The Power of Data Reduction for Matching

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Abstract

Finding maximum-cardinality matchings in undirected graphs is arguably one of the most central graph primitives. For \(m\)-edge and \(n\)-vertex graphs, it is well-known to be solvable in \(O(m\sqrt{n})\) time; however, for several applications this running time is still too slow. We investigate how linear-time (and almost linear-time) data reduction (used as preprocessing) can alleviate the situation. More specifically, we focus on (almost) linear-time kernelization. We start a deeper and systematic study both for general graphs and for bipartite graphs. Our data reduction algorithms easily comply (in form of preprocessing) with every solution strategy (exact, approximate, heuristic), thus making them attractive in various settings.

1 Introduction

“Matching is a powerful piece of algorithmic magic” \cite{18}. In MATCHING, given a graph, one has to compute a maximum-cardinality set of nonoverlapping edges. MATCHING is arguably among the most fundamental graph-algorithmic primitives allowing for a polynomial-time algorithm. More specifically, on an \(n\)-vertex and \(m\)-edge graph a maximum matching can be found in \(O(m\sqrt{n})\) time \cite{17}. Improving this upper time bound, even for bipartite graphs, resisted decades of research. Recently, however, Duan and Pettie \cite{8} presented a linear-time algorithm that computes a \((1-\epsilon)\)-approximate maximum-weight matching, where the running time dependency on \(\epsilon\) is \(\epsilon^{-1}\log(\epsilon^{-1})\). For the unweighted case, the \(O(m\sqrt{n})\) algorithm of Micali and Vazirani \cite{17} implies a linear-time \((1-\epsilon)\)-approximation, where in this case the running time dependency on \(\epsilon\) is \(\epsilon^{-1}\) \cite{8}. We take a different route: First, we do not give up the quest for optimal solutions. Second, we focus on efficient data reduction rules—not solving an instance but significantly shrinking its size before actually solving the problem\(^1\). In the context of decision problems, in parameterized algorithmics this is known as kernelization, a particularly active area of algorithmic research on NP-hard problems.

The spirit behind our approach is thus closer to the identification of efficiently (i.e. linearly) solvable special cases of MATCHING. There is quite some body of work in this direction. For instance, since an augmenting path can be found in linear time \cite{10}, the standard augmenting path-based algorithm runs in \(O(s(n+m))\) time, where \(s\) is the number of edges in the maximum matching. Yuster \cite{20} developed an \(O(rn^2\log n)\)-time algorithm, where \(r\) is the difference between maximum and minimum vertex degree of the input graph. Moreover, there are linear-time algorithms for computing maximum matchings in special graph classes, including convex bipartite \cite{19}, strongly chordal \cite{7}, and chordal bipartite graphs \cite{6}.

All this and the more general spirit of “parameterization for polynomial-time solvable problems” \cite{12} (also referred to as “FPT in P” or “FPTP” for short) forms the starting point of our

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\(^1\)Doing so, however, we focus here on the unweighted case.
Table 1: Our kernelization results.

| Parameter                          | running time | kernel size |
|------------------------------------|--------------|-------------|
| **Results for Matching**           |              |             |
| Feedback edge number               | $O(n + m)$   | $O(k)$      |
| Feedback vertex number             | $O(kn)$      | $2^{O(k)}$  |
| **Results for Bipartite Matching**|              |             |
| Distance to chain graphs           | $O(n + m)$   | $O(k^3)$    |

research. Remarkably, Fomin et al. [9] recently developed an algorithm to compute a maximum matching in graphs of treewidth $k$ in $O(k^4n \log n)$ randomized time.

Following the paradigm of kernelization, that is, provably effective and efficient data reduction, we provide a systematic exploration of the power of polynomial-time data reduction for Matching. Thus, our aim (fitting within FPTP) is to devise problem kernels that are computable in (almost) linear time. A particular motivation for this is that with such very efficient kernelization algorithms it is possible to transform multiplicative ($O(f(k)(n+m))$) into additive ($O(f'(k) + n+m)$) “(almost) linear-time FPTP” algorithms. Furthermore, kernelization algorithms (typically based on data reduction rules) can be used as a preprocessing to heuristics or approximation algorithms with the goal of getting larger matchings.

As kernelization is usually defined for decision problems, we use in the remainder of the paper the decision version of Matching. In a nutshell, a kernelization of a decision problem instance is an algorithm that produces an equivalent instance whose size can solely be upper-bounded by a function in the parameter (preferably a polynomial). The focus on decision problems is justified by the fact that all our results, although formulated for the decision version, in a straightforward way extend to the corresponding optimization version.

(MAXIMUM-CARDINALITY) MATCHING

**Input:** An undirected graph $G = (V, E)$ and a nonnegative integer $s$.

**Question:** Is there a size $s$ subset $M_G \subseteq E$ of nonoverlapping (i.e. disjoint) edges?

Since solving the given instance and returning a trivial yes- or no-instance always produces a constant-size kernel in polynomial time, we are looking for kernelization algorithms that are faster than the algorithms solving the problem. For NP-hard problems, each kernelization algorithm, since running in polynomial time, is (presumably) faster than any solution algorithm. This is, of course, no longer true when applying kernelization to a polynomial-time solvable problem like MATCHING. While the focus of classical kernelization for NP-hard problems is mostly on improving the size of the kernel, we particularly emphasize that for polynomially solvable problems it now becomes crucial to also focus on the running time of the kernelization algorithm. Moreover, the parameterized complexity analysis framework can also be applied to the kernelization algorithm itself. For example, a kernelization algorithm running in $O(k^5n)$ time ($k$ is the “problem specific” parameter) might be preferable to another one running in $O(n^3)$ time. In this paper, we present kernelization algorithms for MATCHING which run in linear time (see Sections 2.1 and 3) or in almost linear time (i.e. in $O(kn)$ time, see Section 2.2).

**Our contributions.** In this paper we present three efficiently computable kernels for MATCHING (see Table 1 for an overview). All our parameterizations can be categorized as “distance to triviality” [13]. They are motivated as follows. First, note that maximum-cardinality matchings be trivially found in linear time on trees (or forests). So we consider the corresponding edge deletion distance (feedback edge number) and vertex deletion distance (feedback vertex number). Notably, there is a trivial linear-time algorithm for computing the feedback edge number and there is a linear-time factor-4 approximation algorithm for the feedback vertex number [1]. We mention in passing that the parameter vertex cover number, which is lower-bounded by the feedback vertex number, has been frequently studied for kernelization [3, 4]. In particular,
Gupta and Peng [14] (implicitly) provided a quadratic-size kernel for MATCHING with respect to the parameter vertex cover number. Coming to bipartite graphs, note that our parameterization by vertex deletion distance to chain graphs is motivated as follows. First, chain graphs form one of the most obvious easy cases for bipartite graphs where MATCHING can be solved in linear time [19]. Second, we show that the vertex deletion distance of any bipartite graph to a chain graph can be 2-approximated in linear time. Moreover, vertex deletion distance to chain graphs lower-bounds the vertex cover number of a bipartite graph.

An overview of our main results is given in Table 1. We study kernelization for MATCHING parameterized by the feedback vertex number, that is, the vertex deletion distance to a forest (see Section 2). As a warm-up we first show that a subset of our data reduction rules for the “feedback vertex set kernel” also yields a linear-time computable linear-size kernel for the typically much larger parameter feedback edge number (see Section 2.1). As for BIPARTITE MATCHING no faster algorithm is known than on general graphs, we kernelize BIPARTITE MATCHING with respect to the vertex deletion distance to chain graphs (see Section 3).

Seen from a high level, our two main results employ the same algorithmic strategy, namely upper-bounding (as a function of the parameter) the number of neighbors in the appropriate vertex deletion set \( X \); that is, in the feedback vertex set or in the deletion set to chain graphs, respectively. To achieve this we develop new “irrelevant edge techniques” tailored to these two kernelization problems. More specifically, whenever a vertex \( v \) of the deletion set \( X \) has large degree, we efficiently detect edges incident to \( v \) whose removal does not change the size of the maximum matching. Then the remaining graph can be further shrunk by scenario-specific data reduction rules. While this approach of removing irrelevant edges is natural, the technical details and the proofs of correctness can become quite technical and combinatorially challenging. In particular, for the case of feedback vertex number \( k \) we could only upper-bound the number of neighbors of each vertex in \( X \) by \( 2^{O(k)} \).

As a technical side remark, we emphasize that in order to achieve an (almost) linear-time kernelization algorithm, we often need to use suitable data structures and to carefully design the appropriate data reduction rules to be exhaustively applicable in linear time, making this form of “algorithm engineering” much more relevant than in the classical setting of mere polynomial-time data reduction rules.

**Notation and Observations.** We use standard notation from graph theory. In particular all paths we consider are simple paths. Two paths in a graph are called *internally vertex-disjoint* if they are either completely vertex-disjoint or they overlap only in their endpoints. A matching in a graph is a set of pairwise disjoint edges. Let \( G = (V, E) \) be a graph and let \( M \subseteq E \) be a matching in \( G \). The degree of a vertex is denoted by \( \deg(v) \). A vertex \( v \in V \) is called *matched* with respect to \( M \) if there is an edge in \( M \) containing \( v \), otherwise \( v \) is called *free* with respect to \( M \). If the matching \( M \) is clear from the context, then we omit “with respect to \( M \)”. An alternating path with respect to \( M \) is a path in \( G \) such that every second edge of the path is in \( M \). An augmenting path is an alternating path whose endpoints are free. It is well known that a matching \( M \) is maximum if and only if there is no augmenting path for it. Let \( M \subseteq E \) and \( M' \subseteq E \) be two matchings in \( G \). We denote by \( G(M, M') := (V, M \triangle M') \) the graph containing only the edges in the symmetric difference of \( M \) and \( M' \), that is, \( M \triangle M' := M \cup M' \setminus (M \cap M') \). Observe that every vertex in \( G(M, M') \) has degree at most two.

For a matching \( M \subseteq E \) for \( G \) we denote by \( M_G^{\text{max}}(M) \) a maximum matching in \( G \) with the largest possible overlap (in number of edges) with \( M \). That is, \( M_G^{\text{max}}(M) \) is a maximum matching in \( G \) such that for each maximum matching \( M' \) for \( G \) it holds that \( |M \triangle M'| \geq |M \triangle M_G^{\text{max}}(M)| \). Observe that, if \( M \) is a maximum matching for \( G \), then \( M_G^{\text{max}}(M) = M \). Furthermore observe that \( G(M, M_G^{\text{max}}(M)) \) consists of only odd-length paths and isolated vertices, and each of these paths is an augmenting path for \( M \). Moreover the paths in \( G(M, M_G^{\text{max}}(M)) \) are as short as possible:

**Observation 1.1.** For any path \( v_1, v_2, \ldots, v_p \) in \( G(M, M_G^{\text{max}}(M)) \) it holds that \( \{v_{2i-1}, v_{2i}\} \not\in E \) for every \( 1 \leq i < j \leq p/2 \).
Proof. Assume that \( \{v_{2i-1}, v_{2j}\} \in E \). Then \( v_1, v_2, \ldots, v_{2i-2}, v_{2i-1}, v_{2j}, v_{2j+1}, \ldots, v_p \) is a shorter path which is also an augmenting path for \( M \) in \( G \). The corresponding maximum matching \( M' \) satisfies \(|M \setminus M_{G^{\text{FES}}}[M]| > |M \setminus M'|\), a contradiction to the definition of \( M_{G^{\text{FES}}}[M] \).

Observation 1.2. Let \( G = (V, E) \) be a graph with a maximum matching \( M_G \), let \( X \subseteq V \) be a vertex subset of size \( k \), and let \( M_{G - X} \) be a maximum matching for \( G - X \). Then, \(|M_{G - X}| \leq |M_G| - |M_G - X| + k\).

Kernelization. A parameterized problem is a set of instances \((I, k)\) where \( I \in \Sigma^* \) for a finite alphabet \( \Sigma \), and \( k \in \mathbb{N} \) is the parameter. We say that two instances \((I, k)\) and \((I', k')\) of parameterized problems \( P \) and \( P' \) are equivalent if \((I, k)\) is a yes-instance for \( P \) if and only if \((I', k')\) is a yes-instance for \( P' \). A kernelization is an algorithm that, given an instance \((I, k)\) of a parameterized problem \( P \), computes in polynomial time an equivalent instance \((I', k')\) of \( P \) (the kernel) such that \(|I'| + k' \leq f(k)\) for some computable function \( f \). We say that \( f \) measures the size of the kernel, and if \( f(k) \in O(1) \), we say that \( P \) admits a polynomial kernel. Often, a kernel is achieved by applying polynomial-time executable data reduction rules. We call a data reduction rule \( R \) correct if the new instance \((I', k')\) that results from applying \( R \) to \((I, k)\) is equivalent to \((I, k)\). An instance is called reduced with respect to some data reduction rule if further application of this rule has no effect on the instance.

2 Kernelization for Matching on General Graphs

In this section, we investigate the possibility of efficient and effective preprocessing for MATCHING. As a warm-up, we first present in Section 2.1 a simple, linear-size kernel for MATCHING with respect to the parameter “feedback edge set”. Exploiting the data reduction rules and ideas used for this kernel, we then present in Section 2.2 the main result of this section: an exponential-size kernel for the smaller parameter “feedback vertex number”.

2.1 Warm-up: Parameter feedback edge number

We provide a linear-time computable linear-size kernel for MATCHING parameterized by the feedback edge number, that is, the size of a minimum feedback edge set. Observe that a minimum feedback edge set can be computed in linear time via a simple depth-first search or breadth-first search. The kernel is based on the next two simple data reduction rules due to Karp and Sipser [16]. They deal with vertices of degree at most two.

Reduction Rule 2.1. Let \( v \in V \). If \( \deg(v) = 0 \), then delete \( v \). If \( \deg(v) = 1 \), then delete \( v \) and its neighbor and decrease the solution size \( s \) by one (\( v \) is matched with its neighbor).

Reduction Rule 2.2. Let \( v \) be a vertex of degree two and let \( u, w \) be its neighbors. Then remove \( v \), merge \( u \) and \( w \), and decrease the solution size \( s \) by one.

The correctness was stated by Karp and Sipser [16]. For completeness, we give a proof.

Lemma 2.1. Reduction Rules 2.1 and 2.2 are correct.

Proof. If \( v \) has degree zero, then clearly \( v \) cannot be in any matching and we can remove \( v \).

If \( v \) has degree one, then let \( u \) be its single neighbor. Let \( M \) be a maximum matching of size at least \( s \) for \( G \). Then \( v \) is matched in \( M \) since otherwise adding the edge \( \{u, v\} \) would increase the size of the matching. Thus, a maximum matching in \( G' = G - u - v \) is of size at least \( s - 1 \). Conversely, a maximum matching of size \( s - 1 \) in \( G' \) can easily be extended by the edge \( \{u, v\} \) to a maximum matching of size \( s \) in \( G \).

If \( v \) has degree two, then let \( u \) and \( w \) be its two neighbors. Let \( M \) a maximum matching of size at least \( s \). If \( v \) is not matched in \( M \), then \( u \) and \( w \) are matched since otherwise adding the edge \( \{u, v\} \) resp. \( \{v, w\} \) would increase the size of the matching. Thus, deleting \( v \) and merging \( u \)
and $w$ decreases the size of $M$ by one ($M$ looses either the edge incident to $v$ or one of the edges incident to $u$ and $w$). Hence, the resulting graph $G'$ has a maximum matching of size at least $s - 1$. Conversely, let $M'$ be a matching of size at least $s - 1$ for $G'$. If the merged vertex $vw$ is free, then $M := M' \cup \{u, v\}$ is a matching of size $s$ in $G$. Otherwise, $vw$ is matched to some vertex $y$. Then matching $y$ in $G$ with either $v$ or $w$ (at least one of the two vertices is a neighbor of $y$) and matching $u$ with the other vertex yields a matching of size $s$ for $G$.

Although Reduction Rules 2.1 and 2.2 are correct, it is not clear whether Reduction Rule 2.2 can be exhaustively applied in linear time. However, for our purpose it suffices to consider the following restricted version which we can exhaustively apply in linear time.

**Reduction Rule 2.3.** Let $v$ be a vertex of degree two and $u, w$ be its neighbors with $u$ and $w$ having degree at most two. Then remove $v$, merge $u$ and $w$, and decrease $s$ by one.

**Lemma 2.2.** Reduction Rules 2.1 and 2.3 can be exhaustively applied in $O(n + m)$ time.

**Proof.** We give an algorithm which exhaustively applies Reduction Rules 2.1 and 2.3 in linear time. First, using bucket sort, sort the vertices by degree and keep three lists containing all degree-zero/one/two vertices. Then one applies Reduction Rules 2.1 and 2.3 in a straightforward way. When a neighbor of a vertex is deleted, then check if the vertex has now degree zero, one, or two. If yes, then add the vertex to the corresponding list.

We next show that this algorithm runs in linear time. First, observe that the deletion of each degree-zero vertex can be done in constant time as no further vertices are affected. Second, consider a degree-one vertex $v$ with a neighbor $u$ and observe that deleting $u$ and $v$ can be done $O(\deg(v))$ time since one needs to update the degrees of all neighbors of $v$. Furthermore, decreasing $s$ by one can be done in constant time for each deleted degree-one vertex. Finally, consider a degree-two vertex $v$ with two neighbors $u$ and $w$, each of degree at most two. Deleting $v$ takes constant time. To merge $u$ and $w$ iterate over all neighbors of $u$ and add them to the neighborhood of $w$. If a neighbor $u'$ of $u$ is already a neighbor of $w$, then decrease the degree of $u'$ by one. Then, relabel $w$ to be the new contracted vertex $uw$.

Overall, the worst-case running time to apply Reduction Rules 2.1 and 2.3 exhaustively can be upper-bounded by $O(n + \sum_{v \in V} \deg(v)) = O(n + m)$. 

**Theorem 2.3.** Matching admits a linear-time computable linear-size kernel with respect to the parameter “feedback edge number” $k$.

**Proof.** Apply Reduction Rules 2.1 and 2.3 exhaustively in linear time (see Lemma 2.2). We claim that the reduced graph $G = (V, E)$ has less than $12k$ vertices and $13k$ edges. Denote with $X \subseteq E$ a feedback edge set for $G$, $|X| \leq k$. Furthermore, denote with $V_{G-X}^1, V_{G-X}^2$, and $V_{G-X}^3$ the vertices that have degree one, two, and more than two in the $G - X$. Thus, $|V_{G-X}^1| \leq 2k$ as each leaf in $G - X$ has to be incident to an edge in $X$. Next, since $G - X$ is a forest (or tree), we have $|V_{G-X}^3| < |V_{G-X}^1|$ and thus $|V_{G-X}^3| < 2k$. Finally, each degree-two vertex in $G$ needs at least one neighbor of degree at least three since $G$ is reduced with respect to Reduction Rule 2.3. Thus, the vertices in $V_{G-X}^2$ are either incident to an edge in $X$ or adjacent to one of the at most $|V_{G-X}^3| + 2k$ vertices in $G$ that have degree at least three. Since the sum over all degrees of vertices in $V_{G-X}^3$ is at most $\sum_{v \in V_{G-X}^3} \deg_G(v) \leq 2|V_{G-X}^3| + |V_{G-X}^1| < 6k$, it follows that $|V_{G-X}^2| \leq 8k$. Thus, the number of vertices in $G$ is $|V_{G-X}^1| + |V_{G-X}^2| + |V_{G-X}^3| \leq 12k$. Since $G - X$ is a forest, it follows that $G$ has at most $|V| + k \leq 13k$ edges.

Applying the $O(m\sqrt{n})$-time algorithm for Matching [17] on the kernel yields:

**Corollary 2.4.** Matching can be solved in $O(n + m + k^{1.5})$ time, where $k$ is the feedback vertex number.
2.2 Parameter feedback vertex number

We next provide for MATCHING a kernel of size $2^{O(k)}$ computable in $O(kn)$ time where $k$ is the “feedback vertex number”. Using a known linear-time factor 4-approximation algorithm [1], we can approximate feedback vertex set and use it in our kernelization algorithm.

Roughly speaking, our kernelization algorithm extends the linear-time computable kernel with respect to the parameter “feedback edge set”. Thus, Reduction Rules 2.1 and 2.3 play an important role in the kernelization. Compared to the other kernels presented in this paper, the kernel presented here comes at the price of higher running time $O(kn)$ and bigger kernel size (exponential size). It remains open whether MATCHING parameterized by the “feedback vertex number” admits a linear-time computable kernel (possibly of exponential size), and whether it admits a polynomial kernel computable in $O(kn)$ time.

Subsequently, we describe our kernelization algorithm which keeps in the kernel all vertices in the given feedback vertex set $X$ and shrinks the size of $G - X$. Before doing so, we need some further notation. In this section, we assume that each tree is rooted at some arbitrary (but fixed) vertex such that we can refer to the parent and children of a vertex. A leaf in $G - X$ is called a bottommost leaf either if it has no siblings or if all its siblings are also leaves. (Here, bottommost refers to the subtree with the root being the parent of the considered leaf.) The outline of the algorithm is as follows (we assume throughout that $k < \log n$ since otherwise the input instance is already a kernel of size $O(2^k)$):

1. Reduce $G$ wrt. Reduction Rules 2.1 and 2.3.
2. Compute a maximum matching $M_{G-X}$ in $G - X$.
3. Modify $M_{G-X}$ in linear time such that only the leaves of $G - X$ are free (Section 2.2.1).
4. Bound the number of free leaves in $G - X$ by $k^2$ (Section 2.2.2).
5. Bound the number of bottommost leaves in $G - X$ by $O(k^2 2^k)$ (Section 2.2.3).
6. Bound the degree of each vertex in $X$ by $O(k^2 2^k)$. Then, use Reduction Rules 2.1 and 2.3 to provide the kernel of size $2^{O(k)}$ (Section 2.2.4).

Whenever we reduce the graph at some step, we also show that the reduction is correct. That is, the given instance is a yes-instance if and only if the reduced one is a yes-instance. The correctness of our kernelization algorithm then follows by the correctness of each step. We discuss in the following some details of each step.

2.2.1 Steps 1 to 3

By Lemma 2.2 we can perform Step 1 in linear time. By Lemma 2.1 this step is correct.

A maximum matching in Step 2 can be computed by repeatedly matching a free leaf to its neighbor and by removing both vertices from the graph (thus effectively applying Reduction Rule 2.1 to $G - X$). By Lemma 2.2, this can be done in linear time.

Step 3 can be done in $O(n)$ time by traversing each tree in $M_{G-X}$ in a BFS manner starting from the root: If a visited inner vertex $v$ is free, then observe that all children are matched since $M_{G-X}$ is maximum. Pick an arbitrary child $u$ of $v$ and match it with $v$. The vertex $w$ that was previously matched to $u$ is now free and since it is a child of $u$, it will be visited in the future. Observe that Steps 2 and 3 do not change the graph but only the auxiliary matching $M_{G-X}$, and thus these steps are correct.

2.2.2 Step 4.

Recall that our goal is to upper-bound the number edges between vertices of $X$ and $V \setminus X$, since we can then use a simple analysis as for the parameter “feedback edge set”. Furthermore, recall that by Observation 1.2 the size of any maximum matching in $G$ is at most $k$ plus the size of $M_{G-X}$.
Now, the crucial observation is that if a vertex \( x \in X \) has at least \( k \) neighbors in \( V \setminus X \) that are free wrt. \( M_{G-X} \), then there exists a maximum matching where \( x \) is matched to one of these \( k \) vertices since at most \( k-1 \) can be “blocked” by other matching edges. This means we can delete all other edges incident to \( x \). Formalizing this idea, we obtain the following reduction rule.

**Reduction Rule 2.4.** Let \( G = (V, E) \) be a graph, let \( X \subseteq V \) be a subset of size \( k \), and let \( M_{G-X} \) be a maximum matching for \( G - X \). If there is a vertex \( x \in X \) with at least \( k \) free neighbors \( V_x = \{v_1, \ldots, v_k\} \subseteq V \setminus X \), then delete all edges from \( x \) to vertices in \( V \setminus X \).

**Lemma 2.5.** Reduction Rule 2.4 is correct and can be exhaustively applied in \( O(n + m) \) time.

**Proof.** We first discuss the correctness and then the running time. Denote by \( s \) the size of a maximum matching in the input graph \( G = (V, E) \) and by \( s' \) the size of a maximum matching in the new graph \( G' = (V', E') \), where some edges incident to \( x \) are deleted. We need to show that \( s = s' \). Since any matching in \( G' \) is also a matching in \( G \), we easily obtain \( s \geq s' \). It remains to show \( s \leq s' \). To this end, let \( M_G := M_{G_{\text{max}}}^{\text{max}}(M_{G-X}) \) be a maximum matching for \( G \) with the maximum overlap with \( M_{G-X} \) (see ??). If \( x \) is free wrt. \( M_G \) or if \( x \) matched to a vertex \( v \) that is also in \( G' \) a neighbor of \( x \), then \( M_G \) is also a matching in \( G' \) \( (M_G \subseteq E') \) and thus we have in this case \( s \leq s' \). Hence, consider the remaining case where \( x \) is matched to some vertex \( v \) such that \( \{v, x\} \notin E' \), that is, the edge \( \{v, x\} \) was deleted by Reduction Rule 2.4. Hence, \( x \) has \( k \) neighbors \( v_1, \ldots, v_k \) in \( V \setminus X \) such that each of these neighbors is free wrt. \( M_{G-X} \) and none of the edges \( \{v_i, x\}, i \in [k], \) was deleted. Observe that by the choice of \( M_G \), the graph \( G(M_{G-X}, M_G) \) (the graph over vertex set \( V \) and the edges that are either in \( M_{G-X} \) or in \( M_G \), see ??) contains exactly \( s - |M_{G-X}| \) paths (we do not consider isolated vertices as paths). Each of these paths is an augmenting path for \( M_{G-X} \). By Observation 1.2, we have \( s - |M_{G-X}| \leq k \). Observe that \( \{v, x\} \) is an edge in one of these augmenting paths; denote this path with \( P \). Thus, there are at most \( k-1 \) paths \( G(M_{G-X}, M_G) \) that do not contain \( x \). Also, each of these paths contains exactly two vertices that are free wrt. \( M_{G-X} \): the endpoints of the path. This means that no vertex in \( X \) is an inner vertex on such a path. Furthermore, since \( M_{G-X} \) is a maximum matching, it follows that for each path at most one of these two endpoints is in \( V \setminus X \). Hence, at most \( k-1 \) vertices of \( v_1, \ldots, v_k \) are contained in the \( k-1 \) paths of \( G(M_{G-X}, M_G) \) except \( P \). Therefore, one of these vertices, say \( v_i \), is free wrt. \( M_G \) and can be matched with \( x \). Thus, by reversing the augmentation along \( P \) and adding the edge \( \{v_i, x\} \) we obtain another matching \( M_G' \) of size \( s \). Observe that \( M_G' \) is a matching for \( G \) and for \( G' \) and thus we have \( s \leq s' \). This completes the proof of correctness.

Now we come to the running time. We exhaustively apply the data reduction rule as follows. First, initialize for each vertex \( x \in X \) a counter with zero. Second, iterate over all free vertices in \( G - X \) in an arbitrary order. For each free vertex \( v \in V \setminus X \) iterate over its neighbors in \( X \). For each neighbor \( x \in X \) do the following: if the counter is less than \( k \), then increase the counter by one and mark the edge \( \{v, x\} \) (initially all edges are unmarked). Third, iterate over all vertices in \( X \). If the counter of the currently considered vertex \( x \) is \( k \), then delete all unmarked edges incident to \( x \). This completes the algorithm. Clearly, it only deletes edges incident to a vertex \( x \in X \) only if \( x \) has \( k \) free neighbors in \( V \setminus X \) and the edges to these \( k \) neighbors are kept. The running time is \( O(n + m) \): When iterating over all free vertices in \( V \setminus X \) we consider each edge at most once. Furthermore, when iterating over the vertices in \( X \), we again consider each edge at most once. \( \square \)

To finish Step 4, we exhaustively apply Reduction Rule 2.4 in linear time. Afterwards, there are at most \( k^2 \) free (wrt. to \( M_{G-X} \)) leaves in \( G - X \) that have at least one neighbor in \( X \) since each of the \( k \) vertices in \( X \) is adjacent to at most \( k \) free leaves. Thus, applying Reduction Rule 2.1 we can remove the remaining free leaves that have no neighbor in \( X \). However, since for each degree-one vertex also its neighbor is removed, we might create new free leaves and need to again apply Reduction Rule 2.4 and update the matching (see Step 3). This process of alternating application of Reduction Rules 2.1 and 2.4 stops after at most \( k \) rounds since the neighborhood of each vertex in \( X \) can be changed by Reduction Rule 2.4 at most once. This shows the running time \( O(k(n + m)) \). We next show how to improve this to \( O(n + m) \) and arrive at the final lemma of this subsection.
Algorithm 1: Reduce$(G, M_{G-X})$.

**Input:** A matching instance $(G = (V,E), s)$ and a feedback vertex set $X \subseteq V$ for $G$ with $|X| = k$.

**Output:** An equivalent matching instance $(G', s')$ such that $X$ is also a feedback vertex set for $G'$ and a maximum matching $M_{G'-X}$ for $G' - X$ such that only at most $k^2$ leaves in $G' - X$ are free.

1. Reduce $G$ wrt. Reduction Rules 2.1 and 2.3
2. Compute a maximum matching $M_{G-X}$ as described in Step 3
3. foreach $x \in X$ do $c(x) \leftarrow 0$ \hspace{1em} // $c(x)$ will store the number of free neighbors for $x$
4. foreach $e \in E$ do marked($e$) $\leftarrow$ False
5. $L \leftarrow$ stack containing all free leaves in $G - X$
6. while $L$ is not empty do
   7. \hspace{1em} $u \leftarrow$ pop($L$)
   8. \hspace{2em} foreach $x \in N_G(u) \cap X$ do \hspace{1em} // Check whether Reduction Rule 2.4 is applicable for $x$
      \hspace{3em} $c(x) \leftarrow c(x) + 1$, marked($\{u,x\}$) $\leftarrow$ True \hspace{1em} // fix $u$ as free neighbor of $x$
   9. \hspace{2em} if $c(x) = k$ then \hspace{1em} // $x$ has enough free neighbors: apply Reduction Rule 2.4
      \hspace{3em} foreach $y \in N_G(x) \cap X$ do delete ($x,y$)
      \hspace{3em} foreach $v \in N_G(x) \setminus X$ do
      \hspace{4em} if marked($\{x,v\}$) $\leftarrow$ False then
      \hspace{5em} delete ($x,v$) \hspace{1em} // Next deal with the case that $v$ becomes a degree-one vertex
      \hspace{4em} if deg$_G(v) = 1$ and $v$ is free then push $v$ on $L$
      \hspace{4em} if deg$_G(v) = 1$ and $v$ is matched then delete $v$ and its neighbor, $s \leftarrow s - 1$
   10. \hspace{2em} if deg$_G(u) = 1$ then \hspace{1em} // $u$ has no neighbors in $x$
       \hspace{3em} $v \leftarrow$ neighbor of $u$ in $G - X$; $w \leftarrow$ matched neighbor of $v$
       \hspace{3em} delete $u$ and $v$ from $G$
       \hspace{3em} $M_{G-X} \leftarrow M_{G-X} \setminus \{v,w\}$, $s \leftarrow s - 1$ \hspace{1em} // update $M_{G-X}$ and $s$
       \hspace{3em} if $w$ is now a leaf in $G - X$ then add $w$ to $L$ \hspace{1em} // $w$ is free, so add $w$ to the list of vertices to check
       \hspace{3em} else push $u'$ on $L$ \hspace{1em} // $u'$ is free, add $u'$ to the list of vertices to check
   11. return $(G, s)$ and $M_{G-X}$.

Lemma 2.6. Given a matching instance $(G, s)$ and a feedback vertex set $X$, Algorithm 1 computes in linear time an instance $(G', s')$ with feedback vertex set $X$ and a maximum matching $M_{G'-X}$ in $G' - X$ such that the following holds.

- There is a matching of size $s$ in $G$ if and only if there is a matching of size $s'$ in $G'$.
- Each vertex that is free wrt. $M_{G'-X}$ is a leaf in $G' - X$.
- There are at most $k^2$ free leaves in $G' - X$.

Proof. In the following, we explain Algorithm 1 which reduces the graph with respect to Reduction Rules 2.1 and 2.4 and updates the matching $M_{G-X}$ as described in Step 3. The algorithm performs in (Lines 1 and 2) Steps 1 to 3. As described in the previous section, this can be done in linear time. Next, Reduction Rule 2.4 is applied in Lines 8 to 17 using the approach described in the proof of Lemma 2.5: For each vertex in $x$ a counter $c(x)$ is maintained. When iterating over the free leaves in $G - X$, these counters will be updated. If a counter $c(x)$ reaches $k$, then the algorithm knows that $x$ has $k$ fixed free neighbors and according to Reduction Rule 2.4
the edges to all other vertices can be deleted (see Line 10). Observe that once the counter c(x) reaches k, the vertex x will never be considered again by the algorithm since its only remaining neighbors are free leaves in G − X that already have been popped from stack L. The only difference from the description in the proof of Lemma 2.5 is that the algorithm reacts if the degree of some vertex v in G − X is decreased to one (see Lines 15 to 17). If v is matched, then simply remove v and its matched neighbor from G and M_{G−X}. Otherwise, add v to the list L of unmatched degree-one vertices and defer dealing with v to a latter stage of the algorithm.

Observe that the matching M_{G−X} still satisfies the property that each free vertex in G − X is a leaf since only matched vertex pairs were deleted so far. When deleting unmatched degree-one vertices and their respective neighbor, the maximum matching M_{G−X} needs to be updated to satisfy this property. The algorithm does this from Lines 18 to 27: Let u be an entry in L such that u has degree one in Line 18, that is, u is a free leaf in G − X and has no neighbors in X. Then, following Reduction Rule 2.1, delete u and its neighbor v and decrease the solution size s by one (see Lines 20 and 21). Let w denote the previously matched neighbor of v. Since v was removed, w is now free. If w is a leaf in G − X, then we can simply add it to L and in this way deal with it later. If w is not a leaf, then we need to update M_{G−X} since only leaves are allowed to be free. To this end, take an arbitrary alternating path P from w to a leaf u′ of the subtree with root w and augment along P (see Lines 25 and 26). This can be done as follows: Pick an arbitrary child w_1 of w. Let w_2 be the matched neighbor of w_1. Since w is the parent of w_1, it follows that w_2 is a child of w_1. Now, remove \{w_1, w_2\} from M_{G−X} and add \{w_1, w\}. If w_2 is a leaf, then the alternating path P is found with u′ = w_2 and augmented. Otherwise, repeat the above procedure with w_2 taking the role of w. This completes the algorithm. Its correctness follows from the fact that it only deletes edges and vertices according to Reduction Rules 2.1 and 2.4.

It remains to show the running time of O(n + m). To this end, we prove that the algorithm considers each edge in E only two times. First, consider the edges incident to a vertex x ∈ X. These edges will be inspected at most twice by the algorithm: Once, when it is marked (see Line 9). The second time is when it is deleted. This bounds the running time in the first part (Lines 8 to 17).

Now consider the remaining edges within G − X. To this end, observe that the algorithm performs two actions on the edges: deleting the edges (Line 20) and finding and augmenting along an alternating path (Lines 25 and 26). Clearly, after deleting an edge it will no longer be considered, so it remains to show that edge is part of at most one alternating path used in Lines 25 and 26. Assume toward a contradiction that the algorithm augments along an edge twice or more. From all the edges that are augmented twice or more let e ∈ E be one that is closest to the root of the tree e is contained in, that is, there is no edge closer to a root. Let w_1 and u′_1 be the endpoints of the first augmenting path P_1 containing e and w_2 and u′_2 the endpoints of the second augmenting path P_2 containing e. Observe that for each augmenting path chosen in Line 25 it holds that one endpoint is a leaf and the other endpoint is an ancestor of this leaf. Assume without loss of generality that u′_1 and u′_2 are the leaves and w_1 and w_2 are their respective ancestors. Let w_1 and v_1 (w_2 and v_2) be the vertices deleted in Line 20 which in turn made w_1 (w_2) free. Observe that e does not contain any of these four vertices w_1, v_1, w_2, v_2 since before augmenting P_1 (P_2) the vertices u_1 and v_1 (w_2 and v_2) are deleted. Since e is contained in both paths, either w_1 is an ancestor of w_2 or vice versa: the case w_1 = w_2 cannot happen since for the second augmenting path the endpoint w_2 = w_1 would not be matched to v_2; a contradiction (see Line 19).

We next consider the case that w_1 is an ancestor of w_2 (the other case will be handled subsequently). Denote with w_2′ the neighbor of w_2 on P_2. Observe that e = \{w_2, w_2′\} since e is chosen as being closest to the root. We next distinguish the two cases whether or not e is initially matched. If e is initially free, then e is matched after augmenting along P_1. Then, by choice of P_2, e is not changed until the augmentation along P_2. This, however, is a contradiction since augmenting along P_2 only happens after the matched edge \{w_2, v_2\} is deleted. Since w_2′ ≠ v_2 and e is matched all the time until w_2 and v_2 are deleted, this means that w_2 would be matched to two vertices. Thus, consider the case that e is initially matched. Then, after augmenting along P_1, e is free and w_2 is matched to its parent p_{w_2}. As a consequence, v_2 is not matched to w_2, neither before
nor after the augmentation of $P_1$. Since the algorithm augments along $P_2$ only after it deleted $u_2$ and $v_2$ where $v_2$ is matched to $w_2$, it follows that the edge $\{w_2, v_2\}$ is augmented before the algorithm augments along $P_2$. Denote with $P_3$ the augmenting path containing the edge $\{w_2, v_2\}$. Since $w_2$ is apparently not a free leaf, it follows that $P_3$ needs to contain the matched neighbor of $w_2$, which is $p_{u_2}$. This means that the edge $\{w_2, p_{u_2}\}$ is augmented at least twice (through $P_1$ and $P_3$). However, $\{w_2, p_{u_2}\}$ is closer to the root than $e = \{w_2, w'_2\}$, a contradiction to the choice of $e$. This completes the case that $w_1$ is an ancestor of $w_2$.

We now consider the remaining case where $w_2$ is an ancestor of $w_1$. In this case we have $e = \{w_1, w'_1\}$ where $w'_1$ is the neighbor of $w_1$ on $P_1$. Observe that $w'_1$ is a child of $w_1$. Furthermore, observe that after the augmentation along $P_1$ the leaf $u'_1$ is free and can be reached by an alternating path from $w_1$. Hence, before and after the augmentation along $P_1$ it holds that $w_1$ can reach exactly one free leaf via an alternating path ($u_1$ and $u'_1$). Observe that this is true even if the algorithm removes $u'_1$ since then a new free leaf will be created. Thus, before deleting $u_2$ and $v_2$ (right before the augmentation along $P_2$), there is an augmenting path in $G - X$ from $u_2$ to $w_1$ and to the free leaf reachable from $w_1$. This is a contradiction to the fact that the matching $M_{G - X}$ is maximum.

We conclude that each edge in $E$ will be augmented at most once. Thus, the algorithm considers each edge at most twice (when augmenting it and when deleting it). Hence, the algorithm runs in linear time.

Summarizing, in Step 4 we apply Algorithm 1 in order to obtain an instance with at most $k^2$ free vertices in $G - X$ that are all leaves. By Lemma 2.6 this can be done in linear time. Furthermore, Lemma 2.6 also shows that the step is correct.

### 2.2.3 Step 5

In this step we reduce the graph in $O(kn)$ time so that at most $k^2(2^k + 1)$ bottommost leaves will remain in the forest $G - X$. We will restrict ourselves to consider leaves that are matched with their parent vertex in $M_{G - X}$ and that do not have a sibling. We call these bottommost leaves interesting. Any sibling of a bottommost leaf is by definition also a leaf. Thus, at most one of these leaves (the bottommost leaf or its siblings) is matched with respect to $M_{G - X}$ and all other leaves are free. Recall that in the previous step we upper-bounded the number of free leaves with respect to $M_{G - X}$ by $k^2$. Hence there are at most $k^2$ bottommost leaves that are not interesting.

Our general strategy for this step is to extend the idea behind Reduction Rule 2.4: We want to keep for each pair of vertices $x, y \in X$ at most $k$ different internally vertex-disjoint augmenting paths from $x$ to $y$. (For ease of notation we keep $k$ paths although keeping $k/2$ is sufficient.) In this step, we only consider augmenting paths of the form $x, u, v, y$ where $v$ is a bottommost leaf and $u$ is $v$’s parent in $G - X$. Assume that the parent $p$ of $v$ is adjacent to some vertex $x \in X$. Observe that in this case any augmenting path starting with the two vertices $x$ and $u$ has to continue to $v$ and end in a neighbor of $v$. Thus, the edge $\{x, u\}$ can be only used in augmenting paths of length three. Furthermore, all these length-three augmenting paths are clearly internally vertex-disjoint. If we do not need the edge $\{x, u\}$ because we kept $k$ augmenting paths from $x$ already, then we can delete $\{x, u\}$. Furthermore, if we deleted the last edge from $u$ to $X$ (or $u$ had no neighbors in $X$ in the beginning), then $u$ is a degree-two vertex in $G$ and can be removed by applying Reduction Rule 2.2. As the child $v$ of $u$ is a leaf in $G - X$, it follows that $v$ has at most $k + 1$ neighbors in $G$. We show below (Lemma 2.7) that the application of Reduction Rule 2.2 to remove $u$ takes $O(k)$ time. As we remove at most $n$ vertices, at most $O(kn)$ time is spent on Reduction Rule 2.2 in this step.

We now show that after a simple preprocessing one application of Reduction Rule 2.2 in the algorithm above can indeed be performed in $O(k)$ time.

**Lemma 2.7.** Let $u$ be a leaf in the tree $G - X$, $v$ be its parent, and let $w$ be the parent of $v$. If $v$ has degree two in $G$, then applying Reduction Rule 2.2 to $v$ (deleting $v$, contracting $u$ and $v$, and setting $s := s - 1$) can be done in $O(k)$ time plus $O(kn)$ time for an initial preprocessing.
\textbf{Algorithm 2: FVS-Kernel Step 5.}

\textbf{Input:} A matching instance \((G = (V,E),s)\), a feedback vertex set \(X \subseteq V\) of size \(k\) for \(G\) with \(k < \log n\), and a maximum matching \(M_{G,X}\) for \(G - X\) with at most \(k^2\) free vertices in \(G - X\) that are all leaves.

\textbf{Output:} An equivalent matching instance \((G',s')\) such that \(X\) is also a feedback vertex set for \(G'\) and \(G - X\) is a tree with at most \(k^2(2^k + 1)\) bottommost leaves, and a maximum matching \(M_{G',X}\) for \(G' - X\) with at most \(k^2\) free vertices in \(G' - X\) that are all leaves.

1. Fix an arbitrary bijection \(f : 2^X \to \{1, \ldots, 2^k\}\).
2. \textbf{foreach} \(v \in V \setminus X\) \textbf{do}
   \begin{enumerate}
   \item Set \(f_X(v) \leftarrow f(N(v) \cap X)\). \hfill // The number \(f_X(v) < n\) can be read in constant time.
   \end{enumerate}
3. Initialize a table Tab of size \(k \cdot 2^k\) with \(\text{Tab}[x,f(Y)] \leftarrow 0\) for all \(x \in X, \emptyset \subseteq Y \subseteq X\).
4. \(P \leftarrow\) List containing all parents of interesting bottommost leaves.
5. \textbf{while} \(P\) is not empty \textbf{do}
   \begin{enumerate}
   \item \(u \leftarrow \text{pop}(P)\)
   \item \(v \leftarrow\) child vertex of \(u\) in \(G - X\).
   \item \textbf{foreach} \(x \in N(u) \cap X\) \textbf{do}
   \begin{enumerate}
   \item if \(\text{Tab}[x,f_X(v)] < k\) then
   \begin{enumerate}
   \item \(\text{Tab}[x,f_X(v)] \leftarrow \text{Tab}[x,f_X(v)] + 1\)
   \end{enumerate}
   \item else
   \begin{enumerate}
   \item delete \(\{x,u\}\)
   \end{enumerate}
   \end{enumerate}
   \item if \(u\) has now degree two in \(G\) then
   \begin{enumerate}
   \item Apply Reduction Rule 2.2 to \(u\). \hfill // This decreases \(s\) by one.
   \end{enumerate}
   \item \(vw \leftarrow\) vertex resulting from the merge of \(v\) and the parent \(w\) of \(u\).
   \item if \(vw\) is now an interesting bottommost leaf then
   \begin{enumerate}
   \item add the parent of \(vw\) to \(P\).
   \end{enumerate}
   \end{enumerate}
\end{enumerate}
\textbf{return} \((G,s)\) and \(M_{G,X}\).

\textbf{Proof.} The preprocessing is to simply create a partial adjacency matrix for \(G\) with the vertices in \(X\) in one dimension and \(V\) in the other dimension. This adjacency matrix has size \(O(kn)\) and can clearly be computed in \(O(kn)\) time.

Now apply Reduction Rule 2.2 to \(v\). Deleting \(v\) takes constant time. To merge \(u\) and \(w\) iterate over all neighbors of \(u\). If a neighbor \(u'\) of \(u\) is already a neighbor of \(w\), then decrease the degree of \(u'\) by one, otherwise add \(u'\) to the neighborhood of \(w\). Then, relabel \(w\) to be the new merged vertex \(uw\).

Since \(u\) is a leaf in \(G - X\) and its only neighbor in \(G - X\), namely \(v\), is deleted, it follows that all remaining neighbors of \(u\) are in \(X\). Thus, using the above adjacency matrix, one can check in constant time whether \(u'\) is a neighbor of \(w\). Hence, the above algorithm runs in \(O(\deg(u)) = O(k)\) time.

The above ideas are used in Algorithm 2 which we use for this step (Step 5). The algorithm is explained in the proof of the following lemma stating the correctness and the running time of Algorithm 2.

\textbf{Lemma 2.8.} Let \((G = (V,E),s)\) be a matching instance, let \(X \subseteq V\) be a feedback vertex set, and let \(M_{G,X}\) be a maximum matching for \(G - X\) with at most \(k^2\) free vertices in \(G - X\) that are all leaves. Then, Algorithm 2 computes in \(O(kn)\) time an instance \((G',s')\) with feedback vertex set \(X\) and a maximum matching \(M_{G',X}\) in \(G' - X\) such that the following holds.

- There is a matching of size \(s\) in \(G\) if and only if there is a matching of size \(s'\) in \(G'\).
- There are at most \(k^2(2^k + 1)\) bottommost leaves in \(G' - X\).
- There are at most \(k^2\) free vertices in \(G' - X\) and they are all leaves.
Proof. We start with describing the basic idea of the algorithm. To this end, let \( \{u, v\} \in E \) be an edge such that \( v \) is an interesting bottommost leaf, that is, without siblings and matched to its parent \( u \) by \( M_{G-X} \). Counting for each pair \( x \in N(u) \cap X \) and \( y \in N(v) \cap X \) one augmenting path gives in a simple worst-case analysis \( O(k^2) \) time per edge, which is too slow for our purposes. Instead, we count for each pair consisting of a vertex \( x \in N(u) \cap X \) and a set \( Y = N(v) \cap X \) one augmenting path. In this way, we know that for each \( y \in Y \) there is one augmenting path from \( x \) to \( y \) without iterating through all \( y \in Y \). This comes at the price of considering up to \( k2^k \) such pairs. However, we will show that we can do the computations in \( O(k) \) time per considered edge in \( G - X \). The main reason for this improved running time is a simple preprocessing that allows for a bottommost vertex \( v \) to determine \( N(v) \cap X \) in constant time.

The preprocessing is as follows (see Lines 1 to 3): First, fix an arbitrary bijection \( f \) between the set of all subsets of \( X \) to the numbers \( \{1, 2, \ldots, 2^k\} \). This can be done for example by representing a set \( Y \subseteq X = \{x_1, \ldots, x_k\} \) by a length-\( k \) binary string (a number) where the \( i \)th position is 1 if and only if \( x_i \in Y \). Given a set \( Y \subseteq X \) such a number can be computed in \( O(k) \) time in a straightforward way. Thus, Lines 1 to 3 can be performed in \( O(kn) \) time. Furthermore, since we assume that \( k < \log n \) (otherwise the input instance is already an exponential kernel), we have that \( f(Y) < n \) for each \( Y \subseteq X \). Thus, reading and comparing these numbers can be done in constant time. Furthermore, in Line 3 the algorithm precomputes for each vertex the number corresponding to its neighborhood in \( X \).

After the preprocessing, the algorithm uses a table \( \text{Tab} \) where it counts an augmenting path from a vertex \( x \in X \) to a set \( Y \subseteq X \) whenever a bottommost leaf \( v \) has exactly \( Y \) as neighborhood in \( X \) and the parent of \( v \) is adjacent to \( x \) (see Lines 4 to 18). To do this in \( O(kn) \) time, the algorithm proceeds as follows: First, it computes in Line 5 the set \( P \) which contains all parents of interesting bottommost leaves. Clearly, this can be done in linear time. Next, the algorithm processes the vertices in \( P \). Observe that further vertices might be added to \( P \) (see Line 18) during this processing. Let \( u \) be the currently processed vertex of \( P \), let \( v \) be its child vertex, and let \( Y \) be the neighborhood of \( v \) in \( X \). For each neighbor \( x \in N(u) \cap X \), the algorithm checks whether there are already \( k \) augmenting paths between \( x \) and \( Y \) with a table lookup in \( \text{Tab} \) (see Line 10). If not, then the table entry is incremented by one (see Line 11) since \( u \) and \( v \) provide another augmenting path. If yes, then the edge \( \{x, u\} \) is deleted in Line 13 (we show below that this does not change the maximum matching size). If \( u \) has degree two after processing all neighbors of \( u \) in \( X \), then by applying \text{Reduction Rule 2.2}, we can remove \( u \) and contract its two neighbors \( v \) and \( w \). It follows from Lemma 2.7 that this application of \text{Reduction Rule 2.2} can be done in \( O(k) \) time. Hence, \text{Algorithm 2} runs in \( O(kn) \) time.

Recall that all vertices in \( G - X \) that are free wrt. \( M_{G-X} \) are leaves. Thus, the changes to \( M_{G-X} \) by applying \text{Reduction Rule 2.2} in Line 15 are as follows: First, the edge \( \{v, u\} \) is removed and second the edge \( \{w, q\} \) is replaced by \( \{wv, q\} \) for some \( q \in V \). Hence, the matching \( M_{G-X} \) after running \text{Algorithm 2} has still at most \( k2^k \) free vertices and all of them are leaves.

It remains to prove that (a) the deletion of the edge \( \{x, u\} \) in Line 13 results in an equivalent instance and (b) that the resulting instance has at most \( O(k2^k) \) bottommost leaves. First, we show (a). To this end, assume towards a contradiction that the new graph \( G' := G - \{x, u\} \) has a smaller maximum matching than \( G \) (clearly, \( G' \) cannot have a larger maximum matching). Thus, any maximum matching \( M_G \) for \( G \) has to contain the edge \( \{x, u\} \). This implies that the child \( v \) of \( u \) in \( G - X \) is matched in \( M_G \) with one of its neighbors (except \( u \)): If \( v \) is free wrt. \( M_G \), then deleting \( \{x, u\} \) from \( M_G \) and adding \( \{v, u\} \) yields another maximum matching not containing \( \{x, u\} \), a contradiction. Recall that \( N(v) = \{u\} \cup Y \) where \( Y \subseteq X \) since \( v \) is a leaf in \( G - X \). Thus, each maximum matching \( M_G \) for \( G \) contains for some \( y \in Y \) the edge \( \{v, y\} \). Observe that \text{Algorithm 2} deletes \( \{x, u\} \) only if there are at least \( k \) other interesting bottommost leaves \( v_1, \ldots, v_k \) in \( G - X \) such that their respective parent is adjacent to \( x \) and \( N(v_i) \cap X = Y \) (see Lines 9 to 13). Since \( |Y| \leq k \), it follows by the pigeon hole principle that at least one of these vertices, say \( v_i \), is not matched to any vertex in \( Y \). Thus, since \( v_i \) is an interesting bottommost leaf, it is matched to its only remaining neighbor: its parent \( u_i \) in \( G - X \). This implies that there
is another maximum matching
\[ M'_G := (M_G \setminus \{(v, y), (x, u), (u, v_i)\}) \cup \{(v, y), (x, u_i), (u, v)\}, \]
a contradiction to the assumption that all maximum matchings for \( G \) have to contain \( \{x, u\} \).

We next show (b) that the resulting instance has at most \( k^2(2^k + 1) \) bottommost leaves. To this end, recall that there are at most \( k^2 \) bottommost leaves that are not interesting (see discussion at the beginning of this subsection). Hence, it remains to upper-bound the number of interesting bottommost leaves. Observe that each parent \( u \) of an interesting bottommost leaf has to be adjacent to a vertex in \( X \) since otherwise \( u \) would have been deleted in Line 15. Furthermore, after running Algorithm 2, each vertex \( x \in X \) is adjacent to at most \( 2k^2 \) parents of interesting bottommost leaves (see Lines 10 to 13). Thus, the number of interesting bottommost leaves is at most \( k^22^k \). Therefore the number of bottommost leaves is upper-bounded by \( k^2(2^k + 1) \).

2.2.4 Step 6

In this subsection, we provide the final step of our kernelization algorithm. Recall that in the previous steps we have upper-bounded the number of bottommost leaves in \( G - X \) by \( O(k^22^k) \), we computed a maximum matching \( M_{G - X} \) for \( G - X \) such that at most \( k^2 \) vertices are free w.r.t. \( M_{G - X} \) and all free vertices are leaves in \( G - X \). Using this, we next show how to reduce \( G \) to a graph of size \( O(k^32^k) \). To this end we need some further notation. A leaf in \( G - X \) that is not bottommost is called a pendant. We define \( T \) to be the pendant-free tree (forest) of \( G - X \), that is, the tree (forest) obtained from \( G - X \) by removing all pendants. The next observation shows that \( G - X \) is not much larger than \( T \). This allows us to restrict ourselves in the following on giving an upper bound on the size of \( T \).

**Observation 2.9.** Let \( G - X \) be as described above with vertex set \( V \setminus X \) and let \( T \) be the pendant-free tree (forest) of \( G - X \) with vertex set \( V_T \). Then, \( |V \setminus X| \leq 2|V_T| + k^2 \).

**Proof.** Observe that \( V \setminus X \) is the union of all pendants in \( G - X \) and \( V_T \). Thus, it suffices to show that \( G - X \) contains at most \( |V_T| + k^2 \) pendants. To this end, recall that we have a maximum matching for \( G - X \) with at most \( k^2 \) free leaves. Thus, there are at most \( k^2 \) leaves in \( G - X \) that have a sibling which is also a leaf since from two leaves with the same parent at most one can be matched. Hence, all but at most \( k^2 \) pendants in \( G - X \) have pairwise different parent vertices. Since all these parent vertices are in \( V_T \), it follows that the number of pendants in \( G - X \) is \( |V_T| + k^2 \).

We use the following observation to provide an upper bound on the number of leaves of \( T \).

**Observation 2.10.** Let \( F \) be a forest, let \( F' \) be the pendant-free forest of \( F \), and let \( B \) be the set of all bottommost leaves in \( F \). Then, the set of leaves in \( F' \) is exactly \( B \).

**Proof.** First observe that each bottommost leaf of \( F \) is a leaf of \( F' \) since we only remove vertices to obtain \( F' \) from \( F \). Thus, it remains to show that each leaf \( v \) in \( F' \) is a bottommost leaf in \( F \).

We distinguish two cases of whether or not \( v \) is a leaf in \( F \): First, assume that \( v \) is not a leaf in \( F \). Thus, all of its child vertices have been removed. Since we only remove pendants to obtain \( F' \) from \( F \), and since each pendant is a leaf, it follows that \( v \) is in \( F \) the parent of one or more leaves \( u_1, \ldots, u_\ell \). Thus, by definition, all these leaves \( u_1, \ldots, u_\ell \) are bottommost leaves, a contradiction to the fact that they were deleted when creating \( F' \).

Second, assume that \( v \) is a leaf in \( F \). If \( v \) is a bottommost leaf, then we are done. Thus, assume that \( v \) is not a bottommost leaf and therefore a pendant. However, since we remove all pendants to obtain \( F' \) from \( F \), it follows that \( v \) is not contained in \( F' \), a contradiction.

From Observation 2.10 it follows that the set \( B \) of bottommost leaves in \( G - X \) is exactly the set of leaves in \( T \). In the previous step we reduced the graph such that \( |B| \leq k^2(2^k + 1) \). Thus, \( T \) has at most \( k^2(2^k + 1) \) vertices of degree one and, since \( T \) is a tree (a forest), \( T \) also has at most \( k^2(2^k + 1) \) vertices of degree at least three. Let \( V^2_T \) be the vertices of degree two in \( T \) and
Lemma 2.11. Let $M_{G-X}$ be a maximum matching in the forest $G - X$. Let $P_{uv}$ be an augmenting path for $M_{G-X}$ in $G$ from $u$ to $v$. Let $P_{wx}$, $P_{wy}$, and $P_{wz}$ be three internally vertex-disjoint augmenting paths from $w$ to $x$, $y$, and $z$, respectively, such that $P_{uv}$ intersects all of them. Then, there exist two vertex-disjoint augmenting paths with endpoints $u$, $v$, $w$, and one of the three vertices $x$, $y$, or $z$.

Proof. Label the vertices in $P_{uv}$ alternating as odd or even with respect to $P_{uv}$ so that no two consecutive vertices have the same label, $u$ is odd, and $v$ is even. Analogously, label the vertices in $P_{wx}$, $P_{wy}$, and $P_{wz}$ as odd and even with respect to $P_{wx}$, $P_{wy}$, and $P_{wz}$ respectively so that $w$ is always odd. Since all these paths are augmenting, it follows that each edge from an even vertex to its succeeding odd vertex is in the matching $M_{G-X}$ and each edge from an odd vertex to its succeeding even vertex is not in the matching. Observe that $P_{uv}$ intersects each of the other paths at least at two consecutive vertices, since every second edge must be an edge in $M_{G-X}$. Since $G - X$ is a forest and all vertices in $X$ are free with respect to $M_{G-X}$, it follows that the intersection of two augmenting paths is connected and thus a path. Since $P_{uv}$ intersects the three augmenting paths from $w$, it follows that at least two of these paths, say $P_{wx}$ and $P_{wy}$, have a “fitting parity”, that is, in the intersections of $P_{uv}$ with $P_{wx}$ and with $P_{wy}$, the even vertices with respect to $P_{uv}$ are either even or odd with respect to both $P_{wx}$ and $P_{wy}$.

Assume without loss of generality that in the intersections of the paths the vertices have the same label with respect to the three paths (if the labels differ, then revert the ordering of the vertices in $P_{uv}$, that is, exchange the names of $u$ and $v$ and change all labels on $P_{uv}$ to its opposite). Denote with $v_1^1$ and $v_1^2$ the first and the last vertex in the intersection of $P_{uv}$ and $P_{wx}$. Analogously, denote with $v_2^1$ and $v_2^2$ the first and the last vertex in the intersection of $P_{uv}$ and $P_{wy}$. Assume without loss of generality that $P_{uv}$ intersects first with $P_{wx}$ and then with $P_{wz}$. Observe that $v_1^1$ and $v_2^1$ are even vertices and $v_1^2$ and $v_2^2$ are odd vertices since the intersections have to start and end with edges in $M_{G-X}$ (see Fig. 1 for an illustration). For an arbitrary path $P$ and for two arbitrary vertices $p_1, p_2$ of $P$, denote by $p_1 - P - p_2$ the subpath of $P$ from $p_1$ to $p_2$. Observe that $u - P_{uv} - v_1^1 - P_{wx} - x$ and $w - P_{wy} - v_2^2 - P_{uv} - v$ are two vertex-disjoint augmenting
Algorithm 3: Algorithm for computing Step 6 of our kernel wrt. the parameter "feedback vertex number".

**Input:** A matching instance \( (G = (V, E), s) \), a feedback vertex set \( X \subseteq V \) of size \( k \) for \( G \) with \( k < \log n \) and at most \( k^2(2^k + 1) \) bottommost leaves in \( G - X \), and a maximum matching \( M_{G - X} \) for \( G - X \) with at most \( k^2 \) free vertices in \( G - X \) that are all leaves.

**Output:** An equivalent matching instance \( (G', s') \) such that \( G' \) contains at most \( O(k^22^k) \) vertices and edges.

1. Fix an arbitrary bijection \( f : 2^X \to \{1, \ldots, 2^k\} \).
2. **foreach** \( v \in V \setminus X \) do
   3. Set \( f_X(v) \leftarrow f(N(v) \cap X) \).
      // The number \( f_X(v) < n \) can be read in constant time.
4. Initialize a table \( \text{Tab} \) of size \( k \cdot 2^k \) with \( \text{Tab}[x, f(Y)] \leftarrow 0 \) for \( x \in X, \emptyset \subseteq Y \subseteq X \).
5. \( T \leftarrow \) pendant-free tree (forest) of \( G - X \).
6. \( V_T^{\geq 3} \leftarrow \) vertices in \( T \) with degree \( \geq 3 \).
7. **foreach** \( x \in X \) do
   8. **foreach** \( v \in N(x) \setminus X \) do
      9. if \( \text{Keep-Edge}(x, v) = \text{false} \) then
          // Is \( \{x, v\} \) needed for an augmenting path?
         10. delete \( \{x, v\} \).
11. exhaustively apply Reduction Rules 2.1 and 2.3
12. return \((G, s)\).

**Function** \( \text{Keep-Edge}(x, v) \). \( x \in X, v \in V \setminus X \).

13. if \( v \) is free wrt. \( M_{G - X} \) or \( v \in V_T^{\geq 3} \) then return true
14. \( w \leftarrow \) matched neighbor of \( v \) in \( M_{G - X} \)
15. if \( w \in V_T^{\geq 3} \) or \( w \) is adjacent to free leaf in \( G - X \) then return true
16. if \( w \) has at least one neighbor in \( X \) and \( \text{Tab}[x, f_X(w)] < 6k^2 \) then
      17. \( \text{Tab}[x, f_X(w)] \leftarrow \text{Tab}[x, f_X(w)] + 1 \)
      18. return true
   19. **foreach** neighbor \( u \neq x \) of \( w \) that is matched wrt. \( M_{G - X} \) and fulfills \( \{u, x\} \notin E \) do
      20. if \( \text{Keep-Edge}(u, x) = \text{true} \) then return true
21. return false

paths.

**Algorithm description.** We now provide the algorithm for Step 6 (see Algorithm 3 for a pseudocode). The algorithm uses the same preprocessing (see Lines 1 to 3) as Algorithm 2. Thus, the algorithm can determine whether two vertices have the same neighborhood in \( X \) in constant time. As in Algorithm 2, Algorithm 3 uses a table \( \text{Tab} \) which has an entry for each vertex \( x \in X \) and each set \( Y \subseteq X \). The table is filled in such a way that the algorithm detects for each \( y \in Y \) at least \( \text{Tab}[x, y] \) internally vertex-disjoint augmenting paths from \( x \) to \( y \).

The main part of the algorithm is the boolean function ‘Keep-Edge’ in Lines 13 to 22 which makes the decision on whether or not to delete an edge \( \{x, v\} \) for \( v \in V \setminus X \) and \( x \in X \). The function works as follows for edge \( \{x, v\} \): Starting at \( v \) the graph will be explored along possible augmenting paths until a “reason” for keeping the edge \( \{x, v\} \) is found or further exploration is possible.

If the vertex \( v \) is free wrt. \( M_{G - X} \), then \( \{x, v\} \) is an augmenting path and we keep \( \{x, v\} \) (see Line 14). Observe that in Step 4 we upper-bounded the number of free vertices by \( k^2 \) and all these vertices are leaves. Thus, we keep a bounded number of edges incident to \( x \) because the corresponding augmenting paths can end at a free leaf. We provide the exact bound below when discussing the size of the graph returned by Algorithm 3. In Line 14, the algorithm stops exploring the graph and keeps the edge \( \{x, v\} \) if \( v \) has degree at least three in \( T \). The reason is to keep the graph exploration simple by following only paths in \( T \). This ensures that the running time for exploring the graph from \( x \) does not exceed \( O(n) \). Since the number of vertices in \( T \) with degree
at least three is bounded (see discussion after Observation 2.10), it follows that only a bounded number of such edges \( \{x, v\} \) are kept.

If \( v \) is not free wrt. \( M_{G \setminus X} \), then it is matched with some vertex \( w \). If \( w \) is adjacent to some leaf \( u \) in \( G \setminus X \) that is free wrt. \( M_{G \setminus X} \), then the path \( x, v, w, u \) is an augmenting path. Thus, the algorithm keeps in this case the edge \( \{x, v\} \), see Line 16. Again, since the number of free leaves is bounded, only a bounded number of edges incident to \( x \) will be kept. If \( w \) has degree at least three in \( T \), then the algorithm stops the graph exploration here and keeps the edge \( \{x, v\} \), see Line 16. Again, this is to keep the running time at \( O(kn) \) overall.

Let \( Y \subseteq X \) denote the neighborhood of \( w \) in \( X \). Thus the partial augmenting path \( x, v, w \) can be extended to each vertex in \( Y \). Thus, if the algorithm did not yet find \( 6k^2 \) paths from \( x \) to vertices whose neighborhood in \( Y \) is also \( Y \), then the table entry \( \text{Tab}[x; f_X(w)] \) (where \( f_X(w) \) encodes the set \( Y = N(w) \cap X \)) is increased by one and the edge \( \{x, v\} \) will be kept (see Lines 18 and 19). (Here we need \( 6k^2 \) paths since these paths might be long and intersect with many other augmenting paths, see proof of Lemma 2.15 for the details of why \( 6k^2 \) is enough.) If the algorithm already found \( 6k^2 \) “augmenting paths” from \( x \) to \( Y \), then the neighborhood of \( w \) in \( X \) is irrelevant for \( x \) and the algorithm continues.

In Line 20, all above discussed cases to keep the edge \( \{x, v\} \) do not apply and the algorithm extends the partial augmenting part \( x, v, w \) by considering the neighbors of \( w \) except \( v \). Since the algorithm dealt with possible extensions to vertices in \( X \) in Lines 17 to 19 and with extensions to free vertices in \( G \setminus X \) in Line 14, it follows that the next vertex on this path has to be a vertex \( u \) that is matched wrt. \( M_{G \setminus X} \). Furthermore, since we want to extend a partial augmenting path from \( x \), we require that \( u \) is not adjacent to \( x \) as otherwise \( x, u \) would be another, shorter partial augmenting path from \( x \) to \( u \) and we do not need the currently stored partial augmenting path.

**Statements on Algorithm 3.** For each edge \( \{x, z\} \) with \( x \in X \) and \( z \in V \setminus X \) we denote by \( P(x, z) \) the induced subgraph of \( G \setminus X \) on the vertices that are explored in the function Keep-Edge when called in Line 9 with \( x \) and \( z \). More precisely, we initialize \( P(x, z) := \emptyset \). Whenever the algorithm reaches Line 14, we add \( v \) to \( P(x, z) \). Furthermore, whenever the algorithm reaches Line 17, we add \( w \) to \( P(x, z) \).

We next show that \( P(x, z) \) is a path or a path with one additional pendant.

**Lemma 2.12.** Let \( x \in X \) and \( z \in V \setminus X \) be two vertices such that \( \{x, z\} \in E \). Then, \( P(x, z) \) is either a path or a tree with exactly one vertex \( z' \) having more than two neighbors in \( P(x, z) \). Furthermore, \( z' \) has degree exactly three and \( z \) is a neighbor of \( z' \).

**Proof.** We first show that all vertices in \( P(x, z) \) except \( z \) and its neighbor \( z' \) have degree at most two in \( P(x, z) \). Observe that having more vertices than \( z \) and \( z' \) in \( P(x, z) \) requires Algorithm 3 to reach Line 20.

Let \( w \) be the currently last vertex when Algorithm 3 continues the graph exploration in Line 20. Observe that the algorithm therefore dealt with the case that \( w \) has degree at least three in \( T \) in Line 16. Thus, \( w \) is either a pendant leaf in \( G \setminus X \) or \( w \notin V_T^{\geq 3} \) (that is, \( w \) has degree at most two in \( T \)). In the first case, there is no candidate to continue and the graph exploration stops. In the second case, \( w \) has degree at most two in \( T \).

We next show that any candidate \( u \) for continuing the graph exploration in Line 21 is not a leaf in \( G \setminus X \). Assume toward a contradiction, that \( u \) is a leaf in \( G \setminus X \). Since the parent \( w \) of \( u \) is matched with some vertex \( v \neq u \) (this is how \( w \) is chosen, see Line 15), it follows that \( u \) is not matched. This implies that the function 'Keep-Edge' would have returned true in Line 16 and would not have reached Line 20, a contradiction. Thus, the graph exploration follows only vertices in \( T \). Furthermore, the above argumentation implies that \( w \) is not adjacent to a leaf unless this leaf is its predecessor \( v \) in the graph exploration.

We now have two cases: Either \( w \) is not adjacent to a leaf in \( G \setminus X \) or \( v = z \) is a leaf and \( w = z' \) is its matched neighbor. In the first case, \( w \) has at most one neighbor \( u \neq v \) since \( w \notin V_T^{\geq 3} \). Hence, \( w \) has degree two in \( P(x, z) \). In the second case, \( w = z' \) has at most two neighbors \( u \neq v \) and \( u' \neq v \). Thus, \( z' \) has degree at most three.
We set for $x \in X$
\[ \mathcal{P}_x := \{ (x,v) \mid \{ x,v \} \in E \land v \in V \setminus X \} \]
to be the union of all induced subgraphs wrt. $x$.

**Lemma 2.13.** There exists a partition of $\mathcal{P}_x$ into $\mathcal{P}_x = \mathcal{P}_x^A \cup \mathcal{P}_x^B$ such that all graphs within $\mathcal{P}_x^A$ and within $\mathcal{P}_x^B$ are pairwise disjoint.

**Proof.** Since $G - X$ is a tree (or forest), $G - X$ is also bipartite. Let $A$ and $B$ be its two color classes (so $A \cup B = V \setminus X$). We define the two parts $\mathcal{P}_x^A$ and $\mathcal{P}_x^B$ as follows: A subgraph $P \in \mathcal{P}_x$ is in $\mathcal{P}_x^A$ if the neighbor $v$ of $x$ in $P$ is contained in $A$, otherwise $P$ is in $\mathcal{P}_x^B$.

We show that all subgraphs in $\mathcal{P}_x^A$ and $\mathcal{P}_x^B$ are pairwise vertex-disjoint. To this end, assume toward a contradiction that two graphs $P, Q \in \mathcal{P}_x^A$ share some vertex. (The case $P, Q \in \mathcal{P}_x^B$ is completely analogous.) Let $p_1$ and $q_1$ be the first vertex in $P$ and $Q$ respectively, that is, $p_1$ and $q_1$ are adjacent to $x$ in $G$. Observe that $p_1 \neq q_1$. Let $u \neq x$ be the first vertex that is in $P$ and in $Q$. By Lemma 2.12, $P$ and $Q$ are paths or trees with at most one vertex of degree more than two and this vertex has degree three and is the neighbor of $p_1$ or $q_1$, respectively. This implies together with $q_1, p_1 \in A$ that either $u = p_1$ or $u = q_1$. Assume without loss of generality that $u = p_1$. Since $p_1 \in A$ and $q_1 \in A$ and $u$ is a vertex in $Q$, it follows that Algorithm 3 followed $u$ in the graph exploration from $q_1$ in Line 21. However, this is a contradiction since the algorithm checks in Line 20 whether the new vertex $u$ in the path is not adjacent to $x$. Thus, all subgraphs in $\mathcal{P}_x^A$ and $\mathcal{P}_x^B$ are pairwise vertex-disjoint. \qed

We next show that if $\text{Tab}[x, f(Y)] = 6k^2$ for some $x \in X$ and $Y \subseteq X$ (recall that $f$ maps $Y$ to a number, see Line 1), then there exist at least $3k$ internally vertex-disjoint augmenting paths from $x$ to $Y$.

**Lemma 2.14.** If in Line 17 of Algorithm 3 it holds for $x \in X$ and $Y \subseteq X$ that $\text{Tab}[x, f(Y)] = 6k^2$, then there exist in $G$ wrt. $M_{G-X}$ at least $3k$ alternating paths from $x$ to vertices $v_1, \ldots, v_{3k^2}$ such that all these paths are pairwise vertex-disjoint (except $x$) and $N(v_i) \cap X = N(u) \cap X$ for all $i \in [3k^2]$.

**Proof.** Note that each time $\text{Tab}[x, f(Y)]$ is increased by one (see Line 18), the algorithm found a vertex $w$ such that there is an alternating path $P$ from $x$ to $w$ and $N(w) \cap X = Y$. Furthermore, since the function $\text{Keep-Edge}$ returns true in this case, the edge from $x$ to its neighbor on $P$ is not deleted in Line 10. Thus, there exist at least $6k^2$ alternating paths from $x$ to vertices whose neighborhood in $X$ is exactly $Y$. By Lemma 2.13, it follows that at least half of these $6k^2$ paths are vertex-disjoint. \qed

The next lemma shows that Algorithm 3 is correct and runs in $O(kn)$ time.

**Lemma 2.15.** Let $(G = (V,E), s)$ be a matching instance, let $X \subseteq V$ be a feedback vertex set of size $k$ with $k < \log n$ and at most $k^2(2^k + 1)$ bottommost leaves in $G - X$, and let $M_{G-X}$ be a maximum matching for $G - X$ with at most $k^2$ free vertices in $G - X$ that are all leaves. Then, Algorithm 3 computes in $O(kn)$ time an equivalent instance $(G', s')$ of size $O(k^2 2^k)$.

**Proof.** We split the proof into three claims, one for the correctness of the algorithm, one for the returned kernel size, and one for the running time.

**Claim 2.16.** The input instance $(G, s)$ is a yes-instance if and only if the instance $(G', s')$ produced by Algorithm 3 is a yes-instance.

**Proof.** Observe that the algorithm changes the input graph only in two lines: Lines 10 and 11. By Lemma 2.1, applying Reduction Rules 2.1 and 2.3 yields an equivalent instance. Thus, it remains to show that deleting the edges in Line 10 is correct, that is, it does not change the size of a maximum matching. To this end, observe that deleting edges does not increase the size of a maximum matching. Thus, we need to show that the size of the maximum matching does not decrease. Assume toward a contradiction that it does.
Let \( \{x, v\} \) be the edge whose deletion decreased the maximum matching size. Redefine \( G \) to be the graph before the deletion of \( \{x, v\} \) and \( G' \) to be the graph after the deletion of \( \{x, v\} \). Recall that Algorithm 3 gets as additional input a maximum matching \( M_{G-X} \) for \( G - X \). Let \( M_G := M_{G,\text{max}}(G - X) \) be a maximum matching for \( G \) with the largest possible overlap with \( M \) and let \( G^M := G(M_{G-X}, M_G) = (V, M_{G-X} \triangle M_G) \) (cf. ??). Since \( \{x, v\} \in M_G \backslash M_{G-X} \) and \( x \) is free wrt. \( M_{G-X} \) it follows that there is a path \( P \) in \( G^M \) with one endpoint being \( x \).

Recall that since \( P \) is a path in \( G^M \) it follows that \( P \) is an augmenting path for \( M_{G-X} \). Since all vertices in \( X \) are free wrt. \( M_{G-X} \), it follows that all vertices in \( P \) except the endpoints are in \( V \setminus X \). Let \( z \) be the second endpoint of this path \( P \). We call a vertex on \( P \) an even (odd) vertex if it has an even (odd) distance to \( x \) on \( P \). (So \( x \) is an even vertex and \( v \) and \( z \) are odd vertices). Observe that \( v \) is the only odd vertex in \( P \) adjacent to \( x \) since otherwise there would be another augmenting path from \( x \) to \( z \) which only uses vertices from \( P \) implying the existence of another maximum matching that does not use \( \{x, v\} \), a contradiction. Let \( u \) be the neighbor of \( z \) in \( P \).

Since no odd vertex on \( P \) except \( v \) is adjacent to \( x \), it follows that the graph exploration in the function Keep-Edge starting from \( x \) and \( v \) in Line 9 either reached \( u \) or returned true before. If \( z \in V \setminus X \), then in both cases, the function Keep-Edge would have returned true in Line 9 and Algorithm 3 would not have deleted \( \{x, v\} \), a contradiction. Thus, assume that \( z \in X \). Therefore, the function Keep-Edge considered the vertex \( u \) in Line 17 but did not keep the edge \( \{x, v\} \). Thus, when considering \( u \), it holds that Tab\([x, f_X(u)]\) = 6 \( k^2 \), where \( f_X(u) \) encodes \( Y := N(u) \cap X \) and \( z \in Y \).

By Lemma 2.14, it follows that there are \( 3k^2 \) pairwise vertex-disjoint (except \( x \)) alternating paths from \( x \) to vertices \( u_1, \ldots, u_k \) with \( N(u_i) \cap X = Y \). Thus, there are \( 3k^2 \) internally vertex-disjoint paths \( Q \) from \( x \) to \( y \) in \( G \). If one of the paths \( Q \in Q \) does not intersect any path in \( G^M \), then reverting the augmentation along \( P \) and augmenting along \( Q \) would results in another maximum matching not containing \( \{x, v\} \), a contradiction. Thus, assume that \( P \) in \( Q \) intersects at least one path in \( G^M \).

For each two paths \( Q_1, Q_2 \in Q \) that are intersected by the same path \( P' \) in \( G^M \) it holds that each further path \( P'' \) in \( G^M \) can intersect at most one of \( Q_1 \) and \( Q_2 \): Assume toward a contradiction that \( P'' \) does. Since no path in \( G^M \) except \( P \) contains \( x \) and \( z \) it follows that all intersections between the paths are within \( G - X \). Since \( P' \) and \( P'' \) are vertex-disjoint and \( Q_1 \) and \( Q_2 \) are internally vertex-disjoint, it follows that there is a cycle in \( G - X \), a contradiction to the fact that \( X \) is a feedback vertex set.

Since \( 3k^2 > 3k + k^2 \), it follows from the pigeon hole principle that there is a path \( P' \in G^M \) that intersects at least three paths \( Q_1, Q_2, Q_3 \in Q \) such that no further path in \( G^M \) intersects them. We can now apply Lemma 2.11 and obtain two vertex-disjoint augmenting paths \( Q \) and \( Q' \). Thus, reverting the augmentation along \( P \) and \( P'' \) and augment along \( Q \) and \( Q'' \) yields another maximum matching for \( G \) which does not contain \( \{x, v\} \), a contradiction. \( \square \)

Claim 2.17. The graph \( G' \) returned by Algorithm 3 has at most \( O(k^32^k) \) vertices and edges.

Proof. We first show that each vertex \( x \in X \) has degree at most \( O(k^22^k) \) in \( G' \). To this end, we need to count the number of neighbors \( v \in N(x) \setminus X \) where the function Keep-Edge returns true in Line 9. By Lemma 2.12, the function Keep-Edge explores the graph along one or two paths (essentially growing from one starting point into two directions). Recall that \( P_x \) denotes the subgraphs induced by the graph exploration of Keep-Edge for the neighbors of \( x \). By Lemma 2.13 there is a partition of \( P_x \) into \( P_x^A \) and \( P_x^B \) such that within each part the subgraphs are pairwise vertex-disjoint. We consider the two parts independently. We start with bounding the number of graphs in \( P_x^A \) where the function 'Keep-Edge' returned true (the analysis is completely analog for \( P_x^B \)).

Since all explored subgraphs are disjoint and all free vertices in \( G - X \) wrt. \( M_{G-X} \) are leaves, it follows that Algorithm 3 returned at most \( k^2 \) times true in Line 16 due to \( w \) being adjacent to a free leaf in \( G - X \). Also, the algorithm returns at most \( k^2 \) times true in Line 14 due to \( v \) being free. Furthermore, the algorithm returns at most \( 6k^2 \cdot 2^k \) times true in Line 19. Finally, we show that the algorithm returns at most \( k^2 \cdot (2^k - 1) \) times true in Lines 14 and 16, respectively. It follows from the discussion below Observation 2.10 that \( T \), the pendent-free tree of \( G - X \), has at
most $k^2(2^k + 1)$ leaves (denoted by $V_{T}^1$) and $k^2(2^k + 1)$ vertices of degree at least three (denoted by $V_{T}^{≥3}$). Let $V_T$ be the vertices of $T$. Since $T$ is a tree (or forest), it has more vertices than edges and hence

$$\sum_{v \in V_T} \deg_T(v) < 2|V_T|$$

which implies

$$\sum_{v \in V_{T}^{≥3}} \deg_T(v) < 2 \cdot |V_{T}^{≥3}| + |V_{T}^{1}|.$$ 

Thus, Algorithm 3 returns at most $2 \cdot |V_{T}^{≥3}| + |V_{T}^{1}| < 3k^2(2^k + 1)$ times true in Line 16 due to $w$ being a vertex in $V_{T}^{≥3}$. Also, Algorithm 3 returns at most $|V_{T}^{≥3}| \leq k^2(2^k + 1)$ times true in Line 14 due to $v$ being a vertex in $V_{T}^{≥3}$.

Summarizing, considering the graph explorations in $\mathcal{P}_x^A$, Algorithm 3 returned at most $k^2 + k^2 + 6k^2 \cdot 2^k + 2k^2(2^k + 1) \in O(k^22^k)$ times true in the function Keep-Edge. Analogously, considering the graph explorations in $\mathcal{P}_x^A$, Algorithm 3 also returned at most $O(k^22^k)$ times true. Hence, each vertex $x \in X$ has degree at most $O(k^22^k)$ in $G'$.

We now show that the exhaustive application of Reduction Rules 2.1 and 2.3 indeed results in a kernel of the claimed size. To this end, denote with $V_{G'}^{3-X}, V_{G'}^{2-X}$, and $V_{G'}^{≥3-X}$ the vertices that have degree one, two, and at least three in $G' - X$. We have $|V_{G'}^{1-X}| \in O(k^32^k)$ since each vertex in $X$ has degree at most $O(k^22^k)$ and $G'$ is reduced wrt. Reduction Rule 2.1. Next, since $G' - X$ is a forest (or tree), we have $|V_{G'}^{2-X}| < |V_{G'}^{1-X}|$ and thus $|V_{G'}^{2-X}| \in O(k^32^k)$. Finally, each degree-two vertex in $G'$ needs at least one neighbor of degree at least three since $G'$ is reduced with respect to Reduction Rule 2.3. Thus, each vertex in $V_{G'}^{2-X}$ is either incident to a vertex in $X$ or adjacent to one of the at most $O(k^22^k)$ vertices in $G' - X$ that have degree at least three. Thus, $|V_{G'}^{2-X}| \in O(k^32^k)$. Summarizing, $G'$ contains at most $O(k^32^k)$ vertices and edges. \hfill \Box

**Claim 2.18.** Algorithm 3 runs in $O(kn)$ time.

**Proof.** First, observe that Lines 1 to 6 can be done in $O(kn)$ time: The preprocessing and table initialization can be done in $O(kn)$ time as discussed in Section 2.2.3. Furthermore, $T$ and $V_{T}^{≥3}$ can clearly be computed in $O(n + m) \leq O(kn)$ time. Second, by Lemma 2.2, applying Reduction Rules 2.1 and 2.3 can be done in $O(n + m)$ time. Thus, it remains to show that each iteration of the foreach-loop in Line 7 can be done in $O(n)$ time.

By Lemma 2.13, the explored graphs $\mathcal{P}_x$ from $x$ can be partitioned into two parts such that within each part all subgraphs are vertex-disjoint. Thus, each vertex in $G' - X$ is visited only twice during the execution of the function Keep-Edge. Furthermore, observe that in Lines 17 and 18 the table can be accessed in constant time. Thus, the function Keep-Edge only checks once whether a vertex in $V \setminus X$ has a neighbor in $X$, namely in Line 20. This single check can be done in constant time. Since the rest of the computation is done on $G - X$ which has less than $|V \setminus X|$ edges, it follows that each iteration of the foreach-loop in Line 7 can indeed be done in $O(n)$ time. \hfill \Box

This completes the proof of Lemma 2.15. \hfill \Box

Our kernelization algorithm for the parameter “feedback vertex number” essentially calls Steps 1 to 6.

**Theorem 2.19.** Matching parameterized by the feedback vertex number $k$ admits a kernel of size $2^{O(k)}$. It can be computed in $O(kn)$ time.

**Proof.** First, using the linear-time factor-four approximation of Bar-Yehuda et al. [1], we compute an approximate feedback vertex set $X$ with $|X| \leq 4k$. Then, we apply Steps 1 to 6. Applying the first three steps is rather straightforward, see Section 2.2.1. For the remaining three steps, we use Algorithms 1 to 3. By Lemmas 2.6, 2.8 and 2.15, this can be done in $O(kn)$ time and results in a kernel of size $O((4k)^32^{4k}) = 2^{O(k)}$. \hfill \Box
Applying the $O(m\sqrt{n})$-time algorithm for Matching [17] on the kernel yields:

**Corollary 2.20.** Matching can be solved in $O(kn + 2^{O(k)})$ time, where $k$ is the feedback vertex number.

## 3 Kernelization for Matching on Bipartite Graphs

In this section, we investigate the possibility of efficient and effective preprocessing for Bipartite Matching. In particular, we show a linear-time computable polynomial-size kernel with respect to the parameter “distance $k$ to chain graphs". In the first part of this section, we provide the definition of chain graphs and describe how to compute the parameter. In the second part, we discuss the kernelization algorithm.

**Definition and computation of the parameter.** We first define chain graphs which are a subclass of bipartite graphs with special monotonicity properties.

**Definition 1 ([5]).** Let $G = (A, B, E)$ be a bipartite graph. Then $G$ is a chain graph if each of its two color classes $A, B$ admits a linear order w.r.t. neighborhood inclusion, that is, $A = \{a_1, \ldots, a_\alpha\}$ and $B = \{b_1, \ldots, b_\beta\}$ where $N(a_i) \subseteq N(a_j)$ and $N(b_i) \subseteq N(b_j)$ whenever $i < j$.

Observe that if the graph $G$ contains twins, then there is more than one linear order w.r.t. neighborhood inclusion. To avoid ambiguities, we fix for the vertices of the color class $A$ (resp. $B$) in a chain graph $G = (A, B, E)$ one linear order $\prec_A$ (resp. $\prec_B$) such that, for two vertices $u, v \in A$ (resp. $u, v \in B$), if $u \prec_A v$ (resp. if $u \prec_B v$) then $N(u) \subseteq N(v)$. In the remainder of the section we consider a bipartite representation of a given chain graph $G = (A, B, E)$ where the vertices of $A$ (resp. $B$) are ordered according to $\prec_A$ (resp. $\prec_B$) from left to right (resp. from right to left), as illustrated in Figure 2. For simplicity of notation we use in the following $\prec$ to denote the orderings $\prec_A$ and $\prec_B$ whenever the color class is clear from the context.

We next show, that we can approximate the parameter and the corresponding vertex subset in linear time. To this end, we use the following characterization of chain graphs.

**Lemma 3.1 ([5]).** A bipartite graph is a chain graph if and only if it does not contain an induced $2K_2$.

**Lemma 3.2.** There is a linear-time factor-4 approximation for the problem of deleting a minimum number of vertices in a bipartite graph in order to obtain a chain graph.

*Proof.* Let $G = (A, B, E)$ be a bipartite graph. We compute a set $S \subseteq A \cup B$ such that $G - S$ is a chain graph and $S$ is at most four times larger than a minimum size of such a set. The algorithm iteratively tries to find a $2K_2$ and deletes the four corresponding vertices until no further $2K_2$ is

![Figure 2: A chain graph. Note that the ordering of the vertices in $A$ is going from left to right while the ordering of the vertices in $B$ is going from right to left. The reason for these two orderings being drawn in different directions is that a maximum matching can be drawn as parallel edges, see e.g. the bold edges. In fact, Algorithm 4 computes such matchings with the matched edges being parallel to each other.](image-url)
Algorithm 4: An algorithm that computes a maximum matching $M$ in the chain graph $G$ such that all edges in $M$ are parallel (see Fig. 2 for a visualization.)

**Input:** A chain graph $G = (V, E)$, $V = A \cup B$, $A = \{a_1, \ldots, a_n\}$ and $B = \{b_1, \ldots, b_m\}$ with $N(a_i) \subseteq N(a_j)$ and $N(b_i) \subseteq N(b_j)$ for $i < j$.

**Output:** A maximum matching of $G$ where all matched edges are parallel.

1. Compute the size $s$ of a maximum matching in $G$ using an algorithm of Steiner and Yeomans [19] and a second time when these vertices are deleted. Thus, using appropriate data structures, the algorithm runs in $O(n + m)$ time.
2. Return $M$.

Found. Since in each $2K_2$, by Lemma 3.1, at least one vertex needs to be removed, the algorithm yields the claimed factor-4 approximation.

The details of the algorithm are as follows: First, it initializes $S = \emptyset$ and sorts the vertices in $A$ and in $B$ by their degree: the vertices in $A = \{a_1, \ldots, a_n\}$ in increasing order and the vertices $B = \{b_1, \ldots, b_m\}$ in decreasing order, that is, $\deg(a_1) \leq \ldots \leq \deg(a_n)$ and $\deg(b_1) \leq \ldots \leq \deg(b_m)$. Since the degree of each vertex is at most $\max\{\alpha, \beta\}$, this can be done in linear time with e.g. Bucket Sort. At any stage the algorithm deletes all vertices of degree zero and all vertices which are adjacent to all vertices in the other partition. The deleted vertices are not added to $S$ since these vertices can not participate in a $2K_2$. Next, the algorithm recursively processes the vertices in $A$ in a nondecreasing order of their degrees. Let $a \in A$ be a minimum-degree vertex and let $b \in B$ be a neighbor of $a$. Since $b$ is not adjacent to all vertices in $A$ (otherwise $b$ would be deleted), there is a vertex $a' \in A$ that is not adjacent to $b$. Since $\deg(a) \leq \deg(a')$ it follows that $a'$ has a neighbor $b'$ that is not adjacent to $a$. Hence, the four vertices $a, a', b, b'$ induce only two edges: $\{a, b\}$ and $\{a', b'\}$ and thus form a $2K_2$. Thus, the algorithm adds the four vertices to $S$, deletes them from the graph, and continues with a vertex in $A$ that has minimum degree. As to the running time, we now show that, after the initial sorting, the algorithm considers each edge only twice: Selecting $a$ and $b$ as described above can be done in $O(1)$ time. To select $a'$, the algorithm simply iterates over all vertices in $A$ until it finds a vertex that is not adjacent to $b$. In this way at most $\deg(b) + 1$ vertices are considered. Similarly, by iterating over the neighbors of $a'$, one finds $b'$. Hence, the edges incident to $a, a', b, b'$ are used once to find the vertices and a second time when these vertices are deleted. Thus, using appropriate data structures, the algorithm runs in $O(n + m)$ time. \hfill \Box

**Kernelization.** In the rest of this section, we provide a linear-time computable kernel for Bipartite Matching with respect to the parameter vertex deletion distance $k$ to chain graphs. The intuitive description of the kernelization is as follows. First we upper bound by $O(k)$ the number of neighbors of each vertex in the deletion set. Then we mark $O(k^2)$ special vertices and we use the monotonicity properties of chain graphs to upper bound the number of vertices that lie between any two consecutive marked edges, thus bounding the total size of the reduced graph to $O(k^3)$ vertices.

Let $G = (A, B, E)$ be the bipartite input graph, where $V = A \cup B$, and let $X \subseteq V$ be a vertex subset such that $G - X$ is a chain graph. By Lemma 3.2, we can compute an approximate $X$ in linear time. The kernelization algorithm is as follows: First, compute a specific maximum matching $M_{G - X} \subseteq E$ in $G - X$ with Algorithm 4 where all edges in $M_{G - X}$ are “parallel” and all matched vertices are consecutive in the ordering $\prec_A$ and $\prec_B$, see also Fig. 2. Since in convex graphs matching is linear-time solvable [19] and convex graphs are a super class of chain graphs, this can be done in $O(n + m)$ time. We use $M_{G - X}$ in our kernelization algorithm to obtain some local information about possible augmenting paths. For example, each augmenting path has at least one endpoint in $X$. Forming this into a data reduction rule, with $s$ denoting the size of a maximum matching, yields the following.

**Reduction Rule 3.1.** If $|M_{G - X}| \geq s$, then return a trivial yes-instance; if $s > |M_{G - X}| + k$, then return a trivial no-instance.
The correctness of the above reduction rule follows from Observation 1.2.

Without loss of generality, we will be able to assume that each vertex in \( X \) is either matched in \( M \) with another vertex in \( X \) or with a “small-degree vertex” in \( G - X \). This means that an augmenting path starting at some vertex in \( X \) will “enter” the chain graph \( G - X \) in a small-degree vertex. We now formalize this concept. For a vertex \( x \in X \) we define \( N_{\text{small}}^{X\setminus X}(x) \) to be the set of the \( k \) neighbors of \( x \) in \( V \setminus X \) with the smallest degree, formally,

\[
N_{\text{small}}^{X\setminus X}(x) := \{ w \in N(x) \setminus X \mid k > |\{ u \in N(x) \setminus X \mid u < w \}| \}.
\]

**Lemma 3.3.** Let \( G = (V, E) \) be a bipartite graph and let \( X \subseteq V \) be a vertex set such that \( G - X \) is a chain graph. Then, there exists a maximum matching \( M_G \) for \( G \) such that every matched vertex \( x \in X \) is matched to a vertex in \( N_{\text{small}}^{X\setminus X}(x) \cup X \).

**Proof.** Assume, towards a contradiction, that there is no such matching \( M_G \). Let \( M_G' \) be a maximum matching for \( G \) that maximizes the number of vertices \( x \in X \) that are matched to a vertex in \( N_{\text{small}}^{X\setminus X}(x) \cup X \), that is, let \( M_G' \) maximize \( |\{ x \in X \mid \{ u, x \} \in M_G' \wedge u \in N_{\text{small}}^{X\setminus X}(x) \cup X \}| \).

Let \( x \in X \) be a vertex that is not matched with any vertex in \( N_{\text{small}}^{X\setminus X}(x) \cup X \), that is, \( x \) is matched to a vertex \( u \in V \setminus (N_{\text{small}}^{X\setminus X}(x) \cup X) \). If there is an unmatched vertex \( w \in N_{\text{small}}^{X\setminus X}(x) \) in \( M_G' \), then the matching \( M_G' := M_G' \cup \{ \{ x, w \} \} \) is a maximum matching with more vertices \( x \in X \) (compared to \( M_G' \)) that are matched to a vertex in \( N_{\text{small}}^{X\setminus X}(x) \cup X \), a contradiction.

Hence, assume that there is no free vertex in \( N_{\text{small}}^{X\setminus X}(x) \). Since \( |N_{\text{small}}^{X\setminus X}(x)| = |X| = k \), it follows that at least one vertex \( w \in N_{\text{small}}^{X\setminus X}(x) \) is matched to a vertex \( v \in V \setminus X \). Observe that, by definition of \( N_{\text{small}}^{X\setminus X}(x) \), we have \( N_{G - X}(w) \subseteq N_{G - X}(u) \). Thus, we have \( \{ u, v \} \in E \) and thus, \( M_G' := M_G' \cup \{ \{ x, w \}, \{ u, v \} \} \setminus \{ \{ x, w \}, \{ u, v \} \} \) is a maximum matching with more vertices in \( X \) (compared to \( M_G' \)) fulfilling the condition of the lemma, a contradiction. \( \Box \)

Based on Lemma 3.3, we can provide our next data-reduction rule.

**Reduction Rule 3.2.** Let \((G, s)\) be an instance reduced with respect to Reduction Rule 3.1 and let \( x \in X \). Then delete all edges between \( x \) and \( V \setminus N_{\text{small}}^{X\setminus X}(x) \).

Clearly, Reduction Rule 3.2 can be exhaustively applied in \( O(n + m) \) time by one iteration over \( A \) and \( B \) in the order \( \prec \).

The high-level idea of our kernel is as follows: Keep in the kernel all vertices of \( X \). For each vertex \( x \in X \) keep all vertices in \( N_{\text{small}}^{X\setminus X}(x) \) and if a kept vertex is matched, then keep also the vertex with whom it is matched. Denote with \( K \) the set of the vertices kept so far in \( V \setminus X \).

Consider an augmenting path \( P = x, a_1, b_1, \ldots, a_k, b_k, y \) from a vertex \( x \in B \cap X \) to a vertex \( y \in A \cap X \). Observe that if \( a_1 \prec a_k \), then also \( \{b_1, a_k\} \in E \) and thus \( P' = x, a_1, b_1, a_k, b_k, y \) is an augmenting path. Furthermore, the vertices in the augmenting path \( P' \) are a subset of \( K \cup X \) and, thus, by keeping these vertices (and the edges between them), we also keep the augmenting path \( P' \) in our kernel. Hence, it remains to consider the more complicated case that \( a_k \prec a_1 \). To this end, we next show that in certain “areas” of the chain graph \( G - X \) the number of augmenting paths “passing through” such an area is bounded.

**Definition 2.** Let \( G = (A, B, E) \) be a chain graph and let \( M \) be a matching in \( G \). Furthermore let \( a \in A, b \in B \) with \( \{a, b\} \in M \). Then \( \#\text{lmv}(b, M) \) (resp. \( \#\text{rmv}(a, M) \)) is the number of neighbors of \( b \) (resp. of \( a \)) that are to the left of \( a \) (resp. to the right of \( b \); formally,

\[
\#\text{lmv}(b, M) := |\{ a' \in N(b) \mid a' \prec a \}|, \quad \#\text{rmv}(a, M) := |\{ b' \in N(a) \mid b' \prec b \}|.
\]

In Definition 2 the terms “left” and “right” refer to the ordering of the vertices of \( A \) and \( B \) in the bipartite representation of \( G \), as illustrated in Figure 2. The abbreviation \( \#\text{rmv} \) (\( \#\text{lmv} \)) stands for “number of vertices right (left) of the matched vertex.” We set \( \#\text{lmv}(a, M) := \#\text{lmv}(b, M) \) and \( \#\text{rmv}(b, M) := \#\text{rmv}(a, M) \). Finally, we define \( \#\text{rmv}(a_1, a_2, M) := \min_{a_1 < a' < a_2} \{ \#\text{rmv}(a', M) \} \) for \( a_1, a_2 \in A \) and \( \#\text{lmv}(b_1, b_2, M) := \min_{b_2 < b' < b_1} \{ \#\text{lmv}(b', M) \} \) for \( b_1, b_2 \in B \).
Lemma 3.4. Let $G = (A, B, E)$ be a chain graph and $M$ be a maximum matching for $G$ computed by Algorithm 4. Let $a, b \in V$ with $\{a, b\} \in M$. Then the number of vertex-disjoint alternating paths that (1) start and end with edges not in $M$ and that (2) have endpoints left of $a$ and right of $b$ is at most $\min\{\#lmv(b, M), \#rmv(a, M)\}$.

Proof. We prove the case $\#lmv(b, M) \leq \#rmv(a, M)$, that is, $\min\{\#lmv(b, M), \#rmv(a, M)\} = \#lmv(b, M)$. The case $\#lmv(b, M) > \#rmv(a, M)$ follows by symmetry (with switched roles of $a$ and $b$). Let $\#aug$ denote the number of vertex-disjoint alternating paths from $\{a' \in A \mid a' \prec a\}$ to $\{b' \in B \mid b' \prec b\}$ such that the first and last edge are not in $M$. Furthermore, let $a_1^b, \ldots, a_{\#lmv(b, M)}^b$ be the neighbors of $b$ that are to the left of $a$, that is, $a_1^b \prec a_2^b \prec \ldots \prec a_{\#lmv(b, M)}^b$ $\prec a$. Since $G$ is a chain graph it follows that no vertex $a' \in A$ with $a' \prec a_1^b$ is adjacent to any vertex $b' \in B$ with $b' \leq b$. Furthermore, for any edge $\{a', b'\} \in E$ with $a \prec a'$ and $b \prec b'$ it follows from the construction of $M$ (see Algorithm 4) that $\{a', b'\} \notin M$. Hence, any of these alternating paths has to contain at least one vertex from $a_1^b, \ldots, a_{\#lmv(b, M)}^b$. Since the alternating paths are vertex-disjoint it follows that $\#aug \leq \#lmv(b, M)$.

From the previous lemma, we directly obtain the following.

Lemma 3.5. Let $G = (A, B, E)$ be a chain graph and let $M$ be the maximum matching for $G$ computed by Algorithm 4. Let $a_1, a_2 \in A$ and $b_1, b_2 \in B$ with $\{a_1, b_1\}, \{a_2, b_2\} \in M$ with $a_1 \prec a_2$. Then there are at most $\#lmv(b_1, b_2, M)$ vertex-disjoint alternating paths that (1) start and end with edges not in $M$ and that (2) have endpoints left of $a_1$ and right of $b_2$.

Lemma 3.5 states that the number of augmenting paths passing through the area between $a_1$ and $a_2$ is bounded. Using this, we want to replace this area by a gadget with $O(k)$ vertices. To this end, we need further notation. For each kept vertex $v \in K$, we may also keep some vertices to the right and to the left of $v$. We call these $k$ vertices the left buffer (right buffer) of $v$.

Definition 3. Let $G = (A, B, E)$ be a chain graph and let $M$ be the maximum matching for $G$ computed by Algorithm 4. Let $a_1, a_2 \in A$ and $b_1, b_2 \in B$ with $\{a_1, b_1\}, \{a_2, b_2\} \in M$ and $a_1 \prec a_2$. Then the (at most) $\#lmv(b_1, b_2, M)$ vertices to the right of $a_1$ form the right buffer $B^r(a_1, M)$ of $a_1$; formally,

$$B^r(a_1, M) := \{a \in A \mid a_1 \prec a \land |\{a' \in A \mid a_1 \prec a' \prec a\}| \leq \min\{\#lmv(b_1, b_2, M), k\}\}.$$ 

Analogously,

$$B^l(a_2, M) := \{a \in A \mid a \prec a_2 \land |\{a' \in A \mid a \prec a' \prec a_2\}| \leq \min\{\#lmv(b_1, b_2, M), k\}\},$$ 

$$B^r(b_1, M) := \{b \in B \mid b \prec b_1 \land |\{b' \in B \mid b \prec b' \prec b_1\}| \leq \min\{\#lmv(b_1, b_2, M), k\}\},$$ 

$$B^l(b_2, M) := \{b \in B \mid b_2 \prec b \land |\{b' \in B \mid b_2 \prec b' \prec b\}| \leq \min\{\#lmv(b_1, b_2, M), k\}\}.$$ 

Note that in Definition 3 each of the sets $B^r(a_1, M)$, $B^l(a_2, M)$, $B^r(b_1, M)$, and $B^l(b_2, M)$ depends on all four vertices $a_1, a_2, b_1, b_2$; we omit these dependencies from the names for the sake of presentation.

Reduction Rule 3.3. Let $(G, s)$ be an instance reduced with respect to Reduction Rule 3.1. Let $a_1, a_2 \in K \cap A$ with $a_1 \prec a_2$ and $\{a_1, b_1\}, \{a_2, b_2\} \in M_{G-}\{a_1 \prec a_2\}$ of size at least $2\min\{\#lmv(b_1, b_2, M), k\} + 1$ and $A' \cap K = \emptyset$. Then delete all vertices in $A' \setminus (B^r(a_1, M_{G-}) \cup B^l(a_2, M_{G-}))$ and their matched neighbors in $B$, add all edges between the vertices in the right buffer of $a_1$ and the vertices in the left buffer of $b_2$, and decrease $s$ by the number of removed matched vertex pairs.

Lemma 3.6. Reduction Rule 3.3 is correct and can be exhaustively applied in $O(n + m)$ time.

Proof. We first introduce some notation and provide some general observations. Let $a_1, a_2, b_1,$ and $b_2$ be as stated in Reduction Rule 3.3. Denote by $A'$ (resp. $B'$) the set of vertices between $a_1$
and $a_2$ (resp. between $b_1$ and $b_2$). Further denote by $A_D' \subseteq A'$ and $B_D' \subseteq B'$ the sets of deleted vertices. Note that $|A'| = |B'|$ and $|A'_D| = |B'_D|$ since $M_{G \setminus X}$ was produced by Algorithm 4. Denote the vertices in the buffers of $a_1, a_2, b_1$, and $b_2$ by $B^x(y_z, M_{G \setminus X}) = \{y_1, \ldots, y_{\max(\deg(y_1, b_2, M), k)}\}$ for $x \in \{r, t\}$, $y \in \{a, b\}$, $z \in \{2\}$, and $x = r \iff z = 1$.

Since the input instance is reduced with respect to Reduction Rule 3.1, it follows that $s - k \leq |M_{G \setminus X}| < s$. Denote by $M'_{G \setminus X} := M_{G \setminus X} \cap E'$ the matching obtained from $M_{G \setminus X}$ by deleting all edges not in the reduced graph $G'$. Recall that $s$ is reduced by the number of matched edges that are removed. We next show in Claims 3.7 and 3.8 that the input instance $(G = (V, E), s)$ is a yes-instance if and only if the produced instance $(G' = (V', E'), s')$ is a yes-instance. Before we present these two claims, observe that there is a perfect matching between the vertices in $A'_D$ and $B'_D$, and thus

$$s - |M_{G \setminus X}| = s' - |M_{G \setminus X}|.$$  

(1)

Claim 3.7. If $(G, s)$ is a yes-instance, then $(G', s')$ is a yes-instance.

Proof. Recall that $M_{G_{max}}(M_{G \setminus X})$ for $G$ minimizes the size of $M_{G \setminus X} \cup M_{G_{max}}(M_{G \setminus X})$. Since $(G, s)$ is a yes-instance it holds that $|M_{G_{max}}(M_{G \setminus X})| \geq s$. For brevity we set $G^M := G(M_{G \setminus X}, M_{G_{max}}(M_{G \setminus X})) = (V, M_{G \setminus X} \cup M_{G_{max}}(M_{G \setminus X}))$. Note that $G^M$ is a graph that only contains odd-length paths. We will show that there are as many vertex-disjoint augmenting paths for $M_{G \setminus X}$ in $G'$ as there are paths in $G^M$. This will show that $G'$ contains a matching of size $|M_{G \setminus X}| + |M_{G_{max}}(M_{G \setminus X})| - |M_{G \setminus X}| \geq |M_{G \setminus X}| + s - |M_{G \setminus X}| \overset{(1)}{=} |M_{G \setminus X}| + s' - |M_{G \setminus X}| = s'$.

To this end, observe that all paths that do not use vertices in $V \setminus V' = A'_D \cup B'_D$ are also contained in $G'$. Thus, consider the paths in $G^M$ that use vertices in $V \setminus V'$. Denote by $P^{D'}$ the set of all paths in $G^M$ using vertices in $V \setminus V'$ and set $t := |P^{D'}|$. Consider now an arbitrary $i \in [t]$, and let $P_i^M \in P^{D'}$. Denote by $v_1^i, v_2^i, \ldots, v_{p_i}^i$ the vertices in $P_i^M$ in the corresponding order, that is, $v_1^i$ and $v_{p_i}^i$ are the endpoints of $P_i^M$ and we have $\{v_{2j}^i, v_{2j+1}^i\} \in M_{G \setminus X}$ for all $j \in [p_i/2]$. Observe that exactly one endpoint of $P_i^M$ is in $A$ and the other endpoint is in $B$, since $P_i^M$ is an odd-length path. Assume without loss of generality that $v_1^i \in A$ and $v_{p_i}^i \in B$. Thus, the vertices in $P_i^M$ with odd (even) index are in $A$ ($B$).

We next show that for any two vertices $v_j^i, v_{\ell}^i$ of $P_i^M$ with $j < \ell < p_i - 1$ and both being in $A \setminus X$, it follows that $v_j^i < v_\ell^i$. First, observe that if $j = 1$, then $v_1^i \in A \setminus X$ is a free vertex wrt. $M_{G \setminus X}$. Since $v_1^i$ is matched wrt. $M_{G \setminus X}$ and since $M_{G \setminus X}$ is computed by Algorithm 4, it follows that $v_1^i < v_2^i$. Thus, assume that $j > 1$ and $\ell > 1$ (thus $j \geq 3$ and $\ell > 1$). Assume toward a contradiction that $v_\ell^i < v_j^i$. Since $\ell < p_i - 1$, we have $v_{\ell+1}^i \in B \setminus X$ and since $G$ is a chain graph, it follows that $\{v_\ell^i, v_{\ell+1}^i\} \in E$, a contradiction to Observation 1.1. Thus, $v_j^i < v_\ell^i$.

We next show that the path $P_i^M$ contains at least one vertex $v_k^i$ left of $a_1$ and at least one vertex $v_{k'}^i$ right of $b_2$. Recall that $M_{G \setminus X}$ was computed by Algorithm 4 and, thus, the free vertices are the smallest wrt. the ordering $\prec$ (see also Fig. 2). Thus, if one endpoint of $P_i^M$ is in $(A \cup B) \setminus X$, then this vertex is either $v_1^i$ or the left of $a_1$ or it is $v_{p_i}^i$ and right of $b_2$. Thus, assume that the endpoints of $P_i^M$ are in $X$. We showed in the previous paragraph that $v_3^i < v_4^i < \ldots < v_{p_i}^i$. Thus, we also have $v_{p_i-2}^i < v_{p_i-4}^i < \ldots < v_2^i$ since $M_{G \setminus X}$ is computed by Algorithm 4. Since we assumed that some vertices of $P_i^M$ to be in $V \setminus V'$, it follows that for at least one vertex $v_{2j+1}^i$ it holds that $a_1 < v_{2j+1}^i < a_2$. Furthermore since by assumption no vertex between $a_1$ and $a_2$ or between $b_1$ and $b_2$ is in $K$, it follows that $v_3^i < a_1$ (since $v_3^i \in K$) and $v_{p_i-2}^i < b_2$ (since $v_{p_i-2}^i \in K$).

For each $i \in [t]$ denote by $a_i^{p_i}$ the last vertex on the path $P_i^M$ that is not right of $a_1$, that is, $a_i^{p_i}$ is the vertex on $P_i^M$ such that for each vertex $a' \in A \setminus X$ that is in $P_i^M$ it holds that $a_1 < a' < a_i^{p_i}$. It follows from the previous paragraph that $a_1^{p_i}$ exists. Analogously to $a_i^{p_i}$, for each $i \in [t]$ denote by $b_i^{p_i}$ the first vertex on the path $P_i^M$ that is not left of $b_2$, that is, $b_i^{p_i}$ is the vertex on $P_i^M$ such that for each vertex $b' \in B$ that is in $P_i^M$ it holds that $b_2 < b' < b_i^{p_i}$. This means that in $G^M$ there is for each $i \in [t]$ an alternating path
from $a^{p_i'}$ to $b^{p_i'}$ starting and ending with non-matched edges and all these paths are pairwise vertex-disjoint. We show that also in $G'$ there are pairwise vertex-disjoint alternating paths from $a^{p_i'}$ to $b^{p_i'}$. Assume without loss of generality that $a^{p_i'} ⪯ a^{p_2'} ⪯ \ldots ⪯ a^{p_{i+1}'}$. Since in each path $P_j^M$, $j \in [t]$, the successor of $a^{p_i'}$ is to the right of $b_1$, it follows that $a^{p_i'}$ has at least $i$ neighbors right of $b_1$. Since the right buffer of $b_1$ contains the $\#lmv(b_1, b_2, M_{G-X}) \geq t$ (see Lemma 3.5) vertices to the right of $b_1$, we have $\{a^{p_i'}, b_j\} \in E$. By symmetry, we have $\{b^{p_i'}, a_j\} \in E$. Recall that $M_{G-X}$ forms a perfect matching between $B'(b_1, M_{G-X})$ and $B'(a_1, M_{G-X})$ as well as between $B'(a_2, M_{G-X})$ and $B'(b_2, M_{G-X})$. Since Reduction Rule 3.3 added all edges between $B'(a_1, M_{G-X})$ and $B'(b_2, M_{G-X})$ to $E'$, it follows that each path $P_j^M$ can be completed as follows: $a^{p_i'}, b_i', a_i', b_i', a_i', b^{p_i'}$; note that exactly the edges $\{b_i', a_i'\}$ and $\{b_i', a_i'\}$ are in $M_{G-X}$. Thus, each path in $P^M$ can be replaced by an augmenting path for $M_{G-X}$ in $G'$ and all these augmenting paths are vertex-disjoint. Thus, there are as many augmenting paths for $M_{G-X}$ in $G'$ as there are paths in $G^M$ and therefore $(G', s')$ is a yes-instance.

Claim 3.8. If $(G', s')$ is a yes-instance, then $(G, s)$ is a yes-instance.

Proof. Let $M_G$ be a matching for $G$. Observe that $|M_G| \geq s'$. We construct a matching $M_G$ for $G$ as follows. First, copy all edges from $M_G \cap E$ into $M_G$. Second, add all edges from $M_{G-X} \cap (\binom{V}{2})^\prime$, that is, a perfect matching between $A_G$ and $B_G^\prime$ is added to $M_G$. Observe that if all edges in $M_G$ are also in $E$, then $M_G$ is a matching of size $s$ in $G$. Thus, assume that some edges in $M_G$ are not in $E$, that is, $\{a_i', b_j\} \in M_{G-X} \setminus E$ for some $i \in \{1, 2\}$ and $j \in [t]$. Since $|M_{G-X}| \geq t$, we have $\#lmv(b_1, b_2, M_{G-X}) > 0$. Thus, we can find augmenting paths $P_1, \ldots, P_t$ as follows: To create the $q$th augmenting path $P_q$, start with some vertex $b_j$. Denote by $v$ the last vertex added to $P_q$ in the beginning we have $v = b_j$. If $v \in A_G$, then add $v$ the neighbor matched to $v$. If $v \in E$, then the following: if $v$ is adjacent to a vertex $a_i \in \{a_i', \ldots, a_i'\}$, then add $a \in E$ to $P_q$, otherwise add the leftmost neighbor of $v$ to $P_q$. Repeat this process until $P_q$ contains a vertex from $\{a_i', \ldots, a_i'\}$. After we found $P_q$, remove all vertices of $P_q$ from $G$. If $q < t$, then continue with $P_{q+1}$. Observe that any two vertices of $P_q$ that are in $A$ have at least $\#lmv(b_1, b_2, M_{G-X}) - 1$ other vertices of $A$ between them (in the ordering of the vertices of $A$, see Figure 2). Thus, after a finite number of steps, $P_q$ will reach a vertex in $\{a_i', \ldots, a_i'\}$ and $b_j$. Furthermore, it follows that after removing the vertices of $P_q$ it holds that $\#lmv(b_1, b_2, M_{G-X})$ is decreased by exactly one: $P_q$ contains for each vertex $b \in B$ at most one vertex among the $\#lmv(b_1, b_2, M_{G-X})$ neighbors of $b$ that are directly to the left of its matched neighbor in $M_{G-X}$. Thus, in each iteration we have $\#lmv(b_1, b_2, M_{G-X}) > 0$. It follows that the above procedure constructs $t$ vertex-disjoint augmenting paths from $\{a_i', \ldots, a_i'\}$ to $\{b_j', \ldots, b_j'\}$. Hence, $G$ contains a matching of size $s$ and thus $(G, s)$ is a yes-instance.

The correctness of the data reduction rule follows from the previous two claims. It remains to prove the running time. To this end, observe that the matching $M_{G-X}$ is given. Computing all degrees of $G$ can be done in $O(n + m)$ time. Also $\#lmv(v, M_{G-X})$ can be computed in linear time: For each vertex $b \in B$ one has to check for each neighbor of $b$ whether it is to the left of $b$'s matched neighbor and to adjust $\#lmv(b, M_{G-X})$ accordingly. Furthermore, computing $\#lmv(b_1, b_2, M_{G-X})$ and removing the vertices in $A_G$ and $B_G$ can be done in $O(\sum_{b \in B} \deg(b) + \sum_{a \in A} \deg(a))$ time. Thus, Reduction Rule 3.3 can be exhaustively applied in $O(n + m)$ time.
that is, $A^K_{\text{free}}$ contains the $k$ rightmost free vertices in $A \setminus X$. Observe that all vertices in $A^K_{\text{free}}$ are left of $M_{G-X}$. Analogously, denote by $B^K_{\text{free}}$ the set containing the $k$ leftmost free vertices in $B \setminus X$.

**Reduction Rule 3.4.** Let $(G, s)$ be an instance reduced with respect to Reduction Rule 3.1. Then delete all vertices in $(A_{\text{free}} \setminus (K \cup A^K_{\text{free}})) \cup (B_{\text{free}} \setminus (K \cup B^K_{\text{free}}))$.

**Lemma 3.9.** Reduction Rule 3.4 is correct an can be applied in $O(n + m)$ time.

**Proof.** The running time is clear. It remains to show the correctness. Let $(G, s)$ be the input instance reduced with respect to Reduction Rule 3.1 and let $(G', s)$ be the instance produced by Reduction Rule 3.4. We show that deleting the vertices in $A_{\text{free}} \setminus (K \cup A^K_{\text{free}})$ yields an equivalent instance. It then follows from symmetry that deleting the vertices in $B_{\text{free}} \setminus (K \cup B^K_{\text{free}})$ yields also an equivalent instance.

We first show that if $(G, s)$ is a yes-instance, then also the produced instance $(G', s)$ is a yes-instance. Let $(G, s)$ be a yes-instance and $M_G$ be a maximum matching for $G$. Clearly, $|M_G| \geq s$. Observe that for each removed vertex $a \in A_{\text{free}} \setminus (K \cup A^K_{\text{free}})$ it holds that every vertex $a' \in A^K_{\text{free}}$ is to the right of $a$, that is, $a \prec a'$ and thus $N_{G-X}(a) \subseteq N_{G-X}(a')$. Since $(G, s)$ is reduced with respect to Reduction Rule 3.1, it follows that $|M_{G-X}| \geq |M_G| - k$. Thus, there exist at most $k$ augmenting paths for $M_{G-X}$ in $G$. If none of these augmenting paths ends in a vertex $a \in A_{\text{free}} \setminus (K \cup A^K_{\text{free}})$, then all augmenting paths exist also in $G'$ and thus $(G', s)$ is a yes-instance. If one of these augmenting paths, say $P$, ends in $a$, then at least one vertex $a' \in A^K_{\text{free}}$ is not endpoint of any of these augmenting paths. Since $a \notin K$ it follows from Lemma 3.3 that the neighbor $b$ of $a$ on $P$ is indeed in $B \setminus X$. Since $N_{G-X}(a) \subseteq N_{G-X}(a')$, it follows that $\{a', b\} \in E$ and thus we can replace $a$ by $a'$ in the augmenting path. By exhaustively applying the above exchange argument, it follows that we can assume that none of the augmenting paths uses a vertex in $A_{\text{free}} \setminus (K \cup A^K_{\text{free}})$. Thus, all augmenting paths are also contained in $G'$ and hence the resulting instance $(G', s)$ is still a yes-instance.

Finally observe that if $(G', s)$ is a yes-instance, then also $(G, s)$ is a yes-instance, since $G'$ is a subgraph of $G$. Thus any matching of size $s$ in $G'$ is also a matching in $G$. Hence $(G, s)$ is a yes-instance.

**Theorem 3.10.** Matching on bipartite graphs admits a cubic-vertex kernel with respect to the vertex deletion distance to chain graphs. The kernel can be computed in linear time.

**Proof.** Let $(G, s)$ be the input instance with $G = (V, E)$, the two partitions $V = A \cup B$, and $X \subseteq V$ such that $G - X$ is a chain graph. If $X$ is not given explicitly, then use the linear-time factor-four approximation provided in Lemma 3.2 to compute $X$. The kernelization is as follows: First, compute $M_{G-X}$ in linear time with Algorithm 4. Next compute the set $K$. Then, apply Reduction Rules 3.1, 3.3 and 3.4. By Lemmas 3.6 and 3.9, this can be done in linear time. Let $b^K_i$ the leftmost vertex in $K \cap B$ and $a^K_i$ rightmost vertex in $A \cap K$. Let $a^K_i$ and $b^K_i$ be their matched neighbors. Since $|K| \leq 2k$ and we reduced the instance with respect to Reduction Rule 3.3, it follows that the number of vertices between $a^K_i$ and $b^K_i$ as well as the number of vertices between $b^K_i$ and $b^K_i$ is at most $2k^3$, respectively. Furthermore, there are at most $2k$ free vertices left in $V \setminus X$ since we reduced the instance with respect to Reduction Rule 3.4. It remains to upper-bound the number of matched vertices left of $b^K_i$ and right of $a^K_i$.

Observe that all vertices left of $b^K_i$ are matched with respect to $M_{G-X}$. If there are more than $2k$ vertices to the left of $b^K_i$, then do the following: Add four vertices $a_t, b_t, x^K_t, x^K_t$ to $V$. The idea is that $\{a_t, b_t\}$ should be an edge in $M_{G-X}$ such that $a_t \in A$ and $b_t \in B$ are in $K$ and there is no vertex left of $b_t$. This means we add these vertices to simulate the situation where the leftmost vertex in $B \setminus X$ is also in $K$. To ensure that $a_t$ and $b_t$ are in $K$ and that they are not matched with some vertices in $G$, we add $x^K_t$ and $x^K_t$ to $X$ and make $x^K_t$ respectively $x^K_t$ to their neighbor. In this way, we ensure that there is maximum matching in the new graph that is exactly two edges larger than the maximum matching in the old graph. In this new graph we can then apply Reduction Rule 3.3 to reduce the number of vertices between $b_t$ and $b^K_i$. Formally, we add the
following edges. Add \( \{a_t, x_t^a\}, \{b_t, x_t^b\} \) to \( E \). Add all edges between \( b_t \) and the vertices in \( B \setminus X \). Let \( a \) be the rightmost vertex in \( A_{free}^t \). Then, add edges between \( a \) and \( N_{G-X}(a) \). Set \( b' \prec b_t \) for each \( b' \in B \setminus X \), set \( a \prec a_t \) for each vertex \( a \in A_{free} \), and set \( a_t \prec a' \) for each matched vertex in \( A \setminus X \). Furthermore, add \( \{a_t, b_t\} \) to \( MG-X \) and add \( a \) and \( b_t \) to \( K \). Finally, increase \( s \) by two. Next, apply Reduction Rule 3.3 in linear time, then remove \( a_t, b_t, x_t^a, x_t^b \) and reduce \( s \) by two. After this procedure, it follows that there are at most \( 2k \) vertices left of \( b_t^K \). If there are more than \( 2k \) vertices right of the rightmost vertex \( a^K_t \) in \( A \cap K \), then use the same procedure as above. Thus, the total number of vertices in the remaining graph is at most \( |X| + 2k+4k^3 = O(k^3) \). Furthermore, observe that adding and removing the four vertices as well as applying Reduction Rule 3.3 can be done in linear time. Thus, the overall running time of the kernelization is \( O(n+m) \).

Applying the \( O(n^{2.5}) \)-time algorithm for Bipartite Matching [15] on the kernel yields:

**Corollary 3.11.** Matching can be solved in \( O(k^{7.5} + n + m) \) time, where \( k \) is the vertex deletion distance to chain graphs.

### 4 Conclusion

We focussed on kernelization results for Matching. There remain numerous challenges for future research as discussed in the end of this concluding section. First, however, let us discuss the closely connected issue of FPTP algorithms for Matching. There is a generic augmenting path-based approach to provide FPTP algorithms for Matching: To begin with, note that one can find an augmenting path in linear time [2, 11, 17]. Now the solving algorithm for Matching parameterized by some vertex deletion distance \( k \) works as follows:

1. Use a constant-factor linear-time (approximation) algorithm to compute a vertex set \( X \) such that \( G-X \) is a “trivial” graph (where Matching is linear-time solvable).
2. Compute in linear time an initial maximum matching \( M \) in \( G-X \).
3. Start with \( M \) as a matching in \( G \) and increase the size at most \( |X| = k \) times to obtain in \( O(k \cdot (n + m)) \) time a maximum matching for \( G \).

From this we can directly derive that Matching can be solved in \( O(k(n + m)) \) time, where \( k \) is one of the following parameters: feedback vertex number, feedback edge number, vertex cover number. Bipartite Matching can be solved in \( O(k(n + m)) \) time, where \( k \) is the vertex deletion distance to chain graphs. Using our kernelization results, the multiplicative dependence of the running time on parameter \( k \) can now be made an additive one. For instance, in this way the running time for Bipartite Matching parameterized by vertex deletion distance to chain graphs “improves” from \( O(k(n + m)) \) to \( O(k^{7.5} + n + m) \).

We conclude with listing some questions and tasks for future research. Can the size or the running time of the kernel with respect to feedback vertex set (see Section 2) be improved? Is there a linear time computable kernel for Matching parameterized by the treewidth \( t \) (assuming that \( t \) is given)? This would complement the recent randomized \( O(t^4 n \log n) \) time algorithm [9]. Can one extend the kernel of Section 3 from Bipartite Matching to Matching parameterized by the distance to chain graphs? These are only three very concrete questions. There are numerous more, including further parameterizations or any form of kernel lower bound results. Finally, will Matching become the “drosophila” of FPTP studies akin to Vertex Cover for classical FPT studies was? We hope we gave some reasons for believing that.

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