, the so-called “Copenhagen vacuum” was proposed.

However, in a strong field configuration a perturbative analysis is unreliable, and unstable configurations can only be analyzed by non-perturbative methods. Therefore, the only possible technique presently available to tackle this problem is the one based on lattice regularization.

In this context it is interesting to note that the one-loop effective potential can be obtained using

\[ V(H) = V_{\text{classic}} + \text{const} \sum \sqrt{\nu} \]  

where \( \nu \) are the eigenvalues of the second derivative of the action with respect to the gauge fields, and the summation is over all the eigenvalues. This expression is a natural consequence of the saddle point approximation:

\[ V(H) = V_{\text{classic}} - \frac{\hbar}{2\Omega} \log \det \left( \delta^2 S \right) + O(\hbar^2) \]  

where \( \Omega \) is the volume factor. The eigenvalue of this problem can be found by realizing that there is the same symmetry as the one of the Landau levels problem, therefore we have exactly the same class of solutions. That yield to the eigenvalues

\[ \nu = k^2 + (2n + 1 + 2S_z)gH \]  

where \( k = k_0^2 + k_z^2 \), \( n = 0, 1, 2, \ldots \) and \( S_z = \pm 1 \) is the gluon polarization.

For the lowest landau mode, i.e. \( n = 0, S_z = -1 \), we have \( \nu < 0 \) that give the imaginary part of the effective potential. Note that with Abelian background it is possible to obtain all eigenvalues positive only by insertion of non-gauge invariant terms in the action.

3. LATTICE RESULTS

Until now we have argued that this lowest Landau mode plays a very particular role on the one loop perturbative expansion. This motivated us to understand what happened to this mode on lattice where the task is non-perturbative.

To extract information on this mode on lattice we noted that, due to the finiteness of the lattice \( k_z \), is quantized as well, with \( k_z = 2m\pi/L_z \), where \( m \) is an integer. Therefore, the lowest inhomogeneous \( z \)-mode, \( m = 1 \), becomes stable for lattices the extent of which in the \( z \)-direction is smaller than

\[ L_z^{\text{critic}} = \frac{2\pi}{\sqrt{gH}}. \]  

This enables us to search for the critical size \( L_z^{\text{critic}} \). The homogeneous mode, \( m = 0 \), which is always unstable, is eliminated by imposing a delta condition in the path integral.

We perform Monte Carlo simulations that generate a background field \( H \) in the \( z \) direction in the presence of an Abelian source of strength \( j \).

We use a heat bath updating procedure with periodic boundary conditions and for the computational technique we address to reference [3]. To eliminate the homogeneous mode \( m = 0 \) we force the Polyakov line in the \( z \) direction to take a fixed value different from zero. Monte Carlo simulations are made with a lattice volume \( L^3 \times L_z \), where \( L \) is the size of the \( x, y, t \) directions. The finite effect due to \( L \) is not so crucial. In fact, for the observable that we are interested, it is sufficient to use \( L = 12 - 16 \). Simulations are made by varying \( L_z \) from 4 to 50.

The expectation value of the plaquette in the 1-2 plane \( P[F_{12}] \) as a function of \( \beta \) and \( j \) is measured. We monitor the quantity

\[ X = \frac{P[F_{12}(\beta, 0)] - P[F_{12}(\beta, j)]}{jP[F_{12}(\beta, 0)]}, \]  

which is proportional to the main contribution of the vacuum energy of the plaquette in the \( z \) direction [3], and is very sensible to the presence of the unstable mode. The total vacuum energy obviously has also contributions from the other oriented plaquette; however, in the region we are
interested, these other contributions are smoothly variable functions, and thus they will not affect the critical behavior.

We evaluated $X$ for different values of $\beta$ and $j$ performing, for large $j$ 4500 sweeps after discharging 500 for thermalization. For smaller $j$ we increase the number of sweeps until a significant amount of data was collected.

Our data show that there is no sign of the unstable mode away from the critical $\beta$ region ($\beta = 2.1 - 2.5$). The situation changes dramatically in the critical region where the instability appears as a decrease of the vacuum energy contribution to the plaquette in the $z$ direction. This effect becomes more evident in the presence of strong sources. The fact that the instability disappears outside of the scaling window is a strong evidence that the instability is a distinguishing feature of the continuum rather than the strong coupling vacuum. In order to study the IR property of the continuum theory we should remain in the region where the instability is manifest. The disappearance of the instability as $\beta \to 0$ may give a clue to understanding the difference between the physics of the continuum and the strong coupling lattice theory.

We systematically analyzed the critical region of $\beta$ for several values of $j$. The values of $L_{\text{crit}}(\beta, j)$ are obtained by interpolating the kink of $X$ and taking the median value. $L_{\text{crit}}$ was found to be dependent on the source $j$ and on $\beta$ in this region according to the renormalization group dependence. It is clear from our data that $L_{\text{crit}}$ belongs to the confinement phase and that there is good agreement with the renormalization group equation. Our data follow the same dependence of the deconfinement transition as should be for a dimensional scale length. Hence the ratio between the $L_{\text{crit}}$ and the deconfinement scale parameter $L_{\text{dec}}$ is independent of $\beta$. The relevant physical quantities must be obtained in the limit of vanishing of the induced field $Q(j) \to 0$. From our data we obtain

$$L_{\text{crit}} / L_{\text{dec}} = 2.5 \pm 0.2.$$  \hspace{1cm} (9)

Using the known value of $L_{\text{dec}}$ we can establish that the lattice system should be greater than $L_{\text{z}} \sim 18$ in order to reflect the richness of the non-perturbative vacuum of the continuum theory.

In addition to this new scale there is a sequence of $m$ modes. In fact for each $m$ we have:

$$\frac{2\pi m}{L_{\text{crit}}(m)} = \sqrt{gH} \Rightarrow L_{\text{z}(m)} \sim \frac{2.5}{T_{\text{dec}}} m . \hspace{1cm} (10)$$

The simulations show that these modes follow with good approximation eq. (10). This is strong evidence for the harmonicity of this modes. This is a quite surprising result because we are in a region of strong non-perturbative effects. It is stimulating to think in terms of a parallel between the situation described above and the integer Quantum Hall Effect. This similarity is based on the dual picture ($z \leftrightarrow t$, $E \leftrightarrow H$, etc...), and on the plateau structures for each mode.

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