HOPF SOLITON SOLUTIONS FROM LOW ENERGY EFFECTIVE ACTION OF SU(2) YANG-MILLS THEORY

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The Skyrme-Faddeev-Niemi (SFN) model which is an O(3) \( \sigma \) model in three dimensional space up to fourth-order in the first derivative possesses topological soliton solutions with torus or knot-like structure. The model was initiated in 70’s \(^1\) and its interest has been extensively growing. The numerical simulations were performed \(^2\) \(^3\) \(^4\) \(^5\) \(^6\) \(^7\) \(^8\) \(^9\) and the integrability was shown \(^1\) and the application to the condensed matter physics \(^8\) and the Weinberg-Salam model \(^9\) were also considered. The recent research especially focuses on the consistency between the SFN and fundamental theories such as QCD \(^10\) \(^11\) \(^12\) \(^13\). In those references, it is claimed that the SFN action should be derived from the SU(2) Yang-Mills (YM) action at low energies.

1. Introduction
The Skyrme-Faddeev-Niemi (SFN) model which is an O(3) \( \sigma \) model in three dimensional space up to fourth-order in the first derivative possesses topological soliton solutions with torus or knot-like structure. The model was initiated in 70’s \(^1\) and its interest has been extensively growing. The numerical simulations were performed \(^2\) \(^3\) \(^4\) \(^5\) \(^6\) \(^7\) \(^8\) \(^9\) and the integrability was shown \(^1\) and the application to the condensed matter physics \(^8\) and the Weinberg-Salam model \(^9\) were also considered. The recent research especially focuses on the consistency between the SFN and fundamental theories such as QCD \(^10\) \(^11\) \(^12\) \(^13\). In those references, it is claimed that the SFN action should be derived from the SU(2) Yang-Mills (YM) action at low energies.
One can also show from the Wilsonian renormalization group argument that the effective action of Yang-Mills theory recovers the SFN in the infrared region [14]. However, the derivative expansion for slowly varying fields $n$ up to quartic order produces an additional fourth-order term in the SFN model, resulting in instability of the soliton solution.

Similar situations can be seen also in various topological soliton models. In the Skyrme model, the chirally invariant lagrangian with quarks exhibits fourth order terms after the derivative expansion and they destabilize the soliton solution [15,16]. To recover the stability of the skyrmion, the author of Ref.17 introduced a large number of higher order terms in the first derivative whose coefficients were determined from the coefficients of the Skyrme model by using the recursion relations. Alternatively, in Ref. 14 Gies pointed out the possibility that the second derivative order term can work as a stabilizer for the soliton. Similar form of the action was also proposed by Forkel with somewhat different expansion scheme [18]. The author also found out the saddle point soliton solutions with hedgehog type. This result is quite encouraging to examine the consistency between the topological soliton physics and the Yang-Mills theory.

In this paper, we compute the extended Hopf soliton solutions from the action proposed by Gies. In section 2 we give an introduction to the Skyrme-Faddeev-Niemi model with its topological property. In section 3 we show how to derive the SFN model action from the SU(2) Yang-Mills theory. In section 4 soliton solutions of this truncated YM action are studied. For this purpose, we introduce a second derivative term which can be derived in a perturbative manner. The naive extremization scheme, however, produces the fourth order differential equation and the model has no stable soliton solution. Failure of finding the soliton is caused by the basic feature of the second derivative field theory. In section 5 the higher derivative theory and Ostrogradski’s formulation are reviewed. We show the absence of the energy bound in the second derivative theory using an example in quantum mechanics and introduce the perturbative treatment for the second derivative theory. In section 6 we present our numerical results. The possibility of finding new topological bound for this extended, perturbative soliton solutions is also discussed. In section 7 are concluding remarks.

2. Skyrme-Faddeev-Niemi model

The Faddeev-Niemi conjecture for the low-energy model of SU(2) Yang-Mills theory is expressed by the following effective action:

$$S_{SFN} = \Lambda \int d^4x \left[ \frac{1}{2} (\partial_\mu n)^2 + \frac{g_1}{8} (\partial_\mu n \times \partial_\nu n)^2 \right]$$

(1)

where $n(x)$ is a three component vector field normalized as $n \cdot n = 1$. The mass scale $\Lambda$ is a free parameter and in this paper we set $\Lambda = 1$. It has been shown that stable soliton solutions exist when $g_1 > 0$. 

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The static field \( n(x) \) maps \( n : \mathbb{R}^3 \rightarrow S^2 \) and the configurations are classified by the topological maps characterized by a topological invariant \( H \) called Hopf charge

\[
H = \frac{1}{32\pi^2} \int A \wedge F, \quad F = dA
\]  

where \( F \) is the field strength and expressed in terms of \( n(x) \) as \( F = (n \cdot d n \wedge d n) \).

The static energy \( E_{\text{stt}} \) from the action \( I \) has a topological lower bound \[E_{\text{stt}} \geq KH^{3/4}\] where \( K = 16\pi^2 \sqrt{g_1} \).

Performing numerical simulations, one can find that the static configurations for \( H = 1, 2 \) have axial symmetry. Thus “the toroidal ansatz” is suitable to impose on these configurations. The ansatz is given by

\[
\begin{align*}
n_1 &= \sqrt{1 - w^2(\eta, \beta)} \cos(N\alpha + v(\eta, \beta)), \\
n_2 &= \sqrt{1 - w^2(\eta, \beta)} \sin(N\alpha + v(\eta, \beta)), \\
n_3 &= w(\eta, \beta),
\end{align*}
\]

where \((\eta, \beta, \alpha)\) is toroidal coordinates which are related to the Gaussian coordinates in \( \mathbb{R}^3 \) as follows:

\[
x = a \sinh \eta \cos \alpha, \quad y = a \sinh \eta \sin \alpha, \quad z = a \sin \beta \]

with \( \tau = \cosh \eta - \cos \beta \). The function \( w(\eta, \beta) \) is subject to the boundary conditions \( w(0, \beta) = 1, w(\infty, \beta) = -1 \) and is periodic in \( \beta \). \( v(\eta, \beta) \) is set to be \( v(\eta, \beta) = M\beta + v_0(\eta, \beta) \) and \( v_0(\cdot, \beta) \) is considered as a constant map. Equation \( 2 \) then gives \( H = NM \).

To obtain soliton solutions with higher derivative terms, we propose a simpler ansatz than \( 4 \), which is defined by

\[
\begin{align*}
n_1 &= \sqrt{1 - w^2(\eta)} \cos(N\alpha + M\beta), \\
n_2 &= \sqrt{1 - w^2(\eta)} \sin(N\alpha + M\beta), \\
n_3 &= w(\eta),
\end{align*}
\]

where \( w(\eta) \) satisfies the boundary conditions \( w(0) = 1, w(\infty) = -1 \). By using \( 6 \), the static energy is written in terms of the function \( w(\eta) \) as

\[
E_{\text{stt}} = 2\pi^2 a \int d\eta \left( \frac{(w')^2}{1 - w^2} + (1 - w^2)U_{M,N}(\eta) + \frac{g_1}{2a^2} \sinh \eta \cosh w^2 U_{M,N}(\eta) \right),
\]

\[
w' = \frac{dw}{d\eta}, \quad U_{M,N}(\eta) \equiv \left( M^2 + \frac{N^2}{\sinh^2 \eta} \right).
\]

The Euler-lagrange equation of motion is then derived as

\[
\frac{w''}{1 - w^2} + \frac{ww'}{1 - w^2} + U_{M,N}(\eta)w + \frac{g_1}{2a^2} \left( -2N^2 \coth^2 \eta w' \right.)
\]

\[
+ (\cosh^2 \eta + \sinh^2 \eta)U_{M,N}(\eta)w' + \sinh \eta \cosh \eta U_{M,N}(\eta)w' = 0.
\]
The variation with respect to \( a \) produces the equation for \( a \). Soliton solutions are obtained by solving the equations for \( a \) as well as for \( w \).

We obtained soliton solutions numerically for both ansatz (4) and (6). The total energies together with the topological lower bound (3) are shown as a function of coupling constant \( g_1 \) in Fig.1. We found that this simple ansatz produces at most 10% errors and it does not affect to the property of the soliton solution.

3. Cho-Faddeev-Niemi-Shabanov decomposition and the effective action in the Yang-Mills theory

In this section, we briefly review how to derive the SFN effective action from the action of SU(2) Yang-Mills theory in the infrared limit \(^{12}\). For the gauge fields \( A_\mu \), the Cho-Faddeev-Niemi-Shabanov decomposition is applied \(^{10,11,12,13}\)

\[
A_\mu = n C_\mu + (\partial_\mu n) \times n + W_\mu. \tag{8}
\]

The first two terms are the “electric” and “magnetic” Abelian connection, and \( W_\mu \) are chosen so as to be orthogonal to \( n \), i.e. \( W_\mu \cdot n = 0 \). Obviously, the degrees of freedom on the left- and right-hand side of Eq.(8) do not match. While the LHS describes \( 3 \text{color} \times 4 \text{Lorentz} = 12 \), the RHS consists of \((C_\mu)4 \text{Lorentz}+(n)2 \text{color}+(W_\mu)3 \text{color} \times 4 \text{Lorentz} - 4 n \cdot W_\mu = 0\) = 14 degrees freedom. Shabanov introduced in his paper \(^{12}\) the following constraint

\[
\chi(n, C_\mu, W_\mu) = 0, \text{ with } \chi \cdot n = 0. \tag{9}
\]

The generating functional of YM theory can be written as

\[
Z = \int Dn D\bar{C} D\bar{W} \delta(\chi) \Delta_{FP} \Delta_S e^{-S_{YM} - S_{gf}}, \tag{10}
\]

\( \Delta_{FP} \) and \( S_{gf} \) are the Faddeev-Popov determinant and the gauge fixing action respectively, and Shabanov introduced another determinant \( \Delta_S \) corresponding to the condition \( \chi = 0 \). YM and the gauge fixing action is given by

\[
S_{YM} + S_{gf} = \int d^4x \left[ \frac{1}{4g^2} F_{\mu\nu} \cdot F_{\mu\nu} + \frac{1}{2\alpha g^2} (\partial_\mu A_\mu)^2 \right].
\]

Inserting Eq.(8) into the action, one obtains the vacuum functional

\[
Z = \int Dn e^{-S_{eff}(n)} = \int Dn e^{-S_{cl}(n)} \int D\bar{C} D\bar{W} \Delta_{FP} \Delta_S e^{-\frac{1}{2g^2} \int [\tilde{C}_\mu M_{\mu\nu} \tilde{C}_\nu + \tilde{W}_\mu M_{\mu\nu} \tilde{W}_\nu - K_\mu (M_{\mu\nu})^{-1} K_\nu + \bar{K}_\mu (\bar{M}_{\mu\nu})^{-1} \bar{K}_\nu] \right]} \tag{11}
\]

with

\[
\tilde{M}_{\mu\nu} := M_{\mu\nu} + Q_{\lambda\mu} (M_{\lambda\nu})^{-1} Q_{\lambda\nu}, \quad \tilde{W}_\mu = W_\mu - (\tilde{M}_{\mu\nu})^{-1} K_\nu, \\
\tilde{C}_\mu = C_\mu + (M_{\mu\lambda})^{-1} Q_{\lambda\nu} \cdot W_\nu + (M_{\mu\nu})^{-1} K_\nu, \tag{12}
\]
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Fig. 1. The total energy of the solitons with (a) $H = 1$, (b) $H = 2$, as a function of the coupling constant $g_1$: the simple ansatz [11], the toroidal ansatz [11] by Gladikowski-Hellmund, and the expected topological lower bound [11].
and

\[ M^C_{\mu\nu} = -\partial^2 \delta_{\mu\nu} + \partial_{\mu} \mathbf{n} \cdot \partial_{\nu} \mathbf{n}, \quad M^W_{\mu\nu} = -\partial^2 \delta_{\mu\nu} - \partial_{\mu} \mathbf{n} \otimes \partial_{\nu} \mathbf{n} + \partial_{\nu} \mathbf{n} \otimes \partial_{\mu} \mathbf{n}, \]

\[ Q^C_{\mu\nu} = \partial_{\mu} \mathbf{n} \partial_{\nu} + \partial_{\nu} \mathbf{n} \partial_{\mu}, \quad K^C_{\mu\nu} = \partial_{\nu} (\mathbf{n} \cdot \partial_{\mu} \mathbf{n} \times \partial_{\mu} \mathbf{n}) + \partial_{\mu} \mathbf{n} \cdot \partial^2 \mathbf{n} \times \mathbf{n}, \]

\[ K^W_{\mu} = \partial_{\mu} (\mathbf{n} \times \partial^2 \mathbf{n}), \quad (\text{in gauge } \alpha_0 = 1). \]  \(13\)

The classical action for \( \mathbf{n} \) including the gauge fixing term is given by

\[ S_{cl} = \int d^4x \left[ \frac{1}{4g^2} (\partial_{\mu} \mathbf{n} \times \partial_{\nu} \mathbf{n})^2 + \frac{1}{2\alpha g^2} (\partial^2 \mathbf{n} \times \mathbf{n})^2 \right]. \]  \(14\)

The \( \delta \) functional is expressed by its Fourier transform

\[ \delta(\mathbf{x}) = \int D\phi e^{-i\int (\mathbf{\phi} \cdot \partial \mathbf{w}_\mu + \phi C_{\nu} \mathbf{n} \times \mathbf{w}_\mu + (\mathbf{\phi} \cdot \mathbf{n})(\partial_{\mu} \mathbf{n} \cdot \mathbf{w}_\mu))} . \]  \(15\)

Integrating over “fast” variables \( C, \mathbf{W}, \phi \), we finally obtain

\[ e^{-S_{cl}} = e^{-S_{cl}} \Delta_{FP} \Delta_{S}(\det M^C)^{-1/2}(\det \bar{M}^W)^{-1/2}(\det -Q^C_{\mu}(\bar{M}^W)^{-1} Q^C_{\mu})^{-1/2} \]

where \( Q^C_{\mu} := i(-\partial_{\mu} + \partial_{\mu} \mathbf{n} \otimes \mathbf{n}) \). Here several nonlocal terms and the higher derivative components have been neglected.

Performing the derivative expansion for the resultant determinants with \( \partial \mathbf{n} \), one obtain the effective action for “slow” variable \( \mathbf{n} \)

\[ S_{eff} = \int d^4x \left[ \frac{1}{2} (\partial_{\mu} \mathbf{n})^2 + \frac{g_1}{8} (\partial_{\mu} \mathbf{n} \times \partial_{\nu} \mathbf{n})^2 + \frac{g_2}{8} (\partial_{\mu} \mathbf{n})^4 \right] . \]  \(17\)

For \( g_1 > 0 \) and \( g_2 = 0 \), the action is identical to the SFN effective action \(11\).

In order for soliton solutions to exist, \( g_2 \) must be positive \(3\). However, \( g_2 \) is found to be negative according to the above analysis. Therefore we consider higher-derivative terms and investigate if the model with the higher-derivatives possess soliton solutions.

### 4. Search for the soliton solutions (1)

The static energy is derived from Eq. \(17\) as

\[ E_{\text{stt}} = \int d^4x \left[ \frac{1}{2} (\partial_{\mu} \mathbf{n})^2 + \frac{g_1}{8} (\partial_{\mu} \mathbf{n} \times \partial_{\nu} \mathbf{n})^2 + \frac{g_2}{8} (\partial_{\mu} \mathbf{n})^4 \right] := E_2(\mathbf{n}) + E_4^{(1)}(\mathbf{n}) + E_4^{(2)}(\mathbf{n}). \]  \(18\)

A spatial scaling behaviour of the static energy, so called Derrick’s scaling argument, can be applied to examine the stability of the soliton \(20\). Considering the map \( \mathbf{x} \rightarrow \mathbf{x'} = \mu \mathbf{x} (\mu > 0) \), with \( \mathbf{n}(\mu) \equiv \mathbf{n}(\mu \mathbf{x}) \), the static energy scales as

\[ e(\mu) = E_{\text{stt}}(\mathbf{n}(\mu)) = E_2(\mathbf{n}(\mu)) + E_4^{(1)}(\mathbf{n}(\mu)) + E_4^{(2)}(\mathbf{n}(\mu)) = \frac{1}{\mu} E_2(\mathbf{n}) + \mu (E_4^{(1)}(\mathbf{n}) + E_4^{(2)}(\mathbf{n})). \]  \(19\)
Derrick’s theorem states that if the function $e(\mu)$ has no stationary point, the theory has no static solutions of the field equation with finite density other than the vacuum. Conversely, if $e(\mu)$ has stationary point, the possibility of having finite energy soliton solutions is not excluded. Eq. (19) is stationary at $\mu = \sqrt{E_2/(E_3^{(1)} + E_3^{(2)})}$. Then, the following inequality

$$g_1(\partial_1 n \times \partial_2 n)^2 + g_2(\partial_3 n)^2 \partial_1 n)$$

$$= g_1(\partial_1 n)^2(\partial_3 n)^2 - g_1(\partial_1 n \cdot \partial_3 n)^2 + g_2(\partial_3 n)^2(\partial_3 n)^2 \geq (\partial_1 n \cdot \partial_3 n)^2$$

ensures the possibility of existence of the stable soliton solutions for $g_2 \geq 0$. As mentioned in the section 3, $g_2$ should be negative at least within our derivative expansion analysis of YM theory.

A promising idea to tackle the problem was suggested by Gies. The author considered the following type of effective action, accompanying an second derivative term

$$S_{\text{eff}} = \int d^4x \left[ \frac{1}{2}(\partial_\mu n)^2 + \frac{g_1}{8}(\partial_\mu n \times \partial_\nu n)^2 - \frac{g_2}{8}(\partial_\mu n)^4 - \frac{g_2}{8}(\partial^2 n \cdot \partial^2 n) \right].$$

Here we choose positive value of $g_2$ and assign the explicit negative sign to the third term. In principle, it is possible to estimate the second derivative term by the derivative expansion without neglecting throughout the calculation. The calculation is, however, very laborious and hence we will show in detail somewhere else.

The static energy of Eq. (24) with the ansatz (6) is written as

$$E_{\text{stt}} = 2\pi^2 a \int d\eta \left[ \frac{(w')^2}{1 - w'^2} + (1 - w^2)U_{M,N}(\eta) + \frac{g_1}{2a^2}\sinh \eta \cosh \eta (w')^2U_{M,N}(\eta) \right]$$

$$+ \frac{g_2}{4a^2} \left[ - \sinh \eta \cosh \eta \left( \frac{(w')^2}{1 - w'^2} + (1 - w^2)U_{M,N}(\eta) \right)^2 \right]$$

$$+ \left( \coth \eta + \sinh^2 \eta - \sinh \eta \cosh \eta \right) \left( \frac{(w')^2}{1 - w'^2} \right)$$

$$+ \left( \sinh \eta \cosh \eta - \sinh^2 \eta \right) (1 - w^2) M^2 + 2 \left\{ \frac{w(w')^3}{(1 - w'^2)^2} + \frac{w'w''}{1 - w'^2} + w w' U_{M,N}(\eta) \right\}$$

$$+ \sinh \eta \cosh \eta \left( \frac{1}{1 - w'^2} \left[ \frac{(w')^2}{1 - w'^2} + w w'' + (1 - w^2)U_{M,N}(\eta) \right]^2 + (w'')^2 \right) \right],$$

where $w'' = \frac{d^2w}{d\eta^2}$. The Euler-Lagrange equation of motion is derived by

$$- \frac{d^2}{d\eta^2} \frac{\partial E_{\text{stt}}}{\partial w'} + \frac{d}{d\eta} \frac{\partial E_{\text{stt}}}{\partial w''} - \frac{\partial E_{\text{stt}}}{\partial w} = 0,$$

which is too complicated to write explicitly. Thus we adopt the following notation

$$f_0(w, w', w'') + g_1f_1(w, w', w'') + g_2f_2(w, w', w'', w^{(3)}, w^{(4)}) = 0.$$
Here $w^{(3)}$, $w^{(4)}$ represent the third and the fourth derivative with respect to $\eta$. The first two terms of Eq. (23) are identical to those in Eq. (7).

In addition to the boundary conditions at the origin and the infinity $\omega(0) = 1, \omega(\infty) = -1$, the regularity condition specifies $\omega'(0) = \omega''(0) = 0$ (see Eq. (21)).

Unfortunately, under these boundary conditions we could not find soliton solutions in Eq. (23) for any value of $g_2$.

Recently, Forkel studied the soft mode action of the Yang-Mills theory and investigated the solitonic, saddle point solution with hedgehog symmetry [15]. His work is quite encouraging to find the class of solutions in the infrared part of the Yang-Mills action. But, the solutions the author obtained are solitons with non-zero topological charge and, in the case of the Hopf solitons which possess zero topological charge, the situation is a little more complicated.

From the identity
\[
\int d^4x \left[ (\partial^2 n \cdot \partial^2 n) - (\partial^\mu n)^4 \right] = \int d^4x (\partial^2 n \times n)^2 ,
\]
(24)

one easily finds that the static energy obtained from the last two terms in Eq. (20)
\[
\tilde{E}_4^{(2)} = \int d^3x (\partial^2 n \times n)^2
\]
(25)
gives the positive contribution. The total static energy is stationary at $\mu = \sqrt{E_2/(E_4^{(1)} + \tilde{E}_4^{(2)})}$ and hence the possibility of existence of soliton solutions is not excluded. And also, the positivity of Eq. (24) does not spoil the lower bound of original SFN and the possibility still remains.

Therefore, we suspect that the failure of finding the stable soliton is caused by the fact that higher derivative theory has no lower bound. We shall investigate the lower bound in the higher derivative theory in detail in the next section.

5. Higher derivative theory

In this section, we address the basic problems in the higher derivative theory, which essentially falls into two categories. The first problem concerns the increase in the number of degrees of freedom. For example, if the theory contains second derivative terms, the equation of motion becomes up to the order in the fourth derivative. Thus, four parameters are required for the initial conditions. If one considers more higher order terms, the situation gets worse. However, this is not serious problem for our study because our concern is the existence of static soliton solutions. The second problem is that the actions of the theory are not bounded from below. This feature makes the higher derivative theories unstable.

The lagrangian and the hamiltonian formalism with higher derivative was firstly developed by Ostrogradski [26]. We consider the lagrangian containing up to nth order derivatives
\[
S = \int dt \mathcal{L}(q, \dot{q}, \cdots, q^{(n)}).
\]
(26)
Taking the variation of the action $\delta S = 0$ leads to the Euler-lagrange equation of motion
\begin{equation}
\sum_{i=0}^{n} (-1)^i \frac{d^i}{dt^i} \left( \frac{\partial L}{\partial q^{(i)}} \right) = 0. \tag{27}
\end{equation}

The hamiltonian is obtained by introducing $n$ generalized momenta
\begin{equation}
p_i = \sum_{j=i+1}^{n} (-1)^{j-i-1} \frac{d^{j-i-1}}{dt^{j-i-1}} \left( \frac{\partial L}{\partial q^{(j)}} \right), \quad i = 1, \cdots, n, \tag{28}
\end{equation}
or
\begin{equation}
p_n = \frac{\partial L}{\partial q^{(n)}}, \quad p_i = \frac{\partial L}{\partial q^{(i)}} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}^{i+1}} , \quad i = 1, \cdots, n-1, \tag{29}
\end{equation}
and $n$ independent variables
\begin{equation}
q_1 \equiv q, \quad q_i \equiv q^{(i-1)}, \quad i = 2, \cdots, n. \tag{30}
\end{equation}

The lagrangian now depends on the $n$ coordinates $q_i$ and on the first derivative $\dot{q}_n = q^{(n)}$. The hamiltonian is defined as
\begin{equation}
\mathcal{H}(q, p) = \sum_{i=1}^{n} p_i \dot{q}_i - L = \sum_{i=1}^{n-1} p_i q_{i+1} + p_n \dot{q}_n - L. \tag{31}
\end{equation}

The canonical equations of motion turn out to be
\begin{equation}
\dot{q}_i = \frac{\partial \mathcal{H}}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial \mathcal{H}}{\partial q_i}. \tag{32}
\end{equation}

Thus, we replace a theory of one coordinate $q$ system obeying $2n$–th differential equation with a set of 1-st order canonical equations for $2n$ phase-space variables $[q_i, p_i]$.

We consider a simple example including a second derivative term defined as
\begin{equation}
L = \frac{1}{2} (1 + \epsilon^2 \omega^2) \dot{q}^2 - \frac{1}{2} \omega^2 q^2 - \frac{1}{2} \epsilon^2 q^2, \tag{33}
\end{equation}
where constant $\epsilon$ works as a coupling constant of second derivative term. The equation of motion is
\begin{equation}
(1 + \epsilon^2 \omega^2) \ddot{q} + \omega^2 q + \epsilon^2 q^{(4)} = 0. \tag{34}
\end{equation}

From Eq. (33), one gets
\begin{equation}
\pi_q = \frac{\partial L}{\partial \dot{q}} = -\epsilon^2 \dot{q}, \quad \pi_q = \frac{\partial L}{\partial \dot{q}} - \frac{d}{dt} \left( \frac{\partial L}{\partial q} \right) = (1 + \epsilon^2 \omega^2) \dot{q} + \epsilon^2 \ddot{q}. \tag{35}
\end{equation}

Thus the hamiltonian becomes
\begin{equation}
\mathcal{H} = \dot{q} \pi_q + \dot{\pi}_q - L = \dot{q} \pi_q - \frac{1}{2 \epsilon^2} \pi_q^2 - \frac{1}{2} (1 + \epsilon^2 \omega^2) q^2 + \frac{1}{2} \omega^2 q^2. \tag{36}
\end{equation}
We introduce the new canonical variables

\[ q_+ = \frac{1}{\omega \sqrt{1 - \varepsilon^2 \omega^2}} (\varepsilon^2 \omega^2 \dot{q} - \pi q), \quad p_+ = \frac{\omega}{\sqrt{1 - \varepsilon^2 \omega^2}} (q - \pi q), \]

\[ q_- = \frac{\varepsilon}{\sqrt{1 - \varepsilon^2 \omega^2}} (\dot{q} - \pi q), \quad p_- = \frac{1}{\varepsilon \sqrt{1 - \varepsilon^2 \omega^2}} (\varepsilon^2 \omega^2 q - \pi q), \]

and the Hamiltonian can be written in terms of these variables as

\[ \mathcal{H} \to \frac{1}{2} (p_+^2 + \omega^2 q_+^2) - \frac{1}{2} (p_-^2 + \frac{1}{\varepsilon^2} q_-^2). \]

The corresponding energy spectra is then given by

\[ E = (n + \frac{1}{2}) \omega - (m + \frac{1}{2}) \frac{1}{\varepsilon}, \quad n, m = 0, 1, 2, \cdots \quad (37) \]

One can see that in the limit \( \varepsilon \to 0 \) the energy goes to negative infinity rather than approaching to the harmonic oscillator energy eigenstates.

To obtain physically meaningful solutions, we employ the perturbative analysis where the solution is expanded in terms of the small coupling constant and the Euler-Lagrange equation of motion is replaced with the corresponding perturbative equation. The solutions of the equations of motion that are ill behaved in the limit \( \varepsilon \to 0 \) are excluded from the very beginning.\[24\] \[25\]

We assume that the solution of Eq. (34) can be written as

\[ q_{\text{pert}}(t) = \sum_{n=0}^{\infty} \varepsilon^n q(t). \quad (38) \]

Substituting Eq. (38) into Eq. (34) and taking time derivatives of these equations, we obtain the constraints for higher derivative terms

\[ O(\varepsilon^0) \]

\[ \text{equation:} \quad \ddot{q}_0 + \omega^2 q_0 = 0, \]

\[ \text{constraints:} \quad \dddot{q}_0 = -\omega^2 \dot{q}_0, \quad \dddot{q}_0 = \omega^4 q_0. \quad (39) \]

\[ O(\varepsilon^2) \]

\[ \text{equation:} \quad \dddot{q}_2 + \omega^2 \dot{q}_0 + \omega^2 q_2 + \omega^4 q_0 = 0, \]

\[ \Rightarrow \dddot{q}_2 + \omega^2 q_2 = 0, \quad \text{(using (40))}, \quad (40) \]

\[ \text{constraints:} \quad \dddot{q}_2 = -\omega^2 \dot{q}_2, \quad \dddot{q}_2 = \omega^4 q_2. \quad (41) \]

\[ O(\varepsilon^4) \]

\[ \text{equation:} \quad \dddot{q}_4 + \omega^2 \dot{q}_2 + \omega^2 q_4 + \omega^4 q_2 = 0, \]

\[ \Rightarrow \dddot{q}_4 + \omega^2 q_4 = 0, \quad \text{(using (42))}, \quad (42) \]

\[ \text{constraints:} \quad \dddot{q}_4 = -\omega^2 \dot{q}_4, \quad \dddot{q}_4 = \omega^4 q_4. \quad (43) \]

Combining these results, we find the perturbative equation of motion up to \( O(\varepsilon^4) \)

\[ \dddot{q}_{\text{pert}} + \omega^2 q_{\text{pert}} = O(\varepsilon^6). \quad (44) \]

which is the equation for harmonic oscillator.
6. Search for the soliton solutions (2) – perturbative analysis –

Let us employ the perturbative method introduced in the last section to our problem. We assume that $g_2$ is relatively small and can be considered as a perturbative coupling constant. Thus, the perturbative solution is written by a power series in $g_2$

$$w(\eta) = \sum_{n=0}^{\infty} g_2^n w_n(\eta).$$  \hfill (46)

Substituting Eq. (46) into Eq. (23), we obtain the classical field equation in $O(g_2^0)$

$$f_0(w_0, w'_0, w''_0) + g_1 f_1(w_0, w'_0, w''_0) = 0.$$  \hfill (47)

Taking derivatives for both sides in Eq. (47) and solving for $w''_0, w^{(3)}_0, w^{(4)}_0$ read the following form of the constraint equations for higher derivatives

$$w^{(i)}_0 = F^{(i)}(w_0, w'_0), \quad i = 2, 3, 4.$$  \hfill (48)

The equation in $O(g_2^1)$ can be written as

$$(f_0 + g_2 f_1)_{O(g_2^1)} + f_2(w_0, w'_0, w''_0, w^{(3)}_0, w^{(4)}_0) = 0.$$  \hfill (49)

Substituting the constraint equations (48) into Eq. (49) and eliminate the higher derivative terms, one can obtain the perturbative equation of motion

$$f_0(w, w', w'') + g_1 f_1(w, w', w'') + g_2 \tilde{f}_2(w, w') = O(g_2^2).$$  \hfill (50)

One can see that Eq. (50) has stable soliton solutions.

The numerical results of the functions $w(\eta)$ are displayed in Fig. 2. In Fig. 3, we show the energy density plot for the original SFN model in the cylindrical coordinates $(\rho, z)$. In Fig. 4, we present the energy density for our extended soliton in the $(\rho, z)$. Both results share the toroidal shape and no notable difference at least for the small coupling constant $g_2$. In Fig. 5, we plot the total energy for $H = 1, 2, g_1 = 0.4$ as a function of the coupling constant $g_2$. As can be seen, the change is moderate and, more interestingly, the energy seems to be linearly dependent on $g_2$ at the region of smaller $g_2$. For larger $g_2$, the data gradually deviate from the linear behavior. The deviation would be due to the fact that our analysis is based on the first order perturbation. Therefore, it is possible to observe the critical values of $g_2$ for each $g_1$ and $H$ in which our simple first order perturbation is valid. Performing linear fitting, one finds

$$E_{H=1} \sim 131 + 121 g_2,$$

$$E_{H=2} \sim 208 + 102 g_2.$$
Fig. 2. The function $w(x)$ of (a) $H = 1, g_1 = 0.4, g_2 = 0.25$, (b) $H = 2, g_1 = 0.4, g_2 = 0.3$ (the rescaling radial coordinate $x = \eta/(1 - \eta)$ is used), together with the results of the original SFN model.
Fig. 3. The energy density of naive SFN model in the cylindrical coordinates \((\rho, z)\) for \(H = 2, g_1 = 0.4\).

In Fig. 6, we plot the energies for various values of \(g_1\) as a function of \(g_2\). They are fitted to the following linear functions

\[
E_{g_1=0.2} \sim 147 + 142g_2, \\
E_{g_1=0.6} \sim 255 + 83g_2, \\
E_{g_1=0.8} \sim 294 + 72g_2.
\]

From these results we can extract formulation of topological bound for the energy of the second derivative term \(E_{2nd}\):

\[
E_{2nd} = \beta \frac{g_2}{\sqrt{g_1}} H^{-1/4}.
\] (51)

If \(\beta \sim 76\) is chosen, all data are well fitted by this single formula. For the energy from standard SFN action, we tentatively employ the topological lower bound (3). Then the topological lower bound including second derivative terms becomes

\[
E_{stt} \geq c_{stt} = \alpha \sqrt{g_1} H^{3/4} + \beta \frac{g_2}{\sqrt{g_1}} H^{-1/4}
\] (52)

with \(\alpha = 16\pi^2\). Though this formula is not based on any theory, it contains interesting information about the spectra of soliton. That is, has the minima
Fig. 4. The energy density of extended action (20) in the cylindrical coordinates \((\rho, z)\) for \(H = 2, g_1 = 0.4, g_2 = 0.3\).

at

\[
\frac{\partial e_{\text{ext}}}{\partial g_1} = 0 \rightarrow g_1 = \frac{\beta g_2}{\alpha H} \tag{53}
\]

for given \(g_2\). The energy bound is thus \(e_{\text{ext}} = 2\alpha \sqrt{g_1 H^{3/4}}\).

Let us summarize the numerical results presented in this section. If we take into account the second derivative term, the obtained soliton mass will become twice value of the one in the naive SFN soliton. In Fig. 4, a schematic plot of Eq. (52) is shown, and the existence of such local minima is observed. Interestingly, though at first glance our lower bound formula (52) has no 3/4-scaling behavior like (3) essential to knotted solitons [27], the energy at local minima recovers this scaling law.

Unfortunately, such minima can not be observed at present within our numerical framework. Because, from the condition (53), the value of \(g_2\) should be almost twice of \(g_1\) for \(H = 1\), and four times for \(H = 2\). Otherwise, we need to take into account the higher order perturbation terms to employ totally different formalism to achieve these values.
7. Summary

In this paper we have studied the Skyrme-Faddeev-Niemi model and its extensions by introducing the reduction scheme of the SU(2) Yang-Mills theory to the corresponding low-energy effective model. To simplify the matter, we proposed an ansatz for $n$. That extensively reduced the computational time and did not affect to the property of the soliton solution. The requirement of consistency between the low-energy effective actions of the YM and the SFN type model leads us to take into account second derivative terms in the action. However, we found that such an action including the second derivative terms does not have stable soliton solutions. This is due to the absence of the energy bound in higher derivative theories. This fact inspired us to employ the perturbative analysis to our effective action. Within the perturbative analysis, we were able to obtain soliton solutions.

It should be noted that our solutions do not much differ from the solution of original SFN model, at least in the perturbative regime. We suspect that an appropriate truncation (for instance “extra fourth order term + second derivative term”) can supply stable solutions that are close to the original SFN model. Thus we conclude that the topological soliton model consisting of the “kinetic term + a
special fourth order term” like SFN model is a good approximation.

Our analysis is based on perturbation and the coupling constant $g_2$ is assumed to be small. However, Wilsonian renormalization analysis of YM theory suggests that the coupling constants $g_1, g_2$ (and the mass scale parameter $\Lambda$) depend on the renormalization group time $t = \log k/\Lambda$ ($k, \Lambda$ are infrared, ultraviolet cutoff parameter) and they are almost comparable. To improve the analysis, we could perform the next order of perturbation, but it is tedious and spoils the simplicity of the SFN model unfortunately.

We found numerically that the energy of the extended soliton solutions is linear in the coupling constant $g_2$ and then extracted a new mass formula for the soliton solutions including second derivative term. We expect that the global minima for the coupling constant $g_2$ exists and the corresponding energy becomes twice the value of the naive SFN action. Of course, this statement is not based on any theory and we have to wait for its theoretical confirmation. Since our mass formula was obtained from numerical study in the perturbative approach, it is uncertain whether such linear behavior is kept for larger coupling constant $g_2$ (like twice of $g_1$ for $H = 1$ and four times for $H = 2$). To confirm that, we should proceed to investigate next order perturbation, or, otherwise, find some analytical evidence of this formula. We
point out that the perturbative treatment is only used for excluding the ill behavior of the second derivative field theory. We believe that applying this prescription should not alter the essential feature of the soliton solutions, e.g., existence of the solutions, linear behavior of the mass spectra, et al.

Finally, let us mention the application of the soliton solutions to the glueball. Obviously this is one of the main interest to study the model and, many authors have given discussions about this. On the other hand, the possibility of the magnetic condensation of the QCD vacuum within the Cho-Faddeev-Niemi-Shabanov decomposed Yang-Mills theory have been studied by Kondo. The author claims the existence of the nonzero off diagonal gluon mass $M_X$, which is induced in terms of the condensation of the magnetic potential part of the decomposition $B_\mu \sim (\partial_\mu n) \times n$, as

$$M_X^2 = \langle B_\mu \cdot B_\mu \rangle = \langle (\partial_\mu n)^2 \rangle \quad \text{(54)}$$

Throughout our calculation, we set the overall coupling constant $\Lambda = 1$ in the action but, in this sense, it should reflect the information of such gluon mass, or the condensation property of the vacuum. After a careful examination of the value of the property of $\Lambda$, we will be able to accomplish the detailed predictions for the glueball mass.
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