BOUNDLEDNESS OF POLARIZED CALABI-YAU FIBRATIONS

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Abstract. In this paper, we investigate the boundedness of good minimal models with intermediate Kodaira dimensions, which has a natural Iitaka fibration to the canonical models. We prove that good minimal models are bounded modulo crepant birational equivalence when the base (canonical models) are bounded and the general fibers of the Iitaka fibration are in a bounded family of polarized Calabi-Yau pairs.

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1. Introduction

We work over the field of complex numbers $\mathbb{C}$.

Despite extensive research into boundedness properties of canonically polarized varieties and Fano varieties, leading to recent advances [HMX18][Bir19][Bir21a], relatively little is known about Calabi-Yau varieties. In this paper, our focus is on the boundedness of varieties $X$ with a log Calabi-Yau fibration structure $f : (X, \Delta) \to Z$, where $K_X + \Delta \sim_{\mathbb{Q}, Z} 0$ and the general fiber $(X_g, \Delta_g)$ is a log Calabi-Yau pair.

Such varieties arise naturally in the minimal model program. For instance, varieties of intermediate Kodaira dimension are predicted by the abundance conjecture to admit a minimal model $X'$, where a suitable positive power of $K_{X'}$ is base point free and defines a morphism $f : X' \to Y$, which is equal to the Iitaka fibration. The fibers of $f$ are Calabi-Yau varieties, and the base $Y$ is naturally endowed with the structure of a general type pair.

Recent work by Birkar has analyzed the case when the Iitaka fibration has fibers of Fano type [Bir18], and the results in [Li20] apply to the study of the base of fibrations of Fano type. The boundedness of varieties with an elliptic fibration is considered in [Bir18], [BDCS20], [CDCH18], and [Fil20].

We define the Iitaka volume of a $\mathbb{Q}$-divisor by analogy with the definition of the volume. Let $X$ be a normal projective variety and $D$ be a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor. When the Iitaka dimension $\kappa(D)$ of $D$ is non-negative, the Iitaka volume of $D$ is defined to be

$$\text{Ivol}(D) := \limsup_{m \to \infty} \frac{\kappa(D) h^0(X, O_X(\lfloor mD \rfloor))}{m^{\kappa(D)}}$$

Definition 1.1. Let $d$ be a positive integer, $\mathcal{I} \subset [0,1] \cap \mathbb{Q}$ a DCC set, and $v, u$ two positive rational numbers. We define $\mathcal{G}_{\text{klt}}(d, \mathcal{I}, v, u)$ to be the set of pairs $(X, \Delta)$ such that

Date: May 23, 2023.
• \((X, \Delta)\) is a \(d\)-dimensional klt pair,
• \(\text{coeff}(\Delta) \subset \mathcal{I}\),
• \(\text{Ivol}(K_X + \Delta) = v\),
• \(K_X + \Delta\) is semi-ample and defines a contraction \(f : X \to Z\), and
• there exists an integral divisor \(A\) on \(X\), such that for a general fiber \(X_g\) of \(f\), \(A_g := A\vert_{X_g}\) is ample, and \(\text{vol}(A_g) = u\).

\textbf{Theorem 1.2.} Let \(d\) be a positive integer, \(\mathcal{I} \subset [0,1] \cap \mathbb{Q}\) a DCC set, and \(v,u\) two positive rational numbers. Then

\[ G_{\text{klt}}(d, \mathcal{I}, v, u) \]

is bounded modulo crepant birational equivalence.

We say a smooth projective variety \(Y\) is Calabi-Yau if it is simply connected, \(K_Y \sim 0\) and \(H^i(Y, \mathcal{O}_Y) = 0\) for \(0 < i < \dim Y\).

\textbf{Theorem 1.3.} Let \(d\) be a positive integer and \(u\) a positive rational number. Then the set of projective varieties \(Y\) such that

• \(Y\) is a smooth projective Calabi-Yau variety of dimension \(d\),
• there exists a contraction \(f : Y \to X\), and
• there is an integral divisor \(A\) on \(Y\), such that for any general fiber \(Y_g\) of \(f\), \(A_g := A\vert_{Y_g}\) is ample, and \(\text{vol}(A_g) = u\).

is bounded modulo flop.

The existence of the divisor \(A\) is a natural condition in many cases, for example, if the general fiber \(X_g\) is Fano, then we can choose \(A\) to be the closure of \(-K_{X_U}\), where \(U\) is an open subset of \(Z\), or if \(f\) is an elliptic fibration with a rational section \(i : Z \rightarrow X\), then we may define \(A\) to be the closure of \(i(Z)\). Therefore, we have the following two direct applications of Theorem 1.2 and Theorem 1.3.

\textbf{Corollary 1.4.} Let \(d, k\) be two positive integers, \(\mathcal{I} \subset [0,1] \cap \mathbb{Q}\) a DCC set, and \(v,u\) two positive rational numbers. Suppose \((X, \Delta)\) is a log pair such that

• \((X, \Delta)\) is a \(d\)-dimensional klt pair,
• \(\text{coeff}(\Delta) \subset \mathcal{I}\),
• \(\text{Ivol}(K_X + \Delta) = v\), and
• \(K_X + \Delta\) is semiample and defines a contraction \(f : X \to Z\).

Suppose either \(-K_{X_g}\) is ample and \(\text{vol}(-K_{X_g}) = u\) for the general fiber \(X_g\), or \(f\) is an elliptic fibration with a rational multi-section of degree \(k\), then \((X, \Delta)\) is bounded modulo crepant birational equivalence.

\textbf{Corollary 1.5.} Let \(d\) be a positive integer and \(u\) a positive rational number. Then the set of projective varieties \(Y\) such that

• \(Y\) is a smooth projective Calabi-Yau variety of dimension \(d\),
• there exists an elliptic fibration \(f : Y \to X\), and
• there is a rational section \(i : X \rightarrow Y\),

is bounded modulo flop.

The relevance of Corollaries 1.4 and 1.5 to [Bir18, Theorem 1.2], [Fil20, Theorem 1.1], and [BDCS20, Theorem 1.2] suggests that Theorems 1.2 and 1.3 constitute notable advancements in the study of birational boundedness regarding log Calabi-Yau fibrations.

\textbf{Acknowledgement.} The author would like to thank his advisor Christopher D. Hacon for his encouragement and constant support. He would like to thank Chuanghao Wei, Jingjun Han, Jihao Liu, and Yupeng Wang for their helpful comments. The author was partially supported by NSF research grant no: DMS-1952522 and by a grant from the Simons Foundation; Award Number: 256202.
2. Preliminaries

2.1. Conventions. We will use the notations in \cite{KM98} and \cite{Laz04}.

Let \(\mathcal{I} \subset \mathbb{R}\) be a subset, we say \(\mathcal{I}\) satisfies the DCC if there is no strictly decreasing subsequence in \(\mathcal{I}\). A fibration means a projective and surjective morphism with connected fibers.

2.2. Divisors. For a birational morphism \(f : Y \to X\) and a divisor \(B\) on \(X\), \(f^{-1}(B)\) denotes the strict transform of \(B\) on \(Y\), and \(\text{Exc}(f)\) denotes the sum of reduced exceptional divisors of \(f\).

For a \(\mathbb{Q}\)-divisor \(D\), a map defined by the linear system \(|D|\) means a map defined by \(||D||\). Given two \(\mathbb{Q}\)-Cartier \(\mathbb{Q}\)-divisors \(A, B\), \(A \sim_{\mathbb{Q}} B\) means that there is an integer \(m > 0\) such that \(m(A - B) \sim 0\).

Let \(D\) be a \(\mathbb{Q}\)-divisor, we write \(D = D_{>0} - D_{<0}\) as the difference of the effective part and negative part. Let \(f : X \to Z\) be an algebraic contraction and \(D\) a \(\mathbb{Q}\)-divisor on \(X\), we write \(D = D_{v} + D_{h}\), where \(D_{v}\) is the \(f\)-vertical part and \(D_{h}\) is the \(f\)-horizontal part. We say a closed subvariety \(W \subset X\) is vertical over \(Z\) if \(f(W) \not\subseteq Z\), horizontal over \(Z\) if \(f(W) = Z\).

Let \(f : X \to Y\) be a contraction of normal varieties, \(D\) a \(\mathbb{Q}\)-divisor on \(X\). We say that \(D\) is \(f\)-very exceptional if \(D\) is \(f\)-vertical and for any prime divisor \(P\) on \(Y\) there is a prime divisor \(Q\) on \(X\) which is not a component of \(D\) but \(f(Q) = P\), i.e. over the generic point of \(P\) we have: \(\text{Supp}f^{*}P \not\subseteq \text{Supp}D\).

2.3. Pairs. A sub-pair \((X, \Delta)\) consists of a normal variety \(X\) and a \(\mathbb{Q}\)-divisor \(\Delta\) on \(X\) such that \(K_X + \Delta\) is \(\mathbb{Q}\)-Cartier. A sub-pair \((X, \Delta)\) is called a pair if \(\Delta \geq 0\). If \(g : Y \to X\) is a birational morphism and \(E\) is a divisor on \(Y\), the discrepancy \(a(E, X, \Delta) = -\text{coeff}_{E}(\Delta)\), where \(K_Y + \Delta_Y := g^{*}(K_X + \Delta)\). A sub-pair \((X, \Delta)\) is call sub-klt (resp. sub-lc, sub-a-lc) if for every birational morphism \(Y \to X\) as above, \(a(E, X, \Delta) > -1\) (resp. \(\geq -1, \geq -1 + a\)) for every divisor \(E\) on \(Y\). A sub-pair \((X, \Delta)\) is called klt (resp. lc, a-lc) if \((X, \Delta)\) is sub-klt (resp. sub-lc, sub-a-lc) and \((X, \Delta)\) is a pair.

Let \((Y, \Delta_Y)\) and \((X, \Delta)\) be two sub-pairs and \(h : Y \to X\) a birational morphism, we say \((Y, \Delta_Y) \to (X, \Delta)\) is a crepant birational morphism if \(K_Y + \Delta_Y \sim_{\mathbb{Q}} h^{*}(K_X + \Delta)\), two sub-pairs \((X_i, \Delta_i), i = 1, 2\), are crepant birationally equivalent if there is a sub-pair \((Y, \Delta_Y)\) and two crepant birational morphisms \((Y, \Delta_Y) \to (X_i, \Delta_i), i = 1, 2\).

A generalized pair \((X, \Delta + M)\) consists of a normal variety \(X\) equipped with a birational morphism \(X' \xrightarrow{f} X\) where \(X'\) is normal, a \(\mathbb{Q}\)-divisor \(\Delta\), and a \(\mathbb{Q}\)-Cartier nef divisor \(M\) on \(X'\) such that \(K_{X'} + \Delta + M\) is \(\mathbb{Q}\)-Cartier, where \(M = f^{*}M'\). Let \(\Delta'\) be the \(\mathbb{Q}\)-divisor such that \(K_{X'} + \Delta' + M' = f^{*}(K_X + \Delta + M)\), we call \((X, \Delta + M)\) a generalized sub-klt (resp. generalized sub-lc, generalized sub-a-lc) pair if \((X', \Delta')\) is sub-klt (resp. sub-lc, sub-a-lc), a generalized klt (resp. generalized lc, generalized a-lc) pair if further \(\Delta \geq 0\).

Let \((X, \Delta)\) be a sub-lc pair (resp. \((X, \Delta)\) is a generalized sub-lc pair). A log canonical place of \((X, \Delta)\) (resp. a generalized log canonical place of \((X, \Delta + M)\)) is a prime divisor \(D\) over \(X\) such that \(a(D, X, \Delta) = -1\) (resp. \(a(D, X, \Delta + M) = -1\)). A log canonical center (resp. a generalized log canonical center) is the image on \(X\) of a log canonical place (resp. a generalized log canonical place). Similarly, a log place of \((X, \Delta)\) (resp. a generalized log place of \((X, \Delta + M)\)) is a prime divisor \(D\) over \(X\) such that \(a(D, X, \Delta) \in [-1, 0)\) (resp. \(a(D, X, \Delta + M) \in [-1, 0)\)). A log center (resp. a generalized log center) is the image on \(X\) of a log place (resp. a generalized log place).

A log smooth pair is a pair \((X, \Delta)\) where \(X\) is smooth and \(\text{Supp} \Delta\) is a simple normal crossing divisor. Assume \((X, \Delta)\) is a log smooth pair and assume \(\text{Supp} \Delta = \bigcup_{i} \Delta_i\), where \(\Delta_i\) are the irreducible components of \(\Delta\). A stratum of \((X, \Delta)\) is a component of \(\bigcap_{i \in I} \Delta_i\), where \(I \subset \{1, ..., r\}\). If \(\text{coeff}(\Delta) = 1\), a stratum is a log canonical center of \((X, \Delta)\).
2.4. b-divisors. Let $X$ be a projective variety, we say that a formal sum $B = \sum a_\nu \nu$, where the sum ranges over all valuations of $X$, is a b-divisor, if the set
\[ F_X = \{ \nu \mid a_\nu \neq 0 \text{ and the center } \nu \text{ on } X \text{ is a divisor} \}, \]
is finite. The trace $B_Y$ of $B$ is the sum $\sum a_\nu B_\nu$, where the sum now ranges over the elements of $F_Y$.

**Definition 2.1.** Let $(X, \Delta)$ be a pair. If $\pi : Y \to X$ is a birational morphism, then we may write
\[ K_Y + \Delta_Y \sim_Q \pi^*(K_X + \Delta) \]
Define a b-divisor $L_\Delta$ by setting $L_{\Delta,Y} = \Delta_{Y,z} = 0$.

2.5. Minimal models. Let $\phi : X \to Y$ be a proper birational contraction of normal quasi-projective varieties (so that in particular $\phi^{-1}$ contracts no divisors). If $D$ is a $\mathbb{Q}$-Cartier divisor on $X$ such that $D' := \phi_* D$ is $\mathbb{Q}$-Cartier then we say that $\phi$ is $D$-non-positive (resp. $D$-negative) if for a common resolution $p : W \to X$ and $q : W \to Y$, we have $p^* D = q^* D' + E$ where $E \geq 0$ and $p_* E$ is $\phi$-exceptional (resp. $\text{Supp}(p_* E) = \text{Exc}(\phi)$). Suppose that $f : X \to S$ and $f^m : X^m \to S$ are projective morphisms, $\phi : X \to X^m$ is a birational contraction and $(X, \Delta)$ and $(X^m, \Delta^m)$ are log canonical pairs, klt pairs or dlt pairs where $\Delta^m = \phi_* \Delta$. If $a(E, X, \Delta) > a(E, X^m, \Delta^m)$ for all $\phi$-exceptional divisors $E \subset X$, $X^m$ is $\mathbb{Q}$-factorial and $K_{X^m} + \Delta^m$ is nef over $S$, then we say that $\phi : X \to X^m$ is a minimal model. If instead $a(E, X, \Delta) \geq a(E, X^m, \Delta^m)$ for all divisors $E$ and $K_{X^m} + \Delta^m$ is nef, then we call $X^m$ a weak log canonical model of $K_X + \Delta$. A weak log canonical model $\phi : X \to X^m$ is called a semi-ample model if $K_{X^m} + \Delta^m$ is semi-ample, and it is called a good minimal model if $\phi$ is also a minimal model. Notice that by the negativity lemma, all semi-ample models are crepant birationally equivalent to each other.

The following are some results on good minimal models.

**Theorem 2.2** ([Bir12, Theorem 1.8]). Let $(X, \Delta)$ be a $\mathbb{Q}$-factorial dlt pair, $f : X \to Z$ a contraction with $K_X + \Delta \sim_Q P$ where $P \geq 0$ is $f$-very exceptional. Then any $K_X + \Delta$-MMP over $Z$ with scaling of an ample divisor over $Z$ terminates with a model $Y$ on which we have $K_Y + \Delta_Y \sim_Q P_Y = 0$.

**Lemma 2.3** ([HX13, Lemma 2.4]). Let $f : X \to S$ be a projective morphism, $(X, \Delta)$ a dlt pair and $\phi : X \to X^m$ be minimal models for $K_X + \Delta$ over $S$. Then
\begin{enumerate}
\item the set of $\phi$-exceptional divisors coincides with the set of divisors contained in $B_-(K_X + \Delta/S)$ and if $\phi$ is a good minimal model for $K_X + \Delta$ over $S$, then this set also coincides with the set of divisors contained in $B(K_X + \Delta/S)$.
\item $X^m \to X_m$ is an isomorphism in codimension 1 such that $a(E, X_m, \phi_* \Delta) = a(E, X^m, \phi^0 \Delta)$ for any divisor $E$ over $X$ and 
\item if $\phi$ is a good minimal model of $K_X + \Delta$ over $S$, then so is $\phi'$.
\end{enumerate}

**Lemma 2.4** ([HX13, Lemma 2.10]). Let $X$ be a projective variety, $(X, \Delta)$ a dlt pair and $\mu : X' \to X$ a proper birational morphism. We write $K_{X'} + \Delta' = \mu^*(K_X + \Delta) + F$, where $\Delta'$ and $F$ are effective with no common components.

Then $(X, \Delta)$ has a good minimal model if and only if $(X', \Delta')$ has a good minimal model.

**Theorem 2.5** ([HMX18, Theorem 1.2]). Suppose that $(X, \Delta)$ is a log pair where the coefficients of $\Delta$ belong to $[0, 1] \cap \mathbb{Q}$. Let $\pi : X \to S$ be a projective morphism to a smooth variety $S$. Suppose that $(X, \Delta)$ is log smooth over $S$.

If there is a closed point $0 \in S$ such that the fiber $(X_0, \Delta_0)$ has a good minimal model then $(X, \Delta)$ has a good minimal model over $S$ and every fiber has a good minimal model.

2.6. Canonical bundle formula. In this subsection, we give a version of the canonical bundle formula that follows from the work of Kawamata, Fujino-Mori, Ambro, and Kollár (cf. [Kaw98], [FM00], [Amb05] and [Kol07]).
Lemma 2.8. Let $(X, \Delta) \to Z$ be an lc-trivial fibration, suppose by the canonical bundle formula,

$$K_X + \Delta \sim_Q f^*(K_Z + B_Z + M_Z).$$

Then

(a) every generalized log canonical center of $(Z, B_Z + M_Z)$ is dominated by a log canonical center of $(X, \Delta)$,

(b) every $f$-vertical log center of $(X, \Delta)$ dominates a generalized log center of $(Z, B_Z + M_Z)$,

(c) $(X, \Delta)$ is sub-lc if and only if $(Z, B_Z + M_Z)$ is generalized sub-lc, and

(d) let $a \in (0,1)$ be a rational number, if $(Z, B_Z + M_Z)$ is generalized sub-$a$-lc, then $a(E, X, \Delta) \geq -1 + a$ for any divisor $E$ over $X$ whose center is vertical over $Z$. 

Theorem 2.6 (The canonical bundle formula). Let $X, Z$ be normal projective varieties and $f : X \to Z$ a dominant morphism. Let $\Delta$ be a $\mathbb{Q}$-divisor on $X$ such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. We call $(X, \Delta) \to Z$ an lc-trivial fibration if

- $K_X + \Delta \sim_{Q, Z} f^*0$, and
- $h^0(X, \mathcal{O}_X([\Delta_{X, g} \leq 0])) = 1$, where $X_g$ is the general fiber of $f$.

Then one can write

$$K_X + \Delta \sim_Q f^*(K_Z + B_Z + M_Z),$$

where $(Z, B_Z + M_Z)$ is a generalized pair satisfying the following:

(a) $M_Z = M(X/Z, \Delta)$ is the moduli part. It depends only on the crepant birational equivalence of $(F, \Delta|_F)$ and $Z$, where $F$ is the generic fiber of $f$. It is a $b$-divisor and it is the push-forward of a nef $\mathbb{Q}$-divisor $M_{Z'} = M(X'/Z', \Delta')$ by some birational morphism $Z' \to Z$, where $X' \to X \times_Z Z'$ is birational onto the main component with $X'$ normal and projective. We say that the $b$-divisor $M$ descends on $Z'$.

Suppose $B$ is a reduced divisor on $Z$ such that if $D$ is prime divisor not contained in $B$, then

- no component of $\Delta$ dominates $D$, and
- $(X, \Delta + f^*D)$ is lc over the generic point of $D$.

Then we have:

(b) If $X \to Z$ and $\Delta, B$ satisfy the standard normal crossing assumptions, see Definition 2.7, then $M_Z$ is nef and

$$M_{Z'} = \rho^* M_Z,$$

for any birational morphism $\rho : Z' \to Z$.

(c) $B_Z$ is a $\mathbb{Q}$-divisor, called the boundary part. It is supported on $B$.

(d) Let $P \subset B$ be an irreducible divisor. Then

$$\text{coeff}_P(B_Z) = \sup_Q \left\{ 1 - \frac{1 + a(Q, X, \Delta)}{\text{mult}_Q f^*P} \right\}$$

where the supremum is taken over all divisors over $X$ that dominate $P$.

This implies the following:

(e) If $P$ is dominated by a divisor $E$ such that $a(E, X, \Delta) < 0$ (resp. $\leq 0$) then $\text{coeff}_P(B_Z) > 0$ (resp. $\geq 0$).

(f) If $\Delta$ is effective then so is $B_Z$.

(g) If $\text{coeff}_P(B_Z) = 1$ if and only if $P$ is dominated by a divisor $E$ such that $a(E, X, \Delta) = -1$. 

Definition 2.7 (Standard normal crossing assumptions). We say that $f : X \to Z$ and the $\mathbb{Q}$-divisors $\Delta, B$ satisfy the standard normal crossing assumptions if the following hold,

- $(X, \Delta + f^*B)$ and $(Z, B)$ are log smooth, and
- $(X, \Delta)$ is log smooth over $Z \setminus B$. 

Lemma 2.8. Let $f : (X, \Delta) \to Z$ be an lc-trivial fibration, suppose by the canonical bundle formula,

$$K_X + \Delta \sim_Q f^*(K_Z + B_Z + M_Z).$$

Then

(a) every generalized log canonical center of $(Z, B_Z + M_Z)$ is dominated by a log canonical center of $(X, \Delta)$,

(b) every $f$-vertical log center of $(X, \Delta)$ dominates a generalized log center of $(Z, B_Z + M_Z)$,

(c) $(X, \Delta)$ is sub-lc if and only if $(Z, B_Z + M_Z)$ is generalized sub-lc, and

(d) let $a \in (0,1)$ be a rational number, if $(Z, B_Z + M_Z)$ is generalized sub-$a$-lc, then $a(E, X, \Delta) \geq -1 + a$ for any divisor $E$ over $X$ whose center is vertical over $Z$. 

Proof. Let $E$ be any divisor over $Z$. By the weak semi-stable reduction theorem of Abramovich and Karu (cf. [AK00]), we have the following diagram

$$
\begin{array}{c}
(X, \Delta) \xrightarrow{\rho_X} (X', \Delta') \\
\downarrow f \downarrow \downarrow f' \\
Z \xrightarrow{\rho} Z'
\end{array}
$$

where

- $E$ is a divisor on $Z'$,
- $\rho$ and $\rho_X$ are birational,
- $K_{X'} + \Delta' \sim_{Q} \rho_X^*(K_X + \Delta)$,
- $Z'$ is smooth, $X'$ is normal projective with quotient singularities, and
- $f'$ is equidimensional.

(a) Suppose $E$ is a divisor over $Z$ such that $a(E, Z, B_Z + M_Z) = -1$. Write $K_{Z'} + B_{Z'} + M_{Z'} \sim_{Q} \rho^*(K_Z + B_Z + M_Z)$, because $a(E, Z, B_Z + M_Z) = -1$, then $\coeff_E(B_{Z'}) = 1$. By the definition of the boundary part,

$$\coeff_E(B_{Z'}) = \sup_p \{1 - \frac{1 + a(P, X', \Delta')}{\mult_P f^*E}\}$$

then there is a divisor $Q$ on $X'$ dominating $E$ such that $a(Q, X', \Delta') = -1$. Also because $Q$ dominates $E$, $\rho_X(Q)$ is an lc center of $(X, \Delta)$ which dominates $\rho(E)$.

(b) Suppose $Q$ is a divisor over $X$ such that $a(Q, X, \Delta) \in [-1, 0)$ and the image $Q$ does not dominate Z, by [KM98, Lemma 2.45], there is a birational morphism $Z' \to Z$ such that the image of $Q$ on $Z'$ has codimension 1. By the same diagram, it is easy to see that $Q$ is a divisor on $X'$. By assumption, $a(Q, X', \Delta') < 0$, suppose $E$ is the prime divisor on $Z'$ dominated by $Q$,

$$\coeff_E(B_{Z'}) = \sup_p \{1 - \frac{1 + a(P, X', \Delta')}{\mult_P f^*E}\} \geq 1 - \frac{1 + a(Q, X', \Delta')}{\mult_Q f^*E} > 0,$$

which means $E$ is a generalized log center of $(Z, B_Z + M_Z)$.

(c) Let $E$ be any divisor over $Z$, $\rho : Z' \to Z$ be a birational morphism such that $E$ is a divisor on $Z'$. Consider the diagram constructed above. Then by the definition of the boundary part

$$\coeff_E(B_{Z'}) = \sup_p \{1 - \frac{1 + a(P, X', \Delta')}{\mult_P f^*E}\}$$

then $a(E, Z, B_Z + M_Z) \geq -1$ if and only if $\coeff_E(B_{Z'}) < -1$ if and only if $a(P, X', \Delta') \geq -1$ for every prime divisor $P$ that dominates $E$. Thus, $(X, \Delta)$ is sub-lc implies $(Z, B_Z + M_Z)$ is generalized sub-lc.

Conversely, let $Q$ be any divisor over $X$. If $Q$ is $f'$-horizontal, then $a(Q, X', \Delta') \geq -1$ by the definition of lc-trivial fibration. If $Q$ is $f'$-vertical, by [KM98, Lemma 2.45], there is a birational morphism $Z' \to Z$ such that the image of $Q$ on $Z'$ is a divisor. Then for the same reason, $(Z, B_Z + M_Z)$ is generalized sub-lc implies that $(X, \Delta)$ is sub-lc.

(d) let $Q$ be any divisor over $X$ whose center does not dominates $Z$, then by [KM98, Lemma 2.45], there is a birational morphism $Z' \to Z$ such that $Z'$ is smooth and $Q$ dominates a prime divisor $E$ on $Z'$. Because $(Z, B_Z + M_Z)$ is generalized sub-a-lc, then

$$1 - a \geq \coeff_E(B_{Z'}) = \sup_p \{1 - \frac{1 + a(P, X', \Delta')}{\mult_P f^*E}\} \geq 1 - \frac{1 + a(Q, X, \Delta)}{\mult_Q f^*E}$$

Because $Z'$ is smooth, $E$ is a Cartier divisor, then $\mult_Q f^*E \geq 1$, $a(Q, X, \Delta) \geq -1 + a$. \hfill $\Box$

3. POLARIZED CALABI-YAU PAIRS

In this section, we recall some definitions and results in the first arxiv version of [Bir23, Chapter 7], also see [Kol23]
A polarized Calabi-Yau pair consists of a connected projective slc Calabi-Yau pair \((X, \Delta)\) and an ample integral divisor \(N \geq 0\) such that \((X, \Delta + uN)\) is slc for some real number \(u > 0\). Notice that when \((X, \Delta)\) is klt, such \(u\) exists naturally. We refer to such a pair by saying \((X, \Delta), N\) is a polarized Calabi-Yau pair. Fix a natural number \(d\) and positive rational numbers \(c, v, t, r\). If in addition \(\dim X = d, \Delta = cD\) for some integral divisor \(D\) and \(\vol(N) = v\), we call \((X, \Delta), N\) a \((d, c, v)\)-polarized Calabi-Yau pair.

**Definition 3.1.** Let \(S\) be a reduced scheme over \(\mathbb{C}\). A \((d, c, v)\)-polarized Calabi-Yau family over \(S\) consists of a projective morphism \(f : X \to S\) of schemes, and a \(\mathbb{Q}\)-divisor \(\Delta\) and an integral divisor \(N\) on \(X\) such that

- \((X, \Delta + uN) \to S\) is a stable family for some rational number \(u > 0\) with fibers of pure dimension \(d\),
- \(\Delta = cD\) where \(D \geq 0\) is a relative Mumford divisor,
- \(N \geq 0\) is a relative Mumford divisor,
- \(K_{X/S} + \Delta \sim_{\mathbb{Q}} 0/S\), and
- for any fibers \(X_s\) of \(f\), \(\vol(N|_{X_s}) = v\).

Let \(d, c, v, t, r\) be as in Lemma 3.3. Let \(n\) be a natural number. To simplify notation, let \(\Xi = (d, c, v, t, r, \mathbb{P}^n)\). Let \(S\) be a reduced scheme. A strongly embedded \(\Xi\)-polarized Calabi-Yau family over \(S\) is a \((d, c, v)\)-polarized Calabi-Yau family \(f : (X, \Delta), N \to S\) together with a closed embedding \(g : X \to \mathbb{P}^n_S\) such that

- \((X, \Delta + tN) \to S\) is a stable family,
- \(f = \pi \circ g\) where \(\pi\) denotes the projection \(\mathbb{P}^n_S \to S\),
- letting \(\mathcal{L} := g^*O_{\mathbb{P}^n_S}(1)\), we have \(R^qf_*\mathcal{L} \cong R^q\pi_*O_{\mathbb{P}^n}(1)\) for all \(q\), and
- for every \(s \in S\), we have \(\mathcal{L}_s \cong O_{X_s}(r(K_{X_s} + \Delta_s + tN_s))\).

Denote the functor \(\mathcal{E}^*\mathcal{PCY}_\Xi\) on the category of reduced schemes by setting

\[\mathcal{E}^*\mathcal{PCY}_\Xi(S) = \{\text{strongly embedded } \Xi\text{-polarized Calabi-Yau families over } S\}\]

**Theorem 3.2.** The functor \(\mathcal{E}^*\mathcal{PCY}_\Xi\) has a fine moduli space \(\mathcal{H}\), which is a reduced separated scheme of finite type, and an universal family \((X \subset \mathbb{P}^n_S, D), N \to S\).

**Lemma 3.3.** Let \(d\) be a natural number and \(c, v\) be positive rational numbers. Then there exists a positive rational number \(t\) and a natural number \(r\) such that \(rc, rt \in \mathbb{N}\) satisfying the following. Assume \((X, \Delta), N\) is a \((d, c, v)\)-polarized Calabi-Yau pair. Then

- \((X, \Delta + tN)\) is slc,
- \(\Delta + tN\) uniquely determines \(\Delta, N\), and
- \(r(K_X + \Delta + tN)\) is very ample with

\[h^j(mr(K_X + \Delta + tN)) = 0\]

for \(m, j > 0\).

**Proof.** This is Lemma 7.7 in the first arxiv version of [Bir23].

**Remark 3.4.** Fix a positive integer \(d\), two positive rational numbers \(c, v\). Let \(f : (X, \Delta) \to Z\) be a fibration, \(N\) a divisor on \(X\), denote the general fiber of \(f\) by \(X_g\). Suppose

- \((X_g, \Delta_g)\) is a \(d\)-dimensional klt pair,
- \(\coeff\Delta_g \subset c\mathbb{N}\),
- \(K_{X_g} + \Delta_g \sim_{\mathbb{Q}, \mathbb{Z}} 0\), and
- \(N_g\) is ample and \(\vol(N_g) = v\).

It is easy to see there is an open subset \(U \subset Z\) such that \((X_U, \Delta_U), N_U \to U\) is a \((d, c, v)\)-polarized Calabi-Yau family over \(U\). By Lemma 3.3, there exists \(t \in \mathbb{Q}_{\geq 0}\) and \(r \in \mathbb{N}\) such that \((X_g, \Delta_g + tN_g)\) is slc and \(r(K_{X_g} + \Delta_g + tN_g)\) is very ample without higher cohomology for every
closed point \( g \in U \). Then by cohomology and base change, \( r(K_{X_U} + \Delta_U + tN_U) \) is relatively very ample, and it defines a closed embedding \( g : X_U \hookrightarrow \mathbb{P}^n_U \). Also because \((X_U, \Delta_U + tN_U) \to U \) is a stable family, \( f_U : (X_U \subset \mathbb{P}^n_U, \Delta_U), N_U \to U \) is a strongly embedded polarized slc Calabi-Yau family over \( U \). Since \( \mathcal{E}^{xy} \mathcal{C}_\text{gen} \) has a fine moduli space with the universal family \((X \subset \mathbb{P}^n_S, D), \mathcal{N} \to \mathcal{S}, \) we have \((X_U, \Delta_U) \cong (X, D) \times_S U \), where \( U \to \mathcal{S} \) is the moduli map defined by \( f_U \).

### 4. LC-Trivial Fibrations

**Theorem 4.1.** Let \((X, \Delta)\) be a log canonical pair, \(f : (X, \Delta) \to Z\) an algebraic contraction to a projective normal \(\mathbb{Q}\)-factorial variety with general fiber \(X_g\). Suppose \(K_{X_g} + \Delta_g \sim_{\mathbb{Q}} 0\). Assume there is a crepant birational morphism \(g : (X', \Delta') \to (X, \Delta)\) and a divisor \(D\) on \(Z\), such that the morphism \(h := f \circ g : (X', \Delta') \to (X, \Delta)\) and \(g\) are an lc-trivial fibration induced by the normalization of the main component of the base change.

Then there is a \(\mathbb{Q}\)-divisor \(\overline{\Delta}_g\) on \(X'\), such that

1. \(\overline{\Delta}_g = \Delta'_g\),
2. \((X', \overline{\Delta})\) is log smooth over \(Z \setminus D\) and,
3. \((X', \overline{\Delta}) \to Z\) is an lc-trivial fibration.

**Proof.** Let \(\eta\) denote the generic point of \(Z\). Since \((X_g, \Delta_g)\) is a Calabi-Yau variety, then \(K_{X_g} + \Delta_g \sim_{\mathbb{Q}} 0\), which means that there exists a vertical \(\mathbb{Q}\)-divisor \(B'\) such that \(K_{X'} + \Delta' + B' \sim_{\mathbb{Q}, Z} 0\).

Suppose \(B' = R + G\), where \(\text{Supp}(R) \not\subset h^{-1}(\text{Supp}(D))\) and \(\text{Supp}(G) \subset h^{-1}(\text{Supp}(D))\). Because \(R\) is vertical, \(Z\) is \(\mathbb{Q}\)-factorial and \(h\) is smooth over \(Z \setminus \text{Supp}(D)\), \(h(R)\) is a Cartier divisor on \(Z\), which is denoted by \(R_Z\). Then there exists a \(\mathbb{Q}\)-divisor \(F_R\) supported on \(h^{-1}(\text{Supp}(D))\), such that \(R + F_R = h^*R_Z\).

We have \(K_{X'} + \Delta' + B' - (R + F_R) \sim_{\mathbb{Q}, Z} 0\). Let \(\overline{\Delta} := \Delta' + B' - (R + F_R)\), then \(K_{X'} + \overline{\Delta} \sim_{\mathbb{Q}, Z} 0\), and \(\overline{\Delta}_g = \Delta'_g\). Write \(\Delta'_g = \Delta'_{g, \geq 0} - \Delta'_{g, \leq 0}\). Since \(\Delta'_{g, \leq 0}\) is \(g\)-exceptional, it is easy to see that \((X', \overline{\Delta}) \to Z\) is an lc-trivial fibration. Because \((X', \Delta')\) is log smooth over \(Z \setminus D\), \(\text{Supp}(F_R) \subset h^{-1}(D)\) and \(\text{Supp}(B' - R) \subset h^{-1}(D)\), then \((X', \overline{\Delta})\) is log smooth over \(Z \setminus D\). \(\square\)

**Proposition 4.2** ([Amb05, Proposition 3.1]). Let \(f : (X, \Delta) \to Z\) be an lc-trivial fibration, \(\rho : Z' \to Z\) be a surjective morphism from a proper normal variety \(Z'\) and let \(f' : (X', \Delta') \to Z'\) be an lc-trivial fibration induced by the normalization of the main component of the base change.

\[
\begin{array}{ccc}
(X, \Delta) & \xrightarrow{\rho_X} & (X', \Delta') \\
\downarrow f & & \downarrow f' \\
Z & \xleftarrow{\rho} & Z'
\end{array}
\]

Let \(M\) and \(M'\) be the corresponding moduli \(b\)-divisors. If \(f\) satisfies the standard normal crossing assumption, then

\[
\rho^*M_Z = M'_{Z'}.\]

**Theorem 4.3** ([Jia21, Theorem 6.4]). Let \((X, \Delta)\) be a log canonical pair, \(f : (X, \Delta) \to Z\) an lc-trivial fibration to a smooth projective variety \(Z\), \(g : (X', \Delta') \to (X, \Delta)\) a crepant birational morphism which is also a log resolution of \((X, \Delta)\). Suppose \(D \subset Z\) is a smooth divisor on \(Z\) such that \(h := g \circ f : (X', \Delta') \to Z\) is log smooth over the generic point \(\eta_D\) of \(D\).

Let \(Y\) be the normalization of the irreducible component of \(f^{-1}(D)\) that dominates \(D\), \(\Delta_Y\) the \(\mathbb{Q}\)-divisor on \(Y\) such that \(K_Y + \Delta_Y = (K_X + \Delta + f^*D)|_Y\). Let \(M_Z\) denote the moduli part of \((X, \Delta) \to Z\). Suppose there is a reduced divisor \(B\) on \(Z\) such that \(B + D\) is a reduced simple normal crossing divisor, the morphism \(h : X' \to Z\) and \(\Delta', B\) satisfy the standard normal crossing assumptions. Then \((Y, \Delta_Y) \to D\) is an lc-trivial fibration and its moduli part \(M_D\) is equal to \(M_Z|_D\).
**Theorem 4.4** ([Amb05, Theorem 3.3] and [Jia21, Theorem 6.5]). Let \( f : (X, \Delta) \to S \) be an lc-trivial fibration such that the general fiber \( X_g \) is projective variety and \( \Delta_g \) is effective, then there exists a diagram

\[
\begin{array}{c}
(X, \Delta) \\
\downarrow f \\
S \\
\leftarrow \tau \leftarrow \delta \leftarrow \rho \leftarrow \phi \\
S^i \\
\downarrow \pi \\
S^* \\
\end{array}
\]

satisfying the following properties:

1. \( f^i : (X^i, \Delta^i) \to S^i \) is an lc-trivial fibration.
2. \( \tau \) and \( \pi \) are generically finite and surjective morphisms, \( \rho \) is surjective.
3. There exists a nonempty open subset \( U \subset S \) and an isomorphism

\[
(X, \Delta) \times_S \bar{S}|_U \cong (X^i, \Delta^i) \times_{S^i} \bar{S}|_U
\]

4. Let \( M, M^i \) be the corresponding moduli-\( b \)-divisors, write \( \rho := \Phi \circ \tau \), then \( M^i \) is \( b \)-nef and big. And if \( M \) descends on \( S \) and \( M^i \) descends on \( S^i \), then \( \tau^* M_S = \rho^* M^i_{S^i} \).
5. There is a rational map \( \Phi : S \to S^* \), which is an extension of the period map defined in [Amb05, Chapter 2], and a rational map \( i : S^i \dashrightarrow S \).

**Remark 4.5.** In this remark, we define some notations which will be used in the proof of Theorem 1.2.

Let \( d \) be a natural number and \( c, v \) positive rational numbers. Let \( t, r \) be the number defined in Lemma 3.3, \((\mathcal{X} \subset \mathbb{P}^n, \mathcal{D}), \mathcal{N} \to S \) the universal family of the functor \( \mathcal{E}^s \mathcal{P} C \mathcal{Y}_\Xi \), where \( \Xi := (d, c, v, t, r, \mathbb{P}^n) \). Because \((\mathcal{X}, \mathcal{D}) \to S \) is a family of log Calabi-Yau pairs, by Theorem 4.4, we have the following diagram

\[
\begin{array}{c}
(\mathcal{X}, \mathcal{D}) \\
\downarrow F \\
S \\
\leftarrow \tau \leftarrow \bar{\delta} \leftarrow \rho \leftarrow \phi \\
(\mathcal{X}^i, \mathcal{D}^i) \\
\downarrow F \\
S^i \\
\downarrow \pi \\
S^* \\
\end{array}
\]

After passing to a stratification of \( S \), we may assume \((\mathcal{X}, \text{Supp} \mathcal{D}) \to S \) has a fiberwise log resolution over \( \mathcal{H} \), where \( \mathcal{H} \) is the disjoint union of the locally closed subset of \( S \) and \( \mathcal{H} \) is smooth. Then we replace \( S \) with a compactification of \( \mathcal{H} \) such that \( \mathcal{B} := S \setminus \mathcal{H} \) is an snc divisor, and \( \Phi \) is a morphism on \( S \).

Define \((\mathcal{X}_{\mathcal{H}}, \mathcal{D}_{\mathcal{H}}) := (\mathcal{X}, \mathcal{D}) \times_{\mathcal{X}} \mathcal{H}, \mathcal{H}^* := \Phi(\mathcal{H}), \mathcal{H}^i := \pi^{-1} \mathcal{H}^*, \bar{\mathcal{H}} := \tau^{-1} \mathcal{H} \cap \rho^{-1} \mathcal{H}^i \) and \((\mathcal{X}^i_{\mathcal{H}}, \mathcal{D}^i_{\mathcal{H}}) := ((\mathcal{X}^i_{S^i}, \mathcal{D}^i_{S^i})) \times_{S^i} \mathcal{H}^i \). By the Stein factorization, after replacing \( \mathcal{H}^i \) by a finite cover, we may assume that the general fiber of \( \rho^i_{\mathcal{H}} : \mathcal{H} \to \mathcal{H}^i \) is irreducible.

Let \( \mathcal{Y}_{\mathcal{H}} \to \mathcal{X}_{\mathcal{H}} \) (resp. \( \mathcal{Y}_{\mathcal{H}}^i \to \mathcal{X}^i_{\mathcal{H}} \)) be a log resolution of \((\mathcal{X}_{\mathcal{H}}, \text{Supp} \mathcal{D}_{\mathcal{H}}) \) (resp. \((\mathcal{X}^i_{\mathcal{H}}, \text{Supp} \mathcal{D}^i_{\mathcal{H}}) \)) and \((\mathcal{Y}_{\mathcal{H}}, \mathcal{B}_{\mathcal{H}}) \to (\mathcal{X}_{\mathcal{H}}, \mathcal{D}_{\mathcal{H}}) \) (resp. \((\mathcal{Y}^i_{\mathcal{H}}, \mathcal{B}^i_{\mathcal{H}}) \to (\mathcal{X}^i_{\mathcal{H}}, \mathcal{D}^i_{\mathcal{H}}) \)) be the crepant birational morphism.

By Theorem 4.5, (3), perhaps after replacing \( \mathcal{H} \) by a stratification, we may assume that for every locally closed subset \( Z \in \mathcal{H} \), the generic fiber of \((\mathcal{Y}_{\mathcal{H}}, \mathcal{B}_{\mathcal{H}}) \times_{\mathcal{H}} \tau^{-1} Z \to \tau^{-1} Z \) is crepant birationally equivalent to the generic fiber of \((\mathcal{Y}^i_{\mathcal{H}}, \mathcal{B}^i_{\mathcal{H}}) \times_{\mathcal{H}} \tau^{-1} Z \to \tau^{-1} Z \).

Define \( \mathcal{X}_{\bar{\mathcal{H}}} := \mathcal{X}_{\mathcal{H}} \times_{\mathcal{H}} \bar{\mathcal{H}} = \mathcal{X}^i_{\mathcal{H}} \times_{\mathcal{H}} \bar{\mathcal{H}} \), by the construction of \( \mathcal{X}^i \), we have \( \mathcal{X}_{\mathcal{H}} \cong \mathcal{X}^i_{\mathcal{H}} \times_{\mathcal{H}} \bar{\mathcal{H}} \). Let \( \tau_{\mathcal{X}}, \rho_{\mathcal{X}} \) denote the natural projections \( \mathcal{X}_{\bar{\mathcal{H}}} \to \mathcal{X}_{\mathcal{H}} \) and \( \mathcal{X}_{\bar{\mathcal{H}}} \to \mathcal{X}^i_{\mathcal{H}} \).
Let \( \{\tilde{\Gamma}_{H,k}^i \}_{k=1,...,d+1} \) be \( d+1 \) general sufficient ample divisors on \( X_{H}^i \) such that \( \cap_{1 \leq k \leq d+1} \text{Supp}(\tilde{\Gamma}_{H,k}^i) = \emptyset \), where \( d \) is the relative dimension. \( \rho^*_X \tilde{\Gamma}_{H,k}^i \) is a divisor on \( \tilde{\mathcal{X}}_H \), we claim the following:

**Claim 4.6.** The divisor \( \Gamma_{H,k} := \tau_{\tilde{\mathcal{X}}_H}^* (\tau_{\tilde{\mathcal{X}}_H})_* \rho^*_X \tilde{\Gamma}_{H,k}^i \) is vertical over \( X_{H}^i \), via the projection \( \tilde{\mathcal{X}}_H = X_{H}^i \times H \rightarrow X_{H}^i \) and \( \cap_{1 \leq k \leq d+1} \text{Supp}(\Gamma_{H,k}) = \emptyset \).

Because the general fiber of \( \tilde{\mathcal{H}} \rightarrow \mathcal{H} \) is irreducible, \( \tilde{\Gamma}_{H,k}^i \) is general and \( \Gamma_{H,k} \) is vertical over \( X_{H}^i \), then the general fiber of \( \rho_X : \tilde{\mathcal{X}}_H \rightarrow X_{H}^i \) is irreducible and \( \Gamma_{H,k} \) is equal to the pullback of a divisor \( \Gamma_{H,k}^i \) on \( X_{H}^i \). And it is easy to see that \( \cap_{1 \leq k \leq d+1} \text{Supp}(\Gamma_{H,k}) = \emptyset \). By definition, \( \tilde{\mathcal{H}} \) is also equal to the pullback of a divisor \( \tilde{\mathcal{X}}_H \) on \( \mathcal{X}_H \), and it is easy to see \( \cap_{1 \leq k \leq d+1} \text{Supp}(\Gamma_{H,k}) = \emptyset \).

Let \( \tilde{\mathcal{C}}_{H,k} \) be the pullback of \( \Gamma_{H,k} \) on \( \tilde{\mathcal{Y}}_H \) and \( \tilde{\mathcal{C}}_{H,k}^i \) be the pullback of \( \Gamma_{H,k}^i \) on \( \tilde{\mathcal{Y}}_{H}^i \), it is easy to see that \( \cap_{1 \leq k \leq d+1} \text{Supp}(\tilde{\mathcal{C}}_{H,k}) = \emptyset \) and \( \cap_{1 \leq k \leq d+1} \text{Supp}(\tilde{\mathcal{C}}_{H,k}^i) = \emptyset \).

We replace \( \mathcal{S} \) by a higher birational model such that the moduli part \( \mathcal{M}_S \) of \( (\mathcal{X}_S^i, \mathcal{E}_S^i) \rightarrow \mathcal{S} \) descends on \( \mathcal{S} \). By Theorem 4.4, \( \mathcal{M}_S \) is nef and big, then by Lemma 5.3, we may choose a member in \( \mathcal{M}_S \) such that

- \( \mathcal{S} \setminus \text{Supp}(\mathcal{M}_S) \subseteq \mathcal{H} \),
- there is a very ample divisor \( \mathcal{A} \) on \( \mathcal{S} \) such that \( \text{Supp}(\mathcal{M}_S) \supset \text{Supp} \mathcal{A} \), and
- \( \mathcal{A} \) is weakly bounded (see Definition 5.2).

We replace \( \mathcal{H} \) by \( \mathcal{A} \), after passing to a stratification of \( \mathcal{H} \) and replacing \( \mathcal{H}, \mathcal{H}^i \) by the preimages of \( \mathcal{H} \), we keep the surjectivity of \( \Phi, \pi, \rho, \tau \).

At last we fix an integer \( l \gg 0 \) such that \( l \mathcal{M}_S \) is Cartier and \( l \mathcal{M}_S = \pi^{-1} \mathcal{A} \geq 0 \).

**Proof of claim.** Let \( h \in \mathcal{H} \) be a closed point and \( h \in \pi^{-1} h^* \) a closed point, \( \mathcal{X}_h \) be the fiber of \( \mathcal{X}_h^i \rightarrow \mathcal{H} \) over \( h \), define \( \mathcal{H}_h := \Phi^{-1} h^* \), \( \mathcal{H}_h := \tau^{-1} \mathcal{H}_h, \mathcal{X}_h \rightarrow \mathcal{H} \) and \( \tilde{\mathcal{X}}_h \rightarrow \mathcal{H}_h \) be the corresponding fiber product. By the property of period map (see Theorem 4.5), it is easy to see that \( \mathcal{X}_h \rightarrow \mathcal{H}_h \) is an isotrivial family with general fiber equal to \( \mathcal{X}_h \) and \( \tilde{\mathcal{X}}_h \rightarrow \mathcal{H}_h \). By the structure of the isotrivial family, there is a Galois étale cover \( \tau^\#_{\tilde{\mathcal{X}}_h} : \mathcal{H}_h^\# \rightarrow \mathcal{H}_h \) with Galois group \( G \) which also acts on \( \mathcal{X}_h^i \), such that \( \mathcal{X}_h^\# := \mathcal{X}_h \times _{\mathcal{H}_h} \mathcal{H}_h^\# \cong \mathcal{X}_h^i \times _{\mathcal{H}_h} \mathcal{H}_h^\# \) and \( \mathcal{X}_h \cong \mathcal{X}_h^i \times _{\mathcal{H}_h^\#} G / G \), where \( G \) acts diagonally on \( \mathcal{X}_h^i \times _{\mathcal{H}_h^\#} G \). Perhaps replace \( \mathcal{H}^\# \) with a higher Galois cover, we may assume that there is a finite cover \( \tau^\#_h : \mathcal{H}_h^\# \rightarrow \mathcal{H}_h \) and \( \tau^\#_h = \tau_h \circ \tau^\#_{\tilde{\mathcal{X}}_h} \).

Denote the natural projections \( \mathcal{X}_h^\# \rightarrow \mathcal{X}_h^i \) and \( \mathcal{X}_h^\# \rightarrow \tilde{\mathcal{X}}_h \) by \( \pi_{\mathcal{X}_h^i} \) and \( \pi_{\tilde{\mathcal{X}}_h} \). Define a divisor \( C_h^\# := \sum_{g \in G} g^*(\pi_{\mathcal{X}_h^i})^*(\rho^*_X \tilde{\Gamma}_{H,k}^i)|_{\mathcal{X}_h^\#} \) on \( \mathcal{X}_h^\# \). It is easy to see that \( C_h^\# \) is \( G \)-invariant. Also because \( \mathcal{X}_h \cong \mathcal{X}_h^i \times _{\mathcal{H}_h^\#} G / G \), there is a divisor \( C_h \) on \( \mathcal{X}_h^i \) such that \( (\pi_{\mathcal{X}_h^i})^* C_h = C_h^\# \). Also because \( G \) acts on \( \mathcal{X}_h^i \times _{\mathcal{H}_h^\#} G \) diagonally, \( C_h^\# \) is vertical over \( \mathcal{X}_h^i \). Since the divisor \( (\tau^\#_{\tilde{\mathcal{X}}_h})^* C_h \) on \( \tilde{\mathcal{X}}_h \) satisfies \( (\tau^\#_{\tilde{\mathcal{X}}_h})^* C_h \leq (\pi_{\mathcal{X}_h^i})^* C_h \), \( (\tau^\#_{\tilde{\mathcal{X}}_h})^* C_h \) is vertical over \( \mathcal{X}_h^i \). It is easy to see that \( \cap_{1 \leq k \leq d+1} \text{Supp}(C_h) = \emptyset \).

Because \( \pi \) is a finite cover, then \( ((\tau_{\tilde{\mathcal{X}}_h})^* \rho^*_X \tilde{\Gamma}_{H,k}^i)|_{\mathcal{X}_h} \leq C_h \), and \( (\tau^\#_{\tilde{\mathcal{X}}_h})^* C_h \) is vertical over \( \mathcal{X}_h^i \) implies that \( (\tau^\#_{\tilde{\mathcal{X}}_h})^* \rho^*_X \tilde{\Gamma}_{H,k}^i)|_{\mathcal{X}_h} \) is vertical over \( \mathcal{X}_h^i \). Because \( \cap_{1 \leq k \leq d+1} C_h = \emptyset \), \( \cap_{1 \leq k \leq d+1} ((\tau_{\tilde{\mathcal{X}}_h})^* \rho^*_X \tilde{\Gamma}_{H,k}^i)|_{\mathcal{X}_h} = \emptyset \). Because \( h \) is any closed point, then \( (\tau_{\tilde{\mathcal{X}}_h})^* \rho^*_X \tilde{\Gamma}_{H,k}^i \) is vertical over \( \mathcal{X}_h^i \) and \( \cap_{1 \leq k \leq d+1} ((\tau_{\tilde{\mathcal{X}}_h})^* \rho^*_X \tilde{\Gamma}_{H,k}^i) = \emptyset \). \( \square \)

5. **Weak boundedness**

The definition of weak boundedness is first introduced in [KL10]. We find this definition useful in proving the birational boundedness of fibrations with bounded general fibers.
Definition 5.1. A \((g, m)\)-curve is a smooth curve \(C^0\) whose smooth compactification \(C\) has genus \(g\) and such that \(C \setminus C^0\) consists of \(m\) closed points.

Definition 5.2. Let \(W\) be a proper \(k\)-scheme with a line bundle \(\mathcal{N}\) and let \(U\) be an open subset of a proper variety. We say a morphism \(\xi : U \to W\) is weakly bounded with respect to \(\mathcal{N}\) if there exists a function \(b_\mathcal{N} : \mathbb{Z}^2_{\geq 0} \to \mathbb{Z}\) such that for every pair \((g, m)\) of non-negative integers, for every \((g, m)\)-curve \(C^0 \subseteq C\), and for every morphism \(C^0 \to U\), one has that \(\deg \xi^* \mathcal{N} \leq b_\mathcal{N}(g, m)\), where \(\xi_C : C \to W\) is the induced morphism. The function \(b_\mathcal{N}\) will be called a weak bound, and we will say that \(\xi\) is weakly bounded by \(b_\mathcal{N}\).

We say an open subset \(U\) of a proper variety is weakly bounded if there exists a compactification \(i : U \hookrightarrow W\), such that \(i : U \to W\) is weakly bounded with respect to an ample line bundle \(\mathcal{N}\) on \(W\).

Lemma 5.3 ([Jia21, Lemma 5.3]). Let \(T\) be a quasi-projective variety. Then we can decompose \(T\) into finitely many locally closed subsets \(\bigcup_{i \in I} T_i\), such that each \(T_i\) is weakly bounded.

The following Theorem says that morphisms from bounded varieties to weakly bounded varieties can be parametrized by a scheme of finite type.

Theorem 5.4 ([KL10, Proposition 2.14]). Let \(T\) be a quasi-compact separated reduced \(\mathbb{C}\)-scheme and \(\mathcal{U} \to T\) a smooth morphism. Given a projective \(T\)-variety and a polarization over \(T\), \((\mathcal{M}, \mathcal{O}_{\mathcal{M}}(1))\), an open subvariety \(\mathcal{M}^0 \hookrightarrow \mathcal{M}\) over \(T\), and a weak bound \(b\), there exists a \(T\)-scheme of finite type \(\mathcal{U}^b\) and a morphism \(\Theta : \mathcal{U}^b \times \mathcal{U} \to \mathcal{M}^0\) such that for every geometric point \(t \in T\) and for every morphism \(\xi : \mathcal{U}_t \to \mathcal{M}^0_t \subset \mathcal{M}_t\) that is weakly bounded by \(b\) there exists a point \(p \in \mathcal{U}^b_t\) such that \(\xi = \Theta|_{\{p\} \times \mathcal{U}_t}\).

In particular, if \(\mathcal{M}^0\) is weakly bounded and \(\mathcal{M}\) is the compactification, by definition, every morphism \(\xi : \mathcal{U}_t \to \mathcal{M}^0_t \subset \mathcal{M}_t\) is weakly bounded by a weak bound \(b\), then \(\xi = \Theta|_{\{p\} \times \mathcal{U}_t}\) for a closed point \(p \in \mathcal{U}^b_t\).

6. Birational boundedness of fibrations

Definition 6.1. Let \(d\) be a positive integer, \(I \subset [0, 1] \cap \mathbb{Q}\) a DCC set and \(v, V\) two positive rational numbers. Let \(\mathcal{P}(d, I, v, V)\) be the set of pairs \((X, \Delta)\) satisfying the following properties:

- \((X, \Delta)\) is a \(d\)-dimensional klt pair,
- \(\text{coeff}(\Delta) \subset I\),
- there is a contraction \(f : X \to Z\) such that \(K_X + \Delta \sim_{Q,Z} 0\), by the canonical bundle formula, we write \(K_X + \Delta \sim_{Q,Z} f^*(K_Z + B_Z + M_Z)\).
- \(\kappa(Z, K_Z + B_Z + M_Z) \geq 0\),
- there is a divisor \(A\) on \(X\) such that \(A_g := A|_{X_g}\) is ample and \(\text{vol}(A_g) = v\), where \(X_g\) is a general fiber of \(f\).
- there is a divisor \(H\) on \(Z\) such that \(\text{vol}(H + K_Z + B_Z + M_Z) \leq V\) and \(|H|\) defines a birational map.

Theorem 6.2. Let \(d\) be a positive integer, \(I \subset [0, 1] \cap \mathbb{Q}\) a DCC set and \(v, V\) two positive rational numbers. Then

\[\mathcal{P}(d, I, v, V)\]

is birationally bounded.

Proof of Theorem 6.2. We use the same notation as in Remark 4.5.

Step 1. In this step, we show that the base \(Z\) is birationally bounded. Fix \((X, \Delta) \in \mathcal{P}(d, I, v, V)\) with a contraction \(f : X \to Z\) and a divisor \(A\). Because \(|H|\) defines a birational map \(h : Z \dashrightarrow W\), let \(p : Z' \to Z, q : Z' \to W\) be a common resolution, then there is a very ample divisor \(H_W\) on \(W\) and an effective divisor \(F\) on \(Z'\) such that

\[(6.1) \quad p^*H = q^*H_W + F.\]
It is easy to see that $\text{vol}(H_W) \leq \text{vol}(H) \leq V$. By the boundedness of Chow varieties, $W$ is in a bounded family. We denote this bounded family by $W \to T$ and the natural relative very ample divisor by $H$, suppose $W \cong W_t$ and $H_W = H_{W_t}$.

After replacing $W$ by a log resolution of the generic fiber of $W \to T$ and passing to a stratification of $T$, we may assume that $W \to T$ is a smooth morphism. Because $H$ is big, we can replace $H_W$ by a very ample divisor on the new family and $H$ by a higher multiple so that equation (6.1) still holds.

**Step 2.** Choose a positive rational number $c$ such that $\mathcal{I} \subset c\mathbb{N}$. It is easy to see that the general fiber $(X_g, \Delta_g)$, $A_g$ of $f$ is a $(d, c, v)$-polarized log Calabi-Yau pair. By Remark 3.4, there is a universal family $(\mathcal{X}, \mathcal{D}) \to \mathcal{S}$ and an open subset $U \subset Z$ such that $(X, \Delta) \times_Z U$ is equal to the pullback of $(\mathcal{X}, \mathcal{D}) \to \mathcal{S}$ via the moduli map $\phi : U \to \mathcal{S}$.

We replace $U$ by an open subset such that $\phi(U) \subset \mathcal{H}$. By Remark 4.5, the pullback of $(\mathcal{Y}_H, \mathcal{B}_H) \to \mathcal{H}$ by $\phi$ is a log resolution of $(X_U, \Delta_U)$, denote it by $(Y_U, B_U)$.

Because $Z$ is birationally equivalent to $W_t$, then $\phi$ is a rational map on $W_t$.

**Claim 6.3.** After passing to a stratification of $T$ and a resolution of the generic fiber of $W \to T$, there exists a reduced divisor $D$ on $W$, such that

- $(\mathcal{W}, \mathcal{D})$ is log smooth over $T$,
- the moduli part of $(X, \Delta) \to Z$ descends on $W_t$, and
- define $\phi^* := \phi \circ \phi$ and $U_t := W_t \setminus D_t$, then $\phi^*$ extends to a morphism on $U_t$ and $\phi^*(U_t) \subset H^*.

**Step 3.** In this step, we show that for every $P(d, \mathcal{I}, v, V)$, the corresponding morphism $\phi^* : U_t \to H^*$ can be parametrized by a finite type scheme.

Because $H^*$ is weakly bounded, by Theorem 5.4, there is a finite type $T$-scheme $\mathcal{W}$ and a morphism $\Theta^* : \mathcal{W} \times U \to H^*$, such that $\phi^* = \Theta^*|_{\{p\} \times U_t}$ for a closed point $p \in \mathcal{W}$. We replace $U \to T$ by $\mathcal{W} \times U \to \mathcal{W} \times T$.

Let $U'_t := U \times_{\mathcal{H}^*} \mathcal{H}_t$, denote the morphism $U \to \mathcal{H}^*$ by $\Theta$, define $\mathcal{Y}'_{U_t}$ to be the pullback of $\mathcal{Y}_{H^*} \to \mathcal{H}^*$ by $\Theta$ and $\mathcal{Y}'_{U_t}$ the fiber over $U'_t$.

**Step 4.** We construct a log general type pair on $\mathcal{Y}'_{U_t}$ with bounded volume in this step.

After passing to a stratification of $T$; we may choose $W'$ to be a smooth compactification of $U'$ over $T$ such that there is a generically finite morphism $\Pi : W' \to W$, and $D' := W' \setminus U'$ is a relatively snc divisor over $T$.

Let $(\mathcal{Y}'_{U_t}, \mathcal{B}'_{U_t} + C_{U_t,k}) \to U'_t$, $k = 1, \ldots, d + 1$ be the pullback of $(\mathcal{Y}_{H^*}, \mathcal{B}_{H^*} + C_{H^*,k}) \to \mathcal{H}^*$ by $\Theta' : U' \to \mathcal{H}^*$. Because $(\mathcal{Y}_{H^*}, \mathcal{B}_{H^*} + C_{H^*,k})$ is log smooth over $\mathcal{H}^*$, $(\mathcal{Y}'_{U_t}, \mathcal{B}'_{U_t} + C_{U_t,k})$ is log smooth over $U'$. After passing to a stratification of $T$, we can choose an extension $(\mathcal{Y}_{W'_t}, \mathcal{B}_{W'_t} + C_{W'_t,k})$ of $(\mathcal{Y}'_{U_t}, \mathcal{B}'_{U_t} + C_{U_t,k})$, such that

- there is a morphism $\bar{\phi}^* : \mathcal{Y}_{W'_t} \to \mathcal{W}'$, which is an extension of $\mathcal{Y}'_{U_t} \to U'$ and
- $(\mathcal{Y}_{W'_t}, \mathcal{B}_{W'_t} + C_{W'_t,k} + \text{red}(\bar{\phi}^*(D'_t)))$ is log smooth sub-lc for every closed point $s \in T$.

Let $\mathcal{L}_s$ be a very ample line bundle on $W$ over $T$ such that $\omega_{W_t} \otimes \mathcal{L}_s$ is big for every $s \in T$. Because $\mathcal{L}_s$ is very ample, we can choose a general member $\mathcal{H}_s \in |\mathcal{L}_s|$ such that $(\mathcal{Y}_{W'_t}, \mathcal{B}_{W'_t} + C_{W'_t,k} + \bar{\phi}^*(\mathcal{H}_s))$ is log smooth sub-lc for every closed point $s \in T$, where $\mathcal{H}^*_s := \Pi_* \mathcal{H}_s$.

Let $\mathcal{M}_{W'_t}$ be the moduli divisor corresponding to the lc-trivial fibration $(\mathcal{Y}_{W'_t}, \mathcal{B}_{W'_t}) \to \mathcal{U}'_t$. Because $(\mathcal{Y}_{W'_t}, \mathcal{B}_{W'_t})$ is log smooth over $U'_t = W'_t \setminus D'_t$ and $(\mathcal{W}', D'_t)$ is log smooth, then $\mathcal{M}_{W'_t}$ descends on $W'_t$. 

Because $\mathcal{H}_s$ is general, $\mathfrak{P}_s^*\mathcal{H}'_s = \text{red}(\mathfrak{P}_s^*\mathcal{H}')$. Also because $(\mathcal{Y}'_{s_1}, \mathcal{B}'_{s_1}) \to \mathcal{H}'_1$ is an lc-trivial fibration and $(\mathcal{W}_s, \mathcal{M}'_{\mathcal{W}_s} + \mathcal{H}'_s + \mathcal{D}'_s)$ is generalized lc, by Lemma 2.8,

$$K_{\mathcal{Y}_{\mathcal{W}_s}} + \mathcal{B}'_{\mathcal{W}_s} + \mathfrak{P}_s^*\mathcal{H}'_s + \text{red}(\mathfrak{P}_s^*\mathcal{D}'_s) \geq \mathfrak{P}_s^*(K_{\mathcal{W}_s} + \mathcal{M}'_{\mathcal{W}_s} + \mathcal{H}_s + \mathcal{D}_s).$$

Since $C_{\mathcal{W}_s,k}$ big over $\mathcal{W}_s$ and $K_{\mathcal{W}_s} + \mathcal{M}'_{\mathcal{W}_s} + \mathcal{H}_s + \mathcal{D}_s$ is big, $(\mathcal{Y}'_{\mathcal{W}_s}, B'_{\mathcal{W}_s} + C'_{\mathcal{W}_s,k} + \mathfrak{P}_s^*\mathcal{H}'_s + \text{red}(\mathfrak{P}_s^*\mathcal{D}'_s))$ is log canonical log general type for every closed points $s \in T$. By the invariance of plurigenera [HMX13, Theorem 1.8], there is a number $C \gg 0$ such that

$$0 < \text{vol}(K_{\mathcal{Y}_{\mathcal{W}_s}} + \mathcal{B}'_{\mathcal{W}_s} + \mathfrak{P}_s^*\mathcal{H}'_s + \text{red}(\mathfrak{P}_s^*\mathcal{D}'_s)) \leq C,$$

for all $s \in T$ and $k = 1, ..., d + 1$.

**Step 5.** In this step, we consider the contraction $f : X \to Z$ and the boundary $\Delta$.

Recall that $X$ is birationally equivalent to the pullback of $\mathcal{Y}_\mathcal{H} \to \mathcal{H}$ by the morphism $\phi : U \to \mathcal{H}$. Let $f_Y : Y \to \mathcal{W}_t$ be an extension of the pullback of $\mathcal{Y}_\mathcal{H} \to \mathcal{H}$ by $\phi : U \to \mathcal{H}$, $B_U$ the pullback of $\mathcal{B}_U$ by $\phi : U \to \mathcal{H}$. Since the general fiber $Y_s$ has a morphism to the general fiber $X_g$, we may assume that there is a birational morphism $h : Y \to X$, let $\Delta_Y$ be the $Q$-divisor such that $K_Y + \Delta_Y \sim_Q f_Y^*(K_{\mathcal{W}_t} + \mathcal{M}_{\mathcal{W}_t} + \mathcal{D}_t)$. Because $(\mathcal{Y}_\mathcal{H}, B_{\mathcal{H} \geq 0})$ is log smooth over $\mathcal{H}$ and coef$\mathcal{B}_\mathcal{H} < 1$, after blowing up some vertical subvarieties, we may assume that $(Y, \Delta_{Y \geq 0})$ is lc. Also because $g_*B_Z \leq \mathcal{D}_t$ and $K_X + \Delta$ is nef, by the negativity lemma,

$$h^*(K_X + \Delta) = h^*f_Y^*(K_Z + B_Z + M_Z) \leq f_Y^*(K_{\mathcal{W}_t} + \mathcal{M}_{\mathcal{W}_t} + \mathcal{D}_t) = K_Y + \Delta_Y.$$

Let $C_k$ denote the closure of the pullback of $C_{\mathcal{H},k}$ by $\phi$ in $Y$. Because $\cap_{1 \leq k \leq d+1} C_{\mathcal{H},k} = \emptyset$, by blowing up some vertical subvarieties, we may assume that $(Y, \Delta_{Y \geq 0} + C_k + f_Y^*H_t)$ is log canonical and $\cap_{1 \leq k \leq d+1} \text{Supp}(C_k) = \emptyset$.

**Step 6.** In this step, we compare the two sub-pairs $(\mathcal{Y}'_{\mathcal{W}_t}, B'_{\mathcal{W}_t} + \text{red}(\mathfrak{P}_s^*\mathcal{D}'_s))$ and $(Y, \Delta_Y)$.

Notice that the pullback $Y \times_{\mathcal{W}_t} \mathcal{W}_t$ may be not birationally equivalent to $\mathcal{Y}'_{\mathcal{W}_t}$, but by Remark 4.5, the pullback of $\mathcal{Y}_\mathcal{H} \to \mathcal{H}$ via $\bar{U} \to \mathcal{H}$ is crepant birationally to the pullback of $\mathcal{Y}'_{\mathcal{H}'} \to \mathcal{H}'$ via $\bar{U} \to \mathcal{H}'$, where $\bar{U} := U \times_{\mathcal{H}} \mathcal{H}$.

We replace $\bar{U}$ by a finite cover and let $\bar{W}$ be a compactification of $\bar{U}$, such that there is a finite morphism $\pi_{\bar{W}} : \bar{W} \to \mathcal{W}_t$, then $Y \times_{\mathcal{W}_t} \bar{W}$ is birationally equivalent to $\mathcal{Y}'_{\mathcal{W}_t} \times_{\mathcal{W}_t} \bar{W}$.

Consider the following diagram.

$$\begin{array}{cccccc}
Y & \xrightarrow{\rho_Y} & \mathcal{Y}_\bar{W} & \xrightarrow{f_Y} & \mathcal{W}_t & \xrightarrow{\pi_{\bar{W}}} & \bar{W} \\
\downarrow \pi_Y & & \downarrow \mathfrak{P}_{\bar{W}} & & \downarrow \mathfrak{P}_{\bar{W}} & & \downarrow \mathfrak{P}_{\bar{W}} \\
Y \times_{\mathcal{W}_t} \mathcal{W}_t & \xrightarrow{f_Y} & \mathcal{Y}_{\mathcal{W}_t} & \xrightarrow{\pi_{\mathcal{W}_t}} & \mathcal{W}_t & \xrightarrow{\mathfrak{P}_{\mathcal{W}_t}} & \mathcal{W}_t
\end{array}$$

where

- every variety in the diagram is normal and projective,
- $\pi_{\bar{W}} : \bar{W} \to \mathcal{W}_t$ is finite and $\pi_{\mathcal{W}_t} : \mathcal{Y}_{\mathcal{W}_t} \to \mathcal{Y}_{\mathcal{W}_t}$ is the normalization of the main component of $\bar{W} \times_{\mathcal{W}_t} \mathcal{Y}_{\mathcal{W}_t}$,
- $\mathcal{Y}$ is a log resolution of the main component of $Y \times_{\mathcal{W}_t} \bar{W}$ such that there is a birational morphism $\rho_{\mathcal{Y}} : \mathcal{Y} \to \mathcal{Y}_{\mathcal{W}_t}$, and
- the generic fiber of $f_{\mathcal{Y}}$ is equal to the generic fiber of $\mathfrak{P}_{\mathcal{W}_t}$ and the pullback of the generic fiber of $f_Y$ and $\mathfrak{P}_{\mathcal{W}_t}$.
Recall that in steps 5 and 6, we have
\[ K_Y + \Delta_Y = f_Y^*(K_{W_t} + M_{W_t} + D_t), \]
and
\[ \mathfrak{s}_i^*(K_{W'_t} + M'_{W'_t} + D'_t) \leq K_{W'_t} + B'_{W'_t} + \text{red}(\mathfrak{s}_i^*D_t). \]
Because \( D'_t \) dominates \( D_t \), both \( (W_t, D_t) \) and \( (W'_t, D'_t) \) are log smooth, \( M \) descends on \( W_t \) and \( M' \) descends on \( W'_t \), by Hurwitz’s formula ([Koll13, 2.41.4]),
\[ \Pi_t^*(K_{W_t} + M_{W_t} + D_t) \leq K_{W'_t} + M'_{W'_t} + D'_t. \]
Therefore, we have
\[(6.2) \quad (\rho_Y)_*\pi_Y^*(K_Y + \Delta_Y) \leq \pi_{W_t}^*(K_{W'_t} + B'_{W'_t} + \text{red}(\mathfrak{s}_i^*D_t)). \]

**Step 7.** In this step, we prove the following inequality
\[(\rho_Y)_*\pi_Y^*(K_Y + \Delta_Y, \geq 0) \leq \pi_{W_t}^*(K_{W'_t} + B'_{W'_t} \geq 0 + \text{red}(\mathfrak{s}_i^*D'_t)). \]
Write \((\rho_Y)_*\pi_Y^*\Delta_{Y, < 0} = D_1 + D_2 + D_3 + D_4\), where \( D_1 \) is horizontal over \( W \), \( \mathfrak{s}_W^*D_2 \) has codimension \( \geq 2 \) in \( W \), \( D_3 \) dominates some irreducible components of \( \pi_{W_t}^*D'_t \) and \( D_4 \) dominates divisors on \( W \) but not components of \( \pi_{W_t}^*D'_t \).

First, we consider \( D_1 \). Because \( \pi_{W_t}^* \) is a finite morphism and \( B'_{W'_t, < 0} \) is horizontal over \( W'_t \), \( \pi_{W_t}^*B'_{W'_t, < 0} \) is horizontal over \( W \). Also because the generic fiber of \( f_Y \) is equal to the generic fiber of \( \mathfrak{s}_W^* \) and the pullback of the generic fiber of \( f_Y \), we have \( D_1 = \pi_{W_t}^*B'_{W'_t, < 0} \). Adding both sides to Equation (6.2), we have
\[(6.3) \quad (\rho_Y)_*\pi_Y^*(K_Y + \Delta_Y) + D_1 \leq \pi_{W_t}^*(K_{W'_t} + B'_{W'_t} \geq 0 + \text{red}(\mathfrak{s}_i^*D'_t)). \]

Secondly, we consider \( D_2 \) and \( D_3 \). Because \( \mathfrak{s}_W^* \) is smooth over \( W'_t \setminus D'_t \) and \( \pi_{W_t}^* : \mathfrak{s}_{W_t}^* \to \mathfrak{Y}_{W_t}^* \) is the normalization of the main component of \( W \setminus W \), \( \mathfrak{Y}_{W_t}^* \), then \( D_2 \) is contained in \( \mathfrak{s}_W^* \pi_{W_t}^*D'_t \).

By Hurwitz’s formula, \( D_3 \) is also contained in \( \mathfrak{s}_W^* \pi_{W_t}^*D'_t \).

Suppose \( R \) is an irreducible component of \( \mathfrak{s}_W^* \pi_{W_t}^*D'_t \). Applying Hurwitz’s formula on the right side of Equation (6.2), locally near \( R \), we have
\[ \pi_{W_t}^*(K_{W'_t} + B'_{W'_t} \geq 0 + \text{red}(\mathfrak{s}_i^*D'_t)) = K_{\mathfrak{s}_W^*} + R. \]
Because \( (Y, \Delta_{Y, \geq 0}) \) is lc, by Hurwitz’s formula, near \( R \) we have \((\rho_Y)_*\pi_Y^*(K_Y + \Delta_Y, \geq 0) \leq K_{\mathfrak{s}_W^*} + R. \)

By comparing the coefficients of each component of \( \mathfrak{s}_W^* \pi_{W_t}^*D'_t \) on both sides of Equation (6.2), we have
\[(6.4) \quad (\rho_Y)_*\pi_Y^*(K_Y + \Delta_Y) + D_2 + D_3 \leq \pi_{W_t}^*(K_{W'_t} + B'_{W'_t} \geq 0 + \text{red}(\mathfrak{s}_i^*D_t)). \]

Thirdly, we consider \( D_4 \). Suppose \( R \) is an irreducible component of \( D_4 \) and it dominates a prime divisor \( P \) on \( W \). By assumption, \( P \) is not contained in \( \pi_{W_t}^*D'_t \). Because \( \mathfrak{s}_W^* \) is smooth over \( W'_t \setminus D'_t \), then \( \mathfrak{s}_W^* \) is smooth over \( W \setminus \pi_{W_t}^*D'_t \), hence smooth over the generic point of \( P \). Not locally on the generic fiber over \( P \), we have \( \mathfrak{s}_W^* \mathfrak{U}_t \).

Since \( \pi_{W_t}^* \) is a finite morphism, \( P \) dominates a prime divisor \( P' \) on \( W'_t \). Because \( \mathfrak{H}_t \to \mathfrak{H}_t \) is a finite morphism, by construction, \( \mathfrak{U}_t' \to \mathfrak{U}_t \) is a finite morphism. Also because \( D'_t := W'_t \setminus \mathfrak{U}_t' \) and \( P' \) is not contained in \( D'_t \), then \( \Pi_t^* \) is finite on the generic point of \( P' \), which means \( \mathfrak{P}_t \) dominates a divisor \( P \) on \( W \). Since \( P \) is not contained in \( \pi_{W_t}^*D'_t \), hence not contained in \( \pi_{W_t}^*\Pi_t^*D_t \), it is easy to see that \( P \) is not contained in \( D_t \).

By construction, \( D_t \) contains the sum of the exceptional divisor of \( W_t \), and strict transform of \( B \), then \( P \) is not contained in \( D_t \) implies that \( P \) is a divisor on \( Z \) and \( \text{coeff}_PBZ = 0 \). Since \( \Delta \geq 0 \), \( K_X + \Delta \sim_{Q} f^*(K_Z + B_Z + M_Z) \) and \( h^*(K_X + \Delta) \leq K_Y + \Delta_Y \), let \( Q^\# \) be a prime divisor on \( X \) dominating \( P \), then by Lemma 2.8, we have \( \text{coeff}_{Q^\#}\Delta = \text{coeff}_{Q^\#}\Delta_Y = 0 \).
Suppose $Q$ is a prime divisor on $Y$ such that $\text{coeff}_R(\rho_Y)_*\pi_Y^*Q > 0$, then it is easy to see that $Q$ dominates $P$ on $W_t$. Next, we show that $Q = Q^#$. Because $\text{coeff}_Q\Delta_Y = 0$ and $\text{coeff}_Q f_Y^*P \subset \mathbb{N}$, by Lemma 2.8, locally near $Q^#$ we have

$$K_Y + Q^# \sim_Q f_Y^*(K_{W_t} + M_{W_t} + P).$$

By applying Hurwitz’s formula on $\pi_Y$, we have

$$K_Y + Q^# = \pi_Y^*(K_Y + Q^#),$$

where $\tilde{Q}^#$ is a prime divisor on $\tilde{Y}$ dominating $Q^#$. On the other hand, by Hurwitz’s formula on $\bar{\mathcal{W}}$, locally near $\bar{P}$, we have

$$K_{\bar{W}} + M_{\bar{W}} + \bar{P} = \pi^*_{\bar{W}}(K_{\bar{W}} + M_{\bar{W}} + \bar{P}).$$

Because $\bar{\mathcal{W}}$ is smooth over the generic point of $\bar{P}$, then $\text{coeff}_{\bar{\mathcal{W}}}Q_{\bar{W}} = 0$, near the generic fiber over $\bar{P}$, we have

$$K_{\bar{W}} + M_{\bar{W}} + R := \bar{\mathcal{W}}^*(K_{\bar{W}} + M_{\bar{W}} + \bar{P}),$$

where $\bar{\mathcal{W}} := \pi_{\bar{W}}^*\mathcal{B}_{\bar{W}}'$ is horizontal over $\bar{W}$. Because $\bar{\mathcal{W}} : (\mathcal{Y}_W', B_{\mathcal{W}}') \to \mathcal{W}_t'$ is log smooth over the generic point of $\bar{P}$, then $(\bar{\mathcal{Y}}_W, \bar{B}_W)$ is log smooth over the generic point of $\bar{P}$. Also because $\text{coeff}_{\bar{\mathcal{W}}}$ is horizontal over $\mathcal{Y}_W$, $\bar{\mathcal{W}} + R$ is sub-plt on the generic fiber over $\bar{P}$, which means $R$ is the only lc center of $(\bar{\mathcal{Y}}_W, \bar{B}_W)$ dominating $\bar{P}$. Recall that locally near $Q^#$, we have $K_Y + Q^# = f_Y^*(K_{\bar{W}} + M_{\bar{W}} + \bar{P}) = \rho_Y^*(K_{\bar{W}} + \bar{B}_W + R)$, also because $R$ is the only lc center of $(\bar{\mathcal{Y}}_W, \bar{B}_W + R)$ dominating $\bar{P}$, it is easy to see $\tilde{Q}^#$ is the strict transform of $R$, then we have $Q = Q^#$. Because $\text{coeff}_Q\Delta_Y = 0$, then $\tilde{Q}^#$ is not contained in $\Delta_{Y,<0}$, which means $\text{coeff}_R(\rho_Y)_*\pi_Y^*\Delta_{Y,<0} = 0$. By comparing the coefficient of each component of $D_4$, we have

$$\rho_Y)_*\pi_Y^*(K_Y + \Delta_Y) + D_4 \leq \pi_{\bar{W}}^*(K_{\bar{W}} + \bar{B}^t_{\mathcal{W}_t,\geq 0} + \text{red}(\bar{\mathcal{W}}^tD_t')).$$

Combining Equation (6.3), (6.4) and (6.5), we have

$$\text{coeff}_{\bar{\mathcal{W}}}(\rho_Y)_*\pi_Y^*(K_Y + \Delta_{Y,h,\geq 0}) \leq \rho_Y)_*\pi_Y^*(K_Y + \Delta_{Y,\geq 0}) \leq \pi_{\bar{W}}^*(K_{\bar{W}} + \mathcal{B}^t_{\mathcal{W}_t,\geq 0} + \text{red}(\bar{\mathcal{W}}^tD_t')).$$

Step 8. In this step, we complete the proof.

Because $C_k$, $k = 1, \ldots, d + 1$ has no intersection point and $\pi_Y$ is generically finite, there exists $k_0 \in \{1, \ldots, d + 1\}$ such that every irreducible component of $\pi_Y^*C_{k_0}$ dominates a divisor on $Y$. Also because $C_{k_0}$ is horizontal over $W_t$, then $\pi_Y^*C_{k_0}$ is horizontal over $\bar{W}$. Because $C_{\bar{H}} \times_h \bar{H} = C_{\bar{H}} \times \bar{H}$, then $\pi_{\bar{W}}^*C_{k_0}|_{\bar{W}_t} = \rho_Y^*\pi_Y^*C_{k_0}|_{\mathcal{W}_t,k_0}|_{\mathcal{W}_t}$, where $\bar{W}_t$ is the generic fiber of $f_Y$. Also because $\pi_Y^*C_{k_0}$ is horizontal over $\bar{W}$ and $\rho_Y^*\pi_Y^*C_{\mathcal{W}_t,k_0}$ is effective, we have $\pi_Y^*C_{k_0} \leq \rho_Y^*\pi_Y^*C_{\mathcal{W}_t,k_0}$. Combining with Equation (6.6) and the equation $(\rho_Y)_*\pi_Y^*(K_Y + \Delta_Y,h,\geq 0 + C_{k_0} + f_Y^*H_t) \leq \pi_{\bar{W}}^*(K_{\bar{W}} + \mathcal{B}^t_{\mathcal{W}_t,\geq 0} + \text{red}(\bar{\mathcal{W}}^tD_t') + C_{\mathcal{W}_t,k_0} + \bar{\mathcal{W}}^tH_t)$.
Now compare their volumes, we have

\[
\begin{align*}
&\deg(\pi_Y) \text{vol}(K_Y + \Delta_{Y,h} + C_{ko} + f_Y^* \mathcal{H}_t) \\
= &\text{vol}(\pi_Y^*(K_Y + \Delta_{Y,h} + C_{ko} + f_Y^* \mathcal{H}_t)) \\
\leq &\text{vol}((\rho_Y)_* \pi_Y^*(K_Y + \Delta_{Y,h} + C_{ko} + f_Y^* \mathcal{H}_t)) \\
\leq &\text{vol}(\pi_{Y'}^*(K_{Y'} + B_{W_t} + \text{red}(\mathcal{D}_{t'}^\phi + C_{W_t} \cap \mathcal{H}_t))) \\
= &\deg(\pi_{Y'}^*) \text{vol}(K_{Y'} + B_{W_t} + \text{red}(\mathcal{D}_{t'}^\phi + C_{W_t} \cap \mathcal{H}_t)) \\
\leq &\deg(\pi_{Y'}^*) \text{vol}(K_{Y'} + B_{W_t} + \text{red}(\mathcal{D}_{t'}^\phi + C_{W_t} \cap \mathcal{H}_t)) \\
\leq &\text{vol}(K_{Y'} + \Delta_{Y,h} + C_{ko} + f_Y^* \mathcal{H}_t)
\end{align*}
\]

(6.7)

Therefore, because \(\deg(\pi_Y^*) = \deg(\pi_{Y'}^*)\), we have that \(\text{vol}(K_Y + \Delta_{Y,h} + C_{ko} + f_Y^* \mathcal{H}_t)\) is bounded from above.

Because \(K_Y + \Delta_{Y,h} + C_{ko} + f_Y^* \mathcal{H}_t\) is big and \(\text{coeff}(\Delta_{Y,h} + C_{ko})\) is in a finite set, by [HMX13, Lemma 2.3.4], [HMX13, Theorem 3.1] and [HMX14, Theorem 1.3], \((Y, \Delta_{Y,h} + C_{ko} + f_Y^* \mathcal{H}_t)\) is log birationally bounded, then \(X\) is birationally bounded.

\[
\square
\]

Proof of the claim. With the same notation as above. Let \(Z'\) be a log resolution of \((Z, B_Z)\) such that \(\phi^*\) extend to a morphism \(Z' \to S^*\) and there is a morphism \(g : Z' \to W_t\). Define \(B_{Z'}\) to be the strict transform of \(B_Z\) plus the exceptional divisors of \(Z' \to Z\), because \((Z', B_{Z'})\) is log smooth, \((Z', B_{Z'})\) is a dlt pair. Define \(B'_{Z'} = \tfrac{1}{3} \text{red}(B_{Z'})\). By the ACC for log canonical thresholds and the construction of the boundary part, \(\text{coeff}(B_{Z'})\) is in a DCC set, in particular, there is a positive rational number \(\delta\) such that \(\text{coeff}(B_{Z'}) \geq \delta\), hence \(\delta B'_{Z'} \leq B_{Z'}\).

Suppose \(\dim Z = n\). By the length of extremal rays, \(K_{W_t} + 3nH_{W_t}\) is ample. Also because \(Z'\) and \(W_t\) are smooth, \(K_{Z'} \geq g^* K_{W_t}\), then \(K_{Z'} + 3ng^* H_{W_t}\) is big. Let \(Z' \to Z_c\) be the canonical model of \(K_{Z'} + B'_{Z'} + 3ng^* H_{W_t} + 3n(\phi^*)^* A^*\), where \(A^*\) is the very ample divisor on \(S^*\) defined in Remark 4.5 and we think of \((Z', B'_{Z'} + 3ng^* H_{W_t} + 3n(\phi^*)^* A^*)\) as a generalized pair with nef part \(3ng^* H_{W_t} + 3n(\phi^*)^* A^*\).

\[
\begin{tikzcd}

Z' \\
W_t \\
Z_c \\
S^*
\end{tikzcd}
\]

By [BZ16, Lemma 4.4], the contraction \(Z' \to Z_c\) is \(g^* H_{W_t}\) and \((\phi^*)^* A^*\)-trivial, so there are two morphisms \(g_c : Z_c \to W_t\) and \(\phi^*_c : Z_c \to S^*\).

By [BZ16, Theorem 8.1], there is a rational number \(e \in (0, 1)\) such that \(K_{Z_c} + e(B'_{Z_c} + 3ng_c^* H_{W_t} + 3n(\phi^*_c)^* A^*)\) is a big \(Q\)-divisor. Also because the \(\text{coeff}(B'_{Z'})\) is equal to \(\tfrac{1}{3}\), \(\text{coeff}(eB'_{Z'})\) is equal to \(\tfrac{e}{3}\), then by [BZ16, Theorem 1.3], there exists an integer \(m \in \mathbb{N}\) such that \(|m(K_{Z_c} + e(B'_{Z_c} + 3ng_c^* H_{W_t} + 3n(\phi^*_c)^* A^*))|\) defines a birational map. We choose \(m\) sufficiently divisible such that both \(\frac{m}{2} = \frac{1}{e}\), \(m \in \mathbb{N}\), then there is an effective divisor

\[
A'_{Z_c} \sim m(K_{Z_c} + e(B'_{Z_c} + 3ng_c^* H_{W_t} + 3n(\phi^*_c)^* A^*)).
\]

Define \(A_{Z_c} := A'_{Z_c} + (1 - e)(B'_{Z_c} + 3ng_c^* H_{W_t} + 3n(\phi^*_c)^* A^*)\), because \(m(1 - e)(B'_{Z_c} + 3ng_c^* H_{W_t} + 3n(\phi^*_c)^* A^*)\) is a divisor, then

\[
A_{Z_c} \sim m(K_{Z_c} + B'_{Z_c} + 3ng_c^* H_{W_t} + 3n(\phi^*_c)^* A^*)
\]

is an effective ample divisor and \(\text{Supp}(B'_{Z_c} + g_c^* H_{W_t} + (\phi^*_c)^* A^*) \subset \text{Supp}(A_{Z_c})\).
Write $N := 2(2n+1)g^*_c H_{W_i}$. By [HMX13, Lemma 3.2],
\[
\text{red}(A_{Z_c}).N^{n-1} \\
\leq 2^n \text{vol}(K_{Z_c} + \text{red}(A_{Z_c}) + N) \\
\leq 2^n \text{vol}(K_{Z_c} + m(K_{Z_c} + B'_{Z_c} + 3n g^*_c H_{W_i} + 3n(\phi^*_c)^*A^* + N)) \\
\leq 2^n \text{vol}(m(1+1)K_{Z_c} + mB'_{Z_c} + (3mn + 2(2n+1))g^*_c H_{W_i} + 3mn(\phi^*_c)^*A^*) \\
\leq (2a)^n \text{vol}(K_{Z_c} + B'_{Z_c} + 3n g^*_c H_{W_i} + 3n(\phi^*_c)^*A^*) \\
\leq (2a)^n \text{vol}(K_{Z_c} + B'_{Z_c} + 3n g^*_c H_{W_i} + 3n(\phi^*_c)^*A^*)
\]
where $a = \max\{m + 1, m + \frac{2(2n+1)}{m}\}$.

Let $\hat{Z}'$ be the normalization of the main component of $Z' \times_S S^1$, because $\mathcal{H}' \to \mathcal{H}'$ is a finite cover and $\phi^*$ maps the generic point of $Z'$ into $S^1$, then $\pi_{Z'} : \hat{Z}' \to Z'$ is also a generically finite cover. Let $\mathcal{M}$ be the moduli b-divisor defined by the pullback of $(X, \Delta) \to Z$ by $\hat{Z}' \to Z$. Because $\mathcal{M}$ descends on $Z'$, by Lemma 4.2, we have $\mathcal{M}_{\hat{Z}'} = \pi^*_Z \mathcal{M}_{Z'}$. Also because $\phi(Z')$ is not contained in $\text{Supp}(\mathcal{M}_S^1)$, by the proof of Lemma 4.5, we have $\mathcal{M}_{\hat{Z}'} = \phi^* \mathcal{M}_S^1$, where $\phi^*$ is the natural morphism $\hat{Z}' \to S^1$. Because $l(\mathcal{M}_S^1 - \pi^*A^* \geq 0$, we have $l(\mathcal{M}_{\hat{Z}'}, \mathcal{M}_S^1 \geq (\phi^*)^*A^*$.

Because $Z' \to Z_c$ is the canonical model of $K_{Z'} + B'_{Z'} + 3ng^*_c H_{W_i} + 3n(\phi^*_c)^*A^*$, then
\[
\text{vol}(K_{Z_c} + B'_{Z_c} + 3ng^*_c H_{W_i} + 3n(\phi^*_c)^*A^*) \\
= \text{vol}(K_{Z'} + B'_{Z'} + 3ng^*_c H_{W_i} + 3n\phi^*A^*) \\
\leq \text{vol}(K_{Z'} + B'_{Z'} + 3ng^*_c H_{W_i} + 3n\mathcal{M}_{Z'})
\]
where the second inequality comes from that $l(\mathcal{M}_{Z'}) \geq (\phi^*)^*A^*$. Consider the following equation
\[
K_{Z'} + \delta B'_{Z'} + 3ng^*_c H_{W_i} + 3n\mathcal{M}_{Z'} \\
= \delta(K_{Z'} + B'_{Z'} + 3ng^*_c H_{W_i} + 3n\mathcal{M}_{Z'}) + (1-\delta)(K_{Z'} + 3ng^*_c H_{W_i} + 3n\mathcal{M}_{Z'}).
\]
Because $K_{Z'} + 3ng^*_c H_{W_i} + 3n\mathcal{M}_{Z'}$ is big, and $\delta B'_{Z'} \leq B_{Z'}$ we have
\[
\text{vol}(K_{Z'} + B'_{Z'} + 3ng^*_c H_{W_i} + 3n\mathcal{M}_{Z'}) \\
\leq (\frac{1}{\delta})^n \text{vol}(K_{Z'} + \delta B'_{Z'} + 3ng^*_c H_{W_i} + 3n\mathcal{M}_{Z'}) \\
\leq (\frac{1}{\delta})^n \text{vol}(K_{Z'} + B'_{Z'} + 3ng^*_c H_{W_i} + 3n\mathcal{M}_{Z'})
\]
By assumption, $K_{Z'} + B_{Z'} + \mathcal{M}_{Z'}$ is big, then $K_{Z'} + B_{Z'} + \mathcal{M}_{Z'} + g^*_c H_{W_i}$ is big. Again by [BZ16, Theorem 8.1], there is a positive rational number $\epsilon' < 1$ such that $K_{Z'} + B_{Z'} + \epsilon'\mathcal{M}_{Z'} + \epsilon'g^*_c H_{W_i}$ is big. Consider the following inequality,
\[
\alpha(K_{Z'} + B_{Z'} + 3ng^*_c H_{W_i} + 3n\mathcal{M}_{Z'}) + (1-\alpha)(K_{Z'} + B_{Z'} + \epsilon'g^*_c H_{W_i} + \epsilon'\mathcal{M}_{Z'}) \\
\leq K_{Z'} + B_{Z'} + \mathcal{M}_{Z'} + g^*_c H_{W_i},
\]
where $\alpha = \frac{1-\epsilon'}{3n-\epsilon'}$. Then
\[
\text{vol}(K_{Z'} + B_{Z'} + 3ng^*_c H_{W_i} + 3n\mathcal{M}_{Z'}) \\
\leq (\frac{1}{\alpha})^n \text{vol}(K_{Z'} + B_{Z'} + 3ng^*_c H_{W_i} + 3n\mathcal{M}_{Z'}) \\
\leq (\frac{1}{\alpha})^n \text{vol}(K_{Z'} + B_{Z'} + 3ng^*_c H_{W_i} + 3n\mathcal{M}_{Z'}) \\
\leq (\frac{1}{\alpha})^n \text{vol}(K_{Z'} + B_{Z'} + 3ng^*_c H_{W_i} + 3n\mathcal{M}_{Z'}) \\
\leq (\frac{1}{\alpha})^n H \\
\leq (\frac{1}{\alpha})^n V,
\]
where the last inequality comes from the definition of $\mathcal{P}(d, I, v, V)$. Then, we have
\[
\text{red}(A_{Z_c}).N^{n-1} \leq (\frac{2a}{\alpha})^n V.
\]
By the boundedness of Chow variety, $(W_i, \text{Supp}(g_c), \text{Supp}(A_{Z_c}))$ is in a log bounded family, write it as $(W, D) \to T$. After passing to a stratification of $T$ and taking a log resolution of the
generic fiber, we may assume that \((\mathcal{W}, \mathcal{D})\) is log smooth over \(T\). Also by the proof of [Jia21, Theorem 1.1], we may assume the moduli part of \((X, \Delta) \to Z\) descends on \(\mathcal{W}_t\).

Because \(A_Z\) is effective and ample, \(\mathcal{W}_t\) is smooth, by the negativity lemma, \(g_\ast(g_c)_\ast A_Z - A_c\) is effective and contains all \(g_c\)-exceptional divisor. Define \(\mathcal{U} := \mathcal{W} \setminus \mathcal{D}\), then \(\mathcal{U}_t = \mathcal{W}_t \setminus \text{Supp}(g_c) \setminus \text{Supp}(A_Z)\). Because \(\mathcal{W}_t\) is smooth, the exceptional locus of \(g_c\) is pure of codimension 1, then

\[ \mathcal{U}_t \subset Z_c \setminus \text{Supp}(A_Z), \]

and \(\phi^\ast_c: Z_c \to S^*\) induce a morphism \(\phi^\ast_c: \mathcal{U}_t \to S^*\). Because \(\text{Supp}((\phi^\ast_c)^\ast A^*) \subset \text{Supp}(A_Z)\) and \(H^* \subset S^* \setminus \text{Supp}(A^*)\), then \(\phi^\ast_c(\mathcal{U}_t) \subset H^*\).

**Remark 6.4.** With the same notation as in the proof of theorem above, suppose \(K_Z + B_Z + M_Z\) is big and there exists a positive integer \(r\) such that \(l(K_X + \Delta)\) is Cartier. Then by Remark [6.4], we may assume the moduli part of \((X, \Delta)\) is Cartier. And because that general fiber of \((X, \Delta)\) has a good minimal model. Since \(\mathcal{Y}_s\) is log birationally bounded, in particular, \((X, \Delta)\) has a good minimal model and its good minimal model is crepant birationally equivalent to \((X, \Delta)\).

**Theorem 6.5.** There is a family of log smooth pairs \((\mathcal{Y}, \mathcal{B}) \to T\) such that for every \((X, \Delta) \in G_{\text{mkt}}(d, I, v, u)\), there is a closed point \(s \in T\), \((\mathcal{Y}_s, B_s)\) has a good minimal model, and its good minimal model is crepant birationally equivalent to \((X, \Delta)\).

**Proof.** Fix \((X, \Delta) \in G_{\text{mkt}}(d, I, v, u)\) with the contraction \(f: X \to Z\) and the divisor \(A\). Let \((Z, B_Z + M_Z)\) be the generalized pairs defined by the canonical bundle formula. Because \(K_Z + B_Z + M_Z\) is big, by [BZ16, Theorem 1.3], there is a natural number \(m\) depending on \(d, l, I, Z\) such that \(m(K_Z + B_Z + M_Z)\) defines a birational map. We may assume \(H = m(K_Z + B_Z + M_Z)\), then \(\text{vol}(H + K_Z + B_Z + M_Z) \leq (m + 1)^n \text{vol}(K_Z + B_Z + M_Z) \leq (m + 1)^n \text{vol}(K_X + \Delta) = (m + 1)^nv\), thus Theorem 6.2 applies. We will use the same notation as in Theorem 6.2.

By the main result of [Bir21b] and [Jia21], there is an integer \(r > 0\) such that \(r(K_Z + B_Z + M_Z)\) is Cartier. Then by [Flor14, Remark 2.10], there exists a natural number \(l\) such that \(l(\Delta_Y + \Delta_Y) \sim l(K_X + \Delta) \sim l(K_Z + B_Z + M_Z)\) is Cartier. Then by Remark 6.4, \((Y, \Delta_Y \geq 0 + C_{k_0})\) is log birationally bounded, in particular, \((Y, \Delta_Y \geq 0)\) is log birationally bounded. Thus there is a family of log smooth varieties \((\mathcal{Y}, Q) \to S\), for any \((X, \Delta) \in G_{\text{mkt}}(d, I, v, u)\), there is a closed point \(s \in S\) and a birational map \(\pi_s: \mathcal{Y}_s \to X\) such that \(L_{\Delta, \mathcal{Y}_s} \leq Q_s\).

Because \(l(K_X + \Delta)\) is Cartier and \((X, \Delta)\) is kl, \(L_{\Delta, \mathcal{Y}_s} \leq (1 - \frac{1}{d})Q_s\). Let \(\chi: \mathcal{Y}' \to (\mathcal{Y}, (1 - \frac{1}{d})Q)\) be a birational morphism such that

1. \(\chi\) only blows up strata of \((\mathcal{Y}, Q)\), and
2. \((\mathcal{Y}', L_{(1 - \frac{1}{d})Q, \mathcal{Y}'}\) has terminal singularities.

Let \(p': \tilde{X} \to \mathcal{Y}'\) and \(q: \tilde{X} \to X\) be a common resolution, \(p := \chi \circ p': \tilde{X} \to \mathcal{Y}_s\). Write \(K_{\tilde{X}} + \tilde{\Delta} \sim_Q q^*(K_X + \Delta)\), then \(L_{\Delta, \mathcal{Y}_s} = p_{\ast}L_{\tilde{\Delta}, \tilde{X}}\). Because \(K_X + \Delta\) is nef, by the negativity lemma, \(K_{\tilde{X}} + \tilde{\Delta} \leq p^\ast(K_{\mathcal{Y}_s} + \Delta) \leq p^\ast(K_{\mathcal{Y}_s} + L_{\Delta, \mathcal{Y}_s})\), also because \(L_{\Delta, \mathcal{Y}_s} \leq (1 - \frac{1}{d})Q_s\), then \(L_{\Delta, \mathcal{Y}_s} \leq L_{(1 - \frac{1}{d})Q, \mathcal{Y}_s}\) and \((\mathcal{Y}', L_{\Delta, \mathcal{Y}_s})\) has terminal singularities. We replace \((\mathcal{Y}, Q)\) by \((\mathcal{Y}', \text{Supp}(L_{(1 - \frac{1}{d})Q, \mathcal{Y}'}))\) and assume that \((\mathcal{Y}_s, L_{\Delta, \mathcal{Y}_s})\) has terminal singularities.

Next, we prove that \((\mathcal{Y}_s, L_{\Delta, \mathcal{Y}_s})\) has a good minimal model and the good minimal model is crepant birationally equivalent to \((X, \Delta)\). Because \(K_X + \Delta\) is semistable, it is easy to see \((X, L_{\Delta, \tilde{X}})\) has a good minimal model.

By the definition of \(b\)-divisor \(L_{\Delta, \mathcal{Y}_s} = p_{\ast}L_{\Delta, \tilde{X}}\). Because \((\mathcal{Y}_s, L_{\Delta, \mathcal{Y}_s})\) has terminal singularities, \(K_{\tilde{X}} + L_{\tilde{X}, \tilde{X}} - p^\ast(K_{\mathcal{Y}_s} + L_{\Delta, \mathcal{Y}_s})\) is effective and \(p\)-exceptional. Then by Lemma 2.4, \((\mathcal{Y}_s, L_{\Delta, \mathcal{Y}_s})\) has a good minimal model since \((X, L_{\Delta, \tilde{X}})\) has a good minimal model. Since adding exceptional divisor does not change global sections, then a good minimal model of \((\mathcal{Y}_s, L_{\Delta, \mathcal{Y}_s})\) is crepant birationally equivalent to \((X, \Delta)\).
By construction $\text{coeff}(L_{\Delta,Y_u}) \subseteq \{0, \frac{1}{l}, \ldots, \frac{l-1}{l}\}$ and $L_{\Delta,Y_u} \leq Q_s$. Write $Q = \sum_{k \in I} Q_k$ as the sum of all irreducible components, and for a vector $a = \{a_k\}_{k \in I}$, define $aQ = \sum_{k \in I} a_k Q_k$. Let $J$ be the set of all vectors $a$ such that the coefficient of each element of $a$ is in $\{0, \frac{1}{l}, \ldots, \frac{l-1}{l}\}$. Then $J$ is a finite set, we define $T := J \times S$, and $B$ to be the $\mathbb{Q}$-divisor supported on $J \times Q$ such that $B|_{(a)} = aQ$ for all $a \in J$.

\[
\text{Proof of Theorem 1.2. Let } (\mathcal{Y}, \mathcal{B}) \to T \text{ be the family defined in Theorem 6.5. After passing to a stratification of } T, \text{ we may assume that there is a dense open subset } T' \subset T \text{ such that for any closed point } u \in T', \text{ there is a pair in } \mathcal{G}(d, l, v, u) \text{ that is crepant birationally equivalent to a good minimal model of } (\mathcal{Y}_u, \mathcal{B}_u).
\]

Let $(X, \Delta) \in \mathcal{G}(d, l, v, u)$, suppose $s \in T$ is the corresponding closed point, then a good minimal model of $(\mathcal{Y}_s, \mathcal{B}_s)$ is crepant birationally equivalent to $(X, \Delta)$.

By Theorem 2.5, $(\mathcal{Y}, \mathcal{B})$ has a good minimal model $(\mathcal{Y}^m, \mathcal{B}^m)$ over $T$, and for every closed point $u \in T$, $(\mathcal{Y}^m_u, \mathcal{B}^m_u)$ is a semi-ample model of $(\mathcal{Y}_u, \mathcal{B}_u)$. Then $(X, \Delta)$ is crepant birationally equivalent to $(\mathcal{Y}^m_s, \mathcal{B}^m_s)$, and $(X, \Delta)$ is bounded modulo crepant birational equivalence.

\[
\text{7. Rationally connected Calabi-Yau pairs}
\]

**Corollary 7.1.** [BCDS20, Corollary 5.1] Let $Y$ be a smooth projective Calabi-Yau variety. Assume that $Y$ is endowed with a morphism $f : Y \to X$ of relative dimension $0 < d < \dim Y$. Then $X$ is rationally connected.

It is easy to see that Theorem 1.3 is a special case of the following theorem.

**Theorem 7.2.** Fix positive integers $d, l$ and a positive rational number $v$. Then the set of projective varieties $Y$ such that

- $Y$ is terminal of dimension $d$,
- $lK_Y \sim 0$,
- $f : Y \to X$ is an algebraic contraction,
- $X$ is rationally connected, and
- there is an integral divisor $A$ on $Y$ such that $A_g := A|_{Y_g}$ is ample and $\text{vol}(A_g) = v$, where $Y_g$ is the general fiber of $f$,

is bounded modulo flops.

**Definition 7.3.** [Bir18, Definition 2.1] Let $d, r$ be natural numbers and $\epsilon$ be a positive rational number. A generalized $(d, r, \epsilon)$-Fano type (log Calabi-Yau) fibration consists of a generalized pair $(X, \Delta + M_X)$ and a contraction $f : X \to Z$ such that

- $(X, \Delta + M_X)$ is a generalized $\epsilon$-lc pair of dimension $d$,
- $K_X + \Delta + M_X \sim_\mathbb{Q} f^* L$ for some $\mathbb{Q}$-divisor $L$,
- $-K_X$ is big over $Z$, i.e. $X$ is of Fano type over $Z$,
- $A$ is very ample divisor on $Z$ with $A^\dim Z \leq r$, and
- $A - L$ is ample.

**Theorem 7.4.** [Bir18, Theorem 2.2] Let $d, r$ be natural numbers and $\epsilon, \tau$ be positive rational numbers. Consider the set of all generalized $(d, r, \epsilon)$-Fano type fibrations $(X, \Delta + M_X) \to Z$ such that

- we have $0 \leq B \leq \Delta$ whose non-zero coefficients are $\geq \tau$, and
- $-(K_X + B)$ is big over $Z$.

Then the set of such pairs $(X, B)$ is log bounded.

**Theorem 7.5.** [BCDS20, Theorem 3.1] Let $(X, \Delta)$ be a projective klt Calabi-Yau pair with $\Delta \neq 0$. Then there exists a birational contraction $\pi : X \dasharrow X'$.
to a $\mathbb{Q}$-factorial Calabi-Yau pair $(X', \Delta' := \pi_* \Delta)$, $\Delta' \neq 0$ and a tower of morphisms

\[ X' = X_0 \xrightarrow{p_0} X_1 \xrightarrow{p_1} X_2 \xrightarrow{p_2} \ldots \xrightarrow{p_{k-1}} X_k \]

such that

1. for any $1 \leq i < k$ there exists a boundary $\Delta_i \neq 0$ on $X_i$ and $(X_i, \Delta_i)$ is a klt Calabi-Yau pair,
2. for any $0 \leq i < k$ the morphism $p_i : X_i \to X_{i+1}$ is a Mori fiber space, with $\rho(X_i/X_{i+1}) = 1$, and
3. either $\dim X_k = 0$, or $\dim X_k > 0$ and $X_k$ is a klt variety with $K_{X_k} \sim_{\mathbb{Q}} 0$.

**Proposition 7.6.** [BCDS20, Proposition 3.7] Let $(Y, D)$ be a klt pair and let $f : Y \to Z$ be a projective contraction of normal varieties. Assume that $K_Y + D \sim_{\mathbb{Q}} 0$ and $Z$ is $\mathbb{Q}$-factorial and let $Z \to Z'$ be a birational contraction of normal projective varieties. Then there exists a $\mathbb{Q}$-factorial klt pair $(Y', D')$ isomorphic to $(Y, D)$ in codimension $1$ and a projective contraction of normal varieties $f' : Y' \to Z'$.

**Theorem 7.7.** [BCDS20, Theorem 4.1] Fix positive integers $d, l$. Consider varieties $X$ such that

1. $X$ is klt projective of dimension $d$,
2. $X$ is rationally connected, and
3. $lK_X \sim 0$.

Then the set of such $X$ is bounded up to flops.

**Theorem 7.8.** Assume Theorem 7.2 in dimension $d - 1$. Fix a positive integer $l$ and a positive rational number $v$. Then there exists a bounded family $X \to T$, such that if $f : Y \to X$ is an algebraic contraction between normal projective varieties with the following properties:

1. $Y$ is klt of dimension $d$.
2. $lK_Y \sim 0$.
3. $X$ is rationally connected.
4. there is an integral divisor $A$ on $Y$ such that $A := A|_{Y_q}$ is ample and $\operatorname{vol}(A) = v$, where $Y_q$ is the general fiber of $f$.

Then there is a closed point $t \in T$ and a birational contraction $X \to X_t$.

**Proof.** By Corollary 7.1, $X$ is rationally connected.

Applying the canonical bundle formula on $f$, there is a generalized klt pair $(X, \Delta + M_X)$ such that

\[ K_X + \Delta + M_X \sim_{\mathbb{Q}} 0. \]

By the main result of [Amb05], $M$ is $b$-nef and $b$-abundant. Also because $(X, \Delta + M_X)$ is generalized klt, then there exists a $\mathbb{Q}$-divisor $B \sim_{\mathbb{Q}} \Delta + M_X$ such that $(X, B)$ is a klt pair.

If $B = 0$, because the general fiber of $f$ is in a bounded family, then the Cartier index $r$ of $M$ is bounded, which implies that there is a natural number $l'$ such that $l'K_X \sim 0$. Then by Theorem 7.7, $X$ is bounded up to flop.

If $B \neq 0$, by Theorem 7.5, there exists a birational contraction

\[ \pi : X \to X' \]

to a $\mathbb{Q}$-factorial Calabi-Yau pair $(X', \Delta' := \pi_* \Delta)$, $\Delta' \neq 0$ and a tower of morphisms

\[ X' = X_0 \xrightarrow{p_0} X_1 \xrightarrow{p_1} X_2 \xrightarrow{p_2} \ldots \xrightarrow{p_{k-1}} X_k = Z. \]

Let $f' : Y' \to X'$, $g : Y' \to Y$ be the closure of the graph of $\pi \circ f : Y \to X'$, write $K_{Y'} + E \sim_{\mathbb{Q}} g^* K_Y + F$, where $E$ and $F$ are effective without common components. Because $X \to X'$ is a birational contraction, the image of $\operatorname{Exc}(g)$ in $X'$ does not contain any codimension $1$ point of $X'$, then, $E + F$ is $f'$-very exceptional. Fix a positive number $\delta \ll 1$, run $K_{Y'} + (1 + \delta)E \sim_{\mathbb{Q}} \delta E + F$-MMP over $X'$, by Theorem 2.2, it terminates with a model $W$ such that $E_W + F_W = 0$, then
Let $e_j$ denote the contraction $W \to X_j$. Since we assume Theorem 7.2 in dimension $d - 1$, the general fiber of $e_j : W \to X_j$ is bounded modulo flop for all $j \leq k - 1$. Then the Cartier index of the moduli part of $e_j$ is bounded. Also because $lK_W \sim 0$, there exists a natural number $l_j$ and a generalized pair $(X_j, B_j + M_{j,X_j})$ such that $l_j(K_{X_j} + B_j + M_{j,X_j}) \sim 0$ and $l_j M_{j}$ is $b$-nef and $b$-Cartier. It is easy to see that there is a positive rational number $\epsilon > 0$ such that $(X_j, B_j + M_{j,X_j})$ is $\epsilon$-lc for each $j = 0, \ldots, k - 1$.

If $\dim Z > 0$, then by the same reason, there is an integer $l_k$ and a generalized pair $(Z, B_Z + M_Z)$ such that $l_k(K_Z + B_Z + M_Z) \sim 0$ and $l_k M_Z$ is Cartier, also by assumption, $K_Z \sim_Q 0$, then $l_k K_Z \sim 0$. By Theorem 7.7, $Z$ is bounded up to flop.

Suppose $X_j$ is bounded up to flop. That is, there exists a $klt$ variety $X'_j$ isomorphic to $X_j$ in codimension 1 which belongs to a bounded family. Because $\phi_j : X'_j \dasharrow X_j$ is an isomorphism in codimension 1 of projective varieties, it is also a birational contraction. Also since $X_j$ is $Q$-factorial, we can apply Proposition 7.6 to the Mori fiber space $X_{j-1} \to X_j$ and obtain a commutative diagram

$$
\begin{array}{ccc}
X'_{j-1} & \rightarrow & X_{j-1} \\
\downarrow & & \downarrow \\
X'_j & \dasharrow & X_j
\end{array}
$$

where the horizontal arrow $X'_{j-1} \dasharrow X_{j-1}$ is an isomorphism in codimension 1 of $Q$-factorial projective varieties. As all horizontal arrows in the diagram are isomorphisms in codimension 1 of $Q$-factorial projective varieties, it follows that $\rho(X_{j-1}/X_j) = \rho(X'_{j-1}/X'_j) = 1$; hence $X_{j-1} \to X'_j$ is a $K_{X'_{j-1}}$-Mori fiber space.

Because $X'_j$ is in a bounded family, there exists a number $r_j$ and a very ample divisor $A_j$ on $X'_j$ such that $A_j^{\dim X'_j} \leq r_j$. Also because $(X_{j-1}, B_{j-1} + M_{j-1,X_{j-1}})$ is $\epsilon$-lc, $X'_j$ is isomorphic to $X_j$ in codimension 1 and $K_{X_{j-1}} + B_{j-1} + M_{j-1,X_{j-1}} \sim_Q 0$, then $(X'_j, (\phi_j)^{-1}(B_j + M_{j,X_j}))$ is $\epsilon$-lc. Therefore, $X'_{j-1} \to X'_j$ is a generalized $(\dim X'_{j-1}, r_j, \epsilon)$-Fano type fibration. By Theorem 7.4, $X'_{j-1}$ is bounded, and it implies $X_{j-1}$ is bounded up to flop.

Since $Z$ is either a point or bounded up to flop. By induction, $X'$ is bounded up to flop. \qed

**Proof of Theorem 7.2.** We prove this by induction on dimension. Assume the result in dimension $d - 1$.

Let $X \to T$ be the bounded family given in Theorem 7.8. By boundedness, we may assume that there is a positive number $V > 0$ and a relative very ample divisor $H$ on $X$ over $T$ such that $\text{vol}(H_s) \leq V$ for all $s \in T$.

Let $M$ be the moduli $b$-divisor of $Y \to X$. Since $K_Y \sim_Q 0$, then

$$
K_X + \Delta + M_X \sim_Q 0.
$$

Let $p : W \to X$, $q : W \to X_t$ be a common resolution of $X \dasharrow X_t$, write

$$
K_W + \Delta_W + M_W \sim_Q p^*(K_X + \Delta + M_X) \sim_Q 0.
$$

Because $\Delta \geq 0$, it is easy to see that $\Delta_W^{\leq 0}$ is $p$-exceptional. Let $\Delta_{X_t} = q_* \Delta_W$, then

$$
K_{X_t} + \Delta_{X_t} + M_{X_t} \sim_Q 0.
$$

Because $X \dasharrow X_t$ is a birational contraction, then $\Delta_W^{\leq 0}$ is $q$-exceptional and $\Delta_{X_t} \geq 0$.

Let $h : Y' \to Y$ be the closure of the graph of $Y \dasharrow X_t$ and write $K_{Y'} + \Delta' \sim_Q h^* K_Y$. Because $X \dasharrow X_t$ is a birational contraction and $Y \to X$ is a contraction, then $\Delta'_{Y',\leq 0}$ is very exceptional over $X_t$. By Theorem 2.2, we can run $K_{Y'} + \Delta'_{Y',\geq 0}$-MMP over $X_t$, which terminates
with a model $Y''$ such that $\Delta_{Y''} \geq 0$, then we have $K_{Y''} + \Delta_{Y''} \sim \mathbb{Q} 0$. Also because $lK_Y \sim 0$, then $l(K_{Y''} + \Delta_{Y''}) \sim 0$ and coeff$\Delta_{Y''}$ is in a finite set.

Notice that

$$\text{vol}(\mathcal{H}_t + K_X + \Delta_X + M_X) = \text{vol}(\mathcal{H}_t) \leq V.$$ 

Also because the general fiber of $Y \to X$ is isomorphic to the general fiber of $Y'' \to X_t$, then by Theorem 6.2, $Y''$ is birationally bounded. Let $\mathcal{Y} \to S$ be the corresponding bounded family, then $Y$ is birational equivalent to $\mathcal{Y}_s$ for a closed point $s \in S$.

After taking a resolution of the generic fiber and passing to a stratification of $S$, we may assume that $\mathcal{Y}_t$ is smooth for all $t \in S$. Because $K_Y \sim \mathbb{Q} 0$, $Y$ and $\mathcal{Y}_s$ have terminal singularities, then $Y$ is a minimal model of $\mathcal{Y}_s$. By [HX13, Theorem 2.12], $\mathcal{Y}$ has a good minimal model over $T$, denote it by $\mathcal{Y}' \to S$. Therefore, by [HX13, Lemma 2.4], $Y \dashrightarrow \mathcal{Y}'_s$ is isomorphic in codimension 1 and a crepant birational equivalence, which means $Y$ is bounded up to flops. \qed

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