The Meissner effect in the ground state of free charged Bosons in a constant magnetic field

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October 1, 2014

Abstract

The model of free charged Bosons in an external constant magnetic field inside a cylinder, one of the few locally gauge covariant systems amenable to analytic treatment, is rigorously investigated in the semi-classical approximation. The model was first studied by Schafroth and is suitable for the description of quasi-bound electron pairs localized in physical space, so-called Schafroth pairs, which occur in certain compounds, but he used perturbation theory on the magnetic field, which is not applicable to the homogeneous fields for which the thermodynamical results may be derived. A simple nonperturbative spectral argument shows that, for sufficiently low values of the magnetic field, the ground state expectation value of the current $\langle \mathbf{j}(\mathbf{x}) \rangle$ is of the London form $\langle \mathbf{j}(\mathbf{x}) \rangle = -\lambda(\mathbf{x})\mathbf{A}(\mathbf{x})$, with $\lambda$ a positive function of $\mathbf{x}$, $\mathbf{A}$ denoting the magnetic vector potential. As a consequence, the magnetic induction inside the sample is given by a non-uniform field, monotonically decreasing from the surface. Under a plausible assumption on the ground state wave function, the Meissner effect is derived, and the results fit the thermodynamics calculated by Schafroth as a finite size correction. We also briefly review the link with relativistic quantum field theory, in particular with London-Schwinger screening.
1 Introduction: motivation of the paper

The Meissner effect is perhaps the prototypical electrodynamic property of superconductors, and is one of the most spectacular effects in physics. It may be stated as follows (see, e.g., [MR04], pg. 162):

Meissner effect If a superconductor sample is submitted to a magnetic (applied) field $\vec{H}$ and then cooled to a temperature $T$ below the transition temperature, there is a critical field $H_c(T) > 0$ such that, if $|\vec{H}| < H_c(T)$, the field is expelled from all points of the sample situated sufficiently far from from the surface, i.e., beyond a certain distance from the surface called the ”penetration depth”.

It follows from the above that the Meissner effect is an extreme instance of the magnetic effects found in substances, called diamagnetic, which are repelled from, e.g., a strong electromagnet with a sharply pointed pole piece - one might call them complete diamagnets [Sch60]. As such, it is necessarily a quantum phenomenon, because classical physics explains neither paramagnetism or diamagnetism in an equilibrium state (thermal or ground state) [Fey63]. The word ”equilibrium” is important here, and therefore it should be mentioned that, in contrast to the vanishing of electrical resistance, the Meissner effect is a phenomenon in thermal equilibrium - see [Sch60], pg. 299 for a full justification of this assertion. An important part of the argument is directly connected to the present paper, and therefore we mention it here: the quantum aspect of the theory, when treated semiclassically, depends only on the expectation value of the current $\vec{j}(\vec{x})$ in the equilibrium state $<.>$ (ground state or thermal), see the forthcoming (2.7), in case the state is an equilibrium state. When no magnetic field is present, this equilibrium state may be assumed to be invariant under the time reversal operator $\tau$. However, $\tau(\vec{j}(\vec{x})) = -\vec{j}(\vec{x})$, hence $< \vec{j}(\vec{x}) >=< \tau(\vec{j}(\vec{x})) >= - < \tau(\vec{j}(\vec{x})) >= 0$. This is Bloch’s theorem (unpublished), cited in [Sch60] and [Kad13]. Thus, the phenomenon of persistent currents, unlike the Meissner effect, cannot be an equilibrium phenomenon. A generalization of this statement beyond the semiclassical approximation for systems carrying a current was given by Sewell in [Sew80], see also [SW09] for the superfluid case.

The theory of Bardeen, Cooper and Schrieffer [JBS57] - see also the textbook accounts, too numerous to cite; we mention only [MR04], [Enz92] and [Zim65], which will be directly referred to in this paper - has been very successful in explaining a variety of phenomena observed in (ordinary) superconductivity, such as the energy of the superconducting ground state and
the energy gap associated with single-particle excitations. An excellent account, to a large extent still up-to-date, is M. R. Schafroth’s review, which appeared shortly before his untimely death [Sch60], of which we quote: ”The most serious failing of the theory of BCS is, however, its failure to account for the electrodynamical properties of superconductors, such as the Meissner effect and the persistent currents.”

Schafroth’s remark was due to the fact that the BCS model does not satisfy the requirement of local gauge invariance (or, more properly, local gauge covariance, see section 3): this fact was observed by the authors of [JBS57] themselves in a footnote in [JBS57]: we shall refer to this problem briefly as the ”gauge problem”. Many very renowned scientists worked on the gauge problem ([And58b], [And58a], [And58c], [Nam60]), see also [Sew02], and further references given there. Today, the best accepted justification seems to be the one by Anderson ([And58b], [And58a], [And58c]), quoted by Kadanoff [Kad13] in his recent historical review: ”oscillations of the gap parameter or, equivalently, of the condensate wave-function, produces extra states of the system, states which rescue the gauge invariance of the BCS theory”. The type of oscillation referred to by Kadanoff are the plasma oscillations, whose inclusion should, according to Anderson, favor the particular gauge adopted by BCS [JBS57] to study the electrodynamical properties of the ground state. However, as remarked by Schafroth [Sch60], pg. 471), since Anderson’s starting point ([And58b], [And58a]) was the truncated Hamiltonian of the BCS theory, ”whose relation to the complete Hamiltonian of the metal electrons (which would include the plasma oscillations) is not well-defined, his considerations can be, at most, of heuristic value”. The more systematic construction of the ”collective excited states” by Bogoliubov [NNBS59], with energies lying in the energy gap of single-particle excitations, is spoiled if one attempts to include the Coulomb interaction ([NNBS59], [And58b], [And58a]), see also the remarks in [Sch60], pp 487-488), showing that Anderson’s justification of local gauge invariance of the BCS model displays the same kind of instability. Moreover, the validity of the concept of ”approximately gauge invariant” used by him ([And58b], [And58a]) is difficult to assess, and the random-phase approximation [And58c] ( see also [MR04] for a particularly lucid discussion) does not provide a way of estimating the errors involved in the approximation. It is certainly impossible to reconcile such approximate arguments with the fact that the BCS model is regarded as one of the few rigorous (mean-field) models in statistical mechanics (see, e.g., [vH78] and references given there).
Although the basic (gauge covariant) model for superconductivity, the H. Froehlich electron-phonon system ([Sch60]), has so far resisted analysis, a serious alternative to the BCS theory arose when Yang [Yan62] proposed that the property of off-diagonal long-range-order (ODLRO), introduced by Penrose and Onsager [PO56] (see [Sew02], 9.3.2, for the precise definition), provides a characterization of the superconductive phase (see also chapter 5 of [Leg06]). This proposal was taken up by Sewell ([Sew02], [Sew90]), who provided a simple and elegant proof of the following fact (here roughly stated: for details, see Prop. 9.3.2, pg 218, of [Sew02] or [Sew90]): when the magnetic induction is uniform and the equilibrium state of the system is covariant under space translations and local gauge transformations and possesses the ODLRO property, the magnetic induction vanishes. Further assumptions related to thermodynamic stability ([Sew02], pp. 223-224) imply the Meissner effect (Prop. 9.4.1, pg. 255, of [Sew02]).

Sewell’s analysis captures the conceptual essence of the Meissner effect, up to one important point: since it concerns the bulk of the material, i.e., infinite systems, it misses the phenomenon of the penetration depth (see, e.g., [MR04], pg. 165). It is our objective in the present paper to fill this gap, at least partially, by revisiting the model of the ground state (g.s) of free charged Bosons in a constant magnetic field, in the semiclassical approximation. In spite of the simplicity of this model, we believe, with Schafroth, who first studied it (also for nonzero temperature) ([Sch60], [Sch55]), that it exhibits (in addition to being locally gauge covariant) several of the essential features of real superconductors: the first one is necessarily the thermodynamics, which yields the (experimentally measurable) $B(H)$-curve (4.29), where $B$ denotes the magnetic induction, in the case of a homogeneous (constant) magnetic field ([Sch60], [Sch55]). On the other hand, in analogy to the charge-screening phenomenon (see, e.g., [MR04], pg. 166 and the conclusion section 6), the existence of a penetration depth is a surface phenomenon, which may be expected to arise as a “finite-size correction” to the thermodynamics, but, in any case, should fit the thermodynamical results. Since the latter have only been obtained for homogeneous fields, in which case (as Schafroth himself remarks) perturbation theory in the magnetic field is not applicable, Schafroth’s use of perturbation theory to investigate the penetration depth may be open to some criticism.

The subject of the present paper may be considered by many to be well understood, even by those who are aware of the “gauge problem”. They
would argue that the macroscopic "slippery" wave function \( \Phi(\vec{x}, t) \), a function varying appreciably only over distances characteristic to variations of the electromagnetic fields themselves ([Kad13], [Leg99], [Leg06]), provides a justification of the Meissner effect, as shown in several textbook treatments (see [MR04], pg. 165 or [Fey65], 21-9)). It is, however, the previously mentioned concept of ODLRO which provides a quantum, i.e., from first principles justification of the macroscopic wave-function (see chapter 5 of [Leg06] or [Kad13]). If one accepts this, there remains, however, a basic unresolved conceptual problem (not just a question of mathematical rigor!): by the previously mentioned theorem by Sewell (Prop. 9.3.2. of [Sew02]) - a nonperturbative result! - ODLRO implies, together with the other standard assumptions of space translation and local gauge invariance) that \( \vec{B} = \vec{0} \). It is, however, essential to consider a \( \vec{B} \neq \vec{0} \) (with \( |\vec{B}| \) sufficiently small) to start with, to show that the field is repelled! This shows that the assumption of a macroscopic wave function is not, a priori, justified.

In addition to statistical mechanics, the Meissner effect has also been considered to belong to the realms of relativistic quantum field theory (rqft), and is even discussed in Weinberg’s famous book [Wei96]. The reason is that the penetration depth is a direct analogue of the screening length in electrostatics, and, although the physical mechanisms of charge screening in rqft are necessarily different in a profound sense (since many-body theory possesses neither vacuum polarization or microcausality (locality)), the Meissner effect does have an element in common with rqft, viz., the London form of the current: a link with the London-Schwinger screening mechanism exists, and will be briefly reviewed in section 5.

Our main contribution is a complete derivation of the London form of the current, with no appeal to perturbation theory. It must be said, however, that the full derivation of the exponential decay and the remaining features of the effect still rely, as in Schafroth’s approach, on an assumption on the behavior of the ground state wave function (assumption A), which we argue to be eminently plausible, but whose proof is very difficult, even for this apparently very simple model. The basic reason for this fact is that the semiclassical approximation defines a pair of coupled equations for the ground state wave function and the magnetic induction, and therefore the (self-consistent) ground state wave function is basically unknown. There are good reasons to believe that the self-consistent g.s. solution is "close" to the free (constant) wave-function (see also [Sch60], [Sch55]), and, indeed, this is
the subject of assumption A, but it is not clear how to give a precise sense to this notion, much less how to prove the assumption. Nevertheless, we hope that a clear exposition of these problems may be interesting and useful.

For a reader less interested in the mathematics, we only present the idea of the arguments in the main text, leaving the details to the appendices A and B.

2 The model: ground state of a free charged Bose gas in an external constant magnetic field in the semiclassical approximation

The model we shall revisit was studied by Schafroth [Sch55], as one of the very few locally gauge invariant systems possibly related to superconductivity. Today, it may be viewed as a model for Schafroth pairs [Sch54], which are known to occur in certain compounds ([NR85], [Leg80]), i.e., quasi-bound electron pairs localized in physical space, i.e., for which the spatial extension of the pair wave function, measured by the coherence length, is small compared with the average distance between pairs. In this case all electrons of the band are paired, and the pairs form a dilute Bose gas (see also the discussion in the book by Enz [Enz92], chapter 4, pg. 180 et seq.). The Hamiltonian may be written ($e = 2e_0$, $e_0$ being the electron charge):

$$H(\vec{A}) = \frac{\hbar^2}{2m} \int_K \left( \nabla + \frac{ie}{\hbar} \vec{A}(\vec{x}) \right) \Psi^*(\vec{x}) \cdot \left( \nabla - \frac{ie}{\hbar} \vec{A}(\vec{x}) \right) \Psi(\vec{x}) d\vec{x}$$

(2.1)

where $\Psi(\vec{x})$ and $\Psi^*(\vec{x})$ are the basic destruction and creation operators on symmetric Fock space $\mathcal{F}_s(\mathcal{H})$, with $\mathcal{H} = L^2(K)$ the (one-particle) Hilbert space of square integrable wave functions on the cylinder $K$ (see, e.g., [MR04] pg. 91 - there the $\Psi$ are denoted by $a$), satisfying the canonical commutation relations

$$[\Psi(\vec{x}), \Psi^*(\vec{x}')] = \delta(\vec{x} - \vec{x}')$$

(2.2.1)

or, in smeared form, with

$$\Psi(f) = \int d\vec{x} f(\vec{x}) \Psi(\vec{x})$$

(2.2.2)
[Ψ(f), Ψ*(g)] = (f, g) \quad (2.2.3)

for \( f, g \in \mathcal{H} \), and \( (f, g) \equiv \int d\vec{x} \bar{f}(\vec{x})g(\vec{x}) \) the inner (scalar) product in the Hilbert space \( \mathcal{H} \), \( \bar{f} \) denoting the complex conjugate and * the Hermitian conjugate or adjoint. \( H(\vec{A}) \) is the (self-adjoint) second quantization of a one-particle operator on \( \mathcal{H} \) ([MR04], pg 101), with certain boundary conditions (see appendix A). The current density operator in this model is [Sch60], pg. 416:

\[
\vec{j}(\vec{x}) = \vec{j}_{\text{mom}}(\vec{x}) + \vec{j}_{\text{Lon}, \vec{B}}(\vec{x}) \quad (2.3)
\]

where

\[
\vec{j}_{\text{mom}}(\vec{x}) = -\frac{i e \hbar}{2m} : \Psi^*(\vec{x}) \nabla \Psi(\vec{x}) - \Psi(\vec{x}) \nabla \Psi(\vec{x}) : \quad (2.4)
\]

and

\[
\vec{j}_{\text{Lon}, \vec{B}}(\vec{x}) = -\frac{e^2}{m} \Psi^*(\vec{x}) \Psi(\vec{x}) \vec{A}(\vec{x}) \quad (2.5)
\]

Above, :: denotes the Wick or normal product (e.g., [MR04], pg 100). since (2.4) is the momentum density operator, we call it the momentum density component of the current; \( \vec{j}_{\text{Lon}, \vec{B}} \) is the London part ([MR04], (5.15), pg. 164 et seq.), about which much will be said later on.

The **semiclassical** (static) model will be defined by the set of Maxwell equations for the magnetic induction \( \vec{B} \) (using the MKS system):

\[
(\nabla \cdot \vec{B})(\vec{x}) = 0 \quad (2.6)
\]

\[
(\nabla \times \vec{B})(\vec{x}) = \mu_0 \Omega_{\vec{B}} (\vec{j}(\vec{x}) \Omega_{\vec{B}}) \quad (2.7)
\]

where

\[
\vec{B}(\vec{x}) = (\nabla \times \vec{A})(\vec{x}) \quad (2.8)
\]

and \( \Omega_{\vec{B}} \) is the ground state of \( H(\vec{A}) \), assumed unique (see later). (2.6), (2.7) is the standard formulation of the "back-reaction" of the quantum field \( \vec{j} \) on the classical field \( \vec{B} \), analogous to the back reaction of the energy-momentum tensor quantum field upon the space-time geometry in the semiclassical Einstein equation ([Wal94], Chapter 4, pg.98). Under conditions analogous to the ones stated in the latter reference - presently, that the fluctuations of \( \vec{j} \) in \( \Omega_{\vec{B}} \) are small compared with \( |(\Omega_{\vec{B}}, \vec{j}\Omega_{\vec{B}})| \) (which may be expected for macroscopic systems, see, e.g., the analogous discussion of the relative fluctuation of the mean number of particles in the BCS ground state in [MR04], pg. 192), and, further, that the fluctuations of the quantized field associated to
\( \bar{B} \) in the corresponding state (quantum vacuum or thermal state) are small in comparison with the average of the square of the field (\cite{Sak67}, pg. 35), the semiclassical model may be expected to be a good approximation. However, as Wald remarks (op. cit. pg. 98), "even the analogous (to Einstein’s equation) justification of the semiclassical Maxwell equation \( \nabla^a F_{ab} = -4\pi < j_b > \) in quantum electrodynamics has not yet been given".

A very important point concerning (2.7) is that it defines an interacting system: the g. s. \( \Omega \bar{B} \) depends, itself, on the solution to (2.7), similarly to the much harder problem of the semiclassical Einstein equation in a cosmological scenario \cite{Pin11}. Even in the present case the problem is very difficult and the ground state wave-function is not explicitly known. It lies, however, very closely "in between" two known problems, which allows us to formulate a very plausible assumption on its behavior (assumption A), which, however, remains open.

A major issue in the Meissner effect and, indeed, in any phenomenon involving currents, is local gauge invariance, our next topic.

3 Local gauge covariance, its role in the semiclassical model, and properties of the current

In classical field theory, local gauge invariance or the gauge principle\cite{MR04}, Ch. 8, pg. 304, \cite{Thi78}, 4.2, pg. 174) allows one to deduce the form of the field-matter interactions, whereby, specializing to global (space-time independent) transformations, local current conservation is obtained \cite{Thi78}, 4.2, pg. 174). In the quantum case, local gauge transformations may be defined by

\[
\begin{align*}
\Psi^*(\vec{x}) & \rightarrow \Psi^*(\vec{x}) \exp(i\epsilon \alpha(\vec{x})/\hbar) \\
\Psi(\vec{x}) & \rightarrow \Psi(\vec{x}) \exp(-i\epsilon \alpha(\vec{x})/\hbar) \\
\vec{A}(\vec{x}) & \rightarrow \vec{A}(\vec{x}) + \nabla \alpha(\vec{x})
\end{align*}
\]

or, in the smeared form (2.2.2),

\[
U^{-1}_\alpha \Psi^*(f) U_\alpha = \Psi^*(\exp(i\alpha \epsilon)f)
\]
with $U_\alpha$ the unitary operator on Fock space given by

$$U_\alpha = \exp(-i \int_K d\bar{x} \alpha(\bar{x}) \rho(\bar{x}))$$  \hspace{1cm} (3.4)

and $\rho$ the density operator

$$\rho(\bar{x}) = \Psi^*(\bar{x})\Psi(\bar{x})$$  \hspace{1cm} (3.5)

(this is analogous to [MR04], pg. 107). **Local gauge covariance** is defined
in the quantum case by ([Sch60], (13.31), pg. 410)

$$U_\alpha^{-1} H(\vec{A}) U_\alpha = H(\vec{A} + \nabla \alpha)$$  \hspace{1cm} (3.6)

which is readily seen (using (2.2.1) to be satisfied by the Hamiltonian $H(\vec{A})$
given by (2.1). Specializing to $\vec{x}$-independent (global) gauge transformations
leads to current conservation

$$\frac{\partial \rho(\vec{x}, t)}{\partial t} + \nabla \cdot \vec{j}(\vec{x}, t) = 0$$  \hspace{1cm} (3.7)

with the time dependent operators defined by the Heisenberg picture evolution
under $H(\vec{A})$, the operators at zero time being given by (2.3)-(2.5) and
(3.5). We have $\frac{\partial \rho(\vec{x}, t)}{\partial t} = i[H(\vec{A}), \rho(\vec{x}, t)]$, from which $(\Omega_{\vec{B}}, \frac{\partial \rho}{\partial t} \Omega_{\vec{B}}) = 0$ and,
thus, by (3.7),

$$\nabla \cdot (\Omega_{\vec{B}}, \vec{j}(\vec{x}) \Omega_{\vec{B}}) = 0$$  \hspace{1cm} (3.8)

(3.8) is, of course, a necessary condition for the validity of (2.7), the basic
equation of the semiclassical theory. It should be remarked that equations
(2.6), (2.7) should be considered in the distributional sense, i.e., as functionals
on $\mathcal{D}^\ast(\Gamma)$, the space of vector-valued functions whose components are $C^\infty$
functions with compact support in (some open region) $\Gamma$ (see, e.g., [Jos76], pp
47-51), with the boundary conditions also taken in the distributional sense.
This is specially important here, because even in the simple case (2.3)-(2.5),
the current is not an operator when taken pointwise, and, thus, (3.7) and
(3.8) are formal calculations, which may be made rigorous by employing the
smeared forms $\rho \rightarrow \rho(\vec{f})$, $\vec{j} \rightarrow \vec{j}(\vec{f})$, where $f$ is a smooth function of compact
support in $\mathbb{R}^3$; $(\nabla \cdot \vec{j})(f) = -\vec{j} \cdot (\nabla f)$. We shall assume all components of
$\vec{f} \in \mathcal{D}(\Gamma)$ equal to a certain $\vec{f}$. The Maxwell equation (2.7) is more precisely
stated in the form $(\nabla \times \vec{B})(f) = \mu_0 (\Omega_{\vec{B}}, \vec{j}(\vec{f}) \Omega_{\vec{B}})$; taking a sequence $\{f_n\}_{n \geq 1}$,
with $f_n \rightarrow \delta(\vec{x})$ in the distributional sense, we may obtain (2.7) in the form

$$(\nabla \times \vec{B})(\vec{x}) = \mu_0 \lim_{n \rightarrow \infty} (\Omega_{\vec{B}}, \vec{j}(f_n) \Omega_{\vec{B}})$$  \hspace{1cm} (2.7)'
in case the limit on the r.h.s. of \((2.7)'\) exists, which will be seen to hold shortly.

For a system of \(N\) Bosons in the volume \(|K| = 2\pi R^2 L\) of the cylinder, supposed of radius \(R\), and height \(L\), centered at the origin, the state \(\Omega_B\) is

\[
|\Omega_B\rangle = \frac{\Psi^*(\phi_0)^N|\Omega_0\rangle}{(N!)^{1/2}}
\]  

(3.9)

where \(|\Omega_0\rangle\) is the Fock vacuum (no-particle state), and \(\phi_0\) denotes the normalized ground-state wave function of the one-particle operator

\[
H_{A}^1 \equiv \frac{(\rho - e\vec{A}(\vec{x}))^2}{2m}
\]  

(3.10)

on the Hilbert space \(\mathcal{H} = L^2(K)\), with the boundary conditions specified in appendix A. Let \((\vec{e}_\rho, \vec{e}_\theta, \vec{e}_z)\) be a positively-oriented orthonormal basis adapted to cylindrical coordinates, and

\[
\vec{A}(\vec{x}) = \frac{B\rho}{2} \vec{e}_\theta
\]  

(3.11)

**Lemma 1** Assume that \(\phi_0(\rho, \theta, z)\) is independent of \(\theta\), i.e., satisfies

\[
\frac{\partial \phi_0}{\partial \theta} = 0
\]  

(3.12)

Then,

\[
\lim_{n \to \infty} (\Omega_B, \vec{J}(f_{n}^{\vec{x}_0})|\Omega_B\rangle = \\
= (\Omega_B, \vec{J}_{Lon,B}(\vec{x}_0)|\Omega_B\rangle = \\
= -e^2 N \rho_0 |\phi_0(\rho_0, z_0)|^2 B \vec{e}_\theta
\]  

(3.13)

for any \(\vec{x}_0 = (\rho_0, \theta_0, z_0) \in K\).

**Proof** We use \((2.7)'\) and write

\[
(\Omega_B, (\Psi^* \nabla_{\vec{e}_\theta} \Psi)(f_{n}^{\vec{x}_0})|\Omega_B\rangle = \\
= -(\Psi(f_{n}^{\vec{x}_0})|\Omega_B\rangle, \Psi(-\nabla_{\vec{e}_\theta} f_{n}^{\vec{x}_0})|\Omega_B\rangle
\]

10
Using (2.2.3) and (3.9),
\[
\Psi(\nabla e_\theta f_n^\infty | \Omega_B) = N^{1/2}(\nabla \theta f_n^\infty, \phi_0) \\
\Psi^*(\phi_0) N^{-1} ((N-1)!)^{1/2} | \Omega_0)
\]
Since \( f_n^\infty \) is a smooth approximation of compact support to \( \delta(\rho-\rho_0)/\rho \delta(\theta-\theta_0) \delta(z-z_0) \), an integration by parts, together with (3.12), yields \( (\nabla e_\theta f_n^\infty, \phi_0) = 0 \). Similarly,
\[
(\Omega_B, (\nabla \nabla \theta \Psi^*) (f_n^\infty) \Omega_B) = 0
\]
and thus
\[
\lim_{n \to \infty} (\Omega_B, \mathcal{j}_{\text{mom}}(f_n^\infty) \Omega_B) = 0 \tag{3.14}
\]
An elementary Fock space computation, together with (3.14), yields (3.13). q.e.d.

The standard choice of operators \( \Psi, \Psi^* \), satisfying (2.2.1), implies a choice of phase \( \alpha(x) = 0 \) in (3.1.1), (3.1.2). Alternatively, from (3.3), this choice corresponds to a choice of a real ground state wave function \( \phi_0 \). Choosing a different \( \alpha = \alpha(x) \) instead, one obtains from (2.4) an additional term on the r.h.s. of (3.14),
\[
\lim_{n \to \infty} (\Omega_B, \mathcal{j}(f_n^\infty) \Omega_B) = \\
= e^2 NB m |\phi_0(\rho_0, z_0)|^2 (\nabla \alpha)(\vec{x}_0)
\]
from which
\[
\lim_{n \to \infty} (\Omega_B, \mathcal{j}(f_n^\infty) \Omega_B) = \\
= - e^2 NB m |\phi_0(\rho_0, z_0)|^2 (\bar{A}(\vec{x}_0) - (\nabla \alpha)(\vec{x}_0))
\]
(3.15.2)
The additional transformation (3.2) yields back the original form (3.13), and, therefore, the result (3.13), under the assumption (3.12), is gauge invariant, although the splitting (2.3)-(2.5) is not. For this reason, (2.5) should be written, more precisely:

$$j_{\text{Lon},\vec{B}}(\vec{x}) = -\frac{e^2}{m} \Psi^*(\vec{x})\Psi(\vec{x})(\vec{A}(\vec{x}) - (\nabla \alpha)(\vec{x})) \quad (2.5')$$

where $\nabla \alpha$ is a gauge function which balances the gauge non-invariance of $\vec{A}$, in order that a gauge invariant combination results.

4 Main results: assumption A and the Meissner effect

4.1 Main results

With the choice (3.11) for the vector potential, which we have shown above not to entail any loss of generality, the one-particle Hamiltonian $H_{\vec{A}}^1$, given by (3.10), may be written:

$$H_{\vec{A}}^1 = \bigoplus_{k \in \mathbb{Z}} H_{\vec{A}}^1(k)$$

where

$$H_{\vec{A}}^1(k) = \frac{\hbar^2}{2m} \left(-\frac{\partial^2}{\partial \rho^2} - \frac{\partial}{\rho \partial \rho} + \frac{|k + \frac{e\rho^2 B}{2\hbar}|^2}{\rho^2} - \frac{\partial^2}{\partial z^2} \right)$$

(4.1)

corresponding to the decomposition of the one-particle Hilbert space $\mathcal{H}_K = \bigoplus_{k \in \mathbb{Z}} \mathcal{H}_K^{(k)}$, with $\mathcal{H}_K^{(k)} = \mathcal{H}^{(k)} \otimes L^2((-L/2, L/2); dz)$, $\mathcal{H}^{(k)}$ being the Hilbert space spanned by the orthonormal basis

$$\{\Psi_{k,j}(\rho, \theta) = (2\pi)^{-1/2} \exp(ik\theta)\Psi_j(\rho)\}_{j \in \mathbb{N}} \quad (4.2)$$
, where \( \{\Psi_j(\rho)\}_{j \in \mathbb{N}} \) is any orthonormal basis of \( L^2((0,R); \rho d\rho) \), the latter denoting the Hilbert space of square integrable functions \( f(\rho) \) relative to the measure \( \rho d\rho \) on \( \mathbb{R}_+ \), with the inner product

\[
< f, g > = \int_0^R \rho \bar{f}(\rho)g(\rho) d\rho
\]

(see appendix A, (A.10) et seq.). Proper boundary conditions (elastic boundary conditions of "repelling" type) must, in addition, be specified: see (A.2), (A.3.1) and (A.3.2) of appendix A.

By (4.1), it is clear that, for \( B \) sufficiently large, the ground state will lie in a sector of large (in this case negative) \( k \), not in the sector \( k = 0 \) in (4.2). Thus, lemma 1 does not apply and both components in (2.3) contribute to the current (note that, by the choice (3.11), only the \( \vec{e}_\theta \)-component of the gradient appears in (2.4), which acts on the \( \exp(ik\theta) \) part in (4.2)). These should cancel each other, being, thus, consistent with \( \nabla \times \vec{B} = \vec{0} \) in equation (2.7), and a constant \( \vec{B} = B\vec{e}_z \) penetrating the sample.

When, however, \( B \) decreases beyond the value given by

\[
a \equiv \lambda R^2 < 1/2 \quad \quad \quad (4.3.1)
\]

with

\[
\lambda = \frac{eB}{2\hbar} \quad \quad \quad (4.3.2)
\]

it follows from the simple inequality

\[
\frac{|k + \frac{eB\rho^2}{2\hbar}|^2}{\rho^2} \geq \frac{\min_{k \in \{0,-1\}} |k + \frac{eB\rho^2}{2\hbar}|^2}{\rho^2} \geq \frac{(\frac{eB}{2\hbar})^2 \rho^2}{\rho^2} = \lambda^2 \rho^2
\]

that \( H^1_{A}(k) \), given by (4.1), satisfies

\[
H^1_{A}(k) \geq \tilde{V}^1_{\sigma,\kappa,0} - \frac{\hbar^2}{2m} \frac{\partial^2}{\partial z^2} \quad \quad \quad (4.5)
\]
where
\[
\tilde{V}_{\sigma,K,0} = \frac{\hbar^2}{2m} \left( -\frac{\partial^2}{\partial \rho^2} - \frac{\partial}{\rho \partial \rho} + \lambda^2 \rho^2 \right)
\]  

is the harmonic oscillator Hamiltonian in two dimensions (the complications introduced by the finite cylinder and the boundary conditions are taken care of in appendix A). Not unexpectedly, the ground state eigenfunction \( \phi_0 \) is a Gaussian, with
\[
\phi_0(\rho, \theta, z)^2 = \frac{\phi_0(\rho)^2}{\pi R^2 L}
\]
with
\[
h(\rho) = \frac{\lambda R^2}{1 - \exp(-\lambda R^2)} \exp(-\lambda \rho^2)
\]
(see lemma A.1 of appendix A for details). Now, however, (3.12) applies, and (3.13), being the r.h.s. of (2.7) and nonzero (as long as \( \vec{B} \neq \vec{0} \)), conflicts with \( \vec{B} = B \vec{e}_z \) inside the sample, which implies \( \nabla \times \vec{B} = \vec{0} \) (see also (B.2),(B.4) of appendix B). We are thus forced to search for a new solution of (2.7), with new \( \vec{B}, \vec{A} \), satisfying (2.8). We shall assume cylindrical symmetry, \( \vec{B} = B(\rho) \vec{e}_z, \vec{A} = a(\rho) \vec{e}_\theta \), with \( \rho' \), \( a'' \) absolutely continuous in \([0, R]\) (and therefore uniformly bounded in this interval), whereby (2.8) becomes
\[
\frac{\partial (\rho a(\rho))}{\rho \partial \rho} = B(\rho)
\]
from which we may choose
\[
a(\rho) = \int_0^\rho \frac{d\rho' \rho' B(\rho')}{\rho}
\]
leading, by (3.13) (see also (B.2)-(B.4) of appendix B) to the integrodifferential equation
\[
\rho \frac{d B}{d \rho} = \nu g(\rho) \int_0^\rho d\rho' \rho' B(\rho')
\]
with the boundary condition
\[
B(R) = B
\]
\[ \nu = \frac{e^2 d \mu_0}{m} \quad (4.10) \]

For homogeneous fields, the g.s. wave-function, given by (4.7), corresponds to the g.s. eigenvalue \( E_0 = \frac{\hbar^2 \lambda}{m} \) of the operator (4.6) (see also (A.20.2)):

\[ \tilde{V}_{\sigma,K,0}^1 \phi_0(\rho) = \frac{\hbar^2}{2m} \left[ -\frac{\partial^2}{\partial \rho^2} \right. \]
\[ \left. - \frac{\partial}{\rho \partial \rho} + \lambda \rho^2 \right] \phi_0(\rho) = E_0 \phi_0(\rho) \quad (4.11.1) \]

For the inhomogeneous problem, the corresponding operator is

\[ \tilde{Q}_{\sigma,K,0}^1 \equiv \left[ \frac{\hbar^2}{2m} \left[ -\frac{\partial^2}{\partial \rho^2} \right. \right. \]
\[ \left. \left. - \frac{\partial}{\rho \partial \rho} + \left( \frac{e}{\hbar} \right)^2 a(\rho)^2 \right] \left. \right. \]
\[ \left. \right. \phi_0(\rho) = \tilde{E}_0 \phi_0(\rho) \quad (4.11.2) \]

and the corresponding eigenvalue equation is

\[ \tilde{Q}_{\sigma,K,0}^1 \Psi_0(\rho) = \tilde{E}_0 \Psi_0(\rho) \quad (4.11.3) \]

where \( \tilde{E}_0 \) is the smallest eigenvalue of \( \tilde{Q}_{\sigma,K,0}^1 \) and \( \Psi_0 \) the g. s. wave-function, such that \( \Psi'_0 \) is absolutely continuous (a.c.) on \([0, R]\). For flexibility we impose the boundary condition (A.1), but with \( \sigma \geq 0 \) left open. By general properties of the Sturm-Liouville operators [Wei80], the g.s. \( \Psi_0 \) is positive and has no nodes, and, by the minimax principle,

\[ 0 \leq \tilde{E}_0 \leq E_0 \quad (4.11.4) \]

We also write in analogy to (4.7),

\[ \Psi_0(\rho)^2 = \frac{g(\rho)}{\pi R^2 L} \quad (4.11.5) \]

Note that \( \Psi_0 \) in (4.11.3) and consequently \( g \) in (4.11.5) depend on \( B(\cdot) \) and, possibly, on the parameters \( B \) and \( R \).
Theorem 1 Equation (4.9.2), with the boundary condition (4.9.2), which we write in the form
\[ B(R) = B \in (0, H_c(R)) \] (4.12)
with
\[ H_c(R) \equiv \frac{\hbar}{eR^2} \] (4.13)
has a unique bounded solution, which we denote by \( B(\cdot) \), satisfying
\[ 0 \leq B(\rho) \leq B \forall 0 \leq \rho \leq R \] (4.14)
and
\[ B(\cdot) \text{ is monotonically decreasing in the interval } [0, R] \] (4.15)
The current expectation value on the right hand side of (2.7) is of the London form
\[ \langle \Omega_{\vec{B}}, \vec{j}_{\text{Lon}}, \vec{B}(\vec{x}) \Omega_{\vec{B}} \rangle = -\frac{e^2}{m} N|\Psi_0(\rho)|^2 \vec{A}(\vec{x}) \]
(4.16.1)
with
\[ \vec{A}(\vec{x}) = a(\rho)\vec{e}_\theta \] (4.16.2)
where \( a(\cdot) \) is given by (4.9.1), with \( B(\cdot) \) the unique solution of (4.9.2).

Proof The operator corresponding to \( H_{\vec{A}}^1(k) \) in (4.1) has, now, by (4.11.2) and (4.11.3),
\[ |k + \frac{eB\rho^2}{2\hbar}|^2 \]
replaced by
\[ |k + \frac{e\rho a(\rho)}{\hbar}|^2 \]
Assuming that
\[ 0 \leq a(\rho) \leq \frac{\rho B}{2} \] (4.17)
the same argument in (4.4) shows that the inequality
\[ |k + \frac{e\rho a(\rho)}{\hbar}|^2 \geq \left( \frac{e}{\hbar} \right)^2 a(\rho)^2 \forall k \in \mathbb{Z} \] (4.18)
holds. Inequality (4.17) follows from the formulation of the boundary value problem (4.9.2), (4.9.3) as fixed point problem (see (B.12) of Theorem B.1 of appendix B, from which (4.14) follows, and thus (4.17) by (4.9.1). Finally, with the Ansatz (4.16.2), lemma 1 (with \( \phi_0 = \Psi_0 \), solution of (4.11.3)) implies (4.16.1), concluding the proof. q.e.d.

Unfortunately, \( \Psi_0 \) is the (unknown) self-consistent solution of the coupled system (4.9.2), (4.9.3), (4.11.3), with \( g \) given by (4.11.5). Some idea of its properties may be obtained by considering the boundary functions of (4.17). For \( a(\rho) = 0 \), the solution of (4.11.3) (with Neumann b.c. \( \sigma = 0 \)) yields \( (g(\rho))^{1/2} = 1 \) in (4.11.5), while, for \( a(\rho) = \frac{\rho R_0}{2} \), by (4.7), (4.8), for \( 0 \leq a = \lambda R^2 < 1/2 \), we have \( (h(R))^{1/2} = 0.8779 \leq (h(\rho))^{1/2} \leq (h(0))^{1/2} = 1.1272 \) for \( a = 1/2 \), while \( (h(\rho))^{1/2} = 1 \) for \( a = 0 \). It is thus extremely plausible that the range of \( \Psi_0 \) lies somewhere in between these values. Furthermore, by the forthcoming theorem, adopting the Gaussian (4.7), (4.8) as initial g.s. wave function, the resulting induction field is exponentially decreasing inside the sample; taking now this induction field as input in the eigenvalue equation (4.11.3), it is plausible that the resulting eigenfunction is approximately constant except near the surface, satisfying assumption A below and leading to an induction field which decays exponentially, by the same theorem, being thus "close" to being self-consistent. On these counts, we pose:

**Assumption A** \( \exists c > 0 \) independent of \( R \) such that

\[
(g(\rho))^{1/2} \geq c > 0 \forall 0 \leq \rho \leq R
\]  

(4.18)

We may now formulate

**Theorem 2** Under assumption A, there exists a number \( 0 < p < \infty \) such that, if

\[
\sqrt{\nu} b > p
\]  

(4.19)

then

\[
B(b) \leq B \exp \left[ \frac{\sqrt{\nu} \int_b^R \rho \rho (g(\rho))^{1/2} \, d\rho}{2} \right] \leq B \exp \left[ -c \sqrt{\nu} (R - b) \right]
\]  

(4.20)

In order to give an idea what is involved in the proof, consider the case

\[
g(\rho) = \text{constant} = c > 0
\]  

(4.21.1)
in (4.9.2). Differentiating (4.9.2), we get

\[ \rho \frac{d^2 B}{d\rho^2} + \frac{dB}{d\rho} = \nu \rho B(\rho) \]  \hspace{1cm} (4.21.2)

which, upon division by \( \rho \), becomes

\[ \frac{d^2 B}{d\rho^2} + \frac{dB}{\rho d\rho} - \nu B(\rho) = 0 \]  \hspace{1cm} (4.21.3)

Equation (4.21.3), together with the boundary condition (4.11.1), has the unique bounded solution

\[ B(\rho) = \frac{B I_0[\nu c^{1/2} \rho]}{I_0[\nu c^{1/2} R]} \]  \hspace{1cm} (4.21.4)

where \( I_0 \) is the modified Bessel function of zero order ([AS65], pg. 374). Since, asymptotically, \( I_0(z) \sim \exp(z) \) as \( z \to \infty \) ([AS65], pg. 377), (4.21.4) implies that the solution decays exponentially. The proof in appendix B follows this idea closely. The quantity \( \nu \), given by (4.10), is such that

\[ \sqrt{\nu} = \sqrt{\frac{e^2 d}{\epsilon_0 c^2 m}} = \delta^{-1} \]  \hspace{1cm} (4.21.5)

Thus,

\[ \delta = \sqrt{\frac{\epsilon_0 mc^2}{de^2}} \]  \hspace{1cm} (4.22)

is the "penetration depth". It is independent of the size of the sample and, with physical data, of the order of 1000 Angstrom, agreeing precisely with the literature ([MR04], pg. 165). It is to be noted that, as usual, with macroscopic data, one is within the asymptotic domain (4.19). For example, with \( R = 10 \text{ cm}, b = R - 10\delta, \) say, \( b/\delta = 10^6 \), in the secure domain of asymptoticity of \( I_0 \), with a decay of \( \exp(-10) \).

4.2 The Meissner effect

The magnetization for \( T = 0 \) is

\[ \vec{M} = -\frac{\partial E_0^N}{V \partial B} \vec{e}_z = -\frac{e \hbar}{m} \vec{e}_z \]  \hspace{1cm} (4.23.1)
by (A.14), where
\[ E_0^N = NE_0 \]  
(4.23.2)
is the ground state energy of the system of \( N \) particles. We recall the defining relation for the (applied) magnetic field \( \vec{H} \),
\[ \vec{B} = \mu_0 \vec{M} + \vec{H} \]  
(4.24)
Let, also,
\[ \vec{B} = B \vec{e}_z \text{ and } \vec{H} = H \vec{e}_z \]  
(4.25)
and
\[ H_0 \equiv \frac{\mu_0 de\hbar}{m} \]  
(4.26)
The following picture emerges from (4.23), (4.24) and theorems 1 and 2:

**Proposition 1 - the Meissner effect** For some
\[ H_{c,R} \geq H_c(R) \]  
(4.27)
where \( H_c(R) \) is given by (4.11.2), the following assertions hold: a.) if \( H_0 < H < H_0 + H_{c,R} \) there is exponential decay of the magnetic induction \( B \) inside the sample, starting from a boundary value \( B_f \in (0, H_{c,R}) \); b.) \( B = 0 \) for all \( H \leq H_0 \); c.) for \( H > (H_0 + H_{c,R}) \), \( B = H - H_0 \).

**Proof** The sign \( > \) in (4.27) follows from the fact that the estimate (4.4) may not be optimal. a.) follows directly from (4.23)-(4.27) and (4.20) of theorem 2. For \( H = H_0 \), the boundary value of the magnetic induction is \( B = B_f = 0 \), which is consistent with (2.7), with r.h.s. given by (3.13). The magnetic induction \( B \) remains zero for \( H < H_0 \), because, assuming \( B \neq 0 \), (4.23) yields \( \vec{M} - -H_0 \vec{e}_z \), contradicting (4.24) and (4.25) if \( H < H_0 \). This concludes the proof of b.), and c.) follows immediately from (4.24). q. e. d.

Proposition 1 implies that the critical field \( H_c(0) \), for zero temperature, defined in the description of the Meissner effect given in the introduction, equals
\[ H_c(0) = H_0 + H_{c,R} \]  
(4.28)
Even if \( H_c(R) \), given by (4.11.2), is not optimal, it is expected that \( H_{c,R} = o(R) \) as \( R \to \infty \), i.e., it should vanish in the thermodynamic limit, being a "finite-size correction" to the thermodynamic results \( (\vec{H}_0 = H_0 \frac{\vec{B}}{|\vec{B}|}) \):
\[ \vec{B}(\vec{H}) = 0 \text{ if } |\vec{H}| < H_0 \text{ and } = \vec{H} - \vec{H}_0 \text{ if } |\vec{H}| > |\vec{H}_0| \]  
(4.49)
obtained by Schafroth ([Sch60], pg. 425; [Sch55], (5.12), pg. 471). The formula for the magnetization implies that the ”active field” $\vec{H}'$ is identified with the microscopic field, i.e., the magnetic induction $\vec{B}$. This is made plausible by Schafroth ([Sch55], pg 471).

5 Analogies with the Higgs effect in relativistic quantum field theory

The basic feature of the analysis of the Meissner effect is the London form of the current (4.17.1), (4.17.2). In the conventional treatment, this form originates in the macroscopic wave-function - ODLRO hypothesis, which, however, is incompatible with a nonzero (constant) magnetic field, as discussed in section 1; in our model it was shown for certain nonzero values of the magnetic field in theorem 1. In the case of a constant ground state wave-function $\phi_0 = \frac{1}{V}$, with $V = \pi R^2 L$ in (4.17.1), we obtain

$$\langle \Omega_{\vec{B}}, \vec{j}_{\text{Lon}, \vec{B}}(\vec{x}) \Omega_{\vec{B}} \rangle = -\frac{e^2}{md} \vec{A}(\vec{x})$$

(5.1)

where $d = N/V$ is the density. Equation (2.7) may then be written

$$\nabla \times \vec{B} = -\frac{\mu_0 e^2 d}{m} \vec{A}(\vec{x}) = \delta^{-2} \vec{A}(\vec{x})$$

(5.2)

with $\delta$ the penetration depth (4.22). The form (5.2) has an analogue in relativistic quantum field theory (rqft), although, as remarked in the introduction, much care must be taken with such analogies, because the many-body problem does not share important properties with rqft, such as vacuum polarization and microcausality: several textbooks expound this feature, one of the clearest being [MR04], pg. 305-311. We follow in this brief review - which does not claim any originality - the introduction in [FJ82], as well as [Jac75] and [JJ73]. We do, however, insert these known results in a (hopefully) coherent context, which might be useful.

In informal language, let $j^\mu$ be the source of a gauge field $F_{\mu\nu}$, with

$$\partial_\mu F^{\mu\nu} = j^\nu$$

(5.3)
with
\[ F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \]
\[ \partial_\mu j^\mu = 0 \]  
(5.4)

It follows that
\[ \Box F_{\mu\nu} = \partial_\mu j_\nu - \partial_\nu j_\mu \]  
(5.5)

If \( F_{\mu\nu} \) is massive, it must satisfy an equation
\[ (\Box F_{\mu\nu} + \mu^2)F_{\mu\nu} = \cdots \]  
(5.6)

where in (5.6) and elsewhere in this chapter the dots stand for nonlinear and nonlocal terms. Comparison with (5.6) and (5.5) yield
\[ \partial_\mu j_\nu - \partial_\nu j_\mu = -\mu^2 F_{\mu\nu} + \cdots \]  
(5.7)

or, equivalently,
\[ j_\mu = -\mu^2 A_\mu + \partial_\mu \alpha \]  
(5.8)

where, as in (2.5)', \( \alpha \) is a function which restores gauge invariance. This is the London form of the current, the relativistic analogue of (5.2), with
\[ \mu = \delta^{-1} \]  
(5.9)

(see the remarks in [MR04], pg. 308)). We shall come back to (5.9) presently, but for the moment only note the well-known fact that the London form of the current (5.8) signals the present of a non-zero mass in the theory. In the following, we comment on some of the nontrivial consequences of this assertion.

In the popular Higgs model of interaction between a massless gauge field \( A_\mu \) and a complex scalar field \( \phi, \phi^* \) (see [MR04], pg. 305), the current is
\[ j_\mu = ie\phi^*(\partial_\mu + ieA_\mu)\phi - ie\phi(\partial_\mu - ieA_\mu)\phi^* \]  
(5.10)

with
\[ < \phi >_0 = \lambda \neq 0 \]  
(5.11)

where \( < \cdot >_0 \) denotes the vacuum expectation value; hence
\[ j_\mu = -2e^2|\lambda|^2 A_\mu + \cdots \]  
(5.12)
which is of the London form (5.8), with

\[ \mu^2 = 2e^2|\lambda|^2 \]  \hspace{1cm} (5.13)

(5.12) is a semiclassical Ansatz, whose justification in a model with interaction of type \( V = -\mu^2\phi\phi^* + g(\phi\phi^*)^2 \) (see, e.g., [MR04], pp. 308-309)) may be given in lowest order perturbation theory (in this respect analogous to the conventional theory of the Meissner effect). There it may be shown that the vacuum polarization tensor has a pole, leading, by a mechanism first shown by Schwinger [Sch62], to the vector meson mass (5.13) ([Jac75], pg. 229). The above procedure has shortcomings: similarly to the Meissner case, as remarked by Jackiw ([Jac75], pg. 229), the Ansatz \( \lambda \neq 0 \) “is an assumption whose validity can only be checked in low orders of perturbation theory. It is by no means obvious that the complete theory possesses solutions with this property”. In addition, and this time in contrast to the Meissner case, (5.13) is not gauge invariant, because (5.11), with \( \lambda \neq 0 \), is violated for special choices of gauge [JFS81]. On the other hand, in the Weinberg (unitary) gauge [Wei73], no symmetry is broken: as Coleman remarks, ”no one presented with the final theory could possibly tell it was a result of the Higgs phenomenon... the Higgs phenomenon leaves no footprints” [Col79].

Starting directly from a massive vector meson theory, Duetsch and Schroer [DS00] showed that, in the Stueckelberg-Bogoliubov-Shirkov-Epstein-Glaser (SBSEG) version of the perturbation theory together with a ghost formalism (see [Sch01] or [GST94] for a brief review), the requirements of renormalizability and physical consistency fix the theory essentially uniquely, requiring a scalar field as envisaged by Higgs, but with no “symmetry-breaking condensate”.

On the other hand, (5.9) does point to deeper analogies between the Meissner effect and charge screening in field theories: the penetration depth is an analogue of the screening length in electrostatics. one specially nice model example of charge screening, now outside the semiclassical approximation, may be found in the two-dimensional Schwinger model, which has been treated from the London-Ansatz point of view by R. Jackiw ([Jac75], see also [JJ73] and, again, [FJ82] which we (mostly) follow. The current is bilinear in the massless Dirac field \( \Psi \),

\[ j_\mu = -e\bar{\Psi}\gamma_\mu\Psi \]  \hspace{1cm} (5.14)
We have

\[ \partial_\mu j_\nu - \partial_\nu j_\mu = -\epsilon_{\mu\nu} \epsilon^{\alpha\beta} \partial_\alpha j_\beta = \epsilon_{\mu\nu} \partial_\alpha j_5^\alpha \]  \hspace{1cm} (5.15.1) 

where

\[ j_5^\alpha = i e \bar{\Psi} \gamma^\alpha \gamma_5 \Psi \]  \hspace{1cm} (5.15.2) 

is the axial vector current, which one might think is conserved in the massless case as a consequence of chiral symmetry. (5.15.1) follows because

\[ \epsilon^{\mu\nu} \gamma_\nu = i \gamma^\mu \gamma_5 \\
\gamma_5 = i \gamma^0 \gamma^\mu \]

which are special to two dimensions. It turns out, however, that the axial vector current is not conserved in spite of the masslessness of the Fermions, but there is an anomaly of topological origin

\[ \partial_\alpha j_5^\alpha = \frac{e^2}{2\pi} \epsilon^{\mu\nu} F_{\mu\nu} \]  \hspace{1cm} (5.16) 

The above fact depends on the careful definition of the current by the point-splitting method

\[ j_5^\alpha = \lim_{\epsilon \to 0} [i \bar{\Psi}(x + \epsilon) \gamma^\alpha \gamma_5 \Psi(x - \epsilon) \exp(i e \int_{x-\epsilon}^{x+\epsilon} A^\mu dz_\mu)] \]

from which it may be shown that (5.16) follows [JJ73]. From (5.15.1) and (5.16), one finds the remarkable fact that the London Ansatz (5.8) is exact in the Schwinger model:

\[ \partial_\mu j_\nu - \partial_\nu j_\mu = -\frac{e^2}{\pi} F_{\mu\nu} \]  \hspace{1cm} (5.17) 

leading to the mass

\[ \mu^2 = \frac{e^2}{\pi} \]  \hspace{1cm} (5.18)

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(note that the gauge function in (5.8) drops upon taking the curl on the l.h.s. of (5.17)). The theory of charge screening in the model is, however, subtle, and has been studied by Lowenstein and Swieca [LS71], who examined the algebra of observables and showed that electron excitations completely disappear from the theory, which is in agreement with Schwinger’s intuitive picture of the total screening of the electronic charge [Sch62], and is isomorphic to that of a massive scalar free field of mass (5.18).

Farhi and Jackiw also present a second model in which, also for topological reasons, the London Ansatz is exact [FJ82]: quantum electrodynamics in three space-time dimensions QED3 [SDT82]. In this theory, there is, however, a regularization ambiguity for the photon mass. We have shown that such is not present in the causal SBSEG theory in [GST94]. Charge screening in this model, as well as any abelian massive gauge theory in space-time dimensions greater than two, follows from a deep, general theorem of Swieca, Buchholz and Fredenhagen ([Swi76], [BF79]), which states, roughly, that in such a theory the charge is screened. Swieca’s argument [Swi76] is not completely rigorous, due to the use of improper particle states of sharp momentum; the full theorem in [BF79] adds important new ideas to arrive at a complete proof: in order to proceed from a positive mass to charge screening requires in field theory a subtle precise definition of ”almost local operators” [BF79], i.e., how ”well-localized” can be operators which create one-particle states from the vacuum.

In a very nice review [Sch], B. Schroer refers (pg.61) to the current (5.8) as the ”Maxwell current”, which is, of course, unrelated to the matter current, e.g., (5.14) in the Schwinger model, and tends to it only in the zero mass limit. He points out that, as above, the vanishing of the corresponding ”charge” $Q = \int j_0(\vec{x}) d\vec{x} = 0$ is characteristic of massive gauge theories, while spontaneous symmetry breaking (s.s.b.) requires, formally, $Q = \int j_0(\vec{x}) d\vec{x} = \infty$, due to the existence of vacuum fluctuations occurring all over space, a consequence of translation invariance. For this reason, the Higgs model bears no relation with s.s.b., which fits our previous description.

To summarize, the link between the Meissner effect and field theory, reviewed in this section, lies in the London (or Maxwell) form of the current, which signals, on the one hand, a ”penetration depth”, on the other a massive theory. If one extrapolates the London form from the semiclassical field theory to the full quantum field theory, the original Schwinger screening argument [Sch62], that a gauge invariant mass is equivalent to a light-like pole in the momentum-space vacuum polarization, has been heuristically reconciled.
with the London Ansatz ([And76], quoted in [FJS2], pg.4): we call this phenomenon, therefore, London-Schwinger screening: in the Schwinger model it holds exactly ([Jac75], pp. 226-228; [LS71]). As also briefly discussed, the deep Swieca-Buchholz-Fredenhagen theorem relates London-Schwinger screening to locality in a precise way.

6 Conclusion and open problems

In this paper we presented a proof (under assumption A) of the Meissner effect in a (necessarily) gauge covariant model, the model of the ground state of free charged Bosons in a constant magnetic field, in the semiclassical approximation, defined by (2.1) (with elastic boundary conditions of ”repelling" type, see appendix A), together with the equations (2.6)-(2.8) (the latter taken in distributional sense, as explained in the text). We summarize here what we found to be the essential basic features underlying the effect, which, in spite of the simplicity of the model, may also be expected to hold in a (still open) more realistic gauge-covariant model with interactions.

The basic features are: a.) the pairing mechanism: Bosons versus Fermions; b.) finite-size corrections to thermodynamics, surface current and magnetization, the form of the penetration depth and the analogy to charge screening; c.) how the London form of the current arises, the ”rigidity” of the ground-state wave function and its relation to exponential decay of the magnetic induction.

Concerning a.), we note that the effect depends crucially on the pairing phenomenon, which shows up in the Bosonic character of the ground state $|\Omega_{\vec{B}}\rangle$ given by (3.9). Indeed, for free Fermions in a constant magnetic field, the expectation values of the momentum-density and London parts of the current (2.4) and (2.5) almost cancel each other, leaving out a weak diamagnetism (see, e.g., [Zim65], pg. 344: a proof proceeds along the same lines by which Landau diamagnetism is proved, see [Zim65], pg. 285).

Point b.) appears in the statement of the Meissner effect, a.) of proposition 1: the effect appears as a finite-size correction to the thermodynamics (4.29). There is an analogy to the charge-screening phenomenon ([MR04], pg. 142): (only) an infinite extended system, being an infinite reservoir of particles, accounts for an excess or deficiency of a local charge, not being in contradiction with the conservation of the number of particles. For a finite system, the creation of a polarization cloud around a given charge is accom-
panied by an accumulation of opposite charges at the surface of the system (surface polarization, see fig. 4.2 of [MR04], pg. 143). In close analogy, (4.16.1), (4.16.2) and (4.9.1), together with the estimate (4.20) of theorem 2 show that the current also decays exponentially, and is, thus, also confined to a region of only a few penetration depths from the surface. Therefore, indeed, the essential physical mechanism responsible for the effect are surface currents which create a field $-\vec{B}$ exactly compensating the contribution of the magnetic induction $\vec{B}$ imposed on the sample, at all points inside the metal sufficiently far from the surface ([MR04], pg. 166). Writing

$$\vec{j}_{\text{Lon}}(\vec{x}) = \nabla \times \vec{M}_0(\vec{x})$$

we see that this shielding corresponds to a surface magnetization $\vec{M}_0$ (analogous to the surface polarization in charge screening), familiar from electrostatics, but obtained here in a quantum context (albeit semiclassical). A very important point is that the penetration depth (4.21) is independent of the size of the sample: it agrees, in fact, very well with the conventional theory ([MR04], pg. 165).

Concerning c.), we have seen that the London form of the current (4.16.1) takes place only for sufficiently small fields (4.11.1), (4.11.2), in the present model, by a nonperturbative spectral argument, the simple estimate (4.4), which may, however, not be optimal.

Section 5 reviews the link with relativistic quantum field theory, in particular with London-Schwinger screening.

Finally, and most importantly, as mentioned in the introduction, we succeeded in establishing the London form of the current, but did not solve the challenging self-consistency problem. However plausible, assumption A needs proof, in order that the present results may be regarded as well understood, even for this simple model.

The free model treated in this paper could be improved by consideration of the weakly interacting dilute Boson gas of Huang, Yang and Luttinger [KHL57] in a magnetic field. In the absence of the field, the model exhibits a gap, which might render it specially interesting as a model of Schafroth pairs in a magnetic field. The present model has been treated from different points of view in [Das73], [RFR91]. Although making no claims to exactness, these references may suggest a different approach to a final completely rigorous solution.
7 Appendix A - Details on the Hamiltonian (2.1)

In this appendix, we provide the mathematical details concerning the precise definition of Hamiltonian $H(\vec{A})$, given by (2.1). Since gauge invariance has been explained and shown to hold in (3.15.1) et seq., we are free to adopt a special form of $\vec{A}(\vec{x})$, which we choose to be (3.11). Let $T_{\sigma,K}^1$ denote the self-adjoint extension of the Laplacian $-(\nabla)^2$ corresponding to boundary conditions

$$\frac{\partial f}{\partial n} = \sigma f \quad (A.1)$$

where $\frac{\partial f}{\partial n}$ denotes the inward normal derivative, and assume that the boundary $\partial K$ is piecewise differentiable ([BR97], pg. 55). This is the case of the cylinder, and we take:

$$\sigma_1 = 0 \text{ for } (\rho, \theta, z); 0 \leq \rho \leq R; 0 \leq \theta \leq 2\pi \text{ and } z = L/2 \text{ or } z = -L/2 \quad (A.2.1)$$

i.e., Neumann b.c. on the upper and lower bases, and

$$\sigma_2 = \lambda R \text{ for } (\rho, \theta, z); \rho = r : 0 \leq \theta \leq 2\pi \text{ and } -L/2 \leq z \leq L/2 \quad (A.2.2)$$

with

$$\lambda = \frac{eB}{2\hbar} \quad (A.3)$$

(see later for the reason). Since $\sigma \geq 0$ by (A.2.1), (A.2.2), $T_{\sigma,K}^1 \geq 0$, i.e., for $f \in D(T_{\sigma,K}^1)$ ($D(A)$ denoting the domain of an operator $A$),

$$\langle f, T_{\sigma,K}^1 f \rangle = \int d\vec{x} |(\nabla f)(\vec{x})|^2 + \int_{\partial K} d\vec{x} \sigma(\vec{x}) |f(\vec{x})|^2 \quad (A.4)$$
The same construction above may be performed for $H(\vec{A})$, given by (2.1), (3.11), which, according to the definition of the creation (destruction) operators $\Psi, \Psi^*$ (in [BR97], pg. 8 et seq., there denoted $a, a^*$), and (3.11), may be defined precisely by

$$H(\vec{A}) = d\Gamma(\tilde{T}_{\sigma,K}^1)$$

where $d\Gamma$ is the second quantization map ([BR97], pg. 8), and, for $f \in D(\tilde{T}_{\sigma,K}^1)$,

$$(f, \tilde{T}_{\sigma,K}^1 f) = \frac{\hbar^2}{2m} \int_K d\vec{x} \left[ (|\nabla \vec{e}_x f|^2 + \frac{\nabla \vec{e}_z}{\rho} f^2 + |\nabla \vec{e}_z f|^2) + \int_{-L/2}^{L/2} dz \int_0^{2\pi} d\theta \sigma_2 |f(R, \theta, z)|^2 \right]$$

(A.5.2)

where

$$d\vec{x} \equiv d\rho d\theta dz$$

(A.6)

Corresponding to (A.4), let

$$H(\vec{A}) = \tilde{T}_{\sigma,K} = \bigoplus_{n \geq 0} \tilde{T}_{\sigma,K}^{(n)}$$

(A.7)

be the decomposition of $\tilde{T}_{\sigma,K}$ associated to the direct sum decomposition of the symmetric Boson Fock space $\mathcal{F}_+(K)$ into the $n$- particle subspaces $\mathcal{H}_K^n = L^2(K)^n_+$ ([BR97], pg. 8 et seq.): $\tilde{T}_{\sigma,K}^{(n)}$ is the non-negative self-adjoint operator associated with the positive closed quadratic form defined as in [BR97], pg. 356, defining uniquely a self-adjoint operator, which is the precise definition of $H(\vec{A})$, by (A.7).

Definition (A.5) shows that $\tilde{T}_{\sigma,K}^1$ allows, itself, a direct sum decomposition

$$\tilde{T}_{\sigma,K}^1 = \bigoplus_{k \in \mathbb{Z}} \tilde{T}_{\sigma,K,k}^1$$

(A.8)

with

$$\tilde{T}_{\sigma,K,k}^1 = \frac{\hbar^2}{2m} \left( -\frac{\partial^2}{\partial \rho^2} - \frac{\partial}{\rho \partial \rho} + \frac{|k + \frac{\epsilon}{i\hbar} \rho^2 B|^2}{\rho^2} - \frac{\partial^2}{\partial z^2} \right)$$
corresponding to the decomposition of the one-particle Hilbert space

$$\mathcal{H}_K = \bigoplus_{k \in \mathbb{Z}} \mathcal{H}_K^{(k)}$$

(A.10.1)

where

$$\mathcal{H}_K^{(k)} = \mathcal{H}^{(k)} \otimes L^2((-L/2, L/2); dz)$$

(A10.2)

with $\mathcal{H}^{(k)}$ being the Hilbert space spanned by the orthonormal basis

$$\{\Psi_{k,j}(\rho, \theta) = (2\pi)^{-1/2} \exp(i k \theta) \Psi_j(\rho)\}_{j \in \mathbb{N}}$$

and $\mathcal{H}^{(k)} = \mathcal{H}^{(k)} \otimes L^2((0, R); \rho d\rho)$, the later denoting the Hilbert space of square integrable functions $f(\rho)$ with the inner product

$$< f, g > = \int_0^R \rho \bar{f}(\rho) g(\rho) d\rho$$

(A.10.3)

which satisfy the boundary condition (A.1) with $\sigma$ defined by (A.2.1), (A.2.2). It is also necessary to check that the functions belong to the domain of the quadratic form defined previously: this will not be a problem for our ground state wave function, which will be seen to satisfy (A.1) in the usual sense of functions (in general, the b.c. are satisfied in the sense of distributions, in order to connect with the definition using quadratic forms).

**Lemma A.1** The spectrum of $\tilde{T}_{\sigma,K}^1$ consists of discrete eigenvalues of finite multiplicity. If

$$a \equiv \lambda R^2 < 1/2$$

(A.11.1)

where

$$\lambda = \frac{eB}{2\hbar}$$

(A.11.2)

there is a unique lowest eigenvalue $E_0$ of $\tilde{T}_{\sigma,K}^1$, which lies in the subspace $\mathcal{H}_K^0$, and is given by

$$E_0 = \frac{eB \hbar}{2m}$$

(A.12)

corresponding to the eigenfunction

$$\phi_0(\rho, \theta, z) = \phi_0(\rho) = \frac{(h(\rho))^{1/2}}{\sqrt{\pi R^2 L}}$$

(A.13)
with

\[ h(\rho)^{1/2} = \sqrt{\frac{\lambda R^2}{1 - \exp(-\lambda R^2) \exp(-\frac{\lambda \rho^2}{2})}} \quad (A.14) \]

**Proof** By (A.8), (A.9) and (A.11.1), (A.11.2),

\[ \tilde{T}^1_{\sigma,K,k} \geq U^1_{\sigma,K} + \frac{|k + \frac{e}{2\hbar^2} B|^2}{\rho^2} \quad (A.15) \]

where \( U^1_{\sigma,K} \) denotes the operator

\[ U^1_{\sigma,K} = \frac{\hbar^2}{2m} \left[ -\frac{\partial^2}{\partial \rho^2} - \frac{\partial}{\rho \partial \rho} - \frac{\partial^2}{\partial z^2} \right] \quad (A.16) \]

with the boundary conditions (A.1), (A.2.1), (A.2.2); more precisely, defined by the quadratic form (A.4) with the middle term in \( \nabla \tilde{e}_\theta \) omitted; this operator is positive. We have, for \( k \in \mathbb{Z} \), under assumptions (A.11.1), (A.11.2),

\[ \frac{|k + \frac{e}{2\hbar^2} B|^2}{\rho^2} \geq \min_{k \in \{0,-1\}} \frac{|k + \frac{e}{2\hbar^2} B|^2}{\rho^2} \geq \left( \frac{eB}{2\hbar} \right)^2 \rho^2 + \frac{eB}{\hbar} \epsilon \quad (A.17) \]

for some \( \epsilon > 0 \). Thus, by (A.17),

\[ \tilde{T}^1_{\sigma,K,k} \geq \tilde{T}^1_{\sigma,K,0} + 2\lambda \epsilon 1 \quad (A.18) \]

with \( \lambda \) given by (A.11.2), and \( 1 \) denoting the identity operator on the Hilbert space \( \mathcal{H}^0_K = L^2((0,R); \rho d\rho) \otimes L^2((-L/2, L/2); dz) \) with boundary condition (A.2.1), (A.2.2). By (A.4),

\[ \tilde{T}^1_{\sigma,K,0} \geq \tilde{U}^1_{0,K} \quad (A.19) \]

where \( \tilde{U}^1_{0,k} \) is the operator \( U^1_{\sigma,K} \), given by (A.16), with \( \sigma = 0 \) (Neumann) b.c.. The eigenfunctions of this operator on \( \mathcal{H}^0_K \) are known explicitly: they are
given by \( \{1 \otimes \Psi_n\}_{n=0,1,...} \) with \( \Psi_n(z) = N \cos\left(\frac{2\pi z}{L}\right) \) where \( N \) is a normalization factor, and hence, by the minimax principle (see, e.g., [BR97], Prop. 6.3.4), (A.18) and (A.19) imply that the spectrum of \( \tilde{T}_{1,\sigma, K} \) consists, for each \( k \in \mathbb{Z} \), of discrete eigenvalues of finite multiplicity, the same following for \( \tilde{T}_{1,\sigma, K} \). Again by the minimax principle and (A.18), the lowest eigenvalues of \( \tilde{T}_{1,\sigma, K} \) lie on the subspace \( H_0^K \). The eigenfunctions of the operator

\[
V_{1,\sigma, K} = \tilde{V}_{1,\sigma, K, 0} - \frac{\hbar^2}{2m} \frac{\partial^2}{\partial z^2}
\]

(A.20.1)

where

\[
\tilde{V}_{1,\sigma, K, 0} = \frac{\hbar^2}{2m} \left[ -\frac{\partial^2}{\partial \rho^2} - \frac{\partial}{\rho \partial \rho} + \lambda^2 \rho^2 \right]
\]

(A.20.2)

are of the product form \( f(\rho)g(z) \), and by the b.c. (A.2.1), the lowest corresponds to the function \( g_0(z) = \frac{1}{\sqrt{L}} \). The spectrum of the self-adjoint operator

\[
W_{1,\sigma, K, 0} = \frac{\hbar^2}{2m} \left[ -\frac{\partial^2}{\partial \rho^2} - \frac{\partial}{\rho \partial \rho} + \lambda^2 \rho^2 \right]
\]

(A.21)

but now on the Hilbert space \( \mathcal{H} = L^2((0, \infty); \rho d\rho) \), is well-known, i.e., that of the harmonic oscillator in two dimensions ([Mes65], Chap. XII). Its lowest eigenvalue is nondegenerate and given by (A.12), and the restriction \( f_0 \) of the corresponding Gaussian eigenfunction to \( L^2((0, R); \rho d\rho) \) satisfies the b.c. (A.2.2). Together with \( g_0 \), this yields (A.13) and (A.14). The remaining assertions of lemma A.1 follow from the above-mentioned property. Indeed, an eigenfunction \( f_1 \) of \( V_{1,\sigma, K, 0} \) with eigenvalue \( E_1 < E_0 \) could be naturally extended to \( \mathcal{H} \), defined as equal to zero on the complement of \((0, R)\). It would lie on the domain of \( W_{1,\sigma, K, 0} \) and correspond to the same eigenvalue, contradicting the explicitly known spectrum of \( W_{1,\sigma, K, 0} \). The uniqueness assertion follows in analogous fashion. q.e.d.
8 Appendix B - Proof of theorems 1 and 2

We start from equation (2.7)

\[ \nabla \times \vec{B} = \mu_0 (\Omega_{\vec{B}}, \vec{j}(\vec{x}) \Omega_{\vec{B}}) \]  

(B.1)

For \( B < H_{c,R} \), where \( H_{c,R} \) is given by (4.11.2), it follows from Lemma A.1, together with lemma 1, that

\[ \vec{j}(\vec{x}) = \vec{j}_{\text{Lon}, \vec{B}}(\vec{x}) = \]

\[ = -\frac{e^2}{m} \Psi^*(\vec{x}) \Psi(\vec{x}) \overrightarrow{A}(\vec{x}) \]

(B.2)

Taking \( \overrightarrow{A}(\vec{x}) \) in the form (3.11), we have

\[ (\Omega_{\vec{B}}, \vec{j}(\vec{x}) \Omega_{\vec{B}}) = \]

\[ = (\Omega_{\vec{B}}, \vec{j}_{\text{Lon}, \vec{B}}(\vec{x}) \Omega_{\vec{B}}) = \]

\[ = -\frac{e^2}{2m} N \rho B |\phi_0(\rho)|^2 \hat{e}_\theta \]

(B.3)

for \( \vec{x} = (\rho, \theta, z) \). By lemma A.1, finally,

\[ (\Omega_{\vec{B}}, \vec{j}(\vec{x}) \Omega_{\vec{B}}) = -\frac{e^2}{2m} dh(\rho) B \rho \hat{e}_\theta \]  

(B.4)

where \( d \) is the density and

\[ h(\rho) = \alpha(B, R) \exp(-\lambda \rho^2) \]

with \( \alpha(B, R) = \frac{\lambda R^2}{1 - \exp(-\lambda R^2)} \)

(B.5)

where \( \lambda \) is defined by (4.3.2). Inserting (B.3) - (B.5) into (B.1), we see that \( \vec{B} = B \vec{e}_z \) cannot be a solution of (B.1), because the r.h.s. of (B.1) is not zero. We must thus search for a new solution of (B.1), with new \( \vec{A} \) and \( \vec{B} \) satisfying (2.8). Assuming cylindrical symmetry, \( \vec{B} = B(\rho) \vec{e}_z, \vec{A} = a(\rho) \vec{e}_\theta \), with \( B' \),
" absolutely continuous on [0, R] (which automatically satisfy (2.6)), (2.8) becomes
\[ B(\rho) = \frac{\partial(\rho a(\rho))}{\rho \partial \rho} \]
which may be solved by
\[ a(\rho) = \int_0^\rho \frac{d \rho B(u)}{\rho} \] (B.6)
where we are assuming that \( |B(\rho)| < \infty \forall 0 \leq \rho \leq R \), i.e., bounded solutions. (B.1) becomes, with (B.6),
\[ -\frac{d B(\rho)}{d \rho} = -\mu_0 \frac{e^2}{m} d g(\rho) \int_0^\rho \frac{d \rho B(u)}{\rho} \] (B.7)
Introducing the constant
\[ \nu = \frac{e^2 d \mu_0}{m} \] (B.8)
(B.7) becomes
\[ \rho \frac{d B(\rho)}{d \rho} = \nu g(\rho) \int_0^\rho d \rho B(u) \] (B.9.1)
If \( g(\rho) = \text{constant} = c > 0 \), we have seen in the main text that the unique bounded solution of (B.8) with the boundary condition
\[ B(R) = B \] (B.9.2)
is given by
\[ B(\rho) = B I_0((\nu c)^{1/2} \rho) \]
This motivates the following approach. We write the integrodifferential equation (B.9.1) together with the boundary condition (B.9.2) in the form
\[ B(\rho) = B - \nu \int_\rho^R d u g(u)/u \int_0^u d v B(v) \] (B.11)
or
\[ (AB)(\rho) \equiv B - \nu \int_\rho^R d u g(u)/u \int_0^u d v B(v) = B(\rho) \] (B.12)
where $A$ is an operator from the complete metric space $M_{B,a}$ of continuous functions $h : [R - a, R] \to \mathbb{R}$ such that

$$0 \leq h(\rho) \leq B \quad (B.13)$$

and

$$||h_1 - h_2|| \equiv \max_{\rho - R \leq a} |h_1(\rho) - h_2(\rho)| \quad (B.14)$$

Let $L$ be such that

$$\max_{0 \leq \rho \leq R} [\nu g(\rho)] \leq L \quad (B.15)$$

and $a$ be fixed and chosen such that

$$a < \min\{1/L, 1/(LB)\} \quad (B.16)$$

Then it follows from the definition (B.12), the positivity of $g$, and (B.13)-(B.16) that $A$ maps $M_{B,a}$ into itself and is a contraction from $M_{B,a}$ to itself, i.e.,

$$||A(h_1 - h_2)|| < ||h_1 - h_2|| \quad \text{for } h_1, h_2 \in M_{B,a} \quad (B.17)$$

Let, now,

$$0 < \delta < a \quad (B.18)$$

and $g_\delta$ be defined as

$$g_\delta = g(R - l\delta)\chi(R - (l + 1)\delta, R - l\delta) \quad \text{for } l = 0, 1, \cdots, K \quad (B.19.1)$$

where $\chi(a, b)$ is the characteristic function of the interval $(a, b)$, defined as

$$\chi(a, b)(\rho) \equiv 1 \quad \text{for } a < \rho \leq b \quad \text{and zero otherwise} \quad (B.19.2)$$

Let $A_\delta$ denote the operator defined as in (B.12), but replacing $g$ by $g_\delta$ there. Note that $(A_\delta B)$ is a continuous function of $\rho$ for $g \in L^1(R - a, R)$ if $B(\cdot)$ is only bounded $L^1$. The equation corresponding to (B.11) may be written

$$B_\delta(\rho) = B - \nu \int_{\rho}^{R} du g_\delta(u)/u \int_{\rho}^{u} dv B_\delta(v) \quad (B.20)$$

or, alternatively,

$$(A_\delta B)(\rho) = B_\delta(\rho) \quad (B.21)$$

The proof of Theorem 1 in the main text is completed by the following result:
**Theorem B.1** Equations (B.11) and (B.20) with the boundary condition (B.9) have both unique solutions in the same space \( M_{B,a} \), for any fixed \( a \) satisfying (B.16) and any \( \delta \) satisfying (B.18), given, respectively, by

\[
B(\rho) = \lim_{m \to \infty} (A^m B)(\rho) \quad (B.22)
\]

and

\[
B_\delta(\rho) = \lim_{m \to \infty} (A^m_\delta B)(\rho) \quad (B.23)
\]

Moreover, the above solutions may be uniquely extended to the whole interval \([0, R]\). Denoting the extensions by the same symbols, there exists a constant \( f > 0 \) such that

\[
\max_{0 \leq \rho \leq R} |B(\rho) - B_\delta(\rho)| \leq f \delta \quad (B.24)
\]

In (B.22), (B.23), \( B \) denotes the constant function, equal to \( B \) for all \( \rho \in [0, R] \), and \( A^m \) denotes the \( m \)-fold composition of \( A \) with itself.

**Proof** Since \( |g_\delta| \leq L \) with the same \( L \) as \( g \) by the definition (B.19.1), (B.19.2), and \( g_\delta \geq 0 \), \( A_\delta \) is also a contraction from \( M_{B,a} \) to itself. The first assertion of theorem B.1, together with (B.22) and (B.23), follow from the contraction mapping principle (see, e.g., [Arn74], Th. 2, Ch. 4, pg. 209). The existence of unique extensions to the whole interval follows from the same theorem.

We now prove (B.24). We have

\[
A^m - A_\delta^m = \sum_{\gamma=1}^{m} A^{m-\gamma}(A - A_\delta)A_\delta^{\gamma-1} \quad (B.25)
\]

for any \( A, A_\delta \) bounded operators, here considered on \( M_{B,a} \). We now apply both members of (B.25) to \( B \) Since \( A \) is a contraction from \( M_{B,a} \) to itself,

\[
\| (A^m - A_\delta^m) B \| \leq \sum_{\gamma=1}^{m} \| A^{m-\gamma} \| \| A - A_\delta \| \| A_\delta^{\gamma-1} B \| \quad (B.26)
\]

where \( \| A \| = \sup_{f \in M_{B,a}} \| Af \|/\| f \| \) is the operator norm on \( M_{B,a} \). By definitions of \( A, A_\delta, g \) and \( g_\delta \), we obtain

\[
\| A - A_\delta \| \leq \nu Br_\delta \quad (B.27)
\]

where

\[
r = \max_{\rho \in [0, R]} |g'(\rho)| \quad B.28)
\]

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which is finite by (4.11.5), $\Psi_0$ being assumed such that $\Psi'_0$ is absolutely continuous on $[0, R]$ as solution of the eigenvalue problem (4.11.3). The result (B.24) follows, with the interval replaced by $[R - a, R]$, from (B.26)-(B.28) and the contractivity of both $A$ and $A_\delta$. By the aforementioned extension, the result follows in general. q.e.d.

As indicated in the main text, the solution of (B.20) is of the form

$$\alpha_0 I_0[(\nu c_0)^{1/2} \rho]$$

with $\alpha_0$ determined by the boundary condition (B.9),

$$\alpha_0 = \frac{B}{I_0[(\nu c_0)^{1/2} R]}$$

(B.30)

In the subsequent intervals $[R - (i-1)\delta, R - i\delta]$, $i = 1, 2, \ldots, N$, the solution is determined by continuity at the upper boundaries and equals

$$B_\delta(R - i\delta) = B \prod_{j=0}^{i-1} \frac{I_0[(\nu c_j)^{1/2} (R - (j+1)\delta)]}{I_0[(\nu c_j)^{1/2} (R - j\delta)]}$$

(B.31)

with

$$c_j = g(R - j\delta)$$

(B.32)

Let, now,

$$R - N\delta = b$$

(B.33)

Theorem 2 of the main text now follows from

**Theorem B.2** Let assumption A hold and let $B(\cdot)$ denote the unique solution of (B.11). Then there exists a $0 < p < \infty$ such that, if

$$(\nu)^{1/2} b > p$$

(B.34)

then

$$|B(b)| \leq B \exp\left[- \frac{(\nu)^{1/2} \int_b^R d\rho(g(\rho))^{1/2}}{2}\right]$$

$$\leq B \exp\left[-c(\nu)^{1/2}(R - b)\right]$$

(B.35)

**Proof** We write in (B.31)

$$I_0[(\nu c_j)^{1/2} (R - (j+1)\delta)] = I_0[(\nu c_j)^{1/2} (R - j\delta)] - I_1(\lambda_j)(\nu c_j)^{1/2} \delta$$
where
\[ \lambda_j \in (\nu c_j)^{1/2}(R - (j + 1)\delta), (\nu c_j)^{1/2}(R - j\delta)) \]

Thus,
\[
\frac{I_0[\nu c_j^{1/2}(R - (j + 1)\delta)]}{I_0[\nu c_j^{1/2}(R - j\delta)]} = 1 - \frac{I_1(\lambda_j)(\nu c_j)^{1/2}\delta}{I_0[\nu c_j^{1/2}(R - j\delta)]} \\
\leq \exp[-\frac{I_1(\lambda_j)(\nu c_j)^{1/2}\delta}{I_0[\nu c_j^{1/2}(R - j\delta)]}]
\]

by the inequality \(1 - x \leq \exp(-x)\), true if \(x \geq 0\). Inserting this estimate in (B.31), we get
\[
|B_\delta(b)| \leq B \exp[-\sum_{j=0}^{N-1} \frac{I_1(\lambda_j)(\nu c_j)^{1/2}\delta}{I_0[\nu c_j^{1/2}(R - j\delta)]}] \tag{B.36}
\]

We now use (B.24) and the fact that we have, by (B.32), a Riemann sum for the function \(\frac{I_1(\nu g(\rho)^{1/2}\rho)}{I_0(\nu g(\rho)^{1/2}\rho)}(\nu g(\rho))^{1/2}\) inside the exponential in (B.36), to obtain
\[
|B(b)| \leq B \exp[-\int_b^R d\rho I_1((\nu g(\rho)^{1/2}\rho))I_0((\nu g(\rho)^{1/2}\rho))^{1/2}] \tag{B.37}
\]

The asymptotic expansions for \(I_0\) and \(I_1\) ([AS65], pg. 377), together with assumption A, now yield that, for given \(0 < p < \infty\), \(\frac{I_1((\nu g(\rho)^{1/2}\rho))}{I_0((\nu g(\rho)^{1/2}\rho))} \geq 0.5\) if \(b\) satisfies (B.34). Employing again assumption A on (B.37) for the term \((\nu g(\rho))^{1/2}\), we obtain the last bound in (B.35). q. e. d.

**Acknowledgement** We should like to thank G. L. Sewell for his interest in this work and for critical remarks on the first version of these notes, which helped to shape the present version.

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