Field Theory on the von Neumann Lattice and the Quantized Hall Conductance of Bloch Electrons

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We construct useful sets of one-particle states in the quantum Hall system based on the von Neumann lattice. Using the set of momentum states, we develop a field-theoretical formalism and apply the formalism to the system subjected to a periodic potential. The topological formula of the Hall conductance written by the winding number of propagator is generalized to Bloch electrons. The relation between the winding number and the Chern number is clarified.

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I. INTRODUCTION

Current semiconductor technology can produce a class of an electric modulation in a two-dimensional system. In a two-dimensional electron system under a magnetic field such a modulation causes quite interesting structure in various observables. It is also used to examine various aspects of quantum Hall systems. The system subjected to a periodic potential which is described by finite number of harmonics has been extensively studied by many authors.

The system subjected to a cosine potential is equivalent to the nearest-neighbor (NN) tight-binding model with a magnetic flux. Hofstadter computed its spectrum of butterfly-shape and discovered a multi-fractal structure. Other periodic potential may also have a self-similar structure in a quantum region. Experimental observation of such a structure is a challenging theme. The Hall conductance is a significant observable to extract physical properties of such a system. Thouless et al. showed that in the system subjected to a periodic potential the Hall conductance is written by the Chern number of the Bloch function in a gap region. In the cosine potential case the Chern number is given by a solution of the Diophantine equation. The spectra and the Hall conductances in the next NN model and in periodic potentials which are described by finite number of harmonics are also obtained. However, the problem of a periodic short-range potential remains to be solved. Recently, a remarkable progress was made. That is, its spectrum in the lowest Landau level was computed with sufficient accuracy to see self-similarity. However, effects of higher Landau levels and Landau level mixing is not known well. In addition the Hall conductance of subbands is unknown. A suitable base function is necessary for this calculation. Many authors have used the Landau function or the eigenfunction of the angular momentum. The former one extends in one direction of spatial coordinates and localized in the other direction. This is suitable for the potential of one-dimensional translational invariance. The latter one is localized in the radial direction and has a rotational invariance. This is suitable for a rotational invariant potential.

The magnetic von Neumann lattice is another basis in a magnetic field and can be constructed independently of the gauge. The basis consists of direct products of coherent states in the guiding center coordinate space and harmonic oscillator eigenstates in the relative coordinate space. The von Neumann lattice basis is a set of localized states. It turns into a set of extended states by the Fourier transformation. This basis has the following desirable properties: (i) A two-dimensional lattice translational invariance exists. (ii) Lattice structure of the von Neumann lattice can be changed according to a problem. (iii) A modular invariance of the von Neumann lattice exists. Owing to the first property, the Hilbert space of one-particle states is specified by the Landau level index and the lattice momentum. The second property is desirable in solving various periodic potential problems. The modular invariance is a key to develop a field theoretical formalism based on the von Neumann lattice. In our previous paper we showed advantages of the von Neumann lattice by studying the spectrum of a periodic short-range potential in the lowest Landau level. A field theoretical formalism is necessary when we study various interactions in a systematic way. In fact the field theoretical formalism based on the von Neumann lattice allow us to prove the exactness of the quantized Hall conductance by a topological formula. The topological formula is written by the winding number of the full propagator in the momentum space. The weak localization, QED effects, and electron-phonon interaction do not alter the quantized Hall conductance. However, a periodic potential is not included in the proof.

In the present paper we study dynamical properties of the system subjected to a periodic potential using the von Neumann lattice. We first construct explicitly three sets of one-particle states based on the von Neumann lattice.
That is, the set of coherent states, the set of momentum states and the set of Wannier states. The coherent state is localized on a lattice site and is not orthogonal. The momentum state is extended and orthogonal. Its analytical expression in the spatial coordinate space was not known. We give it in terms of the theta function for the first time. The Wannier state is localized on a lattice site and is orthogonal. Recently, this state was discussed by two groups. However, their results are restricted to a rectangular von Neumann lattice. We study the analytical form of the Wannier state in a general von Neumann lattice. Using the set of momentum states we develop a field theoretical formalism in a complete and strict manner. We pay attention to a peculiar property of the system in a magnetic field. In this system, we can not diagonalize both the current and the energy with respect to the Landau level index simultaneously. This imply that the usual relation does not hold in this system. However, the gauge invariance allow us to prove a similar relation as the Ward-Takahashi identity. This identity is used to derive the topological formula of the Hall conductance. In a periodic potential system the spectrum consists of Landau subbands and the range of the lattice momentum becomes narrow according to the period of the potential. The propagator has additional index which reflects the subband structure. We prove that in this system the Hall conductance is written by the winding number of the propagator. Furthermore, this winding number can be rewritten by the Chern number of the eigenfunction. This Chern number is the same as the one obtained by Thouless et al. As an example we calculate the Chern number of the system subjected to a periodic short-range potential. The result gives some insights to the Wigner crystal phase of the quantum Hall system.

The content of this paper is as follows. We construct three sets of one-particle states based on the von Neumann lattice in Sec. II. A field-theoretical formalism based on the set of momentum states is developed in Sec. III. The Ward-Takahashi identity and the topological formula of the Hall conductance written by the winding number of full propagator are shown. A periodic potential and defect are studied using the formalism in Sec. IV. The Ward-Takahashi identity and the topological formula are generalized to a system subjected to a periodic potentials in Sec V. The relation between the winding number and the Chern number are clarified. The Hall conductance of the system subjected to a periodic short-range potential are calculated with this formalism. Summary and discussion are given in Sec. VI.

II. THE VON NEUMANN LATTICE BASIS

In a two-dimensional system under a uniform perpendicular magnetic field $B$, we introduce two sets of coordinates, i.e. the guiding center coordinates $(X,Y)$ and the relative coordinates $(\xi,\eta)$:

$$\xi = (eB)^{-1}(p_y + eA_y), \quad \eta = -(eB)^{-1}(p_x + eA_x),$$

$$X = x - \xi, \quad Y = y - \eta,$$  (2.1, 2.2)

where $B = \partial_x A_y - \partial_y A_x$ and $eB > 0$. Each set of coordinates satisfies the canonical commutation relation

$$[\xi,\eta] = -[X,Y] = \frac{a^2}{2\pi i}, \quad a = \sqrt{\frac{2\pi\hbar eB}{c}}$$  (2.3)

and two sets of coordinates are commutative. Using these operators, a one-body Hamiltonian for a free charged particle is written in the form

$$\hat{H}_0 = \frac{1}{2} m\omega_c^2 (\xi^2 + \eta^2),$$  (2.4)

where $\omega_c = eB/m$. $\hat{H}_0$ is equivalent to the Hamiltonian of a harmonic oscillator and the eigenvalue is solved as follows:

$$\hat{H}_0 |f_l\rangle = E_l |f_l\rangle, \quad E_l = \hbar\omega_c(l + \frac{1}{2}), \quad l = 0, 1, 2, \ldots$$  (2.5)

This energy level is called the Landau level. Since $\hat{H}_0$ is independent of $X$ and $Y$, the phase space $(X,Y)$ corresponds to the degeneracy of the Landau level. It is convenient to use the coherent state defined by

$$(X + iY)|\alpha_{mn}\rangle = z_{mn} |\alpha_{mn}\rangle,$$  (2.6)

where $m, n$ are integers and $\omega_x, \omega_y$ are complex numbers which satisfy
\[ \text{Im}[\omega_x^* \omega_y] = 1. \]  

(2.7)

\( z_{mn} \) is a point on the lattice site in the complex plane; an area of the unit cell is \( a^2 \). We call this lattice the magnetic von Neumann lattice. Under the condition (2.7), the completeness of the set \( \{ |\alpha_{mn}\rangle \} \) is ensured. The coherent state \( |\alpha_{mn}\rangle \) is constructed as follows:

\[ |\alpha_{mn}\rangle = e^{i\pi(m+n+mn) + \sqrt{2\pi}(A^\dagger z_{mn} - A z_{mn})} |\alpha_{00}\rangle, \]

(2.8)

\[ A = \sqrt{a}(X + iY), \quad [A, A^\dagger] = 1. \]

The coherent states are not orthogonal, that is,

\[ \langle \alpha_{m+m',n+n'} | \alpha_{m',n'} \rangle = e^{i\pi(m+n+mn) - \frac{\pi}{a} |z_{mn}|^2}. \]

(2.9)

Thus, a translational invariance exists. The Hilbert space of one-particle states is spanned by the state \( |f_l \otimes \alpha_{mn}\rangle \).

The expectation value of the position of this state is a coordinate of the site of the magnetic von Neumann lattice and its mean square deviation is \( (l+1)a^2/2\pi \). Hence, the state \( |f_l \otimes \alpha_{mn}\rangle \) is a localized state. The wave function of the state \( |f_l \otimes \alpha_{mn}\rangle \) in the spatial coordinate space is given by

\[ \langle \mathbf{x} | f_l \otimes \alpha_{mn} \rangle = \frac{1}{a} \sqrt{\frac{\pi}{l!}} \left( \frac{z - z_{mn}}{a} \right)^l e^{-\frac{1}{2a^2} |z - z_{mn}|^2 - i\pi m \frac{\delta}{2} + i\pi n \frac{\delta}{2} + i\lambda(x)}. \]

(2.10)

Here, \( \hat{x}_i = x_j W_{ji}^{-1} \) with the matrix \( W \) defined by

\[ W = \begin{pmatrix} \text{Re}[\omega_x] & \text{Im}[\omega_x] \\ \text{Re}[\omega_y] & \text{Im}[\omega_y] \end{pmatrix}, \quad \det W = 1. \]

(2.11)

In terms of \( \hat{\omega}_x, \hat{\omega}_y \), \( z = x + iy = \hat{\omega}_x + \hat{\omega}_y \). Thus, a site of the von Neumann lattice corresponds to integer \( \hat{x}/a, \hat{y}/a \) value. The function \( \lambda(x) \) represents a gauge degree of freedom of the eigenfunction. For example \( \lambda(x) = 0 \) corresponds to the symmetric gauge.

In Fig.1 we show the probability density \( a^2 |\langle \mathbf{x} | f_l \otimes \alpha_{00} \rangle|^2 \) with \( l = 0, 1 \) for the square von Neumann lattice.

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**FIG. 1.** The probability density \( a^2 |\langle \mathbf{x} | f_l \otimes \alpha_{00} \rangle|^2 \) with \( l = 0, 1 \) for the square von Neumann lattice. Localization around \( \mathbf{x} = \mathbf{0} \) is clearly seen in this figure. Since the probability density is a function of \( \hat{x} - ma \) and \( \hat{y} - na \), the correspondent figure of \( |f_l \otimes \alpha_{mn}\rangle \) is obtained by sliding the coordinate as \( (\hat{x}, \hat{y}) \rightarrow (\hat{x} - am, \hat{y} - an) \).

As Eq. (2.9) has a translational invariance, an orthogonal basis can be obtained in the momentum space. Fourier transformed states denoted by

\[ |\alpha_p\rangle = \sum_{m,n} e^{ip_x m + ip_y n} |\alpha_{m,n}\rangle, \]

(2.12)

are orthogonal, that is,

\[ \langle \alpha_p | \alpha_{p'} \rangle = \alpha(p) \sum_N (2\pi)^2 \delta(p - p' - 2\pi N). \]

(2.13)
Here, \( \mathbf{N} = (N_x, N_y) \) is a vector with integer values and \( \mathbf{p} = (p_x, p_y) \) is a momentum in the Brillouin zone (BZ), that is, \( |p_x|, |p_y| \leq \pi \). The function \( \alpha(\mathbf{p}) \) is the Fourier transform of Eq. (2.9) and calculated by using the Poisson resummation formula as follows:

\[
\alpha(\mathbf{p}) = \beta(\mathbf{p})^* \beta(\mathbf{p}), \tag{2.14}
\]

\[
\beta(\mathbf{p}) = (2\text{Im}\tau)^{1/2} e^{i\pi r_x^2} \vartheta_1(\frac{p_x + \tau p_y}{2\pi}|\tau), \tag{2.15}
\]

where \( \vartheta_1(z|\tau) \) is a theta function and the moduli of the von Neumann lattice is defined by \( \tau = -\omega_x/\omega_y \). The magnetic von Neumann lattice is parameterized by \( \tau \). To indicate the dependence on \( \tau \), we sometimes use a notation such as \( \beta(\mathbf{p}|\tau) \). For \( \tau = i \), the von Neumann lattice becomes a square lattice. For \( \tau = e^{i2\pi/3} \), it becomes a triangular lattice. Some properties of the above functions are presented in Appendix A. Whereas \( \alpha(\mathbf{p}) \) satisfies the periodic boundary condition, \( \beta(\mathbf{p}) \) obeys a nontrivial boundary condition

\[
\beta(\mathbf{p} + 2\pi\mathbf{N}) = e^{i\phi(p,\mathbf{N})/\beta(\mathbf{p})}, \tag{2.16}
\]

where \( \phi(p, N) = \pi(N_x + N_y) - N_y p_y \). We can define the orthogonal state which is normalized with \( \delta \)-function as follows:

\[
|\beta_\mathbf{p}\rangle = \frac{|\alpha_\mathbf{p}\rangle}{\beta(\mathbf{p}) e^{i\chi(\mathbf{p})}}. \tag{2.17}
\]

The function \( \chi(\mathbf{p}) \) reflects an ambiguity with respect to the normalization. This ambiguity implies another gauge symmetry of this system. Later, we identify this symmetry as the gauge symmetry of constant magnetic field in BZ. It should be noted that the state \(|\alpha_\mathbf{0}\rangle\) is a null state, that is, \( \sum_{m,n} |\alpha_{m,n}\rangle = |\mathbf{0}\rangle \), because \( \beta(0) = 0 \). Since \( |\alpha_\mathbf{0}\rangle/\beta(\mathbf{p}) \) is indeterminate in Eq. (2.17), \( |\beta_\mathbf{p}\rangle \) is defined by \( \lim_{\mathbf{p} \to \mathbf{0}} |\beta_\mathbf{p}\rangle \). The Hilbert space of one-particle states is also spanned by the state \(|f_\mathbf{l} \otimes \beta_\mathbf{p}\rangle \). We call the state \(|f_\mathbf{l} \otimes \beta_\mathbf{p}\rangle \) the momentum state of the von Neumann lattice. The wave function of the state \(|f_\mathbf{l} \otimes \beta_\mathbf{p}\rangle \) in the spatial coordinate space is given by

\[
\langle x|f_\mathbf{l} \otimes \beta_\mathbf{p}\rangle = \frac{e^{i\lambda(x)-ix(\mathbf{p})}}{a} \sqrt{\frac{\pi^l}{l!}} \left( \frac{a}{2\pi} \right)^l \left( -2\partial_x \right)^l \left( \frac{\pi}{a^2} z \right)^l \left( \beta^*(p_x - 2\pi \frac{\tilde{y}}{a}, p_y + 2\pi \frac{\tilde{x}}{a}) e^{ip_x \frac{\tilde{x}}{a} + ip_y \frac{\tilde{y}}{a}} \right). \tag{2.18}
\]

The probability density \( \langle x|f_\mathbf{l} \otimes \beta_\mathbf{p}\rangle^2 \) is invariant under the translation \( (\tilde{x}, \tilde{y}) \to (\tilde{x} + aN_x, \tilde{y} + aN_y) \) with integer \( N_x, N_y \). Thus, the momentum state is an extended state.

![FIG. 2. The probability density \( a^2|\langle x|f_\mathbf{l} \otimes \beta_\mathbf{p}\rangle|^2 \) with \( l = 0, 1 \) for the square von Neumann lattice.](image)

In Fig. 2, we show the probability density for the lowest two Landau levels. In Fig. 2 the von Neumann lattice is the square lattice and we choose \( \mathbf{p} = \mathbf{0} \). The probability density with a non-zero momentum \( \mathbf{p} \) is obtained by sliding the coordinate as \( (\tilde{x}, \tilde{y}) \to (\tilde{x} + ap_y/2\pi, \tilde{y} - ap_x/2\pi) \).

Since \( |\beta_\mathbf{p}\rangle \) is orthogonal and normalized with \( \delta \)-function, its inverse Fourier transformation leads to an orthonormal state. That is, the state defined by

\[
|\beta_{m,n}\rangle = \int_{BZ} \frac{d^2 p}{(2\pi)^2} e^{-ip_x m - ip_y n} |\beta_\mathbf{p}\rangle \tag{2.19}
\]
satisfies
\[ \langle \beta_{mn} | \beta_{m' n'} \rangle = \delta_{mn} \delta_{m' n'}. \] (2.20)

The state $|\beta_{mn}\rangle$ is completely different from the coherent state $|\alpha_{mn}\rangle$ due to the normalization factor $\beta(p)e^{i\chi(p)}$. The relation between the coherent state and the orthonormal state is given by
\[ |\beta_{mn}\rangle = \sum_{m'n'} G(m - m', n - n')|\alpha_{m'n'}\rangle, \] (2.21)
where $G(m, n)$ is the inverse Fourier transformation of $1/\beta(p)e^{i\chi(p)}$. It has a long tail proportional to $(m^2 + n^2)^{-1/2}$. The Hilbert space of one-particle states is also spanned by the state $|\beta_{mn}\rangle$ which is an orthonormal localized state and has the center around $z = z_{mn}$ in the spatial coordinate space. However, it also has a long tail away from the center.

We call the state $|\beta_{mn}\rangle$ the Wannier state of the von Neumann lattice. The wave function of the state $|\beta_{mn}\rangle$ in the spatial coordinate space depends strongly on the phase $\chi(p)$. A simple form is obtained if we choose $\chi(p) = p_x p_y/2\pi + (p_y + \pi)/2$. In this case the wave function becomes
\[ \langle x|\beta_{mn}\rangle = -\frac{e^{i\lambda(x)}}{a} \sqrt{\frac{\pi^l}{l!}} \frac{a}{2\pi}(a^2 2\partial_{z^2} + \pi a^2 z^l) \times (2\text{Im}^l)^\frac{1}{2} e^{-i\pi^l (\frac{a}{2} - m)^2} 2\pi i m + i\pi (m + n + \frac{1}{2}) + \pi a^2 \int_{-\pi}^{\pi} \frac{dp_x}{2\pi} e^{i\pi (\frac{p_x}{a} - \frac{a}{2} + n + \frac{1}{2} + \pi^l (\frac{a}{2} - m)).} \] (2.22)

The similar result was obtained by Zak for a rectangular von Neumann lattice in the lowest Landau level. In Fig.3 we show the probability density $a^2|x|\beta_{mn}\rangle|^2$ with $l = 0, 1$ for the square von Neumann lattice.

![Fig. 3. The probability density $a^2|x|\beta_{mn}\rangle|^2$ with $l = 0, 1$ for the square von Neumann lattice.](image)

In Fig.3 we show the probability density $a^2|x|\beta_{mn}\rangle|^2$ with $l = 0, 1$ for the square von Neumann lattice. Asymmetry between $x$ and $y$-direction is owing to the phase choice. The center of the wave function $|x|\beta_{mn}\rangle$ is at $(\tilde{x}, \tilde{y}) = (0, 1/2)a$ in contrast to a naive expectation that it is at $(\tilde{x}, \tilde{y}) = (0, 0)$. This is not restricted to the case $\beta_{00}$. Since the probability density $|\langle x|\beta_{mn}\rangle|^2$ is a function of $\tilde{x} - an, \tilde{y} - an$, the slide of the center occurs for all $\beta_{mn}$. We should note that the Wannier state found by Rashba et al. is slightly different from ours. They used $|\beta(p)|$ as the normalization factor. As a consequence, the probability density of the wave function $|x|\beta_{00}\rangle$ for the square von Neumann lattice has a center at $z = 0$ and is rather symmetric than ours. Their wave function behaves as $1/r^2$ for large $r$ which is the critical behavior in the Thouless’s criterion. Our wave function behaves as $1/x$ for large $x$ and is exponentially dumped for large $y$. This behavior is also the Thouless’s critical behavior.

### III. FIELD THEORETICAL FORMALISM AND TOPOLOGICAL FORMULA OF HALL CONDUCTANCE

In the preceding section we obtain three sets of one-particle states based on the von Neumann lattice, that is, the coherent state $|\alpha_{mn}\rangle$, the momentum state $|\beta_{mn}\rangle$ and the Wannier state $|\beta_{mn}\rangle$. A field theoretical formalism based on the coherent state was developed in [25]. A formalism based on the momentum state or on the Wannier state was partially used in [26]. Here we develop the field theoretical formalism based on the momentum state...
We use the definition of $A_f$ for the spatial coordinate space. Let us define the polynomial.

The reason to prefer our choice ($\chi(p) = 0$) is simplicity of the form for the density operator $\rho(k)$ discussed later. If we use the definition of $\rho$, $\rho(k)$ becomes a complicated form.

We expand the electron field operator in the form

$$\psi(x) = \int_{BZ} \frac{d^2p}{(2\pi)^2} \sum_{l=0}^{\infty} b_l(p) \langle x|l,p\rangle.$$  \hspace{1cm} (3.1)

$b_l(p)$ satisfies the anti-commuting relation

$$\{b_l(p), b_{l'}^\dagger(p')\} = \delta_{l,l'} \sum_{N} (2\pi)^2 \delta(p - p' - 2\pi N)e^{i\phi(p',N)},$$  \hspace{1cm} (3.2)

and the same boundary condition as $\beta(p)$. $b_l^\dagger$ and $b_l$ are creation and annihilation operators which operate on the many-body states. The free Hamiltonian is given by

$$\mathcal{H}_0 = \int d^2x \psi^\dagger(x) \hat{H}_0 \psi(x) = \sum_l \int_{BZ} \frac{d^2p}{(2\pi)^2} E_l b_l^\dagger(p) b_l(p).$$  \hspace{1cm} (3.3)

The density and current operators in the momentum space are

$$\rho(k) = \int d^2x e^{-ik \cdot x} \psi^\dagger(x) \psi(x),$$

$$j(k) = \int d^2x e^{-ik \cdot x} \left(-\frac{i}{m}\right) \psi^\dagger(x)(\frac{\nabla_x}{2} + ieA(x)) \psi(x).$$  \hspace{1cm} (3.4)

Using a modular transformation of $\beta(p)$, which is given in Appendix A, $j^\mu = (\rho, j)$ becomes

$$j^\mu(k) = \int_{BZ} \frac{d^2p}{(2\pi)^2} \sum_{l,l'} b_{l'}^\dagger(p) b_l(p + a\hat{k}) \langle fi,1/2 \{v^\mu, e^{-ik \cdot \xi}\} | fi,1/2 \rangle e^{i\pi a\hat{k} \cdot (2p + a\hat{k}) \cdot \nu},$$  \hspace{1cm} (3.5)

Here, $v^\mu = (1, -\omega, \eta, \omega, \xi)$, and $\hat{k}_i = W_i k_j$. The explicit form of $\langle fi,e^{-ik \cdot \xi}| fi,1/2 \rangle$ is given in Appendix A. The phase factor in Eq. (3.5) can be written as $\exp(i \int \frac{p + a\hat{k}}{2\pi} A_i(p) dp_i)$, where the path is the straight line from $p$ to $p + a\hat{k}$ and $A_i(p) = (p_y/2\pi, 0)$. If we choose a non-zero $\chi(p)$, then $A_i(p)$ becomes $A_i - \partial_i \chi$, i.e. a gauge transformation of $A_i$. Therefore, $A_i(p)$ is a gauge field on BZ and the phase factor in Eq. (3.5) represents a flux on BZ. We note that the simple form of $j^\mu(k)$ is due to our convention $\chi(p) = 0$. For later convenience, we derive the density operator in the spatial coordinate space. Let us define the polynomial $f_{\nu}(k)$ as

$$\langle fi,e^{-ik \cdot \xi}| fi,1/2 \rangle = f_{\nu}(k) e^{-\frac{2\pi^2 \xi}{2}}.$$  \hspace{1cm} (3.6)

The explicit form of $f_{\nu}(k)$ is given in Appendix A. The density operator can be written as

$$\rho(x) = \frac{1}{a^2} \sum_{l,l'} \int_{BZ} \frac{d^2p}{(2\pi)^2} \frac{d^2q}{(2\pi)^2} b_{l'}^\dagger(p) b_l(q) \hat{H}_0(\frac{\partial}{\partial \bar{x}})$$

$$\times \beta(p_x - 2\pi \frac{\bar{y}}{a} p_y + 2\pi \frac{\bar{x}}{a} q_x - 2\pi \frac{\bar{y}}{a} q_y + 2\pi \frac{\bar{x}}{a}) e^{i\bar{x} \cdot \bar{y} \cdot \nu}.$$  \hspace{1cm} (3.7)

Since $\rho(x) = \psi^\dagger(x) \psi(x)$, we obtain a useful identity which relates the normalization factor $\beta(p)$ with the coordinate representation of the state $|l,p\rangle$ as

$$a^2\langle l,p|x\rangle\langle x|l',q\rangle = f_{\nu}(\frac{\partial}{\partial \bar{x}}) \beta(p_x - 2\pi \frac{\bar{y}}{a} p_y + 2\pi \frac{\bar{x}}{a}) e^{i\bar{x} \cdot \bar{y} \cdot \nu}.$$  \hspace{1cm} (3.8)

As a consequence of the above identity, we obtain Eq. (2.18) for example.

The free Hamiltonian $\mathcal{H}_0$ is diagonal in the above basis. However, the density operator is not diagonal with respect to the Landau level index. This basis, which we call the energy basis, is convenient to describe the energy spectrum of the system. In another basis, $\mathcal{H}_0$ is not diagonal and the density operator is diagonal. This basis, which we call the
current basis, is convenient to describe the Ward-Takahashi identity and the topological formula of Hall conductance. There is no basis in which both the Hamiltonian and the density are diagonal. This is one of peculiar features in a magnetic field.

The current basis is constructed as follows. Using a unitary operator, we can diagonalize the density operator with respect to the Landau level index. We define the unitary operator

$$U_{l'}(p) = \langle f_{l'} | e^{ip \cdot \xi/a} - \frac{1}{\text{det} \rho \rho | f_{l'} \rangle.}$$

(3.9)

By introducing a unitary transformed operator $\tilde{b}_{l}(p) = \sum_{l'} U_{l'}(p) b_{l}(p)$, the density operator is written in the diagonal form and the current operator becomes a simple form:

$$\rho(k) = \int_{BZ} \frac{d^2p}{(2\pi)^2} \sum_{l} \tilde{b}_{l}^\dagger(p) \tilde{b}_{l}(p + a\mathbf{k}).$$

$$j(k) = \int_{BZ} \frac{d^2p}{(2\pi)^2} \sum_{l,l'} \tilde{b}_{l}^\dagger(p) \{ v + \frac{a\omega_{\mathbf{c}}}{2\pi} (W^{-1}p + \frac{a}{2} \mathbf{k}) \} _{l,l'} \tilde{b}_{l'}(p + a\mathbf{k}).$$

(3.10)

$\tilde{b}_{l}$ and $\tilde{b}_{l}^\dagger$ satisfy the anti-commutation relation and boundary condition

$$\{ \tilde{b}_{l}(p), \tilde{b}_{l'}^\dagger(p') \} = (2\pi)^2 \delta(p - p' - 2\pi \mathbf{N}) \Lambda_{l\nu}(\mathbf{N}),$$

$$\tilde{b}_{l}(p + 2\pi \mathbf{N}) = \sum_{l'} \Lambda_{l\nu}(\mathbf{N}) \tilde{b}_{l'}(p),$$

(3.11)

where $\Lambda$ is defined by $\Lambda(\mathbf{N}) = (-1)^{N_{x} + N_{y}}U(2\pi \mathbf{N})$ and is independent of $\mathbf{p}$. However, it shuffles the Landau-level index.

Here we review briefly the Ward-Takahashi identity and the topological formula of Hall conductance using the current basis. We consider various interactions perturbatively. It is convenient to define the one-particle irreducible vertex part $\tilde{\Gamma}^{\mu}$ as

$$\langle j^{\mu}(q) \tilde{b}_{l}(p) \tilde{b}_{l'}^\dagger(p') \rangle = (2\pi)^3 \delta(p + Q - p') \tilde{S}_{ll'}(p) \tilde{\Gamma}^{\mu}_{l'l_1 l_2}(p,p + Q) \tilde{S}_{l_1 l_2}(p + Q),$$

(3.12)

where $Q^{\mu} = (q_{0}, a\mathbf{q}_{x}, a\mathbf{q}_{y}) = t^{\nu, \mu} q^{\nu}$ is a linear combination of $q^{\nu}$ and $\tilde{S}$ is the full propagator defined by

$$\langle \tilde{b}_{l}(p) \tilde{b}_{l'}^\dagger(p') \rangle = (2\pi)^3 \delta(p - p') \tilde{S}_{l'}(p).$$

(3.13)

Between $\tilde{S}$ and $\tilde{\Gamma}^{\mu}$ the Ward-Takahashi identity is satisfied. The identity has crucial roles in the following derivation of the topological formula of Hall conductance. The Ward-Takahashi identity in this case becomes

$$\tilde{\Gamma}^{\mu}(p,p) = t^{\nu} \frac{\partial \tilde{S}^{-1}(p)}{\partial p^{\nu}}.$$  

(3.14)

In a theory without a magnetic field, the Ward-Takahashi identity gives a relation that the state of the dispersion $\epsilon(p)$ moves with the velocity $\frac{\partial \epsilon(p)}{\partial p}$. However in a magnetic field, we cannot diagonalize both the current and the energy simultaneously. Therefore, the Ward-Takahashi identity Eq. (3.14) does not imply the relation.

The Hall conductance is the slope of the current correlation function $\pi^{\mu\nu}(q)$ at the origin and is written as

$$\sigma_{xy} = \frac{e^2}{3!} \epsilon^{\mu\nu\rho} \partial_{\rho} \pi_{\mu\nu}(q)|_{q=0}.$$  

(3.15)

If the derivative $\partial_{\rho}$ acts on the vertex with the external line attached, its contribution becomes zero owing to the epsilon tensor. Therefore, the case that the derivative acts on the bare propagator is survived. At this point it is proved that only diagrams of Fig.4 do contribute to $\sigma_{xy}$.
FIG. 4. Feynman diagrams which contribute to $\sigma_{xy}$.

In Fig. 4 the dark disk of the vertex and the dark disk of the propagator are the one-particle irreducible vertex part and the full propagator. These satisfy the Ward-Takahashi identity. Thus, $\sigma_{xy}$ is written as a topologically invariant expression of the full propagator:

$$\sigma_{xy} = \frac{e^2}{h} \frac{1}{24\pi^2} \int_{BZ \times S^1} d^3p \epsilon_{\mu \nu \rho} \text{tr} \left( \partial_\mu \tilde{S}^{-1}(p) \tilde{S}(p) \partial_\nu \tilde{S}^{-1}(p) \tilde{S}(p) \partial_\rho \tilde{S}^{-1}(p) \tilde{S}(p) \right)$$  \hspace{1cm} (3.16)

Here, the trace is taken over the Landau level index and the $p_0$ integral is a contour integral on a closed path which is drawn in Fig. 5.

FIG. 5. The contour line of $p_0$. Bold lines in the real $p_0$ axe represent Landau bands. The Fermi energy $E_F$ is located in the gap region.

Thus, we denote $S^1$ as the integration range.

If the Fermi energy lies in a gap region, the expression of $\sigma_{xy}$ becomes simple. The Coleman-Hill theorem\cite{21} tells us that in this region only the lowest order diagram, i.e. the diagram of Fig. 4 with the bare propagator and the bare vertex, does contribute to $\sigma_{xy}$. Thus, the full propagator $\tilde{S}$ can be replaced by the bare propagator $\tilde{S}_0$. The integral $\frac{1}{24\pi^2} \int tr(\tilde{d} \tilde{S}_0^{-1} \tilde{S}_0)^3$ gives an integer value and in fact counts the number of Landau bands below the Fermi energy. Thus, the Hall conductance is proved to be a integer times $e^2/h$ in a gap region.

IV. PERIODIC POTENTIAL AND DEFECT

In this section, we apply the formalism developed in the previous section to a system subjected to a periodic potential. Generally, a periodic potential lifts the degeneracy of the Landau level and the spectrum consists of Landau bands. As we mentioned in the introduction, Bloch electron in a strong magnetic field is of great interest in some contexts. In what follows, we assume that the potential lattice is formed by two linear-independent basis vectors with integer coefficients. Such a lattice is called a regular lattice. A periodic potential without this property, e.g. a honeycomb lattice potential, is not considered in this paper. A periodic potential with this property is generally written as

$$V(x) = \sum_{N \in \mathbb{Z}} v(x + aN_x w_{pot}^x + aN_y w_{pot}^y),$$  \hspace{1cm} (4.1)

where $w_{pot}^x, w_{pot}^y$ are basis vectors of the potential lattice. As is well-known, a periodic potential problem in a magnetic field is very sensitive to the flux penetrates a unit cell of the potential lattice. The unit cell means a parallelogram spanned by basis vectors of the potential lattice. If the flux $\Phi$ is a rational multiple of the flux quantum $\Phi_0$, that is,
for integers

eigenvalue equation becomes

\[ t = \frac{\Phi}{\Phi_0} = \text{Im}(\omega_y^\text{pot}) \ast \omega_x^\text{pot} = \frac{q}{p}, \quad (4.2) \]

each Landau band splits into \( q \) subbands. The magnetic Brillouin zone (MBZ) is one \( q \)-th of the Brillouin zone. In addition, it was proven that the spectrum in the MBZ is \( p \)-fold degenerate. We re-prove this degeneracy in a simple manner using the previously developed formalism.

When the flux is given by Eq. (1.2), it is convenient to select basis vectors of the von Neumann lattice as \( \omega_x = \omega_x^\text{pot}/q, \omega_y = \omega_y^\text{pot}. \) The moduli of the von Neumann lattice becomes \( \tau = \tau^\text{pot}/pq \), where the moduli of the potential lattice is defined by \( \tau^\text{pot} = -\omega_y^\text{pot}/\omega_x^\text{pot} \). In this case, the potential energy term in the second quantized form becomes

\[ H^\text{pot} = \frac{1}{q} \sum_{l,l',s=0}^{p-1} \sum_{r=0}^{q-1} \int_{\text{BZ}} d^2x v(x) \int d^2p \frac{2\pi}{(2\pi)^2} b_l^\dagger(p) b_{l'}(p_x - \frac{2\pi r}{q}, p_y) \times (l, p_x - 2\pi \left(\frac{y}{a} + \frac{s}{p}\right), p_y + 2\pi \frac{x}{a}) \langle x = 0| \langle 0| l', p_x - 2\pi \left(\frac{y}{a} + \frac{s}{p} + \frac{r}{q}\right), p_y + 2\pi \frac{x}{a}). \quad (4.3) \]

The eigenvalue equation reads

\[ (E - E_l) \psi_{l'}(p) = \frac{1}{q} \sum_{l'=0}^{p-1} \sum_{s=0}^{q-1} \int d^2x v(x) \langle l, p_x - 2\pi \left(\frac{y}{a} + \frac{s}{p}\right) | 0\rangle \langle 0| l', p_x - 2\pi \left(\frac{y}{a} + \frac{s}{p} + \frac{r'}{q}\right), p_y + 2\pi \frac{x}{a} \psi_{l'}(p), \quad (4.4) \]

where \( p \) is a momentum in MBZ, that is, \( |p_x| \leq \pi/q, |p_y| \leq \pi \). The function \( \psi_{l'}(p) \) is given by the following form:

\[ \psi_{l'}(p) = \psi_l(p_x - 2\pi \frac{r}{q}, p_y). \quad (4.5) \]

The equation is solved by diagonalizing a \( Lq \times Lq \) matrix, where \( L \) is the number of Landau levels. Therefore, each Landau band splits into \( q \) subbands generally. It is easy to see that the spectrum of the eigenvalue equation is invariant under the translations \( p_x \rightarrow p_x + 2\pi m/p \) and \( p_x \rightarrow p_x + 2\pi n/q \), where \( m, n \) are integers. Since \( p, q \) are relatively prime integers, the above symmetry reads

\[ E(p) = E(p_x + 2\pi \frac{n}{pq}, p_y). \quad (4.6) \]

Thus, it is proven that the spectrum in MBZ is \( p \)-fold degenerate.

Next, we consider additional defects to a periodic potential. To extract properties in such a system, let us suppose a defect in a periodic short-range potential. That is, the potential is given by

\[ V(x) = \sum_{N \in \mathbb{Z}} V_0 a^2 \delta(x - aN_x \omega_x^\text{pot} - aN_y \omega_y^\text{pot}) + ga^2 \delta(x - aM_x \omega_x^\text{pot} - aM_y \omega_y^\text{pot}), \quad (4.7) \]

for integers \( M_x, M_y \). The second term breaks the periodicity of the potential. If we neglect the second term, the eigenvalue equation becomes

\[ \sum_{l',r'} \frac{V_0}{q} (D_l^\dagger(p) D_{l'}(p))_{rr'} \psi_{l',r'}(p) = (E - E_l) \psi_{l'}(p), \quad (4.8) \]

where \( p \times q \) matrix \( D_l \) is given by

\[ (D_l(p))_{sr} = a \langle 0| l, p_x - 2\pi \left(\frac{s}{p} + \frac{r}{q}\right), p_y \rangle \quad (s = 0, \ldots, p - 1; r = 0, \ldots, q - 1). \quad (4.9) \]

As was discussed in our previous paper, the spectrum in the LLL consists of flat bands and Hofstadter-type bands. If we take account of higher Landau levels and the Landau level mixing, flat bands still exist at original Landau levels and also Hofstadter-type bands exist. The wave function of the flat band at \( E = E_l \) is given by \( \psi_k = \delta_{kl} \text{Ker}(D_l) \).
FIG. 6. The spectrum of the periodic short-range potential problem. The lowest two Landau bands are drawn in this figure. The lowest five Landau bands are taken into account.

In Fig. 6 the spectrum for the square lattice potential is shown. Here, we incorporate lowest five Landau levels and $V_0 / \hbar \omega_c = 0.3$ as an example. We observe that a self-similar pattern and large gaps above flat bands exist in this figure. Hofstadter-type bands above flat bands tend to a set of bound states as $t$ becomes infinity. Detailed explanation of the spectrum and the Hall conductance will be given in our forthcoming article.

Incorporating the defect leads to the following additional term in the L.H.S. of Eq. (4.8):

$$g \sum_{l'm'} \int_{MBZ} \frac{d^2q}{(2\pi)^2} (D_l^\dagger(p))_{rs}(D_{l'}(q))_{s'm'} e^{iqM_y(q_y-p_y)+iS(q_y-p_y)} \psi_{l'm'}(q),$$

where $\text{Mod}(M_y,p) = s$ and $(M_y-s)/p = S$. Apparently, flat bands are still flat even if the defect exists. Furthermore, in each subband gap of the periodic short-range potential problem a bound state appears rearranging eigenstates of Hofstadter-type bands. The equation of bound state energies is given by

$$1 = g \sum_A \int_{MBZ} \frac{d^2p}{(2\pi)^2} |\sum_{l'r} (D_l(p))_{sr} \psi_{l'r} A(p) e^{iqM_y(p_y)+iS(p_y)}|^2$$

where $\psi_{l'r} A(p)$ and $E_A(p)$ are the eigenfunction and the eigenvalue of Eq. (4.8). The R.H.S of the above equation generally becomes infinite when $E$ approaches to the upper edge of the Landau subband. Also, it becomes minus infinite when $E$ approaches to the lower edge. Thus, there exists a solution in each subband gap. The solution corresponds to a bound state trapped at the defect. If many defects exist in the periodic potential, many bound states appear in each the subband gap. As the number of defects increases, subband gaps tend to be filled with bound states and the number of extended states decreases. At last states which correspond to Landau subbands and bound states are shuffled each other and turn into a set of bound states (localized states) which has a finite density of state.

V. QUANTIZED HALL CONDUCTANCE OF BLOCH ELECTRONS

In this section, we generalize results given in section III to a system subjected to a periodic potential. In what follows, we assume the periodic potential has $t = q/p$. Thus, each Landau band splits into $q$ subbands generally.
In this case we can not directly use the Ward-Takahashi identity and thus the topological formula of the Hall conductance discussed in section III. This is because only the reduced momentum, i.e., the momentum in MBZ is conserved in this case. Therefore, we must generalize the Ward-Takahashi identity to the system subjected to a periodic potential first.

Let us suppose that the eigenvalue and the eigenfunction of the one-body Hamiltonian under the periodic potential is given by $E_A(p)$, $\psi_{lr}^A(p)$. Here, $p$ is a momentum in the MBZ, $A$ is the suffix to specify the eigenstate, $l$ is the Landau level index and $r = 0, 1, \ldots, q - 1$. The electron propagator in the current basis is written as

$$
\tilde{S}_{ll'}(p_0, p_x - 2\pi r q, p_y; p'_0, p'_{x} - 2\pi r' q, p'_y)
= (2\pi)^2 \delta(p-p') U_{ll'}(p_0 - E_A(p)) \psi_{lr}^A(p) U_{l'l'}^\dagger(p_0 - E_A(p)) \psi_{l'r'}^A(p)
= (2\pi)^3 \delta(p-p') \tilde{S}_{(l,r)(l',r')}(p).
$$

Thus the momentum in the MBZ is conserved in the propagator. The propagator $\tilde{S}_{(l,r)(l',r')}(p)$ can be regarded as a matrix whose indices run over allowed set of $(l, r)$. Owing to the orthogonality and the completeness of $\psi$, its inverse is given by

$$
\tilde{S}^{-1} = U \psi \frac{p_0 - E_A}{i} \psi^\dagger U^\dagger.
$$

The one-particle irreducible vertex in the current basis is the same as the one in the free theory, because effects of the periodic potential are absorbed in the propagator. Thus, for an infinitesimally small momentum transfer, the vertex becomes

$$
\tilde{\Gamma}_{ll'}^\mu(p_0, p_x - 2\pi r q, p_y; p'_0, p'_{x} - 2\pi r' q, p'_y)
= -i \delta_{ll'} (2\pi)^3 \delta(p + Q - p') \times
\left( e^{ip_0} \frac{1}{2ma} \left[ \delta_{i}^{\mu} \left( \sum_{i=x,y} W_{x_i}^{-1} (2 p_i + Q_i) - W_{y_i}^{-1} 4\pi r q \right) + \delta_{r}^{\mu} \left( \sum_{i=x,y} W_{y_i}^{-1} (2 p_i + Q_i) - W_{x_i}^{-1} 4\pi r q \right) \right] \right)
= (2\pi)^3 \delta(p + Q - p') \tilde{\Gamma}_{(l,r)(l',r')}(p, p + Q).
$$

Between $\tilde{S}$ and $\tilde{\Gamma}$, the Ward-Takahashi identity is satisfied in a generalized form. The Ward-Takahashi identity in this case is simply given by

$$
\tilde{\Gamma}_{(l,r)(l',r')}(p, p) = \epsilon^{\mu\nu} \frac{\partial \tilde{S}_{(l,r)(l',r')}(p)}{\partial p^\nu}.
$$

The current correlation function for infinitesimally small momentum transfers $q, q'$ is given by

$$
\pi^{\mu\nu}(q, q') = \frac{1}{a^2} (2\pi)^3 \delta(q - q') \int \frac{dp_0}{2\pi} \int_{\text{MBZ}} \frac{d^2 p}{(2\pi)^2} Tr \tilde{\Gamma}(p, p + Q) \tilde{S}(p + Q) \tilde{\Gamma}^\nu(p + Q, p) \tilde{S}(p)
= (2\pi)^3 \delta(q - q') \pi^{\mu\nu}(q).
$$

Here, the trace is taken over indices $(l, r)$. In the expression of the Hall conductance Eq. (3.15), the term including $\partial_p \tilde{\Gamma}$ or $\partial_p \tilde{S}$ vanishes when the totally antisymmetric part is taken. The term including $\partial_p \tilde{S}$ survives. Therefore, using the Ward-Takahashi identity, the Hall conductance is written as a topologically invariant expression of the propagator $\tilde{S}$:

$$
\sigma_{xy} = \frac{e^2}{h} \frac{1}{24\pi^2} \int_{\text{MBZ} \times S^1} Tr (d\tilde{S}^{-1})^3.
$$

Here, the $p$ integral is taken over MBZ. Since the integral $\frac{1}{24\pi^2} \int Tr (d\tilde{S}^{-1})^3$ gives the winding number of the propagator which is an integer value under general assumptions, the Hall conductance is proved to be an integer times $e^2/h$ whereas the filling factor is a fraction. This formula of the Hall conductance is a generalization of the topological formula Eq. (3.15) to the system subjected to a periodic potential.
Next, we consider the relation between the winding number and Chern number. Thouless et al. showed that the Hall conductance of Bloch electrons in a magnetic field is given by the Chern number of the Bloch function. This formula of the Hall conductance leads to a surprising result in a cosine potential. That is, in the gap region of Hofstadter bands, the Hall conductance becomes a integer times $e^2/h$ in spite that the system has a fractional filling factor in the gap. Furthermore, the Hall conductance is changed drastically as the Fermi energy is increased across subband gaps. This shows each subband carries a very large mobility. These features of the Hall conductance in a cosine potential are also obtained by the Streda formula\cite{23}. In contrast to the cosine potential the Hall conductance in a periodic short-range potential is not known. Taking account of recent antidot experiments, it is also important to study the Hall conductance in the periodic short-range potential.

To solve these problems, we rewrite the generalized topological formula Eq. (5.6) in a more convenient form. In computation of Eq. (5.6), there is a useful relation due to Polyakov-Wiegmann\cite{24}:

$$I(ST) - I(S) + I(T) + 3 \int Tr dSdTT^{-1},$$

where $I(S)$ is defined by $\int Tr(SdS^{-1})^3$. To utilize the relation, it is convenient to regard $\psi, U, S^{(0)} = i/(p_0 - E_A)$ as matrices of order $Lq$. We regard that $U_{\nu}(p_{\nu} - 2\pi r/q, p_y)$ is diagonal with respect to the index $r$ and that $S^{(0)}$ is diagonal with respect to the index $A$. Therefore, the propagator $\hat{S}$ is regarded as a product of three matrices, $\hat{S}^U = U\psi, S^{(0)}$ and $(\psi^U)^\dagger$. Using Eq. (5.7), the generalized topological formula leads to

$$\sigma_{xy} = \frac{e^2}{h} \frac{i}{2\pi} \sum_{A(E_A < E_F)} \left( \int_{MBZ} d(\psi^A d\psi^A) - 2\pi i \frac{1}{q} \right).$$

This formula is similar to that of Thouless et al.\cite{25} and Kohmoto\cite{26} except that there is the second term proportional to the filling factor. The second term comes from the matrix $U(p)$. However, this term at last cancels with the boundary contribution of the first term. For the first term, we give a topological argument which is similar to the one given in Eq. (5.11).

The boundary condition in BZ implies that the phase of the eigenfunction varies by $2\pi$ as one follows the boundary of BZ counterclockwise. Since BZ is divided into $q$ MBZs, this leads to a constraint given by

$$\sum_{r=0}^{q-1} \oint_{\partial MBZ} d\arg(\psi_{lr}^A) = 2\pi.$$  \hspace{1cm} (5.9)

Under this constraint the configuration which is singular in $\psi^A d\psi^A$ at some $p$ may appears for a band $A$. We call this the vortex. If the vortex appears, $\psi^A d\psi$ is ill-defined in the entire MBZ. Let us assume there is a vortex at $p = p_c$. We remark that there is a gauge degree of freedom in the eigenfunction. That is, the overall phase factor of the eigenfunction $\psi^A(p)$ is not determined. As is well known in the description of monopole, we can define $\psi^A d\psi$ in the entire MBZ using the gauge degree of freedom. In order to define it, we divide MBZ into two regions. One region (MBZ$_1$) does not include $p_c$ and has a boundary of MBZ. The other region (MBZ$_2$) includes $p_c$. The boundary between two regions is a closed line around $p_c$. In MBZ$_2$ using the gauge degree of freedom, we choose a different phase assignment which has no vortex. It is not necessary to satisfy the boundary condition when we extrapolate the wave function to MBZ$_1$. Thus, $\psi^A d\psi$ is completely defined in the entire MBZ. However, there is a phase mismatch at the boundary between two regions. That is, the eigenfunction $\psi^{(1)}$ in MBZ$_1$ and the eigenfunction $\psi^{(2)}$ in MBZ$_2$ are related as $\psi^{(2)} = e^{i\theta} \psi^{(1)}$ at the boundary. For the integral $\int_{MBZ_i} d(\psi^A d\psi)(i = 1, 2)$, we can use the Stokes theorem and obtain

$$\int_{MBZ} d(\psi^A d\psi) = \oint_{\partial MBZ} \psi^A d\psi + i \int_C d\theta,$$  \hspace{1cm} (5.10)

where $C$ is the closed line around $p_c$. The first term in R.H.S. of the above equation is equal to $2\pi i/q$ due to the normalization and the boundary condition of the eigenfunction $\psi$. The second term gives $2\pi i$ times a integer. This integer is called the Chern number. Therefore, starting from the generalized topological formula we obtain the Chern number for the Hall conductance:

$$\sigma_{xy} = -\frac{e^2}{h} \frac{1}{2\pi} \sum_{A(E_A < E_F)} \oint_C d\theta^A.$$  \hspace{1cm} (5.11)

To compare our result with the one obtained by Thouless et al.\cite{25}, we consider the correspondence between the Bloch function $u_k$ and the eigenfunction $\psi_{lr}$ in the von Neumann lattice basis. In terms of the Bloch function the energy eigenstate can be written as
number is the same as the Chern number obtained by Thouless conductance in a gap region is written by the winding number of the propagator Eq. (5.1). Furthermore, the winding and related theorems still hold, results obtained in Sec. 3 can be also applied in this system. Therefore, the Hall perturbatively. Thus, the propagator Eq. (5.1) replaces the “bare” propagator. Since the Ward-Takahashi identity way. In this system we can treat the periodic potential nonperturbatively as discussed above and treat interactions subjected to the periodic potential the winding number formula of the Hall conductance can be applied in the following

\[ n_C = \frac{1}{2\pi i} \int_{\text{MBZ}} dA, \quad (5.14) \]

where \( A \) is the connection 1-form written as

\[ A(k) = \int_{\text{MUC}} d^2x u_k^*(x) d_k u_k(x). \quad (5.15) \]

Substituting Eq. (5.13) into Eq. (5.14), we obtain

\[ n_C = \frac{1}{2\pi i} \int_{\text{MBZ}} d(\psi^* d\psi) - \frac{1}{q}. \quad (5.16) \]

This Chern number is nothing but the one in Eq. (5.8). Therefore, the winding number of the propagator is reduced to the Chern number of Thouless et al. if interactions are neglected.

So far, we neglected interactions and consider only a periodic potential. However, in an interacting system subjected to the periodic potential the winding number formula of the Hall conductance can be applied in the following way. In this system we can treat the periodic potential nonperturbatively as discussed above and treat interactions perturbatively. Thus, the propagator Eq. (5.1) replaces the “bare” propagator. Since the Ward-Takahashi identity and related theorems still hold, results obtained in Sec. 3 can be also applied in this system. Therefore, the Hall conductance in a gap region is written by the winding number of the propagator Eq. (5.1). Furthermore, the winding number is the same as the Chern number obtained by Thouless et al. as discussed above.

Finally we calculate the Hall conductance in the system of a periodic short-range potential. To calculate the Chern number for a general periodic potential is a subtle problem. However, we can calculate the Chern number for some analytically solved potentials. Let us first consider the potential with \( t = 1/p \). This is the case that each Landau band does not split into subbands. In this case, the \( p \)-dependence of \( \psi_1(p) \) can be set as \( \beta(p)/|\beta(p)| \) which satisfies the boundary condition. Apparently, this has a vortex at \( p = 0 \). Thus, we must choose another gauge for the wave function around \( p = 0 \). A simple choice is 1, that is, there is no \( p \)-dependence around \( p = 0 \). Since \( \int_{\text{MBZ}} d\arg\beta(p) = 2\pi \), the Chern number of each Landau band is equal to 1. Thus, the Hall conductance is \( e^2/h \) times the number of Landau bands below the Fermi energy as expected.

Next we consider a \( t = 2 \) case. A periodic potential with \( t = 2 \) can be solved analytically if the number of truncated Landau levels is one or two. Let us consider the periodic short-range potential in the lowest Landau level approximation. In this case the spectrum consists of a flat band and a lifted band. The wave function of the flat band satisfies the equation

\[ \beta^*(p)\psi_0(p) + \beta^*(p_x - \pi, p_y)\psi_1(p) = 0. \quad (5.17) \]

Here, \( \psi_k(p) = \psi(p_x - k\pi, p_y) \) and MBZ is the region \( |p_x| \leq \pi/2, |p_y| \leq \pi \). The following choice of the phase is consistent with the boundary condition of \( y \)-direction:

\[ \left( \begin{array}{c}
\psi_0(p) \\
\psi_1(p)
\end{array} \right) = \frac{1}{\alpha(p) + \alpha(p_x - \pi, p_y)} \left( \begin{array}{c}
|\beta(p_x - \pi, p_y)|e^{i\arg(\beta(p))} + ip_x \\
-|\beta(p)|e^{i\arg(\beta(p_x - \pi, p_y))} + ip_x
\end{array} \right). \quad (5.18) \]

At \( p = 0 \) the upper component of the above eigenfunction has a vortex. Therefore, we must choose another patch around \( p = 0 \). A possible choice of the eigenfunction around \( p = 0 \) is given by

\[ \left( \begin{array}{c}
\psi_0(p) \\
\psi_1(p)
\end{array} \right) = \frac{1}{\alpha(p) + \alpha(p_x - \pi, p_y)} \left( \begin{array}{c}
\beta^*(p_x - \pi, p_y) \\
-\beta^*(p)
\end{array} \right). \quad (5.19) \]
Thus there is a phase mismatch \( \theta = -\arg \beta(p) - \arg (p_x - \pi, p_y) \) at the boundary. Its integral along the boundary is \(-2\pi\). Thus, we conclude that the Hall conductance of the flat band is \( e^2/h \). The eigenvalue equation for the lifted band becomes

\[
\beta(p_x - \pi, p_y) \psi_0(p) - \beta(p) \psi_1(p) = 0. \tag{5.20}
\]

Its solution which is consistent with the boundary condition is

\[
\begin{pmatrix}
\psi_0(p) \\
\psi_1(p)
\end{pmatrix} = \frac{1}{\sqrt{\alpha(p) + \alpha(p_x - \pi, p_y)}} \begin{pmatrix}
\beta(p) \\
\beta(p_x - \pi, p_y)
\end{pmatrix}. \tag{5.21}
\]

This solution has no vortex. Thus, the Hall conductance of the lifted band is 0. We can see that the Wannier state constructed from the flat band is relatively spread than the one from the lifted band.

In the case that \( t \) is an integer larger than 2, the spectrum also consists of a flat band and a lifted band in the lowest Landau level. The eigenfunction of the lifted band

\[
\psi_n(p) = \frac{\beta(p_x - 2\pi p, p_y)}{\sqrt{\sum_{t=1}^{t-1} \alpha(p_x - 2\pi p, p_y)}} \tag{5.22}
\]

satisfies the boundary condition and has no vortex in the MBZ. Thus, the Hall conductance of the lifted band is 0. This result is reasonable because the lifted band becomes a set of bound states and thus has no mobility as \( t \) becomes infinity. Since it is proved that the Hall conductance of the Landau band is \( e^2/h \), the Hall conductance of the flat band is \( e^2/h \). These results are irrespective to the lattice structure of the periodic short-range potential.

**VI. SUMMARY AND DISCUSSION**

We constructed three sets of one-particle states based on the von Neumann lattice in a gauge-independent manner. The first one is the set of coherent states which are localized on the sites of the von Neumann lattice and are orthogonal. The second one is the set of momentum states which are extended, orthogonal and normalized with \( \delta \)-function. The third one is the set of Wannier states which are localized on sites of the von Neumann lattice and are orthonormal. Using the second one we developed a field-theoretical formalism of a quantum Hall system and showed the Ward-Takahashi identity and the topological formula of the Hall conductance. This formalism was applied to a system subjected to a periodic potential. The \( p \)-fold degeneracy of the spectrum in a \( t = q/p \) problem is easily proved using this formalism. The generalized topological formula for the Hall conductance of Bloch electron was obtained. The formula is written as a winding number of the propagator. Relation between the winding number and the Chern number was clarified. As an example we calculated the quantized Hall conductance in a periodic short-range potential.

In this paper we study the system subjected to a periodic potential using the set of momentum states. We believe that our formalism is also useful to study Coulomb interaction in the quantum liquid phase[2] and the Wigner crystal phase[3] of the quantum Hall system. In the Wigner crystal phase, a triangular lattice potential is generated from the mean field of the charge density. In this case, the filling factor \( \nu \) is related to \( t \) with \( \nu = 1/t \). If the mean field contributes to the short-range potential with positive coefficient, the Fermi energy lies in the large energy gap at \( \nu = 1 - 1/t \). That is, the flat band of the lowest Landau level in Fig.6 is completely filled. Then, \( \nu = 1/2 \) state can be a self-consistent solution which has a large energy gap in contrast to the quantum liquid phase in which gap closes at \( \nu = 1/2 \). If the mean field contributes to the short-range potential with negative coefficient, the Fermi energy lies in the large energy gap at \( \nu = 1/t \). That is, the lifted band is completely filled. Then, all states of \( \nu < 1 \) can be a self-consistent solution which has a large energy gap. Since the Hall conductance of the lifted band is zero, the ground state in this approximation is close to the crystal.

Although the short-range potential is not a good approximation for the Coulomb interaction, the mean field treatment starting from the short-range potential may give some insights to the Wigner crystal phase of the quantum Hall system.

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APPENDIX A:

The properties of functions used in this paper are given here. A theta function \( \vartheta_1(z|\tau) \) is defined by

\[
\vartheta_1(z|\tau) = 2 \sum_{n=0}^{\infty} (-1)^n e^{\pi\tau(n+\frac{1}{2})^2} \sin[\pi z(2n + 1)].
\] (A1)

Under the modular transformation \( \beta(p|\tau) \) defined in Eq. (2.15) transforms as follows:

\[
\beta(p, y|c\tau + d) = \eta \sqrt{c\tau + d} |c\tau + d| e^{\frac{i}{4\tau}(abp_x^2 + 2bcp_xyp_y + cdx^2)} \times \beta(dp + bp_y, cp_x + ap_y|\tau),
\] (A2)

\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}),
\]

where \( \eta \) is an eighth-root of unity. \( \alpha(p) \) defined in Eq. (2.14) obeys the periodic boundary condition. \( G = \ln \alpha \) is a Green function of the Laplacian operator on a torus, that is,

\[
\triangle G(z) = 2\pi\delta^{(2)}(z) - \frac{4\pi}{\text{Im}\tau},
\] (A3)

where

\[
\delta^{(2)}(z) = \sum_N \delta^{(2)}(z + N_x + \tau N_y),
\]

\( z = (p_x + \tau p_y)/2\pi \), and \( \triangle = 4\partial_\tau \partial_z \). As mentioned in Sec. II, the isolated singularity of the delta-function at \( p = 0 \) in Eq. (A3) is not involved in our theory. The factor \( \langle f|e^{-ik\cdot\xi}|f'\rangle \) which appears in \( \rho(k) \) is written by

\[
\langle f|e^{-ik\cdot\xi}|f\rangle = \sqrt{\frac{p}{(2\pi)^d}} \left( -\frac{a(k_x + ik_y)}{\sqrt{4\pi}} \right)^n e^{-\frac{(ak)^2}{8\tau}}
\times L_n^{(\frac{\sigma_k}{\sqrt{4\pi}})},
\] (A4)

where \( L_n^{(\frac{\sigma_k}{\sqrt{4\pi}})} \) is the associated Laguerre polynomial.

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