A spectral condition for graphs with all fractional \([a, b]\)-factors

Jiaxin Zheng, Junjie Wang

School of Mathematics, East China University of Science and Technology, Shanghai 200237, China

Abstract  Let \(a < b\) be two positive integers. We say that a graph \(G\) has all fractional \([a, b]\)-factors if it has a fractional \(p\)-factor for every \(p : V(G) \to \mathbb{Z}^+\) such that \(a \leq p(x) \leq b\) for every \(x \in V(G)\). In this paper, we provide a tight spectral radius condition for graphs having all fractional \([a, b]\)-factors.

Keywords: Spectral radius; fractional factor; all fractional \([a, b]\)-factors.

1 Introduction

All graphs considered in this paper are simple and undirected. Let \(G\) be a graph with vertex set \(V(G)\) and edge set \(E(G)\). Let \(e(G) := |E(G)|\) denote the number of edges in \(G\). For any \(v \in V(G)\), let \(d_G(v)\) denote the degree of \(v\) in \(G\), \(N_G(v)\) denote the set of vertices adjacent to \(v\) in \(G\), and \(E_G(v)\) denote the set of edges incident with \(v\) in \(G\). Let \(\delta(G) = \min\{d_G(v) : v \in V(G)\}\) denote the minimum degree of \(G\). For any vertex subset \(S \subseteq V(G)\), we denote by \(G[S]\) the subgraph of \(G\) induced by \(S\), and \(e(S) := e(G[S])\). Also, we denote by \(e(S, T)\) the number of edges between two disjoint subsets \(S\) and \(T\) of \(V(G)\). A vertex set \(S \subseteq V(G)\) is called independent if any two vertices in \(S\) are non-adjacent in \(G\). The join of two graphs \(G_1\) and \(G_2\), denoted by \(G_1 \nabla G_2\), is the graph obtained from \(G_1 \cup G_2\) by adding all possible edges between \(G_1\) and \(G_2\).

The adjacency matrix of \(G\) is defined as \(A(G) = (a_{u,v})_{u,v \in V(G)}\), where \(a_{u,v} = 1\) if \(u\) and \(v\) are adjacent in \(G\), and \(a_{u,v} = 0\) otherwise. The largest eigenvalues of \(A(G)\) is called the spectral radius of \(G\), and denoted by \(\rho(G)\). For some basic results on the spectral radius of graphs, we refer the reader to [5, 20], and references therein.

The theory of graph-factors plays a key role in the study of graph theory [3, 4, 8–11, 13, 15, 17–19, 22, 23, 25]. Let \(g\) and \(f\) be two integer-valued functions defined

*Corresponding author.
E-mail address: junjiewang_ecust@163.com
on $V(G)$ such that $0 \leq g(v) \leq f(v)$ for all $v \in V(G)$. A $(g,f)$-factor of $G$ is a spanning subgraph $F$ of $G$ satisfying $g(v) \leq d_F(v) \leq f(v)$ for all $v \in V(G)$. In particular, an $(f,f)$-factor is called an $f$-factor. In 1952, Tutte [21] established the famous $f$-Factor Theorem, which provides a necessary and sufficient condition for the existence of an $f$-factor in a graph. In 1970, Lovász [12] generalized the conclusion of Tutte’s $f$-Factor Theorem for $(g,f)$-factors in graphs.

**Theorem 1.** (Lovász [12]) A graph $G$ has a $(g,f)$-factor if and only if

$$f(D) - g(S) + d_{G-D}(S) - \hat{q}_G(D,S,g,f) \geq 0$$

for all disjoint sets $D, S \subseteq V$, where $\hat{q}_G(D,S,g,f)$ denotes the number of components $C$ of $G - (D \cup S)$ with $g(v) = f(v)$ for all $v \in V(C)$ and $e_G(V(C), S) + f(V(C)) \equiv 1 \mod 2$.

An alternative approach to $(g,f)$-factor is provided by the concept of fractional $(g,f)$-factor. Let $h : E(G) \rightarrow [0,1]$ be a function defined on $E(G)$ satisfying $g(v) \leq \sum_{e \in E_G(v)} h(e) \leq f(v)$ for all $v \in V(G)$. Setting $F_h = \{e : e \in E(G), h(e) > 0\}$. Then the subgraph of $G$ with vertex set $V(G)$ and edge set $F_h$, denoted by $G[F_h]$, is called a fractional $(g,f)$-factor of $G$ with indicator function $h$. In particular, a fractional $(f,f)$-factor is called a fractional $f$-factor. Anstee [1] gave a necessary and sufficient condition for the existence of a fractional $(g,f)$-factor in a graph, and Liu and Zhang [16] provided a new proof for Anstee’s result.

**Theorem 2.** (Anstee [1]; Liu and Zhang [16]) Let $G$ be a graph and $g, f : V(G) \rightarrow \mathbb{Z}^+$ be two integer functions such that $g(v) \leq f(v)$ for all $v \in V(G)$. Then $G$ has a fractional $(g,f)$-factor if and only if for any subset $S \subseteq V(G)$, we have

$$f(S) - g(T) + \sum_{v \in T} d_{G-S}(v) \geq 0,$$

where $T = \{v \mid v \in V(G) - S$ and $d_{G-S}(v) < g(v)\}$.

We say that $G$ has all fractional $(g,f)$-factors if it has a fractional $p$-factor for every function $p : V(G) \rightarrow \mathbb{Z}^+$ such that $g(v) \leq p(v) \leq f(v)$ for every $v \in V(G)$. Lu [14] provided a characterization for graphs having all fractional $(g,f)$-factors.

**Theorem 3.** (Lu [14]) Let $G$ be a graph and $g, f : V(G) \rightarrow \mathbb{Z}^+$ be two integer functions such that $g(x) \leq f(x)$ for all $x \in V(G)$. Then $G$ has all fractional $(g,f)$-factors if and only if for any subset $S \subseteq V(G)$, we have

$$g(S) - f(T) + \sum_{x \in T} d_{G-S}(x) \geq 0,$$

where $T = \{v \mid v \in V(G) - S$ and $d_{G-S}(v) < f(v)\}$.

Let $a < b$ be two positive integers. If $g \equiv a$ and $f \equiv b$, then we use the notion ‘fractional $[a,b]$-factors’ instead of ‘fractional $(g,f)$-factors’. By Theorem 3, we deduce the following result immediately.
Theorem 4. (Lu [14]) Let $G$ be a graph and $a < b$ be two positive integers. Then $G$ has all fractional $[a, b]$-factors if and only if for any subset $S \subseteq V(G)$, we have
\[ a|S| - b|T| + \sum_{x \in T} d_{G-S}(x) \geq 0, \tag{1} \]
where $T = \{ v | v \in V(G) - S \text{ and } d_{G-S}(v) < b \}$.

As an application of Theorem 4, Lu [14] obtained a sufficient condition for graphs having all fractional $[a, b]$-factors.

Theorem 5. (Lu [14]) Let $a < b$ be two positive integers. Let $G$ be a graph with order $n \geq \frac{(a+b)(a+b-1)}{a}$ and minimum degree $\delta_G \geq \frac{(a+b-1)^2 + 4b}{4a}$. If $|N_G(u) \cup N_G(v)| \geq \frac{bn}{a+b}$ for any two nonadjacent vertices $u$ and $v$ in $G$, then $G$ has all fractional $[a, b]$-factors.

Zhou and Sun [24] posed a new neighborhood union condition for the existence of all fractional $[a, b]$-factors in graphs.

Theorem 6. (Zhou and Sun [24]) Let $a, b, r$ be three integers with $1 \leq a \leq b$ and $r \geq 2$. Let $G$ be a graph of order $n$ with $n > \frac{(a+b)(r(a+b)-2)}{a}$. If $\delta_G \geq \frac{(r-1)b^2}{a}$ and $|N_G(x_1) \cup N_G(x_2) \cup \cdots \cup N_G(x_r)| \geq \frac{bn}{a+b}$ for any independent subset $\{x_1, x_2, \cdots, x_r\}$ in $G$, then $G$ admits all fractional $[a, b]$-factors.

Motivated by the work of Lu [14] and Zhou and Sun [24], it is natural to ask whether there are some spectral conditions for graphs having all fractional $[a, b]$-factors. In this paper, motivated by the work of Lu [14] and Zhou and Sun [24], we provide a tight spectral radius condition for a graph to have all fractional $[a, b]$-factors. For any integers $b$ and $n$ with $2 \leq b \leq n$, we denote $H_{n,b} := K_{b-1} \nabla (K_1 \cup K_{n-b})$. The main result of this paper is as below.

Theorem 7. Let $1 \leq a < b$ be integers, and let $G$ be a graph of order $n \geq 3 + \sqrt{8b^4 + 32b^3 + 4b^2 - 56b + 25}$. If $\rho(G) \geq \rho(H_{n,b})$, then $G$ has all fractional $[a, b]$-factors unless $G \cong H_{n,b}$.
2 Preliminaries

Let $M$ be a real $n \times n$ matrix, and let $\Pi = \{X_1, X_2, \ldots, X_k\}$ be a partition of $[n] = \{1, 2, \ldots, n\}$. Then the matrix $M$ can be written as

$$M = \begin{pmatrix}
M_{1,1} & M_{1,2} & \cdots & M_{1,k} \\
M_{2,1} & M_{2,2} & \cdots & M_{2,k} \\
\vdots & \vdots & \ddots & \vdots \\
M_{k,1} & M_{k,2} & \cdots & M_{k,k}
\end{pmatrix}.$$ 

The quotient matrix of $M$ with respect to $\Pi$ is the matrix $B_\Pi = (b_{i,j})_{i,j=1}^k$ with

$$b_{i,j} = \frac{1}{|X_i|}j^T_{[X_i]}M_{i,j}j_{|X_j|}$$

for all $i, j \in \{1, 2, \ldots, k\}$, where $j_k$ denotes the all ones vector in $\mathbb{R}^k$. If each block $M_{i,j}$ of $M$ has constant row sum $b_{i,j}$, then $\Pi$ is called an equitable partition, and the quotient matrix $B_\Pi$ is called an equitable quotient matrix of $M$. Also, if the eigenvalues of $M$ are all real, we denote them by $\lambda_1(M) \geq \lambda_2(M) \geq \cdots \geq \lambda_n(M)$.

**Lemma 1.** (Brouwer and Haemers [2, p. 30]; Godsil and Royle [6, pp.196–198]) Let $M$ be a real symmetric matrix, and let $B$ be an equitable quotient matrix of $M$. Then the eigenvalues of $B$ are also eigenvalues of $M$. Furthermore, if $M$ is nonnegative and irreducible, then

$$\lambda_1(M) = \lambda_1(B).$$

**Lemma 2.** (Hong [7]) Let $G$ be a connected graph with $n$ vertices and $m$ edges. Then

$$\rho(G) \leq \sqrt{2m - n + 1}.$$ 

**Lemma 3.** Let $b$ and $n$ be positive integers with $n \geq 3 + \sqrt{8b^4 + 32b^3 + 4b^2 - 56b + 25}$. Then

$$\max\{\rho(K_{4b} \nabla (K_2 \cup K_{n-4b-2})), \rho(K_{2b^2+4b-4} \nabla (K_2 \cup K_{n-2b^2-4b+2}))\} < n - 2.$$ 

**Proof.** Suppose $G_1 = K_{2b^2+4b-4} \nabla (K_2 \cup K_{n-2b^2-4b+2})$. Let $V_1 = V(K_2)$, $V_2 = V(K_{2b^2+4b-4})$ and $V_3 = V(K_{n-2b^2-4b+2})$. It is easy to check that the partition $\Pi : V(G_1) = V_1 \cup V_2 \cup V_3$ is an equitable partition of $G_1$, and the corresponding quotient matrix is

$$B_\Pi = \begin{pmatrix}
1 & 2b^2 + 4b - 4 & 0 \\
2 & 2b^2 + 4b - 5 & n - 2b^2 - 4b + 2 \\
0 & 2b^2 + 4b - 4 & n - 2b^2 - 4b + 1
\end{pmatrix}.$$ 

Let $f(x)$ denote the characteristic polynomial of $B_\Pi$. By simple calculation, we get

$$f(n - 2) = |(n-2)I - B_\Pi| = n^2 - 4n - 8b^4 - 32b^3 - 4b^2 + 56b - 21 > 0$$
because \( n \geq 3 + \sqrt{8b^4 + 32b^3 + 4b^2 - 56b + 25} \). We assert that \( \rho_1(B_{11}) < n - 2 \). If not, since \( f(n - 3) = 8(b^2 + 2b - 2)^2 < 0 \), we conclude that \( \rho_3(B_{11}) > n - 3 \), and hence \( \rho_1(B_{11}) + \rho_2(B_{11}) + \rho_3(B_{11}) > 3n - 9 \), contrary to \( \rho_1(B_{11}) + \rho_2(B_{11}) + \rho_3(B_{11}) = \text{trace}(B_{11}) = n - 3 \). For this reason, by Lemma 1,
\[
\rho(G_1) = \rho_1(B_{11}) < n - 2.
\]
Similarly, we can prove that \( \rho(K_{4b}\nabla(K_2 \cup K_{n-4b-2})) < n - 2 \) because \( n \geq 3 + \sqrt{32b^2 - 8b + 4} \) due to \( n \geq 3 + \sqrt{8b^4 + 32b^3 + 4b^2 - 56b + 25} \). This proves the lemma.

**Lemma 4.** Let \( 1 \leq a < b \) and \( n \) be positive integers. If \( n \geq b + 2 \), then \( H_{n,b} \) does not have all fractional \([a, b]\)-factors.

**Proof.** Recall that \( H_{n,b} = K_{b-1} \nabla (K_1 \cup K_{n-b}) \). Let \( V_1 = V(K_1) \), \( V_2 = V(K_{b-1}) \) and \( V_3 = V(K_{n-b}) \). In Theorem 4, take \( S = \emptyset \) and \( T = V_1 \). According to \( n \geq b + 2 \), we have \( T \) contains all vertices of degree at most \( b - 1 \) in \( H_{n,b} - S = H_{n,b} \). Furthermore, we have
\[
a|S| - b|T| + \sum_{x \in T} d_{G-S}(x) = -b + b - 1 = -1 \leq 0,
\]
contrary to (1). Therefore, by Theorem 4, we conclude that \( H_{n,b} \) does not have all fractional \([a, b]\)-factors.

\( \square \)

### 3 Proof of the main result

In this section, we shall give the proof of Theorem 7.

**Proof of Theorem 7.** Since \( n \geq \sqrt{8b^4 + 32b^3 + 4b^2 - 56b + 25} + 3 > b + 2 \), by Lemma 4, we see that \( H_{n,b} \) does not have all fractional \([a, b]\)-factors. Therefore, we always suppose that \( G \not\cong H_{n,b} \). By assumption, we have \( \rho(G) \geq \rho(H_{n,b}) > \rho(K_{n-1}) = n - 2 \) because \( K_{n-1} \) is a proper subgraph of \( H_{n,b} \). We first assert that \( G \) is connected.

If not, suppose that \( G_1, G_2, \ldots, G_m (m \geq 2) \) are the components of \( G \). Hence \( \rho(G) = \max\{\rho(G_1), \rho(G_2), \ldots, \rho(G_m)\} \leq \rho(K_{n-1}) = n - 2 \), which contradicts that \( \rho(G) \geq \rho(H_{n,b}) > \rho(K_{n-1}) = n - 2 \). Suppose that the result does not hold. By Theorem 4, there exists some subset \( S \subseteq V(G) \) such that
\[
a|S| - b|T| + \sum_{x \in T} d_{G-S}(x) \leq -1, \tag{2}
\]
where \( T = \{x : x \in V(G) \backslash S, d_{G-S}(x) < b\} \). Let \( s = |S| \) and \( t = |T| \). Note that \( |T| \neq \emptyset \), otherwise, \( a|S| - b|T| + \sum_{x \in T} d_{G-S}(x) = a|S| \geq 0 \) contradicting to (2). We shall divide the proof into the following two cases.
Case 1. \( t = 1 \).
In this situation, suppose \( T = \{ x_0 \} \). Then (2) gives that \( d_{G-S}(x_0) \leq b-as-1 \), hence that \( d_G(x_0) \leq b-as+s-1 = b+(1-a)s-1 \leq b-1 \) (since \( a \geq 1 \)), and finally that \( G \) is a spanning subgraph of \( H_{n,b} \). As \( G \) is connected and \( G \not\subset H_{n,b} \), we have \( \rho(G) < \rho(H_{n,b}) \), contrary to our assumption.

Case 2. \( t \geq 2 \).
In this situation, we claim that \( t \leq 2b+2 \). By contradiction, suppose that \( t \geq 2b+3 \). According to (2), we have \( \sum_{x \in T} d_{G-S}(x) \leq bt-as-1 \). Let \( T' = V(G) \setminus (S \cup T) \). Then

\[
\begin{align*}
e(G) &= e(S) + e(S,T) + e(S,T') + e(T) + e(T,T') + e(T') \\
&\leq \frac{s(s-1)}{2} + st + s(n-s-t) + \sum_{x \in T} d_{G-S}(x) + \frac{(n-s-t)}{2} (n-s-t-1) \\
&\leq \frac{s(s-1)}{2} + st + s(n-s-t) + (bt-as-1) + \frac{(n-s-t)}{2} (n-s-t-1) \\
&= \frac{(n-2)^2 - n(2t-3) + t^2 + t + 2t + 2st - 2as - 6}{2}.
\end{align*}
\]

Since \( n \geq s + t \) and \( t \geq 2b+3 \), by Lemma 2, we obtain

\[
\rho(G) \leq \sqrt{2e(G) - n + 1} \\
\leq \sqrt{(n-2)^2 - n(2t-2) + t^2 + t + 2t + 2st - 2as - 5} \\
\leq \sqrt{(n-2)^2 - [(2t-2)n - t^2 - (2s + 2b + 1)t + 2as + 5]} \\
\leq \sqrt{(n-2)^2 - [(2t-2)(s+t) - t^2 - (2s + 2b + 1)t + 2as + 5]} \\
\leq \sqrt{(n-2)^2 - [t^2 - (2b+3)t + 5]} \\
< n - 2 \\
< \rho(H_{n,b}),
\]

contrary to our assumption. Hence, \( t \leq 2b+2 \). Now we will deduce some contradictions according to the following two subcases.

Subcase 2.1. \( s < t \).
Let \( G_1 = K_{4b} \nabla (K_2 \cup K_{n-4b-2}) \). As \( |T| = t \geq 2 \), we can choose \( x_1, x_2 \in T \) with \( x_2 \neq x_1 \). Recall that \( |S| = s < t \leq 2b+2 \). Then we have \( |(N_G(x_1) \setminus \{x_2\}) \cup (N_G(x_2) \setminus \{x_1\})| \leq |S| + |(N_{G-S}(x_1) \setminus \{x_2\}) \cup (N_{G-S}(x_2) \setminus \{x_1\})| \leq (2b+2) + 2(b-1) = 4b \), and hence \( G \) is a spanning subgraph of \( G_1 \). Combining this with Lemma 3, we obtain \( \rho(G) \leq \rho(G_1) < n-2 < \rho(H_{n,b}) \), contrary to our assumption.

Subcase 2.2. \( s \geq t \).
Let \( G_2 = K_{2b^2+4b+4} \nabla (K_2 \cup K_{n-2b^2-4b+2}) \). In fact, by definition, every vertex in \( T \) has degree at most \( b-1 \) in \( G-S \). Furthermore, we assert that there exists some vertex \( x_1 \in T \) such that \( d_{G-S}(x_1) \leq b-2 \), since otherwise we can deduce from (2)
that \( as - t \leq -1 \), which is impossible by \( s \geq t \) and the fact that \( a \geq 1 \). In this situation, if \( as \geq bt \), from (2), we can see that this is impossible. If \( as < bt \), recall that \( 2 \leq t \leq 2b + 2 \), then we have \( t \leq s < \frac{bt}{a} \leq \frac{2b^2 + 2b}{a} \). By using the same analysis as above, we deduce that \( G \) is a spanning subgraph of \( G_2 \). Combining this with Lemma 3, we obtain \( \rho(G) \leq \rho(G_2) < n - 2 < \rho(H_{n,b}) \), contrary to our assumption. Therefore, this completes the proof.

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