Taylor Series Method for Continuous Linear-Quadratic Regulators

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Abstract: A simple and efficient algorithm for finite-horizon continuous time LQR feedback control computation is presented. The method uses only basic linear algebra and does not require the solution of algebraic Riccati equation. The developed algorithm is applied to the benchmark problem of double inverted pendulum stabilization.

Keywords: Linear quadratic regulators, Riccati equations

1. INTRODUCTION

A generic mechanical system is described by Lagrange equations

\[ M(q)\ddot{q} + D(q, \dot{q})\dot{q} + K(q, u) = 0 \]  (1)

with \( M \) and \( D \) being the matrices with non-linear entries depending on the generalized coordinates \( q \) and velocities \( \dot{q} \) and \( K \) being the vector of generalized forces. Linearization near the controlled equilibrium point reduces (1) to the control-affine system

\[ \dot{x} = Ax + Bu. \]  (2)

To construct an LQR for a given interval \([0, T]\) one has to solve a Riccati ODE given in Sontag (1998) for the matrix \( P(t) \):

\[ \dot{P} = -A^T P - PA + PBR^{-1}B^T P - Q, \]  (3)

where the constant positive definite matrices \( R \) and \( Q \) define the cost functional

\[ J = \int_0^T x^T Q x + u^T Ru \]  (4)

and the initial condition is \( P(T) = 0 \). The feedback control is then \( u(x) = -R^{-1}B^T P(t)x \).

The core of the method is the determination of \( P(t) \) to which we now turn. To solve (3) we use the Taylor Series Method stated for a general ODE system with polynomial right-hand sides as given by Babadzhanjanz (2006). The next subsection provides a short introduction to the method.

2. TAYLOR SERIES METHOD

According to the general theory of Graça (2008) and by the introduction of additional variables of Babadzhanjanz (2006) a wide class of non-linear ODEs can be written as a system of \( n \) equations with polynomial right-hand sides

\[ \dot{z}_k = f_k + \sum_{m=1}^{N} \sum_{i_1=1}^{n} \cdots \sum_{i_m=1}^{N} f_{k;i_1,i_2,\ldots,i_m} z_{i_1} \cdots z_{i_m}. \]  (5)

The initial conditions for the moment \( t = t_0 \) are

\[ z_k(t_0) = z_{k,0}, \quad k = 1, \ldots, n. \]  (6)

In (5) \( f_k, f_{k;i_1,i_2} \) etc. are the linear, quadratic and all the higher-order constant coefficients of the polynomial. The number \( N \) is the maximum degree of the polynomials in right-hand sides, \( k \) is the index of the equations, the sum \( \sum_{m=1}^{N} \) denotes a sequence of embedded sums. For \( N = 1 \) this is a single sum representing linear terms, for \( N = 2 \) this is a double sum representing quadratic terms and so on.

Here we consider only the \( N = 2 \) case with the quadratic right-hand side

\[ \dot{z}_k = f_k + \sum_{i=1}^{n} f_{k;i} z_i + \sum_{i=1}^{n} \sum_{j=1}^{n} f_{k;i,j} z_i z_j, \]  (7)

since the LQR Riccati Equation (3) has the form (7), which we show in the following section.

The idea of Taylor Series Method is to expand the solution \( z_k(t) \) in power series near \( t = t_0 \)

\[ z_k(t) = \sum_{q=0}^{+\infty} z_{k,q} (t - t_0)^q. \]  (8)

We denote the \( M \)-th order approximation to the solution of (7) as

\[ T_M z_k(t) = \sum_{q=0}^{M} z_{k,q} (t - t_0)^q. \]  (9)

Zero-order coefficients coincide with initial conditions (6) and the higher-order coefficients are calculated using the recurrent expression

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\[
z_{k,q+1} = \frac{1}{q+1} \left[ \delta_q f_k + \sum_{i=1}^{n} f_{k,i} z_{i,q+1} + \sum_{i=1}^{n} \sum_{j=1}^{i} \sum_{l=0}^{q} f_{k,ij} z_{i,l} z_{j,q-l} \right].
\]

To use the method for actual solution of Cauchy problems we need some error estimates which are given in Babadzhanian (2006).

After introducing the norm
\[
||z|| = ||(z_1, \ldots, z_n)|| = \max_{j=1,\ldots,n} |z_j|
\]
and the auxiliary function
\[
s(\gamma) = \max_{k=1,\ldots,n} \left( \sum_{i=1}^{n} \left| f_{k,i} \right| + \gamma \sum_{i=1}^{n} \sum_{j=1}^{i} \left| f_{k,ij} \right| \right)
\]
we have the following estimate on the radius of convergence \( \rho \) for the series (8):
\[
\rho = \frac{1}{2s(||z(t_0)||)}.
\]

The error estimate for each \( t \) in the region of convergence is
\[
||\delta_M(t)|| = ||z(t) - T_M z(t)|| \leq \frac{||z(t_0)||||t||^{M+1}}{(1-t_1)^2},
\]
where
\[
t_1 = \frac{|t-t_0|}{\rho}.
\]

In other words, if we specify the absolute local error \( \varepsilon \), then the error at the \( t = t_0 + h \) is bounded as
\[
|z_j(t_0 + h) - T_M z_j(t_0 + h)| \leq \varepsilon ||z(t_0)||
\]
for \( h = \rho \min \left\{ \frac{1}{2}, \frac{M+1}{2} \right\} \).

3. RICCATI ODE COEFFICIENTS

Let us unroll the matrix notation in (3) and get the explicit equations for individual elements of \( P(t) \). Since \( P \) is an \( n \times n \) matrix, we assume the indices for each sum in the following formulas run from 1 to \( n \).

Introducing a new matrix
\[
C = BR^{-1}B^T,
\]
after some basic linear algebra we get the parts of the right hand side of differential Ricatti Equation (3).
\[
\begin{align*}
\left( A^T P \right)_{ij} &= \sum_{k} a_{ki} p_{kj}, \\
\left( PA \right)_{ij} &= \sum_{k} p_{ik} a_{kj}, \\
P C &= \left\{ \sum_{k} p_{ik} c_{kj} \right\}_{ij}, \\
PCP &= \left\{ \sum_{l} \left( \sum_{k} p_{ik} c_{kj} \right) \times p_{lj} \right\}_{ij} \\
&= \left\{ \sum_{k} \sum_{l} c_{ki} p_{lk} p_{lj} \right\}_{ij}.
\end{align*}
\]

Finally for an element of \( P(t) \) we have the equation
\[
\dot{p}_{ij} = -\sum_{k} a_{ki} p_{kj} - \sum_{k} p_{ik} a_{kj} + \sum_{k} c_{ki} p_{lk} p_{lj} - q_{ij},
\]
\[
(21)
\]

The rest of this section is devoted to the “vectorization” of this equation, i.e., converting the \( p_{ij} \) matrix to the \( n^2 \)-dimensional vector \( p_m \).

Let \( m/n \) denote integer quotient of \( m \) by \( n \) and \( m\%n \) denote the integer remainder of \( m \) divided by \( n \).

First we introduce the linear indexing for \( q_{ij} \) and \( p_{ij} \) components
\[
p_{ij} = p_m, \quad q_{ij} = q_m, \quad m = 1, \ldots, n^2, \quad m = (i-1)n + j, \quad j = m\%n, \quad i = m/n + 1.
\]

Then we rewrite (21) using (18), (19) and (20) as
\[
\dot{p}_m = -\sum_{r=1}^{n} a_{(r-1)n+(m/n)+1} p_m + m\%n - \sum_{r=1}^{n} a_{(r-1)n+(m/n)+1} \sum_{l=1}^{n} \left( c_{l(t-1)+c(t-1)n+l} \times p_{l+n(m/n)+1} - \sum_{r=1}^{n} c_{l(r-1)+r} p_{l+n(m/n)+1} - q_m \right)
\]
\[
(23)
\]
to get the system of equations similar to (7). The right-hand side of equation (23) is a quadratic polynomial with respect to variables \( p_m \) and \( m \) runs from 1 to \( n^2 \).

To use Taylor Series Method and calculate coefficients of the expansion (10) we give explicit formulas for \( f_k \), \( f_{k,i} \) and \( f_{k,ij} \) expressed in terms of \( a_k \) and \( c_k \) values. We compare (21) with (7) and see that
\[
f_k = -q_k.
\]
\[
(24)
\]

For linear terms we have two sums in (21) so
\[
f_{k,i} = f^{(1)}_{k,i} + f^{(2)}_{k,i},
\]
\[
(25)
\]

where
\[
f^{(1)}_{k,i} = a_{1+(i-k\%n)/n}, \quad f^{(2)}_{k,i} = a_{i-k\%n}/n, \quad (k-i)\%n = 0,
\]
\[
(26)
\]

The quadratic terms are given by
\[
f_{k,ij} = \delta(i,j) [q_{i} \delta(t_1, i) + c_{x} \delta(t_2, i, j)],
\]
\[
(28)
\]

where
\[
\delta(q_1 q_2) = \begin{cases} 1, & (1 \leq q_1 - n(m\%n) \leq n) \& (m - q_2), \\ 0, & \text{else} \end{cases}
\]
\[
(29)
\]

The \( c_{x, l} \) is the \((s, l)\) element of the \( C \) matrix in (17). Indices \( s \) and \( l \) are calculated from \( i \) and \( j \):
\[
s = 1 + (j - k\%n)/n, \quad l = i - n(k/n).
\]
\[
(30)
\]
4. THE ALGORITHM

To summarize two previous sections we outline the algorithm for LQR construction.

Algorithm 1 constructs the values of $P(t)$ matrix at the moments $t = t_0, \ldots, t_N = T$ for the given $A$, $B$, $Q$, $R$ matrices, time $T$, approximation order $M$ and time step $h$.

**Algorithm 1 LQR construction**

1: \textbf{procedure} LQRCompute($A,B,Q,R,T,M,h$)
2: \hspace{1em} $t = T$
3: \hspace{1em} $P(T) = 0$
4: \hspace{1em} \textbf{while} $t > 0$ \textbf{do}
5: \hspace{2em} $t \leftarrow t - h$
6: \hspace{2em} Calculate $f_k$, $f_{k;i}$, $f_{k;i}$ from $A$, $B$, $Q$ and $R$
7: \hspace{2em} Calculate $p_{ij}^{(q)}$ using (10) for (3)
8: \hspace{2em} $p_{ij}(t) \leftarrow \sum_{q=0}^{M} p_{ij}^{(q)} h^q$
9: \hspace{1em} \textbf{end while}
10: Return $P(t)$
11: \textbf{end procedure}

5. BENCHMARK EXAMPLE

![Cart–pendulum system](image)

Fig. 1. Cart–pendulum system

For the system in Figure 1 we take $q = (s, \theta_1, \theta_2)$ as the generalized coordinates, $M$ is the mass of the cart, $m$ — mass of each link, $l$ — length of each link, $\theta_i$ is the angle of $i$-th link with the vertical axis, $s$ is the displacement of the cart from the origin.

Using the shorthand notation $s_i = \sin(\theta_i)$ and $s_{i-2} = \sin(\theta_1 - \theta_2)$, the Lagrange equations for this mechanism are

$$D^{-1} = \begin{pmatrix} M & 2c & c \cr 2c & 2cl & cl \cr c & cl & cl \end{pmatrix}^{-1} =
\begin{pmatrix} 1/d & -1/d & 0 \\
-1/d & (c+d)/(dc) & -1/cl \\
0 & -1/cl & 2/cl \end{pmatrix},$$

where $c = ml$ and $d = Ml - 2c$.

Multiplying both sides of (32) by $D^{-1}$ we get

$$\begin{pmatrix} \ddot{s} \\
\ddot{\theta}_1 \\
\ddot{\theta}_2 \end{pmatrix} = \tilde{A} \begin{pmatrix} s \\
\theta_1 \\
\theta_2 \end{pmatrix} + \tilde{B} u$$

(34)

Introducing the matrices $A$ and $B$, state vector $x$ and control $u$ with the following expressions

$$x = \begin{pmatrix} s, \theta_1, \theta_2, s, \theta_1, \theta_2 \end{pmatrix}^T,$n$$

$$A = \begin{pmatrix} 0 & -2g/c & 0 \\
2(d-c)g/d & -g/l & 2g/l \\
0 & -2g/l & 0 \end{pmatrix}, B = \begin{pmatrix} l/d \\
-1/d \end{pmatrix}$$

the Lagrange equations (32) have exactly the form of (2).

Taking $Q$ and $R$ in (4) to be the identity matrices we calculate the $C$ matrix as

$$C = \frac{1}{d^2} \begin{pmatrix} l^2 & -l & 0_{2 \times 4} \\
-l & 1 & 0_{4 \times 2} \\
0_{4 \times 2} & 0_{4 \times 4} \end{pmatrix}.$$  

(35)

Using Algorithm 1, we calculate the control to stabilize the cart–pendulum system near its unstable equilibrium. Figure 2 shows a sequence of frames corresponding to the stabilization process.

![Cart–pendulum stabilization sequence](image)

Fig. 2. Cart–pendulum stabilization sequence

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