On the Dynamics of solitons in the nonlinear Schroedinger equation

Vieri Benci∗ Marco Ghimenti† Anna Maria Micheletti†

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Abstract

We study the behavior of the soliton solutions of the equation

\[ i\frac{\partial \psi}{\partial t} = -\frac{1}{2m}\Delta \psi + \frac{1}{2}W'_\varepsilon(\psi) + V(x)\psi \]

where \( W'_\varepsilon \) is a suitable nonlinear term which is singular for \( \varepsilon = 0 \). We use the “strong” nonlinearity to obtain results on existence, shape, stability and dynamics of the soliton. The main result of this paper (Theorem 1) shows that for \( \varepsilon \to 0 \) the orbit of our soliton approaches the orbit of a classical particle in a potential \( V(x) \).

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∗Dipartimento di Matematica Applicata, Università degli Studi di Pisa, Via F. Buonarroti 1/c, Pisa, ITALY and Department of Mathematics, College of Science, King Saud University, Riyadh, 11451, SAUDI ARABIA. e-mail: benci@dma.unipi.it

†Dipartimento di Matematica Applicata, Università degli Studi di Pisa, Via F. Buonarroti 1/c, Pisa, ITALY. e-mail: marco.ghimenti@dma.unipi.it, a.micheletti@dma.unipi.it
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1 Introduction

Roughly speaking a solitary wave is a solution of a field equation whose energy travels as a localized packet and which preserves this localization in time. A soliton is a solitary wave which exhibits some strong form of stability so that it has a particle-like behavior. In this paper we study the dynamics of solitons arising in the nonlinear Schroedinger equation (NSE):

\[
i \frac{\partial \psi}{\partial t} = -\frac{1}{2m} \Delta \psi + \frac{1}{2} W'_\varepsilon(\psi) + V(x)\psi.
\]  

A solution of our equation can be written as follows

\[
\psi(t, x) = \Psi_\varepsilon(t, x) + \varphi(t, x)
\]  

where \(\varphi(t, x)\) can be considered as a wave and \(\Psi_\varepsilon(t, x)\) is our soliton: a bump of energy concentrated in a ball centered at the point \(q = q_\varepsilon(t)\) with radius \(R_\varepsilon \to 0\) (for \(\varepsilon \to 0\)). Considering this decomposition, the solutions to our equation can be thought as a combination of a wave and a particle. \(\varepsilon\) occurs in the equation as a parameter. The main purpose of this paper is to show that for \(\varepsilon\) and \(\varphi(0, x)\) sufficiently small, our soliton behaves as a classical particle in a potential \(V(x)\). More exactly, we prove that the decomposition (2) holds for all times and the bump follows a dynamics which approaches the dynamics of a pointwise particle moving under the action of the potential \(V\) (Theorem 1); in particular the position \(q_\varepsilon(t)\) of the soliton approaches the position of the particle uniformly on bounded time intervals (Corollary 2).

The attention of the mathematical community on the dynamics of soliton of NSE began with the pioneering paper of Bronski and Jerrard [11]; then Fröhlich, Gustafson, Jonsson, and Sigal faced this problem using a different approach [15]. In the last years, several others works appeared following the first approach ([24], [25], [29], [30], [31]) or the second one ([11], [2], [3], [4], [16], [22], [28]).
In this paper, we have studied equation (1) which gives a different problem with respect to the ones mentioned above. Actually, in equation (1), unlike the other papers, the parameter $\varepsilon$ appears in the nonlinear term and it will be chosen in such a way that $|\psi|^2$ approaches the delta-measure as $\varepsilon \to 0$. Thus equation (1) describes the dynamics of a soliton when its support is small with respect to the other relevant elements (namely $V(x)$, the initial conditions and its $L^2$-size). Also the method employed here is different from those of the paper quoted above and it exploits and develops some ideas of [9]. Basically, we use the “strong” nonlinearity to obtain results on existence, shape, stability and dynamics of the soliton (Theorem 1). Finally, we notice that this method applies to a large class of nonlinearities and we do not make any assumption on the nondegeneracy or the uniqueness of the ground state solution (see the discussion in Section 1.2).

1.1 Notations

In the next we will use the following notations:

\[ \text{Re}(z), \text{Im}(z) \text{ are the real and the imaginary part of } z \]
\[ B_\rho(x_0) = B(x_0, \rho) = \{ x \in \mathbb{R}^N : |x - x_0| \leq \rho \} \]
\[ B(x_0, \rho)^C = \mathbb{R}^N \setminus B(x_0, \rho) \]
\[ S_\sigma = \{ u \in H^1 : ||u||_{L^2} = \sigma \} \]
\[ J(u) = \int \left( \frac{1}{2} |\nabla u|^2 + W(u) \right) dx \]
\[ J^c = \{ u \in H^1 : J(u) < c \} \]
\[ |\partial^\alpha V(x)| = \sup_{i_1, \ldots, i_\alpha} \left| \frac{\partial^{\alpha} V(x)}{\partial x_{i_1} \ldots \partial x_{i_\alpha}} \right| \text{ where } \alpha \in \mathbb{N}, i_1, \ldots, i_\alpha \in \{1, \ldots, N\} \]
\[ I_{\sigma^2} = \inf_{u \in H^1, \|u\|^2 = \sigma^2} J(u) = c \]
\[ |\cdot| \text{ is the euclidean norm both of a vector or of a matrix} \]
\[ \Gamma = \left\{ U \in H^1, J(U) = \inf_{\|U\|_{L^2} = 1} J(U) \right\} \text{ is the set of ground state solutions} \]

1.2 Statement of the problem

First, we focus on the “concentration” properties of a soliton solution of eq. (1) without the potential term $V$. We consider the following Cauchy problem
relative to the NSE:

\[ i \frac{\partial \psi}{\partial t} = -\frac{1}{2m} \Delta \psi + \frac{1}{2} W'_\varepsilon(\psi) \]  

(3)

\[ \psi(0, x) = U_\varepsilon(x - \bar{q}) e^{i\bar{p} \cdot x} \]

(4)

where, with some abuse of notation, we have set

\[ W_\varepsilon(\psi) = \frac{1}{\varepsilon^{N/2}} W(\varepsilon^{N/2} |\psi|) ; \quad W'_\varepsilon(\psi) = \frac{1}{\varepsilon^{N/2+2}} W'(\varepsilon^{N/2} |\psi|) \frac{\psi}{|\psi|} \]

(5)

and \( W : \mathbb{R}^+ \to \mathbb{R} \) is a real function which satisfies suitable assumptions (see (ii) below). \( U \) denotes a ground state solution of the equation

\[ -\frac{1}{2m} \Delta U + \frac{1}{2} W'(U) = \omega_1 U \]

(6)

namely a function such that

\[ \int \left( \frac{|\nabla U|^2}{2m} + W(U) \right) dx = c_0 \]

with

\[ c_0 = \inf_{\|u\|_{L^2} = 1} \int \left( \frac{|\nabla u|^2}{2m} + W(u) \right) dx \]

(7)

It is well known that we can choose \( U \) radially symmetric and positive.

Direct computations show that, by virtue of (5), the function

\[ U_\varepsilon(x) = \frac{1}{\varepsilon^{N/2}} U \left( \frac{x}{\varepsilon} \right) \]

satisfies the equation

\[ -\frac{1}{2m} \Delta U_\varepsilon + \frac{1}{2} W'_\varepsilon(U_\varepsilon) = \omega_\varepsilon U_\varepsilon. \]

(8)

where

\[ \omega_\varepsilon = \frac{\omega_1}{\varepsilon^2}. \]

Moreover \( U_\varepsilon(x) \) is a ground state solution of (8). In many cases, the ground state solution \( U \) is unique up to translations and change of sign, but we do not need this assumption.

Notice that the choice of \( W_\varepsilon \) given by (5) implies that

\[ \|U_\varepsilon\|_{L^2} = 1 \]

4
for every $\varepsilon > 0$.

Direct computation shows that the solution of (3), (4) is given by the following soliton

$$
\psi_{q,\varepsilon}(t, x) = U_\varepsilon(x - \bar{q} - \bar{v}t) e^{i(\bar{p} \cdot x - \omega t)} \quad \text{where} \quad \bar{v} = \frac{\bar{p}}{m}
$$

with

$$
\omega = \omega_\varepsilon + \frac{1}{2}m\bar{v}^2
$$

Thus $\psi_{q,\varepsilon}(t, x)$ behaves as a particle of “radius” $\varepsilon$ living in the point

$$
q = \bar{q} + \bar{v}t
$$

Since $\|\psi_{q,\varepsilon}(t, \cdot)\|_{L^2} = 1$ for every $\varepsilon > 0$, if $\varepsilon \to 0$, we have that

$$
|\psi_{q,\varepsilon}(t, x)|^2 \to \delta(x - \bar{q} - \bar{v}t) \quad \text{in} \quad D'(\mathbb{R}^N) \quad \forall t \in \mathbb{R},
$$

where $\delta(x - x_0)$ denotes the Dirac measure concentrated in the point $x_0$. The energy $E_\varepsilon(\psi)$ of the configuration $\psi$ is given by

$$
E_\varepsilon(\psi) = \int \left[ \frac{1}{2m} |\nabla \psi|^2 + W_\varepsilon(\psi) \right] dx,
$$

so the energy of $\psi_{q,\varepsilon}$ is

$$
E_\varepsilon(\psi_{q,\varepsilon}) = \int \left( \frac{|\nabla U_\varepsilon|^2}{2m} + W_\varepsilon(U_\varepsilon) \right) dx + \frac{1}{2}m\bar{v}^2 \quad (11)
$$

Thus $\psi_{q,\varepsilon}(t, x)$ behaves as a particle of mass $m$: $\bar{p}$ can be interpreted as its momentum, $\frac{1}{2m}\bar{p}^2 = \frac{1}{2}m\bar{v}^2$ as its kinetic energy and

$$
\int \frac{|\nabla U_\varepsilon|^2}{2m} + W_\varepsilon(U_\varepsilon) \, dx = \frac{c_0}{\varepsilon^2}
$$

as the internal energy; here $c_0$ is a constant defined as follows

$$
c_0 := \int \frac{|\nabla U_1|^2}{2m} + W(U_1) \, dx
$$

The aim of this paper is to study the dynamics of the solitons in the presence of a potential $V(x)$ namely to investigate the problem

$$
\begin{cases}
  i\frac{\partial \psi}{\partial t} = -\frac{1}{2m}\Delta \psi + \frac{1}{2}W'_\varepsilon(\psi) + V(x) \psi \\
  \psi(0, x) = \psi_{0,\varepsilon}(x)
\end{cases} \quad (P)
$$
where $\psi_{0,\varepsilon}$ satisfies the following assumptions

$$
\psi_{0,\varepsilon}(x) = U_\varepsilon(x - \bar{q}) e^{i\bar{p} \cdot x} + \varphi_{0,\varepsilon}(x), \quad \varphi_{0,\varepsilon} \in H^1(\mathbb{R}^N); \\
\left\|\psi_{0,\varepsilon}\right\|_{L^2} = 1 \\
E_\varepsilon(\psi_{0,\varepsilon}) \leq \frac{c_0}{\varepsilon^2} + M
$$

with $M > 0$ independent of $\varepsilon$; here $E_\varepsilon(\psi)$ denotes the energy in the presence of the potential $V:

$$
E_\varepsilon(\psi) = \int \left[ \frac{1}{2m} |\nabla \psi|^2 + W_\varepsilon(\psi) + V(x) |\psi|^2 \right] dx.
$$

1.3 The main results

We make the following assumptions:

(i) $W : \mathbb{R}^+ \rightarrow \mathbb{R}$ is a $C^2$ function which satisfies the following assumptions:

$$
W(0) = W'(0) = W''(0) = 0 \quad (W_0)
$$

$$
|W''(s)| \leq c_1|s|^{q-2} + c_2|s|^{p-2} \text{ for some } 2 < q \leq p < 2^* \quad (W_1)
$$

$$
W(s) \geq -c|s|^{\nu}, \ c \geq 0, \ 2 < \nu < 2 + \frac{4}{N} \text{ for } s \text{ large} \quad (W_2)
$$

$$
\exists s_0 \in \mathbb{R}^+ \text{ such that } W(s_0) < 0 \quad (W_3)
$$

(ii) $V : \mathbb{R}^N \rightarrow \mathbb{R}$ is a $C^2$ function with bounded derivatives which satisfies the following assumptions:

$$
0 \leq V(x) \leq V_0 < \infty; \quad (V_0)
$$

The main result of this paper is the following theorem which describes the shape and the dynamics of the soliton $\Psi_\varepsilon(t, x)$:

**Theorem 1** Assume (i) and (ii); then the solution of problem (P) has the following form

$$
\psi_\varepsilon(t, x) = \Psi_\varepsilon(t, x) + \varphi_\varepsilon(t, x) \quad (15)
$$

where $\Psi_\varepsilon(t, x)$ is a function having support in a ball $B_{R_\varepsilon}(q_\varepsilon)$, with radius $R_\varepsilon \rightarrow 0$ and center $q_\varepsilon = q_\varepsilon(t)$. Moreover,

$$
|||\Psi_\varepsilon(t, x)| - U_\varepsilon(x - q_\varepsilon(t))|||_{L^2} \rightarrow 0 \text{ as } \varepsilon \rightarrow 0 \quad (16)
$$

6
uniformly in $t$, where $U_\varepsilon = \frac{1}{\varepsilon^{1/2}} U \left( \frac{x}{\varepsilon} \right)$ and $U$ is a ground state solution of (6).

The dynamics is given by the following equations:

$$
\begin{aligned}
\dot{q}_\varepsilon(t) &= \frac{1}{m_\varepsilon(t)} \ p_\varepsilon(t) + K_\varepsilon(t) \\
\dot{p}_\varepsilon(t) &= -\nabla V(q_\varepsilon(t)) + F_\varepsilon(q_\varepsilon) + H_\varepsilon(t)
\end{aligned}
$$

with initial data

$$
\begin{aligned}
q_\varepsilon(0) &= \bar{q} + o(1) \\
p_\varepsilon(0) &= \bar{p} + o(1)
\end{aligned}
$$

where

• (a) $q_\varepsilon(t)$ is the **barycenter** of the soliton and it has the following form:

$$
q_\varepsilon(t) = \frac{\int x \ |\Psi_\varepsilon|^2 \ dx}{\int |\Psi_\varepsilon|^2 \ dx}
$$

• (b) $m_\varepsilon(t) = m \int |\Psi_\varepsilon(t,x)|^2 \ dx = m + o(1)$ can be interpreted as the **mass** of the soliton,

• (c) $p_\varepsilon(t)$ is the **momentum** of the soliton and it has the following form:

$$
p_\varepsilon(t) = \text{Im} \int \nabla \Psi_\varepsilon(t,x) \ \overline{\Psi_\varepsilon(t,x)} \ dx;
$$

• (d) $K_\varepsilon(t)$ and $H_\varepsilon(t)$ are errors due to the fact that the soliton is not a point and

$$
\sup_{t \in \mathbb{R}} (|H_\varepsilon(t)| + |K_\varepsilon(t)|) \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0.
$$

• (e) $F_\varepsilon(q_\varepsilon)$ is the force due to the pressure of the wave $\varphi_\varepsilon$ on the soliton and $F_\varepsilon \rightarrow 0$ in the space of distributions, more exactly we have that

$$
\forall \tau_0, \tau_1, \left| \int_{\tau_0}^{\tau_1} F_\varepsilon(q_\varepsilon) \ dt \right| \leq c(\varepsilon) \left( 1 + |\tau_1 - \tau_0| \right)
$$

where $c(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$.  


Corollary 2 Let \( q \) and \( p \) be the solution of the following Cauchy problem:

\[
\begin{aligned}
\dot{q}(t) &= \frac{1}{m} p(t) \\
\dot{p}(t) &= -\nabla V(q(t))
\end{aligned}
\]  

with initial data

\[
\begin{aligned}
q(0) &= q_\epsilon(0) \\
p(0) &= p_\epsilon(0)
\end{aligned}
\]  

where \( q_\epsilon(t) \) and \( p_\epsilon(t) \) are as in Th. 1. Then, as \( \varepsilon \to 0 \)

\((q_\epsilon(t), p_\epsilon(t)) \to (q(t), p(t)) \)

uniformly on compact sets.

Let us discuss the set of our assumptions.

Remark 3 The conditions \((W_0)\) and \((V_0)\) are assumed for simplicity; in fact they can be weakened as follows

\[ W(0) = W'(0) = 0, \quad W''(0) = E_0 \]

and

\[ E_1 \leq V(x) \leq V_\infty < +\infty. \]

In fact, in the general case, the solution of the Schroedinger equation is modified only by a phase factor.

Remark 4 In [6] the authors prove that if (ii) holds equation (3) admits orbitally stable solitary waves having the form (4). In particular the authors show that, under assumptions \((W_0)\), \((W_1)\), \((W_2)\) and \((W_3)\), for any \( \sigma \) there exists a minimizer \( U(x) = U_\sigma(x) \) of the functional

\[ J(u) = \int \left( \frac{1}{2} |\nabla u|^2 + W(u) \right) dx \]

on the manifold \( S_\sigma := \{ u \in H^1, \ |u|_{L^2} = \sigma \} \). Such a minimizer satisfies eq. (6) where \( \omega_1 \) is a Lagrange multiplier.

Remark 5 By our assumptions, the problem \((P)\) has a unique solution

\[ \psi \in C^0(\mathbb{R}, H^2(\mathbb{R}^N)) \cap C^1(\mathbb{R}, L^2(\mathbb{R}^N)). \]  

(21)
Let us recall a result on the global existence of solutions of the Cauchy problem $P$ (see [13, 18, 23]). Assume $(W_1)$, $(W_2)$ and $(W_3)$ for $W$. Let $D(A)$ (resp. $D(A^{1/2})$) denote the domain of the self-adjoint operator $A$ (resp. $A^{1/2}$) where

$$A = -\Delta + V : L^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N).$$

If $V \geq 0$, $V \in C^2$ and $|\partial^2 V| \in L^\infty$ and the initial data $\psi(0, x) \in D(A^{1/2})$ then there exists the global solution $\psi$ of $P$ and

$$\psi(t, x) \in C^0(\mathbb{R}, D(A^{1/2})) \cap C^1(\mathbb{R}, H^{-1}(\mathbb{R}^N)).$$

Furthermore, if $\psi(0, x) \in D(A)$ then

$$\psi(t, x) \in C^0(\mathbb{R}, D(A)) \cap C^1(\mathbb{R}, L^2(\mathbb{R}^N)).$$

In this case, since $D(A) \subset H^2(\mathbb{R}^N)$, (21) is satisfied.

2 The dynamics of solitons

In this section, after having stated the main dynamical properties of our system (Subsection 2.1), we define explicitly the splitting (2) and the equations which rules the dynamics of the soliton. In all this section, it is not necessary to assume $\varepsilon$ to be small.

2.1 First integrals of NSE

Equation (P) is the Euler-Lagrange equation relative to the Lagrangian density

$$\mathcal{L} = \text{Re}(i\partial_t \bar{\psi} \psi) - \frac{1}{2m} |\nabla \psi|^2 - W_\varepsilon(\psi) - V(x) |\psi|^2. \quad (22)$$

Sometimes it is useful to write $\psi$ in polar form

$$\psi(t, x) = u(t, x)e^{iS(t, x)}. \quad (23)$$

Thus the state of the system $\psi$ is uniquely defined by the couple of variables $(u, S) \in \mathbb{R}^+ \times [\mathbb{R}/(2\pi\mathbb{Z})]$. Using these variables, the action $S = \int \mathcal{L} dxdt$ takes the form

$$S(u, S) = -\int \left[ \frac{1}{2m} |\nabla u|^2 + W_\varepsilon(u) + \left( \partial_t S + \frac{1}{2m} |\nabla S|^2 + V(x) \right) u^2 \right] dxdt \quad (24)$$

and equation (P) splits in the two following equations:
\[-\frac{1}{2m} \Delta u + W_\epsilon(u) + \left( \partial_t S + \frac{1}{2m} |\nabla S|^2 + V(x) \right) u = 0 \quad (25)\]
\[
\partial_t (u^2) + \nabla \cdot \left( u^2 \frac{\nabla S}{m} \right) = 0 \quad (26)
\]

Noether’s theorem states that any invariance for a one-parameter group of the Lagrangian implies the existence of an integral of motion (see e.g. [17] or [7]). They are derived by a continuity equation.

Now we describe the first integrals which will be relevant for this paper, namely the energy, the “hylenic charge” and the momentum.

**Energy.** The energy, by definition, is the quantity which is preserved by the time invariance of the Lagrangian; it has the following form

\[
E_\epsilon(\psi) = \int \left[ \frac{1}{2m} |\nabla \psi|^2 + W_\epsilon(\psi) + V(x) |\psi|^2 \right] dx. \quad (27)
\]

Using (23) we get:

\[
E_\epsilon(\psi) = \int \left( \frac{1}{2m} |\nabla u|^2 + W_\epsilon(u) \right) dx + \int \left( \frac{1}{2m} |\nabla S|^2 + V(x) \right) u^2 dx \quad (28)
\]

Thus the energy has two components: the *internal energy* (which, sometimes, is also called *binding energy*)

\[
J_\epsilon(u) = \int \left( \frac{1}{2m} |\nabla u|^2 + W_\epsilon(u) \right) dx \quad (29)
\]

and the *dynamical energy*

\[
G(u, S) = \int \left( \frac{1}{2m} |\nabla S|^2 + V(x) \right) u^2 dx \quad (30)
\]

which is composed by the *kinetic energy* \( \frac{1}{2m} \int |\nabla S|^2 u^2 dx \) and the *potential energy* \( \int V(x)u^2 dx \). By our assumptions, the internal energy is bounded from below and the dynamical energy is positive.

**Hylenic charge.** Following [7] the *hylenic charge*, is defined as the quantity which is preserved by the invariance of the Lagrangian with respect to the action

\[
\psi \mapsto e^{i\theta} \psi.
\]
This invariance gives the continuity equation (26). Thus, in this case, the charge is nothing else but the $L^2$ norm, namely:

$$C(\psi) = \int |\psi|^2 \, dx = \int u^2 dx.$$  

**Momentum.** The momentum is constant in time if the Lagrangian is space-translation invariant; this happens when $V$ is a constant. In general we have the equation

$$\partial_t (u^2 \nabla S) = -u^2 \nabla V + \nabla \cdot T$$  \hspace{1cm} (31)

where $T$ is the stress tensor whose components are given by

$$T_{jk} = \text{Re} \left( \partial_{x_j} \psi \partial_{x_k} \bar{\psi} \right)$$

$$-\delta_{jk} \left[ \frac{1}{4m} \Delta |\psi|^2 - \frac{1}{2} W'_{\epsilon}(\psi) \bar{\psi} + W_{\epsilon}(\psi) \right]$$  \hspace{1cm} (32)

$$= \left[ \partial_{x_j} u \partial_{x_k} u + (\partial_{x_j} S \partial_{x_k} S) u^2 - \frac{1}{4m} \delta_{jk} \Delta (u^2) \right]$$

$$+ \delta_{jk} \left( \frac{1}{2} W'_{\epsilon}(u) u - W_{\epsilon}(u) \right)$$  \hspace{1cm} (33)

**Proof:** It is well known that the stress tensor has the following form (see e.g. [7] or [8])

$$T_{jk} = -\text{Re} \left( \frac{\partial \mathcal{L}}{\partial (\partial_{x_j} \psi)} \cdot \partial_{x_k} \bar{\psi} \right) + \delta_{jk} \mathcal{L}$$

Using equation (32) to eliminate $i \partial_t \psi$, we get

$$\mathcal{L} = \text{Re}(i \partial_t \bar{\psi}) - \frac{1}{2m} |\nabla \psi|^2 - W_{\epsilon}(\psi) - V(x) |\psi|^2$$

$$= \text{Re} \left[ \left( -\frac{1}{2m} \Delta \psi + \frac{1}{2} W'_{\epsilon}(\psi) + V(x) \psi \right) \bar{\psi} \right] - \frac{1}{2m} |\nabla \psi|^2 - W_{\epsilon}(\psi) - V(x) |\psi|^2$$

$$= \text{Re} \left( -\frac{1}{2m} \Delta \bar{\psi} \psi \right) - \frac{1}{2m} |\nabla \psi|^2 + \frac{1}{2} W'_{\epsilon}(\psi) \bar{\psi} - W_{\epsilon}(\psi).$$

Moreover, we have that

$$\Delta \psi \bar{\psi} = \Delta \psi_1 \psi_1 + \Delta \psi_2 \psi_2 = \nabla \cdot (\psi_1 \nabla \psi_1) + \nabla \cdot (\psi_2 \nabla \psi_2) - |\nabla \psi|^2$$

$$= \frac{1}{2} \Delta |\psi_1|^2 + \frac{1}{2} \Delta |\psi_2|^2 - |\nabla \psi|^2 = \frac{1}{2} |\psi_1|^2 - |\nabla \psi|^2,$$
Then
\[ \mathcal{L} = -\frac{1}{4m} \Delta |\psi|^2 + \frac{1}{2} W'_\varepsilon(\psi) \bar{\psi} - W_\varepsilon(\psi) \]

We recall that, if \( z = a + ib \) is a complex number, by definition, \( \frac{\partial L}{\partial \bar{z}} = \frac{\partial L}{\partial a} + i \frac{\partial L}{\partial b} \). So
\[
\text{Re} \left( \frac{\partial \mathcal{L}}{\partial (\partial_{x_j} \psi)} \cdot \bar{\psi} \frac{\partial}{\partial x_k} \right) = - \text{Re} \left( \partial_{x_j} \psi \partial_{x_k} \bar{\psi} \right)
\]
and the conclusion follows from direct computation.

\[ \square \]

If \( V = \text{const} \), eq. (31) is a continuity equation and the momentum
\[ \mathbf{P}(\psi) = \int u^2 \nabla S \, dx = \int \text{Im} (\bar{\psi} \nabla \psi) \, dx \tag{34} \]
is constant in time. Notice that, by equation (26), the density of momentum \( u^2 \nabla S \) is nothing else than the flow of hylenic charge.

Let us consider the soliton (9); in this case we have
\[ E_\varepsilon(\psi_{q,\varepsilon}) = \frac{c_0^2}{\varepsilon^2} + \frac{1}{2m} \bar{p}^2 \]
where \( c_0 \) is defined by (7),
\[ C(\psi_\varepsilon) = 1 \]
and
\[ \mathbf{P}(\psi_\varepsilon) = \bar{p}. \]

Now, let us see the rescaling properties of the internal energy and the \( L^2 \) norm of a function \( u(x) \) having the form
\[ u(x) := \frac{1}{\varepsilon^{N/2}} v \left( \frac{x}{\varepsilon} \right). \]
We have
\[ ||u||_{L^2}^2 = \frac{1}{\varepsilon^N} \int v \left( \frac{x}{\varepsilon} \right)^2 \, dx = \int v(\xi)^2 \, d\xi = ||v||_{L^2}^2, \]
and
\[ J_\varepsilon(u) = \int \left[ \frac{1}{2m} |\nabla u|^2 + W_\varepsilon(u) \right] \, dx \]
\[ = \int \left[ \frac{1}{2m} |\nabla u|^2 + \frac{1}{\varepsilon^{N+2}} W(\varepsilon^{N/2} u) \right] \, dx \]
\[ = \int \left[ \frac{1}{2m} \frac{1}{\varepsilon^N} \left| \nabla_x v \left( \frac{x}{\varepsilon} \right) \right|^2 + \frac{1}{\varepsilon^{N+2}} W \left( v \left( \frac{x}{\varepsilon} \right) \right) \right] \, dx \]
\[ = \int \left[ \frac{1}{2m} \frac{1}{\varepsilon^{N+2}} \left| \nabla_\xi v (\xi) \right|^2 + \frac{1}{\varepsilon^{N+2}} W \left( v (\xi) \right) \right] \varepsilon^N \, d\xi \]
\[ = \frac{1}{\varepsilon^2} \int \frac{1}{2m} \left| \nabla_\xi v (\xi) \right|^2 + W \left( v (\xi) \right) \, d\xi = \frac{1}{\varepsilon^2} J_1(u). \]

### 2.2 Definition of the soliton

In this section we want to describe a method to split a solution of eq. \([P]\) in a wave and a soliton as in \([2]\).

If our solution has the following form

\[ \psi_\varepsilon(t, x) = u_\varepsilon e^{iS_\varepsilon} = [U_\varepsilon(x - \xi(t)) + w_\varepsilon(t, x)] e^{iS_\varepsilon(t, x)} \]

where \( w \) is sufficiently small, then a possible choice is to identify the soliton with \( U_\varepsilon(x - \xi(t)) e^{iS_\varepsilon(t, x)} \) and the wave with \( w_\varepsilon(t, x) e^{iS_\varepsilon(t, x)} \). However, we want to give a definition which localize the soliton, namely to assume the function \( \Psi_\varepsilon(t, x) \) to have compact support in space.

Roughly speaking, the soliton can be defined as the part of the field \( \psi_\varepsilon \) where some density function \( \rho_\varepsilon(t, x) \) is sufficiently large (e.g., after a suitable normalization, \( \rho_\varepsilon(t, x) \geq 1 \)).

For the moment we do not define \( \rho_\varepsilon(t, x) \) explicitly. We just require that \( \rho_\varepsilon(t, x) \) satisfies the following assumptions:

- \( \rho_\varepsilon \in C^1(\mathbb{R}^{N+1}) \) and \( \rho_\varepsilon(t, x) \to 0 \) as \( |x| \to \infty \)

- \( \rho_\varepsilon \) satisfies the continuity equation

\[ \partial_t \rho_\varepsilon + \nabla \cdot J_{\rho_\varepsilon} = 0 \]  \quad (35)

for some \( J_{\rho_\varepsilon} \in C^1(\mathbb{R}^{N+1}) \)

In order to fix the ideas you may think of \( \rho_\varepsilon(t, x) \) as a smooth approximation of \( u_\varepsilon(t, x)^2 \). An explicit definition of \( \rho_\varepsilon(t, x) \) is given in Section 3.2. However, in other problems, it might be more useful to make different choices.
of it such as the energy density. We have postponed the choice of $\rho_\varepsilon$ since the results of this section are independent of this choice.

Next we set

$$\chi_\varepsilon(t, x) = \sqrt{\varphi(\rho_\varepsilon(t, x))}$$

where

$$\varphi(s) = \begin{cases} 0 & \text{if } s \leq 1 \\ s - 1 & \text{if } 1 \leq s \leq 2 \\ 1 & \text{if } s \geq 2 \end{cases}$$

So we have that $\chi_\varepsilon(t, x) = 1$ where $\rho_\varepsilon(t, x) \geq 2$ and $\chi_\varepsilon(t, x) = 0$ where $\rho_\varepsilon(t, x) \leq 1$; thus you may think of $\chi_\varepsilon(t, x)$ as a sort of approximation of the characteristic function of the region occupied by the soliton. Finally, we set

$$\Psi_\varepsilon(t, x) = \psi_\varepsilon(t, x) \chi_\varepsilon$$
$$\varphi_\varepsilon(t, x) = \psi_\varepsilon(t, x) [1 - \chi_\varepsilon]$$

$\Psi_\varepsilon(t, x)$ is the soliton and $\varphi_\varepsilon(t, x)$ is the wave; the region

$$\Sigma_{\varepsilon,t} = \{ (t, x) \in \mathbb{R}^{N+1} | 1 < \rho(t, x) < 2 \}$$
$$= \{ (t, x) \in \mathbb{R}^{N+1} | 0 < \chi_\varepsilon(t, x) < 1 \}$$

is the region where the soliton and the wave interact with each other; we will refer to it as the halo of the soliton.

### 2.3 The equation of dynamics of the soliton

**Definition 6** We define the following quantities relative to the soliton:

- **the barycenter:**
  $$q_\varepsilon(t) = \int x |\Psi_\varepsilon|^2 \, dx$$

- **the momentum:**
  $$p_\varepsilon(t) = \int \nabla S_\varepsilon |\Psi_\varepsilon|^2 \, dx$$

- **the mass:**
  $$m_\varepsilon(t) = m \int |\Psi_\varepsilon|^2 \, dx.$$

**Remark 7** Notice that the mass of the soliton $m_\varepsilon(t)$ depends on $t$. The global mass is constant (namely $m$) but it is shared between the soliton and the wave whose mass is $\int u_\varepsilon^2 [1 - \chi_\varepsilon^2] \, dx$.
The next theorem shows the relation between \( q_\varepsilon(t) \) and \( p_\varepsilon(t) \) and their derivatives.

**Theorem 8** The following equations hold

\[
\dot{q}_\varepsilon = \frac{p_\varepsilon}{m_\varepsilon} + \frac{1}{m_\varepsilon} \int_{\Sigma_{\varepsilon,t}} (x - q_\varepsilon) \left[ u_\varepsilon^2 \nabla S_\varepsilon \cdot \nabla \rho_\varepsilon - \nabla \cdot J_{\rho_\varepsilon} \right] dx \quad (40)
\]

\[
\dot{p}_\varepsilon = - \int \nabla V |\Psi_\varepsilon|^2 dx - \int_{\Sigma_{\varepsilon,t}} \left[ T \cdot \nabla \rho_\varepsilon + u_\varepsilon^2 \nabla S_\varepsilon \left( \nabla \cdot J_{\rho_\varepsilon} \right) \right] dx. \quad (41)
\]

**Remark 9** The term \( \int_{\Sigma_{\varepsilon,t}} T \cdot \nabla \rho_\varepsilon dx \) represents the pressure of the wave on the soliton; if \( \varepsilon \to 0 \) and \( \partial \Sigma_{\varepsilon,t} \) is sufficiently regular then

\[
\int_{\Sigma_{\varepsilon,t}} T \cdot \nabla \rho_\varepsilon dx = \int_{\sigma_\varepsilon} T \cdot n d\sigma
\]

where \( \sigma_\varepsilon = \{ x \mid \rho_\varepsilon(x) = 1 \} \) and \( n \) is its outer normal.

**Proof of Th. 8.** We calculate the first derivative of the barycenter.

\[
\dot{q}_\varepsilon(t) = \frac{d}{dt} \left( \frac{\int x |\Psi_\varepsilon|^2 dx}{\int |\Psi_\varepsilon|^2 dx} \right)
\]

\[
= \frac{d}{dt} \int x |\Psi_\varepsilon|^2 dx - \frac{\left( \int x |\Psi_\varepsilon|^2 dx \right) \left( \frac{d}{dt} \int |\Psi_\varepsilon|^2 dx \right)}{\int |\Psi_\varepsilon|^2 dx^2}
\]

\[
= \frac{d}{dt} \int x |\Psi_\varepsilon|^2 dx - q_\varepsilon(t) \frac{d}{dt} \int |\Psi_\varepsilon|^2 dx = \frac{\int (x - q_\varepsilon(t)) \frac{d}{dt} |\Psi_\varepsilon|^2 dx}{\int |\Psi_\varepsilon|^2 dx}.
\]

We have

\[
\nabla \chi^2 = \nabla \varphi(\rho_\varepsilon(t,x)) = \varphi'(\rho_\varepsilon(t,x)) \nabla \rho_\varepsilon = \mathbb{I}_{\Sigma_{\varepsilon,t}} \nabla \rho_\varepsilon
\]

and

\[
\frac{d}{dt} \chi^2 = \frac{d}{dt} \varphi(\rho_\varepsilon(t,x)) = \varphi'(\rho_\varepsilon(t,x)) \partial_t \rho_\varepsilon
\]

\[
= \mathbb{I}_{\Sigma_{\varepsilon,t}} \partial_t \rho_\varepsilon = -\mathbb{I}_{\Sigma_{\varepsilon,t}} \nabla \cdot J_{\rho_\varepsilon}
\]

where \( \mathbb{I}_{\Sigma_{\varepsilon,t}} \) is the characteristic function of \( \Sigma_{\varepsilon,t} \).

So, we have
\[
\dot{q}_\epsilon(t) = \frac{\int (x - q_\epsilon(t)) \frac{d}{dt}(\chi^2 u^2)\,dx}{\int |\Psi_\epsilon|^2\,dx} = \frac{\int (x - q_\epsilon(t)) \left(\chi^2 \frac{d}{dt}u^2 + u^2 \frac{d}{dt}\chi^2\right)\,dx}{\int |\Psi_\epsilon|^2\,dx} = \frac{\int_{\mathbb{R}^N} (x - q_\epsilon(t)) \chi^2 u^2\,dx - \int_{\Sigma_{\epsilon,t}} (x - q_\epsilon(t)) \nabla \cdot J_{\rho_\epsilon}\,dx}{\int_{\mathbb{R}^N} |\Psi_\epsilon|^2\,dx}.
\]

For the first term we use the continuity equation \([26]\). We have

\[
\int (x - q_\epsilon(t)) \chi^2 \frac{d}{dt} u^2\,dx = \int (x - q_\epsilon(t)) \nabla \left( u^2 \frac{\nabla S_\epsilon}{m} \right) \chi^2\,dx = \frac{1}{m} \int (u^2 \nabla S_\epsilon) \chi^2\,dx + \frac{1}{m} \int (x - q_\epsilon(t)) u^2 \nabla S_\epsilon \cdot \nabla \chi^2\,dx = \frac{p_\epsilon(t)}{m} + \frac{1}{m} \int_{\Sigma_{\epsilon,t}} (x - q_\epsilon(t)) u^2 \nabla S_\epsilon \cdot \nabla \rho_\epsilon\,dx.
\]

Concluding, we get the first equation of motion:

\[
\dot{q}_\epsilon(t) = \frac{p_\epsilon(t)}{m_{\epsilon}} + \frac{\int_{\Sigma_{\epsilon,t}} (x - q_\epsilon(t)) \left[u^2 \nabla S_\epsilon \cdot \nabla \rho_\epsilon - \nabla \cdot J_{\rho_\epsilon}\right]\,dx}{m_{\epsilon}}.
\]

Next, we will get the second one. We have that

\[
\dot{p}_\epsilon = \int \left(\frac{\partial}{\partial t} u^2 \nabla S_\epsilon\right) \chi^2\,dx + \int u^2 \nabla S_\epsilon \frac{\partial}{\partial t} \chi^2\,dx.
\]

Now, using \([31]\) we have that

\[
\int \left(\frac{\partial}{\partial t} u^2 \nabla S_\epsilon\right) \chi^2\,dx = -\int \nabla V |\Psi_\epsilon|^2\,dx + \int \nabla \cdot T \chi^2\,dx = -\int \nabla V |\Psi_\epsilon|^2\,dx - \int T \cdot \nabla \chi^2\,dx = -\int \nabla V |\Psi_\epsilon|^2\,dx - \int_{\Sigma_{\epsilon,t}} T \cdot \nabla \rho_\epsilon\,dx.
\]

The second term of eq. \([43]\) takes the form:

\[
\int u^2 \nabla S_\epsilon \frac{\partial}{\partial t} \chi^2\,dx = -\int_{\Sigma_{\epsilon,t}} u^2 \nabla S_\epsilon (\nabla \cdot J_{\rho_\epsilon})\,dx.
\]
It is possible to give a “pictorial” interpretation to equations (40) and (41). We may assume that $u_\varepsilon^2$ represents the density of a fluid; so the soliton is a bump of fluid particles which stick together and the halo $\Sigma_{\varepsilon,t}$ can be regarded as the interface where the soliton and the wave might exchange particles, momentum and energy.

Hence,

- $m_\varepsilon(t)$ is the mass of the soliton
- $\nabla S_\varepsilon/m_\varepsilon$ is the velocity of the fluid particles and $\nabla S_\varepsilon$ is their momentum

So each term of the equations (40) and (41) have the following interpretation

- $p_\varepsilon(t)/m_\varepsilon$ is the average velocity of each particle; in fact
  \[
  \frac{p_\varepsilon(t)}{m_\varepsilon} = \frac{\int \nabla S_\varepsilon |\Psi_\varepsilon|^2 \, dx}{m_\varepsilon} = \frac{\int \nabla S_\varepsilon/m_\varepsilon |\Psi_\varepsilon|^2 \, dx}{\int |\Psi_\varepsilon|^2 \, dx}
  \]

- the “halo term” $\frac{1}{m_\varepsilon} \int_{\Sigma_{\varepsilon,t}} (x-q_\varepsilon) \left[ u_\varepsilon^2 \nabla S_\varepsilon \cdot \nabla \rho_\varepsilon - \nabla \cdot J_{\rho_\varepsilon} \right] \, dx$ describes the change of the average velocity of the soliton due to the exchange of fluid particles

- the term $-\int \nabla V |\Psi_\varepsilon|^2 \, dx$ describes the volume force acting on the soliton

- the term $-\int_{\Sigma_{\varepsilon,t}} T \cdot \nabla \rho_\varepsilon \, dx$ describes the surface force exerted by the wave on the soliton

- the term $-\int_{\Sigma_{\varepsilon,t}} u_\varepsilon^2 \nabla S_\varepsilon (\nabla \cdot J_{\rho_\varepsilon}) \, dx$ describes the change of the momentum of the soliton due to the exchange of fluid particles with the wave.

3 The limit dynamics

In this section, we analyze the dynamics of the soliton as $\varepsilon \to 0$ and we end proving the main theorem i.e. Th. [1]
3.1 Analysis of the concentration point of the soliton

If \( \psi_\varepsilon(t, x) \) is a solution of the problem (P), we say that \( \hat{q}_\varepsilon(t) \) is the concentration point of \( \psi_\varepsilon(t, x) \) if it minimizes the following quantity

\[
\mathbf{f}(q) = \| |\psi_\varepsilon(t, x)| - U_\varepsilon(x - q) \|_{L^2}^2.
\]  

(44)

It is easy to see that \( \mathbf{f}(q) \) has a minimizer; of course, it might happen that it is not unique; in this case we denote by \( \hat{q}_\varepsilon(t) \) one of the minimizers of \( f \) at the time \( t \).

Basically \( \hat{q}_\varepsilon(t) \) is a good candidate for the position of our soliton, but it cannot satisfy an equation of type (17) since in general it is not uniquely defined and \( a \text{ fortiori} \) is not differentiable. \( \hat{q}_\varepsilon(t) \) could be uniquely defined if we make assumptions on the non degeneracy of the ground state, but we do not like to make such assumptions since they are very hard to be verified and in general they do not hold. Actually the position of the soliton is supposed to be \( q_\varepsilon(t) \) as in Def. 6 However, as we will see, \( \hat{q}_\varepsilon \) is useful to recover some estimates on \( q_\varepsilon \). So, in this subsection we will analyze some properties of \( \hat{q}_\varepsilon \).

We start with a variant of a result contained in [6].

**Lemma 10** Given \( u \in H^1 \), we define (if it exists) \( \hat{q} \in \mathbb{R}^N \) to be a minimizer of the function

\[
q \mapsto \| U(x - q) - u(x) \|_{L^2}^2.
\]

For any \( \eta \) there exists a \( \delta(\eta) \) such that, if \( u \in J^{\rho_0 + \delta(\eta)} \cap S_1 \) (see section 1.1), \( \hat{q} \) exists and it holds

\[
\| U(x - \hat{q}) - u \|_{H^1} \leq \eta
\]

(45)

\[
\int_{\mathbb{R}^N \setminus B(\hat{q}, \hat{R}_\eta)} u^2 dx \leq \eta
\]

(46)

where \( \hat{R}_\eta = -C \log(\eta) \) and \( U \in \Gamma \).

**Proof:** The proof of (45) can be found in [6]. If \( U \in \Gamma \), again by [6] we know that, for \( R \) sufficiently large,

\[
\int_{|x| > R} U^2(x) dx < \int_{|x| > R} C_1 e^{-C_2 |x|}.
\]

Thus

\[
\int_{|x| > R} U^2(x) dx = C_3 \int_R^\infty \rho^{N-1} e^{-C_2 \rho} d\rho = C_4 R^N e^{-C_2 R} \leq e^{-C_5 R}
\]

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where the \( C_i \)'s are suitable positive constants. We remark that \( R \) does not depend on \( U \).

Now, it is sufficient to take \( \hat{R}_\eta > -\frac{1}{\bar{c}_5} \log(\eta) \) and by (15) we obtain (16).

We define the set of admissible initial data as follows:

\[
B_{\epsilon,M} = \left\{ \psi(x) = U_{\epsilon}(x - q_0) e^{ip_0 \cdot x} + \varphi(x) : E_{\epsilon}(\psi) \leq \frac{c_0}{\epsilon^2} + M \text{ and } \|\psi\|_{L^2} = 1 \right\}
\]

**Lemma 11** For every \( \eta > 0 \), there exists \( \epsilon = \epsilon(\eta) > 0 \) such that

\[
\int_{\mathbb{R}^N \setminus B(\hat{q}_\epsilon, \hat{R}_\eta)} |\psi_\epsilon(t,x)|^2 \, dx < \eta
\]

where \( \psi_\epsilon(t,x) \) is a solution of problem (\( \mathcal{P} \)), with initial data in \( B_{\epsilon,M} \) and \( \hat{q}_\epsilon \) is the concentration point of \( \psi_\epsilon \).

**Proof.** By the conservation law, the energy \( E_{\epsilon}(\psi_\epsilon(t,x)) \) is constant with respect to \( t \). Then we have, by hypothesis on the initial datum

\[
E_{\epsilon}(\psi_\epsilon(0,x)) = E_{\epsilon}(\psi_\epsilon(t,x)) \leq \frac{c_0}{\epsilon^2} + M.
\]

Thus

\[
J_{\epsilon}(\psi_\epsilon(t,x)) = E_{\epsilon}(\psi(t,x)) - G(\psi(t,x))
\]

\[
= E_{\epsilon}(\psi_\epsilon(t,x)) - \int_{\mathbb{R}^N} \left[ \frac{\|\nabla S_{\epsilon}(t,x)\|^2}{2m} + V(x) \right] u_{\epsilon}(t,x)^2 \, dx
\]

\[
\leq E_{\epsilon}(\psi_\epsilon(t,x)) \leq \frac{c_0}{\epsilon^2} + M
\]

(48)

because \( V \geq 0 \). By rescaling the inequality (48), and setting \( y = x/\epsilon \) we get

\[
J(|\epsilon^{N/2}\psi_\epsilon(t,\epsilon y)|) \leq c_0 + \epsilon^2 M
\]

(49)

We choose \( \epsilon \) small such that \( \epsilon^2 M \leq \delta(\eta) \). Then \( \epsilon^{N/2}\psi_\epsilon(t,\epsilon y) \in J^{c_0 + \delta(\eta)} \cap S_1 \), and so applying Lemma 10,

\[
\int_{\mathbb{R}^N \setminus B(\hat{q}_\epsilon, \hat{R}_\eta)} \epsilon^N |\psi_\epsilon(t,\epsilon y)|^2 \, dy < \eta
\]

(50)

Now, making the change of variable \( x = \epsilon y \), we obtain the desired result.

\( \square \)
Lemma 12 If $\psi_\varepsilon(t, x)$ is a solution of problem (P), with initial data in $B_{\varepsilon, M}$ and $\varepsilon$ sufficiently small, then

$$\int_{\mathbb{R}^N \setminus B(\hat{q}, \varepsilon)} |\psi_\varepsilon(t, x)|^2 \, dx = \eta(\varepsilon) \tag{51}$$

where $\eta(\varepsilon) \to 0$ as $\varepsilon \to 0$.

**Proof.** First we prove that for every $\eta > 0$, there exists $\varepsilon_1(\eta) > 0$ such that, if $\psi_\varepsilon(0, x) \in B_{\varepsilon_1(\eta), M}$, we have

$$\int_{\mathbb{R}^N \setminus B(\hat{q}, \sqrt{\varepsilon_1(\eta)})} |\psi_\varepsilon(t, x)|^2 \, dx < \eta.$$

Arguing as in the proof of Lemma 11 if $\varepsilon_1(\eta) \leq \min \left[ \sqrt{\delta(\eta)} \frac{1}{M R_\eta} \right]$, we get (47). At this point, since $\varepsilon_1(\eta) \leq \frac{1}{R_\eta}$ we have that $\varepsilon_1(\eta) R_\eta \leq \sqrt{\varepsilon_1(\eta)}$.

Now set

$$\varepsilon(\eta) = \min_{\eta \leq \zeta} \varepsilon_1(\zeta).$$

Clearly, $\varepsilon(\eta)$ is a non-increasing function (which might be discontinuous) and $\varepsilon(\eta) \to 0$ as $\eta \to 0$. Then it has a “pseudoinverse” function $\eta(\varepsilon)$ namely a function which is the inverse in the monotonicity points, which is discontinuous where $\varepsilon(\eta)$ is constant and constant where $\varepsilon(\eta)$ is discontinuous. Moreover $\eta(\varepsilon)$ as $\varepsilon \to 0$.

\[\square\]

3.2 Definition of the density $\rho_\varepsilon$

First of all we notice that, in Lemma 12, it is not restrictive to assume that

$$\eta = \eta(\varepsilon) \geq \varepsilon. \tag{52}$$

Now we set

$$\rho_\varepsilon(t, x) = a(x) \ast u(t, x)^2$$

where, $a_\varepsilon(s) \in C^\infty$,

$$a_\varepsilon(s) = \begin{cases} 
3 & |s| \leq \eta^\frac{1}{3} \left(1 - \eta^\frac{1}{3}\right) \\
0 & |s| \geq \eta^\frac{1}{3} \left(1 + \eta^\frac{1}{3}\right)
\end{cases}$$

and

$$|\nabla a_\varepsilon(s)| \leq \eta^{-\frac{1}{3}}. \tag{53}$$
Lemma 13  Take $\psi_\varepsilon$ a solution of (P) with initial data in $B_{\varepsilon,M}$.

If $|x-\hat{q}_\varepsilon(t)| \leq \eta^\frac{1}{3} \left(1 - 2\eta^\frac{1}{3}\right)$ then $\rho_\varepsilon(t,x) \geq 3(1 - \eta)$

if $|x-\hat{q}_\varepsilon(t)| \geq \eta^\frac{1}{3} \left(1 + 2\eta^\frac{1}{3}\right)$ then $\rho_\varepsilon(t,x) \leq 3\eta$

where $\eta = \eta(\varepsilon)$ as in Lemma 12. In particular we have that

$$\Sigma_{\varepsilon,t} \subset B\left(\hat{q}_\varepsilon(t), \eta^\frac{1}{3} \left(1 + 2\eta^\frac{1}{3}\right)\right) \setminus B\left(\hat{q}_\varepsilon(t), \eta^\frac{1}{3} \left(1 - 2\eta^\frac{1}{3}\right)\right) \quad (54)$$

where $\Sigma_{\varepsilon,t}$ is defined by (38).

Proof. If $|x-\hat{q}_\varepsilon| \leq \eta^\frac{1}{3} \left(1 - 2\eta^\frac{1}{3}\right)$, then

$$|x-\hat{q}_\varepsilon| + \sqrt{\varepsilon} \leq \eta^\frac{1}{3} \left(1 - 2\eta^\frac{1}{3}\right) + \sqrt{\eta} \leq \eta^\frac{1}{3} \left(1 - \eta^\frac{1}{3}\right)$$

and hence

$$B(\hat{q}_\varepsilon, \sqrt{\varepsilon}) \subset B\left(x, \eta^\frac{1}{3} \left(1 - \eta^\frac{1}{3}\right)\right).$$

Then, by using Lemma 12,

$$\rho_\varepsilon(t,x) = \int a_\varepsilon(y-x)u_\varepsilon(t,y)^2dy \geq 3 \int_{B(x, \eta^{1/3} - \eta^{1/4})} u_\varepsilon(t,y)^2dy$$

$$\geq 3 \int_{B(\hat{q}_\varepsilon, \varepsilon^{1/2})} u_\varepsilon(t,y)^2dy \geq 3(1 - \eta).$$

If $|x-\hat{q}_\varepsilon(t)| \geq \eta^\frac{1}{3} \left(1 + 2\eta^\frac{1}{3}\right)$,

$$|x-\hat{q}_\varepsilon| - \sqrt{\varepsilon} \geq \eta^\frac{1}{3} \left(1 + 2\eta^\frac{1}{3}\right) - \sqrt{\eta} \geq \eta^\frac{1}{3} \left(1 + \eta^\frac{1}{3}\right)$$

and so

$$B\left(x, \eta^\frac{1}{3} \left(1 + \eta^\frac{1}{3}\right)\right) \subset \mathbb{R}^N \setminus B(\hat{q}_\varepsilon, \sqrt{\varepsilon}).$$

Then, using again Lemma 12,

$$\rho_\varepsilon(t,x) = \int a_\varepsilon(y-x)u_\varepsilon(t,y)^2dy \leq 3 \int_{B(x, \eta^\frac{1}{3}(1+\eta^\frac{1}{3}))} u_\varepsilon(t,y)^2dy$$

$$\leq 3 \int_{\mathbb{R}^N \setminus B(\hat{q}_\varepsilon, \sqrt{\varepsilon})} u_\varepsilon(t,y)^2dy \leq 3\eta$$

$\Box$
Clearly, \( \rho_\varepsilon = a_\varepsilon * u_\varepsilon^2 \in C^1(\mathbb{R}^{N+1}) \) and, by (20), it satisfies the continuity equation (35) with
\[
J_{\rho_\varepsilon} = a_\varepsilon * (u_\varepsilon^2 \nabla S_\varepsilon).
\] (55)

Therefore, the results of Section 2 hold. In particular, we have that the support of \( \Psi_\varepsilon(t,x) \) is contained in \( B(\hat{q}_\varepsilon, \eta^{1/2} (1 + 2\eta^{1/2})) \) when \( \eta \) is sufficiently small (namely \( \eta < 1/3 \)). Moreover, by (54), we see that the size of the halo is an infinitesimal of higher order with respect to the diameter of the soliton.

### 3.3 The equation of dynamics as \( \varepsilon \to 0 \)

**Theorem 14** The following equations hold
\[
\dot{q}_\varepsilon(t) = \frac{p_\varepsilon(t)}{m_\varepsilon(t)} + K_\varepsilon(t)
\] (56)
\[
\dot{p}_\varepsilon = -\nabla V(q_\varepsilon(t)) + F_\varepsilon(q_\varepsilon) + H_\varepsilon(t)
\] (57)
where
\[
\sup_{t \in \mathbb{R}} (|H_\varepsilon(t)| + |K_\varepsilon(t)|) \to 0 \quad \text{as} \quad \varepsilon \to 0
\] (58)
and
\[
F_\varepsilon(q_\varepsilon) = -\int_{\Sigma_\varepsilon,t} T \cdot \nabla \rho_\varepsilon \, dx.
\] (59)

Moreover we have that
\[
\forall \tau_0, \tau_1, \quad \left| \int_{\tau_0}^{\tau_1} F_\varepsilon(q_\varepsilon) \, dt \right| \leq c(\varepsilon) (1 + |\tau_1 - \tau_0|)
\] (60)
where \( c(\varepsilon) \to 0 \) as \( \varepsilon \to 0 \).

**Proof.** We set
\[
K_\varepsilon(t) = \frac{1}{m_\varepsilon} \int_{\Sigma_\varepsilon,t} (x - q_\varepsilon) \left[ u_\varepsilon^2 \nabla S_\varepsilon \cdot \nabla \rho_\varepsilon - \nabla \cdot J_{\rho_\varepsilon} \right] \, dx,
\]
\[
H_{1,\varepsilon}(t) = \int_{\Sigma_\varepsilon,t} u_\varepsilon^2 \nabla S_\varepsilon \left( \nabla \cdot J_{\rho_\varepsilon} \right) \, dx,
\]
\[
H_{2,\varepsilon}(t) = \nabla V(q_\varepsilon(t)) - \int \nabla V(x) |\Psi_\varepsilon|^2 \, dx,
\]
\[
H_\varepsilon(t) = H_{1,\varepsilon}(t) + H_{2,\varepsilon}(t),
\]

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and hence, by Th. 8, we need just to prove (58). We estimate each individual term of $K_\varepsilon$. We have that
\[ \sup_{x \in \Sigma_{\varepsilon,t}} |x - q_\varepsilon| \leq 2 \left( \eta^{1/8} + 2\eta^{1/4} \right) \leq 3\eta^{1/8} \] since $q_\varepsilon(t), x \in B(\hat{q}_\varepsilon, \eta^{1/8} + 2\eta^{1/4})$.

Also, by (53) and well known properties on convolutions,
\[ \sup_{x \in \Sigma_{\varepsilon,t}} |\nabla \rho_\varepsilon| \leq \sup_{x \in \mathbb{R}^N} |\nabla a_\varepsilon(x) * u_\varepsilon(t, x)| \]
\[ \leq \|\nabla a_\varepsilon\|_{L^\infty} \cdot \|u_\varepsilon\|_{L^2} \leq \frac{1}{\eta^{1/4}} \]

If $\psi_\varepsilon(0, x) \in B_{\varepsilon,M}$, by (50), we have
\[ G(\psi) = E_\varepsilon(\psi) - J_\varepsilon(\psi) \leq c_0 \varepsilon^2 + M - c_0 \varepsilon^2 = M; \] so, by Lemma 12,
\[ \int_{\mathbb{R}^N \setminus B(\hat{q}_\varepsilon, \sqrt{\varepsilon})} u_\varepsilon^2 |\nabla S_\varepsilon| \leq \left( \int_{\mathbb{R}^N \setminus B(\hat{q}_\varepsilon, \sqrt{\varepsilon})} u_\varepsilon^2 \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^N \setminus B(\hat{q}_\varepsilon, \sqrt{\varepsilon})} u_\varepsilon^2 |\nabla S_\varepsilon|^2 \right)^{\frac{1}{2}} \]
\[ \leq \eta^{\frac{1}{2}} [2mG(\psi)]^{\frac{1}{2}} \leq \text{const.}\eta^{\frac{1}{2}} \]
and in particular
\[ \int_{\Sigma_{\varepsilon,t}} u_\varepsilon^2 |\nabla S_\varepsilon| \, dx \leq \eta^{\frac{1}{2}} [2mG(\psi)]^{\frac{1}{2}} \leq \text{const.}\eta^{\frac{1}{2}}. \] Finally, by (55)
\[ \sup_{x \in \Sigma_{\varepsilon,t}} |\nabla \cdot J_\rho_\varepsilon| \leq \sup_{x \in \mathbb{R}^N} |(\nabla \cdot a_\varepsilon) \ast (u_\varepsilon^2 |\nabla S_\varepsilon|)| \]
\[ \leq \|\nabla \cdot a_\varepsilon\|_{L^\infty} \cdot \int_{\mathbb{R}^N} u_\varepsilon^2 |\nabla S_\varepsilon| \]
\[ \leq \frac{1}{\eta^{1/4}} \left( \int_{\mathbb{R}^N} u_\varepsilon^2 \right)^{\frac{1}{2}} \cdot \left( \int_{\mathbb{R}^N} u_\varepsilon^2 |\nabla S_\varepsilon| \right)^{\frac{1}{2}} \]
\[ \leq \frac{[2mG(\psi)]^{\frac{1}{2}}}{\eta^{1/4}} = \text{const.}\eta^{-\frac{1}{4}}. \]
|Σε,t| ≤ |B(_accounts, η^\frac{1}{2} (1 + 2η^\frac{1}{2}))| − |B(_accounts, η^\frac{1}{2} (1 − 2η^\frac{1}{2}))|

= ω_N[η^\frac{1}{2} (1 + 2η^\frac{1}{2})]^N − ω_N[η^\frac{1}{2} (1 − 2η^\frac{1}{2})]^N

= ω_Nη^\frac{1}{16} [(1 + 2η^\frac{1}{16})^N − (1 − 2η^\frac{1}{16})^N]

≤ ω_Nη^\frac{1}{16} · 5Nη^\frac{1}{16} ≤ const.η^\frac{1}{8}. \quad (67)

So, by (61), .... (67)

|K_ε(t)| ≤ \int_{Σ_ε,t} |(x - q_ε) \left[ u_ε^2 \nabla S_ε \cdot \nabla ρ_ε - \nabla \cdot J ρ_ε \right]| dx

≤ sup \left| x - q_ε \right| \cdot \left[ \int_{Σ_ε,t} |u_ε^2 \nabla S_ε \cdot \nabla ρ_ε| dx + \int_{Σ_ε,t} |(\nabla \cdot J ρ_ε)| dx \right]

≤ sup \left| x - q_ε \right| \cdot \left[ sup_{x \in Σ_ε,t} |\nabla ρ_ε| \cdot \int_{Σ_ε,t} |u_ε^2 \nabla S_ε| + sup_{x \in Σ_ε,t} |\nabla \cdot J ρ_ε| \cdot \int_{Σ_ε,t} dx \right]

≤ 3η^{1/8} \left[ const.η^{-1/4} \cdot η^{1/2} + const.η^{-1/4} \cdot |Σ_ε,t| \right]

≤ const.η^{1/8} \left[ η^{-1/4} \cdot η^{1/2} + η^{-1/4} \cdot η^\frac{N+1}{8} \right] ≤ const. η^{1/8}.

Then, by Lemma 12,

|K_ε(t)| → 0 \quad (68)

uniformly in t.

Now, let us estimate |H_{1,ε}(t)|; by (66) and (65) we have

|H_{1,ε}(t)| ≤ \int_{Σ_ε,t} |u_ε^2 \nabla S_ε (\nabla \cdot J ρ_ε)| dx

≤ sup \left| x \in Σ_ε,t \right| |\nabla \cdot J ρ_ε| \cdot \int_{Σ_ε,t} |u_ε^2 \nabla S_ε|

≤ const. \frac{1}{η^{1/4}} \cdot η^{1/2} = const. η^{1/4}

By the above estimate,

|H_{1,ε}(t)| → 0. \quad (69)

We recall that

\int |Ψ_ε|^2 = 1 − o(1)
when \( \varepsilon \to 0 \), and that \( \text{supp} \Psi_{\varepsilon} \subset B(\hat{q}_{\varepsilon}, \eta^{1/8} + 2\eta^{1/4}) \). We have

\[
\nabla V(q_\varepsilon(t)) = (1 + o(1)) \int \nabla V(q_\varepsilon(t)) |\Psi_{\varepsilon}|^2
\]

and so

\[
|H_2(t)| = \left| \int \nabla V(x) |\Psi_{\varepsilon}|^2 \, dx - \nabla V(q_\varepsilon(t)) \right|
\]

\[
= \int |\nabla V(x) - \nabla V(q_\varepsilon)| |\Psi_{\varepsilon}|^2 \, dx + o(1) \int \nabla V(q_\varepsilon(t)) |\Psi_{\varepsilon}|^2
\]

\[
\leq \|V''\|_{C^0(\mathbb{R}^N)} \int |x - q_\varepsilon| |\Psi_{\varepsilon}|^2 \, dx + o(1) \int \nabla V(q_\varepsilon(t)) |\Psi_{\varepsilon}|^2
\]

\[
\leq o(1) \left( \|V''\|_{C^0(\mathbb{R}^N)} + \|\nabla V\|_{C^0(\mathbb{R}^N)} \right) = o(1)
\]

for all \( t \). By the above equation, (68) and (69), the (58) follows.

Let \( P = P(\psi_\varepsilon) \) be defined by (31). By the definitions of \( p_\varepsilon \), and (64), for every \( t \in \mathbb{R} \), we have that

\[
|p_\varepsilon(t) - P(t)| = \left| \int \nabla S (|\Psi_{\varepsilon}|^2 - u_\varepsilon^2) \, dx \right|
\]

\[
\leq \int_{\mathbb{R}^N \setminus B(\hat{q}_{\varepsilon}, \sqrt{\varepsilon})} |\nabla S| \, u_\varepsilon^2 \, dx = c_0(\varepsilon) \quad (70)
\]

where \( c_1(\varepsilon) \to 0 \) as \( \varepsilon \to 0 \). By (31)

\[
\dot{P} = \int (-u_\varepsilon^2 \nabla V + \nabla \cdot T) \, dx
\]

and since \( T \in L^1(\mathbb{R}^N) \), \( \dot{P} = -\int u_\varepsilon^2 \nabla V \, dx \). So, by (31) and (59)

\[
\dot{p}_\varepsilon - \dot{P} = \int \nabla V (u_\varepsilon^2 - |\Psi_{\varepsilon}|^2) \, dx - \int_{\Sigma_{\varepsilon,t}} [T \cdot \nabla \rho_\varepsilon + u_\varepsilon^2 \nabla S_\varepsilon (\nabla \cdot J_{\rho_\varepsilon})] \, dx
\]

\[
= \int \nabla V (u_\varepsilon^2 - |\Psi_{\varepsilon}|^2) \, dx + F_\varepsilon(q_\varepsilon) - \int_{\Sigma_{\varepsilon,t}} u_\varepsilon^2 \nabla S_\varepsilon (\nabla \cdot J_{\rho_\varepsilon}) \, dx.
\]

Then, by (69) and Lemma 12

\[
|F_\varepsilon(q_\varepsilon) - (\dot{p}_\varepsilon - \dot{P})| = \left| \int_{\Sigma_{\varepsilon,t}} u_\varepsilon^2 \nabla S_\varepsilon (\nabla \cdot J_{\rho_\varepsilon}) \, dx - \int \nabla V (u_\varepsilon^2 - |\Psi_{\varepsilon}|^2) \, dx \right|
\]

\[
\leq o(1) + \|\nabla V\|_{L^\infty} \int |u_\varepsilon^2 - |\Psi_{\varepsilon}|^2|
\]

\[
\leq o(1) + \|\nabla V\|_{L^\infty} \int_{\mathbb{R}^N \setminus \hat{B}(\varepsilon \sqrt{\varepsilon})} u_\varepsilon^2 \, dx = c_1(\varepsilon)
\]

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where \( c_1(\varepsilon) \to 0 \) as \( \varepsilon \to 0 \). Finally by (70), \( \forall \tau_0, \tau_1 \),
\[
\left| \int_{\tau_0}^{\tau_1} F_\varepsilon(q_\varepsilon) \, dt \right| = \left| \int_{\tau_0}^{\tau_1} (\dot{p}_\varepsilon - \dot{P}) \, dt \right| + c_1(\varepsilon) (\tau_1 - \tau_0)
\leq \left| p_\varepsilon(\tau_1) - P(\tau_1) \right| + \left| p_\varepsilon(\tau_0) - P(\tau_0) \right| + c_1(\varepsilon) (\tau_1 - \tau_0)
\leq 2c_0(\varepsilon) + c_1(\varepsilon) (\tau_1 - \tau_0) \leq c(\varepsilon) (1 + |\tau_1 - \tau_0|)
\]

with a suitable choice of \( c(\varepsilon) \).

\( \square \)

Collecting the previous results, we get our main theorem and Cor. 2

**Proof of Th. 1**

By the def. (36), (37), Lemma 12 and Th. 14, Theorem 1 holds with
\[
R_\varepsilon = \eta^{\frac{1}{2}} \left( 1 + 2\eta^{\frac{1}{2}} \right).
\]

\( \square \)

**Proof of Cor. 2**

We rewrite (17), (19) and (20) in integral form and we get
\[
\begin{align*}
q_\varepsilon(t) &= q_\varepsilon(0) + \int_0^t \frac{p_\varepsilon(s)}{m_\varepsilon(s)} \, ds + \int_0^t K_\varepsilon(s) \, ds \\
p_\varepsilon(t) &= p_\varepsilon(0) - \int_0^t \nabla V(q_\varepsilon(s)) \, ds + \int_0^t [F_\varepsilon(q_\varepsilon) + H_\varepsilon(s)] \, ds 
\end{align*}
\]
and hence, for any \( |t| \leq T \)
\[
|q_\varepsilon(t) - q(t)| \leq \int_0^t \left| \frac{p_\varepsilon(s)}{m_\varepsilon(s)} - \frac{p(s)}{m} \right| \, ds + \int_0^t |K_\varepsilon(s)| \, ds
\leq L_1 \int_0^t |p_\varepsilon(s) - p(s)| \, ds + \alpha_1(\varepsilon)
\]
where, by (58), \( \alpha_1(\varepsilon) \to 0 \) as \( \varepsilon \to 0 \) and
\[
|p_\varepsilon(t) - p(t)| \leq \int_0^t |\nabla V(q_\varepsilon(s)) - \nabla V(q(s))| \, ds + \int_0^t |F_\varepsilon(q_\varepsilon)| \, ds + \int_0^t |H_\varepsilon(s)| \, ds
\leq L_2 \int_0^t |q_\varepsilon(s) - q(s)| \, ds + \alpha_2(\varepsilon)
\]
where, by (58), \( \alpha_2(\varepsilon) \to 0 \) as \( \varepsilon \to 0 \). Then, setting \( z_\varepsilon(t) = |q_\varepsilon(t) - q(t)| + |p_\varepsilon(t) - p(t)| \), we have
\[
z_\varepsilon(t) \leq L \int_0^t z_\varepsilon(s) \, ds + \alpha(\varepsilon)
\]
with a suitable choice of $L$ and $\alpha(\varepsilon)$. Now, by the Gronwall inequality, we have
\[ z_\varepsilon(t) \leq \alpha(\varepsilon)e^{Lt} \]
and from here, we get the conclusion.
\[ \square \]

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