I. INTRODUCTION

The hierarchical three-body problem, in which a binary is orbited by a distant third companion, has wide applications in astrophysics. Triple systems can explain phenomena over a wide range of scales from asteroids to supermassive black holes (SMBHs) [1–13]. A key characteristic of hierarchical triples is the exchange of angular momentum between the inner and outer orbits, which can lead to large inclination and eccentricity oscillations known in the literature as the Lidov-Kozai (LK) resonance [14, 15]. Dynamical models of triples undergoing LK resonant excitations have complemented observations and informed theories about the formation and evolution of these systems, especially in the context of exoplanets and compact object mergers [16–19].

Traditionally, the LK effect is calculated by expanding the orbit-averaged, three-body Newtonian equations of motion as a power series in \( \epsilon = a_1/a_2 \), where \( a_1 \) and \( a_2 \) are the semi-major axes of the inner and outer binary, respectively. Perturbations that accumulate over each orbit (unlike average-free periodic perturbations) are referred to as “secular” perturbations. The leading secular effect, the Newtonian-quadrupole or “quadrupole” for short, arises at order \( \epsilon^3 \) beyond Keplerian forces which scale as \( r^{-2} \). These quadrupole terms facilitate the exchange of orbital angular momentum which induce oscillations in the eccentricity and inclination.

Higher-order perturbations can change the nature of the LK effect. The addition of \( \epsilon^4 \) (octupole) order perturbations can cause orbital flips [20–22], extremely large eccentricities [23–25], and chaotic evolution [20–20]. These behaviors persist through \( \epsilon^5 \) (hexadecapole) order [27].

The implications of two-body relativistic effects in LK triples have been thoroughly studied. In a post-Newtonian expansion of the two-body equations of motion, the leading relativistic effect induces the precession of pericenter and appears at order \( \delta = v^2/c^2 \) (“1pN” order) beyond Keplerian forces, where \( v \) is the velocity of the binary. If the 1pN precession timescale of the inner binary is much shorter than the quadrupole timescale, eccentricity growth is heightened [23, 29]. Dissipative terms appearing at order \( \delta^{5/2} \) (“2.5pN” order) cause the orbit to shrink due to gravitational radiation. Eccentricity peaks induced by the LK effect can drastically increase the efficiency of gravitational radiation, driving the inner binary to merge much faster than if the binary were circular [30–32]. This has exciting implications for compact-object binaries with third companions as potentially eccentric gravitational-wave sources for LIGO and LISA [12, 33].

In comparison, little is known about relativistic three-body effects or how they may alter eccentricity growth in LK triples. Even though the three-body 1pN (3BpN) terms are required for a self-consistent secular evolution through 1pN-quadrupole order, they are generally not included in the majority of analyses of hierarchical triples. The 3BpN effects can be derived with a post-Keplerian, two-parameter expansion which we illustrate in Fig. 1. While the two-body 1pN (2BpN) precession effect appears at order \( \delta \), 3BpN effects appear at mixed orders \( \delta \epsilon^k \). These are referred to as the “interaction terms” by Ref. 29 or “cross terms” by Ref. 31.

One approach to study the 3BpN cross terms is to directly integrate the complete three-body pN equations, as done in Refs. 35–36 through 2.5pN order. While
these numerical solutions are exact, much work is required to gather physical insight. Approaches involving analytic expressions from perturbative calculations can play an important role in interpreting the output of N-body codes and understanding underlying physics. Such a synergy is common in the existing literature on secular effects in LK triples (e.g., the case of orbital flips in hot Jupiter systems [21, 29]). Furthermore, such integrations are typically far more time consuming than an integration of the secular equations, prohibiting a broad exploration of the parameter space.

To our knowledge, only four existing studies [29, 34, 37, 38] investigate 3BpN cross terms on the inner binary with an orbit-averaged, perturbative approach. Although results from these studies suggest that specific 3BpN terms can significantly affect the evolution of the inner binary, they either only consider a subset of the relevant 3BpN terms [29, 38] or derive them restricting the outer orbit to be constant [34, 37].

In this study, we derive the general case for arbitrary masses and orbital parameters, using a set of equations averaged over the inner orbit. We then focus on the specific case where the outer companion is much larger than the inner binary (e.g., a binary BH around a supermassive BH) and identify the specific cross terms that can influence the dynamical evolution of the inner binary. In much of the parameter space for secular hierarchical triples, 3BpN effects are subdominant to 2BpN effects and do not alter the evolution of the triple.

In certain regions of parameter space, the magnitude of 3BpN terms can approach that of 2BpN terms and substantially change the evolution of the inner binary. At the quadrupole level, the 3BpN effects coherently modulate the amplitude of LK oscillations which can lead to a greater range in eccentricity. In systems with initially moderate inclinations, the 3BpN terms can interact with the octupole terms and cause even larger eccentricity growth and significantly reduce merger times.

The outline of this paper is as follows: In Sec. [II] we review the existing literature on relativistic cross terms in hierarchical triples. In Sec. [III] we present a derivation starting with the Einstein-Infeld-Hoffman equations for three bodies. We then conduct a multiple-scale analysis of the Lagrange planetary equations to compute secular effects through 1pN-quadrupole order. The derived 3BpN cross terms are presented in App. [A]. We provide a Mathematica notebook upon request that contains a complete derivation. In Sec. [IV.A] we discuss general features of the 3BpN effects and estimate where in parameter space their effects may be important. In Sec. [IV.B] we present examples of systems where the LK resonance is significantly altered by 3BpN effects at the quadrupole-level of approximation. In Sec. [IV.C] we analyze a population of hierarchical triples including octupole terms and gravitational-wave emission to identify systematic 3BpN effects that impact a population of LK-driven mergers around a SMBH.

II. EXISTING STUDIES ON THIRD-BODY 1PN EFFECTS

Most investigations on relativistic triples take into account post-Newtonian corrections due to binary motion along with Newtonian third-body interactions. In comparison, little is known about three-body relativistic interactions in triples. To date, the authors are aware of four previous studies that consider these effects in hierarchical triples. We summarize these studies below and then comment on how their results motivate our current work.

1. Naoz et al. [29] derives 3BpN cross terms using the orbit-averaged three-body 1PN Hamiltonian, which is calculated by applying two successive canoni-
(2) Will [34] uses the Lagrange planetary equations to calculate post-Keplerian perturbations as we do here. Ref. [34] also discusses how lower-order periodic perturbations generate higher-order secular perturbations. However, this particular analysis does not systematically distinguish between secular and periodic variations, which complicates interpretation of the results [39]. The 3BpN cross terms are also derived by restricting the outer binary’s orbit to be constant, circular, and coplanar and only considering terms to leading order in the tertiary’s mass, $m_3$.

(3) Will [37] revisits the 3BpN cross terms in application to Mercury’s orbit around the Sun. This analysis uses a multiple-scale analysis to systematically account for periodic effects. Similar to Ref. [34], this analysis assumes the outer orbit is constant, circular, and coplanar, and considers effects up to linear order in $m_3$. With these assumptions, the 3BpN terms induce a precession [cf. their Eq. (1)]

$$\Delta \bar{\omega} = \frac{4\pi Gm_3 a^{3/2}}{c^2 R^3} + \frac{3\pi Gm_3 a^2}{4c^2 R^3} \left(28 + 47\varepsilon_1^2\right),$$

where $\bar{\omega}$ is pericenter angle measured from a reference direction, $R$ is the circular radius of the outer tertiary, $a$ is the Mercury-Sun semi-major axis, and $m_3$ is the mass of the tertiary planet. In this paper we will investigate a different limit where the tertiary is more massive than the inner binary.

(4) Liu et al. [38] considers additional relativistic interactions between the spins and orbital angular momenta in triple systems containing a supermassive black hole (SMBH) with masses $m_1 = 30$, $m_2 = 20M_\odot$, and $m_3 \gtrsim 10^8 - 10^9M_\odot$. For the inner and outer orbit they include the 1.5pN spin-orbit (Lense-Thirring) precessions. For point-particle effects, they include (without derivation) the cross-term de Sitter precession of the inner orbital plane. They write the frequency for this cross-precession effect as

$$\Omega_{\text{int-rot}} = \frac{3}{2} \frac{G^{3/2}m_3(4m + 3m_3)}{c^2 \sqrt{Ma_2^{5/2} \left(1 - \frac{c^2}{c_2^2}\right)}}.$$  

While we do find agreement with their formula, we will show that this term is only one of many that falls naturally out of the multiple-scale approach employed here.

Current discrepancies in the literature over the secular 3BpN cross terms exist (e.g. between Refs. [29, 34]), in part, due to differences in how lower-order periodic perturbations are considered in generating higher-order secular perturbations. Therefore, our first aim is to outline a clear procedure that systematically accounts for periodic effects for general hierarchical triple configurations.

### III. Calculating 3BpN Cross Terms

#### A. 1pN Equations of Motion

The Einstein-Infeld-Hoffman (EIH) equations describe the post-Newtonian gravitational dynamics of a system of point-like masses. The equations are expressed in terms of coordinate positions $\mathbf{r}_i = r_i \mathbf{n}_i$ and velocities $v_i$, where $i$ labels each mass. For a system of point-like masses the accelerations are given by

$$\frac{d^2 \mathbf{r}_i}{dt^2} = -\sum_{j \neq i} \frac{Gm_i m_j}{r_{ij}^3} \mathbf{n}_{ij} + \frac{1}{c^2} \left\{ \sum_{j \neq i} \frac{Gm_j}{r_{ij}^5} \times \right.$$

$$\left[ 4 \frac{Gm_i}{r_{ij}^3} \mathbf{r}_{ij} + 5 \sum_{k \neq i,j} \left( \frac{Gm_k}{r_{jk}^3} + 4 \frac{Gm_k}{r_{ik}^3} \right) \mathbf{r}_{jk} \right] - \mathbf{v}_i^2 + 4 \mathbf{v}_i \cdot \mathbf{v}_j - 2 \mathbf{v}_j^2$$

$$+ \frac{3}{2} \left( \mathbf{v}_j \cdot \mathbf{n}_{ij} \right)^2 \right\} - \frac{7}{2} \sum_{j \neq i} \frac{Gm_i}{r_{ij}^3} \sum_{k \neq i,j} \frac{Gm_k}{r_{jk}^3} \mathbf{r}_{jk}$$

$$+ \sum_{j \neq i} \frac{Gm_j}{r_{ij}^3} \mathbf{n}_{ij} \cdot (4\mathbf{v}_i - 3\mathbf{v}_j)(\mathbf{v}_i - \mathbf{v}_j),$$

where $\mathbf{n}_{ij} = \mathbf{n}_i - \mathbf{n}_j$ and $\mathbf{r}_i - \mathbf{r}_j = r_{ij} \mathbf{n}_{ij}$.

In a hierarchical triple, two bodies of mass $m_1$ and $m_2$ constitute an “inner” orbit with separation $r \equiv r_{12}$ and center of mass $r_0$. A tertiary body of mass $m_3$ follows an “outer” orbit about the inner orbit’s center of mass with separation $R \equiv r_3 - r_0$, where $|R| \gg |r|$. For the inner and outer orbits, we define the velocities as $\mathbf{v} \equiv d\mathbf{r}/dt$, $\mathbf{V} \equiv d\mathbf{R}/dt$ and the separation unit vectors as $\mathbf{n} \equiv \mathbf{r}/r$, $\mathbf{N} \equiv \mathbf{R}/R$. In the center of mass frame,

$$\sum_i m_i \mathbf{r}_i = m \mathbf{r}_0 + m_3 \mathbf{r}_3 = \mathcal{O}(c^{-2}),$$

which leads to

$$r_1 = \frac{m_2}{m} r - \frac{m_3}{M} R,$$

$$r_2 = -\frac{m_1}{m} r - \frac{m_3}{M} R,$$

$$r_3 = \frac{m_3}{M} R.$$
where \( m = m_1 + m_2 \) is the total mass of the inner binary and \( M = m + m_3 \) is the total mass of the triple. Post-Newtonian corrections to the center of mass frame are not relevant at 1pN order since only differences of position vectors appear, and also velocities only appear in terms that are already 1pN order \([40]\). In this frame, the 1pN acceleration of the inner orbit’s center of mass \( r_0 \) will also affect \( \mathbf{R} \).

The EIH equations can be rewritten by grouping all post-Keplerian accelerations on the right-hand side,

\[
\frac{d^2 \mathbf{R}}{dt^2} + \frac{GM}{R^2} \mathbf{n} = \mathbf{A},
\]

\[
\frac{d^2 \mathbf{r}}{dt^2} + \frac{Gm}{r^2} \mathbf{n} = \mathbf{a},
\]

where \( \mathbf{a} \) and \( \mathbf{A} \) contain both relativistic and third-body terms. In the absence of post-Keplerian accelerations \( (\mathbf{a}, \mathbf{A} = 0) \), Eqs. (3.4) - (3.5) take on their homogeneous forms leading to Keplerian motion for each orbit.

The post-Keplerian accelerations \( \mathbf{a} \) and \( \mathbf{A} \) contain terms that depend on powers of \( r_{13}, r_{23} \), which can be expanded as a power series in \( \epsilon = a_1/a_2 \) (equivalent to expanding in \( r/R \)). The leading interactions between the inner and outer orbit occur at quadrupole order.

In the limit that \( m \ll M \), perturbations on the outer orbit due to the inner binary are small. Thus, for the outer binary, we only consider Newtonian three-body effects and the 2BpN term for the outer orbit. At quadrupole order, this reads

\[
\mathbf{A} = \mathbf{A}_{1pN} + \mathbf{A}_{\text{quad}}.
\]

In contrast, perturbations on the inner binary due to the SMBH can be significant (e.g. see Ref. \([38]\)) so we include include the 3BpN accelerations up to 1pN-quadrupole order:

\[
\mathbf{a} = \mathbf{a}_{1pN} + \mathbf{a}_{\text{quad}} + \mathbf{a}_{3BpN}.
\]

The quadrupole accelerations scale relative to the Keplerian accelerations as

\[
\mathbf{a}_{\text{quad}} \sim \left( \frac{Gm}{r^2} \right) \times \epsilon^3 \left( \frac{m_3}{m} \right),
\]

\[
\mathbf{A}_{\text{quad}} \sim \left( \frac{GM}{R^2} \right) \times \epsilon^2,
\]

whereas the two-body 1pN accelerations scale as

\[
\mathbf{a}_{1pN} \sim \left( \frac{Gm}{r^2} \right) \times \delta,
\]

\[
\mathbf{A}_{1pN} \sim \left( \frac{GM}{R^2} \right) \times \delta_2,
\]

where \( \delta = v^2/c^2 \sim (Gm/r)/c^2 \) is a parameter characterizing pN perturbations on the inner binary and \( \delta_2 = V^2/c^2 \sim (GM/R)/c^2 \) is a parameter characterizing pN perturbations on the outer binary.

3BpN effects can arise directly from the equations of motion through \( \mathbf{a}_{3BpN} \) or indirectly through the interaction of lower-order effects from \( \mathbf{a}_{\text{quad}}, \mathbf{a}_{1pN}, \mathbf{A}_{\text{quad}}, \) and \( \mathbf{A}_{1pN} \). Due to the coupling between the inner and outer orbit, pN perturbations from both orbits can lead to cross terms. We express all pN corrections in terms of \( \delta \), using

\[
\delta_2 = \delta_\epsilon \left( \frac{M}{m} \right),
\]

Cross terms indirectly generated through the interaction of effects from \( \mathbf{a}_{1pN} \) and \( \mathbf{a}_{\text{quad}} \) are of order \( \delta_\epsilon^3 \sim \delta^4 \). Therefore, we must expand \( \mathbf{a}_{3BpN} \) up to comparable order \( \delta_\epsilon^4 \):

\[
\mathbf{a}_{3BpN} \sim \left( \frac{Gm}{r^2} \right) \times \delta_\epsilon^k \left( \frac{M}{m} \right)^\ell,
\]

where the powers of non-zero terms include

\[
(k, \ell) \in \left\{ (4, 2), (4, 1), (4, 0), (4, -1), \right. \\
(7, 3), (7, 2), (7, 1), (7, 0), (7, -1), (5, 3), (5, 2), (5, 1), (5, 0), (2, 1), (2, 0), (1, 1), (1, 0), (1, -1), \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \}.
\]

Only the \( k \geq 5/2 \) terms generate non-zero secular effects at 1pN-quadrupole order. We verify that our expression for \( \mathbf{a}_{3BpN} \) agrees with Ref. \([34]\) [cf. Eq. (4.7b)] when \( m_3 \ll m \).

**B. Lagrange Planetary Equations**

Equations (3.4) and (3.5) constitute a second order differential equation for the positions and velocities of the two orbits. It is possible to rewrite this as a first order differential equation for the time-dependent osculating orbital elements \( \{p_i, e_i, i, \omega_i, \Omega_i\} \) (e.g. see Ref. \([41]\) ), where \( i = 1, 2 \) labels the inner and outer orbit, respectively. The positions and velocities of each orbit are defined in terms of the orbital elements as

\[
r = p_1 n/[1 + e_1 \cos(f)],
\]

\[
v = \sqrt{\frac{Gm}{p_1}} \left\{ e_1 \sin(f) n + [1 + e_1 \cos(f)] \lambda \right\},
\]

\[
R = p_2 N/[1 + e_2 \cos(F)],
\]

\[
V = \sqrt{\frac{GM}{p_2}} \left\{ e_2 \sin(F) N + [1 + e_2 \cos(F)] \Lambda \right\},
\]

where the bases \( (n, \lambda, h) \) and \( (N, \Lambda, H) \) of the inner and outer orbits, respectively, can be defined with respect to
a reference basis \((e_X, e_Y, e_Z)\) as
\[
\begin{align*}
n &= \left[ \cos \Omega_1 \cos(\omega_1 + f) - \cos \iota_1 \sin \Omega_1 \sin(\omega_1 + f) \right] e_X \\
&\quad + \left[ \sin \Omega_1 \cos(\omega_1 + f) + \cos \iota_1 \cos \Omega_1 \sin(\omega_1 + f) \right] e_Y \\
&\quad + \sin \iota_1 \sin(\omega_1 + f) e_Z, \\
\lambda &= d\iota_1/df, \\
h &= n \times \lambda, \\
N &= \left[ \cos \Omega_2 \cos(\omega_2 + F) - \cos \iota_2 \sin \Omega_2 \sin(\omega_2 + F) \right] e_X \\
&\quad + \left[ \sin \Omega_2 \cos(\omega_2 + F) + \cos \iota_2 \cos \Omega_2 \sin(\omega_2 + F) \right] e_Y \\
&\quad + \sin \iota_2 \sin(\omega_2 + F) e_Z, \\
\Lambda &= dN/dF, \\
H &= N \times \Lambda.
\end{align*}
\]

The basis vector \(e_Z\) is conventionally chosen to align with the total angular momentum of the triple. The true anomalies \(f\) and \(F\) of the inner and outer orbits, respectively, track the phase of each orbit. \(\omega_1\) is the argument of pericenter, and \(\Omega_1\) is the longitude of ascending node.

The dynamical equations recast in terms of the above osculating orbital elements are referred to as the Lagrange planetary equations. For the inner binary, the planetary equations read
\[
\begin{align*}
\frac{dp_1}{dt} &= 2\sqrt{\frac{p_1}{Gm}} rS, \\
\frac{de_1}{dt} &= \sqrt{\frac{p_1}{Gm}} \left[ \sin(f)R + \frac{2\cos(f) + e_1 + e_1 \cos^2(f)}{1 + e_1 \cos(f)} S \right], \\
\frac{dw_1}{dt} &= e_1 \sqrt{\frac{p_1}{Gm}} \left[ -\cos(f)R + \frac{2 + e_1 \cos(f)}{1 + e_1 \cos(f)} S \right] \\
&\quad - e_1 \cot \iota_1 \frac{\cos(\omega_1 + f)}{1 + e_1 \cos(f)} W, \\
\frac{df}{dt} &= \sqrt{\frac{Gmp_1}{r^2}} - \frac{d\omega_1}{dt} - \frac{d\Omega_1}{dt} \cos \iota_1, \\
\end{align*}
\]
where \((R, S, W) = (a, n, \lambda, a, h)\) are the vector components of the perturbation \(a\) projected onto the inner orbit’s basis.

The planetary equations are supplemented by an additional sixth equation that converts between the true anomaly \(f\) and time,
\[
\frac{df}{dt} = \sqrt{\frac{Gmp_1}{r^2}} - \frac{d\omega_1}{dt} - \frac{d\Omega_1}{dt} \cos \iota_1,
\]
where the first term is the usual Keplerian expression and \(-\dot{\omega}_1 - \Omega_1 \cos \iota_1\) is a post-Keplerian correction.

The equations for the outer orbit are the same as Eqs. (3.17) and (3.18), but with the substitutions \(m \rightarrow M, \{f, p_1, e_1, \iota_1, \omega_1, \Omega_1\} \rightarrow \{F, p_2, e_2, \iota_2, \omega_2, \Omega_2\}, \) and \((R, S, W) \rightarrow (R_3, S_3, W_3)\), where \((R_3, S_3, W_3) = (A \cdot N, A \cdot \Lambda, A \cdot H)\) are the vector components of the perturbation \(a\) as projected onto the outer orbit’s basis. Eqs. (3.17) and (3.18) along with the outer orbit’s counterpart equations are exact reformulations of Eqs. (3.4) and (3.5).

From the planetary equations, one can see that the inner binary perturbations that scale as
\[
\alpha \sim \frac{Gm}{r^2} e^{k\delta t},
\]
generate orbital perturbations that scale as
\[
\frac{dc_1}{dt} \sim \frac{1}{P_{\text{in}}^K} e^{k\delta t},
\]
where \(P_{\text{in}}^K\) is the Keplerian expression for the orbital period. Similarly, outer binary perturbations that scale as
\[
A \sim \frac{GM}{R^2} e^{k\delta t},
\]
generate orbital perturbations that scale as
\[
\frac{dc_2}{dt} \sim \frac{1}{P_{\text{out}}^K} e^{k\delta t} = \frac{1}{P_{\text{in}}^K} \left( \frac{M}{m} \right)^{1/2} e^{k+3/2\delta t},
\]
where \(P_{\text{out}}^K\) is the Keplerian expression for the orbital period.

The leading-order secular perturbations on the orbital elements are calculated by taking the orbit average of the planetary equations,
\[
\left\langle \frac{dX_\alpha}{dt} \right\rangle = \lim_{T \to \infty} \frac{1}{T} \int_0^T dX_\alpha dt.
\]

We use \(\alpha = 1, 2, \ldots, 10\) to label the orbital elements. We reserve the first five indices (1 \(\leq \alpha \leq 5\)) for the inner orbit’s elements and the last five (6 \(\leq \alpha \leq 10\)) for the outer’s. In the literature (e.g. Ref. [27]), this integral is evaluated by using the double-orbit average approximation, which uses the fact that each term on the right-hand side of Eq. (3.17) can be rewritten as a sum of products whose factors depend periodically on either \(f\) or \(F\) in addition to the orbital elements \(X_\beta\):
\[
\frac{dX_\alpha}{dt} = \sum_i A_i(X_\beta, f) B_i(X_\beta, F).
\]

With this factorization, the average can be approximated assuming \(P_{\text{in}} \ll P_{\text{out}}\). One first averages over the inner orbit and then subsequently averages over the outer orbit while holding \(X_\beta\) fixed,
\[
\left\langle \frac{dX_\alpha}{dt} \right\rangle \approx \sum_i \frac{1}{P_{\text{in}}} \int_0^{P_{\text{in}}} A_i dt \times \frac{1}{P_{\text{out}}} \int_0^{P_{\text{out}}} B_i dt
\]
\[
= \frac{1}{P_{\text{in}} P_{\text{out}}} \sum_i \int_0^{2\pi} A_i \left( \frac{dt}{df} \right) df \int_0^{2\pi} B_i \left( \frac{dt}{dF} \right) dF.
\]
The post-Keplerian corrections to \((df/dt), (dF/dt)\), \(P_{\alpha}\) and \(P_{\text{out}}\) appearing in Eq. (3.18) generate cross-term order effects and do not play a role in a leading-order analysis.

Before evaluating Eq. (3.25), the factors \(A_1\) and \(B_1\) can be simplified. By substituting Eqs. (3.15) and (3.16) into Eq. (3.17), one can verify that \(A_1\) and \(B_1\) only depend on the ascending nodes \(\Omega_1\) only through powers of \(\cos(\Delta \Omega)\) and \(\sin(\Delta \Omega)\), where \(\Delta \Omega = \Omega_1 - \Omega_2\). The equations greatly simplify by setting \(\Delta \Omega = \pi\). The justification comes in two parts. First, one initially aligns the reference direction \(e_\perp\) with the total orbital angular momentum so that \(\Delta \Omega = \pi\). Also with this choice, Newtonian and 2BpN perturbations lead to \(\Omega_1 = \Omega_2\) at all subsequent times. This is simplification is different from eliminating the nodes in the Hamiltonian, which can lead to the incorrect equations of motion as discussed in Ref. [24]. The simplification we describe here is applied directly to the equations of motion. We adopt the node-eliminated simplified set of equations, but note that the cross term perturbations in general lead to \(\Omega_1 \neq \Omega_2\). However, our quadrupole-order evolutions result in \(\Delta \Omega \approx \pi\) within 10%, which provides a rough consistency check. Including corrections that depend on \(\Delta \Omega\) is left to future work.

In a leading-order analysis, the secular evolution is determined solely from orbit-averaged effects — one neglects any average-free, periodic variations. The only integrals involved are orbit averages [Eq. (3.25)] where the limits of integration span one period. Beyond leading order, periodic variations will also contribute to the secular dynamics. To calculate these periodic variations, one must solve for the instantaneous values of the elements and integrate the planetary equations with respect to an orbital phase. A few choices for the orbital phase include the true, eccentric, and mean anomalies. We use a placeholder \(\phi\) to represent whatever angle is used to re-parametrize the planetary equations, which read,

\[
Q_\alpha(X_\beta, F(\phi), f(\phi)) = \frac{dX_\alpha}{d\phi} = dX_\alpha = dX_\alpha \frac{dt}{d\phi},
\]

The planetary equations for the inner binary, \(\dot{X}_\alpha = dX_\alpha/dt\) [Eq. (3.17)], can be organized as

\[
\dot{X}_\alpha = (\dot{X}_\alpha)_{1\text{PN}} + (\dot{X}_\alpha)_{\text{quad}} + (\dot{X}_\alpha)_{3\text{BpN}},
\]

where each term on the right-hand side is due plugging in \(a_{1\text{PN}}, a_{\text{quad}}, a_{3\text{BpN}}\) into Eq. (3.17), respectively. Due to the scaling with \(\delta\) and \(\epsilon\) for each of these accelerations [Eqs. (3.8), (3.10), and (3.13)], the inner binary terms scale as

\[
\begin{align*}
(\dot{X}_\alpha)_{1\text{PN}} &\sim \frac{\delta}{P_{\text{out}}^5}, \\
(\dot{X}_\alpha)_{\text{quad}} &\sim \frac{\epsilon^2}{P_{\text{in}}^5}, \\
(\dot{X}_\alpha)_{3\text{BpN}} &\sim \frac{\delta \epsilon^k}{P_{\text{in}}^5}.
\end{align*}
\]

Note that for the outer binary [Eqs. (3.9) and (3.11)], the terms scale as

\[
\begin{align*}
(\dot{X}_\alpha)_{1\text{PN}} &\sim \frac{\delta}{P_{\text{out}}^5}, \\
(\dot{X}_\alpha)_{\text{quad}} &\sim \frac{\epsilon^2}{P_{\text{in}}^5},
\end{align*}
\]

We also include cross terms from leading-order 1PN and quadrupole corrections to \(d\phi/dt\),

\[
\frac{dt}{d\phi} = \left(\frac{dt}{d\phi}\right)_K + \left(\frac{dt}{d\phi}\right)_{1\text{PN}} + \left(\frac{dt}{d\phi}\right)_{\text{quad}},
\]

where \((dt/d\phi)_K\) is the Keplerian expression. Combining Eqs. (3.27) and (3.30), we can write the re-parametrized equations \(Q_\alpha\) up to 1PN-quadrupole order as

\[
Q_\alpha = (\dot{X}_\alpha)_{1\text{PN}} \left[\left(\frac{dt}{d\phi}\right)_K + \left(\frac{dt}{d\phi}\right)_{\text{quad}}\right]
\]

\[
+ (\dot{X}_\alpha)_{\text{quad}} \left[\left(\frac{dt}{d\phi}\right)_K + \left(\frac{dt}{d\phi}\right)_{1\text{PN}}\right]
\]

\[
+ (\dot{X}_\alpha)_{3\text{BpN}} \left(\frac{dt}{d\phi}\right)_K,
\]

where the cross terms include

\[
\begin{align*}
(\dot{X}_\alpha)_{1\text{PN}} \left(\frac{dt}{d\phi}\right)_{\text{quad}}, \\
(\dot{X}_\alpha)_{\text{quad}} \left(\frac{dt}{d\phi}\right)_{1\text{PN}}, \\
(\dot{X}_\alpha)_{3\text{BpN}} \left(\frac{dt}{d\phi}\right)_K,
\end{align*}
\]

In addition to cross terms from Eq. (3.32), additional 3BpN cross terms arise from lower-order periodic variations and corrections to \(P_{\alpha}\) and \(P_{\text{out}}\). These additional cross terms can be calculated through a multiple-scale analysis described in Sec. IIIIC.

C. Multiple-scale analysis

The method of multiple scales provides a clear procedure for how to systematically calculate higher-order secular effects due to lower-order periodic effects. We refer the reader to Ref. [42] for a review of the method of multiple scales and Refs. [24, 33, 24] for applications in a post-Keplerian, two-body context. The multiple-scale method has also been applied to post-adiabatic calculations in extreme-mass-ratio inspirals around Kerr black holes [45, 46].

In a multiple-scale analysis of the planetary equations with two bodies, one introduces an additional long-timescale variable, \(\theta \equiv \epsilon \phi\), to artificially separate the secular and average-free parts of the orbital elements with the ansatz \(X_\alpha = X_\alpha(\theta) + \epsilon W_\alpha(\theta, \phi)\), where \(X_\alpha\) is the slowly evolving secular part and \(W_\alpha\) is the average-free
periodic part. \( W_\alpha \) itself is expanded in a power series, 
\( W_\alpha = W_\alpha^{(0)} + \epsilon W_\alpha^{(1)} + \ldots \), which can then be used to solve for \( \tilde{X}_\alpha \) to the desired order.

To calculate cross terms in a three-body context, one must consider perturbations by both relativistic effects and orbital interaction effects. Thus, we introduce two long-timescale variables \( \theta \equiv \epsilon \phi \) and \( \tau \equiv \delta \phi \) such that
\[
\frac{d}{d\phi} = \frac{\partial}{\partial \phi} + \epsilon \frac{\partial}{\partial \theta} + \delta \frac{\partial}{\partial \tau}.
\] (3.33)

The slow changing variables \( \theta \) and \( \tau \) resolve changes occurring over a quadrupole timescale and pN pericenter precession timescale, respectively. The fast changing variable \( \phi \) describes changes occurring over an orbital period. Practical considerations which inform our choice of \( \phi \) are discussed in Sec. IIID.

We introduce an ansatz to Eq. (3.26) which reads
\[
X_\alpha (\tilde{X}_\beta, \theta, \tau) = \tilde{X}_\alpha (\theta, \tau) + W_\alpha (\tilde{X}_\beta, \theta, \tau),
\] (3.34)
where \( \tilde{X}_\beta \) is the average (secular) part of \( X_\alpha \) and \( W_\alpha \) is the average-free part of \( X_\alpha \), defined as
\[
\langle A \rangle_\phi \equiv \frac{1}{2\pi} \int_0^{2\pi} A(\theta, \tau, \phi) \, d\phi,
\] (3.35)
\[
\mathcal{A}F(A) \equiv A(\theta, \tau, \phi) - \langle A \rangle_\phi,
\] (3.36)
with \( \theta \) and \( \tau \) held fixed in the integral.

We expand the average-free part
\[
W_\alpha (\tilde{X}_\beta, \phi) = \sum_{\ell, m=0}^{\infty} \epsilon^\ell \delta^m W_\alpha^{\ell m} (\tilde{X}_\beta, \phi),
\] (3.37)
where \( \langle W_\alpha^{\ell m} \rangle_\phi = 0 \). Note that \( W_\alpha^{00} \) is chosen to enforce constant orbital elements at zeroth order. We substitute the ansatz [Eq. (3.34)] back into the planetary equations [Eq. (3.26)] and separate the average part,
\[
\frac{d\tilde{X}_\alpha}{d\phi} = \langle Q_\alpha \rangle_\phi,
\] (3.38)
from the average-free part,
\[
\sum_{\ell, m=0}^{\infty} \epsilon^\ell \delta^m \frac{\partial W_\alpha^{\ell m}}{\partial \phi} + \epsilon^{\ell+1} \delta^m \frac{\partial W_\alpha^{\ell m}}{\partial \theta} + \epsilon^\ell \delta^{m+1} \frac{\partial W_\alpha^{\ell m}}{\partial \tau} = \mathcal{A}F(Q_\alpha),
\] (3.39)
where the perturbations \( Q_\alpha \) are written in Eq. (3.31) and we use
\[
\frac{d\tilde{X}_\alpha}{d\phi} = \epsilon \frac{\partial X_\alpha}{\partial \theta} + \delta \frac{\partial X_\alpha}{\partial \tau}.
\] (3.40)
We also expand
\[
Q_\alpha (\tilde{X}_\beta + W_\beta, \phi) \equiv \sum_{n=0}^\infty \frac{1}{n!} \frac{\partial^n Q_\alpha^{(0)}}{\partial \tilde{X}_\beta \ldots \partial X_\gamma} W_\beta \ldots W_\gamma,
\] (3.41)
where the periodic parts \( W_\alpha \) are written in Eq. (3.37), repeated indices are summed over all 10 elements, and
\[
Q^{(0)} = Q(\tilde{X}_\beta, \phi).
\] (3.42)

The periodic parts \( W_\alpha \) combine with perturbations \( Q_\alpha \) according to Eq. (3.41) and generate higher-order cross terms.

Written above, Eqs. (3.38), (3.39), and (3.41) are the central equations which can be iteratively solved to obtain the secular evolution in terms of \( \phi \) to desired order in \( \delta^n \epsilon^m \). To calculate the secular time evolution, one must use the conversion
\[
\frac{d\tilde{X}_\alpha}{dt} = \frac{d\tilde{X}_\alpha}{d\phi} \left( \frac{d\phi}{dt} \right)_\phi, \tag{3.43}
\]
where the conversion factor \( (d\phi/dt)_\phi \) also generates cross-term secular effects.

### D. Discussion on orbit averages

Our discussion above is general as we did not specify the short-timescale variable \( \phi \). To solve for the cross-term contributions in Eq. (3.38) we must choose what phase-like variable to use.

In principle, \( \phi \) can be any phase-like variable characterizing the inner or outer orbits. In practice, it is difficult to explicitly write both \( F \) and \( f \) in terms of a single variable \( \phi \). To address these difficulties, we choose \( \phi = F \) and average the perturbations \( Q_\alpha \) over the inner orbit, using the assumption \( P_{in} \ll P_{out} \). This expresses the equations of motion in terms of \( F \) only:
\[
Q_\alpha (X_\beta, f, F) = \frac{dX_\alpha}{dt} \frac{df}{df},
\] (3.44)
where the inner-orbit average is
\[
\langle A \rangle_{in} = \frac{1}{P_{in}} \int_0^{2\pi} A(X_\beta, f, F) \, df,
\] (3.45)
with the inner period is defined as
\[
P_{in} = \int_0^{2\pi} \frac{df}{df}, \tag{3.46}
\]
holding \( F \) fixed. The orbit-average defined in Eq. (3.35) when evaluated with Eq. (3.44) is also consistent with the usual double-orbit average procedure encountered in the literature.

In the inner-orbit average [Eq. (3.45)], we include post-Keplerian corrections to \( (dt/df) \) which lead to the additional cross terms:
\[
\tilde{X}_\alpha^{(1pN)} \quad \frac{dt}{df}, \tag{3.47a}
\]
\[
\tilde{X}_\alpha^{quad} \quad \frac{dt}{df}, \tag{3.47b}
\]
Cross terms also result from leading-order corrections to the orbital period $P_{\text{in}}$ [Eq. (3.45)],

$$P_{\text{in}}^{1\text{pN}} = \int_0^{2\pi} \left( \frac{dt}{df} \right)_{1\text{pN}} df$$ (3.48a)

$$P_{\text{in}}^{\text{quad}} = \int_0^{2\pi} \left( \frac{dt}{df} \right)_{\text{quad}} df,$$ (3.48b)

which are order $\epsilon^3$ and $\delta$ beyond the Keplerian expression $P_{\text{in}}^K$. Collecting the post-Keplerian corrections, the perturbations $Q_\alpha$ can be written up to 1pN-quadrupole order as

$$Q_\alpha(X_\beta, F) = (Q_\alpha)_{1\text{pN}} + (Q_\alpha)_{\text{quad}} + (Q_\alpha)_{3\text{BpN}},$$ (3.49)

where

$$(Q_\alpha)_{\text{quad}} = \frac{1}{P_{\text{in}}^{\text{quad}}} \int_0^{2\pi} \left( \frac{dX_\alpha}{dF} \right)_{\text{quad}} \left( \frac{dF}{dF} \right)_{\text{quad}} df,$$ (3.50)

$$(Q_\alpha)_{1\text{pN}} = \frac{1}{P_{\text{in}}^{\text{quad}}} \int_0^{2\pi} \left( \frac{dX_\alpha}{dF} \right)_{1\text{pN}} \left( \frac{dF}{dF} \right)_{1\text{pN}} df,$$ (3.51)

binary. This leads to the expansion

$$(Q_\alpha)_{1\text{pN}} = (Q_\alpha)_{1\text{pN}}^{(0)} + \sum_{\beta=1}^5 \frac{\partial (Q_\alpha)^{(0)}_{1\text{pN}}}{\partial X_\beta} W_{30}^{\beta}$$

$$+ \sum_{\beta=6}^{10} \frac{\partial (Q_\alpha)^{(0)}_{1\text{pN}}}{\partial X_\beta} W_{70}^{\beta},$$ (3.52)

$$(Q_\alpha)_{\text{quad}} = (Q_\alpha)_{\text{quad}}^{(0)} + \sum_{\beta=1}^5 \frac{\partial (Q_\alpha)_{\text{quad}}^{(0)}}{\partial X_\beta} W_{01}^{\beta}$$

$$+ \sum_{\beta=6}^{10} \frac{\partial (Q_\alpha)_{\text{quad}}^{(0)}}{\partial X_\beta} W_{11}^{\beta},$$ (3.53)

where the lowest-order periodic parts are

$$W_{30}^{\beta} = \int_0^F A\mathcal{F}\left((Q_\alpha)^{(0)}_{\text{quad}}\right) dF' + C,$$ (3.54)

$$W_{01}^{\alpha} = \int_0^F A\mathcal{F}\left((Q_\alpha)^{(0)}_{1\text{pN}}\right) dF' + D, \quad 1 \leq \alpha \leq 5,$$ (3.55)

for the inner binary, and

$$W_{70}^{\beta} = \int_0^F A\mathcal{F}\left((Q_\alpha)^{(0)}_{\text{quad}}\right) dF' + E,$$ (3.56)

$$W_{11}^{\alpha} = \int_0^F A\mathcal{F}\left((Q_\alpha)^{(0)}_{1\text{pN}}\right) dF' + H, \quad 6 \leq \alpha \leq 10,$$ (3.57)

for the outer binary. The integration constants $C, D, E, H$ are determined by $\langle W^{(0)}_{\alpha}\rangle_F = 0$ and are identical to those in Eq. (B11) in Ref. [46].

The total 3BpN secular contribution is
We take into account leading-order corrections to the orbit average approximation \[\text{Eq. (3.45)}\] as 
\[\text{P}^{\text{corrections}}\] will also generate cross terms due to leading-order corrections to \(P_{\text{out}}\), which can be calculated with the single-orbit average approximation \[\text{Eq. (3.45)}\] as 
\[P_{\text{out}} \approx 2\pi \left\langle \frac{d\alpha}{dF} \right\rangle_F = 2\pi \left\langle \frac{dt}{dF} \right\rangle_F \approx \int_0^{2\pi} \frac{dt}{dF} \, dF.\]  
(3.62)

We take into account leading-order corrections to \(P_{\text{out}}\) from the standard corrections to \((dt/dF)\) \[\text{Eq. (3.18)}\] and also periodic variations in \((dt/dF)\).

IV. EFFECTS OF THIRD-BODY 1PN CROSS TERMS DUE TO A SMBH

A. Dominant cross terms around a SMBH

For completeness, we keep cross terms of all powers in \((m/M)\) in the derivation in Sec. II, but we work on the assumption that the inner binary’s mass is small relative to the total mass \[\text{Eqs. (3.6) and (3.7)}\]. In this Section, we closely examine the dominant cross term effects when \(m \ll M\) and locate regions in parameter space where their effects become significant in triples undergoing strong LK oscillations.

We are in particular interested in how the dominant three-body 1PN (3BpN) cross terms interact with other secular effects, including the two-body 1PN (2BpN), quadrupole, and octupole terms. The conventional picture is that 2BpN pericenter precession will quench eccentricity growth if the timescale for precession is much shorter than that of quadrupole (LK) effects \[33, 49\].

However, in some cases, 2BpN effects can instead stimulate eccentricity growth. Refs. \[23\] and \[29\] demonstrate heightened resonant-like eccentricity excitation if the 1PN precession timescale is comparable to the Newtonian (quadrupole and octupole) timescales. Given this resonant-like behavior between 2BpN and Newtonian terms, it may be unsurprising if the 3BpN cross terms also lead to resonant-like behavior when their effective timescale approaches that of 2BpN or Newtonian effects.

The mixed-order \((\delta \epsilon^k)\) cross terms are higher order than the 2BpN \((\delta)\) terms. But as \(q = M/m\) increases, so does the relative strength of cross terms which scale with positive powers of \(q\). Therefore, we consider each cross term by their scaling with \(q = M/m\) and \(\epsilon = r/R\) relative to the the 2BpN pericenter precession term, which reads 
\[\left(\frac{d\omega_1}{dt}\right)_{1\text{pN}} = \frac{G^{3/2}m^{3/2}}{c^2a_1^{5/2}(1-e_1^2)} \sim \frac{\delta}{P_{\text{in}}}.\]  
(4.1)

Organized relative the 1PN precession rate \[\text{Eq. (4.1)}\], the contribution from all cross terms can be written as 
\[\left(\frac{dX_{\alpha}}{dt}\right)_{3\text{BpN}} = \frac{\delta}{P_{\text{in}}} \sum_{\ell,m} f_{\ell m} \epsilon^m = \frac{\delta}{P_{\text{in}}} \sum_{\ell,m} \tilde{f}_{\ell m} X_{\ell m},\]  
(4.2)

where the factor \(f_{\ell m}\) contains numerical factors of order unity and factors including \(e_j, \omega_j, \iota_j\). We will use the scaling magnitude \(X_{\ell m} = q^j \epsilon^m\) as an estimate for which cross terms will be dominant or subdominant given an initial set of triple parameters. Since the semi-latus rectum has dimensions of length, \(f_{\ell m}\) includes an additional factor of \(p_1\) so that a given perturbation leads to the same scaling factor \(X_{\ell m}\) across all elements. By construction, the 2BpN pericenter precession term \[\text{Eq. (4.1)}\] has a scaling magnitude of \(X_{00} = 1\).

We illustrate a comparison of cross-term scaling magnitudes by considering a generic hierarchical triple system with \(m_1 = m_2 = 25M_\odot, m_3 = 4 \times 10^6M_\odot\), and initial semi-major axes \(a_1 = 1\) AU and \(a_2 = 2000\) AU. With these initial masses and separations, one can estimate the relative contribution of each cross term on the system. For a wide portion of parameter space, the dominant 3BpN effect on the inner binary is the geodetic (de Sitter-like) precession of the inner orbit’s vectors (e.g. \(e_1, j_1\)) as they are parallel transported around the SMBH. This de Sitter cross term, which comes directly...
from the EIH equations (a_{3BPN}), induces the orbital element \( \dot{\omega}_1 \equiv \omega_1 + \Omega_1 \cos t_1 \) to precess at the rate

\[
\frac{d\omega_1}{dt}_{3BPN} = \frac{G^{3/2}(4m + 3m_3)m_3}{2M^{1/2}c^2a_1^{3/2}(1 - e_2^2)} \cos t, \tag{4.3}
\]

and has a scaling magnitude \( X_{31/2} = 0.13 \), where we can safely neglect smaller corrections proportional to \( X_{13/2} = 10^{-5}X_{31/2} \).

For the same initial parameters, the second dominant cross term effect perturbs the pericenter at a rate

\[
\frac{d\omega_1}{dt}_{3BPN} = \frac{15G^{3/2}m_3m}{4M^{1/2}a_1^{3/2}c^2} \frac{\ell_1^2(1 + \ell_2 - 2\ell_2^2)}{\ell_1^2(1 + \ell_2)} \times \left( \cos t \cos 2\omega_1 \cos 2\omega_2 + \frac{1 + \cos^2 t}{2} \sin 2\omega_1 \sin 2\omega_2 \right), \tag{4.4}
\]

where \( \ell_1 = \sqrt{1 - e_1^2} \) and \( \ell_2 = \sqrt{1 - e_2^2} \), and has a scaling magnitude \( X_{31/2} = 3 \times 10^{-3} \). In isolation, the cross term effect in Eq. (4.4) will lead to bounded oscillations in \( \omega_1 \). For this reason we refer to Eq. (4.4) as the “libration” cross term. It is derived by considering the leading three-body correction to the usual 1pN pericenter precession on \( \omega_1 \), due to periodic (quadrupole-induced) perturbations on the outer binary. Note that if the 1pN-binary precession [Eq. (4.1)] is dominant, the libration cross term will average out.

The third dominant cross terms are the leading 1pN corrections to the usual quadrupole (LK) effects. Unlike the previous two cross-term effects, these perturbations affect all inner orbital elements, and are presented in App. A as an example, we write the perturbation on eccentricity below:

\[
\frac{de_1}{dt}_{3BPN} = \frac{15G^{3/2}M^2a_1^{3/2}}{32c^2m^{1/2}a_2^{3/2}} \frac{\ell_1^2}{\ell_1} \left( e_2^2 \left[ (3 + \cos 2t) \cos 2\omega_2 \times \text{sin} 2\omega_1 - 4 \cos t \cos 2\omega_1 \sin 2\omega_2 \right] - g_{e_2} \sin^2 t \sin 2\omega_1 \right), \tag{4.5}
\]

where

\[
g_{e_2} = 6(8 + 3e_2^2 + 4e_2^4). \tag{4.6}
\]

These effects have scaling magnitude \( X_{24} = 4 \times 10^{-4} \), and appear as corrections to the quadrupole (LK) terms from 1pN perturbations on the outer binary. Therefore, we call terms that scale as \( X_{24} \) as “relativistic-LK” cross terms. The scaling \( X_{24} \), suggests that the relativistic-LK cross terms will surpass the libration cross terms in magnitude when \( q > e^{-5/3} \). Other cross terms besides Eqs. (4.3) - (4.5) are negligible with scaling magnitudes \( X_{31} \times X_{20} \leq 10^{-5} \).

Although the scaling magnitudes \( X_{1m} \) quoted above are specific to a system with initial parameters \( (a_1, a_2, m, M) = (1AU, 10^4AU, 50M_\odot, 4 \times 10^6M_\odot) \), the general conclusion is the same in much of parameter space: the geotic, libration, and relativistic-LK cross terms [Eqs. (4.3) - (4.5)] represent the dominant relativistic three-body secular effects. For the remainder of the paper, we focus on the effect of these three dominant cross terms and neglect other subdominant cross terms.

Inspired by recent direct detections made by LIGO, we choose \( m_1 = 30M_\odot \) and \( m_2 = 20M_\odot \). As the mass ratio \( q \) increases, so does the region of \( (a_1, a_2) \) parameter space where cross terms are expected to be signif-
icant. When \( q \gtrsim 10^7 \), the resolution required to resolve quadrupole effects becomes computationally burdensome, as the quadrupole timescale goes as \( \sqrt{m/M} \) \[29\].

Given the masses and initial eccentricities, we can identify regions in parameter space where the cross terms are significant by comparing timescales for various effects (Fig. 2). Our primary interest lies in LK triples with the potential for eccentricity growth, so we demand that 1pN-binary effects not squash LK effects [c.f. Eq. (10) in Ref. 38]. Another constraint we impose is that the GW timescale is longer than the LK timescale [c.f. Eq. (31) in Ref. 38]. We must also stay in the region of parameter space where the secular approximation is valid. We use the criterion from Ref. 50 and restrict our initial parameters assuming the maximum eccentricity achieved is \( e_1 = 0.99 \). This limit is somewhat arbitrary, as we verify the secular criterion for each evolution \textit{a posteriori}.

Finally, we estimate where the de Sitter precession rate exceeds the 1pN-binary precession rate, and also where the the librating cross term exceeds the octupole terms. We set \( m_3 = 2 \times 10^7 \) and \( e_2 = 0.8 \), and leave a wider exploration of parameter space and larger \( m_3 \) to future work.

### B. Case study

In this section, we discuss two examples of resonant-like behaviors induced by the 3BpN terms. We demonstrate the effect of these behaviors by comparing evolutions: one with and without 3BpN cross terms. We

---

**FIG. 3.** Three-body 1pN (3BpN) effects for a librating system, including quadrupole (Quad), two-body 1pN (1pN), and 3BpN effects. We plot trajectories in \((\epsilon_1, \omega_1)\) phase space (top) and \((\epsilon_1, \omega_1)\) phase space (bottom) for a triple with \((m_1, m_2, m_3) = (30 M_\odot, 20 M_\odot, 2 \times 10^7 M_\odot)\), \( e_2 = 0.8 \), and \((a_1, a_2) = (0.10 \text{ AU}, 209.84 \text{ AU})\). Each trajectory is initialized with \( \omega_1 = 90^\circ \), \( \omega_2 = 282.27^\circ \), \( \Omega_1 = 192.5^\circ \), and \( \Omega_2 = 12.5^\circ \) but with different initial \( \epsilon_1 \) and \( \epsilon_1 \) such that \( \ell_z = \sqrt{1 - \epsilon_1 \cos \epsilon_1} = -0.6593 \). In the limit that \( m/M \ll 1 \) and \( a_1/a_2 \ll 1 \), when only considering 2BpN and quadrupole effects, the \( z \)-component of the angular momentum of the inner binary \( \ell_z \) [Eq. (4.7)] is nearly constant (left). As a result, all trajectories are closed and either exhibit circulation or libration. 3BpN effects lead to thickening of the phase space trajectories. For librating trajectories inside the separatrix, 3BpN effects can significantly modulate the amplitude of LK oscillations in \( \epsilon_1 \), \( \omega_1 \), and \( \omega_1 \), which reach a maximum when \( \ell_z \) is at a maximum (e.g. near \( t = 350 \text{ yr} \)). We also show the instantaneous magnitudes of quadrupole, 2BpN, and all 3BpN perturbations on \( \dot{\epsilon_1} \) [Eq. (4.8)] in units of \( \text{yr}^{-1} \) (bottom).

---

**FIG. 4.** Evolution of a triple system exhibiting the 3BpN librating resonance. We plot the time evolution (top) including quadrupole and 2BpN effects (red), and the time evolution including quadrupole, 2BpN, and 3BpN effects (blue). With 3BpN effects, the angular momentum component \( \ell_z \) oscillates about its initial value. This induces modulations in the amplitude of LK oscillations in \( \epsilon_1 \), \( \omega_1 \), and \( \omega_1 \), which reach a maximum when \( \ell_z \) is at a maximum (e.g. near \( t = 350 \text{ yr} \)). We also show the instantaneous magnitudes of quadrupole, 2BpN and all 3BpN perturbations on \( \dot{\epsilon_1} \) [Eq. (4.8)] in units of \( \text{yr}^{-1} \) (bottom).
restrict our attention to the three dominant cross-term effects discussed in Sec. IV A and initially neglect octupole effects and GW dissipation. Later in Sec. IV C we discuss the 3BpN effects conjunction with octupole effects and GW dissipation.

In Fig. 3 we plot the various parametrized trajectories in phase space each with the same initial value for

$$\ell_2 = \sqrt{1 - e_1^2 \cos \ell}.$$  \hspace{1cm} (4.7)

When only including quadrupole and 2BpN effects, $\ell_2$ is a constant of motion in the test-particle limit ($m_2 \to 0$). Although we work outside the test-particle limit, we consider systems where the ratio of inner to outer angular momentum is sufficiently small $L_1 / L_2 \sim 0.008$ so that $\ell_2$ is still nearly constant \[24\]. Since $\ell_2$ is nearly constant, the trajectories are closed and exhibit either libration or circulation (see Ref. \[19\] for a review). Circulating trajectories are those for which $\omega_1$ spans all values in $(0, 2\pi)$, increasing or decreasing monotonically with time. Librating trajectories are those for which $\omega_1$ spans a subset of $(0, 2\pi)$, oscillating with a constant amplitude about the fixed point. The separatrix is the trajectory separating the two types of behavior.

The cross terms lead to chaotic thickening of both librating and circulating trajectories. This is due to cross-term induced oscillations in the Newtonian-order angular momentum expression, which causes $\ell_2$ to oscillate. A similar cross-term effect is described in Ref. \[34\]. As $\ell_2$ oscillates in time, the triple’s trajectory in phase space migrates through multiple nearby “closed” trajectories corresponding to different initial $\ell_2$. For trajectories near the separatrix, this causes the system to chaotically switch between circulation and libration (red trajectory in Fig. 3). We note that a similar effect can be seen in triples where the octupole terms have a strong influence on the dynamics (c.f. Fig. 4 in Ref. \[23\]), but is identified here at lower quadrupole order due to the influence of cross terms.

Maximal eccentricity growth occurs when 3BpN effects are comparable to 2BpN effects in magnitude. Fig. 3 (blue
trajectory) shows an example of this behavior in phase space. The addition of cross terms significantly thicken the librating trajectory, completely filling the interior region. The evolution over time for the same system is shown in Fig. 4 (top panel). The amplitude of LK oscillations in $e_1, f_1, \omega_1$ changes as $\ell_z$ modulates about its initial value and is larger when $\ell_z(t) < \ell_z(0)$ and smaller when $\ell_z(t) > \ell_z(0)$. We find that the opposite is true for retrograde systems.

In Fig. 4 (bottom panel), we also plot the individual contribution from each effect towards the perturbation on the inner binary’s orbital vector,

$$\frac{de_1}{dt} = \left| \frac{d(e_1 n)}{dt} \right| \tag{4.8}$$

where $n$ is the unit vector pointing towards the pericenter [Eq. (3.15)]. The three dominant contributions are from the quadrupole (“Quad”), $2BpN$ (“1pN”), and de Sitter $3BpN$ terms (“dS”). The period of LK oscillations is about 10 yr, while the cross-terms induce modulations to the LK oscillations with a period of about 700 yr. Throughout the evolution, the dS cross terms exceed the $2BpN$ perturbations,

$$\left| \frac{de_1}{dt} \right|_{dS} > \left| \frac{de_1}{dt} \right|_{1pN} \tag{4.9}$$

In general, we find that in systems where Eq. (4.9) is true at some point, there is non-trivial addition of the $3BpN$ and $2BpN$ effects leading to resonant-like modulations resembling Fig. 4. On the other hand, when the cross-terms are always subdominant to $2BpN$ terms, the modulations are suppressed.

In Fig. 5, we show an example of a second resonant-like effect for circulating trajectories. Similar to the librating behavior, the phase space trajectory is substantially thickened so the system spans a larger range of inclination and eccentricity. However, unlike the librating effects, the circulating trajectory undergoes LK oscillations where the mean eccentricity changes with $\ell_z$ and the amplitude is roughly constant (Fig. 2 top panel). During the peaks in the LK oscillations, the dS and 1pN perturbations can exceed the quadrupole perturbations, so that

$$\left| \frac{de_1}{dt} \right|_{1pN} > \left| \frac{de_1}{dt} \right|_{quad} \quad \text{and} \quad \left| \frac{de_1}{dt} \right|_{dS} > \left| \frac{de_1}{dt} \right|_{quad} \tag{4.10}$$

For the system plotted in Fig. 6 (bottom panel), this occurs during the high-eccentricity phase of the modulations, when $\ell_z < \ell_z(0)$.

C. $3BpN$ effects on a population of triples

To study how $3BpN$ perturbations systematically affect a population of triples, we focus on the region of parameter space described in Fig. 2, where $3BpN$ effects are expected to be significant. We generate initial separations for 10,000 triples by sampling a log-uniform distribution within this $(a_1, a_2)$ region and set $\epsilon_2 = 0.8$, $m_1 = 30M_\odot$, $m_2 = 20M_\odot$ and $m_3 = 2 \times 10^5 M_\odot$. For the inner eccentricity we assume an initially thermal distribution, uniform in $e_1^2$. We also assume an initially isotropic distribution so that $\Omega_1, \omega_j, \cos i_j$ are uniformly sampled across all possible values. For each evolution we include quadrupole, octupole, and two-body 1pN secular effects on the inner and outer binary, as well as GW dissipation in the inner binary. We evolve each system twice — with and without $3BpN$ cross terms. We integrate the secular equations using GSL, which implements the explicit Dormand-Prince $(8,9)$ method with adaptive timesteps.

We ensure that numerical errors do not impact our overall conclusions by comparing evolutions with different error tolerances, $\epsilon_{rel} = (10^{-15}, 10^{-12})$, which control the timestep.

As the inner binary shrinks due to GW dissipation, it eventually enters a GW-dominated regime and decouples from the outer orbit. We integrate each system until the Keplerian orbital frequency reaches $f_{orb} = 5 \, \text{Hz}$, approximately corresponding to a gravitational wave frequency of $10 \, \text{Hz}$, the lower edge of the LIGO sensitivity range, after which we consider the system “merged”. For the masses we consider, this occurs when

$$a_1 = \left( \frac{G m}{f_{orb}} \right)^{1/3} = 4.3 \times 10^{-5} \, \text{AU} = 44 \, R_g, \tag{4.11}$$

where $R_g = 2Gm/c^2$ is the gravitational radius. All systems in our population merge before a Hubble time,

$$t_{\text{merge}} < t_H = 1.38 \times 10^{10} \, \text{yr}, \tag{4.12}$$

which is expected given that the timescale for GW dissipation is $52$

$$t_{GW} = \frac{a_1}{\langle \langle 1\rangle \rangle_{GW}} = \frac{5 \epsilon_5 \epsilon_1^3 (1 - \epsilon_1)^{7/2}}{64 G^3 m_1 m_2 m} \tag{4.13}$$

For systems that achieve large eccentricities through the LK resonance, $e_1 \gtrsim 0.9$, the merger timescale will decrease by three orders of magnitude. When eccentricity is very large, $e_{1,\text{max}} \gtrsim 0.999$, the evolution becomes non-secular, which we identify using the criterion from Ref. [50]. We neglect these non-secular evolutions in our analysis, which account for 4.0% of all runs.

In Fig. 7 we compare the effect of $3BpN$ terms on the maximum eccentricity $e_{1,\text{max}}$, the merger time $t_{\text{merge}}$, and the residual eccentricity upon entering the LIGO frequency band $e_{LIGO}$ as a function of initial inclination. We define $e_{LIGO}$ as the eccentricity when the frequency of the peak GW harmonic reaches $10 \, \text{Hz}$:

$$f_{GW} = \frac{2f_{orb}(1 + e_1)_{1.1954}^{1.1954}}{(1 - e_1^{3})^{3/2}} = 10 \, \text{Hz}. \tag{4.14}$$
We also plot the fractional decrease in merger time defined as
\[
\frac{\Delta t_{\text{merge}}}{t_{\text{merge}}} = \frac{t_{\text{merge}}^{3BpN} - t_{\text{merge}}^{2BpN}}{\frac{1}{2} (t_{\text{merge}}^{3BpN} + t_{\text{merge}}^{2BpN})}.
\]

When including 3BpN effects, a "shoulder"-like cluster in the maximum eccentricity distribution appears around 0.95 $\lesssim e_{1, \text{max}} \lesssim 0.99$ for systems with initially moderate inclinations $35^\circ \lesssim \iota_0 \lesssim 75^\circ$, where $\iota_0$ is the initial mutual inclination. In these systems, the 3BpN effects lead to a preferential increase in $e_{1, \text{max}}$ and $e_{1, \text{LIGO}}$, and decrease in $t_{\text{merge}}$. This effect is strongest when the 2BpN and 3BpN (de Sitter) perturbations can briefly exceed the quadrupole perturbations [Eq. (4.10)]. The resulting behavior resembles the quadrupole-level effects shown in Fig. 8, where coherent perturbations to the LK oscillations occur with some characteristic amplitude and frequency. For these coherent perturbations to occur, the de Sitter term [Eq. (4.3)] must be the largest cross term, followed by the relativistic-LK terms [Eq. (4.5)],
\[
\left| \frac{de_1}{dt} \right|_{\text{RLK}} < \left| \frac{de_1}{dt} \right|_{\text{ds}} \sim \left| \frac{de_1}{dt} \right|_{1pN} < \left| \frac{de_1}{dt} \right|_{\text{quad}}.
\]

Including the octupole terms can enhance the 3BpN effects and lead to larger eccentricities than with quadrupole terms alone. We show an example of this in Fig. 3, also plotted with a green dot in Fig. 7. Initially, the system undergoes LK oscillations with a period of about 5 yr. During the first few LK cycles, the eccentricity oscillates between 0.2 < $e_1$ < 0.8, while the inclination oscillates between 39° < $\iota_1$ < 60°. The 3BpN cross terms induce periodic modulations to the LK oscillations (similar to Fig. 6) that occur over a period of about 300 yr. Over longer timescales around 0.05 – 0.1 Myr, the octupole terms interact with the 3BpN cross terms leading to cycles of enhanced eccentricity growth, reaching up to $e_{1, \text{max}} = 0.986$. Eventually, the 2BpN precession arrests these octupole modulations near a phase of high eccentricity (around $t \approx 3.6 \times 10^5$ yr) and the system transitions into a GW-dominated regime.

For highly inclined systems $80^\circ \lesssim \iota_0 \lesssim 100^\circ$, the coherent modulations become chaotic if the relative ordering of the various cross terms is different from Eq. (4.10). For instance, if the maximum eccentricity is sufficiently large $e_{1, \text{max}} \gtrsim 0.99$, the libration cross terms, which go as $e_1 e_{1, \text{max}}^{-1}$ [Eq. (4.4)], can become significant. In these systems, the cross terms lead to a systematic suppression of eccentricity growth and delayed merger times (Fig. 7). For moderately retrograde systems $120^\circ \lesssim \iota_0 \lesssim 160^\circ$, the 2BpN and 3BpN perturbations approach the quadrupole perturbations in magnitude, also resulting in chaotic perturbations. Although some systems with chaotic perturbations appear to have large eccentricities, where $e_{1, \text{LIGO}} > 10^{-3}$ (Fig. 8), the long-term evolutions for these systems do not converge with our current code, unlike the coherent behavior observed at moderate prograde inclinations (Fig. 8). We leave further investigation of these chaotic oscillations to future work.
FIG. 8. Evolution of hierarchical triple resulting in an accelerated merger due to significant eccentricity growth from 3BpN and octupole effects. The initial parameter for this system are $(m_1, m_2, m_3) = (30 \, M_\odot, 20 \, M_\odot, 2 \times 10^7 \, M_\odot)$, $a_1 = 0.0713$ AU, $e_1 = 0.808$, $a_2 = 144.694$ AU, and $e = 37.935^\circ$. For the evolution without 3BpN terms, $e_{1,\text{max}} = 0.813$, $e_{1,\text{LIGO}} = 3.81 \times 10^{-6}$ and $t_{\text{merge}} = 1.76 \times 10^7$ yr. For the evolution with 3BpN terms, $e_{1,\text{max}} = 0.986$, $e_{1,\text{LIGO}} = 1.63 \times 10^{-4}$ and $t_{\text{merge}} = 4.86 \times 10^5$ yr.

V. DISCUSSION

In this paper we derived and investigated three-body post-Newtonian (3BpN) effects in hierarchical triples containing a SMBH. We solved the Lagrange planetary equations to 1pN-quadrupole order with a two-parameter perturbative expansion in the pN parameter $\delta = v/c$ and the ratio of semi-major axes $\epsilon = a_1/a_2$. Using a multiple scales method, we derived 3BpN terms that can significantly change the evolution of the inner binary. Upon request, we will provide a Mathematica notebook that contains a complete derivation.

When the mass of the inner binary is relatively small $(m \ll M)$, three dominant 3BpN effects emerge. The main effect is the de Sitter precession (dS), which paral-
lateral transports the inner orbit’s angular momentum vector along its path around the tertiary. Other dominant effects include 1pN corrections to LK oscillations and three-body corrections to relativistic precession. While the dS term comes directly from the EIH equations, the other 3BpN terms arise from the interaction of lower-order perturbations and can be derived with a multiple-scale analysis.

For a population in the parameter space where 3BpN effects are expected to be important, we found systematic eccentricity growth for systems with initially moderate inclinations. The 3BpN effects altered the evolution of these triples by inducing coherent modulations in the quadrupole LK oscillations, which led to a larger range in eccentricity and inclination. The octupole terms enhanced the 3BpN effects inducing greater eccentricity growth for systems with initially modulated eccentricity, and caused systems to merge more rapidly through GW dissipation. At high inclinations, the coherent modulations become chaotic and can suppress eccentricity growth.

With orbit-averaged methods, one can only get an estimate on the merger times and eccentricities. It would also be insightful to compare results from this analysis with integrations from N-body codes that include all 3BpN effects. Given the rich and complex behavior we observe in this analysis, further work is warranted to fully explore the implications of relativistic three-body effects in hierarchical triples.

ACKNOWLEDGMENTS

Our work on this problem was supported at the Massachusetts Institute of Technology (MIT) by the National Science Foundation Grant PH-1707549. Additionally, H. L. was supported by an MIT Dean of Science Graduate Fellowship. C. R. was supported by an MIT Pappalardo Fellowship. We would like thank Clifford M. Will, Smadar Naoz, and Scott A. Hughes for helpful discussions related to this paper.

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Appendix A: Appendix

We present all the three-body pN secular terms through 1pN-quadrupole order (\(e^3/c^2\)) for general masses. We also present the 1pN-octupole order (\(e^4/c^2\)) terms to lowest order in \((m/M)\), as 1pN-octupole terms are generated from the interaction of outer-1pN and quadrupole perturbations which scale as \(\delta_2 \sim \delta e^3\), respectively.

Since the equations are lengthy, we organize their presentation by distinguishing the direct cross terms which come directly from \(a_{3BpN}\) [Eq. (3.13)] inserted into the planetary equations [Eq. (3.17)–(3.18)], which are then orbit averaged [Eqs. (3.25)] using the Keplerian-order expressions for \((dt/df)\), \((dt/dF)\), \(P_{in}\) and \(P_{out}\):

\[
\frac{d\bar{X}_\alpha}{dt} = \frac{1}{P_{out}^{K}} \int_0^{2\pi} \frac{1}{P_{in}^{K}} \int_0^{2\pi} \left( \bar{X}_\alpha \right)_{3BpN} \left( \frac{dt}{df} \right)_K \left( \frac{dt}{dF} \right)_K df \ dF
\]

(A1)

1. Direct cross terms

The direct cross terms come directly from \(a_{3BpN}\) [Eq. (3.13)] inserted into the planetary equations [Eq. (3.17)–(3.18)], which are then orbit averaged [Eqs. (3.25)] using the Keplerian-order expressions for \((dt/df)\), \((dt/dF)\), \(P_{in}\) and \(P_{out}\):

\[
dp_1 = \frac{33e_1^2G^{3/2}\sqrt{mm_3}p_1^{3/2}e_2^3\sin^2(\iota)\sin(2\omega_1)}{4c^2p_2^3\ell_1^2}
\]

(A2)

\[
de_1 = \frac{3G^{3/2}\sqrt{mm_3}\sqrt{p_1}(\ell_1 - 1)^2e_2^3\sin^2(\iota)(\sin(2\omega_1))(\ell_1((12\eta - 23)\ell_1 - 22) - 11)}{8c^2e_1^3p_2^3}
\]

(A3)

\[
d\iota_1 = \frac{33e_1^2G^{3/2}\sqrt{mm_3}\sqrt{p_1}\ell_1^3\sin(2\iota)(\sin(2\omega_1))}{16c^2p_2^3\ell_1^2}
\]

(A4)
\[ \frac{d\omega_1}{dt} = - \frac{G^{3/2} m_3 (4m + 3m_3) \ell_2^3 \csc \ell_1 \sin \ell_2}{2c^2 \sqrt{M \rho_2}} + \frac{G^{3/2} m_3 \ell_2^3 \csc (\ell_1) \sqrt{\mp \rho_1}}{32c^2 \rho_2^5 / \ell_1^2} \left( 2(\ell_1 - 1)^2 \ell_2^3 \sin (\ell_1)((10\eta - 11)(\ell_1 + 1)^2(3 \cos (2\nu) + 1) - 6 \sin^2 (\nu) \cos (2\omega_1) \times (-6\eta + \ell_1(6\eta(\ell_1 - 2) + 5\ell_1 + 34) + 17)) - 6e_1^4 \sin (2\nu) \cos (\ell_1)(-11e_1^2 \cos (2\omega_1) + (4\eta - 5)\ell_1^2 + 11) \right) \]

\[ \frac{d\Omega_1}{dt} = - \frac{G^{3/2} m_3 (4m + 3m_3) \ell_3^3 \csc \ell_1 \sin \ell}{2c^2 \sqrt{M \rho_2}} + \frac{3G^{3/2} \sqrt{m \rho_3} \ell_3^3 \sin (2\nu) \csc (\ell_1)((-11e_1^2 \cos (2\omega_1) + (4\eta - 5)\ell_1^2 + 11))}{16c^2 \rho_2^5 / \ell_1^4} \]

b. 1pN-octupole order terms

\[ \frac{dp_1}{dt} = \frac{3e_1 e_2 \sqrt{1 - 4\eta G^{3/2} M^{3/2} \rho_2^5 / \ell_1^2} \left( \cos (\nu) (5 \cos^2 (\nu) - 1) \sin (\omega_1) \cos (\omega_2) + (3 - 7 \cos^2 (\nu)) \sin (\omega_2) \cos (\omega_1) \right)}{4c^2 \rho_2^5 / \ell_1^2} \]

\[ - \frac{15e_1^2 G^{3/2} M^{2} \rho_2^5 / \ell_1^2} {8c^2 \rho_2^5 / (\ell_1 + 1)^2} \left( \cos (\nu) \sin (\omega_1) \cos (\omega_2)(\cos (\omega_1)(5(1 - 2(\ell_1 - 1)\ell_1) \cos^2 (\nu) + \ell_1(17\ell_1 + 4) + 2) + \sin^2 (\omega_1)(5\ell_1^2 \cos^2 (\nu) + 2(\ell_1 + 7) \ell_1 + 7 - 4(\ell_1 + 1)^2) - \sin (\omega_2) \cos (\omega_1)(\cos (\omega_1)(2\ell_1(\ell_1 + 7) + 7) \times \cos^2 (\nu) + 5\ell_1^2) + \sin^2 (\nu)((\ell_1(17\ell_1 + 4) + 2) \cos^2 (\nu) - 10(\ell_1 - 1)\ell_1 + 5) - 4(\ell_1 + 1)^2) \right) \]

\[ \frac{de_1}{dt} = \frac{-3e_1 e_2 \sqrt{1 - 4\eta G^{3/2} M^{3/2} \rho_2^5 / \ell_1^2} \sin (\nu)((5 \cos (2\nu) - 11) \sin (\omega_1) \cos (\omega_2) - 14 \cos (\nu) \sin (\omega_2) \cos (\omega_1))}{16c^2 \rho_2^5 / \ell_1^2} \]

\[ - \frac{3G^{3/2} M^2 \rho_2^5 / \ell_1^2 \sin (\nu)(e_1^2 \sin (2\omega_2) - 5e_1^2 \cos (2\omega_1) - 3\ell_1^2 - 5 - 5e_1^2 \sin (\omega_2) \sin (\omega_1)(6 - e_1^2 \cos (2\omega_2)))}{16c^2 \rho_2^5 / \ell_1^2} \]

\[ \frac{dw_1}{dt} = \frac{-3e_1 e_2 \sqrt{1 - 4\eta G^{3/2} M^{3/2} \rho_2^5 / \ell_1^2} \left( \sin (\nu) \left( \frac{1}{2} \cos (\omega_1) \cos (\omega_2)(-5(e_1^2 + 2\ell_1^4 - 4e_2^4 + 2\ell_1) \cos (2\omega_1) + 20\ell_1^4 + 20e_1^2 + \ell_1^2 - 18\ell_1 - 9 - 17(\ell_1 - 1)(\ell_1 + 1)^3 \sin (\nu) \cot (\ell_1) \sin (\omega_1) \sin (\omega_2) + \cos (\nu) \sin (\omega_1) \sin (\omega_2)(\ell_1 / (\ell_1 + 1)^3) \sin (\nu) \cot (\ell_1) (\sin (\omega_1) \cos (\omega_2)(\ell_1(17\ell_1 + 44) + 7 - 30) - 15 \sin^2 (\omega_1) + \ell_1(3\ell_1 - 2)\ell_1(9\ell_1 + 14) + 10) - 10 \cos^2 (\omega_1)) + 5 \cos^3 (\nu) + \cos (\omega_1) \cos (\omega_2)(\ell_1(-\ell_1^2 + 4\ell_1^4 + \ell_1^2 - 2) - 1) \sin^2 (\omega_1) + \ell_1^4 \cos^2 (\omega_1)(-7(\ell_1 - 1)(\ell_1 + 1)^3 \sin (\nu) \cot (\ell_1) \times \cos (\omega_1) \sin (\omega_2) + \sin (\omega_1) \sin (\omega_2)(5\ell_1^2 \sin^2 (\omega_1) - 5\ell_1(\ell_1((\ell_1 - 4)\ell_1 - 1) - 1) + 2) + 1) \cos^2 (\omega_1) - 4(\ell_1 + 1)^2(3\ell_1^2 - 2)) \right) \right) \]

\[ - \frac{3G^{3/2} M^2 \rho_2^5 / \ell_1^2 \ell_1^2}{32c^2 \rho_2^5 / \ell_1^2} \left( 10e_1^2 e_2^2 \sin (\nu) \cot (\ell_1) \sin (\omega_1) \sin (2\omega_2) - \ell_1^2 \cos (2\nu) (5 \cos (2\omega_1) - 3)(6 - e_2 \cos (2\omega_2)) + 20e_1^2 e_2^2 \cos (\nu) \sin (2\omega_2) + \sin (2\nu) \cot (\ell_1)(6 - e_1^2 \cos (2\omega_2))(\ell_1^4 \cos (2\omega_1) - 3\ell_1^2 + 5) - 3\ell_1^2 (5 \cos (2\omega_1) + 1)(e_2^2 (- \cos (2\omega_2) - 2)) \right) \]
\[
\begin{align*}
\frac{d\Omega}{dt} &= 3\epsilon_1\epsilon_2\sqrt{1 - 4\eta}G^{3/2}M^{3/2}p_1\ell_2^3\sin(\iota) \left( 17\cos(\iota) \sin(\omega_1) \sin(\omega_2) + 7\cos(\omega_1) \cos(\omega_2) \right) \\
&\quad + \frac{8c^2p_2^2/\ell_1^2}{16c^2/\sqrt{mp_2\ell_1^4}} \left( 3G^{3/2}M^2p_1^{3/2}\ell_2^3\sin(\iota) \left( 5\epsilon_1^2c_2^2 \sin(2\omega_1) \sin(2\omega_2) + \cos(\iota) \left( 6 - c_2^2 \cos(2\omega_2) \right) \left( -5\epsilon_1^2 \cos(2\omega_1) - 3\ell_1^2 + 5 \right) \right) \right)
\end{align*}
\]

2. **Indirect cross terms due to corrections to \((dt/df), (dt/dF), P_{in}, \text{ and } P_{out}\)**

These cross terms come from 1pN and quadrupole corrections to \((dt/df), (dt/dF), P_{in}, \text{ and } P_{out}\), which combine with perturbations from \(a_{\text{quad}}\) and \(a_{1pN}\).

a. **Cross terms from \((dt/dF)_{1pN} \times (X_{\alpha})_{\text{quad}}\)**

These cross terms come from 1pN corrections to \((dt/dF)\) which combine with \((X_{\alpha})_{\text{quad}}\):

\[
\frac{d\dot{X}_{\alpha}}{dt} = \frac{1}{P_{out}} \int_0^{2\pi} \frac{1}{P_{in}} \int_0^{2\pi} \left( \dot{X}_{\alpha} \right)_{\text{quad}} \left( \frac{dt}{df} \right) \left( \frac{dt}{dF} \right)_{1pN} df \ dF.
\]

\[
\frac{dp_1}{dt} = \frac{15\epsilon_1^2G^{3/2}M^2p_1^{5/2}\ell_2^3}{32c^2/\sqrt{mp_2\ell_1^4}} \left( \sin(2\omega_1) \left( (\ell_2^2 + 12) (\cos(2\iota) + 3) \cos(2\omega_2) + 4(\ell_2^2 - 4) \sin^2(\iota) \right) - 4(\ell_2^2 + 12) \cos(\iota) \right) \sin(2\omega_2) \cos(2\omega_1)
\]

\[
\frac{de_1}{dt} = \frac{15\epsilon_1^2M^2\ell_3^3(Gp_1)^{3/2}}{64c^2/\sqrt{mp_2\ell_1^4}} \left( \sin(2\omega_1) \left( (\ell_2^2 + 12) (\cos(2\iota) + 3) \cos(2\omega_2) + 4(\ell_2^2 - 4) \sin^2(\iota) \right) - 4(\ell_2^2 + 12) \cos(\iota) \right) \cos(\iota) \sin(2\omega_2) \cos(2\omega_1)
\]

\[
\frac{dt_1}{dt} = -\frac{3M^2\ell_3^3(Gp_1)^{3/2} \sin(\iota)}{32c^2/\sqrt{mp_2\ell_1^4}} \left( 5\epsilon_1^2\cos(\iota) \sin(2\omega_1) \left( (\ell_2^2 + 12) \cos(2\omega_2) - 2\ell_2^2 + 8 \right) + (\ell_2^2 + 12) \sin(2\omega_2) \left( 5\ell_1^2 - 1 \right) \cos(2\omega_1) + 3\ell_1^2 - 5 \right)
\]

\[
\frac{dA_{12}}{dt} = -\frac{3M^2\ell_3^3(Gp_1)^{3/2} \sin(\iota)}{16c^2/\sqrt{mp_2\ell_1^4}} \left( \sin^2(\omega_1) \left( \cos^2(\omega_2) \left( (\ell_2^2 - 20) \cos(\iota) \left( 4\ell_2^2 \cos(\iota) + (5 - 4\ell_2^2) \sin(\iota) \cot(\iota) \right) \right) - \ell_2^2 \left( 3 \ell_2^2 + 4 \right) \right) + 2 \sin^2(\omega_2) \left( (3\ell_2^2 + 4) \cos(\iota) \left( 4\ell_2^2 \cos(\iota) + (5 - 4\ell_2^2) \sin(\iota) \cot(\iota) \right) - \ell_2^2 \left( 3 \ell_2^2 - 20 \right) \right) \right) \sin(\omega_1) \cos(\omega_2) \cos(\omega_1) \left( 2\ell_1^2 \cos(\iota) - \ell_1^2 - 1 \right) \sin(\iota) \cot(\iota) - 4\ell_1^2 \ell_2^2)
\]

\[
\frac{d\Omega}{dt} = \frac{3M^2\ell_3^3(Gp_1)^{3/2} \sin(\iota) \cos(\iota)}{16c^2/\sqrt{mp_2\ell_1^4}} \left( 5\epsilon_1^2(\ell_2^2 + 12) \sin(\omega_1) \sin(2\omega_2) \cos(\omega_1) - \cos(\iota) \left( (\ell_2^2 + 12) \cos(2\omega_2) - 2\ell_2^2 \right) + 8 \left( (5 - 4\ell_2^2) \sin^2(\omega_1) + \ell_2^2 \cos^2(\omega_1) \right) \right)
\]
b. Cross terms from \((dt/df)_{\text{quad}} \times (\dot{X}_\alpha)_{1pN}\)

These cross terms come from quadrupole corrections to \((dt/df)\) which combine with \((\dot{X}_\alpha)_{1pN}\):

\[
\frac{d\dot{X}_\alpha}{dt} = \frac{1}{P_{\text{out}}} \int_0^{2\pi} \frac{1}{P_{\text{in}}} \int_0^{2\pi} \left(\dot{X}_\alpha\right)_{1pN} \left(\frac{dt}{df}\right)_\text{quad} \frac{dt}{df} \, df \, dF. \tag{A18}
\]

\[
\frac{dp_1}{dt} = \frac{6(\eta - 2)\eta Gm(\ell_1 - 1)^2 \ell_1^2 (\ell_2 - 1)^2 \ell_2^2 \sqrt{Gmp_1}}{c^2 e_1^2 e_2^2 p_2^2} \left(\cos(2\ell_1 + 3) \sin(\omega_1) \cos(2\omega_2) - 2 \cos(\omega_1) \sin(2\omega_2) \right) \times \cos(2\omega_1) \tag{A19}
\]

\[
\frac{de_1}{dt} = -\frac{3\eta (\ell_1 - 1)^2 \ell_1^2 (\ell_2 - 1)^2 \ell_2^2 (Gm)^{3/2} (-5\eta + 11\eta \ell_1 - 14\ell_1 + 10)}{4c^2 e_1^4 e_2^2 \sqrt{p_1^2 p_2^2}} \left(\cos(2\ell_1 + 3) \sin(\omega_1) \cos(2\omega_2) \right)
\]

\[
- 2 \cos(\omega_1) \sin(2\omega_2) \cos(2\omega_1) \tag{A20}
\]

\[
\frac{dw_\omega}{dt} = \frac{3\eta (\ell_1 - 1)^2 \ell_1^2 (\ell_2 - 1)^2 \ell_2^2 (Gm)^{3/2}}{64 c^2 e_1^4 e_2^2 \sqrt{p_1^2 p_2^2}} \left(4(\cos(2\ell_1 + 3) \cos(2\omega_1) \cos(2\omega_2)(7\eta + \ell_1(-22\eta + 3\eta + 2\ell_1 + 28)
\]

\[
- 22) - 8(\ell_1 + 1) \sin^2(\omega_1) \cos(2\omega_2)(-3\eta + 2\eta \ell_1 + 2(\ell_1 + 22) + 16 \cos(\omega_1) \sin(2\omega_1) \sin(2\omega_2)(7\eta + \ell_1(-22\eta
\]

\[
+ 3\eta \ell_1 + 2(\ell_1 + 28) - 22) \right) \tag{A21}
\]

c. Cross terms from \((dt/df)_{1pN} \times (\dot{X}_\alpha)_{\text{quad}}\)

These cross terms come from 1pN corrections to \((dt/df)\) which combine with \((\dot{X}_\alpha)_{\text{quad}}\):

\[
\frac{d\dot{X}_\alpha}{dt} = \frac{1}{P_{\text{out}}} \int_0^{2\pi} \frac{1}{P_{\text{in}}} \int_0^{2\pi} \left(\dot{X}_\alpha\right)_{\text{quad}} \left(\frac{dt}{df}\right)_{1pN} \left(\frac{dt}{df}\right)_K \, df \, dF. \tag{A22}
\]

\[
\frac{dp_1}{dt} = 3G^{3/2} \sqrt{m m_3} p_3^{3/2} \ell_2^2 \sin^2(\ell_1) \sin(2\omega_1) \left(\eta - 3(\eta + 2)\ell_1^6 + 16(\eta - 5)\ell_1^6 - 17(\eta - 10)\ell_1^6 + (3\eta - 122)\ell_1^6 + 38\right)
\]

\[
\frac{8c^2 e_1^2 p_2^2}{\ell_1^2} \tag{A23}
\]

\[
\frac{de_1}{dt} = 3G^{3/2} \sqrt{m m_3} \sqrt{p_1}(\ell_1 - 1)^2 \ell_1^2 \sin^2(\ell_1) \sin(2\omega_1) \left(\eta(\ell_1(3\ell_1^3(-3\ell_1 - 8) + 10) + 2) - 21 + 2\ell_1(\ell_1(\ell_1
\]

\[
	imes (5\ell_1 + 66) - 32) - 38) \tag{A24}
\]

\[
\frac{d\ell_1}{dt} = 3G^{3/2} \sqrt{m m_3} \sqrt{p_1}(\ell_1 - 1)^2 \ell_1^2 \sin(2\omega_1) \left(\eta - 3(\eta + 2)\ell_1^6 + 16(\eta - 5)\ell_1^6 - 17(\eta - 10)\ell_1^6 + (3\eta - 122)\ell_1^6 + 38\right)
\]

\[
\frac{32c^2 e_1^2 p_2^2}{\ell_1^2} \tag{A25}
\]

\[
\frac{d\omega_\ell}{dt} = \frac{G^{3/2} \sqrt{m m_3} \sqrt{p_1}(\ell_1 - 1)^2 \ell_1^2 \ell_1^2}{32c^2 e_1^2 p_2^2} \left(-6e_1^2 \sin^2(\omega_1) \cot(\ell_1) \cos(2\omega_1)(-\eta + \ell_1(-6\eta + 1)(-10\eta + 3(\eta + 2)
\]

\[
x (\ell_1 + 92) + 2(\eta + 38)) - 38) + 6e_1^2 \sin^2(\omega_1) \cos(2\omega_1) \eta(\ell_1 - 2)\ell_1(2\ell_1(3\ell_1 + 8) - 17) + 17 + 2\ell_1(\ell_1(\ell_1
\]

\[
\times (\ell_1 + 14) + 63) - 74 - 74) + \ell_1^2(\ell_1 + 1)(3\cos(2\omega_1 + 1)(3\eta - 10) + \ell_1(3\eta - 10) + 2\ell_1(-4\eta + (3\eta + 2)\ell_1
\]

\[
+ 22)))) - 3(\ell_1 - 1)(\ell_1 + 1)^2 \sin(2\omega_1) \cot(\ell_1)(-\eta + (9\eta + 2)\ell_1^2 - 38) \right) \tag{A26}
\]
\[
\frac{d\Omega_1}{dt} = \frac{3G^{3/2}\sqrt{m_3}\sqrt{p_1}\ell_1^3\sin(\iota)\cos(\iota)\csc(\iota)}{16c^2e_1^2p_2^2_1^2_1} \left((\ell_1 - 1)^2 \cos(2\omega_1)(-\eta + \ell_1(-6\eta + \ell_1(-10\eta + 3(\eta + 2)\ell_1 + 92) + 8) - 2(\eta + 38) - 38) - e_1^4(-\eta + (9\eta + 2)\ell_1^2 - 38)\right)
\]  
(A27)

d. Cross terms from \( (dt/df)_{\text{quad}} \times (\dot{X}_\alpha)_{1\text{pN}} \)

These cross terms come from quadrupole corrections to \( (dt/df) \) which combine with \( (\dot{X}_\alpha)_{1\text{pN}} \):

\[
\frac{d\dot{X}_\alpha}{dt} = \frac{1}{P_{\text{out}}^K} \int_0^{2\pi} \frac{1}{P_{\text{in}}^K} \int_0^{2\pi} (\dot{X}_\alpha)_{1\text{pN}} (dt/df)_{\text{quad}} \frac{dt}{dF} K \, df \, dF
\]  
(A28)

\[
\frac{dp_1}{dt} = \frac{3(\eta - 2)G^{3/2}\sqrt{m_3}p_1^{3/2}((\ell_1(\ell_1 + 4) - 9)\ell_1 + 5\ell_1 - 1)\ell_3^2\sin(2\omega_1)}{2c^2e_1^2p_2^2_1^2_1}
\]  
(A29)

\[
de_1 = \frac{-3G^{3/2}\sqrt{m_3}\sqrt{p_1}(\ell_1 - 1)^2\ell_2^3\sin^2(\iota)\sin(2\omega_1)}{16c^2e_1^2p_2^2_1^2_1} \left(-\eta + \eta\ell_1(\ell_1(5\ell_1 + 28) - 42) - 2 - 2\ell_1(\ell_1(7\ell_1 + 22) - 56) + 6 - 6\right)
\]  
(A30)

\[
\frac{d\omega_1}{dt} = \frac{G^{3/2}\sqrt{m_3}\sqrt{p_1}(\ell_1 - 1)^2\ell_2^2}{32c^2e_1^2p_2^2_1^2_1} \left(6\sin^2(\iota)\cos(2\omega_1)(\eta(\ell_1 - 2)\ell_1(2\ell_1(3\ell_1 + 8) - 17) + 17) + 2\ell_1(\ell_1(2\ell_1 - 1)(\ell_1 + 1) - 74) - 74) + (\ell_1 + 1)(3\cos(2\iota) + 1)(3(\eta - 10) + \ell_1(3(\eta - 10) + 2\ell_1(-4\eta + (3\eta + 2)\ell_1 + 22)))\right)
\]  
(A31)

e. Cross terms from \( P_{\text{in}}^{1\text{pN}} \times (\dot{X}_\alpha)_{\text{quad}} \)

These cross terms come from \( 1\text{pN} \) corrections to \( P_{\text{in}} \) which combine with \( (\dot{X}_\alpha)_{\text{quad}} \):

\[
\frac{d\dot{X}_\alpha}{dt} = \frac{1}{P_{\text{out}}^K} \int_0^{2\pi} \frac{1}{P_{\text{in}}^K} \int_0^{2\pi} (\dot{X}_\alpha)_{1\text{pN}} (dt/df)_{\text{quad}} \frac{dt}{dF} K \, df \, dF
\]  
(A32)

where above we expanded \( 1/P_{\text{in}} \) [Eq. (3.44)] to linear order in \( P_{\text{in}}^{1\text{pN}} \) and

\[
P_{\text{in}}^{1\text{pN}} = \int_0^{2\pi} \left(\frac{dt}{df}\right)_{1\text{pN}} df = P_{\text{in}}^K \left(Gm/p_1c^2\right) \frac{e_1^2((-21\eta + 8)) (\ell_1 + 1) + 21\eta + \ell_1 (21\eta + \ell_1 (80 - \eta (9\ell_1 + 49)) + 8) + 8}{\ell_1 + 1}
\]  
(A33)

There are no corrections to \( P_{\text{in}} \) due to periodic perturbations since \( P_{\text{in}}^K \) only depends on the elements \( e_1 \) and \( p_1 \), which are not perturbed at \( 1\text{pN} \) order. \( P_{\text{in}}^{1\text{pN}} \) does not depend on \( F \) and can be factored outside the outer orbit integral [Eq. (A32)]. As a result, these cross terms are equal to the usual secular quadrupole terms times a multiplicative factor:

\[
\frac{d\dot{X}_\alpha}{dt} = -\frac{P_{\text{in}}^{1\text{pN}}}{P_{\text{in}}^K} \left(\frac{d\dot{X}_\alpha}{dt}\right)_{\text{quad}}
\]  
(A34)

where the secular quadrupole terms can be found in the literature (e.g. Refs. [40, 53]).
f. Cross terms from $P_{\text{in}}^{\text{quad}} \times (\dot{X}_\alpha)_{1\text{pN}}$

These cross terms come from quadrupole corrections to $P_{\text{in}}$ which combine with $(\dot{X}_\alpha)_{1\text{pN}}$:

$$\frac{d\dot{X}_\alpha}{dt} = \frac{1}{P_{\text{out}}^{\text{K}}} \int_0^{2\pi} \left( \dot{X}_\alpha \right)_{1\text{pN}} \left( \frac{dt}{dF} \right)_K \left( \frac{dt}{dF} \right)_K \, dF,$$

(A35)

where above we expanded $1/P_{\text{in}}$ [Eq. (3.44)] to linear order in $P_{\text{in}}^{\text{quad}}$ and

$$P_{\text{in}}^{\text{quad}} = \int_0^{2\pi} \left[ \frac{dt}{dF} \right]_{\text{quad}} + W_{\beta}^{\text{21}} \frac{\partial}{\partial X_{\beta}} \left( \frac{dt}{dF} \right)_K \, dF$$

(A36)

$$= P_{\text{in}}^{\text{K}} \frac{m_3 (1 + e_2 \cos(F))^3 p_1^3}{64 m p_2^2 \ell_1^3} \left( -5 (3 \ell_1^2 - 7) (6 \cos(2\ell_1) \sin^2(F + \omega_2) + 3 \cos(2(F + \omega_2)) + 1) - 3 (17 \ell_1^2 - 49) \right.$$

$$\times \cos(2\omega_1) (-2 \cos(2\ell_1) \sin^2(F + \omega_2) + 3 \cos(2(F + \omega_2)) + 1) + \left. 12 (49 - 17 \ell_1^2) \cos(\iota) \sin(2\omega_1) \sin(2(F + \omega_2)) \right).$$

The periodic contributions average to zero, so the only correction comes from $(dt/df)_{\text{quad}}$, leading to

$$\frac{d\omega_1}{dt} = \frac{3 G^{3/2} m_3^3 p_1^3}{\sqrt{m p_1}} \left( 6 (17 \ell_1^2 - 49) \sin^2(\iota) \cos(2\omega_1) + 5 (3 \ell_1^2 - 7) (3 \cos(2\iota) + 1) \right).$$

(A37)

g. Cross terms from $P_{\text{out}}^{1\text{pN}} \times (\dot{X}_\alpha)_{\text{quad}}$

These cross terms come from 1pN corrections to $P_{\text{out}}$ which combine with $(\dot{X}_\alpha)_{\text{quad}}$:

$$\frac{d\dot{X}_\alpha}{dt} = - P_{\text{out}}^{1\text{pN}} \left( \frac{dt}{dF} \right)_{1\text{pN}} \int_0^{2\pi} \frac{1}{P_{\text{in}}^{\text{K}}} \int_0^{2\pi} \left( \dot{X}_\alpha \right)_{1\text{pN}} \left( \frac{dt}{dF} \right)_K \left( \frac{dt}{dF} \right)_K \, dF,$$

(A38)

where

$$P_{\text{out}}^{1\text{pN}} = \int_0^{2\pi} \left[ \left( \frac{dt}{dF} \right)_{1\text{pN}} + W_{\beta}^{\text{21}} \frac{\partial}{\partial X_{\beta}} \left( \frac{dt}{dF} \right)_K \right] \, dF = P_{\text{out}}^{\text{K}} \frac{3 G M}{2 c^2 p_2} \frac{(-3 + \ell_2^2)(-5 + 2 \ell_2^2)}{\ell_2^2}.$$

(A39)

These cross terms are equal to the usual secular quadrupole terms times a multiplicative factor:

$$\frac{d\dot{X}_\alpha}{dt} = - P_{\text{out}}^{1\text{pN}} \left( \frac{d\dot{X}_\alpha}{dt} \right)_{\text{quad}}$$

(A40)

h. Cross terms from $P_{\text{in}}^{\text{quad}} \times (\dot{X}_\alpha)_{1\text{pN}}$

These cross terms come from quadrupole corrections to $P_{\text{out}}$ which combine with $(\dot{X}_\alpha)_{1\text{pN}}$:

$$\frac{d\dot{X}_\alpha}{dt} = - P_{\text{out}}^{\text{quad}} \left( \frac{dt}{dF} \right)_K \int_0^{2\pi} \frac{1}{P_{\text{in}}^{\text{K}}} \int_0^{2\pi} \left( \dot{X}_\alpha \right)_{1\text{pN}} \left( \frac{dt}{dF} \right)_K \left( \frac{dt}{dF} \right)_K \, dF,$$

(A41)
where

\[ p_{\text{out}}^{\text{quad}} = \int_0^{2\pi} \left( \frac{dt}{dF} \right)_{\text{quad}} + W_{\beta}^{2\hat{0}} \frac{\partial}{\partial X_{\beta}} \left( \frac{dt}{dF} \right)_{K} \] dF \approx \int_0^{2\pi} \left( \frac{dt}{dF} \right)_{\text{quad}} + W_{\beta}^{2\hat{0}} \frac{\partial}{\partial X_{\beta}} \left( \frac{dt}{dF} \right)_{K} \) dF \] (A42)

\[ = \int_0^{2\pi} \frac{1}{P_{\text{in}}} \int_0^{2\pi} \left[ \left( \frac{dt}{dF} \right)_{\text{quad}} + W_{\beta}^{2\hat{0}} \frac{\partial}{\partial X_{\beta}} \left( \frac{dt}{dF} \right)_{K} \right] \left( \frac{dt}{df} \right) dF dF \]

\[ = p_{\text{out}}^{K} \left( \frac{p_1^2}{p_2^2} \right) \left( \frac{\eta}{32\ell_2\ell_1^4(1 + \ell_2^2)} \right) \left( -3(-8(A_1 - A_4)(A_1 + A_4)\ell_1^2\ell_2^2 - 3\ell_1^2(3(A_1^2 + 3A_2^2 + 4)\ell_1^2 - 20) - 6\ell_1^2(3 \times (A_1^2 + 3A_2^4 + 4)\ell_1^2 - 20) + 6\ell_1^2(3(A_1^2 + A_2^2 + 2)\ell_1^2 - 10) + 6(7A_1^2 + 13A_2^2 + 20)\ell_1^2\ell_2 + 3(7A_1^2 + 13A_2^4 + 20)\ell_1^2
\]

\[ - A_2^2(4\ell_1^2 - 5)(\ell_1^2(\ell_2(\ell_2(8\ell_2 - 27) - 54) + 12) + 78) + 39) + A_2^2(4\ell_1^2 - 5)(\ell_1^2(\ell_2(\ell_2(8\ell_2 + 9) + 18)) - 12) - 42) - 21) - 100(\ell_2 + 1) - 6\ell_2^2(-5\ell_2^2(\cos(2\ell_1) + 3)\cos(2\omega_1)\cos(2\omega_2) - 20\ell_2^2\cos(\ell_1)\sin(2\omega_1)\sin(2\omega_2)
\]

\[ + 2(3\ell_2^2 - 5)\sin^2(\ell_1)\cos(2\omega_2)) \right). \]

where

\[ A_1 = \cos \ell \cos \omega_1 \cos \omega_2 + \sin \omega_1 \sin \omega_2 \]

\[ A_2 = \cos \ell \cos \omega_2 \sin \omega_1 - \cos \omega_1 \sin \omega_2 \]

\[ A_3 = \sin \omega_1 \cos \omega_2 + \cos \ell \sin \omega_1 \sin \omega_2 \]

\[ A_4 = \cos \omega_2 \sin \omega_1 - \cos \ell \cos \omega_1 \sin \omega_2. \] (A43)

These cross terms are equal to the usual secular quadrupole terms times a multiplicative factor:

\[ \frac{d\tilde{X}_\alpha}{dt} = -p_{\text{out}}^{\text{quad}} \left( \frac{d\tilde{X}_\alpha}{dt} \right)_{\text{quad}} \] (A44)

3. Indirect cross terms due to periodic 1pN perturbations

These cross terms come from average-free, periodic 1pN perturbations which combine with perturbations from \( a_{\text{quad}} \). The inner binary periodic perturbations do not generate secular effects,

\[ \frac{d\tilde{X}_\alpha}{dt} = \frac{1}{P_{\text{in}}} \int_0^{2\pi} \frac{1}{P_{\text{in}}} \int_0^{2\pi} \sum_{\beta = 1}^{5} W_{\beta}^{01} \frac{\partial (Q_{\alpha (0)}^{(0)})_{\text{quad}}}{\partial X_{\beta}} \] dF dF = 0 \] (A45)

The outer binary periodic perturbations generate secular effects which read

\[ \frac{d\tilde{X}_\alpha}{dt} = \frac{1}{P_{\text{out}}} \int_0^{2\pi} \frac{1}{P_{\text{in}}} \int_0^{2\pi} \sum_{\beta = 6}^{10} W_{\beta}^{51} \frac{\partial (Q_{\alpha (0)}^{(0)})_{\text{quad}}}{\partial X_{\beta}} \] dF dF = 0. (A46)

a. Cross terms from periodic 1pN effects on the outer binary

\[ \frac{dp_1}{dt} = \frac{15e_2^2G^{3/2}m_3p_1^{5/2}e_3}{32c^2\sqrt{mM}p_2^2\ell_1^2} \left( \sin(\omega_1)\cos(\omega_1)(\cos^2(\omega_2)\ell_2 - \cos^2(\ell_2)K_{\ell_2}) + \sin^2(\omega_2)(K_{\ell_2} - \cos^2(\ell_2)\ell_2) \right) \]

\[ - 2F_{\ell_2} \cos(\ell_2)\sin(2\omega_2)\cos(2\omega_1) \] (A47)
\[
\frac{de_1}{dt} = \frac{15e_1 m_3 \ell_3^2 (Gp_1)^{3/2}}{256 \pi^2 M p_1^2 \ell_1^2 \sqrt{m}} \left( 8F_{\ell_2} \cos(\omega) \sin(2\omega_2) \cos(2\omega_1) + \sin(\omega_1) \cos(\omega)(\cos(2\ell_1) + 3) \cos(2\omega_2)(K_{\ell_2} - L_{\ell_2}) - 2 \sin^2(\ell_1)(K_{\ell_2} + L_{\ell_2}) \right)
\]

\[
\frac{d\omega_1}{dt} = \frac{3m_3 \ell_3^2 (Gp_1)^{3/2} \sin(\omega)}{64 \pi^2 \sqrt{m} M p_1^2 \ell_1^2} \left( F_{\ell_2} \sin(2\omega_2)(5e_1^2 \cos(2\omega_1) - 3\ell_1^2 + 5) + \frac{5}{4} e_1^2 \cos(\omega) \sin(2\omega_1)(\cos(2\omega_2)(K_{\ell_2} - L_{\ell_2}) + K_{\ell_2} + L_{\ell_2}) \right)
\]

\[
\frac{d\Omega_1}{dt} = \frac{3m_3 \ell_3^2 (Gp_1)^{3/2} \sin(\omega) \cos(\omega_1)}{64 \pi^2 \sqrt{m} M p_1^2 \ell_1^2} \left( 5e_1^2 F_{\ell_2} \sin(2\omega_1) \sin(2\omega_2) + \frac{1}{4} \cos(\omega)(-5e_1^2 \cos(2\omega_1) - 3\ell_1^2 + 5)(\cos(2\omega_2) \right) \times (K_{\ell_2} - L_{\ell_2}) + K_{\ell_2} + L_{\ell_2}),
\]

where

\[
F_{\ell_2} = M^2(-32 + 6\ell_2^2) + mm_3(29 - 11\ell_2^2)
\]

\[
H_{\ell_2} = 8M^2(2 - 5\ell_2^2) + mm_3(7 + \ell_2^2)
\]

\[
K_{\ell_2} = 4M^2(12 + 7\ell_2^2) + mm_3(-65 + 21\ell_2^2)
\]

\[
L_{\ell_2} = M^2(-80 + 52\ell_2^2) + mm_3(51 - 23\ell_2^2).
\]

4. Indirect cross terms due to periodic quadrupole perturbations

These cross terms come from average-free, periodic quadrupole perturbations which combine with perturbations from \(a_{1PN}\). The secular effects from periodic perturbations on the inner binary are

\[
\frac{d\tilde{X}_\alpha}{dt} = \frac{1}{P_{K}} \left[ \frac{1}{P_{K}^{\text{in}}} \int_{0}^{2\pi} \sum_{\beta=1}^{5} W_{\beta}^{30} \frac{\partial(Q_0^{(0)})_{1PN}}{\partial\tilde{X}_\beta} df \varepsilon F \right]
\]

and on the outer binary are

\[
\frac{d\tilde{X}_\alpha}{dt} = \frac{1}{P_{K}^{\text{out}}} \left[ \frac{1}{P_{K}^{\text{in}}} \int_{0}^{2\pi} \sum_{\beta=6}^{10} W_{\beta}^{50} \frac{\partial(Q_0^{(0)})_{1PN}}{\partial\tilde{X}_\beta} df \varepsilon F \right]
\]

We find that only \(\omega_1\) is affected, with no secular effects on the other elements.
a. Cross terms from periodic quadrupole effects on the outer binary

\[
\frac{d\omega_1}{dt} = \frac{9\eta G^{3/2} m_3^{3/2}}{32c^2 e_2^3 \sqrt{p_1 p_2} \ell_1 \ell_2^2} \left( \ell_2^6 (7A_1^2 - 43A_2^2 - 36) + 60 + \ell_2^4 (3\ell_2^2 (13A_1^2 + 31A_2^2 + 44) - 220) - 3\ell_2^4 \ell_2^2 (17A_1^2 \\
+ 35A_2^2 + 52) + 3\ell_2^4 (7A_1^2 + 13A_2^2 + 20) - 8\ell_2^4 \ell_2^2 (A_1 - A_4)(A_1 + A_4) - 8\ell_2^4 \ell_2^2 (A_1 - A_4)(A_1 + A_4) \\
+ A_2^2 (4\ell_2^2 - 5)(\ell_2 - 1)^2 (\ell_2 (\ell_2 (\ell_2 (8\ell_2 + 9) + 18) - 12) - 42) - 21) - A_3^2 (4\ell_2^2 - 5)(\ell_2 - 1)^2 \\
\times (\ell_2 (\ell_2 (\ell_2 (8\ell_2 - 27) - 54) + 12) + 78) + 39) + 20(13\ell_2^2 - 5) \right),
\]  

where \( A_1, A_2, A_3, \) and \( A_4 \) are defined in Eq. \( [A43] \).

b. Cross terms from periodic quadrupole effects on the inner binary

\[
\frac{d\omega_1}{dt} = \frac{15G^{3/2} m_3 (1 - \ell_2)(1 + 2\ell_2) c_2^2 C_1}{4c^2 \sqrt{M} p_1 p_2 \ell_1 \ell_2 (1 + \ell_2)},
\]

where \( C_1 = A_1 A_3 - A_2 A_4 \).