Quantum Cosmological Multidimensional Einstein-Yang-Mills Model in a $\mathbb{R} \times S^3 \times S^d$ Topology

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ABSTRACT

The quantum cosmological version of the multidimensional Einstein-Yang-Mills model in a $\mathbb{R} \times S^3 \times S^d$ topology is studied in the framework of the Hartle-Hawking proposal. In contrast to previous work in the literature, we consider Yang-Mills field configurations with non-vanishing time-dependent components in both $S^3$ and $S^d$ spaces. We obtain stable compactifying solutions that do correspond to extrema of the Hartle-Hawking wave function of the Universe. Subsequently, we also show that the regions where 4-dimensional metric behaves classically or quantum mechanically (i.e. regions where the metric is Lorentzian or Euclidean) will depend on the number, $d$, of compact space dimensions.
1 Introduction

The issue of compactification is central in multidimensional theories of unification, such as generalized Kaluza – Klein theories, Supergravity and Superstring theories. Consistency with known phenomenology requires that the extra dimensions in these theories are Planck size and stable. A necessary condition for the latter is the presence of matter with repulsive stresses to counterbalance the collapsing thrust of gravity. For this purpose, magnetic monopoles [1], Casimir forces [2] and Yang-Mills fields [3, 4] have been considered. The situation with Yang-Mills fields is particularly interesting as it illustrates well the importance of considering non-vanishing external-space components of the gauge fields, a point that has been disregarded in previous work in the literature. In fact, it was shown in Ref. [4] that it is precisely this feature that renders compactifying solutions classically as well as semiclassically stable.

The main motivation for considering our study of compactification in the context of quantum cosmology lies in ascertaining how this process takes place. Indeed, this is crucial for extracting classical predictions from any multidimensional unifying theories. In fact, no cosmological description can be considered complete till specifying the set of initial conditions for integration of the classical equations of motion. Furthermore, since the quantum cosmological approach of Hartle and Hawking [5] allows for a well defined programme for establishing this set of initial conditions, it is quite natural to extend this approach to the study of the issue of compactification in higher-dimensional theories. This programme has been already applied to many different quantum models of interest such as massive scalar fields [6], Yang-Mills fields [7], massive vector fields [8] as well as in supersymmetric models (see Ref. [9] for a review and a complete set of references) and to the lowest order gravity-dilaton theory arising from string theory [10]. The generalization of the Hartle-Hawking programme to higher spacetime dimensions has been considered previously for the 6-dimensional Einstein-Maxwell theory [11], for gravity coupled with a $(D - 4)$th rank antisymmetric tensor field [12], where the stability of compactification was achieved thanks to the presence of a magnetic monopole type configuration, and also to 11-dimensional supergravity [13].

In this work a rather general and realistic setting to study the compactification process is considered in the context of Einstein-Yang-Mills multidimensional model of Ref. [4] with an $SO(N)$ gauge field in $D = 4 + d$ dimensions and an homogeneous and (partially) isotropic spacetime with a $\mathbb{R} \times S^3 \times S^d$ topology. We aim to study the quantum mechanics of the coset compactification of the $D$-dimensional spacetime $\mathcal{M}^D$

$$\mathcal{M}^D = \mathbb{R} \times G^{\text{ext}}/H^{\text{ext}} \times G^{\text{int}}/H^{\text{int}}, \quad (1)$$

where $G^{\text{ext(int)}} = SO(4)(SO(d + 1))$ and $H^{\text{ext(int)}} = SO(3)(SO(d))$ are respectively the homogeneity and isotropy groups in $3(d)$ dimensions. For this purpose we will seek compactifying solutions of the Wheeler-DeWitt equation for the Einstein-Yang-Mills cosmological model of Ref. [4] in the framework of the Hartle-Hawking proposal.

In contrast to previous work in the literature [11, 12] we consider Yang-Mills field configurations with non-vanishing time-dependent components in both $S^3$ and $S^d$ spaces. We then derive
an effective model by restricting the fields to be homogeneous and isotropic. This construction will allow us to study in detail the issue of compactification, which as discussed in Ref. [4], depends crucially on the contribution of the external gauge field components. Our analysis of the resulting Wheeler-DeWitt equation indicates that the regions where the metric is Lorentzian or Euclidean do depend on the number, $d$, of internal dimensions and on the potentials for the external and internal components of the gauge field. Furthermore, we show that stable compactifying solutions do indeed correspond to the extrema of the wave function of the Universe implying a correlation between compactification of the extra dimensions and expansion of the macroscopic spacetime. We should mention that an attractive feature of our model is that it can be regarded as the bosonic sector of some general unifying theories, implying that possibly most of the conclusions of our quantum mechanical analysis of the compactification process and of its stability will remain valid in those theories as well.

This paper is then organized as follows. In the next section we present our Ans"atze for the metric and for the gauge field (see Refs. [17, 18] for a general discussion) as well as the resulting effective action which is the starting point of our analysis. We also obtain in that section the Wheeler-DeWitt equation of our effective model. In the Section 3 we present and discuss compactifying solutions of the Wheeler-DeWitt equation and in Section 4 we discuss their interpretation. In Section 5 we present our conclusions. We also include an Appendix where the mathematical aspects of extending the Hartle-Hawking proposal to higher-dimensional spacetimes is described, with emphasis in our model where hypersurfaces are of $\Sigma^D_{D-1} \sim S^3 \times S^d$ type.

2 Effective Model and Wheeler-DeWitt Equation

We shall describe in this section our multidimensional Einstein-Yang-Mills quantum cosmological model. Special emphasis will be given to the differences between our model and others present in the literature [11, 12, 13]. Namely, the gauge field in our reduced model will have time-dependent spatial components on the 3-dimensional physical space. This contrasts with previous work on the subject where either static magnetic monopole type configurations, whose only non-vanishing components were the internal $d$–dimensional ones [14, 12], or scalar fields [20] were considered. Our approach provides therefore a somewhat more realistic model to study the influence of higher dimensions on the evolution of the 4-dimensional physical spacetime. In addition, we shall also see how different values for $d$, the number of internal space dimensions, may induce fairly different physical situations.

Our model is derived from the generalized Kaluza-Klein action:

$$S[\hat{g}_{\mu\nu}, \hat{A}_\mu, \hat{\chi}] = S_{\text{gr}}[\hat{g}_{\mu\nu}] + S_{\text{gt}}[\hat{g}_{\mu\nu}, \hat{A}_\mu] + S_{\text{inf}}[\hat{g}_{\mu\nu}, \hat{\chi}] ,$$

with

$$S_{\text{gr}}[\hat{g}_{\mu\nu}] = \frac{1}{16\pi k} \int_{M^D} d\hat{x} \sqrt{-\hat{g}} (\hat{R} - 2\hat{\Lambda}) ,$$

(2)
admiting local coordinates $\hat{\mathcal{M}}$, in this case: 

$$S_{gf}[\hat{g}_{\mu\nu}, \hat{A}_\mu] = \frac{1}{8\ell^2} \int_{\hat{\mathcal{M}}^D} d\hat{x} \sqrt{-\hat{g}} \text{Tr} \hat{F}_{\mu\nu} \hat{F}^{\mu\nu},$$  

(4)

$$S_{\inf}[\hat{g}_{\mu\nu}, \hat{\chi}] = - \int_{\mathcal{M}^D} d\hat{x} \sqrt{-\hat{g}} \left[ \frac{1}{2} (\partial_\mu \hat{\chi})^2 + \hat{U}(\hat{\chi}) \right],$$  

(5)

where $\hat{g}$ is $\det (\hat{g}_{\mu\nu})$, $\hat{g}_{\mu\nu}$ is the $D = 4 + d$ dimensional metric, $\hat{R}$, $\hat{e}$, $\hat{k}$ and $\hat{A}$ are, respectively, the scalar curvature, gauge coupling, gravitational and cosmological constants in $D$ dimensions. In addition, the following filed variables are defined in $\mathcal{M}^D$: $\hat{F}_{\mu\nu} = \partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu + [\hat{A}_\mu, \hat{A}_\nu]$ is the field strenght and $\hat{A}_\mu$ denotes the components of the gauge field, $\hat{\chi}$ is the inflaton responsible for the inflationary expansion of the external space with $\hat{U}(\hat{\chi})$ being the potential for $\hat{\chi}$. We assume that the potential $\hat{U}(\hat{\chi})$ is bounded from below, that it has a global minimum and without loss of generality that $\hat{U}_{\min} = 0$. As first suggested in Refs. [21], the splitting of the internal and external dimensions of space in the generalized Kaluza-Klein theory (4) can have its origin in the spontaneous symmetry breaking process, which is due to vacuum solutions corresponding to a factorization of spacetime in a product of spaces. Assuming that is indeed the case, then:

$$\mathcal{M}^D = M^4 \times I^d,$$  

(6)

$M^4$ being the four-dimensional Minkowski spacetime and $I^d$ a Planck-size $d$–dimensional compact space. For the cosmological setting we are interested in consider instead

$$\mathcal{M}^{4+d} = \mathbf{R} \times G^{\text{ext}}/H^{\text{ext}} \times G^{\text{int}}/H^{\text{int}},$$  

(7)

admiting local coordinates $\hat{x}^\mu = (t, x^i, \xi^m)$, where $\hat{\mu} = 0, 1, \ldots, 3+d; i = 1, 2, 3; m = 4, \ldots, d+3$; where $\mathbf{R}$ denotes a timelike direction and $G^{\text{ext}}/H^{\text{ext}} \left(G^{\text{int}}/H^{\text{int}}\right)$ the space of external (internal) spatial dimensions realized as a coset space of the external (internal) isometry group $G^{\text{ext}} \left(G^{\text{int}}\right)$. We restrict ourselves to spatially homogeneous and (partially) isotropic field configurations, which means that these are symmetric under the action of the group $G^{\text{ext}} \times G^{\text{int}}$. Let the gauge group $\hat{K}$ of the $D$–dimensional theory be a simple Lie group. For definiteness, let us consider the case with the gauge group $\hat{K} = SO(N), N \geq 3 + d$ and

$$\mathcal{M}^{4+d} = \mathbf{R} \times S^3 \times S^d,$$  

(8)

where $S^3(S^d)$ is the $3–(d–)$dimensional sphere. The group of spatial homogeneity and isotropy is, in this case:

$$G^{\text{HI}} = SO(4) \times SO(d + 1),$$  

(9)

while the group of spatial isotropy is

$$H^1 = SO(3) \times SO(d),$$  

(10)

which allows for the alternative realization of $\mathcal{M}^{4+d}$

$$\mathcal{M}^{4+d} = \mathbf{R} \times SO(4)/SO(3) \times SO(d + 1)/SO(d)$$

$$= \mathbf{R} \times [SO(4) \times SO(d + 1)]/[SO(3) \times SO(d)].$$  

(11)
The field configurations associated with the above geometry were described in Ref. \[4\], using the theory of symmetric fields (see also Refs. \[9, 13, 14\]). The most general form of a $SO(4) \times SO(d + 1)$--invariant metric in $E^{d+4}$ as \[8\] reads

$$ \hat{g} = -\tilde{N}^2(t) dt^2 + \tilde{a}^2(t) \sum_{i=1}^{3} \omega_i^2(t) + b^2(t) \Sigma_{m=4}^{d+3} \omega_m^2(t), $$

(12)

where the scale factors $\tilde{a}(t), b(t)$ and the lapse function $\tilde{N}(t)$ are arbitrary non-vanishing functions of time. Moreover, $\omega^\alpha$ denote local moving coframes in $S^3 \times S^d$, $\Sigma_{i=1}^{3} \omega_i^2(t)$ and $\Sigma_{m=4}^{d+3} \omega_m^2(t)$ coincide with the standard metrics $d\Omega^2_3$ and $d\Omega^2_d$ of 3 and $d$--dimensional spheres with local coordinates $(x^i, \xi^m)$, respectively.

The $SO(4) \times SO(d + 1)$--invariant ansatz for the inflaton field $\hat{\chi}$ reads

$$ \hat{\chi}(t, x^i, \xi^m) = \hat{\chi}(t). $$

(13)

As for the $SO(4) \times SO(d + 1)$--symmetric gauge field, the following Ansatz is considered:

$$ \hat{A} = \frac{1}{2} \sum_{p,q=1}^{N-3-d} B^{pq}(t) T_{i}^{(N)}_{3+d+p+3+d+q} dt + \frac{1}{2} \Sigma_{1 \leq i < j \leq 3} T_{ij}^{(N)} \omega_{ij} $$

$$ + \frac{1}{2} \sum_{4 \leq m < n \leq 3} T_{mn}^{(N)} \omega_{m3n3} $$

$$ + \frac{1}{4} \sum_{j,k=1}^{3} \Sigma_{q=1}^{N-3-d} f_{q}(t) T_{j}^{(N)}_{k} $$

$$ + \frac{1}{2} \Sigma_{p=1}^{N-3-d} f_{p}(t) T_{i}^{(N)}_{d+3+p} \omega_{i} $$

$$ + \frac{1}{2} \Sigma_{m=4}^{d+3} \sum_{1 \leq q \leq 3} \Sigma_{p=1}^{N-3-d} g_{q}(t) T_{m}^{(N)}_{d+3+q} \omega_{m} $$

(14)

where $f_{0}(t)f_{p}(t), p = 1, \ldots, N-3-d; g_{q}(t), q = 1, \ldots, N-3-d; B^{pq}(t), 1 \leq p < q \leq N-3-d$ are arbitrary functions and $T_{pq}^{(N)}, 1 \leq p < q \leq N$ are the generators of the gauge group $SO(N)$. We have used the decomposition

$$ \omega = \sum_{\alpha=1}^{d+3} \omega_{\alpha} T_{\alpha} + \sum_{1 \leq i < j \leq 3} \omega_{ij} \frac{T_{ij}^{(4)}}{2} + \Sigma_{1 \leq m < n \leq d} \tilde{\omega}_{mn} \frac{T_{mn}^{(d+1)}}{2} $$

(15)

for the Cartan’s one-form in $S^3 \times S^d$. Here $T_{ij}^{(4)}, T_{mn}^{(d+1)}$ form a basis of the Lie algebra of $G^{\mathrm{HI}}$, $T_{\alpha} = \frac{T_{\alpha}^{(4)}}{2}, \alpha = 1, 2, 3$ and $T_{\alpha} = \frac{T_{\alpha}^{(d+1)}}{2}, \alpha = 4, \ldots, d+3$.

Substituting the Ansätze \[12\], \[13\] and \[14\] into action \[2\] and performing the conformal changes

$$ \tilde{N}^2(t) = \left[ \frac{b_{0}}{b(t)} \right]^d N^2(t), $$

(16)

$$ \tilde{a}^2(t) = \left[ \frac{b_{0}}{b(t)} \right]^d a^2(t), $$

(17)

where $b_{0}$ denotes the equilibrium value of $b$, we obtain a one-dimensional effective reduced action for the functions of time that parametrize the symmetric field configurations \[4\]:

5
\[ S_{\text{eff}} = [a, \psi, f_0, f, g, \chi, N, \hat{B}] = 16\pi^2 \int dtNa^3 \left\{ -\frac{3}{8\pi k} \frac{1}{a^2} \left[ \dot{a} \right]^2 + \frac{3}{32\pi k} \frac{1}{a^2} \left[ \dot{\psi} \right]^2 + \frac{1}{2} \left[ \dot{\chi} \right]^2 \right\} + e^{d\beta\psi} \frac{3}{4e^2a^2} \left( \frac{1}{2} \left[ \frac{f_0}{N} \right]^2 + \frac{1}{2} \left[ D_t f \right]^2 \right) + e^{-2\beta\psi} \frac{d}{dN} \frac{1}{2} \left[ D_t g \right]^2 - W(a, \psi, f_0, f, g, \chi) \right\}, \] (18)

with \( k = \hat{k}/v_d b_0^d, e^2 = \epsilon^2/v_d b_0^d, \beta = \sqrt{16\pi k/d(d + 2)}, v_d \) is the the volume of \( S^d \) for \( b = 1 \), and where we have used \( \dot{\psi} = \beta^{-1} \ln(b/b_0) \) and \( \chi = \sqrt{v_d b_0^d} \chi \) as the dilaton\(^4\) and inflaton fields, respectively. In (18), the dots denote time derivatives and \( D_t \) is the covariant derivative with respect to the remnant \( SO(N - 3 - d) \) gauge field \( \hat{B}(t) \) in \( \mathbb{R} \):

\[ D_t f(t) = \frac{d}{dt} f(t) + \hat{B}(t)f(t), \quad D_t g(t) = \frac{d}{dt} g(t) + \hat{B}(t)g(t). \] (19)

Notice that \( f_0(t), f = \{f_p\} \) represent the gauge field components in the 4-dimensional physical space-time, while \( g = \{g_{pq}\} \) denotes the components in the space \( I^d \) and \( \hat{B} \) is an \( (N - 3 - d) \times (N - 3 - d) \) antisymmetric matrix \( \hat{B} = (B_{pq}) \). The potential \( W \) in (18) is given by

\[ W = e^{-d\beta\psi} \left[ -e^{-2\beta\psi} \frac{1}{16\pi k} \frac{d(d - 1)}{4 b_0^d} + e^{-4\beta\psi} \frac{1}{8e^2} \frac{d(d - 1)}{b_0^d} V_2(g) + \frac{\Lambda}{8\pi k} + U(\chi) \right] + e^{-2\beta\psi} \frac{1}{ab_0^d} \frac{3d}{32e^2} (f \cdot g)^2 + e^{d\beta\psi} \frac{3}{4e^2a^2} V_1(f_0, f), \] (20)

where \( \Lambda = v_d b_0^d \hat{\Lambda}, U(\chi) = v_d b_0^d \hat{U} \left( \dot{\chi}/v_d b_0^d \right) \) and

\[ V_1(f_0, f) = \frac{1}{8} \left[ (f_0^2 + f^2 - 1)^2 + 4 f_0^2 f^2 \right], \]

\[ V_2(g) = \frac{1}{8} \left( g^2 - 1 \right)^2, \] (21) (22)

are related with the external and internal components of the gauge field, respectively. Variables \( N \) and \( \hat{B} \) are Lagrange multipliers associated with the symmetries of the effective action (18). The lapse function \( N \) is associated with the invariance of \( S_{\text{eff}} \) under arbitrary time reparametrizations, while \( \hat{B} \) is connected with the local remnant \( SO(N - d - 3) \) gauge invariance. The equations of motion for the physical variables \( a, \psi, \chi, f_0, f, g \) can be found in Ref.\(^4\).

The canonical conjugate momenta associated with the canonical variables in model (18) are given by:

\[ \pi_a = -\frac{12\pi}{k} a \frac{\dot{a}}{N}, \quad \pi_\psi = 16\pi^2 a^3 \frac{\dot{\psi}}{N}, \quad \pi_\chi = 16\pi^2 a^3 \frac{\dot{\chi}}{N}, \]

\[ \pi_{f_0} = \frac{12\pi^2}{e^2} e^{d\beta\psi} a \frac{\dot{f_0}}{N}, \quad \pi_f = \frac{12\pi^2}{e^2} e^{d\beta\psi} a \frac{D_t f}{N}, \quad \pi_g = \frac{4\pi^2}{e^2 b_0^d} e^{-2\beta\psi} \frac{a^3}{N} D_t g. \] (23) (24)

\(^4\)The scale factor \( b(t) \) of the internal space induces a behaviour similar to the case of a minimally coupled scalar field. In fact, by introducing the field \( \psi \) by \( b \sim \exp \psi \), this quantity corresponds to the scalar field which appears in the harmonic expansion of the Kaluza-Klein theory.
For simplicity we replace the variables \((a, \psi, \chi)\) by the new variables \((\mu, \phi, \xi)\):

\[
a = e^\mu \left( \frac{k}{6\pi} \right)^{\frac{1}{2}}, \quad \psi = \phi \left( \frac{3}{4\pi k} \right)^{\frac{1}{2}}, \quad \chi = \xi \left( \frac{3}{4\pi k} \right)^{\frac{1}{2}}.
\]

The corresponding new conjugate momenta then read

\[
\pi_\mu = -\left( \frac{2k}{3\pi} \right)^{\frac{1}{2}} e^{3\mu} N^\mu, \quad \pi_\phi = \left( \frac{2k}{3\pi} \right)^{\frac{1}{2}} e^{3\mu} N^\phi, \quad \pi_\xi = \left( \frac{2k}{3\pi} \right)^{\frac{1}{2}} e^{3\mu} N^\xi.
\]

The Hamiltonian and \(SO(N-3-d)\) gauge constraints are then obtained by varying (18) with respect to \(N\) and \(\hat{B}\), and in terms of the momenta (26) are given by:

\[
-\pi_\mu^2 - e^{4\mu} + \pi_\phi^2 + \pi_\xi^2 + e^{2\mu-\alpha_\phi} e^2 \left[ \pi_f^2 + \pi_0^2 \right] + e^{2\alpha_\phi} \frac{3e^2b_0^2}{d\pi k} \pi_g^2 + e^{6\mu} \left( \frac{4k^2}{3} \right) W = 0,
\]

\[
\pi_{f_0} f_q + \pi_{g_0} g_q - \pi_{f_q} f_p - \pi_{g_q} g_p = 0,
\]

where \(\alpha = \sqrt{12/d(d+2)}\).

The canonical quantization follows by promoting the conjugate momenta into operators as

\[
\pi_\mu \mapsto -i \frac{\partial}{\partial \mu}, \quad \pi_\phi \mapsto -i \frac{\partial}{\partial \phi}, \quad \pi_\xi \mapsto -i \frac{\partial}{\partial \xi}, \quad \pi_{f_0} \mapsto -i \frac{\partial}{\partial f_0}, \quad \pi_f \mapsto -i \frac{\partial}{\partial f}, \quad \pi_g \mapsto -i \frac{\partial}{\partial g}.
\]

The Hamiltonian constraint (27) is then quantized to yield the Wheeler-DeWitt equation:

\[
\left\{ \frac{\partial^2}{\partial \mu^2} - e^{4\mu} - \frac{\partial^2}{\partial \phi^2} - \frac{\partial^2}{\partial \xi^2} - e^{2\mu-\alpha_\phi} e^2 \left[ \frac{\partial^2}{\partial f_0^2} + \frac{\partial^2}{\partial f^2} \right] - e^{2\alpha_\phi} \frac{3e^2b_0^2}{d\pi k} \frac{\partial^2}{\partial g^2} + e^{6\mu} \left( \frac{4k^2}{3} \right) W \right\} \Psi = 0,
\]

where in the usual parametrization of the factor ordering ambiguity, \(\pi_\mu^2 \mapsto -\mu^{-p} \frac{\partial}{\partial \mu} \left( \mu^p \frac{\partial}{\partial \mu} \right)\), we have set \(p = 0\).

The richness of the effective model (18) and the corresponding Wheeler-DeWitt equation (30) is quite evident. In this reduced model the gauge field has non-vanishing time-dependent components in both the external and internal spaces. Moreover, we have also two time-dependent scalar fields, the dilaton and the inflaton. This contrasts with previous work in the literature, where either static magnetic monopole configurations with non-zero components only in \(I^d\) or scalar fields were present. Our model allows thus to consider several possibilities.

Aiming to study the compactification process we shall focus our analysis on the variables \(\mu\) and \(\phi\) and the contributions to the potential \(W\) from the gauge field. This choice is justifiable as it can be seen from (31) that the kinetic term for the external components of the gauge field

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5The replacement \(\psi \to \phi\) and \(\chi \to \xi\) is a mere rescaling, while introducing \(\mu \to \ln a\) for the scale factor can bring some advantages. In fact, the minisuperspace metric becomes then proportional to \(\text{diag}(1,-1)\) with useful consequences as far as the Wheeler-DeWitt equation is concerned [19].
is suppressed in an expanding Universe, while for the internal components the kinetic term is not relevant as compactifying solutions require $g$ to seat at the extremum of the potential $V_2(g)$ \[4\]. In doing that, we shall keep the inflaton field frozen as it has been shown that this field does not affect the compactification process \[4\]. Of course, we could instead consider taking $\mu$ and $\chi$ as the physically relevant variables and freeze the remaining ones and actually models of this type have been studied in Ref. \[20\].

Hence, in what follows we shall restrict ourselves instead to the study of compactification and hence concentrate our study on the sub-system where the relevant variables are $\mu$ and $\phi$. Hence, it requires solving the Wheeler-DeWitt equation (30) for static vacuum configuration of the gauge and inflaton fields:

$$
\xi = \xi^v, \quad f_0 = f_0^v, \quad f = f^v, \quad g = g^v = 0;
$$

(31)

we also assume that $U(\xi^v) = 0$ and that $f$ and $g$ are orthogonal. The notation $v_1 \equiv V_1(f_0^v, f^v)$ and $v_2 \equiv V_2(g^v) = \frac{1}{8}$ will be used throughout this paper. The Wheeler-DeWitt equation suitable for the study of compactification is the following:

$$
\left[ \frac{\partial^2}{\partial \mu^2} - \frac{\partial^2}{\partial \phi^2} + U(\mu, \phi) \right] \Psi(\mu, \phi) = 0,
$$

(32)

where

$$
U(\mu, \phi) = e^{6\mu} \left( \frac{4k}{3} \right)^2 \Omega(\mu, \phi) - e^{4\mu},
$$

(33)

and

$$
\Omega(\mu, \phi) = e^{-d\alpha\phi} \left[ -e^{-2\alpha\phi} \frac{d(d-1)}{16\pi k} \frac{1}{b_0^4} + e^{-4\alpha\phi} \frac{d(d-1)}{8e^2} v_2 + \frac{\Lambda}{8\pi k} + e^{d\alpha\phi-4\mu} \left( \frac{6\pi}{k} \right)^2 \frac{3}{4e^2} v_1 \right].
$$

(34)

The scenario associated with this choice is analogous to the ones of Refs. \[11, 12, 13, 15\], with the novel feature of taking into account the external components of the gauge field. As it will be seen, the last term in (34) is central in our model and constitutes one of the major differences with respect to, for instance, Ref. \[14\]. Indeed, it is precisely this term sets the dependence of early Universe scenarios ($\mu \ll 0$, i.e. $a \to 0$) on different values of $d$ and $v_1$, brought about by the gauge field components in the 4-dimensional spacetime.

Moreover, as it will be discussed in the next section, it is the term $e^{d\alpha\phi-4\mu} \left( \frac{6\pi}{k} \right)^2 \frac{3}{4e^2} v_1$ in (34) that establishes that the external spatial dimensions and the internal $d$-dimensions are at the same footing in the early Universe prior to compactification, i.e., when $\mu \ll 0$. It is only through the expansion of the external dimensions (increase of $\mu$) that compactification ($b \to b_0$) is achieved. Thus, it is the dynamics of the 3-dimensional physical space which induces the evolution of $I^d$ towards compactification. Furthermore, we shall see how different values for $v_1$ and $d$ do lead to different quantum scenarios, i.e. solutions of the Wheeler-DeWitt equation, whose physical features can be compared with those of Refs. \[11, 12\].
3 Solutions with dynamical compactification

In this section we shall establish the boundary conditions for the Wheeler-DeWitt equation (32)-(34) and obtain solutions with dynamical compactification for certain regions of the $\mu \phi$-plane. Let us first address the latter issue, i.e., the scenario for dynamical compactification in our model.

As discussed in Ref. [4], from the classical point of view, different values for the cosmological constant $\Lambda$ lead to different compactifying scenarios. Indeed, for $\Lambda > \frac{c_2}{16\pi k}$ ($c_2 = \frac{[(d+2)^2(d-1)/(d+4)]e^2/16v_2}$) there are no compactifying solutions and for $\frac{c_1}{16\pi k} < \Lambda < \frac{c_2}{16\pi k}$ (35) ($c_1 = d(d-1)e^2/16v_2$) a compactifying solution exists which is classically stable, but semi-classically unstable. Finally, a value of $\Lambda < \frac{c_1}{16\pi k}$ implies that the value of the effective 4-dimensional cosmological constant, $\Lambda^{(4)} = 8\pi k \Omega(\infty, \phi)$, is negative (see Figure 1). Since the 4-dimensional cosmological constant, $\Lambda^{(4)}$, must satisfy the bound

$$|\Lambda^{(4)}| < 10^{-120} \frac{1}{16\pi k},$$

we are led to choose $\Lambda = \frac{c_1}{16\pi k}$. On the other hand, since we are interested in compactifying solutions, for which $\phi \approx 0$, we shall take $\Lambda$ such that $\phi = 0$ corresponds to the absolute minimum of (33). This corresponds to $b_0^2 = \frac{16\pi k v_2}{e^2}$, and the fine-tuning [4]

$$\Lambda = \frac{d(d-1)}{16b_0^2}. \quad (37)$$

The potential (33) simplifies then to

$$U(\mu, \phi) = e^{6\mu - d\phi} \frac{2k\Lambda}{9\pi} (e^{-2\phi} - 1)^2 - e^{4\mu} + e^{2\mu + d\phi} \frac{3\pi v_1}{k v_2} b_0^2,$$

and its form is shown in Figure 2. Moreover, as can be seen from the plot of $\Omega(\mu = \text{constant}, \phi)$ in Figure 3 (cf. eq. (33)), for $\mu$ greater than a critical value, $\mu_c$, the potential $U(\mu, \phi)$ has a local maximum, $\phi_{\text{max}}$, given approximately by $e^{-2\alpha \phi_{\text{max}}} = d/(d+4)$. This critical value arises from the last term in (33) and in first order approximation is given by

$$a_c^4 \sim e^{4\mu_c} = \frac{-B - \sqrt{B^2 - 4AC}}{2A}, \quad (39)$$

where

$$A = -\alpha A_1 A_2 \left(\frac{2k\Lambda}{9\pi}\right)^2,$$

$$B = d \frac{2\Lambda v_1}{3} b_0^2 \left[ d A_1 e^{2d\alpha\phi_0} \left( \frac{1}{\phi_{\text{max}} - \phi_0} + \alpha d \right) + A_2 e^{2d\alpha\phi_{\text{max}}} \left( \frac{1}{\phi_{\text{max}} - \phi_0} - \alpha d \right) \right],$$

$$C = \alpha d^5 e^{2d\alpha\phi_0} \left( \frac{3\pi v_1 b_0^2}{k v_2} \right)^2,$$
with \( A_1 = -8e^{-2a\phi_{\text{max}}} = \frac{8d}{d+1} \), \( A_2 = \frac{(d + 4)^2e^{4a\phi_0} - d^2)}{2} \) and \( e^{-2a\phi_0} = \frac{(d + 2)^2}{\sqrt{(d + 2)^4 - d^2(d + 4)^2}} \).

We now turn to the discussion of the boundary conditions for the Wheeler-DeWitt equation \( \Psi \). We shall use the path integral representation for the ground-state of the Universe

\[
\Psi[\mu, \phi] = \int e^{-S} \prod_i D\phi_i \exp(-S_E),
\]

which does allow us to evaluate \( \Psi(\mu, \phi) \) close to \( \mu = -\infty \). In here, \( S_E = -iS_{\text{eff}} \) is the Euclidean action, obtained through the effective action \( \{12\} \) and taking \( d\tau = iNdt \)

\[
S_E = \int d\tau \frac{6\pi}{k} \left[ -a\dot{a}^2 + a^3 \dot{\phi}^2 - \frac{a^3}{4} + a^3 e^{-d\phi} (e^{-2a\phi} - 1) \right] + \frac{1}{a} e^{d\phi} \frac{2\pi k}{e^2 v_1}.
\]

To ensure that the sum \( C \) does corresponds to compact \((d + 4)\)-metrics we must impose conditions on \( \dot{a}(t) \) and \( b(t) \) at \( \tau = 0 \) (where \( \tau \) is the euclidean time \( d\tau = iNdt \)), such that the Euclidean metric

\[
\hat{g} = d\tau^2 + a^2(\tau) \sum_{i=1}^3 \omega^i \omega^i + b^2(\tau) \sum_{m=4}^{d+3} \omega^m \omega^m,
\]

is compact. In Ref. \([11]\) the following conditions were suggested: \( \dot{a} = 0, b > 0, \frac{db}{d\tau} = 1 \) and \( \frac{db}{d\tau} = 0 \) at \( \tau = 0 \), which can also be inferred from the regularity of the Euclidean equations of motion \([12]\). Notice that physical reasons, such as the vanishing of the internal gauge field components and of the gravitational coupling in 4-dimensions, prevent the interchange of these conditions. Clearly, this approach to select the boundary conditions to the Hartle-Hawking wave function is not quite correct from the quantum point of view as it implies a simultaneous fixing of both canonical and corresponding conjugated momentum variables.

As far as our reduced model (see eq. \([13]\)) is concerned, consistent boundary conditions can be implemented as follows. Let us first point out that our reduced model is similar to a closed Friedmann-Robertson-Walker model with a scalar field \( \psi \) (or \( \phi \)) \([14]\). Hence, our boundary conditions which are consistent with a 4-geometry closing off in a regular way and with regular field configurations: \( a(0) = 0 \) and \( \frac{db}{d\tau}(0) = 0 \). The next step is to note that the corresponding constraint (Friedmann) equation in our model implies that \( \frac{db}{d\tau}(0) = 1 \) \([22, 23]\), i.e. the condition \( a(0) = 0 \) is equivalent to \( \frac{db}{d\tau}(0) = 1 \). In addition, \([3, 10]\) \( \frac{db}{d\tau}(0) = 0 \) leads, using \( \psi \sim \ln b \), to \( \frac{db}{d\tau}(0) = 0 \) and \( b(0) > 0 \). It is important to realize that the geometries summed over in the path integral will be closed at \( \tau = 0 \) for the 4-dimensional physical spacetime, but generally not regular, and also that the geometries will be regular at \( \tau = 0 \) for the extra \( d \)-dimensional space. \( \Psi \)From the constraint equation, the other condition \( \frac{da}{d\tau}(0) = 1 \) (regularity) will hold at saddle points, and similarly for \( b(0) > 0 \) which will follow from the corresponding regularity of the equations of motion \([12]\).

Thus, integrating \([12]\) from an initial point \( \tau = 0 \) to \( \Delta \tau \), a very close point to \( \tau = 0 \), we get

\[
S_E = \int_0^{\Delta \tau} d\tau \frac{6\pi}{k} \left[ -\frac{\tau}{4} + \frac{\tau^3e^{-d\phi} (e^{-2a\phi} - 1)}{3} + \frac{1}{\tau} e^{d\phi} \frac{2\pi k}{e^2 v_1} \right],
\]

\( ^6 \)

A discussion on the mathematical aspects of generalizing the Hartle-Hawking no-boundary proposal to higher dimensions can be found in the Appendix.
where we used $a \approx \tau$ close to $\tau = 0$. Finally, by setting $a = e^{\mu} \sqrt{k/6\pi}$, the integration yields

$$S_{E} = \begin{cases} -\frac{5}{8} e^{2\mu} + e^{4\mu - d\phi} \left(e^{-2\phi} - 1\right)^{2} \frac{k\Lambda}{t^{2}\pi}, & \text{for } v_{1} = 0, \\ +\infty, & \text{for } v_{1} \neq 0. \end{cases}$$

(45)

Since, with a suitable choice of the metric, we can have $\Psi = e^{-S_{E}}$ near the past null infinity (see ref. [11]), we can easily obtain the boundary conditions. This analysis is simplified by introducing the following new variables:

$$
\begin{align*}
x &= e^{\mu} \sinh \phi \\
y &= e^{\mu} \cosh \phi
\end{align*}
$$

(46)

such that the past null infinity $\mathfrak{N}^{-}$ now corresponds to the lines $x = y$ and $x = -y$. The boundary conditions on $\mathfrak{N}^{-}$, that are shown in Table 1, can be easily obtained from (45). For all over $\mathfrak{N}^{-}$, the normal derivative vanishes, $\partial \Psi / \partial n = 0$.

| $v_{1} = 0$ | $v_{1} \neq 0$ |
|---|---|
| on $\mathfrak{N}^{-}$ and $\phi < 0$ | $\Psi = 0(1)$ for $d < 19$ ($d \geq 19$) |
| on $\mathfrak{N}^{-}$ and $\phi > 0$ | $\Psi = 1$ |

Table 1: Boundary conditions on $\mathfrak{N}^{-}$ for $\Psi$.

Let us now further proceed with our search for solutions to the Wheeler-DeWitt equation. In this situation, one must generally begin by determining the regions where the solution is oscillatory and where it is exponential. This can be heuristically done by examining the regions where for surfaces of constant $U$, the minisuperspace metric $ds^{2} = d\mu^{2} - d\phi^{2}$ is either spacelike ($ds^{2} > 0$) or timelike ($ds^{2} < 0$):

In spacelike regions we can locally perform a Lorentz-type transformation to new coordinates $(\tilde{\mu}, \tilde{\phi})$:

$$
\begin{align*}
\tilde{\mu} &= \mu \cosh \theta - \phi \sinh \theta \\
\tilde{\phi} &= -\mu \sinh \theta + \phi \cosh \theta
\end{align*}
$$

(47)

where $\theta$ is a constant, such that the surfaces of constant $U$ are parallel to the $\tilde{\phi}$ axis. The potential will then depend, at least locally, only on $\tilde{\mu}$ and the Wheeler-DeWitt equation can be rewritten as

$$
\left[ \frac{\partial^{2}}{\partial \tilde{\mu}^{2}} - \frac{\partial^{2}}{\partial \tilde{\phi}^{2}} + U(\tilde{\mu}) \right] \Psi(\tilde{\mu}, \tilde{\phi}) = 0,
$$

(48)

and $\Psi$ will be oscillatory if $U > 0$ and exponential type if $U < 0$, assuming that its dependence on $\tilde{\phi}$ is small.

Similarly, when the surfaces of constant $U$ correspond to timelike regions of the minisuperspace metric, a Lorentz-type transformation can rotate coordinates $(\mu, \phi)$ such that they become
parallel to the $\tilde{\mu}$ axis. The potential, $U$, will then depend only on $\tilde{\phi}$, and $\Psi$ will be exponential type for $U < 0$ and oscillatory type for $U > 0$, assuming now that the wave function dependence on $\tilde{\mu}$ is small. The surfaces $U = 0$ depend on the relation $\frac{\mu}{v_2}$ and are given by the expression

$$e^{2\mu} = \frac{9\pi}{4k\Lambda} \frac{e^{d\alpha\phi}}{(e^{-2\alpha\phi} - 1)^2} \left( 1 \pm \left[ 1 - \frac{d(d-1)}{6} \frac{v_1}{v_2} \left( e^{-2\alpha\phi} - 1 \right)^2 \right]^{1/2} \right).$$  \hspace{1cm} (49)

These surfaces (see Figure 4) provide all points for which a Euclidean solution can be smoothly matched into a Lorentzian one, that is $\dot{\tilde{\mu}} = \dot{\tilde{\phi}}$ (the extrinsic curvature being continuous). For $\frac{\mu}{v_2} = 0$ we recover the result found in Ref. [11]. In order to further characterize the regions where solutions are oscillatory or exponential, we further summarize the asymptotic branches of the surface $U = 0$ as follows:

(i) For $v_1/v_2 = 0$ and $\phi \to +\infty$, $b \to +\infty$, we have $e^{2\mu} \to \frac{9\pi}{2k\Lambda} e^{d\alpha\phi}$, $\tilde{a} \to \sqrt{\frac{3}{4\Lambda}}$.

(ii) When $\phi \to -\infty$, $b \to 0$, we have $e^{2\mu} \to \frac{9\pi}{2k\Lambda} e^{\alpha(d+4)\phi}$, $\tilde{a} \to 0$.

(iii) Finally, when $\phi \to 0$, $b \to b_0$, we obtain $e^{2\mu} \propto \phi^{-1}$, $\tilde{a} \to 0$.

(iv) For $v_1/v_2 \neq 0$ only the asymptotic branch $\phi \to 0$ survives.

However, besides the surfaces of constant $U$ that correspond to timelike or spacelike regions, we have also to look for the curves of constant $U$ surfaces for which the minisuperspace metric is null, $\frac{du}{d\phi} = \pm 1$. The expression for these curves is given by $\frac{dU}{d\mu} = \pm \frac{dU}{d\phi}$, that is

$$e^{2\mu} = \frac{9\pi}{k\Lambda} e^{d\alpha\phi} \left[ 1 \pm \frac{d(d-1)}{96} \frac{v_1}{v_2} \left( e^{-2\alpha\phi} - 1 \right) (2 \mp d\alpha) \left[ e^{-2\alpha\phi} (6 \pm \alpha(d+4)) - (6 \pm d\alpha) \right] \right]^{1/2},$$

where the sign “$\pm$” is independent of the remaining ones appearing in (50). It is quite important to point out that the sign of one of the terms in (50) depends on the number of extra dimensions, $d$:

$$\begin{align*}
6 - \alpha(d+4) &> 0 \quad \text{for } d \geq 4 \\
6 - \alpha(d+4) &< 0 \quad \text{for } d < 4.
\end{align*}$$  \hspace{1cm} (51)

This implies that there will be different solutions for different values of $d$. As far as the asymptotic branches of (51) are concerned, we have the following:

(i) For $\phi \to +\infty$, $b \to \infty$, we have the asymptotic branch $e^{2\mu} \to \frac{9\pi}{k\Lambda} e^{d\alpha\phi} (\frac{1 + \sqrt{C}}{6 - d\alpha})$, $\tilde{a} \propto \Lambda^{-1/2}$, where $C = 1 - \frac{d(d-1)}{96} \frac{v_1}{v_2} (2 \mp d\alpha)(6 \pm d\alpha)$.

(ii) If $v_1/v_2$ verifies the condition $v_1/v_2 < 96/[d(d-1)(2 + d\alpha)(6 - d\alpha)]$, then there are two other asymptotic branches: $e^{2\mu} \to \frac{9\pi}{k\Lambda} e^{d\alpha\phi} (\frac{1 \pm \sqrt{C}}{6 - d\alpha})$, $\tilde{a} \propto \Lambda^{-1/2}$. 

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(iii) For $\phi \to -\infty$, $b \to 0$, and $v_1/v_2 = 0$ we have $e^{2\mu} \to \frac{2\pi}{k\Lambda} e^{(d+4)\alpha \phi} \frac{\bar{a}}{6\pm\alpha(d+4)}$, $\bar{a} \to 0$. The lower sign branch exists only for $d \geq 4$.

(iv) When $\phi \to -\infty$ and $v_1/v_2 \neq 0$, we have $e^{2\mu} \to \frac{2\pi}{k\Lambda} e^{(d+2)\alpha \phi} \sqrt{\frac{d(d-1)}{6\pm\alpha(d+4)} v_1}$, $\bar{a} \to 0$, and the lower sign branch exists only for $d < 4$.

(v) Finally, when $\phi \to 0$, $b \to b_0$, then $e^{2\mu} \propto \phi^{-1}$.

(vi) There is an additional asymptotic branch for $\phi \to \phi_\pm$, where $\exp(-2\alpha \phi_\pm) = (6 \pm d\alpha)/[6 \pm \alpha(d + 4)]$, with $e^{2\mu} \approx |\phi - \phi_\pm|^{-1}$. The lower branch $\phi_-$ exists only for $d \geq 4$.

In Figures 5, 6 and Figures 7, 8 we plot the curves $U = 0$ (dashed lines) together with the ones for which $d\mu/d\phi = \pm 1$ (bold lines) for cases $d = 3$ and $d = 6$. Notice the difference between the $d < 4$ and the $d \geq 4$ cases. For each region we further indicate whether $\Psi$ is expected to be oscillatory (osc.) or exponential (exp.).

In the following subsections we shall analyse in some detail different physical situations and derive the corresponding Hartle-Hawking (no-boundary) wave-function. We shall employ the transformation (47), after which the Wheeler-DeWitt equation takes the general form

$$\left[ \frac{\partial^2}{\partial \tilde{\mu}^2} - \frac{\partial^2}{\partial \tilde{\phi}^2} + U(\tilde{\mu}, \tilde{\phi}) \right] \Psi(\tilde{\mu}, \tilde{\phi}) = 0. \quad (52)$$

We can anticipate that sub-sections 3.4 and 3.5 contain the most interesting physical results as far as the process of compactification is concerned.

### 3.1 Wave function for $\mu > 0$ and $\phi \ll 0$

This case represents the physical situation prior to the compactification process. For $\mu > 0$ (i.e. $a > 0$) and $\phi \ll 0$ (i.e. $b \to 0$ with $U \gg 1$) the potential (38) becomes

$$U(\mu, \phi) \approx \frac{2k\Lambda}{9\pi} e^{6\mu-(d+4)\alpha \phi}, \quad (53)$$

and we can distinguish two situations:

(a) $d < 4$, for which we can choose $\sinh \theta = \frac{6}{\omega}$ and hence $U \approx U(\tilde{\phi}) = \frac{2k\Lambda}{9\pi} e^{-\omega \tilde{\phi}}$

(b) $d \geq 4$, for which we can choose $\cosh \theta = \frac{6}{\omega}$ and hence $U \approx U(\tilde{\mu}) = \frac{2k\Lambda}{9\pi} e^{\omega \tilde{\mu}}$,

where $\omega^2 = |24(-d^2 + d + 8)/d(d + 2)|$, with $\omega > 0$.

We can now solve eq. (52) with (53) by separation of variables to find that for $d < 4$, the solution is a combination of the Bessel functions of the first kind, $I_\nu(z)$, and of the second kind,
where \( K_\nu(z) \). For \( d \geq 4 \), we have a combination of the modified Bessel functions of the first kind, \( J_\nu(z) \), and of the second kind, \( Y_\nu(z) \). The study of the boundary conditions carried out above allows us to pick the appropriate Bessel function:

\[
\Psi(\tilde{\mu}, \tilde{\phi}) \approx e^{\pm \sqrt{\tilde{\mu}}} K_{\frac{d}{2} \sqrt{\epsilon}} \left[ \frac{2}{\omega} \left( \frac{2k\Lambda}{9\pi} \right)^{1/2} e^{-\frac{\phi}{2\epsilon}} \right], \quad \text{for } d < 4 , \tag{54}
\]

\[
\Psi(\tilde{\mu}, \tilde{\phi}) \approx e^{\sqrt{\tilde{\phi}}} J_{\frac{d}{2} \sqrt{\epsilon}} \left[ \frac{2}{\omega} \left( \frac{2k\Lambda}{9\pi} \right)^{1/2} e^{\frac{\phi}{2\epsilon}} \right], \quad \text{for } d \geq 4 , \tag{55}
\]

\[
\Psi(\tilde{\mu}, \tilde{\phi}) \approx J_0 \left[ \frac{2}{\omega} \left( \frac{2k\Lambda}{9\pi} \right)^{1/2} e^{\frac{\phi}{2\epsilon}} \right], \quad \text{for } d \geq 19 \text{ and } v_1 = 0 , \tag{56}
\]

where \( e^{\pm \sqrt{\tilde{\mu}}} \) means a combination of \( e^{\sqrt{\tilde{\mu}}} \) and \( e^{-\sqrt{\tilde{\mu}}} \), and \( \epsilon \) is the separation constant, which is determined by matching this solution onto the solution in the adjacent region (one can also see that \( \epsilon \approx 0 \). In (54)-(56) we have assumed that \( \epsilon \geq 0 \). The case \( \epsilon < 0 \) is not consistent with the Hartle-Hawking boundary conditions for a wave function of the type \( I_{\text{const.}} \sqrt{\tau}(z) \). Notice that, as expected, \( d < 4 \) implies an exponential behaviour, while \( d \geq 4 \) corresponds to an oscillatory one.

### 3.2 Wave function for \( \mu \gg 1 \) and \( \phi \gg 1 \)

This case corresponds to the situation where the radii of the \( S^3 \) and \( S^d \) sections are large. For \( \mu \gg 1 \) and \( \phi \gg 1 \) one has to deal with two regions separated by \( \mu = \frac{d\phi}{2} \), on which different behaviours are expected. On the lower region (1) in Figure 7 (\( \mu < \frac{d\phi}{2} \)) the potential is approximately given by

\[
U(\mu, \phi) \approx \begin{cases} 
-e^{4\mu} & , \text{for } v_1/v_2 = 0 \\
e^{2\mu+d\phi} \frac{3\pi v_1 k}{v_2 b_0^2} & , \text{for } v_1/v_2 \neq 0.
\end{cases} \tag{57}
\]

For \( v_1/v_2 \neq 0 \) we can choose \( \sinh \theta = -\sqrt{(d+2)/2(d-1)} \), so that \( U \approx U(\tilde{\phi}) = e^{\tilde{\phi} \frac{3\pi v_1 k}{v_2 b_0^2}} \), where \( \tilde{\omega} = \sqrt{8(d-1)/(d+2)} \). The solution is then

\[
\Psi(\tilde{\mu}, \tilde{\phi}) \approx e^{\sqrt{\tilde{\mu}}} K_{\frac{d}{2} \sqrt{\epsilon}} \left[ \frac{2}{\omega} \left( \frac{3\pi v_1 k}{v_2 b_0^2} \right)^{1/2} e^{\frac{\tilde{\phi}}{2\epsilon}} \right], \tag{58}
\]

where \( \epsilon \) is the separation constant.

For \( v_1/v_2 = 0 \) the wave function is a combination of \( K_0(z) \) and \( I_0(z) \), with \( z = \frac{1}{2} e^{2\mu} \). These solutions are, as expected, exponential type and are also valid in the region \( \phi \gg 1 \) and \( \mu < 0 \).

For the other case (region (2) in Figure 7), we have \( U \approx e^{6\mu-d\phi} \frac{2k\Lambda}{3\pi} \). Choosing \( \sinh \theta = \sqrt{d/2(d+3)} \) we get \( U \approx U(\tilde{\mu}) = e^{\tilde{\phi} \frac{2k\Lambda}{3\pi}} \), with \( \tilde{\omega} = \sqrt{24(d+3)/(d+2)} \), and \( \Psi \) is a combination
of $J_\nu(z)$ and $Y_\nu(z)$, with $\nu = \frac{2}{\sqrt{\epsilon}}$ and $z = \frac{2}{\omega} \left( \frac{2k\Lambda}{\pi} \right)^{1/2} e^{\frac{2}{\sqrt{\epsilon}} \bar{\mu}}, \epsilon$ being a separation constant. This solution is, as expected, oscillatory.

### 3.3 Wave function for $\mu \ll 0$

This case corresponds to a 4-dimensional physical Universe at a very early stage and with a generic $S^d$ section. In the region $\mu \ll 0$ (i.e., $a(t) \to 0$) and $\phi > 0$ the potential is also given by (57). For $v_1/v_2 \neq 0$ we obtain

$$\Psi \approx e^{\pm \sqrt{\epsilon} \bar{\mu}} I_{\frac{3}{2}} \left( \frac{2}{\omega} \left( \frac{3\pi v_1 b_0}{k v_2} \right)^{1/2} e^{\frac{2}{\sqrt{\epsilon}} \bar{\phi}} \right),$$

while for $v_1/v_2 = 0$ we have $\Psi(\mu, \phi) \approx I_0 \left[ 1/2 e^{2\mu} \right]$. In both cases the behaviour is exponential. These solutions also apply for $\phi < 0$ and $\mu < \frac{\alpha (d+4)}{2} \phi$.

For the particular situation where $\mu \ll 0$ together with $\phi \ll 0$, we further distinguish two different situations:

(a) For $\mu > \frac{(d+2)\alpha}{2} \phi$ we expect a behaviour similar to the one found for $\mu > 0$ and $\phi \ll 0$ (see subsection 3.1).

(b) As for the region $\frac{(d+2)\alpha}{2} \phi < \mu \leq \frac{(d+4)\alpha}{2} \phi$, this is a transition region and one should expect a mixture of the previous wave functions.

### 3.4 Wave function in the neighbourhood of $\phi = \phi_{\text{max}}$

We shall now obtain approximate solutions in the neighbourhood of $\phi = \phi_{\text{max}}$ using the semiclassical approximation to the path integral (41)

$$\Psi(\mu, \phi) \approx A(\mu, \phi) e^{-S_E(\mu, \phi)},$$

where $\phi_{\text{max}}$ is the local maximum of $U(\mu = \text{constant}, \phi)$ and is given approximately by $e^{-2\alpha \phi_{\text{max}}} = d/d + 4$. This corresponds to the physical state of our universe where the extra $d$-dimensional space is at an equilibrium point, corresponding to its maximum value.

Using then the classical field equations of motion obtained from $S_{\text{eff}}$ to integrate the Euclidean action we get for $v_1/v_2 = 0$:

$$S_E = \frac{3}{16k^2 \Omega} \left[ 1 - \left( \frac{4k}{3} e^{2\mu \Omega} \right)^{3/2} - 1 \right],$$

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where the potential $\Omega(\mu, \phi)$, given by
\[
\Omega(\mu, \phi) = e^{-\alpha \phi} \frac{\Lambda}{8\pi k} \left( e^{-2\alpha \phi} - 1 \right)^2 + e^{\alpha \phi - 4\mu} \left( \frac{27\pi b^2 v_1}{16k^3 v_2} \right),
\]
was assumed to be approximately constant near $\phi = \phi_{\text{max}}$. Hence, in the region $U < 0$:
\[
\Psi \approx A(\mu, \phi) \exp \left[ \frac{3}{16k^2 \Omega} \right] \exp \left[ -\frac{3}{16k^2 \Omega} \left( 1 - \left( \frac{4k}{3} \right)^2 e^{2\mu \Omega} \right)^{3/2} \right],
\]
where the prefactor $A$ is such that it verifies the condition $A(-\infty, \phi) = 1$.

In the region $U > 0$ the wave function becomes oscillatory, and the WKB procedure shows that
\[
\Psi \approx B(\mu, \phi) \exp \left[ \frac{3}{16k^2 \Omega} \right] \cos \left[ \frac{3}{16k^2 \Omega} \left( \left( \frac{4k}{3} \right)^2 e^{2\mu \Omega} - 1 \right)^{3/2} - \frac{\pi}{4} \right].
\]
Replacing (64) in the Wheeler-DeWitt equation one obtains the prefactor
\[
B(\mu, \phi) \approx e^{-\mu} \left( \frac{4k}{3} \right)^2 e^{2\mu \Omega} - 1 \right]^{-1/4}.
\]
For $v_1/v_2 \neq 0$ these results are still valid for $\mu > 0$. For $\mu < 0$ we expect the behaviour described in subsection 3.3.

### 3.5 Wave function in the neighbourhood of $\phi = 0$ and large $\mu$

Finally, we consider the case where the 4-dimensional physical Universe is in a stage of large $S^3$ radius and with $b \sim b_0$. In the neighbourhood of $\phi = 0$, at the minimum of $U(\mu = \text{constant}, \phi)$, we consider the dominant term of the potential for large $\mu$:
\[
U(\mu, \phi) \approx e^{6\mu - d\alpha \phi} \frac{2k \Lambda}{9\pi} \left( e^{-2\alpha \phi} - 1 \right)^2 \approx e^{6\mu} \phi^2 \frac{8\alpha^2 k \Lambda}{9\pi}.
\]
Notice that the potential vanishes for $\phi = 0$ and that in (66) we exhibit the dominant term for values of $\phi$ around the minimum. Quadratic potentials of this kind are found in massive scalar field models [1].

We now perform a simple change of variables
\[
x = e^{3\mu}, \\
y = e^{-2\alpha \phi}
\]

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from which yields the Wheeler-DeWitt equation:
\[ 9x^2 \frac{\partial^2}{\partial x^2} - 4\alpha^2 y^2 \frac{\partial^2}{\partial y^2} + 9x \frac{\partial}{\partial x} - 4\alpha^2 y \frac{\partial}{\partial y} + x^2 y^{d/2}(y - 1)^2 \frac{2k\Lambda}{9\pi} \] \( \Psi(x, y) = 0 \). (68)

As we are interested in the limit \( x \ll 1 \) and \( y \approx 1 \), we actually have to solve:
\[ x^2 \frac{\partial^2}{\partial x^2} + x \frac{\partial}{\partial x} + \frac{1}{9} x^2 y^{d/2}(y - 1)^2 \frac{2k\Lambda}{9\pi} \] \( \Psi(x, y) = 0 \). (69)

Thus, choosing \( z = \frac{1}{3} \sqrt{\frac{2k\Lambda}{9\pi} x y^{d/2}|y - 1|} \), one easily sees that \( \Psi \) is a combination of Bessel functions \( J_0(z) \) and \( Y_0(z) \), where \( z = \sqrt{\frac{2k\Lambda}{9\pi} \frac{2\alpha}{3} e^{\beta\mu} |\phi|} \). If \( \Psi \propto J_0(z) \) then, as \( z \to 0 \), the wave function behaves as \( \Psi \approx 1 - z^2/4 \). If, on the other hand, \( \Psi \) also depends on \( Y_0(z) \), then, as \( z \to 0 \), \( \Psi \) behaves asymptotically as \( \Psi \approx \frac{2}{\pi} \ln \frac{2}{z} \). This behaviour is depicted in Figure 9.

According to the standard interpretational rules of quantum cosmology (see for instance Ref. [24]), the probabilistic interpretation of the wave function does make sense in the classical and the semiclassical regions. Therefore, as the large \( \mu \) region corresponds to a classical region, the fact that the wave function is highly peaked around \( \phi = 0 \) means that the most probable configuration does indeed correspond to solutions with compactification for expanding external spacetime. In the next section we shall draw additional physical information concerning some of the solutions in this section.

## 4 Interpretation of the wave function

In order to interpret the wave function we shall use the trace of the square of the extrinsic curvature, \( K^2 = K_{ij} K^{ij} \), to see whether the wave function in the semiclassical limit corresponds to a Lorentzian or to a Euclidean geometry. This is justified as the Wheeler-DeWitt equation is the same from whatever metric (Lorentzian or Euclidean) one derives it. The extrinsic curvature is a measure of the variation of the normal to the hypersurfaces of constant time, and is given by:
\[ K_{ij} = N^{-1} \left( -\frac{1}{2} \frac{\partial h_{ij}}{\partial t} + \nabla_j N_i \right) \],

where \( h_{ij} \) is the \( d + 3 \)-metrics and \( N_i \) are the components of the shift-vector. From (12) and using (26) we obtain
\[ K^2 = -e^{-6\mu + d\omega} \frac{3\pi}{2k} \left[ 9 \frac{\partial^2}{\partial \mu^2} + \left( \frac{d\alpha}{2} \right)^2 \frac{\partial^2}{\partial \phi^2} + 3d\alpha \frac{\partial^2}{\partial \mu \partial \phi} \right] \].

Performing the Lorentz-type transformation (17) with \( \sinh \theta = \sqrt{d/2(d + 3)} \), and using \( \omega = \)
\[ \sqrt{24(d+3)/(d+2)}, \ K^2 \text{ simplifies to} \]

\[ K^2 = -e^{-\omega_0} \frac{9\pi}{k} \left( \frac{d+3}{d+2} \right) \frac{\partial^2}{\partial \mu^2}, \tag{72} \]

and we see that, in regions where the wave function behaves as an exponential the quantity \( K^2 \Psi/\Psi \) is negative. Therefore, in the classical limit, \( K \) is imaginary and we have a Euclidean \((d+4)\)-geometry. When the wave function is oscillatory, the corresponding \( K \) is real, and the \((d+4)\)-geometry is Lorentzian. Note that a Lorentz geometry corresponds to a classical state of the Universe, while a Euclidean one is normally associated to a quantum or tunneling state. As shown in Figures 7 and 8 there exist, for \( d \geq 4 \), well defined Lorentzian regions for \textit{different} values of the ratio \( v_1/v_2 \). These regions are however, inexistent when \( d < 4 \) as depicted in Figures 5 and 6.

In the oscillatory region, the wave function can be further interpreted using the WKB approximation \( \Psi = \text{Re} \left( C e^{iS} \right) \), where \( S \) is a rapidly varying phase and \( C \) a slowly varying prefactor. One chooses \( S \) to satisfy the classical Hamilton-Jacobi equation

\[ -\left( \frac{\partial S}{\partial \mu} \right)^2 + \left( \frac{\partial S}{\partial \phi} \right)^2 + U(\mu, \phi) = 0. \tag{73} \]

The significance of \( S \) becomes evident when operating \( \pi_\mu \) on \( \Psi \) (for \( \pi_\phi \) the procedure is analogous):

\[ \pi_\mu \Psi = \left[ \frac{\partial S}{\partial \mu} - i \frac{\partial}{\partial \mu} \ln C \right] \Psi. \tag{74} \]

Since in the WKB approximation we assume \( |\frac{\partial S}{\partial \mu}| \gg |\frac{\partial S}{\partial \phi} \ln C| \), we have

\[ \pi_\mu = \frac{\partial S}{\partial \mu}, \ \pi_\phi = \frac{\partial S}{\partial \phi}. \tag{75} \]

The wave function corresponds then to a two-parameter subset of solutions which obey (75) and that can be regarded as providing the boundary conditions for the classical solutions. We shall now try to obtain an approximate solution for the Hamilton-Jacobi equation (73) in the region close to the space segment \( U = 0 \) and \( \phi = \phi_{\text{max}} \) as it is there that classical trajectories start. Assuming that \( S \) is separable and that \( |\frac{\partial S}{\partial \mu}| \gg |\frac{\partial S}{\partial \phi}| \) we can use a series expansion around \( \phi = \phi_{\text{max}} \) to obtain

\[ S \approx \pm \frac{e^{3\mu}}{3} \left( \frac{2k\Lambda}{9\pi} \right)^{1/2} \left( \frac{d}{d+4} \right)^{d/4} \left[ \frac{4}{d+4} - E \left( e^{-2\alpha\phi} - \frac{d}{d+4} \right)^2 \right], \tag{76} \]

where \( E = \frac{3}{16} \frac{(d+2)(d+4)}{d} \left[ 1 + \frac{4}{3} \left( \frac{d+4}{d+2} \right)^{1/2} - 1 \right] \). The upper (lower) sign on (76) corresponds to a collapsing (expanding) Universe. This result agrees with (64). Using (75) and (26) we have,
for the gauge $N = 1$,

$$
\dot{\mu} \approx \pm \left( \frac{\Lambda}{3} \right)^{1/2} \left( \frac{d}{d + 4} \right)^{d/4} \left[ \frac{4}{d + 4} - E \left( e^{-2\alpha\phi} - \frac{d}{d + 4} \right)^2 \right],
$$

(77)

$$
\dot{\phi} \approx \pm \left( \frac{\Lambda}{3} \right)^{1/2} \frac{4\alpha}{3} E e^{-2\alpha\phi} \left( e^{-2\alpha\phi} - \frac{d}{d + 4} \right).
$$

(78)

If $\phi_0$, the initial value of $\phi$, is close to $\phi_{\text{max}}$, then $\dot{\phi}$ will be very small and the scale factor $a(t)$ will grow exponentially like $\exp \left[ \left( \frac{\Lambda}{3} \right)^{1/2} \left( \frac{d}{d + 4} \right)^{d/4} \frac{4}{d + 4} t \right]$ for an expanding Universe. Given the fine-tuning (37) this last expression becomes

$$
a(t) \approx \exp \left[ \frac{1}{b_0} \frac{d}{d + 4} \left( \frac{d - 1}{3(d + 4)} \right)^{1/2} t \right],
$$

(79)

which gives $a(t) \approx \exp \left( \frac{1}{b_0 \sqrt{3e}} t \right)$ for $d \to +\infty$.

Thus, we confirm the expectation that $\phi$ configurations to which the main contribution to the potential after compactification is an effective cosmological constant, do correspond, in the semiclassical regime, to inflationary solutions for expanding universes.

### 4.1 Wave function for the vacuum configuration $v_2 = V_2(g_v)$

Throughout the previous sections we have assumed that $v_2 > 0$. This corresponds to the choice $g = 0$ for the potential (22), which is obviously associated to a classically unstable situation. Nevertheless, since the wave function can be interpreted, at least in a semiclassical situation, as giving the probability of a certain configuration, one expects, for consistency, to have the wave function peaked around $g = 0$ when unfreezing $v_2$ and varying $g$. This means that the most probable configuration should correspond to the choice $g = 0$.

The dependence of $\Psi$ on $v_2$ can be seen fixing the value of the gauge coupling constant, $e$, and rewriting the term $\frac{2k\Lambda}{9\pi}$ in potential (38) as $\left( \frac{e}{12\pi} \right)^2 \frac{d(d-1)}{2v_2}$ (where (37) was used). Furthermore, using the value of the radius of compactification, $b_0^2 = \frac{16\pi k v_2}{e^2}$, we can see that the term $b_0^2 \omega = \frac{v_1}{16\pi \epsilon} e^2$ in (38) does not depend on $v_2$. We hence conclude that solutions depending on $\Lambda$ will depend on $v_2^{-1}$.

We are only interested in regions where $\mu > 0$ ($a > 0$), i.e. in regions where the probabilistic interpretation can be unambiguously used, from which implies that we have the following cases:

(a) For $\mu > 0$ and $\phi \ll 0$ (i.e. $b(t) \to 0$), $\Psi \propto K_\nu(\Lambda^{1/2})$ (Figure 10) or $\Psi \propto J_\nu(\Lambda^{1/2})$ (Figure 11) according to the value of $d$ (cf. wave functions (53) and (54)).
(b) For $\mu \gg 1$, $\phi \gg 1$ and $\mu > \frac{d\phi}{2}$, $\Psi$ is a combination of $J_\nu(\Lambda^{1/2})$ and $Y_\nu(\Lambda^{1/2})$. If $g$ is not too large the behavior of $Y_\nu$ is similar to the one of $J_\nu$ (Figure 11).

(c) For $\phi \approx \phi_{\text{max}}$ and $\mu > 0$ the wave function is given by either (33) or (64) (Figure 12).

(d) Finally, for $\phi \approx 0$, the wave function, $\Psi$, is a combination of $J_0(\Lambda^{1/2})$ and $Y_0(\Lambda^{1/2})$, whose behavior is similar to the one depicted in Figure 11.

Notice that when $\Psi$ is oscillatory, the peak for $g = 0$ will disappear for certain values of $\mu$. Nevertheless, we have always $\Psi(|g| = \pm 1) = 0$. We can therefore conclude that we do observe the expected maximum of the wave function for $g = 0$.

5 Conclusions

In this paper we have obtained solutions of the Wheeler-DeWitt equation derived from the effective model that arises from dimensionally reducing to one dimension the Einstein-Yang-Mills generalized Kaluza-Klein theory in $D = 4 + d$ dimensions. We considered a $\mathbb{R} \times S^3 \times S^d$ topology and the corresponding Hartle-Hawking boundary conditions. The dimensional reduction was achieved by restricting the field configurations to be homogeneous and isotropic through coset space compactification as indicated in Sections 1 and 2. This model of compactification has been proposed in Refs. [3, 4]. In particular, the crucial role played by the external space components of the gauge field in order to achieve classically as well as semiclassically stable compactifications was shown in Ref. [4].

In Section 2 we have presented the most salient features of the model and set up the Hamiltonian constraint which allows us to obtain the Wheeler-DeWitt equation to study the compactification process from the quantum mechanical point of view. Notice that in our model the gauge fields associated angular momentum is also constrained to vanish. The richness of our effective model (18) is quite evident. In this reduced model the gauge field has non-vanishing time-dependent components in both the external and internal spaces. Moreover, we have also two time-dependent scalar fields, the dilaton and the inflaton. This contrasts with previous work in the literature, where either static magnetic monopole configurations with non-zero components only in $I^d$ or scalar fields were present.

In section 3 we have obtained no-boundary solutions of the Wheeler-DeWitt equation which exhibit very interesting features. The term $e^{d\phi - 4\mu} \left( \frac{6\pi}{k} \right)^2 \frac{3}{\pi^2} v_1$ in (64) establishes that the external spatial dimensions and the internal $d$-dimensions are at the same footing in the early Universe prior to compactification, i.e. when $\mu \ll 0$. It is only through the expansion of the external dimensions (increase of $\mu$) that compactification $(b \rightarrow b_0)$ is achieved. Thus, it is the dynamics of the 3-dimensional physical space which induces the evolution of $I^d$ towards compactification.

We also find that stable compactifying solutions do correspond to extrema of the wave function of the Universe showing that the process of compactification does indeed takes place
for expanding universes. Furthermore, our analysis indicates that the main properties of the Hartle-Hawking wave function do depend on the following two features. On the one hand, on a non-vanishing contribution to the potential (38) of the external physical space dimensions of the gauge field, a feature already found in the classical analysis of Ref. [4]. On the other hand, also on the number, $d$, of internal space dimensions. In the case we set the contribution of the external space dimensions of the gauge field to the potential (38) to vanish, we find that we recover the main aspects of the discussion of Ref. [11], where compactification was discussed in the framework of an Einstein-Maxwell model with a magnetic monopole configuration whose gauge (Maxwell) field contribution was non-vanishing only for the internal space. The same can be said about Ref. [12], where a stable compactification was achieved through the non-vanishing contribution of the internal components of a $(D - 4)$th rank antisymmetric tensor field. Finally, we also find that for expanding models, inflationary solutions can be predicted, as shown in section 4, if in the semiclassical regime the potential is essentially given by an effective cosmological constant.

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Appendix

Hartle-Hawking proposal and its generalization to higher spacetime dimensions

For clarification purposes, let us briefly outline here the main features of the Hartle-Hawking proposal \[5\] and its generalization to higher spacetime dimensions \[11\] (see also ref. \[12, 13, 14, 15\]). In quantum cosmology it is assumed that the quantum state of a $D=4$ Universe is described by a wave function $\Psi[h_{ij}, \Phi]$, which is a functional of the spatial 3-metric, $h_{ij}$, and matter fields generically denoted by $\Phi$ on a compact 3-dimensional hypersurface $\Sigma$. The hypersurface $\Sigma$ is then regarded as the boundary of a compact 4-manifold $M^4$ on which the 4-metric $g_{\mu\nu}$ and the matter fields $\Phi$ are regular. The metric $g_{\mu\nu}$ and the fields $\Phi$ coincide with $h_{ij}$ and $\Phi_0$ on $\Sigma$ and the wave function is then defined through the path integral over 4-metrics, $^4g$, and matter fields:

$$\Psi[h_{ij}, \Phi_0] = \int_C D[4g] D[\Phi] \exp \left(-S_E[^4g, \Phi]\right), \quad (80)$$

where $S_E$ is the Euclidean action and $C$ is the class of 4-metrics $g_{\mu\nu}$ and regular fields $\Phi$ defined on Euclidean compact manifolds $M^4$ and with no other boundary than $\Sigma$. An extension of the Hartle-Hawking proposal for universes with $D > 4$ dimensions was first discussed in Ref. \[11\]. Let us summarize it, mentioning some of its difficulties and comparing it with the $D=4$ case.

In $D = 4$ the theory of cobordism \[14\] guarantees that for all compact 3-surfaces there always exists a compact 4-dimensional manifold such that $S^3$ is the only boundary, or equivalently, all 3-dimensional compact hypersurfaces are cobordant to zero \[14\]. Let us now consider the case for $D > 4$. In these $D$-dimensional models, the wave function would be a functional of the $(D - 1)$ spatial metric, $h_{IJ}$, and matter fields, $\Phi$, on a $(D - 1)$-hypersurface, $\Sigma_{D-1}$ and is defined as the result of performing a path integral over all compact $D$-metrics and regular matter fields on $M^D$, that match $h_{IJ}$ and the matter fields on $\Sigma_{D-1}$.

Let us then start by assuming that the $(D - 1)$-surface $\Sigma_{D-1}$ does not possess any disconnected parts \[11\]. Would there always be a $D$-dimensional manifold $M^D$ such that $\Sigma_{D-1}$ is the only boundary? In higher dimensional manifolds however, this is not guaranteed. There exist compact $(D - 1)$-hypersurfaces $\Sigma_{D-1}$ for which there is no compact $D$-dimensional manifold such that $\Sigma_{D-1}$ is the only boundary. This seems to indicate that in $D > 4$ dimensions there are configurations which cannot be attained by the sum over histories in the path integral. The wave function for such configurations would therefore be zero. In ref. \[11\] this situation was be circumvented so as to obtain non-zero wave-functions for such configurations, namely by dropping the assumption that the $(D - 1)$-surface $\Sigma_{D-1}$ does not possess any disconnected parts.

As described in \[11\], if one assumes that the hypersurfaces $\Sigma_{D-1}$ consist of any number $n > 1$ of disconnected parts $\Sigma_{D-1}^{(n)}$, then one finds that the path integral for this disconnected configuration involves terms of two types. The first type consists of disconnected $D-$manifolds, each disconnected part of which closes off the $\Sigma_{D-1}^{(n)}$ surfaces separately. These will exist only
if each of the $\Sigma^{(n)}_{D-1}$ are cobordant to zero, but this may not always be the case. There will indeed be a second type of term which consists of connected $D-$manifolds which just plainly joins some of the $\Sigma^{(n)}_{D-1}$ together. This second type of manifold will always exist in any number of dimensions, providing the $\Sigma^{(n)}_{D-1}$ are similar topologically, i.e. have the same characteristic numbers [14]. The wave function of any $\Sigma^{(1)}_{D-1}$ surface which is not cobordant to zero would be different from zero and obtained by assuming the existence of other surfaces of suitable topology and then summing over all compact $D-$manifolds which join these surfaces together. Thus, given a compact ($D-1$) hypersurface $\Sigma_{D-1}$ which is not cobordant to zero, a non-zero amplitude could be obtained by assuming it possesses disconnected parts.

However, the above considerations for disconnected pieces and generic $\Sigma_{D-1}$ surfaces would spoil the Hartle-Hawking prescription since the manifold would have more than one boundary. In other words, the general extension above discussed would imply a description in terms of propagation between such generic $\Sigma_{D-1}$ surfaces. The wave function would then depend on every piece and not on a single one as advocated in [11]. Nevertheless, if we restrict ourselves, as we do in the present paper, to the case of a truncated model with a global topology given by a product of a 3-dimensional manifold to a $d$-dimensional one, then the spacelike sections always form a boundary of a $D$-dimensional manifold with no other boundaries [15]. Since hypersurfaces $S^3 \times S^d$ are always cobordant to zero, it implies that for spacetimes with topology $\mathbb{R} \times S^3 \times S^d$ the Hartle-Hawking proposal can be always implemented, and thus we can consider the original no-boundary proposal in our study.
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Figure Captions

Figure 1
Potential $U(\mu = \text{constant}, \phi)$ for some values of $\Lambda$ and $d = 6$ ((a) $\Lambda > \frac{c_1}{16\pi k}$, (b) $\frac{c_1}{16\pi k} < \Lambda < \frac{c_2}{16\pi k}$, (c) $\Lambda < \frac{c_1}{16\pi k}$).

Figure 2
Potential $U(\mu, \phi)$ for $d = 6$ and large $\mu$ ($\mu > \mu_c$, see Figure 3).

Figure 3
Potential $\Omega(\mu = \text{constant}, \phi)$ for $d = 6$ and some values of $\mu$.

Figure 4
$U = 0$ curves in the $\mu\phi$-plane for $d = 6$ and different values of the ratio $\frac{n_1}{n_2}$ ((a) $\frac{n_1}{n_2} = 0$, (b) $\frac{n_1}{n_2} = \frac{1}{3}$, (c) $\frac{n_1}{n_2} = 1$).

Figure 5
$U = 0$ (dashed) and null curves (bold) in the $\mu\phi$-plane for $d = 3$ and $\frac{n_1}{n_2} = 0$.

Figure 6
$U = 0$ (dashed) and null curves (bold) in the $\mu\phi$-plane for $d = 3$ and $\frac{n_1}{n_2} = 1$.

Figure 7
$U = 0$ (dashed) and null curves (bold) in the $\mu\phi$-plane for $d = 6$ and $\frac{n_1}{n_2} = 0$.

Figure 8
$U = 0$ (dashed) and null curves (bold) in the $\mu\phi$-plane for $d = 6$ and $\frac{n_1}{n_2} = 1$.

Figure 9
Wave function in the neighbourhood of $\phi = 0$.

Figure 10
Module of the wave function for $\mu > 0$ and $\phi \ll 0$.

Figure 11
Module of the wave function in the region (2) of Figure 7.

Figure 12
Module of the wave function in the neighbourhood of $\phi = \phi_{\text{max}}$ and $\mu > 0$. 