EXISTENCE OF MULTIPLE POSITIVE WEAK SOLUTIONS AND ESTIMATES FOR EXTREMAL VALUES FOR A CLASS OF CONCAVE-CONVEX ELLIPTIC PROBLEMS WITH AN INVERSE-SQUARE POTENTIAL

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Abstract. In this paper, variational methods are used to establish some existence and multiplicity results and provide uniform estimates of extremal values for a class of elliptic equations of the form:

$$-\Delta u - \frac{\lambda}{|x|^2} u = h(x)u^q + \mu W(x)u^p, \quad x \in \Omega \setminus \{0\}$$

with Dirichlet boundary conditions, where $0 \in \Omega \subset \mathbb{R}^N (N \geq 3)$ be a bounded domain with smooth boundary $\partial \Omega$, $\mu > 0$ is a parameter, $0 < \lambda < \Lambda = \frac{(N-2)^2}{4}$, $0 < q < 1 < p < 2^* - 1$, $h(x) > 0$ and $W(x)$ is a given function with the set $\{x \in \Omega : W(x) > 0\}$ of positive measure.

1. Introduction. Let $0 \in \Omega \subset \mathbb{R}^N (N \geq 3)$ be a bounded domain with smooth boundary $\partial \Omega$ and consider the following problem:

$$\begin{cases}
-\Delta u - \frac{\lambda}{|x|^2} u = h(x)u^q + \mu W(x)u^p & \text{in } \Omega \setminus \{0\}, \\
u(x) > 0 & \text{in } \Omega \setminus \{0\}, \\
u(x) = 0 & \text{on } \partial \Omega,
\end{cases} \quad (1.1)$$

where $\Delta$ is the Laplace operator and $\mu > 0$ is a real parameter. $0 < \lambda < \Lambda = \frac{(N-2)^2}{4}$ and $0 < q < 1 < p < 2^* - 1 = \frac{2N}{N-2}$, where $2^* = \frac{2N}{N-2}$ is the so-called critical Sobolev exponent. $h(x) > 0$ and $W(x)$ is a given function with the set $\{x \in \Omega : W(x) > 0\}$ of positive measure, that is, $W \geq 0$ or $W$ changes sign. We will assume throughout the paper that $h, W \in C(\bar{\Omega})$. 

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The constant $\Lambda = \frac{(N-2)^2}{4}$ is the best constant in the Hardy inequality (see [4, pp.1879, Theorem 1.2] and apply it with $\gamma = 0$):

$$\int_{\Omega} \frac{u^2}{|x|^2} dx \leq \frac{1}{\Lambda} \int_{\Omega} |\nabla u|^2 dx \quad \text{for all } u \in H^1_0(\Omega).$$

Consequently, when $\lambda > \Lambda$, the operator $-\Delta - \frac{\lambda}{|x|^2}$ is coercive in $H^1_0(\Omega)$. This turns out to be crucial, since [5, pp.124, Theorem 2.2] implies that if $\lambda > \Lambda$, there is no nonnegative $u, \not \equiv 0$ such that

$$-\Delta u - \frac{\lambda}{|x|^2} u \geq 0$$

and hence no solution of problem (1.1), even in the weak sense.

Still few general results are known. In the case $\lambda = 0$, $R(u) = (1 + u)^p$ (that is, $-\Delta u = \mu (1 + u)^p$) and $1 < p \leq 2^* - 1$, F. Gazzola and A. Malchiodi [8, pp.811, Theorem 1] proved that there exists a constant $\mu_1 > 0$ such that the problem has at least two positive solutions $u_\mu$ and $U_\mu$, where $u_\mu$ is minimal (in the sense that $u_\mu(x) \leq v_\mu(x)$ for all $x \in \Omega$ and for any other solution $v_\mu$ of $(P_{\lambda,\mu})$) and $U_\mu$ is a mountain-pass solution when $\mu \in (0, \mu_1)$, has a unique solution $U_\mu$ in $H^1_0(\Omega)$ when $\mu = \mu_1$ and has no solution when $\mu > \mu_1$ even in distributional sense. In the case $\lambda = 0$, $0 < q < 1 < p = 2^* - 1$ and $h, W \equiv 1$, Y. Sun and S. Li [10, pp.1858, Theorem 1 and Theorem 2] proved that there exists a constant $\mu_2 > 0$ such that the problem has at least two positive weak solutions if $0 < \mu < \mu_2$ and has no solution if $\mu > \mu_2$. In the case $\lambda = 0$ and $p, q, h, W$ are under the same assumptions as in (1.1), Y. Sun [9, pp.752, Theorem 1.1] proved that there exists a constant $\mu_3 > 0$ such that the problem has at least two positive weak solutions if $0 < \mu < \mu_3$ and has no solution if $\mu > \mu_3$. Thus, provided $0 < \lambda < \Lambda$, it is natural to ask what the case would be for problem (1.1). Our goal of this paper is to show how variational methods can be used to establish some existence and multiplicity results for problem (1.1).

On $H^1_0(\Omega)$, we use the norm

$$\|u\|_\lambda^2 = \int_{\Omega} \left( |\nabla u|^2 - \frac{\lambda}{|x|^2} u^2 \right) dx.$$

Thanks to the Hardy inequality, the norm $\| \cdot \|_\lambda$ is equivalent to the usual norm $\| \cdot \|_{H^1_0(\Omega)}$. Problem (1.1) is variational in nature, so for $u \in H^1_0(\Omega)$, we define $I_{\lambda,\mu} : H^1_0(\Omega) \to \mathbb{R}$ by

$$I_{\lambda,\mu}(u) = \frac{1}{2} \|u\|_\lambda^2 - \frac{1}{1+q} \int_{\Omega} h(x) |u|^{1+q} dx - \frac{\mu}{p+1} \int_{\Omega} W(x) |u|^{p+1} dx,$$

the energy functional associated to problem (1.1). It is well known that there exists one-to-one correspondence between the weak solutions of problem (1.1) and the critical points of $I_{\lambda,\mu}$ on $H^1_0(\Omega)$. More precisely, we say that $u \in H^1_0(\Omega)$ is a positive weak solution of problem (1.1) we mean a function $u \in H^1_0(\Omega)$ such that $u \geq 0, u \not \equiv 0$ (by the strong maximum principle then $u > 0$ in $\Omega \setminus \{0\}$) and for any $\varphi \in H^1_0(\Omega)$ there holds

$$\langle I'_{\lambda,\mu}(u), \varphi \rangle = \int_{\Omega} \left( \nabla u \nabla \varphi - \frac{\lambda}{|x|^2} u \varphi \right) dx - \int_{\Omega} h(x) u^q \varphi dx - \mu \int_{\Omega} W(x) u^p \varphi dx = 0.$$

A standard regularity argument shows that $u \in C^2(\Omega \setminus \{0\})$, in which case, we say that $u$ satisfies (1.1) in the classical sense. Of course, if such a solution exists, it
must lie in the following Nehari type sets:

\[ F_{\lambda,\mu} = \left\{ u \in H_0^1(\Omega) : \langle I'_{\lambda,\mu}(u), u \rangle = 0 \right\} \]

\[ = \left\{ u \in H_0^1(\Omega) : \|u\|^2_\lambda = \int_\Omega h(x)|u|^{1+q}dx + \mu \int_\Omega W(x)|u|^{p+1}dx \right\}. \]

In order to motivate our results, denote \( J_{\lambda,\mu}(u) = \langle I'_{\lambda,\mu}(u), u \rangle \) and decompose \( F_{\lambda,\mu} \) with \( F^+_{\lambda,\mu}, F^0_{\lambda,\mu}, F^-_{\lambda,\mu} \) defined as follows:

\[
F^+_{\lambda,\mu} = \left\{ u \in F_{\lambda,\mu} : \langle J'_{\lambda,\mu}(u), u \rangle > 0 \right\} = \left\{ u \in F_{\lambda,\mu} : (1 - q)\|u\|^2_\lambda > \mu(p - q) \int_\Omega W(x)|u|^{p+1}dx \right\},
\]

\[
F^0_{\lambda,\mu} = \left\{ u \in F_{\lambda,\mu} : \langle J'_{\lambda,\mu}(u), u \rangle = 0 \right\} = \left\{ u \in F_{\lambda,\mu} : (1 - q)\|u\|^2_\lambda = \mu(p - q) \int_\Omega W(x)|u|^{p+1}dx \right\},
\]

\[
F^-_{\lambda,\mu} = \left\{ u \in F_{\lambda,\mu} : \langle J'_{\lambda,\mu}(u), u \rangle < 0 \right\} = \left\{ u \in F_{\lambda,\mu} : (1 - q)\|u\|^2_\lambda < \mu(p - q) \int_\Omega W(x)|u|^{p+1}dx \right\}.
\]

Our main results are as follows:

First, inspired by [9] and [11], we take full advantage of the connection between the Nehari manifolds and the fibering maps (that is, maps of the form \( t \rightarrow I_{\lambda,\mu}(tu) \)); see [3] and [7] for related results to establish some existence and multiplicity results for problem (1.1):

**Theorem 1.1.** Suppose that \( \mu \in (0, T_\lambda) \), where

\[
T_\lambda = \frac{1}{p - q} \cdot \left( \frac{p - 1}{p - q} \right)^{\frac{p-1}{p}} \cdot \frac{1}{\|W\|_\infty} \cdot \left( \frac{1}{\|h\|_\infty} \right)^{\frac{p-1}{p}} \cdot \left( \frac{S_{\lambda}}{\|\Omega\|^{\frac{1}{2}}} \right)^{\frac{p-1}{2}}, \tag{1.2}
\]

then problem (1.1) has at least two solutions \( u_\lambda \in F^+_{\lambda,\mu}, U_\lambda \in F^-_{\lambda,\mu} \) with \( \|U_\lambda\|_\lambda > \|u_\lambda\|_\lambda \).

Then, as a by-product of the proof of Theorem 1.1, we obtain the blow up behavior of solution \( U_{\lambda,\varepsilon} \in F^-_{\lambda,\mu} \) of problem (1.1) with \( p = 1 + \varepsilon \) as \( \varepsilon \to 0^+ \), like claimed in [8, pp.830, Theorem 15] for a non-singular problem and [9, pp.752, Theorem 1.3] for problem (1.1) provided \( \lambda = 0 \):

**Theorem 1.2.** Suppose that \( U_{\lambda,\varepsilon} \in F^-_{\lambda,\mu} \) is the solution of problem (1.1) with \( p = 1 + \varepsilon \), where \( \mu \in (0, T_\lambda) \), then

\[
\|U_{\lambda,\varepsilon}\|_\lambda > C_{\lambda,\varepsilon} \cdot \left( \frac{T_\lambda}{\mu} \right)^{\frac{1}{2}},
\]

where

\[
C_{\lambda,\varepsilon} = \|\Omega\|^{\frac{1}{4}} \cdot \left( \frac{\|\Omega\|^{\frac{1}{2}}}{\sqrt{S_\lambda}} \right)^{\frac{1}{2(p-1)}} \cdot \left( 1 + \frac{1 - q}{\varepsilon} \right)^{\frac{1}{\varepsilon}} \to \infty \quad \text{as} \quad \varepsilon \to 0^+.
\]

Namely, \( U_{\lambda,\varepsilon} \) blows up faster than exponentially with respect to \( \varepsilon \).

Lastly, inspired by [1], we use the method of sub- and supersolutions to show that
Remark 1. When \( \lambda = \Lambda \), the operator \(-\Delta - \frac{\lambda}{|x|^2}\) is no longer coercive in \( H^1_0(\Omega) \). However, we can still make use of the improved Hardy inequality (see [6, pp.452, Theorem 1.3] or \([12, \text{pp.108, Theorem 2.1}]\) to define a new Hilbert space, in which the operator is coercive, even when \( \lambda = \Lambda \). We conjecture that it is a problem to prove the existence of solutions for problem (1.1) provided \( \lambda = \Lambda \) for further study, although there are still some difficulties.

This paper is organized as follows. Section 2 contains some notations and preliminaries. Section 3 is devoted to the proof of Theorem 1.1. In Section 4, we prove Theorem 1.2. The proof of Theorem 1.3 is contained in Sections 5. In Section 6, we give an appendix, where some proofs are shown.

2. Notations and some preliminaries. In the rest of this paper, we make use of the following notations: \( D^{1,2}(\mathbb{R}^N) \) is the closure of \( C_0^\infty(\mathbb{R}^N) \) under the norm of \( \int_{\mathbb{R}^N} |\nabla\cdot|^2 dx; H^1_0(\Omega), L^1(\Omega) \) are standard Sobolev spaces with the usual norms \( ||\cdot||, ||\cdot||_1 \); The \( \to \) and \( \rightharpoonup \) denote strong and weak convergence, respectively; The \( c, C \) and \( C_i \) \((i = 1,2,3)\) denote (possibly different) suitable positive constants, whose exact values are not important unless otherwise specified; The \( \mu_1 \) denotes the smallest eigenvalue of

\[
-\Delta u - \frac{\lambda}{|x|^2} u = \mu_1 u, \quad x \in \Omega \setminus \{0\}, \quad u \in H^1_0(\Omega) \tag{1}
\]

and \( \varphi_1 \) denotes the corresponding eigenfunction with \( \varphi_1 > 0 \) in \( \Omega \) and \( ||\varphi_1||_2 = 1 \);

The \( S_\lambda \) denotes the best Sobolev constant for the embedding \( H^1_0(\Omega) \hookrightarrow L^{2^*}(\Omega) \), namely

\[
S_\lambda = \inf \left\{ \int_{\mathbb{R}^N} \left( |\nabla u|^2 - \frac{\lambda}{|x|^2} u^2 \right) dx; \ u \in D^{1,2}(\mathbb{R}^N), \ \int_{\mathbb{R}^N} |u|^{2^*} dx = 1 \right\} > 0.
\]

So for any \( u \in H^1_0(\Omega) \setminus \{0\} \), \( S_\lambda \leq \frac{||u||_2^2}{||u||_{2^*}^2} \), which gives:

\[
||u||_{2^*} \leq \frac{||u||_{2^*}}{\sqrt{S_\lambda}}.
\]

Therefore, from the Hölder inequality we have:

\[
\int_{\Omega} |u|^{p+1} dx \leq \left( \int_{\Omega} |u|^{(p+1) \cdot \frac{2^*}{2^* - 1}} dx \right)^{\frac{2^*}{2^* - 1}} \cdot \left( \int_{\Omega} 1 dx \right)^{\frac{2^* - p - 1}{2^* - 1}} = |\Omega|^\frac{2^* - p - 1}{2^* - 1} \cdot ||u||_{2^*}^{p+1} \leq |\Omega|^\frac{2^* - p - 1}{2^* - 1} \cdot \left( \frac{||u||_{2^*}}{\sqrt{S_\lambda}} \right)^{p+1}, \tag{2}
\]
\[
\int_{\Omega} |u|^{1+q} dx \leq \left[ \int_{\Omega} |u|^{(1+q)\frac{2^* q}{2^* - q}} dx \right]^{\frac{2^* - q}{2}} \cdot \left( \int_{\Omega} 1 dx \right)^{\frac{2^* - q}{2^*}} = |\Omega|^{\frac{2^* - q}{2^*}} \cdot \|u\|_2^{1+q} \leq |\Omega|^{\frac{2^* - q}{2^*}} \cdot \left( \frac{\|u\|}{\sqrt{S_\lambda}} \right)^{1+q}, (3)
\]

\[
\int_{\Omega} |u|^{1+q} dx \leq \left[ \int_{\Omega} |u|^{(1+q)\frac{2^* q}{2^* - q}} dx \right]^{\frac{2^* - q}{2}} \cdot \left( \int_{\Omega} 1 dx \right)^{\frac{2^* - q}{2^*}} = |\Omega|^{\frac{2^* - q}{2^*}} \cdot \|u\|_p^{1+q}. (4)
\]

3. **Proof of Theorem 1.1.** To obtain the existence of multiple positive solutions for problem (1.1), several lemmas are in order first. Moreover, the number \(T_\lambda\) defined in (1.2) is determined by Lemma 3.1, exactly, by (5):

**Lemma 3.1.** Suppose that \(\mu \in (0, T_\lambda)\), then \(F^{\pm}_{\lambda, \mu} \neq \emptyset\) and \(F^0_{\lambda, \mu} = \{0\}\).

**Proof.** (1) Since \(h(x) > 0\) and the set \(\{ x \in \Omega : W(x) > 0 \}\) has positive measure, we can choose \(u_0 \in H_0^1(\Omega)\) such that \(\int_{\Omega} h(x)|u_0|^{1+q} dx > 0\) and \(\int_{\Omega} W(x)|u_0|^{p+1} dx > 0\) and further define \(\varphi_\lambda : (0, +\infty) \to \mathbb{R}\) by

\[
\varphi_\lambda(t) = \frac{1}{t^{p-q}} \cdot \frac{d}{dt} F_{\lambda, \mu}(tu_0) = t^{1-p} \|u_0\|_A^2 - t^{q-p} \int_{\Omega} h(x)|u_0|^{1+q} dx - \mu \int_{\Omega} W(x)|u_0|^{p+1} dx.
\]

Computations show that \(\varphi_\lambda(t) \to -\infty\) as \(t \to 0\) and \(\varphi_\lambda(t) \to -\mu \int_{\Omega} W(x)|u_0|^{p+1} dx\) as \(t \to \infty\), and Appendix (1) in Section 6 shows that \(\varphi_\lambda(t)\) achieves its maximum at

\[
t_{\lambda, \max} = \left[ \frac{(p-1) \|u_0\|_A^2}{(p-q) \int_{\Omega} h(x)|u_0|^{1+q} dx} \right]^{\frac{1}{p-q}}.
\]

It is now deduced from (2) and (3) that

\[
\varphi_\lambda(t_{\lambda, \max}) = 1 - q \left( \frac{p-1}{p-q} \right) \frac{\|u_0\|_A^{2+p+1}}{\int_{\Omega} h(x)|u_0|^{1+q} dx} - \mu \int_{\Omega} W(x)|u_0|^{p+1} dx
\]

\[
\geq 1 - q \left( \frac{p-1}{p-q} \right) \frac{\|u_0\|_A^{2+p+1}}{\int_{\Omega} h(x)|u_0|^{1+q} dx} - \mu \int_{\Omega} W(x)|u_0|^{p+1} dx
\]

\[
- \mu \|W\|_\infty \cdot |\Omega|^{\frac{2^*-q-1}{2^*}} \left( \|u_0\|_A \right)^{p+1}
\]

\[
= \left[ 1 - q \left( \frac{p-1}{p-q} \right) \frac{\|u_0\|_A^{2+p+1}}{\int_{\Omega} h(x)|u_0|^{1+q} dx} - \mu \|W\|_\infty \left( \frac{p+1}{2^*} \right) \right]^{\frac{1}{p+1}} \cdot \|u_0\|_A^{p+1}
\]

\[
= E(\lambda, \mu) \cdot \|u_0\|_A^{p+1}.
\]
Appendix (2) in Section 6 shows that $E(\lambda, \mu) = 0$ if and only if $\mu = T_\lambda$. Since $\mu \in (0, T_\lambda)$, one has $E(\lambda, \mu) > 0$, and therefore it follows from (5) that $\varphi^\prime_\lambda(t_{\lambda, \mu}) > 0$. Consequently, $\varphi_\lambda(t)$ has exactly two points $0 < t_{\lambda, \mu}^- < t_{\lambda, \mu}^+ < t_{\lambda, \mu}^\prime$ such that

$$\varphi_\lambda(t_{\lambda, \mu}^-) = 0 = \varphi_\lambda(t_{\lambda, \mu}^+)$$

and

$$\varphi^\prime_\lambda(t_{\lambda, \mu}^-) > 0 > \varphi^\prime_\lambda(t_{\lambda, \mu}^+)$$

Appendix (3) and (4) in Section 6 show that $t_{\lambda, \mu}^- u_0 \in F_{\lambda, \mu}^+$ and $t_{\lambda, \mu}^+ u_0 \in F_{\lambda, \mu}^-$. In conclusion, both $F_{\lambda, \mu}^+$ and $F_{\lambda, \mu}^-$ are non-empty sets whenever $\mu \in (0, T_\lambda)$.

(2) At this point, we can claim that $F_{\lambda, \mu}^0 = \{0\}$. Let us prove this claim by a contradiction. Assume that there exists $u_{\lambda, *} \in F_{\lambda, \mu}^0$, $u_{\lambda, *} \neq 0$ such that

$$(1 - q)\|u_{\lambda, *}\|_\lambda^2 = \mu(p - q) \int_\Omega W(x)|u_{\lambda, *}|^{p + 1} dx.$$

Thus,

$$0 = \|u_{\lambda, *}\|_\lambda^2 - \int_\Omega h(x)|u_{\lambda, *}|^{1 + q} dx - \mu \int_\Omega W(x)|u_{\lambda, *}|^{p + 1} dx = \|u_{\lambda, *}\|_\lambda^2 - \int_\Omega h(x)|u_{\lambda, *}|^{1 + q} dx - \frac{1 - q}{p - q}\|u_{\lambda, *}\|_\lambda^2 = \frac{p - 1}{p - q}\|u_{\lambda, *}\|_\lambda^2 - \int_\Omega h(x)|u_{\lambda, *}|^{1 + q} dx.$$ 

Therefore, as $\mu \in (0, T_\lambda)$ and $u_{\lambda, *} \neq 0$, similar to (5) we deduce that

$$0 < E(\lambda, \mu) \cdot \|u_{\lambda, *}\|_\lambda^{p + 1} \leq \frac{1 - q}{p - q} \left( \frac{p - 1}{p - q} \right) \frac{\|u_{\lambda, *}\|_\lambda^{2 \frac{p - 1}{p - q}}}{\int_\Omega h(x)|u_{\lambda, *}|^{1 + q} dx} \cdot \left( \int_\Omega W(x)|u_{\lambda, *}|^{p + 1} dx \right)^{-\frac{p - 1}{p - q}} - \frac{1 - q}{p - q}\|u_{\lambda, *}\|_\lambda^2 = 0,$$

a contradiction. Hence, $u_{\lambda, *} = 0$. That is, $F_{\lambda, \mu}^0 = \{0\}$.

This completes the proof of Lemma 3.1.

We note that $T_\lambda$ is also related to a gap structure in $F_{\lambda, \mu}$:

**Lemma 3.2.** Suppose that $\mu \in (0, T_\lambda)$, then for all $u \in F_{\lambda, \mu}^+$, $U \in F_{\lambda, \mu}^-$,

$$\|U\|_\lambda > A_\lambda(\mu) > A_{\lambda, 0} > \|u\|_\lambda, \|U\|_{p + 1} > B_\lambda(\mu) > B_{\lambda, 0} > \|u\|_{p + 1},$$

where

$$A_{\lambda, 0} = \left[ \frac{p - 1}{p - q} \cdot \|h\|_\infty \cdot \frac{|\Omega|^{\frac{2 - q}{2 - p}}}{(\sqrt{\lambda})^{1 + q}} \right]^{-\frac{1}{p - 1}}.$$
Proof. If \( u \in F_{\lambda,\mu}^{+} \subset F_{\lambda,\mu} \), then necessarily

\[
0 < (1 - q)\|u\|_\lambda^2 - \mu(p - q) \int_\Omega W(x)|u|^{p+1} dx
= (1 - q)\|u\|_\lambda^2 - (p - q) \left[ \|u\|_\lambda^2 - \int_\Omega h(x)|u|^{1+q} dx \right]
= (1 - p)\|u\|_\lambda^2 + (p - q) \int_\Omega h(x)|u|^{1+q} dx.
\]

Hence, it follows from (3) that

\[
(p - 1)\|u\|_\lambda^2 < (p - q) \int_\Omega h(x)|u|^{1+q} dx \leq (p - q) \cdot \|h\|_\infty \cdot |\Omega|^{\frac{p-1}{p+1}} \cdot \left( \frac{\|u\|_\lambda}{\sqrt[2+p/q]{S_\lambda}} \right)^{1+q},
\]

which yields:

\[
\|u\|_\lambda < \left[ \frac{p-q}{p-1} \cdot \|h\|_\infty \cdot |\Omega|^{\frac{p-1}{p+1}} \cdot \left( \frac{\|u\|_\lambda}{\sqrt[2+p/q]{S_\lambda}} \right)^{1+q} \right]^{\frac{1}{p-1}} = A_{\lambda,0};
\]

it follows from (2) and (4) that

\[
(p - 1) \frac{S_\lambda}{|\Omega|^{2 + \frac{p-1}{2(p+1)}}} \|u\|_{p+1}^2 \leq (p - 1) \frac{S_\lambda}{|\Omega|^{2 + \frac{p-1}{2(p+1)}}} \left[ |\Omega|^{\frac{p-1}{2+q}} \cdot \left( \frac{\|u\|_\lambda}{\sqrt[2+p/q]{S_\lambda}} \right)^{p+1} \right]^{\frac{2}{p+1}}
= (p - 1)\|u\|_\lambda^2
< (p - q) \int_\Omega h(x)|u|^{1+q} dx
\leq (p - q)\|h\|_\infty |\Omega|^{\frac{p-1}{p+1}} \|u\|_{p+1}^{1+q},
\]

which yields:

\[
\|u\|_{p+1} < \left[ \frac{p-q}{p-1} \cdot \|h\|_\infty \cdot |\Omega|^{\frac{p-1}{p+1}} \cdot \left( \frac{\|u\|_\lambda}{\sqrt[2+p/q]{S_\lambda}} \right) \right]^{\frac{1}{p-1}} = B_{\lambda,0}.
\]

Moreover, for \( U \in F_{\lambda,\mu}^{-} \), it follows from (2) that

\[
(1 - q)\|U\|_\lambda^2 < \mu(p - q) \int_\Omega W(x)|U|^{p+1} dx
\leq \mu(p - q) \cdot \|W\|_\infty \cdot |\Omega|^{\frac{p-1}{2}} \cdot \left( \frac{\|U\|_\lambda}{\sqrt[2+p/q]{S_\lambda}} \right)^{p+1},
\]
which yields:

$$\|U\|_\lambda \geq \left[ \frac{1-q}{\mu(p-q)} \cdot \|W\|_\infty \cdot \frac{1}{\|\Omega\|^{\frac{1}{2} - \frac{2\lambda}{2(p-1)}}} \right]^{\frac{1}{p-1}} \equiv A_\lambda(\mu);$$

and

$$(1-q) \frac{S_\lambda}{|\Omega|^{\frac{2-\beta}{2(p-1)}}} \|U\|_{p+1}^2 \leq (1-q) \frac{S_\lambda}{\|\Omega\|^{\frac{2-\beta}{2(p-1)}}} \left[ \|U\|_\lambda^{p+1} \right]^{\frac{2}{p-1}}$$

$$\leq (1-q)\|U\|_\lambda^2 \mu(p-q) \int_\Omega W(x)|U|^{p+1}dx$$

$$\leq \mu(p-q)\|W\|_{\infty}\|U\|_{p+1}^{p+1},$$

which yields:

$$\|U\|_{p+1} > \left[ \frac{1-q}{\mu(p-q)} \cdot \|W\|_\infty \cdot \frac{S_\lambda}{\|\Omega\|^{\frac{2-\beta}{2(p-1)}}} \right]^{\frac{1}{p-1}} \equiv B_\lambda(\mu).$$

Appendix (5) and (6) in Section 6 show that $A_\lambda(\mu) = A_{\lambda,0}$ if and only if $\mu = T_\lambda$ and $B_\lambda(\mu) = B_{\lambda,0}$ if and only if $\mu = T_\lambda$, respectively. So for all $\mu \in (0, T_\lambda)$, we can conclude that for all $u \in F_{\lambda,\mu}^+$, $U \in F_{\lambda,\mu}$,

$$\|U\|_\lambda > A_\lambda(\mu) > A_{\lambda,0} > \|u\|_\lambda, \; \|U\|_{p+1} > B_\lambda(\mu) > B_{\lambda,0} > \|u\|_{p+1}.$$

This completes the proof of Lemma 3.2. □

**Lemma 3.3.** Suppose that $\mu \in (0, T_\lambda)$, then $F_{\lambda,\mu}^-$ is a closed set in $H_0^1(\Omega)$-topology.

**Proof.** Let $\{U_n\}$ be a sequence in $F_{\lambda,\mu}^-$ with $U_n \to U_0$ in $H_0^1(\Omega)$. Clearly, $U_0 \in F_{\lambda,\mu}^-$ $\cup$ $F_{\lambda,\mu}^0$. Since $\{U_n\} \subset F_{\lambda,\mu}^-$, from Lemma 3.2 we have that

$$\|U_0\|_\lambda = \lim_{n \to \infty} \|U_n\|_\lambda \geq A_{\lambda,0} > 0,$$

that is, $U_0 \neq 0$. It follows from lemma 3.1 that $U_0 \notin F_{\lambda,\mu}^0$ for any $\mu \in (0, T_\lambda)$. In turn, $U_0 \in F_{\lambda,\mu}^-$. That is, $F_{\lambda,\mu}^-$ is a closed set in $H_0^1(\Omega)$-topology.

This completes the proof of Lemma 3.3. □

**Lemma 3.4.** Given $u \in F_{\lambda,\mu}^\pm$ and $u \geq 0$, then there exist a number $\varepsilon > 0$ and a continuous function $f(w) > 0$, $w \in H_0^1(\Omega)$, $\|w\| < \varepsilon$ satisfying the following:

$$f(0) = 1, \; f(w)(u + w) \in F_{\lambda,\mu}^\pm, \; \forall \; w \in H_0^1(\Omega), \; \|w\| < \varepsilon$$

and

$$\langle f'(0), \varphi \rangle = \int_\Omega \left[ -2(\nabla u \nabla \varphi - \frac{\lambda}{|x|^2} u \varphi) + (1+q)h(x)u^q \varphi + \mu(p+1)W(x)u^p \varphi \right] dx$$

$$(1-q)\|u\|_\lambda^2 - \mu(p-q) \int_\Omega W(x)u^{p+1} dx$$

for any $\varphi \in H_0^1(\Omega)$.

**Proof.** (1) Given $u \in F_{\lambda,\mu}^+$ and $u \geq 0$, we have from Lemma 3.1 that $u \neq 0$. Define $F : H_0^1(\Omega) \times \mathbb{R}^+ \to \mathbb{R}$ as follows:

$$F(w, r) = r^{1-q}\|u+w\|_\lambda^2 - \int_\Omega h(x)|u+w|^{1+q}dx - \mu r^{p-q} \int_\Omega W(x)|u+w|^{p+1}dx.$$
Since $u \in F_{\lambda,\mu}^+ (\subset F_{\lambda,\mu})$, we have that
\[
F(0, 1) = \|u\|_\lambda^2 - \int_{\Omega} h(x)|u|^{1+q}dx - \mu \int_{\Omega} W(x)|u|^{p+1}dx = 0
\]
and
\[
F_r(0, 1) = (1 - q)\|u\|_\lambda^2 - \mu(p - q) \int_{\Omega} W(x)|u|^{p+1}dx > 0.
\]
Applying the implicit function theorem at the point $(0, 1)$, then there exist $\varepsilon > 0$ such that for $w \in H_0^1(\Omega)$, $\|w\| < \varepsilon$, the equation $F(w, r) = 0$ has a unique continuous solution $r = f(w) > 0$. It follows from $F(0, 1) = 0$ that $f(0) = 1$, and from $F(w, f(w)) = 0$ that
\[
f^{1-q}(w)\|u + w\|_\lambda^2 - \int_{\Omega} h(x)|u + w|^{1+q}dx - \mu f^{p-q}(w) \int_{\Omega} W(x)|u + w|^{p+1}dx = 0,
\]
that is,
\[
f(w)(u + w) \in F_{\lambda,\mu}, \quad \forall w \in H_0^1(\Omega), \quad \|w\| < \varepsilon.
\]
Since $F_r(0, 1) > 0$ and
\[
F_r(w, f(w)) = (1 - q) f^{-q}(w)\|u + w\|_\lambda^2 - \mu(p - q) f^{p-q-1}(w) \int_{\Omega} W(x)|u + w|^{p+1}dx
\]
\[
= (1 - q)\|f(w)(u + w)\|_\lambda^2 - \mu(p - q) \int_{\Omega} W(x)f(w)(u + w)|^{p+1}dx\frac{f^{p+q}(w)}{f^{2+q}(w)},
\]
we can take $\varepsilon > 0$ possibly smaller ($\varepsilon < \varepsilon$) such that for any $w \in H_0^1(\Omega)$, $\|w\| < \varepsilon$,
\[
(1 - q)\|f(w)(u + w)\|_\lambda^2 - \mu(p - q) \int_{\Omega} W(x)f(w)(u + w)|^{p+1}dx > 0,
\]
that is,
\[
f(w)(u + w) \in F_{\lambda,\mu}^+, \quad \forall w \in H_0^1(\Omega), \quad \|w\| < \varepsilon.
\]
Moreover, for any $\varphi \in H_0^1(\Omega)$, $v > 0$, from
\[
F(0 + v\varphi, 1) - F(0, 1)
\]
\[
= \int_{\Omega} \left[|\nabla(u + v\varphi)|^2 - \frac{\lambda}{|x|^2}(u + v\varphi)^2\right]dx - \int_{\Omega} h(x)|u + v\varphi|^{1+q}dx
\]
\[- \mu \int_{\Omega} W(x)|u + v\varphi|^{p+1}dx - \int_{\Omega} \left(|\nabla u|^2 - \frac{\lambda}{|x|^2}u^2\right)dx
\]
\[+ \int_{\Omega} h(x)|u|^{1+q}dx + \mu \int_{\Omega} W(x)|u|^{p+1}dx
\]
\[= \int_{\Omega} \left[2v \cdot \nabla u \nabla \varphi + v^2 \cdot |\nabla \varphi|^2 - \frac{\lambda}{|x|^2} (2v \cdot u \varphi + v^2 \cdot \varphi^2)\right]dx
\]
\[+ \int_{\Omega} h(x) \left(|u + v\varphi|^{1+q} - |u|^{1+q}\right)dx - \mu \int_{\Omega} W(x) \left(|u + v\varphi|^{p+1} - |u|^{p+1}\right)dx
\]
we have that
\[
\langle F_w, \varphi \rangle |_{w=0,r=1} = \lim_{v \to 0} \frac{F(0 + v \varphi, 1) - F(0, 1)}{v}
\]
\[
= 2 \int_{\Omega} \left( \nabla u \nabla \varphi - \frac{\lambda}{|x|^2} u \varphi \right) dx - (1 + q) \int_{\Omega} h(x)|u|^{q-1} u \varphi dx
\]
\[
- \mu(p + 1) \int_{\Omega} W(x)|u|^{p-1} u \varphi dx
\]
\[
= 2 \int_{\Omega} \left( \nabla u \nabla \varphi - \frac{\lambda}{|x|^2} u \varphi \right) dx - (1 + q) \int_{\Omega} h(x)|u|^q \varphi dx
\]
\[
- \mu(p + 1) \int_{\Omega} W(x)|u|^p \varphi dx.
\]
Therefore, we have that
\[
\langle f'(0), \varphi \rangle
\]
\[
= - \frac{\langle F_w, \varphi \rangle}{F_r} |_{w=0,r=1}
\]
\[
= \int_{\Omega} \left[ -2 \left( \nabla u \nabla \varphi - \frac{\lambda}{|x|^2} u \varphi \right) + (1 + q)h(x)|u|^q + \mu(p + 1)W(x)|u|^p \right] dx
\]
\[
\times (1 - q)\|u\|_{\lambda}^2 - \mu(p - q) \int_{\Omega} W(x)|u|^{p+1} dx.
\]
(2) The case \(u \in F_{\lambda,\mu}^-\) may be preceded exactly, so we omit the details here.
This completes the proof of Lemma 3.4. \(\square\)

**Proof of Theorem 1.1.** For any \(u \in F_{\lambda,\mu}\), we easily derive from (3) that
\[
I_{\lambda,\mu}(u) - \frac{1}{2} \|u\|_{\lambda}^2 - \frac{1}{1 + q} \int_{\Omega} h(x)|u|^{1+q} dx - \frac{\mu}{p + 1} \int_{\Omega} W(x)|u|^{p+1} dx
\]
\[
= \frac{1}{2} \|u\|_{\lambda}^2 - \frac{1}{1 + q} \int_{\Omega} h(x)|u|^{1+q} dx - \frac{1}{p + 1} \left[ \|u\|_{\lambda}^2 - \int_{\Omega} h(x)|u|^{1+q} dx \right]
\]
\[
= \left( \frac{1}{2} - \frac{1}{p + 1} \right) \|u\|_{\lambda}^2 - \left( \frac{1}{1 + q} - \frac{1}{p + 1} \right) \int_{\Omega} h(x)|u|^{1+q} dx
\]
\[
\geq \left( \frac{1}{2} - \frac{1}{p + 1} \right) \|u\|_{\lambda}^2 - \left( \frac{1}{1 + q} - \frac{1}{p + 1} \right) \cdot \|h\|_{\infty} \cdot |\Omega|^{\frac{2^* - 1 - q}{2}} \cdot \left( \frac{\|u\|_{\lambda}}{\sqrt{\lambda}} \right)^{1+q}
\]
\[
= \left( \frac{1}{2} - \frac{1}{p + 1} \right) \|u\|_{\lambda}^2 - \left( \frac{1}{1 + q} - \frac{1}{p + 1} \right) \cdot \|h\|_{\infty} \cdot |\Omega|^{\frac{2^* - 1 - q}{2}} \cdot \frac{\|u\|_{\lambda}^{1+q}}{(\sqrt{\lambda})^{1+q}}.
\]
Therefore, Appendix (7) in Section 6 shows that \(I_{\lambda,\mu}\) is coercive and bounded below on \(F_{\lambda,\mu}\). From Lemma 3.3, the Ekeland variational principle (see [2, Corollary 5.3.2]) can be applied to the problem of finding the infimum of \(I_{\lambda,\mu}\) on \(F_{\lambda,\mu}^+ \cup F_{\lambda,\mu}^0\). Consider \(\{u_n\} \subseteq F_{\lambda,\mu}^+ \cup F_{\lambda,\mu}^0\) with the properties:
(i): \(I_{\lambda,\mu}(u_n) < \inf_{F_{\lambda,\mu}^+ \cup F_{\lambda,\mu}^0} I_{\lambda,\mu} + \frac{1}{n}\);
(ii): \(I_{\lambda,\mu}(u) \geq I_{\lambda,\mu}(u_n) - \frac{1}{n} \|u - u_n\|, \forall u \in F_{\lambda,\mu}^+ \cup F_{\lambda,\mu}^0\).
Since \(I_{\lambda,\mu}(\|u\|) = I_{\lambda,\mu}(u)\), we may assume that \(u_n \geq 0\) on \(\Omega\). Since \(I_{\lambda,\mu}\) is bounded
below on $F_{\lambda,\mu}$, it follows from condition (i) that $\{u_n\}$ is bounded in $H^1_0(\Omega)$. Going if necessary to a subsequence, we can assume that

$$u_n \rightharpoonup u_\lambda \text{ in } H^1_0(\Omega),$$
$$u_n \rightarrow u_\lambda \text{ a.e. in } \Omega,$$
$$u_n \rightarrow u_\lambda \text{ in } L^{1+q}(\Omega) \text{ and } L^{p+1}(\Omega).$$

For $u \in F_{\lambda,\mu}^+ (\subset F_{\lambda,\mu})$, we have:

$$I_{\lambda,\mu}(u) = \frac{1}{2} \|u\|_\lambda^2 - \frac{1}{1+q} \int_\Omega h(x)|u|^{1+q}dx - \frac{\mu}{p+1} \int_\Omega W(x)|u|^{p+1}dx$$

$$= \frac{1}{2} \|u\|_\lambda^2 - \frac{1}{1+q} \left[ \|u\|_\lambda^2 - \mu \int_\Omega W(x)|u|^{p+1}dx \right] - \frac{\mu}{p+1} \int_\Omega W(x)|u|^{p+1}dx$$

$$= \left( \frac{1}{2} - \frac{1}{1+q} \right) \|u\|_\lambda^2 + \left( \frac{1}{1+q} - \frac{1}{1+q} - \frac{\mu}{p+1} \right) \cdot \mu \int_\Omega W(x)|u|^{p+1}dx$$

$$< \left( \frac{1}{2} - \frac{1}{1+q} \right) \|u\|_\lambda^2 + \left( \frac{1}{1+q} - \frac{1}{1+q} + \frac{\mu}{p+1} \right) \cdot \frac{1-q}{p-q} \|u\|_\lambda^2$$

$$= \frac{1-q}{1+q} \left( \frac{1}{p+1} - \frac{1}{2} \right) \|u\|_\lambda^2 < 0 \text{ (since } p > 1),$$

which means that $\inf_{F_{\lambda,\mu}^+} I_{\lambda,\mu} < 0$; given $\mu \in (0,T_\lambda)$, from lemma 3.1 we get that $F_{\lambda,\mu}^0 = \{0\}$. Together, these imply that $u_n \in F_{\lambda,\mu}^+$ and

$$\inf_{F_{\lambda,\mu}^+ \cup F_{\lambda,\mu}^0} I_{\lambda,\mu} = \inf_{F_{\lambda,\mu}^+} I_{\lambda,\mu} < 0.$$

While by the weak lower semi-continuity of norm,

$$I_{\lambda,\mu}(u_\lambda) \leq \liminf_{n \rightarrow \infty} I_{\lambda,\mu}(u_n) = \inf_{F_{\lambda,\mu}^+ \cup F_{\lambda,\mu}^0} I_{\lambda,\mu} < 0,$$

we see that $u_\lambda \geq 0$ and $u_\lambda \not\equiv 0$.

It follows from $\{u_n\} \subset F_{\lambda,\mu}^+ (\subset F_{\lambda,\mu})$ that

$$(1-q) \int_\Omega h(x)u_n^{1+q}dx - \mu(p-1) \int_\Omega W(x)u_n^{p+1}dx$$

$$= \lim_{n \rightarrow \infty} \left[ (1-q) \int_\Omega h(x)u_n^{1+q}dx - \mu(p-1) \int_\Omega W(x)u_n^{p+1}dx \right]$$

$$= \lim_{n \rightarrow \infty} \left[ (1-q) \|u_n\|_\lambda^2 - \mu(p-q) \int_\Omega W(x)u_n^{p+1}dx \right] \geq 0.$$
Since \( u_n \in F_{\lambda, \mu} \), it follows from (7) that
\[
0 = \lim_{n \to \infty} \left[ \|u_n\|_{\lambda}^2 - \int_{\Omega} h(x)u_n^{1+q}dx - \mu \int_{\Omega} W(x)u_n^{p+1}dx \right]
\geq \|u_\lambda\|_{\lambda}^2 - \int_{\Omega} h(x)u_\lambda^{1+q}dx - \mu \int_{\Omega} W(x)u_\lambda^{p+1}dx
= \begin{cases} 
\|u_\lambda\|_{\lambda}^2 - \mu \cdot \frac{p-q}{p-q} \int_{\Omega} W(x)u_\lambda^{p+1}dx, \\
\|u_\lambda\|_{\lambda}^2 - \frac{p-q}{p-q} \int_{\Omega} h(x)u_\lambda^{1+q}dx.
\end{cases}
\]
Thus, for \( \mu \in (0, T_\lambda) \) and \( u_\lambda \not= 0 \), by the same arguments as those in (5), we have that
\[
0 < E(\lambda, \mu) \cdot \|u_\lambda\|_{\lambda}^{p+1}
\leq \frac{1-q}{p-q} \left( \frac{p-1}{p-q} \right)^{\frac{p-1}{p-q}} \cdot \frac{\|u_\lambda\|_{\lambda}^2 \cdot \frac{p-q}{p-q}}{\left[ \int_{\Omega} h(x)u_\lambda^{1+q}dx \right]^{\frac{p-1}{p-q}}} - \mu \int_{\Omega} W(x)u_\lambda^{p+1}dx
\leq \frac{1-q}{p-q} \cdot \left( \frac{p-1}{p-q} \right)^{\frac{p-1}{p-q}} \cdot \frac{\|u_\lambda\|_{\lambda}^2 \cdot \frac{p-q}{p-q}}{\left( \frac{p-1}{p-q} \|u_\lambda\|_{\lambda}^2 \right)^{\frac{p-1}{p-q}}} - \frac{1-q}{p-q} \|u_\lambda\|_{\lambda}^2 = 0,
\]
which is clearly impossible. So by (6), we get that
\[
(1-q) \int_{\Omega} h(x)u_n^{1+q}dx - \mu(p-1) \int_{\Omega} W(x)u_n^{p+1}dx \geq C_1
\]
for \( n \) sufficiently large and a suitable positive constant \( C_1 \). This, together with the fact \( u_n \in F_{\lambda, \mu} \) implies
\[
(1-q)\|u_n\|_{\lambda}^2 - \mu(p-1) \int_{\Omega} W(x)u_n^{p+1}dx \geq C_1. \tag{8}
\]
Fixing \( \varphi \in H_0^1(\Omega) \) and using Lemma 3.4 with \( u = u_n \in F_{\lambda, \mu}^+ \) and \( w = t\varphi, \ t > 0 \) sufficiently small, we find \( f_n = f_n(t\varphi) > 0 \) with \( f_n(0) = 1 \) and \( f_n(t\varphi)(u_n + t\varphi) \in F_{\lambda, \mu}^+ \). Therefore, applying condition (ii) with \( u = f_n(t\varphi)(u_n + t\varphi) \in F_{\lambda, \mu}^+ \), we have:
\[
\begin{align*}
&\frac{1}{n} \left[ |f_n(t\varphi) - 1| \cdot \|u_n\| + tf_n(t\varphi) \cdot \|\varphi\| \right] \\
&\geq \frac{1}{n} \|f_n(t\varphi) - 1\| \cdot u_n + tf_n(t\varphi) \cdot \varphi \\
&= \frac{1}{n} \|f_n(t\varphi)(u_n + t\varphi) - u_n\| \\
&\geq I_{\lambda, \mu}(u_n) - I_{\lambda, \mu}(f_n(t\varphi)(u_n + t\varphi)) \\
&= \frac{1}{2} \|u_n\|_{\lambda}^2 - \frac{1}{1+q} \int_{\Omega} h(x)|u_n|^{1+q}dx - \frac{\mu}{p+1} \int_{\Omega} W(x)|u_n|^{p+1}dx \\
&- \frac{1}{2} \|f_n(t\varphi)(u_n + t\varphi)\|_{\lambda}^2 + \frac{1}{1+q} \int_{\Omega} h(x)|f_n(t\varphi)(u_n + t\varphi)|^{1+q}dx \\
&+ \frac{\mu}{p+1} \int_{\Omega} W(x)|f_n(t\varphi)(u_n + t\varphi)|^{p+1}dx
\end{align*}
\]
we conclude:

For taking into account (8), we get that

Note that, from Lemma 3.4 it follows:

Dividing by $t > 0$ and passing to the limit for $t \to 0^+$, we get that

\[
\frac{1}{n} \left[ |f'_n(0)\varphi| \cdot \|u_n\| + \|\varphi\| \right] \\
\geq - f'_n(0)\varphi \cdot \|u_n\|^2 - \int_\Omega \left( \nabla u_n \cdot \nabla \varphi - \frac{\lambda}{|x|^2} u_n \varphi \right) dx + f'_n(0)\varphi \int_\Omega h(x)u_1^{1+q}dx \\
+ \int_\Omega h(x)u_n^p \varphi dx + \mu f'_n(0)\varphi \int_\Omega W(x)u_n^{p+1}dx + \mu \int_\Omega W(x)u_n^p \varphi dx \\
= - f'_n(0)\varphi \left[ \|u_n\|^2 - \int_\Omega h(x)u_n^{1+q}dx - \mu \int_\Omega W(x)u_n^{p+1}dx \right] \\
- \int_\Omega \left( \nabla u_n \nabla \varphi - \frac{\lambda}{|x|^2} u_n \varphi \right) dx + \int_\Omega h(x)u_n^p \varphi dx + \mu \int_\Omega W(x)u_n^p \varphi dx \\
= - \int_\Omega \left( \nabla u_n \nabla \varphi - \frac{\lambda}{|x|^2} u_n \varphi \right) dx + \int_\Omega h(x)u_n^p \varphi dx + \mu \int_\Omega W(x)u_n^p \varphi dx. \tag{9}
\]

Note that, from Lemma 3.4 it follows:

\[
|f'_n(0)\varphi| \\
= \left| \int_\Omega \left[ -2 \left( \nabla u_n \nabla \varphi - \frac{\lambda}{|x|^2} u_n \varphi \right) + (1 + q)h(x)u_n^p \varphi + \mu(p + 1)W(x)u_n^p \varphi \right] dx \right| \\
\frac{(1 - q)\|u_n\|^2 - \mu(p - q)\int_\Omega W(x)u_n^{p+1}dx}{\int_\Omega h(x)u_n^{1+q}dx}.
\]

Taking into account (8), we get that

\[
|f'_n(0)\varphi| \leq C_2
\]

for $n = 1, 2, \ldots$ and a suitable positive constant $C_2$. Therefore, letting $n \to \infty$ from (9) we conclude:

\[
\int_\Omega \left( \nabla u_\lambda \nabla \varphi - \frac{\lambda}{|x|^2} u_\lambda \varphi \right) dx - \int_\Omega h(x)u_\lambda^p \varphi dx - \mu \int_\Omega W(x)u_\lambda^p \varphi dx \geq 0.
\]

Since this holds equally well for $-\varphi$, it follows that $u_\lambda$ is indeed a solution of problem (1.1), that is, for all $\varphi \in H^1_0(\Omega)$, there holds:

\[
\int_\Omega \left( \nabla u_\lambda \nabla \varphi - \frac{\lambda}{|x|^2} u_\lambda \varphi \right) dx - \int_\Omega h(x)u_\lambda^p \varphi dx - \mu \int_\Omega W(x)u_\lambda^p \varphi dx = 0. \tag{10}
\]
In particular, \( u_\lambda \in F_{\lambda,\mu} \) (use (10) with \( \varphi = u_\lambda \)). Moreover, from (6) it follows necessarily:
\[
(1-q)\|u_\lambda\|_X^2 - \mu(p-q) \int_\Omega W(x)u_{\lambda}^{p+1}dx = (1-q) \left[ \int_\Omega h(x)u_{\lambda}^{1+q}dx + \mu \int_\Omega W(x)u_{\lambda}^{p+1}dx \right] - \mu(p-q) \int_\Omega W(x)u_{\lambda}^{p+1}dx
\]
\[
= (1-q) \int_\Omega h(x)u_{\lambda}^{1+q}dx - \mu(p-1) \int_\Omega W(x)u_{\lambda}^{p+1}dx > 0,
\]
that is, \( u_\lambda \in F_{\lambda,\mu}^+ \). Also, we can get that
\[
\lim_{n \to \infty} \|u_n\|_X^2 = \lim_{n \to \infty} \left[ \int_\Omega h(x)u_n^{1+q}dx + \mu \int_\Omega W(x)u_n^{p+1}dx \right] = \int_\Omega h(x)u_\lambda^{1+q}dx + \mu \int_\Omega W(x)u_\lambda^{p+1}dx = \|u_\lambda\|_X^2.
\]
Consequently, \( u_n \to u_\lambda \) in \( H^1_0(\Omega) \).

Since \( I_{\lambda,\mu}(u) = I_{\lambda,\mu}(u) \) and \( F_{\lambda,\mu}^- \) is closed whenever \( \mu \in (0,T_\lambda) \) (see Lemma 3.3), using the Ekeland variational principle again, we may find a minimizing sequence \( \{u_n\} \subset F_{\lambda,\mu}^- \), \( u_n \geq 0 \) for the minimization problem \( \inf_{F_{\lambda,\mu}^-} I_{\lambda,\mu} \) such that for \( u_n \in H^1_0(\Omega) \), we have \( u_n \to u_\lambda \) in \( H^1_0(\Omega) \) and pointwise a.e. in \( \Omega \). It from Lemma 3.2 that \( \|u_\lambda\|_X \geq A_{\lambda,\mu} > 0 \), and thus \( U_{\lambda} \neq 0 \). We can now repeat the arguments used between (6) and (8) to derive that when \( \mu \in (0,T_\lambda) \),
\[
(1-q) \int_\Omega h(x)U_{\lambda}^{1+q}dx - \mu(p-1) \int_\Omega W(x)U_{\lambda}^{p+1}dx < 0,
\]
which yields:
\[
(1-q) \int_\Omega h(x)U_n^{1+q}dx - \mu(p-1) \int_\Omega W(x)U_n^{p+1}dx \leq -C_3
\]
for \( n \) sufficiently large and a suitable positive constant \( C_3 \). At this point we may proceed exactly as above and conclude that \( U_{\lambda} \) is a solution of problem (1.1).

In particular \( U_{\lambda} \in F_{\lambda,\mu} \). Moreover, from (11) it follows that
\[
(1-q)\|U_{\lambda}\|_X^2 - \mu(p-q) \int_\Omega W(x)U_{\lambda}^{p+1}dx = (1-q) \left[ \int_\Omega h(x)U_{\lambda}^{1+q}dx + \mu \int_\Omega W(x)U_{\lambda}^{p+1}dx \right] - \mu(p-q) \int_\Omega W(x)U_{\lambda}^{p+1}dx
\]
\[
= (1-q) \int_\Omega h(x)U_{\lambda}^{1+q}dx - \mu(p-1) \int_\Omega W(x)U_{\lambda}^{p+1}dx < 0,
\]
that is, \( U_{\lambda} \in F_{\lambda,\mu}^- \).

From Lemma 3.2, we can conclude that problem (1.1) has at least two solutions \( u_\lambda \in F_{\lambda,\mu}^+, U_{\lambda} \in F_{\lambda,\mu}^- \) with \( \|U_{\lambda}\|_X > A_{\lambda,\mu} > A_{\lambda,0} > \|u_\lambda\|_X \) for any \( \mu \in (0,T_\lambda) \).

This completes the proof of Theorem 1.1. \( \square \)
4. Proof of Theorem 1.2.

Proof. For any $U \in F_{\lambda,\mu}^-$, it follows from Lemma 3.2 that

$$
\|U\|_{\lambda} > A_{\lambda}(\mu) = \left[ \frac{1 - q}{\mu(p - q)} \cdot \frac{1}{\|W\|_{\infty}} \cdot \frac{(\sqrt{S_{\lambda}})^{p+1}}{|\Omega|^{\frac{2^* - p - 1}{2(p - 1)}}} \right]^\frac{1}{p-1}
$$

$$
= \left( \frac{1}{\mu} \right)^\frac{1}{p-1} \cdot \left( \frac{1 - q}{p - q} \right)^\frac{1}{p-1} \cdot \left( \frac{1}{\|W\|_{\infty}} \right)^\frac{1}{p-1} \cdot \left( \frac{(\sqrt{S_{\lambda}})^{p+1}}{|\Omega|^{\frac{2^* - p - 1}{2(p - 1)}}} \right)^\frac{1}{p-1} \cdot \left( T_{\lambda} \right)^\frac{1}{p-1}
$$

Thus, by the definition of $T_{\lambda}$ we obtain that

$$
\|U\|_{\lambda} > \left( \frac{p - q}{1 - q} \right)^\frac{1}{p-1} \cdot \left( \frac{p - q}{p - 1} \right)^\frac{1}{p-1} \cdot \left( \frac{(\sqrt{S_{\lambda}})^{p+1}}{|\Omega|^{\frac{2^* - p - 1}{2(p - 1)}}} \right)^\frac{1}{p-1} \cdot \left( T_{\lambda} \right)^\frac{1}{p-1}
$$

$$
= \left( \frac{p - q}{p - 1} \right)^\frac{1}{p-1} \cdot \left( \frac{(\sqrt{S_{\lambda}})^{p+1}}{|\Omega|^{\frac{2^* - p - 1}{2(p - 1)}}} \right)^\frac{1}{p-1} \cdot \left( T_{\lambda} \right)^\frac{1}{p-1}
$$

$$
= \left| \frac{1}{\sqrt{S_{\lambda}}} \right| \cdot \left( \frac{1}{p - 1} \right)^\frac{1}{p-1} \cdot \left( 1 + \frac{1 - q}{p - 1} \right) \cdot \left( \frac{(\sqrt{S_{\lambda}})^{p+1}}{|\Omega|^{\frac{2^* - p - 1}{2(p - 1)}}} \right)^\frac{1}{p-1} \cdot \left( T_{\lambda} \right)^\frac{1}{p-1},
$$

where we use the relations

$$
\frac{2}{N} \cdot \frac{p - q}{(1 - q)(p - 1)} - \frac{2^* - p - 1}{2^*(p - 1)} - \frac{2^* - p}{(1 - q)(p - 1)} = \frac{2^* - 2}{2^*(p - 1)} = \frac{p - q}{(1 - q)(p - 1)} - \frac{2^* - p - 1}{2^*(p - 1)}
$$

and

$$
\frac{2}{N} \cdot \frac{p - q}{(1 - q)(p - 1)} - \frac{p + 1}{p - 1} = \frac{p - q + pq - 1}{(1 - q)(p - 1)} = \frac{(p - 1)(1 + q)}{(1 - q)(p - 1)} = \frac{1 + q}{1 - q}
$$

Hence, letting $U_{\lambda,\varepsilon} \in F_{\lambda,\mu}^-$ be the solution of problem (1.1) with $p = 1 + \varepsilon$, where $\mu \in (0, T_{\lambda})$, we have that

$$
\|U_{\lambda,\varepsilon}\|_{\lambda} > C_{\lambda,\varepsilon} \cdot \left( \frac{T_{\lambda}}{\mu} \right)^\frac{1}{p-1}
$$

with

$$
C_{\lambda,\varepsilon} = \left| \Omega \right|^{\frac{1}{2}} \cdot \left( \frac{(\sqrt{S_{\lambda}})^{p+1}}{|\Omega|^{\frac{2^* - p - 1}{2(p - 1)}}} \right)^\frac{1}{p-1} \cdot \left( 1 + \frac{1 - q}{\varepsilon} \right)^\frac{1}{p-1} \to \infty \text{ as } \varepsilon \to 0^+.
$$

This completes the proof of Theorem 1.2. \(\square\)
5. **Proof of Theorem 1.3.** In this section, we prove Theorem 1.3. Let us define

$$\mu^* = \mu^*(N, \Omega, \lambda, q, p) = \sup \{ \mu > 0 : \text{problem (1.3)}_{\mu} \text{ has a positive solution} \}.$$ 

Then we provide:

**Lemma 5.1.** For $N \geq 3$, $\Omega \in L$, where $L = \{ \Omega \subset \mathbb{R}^N : \Omega \text{ is open and bounded domain} \}$, $0 < \lambda < \Lambda$ and $0 < q < 1 < p < 2^* - 1$, we have that

$$0 < \mu^* \leq \mu^+ < \infty,$$

where

$$\mu^- = \frac{1 - q}{p - q} \left( \frac{p - 1}{p - q} \right)^{\frac{p - 1}{q - 1}} \left[ \frac{S_{\lambda}^N}{|\Omega|^{\frac{p}{q}}} \right]^{\frac{q - q}{q - 1}} (T_\lambda \text{ provided } h, W \equiv 1)$$

and

$$\mu^+ = \mu_1 \left( 1 - q \right) \left[ \frac{\mu_1 (p - 1)}{p - q} \right]^{\frac{p - 1}{q - 1}} + 1 \ (\mu_1 \text{ is defined in (1)}).$$

**Proof.** (1) Suppose that $\mu \in (0, \mu^-)$, then problem (1.3)$_{\mu}$ has at least two solutions (apply Theorem 1.1 with $h, W \equiv 1$). Thus, it follows from the definition of $\mu^*$ that $\mu^* \geq \mu^- > 0$.

(2) From Appendix (8)-(i) in Section 6, we can choose $\mu^+ > 0$ such that

$$p^q + \mu^+ t^p > \mu_1 t, \ \forall \ t > 0. \quad (12)$$

If any $\mu$ is such that problem (1.3)$_{\mu}$ has a positive solution $u$, multiplying (1.3)$_{\mu}$ by $\varphi_1$ and integrating over $\Omega$ we find:

$$\mu_1 \int_{\Omega} u \varphi_1 dx = \int_{\Omega} \left( -\Delta u - \frac{\lambda}{|x|^2} u \right) \varphi_1 dx = \int_{\Omega} u^q \varphi_1 dx + \mu \int_{\Omega} u^p \varphi_1 dx.$$

Applying (12) with $t = u$, multiplying (12) by $\varphi_1$ and integrating over $\Omega$ we have:

$$\int_{\Omega} u^q \varphi_1 dx + \mu^+ \int_{\Omega} u^p \varphi_1 dx > \mu_1 \int_{\Omega} u^q \varphi_1 dx,$$

which immediately implies that $\mu < \mu^+$. Thus, $\mu^* \leq \mu^+ < \infty$.

This completes the proof of Lemma 5.1.

**Proof of Theorem 1.3.** (1) Given $0 < \mu < \mu^*$, from the definition of $\mu^*$, there exists $\mu < \theta < \mu^*$ such that problem (1.3)$_{\theta}$:

$$-\Delta u - \frac{\lambda}{|x|^2} u = u^q + \theta u^p$$

has a positive solution, denoted by $u_{\theta}$. Then we have:

$$-\Delta u_{\theta} - \frac{\lambda}{|x|^2} u_{\theta} = u_{\theta}^q + \theta u_{\theta}^p \geq u_{\theta}^q + \mu u_{\theta}^p,$$

that is, $u_{\theta}$ is a super-solution of (1.3)$_{\mu}$. Moreover, from (1) and Appendix (8)-(ii) in Section 6, we can find $\varepsilon > 0$ sufficiently small such that $\varepsilon \varphi_1 < u_{\theta}$ and

$$-\Delta (\varepsilon \varphi_1) - \frac{\lambda}{|x|^2} \varepsilon \varphi_1 = \mu_1 \cdot \varepsilon \varphi_1 \leq (\varepsilon \varphi_1)^q + \mu (\varepsilon \varphi_1)^p \quad \text{for all } \mu > 0, \quad (13)$$

that is, the function $\varepsilon \varphi_1$ is a sub-solution of problem (1.3)$_{\mu}$. Since $-\Delta - \frac{\lambda}{|x|^2}$ is a monotone operator, it follows that problem (1.3)$_{\mu}$ has a positive solution $u_{\mu}$ satisfying $\varepsilon \varphi_1 \leq u_{\mu} \leq u_{\theta}$.

(2) For $\mu > \mu^*$, the proof follows from the definition of $\mu^*$.
This completes the proof of Theorem 1.3. □

Remark 2. For \( \mu = \mu^* \), let \( \{ \mu_n \} \) be a sequence such that \( \mu_n \uparrow \mu^* \) and \( u_n = u_{\mu_n} \) be a solution of problem (1.3)\( _{\mu_n} \), that is, for any \( \varphi \in H^1_0(\Omega) \), there holds

\[
\int_\Omega \left( \nabla u_n \nabla \varphi - \frac{\lambda}{|x|^2} u_n \varphi \right) dx - \int_\Omega u_n^q \varphi dx - \mu_n \int_\Omega u_n^p \varphi dx = 0. \tag{14}
\]

If one can show that \( \{ u_n \} \) is bounded in \( H^1_0(\Omega) \) (there are still some difficulties), then going if necessary to subsequence, we can assume that

\[
\begin{align*}
&u_n \rightharpoonup u^* \text{ in } H^1_0(\Omega), \\
u_n \rightarrow u^* \text{ a.e. in } \Omega, \\
u_n \rightarrow u^* \text{ in } L^{1+q}(\Omega) \text{ and } L^{p+1}(\Omega).
\end{align*}
\]

Letting \( n \rightarrow \infty \) from (14), we get that such a \( u^* \) is thus a weak solution of problem (1.3)\( _{\mu^*} \), that is, for \( \mu = \mu^* \), problem (1.3)\( _{\mu} \) has at least a solution.

6. Appendix.

(1) \( \varphi_{\lambda}(t) \) achieves its maximum at \( t_{\lambda, \max} \).

Proof. Let

\[
f(t) = at^{1-p} - bt^{q-p} - \mu c, \ t > 0,
\]

where \( a, b, c \) are positive constants. Then \( f(t) \) achieves its maximum at \( t_{\max} \):

\[
f'(t) = (at^{1-p} - bt^{q-p} - \mu c)' = a(1-p)t^{p-1} - b(q-p)t^{q-p-1}
\]

\[
\Rightarrow \begin{cases}
\text{Let } f'(t) = 0, \text{ then } t = \left[ \frac{a(p-1)}{b(p-q)} \right]^{-\frac{1}{p-q}} \equiv t_{\max} > 0, \\
f''(t) = -ap(1-p)t^{p-2} - b(q-p)(q-p-1)t^{q-p-2}
\end{cases}
\]

\[
\Rightarrow f''(t_{\max}) = ap(1-p) \left[ \frac{a(p-1)}{b(p-q)} \right]^{\frac{1}{p-q}} - b(p-q)(q-p+1) \left[ \frac{a(p-1)}{b(p-q)} \right]^{\frac{p+q+2}{p-q}} < 0.
\]

Applying \( f(t) \) and \( t_{\max} \) with

\[
a = \| u_0 \|_{\lambda}^2, \ b = \int_\Omega h(x)|u_0|^{1+q} dx \text{ and } c = \int_\Omega W(x)|u_0|^{p+1} dx,
\]

we have that

\[
\varphi_{\lambda}(t) = t^{1-p} \| u_0 \|_{\lambda}^2 - t^{q-p} \int_\Omega h(x)|u_0|^{1+q} dx - \mu \int_\Omega W(x)|u_0|^{p+1} dx
\]

achieves its maximum at

\[
t_{\lambda, \max} = \left[ \frac{(p-1) \| u_0 \|_{\lambda}^2}{(p-q) \int_\Omega h(x)|u_0|^{1+q} dx} \right]^{\frac{1}{1-q}}.
\]

This completes the proof of Appendix (1). □

(2) \( E(\lambda, \mu) = 0 \iff \mu = T_\lambda. \)
Proof. One can show that

\[ E(\lambda, \mu) = \frac{1 - q}{p - q} \cdot \left( \frac{p - 1}{p - q} \right)^{\frac{p-1}{q}} \cdot \left( \frac{1}{\| h \|_\infty} \right)^{\frac{p-1}{q}} \cdot \left( \frac{\sqrt{S_\lambda}}{\| \Omega \|} \right)^{(p-1)(1+q) \frac{1}{1-q}} \cdot \left( \frac{\| \Omega \|}{\Omega} \right)^{\frac{2^*-p-1}{2^*(1-q)}} = 0 \]

\[ \iff \mu = \frac{1 - q}{p - q} \cdot \left( \frac{p - 1}{p - q} \right)^{\frac{p-1}{q}} \cdot \left( \frac{1}{\| h \|_\infty} \right)^{\frac{p-1}{q}} \cdot \left( \frac{\sqrt{S_\lambda}}{\| \Omega \|} \right)^{(p-1)(1+q) \frac{1}{1-q}} \cdot \left( \frac{\| \Omega \|}{\Omega} \right)^{\frac{2^*-p-1}{2^*(1-q)}} \]

where we use the relations

\[ \frac{(p-1)(1+q)}{1-q} + p + 1 = \frac{(p-1)(1+q) + (p+1)(1-q)}{1-q} = 2 \cdot \frac{p - q}{1 - q} \]

and

\[ \frac{(2^*-1-q)(p-1)}{2^*(1-q)} + \frac{2^*-p-1}{2^*} = \frac{(2^*-2)(p-q)}{2^*(1-q)} = \frac{2}{N} \cdot \frac{p - q}{1 - q} \]

This completes the proof of Appendix (2).

(3) **Suppose that** \( \varphi_\lambda(t) = 0 \) **and** \( \varphi'_\lambda(t) > 0 \), **then** \( tu_0 \in F_{\lambda,\mu}^+ \).

**Proof.** We have that

\[ \varphi_\lambda(t) = 0 \Rightarrow t^{1-p}\| u_0 \|_2 - t^{q-p} \int_\Omega h(x)|u_0|^{1+q}dx - \mu \int_\Omega W(x)|u_0|^{p+1}dx = 0 \]

\[ \Rightarrow \| tu_0 \|_\Delta^2 = \int_\Omega h(x)|tu_0|^{1+q}dx + \mu \int_\Omega W(x)|tu_0|^{p+1}dx \]

\[ \Rightarrow tu_0 \in F_{\lambda,\mu}, \]

and therefore

\[ \varphi'_\lambda(t) > 0 \Rightarrow (1-p)t^{1-p}\| u_0 \|_2^2 - (q-p)t^{q-p-1} \int_\Omega h(x)|u_0|^{1+q}dx > 0 \]

\[ \Rightarrow (1-p)\| tu_0 \|_\Delta^2 - (q-p) \int_\Omega h(x)|tu_0|^{1+q}dx > 0 \]

\[ \Rightarrow (1-p)\| tu_0 \|_\Delta^2 - (q-p) \left[ \| tu_0 \|_\Delta^2 - \mu \int_\Omega W(x)|tu_0|^{p+1}dx \right] > 0 \]

\[ \Rightarrow (1-q)\| tu_0 \|_\Delta^2 - \mu(p-q) \int_\Omega W(x)|tu_0|^{p+1}dx > 0 \]

\[ \Rightarrow tu_0 \in F_{\lambda,\mu}^+. \]

This completes the proof of Appendix (3).

(4) **Suppose that** \( \varphi_\lambda(t) = 0 \) **and** \( \varphi'_\lambda(t) < 0 \), **then** \( tu_0 \in F_{\lambda,\mu}^- \).
Proof. We may proceed as Appendix (3) above and concludes the proof of Appendix (4).

(5) \( \mu = T_\lambda \iff A_\lambda(\mu) = A_{\lambda,0} \).

Proof. One can show that

\[
\begin{align*}
\mu = T_\lambda &= \frac{1 - q}{p - q} \cdot \left( \frac{p - 1}{p - q} \right)^{\frac{1}{p-q}} \cdot \frac{1}{\|W\|_\infty} \cdot \left( \frac{1}{\|h\|_\infty} \right)^{\frac{p-1}{p-q}} \cdot \left[ \frac{S_\lambda}{|\Omega|} \right]^{\frac{1}{p-q}} \\
\iff A_\lambda(\mu) &= \left[ \frac{1 - q}{\mu(p - q)} \cdot \frac{1}{\|W\|_\infty} \cdot \frac{(\sqrt{S_\lambda})^{p+1}}{|\Omega|^{\frac{2}{p-(p-1)}}} \right]^{\frac{1}{p-q}} \\
&= \mu^{-\frac{1}{p-q}} \cdot \left( \frac{1 - q}{p - q} \right)^{\frac{1}{p-q}} \cdot \frac{1}{\|W\|_\infty} \cdot \left( \frac{1}{\|h\|_\infty} \right)^{\frac{1}{p-q}} \cdot \left( \frac{\sqrt{S_\lambda}}{|\Omega|^{\frac{2}{p-(p-1)}}} \right)^{\frac{1}{p-q}} \\
&= \left( \frac{p - q}{1 - q} \right)^{\frac{1}{1-q}} \cdot \left( \frac{1 - q}{p - q} \right)^{\frac{1}{p-q}} \cdot \left( \frac{1}{\|W\|_\infty} \right)^{\frac{1}{p-q}} \cdot \left( \frac{1}{\|h\|_\infty} \right)^{\frac{1}{p-q}} \cdot \left( \frac{\sqrt{S_\lambda}}{|\Omega|^{\frac{2}{p-(p-1)}}} \right)^{\frac{1}{p-q}} \\
&= \left( \frac{p - q}{1 - q} \right)^{\frac{1}{1-q}} \cdot \left( \frac{1 - q}{p - q} \right)^{\frac{1}{p-q}} \cdot \left( \frac{\sqrt{S_\lambda}}{|\Omega|^{\frac{2}{p-(p-1)}}} \right)^{\frac{1}{p-q}} \\
&= \left[ \frac{p - q}{1 - q} \cdot \frac{\sqrt{S_\lambda}}{|\Omega|^\frac{2}{p-(p-1)}} \right]^{\frac{1}{p-q}} = A_{\lambda,0},
\end{align*}
\]

where we use the relations

\[
2 \cdot \frac{p - q}{(1 - q)(p - 1)} - \frac{2^* - p - 1}{2^*(p - 1)} = \frac{2^* - p - 1}{2^*(p - 1)} - \frac{2^* - 1}{2^*(1 - q)(p - 1)} = \frac{2^* - 1}{2^*(1 - q)} = \frac{2^* - 1 - q}{2^*(1 - q)}
\]

and

\[
2 \cdot \frac{p - q}{(1 - q)(p - 1)} - \frac{p + 1}{p - 1} = \frac{(1 + q)(p - 1)}{(1 - q)(p - 1)} = \frac{1 + q}{1 - q}.
\]

This completes the proof of Appendix (5).

(6) \( \mu = T_\lambda \iff B_\lambda(\mu) = B_{\lambda,0} \).

Proof. Similar to Appendix (5), we have

\[
\begin{align*}
\mu = T_\lambda &= \frac{1 - q}{p - q} \cdot \left( \frac{p - 1}{p - q} \right)^{\frac{1}{p-q}} \cdot \frac{1}{\|W\|_\infty} \cdot \left( \frac{1}{\|h\|_\infty} \right)^{\frac{p-1}{p-q}} \cdot \left[ \frac{S_\lambda}{|\Omega|} \right]^{\frac{1}{p-q}} \\
\iff B_\lambda(\mu) &= \left[ \frac{1 - q}{\mu(p - q)} \cdot \frac{1}{\|W\|_\infty} \cdot \frac{S_\lambda}{|\Omega|^{\frac{2}{p-(p-1)}}} \right]^{\frac{1}{p-q}} \\
&= \mu^{-\frac{1}{p-q}} \cdot \left( \frac{1 - q}{p - q} \right)^{\frac{1}{p-q}} \cdot \left( \frac{1}{\|W\|_\infty} \right)^{\frac{1}{p-q}} \cdot \left( \frac{S_\lambda}{|\Omega|^{\frac{2}{p-(p-1)}}} \right)^{\frac{1}{p-q}}
\end{align*}
\]
= \left(\frac{p-q}{1-q}\right)^{\frac{1}{\sigma}} \cdot \left(\frac{p-q}{p-1}\right)^{\frac{1}{\tau}} \cdot (\|W\|_\infty)^{\frac{1}{\sigma}} \cdot (\|h\|_\infty)^{\frac{1}{\tau}}
\cdot \frac{|\Omega|^2 \frac{\gamma - \delta}{\gamma - \delta + 1}}{(S_{\lambda})^{\frac{\gamma - \delta}{\gamma - \delta + 1}}}, \frac{(1-q)^{\frac{1}{\sigma}} \cdot \left(\frac{1}{p-q}\right)^{\frac{1}{\tau}} \cdot \left(\frac{1}{\|W\|_\infty}\right)^{\frac{1}{\sigma}} \cdot \left(S_{\lambda}\right)^{\frac{1}{\tau}}}{|\Omega|^{\frac{\gamma - \delta}{\gamma - \delta + 1}}}.
\end{equation}

where we use the relations
\begin{align*}
&\frac{2}{N} \cdot \frac{p-q}{(1-q)(p-1)} = 2 \cdot \frac{2^* - p - 1}{2^*(p+1)(p-1)} \\
&= \frac{2^* - 2}{2^*} \cdot \frac{p-q}{(1-q)(p-1)} - 2 \cdot \frac{2^* - p - 1}{2^*(p+1)(p-1)} \\
&= \frac{2^*(p^2 - p - pq + q) + 2(2^* - p - 2^* - q^2 + 1)}{2^*(p+1)(p-1)(1-q)} \\
&= \frac{2^*(p-q) + 2(2^* - p - 1)}{2^*(p+1)} \cdot \frac{1}{1-q} = \left[\frac{p-q}{p+1} + 2 \cdot \frac{2^* - p - 1}{2^*(p+1)}\right] \cdot \frac{1}{1-q}
\end{align*}

and
\begin{equation}
\frac{p-q}{(1-q)(p-1)} - \frac{1}{p-1} = \frac{p-q - (1-q)}{(1-q)(p-1)} = \frac{1}{1-q}.
\end{equation}

This completes the proof of Appendix (6).

\(\square\)

(7) \(I_{\lambda,\mu}\) is coercive and bounded below on \(F_{\lambda,\mu}\).

\textbf{Proof.} Let
\begin{equation}
h(t) = at^2 - bt^{1+q}, \quad t > 0,
\end{equation}

where \(a, b\) are both positive constants. Easy computations show that \(h(t) \to 0\) as \(t \to 0^+\) and \(h(t) \to \infty\) as \(t \to \infty\), and similar arguments as those in Appendix (1) show that \(h(t)\) achieves its minimum at
\begin{equation}
t_{\text{min}} = \left[\frac{b(1+q)}{2a}\right]^{\frac{1}{1+q}}
\end{equation}

and
\begin{equation}
h(t_{\text{min}}) = a \left[\frac{b(1+q)}{2a}\right]^{\frac{1}{1+q}} - b \left[\frac{b(1+q)}{2a}\right]^{\frac{1}{1+q}} = -\frac{1}{2} \cdot b \cdot \frac{2}{2^*} \cdot \left(\frac{1 + q}{2a}\right)^{\frac{1+q}{1+q}}.
\end{equation}

Applying \(h(t)\) with
\begin{equation}
a = \left(\frac{1}{2} - \frac{1}{p+1}\right), \quad b = \left(\frac{1}{1+q} - \frac{1}{p+1}\right) C, \quad t = \|u\|_\lambda \quad (u \in F_{\lambda,\mu}),
\end{equation}

we obtain:
\begin{equation}
\lim_{\|u\|_\lambda \to \infty} I_{\lambda,\mu}(u) \geq \lim_{t \to \infty} h(t) = \infty,
\end{equation}

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that is, $I_{\lambda,\mu}$ is coercive on $F_{\lambda,\mu}$; and

\[ I_{\lambda,\mu}(u) \geq h(t) \geq h(t_{\min}) \quad \text{(a constant)}, \]

that is, $I_{\lambda,\mu}$ is bounded below on $F_{\lambda,\mu}$.

This completes the proof of Appendix (7). \qed

(8) Proofs of (12) and (13).

Proof. Let

\[ g_{\mu}(t) = t^p + \mu t^p - \mu_1 t = t^p (t^{q-p} - \mu_1 t^{1-p} + \mu) \equiv t^p \cdot r_\mu(t), \quad t > 0. \]

(i) Similar arguments as those in Appendix (1) show that $r_{\mu}(t)$ achieves its minimum at

\[ t_{\min} = \left[ \frac{\mu_1(p-1)}{p-q} \right]^{\frac{1}{p-q}} \]

and

\[ r_{\mu}(t_{\min}) = \left[ \frac{\mu_1(p-1)}{p-q} \right]^{\frac{1}{p-q}} - \mu_1 \left[ \frac{\mu_1(p-1)}{p-q} \right]^{\frac{p-1}{p-q}} + \mu \]

\[ = - \frac{\mu_1(1-q)}{p-q} \cdot \left[ \frac{\mu_1(p-1)}{p-q} \right]^{\frac{p-1}{p-q}} + \mu. \]

We can choose

\[ \mu = \frac{\mu_1(1-q)}{p-q} \cdot \left[ \frac{\mu_1(p-1)}{p-q} \right]^{\frac{p-1}{p-q}} + 1 \equiv \mu^+ > 0 \]

such that

\[ r_{\mu^+}(t) \geq r_{\mu^+}(t_{\min}) = 1 > 0, \quad \forall \ t > 0. \]

Therefore,

\[ g_{\mu^+}(t) = t^q + \mu^+ t^p - \mu_1 t = t^p \cdot r_{\mu^+}(t) > 0, \quad \forall \ t > 0, \]

that is, (12) holds.

(ii) For all $\mu > 0$, since

\[ \lim_{t \to 0^+} r_{\mu}(t) = \lim_{t \to 0^+} \left( t^{p-q} - \mu_1 t^{1-p} + \mu \right) = \lim_{t \to 0^+} \frac{1 - \mu_1 t^{1-q} + \mu t^{p-q}}{t^{p-q}} = \infty, \]

we can choose $\varepsilon > 0$ small enough such that $r_{\mu}(\varepsilon \varphi_1) \geq 0$. Therefore,

\[ g_{\mu}(\varepsilon \varphi_1) = (\varepsilon \varphi_1)^q + \mu(\varepsilon \varphi_1)^p - \mu_1 \cdot \varepsilon \varphi_1 = (\varepsilon \varphi_1)^p \cdot r_{\mu}(\varepsilon \varphi_1) \geq 0 \quad \text{for all } \mu > 0, \]

that is, (13) holds.

This completes the proof of Appendix (8). \qed

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