Tensor categories: A selective guided tour*

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Abstract

These are the – only lightly edited – lecture notes for a short course on tensor categories. The coverage in these notes is relatively non-technical, focusing on the essential ideas. They are meant to be accessible for beginners, but it is hoped that also some of the experts will find something interesting in them.

Once the basic definitions are given, the focus is mainly on \( k \)-linear categories with finite dimensional hom-spaces. Connections with quantum groups and low dimensional topology are pointed out, but these notes have no pretension to cover the latter subjects at any depth. Essentially, these notes should be considered as annotations to the extensive bibliography. We also recommend the recent review [33], which covers less ground in a deeper way.

1 Tensor categories

1.1 Strict tensor categories

- Assumed: Categories, functors, natural transformations (Eilenberg, Mac Lane). \([\rightarrow \text{2-category } \mathcal{C}, \mathcal{A}, \mathcal{T}]\)
- We want “categories with multiplication” (title of a paper \([16]\) by Bénabou 1963),(Mac Lane 1963 \([148]\)). Later ‘monoidal categories’ or ‘tensor categories’. (We use these synonymously.) Why did they appear so late, twenty years after categories?
- A strict tensor category (strict monoidal category) is a triple \((\mathcal{C}, \otimes, 1)\), where \(\mathcal{C}\) is a category, \(1\) a distinguished object and \(\otimes: \mathcal{C} \times \mathcal{C} \to \mathcal{C}\) is a functor, satisfying

\[
(X \otimes Y) \otimes Z = X \otimes (Y \otimes Z), \quad X \otimes 1 = X = 1 \otimes X \quad \forall X, Y, Z.
\]

If \((\mathcal{C}, \otimes, 1), (\mathcal{C}', \otimes', 1')\) are tensor categories, a strict tensor functor \(\mathcal{C} \to \mathcal{C}'\) is a functor \(F: \mathcal{C} \to \mathcal{C}'\) such that

\[
F(X \otimes Y) = F(X) \otimes' F(Y), \quad F(1) = 1'.
\]

If \(F, F': \mathcal{C} \to \mathcal{C}'\) are strict tensor functors, a natural transformation \(\alpha : F \to F'\) is monoidal iff \(\alpha_1 = \text{id}_{1'}\) and

\[
\alpha_{X \otimes Y} = \alpha_X \otimes \alpha_Y \quad \forall X, Y \in \mathcal{C}.
\]

(Both sides live in \(\text{Hom}(F(X \otimes Y), F'(X \otimes Y)) = \text{Hom}(F(X) \otimes' F(Y), F'(X) \otimes' F'(Y))\).)

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• WARNING: We will see that, in a sense, strict tensor categories are sufficient for all purposes. But even when dealing with strict tensor categories, one needs non-strict tensor functors!

• Examples:
  – $\mathcal{C}$ any category. Then $\text{End}\mathcal{C} = \{ \text{functors } \mathcal{C} \to \mathcal{C} \text{ and their natural transformations} \}$ is strict $\otimes$-category. Also denoted ‘center’ $Z_0(\mathcal{C})$.
  – Discrete tensor category $\mathcal{C}(G)$ for $G$ a group:
    \[
    \text{Obj } \mathcal{C}(G) = G, \quad \text{Hom}(g, h) = \begin{cases} 
      \{ \text{id}_g \} & g = h \\
      \emptyset & g \neq h
    \end{cases}, \quad g \otimes h = gh.
    \]
  – The symmetric category $S$:
    \[
    \text{Obj } S = \mathbb{Z}_+, \quad \text{Hom}(n, m) = \begin{cases} 
      S_n & n = m \\
      \emptyset & n \neq m
    \end{cases}, \quad n \otimes m = n + m
    \]
    with tensor product of morphisms given by the obvious map $S_n \times S_m \to S_{n+m}$.
    Remark: 1. $S$ is equivalent to the category of finite sets and bijective maps.
    2. This construction works with any family $(G_i)$ of groups with an associative composition $G_i \times G_j \to G_{i+j}$.
  – $\text{End}\mathcal{A}$ for unital algebra $\mathcal{A}$ over some field. (Objects: unital algebra homomorphisms $\mathcal{A} \to \mathcal{A}$. Morphisms: Exercise.) Important in subfactor theory [139] and (algebraic) quantum field theory [53, 72]. Yamagami [236]: every countably generated $C^*$-tensor category with conjugates embeds fully into $\text{End}\mathcal{A}$ for some $C^*$-algebra $\mathcal{A} = \mathcal{A}(\mathcal{C})$. (See the final section on conjectures for an algebra that should work for all such categories.)
  – The Temperley-Lieb categories $\mathcal{T L}(\tau)$. (Cf. e.g. [88].) Let $k$ be a field, $\tau \in k^*$:
    \[
    \text{Obj } \mathcal{T L}(\tau) = \mathbb{Z}^+, \quad n \otimes m = n + m,
    \]
    \[
    \text{Hom}(n, m) = \text{span}_k \{ \text{Isotopy classes of } (n, m)\text{-TL diagrams} \}.
    \]
    Ex: A (7,5)-TL diagram:

Tensor product of morphisms: horizontal juxtaposition. Composition of morphisms: vertical juxtaposition, and substitution of each circle by a factor $\tau$.

Remark: 1. The Temperley-Lieb algebras $\mathcal{T L}(n, \tau) = \text{End}_{\mathcal{T L}(\tau)}(n)$ first appeared in the theory of exactly soluble lattice models of statistical mechanics. They, as well as $\mathcal{T L}(\tau)$, are closely related to the Jones polynomial [103] and the quantum group $SL_q(2)$. Cf. [218], Chapter XII.

2. The Temperley-Lieb algebras, as well as the categories $\mathcal{T L}(\tau)$ can be defined purely algebraically in terms of generators and relations.
Graphical notation: It is often convenient to denote expressions involving many morphisms in a tensor category graphically:

$$s : X \rightarrow Y \iff \begin{array}{c}
  Y \\
  \downarrow \\
  X \\
  \uparrow \\
  X
\end{array}$$

If $s : X \rightarrow Y$, $t : Y \rightarrow Z$, $u : Z \rightarrow W$ then we write

$$t \circ s : X \rightarrow Z \iff \begin{array}{c}
  Z \\
  \downarrow \\
  X \\
  \uparrow \\
  X
\end{array} \quad s \otimes u : X \otimes Z \rightarrow Y \otimes W \iff \begin{array}{c}
  Y \otimes W \\
  \downarrow \\
  X \\
  \uparrow \\
  X
\end{array}$$

The usefulness becomes apparent when there are morphisms with 'different numbers of inputs and outputs': Let $a : X \rightarrow S \otimes T$, $b : 1 \rightarrow U \otimes Z$, $c : S \rightarrow 1$, $d : T \otimes U \rightarrow V$, $e : Z \otimes Y \rightarrow W$ and consider the formula

$$c \otimes d \otimes e \circ a \otimes b \otimes \text{id}_Y : X \otimes Y \rightarrow V \otimes W$$

This formula is almost unintelligible. It is not even clear whether it represents a morphism in the category. This is immediately obvious from the diagram:

The graphical notation becomes even more useful in the context of rigid and braided tensor categories. For more details see e.g. [111].

1.2 Non-strict tensor categories

- Strict tensor categories are not general enough:
  - From categorical point of view it is unnatural to require equality of objects as in $(X \otimes Y) \otimes Z = X \otimes (Y \otimes Z)$.
  - Many would-be tensor categories are not strict: Vect, Rep, etc.

- Minimal modification: require only existence of isomorphisms $(X \otimes Y) \otimes Z \cong X \otimes (Y \otimes Z)$ for all $X, Y, Z$ and $1 \otimes X \cong X \cong X \otimes 1$ for all $X$. This is too weak.
• Not-necessarily-strict tensor category (Bénabou [16]): A sixtuple \((C, \otimes, 1, \alpha, \lambda, \rho)\), where \(C\) is a category, \(\otimes : C \times C \to C\) a functor, \(1\) an object, and \(\alpha : \otimes \circ (\otimes \times \text{id}) \to \otimes \circ (\text{id} \times \otimes)\), \(\lambda : 1 \otimes - \to \text{id}\), \(\rho : - \otimes 1 \to \text{id}\) are natural isomorphisms (i.e., for all \(X, Y, Z\) we have isomorphisms \(\alpha_{X,Y,Z} : (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)\) and \(\lambda_X : 1 \otimes X \to X\), \(\rho_X : X \otimes 1 \to X\)) such that all morphisms between the same pair of objects that can be built from \(\alpha, \lambda, \rho\) coincide. (What does this mean? Examples: Commutativity of the two following diagrams.)

• Coherence theorem A (Mac Lane [148, 149]): all morphisms built from \(\alpha, \lambda, \rho\) are unique iff \(\alpha\) satisfies the pentagon identity, i.e. commutativity of

\[
\begin{array}{c}
\begin{array}{ccc}

((X \otimes Y) \otimes Z) \otimes T & \xrightarrow{\alpha_{X,Y,Z}} & (X \otimes (Y \otimes Z)) \otimes T & \xrightarrow{\alpha_{X,Y}} & X \otimes ((Y \otimes Z) \otimes T)
\end{array}

\end{array}
\]

and \(\lambda, \rho\) satisfy the unit identity

\[
\begin{array}{c}
\begin{array}{ccc}

(X \otimes 1) \otimes Y & \xrightarrow{\alpha_{X,Y,1}} & X \otimes (1 \otimes Y)
\end{array}

\end{array}
\]

\[
\begin{array}{c}
\begin{array}{ccc}

\rho_X \otimes \text{id}_Y & \xrightarrow{\lambda_X} & \text{id}_X \otimes \lambda_Y
\end{array}

\end{array}
\]

\[
\begin{array}{c}
\begin{array}{ccc}

X \otimes Y & \equiv & X \otimes Y
\end{array}
\end{array}
\]

For modern expositions of the coherence theorem see [149, 111]. (Notice that the definition of non-strict tensor categories in [148] was modified in [120, 121].)

• Examples of non-strict tensor categories:

– Let \(C\) be a category with products and terminal object \(T\). Define \(X \otimes Y = X \prod Y\) (choose a product, non-unique) and \(1 = T\). Then \((C, \otimes, 1)\) is non-strict tensor category.

– \(\text{Vect}_k\) with \(\alpha_{U,V,W} : (u \otimes v) \otimes w \mapsto u \otimes (v \otimes w)\). NB: This trivially satisfies the pentagon identity, but the other choice \((u \otimes v) \otimes w \mapsto -u \otimes (v \otimes w)\) does not!

– \(G\) a group, \(A\) an abelian group (written multiplicatively), \(\omega \in Z^3(G, A)\), i.e.

\[
\omega(h, k, l) \omega(g, hk, l) \omega(g, h, k) = \omega(gh, k, l) \omega(g, h, kl) \quad \forall g, h, k, l \in G.
\]

Define \(C(G, \omega)\) by

\[
\text{Obj} C = G, \quad \text{Hom}(g, h) = \begin{cases} A & g = h \\ \emptyset & g \neq h \end{cases}, \quad g \otimes h = gh.
\]

with associator \(\alpha = \omega\). (Due to Sinh, student of Grothendieck.) If \(k\) is a field, \(A = k^*\), one has a \(k\)-linear version where \(\text{Hom}(g, h) = \begin{cases} k & g = h \\ \{0\} & g \neq h \end{cases}\). I denote this by \(C_k(G, \omega)\).

Also called \(\text{Vect}_C^\omega\).

Importance: shows relations between categories and cohomology (reinforced by ‘higher category theory’), but also the concrete example is relevant for the classification of fusion categories, at least the large class of ‘group theoretical categories’ (Ostrik, ENO). See below.
A categorical group is a tensor category that is a groupoid (all morphisms are invertible) and every object has a tensor-inverse, i.e., for every $X$ there is an object $X^\circ$ such that $X \otimes X^\circ \cong 1$. The categories $C(G, \omega)$ are just the skeletal categorical groups.

- General definition of tensor functors (between non-strict tensor categories or non-strict tensor functors between strict tensor categories): A tensor functor between tensor categories $\mathcal{C}$ and $\mathcal{C}'$ is a family of natural isomorphisms $\alpha_{X,Y}: F(X) \otimes F(Y) \to F(X \otimes Y)$, satisfying

$$d^F_{X,Y}: F(X) \otimes F(Y) \to F(X \otimes Y), \quad e^F: F(1) \to 1',$$

satisfying

$$\begin{array}{c}
(F(X) \otimes' F(Y)) \otimes' F(Z) \\
\alpha_{F(X),F(Y),F(Z)}
\end{array}
\begin{array}{c}
F(X) \otimes' (F(Y) \otimes' F(Z)) \\
\alpha_{X,Y,Z}
\end{array}
\begin{array}{c}
\alpha_{X,Y} \otimes \id \\
d_{X,Y,Z}
\end{array}
\begin{array}{c}
F(X \otimes Y) \otimes Z \\
\id \otimes d_{Y,Z}
\end{array}
\begin{array}{c}
\id \\
d_{X,Y \otimes Z}
\end{array}
\begin{array}{c}
F(X \otimes Y \otimes Z) \\
F(X \otimes (Y \otimes Z))
\end{array}$$

In particular when $\alpha \equiv \id$) and

$$\begin{array}{c}
F(X) \otimes F(1) \\
\id \otimes e^F
\end{array}
\begin{array}{c}
\id \\
d^F_{X,1}
\end{array}
\begin{array}{c}
\rho_{F(X)} \\
\rho'_{F(X)}
\end{array}
\begin{array}{c}
F(X \otimes 1) \\
F(1)
\end{array}$$

Remark: Occasionally, functors as defined above are called strong tensor functors in order to distinguish them from the lax variant, where the $d^F_{X,Y}$ and $e^F$ are not required to be isomorphisms. (In this case it also makes sense to consider $d^F, e^F$ with source and target exchanged.)

- Let $(\mathcal{C}, \otimes, 1, \alpha, \lambda, \rho), (\mathcal{C}', \otimes', 1', \alpha', \lambda', \rho')$ be tensor categories, $(F, d, e), (F', d', e') : \mathcal{C} \to \mathcal{C}'$ tensor functors. Then a natural transformation $\alpha : F \to F'$ is monoidal if

$$\begin{array}{c}
F(X) \otimes' F(Y) \\
\alpha_{X,Y}
\end{array}
\begin{array}{c}
\alpha_{X,Y} \\
d_{X,Y}
\end{array}
\begin{array}{c}
F'(X) \otimes' F'(Y) \\
\alpha_{X,Y}
\end{array}$$

For strict tensor functors, we have $d \equiv \id \equiv d'$, and we obtain the earlier condition.

- A tensor functor $(\mathcal{C}, \otimes, 1, \alpha, \lambda, \rho) \to (\mathcal{C}', \otimes', 1', \alpha', \lambda', \rho')$ is called an equivalence if there exist a tensor functor $G : \mathcal{C}' \to \mathcal{C}$ and natural monoidal isomorphisms $\alpha : G \circ F \to \id_{\mathcal{C}}$ and $\beta : F \circ G \to \id_{\mathcal{C}'}$. For the existence of such a $G$ it is necessary and sufficient that $F$ be full, faithful and essentially surjective (and of course monoidal), cf. [200].

- Fact: Given a group $G$ and $\omega, \omega' \in Z^3(G, A)$, the identity functor is part of a monoidal equivalence $\mathcal{C}(G, \omega) \to \mathcal{C}(G, \omega')$ iff $[\omega] = [\omega']$ in $H^3(G, A)$. More generally: $\mathcal{C}(G, \omega) \simeq \mathcal{C}(G, \omega')$ as tensor categories iff there is a $\gamma \in \text{Aut}(G)$ such that $[\omega^\gamma] = [\omega']$.

Thus: categorical groups $\mathcal{G}$ are classified by pairs consisting of the group $G = \text{Obj} \mathcal{G} / \cong$ and an element of $H^3(G, A)/\text{Aut} G$, where $A \cong \text{End} g$ for any $g$. 

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• Coherence theorem B: Every tensor category is monoidally equivalent to a strict one. This allows us to pretend that all tensor categories are strict. (But we cannot restrict ourselves to strict tensor functors!)

• What is the strictification of \( \mathcal{C}(G, \omega) \)? Pretty complicated! Thus: sometimes better to work with non-strict categories!

• As shown in [203], many non-strict tensor categories can be turned into equivalent strict ones by changing only the tensor functor \( \otimes \), but leaving the underlying category unchanged.

• Important fact: For any tensor category, the monoid \( \text{End} \) is commutative by the Eckmann-Hilton argument: If a set has two monoid structures \( \star_1, \star_2 \) satisfying \( (a \star_2 b) \star_1 (c \star_2 d) = (a \star_1 c) \star_2 (b \star_1 d) \) with the same unit, the two products coincide and are commutative. Thus in the Ab- \((k\text{-linear})\) case, \( \text{End} \) is a commutative unital ring (\(k\)-algebra).

1.3 Generalization: 2-categories and bicategories

• As noted, \( \mathcal{C}AT \) is a 2-category. A 2-category consists \( \mathcal{E} \) of a set (class) of objects and, for every \( X, Y \in \text{Obj} \mathcal{E} \), a category \( \text{HOM}(X, Y) \). The objects (morphisms) in \( \text{HOM}(X, Y) \) are called 1-morphisms (2-morphisms) of \( \mathcal{E} \). Axioms: In particular, we have functors \( \circ : \text{HOM}(\mathfrak{A}, \mathfrak{B}) \times \text{HOM}(\mathfrak{B}, \mathfrak{C}) \to \text{HOM}(\mathfrak{A}, \mathfrak{C}), \) and \( \circ \) is associative (on the nose). Notice: If \( \mathcal{E} \) is a 2-category and \( X \in \text{Obj} \mathcal{E} \), then \( \text{END}(X) = \text{HOM}(X, X) \) is a strict tensor category.

• This leads to the generalization called bicategories: we replace the associativity of the composition \( \circ \) of 1-morphisms by the existence of isomorphisms \( (X \circ Y) \circ Z = X \circ (Y \circ Z) \) satisfying some axioms. Now: If \( \mathcal{E} \) is a bicategory and \( X \in \text{Obj} \mathcal{E} \), then \( \text{END}(X) = \text{HOM}(X, X) \) is a (non-strict) tensor category. Bicategories are a very important generalization of tensor categories, and we’ll meet them again. Also the relation between bicategories and tensor categories is prototypical for ‘higher category theory’.

References: [124] for 2-categories and [18] for bicategories.

1.4 Categorification of monoids

Tensor categories (or monoidal categories) can be considered as the categorification of the notion of a monoid. This has interesting consequences:

• (A) Monoids in monoidal categories:
  
  Let \( (\mathcal{C}, \otimes, 1) \) be a strict \( \otimes \)-category. A monoid in \( \mathcal{C} \) (Bénabou [17]) is a triple \((A, m, \eta)\) with \( A \in \mathcal{C} \), \( m : A \otimes A \to A \), \( \eta : 1 \to A \) satisfying

\[
m \circ m \otimes \text{id}_A = m \circ \text{id}_A \otimes m , \quad m \circ \eta \otimes \text{id}_A = \text{id}_A = m \circ \text{id}_A \otimes \eta.
\]

(In the non-strict case, insert associator.) NB: A monoid in \( \text{Ab} (\text{Vect}_k) \) is a ring \((k\text{-algebra})\). Therefore, in the recent literature monoids are often called ‘algebras’.

• If \( \mathcal{C} \) is any category, monoids in the tensor category \( \text{End}\mathcal{C} \) are known as ‘monads’. (As such they are older than tensor categories!)

• If \((A, m, \eta)\) is a monoid in the strict tensor category \( \mathcal{C} \), a left \( A \)-module is a pair \((X, \mu)\), where \( X \in \mathcal{C} \) and \( \mu : A \otimes X \to X \) satisfies

\[
\mu \circ m \otimes \text{id}_X = \mu \circ \text{id}_A \otimes \mu , \quad \mu \circ \eta \otimes \text{id}_X = \text{id}_X.
\]

Together with the obvious notion of \( A \)-module morphism

\[
\text{Hom}_{A-\text{Mod}}((X, \mu), (X', \mu')) = \{ s \in \text{Hom}_\mathcal{C}(X, X') \mid s \circ \mu = \mu' \circ \text{id}_A \otimes s \},
\]

\(A\)-modules form a category. Right \( A \)-modules and \( A \to A \) bimodules are defined analogously.

Free \( A \)-module of rank 1: \((A, m)\).
Let \((A,m,\eta)\) be an algebra in \(\mathcal{C}\). An ideal in \(A\) is an \(A\)-module \((X,\mu)\) together with a monic \((X,\mu)\to (A,m)\). Much as in ordinary algebra, one can define a quotient algebra \(A/I\). Furthermore, every ideal is contained in maximal ideal, and an ideal \(I\subset A\) in a commutative monoid is maximal iff the ring \(\Gamma_{A/I}\) is a field. (Cf. [167].)

**Coalgebras** and their comodules are defined analogously. In a symmetric/braided category it makes sense to say that an (co)algebra is (co)commutative.

(B) Just as monoids can act on sets, tensor categories can act on categories:

Let \(\mathcal{C}\) be a tensor category. A left \(\mathcal{C}\)-module category is a pair \((\mathcal{M},F)\) where \(\mathcal{M}\) is a category and \(F:\mathcal{C}\to \text{End}\mathcal{M}\) is a tensor functor.

In other words, we have a functor \(F':\mathcal{C}\times \mathcal{M}\to \mathcal{M}\), natural isomorphisms \(\beta_{X,Y,A}:F'(X\otimes Y,A)\to F(X,F(Y,A))\) satisfying a pentagon-type coherence law, unit constraints, etc.

Now one can define indecomposable module categories, etc. (Ostrik [185]).

Connection between module categories and categories of modules:

If \((A,m,\eta)\) is an algebra in \(\mathcal{C}\), then there is an natural right \(\mathcal{C}\)-module structure on the category \(A\to \text{Mod}_{\mathcal{C}}\) of left \(A\)-modules:

\[ F': A \to \text{Mod}_{\mathcal{C}} \times \mathcal{C}, \quad (M,\mu) \times X \mapsto (M \otimes X, \mu \otimes \text{id}_X). \]

(In the case where \((M,\mu)\) is the free rank-one module \((A,m)\), this gives the free \(A\)-modules \(F'((A,m),X) = (A \otimes X, m \otimes \text{id}_X)\).)

For a fusion category (cf. below), every semisimple, indecomposable left \(\mathcal{C}\)-module category arises in this way from an algebra in \(\mathcal{C}\) (Ostrik [185]).

### 1.5 Duality in tensor categories I

\(G\) a finite (or compact) group, \(\pi \in \text{Rep}_f G\). Then there exists a dual/conjugate representation \(\overline{\pi} \in \text{Rep}_f G\) (defined by \(\overline{\pi}(g) = \pi(g^{-1})'\)) such that \(\pi \otimes \overline{\pi} \geq \pi_0\), where \(\pi_0\) is the trivial representation. If \(\pi\) is irreducible, then so is \(\overline{\pi}\) and the multiplicity of \(\pi_0\) in \(\pi \otimes \overline{\pi}\) is one. (Proof uses existence of Haar measure with \(\mu(G) < \infty\).)

This discussion is quite specific to the group situation and needs to be generalized!

Let \((\mathcal{C},\otimes,\mathbf{1})\) be a strict tensor category and \(X,Y \in \mathcal{C}\). We say that \(Y\) is a left dual of \(X\) if there are morphisms \(e: Y \otimes X \to \mathbf{1}\) and \(d: \mathbf{1} \to X \otimes Y\) satisfying

\[ \text{id}_X \otimes e \circ d \otimes \text{id}_X = \text{id}_X, \quad e \otimes \text{id}_Y \circ \text{id}_Y \otimes d = \text{id}_Y, \]

or, representing \(e: Y \otimes X \to \mathbf{1}\) and \(d: \mathbf{1} \to X \otimes Y\) by \(\otimes\) and \(\sum\), respectively,

\[
\begin{align*}
\begin{array}{ccc}
X & \overset{Y}{\otimes} & X \\
\downarrow & & \\
Y & \overset{\mathbf{1}}{\otimes} & X \\
\end{array}
\end{align*}
\]

\[
\begin{align*}
\begin{array}{ccc}
Y & \overset{X}{\otimes} & Y \\
\downarrow & & \\
X & \overset{\mathbf{1}}{\otimes} & Y \\
\end{array}
\end{align*}
\]

(\(e\) stands for ‘evaluation’ and \(d\) for ‘dual’.). In this situation, \(X\) is called a right dual of \(Y\).

Example: \(\mathcal{C} = \text{Vect}_k^{\text{fin}},\) \(X \in \mathcal{C}\). Let \(Y = X^*\), the dual vector space. Then \(e: Y \otimes X \to \mathbf{1}\) is the usual pairing. With the canonical isomorphism \(f: X^* \otimes X \xrightarrow{\sim} \text{End} X\), we have \(d = f^{-1}(\text{id}_X)\).

Facts:
1. Whether an object $X$ admits a left or right dual is not for us to choose. It is a property of the tensor category.

2. If $Y, Y'$ are left (or right) duals of $X$ then $Y \cong Y'$.

3. If $^\vee A, ^\vee B$ are left duals of $A, B$, resp. then $^\vee B \odot ^\vee A$ is a left dual for $A \odot B$. Similar for right duals.

4. If $X$ has a left dual $Y$ and a right dual $Z$, we may or may not have $Y \cong Z$! (Again, that is a property of $X$.)

While duals, if they exist, are unique up to isomorphisms, it is often convenient to make choices. Thus: A left duality of a strict tensor category $(\mathcal{C}, \otimes, 1)$ assigns to each object $X$ a left dual $^\vee X$ and morphisms $e_X : ^\vee X \otimes X \to 1$ and $d_X : 1 \to X \otimes ^\vee X$ satisfying the above identities.

Given a left duality and a morphism, $s : X \to Y$ we define

$$^\vee s = e_Y \otimes \text{id}_X \circ \text{id}_Y \otimes s \circ \text{id}_X \otimes d_X.$$

Then $(X \mapsto ^\vee X, s \mapsto ^\vee s)$ is a contravariant functor. (We cannot recover the $e$'s and $d$'s from the functor!) It can be equipped with a natural (anti-)monoidal isomorphism $^\vee (A \otimes B) \to ^\vee B \otimes ^\vee A$, $^\vee 1 \to 1$. Often, the duality functor comes with a given anti-monoidal structure, e.g. in the case of pivotal categories, cf. below.

- A chosen right duality $X \mapsto (X^\vee, e^\vee_X : X \otimes X^\vee \to 1$, $d^\vee_X : 1 \to X^\vee \otimes X)$ also give rise to a contravariant anti-monoidal functor $X \mapsto X^\vee$.

- Categories equipped with a left (right) duality are called left (right) rigid (or autonomous).

Categories with left and right duality are called rigid (or autonomous).

- Examples: $\text{Vect}_k^{\text{fin}}, \text{Rep}_k G$ are rigid.

- NB: We have $^\vee ^\vee X \cong X$ iff $^\vee X \cong X^\vee$, for which there is no general reason.

- If every object $X \in \mathcal{C}$ admits a left dual $^\vee X$ and a right dual $X^\vee$, and both are isomorphic, we say that $\mathcal{C}$ has two-sided duals and write $\overline{\mathcal{C}}$. We will only consider such categories, but we will need stronger axioms.

- Let $\mathcal{C}$ be a $*$-category (cf. below) with left duality. If $(^\vee X, e_X, d_X)$ is a left dual of $X \in \mathcal{C}$ then $(X^\vee = ^\vee X, d^\vee_X, e^\vee_X)$ is a right dual. Thus duals in $*$-categories are automatically two-sided.

For this reason, duals in $*$-category are often axiomatized in a symmetric fashion by saying that a conjugate, cf. 

\[ [55, 142] \], of an object $X$ is a triple $(\overline{X}, r, \overline{r})$, where $r : 1 \to \overline{X} \otimes X$, $\overline{r} : 1 \to X \otimes \overline{X}$ satisfy

\[ \text{id}_X \otimes r^* \circ \overline{r} \circ \text{id}_X = \text{id}_X, \quad \text{id}_{\overline{X}} \otimes \overline{r}^* \circ r \otimes \text{id}_{\overline{X}} = \text{id}_{\overline{X}}. \]

It is clear that then $(\overline{X}, r^*, \overline{r})$ is a left dual and $(\overline{X}, \overline{r}^*, r)$ a right dual.

- Beware of the Babylonian inflation of terms involving duals (and braidings): rigid, autonomous, sovereign, pivotal, spherical, ribbon, tortile, balanced, closed, category with conjugates, . . . To make things worse, these terms are not always used in the same way!

- Before we continue the discussion of duality in tensor categories, we discuss symmetries. For symmetric tensor categories, the discussion of duality is somewhat simpler than in the general case. Proceeding like this seems justified since symmetric (tensor) categories already appeared in the second paper (143) 1963 on tensor categories.
1.6 Additive structure

- The discussion so far is incomplete: typically categories have more structure.
- Ab-categories: Each \( \text{Hom}(X, Y) \) is an abelian group, and \( \circ \) is bi-additive. Example: Ab. In \( \otimes \)-categories, also \( \otimes \) must be bi-additive. Functors of Ab-tensor categories required to be additive on hom-sets.
- \( k \)-linear categories: Each \( \text{Hom}(X, Y) \) is \( k \)-vector space (often required finite dimensional), and \( \circ \) and \( \otimes \) is bilinear. (Functors additive.) Example: \( \text{Vect}_k \).
- \( \ast \)-categories: A \( \ast \)-operation on a \( \mathbb{C} \)-linear category \( \mathcal{C} \) is a functor \( \ast : \mathcal{C} \to \mathcal{C} \), \( \ast (X) = X \forall X \), contravariant \( \ast : \text{Hom}(X, Y) \to \text{Hom}(Y, X) \), \( (s \circ t)\ast = t\ast \circ s\ast \), involutive \( (s\ast \ast = s) \) (and monoidal: \( (s \otimes t)\ast = s\ast \otimes t\ast \)). A \( \ast \)-operation is called positive if \( s\ast \circ s = 0 \) implies \( s = 0 \). Categories with \( \ast \)-operation are also called hermitean (unitary). (For me: \( \ast \)-category = unitary category.) Example: \( \mathcal{HILB} \) with \( \ast \) = adjoint.
- Fact: A finite dimensional \( \mathbb{C} \)-algebra with \( \ast \)-operation is semisimple. Thus: A unitary category with finite dimensional hom-sets and splitting idempotents is automatically semisimple, cf. below.
- \( C^\ast \)-categories: \( \ast \)-category, where each \( \text{Hom}(X, Y) \) is a Banach space, and the norms satisfy
  \[ \| s \circ t \| \leq \| s \| \| t \|, \qquad \| s\ast \circ s \| = \| s \|^2. \]
  Remark: Each \( \text{End}(X) \) is a \( C^\ast \)-algebra. Just as an additive category is a ‘ring with several objects’, a \( C^\ast \)-category is a “\( C^\ast \)”-algebra with several objects”. Relevant for applications to K-theory (Karoubi), \( L^2 \)-cohomology (e.g. Lück), representation theory of non-compact groups, subfactors, quantum groups (Woronowicz), QFT, …
  For \( \ast \)-categories and \( C^\ast /W^\ast \)-categories cf. [87, 55, 142].
  Remark: A \( \ast \)-category with finite dimensional hom-spaces and \( \text{End} 1 = \mathbb{C} \) automatically is a \( C^\ast \)-category in a unique way. (Cf. [159].)
  If \( \mathcal{C} \) is a \( C^\ast \)-tensor category, \( \text{End} 1 \) is a commutative \( C^\ast \)-algebra, thus \( \cong C(S) \) for some compact Hausdorff space \( S \). Under certain technical conditions, the spaces \( \text{Hom}(X, Y) \) can be considered as vector bundles over \( S \), or at least as (semi)continuous fields of vector spaces. (Work by Zito [243] and Vasselli [226].) In the case where \( \text{End} 1 \) is finite dimensional, this boils down to a direct sum decomposition of \( \mathcal{C} = \bigoplus \mathcal{C}_i \), where each \( \mathcal{C}_i \) is a tensor category with \( \text{End}_{\mathcal{C}_i}(1_{\mathcal{C}_i}) = \mathbb{C} \). (E.g. Baez on finite groupoids [4].)
- Abelian categories: usually the functors \( - \otimes X \) and \( X \otimes - \) are required to be exact. (Automatic in the presence of duals.)
- Semisimple category: abelian category where every short exact sequence splits.
  Essentially equivalent to: Additive category with direct sums and splitting idempotents admitting a family of objects \( X_i, i \in I \) such that every \( X \in \mathcal{C} \) is a finite direct sum of objects \( X_i \).
  Standard examples: \( \text{Rep}_G \) for compact group \( G \), \( H - \text{Mod} \) for f.d. semisimple Hopf algebra \( H \).
  A fusion category is a semisimple \( k \)-linear category with finite dimensional hom-sets, finitely many isomorphism classes of simple objects and \( \text{End} 1 = k \). We also require that \( \mathcal{C} \) has ‘duals’.
- A finite tensor category (Etingof, Ostrik [68]) is a \( k \)-linear tensor category with \( \text{End} 1 = k \) that is equivalent (as a category) to the category of modules over a finite dimensional \( k \)-algebra. (There is a more intrinsic characterization.) NB: Semisimplicity is not assumed.
  Dropping the condition \( \text{End} 1 = k \text{id}_1 \), one arrives at a multi-fusion category (Etingof/ Nikshych/ Ostrik [67]).
  Despite the recent work on generalizations, most of these lectures will be concerned with semisimple \( k \)-linear categories satisfying \( \text{End} 1 = k \text{id}_1 \), including infinite ones!
• If \( \mathcal{C} \) is a semisimple tensor category, one can choose representers \( \{X_i, i \in I\} \) of the simple isomorphism classes and define \( N^k_{i,j} \in \mathbb{Z}_+ \) by

\[
X_i \otimes X_j \cong \bigoplus_{k \in I} N^k_{i,j} X_k.
\]

There is a distinguished element \( 0 \in I \) such that \( X_0 \cong I \), thus \( N^0_{i,0} = N^k_{0,i} = \delta_{i,k} \). By associativity of \( \otimes \) (up to isomorphism)

\[
\sum_n N^n_{i,j} N^l_{n,k} = \sum N^n_{i,m} N^l_{j,k} \quad \forall i, j, k, l \in I.
\]

If \( \mathcal{C} \) has two-sided duals, there is an involution \( i \mapsto \tau \) such that \( \sum_i \cong X_\tau \). One has \( N^0_{i,j} = \delta_{i,j} \).

The quadruple \( (I, \{N^k_{i,j}\}, 0, i \mapsto \tau) \) is called the \textbf{fusion ring} or \textbf{fusion hypergroup} of \( \mathcal{C} \).

\[\text{The above does not work when} \quad \mathcal{C} \text{ is not semisimple. But: In any tensor category (not necessarily semisimple), one can consider the \textbf{Grothendieck ring} } \mathcal{R}(\mathcal{C}), \text{ the free abelian group generated by the isomorphism classes } [X]\text{ of objects in } \mathcal{C}, \text{ with the relations}
\]

\[\mathcal{X}[X] + \mathcal{X}[Y] = \mathcal{X}[X \otimes Y], \quad \mathcal{X}[X] \cdot \mathcal{X}[Y] = \mathcal{X}[X \otimes Y].\]

In the semisimple case, the Grothendieck ring has \( \{[X_i], i \in I\} \) as \( \mathbb{Z} \)-basis and \( [X_i] \cdot [X_j] = \sum_k N^0_{i,j} [X_k] \).

I prefer working with the fusion hypergroup rather than the Grothendieck (semi)ring, since there can be spurious isomorphisms of Grothendieck rings that don’t arise from an isomorphism of the hypergroups. (This can be also ruled out by talking about the Grothendieck \textbf{semiring} or the \textbf{ordered} Grothendieck ring.)

Back to hypergroups:

• The hypergroup contains important information, but it misses that encoded in the associativity constraint. Well known: there are finite groups with isomorphic fusion hypergroups but inequivalent tensor categories of representations. (By the way, the hypergroup of \( \text{Rep} G \) contains exactly the same information as the character table of \( G \).)

On the positive side: (1) If a finite group \( G \) has the same fusion hypergroup (or character table) as a finite simple group \( G' \), then \( G \cong G' \), cf. \([39]\). (The proof uses the classification of finite simple groups.) (2) Compact groups that are abelian or connected are determined by their fusion rings (by Pontrjagin duality resp. a result of McMullen \([157]\) and Handelman \([93]\). The latter is first proven for simple compact Lie groups and then one deduces the general result via the structure theorem for connected compact groups.)

• If all objects in a semisimple category \( \mathcal{C} \) are invertible, the fusion hypergroup becomes a group.

Such fusion categories are called \textbf{pointed} and are just the linear versions of the categorical groups encountered earlier. This situation is very special, but:

• To each hypergroup \( \{I, N, 0, i \mapsto \tau\} \) one can associate a group \( G(I) \) as follows: Let \( \sim \) be the smallest equivalence relation on \( I \) such that

\[ i \sim j \quad \text{whenever} \quad \exists m, n \in I : \quad i < mn \succ j \quad (\text{i.e.} \quad N^n_{i,m} \neq 0 \neq N^n_{j,m}). \]

Now let \( G(I) = I/\sim \) and define

\[ [i] \cdot [j] = [k] \quad \text{for any} \quad k \prec ij, \quad [i]^{-1} = [\overline{k}], \quad e = [0]. \]

Then \( G(I) \) is a group, and it has the universal property that every map \( p : I \to K, K \) a group, satisfying \( p(k) = p(i)p(j) \) when \( k \prec ij \) factors through the map \( I \to G(I), \quad i \mapsto [i] \).

In analogy to the abelianization of a non-abelian group, \( G(I) \) should perhaps be called the \textbf{groupification} of the hypergroup \( I \). But it was called the \textbf{universal grading group} by Gelaki/Nikshych \([51]\), to which this is due in the above generality, since every group-grading on the objects of a fusion category having fusion hypergroup \( I \) factors through the map \( I \to G(I) \).
• NB: In the symmetric case (where \( I \) and \( G(I) \) are abelian, but everything else as above) this is due to Baumgärtel/Lledó [13], who spoke of the ‘chain group’. They conjectured (and I proved [165]) the following: If \( K \) is a compact group, then the (discrete) universal grading group \( G(\text{Rep} K) \) of \( \text{Rep} K \) is the Pontrjagin dual of the (compact) center \( Z(K) \). Thus: The center of a compact group \( K \) can be recovered from the fusion ring of \( K \), even if \( K \) itself in general cannot!

Example: The representations of \( K = SU(2) \) are labelled by \( \mathbb{Z}_+ \) with

\[
i \otimes j = |i - j| \oplus \cdots \oplus i + j - 2 \oplus i + j.
\]

From this one easily sees that there are two \(~\)-equivalence classes, consisting of the even and odd integers. This is compatible with \( Z(SU(2)) = \mathbb{Z}/2\mathbb{Z} \). Cf. [13].

• Other application of \( G(C) \): If \( C \) is k-linear semisimple then group of natural monoidal isomorphisms of \( \text{id}_C \) is given by \( \text{Aut}_C(\text{id}_C) \cong \text{Hom}(G(C), k^*) \).

• Given a fusion category \( C \) (where we have two-sided duals \( X \)), Gelaki/Nikshych [84] define the full subcategory \( C_{ad} \subset C \) to be the generated by the objects \( X \otimes X \) where \( X \) runs through the simple objects.

Notice: \( C_{ad} \) is just the full subcategory of objects of universal grading zero.

Ex: \( G \) a compact group \( \Rightarrow (\text{Rep} G)_{ad} = \text{Rep}(G/Z(G)) \).

A fusion category \( C \) is called nilpotent [84] when its upper central series

\[
C \supset C_{ad} \supset (C_{ad})_{ad} \supset \cdots
\]

leads to the trivial category after finitely many steps.

Ex: \( G \) a finite group \( \Rightarrow \text{Rep}_f G \) is nilpotent iff \( G \) is nilpotent.

• Let \( I \) be a finite hypergroup. For \( i \in I \), define \( N_i \in \text{Mat}(|I| \times |I|) \) by \( (N_i)_{jk} = N_{i,j}^k \). The Frobenius-Perron dimension \( d_{FP}(i) \) of \( i \in I \) is the PF-eigenvalue of \( N_i \). Then:

\[
d_{FP}(i)d_{FP}(j) = \sum_k N_{i,j}^k d_{FP}(k).
\]

Also \( I \) has a Frobenius-Perron dimension: \( FP - \dim(I) = \sum_i d_{FP}(i)^2 \). This also defines the FP-dimension of a fusion category. [67]

• Problem: Characterize hypergroups arising from a fusion category. (Probably hopeless.)

• Ocneanu rigidity: Up to equivalence there are only finitely many fusion categories with given fusion hypergroup. (Related results: Stefan: The number of isomorphism classes of s.s.&co-s.s. Hopf algebras of given finite dimension is finite. For Hopf \(*\)-algebras, Blanchard even proved a bound on the number of iso-classes in terms of the dimension. There also is an upper bound on the number of iso-classes of semisimple Hopf algebras with given number of irreducible representations, cf. Etingof’s appendix to [187].) General statement announced by Blanchard/A. Wassermann. Proof: Etingof/Nikshych/Ostrik [67], using the deformation cohomology theory of Davydov [40] and Yetter [242].

• There is an enormous literature on hypergroups. Much of this concerns harmonic analysis on the latter and is not too relevant to tensor categories. But the notion of amenability of hypergroups does have such applications, cf. e.g. [99].

• A considerable fraction of the literature on tensor category is devoted to categories that are \( k \)-linear over a field \( k \) with finite dimensional Hom-spaces. Clearly this a rather restrictive condition. It is therefore very remarkable that \( k \)-linearity can actually be deduced from the other conditions on fusion categories. Cf. [134].

11
2 Symmetric tensor categories

- A symmetry on a tensor category \((\mathcal{C}, \otimes, 1, \alpha, \rho, \lambda)\) is a natural isomorphism \(c : \otimes \rightarrow \otimes \circ \sigma\), where \(\sigma : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C} \times \mathcal{C}\) is the flip automorphism of \(\mathcal{C} \times \mathcal{C}\), such that \(c^2 = \text{id}\), i.e. \(c_{X,Y} \circ c_{X,Y} = \text{id}_{\otimes Y}\) for all \(X,Y\), and “all diagrams commute”, i.e. the category is coherent. A symmetric tensor category (STC) is a tensor category equipped with a symmetry.

We represent the symmetry graphically by

\[
\begin{array}{c}
\alpha \\
X \\
Y \\
\end{array} \quad \begin{array}{c}
\alpha \\
X \\
Y \\
\end{array}
\]

- Coherence Theorem A (Mac Lane [148]) Let \((\mathcal{C}, \otimes, 1, \alpha, \rho, \lambda)\) be a tensor category. Then a natural isomorphism \(c : \otimes \rightarrow \otimes \circ \sigma\) satisfying \(c^2 = \text{id}\) is a symmetry iff

\[
\begin{array}{ccc}
\alpha_{X,Y,Z} & \quad & (X \otimes Y) \otimes Z \\
\alpha_{Y,Z,X} & \quad & (Y \otimes X) \otimes Z \\
\end{array}
\]

commutes. (In the strict case, this reduces to \(c_{X,Y \otimes Z} = \text{id}_Y \circ c_{X,Z} \circ c_{X,Y} \circ \text{id}_Z\).)

A symmetric tensor functor is a tensor functor \(F\) such that \(F(c_{X,Y}) = c'_{F(X),F(Y)}\). NB: A natural transformation between symmetric tensor functors is just a monoidal natural transformation.

Examples:
- The category \(\mathbb{S}\) defined earlier, when \(c_{n,m} : n + m \rightarrow n + m\) is taken to be the element of \(S_{n+m}\) defined by \((1, \ldots, n+m) \mapsto (n+1, \ldots, n+m, 1, \ldots n)\). It is the free symmetric monoidal category generated by one object.
- \(\mathcal{C}(G)\) for \(G\) abelian. The pairs \((\alpha, c)\) = (associator, symmetry) on this category are classified by \(H^3_{ab}(G, A)\). (Eilenberg-Mac Lane cohomology theory for abelian groups [147].)
- \(\text{Vect}_k\), \(\text{Rep} G\): we have the canonical \(c_{X,Y} : X \otimes Y \rightarrow Y \otimes X\), \(x \otimes y \mapsto y \otimes x\).
- The tensor categories obtained using products or coproducts are symmetric.

- Coherence theorem B: Every symmetric tensor category is equivalent (by a symmetric tensor functor) to a strict one.
- \(\mathcal{C}\) strict STC, \(X \in \mathcal{C}\), \(n \in \mathbb{N}\). There is a homomorphism

\[
\Pi^X_n : S_n \rightarrow \text{Aut} X^\otimes n : \quad \sigma_i \mapsto \text{id}_{X^\otimes (n-i)} \otimes c_{X,X} \otimes \text{id}_{X^\otimes (n-i-1)}.
\]

Proof: Immediate by the definition of STCs and the presentation

\[
S_n = \{ \sigma_1, \ldots, \sigma_{n-1} \mid \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \ \sigma_i \sigma_j = \sigma_j \sigma_i \ \text{when} \ |i - j| > 1, \ \sigma_i^2 = 1 \}
\]

of the symmetric groups.

These homomorphisms in fact combine to a symmetric tensor functor \(F : \mathbb{S} \rightarrow \mathcal{C}\) such that \(F(n) = X^\otimes n\).

- In the \(\otimes\)-category \(\mathcal{C} = \text{Vect}^\text{fin}_k\), \(\text{Hom}(V,W)\) is itself an object of \(\mathcal{C}\), giving rise to an internal hom-functor: \(\mathcal{C}^\text{op} \times \mathcal{C} \rightarrow \mathcal{C}\), \(X \times Y \mapsto [X,Y] = \text{Hom}(X,Y)\) satisfying some axioms. In the older literature, a symmetric tensor category with such an internal-hom functor is called a closed category. There are coherence theorems for closed categories. [123] [122].
Since in Vect\textsuperscript{fin} \(k\) we have \(\text{Hom}(V, W) \cong V^* \otimes W\), it is sufficient – and more transparent – to axiomatize duals \(V \mapsto V^\vee\), as is customary in the more recent literature. We won’t mention ‘closed’ categories again. (Which doesn’t mean that they have no uses!)

- We have seen that, even if a tensor category has left and right duals \(V^\vee, X^\vee\) for every object, they don’t need to be isomorphic. But: If \(C\) is symmetric and \(X \mapsto (X^\vee, e_X, d_X)\) is a left duality, then defining

\[
X^\vee = X, \quad e'_X = e_X \circ c_X, \quad d'_X = c_X \circ d_X,
\]

one easily checks that \(X \mapsto (X^\vee, e'_X, d'_X)\) defines a right duality. We can thus take \(\vee X = X^\vee\) and denote this more symmetrically by \(X\).

- \(C\) symmetric with left duals, right duals as just defined, \(X \in C\). Define the (left) trace \(\text{Tr}_X : \text{End} X \to \text{End} 1\) by

\[
\text{Tr}_X(s) = e_X \circ \text{id}_X \otimes s \circ d'_X = \left(\begin{array}{c}
\text{Tr}_X(s) = e_X \circ \text{id}_X \\
\text{id}_X \circ s \circ d'_X
\end{array}\right)
\]

**Remark:** For more on traces in tensor categories cf. e.g. [109, 154].

- Define **dimension**: \(d(X) = \text{Tr}_X(\text{id}_X) \in \text{End} 1\). If \(\text{End} 1 = \text{id}_{1}\), we have \(d(X) \in k\).

With this dimension and the usual symmetry and duality on Vect\textsuperscript{fin} \(k\), we have \(d(V) = \dim_k V 1_k\).

However, in the category SVect\(k\) of super-vector spaces (which coincides with Vect\(k\), but has the symmetry modified by the Koszul rule) it gives the super-dimension, which can be negative, while one might prefer the total dimension. Such situations can be taken care of (without changing the symmetry) by introducing twists.

- If \((C, \otimes, 1)\) is strict symmetric, we define a **twist** to be natural family \(\{\Theta_X \in \text{End} X, X \in C\}\) of isomorphisms satisfying

\[
\Theta_{X \otimes Y} = \Theta_X \otimes \Theta_Y, \quad \Theta_1 = \text{id}_1
\]

i.e., \(\Theta\) is a monoidal natural isomorphism of the functor \(\text{id}_C\). If \(C\) has a left duality, we also require

\[
\Theta_X^\vee = \Theta_X \circ d_X
\]

The second condition implies \(\Theta_X^\vee = \text{id}\) Notice that \(\Theta_X = \text{id}_X \forall X\) is a legal choice. This will not remain true in braided tensor categories!

**Example:** If \(G\) is a compact group and \(C = \text{Rep} G\), then the \(\Theta\) satisfying only [1] are in bijection with \(Z(G)\). The second condition reduces this to central elements of order two. (Cf. e.g. [167].)

- Given a strict symmetric tensor category with left duality and a twist, we can define a right duality by \(X^\vee = X^\vee, \text{writing } \overline{X} = X^\vee = X^\vee\), but now

\[
e'_X = e_x \circ c_{x, \overline{x}} \circ \Theta_X \otimes \text{id}_{\overline{x}}, \quad d'_X = \text{id}_{\overline{x}} \circ \Theta_X \circ c_{x, \overline{x}} \circ d_X,
\]

still defining a right duality and the maps \(\text{Tr}_X : \text{End} X \to \text{End} 1\) still are traces.
Conversely, the twist can be recovered from $X \mapsto (\overline{X}, e_X, d_X, e'_X, d'_X)$ by

$$\Theta_X = (\text{Tr}_X \otimes \text{id})(c_{X,X}) = \begin{bmatrix} e_X & 0 \\ 0 & d'_X \end{bmatrix}$$

Thus: Given a symmetric tensor category with fixed left duality, every twist gives rise to a right duality, and every right duality that is ‘compatible’ with the left duality gives a twist. (The trivial twist $\Theta \equiv \text{id}$ corresponds to the original definition of right duality. The latter does not work in proper braided categories!) This compatibility makes sense even for categories without symmetry (or braiding) and will be discussed later ($\leadsto$ pivotal categories).

- The symmetric categories with $\Theta \equiv \text{id}$ are now called **even**.
- The category of super-vector spaces with $\Theta$ defined in terms of the $\mathbb{Z}_2$-grading now satisfy $\dim(V) \geq 0$ for all $V$.
- The standard examples for STCs are $	ext{Vect}_k$, $S\text{Vect}_k$, $	ext{Rep}_G$ and the representation categories of supergroups.
  In fact, rigid STCs try hard to be a representation categories! (But not all succeed, cf. e.g. [53] for examples of non-tannakian symmetric categories.)

- A category $\mathcal{C}$ is called **concrete** if its objects are sets and $\text{Hom}_{\mathcal{C}}(X,Y) \subset \text{Hom}_{\text{Sets}}(X,Y)$. A $k$-linear category is called concrete if the objects are fin. dim. vector spaces over $k$ and $\text{Hom}_{\mathcal{C}}(X,Y) \subset \text{Hom}_{\text{Vect}_k}(X,Y)$.
- Better way of thinking of concrete categories: Category $\mathcal{C}$ equipped with faithful functor $E : \mathcal{C} \to \text{Sets}$, resp. $E : \mathcal{C} \to \text{Vect}_k$, required monoidal if $\mathcal{C}$ is $\otimes$-category. $E$ is called **fiber functor**.
- Example: $G$ a group. Then $\mathcal{C} := \text{Rep}_k G$ should be considered as an abstract $k$-linear $\otimes$-category together with a faithful $\otimes$-functor $E : \mathcal{C} \to \text{Vect}_k$.
- The point of this: A category $\mathcal{C}$ may have inequivalent fiber functors!!
- But: $k$ alg, closed of char. zero, $\mathcal{C}$ rigid symmetric $k$-linear, $\text{End} \mathbb{I} = k$ and $F, F' \text{ symmetric}$ fiber functors $\Rightarrow F \cong F'$ (as $\otimes$-functors). (Saavedra Rivano 200 [49].)
- Perhaps first non-trivial application of (symmetric) tensor categories: Theorems of Tannaka [213, 1939!] and Saavedra-Rivano 200 [49].

Let $k$ be algebraically closed. Let $\mathcal{C}$ be rigid symmetric $k$-linear with $\text{End} \mathbb{I} = k$ and $F, F' \text{ symmetric}$ fiber functors $\Rightarrow F \cong F'$ (as $\otimes$-functors). [Tannaka: $k = \mathbb{C}$, $\mathcal{C}$ *-category, $E$ *-preserving.] Let $G = \text{Aut}_F F$ be the group of natural monoidal [unitary] automorphisms of $F$. Define a functor $F : \mathcal{C} \to \text{Rep} G$ [unitary representations] by

$$F(X) = (E(X), \pi_X), \quad \pi_X(g) = g_X \quad (g \in G).$$

Then $G$ is pro-algebraic [compact] and $F$ is an equivalence of symmetric tensor $[*]$categories.

Proof: Idea (Grothendieck, Saavedra, cf. Bichon [49]): Let $E_1, E_2 : \mathcal{C} \to \text{Vect}_k$ be fiber functors. Define a unital $k$-algebra $A_0(E_1, E_2)$ by

$$A_0(E_1, E_2) = \bigoplus_{X \in \mathcal{C}} \text{Hom}_{\text{Vect}_k}(E_2(X), E_1(X)),$$
spanned by elements \([X, s], \ X \in \mathcal{C}, \ s \in \text{Hom}(E_2(X), E_1(X))\), with \([X, s] : [Y, t] = [X \otimes Y, u]\), where \(u\) is the composite
\[
E_2(X \otimes Y) \xrightarrow{\left(\frac{d^2_{X,Y}}{d^1_{X,Y}}\right)^{-1}} E_2(X) \otimes E_2(Y) \xrightarrow{s \otimes t} E_1(X) \otimes E_1(Y) \xrightarrow{d^1_{X,Y}} E_1(X \otimes Y).
\]
This is a unital associative algebra, and \(A(E_1, E_2)\) is defined as the quotient by the ideal generated by the elements \([X, a \circ E_2(s)] - [Y, E_1(s) \circ a]\), where \(s \in \text{Hom}_\mathcal{C}(X, Y), \ a \in \text{Hom}_{\text{Vect}}(E_2(Y), E_1(X))\).

- **Remark:** Let \(E_1, E_2 : \mathcal{C} \to \text{Vect}_k\) be fiber functors as above. Then the map
  \[
  X \times Y \mapsto \text{Hom}_{\text{Vect}_k}(E_2(Y), E_1(X))
  \]
extends to a functor \(F : \mathcal{C} \times \mathcal{C}^{\text{op}} \to \text{Vect}_k\). Now the algebra \(A(E_1, E_2)\) is just the coend \(\int^X F(X, X)\) of \(F\), a universal object. Coends are a categorical, non-linear version of traces, but we refrain from going into them since it takes some time to appreciate the concept. (Cf. [149].)

- **One proves** [19, 167]:
  - \(E_1, E_2\) are symmetric tensor functors \(\Rightarrow A(E_1, E_2)\) is commutative.
  - \(\mathcal{C}\) is \(*\)-category and \(E_1, E_2\) are \(*\)-preserving \(\Rightarrow A(E_1, E_2)\) is a \(*\)-algebra and has a \(\mathcal{C}^*\)-completion.
  - \(\mathcal{C}\) is finitely generated (i.e. \(\exists Z \in \mathcal{C}\) such that every \(X \in \mathcal{C}\) is direct summand of some \(Z^{\otimes N}\) \(\Rightarrow A(E_1, E_2)\) is finitely generated.
  - There is a bijection between natural monoidal (unitary) isomorphisms \(\alpha : E_1 \to E_2\) and \((\ast\ast)\)-characters on \(A(E_1, E_2)\).

Thus: If \(E_1, E_2\) are symmetric and either \(\mathcal{C}\) is finite generated or \(\mathcal{C}\) is a \(*\)-category, the algebra \(A(E_1, E_2)\) has characters (by Gelfand/Naimark or the Nullstellensatz), thus \(E_1 \cong E_2\). Also: \(G = \text{Aut}_\mathcal{C} E \cong (\ast\ast)\text{Char}(A(E, E))\) and \(A(E) = \text{Fun}(G)\) (representative vs. continuous functions). This is used to prove that \(F : \mathcal{C} \to \text{Rep} G\) is an equivalence.

- **Rem 1:** While it has become customary to speak of Tannakian categories, the work of Krein, cf. [151], [98, Section 30], should also be mentioned since it can be considered as a model for the later generalizations to non-symmetric categories, in particular in Woronowicz’s approach.

- **Rem 2:** Symm. fiber functor \(E\) unique \((/\cong) \Rightarrow G\) unique up to iso.

- **Rem 3:** For the above construction we need to have a fiber functor. Doplicher-Roberts [59], Deligne [13] (indep., around 1989): construct one under weak assumptions on \(\mathcal{C}\). See below.

- **Rem 4:** The uniqueness proof fails if either of \(E_1, E_2\) is not symmetric (or \(\mathcal{C}\) is not symmetric). Given a group \(G\), there is a tautological fiber functor \(E\). The fact that there may be (non-symmetric) fiber functors that are not naturally isomorphic to \(E\) reflects the fact that there can be groups \(G'\) such that \(\text{Rep} G \cong \text{Rep} G'\) as tensor categories, but not as symmetric tensor categories!!! This phenomenon was discovered by Etingof/Gelaki [64], who called such \(G, G'\) isocategorical and produced examples of isocategorical but non-isomorphic finite groups. Their treatment relies on the fact that if \(G, G'\) are isocategorical then \(CG' \cong CJ\) for some twist \(J\). See also [12, Coro. 6.2]. More on this (e.g. extension to compact groups, alternative approach) in progress (MM).

A group \(G\) is called categorically rigid if every \(G'\) isocategorical to \(G\) is actually isomorphic to \(G\). (Compact groups that are abelian or connected are categorically rigid in a strong sense since they are determined already by their fusion hypergroups.)

- The above results already establish strong connections between tensor categories and representation theory, but there is much more to say.
3 Back to general tensor categories

- In a general tensor category, left and right duals need not coincide. Ex: $H - \text{Mod}$, where $H$ is Hopf algebra. Has left and right duals, related to $S$ and $S^{-1}$. ($S$ must be invertible, but can be aperiodic!) They coincide when $S^2 = u \cdot u^{-1}$ with $u \in H$.
- We only consider tensor categories that have isomorphic left and right duals, i.e. two-sided duals, which we denote $X^\vee$.
- If $C$ is $k$-linear with $\text{End} \ 1 = \text{id}$ and $\text{End} \ X = \text{id}$ ($X$ is simple/irreducible), one can canonically define $d^2(X) \in k$:

$$d^2(X) = (e_X \circ d'_X) \cdot (e'_X \circ d_X) \in \text{End} \ 1.$$  

(Since $X$ is simple, $d, d', e, e'$ are unique up to scalars, and well-definedness of $d^2$ follows from the equations involving $(d, e), (d', e')$ bilinearly.) Cf. [161].
- $C$ a fusion category: Dimension $\dim C = \sum_i d^2(X_i)$.
- $H$ fin.dim. ss.&co-ss. Hopf algebra $\Rightarrow \dim H - \text{Mod} = \dim_k H$.
- Even if $C$ is semisimple, it is not clear whether one can choose roots $d(X)$ such that $d$ is additive and multiplicative!
- In pivotal categories this can be done. A strict pivotal category [73, 76] is a strict left rigid category with a monoidal structure on the functor $X \mapsto X^\vee$ and a monoidal equivalence of the functors $\text{id}_C$ and $X \mapsto \vee^\vee X$. As a consequence, one can define a right duality satisfying $X^\vee = \vee X$.
- In a strict pivotal categories we can define left and right traces for every endomorphism:

$$\text{Tr}_L^X(s) = \begin{array}{c}
\circlearrowright \\
\circlearrowleft
\end{array} s$$

$$\text{Tr}_R^X(s) = \begin{array}{c}
\circlearrowright \\
\circlearrowleft
\end{array} s$$

Notice: In general $\text{Tr}_L^X(s) \neq \text{Tr}_R^X(s)$.
- We define $d(X) = \text{Tr}_L^X(\text{id}_X) \in \text{End} \ 1$. Notice: $d(X) = \text{Tr}_R^X(\text{id}_X)$, which can differ from $d(X)$, but for simple $X$ we have $d(X)d(\overline{X}) = d^2(X)$ with $d^2(X)$ as above.
- A strict spherical category [12] is a pivotal category where the left and right traces coincide. Equivalently, it is a strict autonomous category (i.e. a tensor category equipped with a left and a right duality) for which the resulting functors $X \mapsto X^\vee$ and $X \mapsto \vee^\vee X$ coincide. Sphericity implies $d(X) = d(\overline{X})$, and the converse is true for semisimple $C$.
- The Temperley-Lieb categories $\mathcal{T}(\tau)$ are spherical.
- A finite dimensional involutive Hopf algebra ($S^2 = \text{id}$) gives rise to a spherical category. (Thus the semisimple and co-semisimple ones.) More generally: ‘spherical Hopf algebras’ ($S^2(x) = a a c^{-1}$) [12].
- In a $\ast$-category with conjugates, traces of endomorphisms, in particular dimensions of objects, can be defined uniquely without choosing a spherical structure, cf. [55, 142]. The dimension satisfies $d(X) \geq 1$ for every non-zero $X$, with $d(X) = 1$ iff $X$ is invertible. Furthermore,

$$d(X) \in \left\{2 \cos \frac{\pi}{n}, \ n = 3, 4, \ldots\right\} \cup [2, \infty).$$
This is just the $*$-categorical version of the quantization of the Jones index \[108\].

On the other hand, every tensor $*$-category can be equipped \[225\] with an (essentially) unique spherical structure such the traces and dimension defined using the latter coincide with those of \[112\].

- In a $\mathbb{C}$-linear fusion category one has $d^2(X) > 0$ for all $X$. (Etingof/Nikshych/Ostrik)

Application: If $A \subset B$ is a full inclusion then $\dim A \leq \dim B$, and equality holds iff $A \simeq B$.

- In a unitary category, $\dim C = FP - \dim C$. Categories with the latter property are called \textbf{pseudo-unitary} in \[67\], where it is shown that every pseudo-unitary category admits a unique spherical structure such that $FP - d(X) = d(X)$ for all $X$.

- Tannaka theorem for not necessarily symmetric categories (Ulbrich \[223\], Yetter \[240\], Schauenburg \[201\]): Let $C$ be a $k$-linear pivotal category, $\text{End}1 = \text{kid}_1$ and $E : C \to \text{Vect}_k$ a fiber functor. Then the algebra $A(E)$ defined as above admits a coproduct and an antipode, thus the structure of a Hopf algebra $H$, and an equivalence $F : C \to \text{Comod} H$ such that $E = K \circ F$, where $K : \text{Comod} H \to \text{Vect}_k$ is the forgetful functor.

(If $C$ and $E$ are symmetric, $H$ is a commutative Hopf algebra of functions on $G$.) Woronowicz proved a similar result \[238\] for *-categories, obtaining a compact quantum group (as defined by him \[232\], \[234\]). Commutative compact quantum groups are just algebras $C(G)$ for a compact group, thus one recovers Tannaka’s theorem. Cf. \[106\] for an excellent introduction to the area of Tamakas-Krein reconstruction.

- Given a fiber functor, can one find an algebraic structure whose \textit{representations} (rather than corepresentations) are equivalent to $C$? Yes, if one uses a slight generalization of Hopf algebras: A. van Daele’s ‘Algebraic Quantum Groups’ \[224\], \[225\] (or ‘Multiplier Hopf algebras with Haar functional!’): not necessarily unital, coproduct $\Delta$ takes values in multiplier algebra $M(A \otimes A)$, Haar-functional $\mu \in A^*$ part of the data.

Thm \[168\]: Let $C$ be a semisimple spherical (*-)category and $E$ a (*-)fiber functor. Then there is a discrete multiplier Hopf (*-)algebra $(A, \Delta)$ and an equivalence $F : C \to \text{Rep}(A, \Delta)$ such that $K \circ F = E$, where $K : \text{Rep}(A, \Delta) \to \text{Vect}$ is the forgetful functor. (This $(A, \Delta)$ is the Pontrjagin dual of the $A(E)$ above.) This theory exploits the semisimplicity from the very beginning, which makes it quite transparent:

$$A = \bigoplus_{i \in I} \text{End} E(X_i) \quad \text{and} \quad M(A) = \prod_{i \in I} \text{End} E(X_i) \cong \text{Nat} E.$$

Now the tensor structures of $C$ and $E$ give rise to a coproduct $\Delta : A \to M(A \otimes A)$.

Notice: This reconstruction is related to the preceding one as follows. Since $H - \text{comod} \simeq C$ is semisimple, the Hopf algebra $H$ has a left-invariant integral $\mu$, thus $(H, \mu)$ is a compact algebraic quantum group, and the discrete algebraic quantum group $(A, \Delta)$ is just the Pontrjagin dual of the latter.

- In this situation, there is a bijection between braidings on $C$ and R-matrices (in $M(A \otimes A)$).

But: The braiding on $C$ plays no essential rôle in the reconstruction.

- Thus: Linear [braided] tensor categories admitting a fiber functor are (co)representation categories of [(co)quasi-triangular] discrete (compact) quantum groups.

NB: Here ‘Quantum groups’ refers to Hopf algebras and suitable generalizations thereof, but not necessarily to q-deformations of something!!

- \textbf{CAVEAT}: The non-uniquess of fiber functors means that there can be non-isomorphic quantum groups whose (co)representation categories are equivalent to the given $C$!!

The study of this phenomenon leads to Hopf-Galois theory and is connected (in the $*$-case) to the study of ergodic actions of quantum groups on $C^*$-algebras. (Bichon, Vaes et al. \[20\])

- Anyway: Can one \textit{intrinsically} characterize the tensor categories admitting a fiber functor, thus being related to quantum groups? (Existence of a fiber functor is extrinsic.)
• The left regular representation $\pi_l$ of a compact group $G$ (living on $L^2(G)$) has the well known properties

$$\pi_l \cong \bigoplus_{\pi \in \mathcal{G}} d(\pi) \cdot \pi, \quad \text{(Peter-Weyl)}$$

$$\pi_l \otimes \pi \cong d(\pi) \cdot \pi_l \quad \forall \pi \in \text{Rep } G. \quad \text{(absorbing property).}$$

• The second property generalizes to any algebraic quantum group' $(A, \Delta)$ (MM/Tuset [169]):

1. Let $\Gamma = \pi_l$ be the left regular representation. If $(A, \Delta)$ is discrete, then $\Gamma$ carries a monoid structure $(\Gamma, m, \eta)$ with $\dim \text{Hom}(1, \Gamma) = 1$, which we call the regular monoid. (Algebras in $k$-linear tensor categories satisfying $\dim \text{Hom}(1, \Gamma) = 1$ have been called ‘simple’ or ‘haploid’.) If $(A, \Delta)$ is compact, $\Gamma$ has a comonoid structure. (And in the finite (=compact + discrete) case, the algebra and coalgebra structures combine to a Frobenius algebra, MM 2001, cf. below.)

2. If $(A, \Delta)$ is a discrete algebraic quantum, one has a monoid version of the absorbing property: For every $X \in \text{Rep}(f, A, \Delta)$ one has an isomorphism

$$(\Gamma \otimes X, m \otimes \text{id}_X) \cong n(X) \cdot (\Gamma, m)$$

of $(\Gamma, m, \eta)$-modules in $\text{Rep}(A, \Delta)$. (Here $n(X) \in \mathbb{N}$ is the dimension of the vector space of the representation $X$.)

• Theorem (Essentially in Deligne [43]. Cf. also [169]): Let $\mathcal{C}$ be a $k$-linear category and $(\Gamma, m, \eta)$ a monoid in $\mathcal{C}$ (more generally $\text{Ind } \mathcal{C}$) satisfying $\dim \text{Hom}(1, \Gamma) = 1$ and $\oplus$ for some function $n : \text{Obj } \mathcal{C} \rightarrow \mathbb{N}$. Then

$$E(X) = \text{Hom}_{\text{Vect}_k}(1, \Gamma \otimes X)$$

defines a faithful $\otimes$-functor $E : \mathcal{C} \rightarrow \text{Vect}_k$, i.e. a fiber functor.

If $\mathcal{C}$ is symmetric and $(\Gamma, m, \eta)$ commutative (i.e. $m \circ c_{\Gamma,1} = m$), then $E$ is symmetric.

Notice: $\dim E(X) = n(X) \forall X$ and $\Gamma \cong \oplus_i n(X_i)X_i$.

• Thus:

There is a discrete AQG $(A, \Delta)$ such that $\mathcal{C} \simeq \text{Rep}_f(A, \Delta)$

Thus: (Semi)Intrinsic characterization of quantum group categories. (The case of finite $\ast$-categories was treated in [140], using subfactor theory and a functional analysis.)

Remark: 1. This result is quite unsatisfactory, but I doubt that a better result can be obtained without restriction to special classes of categories or generalization of the notion of quantum groups. Examples for both will be given below.

2. For a different approach, also in terms of the regular representation, cf. [54].
Notice that having an absorbing monoid in $C$ (or rather $\text{Ind}(C)$) means having an $\mathbb{N}$-valued dimension function $n$ on the hypergroup $I(C)$ and an associative product on the object $\Gamma = \oplus_{i \in I} I_i X_i$. The latter is a cohomological condition.

If $C$ is finite, there is only one dimension function, namely the intrinsic one $i \mapsto d(X_i)$. Thus a finite category with non-integer intrinsic dimensions cannot be tannakian (in the above sense).

Very beautiful result: Deligne [43] (simplified by Bichon [19]):

Let $C$ be a semisimple $k$-linear rigid even symmetric category satisfying $\text{End} \, 1 = k$, where $k$ be alg. closed of char. zero. Then there is an absorbing commutative monoid as above. (Thus we have a symmetric fiber functor, and $C \simeq \text{Rep} G$.)

Sketch: The homomorphisms $\Pi^X_n : S_n \to \text{Aut} X^{\otimes n}$ allow to define the idempotents

$$P_\pm (X, n) = \frac{1}{n!} \sum_{\sigma \in S_n} (\pm 1)^{\text{sgn}(\sigma)} \Pi^X_n(\sigma) \in \text{End}(X^{\otimes n})$$

and their images $S^n(X), A^n(X)$, which are direct summands of $X^{\otimes n}$. Making crucial use of the evenness assumption on $C$, one proves

$$d(A^n(X)) = \frac{d(X)(d(X) - 1) \cdots (d(X) - n + 1)}{n!} \quad \forall n \in \mathbb{N}.$$

In a $*$-category, this must be non-negative $\forall n$, implying $d(X) \in \mathbb{N}$. (Cf. [55].) Using or assuming this as in [43], one has $d(A^{d(X)}(X)) = 1$, and $A^{d(X)}$ is called the determinant of $X$. On the other hand, one can define a commutative monoid structure on

$$S(X) = \bigoplus_{n=0}^{\infty} S^n(X),$$

obtaining the symmetric algebra $(S(X), m, \eta)$ of $X$. Let $Z$ be a $\otimes$-generator $Z$ of $C$ satisfying $\det Z = 1$. Then the ‘interaction’ between symmetrization (symmetric algebra) and antisymmetrization (determinants) allows to construct a maximal ideal $I$ in the commutative algebra $S(Z)$ such that the quotient algebra $A = S(Z)/I$ has all desired properties: it is commutative, absorbing and satisfies $\dim \text{Hom}(1, A) = 1$. QED.

Remarks: 1. The absorbing monoid $A$ constructed in [43] did not satisfy $\dim \text{Hom}(1, A) = 1$. Therefore the construction considered above does not give a fiber functor to $\text{Vec}_C$, but to $\Gamma_A - \text{Mod}$, and one needs to quotient by a maximal ideal in $\Gamma_A$. Showing that one can achieve $\dim \text{Hom}(1, A) = 1$ was perhaps the main innovation of [167]. This has the advantage that $(A, m, \eta)$ actually is (isomorphic to) the regular monoid of the group $G = \text{Nat}_{\otimes E}$. As a consequence, the latter can be obtained simply as the automorphism group

$$\text{Aut}(\Gamma, m, \eta) \equiv \{ g \in \text{Aut} \Gamma \mid g \circ m = m \circ g \otimes g, \ g \circ \eta = \eta \}$$

of the monoid!

2. If $C$ is not even, its symmetry can be ‘bosonized’ into an even one, cf. [55]. Then one applies the above result and obtains a group $G$. The $\mathbb{Z}_2$-grading on $C$ given by the twist gives rise to an element $k \in Z(G)$ satisfying $k^2 = e$. Thus $C \simeq \text{Rep}(G, k)$ as symmetric category. Cf. also [43].

Applications: Motives [3] [137], differential Galois theory & Riemann Hilbert problem [193]. Modularization of braided tensor categories (cf. below). Classification of triangular Hopf algebras in terms of Drinfeld twists of group algebras (Etingof/Gelaki, cf. [82] and references therein).

Quantum field theory in $\geq 2 + 1$ dimensions: ‘Galois theory’ of quantum fields [50]. (Cf. also [82].)
• Thus, at least in char. zero, rigid symmetric categories with $\text{End} 1 = \text{id}_1$ are reasonably well understood. What about relaxing the last condition? The category of a representations (on continuous fields of Hilbert spaces) of a compact groupoid $G$ is a symmetric $C^*$-tensor category. Since a lot of information is lost in passing from $G$ to $\text{Rep} G$, there is no hope of reconstructing $G$, but one may hope to find a compact group bundle giving rise to the given category and proving that it is Morita equivalent to $G$. However, there seem to be topological obstructions (Vasselli [227]).

• Earlier related work by Bruguieres/Maltsiniotis [153, 31, 28] on quasi quantum groupoids (purely algebraic).

• As promised: Characterization of certain special classes of tensor categories:

• Prototype: Combining Doplicher/Roberts with McMullen/Handelman one has:

$$ \text{If } \mathcal{C} \text{ is an even symmetric tensor *-category with conjugates, } \text{End} 1 = \mathbb{C} \text{ and fusion hypergroup isomorphic to that of a connected compact Lie group } G, \text{ then } \mathcal{C} \simeq \text{Rep}_G. $$

• Kazhdan/Wenzl [119]: Let $\mathcal{C}$ be a semisimple $\mathbb{C}$-linear spherical $\otimes$-category with $\text{End} 1 = \mathbb{C}$, whose fusion hypergroup is isomorphic to that of $\mathfrak{s}(N)$. Then there is a $q \in \mathbb{C}^*$ such that $\mathcal{C}$ is equivalent (as a tensor category) to the representation category of the Drinfeld/Jimbo quantum group $SL_q(N)$ or (one of finitely many twisted versions of it). Here $q$ is either 1 or not a root of unity and unique up to $q \rightarrow q^{-1}$. For another approach to a characterization of the $SL_q(N)$-categories, excluding the root of unity case, cf. [191].

Let $\mathcal{C}$ be a semisimple $\mathbb{C}$-linear rigid $\otimes$-category with $\text{End} 1 = \mathbb{C}$, whose fusion hypergroup is isomorphic to that of the (finite!) representation category of $SL_q(N)$, where $q$ is a primitive root of unity of order $\ell > N$. Then $\mathcal{C}$ is equivalent to $\text{Rep} SL_q(N)$ (or one of finitely many twisted versions).

We will say (a bit) more on quantum groups later. The reason that we mention the Kazhdan/Wenzl result already here is that it does not require $\mathcal{C}$ to come with a braiding. Unfortunately, the proof is not independent of quantum group theory, nor does it provide a construction of the categories.

Beginning of proof: The assumption on the fusion rules imply that $\mathcal{C}$ has a multiplicative generator $Z$. Consider the full monoidal subcategory $\mathcal{C}_0$ with objects $\{Z^\otimes n, n \in \mathbb{Z}_+\}$. Now $\mathcal{C}$ is equivalent to the idempotent completion ("Karoubification") of $\mathcal{C}_0$. (Aside: Tensor categories with objects $\mathbb{N}_+$ and $\otimes = +$ for objects appear often: the symmetric categ $S$, the braid categ $\mathbb{B}$, PROPs.) A semisimple $k$-linear categ with objects $\mathbb{Z}_+$ is called a monoidal algebra, and is equivalent to having a family $A = \{A_{n,m}\}$ of vector spaces together with semisimple algebra structures on $A_n = A_{n,n}$ and bilinear operations $\circ : A_{n,m} \times A_{m,p} \rightarrow A_{n,p}$ and $\otimes : A_{n,m} \times A_{p,q} \rightarrow A_{n+p,m+q}$ satisfying obvious axioms. A monoidal algebra is of diagonal if $A_{n,m} = 0$ for $n \neq m$ and of type $N$ if $\dim A(0,n) = \dim A(n,0) = 1$ and $A_{n,m} = 0$ unless $n \equiv m (\text{mod } N)$. If $A$ is of type $N$, there are exactly $N$ monoidal algebras with the same diagonal. The possible diagonals arising from type $N$ monoidal algebras can be classified, using Hecke algebras $H_n(q)$ (defined later).

• There is an analogous result (Tuba/Wenzl [215]) for categories with the other classical (BCD) fusion rings, but that does require the categories to come with a braiding.

• For fusion categories, there are a number of classifications in the case of low rank (number of simple objects) (Ostrik: fusion categories of rank 2 [187], braided fusion categories of rank 3 [188] or special dimensions, like $p$ or $pq$ (Etingof/Gelaki/Ostrik [63]). Furthermore, one can classify near group categories, i.e. fusion categories with all simple objects but one invertible (Tambara/Yamagami [212], Siehler [205]).

• Other direction: Represent more tensor categories as module categories by generalizing the notion of Hopf algebras. We have already encountered a very modest (but useful!) generalization, to wit Van Daele’s multiplier Hopf algebras. (But the main rationale for the latter was to have a category with a duality, generalizing Pontrjagin duality, which holds for finite dimensional Hopf algebras and fails for infinite dimensional ones.)
Drinfeld’s quasi-Hopf algebras \[185\] go in a different direction: Associative unital algebra \(H\), unital algebra homomorphism \(\Delta : H \to H \otimes H\), but coassociativity holds only up to conjugation with an invertible element \(\phi \in H \otimes H \otimes H\):

\[
\text{id} \otimes \Delta \circ \Delta(x) = \phi(\Delta \otimes \text{id} \circ \Delta(x))\phi^{-1},
\]

where \((\Delta, \phi)\) must satisfy some identity in order for \(\text{Rep} H\) with the tensor product defined in terms of \(\Delta\) to be (non-strict) monoidal. Disadvantage: Duals are not quasi-Hopf algebras. But useful, even for the proof of results concerning ordinary Hopf algebras, like the Kohno-Drinfeld theorem for \(U_q(\mathfrak{g})\).

Examples: Given a finite groups \(G\) and \(\omega \in Z^3(G, \omega)\), there is a finite dimensional quasi Hopf algebra \(D^\omega(G)\), the twisted quantum double of Dijkgraaf/Pasquier/Roche \[51\]. (We’ll later define its representation category in purely categorical way.) Recently, Naidu/Nikshych \[171\] have given necessary and sufficient conditions on pairs \((G, [\omega]), (G', [\omega'])\) for \(D^\omega(G) - \text{Mod}\), \(D^\omega'(G') - \text{Mod}\) to be equivalent as braided tensor categories. But the question for which pairs \((G, [\omega])\) \(D^\omega(G) - \text{Mod}\) is tannakian (i.e. admits a fiber functor) seems to be still open.

Various attempts at proving generalized Tannaka reconstruction theorems in terms of quasi-Hopf algebras \[151\] and “weak quasi-Hopf algebras”. (Cf. e.g. \[146\].) As it turned out, it is sufficient to consider ‘weak’, but ‘non-quasi’ Hopf algebras:

Hayashi (‘face algebras’ \[95\]), Böhm/Szlachányi \[26\], then Nikshych, Vainerman, L. Kadison, . . . : (fin-dim.) **weak Hopf algebras** (\(\sim\) fin-dim. quantum groupoids): Associative unital algebra \(A\), algebra homomorphism \(\Delta : A \to A \otimes A\) which is coassociative but \textit{not necessarily unital}. I.e. can have \(\Delta(1) \neq 1 \otimes 1\) (same for \(\varepsilon\)).

Weak Hopf algebras do have dual weak Hopf algebras and Pontrjagin duality holds! Most importantly, objects of \(\text{Rep} A\) can have non-integer dimensions.

Weak Hopf algebras are closely related to Hopf algebroids.

Ostrik \[185\]: Every fusion category is the module category of a semisimple weak Hopf algebra. (Earlier work by Hayashi \[95\], using his face algebras \[95\].)

Proof idea: An \(R\)-fiber functor on a fusion category \(\mathcal{C}\) is a faithful tensor functor \(\mathcal{C} \to \text{Bimod } R\), where \(R\) is a finite direct sum of matrix algebras. Szlachányi \[210\]: An \(R\)-fiber functor on \(\mathcal{C}\) gives rise to an equivalence \(\mathcal{C} \simeq A - \text{Mod}\) for a weak Hopf algebra (with base \(R\)). (Cf. also \[91\].) How to construct an \(R\)-fiber functor?

Since \(\mathcal{C}\) is semisimple, we can choose an algebra \(R\) such that \(\mathcal{C} \simeq R - \text{Mod}\) (as abelian categories). Since \(\mathcal{C}\) is a module category over itself, we have a \(\mathcal{C}\)-module structure on \(R - \text{Mod}\). Now use that, for \(\mathcal{C}, R\) as above, there is a bijection between \(R\)-fiber functors and \(\mathcal{C}\)-module category structures on \(R - \text{Mod}\) (i.e. tensor functors \(\mathcal{C} \to \text{End}(R - \text{Mod})\).

Rem.: \(R\) is very non-unique: the only requirement was that the number of simple direct summands equals the number of simple objects of \(\mathcal{C}\). (\(\exists\) unique commutative \(R\), but even for that, there is no uniqueness of \(R\)-fiber functors.)

The above proof uses semisimplicity. Non-semisimple generalization announced by Brugières/Virelizier, 2008.

Question: Is there a version for infinite (semisimple) categories, perhaps using the quantum groupoids defined by Lesieur and Enock?

Let \(\mathcal{C}\) be fusion category and \(A\) a weak Hopf algebra such that \(\mathcal{C} \simeq A - \text{Mod}\). Since there is a weak Hopf algebra \(\hat{A}\), one wonders how \(\hat{A} - \text{Mod}\) is related to \(\mathcal{C}\). (One may call such categories dual to \(\mathcal{C}\), but must keep in mind that there is one for each \(A\)!

Answer: \(\hat{A} - \text{Mod}\) is (weakly monoidally) Morita equivalent to \(\mathcal{C}\). This notion (MM \[161\]) was inspired by subfactor theory, in particular ideas of Ocneanu, cf. \[179\] \[180\]. For this we need the following:
• A Frobenius algebra in a strict tensor category is a quintuple \((A, m, \eta, \Delta, \varepsilon)\), where \((A, m, \eta)\) is an algebra, \((A, \Delta, \varepsilon)\) is a coalgebra and
\[
m \otimes \text{id}_A \circ \text{id}_A \otimes \Delta = \Delta \circ m = \text{id}_A \otimes m \circ \Delta \otimes \text{id}_A.
\]
A Frobenius algebra in a linear category is called strongly separable if
\[
\varepsilon \circ \eta = \alpha \text{id}_1, \quad m \circ \Delta = \beta \text{id}_\Gamma, \quad \alpha \beta \neq 0.
\]

Quite old roots, then F. Quinn [194]: ‘ambialgebras’. L. Abrams [1]: Frobenius algebras in \(\text{Vect}^\text{fin}\) are the usual Frobenius algebras, i.e. \(k\)-algebras \(V\) equipped with a \(\phi \in V^*\) such that \((x, y) \mapsto \phi(xy)\) is non-degenerate.

• Frobenius algebras from duals: \(\mathcal{C}\) a tensor category, \(X \in \mathcal{C}\) with two-sided dual \(\overline{X}\). Define \(\Gamma = X \otimes \overline{X}\). Then \(\Gamma\) carries a Frobenius algebra structure:
\[
m = \begin{array}{c}
X \\
X \\
X \\
X \\
X
\end{array}, \quad \Delta = \begin{array}{c}
X \\
X \\
\epsilon \quad \epsilon \\
\epsilon \\
\epsilon
\end{array}, \quad \eta = \begin{array}{c}
X \\
X \\
\eta \quad \eta \\
\eta \\
\eta
\end{array}, \quad \varepsilon = \begin{array}{c}
X \\
X \\
\varepsilon \quad \varepsilon \\
\varepsilon \\
\varepsilon
\end{array}
\]
Verifying the Frobenius identities and strong separability is a trivial exercise! [161]

• Question: Does every (strongly separable) Frobenius algebra in a \(\otimes\)-category arise in this way?
Not quite, but: Let \(\Gamma\) be a strongly separable Frobenius algebra in a \(k\)-linear spherical tensor category \(\mathcal{A}\). Then there exist
– a spherical \(k\)-linear 2-category \(\mathcal{E}\) with two objects \(\{\mathfrak{A}, \mathfrak{B}\}\),
– a 1-morphism \(X \in \text{Hom}_\mathcal{E}(\mathfrak{B}, \mathfrak{A})\) with 2-sided dual \(\overline{X} \in \text{Hom}_\mathcal{E}(\mathfrak{A}, \mathfrak{B})\), and therefore a Frobenius algebra \(X \circ \overline{X}\) in the \(\otimes\)-category \(\text{End}_\mathcal{E}(\mathfrak{A})\),
– a \(\otimes\)-equivalence \(\text{End}_\mathcal{E}(\mathfrak{A}) \to \mathcal{A}\) mapping the the Frobenius algebra \(X \circ \overline{X}\) to \(\Gamma\).

Thus every Frobenius algebra in \(\mathcal{A}\) arises from a 1-morphism in a bicategory \(\mathcal{E}\) containing \(\mathcal{A}\) as a corner. In this situation, the tensor category \(\mathcal{B} = \text{End}_\mathcal{E}(\mathfrak{B})\) is called weakly monoidally Morita equivalent to \(\mathcal{A}\) and the bicategory \(\mathcal{E}\) is called a Morita context.

• The original proof (MM [161]) was tedious. Assuming mild technical conditions on \(\mathcal{A}\) and strong separability of \(\Gamma\), the bicategory \(\mathcal{E}\) can simply be obtained as follows:
\[
\text{Hom}_\mathcal{E}(\mathfrak{A}, \mathfrak{A}) = \mathcal{A},
\text{Hom}_\mathcal{E}(\mathfrak{A}, \mathfrak{B}) = \Gamma - \text{Mod}_\mathcal{A},
\text{Hom}_\mathcal{E}(\mathfrak{B}, \mathfrak{A}) = \text{Mod}_\mathcal{A} - \Gamma,
\text{Hom}_\mathcal{E}(\mathfrak{B}, \mathfrak{B}) = \Gamma - \text{Mod}_\mathcal{A} - \Gamma,
\]
with the composition of 1-morphisms given by the usual tensor products of (left and right) \(\Gamma\)-modules. Cf. [237]. (A discussion free of any technical assumptions on \(\mathcal{A}\) was recently given in [135].)

• The above situation has an interpretation in terms of module categories: If \(\mathfrak{A}, \mathfrak{B}\) are objects in a bicategory \(\mathcal{E}\) as above, the category \(\text{Hom}_\mathcal{E}(\mathfrak{A}, \mathfrak{B})\) is a left module category over the tensor category \(\text{End}_\mathcal{E}(\mathfrak{B})\) and a right module category over \(\mathcal{A} = \text{End}_\mathcal{E}(\mathfrak{A})\). In fact, the whole structure can be formulated in terms of module categories, thereby getting rid of the Frobenius algebras (Etingof, Ostrik [68, 67]): Writing \(\mathcal{M} = \text{Hom}_\mathcal{E}(\mathfrak{A}, \mathfrak{B})\), the dual category
$B = \text{End}_A(\mathcal{B})$ can be obtained as the tensor category $\text{HOM}_A(\mathcal{M}, \mathcal{M})$ of right $A$-module functors from $\mathcal{M}$ to itself. EO denote this by $A^*_\mathcal{M}$.

Since the two pictures are essentially equivalent, the choice is a matter of taste. The picture with Frobenius algebras and the bicategory $\mathcal{E}$ is closer to subfactor theory. What speaks in favor of the module category picture is the fact that non-isomorphic algebras in $\mathcal{A}$ can have equivalent module categories, thus give rise to the same $\mathcal{A}$-module category. (But not in the case of commutative algebras!)

- Morita equivalence is an equivalence relation, denoted $\simeq$. (In particular, $B$ contains a strongly separable Frobenius algebra $\overline{\Gamma}$ such that $\overline{\Gamma} - \text{Mod}_B - \overline{\Gamma} \simeq \mathcal{A}$.)

- As mentioned earlier, the left regular representation of a finite dimensional Hopf algebra $H$ gives rise to a Frobenius algebra $\Gamma$ in $H - \text{Mod}$. $\Gamma$ is strongly separable iff $H$ is semisimple and cosemisimple. In this case, one finds for the ensuing Morita equivalent category:

$$B = \Gamma - \text{Mod}_{H - \text{Mod}} - \Gamma \simeq \hat{H} - \text{Mod}.$$ 

(This is a situation encountered earlier in subfactor theory.) Actually, in this case the Morita context $\mathcal{E}$ had been defined independently by Tambara [211].

The same works for weak Hopf algebras, thus for any s.s.&co-s.s. weak Hopf algebra we have $A - \text{Mod} \simeq A - \text{Mod}$, provided the weak Hopf algebra is Frobenius, i.e. has a non-degenerate integral. (It is unknown whether every weak Hopf algebra is Frobenius.)

- Implications of Morita equivalence: Let $C_1, C_2$ be Morita equivalent (spherical) fusion categories. Then:
  1. $\dim C_1 = \dim C_2$.
  2. $C_1$ and $C_2$ give rise to the same triangulation TQFT in 2+1 dimensions (as defined by Barrett/Westbury [11] and S. Gelfand/Kazhdan [89], generalizing the Turaev/Viro TQFT [220] [218] to non-braided categories. Cf. also Ocneanu [181].) This fits nicely with the known fact (Kuperberg [132], Barrett/Westbury [10]) that, the spherical categories $H - \text{Mod}$ and $\hat{H} - \text{Mod}$ (for a s.s.&co-s.s. Hopf algebra $H$) give rise to the same triangulation TQFT.
  3. The centers $Z_1(C_1), Z_1(C_2)$ are equivalent as braided tensor categories (immediate by a result of Schauenburg [202]). More on $Z_1$ later.

- Emphasize: A fusion category can contain many (strongly separable) Frobenius algebras, thus it can be Morita equivalent to many other tensor categories.

Thus: Important to study (Frobenius) algebras in fusion categories! (In particular, in braided case.)

- Example: Commutative algebras in $\text{Rep} G$ are the same as commutative algebras carrying a $G$-action by algebra automorphisms. The condition $\dim \text{Hom}(1, \Gamma) = 1$ means that the $G$-action is ergodic. Such algebras correspond to closed subgroups $H \subset G$ via $\Gamma_H = C(G/H)$. (Kirillov/Ostrik [129].)

- Ostrik: Algebras in module categories over $\mathcal{C}_k(G, \omega)$.

- A group theoretical category is a fusion category that is weakly Morita equivalent (or ‘dual’) to a pointed fusion category, i.e. one of the form $\mathcal{C}_k(G, \omega)$ ($G$ finite, $[\omega] \in H^3(G, T)$).

- Connection with subfactors: A factor is a von Neumann algebra with center $\mathbb{C}1$. For an inclusion $N \subset M$ of factors, there is a notion of index $[M : N] \in [1, +\infty]$ (not necessarily integer!!), cf. [103] [139]. One has $[M : N] < \infty$ iff the canonical N-M-bimodule $X$ has a dual
1-morphism $\overline{X}$ in the bicategory of von Neumann algebras, bimodules and their intertwiners. In this case, the bicategory with the objects $\{N, M\}$ and bimodules generated by $X, \overline{X}$ is a Morita context. (Motivated by Ocnelan’s bimodule picture of subfactors [179] [180]). On the other hand, a single factor $M$ gives rise to a certain tensor $\ast$-category $\mathcal{C} (M - M$-bimodules or End $M$) such that the Frobenius algebras (“Q-systems”) in $\mathcal{C}$ are (roughly) in bijection with the subfactors $N \subset M$ with $[M : N] < \infty$. (Longo [140]. Cf. also the introduction of [101].)

- (Higher) Various other notions familiar from finite group theory can be generalized to fusion categories: (Higher) Frobenius-Schur indicators and exponents for pivotal/spherical categories (Ng/Schauenburg [175] [176], Natale [172] [173]).

4 Braided tensor categories

- The symmetric groups have the well known presentations

$$S_n = \{\sigma_1, \ldots, \sigma_{n-1} \mid \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \ \sigma_i \sigma_j = \sigma_j \sigma_i \text{ when } |i - j| > 1, \ \sigma_i^2 = 1\}$$

Artin (1928) Braid groups:

$$B_n = \{\sigma_1, \ldots, \sigma_{n-1} \mid \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \ \sigma_i \sigma_j = \sigma_j \sigma_i \text{ when } |i - j| > 1\}$$

Geometric interpretation:

$$\sigma_1 = \begin{array}{ccccc} \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \end{array} \quad \sigma_2 = \begin{array}{ccccc} \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \end{array} \quad \sigma_{n-1} = \begin{array}{ccccc} \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \end{array}$$

Note: $B_n$ is infinite for all $n \geq 2$, $B_2 \cong \mathbb{Z}$. The representation theory of the $B_n$ is difficult. Known: all $B_n$ are linear, i.e. $B_n \hookrightarrow GL(m, \mathbb{C})$ for suitable $m = m(n)$. Cf.: Kassel/Turaev [115].

- Joyal/Street [107]: Braiding on a tensor category: Like a symmetry, a braiding is a family of natural isomorphisms $c_{X,Y} : X \otimes Y \to Y \otimes X$ satisfying two hexagon identities, but we drop the condition $c_i^2 = id$, i.e. $c_{Y,X} \circ c_{X,Y} = id_{X \otimes Y}$.

NB: One then needs a second hexagon, obtained from the earlier one by the replacement $c_{X,Y} \mapsto c_{Y,X}^{-1}$ (which does nothing when $c_i^2 = id$). This is the non-strict generalization of $c_{X \otimes Y, Z} = c_{Z,Y,X} \otimes id_Y \circ id_X \otimes c_{Y,Z}$. A braided tensor category (BTC) is a tensor category equipped with a braiding.

- In analogy to the symmetric case, given a BTC $\mathcal{C}$, $X \in \mathbb{N}$, $n \in \mathbb{Z}_+$, one has a homomorphism $\Pi_n^X : B_n \to \text{Aut}(X^\otimes n)$.

- Only immediately obvious example of BTC that is not symmetric: Braid category $\mathcal{B}$. $\text{Obj } \mathcal{B} = \mathbb{Z}_+$, $\text{End}(n) = B_n, n \otimes m = n + m, \otimes$ on morphisms: juxtaposition of braid diagrams (obvious associative family of homomorphism $B_n \times B_m \to B_{n+m}$) and braiding $c_{n,m} \in \text{End}(n + m) = B_{n+m}$ given by

$$c_{n,m} = \left( \begin{array}{cccccc} \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \end{array} \right) \quad (n, m) = (3, 2)$$
If $C$ is a strict BTC and $X \in C$, there is a unique braided tensor functor $F : B \to C$ such that $F(1) = X$ and $F(c_{2,2}) = c_{X,X}$. Thus $B$ is the free braided tensor category generated by one object.

Centralizer and center $Z_2$:
Let $C$ be a BTC, $D \subset C$ any subcategory (or just subset of Obj $C$). Then the centralizer $C \cap D' \subset C$ is the full subcategory defined by 
\[ \text{Obj}(C \cap D') = \{ X \in C \mid c_{Y,X} \circ c_{X,Y} = id_{X \otimes Y} \ \forall Y \in D \}. \]

Now, the center $Z_2(C)$ is 
\[ Z_2(C) = C \cap C'. \]

$C \cap D'$ is monoidal and $Z_2(C)$ is symmetric! In fact: A BTC $C$ is symmetric iff $C = Z_2(C)$. The objects of $Z_2(C)$ have been called ‘degenerate’ [195], ‘transparent’ [20] or ‘central’.

Thus: STC are maximally commutative BTCs. What about maximally non-commutative BTCs? Example: Obj $Z_2(B) = \{0\}$. Braided fusion categories with ‘trivial’ center $= Turaev$’s modular categories, cf. below.

The definition of BTCs is quite natural if one knows the braid groups. So why did they appear so late? Lack of examples! They finally arrived in the same year as Drinfeld’s definition of quasi-triangular Hopf algebras (below)!

Duality: We want ribbon or spherical structures. Contrary to the symmetric case, it is not enough to have a left duality and a braiding! (If we define a right duality in terms of a left duality and a braiding, the result in general is not pivotal.)

A twist for a braided category with left duality is a natural family \( \{\Theta_X \in \text{End}_X, X \in C\} \) of isomorphisms (i.e. a natural isomorphism of the functor $id_C$) satisfying 
\[ \Theta_{X \otimes Y} = \Theta_X \otimes \Theta_Y \circ c_{Y,X} \circ c_{X,Y}, \quad \Theta_1 = id_1, \quad \forall (\Theta_X) = \Theta_\cdot X. \]

Notice: If $c_{Y,X} \circ c_{X,Y} \not\equiv id$ then the natural isomorphism $\Theta$ is not monoidal and $\Theta = id$ is not a legal twist!

A ribbon category is a strict braided tensor category equipped with a left duality and a twist.

A ribbon category is spherical (with right duality defined in terms of left duality, braiding and twist as above). Conversely, if $C$ is spherical and braided, then defining 
\[ \Theta_X = (\text{Tr}_X \otimes id_X)(c_{X,X}), \]
$\{\Theta_X, X \in C\}$ gives a twist, thus a ribbon structure. (Deligne, Yetter [211], Barrett/Westbury [12].)

I prefer to consider the twist as a derived structure, thus talking about spherical categories with a braiding, rather than ribbon categories. (This is often done implicitly, e.g. in defining the twist of tangle categories that we consider now.)

While the usual representation category of a group is symmetric, the category of representations of the general linear group $GL_n(\mathbb{F}_q)$ over a finite field with the external tensor product of representations turns out to be braided and non-symmetric, cf. [108].

Combining the ideas behind the categories $TL(\tau)$ (which has duals) and $B$ (which is braided), one arrives at the categories of tangles (Turaev [216], Yetter [239]. See also [218, 111]).

Unoriented tangles: Obj $U - TAN = Z_+, n \otimes m = n + m$, morphisms:
Notice: Contrary to $TL(\tau)$, disconnected loops are allowed!

Oriented tangles: Obj $O - TAN = \{+, -\}^*$ (finite words in $\pm$, $1 = \emptyset$). Morphisms: Similar to $U - TAN$, but oriented, compatible with the signs of the objects.

NB: The morphisms in $\text{Hom}(1,1)$ in $U - TAN (O - TAN)$ are just the unoriented (oriented) links.
• The tangle categs are pivotal, in fact spherical, thus ribbon categories. $O - \mathcal{TAN}$ is the free ribbon category generated by one element, cf. [204].

• Let $\mathcal{C}$ be a ribbon category. Then one can define a category $\mathcal{C} - \mathcal{TAN}$ of $\mathcal{C}$-labeled oriented tangles and a ribbon tensor functor $F_{\mathcal{C}} : \mathcal{C} - \mathcal{TAN} \to \mathcal{C}$. (This is the rigorous rationale behind the diagrammatic calculus for braided tensor categories.)

Let $\mathcal{C}$ be a ribbon category and $X$ a self-dual object. Given an unoriented tangle, we can label every edge by $X$. This gives a composite map

$$\{\text{links}\} \cong \text{Hom}_{U - \mathcal{TAN}}(0, 0) \to \text{Hom}_{\mathcal{C} - \mathcal{TAN}}(0, 0) \xrightarrow{F_{\mathcal{C}}} \text{End}_C(1).$$

In particular, if $\mathcal{C}$ is $k$-linear with $\text{End} 1 = \text{id}$, we obtain a map from $\{\text{links}\}$ to $k$, which is easily seen to be a knot invariant. If $\mathcal{C} = U_q(\mathfrak{sl}(2)) - \text{Mod}$ and $X$ is the fundamental object, one essentially obtains the Jones polynomial. Cf. [216, 196]. (The other objects of $\mathcal{C}$ give rise to the colored Jones polynomials, which are much studied in the context of the volume conjecture for hyperbolic knots.)

• So far, all our examples of braided categories come from topology. In a sense, they are quite trivial, since they are just universal categories freely generated by one object. Furthermore, we are primarily interested in linear categories. Of course, we can apply the linearization $\text{CAT} \to \text{lin.CAT}$. But the categories we obtain have infinite dimensional hom-sets and are not more interesting than the original ones.

• Analogy: Braid group $B_n$ ($n > 1$) is infinite, thus the group algebra $CB_n$ is infinite dimensional. But it has finite dimensional quotients, e.g. the Hecke algebra $H_n(q)$, the unital $\mathbb{C}$-algebra generated by $\sigma_1, \ldots, \sigma_{n-1}$, modulo the relations

$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, \quad \sigma_i\sigma_j = \sigma_j\sigma_i \quad \text{when} \quad |i - j| > 1, \quad \sigma_i^2 = (q - 1)\sigma_i + q1.$$  

For $q = 1$, we have $H_n(1) \cong \mathbb{C}S_n$, which is finite dimensional. In fact, $H_n(q)$ is finite dimensional for all $q \in \mathbb{C}$ and semisimple whenever $q$ is not a root of unity. Cf. e.g. [136].

The idea now is to do a similar thing on the level of categories, or to ‘categorify’ the Hecke algebras or other quotients of $CB_n$ like the Birman/Murakami/Wenzl- (BMW-)algebras [21].

• We have seen that ribbon categories give rise to knot invariants. One can go the other way (Turaev [218], Turaev/Wenzl [222]):

A $k$-valued link invariant $G$ is said to admit functorial extension to tangles if there exists a tensor functor $F : U - \mathcal{TAN} \to k - \text{Mod}$ whose restriction to $\text{End}_{U - \mathcal{TAN}}(0) \cong \{\text{links}\}$ equals $G$.

For any $X \in U - \mathcal{TAN}$, $f \in \text{End}(X)$, let $L_f$ be the link obtained by closing $f$ on the right, and define $\text{Tr}_G(f) = G(L_f)$. If $\mathcal{C}$ is the $k$-linearization of $U - \mathcal{TAN}$, one defines an ideal $I$ in $\mathcal{C}$ as consisting of the negligible morphisms. ($s : X \to Y$ is negligible if $\text{Tr}_G(s \circ t) = 0$ for all $t : Y \to X$.) Then under weak assumptions on $G$, the idempotent and direct sum completion of the quotient $\mathcal{C}/I$ is a semisimple ribbon category (with finite dimensional hom-sets)!

Example: Applying the above procedure $G = V_1$, the Jones polynomial, one obtains a Temperley-Lieb category $T\mathbb{L}_r$, which in turn is equivalent to a category $U_q(\mathfrak{sl}(2)) - \text{Mod}$. 

Figure 1: An unoriented 3-5 tangle
Cf. [218, Chapter XII]. Applying it to the Kauffman polynomial [116], one obtains the quantized BTCs of types BCD, cf. [222]. (The general theory in [222] is very satisfactory, but its application to the Kauffman polynomial used input from quantum group theory for the proof of functorial extension to tangles and of modularity. This defect was repaired by Beliakov/Blanchet, cf. [14, 15].)

Blanchet (2000): Similar construction with HOMFLY polynomial [74], obtaining the type A categories. (The HOMFLY polynomial is an invariant for oriented links, thus one must work with $O - \mathcal{TAN}$. Cf. [22].

Remark: The ribbon categories of BCD type arising from the Kauffman polynomial give rise to topological quantum field theories. The latter can even be constructed directly from the Kauffman bracket, bypassing the categories, cf. [23]. This construction actually preceded those mentioned above.

• The preceding constructions reinforce the close connection between braided categories and knot invariants. It is important to realize that this reasoning is not circular, since the polynomials of Jones, HOMFLY, Kauffman can (nowadays) be constructed in rather elementary ways, independently of categories and quantum groups! (Cf. e.g. [138].) Since the knot polynomials are defined in terms of skein relations, we speak of the skein construction of the quantum categories, which arguably is the simplest known so far (but not the most conceptual).

• The skein constructions of the ABCD categories also cover the case $q = 1$, where they reproduce the classical categories. (This is just classical invariant theory.) $q = 1$ corresponds to parameters in the knot polynomials for which they fail to distinguish over- from under-crossings. Then one can replace the tangle categories by categories of cobordisms without embedding into $\mathbb{R}^3$. Cf. Deligne [14] for an explicit description of the constructions. (Taking the parameter $t$ to be non-integer, this construction provides many more non-tannakian rigid symmetric categories.)

• Concerning the exceptional Lie algebras and their quantum categories, inspired by [228] Deligne conjectured [14] that there is a one parameter family of symmetric tensor categories $\mathcal{C}_t$ specializing to $\text{Rep } G$ for the exceptional Lie groups. This is still unproven, but see [57, 18, 47] for work resulting from this conjecture. (For the $E_n$-categories, including the $q$-deformed ones, cf. [251].)

In a similar vein, Deligne defined [14] a one parameter family of rigid symmetric tensor categories $\mathcal{C}_t$ such that $\mathcal{C}_t \simeq \text{Rep } S_t^n$ for $t \in \mathbb{N}$. (Recall that $S_n$ is considered as the $SL_n(\mathbb{F}_1)$ where $\mathbb{F}_1$ is the ‘field with one element’, cf. [207].)

• More generally: Define linear categories by generators and relations (Kuperberg spiders [133], etc.)

• Apart from the topological route (A), there are two major methods of obtaining non-trivially braided categories:
  (B) Quantum doubles / centers (“non-perturbative approach”).
  (C) Deformation (‘quantization’) of symmetric categories (“perturbative approach”).

We begin with route (B).

• **Quasi-triangular** Hopf algebras (Drinfeld 1986 [57]): If $H$ is a Hopf algebra, $R \in (H \otimes H)^*$ (possibly completed), satisfying

$$R \Delta(\cdot) R^{-1} = \sigma \circ \Delta(\cdot), \quad \sigma(x \otimes y) = y \otimes x,$$

$$\left(\Delta \otimes \text{id}\right)(R) = R_{13} R_{23}, \quad \left(\text{id} \otimes \Delta\right)(R) = R_{13} R_{12}.$$

If $(V, \pi), (V', \pi') \in H - \text{Mod}$, we define $c_{(V, \pi), (V', \pi')} = \Sigma_{V, V'}(\pi \otimes \pi')(R)$. This is a braiding for $H - \text{Mod}$.

• But this has only shifted the problem: how to get quasi-triangular Hopf algebras?? Drinfeld: quantum double $H \leadsto D(H)$ [57].

27
• Center construction \( Z_1 \) (Drinfeld (unpubl.), Joyal/Street \[105\], Majid \[150\]).

Let \( \mathcal{C} \) be a strict tensor category and let \( X \in \mathcal{C} \). A half braiding \( e_X \) for \( X \) is a family \( \{ e_X(Y) \in \text{Hom}_{\mathcal{C}}(X \otimes Y, Y \otimes X), \ Y \in \mathcal{C} \} \) of isomorphisms, natural w.r.t. \( Y \), satisfying \( e_X(1) = \text{id}_X \) and

\[
e_X(Y \otimes Z) = \text{id}_Y \otimes e_X(Z) \circ e_X(Y) \otimes \text{id}_Z \quad \forall Y,Z \in \mathcal{C}.
\]

Now, the center \( Z_1(\mathcal{C}) \) of \( \mathcal{C} \) has as objects pairs \( (X,e_X) \), where \( X \in \mathcal{C} \) and \( e_X \) is a half braiding for \( X \). The morphisms are given by

\[
\text{Hom}_{Z_1(\mathcal{C})}((X,e_X),(Y,e_Y)) = \{ t \in \text{Hom}_{\mathcal{C}}(X,Y) \mid \text{id}_X \otimes t \circ e_X(Z) = e_Y(Z) \circ t \otimes \text{id}_X \quad \forall Z \in \mathcal{C} \}.
\]

The tensor product of objects is given by \( (X,e_X) \otimes (Y,e_Y) = (X \otimes Y, e_X \otimes e_Y) \), where

\[
e_{X \otimes Y}(Z) = e_X(Z) \otimes \text{id}_Y \circ \text{id}_X \otimes e_Y(Z).
\]

The tensor unit is \( (1,e_1) \) where \( e_1(X) = \text{id}_X \). The composition and tensor product of morphisms are inherited from \( \mathcal{C} \). The braiding is given by

\[
c((X,e_X),(Y,e_Y)) = e_X(Y).
\]

(This definition is much more transparent than that of \( D(H) \).)

• Just as \( \mathcal{C} \cap \mathcal{D}' \) generalizes \( Z_2(\mathcal{C}) = \mathcal{C} \cap \mathcal{C}' \), there is a version of \( Z_1 \) relative to a subcategory \( \mathcal{D} \subset \mathcal{C} \). (Majid \[150\].)

• \( Z_1(\mathcal{C}) \) is categorical version (generalization) of Hopf algebra quantum double:

Thm: If \( H \) is a finite dimensional Hopf algebra, there is an equivalence

\[
Z_1(H-\text{Mod}) \simeq D(H)-\text{Mod}
\]

of braided tensor categories, cf. e.g. \[111\]. (If \( H \) is infinite dimensional, one still has an equivalence between \( Z_1(H-\text{Mod}) \) and the category of Yetter-Drinfeld modules over \( H \).)

• Abstract rationale for btc (A): A second, compatible, multiplication functor on a tensor category gives rise to a braiding, and conversely (Joyal&Street \[107\]). (This is a higher dimensional version of the Eckmann-Hilton argument.)

Abstract rationale for btc (B): Tensor categories \( \cong \) bicategories with one object. Braided tensor categ \( \cong \) monoidal bicategories with one object \( \cong \) weak 3-categories with one object and one 1-morphism.

• Baez-Dolan \[8\] periodic table (conjectural) of ‘\( k \)-tuply monoidal \( n \)-categories’:

| \( k = 0 \) | \( n = 0 \) | \( n = 1 \) | \( n = 2 \) | \( n = 3 \) | \( n = 4 \) |
|---|---|---|---|---|---|
| sets | categories | 2-categories | 3-categories | \ldots |
| \( k = 1 \) | monoids | monoidal categories | monoidal 2-categories | monoidal 3-categories | \ldots |
| \( k = 2 \) | commutative monoids | braided monoidal categories | braided monoidal 2-categories | braided monoidal 3-categories | \ldots |
| \( k = 3 \) | " | symmetric monoidal categories | "sylleptic" monoidal 2-categories | ? | \ldots |
| \( k = 4 \) | " | " | symmetric monoidal 2-categories | ? | \ldots |
| \( k = 5 \) | " | " | " | symmetric monoidal 3-categories | \ldots |
| \( k = 6 \) | " | " | " | " | \ldots |
In particular, one expects to find ‘center constructions’ from each structure in the table to the one underneath it. For the column $n = 1$ these are the centers $Z_0, Z_1, Z_2$ discussed above. For $n = 0$ they are given by the endomorphism monoid of a set and the ordinary center of a monoid. The column $n = 2$ is also relatively well understood, cf. Craw 35. There is an accepted notion of a non-strict 3-category (i.e. $n = 3, k = 0$) (Gordon/Power/Street 39), but there are many competing definitions of weak higher categories. We refrain from moving any further into this subject.

- **High-brow interpretation of $Z_1$ (JS):** Let $C$ be tensor category, $\Sigma C$ the corresponding bicategory with one object. Then the category $\text{End}(\Sigma C)$ of endofunctors of $\Sigma C$ is a monoidal bicategory (with natural transformations as 1-morphisms and ‘modifications’ as 2-morphisms).
  
  Now $D = \text{End}_{\text{End}(\Sigma C)}(1)$ is a tensor category with two compatible $\otimes$-structures (categorifying $\text{End} 1$ in a tensor category), thus braided, and it is equivalent to $Z_1(C)$.

- For further abstract considerations on the center cf. 208, 209 and the ongoing work by Bruguières/Virelizier 32.

If $C$ is braided: braided embedding $\iota_1 : C \hookrightarrow Z_1(C)$, $X \mapsto (X, e_X)$, where $e_X(Y) = c(X, Y)$. Defining $\tilde{C}$ to be the tensor category $C$ with braiding $\tilde{c}_{X,Y} = c_{Y,X}^{-1}$, there is an analogous embedding $\tilde{i} : C \hookrightarrow Z_1(\tilde{C})$. In fact:

$$Z_1(C) \cap \iota(C)' = \tilde{i} (\tilde{C}), \quad Z_1(C) \cap \tilde{i}(\tilde{C})' = \iota(C).$$

Cf. 162. On the one hand, this is an instance of the double commutant principle (more later)

$$\iota(C) \cap \tilde{i}(\tilde{C}) = \iota(Z_2(C)) = \tilde{i}(Z_2(\tilde{C})), $$

on the other hand, this establishes one connection between $Z_1$ and $Z_2$, and it suggests that

"$Z_1(C) \simeq C \times \tilde{C}$" when $Z_2(C)$ is “trivial”. A version of this will be given later.

- **Route (C):** Deformation approach to construction of braided tensor categories. We distinguish between the deformation of Hopf algebras related to groups and direct deformation of symmetric tensor categories.

- **(C1):** Deformation of Hopf algebras related to groups: Drinfeld (formal) vs. Jimbo (non-formal) ‘quantum groups’ $U_q(\mathfrak{g})$ and (Leningrad school, Soibelman,..., Woronowicz) ‘algebra of functions on a quantum group’.

“Definition” of $U_q(\mathfrak{g})$: Let $\mathfrak{g}$ be a simple complex Lie algebra, $U(\mathfrak{g})$ its universal enveloping algebra. Write $U(\mathfrak{g})$ in terms of appropriate generators and relations, then insert factors $q$ in suitable places. Now consider $q \neq 1$.

For $q \in \mathbb{C} \setminus \{0\}$ not a root of unity, the quantized universal enveloping algebra $U_q(\mathfrak{g})$ is generated by elements $E_i, F_i, K_i, K_i^{-1}, 1 \leq i \leq r$, satisfying the relations

$$K_i K_i^{-1} = K_i^{-1} K_i = 1, \quad K_i K_j = K_j K_i, \quad K_i E_j K_i^{-1} = q_i^{a_{ij}} E_j, \quad K_i F_j K_i^{-1} = q_i^{-a_{ij}} F_j, \quad E_i F_j - F_j E_i = \delta_{ij} K_i - \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}},$$

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \binom{1-a_{ij}}{k}_{q_i} E_i^k E_j E_i^{1-a_{ij}-k} = 0, \quad \sum_{k=0}^{1-a_{ij}} (-1)^k \binom{1-a_{ij}}{k}_{q_i} F_i^k F_j F_i^{1-a_{ij}-k} = 0,$$

where $\left[ \frac{m}{k} \right]_{q_i} = \frac{[m]_{q_i}!}{[k]_{q_i}! [m-k]_{q_i}!}, \quad [m]_{q_i} = [m]_{q_i} [m+1]_{q_i} \cdots [1]_{q_i}, \quad [n]_{q_i} = \frac{q_i^n - q_i^{-n}}{q_i - q_i^{-1}}$, and $q_i = q^{d_i}$.

This is a Hopf algebra with coproduct $\Delta$ and counit $\varepsilon$ defined by

$$\Delta(K_i) = K_i \otimes K_i, \quad \Delta(E_i) = E_i \otimes 1 + K_i \otimes E_i, \quad \Delta(F_i) = F_i \otimes K_i^{-1} + 1 \otimes F_i,$$
Three approaches:

Drinfeld [57]: Let $q = e^h$ and define $U_h(g)$ as a Hopf algebra over $\mathbb{C}[[h]]$ such that $U_h(g) \equiv U(g)(\text{mod } h)$. Cf. [57].

Jimbo (Lusztig, ...): Define $U_q(g)$ as Hopf algebra over $\mathbb{C}$ for every $q \in \mathbb{C}^*$.

Faddeev school, Woronowicz [232]: Dual construction starting from the algebra of continuous functions on a simply connected Lie group.

There is no time to say anything substantial on either $U_h(g)$ or $U_q(g)$. Cf. e.g. [111] [143] [36] [153] [102]. Here our point of view is that the representation categories are more fundamental than the Hopf algebras and we limit ourselves to mentioning alternative approaches to the construction of the categories.

• (C2): Formal deformation quantization of STCs (Cartier [35], Kassel/Turaev [111] Appendix).

Let $\mathcal{C}$ be a strict symmetric Ab-category. An infinitesimal braiding on $\mathcal{C}$ is a natural family of isomorphisms $t_{X,Y} : X \otimes Y \to X \otimes Y$ satisfying

$$c_{X,Y} \circ t_{X,Y} = t_{Y,X} \circ c_{X,Y} \quad \forall X,Y,$$

$$t_{X,Y \otimes Z} = t_{X,Y} \otimes \text{id}_Z + c_{X,Y}^{-1} \otimes \text{id}_Z \circ \text{id}_Y \otimes t_{X,Z} \circ c_{X,Y} \otimes \text{id}_Z \quad \forall X,Y,Z.$$

Strict symmetric Ab-categories equipped with an infinitesimal braiding might be called infinitesimally braided. (Originally: infinitesimal symmetric.)

Example: If $H$ is a Hopf algebra, there is a bijection between infinitesimal braidings $t$ on $S = H - \text{Mod}$ and elements $t \in \text{Prim}(H) \otimes \text{Prim}(H)$ satisfying $t_{21} = t$ and $[t, \Delta(H)] = 0$, given by $t_{X,Y} = (\pi_X \otimes \pi_Y)(t)$.

Now let $\mathcal{C}$ in addition be $\mathcal{C}$-linear with fin.dim. hom-sets and write $\mathcal{C}[[h]]$ for the $\mathbb{C}[[h]]$-linear category obtained by extension of scalars. For $X, Y, Z \in S$ define

$$\alpha_{X,Y,Z} = \Theta_{KZ}(h t_{X,Y} \otimes \text{id}_Z, h \text{id}_X \otimes t_{Y,Z}), \quad \bar{c}_{X,Y} = c_{X,Y} \circ e^{ht_{X,Y}/2}.$$

(Here $\Theta_{KZ}$ is the Drinfeld associator [58], a certain formal power series

$$\Theta_{KZ}(A, B) = \sum_{w \in \{A,B\}^*} c_w w$$

in two non-commuting variables $A, B$, where $c_w \in \mathbb{C}$.) Then $(S[[h]], \otimes, 1, \alpha)$ is a (non-strict) tensor category (with trivial unit constraints) and $\bar{c}$ a braiding. If $\mathcal{C}$ is rigid, then $(S[[h]], \otimes, 1, \alpha, \bar{c})$ admits a ribbon structure.

Application: Let $g$ be a simple Lie algebra/$\mathbb{C}$. Let $S = g - \text{Mod}$ and define $\{t_{X,Y}\}$ be as in the example, corresponding to $t = (\sum x_i \otimes x_i + x_i \otimes x_i)/2$, where $x_i, x_i$ are dual bases of $g$ w.r.t. the Killing form. Then $[t, \Delta(\cdot)] = 0$ and

$$(S[[h]], \otimes, 1, \alpha, \bar{c}) \simeq U_h(g) - \text{Mod} \quad (3)$$

as $\mathbb{C}[[h]]$-linear ribbon categories. (The proof is a corollary of the proof of the Kohno-Drinfeld theorem [28] [59].)

Remark: 1. Obviously, we have cheated: The main difficulty resides in the definition of $\Theta_{KZ}$!

(Giving the latter and proving its properties requires ca. 10-15 pages of technical ODE/PDE stuff, but no Lie theory. Drinfeld also proved the existence of an associator over $\mathbb{Q}$, cf. [59].)

2. The above is relevant for a more conceptual approach to the theory of finite-type knot invariants (Vassiliev invariants), cf. [35] [114].
3. Disadvantage: We get only a formal deformation of $\mathcal{C}$. By [3], in the special case of $\mathcal{C} = \mathfrak{g} - \text{Mod}$ and the given $t$, we obtain the $\mathbb{C}[\hbar]$-category $U_\hbar(\mathfrak{g}) - \text{Mod}$. We know (Jimbo, Lusztig, ...) that there is a non-formal version $U_q(\mathfrak{g})$ of the quantum group with $\mathbb{C}$-linear representation category. In fact, for numerical $q \in \mathbb{C} \setminus \mathbb{Q}$, with some more analytical effort one can make sense of $\Theta_{KZ}(h t X, Y \otimes \text{id}_Z, h \text{id}_X \otimes t_Y Z)$ as an element of $\text{End}(X \otimes Y \otimes Z)$ and define a non-formal, $\mathbb{C}$-linear version of the category $\mathcal{C}(\mathfrak{g}, q)$ (Kazhdan, Lusztig [119]) and an equivalence

$$\mathcal{C}(\mathfrak{g}, q) = (S, \otimes, 1, \alpha, \beta) \simeq U_q(\mathfrak{g}) - \text{Mod}$$

of $\mathbb{C}$-linear ribbon categories. (Nice recent exposition by Neshveyev, Tuset [174].)

Question: Is there a non-formal version of the general quantization of an infinitesimally braided category, perhaps with some technical conditions?

- Fact: If $q \in \mathbb{C}^*$ generic, i.e. not a root of unity, then $\mathcal{C}(\mathfrak{g}, q) := U_q(\mathfrak{g}) - \text{Mod}$ is a semisimple braided ribbon category whose fusion hypergroup is isomorphic to that of $U_\hbar(\mathfrak{g})$, thus of the category of $\mathfrak{g}$-modules! But not symmetric for $q \neq 1$, thus certainly not equivalent to the latter. In fact, $U_q(\mathfrak{g}) - \text{Mod}$ and $U_\hbar(\mathfrak{g}) - \text{Mod}$ are already inequivalent as $\otimes$-categories, thus they have inequivalent associativity constraints. (Lusztig 1988).

- Tuba/Wenzl [215]: A semisimple ribbon category with the fusion hypergroup isomorphic to that of a classical group of BCD type is equivalent to the category $\mathcal{C}(\mathfrak{g}, q)$, with $q = 1$ or not a root of unity, or one of finitely many twisted versions thereof. Notice that in contrast to the Kazhdan/Wenzl result [119], this needs the category to be braided! (Again, this is a characterization, not a construction of the categories.)

- Finkelberg [21]: Braided equivalence between $\mathcal{C}(\mathfrak{g}, q)$, $q = e^{i\pi/m\kappa}$ where $m = 1 : ADE, m = 2 : BCD, m = 3 : G_2$, and the ribbon category $\mathcal{O}_\kappa$ of integrable representations of the affine Lie algebra $\hat{\mathfrak{g}}$ of central charge $c = \kappa - \hbar$, where $\hbar$ is the dual Coxeter number of $\mathfrak{g}$.

The category $\mathcal{O}_\kappa$ plays an important rôle in conformal field theory, either directly (VOAs) or via representations of loop groups (Wassermann [229], Toledano-Laredo [214]). This is the main reason for the relevance of quantum groups to CFT. (Of course, historically things didn’t go this way.)

- Connection between route (B) and (C) to BTCs: In order to find an $R$-matrix for the Hopf algebra $U_q(\mathfrak{g})$ one traditionally uses the quantum double, appealing to $U_q(\mathfrak{g}) \cong D(B_q(\mathfrak{g}))/I$, where $B_q(\mathfrak{g})$ is the $q$-deformation of a Borel subalgebra of $\mathfrak{g}$ and $I$ an ideal in $D(B_q(\mathfrak{g}))$. Now $R_{U_q(\mathfrak{g})} = (\phi \otimes \phi)(R_{D(B_q(\mathfrak{g}))})$, where $\phi$ is the quotient map. Since a surjective Hopf algebra homomorphism $H_1 \rightarrow H_2$ corresponds to a full monoidal inclusion $H_2 - \text{Mod} \hookrightarrow H_1 - \text{Mod}$, we conclude that the BTC $U_q(\mathfrak{g}) - \text{Mod}$ is a full $\otimes$-subcategory of $Z_1(B_q(\mathfrak{g}) - \text{Mod})$ (with the inherited braiding). Thus: Also in the deformation approach, the braiding can be understood as arising from the $Z_1$ center construction.

Question: Does a similar observation also hold for $q$ a root of unity? I.e., can the modular categories $\mathcal{C}(\mathfrak{g}, q)$, for $q$ a root of unity, be understood as full $\otimes$-subcategories of $Z_1(D)$, where $D$ is a fusion category corresponding to the deformed Borel subalgebra $B_q(\mathfrak{g})$? Cf. the section on modular categories.

- We have briefly discussed the formal deformation quantization of symmetric categories equipped with an infinitesimal braiding (Cartier/Kassel/Turaev). There is a cohomology theory for Ab-tensor categories and tensor functors that classifies deformations (Davydov [32], Yetter [242]).

Definition: Let $F : \mathcal{C} \rightarrow \mathcal{C}'$ a tensor functor. Define $T_n : \mathcal{C}^n \rightarrow \mathcal{C}$ by $X_1 \times \cdots \times X_n \rightarrow X_1 \otimes \cdots \otimes X_n$. $(T_0(\emptyset) = 1, T_1 = \text{id})$. Let $C^{n}_{\mathfrak{c}}(\mathcal{C}) = \text{End}(T_n \circ F^{\otimes n})$. $(C^{0}_{\mathfrak{c}}(\mathcal{C}) = \text{End} 1')$. For a fusion category, this is finite dimensional. Define $d : C^{n}_{\mathfrak{c}}(\mathcal{C}) \rightarrow C^{n+1}_{\mathfrak{c}}(\mathcal{C})$ by

$$df = \text{id} \otimes f_{2,\ldots,n+1} - f_{1,\ldots,n+1} + f_{1,2,\ldots,n+1} - \cdots + (-1)^n f_{1,\ldots,n,n+1} + (-1)^{n+1} f_{1,\ldots,n} \otimes \text{id},$$

where, e.g., $f_{1,2,\ldots,n+1}$ is defined in terms of $f$ using the isomorphism $d^{k}_{X_1,X_2} : F(X_1) \otimes F(X_2) \rightarrow F(X_1 \otimes F_2)$ coming with the tensor functor $F$. 31
One has \( d^2 = 0 \), thus \((C^i, d)\) is a complex. Now \( H_F^k(C)\) is the cohomology of this complex, and \( H^1_C = H^1_F(C)\) for \( F = \text{id}_C\).

Low dimensions: \( H^1_C \) classifies derivations of the tensor functor \( F \), \( H^2_C \) classifies deformations of the tensor structure \( \{d^F_{X,Y}\} \) of \( F \). \( H^3(C)\) classifies deformations of the associativity constraint \( \alpha \) of \( C\).

Examples: 1. If \( C\) is fusion then \( H^i(C) = 0 \) \( \forall i > 0 \). This implies Ocneanu rigidity. \[67\].
2. If \( g\) is a reductive algebraic group with Lie algebra \( g\) and \( C = \text{Rep}_G \) (algebraic representations). Then \( H^i(C) \cong (N^g)^C \) \( \forall i \). If \( g\) is simple then \( H^1(C) = H^2(C) = 0\), but \( H^3(C)\) is one-dimensional, corresponding to a one-parameter family of deformations \( C\), namely the categories \( U_q(g) - \text{Mod}\). Cf. \[67\].

5 Modular categories

- Turaev \[217, 218\]: A modular category is a fusion category that is ribbon (alternatively, spherical and braided) such that the matrix \( S = (S_{i,j}) \)

\[
S_{i,j} = \text{Tr}_{X \otimes Y}(c_{Y,X} \circ c_{X,Y}), \quad i,j \in I(C)
\]

is invertible.

- A fusion category that is ribbon (alternatively, spherical and braided) is modular iff \( \dim C \neq 0 \) and the center \( Z_2(C)\) is trivial. (In the sense of consisting only of objects \( 1 \oplus \cdots \oplus 1\).) (Rehren \[195\] for \(*\)-categories, Beliakova/Blanchet \[15\] in general.)

Thus: Modular categories are braided fusion categories with trivial center, i.e. the maximally non-symmetric ones. (Better than the original definition.)

- Why ‘modular’? Let \( S\) as above and \( T = \text{diag}(\omega_i)\), where \( \Theta_{X_i} = \omega_i \text{id}_{X_i}, \ i \in I \). Then

\[
S^2 = \alpha C, \quad (ST)^3 = \beta C, \quad (\alpha \beta \neq 0)
\]

where \( C_{i,j} = \delta_i \gamma_j\), thus \( S,T\) give rise to a projective representation of the modular group \( SL(2,\mathbb{Z})\) (which has a presentation \( \{s, t \mid (st)^3 = s^2 = c, \ c^2 = e\} \)). Cf. \[195, 218\].

- This is somewhat mysterious. Notice: \( SL(2,\mathbb{Z})\) is the mapping class group of the 2-torus \( S^1 \times S^1\). Now every modular category gives rise to a topological quantum field theory in 2 + 1 dimensions. (Reshetikhin/Turaev, rigorous version of ideas of Witten.) Every such TQFT gives rise to projective representations of the mapping class groups of all closed surfaces, and for the torus this is just the above representation of \( SL(2,\mathbb{Z})\). Cf. \[218, 9\]. We don’t have the time to say more about TQFTs.

- Turaev’s motivation came from conformal field theory (CFT). (Moore-Seiberg \[158\]). In fact:

- There is a (rigorous) definition of rational chiral CFTs (using von Neumann algebras) and their representations, for which one can prove that the latter are unitary modular (Kawahigashi, Longo, MM \[117\]). Most of the examples considered in the (heuristic) physics literature fit into this scheme. (Loop group models: Wassermann \[229\], F. Xu \[235\], minimal Virasoro models with \( c < 1 \) \[Loke\].)

Similar result for vertex operator algebras (Huang \[101\]).

- Less complicated ways to produce modular categories?

- Route (A): Recall that the ABCD categories at roots of unity can be obtained from the linearized tangle categories (A: oriented, BCD: unoriented), dividing by ideals defined in terms of the knot polynomials of HOMFLY and Kaufmann. (Turaev/Wenzl \[222\], Beliakova/Blanchet \[22, 15\])
• Route (C₁): H. Andersen et al., Turaev/Wenzl [221] (and others): Let \( g \) be a simple Lie algebra, \( q \) a primitive root of unity. Then \( U_q(g) \) gives rise to a modular category \( C(g,q) \).

(Using tilting modules, dividing by negligible morphisms, etc.)

Remark: In a \( k \)-linear category with direct sums and splitting idempotents, absence of non-zero negligible morphisms is equivalent to semisimplicity: A finite dimensional algebra (over an algebraically closed field) is semisimple iff it admits a non-degenerate trace. Thus dividing by the ideal of negligible morphisms, one obtains a semisimple category.

• Let \( q \) be primitive root of unity of order \( \ell \). Then \( C(g,q) \) has a positive \(*\)-operation (i.e. is unitary) if \( \ell \) is even (Kirillov Jr. [120], Wenzl [230]) and is not unitarizable for odd \( \ell \) (Rowell [198]).

• Characterization theorem: A braided fusion category with the fusion hypergroup of \( C(g,q) \), where \( g \) is a simple Lie algebra of BCD type and \( q \) a root of unity, is equivalent to \( C(g,q) \) or one of finitely many twisted versions. (Tuba/Wenzl [215]).

• Before we reconsider Route (B), we assume that we already have a braided fusion category, or pre-modular category.

Failure of modularity is due to non-trivial center \( Z_2(C) \). Idea: Given a braided (but not symmetric) category with even center \( Z_2(C) \), kill the latter, using the Deligne/ Doplicher/ Roberts theorem: \( Z_2(C) \cong \text{Rep } G \). The latter contains a commutative (Frobenius) algebra \( \Gamma \) corresponding to the regular representation of \( G \). Now \( \Gamma \) is modular. (Bruguieres [39], MM [159]). Interpretation in terms of Galois theory for BTCs and Galois closure [159].

• Route (B): Quantum doubles: \( G \) finite group \( \Rightarrow D(G) \) – Mod and \( D^\omega(G) \) – Mod modular (Altschuler/ Coste [2]). \( H \) fin.dim. semisimple \& cosemisimple Hopf algebra \( \Rightarrow D(H) \) – Mod modular (Etingof/Gelaki [63]). A fin.dim. weak Hopf algebra \( \Rightarrow D(A) \) – Mod modular (Nikshych/ Turaev/ Vainerman [178]).

• The center \( Z_1 \) of a left/right rigid, pivotal, spherical category has the same properties. In particular, the center of a spherical category is a ribbon category. Under weaker assumptions, this is not true, and existence of a twist for the center, if desired, must be enforced by a categorical version of the ribbonization of a Hopf algebra, cf. [173].

• \( Z_1: C \) spherical fusion category, \( \dim C \neq 0 \Rightarrow Z_1(C) \) is modular and \( \dim Z_1(C) = (\dim C)^2 \).

(MM [102].)

Comments: Semisimplicity not difficult. Then one finds a Frobenius algebra \( \Gamma \) in \( D = C \boxtimes C^\text{op} \) such that the dual category \( \Gamma \cong \text{Mod}_D \) – \( \Gamma \) is equivalent to \( Z_1(C) \), implying \( \dim Z_2(C) = (\dim C)^2 \). Notice: \( \Gamma = \oplus_i X_i \boxtimes X_i^\text{op} \), which is again a coend and can exist also in non-semisimple categories.

• This contains all the earlier modularity results on \( D(G) \) – Mod and \( D(H) \) – Mod, but also for \( D^\omega(G) \) – Mod since:

\[
D^\omega(G) \cong Z_1(C_k(G,\omega)).
\]

(Using work by Hausser/Nill or Panaite [189] on quantum double of quasi Hopf-algebras.)

• Modularity of \( Z_1(C) \) also follows by combination of Ostrik’s result that every fusion category arises from a weak Hopf algebra \( A \), combined with modularity of \( D(A) \) – Mod [173], provided one proves \( D(A) \) – Mod \( \cong Z_1(A – \text{Mod}) \), generalizing the known result for Hopf algebras.

But: Purely categorical proof avoiding weak Hopf algebras seems preferable.

• In the Morita context having \( C \boxtimes C^\text{op} \) and \( Z_1(C) \) as its corners, the two off-diagonal categories are equivalent to \( C \) and \( C^\text{op} \), and their structures as \( C \boxtimes C^\text{op} \)-module categories are the obvious ones. Therefore, the center can also be understood as (using the notation of EO):

\[
Z_1(C) \cong (C \boxtimes C^\text{op})_C^\omega.
\]

A (somewhat sketchy) proof of this equivalence can be found in [186] Prop. 2.5].
- Other example for a purely categorical result, proven using weak Hopf algebras: Radford’s formula for $S^4$ has a generalization to weak Hopf algebras [177], and this can be used to prove that in every fusion category, there exists an isomorphism of tensor functors $id \to \ast \ast \ast$, cf. [68]. (NB: In every pivotal category we have $id \cong \ast \ast$, thus here it is important that we understand ‘fusion’ just to mean existence of two-sided duals. But: ENO conjecture that every fusion category has a pivotal structure.)

- $\mathcal{C}$ modular $\Rightarrow$ $Z_1(\mathcal{C}) \simeq \mathcal{C}\mathcal{C}^{\text{op}}$ [162] Thus: every modular categ $\mathcal{M}$ is full subcategory of $Z_1(\mathcal{C})$ for some fusion categ. (Probably not useful for classification of modular categories, since there are ‘more fusion categories than modular categories’: $C_1 \approx C_2 \Rightarrow Z_1(C_1) \simeq Z_1(C_2)$. (For converse, see below.)

- “Double commutant theorem” for modular catories (MM [163], inspired by Ocneanu [182]):

  1. $(\mathcal{M} \cap (\mathcal{M} \cap \mathcal{C}'')) = \mathcal{C}$.
  2. $\dim \mathcal{C} \cdot \dim (\mathcal{M} \cap \mathcal{C}') = \dim \mathcal{M}$.
  3. If, in addition $\mathcal{C}$ is modular, then also $\mathcal{D} = \mathcal{M} \cap \mathcal{C}'$ is modular and $\mathcal{M} \simeq \mathcal{C} \boxtimes \mathcal{D}$.
  4. Thus: Every modular category is direct product of prime ones, having no proper modular subcategories. (Non-modular subcategories can’t be ruled out.)

Moral: Modular categories behave nicer than finite groups: “All short exact sequences of modular categories split”, thus there is no problem of classifying extensions.

- Corollary: Let $\mathcal{M}$ be modular and $\mathcal{S} \subset \mathcal{M}$ symmetric. Then $\mathcal{S} \subset \mathcal{M} \cap \mathcal{S}'$. Thus

$$\left(\dim \mathcal{S}\right)^2 \leq \dim \mathcal{S} \cdot \dim (\mathcal{M} \cap \mathcal{S}') = \dim \mathcal{M},$$

thus $\dim \mathcal{S} \leq \sqrt{\dim \mathcal{M}}$. Notice that the bound is satisfied by $\text{Rep} G \subset D^{(\omega)}(G) - \text{Mod}$. In fact, attainment of this bound characterizes the representation categories of twisted doubles, cf. below.

- On the other hand, consider $\mathcal{C} \subset \mathcal{M}$ with $\mathcal{M}$ modular. We have $\mathcal{M} \cap \mathcal{C}' \supset Z_2(\mathcal{C})$, implying $\dim \mathcal{M} \geq \dim \mathcal{C} \cdot \dim Z_2(\mathcal{C})$. This provides a lower bound on the dimension of a modular category containing a given pre-modular subcategory as a full tensor subcategory.

Conjecture [163]: This bound can always be attained. (Work in progress.)

- So what about the quantum doubles of the finite simple groups:

  $G = \mathbb{Z}/p\mathbb{Z}$: $p = 2$: $D(G) - \text{Mod}$ is prime, $p$ odd prime: $D(G) - \text{Mod}$ has two prime factors, both of which are modular categories with $p$ invertible objects. [163]

  $G$ finite simple non-abelian: $D(G) - \text{Mod}$ is prime. In fact: it has only one replete full tensor subcateg, namely $\text{Rep} G$. Thus all these categories are mutually inequivalent: The classification of prime modular categories contains that of finite simple groups.

- Fact: If $\mathcal{C}$ is symmetric and $(\Gamma, m, \eta)$ a commutative algebra, then $\Gamma - \text{Mod}_\mathcal{C}$ is again symmetric and

$$\dim \Gamma - \text{Mod}_\mathcal{C} = \frac{\dim \mathcal{C}}{d(\Gamma)}.$$ (4)

If $\mathcal{C}$ is only braided, $\Gamma - \text{Mod}_\mathcal{C}$ is a fusion category satisfying [1], but in general it fails to be braided! (Unless $\Gamma \in Z_2(\mathcal{C})$, as was the case in the center of modularization.)

- Example: Given BTC $\mathcal{C} \supset \mathcal{S} \simeq \text{Rep} G$. Let $\Gamma$ be the regular monoid in $\mathcal{S}$. Then $\mathcal{C} \rtimes \mathcal{S} := \Gamma - \text{Mod}_\mathcal{C}$ is fusion category, but it is braided only if $\mathcal{S} \subset Z_2(\mathcal{C})$, as in the discussion of modularization. In general, one obtains a braided crossed $G$-category (Turaev [219], Carrasco/Moreno [33]), i.e. a tensor category with $G$-grading $\partial$ on the objects, a $G$-action $\gamma$ such that $\partial(\gamma_g(X)) = g\partial X g^{-1}$ and a ‘braiding’ $\epsilon_{X,Y} : X \otimes Y \xrightarrow{\cong} \gamma_{\partial X}(Y) \otimes X$. The degree zero part is $\Gamma - \text{Mod}_{\mathcal{C},\mathcal{S}'} \simeq \Gamma - \text{Mod}_\mathcal{C}^{(0)}$ (cf. below). (Kirillov Jr. [127] [128], MM [164])

Connection to conformal orbifold models (MM [160] [166]).
• But: There is a full tensor subcategory $\Gamma -\text{Mod}_C^0 \subset \Gamma -\text{Mod}_C$ that is braided: A module $(X, \mu) \in \Gamma -\text{Mod}_C$ is called dyslexic if

$$\mu \circ c_{X, \Gamma} = \mu \circ c_{\Gamma, X}^{-1}.$$ 

The full subcategory $\Gamma -\text{Mod}_C^0$ of dyslexic modules is monoidal, and it inherits a braiding from $C$. (Pareigis [190], Kirillov/Ostrik [129]). KO also proved:

$$\dim \Gamma -\text{Mod}_C^0 \cong \dim C,$$

Remark: Analogous results were previously obtained by Böckenhauer/Evans/Kawahigashi [25] in an operator algebraic context, whose generalization to tensor $\ast$-categories is immediate. But removing the $\ast$-assumption requires some work.

• The above implies (for $\ast$-categories, but also in general over $\C$ by ENO):

$$d(\Gamma) \leq \sqrt{\dim C},$$

for commutative Frobenius algebras in modular categories. (The above bound on $\dim S$ follows from this, since $S$ contains a commutative Frobenius algebra $\Gamma$ with $d(\Gamma) = \dim S$, the regular representation.)

• All these facts have applications to chiral conformal field theories:

Longo/Rehren [141]: Finite local extensions of a CFT $A$ are classified by the ‘local Q-systems’ in $\text{Rep} A$, which is a $\ast$-BTC.

Böckenhauer/Evans [24], MM (unpubl.): If $B \supset A$ is the finite local extension corresponding to the commutative Frobenius algebra $\Gamma \in \text{Rep} A$, then $\text{Rep} B \cong \Gamma -\text{Mod}_{\text{Rep}_A}^0$.

Analogous results for VOAs: Kirillov/Ostrik [129].

It is perhaps not completely absurd to compare these results to local class field theory, where finite Galois extensions of a local field $k$ are shown to be in bijection to finite index subgroups of $k^\ast$. (Over $\C$, the condition is equivalent to $d(\Gamma)^2 = \dim M$.)

• Characterization of centers of fusion categories (Drinfeld/Gelaki/Nikshych/Ostrik [60], Kitaev+MM):

Every commutative Frobenius algebra $\Gamma$ in a modular category $\mathcal{M}$ gives rise to an equivalence

$$\mathcal{M} \cong Z_1(\Gamma -\text{Mod}_\mathcal{M}) \cong \Gamma -\text{Mod}_C^0.$$ 

Thus if $\Gamma -\text{Mod}_\mathcal{M}$ is trivial then $\mathcal{M} \cong Z_1(\Gamma -\text{Mod}_\mathcal{M})$, i.e. it is the center of a fusion category. (Over $\C$, the condition is equivalent to $d(\Gamma)^2 = \dim \mathcal{M}$.)

In a sense, this is an answer to the question raised earlier, as to whether a modular category $\mathcal{M}$ can be considered as a full subcategory (and therefore direct factor) of the center of a not-too-big fusion category $\mathcal{C}$. The latter can be made the smaller the bigger one can find a commutative algebra in $\mathcal{M}$.

• Application: Let $\mathcal{M}$ be modular and $\mathcal{S} \subset \mathcal{M}$ even symmetric such that $\dim \mathcal{S} = \sqrt{\dim \mathcal{M}}$. Then $\mathcal{M} \cong D^\omega(G) -\text{Mod}$, where $\mathcal{S} \cong \text{Rep}_\omega G$ and $\omega \in Z^3(G, \mathbb{T})$.

Application in CFT: If $A$ is a chiral CFT with trivial $\text{Rep} A$, acted upon by finite group $G$. Then $\text{Rep} A^G \cong D^\omega(G) -\text{Mod}$. (Together with the results of [117], this proves the folk conjecture, having its roots in [52, 51], that the representation category of a ‘holomorphic chiral orbifold CFT’ is given by a category $D^\omega(G) -\text{Mod}$.)

• Converse (MM/Kitaev, unpubl.): If $\mathcal{C}$ is fusion then $Z_1(\mathcal{C})$ contains a commutative Frobenius algebra $\Gamma$ such that

$$\Gamma -\text{Mod}_{Z_1(\mathcal{C})}^0 \text{ trivial,} \quad \Gamma -\text{Mod}_{Z_1(\mathcal{C})} \cong \mathcal{C}.$$ 

• $\mathcal{C}_1 \cong \mathcal{C}_1 \to Z_1(\mathcal{C}_1) \cong Z_1(\mathcal{C}_2)$. (MM, immediate corollary of definition of $\cong$ and [202].)

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• The converse is true for group theoretical categories (Naidu/Nikshych [171]) and a general proof is announced by Nikshych.

• By definition, a group theoretical category $\mathcal{C}$ is weakly Morita equivalent (dual) to $\mathcal{C}_k(G,\omega)$ for a finite group $G$ and $[\omega] \in H^3(G, T)$. Thus $Z_1(\mathcal{C}) \simeq Z_1(\mathcal{C}_k(G,\omega)) \simeq D^2(G) - \text{Mod}$. The converse is also true. Therefore, (with $\mathcal{M}$ modular, $\mathcal{C}$ fusion) we have:

| contains | $\mathcal{M}$ | $Z_1(\mathcal{C})$ |
|----------|---------------|-------------------|
| maximal STC $S$ | $\mathcal{M} \simeq D^2(G) - \text{Mod}$ | $\mathcal{C}$ is group theoretical |
| maximal cFA $\Gamma$ | $\mathcal{M} \simeq Z_1(\mathcal{C})$ | always true |

• What about non-commutative (Frobenius) algebras in modular categories?

Let $\mathcal{C}$ be a rigid symmetric $k$-linear tensor category and $\Gamma$ a strongly separable Frobenius algebra in $\mathcal{C}$. Define $p \in \text{End} \Gamma$ by

$$p = \begin{array}{c}
\Gamma \\
\downarrow \\
\Gamma \\
\downarrow \\
\Gamma \\
\end{array}$$

Then $p$ is idempotent and its kernel is an ideal, thus its image is a commutative Frobenius subalgebra of $\Gamma$. The latter is called the center of $\Gamma$ since it is the ordinary center in the case $\mathcal{C} = \text{Vect}^\text{fin}_k$.

[Application to TQFT: Every finite dimensional semisimple $k$-algebra $A$ gives rise to a TQFT in $1+1$ dimensions via triangulation (Fukuma/Hosono/Kawai [81]). By the classification of TQFTs in $1+1$ dimensions [50, 1, 130], this TQFT corresponds to a commutative Frobenius algebra $B$ (in $\text{Vect}^\text{fin}_k$), with $A = V(S^1)$ and the product arising from the pants cobordism. The latter is given by the vector space associated with the circle and the multiplication is given by the pants cobordism. One finds $B = Z(A)$, and $B$ arises exactly as the image of $A$ under the above projection $p$. (This works since every semisimple algebra is a Frobenius algebra.)]

• If $\mathcal{C}$ is braided, but not symmetric, we must choose between $c_{\Gamma,\Gamma}$ and $c_{\Gamma,\Gamma}^{-1}$ in the definition of the idempotent $p$. This implies that a non-commutative Frobenius algebra will typically have two different centers, called the left and right centers $\Gamma_l, \Gamma_r$. Remarkably, one also obtains an equivalence

$$E : \Gamma_l - \text{Mod}_c^0 \simeq \Gamma_r - \text{Mod}_c^0$$

of modular categories (Böckenhauer/Evans/Kawahigashi [25], Ostrik [185], Fröhlich/ Fuchs/ Runkel/ Schweigert [80, 77]).

This is relevant for the classification of CFTs in two dimensions: The latter are constructed from a pair of chiral CFTs $+$ some piece of data specifying how the two chiral CFTs are glued together (‘modular invariant’). It is now believed that triples $(\Gamma_l, \Gamma_r, E)$ as above are the right datum. Cf. the topological construction of “topological 2d-CFT” by Fuchs/ Runkel/ Schweigert, cf. [79] and sequels.

Thus: We should classify also non-commutative Frobenius algebras in modular categories, or rather their Morita classes, i.e. the module categories to which they give rise.
• Classification of Frobenius algebras in module categories of $SU_q(2) - \text{Mod}$ in terms of ADE graphs. (Quantum MacKay correspondence.) (Böckenhauer/Evans [24], Kirillov Jr./Ostrik [129], Etingof/Ostrik [69]).

• Extension to other Lie groups? If $SU(2)$ already leads to the ubiquitous ADE graphs, cf. [97], the other classical groups should give rise to very interesting algebraic/combinatorial structures, cf. [183, 184].

• Three more reasons why modular categories are interesting:
  1. Many connections with number theory:
     - Rehren [195], Turaev [218]: $\sum_i d_i^2 = |\sum_i d_i^2 \omega_i|^2$. Pointed case: $|\sum_i \omega_i| = \pm \sqrt{|I|}$. Generalizes Gauss’ sums. (Gauss actually also computed the sign...)
     - Elements of $T$ matrix are roots of unity, elements of $S$ are cyclotomic integers [27, 62].
     - Related integrality properties (Masbaum, Roberts, Wenzl [155, 156], Bruguières [29]) in TQFTs.
     - Congruence subgroup conjecture: Let $N = \text{ord} T(< \infty)$. Then
       $$\ker(\pi : SL(2, \mathbb{Z}) \rightarrow GL(|I|, \mathbb{C})) \supset \Gamma(N) = \ker(SL(2, \mathbb{Z}) \rightarrow SL(2, \mathbb{Z}/N\mathbb{Z})).$$
       Proof announced by Ng/Schauenburg, using a categorical version [176] of the higher Frobenius-Schur indicators for Hopf algebras defined in [110] and adapting the strategy of proof in [206].

  2. A modular category $\mathcal{M}$ gives rise to a surgery TQFT in $2 + 1$ dimensions (Reshetikhin/Turaev [197, 218]). Conjecturally, if $\mathcal{M} = Z_1(\mathcal{C})$ with $\mathcal{C}$ spherical fusion, then there is an isomorphism $RT_{\mathcal{M}} = BWGK_{\mathcal{C}}$ of TQFTs, where BWGK denotes the triangulation TQFT [11, 86]. (Even if this is true, the surgery construction provides more TQFTs than the triangulation approach, since not all modular categories are centers.)

  3. ‘Application’ to topological quantum computing [73]. M. Freedman: Use TQFT, A. Kitaev: Use $d = 2$ quantum spin systems. In both proposals, the modular representation categories are central. Cf. also Z. Wang, E. Rowell et al. [100, 199].

6 Some open problems

1. Classify all prime modular categories. (The next challenge after the classification of finite simple groups...)

2. Give a direct construction of the fusion categories associated with the two Haagerup subfactors [80, 5].

3. Prove that every braided fusion category $\mathcal{C}/\mathcal{C}$ embeds fully into a modular category $\mathcal{M}$ with $\dim \mathcal{M} = \dim \mathcal{C} \cdot \dim Z_2(\mathcal{C})$. (This is the optimum allowed by the double commutant theorem, cf. [163].)

4. Find the most general context in which an analytic (i.e. non-formal) version of the Cartier/Kassel/Turaev [11, 114] formal deformation quantization of a symmetric tensor category $\mathcal{S}$ with infinitesimal braiding can be given. (I.e. give an abstract version of the Kazhdan/Lusztig construction of Drinfeld’s category [113] that does not suppose $\mathcal{S} = \text{Rep}(G)$.)

5. Let $\mathcal{C}$ be a (semisimple) fusion category, thus $Z_1(\mathcal{C})$ modular. Prove the isomorphism of the surgery TQFT $RT_{Z_1(\mathcal{C})}$ with the non-braided version [11, 86] of the Turaev-Viro state-sum TQFT.

6. Generalize the proof of modularity of $Z_1(\mathcal{C})$ for (semisimple) fusion categories to the finite categories. (Using Lyubashenko’s definition [144] of modularity.)
7. Likewise for the triangulation TQFT. Generalize the relation to surgery TQFT to the non-semisimple case. (For the non-semisimple version of the RT-TQFT cf. [125].)

8. Hard non-commutative analysis: Every countable $C^*$-tensor category with conjugates and $\text{End} 1 = \mathbb{C}$ embeds fully into the $C^*$-tensor category of bimodules over $L(F_\infty)$ and, for any infinite factor $M$, into $\text{End}(L(F_\infty) \otimes M)$. Here $F_\infty$ is the free group with countably many generators and $L(F_\infty)$ the type $II_1$ factor associated to its left regular representation. (This would extend and conceptualize the results of Popa/Shlyakhtenko [192] on the universality of the factor $L(F_\infty)$ in subfactor theory.)

9. Find satisfactory categorical interpretations of dynamical quantum groups and Toledano-Laredo’s quasi-Coxeter algebras.

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Disclaimer: While the following bibliography is quite extensive, it should be clear that it has no pretense whatsoever at completeness. Therefore the absence of this or that reference should not be construed as a judgment of its relevance. The choice of references was guided by the principal thrust of these lectures, namely linear categories. This means that the subjects of quantum groups and low dimensional topology, but also general categorical algebra are touched upon only tangentially.

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