AN OPTIMAL CONTROL PROBLEM OF MONETARY POLICY

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Abstract. This paper analyses an optimal monetary policy under a non-linear Phillips curve and linear GDP dynamics. A central bank controls the inflation and the GDP trends through the adjustment of the interest rate to prevent shocks and deviations from the long-run optimal targets. The optimal control path for the monetary instrument, the interest rate, is the result of a dynamic minimization problem in a continuous-time fashion. The model allows considering various economic dynamics ranging from hyperinflation to disinflation, sustained growth and recession. The outcomes provide useful monetary policy insights and reveal the dilemma between objectives faced by the monetary authority in trade-off scenarios.

1. Introduction. Most macroeconomists and central bankers agree that the main aim of monetary policy should be to control long-run inflation (e.g. [22, 21, 13]). However, many also believe that monetary policy can have a short-run role in helping to stabilize business cycles (e.g. [16, 14]). Thus, the ultimate target of a Central Bank (hereinafter CB) is often a mixed-signal based both on inflation and the business cycle, generating a potential conflict between short-run and long-run goals.

One of the objectives of the monetary authority is to minimize any deviation, or gap, of the gross domestic product (hereinafter GDP) from its long-term level, i.e. the potential output. As GDP can rise or fall, the output gap can be either positive or negative. On the one hand, a positive output gap, occurs in periods characterized by particularly high levels of demand so that firms and workers operate close or even above their efficient capacity frontier to meet demand. This situation generates upward pressures on prices leading to a rise in inflation and subsequent negative consequences for the economy. Indeed, high inflation can lead to a spiral of increasing prices, limiting households purchasing power and making firms investment more complicated. On the other hand, a negative output gap occurs when the actual output is less than what an economy could potentially produce, given the technology and the production factors, among others. It means that there is a spare capacity or a slack in the economy due to a weak demand compared to the supply,
or to frictions and market imperfections. In this case, the economic downturn might drive down inflation, and the latter may create negative feedback loops with the real economy leading to a spiral of falling prices. The burden for debt servicing rises creating additional negative feedback loops between the real economy and the price level as long as firms and households postpone their investment and consumption decisions.

Output fluctuations are intrinsic to any economy and often they do not exhibit uniform or predictable dynamics. One of the major tasks of the monetary policy should be to mitigate these temporary downward and upward movements of GDP around its long-term growth trend through expansionary or contractionary monetary measures. Along with the so-called Keynesian counter-cyclical objective, the core aim of a central bank remains that of stabilizing prices in order to prevent whatsoever relevant deviation from the long-run inflation target.

The existing literature in this area has mainly focused on simple monetary policy rules that are generalizations of the Taylor Rule ([27]) and has not fully been drawn to the idea of applying optimal control theory to the problem of monetary policy, with few exceptions ([23, 17]). In this paper we improve with respect to the previous literature by addressing the issue of the impact of monetary policy on prices and output by means of a dynamic model based on optimal control theory to identify an appropriate monetary policy path.

We start by defining the aim of the CB as the minimization of a loss function that depends both on output and inflation gaps, the two main objectives. Then, we model GDP and inflation dynamics.

A key novelty of the model is the assumption of an augmented nonlinear Phillips curve, to take into account the possibility that the response of inflation to changes in the economic activity may be asymmetric. Indeed, recent studies have argued that the dynamics of inflation have changed substantially in many, if not all, advanced economies over the past four decades, showing that various types of structural changes have affected the statistical properties of inflation, making increasingly complex the modeling of inflation dynamics ([18]).

We obtain a dynamic system that presents several non-linearities, as it is formed by four differential equations, meaning that is a 4-th order or 4-th dimensional system. We explore the model dynamics by means of numerical simulations, and the main findings of the model can be described as follows. Firstly, our model allows considering different phenomena such as inflation and deflation as well as situations of sustained economic growth and periods of recession. Secondly, the model simulations well represent different economic scenarios and trade-off situations. Thirdly, the model reveals the CB difficulties to reach both the output and the inflation targets with only one instrument, i.e. the interest rate. Finally, the dynamics of the variables heavily depend on the value of the parameters, and the relative priority assigned by the CB to one target relative to the other.

The paper is organized as follows. In the next Section we describe the model, while Section 3 solves the optimal control problem. Section 4 analyzes the model dynamics employing numerical simulations, and Section 5 concludes.

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1Some of the pioneer works in this field are Chow (1976)[7], Tabellini (1986)[26], Svensson (1997)[25] and more recently Evans et al. (2001)[10], Bischi and Marimon (2001)[3], Ferrero (2007)[11], Orphanides and Williams (2008)[20], which addressed the issue even if with different methodologies.
2. The model. The CB minimizes both the output and the inflation gap, over a defined time period \([0, T]\), where 0 can be assumed as the time when the Board of governors is appointed, and T the end of the term.\(^2\) The two objectives can be conflicting not only in attainment but also in time.

Variables are expressed as percentage variation from one period of time to the other (i.e. years). Therefore, the current GDP rate of growth and inflation rate are defined, respectively, as follows:

\[
y_t = \frac{GDP_t - GDP_{t-1}}{GDP_{t-1}}, \quad \pi_t = \frac{p_t - p_{t-1}}{p_{t-1}}.
\]

Then, inflation and output gap at time \(t\) are defined as:

\[
\pi_t = \pi_t - \bar{\pi} \quad \text{and} \quad y_t = y_t - \bar{y},
\]

where \(\bar{\pi}\) and \(\bar{y}\) are the inflation target and the potential or natural output growth, which we assume exogenous to the model.

Similarly, the interest rate \(i\) is the monetary policy instrument (the control variable of the problem), and might be as well expressed in terms of deviation from the long-run optimal value \(\tilde{i}\) prevailing in the economy. Thus, \(i_t = i_t - \tilde{i}\), where the variable with tilde above indicates the current value of the interest rate at time \(t\).

2.1. The loss function. Tinberger (1956)[28] addressed the issue of the controllability of a fixed set of targets by a policymaker endowed with given instruments, stating that when targets exceed instruments, the system is overdetermined and the economic policy model might not allow for solutions. One way to overcome this problem is to reduce the number of targets and make them flexible. Thus, in our model, the authorities can reduce the two targets to a single one by channelling inflation and output into a single function, the ‘Loss function’ of the policymaker.

We assume a quadratic Loss Function specified as follows:\(^3\)

\[
L = \frac{1}{2} \alpha \pi^2 + \frac{1}{2} \beta y^2
\] (1)

In equation (1) inflation and output gaps are squared, so that positive and negative deviations are counted the same way. This is because, as stated in the introduction, both can be potentially detrimental for the economy.

However, \(\pi\) and \(y\) do enter the loss function with different weights, \(\alpha\) and \(\beta\), respectively. The latter parameters reflect the different degrees of preference by CB for the two main goals. Accordingly, \(0 \leq \alpha, \beta \leq 1\) always sum to 1, and can be interpreted as behavioral parameters that depend on the policy will of the Board of governors. In the next Sections, we assume that the political stance on the two final objectives, the control over inflation and the mitigation of output fluctuations, remains unchanged during the period the Board of governors is in charge.

If the CB ascribes more relevance to the price stabilization objective rather than the mitigation of output fluctuations, \(\alpha\) will be greater than 0.5; while \(\beta > 0.5\) if the economic stimulus becomes primary in the CB agenda. In the scenario in which

\(^2\)The Board of governors of a CB is the executive committee composed of senior members and responsible for the monetary policy stance of the bank. Generally, it remains in charge for a defined period of time that varies from 3 years up to 8 or 10 years, depending on the statute and regulation of each CB.

\(^3\)This loss function is known as LQR (linear quadratic regulator) for optimal control systems in engineering theory. It has been widely used also in economic dynamics, particularly in monetary policy problems. Some examples are Svensson (1997)[25], where the same loss function is used in a linear discrete-time optimal control problem and Tabellini (1986)[26] that models a linear-quadratic game between the fiscal and monetary authorities. In our case, we use this quadratic regulator (i.e. loss function), but with the additional complexity of a non-linear differential equation.
CB gives the same priority, \( \alpha \) and \( \beta \) are equal to 0.5, while in the extreme cases where only one objective is pursued at the expense of the other the weight is set to 1 (and the complementary will be zero, not entering the functional).

Aside from the monetary policy trade-off in terms of preference, the two variables under consideration are also linked to each other, as described in the next Sections.

2.2. The inflation dynamics: A non-linear Phillips curve. The differential equation that describes the evolution of \( \pi \) consists of two parts, as follows:

\[
\dot{\pi} = \gamma y^3 - \frac{1}{2} \omega i^2 - i \tag{2}
\]

with \( \gamma, \omega > 0 \).

The first part of equation 2 derives from the non-linear Phillips curve specification proposed by Filardo (1998)[12], which relates the acceleration of the inflation (\( \dot{\pi} \)) with the deviation of growth from its natural value (\( y \)) through a third-degree equation. When the output gap is \( y = 0 \), the economy is at its potential output growth \( \bar{y} \), that can be interpreted as the level of growth needed to maintain the economy at its optimal level of production given institutional and natural constraints. This level of growth is associated with the NAIRU, namely “Non-accelerating inflation rate of unemployment” (Okun, 1975). As the acronym means, for this particular economic equilibrium value the inflation rate does not accelerate, thus \( \dot{\pi} = 0 \), given a constant monetary policy.

The shape of this nonlinear Phillips curve implies that when the output gap is positive \( (y > 0) \), the Phillips curve is convex and when the output gap is negative \( (y < 0) \), the Phillips curve is concave, as shown in Figure 1.

The convex-concave Phillips curve implies that the cost of fighting inflation or disinflation rises when the output gap increases, both in positive and negative terms.\(^4\)

The positive parameter \( \gamma \) measures the cost of fighting inflation gaps. It can be also interpreted as the sensitivity of the rate of change of inflation to the output deviations, capturing the impact of the real variable \( y \) on prices.

In a Keynesian vision, we argue that the higher is the output and the associated aggregate demand in the short-run, the greater will be the upward pressure on prices. On the contrary, in a period characterized by a slowdown of growth, the uncertainty and the lack of trust of consumers and economic agents lead to a postponement of the decisions of consumption and investment, which results in a downward pressure on prices causing disinflation, or even deflation when the percentage change in prices becomes negative.

For positive output gaps the convex shape of the Phillips curve is consistent with an economy subject to capacity constraints. Indeed, as the economy becomes stronger and capacity constraints restrict firms’ ability to expand output, an increase in demand is more likely to show up as higher inflation than as higher output. Moreover, if firms have some market power, they could benefit of this growing aggregate demand setting increasingly higher prices and mark-ups on their final product, on the margin ([30, 8, 9]).

\(^4\)Filardo (1998)[12] explores the problem of the shape of the Phillips curve theoretically and empirically. He proposes that the curve is not purely convex or concave, but a combination of both, namely, the convex-concave curve.
For negative output gaps the concave shape of the Phillips curve describes an
economy where agents are not purely competitive. In this case, price-maker firms
will decide to reduce prices in order to stimulate demand of their products during
economic downturn ([24]).

The second part of equation 2 captures the impact of the monetary policy instru-
ment, the interest rate $i$, on the acceleration of inflation (Figure 2). This introduces
a second non-linearity given that the instrument enters in a quadratic form in differ-
ential equation 2.

The positive parameter $\omega > 0$ measures the strength of the monetary policy on
inflation variations, modifying the steepness of the parabola in Figure 2.

An interest rate gap equal to $i = 0$ means that the CB sets the interest rate
at the value $\bar{i}$ coherent to the long-run potential growth of the economy $\bar{y}$ and the
steady optimal inflation rate $\bar{\pi}$.

In accordance with the monetarist theory ([13]), we assume a negative relation-
ship between $\dot{\pi}$ and $i$. However, we postulate that the impact of a contractionary
or expansionary monetary policy on inflation is asymmetric. Indeed, the second
part of equation (2) captures the assumption that for the CB is relatively more
difficult/costly to stimulate a price increase with an expansionary measure than the
opposite (i.e restrictive policy). A monetary contractionary policy (an increase of
the interest rate over $\bar{i}$) means a positive interest rate gap ($i > 0$), which implies a
reduction of the amount of liquidity in circulation in the economy and, as a result,
weakens price level over long periods. A monetary expansionary policy (a reduction
of the interest rate under $\bar{i}$) means a negative interest rate gap ($i < 0$). This implies

Figure 1. $\gamma = 0.1$
a boost of the amount of liquidity in circulation in the economy and an increase of price level over long periods, even though with a lower magnitude than a restrictive policy.

The asymmetric effect is in line with what many advanced economies (especially Europe) have experienced in the aftermath of the financial crisis, where albeit very low interest rates set by the monetary institutions (almost near to minimum bound of zero), inflation remains at low values during the 2014-2017 period.

When the interest rate \( \tilde{i} \) reaches the minimum level called zero lower bound (ZLB), it cannot be further reduced. In this situation, the negative effect of a lower interest rate on bank profits may lead to a contraction in lending and economic activity ([5]).

In our model the ZLB is defined by the maximum (or apex) of the parabola depicted in Figure 2 (\( i_{\text{min}} \)). The parameter \( \omega \), acting on the shape of the parabola, affects both the value of \( i_{\text{min}} \) and \( \tilde{i} \). Coherently with the empirical evidence ([1]), an ultra expansive monetary measure, with a reduction of the interest rate near to zero lower bound \( i_{\text{min}} \), decreasingly boosts inflation \( \frac{\partial^2 \pi}{\partial i^2} < 0 \).

In conclusion, equation 2 allows us to embrace the two major macroeconomic contributions and theories that describe the inflation dynamics over time.

2.3. The GDP dynamics. We assume that the GDP dynamics depends on inflation and interest rate, as follows:
\[ \dot{y} = \eta \pi - \phi i \] 

The first term in equation 3 assumes that the price dynamics can linearly affect the GDP rate of growth, at least in the short run.

An inflation moderately above the reference \((\pi > 0)\), especially if not forecasted by economic agents, may represent a stimulus in the short-run to anticipate some decisions of consumption. In fact, in the cases of positive inflation deviations, it is more convenient for privates and firms to purchase goods, production factors and services now than later at higher prices. This extra consumption and investment leads to an increase in the aggregate demand and ultimately in GDP. In the cases of negative inflation deviations \((\pi < 0)\), the opposite happens: households postpone their consumption decisions waiting for a further decrease in prices, and firms, similarly, delay investment. The parameter \(\eta > 0\) measures the magnitude of this impact.

The second term of equation 3 captures the interaction between monetary and real variables. The traditional counter-cyclical role of the monetary policy indicates that a boost in the money supply (i.e. reduction of the interest rate, so \(i < 0\)) brings a stimulus to the economy. On the contrary, a monetary tightening (i.e. increase of the interest rate, so \(i > 0\)) causes a reduction in the aggregate demand and a slowdown of growth in the short-run. The parameter \(\phi > 0\) indicates the effectiveness of the monetary policy on output variations.

2.4. The minimization problem. The CB faces the following optimal control problem in which it minimizes the Loss Function (1) subject to inflation dynamics (2) and GDP dynamics (3):

\[ \min_i J(t) = \int_0^T \left( \frac{1}{2} \alpha \pi^2 + \frac{1}{2} \beta y^2 \right) dt \] 

subject to

\[ \dot{\pi} = \gamma y^3 - \frac{1}{2} \omega i^2 - i, \]

\[ \dot{y} = \eta \pi - \phi i \]

where the interest rate \(i\) is the control variable that influence the dynamics of both

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5 However, a clarification must be made. In the long-run, an inflation well over the target and persistent in time might be costly in term of growth because it creates uncertainty over relative prices generating resource misallocations and distortions in economic choices. Some of these costs involve the postponement of relevant investment’s decisions (i.e. a reduction of investment) that could impair the growth potential of an economy [(4)], thus highlighting a negative relationship between the variables.

Some empirical works have also tried to estimate this cost. Bruno and Easterly (1996)[6] have demonstrated that growth could relevantly decrease when inflation gap \(\pi\) is positive and over 20 percent values (hyperinflation), as witnessed by several Latin America developing economies during the early 90’s. Barro (1997)[2], similarly, has shown that for very high inflation values (> +10%) the lack of growth could be between 3 and 4 percentage points.
the state variables \( \pi \) and \( y \), and with the following initial and final conditions:

\[
\begin{align*}
\pi(0) &= \pi_0 & \pi(T) & \text{free} & \pi_0, & T \text{ given} \\
y(0) &= y_0 & y(T) & \text{free} & y_0, & T \text{ given}
\end{align*}
\]

Given that it is a free-terminal-state problem for the two state variables \( \pi \) and \( y \), we must define the transversality conditions needed to solve it. Such conditions only concern what happens at the terminal time \( T \) and involve the co-state variables:

\[
\lambda_1(T) = 0 \quad \lambda_2(T) = 0
\]

In this vertical-terminal-line problem, \( i \), namely the control variable, represents the principal instrument at disposal of CB to achieve the two main goals of stabilizing prices and stimulating growth in the short-run, especially in the event of unexpected negative shocks. The variation of nominal interest rate \( i \) is directly controlled by the monetary authority through open-market operations, interbank rate interventions, reserve requirements.

For the sake of realism, hereinafter, we assume 5 years as time span \( T \), an average between the duration in charge of the Board of governors of the main central banks (Federal Reserve 4 years, ECB 8 y., Bank of England 8 y., etc.).

3. The solution. The Hamiltonian of the optimal control problem is defined as:

\[
H(t, \pi, y, i, \lambda_1, \lambda_2) = \frac{1}{2} \alpha \pi^2 + \frac{1}{2} \beta y^2 + \lambda_1 (\gamma y^3 - \frac{1}{2} \omega i^2 - i) + \lambda_2 (\eta \pi - \phi i) \quad (5)
\]

We start by deriving (5) with respect to \( i(t) \) \(^6\), the control variable:

\[
\frac{\partial H}{\partial i} = 0 \quad -\omega \lambda_1 i - \lambda_1 - \phi \lambda_2 = 0
\]

\[
i^*(t) = -\frac{1}{\omega} - \frac{\phi}{\omega} \frac{\lambda_2}{\lambda_1}
\]

Substituting the optimal control (6) into the equation of motion for \( \pi \) in (4), after some algebrical steps we can obtain the following differential equation for the first state variable:

\[
\dot{\pi} = \gamma y^3 + \frac{1}{2} \omega - \frac{\phi^2}{2 \omega} \frac{\lambda_2}{\lambda_1^2}
\]

Considering the law of motion of \( y \) in (4) and repeating the substitution of the optimal control (6), we achieve the differential equation for the second state variable:

\[
\dot{y} = \eta \pi + \phi \frac{\lambda_2}{\lambda_1}
\]

The optimal co-state paths are derived from the following conditions:

\[
\dot{\lambda}_1 = -\frac{\partial H}{\partial \pi} = -\alpha \pi - \eta \lambda_2
\]

\(^6\)We should recall that the variables of the problem \( \pi, y, \lambda_1, \lambda_2, i \) are dynamical and hence function of time, for conciseness we decided to omit throughout the term \( t \).
We apply the Pontryagin’s maximum principle to obtain a system of four differential equations, two for the state variables, (7) and (8), and two for the associated co-state variables, (9) and (10):

\[
\begin{align*}
\dot{\pi} &= \gamma y^3 + \frac{1}{2\omega} - \frac{\phi^2}{2\omega} \frac{\lambda_2}{\lambda_1}^2 \\
\dot{y} &= \eta \pi + \frac{\phi^2}{2\omega} \frac{\lambda_2}{\lambda_1} \\
\dot{\lambda}_1 &= -\alpha \pi - \eta \lambda_2 \\
\dot{\lambda}_2 &= -\beta y - 3 \gamma \lambda_1 y^2
\end{align*}
\] (10)

with again the following initial and final conditions:

\[
\begin{align*}
\pi(0) &= \pi_0 & \pi(T) \text{ free} & \pi_0, T \text{ given} \\
y(0) &= y_0 & y(T) \text{ free} & y_0, T \text{ given}
\end{align*}
\]

and the transversality conditions associated to the two free-terminal state variables:

\[
\lambda_1(T) = 0 \quad \lambda_2(T) = 0.
\]

From the solution of this ordinary differential equation (ODE) system, it is possible to trace back the optimal paths for the two state variables \(\pi\) and \(y\), as well as for the two auxiliary variables \(\lambda_1\) and \(\lambda_2\). In addition, it allows us to define from equation 6 the optimal control path for \(i\). This path represents the optimal dynamic monetary instrument or measure implemented by the CB to drive, in time, the initial deviations/shocks of inflation and growth toward zero.

This system presents several non-linearities and is formed by four differential equations, meaning that is a 4-th order or 4-th dimensional system. Besides, the two transversality conditions on the lambdas make the solutions even more difficult to obtain. In mathematical research, these are known as boundary value problems (BVPs).

The long-run equilibrium of the economy is obtained for \(\pi = 0\), \(y = 0\) and \(i = 0\), when neither the GDP, nor the inflation accelerate (i.e. they remain steady at their pre-fixed target rate of growth), and the CB pursue the optimal long-run monetary policy. Consequently, we are interested in disequilibrium situations where for some reasons (shocks, business cycle fluctuations etc.) inflation and GDP growth are not in the long-run steady-state, calling for an intervention of the CB to restore the long-run trends of the economy.

Therefore, in the next section, we provide numerical simulations to try to understand the behavior of the model and to cover different economic scenarios.\(^7\)

\(^7\)Numerical simulations are performed by means of an algorithm in the software Matlab. In particular, the function 'bvp5c' allows to perform numerical computations for solving (non-linear) ODE system in all those cases in which there are final/transversality conditions on the variables, by using an educated guess of their starting values. These algorithms work for successive approximation through thousands iterations of the derivatives at each point of time (Runge-Kutta methods). As for all the numerical computations the results are not exact, nevertheless, it is possible to achieve on average an accuracy of 0.01 percent with an estimated maximum error lower than 0.02 percent.
4. Dynamics and numerical simulations. As previously mentioned, the two main objectives for a monetary policy are to control the evolution of prices, avoiding any detrimental effect caused by acceleration or deceleration of inflation in the long-run, and, at the same time, to influence the macro-economic condition in the short-run. This latter function is now widely recognized among economists and policymakers. It has been pursued by the vast majority of central banks in the aftermath of the Great Recession and also recently, during the Coronavirus recession, to alleviate the financial and economic distress through a massive injection of liquidity in the economy and extraordinary expansive monetary measures.

As a consequence, it becomes interesting to analyze situations characterized by an economy located far from its long-run employment and growth equilibrium. This might occur frequently due to the number of shocks and cyclical fluctuations typical of the economic cycle. Therefore, in this section, we provide some numerical simulations to study the effects of different economic conditions and/or monetary policies on the dynamics of the economic system.

To this purpose, we start by defining inflation and output targets, taking the Eurozone as a reference area. The European Central Bank (ECB) aims at inflation rates below but close to 2% over the medium term, whereas a plausible natural value for annual growth in the Eurozone might be in a range between 1% and 2%. The target nominal interest rate may be associated with the long-run inflation target and the natural rate of growth.

These optimal values for inflation, nominal interest rate and output can be interpreted as the long-run optimum in absence of any shock in the economy. Coherently, in the next figures, all variables are intended as deviations from these target values ($\bar{\pi}, \bar{y}, \bar{i}$), here normalized at zero for convenience.

In the following simulations, we assume that at time $t = 0$ a shock affects the economy, so that it moves from its long-run path. Then, we study the proper dynamic monetary response in time $i(t)$ to bring back GDP and inflation to their long-run values.

The blue curve represents the evolution of the inflation gap, while the dynamics of the business cycle and the nominal interest rate are represented by the yellow and orange curves, respectively.

4.1. Sustained economic growth and inflation. Figure 3 represents a situation where both inflation and GDP are 2 percent point over the pre-established target, the CB gives the same priority to the objectives and $\phi$ is relatively low.

As can be expected, as this situation might be related to economic bubbles, the CB reacts by undertaking a restrictive monetary measure, motivated by the need to mitigate the negative effect of an “overheating” situation, maintaining the interest rate above the long-run optimal value for the whole time span considered (from a positive 3.95% gap at the initial moment $t = 0$ of the shock to a +0.1% in the last years).

The CB effort leads to a strong reduction of aggregate demand that in turn helps to mitigate the initial over-inflation and the excessive economic growth. The result is satisfactory as the macroeconomic gaps approach zero, even though GDP remains 0.93% higher than the target and inflation ends up slightly under the reference ($-0.44\%$) due to also the output slowdown.

A different time period from the 5 years used as benchmark, does not qualitatively affect the evolution of the economic variables. A longer time span $T$ simply allows
CB to have a more gradual approach in the change of the monetary instrument \( i \) to achieve its goals.

4.2. **Recession and deflation.** Figure 4 simulates an opposite scenario, a negative shock carries the economy into a severe recession (−2.5 %) and to an inflation slowdown (−1 %).

As a consequence, an injection of liquidity through a reduction of the interest rate (at \( T = 0, \ i = -3.19 \ % \)) brings back the values closer to their targets. In this case, however, the variables follow a non-linear path.

In particular, an ultra expansive monetary policy, while leading to a relevant increase of output, affects the inflation dynamics both directly and indirectly. On the one side, we have a direct effect resulting from an expansion of liquidity that boosts prices over time. On the other, there is an indirect effect arising from the growth in time of GDP and aggregate demand, which, approaching to the zero value, bears to a rise in money demand for transactional scope and can ultimately strengthen inflation. As can be noticed, thanks to these two positive effects, the initial disinflation moderately improves and ends up only to 0.1 point under the normalized target of zero.

However, as already pointed out the CB effort required to recover from negative inflation deviations is relatively greater than in contexts characterized by positive inflation gaps (such as Figure 3). Coherently, in Figure 4, the improvement is relatively slower and lower in magnitude.
Finally, given the strong original shock on GDP (−2.5%), the economy does not fully recover. Despite the strong upswing brings by the extraordinary expansionary measure, it ends more than a percentage point under the potential growth value (−1.21%).

This confirms and supports the evidence that, in the most severe scenarios, the monetary policy alone is not sufficient to restore the long-run equilibrium trends of the economy and should be accompanied and supported by other types of interventions, such as an adequate counter-cyclical fiscal policy.

4.3. Trade-off scenarios. In Figure 5 we highlight a trade-off scenario characterized by an inflation two points under the target and a positive output gap at $t = 0$. This simulation might be particularly relevant to understand the difficulties faced by the monetary authority to achieve two conflicting objectives with a unique instrument at disposal.

Furthermore, right after the global financial crisis and the EU debt sovereign crisis, Europe and also many other advanced economies experienced a situation that can be well stylized by Figure 5. In fact, from 2015 onward, inflation remained well under the target for a long period in many OECD countries (including the United States and Japan), while there was an economic upturn after several years of recessions.

\[ \pi_0 = -1, \ y_0 = -2.5, \ T = 5, \ \alpha = 0.5, \ \beta = 0.5, \ \gamma = 0.1, \ \eta = 0.1, \ \omega = 0.3, \ \phi = 0.5 \]
Many economists, such as Paul Krugman, started to debate about a possible liquidity trap. Theorized by Keynes in his 1936 General Theory, a liquidity trap is a situation where interest rates are close to zero and changes in the money supply fail to translate into movements of the price level. Among the causes, he argued, there is a climate of general uncertainty that pushes people to hoard cash in the fear of an adverse event such as deflation, insufficient aggregate demand, or war.

In this context, the exit from a period of deep recession and of prolonged low inflation values (2 percentage points under the target, which means zero inflation, or even deflation) has caused the postponement of several relevant economic decisions that are vital for the wellbeing of an economy.

Figure 5 is rather explicative on how it can be difficult to find a compromise between the need to slowdown the positive output gap and the need of revitalizing inflation. It shows the powerlessness of the CB in achieving both objectives. Initially, it sets a expansive monetary measure $i < 0$ that improves inflation, but slightly worsens the output gap. Subsequently (after the first year to almost the fifth), the CB changes monetary policy stance through a prolonged tightening measure $i > 0$ that helps reducing only in part the positive output gap, but prevents inflation to further improve. Inflation does not worsen and remains steady at $-0.7\%$ only thanks to the enduring positive output gap (from the non-linear Phillips curve in equation 2).

In the end, none of the two objectives is entirely met.

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9Theorized by Keynes in his 1936 General Theory, a liquidity trap is a situation where interest rates are close to zero and changes in the money supply fail to translate into movements of the price level. Among the causes, he argued, there is a climate of general uncertainty that pushes people to hoard cash in the fear of an adverse event such as deflation, insufficient aggregate demand, or war.
The GDP remains at 1.43% over the long-run optimal, and the prominent disinflation slightly improves, but not sufficiently to reach the target (−0.42%). Consequently, Figure 5 incontrovertibly reveals the dilemma faced by the CB in these scenarios.

Figure 6 shows another relevant trade-off, characterized by an inflation over target (+2%) and a growth below potential (−2%), that is a situation of stagflation. We can trace back a similar situation during the 1970s energy crisis when for the first time was coined the term stagflation to depict this unusual economic condition. A swift increase in the price of oil and its derivative, in absence of valid energy alternatives, brought to a negative spiral of rising inflation and economic stagnation. Before the supply shocks of the ’70s, inflation and recession were usually regarded as mutually exclusive, the relationship between the two being described by the classical Phillips curve ([22]).

Nowadays, stagflation or recession-inflation is recognized as a situation in which the inflation rate is high, the economic growth rate slows, and unemployment remains steadily high. It presents a dilemma for economic policy, since actions intended to lower inflation may exacerbate unemployment.

From Figure 6 we can better appreciate the difficulties and the consequent policy indecision that may arise for the CB.

An initial tightening measure (\(i_0 = +2.9\%\)) improves inflation partly reducing the upward pressure on prices, but at the cost of worsening the output gap to a
value slightly lower than the initial shock. A subsequent mild expansionary policy brings a little stimulus to the output, but prevents inflation to improve further.

Also in this scenario, after the 5-years period considered, both the macroeconomic variables end up far from the objective ($\pi_T = +1 \%$ and $y_T = -1.4 \%$).

This again reveals the trade-off pointed out by the political economy theory about policy instruments. When we specifically refer to monetary policy, a situation of dilemma could arise, for the central bank, every time the gap of the two state variables ($\pi$, $y$) is not concordant in sign (i.e. asymmetric shock). In these contexts, a given monetary policy measure (expansive or restrictive) can improve (i.e. minimize) only one of the two initial deviations/shocks on the state variables, and this is unfortunately obtained at the expenses of the other variable.

Figures 5 and 6 clearly demonstrate this.

In the next two graphs (7 and 8), starting from the same initial scenario ($\pi_0 = +2\%, y_0 = -2\%$), we try to understand if changing the relative CB priority on goals might help to reach better at least one of them.

In Figure 7, the monetary authority decides to ascribe more relevance to the inflation target ($\alpha = 0.9$, $\beta = 0.1$), ceteris paribus.

Coherently, the stronger tightening policy ($i = +4.83\%$) leads to an improvement in the price rate of change, which reduces following a non-linear path to end at just +0.61\% points above the target compared to +1\% of Figure 6.

![Figure 7](image-url)
This positive result is offset by a relatively worst output path with respect to the previous Figure 6 (from $-1.40\%$ to $-1.46\%$ at time $T$) accordingly to the minor importance ascribed to this objective.

In Figure 8, the CB shifts the priority on the output target ($\alpha = 0.2$, $\beta = 0.8$), ceteris paribus. As can be expected, the higher priority given to stabilizing GDP, results in a less restrictive measure ($i = +1.40\%$) that helps $y$ to get closer to the objective after the 5-years considered ($-1.16\%$), but at the expenses of the other macroeconomic variable $\pi$ that deteriorates with respect to Figure 6 towards a serious and persistent inflation issue ($+1.3\%$).

5. Conclusion. The present paper has analyzed the effects of monetary policy, as captured by changes in the nominal interest rate, on the dynamics of GDP and inflation utilizing an optimal control model.

With respect to the previous literature, we improved in at least two directions. First, in the literature, monetary policy macroeconomic models are often presented in a discontinuous time fashion ([14, 29]), while we propose a continuous-time version of a monetary policy model, investigating the impact of the CB instrument (i.e. interest rate) according to a specific loss function. Secondly, we propose an augmented non-linear Phillips curve to take into account the direct impact of monetary policy on price dynamics.

The results of the analysis are worth stressing.
First of all, the model allows considering in a unified manner different phenomena such as inflation and deflation, as well as situations of sustained economic growth and periods of recession. In the second place, the simulations show that the model might well represent different economic scenarios and trade-off situations, underlying the difficulties of the CB in reaching both the output and inflation targets employing only one instrument.

Furthermore, the dynamics of the variables depend on the value of the parameters and the relative relevance assigned by the CB to one target relative to the other. Coherently, a different priority in the monetary agenda can change the outcomes towards a direction or another reflecting the potential diverse preference of each policymaker.

Future research could focus on at least two directions.

Firstly, the model could specifically consider fiscal policy in order to explicit the interactions between fiscal and monetary policies, and the relationship between interest rates, public deficit, and the two macroeconomic variables considered. Secondly, a line of future research may study the impact of evolving forecasting rules on the model dynamics.

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