10 D Euclidean dynamical triangulations.

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Abstract

We investigate numerically 10 - dimensional Euclidean quantum gravity (with discretized Einstein - Hilbert action) in the framework of the dynamical triangulation approach. For the considered values of the gravitational coupling we observed two phases, the behavior of which is found to be similar to that of the crumpled and elongated phases of 3, 4 and 5 dimensional models. Surprisingly, (for the observed lattice sizes) the natural state of the 10 D system (when the Einstein - Hilbert action is turned off) is found to resemble branched polymer while in the low dimensional systems the natural state belongs to the crumpled phase.
1 Introduction

The attempts to unify fundamental interactions have produced a lot of various models (see, for example, [1] and references therein). All of them are of rich mathematical structure and most of them are to some degree based on the Riemannian geometry. That’s why we believe that the quantization of the latter is rather important. Even if it has nothing to do with the real gravity, it may play an important role in the further construction of the realistic unified theory.

Recently it has been paid much attention to the dynamical triangulation (DT) approach to quantization of Riemannian geometry [2] in 2, 3 and 4 dimensions\(^1\). For \(D = 2\) the DT model has a well-defined continuum limit consistent with the predictions of the continuum theory [3]. At \(D = 3, 4\) the Euclidean DT models with the discretized Einstein action have two phases: the crumpled phase with infinite fractal dimension and the elongated one, which resembles branched polymer model with the fractal dimension close to 2. It appears that the introducing of the causal structure to these models (correspondent to the transition from Euclidean to Lorentzian quantum gravity) or coupling them to matter changes their behavior and make them more realistic [5, 6]. Nevertheless, the pure Euclidean gravity is still of interest as an area of developing the investigation methods. Moreover, mechanisms observed within this model can be related to more realistic models.

The mathematical structures related to the unification of fundamental forces may include the concept of high dimensional (\(D > 4\)) space. In particular, all known superstring models become unambiguous only in the 10-dimensional space-time. Unfortunately, most of information about their structure comes from the perturbative methods and the (incomplete) investigation of certain special excitations (such as \(D\)-branes). It seems that the nonperturbative investigation of those models would become exhaustive only if the numerical lattice methods are used. The modern lattice theory has an experience of dealing with quantum gravity models in lower dimensions. In particular, the dynamical triangulation method was applied to the systems of \(D = 2, 3, 4\) dimensions. Therefore we guess that it is reasonable to apply this method to the higher dimensional problems. The investigation of the pure Einstein Euclidean gravity can become the first step. The work related to this step is the content of the present paper.

\(^1\)5-dimensional Euclidean quantum gravity was considered in [4].
Namely, we consider the ten-dimensional Euclidean dynamical triangulations of spherical topology. Their behavior was expected to be similar to that of 3, 4 and 5-dimensional systems. This supposition was confirmed partially. However, it turns out that there are a few features that are not present in the lower dimensional models. For example, the 10D model does have two phases that are similar to the crumpled and elongated phases of 3, 4 and 5-dimensional models. But in our case the phase transition (at least for the observed volumes \( V = 8000 \) and \( V = 32000 \)) is at negative gravitational coupling constant. This means that at ”physical” positive gravitational constant \( G \) the considered 10-dimensional system cannot exist in the crumpled phase unlike 3, 4 and 5-dimensional systems. Also this means that for the observed lattice sizes the natural state of the 10D system (when the Einstein-Hilbert action is turned off) resembles branched polymer while in the low-dimensional systems the natural state belongs to the crumpled phase.

It is worth mentioning that the considered volumes are not consistent with the system of ”physical” dimension \( D = 10 \). It follows from the observation that already at linear size 3 the analogous rectangular 10D lattice has the volume \( V = 3^{10} = 59049 \). However, the observed effective (fractal) dimension of the triangulated space, say, at \( V = 32000 \) is \( \sim 4 \) in the crumpled phase and \( \sim 2 \) in the elongated one. So, the linear size of the system is expected to be of the order of \( \sim (32000)^{\frac{1}{4}} \sim 10 \) and \( \sim (32000)^{\frac{1}{2}} \sim 100 \) respectively. This is confirmed by the direct measurement of the linear extent.

2 The model.

In this section we shall remind briefly the definition of the considered model and the description of the numerical algorithm. For the complete review of the dynamical triangulations see [2] and references therein. For the full description of the numerical algorithm for arbitrary dimension see [7].

In the dynamical triangulation approach the Riemannian manifold (of Euclidian signature) is approximated by the simplicial complex obtained by gluing the \( D \)-dimensional simplices. Each simplex has \( D+1 \) vertices. All links are assumed to have the same length \( a \). The metrics inside each simplex is supposed to be flat. Therefore the deviation from the flatness is concentrated on the boundaries of the simplices. The scalar curvature \( R \) is zero everywhere except the bones (\( D-2 \)-dimensional subsimplices of the triangulation simplices). The Einstein-Hilbert action can be expressed through
the number of bones and the number of simplices of the given triangulated manyfold:

\[ S = -\frac{1}{16\pi G} \int R(x) \sqrt{g} d^D x = -\frac{\text{Vol}_{D-2}}{16\pi G} \sum_{\text{bones}} (2\pi - O(\text{bone}) \cos^{-1}(\frac{1}{D})) \]

\[ = -\frac{\text{Vol}_{D-2}}{8G} (N_{\text{bones}} - \frac{D(D+1)}{4\pi} N_{\text{simplices}} \cos^{-1}(\frac{1}{D})) , \]  

(1)

where \( O(\text{bone}) \) is the number of simplices sharing the given bone, \( \text{Vol}_j = \frac{a^j \sqrt{j+1}}{j! \sqrt{2}^j} \) is the volume of a \( j \) - dimensional simplex with the edges of length \( a \), \( N_{\text{bones}} \) is the total number of bones and \( N_{\text{simplices}} \) is the total number of simplices.

The metric of the triangulated manyfold is completely defined by the way the simplices are glued together. Therefore, the functional integral over \( Dg \) in this approach is changed by the summation over the different triangulations (we restrict ourselves with the spherical topology only):

\[ \int Dg \to \sum_T \frac{1}{C_T} \]  

(2)

Where the sum is over the triangulations \( T \) that approximate different Riemannian manyfolds\(^2\) and \( C_T \) is the symmetry factor of the triangulation itself (the order of its automorphism group).

We consider the model, in which the fluctuations of the global invariant \( D \) - volume are suppressed. The partition function has the form:

\[ Z_V = \sum_T \frac{1}{C_T} \exp(-S(T)) = \sum_T \frac{1}{C_T} \exp(\kappa_{D-2} N_{D-2} - \kappa_D N_D - \gamma (N_D - V)^2) \]

(3)

where we denoted \( N_D = N_{\text{simplices}} \), \( N_{D-2} = N_{\text{bones}} \), and \( \kappa_{D-2} = \frac{\text{Vol}_{D-2}}{8G} \). Unfortunately, it is not possible to construct an algorithm that generates the sum over the triangulations of the same volume. So the constant \( \gamma \) is kept finite. \( \kappa_D(V, \kappa_{D-2}) \) is chosen in such a way that

\[ <N_D>_V = \sum_T \frac{1}{C_T} \exp(\kappa_{D-2} N_{D-2} - \kappa_D N_D - \gamma (N_D - V)^2) N_D = V \]  

(4)

\(^2\)Two formally different triangulations may approximate the same Riemannian manyfold. Therefore the correspondent factorization should be implemented.
This provides that the volume fluctuates around the required value $V$. The fluctuations are of the order of $\delta V \sim \frac{1}{\sqrt{\gamma}}$. In order to approach the model with constant $N_D$ we must keep $\frac{\delta V}{V} \ll 1$. In practice we use $\gamma = 0.005$ for $V = 16000, 32000$. So $\frac{\delta V}{V} \sim 10^{-3}$.

For the numerical investigations we used Metropolis algorithm in its form described in [7]. It is based on the following Markov chain. Each step of the chain is the proposal of a deformation $T_i \rightarrow T_f$ of the given triangulation $T_i$, which is accepted or rejected with the probability $P(T_i \rightarrow T_f)$ that satisfies the detailed balance condition

$$\exp(-S(T_i))P(T_i \rightarrow T_f) = \exp(-S(T_f))P(T_f \rightarrow T_i) \tag{5}$$

The definition of the proposed deformations is based on the following idea. Let us consider some closed $D$-dimensional simplicial manifold of the topology of a $D$-dimensional sphere. Then, if a connected piece of our original triangulation is equal to a piece of this manifold, we can replace it by the remaining part of the given manifold. Thus we obtain the deformed triangulation with the same topology as the original one. Further, let us choose the boundary $\partial s_{D+1}$ of a $D+1$-dimensional simplex $s_{D+1}$ as the mentioned above manifold. There are $D+1$ opportunities to distinguish a piece of $\partial s_{D+1}$ correspondent to $p$-dimensional subsimplices of $s_{D+1}$ ($p = 0, ..., D$). The resulting deformation is called $(p, D-p)$ move [7].

It has been shown that via such moves it is possible starting from any triangulation to reach a triangulation that is combinatorially equivalent\textsuperscript{3} to an arbitrarily chosen other triangulation [9]. This property is called ergodicity. Due to this property starting from the triangulation that approximates the given Riemannian manifold it is possible to reach a triangulation that approximates almost any other Riemannian manifold. The exceptional cases, in which this is impossible, are commonly believed not to affect physical results.

In practise we choose randomly the type of the move ($p \in \{0, ..., D\}$), the simplex of a triangulation and its $p$-dimensional subsimplex. After that we check is there a vicinity of this $p$-dimensional simplex that is equivalent to the required piece of $\partial s_{D+1}$. If so, the suggested move (and the correspondent

\textsuperscript{3}Two triangulations are combinatorially equivalent if they have subdivisions that are equivalent to each other up to the relabeling of (sub)simplices. The subdivision $T_s$ of the given triangulation $T$ is another triangulation such that the set of its vertices contains all vertices of $T$ and any simplex $s \in T_s$ either belongs to $T$ or belongs to some simplex of $T$.}
subsimplex) is called legal and we proceed with checking the possibility to perform the move \(^4\). If the move is allowed we accept or reject it with the following probability:

\[
p(T_i \rightarrow T_f) = \frac{1}{1 + (1 + \frac{N_D(T_f) - N_D(T_i)}{N_D(T_i)}) \exp(S(T_f) - S(T_i))} \tag{6}
\]

In our calculations we start from the triangulation of minimal size that is \(\partial s_{D+1}\). Then we allow it to grow randomly until the volume reaches the given value \(V\). After that we proceed with the normal Metropolis process. During this process the coupling \(\kappa_D\) is self-tuned automatically in order to get its required value satisfying (4). This is done via redefining it as \(\kappa \rightarrow \kappa + 2\gamma(\langle N_D \rangle - V)\) after each 10 sweeps \(^5\).

In order to make a check of our results, we have made two independent program codes. The main program was written in C++ using modern methods of object-oriented programming. The algorithm is partially based on the one described in [7] and on the ideas suggested in [10]. The second program was written in Fortran 77 and was used for checking the main program at small lattice sizes. The calculations were made within the parallel programming environment using the computation facilities of Joint Supercomputer Center (Moscow).

3 Numerical results.

We investigate the behavior of 10D model for \(V = 8000\) and \(V = 32000\). We considered the values of \(\kappa_D\) varying from \(-0.1\) to \(0.1\), where the system is found to exist in two phases. The phase transition point is at \(\kappa_D \sim -0.03\) at \(V = 8000\) and at \(\kappa_D \sim -0.01\) at \(V = 32000\).

The self-tuned value of \(\kappa_{10}\) appears to be independent of \(V\). In accordance with large volume asymptotics obtained in [8] its dependence upon \(\kappa_8\) is linear with a good accuracy. The best fit is:

\[
\kappa_{10} = 15.57(1)\kappa_8 + 0.42(1) \tag{7}
\]

We investigate the following variables, which reflect the properties of the triangulated manyfold.

\(^4\)The move is not geometrically allowed if the resulting new simplex already exists in the given triangulation.

\(^5\)One sweep is \(V\) suggestions of legal moves.
1. The mean curvature carried by a simplex. The normalization is chosen in such a way that it is defined as

\[
R = \frac{4\pi N_{\text{bones}}}{D(D+1)N_{\text{simplices}} \cos^{-1}\left(\frac{1}{D}\right)} - 1
\]  
(8)

The results for \(V = 8000\) and \(V = 32000\) coincide with a good accuracy. They are represented in the figure 1. In contrast with the four dimensional case (at the observed values of \(\kappa_D\)) \(R > 0\) (in the four dimensional case it becomes negative at the value of \(\kappa_{D-2}\) close to 0). The best fit to the curve of the figure 1 is:

\[
R = 0.218(1) + 0.654(2)\kappa_D - 1.97(3)\kappa_D^2
\]  
(9)

2. The geodesic distance between two simplices is the length of the shortest path that connects them. The points of this path correspond to the simplices. The links of the path correspond to the pairs of neighboring simplices. We denote the geodesic distance between the simplices \(u\) and \(v\) by \(\rho(u, v)\). Let us fix a simplex \(s\). Then the ball \(B_R(s)\) of radius \(R \in \mathbb{Z}\) is consisted of the simplices \(u\) such that \(\rho(u, s)\) is less than or equal to \(R\). The volume of the ball is defined as the number of simplices contained in it.

One of the most informative characteristics of the triangulated manifold is the average volume \(V(R)\) of the balls (of the constant radius \(R\)) as a function of this radius. In practise during each measurement we calculate \(V_R(s)\) for arbitrary chosen simplex \(s\). Then we perform its averaging over the measurements (separately for each \(R\)). \(V(R)\) becomes constant at some value of \(R\). This value is the averaged largest distance between two simplices of the manifold, which is also called the diameter \(d\). We found that for \(V = 8000\) at \(1 < R < \frac{d}{2}\) the dependence of \(\log V\) on \(\log R\) is linear. For \(V = 32000\) the same takes place at \(4 < R < \frac{d}{2}\). The slopes give us the definition of the fractal (Hausdorff) dimension of the manifold: \(\log V = \text{const} + D\log R\).

3. The mean fractal dimension \(D(V)\) of the manifold for the observed volumes \(V\) as a function of \(\kappa_{D-2}\) is represented on the Fig. 2. The figure indicates that there is a phase transition at critical \(\kappa_{D-2} = \kappa_c(V)\). For the observed volumes \(\kappa_c\) appears to be small and negative (\(\kappa_c(8000) \sim -0.3\) and \(\kappa_c(32000) \sim -0.1\)). For \(\kappa > \kappa_c\) the fractal dimension is close to two in accordance with the supposition that similar to the 4-dimensional case the correspondent phase resembles branched polymers. For \(\kappa < \kappa_c\) \(D(8000) \sim 3.2\) and \(D(32000) \sim 4\). This is in accordance with the expectations that this
phase has a singular nature and corresponds to $D(\infty) = \infty$. Further we shall call these two phases crumpled and elongated respectively. Our results give us arguments in favor of view that the nature of these phases is similar to that of the phases of the lower dimensional models. We must notice, however, that the complete proof has not yet been obtained.

4. The linear size of the system can be evaluated as $V^\frac{1}{D}$. So, we expect that in the crumpled phase the diameter $d \sim 10$ while in the elongated phase $d \sim 100$. These expectations are in accordance with the direct measurements of the diameter as well as another parameter called linear extent (see [7]).

The linear extent $L$ is defined as the average distance between two simplices of the triangulation:

$$L = \frac{1}{V^2} \sum_{u,v} < \rho(u,v)>$$

Due to its construction $L$ should be close to half a diameter. In our measurements we calculate the linear extent using the slightly different definition (that, anyway, should lead to the same result after averaging over the measurements). Namely, we calculate

$$L(s) = \frac{1}{V} \sum_v < \rho(s,v)> = \frac{1}{V} \sum_R R(V_R(s) - V_{R-1}(s)).$$

Performing the averaging over the measurements we obtain the required value of the mean linear extent. Our results on $L$ are represented in the figure. One can see that the linear size of the manyfold is increasing very fast in the elongated phase, while in the crumpled phase it remains almost constant (and close to that of 3, 4 and 5 dimensional models [2]). We found that the fluctuations of the linear extent almost absent in the crumpled phase and are of the order of the mean size of the manyfold in the elongated phase.

4 Discussion.

The force between two particles (of masses $m$) in the classical mechanics is equal to

$$F = -\frac{Gm^2}{R^2},$$

where $R$ is the distance between them. Therefore negative gravitation coupling $G$ and, correspondingly, negative $\kappa_2$ in the four-dimensional model
would correspond to repulsion instead of attraction. The same picture is, of course, valid for the negative $\kappa_{D-2}$ in higher-dimensional models. This means that the description of gravity in the dynamical triangulation approach may appear only for the positive $\kappa_{D-2}$. The phase transition in the pure Euclidean four-dimensional gravity is at $\kappa_2 \sim 1$ and the model exists at physical couplings both in the crumpled and in the elongated phases. In the ten-dimensional model for the observed volumes the phase transition point is $\sim -0.01$. So at physical gravitational couplings the system cannot exist in the crumpled phase.

The fractal dimension at $G > 0$ is found to be close to 2. This means that in order to construct a dynamical triangulation model that corresponds to the dynamics of Riemannian manifold (of Euclidean metrics) of the dimension $D > 2$, it is necessary to add something to the pure Euclidean gravity. However, the investigation of the latter seems sensible, because mechanisms that appear in this model may play a role in more realistic models. From this point of view the most interesting is the mechanism of compactification. We started from the model, which has $\frac{D(D-1)}{2} = 45$ local degrees of freedom corresponding to the independent components of the metric tensor $g$. Finally we arrive (in the elongated phase) at the system, which has the fractal dimension close to two. We expect that similar to the lower dimensional cases this system can be effectively approximated by a tree graph. In the correspondent branched polymer model the lengths of the linear pieces of the graph and the way of their gluing together are the dynamical variables. What does then happen with the remaining degrees of freedom? They correspond to the small fluctuations of the pieces of the original manifold represented by the links of the graph. Those fluctuations, obviously, become the internal degrees of freedom living on the polymer. This dimensional reduction should, in principal, resemble the dimensional reduction of the Kaluza-Klein models. However, the concrete realization of the mechanism has to be investigated. We expect that more realistic realizations of the dynamical triangulation approach should possess the dimensional reduction, which may be similar to that of the pure Euclidean gravity.

The investigation of the 5-dimensional model (see [4]) indicates that it has a complicated phase structure. Namely, at $\kappa_{D-2} < -5$ several different vacua appear. And the system possesses tunnelling between them. In our research of the 10-dimensional gravity we restricted ourselves by the ranges of $\kappa_8 \in (-0.1, 0.1)$, where the phase transition between crumpled and elongated phases is observed. However, taking into account the above mentioned
property of the $5D$ model we expect that the phase structure of the considered model is, probably, not limited by the observed two phases. Another open question is the order of the observed phase transition. Although we have some indications that it is of the first order, the complete investigation of the subject has not been performed.

5 Conclusions.

In this paper we report our results on the numerical investigation of the 10-dimensional Euclidean quantum gravity in the framework of the dynamical triangulation approach. Our summary is as follows.

1. The considered model contains the phases that we call crumpled and elongated. The arguments in favor of view that those phases resemble the correspondent phases of lower dimensional models were obtained.

2. For the observed volumes the phase transition between the mentioned phases corresponds to small negative values of $\kappa_8$. Therefore at the ”physical” positive gravitational coupling $G \sim \frac{1}{\kappa_8}$ the model does not contain the crumpled phase contrary to the lower dimensional cases, where critical value of $\kappa_{D-2}$ is positive. This also means that the natural state of the collection of 10 D simplices (that appears when the Einstein - Hilbert action is turned off and $\kappa_8 = 0$) belongs to the elongated phase (at least for the observed volumes). This is in the contrast with the behavior of the low dimensional models: at $D = 3, 4$ and $\kappa_{D-2} = 0$ the systems exist in the crumpled phase.

3. The average scalar curvature appears to be positive at the observed gravitational couplings ($\kappa_8 \in (-0.1, 0.1)$) in contrast with, say, the 4-dimensional case, where it becomes negative at $\kappa_2$ close to 0.

4. The fractal dimension of the model is close to 2 in the elongated phase for both considered volumes ($V = 8000$ and $V = 32000$). In the crumpled phase the fractal dimension appears to be $\sim 3.2$ for $V = 8000$ and $D \sim 4$ for $V = 32000$.

5. The linear size of the system in the crumpled phase appears to be close to that of the lower dimensional models (the linear extent is $\sim 10$). In the elongated phase the mean linear extent and its fluctuations appear to be sufficiently larger than in the crumpled phase.

6. It is expected that the considered phase transition is of the first order. However, the investigation of this subject has not been performed. So the question about the order of the phase transition remains open.
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References

[1] O. Aharony, S.S. Gubser, J. Maldacena, H. Ooguri, Y. Oz, Phys.Rept. 323 (2000) 183-386

[2] J.Ambjorn, hep-th/9411179

[3] J. Ambjorn, K. N. Anagnostopoulos, U. Magnea, G. Thorleifsson, Phys.Lett. B388 (1996) 713-719

[4] Alun George, hep-lat/9909033

[5] J. Ambjorn, J. Jurkiewicz, R. Loll, Nucl.Phys. B610 (2001) 347-382

[6] H.S.Egawa, S.Horata, T.Yukawa, Nucl.Phys.Proc.Suppl. 106 (2002) 971-973

[7] S. Catterall, Comput.Phys.Commun. 87 (1995) 409-415

[8] J. Ambjorn, M. Carfora, A. Marzuoli, hep-th/9612069

[9] U.Pachner, Europe J. Combinatorics, 12 (1991) 129 - 145

[10] B.V. de Bakker, hep-lat/9508006

[11] J. M. Overduin, P. S. Wesson, Phys.Rept. 283 (1997) 303
Figure 1: The curvature.
Figure 2: The fractal dimension. Data for $V = 8000$ is represented by the points and the solid line. Data for $V = 32000$ is represented by the circles.
Figure 3: The linear extent. Data for $V = 8000$ is represented by the points and the solid line. Data for $V = 32000$ is represented by the circles and the dashed line.