Polynomial Entropy and Expansivity

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Abstract

In this paper we study the polynomial entropy of homeomorphism on compact metric space. We construct a homeomorphism on a compact metric space with vanishing polynomial entropy that it is not equicontinuous. Also we give examples with arbitrarily small polynomial entropy. Finally, we show that expansive homeomorphisms and positively expansive maps of compact metric spaces with infinitely many points have polynomial entropy greater or equal than 1.

1 Introduction

The study of chaotic dynamical systems presents several difficult problems as, for example, to estimate the topological entropy as can be seen in [4, 5, 11, 21] and, with more recent approaches, in [1, 2, 6]. Some systems are known to have positive topological entropy. For instance, in [9] it is shown that every expansive homeomorphism on a compact metric space of positive topological dimension has positive topological entropy. This result is extended in [13] for continuum-wise expansivity.

For dynamical systems with vanishing topological entropy the polynomial entropy becomes an interesting object to measure the complexity of the orbit structure. The concept of polynomial entropy, that is recalled in §2, was considered in [19] with a detailed analysis of the basic properties and examples from Hamiltonian dynamics. In [14, 15] it is considered the problem of finding Riemannian metrics with minimal polynomial entropy.

It is easy to prove that the polynomial entropy of an equicontinuous homeomorphism is zero. In fact, a stronger result is true: a homeomorphism is equicontinuous if and only if it has bounded complexity, see [3, Proposition 2.2] and [20, Corollary 1.4]. Recall that $f : M \to M$ is equicontinuous if for all $\varepsilon > 0$ there is $\delta > 0$ such that if $x, y \in X$, $\text{dist}(x, y) < \delta$ then $\text{dist}(f^n(x), f^n(y)) < \varepsilon$ for all $n \in \mathbb{Z}$. For an equicontinuous homeomorphism of a compact metric space there is a compatible metric that makes it an isometry. Such metric can be defined as $\text{dist}'(x, y) = \sup_{n \in \mathbb{Z}} \text{dist}(f^n(x), f^n(y))$, for all $x, y \in X$. On the circle the converse is true (see [16, Theorem 1]): if a homeomorphism of the circle has vanishing polynomial entropy then it is equicontinuous (i.e., conjugate to a rotation). In §3.1 we give an example of a homeomorphism of a compact
metric space with vanishing polynomial entropy that is not equicontinuous. In [7, Theoreme 6.1], for every \( \alpha \in (1, 2) \) a subshift \( \sigma \) with \( h_{pol}(\sigma) = \alpha \) is constructed; taking products, we obtain that every real number larger than 1 is the polynomial entropy of an expansive homeomorphism (note that subshifts are expansive). In §3.2 for all \( \varepsilon > 0 \) we construct a (non-expansive) homeomorphism \( f \) of a compact metric with \( h_{pol}(f) \in (0, \varepsilon) \); so, given \( 0 < a < b \) there exists \( g \) with \( h_{pol}(g) \in (a, b) \).

In §4 we show that the polynomial entropy of an expansive homeomorphism on a compact metric space with infinitely many points is greater or equal to 1. We also sketch the proof of the corresponding result for positively expansive maps, extending [20, Theorem 1.2] where it is shown that the complexity is unbounded.

2 Basic properties of polynomial entropy

In this section we recall the definitions of entropy that we will use. Also, we prove a simple criterion to obtain polynomial entropy equal or greater than 1.

Let \((X, \text{dist})\) be a compact metric space and consider a homeomorphism \( f : X \to X \). For \( E \subset X \), \( n \in \mathbb{Z} \) and \( \varepsilon > 0 \) we say that \( E \) \((n, \varepsilon)\)-spans with respect to \( f \), if for each \( y \in X \) there is \( x \in E \) such that \( \text{dist}(f^k(x), f^k(y)) \leq \varepsilon \) for all \( 0 \leq k \leq n - 1 \). Let \( r_n(\varepsilon) \) denote the minimum cardinality of the set which \((n, \varepsilon)\)-spans. Let us consider

\[
\hat{h}^{\varepsilon}_{\text{top}}(f) = \limsup_{n \to \infty} \frac{1}{n} \log r_n(\varepsilon).
\]

Following [4], the topological entropy of \( f \) is defined as

\[
h_{\text{top}}(f) = \lim_{\varepsilon \to 0} \hat{h}^{\varepsilon}_{\text{top}}(f).
\]

Analogously,

\[
h^{\varepsilon}_{\text{pol}}(f) = \limsup_{n \to \infty} \frac{1}{\log n} \log r_n(\varepsilon).
\]

According to [19], the polynomial entropy of \( f \) is defined by

\[
h_{\text{pol}}(f) = \lim_{\varepsilon \to 0} h^{\varepsilon}_{\text{pol}}(f).
\]

A set \( F \subset X \) is \((n, \varepsilon)\)-separated if for different points \( x, y \in F \) there is \( 0 \leq k < n \) such that \( \text{dist}(f^k(x), f^k(y)) \geq \varepsilon \). We denote by \( s_n(\varepsilon) \) the maximal cardinality of an \((n, \varepsilon)\)-separated subset of \( Y \). The topological and polynomial entropies can be defined equivalently in terms of separated sets (see [5, 19]). Note that the subtle change of \( n \) by \( \log(n) \) (as in the classical topological entropy), gives completely different values for the entropy of the system.

For \( x \in X \) and \( \varepsilon > 0 \) we denote by \( B_\varepsilon(x) \) the open ball of radius \( \varepsilon \) and centered at \( x \). The following result is a generalization of [16, Proposition 2.1] and [19, the end of the proof of Proposition 5]. We recall that a point \( x \in X \) is recurrent if for all \( \varepsilon > 0 \) there is \( m \geq 1 \) such that \( f^m(x) \in B_\varepsilon(x) \).
exists a sequence $q_n$ of homeomorphism $f$ and $g$ such that for all $x \in X$ there is $n \in \mathbb{Z}$ such that $f^n(x) \in B_{\epsilon}(x)$ and $0 < n \leq m$. In particular, if $h_{pol}(f) < 1$ then every point $x \in X$ is recurrent.

Proof. Arguing by contradiction, assume that $h_{pol}(f) < 1$ and there is $\epsilon > 0$ such that for all $m \geq 1$ there is $x \in X$ for which $f^n(x) \notin B_\epsilon(x)$ for all $n = 1, 2, \ldots, m$. Define the set $F = \{x, f^{-1}(x), \ldots, f^{-m}(x)\}$. Given two different points $u = f^{-j}(x), v = f^{-k}(x) \in F$ with $0 \leq j < k \leq m$, we have that $f^k(v) = x$ and $f^k(u) = f^{k-j}(f^j(u)) = f^{k-j}(x)$. As $0 < k - j \leq m$ we conclude that $f^k(u) \notin B_\epsilon(x)$ and $\text{dist}(f^k(u), f^k(v)) > \epsilon$. This proves that $F$ is an $(m, \epsilon)$-separated set with $m+1$ elements. Consequently, $s_m(\epsilon) \geq m+1$ and $h_{pol}(f) \geq 1$. This contradiction finishes the proof. \hfill \square

3 Calculating the polynomial entropy

In this section we will construct a family of examples depending on a sequence $a_n$. For $a_n = e^{-n}$ we will obtain a homeomorphism with vanishing polynomial entropy that is not equicontinuous. For $a_n = 1/n^c$ with $c > 1$ we will obtain an example with non-integer polynomial entropy.

We start with the general properties of our construction. Consider the circle $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ with the usual distance $\text{dist}_S$. Given $\{a_n\}_{n \geq 1}$ a decreasing sequence of positive real numbers such that $a_n \to 0$, take

$$M = \mathbb{S}^1 \times (\{a_n: n \in \mathbb{N}\} \cup \{0\}),$$

equipped with the distance $\text{dist}$ defined as

$$\text{dist}((x_1, y_1), (x_2, y_2)) = \max\{\text{dist}_S(x_1, x_2), |y_2 - y_1|\}.$$

The space $M$ is a countable union of circles. Let $f: M \to M$ be the function such that

$$f(x, y) = (x + y, y). \quad (3.1)$$

In this way, $f$ acts as a rotation of angle $a_n$ on each circle of $M$.

Remark 3.1. Observe that $f$ is not equicontinuous. This can be seen using the following property: if $g: X \to X$ is equicontinuous, $p \in X$ is a fixed point of $g$ and $q_n \to p$ then $\lim_{n \to \infty} g^n(q_n) \to p$. In our case, we have that there exists a sequence $q_n$ that converges to $(0, 0)$ but $f^n(q_n) \not\to (0, 0)$. Then, the homeomorphism $f$ we are considering is not equicontinuous.

Given $N \in \mathbb{N}$ and $\epsilon > 0$, we will find an $(N, \epsilon)$-spanning set for $f$. We define

$$H = \min\{n \in \mathbb{N}: Na_n < \epsilon\}$$

and

$$r = \left\lfloor \frac{1}{\epsilon} \right\rfloor + 1.$$
We can take a set with $r$ elements $P = \{p_1, \ldots, p_r\} \subset S^1$ such that
\[
\text{dist}_S(p_i, p_{i+1}) < \varepsilon
\]
for all $i = 1, \ldots, r$. Define $A(N, \varepsilon) = P \times (\{a_n : n \leq H\} \cup \{0\})$.

**Proposition 3.2.** The set $A(N, \varepsilon)$ is an $(N, \varepsilon)$-spanning set for $f$ with
\[
\#A(N, \varepsilon) = (\lfloor 1/\varepsilon \rfloor + 1)(H + 1).
\]

**Proof.** Let $q = (q_1, q_2) \in M$. If $q_2 \geq a_H$ then there exists $p = (x, q_2) \in A(N, \varepsilon)$ such that $\text{dist}_S(x, q_1) < \varepsilon$. As $f$ is a rotation in $S^1 \times \{q_1\}$, then
\[
\text{dist}(f^k(p), f^k(q)) = \text{dist}_S(x, q_1) < \varepsilon
\]
for all $k \in \mathbb{Z}$ (in particular, if $0 \leq k < N$).

If $q_2 < a_H$ then $Nq_2 < Na_H < \varepsilon$. Then, there is an element $p = (x, 0) \in A(N, \varepsilon)$ such that $\text{dist}_S(x, q_1 + kq_2) < \varepsilon$ for all $k = 0, \ldots, N - 1$. Therefore,
\[
\text{dist}(f^k(x, 0), f^k(q_1, q_2)) = \text{dist}((x, 0), (q_1 + kq_2, q_2)) = \max\{\text{dist}_S(x, q_1 + kq_2), q_2\} < \varepsilon
\]
for all $k = 0, \ldots, N - 1$. This proves that $A(N, \varepsilon)$ is an $(N, \varepsilon)$-spanning set for $f$. The cardinality of $A(N, \varepsilon)$ is easily calculated from definitions.

### 3.1 Vanishing polynomial entropy

The next result gives an example of a homeomorphism with vanishing polynomial entropy that is not equicontinuous.

**Proposition 3.3.** If $a_n = e^{-n}$ then $h_{pol}(f) = 0$.

**Proof.** In this case $H = \lceil -\log(\frac{\varepsilon}{N}) \rceil$. By Proposition 3.2 we have that $A(N, \varepsilon)$ is an $(N, \varepsilon)$-spanning set with
\[
\#A(N, \varepsilon) = (\lfloor 1/\varepsilon \rfloor + 1)(\lceil -\log(\varepsilon/N) \rceil + 1)
\]
Then, for all $\varepsilon > 0$ it holds that
\[
\lim_{N \to \infty} \frac{\log(\#A(N, \varepsilon))}{\log(N)} = \lim_{N \to \infty} \frac{\log(\lfloor 1/\varepsilon \rfloor + 1) + \log(\lceil -\log(\varepsilon/N) \rceil + 1)}{\log(N)} = 0.
\]
This proves that $h_{pol}(f) = 0$.

The example given in Proposition 3.3 is defined in the space $M$ that is a countable union of circles. In particular, the space is not connected.

**Problem 1.** If $f$ is a homeomorphism of a connected compact metric space with $h_{pol}(f) = 0$ is it true that $f$ is equicontinuous?
A homeomorphism $f$ is distal if for all $x \neq y$ it holds that
\[
\inf_{n \in \mathbb{Z}} \text{dist}(f^n(x), f^n(y)) > 0.
\]
Note that equicontinuous homeomorphisms are distal. As remarked above, our homeomorphism $f$ is not equicontinuous. It is easy to see that it is distal.

**Problem 2.** If $f$ is a homeomorphism of a compact metric space such that $h_{pol}(f) = 0$ must $f$ be distal?. It would be interesting to know a dynamical characterization of those homeomorphisms with vanishing polynomial entropy.

### 3.2 Non-integer polynomial entropy

Now we will show that the polynomial entropy may not be an integer number.

**Remark 3.4.** In the case of the topological entropy we can easily construct a homeomorphism with non-integer topological entropy from a homeomorphism with non-vanishing integer topological entropy. Suppose that $f : X \to X$ is a homeomorphisms. Consider $Y$ as a disjoint union of $n$ copies of $X$. Formally, $Y = \{1, \ldots, n\} \times X$. Define $g : Y \to Y$ as $g(i, x) = (i + 1, x)$ if $i = 1, \ldots, n - 1$ and $g(n, x) = (0, f(x))$. Then, $g^n$ restricted to $\{i\} \times X$ is conjugate to $f$, for all $i = 1, \ldots, n$. This implies that $h_{top}(g) = h_{top}(f)/n$. Then, if $h_{top}(f)$ is an integer we can take $n \in \mathbb{Z}^+$ such that $h_{top}(f)/n$ is not an integer and we obtain a homeomorphism $g$ with non-integer topological entropy. For the polynomial entropy we have that $h_{pol}(f^n) = h_{pol}(f)$ for all positive integer $n$. This is proved in [19, Proposition 2]. Then, this trick does not work to obtain a homeomorphism with non-integer polynomial entropy.

**Proposition 3.5.** If $a_n = \frac{1}{n^c}$, with $c \geq 1$, then $\frac{1}{c+1} \leq h_{pol}(f) \leq \frac{1}{c}$, where $f : M \to M$ is the homeomorphism given by (3.1).

**Proof.** In this case $H = [(\frac{N}{c})^+]$ and for all $\varepsilon > 0$ it holds that
\[
\lim_{N \to \infty} \frac{\log(#A(N, \varepsilon))}{\log(N)} = \lim_{N \to \infty} \frac{\log([\frac{1}{c}] + 1) + \log([\frac{N}{c}]^+))}{\log(N)} = \frac{1}{c}.
\]
This proves that $h_{pol}(f) \leq \frac{1}{c}$.

To prove the other inequality we will construct a separating set. We can take a set $Y \subset S^1$ with $r - 1$ elements such that $\dist_S(z, w) \geq \varepsilon$ for all $z, w \in Y$, $z \neq w$. Define
\[
S(N, \varepsilon) = Y \times \{a_n : n < D\},
\]
where $D = (\frac{\varepsilon N}{c})^+$. Let $p = (x, \frac{1}{n})$ and $q = (y, \frac{1}{m})$ be different points of $S(N, \varepsilon)$, with $n \geq m$. If $x \neq y$ then $\dist(p, q) \geq \dist_S(x, y) \geq \varepsilon$. If $x = y$ then
\[
\dist(f^k(p), f^k(q)) = \dist((x + \frac{k}{n}, \frac{1}{n}), (x + \frac{k}{m}, \frac{1}{m})) \geq \dist_S(x + \frac{k}{n}, x + \frac{k}{m}).
\]

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Applying the mean value theorem we obtain
\[
\text{dist}_S \left( x + \frac{k}{n^c}, x + \frac{k}{m^c} \right) = k \left| \frac{1}{n^c} - \frac{1}{m^c} \right| = k \frac{\theta^{c-1}}{m^c} \left| \frac{1}{n} - \frac{1}{m} \right|
\]
for some \( \frac{1}{n} < \theta < \frac{1}{m} \).

As \( n < D \), we have for some \( k \leq N \):
\[
\text{dist}(f^k(p), f^k(q)) > k c \left( \frac{1}{n} \right)^{c-1} \frac{m - n}{mn} \geq \frac{k c}{n^{c+1}} > \varepsilon.
\]

We conclude
\[
\lim_{N \to \infty} \frac{\log(\#S(N, \varepsilon))}{\log(N)} = \lim_{N \to \infty} \frac{\log(\left\lfloor \frac{1}{\theta} \right\rfloor) + \log(\left\lfloor \frac{cN}{n^{c+1}} \right\rfloor)}{\log(N)} = \frac{1}{c + 1},
\]
Therefore \( h_{pol}(f) \geq \frac{1}{c + 1} \) and the proof ends.

**Remark 3.6.** Applying [19, Proposition 2] we know that if \( f \) and \( g \) are homeomorphisms of compact metric spaces then \( h_{pol}(f \times g) = h_{pol}(f) + h_{pol}(g) \). This and Proposition 3.5 give us that the set \( \{ \alpha \in \mathbb{R} : \alpha = h(f) \text{ for some homeomorphism } f \} \) is dense in \( \mathbb{R}^+ \).

**Problem 3.** Is every positive real number the polynomial entropy of a homeomorphism of a compact metric space?

**Problem 4.** If \( f \) is a homeomorphism of a connected compact metric space with \( h_{pol}(f) \) finite, is it true that \( h_{pol}(f) \) is an integer number?

**Remark 3.7.** The example of Proposition 3.5 shows that distality does not imply vanishing polynomial entropy.

## 4 Expansive systems

In this section we will show that expansive homeomorphisms and positively expansive maps of compact metric spaces with infinitely many points have positive polynomial entropy. These results are true and well known in the case of topological entropy if the space has positive topological dimension.

### 4.1 Expansive homeomorphisms

A homeomorphism \( f : X \to X \) of a compact metric space \((X, \text{dist})\) is expansive if there exists \( \delta > 0 \) (an expansivity constant) such that if \( \text{dist}(f^n(x), f^n(y)) < \delta \) for all \( n \in \mathbb{Z} \) then \( x = y \). It is known that for an expansive homeomorphism it holds that \( h_{top}(f) = h_{top}(f, \delta) \), see [5, Proposition 2.5.7] (also [4]). We remark that
\[
h_{pol}(f) = h_{pol}(f, \delta)
\]
also holds for the polynomial entropy, if $\delta$ is an expansivity constant. The proof is analogous.

As we said, the topological entropy of an expansive homeomorphism on a compact metric space of positive topological dimension is positive, see [9, 13]. Also, there are expansive homeomorphisms with vanishing topological entropy, for example, expansive homeomorphisms of a countable spaces and the non-wandering set of a Denjoy circle diffeomorphism. We will show that the polynomial entropy of every expansive homeomorphism of an infinite compact metric space is greater than or equal to 1 (in particular, positive).

Let $\sigma : \Sigma \to \Sigma$ be a two-sided full shift on $l \geq 2$ symbols. We say that $\mathcal{M} \subset \Sigma$ is a minimal set if $\mathcal{M}$ is the closure of the orbit of each $x \in \mathcal{M}$.

**Theorem 4.1.** If $f$ is an expansive homeomorphism of a compact metric space with infinitely many points then $h_{\text{pol}}(f) \geq 1$.

**Proof.** At first, we will prove the theorem in the case $f$ is the restriction of a shift $\sigma : \Sigma \to \Sigma$ to a minimal set $\mathcal{M}$.

Let $\delta$ be an expansivity constant for $\sigma$, $x \in \Sigma$ be a not periodic point for $\sigma$. From [12, Theorem 7.3] (or [10, Proposition 2]) we know that for every $n \geq 1$ there exists $C \subset \Sigma$ with $n + 1$ elements, included in the orbit of $x$, such that if $a, b \in C$ with $a \neq b$ then $\text{dist}(\sigma^i(a), \sigma^i(b)) > \delta$ for some $0 \leq i \leq n$. Now, taking $x \in \mathcal{M}$, we can conclude that for all $\varepsilon \leq \delta$ and for all $n \geq 1$ there exists an $(n, \varepsilon)$-separated set included in $\mathcal{M}$ with $n + 1$ elements. So, $r_n(\varepsilon) > n$ for all $\varepsilon \leq \delta$ and for all $n \geq 1$. From the definition, this implies that $h_{\text{pol}}(\sigma|_{\mathcal{M}}) \geq 1$.

Now we will see the general case, that is, when the domain $X$ of $f$ is any compact metric space. Arguing by contradiction assume that $h_{\text{pol}}(f) < 1$. Proposition 2.1 implies that every point is recurrent.

Suppose that $p \in X$ is a periodic point, say $f^m(p) = p$. We will show that there is $r > 0$ such that $B_r(p) = \{p\}$. If this is not the case we can take $x_n \to p$ with $x_n \neq p$ for all $n \geq 1$. Since $f^m$ is expansive there is $\delta_0$ such that for all $n \geq 1$ there is $k_n \in \mathbb{Z}$ with

$$f^{mk_n}(x_n) \notin B_{\delta_0}(p). \quad (4.1)$$

Taking a subsequence we can suppose that $k_n \geq 0$ for all $n \geq 1$ (the other case is similar). We can also assume that $k_n$ is the first number satisfying (4.1), that is, $f^{mj}(x_n) \in B_{\delta_0}(p)$ for all $j = 0, 1, \ldots, k_n - 1$. Taking a subsequence we can suppose that $f^{mk_n-1}(x_n) \to x$. The continuity of $f$ implies that $f^{-mj}(x) \in \text{cl}(B_{\delta_0}(p))$ for all $j \geq 0$. Expansivity implies that $f^{-mj}(x) \to p$ as $j \to +\infty$. This contradicts that $x$ is recurrent and proves that every periodic point is an isolated point of the space.

From [18] we know that every minimal subset of $X$ is conjugate to a subshift. As we know the theorem is true in this case, we conclude that every minimal subset of $X$ is a periodic orbit. Since every point is recurrent, we conclude that every point is periodic, and then, every point is isolated. As the space is compact, it must be finite. This contradiction proves the result. $\square$
Remark 4.2. For a Sturmian minimal subshift it holds that \( r_n(M, \delta) = n + 1 \) and its polynomial entropy equals 1. In [8] it is shown an example of a subshift with vanishing topological entropy and infinite polynomial entropy.

4.2 Positively Expansive Maps

Let \( f : X \to X \) be a continuous map of a compact metric space \((X, \text{dist})\). We say that \( f \) is positively expansive if there is \( \delta > 0 \) such that if \( \text{dist}(f^n(x), f^n(y)) < \delta \) for all \( n \geq 0 \) then \( x = y \).

Remark 4.3. If \( f : X \to X \) is a positively expansive map and the topological dimension of \( X \) is positive then \( h_{\text{top}}(f) > 0 \), hence \( h_{\text{pol}}(f) = \infty \). This can be proved with the techniques of [9,13].

We recall that \( X \) has positive topological dimension if and only if it is not totally disconnected. In [20, Theorem 1.2] it is shown that for every positively expansive map there is \( \varepsilon > 0 \) such that \( \lim_{n \to +\infty} s_n(X, \varepsilon) = +\infty \). We extend this result as follows.

**Theorem 4.4.** If \( f : X \to X \) is a positively expansive map and \( \text{card}(X) = \infty \) then \( h_{\text{pol}}(f) \geq 1 \).

**Sketch of the proof.** As explained in Remark 4.3, if \( X \) has positive dimension then \( h_{\text{pol}}(f) = \infty \). If \( X \) is totally disconnected it is known that \( f \) is conjugate with a one-sided subshift. Therefore, the proof follows applying [12, Theorem 7.3] as in the proof of Theorem 4.1.

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