On initial direction, orientation and discreteness in the analysis of circular variables

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Abstract

In this paper, we propose a discrete circular distribution obtained by extending the wrapped Poisson distribution. This new distribution, the Invariant Wrapped Poisson (IWP), enjoys numerous advantages: simple tractable density, parameter-parsimony and interpretability, good circular dependence structure and easy random number generation thanks to known marginal/conditional distributions. Existing discrete circular distributions strongly depend on the initial direction and orientation, i.e. a change of the reference system on the circle may lead to misleading inferential results. We investigate the invariance properties, i.e. invariance under change of initial direction and of the reference system orientation, for several continuous and discrete distributions. We prove that the introduced IWP distribution satisfies these two crucial properties. We estimate parameters in a Bayesian framework and provide all computational details to implement the algorithm. Inferential issues related to the invariance properties are discussed through numerical examples on artificial and real data.

Keywords: Circular data, Bayesian inference, Wrapped Poisson distribution
1 Introduction

Circular data (for a review see e.g. [Lee, 2010]) arise naturally in many scientific fields where observations are recorded as directions or angles. Examples of circular data include wind and wave directions ([Bulla et al., 2012; Lagona et al., 2014; Wang and Gelfand, 2014; Lagona et al., 2014, 2015; Mastrantonio et al., 2015a,b]), animal movements ([Eckert et al., 2008; Langrock et al., 2012, 2014; McLellan et al., 2015]), social science ([Gill and Hangartner, 2010]) and auditory localization data ([McMillan et al., 2013]). Standard statistical distributions cannot be used to analyze circular data because of the finite and on the unit circle \([0, 2\pi]\) support and to dependence of descriptive and inferential analyses on the initial value and orientation on the unit circle. These features make circular data special, so that ad-hoc method and distributions have been developed in the literature.

Dating back to [Von Mises, 1918], the attention over circular data has increased over time ([Mardia, 1972; Fisher, 1996; Mardia and Jupp, 1999; Jammalamadaka and SenGupta, 2001; Pewsey et al., 2013]), leading to important probability distributions theory and inferential results. Nevertheless, not all the introduced circular distributions hold/satisfy crucial properties required to avoid misleading inference, as their parameters strongly depend on the chosen initial direction and sense of orientation (clockwise or anti-clockwise) on the unit circle. Indeed, in general, any circular distribution should have good fitting characteristics, a tractable form, be parsimonious in terms of parameters at play and invariant with respect to the chosen reference system.

All these conditions are fulfilled by the new distribution we propose in the present paper. Motivated by real-data examples, we introduce a new discrete circular distribution, the Invariant Wrapped Poisson (IWP), and investigate its properties along with a discussion on the interpretation of parameters. The literature on discrete circular data modelling has been often overlooked as discrete circular distributions have several drawbacks, as we will discuss later in this paper. However, the increased availability of discrete data on the circle has recently led to a series of researches introducing distributions able to account for this specific data feature ([Girija et al., 2014; Jayakumar and Jacob, 2012]). We would contribute to extend this branch of literature and provide a reference distribution for the analysis of discrete circular data. The Poisson distribution is recognized as an important tool to
analyze count data, and its correspondent on the circle, the Wrapped Poisson distribution, is often considered for count data analysis on the circle. Nevertheless, as the Poisson distribution is restrictive because it assumes a unit, variance-to-mean ratio the Wrapped Poisson ([Mardia and Jupp 1999] shows a strict functional relation between circular mean and variance, and strongly depends on the reference system orientation adopted. To avoid these issues, and recognizing the importance of relaying on a Poisson-based distribution, we introduce the IWP which is invariant with respect to the initial origin and the reference system and, furthermore, allows for relaxing the strong imposed linkage between circular mean and variance. Indeed, by deriving IWP’s trigonometric moments, it is possible to show that the circular mean and variance are independent.

We estimate the model under a Bayesian framework proposing two Markov chain Monte Carlo (MCMC) algorithms. An exact algorithm is considered and, to increase efficiency, an approximated algorithm is introduced based on IWP density approximation. The approximation is obtained investigating the link between the wrapped Normal and the IWP densities and following the truncation strategy proposed by [Jona Lasinio et al. 2012] for the wrapped Normal distribution. Both algorithms use the well known latent variable approach for wrapped distribution of [Coles and Casson 1998] and a specific prior distribution for one of the parameters. The approximated algorithm requires a re-parametrization of the latent variable to ensure computational efficiency and avoid local optima.

The invariance properties in circular distributions are formally introduced in Section 2, along with the notation used throughout the paper. Proper definitions of invariance under changes of initial direction and reference system orientation are provided along with necessary and sufficient conditions that circular distributions must hold to be invariant. Section 3 is devoted to the discussion of these properties in widely used (continuous and discrete) circular distribution. In the same section we shall show that all the known discrete distributions do not hold the two invariance properties. The IWP is introduced in Section 4. Its good properties are widely discussed and all computational details provided. Model inference in a Bayesian setting is provided in Section 5. The good properties of the IWP distribution will be studied on artificial and real data from different empirical contexts, and compared with well-known models from the literature in Section 6. Section 7 summarized
obtained inferential results and provides a discussion of possible applications and extensions of the IWP distribution.

2 Invariance in circular distributions

Let \( \{S, A, P\} \) be a probability space, where the sample space \( S = \{(x, y) : x^2 + y^2 = 1\} \) is the unit circle, \( A \) is the \( \sigma \)-algebra on \( S \) and \( P : S \rightarrow [0, 1] \) is the normalized Lebesgue measure on the measurable space \( \{S, A\} \). Let \( D \) be a subset of \( \mathbb{R} \) such that its length is \( 2\pi \), for example if \( D = [a, b) \) we have \( b - a = 2\pi \).

Let us consider the measurable function \( \Theta : S \rightarrow D \), with \( \Theta^{-1}(d) = (x, y) = (\cos d, \sin d) \), \( d \in D \), and let \( D = \sigma(D) \) be the sigma algebra of \( D \) induced by \( \Theta \) and \( A_{\Theta,D} \equiv \{(x, y) : \Theta(x, y) \in D\} \) and \( \mathbb{P}_\Theta(D) = P(\Theta^{-1}(D)) = P(A_{\Theta,D}), \forall D \in \mathcal{D} \). Then the measurable space induced by \( \Theta \) is \( (D, \mathcal{D}, \mathbb{P}_\Theta) \) with

1. \( \mathbb{P}_\Theta(D) = P(A_{\Theta,D}) \geq 0, \forall D \in \mathcal{D}; \)
2. \( \mathbb{P}_\Theta(D) = P(A_{\Theta,D}) = 1; \)
3. for any countable sequence of disjoint sets \( \{D_j\}_{j=1}^\infty \) of \( \mathcal{D} \), \( \mathbb{P}_\Theta \left( \bigcup_{j=1}^\infty D_j \right) = \sum_{j=1}^\infty P(A_{\Theta,D_j}) = \sum_{j=1}^\infty \mathbb{P}_\Theta(D_j) \),

i.e. \((D, \mathcal{D}, \mathbb{P}_\Theta)\) is a probability space. It follows that \( \Theta \) is a random variable and \( \mathbb{P}_\Theta \) is its probability distribution. \( \Theta \) represents an angle over the unit circle and it is called a circular random variable. Accordingly, for all \( d \in D \), \( \Theta^{-1}(d) = \Theta^{-1}(d \mod 2\pi) \) and, without loss of generality, we can represent any circular variable in \([0, 2\pi)\). \( D \) can be either continuous or discrete. In the latter case, we assume that it is composed of \( l \) distinct points equally spaced between two extremes again denoted \( a \) and \( b \) and we write \( D \equiv \{a + 2\pi j/l\}_{j=0}^{l-1} \).

If \( D \) is a continuous domain, \( \Theta \) is a continuous circular variable and \( \mathbb{P}_\Theta \) is the Lebesgue measure; on the other hand, if \( D \) is discrete, \( \Theta \) is a discrete circular variable or a lattice variable (see [Mardia and Jupp, 1999]), and \( \mathbb{P}_\Theta \) is the counting measure. In both cases we indicate with \( f_\Theta = d\mathbb{P}_\Theta/dP_\Theta : D \rightarrow \mathbb{R}^+ \) the Radon-Nicodym derivative of \( \mathbb{P}_\Theta \) (rnd), i.e. \( \mathbb{P}_\Theta(D) = \int_D f_\Theta dP_\Theta \). Hence, we denote by \( f_\Theta \) the probability density function (pdf) of \( \Theta \) including both situations, continuous and discrete.
Figure 1: Probability density functions of a wrapped skew normal (a-c) and a wrapped exponential (d-f) under different initial directions and orientations. The arrows indicate the axis orientation.

In the representation of circular variables a key role is played by the the initial direction (the angle $\theta$) and the sense of orientation (clockwise or anti-clockwise) of the domain. Both are uniquely determined by the choice of the orthogonal reference system on the plane. Any statistical tool for circular variables should be invariant with respect to different choices of the reference orientation system to avoid conflicting or misleading conclusions. Accordingly, $f_{\Theta}(-\psi)$ must be invariant for changes in the orientation of the system’s axis and initial direction. The following Definitions formalize the invariance properties with respect to the reference system on the plane.

**Definition 1.** We say that $f_{\Theta}$ is invariant under change of initial direction (ICID) if $\forall \xi \in \mathbb{D}, \forall \theta \in \mathbb{D}$ and $\forall \psi \in \Psi$ there exists $\psi^* \in \Psi$ such that $f_{\Theta}(\theta|\psi) = f_{\Theta}(\theta - \xi|\psi^*)$. 

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Definition 2. We say that \( f_\theta \) is invariant under changes of the reference system orientation (ICO) if \( \forall \xi \in \mathbb{D}, \forall \theta \in \mathbb{D} \) and \( \forall \psi \in \Psi \) there exists \( \psi^* \in \Psi \) such that: 
\[
f_\Theta(\xi - \theta | \psi) = f_\Theta(\xi + \theta | \psi^*).
\]

A simple example clarifies the two definition above. In Figure 1 (a), a wrapped skew normal density (Pewsey, 2000), is plotted on a sample of circular measurements. Most of the probability mass is in the first quadrant of the current reference system. The origin (initial direction) is set to zero radiant and the orientation is anti-clockwise. Figure 1 (b) is obtained by adding \( \pi \) to Figure 1 (a), i.e. changing the system origin, and Figure 1 (c) is built as \(-1 \times\) Figure 1 (a) i.e. changing the system orientation. The densities in Figures 1 (b) and (c) are still skew normal. The distribution in 1 is ICID and ICO.

We would remark that a necessary, but not sufficient, condition to be ICID and ICO is that distributions under different reference systems must have the same circular variance (Mardia and Jupp, 1999), as for example shown in the three densities in Figures 1 (a), (b) and (c). To further corroborate this result, let us provide another example. In figure 1 (d) a wrapped exponential distribution (Jammalamadaka and Kozubowski, 2004) is shown. Let us transform circular measurements as before, in the wrapped skew normal examples. Although transformed densities in Figures 1 (e) and (f) have the same variance as Figures 1 (d), both look different from 1 (d) at first glance. Indeed, the wrapped exponential is not ICID and ICO, as we will discuss in depth in the next section.

It is clear that a graphical inspection could reveal some indications, but we need to formally prove whether a circular distribution is ICID and ICO or not. Theorem 1 helps us to define distributions that are both ICID and ICO.

Theorem 1. Let \( f_\theta(\cdot | \psi) \) be the pdf of the circular variable \( \Theta \in \mathbb{D} \) with \( \psi \in \Psi \) and let \( \Theta^* = g(\theta) = \delta(\theta + \xi) \) with scalar \( \delta = \{-1, 1\} \), \( \xi \in \mathbb{D} \) and pdf \( f_{\Theta^*}(\cdot | \psi^*) \) and \( \psi^* \in \Psi \). \( f_\theta \) is ICID and ICO iff
\[
f_{\Theta^*}(\theta^* | \psi^*) = f_\Theta(\theta^* | \psi^*).
\]
i.e. \( f_\Theta \) and \( f_{\Theta^*} \) belong to the same parametric family.

Proof. Let us focus on \( f_{\Theta^*} \). Notice that \( g(\cdot) \) is a linear mapping such that \( \mathbb{D} \rightarrow \mathbb{D} \) and by rule of variable transformation we have that
\[
f_{\Theta^*}(\theta^* | \psi^*) = f_\Theta(\delta \theta^* - \xi | \psi),
\]
where the vector of parameters $\psi^*$ is a function of $\psi$ and $(\delta, \xi)$.

Now we prove that if $f_\Theta$ and $f_{\Theta^*}$ belong to the same parametric family this implies that $f_\Theta$ is ICID and ICO.

1. **If $f_{\Theta^*}(\theta^*|\psi^*) = f_\Theta(\theta^*|\psi^*)$ then** (1) **implies**

\[
f_\Theta(\theta^*|\psi^*) = f_\Theta(\delta \theta^* - \xi | \psi),
\]

Equation (2) is true for all $\theta^* \in \mathbb{D}$, $\xi \in \mathbb{D}$, $\delta \in \{-1, 1\}$ and moreover must be that $\psi^* \in \Psi$ since the left and right sides of (3) belong to the same parametric family. To prove that $f_\Theta$ is ICID it is sufficient to set $\delta = 1$ in (3) and we obtain Definition 1. For simplicity let in (3) $\xi = 0$ and $\delta = -1$. Then we immediately obtain Definition 2

\[
f_{\Theta^*}(\theta^*|\psi^*) = f_\Theta(-\theta^*|\psi),
\]

As $f_\Theta$ is ICID what holds in $\xi = 0$ is going to hold for all $\xi \in \mathbb{D}$ then:

\[
f_\Theta(\theta^*|\psi^*) = f_\Theta(\xi^* + \theta^*|\psi^{**}) = f_\Theta(\xi^* - \theta^*|\psi^{***}) = f_\Theta(-\theta^*|\psi).
\]

The central part of (4) is Definition 2 and proves that $f_\Theta$ is also ICO.

2. **If $f_\Theta$ is ICID and ICO then** $f_{\Theta^*}(\theta^*|\psi^*) = f_\Theta(\theta^*|\psi^*)$.

Since $f_\Theta$ is ICID and ICO we can write $f_\Theta(\delta \theta^* - \xi | \psi) = f_\Theta(\theta^*|\psi^{**})$ with $\psi^{**} \in \Psi$. Using this equivalence with equation (2) we can write $f_{\Theta^*}(\theta^*|\psi^*) = f_\Theta(\theta^*|\psi^{**})$. Since $f_{\Theta^*}$ and $f_\Theta$ assume the same value at the same point for all $\theta \in \mathbb{D}$ they must be the same function with $\psi^* \equiv \psi^{**}$.

It is always possible to transform any circular pdf so to obtain its ICID and ICO version.

**Proposition 1.** *If $f_\Theta(\cdot|\psi)$ is not ICID and ICO, we can define $f_{\Theta^*}(\cdot|\psi^*)$, where $\Theta^* = \delta(\Theta + \xi)$ and $\psi^* = \{\psi, \delta, \xi\}$ such that $f_{\Theta^*}(\cdot|\psi^*)$ is ICID and ICO.*

**Proof.** Let $g(\Theta) = \delta(\Theta + \xi)$, we have that $f_{\Theta^*}(\theta^*|\psi^*) = f_\Theta(\delta \theta^* - \xi | \psi)$.

Let us define $\Theta^{**} = \delta^*(\Theta^* + \xi^*)$, following Theorem 1 if $f_{\Theta^{**}}$ belongs to the same parametric family as $f_{\Theta^*}$, then $f_{\Theta^*}$ is ICID and ICO. As $f_{\Theta^*}(\theta^*|\psi^{**}) = f_{\Theta^*}(\delta^* \theta^{**} - \xi^* | \psi^*)$ we have
the following identity: \( f_{\Theta^*}((\theta^*|\psi^*)) = f_{\Theta^*}((\delta^*\theta^* - \xi^*|\psi^*)) = f_{\Theta}(\delta^*(\theta^* - \xi^*) - \xi|\psi) \). Now let \( \delta^* = \delta \delta^* \) and \( \xi^* = (\delta \xi + \xi) \) we can write \( f_{\Theta^*}((\theta^*|\psi^*)) = f_{\Theta^*}((\delta^*\theta^* - \xi^*|\psi^*)) = f_{\Theta}(\delta^*\theta^* - \xi^*|\psi), \) i.e. \( f_{\Theta^*} \) is of the same functional form as \( f_{\Theta} \) and this proves theorem \( \square \)

Distributions obtained from Proposition \( \square \) are said to be invariant. In other words, by applying the previous proposition, we get the invariant version of the density of \( \Theta \).

### 3 Circular distributions: Examples

In this section conventional circular distributions are investigated, focusing on the invariance properties previously introduced. To simplify the notation, we avoid the use of the modulus 2\( \pi \) operator, and define \( \Theta^* = \delta (\Theta + \xi) \in \delta (\mathbb{D} + \xi) \).

#### 3.1 Continuous distributions

**Wrapped Normal**

A circular variable \( \Theta \) is said to have wrapped normal distribution with parameters \( \mu \) and \( \sigma^2 \) (\( \Theta \sim WN(\mu, \sigma^2) \)) if

\[
 f_{\Theta}(\theta | \mu, \sigma^2) = \sum_{k=-\infty}^{\infty} \left(2\pi\sigma^2\right)^{-\frac{1}{2}} e^{-\frac{(\theta - 2\pi k - \mu)^2}{2\sigma^2}}. \tag{5}
\]

The random variable \( \Theta^* \) has density

\[
 f_{\Theta^*}(\theta^* | \mu, \sigma^2, \delta, \xi) = \sum_{k=-\infty}^{\infty} \left(2\pi\sigma^2\right)^{-\frac{1}{2}} e^{-\frac{(\theta^* - 2\pi k - \delta - \xi - \mu)^2}{2\sigma^2}} = \sum_{k=-\infty}^{\infty} \left(2\pi\sigma^2\right)^{-\frac{1}{2}} e^{-\frac{(\theta^* - 2\pi k - \delta - (\xi + \mu))^2}{2\sigma^2}}. \tag{6}
\]

To prove that (5) is ICID and ICO, following Theorem \( \square \) we need to prove that (6) is still a wrapped normal density, i.e. there must exist two parameters \( \mu^* \) and \( \sigma^{2*} \), apart from the trivial case \( \mu^* = \mu \) and \( \sigma^{2*} = \sigma^2 \), that are such that:

\[
 \sum_{k=-\infty}^{\infty} \left(2\pi\sigma^2\right)^{-\frac{1}{2}} e^{-\frac{(\theta^* - 2\pi k - \mu^*)^2}{2\sigma^{2*}}} = \sum_{k=-\infty}^{\infty} \left(2\pi\sigma^{2*}\right)^{-\frac{1}{2}} e^{-\frac{(\theta^* - 2\pi k - \delta - (\xi + \mu))^2}{2\sigma^{2*}}}. \tag{7}
\]

Clearly (7) is true for \( \mu^* = \delta (\xi + \mu) \) and \( \sigma^{2*} = \sigma^2 \), i.e. the wrapped normal is ICID and ICO.
Von Mises

A circular variable $\Theta$ is said to follow a von Mises distribution with parameters $\mu$ and $\kappa$ ($\Theta \sim \text{vM}(\mu, \kappa)$) if

$$f_{\Theta}(\theta|\mu, \kappa) = \frac{e^{\kappa \cos(\theta - \mu)}}{2\pi I_0(\kappa)}$$

where $I_0(\kappa)$ is the modified Bessel function of order 0. As we did above for the wrapped normal, we find the density of $\Theta^*$, that is

$$f_{\Theta^*}(\theta^*|\mu, \kappa, \delta, \xi) = \frac{e^{\kappa^* \cos(\delta \theta^* - \xi - \mu)}}{2\pi I_0(\kappa^*)},$$

and since $\mu^* = \delta(\xi + \mu)$ and $\kappa^* = \kappa$ are such that:

$$e^{\kappa \cos(\delta \theta - \xi - \mu)} = e^{\kappa^* \cos(\delta \theta^* - \xi - \mu)}$$

then the von Mises is ICID and ICO

Wrapped skew normal

Suppose that $\Theta$ is distributed as a wrapped skew normal with parameters $\mu$, $\sigma^2$ and $\lambda$ ($\Theta \sim \text{WSN}(\mu, \sigma^2, \alpha)$), then

$$f_{\Theta}(\theta|\mu, \sigma^2, \alpha) = \sum_{k=-\infty}^{\infty} \frac{1}{\pi\sigma} e^{\frac{(\theta+2\pi k-\mu)^2}{2\sigma^2}} \int_{-\infty}^{\alpha} e^{\frac{(\theta+2\pi k-\mu)}{\sigma}} e^{-t^2} dt. \quad (8)$$

Since the distribution of $\Theta^*$ is

$$f_{\Theta^*}(\theta^*|\mu, \sigma^2, \alpha) = \sum_{k=-\infty}^{\infty} \frac{1}{\pi\sigma} e^{\frac{(\delta \theta^* - \xi + 2\pi k - \mu)^2}{2\sigma^2}} \int_{-\infty}^{\alpha} e^{\frac{(\delta \theta^* - \xi + 2\pi k - \mu)}{\sigma}} e^{-t^2} dt,$$

we can easily see that $\Theta^* \sim \text{WSK}(\delta(\xi + \mu), \sigma^2, \alpha)$ and then (8) is ICO and ICID.

Wrapped Exponential

Not all distributions for continuous circular variables have the two required properties. For example the wrapped exponential of Jammalamadaka and Kozubowski (2004) with density

$$f_{\Theta}(\theta|\lambda) = \frac{\lambda e^{-\lambda \theta}}{1 - e^{-2\pi \lambda}}, \lambda > 0 \quad (9)$$

it is such that the density of $\Theta^*$ is

$$f_{\Theta^*}(\theta^*|\lambda, \delta, \xi) = \frac{\lambda e^{-\lambda(\delta \theta^* - \xi)} e^{-\lambda \theta}}{1 - e^{-2\pi \lambda}}. \quad (10)$$

To see that the wrapped exponential is not ICID and ICO we need to prove that does not exist a $\lambda^*$ such that

$$\frac{\lambda e^{-\lambda(\delta \theta^* - \xi)}}{1 - e^{-2\pi \lambda}} = \frac{\lambda^* e^{-\lambda^* (\theta^* \mod 2\pi)}}{1 - e^{-2\pi \lambda^*}}. \quad (11)$$

The modulus on the right side of (11) is needed because the definition of the wrapped exponential given in equation (9) assumes that the circular variable belongs to $\mathbb{D}$ while
\( \Theta^* \in \delta(\mathbb{D} + \xi) \). It is not easy to solve (11) for \( \lambda^* \) but we can simplify the demonstration by noticing that all the distributions can be expressed as the product of a normalization constant and a kernel. If two density are equal, their kernel must be identical. Then it is sufficient to prove that it does not exist \( \lambda^* \) such that the following relation is true:

\[
e^{-\lambda \delta \Theta^*} = e^{-\lambda^*(\theta^* \text{ mod } 2\pi)}.
\] (12)

If we can prove that (12) is not true for a particular choice of \( \xi \) and \( \delta \) we have shown that the wrapped exponential is not ICID and ICO. We consider \( \delta = 1 \) and \( \xi > 0 \) and we evaluate the relation (12) at \( \Theta^* = 2\pi \) and \( \Theta^* = \xi \), note that for the particular set of parameters we chose, the points \( 2\pi \) and \( \xi \) belong to the domain of \( \Theta^* \). We have \( \lambda 2\pi = 0 \) if \( \Theta = 2\pi \), and \( \lambda \xi = \lambda^* \xi \), if \( \Theta = \xi \). Then (12) is true only if \( \lambda = \lambda^* = 0 \) but \( \lambda > 0 \) and then it follows that (12) is never verified for a valid value of \( \lambda \). This proves that the wrapped exponential is not ICID and ICO.

Note that using Preposition 1 we obtain the density in equation (10), that is the density of the invariant wrapped exponential.

**Wrapped Weibull**

Sarma et al. (2011) propose the wrapped Weibull:

\[
f_{\Theta}(\theta | \lambda) = \sum_{k=0}^{\infty} \lambda(\theta + 2\pi k)^{\lambda-1} e^{-(\theta + 2\pi k)^{\lambda}}.
\] (13)

To show that (13) does not verify ICID and ICO, we use the characteristic function:

\[
\varphi_{\Theta}(p) = i \sum_{k \in \mathbb{Z}_{odd}^+} \frac{p^k}{k!} \left( 1 + \frac{k}{\lambda} \right) - \sum_{k \in \mathbb{Z}_{even}^+} \frac{p^k}{k!} \left( 1 + \frac{k}{\lambda} \right),
\] (14)

where \( \mathbb{Z}_{odd}^+ \) and \( \mathbb{Z}_{even}^+ \) are respectively the even (zero included) and odd integer numbers.

If \( \Theta \) and \( \Theta^* \) have the same distribution their characteristic function must be of the same
functional form. Since $\Theta^*$ is a linear transformation of $\Theta$ its characteristic function is

$$\varphi_\Theta^*(p) = e^{i\delta p} \varphi_\Theta(\delta p) = \cos(\delta \xi p) \varphi_\Theta(\delta p) + i \sin(\delta \xi p) \varphi_\Theta(\delta p)$$

(15)

$$= \sin(\delta \xi p) \sum_{k \in \mathbb{Z}^{+}_{\text{odd}}} \frac{p^k}{k!} \left(1 + \frac{k}{\lambda}\right) - \cos(\delta \xi p) \sum_{k \in \mathbb{Z}^{+}_{\text{even}}} \frac{(\delta p)^k}{k!} \left(1 + \frac{k}{\lambda}\right)$$

$$+ i \left( \cos(\delta \xi p) \sum_{k \in \mathbb{Z}^{+}_{\text{odd}}} \frac{p^k}{k!} \left(1 + \frac{k}{\lambda}\right) - \sin(\delta \xi p) \sum_{k \in \mathbb{Z}^{+}_{\text{even}}} \frac{(\delta p)^k}{k!} \left(1 + \frac{k}{\lambda}\right) \right)$$

(16)

The real and imaginary parts of (14) and (16) differ unless $\delta = 1$ and $\xi = 0$, and then the wrapped Weibull is not ICID and ICO.

**Wrapped Lévy**

Following Fisher (1996), if a circular random variable $\Theta$ has pdf

$$f_\Theta(\theta|\mu,\sigma^2) = \sum_{k:0+2kl-\mu>0} \sqrt{\frac{\sigma^2}{2\pi}} \frac{e^{-\frac{1}{2} \left(\frac{\theta - 2\pi k - \mu}{\sigma}\right)^2}}{2\pi \left(\frac{2\pi + 2kl - \mu}{\sigma}\right)^\frac{3}{2}}, \mu \in \mathbb{R}, \sigma^2 > 0$$

(17)

it is said to be distributed as a wrapped Lévy.

Here again to prove that (17) does not hold the ICID and ICO properties we exploit the characteristic function of $\Theta$ and $\Theta^*$ that are respectively $\varphi_\Theta(p) = e^{i\mu p - \sqrt{-2\lambda^2 p}}$ and $\varphi_{\Theta^*}(p) = e^{ip(\mu + \xi) - \sqrt{-2\lambda^2 \delta p}}$. The two characteristic functions are of the same kind if the parameters of the distribution of $\Theta^*$ are $\mu^* = \delta(\mu + \xi)$ and $\sigma^{2*} = \sigma^2 \delta$ but if $\delta = -1$ then $\sigma^{2*}$ is negative while, by definition of the Wrapped Lévy distribution, it must be positive, then the distribution is not ICID and ICO.

### 3.2 Discrete distributions

We are particularly interested in discrete circular distributions since we did not found in the literature discrete circular distributions that satisfy the ICID and ICO properties (definitions 1 and 2), apart the trivial case of the circular uniform.

**Discrete circular uniform**

The discrete circular uniform density (Mardia and Jupp, 1999) for $\Theta$ is of the form: $f_\Theta = \frac{1}{l}$ where $l$ is the number of $l$ distinct points equally spaced in $\mathbb{D}$. Clearly $\Theta^*$ has
the same exact density and then the discrete circular uniform is ICID and ICO.

**Wrapped Poisson**

A discrete circular variable $\Theta$ is said to follow a wrapped Poisson distribution (Mardia and Jupp, 1999) if it has pdf $f_{\Theta}(\theta|\lambda) = \sum_{k=0}^{\infty} \frac{\lambda^k e^{-\lambda}}{\theta + kl)!}$. The wrapped Poisson is not ICID and ICO and to see that we use its characteristic function. From (Girija et al., 2014) we have that $\varphi_{\Theta}(p) = e^{-\lambda \left(1 - e^{-2\pi p} \right)} = e^{-\lambda \left(1 - \cos \frac{2\pi p}{\lambda} \right)} e^{i\lambda \sin \frac{2\pi p}{\lambda}}$ and using equation (15) we can compute

$$\varphi_{\Theta^*}(p) = e^{i\delta\xi_p} e^{-\lambda \left(1 - e^{-2\pi p} \right)} = e^{-\lambda \left(1 - \cos \frac{2\pi p}{\delta\xi + \lambda\sin \frac{2\pi p}{\lambda}} \right)}.$$  (18)

The imaginary part of the two characteristic function are equal only if $\delta = 1$ and $\xi = 0$, i.e. $\Theta^* = \Theta$, then it follows that the wrapped Poisson is not ICID and ICO.

**Wrapped geometric**

Another example is the wrapped geometric proposed by Jayakumar and Jacob (2012). The pdf of $\Theta$ is

$$f_{\Theta}(\theta|\lambda) = \frac{\lambda (1 - \lambda)}{1 - (1 - \lambda)l^{-1}}.$$  (19)

We can see that (19) has not the two properties because it is a decreasing function in $\Theta$ and then its maximum value is reached always at $\Theta = 0$ and it cannot satisfy the two properties given in definitions 1 and 2.

**Wrapped skew Laplace on integers**

The wrapped skew Laplace on integers of Jayakumar and Jacob (2012) has pdf $f_{\Theta}(\theta|d, q) = \sum_{k=0}^{\infty} \frac{(1-d)(1-q)}{1-dq} \left( q^{d/2\pi + 2kl} + q^{-d/2\pi - 2kl} \right)$, $d, q \in (0, 1)$. In each interval $[l(k-1), l(k-1) + (l-1)]$ the maximum value of the term inside the sum is reached at $l(k-1)$ that, over the circle, corresponds to the point $\Theta = 0$. Here again, as with the wrapped geometric example, since the maximum value of the density is fixed on the point $\Theta = 0$, the density cannot be ICID and ICO.
Wrapped binomial

The wrapped binomial of Girija et al. (2014) has pdf
\[
f_\Theta(\theta|q, n) = \sum_{k=0}^{k:n-(\theta \frac{1}{2\pi} + kl) \geq 0} \binom{n}{\theta \frac{1}{2\pi} + kl} q^{\theta \frac{1}{2\pi} + kl} (1 - q)^{n-(\theta \frac{1}{2\pi} + kl)},
\]
and characteristic function
\[
\varphi_\Theta(p) = (1 - q + qe^{ip})^n. \tag{20}
\]
while the characteristic function of \(\Theta^*\) is
\[
\varphi_{\Theta^*}(p) = e^{i\delta p}(1 - q + qe^{ip})^n. \tag{21}
\]
Equations (20) and (21) are not of the same functional form and then the wrapped binomial is not ICID and ICO.

4 A new discrete circular distribution: The Invariant wrapped Poisson

In this Section, relying on Proposition 1, we build a new discrete circular distribution that is ICID and ICO. Among the non-invariant distributions discussed in the previous Section, we chose the wrapped Poisson for several reasons. First of all the resulting distribution has a wide range of possible shapes, see Figure 2, ranging from a highly concentrated distribution to the discrete circular uniform as limit case (Section 4.2). Second its trigonometric moments can be written in closed form, allowing for an easy computation of the circular concentration, directional mean and circular skewness (again see Section 4.2). Third, parameters estimates can be easily estimated in a Bayesian framework by introducing some latent variables (see Section 5).

4.1 The Invariant wrapped Poisson distribution

Let \(X \sim Pois(\lambda)\) and let \(\Theta^* = (2\pi X/l) \mod 2\pi\) be its circular counterpart, assuming \(l\) distinct values over the unit circle. \(\Theta^*\) is then distributed as a wrapped Poisson \(WPois(\lambda)\).
Proposition 1 tells us that the random variable $\Theta = \delta (\Theta^* + \xi)$ has pdf that verifies the two desired properties.

We include $\delta$ and $\xi$ in the set of characterizing parameters without restricting $\Theta$ to $\mathbb{D}$ (i.e. we do not apply the module $2\pi$ operation in the transformation $\Theta = \delta (\Theta^* + \xi)$) then the distribution we propose, the Invariant Wrapped Poisson distribution, has pdf:

$$f_\Theta(\theta | \lambda, \delta, \xi) = \sum_{k=0}^{\infty} \frac{\lambda^{(\delta\theta - \xi)/l/(2\pi) + kl}e^{-\lambda}}{((\delta\theta - \xi)/l/(2\pi) + kl)!}, \Theta \in \delta(\mathbb{D} + \xi)$$ (22)

This formulation of the IWP produces a distribution with domain dependent on both $\delta$ and $\xi$ this can be problematic for model fitting. On the other hand we mentioned above that we are interested in restricting $\Theta$ to $[0, 2\pi)$. One advantage of this choice is that by applying the modulus operation when computing $\Theta = \delta (\Theta^* + \xi)$ we obtain a random variable with domain independent from the characterizing parameters. In this setting the IWP pdf becomes:

$$f_\Theta(\theta | \lambda, \delta, \xi) = \sum_{k=0}^{\infty} \frac{\lambda^{(\delta\theta - \xi) \text{ mod } (2\pi)/l/(2\pi) + kl}e^{-\lambda}}{((\delta\theta - \xi) \text{ mod } (2\pi)/l/(2\pi) + kl)!}$$ (23)

In general we say that if a random variable $\Theta$ has pdf (23) (or (22)) it has an invariant wrapped Poisson distribution with $l$ values (IWP$_l$). If $\delta = 1$ and $\xi = 0$ then the IWP$_l$ reduces to the standard WP$_l$.

4.2 Trigonometric moments

The trigonometric moments, defined as $\alpha_p = E \cos p\Theta$ and $\beta_p = E \sin p\Theta$, are quantities used to inform about many features of circular distributions, for example the mean direction, $\mu = \tan^* \frac{\beta_1}{\alpha_1}$ (the function $\tan^*$ is the modified atan function, see for example Jammalamadaka and SenGupta (2001)), the circular concentration, $c = \sqrt{\alpha_1^2 + \beta_1^2}$, and the circular skewness, $s = \frac{c \sin(\mu_2 - 2\mu)}{(1-c)^2}$ where $\mu_2 = \tan^* \frac{\beta_2}{\alpha_2}$ (see Mardia and Jupp, 1999). $\alpha_p$ and $\beta_p$ are related to the characteristic function of the circular distribution, defined only on integer $p$ (see again Mardia and Jupp, 1999): $\varphi_\theta(p) = E(e^{ip\Theta}) = \alpha_p + i\beta_p$.

The characteristic function in equation (18), is the characteristic function of the random variable that arises applying Proposition 1 to the wrapped Poisson distribution, then (18) is the characteristic function of the invariant version of the wrapped Poisson, i.e. the
The characteristic function of the IWP. Hence \( \varphi(p) = e^{-\lambda|1 - \cos \frac{2\pi p}{\lambda}|} e^{i(\delta \xi_p + \lambda \sin \delta \frac{2\pi p}{\lambda})} \) and since \( e^{i(\delta \xi_p + \lambda \sin \delta \frac{2\pi p}{\lambda})} = \cos (\delta \xi_p + \lambda \sin \delta \frac{2\pi p}{\lambda}) + i \sin (\delta \xi_p + \lambda \sin \delta \frac{2\pi p}{\lambda}) \) we have \( \alpha_p = e^{-\lambda|1 - \cos \frac{2\pi p}{\lambda}|} \cos (\delta \xi_p + \lambda \sin \delta \frac{2\pi p}{\lambda}) \), \( \beta_p = e^{-\lambda|1 - \cos \frac{2\pi p}{\lambda}|} \sin (\delta \xi_p + \lambda \sin \delta \frac{2\pi p}{\lambda}) \). The mean direction, the circular concentration and the circular skewness are respectively

\[
\mu = \delta \xi + \lambda \sin \frac{2\pi}{l},
\]

\[
\gamma = e^{-\lambda|1 - \cos \frac{2\pi}{l}|},
\]

\[
s = \frac{c \sin \left( -2\lambda \sin \left( \frac{2\pi}{l} \right) \left( 1 - \cos \left( \frac{2\pi}{l} \right) \right) \right)}{(1 - c)^{\frac{1}{4}}}
\]

The circular concentration is a function of \( \lambda \) and \( l \), its maximum value, \( c = 1 \), is reached at \( \lambda = 0 \) and it is inversely proportional to \( \lambda \). As we expected, \( c \) does not depend on \( \delta \).
and $\xi$. Note that as $\lambda \to \infty$ the concentration goes to zero and then the IWP$_l$ becomes a discrete uniform distribution.

The directional mean is function of all three parameters and note that if we change $\delta$ the directional mean shift to its specular circular value with respect to initial direction, e.g. if with $\delta = 1$ we have $\mu = d$, then with $\delta = -1$ we have $\mu = l - d$.

Note that since the circular concentration depends only on $\lambda$ while the directional mean is a function of all the three parameters, for a given value of $c$, i.e. a given value of $\lambda$, we can change the directional mean and let $c$ unchanged by modifying the value of $\xi$. Hence the IWP circular mean and concentration are independent, breaking the dependence between directional mean and circular concentration that the wrapped Poisson, inherits from the dependence between the mean and the variance of the Poisson.

The skewness depends on both $\lambda$ and $\delta$ with a complex functional form and we cannot say much about its value, apart that $s \to 0$ as $\lambda \to \infty$ since $c \to 0$. It is of interest to understand how the sign of $s$ changes. Since $c \in [0, 1]$ the sign of $s$ depends only on $\sin \left( -2\lambda \sin \left( \frac{\delta \pi}{T} \right) \left( 1 - \cos \left( \frac{2\pi}{T} \right) \right) \right)$ then let $h_r = \frac{r\pi}{2\sin \left( \frac{\delta \pi}{T} \right)(1-\cos \left( \frac{2\pi}{T} \right))}$, $r = 0, 1, \ldots$, and consider the intervals

$$h_r \leq \lambda < h_{r+1}. \quad (26)$$

Inside the intervals in equation (26) the sign of $s$ depends only on $\delta$. Precisely if $r$ is even the sign of $s$ is positive if $\delta = -1$ and negative otherwise, while if $r$ is odd $s \leq 0$ if $\delta = -1$ and $s > 0$ otherwise. The sign of the skewness depends on both $\lambda$ and $\delta$. Remark that at the right limit of the intervals (26) the circular concentration becomes

$$\exp \left\{ -\frac{(r + 1)\pi}{2\sin \left( \frac{2\pi}{T} \right)} \right\}. \quad (27)$$

(27) is inversely proportional to $r$ and it decreases with $l$ as soon as $l \geq 4$. For $l$ large enough (indicatively $l \geq 20$), at the right limit of the first interval ($h_0 \leq \lambda < h_1$) the circular concentration (27) is smaller than 0.007, the invariant wrapped Poisson starts to become indistinguishable from the discrete uniform and the skewness of the distribution is no more a useful information. Then if $l \geq 20$, in the range of values of $\lambda$ where $s$ gives information on the shape of the distribution, the sign of the skewness depends only on $\delta$. 

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4.3 Random number generation

We have two different, and easy, ways to generate a value from a $IWP_1(\lambda, \delta, \xi)$.

In the first method we simulate $Z$ from a Poisson with parameter $\lambda$ and then we obtain the associated wrapped circular variable $\Theta = (Z \frac{2\pi}{l}) \mod 2\pi$ that is distributed as a $WP_1(\lambda)$.

The discrete circular random variable $\Theta^* = \delta(\Theta + \xi)$ is then distributed as a $IWP_1(\lambda, \delta, \xi)$.

To introduce the second method, note that, if $Z \sim P(\lambda)$, the density of the random variable $Y = \delta(\lambda Z \frac{2\pi}{l} + \xi)$ is

$$f_Y(y|\lambda, \delta, \xi) = \frac{\lambda^{(\delta y - \xi)} e^{-\lambda}}{(\delta y - \xi)^{2\pi l}},$$

and we can see that the pdf of $\Theta$, given in equation (22), arises by wrapping the random variable $Y$. Then if $Z$ is a drawn from a $P(\lambda)$, the random variable $\Theta = Y \mod 2\pi$, where $Y = \delta(\lambda Z \frac{2\pi}{l} + \xi)$, is invariant wrapped Poisson distributed with parameter $\lambda, \delta$ and $\xi$.

The two proposed methods can be applied to the generation of any invariant wrapped distribution, for example if we use the first method and we first generate $X$ from a geometric distribution with parameter $\lambda$, then $\Theta = (X \frac{2\pi}{l} \mod 2\pi$ is distributed as a wrapped geometric and $\Theta^* = \delta(\Theta + \xi)$ follows an invariant wrapped geometric with parameters $\lambda$, $\delta$ and $\xi$.

4.4 Relation with the wrapped normal

It is well know that a Poisson distribution can be well approximated by a normal when the Poisson parameter when is “large’ enough”. Then, when variability is high, if $Z \sim P(\lambda)$, we have $f_Z(z|\lambda) \approx \phi(z|\lambda - 0.5, \lambda)$, where $\phi(\cdot|\cdot, \cdot)$ is the normal pdf. It follows that the density of $Y = \delta(\lambda Z \frac{2\pi}{l} + \xi)$ can be well approximated by $\phi \left( y|\lambda((\gamma - 0.5) \frac{2\pi}{l} + \xi, \lambda \left(\frac{2\pi}{l}\right)^2 \right)$. As described in Section 4.3 the invariant wrapped Poisson can be obtained by wrapping $Y$: $f_{\Theta}(\theta|\lambda, \delta, \xi) = \sum_{k=0}^{\infty} f_Y(\theta + 2\pi k|\lambda, \delta, \xi)$, and since $f_Y(y|\lambda, \delta, \xi) \approx \phi \left( y|\lambda((\gamma - 0.5) \frac{2\pi}{l} + \xi, \lambda \left(\frac{2\pi}{l}\right)^2 \right) = \phi \left( \theta + 2\pi k|\lambda((\gamma - 0.5) \frac{2\pi}{l} + \xi, \lambda \left(\frac{2\pi}{l}\right)^2 \right)$ we have

$$f_{\Theta}(\theta|\lambda, \delta, \xi) \approx \sum_{k=0}^{\infty} \phi \left( \theta + 2\pi k|\lambda((\gamma - 0.5) \frac{2\pi}{l} + \xi, \lambda \left(\frac{2\pi}{l}\right)^2 \right).$$

The sum (28) is the density of a wrapped normal that well approximate $f_{\Theta}(\theta|\lambda, \delta, \xi)$ when $\lambda$ is large enough and then the mean of the normal density in (28) is far away from 0. When $l$ is relatively small a continuity correction must be added to account for the different type
Figure 3: Sup distance between Wrapped Normal approximation and Invariant Wrapped Poisson as a function of $\lambda$, dashed lines correspond to different values of $l = 5, 10, 50, 100, 200, 300$ of distributions support.

Remark that when the ratio $\lambda/l^2$ is large the WN in (28) is indistinguishable from a circular uniform distribution (see Jona Lasinio et al., 2012, for details), in the same way the IWP is indistinguishable from a circular discrete uniform. As an example, in Figure 3 the sup distance between the IWP, with $\delta = 1, \xi = 0$ and the corresponding approximating WN is reported for several values of $\lambda$ and $l$. It is clear that the approximation is accurate at the third decimal figure even with relatively small $\lambda/l$ ratios. However for the use we are envisioning in the sequel we consider “acceptable” the approximation when the sup distance between the two distributions is less than 0.001. In the next section we are going to use this approximation to derive an efficient Markov chain Monte Carlo (MCMC) algorithm.

5 Statistical inference: Bayesian estimation

In this section we illustrate how to obtain estimates of the IWP parameters and prediction from the same distribution in a Bayesian framework. Notice that wrapped distributions have convenient inferential properties because, by introducing a latent variable in a data augmentation approach (see for example Coles 1998), we can linearize the circular variable
avoiding the evaluation of the infinite sum involved in their definitions. Here we consider the random variable \( X = (\delta \Theta - \xi) \mod (2\pi)l/(2\pi) + Kl \), where in this case \( K \) is a latent random variable. The joint pdf of \((\Theta, K)\) is the summand of (23) and a marginalization over \( K \) gives \( f_\Theta: f_\Theta(\theta|\lambda, \delta, \xi) = \sum_{k=0}^{\infty} f_{\Theta,K}(\theta, k|\lambda, \delta, \xi) \). Now suppose to observe a sample of \( T \) values from a discrete circular variable \( \Theta_t \sim IWP_t(\lambda, \delta, \xi), t = 1, \ldots, T \), we want to run posterior and predictive inference using this sample. The parameters are \( \psi = (\lambda, \delta, \xi) \) and to facilitate estimation we introduce the latent variables \( K_t \in \mathbb{Z} \). Let \( X_t = (\delta \Theta_t - \xi) \mod (2\pi)l/(2\pi) + Kl, \ X = \{X_t\}_{t=1}^T, \ \Theta = \{\Theta_t\}_{t=1}^T \) and \( K = \{K_t\}_{t=1}^T \) we can write the (augmented) likelihood of \((\Theta, K)\) as \( f_{\Theta,K}(\theta, k|\psi) = \prod_{t=1}^{T} \frac{\lambda x_{t+1} - \lambda}{x_t!} = \prod_{t=1}^{T} \frac{\lambda(\delta \Theta_t - \xi) \mod (2\pi)l/(2\pi) + kl}{(\delta \Theta_t - \xi) \mod (2\pi)l/(2\pi) + kl} \). As usual in the Bayesian framework, to estimate the posterior distribution of \( k, \psi|\theta \) we obtain posterior samples using a Markov chain Monte Carlo (MCMC) algorithm. From (24) it is clear that all parameters are involved in the evaluation of the directional mean. This implies that a component-wise simulation scheme based on parameters full conditional distributions, could be highly inefficient. A typical solution to this problem is to block sample (Finley et al., 2013) the parameters. Then by choosing the appropriate prior distributions we can update \{\lambda, \delta, \xi\} jointly. We assume independence between \( \lambda \) and \((\delta, \xi)\) and as prior distribution for \( \lambda \) a Gamma, \( \lambda \sim G(v, m) \) while no specific requirement are needed for modeling prior information on \( \delta \) and \( \xi \), indeed the distribution of \( \xi \) can be used to model prior information on the sign of the skewness if \( l \geq 20 \), see Section 4.2.

Using composition sample, to draw a sample from \( \lambda, \delta, \xi|k, \theta \) we can first sample from \( \delta, \xi|k, \theta \) and then from \( \lambda|\delta, \xi, k, \theta \). Using the linear variables \( X_t \)'s and the conjugacy between the Poisson distribution and the Gamma we can find the distribution of \( \delta, \xi|k, \theta \):

\[
f_{\delta,\xi|k,\theta}(\delta, \xi|k, \theta) \propto \frac{\Gamma(\sum_{t=1}^{T} x_t + v)}{(m + T)\sum_{t=1}^{T} x_t + v} \prod_{t=1}^{T} (x_t! \lambda^{\delta x_t - \lambda}) f_{\delta,\xi}(\delta, \xi) \tag{29}
\]

We evaluate (29) for all the possible combinations of values \((\delta, \xi)\), and then we can sample from the discrete distribution \( f_{\delta,\xi|k,\theta}(\delta, \xi|k, \theta) \). The full conditional of \( \lambda \) is a \( G(\sum_{t=1}^{T} x_t + v, T + m) \) and then we can easily simulate from it. To conclude the MCMC specification the latent variable \( k_t \) can be simulated with a Metropolis step.

We occasionally experienced slow mixing determined by the separate sampling of \( \lambda \) and \( k_t \)'s. Block sampling of \( \lambda, k \) is computationally unfeasible. Hence, to solve the problem, we need a different strategy: we are going to use the density approximation described in
sections 4.4 to “appropriately” limiting $\lambda$ and $k_t$’s ranges and we adopt a different formalization for the winding numbers to be used in the MCMC algorithm.

We know that the IWP can be well approximated by the WN when the ratio $\lambda/l$ is relatively large, and then both distributions are closed to the circular uniform (discrete and continuos respectively). [Jona Lasinio et al. (2012)] show that, for a given sample size we can find, through simulation, the maximum value of the wrapped normal variance, $\sigma_{\text{max}}^2$ at which a wrapped normal and a circular uniform are not significantly different. Let $\lambda_{\text{max}} = \left(\frac{\mu}{2\pi}\right)^2 \sigma_{\text{max}}^2$, then [28] with $\lambda = \lambda_{\text{max}}$ becomes $f_\theta(\theta|\lambda_{\text{max}}, \delta, \xi) \approx \sum_{k=0}^{\infty} \phi(\theta + 2\pi k | (\lambda_{\text{max}} - 0.5)\frac{2\pi}{l} + \xi, \sigma_{\text{max}}^2)$ the sum describes a wrapped normal density evaluated at $\sigma^2 \approx \sigma_{\text{max}}^2$ implying that the WN is almost a circular uniform. Again the IWP($\lambda_{\text{max}}, \delta, \xi$) and the WN are indistinguishable from a (discrete and continuos) circular uniform and we are not interested in $\lambda > \lambda_{\text{max}}$. Then we can truncate $\lambda$ domain to $(0, \lambda_{\text{max}})$. The constrained domain of $\lambda$ and again a result from [Jona Lasinio et al. (2012)], let us define a limited range for $k_t$. We define $k_{\text{max}}$, such that for any value of $\lambda$, the equation

$$
\sum_{k_t=0}^{k_{\text{max}}} \lambda^{(\delta \theta_t - \xi) \bmod (2\pi)l/(2\pi) + k_t l} e^{-\lambda} 
$$

(30)

well approximate the density of the IWP.

[Jona Lasinio et al. (2012)] show that we can approximate the wrapped normal density if we let $k_t \in [\hat{k}_{\text{low}}, \hat{k}_{\text{up}}]$, with $\hat{k}_{\text{low}} > -\frac{3\sigma}{2\pi} + \frac{\mu}{2\pi} + \frac{1}{2}$ and $\hat{k}_{\text{up}} < \frac{3\sigma}{2\pi} + \frac{\mu}{2\pi} - \frac{1}{2}$ where $\mu$ is the mean of the WN. Notice that both $\hat{k}_{\text{low}}, \hat{k}_{\text{up}}$ depend on the WN parameters that in turn depend directly on $\lambda$ in (28). Here we are looking for an upper bound for $k_t$ and then we can focus on $\hat{k}_{\text{up}}$. If we set $\lambda = \lambda_{\text{max}}$ we obtain the desired upper bound for $k_t$ ($k_{\text{max}}$) Then let $k_{\text{max}} = \lfloor \frac{3\sqrt{\lambda_{\text{max}}}}{l} + \frac{\lambda_{\text{max}}}{l} - \frac{1}{2} \rfloor$ and using the result in [Jona Lasinio et al. (2012)], the probability mass captured by the approximation (30) is

$$
\sum_{\theta_t \in \Theta} \sum_{k_t=0}^{k_{\text{max}}} \lambda^{(\delta \theta_t - \xi) \bmod (2\pi)l/(2\pi) + k_t l} e^{-\lambda} > 0.997, \forall \lambda \in [0, \lambda_{\text{max}}].
$$

(31)

and (31) becomes close to 1 when $\lambda << \lambda_{\text{max}}$. Now we decompose the winding numbers in two components: $k_t = (\hat{k} + \tilde{k}_t) \bmod k_{\text{max}}$, one ($\tilde{k}_t \in [0, k_{\text{max}}]$) specific to the $t^{th}$ observation and one ($\hat{k} \in [0, k_{\text{max}}]$) common to all $k_t$s. Let $\mathbf{k} = \{\hat{k}_t\}_{t=1}^T$, the pmf of $\theta, \hat{k}, \mathbf{k}$, based on the approximated density (30), is

$$
\prod_{t=1}^{T} \frac{1}{k_{\text{max}}} \lambda^{(\delta \theta_t - \xi) \bmod (2\pi)l/(2\pi) + (\tilde{k} + \tilde{k}_t) \bmod (k_{\text{max}})} e^{-\lambda}.
$$

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Figure 4: Circular barplot of simulated (solid), posterior from WP (dashed) and posterior from IWP (dash-dots) values (a) example 1, (b) example 2: same data as in (a) but with different reference system.

\[ \prod_{t=1}^{T} \frac{x_t^e e^{-\lambda}}{x_t^{k_{\max}}} \] where \( x_t = (\delta \theta_t - \xi) \mod (2\pi) l/(2\pi) + (\tilde{k} + \bar{k}_t) \mod (k_{\max}) l \). In this setting we can block sample \((\lambda, \delta, \xi, \tilde{k})\) by first sampling from \( f_{\delta,\xi,\tilde{k}|k,\theta}(\delta, \xi, \tilde{k}|k, \theta) \) and then from the full conditional of \( \lambda \). Remark that we can direct sample from \( f_{\delta,\xi,\tilde{k}|k,\theta}(\delta, \xi, \tilde{k}|k, \theta) \) only because \( \tilde{k} \) has a limited domain. The functional form of \( f_{\delta,\xi,\tilde{k}|k,\theta}(\delta, \xi, \tilde{k}|k, \theta) \) is the same as in (29) but now it must be seen as a function of \( (\delta, \xi, \tilde{k}) \) and it has to be evaluated for all the \( 2lk_{\max} \) possible values of \( (\delta, \xi, \tilde{k}) \). As a final remark observe that \( \tilde{k} \) and \( \bar{k} \) are not identifiable but we can always transform them back to the associated \( k_t \)s that are identifiable. The full conditional of \( \lambda \) is \( G(\sum_{t=1}^{T} +v, T + m)I(0, \lambda_{\max}) \). The latent variable \( \bar{k}_t \) can be updated using a Gibbs step since its full conditional is proportional to

\[ \frac{\lambda(\delta \theta_t - \xi) \mod (2\pi) l/(2\pi) + (\tilde{k} + \bar{k}_t) \mod (k_{\max}) l e^{-\lambda}}{(\delta \theta_t - \xi) \mod (2\pi) l/(2\pi) + (k + k_{\max}) l \mod (k_{\max}) l} \] and has to be evaluated only for a finite set of values.

6 Numerical examples

6.1 Artificial data

By means of a simulated example, we show the consequences of the lack of ICO and ICID properties on model inference. Let us simulate data from a Wrapped Poisson with
parameter \( \lambda \), i.e. an IWP with \( \delta = 1 \) and \( \xi = 0 \). Then we estimate on one hand the posterior distribution of \( \lambda \) using the WP and on the other hand the set of parameters \( \{ \lambda, \delta, \xi \} \) using the IWP. We compute the posterior estimate of the mean direction, circular concentration and the predictive density. Then we modify the orthogonal system and we show how the posterior estimates change.

We simulate \( T = 200 \) realizations from a wrapped Poisson with 36 values over the unit circle and parameter \( \lambda = 0.5 \) (\( \Theta \sim WP_{36}(\lambda) \)) and we choose a weakly informative prior for \( \lambda \) i.e. \( \lambda \sim G(1, 0.0005)I(0, 500) \) that is roughly uniform between \([0, 500] \). A circular barplot of the simulated data is given in Figure 4 (a). Using Equations (24) and (25) we can compute the true directional mean and concentration that are 0.087 and 0.992.

We estimate the model using the approximated MCMC with 10000 iteration, burnin 4000 and thinning 2, i.e. we estimated the posterior distribution using 3000 samples. Under a WP model the posterior mean estimate of \( \lambda \) is 0.519, with 95\% credible interval (CI) [0.423, 0.627] and the posterior mean direction and concentration are 0.090 (CI [0.073, 0.109]) and 0.992 (CI [0.991, 0.994]). The posterior predictive distribution, reported in Figure 4 (a), is close to the barplot of Figure 4 (a). Under the IWP model we obtain very similar results with \( \lambda = 0.519 \) (CI [0.423, 0.627]), while \( \delta = 1 \) and \( \xi = 0 \) along all simulations. The circular mean is 0.090 (CI [0.073, 0.109]) and the concentration 0.992 (CI [0.991, 0.994]).

We then changed the orientation and the initial direction of the reference system. In Figure 4 (b) the data are depicted in the new system. Data are obtained as \( \Theta^* = (-(\Theta + \pi/2)) \mod 2\pi \). Note that \( \Theta^* \) can be seen as a draw from a IWP_{36}(0.5, -1, \pi/2) with directional mean 4.626 and unchanged circular concentration (0.992). Under the WP model the posterior mean estimate of \( \lambda \) is 26.483 with 95\% CI [25.762, 27.200]. The posterior directional mean is 4.599 (CI [4.474, 4.723]), fairly close to the “true” value, but the circular concentration is 0.669 (CI [0.662, 0.676]), that is seriously underestimated and then the predictive density, the dashed line in Figure 4 (b), is less concentrated and starts to resemble the circular uniform. Under the IWP model the estimate of \( \lambda \) as well as its credible interval does not change with respect to the first example. Estimates of \( \delta \) and \( \xi \)

\footnote{For the circular variable the 95\% CI is computed as the shorter arc that contains the 95\% of posterior samples}
Figure 5: Circular barplot of observed (solid) and predicted (dashed) values (a) wind data and (b) gun crime data are -1 and $\frac{\pi}{2}$ respectively in all simulations. The circular mean is 4.622 (CI [4.603, 4.639]) and concentration is 0.992 (CI [0.991, 0.994]). Figure 4 (b) illustrates these results.

6.2 Wind direction

This section describes data on wind directions recorded on January 2000 at the monitoring station of Capo Palinuro. The monitoring station of Capo Palinuro (WMO code 16310) is one of the coastal stations managed by the Meteorological Service of the Italian Air Force. The station is located on the rocky cape of Capo Palinuro, in the town of Centola in the province of Salerno, South Italy. The cape goes for two kilometers in the Tyrrhenian Sea, between the Velia gulf and the Policastro gulf. Its geographical coordinates are $40^\circ01'31.06''$N $15^\circ16'50.71''$E GW and the station elevation is 185 meters above sea level. To the east of the station there are mountains following a North West South East direction. Accordingly, it is plausible that the orography affects the observed measurements, leading to a unimodal distribution indicating a prevalent direction.

Wind directions are monitored and routinely collected by several environmental agencies. Analyzed data come from reports prepared at the station and provided by the National
Center of Aeronautical Meteorology and Climatology (C.N.M.C.A.), special office of the Meteorological Service of the Italian Air Force. The database includes date and time of registration, direction of the wind in degrees, with eight daily measurements (every three hours), i.e. we have 240 observations. The measuring instrument, anemometer, is placed away from obstacles and at a height of 10 meters above ground. A relevant issue with this recordings is that the measurement instrument asses wind directions on a discrete scale dividing the circle into ten-degree intervals \( l = 36 \). Accordingly, the need for a discrete circular distribution to proper model these data. The barplot of the available data is provided in Figure \( 5(a) \).

To get parameters estimates, we use the approximated MCMC illustrated in section \( 5 \) with 10000 iterations, burnin 4000 and thinning 2.

The marginal posterior distribution of \( \delta \) returns a probability value of 0.23 for \( \delta = -1 \) and of 0.77 for \( \delta = 1 \) suggesting a negative skewness \( s \), since, as we noted in Section \( 4 \) with \( l \geq 20 \) the sign of the skewness depends only on \( \delta \). The correspondent posterior distribution of \( s \) is bimodal with one mode on the negative axis and a smaller one on the positive one; the posterior mean of the skewness is -0.192, the 95% CI is \([-0.198, 0.196]\) ( \([-0.198, -0.187]\) conditioning to \( \delta = 1 \) and \([0.184, 0.197]\) conditioning to \( \delta = -1 \)), suggesting that the predictive distribution is almost symmetric, see the predictive distribution in Figure \( 5(a) \). The posterior 95% CI of \( \lambda \) is \([47.809, 59.442]\) with posterior mean 51.744 that is far away from 0 and then we expect a fairly dispersed predictive distribution. Indeed the posterior distribution of the circular concentration has mean value 0.456 and CI \([0.405, 0.484]\) (recall that \( c_{\text{max}} = 1 \)). The posterior distribution of \( \xi \) is negatively skewed assigning zero probability below \( \xi = 12\frac{2\pi}{36} \) and above \( \xi = 25\frac{2\pi}{36} \), its modal value is \( 24\frac{2\pi}{36} \) reached with probability 0.123. The posterior mean estimates of the directional mean is 0.164 (closed to the geographical North), with CI \([6.248, 0.366]\). To evaluate the goodness of fit of the model we use the average prediction error proposed in Jona Lasinio et al. (2012) that is computed as \( \frac{1}{B} \sum_{b=1}^{B} \sum_{t=1}^{n}(1 - \cos(\theta_{t} - \theta_{t}^{b})) \), where \( B \) is the number of posterior samples, \( T \) is the number of observations used to estimate the model, \( \theta_{t} \) is the \( t^{th} \) observation and \( \theta_{t}^{b} \) is a posterior simulation of the \( t^{th} \) observation using as parameters the ones of the \( b^{th} \) posterior sample; in this example we obtain a value 0.780, indicating a reasonable fit given
the large variability of the observations (maximum value for the ape is 2).

6.3 Gun crime data

In a number of areas of police resource management, analysis of the relative frequency of events occurring at different times of day plays an important role. Identifying times of day when certain kinds of crime or disorder are more likely to occur offers a first step towards prioritising resources and improved targeting.

The data set described in this section relates to reports of crimes committed with a gun in Pittsburgh, Pennsylvania, recorded over the period 1987-1998 [Cohen and Gorr, 2006; Gill and Hangartner, 2010]. To avoid under- or over-reporting related to the time lag between the occurrence of the crime and the time of its reporting, a discrete scale is considered, i.e. data are collected on hourly basis. We can think of the data as circular discrete data with 24 points over the unit circle. The barplot of the data is shown in Figure 5 (b), where the initial direction is midnight, and the sense of orientation is clockwise. The data contains information on 15831 crimes. An initial inspection of the data reveals an unimodal distribution, where frequency of reporting times peaks during the night, and falls off during the day.

The posterior directional mean is 6.166 with 95% CI [6.138, 6.193] that on the 24 hours scale, corresponds to 23:33 ([23:27,23:39]), close to midnight. The posterior estimate of $\lambda$ is 33.834 (CI [33.727, 33.941]) implying a large variability of the data ($\lambda >> 0$), confirmed by the estimate of the circular concentration 0.316 with 95% CI [0.315, 0.317]. $\delta = -1$ with probability 1 and it follows that the posterior distribution of the skewness is non-zero over the positive axis, with a point estimate of 0.236 and 95% CI [0.235, 0.236]. Again the value of the skewness is small suggesting an almost symmetric predictive density, see Figure 5 (b), The posterior of $\xi$ is concentrated on $15 \frac{2\pi}{24}$. In this example the average prediction error is 0.838 again a reasonable fit given the observed variability.
7 Summary and concluding remarks

In this paper we developed a method to build circular distribution invariant for changes in the reference system. In particular we discuss and analyze in details a new discrete circular distribution that helps the modeling of several phenomena. Simulated examples illustrated how misleading can be the use of not invariant distributions. Real world examples show that from the description of natural processes to the investigation of social phenomena using discrete circular data we can benefit from an appropriate statistical modeling.

Future work will be focused on classification problems using hidden markov models based on discrete invariant circular distribution being the wind data our motivating example. Further developments will involve the study of these type of distributions with dependent (in both time and space) phenomena.

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