Lit-only sigma-game on nondegenerate graphs

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Abstract A configuration of the lit-only \( \sigma \)-game on a graph \( \Gamma \) is an assignment of one of two states, on or off, to each vertex of \( \Gamma \). Given a configuration, a move of the lit-only \( \sigma \)-game on \( \Gamma \) allows the player to choose an on vertex \( s \) of \( \Gamma \) and change the states of all neighbors of \( s \). Given an integer \( k \), the underlying graph \( \Gamma \) is said to be \( k \)-lit if for any configuration, the number of on vertices can be reduced to at most \( k \) by a finite sequence of moves. We give a description of the orbits of the lit-only \( \sigma \)-game on nondegenerate graphs \( \Gamma \) which are not line graphs. We show that these graphs \( \Gamma \) are 2-lit and provide a linear algebraic criterion for \( \Gamma \) to be 1-lit.

Keywords Group action · Lit-only \( \sigma \)-game · Nondegenerate graph

Mathematics Subject Classification Primary 05C57; Secondary 15A63 · 20F55

1 Introduction

The notion of the \( \sigma \)-game on finite graphs \( \Gamma \) was first introduced by Sutner [17, 18] around 1989. A configuration of the \( \sigma \)-game on \( \Gamma \) is an assignment of one of two states, on or off, to each vertex of \( \Gamma \). Given a configuration, a move consists of choosing a vertex of \( \Gamma \), followed by changing the states of all of its neighbors. If only on vertices can be chosen in each move, we come to the variation: lit-only \( \sigma \)-game. Starting from an initial configuration, the goal of the lit-only \( \sigma \)-game on \( \Gamma \) is to minimize the number of on vertices of \( \Gamma \), or to reach an assigned configuration by a finite sequence of moves.

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Given an integer \( k \), the underlying graph \( \Gamma \) is said to be \( k \)-lit if for any configuration, the number of on vertices can be reduced to at most \( k \) by a finite sequence of moves. More precisely, we are interested in the orbits of the lit-only \( \sigma \)-game on \( \Gamma \) and the smallest integer \( k \), the minimum light number of \( \Gamma \) [19], for which \( \Gamma \) is \( k \)-lit. The notion of lit-only \( \sigma \)-games occurred implicitly in the study of equivalence classes of Vogan diagrams. The Borel-de Siebenthal theorem [2] showed that every Vogan diagram is equivalent to one with a single-painted vertex, which implies that each simply-laced Dynkin diagram is 1-lit. The equivalence classes of Vogan diagrams were described by Chuah and Hu [7]. A conjecture made by Chang [5, 6] that any tree with \( k \) leaves is \( \lceil k/2 \rceil \)-lit was confirmed by Wang and Wu [19], where the name “lit-only \( \sigma \)-game” was coined.

The lit-only \( \sigma \)-game on a simple graph \( \Gamma \) is simply the natural action of a certain subgroup \( H_\Gamma \) of the general linear group over \( \mathbb{F}_2 \) [19]. Under the assumption that \( \Gamma \) is the line graph of a simple graph \( G \), Wu [21] described the orbits of the lit-only \( \sigma \)-game on \( \Gamma \) and gave a characterization for the minimum light number of \( \Gamma \). Moreover, if \( G \) is a tree of order \( n \geq 3 \), Wu showed that \( H_\Gamma \) is isomorphic to the symmetric group on \( n \) letters. Weng and the author [13] determined the structure of \( H_\Gamma \) without any assumption on \( G \). The lit-only \( \sigma \)-game on a simple graph \( \Gamma \) can also be considered as a representation \( \kappa_\Gamma \) of the simply-laced Coxeter group \( W_\Gamma \) over \( \mathbb{F}_2 \) [12]. The dual representation of \( \kappa_\Gamma \) preserves a certain symplectic form \( B_\Gamma \). The two representations are equivalent whenever the form \( B_\Gamma \) is nondegenerate. From this viewpoint it is natural to partition simple connected graphs into two classes according as \( B_\Gamma \) is degenerate or nondegenerate.

In this paper, we treat nondegenerate graphs \( \Gamma \) which are not line graphs. We show that \( H_\Gamma \) is isomorphic to an orthogonal group, followed by a description of the orbits of lit-only \( \sigma \)-game on \( \Gamma \) (Theorem 3.1). Moreover, we show that these graphs \( \Gamma \) are 2-lit and provide a linear algebraic criterion for \( \Gamma \) to be 1-lit (Theorem 3.2). Combining Theorem 3.1, Theorem 3.2, and those in [13, 21], the study of the lit-only \( \sigma \)-game on nondegenerate graphs is quite completed, and the focus for further research is on degenerate graphs.

## 2 Preliminaries

From now on, let \( \Gamma = (S, R) \) denote a finite simple connected graph with vertex set \( S \) and edge set \( R \). Let \( \mathbb{F}_2 \) denote the two-element field \( \{0, 1\} \). Let \( V \) denote an \( \mathbb{F}_2 \)-vector space that has a basis \( \{ \alpha_s \mid s \in S \} \) in one-to-one correspondence with \( S \). Let \( V^* \) denote the dual space of \( V \). For each \( s \in S \), we define \( f_s \in V^* \) by

\[
    f_s(\alpha_t) = \begin{cases} 
        1 & \text{if } s = t, \\
        0 & \text{else}
    \end{cases}
\]

for all \( t \in S \). The set \( \{ f_s \mid s \in S \} \) forms a basis of \( V^* \) and is called the \textit{basis of } \( V^* \text{ dual to } \{ \alpha_s \mid s \in S \} \). Each configuration \( f \) of the lit-only \( \sigma \)-game on \( \Gamma \) is interpreted as the vector

\[
    \sum_{\text{on vertices } s} f_s \in V^*.
\]
If all vertices of $\Gamma$ are assigned the off state by $f$, we interpret (2) as the zero vector of $V^*$. Given $s \in S$ and $f \in V^*$ observe that $f(\alpha_s) = 1$ (resp. 0) if and only if the vertex $s$ is assigned the on (resp. off) state by $f$.

For each $s \in S$ define a linear transformation $\kappa_s : V^* \to V^*$ by

$$\kappa_s f = f + f(\alpha_s) \sum_{st \in R} f_t$$

for all $f \in V^*$. (3)

Fix a vertex $s$ of $\Gamma$. Given any $f \in V^*$, if the state of $s$ is on, then $\kappa_s f$ is obtained from $f$ by changing the states of all neighbors of $s$, and $\kappa_s f = f$ otherwise. Therefore, we may view $\kappa_s$ as the move of the lit-only $\sigma$-game on $\Gamma$ for which we choose the vertex $s$ and change the states of all neighbors of $s$ if the state of $s$ is on. In particular $\kappa_s^2 = 1$. For any vector space $U$, let $GL(U)$ denote the general linear group of $U$. Then $\kappa_s \in GL(V^*)$ for all $s \in S$. The subgroup $H = H_{\Gamma}$ of $GL(V^*)$ generated by the $\kappa_s$ for all $s \in S$ was first mentioned by Wu [19], which is called the flipping group of $\Gamma$ in [12] and the lit-only group of $\Gamma$ in [21].

The lit-only groups are closely related to the simply-laced Coxeter groups in the following way. Recall that the simply-laced Coxeter group $W = W_{\Gamma}$ associated with $\Gamma = (S, R)$ is the group generated by all elements $s \in S$ subject to the relations

$$s^2 = 1,$$

$$(st)^2 = 1 \quad \text{if } st \notin R,$$

$$(st)^3 = 1 \quad \text{if } st \in R$$

for all $s, t \in S$. By [12, Theorem 3.2], there exists a unique representation $\kappa = \kappa_{\Gamma} : W \to GL(V^*)$ such that $\kappa(s) = \kappa_s$ for all $s \in S$. Clearly $\kappa(W) = H$. Given any $f, g \in V^*$ observe that $g$ can be obtained from $f$ by a finite sequence of moves of the lit-only $\sigma$-game on $\Gamma$ if and only if there exists $w \in W$ such that $g = \kappa(w)f$. Given an integer $k$, the underlying graph $\Gamma$ is $k$-lit if and only if for each $\kappa(W)$-orbit $O$ on $V^*$, there exists a subset $K$ of $S$ with size at most $k$ such that $\sum_{s \in K} f_s \in O$.

We now give the definitions of degenerate and nondegenerate graphs. Let $B = B_{\Gamma}$ denote the symplectic form on $V$ defined by

$$B(\alpha_s, \alpha_t) = \begin{cases} 1 & \text{if } st \in R, \\ 0 & \text{else} \end{cases}$$

for all $s, t \in S$ [16]. The radical of $V$ (relative to $B$) is the subspace of $V$ consisting of the vectors $\alpha$ that satisfy $B(\alpha, \beta) = 0$ for all $\beta \in V$. The form $B$ is said to be degenerate whenever the radical of $V$ is nonzero and nondegenerate otherwise. The graph $\Gamma$ is said to be degenerate whenever the form $B$ is degenerate, and nondegenerate otherwise. The form $B$ induces a linear map $\theta : V \to V^*$ given by

$$\theta(\alpha) \beta = B(\alpha, \beta)$$

for all $\alpha, \beta \in V$. (5)
Since the kernel of $\theta$ is the radical of $V$ and the matrix representing $B$ with respect to the basis $\{\alpha_s \mid s \in S\}$ is the adjacency matrix of $\Gamma$ over $\mathbb{F}_2$, the following lemma is straightforward.

**Lemma 2.1** Let $A$ denote the adjacency matrix of $\Gamma$ over $\mathbb{F}_2$. Then the following are equivalent:

(i) $\Gamma$ is a nondegenerate graph.
(ii) $\theta$ is an isomorphism of vector spaces.
(iii) $A$ is invertible.

Recall that given a simple graph $G$, the line graph of $G$ is a simple graph that has a vertex for each edge of $G$, and two of these vertices are adjacent whenever the corresponding edges in $G$ have a common vertex. The purpose of this paper is to investigate the lit-only $\sigma$-game on nondegenerate graphs which are not line graphs. Thus, it is natural to ask how to determine if a nondegenerate graph is a line graph. We will give two characterizations of nondegenerate line graphs as Proposition 2.4 below.

**Lemma 2.2** Let $G$ denote a finite simple connected graph of order $n$. Assume that $\Gamma_1$ is the line graph of $G$. Then $\theta(V)$ has dimension $n - 1$ if $n$ is odd and has dimension $n - 2$ if $n$ is even.

**Proof** Let $U$ denote the vertex space of $G$ over $\mathbb{F}_2$. Define a linear map $\mu : V \to U$ by

$$\mu(\alpha_s) = u + v$$

for all $s \in S$, where $u$ and $v$ are the two endpoints of $s$ in $G$. Since $G$ is connected, the image of $\mu$ is the subspace of $U$ consisting of these vectors each of which equals the sum of an even number of vertices of $U$. Define a linear map $\lambda : U \to V^*$ by

$$\lambda(u)\alpha_s = \begin{cases} 
1 & \text{if } u \text{ is incident to } s \text{ in } G, \\
0 & \text{else}
\end{cases}$$

for all $u \in U$ and for all $s \in S$. There is only one nonzero vector, the sum of all vertices of $G$, in the kernel of $\lambda$. Since $\theta = \lambda \circ \mu$ and by the above comments, the result follows. \hfill \Box

A *claw* is a tree with one internal vertex and three leaves. A simple graph is said to be *claw-free* if it does not contain a claw as an induced subgraph. A *cut-vertex* of $\Gamma$ is a vertex of $\Gamma$ whose deletion increases the number of components. A *block* of $\Gamma$ is a maximal connected subgraph of $\Gamma$ without cut-vertices. A *block graph* is a simple connected graph in which every block is a complete graph.

**Lemma 2.3** [10, Theorem 8.5]. Let $\Gamma_1$ denote a simple connected graph. Then $\Gamma_1$ is the line graph of a tree if and only if $\Gamma_1$ is a claw-free block graph.

The following proposition follows by combining Lemmas 2.1–2.3.
Proposition 2.4 Let $\Gamma$ denote a simple connected graph. Then the following are equivalent:

(i) $\Gamma$ is a nondegenerate line graph.
(ii) $\Gamma$ is the line graph of an odd-order tree.
(iii) $\Gamma$ is a claw-free block graph of even order.

3 Main results

A quadratic form $Q$ on $V$ is a function $Q : V \rightarrow \mathbb{F}_2$ satisfying

$$Q(\alpha + \beta) = Q(\alpha) + Q(\beta) + B(\alpha, \beta) \quad \text{for all } \alpha, \beta \in V.$$  \hspace{1cm} (6)

Given a quadratic form $Q$ on $V$, the orthogonal group with respect to $Q$ is the subgroup of $\text{GL}(V)$ consisting of all $\sigma \in \text{GL}(V)$ such that $Q(\sigma \alpha) = Q(\alpha)$ for all $\alpha \in V$. Given a basis $P$ of $V$ we define $Q_P$ to be the unique quadratic form on $V$ with $Q_P(\alpha) = 1$ for all $\alpha \in P$.

For the rest of this paper, the form $B$ is assumed to be nondegenerate. Moreover, let $Q = Q_P$ where $P = \{\alpha_s \mid s \in S\}$ and let $O(V)$ denote the orthogonal group with respect to $Q$. By (6), for any $T \subseteq S$ a combinatorial interpretation of $Q(\sum_{s \in T} \alpha_s)$ is the parity of the number of vertices and edges on the subgraph of $\Gamma$ induced by $T$.

We now can state the main results of this paper, which are Theorem 3.1, Theorem 3.2, and Corollary 3.3.

Theorem 3.1 Assume that $\Gamma$ is a nondegenerate graph, but not a line graph. Then $\kappa(W)$ is isomorphic to $O(V)$. Moreover, the $\kappa(W)$-orbits on $V^*$ are

$$\{0\}, \theta(Q^{-1}(0) \setminus \{0\}), \theta(Q^{-1}(1)).$$

Under the assumption that $B$ is nondegenerate, the number $|S| = 2m$ is even and there exists a basis $\{\beta_1, \gamma_1, \ldots, \beta_m, \gamma_m\}$ of $V$ such that $B(\beta_i, \beta_j) = 0$, $B(\gamma_i, \gamma_j) = 0$ and

$$B(\beta_i, \gamma_j) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{else} \end{cases}$$

for all $1 \leq i, j \leq m$. Such a basis $\{\beta_1, \gamma_1, \ldots, \beta_m, \gamma_m\}$ of $V$ is called a symplectic basis of $V$. The Arf invariant of $Q$ is defined to be

$$\text{Arf}(Q) = \sum_{i=1}^{m} Q(\beta_i)Q(\gamma_i),$$

which is independent of the choice of the symplectic basis $\{\beta_1, \gamma_1, \ldots, \beta_m, \gamma_m\}$ of $V$ (for example see [1] or [9, Theorem 13.13]). Any two quadratic forms over $\mathbb{F}_2$ are equivalent if and only if they have the same Arf invariant and the underlying spaces have the same dimension (for example see [1] or [9, Proposition 13.14]). The order of
$O(V)$ and the sizes of nontrivial $O(V)$-orbits on $V$ are as follows (cf. [9, Chapter 14]). If $\operatorname{Arf}(Q) = 0$ then

\[
|O(V)| = 2^{m^2} - m + 1(2^m - 1)(2^2 - 1)(2^4 - 1) \cdots (2^{2m-2} - 1),
\]
\[
|Q^{-1}(1)| = 2^{2m-1} - 2^{m-1},
\]
\[
|Q^{-1}(0)\setminus\{0\}| = 2^{2m-1} + 2^{m-1} - 1.
\]

If $\operatorname{Arf}(Q) = 1$ then

\[
|O(V)| = 2^{m^2} - m + 1(2^m + 1)(2^2 - 1)(2^4 - 1) \cdots (2^{2m-2} - 1),
\]
\[
|Q^{-1}(1)| = 2^{2m-1} + 2^{m-1},
\]
\[
|Q^{-1}(0)\setminus\{0\}| = 2^{2m-1} - 2^{m-1} - 1.
\]

For each $s \in S$, there exists $\alpha_s^\vee \in V$ such that

\[
B(\alpha_s^\vee, \alpha_t) = \begin{cases} 1 & \text{if } s = t, \\ 0 & \text{else} \end{cases}
\] (7)

for all $t \in S$. The set $\{\alpha_s^\vee | s \in S\}$ forms a basis of $V$ and is called the basis of $V$ dual to $\{\alpha_s | s \in S\}$ (with respect to $B$).

**Theorem 3.2** Assume that $\Gamma = (S, R)$ is a nondegenerate graph, but not a line graph. Then $\Gamma$ is 2-lit. Moreover, the following are equivalent:

(i) $\Gamma$ is 1-lit.

(ii) The restriction of $Q$ to $\{\alpha_s^\vee | s \in S\}$ is surjective.

When the nondegenerate graph $\Gamma$ is bipartite, Theorem 3.2 can be improved as follows.

**Corollary 3.3** Assume that $\Gamma$ is a nondegenerate bipartite graph. Then $\Gamma$ is 2-lit. Moreover, the following are equivalent:

(i) $\Gamma$ is 1-lit

(ii) $\Gamma$ contains a vertex with even degree or $\Gamma$ is a single edge.

As consequences of Corollary 3.3, we obtain two families of 1-lit graphs as follows.

- A tree is nondegenerate if and only if it has a perfect matching. By [11, Lemma 2.4], a tree with a perfect matching satisfies Corollary 3.3(ii) and is therefore 1-lit (cf. [14, Theorem 1.1]). This result gives a partial affirmative answer for [20, Conjecture 7].

- For any two positive integers $m$ and $n$, the $m \times n$ grid is nondegenerate if and only if $m + 1$ and $n + 1$ are coprime [18]. By Corollary 3.3 any such $m \times n$ grid is 1-lit. This result partially improves [8, Theorem 26].
The following example shows that Corollary 3.3 is no longer true if the assumption of $\Gamma$ is the same as that of Theorem 3.2. Consider the graph $\Gamma = (S, R)$ as below.

The graph $\Gamma = (S, R)$ is nondegenerate and not a block graph. Therefore $\Gamma$ is not a line graph by Proposition 2.4. The basis $\{\alpha_1^\vee, \alpha_2^\vee, \ldots, \alpha_6^\vee\}$ of $V$ dual to $\{\alpha_1, \alpha_2, \ldots, \alpha_6\}$ can be expressed as follows.

\[
\begin{align*}
\alpha_1^\vee &= \alpha_2 + \alpha_6, \\
\alpha_2^\vee &= \alpha_1 + \alpha_3 + \alpha_5 + \alpha_6, \\
\alpha_3^\vee &= \alpha_2 + \alpha_4 + \alpha_5,
\end{align*}
\]
\[
\begin{align*}
\alpha_4^\vee &= \alpha_3 + \alpha_5, \\
\alpha_5^\vee &= \alpha_2 + \alpha_3 + \alpha_4 + \alpha_6, \\
\alpha_6^\vee &= \alpha_1 + \alpha_2 + \alpha_5.
\end{align*}
\]

A direct computation shows that $Q(\alpha_s^\vee) = 0$ for all $s \in S$. Therefore $\Gamma$ is not 1-lit by Theorem 3.2, but the vertices 2,5 have even degree in $\Gamma$.

4 Proof of Theorem 3.1

To prove Theorem 3.1, we consider a family of linear transformations on $V$ defined as follows. For $\alpha \in V$, the transvection on $V$ with direction $\alpha$ is a linear transformation $\tau_\alpha : V \to V$ defined by

\[
\tau_\alpha \beta = \beta + B(\beta, \alpha)\alpha \quad \text{for all } \beta \in V.
\]

Observe that $\tau_\alpha$ preserves the form $B$ and that $\tau_\alpha \in \text{GL}(V)$ since $\tau_\alpha^2 = 1$.

For a subset $P$ of $V$ define $T_\alpha(P)$ to be the subgroup of $\text{GL}(V)$ generated by $\tau_\alpha$ for $\alpha \in P$, and define $G(P)$ to be the simple graph whose vertex set is $P$ and where $\alpha, \beta$ in $P$ form an edge if and only if $B(\alpha, \beta) = 1$. For any two linearly independent sets $P$ and $P'$ of $V$, we say that $P'$ is elementary $t$-equivalent to $P$ whenever there exist $\alpha, \beta \in P$ such that $P'$ is obtained from $P$ by changing $\beta$ to $\tau_\alpha \beta$. The equivalence relation generated by the elementary $t$-equivalence relation is called the $t$-equivalence relation [3].

Lemma 4.1 [3, Theorem 3.3]. Let $P$ denote a linearly independent set of $V$. Assume that $G(P)$ is a connected graph. Then there exists $P'$ in $t$-equivalence class of $P$ for which $G(P')$ is a tree.

Lemma 4.2 [15, Lemma 3.7]. Let $P$ denote a linearly independent set of $V$. Assume that $G(P)$ is the line graph of a tree. Then, for each $P'$ in the $t$-equivalence class of $P$, the graph $G(P')$ is the line graph of a tree.
A basis $P$ of $V$ is said to have orthogonal type [4] if $P$ is $t$-equivalent to some $P'$ for which $G(P')$ is a tree containing the graph as a subgraph.

**Lemma 4.3** Assume that $P$ is a basis of $V$ for which $G(P)$ is a tree, but not a path. Then $P$ is of orthogonal type.

**Proof** Since $G(P)$ is not a path it contains a vertex $\alpha$ with degree at least three. If any two neighbors of $\alpha$, say $\beta$ and $\gamma$, are leaves of $G(P)$, then $\beta + \gamma$ lies in the radical of $V$, which contradicts that $B$ is nondegenerate. Therefore, at most one neighbor of $\alpha$ is a leaf in $G(P)$ and so $P$ is of orthogonal type.

**Lemma 4.4** [4, Section 10]. Let $P$ denote a basis of $V$ which is of orthogonal type. Then $Tv(P)$ is the orthogonal group with respect to $Q_P$. Moreover, the $Tv(P)$-orbits on $V$ are

$$\{0\}, \quad Q_P^{-1}(0) \setminus \{0\}, \quad Q_P^{-1}(1).$$

**Proof of Theorem 3.1.** For each $s \in S$, let $\tau_s$ denote the transvection on $V$ with direction $\alpha_s$. By [16, Section 5], there exists a unique representation $\tau = \tau_\Gamma : W \to \text{GL}(V)$ such that $\tau(s) = \tau_s$ for all $s \in S$. For each $w \in W$ the transpose of $\tau(w^{-1})$ is equal to $\kappa(w)$. Therefore $\kappa$ is the dual representation of $\tau$. Since $\tau$ preserves the form $B$ we have

$$\theta \circ \tau(w) = \kappa(w) \circ \theta \quad \text{for all } w \in W. \tag{8}$$

Let $P = \{\alpha_s \mid s \in S\}$. Clearly $Tv(P) = \tau(W)$ and $G(P)$ is (isomorphic to) $\Gamma$. By Lemma 4.1 there exists $P'$ in $t$-equivalence class of $P$ for which $G(P')$ is a tree. Since $G(P)$ is not a line graph, the tree $G(P')$ is not a path by Lemma 4.2. By Lemma 4.3 the basis $P'$ of $V$, as well as $P$, is of orthogonal type. By Lemma 4.4, the group $\tau(W) = O(V)$ and the $\tau(W)$-orbits on $V$ are $\{0\}, Q^{-1}(0) \setminus \{0\}$, and $Q^{-1}(1)$. Applying (8) and since $\theta$ is an isomorphism by Lemma 2.1, the result follows. \hfill \Box

## 5 Proof of Theorem 3.2 and Corollary 3.3

Recall the basis $\{\alpha^\wedge_s \mid s \in S\}$ of $V$ from (7). To prove Theorem 3.2 and Corollary 3.3, we introduce a simple graph which includes the information of the values $B(\alpha^\wedge_s, \alpha^\wedge_t)$ for all $s, t \in S$ as follows.

Define $R^\wedge$ to be the set consisting of all two-element subsets $\{s, t\}$ of $S$ with $B(\alpha^\wedge_s, \alpha^\wedge_t) = 1$. Define $\Gamma^\wedge$ to be the simple graph with vertex set $S$ and edge set $R^\wedge$. We will refer to $\Gamma^\wedge$ as the dual graph of $\Gamma$. Note that the notion of dual graphs defined above is different from the usual ones in graph theory. The following lemma suggests why the graph $\Gamma^\wedge$ is of interest.
Lemma 5.1  For each \( s \in S \) we have \( \theta(\alpha_s^\vee) = f_s \).

Proof  Let \( s, t \in S \) be given. Using (5) and (7), we have \( \theta(\alpha_s^\vee)\alpha_t = 1 \) whenever \( s = t \) and otherwise \( \theta(\alpha_s^\vee)\alpha_t = 0 \). Comparing this with (1) the result follows. \( \square \)

Lemma 5.2  For each \( s \in S \) we have

\[
\alpha_s = \sum_{st \in R} \alpha_t^\vee.
\]

Proof  Fix \( s \in S \). By (1), (4), and (5), the vector \( \theta(\alpha_s) \) is equal to

\[
\sum_{st \in R} f_t.
\]

By Lemma 5.1 the above is equal to

\[
\theta\left( \sum_{st \in R} \alpha_t^\vee \right).
\]

Now, by Lemma 2.1(ii) this lemma follows. \( \square \)

Observe that \( B_{\Gamma^\vee} \) is equivalent to \( B \). Therefore \( \Gamma^\vee \) is a nondegenerate graph. Since \( \{ \alpha_s \mid s \in S \} \) is the basis of \( V \) dual to \( \{ \alpha_s^\vee \mid s \in S \} \), the graph \( \Gamma \) is the dual graph of \( \Gamma^\vee \). By duality Lemma 5.2 implies that

Lemma 5.3  For each \( s \in S \) we have

\[
\alpha_s^\vee = \sum_{st \in R^\vee} \alpha_t^\vee.
\]

Lemma 5.4  Let \( A \) and \( A^\vee \) denote the adjacency matrices of \( \Gamma \) and \( \Gamma^\vee \) over \( \mathbb{F}_2 \), respectively. Then \( A \) and \( A^\vee \) are inverses of each other.

Proof  We show that \( A^\vee A \) is equal to the identity matrix. Let \( s, t \in S \) be given. By the comment below Lemma 5.1 the \((s, t)\)-entry of \( A \) (resp. \( A^\vee \)) is equal to \( B(\alpha_s, \alpha_t) \) (resp. \( B(\alpha_s^\vee, \alpha_t^\vee) \)). By the definition of \( \Gamma^\vee \) the \((s, t)\)-entry of \( A^\vee A \) is equal to

\[
B\left( \sum_{su \in R^\vee} \alpha_u, \alpha_t \right).
\]

By Lemma 5.3 the vector in the first coordinate of (9) is equal to \( \alpha_s^\vee \). Therefore (9) is equal to 1 if and only if \( s = t \) by (7). The result follows. \( \square \)

We are now ready to prove Theorem 3.2.

Proof of Theorem 3.2  In Lemma 5.1 we saw that \( \theta(\alpha_s^\vee) = f_s \) for all \( s \in S \). Therefore (i) and (ii) are equivalent by Theorem 3.1. To show that \( \Gamma \) is 2-lit, it is now enough to consider the two cases: (a) \( Q(\alpha_s^\vee) = 0 \) for all \( s \in S \); (b) \( Q(\alpha_s^\vee) = 1 \) for all \( s \in S \).
(a) It suffices to show that there exist \( s, t \in S \) such that \( Q(\alpha^+_s + \alpha^-_t) = 1 \). Since the form \( B \) is non-trivial there exist \( s, t \in S \) such that \( B(\alpha^+_s, \alpha^-_t) = 1 \). Then the \( s \) and \( t \) are the desired elements in \( S \).

(b) It suffices to show that there exist two distinct \( s, t \in S \) such that \( Q(\alpha^+_s + \alpha^-_t) = 0 \). By our assumption, the graph \( \Gamma^\vee \) is not a complete graph. Using Lemma 5.4, we deduce that \( \Gamma^\vee \) is not a complete graph. Therefore there exist two distinct \( s, t \in S \) such that \( B(\alpha^+_s, \alpha^-_t) = 0 \). Such \( s \) and \( t \) are the desired elements in \( S \).

To prove Corollary 3.3, we give a sufficient condition for Theorem 3.2(ii).

**Lemma 5.5** Let \( \Gamma = (S, R) \) denote a nondegenerate graph. Assume that there exists \( s \in S \) with even degree in \( \Gamma \) such that

\[
\sum_{\{u, v\} \subseteq S \atop su, sv \in R} B(\alpha^+_u, \alpha^-_v) = 0,
\]

where the sum is over all two-element subsets \( \{u, v\} \) of \( S \) with \( su, sv \in R \). Then the restriction of \( Q \) to \( \{\alpha^-_t \mid st \in R\} \) is surjective.

**Proof** Apply \( Q \) to either side of the equation in Lemma 5.2. Using (6), (10) and \( Q(\alpha^-_s) = 1 \) to evaluate the resulting equation, we obtain that

\[
\sum_{st \in R} Q(\alpha^-_t) = 1.
\]

By (11) there exists a neighbor \( u \) of \( s \) for which \( Q(\alpha^-_u) = 1 \). Since \( s \) has even degree in \( \Gamma \) there exists a neighbor \( v \) of \( s \) for which \( Q(\alpha^-_v) = 0 \). The result follows. \( \square \)

**Proof of Corollary 3.3.** By Proposition 2.4 a nondegenerate bipartite graph \( \Gamma \) is a line graph if and only if \( \Gamma^\vee \) is a path of even order. Since every path is 1-lit, this corollary holds for \( \Gamma^\vee \) as a line graph. We thus assume that \( \Gamma^\vee \) is not a line graph. By Theorem 3.2 the graph \( \Gamma \) is 2-lit. By Lemma 5.4 we deduce that the graph \( \Gamma^\vee \) is bipartite with the same bipartition that of \( \Gamma \). We use this to show that (i) and (ii) are equivalent.

(ii) \( \Rightarrow \) (i): Let \( s \) denote a vertex of \( \Gamma \) with even degree. Since \( \Gamma \) and \( \Gamma^\vee \) are bipartite graphs with same bipartition, we deduce that \( B(\alpha^+_u, \alpha^-_v) = 0 \) for any neighbors \( u, v \) of \( s \) in \( \Gamma \). Therefore (10) holds. By Lemma 5.5 the restriction of \( Q \) to \( \{\alpha^-_t \mid st \in R\} \) is onto. Therefore \( \Gamma \) is 1-lit by Theorem 3.2.

(i) \( \Rightarrow \) (ii): Suppose on the contrary that each vertex of \( \Gamma \) has odd degree. Using Lemma 5.4, we deduce that each vertex of \( \Gamma^\vee \) has odd degree. Let \( s \) denote any element of \( S \). By Lemma 5.3, \( Q(\alpha^+_s) \) is equal to

\[
Q\left(\sum_{st \in R^\vee} \alpha^-_t\right).
\]

Since the bipartite graphs \( \Gamma \) and \( \Gamma^\vee \) have the same bipartition, we deduce that \( B(\alpha^+_u, \alpha^-_v) = 0 \) for any neighbors \( u, v \) of \( s \) in \( \Gamma^\vee \). By (6), the summation in (12)
can be moved out front. Since \( Q(\alpha_s) = 1 \) for all \( s \in S \), it follows that (12) is equal to 1, contradicting Theorem 3.2(ii). 

\[ \square \]

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