A review of the relationships between matrix models and noncommutative gauge theory is presented. A lattice version of noncommutative Yang-Mills theory is constructed and used to examine some generic properties of noncommutative quantum field theory, such as UV/IR mixing and the appearance of gauge-invariant open Wilson line operators. Morita equivalence in this class of models is derived and used to establish the generic relation between noncommutative gauge theory and twisted reduced models. Finite dimensional representations of the quotient conditions for toroidal compactification of matrix models are thereby exhibited. The coupling of noncommutative gauge fields to fundamental matter fields is considered and a large mass expansion is used to study properties of gauge-invariant observables. Morita equivalence with fundamental matter is also presented and used to prove the equivalence between the planar loop renormalizations in commutative and noncommutative quantum chromodynamics.

1. Matrix Models and Noncommutative Gauge Theory

In this article we will discuss the intimate relationship that exists between Yang-Mills theory on a noncommutative space and large $N$ matrix models which are conjectured to provide nonperturbative definitions of string theory and M-Theory. This paper is based on the articles [1]–[4]. A related review can be found in [1]. We will begin by recalling how noncommutative gauge theory first appeared within the context of nonperturbative string theory.

The IKKT Matrix Model

The IKKT matrix model [3] is defined as the dimensional reduction to a point of ten-dimensional maximally supersymmetric Yang-Mills theory. The action is

$$S_{IKKT} = -\frac{1}{g^2} \text{Tr} \left( \frac{1}{4} [X^i, X^j]^2 + \frac{1}{2} \bar{\Psi} \gamma_i [X^i, \Psi] \right),$$

(1.1)

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where $X^i, i = 1, \ldots, 10$, are $N \times N$ Hermitian matrices, whose eigenvalues represent the coordinates of a D-instanton in ten dimensional spacetime, and $\Psi$ are $N \times N$ Hermitian matrices which are Majorana-Weyl spinors in ten dimensions. The $X^i$‘s are the reductions of the ten dimensional gauge fields and $\Psi$ the reductions of their superpartners. In the double scaling limit $N \to \infty, g^2 \to 0$ with the product $Ng^2$ finite, the model (1.1) is conjectured to provide a nonperturbative definition of Type IIB superstring theory \cite{6}. Precisely, it is related to the Green-Schwarz formulation of the Type IIB string in the Schild gauge. It is also related to various other matrix models in string theory. For instance, by compactifying one of the directions on a circle $S^1$ one can recover the BFSS matrix model \cite{7} which is conjectured to provide a nonperturbative definition of M-Theory. By compactifying two of the directions on a torus $T^2 = S^1 \times S^1$, one can arrive at the DVV matrix string theory which proposes a nonperturbative definition of Type IIA superstring theory \cite{8}.

Toroidal Compactification

The fact that the spacetime of the model (1.1) is described by mutually noncommuting matrices $X^i$ suggests that it should be related in some way to noncommutative geometry \cite{9}. This feature was first made precise in \cite{10} in the following way. Let us compactify the coordinates $X^i$ on a hypercubic $D$-dimensional torus $T^D$ of sides $R^i$ for $i = 1, \ldots, D$. Because of the $U(N)$ gauge symmetry of the action (1.1), it should be invariant under periodic shifts of the matrices $X^i, i = 1, \ldots, D$ around the cycles of this torus up to unitary conjugation. Thus the toroidal compactification of the IKKT matrix model is tantamount to finding unitary matrices $U_i$ such that

$$X^i + 2\pi R^i \delta^i_j \mathbb{I}_N = U_j^{-1} X^i U_j$$

(1.2)

for each $i, j = 1, \ldots, D$. Taking the trace of both sides of the equations (1.2) shows that they cannot be solved by finite dimensional matrices. It is, however, straightforward to solve them by operators which act on an infinite dimensional Hilbert space, i.e. by setting $N \to \infty$. By applying the translations (1.2) in two different directions it is easily seen that the unitary operators $U_i$ must satisfy the consistency conditions $[U_i U_j U_i^{-1} U_j^{-1}, X^k] = 0$ for all $i, j, k = 1, \ldots, D$, which imply that the unitary operators $U_i U_j U_i^{-1} U_j^{-1}$ can be represented via multiplication by some phases $e^{-2\pi i \Theta^{ij}}, \Theta^{ij} \in \mathbb{R}$, on this Hilbert space. The operators $U_i$ thereby obey the commutation relations

$$U_i U_j = e^{-2\pi i \Theta^{ij}} U_j U_i.$$

(1.3)

We will assume throughout that the antisymmetric $D \times D$ matrix $\Theta^{ij}$ is invertible (so that $D = 2d$ is even).

The algebraic relations (1.3) are the defining presentation of the noncommutative torus $T^D_{\Theta}$ \cite{10}. The operators $U_i$ generate the algebra of smooth functions on $T^D_{\Theta}$ through the generalized Fourier series expansions

$$f = \sum_{\vec{m} \in \mathbb{Z}^D} f_{\vec{m}} U_1^{m_1} \cdots U_D^{m_D}.$$  

(1.4)
where \( f_{\tilde{m}} \) lives in an appropriate Schwartz space of sequences of sufficiently rapid decrease. The \( U_i \)'s may be represented in terms of Hermitian coordinate operators \( \hat{x}^i \) as

\[
U_i = e^{2\pi i \hat{x}^i/R^i}, \quad [\hat{x}^i, \hat{x}^j] = \frac{i R^i R^j \Theta^{ij}}{2\pi}.
\]

In the limit \( \Theta \to 0 \) the expansion \( (1.4) \) becomes the usual Fourier mode expansion for functions on the ordinary torus \( T^D \).

The Moyal Product

The algebra \( (1.3,1.4) \) can be alternatively represented by deforming the usual pointwise multiplication in the algebra \( C^\infty(T^D) \) of smooth functions on the ordinary torus to the associative, non-local star-product

\[
f(x) \star g(x) = f(x) \exp \left( \frac{i}{2} \left( \frac{\partial}{\partial x^i} \Theta^{ij} \frac{\partial}{\partial x^j} \right) \right) g(x) = f(x) g(x) + i \Theta^{ij} \partial_i f(x) \partial_j g(x) + \mathcal{O}(\theta^2) = \frac{1}{\pi^D |\det \theta|} \int d^D y \int d^D z f(x+y) g(x+z) e^{-2i(\theta^{-1})_{ij} y^i z^j}. \tag{1.6}
\]

On the torus \( T^D \), where \( \theta^{ij} = R^i R^j \Theta^{ij}/2\pi \), we can in addition represent this product by the Fourier series expansion

\[
f(x) \star g(x) = \sum_{\vec{m},\vec{n} \in \mathbb{Z}^D} f_{\vec{m}} g_{\vec{n}-\vec{m}} \omega^{-i \Theta^{ij} m_i n_j} e^{i n_i x^i/R^i}. \tag{1.7}
\]

In particular, via an integration by parts one can find that the natural trace of products of functions in this deformed algebra coincides with that of the undeformed algebra,

\[
\text{Tr} f \star g = \int d^D x \ f(x) \star g(x) = \int d^D x \ f(x) \ g(x). \tag{1.8}
\]

Solving Quotient Conditions

To solve the quotient conditions \( (1.2) \) for toroidal compactification of the IKKT matrix model, we introduce anti-Hermitian linear derivations \( \hat{\partial}_i \) which, together with the commutators of the coordinate operators in \( (1.5) \), obey the commutation relations

\[
[\hat{\partial}_i, \hat{x}^j] = \delta_i^j, \quad [\hat{\partial}_i, \hat{\partial}_j] = f_{ij}, \tag{1.9}
\]

where \( f_{ij} \) is an antisymmetric c-number tensor. Since then \( \hat{\partial}_i U_j = U_j (\hat{\partial}_i + 2\pi i \delta_{ij} / R^i) \), these derivations constitute a particular solution to the equations \( (1.2) \). A solution of the corresponding homogeneous equation can be obtained from any function \( A_i(\bar{U}) \) of the generators \( \bar{U}_i \) of the commutant of the algebra \( C^\infty(T^D) \), i.e. \( [\bar{U}_i, U_j] = 0 \ \forall i, j \). They themselves generate a related noncommutative \( D \)-torus. Therefore, the most general solution to the quotient conditions is of the form

\[
X^i = -i \left( R^i \right)^2 \hat{\partial}_i + A_i(\bar{U}). \tag{1.10}
\]
The infinite dimensional operators (1.10) represent a connection of a gauge bundle (of topological charges \( f_{ij} \)) over the noncommutative torus. When they are substituted back into the action (1.1), one arrives at a field theory which can be obtained from ordinary Yang-Mills gauge theory on \( T^D \) by replacing all products of fields by the Moyal product (1.6). This field theory is known as noncommutative gauge theory.

2. Noncommutative Yang-Mills Theory

The gauge theory obtained in the previous section is Yang-Mills theory on the noncommutative torus which is defined by the action

\[
S_{NCYM} = -\frac{1}{g^2} \int d^D x \, F_{ij}(x) \star F^{ij}(x) , \tag{2.1}
\]

where

\[
F_{ij} = \partial_i A_j - \partial_j A_i - i A_i \star A_j + i A_j \star A_i . \tag{2.2}
\]

This field theory possesses the noncommutative gauge symmetry

\[
\delta_{\lambda} A_i = \partial_i \lambda + i \lambda \star A_i - i A_i \star \lambda , \quad \delta_{\lambda} F_{ij} = i \lambda \star F_{ij} - i F_{ij} \star \lambda , \tag{2.3}
\]

where \( \lambda \in C^\infty(T^D) \). Although the Moyal product leads to an infinitely non-local interaction, this deformation of ordinary Yang-Mills theory leads to a sensible quantum field theory. In fact, it is the unique associative deformation which reduces to commutative Yang-Mills theory. It can be analysed in perturbation theory by replacing the structure constants \( f_{abc} \) in the Feynman rules for ordinary non-abelian gauge theory everywhere by the oscillatory momentum-dependent functions \( 2i \sin(\frac{1}{2} \theta_{ij} p_a p_b) \), where \( p_a, p_b \) and \( p_c \) are the incoming momenta of a three-gluon vertex. Many of the perturbative properties of this theory can be thereby analysed.

While the effective noncommutative field theory so obtained has very natural interpretations in both string theory and 11-dimensional supergravity, it has required a stringent large \( N \) limit to be taken in the IKKT matrix model and the information originally encoded by the matrix dynamics has been lost. It would be interesting to see if the origins of noncommutative geometry persists at the level of a finite dimensional matrix model. This would enable, at least in principal, the usage of standard numerical and analytical methods from matrix model technology to solve a host of problems in string theory. In [1]–[4] it was shown precisely how to do this via a lattice version of noncommutative gauge theory. In addition to providing finite dimensional versions of the above construction, it allows one to analyse various properties of noncommutative field theory and also to explain why noncommutative gauge theory arises so naturally from reduced models of Yang-Mills theory. A key feature of this analysis is that it is carried out within the framework of regulated quantum field theory and hence all results thereby obtained are rigorous.
3. Properties of Noncommutative Quantum Field Theory

Let us first summarize some of the basic properties of noncommutative field theories that will be analysed within the lattice approach to noncommutative gauge theory in subsequent sections.

UV/IR Mixing

There is a well-known distinction between planar and non-planar Feynman graphs in noncommutative perturbation theory, analogous to that which arises in multi-colour quantum chromodynamics. For illustration, consider massive noncommutative \( \phi^4 \)-theory in four dimensions, which is defined by the interaction

\[
\int \! d^4x \, \phi(x) \star \phi(x) \star \phi(x) \star \phi(x) = \prod_{a=1}^{4} \int \! \frac{d^4k_a}{(2\pi)^4} \, \phi(k_a) \, \delta^{(4)}(\sum_b k_b) \, V(k_1, k_2, k_3, k_4) \quad (3.1)
\]

with momentum space vertex function

\[
V(k_1, k_2, k_3, k_4) = \prod_{a<b} e^{-\frac{i}{2} k_{ai} \theta^{ij} k_{bj}}. \quad (3.2)
\]

The momentum dependent phase factor (3.2) depends on the cyclic ordering of the vertex momenta \( k_a \), and thereby contributes non-trivially only to non-planar Feynman graphs. For instance, the one-loop planar and non-planar contributions to the mass renormalization in this theory can be written symbolically as (neglecting overall numerical factors)

\[
\begin{align*}
\begin{array}{c}
\text{_planar} \\
\text{non-planar}
\end{array}
\end{align*}
\]

\[
\begin{align*}
\int \! d^4k \, \frac{1}{k^2 + \mu^2},
\int \! d^4k \, \frac{e^{ik_{p_j} \theta^{ij}}}{k^2 + \mu^2}.
\end{align*}
\]

From (3.3) it is evident that the renormalizability properties of planar noncommutative diagrams are the same as those of the corresponding commutative theory (obtained by setting \( \theta = 0 \)) [11]–[13]. This dispels the old belief that noncommutativity would generically serve as a regulator of ultraviolet divergences in quantum field theory (at least for this class of noncommutative geometries that arises naturally in string theory and in M-Theory). On the other hand, the non-planar diagrams (3.4) exhibit the characteristic mixing of ultraviolet and infrared modes in noncommutative perturbation theory. Namely, although for finite external momentum \( p \) the Feynman integral (3.4) is ultraviolet convergent due to the oscillatory momentum-dependent phase factor, in the infrared limit \( p \to 0 \) the integral collapses to (3.3) which is ultraviolet divergent. The ultraviolet divergences in non-planar graphs have been regulated by the noncommutativity, but these have reappeared as infrared divergences. This is not completely surprising, since at \( \theta = 0 \) these divergences must reappear in some way, and this reemergence is characterized by the low energy sector of the quantum field theory. It is a highly unexpected result to
have found infrared divergences in a massive quantum field theory. Notice also that the \( \theta \to 0 \) limit is not a smooth one \[13\]. One of the questions we will address in the following is whether this property is an artifact of perturbation theory or if it manifests itself in nonperturbative properties of the field theory.

This mixing of ultraviolet and infrared modes can in fact be argued for at a heuristic level just from the existence of noncommutativity. Indeed, the noncommutativity parameters lead to a correlation between the position uncertainties in a given pair \( i \neq j \) of spacetime directions of the form \( \Delta x^i = \theta^{ij} / \Delta x^j \). On the other hand, from the ordinary Heisenberg uncertainty principle of quantum mechanics we have \( \Delta x^j = 1 / \Delta p^j \), and therefore the spatial extension of a particle in a direction grows with its momentum in the transverse noncommutative directions. The growth in the size of an object with its energy \( E_i \) is characteristic of string-modified uncertainty relations which have the form \[14\]

\[
\Delta x^i = \frac{1}{E_i} + \alpha' E_i ,
\]

and so the UV/IR mixing property of noncommutative quantum field theory can be regarded as an intrinsically “stringy” feature of these models. This has been at least part of the reason for the huge surge in activity in this models, because as quantum field theories they are believed to capture many of the generic properties of string theory.

**Origins of Noncommutative Yang-Mills Theory**

Besides the manner outlined in section 1, there are two other ways that noncommutative Yang-Mills theory arises as an effective field theory of string dynamics. The first is as a description of the low-energy dynamics of open strings on D-branes in background magnetic fields \[15\]. Formally, the dynamics of the endpoints of the open strings is analogous to that described by the quantum mechanical Landau Lagrangian

\[
L_L = \frac{\mu^2}{2} (\dot{x}^i)^2 + \frac{B}{2} \epsilon_{ij} \dot{x}^i \dot{x}^j
\]

which describes the motion of electrons in the plane \((x^1, x^2)\) in the presence of a constant perpendicular magnetic field of strength \(B\). In the low energy limit \(\mu \to 0\), the coordinates become canonically conjugate operators with commutator \([x^1, x^2] = i \theta\), where \(\theta = 1/B\). Thus the configuration space of this simple model is deformed into a noncommutative space by the presence of the magnetic field. In this same sort of low-energy limit, the effective field theory governing the dynamics of D-branes in magnetic fields is noncommutative gauge theory \[15\].

The second appearance of noncommutative Yang-Mills theory in string theory comes from expanding the spacetime variables \(X^i\) of the IKKT matrix model \(1.1\) about a D-brane background, which is characterized by a particular noncommuting configuration of the variables with constant curvature \[16\]. Again this background can only be represented by infinite-dimensional matrices. This derivation has been used to suggest that twisted large \(N\) reduced models may provide a concrete, non-perturbative definition of noncommutative gauge theory. In the following we shall examine, within a much more general framework, the reasons why this proposal appears to be correct.
One-Loop Renormalization of Noncommutative Gauge Theory

The quadratic part of the one-loop effective action in momentum space for noncommutative quantum electrodynamics in four dimensions has been computed to be [17]

\[
\Gamma_{\text{quad}} = \frac{1}{4} \int \frac{d^4k}{(2\pi)^4} \left\{ \frac{1}{g^2} - \frac{1}{8\pi^2} \left( \frac{11}{3} - \frac{2}{3} n_f \right) \ln \left( \frac{\Lambda^2}{k^2} \right) \right. \\
+ \frac{11}{24\pi^2} \ln \left( \frac{1}{k^2 \left[ k_i (\theta^{ij})^2 k_j \right]} \right) \left\} F^2 ,
\]

where \( \Lambda \) is an ultraviolet cutoff and \( F \) is the noncommutative field strength tensor (2.2). The second term in (3.7) comes from the planar one-loop Feynman diagrams while the third one is due to the non-planar contributions. This result shows that, at the level of planar noncommutative diagrams, i.e. in the limit \( \theta \to \infty \), the noncommutative \( U(1) \) gauge theory has a running coupling constant which coincides with that of ordinary \( SU(\infty) \) Yang-Mills theory. In particular, ignoring issues related to the center \( U(1) \) of the gauge group when \( N > 1 \) [18], we see that \( U(N) \) noncommutative Yang-Mills theory with \( n_f \) flavours of fundamental (fermion) matter fields is equivalent to ordinary \( U(\infty) \) Yang-Mills theory coupled to \( n_f \cdot N \) flavours of matter. As we will explain in the following, the feature that noncommutative Yang-Mills theory at large \( \theta \) coincides with large \( N \) commutative gauge theory is a consequence of Morita equivalence of noncommutative gauge theories, which can also be thought of as a stringy characteristic within the present class of quantum field theories.

4. Lattice Regularization

The questions raised thus far will be answered via a lattice regularization of noncommutative field theory. Since, as we have discussed, constant noncommutativity parameters do not cure a quantum field theory of its ultraviolet divergences, this formalism will also provide a natural ultraviolet regulator for these models. Our starting point will be the definition of a noncommutative space of dimension \( D \) which is defined by Hermitian coordinate operators \( \hat{x}^i \) that obey the commutation relations

\[
\left[ \hat{x}^i , \hat{x}^j \right] = i \theta^{ij}
\]

of noncommutative \( \mathbb{R}^D \), where \( i, j = 1, \ldots, D \). The lattice discretization is defined by restricting the spacetime points of \( \mathbb{R}^D \) to the discrete values \( x^i \in \epsilon \mathbb{Z} \), where \( \epsilon \) is the lattice spacing. This leads to a compact momentum space with periodically identified momenta \( k_i \sim k_i + \frac{2\pi}{\epsilon} \delta_{ij} \) for \( j = 1, \ldots, D \). As a consequence, the lattice discretization of noncommutative spacetime requires that the operators \( \hat{x}^i \) obey the constraint

\[
e^{i(k_i + \frac{2\pi}{\epsilon} \delta_{ij})\hat{x}^i} = e^{ik_i \hat{x}^i}
\]

for each \( j = 1, \ldots, D \). By multiplying both sides of (4.2) by the operator \( e^{-ik_i \hat{x}^i} \) we learn two things. First of all, \( e^{2\pi i \hat{x}^i / \epsilon} = \mathbb{I} \) for \( i = 1, \ldots, D \). This is just the usual constraint that arises in lattice field theory, stating that the spacetime discretization
must be compatible with the spectra of the position operators. However, because of (4.1), the Baker-Campbell-Hausdorff formula produces a non-trivial c-number phase factor which leads to the constraint $\theta_{ij} k_j \in 2\epsilon \mathbb{Z}$, which when combined with the periodicity of momentum space implies that the numbers $\frac{2}{\epsilon} \theta_{ij}$ are integral for each $i, j = 1, \ldots, D$.

This means that the momentum space is also discrete, $k_i = \frac{2\pi}{\epsilon} (\Sigma^{-1})^a_i m_a$, where $m_a \in \mathbb{Z}, a = 1, \ldots, D$, have the periodicities $m_a \sim m_a + \frac{1}{\epsilon} \Sigma^a_i$ for $i = 1, \ldots, D$. Thus the spacetime coordinates are effectively restricted to lie on a periodic lattice

$$x^i \sim x^i + \Sigma^i_a, \quad a = 1, \ldots, D,$$

where the $D \times D$ period matrix $\Sigma$ is defined by

$$M^{ia} \Sigma^j_a - M^{ja} \Sigma^i_a = \frac{2\pi}{\epsilon} \theta^{ij}$$

for some integral $D \times D$ matrix $M$. We have therefore found that lattice regularization of noncommutative field theory implies that spacetime is necessarily compact. Notice that the continuum limit $\epsilon \to 0$ does not commute with the commutative limit $\theta^{ij} \to 0$, since the former limit restores the infinite (noncommutative) spacetime $\mathbb{R}^D$ while the latter limit shrinks it to a point. This discrete compactification is therefore just the UV/IR mixing phenomenon that we encountered perturbatively in the previous section. Here we have found that it arises at a completely non-perturbative level.

It is possible to take a continuum limit to a noncommutative torus in this case by letting the matrix elements $M^{ia} \to \infty$. In the general case, however, we obtain a discrete representation of the noncommutative torus characterized by the algebra

$$\hat{Z}_a \hat{Z}_b = e^{-2\pi i \Theta^{ab}} \hat{Z}_b \hat{Z}_a,$$

where we have defined the single-valued coordinate operators

$$\hat{Z}_a = e^{2\pi i (\Sigma^{-1})^a_i \hat{x}^i},$$

and

$$\Theta^{ab} = 2\pi \left( (\Sigma^{-1})^a_i \theta^{ij} (\Sigma^{-1})^b_j \right) = \epsilon \left( (\Sigma^{-1})^a_i M^{ib} - \epsilon \left( (\Sigma^{-1})^b_j M^{ia} \right) \right)$$

are the corresponding dimensionless noncommutativity parameters which are rational-valued. Furthermore, on the lattice one should as usual consider not the linear derivations $\hat{\partial}_i$ defined by (1.9), but rather the shift operators $\hat{D}_i = e^{\epsilon \hat{\partial}_i}$ which affect translations in units of the lattice spacing $\epsilon$. They act on the coordinate operators (4.6) as

$$\hat{D}_i \hat{Z}_a \hat{D}_i^\dagger = e^{2\pi i \epsilon (\Sigma^{-1})^a_i \hat{x}^i} \hat{Z}_a.$$

To construct a field theory on the noncommutative lattice, we introduce a map $\hat{\Delta}(x)$ which provides an isomorphism between the algebra of finite dimensional Weyl operators $\hat{f}$ and the noncommutative algebra of lattice fields $f(x)$ with a discrete version of the Moyal product. If $\hat{f}_{\vec{m}}$ denotes the periodic lattice Fourier transform of the lattice field $f(x)$, then one can define its corresponding Weyl operator by the Fourier series

$$\hat{f} = \frac{1}{|\det \frac{1}{\epsilon} \Sigma|} \sum_{\vec{m}} e^{2\pi i (\Sigma^{-1})^a_i m_a \hat{x}^i} \hat{f}_{\vec{m}},$$

(4.9)
where the sum runs over all integral vectors $\vec{m} \in \mathbb{Z}^D$ modulo the periodicity of momentum space. The expression (4.9) can be written as

$$f = \sum_x f(x) \hat{\Delta}(x),$$

(4.10)

where the sum runs over all lattice points $x \in \epsilon \mathbb{Z}^D$ modulo the periodicity (4.3) and

$$\hat{\Delta}(x) = \frac{1}{|\det \frac{1}{\epsilon} \Sigma|} \sum_{\vec{m}} \left[ \prod_{a=1}^D \hat{Z}_a^{m_a} \prod_{b < a} e^{\pi i m_a \Theta^{ab} m_b} \right] e^{-2\pi i (\Sigma^{-1})_a^i m_a x^i}.$$

(4.11)

Because of the identity

$$\frac{1}{|\det \frac{1}{\epsilon} \Sigma|} \sum_{\vec{m}} e^{-2\pi i (\Sigma^{-1})_a^i m_a x^i} = \delta_{x,0 (\text{mod} \Sigma)},$$

(4.12)

it follows that at $\theta = 0$ the map (4.11) reduces trivially to $\delta(x - \hat{x})$. However, in the generic noncommutative case it defines a very complicated transformation.

If $\text{Tr}$ is any suitably normalized trace on the algebra of Weyl operators, then it is straightforward to show that the trace of the map (4.11) is $\text{Tr} \Delta(x) = 1$. Furthermore, these maps form a mutually orthonormal system of operators on the lattice, $\text{Tr} \hat{\Delta}(x) \hat{\Delta}(y) = \delta_{xy}$. From these properties it follows that the operator trace $\text{Tr}$ may be uniquely represented in terms of a lattice sum as

$$\text{Tr} \hat{f} = \sum_x f(x),$$

(4.13)

and one can construct the Wigner map

$$f(x) = \text{Tr} \left( \hat{f} \hat{\Delta}(x) \right)$$

(4.14)

which is the inverse of the Weyl map (4.10). In particular, applying the Wigner map (4.14) to the product $\hat{f} \hat{g}$ of two Weyl operators leads to the definition of the lattice star-product,

$$f(x) \star g(x) \equiv \text{Tr} \left( \hat{f} \hat{g} \hat{\Delta}(x) \right) = \frac{1}{|\det \frac{1}{\epsilon} \Sigma|} \sum_{y,z} e^{-2\pi i (\Sigma^{-1})_i^j y^i z^j} f(x + y) g(x + z),$$

(4.15)

where in the second equality we have assumed that $M^{-1}$ is an integral matrix. The expression (4.13) is the lattice version of the third equality in (1.4).

5. Discrete Noncommutative Yang-Mills Theory

The construction of the previous section can be used to systematically construct a lattice version of noncommutative gauge theory. As in ordinary lattice gauge theory, the gauge fields are placed on links of the lattice, which in the present case leads to the definition of the Weyl operators

$$\hat{U}_i = \sum_x \hat{\Delta}(x) \otimes U_i(x),$$

(5.1)
where $U_i(x)$ are $N \times N$ matrices. Unitarity of the Weyl operators (5.1), $\hat{U}_i \hat{U}_i = \mathbb{1}$, is then equivalent to star-unitarity of the corresponding lattice fields, $U_i(x) \star U_i(x) = \mathbb{1}_N$.

A natural plaquette action may then be constructed from these fields as

$$S_D = -\frac{1}{g^2} \sum_{i \neq j} \text{Tr} \text{tr}_{(N)} \left[ \hat{U}_i \left( \hat{D}_i \hat{D}_j \right) \left( \hat{D}_j \hat{D}_i \right) \right]$$

(5.2)

$$= -\frac{1}{g^2} \sum_N x \sum_{i \neq j} \text{tr}_{(N)} \left[ U_i(x) \star U_j(x + \epsilon \hat{i}) \star U_i(x + \epsilon \hat{j} \dagger) \star U_j(x) \right],$$

(5.3)

where $\text{tr}_{(N)}$ is the trace in the fundamental representation of the $N \times N$ unitary group $U(N)$, and $\hat{i}$ is a unit vector in the $i$th direction of the lattice. The action (5.2, 5.3) is invariant under the noncommutative gauge transformations

$$\hat{U}_i \mapsto \hat{\omega} \hat{U}_i \left( \hat{D}_i \hat{\omega}^\dagger \hat{D}_i \right), \quad U_i(x) \mapsto \omega(x) \star U_i(x) \star \omega(x + \epsilon \hat{i} \dagger),$$

(5.4)

where the gauge operator $\hat{\omega}$ is unitary, or equivalently the $N \times N$ gauge function $\omega(x)$ is star-unitary.

Note that this construction displays the manner in which the colour and spacetime degrees of freedom are treated on equal footing in noncommutative gauge theory. As we will see, this will be the essence of the relationship between these models and large $N$ reduced models. The lattice regularization now renders the quantum theory corresponding to the action (5.3) rigorously defined. The lattice noncommutative gauge transformations (5.4) form a finite-dimensional Lie group, and so the path integral measure can be taken to be the corresponding Haar measure, i.e. that which is invariant under left and right multiplications by elements of the noncommutative gauge group, $\hat{U}_i \mapsto \hat{\omega} \hat{U}_i$ and $\hat{U}_i \mapsto \hat{U}_i \hat{\omega}$. In the continuum limit $\epsilon \to 0$, we write $U_i(x) = \exp_{\star} i \epsilon A_i(x)$, where the star-exponential $\exp_{\star}$ is defined by replacing ordinary products in the Taylor series expansion of the exponential function by star-products. Then one can easily work out the star-product of lattice gauge fields around a plaquette to be

$$U_i(x) \star U_j(x + \epsilon \hat{i}) \star U_i(x + \epsilon \hat{j} \dagger) \star U_j(x) = \exp_{\star} i \epsilon^2 F_{ij}(x),$$

(5.5)

where $F_{ij}$ is the noncommutative field strength tensor (2.2). Thus in the continuum limit the action (5.3) reduces to the usual continuum noncommutative Yang-Mills action functional (2.1), and as such it represents the natural noncommutative version of the Wilson plaquette action for ordinary lattice gauge theory (2.0).

Let us now describe the observables of lattice noncommutative gauge theory [1, 3, 4, 12]. In analogy with the commutative case, let us choose an oriented contour $\mathcal{C} = \{i_1, i_2, \ldots, i_L\}$ on the lattice which is specified by $L$ links $i_a = \pm 1, \pm 2, \ldots, \pm D$, $a = 1, \ldots, L$, which start from the origin $x = 0$ and end at the point $\ell = \epsilon \sum_a i_a$. We then introduce the noncommutative version of the lattice parallel transport operator along the contour $\mathcal{C}$,

$$U(x; \mathcal{C}) = U_{i_1}(x) \star U_{i_2}(x + \epsilon \hat{i}_1) \star \cdots \star U_{i_L}(x + \epsilon \sum_{a=1}^{L-1} \hat{i}_a),$$

(5.6)

which transforms under noncommutative gauge transformations (5.4) in the expected way as

$$U(x; \mathcal{C}) \mapsto \omega(x) \star U(x; \mathcal{C}) \star \omega(x + \ell \dagger).$$

(5.7)
To construct star-gauge invariant operators from (5.6), we note that a simple application of the Baker-Campbell-Hausdorff formula gives
\[
e^{2\pi i (\Sigma^{-1})^a_m x^i} \star e^{2\pi i (\Sigma^{-1})^a_n a x^i} \star e^{-2\pi i (\Sigma^{-1})^a_m x^i} = e^{2\pi i (\Sigma^{-1})^a_n a x^i} \star e^{2\pi i (\Sigma^{-1})^a_m x^i} e^{2\pi i n a \Theta^a b m b},
\]
which via Fourier transformation implies that plane waves in noncommutative geometry affect the translations
\[
e^{2\pi i (\Sigma^{-1})^a_m x^i} \star \omega(x) \star e^{-2\pi i (\Sigma^{-1})^a_m x^i} = \omega(x + \ell \vec{m})
\]
on arbitrary functions \(\omega(x)\), where
\[
\ell_i \vec{m} = 2\pi \theta^{ij} (\Sigma^{-1})^j_a m_a + \Sigma_b w^a
\]
with \(w^a \in \mathbb{Z}\). This property, that unitary conjugation can induce spacetime translations of a function, is particular to the noncommutative Moyal product. It is related to the fact that noncommutative gauge theories are intimately connected with general relativity, another feature of them which is tied to their stringy nature.

From these properties it is straightforward to construct a gauge invariant observable associated with the contour \(C\) as
\[
O(C) = \sum_x \text{tr}(N) U(x; C) \star e^{2\pi i (\Sigma^{-1})^a_m x^i}.
\]
Its invariance follows from (5.7) and (5.9). The integer vector \(\vec{m}\) can be interpreted as the total momentum of the line \(C\) and it is related to the separation of its endpoints by (5.10). The most remarkable aspect of this construction is that we haven’t had to assume that the contour \(C\) is closed. Therefore, in marked contrast to the commutative case, in noncommutative gauge theory there are gauge-invariant observables associated with open contours. The price to pay for this extra class of operators is that they are necessarily non-local, as the noncommutative gauge symmetry is a geometrical one and so requires the usage of the spacetime trace Tr in addition to the colour trace \(\text{tr}(N)\) to define star-gauge invariant operators. In the commutative limit \(\theta = 0\), we recover the well-known fact that on a compact space the only gauge invariant observables are the Polyakov lines which wind \(w^a\) times around the compact direction \(a\). In that case, any function can be convoluted with the parallel transport operator in (5.11) and we recover the usual local observables of Yang-Mills theory. Notice also the feature that the size \(\ell\) of the contour grows with its momentum according to (5.10). This is just another manifestation of the UV/IR mixing property that generically persists in noncommutative field theories.

6. Morita Equivalence

We will now describe a remarkable symmetry on the space of noncommutative (lattice) gauge theories. Consider commutative \(U(N)\) lattice gauge theory with ’t Hooft flux in dimension \(D = 2d\). The action is the usual Wilson plaquette action \([20]\)
\[
S_W = -\frac{1}{g^2} \sum_x \sum_{i \neq j} \text{tr}(N) \left[ U_i(x) U_j(x + \epsilon \hat{i}) U_i(x + \epsilon \hat{j})^\dagger U_j(x)^\dagger \right], \quad (6.1)
\]
with the gauge fields obeying the twisted boundary conditions

$$U_i(x + \Sigma_j \hat{J}^j) = \Omega_a(x) U_i(x) \Omega_a(x + \epsilon \hat{i}) .$$  \hspace{1cm} (6.2)

Gauge fields which are multi-valued according to (6.2) are only defined on the universal covering space of the lattice because they carry a non-vanishing flux. By writing the large gauge transformation (6.2) along two different directions, we find that the transition functions $\Omega_a(x)$ must satisfy the cocycle conditions

$$\Omega_a(x + \Sigma_b \hat{J}^b) \Omega_b(x) = Z_{ab} \Omega_b(x + \Sigma_a \hat{J}^a) \Omega_a(x) .$$  \hspace{1cm} (6.3)

The phase factor $Z_{ab} = e^{2\pi i Q_{ab}/N} \in \mathbb{Z}_N$ is called the “twist”, where $Q_{ab}$ is the integer ’t Hooft flux through the $(ab)$-th two-cycle of the torus [21]. If we choose $\Omega_a(x) = \Gamma_a$ to be constant $SU(N)$ matrices, then (6.3) implies that the transition functions obey the Weyl-'t Hooft algebra

$$\Gamma_a \Gamma_b = e^{2\pi i Q_{ab}/N} \Gamma_b \Gamma_a$$  \hspace{1cm} (6.4)

which determines them as twist-eating solutions for $SU(N)$. Our task is to now find the general form of the gauge fields which solve the twisted boundary conditions (6.2). For this, we map the problem onto an equivalent one involving (commutative) Weyl operators. We introduce the map (5.1) and rewrite the action (6.1) as in (5.2). The large gauge transformations (6.2) may then be expressed in terms of Weyl operators as

$$\left(\hat{D}_j\right)^{\Sigma_j \epsilon / \epsilon} U_i \left(\hat{D}_j\right)^{\Sigma_j \epsilon / \epsilon} = \Gamma_a \hat{U}_i \Gamma_a^\dagger .$$  \hspace{1cm} (6.5)

Irreducible Representation of Twist Eaters

Before solving (6.5), we first need to digress a bit and describe the representation theory of the Weyl-'t Hooft algebra (6.4) [22]. For this, we use the discrete $SL(D, \mathbb{Z})$ automorphism symmetry group of the torus to rotate $Q_{ab}$ into a canonical skew-diagonal form with skew-eigenvalues $q_{\alpha} \in \mathbb{Z}$, $\alpha = 1, \ldots , d$, and define the three sets of $d$ integers

$$N_\alpha = \text{gcd}(q_{\alpha}, N) , \ \ \ \tilde{N}_\alpha = \frac{N}{N_\alpha} , \ \ \ \check{q}_{\alpha} = \frac{q_{\alpha}}{N_\alpha} .$$  \hspace{1cm} (6.6)

A necessary and sufficient condition for the existence of twist-eating solutions $\Gamma_a$ is that there exists an integer $\tilde{N}_0$ such that $N = \tilde{N}_0 \cdot \tilde{N}_1 \cdots \tilde{N}_d$. The integer $\tilde{N}_1 \cdots \tilde{N}_d$ is then the dimension of the irreducible representation of the Weyl-'t Hooft algebra (5.4), and the twist eaters $\Gamma_a$ may be defined on the $SU(N)$ subgroup $SU(\tilde{N}_1) \otimes \cdots \otimes SU(\tilde{N}_d) \otimes SU(\tilde{N}_0)$ as

$$\Gamma_{2a-1} = \mathbb{I}_{\tilde{N}_1} \otimes \cdots \otimes V_{\tilde{N}_\alpha} \otimes \cdots \otimes \mathbb{I}_{\tilde{N}_d} \otimes \mathbb{I}_{\tilde{N}_0} , \ \ \ \Gamma_{2a} = \mathbb{I}_{\tilde{N}_1} \otimes \cdots \otimes (W_{\tilde{N}_\alpha})^{\check{q}_{\alpha}} \otimes \cdots \otimes \mathbb{I}_{\tilde{N}_d} \otimes \mathbb{I}_{\tilde{N}_0} ,$$  \hspace{1cm} (6.7)

for $\alpha = 1, \ldots , d$. Here $(V_N)_{ab} = \delta_{a,b-1}$ and $(W_N)_{ab} = e^{2\pi i (a-1)/N} \delta_{ab}$ are the cyclic shift and clock matrices of $SU(N)$ which obey the commutation relations

$$V_N W_N = e^{2\pi i / N} W_N V_N .$$  \hspace{1cm} (6.8)
Note that the matrices (6.7) commute with the $SU(\tilde{N}_0)$ subgroup of $SU(N)$ consisting of matrices of the form $\mathbb{1}_{\tilde{N}_1} \otimes \cdots \otimes \mathbb{1}_{\tilde{N}_d} \otimes Z_0$, $Z_0 \in SU(\tilde{N}_0)$.

By construction, for each $\alpha = 1, \ldots, d$ the integers $\tilde{N}_\alpha$ and $\tilde{q}_\alpha$ are relatively prime, and so there exist integers $a_\alpha$ and $b_\alpha$ such that $a_\alpha \tilde{N}_\alpha + b_\alpha \tilde{q}_\alpha = 1$. We now introduce two diagonal $D \times D$ matrices $A$ and $\tilde{N}$ whose diagonal elements are the integers $a_\alpha$ and $\tilde{N}_\alpha$, respectively, each of which appear with multiplicity two. Likewise, we define two skew-diagonal matrices $B$ and $\tilde{Q}$ whose skew-diagonal elements are the integers $b_\alpha$ and $\tilde{q}_\alpha$. After using the $SL(D, \mathbb{Z})$ automorphism group to rotate the flux matrix $Q$ back to general form, we find that the resulting four integral $D \times D$ matrices $A$, $B$, $\tilde{N}$ and $\tilde{Q}$ are all mutually commutating and obey the matrix identity

$$A\tilde{N} + B\tilde{Q} = \mathbb{1}_D.$$  \hspace{1cm} (6.9)

As a consequence, these four matrices naturally produce a block matrix

$$\begin{pmatrix} A & B \\ -\tilde{Q} & \tilde{N} \end{pmatrix} \in SO(D, D; \mathbb{Z}).$$  \hspace{1cm} (6.10)

### Solving Twisted Boundary Conditions

Using the above construction, it is now straightforward to write down the general solution to the twisted boundary conditions (6.5). The details can be found in [4], and one finds

$$\hat{U}_i = \sum_{\vec{m}} \left\{ \prod_{a=1}^{D} \hat{Z}_a^{m_a} \prod_{b<a} e^{\pi i m_a \Theta_{ab} m_b} \right\} \otimes u_i(\vec{m}),$$  \hspace{1cm} (6.11)

where $u_i(\vec{m})$ is an $\tilde{N}_0 \times \tilde{N}_0$ matrix and the new coordinate operators are

$$\hat{Z}_a' = e^{2\pi i (\Sigma^{-1})_a^i x^i} \otimes \prod_{b=1}^{D} (\Gamma_b)^{B_{ab}}.$$  \hspace{1cm} (6.12)

Because of the algebra (6.4), the operators (6.12) obey the commutation relations (4.1) with noncommutativity parameter matrix

$$\Theta = -\tilde{N}^{-1} B^\top$$  \hspace{1cm} (6.13)

and also the relations (4.8) with period matrix $\Sigma' = \Sigma \tilde{N}$. We see therefore that by absorbing the 't Hooft flux $Q$ and some of the colour degrees of freedom into the coordinate operators (6.12), we have effectively arrived at a new toroidal lattice which is now a noncommutative space.

We can now introduce a new Weyl-Wigner correspondence map $\hat{\Delta}'(x')$ by substituting (6.12) into (4.11), where $x'$ are coordinates on the new (noncommutative) torus. Using this new map, we may then expand the operator $\hat{U}_i$ in terms of a new field $U'_i(x')$ which is a single-valued $\tilde{N}_0 \times \tilde{N}_0$ star-unitary matrix field on a toroidal lattice with periods $\Sigma'$. The commutative Wilson plaquette action (6.1) then becomes the discrete noncommutative Yang-Mills action (5.3) of reduced gauge group rank $\tilde{N}_0$, with rational-valued noncommutativity parameters (6.13), and with new Yang-Mills coupling constant $g'^2 = Ng^2/\tilde{N}_0$.  

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We have thereby found that $U(N)$ commutative Yang-Mills theory with 't Hooft flux is equivalent to a noncommutative $U(N_0)$ gauge theory with single-valued gauge fields. This transformation is an example of Morita equivalence. Note that it reduces the rank of the gauge group and increases the size of the torus. In fact, when $N_0 = 1$ (so that gauge group rank $N$ itself is the dimension of the desired irreducible representation of the Weyl-'t Hooft algebra), we see that the colour degrees of freedom of any non-abelian gauge theory can be completely absorbed into the noncommutativity of spacetime. This explains in part the equivalences between ordinary and noncommutative quantum gauge theories that we observed in section 3.

Morita equivalence is a standard equivalence relation among certain noncommutative spaces. Here we have found that it is simply a result of a change of basis $\hat{\Delta}(x) \leftrightarrow \hat{\Delta}'(x')$ for the mapping between Weyl operators and lattice fields. This Morita equivalence is in fact the noncommutative field theory version of the $SO(D, D; \mathbb{Z}) T$-duality symmetry of toroidally compactified string theory [23]. It is a remarkable fact that this string theoretical duality is manifested explicitly at a field theoretical level in noncommutative gauge theory. Furthermore, this equivalence can be shown to hold also at the level of all quantum correlation functions, and hence at the level of the full quantum field theory [4]. For example, under this duality the Polyakov lines of ordinary gauge theory (with non-vanishing winding number $w^a$) on a torus are mapped onto the open Wilson lines of the Morita equivalent noncommutative gauge theory (with non-vanishing momentum $m_a$). Generally, at the level of observables, the Morita equivalence acts precisely like a $T$-duality, in that it interchanges momentum and winding modes $m_a \leftrightarrow w^a$.

7. Twisted Eguchi-Kawai Model

We are now ready to naturally describe the link between noncommutative gauge theories and twisted reduced models. Let us reduce the Wilson action (6.1) to a single plaquette, leaving a one-site $U(N)$ lattice gauge theory, i.e. $\Sigma = \epsilon \mathbb{I}_D$, with 't Hooft flux. By reducing the model to a point $x = 0$, we can obtain the values of gauge fields at the other three sites of the plaquette by using the twisted boundary conditions (6.2) to get $U_i(\epsilon \delta_j^a \hat{j}) = \Gamma_a U_i(0) \Gamma_a^\dagger$. The action (6.1) then reduces to

$$S_{TEK} = -\frac{1}{g^2} \sum_{i \neq j} Z_{ij} \text{tr} (N) \left( V_i V_j V_i^\dagger V_j^\dagger \right),$$

where $V_i = U_i(0) \Gamma_i$ are $N \times N$ unitary matrices and the phase factor $Z_{ij} = e^{2\pi i Q_{ij}/N}$ is the twist that appeared in (6.3). This unitary matrix model is known as the (twisted) Eguchi-Kawai model [24]. Such reduced models of gauge theory where originally introduced as matrix models whereby the spacetime dependence of gauge fields is absorbed into colour degrees of freedom, necessitating a large $N$ limit to be taken. The twist is required so that the reduced model is equivalent to the 't Hooft limit of large $N$ quantum field theory on the continuum spacetime. This connection with noncommutative gauge theories is very natural, since both sorts of models describe a mixing of colour and spacetime degrees of freedom. In order to reproduce continuum field theories, the Eguchi-Kawai model was always studied in the large $N$ limit. But now we have unravelled a much deeper connection
between the finite $N$ version of the twisted Eguchi-Kawai model and noncommutative lattice gauge theory, i.e. the two models are Morita equivalent. In the continuum limit, which requires $N \to \infty$, we now see why the reason for the intimate relationship between noncommutative Yang-Mills theory and reduced models – it is the simplest instance of Morita equivalence of noncommutative geometries. Note that the twist factor is related to a background flux, consistent with the fact that noncommutative field theory arises from string theory in background magnetic fields. We emphasize that the connection just presented between matrix models and noncommutative gauge theory is rigorous because it is derived from regulated quantum field theory.

A particularly interesting feature of the reduced model is that it yields a finite dimensional representation of the noncommutative torus. The algebraic relations (4.5,4.8) are satisfied by the $N \times N$ matrices

$$\hat{Z}_i = \prod_{j=1}^{D} (\Gamma_j)^{R_{ij}}, \quad \hat{D}_i = \Gamma_i, \quad (7.2)$$

with rational noncommutativity parameters (6.13) and period matrix $\Sigma = \epsilon \tilde{N}$. The problem of representing a continuum noncommutative torus using such a finite dimensional approximation is a little subtle and technically involved. The detailed, rigorous construction can be found in [2]. However, given the matrix model (7.1), which is the unitary version of the IKKT matrix model obtained by exponentiating the Hermitian matrices of (1.1) according to $V_i = e^{iX_i/\epsilon}$, we can now write down a finite-dimensional version of the quotient conditions for toroidal compactification of the matrix model. Exponentiation of (1.2) leads to the conditions

$$\hat{Z}_j V_i \hat{Z}_j^\dagger = e^{2\pi i \delta_{ij} \tilde{N}_i/N} V_i \quad (7.3)$$

which define the compactification of the twisted Eguchi-Kawai model (7.1) on a rectangular torus $T^D$ of sides $\epsilon \tilde{N}_i, \ i = 1, \ldots, D$. Now taking the trace of both sides of the equation (7.3) only implies that $V_i$ is a traceless unitary matrix, which is a condition obeyed by the twist-eating solutions $\Gamma_i$. Moreover, the consistency condition generated by (7.3) yields precisely the defining relations (1.4) of the noncommutative torus. Because all parameters involved are rational numbers, finite-dimensional representations of all these quantities exist, and so it is possible to maintain a finite $N$ version of the noncommutative geometry described in section 1. A particular solution to the quotient conditions comes from the shift operators $\hat{D}_i$ in (7.2), which modulo gauge transformations is the vacuum configuration of the matrix model (7.1). The general solution to (7.3) is then given by

$$V_i = \hat{\Lambda}_i \Gamma_i, \quad (7.4)$$

where the unitary matrices $\hat{\Lambda}_i$ generate the commutant of the noncommutative torus (1.5), i.e. $[\hat{\Lambda}_i, \hat{Z}_j] = 0$. The unitary matrices (7.3) represent a finite-dimensional approximation to a generic gauge connection on the noncommutative torus of topological charges $Q_{ij}$ [2]. The general solutions of the commutant condition can be expanded in the Weyl basis of $gl(N, \mathbb{C})$, whose lattice Fourier transform provides the map between finite dimensional matrices and lattice fields. The corresponding expansion coefficients $U_i(\tilde{m})$ of $\hat{\Lambda}_i$ are now interpreted as the Fourier coefficients of some fields $U_i(x)$ defined on a periodic lattice. When (7.4) is then substituted back into the Eguchi-Kawai action (7.1), one arrives at the
noncommutative lattice Yang-Mills action (5.3) for gauge group $U(1)$. For more details of this construction, see [1, 5]. In this way we have arrived at another derivation of the relationship between reduced models and noncommutative gauge theory. The two approaches are equivalent, because the commutant algebra generated by the matrices $\hat{\Lambda}_i$ in fact defines a Morita equivalent noncommutative torus [1, 10].

8. Coupling to Fundamental Matter Fields

We will now examine to what extent the observables of noncommutative gauge theory can be regarded as fundamental. For this, we minimally couple the noncommutative gauge theory (5.3) to complex scalar fields $\phi(x)$ in the fundamental representation of the $U(N)$ gauge group. The matter-coupled action is

$$S_m = -\sum_{x,i} \phi(x) \dagger \ast U_i(x) \ast \phi(x + \epsilon \hat{i}) + \mu^2 \sum_x \phi(x) \dagger \phi(x) \ , \quad (8.1)$$

and, together with (5.4), it is invariant under the star-gauge transformations

$$\phi(x) \mapsto \omega(x) \ast \phi(x) \ , \quad \phi(x) \dagger \mapsto \phi(x) \dagger \ast \omega(x) \dagger \ . \quad (8.2)$$

The free scalar field propagator is

$$\langle \phi_a(x)^\dagger \phi_b(y) \rangle_0 = \frac{1}{\mu^2} \delta_{ab} \delta_{xy} \ . \quad (8.3)$$

where all averages in the following will denote those for a fixed gauge background, i.e. the gauge fields are not integrated over.

A real technical advantage of the lattice formulation of noncommutative gauge theory now comes into play. The scalar matter fields in the present lattice field theory may be integrated out analytically using the standard large mass expansion, in powers of $\frac{1}{\mu^2}$, of lattice gauge theory. For instance, we can evaluate the effective action functional thereby induced for the gauge fields, which is defined by the perturbation series

$$S_{\text{eff}}[U] = -\ln \left[ \sum_{n=0}^{\infty} \frac{1}{n!} \left\langle \left( \sum_{x,i} \phi(x) \dagger \ast U_i(x) \ast \phi(x + \epsilon \hat{i}) \right)^n \right\rangle_0 \right] . \quad (8.4)$$

The expectation values in (8.4) may be computed by using the standard Wick expansion and the propagator (8.3). The series (8.4) can thereby be reduced to a geometric sum over contours on the lattice of the same type that appears in ordinary lattice gauge theory, and we have

$$S_{\text{eff}}[U] = \sum_{C: \text{closed}} \frac{\mu^{-2L(C)}}{L(C)} \sum_x \text{tr}_{(N)} U(x; C) \ , \quad (8.5)$$

where $L(C)$ is the number of links in the contour $C$. In this way we recover the closed loop observables (5.11) (of momentum $\vec{m} = \vec{0}$) of noncommutative Yang-Mills theory.
To see how the star-gauge invariant observables associated with open Wilson lines arise, consider a two-point function of the form
\[ G[F] = \left\langle \sum_x \bar{\phi}(x)^\dagger \star \bar{\phi}(x) \star F(x) \right\rangle, \quad (8.6) \]

where \( F(x) \) can be regarded as the wavefunction of the composite operator \( \bar{\phi}(x)^\dagger \star \bar{\phi}(x) \).

By Fourier transforming the function \( F(x) \) to momentum space, we can write (8.6) as
\[ G[F] = \sum_{\vec{m}} F_{\vec{m}} \left\langle \sum_x \bar{\phi}(x)^\dagger \star e^{2\pi i(S^{-1})_a^i m_a x^i} \star \bar{\phi}(x + \ell_{\vec{m}}) \right\rangle, \quad (8.7) \]

where \( \ell_{\vec{m}} \) is defined in (5.10) and we have used (5.9). The matter correlators in (8.7) are now straightforward to evaluate and are again given geometrically by a standard sum-over-paths representation. This enables us to write
\[ G[F] = \sum_{\vec{m}} F_{\vec{m}} \sum_{\mathcal{C} : i_1 + \ldots + i_{L(C)} = \ell_{\vec{m}}} \mu^{-2L(C)} \sum_x \text{tr}_{(N)} U(x; \mathcal{C}) \star e^{2\pi i(S^{-1})_a^i m_a x^i}, \quad (8.8) \]

and so we recover the open Wilson line observables (of arbitrary momentum \( \vec{m} \)) of noncommutative gauge theory. Note that in the commutative limit \( \theta = 0 \), only closed contours \( \mathcal{C} \) contribute to the sum in (8.8) and are thereby independent of the integer vector \( \vec{m} \). Then, the Fourier sum in (8.8) can be done explicitly, reinstating the wavefunction \( F(x) \) and thereby recovering the closed Wilson line observables of ordinary gauge theory. Thus the open Wilson line observables of noncommutative gauge theory play just as fundamental a role as the closed ones do in ordinary Yang-Mills theory. The present demonstration of this feature is in fact identical to the way that Wilson loops were originally discovered [21].

**Morita Equivalence with Fundamental Matter**

Let us now consider commutative \( U(N) \) gauge theory minimally coupled to \( N_f \) flavours of fundamental scalar fields \( \Phi(x)_{a,\alpha} \), where the index \( a = 1, \ldots, N \) labels colour and \( \alpha = 1, \ldots, N_f \) indexes flavour. If we take \( N_f = N \), then we may regard \( \Phi(x) \) as an \( N \times N \) complex matrix field. The action obtained by summing \( N_f \) commutative actions analogous to (8.1) can then be written as
\[ S_m^{(N_f)} = -\sum_{x,\dot{i}} \text{tr}_{(N)} \Phi(x)^\dagger U_i(x) \Phi(x + \epsilon \dot{i}) + \mu^2 \sum_x \text{tr}_{(N)} \Phi(x)^\dagger \Phi(x). \quad (8.9) \]

It possesses the local left \( U(N) \) colour symmetry analogous to (8.2), and also a global right \( U(N) \) flavour symmetry \( \Phi(x) \mapsto \Phi(x) h, \ h \in U(N_f) \). We can thereby use the global \( SU(N_f) \) flavour symmetry to mimic the adjoint \( U(N) \) representation for these fundamental matter fields. In particular, as in (5.2) we can impose twisted boundary conditions on the fields \( \Phi(x) \) in the form \( \Phi(x + \Sigma_i^a \dot{i}) = \Gamma_a \Phi(x) \Gamma_a^\dagger \), where here \( \Gamma_a \) represents a large gauge transformation while \( \Gamma_a^\dagger \) represents a rotation in flavour space. We may now repeat the procedure described in section 6 of obtaining a Morita equivalent matter-coupled noncommutative gauge theory. One finds generally that ordinary \( U(N) \) gauge theory coupled to \( n_f \cdot N \) flavours of fundamental matter fields is equivalent to \( U(N_0) \) noncommutative gauge
theory with $n_f \cdot \tilde{N}_0$ flavours of fundamental matter. In the case $\tilde{N}_0 = 1$, we have thereby unveiled a proper explanation for the equivalences of commutative and noncommutative matter-coupled quantum gauge theories mentioned at the end of section 3. We stress once more that these equivalences are completely rigorous within the present setting, since they are obtained in a regularized quantum field theory.

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