We present a systematic construction of quantum circuits implementing Grover’s database search algorithm for arbitrary number of targets. We introduce a new operator which flips the sign of the targets and evaluate its circuit complexity. We find the condition under which the circuit complexity of the database search algorithm based on this operator is less than that of the conventional one.

Keywords: Quantum search, quantum circuits

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1. Introduction

Efficiency of quantum algorithm, e.g. speed-up, over its classical counterpart can be found typically in two algorithms: factorization of large numbers and database search. Shor’s algorithm\(^1\) solves the former problem with exponential efficiency compared to its classical counterpart, while Grover’s algorithm (GA)\(^2\)\(^-\)\(^7\) solves the latter with quadratic efficiency. GA has been a fascinating one to be implemented, since a lot of informational problems are translated into the database search problem, in spite of its moderate speed-up.

There are two different classes in the database search problem. One is a purely information theoretic problem such as satisfiability problem, where a Boolean function \(f\) is given and bit strings satisfying \(f\) should be identified. The other is a rather practical unstructured database search problem, e.g. to find the owner of a telephone in a telephone directory, given a phone number, as stated in Ref. 2. It should be stressed that, in contrast to the former, the player is allowed to know what should be searched (the owner of a telephone) in advance in the latter case and searches the target under the reference of this information. We focus on the latter situation in this paper, motivated by the fact that many experimental demonstrations and proposals of implementation of GA\(^7\)\(^-\)\(^16\) implicitly assume the latter situation. Also, these implementations so far focus only on GA with a single target and there seems no systematic construction of a quantum circuit implementing GA with multiple targets. The detailed analysis of the circuit complexity has not been presented for this case.

In this paper, by a simple way, we provide a systematic construction of quantum circuits which implements GA with multiple targets. First, we propose a unitary operator which replaces the conventional oracle operator of GA (Theorem 1). We show that the construction of a quantum circuit of our oracle operator amounts to implementing a unitary operator which yields an equal weight superposition state of the target states from \(|0\rangle^\otimes n\). Utilizing the dichotomy on each bit in the target sequences, we obtain a simple recursion relation with which the quantum circuit of the unitary operator is designed (Lemma 2 and Theorem 2). Also, we propose another database search method (Theorem 3), in which circuit complexity is reduced considerably (Proposition 2). Hence, the present systematic and explicit ‘algorithm’ to build the oracle in the quantum search enables us to estimate the circuit complexity.

This paper is organized as follows. In Sec. 2, we show the equivalence between the proposed oracle operator and the conventional one. Quantum circuit design of the former one and its analysis from the viewpoint of the circuit complexity are provided in Sec. 3 and possibility of further reduction of the gate complexity is discussed in Sec. 4. Simple examples are given in Sec. 5. Section 6 is devoted to conclusion and discussions.
2. Equivalent Quantum Algorithms

Consider a database which consists of $2^n$ elements each of which is labeled by

$$x := \sum_{i=0}^{n-1} x_i 2^i,$$

where $x_i \in \{0, 1\}$ for all $i$. The set of all labels is denoted by $N := \{0, 1, \ldots, 2^n - 1\}$. A nonempty subset $S$ of $N$ is made of the set of the targets of the database search problem. We further introduce the complementary subset $\bar{S} := N \setminus S$. Utilizing the binary representation (1) of $x$, we can represent each element $x \in N$ as an $n$-qubit normalized quantum state in $H := \bigotimes_{i=1}^{n} H_i$ with $H_i = \mathbb{C}^2$ for all $i$,

$$|x\rangle := |x_{n-1}\rangle_1 \otimes |x_{n-2}\rangle_2 \otimes \cdots \otimes |x_0\rangle_n, \quad |x_{n-1}\rangle_i \in H_i,$$

where

$$n-i \langle x_i|y_i\rangle_{n-i} = \delta_{x_i,y_i}, \quad x_i, y_i \in \{0, 1\}$$

for all $i$ and $|x_i\rangle_{n-i} \in H_{n-i}$ is a normalized eigenvector of the Pauli matrix $\sigma_z$ with the eigenvalue $(-1)^{x_i}$. Note that the set $\{|x\rangle \mid x \in N\}$ constitutes a complete orthonormal basis in $H$. Let

$$|\psi\rangle := |N\rangle^{-1/2} \sum_{x \in N} |x\rangle,$$

be the uniform superposition of all the basis vectors of $H$, where $|A|$ denotes the cardinality of a set $A$. The state $|\psi\rangle$ is the initial state in GA.

Let us define two unitary operators essential for GA, the inversion operator $D$ with respect to the mean and the conventional oracle operator $O_{\text{conv}}(S)$, respectively, as

$$D := -1^\otimes n + 2|\psi\rangle\langle\psi|,$$

and

$$O_{\text{conv}}(S) := 1^\otimes n - 2 \sum_{x \in S} |x\rangle\langle x|,$$
where $1$ is the identity operator on $C^2$. Note that the oracle operator $O_{\text{conv}}(S)$ distinguishes the targets in the set $N$ as
\[
O_{\text{conv}}(S)|x\rangle = \begin{cases} 
-|x\rangle & x \in S, \\
|x\rangle & x \in \overline{S}.
\end{cases}
\]

Then, measurements of the state
\[
|\psi\rangle := [DO_{\text{conv}}(S)]^k|\psi\rangle
\]
with $k \approx O(\sqrt{|N|/|\overline{S}|})$ yield elements of $S$ with high probability$^{3-7}$. Figure 1 shows a quantum circuit which implements $D$. Although the oracle operator $O_{\text{conv}}(S)$ has been utilized as a de facto standard, we introduce another oracle operator. Let $|S\rangle := |S\rangle - 1/2 \sum_{x \in \overline{S}} c_x |x\rangle$ (6) and define
\[
O(S) := 1^\otimes n - 2|S\rangle\langle S|,
\]
which distinguishes the targets from the others and plays the central rôle in the present work. The operator $O(S)$ replaces $O_{\text{conv}}(S)$ as we show next.

Let us first show that the action of $O(S)$ is equivalent to that of $O_{\text{conv}}(S)$ on the specific state $|\psi\rangle$ in Eq. (5). To this end, we prove

**Lemma 1.** The state $|\phi\rangle \in H$ is an element of the kernel of $O(S) - O_{\text{conv}}(S)$ if and only if $|\phi\rangle$ can be written as
\[
|\phi\rangle = c|S\rangle + \sum_{x \in \overline{S}} c_x |x\rangle,
\]
which implies
\[
\text{Ker}[O(S) - O_{\text{conv}}(S)] = C^{|\overline{S}|+1}.
\]

**Proof.** Let
\[
|\phi\rangle = \sum_{x \in N} c_x |x\rangle
\]
be any vector in $H$. Then, we find
\[
O(S)|\phi\rangle = |\phi\rangle - 2|S\rangle^{-1/2} \sum_{x \in S} c_x |S\rangle
\]
and
\[
O_{\text{conv}}(S)|\phi\rangle = |\phi\rangle - 2 \sum_{x \in S} c_x |x\rangle.
\]
Let $|\phi\rangle$ be an element of the kernel of $O(S) - O_{\text{conv}}(S)$, that is,
\[
\langle y|[O(S) - O_{\text{conv}}(S)]|\phi\rangle = 0
\]
for all $y \in \mathbb{N}$. Then we obtain for any $z \in S$

$$\sum_{x \in S} c_x = |S| c_z,$$

from which we find

$$|\phi\rangle \in \text{Ker}[O(S) - O_{\text{conv}}(S)] \Rightarrow |\phi\rangle = c|S\rangle + \sum_{x \in S} c_x |x\rangle,$$

where $c$ is the common coefficient of $|x\rangle, x \in S$, in $|\phi\rangle$. The converse is trivial. 

At this stage, we note that, introducing real parameters $\varphi_k$ and the state $|\bar{S}\rangle$ constructed by the same fashion as in Eq. (6), the state $|\psi^{(k)}\rangle$ can be rewritten as

$$|\psi^{(k)}\rangle = \sin \varphi_k |S\rangle + \cos \varphi_k |\bar{S}\rangle$$

as shown in Ref. 4. Thus, we immediately find

**Corollary 1.** The actions of $O(S)$ and $O_{\text{conv}}(S)$ on $|\psi^{(k)}\rangle$ result in the same state:

$$O(S)|\psi^{(k)}\rangle = O_{\text{conv}}(S)|\psi^{(k)}\rangle$$

for any positive integer $k$.

Since this Corollary is valid for all $k \in \mathbb{N}$, we may replace all actions of $O_{\text{conv}}(S)$ by $O(S)$ during the iteration processes in GA. Hence, we conclude that

**Theorem 1.** Grover’s algorithm can be realized by the iterations of $DO(S)$, that is,

$$|\psi^{(k)}\rangle = [DO(S)]^k |\psi\rangle.$$

This theorem suggests non-uniqueness of the way to realize quantum information processing.

### 3. Building Quantum Circuits by Dichotomy

We turn into an explicit and systematic construction of the oracle $O(S)$ via elementary quantum gates. The key observation is that $O(S)$ is unitarily equivalent to a conventional oracle operator in GA with a unique target $|0\rangle^\otimes n \in \mathcal{H}$. Let

$$P := 1^\otimes n - 2(|0\rangle \langle 0|)^\otimes n$$

be this conventional oracle operator and define $U(S)$ by

$$|S\rangle = U(S)|0\rangle^\otimes n.$$

Then we find that $O(S)$ and $P$ are related by $U(S)$ as

$$O(S) = U(S)P U(S)^\dagger,$$

which can be interpreted as a decomposition of $O(S)$ in terms of $U(S)$ and $P$. 
We address two features of the decomposition (9). First, the unitary operator $U(S)$ in Eq. (8) is not uniquely determined since it does not specify how basis vectors other than $|0\rangle^\otimes n$ are mapped under $U(S)$. Thus, we define a set $S(S)$ whose elements are $U(S)$:

$$S(S) := \{ U(S) \mid |S\rangle = U(S)|0\rangle^\otimes n \} \simeq U(2^n - 1),$$

which has $(2^n-1)^2$ free parameters. Second, since the construction of $D$ and $P$ from the universal gate set (local unitary operators and CNOT gates) has been clarified in Refs. 7 and 10 (See Fig. 2), the construction of quantum circuits for GA boils down to that of $U(S)$. Note that from Fig. 2 and Ref. 17, the circuit complexity for $P$ is at most $O(n^2)$. Utilizing the operator $P$, we also find a circuit diagram of the conventional oracle $O_{\text{conv}}(S)$ with the universal gate set as a biproduct, which is given in Appendix.

To proceed, let us classify elements of $S$ by dichotomy on the value of each bit. Prior to this, let us define, for a given $x \in N$,

$$x^{(m)} := \sum_{i=1}^{m} x_{n-i}2^{m-i}$$

for an integer $m = 1, 2, \ldots, n$, which can be embedded in $\bigotimes_{i=1}^{m} \mathcal{H}_i$ as a normalized vector

$$|x^{(m)}\rangle := |x_{n-1}\rangle_1 \otimes |x_{n-2}\rangle_2 \otimes \cdots \otimes |x_{n-m}\rangle_m.$$ 

This vector $|x^{(m)}\rangle$ is constructed by picking up the first $m$ bits of the binary representation of $x$ and utilizing the correspondence similar to Eq. (2). Note that $x^{(n)} = x$. We introduce two types of subsets of $S$ with a help of $x^{(m)}$. One is

$$S_m(\alpha_m) := \{ x \mid x^{(m)} = \alpha_m, x \in S \}$$

for $m = 1, 2, \ldots, n$ and $\alpha_m = 0, 1, \ldots, 2^m - 1$. The other results from the dichotomy:

$$S_{m+1}(i, \alpha_m) := \{ x \mid x_{n-m-1} = i, x \in S_m(\alpha_m) \},$$

for $i = 0, 1$ and $m = 1, 2, \ldots, n - 1$.  

![Quantum circuit for $P$.](image-url)
Next, we focus on relations between the cardinalities of the above two sets $S_m(\alpha_m)$ and $S_{m+1}(i, \alpha_m)$. The following three identities are easy to verify:

\begin{align}
|S_m(0)| + |S_m(1)| + \cdots + |S_m(2^m - 1)| &= |S|, \quad (11a) \\
|S_{m+1}(0, \alpha_m)| + |S_{m+1}(1, \alpha_m)| &= |S_m(\alpha_m)|, \quad (11b) \\
|S_{m+1}(i, \alpha_m)| &= |S_{m+1}(2\alpha_m + i)|. \quad (11c)
\end{align}

We can formally introduce a “probability” distribution \(
P_m(\alpha_m)\) for \(\alpha_m = 0, 1, \ldots, 2^m - 1\), from Eq. (11a), and a conditional probability distribution \(p_{m+1}(i|\alpha_m)\) for \(i = 0, 1\), from Eq. (11b). Note that Eq. (11c) implies

\[
p_{m+1}(i|\alpha_m)p_m(\alpha_m) = p_{m+1}(2\alpha_m + i). \quad (12)
\]

Let \(A_m\) be a set satisfying

\[
A_m := \{\alpha_m \mid S_m(\alpha_m) \neq \emptyset\}.
\]

Then, we have

\[
p_m(\alpha_m) = 0 \quad \text{for} \quad \alpha_m \notin A_m. \quad (13)
\]

We show a recursion relation to construct \(U(S)\). First, on the basis of the above probability distributions, we define two kinds of unitary operators on \(\mathbb{C}^2\) by

\[
V_1 := \sqrt{p_1(0)}1 - i\sqrt{p_1(1)}\sigma_y
\]

and

\[
V_{m+1}(\alpha_m) = \sqrt{p_{m+1}(0|\alpha_m)}1 - i\sqrt{p_{m+1}(1|\alpha_m)}\sigma_y,
\]

for \(\alpha_m \in A_m\), whereas we define \(V_{m+1}(\alpha_m) = 1\) for \(\alpha_m \notin A_m\). These one qubit unitary operators are followed by unitary operators

\[
U_1 := V_1 \otimes 1^{\otimes (n-1)}
\]

and

\[
U_{m+1}(\alpha_m) := |\alpha_m\rangle \langle \alpha_m| \otimes V_{m+1}(\alpha_m) \otimes 1^{\otimes (n-m-1)}
\]

\[
+ (1^{\otimes m} - |\alpha_m\rangle \langle \alpha_m|) \otimes 1^{\otimes (n-m)},
\]

respectively. Here, \(|\alpha_m\rangle\) is a normalized vector in \(\bigotimes_{i=1}^m \mathcal{H}_i\) constructed from \(\alpha_m\) by the same way as \(|x^{(m)}\rangle\). Note that the elements of the set \(\{U_{m+1}(\alpha_m)\}_{\alpha_m}\) are commutative with each other:

\[
[U_{m+1}(\alpha_m), U_{m+1}(\beta_m)] = 0.
\]
Following this, let us define
\[ U_{m+1} := \prod_{\alpha_m=0}^{2^m-1} U_m(\alpha_m) = \sum_{\alpha_m=0}^{2^m-1} |\alpha_m\rangle\langle\alpha_m| \otimes V_{m+1}(\alpha_m) \otimes 1^{\otimes(n-m-1)} \] (17)
for \( m = 1, 2, \ldots, n-1 \). Operators \( U_{m+1} \) are nothing but uniformly controlled gates introduced in Refs. 18 and 19. Then, we have

**Lemma 2.**
\[ U_{m+1} U_m \cdots U_1 |0\rangle^{\otimes n} = \sum_{\alpha_{m+1}=0}^{2^{m+1}-1} \sqrt{p_{m+1}(\alpha_{m+1})} |\alpha_{m+1}\rangle \otimes |0\rangle^{\otimes(n-m-1)} \] (18)
holds for \( m = 1, 2, \ldots, n-1 \).

**Proof.** We prove this claim by induction. For \( m = 1 \), we can easily check the validity of Eq. (18). Next, suppose that Eq. (18) holds for \( m = k-1 \). Then using Eqs. (17), (13), (15) and (12), we obtain
\[
U_{k+1} U_k \cdots U_1 |0\rangle^{\otimes n} = U_{k+1} \sum_{\alpha_k=0}^{2^k-1} \sqrt{p_k(\alpha_k)} |\alpha_k\rangle \otimes |0\rangle^{\otimes(n-k)}
\]
\[ = \sum_{\alpha_k \in A_k} \sum_{i=0,1} \sqrt{p_{k+1}(i\alpha_k) p_k(\alpha_k)} |\alpha_k\rangle \otimes |i\rangle \otimes |0\rangle^{\otimes(n-k-1)}
\]
\[ = \sum_{\alpha_k \in A_k} \sum_{i=0,1} \sqrt{p_{k+1}(2\alpha_k + i) p_k(\alpha_k)} |\alpha_k\rangle \otimes |i\rangle \otimes |0\rangle^{\otimes(n-k-1)}.
\]
We find Eq. (18) with \( m = k \) by introducing \( \alpha_{k+1} = 2\alpha_k + i \) and \( |\alpha_{k+1}\rangle = |\alpha_k\rangle \otimes |i\rangle \), since the probability distribution \( p_{k+1}(2\alpha_k + i) \) is normalized by Eq. (12). This completes the proof. \( \square \)

Let us construct
\[ U := U_n U_{n-1} \cdots U_1, \]
whose action on \( |0\rangle^{\otimes n} \) is found from Lemma 2 as
\[ U|0\rangle^{\otimes n} = \sum_{x \in \mathbb{N}} \sqrt{p_n(x)} |x\rangle. \]
Since
\[ p_n(x) = \begin{cases} 
0 & \text{for } x \in \bar{S}, \\
|S|^{-1} & \text{for } x \in S,
\end{cases} \]
Quantum Oracles in Terms of Universal Gate Set

is valid, we observe

$$U|0\rangle^{\otimes n} = |S|^{-1/2} \sum_{x \in S} |x\rangle = |S\rangle,$$

which manifestly proves

**Theorem 2.** The operator $U = U_n U_{n-1} \cdots U_1$ belongs to $\mathcal{S}(S)$, that is,

$$U \in \mathcal{S}(S).$$

Thus, hereafter we may put

$$U = U(S),$$

whose quantum circuit is given in Fig. 3.

To close this section, let us estimate circuit complexity of $U(S)$, i.e., the number of elements in the universal gate set necessary to emulate $U(S)$. This is done by noticing

$$A_n = S, \quad (19)$$

which leads us to

**Proposition 1.** The circuit complexity of $U(S)$ has an upper bound $\mathcal{O}(n^3 |S|/6)$.

**Proof.** Consider an element $\alpha_n \in A_n$. Then, there exists an appropriate doublet $(\alpha_n - 1, i)$ satisfying

$$\alpha_n = 2\alpha_n - 1 + i.$$

Using Eqs. (13) and (14), we find that the existence of such a doublet implies $\alpha_n - 1 \in A_{n-1}$, which means that the operator (15) is well-defined and might be non-trivial. Since this is true for all elements in $A_n$, from Eqs. (10) and (19), we obtain an upper bound $|S|$ for the number of non-trivial $U_n(\alpha_n)$. Also, by induction, we find that $|S|$ also becomes the upper bound of the number of the non-trivial $U_m(\alpha_m)$ for any $m$.

Combining this bound $|S|$ with the fact that one-qubit unitary gates with $m$ control qubits can be emulated by $\mathcal{O}(m^2)$ elements of the universal gate set, we...
have an upper bound $O(m^2|S|)$ on the gate complexity of $U_{m+1}$. Summing up them from $m = 1$ to $m = n - 1$, we find the upper bound $O(n^3|S|/6)$ for the circuit complexity of $U(S)$.

We address that this bound is not so tight, since for the derivation of this bound we have not taken into account the possibilities that $p_{m+1}(1|\alpha_m) = 0$ for $\alpha_m \in A_m$, which implies that $U_{m+1}(\alpha_m)$ becomes the identity operator.

4. Reducing the Circuit Complexity

Following the results in the previous section, let us present another construction of the quantum circuit and estimate its circuit complexity. For this purpose, we define a set $\tilde{S}$ by

$$\tilde{S} = \{0, 1, \ldots, |S| - 1\}$$

which is related to the given $S$ by an appropriate element $\sigma$ of the symmetric group $\mathfrak{S}_{|N|}$, that is,

$$\sigma \tilde{S} = S. \quad (20)$$

Note that $|\tilde{S}| = |S|$. We define $|\tilde{S}\rangle$ and $O(\tilde{S})$ by replacing $S$ with $\tilde{S}$ in Eqs. (6) and (7), respectively. Also note that $\sigma$ satisfying Eq. (20) is not uniquely determined since we do not specify how elements in $\tilde{S}$ are mapped under $\sigma$.

Let $\pi_\sigma$ be an $|N|$-dimensional unitary representation of $\sigma$. Then we observe

**Theorem 3.** Grover’s algorithm to amplify the target state $|S\rangle$ can be realized by the invocations of the oracle $O(\tilde{S})$, that is,

$$|\psi^{(k)}\rangle = \pi_\sigma[D O(\tilde{S})]^k \pi_\sigma^\dagger |\psi\rangle. \quad (21)$$

**Proof.** By the definition of $\tilde{S}$ and $\sigma$, there exists a unitary operator $\pi_\sigma$ satisfying

$$\pi_\sigma U(\tilde{S}) \in S(S),$$

which implies

$$\pi_\sigma |\tilde{S}\rangle = |S\rangle \quad \text{and} \quad \pi_\sigma O(\tilde{S}) \pi_\sigma^\dagger = O(S).$$

Since $|\psi\rangle$ in Eq. (3) is manifestly a one-dimensional symmetric representation of $\mathfrak{S}_{|N|}$, we obtain, by taking Eq. (4) into account,

$$\pi_\tau |\psi\rangle = |\psi\rangle$$

and

$$[\pi_\tau, D] = 0$$

for all $\tau \in \mathfrak{S}_{|N|}$. Then, we observe

$$DO(S) = \pi_\sigma DO(\tilde{S}) \pi_\sigma^\dagger.$$
Using $\pi^\dagger \pi_\sigma = 1^\otimes n$, we find
\[
\pi_\sigma [DO(\tilde{S})]^k \pi^\dagger_\sigma |\psi\rangle = [\pi_\sigma DO(\tilde{S})^k \pi^\dagger_\sigma] |\psi(k)\rangle,
\]
which completes the proof.

4.1. Construction of $O(\tilde{S})$

Since one can learn from the above theorem that amplification of $|S\rangle$ can be performed by the invocations of the oracle $O(\tilde{S})$, let us investigate detailed properties of $O(\tilde{S})$ from the viewpoint of circuit complexity. For this purpose, it is convenient to introduce a number $l \in \mathbb{Z}$ satisfying
\[
l - 1 \leq \log_2 |\tilde{S}| < l,
\]
which is the minimal number of bits required for binary representation of all the elements of $\tilde{S}$. Thus, the $n$-bit representation $[1]$ of each label of the element $x \in \tilde{S}$ should be
\[
x_l = x_{l+1} = \cdots = x_{n-1} = 0.
\]
Then, from Eq. (10), we find
\[
S_m(\alpha_m) = \begin{cases} 
\tilde{S} & \text{for } \alpha_m = 0, \\
\emptyset & \text{otherwise}
\end{cases}
\]
for $m = 1, 2, \ldots, n - l + 1$. Thus, if we employ the construction of $U(\tilde{S})$ introduced in the previous section, we observe from Eqs. (14) and (15) that
\[
U_1 = U_2 = \cdots = U_{n-l} = 1^\otimes n
\]
for $\tilde{S}$. This implies that the circuit complexity of $U(\tilde{S})$ is considerably less than that of $U(S)$ for generic $S$.

Remarkably, there exists further reduction of the circuit complexity as we will show now. First, we fix notation. Associated with the subscript $m$ in Eq. (10), we employ a shifted index $m' := m - (n - l)$. A normalized vector $|\alpha_{m'}\rangle \in \bigotimes_{i=m-l+1}^n \mathcal{H}_i$ is defined by a similar manner to $|\alpha_m\rangle$, where $\alpha_{m'} = 0, 1, \ldots, 2^{m'} - 1$.

Next, for a given $|\alpha_m\rangle$ with $m \geq n - l$, we seek $|\alpha_{m'}\rangle$ satisfying
\[
|\alpha_m\rangle = |0\rangle_1 \otimes |0\rangle_2 \otimes \cdots \otimes |0\rangle_{n-l} \otimes |\alpha_{m'}\rangle.
\]
Depending on a type of the solutions of Eq. (23), we prepare a controlled unitary gate, $U'_{m+1}(\alpha_m)$: (i) If there exists $|\alpha_{m'}\rangle$ satisfying Eq. (23) for $m' > 0$, then
\[
U'_{m+1}(\alpha_m) := 1^\otimes (n-l) \otimes |\alpha_{m'}\rangle \langle \alpha_{m'}| \otimes V_{m+1}(\alpha_m) \otimes 1^\otimes (n-m-1) + 1^\otimes (n-l) \otimes (1^\otimes m' - |\alpha_{m'}\rangle \langle \alpha_{m'}|) \otimes 1^\otimes (n-m).
\]
(ii) If there exists $|\alpha_{m'}\rangle$ satisfying Eq. (23) for $m' = 0$, then
\[
U'_{m+1}(\alpha_m) := 1^\otimes (n-l) \otimes V_{m+1}(\alpha_m) \otimes 1^\otimes (l-1).
\]
If there is no $|\alpha_{m'}\rangle$ satisfying Eq. (23), then
\[
U'_{m+1}(\alpha_m) := 1 \otimes^n.
\] (24c)

Under these preparations, we find

**Lemma 3.** The controlled gates $U_{m+1}(\alpha_m)$ for $\tilde{S}$ can be replaced with $U'_{m+1}(\alpha_m)$ for $m > n - l$.

**Proof.** Let us focus on the controlled gates (16) with $m \geq n - l$ and consider the actions of $U(\tilde{S})$ on $|0\rangle \otimes^n$. Since we have Eq. (22), we find that the first $(n - l)$ qubits only play the role of the control qubits for Eq. (16). Besides, since the initial state $|0\rangle \otimes^n$ is separable, these first $(n - l)$ qubits remain unchanged under the actions of Eq. (16) with $m \geq n - l$. This observation implies that the controlled gate (16) for $\alpha_m$ becomes trivial ($1 \otimes^n$) unless there exists an appropriate state $|\alpha_{m'}\rangle \in \bigotimes_{i=n-l+1}^m H_i$, by which the vector of the control qubits $|\alpha_m\rangle$ can be written as Eq. (23).

If $|\alpha_m\rangle$ is written as Eq. (23) and hence the first $n - l$ qubits are in the state $|0\rangle \otimes^{(n-l)}$, these $n - l$ qubits play no role as control qubits. Thus, we can replace $U_{m+1}(\alpha_m)$ by $U'_{m+1}(\alpha_m)$ as is claimed.

Parallel to the previous section, we define
\[
U'_{m+1} := \prod_{\alpha_m=0}^{2^m-1} U'_{m+1}(\alpha_m)
\]
for $m > n - l$ and
\[
U' := U''_n U'_{n-1} \cdots U'_{n-l+1}.
\]
Then, from the above lemma, we immediately see $U' \in S(\tilde{S})$. Thus, hereafter we reset
\[
U(\tilde{S}) = U'.
\]

We estimate the circuit complexity for this $U(\tilde{S})$.

**Proposition 2.** The circuit complexity of $U(\tilde{S})$ has an upper bound $O(l^2 2^l)$.

**Proof.** Obviously there remain $m'$ qubits for each possibly non-trivial $U'_{m+1}(\alpha_m)$ as free control qubits. Since we can place either $|0\rangle$ or $|1\rangle$ in each qubit in $|\alpha_{m'}\rangle$, the number of non-trivial $U'_{m+1}(\alpha_m)$ is at most $2^{m'}$ for each $m$. Also, the number of those control qubits in each $U'_{m+1}(\alpha_m)$ is manifestly $m'$. Thus, utilizing the results of Ref. 17, we estimate the circuit complexity of $U(\tilde{S})$ as
\[
1 + \sum_{m'=1}^{l-1} m'^2 2^m \approx l^2 2^l.
\]
4.2. Construction of $\pi_\sigma$

Now, we turn to the implementation of the unitary operator $\pi_\sigma$ in terms of the elementary gates. To this end, we first introduce

$$B := \{x \mid x \in S \setminus (S \cap \tilde{S})\} \quad \text{and} \quad C := \{x \mid x \in \tilde{S} \setminus (S \cap \tilde{S})\}.$$  \hfill (25)

It immediately follows from $|S| = |\tilde{S}|$ that

$$|B| = |C| \leq |S|,$$  \hfill (26)

which implies that we can introduce a bijection between $B$ and $C$. Based on this observation, we define a transposition $(x \ y)$ for any pair $x \in B$ and $y \in C$. Since these transpositions commute with each other, $\sigma$ can be realized as a product of transpositions for all distinct pairs.

Utilizing the methodology given in Ref. 17, which is based on the grey code and the Hamming distance, the quantum gate of such transposition is implemented with at most $n$ controlled gates with $n - 1$ control qubits. Thus, the circuit complexity of the transposition is $O(n(n - 1)^2) = O(n^3)$. Combining this observation and the inequality (26) with the fact that we need $|B|$ transpositions to implement $\sigma$, we conclude that

**Proposition 3.** The circuit complexity of $\pi_\sigma$ satisfying Eq. (20) has an upper bound $O(n^3|S|)$.

4.3. Circuit Complexity

So far, we clarified the circuit complexity of each ingredient of Eq. (21), which further yields
Corollary 2. The order of circuit complexity to obtain $|\psi^{(k)}\rangle$ by the algorithm (27) is

$$
\mathcal{O}(2n^3|S| + 2kn^2 + 2kl^2 l^4).
$$

Proof. The first term in the argument of Eq. (27) comes from Proposition 3. The second term results from the fact that the circuit complexity of $D$ and $P$ is $\mathcal{O}(n^2)$ and the total number of $D$ and $P$ in the r.h.s. of Eq. (21) is equal to $2k$. Proposition 2 and the number of $U(\tilde{S})$ in Eq. (21) yield the third term. \hfill \Box

Let us compare the circuit complexity given by Eq. (27) with that for Eq. (5) by setting $k \approx \sqrt{|N|/|S|}$. For this purpose, it is convenient to introduce the ratio

$$
\Gamma := \frac{2n^3|S| + 2(n^2 + l^2 l^4) \sqrt{|N|/|S|}}{n^2(|S| + 1) \sqrt{|N|/|S|}},
$$

where the numerator comes from Eq. (27) and the denominator comes from the circuit complexity $\mathcal{O}(n^2|S|)$ of $O_{\text{conv}}(S)$. Note that the circuit complexity of $O_{\text{conv}}(S)$ cannot be improved even if we employ the method introduce in this section (See Appendix). From the viewpoint of the circuit complexity, we conclude that Eq. (21) dominates when $\Gamma < 1$. Recalling $|N| = 2^n$, $|S| = |\tilde{S}| < 2^l$ and introducing a ratio $\gamma := l/n$, the ratio $\Gamma$ is approximated as

$$
\Gamma \approx 2 \left[n2^{-\frac{5}{2}(1-\gamma)} + (2^{-n\gamma} + \gamma^2)\right].
$$

If $\gamma$ is fixed and we take the limit $n \to \infty$, we obtain $\Gamma \to 2\gamma^2$ from which we find that the algorithm (21) is preferable if $\gamma$ satisfies

$$
\gamma < \frac{1}{\sqrt{2}} \approx 0.71.
$$

Further, Fig. 4 tells us that the above bound $\gamma = 1/\sqrt{2}$ is universal if the number of qubits is sufficiently large. Thus, we may conclude for a sufficiently large database that the proposed algorithm is preferable than the conventional one from the viewpoint of the circuit complexity, provided that the number of the target satisfies

$$
|S| \lesssim |N|^{0.71}.
$$

5. Examples

In this section, we give two simple examples to demonstrate the difference between the implementations of $U(S)$ and $U(\tilde{S})$. For brevity, we utilize the binary representations to describe the elements of sets $S$, $\tilde{S}$ and the sets derived from them. For the later convenience, we also introduce a one-qubit gate $V := \frac{1}{\sqrt{2}}(1 - i\sigma_y)$. 

\[ \text{Unitary Gate: } V = \frac{1}{\sqrt{2}}(1 - i\sigma_y). \]
Note that $V|0\rangle = H|0\rangle$, where $H$ is the Hadamard gate
\begin{equation}
H := \frac{1}{\sqrt{2}} (\sigma_x + \sigma_z).
\end{equation}

### 5.1. Circuit for $U(S)$

As a demonstration of our algorithm to design the relevant quantum circuit, let us consider a database search problem to extract four three-bit strings
\begin{equation}
S = \{000, 001, 010, 100\}.
\end{equation}

The classification of whose elements according to the dichotomy yields
\begin{align*}
S_1(0) &= \{000, 001, 010\}, & S_1(1) &= \{100\},
\end{align*}
from which we find
\begin{equation}
V_1 = \frac{1}{2} \left( \sqrt{3} \times 1 - i\sigma_y \right).
\end{equation}

Further, from the dichotomy on the elements in $S_1(0)$ and $S_1(1)$, we obtain
\begin{align*}
S_2(0, 0) &= \{000, 001\} & S_2(1, 0) &= \{010\}, \\
S_2(0, 1) &= \{100\} & S_2(1, 1) &= \emptyset,
\end{align*}
respectively. These enable us to construct
\begin{equation}
V_2(0) = \frac{1}{\sqrt{3}} \left( \sqrt{2} \times 1 - i\sigma_y \right) \quad \text{and} \quad V_2(1) = 1.
\end{equation}

By the same way, we find
\begin{align*}
S_3(0, 0) &= \{000\}, & S_3(1, 0) &= \{001\}, \\
S_3(0, 1) &= \{010\}, & S_3(1, 1) &= \emptyset, \\
S_3(0, 2) &= \{100\}, & S_3(1, 2) &= \emptyset, \\
S_3(0, 3) &= S_3(1, 3) = \emptyset.
\end{align*}
Based on these sets, we have
\[ V_3(0) = V, \quad V_3(1) = V_3(2) = V_3(3) = 1. \] (32)
These results are summarized as the quantum circuit depicted in Fig. 5.

5.2. Circuit for $U(\tilde{S})$

From the given set $S$ in Eq. (29), we construct the set $\tilde{S}$ as
\[ \tilde{S} = \{000, 001, 010, 011\}. \] (33)
By the same method utilized in the previous subsection, we find
\[ V_1 = 1 \] (34)
as is expected from Eq. (22). Also, further dichotomy reveals
\[ V_2(0) = V_3(0) = V_3(1) = V, \]
\[ V_2(1) = V_3(2) = V_3(3) = 1. \] (36)
Thus, the non-trivial $U_{m+1}(\alpha_m)$ are found from Eq. (16) as
\[ U_2(0) = (|0\rangle\langle 0| \otimes V + |1\rangle\langle 1| \otimes 1) \otimes 1, \]
\[ U_3(0) = |00\rangle\langle 00| \otimes V + (1^{\otimes 2} - |00\rangle\langle 00|) \otimes 1, \]
\[ U_3(1) = |01\rangle\langle 01| \otimes V + (1^{\otimes 2} - |01\rangle\langle 01|) \otimes 1, \]
by the help of the resolution of identity. On the other hand, following Eqs. (24a), (24b) and (24c), we introduce
\[ U'_2(0) = 1 \otimes V \otimes 1, \]
\[ U'_3(0) = 1 \otimes (|0\rangle\langle 0| \otimes V + |1\rangle\langle 1| \otimes 1), \]
\[ U'_3(1) = 1 \otimes (|1\rangle\langle 1| \otimes V + |0\rangle\langle 0| \otimes 1). \]
Then one can verify, by direct calculation, that
\[ U_3(1)U_3(0)U_2(0)|0\rangle^{\otimes 3} = U'_3(1)U'_3(0)U'_2(0)|0\rangle^{\otimes 3}, \]
which is the key for the reduction of the circuit complexity. The quantum circuit obtained by this replacement is shown in Fig. 6. We further observe that
\[ U'_3(1)U'_3(0)U'_2(0)|0\rangle^{\otimes 3} = 1 \otimes V \otimes V|0\rangle^{\otimes 3} = 1 \otimes H \otimes H|0\rangle^{\otimes 3}, \]
which implies that our algorithm yields an intuitive way to produce |\tilde{S}\rangle.

5.3. Circuit for \( \pi_\sigma \)

For \( S \) and \( \tilde{S} \) in this section, we find
\[ B = \{100\} \quad \text{and} \quad C = \{011\} \]
from Eq. (25). Then, the permutation \( \sigma \) associated with \( B \) and \( C \) is uniquely determined, and we find
\[ \sigma = (100\ 011). \]
We design a sequence of bit strings from 100 to 011 so that the Hamming distance between the neighbours is equal to the identity, e.g.,
\[ 100, 101, 111, 011. \]
This sequence shows that the permutation operator \( \pi_\sigma \) is rewritten as
\[ \sigma = (100\ 101)(101\ 111)(111\ 011). \quad (37) \]
Since every transposition in the RHS of Eq. (37) is realized by a controlled-controlled-NOT gate, we can materialize \( \pi_\sigma \) as a quantum circuit of these controlled-controlled-NOT gates (see Fig. 7).
and it is proven that it works as an oracle for GA under a certain condition on the initial state (Theorem 1). Since the modified oracle $O(S)$ is decomposed into operators $P$ and $U(S)$, where $P$ is the conventional oracle operator with a unique target $|0\rangle^\otimes n$, we have shown that the quantum circuit of the oracle built by the elementary gate set can be designed if that of $U(S)$ was given. The construction of the quantum circuit for $U(S)$ was accomplished by employing Lemma 2 and Theorem 2, which were derived through dichotomy on the value of each qubit. We should emphasize that our work takes advantage of the non-uniqueness of the implementation of a quantum information processing.

Also, we found another algorithm to perform the same database search (Theorem 3). One of the advantages of utilizing this is that we can reduce the circuit complexity considerably (Proposition 2). We showed that there existed a simple condition under which our algorithm was more advantageous than the conventional algorithm with the oracle $O_{\text{conv}}(S)$ from the viewpoint of the circuit complexity.

The modified oracle $O(S)$ fails to sort the targets with a good precision when noise and errors are present. This deviation from the targets will be circumvented to some extent by fault tolerant quantum error correction codes (QECC)\textsuperscript{20}, as in the case of the conventional sorting algorithm with $O_{\text{conv}}(S)$. QECC can be built-in in $O(S)$ similarly to $O_{\text{conv}}(S)$, since QECCs are independent of the system size and actual forms of quantum algorithms. Note, however, that the algorithm is probabilistic in nature. We have to verify the targets obtained to make sure that we got correct results.

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**Appendix.**

In this Appendix, we decompose the operator $O_{\text{conv}}(S)$ into the elementary gates. To this end, we note that $\langle x|y \rangle = \delta_{xy}$. Due to this orthogonality, we have

$$O_{\text{conv}}(S) = 1^\otimes n - 2\sum_{x \in S} |x\rangle\langle x| = \prod_{x \in S} (1^\otimes n - 2|x\rangle\langle x|) = \prod_{x \in S} U_x P U_x^\dagger, \quad (38)$$

where $U_x$ is a unitary operator satisfying

$$U_x |0\rangle^\otimes n = |x\rangle. \quad (39)$$

It is obvious that $U_x$ is far from unique. Nonetheless, in view of Eq. (39), there exist local unitary operators satisfying Eq. (38), e.g.,

$$U_x = \bigotimes_{i=0}^{n-1} (1\delta_{x,0} + \sigma_x \delta_{x,1}). \quad (40)$$
We estimate the circuit complexity of $O_{\text{conv}}(S)$ using Eq. (40). Since the number of (non-trivial) one-qubit gates to construct $U_x$ is bounded by $\sum x_i \leq n$ from above, plugging this into the number of gates for $P$, we conclude that we need $O(n^2 |S|)$ elementary gates to implement $O_{\text{conv}}(S)$.

Let us finally comment on the circuit complexity of $O_{\text{conv}}(\tilde{S})$, where $\tilde{S}$ is given by Eq. (20). Since we have $|S| = |\tilde{S}|$ and the upper bound of the circuit complexity of $U_x$ is $n$, the circuit complexity is left unchanged even if we employ the decomposition (38) for $\tilde{S}$. This observation implies that the circuit complexity cannot be improved by the method proposed in Sec. IV as far as the decomposition (38) for $O_{\text{conv}}(S)$ is utilized.

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