FIRST PASSAGE PERCOLATION ON NILPOTENT CAYLEY GRAPHS
AND BEYOND

ITAI BENJAMINI AND ROMAIN TESSERA

Abstract. We prove an asymptotic shape theorem for first-passage percolation on Cayley graphs of virtually nilpotent groups. By a theorem of Pansu, the asymptotic cone of a finitely generated nilpotent group is isometric to a simply connected nilpotent Lie group equipped with some left-invariant Carnot-Carathéodory metric. Our main result is an extension of Pansu’s theorem to random metrics, where the edges of the Cayley are i.i.d. random variable with some finite exponential moment. Based on the companion work [Te14], the proof relies on Talagrand’s concentration inequality, and on Pansu’s theorem. Adapting an argument from [BKS03] we prove a sublinear estimate on the variance for virtually nilpotent groups which are not virtually isomorphic to $\mathbb{Z}$. We further discuss the asymptotic cones of first-passage percolation on general infinite connected graphs: we prove that the asymptotic cones are a.e. deterministic if and only the volume growth is subexponential.

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1. Introduction

First passage percolation is a model of random perturbation of a given geometry. In this paper, we shall restrict to the simplest model, where random i.i.d lengths are assigned to the edges of a fixed graph. We refer to [GK12, Ke86] for background and references. A fundamental result (the shape theorem) states that the random metric on Euclidean lattices when rescaled by $1/n$, almost surely converges to a deterministic invariant metric on the Euclidean space [CD81, Ke86]. Underling this theorem is the simple fact that the graph metric on the Euclidean grid, when rescaled, converges to the metric associated to the $\ell^1$-norm on the Euclidean space. In the world of Cayley graphs, a version of this last fact holds and characterizes polynomial growth: by a theorem of Gromov [Gr81], groups of polynomial growth are virtually nilpotent, and by a theorem of Pansu [Pa], the rescaled sequence converges in the pointed Gromov-Hausdorff topology to a simply connected nilpotent Lie group equipped with some left-invariant Carnot-Caratheodory metric. It is therefore natural to ask if when assigning random i.i.d. lengths to Cayley graph of polynomial growth, the rescaled metric almost surely converge to a deterministic metric on the Lie group. Establishing this was the original goal of this note. As we will see below, we end up getting a very general statement (though for a weaker notion of convergence) for first-passage percolation (FPP for short) on general graphs with bounded degree.

Before stating our main results for nilpotent groups, let us illustrate it in a concrete case. The notion of (pointed) Gromov-Hausdorff convergence will be informally recalled along the way (see [BB10] for a more thorough introduction to this notion).
1.1. A motivating example: the Heisenberg group. Recall that the real Heisenberg group $\mathbb{H}(\mathbb{R})$ is defined as the matrix group

$$\mathbb{H}(\mathbb{R}) = \left\{ \begin{pmatrix} 1 & u & w \\ 0 & 1 & v \\ 0 & 0 & 1 \end{pmatrix} ; u, v, w \in \mathbb{R} \right\},$$

and that the discrete Heisenberg $\mathbb{H}(\mathbb{Z})$ sits inside $\mathbb{H}(\mathbb{R})$ as the cocompact discrete subgroup consisting of unipotent matrices with integral coefficients. We equip the group $\mathbb{H}(\mathbb{Z})$ with the word metric associated with the finite generating set $S = \{a^\pm 1, b^\pm 1\}$, where

$$a = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

By Pansu’s theorem [Pa], the sequence of pointed metric spaces $(\mathbb{H}(\mathbb{Z}), e, d_\mathbb{Z}/n)$ converges in the Gromov-Hausdorff sense to $\mathbb{H}(\mathbb{R})$ equipped with the (unique up to isometry) Carnot-Caratheory metric which projects to the $\ell^1$-metric on $\mathbb{R}^2$. More explicitly, consider the one-parameter group $(\delta_t)_{t \in \mathbb{R}^*_+}$ of automorphisms of $\mathbb{H}(\mathbb{R})$ defined as follows

$$\delta_t \begin{pmatrix} 1 & u & w \\ 0 & 1 & v \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & tu & t^2w \\ 0 & 1 & tv \\ 0 & 0 & 1 \end{pmatrix}.$$

It is a standard fact that given a norm $\| \cdot \|$ on $\mathbb{R}^2$, there exists a unique left-invariant Carnot-Caratheory $d_{cc}$ metric on $\mathbb{H}(\mathbb{R})$ that projects to $\| \cdot \|$ and that is scaled by $\delta_t$, i.e. such that $d_{cc}(e, \delta_t(g)) = td_{cc}(e, g)$ for all $t \in \mathbb{R}^*_+$ and all $g \in \mathbb{H}(\mathbb{R})$. Such an automorphism $\delta_t$ (for $t > 1$) is called a dilation.

Now, consider the sequence of embeddings of $\delta_{1/n} \circ i : \mathbb{H}(\mathbb{Z}) \to \mathbb{H}(\mathbb{R})$, where $i$ is the standard embedding described above. This can be interpreted as a sequence of maps $\phi_n$ of pointed metric spaces $(\mathbb{H}(\mathbb{Z}), d_\mathbb{Z}/n, e)$ to $(\mathbb{H}(\mathbb{R}), d_{cc}, e)$. Pansu’s Theorem says that $\phi_n$ is a sequence of Gromov-Hausdorff approximations: i.e. for all $\varepsilon > 0$, there exists $n_0$ such that for all $n \geq n_0$,

- for all $g, h \in B_\varepsilon(e, n/\varepsilon)$,
  $$(1 - \varepsilon)d_\mathbb{Z}(g, h)/n - \varepsilon \leq d_{cc}(\phi_n(g), \phi_n(h)) \leq (1 + \varepsilon)d_\mathbb{Z}(g, h)/n + \varepsilon;$$
- for every $k \in \mathbb{H}(\mathbb{R})$ such that $d_{cc}(e, k) \leq n/\varepsilon$, there exists $g \in \mathbb{H}(\mathbb{Z})$ such that $d_{cc}(\phi_n(g), k) \leq \varepsilon$.

In particular, one deduces the so-called “asymptotic shape theorem” saying that for every $r > 0$ the sequence $K_n = \phi_n(B_\varepsilon(e, rm))$ of compact subsets of $(\mathbb{H}(\mathbb{R}), d_{cc})$ converges for the Hausdorff metric to $B_{\varepsilon}(e, r)$. 
Now assume that the length of each edge \( f \) in the Cayley graph \((H(\mathbb{Z}), S)\) is given by an i.i.d. random variable which – says equals 1 or 2 with equal probability –. This provides a family of metrics on \( H(\mathbb{Z}) \) indexed by a probability space, namely \( ([1, 2]^{H(\mathbb{Z})}, P) \) where \( P \) is the product measure. What can be said of the sequence of pointed metric spaces \((G, e, d_\omega/n)\) for a “typical \( \omega \)?

1.2. **An asymptotic shape theorem for Heisenberg’s group.** Before stating our precise result, let us describe our general set up. We consider a connected non-oriented graph \( X \), whose set of vertices (resp. edges) is denoted by \( V \) (resp. \( E \)). For every function \( \omega : E \to (0, \infty) \), we equip \( V \) with the weighted graph metric \( d_\omega \), where each edge \( e \) has weight \( \omega(e) \). In other words, for every \( v_1, v_2 \in V \),

\[
d_\omega(v_1, v_2) = \inf_{p = (e_1, \ldots, e_m)} \ell_f(p) \defeq \sum_{i=1}^m \omega(e_i).
\]

Observe that the simplicial metric on \( V \) corresponds to the case where \( \omega \) is constant equal to 1, we shall simply denote it by \( d \). We will now consider a probability measure on the set of all weight functions \( \omega \). We let \( \nu \) be a probability measure supported on \([0, \infty)\). Our model consists in choosing independently at random the weights \( \omega(e) \) according to \( \nu \). More formally, we equip the space \( \Omega = [0, \infty)^E \) with the product probability that we denote by \( P \).

A central result in first passage percolation is the following Gaussian concentration inequality due to Talagrand.

**Theorem 1.1.** [Tal95, Proposition 8.3]). Suppose that \( \omega(e) \) has an exponential moment: i.e. there exists \( c > 0 \) such that \( \mathbb{E} \exp(c \omega(e)) < \infty \). Then there exists \( C_1 \) and \( C_2 \) such that for every graph \( X = (V, E) \), for every pair of vertices \( x, y \), and for every \( 0 \leq u \leq d(x, y) \),

\[
P \left( |d_\omega(x, y) - \bar{d}(x, y)| \geq u \right) \leq C_1 \exp \left( -C_2 \min \left( \frac{u^2}{\bar{d}(x, y)}, u \right) \right).
\]

In the sequel we shall make the following assumptions on \( \nu \).

- **(A1)** We assume that \( \nu \) has an exponential moment.
- **(A2)** We also suppose that there exists \( a > 0 \) such that \( \bar{d}(x, y) \geq ad(x, y) \) for all \( x, y \in V \).

It turns out that the second condition is fulfilled provided that \( \nu([0]) < 1/k \), where \( k \) is an upper bound on the degree of the graph [Te14, Corollary A2]. On the other hand, one has \( \bar{d}(x, y) \leq bd(x, y) \), where \( b = \mathbb{E}(\omega(e)) \). It follows that under condition (A2), \( d \) and \( \bar{d} \) are bi-Lipschitz equivalent.

In what follows, we consider as a subset of \( H(\mathbb{R}) \) equipped with the left-invariant Carnot-Caratheodory metric \( d_{cc} \) obtained as the limit of \( (G, d_S, e) \) in
Pansu’s theorem. We let $B_{cc}(g, r)$ denote the ball of radius $r$ centred at $g$ for this metric. Recall that the Hausdorff distance between two compact subsets $A$ and $B$ of $H(\mathbb{R})$ is defined as
\[
d_H(A, B) = \sup \{ r > 0, A \subset \bar{B}_r, B \subset \bar{A}_r \},
\]
where $\bar{A}_r$ denotes the set of points of $H(\mathbb{R})$ at distance at most $r$ from $A$. Observe that here, $\bar{A}_r = AB_{cc}(0, r)$, where we adopt the notation $AB = \{ ab, (a, b) \in A \times B \}$.

The following theorem is the analogue of the shape theorem in $\mathbb{Z}^d$ [CD81, Ke86].

**Theorem 1.2. (asymptotic shape theorem for Heisenberg)** We consider FPP on some Cayley graph of the discrete Heisenberg group $G$, and assume $(A_1)$ and $(A_2)$ are satisfied. There exists a left-invariant Carnot-Caratheodory metric $d'_{cc}$ on $H(\mathbb{R})$ for a.e. $\omega$, $(G, d_{\omega}/n, e)$ converges in the GH-topology to $(H(\mathbb{R}), d'_{cc}, e)$. Moreover, for a.e. $\omega$,
\[
d_H(\delta_{1/n}(B_\omega(e, n)), B_{cc}(e, 1)) \to 0.
\]

1.3. **A general result for virtually nilpotent groups.** In the sequel, we let $G$ be a finitely generated group, $S$ be a finite generating subset, and $(G, S)$ be the corresponding Cayley graph of $G$. We first recall Pansu’s theorem in complete generality

**Theorem 1.3. [Pa]** Let $G$ be a finitely generated virtually nilpotent group equipped with some finite generating set $S$. We assume $(A_1)$ and $(A_2)$ are satisfied. Then $(G, d_S/n, 1_G)$ converges in the pointed Gromov-Hausdorff topology to some simply connected (Carnot) nilpotent Lie group $N_\mathbb{R}$ equipped with some left-invariant Carnot-Caratheodory metric $d_{cc}$.

Our main result is an extension of this theorem to FPP.

**Theorem 1.4. (Asymptotic shape theorem for nilpotent groups)** Assume that $G$ is virtually nilpotent and we let $(N_\mathbb{R}, d_{cc})$ be the limit of $(G, d/n, 1_G)$ as in Theorem 1.3. We assume $(A_1)$ and $(A_2)$ are satisfied. There exists a left-invariant Carnot-Caratheodory metric $d'_{cc}$ on $N_\mathbb{R}$, which is bi-Lipschitz equivalent to $d_{cc}$ such that for a.e. $\omega \in \Omega$, $(G, d_{\omega}/n, 1_G)$ converges in the pointed Gromov-Hausdorff topology to $(N_\mathbb{R}, d'_{cc})$.

As in the case of the Heisenberg group discussed above, one can explicitly produce (both for Theorem 1.3 and Theorem 1.4) a sequence of Gromov-Hausdorff approximations involving a 1-parameter subgroup of dilation of $N_\mathbb{R}$,
and accordingly deduce an asymptotic shape theorem of the same flavour. The
precise statements can be found in [Pa] or [B] in the deterministic setting, and it
is straightforward to deduce from our proof of Theorem 1.4 the corresponding
statements for first passage percolation.

The proof of Theorem 1.4 goes in two steps: first we use Theorem 1.1 to show
that the identity \((G, d_\omega/n, 1_G) \to (G, \bar{d}/n, 1_G)\), where \(\bar{d} = \mathbb{E}d_\omega\) is almost surely a
sequence of Gromov-Hausdorff approximation. This step is completely gen-
eral: the only geometric property that is used is the fact that the volume of balls
in \((G, S)\) grows subexponentially (see §3). The second step consists in showing
that \(\bar{d}\) is sufficiently close to being geodesic to apply Pansu’s theorem to the
sequence \((G, \bar{d}/n, 1_G)\).

We point out that the proof of Theorem 1.4 (except for Pansu’s theorem) is
essentially contained in [Te14].

1.4. Asymptotic cone of FPP on graphs with bounded degree. The first con-
tdition to obtain a limit shape theorem is to have relative compactness for the
Gromov-Hausdorff topology, which restricts our investigations to graphs with
polynomial growth. In order to treat more general situations, one needs the
notion of asymptotic cone, which is some way to force the scaling limit to exist
(using some non-principal ultrafilter). These notions are recalled in §3. One
can then prove a very general result which in some sense is a far-reaching
generalization of the phenomenon observed in Theorem 1.4.

Theorem 1.5. Let \(X = (V, E)\) be a graph with degree at most \(k \in \mathbb{N}\), let \(o_n\) be a
sequence of vertices, \(r_n \in \mathbb{N}\) be an increasing sequence, and let \(\eta\) be a non-principal
ultrafilter. We assume that \(\nu\) is supported on \([a, b]\), with \(0 < a < b < \infty\) and that
\(\nu([a]) < 1/k\). Then “the asymptotic cone is almost surely deterministic”, i.e. for a.e. \(\omega\),

\[
\lim_{\eta} (X, d_\omega/r_n, o_n) = \lim_{\eta} (X, \bar{d}/r_n, o_n),
\]

if and only if for every \(\varepsilon > 0\),

\[
\lim_{\eta} \frac{\log |B(o_n, r_n/\varepsilon)|}{r_n} = 0.
\]

Saying that the asymptotic cone is almost surely deterministic amounts to
saying that the fluctuations of the metric in the ball of radius \(r\) are almost surely
“sublinear”, i.e. in \(o(r)\). For those who do not like ultrafilters and asymptotic
cones, we recommend to read the statements of Propositions 1.5 and 3.5, which
are written in terms of fluctuations.
Theorem 1.5 is the combination of two independent statements: one dealing with the subexponential growth case, and one with the exponential growth case (see Remark 3.8). The first statement (Corollary 3.2) is a consequence of Talagrand’s Theorem 1.1, while the second one (Corollary 3.6) is completely elementary. The conclusion of Corollary 3.6 is actually stronger than the statement of Theorem 1.5: roughly speaking it says that the $\omega$-distance in the ball $B(o_n, r_n)$ a.s. admits fluctuations of size of the order of $r_n$ about the average distance. Note that we do not know whether this remains true for the distance to the origin.

1.5. **Sublinear upper bound on the variance.** A straightforward and well-known consequence of Theorem 1.1 is a linear bound on the variance $\var(d_\omega(x, y)) = O(d(x, y))$ valid for any graph, and sharp for $\mathbb{Z}$ (Kesten first proved it for FPP on $\mathbb{Z}^d$ using martingales [Ke93]). In [BKS03], the authors manage to improve this linear bound on $\mathbb{Z}^d$, for $d \geq 2$:

$$\var(d_\omega(x, y)) \leq C \frac{d(x, y)}{\log(1 + d(x, y))}.$$ 

To be more precise, they prove it under the assumption that $\nu(\{a\}) = \nu(\{b\}) = 1/2$. However, in [BR07, Theorem 4.4], the same result is proved under much more general assumptions on $\nu$ (including e.g. exponential laws). In a subsequent paper, these authors prove a concentration inequality as well [BR08, Theorem 5.4]. All these results rely on the same geometric trick from [BKS03]. Therefore they can all be generalized to the setting of Theorem 1.6 below. However, to simplify the exposition and avoid useless repetitions, we shall only focus on the original statement of [BKS03].

**Theorem 1.6.** Assume that $\nu(\{a\}) = \nu(\{b\}) = 1/2$ (or more generally the assumptions of [BR07, Theorem 4.4]) and consider FPP on some Cayley graph $(G, S)$. Assume that $G$ has a finite index subgroup $G' < G$ whose center $Z(G')$ satisfies the following property: there exists $\delta > 1$ and $c > 0$ such that for all $n$

$$(1.1) \quad |Z(G')| \cap B_S(e, n) \geq cn^\delta.$$ 

Then there exists $C > 0$ such that for all $x, y \in G$, one has

$$(1.2) \quad \var(d_\omega(x, y)) \leq C \frac{d(x, y)}{\log(1 + d(x, y))}.$$
Let us examine the case of the Heisenberg group: its center is isomorphic to the cyclic subgroup generated by the matrix
\[
c = \begin{pmatrix}
1 & 0 & 1 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}.
\]

Note that \([a^k, b^n] = a^{-k}b^{-n}a^kb^n = c^{nk}\), from which one easily deduces that \(|Z(H(Z))| \cap B_S(e, n)| \geq \alpha n^2\) for some constant \(\alpha > 0\). Therefore the previous theorem applies to \(H(Z)\). More generally, it is well-known (see e.g. [Gui73]) that non-virtually abelian nilpotent groups satisfy (1.1) with some \(\delta \geq 2\). So Theorem 1.6 applies to Cayley graphs of virtually nilpotent groups which are not virtually isomorphic to \(\mathbb{Z}\).

2. Asymptotic shape theorem for FPP nilpotent groups

Theorem 1.4 relies on the following proposition (which is [Te14, Proposition 1.2]).

Proposition 2.1. (Fluctuations about the average distance) We let \(X : (V, E)\) be a graph, and we assume that there exists \(C > 0\) and \(\bar{d} > 0\) for all \(o \in V\) and all \(r > 0\), \(|B(o, r)| \leq Cr^d\). We assume \((A_1)\) and \((A_2)\) are satisfied. Then there exists \(C' > 0\) such that for a.e. \(\omega\), there exists \(r_0\) such that for \(r \geq r_0\),
\[
\sup_{x,y \in B(o,r)} |d_{\omega}(x,y) - \bar{d}(x,y)| \leq C'(r \log r)^{1/2}.
\]

We deduce from the previous proposition that there exists \(C'\) such that for a.e. \(\omega\) there exists \(n_0\) such that for all \(n \geq n_0\), one has
\[
\bar{B} \left(0, r - C'(r \log r)^{1/2}\right) \subset B_{\omega}(0, r) \subset \bar{B} \left(0, r + C'(r \log r)^{1/2}\right).
\]

Corollary 2.2. Under the assumptions of Proposition 2.1, there exists a measurable subset of full measure \(\Omega' \subset \Omega\) such that for all \(\varepsilon > 0\), and all \(\omega \in \Omega'\), there exists \(r_0\) such that one has
\[
\bar{B} \left(o, (1 - \varepsilon)r\right) \subset B_{\omega}(o, r) \subset \bar{B} \left(o, (1 + \varepsilon)r\right),
\]
for all \(r \geq r_0\), and all \(o \in V\).

In order to prove Theorem 1.4 it is therefore enough to show the following result.

Theorem 2.3. With the assumptions of Theorem 1.4, the sequence of pointed metric spaces \((G, \bar{d}/n, 1_G)\) converges in the pointed Gromov-Hausdorff topology to \((N_{\mathbb{R}}, d_{cc}')\).
This relies on the following proposition:

**Proposition 2.4.** [Te14, Proposition 1.5] Under the assumptions of Proposition 2.1, the metric \( \overline{d} \) is ‘inner’ in the sense of [Pa]: for all \( \varepsilon > 0 \), there exists \( \alpha > 0 \) such that for all \( x, y \in X \), there exists a sequence \( x = x_1 \ldots x_{k+1} = y \) satisfying, for all \( 1 \leq i \leq k-1 \),

\[
\overline{d}(x_i, x_{i+1}) \leq \alpha, \\
\quad \text{and} \\
\overline{d}(x, y) \geq (1 - \varepsilon) \sum_{i=1}^{k-1} \overline{d}(x_i, x_{i+1}).
\]

Theorem 2.3 is now a direct consequence of Pansu’s theorem [Pa], which holds for inner left-invariant metrics on \( G \).

3. Asymptotic cones of FPP on graphs with bounded degree

3.1. Graphs with subexponential growth: “fluctuations vanish in the asymptotic cone”. We have the following consequence of Talagrand’s concentration inequality.

**Proposition 3.1.** (Sublinear fluctuations) Let \( X = (V, E) \) be a graph. We assume \((A_1)\) and \((A_2)\) are satisfied. Suppose in addition that there exists a sequence of vertices \( o_n \) and \( r_n \geq 0 \), an increasing sequence of integers such that for all \( \varepsilon > 0 \),

\[
\log |B(o_n, r_n/\varepsilon)|/r_n = o(r_n).
\]

There exists a measurable subset of full measure \( \Omega' \subset \Omega \) such that for all \( \varepsilon > 0 \), and all \( \omega \in \Omega' \), there exists \( n_0 = n_0(\omega, \varepsilon) \) such that for all \( n \geq n_0 \), for all \( x, y \in B(o_n, r_n/\varepsilon) \), one has

\[
|d(x, y) - \overline{d}(x, y)| \leq \varepsilon r_n.
\]

**Proof.** Note that (3.1) is equivalent to the following condition: for all \( C, c > 0 \),

\[
\sum_n e^{-c r_n} |B(o_n, C r_n)|^2 < \infty.
\]

Let \( 0 < \varepsilon \leq 1 \). Applying Talagrand’s theorem, we obtain that for all pairs of vertices \( x, y \in B(o, r_n/\varepsilon) \),

\[
P( |d(x, y) - \overline{d}(x, y)| \geq \varepsilon r_n ) \leq C_1 \exp \left( -C_2 \min \left\{ \varepsilon^3 r_n, \varepsilon r_n \right\} \right) \\
\leq C_1 \exp \left( -C_2 \varepsilon^3 r_n \right).
\]

\(^1\)The actual statement of [Te14, Proposition 1.5] is that the metric is strongly asymptotically geodesic.

\(^2\)This property is called asymptotic geodesicity in [B].
Given \( n \in \mathbb{N} \), we deduce that the probability that \( |d(\omega, x, y) - \bar{d}(x, y)| \leq \varepsilon_n \) for all \( x, y \in B(o, r_n \varepsilon) \), is at most

\[
1 - |B(o, r_n / \varepsilon)| C_1 \exp \left( -C_2 \varepsilon^3 r_n \right).
\]

Let \( \varepsilon_k \) be a sequence converging to 0. For all \( n_0, k \in \mathbb{N} \), we let \( \Omega_{n_0, k} \) be the event that for all \( n \geq n_0 \), and for all \( x, y \in B(o_n, r_n / \varepsilon_k) \) such that \( d(x, y) \geq \varepsilon_k r_n \) one has

\[
(1 - \varepsilon_k) \bar{d}(x, y) \leq d(\omega, x, y) \leq (1 + \varepsilon_k) \bar{d}(x, y).
\]

Clearly, it is enough to show that \( \Omega' = \bigcap_k \left( \bigcup_{n_0} \Omega_{n_0, k} \right) \), has full measure. This is equivalent to showing that for every \( k \in \mathbb{N} \), \( \bigcup_{n_0} \Omega_{n_0, k} \) has full measure. For this, it is enough to show that for every \( k \in \mathbb{N} \), \( P(\Omega_{n_0, k}) \to 1 \) as \( n_0 \to \infty \). Observe that \( 1 - P(\Omega_{n_0, k}) \) is the probability that there exists \( n \geq n_0 \) such that (3.3) is not satisfied for some \( x, y \in B(e, r_n / \varepsilon_k) \) such that \( d(x, y) \geq \varepsilon_k r_n \). It follows from Lemma 3.4 that for \( n_0 = n_0(k) \) large enough,

\[
1 - P(\Omega_{n_0, k}) \leq \sum_{n \geq n_0} |B(o, r_n / \varepsilon_k)| C_1 \exp \left( -C_2 \varepsilon^3 r_n \right),
\]

which tends to zero as \( n_0 \to \infty \) by (3.2). \( \square \)

Let us mention an immediate consequence of Proposition 3.1. We let \( \eta \) be a non-principal ultrafilter on \( \mathbb{N} \). Recall that given a sequence of pointed metric spaces \( (X_n, d_n, o_n) \), one can define its ultra-limit according to \( \eta \) as follows:

\[
X_\eta = \lim_{\eta} (X_n, d_n, o_n) := \{(x_n), d(o_n, x_n) = O(1)\}/\sim
\]

where \((x_n) \sim (y_n)\) if and only if \( \lim_{\eta} d_n(x_n, y_n) = 0 \). The distance on \( X_\eta \) is defined by the formula \( d_\eta((x_n), (y_n)) := \lim_{\eta} d(x_n, y_n) \). Now, given a fixed metric space \( X \), a sequence of points \( o_n \), a sequence \( r_n \to \infty \), and a non-principal ultrafilter \( \eta \), we call asymptotic cone (with respect to these data) the ultralimit \( \lim_{\eta} (X, d / r_n, o_n) \).

**Corollary 3.2.** Under the assumption of Proposition 3.1, there exists a measurable subset \( \Omega' \) of full measure such that for all \( \omega \in \Omega' \) and all non-principal ultrafilter \( \eta \),

\[
\lim_{\eta} d_\omega(x_n, y_n) / r_n = \lim_{\eta} \bar{d}(x_n, y_n) / r_n.
\]

In other words, “the asymptotic cone is almost surely deterministic”.
As a special case of the previous corollary, we deduce that the asymptotic cone of FPP on a Cayley graph with subexponential growth is almost surely deterministic.

It is important to make a clear distinction between the strong statement of Corollary 3.2 and the following much weaker one, which is true on any graph.

**Proposition 3.3.** Let $X = (V, E)$ be a graph, and for every $n \in \mathbb{N}$, let $o_n \in V$ and $r_n \geq 0$, and let $\eta$ be a non-principal ultrafilter. We assume $(A_1)$ and $(A_2)$ are satisfied. Then for all $x_n, y_n \in V$ such that $d(o_n, x_n) = O(r_n)$ and $d(o_n, y_n) = O(r_n)$, there exists a measurable subset of full measure $\Omega'$ (depending on the sequence) such that for all $\omega \in \Omega'$,

$$\lim_{\eta} d(\omega)(x, y)/r_n = \lim_{\eta} \bar{d}(x_n, y_n)/r_n.$$

**Proof.** This follows from the following lemma, which is an immediate consequence of Talagrand’s inequality. □

**Lemma 3.4.** Let $X = (V, E)$ be a graph, and let $o_n$ be a sequence of vertices. We assume $(A_1)$ and $(A_2)$ are satisfied. There exists constants $C'_1, C'_2$ such that the following holds. Let $r_n \geq 0$ be an increasing sequence of integers, and let $\epsilon > 0$. Then for all $x_n, y_n \in B(o_n, r_n/\epsilon)$, the probability that

$$|d_\omega(x_n, y_n) - \bar{d}(x_n, y_n)| \leq \epsilon r_n,$$

is at least $1 - C'_1 \exp \left(-C'_2\epsilon^3 r_n\right)$.

3.2. **Graphs with exponential growth:** “fluctuations remain non-trivial in the asymptotic cone”. In this subsection, we shall make the assumption that the minimal interval containing the support of $\nu$ is of the form $[a, b]$ with $0 \leq a < b < \infty$.

Note that for first-passage percolation on the $r$-regular tree for $r \geq 3$, it is easy to see that the asymptotic cone of FPP is not deterministic. We can use the fact that the random distance between two vertices $x, y$ in the tree is only determined by the edges along the unique geodesic between them: this distance is therefore the sum of $n := d(x, y)$ independent random variables. The average distance $\bar{d}(x, y)$ is equal to $cd(x, y)$, where $c \in (a, b)$ is the expected length of a given edge. The probability that $d_\omega(x, y)$ is less than $-\epsilon(a + c)d(x, y)/2$ (resp. more than $(c + b)d(x, y)/2$) decays (at most) exponentially with $n$. On the other hand, there are at least exponentially many pairs of disjoint geodesics of length $n$ in a ball of radius $kn$, for $k \geq 2$ (the exponential exponent can be made as large as we want by increasing $k$): for instance, for all $x$ in the sphere of radius $(k - 1)n$, pick a geodesic joining $x$ to a point of the sphere of radius $kn$. It
follows for a.e. \( \omega \), one can find in the asymptotic cone a pair of distinct points whose \( \omega \)-distance is strictly less (or strictly larger) than the average distance.

The same argument adapts to non-elementary hyperbolic groups. To generalize the previous argument, one uses the fact that there exists \( C > 0 \) such that for all \( \omega \) and for every pair of points \( x, y \), there is a geodesic (say for the word metric) \( \gamma \) between \( x \) and \( y \) whose \( C \)-neighborhood contains any \( d_\omega \)-geodesic between \( x \) and \( y \). To conclude that there exist fluctuations of linear size (both above and below the average distance), one needs to produce exponentially many “independent” pairs of points at distance \( n \) in a ball of radius \( \lesssim n \): this follows for instance by considering a quasi-isometrically embedded 3-regular tree.

For general graphs of exponential growth (even Cayley graphs), we do not know whether it is possible to exhibit fluctuations above the average distance in the asymptotic cone. However, it is possible to show that it always has fluctuations below the average distance: More precisely, the following proposition says that if the growth is exponential, then a.s. one can find in the asymptotic cone pairs of distinct points whose \( \omega \)-distance are “as close as possible to the minimal possible distance \( a(dx, y) \)”. Provided that the average distance is bounded away from this minimal distance (see Lemma 3.7), this implies that FPP admits “random fluctuations of linear size”, which are visible in the asymptotic cone.

**Proposition 3.5.** Let \( X = (V, E) \) be a (not necessarily connected) graph with degree at most \( q \), let \( o_n \) be a sequence of vertices. Assume that there exists an increasing sequence \( r_n \in \mathbb{N} \) such that \( \log |B(o_n, r_n)| \geq cr_n \) for some constant \( c > 0 \). Then there exists a measurable subset \( \Omega'' \) of full measure with the following properties. For all \( \omega \in \Omega'' \), for all \( \varepsilon > 0 \), there exists \( \tau > 0 \) and \( x_n, y_n \in V \) such that \( d(o_n, x_n) = O(r_n) \) and \( d(o_n, y_n) = O(r_n) \) and such that for all \( n \) large enough,

\[
\tau r_n \leq d_\omega(x_n, y_n) \leq (a + \varepsilon)d(x_n, y_n).
\]

Moreover, if \( \nu(|a|) > 0 \), then one can take \( \varepsilon = 0 \).

Before proving this proposition, let us restate it in terms of asymptotic cones.

**Corollary 3.6.** Let \( X = (V, E) \) be a (not necessarily connected) graph with degree at most \( q \), let \( o_n \) be a sequence of vertices and let \( \eta \) be a non-principal ultrafilter. Assume that there exists an increasing sequence \( r_n \in \mathbb{N} \) such that \( \log |B(o_n, r_n)| \geq cr_n \) for some constant \( c > 0 \). Then there exists a measurable subset \( \Omega'' \) of full measure with the following properties. For all \( \omega \in \Omega'' \), for all \( \varepsilon > 0 \), there exists \( \alpha > 0 \) and \( x_n, y_n \in V \)
such that $d(o_n, x_n) = O(r_n)$ and $d(o_n, y_n) = O(r_n)$ and such that,

$$0 < \lim_{\eta} d_{\omega}(x_n, y_n)/r_n \leq (a + \varepsilon) \lim_{\eta} d(x_n, y_n)/r_n.$$  

Moreover, if $\nu(|a|) > 0$, then one can take $\varepsilon = 0$.

**Proof.** Note that since the degree of $X$ is bounded, there exists $C$ such that

$$(3.4) \quad e^{cn} \leq |B(o_n, r_n)| \leq e^{Cn}.$$

Let $\lambda = c/2C$, so that $|B(o_n, \lambda r_n)| \leq e^{cr_n}/2$. We now consider a subset $X_n$ of $B(o_n, r_n)$ whose points are pairwise at distance at least $(c/4C)r_n$ apart and which is maximal for this property. It follows that

$$B(o_n r_n) \subset \bigcup_{x \in X_n} B(x, c/2C)r_n,$$

from which we deduce that

$$|B(o_n, r_n)| \leq |X_n| e^{cr_n}/2.$$

Thus we deduce that

$$|X_n| \geq e^{cr_n}/2.$$  

We let $\lambda > 0$ to be determined later and let $k_n = \lfloor \lambda(c/8C)r_n \rfloor$. Observe that the balls $B(x, k_n)$, for $x \in X_n$ are pairwise disjoint. So one can pick for every $x \in X_n$ a point $y_x$ at distance $k_n$ from $x$, and a geodesic $\gamma_x$ between them. The probability that all edges of $\gamma_x$ have $\omega$-length at most $(1 + \varepsilon)a$ is at least $\nu([a, a(1 + \varepsilon)])^{k_n}$. Since the paths $\gamma_x$ are disjoint, these events are independent, so that the probability that one of them has $\omega$-length at most $(a + \varepsilon)k_n$ is at least

$$1 - (1 - \nu([a, a(1 + \varepsilon)])^{k_n})^{|X_n|} \geq 1 - (1 - \nu([a, a(1 + \varepsilon)])^{\lambda(c/8C)r_n})^{\exp(cr_n/2)}.$$  

Recall that given two sequences such that $u_n \to 0$ and $v_n \to \infty$, one has $(1 - u_n)^{v_n} \leq \exp(-u_nv_n)$. On the other hand, by taking $\lambda$ small enough (depending on $\varepsilon$, unless $\nu(|a|) > 0$), one can ensure that $e^{cr_n/2}v([a, a(1 + \varepsilon)])^{\lambda(c/8C)r_n} \geq \exp(c'n)$ for some $c' > 0$. Therefore for this choice of $\lambda$, the above probability tends to 1 as $n$ tends to infinity very quickly (in particular the probability of the complement event is summable). This ensures the existence of a measurable subset of full measure $\Omega''$ such that for all $\omega \in \Omega''$, there is a sequence $x_n \in X_n$ such that for $n$ large enough, $d_\omega(y_{x_n}, x_n) \leq (a + \varepsilon)k_n' = a(1 + \varepsilon)d(x_n, y_n).$ This proves the first part of the proposition with $y_n = y_{x_n}$. \hfill $\Box$

To finish the proof of Theorem 1.5, we need the following lemma.
Lemma 3.7. [Te14 Lemma A.1] Let \( X = (V, E) \) be a graph of degree \( \leq q \). Assume that \( v \) is supported on the interval \( [a, \infty) \) and that \( \nu([a]) < 1/q \). Then there exists \( a' > a \) and \( r_0 \geq 0 \), such that \( d(x, y) \geq a'd(x, y) \) for all \( x, y \in V \) such that \( d(x, y) \geq r_0 \).

Remark 3.8. To conclude the proof of Theorem 1.5, let us remark that in the proof of Corollary 3.2 (resp. in Corollary 3.6) the condition \( \log |B(o_n, r_n)| = o(r_n) \) (resp. \( \log |B(o_n, r_n)| \geq cr_n \)) only needs to hold \( \eta \)-almost surely.

4. Upper bound on the variance

The proof of Theorem 1.6 is a simple generalization of the proof of [BKS03 Theorem 1] (which deals with the case of \( \mathbb{Z}^d, d \geq 2 \)). We shall sketch its proof, following the same order as in [BKS03], but only providing justifications when the argument needs to be adapted to our more general setting. To simplify the exposition, we shall assume that \( \delta \geq 2 \). In this section, we will denote \( 1_G \) for the neutral element of \( G \), keeping the letter \( e \) for the edges. Remember, since this will play a crucial role in this proof that the graph structure on \((G, S)\) is defined by saying that two elements (i.e. vertices) \( g \) and \( g' \) and joined by an edge if there exists \( s \in S \) such that \( g' = gs^{\pm 1} \). Hence, the action by left-translations of \( G \) on itself preserves the graph structure and thus the metric.

Following [BKS03], let us fix \( g \in G \), and consider the random variable \( f(\omega) := |g|_\omega \), where \( |g|_\omega \) denotes the \( \omega \)-distance from the neutral element to \( g \). We shall also denote \( |g| = d(1_G, g) \), where \( d \) is the word metric on \( G \). For every \( \omega \), we pick some \( \omega \)-geodesic \( \gamma \) from \( 1_G \) to \( g \). For every \( \omega \) and every edge \( e \in E \) we denote \( \sigma_{\omega, e} \) the configuration which is different from \( \omega \) only in the \( e \)-coordinate. We start remarking that

\[
\sum_{e \in E} P(e \in \gamma) \leq (b/a)|g|.
\]

We then fix \( m = [d(1_G, g)^{1/4}] \) and consider the function \( g_m: \{a, b\}^m \to \{1, \ldots, m\} \) constructed in [BKS03 Lemma 3]. Let \( \Sigma = \{0, 1\} \times \{1, \ldots, m^2\} \), \( \bar{\Omega} := \{a, b\}^\Sigma \). We define an injective map \( \psi: \bar{\Omega} \to \{1, \ldots, m\}^2 \) by

\[
\psi(x) = (g_m(x_{0, 1}, \ldots, x_{0,m}), g_m(x_{1, 1}, \ldots, x_{1,m})).
\]

We let \( C \geq 1 \) be large enough so that \( |Z(G') \cap B_2(1_G, Cm)| \geq m^2 \), and we pick some injective map from \( j: \{1, \ldots, m^2\} \to Z(G') \cap B_2(1_G, Cm) \). Let \( z := j \circ \psi \). We can now define \( \tilde{f} \) as a map from \( \{a, b\}^\Sigma \times \{a, b\}^\Sigma \) to \( Z(G') \cap B_2(1_G, Cm) \) by

\[
\tilde{f}(x, \omega) = d_\omega(z(x), z(x)g).
\]
The first important estimate from \[BKS03\] is
\[
(4.2) \quad \text{var}(f) \leq \text{var}(\tilde{f}) + O\left( m \sqrt{\text{var} \tilde{f}} \right) + O(m^2).
\]

If \( z \) commutes with \( g \), as \( d(1_G, z) = d(g, gz) \leq Cm \), we deduce by triangular inequality that \( |f - \tilde{f}| \leq 2bCm \), which implies (4.2). Otherwise, one needs that \( gz = z'g \), for some \( z' \in Z(G') \) such that \( d_S(e, z') = O(m) \). This is guarantied by the following lemma, after noticing that up to replacing \( G' \) with the intersection of all its conjugates, we can assume that \( G' \) is a characteristic subgroup of \( G \), whose center is therefore normal in \( G \).

**Lemma 4.1.** Assume \( G' \) is characteristic. There exists some constant \( C \), such that for all \( g \in G \) and \( z \in Z(G') \), \( d_S(e, gzg^{-1}) \leq C d_S(e, z) \).

**Proof.** Note that the action by conjugation of \( G \) on \( Z(G') \) factors through \( G/G' \) which is finite. Let \( F \subset G \) be a set of representatives of \( G/G' \), and let \( C = \max_{g \in F, s \in S} d_S(e, gsg^{-1}) \). Let \( z \in Z(G') \) of length \( n \), and let \( z = s_1 \ldots s_n \), where \( s_i \in S \). Given \( g \in G \), there exists \( h \in F \) such that \( gzg^{-1} = hzh^{-1} \). Thus we have
\[
g^{-1}zg = (hs_1h^{-1}) \ldots (hs_nh^{-1}),
\]
so the lemma follows by triangular inequality. \( \square \)

Define
\[
I_e(\tilde{f}) := P \left( \tilde{f}(x, \omega) \neq \tilde{f}(x, \sigma_e(\omega)) \right) = 2P \left( \hat{f}(x, \omega) < \hat{f}(x, \sigma_e(\omega)) \right).
\]

Then one needs to show that
\[
(4.3) \quad I_e(\hat{f}) = O(d(1_G, g)^{-1/4}),
\]
and
\[
(4.4) \quad \sum_e I_e(\hat{f}) = O(d(1_G, g)).
\]

The rest of the proof is identical to \[BKS03\] so we will not repeat it. Note that if the pair \((x, \omega) \in \hat{\Omega}\) satisfies \( \hat{f}(x, \sigma_e(\omega)) > \hat{f}(x, \omega) \), then \( e \) must belong to every geodesic between \( z \) and \( zg \), so in particular it belongs to \( z\gamma \). So we get
\[
(4.5) \quad I_e(\hat{f}) \leq 2P \left( z^{-1}e \in \gamma \right).
\]

Let \( Q \) be the set of edges \( e' \) such that \( P(z^{-1}e = e') > 0 \). Note that \( e' \) lies in the \( B(1_G, Cm) \)-orbit of \( e \), so that once again the lemma ensures that \( Q \) has diameter
in $O(m)$. It results that $\gamma \cap Q$ contains $O(m)$ edges. We now need the following property of $g_m$ ([BKS03, Lemma 3]):

$$\max_y P(g_m(x) = y) = O(1/m),$$

from which we deduce that

$$\max_{z_0} P(z = z_0) = O(1/m^2).$$

Conditioning on $\gamma$ and summing over the edges in $\gamma \cap Q$, we get

$$P(e \in z\gamma | \gamma) = O(1/m).$$

Consequently (4.5) and the choice of $m$ give (4.3). Also, (4.1) implies

$$\sum_{e \in E} P(z^{-1}e \in \gamma | z) \leq (b/a)|g|.$$

Combining this with (4.5) gives (4.4), so we are done.

5. Remarks and questions

5.1. More general distributions. It would be interesting to investigate whether our results survive to non-trivial correlations between edges lengths. Note that in some sense, Talagrand’s exponential concentration estimate is far too strong for Theorem 1.4: actually a polynomial decay with a large exponent would be enough to beat the (polynomial) growth rate of the group. This suggests that one should be able to use a weaker moment condition, and possibly allowing some weak correlations.

For groups, one can consider a different type of generalization: given an ergodic $G$-probability space $(\Omega, P)$, an invariant random metric (IRM) on $G$ is a measurable map $G \times G \times \Omega \rightarrow \mathbb{R}_+, (g, h, \omega) \rightarrow d_\omega(g, h)$, such that for a.e. $\omega \in \Omega$, $d_\omega(\cdot, \cdot)$ is a distance on $G$, and that satisfies the equivariance condition: for a.e. $\omega$, and all $g, h_1, h_2 \in G$,

$$d_{g\omega}(gh_1, gh_2) = d_\omega(h_1, h_2).$$

Clearly FPP is a special case of IRM, where the space $\Omega$ is $[a, b]^E$ equipped with the product probability. Observe that in this case, the action of $G$ on $\Omega$, induced by its action of $E$, is ergodic (actually even mixing).

One may wonder under what conditions on an IRM is the asymptotic cone of $(G, d_\omega, e)$ almost surely deterministic. In the special case of virtually nilpotent groups, one may ask whether $(G, d_\omega, e)$ converges in the pointed Gromov-Hausdorff topology to a connected Lie group equipped with an invariant Carnot-Carathéodory metric. Classical proofs of the limit shape theorem for
$Z^d$ are based on the subadditive ergodic theorem, which allows to treat very general IRM (see [Bj10] for the most general known statement). Unfortunately, we were not able to exploit the subadditive ergodic theorem for non-virtually abelian nilpotent groups: this only gives us that distances along certain “horizontal” directions are asymptotically deterministic, but for instance in the case of Heisenberg, it is not clear under what conditions distances in the direction of the center do not have large fluctuations.

Let us end with a last remark. Recall that the proof of Theorem 1.4 splits into two independent parts: one consists in proving a concentration phenomenon, namely that the identity map $(G, d_ω/n, e) → (G, ̅d/n, e)$ induces a sequence of Gromov-Hausdorff approximations (recall that $̅d = E_dω$). This might remain true under very general assumptions on $d_ω$, and in particular it may not require $d_ω$ to be geodesic, not even in a weak sense. This contrasts with the second step, consisting in proving that $(G, ̅d/n, e)$ converges, which does require $̅d$ to be asymptotically geodesic: indeed, conversely, if $(G, ̅d/n, e)$ converges to some geodesic metric space, then $̅d$ must be asymptotically geodesic. On the other hand one can exhibit invariant metrics on the Heisenberg group which are not asymptotically geodesic and yet quasi-isometric to the word metric. Moreover such a metric $d$ can be chosen so that $(G, d/n, e)$ does not converge at all [CT1, Remark A.6].

5.2. Sublinear variance. The proof of the sublinear estimate on the variance (Theorem 1.6) uses the fact that the group has a large center. By contrast, we know that for $Z$, or more generally on a tree, the variance grows linearly (this can easily be extended to Gromov-hyperbolic graphs). We suspect that—at least in the context of Cayley graph—the fact that the variance is sublinear might be related to the fact that no asymptotic cone has cut points (a cut point has the property that when we remove it, the space becomes disconnected). We propose the following more modest conjecture

**Conjecture 5.1.** Suppose $G$ is the direct product of two infinite finitely generated groups, then (1.2) is satisfied for all Cayley graphs of $G$.

A particularly interesting case is the direct product of the 3-regular tree $T$ with $Z$: in this case, [BM13] have managed to prove that $E(|d_ω − ̅d|)$ is tight in the $Z$-direction. There is some reason to believe that in the $T$-direction the variance should behave as for $Z^2$ (since geodesics are likely to remain at bounded distance from the direct product of a geodesic in $T$ times $Z$). Overall, the variance should be even smaller for $T × Z$ than for $Z^2$, where it is classically
conjectured to be of the order of $n^{2/3}$ (we refer to \cite{BKS03} and \cite{GK12} for a more detailed discussion concerning $\mathbb{Z}^2$). Another interesting example is the product of two 3-regular trees, for which no sublinear estimate is known at the moment.

5.3. **RWRE on virtually nilpotent Cayley graphs.** The FPP shape theorem and the rate of convergence are statements regarding large scale metric homogenization of local random metric perturbations. Similarly to the path we took here for FPP, it is of interest to consider the random walk, heat kernel and Green functions homogenization in the context of virtually nilpotent Cayley graphs, extending the work from lattices in Euclidean spaces, studied in PDE under the name of homogenization and in probability theory under the name RWRE (random walk in random environment) see e.g. \cite{Ba}.

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The Weizmann Institute, Rehovot, Israel
E-mail address: itai.benjamini@gmail.com

Laboratoire de Mathématiques, Bâtiment 425, Université Paris-Sud 11, 91405 Orsay, France
E-mail address: romain.tessera@math.u-psud.fr