A new blow-up criterion for the $N - abc$ family of Camassa-Holm type equation with both dissipation and dispersion

Abstract: In this paper, we investigate the Cauchy problem for the $N - abc$ family of Camassa-Holm type equation with both dissipation and dispersion. Furthermore, we establish the blow-up result of the positive solutions in finite time under certain conditions on the initial datum. This result complements the early one in the literature, such as [E. Novruzov, Blow-up phenomena for the weakly dissipative Dullin-Gottwald-Holm equation, J. Math. Phys. 54 (2013), no. 9, 092703, DOI 10.1063/1.4820786] and [Z.Y. Zhang, J.H. Huang, and M.B. Sun, Blow-up phenomena for the weakly dissipative Dullin-Gottwald-Holm equation revisited, J. Math. Phys. 56 (2015), no. 9, 092701, DOI 10.1063/1.4930198].

Keywords: blow-up, $N - abc$ family of Camassa-Holm type equation, dissipation, dispersion

1 Introduction

Differential equations and dynamical modeling have attracted some attention from many researchers as a result of their potential applications in fields of biology [3–6], physics [7–9], engineering [10, 11], information technology and so forth [12–14]. Since the seminal work by Camassa and Holm [15], Camassa-Holm type equations have been intensively investigated. In this paper, we consider the Cauchy problem for the $N - abc$ family of Camassa-Holm type equation with both dissipation and dispersion

$$\begin{align*}
    u_t - u_{txx} - cu^N u_{xxx} - bu^{N-1}u_x u_{xx} + au^N u_x + k(1 - \partial_x^2)u_x + \lambda(1 - \partial_x^2)u &= 0, \quad x \in \mathbb{R}, \quad t > 0, \\
    u(x, 0) &= u_0(x), \quad x \in \mathbb{R},
\end{align*}$$

where $N \in \mathbb{Z}^+$, $N \geq 2$, $k, \lambda \geq 0$, and $k, \lambda$ are dissipation and dispersion coefficients respectively. $a, b, c$ are positive constants and $a = b + c$.

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When \( c = 1, a = b + 1, k = \lambda = 0 \), the first equation of (1.1) becomes
\[
u_t - u_{txx} - u^N u_{xxxx} - bu^{N-1}u_x u_{xx} + (b+1)u^N u_x = 0, \quad x \in \mathbb{R}, \ t > 0.
\]
(1.2)

Eq. (1.2) was first investigated by Himonas and Holliman [16] and they proved the local well-posedness and the nonuniform dependence of its Cauchy problem in Sobolev space \( H^s \) with \( s > \frac{3}{2} \). In [17], Zhou and Mu studied the persistence properties of strong solutions and the existence of its weak solutions of (1.2). Later on, Himonas and Mantzavinos [18] showed well-posedness in \( H^s \) with \( s > \frac{3}{2} \). They also provided a sharpness result on the data-to-solution map and proved that it is not uniformly continuous from any bounded subset of \( H^s \) into \( C([0, T); H^s) \). Eq. (1.2) was also studied by Barostichi, Himonas and Petronilho [19] and they exhibited a power series method in abstract Banach spaces equipped with analytic initial data, and established a Cauchy-Kovalevsky type theorem.

It is important to note that (1.1) is an evolution equation with \((N + 1)\)-order nonlinearities and includes three famous integrable dispersive equations: the Camassa-Holm (CH) equation, the Degasperis-Procesi (DP) equation and the Novikov equation (NE).

As \( c = 1, N = 1, b = 2, a = 3, k = \lambda = 0 \), (1.1) becomes the well-known CH equation. The local well-posedness of Cauchy problem of the CH equation has extensively been investigated in [20]. It was shown that there exist global strong solutions to the CH equation [20] and finite time blow-up strong solutions to the CH equation [20, 21]. The existence and uniqueness of global weak solutions to the CH equation were studied in [22].

As \( c = 1, N = 1, b = 3, a = 4, k = \lambda = 0 \), (1.1) reads the DP equation in [23]. It is another integrable peakon model with quadratic nonlinearity, but with \( 3 \times 3 \) Lax pairs [24]. The local well-posedness, global existence and blow-up phenomena of the DP equation was studied in [25–28].

As \( c = 1, N = 2, b = 3, a = 4, k = \lambda = 0 \), (1.1) reads the NE in [29], which is also integrable peakon model with \( 3 \times 3 \) Lax pairs and the peakon solution \( u(x, t) = \sqrt{\alpha} e^{-|x-ct|} \) with \( c > 0 \). The most difference between the NE and the CH and DP equations is that the former one has cubic nonlinearity and the latter ones have quadratic nonlinearity. The local well-posedness, global existence and blow-up phenomena of the NE was studied in [29–34].

It is well known that it is difficulty to avoid energy dissipation in a real world. Thus it is reasonable to study the model with energy dissipation in propagation of nonlinear waves, see [35–38]. Recently, Wu and Yin [39] investigated the blow-up, blow-up rate and decay of solutions to the weakly dissipative periodic CH equation (i.e (1.1) with \( N = 1, c = 1, b = 2, a = 3, k = 0 \)). Thereafter, they also studied the blow-up and decay of solutions to weakly dissipative non-periodic CH equation (i.e (1.1) with \( N = 1, c = 1, b = 3, a = 4, k = 0 \) [40]. Hu and Yin [41] investigated the blow-up, blow-up rate of solutions to weakly dissipative periodic rod equation. Later on, Hu [42] discussed the global existence and blow-up phenomena for a weakly dissipative periodic two component CH system. Zhou, Mu and Wang [43] considered the weakly dissipative gcH equation (i.e (1.1) with \( c = 1, a = b + 1, k = 0 \)). Recently, Novruzov [1] studied the Cauchy problem for the weakly dissipative Dullin-Gottwald-Holm (DGH) equation (i.e (1.1) with \( N = 1, c = 1, b = 2, a = 3 \)) and establish certain conditions on the initial datum to guarantee that the corresponding positive strong solutions blow up in finite time. The same equation for arbitrary solution has been considered in [44]. Authors showed the simple conditions on the initial data that lead to the blow-up of the solutions in finite time or guarantee that the solutions exist globally. Later on, Zhang et al. [2] improved the results of [1]. In [45], Novruzov extended the obtained "blow-up" result to the DGH equation under some conditions on the initial data. This issue is extensively studied, e.g. in [46–48].

2 Preliminaries

In this section, we recall some useful results in order to achieve our aim.
Let us first present the local well-posedness of Cauchy problem for (1.1). Thus, we can rewrite (1.1) in the equivalent form. Let \( y = u - u_{xx} \). Then (1.1) becomes

\[
\begin{align*}
  y_t + y_x(cu^N + k) + \frac{b}{N}y(u^N)_x + \lambda y &= 0, \quad x \in R, \quad t > 0, \\
  u(x, 0) &= u_0(x), \quad x \in R.
\end{align*}
\] (2.1)

Notice that \( G(x) = \frac{1}{2}e^{-|x|} \) is the kernel of \((1 - \partial_x^2)^{-1}\). Then \((1 - \partial_x^2)^{-1}f = G \ast f \) for all \( f \in L^2(R) \) and \( G \ast y = u \). Hence, (2.1) can be reformulated in the form as follows:

\[
\begin{align*}
  u_t + (cu^N + k)u_x + \partial_x G \ast h + G \ast h &= 0, \quad x \in R, \quad t > 0, \\
  u(x, 0) &= u_0(x), \quad x \in R,
\end{align*}
\] (2.2)

where

\[
\begin{align*}
  h &= \frac{b}{N+1}u^{N+1} + \frac{3cN - b}{2}u^{-1}u_x^2 - \lambda u, \\
  g &= \frac{(N-1)(b - cN)}{2}u^{N-2}u_x^3 + \lambda u.
\end{align*}
\]

The local well-posedness of Cauchy problem for (1.1) with the initial data \( u_0(x) \in H^s, \ s > \frac{1}{2} \), can be obtained by applying the Kato’s theory, see [2, 49]. It is easy to see that some results hold for (1.1). So, we omit the further details and show corresponding result directly.

**Lemma 2.1.** Given \( u_0(x) \in H^s, \ s > \frac{1}{2} \), there exist a maximal \( T = T(u_0, k) > 0 \) and a unique solution \( u \) to (1.1), such that \( u = u(\cdot, u_0) \in C([0, T); H^s(R)) \cap C^1([0, T]; H^{s-1}(R)) \). Moreover, the solution depends continuously on the initial data, i.e., the mapping

\[
u_0 \rightarrow u(\cdot, u_0) : H^s \rightarrow C([0, T); H^s(R)) \cap C^1([0, T); H^{s-1}(R))
\]
is continuous and the maximal time of existence \( T > 0 \) can be chosen to be independent of index \( s \).

The following lemma gives necessary and sufficient condition for the blow-up of the solution.

**Lemma 2.2.** Given \( u_0(x) \in H^s, \ s > \frac{1}{2} \), then the solution \( u \) of (1.1) blows up in the finite time \( T < +\infty \), if and only if

\[
\lim_{t \to T^-} \inf \left\{ \inf_{x \in R} u(t, \cdot) \right\} \rightarrow +\infty,
\]
or

\[
\lim_{t \to T^-} \inf \left\{ \inf_{x \in R} u_x(t, \cdot) \right\} \rightarrow +\infty.
\]

**Proof.** Indeed, the above result follows by standard manner in [49]. Assume \( u_0 \in H^s \) for some \( s \in N, \ s \geq 2 \). Multiplying both sides of the first equation of (2.1) by \( 2y = 2u - 2u_{xx} \) and integrating by parts with respect to \( x \), we get

\[
\begin{align*}
  2\int_{\mathbb{R}} yy_t dx + 2\int_{\mathbb{R}} y y_x (cu^N + k) dx &= 2\int_{\mathbb{R}} \frac{b}{N}y^2(u^N)_x dx + 2\lambda \int_{\mathbb{R}} y^2 dx = 0,
\end{align*}
\]

that is,

\[
\begin{align*}
  2\int_{\mathbb{R}} yy_t dx &= -2c\int_{\mathbb{R}} yy_x u^N dx - 2\int_{\mathbb{R}} \frac{b}{N}y^2(u^N)_x dx - 2k\int_{\mathbb{R}} yy_x dx - 2\lambda \int_{\mathbb{R}} y^2 dx \\
  &= (1 - \frac{2b}{N})\int_{\mathbb{R}} y^2(u^N)_x dx - 2\lambda \int_{\mathbb{R}} y^2 dx \\
  &= (N - 2b)\int_{\mathbb{R}} y^2 u^{N-1} u_x dx - 2\lambda \int_{\mathbb{R}} y^2 dx,
\end{align*}
\] (2.3)

which implies the following result:

- If \( \|u\|_{L^\infty} \) and \( \|u_x\|_{L^\infty} \) are bounded on \([0, T]\), that is, there exists a positive constant \( K \), such that \( \|u\|_{L^\infty}, \|u_x\|_{L^\infty} \leq K \).

Then, based on the above arguments and noticing (2.3), we have

\[
\frac{d}{dt} \int_{\mathbb{R}} y^2 dx \leq C(K^N + 1) \int_{\mathbb{R}} y^2 dx,
\] (2.4)

where \( C \) is a positive constant. On one hand, we observe that

\[
\|u\|_{H^s}^2 \leq \|y\|_{L^2}^2 \leq 2\|u\|_{H^s}^2.
\] (2.5)
Using the Gronwalls inequality, from (2.4) and (2.5), we obtain

$$||u||_{H^2}^2 \leq ||y||_{L^2}^2 \leq 2e^{C(R^s+1)}||u_0||_{L^2}^2,$$

which implies that the $H^2$-norm of the solution to Eq. (2.1) does not blow up in a finite time. On the other hand, by Sobolev’s imbedding theorem, if

$$\lim \inf_{t \to T} \inf_{x \in R} u(t, \cdot) \to +\infty, \text{ or } \lim \inf_{t \to T} \inf_{x \in R} u_s(t, \cdot) \to +\infty.$$

Then the solution will blow up in a finite time. By density argument, we know that Lemma 2.2 holds for all $s > \frac{1}{2}$. Thus, this finishes the proof of Lemma 2.2.

Remark 2.1. When $N = 1, 2$, we refer to the proof of Theorem 3.1 in [30]. In this sense, when $N \geq 2$, this result improves their results.

Remark 2.2. Lemma 2.2 covers Lemma 5.1 in [50].

Consider now the following initial value problem

$$\begin{align*}
q_t(t, x) &= cu^N(t, x) + k, x \in R, t \in [0, T), \\
u(x, 0) &= u_0(x), x \in R,
\end{align*}$$

(2.6)

where $u(x, t)$ is the corresponding strong solution to (1.1).

After simple computations and solving (2.6), we get the following lemma.

Lemma 2.3. Let $u_0(x) \in H^s, s \geq 3$, and let $T > 0$ be the maximal existence time of the solution $u$ to (1.1). Then, we have

$$y(t, q_t(t, x))q_x^b(t, x) = y_0(x)e^{-\lambda t}$$

which implies

$$e^{-\lambda t}||y_0||_{L^\infty}^\frac{n}{b} = ||y||_{L^\infty}^\frac{n}{b}.$$

In particular, if $N = 2b$, we have

$$e^{-\lambda t}||y_0||_{L^2} = ||y||_{L^2}.$$

Proof. It follow from (1.1) and (2.1) that

$$\begin{align*}
\frac{d}{dt} y(t, q_t(t, x))q_x^b(t, x) &= (y_t + y_xq_t)q_x^b + \frac{b}{N}yq_{x}^{b-1} q_t \\
&= (y_t + y_xq_t)q_x^b + \frac{b}{N}yq_{x}^{b-1}(u^N)xq_x \\
&= (y_t + y_x(cu^N + k) + \frac{b}{N}y(u^N)_xq_x^b \\
&= -\lambda yq_x^b,
\end{align*}$$

which implies

$$y(t, q_t(t, x))q_x^b(t, x) = y_0(x)e^{-\lambda t}.$$

Thus, setting $\xi = q(t, x)$, we arrive at

$$e^{-\lambda t}||y_0||_{L^\infty}^\frac{n}{b} = ||y(t, q(t, \cdot))q_x^b(t, \cdot)||_{L^\infty}^\frac{n}{b} = \int_R |y(t, q(t, x))q_x^b(t, x)|dx = \int_R |y(t, \xi)|^\frac{n}{b}d\xi.$$

Obviously, letting $N = 2b$ leads to $e^{-\lambda t}||y_0||_{L^2} = ||y||_{L^2}$. This completes the proof of Lemma 2.3.

Finally, let us now give the following lemma which will be used in the sequel.

Lemma 2.4. Let $u_0(x) \in H^s, \ s \geq 3, c = \frac{b}{N+1}$ and let $T > 0$ be the maximal existence time of the solution $u$ to (1.1). Then, we have

$$||u||_{H^1} = e^{-2\lambda t}||u_0||_{H^1}.$$
Proof. Multiplying both sides of Eq. (2.1) by \( u \), we get
\[
\int_R u y_t dx + c \int_R u u^N y_x dx + k \int_R u y_x dx + b \int_R u(u^N)_x y dx + \lambda \int_R u y dx = 0. \tag{2.7}
\]
Noticing that
\[
k \int_R u y_x dx = 0
\]
and
\[
c \int_R u u^N y_x dx + b \int_R u(u^N)_x y dx = (c - b/N + 1) \int_R (u^N) u_x y dx = 0,
\]
we obtain
\[
\frac{1}{2} \frac{d}{dt} \int_R (u^2 + u_x^2) dx + \lambda \int_R (u^2 + u_x^2) dx = 0 \Leftrightarrow \frac{d}{dt} ||u||^2_{H^1} + 2\lambda ||u||^2_{H^1} = 0,
\]
which implies the desired result in the lemma.

3 Main result

We are now in position to state our main result.

Theorem 3.1. Let \( b = c(N + 1) \). \( u_0(x) \in H^s \), \( s \geq 3 \), is such that \( y_0 = (1 - \partial^2_x)u_0 \) satisfies \( y_0(x) \leq 0 \) on \( [x_0, \infty) \) for some point \( x_0 \in R \) and condition
\[
(\int_{x_0}^{\infty} |u_0|^{1-a} e^{x_0-x} dx)^{\frac{1}{a-1}} < \frac{1 + \alpha}{2(\lambda v^N + \lambda(\alpha - 1))} \tag{3.1}
\]
is satisfied. Then the corresponding positive solution \( u(t, x) \) to (1.1) blows up in finite time. Here
\[
\nu \leq \sqrt{\frac{1}{2} ||u_0||_{H^1}}, 1 < \alpha < 2.
\]

Proof. We shall give the proof by contradiction. We assume that it is not true and solutions exist globally. That is, there exists constant \( C \) such that \( u_x \geq -C \) (due to Lemma 2.2).

Observing that \( u = G \ast y \) with \( G(x) = \frac{1}{2} e^{-|x|}, x \in R \), we have
\[
u(t, x) = \frac{1}{2} e^{-x} \int_{-\infty}^{x} e^{\xi} y(t, \xi) d\xi + \frac{1}{2} e^{x} \int_{x}^{\infty} e^{-\xi} y(t, \xi) d\xi,
\]
\[
u_x(t, x) = \frac{1}{2} e^{-x} \int_{-\infty}^{x} e^{\xi} y(t, \xi) d\xi + \frac{1}{2} e^{x} \int_{x}^{\infty} e^{-\xi} y(t, \xi) d\xi.
\]
So, we have
\[
u_x(t, x) = u(t, x) + e^{x} \int_{x}^{\infty} e^{-\xi} y(t, \xi) d\xi.
\]
For \( t \in [0, T) \), \( q(t, \cdot) \) is the increasing diffeomorphism of the line. From Lemma 2.3, we deduce that \( y(t, x) \leq 0 \), \( x \geq q(t, x_0) \). Hence, we conclude that \(-u_x(t, x) \succeq u(t, x) \geq 0 \), for \( x \geq q(t, x_0) \). Due to \( u_x \succeq -C \), we get
\[
C \succeq -u_x(t, x) \succeq u(t, x) > 0. \tag{3.2}
\]
Differentiating (2.2) with respect to \( x \) and noticing that
\[
\partial^2_x G \ast f = G \ast f - f,
\]
we have
\[ \frac{d}{dt}(-u_x) = (c - \frac{3N-b}{2})u^{N-1}u_x^2 + cu^nu_{xx} - k(-u_x)_x - \frac{b}{N+2}u^{N+1} + G \ast h + \partial_x G \ast g + \lambda u_x = 0. \] (3.3)

Multiplying (3.3) by \((1-\alpha)e^{-\alpha}(u_x)^a(1 < \alpha < 2)\) and integrating over \((q(t, x_0), \theta) (\theta < \infty)\), we have
\[
(1-\alpha) \int_{q(l,x_0)}^\theta (u_x)^a e^{-\alpha} \frac{d}{dt}(-u_x) \, dx
- (1-\alpha)c \int_{q(l,x_0)}^\theta (u_x)^a (-u_x)_x e^{-\alpha} \, dx
= (1-\alpha) \int_{q(l,x_0)}^\theta \left[ (\lambda u - \frac{b}{N+2})u^{N+1}(-u_x)^2 - \alpha e^{-\alpha} \right] \, dx
- (1-\alpha) \int_{q(l,x_0)}^\theta \left[ (\alpha e^{-\alpha} - \lambda u)(-u_x)_x \right] \, dx
+ \lambda(1-\alpha) \int_{q(l,x_0)}^\theta u(-u_x)^a e^{-\alpha} \, dx
\]

Next, we shall estimate \(I_j(j = 1, 2, \ldots, 7)\) in (10). First, integrating by parts, we have
\[
I_2 = -(1-\alpha)c \int_{q(l,x_0)}^\theta \left[ (\lambda u - \frac{b}{N+2})u^{N+1}(-u_x)_x e^{-\alpha} \right] \, dx
- (1-\alpha) \int_{q(l,x_0)}^\theta \left[ (\alpha e^{-\alpha} - \lambda u)(-u_x)_x \right] \, dx
+ \lambda(1-\alpha) \int_{q(l,x_0)}^\theta u(-u_x)^a e^{-\alpha} \, dx
\]

By (3.2) and \(\|G\|_2 \leq \|\partial_x G\|_{L^1} = 1\), we get
\[
I_5 = -(1-\alpha) \int_{q(l,x_0)}^\theta \left[ (\lambda u - \frac{b}{N+2})u^{N+1}(-u_x)_x e^{-\alpha} \right] \, dx
+ \lambda(1-\alpha) \int_{q(l,x_0)}^\theta u(-u_x)^a e^{-\alpha} \, dx
\]

Substituting (3.5)-(3.9) into (3.4) and observing that
\[ q(t, x_0) = cu^N(q(t, x_0)) + k \] (by (2.6)),
we conclude
\[
\frac{d}{dt} \int_{q(l,x_0)} u(-u_x)^a e^{-\alpha} \, dx \leq [(1-\alpha) - c - N - c + (1-\alpha) \frac{(N-1)(b-cN)}{2} \int_{q(l,x_0)}^\theta u^{N+1}(-u_x)^2 e^{-\alpha} \, dx
+ \lambda(1-\alpha) \int_{q(l,x_0)}^\theta u(-u_x)^a e^{-\alpha} \, dx
\]

Let \(\beta = k - \lambda(\alpha - 1)\). By \(c = \frac{b}{N+2} \iff b = c(N+1)\) (see Lemma 2.4) and \(0 < u \leq \|u\|_{L^\infty}\), we have
\[
\frac{d}{dt} \int_{q(l,x_0)} u(-u_x)^a e^{-\alpha} \, dx + \beta \int_{q(l,x_0)} e^{-\alpha} \, dx \leq \gamma \int_{q(l,x_0)} e^{-\alpha} \, dx
\] (3.10)
Note that $\gamma = c(N + 1)\|u\|_{L^\infty}^{N-1}$.

Setting
\[
J(t) = \frac{\theta}{q(t,x_0)} (-u_x)^{1-a} e^{-x} dx,
\]
\[
K(t) = \left( \int_{q(t,x_0)}^{\theta} e^{-x} dx \right)^{-1} = (e^{-q(t,x_0)} - e^{-\theta})^{-1}.
\]

By (2.6), we have
\[
K = (e^{-q(t,x_0)} - e^{-\theta})^{-1} = (e^{-c \int_{t}^{\theta} u^\alpha(r,q(t,x_0))dr - kt - x_0} - e^{-\theta})^{-1}
\leq (e^{-c t \max u^\alpha - kt - x_0} - e^{-\theta})^{-1}.
\]

Note that $\theta$ can be taken to be a sufficient large. Due to
\[
\frac{2 - a}{1 - a} < 0 \big( 1 < a < 2 \big)
\]
and
\[
\int_{q(t,x_0)}^{\theta} K(t) e^{-x} dx = 1
\]
and by Jensen’s inequality, we arrive at
\[
\left( \int_{q(t,x_0)}^{\theta} (-u_x)^{1-a} K(t) e^{-x} dx \right)^{\frac{2-a}{1-a}} \leq \int_{q(t,x_0)}^{\theta} (-u_x)^{2-a} K(t) e^{-x} dx.
\]

Thus, we conclude that
\[
\frac{d}{dt} I + \beta J \leq -\gamma \frac{1+a}{2} \frac{1}{K} \int_{q(t,x_0)}^{\theta} (-u_x)^{2-a} e^{-x} dx
\leq -\gamma \frac{1+a}{2} \int_{q(t,x_0)}^{\theta} (-u_x)^{1-a} Ke^{-x} dx \frac{1}{K}^{\frac{a}{2-a}}
\leq -\gamma \frac{1+a}{2} \frac{1}{K} \frac{e^{\theta t}}{t^{\frac{a}{2-a}}}.
\]

Multiplying both sides of (3.11) by $\frac{e^{\theta t}}{t^{\frac{a}{2-a}}}$, we obtain
\[
\frac{d}{dt} \int_{q(t,x_0)}^{\theta} e^{\frac{\beta}{\alpha-1} t} + \frac{\beta}{\alpha-1} \int_{q(t,x_0)}^{\theta} \leq -\gamma \frac{1+a}{2(\alpha-1)} \frac{1}{K} \frac{1}{t^{\frac{a}{2-a}}}.
\]

That is,
\[
\frac{d}{dt} \frac{e^{\theta t}}{t^{\frac{a}{2-a}}} \leq -\gamma \frac{1+a}{2(\alpha-1)} \frac{e^{\theta t}}{K} \left( \frac{1}{t^{\frac{a}{2-a}}} \right).
\]

Integrating with respect to $t$, we have
\[
e^{\frac{\beta}{\alpha-1} t} \leq -\gamma \frac{1+a}{2} \int_{0}^{t} \frac{e^{\theta t}}{K} \left( \frac{1}{t^{\frac{a}{2-a}}} \right) d\tau + e^{\frac{\beta}{\alpha-1} t}(0).
\]

Noticing that $\theta$ can tend to $\infty$, we obtain
\[
L(t) \leq e^{-\frac{\alpha}{2-a}} t \left[ -\gamma \frac{1+a}{2} \int_{0}^{t} \frac{e^{\theta t}}{e^{\frac{\beta}{\alpha-1} (\alpha-1)}} d\tau + L(0) \right],
\]
where $L(t) = \int_{q(t,x_0)}^{\theta} (-u_x)^{1-a} e^{-x} dx$.  

(3.12)
Hence, from (3.13), we have
\[ L^\tau\gamma \leq e^{-\frac{\beta}{\alpha}\tau} \left[ -\frac{1}{2} e^{\frac{\beta}{2}\tau} \int_0^T e^{\frac{\beta}{2}\tau} \cdot \frac{1}{\alpha} d\tau + L^\tau\gamma(0) \right] \]
\[ \leq e^{-\frac{\beta}{\alpha}\tau} \left[ -\frac{1}{2} e^{\frac{\beta}{2}\tau} \int_0^T e^{\frac{\beta}{2}\tau} \cdot \frac{1}{\alpha} d\tau + L^\tau\gamma(0) \right] \]
\[ = e^{-\frac{\beta}{\alpha}\tau} \left[ -\frac{1}{2} e^{\frac{\beta}{2}\tau} \int_0^T e^{\frac{\beta}{2}\tau} \cdot \frac{1}{\alpha} d\tau + L^\tau\gamma(0) \right] \]
\[ \leq e^{-\frac{\beta}{\alpha}\tau} \left[ -\frac{1}{2} e^{\frac{\beta}{2}\tau} \cdot e^{\alpha/\gamma} \right] , \]
where \( \nu \leq \sqrt{\frac{1}{2}||u_0||^2} \) by Lemma 2.4.

On the other hand,
\[ \frac{1}{2} e^{-\frac{\beta}{2}\tau} \cdot e^{\frac{\beta}{2}\tau} = \frac{1}{2} \rightarrow \frac{1}{2} \cdot e^{-\frac{\beta}{2}\tau} \cdot e^{\frac{\beta}{2}\tau} = \frac{1}{2} \alpha - \frac{1}{2} \frac{1}{e^{\alpha/\gamma}} \]
and by condition (3.1). Thus, we get
\[ L^\tau\gamma(0) \leq \frac{(1 + \alpha)^{\gamma}}{2(cv^N + \lambda(\alpha - 1))} e^{\frac{\beta}{2}\tau} . \]

Hence, from (3.13), we have \( L \to 0 \) as
\[ t \to \frac{1}{cv^N + \lambda(\alpha - 1)} \ln\left[ 1 - \frac{2}{(1 + \alpha)^{\gamma}} \right] L^\tau\gamma(0)(cv^N + \lambda(\alpha - 1)) e^{\frac{\beta}{2}\tau} ] \]
That is, there exists a sequence \( (t_n, x_n) \) such that \( -u_x(t_n, x_n) \to -\infty \) as \( t \to T \) which contradicts with (3.2).
Thus, our main result is completed.

Acknowledgement: This work was supported by Scientific Research Fund of Hunan Provincial Education Department Nos.18A325, 17A087, 17B113, 17C0711, NNSF of China Grant Nos. 11671101, 11926205, Natural Science Foundation of Hainan Province No. 119M5036. Also, this work was partially supported by Hunan Provincial Key Laboratory of Mathematical Modeling and Analysis in Engineering of Changsha University of Science and Technology Grant No. 018MMAEZD191 and NNSF of China Grant Nos. 71471020, 51839002.

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