Quantum walks on cycles

Małgorzata Bednarska,1 Andrzej Grudka,2 Paweł Kurzyński,2 Tomasz Łuczak,1 and Antoni Wójcik2

1Faculty of Mathematics and Computer Science, Adam Mickiewicz University, Umultowska 87, 61-614 Poznań, Poland.
2Faculty of Physics, Adam Mickiewicz University, Umultowska 85, 61-614 Poznań, Poland.

Abstract

We consider asymptotic behaviour of a Hadamard walk on a cycle. For a walk which starts with a state in which all the probability is concentrated on one node, we find the explicit formula for the limiting distribution and discuss its asymptotic behaviour when the length of the cycle tends to infinity. We also demonstrate that for a carefully chosen initial state, the limiting distribution of a quantum walk on cycle can lie further away from the uniform distribution than its initial state.

PACS numbers: 03.67.Lx
I. INTRODUCTION

Celebrated results of Shor 1 and Grover 2 started a quest for algorithms based on quantum mechanics which can surpass corresponding classical algorithms. As many classical algorithms employ properties of random walks on graphs, several groups of researchers have begun to study properties of quantum analogues of classical random walks (for a survey see 3). Two models of quantum random walks have been proposed by Aharonov et al. 4 and Farhi and Gutmann 5; in the paper we consider only a discrete quantum walk as defined in 4. The behavior of quantum random walks have been shown to differ greatly from their classical counterparts. Ambainis et al. 6 proved that the spreading time in the quantum walk on line scales linearly with the number of steps, while Aharonov et al. 4 showed that the mixing time for a walk on a cycle grows linearly with the cycle length. Even more spectacular exponential speed up was discovered by Kempe et al. 7 who studied a quantum random walk on hypercube; this result led to the construction of the first quantum algorithm based on a random walk by Shenvi et al. 8. The fact that in quantum walks on graphs the probability function spreads out much faster than in classical case is not the only factor which can be explored in designing new quantum algorithms. In 4 the authors remarked that ‘one may try to use quantum walks which converge to limiting distribution which are different than those of the corresponding classical walks’. In this note we follow this suggestion and study the limiting distribution of a random walk on a cycle; it turns out that it depends on the length of the cycle in a somewhat surprising way. We also give an example of a quantum walk in which the distance from the initial state to the uniform distribution is larger than the distance between the uniform distribution and the initial state. Finally, we remark that recently Travaglione and Milburn 9 and Dür et al. 10 have proposed a scheme of experiment which realizes a quantum walk on a cycle.

II. MODEL

We study a quantum random walk on a cycle with \(d\) nodes. In the model of such a walk proposed in 4 nodes are represented by vectors \(|v\rangle, v = 0, 1, \ldots, d - 1\), which form an orthonormal basis of the Hilbert space \(H_V\). An auxiliary two-dimensional Hilbert space \(H_A\) (coin space) is spanned by vectors \(|s\rangle, s = 0, 1\). The initial state of the walk is a normalized
vector
\[ |\Psi_0\rangle = \sum_{s,v} \alpha_{sv} |s,v\rangle = \sum_{s,v} \alpha_{sv} |s\rangle |v\rangle \]  \tag{1}
from the tensor product \( H = H_A \otimes H_V \). In a single step of the walk the state changes according to the equation
\[ |\Psi_{n+1}\rangle = U |\Psi_n\rangle, \]  \tag{2}
where the operation \( U = S(H \otimes I) \) first applies the Hadamard gate operator \( H = \sum_{s,s'} (-1)^{ss'} |s\rangle \langle s'| \) to the vector from \( H_A \), and then shifts the state by the operator
\[ S = \sum_{s,v} |s\rangle \langle s| \otimes |v + 2s - 1(\text{mod } d)\rangle \langle v|. \]  \tag{3}

The probability distribution on the nodes of the cycle after the first \( n \) steps of the walk is given by
\[ p_n(v) = \sum_s |\langle s,v|\Psi_n\rangle|^2. \]  \tag{4}
However, as was observed by Aharonov at el. [4], for a fixed \( v \), the probability \( p_n(v) \) is ‘quasi-periodic’ as a function of \( n \) and thus, typically, it does not converge to a limit. Thus, instead of \( p_n(v) \) the authors of [4] considered
\[ \tilde{p}_n(v) = \frac{1}{n} \sum_{i=1}^{n} p_n(v), \]  \tag{5}
and proved that for any initial state \( |\Psi_0\rangle \) and every node \( v = 0, 1, \ldots, d - 1 \), the sequence \( \tilde{p}_n(v) \) converges to the limiting distribution
\[ \pi(v) = \sum_{a,a'} \sum_s \Gamma_{aa'} |\langle \phi_a|\Psi_0\rangle \langle \Psi_0|\phi_{a'}\rangle \langle s,v|\phi_a\rangle \langle \phi_{a'}|s,v\rangle|, \]  \tag{6}
where \( c_a \) are the eigenvalues of \( U \), \( |\phi_a\rangle \) stand for the eigenvectors of \( U \), and
\[ \Gamma_{aa'} = \begin{cases} 1 & \text{if } c_a = c_{a'} \\ 0 & \text{if } c_a \neq c_{a'} \end{cases} \]  \tag{7}

In the case of the Hadamard walk on cycle, for \( j = 0, 1 \) and \( k = 0, 1, \ldots, d - 1 \), we get
\[ c_{jk} = \frac{1}{\sqrt{2}} ((-1)^k \sqrt{1 + \cos^2(2\pi j/d)} - i \sin(2\pi j/d)), \]  \tag{8}
and
\[ |\phi_{jk}\rangle = (a_{jk}|0\rangle + a_{jk}b_{jk}|1\rangle) \otimes \sum_v \omega_{av}^j |v\rangle, \]  \tag{9}
where $\omega_d = e^{2\pi i / d}$,

$$a_{jk} = 1 / \sqrt{d(1 + |b_{jk}|^2)},$$

$$b_{jk} = \omega_d^j((-1)^k \sqrt{1 + \cos^2(2\pi j/d)} - \cos(2\pi j/d)).$$

Note that $c_{j0} \neq c_{j'1}$ so $\Gamma_{jk,j'k'} = \gamma_{jj'} \delta_{kk'}$, where $\delta_{kk'}$ is the Kronecker delta. Thus, (6) becomes

$$\pi(v) = \sum_{j,j'} \sum_k \gamma_{jj'} |\langle \phi_{jk} | \Psi_0 \rangle |^2 \omega^{v(j-j')},$$

where $A_{jj'k} = a_{jk} a_{j'k}^*(1 + b_{jk} b_{j'k}^*)$. If $d$ is odd, then all eigenvalues are distinct, $\gamma_{jj'} = \delta_{jj'}$, and $A_{jjk} = 1/d$; consequently, $\pi(v) = (1/d) \sum_j \sum_k |\langle \phi_{jk} | \Psi_0 \rangle|^2 = 1/d$. It comes as no surprise, since as was proved in Aharonov et al. [4], the limiting distribution $\pi$ is always uniform in a non-degenerate case. Thus we concentrate on more interesting case of even $d$. Then, because of symmetries $c_{d/2-j,k} = c_{j,k}$ and $c_{d/2+j,k} = c_{j-k}^* = c_{d-j,k}$, the coefficient $\gamma_{jj'}$ does not vanish when one of the following conditions hold:

(i) $j = j'$;

(ii) $j = 0$, $j' = d/2$;

(iii) $j = t$, $j' = d/2 - t$, for $t = 1, 2, \ldots, t_{\text{max}}$;

(iv) $j = d - t$, $j' = d/2 + t$, for $t = 1, 2, \ldots, t_{\text{max}}$;

where here and below $t_{\text{max}} = \lfloor (d - 2)/4 \rfloor$.

**III. RESULT AND DISCUSSION**

We study in detail the limiting distribution $\pi$ for a quantum walk for which starts with a state in which with probability one a particle is at a node $v_0$; more specifically we set

$$|\Psi_0 \rangle = \frac{1}{\sqrt{2}} (|0, v_0 \rangle + i|1, v_0 \rangle).$$

Then $\langle \phi_{jk} | \Psi_0 \rangle = g_{jk} \omega_d^{-v_0 j}$, where $g_{jk} = a_{jk} (1 + \text{sign}(b_{jk})) / \sqrt{2}$, and

$$\pi(v) = \sum_{j,j'} \sum_k \gamma_{jj'} f(j,j',k) \omega_d^{(v-v_0)(j-j')},$$

where $\text{sign}(b_{jk}) = 1 + b_{jk} b_{j'k}^*$. If $d$ is odd, then all eigenvalues are distinct, $\gamma_{jj'} = \delta_{jj'}$, and $A_{jjk} = 1/d$; consequently, $\pi(v) = (1/d) \sum_j \sum_k |\langle \phi_{jk} | \Psi_0 \rangle|^2 = 1/d$. It comes as no surprise, since as was proved in Aharonov et al. [4], the limiting distribution $\pi$ is always uniform in a non-degenerate case. Thus we concentrate on more interesting case of even $d$. Then, because of symmetries $c_{d/2-j,k} = c_{j,k}$ and $c_{d/2+j,k} = c_{j-k}^* = c_{d-j,k}$, the coefficient $\gamma_{jj'}$ does not vanish when one of the following conditions hold:

(i) $j = j'$;

(ii) $j = 0$, $j' = d/2$;

(iii) $j = t$, $j' = d/2 - t$, for $t = 1, 2, \ldots, t_{\text{max}}$;

(iv) $j = d - t$, $j' = d/2 + t$, for $t = 1, 2, \ldots, t_{\text{max}}$;

where here and below $t_{\text{max}} = \lfloor (d - 2)/4 \rfloor$.
with \( f(j, j', k) = g_{jk}g_{j'k}^* A_{jj'k} \). Since \( \gamma_{jj'} = 0 \) except of the four cases (i)–(iv) described above, we get

\[
\pi(v) = F + (-1)^\Delta 2\text{Re}(F_0) + (-1)^\Delta \text{Re}\left( \sum_{i=1}^{t_{\text{max}}} F_i \right),
\]

(15)

where by

\[
\Delta = \Delta(v) = \min\{|v - v_0|, d - |v - v_0|\}
\]

denotes the distance between nodes \( v_0 \) and \( v \), and

\[
F = \sum_{j,k} f(j, j, k),
\]

\[
F_0 = \sum_k f(0, d/2, k),
\]

\[
F_t = \sum_k f(t, d/2 - t, k) + f(d/2 + t, d - t, k).
\]

(17)

After some elementary but not very exciting calculations (17) reduces to

\[
F = \frac{1}{d},
\]

\[
F_0 = 0,
\]

\[
F_t = \frac{2}{d^2} \frac{\sin^2(2\pi t/d)}{1 + \cos^2(2\pi t/d)}.
\]

(18)

We remark that \( F_t = 0 \) whenever \( t_{\text{max}} = 0 \), i.e., for \( d = 2 \) and \( d = 4 \). Hence the limiting distribution for even cycles of sizes two and four are uniform, which has also been observed by Travaglione and Milburn [9], who analyzed a quantum walk on cycle of length four step by step. However, for \( d \geq 6 \) we have

\[
\pi(v) = (1 + \Pi(v))/d,
\]

(19)

with the ‘correction’ term \( \Pi(v) = (-1)^\Delta 4S/d \), where

\[
S = \sum_{t=1}^{t_{\text{max}}} \cos(4\pi t\Delta/d) \left( \frac{2}{1 + \cos^2(2\pi t/d)} - 1 \right).
\]

(20)

Setting \( z = 3 - 2\sqrt{2} \sim 0.17157 \ldots \), one can write (20) as

\[
S = \left( \frac{8z}{1 - z^2} - 1 \right) \sum_{t=1}^{t_{\text{max}}} \cos(4\pi t\Delta/d) + \frac{8z}{1 - z^2} \sum_{m=1}^{\infty} (-z)^m \sum_{t=1}^{t_{\text{max}}} \cos(4\pi(\Delta + m)t/d)
\]

\[
+ \frac{8z}{1 - z^2} \sum_{m=1}^{\infty} (-z)^m \sum_{t=1}^{t_{\text{max}}} \cos(4\pi(\Delta - m)t/d),
\]

(21)
FIG. 1: The limiting distributions for the cases of $d = 24$ and $d = 26$. The walk starts with the state where with probability one a particle is at the node $v_0 = 5$.

which, in turn, transforms to

$$
S = -\frac{d}{4}(\delta_{\Delta,0} + \delta_{\Delta,d/2}) + (-1)^\Delta \frac{d}{4} \frac{8z}{1 - z^2} \frac{z^{-\Delta}(-z)^{d/2} + z^\Delta}{1 - (-z)^{d/2}}
$$

$$+ \frac{1 + (-1)^\Delta \xi}{2} - \frac{4z}{(1 - z)^2} - \frac{4z(-1)^\Delta \xi}{(1 + z)^2},
$$

(22)

where $\xi = (1 + (-1)^{d/2})/2$, i.e., $\xi = 1$ when $d/2$ is even, and $\xi = 0$ if $d/2$ is odd. Thus, we arrive at

$$
\Pi(v) = (-1)^{\Delta + 1}(\delta_{\Delta,0} + \delta_{\Delta,d/2}) + \frac{8z}{1 - z^2} \frac{z^{-\Delta}(-z)^{d/2} + z^\Delta}{1 - (-z)^{d/2}}
$$

$$+ \frac{4}{d} \left(\frac{\xi + (-1)^\Delta}{2} - \frac{4(-1)^\Delta z}{(1 - z)^2} - \frac{4z\xi}{(1 + z)^2}\right).
$$

(23)

In Figure 1 we pictured the resulting limiting distributions for $d = 24$ and $d = 26$. It is easy to see they are almost uniform except for the nodes which lie next to the initially populated node $v_0$ and the opposite node $\hat{v}_0 = v_0 + d/2$ (mod $d$). Indeed, as $d \to \infty$, the last term of (23) vanishes and

$$
\bar{\Pi}(v) = \lim_{d \to \infty} \Pi(v) = \eta(\Delta) - (-1)^\xi \eta(\Delta'),
$$

(24)
FIG. 2: The limiting distribution for cycles of lengths 2000, 2001 and 2002. In each case the walk starts with a state in which with probability one a particle is at \( v_0 = 0 \).

where

\[
\eta(x) = \frac{8z^{1+x}}{1 - z^2} - \delta_{x0},
\]

and \( \Delta' \) is the distance between nodes \( v \) and \( \hat{v}_0 \). The equation (24) shows that the correction term \( \Pi'(v) \) is significant only for \( v \) which lie close to either \( v_0 \) or \( \hat{v}_0 \) and decreases exponentially with the distance between \( v \) and \( v_0 \) and \( \hat{v}_0 \). Note that the shapes of the cusps near \( v_0 \) and \( \hat{v}_0 \) does not depend very much on \( d \) for large \( d \) (except of the scaling factor \( 1/d \)). However, if \( d/2 \) is odd then the limiting distribution \( \pi(v) \) has a minimum at \( \hat{v}_0 \), while if \( d/2 \) is even the distribution has a peak \( \hat{v}_0 \), virtually identical with that which appear at \( v_0 \) (see Figures 1 and 2). The fact that such a local behaviour of the limiting distribution \( \pi \) depends so strongly on ‘global’ properties of space, as the parity of \( d/2 \), is somewhat surprising. We hope that this and/or analogous phenomena can be used in constructing efficient quantum algorithms.

It is easy to see that, despite of a narrow cusps near nodes \( v_0 \) and \( \hat{v}_0 \), the total variation distance \( d_{TV} \) between the limiting distribution given by (19) and (23) and the uniform distribution tends to 0 as \( d \to \infty \). However it is not hard as well to construct ‘highly non-uniform’ distributions which remain invariant during the walk: one can simply take as the initial state a superposition of two degenerated eigenvectors (the distribution corresponding to a single eigenvector is always a uniform one). An example of such an invariant nonuniform probability distribution is presented on Figure 3.

With a slightly more work one can construct the initial states which indicate differences between quantum and classical walks on cycles even more distinctively. One such example is shown on Figure 4. Here the total variation distance from the uniform distribution, defined
FIG. 3: An example of a non-uniform invariant probability distribution for an initial state of the form $|\Psi_0\rangle = \frac{1}{\sqrt{2}}(|\phi_{5,0}\rangle + |\phi_{7,0}\rangle)$.

FIG. 4: The initial and limiting distributions for the initial state $|\Psi_0\rangle = \frac{1}{2}(|\phi_{3,0}\rangle + |\phi_{9,0}\rangle - |\phi_{15,0}\rangle - |\phi_{21,0}\rangle)$.

as

$$\frac{1}{2} d - \frac{1}{2} \sum_{v=0}^{d-1} \left| p(v) - \frac{1}{d} \right|$$

is equal 0.046 for the initial state, while for the limiting distribution it grows to 0.204.
IV. CONCLUSION

We study the properties of a Hadamard quantum walk on a cycle with \( d \) nodes. In the case of a walk starting from a single node we give an explicit formula for the limiting distribution and show that it is very sensitive to the arithmetic properties \( d \). We hope that this or an analogous mode of behaviour can be used in construction of efficient quantum algorithms.

Furthermore, we present an example of a quantum walk on a cycle, for which the total variation distance between the initial distribution and the uniform distribution is much smaller than the distance between limiting distribution and the uniform one.

Acknowledgments

We wish to thank the State Committee for Scientific Research (KBN) for its support: M.B. and T.L. were supported by grant 2 P03A 016 23; A.G. and A.W. by grant 0 T00A 003 23.

[1] P. Shor, Proceedings of the 35th Annual Symposium on Foundations of Computer Science (IEEE Computer Society Press, Los Alamitos, CA 1994) 124 (1994).
[2] L. Grover, Phys. Rev. Lett. 79, 325 (1977).
[3] J. Kempe, quant-ph/0303081.
[4] D. Aharonov, A. Ambainis, J. Kempe, and U. Vazirani, Proceedings of the 30th Annual ACM Symposium on Theory of Computation (ACM Press, New York, 2001) 50 (2001).
[5] E. Farhi, S. Gutmann, Phys. Rev. A 58, 915 (1998).
[6] A. Ambainis, E. Bach, A. Nayak, A. Vishwanath, and J. Watrous, Proceedings of the ACM Symposium on Theory of Computation (ACM Press, New York, 2001) 37 (2001).
[7] J. Kempe, quant-ph/0205083.
[8] N. Shenvi, J. Kempe, and K. Whaley, quant-ph/0210064.
[9] B. Travaglione and G. Milburn, Phys. Rev. A 65, 032310 (2002).
[10] W. Dürr, R. Raussendorf, V.M. Kendon, and H.-J. Briegel, Phys. Rev. A 66, 052319 (2002).