Scattering theory for Laguerre operators

D. R. Yafaev

To Ari Laptev on the occasion of his 70th birthday

Abstract

We study Jacobi operators $J_p$, $p > -1$, whose eigenfunctions are Laguerre polynomials. All operators $J_p$ have absolutely continuous simple spectra coinciding with the positive half-axis. This fact, however, by no means imply that the wave operators for the pairs $J_p, J_q$ where $p \neq q$ exist. Our goal is to show that, nevertheless, this is true and to find explicit expressions for these wave operators. We also study the time evolution of $(e^{-Jt}f)_n$ as $|t| \to \infty$ for Jacobi operators $J$ whose eigenfunctions are different classical polynomials. For Laguerre polynomials, it turns out that the evolution $(e^{-J_p t}f)_n$ is concentrated in the region where $n \sim t^2$ instead of $n \sim |t|$ as happens in standard situations.

As a by-product of our considerations, we obtain universal relations between amplitudes and phases in asymptotic formulas for general orthogonal polynomials.

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1 Introduction

1.1 Jacobi operators

Jacobi operators $J$ are discrete analogues of one-dimensional differential operators. They are defined in the space $\ell^2(\mathbb{Z}_+)$ by the formula

$$(Jf)_n = a_{n-1}f_{n-1} + b_nf_n + a_nf_{n+1}, \quad n \in \mathbb{Z}_+, \quad a_{-1} = 0. \quad (1.1)$$

We always suppose that $a_n > 0$ and $b_n = \bar{b}_n$. Of course spectral properties of Jacobi operators depend crucially on a behavior of the coefficients $a_n$ and $b_n$ as $n \to \infty$. 

Univ Rennes, CNRS, IRMAR-UMR 6625, F-35000 Rennes, France and SPGU, Univ. Nab. 7/9, Saint Petersburg, 199034 Russia; email: yafaev@univ-rennes1.fr
In the simplest case $a_n = a_\infty > 0$, $b_n = 0$, the operator $J$ (known as the “free”
discrete Schrödinger operator) has the absolutely continuous spectrum $[-2a_\infty, 2a_\infty]$, and its
eigenfunctions are expressed via Chebyshev polynomials of second kind. Operators $J$ whose
eigenfunctions are Jacobi polynomials are natural generalizations of this operator.

The situation $a_n \to \infty$ as $n \to \infty$ is also quite common in applications. We discuss
only the case where the Carleman condition

$$\sum_{n=0}^{\infty} a_n^{-1} = \infty$$

(1.2)
is satisfied. Suppose that $b_n/2a_n \to \gamma$ as $n \to \infty$. If $|\gamma| < 1$, then the spectrum of the
operator $J$ is purely absolutely continuous and coincides with the whole axis $\mathbb{R}$; see
[5, 2]. A famous example is

$$a_n = \sqrt{(n + 1)/2}, \quad b_n = 0.$$  

(1.3)

Eigenfunctions of the corresponding operator $J$ are the Hermite polynomials. If $|\gamma| > 1$
($|\gamma| = \infty$ is admitted), then the spectrum of $J$ is discrete. The case $|\gamma| = 1$ is critical,
and the spectral properties of $J$ depend on details of the behavior of $b_n/(2a_n) - \gamma$ as
$n \to \infty$. The results pertaining to this situation are scarce. We mention only papers
[6, 7] and references therein.

Here we study an important particular case

$$a_n = a_n^{(p)} = \sqrt{(n + 1)(n + 1 + p)} \quad \text{and} \quad b_n = b_n^{(p)} = 2n + p + 1, \quad p > -1.$$  

(1.4)

Thus we have $\gamma = 1$. For all $p$, the Jacobi operators $J = J_p$ (we call them the Laguerre
operators) with the recurrence coefficients (1.4) have absolutely continuous spectra
coinciding with $[0, \infty)$. Eigenfunctions of these operators are the Laguerre polynomials
$L_n^{(p)}(\lambda)$. Our goal is to investigate an asymptotic behavior of the unitary group $e^{-iJ_p t}$
as $t \to \pm \infty$. We show that, for different $p$, these asymptotics are essentially the same
although the operators $J_q - J_p$ are not even compact unless $q = p$.

Another goal of the paper is to obtain detailed asymptotic formulas for $(e^{-iJ_p t} f)_n$
as $|t| \to \infty$ for sufficiently arbitrary Jacobi operators $J$. Here we suppose that an
asymptotic behavior of the corresponding orthogonal polynomials $P_n(\lambda)$ as $n \to \infty$ is
known. These general results are illustrated on examples of the classical polynomials.
In particular, for Laguerre operators $J_p$, we show that the evolution $(e^{-J_p t} f)_n$ is
concentrated in the region where $n \sim t^2$ instead of $n \sim |t|$ as happens in standard
situations.

1.2 Scattering theory

We work in the framework of scattering theory. Let us briefly recall its basic notions. We
refer, e.g., to the book [10] for a detailed presentation. Consider a couple of self-adjoint
operators $A$ and $B$ in some Hilbert space $\mathcal{H}$. In view of our applications, we suppose that both of these operators are absolutely continuous. Scattering theory studies the strong limits

$$W_{\pm} = W_{\pm}(B, A) = \text{s-lim}_{t \to \pm \infty} e^{iBt} e^{-iAt}$$

(1.5)

known as the wave operators for the pair $A, B$. If the limits (1.5) exist, then the wave operators possess several useful features. In particular, they are isometric and enjoy the intertwining property $BW_{\pm} = W_{\pm}A$. It follows that the restriction of the operator $B$ on the image $\text{Ran} \ W_{\pm}$ of $W_{\pm}$ is unitary equivalent to the operator $A$. If both wave operators $W_{\pm}(B, A)$ and $W_{\pm}(A, B)$ exist, then the operators $A$ and $B$ are unitarily equivalent. In this case the spectra of the operators $A$ and $B$ coincide.

The existence of the limits (1.5) is a non-trivial problem. We emphasize that the unitary equivalence of operators $A$ and $B$ does not imply the existence of the wave operators (1.5). A notorious example is given by the pair of multiplication $A, (Au)(x) = xu(x)$, and differential $B = -id/dx$ operators in the space $L^2(\mathbb{R})$. Generally speaking, the limits (1.5) (or their appropriate modifications) exist if the perturbation $B - A$ is in some sense small. This might mean different things. For example, it suffices to assume that the operator $B - A$ is trace class or that it acts as an integral operator with smooth kernel in the diagonal representation of the operator $A$ (or $B$).

Our aim is to develop scattering theory for pairs of the operators $J_p, J_q$. According to (1.4) we have

$$a_n^{(q)} - a_n^{(p)} = (q - p)/2 + O(n^{-1}), \quad n \to \infty, \quad \text{and} \quad b_n^{(q)} - b_n^{(p)} = q - p,$$

(1.6)

so that the operator $J_q - J_p$ is not even compact. Therefore standard methods of scattering theory do not work in this case. It turns out however that, from the viewpoint of scattering theory, the diagonal and off-diagonal terms in (1.6) compensate each other. Note that if only one of the coefficients $a_n$ or $b_n$ is changed, then the spectrum of the operator $J_p$ is shifted. In this case the wave operators cannot exist.

Although the difference $J_q - J_p$ is by no means small, there exists a natural one-to-one correspondence between eigenfunctions of the operators $J_p$ for different $p$. Their asymptotics as $n \to \infty$ differ by a phase shift only. This allows us to show that the wave operators $W_{\pm}(J_q, J_p)$ exist for all $p, q > -1$. This result is obtained by a direct calculation which yields also explicit expressions for the wave operators.

### 1.3 Structure of the paper

Jacobi operators and associated orthogonal polynomials, in particular, Laguerre polynomials, are discussed in Sect. 2.

In Sect. 3, we prove the existence of the wave operators $W_{\pm}(J_q, J_p)$ (Theorem 3.1). We also construct scattering theory for pairs $J, \bar{J}$ where $J = J_p$ for some $p > -1$ and the coefficients of a Jacobi operator $\bar{J}$ are sufficiently close to those of $J$ (Theorem 3.8).

In Sect. 4.1, we exhibit a link between a large time behavior of $(e^{-iJt} f)_n$ and asymptotics of associated orthogonal polynomials $P_n(\lambda)$ as $n \to \infty$. This leads to
universal relations between amplitudes and phases in asymptotic formulas for $P_n(\lambda)$. These results are illustrated in Sect. 4.2 – 4.4 on examples of classical polynomials. The results for Laguerre, Jacobi and Hermite polynomials are stated as Theorems 4.5, 4.9 and 4.10, respectively. The Hermite operator $J$ is somewhat exceptional since the evolution $e^{-iJt} f$ is dispersionless in this case.

Below $\| \cdot \|$ is the norm in the space $\ell^2(\mathbb{Z}_+)$; $I$ is the identity operator; $C$ are different positive constants whose precise values are of no importance.

## 2 Jacobi operators and orthogonal polynomials

### 2.1 Orthogonal polynomials

Here we collect necessary information about the Jacobi operators $J$ given by formula (1.1) and associated orthogonal polynomials $P_n(z)$; see the books [1, 8] for a comprehensive presentation. Given coefficients $a_n > 0$, $b_n = \bar{b}_n$, $n \in \mathbb{Z}_+$, one constructs $P_n(z)$ by the recurrence relation

$$a_{n-1} P_{n-1}(z) + b_n P_n(z) + a_n P_{n+1}(z) = z P_n(z), \quad n \in \mathbb{Z}_+, \quad z \in \mathbb{C}, \quad (2.1)$$

and the boundary conditions $P_{-1}(z) = 0$, $P_0(z) = 1$. Then $P_n(z)$ is a polynomial of degree $n$. Obviously, $P(z) = \{P_n(z)\}_{n=0}^\infty$ satisfies the equation $J P(z) = z P(z)$, that is, it is an “eigenvector” of the operator $J$.

We consider the operator $J$ in the space $\ell^2(\mathbb{Z}_+)$. Let us denote by $J_0$ the minimal operator defined by formula (1.1) on a set $D$ of vectors $f = \{f_n\}_{n=0}^\infty$ with only a finite number of non-zero components. This operator is symmetric; moreover, it is essentially self-adjoint if the Carleman condition (1.2) is satisfied. In particular, condition (1.2) holds true for the coefficients (1.4). For essentially self-adjoint operators, the domain $D(J)$ of the closure $J$ of the operator $J_0$ consists of all vectors $f \in \ell^2(\mathbb{Z}_+)$ such that $J f \in \ell^2(\mathbb{Z}_+)$. The spectrum of the self-adjoint operator $J$ is simple with $\epsilon_0 = (1, 0, 0, \ldots)^T$ being a generating vector. Therefore it is natural to define the spectral measure of $J$ by the relation $d\rho(\lambda) = d(E(\lambda)\epsilon_0, \epsilon_0)$ where $E(\lambda)$ is the spectral family of the operator $J$. The polynomials $P_n(\lambda)$ (we call them orthonormal) are orthogonal and normalized in the spaces $L^2(\mathbb{R}; d\rho)$:

$$\int_{-\infty}^\infty P_n(\lambda) P_m(\lambda) d\rho(\lambda) = \delta_{n,m}, \quad (2.2)$$

as usual, $\delta_{n,n} = 1$ and $\delta_{n,m} = 0$ for $n \neq m$.

Alternatively, given a probability measure $d\rho(\lambda)$, the polynomials $P_0(\lambda), P_1(\lambda), \ldots, P_n(\lambda), \ldots$ can be obtained by the Gram-Schmidt orthonormalization of the monomials $1, \lambda, \ldots, \lambda^n, \ldots$ in the space $L^2(\mathbb{R}_+; d\rho)$. It is easy to see that $P_n(\lambda)$ is a polynomial of degree $n$, that is, $P_n(\lambda) = k_n (\lambda^n + r_n \lambda^{n-1} + \cdots)$ with $k_n \neq 0$. One usually requires $k_n > 0$. The recurrence coefficients $a_n, b_n$ can be recovered by the formulas $a_n = k_n / k_{n+1}$, $b_n = r_n - r_{n+1}$. 

$D. R. Yafaev$
One defines a mapping \( \Phi : \ell^2(\mathbb{Z}_+) \to L^2(\mathbb{R}; d\rho) \) by the formula

\[
(\Phi f)(\lambda) = \sum_{n=0}^{\infty} P_n(\lambda) f_n, \quad f = \{f_n\}_{n=0}^{\infty} \in \mathcal{D}.
\]  

(2.3)

This mapping is isometric according to (2.2). It is also unitary if the set of all polynomials \( P_n(\lambda), n \in \mathbb{Z}_+ \), is dense in \( L^2(\mathbb{R}; d\rho) \). This condition is satisfied if the operator \( J_0 \) is essentially self-adjoint. Putting together definitions (1.1), (2.1) and (2.3), it is easy to check the intertwining property \( (\Phi J f)(\lambda) = \lambda (\Phi f)(\lambda) \).

Typically, on the absolutely continuous spectrum of a Jacobi operator \( J \) when \( d\rho(\lambda) = \tau(\lambda)d\lambda \), the orthonormal polynomials \( P_n(\lambda) \) have oscillating asymptotics

\[
P_n(\lambda) = 2\kappa(\lambda)n^{-r} \cos \Omega_n(\lambda) + o(n^{-r}),
\]

(2.4)

\[
\Omega'_n(\lambda) = \omega(\lambda)n^s + o(n^s)
\]

(2.5)

where \( \kappa(\lambda) > 0, s > 0 \) and we can suppose that \( \omega(\lambda) > 0 \). The exponents \( r, s \), the amplitude \( \kappa(\lambda) \) and the phase \( \omega(\lambda) \) are determined by the recurrence coefficients \( a_n \) and \( b_n \). Our considerations (see Sect. 4.1) show that these quantities are necessarily linked by universal relations:

\[
2r + s = 1,
\]

(2.6)

\[
2\pi \tau(\lambda) \kappa^2(\lambda) = s\omega(\lambda).
\]

(2.7)

### 2.2 Laguerre operators

Suppose now that the recurrence coefficients \( a_n, b_n \) are given by formulas (1.4). In this case the orthogonal polynomials \( L_n^{(p)}(z) \) defined by relations (2.1) with the boundary conditions \( L_{-1}^{(p)}(z) = 0, L_0^{(p)}(z) = 1 \) are known as the Laguerre polynomials. Note that the normalized polynomials \( \bar{L}_n^{(p)}(z) \) we consider here are related to the Laguerre polynomials \( L_n^{(p)}(z) \) defined in §10.12 of the book [3] or in §5.1 of the book [9] by the equality

\[
\bar{L}_n^{(p)}(z) = (-1)^n \sqrt{\frac{\Gamma(1 + n)\Gamma(1 + p)}{\Gamma(1 + n + p)}} L_n^{(p)}(z).
\]

(2.8)

According to asymptotic formula (10.15.1) in [3] for positive \( \lambda \), we have

\[
L_n^{(p)}(\lambda) = (-1)^n \sqrt{\frac{\Gamma(1 + p)}{\pi}} \lambda^{-p/2 - 1/4} e^{\lambda^{1/2} n^{-1/4} \cos \left(2\sqrt{n\lambda} - \frac{2p + 1}{4} \pi\right)} + O(n^{-3/4})
\]

(2.9)

as \( n \to \infty \). This asymptotics is uniform in \( \lambda \in [\lambda_0, \lambda_1] \) if \( 0 < \lambda_0 < \lambda_1 < \infty \).

Let us now consider the Laguerre operators \( J_p \) defined by formula (1.1) where \( a_n, b_n \) are given by (1.4). The spectral measures of the operators \( J_p \) are supported on the
half-axis \([0, \infty)\), they are absolutely continuous and are given by the relation (see, e.g., formula (10.12.1) in [3])

\[
d\rho_p(\lambda) = \tau_p(\lambda)d\lambda \quad \text{where} \quad \tau_p(\lambda) = \frac{1}{\Gamma(p+1)} \lambda^p e^{-\lambda}, \quad \lambda \in \mathbb{R}_+.
\] (2.10)

Since the measure \(d\rho_p(\lambda)\) is absolutely continuous, it is convenient to reduce the Jacobi operator \(J_p\) to the operator \(A\) of multiplication by \(\lambda\) in the space \(L^2(\mathbb{R}_+)\). To that end, we put

\[
\varphi_n^{(p)}(\lambda) = \sqrt{\tau_p(\lambda)} L_n^{(p)}(\lambda), \quad \lambda \in \mathbb{R}_+.
\] (2.11)

and introduce a mapping \(\Phi_p : \ell^2(\mathbb{Z}_+) \to L^2(\mathbb{R}_+)\) by the formula (cf. (2.3))

\[
(\Phi_p f)(\lambda) = \sum_{n=0}^{\infty} \varphi_n^{(p)}(\lambda) f_n, \quad f = \{f_n\}_{n=0}^{\infty} \in \mathcal{D}, \quad \lambda \in \mathbb{R}_+.
\] (2.12)

The operator \(\Phi_p^* : L^2(\mathbb{R}_+) \to \ell^2(\mathbb{Z}_+)\) adjoint to \(\Phi_p\) is given by the equality

\[
(\Phi_p^* g)_n = \int_{0}^{\infty} \varphi_n^{(p)}(\lambda) g(\lambda) d\lambda, \quad n \in \mathbb{Z}_+.
\]

The operator \(\Phi_p\) is unitary, that is,

\[
\Phi_p^* \Phi_p = I, \quad \Phi_p \Phi_p^* = I,
\] (2.13)

and enjoys the intertwining property

\[
\Phi_p J_p = A \Phi_p.
\] (2.14)

3 Wave operators

3.1 Two Laguerre operators

One of our main results is stated as follows.

**Theorem 3.1.** Let \(J_p\) be a Jacobi operator with matrix elements (1.4) in the space \(\ell^2(\mathbb{Z}_+)\). Define the unitary operators \(\Phi_p : \ell^2(\mathbb{Z}_+) \to L^2(\mathbb{R}_+)\) by formulas (2.11) and (2.12). Then for all \(p, q > -1\), the wave operators \(W_\pm(J_q, J_p)\) exist and

\[
W_\pm(J_q, J_p) = e^{\pm i(q-p)\pi/2} \Phi_q^* \Phi_p^*.
\]

We start a proof with a simple standard statement.

**Lemma 3.2.** The claim of Theorem 3.1 is equivalent to the relation

\[
s-lim_{t \to \pm\infty} (\Phi_p^* - \mu_\pm \Phi_q^*) e^{-iAt} = 0, \quad \mu_\pm = e^{\pm i(q-p)\pi/2}.
\] (3.1)
Proof. Let $f$ be an arbitrary element of the space $\ell^2(\mathbb{Z}_+)$ and $g = \Phi_pf$. In view of the properties (2.13) and (2.14), we have
$$
\|e^{iJq_t}e^{-iJp_t}f - \mu\Phi_q^*\Phi_pf\| = \|e^{-iJp_t}\Phi_p^*g - \mu\pm e^{-iJq_t}\Phi_q^*g\| = \|\Phi_p^* - \mu\Phi_q^*\|e^{-iAt}g\|.
$$
Since the left- and right-hand sides here tend to zero at the same time, this concludes the proof. \hfill \Box

It suffices to check (3.1) on a set $C^\infty_0(\mathbb{R}_+)$ dense in $L^2(\mathbb{R}_+)$. Let the function $\varphi(p)_n(\lambda)$ be defined by equalities (2.10) and (2.11). It follows from asymptotic formula (2.9) that
$$
\varphi_n(p)(\lambda) = (-1)^n2^{-1/2}\pi^{-1/4}(n + 1)^{-1/4}(\nu_p e^{2i\sqrt{n}l} + \bar{\nu}_p e^{-2i\sqrt{n}l}) + r_n(p)(\lambda) \quad (3.2)
$$
where $\nu_p = e^{-i\pi(2p+1)/4}$ and
$$
|r_n(p)(\lambda)| \leq C(n + 1)^{-3/4} \quad (3.3)
$$
uniformly on every compact subinterval of $\mathbb{R}_+$.

Let us define mappings $V_\pm: C^\infty_0(\mathbb{R}_+) \to \ell^2(\mathbb{Z}_+)$ by the formula
$$
(V_\pm g)_n = (-1)^n2^{-1/2}\pi^{-1/4}(n + 1)^{-1/4}\int_0^{\infty} e^{\pm 2i\sqrt{n}l}l^{-1/4}g(\lambda)d\lambda. \quad (3.4)
$$
Equality (3.2) implies that
$$
\Phi_p^*e^{-iAt}g = \nu_p V_+e^{-iAt}g + \bar{\nu}_p V_-e^{-iAt}g + R_p(t)g \quad (3.5)
$$
where
$$
(R_p(t)g)_n = \int_0^{\infty} r_n(p)(\lambda)e^{-iAt}g(\lambda)d\lambda.
$$

First, we check that the remainder in (3.5) is negligible.

Lemma 3.3. Let $g \in C^\infty_0(\mathbb{R}_+)$. Then
$$
\lim_{|t| \to \infty} \|R_p(t)g\| = 0. \quad (3.6)
$$
Proof. By the Riemann-Lebesgue lemma, every integral in the sum
$$
\|R_p(t)g\|^2 = \sum_{n=0}^{\infty} \left| \int_0^{\infty} r_n(p)(\lambda)e^{-iAt}g(\lambda)d\lambda \right|^2 \quad (3.7)
$$
tends to zero as $|t| \to \infty$. Estimate (3.3) allows us to use the dominated convergence theorem. Therefore the sum (3.7) tends to zero as $|t| \to \infty$. \hfill \Box
Note a formula
\[
\int_0^\infty e^{\pm 2i\sqrt{n}\lambda - iAt} G(\lambda) d\lambda = \int_0^\infty e^{\pm 2i\sqrt{n}\lambda - iAt} \left( \frac{G(\lambda)}{\pm \sqrt{n}\lambda^{1/2} - t} \right)' d\lambda
\]
(3.8)
which can be verified by a direct integration by parts. Using it, we obtain the following elementary assertion.

**Lemma 3.4.** For an arbitrary \( G \in C_0^\infty(\mathbb{R}_+) \), an estimate
\[
\left| \int_0^\infty e^{\pm 2i\sqrt{n}\lambda - iAt} G(\lambda) d\lambda \right| \leq C_k(\sqrt{n} + |t|)^{-k}, \quad \forall t > 0,
\]
(3.9)
is true for all \( k \in \mathbb{Z}_+ \).

**Proof.** Suppose that \( \text{supp} \, G \subset [\lambda_1, \lambda_2] \). Then
\[
\sqrt{n}\lambda^{1/2} + |t| \geq \sqrt{n}\lambda_2^{1/2} + |t|
\]
and \( \lambda^{3/2} \leq \lambda_2^{3/2} \). Therefore the right-hand side of (3.8) is estimated by \( C_1(\sqrt{n} + |t|)^{-1} \) which proves (3.9) for \( k = 1 \). Further integrations by parts in (3.8), yield (3.9) for an arbitrary \( k \).

**Corollary 3.5.** For the operators (3.4) and all \( g \in C_0^\infty(\mathbb{R}_+) \), we have
\[
\lim_{t \to \pm \infty} \|V_{\pm} e^{-iAt} g\| = 0.
\]
(3.10)

Using now relation (3.5) and taking into account Lemma 3.3, we arrive at the following result.

**Lemma 3.6.** Let \( g \in C_0^\infty(\mathbb{R}_+) \). Then
\[
\lim_{t \to \pm \infty} \|\Phi^*_p e^{-iAt} g - \nu_p^\pm V_{\pm} e^{-iAt} g\| = 0.
\]
(3.11)

The same result is of course true for \( \Phi^*_q e^{-iAt} g \). This yields relation (3.1) with \( \mu_{\pm} = (\nu_p \bar{\nu}_q)^\pm 1 \). Using Lemma 3.2, we conclude the proof of Theorem 3.1. \( \square \)

For the scattering operator \( S = W^*_+ W_- \), we obtain a very simple expression.

**Proposition 3.7.** Under the assumptions of Theorem 3.1, we have \( S = e^{i(p-q)\pi I} \).

Recall that the wave operators \( W_{\pm}(B, A) \) for a couple of self-adjoint operators \( A \) and \( B \) can be represented as products of an appropriate Fourier transform for the operator \( A \) and the inverse transform corresponding to \( B \). For Schrödinger operators in the space \( L^2(\mathbb{R}_+) \), this is discussed, for example, in Section 4.2 (see formula (2.30)) of the book [11]. Normally, there are two natural sets of eigenfunctions of the operators \( A \) and \( B \). This leads to two wave operators. In our case these sets of eigenfunctions almost coincide so that the wave operators \( W_{\pm}(J_q, J_p) \) differ by a phase factor only whence the scattering operator is almost trivial.
3.2 Perturbation theory

Here we choose some \( p > -1 \) and construct scattering theory for the pair \( J = J_p, \ \overline{J} = J + V \) where the operator \( V \) is in some sense small. We do not assume that \( V \) is a Jacobi operator, but, in particular, our results apply to Jacobi operators. We denote by \( \mathcal{H}_{ac} \) the absolutely continuous subspace of the operator \( \overline{J} \).

Let us define an operator \( \mathcal{N} \) in the space \( \ell^2(\mathbb{Z}_+) \) by the formula
\[
(\mathcal{N} f)_n = (n + 1) f_n.
\]

Our goal is to prove the following result.

**Theorem 3.8.** Let \( J = J_p, \ p > -1, \) be the Laguerre operator with matrix elements (1.4), and let \( \overline{J} = J + V \) where \( V \) is a symmetric operator such that
\[
V = N^{-r} TN^{-r_0}
\]
(3.12)
for some bounded operator \( T \).

1° If \( r_0 > 1/4, \ r > 1/4 \), then the wave operators \( W_{\pm}(\overline{J}, J) \) exist and are complete, that is, \( \text{Ran} \ W_{\pm}(\overline{J}, J) = \mathcal{H}_{ac} \).

2° If \( r_0 > 1/4, \ r > 1/2 \), then the singular spectrum of the operator \( \overline{J} \) consists of eigenvalues of finite multiplicities that may accumulate to the point 0 only.

**Corollary 3.9.** Let \( a_n, b_n, \) be defined by formulas (1.4), and let \( \overline{J} \) be a self-adjoint Jacobi operator with matrix elements \( \tilde{a}_n, \tilde{b}_n \) such that
\[
\tilde{a}_n - a_n = O(n^{-\rho}), \quad \tilde{b}_n - b_n = O(n^{-\rho}).
\]
(3.13)

1° If \( \rho > 1/2 \), then the wave operators \( W_{\pm}(\overline{J}, J) \) exist and are complete.

2° If \( \rho > 3/4 \), then the singular spectrum of the operator \( \overline{J} \) consists of eigenvalues that may accumulate to the point 0 only.

Let us deduce Corollary 3.9 from Theorem 3.8. We introduce diagonal matrices \( A \) and \( B \) with the elements \( a_n \) and \( b_n \) and the shift \( S: (Sf)_n = f_{n+1}, \ n \in \mathbb{Z}_+ \). Then \( J = AS + S^* A + B \) and with obvious notation, we have
\[
\overline{J} - J = (\tilde{A} - A)S + S^*(\tilde{A} - A) + (\tilde{B} - B).
\]
The operators \( N^{r}(\tilde{A} - A)N^{r_0} \) and \( N^{r}(\tilde{B} - B)N^{r_0} \) are bounded if \( r_0 + r = \rho \). Since the operator \( N^{-r} S N^r \) is also bounded, we see that \( N^{r}(\overline{J} - J)N^{r_0} \) is bounded as long as \( r_0 + r = \rho \). For the proof of the first statement of Corollary 3.9, we set \( r_0 = r = \rho/2 \).

The second statement follows if \( r_0 = (\rho - 1/4)/2 \) and \( r = (\rho + 1/4)/2 \).

**Remark.** Under the assumptions of Corollary 3.9 one can find asymptotics of the associated orthogonal polynomials. Essentially, it is the same as for the Laguerre polynomials, that is, given (up to a phase shift) by formula (2.9).
Example 3.10. Consider the Jacobi operator $\tilde{J}$ with the coefficients
\[ \tilde{a}_n = n + \alpha, \quad \tilde{b}_n = 2n + 2\alpha - 1, \quad \alpha > 1/2. \]
Up to a shift by $2\alpha - 1$, this operator is related to the birth and death processes (see §5.2 of the book [4]). Now conditions (3.13) with $\rho = 1$ are satisfied for $a_n = a^{(p)}_n$, $b_n = b^{(p)}_n$ where $p = 2(\alpha - 1)$.

Note that under the assumptions of Theorem 3.8 or Corollary 3.9, the operator $V = \tilde{J} - J$ belongs to the Hilbert-Schmidt but not to the trace class. Therefore the assertion about the wave operators $W_{\pm}(\tilde{J}, J)$ does not follow from the classical Kato-Rosenblum theorem.

3.3 Strong smoothness

Our proof of Theorem 3.8 relies on a notion of strong smoothness (see Definition 4.4.5 in the book [10]). We will check strong $J$-smoothness of the operator $N - r$ for $r > 1/4$.

Recall that the operator $\Phi$ is defined by formula (2.12) where the functions $\varphi_n(\lambda)$ are linked to the Laguerre polynomials $L_n(\lambda)$ by equalities (2.10), (2.11).

Lemma 3.11. Let $\Lambda$ be a compact subinterval of $\mathbb{R}_+$ and $r > 1/4$. Then
\[ |(\Phi N^{-r} f)(\lambda)| \leq C \|f\|. \tag{3.14} \]
Moreover,
\[ |(\Phi N^{-r} f)(\mu) - (\Phi N^{-r} f)(\lambda)| \leq C |\mu - \lambda|^s \|f\| \tag{3.15} \]
if $s < 2r - 1/2$ and $s \leq 1$. The constants $C$ in (3.14) and (3.15) do not depend on $f \in L^2(\mathbb{Z}_+)$ and $\lambda, \mu \in \Lambda$.

Proof. It follows from asymptotics (2.9) of $L_n^{(p)}(\lambda)$ that
\[ |\varphi_n(\lambda)| \leq C (1 + n)^{-1/4} \tag{3.16} \]
where the constant $C$ does not depend on $n \in \mathbb{Z}_+$ and on $\lambda \in \Lambda$. So, by the Schwarz inequality, we have
\[ |(\Phi N^{-r} f)(\lambda)|^2 \leq C \left( \sum_{n=0}^{\infty} (1 + n)^{-1/4 - r} |f_n| \right)^2 \leq C \sum_{n=0}^{\infty} (1 + n)^{-1/2 - 2r} \|f\|^2 \]
where the series is convergent if $2r > 1/2$. This proves (3.14).

For a proof of (3.15), we need an estimate on derivatives of $dL_n^{(p)}(\lambda)/d\lambda$ of Laguerre polynomials for large $n$. Let us use formula (5.1.14) of the book [9] for Laguerre polynomials $L_n^{(p)}(\lambda)$ linked to $L_{n-1}^{(p+1)}(\lambda)$ by equality (2.8):
\[ \frac{d}{d\lambda} L_n^{(p)}(\lambda) = -L_{n-1}^{(p+1)}(\lambda) \quad \text{whence} \quad \frac{d}{d\lambda} L_n^{(p)}(\lambda) = \sqrt{\frac{n}{p+1}} L_{n-1}^{(p+1)}(\lambda). \]
It now follows from estimate (3.16) that
\[ |\varphi_n'(\lambda)| \leq C (1 + n)^{1/4}. \quad (3.17) \]

Note that
\[ |\varphi_n(\mu) - \varphi_n(\lambda)| \leq (2 \sup_{x \in \Lambda} |\varphi_n(x)|)^{1-s} \sup_{x \in \Lambda} |\varphi_n'(x)||\mu - \lambda|^s \]
for any \( s \in [0, 1] \). Therefore using (3.16) and (3.17), we see that
\[ |\varphi_n(\mu) - \varphi_n(\lambda)| \leq C (1 + n)^{-1/4 + s/2} |\mu - \lambda|^s. \]

This yields an estimate
\[
|\Phi N^{-r} f)(\mu) - (\Phi N^{-r} f)(\lambda)| \leq \sum_{n=0}^{\infty} |\varphi_n(\mu) - \varphi_n(\lambda)|(1 + n)^{-r} |f_n| \\
\leq \|f\| \sqrt{\sum_{n=0}^{\infty} |\varphi_n(\mu) - \varphi_n(\lambda)|^2 (1 + n)^{-2r}} \leq C |\mu - \lambda|^s \|f\|
\]
provided \( s < 2r - 1/2 \). Thus we get (3.15).

The operator \( N^{-r} \) satisfying estimates (3.14) and (3.15) is called strongly \( J \)-smooth with exponent \( s \in (0, 1] \) on the interval \( \Lambda \).

Theorem 4.6.4 of [10] states that if a perturbation \( V \) admits representation (3.12) with the operators \( N^{-r_0} \) and \( N^{-r} \) strongly \( J \)-smooth (with some exponents \( s_0, s > 0 \)) on all compact subintervals \( \Lambda \) of \( \mathbb{R}_+ \), then the wave operators \( W_{\pm}(J, J) \) exist and are complete. This is part 1\(^0 \) of Theorem 3.8.

Theorems 4.7.9 and 4.7.10 of [10] state that all spectral results enumerated in part 2\(^0 \) of Theorem 3.8 are true provided \( s > 1/2 \) (and \( s_0 > 0 \)). According to Lemma 3.11 we can choose \( s > 1/2 \) if \( r > 1/2 \). This concludes the proof of part 2\(^0 \) of Theorem 3.8.

Finally, we note that an unusually weak assumption \( \rho > 1/2 \) (instead of the standard \( \rho > 1 \) in (3.13)) is explained by a decay (3.16) of eigenfunctions of \( J \).

4 Time evolution

It is a common wisdom that an asymptotic behavior of \( e^{-iJt} f \) as \( t \to \infty \) is determined by spectral properties of the Jacobi operator \( J \) and by asymptotics of the corresponding orthonormal polynomials \( P_n(\lambda) \) as \( n \to \infty \). We first discuss this general idea at a heuristic level and derive new relations between amplitudes and phases in asymptotic formulas for \( P_n(\lambda) \). Then we illustrate the formulas obtained on examples of the classical polynomials.
4.1 Universal asymptotic relations

Assume that the spectrum of a Jacobi operator $J$ is absolutely continuous on an interval $\Lambda$ and the corresponding measure $d\rho(\lambda) = \tau(\lambda)d\lambda$ has a smooth weight $\tau(\lambda)$ for $\lambda \in \Lambda$. Let the operator $\Phi$ diagonalizing $J$ be defined by formula (2.3). Choose $f$ such that $f = E(\Lambda)f$ and set $g(\lambda) = \sqrt{\tau(\lambda)}(\Phi f)(\lambda)$. Clearly, $\|g\|_{L^2(\Lambda)} = \|f\|$. If $P_n(\lambda)$ satisfy asymptotic relation (2.4), then

$$
(e^{-iJt}f)_n = (n + 1)^{-r} \int_{\Lambda} \kappa(\lambda)\left(e^{i\Omega_n(\lambda) - i\lambda t} + e^{-i\Omega_n(\lambda) - i\lambda t}\right)g(\lambda)d\lambda
$$

(4.1)

where $\kappa(\lambda) = \sqrt{\tau(\lambda)}\kappa(\lambda)$. Here and below we keep track only of leading terms in asymptotic formulas. We suppose that the phase $\Omega_n(\lambda)$ obeys condition (2.5) where $\omega'(\lambda) \neq 0$ for $\lambda \in \Lambda$ and that $g \in C^\infty_0(\Lambda)$.

Stationary points of the integrals (4.1) are determined by the equations

$$
\pm \omega(\lambda)n^s = t.
$$

(4.2)

Suppose, for definiteness, that $t \to +\infty$. Then equation (4.2) may have a solution (necessary unique) for the sign “+” only. Let $\sigma = s^{-1}$, $\xi = n/t^\sigma$, and let $\lambda = \lambda(\xi)$ be the solution of the equation

$$
\omega(\lambda) = \xi^{-s}.
$$

(4.3)

Applying the stationary phase method to integrals (4.1), we see that

$$
(e^{-iJt}f)_n = (2\pi)^{1/2}n^{-r-s/2}e^{i\phi(\xi, t)}|\omega'\lambda(\xi)|^{-1/2}\kappa(\lambda(\xi))g(\lambda(\xi))
$$

(4.4)

where

$$
\phi(\xi, t) = \pm \pi/4 + \Omega \xi t^\sigma (\lambda(\xi))t - \lambda(\xi)t \quad \text{if} \quad \pm \omega'(\lambda) > 0.
$$

Let

$$
h(\xi) = \xi^{-r-s/2}|\omega'\lambda(\xi)|^{-1/2}\kappa(\lambda(\xi))g(\lambda(\xi))
$$

(4.5)

so that (4.4) reads as

$$
(e^{-iJt}f)_n = (2\pi)^{1/2}t^{-(2r+s)\sigma/2}e^{i\phi(n/t^\sigma, t)}h(n/t^\sigma).
$$

(4.6)

Since the operators $e^{-iJt}$ are unitary, it follows from (4.6) that

$$
2\pi \lim_{t \to \infty} \left(t^{-(2r+s)\sigma} \sum_{n=0}^{\infty} |h(n/t^\sigma)|^2 \right) = \|f\|^2.
$$

(4.7)

Observe that the integral sums

$$
N^{-1} \sum_{n=0}^{\infty} |h(n/N)|^2 \to \int_{\Lambda^{-1}(\Lambda)} |h(\xi)|^2 d\xi
$$

(4.8)
as \( N \to \infty \), by the definition of the integral. Let us here set \( N = t^{\sigma} \) and compare (4.8) with (4.7). First, we obtain relation (2.6) since the powers of \( t \) in the left-hand sides of (4.7) and (4.8) should be the same. It follows that \( \xi^{-r-s/2} \) in (4.5) can be replaced by \( \xi^{-1/2} \). Second, comparing the right-hand sides, using definition (4.5) and taking into account that \( \|f\| = \|g\|_{L^2(\Lambda)} \), we see that

\[
2\pi \int_{\Lambda^1(\Lambda)} \xi^{-1} |\omega'(\lambda(\xi))|^{-1} |\xi^2(\lambda(\xi))| |g(\lambda(\xi))|^2 d\xi = \int_{\Lambda} |g(\lambda)|^2 d\lambda. \tag{4.9}
\]

Differentiating relation (4.3), we find that \( \omega'(\lambda(\xi))\lambda'(\xi) = -s\xi^{-s-1} = -s\xi^{-1}\omega(\lambda(\xi)) \). Substituting this expression for \( \omega'(\lambda(\xi)) \) into the left-hand side of (4.9), we rewrite equality (4.9) as

\[
2\pi s^{-1} \int_{\Lambda} \xi^2(\lambda) \omega(\lambda)^{-1} |g(\lambda)|^2 d\lambda = \int_{\Lambda} |g(\lambda)|^2 d\lambda.
\]

Since \( g \in C^\infty_0(\Lambda) \) is arbitrary, this yields relation (2.7) between asymptotic coefficients in (2.4), (2.5) and the spectral measure.

According to formula (4.4) the functions \((e^{-iHt} f)_n\) “live” in the region where \( n \sim |t|^\sigma \). For example, for the Laguerre polynomials, it follows from (2.9) that \( \sigma = 2 \). This is fairly unusual since scattering states are normally concentrated in the region where \( n \sim |t| \). Similarly, for continuous operators of the Schrödinger type we have the relation \( x \sim |t| \). Indeed, consider, for example, the operator \( H = -d^2/dx^2 \) in the space \( L^2(\mathbb{R}) \). In this case, we have

\[
(e^{-iHt} f)(x) = e^{\mp n^2/4(2|t|)^{-1/2}e^{ix^2/(4t)}} f(x/(2t)) + \varepsilon(x,t)
\]

where \( f(x) = (2\pi)^{-1/2} \int_\infty^{-\infty} e^{-ix\xi} f(x)dx \) is the Fourier transform of \( f(x) \) and the norm in \( L^2(\mathbb{R}) \) of the term \( \varepsilon(x,t) \) tends to zero as \( |t| \to \infty \).

The above arguments relied strongly on the stationary phase method and used the assumption \( \omega'(\lambda) \neq 0 \). Let us now consider, at a very heuristic level, the dispersionless case \( \omega'(\lambda) = 0 \). We suppose that condition (2.4) is satisfied with \( \Omega_n(\lambda) = \nu_n \lambda + \delta_n \) where \( \lambda \in \mathbb{R} \), \( \nu_n = \omega n^s + o(n^s) \), \( s \in (0,1) \) and \( \delta_n \) do not depend on \( \lambda \). Then it follows from (4.1) where \( \Lambda = \mathbb{R} \) that, up to a term which tends to zero in \( \ell^2(\mathbb{Z}_+) \) as \( |t| \to \infty \),

\[
(e^{-iHt} f)_n = (2\pi)^{1/2} (n+1)^{-r} (e^{i\delta_n \hat{G}(t-v_n)} + e^{-i\delta_n \hat{G}(t+v_n)}) \tag{4.10}
\]

where \( \hat{G}(t) \) is the Fourier transform of \( G(\lambda) = \nu(\lambda)g(\lambda) \). If \( \omega > 0 \) and \( t \to +\infty \), the second term in the right-hand side is negligible. The operators \( e^{-iHt} \) being unitary, it follows from (4.10) that

\[
2\pi \sum_{n=0}^\infty (n+1)^{-2r} |\hat{G}(t-v_n)|^2 \to \|f\|^2 = \int_{-\infty}^\infty |g(\lambda)|^2 d\lambda \quad \text{as} \quad t \to +\infty. \tag{4.11}
\]

It is natural to expect that the limit of the left-hand side here is determined by \( n \) such that \( \omega n^s \sim t \) whence \( n^{-2r} \sim (t/\omega)^{-2\sigma r} \). Let us set \( m = n - (t/\omega)^{\sigma r} \) and use that
$t - \nu_n \sim -st(\omega/t)^\sigma m$. Then (4.11) implies that

$$2\pi t^{-\sigma-1}\omega^2 |G(-m/N)|^2 \rightarrow \int_{-\infty}^{\infty} |g(\lambda)|^2 d\lambda$$

where $N = s^{-1}t^{\sigma-1}\omega^{-\sigma} \rightarrow \infty$. According to (4.8) and the Parseval identity the second factor in the left-hand side has a finite limit $\int_{-\infty}^{\infty} |G(\lambda)|^2 d\lambda$. Therefore the power of $t$ in the first factor should be zero which yields equality (2.6). Now relation (4.12) shows that

$$2\pi \int_{-\infty}^{\infty} |G(\lambda)|^2 d\lambda = \omega_s \int_{-\infty}^{\infty} |g(\lambda)|^2 d\lambda.$$  

Since $G(\lambda) = \kappa(\lambda)g(\lambda)$ and $g \in C^\infty_0(\mathbb{R})$ is arbitrary, we again arrive at equality (2.7) where $\omega(\lambda) = \omega$ does not depend on $\lambda$.

Let us summarize the results obtained. Suppose that the orthonormal polynomials $P_n(\lambda)$ have asymptotic behavior (2.4) with the phase $\Omega_n(\lambda)$ satisfying (2.5). Then, necessarily relations (2.6) and (2.7) hold true. Precise conditions guaranteeing (2.6) and (2.7) and proofs of these relations will be published elsewhere.

### 4.2 Laguerre polynomials

Let the operators $J_p$ be defined by formula (1.1) with the coefficients (1.4). Theorem 3.1 shows that, for all $p, q > -1$,

$$\lim_{t \rightarrow \pm \infty} ||e^{-iJ_p t}f - e^{-iJ_q t}f_\pm|| = 0 \quad \text{if} \quad f_\pm = W_\pm(J_q, J_p)f,$$

that is, the time evolution of $e^{-iJ_p t}f$ is the same for all $p > -1$; only the initial data are changed.

Our goal here is to obtain detailed asymptotic formulas for $(e^{-iJ_p t}f)_n$ as $t \rightarrow \pm \infty$. We choose $f$ from the set $\Phi^*_p C^\infty_0(\mathbb{R}_+)$ dense in $L^2(\mathbb{R}_+)$. It turns out that the asymptotics of $(e^{-iJ_p t}f)_n$ depends crucially on the ratio $n/t^2$. Below we omit the index $p$. Since $e^{-iJ_1 t}f = \Phi^* e^{-iAt} \Phi f$, Lemma 3.6 can be reformulated as follows.

**Lemma 4.1.** Let $g = \Phi f \in C^\infty_0(\mathbb{R}_+)$, and let the operators $V_\pm$ be defined by formula (3.4). Then

$$e^{-iJ_1 t}f = v^\pm_1 V_\pm e^{-iAt}g + \varepsilon_\pm(t)$$

where $||\varepsilon_\pm(t)|| \rightarrow 0$ as $t \rightarrow \pm \infty$.

Thus, we have to find the asymptotics of

$$(V_\pm e^{-iAt}g)_n = (-1)^n 2^{-1} n^{-1/2}(n + 1)^{-1/4} \int_0^\infty e^{\pm 2i\sqrt{n} - iAt} G(\lambda) d\lambda,$$  

where $G(\lambda) = \lambda^{-1/4} g(\lambda)$ as $t \rightarrow \pm \infty$.

The first assertion shows that these functions are small as $|t| \rightarrow \infty$ both for relatively “small” and “large” $n$. 

Lemma 4.2. Let the operators $V_k$ be defined by formula (3.4), and let $G \in C_0^\infty(\mathbb{R}_+)$. Suppose that $\text{supp } G \subset [\lambda_1, \lambda_2]$ and choose $\mu_1 < \lambda_1$, $\mu_2 > \lambda_2$. Then for all $n \leq \mu_1 t^2$ and $n \geq \mu_2 t^2$, all $k \in \mathbb{Z}_+$ and some constants $C_k$, we have estimates

$$\left|(V_k e^{-i\lambda t} g)_n\right| \leq C_k(\sqrt{n} + |t|)^{-k}, \quad \forall t \in \mathbb{R}. \quad (4.14)$$

Proof. We proceed from formula (3.8) and estimate its right-hand side. If $n \leq \mu_1 t^2$, then

$$|\sqrt{n}\lambda^{-1/2} - t| \geq |t| - \sqrt{n}\lambda_1^{-1/2} \geq |t|(1 - (\mu_1 / \lambda_1)^{1/2}) \geq c(\sqrt{n} + |t|)$$

where $c(\sqrt{\mu_1} + 1) = 1 - \sqrt{\mu_1} / \lambda_1 > 0$. Quite similarly, if $n \geq \mu_2 t^2$, then

$$|\sqrt{n}\lambda^{-1/2} - t| \geq |t| - \sqrt{n}\lambda_2^{-1/2} \geq |t|(\lambda_2^{-1/2} - \mu_2^{-1/2}) \geq c(\sqrt{n} + |t|)$$

where $c(\sqrt{\mu_2} + 1) = \sqrt{\mu_2} / \lambda_2 - 1 > 0$. According to (3.8) this proves (4.14) for $k = 1$. Integrating by parts $k$ times in (3.8), we obtain (4.14).

To find asymptotics of the integral in (4.13), we use the stationary phase method. Put

$$\xi = \sqrt{n}t^{-1} \quad \text{and} \quad \theta(\lambda, \xi) = -2\xi\sqrt{\lambda} + \lambda.$$

Then

$$\int_0^\infty e^{2i\sqrt{n}\lambda^{-1/2} - i\lambda t} G(\lambda) d\lambda = \int_0^\infty e^{-i\theta(\lambda, \xi)t} G(\lambda) d\lambda =: I(t, \xi). \quad (4.15)$$

Differentiating $\theta(\lambda, \xi)$ in $\lambda$, we find that

$$\theta'(\lambda, \xi) = -\xi\lambda^{-1/2} + 1 \quad \text{and} \quad \theta''(\lambda, \xi) = 2^{-1}\xi\lambda^{-3/2}.$$

The stationary point $\lambda_0 = \lambda_0(\xi)$ of the phase $\theta(\lambda, \xi)$ is determined by the equation $\theta'(\lambda_0, \xi) = 0$ whence $\lambda_0 = \xi^2$. Therefore the stationary phase method yields

$$I(t, \xi) = e^{\pi i \xi^4 / 4} e^{-i\theta(\lambda_0, \xi)t} \frac{2\pi}{|\theta''(\lambda_0, \xi)t|} G(\lambda_0) + o(|t|^{-1/2})$$

as $t \to \pm \infty$. Since $\theta(\lambda_0, \xi) = -\xi^2$ and $\theta''(\lambda_0, \xi) = 2^{-1}\xi^{-2}$, we arrive at the following intermediary result.

Lemma 4.3. Let $G \in C_0^\infty(\mathbb{R}_+)$. Then the integral (4.15) has asymptotics

$$I(t, \xi) = 2\pi^{1/2} e^{\pi i \xi^4 / 4} e^{i\xi^2 t |t|^{-1} |\xi| G(\xi^2)} + o(|t|^{-1}), \quad t \to \pm \infty, \quad (4.16)$$

with the estimate of the remainder uniform in $\xi$ from compact subintervals of $\mathbb{R} \setminus \{0\}$.

Let us come back to formula (4.13). Set

$$(U(t)g)_n = (-1)^n e^{in/t} |t|^{-1} g(n/t^2), \quad t \neq 0. \quad (4.17)$$
Lemma 4.4. Let \( G \in C_0^\infty(\mathbb{R}_+) \) and \( 0 < \mu_1 < \mu_2 < \infty \). Then
\[
\sup_{n \in (\mu_1 t^2, \mu_2 t^2)} \left| (V \chi e^{-iAt} g)_n - e^{\pi i/4} (U(t)g)_n \right| = o(|t|^{-1}), \quad t \to \pm \infty.
\] (4.18)

Proof. Let us set in (4.16) \( G(\lambda) = \lambda^{-1/4} g(\lambda) \) and \( \xi = \sqrt{n} t^{-1} \) so that
\[
|\xi| G(\xi^2) = |\xi|^{1/2} g(\xi^2) = n^{1/4} |t|^{-1/2} g(n/t^2).
\]
Thus, it follows from Lemma 4.3 that
\[
\int_0^\infty e^{\pm i \sqrt{n} t^{-1} \lambda^{-1/4}} G(\lambda) d\lambda = 2 \pi^{1/2} e^{\pi i/4} n^{1/4} e^{i n/t} |t|^{-1/2} g(n/t^2) + o(|t|^{-1}), \quad t \to \pm \infty,
\]
as long as \( n \in (\mu_1 t^2, \mu_2 t^2) \). Putting together this relation with (4.13), we arrive at (4.18). \( \square \)

Now we are in a position to obtain an asymptotic formula for \( e^{-iJt} f \) as \( t \to \pm \infty \).

Theorem 4.5. Let \( J_p, p > -1 \), be a Jacobi operator with matrix elements (1.4), and let the operator \( U(t) \) be given by formula (4.17). Define the operators \( \Phi_p \) by formulas (2.11) and (2.12) and suppose that \( \Phi_p f \in C_0^\infty(\mathbb{R}_+) \). Then
\[
\lim_{t \to \pm \infty} \| e^{-iJt} f - e^{\pi i(p+1/2) \pi/2} U(t) \Phi_p f \| = 0.
\] (4.19)

Proof. According to Lemma 4.1 we can replace here \( e^{-iJt} f \) by \( V \chi e^{-iAt} g \) where \( g = \Phi f \). Suppose that \( \text{supp} \, g \subset [\lambda_1, \lambda_2] \) and choose \( \mu_1 < \lambda_1, \mu_2 > \lambda_2 \). It follows from Lemma 4.2 that
\[
( \sum_{n \leq \mu_1 t^2} + \sum_{n \geq \mu_2 t^2} ) |(V \chi e^{-iAt} g)_n|^2 \to 0
\] (4.20)
as \( t \to \pm \infty \). According to (4.18) we also have
\[
\sum_{\mu_1 t^2 \leq n \leq \mu_2 t^2} |(V \chi e^{-iAt} g)_n - e^{\pi i/4} (U(t)g)_n|^2 = o(1).
\]
Combined with (4.20) this yields relation (4.19). \( \square \)

According to (2.9) for the Laguerre polynomials, we have \( \Lambda = \mathbb{R}_+, r = 1/4, s = 1/2 \) and
\[
\kappa(\lambda) = \frac{1}{2} \sqrt{\frac{\Gamma(1+p)}{\pi}} \lambda^{-p/2-1/4} e^{\lambda/2}, \quad \Omega_n(\lambda) = \pi n + 2 \sqrt{n \lambda} - \frac{2p+1}{4} \pi, \quad \omega(\lambda) = \lambda^{-1/2}.
\]
Since \( \tau(\lambda) \) is given by (2.10), the identity (2.7) is satisfied.
4.3 Jacobi polynomials

In this subsection we define a Jacobi operator by its spectral measure \( d\rho(\lambda) \). We suppose that this measure is supported on the interval \([-1, 1]\) and

\[
d\rho(\lambda) = \tau(\lambda)d\lambda, \quad \lambda \in (-1, 1),
\]

where

\[
\tau(\lambda) = k(1 - \lambda)\alpha(1 + \lambda)\beta, \quad \alpha, \beta > -1.
\]

The weight function \( \tau(\lambda) = \tau_{\alpha,\beta}(\lambda) \) (as well as all other objects discussed below) depends on \( \alpha \) and \( \beta \), but these parameters are often omitted in notation. The constant \( k = k_{\alpha,\beta} \) is chosen in such a way that the measure (4.21) is normalized, i.e., \( \rho(\mathbb{R}) = \rho((-1, 1)) = 1 \). The orthonormal polynomials \( G_n(\lambda) = G_n^{(\alpha,\beta)}(\lambda) \) determined by the measure (4.21), (4.22) are known as the Jacobi polynomials.

Let \( J = J_{\alpha,\beta} \) be the Jacobi operator with the spectral measure \( d\rho(\lambda) = d\rho_{\alpha,\beta}(\lambda) \). Explicit expressions for its matrix elements \( a_n, b_n \) can be found, for example, in the books \([3, 9]\), but we do not need them. We here note only asymptotic formulas

\[
a_n = \frac{1}{2} + 2^{-4}(1 - 2\alpha^2 - 2\beta^2)n^{-2} + O(n^{-3}), \quad b_n = 2^{-2}(\beta^2 - \alpha^2)n^{-2} + O(n^{-3}) \quad (4.23)
\]

for the matrix elements and (see formula (8.21.10) in the book \([9]\))

\[
G_n(\lambda) = 2^{1/2}(\pi k)^{-1/2}(1 - \lambda)^{-(1+2\alpha)/4}(1 + \lambda)^{-(1+2\beta)/4} \times \cos \left((n + \gamma)\arcsin\lambda - \pi(2n + \beta - \alpha)/4\right) + O(n^{-1}), \quad \gamma = (\alpha + \beta + 1)/2, \quad (4.24)
\]

for the orthonormal polynomials. Estimate of the remainder in (4.24) is uniform in \( \lambda \) from compact subsets of \((-1, 1)\). Similarly to the cases of the Laguerre polynomials, we set \( \varphi_n(\lambda) = \sqrt{\tau(\lambda)}G_n(\lambda) \) and define the mapping \( \Phi : \ell^2(\mathbb{Z}_+) \to L^2(-1, 1) \) by formula (2.12) where \( \lambda \in (-1, 1) \). Using (4.22) and (4.24), we obtain the representation

\[
e^{-iJt}f = \Phi^*e^{-i\lambda t}g = V_+e^{-i\lambda t}g + V_-e^{-i\lambda t}g + R(t)g, \quad g = \Phi f \in C_0^\infty(-1, 1), \quad (4.25)
\]

where

\[
(V_\pm g)_n = (2\pi)^{-1/2}i\pi n e^{\pm i(\alpha - \beta)\pi/4} \int_{-1}^1 e^{\pm i(n + \gamma)}\arcsin\lambda(1 - \lambda^2)^{-1/4}g(\lambda)d\lambda,
\]

and the remainder \( R(t)g \) satisfies condition (3.6).

Let us state analogues of Lemmas 3.4 and 4.2.

**Lemma 4.6.** For an arbitrary \( G \in C_0^\infty(-1, 1) \), an estimate

\[
\left| \int_{-1}^1 e^{\pm in\arcsin\lambda - i\lambda t}G(\lambda)d\lambda \right| \leq C_k(n + |t|)^{-k} \quad (4.26)
\]

is true for all \( k \in \mathbb{Z}_+ \) if \( \mp t > 0 \).
Proof. Integrating by parts, we see that
\[
\int_{-1}^{1} e^{\pm i n \arcsin \lambda - i \lambda t} G(\lambda) d\lambda = \pm i \int_{-1}^{1} e^{\pm i n \arcsin \lambda - i \lambda t} \left( \frac{G(\lambda)}{n(1 - \lambda^2)^{-1/2} \mp t} \right) d\lambda.
\]
This yields (4.26) for \( k = 1 \) because \( \mp t = |t| \) and \( |\lambda| \leq c < 1 \) on the support of \( G \). Further integrations by parts lead to estimates (4.26) for all \( k \).

Quite similarly, we obtain also the following result.

Lemma 4.7. Let \( G \in C_0^\infty((-1, 1) \setminus \{0\}) \). Then estimates (4.26) hold for all \( t \in \mathbb{R} \) and all \( k \in \mathbb{Z}_+ \) if either \( n \leq \delta |t| \) or \( n \geq (1 - \delta)|t| \) for a sufficiently small number \( \delta \) depending on \( \supp G \).

We use these lemmas with \( G \pm (\lambda) = e^{\pm i \gamma \arcsin \lambda} \left( 1 - \lambda^2 \right)^{-1/4} G(\lambda) \) and take equality (4.25) into account. According to Lemma 4.6 relation (3.10) is satisfied so that it suffices to consider \( V \pm e^{-iAt} g \) as \( t \to \pm \infty \). According to Lemma 4.7 we only have to study
\[
(V \pm e^{-iAt} g)_n = (2\pi)^{-1/2} \mp n e^{i(\alpha - \beta)\pi/4} \int_{-1}^{1} e^{\pm i n \arcsin \lambda - i \lambda t} G_\pm(\lambda) d\lambda
\]
for \( \delta |t| \leq n \leq (1 - \delta)|t| \) where \( \delta > 0 \).

Let us now set
\[
\xi = n|t|^{-1}, \quad \theta(\lambda, \xi) = \xi \arcsin \lambda - \lambda
\]
and consider an integral
\[
I(t, \xi) = \int_{-1}^{1} e^{i\theta(\lambda, \xi)t} G(\lambda) d\lambda, \quad \xi \in (0, 1),
\]
where \( G \in C_0^\infty((-1, 1) \setminus \{0\}) \). Differentiating \( \theta(\lambda, \xi) \) in \( \lambda \), we see that
\[
\theta'(\lambda, \xi) = \xi(1 - \lambda^2)^{-1/2} - 1 \quad \text{and} \quad \theta''(\lambda, \xi) = \xi \lambda(1 - \lambda^2)^{-3/2}.
\]

The stationary points \( \lambda_\pm = \lambda_\pm(\xi) \) of the phase \( \theta(\lambda, \xi) \) are determined by the equation \( \theta'(\lambda_\pm, \xi) = 0 \) whence
\[
\lambda_\pm = \pm \lambda_0, \quad \lambda_0 = \sqrt{1 - \xi^2}
\]
and
\[
\theta''(\lambda_\pm, \xi) = \pm \xi^{-2} \sqrt{1 - \xi^2}.
\]

Note also that
\[
\theta(\lambda_0, \xi) = \xi \arccos \xi - \sqrt{1 - \xi^2} =: \psi(\xi).
\]
Therefore the stationary phase method yields the following intermediary result.
Lemma 4.8. Let the phases $\theta(\lambda, \xi)$ and $\psi(\xi)$ be given by formulas (4.28) and (4.30), respectively. Then the integral (4.29) has asymptotics

$$I(t, \xi) = (2\pi)^{1/2}|t|^{-1/2}(1 - \xi^2)^{-1/4}$$

$$\times \left( e^{\pm i\pi/4}e^{i\psi(\xi)r} G(\sqrt{1 - \xi^2}) + e^{\mp i\pi/4}e^{-i\psi(\xi)r} G(-\sqrt{1 - \xi^2}) \right) + o(|t|^{-1/2}) \quad (4.31)$$

as $t \to \pm \infty$. The estimate of the remainder in (4.31) is uniform in $\xi$ from compact subintervals of $(0, 1)$.

Let us now apply Lemma 4.8 to functions (4.27). We set

$$h_{\pm}(\xi) = e^{\pm i\pi/4}e^{\pm iy} \arccos \xi (1 - \xi^2)^{-1/4} g(\pm \sqrt{1 - \xi^2}) \quad (4.32)$$

and

$$(U(t)g)_n = i^{2n}e^{\pm i(\alpha-\beta)\pi/4}|t|^{-1/2} \left( e^{i\psi(\xi)t} h_+(\xi) + e^{-i\psi(\xi)t} h_-(\xi) \right) \quad (4.33)$$

where $\xi = n|t|^{-1} \in (0, 1)$, $\pm t > 0$ and the function $\psi(\xi)$ is defined by formula (4.30). For $n \geq |t|$, we set $(U(t)g)_n = 0$.

Putting the results obtained together, we state our final result.

Theorem 4.9. Let $J = J_{\alpha, \beta}$ be the Jacobi operator corresponding to the weight function (4.21). Define the operator $\Phi : \ell^2(\mathbb{Z}_+) \to L^2((−1, 1), 1)$ by formula (2.12) and suppose that $g = \Phi f \in C_0^\infty((−1, 1) \setminus \{0\})$. Let the operator $U(t)$ be given by equalities (4.32) and (4.33). Then

$$\lim_{t \to \pm \infty} ||e^{-iJt}f - U(t)\Phi f|| = 0.$$
4.4 Hermite polynomials

The Hermite polynomials $H_n(z)$ are determined by the recurrence coefficients (1.3). As usual, relations (2.1) for $H_n(z)$ are complemented by the boundary conditions $H_{-1}(z) = 0$, $H_0(z) = 1$. According to asymptotic formula (10.15.18) in [3], we have

$$H_n(\lambda) = 2^{1/2} n^{-1/4} e^{1/2} (2n+1)^{-1/4} \cos \left( \sqrt{2n+1} \lambda - \pi n/2 \right) + O(n^{-3/4})$$

(4.34)

as $n \to \infty$. This asymptotics is uniform in $\lambda \in \mathbb{R}$ on compact subintervals.

Let us consider the Jacobi operators $J$ defined by formula (1.1) where $a_n$, $b_n$ are given by (1.3). The spectral measure of $J$ equals $d\rho(\lambda) = \pi^{-1/2} e^{-\lambda^2} d\lambda$ where $\lambda \in \mathbb{R}$ (see, e.g., formula (10.13.1) in [3]). Thus, $d\rho(\lambda)$ is absolutely continuous and its support is the whole axis $\mathbb{R}$.

Following the scheme exposed in Sect. 2.2, we reduce the Jacobi operator $J$ to the operator $A$ of multiplication by $\lambda$ in the space $L^2(\mathbb{R})$. To that end, we introduce a mapping $\Phi : \ell^2(\mathbb{Z}_+) \to L^2(\mathbb{R})$ by the formula

$$(\Phi f)(\lambda) = \pi^{-1/4} \sum_{n=0}^{\infty} e^{-\lambda^2/2} H_n(\lambda) f_n, \quad f = \{f_n\}_{n=0}^{\infty} \in \mathcal{D}, \quad \lambda \in \mathbb{R}.$$  

(4.35)

The operator $\Phi$ is unitary and enjoys the intertwining property $\Phi J = A \Phi$.

Putting together formulas (4.34) and (4.35), we obtain representation (4.25) where

$$(V_\pm g)_n = i^{\pi n} (2\pi)^{-1/2} (2n+1)^{-1/4} \int_{-\infty}^{\infty} e^{\pm t \sqrt{2n+1}} \hat{g}(\lambda) d\lambda$$

and the remainder $R(t)g$ satisfies condition (3.6). Then

$$(V_\pm e^{-iAt} g)_n = i^{\pi n} (2n+1)^{-1/4} \hat{g}(t \pm \sqrt{2n+1})$$

where $\hat{g}(x)$ is the Fourier transform of $g$. Since $\hat{g}(x) = O(x^{-k})$ as $|x| \to \infty$ for all $k > 0$, we see that

$$\lim_{t \to \pm \infty} \sum_{n=0}^{\infty} (2n+1)^{-1/2} |\hat{g}(t \pm \sqrt{2n+1})|^2 = 0,$$

whence relation (3.10) follows. Thus, representation (4.25) implies the result below.

**Theorem 4.10.** Let $J$ be a Jacobi operator with matrix elements (1.3), and let the operator $U(t)$ be given by the formula

$$(U(t)g)_n = i^{\pi n} (2n+1)^{-1/4} \hat{g}(t \pm \sqrt{2n+1}), \quad \pm t > 0.$$  

(4.36)

Define the operator $\Phi$ by formula (4.35) and suppose that $\Phi f \in C^0(\mathbb{R})$. Then

$$\lim_{t \to \pm \infty} ||e^{-iJt} f - U(t)\Phi f|| = 0.$$
Corollary 4.11. For all \( g \in C_0^\infty(\mathbb{R}) \), we have

\[
\lim_{t \to \pm \infty} \sum_{n=0}^{\infty} (2n+1)^{-1/2} |\hat{g}(t \mp \sqrt{2n+1})|^2 = \|\hat{g}\|_{L_2(\mathbb{R})}^2.
\] (4.37)

According to (4.36) the evolution \( e^{-itJ} f \) is dispersionless. Clearly, it is similar to time evolutions for first order differential operators.

Finally, we note that for the Hermite polynomials, \( \Lambda = \mathbb{R} \), \( r = 1/4 \), \( s = 1/2 \), \( \Omega_n(\lambda) = \sqrt{2n+1} \lambda - \pi n/2 \), \( \omega(\lambda) = \sqrt{2} \), \( \kappa(\lambda) = 2^{-3/4} \pi^{-1/4} e^{\lambda^2/2} \). Thus, the identity (2.7) remains true. Evolution (4.36) is dispersionless now, and relation (4.37) is a particular case of (4.11).

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References

[1] N. Akhiezer, The classical moment problem and some related questions in analysis, Oliver and Boyd, Edinburgh and London, 1965.

[2] A. I. Aptekarev and J. S. Geronimo, Measures for orthogonal polynomials with unbounded recurrence coefficients, J. Approx. Theory 207 (2016), 339-347.

[3] A. Erdélyi, W. Magnus, F. Oberhettinger, and F. G. Tricomi, Higher transcendental functions, Vol. 1, 2, McGraw-Hill, New York-Toronto-London, 1953.

[4] M. E. H. Ismail, Classical and quantum orthogonal polynomials in one variable, Cambridge University Press, Cambridge, 2005.

[5] J. Janas and S. Naboko, Jacobi matrices with power-like weights – grouping in blocks approach, J. Funct. Anal. 166 (1999), 218-243.

[6] J. Janas, S. Naboko, and E. Shcherbina, Asymptotic behavior of generalized eigenvectors of Jacobi matrices in the critical ("double root") case, Z. Anal. Anwend. 28 (4) (2009), 411-430.

[7] S. Naboko and S. Simonov, Titchmarsh-Weyl formula for the spectral density of a class of Jacobi matrices in the critical case, arXiv: 1911.10282 (2019).

[8] P. G. Nevai, Orthogonal polynomials, Memoirs of the AMS 18, No. 213, Providence, R. I., 1979.

[9] G. Szegő, Orthogonal polynomials, Amer. Math. Soc., Providence, R. I., 1978.

[10] D. R. Yafaev, Mathematical scattering theory: General theory, Amer. Math. Soc., Providence, R. I., 1992.

[11] D. R. Yafaev, Mathematical scattering theory: Analytic theory, Amer. Math. Soc., Providence, R. I., 2010.

[12] D. R. Yafaev, Analytic scattering theory for Jacobi operators and Bernstein-Szegő asymptotics of orthogonal polynomials, Rev. Math. Phys. 30, No. 8 (2018), 1840019.