Bimaximal mixing and large $\theta_{13}$ in a SUSY SU(5) model based on $S_4$

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Abstract

The recent analyses of the world neutrino data, including the T2K and MINOS results, point toward a statistically significant deviation of $\theta_{13}$ from zero. In this paper we present a SUSY SU(5) model based on the discrete $S_4$ group which predicts a large $\theta_{13} \sim O(\lambda_C)$, $\lambda_C$ being the Cabibbo angle. The other mixing angles in the neutrino sector are all compatible with current experimental data. In the quark sector, the entries of the CKM mixing matrix as well as the mass hierarchies in both up and down quark sectors are well reproduced and only a small enhancement is needed to reproduce $\lambda_C$.
1 Introduction and description of the model

The T2K experiment has recently observed six events which, after all selection criteria at the far detector, are a strong indication of $\nu_\mu \rightarrow \nu_e$ flavour transition \[1\]. In a three flavour scenario with $|\Delta m^2_{32}| = 2.4 \times 10^{-3}$ eV$^2$ and maximal atmospheric mixing, these data are consistent with a non-vanishing $\theta_{13}$ at $2.5\sigma$, with $0.03(0.04) < \sin^2 2\theta_{13} < 0.28(0.34)$ for normal (inverted) hierarchy and best fit value at 0.11 (0.14). This result goes in the same direction of other, statistically less significant, analyses where a not-so-small reactor angle emerged from global fits to the available neutrino data \[2\] \[3\] \[4\].

Hints for $\theta_{13} > 0$ have been confirmed in \[5\], where an analysis of global neutrino data, including the latest T2K and MINOS results \[6\], provides

$$\sin^2 \theta_{13} = \begin{cases} 
0.021 \pm 0.007, & \text{old reactor fluxes} \\
0.025 \pm 0.007, & \text{new reactor fluxes} \end{cases} (1\sigma),$$

which corresponds to a $> 3\sigma$ evidence for a non-vanishing reactor angle, for both old and new reactor neutrino fluxes \[7\]. A large value of $\theta_{13}$ seems to disfavor the picture where the tri-bimaximal (TBM) mixing pattern \[8\],\[9\],\[10\],\[11\],\[12\],\[13\],\[14\],\[15\],\[16\] (tan$^2 \theta_{23} = 1$, tan$^2 \theta_{13} = \frac{1}{2}$ and sin $\theta_{13} = 0$), is a good first order description of the data. In fact, models based on $A_4$, $S_4$, $T_7$, $T'$ and so on (see \[9\] for a review) have the common feature that, at the next level of approximation, all the three mixing angles receive corrections of the same order of magnitude, which is basically fixed by the experimentally allowed departures of $\theta_{12}$ from its TBM value, at the level of $O(\lambda_C^2)$, with $\lambda_C$ being the Cabibbo angle. As a consequence, $\theta_{13}$ (and also the deviation of $\theta_{23}$ from the maximal value) is expected to also be at most of $O(\lambda_C^2)$, which is now only marginally allowed (but see \[10\] for some examples of large $\theta_{13}$ in the context of TBM). As it has been pointed out in \[11\], a large value of the reactor angle can be achieved if the Bimaximal (BM) mixing option, tan$^2 \theta_{23} = 1$, tan$^2 \theta_{13} = 1$, sin $\theta_{13} = 0$, is the correct first order approximation to describe the neutrino mixings; corrections from the charged lepton sector must be large enough, of $O(\lambda_C)$, to move $\theta_{13}$ toward large values and to reconcile the value of $\theta_{12}$ with the experiments but much smaller in the atmospheric sector in order not to destroy the agreement with maximal mixing. This pattern of subleading corrections is in fact realized in \[11\], giving:

$$\sin^2 \theta_{12} \sim \frac{1}{2} + O(\lambda_C) \quad \sin^2 \theta_{23} \sim \frac{1}{2} + O(\lambda_C^2) \quad \sin \theta_{13} \sim O(\lambda_C).$$

In this paper we consider a GUT extension of such a model, based on the SU(5) gauge group, with the aim of extending the $S_4$ symmetry to describe the quark sector and maintaining at the same time the relations in eq.(2). Earlier attempts to describe quarks and leptons in the context of SU(5)×$S_4$ can be found in \[14\] where, however, the main goal was to reproduce the TBM. Similar tentative in SO(10) and Pati-Salam have been discussed in \[12\] and \[13\], respectively. The model proposed here is the first attempt to get the BM pattern in a SU(5) context based on $S_4$. In order to formulate a realistic GUT model, we work in a supersymmetric scenario in 4+1 dimensions where problems related to the breaking of a grand unified symmetry (like the doublet-triplet splitting problem and the proton decay) can be efficiently solved \[15\]. In the simplest setting, the fifth dimension is compactified on a circle $S^1$ of radius $R$ in such a way that the gauge fields, living in the whole 5D space-time, are assumed to be periodic along the extra dimension only up to a discrete parity transformation $\Omega$ under which the gauge fields of the SU(3)×SU(2)×U(1) subgroup are periodic and possess a zero mode while those of the coset SU(5)/SU(3)×SU(2)×U(1) are antiperiodic and form a Kaluza-Klein tower starting at the mass level $1/R$. For a 4D observer, these boundary
conditions break SU(5) down to the Standard Model (SM) gauge group at a GUT scale of order $1/R$. The doublet-triplet splitting problem is solved if the parity $\Omega$ is extended to the Higgs multiplets $H_5$ and $H_\tau$, also assumed to live in the bulk, in such a way that the electroweak doublets are periodic (and then have zero modes), whereas the colour triplets are antiperiodic (getting masses of order $1/R$). To reduce $N = 2$ SUSY (induced by the original $N = 1$ SUSY in five dimensions) down to $N = 1$ it is necessary to compactify the fifth dimension on the orbifold $S^1/Z_2$ rather than on the circle $S^1$. The orbifold projection eliminates all the zero modes of the extra states belonging to $N = 2$ SUSY and also those of the fifth component of the gauge vector bosons. The zero modes we are left with are the 4D gauge bosons of the SM, two electroweak doublets and their $N = 1$ SUSY partners. For the gauge vector bosons and the Higgses we adopt this setup, which is described in detail in refs. [16]. For the remaining fields we have much more freedom [16, 17]: they can be located in the bulk, at the SU(5) preserving brane $y = 0$, or at the SU(5) breaking brane $y = \pi R$. In our construction based on $S_4$, the three $\bar{5}$ are grouped into the $S_4$ triplet $F$, while the tenplets $T_1, T_2$ and $T_3$ are assigned to the singlet 1 of $S_4$. We choose to put the tenplet of the first two families $T_1, T_2$ in the bulk and all remaining $N = 1$ supermultiplets on the SU(5) preserving brane at $y = 0$. To obtain the correct zero mode spectrum with intrinsic parities compatible with symmetry and orbifolding, one must introduce two copies of each multiplet with opposite parity $\Omega$ in the bulk. Therefore $T_{1,2}$ is a short notation for the copies $T_{1,2}$; the zero modes of $T_{1,2}$ are the SU(2) quark doublets $Q_{1,2}$, while those of $T'_{1,2}$ are $U^c_{1,2}$ and $E^c_{1,2}$. This setup has a two-fold advantage: an automatic suppression of the Yukawa couplings for the fields living in the bulk and the breaking of the mass relation $m_e = m^T_d$ for the first two fermion families[1]. The latter is a consequence of using the copies $T_{1,2}$ and $T'_{1,2}$ for the down quarks and charged leptons, respectively, whereas the former originates from the fact that a bulk field $B$ and its zero mode $B^0$ are related by:

$$B = \frac{1}{\sqrt{\pi R}}B^0 + \ldots$$

(3)

where dots stand for the higher modes. This expansion produces a (geometrical) suppression factor

$$s \equiv \frac{1}{\sqrt{\pi R A}} < 1$$

(4)

entering the Yukawa couplings depending on the field $B^0$. This applies also to the Higgs vevs; since all the matter fields (but $T_{1,2}$) are localized at $y = 0$, what matter for the Yukawa couplings are the values of the vevs at $y = 0$:

$$\langle H_5(0) \rangle = \frac{v_u^0}{\sqrt{\pi R}}, \quad \langle H_\tau(0) \rangle = \frac{v_d^0}{\sqrt{\pi R}}.$$  

(5)

A similar setup (to which we refer for further details) has been used in [18] in the context of a SUSY SU(5)$\times A_4$. To break the $S_4$ symmetry, we consider a set of SU(5)-invariant flavon supermultiplets: three triplets $\varphi_\ell, \varphi_\nu (3_1), \chi_\ell (3_2)$ and one singlet $\xi_\nu$. The alignment of their vacuum expectation values (vevs) along appropriate directions in flavour space will be the source of BM lepton mixing. For this to work, we employ a cyclic $Z_3$ symmetry which allows the fields $\varphi_\nu$ and $\xi_\nu$ to be the only ones responsible for neutrino masses at leading order.

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1It is interesting to observe that, to break the SU(5) relation $m_e = m^T_d$ in the case $T_{1,2}$ are also localized on the brane, one could introduce a $\mathbf{45}$ representation for the Higgs fields, also propagating in the bulk. However, it turns out that this representation not only contains zero modes for the electroweak doublet but also for colored Higgses, thus reintroducing a sort of doublet-triplet splitting problem.
The GUT Higgs fields $H_5$ and $H_{\tau}$ are singlets under the family symmetry but charged in the same way under $Z_3$, so that they are distinguished only by their SU(5) transformation properties. To achieve a realistic mass spectrum, beside the geometrical suppression factors, we also exploit the Froggatt-Nielsen mechanism. The tenplets $T_1$ and $T_2$ are charged under a $U(1)_{FN}$ flavour group, spontaneously broken by the vevs of two fields $\theta$ and $\theta'$ both carrying $U(1)_{FN}$ charges $-1$ and transforming as a singlet of $S_4$. The tenplets $T_1$ and $T_2$ are charged under a $U(1)_{FN}$ flavour group, spontaneously broken by the vevs of two fields $\theta$ and $\theta'$ both carrying $U(1)_{FN}$ charges $-1$ and transforming as a singlet of $S_4$. The assignment of the fields under SU(5) and the discrete group $S_4 \times Z_3$ is summarized in Tab.1. Before closing this section, it

| Field | $F$ | $T_1$ | $T_2$ | $T_3$ | $H_5$ | $H_{\tau}$ | $\varphi_\nu$ | $\varphi_\ell$ | $\chi_\nu$ | $\theta$ | $\theta'$ | $\varphi_\nu^0$ | $\varphi_\ell^0$ | $\chi_\nu^0$ | $\psi_\nu^0$ | $\psi_\ell^0$ | $\chi_\ell^0$ |
|-------|-----|-------|-------|-------|-------|------------|---------------|---------------|------------|---------|---------|----------------|---------------|--------------|-------------|-------------|------------|
| SU(5) | 5   | 10    | 10    | 5     | 5     | 1          | 1             | 1             | 1          | 1        | 1       | 1                 | 1             | 1             | 1           | 1           | 1          |
| $S_4$ | 3   | 1     | 1     | 1     | 3     | 1          | 1             | 1             | 1          | 3        | 1       | 3                 | 2             | 2             | 2           | 2           | 2          |
| $Z_3$ | $\omega$ | $\omega$ | $\omega^2$ | $\omega^2$ | $\omega$ | $\omega$ | $\omega$ | $\omega$ | $\omega$ | $\omega$ | $\omega$ | $\omega$ | $\omega$ | $\omega$ | $\omega$ | $\omega$ |
| $U(1)_R$ | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $U(1)_{FN}$ | 0 | 3 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| br | bu | bu | br | bu | bu | br | br | br | br | br | br | br | br | br | br | br |

Table 1: Matter assignment of the model. The symbol br (bu) indicates that the corresponding fields live on the brane (bulk).

is interesting to outline that, as long as the $U(1)_R$ symmetry remains unbroken, dangerous operators that could spoil the solution of the doublet-triplet splitting problem or mediate proton decay are forbidden to all orders. In fact, given the $U(1)_R$ assignments in Tab.1 the mass term $H_5H_\bar{5}$ has $U(1)_R = 0$ and cannot be included in the superpotential of the effective $N = 1$ SUSY, which should have $U(1)_R$ charge $+2$, to compensate the $R$-charge $-2$ coming from the Grassmann integration measure $d^2\theta$. Also, all renormalizable baryon and lepton number violating operators, such as $FH_5$ and $FFT$, are not allowed, and the dimension five operator $FTTT$, leading to proton decay, has $R$-charge $+4$ and therefore is absent. The paper is organized as follows: in Sect.2 we discuss the vacuum alignments of the flavon fields at LO and NLO, since they are the necessary ingredients to build the mass matrices for charged and neutral fermions; in Sect.3 we show how to get a realistic pattern of the mass ratios in the down quark and charged leptons, also computing the left-handed rotations in both sectors, needed to build the CKM matrix and correct the neutrino mixing matrix. Sect.4 is devoted to the up-type quarks whereas in sect.5 we discuss the neutrino sector and some phenomenological implications of the LO and NLO mass matrices. In Sect.6 we draw our conclusions.

2 Vacuum alignment

We solve the vacuum alignment problem using the method first introduced in [19]. Within this approach, a continuous $U(1)_R$ symmetry is introduced, under which matter fields have $R = +1$, while Higgses and flavon fields have $R = 0$. Such a symmetry will be eventually broken down to the R-parity by small SUSY breaking effects which can be neglected in the first approximation. The required vacuum alignment is obtained introducing the so-called driving fields with $R = +2$, which enter linearly into the superpotential. We use here the same flavon content and $S_4$ property transformations as in [11], that is two triplets $\varphi_\nu$ and $\chi_\nu$, one doublet $\psi_\nu$ and one singlet $\xi_\nu$. We assume that the family symmetry is broken at an energy scale where SUSY is still an exact symmetry; this allows to deduce the alignment of the flavon fields from equations arising from setting the F-terms of the driving fields to zero.
All the multiplets but the flavon ones have vanishing vevs and set to zero for the present discussion. We regard the $U(1)_{FN}$ Froggatt-Nielsen flavour symmetry as a local symmetry, assuming that other vector-like multiplets (not specified here) are introduced to remove the anomaly associated with $U(1)_{FN}$. Within these assumptions the relevant part of the scalar potential of the model is given by the sum of the F-terms and of a D-term:

$$V = V_F + V_D$$

with

$$V_F = \sum_i \left| \frac{\partial w}{\partial \varphi_i} \right|^2$$

2.1 Leading order

The LO superpotential responsible for the vacuum alignment is equal to the one quoted in [11]:

$$w_d = M_\varphi (\varphi_0 \varphi_\nu) + g_1 (\varphi_\nu (\varphi_\nu \varphi_\nu)_{3_1}) + g_2 (\varphi_\nu \varphi_\nu) \xi_\nu +$$
$$+ \xi_\nu [M_\xi^2 + M_\xi' \xi_\nu + g_3 (\varphi_\nu \varphi_\nu) + g_4 \xi_\nu \xi_\nu] +$$
$$+ f_1 (\psi_\nu^0 (\varphi_\nu \varphi_\nu)_{2}) + f_2 (\psi_\nu^0 (\chi_\nu \chi_\nu)_{2}) + f_3 (\psi_\nu^0 (\varphi_\nu \chi_\nu)_{2}) +$$
$$+ f_4 (\chi_\nu^0 (\varphi_\nu \chi_\nu)_{3_2})$$

We parametrize the triplet flavon vevs as

$$\langle \phi \rangle = v_\phi \left( \begin{array}{c} \phi_1 \\ \phi_2 \\ \phi_3 \end{array} \right)$$

In the SUSY limit, the vacuum configuration is determined by the vanishing of the derivative of $w_d$ with respect to each component of the driving fields. The set of such equations for the minimum of the potential can be divided into two decoupled parts: one for the neutrino sector (involving $\varphi_\nu$ and $\xi_\nu$) and one for the charged lepton sector (driven by $\varphi_\ell$ and $\chi_\ell$). We do not report here such equations (since they are equal to [11]) and only quote the final results:

$$\langle \varphi_\ell \rangle = v_{\varphi_\ell} \left( \begin{array}{c} 0 \\ 1 \\ 0 \end{array} \right) \quad \langle \chi_\ell \rangle = v_{\chi} \left( \begin{array}{c} 0 \\ 0 \\ 1 \end{array} \right)$$

$$\langle \varphi_\nu \rangle = v_{\varphi_\nu} \left( \begin{array}{c} 0 \\ 1 \\ -1 \end{array} \right) \quad \langle \xi_\nu \rangle = v_{\xi}$$

where the various $v_\phi$’s obey:

$$2g_3 v_{\varphi_\nu}^2 = \frac{g_4 M_\varphi^2}{g_2^2} - \frac{M_\xi' M_\varphi}{g_2} + M_\xi^2$$
$$\sqrt{3} f_1 v_{\varphi_\nu}^2 = -v_{\chi} \left( \sqrt{3} f_2 v_{\chi} + f_3 v_{\varphi_\ell} \right)$$
$$v_{\xi} = -\frac{M_\varphi}{g_2}$$
with $v_\chi$ undetermined. The D-term is given by:

$$V_D = \frac{1}{2}(M_{FI}^2 - g_{FN}|\theta|^2 - g_{FN}|\theta'|^2 + ...)^2$$

where $g_{FN}$ is the gauge coupling constant of $U(1)_{FN}$ and $M_{FI}^2$ denotes the contribution of the Fayet-Iliopoulos term. We have omitted the SU(5) contribution to the D-term, whose vev is zero. There are SUSY minima such that $V_F = V_D = 0$. The vanishing of $V_D$ requires

$$g_{FN}|\theta|^2 + g_{FN}|\theta'|^2 = M_{FI}^2$$

If the parameter $M_{FI}^2$ is positive, the above condition determines a non-vanishing vev for a combination of $\theta$ and $\theta'$.

### 2.2 Next to leading order

The next level of approximation is different from \[11\]. The corrections to the flavon superpotential can be expressed as follows:

$$\Lambda \Delta w_d = \sum_{i=1}^{15} \alpha_i I_i^{\nu_0} + \sum_{i=1}^{5} \beta_i I_i^{\nu_0} + \sum_{i=1}^{7} \gamma_i I_i^{\nu_0} + \sum_{i=1}^{9} \delta_i I_i^{\nu_0},$$

where the following operators contribute to the quartic invariants $\{I_i^{\nu_0}, I_i^{\nu_0}, I_i^{\nu_0}, I_i^{\nu_0}\}$ (we do not specify here the different contractions among the fields, see appendix A for details):

$$
\begin{align*}
I_i^{\nu_0} & : \quad \varphi_\nu^3, \varphi_\ell^3, \chi^3, \varphi_\nu^2 \xi_\nu, \varphi_\ell^2 \chi_\ell, \chi_\ell^2 \varphi_\ell, \varphi_\nu \xi_\nu^2 \\
I_i^{\nu_0} & : \quad \varphi_\nu^3, \xi^3, \varphi_\ell^3, \varphi_\nu^2 \xi_\nu, \chi_\ell^2 \varphi_\ell \\
I_i^{\nu_0} & : \quad \varphi_\nu^2 \varphi_\nu, \varphi_\ell^2 \xi_\nu, \chi_\ell^2 \varphi_\nu, \varphi_\nu \varphi_\ell \chi_\ell, \xi_\nu \varphi_\ell \chi_\ell \\
I_i^{\nu_0} & : \quad \varphi_\nu^2 \varphi_\nu, \chi_\ell^2 \varphi_\nu, \varphi_\nu \varphi_\ell \chi_\ell, \xi_\nu \varphi_\ell \chi_\ell. \\
\end{align*}
$$

The relevant feature of such corrections is the presence of terms where flavons of the neutrino and charged lepton sectors mix to each other. To get the NLO vacua, we parametrize the vev shifts according to:

$$
\begin{align*}
\langle \varphi_\ell \rangle &= v_{\varphi_\ell} \begin{pmatrix} \delta_{\varphi_1} \\ 1 + \delta_{\varphi_2} \\ \delta_{\varphi_3} \end{pmatrix} & \langle \chi_\ell \rangle &= v_\chi \begin{pmatrix} \delta_{\chi_1} \\ \delta_{\chi_2} \\ 1 \end{pmatrix} \\
\langle \varphi_\nu \rangle &= v_{\varphi_\nu} \begin{pmatrix} \delta_{\varphi_1} \\ 1 + \delta_{\varphi_2} \\ -1 + \delta_{\varphi_3} \end{pmatrix} & \langle \xi_\nu \rangle &= v_\xi (1 + \delta_{\nu_\xi}).
\end{align*}
$$

where all $\delta_i$ are smaller than 1, and look for solutions by imposing the vanishing of the first derivative of $w_d + \Delta w_d$ with respect to the driving fields. After some algebra, we get the following results:

$$
\begin{align*}
\delta_{\varphi_1} & \neq 0 & \delta_{\varphi_2} & \neq 0 & \delta_{\varphi_3} &= 0 \\
\delta_{\chi_1} & \neq 0 & \delta_{\chi_2} &= 0 \\
\delta_{\nu_1} & \neq 0 & \delta_{\nu_2} &= -\delta_{\nu_3},
\end{align*}
$$
where all the non-vanishing $\delta_i$ are proportional to a combination of $\nu/\Lambda$; this ratio also fixes the relative magnitude between the leading order components of the flavon vevs and the $\delta_i$’s. To make this more transparent, we rescale the perturbations $\delta_i \rightarrow \epsilon' \delta_i$, with now $\delta_i \sim O(1)$. It is important to observe that the corrections to the second and third component of $\varphi_\ell$ are opposite to each other and can be reabsorbed into the leading order results. The same happens for the non-vanishing corrections to the second component of $\varphi_\nu$. According to what discussed, the flavon vev structure that will be used to determine the fermion masses can be summarized as follows:

$$
\langle \varphi_\ell \rangle = v_\varphi \begin{pmatrix} \epsilon' \delta_\varphi_1 \\ 1 \\ 0 \end{pmatrix} \\
\langle \chi_\ell \rangle = v_\chi \begin{pmatrix} \epsilon' \delta_\chi_1 \\ 0 \\ 1 \end{pmatrix} \\
\langle \varphi_\nu \rangle = v_\varphi \nu \begin{pmatrix} \epsilon' \delta_\nu_1 \\ 1 \\ -1 \end{pmatrix} ; \quad (20)
$$

The magnitude of the flavon vevs as well as of the $\epsilon'$ perturbations by will be discussed in the next section.

### 3 Down quarks and charged lepton mass matrices

The superpotential built with operators with two-flavon insertions (beside the flavons carrying Froggatt-Nielsen charges) allows to determine the relevant features of the down quarks and charged lepton mass matrices. According to the discussion in the Introduction, we use the same notation for the couplings involving the tentplet $T_{1,2}$, although it should be understood that they are different for down quarks and charged leptons. The superpotential in this sector reads as follows:

$$
\begin{align*}
\mathcal{W}_\ell &= FT_3 H_3 \left[ \frac{\alpha_6}{\Lambda^{3/2}} \varphi_\ell + \frac{\alpha_1}{\Lambda^{5/2}} (\varphi_\nu \varphi_\ell)_3 + \frac{\alpha_2}{\Lambda^{5/2}} (\varphi_\nu \chi_\ell)_3 + \frac{\alpha_3}{\Lambda^{5/2}} \varphi_\ell \xi_\nu + \right] + \\
&\quad FT_2 H_5 \theta \left[ \frac{\beta_1}{\Lambda^3} \varphi_\nu + \frac{\beta_2}{\Lambda^3} (\varphi_\nu^2)_3 + \frac{\beta_3}{\Lambda^4} \varphi_\nu \xi_\nu \right] + \\
&\quad FT_2 H_5 \theta' \left[ \frac{\beta_4}{\Lambda^4} (\varphi_\ell^2)_3 + \frac{\beta_5}{\Lambda^4} (\chi_\ell^2)_3 + \frac{\beta_6}{\Lambda^4} (\chi_\ell \chi_\ell^2)_3 \right] + \\
&\quad FT_2 H_5 \theta^2 \left[ \frac{\gamma_1}{\Lambda^4} (\varphi_\ell^2)_3 + \frac{\gamma_2}{\Lambda^4} (\varphi_\ell \xi_\ell)_3 + \frac{\gamma_3}{\Lambda^4} (\varphi_\ell \chi_\ell)_3 \right] + \\
&\quad FT_2 H_5 \theta^2 \left[ \frac{\gamma_4}{\Lambda^4} (\varphi_\ell^2)_3 + \frac{\gamma_5}{\Lambda^4} (\varphi_\nu \varphi_\ell)_3 + \frac{\gamma_6}{\Lambda^4} (\varphi_\nu \chi_\ell)_3 + \frac{\gamma_7}{\Lambda^4} \xi_\ell \varphi_\ell \right] + \\
&\quad FT_2 H_5 \theta^2 \left[ \frac{\gamma_8}{\Lambda^4} (\varphi_\ell^2)_3 + \frac{\gamma_9}{\Lambda^4} (\varphi_\ell \xi_\ell)_3 + \frac{\gamma_{10}}{\Lambda^4} \xi_\ell \varphi_\ell \right] + \\
&\quad FT_2 H_5 \theta^3 \left[ \frac{\gamma_{11}}{\Lambda^4} (\varphi_\ell^2)_3 + \frac{\gamma_{12}}{\Lambda^4} (\varphi_\ell \chi_\ell)_3 + \frac{\gamma_{13}}{\Lambda^4} (\varphi_\ell \chi_\ell^2)_3 \right],
\end{align*}
$$

where we took into account that the fields living in the bulk have mass dimension $3/2$. From the superpotential in eq.\((\text{\textsuperscript{20}})\) we expect a common order of magnitude for the vev’s (scaled by the cutoff $\Lambda$):

$$
v_\varphi \sim v_\chi \sim \epsilon \quad v_\varphi \nu \sim v_\nu \sim \epsilon . \quad (22)
$$

although, due to the different minimization conditions that determine $(v_\varphi \ell, v_\chi)$ and $(v_\varphi \nu, v_\nu)$, we may tolerate a moderate hierarchy between $\epsilon$ and $\epsilon$. Similarly the order of magnitude of
\(\langle \theta \rangle\) and \(\langle \theta' \rangle\) is in principle unrelated to those of \(\varepsilon\) and \(\epsilon\). In the following, we assume that \(\varepsilon = \epsilon\) and use the short-hand notations for the ratio among flavon vevs and \(\Lambda\):

\[
\frac{v_{\varphi_{\ell}}}{\Lambda} = \frac{v_{X}}{\Lambda} = \frac{v_{\varphi_{\nu}}}{\Lambda} = \frac{v_{\chi}}{\Lambda} = \varepsilon \quad \frac{\langle \theta \rangle}{\Lambda} = t \quad \frac{\langle \theta' \rangle}{\Lambda} = t'.
\]

Disregarding for the moment all \(O(1)\) coefficients and taking the vev \(\langle H^5_5 \rangle = v_d\), the mass matrix (in the \(\bar{\psi}_Lm_d\bar{\psi}_R\) convention) obtained from this Lagrangian is:

\[
m_d = v_d s \varepsilon \begin{pmatrix}
 s \varepsilon (t^3 + t^2 t' + t'^3) & s t t'^2 & s t t' (t + t') \\
 s \varepsilon t' & s t & s t \\
 \varepsilon & 0 & 1
\end{pmatrix},
\]

where \(s\) is the suppression volume factor. If \(\varepsilon, t\) and \(t'\) are relatively close in magnitude, we can easily recognize that the down quark and charged lepton mass hierarchies are given by:

\[
m_b : m_s : m_d \sim m_\tau : m_\mu : m_e \sim 1 : s \varepsilon (t^3 + t^2 t' + t'^3)
\]

whereas the mixing angles in the down sector can be estimated to be:

\[
\theta_{12}^d \sim t^2 \quad \theta_{13}^d \sim s t t' (t + t') \quad \theta_{23}^d \sim s t.
\]

Notice that, by transposition, we also get an estimate of the charged lepton mixings, given by:

\[
\theta_{12}^e \sim \varepsilon \left( \frac{t'}{t} \right) \quad \theta_{13}^e \sim \varepsilon \quad \theta_{23}^e \sim 0.
\]

Then, a realistic pattern of fermion masses and mixings can be achieved requiring that

\[
s \sim \varepsilon \sim t \sim t' \sim \lambda,
\]

where \(\lambda \equiv \lambda_C\). Given that \(\varepsilon' \delta_i = v_i / \Lambda\), we can also fix \(\varepsilon' \sim \lambda\). Before discussing in details the mass matrices obtained in such a situation, it is useful to consider the relevant corrections coming from three-flavon insertion operators and from the vev shifts quoted in eq.(20). The relevant feature of such corrections is the filling of the vanishing element of the mass matrix in eq.(24), which turns out to be \(s \varepsilon^3\), that is of \(O(\lambda^4)\). This allows to shift \(\theta_{23}^e\) away from zero by a quantity of \(O(\lambda^2)\). The relevant operators are of the form \(FT_3H_5 (\varphi_{\ell} \varphi_{\nu}^2 + \chi e \varphi_{\nu}^2)\) computed with leading order vevs and \(FT_3H_5 (\varphi_{\nu} \varphi_{\ell} + \varphi_{\nu} \chi_{\ell})\) with flavon vevs at the next to leading order (see appendix B for details). Putting together all these elements we are now in the position to write down the most general mass matrix allowed in our model:

\[
m_d \sim v_d \lambda^2 \begin{pmatrix} 
 \lambda^5 & \lambda^4 & \lambda^4 \\
 \lambda^3 & \lambda^2 & \lambda^2 \\
 \lambda & \lambda^2 & 1
\end{pmatrix}.
\]

The unitary left-handed rotation \(U_d\) is obtained diagonalizing \(m_d m_d^\dagger\) whereas the right-handed one \(U_{\ell}\) is the charged lepton rotation. In terms of the Cabibbo angle they are given by:

\[
U_d \sim \begin{pmatrix} 
 1 & \lambda^2 & \lambda^4 \\
 \lambda^2 & 1 & \lambda^2 \\
 \lambda^4 & \lambda^2 & 1
\end{pmatrix} \quad U_{\ell} \sim \begin{pmatrix} 
 1 & \lambda & \lambda \\
 \lambda & 1 & \lambda^2 \\
 \lambda^2 & \lambda & 1
\end{pmatrix}.
\]
We see that $U_d$ perfectly reproduces the correct order of magnitude of $V_{ub}$ and $V_{cb}$, whereas $V_{us}$ turns out to be a bit smaller than expected, as in many other models based on non-abelian discrete symmetries. On the other hand, in the construction of the neutrino mixing matrix $U_{PMNS}$ [20], $U_\ell$ will induce corrections of $O(\lambda)$ to $\theta_{12}$ and $\theta_{13}$ and $O(\lambda^2)$ to $\theta_{23}$. These corrections turns out to be relevant to shift the solar and reactor angles from their BM values to the experimental ones.

For the sake of completeness, we recompute the previous quantities including all $O(1)$ coefficients. The mass matrix of the down quarks is now given by:

$$m_d \sim \alpha_b v_d \lambda^2 \begin{pmatrix} x_1 \lambda^5 & x_2 \lambda^4 & x_3 \lambda^4 \\ x_4 \lambda^3 & x_5 \lambda^2 & -x_5 \lambda^2 \\ x_6 \lambda & x_7 \lambda^2 & 1 \end{pmatrix},$$

(31)

where $\alpha_b$ is the bottom Yukawa coupling from the first operator in eq.(21) and $x_i$ are linear combinations of the other Yukawa couplings of eq.(21), rescaled by $\alpha_b$:

$$\begin{align*}
x_1 &= -\gamma_1 + \gamma_2 - \gamma_3 - \gamma_5 - \gamma_6 - \gamma_{11} + \gamma_{12} - \gamma_{13} + \gamma_8 \delta_{v_1} + \gamma_4 \delta_{\phi_1} \\
x_2 &= -\gamma_8 \\
x_3 &= \gamma_4 + \gamma_8 \\
x_4 &= -\beta_4 + \beta_5 - \beta_6 + \beta_1 \delta_{\nu_1} \\
x_5 &= -\beta_1 \\
x_6 &= -\alpha_1 - \alpha_2 + \alpha_6 \delta_{\phi_1} \\
2x_7 &= 3\alpha_5 + \sqrt{3}\alpha_7 - 2\alpha_1(\delta_{\nu_1} + \delta_{\phi_1}) + 2\alpha_2(\delta_{\nu_1} + \delta_{\chi_1}).
\end{align*}$$

(32)

Notice that the (22) and (23) elements are opposite to each other; this equality is broken only at $O(\lambda^6)$, for example, by operators of the form $FT_2 \theta (\phi^2)_{2} \phi_\ell$. From eq.(31) it is easy to obtain the expression for the charged lepton mass matrix by transposition and changing accordingly $x_1 - x_5$ with $x'_1 - x'_5$, which differ from the previous ones by $O(1)$ coefficients. This is the effect of introducing the copies $T'_{1,2}$ of the first two tenplet fields, whose zero modes are different from those of $T_{1,2}$ and couple with the charged leptons only. Summarizing, the mass matrix of the charged leptons is:

$$m_e \sim \alpha_b v_d \lambda^2 \begin{pmatrix} x'_1 \lambda^5 & x'_4 \lambda^4 & x'_6 \lambda \\
x'_4 \lambda^3 & x'_5 \lambda^2 & x'_7 \lambda^2 \\
x'_3 \lambda^4 & -x'_5 \lambda^2 & 1 \end{pmatrix}.$$  

(33)

Working for simplicity in the limit of real coefficients, the quark and charged lepton masses in unit of $\alpha_b v_d \lambda^2$ are explicitly given by:

$$\begin{align*}
m_b &= m_\tau \\
m_s &= |x_5| \lambda^2 \\
m_\mu &= |x'_5| \lambda^2 \\
m_d &= \left|x_1 - (x_2 + x_3)x_6 - \frac{x_2 x_4}{x_5}\right| \lambda^5 \\
m_e &= \left|x'_1 - (x'_2 + x'_3)x_6 - \frac{x'_2 x'_4}{x'_5}\right| \lambda^5.
\end{align*}$$

(34)

We see that, at the GUT scale, the $b - \tau$ unification is recovered (up to very small corrections of $O(\lambda^4)$ not listed here); also, the other two mass ratios $m_d/m_e$ and $m_s/m_\mu$ are both of the same order of magnitude but not strictly equal because of different $O(1)$ coefficients.
Keeping only the leading order for each matrix elements, the left-handed rotation $U_l$ for charged leptons is given by:

$$
U_l = \begin{pmatrix}
1 & \left(\frac{x'_4 + x_6}{x_5}\right)\lambda & x_6\lambda \\
-x_6\lambda & 1 & x_7\lambda^2 \\
-x_6\lambda & -\left(x^2_6 + x'_6 + \frac{x'_4 x_6}{x_5}\right)\lambda^2 & 1
\end{pmatrix}
$$

(35)

whereas that for down quarks $U_d$ is:

$$
U_d = \begin{pmatrix}
1 & \left(\frac{x_2}{x_5}\right)\lambda^2 & x_3\lambda^4 \\
-(\frac{x_2}{x_5})\lambda^2 & 1 & -x_5\lambda^2 \\
-(x_2 + x_3)\lambda^4 & x_5\lambda^2 & 1
\end{pmatrix}.
$$

(36)

4 Up quarks mass matrix

The up quark mass matrix is completely determined by operators with no more than two-flavon insertions. In the following we list the relevant non-vanishing terms in the superpotential disregarding those flavon contractions which will give contributions only at the NLO, not relevant for our discussion:

$$
w_{up} = \frac{\alpha_t}{\Lambda^{1/2}} T_3 T_3 H_5 + \frac{\delta}{\Lambda^4} T_2 T_3 H_5 \theta' (\varphi_u \varphi_e) + \frac{\sigma}{\Lambda^4} T_1 T_3 H_5 \theta^2 \theta' +
\frac{1}{\Lambda^{11/2}} T_1 T_2 H_5 (\tau_1 \theta^4 + \tau_2 \theta \theta^3) +
\frac{\rho}{\Lambda^{13/2}} T_2 T_2 H_5 \theta^3 +
\frac{1}{\Lambda^{15/2}} T_1 T_1 H_5 (\eta_1 \theta^4 \theta^2 + \eta_2 \theta \theta \theta^5).
$$

(37)

Taking into account the conditions in eq. (28), the corresponding mass matrix reads:

$$
m_{up} = v_u^0 \alpha_t \lambda \begin{pmatrix}
(\eta_1 + \eta_2)\lambda^8 & (\tau_1 + \tau_2)\lambda^6 & \sigma\lambda^4 \\
(\tau_1 + \tau_2)\lambda^6 & \rho\lambda^4 & -\delta\lambda^4 \\
\sigma\lambda^4 & -\delta\lambda^4 & 1
\end{pmatrix}
$$

(38)

where, once again, we have rescaled the Lagrangian coefficients by $\alpha_t$ and used the same symbols. It is easy now to read masses and mixing matrix; for the first we have:

$$
m_t : m_c : m_u \sim 1 : \lambda^4 : \lambda^8
$$

(39)

which perfectly matches with the GUT expectations. To avoid large dimensionless coefficients, we assume here that $v_{u,d} \approx \lambda v_{u,d}^0$. We are allowed to do that because of the freedom related to the boundary values $v_{u,d}^0$, in fact, the electroweak scale is determined by the relations:

$$
v_u^2 + v_d^2 \approx (174 \text{ GeV})^2 \quad v_u^2 \equiv \int_0^{\pi R} dy |\langle H_5(y)\rangle|^2 \quad v_d^2 \equiv \int_0^{\pi R} dy |\langle \bar{H}_5(y)\rangle|^2
$$

(40)

and, unless $\langle H_5(y)\rangle$ are constant in the fifth coordinate, one would expect $v_{u,d}^0 \neq v_{u,d}$. With this assumption, the Yukawa coupling of the top quark is of order one and, thanks
to eqs.\,[34,38], also all the other couplings are of the same order. Up to $\mathcal{O}(\lambda^4)$ the matrix diagonalizing $m_{up}m_{up}^\dagger$ is

$$U_a = \begin{pmatrix} 1 & \left(\frac{\tau_1 + \tau_2}{\rho}\right) \lambda^2 & \sigma \lambda^4 \\ -\left(\frac{\tau_1 + \tau_2}{\rho}\right) \lambda^2 & 1 & -\delta \lambda^4 \\ -\sigma \lambda^4 & \delta \lambda^4 & 1 \end{pmatrix}.$$  \hspace{1cm} (41)

We see that the up sector contributes to both $(V_{us} - V_{cd})$ and $(V_{ub} - V_{td})$ of the CKM, which in turn results to be:

$$V_{CKM} = U_d^\dagger U_u = \begin{pmatrix} 1 & \left(\frac{\tau_1 + \tau_2}{\rho} - \frac{\tau_2}{x_5}\right) \lambda^2 & (\sigma - x_2 - x_3) \lambda^4 \\ -\left(\frac{\tau_1 + \tau_2}{\rho} - \frac{\tau_2}{x_5}\right) \lambda^2 & 1 & x_5 \lambda^2 \\ [x_3 - \sigma + \frac{\tau_2}{\rho}(\tau_1 + \tau_2)] \lambda^4 & -x_5 \lambda^2 & 1 \end{pmatrix}.$$ \hspace{1cm} (42)

It is interesting to observe that the different coefficients of the (1 3) and (3 1) elements can explain the experimental difference among these matrix elements. As a final comment of this section, we observe that:

$$\frac{m_b}{m_t} \sim \left(\frac{v_d^0}{v_u^0}\right) \left(\frac{\alpha_b}{\alpha_t}\right) \lambda$$ \hspace{1cm} (43)

which can easily reproduce the experimental value $m_b/m_t \sim \lambda^2$ for $v_u^0/v_d^0 \sim 1/\lambda$ (and $\mathcal{O}(1)$ Yukawa couplings). Considering the scaling $v_{u,d} \approx \lambda v_{u,d}^0$, the condition $v_u^0/v_d^0 \sim 1/\lambda$ implies $\tan \beta \sim 5$.

5 The neutrino sector

In this section we show how to get a description of the neutrino masses and mixings based on the $S_4$ symmetry. As already outlined in the Introduction, the mixing matrix at leading order will have the BM structure. Our main issue here is to show that dimension 5 Weinberg operators are enough to get the BM mixing matrix at LO and a mass spectrum compatible with the data. Since a see-saw version (with subdominant D=5 contributions) has been carefully studied in \cite{11}, in the second part of this section we limit ourselves to show that such a mechanism can also be successfully employed in the GUT version of the model, with identical LO results.

5.1 The neutrino sector from effective operators only

We start writing the leading and next to leading order contributions to neutrino masses from the Weinberg operator:

$$w_w = \frac{y_w}{\Lambda^2} (FF)_1 H_5 H_5 + \frac{y_{w1}}{\Lambda^3} (FF)_3, H_5 H_5 \varphi_\nu + \frac{y_{w2}}{\Lambda^3} (FF)_1 H_5 H_5 \xi_\nu,$$ \hspace{1cm} (44)

\footnote{In addition, to reproduce the correct Cabibbo angle with $\mathcal{O}(1)$ coefficients one has to ask, for example, for a negative $x_2$ or $x_5$. In this case, it is enough to take the absolute value of these coefficients equal to the unity to gain an enhancement of a factor of 3.}
from which the mass matrix is (we use the symbol $\lambda$ according to eqs. (23)-(28)):

$$m_w = \frac{s^2(v_\nu^0)^2}{\Lambda} \begin{pmatrix}
y_w + y_{w2}\lambda & -y_{w1}\lambda & -y_{w1}\lambda \\
-y_{w1}\lambda & 0 & y_w + y_{w2}\lambda \\
y-w_1\lambda & y_w + y_{w2}\lambda & 0
\end{pmatrix}.\quad (45)$$

This matrix is exactly diagonalized by BM and the eigenvalues are (replacing $sv^\nu$ with $v^\nu_\nu$):

$$m_1 = \left(\frac{v^2_u}{\Lambda}\right) [y_w + (y_{w2} - \sqrt{2}y_{w1})\lambda]$$

$$m_2 = \left(\frac{v^2_u}{\Lambda}\right) [y_w + (y_{w2} + \sqrt{2}y_{w1})\lambda]$$

$$m_3 = -\left(\frac{v^2_u}{\Lambda}\right) (y_w + y_{w2}\lambda).\quad (46)$$

Notice that they satisfy the sum-rule:

$$m_1 + m_2 + 2m_3 = 0,$$

which allows to reduce the number of independent parameters in the neutrino mass matrix.

We can reparametrize the masses in terms of two different complex parameters:

$$y_w + y_{w2}\lambda = A = ae^{i\phi_a}$$

$$\sqrt{2}y_{w1}\lambda = B = be^{i\phi_b} \quad a, b > 0$$

from which:

$$|m_1|^2 = \left(\frac{v^2_u}{\Lambda}\right)^2 (a^2 + b^2 - 2ab\cos\Delta)$$

$$|m_2|^2 = \left(\frac{v^2_u}{\Lambda}\right)^2 (a^2 + b^2 + 2ab\cos\Delta)$$

$$|m_3|^2 = \left(\frac{v^2_u}{\Lambda}\right)^2 a^2,$$

where $\Delta = \phi_a - \phi_b$. Since the solar mass difference is given by:

$$\Delta m^2_{sol} = |m_2|^2 - |m_1|^2 = 4\left(\frac{v^2_u}{\Lambda}\right)^2 a\ b\ \cos\Delta,$$

the condition $\cos\Delta > 0$ must be fulfilled. This implies that the spectrum is of inverted type because the condition $|m_3| > |m_2|$ cannot be satisfied. We then define the quantity

$$\Delta m^2_{atm} = |m_1|^2 - |m_3|^2 = \left(\frac{v^2_u}{\Lambda}\right)^2 b\ (b - 2a\cos\Delta)$$

which is positive for

$$\cos\Delta < \frac{b}{2a}.$$\quad (52)

The value of $r$

$$r = \frac{\Delta m^2_{sol}}{\Delta m^2_{atm}} = \frac{4a\cos\Delta}{b - 2a\cos\Delta}$$\quad (53)
can be made small only if \( \cos \Delta \sim 0 \) because the other possible conditions \( b >> 2a \cos \Delta \) reduces to small \( \cos \Delta \) and \( b << 2a \cos \Delta \) is in conflict with eq. (52). We do not consider here the possibility of \( a << 1 \) because this parameter is dominated by \( y_w \) and we prefer to work with \( \mathcal{O}(1) \) coefficients in the superpotential.

The experimental value \( r \sim 1/30 \) is reproduced if

\[
\cos \Delta = \frac{b}{2a(2 + r)} \sim 10^{-2}
\]

for \( a \sim b \sim \mathcal{O}(1) \). Notice that, using this relation into eq. (51), we can estimate the order of magnitude of the scale \( \Lambda \), which turns out to be:

\[
\Lambda \sim \frac{b v_u^2}{\sqrt{\Delta m_{atm}^2}} \sim 10^{14} \text{ GeV},
\]

for \( v_u = 100 \) GeV and \( \Delta m_{atm}^2 = 2.4 \times 10^{-3} \) eV\(^2\).

At the next to leading order, we have to consider two-flavon insertion operators and the shift of the flavon \( \varphi_\nu \) of eq. (16), that is \( \delta_{\nu_1} \). The contributions of the new operators to the superpotential is:

\[
\delta w_w = \frac{y_{w_3}}{\Lambda^4} (FF)_1 (\varphi_\nu \varphi_\nu)_1 H_5 H_5 + \frac{y_{w_4}}{\Lambda^4} (FF)_2 (\varphi_\nu \varphi_\nu)_2 H_5 H_5 + \frac{y_{w_5}}{\Lambda^4} (FF)_3 (\varphi_\nu \varphi_\nu)_3 H_5 H_5 + \frac{y_{w_6}}{\Lambda^4} (FF)_4 (\varphi_\nu \varphi_\nu)_4 H_5 H_5,
\]

with the corresponding mass matrix given by:

\[
m_w = \frac{v_u^2}{\Lambda} \begin{pmatrix}
    y_w + y_{w_2} \lambda + y_{w_4} \lambda^2 & -y_{w_1} \lambda - y_{w_4} \lambda^2 & -y_{w_1} \lambda - y_{w_7} \lambda^2 \\
    -y_{w_1} \lambda - y_{w_7} \lambda^2 & y_w + y_{w_2} \lambda + y_{w_4} \lambda^2 & \frac{1}{2} (3y_{w_4} - 2y_{w_1} \delta_{\nu_1}) \lambda^2 \\
    -y_{w_1} \lambda - y_{w_7} \lambda^2 & \frac{1}{2} (3y_{w_4} + 2y_{w_1} \delta_{\nu_1}) \lambda^2 & y_w + y_{w_4} \lambda + y_{w_6} \lambda^2
\end{pmatrix},
\]

where \( y'_{w_3} = -2y_{w_3} + y_{w_4} + y_{w_6} \) and \( y''_{w} = 1/2(-4y_{w_3} - y_{w_4} + 2y_{w_6}) \). It is easy to show that, in the limit \( \delta_{\nu_1} \to 0 \), the mass matrix is still diagonalized by BM. For \( \delta_{\nu_1} \neq 0 \), we observe that \( \theta_{13}^\nu \) and \( \theta_{12}^\nu \) are not corrected whereas \( \theta_{23}^\nu \) is shifted by a quantity of \( \mathcal{O}(\lambda^2) \):

\[
\theta_{23}^\nu = \frac{\pi}{4} + \left( \frac{y_{w_3}}{2y_w} \right) \delta_{\nu_1} \lambda^2.
\]

To get the neutrino mixing matrix, we also have to take into account the corrections coming from the charged leptons, eq. (53); to be consistent up to \( \mathcal{O}(\lambda^2) \), we have to include all \( \mathcal{O}(\lambda^2) \) terms in eq. (53), which mainly affect the diagonal elements (the elements of \( \mathcal{O}(\lambda) \) are not modified at this order). Since such corrections are important for \( \theta_{23} \), what matter for us are the \((U_\ell)_{22}\) and \((U_\ell)_{33}\) elements, which turn out to be \((U_\ell)_{22} = 1 - (x'_4 + x'_5 x_6)^2/(2x^2_2) \lambda^2\) and \((U_\ell)_{33} = 1 - (x^2_6/2) \lambda^2\). With this in mind, the neutrino mixing angles can be written as:

\[
\begin{align*}
\theta_{13}^\nu &= \frac{1}{\sqrt{2}} \left( \frac{x'_4}{x'_5} \right) \lambda \\
\theta_{23} &= \frac{\pi}{4} + \left[ -\frac{x'_4}{4x'_5} + \frac{x'_6}{2} + x_7 + \delta_{\nu_1} \left( \frac{y_{w_3}}{2y_w} \right) \right] \lambda^2 \\
\theta_{12} &= \frac{\pi}{4} - \frac{1}{\sqrt{2}} \left( \frac{x'_4 + 2x'_5 x_6}{x'_5} \right) \lambda.
\end{align*}
\]
We explicitly realized the pattern in eq. (2), with all corrections computed analytically. These relations show a certain amount of correlation; for example, the solar angle can be written as:

$$\theta_{12} = \frac{\pi}{4} - \theta_{13} - \sqrt{2} x_6 \lambda$$  \hspace{1cm} (60)

so that we expect a large deviation of $\theta_{12}$ from maximal mixing when $\theta_{13}$ is significantly different from zeros, unless some cancellations with the $x_6$ parameter is at work. This is confirmed by a numerical analysis by treating the parameters in eqs. (35) and (57) as random complex numbers. Since it is crucial that the corrections from the charged lepton sectors have coefficients of $\mathcal{O}(1)$, we restrict the absolute values of the $x_i$ between $1/2$ and $2$; all the other parameters are extracted in the range of absolute values between 0 and 3. Notice that the results in eqs. (35) and (57) are valid at the GUT scale; however, since the model predicts inverted hierarchy, the effect of the running to the SUSY scale can be safely neglected.

Figure 1: $\sin^2 \theta_{12}$ as a function of $\sin^2 \theta_{13}$ (left panel) and $\sin^2 \theta_{23}$ vs $\sin^2 \theta_{13}$ (right panel). The gray bands represents the regions excluded by the experimental data at $2\sigma$ on $\sin^2 \theta_{12}$ and $\sin^2 \theta_{23}$ [5] whereas the vertical lines enclose the $2\sigma$ range on $\sin^2 \theta_{13}$ as obtained in [5]. See text for further details.
region close to $|m_3| \sim \sqrt{\Delta m_{\text{atm}}^2}$, as it can be understood from eq.(19) and recalling that $\nu_u^2/\Lambda \sim \sqrt{\Delta m_{\text{atm}}^2}$. As a consequence, much smaller values of $|m_3|$ can only be obtained if, thanks to additional fine tunings of the parameters, a cancellation between the NLO and LO contributions takes place. Having obtained this only for very few points, we can consider the indication for a lower bound on $|m_3|$ around $5 \cdot 10^{-3}$ eV as a reasonable prediction of the model.

Figure 2: $|m_{ee}|$ as a function of the lightest mass $m_3$. The shaded area corresponds to the region allowed by current neutrino data, for a mass ordering of inverted type. The vertical band corresponds to the future sensitivity on the lightest neutrino mass of 0.2 eV from the KATRIN experiment and the horizontal line to the future sensitivity of 15 meV of CUORE.

5.2 The see-saw version

We introduce three right-handed neutrinos transforming under SU(5) $\times S_4 \times Z_3$ as $(1, 3_1, 1)$ and carrying $U(1)_R = +1$. At leading order the superpotential reads as follows:

$$w_\nu = \frac{y_\nu}{\Lambda^{1/2}} F \nu^c H_5 + M_\nu \Lambda \nu^c \nu^c + a(\nu^c \nu^c)_{3_1} \varphi_\nu + b(\nu^c \nu^c)_{11} \xi_\nu$$

(61)

and the resulting Dirac and Majorana mass matrices are:

$$m_\nu^D = sy\nu^0\left(\begin{array}{ccc}1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0\end{array}\right) \quad M_N = \left(\begin{array}{ccc}M + b \lambda & -a \lambda & -a \lambda \\ -a \lambda & 0 & M + b \lambda \\ -a \lambda & M + b \lambda & 0\end{array}\right) \Lambda,$$

(62)

similar to those quoted in [11]. In particular, the expression of the eigenvalues are the same and the spectrum results compatible with normal hierarchy only. We refer to that paper for a detailed discussion of the leading order results implied by eq.(62) (and, in particular, of a proof that D=5 operators can be made subdominant). At the next level of approximation,
the additional contributions to \( w_\nu \) are:

\[
\delta w_\nu = \frac{y_1}{\Lambda^{3/2}} (F^e)_3 H_5 \varphi_\nu + \frac{y_2}{\Lambda^{3/2}} (F^e)_1 H_5 \xi_\nu + \\
+ \frac{a_1}{\Lambda} (\nu^e \nu^e)(\varphi_\nu \varphi_\nu) + \frac{a_2}{\Lambda} ((\nu^e \nu^e)_2 (\varphi_\nu \varphi_\nu)_2) + \\
+ \frac{a_3}{\Lambda} ((\nu^e \nu^e)_3 (\varphi_\nu \varphi_\nu)_3) + \frac{a_4}{\Lambda} (\nu^e \nu^e \xi_\nu) + h.c. ,
\]

(63)

Since there is no mixing between the flavon fields of different sectors, these contributions only modify the expressions of the neutrino masses but not the mixing matrix, which is still of BM form. However, as shown in eq. (20), the flavon \( \varphi_\nu \) takes a non-vanishing component in the first element, which is the source of the BM breaking. This affects the third operator in eq.(61) and the first operator in eq.(63) and it is of the same order of magnitude in \( \lambda \) of the contributions of the operators in eq.(63) with coefficients \( a_1, ..., a_4 \). To be more explicit and put in evidence the sources of the BM breaking, we write the leading deviation of the light neutrino mass matrix in the following way:

\[
m_\nu = s^2 (v_\nu^0)^2 \left( \begin{array}{ccc}
x & y & y \\
y & z + A & x - z \\
y & x - z & z - A \end{array} \right),
\]

(64)

where

\[
A = \frac{y_\nu}{M^2_\nu} (2M_\nu y_1 - ay_\nu) \delta_{\alpha} \lambda^2 .
\]

(65)

and \( x, y, z \) are linear combinations of the superpotential parameters. It is clear that the limit \( A \to 0 \) reproduces a mass matrix diagonalized by BM. A straightforward calculation of the eigenvectors up to \( O(\lambda^2) \) shows that the solar and reactor angles do not receive any corrections whereas \( \theta_{23} \) deviates from maximal mixing by a quantity proportional to \( \delta_{\nu} \lambda^2 \).

Taking into account the corrections from the charged leptons, we arrive at a final result similar to eq. (59):

\[
\theta_{13} = \frac{1}{\sqrt{2}} \left( \frac{x_4'}{x_5'} \right) \lambda
\]

\[
\theta_{23} = \frac{\pi}{4} + \left[ -\frac{x_4'^2}{4x_5'^2} + \frac{x_6'^2}{2} + x_7 - \delta_{\nu} \left( \frac{a}{2M_\nu} - \frac{y_1}{y_\nu} \right) \right] \lambda^2
\]

(66)

\[
\theta_{12} = \frac{\pi}{4} - \frac{1}{\sqrt{2}} \left( \frac{x_4' + 2x_5'x_6'}{x_5'} \right) \lambda.
\]

6 Conclusions

In this paper we have constructed an SU(5) model based on the flavour symmetry \( S_4 \times Z_3 \times U(1)_{FN} \), where the BM mixing is realized at the LO in a natural way. In order to get a realistic pattern of fermion masses and mixing and to get rid of the usual problems in SU(5) constructions, like the rigid mass relation \( m_e = m_d^T \) and the doublet-triplet splitting problem, we have embedded the model in five dimensions, with the fifth dimension compactified on the orbifold \( S^1/Z_2 \). In this way, we were able to reproduce the correct mass ratios in both charged lepton and quark sectors as well as a good agreement with the entries of the CKM
mixing matrix, with only a moderate fine-tuning needed to describe the Cabibbo angle. Since exact BM mixing is not confirmed by neutrino oscillation data, large corrections are needed in order to lower the value of the solar angle. Such corrections arise at NLO through the diagonalization of the charged lepton mass matrix while, at the same time, the reactor angle receives corrections of $O(\lambda_C)$, which are perfectly in agreement with a large $\theta_{13}$ emerging from global fits to the world neutrino data. An important feature of our model is that the shift of $\sin^2 \theta_{23}$ from the maximal mixing value of $1/2$ is expected to be of $O(\lambda_C^2)$ at most. In order to reproduce the experimental value of the small parameter $r = \Delta m_{\text{sol}}^2/\Delta m_{\text{atm}}^2$ we need some amount of fine tuning. Here we choose to tune the phases of two complex numbers entering the neutrino mass matrix at leading order; this implies that the neutrino spectrum is mainly of the inverted hierarchy type (or moderately degenerate) and that the smallest light neutrino mass and the $0\nu\beta\beta$ parameter $|m_{ee}|$ are expected to be larger than about 0.01 eV, as we obtained from a numerical analysis with mass matrices at the NLO.

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Appendix A: corrections to the leading order vacuum alignment

In this appendix we explicitly compute the correction to the vacuum alignment of eq.(9). For the sake of completeness, we use the following parametrization of the flavon vevs:

$$\langle \varphi_\ell \rangle = v_{\varphi_\ell} \begin{pmatrix} \delta_{\varphi_1} \\ 1 + \delta_{\varphi_2} \\ \delta_{\varphi_3} \end{pmatrix} \quad \langle \chi_\ell \rangle = v_{\chi} \begin{pmatrix} \delta_{\chi_1} \\ \delta_{\chi_2} \\ 1 \end{pmatrix}$$

$$\langle \varphi_\nu \rangle = v_{\varphi_\nu} \begin{pmatrix} \delta_{\varphi_1} \\ 1 + \delta_{\varphi_2} \\ -1 + \delta_{\varphi_3} \end{pmatrix} \quad \langle \xi_\nu \rangle = v_{\xi} (1 + \delta_{\xi})$$
The superpotential responsible for the vev shifts reads as follows:

\[ \Lambda \Delta w_d = \nu^0 \left[ \alpha_1 (\varphi^0_\nu) \varphi_\nu + \alpha_2 (\varphi^0_\nu) \varphi_\nu + \alpha_3 (\varphi^0_\nu) \varphi_\nu + \alpha_4 (\varphi^0_\nu) \varphi_\nu + \alpha_5 (\varphi^0_\nu) \varphi_\nu + \alpha_6 (\varphi^0_\nu) \varphi_\nu + \right. \\
\left. \alpha_7 (\chi^0_\ell) \chi_\ell + \alpha_8 (\chi^0_\ell) \chi_\ell + \alpha_9 (\chi^0_\ell) \chi_\ell + \alpha_{10} (\chi^0_\ell) \chi_\ell + \alpha_{11} (\chi^0_\ell) \chi_\ell + \right. \\
\left. \alpha_{12} (\chi^0_\ell) \varphi_\nu + \alpha_{13} (\chi^0_\ell) \varphi_\nu + \alpha_{14} (\chi^0_\ell) \varphi_\nu + \alpha_{15} (\chi^0_\ell) \varphi_\nu + \right. \\
\left. \xi^0_\nu \left[ \beta_1 (\varphi^0_\nu) \varphi_\nu + \beta_2 (\varphi^0_\nu) \varphi_\nu + \beta_3 (\varphi^0_\nu) \varphi_\nu + \beta_4 (\varphi^0_\nu) \varphi_\nu + \beta_5 (\chi^0_\ell) \chi_\ell \right] + \right. \\
\left. \psi^0 \left[ \gamma_1 (\varphi^0_\nu) \varphi_\nu + \gamma_2 (\varphi^0_\nu) \varphi_\nu + \gamma_3 (\chi^0_\ell) \chi_\ell + \gamma_4 (\chi^0_\ell) \chi_\ell \right] + \right. \\
\left. \chi^0_\ell \left[ \delta_1 (\varphi^0_\nu) \varphi_\nu + \delta_2 (\chi^0_\ell) \chi_\ell + \delta_3 (\chi^0_\ell) \chi_\ell + \delta_4 (\chi^0_\ell) \chi_\ell \right] + \right. \\
\left. \delta_5 (\varphi_\nu \varphi_\ell) \right]. \tag{67} \\

Even after considering the leading order relations in eq. (10), some of the minimizing equations are still cumbersome. In the following we concentrate on those which easily allow to extract the relevant information on the vev shifts:

\[ \frac{\partial \Delta w_d}{\partial \nu^0_\nu} = 2g_1 \nu^2_\nu (\delta_{\nu_2} + \delta_{\nu_3}) = 0 \quad \tag{68} \]

\[ \frac{\partial \Delta w_d}{\partial \psi^0_\ell} = v_\chi \left[ \sqrt{3} f_3 \nu_\ell (\delta_{\psi_3} + \delta_{\chi_2}) - 2f_2 \delta_{\chi_2} \nu_\chi \right] - 2f_1 \delta_{\psi_3} \nu^2_\psi_\ell \quad \tag{69} \]

\[ \frac{\partial \Delta w_d}{\partial \varphi^0_\ell} = 4v_\varphi \nu_\chi (\delta_{\psi_3} - \delta_{\chi_2}) = 0. \quad \tag{70} \]

We see that the last equation implies \( \delta_{\varphi_3} = \delta_{\chi_2} \) which, using eq. (69), forces \( \delta_{\varphi_3} = \delta_{\chi_2} = 0 \). The first equation tells us that \( \delta_{\nu_2} = -\delta_{\nu_3} \) and that they are undetermined. Using the previous relations, the remaining minimizing equations allow to fix the magnitude of the other vev shifts in terms of the superpotential parameters. In particular, from \( \partial \Delta w_d / \partial \nu^0_{\nu_2} \) and \( \partial \Delta w_d / \partial \nu^0_{\nu_3} \) we get a set of conditions on \( \delta_{\nu_1} \) and \( \delta_{\nu_\ell} \), which result both different from zero. From \( \partial \Delta w_d / \partial \chi^0_{\chi_3} = 0 \) we obtain an expression for \( \delta_{\chi_1} \) and \( \delta_{\varphi_1} \), respectively; the last equation \( \partial \Delta w_d / \psi^0_{\ell_2} = 0 \) depends on \( \delta_{\varphi_2} \), which is also completely determined by the superpotential parameters.
Appendix B: corrections to the fermion mass matrices

In this appendix we quote all the operators with three-flavon insertions responsible for the down quark sector:

\[ FT_3 H_5 : \quad (\varphi_\nu^2)_1 \varphi_\nu, (\varphi_\nu^2)_2 \varphi_\nu, (\varphi_\nu^2)_3 \varphi_\nu, (\varphi_\nu^2)_4 \varphi_\nu \]

\[ (\varphi_\nu^2)_5 \chi_\ell, \xi_\nu^2 \varphi_\nu, (\varphi_\nu \varphi_\nu)_3 \xi_\nu, (\varphi_\nu \xi_\ell)_3 \xi_\nu \]

\[ FT_2 H_5 \theta : \quad (\varphi_\nu^2)_1 \varphi_\nu, (\varphi_\nu^2)_2 \varphi_\nu, (\varphi_\nu^2)_3 \varphi_\nu, (\varphi_\nu^2)_4 \varphi_\nu \]

\[ (\varphi_\nu^2)_5 \chi_\ell, (\varphi_\nu^2)_3 \chi_\ell, (\chi_\ell^2)_2 \chi_\ell, (\chi_\ell^2)_3 \chi_\ell, (\varphi_\nu^2)_3 \xi_\nu, \xi_\nu^2 \varphi_\nu, \]

\[ (\varphi_\nu^2)_5 \chi_\ell, (\varphi_\nu^2)_3 \chi_\ell, (\chi_\ell^2)_2 \chi_\ell, (\chi_\ell^2)_3 \chi_\ell \]

\[ FT_2 H_5 \theta' : \quad (\varphi_\nu^2)_1 \varphi_\nu, (\varphi_\nu^2)_2 \varphi_\nu, (\varphi_\nu^2)_3 \varphi_\nu, (\varphi_\nu^2)_4 \varphi_\nu \]

\[ (\chi_\ell^2)_3 \varphi_\nu, (\chi_\ell^2)_3 \xi_\nu, (\varphi_\nu \varphi_\nu)_2 \chi_\ell, (\varphi_\nu \varphi_\nu)_3 \chi_\ell, \]

\[ (\varphi_\ell \varphi_\nu)_3 \chi_\ell, (\varphi_\ell \chi_\ell)_3 \xi_\nu \]

\[ FT_1 H_5 \theta^3 : \quad \text{the same as } FT_2 H_5 \theta' \]

\[ FT_1 H_5 \theta^2 \theta' : \quad \text{the same as } FT_3 H_5 \]

\[ FT_1 H_5 \theta \theta^2 : \quad \text{the same as } FT_2 H_5 \theta \]

\[ FT_1 H_5 \theta^3 : \quad \text{the same as } FT_2 H_5 \theta' \]

Appendix C: the group \( S_4 \)

We report here the multiplication table for \( S_4 \) and we list the Clebsch-Gordan coefficients in the basis used in the paper [11]. In the following we use \( \alpha_i \) to indicate the elements of the first representation of the product and \( \beta_i \) to indicate those of the second representation. Explicit forms of \( S \) and \( T \) are as follows. In the representation 1 we have \( T = 1 \) and \( S = 1 \), while \( T = -1 \) and \( S = -1 \) in \( 1' \). In the representation 2 we have:

\[ T = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad S = \frac{1}{2} \begin{pmatrix} -1 & \sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix} \]  \hspace{1cm} (72)

For the representation 3, the generators are:

\[ T = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & i \end{pmatrix} \quad S = \begin{pmatrix} 0 & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \]  \hspace{1cm} (73)
In the representation $3'$ the generators $S$ and $T$ are simply opposite in sign with respect to those in the $3$.

We start with all the multiplication rules which include the 1-dimensional representations:

$1 \otimes \text{Rep} = \text{Rep} \otimes 1 = \text{Rep}$ \quad with Rep any representation

$1_2 \otimes 1_2 = 1 \sim \alpha \beta$

$1_2 \otimes 2 = 2 \sim \begin{pmatrix} \alpha \beta_2 \\ -\alpha \beta_1 \end{pmatrix}$

$1_2 \otimes 3_1 = 3_2 \sim \begin{pmatrix} \alpha \beta_2 \\ \alpha \beta_1 \\ \alpha \beta_3 \end{pmatrix}$

$1_2 \otimes 3_2 = 3_1 \sim \begin{pmatrix} \alpha \beta_1 \\ \alpha \beta_2 \\ \alpha \beta_3 \end{pmatrix}$

The multiplication rules with the 2-dimensional representation are the following ones:

$2 \otimes 2 = 1 \oplus 1_2 \oplus 2$ \quad with \quad \begin{align*}
1 & \sim \alpha_1 \beta_1 + \alpha_2 \beta_2 \\
1_2 & \sim \alpha_1 \beta_2 - \alpha_2 \beta_1 \\
2 & \sim \begin{pmatrix} \alpha_2 \beta_2 - \alpha_1 \beta_1 \\ \alpha_1 \beta_2 + \alpha_2 \beta_1 \end{pmatrix}
\end{align*}

$2 \otimes 3_1 = 3_1 \oplus 3_2$ \quad with \quad \begin{align*}
3_1 & \sim \begin{pmatrix} \alpha_1 \beta_1 \\ \frac{\sqrt{3}}{2} \alpha_2 \beta_3 - \frac{1}{2} \alpha_1 \beta_2 \\ \frac{\sqrt{3}}{2} \alpha_2 \beta_2 - \frac{1}{2} \alpha_1 \beta_3 \end{pmatrix}
\end{align*}

$2 \otimes 3_2 = 3_1 \oplus 3_2$ \quad with \quad \begin{align*}
3_1 & \sim \begin{pmatrix} -\alpha_2 \beta_1 \\ \frac{\sqrt{3}}{2} \alpha_1 \beta_3 + \frac{1}{2} \alpha_2 \beta_2 \\ \frac{\sqrt{3}}{2} \alpha_1 \beta_2 + \frac{1}{2} \alpha_2 \beta_3 \end{pmatrix}
\end{align*}
The multiplication rules involving the 3-dimensional representations are:

\[ 3_1 \otimes 3_1 = 3_2 \otimes 3_2 = 1 \oplus 2 \oplus 3_1 \oplus 3_2 \quad \text{with} \quad \begin{cases} 
1 \sim \alpha_1 \beta_1 + \alpha_2 \beta_3 + \alpha_3 \beta_2 \\
2 \sim \left( \frac{\sqrt{3}}{2} (\alpha_2 \beta_3 + \alpha_3 \beta_2) - \frac{1}{2} (\alpha_2 \beta_3 + \alpha_3 \beta_2) \right) \\
3_1 \sim \left( \begin{array}{c} 
\alpha_3 \beta_3 - \alpha_2 \beta_2 \\
\alpha_1 \beta_3 + \alpha_3 \beta_1 \\
-\alpha_1 \beta_2 - \alpha_2 \beta_1
\end{array} \right) \\
3_2 \sim \left( \begin{array}{c} 
\alpha_3 \beta_2 - \alpha_2 \beta_3 \\
\alpha_2 \beta_1 - \alpha_1 \beta_2 \\
\alpha_1 \beta_3 - \alpha_3 \beta_1
\end{array} \right)
\end{cases} \]

\[ 3_1 \otimes 3_2 = 1_2 \oplus 2 \oplus 3_1 \oplus 3_2 \quad \text{with} \quad \begin{cases} 
1_2 \sim \alpha_1 \beta_1 + \alpha_2 \beta_3 + \alpha_3 \beta_2 \\
2 \sim \left( \frac{\sqrt{3}}{2} \left( \alpha_2 \beta_2 + \alpha_3 \beta_3 \right) - \frac{1}{2} \left( \alpha_2 \beta_2 + \alpha_3 \beta_3 \right) \right) \\
3_1 \sim \left( \begin{array}{c} 
\alpha_3 \beta_2 - \alpha_2 \beta_3 \\
\alpha_2 \beta_1 - \alpha_1 \beta_2 \\
\alpha_1 \beta_3 - \alpha_3 \beta_1
\end{array} \right) \\
3_2 \sim \left( \begin{array}{c} 
\alpha_3 \beta_3 - \alpha_2 \beta_2 \\
\alpha_1 \beta_3 + \alpha_3 \beta_1 \\
-\alpha_1 \beta_2 - \alpha_2 \beta_1
\end{array} \right)
\end{cases} \]
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