Superconformal $N = 1$ Gauge Theories, $\beta$-Function Invariants and their Behavior under Seiberg Duality

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Abstract

In this paper we discuss some aspects of the behavior of superconformal $N = 1$ models under Seiberg's duality. Our claim is that if an electric gauge theory is superconformal on some marginal subspace of all coupling constants then its magnetic dual must be also superconformal on a corresponding moduli space of dual couplings. However this does not imply that the magnetic dual of a completely finite $N = 1$ gauge theory is again finite. Moreover we generalize this statement conjecturing that also for non-superconformal $N = 1$ models the determinant of the $\beta$-function equations is invariant under Seiberg duality. During the course of this investigation we construct some superconformal $N = 1$ gauge theories which were not yet discussed before.

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Seiberg [1] has conjectured that two different, dual $N = 1$ supersymmetric gauge theories lead to the same physics in the infrared, where the strong coupling region of one gauge theory corresponds to the weak coupling region of the other, and vice versa. For example, $N = 1$ SUSY QCD with gauge group $SU(N_c)$ and $N_F$ quark flavors $Q_i, ar{Q}_i$ ($i = 1, \ldots, N_F$) in the fundamental respectively anti-fundamental representations of the gauge group is dual to the $SU(N_f - N_c)$ SUSY QCD again with $N_F$ fundamental quarks $q_i, \bar{q}_i$ plus gauge singlets $M_{ij}$. Many other examples, based on $SO(N_c)$ and $SP(N_c)$ gauge groups are discussed in the literature. To claim that two $N = 1$ gauge theories are dual in above mentioned sense, certain physical requirements have to be satisfied. Most notably these are

(i) Agreement of global symmetries and global anomalies for dual pairs.

(ii) Same gauge invariant operators (baryons, mesons).

(iii) Same behavior under deformations of the theory (turning on/off vevs or mass terms).

In this note we shall discuss a new constraint which two dual models should fulfill, and which to our knowledge has not been discussed before in the literature. To be more specific, we will discuss the all loop-order gauge plus Yukawa $\beta$-functions of $N = 1$ gauge theories and argue that a certain $\beta$-function determinant, which is build from the coefficients of the $\beta$-function equations, is an invariant under Seiberg’s duality, i.e. should be the same for dual pairs. Therefore the new ingredient emerging from the present discussion is that even though the $\beta$-functions change under Seiberg’s duality, its determinant is unchanged.

This discussion is particularly relevant for $N = 1$ models which are completely finite (like $N = 4$ gauge theories), with vanishing $\beta$-functions as well as vanishing anomalous dimensions. A bigger class of models is given by those which contain at least one or several exactly marginal operators, which were first discussed by Leigh and Strassler [2]. (Of course the models with marginal operators include the completely finite models.) The existence of an exactly marginal operator corresponds to having an arbitrary coupling in the theory, called modulus. On the fixed line of marginal couplings the theory is superconformal, i.e. all $\beta$-functions are vanishing. Following the work of Leigh and Strassler [2], a simple criterion for the existence of exactly marginal operators is given by the vanishing of the $\beta$-function determinant, which we mentioned above.

For dual models, the dimensions of the moduli space should agree, since in the infrared there should be the same number of marginal operators which can be used to deform the theory. Hence, having found a marginal operator in one $N = 1$ gauge theory, there must be also the corresponding marginal operator in the dual gauge theory. Therefore the $\beta$-function determinant must vanish in the dual theory, if it was zero in the original model. In other words, the dual magnetic model must be also superconformal on the line of fixed points if the electric model is superconformal. More generally, we will argue that even in case of non-vanishing $\beta$-function determinants, they nevertheless should be the same for dual models; this statement will be shown to be true for several examples.
We will also discuss that the $\beta$-function determinants are invariant under adding massive fields to the theory or under integrating out fields via their equation of motions.

In a recent paper [3] we have investigated the Seiberg duals of several types of all loop finite $N = 1$ gauge theories, including the finite $SU(5)$ GUT models. As a result of this investigation we have seen that the dual of a finite model is in general non-finite [2, 3] (We found that for one particular finite model, namely $SO(10)$ with matter fields in $N_f = 8$ vector and $N_q = 8$ spinor representations, the dual is also supposed to be finite.) However, as we discussed above, the marginal operators should be preserved under Seiberg’s duality, and consequently, the dual of a finite $N = 1$ must have at least one marginal operator. In this way, non-finite $N = 1$ gauge theories with marginal operators may belong to the same universality class as finite $N = 1$ models. Both, the electric theory as well its magnetic dual are superconformal on the moduli space of marginal couplings.

Recently, a class of all order finite $N = 1$ models were realized as gauge theories on D3 branes with an orbifold action on the space transversal to the branes [5]. In this context the finiteness of these models follows from a duality to type IIB string theory on $AdS_5 \times S^5$ geometry, which should hold in the large $N$ limit of the gauge theory [6]. The $SO(4,2)$ symmetry of $AdS_5$ translates into the superconformal group of the gauge theory which lives at the four-dimensional boundary of $AdS_5$. We will analyze the $\beta$-function determinants and the Seiberg duals for the type of $N = 1$ models constructed in [5]. Also note that in the brane picture of Hanany and Witten [8], which is $T$-dual to construction via D3 branes, it was recently shown [9] that in finite theories with vanishing $\beta$-functions the branes are not bent.

Consider a $N = 1$ supersymmetric gauge theory based on the gauge group $G = \prod_{l=1}^{N} G_l$ with corresponding gauge couplings $g_l$ and $M$ chiral superfields $\phi_i$ ($i = 1, \ldots, M$) in the $R_{i,l} = (R_{i,1}, R_{i,2}, \ldots, R_{i,N})$ representation of the group $G$. The, in general non-renormalizable, superpotential $W$ is of the form

$$W = \sum_{\alpha} h_{\alpha} \prod_{i} \phi_{i}^{n_{i,\alpha}}, \quad \alpha = 1, \ldots, L, \quad (1)$$

where $n_{i,\alpha}$ is the number of superfields $\phi_i$ in each term of $W$. Then the exact, i.e. all orders $\beta$-functions $\beta_m$, $m = 1, \ldots, N, N + 1, \ldots, N + L$, for the gauge couplings $g_l$ and the Yukawa couplings $h_{\alpha}$ are given by the following set of equations [10]:

$$\beta_{g_l} = - \left(3C_2(G_l) - \sum_i T(R_{i,l})\right) - \sum_i T(R_{i,l})\gamma_i = B_{l,0} + \sum_i B_{l,i}\gamma_i,$$

$$\beta_{h_{\alpha}} = \left(-3 + \sum_i n_{i,\alpha}\right) + \frac{1}{2} \sum_i n_{i,\alpha}\gamma_i = B_{\alpha,0} + \sum_i B_{\alpha,i}\gamma_i. \quad (2)$$

Here, $\gamma_i$ is the anomalous dimension of the superfield $\phi_i$, $C_2(G_l)$ is the quadratic Casimir of the adjoint representation of $G_l$, and $T(R_{i,l})$ is the index of the representation $R_i$.

\(^d\)The finite $SU(5)$ GUT models successfully predict, among others, the top quark mass [4].

\(^e\)For a construction of these models by string compactification, see [5].
with respect to $G_l$. The anomalous dimensions are in general unknown functions of the couplings constants $g_l$ and $h_\alpha$.

We see that the $\beta$-functions are given by a set of linear equations in the anomalous dimensions $\gamma_i$ with a $[(N+L)\times(M+1)]$-dimensional coefficient matrix $B_{m,n}$, $n = 0, \ldots, M$. Note that the number of columns can be smaller than $M+1$ in case some of the superfields $\phi_i$ have the same anomalous dimensions; this situation will apply if the superpotential is invariant under some global symmetries. The matrix elements $B_{l,0}$, $l = 1, \ldots, N$, are nothing else than the one loop $\beta$-function coefficients $\beta_l(1)$ of the gauge couplings. On the other hand, the elements $B_{N+\alpha,0}$, $\alpha = 1, \ldots, L$, denote mass dimensions of the coupling constants $h_\alpha$.

The condition for the existence of marginal operators $[2]$ is that all $N + L$ $\beta$-function equations $\beta_m$ in eq.(2) simultaneously vanish on some particular locus in the space of coupling constants $(g_l, h_\alpha)$. The anomalous dimensions are functions of the coupling constants, and therefore the vanishing of the $\beta$-functions puts $N + L$ conditions on the $N + L$ couplings. Since we are not looking for isolated fixed point solutions of the equations $\beta_m = 0$ but for manifolds of fixed points, marginal operators are present if the $\beta_m$ are linearly dependent functions in the anomalous dimensions $\gamma_i$. More precisely, if we find that $r$ $\beta$-function equations are linearly dependent, then there is a $r$-dimensional moduli space of marginal couplings $g_k^{(0)}$, $k = 1, \ldots, r$, which can be arbitrarily chosen. On this manifold of fixed points the theory is supposed to be superconformal. The $r$ equations

$$\sum_m c_{r,m} \beta_m = 0$$

with non-vanishing coefficients $c_{r,m}$ define the marginal couplings as functions of the couplings $g_l$ and $h_\alpha$, but only implicitly, since we do not know the precise form of the all order functions $\gamma_i(g_l, h_\alpha)$. One might expect that an $S$-duality group is acting on the marginal couplings $g_k^{(0)}$, which relates strong and weak coupling in the manifold of marginal operators.

For simplicity consider now the case that $B_{m,n}$ is a square matrix of dimension $(N+L) \times (N+L)$, which means that we assume that the number of different anomalous dimensions of chiral superfields is one less than the number of different couplings. Then the condition for the existence of marginal operators can be simply stated as follows $[4]$

$$\det B = 0.$$  \hspace{1cm} (4)

The number of zero eigenvalues of $B$ coincides with the number $r$ of marginal operators.

Now consider all order finite $N = 1$ gauge theories, which are a subclass of the models with marginal operators. In the present framework $N = 1$ gauge theories are finite if all $\beta$-functions $\beta_m$ and all anomalous dimensions $\gamma_i$ vanish to all orders in perturbation theory. Then the $\beta$-functions eq.(2) immediately imply that all constant terms in these equations

$[4]$If $B$ is not a square matrix this condition is generalized to $\det(BB^T) = 0$.  

4
must be zero. In other words, finiteness requires that all one loop gauge \(\beta\)-functions \(\beta_l^{(1)} = B_{l,0}\) vanish, and that all classical couplings in the superpotential are dimensionless, i.e. \(B_{N+\alpha,0} = 0\). If these conditions are satisfied the vanishing of the \(\beta\)-functions provide an homogeneous system of linear equations in the anomalous dimensions \(\gamma_i\), which puts, as in the case of marginal operators, \(N + L\) conditions on \(N + L\) coupling constants. If \(r\) of these linear equations are linearly dependent, the theory is supposed to be finite on the \(r\)-dimensional submanifold of free couplings \(\tilde{g}_k^{(0)}\). The remaining couplings are not any more independent, but are functions of the \(\tilde{g}_k^{(0)}\). So as a criterion for finite \(N = 1\) theories we require that

\[
\det B = 0, \quad \text{with} \quad B_{l,0} = 0, \quad B_{N+\alpha,0} = 0. \quad (5)
\]

This condition is in agreement with a theorem \cite{11} which states that a \(N = 1\) gauge theory is all orders finite if the one loop \(\beta\)-functions and the one loop anomalous dimensions are vanishing and if there exist non-degenerate solutions of the coupling constant reduction equations \cite{13}. So for finite \(N = 1\) theories, the number of free coupling constants is in general reduced, which just means that we are moving in the \(r\)-dimensional subspace of marginal operators. Also note that the vanishing of the one loop \(\beta\)-functions \(\beta^{(1)}\) and one loop anomalous dimensions \(\gamma_i^{(1)}\) ensures the finiteness at the two loop level, too \cite{14}.

Now let us discuss the presence of marginal operators in the dual Seiberg picture. The dual model depends very much on the details of the original gauge theory. Let us denote the dual gauge group by \(\bar{G} = \prod_{l=1}^{N} \bar{G}_l\) with matter fields \(\bar{\phi}_i (i = 1, \ldots, \bar{M})\), couplings \(\bar{g}_l\) and \(\bar{h}_\alpha (\alpha = 1, \ldots, \bar{L})\), anomalous dimensions \(\bar{\gamma}_i\) and \(\beta\)-functions \(\bar{\beta}_l, \bar{\beta}_\alpha\) with coefficient matrix \(\bar{B}\). \(N = 1\) Seiberg duality states that two dual models flow to the same infrared fixed points, which means that in the infrared they are physically equivalent. So in the infrared, also the dual theory must have the same marginal operators with couplings \(\tilde{g}_k^{(0)}(k = 1, \ldots, r)\) as the original model. However this requirement should not be restricted to the infrared since the marginal couplings do not run. Therefore, for consistency one has to demand that

\[
\det \bar{B} = 0 \quad (6)
\]

in case \(\det B = 0\) in the original model. However this requirement does not mean that the dual of a finite \(N = 1\) theory is again a finite model, as it was already observed in \cite{13}. It only means that the reduced subspace of marginal couplings must agree for dual pairs. Therefore making a statement concerning invariants under Seiberg’s duality it is not the notion of finiteness but only the notion of being superconformal on the manifold of fixed lines which remains invariant. In the infrared, where the electric and the dual magnetic model describe the same physics, there is no difference between a superconformal and a completely finite \(N = 1\) gauge theory. Comparing the electric theory and its magnetic dual, there should be a simple relation which maps the marginal couplings in two dual pairs onto each other:

\[
\bar{g}^{(0)} = \tilde{g}_k^{(0)}(g_k^{(0)}). \quad (7)
\]

\(^9\)See also \cite{12}.
However the couplings with non-vanishing $\beta$-functions are not immediately related to each other, since in the ultraviolet the two theories are not supposed to be identical.

Now we want to go one step further, conjecturing that the $\beta$-function determinants $B$ and $\tilde{B}$ are in fact invariant (up to a group theoretical factor) under Seiberg duality even in the case they do not vanish, i.e. in the case the model is not superconformal and the reduction of coupling constants is not possible. We do not have a firm proof for this assertion, but we will show in the following that it holds for many explicitly constructed dual pairs. The idea behind this conjecture is that $\text{det } B$ corresponds in some sense to an overall coupling constant whose $\beta$-function is not changed by the Seiberg duality. If the gauge theory can be derived from string theory, this overall coupling might be directly related to the string coupling constant.

As the first and simplest example consider supersymmetric QCD (SQCD) with $N_f$ flavors of quarks in the $(N_c + \bar{N}_c)$ representation of the gauge group $G = SU(N_c)$ with gauge coupling $g$. The corresponding chiral superfields will be denoted by $Q_i$ and $\bar{Q}_i$, $i = 1, \ldots, N_f$. For concreteness we assume that the superpotential contains only one Yukawa coupling constant $h$ and consists of all possible $SU(N_f)$-flavor symmetric combinations of the baryons $B \sim Q^{N_c} (N_f \geq N_c)$:

$$ W = h(Q_{i_1}Q_{i_2}\ldots Q_{i_{N_c}} + \bar{Q}_{i_1}\bar{Q}_{i_2}\ldots \bar{Q}_{i_{N_c}} + \ldots). $$

Furthermore, the $SU(N_f)$ flavor symmetry implies that all anomalous dimensions are the same: $\gamma_i = \gamma$. Then the corresponding two $\beta$-function equations lead to the following matrix $B$:

$$ B = \begin{pmatrix} N_f - 3N_c & -N_f \\ N_c - 3 & -N_f^2 \\ \frac{N_f}{2} \end{pmatrix}. $$

Its determinant is given by

$$ \text{det } B = \frac{3}{2}N_cN_f - \frac{3}{2}N_c^2 - 3N_f. $$

The condition for the existence of a marginal operator, $\text{det } B = 0$, is a diophantic equation in $N_c$ and $N_f$, and we found solutions for

(i) $N_c = 3, N_f = 9,$

(ii) $N_c = 4, N_f = 8,$

(iii) $N_c = 6, N_f = 9.$

The first case (i) corresponds to a finite model upon reduction to one coupling constant; it is Seiberg dual to the non-finite case (iii). The second solution (ii) is non-finite but self-dual under Seiberg duality.

Let us consider the Seiberg dual of SQCD in more detail. It is given by the dual gauge group $\tilde{G} = SU(N_f - N_c)$ and $N_f$ fundamental quarks $q_i, \bar{q}_i$ plus gauge singlet mesons $M_{ij}$. 
The dual superpotential is given by
\[ \tilde{W} = M_{ij} \tilde{q}_i \tilde{q}_j + \tilde{h}(q_{i_1} q_{i_2} \ldots q_{i_{N_f-N_c}} + \bar{q}_{i_1} \bar{q}_{i_2} \ldots \bar{q}_{i_{N_f-N_c}} + \ldots). \] (11)

The dual β-functions now provide the following matrix \( \tilde{B} \) with respect to the two couplings \( \tilde{g}, \tilde{h} \) and the anomalous dimension \( \tilde{\gamma} \) of the fields \( q_i \) and \( \bar{q}_i \):
\[ \tilde{B} = \begin{pmatrix} 3N_c - 2N_f & -N_f \\ N_f - N_c - 3 & \frac{N_f - N_c}{2} \end{pmatrix}. \] (12)

It is easy to show that
\[ \det B = \det \tilde{B}, \] (13)
as advocated before.

Next let us discuss the β-function determinants and Seiberg duality \([1, 16]\) for the gauge group \( G = SO(N_c) \) with \( N_f \) quarks \( Q_i \) in the fundamental vector representation. We discuss two types of superpotentials. The first type is given, like for the \( SU(N_c) \) case, by the baryon superfield \( B \sim Q^{N_c} \):
\[ W = h(Q_{i_1}Q_{i_2} \ldots Q_{i_{N_c}} + \ldots). \] (14)

Then we derive the following β-function matrix:
\[ B = \begin{pmatrix} N_f - 3(N_c - 2) & -N_f \\ N_c - 3 & \frac{N_c}{2} \end{pmatrix}. \] (15)

Its determinant is given by
\[ \det B = \frac{3}{2} N_c N_f - \frac{3}{2} N_c^2 - 3N_f + 3N_c. \] (16)

One marginal operator is present, if \( N_c = N_f \), and the model is finite if \( N_c = N_f = 3 \).

The dual theory is based on the gauge group \( \tilde{G} = SO(N_f - N_c + 4) \) and contains \( N_f \) fundamental quarks \( q_i \) and the gauge singlet meson fields \( M_{ij} \). To obtain the dual superpotential it is important to remember that the gauge invariant baryon operators \( B \sim Q^{N_c} \) in the electric theory are mapped to the following gauge invariant operator on the magnetic side: \( B \rightarrow W^2_a q^{N_f - N_c} \). Here, \( W_a \) is the chiral gauge field strength superfield. Therefore the dual superpotential takes the following form:
\[ \tilde{W} = M_{ij} q_i \bar{q}_j + \tilde{h}(W^2_a q_{i_1} q_{i_2} \ldots q_{i_{N_f-N_c}} + \ldots). \] (17)

\(^h\)One could also include the coupling constant of term \( M_{ij} q_i \bar{q}_j \) plus the anomalous dimension of the fields \( M_{ij} \) in the matrix \( \tilde{B} \). However the determinant of the corresponding 3 × 3 matrix \( \tilde{B} \) is only changed by an irrelevant numerical factor compared to eq. (12). Since the coupling of \( M_{ij} q_i \bar{q}_j \) is not independent anyway, we prefer not to include this term (see the discussion in \([13]\) ).
Since the field $W_\alpha$ has mass dimension $\frac{3}{2}$, the dual $\beta$-functions are encoded in the following matrix $\tilde{B}$:

$$\tilde{B} = \begin{pmatrix}
3N_c - 2N_f - 6 & -N_f \\
N_f - N_c & \frac{N_f - N_c}{2}
\end{pmatrix}. \quad (18)$$

It is easy to show that again the dual magnetic determinant agrees with the electric determinant eq. (16).

We can also start in the electric $SO(N_c)$ gauge theory with the superpotential

$$W = h(W_\alpha Q_{i_1} Q_{i_2} \ldots Q_{i_{N_c-4}} + \ldots). \quad (19)$$

Now the $\beta$-function matrix looks like

$$B = \begin{pmatrix}
N_f - 3(N_c - 2) & -N_f \\
N_c - 4 & \frac{N_f - 4}{2}
\end{pmatrix}, \quad (20)$$

and the corresponding determinant is given by

$$\det B = \frac{3}{2} N_c N_f - \frac{3}{2} N_c^2 - 6N_f + 9N_c - 12. \quad (21)$$

We see that with this superpotential, which includes the gauge field strength $W_\alpha$, the gauge theory possesses a marginal operator for $N_c = 4$ and $N_f$ arbitrary. The model is finite for $N_c = 4$ and $N_f = 6$. On the dual magnetic side the superpotential is given in terms of the baryonic operators:

$$\tilde{W} = M_{ij} q_i \bar{q}_j + h(q_{i_1} q_{i_2} \ldots q_{i_{N_f-N_c+4}} + \ldots). \quad (22)$$

This leads to the matrix

$$\tilde{B} = \begin{pmatrix}
3N_c - 2N_f - 6 & -N_f \\
N_f - N_c + 1 & \frac{N_f - N_c + 4}{2}
\end{pmatrix}, \quad (23)$$

whose determinant agrees with eq. (21).

The next example we are considering is given by the product gauge group $G = SU(N_c) \times SU(N_c) \times SU(N_c)$ with gauge couplings $g_1$, $g_2$, and $g_3$. As matter fields we take $N_f$ fields in the representations

$$Q_1 = (N_c, \bar{N}_c, 1), \quad Q_2 = (1, N_c, \bar{N}_c), \quad Q_3 = (\bar{N}_c, 1, N_c), \quad (24)$$

with anomalous dimensions $\gamma_1$, $\gamma_2$, $\gamma_3$. The cubic superpotential is given by

$$W = h(Q_1 Q_2 Q_3 + \ldots). \quad (25)$$

Computing the $\beta$-function equations for the four couplings $g_1$, $g_2$, $g_3$ and $h$ we derive the following matrix $B$ (the first three rows correspond to the gauge $\beta$-functions, the last three columns belong to the three $\gamma_i$):

$$B = \begin{pmatrix}
N_c N_f - 3N_c & -\frac{1}{2} N_c N_f & 0 & -\frac{1}{2} N_c N_f \\
N_c N_f - 3N_c & -\frac{1}{2} N_c N_f & -\frac{1}{2} N_c N_f & 0 \\
N_c N_f - 3N_c & 0 & -\frac{1}{2} N_c N_f & -\frac{1}{2} N_c N_f \\
0 & 0 & 0 & \frac{1}{2}
\end{pmatrix}. \quad (26)$$
The corresponding determinant looks like
\[
\det B = \frac{3}{8}(N_f - 3)N_c^3 N_f^2. \tag{27}
\]

We see that \(\det B\) vanishes for \(N_f = 3\). Then the model is completely finite. The case \(N_f = 3\) can be obtained by \(Z_3\)-orbifolding an \(N = 4, SU(N_c)\) gauge theory and was recently discussed in the context of \(AdS^5 \times S^5\) geometry in type IIB superstrings [5].

Now let us now construct the Seiberg dual of this model applying the procedure outlined in [17]. As an abbreviation we introduce the following parameters:
\[
a = N_f - 1, \quad b = N_f^2 - N_f - 1, \quad c = (N_f^2 - N_f - 1)(N_f - 1)N_f - 1.
\]

The dual gauge group is then given by \(\tilde{G} = SU(aN_c) \times SU(bN_c) \times SU(cN_c)\). The massless dual matter fields transform under this gauge group as
\[
\begin{align*}
cN_f & \times [q_1 = (aN_c, bN_c, 1)] \\
bN_f & \times [q_2 = (aN_c, 1, cN_c)] \\
aN_f & \times [q_3 = (1, bN_c, cN_c)].
\end{align*}
\tag{28}
\]

Note that in deriving this dual spectrum we used several mass terms between mesons fields and quarks to decouple these states from the massless spectrum. The dual superpotential is given as
\[
\tilde{W} = \tilde{h}(q_1 q_2 q_3 + \ldots). \tag{29}
\]

The dual \(\beta\)-functions lead to the following matrix \(\tilde{B}\):
\[
\tilde{B} = \begin{pmatrix}
N_c N_f bc - 3N_c a & -\frac{1}{2}N_c N_f bc & 0 & -\frac{1}{2}N_c N_f bc \\
N_c N_f ac - 3N_c b & -\frac{1}{2}N_c N_f ac & -\frac{1}{2}N_c N_f ac & 0 \\
N_c N_f ab - 3N_c c & 0 & -\frac{1}{2}N_c N_f ab & -\frac{1}{2}N_c N_f ab \\
0 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2}
\end{pmatrix}.
\tag{30}
\]

Then the dual determinant is derived to be
\[
\det \tilde{B} = \frac{3}{8}(N_f - 3)N_c^3 N_f^2 abc. \tag{31}
\]

Up to the group theoretical factor \((abc)\) the dual determinant \(\det \tilde{B}\) agrees with \(\det B\).

For \(N_f = 2\) the theory is selfdual. Note that in this case the determinant does not vanish. This is in some way a very surprising result, since it has been believed that selfduality only appears in connection with marginal lines connecting the two theories.

As a last example consider the model discussed in [18]. The duality relates \(Spin(10)\) gauge theory with vectors and spinors to an \(SU \times Sp\) product gauge group with a symmetric tensor representation for the \(SU\) factor.

The electric theory has gauge group \(G = Spin(10)\) with matter fields transforming as
\[
V = N_f \cdot 10, \quad S = N_q \cdot 16. \tag{32}
\]
The dual theory has $G = SU(N_c = N_f + 2N_q - 7) \times Sp(2N_q - 2)$ gauge group and matter fields transforming as

$q = N_f(\mathbf{1}, \mathbf{1}), \quad q' = 2(\mathbf{1}, \mathbf{1}), \quad \bar{q} = (2N_q-1)\cdot(\mathbf{1}, \mathbf{1}), \quad s = (\text{sym.}, \mathbf{1}), \quad t = (2N_q-2)\cdot(1, \mathbf{1});$

\[ M = (N_f + 1)N_f/2 \cdot (1, 1), \quad N = N_f(2N_q - 1) \cdot (1, 1) \]  

(33)

The dual theory has a superpotential given by

\[ \tilde{W} = \lambda_1MQsq + \lambda_2Nq\bar{q} + \lambda_3q'sq' + \lambda_4\bar{q}q't \]  

(34)

In order to obtain a quadratic matrix $B$ at the electric side we should consider a superpotential with two coupling constants, for example any of the following three

\[ W_1 = h_1S^4 + h_2S^2V^3 \]  
\[ W_2 = h_1S^2V^5 + h_2S^2V^3 \]  
\[ W_3 = h_1S^4 + h_2S^2V. \]  

(35) \hspace{1cm} (36) \hspace{1cm} (37)

According to the operator mapping of \cite{18} the corresponding dual operators are

\[ \tilde{W}_1 = \tilde{h}_1t^2 + \tilde{h}_2q^{N_f-3}q'^{2N_q-4} \]  
\[ \tilde{W}_2 = \tilde{h}_1q^{N_f-5}q'^{2N_q-2} + \tilde{h}_2q^{N_f-3}q'^{2N_q-4} \]  
\[ \tilde{W}_3 = \tilde{h}_1t^2 + \tilde{h}_2N. \]  

(38) \hspace{1cm} (39) \hspace{1cm} (40)

Now it is easy to calculate the determinants on both sides, which are determinants of 3 by 3 matrices in the electric theory, while on the magnetic side we deal with 8 by 8 matrices.\footnote{Note that we have included here the couplings $\lambda_1$ and $\lambda_2$ of the meson fields $M$ and $N$.}

According to the operator mapping of \cite{18} the corresponding dual operators are

\[ \det B_1 = 6(-24 + 2N_f + 3N_q) \quad \det \tilde{B}_1 = \frac{3}{16}(24 - 2N_f - 3N_q)\tilde{N}_c \]  
\[ \det B_2 = 6(8 - N_f) \quad \det \tilde{B}_2 = \frac{3}{16}(N_f - 8)\tilde{N}_c \]  
\[ \det B_3 = 6(8 - N_q) \quad \det \tilde{B}_3 = \frac{3}{16}(N_q - 8)\tilde{N}_c \]  

(41) \hspace{1cm} (42) \hspace{1cm} (43)

Note that the models are superconformal, but not completely finite for those values of $N_f$ and $N_q$ which lead to vanishing determinants $\det B_{1,2,3}$.\footnote{One can obtain a completely finite $SO(10)$ model with $N_f = N_q = 8$ adding the superpotential $W \sim SSV$ on the electric side. After some symmetry breaking also the magnetic dual of this model is supposed to be completely finite \cite{3}.}

Another interesting case to consider is the decoupling of massive fields. Consider an arbitrary gauge theory with 2 kinds of fields, which we call $Q$ and $A$. The superpotential is given by

\[ W = hQA^p + mQ^2. \]  

We now will show that the determinat is the same (up
to a factor of 2) whether we keep the massive quark and the mass term or integrate it out, as it should be. The massive quark cannot affect the IR behaviour. This is another generalization of the results of Leigh and Strassler, who already found that the existence of marginal operators (that is the zeros of the determinant) remain under integrating out massive fields.

Before integrating out the fields we have three couplings, the gauge coupling, $h$ and $m$. The $B$ matrix is hence given as

$$B = \begin{pmatrix} C - \mu_Q - \mu_A & \mu_Q & \mu_A \\ p - 2 & 1/2 & p/2 \\ -1 & 1 & 0 \end{pmatrix}$$

(44)

where $C$ is the contribution of the gauge fields to the 1-loop $\beta$ function and the $\mu_{Q,A}$ denote the indeces of the representations of $Q$ and $A$. With this one obtains

$$\det B = -\frac{p}{2}C + \frac{3}{2}\mu_Q(p-1).$$

(45)

After integrating out we are just left with the field $A$ and an effective superpotential $W = -\frac{h}{4m}A^{2p}$, yielding the reduced matrix

$$B = \begin{pmatrix} C - \mu_A & \mu_a \\ 2p - 3 & p \end{pmatrix}$$

(46)

and hence

$$\det B = -pC + 3\mu_Q(p-1).$$

(47)

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