On spin 3 interacting with gravity

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Received 20 August 2008, in final form 5 November 2008
Published 19 January 2009
Online at stacks.iop.org/CQG/26/035022

Abstract
Recently Boulanger and Leclercq have constructed a cubic four derivative \( 3 - 3 - 2 \) vertex for the interaction of spin 3 and spin 2 particles. This vertex is trivially invariant under the gauge transformations of the spin 2 field, so it seemed that it could be expressed in terms of the (linearized) Riemann tensor. And indeed in this paper we managed to reproduce this vertex in the form \( R \partial/\Phi_1 \partial/\Phi_1 \), where \( R \) is the linearized Riemann tensor and \( \Phi_1 \) is the completely symmetric third rank tensor. Then we consider the deformation of this vertex to (\( A \)dS) space and show that such deformation produces a 'standard' gravitational interaction for spin 3 particles (in the linear approximation) in agreement with general construction of Fradkin and Vasiliev. Then we turn to the massive case and show that the same higher derivative terms allow one to extend the gauge invariant description of a massive spin 3 particle from constant curvature spaces to arbitrary gravitational backgrounds satisfying \( R_{\mu\nu} = 0 \).

PACS numbers: 04.40.-b, 04.40.-h

1. Introduction

The problem of constructing consistent interactions for higher spin particles is an old but still unsolved one (see, e.g., reviews in [1–3]). In this, one of the classical and important tasks is investigation of gravitational interactions for such particles. It was known for a long time that it is not possible to construct a standard gravitational interaction for massless higher spin \( s \geq 5/2 \) particles in flat Minkowski space [4–6] (see also recent discussion in [7]). At the same time, it has been shown [8, 9] that this task indeed has a solution in (\( A \)dS) space with a non-zero cosmological term. The reason is that gauge invariance, which turns out to be broken when one replaces ordinary partial derivatives by the gravitational covariant ones, could be restored by the introduction of higher derivative corrections containing the gauge invariant Riemann tensor. These corrections have coefficients proportional to inverse powers of cosmological constant so that such theories do not have a naive flat limit. But it is perfectly possible to have
a limit when both the cosmological term and gravitational coupling constant simultaneously
tend to zero in such a way that only interactions with highest number of derivatives survive.

It is natural to suggest that in any realistic higher spin theory (like in superstring) most of
higher spin particles must be massive and their gauge symmetries spontaneously broken. So a
very important, interesting and until now poorly investigated problem is to find a mechanism
of spontaneous gauge symmetry breaking which could deform massless particles in (A)dS
space into a massive one in flat Minkowski space. But if such a mechanism does exist, one
can apply the same line of reasoning about a massless limit in the resulting theory. Namely,
it is natural to suggest that there exists a limit when both mass and gravitational coupling
constants simultaneously tend to zero so that only some interactions containing the Riemann
tensor survive.

In both cases above the crucial point is the existence of a cubic higher derivative spin
$s - s - 2$ vertex containing the (linearized) Riemann tensor and two massless spin $s$ particles
in flat Minkowski space. While it is always possible to construct a higher derivative vertex out
of gauge invariant ‘field strengths’ [10], what we need here is a vertex (with less derivatives)
with non-trivial deformation of gauge transformations. For the spin $s = 3$ case an appropriate
candidate has been constructed recently in [11]. Really, the Lagrangian for the four derivative
$3 - 3 - 2$ vertex given in this paper does not look to be the one we need here, but it is
trivially invariant under the spin 2 gauge transformations so it can be expressed in terms of the
(linearized) Riemann tensor. And indeed in the following section, we reproduce this vertex in
the desired form. Then we consider the deformation of this vertex into (A)dS space and show
that such deformation does produce a standard gravitational interaction for a massless spin 3
particle (in the linear approximation).

After that we turn to the main question we are interested in: could non-zero mass play
the same role as non-zero cosmological constant (for the massive spin 5/2 case see [12]). In
our opinion the most natural and convenient formalism for the investigation of spontaneous
gauge symmetry breaking is the gauge invariant description of massive higher spin particles,
which nicely works both in flat Minkowski space as well as in (A)dS ones [13–15]. Due to
large number of fields involved (four in the case of a massive spin $s = 3$ particle) complete
investigation of the gravitational interaction for massive spin 3 requires a lot of work and we
leave it for the future (an example of the massive spin $s = 2$ case was considered in [16]), so
in this paper as a first step we consider a massive spin 3 particle in an arbitrary gravitational
background satisfying its free equations, i.e. $R_{\mu\nu} = 0$. In this, only terms containing the full
Riemann tensor survive which greatly simplifies all calculations.

In section 4, we begin with the gauge invariant description of a massive spin 3 particle
in flat Minkowski space. In such a formalism the problem of switching on a gravitational
interaction looks exactly the same as for massless particles. Namely, when one replaces
ordinary partial derivatives by the covariant ones the gauge invariance of the Lagrangian turns
out to be broken, but this non-invariance contains terms with curvature tensor only. This leaves
us the possibility of restoring gauge invariance with the help of higher derivative corrections.
And in section 5, we show that such restoration is indeed possible (at least in the linear
approximation), in this the same cubic four derivative $3 - 3 - 2$ vertex plays the main role.

2. Cubic vertex $3 - 3 - 2$ with four derivatives

As we have already mentioned, the crucial point is the existence of some cubic higher derivative
vertex in flat Minkowski space containing the (linearized) curvature tensor for the spin 2 field.
Appropriate vertex has been constructed recently in [11]. The form of the Lagrangian written
in this paper does not seem to be the one we are looking for, but its trivial invariance under
the spin 2 gauge transformations shows that it can be reduced to the required form. In this section, we reconstruct this vertex by using the brute force method, namely we begin with the sum of free Lagrangians for massless spin 3 and spin 2 particles and their usual gauge transformations and consider the most general four derivative cubic vertex and appropriate corrections to gauge transformations.

The usual free Lagrangian for the symmetric third rank tensor \( \Phi_{\mu\nu\alpha} \) has the form

\[
L_0 = -\frac{1}{2} \partial^\mu \Phi_{\nu\alpha} \partial^\mu \Phi_{\nu\alpha} + \frac{3}{2} \partial^\mu \Phi_{\nu\alpha} \partial^\mu \Phi_{\nu\alpha} - \frac{3}{2} \partial^\mu \Phi_{\nu\alpha} \partial^\mu \Phi_{\nu\alpha} + \frac{3}{4} \partial^\mu \Phi_{\nu\alpha} \partial^\mu \Phi_{\nu\alpha},
\]

(1)

where \( \partial^\mu \Phi_{\nu\alpha} = \partial^\alpha \Phi_{\nu\mu} \) and \( \Phi_{\mu\nu\alpha} = g^{\alpha\beta} \Phi_{\mu\nu\beta} \). This Lagrangian is invariant under the following gauge transformations:

\[
\delta \Phi_{\mu\nu\alpha} = \partial^\mu \xi_{\nu\alpha} + \partial^\nu \xi_{\mu\alpha} + \partial^\alpha \xi_{\mu\nu}
\]

(2)

where the parameter \( \xi_{\mu\nu} \) is symmetric \( \xi_{\mu\nu} = \xi_{\nu\mu} \) and traceless \( \xi_{\mu\mu} = 0 \). Analogously, the well-known Lagrangian for the symmetric second rank tensor \( h_{\mu\nu} \) looks as follows:

\[
L_0 = \frac{1}{2} \partial^\mu h_{\alpha\beta} \partial^\mu h_{\alpha\beta} - (\partial h)^\mu (\partial h)_\mu + (\partial h)^\mu \partial^\mu h - \frac{1}{4} \partial^\mu h \partial^\mu h
\]

(3)

being invariant under the gauge transformations with the vector parameter

\[
\delta h_{\mu\nu} = \partial^\mu \xi_{\nu} + \partial^\nu \xi_{\mu}.
\]

(4)

The most general four derivative cubic vertex, which is trivially invariant under the spin 2 gauge transformations, can be written (symbolically) as

\[
L \sim R^2 \Phi \oplus R \Phi \Phi \Phi
\]

where \( R \) denotes the linearized Riemann tensor

\[
R_{\mu\nu,\alpha\beta} = \partial_\mu \partial_\alpha h_{\nu\beta} - \partial_\mu \partial_\beta h_{\nu\alpha} - \partial_\nu \partial_\alpha h_{\mu\beta} + \partial_\nu \partial_\beta h_{\mu\alpha}.
\]

(5)

In this, the most general ansatz for the gauge transformation corrections has the form

\[
\delta \Phi \sim \partial R \xi \oplus R \partial \xi \quad \delta h \sim R^2 \Phi \Phi \oplus \partial^2 \Phi \Phi \xi.
\]

Note that we will not introduce corrections to the gauge transformations containing higher derivatives of the gauge transformation parameter \( \xi \). Such terms (in general) could change the whole structure of constraints and, as a result, change the number of physical degrees of freedom.

There are a number of ambiguities which arise in such a straightforward construction (see, e.g., discussion in [17]):

- First of all, the interacting Lagrangian and gauge transformations are always defined up to possible field redefinitions, which do not change the physical content of the theory. In this particular case, we have a number of field redefinitions of the form

\[
\Phi \Rightarrow \Phi \oplus R \Phi.
\]

- Due to gauge invariance of free Lagrangians, gauge transformations in this approximation are defined up total divergency

\[
\delta \Phi \sim \partial (R \xi), \quad \delta h \sim \partial (R^2 \Phi \Phi \xi).
\]

- Working with higher derivative gauge transformations one always faces a number of trivial symmetries, i.e. gauge transformations leaving a sum of free Lagrangians invariant, which do not correspond to any non-trivial interactions [18].

- At last, not all the terms in the interacting Lagrangians which could be constructed are independent because as usual such Lagrangians are defined up to total divergency and there are many groups of terms which could be combined into total divergency.
We resolve these ambiguities in a threefold way. First of all, we try to lower the number of derivatives in the equations of motions. In general, working with four derivative Lagrangians we get equations containing up to the four derivatives on the fields, but in this case we managed to bring resulting Lagrangian into the form when equations contain at most three derivatives. Then we use remaining freedom to make gauge transformations as simple as possible. At last, some freedom that still remains we will use in the following section to bring two derivative terms into the form of the ‘standard’ gravitational interaction.

Now we require that the total Lagrangian be invariant under the corrected gauge transformations up to the terms bilinear in fields and also that algebra of gauge transformations be closed in the lowest order. The final Lagrangian (a result of very lengthy calculations which we will not reproduce here) has the form

\[ 48M^3 \mathcal{L}_1 = 8R_{\mu\nu,\rho\sigma}[\partial^\mu \Phi^{\nu\rho\sigma} \partial^\rho \Phi^{\mu\sigma} + 2\partial^\mu \Phi^{\rho\sigma\nu} \partial^\rho \Phi^{\sigma\mu\nu} - \partial^\rho \Phi^{\sigma\mu\nu} \partial^\rho \Phi^{\sigma\mu\nu} - \partial^\sigma \Phi^{\mu\rho\nu} \partial^\rho \Phi^{\mu\sigma\nu} - \partial^\rho \Phi^{\mu\sigma\nu} \partial^\rho \Phi^{\sigma\mu\nu}]
\]

\[ + 4R_{\mu\nu}[\partial^\rho \Phi^{\mu\nu\rho} - \partial^\rho \Phi^{\mu\nu\rho} + 6\partial^\mu \Phi^{\rho\sigma\nu} \partial^\sigma \Phi^{\mu\rho\nu} - 2\partial^\mu \Phi^{\rho\sigma\nu} \partial^\rho \Phi^{\mu\sigma\nu} + \partial^\mu \Phi^{\rho\sigma\nu} \partial^\rho \Phi^{\mu\sigma\nu} - 4\partial^\mu \Phi^{\rho\sigma\nu} \partial^\rho \Phi^{\mu\sigma\nu} - 6\partial^\mu \Phi^{\rho\sigma\nu} \partial^\rho \Phi^{\mu\sigma\nu} - 10\partial^\mu \Phi^{\rho\sigma\nu} \partial^\rho \Phi^{\mu\sigma\nu}]
\]

\[ + 4\partial^\mu \Phi^{\rho\sigma\nu} \partial^\rho \Phi^{\mu\sigma\nu} + 9\partial^\mu \Phi^{\rho\sigma\nu} (\partial \Phi) - \partial^\mu \Phi^{\rho\sigma\nu} (\partial \Phi)
\]

\[ - 2(\partial \Phi)^{\mu\nu} \partial^\rho \Phi^{\mu\nu} + 2(\partial \Phi)^{\mu\nu} \partial^\rho \Phi^{\mu\nu} + 12(\partial \Phi)^{\mu\nu} \partial^\rho \Phi^{\mu\nu} + 4(\partial \Phi)^{\mu\nu} \partial^\rho \Phi^{\mu\nu} - 9(\partial \Phi)^{\mu\nu} \partial^\rho \Phi^{\mu\nu}]
\]

(6)

(let us stress once again that it is not in any way unique). Here, using the fact that a cubic four derivative vertex for bosonic fields must have dimension \(1/m^3\), we introduce the characteristic scale \(M\). In this, appropriate corrections to gauge transformations look as follows:

\[ 6M^3 \delta \Phi^{\mu\nu} = 2R_{\rho\sigma,\mu\nu}[\partial_{\rho} \eta_{\sigma\mu\nu}] - g_{\mu\nu} R_{\rho\sigma,\sigma\mu\nu}\partial^\rho \eta_{\sigma\mu\nu} + (\mu\nu\alpha)
\]

\[ 3M^3 \delta h_{\mu\nu} = [\partial_{\mu} \partial_{\nu} \Phi^{\rho\sigma\nu} - \partial_{\mu} \partial_{\nu} \Phi^{\rho\sigma\nu} - \partial_{\mu} \partial_{\nu} \Phi^{\rho\sigma\nu}] \partial^\rho \eta_{\sigma\mu\nu}^\rho.
\]

(7)

(8)

Note that the \(h\)-transformations have exactly the same form as in [11], while \(\Phi\)-ones differ by total divergency. Thus, we indeed reproduce cubic vertex of this paper in the form very well suitable for investigations of gravitational spin 3 interactions.

3. Deformation to \((A)dS\) space

In this section, we consider the deformation of a four derivative vertex presented above into \((A)dS\) space. \((A)dS\) space is a constant curvature space without torsion or non-metricity, so the main difference from Minkowski space is that (covariant) derivatives do not commute any more. Our conventions will be

\[ [D_{\mu}, D_{\nu}] \omega_\alpha = -\kappa (g_{\mu\alpha} v_\nu - g_{\nu\alpha} v_\mu), \quad \kappa = \frac{2\Lambda}{(d-1)(d-2)}
\]

(9)

For simplicity, in this section we restrict ourselves with \(d = 4\) space. Non-commutativity of derivatives leads to some ambiguity even in the form of free Lagrangians and also requires

\[ ^{1}\text{As far as we know, for the first time this task was considered by M A Vasiliev in the middle of 1980s (unpublished).} \]
addition to Lagrangian some mass-like terms to keep gauge invariance intact. We will use the following concrete form for the massless spin 3 Lagrangian in (A)dS_4 space

\[ L_0 = - \frac{1}{2} D^\mu \Phi^\alpha \beta D_\mu \Phi_{\alpha \beta} + \frac{3}{2} (D \Phi)^{\mu \nu} (D \Phi)_{\mu \nu} - 3 (D \Phi)^{\mu \alpha} D_\mu \dot{\Phi} \]

as well as the following one for a massless spin 2 particle:

\[ L_0 = \frac{1}{2} D^\mu h^{\alpha \beta} D_\mu h_{\alpha \beta} - \frac{1}{2} (D h)^{\mu} (D h)_{\mu} - \frac{1}{2} D^\mu h^{\alpha \beta} D_\alpha h_{\mu \beta} + (D h)^{\mu} D_\mu h - \frac{1}{2} D^\mu h D_\mu h + \kappa (h^{\alpha \beta} h_{\mu \nu} - h^2). \]

Next, we will use the following generalization of the linearized Riemann tensor:

\[ 2 R^{\alpha \beta, \mu \nu} = D^\mu D^\alpha h^\nu \beta - D^\nu D^\alpha h^\mu \beta - D_\mu D_\beta h_{\alpha \beta} + D_\nu D_\beta h_{\alpha \beta} + D_\alpha D^\nu h_{\mu \beta} - D_\alpha D^\mu h_{\nu \beta} + 2 \kappa (g^{\mu \alpha} h_{\nu \beta} - g^{\nu \alpha} h_{\mu \beta} - g^{\mu \beta} h_{\nu \alpha} + g^{\nu \beta} h_{\mu \alpha}). \]

Besides being gauge invariant under covariant gauge transformations \( \delta h_{\mu \nu} = D_\mu \xi_\nu + D_\nu \xi_\mu \), such definition allows us to keep usual properties of the Riemann tensor such as

\[ R^{\alpha \beta, \mu \nu} = R^{\alpha \beta, \nu \mu}, \quad D^\mu R^{\alpha \beta, \mu \nu} = D_\mu R^{\alpha \beta, \nu \mu} - D_\nu R^{\alpha \beta, \mu \mu}. \]

At last, there is an ambiguity in the generalization of h-transformations containing second derivatives of \( \Phi \), but there is no evident choice here. So we leave this transformation as is and take this ambiguity into account introducing explicit corrections of the form \( \delta h \sim \Phi D^2 \xi \) later on.

As usual, if we replace all derivatives in our four derivative vertex and gauge transformations by the covariant ones, we lose gauge invariance. This non-invariance appears due to non-zero commutator of covariant derivatives and as a result contains lower derivative terms. Explicitly it looks as follows (again this results from some lengthy calculations we omit here):

\[ 6 \frac{M^3}{\kappa} \delta L_1 = D_\mu R_{\alpha \beta \mu \nu} (14 \Phi^{\alpha \beta \mu \nu} - 28 \Phi^{\mu \alpha \nu} \xi^\beta + 20 \Phi^{\mu} \xi^{\alpha \beta} + 4 \Phi^{\mu} \xi^{\mu \beta}) \]

\[ - 16 D_\mu R_{\alpha \beta \mu \nu} \Phi^{\alpha \beta \mu \nu} + D_\mu R (4 \Phi^{\mu \alpha \nu} \xi^\alpha - 11 \Phi_{\nu} \xi^{\alpha \nu}) + R_{\nu \alpha \beta \mu \nu} (10 \Phi^{\alpha \beta \mu \nu} - 40 \Phi^{\beta \mu \nu} \xi_{\alpha \beta} + 42 \Phi^{\mu} \xi^{\alpha \beta} \xi^\nu) + R_{\mu \nu} (28 \Phi^{\alpha \beta \mu \nu} \xi_{\alpha \beta} - 24 \Phi^{\mu} \Phi^{\alpha \beta \nu} \xi_{\alpha \beta} + 4 D_{\mu} \Phi^{\alpha \beta \mu \nu} + 4 D_{\mu} \Phi^{\alpha \beta \mu \nu} \xi_{\alpha \beta} - 24 \Phi^{\mu \alpha \nu} \xi_{\mu \nu} + 11 (\Phi^{\mu \alpha \nu} \xi_{\mu \nu}) + R (18 \Phi^{\mu \alpha \nu} \xi_{\mu \nu} - 22 \Phi^{\mu \alpha \nu} \xi_{\mu \nu}). \]

Now to compensate this non-invariance we have to introduce lower derivative corrections to cubic vertex as well as to gauge transformations. From the last formula we see that ‘effective’ gravitational coupling constant turns out to be \( \kappa / M^3 \sim 1/\mu_p^2 \). In what follows we set this constant to be 1. We proceed as follows. First of all, we introduce all corrections which appear when one replaces background metric by the ‘dynamical’ one:

\[ g_{\mu \nu} \rightarrow g_{\mu \nu} + h_{\mu \nu}. \]

In particular, this requires generalization of tracelessness condition for the parameter \( \xi_{\mu \nu} \):

\[ \xi_{\mu \mu} = 0 \rightarrow \tilde{\xi}_{\mu \mu} - h^{\mu \nu} \xi_{\mu \nu} = 0. \]

Then we replace all derivatives by the ‘fully’ covariant ones, e.g.

\[ D_\mu \xi_{\nu \alpha} \rightarrow D_\mu \xi_{\nu \alpha} - \Gamma_\nu^\beta \xi_{\mu \alpha} - \Gamma_\mu^\beta \xi_{\nu \alpha} \]

(16)
where we introduce linearized Christoffel symbols
\[ \Gamma_{\mu\nu\alpha} = \frac{1}{2} (D_\mu h_{\nu\alpha} + D_\nu h_{\alpha\mu} - D_\alpha h_{\mu\nu}). \] (17)
Also we introduce transformations for the \( \Phi \) field corresponding to standard general coordinate transformations for the third rank tensor
\[ \delta \Phi_{\mu\nu\alpha} = \xi^\beta D_\beta \Phi_{\mu\nu\alpha} + D_\mu \xi^\beta \Phi_{\nu\alpha\beta} + D_\nu \xi^\beta \Phi_{\alpha\beta\mu} + D_\alpha \xi^\beta \Phi_{\mu\nu\beta}. \] (18)
Besides these corrections reproducing (in the linear approximation) the standard gravitational interaction for a massless spin 3 particle we need two more corrections to achieve full gauge invariance under both \( \xi_{\mu\nu} \) and \( \xi_\mu \) transformations. At first, we have to add non-minimal terms
\[ \Delta L = -4 R_{\mu\nu\alpha\beta} \Phi^{\mu\rho\sigma} + \frac{1}{8} R (14 \Phi_{\mu\nu\alpha}^2 - 19 \tilde{\Phi}_\mu^2 \) \] (19)
At last, we need some corrections to \( h \)-transformations we mentioned earlier
\[ \delta h_{\mu\nu} = - \Phi_{\mu\nu\beta} D_\beta \xi^\alpha + (\mu \leftrightarrow \nu) \] (20)
4. A gauge invariant description of massive spin 3

In what follows we will use a gauge invariant description of massive spin 3 particles in Minkowski space [13], though it is possible to generalize these results to (A)dS space as well [14]. Besides the third rank tensor \( \Phi_{\mu\nu\alpha} \) we need three more fields now: the symmetric second rank tensor \( f_{\mu\nu} \), vector \( A_\mu \) and scalar \( \phi \). We begin with the sum of massless Lagrangians for all four fields (we use non-canonical normalization of kinetic terms to simplify mass terms and gauge transformations)
\[ \mathcal{L}_{02} = -\frac{1}{2} \partial^\mu \Phi^{\rho\sigma} \partial_\mu \Phi_{\rho\sigma} + \frac{3}{2} \partial^\mu \Phi^{\rho\nu} \partial_\mu \Phi_{\rho\nu} - 3 \partial^\mu \Phi^{\rho\nu} \partial_\nu \Phi_{\rho} + \frac{3}{2} \partial^\mu \Phi^{\rho\nu} \partial_\nu \Phi_{\rho} + \frac{3}{4} \partial^\mu \Phi \partial_\mu \Phi \] (21)
here and in what follows \( d \geq 4 \) and their usual gauge transformations
\[ \delta_{01} \Phi_{\mu\nu\alpha} = \partial_\mu (\xi_{\nu\alpha}), \quad \delta_{01} f_{\mu\nu} = \partial_\mu (\xi_{\nu}), \quad \delta_{03} A_\mu = \partial_\mu (\xi). \] (22)
In this and in the following section, we use two index notations for Lagrangians and gauge transformations where the first index denotes approximation (0—free theory, 1—the linear approximation and so on), while the second index shows the number of derivatives. To obtain the gauge invariant description of a massive spin 3 particle we introduce cross terms with one derivative (here we use the results of [14] in slightly different normalization)
\[ \frac{1}{m} \mathcal{L}_{01} = \frac{3}{2} [2 \Phi^{\mu\alpha} \partial_\mu f_{\nu\alpha} - 4 \Phi^{\mu\nu} \partial_\mu \Phi] + \frac{12(d + 1)}{d} (f^{\mu\nu} \partial_\mu A_\nu - f (\partial A)) - \frac{36(d + 1)^2}{(d - 2)^2} A_\mu \partial_\mu \phi \] (23)
as well as appropriate mass terms into the Lagrangian
\[ \frac{1}{m^2} \mathcal{L}_{00} = \frac{1}{2} \Phi_{\mu\nu\alpha}^2 - \frac{3}{2} \Phi^{\mu\nu} + \frac{6(d + 1)}{d} \Phi^{\mu} A_\mu - \frac{6(d + 1)(d + 2)}{d^2} A_\mu^2 \] (24)
and the following corrections to gauge transformations:

\[ \delta_{00}\Phi_{\mu\nu} = \frac{2m}{d}(g_{\mu\nu}\xi_\alpha + g_{\mu\alpha}\xi_\nu + g_{\nu\alpha}\xi_\mu), \quad \delta_{00}A_\mu = m\xi_\mu \]

\[ \delta_{00}f_{\mu\nu} = m\xi_{\mu\nu} + \frac{4(d+1)}{d(d-1)}mg_{\mu\nu}\xi, \quad \delta_{00}\phi = m\xi. \]

Let us recall a relation of such gauge invariant description with the usual one. Using gauge transformations with parameters \( \xi_{\mu\nu}, \xi_\mu \) and \( \xi \) one can always choose a gauge where \( f_{\mu\nu} = 0, A_\mu = 0 \) and \( \phi = 0. \) This leaves us with the main field \( \Phi_{\mu\nu\alpha} \) and one scalar auxiliary field \( f \) (the trace of \( f_{\mu\nu} \)) with the Lagrangian

\[ L = -\frac{1}{2}\partial_{\mu}\Phi^{\nu\rho\sigma}D_{\nu}\Phi^{\rho\sigma} + \frac{3}{2}(\partial\Phi)^{\nu\rho}(\partial\Phi)_{\nu\rho} - 3(\partial\Phi)^{\nu\rho}\partial_{\nu}\Phi^{\rho} - 2f^{\mu\nu}\xi^{\nu} - 2D_{\nu}f^{\mu\nu}\xi_{\nu}. \]

Let us recall a relation of such gauge invariant description with the usual one. Using gauge transformations with parameters \( \xi_{\mu\nu}, \xi_\mu \) and \( \xi \) one can always choose a gauge where \( f_{\mu\nu} = 0, A_\mu = 0 \) and \( \phi = 0. \) This leaves us with the main field \( \Phi_{\mu\nu\alpha} \) and one scalar auxiliary field \( f \) (the trace of \( f_{\mu\nu} \)) with the Lagrangian

\[ L = -\frac{1}{2}\partial_{\mu}\Phi^{\nu\rho\sigma}D_{\nu}\Phi^{\rho\sigma} + \frac{3}{2}(\partial\Phi)^{\nu\rho}(\partial\Phi)_{\nu\rho} - 3(\partial\Phi)^{\nu\rho}\partial_{\nu}\Phi^{\rho} - 2f^{\mu\nu}\xi^{\nu} - 2D_{\nu}f^{\mu\nu}\xi_{\nu}. \]

Now if we move to a non-trivial gravitational background\(^2\) satisfying \( R_{\mu\nu} = 0, \)

\[ \left[ D_{\mu}, D_{\nu}\right]v_\alpha = R_{\mu\nu,\alpha\beta}v_\beta \]

we lose gauge invariance

\[ \delta L_0 = 3R_{\mu\nu,\alpha\beta}(4D^{\nu}\Phi^{\mu\rho^\sigma\phi\sigma} - 2(D\Phi)^{\mu\rho^\sigma\phi\sigma} + 4D^{\nu}\Phi^{\alpha\rho^\sigma\phi\sigma} - 2f^{\mu\alpha}\xi^{\phi\sigma} - 2D_{\nu}f^{\mu\alpha}\xi^{\phi}). \]

One could try to restore gauge invariance by adding non-minimal terms, which in this simple case are just

\[ \Delta L_0 = R_{\mu\nu,\alpha\beta}(b_1\Phi^{\mu\alpha\rho}\Phi^{\nu\rho\phi} + b_2f^{\mu\alpha}\nu\phi), \]

but it is easy to check that it is impossible.

5. Massive spin 3 in a gravitational background

In this section, we show that introducing higher derivative corrections to Lagrangian and gauge transformations it is possible to obtain the gauge invariant description of the massive spin 3 particle in an arbitrary gravitational background satisfying \( R_{\mu\nu} = 0. \) By analogy with the massless case we restrict ourselves to terms containing not more than two explicit derivatives (here we treat \( R_{\mu\nu,\alpha\beta} \) as just an external background field).

We start by investigating all possible two derivative vertexes containing the full four-indexed Riemann tensor and any pair of our four fields: \( \Phi_{\mu\nu\alpha}, f_{\mu\nu}, A_\mu \) and \( \phi. \)

**Vertex** \( RD\Phi D\Phi. \) Surely we have to introduce this vertex which was crucial for the massless case. Here we need only the part containing the full Riemann tensor

\[ 2m^2L_{12} = a_0R_{\mu\nu,\alpha\beta}[D^\alpha\Phi^{\rho^\sigma\phi\sigma}D^\beta\Phi^{\rho^\sigma\phi\sigma} + 2D^\alpha\Phi^{\rho^\sigma\phi\sigma}D^\beta\Phi^{\rho^\sigma} - D^\beta\Phi^{\rho^\sigma\mu\alpha}D^\alpha\Phi^{\rho^\sigma\phi\sigma} - D^\alpha\Phi^{\rho^\sigma\mu\alpha}D^\beta\Phi^{\rho^\sigma\phi\sigma} + (D\Phi)^{\mu\alpha}(D\Phi)^{\nu\beta} - 3(D\Phi)^{\mu\alpha}D^\alpha\Phi^{\nu\beta}]. \]

and \( \Phi \) field gauge transformations

\[ m^2\delta_{11}\Phi_{\mu\nu} = a_0\left[R_{\mu\nu,\rho\sigma}D_{\rho\eta\sigma\phi} - \frac{2}{d}g_{\mu\nu}R_{\rho\sigma\phi\sigma}D^\phi\eta^{\phi\phi} + (\mu\nu\phi)\right]. \]

\(^2\) For massive spin 2 in a gravitational background see [19–21].
Vertex $R \Phi D^2 A$. Complete investigation of this four derivative $3 - 2 - 1$ vertex will be given elsewhere, while the part containing the full Riemann tensor only is relatively easy to construct. It has a form

$$m^2 \Delta_1 L_{12} = \frac{6b_0(d + 1)}{d} R_{\mu \nu, \alpha \beta} [2 \Phi^{\alpha \rho} D^\rho D_\beta A^\alpha - 2 \Phi^{\mu \alpha \rho} D^\rho D_\alpha A^\beta - D^\mu D^\rho A^\alpha A^\gamma \Phi^\beta]$$

and trivially invariant under $A_\mu$ field gauge transformations, while invariance under $\Phi$ gauge transformations requires that

$$m^2 \delta_{11} A_\mu = b_0 R_{\mu \nu, \alpha \beta} D^\alpha \xi_\beta.$$  

At last, there are three more cubic vertexes which are trivially gauge invariant

$$m^2 \Delta_2 L_{12} = R_{\mu \nu, \alpha \beta} [d_0 (4 F^{\mu \nu, \alpha \rho} f_\beta \rho - 4 F^{\mu \alpha} f_\nu \beta - F^{\mu \nu, \alpha \beta} f) + c_0 F^{\mu \nu, \alpha \beta} \phi + e_0 D^\mu A^\nu D^\alpha A_\beta]$$

Here $F_{\mu \nu, \alpha \beta}$ is the linearized 'curvature' tensor for the $f_{\mu \nu}$ field. The vertex with coefficient $c_0$ is an analog of Gauss–Bonnet vertex multiplied by the scalar field but with two different spin 2 fields. The one with coefficient $d_0$ also comes as the first non-trivial term from such double spin-2 Gauss–Bonnet vertex. Note, that in $d = 4$ this vertex is a total divergency and as a result, as we will see at the end of this section, this $d = 4$ case is indeed special. The last vertex (with coefficient $e_0$) is rather well known one containing two gauge invariant field strengths for vector field $A_\mu$.

Thus we have achieved cancellation of all variations with three derivatives

$$\delta_{01} L_{12} + \delta_{11} L_{02} = 0.$$ 

But when $m \neq 0$ we get variations with two derivatives coming from

$$\delta_{10} L_{12} + \delta_{11} L_{01} \neq 0.$$ 

So we proceed introducing all possible cubic terms with one derivative into the Lagrangian

$$m L_{11} = R_{\mu \nu, \alpha \beta} [a_1 \Phi^{\mu \rho} D_\rho f_\beta \beta + a_2 \Phi^\beta D^\nu f^{\mu \alpha} + a_3 \Phi^{\mu \alpha \rho} D^\nu f^{\beta \rho} + a_4 (D \Phi)^{\mu \alpha \rho} f^{\beta \rho} + a_5 f^{\mu \alpha} f^{\nu \beta} A^\alpha A^\beta]$$

as well as the only possible correction to gauge transformations

$$m \delta_{10} f_{\mu \nu} = c_1 R_{\mu \alpha, \nu \beta} \xi^{\alpha \beta}.$$ 

Now variations with two derivatives also come from $\delta_{01} L_{11}$ as well as from $\delta_{10} L_{02}$. Then requiring that all variations with two derivatives cancel

$$\delta_{00} L_{12} + \delta_{11} L_{01} + \delta_{01} L_{11} + \delta_{10} L_{02} = 0,$$

we obtain a number of relations on the coefficients

$$b_0 = \frac{(d - 2)a_0 + 8dd_0}{6(d + 1)}, \quad c_0 = -6a_0 - \frac{48d}{d - 2} d_0,$$

$$e_0 = \frac{2(d + 4)(d - 1)}{d^2} a_0 + \frac{32(d + 2)}{d} d_0$$

$$a_1 = a_0 + 16d_0, \quad a_2 = -3a_0 - 8d_0, \quad a_3 = -a_0 - 16d_0, \quad a_4 = \frac{3}{2} a_0, \quad a_5 = \frac{16(5d + 4)}{d} d_0, \quad c_1 = \frac{a_0}{2} - 8d_0.$$ 

At last, to achieve cancellation of remaining variations with one derivative and without derivatives we add the last correction to the Lagrangian

$$L_{10} = R_{\mu \nu, \alpha \beta} (b_1 \Phi^{\mu \alpha \rho} f^{\nu \beta \rho} + b_2 f^{\mu \alpha} f^{\nu \beta}).$$
And indeed all variations cancel provided
\[ b_1 = 2, \quad b_2 = \frac{9}{2}, \quad a_0 = -4, \quad d_0 = \frac{1}{4}. \]

Collecting all pieces together we obtain the final cubic vertex
\[
m^2 \mathcal{L}_1 = R_{\mu\nu,\alpha\beta} \left\{ -4(D^\nu \Phi^{\rho\sigma}) D^\sigma \Phi^{\rho\sigma} + 2 D^\nu \Phi^{\rho\sigma} D^\rho \Phi^{\sigma\beta} - D^\rho \Phi^{\rho\mu\alpha} D^\sigma \Phi^{\rho\nu\beta} - D^\nu \Phi^{\rho\mu\alpha} D^\rho \Phi^{\rho\nu\beta} + (D \Phi)^{\mu\alpha} (D \Phi)^{\nu\beta} - 3 (D \Phi)^{\mu\alpha} D^\nu \Phi^{\sigma\beta} \right\}
\]
\[
- \frac{2(d - 4)}{d} \left[ 2 \Phi^{\mu\rho\sigma} D^\rho A^\sigma - 2 \Phi^{\mu\rho\alpha} D^\rho A^\sigma - D^\nu A^\rho \tilde{\Phi}^{\mu\alpha\beta} - D^\nu A^\rho \tilde{\Phi}^{\mu\beta\alpha} \right]
\]
\[
+ 4(d - 4) \left[ \frac{3}{(d - 2)} F^{\mu\nu,\alpha\beta} - \frac{2}{d^2} D^\alpha A^\nu A^\beta \right]
\]
\[
+ \left[ F^{\mu\nu,\alpha\beta} - F^{\mu\alpha} f^{\nu\beta} - \frac{1}{4} F^{\mu\nu,\alpha\beta} f \right]
\]
\[
+ 2m \left[ 5 \Phi^{\rho\sigma} D^\rho f^{\mu\alpha} + 3 (D \Phi)^{\mu\alpha} f^{\nu\beta} - \frac{2(d - 4)}{d} f^{\mu\alpha} A^\nu A^\beta \right]
\]
\[
+ m^2 \left( 2 \Phi^{\mu\rho\sigma} f^{\nu\rho\sigma} + \frac{9}{2} f^{\mu\alpha} f^{\nu\beta} \right). \tag{38}
\]

Note that for \( d = 4 \) this result turns out to be much simpler
\[
m^2 \mathcal{L}_1 = R_{\mu\nu,\alpha\beta} \left\{ -4(D^\nu \Phi^{\rho\sigma}) D^\sigma \Phi^{\rho\sigma} + 2 D^\nu \Phi^{\rho\sigma} D^\rho \Phi^{\sigma\beta} - D^\rho \Phi^{\rho\mu\alpha} D^\sigma \Phi^{\rho\nu\beta} - D^\nu \Phi^{\rho\mu\alpha} D^\rho \Phi^{\rho\nu\beta} + (D \Phi)^{\mu\alpha} (D \Phi)^{\nu\beta} - 3 (D \Phi)^{\mu\alpha} D^\nu \Phi^{\sigma\beta} \right\}
\]
\[
- \frac{2(d - 4)}{d} \left[ 2 \Phi^{\mu\rho\sigma} D^\rho A^\sigma - 2 \Phi^{\mu\rho\alpha} D^\rho A^\sigma - D^\nu A^\rho \tilde{\Phi}^{\mu\alpha\beta} - D^\nu A^\rho \tilde{\Phi}^{\mu\beta\alpha} \right]
\]
\[
+ 4(d - 4) \left[ \frac{3}{(d - 2)} F^{\mu\nu,\alpha\beta} - \frac{2}{d^2} D^\alpha A^\nu A^\beta \right]
\]
\[
+ \left[ F^{\mu\nu,\alpha\beta} - F^{\mu\alpha} f^{\nu\beta} - \frac{1}{4} F^{\mu\nu,\alpha\beta} f \right]
\]
\[
+ 2m \left[ 5 \Phi^{\rho\sigma} D^\rho f^{\mu\alpha} + 3 (D \Phi)^{\mu\alpha} f^{\nu\beta} - \frac{2(d - 4)}{d} f^{\mu\alpha} A^\nu A^\beta \right]
\]
\[
+ m^2 \left( 2 \Phi^{\mu\rho\sigma} f^{\nu\rho\sigma} + \frac{9}{2} f^{\mu\alpha} f^{\nu\beta} \right). \tag{39}
\]

6. Conclusion

Thus it is indeed possible (at least at the linear approximation) to extend the gauge invariant description of massive spin 3 particles from constant curvature spaces (such as flat Minkowski space or \((A)dS\)) to an arbitrary gravitational background satisfying \( R_{\mu\nu} = 0 \). Let us stress that this is a first but necessary step for the construction of gravitational interactions for such massive spin 3 particles. It is important that the main four derivatives 3–3–2 vertex we need here is exactly the same that one needs for the construction of the gravitational interaction for massless spin 3 particles in \((A)dS\) space. Thus we hope that the gauge invariant description of massive particles is a right tool for the investigation of the mechanism of spontaneous gauge symmetry breaking that allows us to deform massless higher spin theories in \((A)dS\) spaces into massive theories in Minkowski space.

It is instructive to compare our results with those obtained from the requirement of tree level unitarity [22] (see also recent paper [23]). Due to large ambiguity one faces working with higher derivative interactions it is not an easy task to make a direct comparison of different formulations (compare, e.g., our Lagrangian (6) with the Lagrangian (4.28) in [11]). Note
only that the four derivative corrections given in [22] become zero then one puts a ‘gauge’
\((\partial/\Phi_1)_{\mu\nu} = 0\) and \(\tilde{\Phi}_\mu = 0\), while it is not to be the case here. Note also that these corrections
have a universal form for all spins \(s \geq 3\), while the general construction for massless particles
in \((A)dS\) [8, 9] requires that the number of derivatives grows linearly with spin. And indeed,
it is not possible to construct a gauge invariant cubic \(s - s - 2\) vertex for spin \(s > 3\) with just
four derivatives, so the vertex \(3 - 3 - 2\) constructed in [11] and used here is specific namely
for \(s = 3\).

Note added. After the first version of this paper has been submitted into arXiv, there appeared the paper [24] where
the same \(3 - 2 - 2\) vertex was reconstructed in terms of the Weyl tensor (again due to usual ambiguities their Lagrangian
(14) differs from our Lagrangian (6)). Also an appropriate vertex for spin \(s = 4\) has been constructed.

Acknowledgment

Author is grateful to M A Vasiliev for stimulating discussions and correspondence.

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