On the multivariable generalization of Anderson-Apostol sums

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Abstract

In this paper, we give various identities for the weighted average of the product of generalized Anderson-Apostol sums with weights concerning completely multiplicative function, completely additive function, logarithms, the Gamma function, the Bernoulli polynomials and binomial coefficients.

1 Introduction

Let \( \mathbb{N} \) be the set of positive integers and \( \mathbb{N}_0 = \mathbb{N} \cup \{0\} \). For \( j, k \in \mathbb{N} \), let \( \gcd(j, k) \) denote their greatest common divisor. The Ramanujan sum \( c_k \) for \( k \in \mathbb{N} \) is an arithmetic function which is defined as

\[
c_k(j) = \sum_{d \mid \gcd(j, k)} d \mu \left( \frac{k}{d} \right) = \sum_{m=1}^{k} \exp \left( 2\pi i m j / k \right), \quad \text{for} \ j \in \mathbb{N},
\]

where \( \mu \) is the Möbius function and \( i = \sqrt{-1} \). This arithmetic function has received attention due to its connections to many problems in number theory, such as the proof of Vinogradov’s theorem which implies that any sufficiently large odd integer can be written as a sum of three prime numbers.

Nowadays various generalizations of the Ramanujan sum are studied. One of these is according to Cohen \cite{C1, C2, C3} who defined the arithmetic function \( c_k^{(a)} \) for \( k, a \in \mathbb{N} \) by

\[
c_k^{(a)}(j) := \sum_{d \mid k} d^a \mu \left( \frac{k}{d} \right) = \sum_{m=1}^{k^a} \exp \left( 2\pi i m j / k^a \right), \quad \text{for} \ j \in \mathbb{N}, \tag{1}
\]

where \( (m, k^a) = 1 \) means that no positive integer satisfies \( d \mid k \) and \( d^a \mid m \). Many interesting properties and useful formulas of Ramanujan-Cohen sum Eq. (1) have been given by McCarthy \cite{McCarthy2, McCarthy3, McCarthy4}, Kühn and Robles \cite{KuehnRobles}, Robles and Roy \cite{RoblesRoy}, and others.

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In 2006, Mednykh and Nedela [15] studied the average of products of Ramanujan sums with the aim to handle certain problems of enumerative combinatorics. To this end they established the function $E$ of $n$ variables given by

$$E(k_1, \ldots, k_n) = \frac{1}{K} \sum_{j=1}^{K} c_{k_1}(j) \cdots c_{k_n}(j), \quad \text{for } k_1, \ldots, k_n \in \mathbb{N},$$

(2)

where here and throughout this paper the integer $K$ denotes the least common multiple of given $k_1, \ldots, k_n \in \mathbb{N}$, i.e.,

$$K := \text{lcm}(k_1, \ldots, k_n).$$

(3)

Furthermore, for later use we mention already here that if integers $d_1 | k_1, \ldots, d_n | k_n$, then we will write

$$L := \frac{\text{lcm}(k_1, \ldots, k_n)}{\text{lcm}(d_1, \ldots, d_n)} = \frac{K}{\text{lcm}(d_1, \ldots, d_n)}.$$ 

(4)

It is shown in [15] that $E(k_1, \ldots, k_n) \in \mathbb{N}$ for all $(k_1, \ldots, k_n) \in \mathbb{N}^n$. Arithmetic properties of $E$ have been studied by Liskovets [11], who called $E$ the “orbicyclic” arithmetic function, and by Tóth [20]. In particular, Tóth proved that $E$ is a multiplicative function which can also be represented in the form

$$E(k_1, \ldots, k_n) = \sum_{d_1 | k_1, \ldots, d_n | k_n} \frac{d_1 \mu(k_1/d_1) \cdots d_n \mu(k_n/d_n)}{\text{lcm}(d_1, \ldots, d_n)}.$$

Recently, Tóth [21] extended the definition of $E$ in Eq. (2) and considered the weighted average

$$S_r(k_1, \ldots, k_n) := \frac{1}{K^{r+1}} \sum_{j=1}^{K} j^r c_{k_1}(j) \cdots c_{k_n}(j),$$

(5)

where $r \in \mathbb{N}_0$. Clearly, $S_0 = E$. He proved that

$$S_r(k_1, \ldots, k_n) = \frac{1}{2K} \prod_{j=1}^{n} \phi(k_j) + \frac{1}{r+1} \sum_{m=0}^{\lfloor r/2 \rfloor} \binom{r+1}{2m} \frac{B_{2m}}{K^{2m}} g_m(k_1, \ldots, k_n),$$

(6)

where $\phi$ and $B_m$ with $m \in \mathbb{N}_0$ are the Euler totient function and the Bernoulli numbers, respectively, and the functions $g_m(k_1, \ldots, k_n)$ are given by

$$g_m(k_1, \ldots, k_n) = \sum_{d_1 | k_1, \ldots, d_n | k_n} \frac{d_1 \mu(k_1/d_1) \cdots d_n \mu(k_n/d_n)}{(\text{lcm}(d_1, \ldots, d_n))^{1-2m}}.$$

Furthermore, for $r = 1$, Tóth showed that

$$S_1(k_1, \ldots, k_n) = \frac{1}{2K} \prod_{j=1}^{n} \phi(k_j) + \frac{E(k_1, \ldots, k_n)}{2}.$$
Another generalization of the Ramanujan sum \( c_k(j) \) has been given by Anderson and Apostol [1] who introduced the arithmetic function \( s_k \) for \( k \in \mathbb{N} \) defined by the identity

\[
s_k(j) = \sum_{d \mid \gcd(k,j)} f(d) \left( \frac{k}{d} \right), \quad \text{for } j \in \mathbb{N}, \tag{7}
\]

with arithmetic functions \( f \) and \( g \). Then one can generalize the function \( E(k_1, \ldots, k_n) \) in Eq. (2) to

\[
\tilde{E}(k_1, \ldots, k_n) = \frac{1}{K} \sum_{j=1}^{K} s_{k_1}(j) \cdots s_{k_n}(j). \tag{8}
\]

In [20], Tóth investigated this function and proved that if \( f \) and \( g \) are multiplicative, then the function \( \tilde{E} \) is multiplicative as well.

Generalizing Eq. (5) we obtain a weighted average of products of \( s_{k_i} \) of the form

\[
\tilde{S}_r(k_1, \ldots, k_n) := \frac{1}{K^{r+1}} \sum_{j=1}^{K^r} j^r s_{k_1}(j) \cdots s_{k_n}(j). \tag{9}
\]

Recently Ikeda, Kiuchi and Matsuoka [7] proved that

\[
\tilde{S}_r(k_1, \ldots, k_n) = \frac{1}{2K} \prod_{j=1}^{n} f_j * g_j(k_j) + \frac{1}{r+1} \sum_{m=0}^{\lfloor r/2 \rfloor} \binom{r+1}{2m} \frac{B_{2m}}{K^{2m}} \tilde{g}_m(k_1, \ldots, k_n),
\]

where the symbol \(*\) denotes the Dirichlet convolution of arithmetic functions

\[
g * f(n) = \sum_{d \mid n} f(d) g \left( \frac{n}{d} \right),
\]

and where

\[
\tilde{g}_m(k_1, \ldots, k_n) = \sum_{d_1 \mid k_1, \ldots, d_n \mid k_n} \frac{f_1(d_1) \cdots f_n(d_n)}{(\text{lcm}(d_1, \ldots, d_n))^{1-2m}} g_1 \left( \frac{k_1}{d_1} \right) \cdots g_n \left( \frac{k_n}{d_n} \right).
\]

Compare this result with Eq. (6).

We propose a further generalization of Anderson-Apostol sums which will be given in the following definition:

**Definition 1.** For fixed \( a \in \mathbb{N} \) and for given arithmetic functions \( f, g \) and \( h \), define the functions \( s_{f,g,h}^{(a)} \) by

\[
s_{f,g,h}^{(a)}(k,j) = \sum_{d \mid k, d^a \mid j} f(d) g \left( \frac{k}{d} \right) h \left( \frac{j}{d^a} \right) \quad \text{for } k, j \in \mathbb{N}. \tag{10}
\]

We call these function generalized Anderson-Apostol sums.
Let the arithmetic function \(1: \mathbb{N} \rightarrow \mathbb{N}\) be given by \(1(n) = 1\) for all \(n\). In [8], the first author investigated the weighted average of \(s_{f,g,1}^{(a)}(k,j) = s_k^{(a)}(j)\) with the weight function \(\text{id}_r(j) \coloneqq j^r\). He proved that

\[
\frac{1}{k^a} \sum_{j=1}^{k^a} j^r s_k^{(a)}(j) = \frac{1}{2} f \ast g(k) + \frac{1}{r + 1} \sum_{m=0}^{|r/2|} \binom{r + 1}{2m} B_{2m} \left( f \ast \text{id}_{a(1-2m)} \cdot g \right)(k).
\]

Now it is the aim of the present paper to study weighted averages of products

\[
s_{f_1,g_1,h_1}^{(a)}(k,j) \cdots s_{f_n,g_n,h_n}^{(a)}(k,j)
\]

for sequences of arithmetic functions \((f_j)_j\), \((g_j)_j\), and \((h_j)_j\), with weight functions \(\omega\) that are either completely multiplicative or completely additive. Put

\[
U_{\omega}^{(a)}(k_1, \ldots, k_n) := \sum_{j=1}^{K^a} \omega(j) s_{f_1,g_1,h_1}^{(a)}(k_1,j) \cdots s_{f_n,g_n,h_n}^{(a)}(k_n,j).
\]

If \(h_1 = \ldots = h_n = 1\), then we write

\[
\tilde{U}_{\omega}^{(a)}(k_1, \ldots, k_n) := \sum_{j=1}^{K^a} \omega(j) s_{f_1,g_1,1}^{(a)}(k_1,j) \cdots s_{f_n,g_n,1}^{(a)}(k_n,j).
\]

Furthermore, we also derive identities for \(U_{\omega}^{(a)}(k_1, \ldots, k_n)\) with weights being the logarithms, the Gamma function, the Bernoulli polynomials and binomials coefficients. Our results are a generalization of identities due to the Kiuchi, Minamide and Ueda [9].

## 2 The function \(U_{\omega}^{(a)}(k_1, \ldots, k_n)\)

In this section, we derive certain identities for \(U_{\omega}^{(a)}(k_1, \ldots, k_n)\) when the weight \(\omega\) is completely multiplicative or completely additive, respectively.

### 2.1 Completely multiplicative \(\omega\)

We are going to state the first main result of the present paper:

**Theorem 1.** Let \((f_i)_{i=1}^n\), \((g_i)_{i=1}^n\) and \((h_i)_{i=1}^n\) be finite sequences of any arithmetic functions and let \(\omega\) be a completely multiplicative function. Then we have

\[
U_{\omega}^{(a)}(k_1, \ldots, k_n) = \sum_{d_1|k_1, \ldots, d_n|k_n} \omega^a(\text{lcm}(d_1, \ldots, d_n)) \prod_{i=1}^n f_i(d_i) g_i \left( \frac{k_i}{d_i} \right) \times
\]

\[
\times \sum_{\ell=1}^{L^a} \omega(\ell) h_i \left( \left( \frac{\text{lcm}(d_1, \ldots, d_n)}{d_i} \right)^a \ell \right),
\]

\[(12)\]
where $L$ is given by Eq. (4) and, as usual, $\omega^a(x) = \omega(x)^a$.

In addition, if $h_1, \ldots, h_n$ are completely multiplicative functions, then

$$U_{\omega}^{(a)}(k_1, \ldots, k_n) = \sum_{d_1|k_1, \ldots, d_n|k_n} \omega^a((\text{lcm}(d_1, \ldots, d_n))) \prod_{i=1}^{n} f_i(d_i) g_i \left( \frac{k_i}{d_i} \right) \times$$

$$\times h_i^a \left( \frac{\text{lcm}(d_1, \ldots, d_n)}{d_i} \right) \sum_{\ell=1}^{L} \omega(\ell) h_i(\ell). \quad (13)$$

We deduce several corollaries for $\widetilde{U}_{\omega}^{(a)}$ and $\omega = \text{id}_r$, namely:

**Corollary 1.** With the above notation we have

$$\widetilde{U}_{\text{id}_r}^{(a)}(k_1, \ldots, k_n) = \frac{K^a r}{2} \prod_{j=1}^{n} f_j \ast g_j(k_j)$$

$$+ \frac{K^a(r+1)}{r+1} \sum_{m=0}^{\lfloor r/2 \rfloor} \left( \frac{r+1}{2m} \right) \frac{B_{2m}}{K^{2am}} \sum_{d_1|k_1, \ldots, d_n|k_n} (\text{lcm}(d_1, \ldots, d_n))^{a(2m-1)} \prod_{i=1}^{n} f_i(d_i) g_i \left( \frac{k_i}{d_i} \right). \quad (14)$$

From Eq. (14), we get some identities for the weighted average of the products of gcd-sum functions. Taking $f_i \ast \mu$ in place of $f_i$ and $g_1 = \cdots = g_n = 1$ in Eq. (14), we deduce:

**Corollary 2.** With the above notation we have

$$\frac{1}{K^a} \sum_{j=1}^{K^a} j^r \prod_{i=1}^{n} \left( \sum_{d_i|k_i \atop d_i \neq j} f_i \ast \mu(d_i) \right) = \frac{1}{2} \prod_{i=1}^{n} f_i(k_i)$$

$$+ \frac{1}{r+1} \sum_{m=0}^{\lfloor r/2 \rfloor} \left( \frac{r+1}{2m} \right) \frac{B_{2m}}{K^{a(2m-1)}} \sum_{d_1|k_1, \ldots, d_n|k_n} (\text{lcm}(d_1, \ldots, d_n))^{a(2m-1)} \prod_{i=1}^{n} f_i \ast \mu(d_i). \quad (15)$$

Taking $a = 1$, the left-hand side of Eq. (15) is then the sum

$$\frac{1}{K^r} \sum_{j=1}^{K^r} j^r \prod_{i=1}^{n} f_i(\text{gcd}(k_i, j)).$$

Similarly, taking $f_1 = \cdots = f_n = 1$ and $g_i \ast \mu$ in place of $g_i$ in Eq. (14), we deduce:

**Corollary 3.** With the above notation we have
Corollary 4. With the above notation we have

\[
\frac{1}{K^{ar}} \sum_{j=1}^{K^a} \prod_{i=1}^{n} \left( \sum_{d_i|k_i} \frac{g_i \ast \mu \left( \frac{k_i}{d_i} \right)}{d_i} \right) = \frac{1}{2} \prod_{i=1}^{n} g_i(k_i) \\
+ \frac{1}{r+1} \sum_{m=0}^{\lfloor r/2 \rfloor} \frac{B_{2m}}{K^{a(2m-1)}} \sum_{d_i|k_1, \ldots, d_n|k_n} \left( \frac{\text{lcm}(d_1, \ldots, d_n)^{a(2m-1)}}{d_i} \right) \prod_{i=1}^{n} g_i \ast \mu \left( \frac{k_i}{d_i} \right).
\]

### 2.2 Completely additive \( \omega \)

In the following theorem we state our second main result:

**Theorem 2.** Let \((f_i)_{i=1}^{n}\), \((g_i)_{i=1}^{n}\) and \((h_i)_{i=1}^{n}\) be finite sequences of any arithmetic functions and let \( \omega \) be a completely additive function. Then we have

\[
U_{\omega}^{(a)}(k_1, \ldots, k_n) \\
= \sum_{d_i|k_1, \ldots, d_n|k_n} \omega \left( \left( \text{lcm}(d_1, \ldots, d_n) \right)^{a} \right) \prod_{i=1}^{n} f_i(d_i) \left( \frac{k_i}{d_i} \right) \sum_{\ell=1}^{L^a} h_i \left( \frac{\text{lcm}(d_1, \ldots, d_n)^{a}}{d_i} \right) \ell \\
+ \sum_{d_i|k_1, \ldots, d_n|k_n} \prod_{i=1}^{n} f_i(d_i) \left( \frac{k_i}{d_i} \right) \sum_{\ell=1}^{L^a} \omega(\ell) h_i \left( \frac{\text{lcm}(d_1, \ldots, d_n)^{a}}{d_i} \right) \ell, \quad (16)
\]

where \( L \) is given by Eq. (4).

In addition, if \( h_1, \ldots, h_n \) are completely multiplicative functions, then

\[
U_{\omega}^{(a)}(k_1, \ldots, k_n) \\
= \sum_{d_i|k_1, \ldots, d_n|k_n} \omega \left( \left( \text{lcm}(d_1, \ldots, d_n) \right)^{a} \right) \prod_{i=1}^{n} f_i(d_i) \left( \frac{k_i}{d_i} \right) h_i \left( \frac{\text{lcm}(d_1, \ldots, d_n)^{a}}{d_i} \right) \sum_{\ell=1}^{L^a} h_i(\ell) \\
+ \sum_{d_i|k_1, \ldots, d_n|k_n} \prod_{i=1}^{n} f_i(d_i) \left( \frac{k_i}{d_i} \right) h_i \left( \frac{\text{lcm}(d_1, \ldots, d_n)^{a}}{d_i} \right) \sum_{\ell=1}^{L^a} \omega(\ell) h_i(\ell). \quad (17)
\]

Again we deduce several corollaries for \( \widetilde{U}_{\omega}^{(a)} \) by taking the weight \( \omega = \log \). We have

**Corollary 4.** With the above notation we have

\[
\widetilde{U}_{\log}^{(a)}(k_1, \ldots, k_n) = K^a (\log K^a - 1) \sum_{d_i|k_1, \ldots, d_n|k_n} \prod_{i=1}^{n} f_i(d_i) g_i \left( \frac{k_i}{d_i} \right) \left( \frac{\text{lcm}(d_1, \ldots, d_n)^{a}}{d_i} \right) \\
- \frac{a}{2} \sum_{d_i|k_1, \ldots, d_n|k_n} \log \left( \text{lcm}(d_1, \ldots, d_n) \right) \prod_{i=1}^{n} f_i(d_i) g_i \left( \frac{k_i}{d_i} \right) \\
+ \log(\sqrt{2\pi K^a}) \sum_{d_i|k_1, \ldots, d_n|k_n} \prod_{i=1}^{n} f_i(d_i) \left( \frac{k_i}{d_i} \right)
\]
\[ + \frac{\theta}{12K^a} \sum_{d_1|k_1, \ldots, d_n|k_n} (\text{lcm}(d_1, \ldots, d_n))^a \prod_{i=1}^n f_i(d_i)g_i \left( \frac{k_i}{d_i} \right), \]

for some \( \theta \in (0, 1) \).

Taking \( g_1 = \cdots = g_n = 1 \) and \( f_1 = \cdots = f_n = \phi \) into Eq. (18), we deduce an identity for the weighted average of generalized gcd-sum functions, namely:

**Corollary 5.** With the above notation we have

\[
\sum_{j=1}^{K^a} (\log j) \prod_{i=1}^n \left( \sum_{d_i|k_i, d_i^a|j} \phi(d_i) \right) = K^a(\log K^a - 1) \sum_{d_1|k_1, \ldots, d_n|k_n} \left( \prod_{i=1}^n \phi(d_i) \right) \left( \text{lcm}(d_1, \ldots, d_n)^a \right)
\]

\[
- \frac{a}{2} \log (\text{lcm}(d_1, \ldots, d_n)) \prod_{i=1}^n \phi(d_i) + \log(\sqrt{2\pi K^a}) \prod_{i=1}^n \phi(k_i)
\]

\[
+ \theta \sum_{d_1|k_1, \ldots, d_n|k_n} (\text{lcm}(d_1, \ldots, d_n))^a \prod_{i=1}^n \phi(d_i),
\]

for some \( \theta \in (0, 1) \).

When \( a = 1 \), the left-hand side of Eq. (18) becomes

\[
\sum_{j=1}^{K} (\log j) \prod_{i=1}^n \text{gcd}(k_i, j).
\]

### 3 Another representation of \( \tilde{U}_\omega^{(a)} \)

For any two integers \( k \) and \( j \), the generalized gcd function \( (j, k^a)_\omega \) is defined as the largest \( d \in \mathbb{N} \) such that \( d|k \) and \( d^a|j \). When \( a = 1 \) the generalized gcd function becomes the usual gcd function, see [16, Definition 3]. In this section, we use the Dirichlet convolution to give another representation of the function \( \tilde{U}_\omega^{(a)} \).

**Theorem 3.** Let \( k_1, \ldots, k_n \in \mathbb{N} \) and let \( \omega \) be a completely multiplicative function, then we have

\[
\tilde{U}_\omega^{(a)}(k_1, \ldots, k_n) = \left( \omega \prod_{i=1}^n s_{f_i, g_i, 1}^{(a)}(k_i) \right) * \Psi(K^a),
\]

where

\[
\Psi(m) = \sum_{\ell=1}^m \frac{\omega(\ell)}{(\ell, m)}.
\]
If \( \omega \) is a completely additive function, then we have
\[
\tilde{U}_\omega(a_1, \ldots, a_n) = \left( \omega \prod_{i=1}^{n} s_{f_i, g_i, 1}(a_i, \cdot) \right) * \phi(K^a) + \left( \prod_{i=1}^{n} s_{f_i, g_i, 1}(a_i, \cdot) \right) * \Psi(K^a). \tag{20}
\]

For the following we need the Jordan totient function of order \( m \in \mathbb{N} \) which is defined as
\[
\phi_m(n) = n^m \prod_{p \mid n} \left( 1 - \frac{1}{p^m} \right) = \sum_{d \mid n} d^m \mu \left( \frac{n}{d} \right).
\]

Using Theorem 3 with \( \omega = id_r \) and \( \omega = \log \), respectively, we get:

\textbf{Corollary 6.} We have
\[
\frac{1}{K_a} \sum_{j=1}^{K^n} j^r \prod_{i=1}^{n} s_{f_i, g_i, 1}(k_i, \cdot) = \frac{1}{2} \prod_{i=1}^{n} s_{f_i, g_i, 1}(k_i, K^a) + \frac{1}{r + 1} \sum_{m=0}^{\lfloor r/2 \rfloor} \left( r + 1 - 2m \right) B_{2m} \sum_{d \mid K^a} \phi_1(d) \prod_{i=1}^{n} s_{f_i, g_i, 1} \left( k_i, \frac{K^a}{d} \right), \tag{21}
\]

and
\[
\sum_{j=1}^{K^n} (\log j) \prod_{i=1}^{n} s_{f_i, g_i, 1}(k_i, \cdot) = \left( \log \prod_{i=1}^{n} s_{f_i, g_i, 1}(k_i, \cdot) \right) * \phi(K^a) + \sum_{d \mid K^a} \prod_{i=1}^{n} s_{f_i, g_i, 1} \left( k_i, \frac{K^a}{d} \right) \left( \sum_{z \mid d} \mu \left( \frac{d}{z} \right) \log(z!) - \phi(d) \sum_{p \mid d} \frac{\log p}{p - 1} \right), \tag{22}
\]

where the last sum above is extended over all prime numbers \( p \) such that \( p \mid d \).

Take \( f_1 = \cdots = f_n = \phi \) and \( g_1 = \cdots = g_n = 1 \) in Eq. (21) and use the identity \( \sum_{d \mid N} \phi(d) = N \) to get:

\textbf{Corollary 7.} We have
\[
\frac{1}{K^a} \sum_{j=1}^{K^n} j^r \prod_{i=1}^{n} \phi(d) = \frac{1}{2} \prod_{i=1}^{n} k_i + \frac{1}{r + 1} \sum_{m=0}^{\lfloor r/2 \rfloor} \left( r + 1 - 2m \right) B_{2m} \sum_{d \mid K^a} \phi_1(d) \prod_{i=1}^{n} \text{gcd} \left( k_i, \frac{K^a}{d} \right).
\]
4 Other weighted averages

For $x > 0$, the Gamma function $\Gamma$ is defined by

$$\Gamma(x) = \int_0^\infty e^{-u} u^{x-1} \, du.$$ 

We recall that the Bernoulli polynomials are defined by the generating function

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!},$$

for $|t| < 2\pi$. For $x = 0$, the numbers $B_n = B_n(0)$ are the Bernoulli numbers. In this section, we consider weighted averages of the product $s_{f_1, g_1, h_1}^{(a)}(k, j) \cdots s_{f_n, g_n, h_n}^{(a)}(k, j)$ with weights being the Gamma function, Binomial coefficients, and Bernoulli polynomials. We prove that:

**Theorem 4.** Let $k_1, \ldots, k_n \in \mathbb{N}$ and let $a \geq 2$ be a fixed integer. Then we have the following formulas:

$$\sum_{j=1}^{K^a} \log(\Gamma\left(\frac{K^a j}{K^a}\right)) \prod_{i=1}^{n} s_{f_i, g_i, h_i}^{(a)}(k_i, j)$$

$$= \frac{K^a}{2} \log(2\pi) \sum_{d_1|k_1, \ldots, d_n|k_n} \frac{\prod_{i=1}^{n} f_i(d_i) g_i(k_i/d_i)}{(\text{lcm}(d_1, \ldots, d_n))^a} - \frac{1}{2} \log(2\pi K^a) \prod_{i=1}^{n} f_i \ast g_i(k_i)$$

$$+ \frac{a}{2} \sum_{d_1|k_1, \ldots, d_n|k_n} \log(\text{lcm}(d_1, \ldots, d_n)) \prod_{i=1}^{n} f_i(d_i) g_i\left(\frac{k_i}{d_i}\right),$$

$$\sum_{j=0}^{K^a} \binom{K^a}{j} \prod_{i=1}^{n} s_{f_i, g_i, h_i}^{(a)}(k_i, j)$$

$$= 2^{K^a} \sum_{d_1|k_1, \ldots, d_n|k_n} \frac{\prod_{i=1}^{n} f_i(d_i) g_i(k_i/d_i)}{(\text{lcm}(d_1, \ldots, d_n))^a} \sum_{\ell=1}^{(\text{lcm}(d_1, \ldots, d_n))^a} (-1)^{\ell L^a} \cos^{K^a} \left(\frac{\pi \ell}{(\text{lcm}(d_1, \ldots, d_n))^a}\right),$$

and

$$\sum_{j=0}^{K^a-1} B_m \left(\frac{K^a}{a}\right) \prod_{i=1}^{n} s_{f_i, g_i, h_i}^{(a)}(k_i, j) = \frac{B_m}{K^a(m-1)} \sum_{d_1|k_1, \ldots, d_n|k_n} \frac{\prod_{i=1}^{n} f_i(d_i) g_i(k_i/d_i)}{(\text{lcm}(d_1, \ldots, d_n))^a(l-m)}.$$

Now we provide an application of Theorem 4. In particular, we derive formulas for weighted averages of the product of the gcd-sum function with weights being the Gamma function, Binomial coefficients, and Bernoulli polynomials by taking $f_i = \phi$ and $g_i = 1$ for all $i = 1, \ldots, n$ into Eqs. (23), (24) and (25) to obtain the following results:
Corollary 8. Let $k_1, \ldots, k_n \in \mathbb{N}$ and let $a \geq 2$ be a fixed integer. Then we have:

$$
\sum_{j=1}^{K^a} \log \Gamma \left( \frac{j}{K^a} \right) \prod_{i=1}^{n} \left( \sum_{d_i \mid k_i} \phi(d_i) \right)
= \frac{K^a}{2} \log(2\pi) \sum_{d_1 \mid k_1, \ldots, d_n \mid k_n} \frac{\prod_{i=1}^{n} \phi(d_i)}{(\text{lcm}(d_1, \ldots, d_n))^a} - \frac{k_1 \cdots k_n}{2} \log(2\pi K^a) + \frac{a}{2} \sum_{d_1 \mid k_1, \ldots, d_n \mid k_n} \log (\text{lcm}(d_1, \ldots, d_n)) \prod_{i=1}^{n} \phi(d_i),
$$

and

$$
\sum_{j=0}^{K^a} \left( \begin{array}{c} K^a \\ j \end{array} \right) \prod_{i=1}^{n} \left( \sum_{d_i \mid k_i} \phi(d_i) \right) = 2^{K^a} \sum_{d_1 \mid k_1, \ldots, d_n \mid k_n} \frac{\phi(d_1) \cdots \phi(d_n)}{(\text{lcm}(d_1, \ldots, d_n))^a} \sum_{\ell=1}^{(\text{lcm}(d_1, \ldots, d_n))^a} (-1)^{\ell} L^a \cos K^a \left( \frac{\pi \ell}{(\text{lcm}(d_1, \ldots, d_n))^a} \right),
$$

In the case $m = 1$, we have

$$
\sum_{j=0}^{K^a-1} B_m \left( \begin{array}{c} j \\ K^a \end{array} \right) \prod_{i=1}^{n} \left( \sum_{d_i \mid k_i} \phi(d_i) \right) = \frac{B_m}{K^{a(m-1)}} \sum_{d_1 \mid k_1, \ldots, d_n \mid k_n} \frac{\phi(d_1) \cdots \phi(d_n)}{(\text{lcm}(d_1, \ldots, d_n))^{a(1-m)}} - \frac{k_1 \cdots k_n}{2}.
$$

5 Proofs

Proof of Theorem 1. We recall that

$$
S_{f,g,h}^{(a)}(k, j) = \sum_{d \mid k} f(d) g \left( \frac{k}{d} \right) h \left( \frac{j}{d^a} \right).
$$

Then we get with Eq. (11)

$$
U_{\omega}^{(a)}(k_1, \ldots, k_n) = \sum_{j=1}^{K^a} \omega(j) \sum_{d_1 \mid k_1, \ldots, d_n \mid k_n} \prod_{i=1}^{n} f_i(d_i) g_i \left( \frac{k_i}{d_i} \right) h_i \left( \frac{j}{d_i^a} \right).
$$
\[
\sum_{d_1|k_1,\ldots,d_n|k_n} \prod_{i=1}^{n} f_i(d_i) g_i \left( \frac{k_i}{d_i} \right) \sum_{j=1}^{K^a} \omega(j) \prod_{i=1}^{n} h_i \left( \frac{j}{d_i^a} \right). \tag{26}
\]

Using the fact that \(\omega\) is a completely multiplicative function, we obtain

\[
U^{(a)}(k_1,\ldots,k_n) = \sum_{d_1|k_1,\ldots,d_n|k_n} \omega(\text{lcm}(d_1,\ldots,d_n)) \prod_{i=1}^{n} f_i(d_i) g_i \left( \frac{k_i}{d_i} \right) \sum_{\ell=1}^{L^a} \omega(\ell) h_i \left( \frac{\text{lcm}(d_1,\ldots,d_n)\ell}{d_i^a} \right). 
\]

This proves Eq. (12).

In case that all \(h_i, i=1,\ldots,n\), are completely multiplicative, the latter function can be rewritten as

\[
U^{(a)}(k_1,\ldots,k_n) = \sum_{d_1|k_1,\ldots,d_n|k_n} \omega(\text{lcm}(d_1,\ldots,d_n)) \prod_{i=1}^{n} f_i(d_i) g_i \left( \frac{k_i}{d_i} \right) h^a \left( \frac{\text{lcm}(d_1,\ldots,d_n)}{d_i} \right) \sum_{\ell=1}^{L^a} \omega(\ell) h_i(\ell).
\]

This completes the proof of Theorem 1.

Theorem 2 can be proven in the same way.

**Proof of Corollary 1.** We recall that

\[
s^{(a)}(k,j) = \sum_{d|k} f(d) g \left( \frac{k}{d} \right).
\]

Substituting \(\omega = \text{id}_r\) and \(h_1 = \ldots = h_n = 1\) into Eq. (13), we get

\[
\tilde{U}^{(a)}(k_1,\ldots,k_n) = \sum_{i=1}^{K^a} j^r \sum_{d_1|k_1,\ldots,d_n|k_n} \prod_{i=1}^{n} f_i(d_i) g_i \left( \frac{k_i}{d_i} \right)
\]

\[
= \sum_{d_1|k_1,\ldots,d_n|k_n} \prod_{i=1}^{n} f_i(d_i) g_i \left( \frac{k_i}{d_i} \right) \sum_{j=1}^{K^a} j^r
\]

\[
= \sum_{d_1|k_1,\ldots,d_n|k_n} \prod_{i=1}^{n} f_i(d_i) g_i \left( \frac{k_i}{d_i} \right) (\text{lcm}(d_1,\ldots,d_n))^a \sum_{\ell=1}^{L^a} \ell^r,
\]

where \(L\) is defined by Eq. (4). Using the fact that

\[
\sum_{m=1}^{N} m^r = \frac{N^r}{2} + \frac{1}{r+1} \sum_{m=0}^{r/2} \binom{r+1}{2m} B_{2m} N^{r+1-2m},
\]

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for any integer $N \geq 2$, see [3 Proposition 9.2.12] or [6 Section 3.9], we deduce that

\[
\tilde{U}_{id}^{(a)}(k_1, \ldots, k_n) = \frac{K^{ra}}{2} \prod_{i=1}^{n} f_i * g_i(k_i)
+ \frac{K^{a(r+1)}}{r+1} \sum_{m=0}^{[r/2]} \left( \frac{r+1}{2m} \right) \frac{B_{2m}}{K^{2ma}} \sum_{d_1|k_1, \ldots, d_n|k_n} \prod_{i=1}^{n} f_i(d_i) g_i \left( \frac{k_i}{d_i} \right) (\text{lcm}(d_1, \ldots, d_n))^{a(2m-1)}.
\]

This completes the proof. \qed

**Proof of Corollary** [4] Substituting $\omega = \log$ and $h_1 = \cdots = h_n = 1$ into Eq. (16), we obtain

\[
\tilde{U}_{\log}^{(a)}(k_1, \ldots, k_n) = K^{a} \sum_{d_1|k_1, \ldots, d_n|k_n} \prod_{i=1}^{n} f_i(d_i) g_i \left( \frac{k_i}{d_i} \right) 
\log \left( \frac{\text{lcm}(d_1, \ldots, d_n)}{\text{lcm}(d_1, \ldots, d_n)^{a}} \right)
+ \sum_{d_1|k_1, \ldots, d_n|k_n} \prod_{i=1}^{n} f_i(d_i) g_i \left( \frac{k_i}{d_i} \right) \sum_{\ell=1}^{L^{a}} \log \ell
\]

\[
= aK^{a} \sum_{d_1|k_1, \ldots, d_n|k_n} \prod_{i=1}^{n} f_i(d_i) g_i \left( \frac{k_i}{d_i} \right) 
\log \left( \frac{\text{lcm}(d_1, \ldots, d_n)}{\text{lcm}(d_1, \ldots, d_n)^{a}} \right)
+ \sum_{d_1|k_1, \ldots, d_n|k_n} \prod_{i=1}^{n} f_i(d_i) g_i \left( \frac{k_i}{d_i} \right) \log (L^{a}!).
\]

Using the Stirling formula, see [17] p. 91,

\[
\log(\ell !) = \ell \log \ell - \ell + \frac{1}{2} \log \ell + \log \sqrt{2\pi} + \frac{\theta}{12\ell},
\]

the formula (18) is proved. \qed

**Proof of Theorem** [3] Since $K = \text{lcm}(k_1, \ldots, k_n)$, we can write

\[
s_{f,g,1}^{(a)}(k_i, j) = \sum_{d | k_i, d | K} f(d) g \left( \frac{k_i}{d} \right) = \sum_{d | k_i, d | K} f(d) g \left( \frac{k_i}{d} \right) = s_{f,g,1}^{(a)}(k_i, (j, K^a)).
\]

Hence

\[
\tilde{U}_{\omega}^{(a)}(k_1, \ldots, k_n) = \sum_{j=1}^{K^{a}} \omega(j) \prod_{i=1}^{n} s_{f,g,1}^{(a)}(k_i, (j, K^a)).
\]

Now we split the range of summation of the first sum over $j$ according to the value of $(j, K^a)$. This gives

\[
\tilde{U}_{\omega}^{(a)}(k_1, \ldots, k_n) = \sum_{d | K^a} \prod_{i=1}^{n} s_{f,g,1}^{(a)}(k_i, d) \sum_{j=1}^{K^{a}} \omega(j)
\]

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\[
\sum_{d \mid K^a} \prod_{i=1}^{n} s_{f_i,g_i,1}^{(a)}(k_i, d) \sum_{\ell=1}^{K^a/d} \omega(\ell d).
\]

Now we use that \(\omega\) is completely multiplicative. This way we obtain

\[
\tilde{U}_{\omega}^{(a)}(k_1, \ldots, k_n) = \sum_{d \mid K^a} \omega(d) \prod_{i=1}^{n} s_{f_i,g_i,1}^{(a)}(k_i, d) \sum_{\ell=1}^{K^a/d} \omega(\ell)
\]

\[
= \sum_{d \mid K^a} \omega(d) \prod_{i=1}^{n} s_{f_i,g_i,1}^{(a)}(k_i, d) \Psi \left( \frac{K^a}{d} \right)
\]

\[
= \left( \omega \prod_{i=1}^{n} s_{f_i,g_i,1}^{(a)}(k_i, \cdot) \right) * \Psi(K^a).
\]

This proves Eq. (19). Eq. (20) is shown in the same way. \(\square\)

Remark 1. With the same method we can also proof the following representations: if \(\omega\) is completely multiplicative, then

\[
\tilde{U}_{\omega}^{(a)}(k_1, \ldots, k_n) = \left( \omega^{a} \prod_{i=1}^{n} s_{f_i,g_i,1}^{(1)}(k_i, \cdot) \right) * \Psi^{(a)}(K)
\]

with \(\omega^{a}(d) = (\omega(d))^a\) and for \(N \in \mathbb{N}\),

\[
\Psi^{(a)}(N) = \sum_{\ell=1}^{N^a} \omega(\ell).
\]

If \(\omega\) is a completely additive function, then we have

\[
\tilde{U}_{\omega}^{(a)}(k_1, \ldots, k_n) = \left( \omega^{a} \prod_{i=1}^{n} s_{f_i,g_i,1}^{(1)}(k_i, \cdot) \right) * \Phi^{(a)}(K) + \left( \prod_{i=1}^{n} s_{f_i,g_i,1}^{(1)}(k_i, \cdot) \right) * \Psi^{(a)}(K),
\]

with \(\omega^{(a)}(d) = \omega(d^a)\) and for \(N \in \mathbb{N}\),

\[
\Phi^{(a)}(N) = \sum_{\ell=1}^{N^a} 1 = \# \{ \ell \in \{1, \ldots, N^a\} : (\ell, N^a)_{a} = 1 \}.
\]

Proof of Corollary 1. Substituting \(\omega = \text{id}_r\) into Eq. (19), we get

\[
\sum_{j=1}^{K^a} \prod_{i=1}^{n} s_{f_i,g_i,1}^{(a)}(k_i, j) = \sum_{d \mid K^a} d^{\epsilon} \prod_{i=1}^{n} s_{f_i,g_i,1}^{(a)}(k_i, d) \sum_{\ell=1}^{K^a/d} \ell^\epsilon.
\]
Applying the following formula, see [19, Corollary 4], to the last sum above
\[
\sum_{m=1}^{N} m^{r} = \frac{N^{r+1} \sum_{m=0}^{|r/2|} \binom{r+1}{2m}}{r+1} B_{2m} \phi_{1-2m}(N),
\]
for any positive integer \(N > 1\), we find that
\[
\sum_{j=1}^{K^{a}} \prod_{i=1}^{n} s_{j,i,g_{i,1}}^{(a)}(k_{i}, j)
\]
\[=\quad K^{ra} \prod_{i=1}^{n} s_{j,i,g_{i,1}}^{(a)}(k_{i}, K^{a}) + \frac{K^{ra}}{r+1} \sum_{d|K^{a}} \prod_{i=1}^{n} s_{j,i,g_{i,1}}^{(a)}(k_{i}, d) \sum_{m=0}^{|r/2|} \binom{r+1}{2m} B_{2m} \phi_{1-2m}(1/2) \prod_{i=1}^{n} s_{j,i,g_{i,1}}^{(a)}(k_{i}, K^{a}/d)
\]
\[=\quad K^{ra} \prod_{i=1}^{n} s_{j,i,g_{i,1}}^{(a)}(k_{i}, K^{a}) - \frac{K^{ra}}{r+1} \prod_{i=1}^{n} s_{j,i,g_{i,1}}^{(a)}(k_{i}, K^{a}) \sum_{m=0}^{|r/2|} \binom{r+1}{2m} B_{2m}
\]
\[+ \frac{K^{ra}}{r+1} \sum_{m=0}^{|r/2|} \binom{r+1}{2m} B_{2m} \sum_{\ell|K^{a}} \phi_{1-2m}(\ell) \prod_{i=1}^{n} s_{j,i,g_{i,1}}^{(a)}(k_{i}, K^{a}/\ell).
\]
Using the well-known formula,
\[
\sum_{m=0}^{|r/2|} \binom{r+1}{2m} B_{2m} = \frac{r+1}{2},
\]
we obtain the desired formula (21).

In order to prove Eq. (22) we substitute \(\omega = \log\) into Eq. (20) and use the formula, see [7, Lemma 12],
\[
\sum_{\ell=1}^{N} \frac{\log \ell}{(\ell, N)=1} = \sum_{d|N} \mu\left(\frac{N}{d}\right) \log(d!) - \phi(N) \sum_{p|N} \frac{\log p}{p-1},
\]
for any positive integer \(N\), where the last sum above is extended over all prime numbers \(p\) such that \(p|N\). This gives
\[
\sum_{j=1}^{K^{a}} (\log j) \prod_{i=1}^{n} s_{j,i,g_{i,1}}^{(a)}(k_{i}, j)
\]
\[=\quad (\log \prod_{i=1}^{n} s_{j,i,g_{i,1}}^{(a)}(k_{i}, j)) * \phi(K^{a}) + \prod_{i=1}^{n} s_{j,i,g_{i,1}}^{(a)}(k_{i}, j) * \Psi(K^{a})
\].
Proof of Theorem 4. Substituting \( \omega(d) = \log \Gamma(d/K^a) \) and \( h_1 = \ldots = h_n = 1 \) into Eq. (26), we get

\[
\sum_{j=1}^{K^a} \log \Gamma \left( \frac{j}{K^a} \right) \prod_{i=1}^{n} s_{f_i, g_i, 1}^{(a)}(k_i, j) = \sum_{d_1 | k_1, \ldots, d_n | k_n} \prod_{i=1}^{n} f_i(d_i) g_i \left( \frac{k_i}{d_i} \right) \sum_{\ell=1}^{L^a} \log \Gamma \left( \frac{\ell}{L^a} \right),
\]

where \( K \) and \( L \) are given by Eqs. (3) and (4), respectively. From the Gauss-Legendre multiplication formula for the Gamma function is known that

\[
\prod_{j=1}^{n} \Gamma \left( \frac{j}{n} \right) = \left( \frac{2\pi}{n} \right)^{\frac{m}{2}} \text{ for all } n \in \mathbb{N}.
\]

Hence it follows that

\[
\sum_{j=1}^{K^a} \log \Gamma \left( \frac{j}{K^a} \right) \prod_{i=1}^{n} s_{f_i, g_i, 1}^{(a)}(k_i, j) = \sum_{d_1 | k_1, \ldots, d_n | k_n} \prod_{i=1}^{n} f_i(d_i) g_i \left( \frac{k_i}{d_i} \right) \left[ \frac{L^a}{2} \log(2\pi) - \frac{1}{2} \log(2\pi K^a) + \frac{a}{2} \log(\text{lcm}(d_1, \ldots, d_n)) \right]
\]

\[
= \frac{K^a}{2} \log(2\pi) \sum_{d_1 | k_1, \ldots, d_n | k_n} \frac{\prod_{i=1}^{n} f_i(d_i) g_i(k_i/d_i)}{\text{lcm}(d_1, \ldots, d_n)^{a}} - \frac{1}{2} \log(2\pi K^a) \prod_{i=1}^{n} f_i \ast g_i(k_i)
\]

\[
+ \frac{a}{2} \sum_{d_1 | k_1, \ldots, d_n | k_n} \log(\text{lcm}(d_1, \ldots, d_n)) \prod_{i=1}^{n} f_i(d_i) g_i \left( \frac{k_i}{d_i} \right).
\]

This is precisely Eq. (23).

Now set \( w(j) = \left( \frac{K^a}{j} \right) \) and \( h_1 = \ldots = h_n = 1 \) into Eq. (26) to get

\[
\sum_{j=0}^{K^a} \left( \frac{K^a}{j} \right) \prod_{i=1}^{n} s_{f_i, g_i, 1}^{(a)}(k_i, j) = \sum_{d_1 | k_1, \ldots, d_n | k_n} \prod_{i=1}^{n} f_i(d_i) g_i \left( \frac{k_i}{d_i} \right) \sum_{m=0}^{L^a} \left( m \text{lcm}(d_1^m, \ldots, d_n^m) \right).
\]
For any positive integers $r$ and $n$ it is known that
\[
\sum_{m=0}^{\lfloor n/r \rfloor} \binom{n}{mr} = 2^n \sum_{\ell=1}^{r} \cos\left(\frac{\pi \ell}{r}\right) \cos\left(\frac{\pi \ell n}{r}\right),
\]
see [22, Eq. (27)]. Hence
\[
\sum_{j=0}^{K^n} \binom{K^n}{j} \prod_{i=1}^{n} s_{f_i,g_i,1}(k_i, j)
\]
\[
= 2^{K^n} \sum_{d_1|k_1,\ldots,d_n|k_n} \prod_{i=1}^{n} f_i(d_i) g_i \left(\frac{k_i}{d_i}\right) \frac{\lcm(d_1,\ldots,d_n)}{\lcm(d_1^a,\ldots,d_n^a)} \sum_{\ell=1}^{\lcm(d_1^a,\ldots,d_n^a)} \cos^{K^n} \left(\frac{\pi \ell}{\lcm(d_1^a,\ldots,d_n^a)}\right) \cos\left(\pi \ell L^a\right)
\]
\[
= 2^{K^n} \sum_{d_1|k_1,\ldots,d_n|k_n} \prod_{i=1}^{n} f_i(d_i) g_i \left(\frac{k_i}{d_i}\right) \frac{\lcm(d_1,\ldots,d_n)^a}{\lcm(d_1^a,\ldots,d_n^a)} \sum_{\ell=1}^{\lcm(d_1^a,\ldots,d_n^a)} (-1)^{\ell L^a} \cos^{K^n} \left(\frac{\pi \ell}{\lcm(d_1,\ldots,d_n)^a}\right).
\]
This completes the proof of Eq. (24).

Finally, put $w(j) = B_m(j/K^n)$ and $h_1 = \ldots = h_n = 1$ in Eq. (26). Then
\[
\sum_{j=0}^{K^n-1} B_m \left(\frac{j}{K^n}\right) \prod_{i=1}^{n} s_{f_i,g_i,1}(k_i, j) = \sum_{d_1|k_1,\ldots,d_n|k_n} \prod_{i=1}^{n} f_i(d_i) g_i \left(\frac{k_i}{d_i}\right) \sum_{\ell=0}^{\ell L^a-1} B_m \left(\frac{\ell}{L^a}\right).
\]

Using the following property of the Bernoulli polynomials, see [5, Proposition 9.1.3],
\[
\sum_{\ell=0}^{k-1} B_m \left(\frac{\ell}{k}\right) = \frac{B_m}{k^{m-1}},
\]
we find that
\[
\sum_{j=0}^{K^n-1} B_m \left(\frac{j}{K^n}\right) \prod_{i=1}^{n} s_{f_i,g_i,1}(k_i, j)
\]
\[
= \frac{B_m}{K^{a(m-1)}} \sum_{d_1|k_1,\ldots,d_n|k_n} \lcm(d_1,\ldots,d_n)^{a(m-1)} \prod_{i=1}^{n} f_i(d_i) g_i \left(\frac{k_i}{d_i}\right).
\]
This completes the proof of Theorem 4. 

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