Numerical evidence for universality in the excited instability spectrum of magnetically charged Reissner-Nordström black holes

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It is well-known that the SU(2) Reissner-Nordström black-hole solutions of the Einstein-Yang-Mills theory are characterized by an infinite set of unstable (imaginary) eigenvalues \(\{\omega_n(T_{BH})\}_{n=0}^{\infty}\) (here \(T_{BH}\) is the black-hole temperature). In this paper we analyze the excited instability spectrum of these magnetically charged black holes. The numerical results suggest the existence of a universal behavior for these black-hole excited eigenvalues. In particular, we show that unstable eigenvalues in the regime \(\omega_n \ll T_{BH}\) are characterized, to a very good degree of accuracy, by the simple universal relation \(\omega_n(r_+ - r_-) = \text{constant}\), where \(r_\pm\) are the horizon radii of the black hole.

I. INTRODUCTION

The familiar U(1) Reissner-Nordström spacetime is known to describe a stable black-hole solution of the coupled Einstein-Maxwell equations \([1]\) and the coupled Einstein-Maxwell-scalar equations \([2]\). Yasskin \([3]\) has proved that the Einstein-Yang-Mills theory also admits an explicit black-hole solution which is described by the magnetically charged SU(2) Reissner-Nordström spacetime. However, the SU(2) Reissner-Nordström black-hole solution of the coupled Einstein-Yang-Mills equations is known to be unstable \([4-6]\). In fact, it was proved in \([7]\) that the magnetically charged Reissner-Nordström black-hole spacetime is characterized by an infinite family of unstable (growing in time) perturbation modes.

The recent numerical work of Rinne \([8]\) has revealed that these unstable SU(2) Reissner-Nordström black-hole spacetimes play the role of approximate \([9]\) codimension-two intermediate attractors (that is, nonlinear critical solutions \([10]\)) in the dynamical gravitational collapse of the Yang-Mills field \([11]\). In particular, this interesting numerical study \([8]\) has explicitly demonstrated that, during a near-critical evolution of the Yang-Mills field, the time spent in the vicinity of an unstable SU(2) Reissner-Nordström black-hole solution is characterized by the critical scaling law \([12]\)

\[
\tau = \text{const} - \gamma \ln |p - p^*| .
\]

Interestingly, the critical exponents of the scaling law \([11]\) are directly related to the characteristic instability eigenvalues of the corresponding SU(2) Reissner-Nordström black holes \([8]\):

\[
\gamma = 1/\omega_{\text{instability}} .
\]

It is therefore of physical interest to explore the instability spectrum \(\{\omega_n\}_{n=0}^{\infty}\) of the SU(2) Reissner-Nordström black holes. Indeed, Rinne \([8]\) has recently computed numerically the characteristic unstable eigenvalues of these magnetically charged black-hole solutions of the Einstein-Yang-Mills theory \([13]\).

In the present paper we shall analyze these numerically computed black-hole eigenvalues in an attempt to identify a possible hidden pattern which characterizes the black-hole instability spectrum. As we shall show below, the numerical results indeed suggest the existence of a universal behavior for these black-hole unstable eigenvalues.

II. DESCRIPTION OF THE SYSTEM

The Reissner-Nordström black-hole solution of the Einstein-Yang-Mills theory with unit magnetic charge is described by the line element \([8]\)

\[
ds^2 = -\left(1 - \frac{2m}{r}\right)dt^2 + \left(1 - \frac{2m}{r}\right)^{-1}dr^2 + r^2(\sin^2 \theta d\phi^2 ) ,
\]

where the mass function \(m = m(r)\) is given by \([14]\)

\[
m(r) = M - \frac{1}{2r} .
\]
The black-hole temperature is given by

\[ T_{\text{BH}} = \frac{r_+ - r_-}{4\pi r_+^2}, \]  

where

\[ r_{\pm} = M \pm \sqrt{M^2 - 1} \]

are the (outer and inner) horizons of the black hole.

Linearized perturbations \( \xi(r)e^{-i\omega t} \) of the magnetically charged black-hole spacetime are governed by the Schrödinger-like wave equation \[ \left\{ \frac{d^2}{dx^2} + \omega^2 + \frac{1}{r^2}\left[1 - \frac{2m(r)}{r}\right]\right\}\xi = 0, \]

where the “tortoise” radial coordinate \( x \) is defined by the relation \[ dx/dr = [1 - \frac{2m(r)}{r}]^{-1}. \]

Well-behaved (spatially bounded) perturbation modes are characterized by the boundary conditions

\[ \xi(x \to -\infty) \sim e^{i|\omega|x} \to 0 \]  
and

\[ \xi(x \to \infty) \sim xe^{-|\omega|x} \to 0, \]

where \( \omega = i|\omega| \). As shown in [5, 7], these boundary conditions single out a discrete set of unstable \((3\omega > 0)\) black-hole eigenvalues \( \{\omega_n(r_+ )\}_{n=0}^{n=\infty} \).

### III. NUMERICAL EVIDENCE FOR UNIVERSALITY IN THE EXCITED INSTABILITY SPECTRUM

Most recently, Rinne [8] computed numerically the first three instability eigenvalues which characterize the SU(2) Reissner-Nordström black-hole solutions of the coupled Einstein-Yang-Mills equations. We have examined these numerically computed eigenvalues in an attempt to reveal a possible hidden pattern which characterizes the black-hole instability spectrum.

In Table I we present the first excited instability eigenvalues \( \{\omega_1(r_+)\} \) of the magnetically charged SU(2) Reissner-Nordström black holes. In particular, we display the dimensionless ratio \( \omega_1(r_+)/\pi T_{\text{BH}} \), where the black-hole temperature \( T_{\text{BH}} \) is given by [4]. We also display the ratio between the dimensionless quantity \( \omega_1(r_+) \times (r_+ - r_-) \) for generic SU(2) Reissner-Nordström black holes and the corresponding quantity \( \omega_1(r_+ = 10) \times (10 - 1/10) \) for the weakly-magnetized Reissner-Nordström black hole with \( r_+ = 10 \) [18]. Remarkably, the numerical data presented in Table I reveals that the black-hole instability eigenvalues in the regime \( \omega_1(r_+)/T_{\text{BH}} \ll 1 \) are characterized, to a good degree of accuracy, by the universal relation \[ \omega_1(r_+ - r_-) = \lambda_1 ; \quad \lambda_1 = \text{constant}. \]  

In order to support this intriguing finding, we display in Table II the second excited instability eigenvalues \( \{\omega_2(r_+)\} \) of the SU(2) Reissner-Nordström black holes. Remarkably, the numerical data presented in Table II provide compelling evidence for the validity of the suggested universal behavior of the black-hole instability eigenvalues in the regime \( \omega_2(r_+)/T_{\text{BH}} \ll 1 \). In particular, one finds [20]

\[ \omega_2(r_+ - r_-) = \lambda_2 ; \quad \lambda_2 = \text{constant}. \]
TABLE I: The instability eigenvalues of SU(2) Reissner-Nordström black holes. The data shown refers to the first excited eigenvalues \{\omega_1(r_+)\} of these magnetically charged black holes. We display the dimensionless ratio \(\omega_1(r_+) / \pi T_{\text{BH}}\), where \(T_{\text{BH}}\) is the black-hole temperature. Also shown is the ratio between the dimensionless quantity \(\omega_1(r_+) \times (r_+ - r_-)\) for generic SU(2) Reissner-Nordström black holes and the corresponding quantity \(\omega_1(r_+ = 10) \times (10 - 1/10)\) for the weakly-magnetized Reissner-Nordström black hole with \(r_+ = 10\). One finds that the instability eigenvalues in the regime \(\omega_1(r_+) / \pi T_{\text{BH}} < 0.1\) are characterized, to a good degree of accuracy, by the universal relation \(\omega_1(r_+ - r_-) = \text{constant}\).

| \(r_+\) | \(\omega_1(r_+) / \pi T_{\text{BH}}\) | \(\omega_1(r_+) \times (r_+ - r_-)\) |
|---|---|---|
| 9.0 | 9.86 \times 10^{-2} | 0.999 |
| 8.0 | 9.91 \times 10^{-2} | 0.999 |
| 7.0 | 9.99 \times 10^{-2} | 0.998 |
| 6.0 | 1.01 \times 10^{-1} | 0.996 |
| 5.0 | 1.04 \times 10^{-1} | 0.993 |
| 4.0 | 1.08 \times 10^{-1} | 0.987 |
| 3.0 | 1.18 \times 10^{-1} | 0.973 |
| 2.0 | 1.58 \times 10^{-1} | 0.925 |
| 1.5 | 2.57 \times 10^{-1} | 0.824 |
| 1.2 | 6.04 \times 10^{-1} | 0.586 |

TABLE II: The instability eigenvalues of SU(2) Reissner-Nordström black holes. The data shown refers to the second excited eigenvalues \{\omega_2(r_+)\} of these magnetically charged black holes. We display the dimensionless ratio \(\omega_2(r_+) / \pi T_{\text{BH}}\), where \(T_{\text{BH}}\) is the black-hole temperature. Also shown is the ratio between the dimensionless quantity \(\omega_2(r_+) \times (r_+ - r_-)\) for generic SU(2) Reissner-Nordström black holes and the corresponding quantity \(\omega_2(r_+ = 10) \times (10 - 1/10)\) for the weakly-magnetized Reissner-Nordström black hole with \(r_+ = 10\). One finds that the instability eigenvalues in the regime \(\omega_2(r_+) / \pi T_{\text{BH}} \ll 1\) are characterized, to a good degree of accuracy, by the universal relation \(\omega_2(r_+ - r_-) = \text{constant}\).

| \(r_+\) | \(\omega_2(r_+) / \pi T_{\text{BH}}\) | \(\omega_2(r_+) \times (r_+ - r_-)\) |
|---|---|---|
| 9.0 | 2.90 \times 10^{-3} | 1.011 |
| 8.0 | 2.95 \times 10^{-3} | 1.021 |
| 7.0 | 3.00 \times 10^{-3} | 1.029 |
| 6.0 | 3.06 \times 10^{-3} | 1.034 |
| 5.0 | 3.15 \times 10^{-3} | 1.038 |
| 4.0 | 3.31 \times 10^{-3} | 1.040 |
| 3.0 | 3.68 \times 10^{-3} | 1.041 |
| 2.0 | 5.17 \times 10^{-3} | 1.039 |
| 1.5 | 9.37 \times 10^{-3} | 1.034 |
| 1.2 | 3.03 \times 10^{-2} | 1.011 |

IV. SUMMARY

The U(1) Reissner-Nordström black holes are known to be stable within the framework of the coupled Einstein-Maxwell theory \[1, 2\]. This stability property of the black holes manifests itself in the form of an infinite spectrum of damped quasi-normal resonances \[21\]. To the best of our knowledge, for generic U(1) Reissner-Nordström black holes, there is no simple universal formula which describes the infinite family of these damped black-hole quasi-normal resonances.

On the other hand, the SU(2) Reissner-Nordström black holes are known to be unstable within the framework of the coupled Einstein-Yang-Mills theory \[4, 5\]. This instability property of the magnetically charged black holes manifests itself in the form of an infinite spectrum of exponentially growing black-hole resonances \[7\]. In this paper we have provided compelling numerical evidence that the infinite family of these unstable black-hole resonances can be described, to a very good degree of accuracy, by the simple universal formula

\[
\omega_n(r_+ - r_-) = \text{constant}_n \quad \text{for} \quad \omega_n \ll T_{\text{BH}}.
\]  \hspace{1cm} (13)

We believe that it would be highly interesting to find an analytical explanation for this numerically suggested universal behavior.
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[9] As emphasized in [8], the magnetically charged Reissner-Nordström black-hole spacetime is only an approximate intermediate attractor because it is characterized by an infinite set of unstable (growing in time) modes.
[10] For an excellent review on the critical phenomena in gravitational collapse, see C. Gundlach and J. M. Martín-García, Living Rev. Relativity 10 (2007).
[11] This fact refers to type I and Type III critical behaviors, see [8] for details.
[12] Here $|p - p^*|$ is a measure for the distance of the initial data from the threshold (critical) solution [10].
[13] As emphasized above, the magnetically charged Reissner-Nordström black-hole solution of the Einstein-Yang-Mills theory is characterized by an infinite family of unstable perturbation modes [7]. Reference [8] provides, for the first time, detailed numerical results for the first three instability eigenvalues.
[14] We use natural units in which $G = c = \hbar = 1$.
[15] Note that unstable (growing in time) modes are characterized by $\Im \omega > 0$.
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[17] Note that the near-horizon limit $r \to r_+$ corresponds to $x \to -\infty$, whereas the large-$r$ limit $r \to \infty$ corresponds to $x \to \infty$.
[18] The weakly-magnetized SU(2) Reissner-Nordström black hole with horizon radius $r_+ = 10$ is the largest black-hole solution studied numerically in [8].
[19] It is worth emphasizing that, in the regime $\omega_1(r_+)/T_{BH} \ll 1$, the value of $\lambda_1$ is almost independent of the black-hole horizon radius $r_+$.
[20] It is worth emphasizing that, in the regime $\omega_2(r_+)/T_{BH} \ll 1$, the value of $\lambda_2$ is almost independent of the black-hole horizon radius $r_+$.
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