Implicitization of rational surfaces using toric varieties

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Abstract

A parameterized surface can be represented as a projection from a certain toric surface. This generalizes the classical homogeneous and bihomogeneous parameterizations. We extend to the toric case two methods for computing the implicit equation of such a rational parameterized surface. The first approach uses resultant matrices and gives an exact determinantal formula for the implicit equation if the parameterization has no base points. In the case the base points are isolated local complete intersections, we show that the implicit equation can still be recovered by computing any non-zero maximal minor of this matrix.

The second method is the toric extension of the method of moving surfaces, and involves finding linear and quadratic relations (syzygies) among the input polynomials. When there are no base points, we show that these can be put together into a square matrix whose determinant is the implicit equation. Its extension to the case where there are base points is also explored.

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1 Introduction

A rationally parameterized surface $\Phi(s, t)$ in affine three space is defined by a map $\phi: \mathbb{C}^2 \rightarrow \mathbb{C}^3$ given by three rational components:

\[
\begin{align*}
X_1 &= \frac{x_1(s,t)}{x_4(s,t)} \\
X_2 &= \frac{x_2(s,t)}{x_4(s,t)} \\
X_3 &= \frac{x_3(s,t)}{x_4(s,t)}
\end{align*}
\]

Here $x_1, x_2, x_3, x_4$ are (Laurent) polynomials in two variables $s$ and $t$ with coefficients in $\mathbb{C}$ (or $\mathbb{R}$ or $\mathbb{Q}$ or any arbitrary subfield $\mathbb{K}$ of $\mathbb{C}$). Let $\Phi \subset \mathbb{C}^3$ be the smallest algebraic surface containing (1). The implicitization problem [13,11] is to compute the polynomial equation $P(X_1, X_2, X_3)$ defining $\Phi$. At times we will also consider the same surface in projective space, where there are four coordinates with equations given by $X_1, X_2, X_3,$ and $X_4$.

The last few decades have witnessed a rise of interest in the implicitization problem for geometric objects motivated by applications in computer aided geometric design and geometric modelling ([1,3,5,7,9,11,12,18,21,27,29]). A very common approach is to write the implicit equation as the determinant of a matrix whose entries are easy to compute.

Our approach is also to look for matrix formulas, but we recast the parameterization in terms of a projection from a certain toric surface built out of the specific monomials which appear in $x_1, x_2, x_3, x_4$. This generalizes the standard approaches of projections from tensor product surfaces (Segre embeddings of $\mathbb{P}^1 \times \mathbb{P}^1$) or from total degree surfaces (Veronese embeddings of $\mathbb{P}^2$). So while previously $x_1, x_2, x_3, x_4$ have been considered only as “generic” homogeneous or bihomogeneous polynomials, we can exploit sparsity present in the parameterization.

Standard homogenization of sparse polynomials can result in numerous spurious base points of the projection at infinity. By using the more general toric surface, customized for the equations on hand, many of these extraneous base points at infinity can be avoided. This results in smaller matrices and fewer extraneous factors in the computation of the implicit equation. Toric projections can also be exploited in the construction of the parameterization. The work of Krasauskas [24] shows how “toric surface patches” can be used to parametrize regions on a surface shaped like arbitrary sided polygons.

In this article we extend to the toric case two methods for computing the implicit equation: computing a Chow form and computing syzygies on the
input polynomials \( x_1, x_2, x_3, x_4 \).

A classical method for finding the implicit equation is to compute the bivariate resultant or Chow form of the three polynomials

\[
\begin{align*}
    f_1 &= x_1(s, t) - X_1 x_4(s, t) \\
    f_2 &= x_2(s, t) - X_2 x_4(s, t) \\
    f_3 &= x_3(s, t) - X_3 x_4(s, t)
\end{align*}
\]

(2)

Our first approach essentially follows the classical method using the sparse resultant in place of the classical bivariate resultant. Formally, we will reduce the computation of \( F \) to the computation of the Chow form of a toric surface which projects onto \( \Phi \). Exact matrix formulas for computing this Chow form were found by the first author in [22].

We show that if the projection has no base points, points on the toric variety such that \( f_1, f_2, f_3 \) are simultaneously zero, the matrix constructed gives an exact determinantal formula for the implicit equation. New to our approach is an analysis when base points are present. We show that if the base points are isolated local complete intersections, the implicit equation can still be recovered by computing a non-zero maximal minor of this matrix.

The second method involves finding linear and quadratic relations (syzygies) among the polynomials \( x_1, x_2, x_3, x_4 \) of a certain fixed type. When there are no base points, we will see how these can be put together into a square matrix whose determinant is exactly the implicit equation. This is precisely the technique used in the method of moving surfaces for tensor product or total degree surfaces ([30,5,12,11,29]). Our contribution is to exploit the structure of the sparsity of the polynomials to avoid extra base points. Moreover, we present a novel proof of the validity of the method of moving surfaces which ties together the complexes of moving planes and quadrics with the resultant complex in a natural way. This is described in Section 5.

The method of moving surfaces can also be applied in the presence of base-points and often still produces the correct implicit equation. Validity in the presence of basepoints was proved under certain conditions in the total degree [5] and tensor product [2] situations. We do not have a proof in the general toric setting but we illustrate the situation with a few examples.

The paper is organized as follows: in Section 2, we recall some properties of toric surfaces and introduce some notation. In Section 3, we present the first of our methods and show that it works if the base points are a local complete intersection. Next, we present in Section 4 the method of moving quadrics and shows its validity in the case where there are no base points. We also give some
2 Toric varieties, parameterizations, and base points

Let \( \mathcal{A} = \{\alpha_1, \ldots, \alpha_N\} \subset \mathbb{Z}^2 \), a finite subset of points, and \( Q \) the convex hull of the points in \( \mathcal{A} \). The toric variety \( X_\mathcal{A} \) associated with \( \mathcal{A} \) is defined as the Zariski closure of the set of points \( (x^{\alpha_1} : \cdots : x^{\alpha_N}) \) in \( \mathbb{P}^{N-1} \) where \( x \) ranges over \( (\mathbb{C}^*)^2 \) (the “algebraic” torus). See [20,14] for details.

If each of the polynomials \( x_i \) has its support contained in \( \mathcal{A} \), then it is a linear combination of monomials in \( \mathcal{A} \), hence can be thought of as a linear functional on \( \mathbb{P}^{N-1} \) defining a hyperplane section of \( X_\mathcal{A} \). If each \( x_i \) has a different support \( \mathcal{A}_i \) we will define \( \mathcal{A} \) as the union of the supports; that is:

\[
\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{A}_3 \cup \mathcal{A}_4
\]

Therefore we can consider the set of zeros \( Z \) of \( x_1, x_2, x_3, x_4 \) in \( X_\mathcal{A} \). Note that \( Z \) will contain those common zeros of the \( x_i \) in \( (\mathbb{C}^*)^2 \). Now the map \( \phi \) can be realized as (the affine part of) a projection from \( X_\mathcal{A} \) to \( \mathbb{P}^3 \) via the hyperplane sections \( x_1, x_2, x_3, x_4 \).

The points in \( Z \) correspond to basepoints of this projection. We will assume that \( \text{gcd}(x_1, x_2, x_3, x_4) = 1 \), so that \( Z \) is finite. For each \( p \in Z \), we get a certain multiplicity \( e(\mathcal{I}_{Z,p}, \mathcal{O}_{X_\mathcal{A},p}) \). The degree of the parameterization \( \phi \) is the generic number of points in \( X_\mathcal{A} \) which map to a point in \( \Phi \). The degree of \( \Phi \) is the total degree of its implicit equation.

Now, as in [11, Appendix], we have the following degree formula:

**Proposition 1**

\[
\text{deg}(\phi) \text{deg}(\Phi) = \text{Area}(Q) - \sum_{p \in Z} e(\mathcal{I}_{Z,p}, \mathcal{O}_{X_\mathcal{A},p})
\]

where \( \text{Area}(Q) \) is the normalized area of the polygon \( Q \) equal to twice its usual Euclidean area (in particular \( \text{Area}(Q) \) is always an integer).

In the next section we will consider the case when there are no basepoints, that is \( Z = \emptyset \). If \( x_1, x_2, x_3, x_4 \) are each generic with respect to their supports \( \mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4 \) then this will be the case provided a certain geometric condition on the supports \( \mathcal{A}_i \) holds. This is expressed in the next result.

**Proposition 2** Fix subsets \( \mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_k \subset \mathbb{Z}^2 \) with \( k \geq 3 \) and define \( \mathcal{A} = \mathcal{A}_1 \cup \cdots \cup \mathcal{A}_k \). Let \( x_1, \ldots, x_k \) be generic polynomials with \( x_i \) supported on \( \mathcal{A}_i \). Let
Let \( Q_i = \text{Conv}(A_i) \) and \( Q = \text{Conv}(A) \) be the associated polytopes. Assume that \( \dim(Q) = 2 \). The polynomials \( x_i \) viewed as sections of \( X_A \) have no common zeros if and only if every edge of \( Q \) intersects at least two of the polytopes \( Q_i \).

**PROOF.** The torus orbits of \( X_A \) correspond to the faces of \( Q_A \). The restriction of \( x_i \) to a particular orbit corresponds to intersecting \( Q_i \) with the corresponding face of \( Q \) and setting all terms of \( x_i \) not in the intersection to 0. A zero-dimensional face of \( Q \) corresponds to a vertex, which by construction must be a vertex of some \( Q_i \). The corresponding \( x_i \) restricts to a single non-zero monomial with a generic (non-zero) coefficient, hence does not vanish at this point. On the orbit corresponding to the dense 2 dimensional torus, any 3 of the polynomials do not generically have a common zero. Finally, a one-dimensional orbit corresponds to an edge of \( Q \). By hypothesis, intersecting with the \( Q_i \) yields at least two non-zero polynomials with generic coefficients which do not have a common zero on the one-dimensional space. Conversely, if an edge of \( Q \) intersects only one of the \( Q_i \) then it must be an edge of that \( Q_i \), so that \( x_i \) restricts to a polynomial in one variable, while all other \( x_j \) restrict to zero. Thus every root of this restriction of \( x_i \) is a common zero of all of the \( x_j \).

Of course if all of the \( x_i \) have the same support, the case most often of interest for implicitization, then the condition above is automatically satisfied. In general it corresponds to a mild geometric compatibility of the supports.

### 3 Implicitization from the Chow Form

The *Chow form* of \( X_A \) is a polynomial \( Ch_A \) in the coefficients of three linear sections \( f_1, f_2, f_3 \) which is zero whenever \( f_1, f_2, f_3 \) have a common root on \( X_A \). In the case where \( Z = \emptyset \), we get the following result.

**Theorem 3** Let \( x_1, x_2, x_3, x_4 \) be Laurent polynomials with complex coefficients. Let \( Z \) be the set of common zeros on the toric variety \( X_A \) corresponding to the union of their supports. Let \( Ch_A \) be the Chow form of the toric variety \( X_A \). Let \( f_1, f_2, f_3 \) be as in (2) and \( P(X_1, X_2, X_3) \) the implicit equation of \( \Phi \). If \( Z = \emptyset \), then there exists a nonzero constant \( c \in \mathbb{C} \) such that

\[
Ch_A(f_1, f_2, f_3) = c P^{\text{deg}(\phi)}.
\]

**PROOF.** Let \( G(X_1, X_2, X_3) \) be the left-hand side of (3). For a generic point on the surface there is an associated common zero of \( f_1, f_2, f_3 \). Conversely if
$X_1, X_2, X_3$ are such that $f_1, f_2, f_3$ have a common zero $(s, t)$ then as $Z = \emptyset$, $x_4(s, t) \neq 0$ and thus $(X_1, X_2, X_3)$ is a point on the surface. As $P$ is irreducible, it follows that $G = cP^d$, with $c \neq 0$ and $d \in \mathbb{N}$. In order to verify that $d = \deg(\phi)$, it is enough to see that the degree of $P$ is $\text{Area}(Q)$. This follows easily by applying the Chow form to the dual Plücker coordinates of the polynomials (2) (see [20]) and by noting that the dual Plücker coordinates have degree one in $X_1, X_2, X_3$. The degree of $\text{Ch}_A$ in the Plücker coordinates is $\text{Area}(Q)$.

In [23] there is a construction for computing the Chow form of any toric surface. Given a toric surface $X_A$ with $Q = \text{conv}(A)$, and three sections $f_1, f_2, f_3$ with $f_i = \sum_{a \in A} C_{ia} x^a$. Then $\text{Ch}_A(f_1, f_2, f_3)$ is the determinant of a matrix of the following block form:

$$
\begin{pmatrix}
B & L \\
\tilde{L} & 0
\end{pmatrix},
$$

Here the entries of $L$ and $\tilde{L}$ are linear forms, and the entries of $B$ are cubic forms in the coefficients $C_{ia}$, as described below.

The columns of $B$ and $\tilde{L}$ are indexed by the lattice points in $Q$, the rows of $B$ and $L$ are indexed by the interior lattice points in $2 \cdot Q$, the matrix $\tilde{L}$ has three rows indexed by $\{f_1, f_2, f_3\}$, and the columns of the matrix $L$ are indexed by pairs $(f_i, a)$ where $i \in \{1, 2, 3\}$ and $a$ runs over the interior lattice points of $Q$. Each entry of $L$ and $\tilde{L}$ is either zero or is a coefficient of some $f_i$ and is determined in the following straightforward manner. The entry of $\tilde{L}$ in row $f_i$ and column $a$ is the coefficient of $x^a$ in $f_i$. The entry of $L$ in row $b$ and column $(f_i, a)$ is the coefficient of $x^{b-a}$ in $f_i$. The entries of the matrix $B$ are linear forms in \textit{bracket variables}. A bracket variable is defined as

$$
[abc] = \text{det}
\begin{bmatrix}
C_{1a} & C_{1b} & C_{1c} \\
C_{2a} & C_{2b} & C_{2c} \\
C_{3a} & C_{3b} & C_{3c}
\end{bmatrix},
$$

There is an explicit, combinatorial construction of the matrix $B$ given in [22]. By virtue of Theorem 3 above we get the immediate corollary.

\textbf{Corollary 4} If $Z = \emptyset$, then there is a determinantal formula $M_A$ for computing $P^{\deg(\phi)}$. 

6
Example 5  Consider the surface parameterized by

\[
\begin{align*}
    x_1 &= s^3 + t^2 \\
    x_2 &= s^2 + t^3 \\
    x_3 &= s^2t + st^2 \\
    x_4 &= st
\end{align*}
\]

The associated polygon $Q$ is a quadrilateral in the first quadrant with vertices $(2, 0), (3, 0), (0, 2), (0, 3)$. So we compute the Chow form, with respect to this polygon of $s^3 + t^2 - X_1st, s^2 + t^3 - X_2st, s^2t + st^2 - X_3st$ which results in the following $7 \times 7$ matrix:

\[
\begin{bmatrix}
0 & 1 & -X_1 & 0 & 1 & 0 & 0 \\
1 & 0 & -X_2 & 0 & 0 & 0 & 1 \\
0 & 0 & -X_3 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & -X_3 & -X_2 & X_1 & X_1 - 1 \\
0 & 0 & 0 & 0 & X_3 & -1 & -1 \\
0 & 0 & -X_1 & X_2 - 1 & 1 & X_2 & -X_3 \\
0 & 0 & 1 & -X_1 & X_2 & 1 - X_2 & -X_3 & X_1
\end{bmatrix}
\]

The determinant of this matrix is:

\[
2 + X_1 - 5X_3^3 - X_1^2X_2 - X_2^2X_1 + X_3X_1^3 + X_2X_3^3 + X_3^2 + X_2 + 5X_3 \\
+4X_2X_3^2 - X_3X_1 - 2X_2X_1 - X_2X_3 - 3X_2X_3^2 + X_2^3X_3 - 3X_2X_1X_3^3 \\
-X_2^2X_3^2X_1 + 4X_1X_3^2 - 3X_1X_3^2 + X_2^2X_1X_3 + X_3X_1X_2 - X_1X_3^2X_2 \\
+2X_2X_3^2 + X_1^2X_3 - 5X_1X_2X_3 + 2X_1X_3^2
\]

This is the degree 5 (equal to $\text{Area}(Q)$) affine implicit equation.

It is an immediate consequence from the proof of Theorem 3 that when $Z \neq \emptyset$ the Chow form $Ch_A$ is identically zero. However, we shall see in the next section that the implicit equation can still be recovered from maximal minors of the resultant matrix. This shows how a matrix resultant formula encodes much more information than just the Chow form.
3.1 Base Points

In this section we take a closer look at the Chow form matrix described above in order to determine what happens in the presence of base points. Throughout this section we will assume $x_1, x_2, x_3, x_4$ are specific choices of polynomials supported on $\mathcal{A}$ which may in particular have base points.

We will see that we can always get a matrix whose determinant is a non-trivial multiple of the implicit equation. In order to still get an exact formula we will need a hypothesis on the structure of the basepoints. By the construction of $\mathcal{A}$, we will always be able to assume that the points in $Z$ are smooth points of $X$ (see the explanation in the proof of Theorem 8). In that case the local ring $\mathcal{O}_{X,p}$ is just the localized polynomial ring in two variables $x, y$.

**Definition 6** Let $X$ be a variety of dimension $n$. A zero dimensional local complete intersection (LCI) is a subscheme $Z$ in the smooth locus of $X$, such that for each point $p$ in $Z$, the ideal $I_{Z,p}$ of the local ring $\mathcal{O}_{X,p}$ is defined by $n$ equations.

The main property of local complete intersections that we will use is contained in the next proposition.

**Proposition 7** If $Z$ is a local complete intersection then the multiplicity $e(I_{Z,p}, \mathcal{O}_{X,p})$ is equal to the vector space dimension of the finite local algebra $\mathcal{O}_{Z,p} = \mathcal{O}_{X,p}/I_{Z,p}$. In particular $\sum_{p \in Z} e(I_{Z,p}, \mathcal{O}_{X,p})$ is equal to the vector space dimension of the affine coordinate ring of $Z$.

This proposition is a consequence of [6, Theorem 4.7.4] as $I_{Z,p}$ is generated by a regular sequence. Hence, the Euler characteristic is just the length of $\mathcal{O}_{X,p}/I_{Z,p}$, which is the vector space dimension in the zero-dimensional case.

We can now state the main result of this section:

**Theorem 8** Let $\pi : X_A \to \mathbb{P}^3$ be a projection onto a surface $\Phi$ parameterized by $x_1, x_2, x_3, x_4$ with no common factor such that $\mathcal{A}$ is the union of the supports of the $x_i$. Let $Z \subset X_A$ be the finite set of basepoints of $\pi$. Now, let $M_A$ be the determinantal formula for $Ch_A$ from [22], where $f_1, f_2, f_3$ are the polynomials $x_1(s,t) - X_1x_4(s,t), x_2(s,t) - X_2x_4(s,t),$ and $x_3(s,t) - X_3x_4(s,t)$ respectively. Then the implicit equation $P^{\deg(\phi)}$ divides any maximal minor of $M_A$.

Moreover, a maximal minor of $Ch_A(f_1, f_2, f_3)$ using all of the Sylvester rows and columns exists and has determinant equal to exactly $P^{\deg(\phi)}$ if:

1. $Z$ is a local complete intersection on $X_A$.
2. The Sylvester columns in $L$, indexed by $\text{int}(Q)$, are linearly independent.
for generic choices of $X_1, X_2, X_3$. Equivalently, $f_1, f_2, f_3$ have no syzygies supported on $\text{int}(Q)$ with coefficients in $\mathbb{C}[X_1, X_2, X_3]$.

The LCI hypothesis seems to be ubiquitous in implicitization [4,5]. Note that in particular isolated basepoints are always LCI so that “generically” even if $x_1, x_2, x_3, x_4$ have basepoints, e.g. if the geometric condition of Theorem 2 is not satisfied, the basepoints they do have will be LCI.

The second condition is somewhat more subtle and is not really well understood except that it was quite difficult to construct examples for which it fails (see Example 3.5). It can be compared with the Assumption 5.1 in the method of moving surfaces, i.e. that the “moving plane” matrix $MP$ is of maximal rank.

Even if the second condition fails we can still recover the implicit equation as the GCD of the maximal minors. This is not true if the first condition fails as illustrated by Example 3.4.

**Example 9** Consider the surface parameterized by:

\[
\begin{align*}
x_1 &= 1 + s - t + st - s^2 t - st^2 \\
x_2 &= 1 + s - t - st + s^2 t - st^2 \\
x_3 &= 1 - s + t - st - s^2 t + st^2 \\
x_4 &= 1 - s - t + st - s^2 t + st^2 
\end{align*}
\]

There is one basepoint at $(s, t) = (1, 1)$. The corresponding polygon $Q$ is a pentagon with vertices $(0, 0), (1, 0), (0, 1), (2, 1), (1, 2)$. Computing the Chow form matrix gives a singular $9 \times 9$ matrix. However, we can remove one row and column to get the $8 \times 8$ matrix below:

\[
\begin{bmatrix}
0 & 0 & 0 & 1 - X_1 & 1 + X_1 & -1 + X_1 & 1 - X_1 & -1 + X_1 \\
0 & 0 & 0 & 1 - X_2 & 1 + X_2 & -1 + X_2 & 1 - X_2 & -1 + X_2 \\
0 & 0 & 0 & 1 - X_3 & -1 + X_3 & 1 + X_3 & -1 - X_3 & -1 + X_3 \\
1 - X_1 & -1 - X_2 & -1 - X_3 & 0 & 4X_3 - 4X_2 + 4X_1 - 4 & 0 & -4X_3 - 4X_1 & 4X_2 + 4X_3 \\
1 - X_1 & -1 - X_2 & -1 + X_3 & 0 & -4 + 4X_1 & 0 & 8 - 4X_3 - 4X_1 & 0 \\
-1 + X_1 & -1 + X_2 & 1 + X_3 & 0 & 4 - 4X_1 & 0 & 4X_3 - 4X_2 + 4X_1 - 4 & -4X_3 + 4X_1 \\
1 + X_1 & 1 + X_2 & -1 + X_3 & 0 & 0 & 0 & 0 & 0 \\
-1 + X_1 & 1 + X_2 & -1 + X_3 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

The determinant is 256 times the irreducible implicit equation which is:

\[
2X_1 - X_2 + X_3 - X_3^3X_1 - X_3^2X_1^2 + X_3X_1^2X_2 - 5X_3X_1 + 3X_2X_1 \\
- 2X_3X_1X_2 - 2X_1^2 - 3X_3^2X_2 + 2X_3X_2 - 2X_2^2 + 4X_2^2X_1 + X_3X_1^2 - X_2^3 \\
+ 3X_3X_2^2 - 2X_3^2 - X_3^2X_2 + 2X_2^3X_1 - X_2^4 - 2X_2^3X_2 - X_3X_2^3 \\
+ X_3^2X_1X_2 + X_3^3 + 4X_3^2X_1
\]
This has degree 4 since Area\(Q\) = 5 and there is one basepoint of multiplicity 1.

**Example 10** Let us now consider an example where the basepoint has multiplicity:

\[
\begin{align*}
x_1 &= (t + t^2)(s - 1)^2 + (1 + st - s^2t)(t - 1)^2 \\
x_2 &= (-t - t^2)(s - 1)^2 + (-1 + st + s^2t)(t - 1)^2 \\
x_3 &= (t - t^2)(s - 1)^2 + (-1 - st + s^2t)(t - 1)^2 \\
x_4 &= (t + t^2)(s - 1)^2 + (-1 - st - s^2t)(t - 1)^2
\end{align*}
\]

Once again there is a single basepoint at \((s, t) = (1, 1)\). But since, locally the ideal at this basepoint is generated by \((s - 1)^2, (t - 1)^2\) the basepoint is an LCI. So applying the method above we get a \(15 \times 15\) matrix and an \(11 \times 11\) maximal minor.

The determinant, after removing the integer constant, is

\[
-12 - 4X_1 - 9X_2 + 5X_3 - X_5^2 - 4X_2^2 - 20X_3^2X_5^2 - 16X_2^2X_5 - 32X_3^3X_5^2 - 12X_3^2X_5^2 - 12X_3^2X_5^2 + 8X_3X_5^4 - 12X_3^2X_5X_1 - 36X_3^2X_2X_1 - 48X_3^2X_2X_1 + 2X_5^3 - 6X_3X_5^2 + 11X_2X_5^2 + X_5^2 - 13X_2X_1 - 3X_1^4 - 3X_3X_2 + 14X_2X_1^2 - 9X_2X_1 - 16X_2^2 - 7X_1X_2 - X_3X_1^4 - X_5^2 - 19X_3X_5X_2 + 9X_3X_1 - 15X_3X_1^3 + 19X_3X_2^2 - 11X_2X_1^3 - 43X_3X_1^2X_2 + 27X_3X_2X_1 - 6X_4^2 + 3X_2^4X_1 + 4X_2^3X_1 - 14X_2^3 + 33X_3X_2^2 + 10X_3X_1X_2^2 - 43X_3X_1^2X_2^2 + 28X_3^2X_2 - 12X_3X_2^3X_1
\]

The degree of this equation is 5 and the area of the support polygon \(Q\) is 9.

**Example 11** Let us now modify the above example so that the basepoints no longer form an LCI. We will see that we can no longer recover the implicit equation exactly from our Chow form matrix.

\[
\begin{align*}
x_1 &= (t + t^2)(s - 1)^2 + (1 + st - s^2t)(t - 1)^2 + (t + st + st^2)(s - 1)(t - 1) \\
x_2 &= (-t - t^2)(s - 1)^2 + (-1 + st + s^2t)(t - 1)^2 + (t + st + st^2)(s - 1)(t - 1) \\
x_3 &= (t - t^2)(s - 1)^2 + (-1 - st + s^2t)(t - 1)^2 + (t + st + st^2)(s - 1)(t - 1) \\
x_4 &= (t + t^2)(s - 1)^2 + (-1 - st - s^2t)(t - 1)^2 + (t + st + st^2)(s - 1)(t - 1)
\end{align*}
\]
Because of the additional $(s-1)(t-1)$ term, the degree of the basepoint at $(1,1)$ drops to 3, however, the multiplicity remains 4. Indeed, a maximal minor of the $15 \times 15$ Chow form matrix now has rank 12. And the determinant of any maximal minor is (up to a constant):

$$(-X_2 + 2X_3 - 1)(101 - 224X_5^5 + 8X_1^5 - 525X_1 + 75X_2 + 2689X_1X_3 - 573X_3
+ 5519X_1^3X_1^2 + 3830X_3^3X_1 + 2948X_1^3X_3 + 1310X_3^3X_3 + 155X_1X_2^2X_3^2
- 169X_1X_3X_3 - 1970X_3X_3 - 2308X_3^3X_1 - 487X_3X_3^2X_1 - 1182X_3^3X_1
- 2296X_3^3X_1 + 1707X_1X_3X_2 + 1006X_1X_3^3X_2 + 1487X_3^3X_2^2
+ 956X_1^3X_2 - 1512X_3^3 - 4795X_1X_3 - 2118X_1X_3X_2 - 624X_1X_3 + X_2^5
- 13X_2^4 - 88X_2^3 - 76X_2^3 - 948X_1^3X_2 + 244X_1^3X_2 - 646X_3X_2 - 513X_1X_2
- 211X_2^3X_2^2 + 191X_3X_2^2 - 105X_3^2X_2 + 1140X_3^2X_2 + 185X_3^2X_2^2
+ 143X_3^2X_2 + 255X_4X_2 + 3X_3^2X_2 - 42X_3^2X_2^2 + 19X_1X_2^4 + 264X_1X_2^2
- 214X_2^4X_2^2 + 48X_2^4X_2^2 - 385X_3X_2 + 248X_3^2X_2 + 729X_3^2X_2 + 18X_3^2X_2^2
+ 337X_4^2 + 1050X_3^2 + 898X_3^2 - 4445X_1^2X_3 + 1133X_1^2$$

The second factor, of degree 5 is the desired implicit equation.

In the last example, there is a linear extraneous factor of $-X_2 + 2X_3 - 1$. One can show that this extraneous factor divides every maximal minor of $M_A$. Hence, the extraneous factor is somehow intrinsic to the resultant matrix and cannot be removed. It would be interesting to have some theoretical explanation for this factor.

We conclude with an example where the Sylvester rows are not linearly independent.

**Example 12**

$$x_1 = s + s^2 + s^3t + s^2t^2 + st^3$$
$$x_2 = t^2(s + 1)$$
$$x_3 = st(s + 1)$$
$$x_4 = t(s + 1)$$

The Newton polygon has three interior points $st, s^2t, st^2$. This system turns out to have a degree 7 LCI basepoint locus on $X_A$. However, one can easily check that $(s^2t - stX_3)(x_2 - X_2x_3) = (st^2 - stX_2)(x_3 - X_3x_4)$ so that there is indeed a syzygy of $f_1, f_2, f_3$ supported in $\text{int}(Q)$. So there is no maximal minor using all of the Sylvester columns. We can still construct maximal minors using as many Sylvester columns as possible, in this case 8 of the 9. The determinant of such a minor depends on which choice of Sylvester columns we remove. If
we remove the column in the Sylvester block corresponding to \( \text{st} \cdot f_3 \) we get a matrix whose determinant is

\[
X_2(X_2^3X_1 + X_1^2X_2^2 + X_3^2X_2^2 - X_1X_2X_3 + X_2X_3^3 - X_1X_3^2).
\]

If, on the other hand, we remove a column corresponding to \( s^2t \cdot f_3 \) the determinant is exactly the implicit equation without the extraneous factor of \( X_2 \).

3.2 Proof of Theorem 8

In this section we prove theorem 8. The Chow form matrix described above, and indeed most of the formulas for Chow forms in the literature, are applications of a general setup due to Weyman [32]. A constructive approach using exterior algebras was described by Eisenbud, Schreyer and Weyman [19]. They start with an arbitrary projective variety \( X \subset \mathbb{P}^N \) of dimension \( n \) and try to compute its Chow form. Hence they consider the incidence correspondence:

\[
\begin{aligned}
V & \subset X \times G_{n+1} \\
X & \subset \mathbb{P}^N \\
G_{n+1} & \text{Grassmanian of codimension } n+1 \text{ planes in } \mathbb{P}^N
\end{aligned}
\]

Here \( G_{n+1} \) is the Grassmanian of codimension \( n+1 \) planes in \( \mathbb{P}^N \) and \( V = \{(x, F) : F(x) = 0\} \) the incidence subvariety of \( X \times G_{n+1} \). Now given any sheaf \( \mathcal{F} \) supported on \( X \) which is generically a vector bundle, there is a complex, denoted \( U_{n+1}(\mathcal{F}) \) in [19], of vector bundles on \( G_{n+1} \) equivalent in the derived category to \( R(\pi_2)_* \pi_1^* \mathcal{F} \). This leads to the following completely general result.

**Theorem 13** Let \( X \subset \mathbb{P}^N \) be any variety of dimension \( n \). Let \( \mathcal{F} \) be any sheaf supported on \( X \) that is generically of rank 1. Let \( F_0, \ldots, F_n \) be any linearly independent sections of \( \mathbb{P}^N \) which simultaneously meet \( X \) only at finitely many points at all of which \( \mathcal{F} \) is of rank 1. The last map in the complex \( U_{n+1}(\mathcal{F}) \) has cokernel of rank equal to the degree of the zero-dimensional subscheme of \( X \) cut out by \( F \).

**PROOF.**

Consider again the incidence correspondence. As \( U_{n+1}(\mathcal{F}) \) is isomorphic in the derived category to \( R(\pi_2)_* \pi_1^* \mathcal{F} \), the cokernel of the last map in particular is
just \((\pi_2)_*\pi_1^*\mathcal{F}\) itself. So all we need to show is that the dimension of the the fiber of this sheaf at a point \(F \in G_{n+1}\) satisfying the above properties is the degree of the subscheme \(X_F\) of \(X\) defined by \(F\).

First consider the fiber of the morphism \(\pi_2\) over \(F\). Let \(R\) be the coordinate ring of \(X\) and \(S\) the Stiefel coordinate ring of \(G_{n+1}\) with variables \(a\). The ideal of \(V\) in \(R \otimes S\) is denoted \(I(a)\). Now, by definition the fiber over the point \(F\) defined by a choice \(a = a\) with corresponding maximal ideal \(m_a\) in \(S\) is \((R \otimes S)/I(a) \otimes_S S/m_a\). But this is just \(R/I(a)\) which is the coordinate ring of \(X_F\). Hence the fiber of \(\pi_2\) over \(F\) is \(X_F \times F\). (Note that different choices of \(a\) realizing the same point \(F\) give the same ideal \(I(a)\)).

Next, since \(X_F\) is a zero dimensional subscheme of the generic locus of \(\mathcal{F}\) it is actually affine and \(\mathcal{F}\) is trivial on \(X_F\). Let \(R/I(a)\), as above, be the (dehomogenized) coordinate ring of \(X_F\) and hence also of \(X_F \times F\). As our sheaf was trivial, the pushforward onto the closed point \(F\) is just \(R/I(a)\) itself viewed as a vector space over the residue field of \(F\). The dimension of this vector space is by definition the degree of \(X_F\) as desired.

We can now prove Theorem 8 as a corollary.

**PROOF.**

We consider, in this case, \(\mathcal{F} = \mathcal{O}(\text{int}(2Q))\) the divisor corresponding to the interior of the polytope \(2Q\). In [22], it was shown that \(U_3(\mathcal{F})\) reduced to a two term complex with matrix exactly as described above. The sheaf \(\mathcal{F}\) is locally free of rank 1, except possibly on the singular points of \(X_\mathcal{A}\), which only occur on the vertices of \(Q\). By the construction of \(\mathcal{A}\), at least one of \(x_1, \ldots, x_4\) does not vanish on each vertex, hence the base point locus always misses the singular locus.

Now, we can apply Theorem 13. Pick a maximal minor of our matrix. For a generic \(X_1, X_2, X_3\) not on the surface \(S\), this remains a maximal minor of the specialization. Moreover, the corank of this minor is the degree of \(I(f_1(X_1), f_2(X_2), f_3(X_3))\). However, for a point \(X_1, X_2, X_3\) on the surface, the number of basepoints increases, therefore the rank of our matrix \(M\) decreases, hence the determinant of our chosen minor must be zero. Moreover, the rank drop of the minor for a generic point on the surface is exactly \(\deg \phi\) (the number of “new basepoints” mapping on to our point). Since any order \(k\) derivative of the determinant of a matrix of linear forms is in the ideal of corank \(k\) minors (easy to see from the expansion of determinant), the first \(\deg \phi - 1\) derivatives of the determinant are also zero for a generic point on the surface. Since \(P\) was irreducible, \(P^{\deg \phi}\) must divide our chosen maximal minor.
For the second part, in the case of an LCI, the corank of our maximal minor, i.e. the degree of the base point locus, is the same as the sum of the multiplicities of our base points. If moreover, the maximal minor is chosen to contain all Sylvester rows and columns, only Bézout rows and columns are removed, each of which drops the degree by 1. Thus the degree of our determinant is equal to the degree of $P^{\deg \phi}$ and so they must be equal up to a constant.

4 The method of moving surfaces

We now switch gears and present an entirely different method for constructing matrix formulas in implicitization. For the rest of this paper we will work with the projective surface $\Phi \subset \mathbb{P}^3$ defined by the four coordinates $X_1, X_2, X_3, X_4$.

The idea will be to construct linear and quadratic syzygies on the polynomials $x_1, x_2, x_3, x_4$ and put them together into a matrix of linear and quadratic forms in the $X_i$. For the case of homogeneous and bihomogeneous polynomials, this is the method of moving planes and surfaces introduced by Sederberg and Chen [30]. However the proof we present in Section 5 is quite different, and in our opinion more insightful, than the ones in the literature. Our goal will to be to extend the method to general toric surfaces which will require looking at certain “degrees” of the homogeneous coordinate ring of the toric variety.

We shall see that the syzygy method has certain advantages and disadvantages to the Chow form/resultant method described above. It will always give smaller matrices due to the fact that some of the entries are quadratic in the $X_i$. Second, the algorithm will be relatively easy to describe and efficient in practice; all of the computations are just numerical linear algebra. Finally, the method appears to be surprisingly flexible in the presence of base points. We shall see empirical evidence supporting this at the end of the section.

On the other hand, rigorous proofs of the method in any of the more complicated situations have been hard to come by. Also, as pointed out above, all of the computations are linear algebra in the coefficients of the $x_i$. In particular, the method becomes much more inefficient with a generic parameterization or whenever the coefficients of the $x_i$ are not numerical. The Chow form matrix constructed above, on the other hand, works the same for arbitrary coefficients and is therefore preferred when implicitizing a family of surfaces.
4.1 Moving planes and quadrics

Given a rational surface $\Phi$ parameterized by

\[
\begin{align*}
X_1 &= x_1(s, t) \\
X_2 &= x_2(s, t) \\
X_3 &= x_3(s, t) \\
X_4 &= x_4(s, t)
\end{align*}
\]

(4)

A moving plane is a syzygy on $I = \langle x_1, x_2, x_3, x_4 \rangle$, i.e an equation of the form

\[
A_1(s, t)X_1 + A_2(s, t)X_2 + A_3(s, t)X_3 + A_4(s, t)X_4
\]

which is identically zero as a polynomial in $s$ and $t$ after the specialization $X_i \mapsto x_i$. Notice that each particular choice of $(s, t)$ gives the equation of a plane which intersects the surface $\Phi$ at the point $(x_1(s, t), \ldots, x_4(s, t))$. Hence, this is said to be a plane that follows the surface $\Phi$ and justifies the terminology moving plane.

Similarly, a moving quadric is a syzygy on $I^2$:

\[
A(s, t)X_1^2 + B(s, t)X_1X_2 + \cdots + J(s, t)X_4^2
\]

Once again a choice of $(s, t)$ gives the equation of a quadric meeting the surface $\Phi$. Hence, the moving quadric is said to follow the surface.

If we rewrite the moving planes and quadrics in terms of the monomial bases in $s$ and $t$ we get vectors of linear or quadratic forms in the $X_i$. Clearly multiplying each moving plane by $X_1, X_2, X_3, X_4$ gives a moving quadric. Therefore, we will only look for “new” moving quadrics. If we can now get enough of these vectors, we may be able to build a square matrix out of them. The determinant of this square matrix will hopefully be equal to the implicit equation of $S$. The following well known result is our starting point.

Proposition 14 Let $M(X_1, X_2, X_3, X_4)$ be any square matrix constructed from moving planes and quadrics as above. Then $\det(M(x_1, x_2, x_3, x_4)) = 0$. In particular the implicit equation always divides the determinant of $M$ (which may, quite possibly, be identically 0).
PROOF. This has been proved in even more generality in [30].

The big question is now, of course, how should the moving planes and quadrics be chosen? In the case of homogeneous polynomials they were chosen to also be homogenous of an appropriate degree. In the case of bihomogeneous polynomials, the moving planes and quadrics can be chosen to be bihomogeneous. In the more general toric setting we will need to work in appropriate homogeneous coordinates for the set \( \mathcal{A} \).

4.2 Homogeneous coordinate ring of \( X_{\mathcal{A}} \)

Let \( \mathcal{A} \) be the union of monomials in the \( x_i \) as before and \( Q = \text{conv}(\mathcal{A}) \) the associated polygon. Let \( E_1, \ldots, E_s \) be the edges of \( Q \) and \( \eta_1, \ldots, \eta_s \) the corresponding inner normals.

We can therefore define \( Q \) by its facet inequalities.

\[
Q = \{ m \in \mathbb{R}^2 : \langle m, \eta_i \rangle \geq -a_i \text{ for } i = 1, \ldots, s \}
\]

For some \((a_1, \ldots, a_s) \in \mathbb{Z}^s\).

\( X_{\mathcal{A}} \) is a toric variety with a given very ample line bundle determined by the polytope \( Q \). We will need to consider other divisors on \( X_{\mathcal{A}} \). David Cox [10] defined a single ring that encapsulates all torus invariant divisors on \( X_{\mathcal{A}} \).

**Definition 15** The homogeneous coordinate ring for \( X = X_{\mathcal{A}} \) is the polynomial ring \( S_X = \mathbb{K}[y_1, \ldots, y_s] \) where the monomials are graded as described below.

Consider the exact sequence of maps:

\[
0 \to \mathbb{Z}^2 \xrightarrow{\phi} \mathbb{Z}^s \xrightarrow{\pi} G \to 0
\]

Here \( \phi \) is the map \( m \to (\langle m, \eta_1 \rangle, \ldots, \langle m, \eta_s \rangle) \). The ring \( S_X \) is graded by elements of \( G \) where \( \deg y^\alpha = \pi(\alpha) \).

The graded pieces of this ring have bases corresponding to lattice points in polygons. More precisely the monomials in \( S_{\pi(b)} \) are in one to one correspondence with the lattice points in \( Q_b = \{ m \in \mathbb{R}^2 : \langle m, \eta_i \rangle \geq -b_i \} \). And moreover, \( \pi(b) = \pi(b') \) iff \( Q_b \) is a translate of \( Q_{b'} \).

So it will make sense to talk about \( S_{Q_b} \), the graded piece of \( S \) defined by \( Q_b \).
Remark 16 In truth the divisors and homogeneous coordinate ring are really defined for the normal toric variety $X_Q$ obtained from the normal fan of $Q$. This variety is the normalization of our $X_A$. The projection and all prior and subsequent results can be lifted up to $X_Q$ without affecting any of the calculations.

4.3 Picking moving planes and quadrics

Also associated to the polygon $Q$ is a certain polynomial $E(k)$, the Ehrhart polynomial defined in [31], which counts the number of lattice points in $k \cdot Q$. In the case $Q$ is two dimensional, it turns out that

$$E(x) = Ax^2 + \frac{B}{2}x + 1$$

where $A = \frac{\text{Area}(Q)}{2}$ and $B$ equals the number of boundary points.

Let $I$ be a nonempty proper subset of $\{1, \ldots, s\}$ such that the corresponding edges form a connected set. Let $E_I$ be this connected set of edges of $Q$, let $B_I$ be the sum of the lattice edge lengths of $E_I$. It is easy to see that the number of lattice points in the set of edges $E_I$ in $k \cdot Q$ is $B_I k + 1$.

Assumption 17 We choose $E_I$ in such a way that $B \geq 2B_I$.

Remark 18 Observe that this can always be done, for instance by taking as $E_I$ the shortest edge of $Q$. In practice, we will want to pick $E_I$ in such a way that $B_I$ is as big as possible consistent with Assumption 17.

Now we can define a degree of $S$ denoted $S_{Q \setminus E_I}$ obtained by “pushing in” all of the edges of $Q$ in $E_I$ by one, whose monomial basis consists of all lattice points in $Q$ not on any of the edges $E_I$. In the case of homogenous polynomials of degree $n$, the only $E_I$ satisfying Assumption 17 consist of a single edge and the degree in question in just $n - 1$. In the case of bihomogeneous polynomials of bidegree $(m, n)$, we can take $E_I$ to be two consecutive edges and the degree is $(m - 1, n - 1)$. Note that in the latter case $B - 2B_I = 0$ which, as we shall see, means that we will not need to take any moving planes and can build a matrix entirely out of moving quadrics. We now formally define what we mean by moving planes and quadrics of this degree.

Consider the following $\mathbb{K}$-linear map

$$\psi_1 : \quad S_{Q \setminus E_I}^4 \rightarrow S_{2Q \setminus E_I}$$

$$(p_1, p_2, p_3, p_4) \mapsto \sum_{i=1}^4 p_ix_i,$$
and let $MP$ be the matrix of this map in the monomial bases.

**Definition 19** As in [12], any element of the form $(A_1, A_2, A_3, A_4) \in \ker(\psi)$ will be called a moving plane of “degree” $Q \setminus E_I$ that follows the surface (1). Sometimes, we will write moving planes as $A_1X_1 + A_2X_2 + A_3X_3 + A_4X_4$.

Now for moving quadrics we consider the following map:

$$\psi_2 : S_{Q \setminus E_I}^{10} \rightarrow S_{3Q \setminus E_I}$$

$$(A_{i,j,k,l})_{i+j+k+l=2} \mapsto \sum_{i+j+k+l=2} A_{i,j,k,l} x_1^i x_2^j x_3^k x_4^l,$$

and let $MQ$ be the matrix of $\psi_2$ in the monomial bases. Then

$$\text{# rows of } MQ = \# (3Q \setminus E_I) \cap \mathbb{Z}^2 =$$

$$= (9A + \frac{3}{2}B + 1) - (3B_I + 1) = 9A + \frac{3}{2}B - 3B_I$$

and

$$\text{# columns of } MQ = 10 \# (Q \setminus E_I) \cap \mathbb{Z}^2 =$$

$$= 10 (A + (B - 2B_I)/2) = 10A + 5B - 10B_I.$$

Now a **moving quadric** of degree $Q \setminus E_I$ which follows our surface $S$ is just an element of the kernel of $MQ$.

We now describe the method of moving quadrics. It differs from the presentations in the literature not only in its application to general toric surfaces but also in that we allow the bases of moving planes and quadrics to be chosen freely. Earlier papers specify that moving planes and quadrics be chosen of a specific form to ensure that the resulting matrix has determinant non-zero. Our more intrinsic proof of Section 5.1 makes this unnecessary.

- Compute a basis $P$ of the kernel of $MP$. The entries are $P_i = A_i^1X_1 + A_i^2X_2 + A_i^3X_3 + A_i^4X_4$. Where the $A_i^j$ are polynomials in $S_{Q \setminus E_I}$.
- Each $P_i \cdot X_j$ for $j = 1, \ldots, 4$ is in the kernel of $MQ$. We will see that these are linearly independent. Extend this set to an entire basis for the kernel of $MQ$. Let $Q_1, \ldots, Q_d$ be the new moving quadrics in this basis.
- Construct a matrix $M$ out of the $P_i$ and $Q_j$ such that the columns correspond to the monomial basis of $S_{Q \setminus E_i}$ and the entries are the linear (or quadratic) polynomial in $X_1, \ldots, X_4$ corresponding to the coefficient of that monomial in $P_i$ (or $Q_j$).

Our hope is that the resulting matrix will be square and that the determinant is the implicit equation. To start with, by Theorem 14, if the matrix $M$ has
more rows than columns, then the determinant of any maximal minor (possibly 0) is divisible by the implicit equation.

5 Validity of the method of moving quadrics without basepoints

In this section we verify, in the absence of basepoints, that the method of moving quadrics gives a square, nonsingular matrix whose determinant is exactly the implicit equation raised to the power the degree of the parameterization. We will need to make one assumption:

**Assumption 20** The moving plane matrix MP, or the map ψ₁, has maximal rank.

This assumption also appears in the papers by Cox, Goldman, and Zhang [12] and D’Andrea [16]. Empirical evidence suggests that it is almost always satisfied. It appears that for a fixed Q and E₁, and any generic set of x₁, x₂, x₃, x₄ without basepoints, MP has maximal rank.

We now build a complex containing both the moving plane map ψ₁ and the moving quadric map ψ₂ in Figure 1.
The terms $K_1$ and $K_2$ are the kernels of the moving plane and moving quadric maps $\psi_1$ and $\psi_2$ respectively. The term $\tilde{K}_2$ is the cokernel of the map $X$ of $K_1^4$ into $K_2$, generated precisely by a basis of $K_2$ extending the image of moving planes multiplied by linear forms. In this new language a matrix $\mathbb{M}$ of moving planes and quadrics is a basis for $K_1^4 \oplus \tilde{K}_2$ taken as vectors in $(S_{Q\setminus E})$ with coefficients that are linear or quadratic forms in $(X_1, X_2, X_3, X_4)$.

We now prove out two main theorems that together prove the validity of the method of moving quadrics.

**Theorem 21** If $MP$ has maximal rank, then $\dim(K_1) + \dim(\tilde{K}_2) = \dim(S_{Q\setminus E})$ and $\dim(K_1) + 2 \dim(\tilde{K}_2) = \text{Area}(Q)$. Therefore, the method of moving quadrics yields a square matrix with determinant of degree equal to the implicit equation.

**Theorem 22** Let $p = (p_1, p_2, p_3, p_4)$ be a point not on the surface $X$. The moving plane matrix $\mathbb{M}$ is nonsingular at $p$. Consequently, if $\psi_1$ has maximal rank $\det(\mathbb{M}) = P^\deg(\phi)$ where $P$ is the implicit equation as desired.

Before proceeding we further describe the maps in the complex. The second row consists of four copies of the moving plane complex. An element of $(S_{Q\setminus E})^{16}$ is represented as a four tuple of linear forms in $X_1, X_2, X_3, X_4$ with coefficients in $S_{Q\setminus E}$. Similarly the bottom row is the moving quadric complex. An element of $(S_{Q\setminus E})^{10}$ is a quadratic form in $X_1, X_2, X_3, X_4$ with coefficients in $S_{Q\setminus E}$, generated by the 10 monomials $X_iX_j$ with $i \leq j$. The map $X$, multiplication by $(X_1, X_2, X_3, X_4)$, sends the four tuple $(u_1, u_2, u_3, u_4)$ of linear forms to the quadratic form $\sum u_iX_i$. This has the effect of sending $X_i$ in position $j$ and $X_j$ in position $i$ both to $X_iX_j$.

The kernel of $X$ is isomorphic to $(S_{Q\setminus E})^{16}$ indexed by pairs $(i, j)$ with $i < j$. The injection $i$ sends the term $p_{ij}$ to $(0, \ldots, p_{ij}X_j, \ldots, -p_{ij}X_i, \ldots, 0)$ with $X_j$ in position $i$ and $-X_i$ in position $j$. The rightmost column is a graded piece of the Koszul complex on $(x_1, x_2, x_3, x_4)$, with $x$ mapping a four tuple $(s_1, s_2, s_3, s_4)$ to $\sum s_i x_i$ and $x'$ sending $p_{ij}$ with $i < j$ to $(0, \ldots, p_{ij}x_j, \ldots, -p_{ij}x_i, \ldots, 0)$.

Commutativity of the diagram is immediate. The rows are all exact by construction. The second column is also clearly exact. The rightmost column is more interesting. When $(x_1, x_2, x_3, x_4)$ have no basepoints, the map $x'$ is injective and $x$ is surjective. This can be seen by investigating the complex $U_4(\mathcal{O}(3Q \setminus E))$ arising from the Tate resolution in the theory of [19]. However, the spot in the middle is not exact. We shall see later that obstruction to exactness comes from a certain 'Bezoutian' map determined exactly by elements of $\tilde{K}_2$.

Now, to prove Theorem 21 we will need three lemmas:
Lemma 23 If $MP$ has maximal rank, then the number of linearly independent moving planes of degree $Q \backslash E_I$ which follow the surface is $B - 2B_I$.

PROOF. There are
\[(4A + B + 1) - (2B_I + 1) = 4A + B - 2B_I\]
integer points in $2Q \backslash E_I$, and
\[\left( A + \frac{B}{2} + 1 \right) - (B_I + 1) = A + (B - 2B_I)/2\]
integer points in $Q \backslash E_I$. If $MP$ has maximal rank, then the number we want to compute is the dimension of the kernel of $\psi_1$ which equals
\[4 \left( A + (B - 2B_I)/2 \right) - (4A + B - 2B_I) = B - 2B_I\]
as claimed.

Lemma 24 If $\psi_1$ is surjective then so is $\psi_2$.

PROOF. This is an easy diagram chase. Given $s \in S_{3Q \backslash E_I}$ pull it back to $S_{2Q \backslash E_I}^4$ and then to $(S_{Q \backslash E_I})^{16}$ via the surjectivity of the corresponding maps. Finally map this down to $t \in (S_{Q \backslash E_I})^{10}$. Commutativity of the diagram yields $\psi_2(t) = s$.

Lemma 25 The map $X$ from $K_1^4$ to $K_2$ is injective.

PROOF. Given $k$ in the kernel, it is a non-zero element of $(S_{Q \backslash E_I})^{16}$ mapping to zero in $(S_{Q \backslash E_I})^{10}$. By exactness it comes from a nonzero element in $S_{Q \backslash E_I}^6$ mapping to a nonzero element $k'$ in $(S_{2Q \backslash E_I})^4$. But commutativity implies $k' = \psi_1(k) = 0$, a contradiction.

We are now ready for the proof of Theorem 21.

PROOF. By Lemma 24 $MQ$ has maximal rank. From the computations of the last section, the dimension of $(S_{Q \backslash E_I})^{10}$ is $10A + 5B - 10B_I$, while the dimension of $S_{3Q \backslash E_I}$ is $9A + \frac{3}{2}B - 3B_I$. Thus the rank of $K_2$ is $A + \frac{3}{2}B - 7B_I$. By Lemma 23 the rank of $K_1^4$ is $4(B - 2B_I)$, so by Lemma 25 the rank of $\tilde{K}_2$ is $A - \frac{B}{2} + B_I$.

So, the sum of the ranks of $K_1$ and $\tilde{K}_2$ is
\[ B - 2B_I + (A - \frac{B}{2} + B_I) = A + \frac{B}{2} - B_I \]
\[ = \dim S_{Q\setminus E_I} \]

Moreover, the total degree of the determinant is

\[ B - 2B_I + 2(A - \frac{B}{2} + B_I) = 2A \]

This is twice the Euclidean area, hence equal to the normalized area of \( Q \) as desired.

The theorem just proved shows that \( M \) is square of the right rank. Theorem 22 will show that its determinant does not vanish outside of the surface.

Let \( p = (p_1, p_2, p_3, p_4) \in \mathbb{P}^3 \) be a point not on the surface parametrized by \( X \). WLOG assume that \( p_4 = 1 \). Make a change of coordinates \( X'_1 = X_1 - p_1 X_4, X'_2 = X_2 - p_2 X_4, X'_3 = X_3 - p_3 X_4 \) and \( X'_4 = X_4 \). The point \((p_1, p_2, p_3, p_4)\) is transformed to \((0, 0, 0, 1)\). Since the parameterization has no base points, the \( \text{Ch}_A(x'_1, x'_2, x'_3) \neq 0 \) by Theorem 3.

We now use two facts arising from resultant complexes.

**Lemma 26** The restricted map \( \tilde{\psi}_1 : S^3_{Q\setminus E_I} \to S^2_{Q\setminus E_I} \), given by \((s_1, s_2, s_3) \to \sum s_i x'_i \) is injective. In particular no moving plane \( A_1 X'_1 + A_2 X'_2 + A_3 X'_3 + A_4 X'_4 \) vanishes at \((X'_1, X'_2, X'_3, X'_4) = (0, 0, 0, 1)\).

**PROOF.** In [22, Theorem 3.4.1], a matrix whose determinant gives \( \text{Ch}_A(x_1, x_2, x_3) \) is constructed, and this matrix has a Sylvester part coming from \( \tilde{\psi}_1 \). As we have \( \text{Ch}_A(x'_1, x'_2, x'_3) \neq 0 \), it turns out that \( \tilde{\psi}_1 \) must be injective. Any vanishing moving plane as above has \( A_4 = 0 \) so must in fact be in the kernel of \( \psi_1 \).

**Lemma 27** The restriction of the last column:

\[ 0 \to S^3_{Q\setminus E_I} \to S^3_{2Q\setminus E_I} \to S^3_{3Q\setminus E_I} \to 0 \]

which is just the Koszul complex on \( x'_1, x'_2, x'_3 \), is exact.

**PROOF.** We may consider \( x'_1, x'_2, x'_3 \) as sections of sheaves on the toric variety \( X_A \). As in [17, Section 4], we start with the Koszul complex of these sheaves in degree \( 3\beta - \beta_I \), where \( \beta \) is the degree associated to \( Q \cap \mathbb{Z}^2 \) and \( \beta_I \) the divisor.
associated to all the edges whose union equals $E$. As in the proof of Theorem 3.1 in [17], one can see that we can apply the Weyman’s complex (see [20, Section 3.4.E]) to this complex. By the toric version of Kodaira vanishing (see [28]), all higher cohomology terms vanish and we get that the complex above is generically exact. Indeed, the determinant of the complex equals the Chow form of $x'_1, x'_2, x'_3$.

**Corollary 28** Any moving quadric $\sum_{1 \leq i \leq j \leq 4} A_{ij} X'_i X'_j$ vanishing at $(0, 0, 0, 1)$ is in the image of $K^4$ under the multiplication map $X$.

**PROOF.** Start with such a vanishing moving quadric $q$. Plugging in we see that $A_{44} = 0$. Hence $q = q_1 X'_1 + q_2 X'_2 + q_3 X'_3$ where $q_1 = A_{11} X'_1 + A_{12} X'_2 + A_{13} X'_3 + A_{14} X'_4$, $q_2 = A_{22} X'_2 + A_{23} X'_3 + A_{24} X_4$ and $q_3 = A_{33} X'_3 + A_{34} X'_4$.

Pulling back to $q' = (q_1, q_2, q_3, 0) \in (S_{Q \setminus E_1})^{16}$ and mapping to $(S_{2Q \setminus E_1})^4$ by substituting $x'_1, x'_2, x'_3$ into $q_1, q_2, q_3$ we get an element of the subspace $(S_{2Q \setminus E_1})^3$ as in the restricted complex above which is still in the kernel of $X$. Thus, by Lemma 24 we can pull back via $X'$ to $S_{Q \setminus E_1}^3 \subset S_{Q \setminus E_1}^6$. Let $q''$ be the image of this element in $S_{Q \setminus E_1}^{16}$. By construction $\psi_i^4(q' - q'') = 0$ and $X(q' - q'') = q$. But now we can pull back $q' - q''$ to $k = (k_1, k_2, k_3, k_4)$ with $X(k) = q$ as desired.

It is now straightforward to finish the proof of Theorem 22.

**PROOF.** Suppose $(u(p), v(p))$ is in the kernel. Write $u = \sum u_i X'_i$ and $v = \sum v_{ij} X'_i X'_j$. Substituting in for $p$ we have $u_4 + v_{44} = 0$. Therefore the moving quadric $X'_4 u + v$ has no $(X'_4)^2$ term and thus vanishes at $p$. By Corollary 28 this must be in the image of $K^4$ so we must have $v = 0$. But now $u(p) = 0$ violating Lemma 26. Hence $M$ is singular only on points of $X$. If $\psi_1$ is maximal rank then $M$ is square, hence its determinant is a power of the implicit equation. Since the degree of $\det(M) = \text{Area}(Q)$, the exponent must be $\deg(\phi)$.

**Example 29** Consider the system from Example 5:

$$
\begin{align*}
    x_1 &= s^3 + t^2 \\
    x_2 &= s^2 + t^3 \\
    x_3 &= s^2 t + st^2 \\
    x_4 &= st
\end{align*}
$$

The total boundary length of the quadrilateral $Q$ is 7. We can pick $E_1$ to be the long edge of length 3. Hence $B - 2B_1 = 1$. Applying the method of moving quadrics then gives a matrix with one moving plane and two moving quadrics:
\[
\begin{bmatrix}
-X_1 - X_2 - X_3 & X_3 + X_4 & X_3 + X_4 \\
X_1 X_3 - X_2 X_4 + X_4^2 & X_1 X_3 - X_2 X_4 - X_3 X_4 & -X_3^2 + X_4^2 \\
-X_1^2 - X_1 X_2 - 3X_1 X_3 & X_1 X_4 + X_2 X_4 + X_3 X_4 & X_1 X_2 + X_2 X_3 + 2X_3^2 \\
+2X_2 X_4 - X_3^2 + X_3 X_4 & -X_3 X_4 - 2X_4^2 
\end{bmatrix}
\]

The determinant is exactly the degree 5 implicit equation.

5.1 Moving quadrics in the presence of base points

In the case of homogeneous parameterizations \((X_A = \mathbb{P}^2)\), [5] gives a series of conditions for when the method of moving quadrics works even with basepoints. The conditions are labelled (BP1)-(BP5) but essentially they boil down to assuming the basepoints form an LCI, there are no syzygies on linear combinations of \(x_1, x_2, x_3, x_4\) of the desired degree, and that there are the “right number” of moving planes of the degree in question.

The last assumption can be rephrased into a regularity assumption on the ideal of basepoints \(I\). Using commutative algebra on graded rings they deduce a corresponding regularity bound on \(I^2\) which implies that there are also the “right number” of linearly independent moving quadrics.

To extend these conditions to the toric setting would seem to require a notion of “toric regularity” using the homogeneous coordinate ring \(S_X\) in place of the usual graded polynomial ring. Perhaps the definition proposed by Maclagan and Smith [25,26] can be applied here. Instead of delving into the theory of toric commutative algebra and what does and does not extend, we simply present some examples to illustrate how the toric method of moving quadrics can often work in the presence of basepoints.

Example 30 We repeat Example 9 using moving quadrics.

\[
x_1 = 1 + s - t + st - s^2 t - st^2 \\
x_2 = 1 + s - t - st + s^2 t - st^2 \\
x_3 = 1 - s + t - st - s^2 t + st^2 \\
x_4 = 1 - s - t + st - s^2 t + st^2 
\]
Recall that we have one basepoint at \((1,1)\) with multiplicity 1. If there were no basepoints then we can choose \(B - 2B_1 = 1\) and we would expect one moving plane and two moving quadrics. Applying the algorithm gives two planes and one quadric but still a \(3 \times 3\) square matrix:

\[
\begin{bmatrix}
-X_3 + X_4 & 0 & X_1 - X_2 \\
X_2 - X_3 + 2X_4 & X_2 + X_3 & -X_2 - X_3 + 2X_4 \\
X_2X_1 + X_3X_1 & X_3X_1 - X_1X_4 + X_2^2 + X_2X_4 - 2X_1^2 + X_2^2 + X_2X_4 - X_3X_4 + X_4^2
\end{bmatrix}
\]

The method of moving quadrics works perfectly here and gives the implicit equation of degree 4.

Example 10 which had an LCI basepoint of multiplicity 4 also works with the method of moving quadrics. In this case we get 5 moving planes and no moving quadrics. The implicit equation is recovered as the determinant of the corresponding \(5 \times 5\) matrix of linear forms.

Example 11 has a basepoint which is not an LCI. In this case, the moving quadric matrix was not square. Indeed there were four moving planes and two moving quadrics on a space of five monomials.

However, taking the maximal minor consisting of the four planes and either one of the two quadrics gives the implicit equation with a linear extraneous factor. Unlike, the Chow form matrix of Example 11, this extraneous factor is not intrinsic to the construction. The two different maximal minors give different extraneous factors, hence the implicit equation is the gcd of the maximal minors.

6 Conclusion

In this paper we extend two of the most important implicitization techniques, resultants and syzygies, to general toric surfaces. There are a couple of interesting open questions remaining.

For the resultant method, when the basepoints are not an LCI every maximal minor of the resultant matrix will have an extraneous factor. Is there a way to compute this extraneous factor apriori?

For the syzygy method, the biggest open question is how to extend the method when basepoints are present. Our examples show that the method may often
still work. The second open problem is an understanding of exactly when the moving plane matrix has maximal rank.

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