Abstract. We establish a connection between the inverse scattering problem and the determination of the distribution of the position of a Lévy process at the exit time of a bounded interval in terms of its Lévy exponent.

1. Introduction

It is well known that the fluctuation properties of a Lévy process are intimately related to the Wiener-Hopf factorization of its Lévy exponent $\phi$. The observation (initiated by Spitzer [Sp] for random walks) that the factorization of $(q + \phi(iu))^{-1}$, where $q$ is a positive constant, can be interpreted as the independence of the past and pre minimum parts of the Lévy process killed at an independent exponential time. The Wiener-Hopf factors yield the distributions of the minimum and the maximum of the killed process. Also the problem of exit from a semi-infinite interval which consists in determining the joint distribution of the exit time of an interval $[-\infty, x]$ and of the position at the exit time, can be reduced to finding the Wiener-Hopf factorization, see [B][D][S]. In the theory of analytic functions, Wiener-Hopf factorization is the simplest of a large class of factorization problems known as Riemann-Hilbert problems. In this paper we establish a connection between the joint distribution of the maximum, minimum and final value of the Lévy process killed at an independent exponential time and a certain Riemann-Hilbert factorization problem. The determination of this joint distribution allows to compute the joint law of the position and the time of the exit of a Lévy process from a bounded interval. More precisely, we prove that this problem reduces to the factorization of the matrix

$$\begin{pmatrix} 0 & -1 \\ 1 & \phi(iu) + q \end{pmatrix}$$

where $\phi$ is the Lévy exponent of the process, into a product $A(x, iu)B(x, iu)$ where $A$ and $B$ are matrices which are boundary values, on the imaginary axis, of analytic functions, defined respectively on the left and on the right complex half planes, satisfying a normalization at infinity involving a positive real parameter $x$. In order to obtain this result we establish that a set of functions, defined from the Laplace transforms of random variables involving the maximum and minimum processes associated with the Lévy process, satisfy a certain system of integral equations. We show that this system of integral equation is analogous to the basic differential system appearing in scattering theory on the real line. Our system does not reduce to the usual problem of scattering theory, since the associated potential is very singular compared to the potentials considered in this theory. The analogy however is sufficiently good that we can apply similar arguments as in the work of Shabat [Sh], and show the equivalence of our system of integral equations with a Riemann-Hilbert problem.

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This paper is organized as follows. In section 2 we recall some basic facts on Lévy processes and the Wiener-Hopf factorization. Then we introduce the main functions of our work and state the two main theorems. The first one gives a system of integral equations satisfied by the functions, and the second one states that this system is equivalent to a certain Riemann-Hilbert problem. We explain why the first theorem is related to the differential equation and the direct problem of the scattering theory while the second one is related to the inverse scattering problem. In section 3 we explain the connection between the main results and the exit problem from an interval and related questions. Section 4 contains a preliminary result on the conditional independence of the pre and post minimum process, knowing the amplitude. In section 5 we establish the first equation of the integral system. Section 6 deals with the second group of equations. In this part we introduce two Markov chains, built from the successive minima and maxima of the process, which play a key role in the proof. In section 7 we establish the second theorem, about the Riemann-Hilbert problem, by adapting the arguments of Shabat [Sh] to our setting. We give some probabilistic interpretations of the factorization in the Riemann-Hilbert problem in section 8 in terms of Wiener-Hopf factorization of certain auxiliary Lévy processes. Finally, section 9 is devoted to apply our results to stable processes and to Lévy processes without positive jumps. This allows us to give some precisions on results obtained by Rogozin [R] for the stable case and by Takacs [T] for Lévy processes without positive jumps.

2. Notations and main results

Let \((\Omega, \mathcal{F})\) be the space of functions defined on \([0, +\infty[\) with values in \(\mathbb{R} \cup \{\delta\}\) where \(\delta\) is a cemetery point, and let \(X\) denote the canonical process \(X_t(\omega) = \omega(t)\). In this paper \(P\) will be the law on \((\Omega, \mathcal{F})\) of a Lévy process started at 0 with Lévy exponent \(\phi\). More precisely, we have:

\[
P(e^{-iuX_t}) = e^{-t\phi(iu)} \quad (i u \in i\mathbb{R})
\]

2.1. Some facts on Wiener-Hopf factorization. We start by recalling some standard facts on Wiener-Hopf factorization and fluctuations of Lévy processes for which we refer to [B] chapter 6. Let \(S_t\) et \(I_t\) be the past maximum and past minimum processes, namely:

\[
S_t := \sup\{X_s, 0 \leq s \leq t\} \quad I_t := \inf\{X_s, 0 \leq s \leq t\}
\]

We introduce now local times at 0 of the reflected processes \(S - X\) et \(X - I\) and the associated Wiener-Hopf factors. The definition of these local times depends on the regularity of \([0, +\infty[\) or \([-\infty, 0]\) for the Lévy process. If \([0, +\infty[\) is regular, i.e. \(T^0 = \inf\{t > 0, X_t > 0\} = 0\) a.s. (resp. \([-\infty, 0[\) is regular \(T_0 = \inf\{t > 0, X_t < 0\} = 0\) a.s.) then 0 is a regular point for the Markov process \(X - S\) (resp. \(X - I\)) and \(L^s\) (resp. \(L^i\)) denotes any local time at 0 of this process. In this case, \(t \mapsto L^s_t[0,t]\) (resp. \(t \mapsto L^i_t[0,t]\)) is an increasing continuous process, we denote by \(L^{s,-1}\) (resp.\(L^{i,-1}\)) its right continuous inverse. The pair \((L^{s,-1},S_{L^{s,-1}})\) (resp. \((L^{i,-1},I_{L^{i,-1}})\)) is a bi-variate Lévy process which may have a finite life time if \(\lim_{t \to +\infty} X_t = -\infty\) a.s. (resp. \(\lim_{t \to +\infty} X_t = +\infty\) a.s.). The Wiener-Hopf factors are the Lévy exponents of this process, more precisely

\[
P(e^{-\lambda S_{L^{s,-1}} - qL^{s,-1}_t}; t < L^s_\infty) = e^{-t\psi_q(\lambda)} \quad \Re(\lambda) \geq 0, q \in [0, +\infty[\]
\]

Respectively,

\[
P(e^{-\lambda I_{L^{i,-1}} - qL^{i,-1}_t}; t < L^i_\infty) = e^{-t\hat{\psi}_q(\lambda)} \quad \Re(\lambda) \leq 0, q \in [0, +\infty[\]
\]

If \([0, +\infty[\) is irregular (this means that the time \(T^{0-} = \inf\{t > 0, X_t \geq 0\}\) is positive a.s.) , (respectively if \([-\infty, 0[\) is irregular , \(T^{0-} = \inf\{t > 0, X_t \leq 0\}\) is positive a.s.) then the set \(\{t; S_t = X_t\}\) (resp. \(\{t; X_t = I_t\}\)
is a.s. discrete and the local time $L^s$ (respectively, $L^i$) is the random point measure
\[ L^s(dt) := \sum_u 1_{X_u = s_u} \delta_u(dt) \]
Respectively,
\[ L^i(dt) := \sum_u 1_{X_u = I_u} \delta_u(dt) \]
The Wiener-Hopf factors are
\[ \psi_q(\lambda) := 1 - P(e^{-\lambda X_T^q} - qT^q; T^q < +\infty) \quad \Re(\lambda) \geq 0, q \in [0, +\infty[ \]
Respectively,
\[ \check{\psi}_q(\lambda) := 1 - P(e^{-\lambda X_{T_0}^q} - qT_0; T_0 < +\infty) \quad \Re(\lambda) \leq 0, q \in [0, +\infty[ \]
Notice that in this case $\psi$ (resp. $\check{\psi}$) is the Lévy exponent of a compound Poisson process and that according to proposition 4 of [B] chapter 6, $X_T^q > 0$ a.s. (resp. $X_{T_0}^q < 0$ a.s. ), thus the times $T^q$ and $T_0 := \inf\{t; X_t > 0\}$ are equal a.s. (resp. $T_{0-} = T_0 := \inf\{t; X_t < 0\}$ a.s.).
Finally, if neither condition is fulfilled, then $P$ is the law a compound Poisson process. In this case, for reasons which will appear later, it is necessary to use a dissymetric definition of local times: We denote by $L^s$ the random measure
\[ L^s(dt) := 1_{S_t = X_t} dt \]
And $L^i$ will be the random point measure :
\[ L^i(dt) := \delta_0(dt) + \sum_{u>0} 1_{I_u > t} 1_{X_u = I_u} \delta_u(dt) \]
As above the Wiener-Hopf factor $\psi_q(\lambda)$ is the Lévy exponent of the bi-variate Lévy process $(L^{s,-1}, S_{L^{s,-1}})$, namely :
\[ P(e^{-\lambda S_{L^{s,-1}}^q - qL^{s,-1}^q}) := e^{-t\psi_q(\lambda)} \quad \Re(\lambda) \geq 0, q \in [0, +\infty[ \]
The Wiener-Hopf factor $\check{\psi}_q(\lambda)$ is the fonction :
\[ \check{\psi}_q(\lambda) := 1 - P(e^{-\lambda X_{T_0}^q} - qT_0; T_0 < +\infty) \quad T_0 = \inf\{t; X_t < 0\} \]
Note that in all cases, one has :
\[ \frac{1}{\psi_q(\lambda)} = P(\int_{0, +\infty[} e^{-\lambda S_t - qt} L^s(dt)) \quad \Re(\lambda) \geq 0, q \in [0, +\infty[, q\Re(\lambda) \neq 0 \]
and
\[ \frac{1}{\check{\psi}_q(\lambda)} = P(\int_{0, +\infty[} e^{-\lambda I_t - qt} L^i(dt)) \quad \Re(\lambda) \leq 0, q \in [0, +\infty[, q\Re(\lambda) \neq 0 \]
It is possible to normalize the local times so that the following Wiener-Hopf factorization holds (see e.g. [B],[S]), and we shall assume in the sequel that it is the case. For the compound Poisson process this follows from the convention we have choosen.

**Proposition 2.1.** The pair $(\psi_q(\lambda), \check{\psi}_q(\lambda))$ satisfies the following Wiener-Hopf identity
\[ \check{\psi}_q(iu)\psi_q(iu) = \phi(iu) + q \quad iu \in i\mathbb{R}, q \in [0, +\infty[ \]
2.2. The main functions. In next proposition we define the so called excursions measures $N$ and $\hat{N}$ associated to local times $L^i$ and $L^s$ by the compensation formula (see for example chapter 4 of [B]).

**Proposition 2.2. Compensation formula**

There exists a unique measure on $(\Omega, \mathcal{F})$, $N$ (resp. $\hat{N}$) such that

$$
P\left( \sum_{[g,d] \in C} 1_{(I_g - X_{(g,s)-})_{x \geq 0} \in dw_1} 1_{(X_{g+1} - I_g)_{0 \leq t < d-g \in dw_2}} \right) = P\left( \int_{[0, +\infty[} 1_{(I_t - X_{(t,s)-})_{x \geq 0} \in dw_1} L^i(dt) \right) N(dw_2)
$$

respectively,

$$
P\left( \sum_{[g,d] \in C} 1_{(S_g - X_{(g,s)-})_{x \geq 0} \in dw_1} 1_{(X_{g+1} - S_g)_{0 \leq t < d-g \in dw_2}} \right) = P\left( \int_{[0, +\infty[} 1_{(S_t - X_{(t,s)-})_{x \geq 0} \in dw_1} L^s(dt) \right) \hat{N}(dw_2)
$$

where $C$ (resp. $\hat{C}$) is the set of connected components of the complement of the support of $L^i(dt)$ (resp. $L^s(dt)$).

Notice that if $[0, +\infty[$ (resp. $]-\infty, 0[)$ is regular, then the state 0 is a regular point of the Markov process $X - S$ (resp. $X - I$) and $\hat{N}$ (resp. $N$) is the usual excursion measure from 0 of this process. If $[0, +\infty[$ (resp. $]-\infty, 0[)$ is irregular, then $\hat{N}$ (resp. $N$) is the distribution under $P$ of the canonical process $X$ killed at time $T^0$ (resp. $T^0_0$).

We can now introduce the main functions of this paper. First define the random stopping times for every $x \in [0, +\infty[$ :

$$
T^x := \inf \{ t; X_t > x \} \quad T^x_* := \inf \{ t; X_t < -x \}
$$

$$
T^x_s := \inf \{ t; X_t - S_t < -x \} \quad T^x_i := \inf \{ t; X_t - I_t > x \}
$$

Define the right continuous left limited functions of $x$ ($x \in [0, +\infty[$):

$$
A_q(x, \lambda) := P\left( \int_{[0, +\infty[} 1_{S_t - I_t \leq x} e^{-\lambda S_t - qL^i(t)} dt \right) \quad \lambda \in C, q \in [0, +\infty[^
$$

$$
\hat{A}_q(x, \lambda) := P\left( \int_{[0, +\infty[} 1_{S_t - I_t \leq x} e^{-\lambda S_t - qL^i(t)} dt \right) \quad \lambda \in C, q \in [0, +\infty[^
$$

$$
C_q(x, \lambda) := N\left( e^{-\lambda X_T - qT}; T < +\infty \right) \quad \Re(\lambda) \geq 0, q \in [0, +\infty[^
$$

$$
\hat{C}_q(x, \lambda) := \hat{N}\left( e^{-\lambda X_T - qT}; T < +\infty \right) \quad \Re(\lambda) \leq 0, q \in [0, +\infty[^
$$

If $[0, +\infty[$ (resp. $]-\infty, 0[)$ is regular, it is easy to check that the process $((L_{L^i_{T^i_x}}^{-1}; S_{L^i_{T^i_x}}^{-1}); 0 \leq t < L_{T^i_x}^i)$ (resp. $((L_{L^i_{T^i_x}}^{-1}; I_{L^i_{T^i_x}}^{-1}); 0 \leq t < L_{T^i_x}^i)$) is a killed Lévy process and we denote by $B_q(x, \lambda)$ (resp. $\hat{B}_q(x, \lambda)$) its Lévy exponent, more precisely :

$$
e^{-tB_q(x, \lambda)} := P\left( e^{-\lambda S_{L^i_{T^i_x}}^{-1} - qL_{T^i_x}^i}; t < L^i_{T^i_x} \right) \quad \Re(\lambda) \geq 0, q \in [0, +\infty[^
$$

$$
e^{-tB_q(x, \lambda)} := P\left( e^{-\lambda S_{L^i_{T^i_x}}^{-1} - qL_{T^i_x}^i}; t < L^i_{T^i_x} \right) \quad \Re(\lambda) \leq 0, q \in [0, +\infty[^
$$

If $[0, +\infty[$ (resp. $]-\infty, 0[)$ is irregular, then

$$
B_q(x, \lambda) := 1 - P\left( e^{-\lambda S_T - qT}; T < T^i_x \right) \quad \Re(\lambda) \geq 0, q \in [0, +\infty[^
$$

$$
\hat{B}_q(x, \lambda) := 1 - P\left( e^{-\lambda S_T - qT}; T < T^i_x \right) \quad \Re(\lambda) \leq 0, q \in [0, +\infty[^
$$
respectively,  
\[ \hat{B}_q(x, \lambda) := 1 - P(e^{-\lambda T_0^0 - q T_0^0}; T_0 < T_i^0) \quad \Re(\lambda) \leq 0, q \in [0, +\infty[ \]

Notice that \( T^0 < T^0_i \) if and only if \( T^0 < T_2 \) (resp. \( T_0 < T^0_i \) if and only if \( T_0 < T^0 \)). Notice also that the function \((q, \lambda) \mapsto B_q(x, \lambda)\) (resp. \((q, \lambda) \mapsto \hat{B}_q(x, \lambda)\)) is the Lévy exponent of a compound Poisson process.

In all cases one gets
\[ \frac{1}{B_q(x, \lambda)} = P(\int_{[0,T_i^+]} e^{-\lambda S_t - qt} L^s(dt)) \quad \Re(\lambda) \geq 0, q \in [0, +\infty[ \]
\[ \frac{1}{\hat{B}_q(x, \lambda)} = P(\int_{[0,T_i^+]} e^{-\lambda I_t - qt} L^i(dt)) \quad \Re(\lambda) \leq 0, q \in [0, +\infty[ \]

Using the previous functions, we now define the following ones :
\[ H_q(x) := A_q(x, 0) = P(\int_{[0,+\infty]} 1_{S_t \leq x} e^{-qt} L^s(dt)) \]
\[ \hat{H}_q(x) := \hat{A}_q(x, 0) = P(\int_{[0,+\infty]} 1_{I_t \leq x} e^{-qt} L^i(dt)) \]

We shall denote \( H_q(dx) \) and \( \hat{H}_q(dx) \) the Stieltjes measures associated to these increasing functions. Since \( L^s[0, \varepsilon] \) and \( L^i[0, \varepsilon] \) are positive for every \( \varepsilon > 0 \), \( H_q(x) \) and \( \hat{H}_q(x) \) do not vanish. Furthermore, one has :
\[ H_q(x) \leq P(\int_{[0,+\infty]} 1_{S_t \leq x} e^{-qt} L^s(dt)) \quad \text{and} \quad \hat{H}_q(x) \leq P(\int_{[0,+\infty]} 1_{I_t \leq x} e^{-qt} L^i(dt)) \]

For \( q = 0 \), the right hand sides of the first (resp. second) inequality is the so called renewal function of the subordinator with Lévy exponent \( \psi_0 \) (resp. \( \psi_0 \)) (see [B] chapter 3). Therefore it is finite and \( H_q(x) \) and \( \hat{H}_q(x) \) are finite too. It is also true obviously when \([0, +\infty[ \) (resp. \([-\infty, 0[\)) is irregular. Let us mention that the following inequalities have been proved in [F] :
\[ P(\int_{[0,+\infty]} 1_{S_t \leq x} e^{-qt} L^s(dt)) \leq 4H_q(x) \quad \text{and} \quad P(\int_{[0,+\infty]} 1_{I_t \leq x} e^{-qt} L^i(dt)) \leq 4\hat{H}_q(x) \]

### 2.3. The main results.

**Theorem 2.3.** For all \( x \in [0, +\infty[, q \in [0, +\infty[, \) one has :

1) For all complex \( \lambda \), the functions \( A_q \) and \( \hat{A}_q \) satisfy the integral equations
\[ A_q(x^-, \lambda) = H_q(0) + \int_{[0,x]} e^{-\lambda y} \hat{A}_q(y, \lambda) \frac{H_q(dy)}{H_q(y)} \]
\[ \hat{A}_q(x, \lambda) = \hat{H}_q(0) + \int_{[0,x]} e^{\lambda y} A_q(y^-, \lambda) \frac{\hat{H}_q(dy)}{\hat{H}_q(y^-)} \]

2) For all complex \( \lambda \) with \( \Re(\lambda) > 0 \) (and \( \Re(\lambda) = 0 \) if \( q > 0 \) or \( \lim X_t = -\infty \)) the functions \( C_q \) and \( B_q \) satisfy the integral equations
\[ C_q(x^-, \lambda) = \int_{[x, +\infty]} e^{-\lambda y} B_q(y, \lambda) \frac{H_q(dy)}{H_q(y)} \]
\[ B_q(x, \lambda) = \psi_q(\lambda) + \int_{[x, +\infty]} e^{\lambda y} C_q(y^-, \lambda) \frac{H_q(dy)}{H_q(y^-)} \]
Moreover, one has

\[ A_q(x^-, \lambda)B_q(x, \lambda) + \hat{A}_q(x, \lambda)C_q(x^-, \lambda) = 1 \]

3) For all complex \( \lambda \) with \( \Re(\lambda) < 0 \) (and \( \Re(\lambda) = 0 \) if \( q > 0 \) or \( \lim_{t \to +\infty} X_t = +\infty \)) the functions \( \hat{C}_q \) and \( \hat{B}_q \) satisfy the integral equations

\[
\hat{C}_q(x, \lambda) = \int_{|x|, +\infty} e^{\lambda y} \hat{B}_q(y^-, \lambda) \frac{\hat{H}_q(dy)}{H_q(y^-)} \]

\[
\hat{B}_q(x^-, \lambda) = \hat{\psi}_q(\lambda) + \int_{|x|, +\infty} e^{-\lambda y} \hat{C}_q(y, \lambda) \frac{H_q(dy)}{H_q(y)}
\]

Moreover, one has

\[ \hat{A}_q(x, \lambda)\hat{B}_q(x^-, \lambda) + A_q(x^-, \lambda)\hat{C}_q(x, \lambda) = 1 \]

For all \( x \in ]0, +\infty[ \), \( q \in ]0, +\infty[ \) define:

\[
M_q(x, \lambda) = \begin{pmatrix}
A_q(x^-, \lambda) & -C_q(x^-, \lambda) \\
\hat{A}_q(x, \lambda) & B_q(x, \lambda)
\end{pmatrix}
\]

if \( \Re(\lambda) > 0 \)

\[
M_q(x, \lambda) = \begin{pmatrix}
\hat{B}_q(x^-, \lambda) & A_q(x^-, \lambda) \\
-C_q(x, \lambda) & \hat{A}_q(x, \lambda)
\end{pmatrix}
\]

if \( \Re(\lambda) < 0 \)

Observe that for all \( \lambda = iu \in i\mathbb{R} \), the following limits exist:

\[
M_q^+(x, iu) := \lim_{\lambda \to iu, \Re(\lambda) > 0} M(x, \lambda) = \begin{pmatrix}
A_q(x^-, iu) & -C_q(x^-, iu) \\
\hat{A}_q(x, iu) & B_q(x, iu)
\end{pmatrix}
\]

\[
M_q^-(x, iu) := \lim_{\lambda \to iu, \Re(\lambda) < 0} M(x, \lambda) = \begin{pmatrix}
\hat{B}_q(x^-, iu) & A_q(x^-, iu) \\
-C_q(x, iu) & \hat{A}_q(x, iu)
\end{pmatrix}
\]

The following result gives a Riemann-Hilbert characterization of the matrix \( M_q \).

**Theorem 2.4.** For all \( x \in ]0, +\infty[ \), \( q \in ]0, +\infty[ \), \( \lambda \mapsto M_q(x, \lambda) \) is the unique function satisfying the following properties

1) \( \lambda \mapsto M_q(x, \lambda) \) is analytic on the two half-planes \( \{ \Re(\lambda) > 0 \} \) and \( \{ \Re(\lambda) < 0 \} \).

2) \( \lambda \mapsto M_q(x, \lambda) \) has a right limit \( (M_q^+(x, iu)) \) and a left limit \( (M_q^-(x, iu)) \) at every point \( iu \in i\mathbb{R} \) and these two limits satisfy the equation:

\[
M_q^+(x, iu) = M_q^-(x, iu) \begin{pmatrix}
0 & -1 \\
1 & \phi(iu) + q
\end{pmatrix}
\]

3) The matrix

\[
\begin{pmatrix}
\frac{M_{11}}{|\lambda| + 1} & e^{\lambda x}M_{12} \\
e^{-\lambda x}M_{21} & \frac{M_{22}}{|\lambda| + 1}
\end{pmatrix}
\]

is bounded in \( \{ \Re(\lambda) > 0 \} \cup \{ \Re(\lambda) < 0 \} \).

4) The following limits are valid for \( \Re(\lambda) \to -\infty \):

\[
\frac{M_{11}(x, \lambda)}{\psi_0(\lambda)} \to 1 \quad \hat{\psi}_0(\lambda)M_{22}(x, \lambda) \to 1 \quad e^{\lambda x}M_{12} \to 0
\]

and \( e^{-\lambda x}M_{21}(x, \lambda) \to 0 \) if \( -\infty, 0[ \) is irregular.
2.4. Connections with scattering theory. The integral equations of theorem 2.3 can be rewritten as the following distribution theoretic differential equation; for all \( \lambda \in \mathbb{C} \setminus \mathbb{R} \),

\[
M'_q(x, \lambda) = \left( \begin{array}{cc} 0 & e^{-\lambda x} \frac{H_q(dx)}{H_q(x)} \\ e^{\lambda x} \frac{H_q(dx)}{H_q(x)} & 0 \end{array} \right) M_q(x, \lambda)
\]

This differential equation is a non standard form of the classical equation of the scattering theory on the line, with a measure valued potential matrix \( \left( \begin{array}{cc} H_q(dx) & 0 \\ H_q(dx) & H_q(x) \end{array} \right) \). Remark that these measures are unbounded in general.

Let us recall the basics of scattering theory, as expounded in [BDZ]. One considers a potential matrix \( \left( \begin{array}{cc} 0 & v \\ \hat{v} & 0 \end{array} \right) \) where \( v \) et \( \hat{v} \) are real functions of real variable satisfying some regularity assumptions, in particular they are integrable. To this matrix is associated the differential equation :

\[
Y'(x, \lambda) = \left( \begin{array}{cc} 0 & e^{-\lambda x} v \\ e^{\lambda x} \hat{v} & 0 \end{array} \right) Y(x, \lambda)
\]

where \( \lambda \) is a complex parameter.

Denotes by \( J \) the matrix \( J := \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right) \). One can prove that for any imaginary complex number \( \lambda = iu \) and any matrix solution \( x \mapsto Y(x, iu) \), the matrix \( e^{\frac{i\pi}{2}J}Y(x, iu)e^{-\frac{i\pi}{2}J} \) converges as \( x \to +\infty \) and \( x \to -\infty \) to the limits \( y^+(iu) \) and \( y^-(iu) \). The scattering matrix associated to the potential is given by \( S(iu) = y^+(iu)[y^-(iu)]^{-1} \). Obviously this matrix doesn’t depend on the particular solution \( x \mapsto Y(x, \lambda) \). On the other hand there exists a unique solution \( x \mapsto \mathcal{V}(x, \lambda) \) such that \( e^{\frac{i\pi}{2}J}\mathcal{V}(x, \lambda)e^{-\frac{i\pi}{2}J} \) converges to the identity matrix for \( x \to -\infty \).

Shabat [Sh], see also [BDZ], has proved that for any \( x \in \mathbb{R} \), the function \( \lambda \mapsto \mathcal{V}(x, \lambda) \) is analytic on the half planes \( \{ \Re(\lambda) > 0 \} \) and \( \{ \Re(\lambda) < 0 \} \), has right limits \( \mathcal{V}(x, (iu)^+) \) and left limits \( \mathcal{V}(x, (iu)^-) \) at any point \( iu \) on the imaginary axis and the jumps matrix \( \mathcal{V}(x, (iu)^+)[\mathcal{V}(x, (iu)^-)]^{-1} \) is equal to \( S(iu) \). Moreover, \( e^{\frac{i\pi}{2}J}\mathcal{V}(x, \lambda)e^{-\frac{i\pi}{2}J} \) converges to the identity matrix for \( |\lambda| \to +\infty \). It is then clear that the function \( \lambda \mapsto \mathcal{V}(x, \lambda) \) is entirely determined by these properties. The determination of the matrix \( \mathcal{V}(x, \lambda) \) and consequently of the potential matrix from the scattering matrix (“the inverse problem”) is reduced to the solution of this Riemann-Hilbert problem.

In our problem, after extending the potential by zero on \( ]-\infty, 0[ \), one can see that the potential is not given by a function but rather by a measure, furthermore it is not integrable on \( \mathbb{R} \) unless \( P \) is the distribution of a compound Poisson process and \( q \) is positive. Consequently, none of the solutions of the associated differential equation is regular both for \( x \to +\infty \) and \( x \to -\infty \) and one does not have any scattering matrix in the classical sens (i.e. \( S(iu) = y^+(iu)[y^-(iu)]^{-1} \)). Moreover there is no solution of the differential equation \( x \mapsto Y(x, \lambda) \) such that the matrix \( e^{\frac{i\pi}{2}J}Y(x, \lambda)e^{-\frac{i\pi}{2}J} \) converges to the identity matrix for \( x \to -\infty \). We have chosen the solution of the differential equation (Sc) that is the most convenient from a probabilistic point of view. Theorem 2.4 tells us that the determination of this solution (and consequently of all the others and of the potential matrix) is reduced to the solution of a Riemann-Hilbert problem, as in classical scattering theory. In our setting the role of the scattering matrix is played by the matrix \( \left( \begin{array}{cc} 0 & -1 \\ 1 & \phi(iu) + q \end{array} \right) \).
3. Connection with the exit time from an interval and related variables

Let us now explain how distributions of certain random variables related to the exit time of a Lévy process from an interval are related to the matrix $M_q(x, \lambda)$.

Let us denote $\text{supp}(L^*)$ and $\text{supp}(L^i)$ the supports of the random measures $L^*(dt)$ and $L^i(dt)$, and for all positive time $t$,

$$g_t^i = \sup ([0, t] \cap \text{supp}(L^i)) \quad g_t^* = \sup ([0, t] \cap \text{supp}(L^*))$$

**Proposition 3.1.** For all $\lambda_1, \lambda_2 \in C$, $q_1, q_2 \in [0, +\infty]$, $x \in [0, +\infty]$, one has

$$\tilde{A}_{q_1}(x, \lambda_1) A_{q_2}(x, \lambda_2) = P \left( \int_{[0, +\infty]} e^{-\lambda_1 I_t - q_1 t} 1_{S_t - I_t \leq x} e^{-\lambda_2 (X_t - I_t) - q_2 (t - s_t)} dt \right)$$

$$= P \left( \int_{[0, +\infty]} e^{-\lambda_2 S_t - q_2 t} 1_{S_t - I_t \leq x} e^{-\lambda_1 (X_t - S_t) - q_1 (t - s_t)} dt \right)$$

This proposition indicates in particular, for $q_1 = q_2 = q$ that the pair of functions $(A_q, \tilde{A}_q)$ determines the resolvent of the trivariate Markov process $(S, X, I)$. We shall see that this result is an immediate consequence of a path decomposition stated in proposition 4.1 and we will prove it later.

The next propositions give expressions of the distribution of some random variables related to fluctuations of the Lévy process in terms of our functions $A, B, C, \tilde{A}, \tilde{B}, \tilde{C}$. They are immediate consequences of the definition of these functions and of the compensation formula stated in proposition 2.2. We leave the proofs to the reader.

**Proposition 3.2.** For all $\Re(\mu_1) \geq 0$, $\Re(\mu_2) \leq 0$, $q_1, q_2 \in [0, +\infty]$, $x \in [0, +\infty]$, one has

$$P \left( e^{-\mu_1 S_{t*} - q_1 g_{t*}} e^{-\mu_2 (X_{t*} - S_{t*}) - q_2 (T_{t*} - g_{t*})} \right) = \frac{1}{\tilde{B}_{q_1}(x, \mu_1)} C_{q_1}(x, \mu_2)$$

$$P \left( e^{-\mu_2 I_{t*} - q_2 g_{t*}} e^{-\mu_1 (X_{t*} - I_{t*}) - q_1 (T_{t*} - g_{t*})} \right) = \frac{1}{\tilde{B}_{q_1}(x, \mu_2)} C_{q_1}(x, \mu_1)$$

Let us denote

$$U^x := \inf \{ t; S_t - I_t \geq x \} \land \inf \{ t; S_t - I_t > x \} \land \inf \{ t; X_t = I_t \}$$

**Proposition 3.3.** For all $\lambda \in C$, $\Re(\mu_1) \geq 0$, $\Re(\mu_2) \leq 0$, $q_1, q_2 \in [0, +\infty]$, $x \in [0, +\infty]$

$$P \left( e^{-\lambda U_x - q_1 g_{U_x}} e^{-\mu_1 (X_{U_x} - I_{U_x}) - q_2 (U_x - g_{U_x})} ; S_{U_x} = X_{U_x} \right) = \tilde{A}_{q_1}(x, \lambda) C_{q_1}(x, \mu_1)$$

$$P \left( e^{-\lambda S_{U_x} - q_1 g_{U_x}} e^{-\mu_2 (X_{U_x} - S_{U_x}) - q_2 (U_x - g_{U_x})} ; I_{U_x} = X_{U_x} \right) = \tilde{A}_{q_1}(x, \lambda) C_{q_1}(x, \mu_2)$$

Let $A_q(x, dy)$ be the measure with Laplace transform $A_q(x, \lambda)$ $(A_q(x, dy) := P(\int_0^{+\infty} 1_{S_t - I_t \leq x} 1_{S_t \in dy} L^*(dt)))$. This measure is absolutely continuous with respect to the measure $G_q(dy) := P(\int_{[0, +\infty]} 1_{S_t \in dy} e^{-q t} L^*(dt))$, with a density $\alpha_q(x, y)$, this entitles us to define

$$A_q(b + y, dy) := \alpha_q(b + y, y) G_q(dy) = P(\int_{[0, +\infty]} 1_{S_t \leq a} 1_{S_t \in dy} e^{-q t} L^*(dt))$$

Similarly, denote : $\tilde{A}_q(a - y, dy)$ the measure :

$$\tilde{A}_q(a - y, dy) := P(\int_{[0, +\infty]} 1_{S_t \leq a} 1_{I_t \in dy} e^{-q t} L^i(dt))$$
For all positive reals $a, b$, denote $T^a_{b} := \inf\{\{t; X_t \notin [-b, a]\}$.

**Proposition 3.4.** For all $\lambda \in \mathbb{C}$, $\mathbb{R}(\mu_1) \geq 0$, $\mathbb{R}(\mu_2) \leq 0$, $q_1, q_2 \in [0, +\infty[$,

\[
P(e^{-\lambda T^a_{b} - q_1 g^T_{b}} e^{-\mu_1 (X_{T^a_{b}} - T^a_{b}) - q_2 (T^a_{b} - g^T_{b})}; X_{T^a_{b}} = S_{T^a_{b}}) = \int_{[-b, 0]} e^{-\lambda y} C_{q_2} (a - y, \mu_1) A_{q_1} (a - y, dy)
\]

\[
P(e^{-\lambda S_{T^a_{b}} - q_1 g^T_{b}} e^{-\mu_2 (X_{T^a_{b}} - S_{T^a_{b}}) - q_2 (S_{T^a_{b}} - g^T_{b})}; X_{T^a_{b}} = I_{T^a_{b}}) = \int_{[0, a]} e^{-\lambda y} C_{q_2} (b + y, \mu_2) A_{q_1} (b + y, dy)
\]

4. Independence of past and post minimum process ”conditionnally on the amplitude” and proof of proposition 3.1

Let us first introduce some more notations : For all $q \in [0, +\infty[$, let $Q_q, Q^+_q$ and $Q^-_q$ be the measures on $(\Omega, F, P)$ defined as follows :

\[
Q_q(dw) := P(\int_0^{+\infty} 1_{(X_s)_{0 \leq s < \epsilon} \in dw} e^{-qt} dt)
\]

\[
Q^+_q(dw) := P(\int_{0, +\infty} 1_{(S_1 - X_{(t-s)})^{-\lambda \geq 0}} dw) e^{-qt} L^s(dt)
\]

\[
Q^-_q(dw) := P(\int_{0, +\infty} 1_{(I_1 - X_{(t-s)})^{-\lambda \geq 0}} dw) e^{-qt} L^s(dt)
\]

All these measures are supported by the set of paths with finite life time. Note that for a positive $q$, $P_q := qQ_q$ is the distribution of the Lévy process $X$ under $P$ killed at an independent exponential time with parameter $q$ and that the measure $Q^-_q$ is finite if $\lim_{t \rightarrow +\infty} X_t = -\infty$ ($P$-a.s.), and $Q^+_q$ is finite if $\lim_{t \rightarrow +\infty} X_t = +\infty$ ($P$-a.s.). The measures $Q^+_q$ and $Q^0_q$ are infinite in the other cases and $Q_0$ is infinite in all cases.

Denote $\zeta$ the life time of the canonical process $X$, and $F, M, m$ the final values of $X, S, I$ :

\[
\zeta := \sup\{t; X_t \in \mathbb{R}\} \quad F := X_{\zeta} \quad M := S_{\zeta} \quad m := I_{\zeta}
\]

Denote $\sigma$ and $\rho$ respectively the last time $X$ takes its maximal value and the first time $X$ takes its minimal value :

\[
\sigma := \sup\{t \leq \zeta; X_t \vee X_{t-} = M\} \quad \rho := \inf\{t \geq 0; X_t \wedge X_{t-} = m\}
\]

We make the convention that $\zeta, \sigma, \rho, F, M, m$ are zero when the path is constantly equal the cemetery point $\delta$.

Note that under $Q_q$, for every $q \in [0, +\infty[$, the time $\sigma$ (respectively $\rho$) is respectively the unique time at which the process $X_t \vee X_{t-}$ (respectively $X_t \wedge X_{t-}$), takes its maximal value, (respectively its minimal value) unless when $P$ is the distribution of a compound Poisson process. Note also that when $q$ is positive, the measures $Q^+_q$ and $Q^0_q$ are respectively the law of the process $(M - X_{(\sigma - s)}; s \geq 0)$ and the law of $(m - X_{(\rho - s)}; s \geq 0)$ under $P_q := qQ_q$. This last property is a standard consequence of the compensation formula of proposition 2.2.

The next identities follow directly from the definitions:

\[
A_q(x, \lambda) = Q^+_q(e^{-\lambda F}; M \leq x) = Q^0_q(e^{-\lambda F - q\zeta}; M \leq x)
\]

\[
H_q(x) = Q^+_q(M \leq x) = Q^0_q(e^{-q\zeta}; M \leq x)
\]

\[
\frac{1}{\psi_q(\lambda)} = Q^+_q(e^{-\lambda F}) = Q^0_q(e^{-\lambda F - q\zeta})
\]
On the other hand, one gets

\[
\tilde{A}_q(x, \lambda) = Q^\perp_q(e^{-\lambda F} \mathbb{1} - m \leq x) = Q^\perp_0(e^{-\lambda F} - \mathbb{1} - m \leq x)
\]

\[
\frac{1}{\psi_q(\lambda)} = Q^\perp_q(e^{-\lambda F}) = Q^\perp_0(e^{-\lambda F} - \mathbb{1})
\]

\[
\tilde{H}_q(x) = Q^\perp_q(-m \leq x) = Q^\perp_0(e^{-\mathbb{1} - \lambda}; -m \leq x)
\]

**Proposition 4.1.** For every \( q \in [0, +\infty[, \ x \in]0, +\infty[ \),

\[
Q_q((m - X_{(\rho - \iota)}) \mathbb{1}; t \geq 0) \in dw_1; M - m \leq x, (X_{t + \rho} - m; t \geq 0) \in dw_2)
\]

\[
= Q_q((M - X_{(\rho - \iota)}) \mathbb{1}; t \geq 0) \in dw_1; M - m \leq x, (X_{t + \rho} - m; t \geq 0) \in dw_2)
\]

\[
= Q_q((m - X_{(\rho - \iota)}) \mathbb{1}; t \geq 0) \in dw_1) P_q((M - X_{(\rho - \iota)}) \mathbb{1}; t \geq 0) \in dw_2)
\]

\[
Q^\perp_q(dw_1) Q^\perp_q(dw_2)
\]

\[
Q^\perp_q(\Omega)
\]

The first equality follows from the well known independence of the past and post minimum processes (see [B] lemma 6 chapter 6 for example), the second from the fact, again well known (see [B] lemma 2 chapter 2), that the process \( (F - X_{(\rho - \iota)}); t \geq 0) \) has the same law as \( X \) under \( P_q \). The last one follows from the compensation formula, as we have already mentioned.

On the other hand, one gets

\[
Q^\perp_q(\Omega) = \frac{1}{\psi_q(0)} \frac{1}{\psi_q(0)} = \frac{1}{q}
\]

The first identity follows from the definitions of \( Q^\perp_q \) and \( Q^\perp_0 \), the second one from the Wiener-Hopf equation of proposition 2.1 for \( iu = 0 \).

Remember that \( P_q = qQ_q \) and simplify the previous identities by \( \frac{1}{q} \) to get

\[
Q_q((m - X_{(\rho - \iota)}) \mathbb{1}; t \geq 0) \in dw_1; (X_{t + \rho} - m; t \geq 0) \in dw_2) = Q^\perp_q(dw_1) Q^\perp_q(dw_2)
\]

Letting \( q \) go to 0, we get the same identity for \( q = 0 \).

The events \( \{M - m \leq x\} \) can be written as the intersection :

\[
\{M - m \leq x\} = \{\min|m - X_{(\rho - \iota)}; t \geq 0| \leq x\} \cap \{\max[X_{t + \rho} - m; t \geq 0] \leq x\}
\]

this yields

\[
Q_q((m - X_{(\rho - \iota)}; t \geq 0) \in dw_1; M - m \leq x; (X_{t + \rho} - m; t \geq 0) \in dw_2)
\]

\[
= Q^\perp_q(dw_1; -m \leq x) Q^\perp_q(dw_2; M \leq x)
\]

Using the identity in law of the process \( (F - X_{(\rho - \iota)}); t \geq 0) \) and \( X \) one gets the other identity of the proposition \( \square \)

In the sequel, unless explicitly mentioned, all the properties hold for every non negative \( q \) and we shall omit to mention it
Proof of proposition 3.1
Denote $X^\uparrow$ and $X^\downarrow$ the processes $(X_{t+\rho} - m; t \geq 0)$ and $(m - X_{(\rho-t)^-}; t \geq 0)$ respectively. One gets
\[
P \left( \int_{0, +\infty} e^{-\lambda_1 t - q_1 g_1 1_{S_t}} dt \right) = Q_0 (e^{-\lambda_1 m - q_1 \rho 1_{M-m} \leq e^{-\lambda_2 (F-m) - q_2 (\rho - \rho)})
\]
\[
= Q_0 (e^{-\lambda_1 F(X^\uparrow) - q_1 \zeta(X^\uparrow) 1_{M(X^\uparrow) \leq x} e^{-\lambda_2 F(X^\uparrow) - q_2 \zeta(X^\uparrow)})
\]
\[
= Q_0 (e^{-\lambda_1 F(X^\uparrow) - q_1 \zeta; -m \leq x} Q_0 (e^{-\lambda_2 F(X^\uparrow) - q_2 \zeta; M \leq x} = \tilde{A}_q(x, \lambda_1) A_{q_2}(x, \lambda_2)
\]
The first, second and fourth equalities follow from the definitions. The third one follows from proposition 4.1. Similarly, the second assertion of proposition 3.1 follows from the second assertion of proposition 4.1. □

5. Proof of property 1) of theorem 2.3
Denote for any $x \in [0, +\infty]$, 
\[
Q_q^{1x} := Q_q^1(dw - m \leq x) \quad Q_q^{1x} := Q_q^1(dw | M < x)
\]
Note that when $Q_q^1(m = 0) > 0$, the mesure $Q_q^1(dw - m = 0)$ is the Dirac mass on the path constantly equal to $\delta$. We make the convention that $Q_q^{10}$ and $Q_q^{10}$ are this Dirac mass.

Lemma 5.1.
\[
Q_q^1((X_{t+\rho} - M; t \geq 0) \in dw | M - X_{(\rho-t)^-}; t \geq 0) = Q_q^{1M}(dw)
\]
\[
Q_q^1((X_{t+\rho} - m; t \geq 0) \in dw | M - X_{(\rho-t)^-}; t \geq 0) = Q_q^{1m}(dw)
\]

Proof Let us discuss few facts about the event $\{\rho \leq \sigma\}$. First of all, one has
\[
\{\rho \leq \sigma\} = \{I_\sigma \leq \inf(X_s; s \geq \sigma)\}
\]
and
\[
\{\rho \leq \sigma\} = \{S_\rho^\sigma \leq \sup(X_s; s \geq \sigma) \vee X_{\rho^-}\}
\]
If $] - \infty, 0[$ is regular then $Q_q$-a.s. the canonical process $X$ has no negative jump at the times $\rho$ and $\sigma$. Consequently, $I_\sigma = I_{\sigma^-}$ and $\sup(X_s; s \geq \rho) \vee X_{\rho^-} = \sup(X_s; s \geq \rho)$ and one gets:
\[
\{\rho \leq \sigma\} = \{I_\rho \leq \inf(X_s; s \geq \sigma)\} = \{S_\rho \leq \sup(X_s; s \geq \rho)\}
\]
If $] - \infty, 0]$ is irregular then, $Q_q$-a.s., either $\rho$ is zero either it is a time when $X$ has a negative jump and $\sigma$ is a time when $X$ has a negative jump. On the other hand, when $\rho = \sigma$, we have $I_{\sigma^-} > \inf(X_s; s \geq \sigma)$ and $S_\rho \sup(X_s; s \geq \rho)$ Thus
\[
\{\rho < \sigma\} = \{I_\rho \leq \inf(X_s; s \geq \sigma)\} = \{S_\rho \leq \sup(X_s; s \geq \rho)\}
\]
For the needs of next proof, let us denote $A$ the event
\[
A := \{I_\rho \leq \inf(X_s; s \geq \sigma)\} = \{S_\rho \leq \sup(X_s; s \geq \rho)\}
\]
By the preceding discussion, the event $A$ is either $\{\rho \leq \sigma\}$ or $\{\rho < \sigma\}$.
Let $X^\downarrow$ and $X^\uparrow$ denote respectively the processes $(X_{t+\rho} - M; t \geq 0)$ and $(X_{\rho-t} - m; t \geq 0)$, we add $\downarrow$ or $\uparrow$ to the corresponding objects.
One has
\[ A = \{ I_{\sigma^-} - M \leq m(X^\uparrow) \} = \{ S_{\rho}^- - m \leq M(X^\uparrow) \} \]

Denote \( \mathcal{F}_\sigma \) the \( \sigma \)-field generated by the pre-maximum process \( M - X_{(\sigma^-)'} \), \( t \geq 0 \). Notice that the trace on the event \( A \) of the \( \sigma \)-field \( \mathcal{F}_\sigma \) contains the random variable \( M - m1_A \) and consequently, this trace is \( \sigma \)-finite for \( Q_q \). The independence under \( Q_q \) of the process \( X^\uparrow \) and the \( \sigma \)-field \( \mathcal{F}_\sigma \) given in proposition 4.1 gives us:

\[ Q_q(X^\uparrow \in dw | \mathcal{F}_\sigma \cap \sigma(A)) = Q_q(X^\uparrow \in dw | \sigma(A)) = Q_q^{\{I_{\sigma^-} - M \leq m(X^\uparrow)\}}(dw) \quad \text{on} \quad A \]

Moreover, the trace on \( A \) of the \( \sigma \)-field \( \mathcal{F}_\sigma \) contains the \( \sigma \)-field \( \mathcal{F}_\sigma \cap \sigma(S_{\rho}^- - m) \), and

\[ (X^\uparrow)^\downarrow = X^\downarrow \quad \text{and} \quad M - I_{\sigma^-} = M(X^\uparrow) \quad \text{on} \quad A \]

The previous identity gives then the following:

\[ Q_q((X^\uparrow)^\downarrow \in dw | \mathcal{F}_\sigma \cap \sigma(S_{\rho}^- - m)) = Q_q^{\{M(X^\uparrow)\}}(dw) \quad \text{on} \quad A = \{ M(X^\uparrow) \geq S_{\rho}^- - m \} \]

Using the independence of the process \( X^\uparrow \) and the random variable \( S_{\rho}^- - m \) given by proposition 4.1, one then deduces easily the first identity of the lemma.

One gets the second identity similarly with using \( cA \) instead of \( A \).

\[ \square \]

**Proof of property 1) of theorem 2.3** For all \( x \in ]0, +\infty[ \) and \( \lambda \in \mathbb{C} \), one has

\[ A_q(x, \lambda) = Q_q(M < x, e^{-\lambda F}) = H_q(0) + \int_{]0,x]} Q_q(e^{-\lambda F} | M = y) H_q(dy) \]

\[ = H_q(0) + \int_{]0,x]} e^{-\lambda y} Q_q(e^{-\lambda F} - m = y) H_q(dy) \]

\[ = H_q(0) + \int_{]0,x]} e^{-\lambda y} Q_q(e^{-\lambda F} - M + t \geq 0 | M = y) H_q(dy) \]

Using Lemma 5.1, this last quantity is equal to

\[ = H_q(0) + \int_{]0,x]} e^{-\lambda y} Q_q(e^{-\lambda F} - m \leq y) H_q(dy) = H_q(0) + \int_{]0,x]} e^{-\lambda y} Q_q(e^{-\lambda F} - m \leq y) \frac{H_q(dy)}{H_q(y)} \]

\[ = H_q(0) + \int_{]0,x]} e^{-\lambda y} A_q(y, \lambda) \frac{H_q(dy)}{H_q(y)} \]

We get the identity

\[ A_q(x, \lambda) = \hat{H}_q(0) + \int_{]0,x]} e^{-\lambda y} A_q(y, \lambda) \frac{H_q(dy)}{H_q(y)} \]

in a similar way.

\[ \square \]

6. **Proof of properties 2 and 3 of theorem 2.3**

6.1. **Two Markov chains.** We now define the successive minima and maxima. We let first

\[ M_1 := M \quad \text{and} \quad T_1 := \sigma \]

\[ m_2 := \inf\{ X_t; t \geq T_1 \} \quad \text{and} \quad T_2 := \inf\{ t \geq T_1; X_t \wedge X_{t-} = m_2 \} \]

then we define inductively

\[ M_{2n+1} := \sup\{ X_t; t \geq T_{2n} \} \quad \text{and} \quad T_{2n+1} := \sup\{ t \geq T_{2n}; X_t \vee X_{t-} = M_{2n+1} \} \]

\[ m_{2n+2} := \inf\{ X_t; t \geq T_{2n+1} \} \quad \text{and} \quad T_{2n+2} := \inf\{ t \geq T_{2n+1}; X_t \wedge X_{t-} = m_{2n+2} \} \]

and

\[ Z_{2n+1} := M_{2n+1} - m_{2n} \quad Z_{2n+2} := m_{2n+2} - M_{2n+1} \]
If $T_n = \zeta$ put $Z_{n+1} := 0$ and $T_{n+1} := \zeta$. Notice that $F = \sum_1^{+\infty} Z_n$.

Below is picture of our sequence.

Similarly, let

- $m_1 := m$
- $S_1 := \rho$
- $M_2 := \sup\{X_t; t \geq T_1\}$
- $S_2 := \sup\{t \geq T_1; X_t \vee X_{t^+} = I_2\}$
- $m_{2n+1} := \inf\{X_t; t \geq S_{2n}\}$
- $S_{2n+1} := \inf\{t \geq S_{2n}; X_t \wedge X_{t^+} = m_{2n+1}\}$
- $M_{2n+2} := \sup\{X_t; t \geq S_{2n+1}\}$
- $S_{2n+2} := \sup\{t \geq S_{2n+1}; X_t \vee X_{t^+} = M_{2n+2}\}$

and

$$Y_{2n+1} := m_{2n+1} - M_{2n} \quad Y_{2n+2} := M_{2n+2} - m_{2n+1}$$

If $S_n = \zeta$ put $Y_{n+1} := 0$ and $S_{n+1} := \zeta$. Notice that $F = \sum_1^{+\infty} Y_n$.

**Lemma 6.1.** Under $Q_q$ the sequence $(Z_1, Z_2, \ldots)$ is a Markov chain with transition kernel given by

For $x \neq 0$

$$P_q(x, dy) = \frac{H_q(dy)}{H_q((-x)^-)} 1_{y \in [0,-x]} + \frac{\bar{H}_q(-dy)}{H_q(x)} 1_{y \in [-x,0]}$$

$P_q(0, \{0\}) = 1$
and with initial law:

\[ \mathbf{Q}^{\uparrow}_q(Z_1 \in dy) = \mathcal{H}_q(dy) \]

Under \( \mathbf{Q}^{\uparrow}_q \) the sequence \((Y_1, Y_2, \ldots)\) is a Markov chain with the same transition kernel and with initial law

\[ \mathbf{Q}^{\uparrow}_q(Y_1 \in dy) = \mathcal{H}_q(dy) \]

**Proof** One has:

\[ \mathbf{Q}^{\uparrow}_q(Z_1 \in dy) = \mathcal{H}_q(dy) \]

\[ \mathbf{Q}^{\uparrow}_q(Z_2 \in dy|Z_1 = x) = \mathbf{Q}^{\uparrow}_q(\min[X_{n+1} - M; t \geq 0] \in dy|M = x) \]

\[ = \mathbf{Q}^{\uparrow}_q(m \in dy - m \leq x) = \frac{\mathcal{H}_q(dy)}{\mathcal{H}_q(x)} 1_{y \in [-x, 0]} \]

The third identity comes from Lemma 5.1, the others follow from definitions of the random variables \(Z_1\) and \(Z_2\).

The same arguments yields:

\[ \mathbf{Q}^{\uparrow}_q(Y_2 \in dy|Y_1 = x) = \frac{\mathcal{H}_q(dy)}{\mathcal{H}_q(x)} 1_{y \in [0, x]} \]

A straightforward induction gives us the rest of the lemma. \(\square\)

Let

\[
U_q(dy) := \sum_{1}^{+\infty} \mathbf{Q}^{\uparrow}_q(Z_n \in dy) \quad V_q(dy) := \sum_{1}^{+\infty} \mathbf{Q}^{\uparrow}_q(Y_n \in dy)
\]

\[ \tau_x := \sup\{n; Z_n \notin [-x, x]\} \quad (\tau_x := 0 \text{ if } Z_1 < x) \]

\[ \nu_x := \sup\{n; Y_n \notin [-x, x]\} \quad (\nu_x := 0 \text{ if } Y_1 \geq -x) \]

**Lemma 6.2.** Under \( \mathbf{Q}^{\uparrow}_q(dw; \tau_x > 0) \) for every \( x \in [0, +\infty] \), the sequence \((Z_{\tau_x - n})_{0 \leq n < \tau_x}\) is a sub-Markov chain with values in \( \mathbb{R}^* \) and with initial law:

\[
\mathbf{Q}^{\uparrow}_q(Z_{\tau_x} \in dy; \tau_x > 0) = \left[ 1_{y \in [-\infty, -x]} \frac{\mathcal{H}_q(x^-)}{\mathcal{H}_q(-y^+)} + 1_{y \in [x, +\infty]} \frac{\mathcal{H}_q(x^-)}{\mathcal{H}_q(y^+)} \right] V_q(dy)
\]

Its transition kernel \( R_q \) does not depend on \( x \) and satisfies the equation:

\[
U_q(dz)P_q(z, dy) = R_q(y, dz)U_q(dy) \quad \text{on } \mathbb{R}^* \times \mathbb{R}^*
\]

Under \( \mathbf{Q}^{\uparrow}_q(dw; \nu_x > 0) \), the sequence \((Y_{\nu_x - n})_{0 \leq n < \nu_x}\) is a sub-Markov chain with values in \( \mathbb{R}^* \) and with initial law:

\[
\mathbf{Q}^{\uparrow}_q(Y_{\nu_x} \in dy; \nu_x > 0) = \left[ 1_{y \in [-\infty, -x]} \frac{\mathcal{H}_q(x^-)}{\mathcal{H}_q(-y^-)} + 1_{y \in [x, +\infty]} \frac{\mathcal{H}_q(x^-)}{\mathcal{H}_q(y^-)} \right] V_q(dy)
\]

Its transition kernel \( S_q \) does not depend on \( x \) and satisfies the equation:

\[
V_q(dz)P_q(z, dy) = S_q(y, dz)V_q(dy) \quad \text{on } \mathbb{R}^* \times \mathbb{R}^*
\]

**Proof** It is a standard fact from the theory of time reversal of Markov chains (or processes) that the time \( \tau_x \) is a so called ”return time” of the Markov chain \((Z_n)\) and consequently that \((Z_{\tau_x - n})_{0 \leq n < \tau_x}\) is a sub-Markov chain with a transition kernel related to the one of \((Z_n)\) by the so called ”duality identity”

\[
U_q(dz)P_q(z, dy) = R_q(y, dz)U_q(dy) \quad \text{on } \mathbb{R}^* \times \mathbb{R}^*
\]

In particular, this transition kernel does not depend on the particular return time \( \tau_x \). Since we lack an adequate reference we check this property in our particular case.
For every $y \in \mathbb{R}^*$, let $z \mapsto p_q(y, z)$ be a density of the measure $P_q(y, dz)$ relatively to $U_q(dz)$, one has for every integer $m$ (measures involved here are on $\mathbb{R}^*$)

$$Q^q_0(Z_{\tau_x-m} \in dy_m, Z_{\tau_x-(m-1)} \in dy_{m-1}, \ldots, Z_{\tau_x} \in dy_0)$$

$$= \sum_{n=m+1}^{+\infty} Q^q_0(Z_{n-m} \in dy_m, Z_{n-(m-1)} \in dy_{m-1}, Z_{n-(m-2)} \in dy_{m-2}, \ldots, Z_{n} \in dy_0, \tau_x = n)$$

$$= \sum_{n=m+1}^{+\infty} Q^q_0(Z_{n-m} \in dy_m, Z_{n-(m-1)} \in dy_{m-1}, Z_{n-(m-2)} \in dy_{m-2}, \ldots, Z_{n} \in dy_0, Z_{n+1} \in [-x, x])1_{y_0 \notin [-x, x]}$$

$$= \sum_{n=m+1}^{+\infty} Q^q_0(Z_{n-m} \in dy_m)p_q(y_m, y_{m-1})U_q(dy_{m-1})P_q(y_{m-1}, dy_{m-2}) \ldots P_q(y_1, dy_0)P_q(y_0, [-x, x])1_{y_0 \notin [-x, x]}$$

Thus the sequence $\{Q^q_n\}_{n \geq m}$ comes from the value of $\tau_x$ and from the fact that the sequence $|Z_n|$ is nonincreasing.

One can deduce from the last equalities that:

$$Q^q_0(Z_{\tau_x-m} \in dy_m|Z_{\tau_x-(m-1)}, \ldots, Z_{\tau_x}) = U_q(dy_m)p_q(y_m, Z_{\tau_x-(m-1)})$$

Thus the sequence $(Z_{\tau_x-n})_{0 \leq n < \tau_x}$ is a sub-Markov chain and its transition kernel is $R_q(z, dy) = p_q(y, z)U_q(dy)$. Multiplying the last identity by the measure $U_q(dz)$, one recognizes the duality equation.

Let us compute the law of $(Z_{\tau_x-n})_{0 \leq n < \tau_x}$. Put $m = 0$ in the preceding equalities and get

$$Q^q_0(Z_{\tau_x} \in dy_0) = 1_{y_0 \notin [-x, x]}\sum_{n=1}^{+\infty} Q^q_0(Z_n \in dy_0; Z_{n+1} \in [-x, x])$$

$$= 1_{y_0 \notin [-x, x]}\sum_{n=1}^{+\infty} Q^q_0(Z_n \in dy_0)P_q(y_0, [-x, x])$$

$$= \left(1_{y_0 \in [x, +\infty]}H_q(x)H_q(y_0) + 1_{y_0 \in [-\infty, -x]}H_q(x^{-})H_q((-y_0)^{-})\right)U_q(dy_0)$$

The last identity comes from the value of $P_q$ given by lemma 6.1. One gets the results for the sequence $(Y_{\nu_x-n})_{0 \leq n < \nu_x}$ similary. \(\square\)

Let

$$c_q(x, \lambda) := \frac{1}{H_q(x)}Q^q_0(e^{-\lambda \sum_{n=1}^{+\infty} Z_n}; \tau_x \text{ is odd}) = \frac{1}{H_q(x)}Q^q_0(e^{-\lambda M_x}; \tau_x \text{ is odd})$$

$$d_q(x, \lambda) := \frac{1}{H_q(x^{-})}Q^q_0(e^{-\lambda \sum_{n=1}^{+\infty} Z_n}; \tau_x \text{ is even}) = \frac{1}{H_q(x^{-})}Q^q_0(e^{-\lambda M_x}; \tau_x \text{ is even})$$

$$e_q(x, \lambda) := \frac{1}{H_q(x)}Q^q_0(e^{-\lambda \sum_{n=1}^{+\infty} Y_n}; \nu_x \text{ is odd}) = \frac{1}{H_q(x)}Q^q_0(e^{-\lambda M_x}; \nu_x \text{ is odd})$$

$$d_q(x, \lambda) := \frac{1}{H_q(x)}Q^q_0(e^{-\lambda \sum_{n=1}^{+\infty} Y_n}; \nu_x \text{ is even}) = \frac{1}{H_q(x)}Q^q_0(e^{-\lambda M_x}; \nu_x \text{ is even})$$
LEMMA 6.3. For \( \Re(\lambda) > 0 \) (and \( \Re(\lambda) = 0 \) if \( q > 0 \) or \( \lim X_t = -\infty \) \( \mathbb{P} \)-a.s.), one has
\[
|c_q(x, \lambda)| < +\infty \quad |b_q(x, \lambda)| < +\infty
\]
and
\[
A_q(x^-, \lambda)b_q(x, \lambda) + \tilde{A}_q(x, \lambda)c_q(x, \lambda) = \frac{1}{\psi_q(\lambda)}
\]
For \( \Re(\lambda) < 0 \) (and for \( \Re(\lambda) = 0 \) if \( q > 0 \) or \( \lim X_t = +\infty \) \( \mathbb{P} \)-a.s.), one has
\[
|\tilde{c}_q(x, \lambda)| < +\infty \quad |\tilde{b}_q(x, \lambda)| < +\infty
\]
and
\[
A_q(x^-, \lambda)\tilde{b}_q(x, \lambda) + \tilde{A}_q(x, \lambda)\tilde{c}_q(x, \lambda) = \frac{1}{\psi_q(\lambda)}
\]

**Proof** Denote \( S_x := \inf\{n, Z_n \in [-x,x]\} \), clearly \( S_x = \tau_x + 1 \). One has
\[
Q^\tau_q(\tau_x = 0, Z_{S_x} \in dy) + \sum_{n=1}^{\infty} Q^\tau_q(Z_1 \in dx_1 \ldots Z_n \in dx_n \tau_x = n, Z_{S_x} \in dy)
\]
\[
= Q^\tau_q(\tau_x = 0, Z_{S_x} \in dy) + \sum_{n=1}^{\infty} 1_{x_n \notin [-x,x]}1_{y \in [-x,x]} Q^\tau_q(Z_1 \in dx_1 \ldots Z_n \in dx_n, Z_{n+1} \in dy)
\]
\[
= H_q(dy) 1_{y \in [0,x]} 1 + \sum_{n \text{ is even}, n \geq 2} 1_{x_n \in [-x,x]} Q^\tau_q(Z_1 \in dx_1 \ldots Z_n \in dx_n) \frac{1}{H_q((x^-))}
\]
\[
+ H_q(-dy) 1_{y \in [-x,0]} \sum_{n \text{ is odd}} 1_{x_n \in [x,x]} Q^\tau_q(Z_1 \in dx_1 \ldots Z_n \in dx_n) \frac{1}{H_q(x^-)}
\]
Again, the first identity is a consequence of the fact that the sequence \( |Z_n| \) is nonincreasing and the second from the value of the transition kernel of the Markov chain given in lemma 6.1.

Now, let us check that the sigma field \( \mathcal{G}_{\tau_x} := \sigma(Z_n 1_{\tau_x \neq 0} + 1_{\tau_x = 0}) \) is \( \sigma \)-finite. Indeed the variable \( Z_1 1_{\tau_x \neq 0} + 1_{\tau_x = 0} \) is positive and is \( \mathcal{G}_{\tau_x} \)-measurable. The variable \( Z_1 1_{\tau_x \neq 0} \) has law \( H_q(dy) 1_{y \in [x,\infty]} \) which is sigma finite and the event \( \{\tau_x = 0\} \) has measure \( H_q(x^-) \). So one can deduce the conditional law of the variable \( Z_{\tau_x} \) on \( \mathcal{G}_{\tau_x} \) from the preceding identities
\[
Q^\tau_q(Z_{S_x} \in dy | \mathcal{G}_{\tau_x}) = \begin{cases} 1_{\tau_x \text{ is odd}} \frac{H_q(-dy)}{H_q(x^-)} 1_{y \in [-x,0]} + 1_{\tau_x \text{ is even}} \frac{H_q(dy)}{H_q(x^-)} 1_{y \in [0,x]} \end{cases}
\]
On the other hand, one has
\[
A_q(x^-, \lambda) = Q^\tau_q(e^{-\lambda E}; M < x) = \int_{[0,x]} Q^\tau_q(e^{-\lambda \sum_{i=1}^{\infty} Z_n}; Z_1 < x) \int_{[0,\infty]} Q^\tau_q(e^{-\lambda \sum_{i=1}^{\infty} Z_n} | Z_1 = y) H_q(dy)
\]
and similarly,
\[
\tilde{A}_q(x, \lambda) = \int_{[0,x]} Q^\tau_q(e^{-\lambda \sum_{i=1}^{\infty} Y_n} | Y_1 = y) \tilde{H}_q(dy)
\]
Since \( (Z_n) \) is a Markov chain under \( Q^\tau_q \) with same transition kernel as the chain \( (Y_n) \) under \( Q^\tau_q \), and since \( S_x \) is a stopping time of the chain \( (Z_n) \) and since the sigma-field \( \sigma(Z_n 1_{S_x} \leq S_x) \) contains \( \mathcal{G}_{\tau_x} \), one can deduce from previous identities that :
\[
Q^\tau_q(e^{-\lambda \sum_{i=1}^{\infty} Z_n} | \mathcal{G}_{\tau_x}) = \begin{cases} 1_{\tau_x \text{ is odd}} \frac{\tilde{A}_q(x, \lambda)}{H_q(x^-)} + 1_{\tau_x \text{ is even}} \frac{A_q(x^-, \lambda)}{H_q(x^-)} \end{cases}
\]
Finally, one gets
\[
\frac{1}{\psi_q(\lambda)} = Q_q^x(e^{-\lambda F}) = Q_q^x(e^{-\lambda \sum_{i=1}^{+\infty} Z_n}) = Q_q^x(e^{-\lambda \sum_{i=1}^{+\infty} Z_n} e^{-\lambda \sum_{i=1}^{+\infty} Z_n})
\]
\[
= Q_q^x(e^{-\lambda \sum_{i=1}^{+\infty} Z_n} 1_{\tau_x} \text{ is even}) \frac{A_q(x, \lambda)}{H_q(x)} + Q_q^x(e^{-\lambda \sum_{i=1}^{+\infty} Z_n} 1_{\tau_x} \text{ is odd}) \frac{\tilde{A}_q(x, \lambda)}{H_q(x)}
\]

The single case for which one has to check that \(c_q(x, \lambda)\) and \(d_q(x, \lambda)\) are actually finite is the case \(q = 0\) and \(\Re(\lambda) > 0\). It is enough to check the property for positive real \(\lambda\). In this case the real numbers \(A_q(x, \lambda), \tilde{A}_q(x, \lambda)\) are positive and \(\frac{1}{\psi_q(\lambda)}\) is finite. Thus the previous identity allows us to conclude that the (positive but possibly infinite a priori) \(c_q(x, \lambda)\) and \(d_q(x, \lambda)\) are actually finite.

One gets the second part of the lemma similarly.

**Lemma 6.4.** For \(\Re(\lambda) > 0\) (and for \(\Re(\lambda) = 0\) if \(q > 0\) or if \(\lim X_t = -\infty\) P.-a.s.), one has

\[
c_q(x, \lambda) = \int_{[x, +\infty[} e^{-\lambda y} b_q(y, \lambda) \frac{H_q(dy)}{H_q(y)}
\]

\[
b_q(x, \lambda) = 1 + \int_{[x, +\infty[} e^{\lambda y} c_q(y, \lambda) \frac{\tilde{H}_q(dy)}{\tilde{H}_q(y^-)}
\]

For \(\Re(\lambda) < 0\) (and for \(\Re(\lambda) = 0\) if \(q > 0\) or if \(\lim X_t = +\infty\) P.-a.s.), one has

\[
\tilde{c}_q(x, \lambda) = \int_{[x, +\infty[} e^{-\lambda y} \tilde{b}_q(y, \lambda) \frac{H_q(dy)}{H_q(y^-)}
\]

\[
\tilde{b}_q(x, \lambda) = 1 + \int_{[x, +\infty[} e^{-\lambda y} \tilde{c}_q(y, \lambda) \frac{\tilde{H}_q(dy)}{\tilde{H}_q(y)}
\]

**Proof** Extend the transition kernel of \((Z_n), R_q\), by setting \(R_q(y, \{-\infty\}) := 1 - R_q(y, R^*)\) for \(y \neq 0\). One gets from the duality identity \(1_{y \not\in \mathbb{Z}} U_q(dy) P_{x}(z, dy) = 1_{y \not\in \mathbb{Z}} U_q(dy) R_q(y, dz)\)

\[
1_{y \in \mathbb{R}^*} U_q(y, R^*) U_q(dy) = 1_{y \in \mathbb{R}^*} \int_{\mathbb{R}^*} U_q(dz) P_{x}(z, dy) = 1_{y \in \mathbb{R}^*} Q_q^x \left( \sum_{n \in \mathbb{N}} 1_{Z_n \in dy} \right) = 1_{y \in \mathbb{R}^*} (U_q(dy) - H_q(dy))
\]

and deduce:

\[
R_q(y, \{-\infty\}) U_q(dy) = H_q(dy) \quad \text{on} \quad \mathbb{R}^*
\]

For \(y \in \mathbb{R}^*\), denote \(p^y\) the distribution of the Markov chain starting from \(y\) and with transition kernel \(R_q, (U_n)\) the canonical process of the space \((R^* \cup \{-\infty\})^\mathbb{N}\) where \(-\infty\) is the cemetery point and \(\xi\) the life time of the process \((U_n)\). One can deduce from lemma 6.2, the identities:

\[
c_q(x, \lambda) = \frac{1}{H_q(x)} Q_q^x \left( e^{-\lambda \sum_{n=1}^{+\infty} Z_n}; \tau_x \text{ is odd} \right) = \int_{[x, +\infty[} p^y \left( e^{-\lambda \sum_{n=1}^{+\infty} U_n} U_q(dy) \right) H_q(y)
\]

and

\[
b_q(x, \lambda) = \frac{1}{H_q(x)} Q_q^x \left( e^{-\lambda \sum_{n=1}^{+\infty} Z_n}; \tau_x \text{ is even} \right)
\]

\[
= \frac{1}{H_q(x^-)} Q_q^x (\tau_x = 0) + \int_{[-\infty, -x]} p^y \left( e^{-\lambda \sum_{n=1}^{+\infty} U_n}, U_q(dy) \right) \frac{H_q(x^-)}{H_q((-y)^-)}
\]

\[
= 1 + \int_{[-\infty, -x]} p^y \left( e^{-\lambda \sum_{n=1}^{+\infty} U_n}, U_q(dy) \right) \frac{H_q((-y)^-)}{H_q((-y)^-)}
\]
In the other hand, one gets (measures involved here are on $\mathbb{R}^*$)

\[
p^y(e^{-\lambda \sum_{n=1}^{q} U_n})U_q(dy) = e^{-\lambda y} \left[ \int_{-\infty, +\infty \setminus \{0\}} p^x(e^{-\lambda \sum_{n=1}^{q} U_n})R_q(y, dz)U_q(dy) \right]
\]

\[
= e^{-\lambda y}R_q(y, \{ -\infty \})U_q(dy) + \int_{-\infty, +\infty \setminus \{0\}} p^x(e^{-\lambda \sum_{n=1}^{q} U_n})U_q(dz)P_q(z, dy)
\]

\[
= e^{-\lambda y}[1 + \int_{-\infty, -y[} p^x(e^{-\lambda \sum_{n=1}^{q} U_n}) \frac{U_q(dz)}{H_q(( -z ) - )}]H_q(dy) = e^{-\lambda y}b_q(y, \lambda)H_q(dy)
\]

The first identity comes from the Markov property of $(U_n)$ under $p^x$ with transition kernel $R_q$, the second one from the duality identity $R_q(y, dz)U_q(dy) = U_q(dz)P_q(z, dy)$, the third one from the value of $P_q(z, dy)$ given in lemma 6.1 and the identity $R_q(y, \{ -\infty \})U_q(dy) = H_q(dy)$, and the fourth one from the expression of $b_q(y, \lambda)$ given below. Replacing the preceding identity into the expression of $c_q(x, \lambda)$ given previously, one gets

\[
c_q(x, \lambda) = \int_{[x, +\infty[} e^{-\lambda y}b_q(y, \lambda)\frac{H_q(dy)}{H_q(y)}
\]

One proves the other assertions of the lemma in a similar way.

**Lemma 6.5.** For all $x \in [0, +\infty[, \mathbb{R}(\lambda) > 0$ (and $\mathbb{R}(\lambda) = 0$ if $q > 0$ or $\lim_{t \to +\infty} X_t = -\infty$, $\mathbb{P}$-ps) one has

\[
\psi_q(\lambda) c_q(x, \lambda) = C_q(x^-), \lambda \quad \text{and} \quad \psi_q(\lambda) b_q(x, \lambda) = B_q(x, \lambda)
\]

For all $x \in [0, +\infty[, \mathbb{R}(\lambda) < 0$ (and $\mathbb{R}(\lambda) = 0$ if $q > 0$ or $\lim_{t \to +\infty} X_t = +\infty$, $\mathbb{P}$-ps), one has

\[
\tilde{\psi}_q(\lambda) c_q(x, \lambda) = \tilde{C}_q(x, \lambda) \quad \text{and} \quad \tilde{\psi}_q(\lambda) b_q(x, \lambda) = \tilde{B}_q(x^-), \lambda
\]

**Proof** Remember that the time $U_x$ is defined as

\[
U^x = \inf \{ t; S_t - I_t \geq x \text{ and } S_t \neq X_t \} \land \inf \{ t; S_t - I_t > x \text{ and } X_t = I_t \}
\]

Let us denote $B^x$ the event $B^x = \{ X_{U^x} = S_{U^x}; U < +\infty \}$ and denote $\hat{X}$ the process $\hat{X}(w) = F - X_t(w)$. Add a $\circ$ for the corresponding objects. Denote $X^\uparrow$ and $X^\downarrow$ the processes $(X_t + m; t \geq 0)$ and $(X_t - m; t \geq 0)$ respectively.

A quick look at the picture will convince the reader that the event \( \{ \tau_\circ(X^\uparrow) \text{ is odd} \} \) is equal to the event $\hat{B}^x$ and on this event, one has : $\sum_{n=1}^{\tau_\circ(X^\uparrow)} Z_n(X^\uparrow) = \hat{M} - \hat{I}_{U^x}$. Take any event $A$ such that $0 < Q_q^\circ(A) < +\infty$ and get

\[
Q_q^\circ(A)\hat{H}_q(x)c_q(x, \lambda) = Q_q^\circ(A)Q_q^\circ(e^{-\lambda \sum_{n=1}^{\tau_\circ(X^\uparrow)} Z_n; \tau_\circ \text{is odd}})
\]

\[
= Q_q(A(m - X(^\circ - t)\circ; t \geq 0); e^{-\lambda \sum_{n=1}^{\tau_\circ(X^\uparrow)} Z_n; \tau_\circ(X^\uparrow) \text{ is odd}})
\]

\[
= Q_q(A(\hat{X}^\circ), e^{-\lambda (M - I_{U^x})}; \hat{B}_{x^\circ} = Q_q(e^{-\lambda(M - I_{U^x})}; B_{x^\circ}, A(X^\uparrow))
\]

The first identity follows from the definition of $c_q$, the second one from proposition 4.1, the third one from what we have just said, and the fourth one from the identity in law of $\hat{X}$ and $X$ under $Q_q$.

On the other hand, $U_x$ is a stopping time and it is smaller than $\sigma$ on the event $B^x$, therefore we get (denote $\theta_{U^x}$ the path $(X_{t+U^x} - X_{U^x}; t \geq 0)$)

\[
M - I_{U^x} = (X_{U^x} - I_{U^x}) + M \circ \theta_{U^x} \quad \text{and} \quad A(X^\uparrow) = A(X^\uparrow \circ \theta_{U^x}) \text{ on } B^x
\]

We deduce then the following identities from the properties of independent increments at the stopping time $U_x$ :

\[
Q_q(e^{-\lambda (M - I_{U^x})}; B^x; A(X^\uparrow)) = Q_q(e^{-\lambda(X_{U^x} - I_{U^x})}; B^x; e^{-\lambda M \circ \theta_{U^x}}; A(X^\uparrow \circ \theta_{U^x}^\circ))
\]
In order to prove the second identity of the lemma, let us remind that

\( Q \) 

First identity of proposition 3.3 for the identity already obtained

\( C \) 

When comparing with the identity obtained at lemma 6.3:

\( P_q(e^{-\lambda (X_{U^x} - I_{U^x}); B^x}) = \bar{H}_q(x)Q_q(x^-, \lambda) \)

and proposition 4.1 gives us:

\[
\begin{align*}
Q_q(e^{-\lambda M; A(X^+)}) &= Q_q^1(e^{-\lambda x})Q_q^2(A) = \frac{1}{\psi_q(\lambda)}Q_q^2(A)
\end{align*}
\]

Putting together the previous identities and simplifying by \( Q_q^2(A)\bar{H}_q(x) \) on gets the following

\( C_q(x, \lambda) = \psi_q(\lambda)c_q(x, \lambda) \)

In order to prove the second identity of the lemma, let us remind that

\[
T^x_z = \inf\{t; X_t - S_t < -x\} \quad \frac{1}{B_q(x, \lambda)} = P_q(\int_{[0,T^x_z]} e^{-\lambda s_1 L^s(dt)}) \quad A_q(x^-, \lambda) = P_q(\int_{[0,U^x]} e^{-\lambda s_1 L^s(dt)})
\]

Notice that if \( U^x < +\infty \) and \( I_{U^x} = X_{U^x} \) then \( T^x_z = U^x \) and if \( U^x = +\infty \) then \( T^x_z = +\infty \). In other cases, one has \( X_{U^x} = S_{U^x} \) and (still denoting \( \theta^0_{U^x} \) the path \( (X_{s+U^x} - X_s; s \geq 0) \)):

\[
T^x_z = U^x + T^x_z \circ \theta^0_{U^x} \quad L^s(w, dt + U^x)1_{t>0} = L^s(\theta^0_{U^x}(w), dt)
\]

\( S_{U^x+t} = X_{U^x} + S_t \circ \theta^0_{U^x} \quad \text{for all } t \in [0, +\infty[ \)

One gets the identity of random variables:

\[
\int_{[0,T^x_z]} e^{-\lambda s_1 L^s(dt)} = \int_{[0,U^x]} e^{-\lambda s_1 L^s(dt)} + 1\{X_{U^x} = S_{U^x}; U^x < +\infty\} e^{-\lambda X_{U^x}} \int_{[0,T^x_z]} e^{-\lambda s_1 L^s(dt)} \circ \theta^0_{U^x}
\]

Take the expectation with respect to \( P_q \) and use the property of independence of increments at time \( U^x \) to get

\[
\frac{1}{B_q(x, \lambda)} = A_q(x^-, \lambda) + P_q(e^{-\lambda X_{U^x}}; X_{U^x} = S_{U^x}; U^x < +\infty)\frac{1}{B_q(x, \lambda)}
\]

In the other hand, use first identity of proposition 3.3 for \( \mu_1 = \lambda \) and \( q_1 = q_2 = q \) and get

\[
P_q(e^{-\lambda X_{U^x}}; X_{U^x} = S_{U^x}; U^x < +\infty) = \bar{A}_q(x, \lambda)C_q(x^-, \lambda)
\]

One deduces

\[
1 = B_q(x, \lambda)A_q(x^-, \lambda) + \bar{A}_q(x, \lambda)C_q(x^-, \lambda)
\]

When comparing with the identity obtained at lemma 6.3: \( A_q(x^-, \lambda)b_q(x, \lambda) + \bar{A}_q(x, \lambda)c_q(x, \lambda) = \frac{1}{\psi_q(x, \lambda)} \) and the identity already obtained \( C_q(x^-, \lambda) = \psi_q(x, \lambda)c_q(x, \lambda) \), we get the following

\( B_q(x, \lambda) = \psi_q(\lambda)b_q(x, \lambda) \)

A similar proof works to obtain the two other identities of the lemma. □

Properties 2 and 3 of theorem 2.3 follow from lemmas 6.3, 6.4 and 6.5.
7. Proof of theorem 2.4

7.1. Two lemmas.

**Lemma 7.1.** For all \( x \in [0, +\infty[ \) and \( iu \in i\mathbb{R}, \) one has

\[ B_q(x, iu) \tilde{B}_q(x^-, iu) - C_q(x^-, iu) \tilde{C}_q(x, iu) = \phi_q(iu) \]

\[ \tilde{B}_q(x, iu) = (\phi(iu) + q)A_q(x, iu) + C_q(x, iu) \]

\[ B_q(x, iu) = (\phi(iu) + q)A_q(x, iu) + \tilde{C}_q(x, iu) \]

**Comments** According to theorem 2.3 the three pairs of functions \((A_q(x^-, iu), \tilde{A}_q(x, iu)), (C_q(x^-, iu), B_q(x, iu))\) and \((\tilde{B}_q(x^-, iu), \tilde{C}_q(x, iu))\) satisfy the same differential equation in \( x, \) in the sense of distribution theory. If the coefficients of this equation were sufficiently regular, the second and the third identities of lemma 6.1 would be an easy consequence of 2- dimensionality of the space of solution of this differential equation and the first identity would come from the wronskian identity. In the next proof, we check that these results still hold in our setting by using Stieljes integral calculus.

**Proof** First, notice that every complex valued function defined on \([0, +\infty[\) which is bounded and have bounded variations on every half line \([x, +\infty[\) (respectively on every interval \([0, x[\) where \( x \) is a positive real can be written \( \int_{[x, +\infty[} f(t)\nu(dt) \) (resp. \( \int_{[0, x[} f(t)\nu(dt) \)) if it is right continuous or \( \int_{[x, +\infty[} f(t)\nu(dt) \) (resp. \( \int_{[0, x[} f(t)\nu(dt) \)) if it is left continuous where \( \nu \) is a positive measure on \([0, +\infty[\) and \( f \) is a complex valued function. We shall denote \( \mu[x, +\infty[\) or \( \mu[x, +\infty[\) (resp. \( \mu[0, x[\) or \( \mu[0, x[\)) these functions in the sequel and \( \mu(dx) \) will be the associated complex measure \( \mu(dx) = f(x)\nu(dx) \). Here are few easy facts about Stieljes integral calculus.

An easy application of Fubini’s theorem gives us, for all positive \( x, \)

\[ \mu_1[x, +\infty[\mu_2[x, +\infty[ = \int_{[x, +\infty[} \mu_1[y, +\infty[\mu_2(dy) + \int_{[x, +\infty[} \mu_2[y, +\infty[\mu_1(dy) \]

(1)

If the complex measure \( \mu_1(dx) \) is also integrable in the neighborhood of 0, one has :

\[ \mu_1[0, x[\mu_2[x, +\infty[ = \int_{[x, +\infty[} \mu_1[0, y[\mu_2(dy) - \int_{[x, +\infty[} \mu_2[y, +\infty[\mu_1(dy) \]

(2)

[To establish this one, add it with previous one]. When adding the term \( \sum_{y \in [x, +\infty[} \mu_1(\{y\})\mu_2(\{y\}) \) in both integrals, one gets :

\[ \mu_1[0, x[\mu_2[x, +\infty[ = \int_{[x, +\infty[} \mu_1[0, y]\mu_2(dy) - \int_{[x, +\infty[} \mu_2[y, +\infty[\mu_1(dy) + \mu_2(\{x\})\mu_1(\{x\}) \]

And by regularisation on the right, one has

\[ \mu_1[0, x[\mu_2[x, +\infty[ = \int_{[x, +\infty[} \mu_1[0, y]\mu_2(dy) - \int_{[x, +\infty[} \mu_2[y, +\infty[\mu_1(dy) \]

(2)

Applying the previous equation to \( \mu_1[0, x[ := \frac{1}{\mu_2[x, +\infty[} \) when \( \frac{1}{\mu_2[x, +\infty[} \) has bounded variations on \([0, x[\) for every \( x, \) one gets the identity of complex measures

\[ \mu_1[0, y]\mu_2(dy) = \mu_2[y, +\infty[\mu_1(dy) \quad \text{on } [0, +\infty[ \]

and one deduces

\[ \mu_1[0, x[ = \frac{1}{\mu_2[x, +\infty[} \Rightarrow \mu_1(dy) = \frac{\mu_2(dy)}{\mu_2[y, +\infty[\mu_2[y, +\infty[} \quad \text{on } [0, +\infty[ \]

(3)
Let us establish now the first identity of the lemma 7.1. When applying identity (1) to
\[ \mu_1[x, +\infty] := \dot{B}_q(x^-, iu) = \dot{\psi}_q(iu) + \int_{[x, +\infty]} C_q(y, iu)e^{-iyu}\frac{H_q(dy)}{H_q(y)} \]
and
\[ \mu_2[x, +\infty] := B_q(x, iu) = \psi_q(iu) + \int_{[x, +\infty]} C_q(y^-, iu)e^{iyu}\frac{\dot{H}_q(dy)}{H_q(y^-)} \]
one gets
\[ \dot{B}_q(x^-, iu)B_q(x, iu) = \dot{\psi}_q(iu)\psi_q(iu) + \int_{[x, +\infty]} \dot{B}(y^-, iu)C(y^-, iu)e^{iyu}\frac{\dot{H}_q(dy)}{H_q(y^-)} \]
\[ + \int_{[x, +\infty]} B(y, iu)\dot{C}(y, iu)e^{-iyu}\frac{H_q(dy)}{H_q(y)} \]
when applying this identity to
\[ \mu_1[x, +\infty] := C_q(x^-, iu) = \int_{[x, +\infty]} B_q(y, iu)e^{-iyu}\frac{H_q(dy)}{H_q(y)} \]
and
\[ \mu_2[x, +\infty] := \dot{C}_q(x, iu) = \int_{[x, +\infty]} \dot{B}_q(y^-, iu)e^{iyu}\frac{\dot{H}_q(dy)}{H_q(y^-)} \]
one gets
\[ C_q(x^-, iu)\dot{C}_q(x, iu) = \int_{[x, +\infty]} C_q(y^-, iu)\dot{B}_q(y^-, iu)e^{iyu}\frac{\dot{H}_q(dy)}{H_q(y^-)} + \int_{[x, +\infty]} \dot{C}_q(y, iu)B_q(y, iu)e^{-iyu}\frac{H_q(dy)}{H_q(y)} \]
When substracting, these equations, one obtains
\[ \dot{B}_q(x^-, iu)B_q(x, iu) - C_q(x^-, iu)\dot{C}_q(x, iu) = \dot{\psi}_q(iu)\psi_q(iu) = \phi(iu) + q \]
Let us now establish the second identity of lemma 7.1. When multiplying by \( \phi(iu) + q \) the identity obtains in theorem 2.3 (\( \dot{A}_q(x, iu)\dot{B}_q(x^-, iu) + A_q(x^-, iu)\dot{C}_q(x, iu) = 1 \)) and substracting it from the previous one, one gets
\[ \dot{B}_q(x^-, iu)[B_q(x, iu) - (\phi(iu) + q)\dot{A}_q(x, iu)] = \dot{C}_q(x, iu)[C_q(x^-, iu) + (\phi(iu) + q)A_q(x^-, iu)] = 0 \]
Multiplying this last equation by the measure \( \frac{1}{B_q(x^-, iu)B_q(x, iu)}e^{-ixu}\frac{H_q(dx)}{H_q(x)} \), one obtains
\[ \left[ \frac{1}{B_q(x, iu)} \right] \left[ (B_q(x, iu) - (\phi(iu) + q)\dot{A}_q(x, iu))e^{-ixu}\frac{H_q(dx)}{H_q(x)} \right] \]
\[ - C_q(x^-, iu) + (\phi(iu) + q)A_q(x^-, iu) \right] \left[ \frac{\dot{C}_q(x, iu)}{B_q(x^-, iu)B_q(x, iu)}e^{-ixu}\frac{H_q(dx)}{H_q(x)} \right] = 0 \]  \hspace{1cm} (4)
Put \( q > 0 \) and let us justify that equation (4) is of the form \( \mu_1[0, x]\mu_2(dx) - \mu_2[x, +\infty]\mu_1(dx) = 0 \) where
\[ \mu_1[0, x] := \frac{1}{B_q(x, iu)} \quad \text{et} \quad \mu_2[x, +\infty] := C_q(x^-, iu) + (\phi(iu) + q)A_q(x^-, iu) \]
Indeed, bounded variations on \([x, +\infty] of C_q(x^-, iu) + (\phi(iu) + q)A_q(x^-, iu) is justified as follows : The measures \( Q^1_q(dw) \) and \( H_q(dx) \) are finite because \( q \) is positive, thus the function \( A_q(x^-, iu) = Q^1_q(M < x; e^{-iuF}) \) tends to \( Q^1_q(e^{-iuF}) = \frac{1}{\psi_q(iu)} \) when \( x \) goes to \( +\infty \). In the other hand, the function of \( x, \frac{\dot{A}_q(x, iu)}{H_q(x)} \) is bounded by 1 and
so it is $H_q(dx)$ integrable. We deduce from these facts and from first identity of theorem 2.3 ($A_q(x^-, iu) = H_q(0) + \int_{x^+, iu} H_q(dy)$), the next one

$$A_q(x^-, iu) = \frac{1}{\psi_q(iu)} - \int_{|x^+, iu|} e^{-iu\lambda} \hat{A}_q(y, iu) \frac{H_q(dy)}{H_q(y)}$$

Consequently, this equation and the second identity of theorem 2.3 ($C_q(x^-, iu) = \int_{|x^+, iu|} e^{-iu\lambda} B_q(x, iu) \frac{H_q(dy)}{H_q(y)}$) gives us that

$$\mu_2[x, +\infty] := C_q(x^-, iu) + (\phi(iu) + q)A_q(x^-, iu)$$

has bounded variations on $[x, +\infty]$ for every positive real $x$ and

$$\mu_2(dx) = (B_q(x, iu) - (\phi(iu) + q)\hat{A}_q(x, iu)) e^{-iu\lambda} \frac{H_q(dx)}{H_q(x)} \text{ on } [0, +\infty]$$

In the other hand, bounded variations on $[0, x]$ of $\frac{1}{B_q(x, iu)}$ follow from the identity

$$\frac{1}{B_q(x, iu)} = \mathcal{P}(\int_{[0, T^i]} e^{-iu\lambda t} L^i(dt))$$

Apply formula (3) to $\mu_1[0, x] := \frac{1}{B_q(x, iu)} = \frac{1}{\mu_2[x, +\infty]}$ and use the identity of theorem 2.3 ($\hat{B}_q(x^-, iu) = \int_{x^+, +\infty} e^{-iu\lambda} \hat{C}_q(x, iu) \frac{H_q(dy)}{H_q(y)}$) and get

$$\mu_1(dx) = \frac{\hat{C}_q(x, iu)}{\hat{B}_q(x^-, iu) \hat{B}_q(x, iu)} e^{-iu\lambda} \frac{H_q(dx)}{H_q(x)} \text{ on } [0, +\infty]$$

Now, one can integrate equation (4) over $[x, +\infty]$ and use equation (2) and get that the function

$$\frac{C_q(x, iu) + (\phi(iu) + q)A_q(x, iu)}{B_q(x, iu)}$$

is a constant function of $x$ on $[0, +\infty]$. On the other hand,

$$\lim_{x \to +\infty} C_q(x, iu) = 0 \quad \lim_{x \to +\infty} A_q(x, iu) = \frac{1}{\psi_q(iu)} \quad \lim_{x \to +\infty} \hat{B}_q(x, iu) = \hat{\psi}_q(iu)$$

Thus

$$\lim_{x \to +\infty} \frac{C_q(x, iu) + (\phi(iu) + q)A_q(x, iu)}{B_q(x, iu)} = \frac{\phi(iu) + q}{\psi_q(iu)\hat{\psi}_q(iu)} = 1$$

This enables us to deduce the second identity of the lemma 7.1

$$\hat{B}_q(x, iu) = C_q(x, iu) + (\phi(iu) + q)A_q(x, iu)$$

The third identity can be proved similarly or by regularizing the second one at the left and integrating it over $[x, +\infty]$ with respect to the measure $e^{iu\lambda} \frac{H_q(dx)}{H_q(x)}$. To get the second and the third identity when $q = 0$, take a limit.

Enlarge the probability space $(\Omega, \mathcal{F}, \mathcal{P})$ in order that it contains a random variable $\xi_q$ independent of the canonical process $X$ and which has an exponential distribution with parameter $q$ and still denote by the same notation this enlarged distribution. For $t > 0$, denote

$$D^t_t = \text{inf} \{t, +\infty| \text{supp } (L^t)\}$$

**Lemma 7.2.** For $\Re(\lambda) \geq 0$, one gets

$$1 - \hat{\psi}_q(iu)\hat{A}_q(x, \lambda) = \mathcal{P}(e^{-\lambda D^t_{t^+} \wedge \xi_q}; D_{t^+} \wedge \xi_q < +\infty)$$
The function $\lambda$ (subordinator) killed at an exponential independent time, thus the function $\lambda$ functions

$$\int_{[0,\infty[} e^{-\lambda t} L'(dt) = \int_{[0,\infty[} e^{-\lambda t} L'(dt) + 1_{D_{Ux\wedge \xi_q}^i < +\infty} e^{-\lambda I_{D_{Ux\wedge \xi_q}^i}} \int_{[0,\infty[} e^{-\lambda t} L'(dt) \circ \theta_{D_{Ux\wedge \xi_q}^i}^\circ \theta_{D_{Ux\wedge \xi_q}^i}

Take the expectation of this equation with respect to $P$, use the independent increments at stopping time $D_{Ux\wedge \xi_q}^i$ and get

$$\frac{1}{\psi_0(\lambda)} = \tilde{A}_q(x,\lambda) + P(e^{-\lambda D_{Ux\wedge \xi_q}^i}; D_{Ux\wedge \xi_q}^i < +\infty) \frac{1}{\psi_0(\lambda)}$

Thus

$$1 - \tilde{\psi}_0(\lambda) \tilde{A}_q(x,\lambda) = P(e^{-\lambda D_{Ux\wedge \xi_q}^i}; D_{Ux\wedge \xi_q}^i < +\infty)$$



7.2. The matrix $M_q$ satisfies properties of theorem 2.4. Property 1) is obvious. Property 2) follows from the second and the third identities of lemma 7.1. Let us check property 3):

The functions $\lambda \mapsto A_q(x^-,-), \lambda \mapsto \tilde{A}_q(x,\lambda), \lambda \mapsto C_q(x^-,-), \lambda \mapsto \tilde{C}_q(x,\lambda)$ are respectively Laplace transforms of finite measures supported respectively by sets $[0,x], [-x,0], [x, +\infty]$ and $-\infty$. Consequently, the functions $\lambda \mapsto A_q(x^-,-), \lambda \mapsto e^{-\lambda x} \tilde{A}_q(x,\lambda), \lambda \mapsto e^{-\lambda x} C_q(x^-,-), \lambda \mapsto e^{-\lambda x} \tilde{C}_q(x,\lambda)$ are bounded on the half plane $\{R(\lambda) > 0\}$ and the functions $\lambda \mapsto e^{-\lambda x} A_q(x^-,-), \lambda \mapsto \tilde{A}_q(x,\lambda), \lambda \mapsto e^{-\lambda x} C_q(x^-,-), \lambda \mapsto e^{-\lambda x} \tilde{C}_q(x,\lambda)$ are bounded on the half plane $\{R(\lambda) < 0\}$.

The function $\lambda \mapsto B_q(x,\lambda)$ (resp. $\lambda \mapsto B_q(x^-,-)$) is the Lévy exponent of a subordinator (resp. opposite of a subordinator) killed at an exponential independent time, thus the function $\lambda \mapsto \frac{B_q(x,\lambda)}{|\lambda| + 1}$ (resp. $\lambda \mapsto \frac{B_q(x^-,-)}{|\lambda| + 1}$) is bounded on the half plane $\{R(\lambda) < 0\}$ (resp. $\{R(\lambda) > 0\}$).

Let us check now property 4). The function $\lambda \mapsto e^{\lambda x} A_q(x^-,-)$ is the Laplace transform of a finite measure supported by $[-x,0]$ thus

$$\lim_{R(\lambda) \to -\infty} e^{\lambda x} A_q(x^-,-) = 0$$

Note that the random variable $I_{D_{Ux\wedge \xi_q}^i}$ is negative (If $P$ is not the distribution of a compound Poisson process, $I_t$ is negative on supp $L^i \setminus \{0\}$ as we have already said in part I of the paper and if $P$ is the distribution of a compound Poisson process, it is still the case by definition of the local time $L^i$), thus the limit

$$\lim_{R(\lambda) \to -\infty} \tilde{\psi}_0(\lambda) \tilde{A}_q(x,\lambda) = 1$$

follows from lemma 7.2.

The limit

$$\lim_{R(\lambda) \to -\infty} \frac{\tilde{B}_q(x^-,-)}{\psi_0(\lambda)} = 1$$

follows from the two previous limits, the boundedness of $\lambda \mapsto e^{-\lambda x} C_q(x,\lambda)$ and the identity $\tilde{A}_q(x,\lambda) \tilde{B}_q(x^-,-) + A_q(x^-,-) \tilde{C}_q(x,\lambda) = 1$ stated in theorem 2.3.

If $-\infty,0]$ is irregular then for every positive stopping time $T$, either $P$ is the distribution of compound Poisson process and the set $\{s; inf_{0 \leq u \leq s} X_{u+t} - X_{u} = X_{s+t} + X_{T-} - X_{T-} \}$ is $(P-a.s.)$ a discrete union of intervals where the process $X$ is constant, either the set $\{s; inf_{0 \leq u \leq s} X_{u+t} - X_{u} = X_{s+t} + X_{T-} - X_{T-} \}$ is discrete. One can easily deduce that the set $\{t; I_t = X_t \}$ has the same property $\tilde{N}$-a.s. On the other hand, the time $T_x$ is defined to be the first time when $X_t$ is strictly smaller than $-x$. One deduces that $X_{T_x} + X_{T_x} < x$ on $T_x < +\infty \tilde{N}$-a.s., and
\( \lambda \mapsto \hat{C}_q(x, \lambda) = \hat{N}(e^{-\lambda \bar{T}_x - q \bar{T}_x}; T_x < +\infty) \) is the Laplace transform of a finite measure supported by \([-\infty, x[\), thus

\[
\lim_{\Re(\lambda) \to -\infty} e^{-\lambda x} \hat{C}_q(x, \lambda) = 0
\]

### 7.3. Proof of the uniqueness part of theorem 2.4

Note first that the identities

\[
A_q(x^-, \lambda) B_q(x, \lambda) + \hat{A}_q(x, \lambda) C_q(x^-, \lambda) = 1
\]

and

\[
\hat{A}_q(x, \lambda) B_q(x^-, \lambda) + \hat{A}_q(x^-, \lambda) \hat{C}_q(x, \lambda) = 1
\]

stated in theorem 2.3 imply that \( \det M_q(x, \lambda) = 1 \) for every \( \lambda \in \mathbb{C} \setminus \mathbb{R} \). Let now \( N(\lambda) \) be another matrix having the same property as \( M_q(\lambda) \) and let us check that \( N(\lambda)[M_q(\lambda)]^{-1} \) is equal to the identity matrix. For simplicity we omit the parameters \( x \) and \( \lambda \) in the notation in the sequel.

By property 1) the matrix \( N(\lambda)[M(\lambda)]^{-1} \) is analytic on the two half-plane \( \{ \Re(\lambda) > 0 \} \) and \( \{ \Re(\lambda) < 0 \} \), and by property 2) it can be extended by continuity to every point of the imaginary axis.

Property 3) allows us to state that \( N(\lambda) M(\lambda)^{-1} \) is bounded on every compact set of \( C \). Thus a standard argument gives us that the extended matrix is entire.

Property 3) gives us also that the matrix \( \frac{1}{(1 + x)} e^{\lambda \hat{J}} N(\lambda)[M(\lambda)]^{-1} e^{-\lambda \hat{J}} \) is bounded on \( C \) (remember that \( J \) denotes the matrix \( \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \)), thus the components of matrix \( e^{\lambda \hat{J}} N(\lambda)[M(\lambda)]^{-1} e^{-\lambda \hat{J}} \) are polynomials.

In the other hand, if \([ -\infty, 0[ \) is irregular then property 4) gives us \( e^{\lambda \hat{J}} N(\lambda)[M(\lambda)]^{-1} e^{-\lambda \hat{J}} \) is equivalent, for \( \Re(\lambda) \to -\infty \), to

\[
\begin{pmatrix}
\hat{\psi}_0(\lambda) & 0 & 1 \\
0 & \psi_0(\lambda) & 0 \\
1 & 0 & \frac{1}{\psi_0(\lambda)}
\end{pmatrix}
\begin{pmatrix}
\hat{\psi}_0(\lambda) & 0 & 1 \\
0 & \psi_0(\lambda) & 0 \\
1 & 0 & \frac{1}{\psi_0(\lambda)}
\end{pmatrix}^{-1}
\]

One deduces that \( e^{\lambda \hat{J}} N(\lambda)[M(\lambda)]^{-1} e^{-\lambda \hat{J}} \) is constantly equal to the identity matrix and \( N(\lambda) = M(\lambda) \).

If \([ -\infty, 0[ \) is regular then the matrix \( e^{\lambda \hat{J}} N(\lambda)[M(\lambda)]^{-1} e^{-\lambda \hat{J}} \) is equivalent, for \( \Re(\lambda) \to -\infty \), to

\[
\begin{pmatrix}
\hat{\psi}_0(\lambda) & 0 & 1 \\
e^{-\lambda \bar{T}_{21}(\lambda)} & \psi_0(\lambda) & 0 \\
e^{-\lambda \bar{T}_{21}(\lambda)} & 0 & \frac{1}{\psi_0(\lambda)}
\end{pmatrix}
\begin{pmatrix}
\hat{\psi}_0(\lambda) & 0 & 1 \\
e^{-\lambda \bar{T}_{21}(\lambda)} & \psi_0(\lambda) & 0 \\
e^{-\lambda \bar{T}_{21}(\lambda)} & 0 & \frac{1}{\psi_0(\lambda)}
\end{pmatrix}^{-1} = \begin{pmatrix} 1 \\ \frac{1}{\psi_0(\lambda)} \left( e^{-\lambda \bar{T}_{21}(\lambda)} - e^{-\lambda \bar{T}_{21}(\lambda)} \right) \end{pmatrix}
\begin{pmatrix} 0 \\ 1 \end{pmatrix}
\]

The function \( \hat{\psi}_0(\lambda) \) goes to \(+\infty\) because \( \hat{\psi}_0(\lambda) \) is the Lévy exponent of the opposite of a subordinator (possibly with a finite life time) which is not a compound Poisson process. On the other hand, the function \( e^{-\lambda \bar{T}_{21}(\lambda)} - e^{-\lambda \bar{T}_{21}(\lambda)} \) stays bounded. Thus \( e^{\lambda \hat{J}} N(\lambda)[M(\lambda)]^{-1} e^{-\lambda \hat{J}} \) converges to the identity matrix again and \( N(\lambda) = M(\lambda) \).

### 8. Probabilistic interpretation of the identities between the six functions

First let us introduce some terminology on Wiener-Hopf factorization. Let us denote \((W, \mathcal{G}, Q)\) any probabilistic space. Let \( Z \) be a real process defined on \((W, \mathcal{G})\) with a life time that may be finite. If the distribution of \( Z \) under \( Q \) is \( P_q \), we shall say that \( \phi(iu) + q \) is the Lévy exponent of \( Z \) under \( Q \) and its spatial Wiener-Hopf factors are the functions \( \hat{\psi}_q \) and \( \psi_q \).

Let \( S = (0, S_1, \ldots, S_n, \ldots) \) be a sequence of real random variables defined on \((W, \mathcal{G})\) and having under \( Q \) the distribution of a random walk possibly killed at an independent geometric time. We shall identify its distribution to the distribution of the compound Poisson process with Lévy exponent \( 1 - Q(e^{-iuS_1}; 1 < \xi) \)
denotes the life time of $S$). We shall say that this function is the Lévy exponent of the random walk $S$ and that its spatial Wiener-Hopf factors are the ones of that compound Poisson process. One can easily check that these factors are respectively $1 - Q(e^{-iuS_{i}; V_{0} < \xi})$ and $1 - Q(e^{-iuS_{i}; V_{0}^{-} < \xi})$. $(V_{0} := \inf\{n \geq 1; S_{n} < 0\})$ and $V_{0}^{-} := \inf\{n \geq 1; S_{n} \geq 0\}$.

Let us now go back to our space $(\Omega, \mathcal{F}, \mathbb{P})$ and denote $\theta_{q}^{T}$ the path $(X_{s+t} - X_{t}; s \geq 0)$ for every time $t$. The reader can easily convince him/herself of the next assertions:

8.1. Interpretation of the identity $\hat{B}_{q}(x, iu) = (\phi(iu) + q)A_{q}(x, iu) + C_{q}(x, iu)$.

If $]-\infty, 0[$ is regular then define:

$$
T_{0} := 0 \quad T_{1} := T_{1}^{x} \quad T_{n+1} = T_{n} + T_{1}^{x} \circ \theta_{T_{n}}^{x} \quad (T_{n+1} := +\infty \text{ if } T_{n} = +\infty)
$$

$$
\hat{L}^{x}(dt) := \sum_{n=0}^{+\infty} 1_{T_{n} \leq t < T_{n+1}} L^{x}(dt - T_{n}, \theta_{T_{n}}^{x})
$$

$$
Y_{t} := X_{\hat{L}^{x}_{t} - 1}
$$

where $\hat{L}^{x}_{t}$ denotes the right inverse of the continuous increasing function $t \mapsto \hat{L}^{x}_{t}[0, t]$.

Under $\mathbb{P}_{q}$, $V$ is a Lévy process, its Lévy exponent is $\hat{B}_{q}(x, iu) - C_{q}(x, iu)$ and its Wiener-Hopf spatial factors are $\hat{\psi}_{q}(iu)$ and $\hat{\psi}_{q}(iu)A_{q}(x, iu)$.

If $]-\infty, 0[$ is irregular then let us define the sequence of stopping time $(T_{n}; n \geq 1)$ as follows:

$$
T_{1} := T_{1}^{x} = \inf\{t, X_{t} \notin [0, x]\} \quad T_{n+1} := T_{n} + T_{1}^{x} \circ \theta_{T_{n}}^{x} \quad (T_{n+1} := +\infty \text{ if } T_{n} = +\infty)
$$

Under $\mathbb{P}_{q}$, the sequence $(0, X_{T_{1}}, \ldots, X_{T_{n}}, \ldots)$ is a random walk. Its Lévy exponent is $\hat{B}_{q}(x, iu) - C_{q}(x, iu)$ and its Wiener-Hopf spatial factors are $\hat{\psi}_{q}(iu)$ and $\hat{\psi}_{q}(iu)A_{q}(x, iu)$.

The interpretation of the identity $B_{q}(x, iu) = (\phi(iu) + q)\hat{A}_{q}(x, iu) + C_{q}(x, iu)$ is similar.

8.2. Interpretation of the identity $(\phi(iu) + q)\hat{A}_{q}(x, iu)A_{q}(x^{-}, iu) + C_{q}(x^{-}, iu)\hat{A}_{q}(x, iu) + \check{C}_{q}(x, iu)A_{q}(x^{-}, iu) = 1$.

This identity is obtained from

$$
B_{q}(x, iu)A_{q}(x^{-}, iu) + C_{q}(x^{-}, iu)\hat{A}_{q}(x, iu) = 1 \quad \text{and} \quad B_{q}(x, iu) = (\phi(iu) + q)\hat{A}_{q}(x, iu) + \check{C}_{q}(x, iu)
$$

Denote $(U_{n}; n \geq 1)$ the sequence of stopping times:

$$
U_{1} := U^{x} \quad U_{n+1} := U_{n} + U^{x} \circ \theta_{U_{n}}^{x} \quad (U_{n+1} := +\infty \text{ if } U_{n} = +\infty)
$$

Under $\mathbb{P}_{q}$, the sequence $(0, X_{U_{1}}, \ldots, X_{U_{n}}, \ldots)$ is a random walk, its Lévy exponent is $1 - C_{q}(x^{-}, iu)\hat{A}_{q}(x, iu) - \check{C}_{q}(x, iu)A_{q}(x^{-}, iu)$ and its Wiener-Hopf spatial factors are $\hat{\psi}_{q}(iu)\hat{A}_{q}(x^{-}, iu)$ and $\hat{\psi}_{q}(iu)A_{q}(x^{-}, iu)$.

8.3. Interpretation of the identity $\hat{B}_{q}(x^{-}, iu)B_{q}(x, iu) - C_{q}(x^{-}, iu)\check{C}_{q}(x, iu) = \phi(iu) + q$.

Let $S = (0, S_{1}, \ldots, S_{n}, \ldots)$ be a real valued random walk (with infinite life time) on any probability space $(W, \mathcal{G}, Q)$. Denote for all $x \in [0, +\infty[$,

$$
V_{0}^{-} = \inf\{n \geq 1; S_{n} \notin [0, x]\} \quad V_{0}^{x} = \inf\{n \geq 1; S_{n} \notin [-x, 0]\}
$$

When applying the identity : $\hat{B}_{q}(x^{-}, iu)B_{q}(x, iu) - \check{C}_{q}(x, iu)C_{q}(x^{-}, iu) = \phi(iu) + q$ to the compound Poisson process which of Lévy exponent $1 - Q(e^{-iuS_{1}})$, on gets : For all $iu \in i\mathbb{R}$ and $s \in [0, 1]$,
\[ [1 - Q(e^{-iuS_{x}^0} - s^{x^0}; S_{x}^0 < 0)][1 - Q(e^{-iuS_{x}^0} + s^{x^0}; S_{x}^0 \geq 0)] - Q(e^{-iuS_{x}^0} - s^{x^0}; S_{x}^0 \geq x)Q(e^{-iuS_{x}^0} + s^{x^0}; S_{x}^0 < -x) = 1 - sQ(e^{-iuS_{x}}) \]

9. Examples

In this section, we treat two examples of Lévy processes: stable processes which are not killed \([q = 0] and Lévy processes without positive jumps. The "bilateral problem" has been essentially solved for these processes, by Rogozin [R] in the first case and by Takacs [T] in the second one. The reader can find also in the recent work [KK] other cases for which this problem is solved. Replacing in our context allows us to use the integral equations satisfied by our functions given in theorem 2.3. This allows us to give some further properties, and known results follow straightforwardly.

9.1. Stable processes. Let \(P\) be the law of a normalized stable process with index \(\alpha\) and asymmetry parameter \(\rho\), (i.e. \(\rho = P(X_1 > 0)\)). Let us \(\gamma := \alpha \rho\) and \(\delta := \alpha (1 - \rho)\), then \(\gamma\) and \(\delta\) belong to the interval \([0, 1]\). The cases \(\gamma = 0\) and \(\delta = 0\) corresponding to subordinators or opposite of subordinators are excluded in the sequel. The case \(\gamma = 1\) corresponds to stable processes without positive jumps. The Lévy exponent \(\phi\) can be written as follows (we use the principal part of the power functions):

\[ \phi(iu) = e^{i\frac{\pi}{2}(\gamma - \delta)e(u)|u|^\alpha} = (-iu)^\delta (iu)^\gamma \]

We choose the spatial Wiener-Hopf factors

\[ \tilde{\psi}_0(iu) = (-iu)^\delta \quad \psi_0(iu) = (iu)^\gamma \]

Theorem 9.1. For all \(x \in [0, +\infty]\), one has

\[ A_0(x, \lambda) = \frac{1}{\Gamma(\gamma)} \int_0^x e^{-\lambda y} y^{\gamma - 1}(1 - \frac{y}{x})^\delta dy \quad (\lambda \in C) \]

for \(\gamma \in [0, 1]\),

\[ C_0(x, \lambda) = \frac{\Gamma(\delta + 1)}{\Gamma(1 - \gamma)\Gamma(\gamma)} \int_x^{+\infty} e^{-\lambda y} (\frac{y}{x} - 1)^{-\gamma} y^{-\delta - 1} dy \quad (\Re(\lambda) \geq 0) \]

\[ B_0(x, \lambda) = \frac{\Gamma(\delta)}{\Gamma(1 - \gamma)\Gamma(\gamma)} \int_0^{x} (1 - e^{-\lambda y})(\frac{y}{x} + 1)^{\gamma} y^{-\delta - 1} dy + \frac{\Gamma(\alpha)}{\Gamma(\delta)} x^{-\gamma} \quad (\Re(\lambda) \geq 0) \]

and for \(\gamma = 1\) (consequently \(\delta = \alpha - 1\),

\[ C_0(x, \lambda) = \Gamma(\alpha)e^{-\lambda x} \quad (\Re(\lambda) \geq 0) \]

\[ B_0(x, \lambda) = \lambda + \frac{\delta}{x} \quad (\Re(\lambda) \geq 0) \]

The functions \(A_0, C_0\) and \(B_0\) are obtained by duality.

Sketch of proof Stability property easily gives us that the functions \(H_0\) and \(\tilde{H}_0\) are of the form:

\[ H_0(x) = k_+ x^\gamma \quad \tilde{H}_0(x) = k_- x^\delta \]

where \(k_+\) and \(k_-\) are positive constants. Consequently, upon differentiating twice the integral equation of theorem 2.3, one finds that the differential equation, in \(x\), satisfied by the three functions \(A_0(x, \lambda), C_0(x, \lambda)\) and \(B_0(x, \lambda)\) is the hypergeometric confluent equation. This fact and the behavior of these functions when \(x\) goes to \(+\infty\) allows us to compute them. We shall leave to the interested reader to check that one recovers in this way the results of Rogozin [R] cited above.
9.2. Lévy processes without positive jumps. In this section $P$ is the distribution of a Lévy process without positive jumps. We refer to the book [B] chapter 7 for the results we remind here. The Lévy exponent is of the form

$$\phi(iu) = \sigma^2 u^2 + au + \int_{|x| \to 0} (1 - e^{-iu \nu} - iux \mathbb{1}_{x>0}) \pi(dx)$$

This function can be extended to define an analytic function on the half plane $\{\Re(\lambda) < 0\}$ which is continuous when approaching the imaginary axis on the left. Still denote $\phi$ this function. Moreover, there exists a unique function $\Psi$ defined on $[0, +\infty]$ such that $\phi(-\Psi(q)) = -q$ for $q \geq 0$, and one can take as Wiener-Hopf factor $\psi_q$ the function:

$$\psi_q(\lambda) = \lambda + \Psi(q)$$

This Wiener-Hopf factor corresponds to the choice of the local time $L^q(dt) = dS_t$. Notice that in case $P$ is the law of a Lévy process of the form $\alpha - Y_t$ where $\alpha$ is a positive real and $Y$ is a subordinator without drift, one has to take $L^q(dt) = \sum \mathbb{1}_{\gamma_i = X, \delta_i}$ instead of $L^q(dt) = \sum \mathbb{1}_{\gamma_i = X, \delta_i}$ as we have stated at the beginning but this doesn’t change previous results.

Denote $W_q$ the so called scale function: It is the unique increasing right continuous function defined for $q \in [0, +\infty[$ and for $x \in [0, +\infty[$ and satisfying the identity:

$$\int_0^{+\infty} e^{-\lambda x} W_q(x) dx = \frac{1}{-\phi(-\lambda) - q} \quad (\lambda > \Psi(q))$$

The associated Stieltjes measure $W_q(dx)$ admits a right continuous density $w_q$ on $[0, +\infty[$ (we shall reprove this property in next proof):

$$W_q(x) = W_q(0) + \int_{[0,x]} w_q(y) dy$$

**Theorem 9.2.** For all $q \in [0, +\infty[$ and $x \in [0, +\infty[$, one has

$$A_q(x, \lambda) = \int_0^x e^{-\lambda y} \frac{W_q(x-y)}{W_q(x)} dy \quad (\lambda \in \mathbb{C})$$

$$\hat{A}_q(x, \lambda) = W_q(0) + \int_{-x}^0 e^{-\lambda y} [w_q(-y) - \frac{w_q(x)}{W_q(x)} W_q(-y)] dy \quad (\lambda \in \mathbb{C})$$

The functions $C_q$ and $B_q$ can be extended to the whole complex plane and

$$C_q(x, \lambda) = \frac{e^{-\lambda x}}{W_q(x)} \quad (\lambda \in \mathbb{C})$$

$$B_q(x, \lambda) = \lambda + \frac{w_q(x)}{W_q(x)} \quad (\lambda \in \mathbb{C})$$

$$\hat{C}_q(x, \lambda) = B_q(x, \lambda) - (\phi(\lambda) + q) \hat{A}_q(x, \lambda) \quad (\Re(\lambda) \leq 0)$$

$$\hat{B}_q(x, \lambda) = (\phi(\lambda) + q) \hat{A}_q(x, \lambda) + C_q(x, \lambda) \quad (\Re(\lambda) \leq 0)$$

**Sketch of proof** The absence of positive jumps gives us immediately that

$$C_q(x^{-}, \lambda) = e^{-\lambda x} N(S_{\xi_q} \geq x) \quad (5)$$

$$B_q(x, \lambda) = \lambda + \hat{N}(\xi_q < \zeta \text{ or } I_{\xi_q} < -x)$$

where $\xi_q$ denotes an independent exponential time after having enlarged the measure $N$ and $\hat{N}$ in order to contain such a variable.
Derive the integral equation of theorem 2.3
\[ C_q(x^-, \lambda) = \int_{[x, +\infty]} e^{-\lambda y} B_q(y, \lambda) \frac{H_q(dy)}{H_q(y)} \]  
(6)

compare with the previous expression of \( C_q \) and \( B_q \) and deduce
\[ \frac{H_q(dx)}{H_q(x)} = N(S_{\xi_q} > x)dx \quad \text{on } [0, +\infty] \]  
(7)

and the measure \( N(S_{\xi_q} \in dx) \) admits a right continuous density \( n_q \) on \([0, +\infty[ \) and
\[ N(\xi_q < \zeta \text{ or } I_{\xi_q} < -x) = \frac{n_q(x)}{N(S_{\xi_q} > x)} \quad (x \in [0, +\infty[) \]

In particular, \( H_q(x), N(S_{\xi_q} > x) \) and \( C_q(x, \lambda) \) are continuous functions of \( x \) on \([0, +\infty[ \). Consequently, the function \( \frac{n_q(x)}{N(S_{\xi_q} > x)} \) is nonincreasing and converges to \( \Psi(q) \) when \( x \) goes to +\( \infty \).

Remember then the two following identities of theorem 2.3 :
\[ A_q(x^-, \lambda) = H_q(0) + \int_{[0, x]} e^{-\lambda y} \tilde{A}_q(y, \lambda) \frac{H_q(dy)}{H_q(y)} \]  
(8)

\[ B_q(x, \lambda)A_q(x^-, \lambda) + C_q(x^-, \lambda) \tilde{A}_q(x, \lambda) = 1 \]  
(9)

One first deduces from (7) and (8) that \( A_q(x, \lambda) \) is a continuous function of \( x \) (Notice that \( A_q(x, 0) = H_q(0) = 0 \) because \([0, +\infty[ \) is regular) and admits a right derivative. One then can compute the right derivative of the function \( \frac{A_q(x, \lambda)}{C_q(x, \lambda)} \) with the help of identities (5), (6), (7), (8), (9) and get :
\[
\left[ \frac{A_q(x, \lambda)}{C_q(x, \lambda)} \right] = \frac{B_q(x, \lambda)A_q(x^-, \lambda) + C_q(x^-, \lambda) \tilde{A}_q(x, \lambda)}{(e^{-\lambda x} N(S_{\xi_q} > x))^2} \times e^{-\lambda x} N(S_{\xi_q} > x) = e^{\lambda x} \frac{1}{N(S_{\xi_q} > x)}
\]

Integrate this equation over \([0, x]\) and use equation (5) and get :
\[ A_q(x, \lambda) = \int_0^x e^{-\lambda(x-y)} \frac{N(S_{\xi_q} > x)}{N(S_{\xi_q} > y)} dy \]

Derive this equation and use equation (7) and (8) and get
\[ \tilde{A}_q(x, \lambda) = \frac{1}{N(S_{\xi_q} > 0)} + \int_0^x e^{\lambda y} \left( \frac{n_q(y)}{N(S_{\xi_q} > y)^2} - \frac{1}{N(S_{\xi_q} > y)} \frac{n_q(x)}{N(S_{\xi_q} > x)} \right) dy \]

For \( \Re(\lambda) < 0 \), when \( x \) goes to +\( \infty \), \( \tilde{A}_q(x, \lambda) \) goes to \( \frac{1}{\phi(\lambda) + q} \) and one gets :
\[ \frac{\lambda + \Psi(q)}{\phi(\lambda) + q} = \frac{1}{N(S_{\xi_q} > 0)} + \int_0^{+\infty} e^{\lambda y} \left( \frac{n_q(y)}{N(S_{\xi_q} > y)^2} - \frac{\Psi(q)}{N(S_{\xi_q} > y)} \right) dy \]

This equation allows us to identify the function \( \frac{1}{N(S_{\xi_q} > x)} \) to the scale function \( W_q(x) \) and the statements of the theorem follow. \( \square \)
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