Cliques in the Union of $C_4$-Free Graphs

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Abstract Let $B$ and $R$ be two simple $C_4$-free graphs with the same vertex set $V$, and let $B \vee R$ be the simple graph with vertex set $V$ and edge set $E(B) \cup E(R)$. We prove that if $B \vee R$ is a complete graph, then there exists a $B$-clique $X$, an $R$-clique $Y$ and a set $Z$ which is a clique both in $B$ and in $R$, such that $V = X \cup Y \cup Z$. For general $B$ and $R$, not necessarily forming together a complete graph, we obtain that

$$\omega(B \vee R) \leq \omega(B) + \omega(R) + \frac{1}{2} \min(\omega(B), \omega(R))$$

and

$$\omega(B \vee R) \leq \omega(B) + \omega(R) + \omega(B \wedge R)$$

where $B \wedge R$ is the simple graph with vertex set $V$ and edge set $E(B) \cap E(R)$.

Keywords $C_4$-free graphs · Cliques · Obedient sets

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1 Introduction

Let $B$ (for “Blue”) and $R$ (for “Red”) be two simple graphs on the same vertex set $V$. Denote by $B \lor R$ the simple graph with vertex set $V$, in which two vertices are adjacent if they are adjacent in $B$ or in $R$. Similarly, we denote by $B \land R$ the simple graph with vertex set $V$, in which two vertices are adjacent if they are adjacent in $B$ and in $R$. Recall that a clique in a graph $G$ is a set of pairwise adjacent vertices and $\omega(G)$ denotes the maximum size of a clique. A natural question arising from Ramsey theory is the relation between $\omega(B \lor R)$, $\omega(B)$ and $\omega(R)$. In particular, we want to ask for sufficient conditions for $\omega(B \lor R)$ to be a linear function of $\omega(B)$ and $\omega(R)$. Some case that got special attention is the case where $\omega(B \lor R) \leq \omega(B) + \omega(R)$, in the strong sense that every clique of $B \lor R$ is the union of a clique in $B$ and a clique in $R$. The following definition appeared in [1].

**Definition 1.1** We say that a subset $U \subseteq V$ is obedient if there exist an $R$-clique $X$ and a $B$-clique $Y$ such that $U = X \cup Y$.

So in fact we look for sufficient conditions for every clique of $B \lor R$ to be obedient. By restricting our attention to one clique of $B \lor R$, we may in fact assume that $B \lor R$ is a complete graph.

In [2], Gyárfás and Lehel, proved the following theorem.

**Theorem 1.2** (Theorem 3 of [2]) Assume that $B \lor R$ is a complete graph. If $B$ and $R$ are $C_k$-free for $k = 4$ and $k = 5$, then $V$ is obedient.

(By $H$-free we means not having an induced copy of the graph $H$.) The methods used in [2] are combinatorial. Using topological methods, Berger [3] proved the following theorem.

**Theorem 1.3** [3] Assume that $B \lor R$ is a complete graph. If $B$ is chordal and $R$ is $C_4$-free, then $V$ is obedient.

Recently, Aharoni, Berger, Chudnovsky and Ziani [1] generalized Theorems 1.2 and 1.3 as follows.

**Theorem 1.4** (Theorem 1.7 of [1]) Let $B$ and $R$ be $C_4$-free graphs with vertex set $V$ and suppose that $R$ is also $C_5$-free. If $B \lor R$ is a complete graph, then $V$ is obedient.

**Definition 1.5** For $B$ and $R$ as above, a double $C_5$ is a pair of complementary $C_5$’s, one in $B$ and one in $R$, on the same 5 vertices.

**Theorem 1.6** (Theorem 3 of [4]) Let $B$ and $R$ be $C_4$-free graphs with vertex set $V$, and suppose that $B \lor R$ does not contains a double $C_5$. If $B \lor R$ is a complete graph, then $V$ is obedient.

In this paper we use this result to prove

**Theorem 1.7** Let $B$ and $R$ be two $C_4$-free graphs on the same vertex set $V$ and assume that $B \lor R$ is a complete graph. Then there exist a $B$-clique $X$, an $R$-clique $Y$ and an $B \land R$-clique $Z$ such that $V = X \cup Y \cup Z$. Furthermore, if $Z \neq \emptyset$ then every $x \in Z$ is one of the vertices of some double $C_5$ in $B \lor R$. 

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We now recall some basic definitions. Let $G = (V, E)$ be a simple graph, i.e., an undirected graph containing no graph loops or multiple edges. A clique in $G$ (or a $G$-clique) is a subset $C \subseteq V$, such that for every two vertices in $C$, there exists an edge connecting the two. The clique number $\omega(G)$ of $G$ is the number of vertices in a maximum clique in $G$, where a maximum clique is a clique of the largest possible size. We say that $G = (V, E)$ is a complete graph if there is an edge between every two distinct vertices. We denote by $K_n$ the complete graph with $n$ vertices. An induced subgraph on a subset $S$ of $V$, denoted by $G[S]$, is a graph whose vertex set is $S$ and whose edge set is $\{uv \mid u, v \in S \text{ and } uv \in E\}$.

2 Pairs of Graphs with Double $C_5$

In this section, we prove Theorem 1.7.

Lemma 2.1 Let $B$ and $R$ be two $C_4$-free graphs and let $C : v_1-v_2-v_3-v_4-v_5-v_1$ be a double $C_5$ in $B \lor R$ with blue edges and red diagonals. Assume that $B \lor R$ is a complete graph and $x \in V \setminus \{v_1, \ldots, v_5\}$. Then one of the following holds:

(1) $x$ is connected in $B$ to all the vertices of $C$.
(2) $x$ is connected in $R$ to all the vertices of $C$.
(3) there exists $i$ such that
   - $xv_i$ is an edge in $B$ and $R$.
   - In $B \setminus R$: $x$ is connected to the two neighbors of $v_i$ in $C$.
   - In $R \setminus B$: $x$ is connected to the two neighbors of $v_i$ in $C$.

In this case we say that $xv_i$ is the shared edge of $x$.

Proof Let $T = \{v_1, \ldots, v_5, x\}$. If $\deg_{B \lor T}(x) = 0$, then condition (2) holds. Assume that $\deg_{B \lor T}(x) = 1$. Without loss of generality, assume that $xv_1 \in E(B)$. Since $B \lor R$ is a complete graph, it follows that $xv_2, \ldots, xv_5 \in E(R)$. Since $R$ is $C_4$-free, it follows that $xv_1 \in E(R)$. For otherwise we have that $x - v_3 - v_1 - v_4 - x$ is an induced $C_4$ in $R$. So $\deg_{B \lor T}(x) = 5$ and condition (2) holds. Assume that $\deg_{B \lor T}(x) = 2$. If $xv_i, xv_j \in E(B)$, where $v_i, v_j \in E(R \lor B)$, then $x - v_i - v_m - v_j - x$ is an induced $C_4$ in $B$, where $v_m$ is the common neighbor of $v_i$ and $v_j$ in the cycle $v_1 - v_2 - v_3 - v_4 - v_5 - v_1$, a contradiction. It follows that $x$ is connected to two successive vertices of the cycle $v_1 - v_2 - v_3 - v_4 - v_5 - v_1$. Without loss of generality, assume that $xv_1, xv_2 \in E(B)$. It follows that $xv_3, xv_4, xv_5 \in E(R)$. Since $R$ is $C_4$-free, it follows that $xv_1 \in E(R)$. For otherwise we have that $x - v_3 - v_1 - v_4 - x$ is an induced $C_4$ in $R$. Since $R$ is $C_4$-free, it follows that $xv_2 \in E(R)$. For otherwise we have that $x - v_4 - v_2 - v_5 - x$ is an induced $C_4$ in $R$. So $\deg_{B \lor T}(x) = 5$ and condition (2) holds. Assume that $\deg_{B \lor T}(x) = 3$. A similar argument shows that $x$ is connected to three successive vertices of the cycle. Without loss of generality, assume that $xv_1, xv_2, xv_3 \in E(B)$. It follows that $xv_4, xv_5 \in E(R)$. Since $R$ is $C_4$-free, it follows that $xv_2 \in E(R)$. For otherwise we have that $x - v_4 - v_2 - v_5 - x$ is an induced $C_4$ in $R$. If $xv_1 \notin E(R)$ and $xv_3 \notin E(R)$, then condition (3) holds. If $xv_1 \in E(R)$, then $xv_3 \in E(R)$. For otherwise, we have that $x - v_1 - v_3 - v_5 - x$ is an induced $C_4$ in $R$. So condition (2) holds. Similarly, if $xv_3 \in E(R)$, then condition (2) holds. Assume that $\deg_{B \lor T}(x) = 4$. Without loss of generality, assume that $xv_1, xv_2, xv_3, xv_4 \in E(R)$.
$E(B)$. Since $B$ is $C_4$-free, it follows that $xv_5 \in E(B)$. For otherwise we have that $x - v_1 - v_5 - v_4 - x$ is an induced $C_4$ in $B$, a contradiction. So $\deg_{B|T}(x) = 5$ and condition (1) holds.

\textbf{Lemma 2.2} Let $H$ be a $C_4$-free graph and $v_1 - v_2 - v_3 - v_4 - v_5 - v_1$ be an induced $C_5$ cycle in $H$. If $x$ and $y$ are two distinct vertices not belonging to $\{v_1, \ldots, v_5\}$ and connected to the same three successive vertices of the cycle, then $xy \in E(H)$.

\textit{Proof} Without loss of generality, assume that $\{x, y\}$ is $H$-complete to $\{v_1, v_2, v_3\}$.

If $xy \notin E(H)$, then $v_1 - y - v_3 - x - v_1$ is an induced $C_4$ in $H$, a contradiction. It follows that $xy \in E(H)$.

Now, we prove the main result.

\textit{Proof of Theorem 1.6} If $B \lor R$ does not contain a double $C_5$, then by Theorem 1.6 there exists a $B$-clique $X$, and an $R$-clique $Y$ such that $V = X \cup Y$. By choosing $Z = \emptyset$, we are done. So assume that $B \lor R$ contains a double $C_5$ say $v_1 - v_2 - v_3 - v_4 - v_5 - v_1$ with blue edges and red diagonals. If $V = \{v_1, \ldots, v_5\}$, then choosing $X = \{v_1, v_2\}$, $Y = \{v_3, v_5\}$ and $Z = \{v_4\}$, satisfies the requirements of the theorem. So assume that $V \setminus \{v_1, \ldots, v_5\} \neq \emptyset$. We define the following sets:

\begin{align*}
M &= \{p \mid p \notin \{v_1, \ldots, v_5\} \text{ and } pv_i \in E(B) \text{ for all } 1 \leq i \leq 5\}, \\
N &= \{p \mid p \notin \{v_1, \ldots, v_5\} \text{ and } pv_i \in E(R) \text{ for all } 1 \leq i \leq 5\}, \\
A_j &= \{p \mid p \notin \{v_1, \ldots, v_5\} \text{ and } pv_j \text{ is the shared edge of } p \} \text{ for all } 1 \leq j \leq 5.
\end{align*}

By Lemma 2.1, we have $V \setminus \{v_1, \ldots, v_5\} = M \cup N \cup A_1 \cdots \cup A_5$.

If $p_1, p_2 \in M$, then $p_1v_i \in E(B)$ and $p_2v_i \in E(B)$ for all $1 \leq i \leq 5$. In particular, $p_1$ and $p_2$ are connected in $B$ to the same three successive vertices of the induced cycle $v_1 - v_2 - v_3 - v_4 - v_5 - v_1$. Since $B$ is $C_4$-free, by Lemma 2.2 we conclude...
that $p_1p_2 \in E(B)$. It follows that $M$ is an $B$-clique. Similarly, if $p_1, p_2 \in N$, then $p_1$ and $p_2$ are connected in $R$ to the same three successive vertices of the induced cycle $v_1 - v_3 - v_5 - v_2 - v_4 - v_1$. By Lemma 2.2, we obtain that $N$ is an $R$-clique.

Let $p_1, p_2 \in A_1$. Note that $\{p_1, p_2\}$ is $B$-complete to $\{v_1, v_2, v_5\}$ and $v_1, v_2, v_5$ are successive vertices in the induced cycle $v_1 - v_2 - v_3 - v_4 - v_5 - v_1$ in $B$. By Lemma 2.2, we obtain that $p_1p_2 \in E(B)$. Note also that $\{p_1, p_2\}$ is $R$-complete to $\{v_1, v_3, v_4\}$ and $v_1, v_3, v_4$ are successive vertices in the induced cycle $v_1 - v_3 - v_5 - v_2 - v_4 - v_1$ in $R$. By Lemma 2.2, we obtain that $p_1p_2 \in E(R)$. It follows that $A_1$ is a clique both in $B$ and in $R$. Similarly, $A_j$ is a clique both in $B$ and in $R$ for all $2 \leq j \leq 5$.

A similar argument shows that $M \cup A_j$ is an $B$-clique for all $1 \leq j \leq 5$ and that $N \cup A_j$ is an $R$-clique for all $1 \leq j \leq 5$.

Claim 1 $M \cup A_1 \cup A_2 \cup \{v_1, v_2\}$ is a clique in $B$.

Proof of the claim If $x \in M \cup A_1$, then by the definition of $M$ and $A_1$, we obtain that $xv_1 \in E(B)$. If $x \in A_2$, then $xv_2$ is the shared edge of $x$. Since $v_1$ is a neighbour (in $B \setminus R$) of $v_2$ in the induced cycle $v_1 - v_2 - v_3 - v_4 - v_5 - v_1$, it follows that $xv_1 \in E(B)$. So $\{v_1\}$ is $B$-complete to $M \cup A_1 \cup A_2$. Similarly, $\{v_2\}$ is $B$-complete to $M \cup A_1 \cup A_2$. We finish the proof of the claim if we show that $A_1$ is $B$-complete to $A_2$. Let $x_1 \in A_1$ and $x_2 \in A_2$. If $x_1x_2 \in E(R)$, then $x_1 - x_2 - v_5 - v_3 - x_1$ is an induced $C_4$ in $R$, a contradiction. Since $B \cup R$ is a complete graph it follows that $x_1x_2 \in E(B)$. This proves the claim. $\square$

Claim 2 $N \cup A_3 \cup A_5 \cup \{v_3, v_5\}$ is a clique in $R$.

Proof of the claim If $x \in N \cup A_3$, then by the definition of $N$ and $A_3$, we obtain that $xv_3 \in E(R)$. If $x \in A_5$, then $xv_5$ is the shared edge of $x$. Since, $v_3$ is a neighbour (in $R \setminus B$) of $v_5$ in the induced cycle $v_1 - v_3 - v_5 - v_2 - v_4 - v_1$, it follows that $xv_3 \in E(R)$. So $\{v_3\}$ is $R$-complete to $N \cup A_3 \cup A_5$. Similarly, $\{v_5\}$ is $R$-complete to $N \cup A_3 \cup A_5$. We finish the proof of the claim if we show that $A_3$ is $R$-complete to $A_5$. Let $x_3 \in A_3$ and $x_5 \in A_5$. If $x_3x_5 \in E(B)$, then $x_3 - v_2 - v_1 - x_5 - x_3$ is an induced $C_4$ in $B$, a contradiction. Since $B \cup R$ is a complete graph it follows that $x_3x_5 \in E(R)$. This proves the claim. $\square$

Claim 3 $A_4 \cup \{v_4\}$ is a clique both in $B$ and in $R$.

Proof of the claim If $x \in A_4$, then $xv_4$ is the shared edge to $x$. So $xv_4$ is an edge both in $B$ and in $R$. Thus, the claim follows from the definition of $A_4$.

Let $X = M \cup A_1 \cup A_2 \cup \{v_1, v_2\}$, $Y = N \cup A_3 \cup A_5 \cup \{v_3, v_5\}$, $Z = A_4 \cup \{v_4\}$.

By the above claims and Lemma 2.1, we obtain that $X$ is a clique in $B$, $Y$ is a clique in $R$ and $Z$ is a clique both in $B$ and in $R$, with $V = X \cup Y \cup Z$.

Let $x \in Z$. If $x = v_4$, then $v_1 - v_2 - v_3 - v_4 - v_5 - v_1$ is a double $C_5$ that contains $v_4$ as a vertex. If $x \neq v_4$, then $x - v_5 - v_1 - v_2 - v_3 - x$ is a double $C_5$ in $B \cup R$. Hence, if $x \in Z$ then $x$ is one of the vertices of some double $C_5$ in $B \cup R$. $\square$
3 Two Corollaries

As a corollary of Theorem 1.7, we obtain the following.

**Corollary 3.1** If $B$ and $R$ are two $C_4$-free graphs on the same vertex set $V$ then

$$\omega(B \lor R) \leq \omega(B) + \omega(R) + \omega(B \land R).$$

**Proof** Let $T$ be a maximum clique in $B \lor R$. So $B|T$ and $R|T$ are two $C_4$-free graphs on the same vertex set $T$ such that $G(B|T, R|T)$ is a complete graph. By Theorem 1.7, there exists a $B|T$-clique $X$, an $R|T$-clique $Y$ and a clique $Z$ in both $B|T$ and $R|T$, such that $T = X \cup Y \cup Z$. So

$$\omega(B \lor R) = |T| \leq |X| + |Y| + |Z| \leq \omega(B) + \omega(R) + \omega(B \land R).$$

$\square$

Also, we have the following additional corollary of Theorem 1.7.

**Corollary 3.2** If $B$ and $R$ are two $C_4$-free graphs on the same vertex set $V$ then

$$\omega(B \lor R) \leq \omega(B) + \omega(R) + \frac{1}{2} \min(\omega(B), \omega(R)).$$

**Proof** Let $T$ be a maximum clique in $B \lor R$. So $B|T$ and $R|T$ are two $C_4$-free graphs on the same vertex set $T$ such that $G(B|T, R|T)$ does not contain a double $C_5$, then by Theorem 1.6, $T$ is the union of a clique in $B|T$ and a clique in $R|T$. It follows that $\omega(B \lor R) \leq \omega(B) + \omega(R)$ and the theorem holds. Assume that $G(B|T, R|T)$ contains a double $C_5$. Following the proof of Theorem 1.7, we may assume that $A_4$ is minimal so that $|A_4| \leq |A_i|$ for all $1 \leq i \leq 5$. We obtain that $2|Z| = 2|A_4| + 2 \leq |A_1| + |A_2| + |\{v_1, v_2\}| = |A_1 \cup A_2 \cup \{v_1, v_2\}| \leq \omega(B|T) \leq \omega(B)$. So $|Z| \leq \frac{1}{2} \omega(B)$. Similarly, $|Z| \leq \frac{1}{2} \omega(R)$. It follows that

$$\omega(B \lor R) \leq \omega(B) + \omega(R) + \frac{1}{2} \min(\omega(B), \omega(R)).$$

$\square$

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