General Formula for the Momentum Imparted to Test Particles in Arbitrary Spacetimes

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Abstract

Ehlers and Kundt have provided an approximate procedure to demonstrate that gravitational waves impart momentum to test particles. This was extended to cylindrical gravitational waves by Weber and Wheeler. Here a general, exact, formula for the momentum imparted to test particles in arbitrary spacetimes is presented.

1 Introduction

There has been debate whether gravitational waves really exist [1,2]. To demonstrate the "reality" of gravitational waves, Ehlers and Kundt [1] considered a sphere of test particles in the path of a plane-fronted gravitational wave and showed that a constant momentum was imparted to the test particles. A similar discussion was given by Weber and Wheeler [2,3] for cylindrical gravitational waves. An operational procedure embodying the same principle has been used [4] to express the consequences of relativity in terms of the Newtonian concept of gravitational force. The pseudo-Newtonian (ψN) "gravitational force" is defined as the vector whose intrinsic derivative along the separation vector is the maximum tidal force, which is given by the acceleration vector for a preferred class of observers. The proper time integral of the force four-vector will be the momentum four-vector. We will not, here, discuss this formalism itself. Rather, we verify that this procedure gives
the Ehlers-Kundt result for plane-fronted gravitational waves, as may be expected on account of the similarity of the two ideas. When we apply it to cylindrical gravitational waves it is found that the result so obtained is physically reasonable and gives an exact expression for the momentum imparted to the rest particle, corresponding to the approximation given by Weber and Wheeler.

In the free fall rest-frame the extended $\psi N (e\psi N)$ force four-vector is given \[5\] by

$$F_0 = m[(\ln A)f,0 + g^{ik}g_{jk,0}g^{jl}g_{il,0}/4A], \quad F_i = m(\ln \sqrt{g_{00}}),$$

(1)

where $A = (\ln \sqrt{-g}),0$ and $g = det(g_{ij})$. Thus the momentum four-vector, $p_\mu$, is

$$p_\mu = \int F_\mu dt.$$  

(2)

2 Plane-Fronted Waves

The metric for plane-fronted gravitational waves is \[6,7\]

$$ds^2 = dt^2 - dx^2 - L^2(t, x)\{\exp[2\beta(t, x)]dy^2 + \exp[-2\beta(t, x)]dz^2\},$$

(3)

where $L$ and $\beta$ are arbitrary functions subject to the vacuum Einstein equations,

$$L_{,\alpha\alpha} + L_{,\beta}^2 = 0, \quad \alpha = 0, 1.$$  

(4)

Since $L$ and $\beta$ are functions of $u = t - x$, eqs.(4) reduce to the single equation

$$L_{uu} + L_{,\beta}^2 = 0$$

(5)

From eqs.(1) and (5)

$$F_0 = -m(\ddot{L} + \beta^2 L)/\dot{L} = 0, \quad F_1 = 0,$$

(6)

where a dot denotes differentiation with respect to $t$. Consequently the momentum four-vector becomes $p_\mu = constant$. Thus there is a constant energy and momentum. The constant, here, determines the "strength" of the wave. This exactly coincides with the Ehlers-Kundt method in which they demonstrate that the test particles acquire a constant momentum and hence a constant energy, from the plane gravitational wave.
3 Cylindrical Gravitational Waves

The metric for cylinder gravitational waves is [2,3]

\[ ds^2 = \exp[2(\gamma - \psi)]dt^2 - \exp[2(\gamma - \psi)]d\rho^2 - \rho^2 \exp(-2\psi)d\phi^2 - \exp(2\psi)dz^2, \] (7)

where \( \gamma \) and \( \psi \) are arbitrary functions of the time and radial coordinates, \( t \) and \( \rho \), subject to the vacuum Einstein equations,

\[ \psi'' + \frac{1}{\rho} \psi' - \ddot{\psi} = 0, \quad \gamma' = \rho(\rho^2 + \dot{\psi}^2), \quad \dot{\gamma} = 2\rho\dot{\psi}\psi'. \] (8)

where a dot denotes differentiation with respect to time and a prime differentiation with respect to \( \rho \).

The solution of eqs.(8) is given by [3]

\[ \psi = AJ_0(\omega \rho) \cos(\omega t) + BY_0(\omega \rho) \sin(\omega t), \]
\[ \gamma = \frac{1}{2}\omega \rho\{(A^2J_0J_0' - B^2Y_0Y_0') \cos(2\omega t) - AB[(J_0Y_0' + Y_0J_0') \sin(2\omega t)
- 2(J_0Y_0' - Y_0J_0')\omega t]\}, \] (9)

where \( A \) and \( B \) are arbitrary constants corresponding to the strength of the cylindrical gravitational waves, \( J_0 \) and \( Y_0 \) are the Bessel function and the Neumann function of zero order respectively. Here a prime denotes differentiation with respect to \( \omega \rho \), \( \omega \) being the angular frequency.

From eqs.(1)

\[ F_0 = -m\{\omega[AJ_0 \cos(\omega t) + BY_0 \sin(\omega t)] - 2\rho\omega[(A^2J_0J_0' - B^2Y_0Y_0') \cos(2\omega t)
- AB(J_0Y_0' + Y_0J_0') \sin(2\omega t)] + 2(AJ_0 \sin(\omega t)
- BY_0 \cos(\omega t))^2/AJ_0 \sin(\omega t) - BY_0 \cos(\omega t)
+ 2\omega \rho[AJ_0 \sin(\omega t) - BY_0 \cos(\omega t)](AJ_0 \cos(\omega t) + BY_0' \sin(\omega t)]\}; \] (10)

\[ F_1 = m\{AJ_0 \cos(\omega t) + BY_0' \sin(\omega t) - \frac{1}{2}[(A^2J_0J_0' - B^2Y_0Y_0')\omega t)
+ \omega \rho(A^2J_0J_0' - B^2Y_0Y_0') \cos(2\omega t),
- \frac{1}{2}AB[2(J_0Y_0' + Y_0J_0') - \omega \rho(J_0Y_0' + Y_0J_0') \sin(2\omega t)
- \frac{1}{2}AB[4(J_0Y_0' - Y_0J_0') + 2\omega \rho(J_0Y_0' - Y_0J_0')]\omega t\}. \] (11)
The corresponding $p_0$ and momentum, $p_1$, imparted to the test particle is

\[
p_0 = -m[\ln AJ_0 \sin(\omega t) - BY_0 \cos(\omega t)] + \ln |1 - 2\omega \rho [AJ_0' \cos(\omega t)]
+ BY_0' \sin(\omega t)](1 + \frac{A^2 J_0 J_0' + B^2 Y_0 Y_0'}{\omega \rho (A J_0^2 + B Y_0^2)})
- \frac{AB(AJ_0 Y_0' + Y_0 J_0' - A \omega \rho J_0 J_0' Y_0' - J_0 Y_0')}{\omega \rho (A J_0^2 + B Y_0^2) \sqrt{1 - 4\omega^2 \rho^2(A^2 J_0^2 + B^2 Y_0^2)}}
\times \tan^{-1}\left[\frac{(1 + 2A \omega \rho J_0') \tan(\frac{1}{2} \omega t) - 2B \omega \rho Y_0'}{\sqrt{1 - 4\omega^2 \rho^2(A^2 J_0^2 + B^2 Y_0^2)}}\right]
+ \frac{AB(J_0 Y_0' - J_0 Y_0')}{\rho (A^2 J_0^2 + B^2 Y_0^2)} + f_1(\omega \rho), \tag{12}
\]

\[
p_1 = \frac{m}{\omega} \{[AJ_0' \sin(\omega t) - BY_0' \cos(\omega t)] - \frac{1}{4}[A^2 J_0 J_0' - B^2 Y_0 Y_0']
+ \rho \omega (A^2 J_0 J_0' - B^2 Y_0 Y_0')] \sin(2\omega t) - \frac{1}{2} AB[J_0 Y_0' + Y_0 J_0']
+ \rho \omega (J_0 Y_0' + Y_0 J_0') \cos(2\omega t) - AB \omega^2 t^2[(J_0 Y_0' - Y_0 J_0')
- \rho \omega (J_0 Y_0' - Y_0 J_0')] + f_2(\rho \omega)\}. \tag{13}
\]

where $f_1$ and $f_2$ are arbitrary constants of integration. Weber and Wheeler [2,3] exclude solutions that contain the irregular Bessel function, $Y_0(\omega \rho)$ as not well defined at the origin. Taking the Weber-Wheeler solution, eqs.(12) and (13) reduce to

\[
p_0 = -m[\ln |AJ_0 \sin(\omega t)| + (1 + AK_0/\omega \rho J_0') \ln |1 - 2\omega \rho AJ_0' \cos(\omega t)|
+ f_1(\omega \rho)], \tag{14}
\]

\[
p_1 = \frac{m}{\omega} \{AJ_0' \sin(\omega t) - \frac{1}{4} A^2 [J_0 J_0' + \rho \omega (J_0 J_0')'] \sin(2\omega t) + f_2(\rho \omega)\}. \tag{15}
\]

We see that the quantity $p_1$ given by eq.(15) can be made zero for the small and large $\rho$ limits by choosing $f_2 = 0$. This expression coincides with the approximate momentum expression given by Weber and Wheeler in the large and small $\rho$ limits. Hence this is a physically reasonable expression for the momentum imparted to test particles by cylindrical gravitational waves. The quantity $p_0$ given by eq.(14) remains finite for small $\rho$ and can also be made finite for large $\rho$ by choosing $f_1 = -\ln(J_0)$. However, there is a singularity at $\omega t = n\pi$. This problem does not arise in the general expression given by eq.(12). However, in that case there appears a term linear in time which
creates interpretational problems. Also \( p_0 \) and \( p_1 \) become singular at \( \rho = 0 \) if \( B \neq 0 \).

4 Conclusion

The Ehlers-Kundt method gives the physically reasonable result that plane gravitational waves impart a constant energy and momentum to test particles in their path. However, it does not provide a simple formula that can be applied to other cases. Since the essential idea there is exactly embodied in the \( e\psi N \)-formalism [5] it is reasonable to expect that it would give the same result. We have seen that the \( e\psi N \)-formalism gives a physically acceptable expression for the energy and momentum imparted to test particles by gravitational waves which coincides with the Ehlers-Kundt approach for the plane-fronted waves. The expression for momentum coincides with the Weber-Wheeler [2,3] approximate result for cylindrical waves for small and large values of \( \rho \). In effect it provides a general formula for the momentum imparted to test particles in arbitrary spacetimes.

The inclusion of \( Y_0 \) in the solution increases the energy at the source and thus provides a singularity at \( \rho = 0 \). If we do not include the \( Y_0 \) term in the solution, \( p_0 \) becomes infinite at various times. One would have normally taken \( p_0 \) to be the energy of the test particle. However, that should be given by \((m^2 + p_1^2)^{1/2}\). As such the significance of \( p_0 \) is not clear. As this is the quantity which has the problem of becoming singular, while \( p_1 \) is well behaved we are ignoring the problem. However, that problem does need to be resolved and an interpretation for \( p_0 \) be provided.

There is a problem of defining energy in general relativity which is most severe when trying to understand the energy content of gravitational waves. As gravitational waves are solutions of the vacuum field equations the stress-energy tensor is zero. This is the problem regarding the reality of gravitational waves. On the one hand it seems that gravitational waves do not carry energy but on the other hand detectors are built to extract energy from them. If we linearize the Einstein field equations [3,6], the complete non-linear equations can be rewritten with the linear part on the left side and all the non-linear terms on the right side. The non-linear part can then be regarded as the stress-energy tensor for the linearized wave equation [6]. Also, a \( 3+1 \) split, such as that of Arnowitt, Deser and Misner [8], provides an expression for the energy density in gravitational waves. We are only provid-
ing an expression for the energy imparted to a test particle. This expression is based on an operational $3+1$ split. Hence this procedure might be able to provide a handle to the problem of the energy content in gravitational waves in particular and the definition of energy in general. To be able to apply it successfully the problem of $p_0$ would have to be resolved.

**Acknowledgement**

We would like to thank the Pakistan Science Foundation for the financial support during this research under the project PSF/RES/C-QU/MATHS.(16).

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