Conditions to the existence of center in planar systems and center for Abel equations

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Abstract

Abel equations of the form \(x'(t) = f(t)x^3(t) + g(t)x^2(t),\ t \in [-a, a]\), where \(a > 0\) is a constant, \(f\) and \(g\) are continuous functions, are of interest because of their close relation to planar vector fields. If \(f\) and \(g\) are odd functions, we prove, in this paper, that the Abel equation has a center at the origin. We also consider a class of polynomial differential equations \(\dot{x} = -y + P_n(x, y)\) and \(\dot{y} = x + Q_n(x, y)\), where \(P_n\) and \(Q_n\) are homogeneous polynomials of degree \(n\). Using the results obtained for Abel’s equation, we obtain a new subclass of systems having a center at the origin.

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1. Introduction and main results

1.1. Historical Aspect

Let the planar system

\[
\begin{align*}
\dot{x} &= -y + P(x, y) \\
\dot{y} &= x + Q(x, y),
\end{align*}
\]

(1.1)

where \(P(x, y)\) and \(Q(x, y)\) are polynomials, without constant term, of maximum degree \(n\). The singular point \((0, 0)\) is a center, if surrounded by closed trajectories; or a focus, if surrounded by spirals. The classical center-focus problem consists in distinguishing when a singular point is either a center or...
a focus. The problem started with Poincaré [16] and Dulac [6], and, in the present days, many questions remain open. The basic results were obtained by A. M. Lyapunov [11]. He proved that if $P(x, y)$ and $Q(x, y)$ satisfy an infinite sequence of recursive conditions, then (1.1) has a center to the origin. He also presented conditions for the origin of the system (1.1) to be a focus.

If we write $P(x, y) = \sum_{i=1}^{l} P_{m_i}(x, y)$ and $Q(x, y) = \sum_{i=1}^{l} Q_{m_i}(x, y)$, where $P_{m_i}(x, y)$ and $Q_{m_i}(x, y)$ are homogeneous polynomials of degree $m_i \geq 1$, then, from Hilbert’s theorem on the finiteness of basis of polynomial ideals (8, Theorem 87, p. 58), it follows that in the mentioned infinite sequence of recursive conditions only a finite number of conditions for center are essential. The others result from them.

In this paper, we study a particular case of (1.1). Namely

$$
\begin{align*}
\dot{x} &= -y + P_n(x, y) \\
\dot{y} &= x + Q_n(x, y),
\end{align*}
$$

(1.2)

where $P_n(x, y)$ and $Q_n(x, y)$ are homogeneous polynomials of degree $n$.

When $n = 2$, systems (1.2) are quadratic polynomial differential systems (or simply quadratic systems in what follows). Quadratic systems have been intensively studied over the last 30 years, and more than a thousand papers on this issue have been published (see, for example, the bibliographical survey of Reyn [17]).

A method for investigating if (1.2) has a center at the origin is to transform the planar system into an Abel equation. In polar coordinates $(r, \theta)$ defined by $x = r \cos \theta, y = r \sin \theta$, the system (1.2) becomes

$$
\begin{align*}
\dot{r} &= A(\theta)r^n \\
\dot{\theta} &= 1 + B(\theta)r^{n-1},
\end{align*}
$$

(1.3)

where

$$
\begin{align*}
A(\theta) &= \cos \theta P_n(\cos \theta, \sin \theta) + \sin \theta Q_n(\cos \theta, \sin \theta), \\
B(\theta) &= \cos \theta Q_n(\cos \theta, \sin \theta) - \sin \theta P_n(\cos \theta, \sin \theta).
\end{align*}
$$

(1.4)

We remark that $A$ and $B$ are homogeneous polynomials of degree $n + 1$ in the variables $\cos \theta$ and $\sin \theta$. In the region

$$
R = \{(r, \theta) : 1 + B(\theta)r^{n-1} > 0\},
$$

the differential system (1.3) is equivalent to the differential equation

$$
\frac{dr}{d\theta} = \frac{A(\theta)r^n}{1 + B(\theta)r^{n-1}}.
$$

(1.5)
It is known that the periodic orbits surrounding the origin of the system (1.3) do not intersect the curve \( \theta = 0 \) (see the Appendix of [3]). Therefore, these periodic orbits are contained in the region \( R \). Consequently, they are also periodic orbits of equation (1.5).

The transformation \((r, \theta) \rightarrow (\gamma, \theta)\) with

\[
\gamma = \frac{r^{n-1}}{1 + B(\theta) r^{n-1}}
\]

is a diffeomorphism from the region \( R \) into its image. As far as we know, Cherkas was the first to use this transformation (see [4]). If we write equation (1.5) in the variable \( \gamma \), we obtain

\[
\frac{d\gamma}{d\theta} = -(n-1)A(\theta)B(\theta)\gamma^3 + [(n-1)A(\theta) - B'(\theta)]\gamma^2,
\]

which is a particular case of an Abel differential equation. We notice that \( f(\theta) = -(n-1)A(\theta)B(\theta) \) and \( g(\theta) = (n-1)A(\theta) - B'(\theta) \) are homogeneous trigonometric polynomials of degree \( 2(n+1) \) and \( n+1 \), respectively.

Now the Center-Focus problem of planar system (1.2) has a translation in equation (1.7), that is, given \( \gamma_0 \) small enough, we look for necessary and sufficient conditions on \( f(\theta) \) and \( g(\theta) \) in order to assure that the solution of equation (1.7), with the initial condition \( \gamma(0) = \gamma_0 \), has the property that \( \gamma(0) = \gamma(2\pi) \). We observe that this condition implies the periodicity of this solution, in particular, one has \( \gamma(-\pi) = \gamma(\pi) \).

The equation \( \frac{dx}{dt} = a(t)x^3 + b(t)x^2 \) was studied in [1], where necessary and sufficient conditions were obtained for this equation has a center at the origin, but with \( a(t) \) and \( b(t) \), particular continuous functions (see Example 1.9). More results that ensure the existence a center at the origin for some subclasses of planar systems and for Abel equations were obtained in [10, 9].

In this paper, some new results are obtained for planar systems (1.2) and Abel equations (1.7).

1.2. Results on planar systems

Consider the planar system of differential equations (1.2). We obtain sufficient conditions on \( P_n(x, y) \) and \( Q_n(x, y) \) that allow the system (1.2) to have a center at the origin.
**Theorem 1.1.** Let the planar system
\[
\begin{align*}
\dot{x} &= -y + P_n(x, y) \\
\dot{y} &= x + Q_n(x, y),
\end{align*}
\] (1.8)
where \(P_n(\cos \theta, \sin \theta)\) is an odd function and \(Q_n(\cos \theta, \sin \theta)\) is an even function in \(C([0, 2\pi])\). Then, the system (1.8) has a center at the origin.

**Example 1.2.** Let the planar system
\[
\begin{align*}
\dot{x} &= -y + ax^{N_1}y^{M_1} \\
\dot{y} &= x + bx^{N_2}y^{M_2},
\end{align*}
\] (1.9)
with \(N_1 + M_1 = N_2 + M_2\), \(N_1, N_2\) be any nonnegative integers numbers, \(M_1\) is an odd natural number, \(M_2\) is an even natural number and \(a, b\) are real numbers. Then, the planar system (1.9) has a center at the origin. Indeed, it is easy to see that \(P_n(\cos \theta, \sin \theta) = a \cos^{N_1} \theta \sin^{M_1} \theta\) is an odd function and \(Q_n(\cos \theta, \sin \theta) = b \sin^{M_2} \theta \cos^{N_2} \theta\) is an even function in \(C([0, 2\pi])\). Thus, the result is a consequence of the Theorem 1.1. Now, we observe that the system (1.9) does not satisfy the reversibility criterion of Poincaré, when \(N_2 = 0\).

A question on Theorem 1.1 is if the conditions \(P_n(\cos \theta, \sin \theta)\) is an odd function and \(Q_n(\cos \theta, \sin \theta)\) is an even function are also necessary conditions for (1.8) to have a center at the origin. The answer for this question is no, as we show in the example below.

**Example 1.3.** Let
\[
\begin{align*}
\dot{x} &= -y + P_n(x, y) \\
\dot{y} &= x + Q_n(x, y),
\end{align*}
\] (1.10)
where \(P_n(x, y) = yP_{n-1}(x, y), \ Q_n(x, y) = -xP_{n-1}(x, y)\) and \(P_{n-1}(1, y)\) has all monomials with odd degree. We observe that \(P_n(\cos \theta, \sin \theta)\) is an even function and \(Q_n(\cos \theta, \sin \theta)\) is an odd function. As already discussed, the planar system (1.10) becomes
\[
\frac{d\gamma}{d\theta} = -(n - 1)A(\theta)B(\theta)\gamma^3 + [(n - 1)A(\theta) - B'(\theta)]\gamma^2, \tag{1.11}
\]
where
\[
\begin{align*}
A(\theta) &= \cos \theta \sin \theta P_{n-1}(\cos \theta, \sin \theta) - \cos \theta \sin \theta P_{n-1}(\cos \theta, \sin \theta) = 0, \\
B(\theta) &= -\cos^2 \theta P_{n-1}(\cos \theta, \sin \theta) - \sin^2 \theta P_{n-1}(\cos \theta, \sin \theta) \\
        &= -P_{n-1}(\cos \theta, \sin \theta). \tag{1.12}
\end{align*}
\]
According to [9], Proposition 2.2 (a), the planar system (1.10) has a center at the origin. This shows that the conditions $P_n(\cos \theta, \sin \theta)$ is an odd function and $Q_n(\cos \theta, \sin \theta)$ is an even function are not necessary conditions for (1.8) to have a center at the origin.

In [10], Llibre et al showed a class of planar systems that have center in the origin. One of the conditions for a system to belong to this class is

$$f'(\theta)g(\theta) - f(\theta)g'(\theta) = ag(\theta)^3,$$  \hspace{1cm} (1.13)

for some $a \in \mathbb{R}$. The example below shows that some planar systems satisfy the hypothesis of the Theorem 1.1, but do not satisfy (1.13).

**Example 1.4.** Let the planar system

$$\begin{align*}
\dot{x} &= -y + 2x^2y \\
\dot{y} &= x + xy^2.
\end{align*}$$  \hspace{1cm} (1.14)

In this case, after calculations, we obtain

$$f'(\theta)g(\theta) - f(\theta)g'(\theta) = -336 \sin(\theta) \cos(\theta)^9 + 144 \sin(\theta) \cos(\theta)^7 + 192 \sin(\theta) \cos(\theta)^{11}$$

and

$$g(\theta)^3 = (6 \cos(\theta)^3 \sin(\theta))^3.$$ 

It is easy to verify that $f'(\theta)g(\theta) - f(\theta)g'(\theta) \neq ag(\theta)^3$ for all $a \in \mathbb{R}$.

**Remark 1.5.** Note that, for $A(\theta)$ in the assumptions of Theorem 1.1 we have $\int_0^{2\pi} A(\theta)d\theta = 0$. If this integral is different from zero, then the system (1.2) has a focus at the origin. Indeed, according to [5], the system (1.2) is non-degenerate and quasi homogeneous. In the same paper, Conti [3, Theorem 7.1, p. 219] proved the origin of (1.2) is a center or a focus. Moreover, using the classical results of Alwash and Lloyd [2], it is well known that for (1.7), a necessary condition to have a center is $\int_0^{2\pi} A(\theta)d\theta = 0$.

### 1.3. Results on Abel equations

Consider the Abel equation

$$x'(t) = f(t)x^3(t) + g(t)x^2(t),$$  \hspace{1cm} (1.15)

t $\in [-a, a]$, where $a > 0$ is a constant, $f$ and $g$ are continuous functions. We obtain conditions on $f$ and $g$ coefficients of Abel equation that ensure the existence of a center in $x = 0$.

We state the main results of this section.
**Theorem 1.6.** Suppose that \( f \) is an odd continuous function in \( C([-a,a]) \). Then, there are infinitely many closed even solutions for (1.15) near the zero solution if and only if \( g \) is an odd continuous function in \( C([-a,a]) \).

**Theorem 1.7.** Suppose that \( g \) is an odd continuous function in \( C([-a,a]) \). Then, there are infinitely many closed even solutions for (1.15) near the zero solution if and only if \( f \) is an odd continuous function in \( C([-a,a]) \).

A conclusion which follows these theorems is that if \( f \) and \( g \) are odd continuous functions, each solution for (1.15) near the solution \( x = 0 \) is a closed even solution.

The next theorem proves a sufficient condition for the existence of a center to the Abel equation (1.15).

**Theorem 1.8.** Suppose that \( f \) and \( g \) are odd continuous function in \( C([-a,a]) \). Then, the origin \( x = 0 \) is a center of the Abel equation (1.15).

To the particular case where \( g(t) = 2t \) and \( f(t) \) are odd polynomials in \([-a,a] = [-1,1]\), this result is a consequence of [12, Theorem 55, p.110], where the authors prove the equivalence to the existence of a center. Our Theorem 1.8 is more general, since the assumptions on \( g \) and \( f \) include the polynomial case, but we prove only the sufficient conditions on \( g \) and \( f \) for the existence of a center. Indeed, as seen in the example below, the converse is not true.

**Example 1.9.** In [1], Alvarez, Gasull and Giacomini proved the following result. Consider the Abel equation

\[
x' = (a_0 + a_1 \cos(2\pi t) + a_2 \sin(2\pi t))x^3 + (b_0 + b_1 \cos(2\pi t) + b_2 \sin(2\pi t))x^2
\]

(1.16)

where \( a_0, a_1, a_2, b_0, b_1 \) and \( b_2 \) are arbitrary real numbers. According to [1], for \( a_0 = b_0 = a_2 b_1 - a_1 b_2 = 0 \), the equation (1.16) has a center at \( x = 0 \).

Considering \( a_0 = b_0 = a_2 = b_2 = 0 \) and \( a_1 = b_1 = 1 \), we obtain the following Abel equation

\[
x' = \cos(2\pi t)x^3 + \cos(2\pi t)x^2
\]

(1.17)

Note that, in this case, the coefficients of Abel equation, \( f(t) = \cos(2\pi t) \) and \( g(t) = \cos(2\pi t) \), are both even functions. This shows that the reciprocal of Theorem 1.8 is not true.
When \( f \) and \( g \) are odd polynomials, then it can be written \( f(t) = t\hat{f}(t^2) \) and \( g(t) = t\hat{g}(t^2) \) and the results of Theorem 1.8 is a consequence of the results, due to Alwash and Lloyd [2], which we present below.

**Proposition 1.10** ([2]). Assume \( f, g \in C([a, b]) \) to be expressed by

\[
f(t) = \hat{f}(\sigma(t))\sigma'(t), \quad g(t) = \hat{g}(\sigma(t))\sigma'(t)
\]

(1.18)

for some continuous functions \( \hat{f}, \hat{g} \) and a continuously differentiable function \( \sigma \), which is closed, i.e., \( \sigma(a) = \sigma(b) \). Then, the Abel equation

\[
x'(t) = f(t)x^3(t) + g(t)x^2(t), \quad t \in [a, b]
\]

has a center \( x = 0 \).

In the proof of the Theorem 1.8 we do not use the composition condition (1.18). The proof of the Theorem 1.8 follows of the Theorem 1.6 and Lemma 2.5 when \( f \) and \( g \) are only odd continuous functions. Thus, we have another result with conclusions similar to those of Alwash and Lloyd.

2. Proof of the theorems

2.1. Preliminary results

We claim that a solution of (1.15) is equivalent to a solution of the integral equation

\[
x(t) = \frac{\rho}{1 - \rho \int_{-a}^{t} (f(s)x(s) + g(s))ds}, \quad t \in [-a, a]
\]

where \( x(-a) = \rho \), for \( \rho \) small enough, such that \( \rho \int_{-a}^{t} (f(s)x(s) + g(s))ds < 1 \) for all \( t \in [-a, a] \). Indeed, the equation (1.15) is equivalent to

\[
-(x^{-1}(t))' = f(t)x(t) + g(t).
\]

By integration from \(-a\) to \( t \) with \( t \in [-a, a] \) we get

\[
\frac{1}{x(t)} - \frac{1}{x(-a)} = -\int_{-a}^{t} (f(s)x(s) + g(s))ds.
\]

After some computations, we obtain

\[
x(-a) = x(t) \left[ 1 - x(-a) \int_{-a}^{t} (f(s)x(s) + g(s))ds \right].
\]
If \( x(-a) = \rho \) satisfies \( \rho \int_{-a}^{t} (f(s)x(s) + g(s))ds < 1 \), for each \( t \in [-a, a] \), it follows the claim.

Let \( B_M(0) = \{ x \in C([-a, a]); \| x \|_{\infty} \leq M \} \) be the closed ball in \( C([-a, a]) \), \( F = \max_{t \in [-a, a]} |f(t)| \) and \( G = \max_{t \in [-a, a]} |g(t)| \). Now, we define the operator \( \Omega : C([-a, a]) \to C([-a, a]) \) by
\[
\Omega(x)(t) = \frac{\rho}{1 - \rho \int_{-a}^{t} (f(s)x(s) + g(s))ds}, \quad t \in [-a, a]. \tag{2.1}
\]

If
\[
0 \leq \rho < \frac{1}{4a(FM + G)}, \tag{2.2}
\]
the operator (2.1) is well defined. Indeed, by (2.2)
\[
\rho \int_{-a}^{t} (f(s)x(s) + g(s))ds \leq 2a\rho(FM + G) < \frac{1}{2},
\]
Therefore,
\[
\frac{1}{2} < 1 - \rho \int_{-a}^{t} (f(s)x(s) + g(s))ds \tag{2.3}
\]
and \( \Omega \) is well defined.

Note that a fixed point of \( \Omega \) is a solution of (1.15).

**Lemma 2.1.** The operator \( \Omega \) is continuous.

**Proof.** For each \( x, y \in C([-a, a]) \), we have
\[
|\Omega(x)(t) - \Omega(y)(t)| = \left| \frac{\rho^2}{(1 - \rho \int_{-a}^{t} (f(s)x(s) + g(s))ds)(1 - \rho \int_{-a}^{t} (f(s)y(s) + g(s))ds)} \right| \times \left| \int_{-a}^{t} (f(s)y(s) + g(s))ds - \int_{-a}^{t} (f(s)x(s) + g(s))ds \right| \leq 4\rho^2 \int_{-a}^{a} |f(s)(y(s) - x(s))|ds \leq 8a\rho^2 F \| y - x \|_{\infty}
\]
for each \( t \in [-a, a] \). Hence
\[
\| \Omega(x) - \Omega(y) \|_{\infty} \leq 8a\rho^2 F \| y - x \|_{\infty}, \quad \forall x, y \in C([-a, a]),
\]
for each \( t \in [-a, a] \). Hence
\[
\| \Omega(x) - \Omega(y) \|_{\infty} \leq 8a\rho^2 F \| y - x \|_{\infty}, \quad \forall x, y \in C([-a, a]),
\]
for each \( t \in [-a, a] \). Hence
\[
\| \Omega(x) - \Omega(y) \|_{\infty} \leq 8a\rho^2 F \| y - x \|_{\infty}, \quad \forall x, y \in C([-a, a]),
\]
for each \( t \in [-a, a] \). Hence
and this proved the continuity.

Now, we define the restriction $\Omega_M = \Omega|_{B_M(0)} : B_M(0) \to C([-a, a])$. By Lemma 2.1, $\Omega_M$ is continuous.

**Lemma 2.2.** We have that $\Omega_M : B_M(0) \to C([-a, a])$ is compact and if

$$0 < \rho < \min \left\{ \frac{M}{2}, \frac{1}{4a(FM + G)} \right\},$$

we have

$$\Omega_M(B_M(0)) \subset B_M(0).$$

**Proof.** We have that $\Omega_M : B_M(0) \to C([-a, a])$ is bounded. Indeed, by (2.3) and (2.4), we obtain

$$|\Omega_M(x)(t)| \leq 2\rho, \forall x \in B_M(0), t \in [-a, a].$$

Hence

$$\|\Omega_M(x)\|_\infty \leq 2\rho, \forall x \in B_M(0).$$

(2.5)

For each $t, \xi \in [-a, a]$, that we can consider $t > \xi$, we have

$$|\Omega_M(x)(t) - \Omega_M(x)(\xi)|$$

$$= \left| \frac{\rho^2}{\left(1 - \rho \int_{-a}^{\xi} (f(s)x(s) + g(s))ds\right) \left(1 - \rho \int_{-a}^{\xi} (f(s)x(s) + g(s))ds\right)} \right| \times \left[ \int_{-a}^{t} (f(s)x(s) + g(s))ds - \int_{-a}^{\xi} (f(s)x(s) + g(s))ds \right]$$

$$\leq 4\rho^2 \int_{-a}^{\xi} |f(s)x(s) + g(s)|ds$$

$$\leq 4\rho^2(FM + G)|t - \xi|, \forall x \in B_M(0).$$

Therefore, $\Omega_M(B_M(0))$ is an equicontinuous subset of $C([-a, a])$. By Ascoli-Arzelà Theorem, see [18, p.772], $\Omega_M : B_M(0) \to C([-a, a])$ is compact.

Now, by (2.4)

$$\|\Omega_M(x)\|_\infty \leq 2\rho < M, \forall x \in B_M(0).$$

Therefore, $\Omega_M : B_M(0) \to B_M(0)$ is well defined.
Now, we define the closed subspace of $C([-a, a])$ defined by
\[
E = \{ x \in C([-a, a]); \ x \text{ is even} \}.
\]
Also, we define the restriction $\Omega_E = \Omega|_E : E \to C([-a, a])$. Let $B^E_M(0) = \{ x \in E; \|x\|_\infty \leq M \}$ be the closed ball in $E$. By Lemma 2.1
\[
\Omega_E : B^E_M(0) \to C([-a, a])
\]
is continuous.

**Lemma 2.3.** We have that $\Omega_E : B^E_M(0) \to C([-a, a])$ is compact and if
\[
0 \leq \rho < \min \left\{ \frac{M}{2}, \frac{1}{4a(FM + G)} \right\}, \tag{2.6}
\]
we have
\[
\Omega_E(B^E_M(0)) \subset B^E_M(0).
\]

**Proof.** We have that $\Omega_E : B^E_M(0) \to C([-a, a])$ is bounded. Indeed, by (2.3) and (2.6), we obtain
\[
|\Omega_E(x)(t)| \leq 2\rho, \forall x \in B^E_M(0), t \in [-a, a].
\]
Hence
\[
\|\Omega_E(x)\|_\infty \leq 2\rho, \forall x \in B^E_M(0). \tag{2.7}
\]
For each $t, \xi \in [-a, a]$, that we can consider $t > \xi$, we have
\[
|\Omega_E(x)(t) - \Omega_E(x)(\xi)|
\]
\[
= \left| \frac{\rho^2}{\left(1 - \rho \int_{-a}^{t}(f(s)x(s) + g(s))ds\right) \left(1 - \rho \int_{-a}^{\xi}(f(s)x(s) + g(s))ds\right)} \right|
\]
\[
\times \left[ \int_{-a}^{t} (f(s)x(s) + g(s))ds - \int_{-a}^{\xi} (f(s)x(s) + g(s))ds \right]
\]
\[
\leq 4\rho^2 \int_{\xi}^{t} |f(s)x(s) + g(s)|ds
\]
\[
\leq 4\rho^2 (FM + G)|t - \xi|, \ \forall x \in B^E_M(0).
\]
Therefore, $\Omega_E(B^E_M(0))$ is an equicontinuous subset of $C([-a, a])$. By Ascoli-Arzelà Theorem, see [13, p.772], $\Omega_E : B^E_M(0) \to C([-a, a])$ is compact.
As the functions $f, g$ are odd and $x$ is even, we have that $fx + g$ is an odd function. Therefore,
\[ \int_{-a}^{t} (f(s)x(s) + g(s))ds \]
is an even function. Hence, for each $x \in E$,
\[ \Omega_E(x)(t) = \frac{\rho}{1 - \rho \int_{-a}^{t} (f(s)x(s) + g(s))ds}, \]
is an even function in $[-a, a]$, that is, $\Omega_E(x) \in E$ for each $x \in E$. Now, by (2.6)
\[ \|\Omega_E(x)\|_\infty \leq 2\rho < M, \forall x \in B^E_M(0). \]
Therefore, $\Omega_E : B^E_M(0) \to B^E_M(0)$ is well defined.

\[\square\]

2.2. Proof of the Theorem 1.6

Suppose that $g$ is an odd function. Let us suppose,
\[ 0 \leq \rho < \min \left\{ \frac{M}{2}, \frac{1}{4a(FM + G)} \right\}. \]

It follows from Lemmas 2.1 and 2.3 that $\Omega_E : B^E_M(0) \to B^E_M(0)$ is well defined, continuous and compact, where
\[ \Omega_E : B^E_M(0) \to C([-a, a]) \]
and
\[ B^E_M(0) = \{ x \in E; \|x\|_\infty \leq M \}. \]
By the Schauder fixed point Theorem, see [18, p.56], $\Omega_E$ has a fixed point $x$, such that,
\[ \Omega_E(x)(t) = x(t) = \frac{\rho}{1 - \rho \int_{-a}^{t} (f(s)x(s) + g(s))ds} \]
and
\[ x(-a) = \rho, \]
for each \( \rho \in \left[0, \min \left\{ \frac{M}{2}, \frac{1}{4a(FM + G)} \right\} \right) \). Since \( \Omega_E(x) \) is an even function, we have
\[
x(-a) = x(a)
\]
and there are infinitely many closed even solutions for (1.15) near the zero solution.

Now, suppose that there are infinitely many closed even solutions for (1.15) near the zero solution. Note that a solution of (1.15) is equivalent to the solution of the integral equation
\[
x(t) = \frac{\rho}{1 - \rho \int_{-a}^{t} (f(s)x(s) + g(s))ds}
\]
where \( x(-a) = \rho \) is small enough, which is equivalent to
\[
\int_{-a}^{t} g(s)ds = -\frac{1}{x(t)} + \frac{1}{\rho} - \int_{-a}^{t} f(s)x(s)ds.
\]
Since \( f \) is odd and \( x(t) \) is even, we obtain \( \int_{-a}^{t} f(s)x(s)ds \) is even. Therefore, \( \int_{-a}^{t} g(s)ds \) is even and consequently we obtain that \( g(t) \) is odd, and the theorem is proved.

**Remark 2.4.** Similar arguments are true when
\[
-\min \left\{ \frac{M}{2}, \frac{1}{4a(FM + G)} \right\} < \rho \leq 0.
\]

### 2.3. Proofs of the Theorems 1.7 and 1.8

To the proof of the Theorem 1.8, we apply the following result of Yang Lijun and Tang Yun [12, Lemma 5.2, p 108].

**Lemma 2.5.** The origin \( x = 0 \) is a center of the Abel equation (1.15), if and only if
\[
\int_{-a}^{a} g(t)dt = 0 \quad \text{and} \quad \int_{-a}^{a} f(t)x(t, \rho)dt = 0, \quad |\rho| < \rho_0 \quad (2.8)
\]
to \( \rho_0 \) small enough, where
\[
x(t, \rho) = \frac{\rho}{1 - \rho \int_{-a}^{t} (f(s)x(s) + g(s))ds}.
\]
Proof. of Theorem 1.7. Suppose that \( f \) is an odd function. The proof that there are infinitely many closed even solutions for (1.15) near the zero solution is analogous to the Theorem 1.6.

Now, suppose that there are infinitely many closed even solutions for (1.15) near the zero solution. Note that a solution of (1.15) is equivalent to the solution of the integral equation

\[
x(t) = \frac{\rho}{1 - \rho \int_{-a}^{t} (f(s)x(s) + g(s))ds}
\]

where \( x(-a) = \rho \) is small enough, which is equivalent to

\[
\int_{-a}^{t} f(s)x(s)ds = -\frac{1}{x(t)} + \frac{1}{\rho} \int_{-a}^{t} g(s)ds.
\]

Since \( g \) is odd and \( x(t) \) is even, we obtain \( \int_{-a}^{t} g(s)ds \) is even. Therefore, \( \int_{-a}^{t} f(s)x(s)ds \) is even and consequently, we obtain that \( f(t)x(t) \) is odd. Since \( x(t) \) is even, we conclude that \( f(t) \) is odd and the theorem is proved.

\[\square\]

Proof. of Theorem 1.8. It follows from Theorem 1.6 that there are infinitely many closed even solutions for (1.15) near the zero solution. Now, we claim that, if \( y(t) \) is a solution for (1.15) and satisfies \( y(-a) = \rho \), with \( 0 \leq \rho < \min \left\{ \frac{M}{2}, \frac{1}{4a(FM + G)} \right\} \), then \( y(t) \) is a closed even solution. Indeed, by Theorem 1.6, there is \( \bar{x}(t) \) a closed even solution for (1.15) such that \( \bar{x}(-a) = \rho \). Therefore, \( y(-a) = \bar{x}(-a) \) and, by uniqueness, we obtain that \( y = \bar{x} \). Consequently, this proves the claim. In conclusion, if \( f \) and \( g \) are odd continuous functions, each solution for (1.15) near the solution \( x = 0 \) is a closed even solution.

Since

\[
\int_{-a}^{a} g(t)dt = 0 \text{ and } \int_{-a}^{a} f(t)x(t, \rho)dt = 0, \ |\rho| < \rho_0 = \min \left\{ \frac{M}{2}, \frac{1}{4a(FM + G)} \right\}
\]

it follows from Lemma 2.5 that the zero solution is a center of the Abel equation (1.15).

\[\square\]
2.4. Proofs of the Theorem

Recall that the Abel equation associated to the planar system (1.8) is

\[
\frac{d\gamma}{d\theta} = -(n-1)A(\theta)B(\theta)\gamma^3 + [(n-1)A(\theta) - B'(\theta)]\gamma^2,
\]  

(2.9)

where

\[
A(\theta) = \cos \theta P_n(\cos \theta, \sin \theta) + \sin \theta Q_n(\cos \theta, \sin \theta),
\]

\[
B(\theta) = \cos \theta Q_n(\cos \theta, \sin \theta) - \sin \theta P_n(\cos \theta, \sin \theta).
\]

(2.10)

As \(P_n(\cos \theta, \sin \theta)\) is odd and \(Q_n(\cos \theta, \sin \theta)\) is even, it follows that \(f(\theta) = -(n-1)A(\theta)B(\theta)\) and \(g(\theta) = (n-1)A(\theta) - B'(\theta)\) are odd continuous functions in \(C([0, 2\pi])\). By Theorem 1.8, \(\gamma = 0\) is a center of the equation (2.9). By the equivalence stated at the introduction, the planar system (1.8) has a center at the origin.

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