Maxwell symmetries and some applications

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Abstract

The Maxwell algebra is the result of enlarging the Poincaré algebra by six additional tensorial Abelian generators that make the fourmomenta non-commutative. We present a local gauge theory based on the Maxwell algebra with vierbein, spin connection and six additional geometric Abelian gauge fields. We apply this geometric framework to the construction of Maxwell gravity, which is described by the Einstein action plus a generalized cosmological term. We mention a Friedman-Robertson-Walker cosmological approximation to the Maxwell gravity field equations, with two scalar fields obtained from the additional gauge fields. Finally, we outline further developments of the Maxwell symmetries framework.
1 Introduction

Maxwell symmetry was introduced around 40 years ago \[1, 2\], but it is only recently that has attracted more attention. The \(D = 4\) Maxwell algebra, with sixteen generators \((P_a, M_{ab}, Z_{ab})\), is obtained from Poincaré algebra if we replace its commuting fourmomenta by noncommuting ones

\[
[P_a, P_b] = \Lambda Z_{ab} \quad , \quad [P_a, Z_{bc}] = 0 \quad , \quad a, b = 0, 1, 2, 3 ,
\]

where the six additional generators \(Z_{ab} = -Z_{ba}\) are Abelian and define a Lorentz-covariant tensor,

\[
[M_{ab}, Z_{cd}] = -\left(\eta_{c[a}Z_{b]d} - \eta_{d[a}Z_{b]c}\right).
\]

Further, we assume that the \(Z_{ab}\) are dimensionless, which implies that the parameter \(\Lambda\) has mass dimensions \([\Lambda] = \Lambda^2\). The Maxwell algebra \( \mathcal{M} \) is the semidirect sum \(\mathcal{M} = \text{so}(3,1) \oplus \mathcal{M}^3\), where \(\mathcal{M}^3\) is the Maxwell ideal generated by \(P_a, Z_{ab}\) (eq. (1)), in which \(\Lambda\) is the central extension parameter. \(\mathcal{M}\) can be obtained from the \(\text{so}(3,2)\) algebra \((\mathcal{M}_{AB}; \ A, B = 0, 1, 2, 3, 4)\) after the rescaling \(M_{ab} = \beta^2 Z_{ab}; \ a, b = 0, 1, 2, 3, M_{a4} = \beta \Lambda^{-\frac{1}{2}} P_a\) in the contraction limit \(\beta \to \infty\) (\(\beta\) is a dimensionless parameter).

The global Maxwell symmetries have been introduced in order to describe Minkowski space with constant e.m. background \[1-4\] in models of relativistic particles interacting with a constant e.m. field\[4\]. In this paper, following \[6\], we present the construction of a local \(D = 4\) gauge theory based on the Maxwell algebra (eqs. (1,2)) and apply it to generalize Einstein gravity. Such a theory will accordingly contain six additional geometric Abelian gauge fields, playing the role of vectorial inflatons\[2\] and which in Maxwell gravity contribute to a generalization of the cosmological term. Further, we shall mention a one-dimensional FRW cosmological approximation describing the cosmic scale factor \(a(t)\) in Maxwell gravity and comment briefly on other uses of Maxwell symmetries.

2 Gauging the Maxwell algebra to generalize Einstein gravity

In order to introduce geometrically the Maxwell gauge vector fields we consider the Maxwell algebra-valued one-form \(h = h_\mu dx^\mu\), where

\[
h_\mu = e^a_\mu P_a + \frac{1}{2} \omega^{ab}_\mu M_{ab} + \frac{1}{2} A^{ab}_\mu Z_{ab} .
\]

The Maxwell multiplet \((e^a_\mu(x), \omega^{ab}_\mu(x), A^{ab}_\mu(x))\) includes the vierbein, the spin connection and the new Abelian gauge fields \(A^{ab}_\mu\), which we interpret as geometrical inflaton vector

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\[1\] For the non-relativistic case see also \[5\], Sec. 8.3.

\[2\] For vector inflatons described by \(SU(2)\) gauge fields and non-abelian ‘gauge-flation’ models see e.g. refs. \[7\]-\[10\].
fields. Their associated curvatures are the components of the $\mathcal{M}$-valued curvature two-form $R = R_{\mu\nu} dx^\mu \wedge dx^\nu$:

$$R_{\mu\nu} = T^a_{\mu\nu} P_a + \frac{1}{2} R^{ab}_{\mu\nu} M_{ab} + \frac{1}{2} F^{ab}_{\mu\nu} Z_{ab},$$ (4)

which defines the two-forms corresponding to the torsion $T^a$, the Lorentz curvature $R^{ab}$, and the field strength $F^{ab}$ for $A^{ab}$, with dimensions $[T] = M^{-1}$, $[R] = M^0 = [F]$ (the one-forms $e^a = e^a_\mu dx$, $\omega^{ab} = \omega^{ab}_\mu dx^\mu$, $A^{ab} = A^{ab}_\mu dx^\mu$ have dimensions $[e^a] = M^{-1}$, $[\omega^{ab}] = M^0 = [A^{ab}]$). Explicitly, it follows from (3) and $R = dh + \frac{i}{2} [h, h]$ that the $T^a$, $R^{ab}$, $F^{ab}$ spacetime components are given by

$$T^a_{\mu\nu} = \partial_{[\mu} e^{a}_{\nu]\rangle + \omega^{a}_{b[\mu} e^{b}_{\nu]\rangle \equiv D^{a}_{b[\mu} e^{b}_{\nu]\rangle}, \quad (5)$$

$$R^{ab}_{\mu\nu} = \partial_{[\mu} \omega^{ab}_{\nu]\rangle + \omega^{a}_{c[\mu} \omega^{b}_{\nu]\rangle = (D\omega^{ab})_{\mu\nu} = -R^{ba}_{\mu\nu}, \quad (6)$$

$$F^{ab}_{\mu\nu} = D^{a}_{c[\mu} A^{b}_{\nu]\rangle + \Lambda e^{a}_{[\mu} e^{b}_{\nu]\rangle, \quad (7)$$

where $D^{a}_{b\mu} = \delta^{a}_{b} \partial_\mu + \omega^{a}_{b\mu}$. We observe that the torsion and the Lorentz curvature are the same as in standard Einstein gravity, which is the particular choice of the Einstein-Cartan gravity described by the Poincaré gauge theory; the $\Lambda$-dependent additional term, which recalls the contribution to $R^{ab}$ in (A)dS gravity, enters through the new gauge curvature $F^{ab}$.

The following two geometric (metric independent) Lagrangian densities were considered in detail in ref. [6], namely

1) The Lagrangian density that leads to Einstein gravity

$$\mathcal{L}_2 = -\frac{1}{2\kappa\Lambda} \varepsilon_{abcd} R^{ab} \wedge F^{cd}. \quad (8)$$

Indeed, using

$$\varepsilon_{abcd} R^{ab} \wedge (DA)^{cd} = d (\varepsilon_{abcd} R^{ab} \wedge A^{cd}), \quad (9)$$

it follows that, modulo the above boundary term, the $\mathcal{L}_2$ in eq. (8) gives the Einstein gravity Lagrangian $\mathcal{L}_E$,

$$\mathcal{L}_2 \simeq \mathcal{L}_E \equiv -\frac{1}{2\kappa} \varepsilon_{abcd} R^{ab} \wedge e^c \wedge e^d. \quad (10)$$

2) The generalized cosmological term.

Let us recall that the Lagrangian density for the standard geometric Einstein cosmological (EC) term, proportional to the cosmological constant $\lambda$ ($[\lambda] = M^2$), is

$$\mathcal{L}_{EC} = \frac{\lambda}{4\kappa} \varepsilon_{abcd} e^a \wedge e^b \wedge e^c \wedge e^d. \quad (11)$$
The generalized cosmological term depends both on the standard cosmological constant \( \lambda \) and on the parameter \( \Lambda \) in eq. (11). It is defined by

\[
L_C = \frac{\lambda}{4\kappa\Lambda^2} \varepsilon_{abcd} F^{ab} \wedge F^{cd} = L_{EC} + \Delta L_C ,
\]

where the additional piece \( \Delta L_C \) with respect to eq. (11) is given by

\[
\Delta L_C = \frac{\lambda}{4\kappa\Lambda^2} \varepsilon_{abcd} [(DA)^{ab} \wedge (DA)^{cd} + 2\Lambda (DA)^{ab} \wedge e^c \wedge e^d] .
\]

Our basic Maxwell gravity action \( L_M \) is then given [6] by the Lagrangian density

\[
L_M = L_E + L_C = L_E + L_{EC} + \Delta L_C ,
\]

which includes a new contribution, \( \Delta L_C \), to the standard cosmological term (11). By varying with respect to the independent field variables the following three equations of motion are obtained

\[
\begin{align*}
\delta \omega^{ab} : & \quad T^{[a} \wedge e^{b]} + \frac{\lambda}{\Lambda^2} F^{[a} e^{c} \wedge A^{c]} = 0 , \\
\delta e^a : & \quad \varepsilon_{abcd} e^b \wedge \left( R^{cd} - \frac{\lambda}{\Lambda} F^{cd} \right) = 0 , \\
\delta A^{ab} : & \quad (De)^{[a} e^{b]} + \frac{1}{\Lambda} R^{[a} e^{c} \wedge A^{c]} = 0 .
\end{align*}
\]

It is useful to introduce a shifted Lorentz curvature by

\[
J^{cd} = R^{cd} - \frac{\lambda}{\Lambda} F^{cd} .
\]

If we assume \( J^{ab} = 0 \) the equations (15) and (17) become identical, but this assumption describes only special solutions of these equations. In the general \( J^{ab} \neq 0 \) case, the field equations (15,17) can be rewritten in simpler form. Using \( J^{ab} \), one obtains

\[
\begin{align*}
\delta \tilde{\omega}^{ab} : & \quad (\tilde{D}J)^{ab} \equiv dJ^{ab} + \tilde{\omega}^{[a} e^c J^{b]}_c = 0 , \\
\delta e^a : & \quad \varepsilon_{abcd} e^b \wedge J^{cd} = 0 , \\
\delta A^{de} : & \quad \varepsilon_{abcd} J^{ab} A^{c]_e} = 0 .
\end{align*}
\]

Equation (20) is the generalization of the Einstein equation.
where $J^\mu_\rho \equiv J^{\mu \nu}_{\rho \nu}$, $J \equiv J^\mu_\mu$, and similarly, $R^\mu_\rho \equiv R^{\mu \nu}_{\rho \nu}$, $R \equiv R^\mu_\mu$, $F^\mu_\rho \equiv F^{\mu \nu}_{\rho \nu}$, $F \equiv F^\mu_\mu$. More explicitly, eq. (22) can be written in a more familiar form as

$$
R^\mu_\nu - \frac{1}{2} R \delta^\mu_\nu - 3 \lambda \delta^\mu_\nu = 
= \frac{\lambda}{\Lambda} \left( e^a_\mu e^b_\sigma (D_{[\nu} A_{\sigma]})^{ab} - \delta^\mu_\nu e^a_\rho e^b_\sigma (D_{\rho} A_\sigma)^{ab} \right) .
$$

We see that the source added to the standard gravity equations with cosmological constant $\lambda$ contains linear contributions from the new gauge fields. The second term in the rhs of (23) provides a field-dependent modification of the cosmological constant at the lhs of the equation.

We wish to add that:

1) Using equations (15) and (16), the spin connection may be expressed as a function $\omega^{ab}(e, A)$ of the vierbein and the new gauge fields (for perturbative solutions see [6], Appendix). In such a way we obtain the second order formulation of Maxwell gravity, with independent fields $e^a_\mu$ and $A^{ab}_\mu$.

2) Using eqs. (15), (17) and the Bianchi identities for the $F^{ab}$ curvature, we obtain the free field equation for new gauge fields $A^{ab}_\mu$,

$$
(DF)^{ab} = 0 .
$$

(24)

Eq. (24) can be modified by a source term if we add to the Lagrangian (14) non-geometrical contributions containing the fields $A^{ab}_\mu$ as e.g., a kinematical term proportional to the density $F^{ab} \wedge *F_{ab}$, similar to the free Lagrangian $-\frac{1}{2} F \wedge *F$ of Maxwell electrodynamics.

3) In order to estimate the effect of the new gauge fields on the dynamics of the universe we have considered recently [11] a cosmological FRW approximation to the field equations of Maxwell gravity. After introducing a function $a(t)$ describing the time dependence of the cosmic scale factor

$$
\epsilon^a_\mu(x) = (\epsilon^a_\mu(x), \epsilon^a_0(x)) \xrightarrow{FRW} (\delta^a_i a(t), 0) ,
$$

(25)

where $a = 0, 1, 2, 3$ and $i, j = 1, 2, 3$, the six Abelian gauge fields $A^{ab}_\mu$ are approximated in terms of the one-dimensional inflaton fields $\phi_1, \phi_2$, as follows

$$
A^r_{s \mu}(x) = (A^r_{s i}(x), A^r_{s 0}(x)) \xrightarrow{FRW} (\epsilon^r_i \phi_1(t), 0) ,
$$

$$
A^0_{r \mu}(x) = (A^0_{r i}(x), A^0_{r 0}(x)) \xrightarrow{FRW} (\delta^0_i \phi_2(t), 0) .
$$

(26)

The usual way of introducing the vector inflaton fields is based on Yang-Mills gauge fields [9,10] with internal symmetry indices. In our case these internal indices are replaced by tangent spacetime indices, and the three-dimensional tensors appearing in formula (26), $\epsilon^r i$ and $\delta^r_i$ (for $a = 1, 2, 3$), are genuine three-dimensional $so(3)$ tensors.
3 Outlook

To conclude, we make the following comments:

a) The action defining Maxwell gravity was chosen to obtain a generalization of the cosmological term. Nevertheless, as for the standard Einstein-Hilbert Lagrangian (10), the action that follows from (14) is only invariant under local Lorentz transformations and spacetime diffeomorphisms, not under the full local Maxwell algebra.

We would like to mention at this point that other Maxwell generalizations of Einstein gravity, invariant under the local Abelian gauge symmetries associated with the $Z_{ab}$ generators, have been proposed recently. The locally Maxwell-invariant gravity model in ref. [12] contains a rather controversial torsion squared term, and the Maxwell-invariant extensions proposed in ref. [13] differ from the Einstein Lagrangian (10) only by a topological term i.e., the Einstein field equations remain unaltered. We also note that another modification of Einstein gravity (see ref. [14]), obtained by gauging a deformation of the Maxwell algebra $\mathfrak{so}(3,1) \oplus \mathfrak{so}(3,2)$, has been proposed recently. The action of the deformed Maxwell gravity in [14] is invariant under local deformed Maxwell transformations, but in the contraction limit that leads to the Maxwell Lie algebra the Abelian local Maxwell symmetries are also broken, as in our case.

b) The Maxwell symmetries have been generalized to Maxwell supersymmetries [16]; in particular, the $N$-extended Maxwell superalgebras were recently described in detail in [17]. When $N=1$ one obtains three different models of lowest dimensional Maxwell superalgebras, containing a pair of two-component Weyl charges.

c) It is known that the higher spin (HS) free fields can be described as a free field theory on enlarged, tensorial spaces which contains the $D$-dimensional ‘physical’ spacetime as a submanifold [18–22]. It turns out that for $D=4$ the free HS fields can be obtained from the first quantization of spinorial particle model on a ten-dimensional tensorial space [19–21]. However, the ten-dimensional group manifold generated by the Maxwell ideal $\mathcal{M}^D$ (eq. (11)) of the Maxwell Lie algebra also defines a ten-dimensional extended $D=4$ spacetime that we call Maxwell $D=4$ tensorial space. One can consider as well a spinorial particle model on this new Maxwell tensorial space which, after first quantization, should also provide an infinite-dimensional multiplet of $D=4$ HS fields; such a model is under consideration [23].

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3For the classification of the Maxwell algebra deformations see ref. [15].

4Such a model can be called AdS-Maxwell gravity.
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