Some Approximate Results of Value-at-Risk for Dependent Compound Stochastic Sums of Heavy-Tailed Risks

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ABSTRACT

According to in-depth research, a wide range of problems in applied science involve estimating the probability of compound stochastic sums of heavy-tailed risks over a large threshold. Many researchers have explored this issue from different aspects in recent times. There are two main difficulties here: one is how to deal with the heavy tail of risk, and another is how to handle the dependence of the aggregated processes. Aimed at these two main problems, we investigate the asymptotic properties of the tail of compound stochastic sums of heavy-tailed risks in a general dependence framework, and some approximate bounds and key characteristics related to value-at-risk are also derived. Several practical examples are given to demonstrate the effectiveness of the approximation results. Furthermore, the main results in this paper can be applied to studies of stochastic models in finance and econometrics and studies of dependent netput processes of the M/G/1 queuing systems, etc.

INDEX TERMS
Multivariate dependence, operational risk, stochastic models, subexponential distribution, value-at-risk.

I. INTRODUCTION

A wide range of problems in applied science involve estimating the probability of compound stochastic sums of heavy-tailed risks over a large threshold. In queuing systems, the netput process of the M/G/1 workload process is often considered as a compound Poisson process with drift (see Asmussen [1]). Moreover, the large deviation of compound stochastic sums is traditionally the focus of research in applied probability theory. Furthermore, many scenarios, including the determination of risk measures for a given portfolio of risks, the number of claims on a company during a given time interval in the classical risk model, the estimation of credit risk, the operational losses in a given risk type/business line cell in a certain bank, etc., are concerned with estimation or modelling the probability of a rare event occurring, i.e., compound stochastic sums of heavy-tailed risks over a large threshold.

In terms of operational risk management of commercial banks, an important problem is how to deal with commercial banks’ risk exposure. And the advanced measurement approach (AMA) allows banks to develop their own model for assessing the regulatory capital that covers their yearly operational risk exposure within a confidence interval of 99.9%. The problem has arisen from the New Accord on Capital Adequacy, or Basel II (see BCBS [2], [3]). Moreover, a new kind of risk has emerged, i.e., an operational risk is proposed in this new agreement: “Operational risk is the risk of loss resulting from inadequate or failed internal processes, people and systems or from external events.”

Over the last few years, several results in this direction have been developed. We can refer to Kuhn and Neu [4] and Alexander [5] for the internal measurement approach, and the scored approach can be found in Anders [6]. Furthermore, Giudici and Bilotta [7] and Valle et al. [8] introduced the Bayesian approach to deal with the comparatively few data in empirical operational research. A well-known advantage of using the Bayesian approach is that it allows us to integrate the scarce and sometimes inaccurate quantitative data collected by the bank with prior information provided by experts. However, the loss distribution approach (LDA) that is particularly suggested by the Basel II Accord has been discussed by
many authors, such as Frachot et al. [9]–[11]. The first paper introduced the LDA for computing the capital charge of a bank in detail, and the second paper aimed at describing step by step how a full LDA can be implemented in operational practice and how both quantitative and qualitative points of view can be reconciled. In addition, Frachot et al. [11] stressed that there are strong arguments in favour of low levels of correlation between aggregate losses partly based on Credit Lyonnais data. Moreover, Chapelle et al. [12] developed a procedure that addressed the main issues, i.e., managing operational risk faced by banks in the implementation of the LDA. For more details on this issue, we refer the interested readers to Haubenstock and Hardin [13], Embrechts et al. [14], [15], Chavez-Demoulin et al. [16], Degen et al. [17], Valle et al. [8], Böcker et al. [18], Biagini et al. [19], Nash [20] and Panjer [21].

As is known, another main problem apart from the lack of loss data is establishing an accurate understanding of the degrees of dependence among losses in various units of measures and implementing it in practice. It has been found by Frachot [11] that the number of external fraud events is also high (respectively low) when the number of internal fraud events is high (respectively low) in the study of Credit Lyonnais data. This situation likely happens when both loss frequencies share a common dependence with respect to some variables including the gross income, the economic cycle, the size of the business, etc. Furthermore, this kind of phenomenon is more severe in China than it is in the normative international banks overseas. As reported in Tables I and 2 in Lu [22], the data were collected from the operational losses of 90 publicly reported events in the newspaper and media from 1999 to the first half of 2006 in China, and the operational losses varied from over tens of thousands of yuan (e.g., 91 thousand) up to tens of billions of yuan, and a positive correlation was shown to exist between the number of internal fraud events and the number of external fraud events. Other recent advances in the dependence of operational risks can be found in Cope et al. [23], Resheter et al. [24], Lu [25], [26], Jaume et al. [27], Carlos et al. [28], and Andrew et al. [29].

The rest of this paper is organized as follows. Section II introduces the model applied in the present work and explains the motivation behind our research. Section III first gives explicit results under more specific assumptions on the tail distributions and the underlying dependence structure. Then, several important examples are provided to illustrate the efficiency of the approximation results. We furnish the complete proof of the main results in Section IV. At the end, we summarize the main results of the full text and point out further research directions in Section V.

II. THE STOCHASTIC MODEL

We first introduce the specific model under the framework of LDA that can be depicted by the ‘loss frequency’ (the number of loss events during a given time period) and ‘loss severity’ (the impact of the event in terms of financial loss). As is known, a matrix with 8 business lines in the rows and 7 event types in the columns is the so-called Basel risk matrix, which consists of 56 separate operational risk cells. We assume that the aggregated loss processes are defined as

\[ S_1(t) := \sum_{i=1}^{N_1(t)} X_i \]

in the first risk cell, and the aggregated loss processes are defined as

\[ S_2(t) := \sum_{j=1}^{N_2(t)} Y_j \]

in the second risk cell.

1. The severity processes: The severities are modelled by a sequence of positive i.i.d. random variables \( \{X_i, i \in \mathbb{N}\} \) in the first risk cell. Let \( F \) be the distribution function (df) of the \( X_0 \) and assume that \( EX_0 = 0 \) for the simplification of the problem. In the same manner, the severities are modelled by a sequence of positive i.i.d. random variables \( \{Y_j, j \in \mathbb{N}\} \) in the first risk cell. Let \( G \) be the distribution function (df) of the \( Y_0 \) and assume that \( EY_0 = 0 \) for the simplification of the problem.

2. The frequency processes: the random variable \( N_1(t) \) of losses in the time interval \([0,t]\) is a nonnegative integer-based process, and we also assume that \( \lambda_1(t) := EN_1(t) < \infty \) for any \( t > 0 \) and \( \lambda_1(t) \to \infty \) as \( t \to \infty \). At the same time, the random variable \( N_2(t) \) of losses in the time interval \([0,t]\) is a nonnegative integer-based process, and we also assume that \( \lambda_2(t) := EN_2(t) < \infty \) for any \( t > 0 \) and \( \lambda_2(t) \to \infty \) as \( t \to \infty \).

3. The severity processes and the frequency processes are assumed to be independent in the two risk cells. The dependence between cells is modelled by a general copula.

4. The aggregated loss processes are defined as

\[ S(t) := S_1(t) + S_2(t). \]

Notably, \( S(t) \) is the sum of the aggregated loss processes in different risk cells, but the relationship between them is not necessarily mutually independent. In fact, it may be completely dependent, comonotonic, positive/negative dependent, independent, etc. The research results also have applications in the normal copula, the t copula, and in an important kind of Archimedean copula, including the Clayton copula, the Ali-Mikhail-Haq copula, the Gumbel-Hougaard copula, the Frank copula, etc.

What needs to be emphasized is that the dependency structure considered in this article has important practical significance. For example, rising house prices will pull Chinese M2 (broad currency) up, and falling house prices to a certain extent will also pull Chinese M2 (broad currency) down. Then, we can choose some suitable copula to measure the dependence structure between the two. Similarly, this method is also applicable to measure the negative dependence between gold prices and the actual federal funds rate (federal
funds rate minus inflation rate). There are many such practical examples, so they will not be repeated here.

In the following, we specify that the symbol \( \lim f(x) \) denotes the upper limit and lower limit, respectively, i.e., \( \lim f(x) = \limsup f(x) \) and \( \liminf f(x) = \liminf f(x) \).

**Definition 1 (Regularly Varying Distribution Class):** Let \( X \) be a positive random variable with the distribution tail \( \bar{F}(x) := 1 - F(x) = P(X > x) \) for \( x > 0 \). If for some \( \alpha \geq 0 \),
\[
\lim_{x \to \infty} \frac{\bar{F}(x)}{\bar{F}(x)} = x^{-\alpha}, \quad x > 0,
\]
then, \( \bar{F} \) is called regularly varying with index \(-\alpha\), denoted by
\[
F \in \mathcal{R}_{-\alpha}.
\]

Notably, we assume that the random variables \( X_i, i = 1, 2, \ldots, n \) are positive. It is mainly because the random variables are selected to model the loss severity in the risk cell.

**Definition 2 (Extended Regular Variation (ERV) Class):** It can be defined that the cumulative distribution function (df for short) \( F \in \text{ERV}(\alpha, \beta) \) for \( 1 < \alpha \leq \beta < \infty \) if \( F \) satisfies that for any \( y > 1 \)
\[
y^{-\beta} \leq \liminf_{x \to \infty} \frac{F(x+y)}{F(x)} \leq \limsup_{x \to \infty} \frac{F(x+y)}{F(x)} \leq y^{-\alpha}.
\] (2:1)

Moreover, for any \( \nu > 1 \), this is equivalent to
\[
y^{\nu} \leq \liminf_{x \to \infty} \frac{F(x\nu)}{F(x)} \leq \limsup_{x \to \infty} \frac{F(x\nu)}{F(x)} \leq y^{\nu}.
\] (2:2)

**Definition 3 (Long-Tailed Distribution Class):** Let \( X \) be a positive random variable with the distribution tail \( \bar{F}(x) := 1 - F(x) = P(X > x) \) for \( x > 0 \). If for all \( y \in \mathbb{R} \),
\[
\lim_{x \to \infty} \frac{\bar{F}(x-y)}{\bar{F}(x)} = 1,
\]
then, \( \bar{F} \) is called long-tailed distribution, denoted by \( F \in \mathcal{L} \).

**Definition 4 (Subexponential Distribution):** Under the condition that \( F \in \mathcal{L} \), a distribution function \( F \) with \( F(x) = 0 \) for \( x < 0 \) is called subexponential (denoted by \( F \in \mathcal{S} \)) if
\[
\lim_{x \to \infty} \frac{1 - F^{*2}(x)}{1 - F(x)} = 2
\]
where \( F^{*2} \) is the convolution of \( F \) with itself.

Notably, if we assume that \( \{X_k, k \in \mathbb{N}\} \) are i.i.d. random variables with df \( F \) with \( F(x) = 0 \) for \( x < 0 \), then \( F \) is a subexponential distribution function \( (F \in \mathcal{S}) \) equivalent to the following expression that holds, i.e.,
\[
\lim_{x \to \infty} \frac{P(X_1 + X_2 + \cdots + X_n > x)}{P(\max(X_1, X_2, \cdots, X_n) > x)} = 1
\]
for some \( n \geq 2 \).

Examples of subexponential distributions are Pareto, Weibull and Lognormal.

**Condition 5 (Assumption A):** Suppose that \( \{N(t), t \geq 0\} \) satisfies
\[
E[(N(t))^{\beta+1}I_{(N(t))>(1+\delta)\lambda(t))}] = O(\lambda(t))
\]
for any fixed \( \delta > 0 \) and some small \( \epsilon = \epsilon(\delta) > 0 \).

**Condition 6 (Assumption B):** Suppose that \( \{N(t), t \geq 0\} \) satisfies
\[
\sum_{n=0}^{\infty} (1 + \epsilon)^n P(N(t) = n) < \infty
\]
for some \( \epsilon > 0 \).

### III. MAIN RESULTS AND SEVERAL IMPORTANT EXAMPLES

Considerable effort has been devoted to the study of asymptotic properties, in particular by means of Value-at-Risk (VaR), and the definition is as follows: Let \( G_t \) be the distribution function of the aggregated loss process \( S(t) \) in a univariate LDA model. The operational value at risk until time \( t \) is expressed as \( VaR_t(k) = \inf \{x \in \mathbb{R} : G_t(x) \geq k\} \).

A well-known shortcoming of using VaR as the risk limit is that it stimulates concentration of risk because it is insensitive to the actual level of the worst losses that can be ignored under a given confidence level. The aim of this paper, however, is not primarily related to the controversy of value-at-risk vs. expected shortfall. For more details on Expected Shortfall, see e.g. 

**A. MAIN RESULTS**

**Theorem 7:** Assume that the aggregated loss processes \( S_1(t), S_2(t) \) in the two risk cells are dependent. In addition, their dependence structure is estimated by an arbitrary absolutely continuous copula function \( C(a, b) \) whose partial derivative \( c_{ab}(a, b) := \frac{\partial C(a, b)}{\partial a} \) exists. Assume that \( F \in \text{ERV}(\alpha_1, \beta_1) \) for \( 1 < \alpha_1 \leq \beta_1 < \infty \); we also suppose that \( N_1(t) \) satisfies Assumption A in Condition 5. Then, for some \( r > 0 \), the following limit exists for all sufficiently large \( x \) values, i.e.,
\[
P(S_1(t) + S_2(t) > x)
\]
\[
\sim \lambda_1(t) \bar{F}(x) - E(min(N_1(t), N_2(t)))
\]
\[
\times \int_0^x (-1 + c_{ab}(F(z), G(x-z)))F(dz)].
\]

Notably, we assume that \( N_1(t) \) satisfies Assumption A in Condition 5 but \( N_2(t) \) need not satisfy this assumption. It is mainly because the distribution function of aggregated loss processes \( S_i(t) \) is the object for comparison. By this condition, one can deduce the important result (serving as a connecting link between the preceding and the following), i.e., for any fixed \( \gamma > 0 \),
\[
P(S_1(t) > x) \sim \lambda_1(t) \bar{F}(x)
\]
holds uniformly for \( x \geq \gamma \lambda_1(t) \).

It should also be pointed out that the condition requirements about an arbitrary absolutely continuous copula function \( C(a, b) \) in Theorem 7 are very weak, and they are widely
Corollary 9: Assume that the aggregated loss processes $S_1(t), S_2(t)$ in the two risk cells are dependent. Furthermore, their dependence structure is estimated by an arbitrary absolutely continuous copula function $C(a, b)$ whose partial derivative $c_a(a, b) := \frac{\partial C(a, b)}{\partial a}$ exists. In addition, we suppose that $N_1(t)/\lambda_1(t) \xrightarrow{p} 1$ as $t \to \infty$ and there exist $\epsilon, \delta > 0$ such that $\lambda(t) \to \infty$ and

$$\sum_{k>(1+\delta)\lambda_1(t)} (1+\epsilon)^k P(N(t) > k) \to 0, \quad t \to \infty.$$  

Suppose that $F$ belongs to one of the following classes: $ERV(-\alpha_1, -\beta_1)$ for $1 < \alpha_1 \leq \beta_1 < \infty$, $LN(\gamma)$, or $WE(\alpha)$ with $\alpha < 0.5$. Then, as $t \to \infty$, for sufficiently large $x$, the results of Theorem 7 and Corollary 8 also hold.

Theorem 10: Assume that $F \in \mathcal{L}$ is absolutely continuous with density $f(x)$, $G$ is absolutely continuous with density $g(y)$, and let $S_1(t)$ and $S_2(t)$ be dependent according to an absolutely continuous copula function $C(a, b)$ where $c_{ab}(a, b) := \frac{\partial C(a, b)}{\partial a} < \infty$ is continuous at $b=1$ a.s. (with respect to the Lebesgue measure). In addition, suppose that $N_1(t)$ satisfies Assumption A in Condition 5 and $\hat{c} = \lim \inf_{x \to \infty} \frac{g(x)}{f(x)} < \infty$ exists. Then,

$$\lim \inf_{x \to \infty} \frac{P(S_1(t) + S_2(t) > x)}{F(x)} \geq \lambda_1(t) \left(1 + E(\min(N_1(t), N_2(t))) \cdot \hat{c} \cdot \int_0^\infty c_{ab}(F(z), 1)F(dz)\right).$$

Theorem 11: Assume that $F \in \mathcal{S}$ is absolutely continuous with density $f(x)$, $G$ is absolutely continuous with density $g(y)$, and let $S_1(t)$ and $S_2(t)$ be dependent according to an absolutely continuous copula function $C(a, b)$, and there exists $M > 0$ such that $c_{ab}(a, b) < M$ for all $(a, b) \in [0, 1] \times [b_0, 1]$ with $b_0 < 1$. In addition, suppose that $N_1(t)$ satisfies Assumption B in Condition 6 and $\hat{c} = \lim \inf_{x \to \infty} \frac{g(x)}{f(x)} < \infty$ exists. Then,

$$\lim_{x \to \infty} \frac{P(S_1(t) + S_2(t) > x)}{F(x)} = \lambda_1(t) \left(1 + E(\min(N_1(t), N_2(t))) \cdot \hat{c} \cdot \int_0^\infty c_{ab}(F(z), 1)F(dz)\right).$$

Corollary 12: Under the conditions of Theorem 11, one can obtain that for large $k \to 1$,

$$VaR_k(k) \sim F^-(1 - \frac{1 - k}{\lambda_1(t) \times l}),$$

where $l$ is equal to

$$1 + E(\min(N_1(t), N_2(t)))\hat{c} \int_0^\infty c_{ab}(F(z), 1)F(dz).$$

Theorem 13: Assume that $S_1(t)$ and $S_2(t)$ are dependent random variables with absolutely continuous marginal distributions. In addition, let $S_1(t)$ and $S_2(t)$ be dependent according to an absolutely continuous copula function $C(a, b)$, and
there exist $M > 0$ and $x_0 < 1$ such that $c_{ab}(a, b) < M$ for all $(a, b) \in [x_0, 1] \times [x_0, 1]$. Then,

$$\lim_{x \to \infty} P(S_2(t) > x | S_1(t) > x) = 0.$$  

**Theorem 14:** Assume that $F \in S$ and $G \in S$. In addition, $S_1(t)$ and $S_2(t)$ are dependent random variables according to an absolutely continuous copula function $C(a, b)$, and there exist $0 < M < \infty$ and $x_0 < 1$ such that $c_{ab}(a, b) < M$ for all $(a, b) \in [x_0, 1] \times [x_0, 1]$. Moreover, we suppose that

$$c = \lim_{x \to \infty} \frac{\tilde{G}(x)}{\tilde{F}(x)} < \infty$$

exists. In addition, we suppose that $\{N_i(t), t \geq 0 \} \ (i = 1, 2)$ satisfy Assumption B in Condition 6. Then, for sufficiently large $x$

$$P(S_1(t) + S_2(t) > x) \sim \tilde{F}(x)[\lambda_1(t) + c\lambda_2(t)].$$

**Corollary 15:** Under the conditions of Theorem 14, one can obtain that

$$\text{VaR}_t(k) \sim F^{-1}(1 - \frac{1 - k}{\lambda_1(t) + c\lambda_2(t)}), \ k \to 1.$$  

**B. SEVERAL IMPORTANT EXAMPLES**

**Example 16:** If the loss severities in the first risk cell are Pareto distributed, i.e., with the distribution function $F(x) = 1 - (1 + \frac{x}{\theta})^{-\alpha}, \ \alpha, \theta, x > 0.$ Then, $\tilde{F}$ is regularly varying with index $-\alpha$ and has an ultimately decreasing Lebesgue density; hence, $\tilde{F}$ is also a subexponential distribution. Under the conditions of Theorem 11, we can obtain for large $k$

$$\text{VaR}_t(k) \sim \theta[(\frac{\lambda_1(t)}{1 - k})^{\frac{1}{\alpha}} - 1].$$

where $l$ is equal to

$$1 + E(\min(N_1(t), N_2(t)))\tilde{F}(x) \int_{\theta}^{\infty} c_{ab}(F(z), 1)F(dz).$$

**Example 17:** Let

$$F(x) = 1 - (1 + \frac{x^\tau}{\theta})^{-\alpha}, \ \tau, \alpha, \theta, x > 0$$

be the Burr distribution function. Then, $\tilde{F} \in R_{-\alpha \tau}$ because for $t > 0$

$$\lim_{x \to \infty} \frac{\tilde{F}(xt)}{\tilde{F}(x)} = \lim_{x \to \infty} \frac{\theta + (xt)\tau}{\theta + x^\tau} = t^{-\alpha \tau}.$$

Then, under the conditions of Theorem 14, we can obtain that, if $\alpha \tau > 1$

$$\text{VaR}_t(k) \sim [\theta(\frac{\lambda_1(t) + c\lambda_2(t)}{1 - k})^{\frac{1}{\alpha \tau}} - 1]^{\frac{1}{\alpha \tau}} , \ k \to 1.$$  

**IV. PROOF OF MAIN RESULTS**

**Proof of Theorem 18:** From Proposition 2.3 in Albrecher [32], we can obtain that

$$\tilde{F}(S_1(t)) = 1 + \int_0^t [1 - c_\alpha(F_S(z), F_S(z - x))]F_S(dz),$$

(4.1)

where $F_S(x)$ and $F_S(x)$ stand for the distribution function of aggregated loss processes $S_1(t)$ and $S_2(t)$.

According to Theorem 1.1 in Su et al. [33], we can derive that for any fixed $\gamma > 0,$

$$P(S_1(t) > x) \sim \lambda_1(t)\tilde{F}(x)$$

holds uniformly for $x \geq \gamma \lambda_1(t).$ Then, for any $x \geq \gamma \lambda_1(t)$

$$P(S_1(t) + S_2(t) > x) \sim \lambda_1(t)\tilde{F}(x)$$

In fact, $\tilde{F}(S_1(z), F_S(z - x))F_S(dz)$

$$\sim 1 + \int_0^t [1 - c_\alpha(F_S(z), F_S(z - x))]F_S(dz) \lambda_1(t)\tilde{F}(x).$$

Then, we only need to consider how to simplify $1 - c_\alpha(F_S(z), F_S(z - x)).$ In fact,

$$C_\alpha(F_S(z), F_S(z - x)) = F_S(z) + F_S(z - x)$$

holds for sufficiently large $z$ and larger $x$.

According to Theorem 2.5 in Omey [34], we can conclude, under appropriate conditions, that

$$P(S_1(t) > z, S_2(t) > x-z) \sim E(\min(N_1(t), N_2(t))) \cdot P(X > z, Y > x-z)$$

for sufficiently large $z$ and larger $x$.

Then, one can derive that

$$C_\alpha(F_S(z), F_S(z - x))$$

$$= \frac{\partial}{\partial \alpha} C(F_S(z), F_S(z - x))$$

$$= 1 + \frac{\partial}{\partial \alpha} P(S_1 > z, S_2 > x-z)$$

$$\sim 1 + E(\min(N_1(t), N_2(t))) \frac{\partial}{\partial \alpha} P(X > z, Y > x-z)$$

$$= 1 + E(\min(N_1(t), N_2(t))) \times$$

$$\times \frac{\partial}{\partial \alpha} [\tilde{F}(z) + \tilde{G}(z - x) - 1 + C(F(z), G(x - z))]$$

$$= 1 + E(\min(N_1(t), N_2(t)))[1 - c_\alpha(F(z), G(x - z))]$$

for sufficiently large $z$ and larger $x.$ Hence, we can prove the result, i.e.,

$$P(S_1 + S_2 > x) \sim \lambda_1(t)\tilde{F}(x) + \int_0^x [1 - c_\alpha(F_S(z), F_S(z - x))]F_S(dz)$$

$$= \lambda_1(t)[\tilde{F}(x) - E(\min(N_1(t), N_2(t))) \times$$

$$\times \int_0^x (-1 + c_\alpha(F(z), G(x - z)))F(dz)]$$

for sufficiently large $z$ and larger $x.$
Proof of Corollary 19:

VaR,(k)
\[
= \inf\{x \in R : P(S_1(t) + S_2(t) \leq x) > k\}
\]
= \inf\{x \in R : P(S_1(t) + S_2(t) > x) \leq 1 - k\}
\sim \inf\{x \in R : \lambda_1(t)
\times [\bar{F}(x) - E(\text{min}(N_1(t), N_2(t)))
\times \int_0^x (-1 + c_\theta(F(z), G(x - z)))F(dz)] \leq 1 - k\}
= \inf\{x \in R : \bar{F}(x) \leq \frac{1 - k}{\lambda_1(t)} + E(\text{min}(N_1(t), N_2(t)))
\times \int_0^x (-1 + c_\theta(F(z), G(x - z)))F(dz)\}
\]

Proof of Corollary 20: We refer the interested readers to Proposition 7.1 in Mikosch [35].

Proof of Theorem 21: From Theorem 7, Fatou’s lemma and L’Hôpital’s rule, one can derive that
\[
\lim_{x \to \infty} P(S_1(t) + S_2(t) > x)
= \lambda_1(t)[1 - E(\text{min}(N_1(t), N_2(t)))
\times \int_0^x (-1 + c_\theta(F(z), G(x - z)))F(dz)]
\geq \lambda_1(t)[1 - E(\text{min}(N_1(t), N_2(t)))
\times \int_0^x (-1 + c_\theta(F(z), G(x - z)))F(dz)]
= \lambda_1(t)[1 - E(\text{min}(N_1(t), N_2(t)))
\times \int_0^x c_\theta(F(z), 1) \lim_{x \to \infty} I_{[z \leq z]} g(x - z) f(x - z) f(x) F(dz)]
\]
for sufficiently large x.

Proof of Theorem 22: According to (4.1) in the proof of Theorem 7, one can derive that
\[
\lim_{x \to \infty} P(S_1(t) + S_2(t) > x)
= 1 + \lim_{x \to \infty} \int_0^x (-1 + c_\theta(F_1(z), F_2(z - x))) F(dz)
= 1 + E(\text{min}(N_1(t), N_2(t))) \cdot \lambda_1(t) \cdot \lim_{x \to \infty} \int_0^x (1 - c_\theta(F(z), G(x - z)))F(dz)
\]

Taking into account that for any fixed \( \gamma > 0 \)
P(\( S_1(t) > x \)) \sim \lambda_1(t)\bar{F}(x)
holds uniformly for \( x \geq \gamma \lambda_1(t) \), we can deduce that
\[
\lim_{x \to \infty} P(S_1(t) + S_2(t) > x)
= \lambda_1(t)[1 + E(\text{min}(N_1(t), N_2(t)))
\times \int_0^x (-1 + c_\theta(F(z), G(x - z)))F(dz)]
\]
Last, we can deduce that
\[
\int_0^{\infty} g(x)F(dx) \to \hat{c} \int_0^{\infty} c_{ab}(F(z), 1)F(dx)
\]
as \(x \to \infty\), and
\[
\int_0^{\infty} G(x)F(dx) \to \max\{\frac{1}{1 - b_0}, M\} \cdot c, \ x \to \infty.
\]
According to Pratt’s lemma, we can derive that
\[
\lim_{x \to \infty} \frac{1 - c_{ab}(F(z), G(x - z))}{F(x)} \cdot F(dx) \leq \hat{c} \int_0^{\infty} c_{ab}(F(z), 1)F(dx).
\]
Hence, we have completed the proof.

**Proof of Corollary 23:**

\(VaR(k)\)

\[
= \inf\{x \in R : P(S_1(t) + S_2(t) \leq x) > k\}
\]
\[
= \inf\{x \in R : P(S_1(t) + S_2(t) > x) \leq 1 - k\}
\]
\[
\sim \inf\{x \in R : \lambda_1(t) \tilde{F}(x)\{1 + E(\min(N_1(t), N_2(t)))
\]
\[
\times \hat{c} \int_0^{\infty} c_{ab}(F(z), 1)F(dx) \leq 1 - k\}
\]
\[
= \inf\{x \in R : \tilde{F}(x) \leq \frac{1 + k}{\lambda_1(t)}
\]
\[
\cdot \frac{1}{\{1 + E(\min(N_1(t), N_2(t)))\hat{c} \int_0^{\infty} c_{ab}(F(z), 1)F(dx)\}}\}
\]
\[
= \inf\{x \in R : F(x) \geq \frac{1 - k}{\lambda_1(t)}
\]
\[
\cdot \frac{1}{\{1 + E(\min(N_1(t), N_2(t)))\hat{c} \int_0^{\infty} c_{ab}(F(z), 1)F(dx)\}}\}
\]
\[
= F^{-1}(1 - \frac{1 - k}{\lambda_1(t)}
\]
\[
\cdot \frac{1}{\{1 + E(\min(N_1(t), N_2(t)))\hat{c} \int_0^{\infty} c_{ab}(F(z), 1)F(dx)\}}\}
\]

**Proof of Theorem 24:** Assume that \(S_1^*\) and \(S_2^*\) are independent and that they are identically distributed as \(S_1\) and \(S_2\), respectively. When \(\min(F_{S_1}, F_{S_2}) > x_0\), we can deduce that
\[
\lim_{x \to \infty} P(S_2(t) > x|S_1(t) > x)
\]
\[
= \lim_{x \to \infty} \frac{1}{P(S_1(t) > x)}
\]
\[
\cdot \int_x^{\infty} \int_x^{\infty} c_{ab}(F_{S_1}(u_1), F_{S_2}(u_2))F_{S_1}(du_1)F_{S_2}(du_2)
\]
\[
\leq M \lim_{x \to \infty} \frac{1}{P(S_1(t) > x)} \int_0^{\infty} \int_0^{\infty} F_{S_1}(du_1)F_{S_2}(du_2)
\]
\[
= M \lim_{x \to \infty} P(S_1^* > x)
\]
\[
= M \lim_{x \to \infty} P(S_2^* > x)
\]
\[
= 0.
\]

**Proof of Theorem 25:** Taking account of Theorem 13 and that
\[
P(S_1 + S_2 > x)
\]
\[
\geq P(\max\{S_1, S_2\} > x)
\]
\[
= P(S_1 > x) + P(S_2 > x) - P(S_1 > x, S_2 > x)
\]
\[
= \tilde{F}_{S_1}(x) + \tilde{F}_{S_2}(x) - \tilde{F}_{S_1}(x)P(S_2 > x|S_1 > x),
\]
one can derive that
\[
\lim_{x \to \infty} \frac{P(S_1(t) + S_2(t) > x)}{\tilde{F}_{S_1}(x)} \geq 1 + \lim_{x \to \infty} \frac{\tilde{F}_{S_2}(x)}{\tilde{F}_{S_1}(x)}.
\]
On the other hand, if we can prove that
\[
\lim \sup_{x \to \infty} \frac{P(S_1(t) + S_2(t) > x)}{\tilde{F}_{S_1}(x)} \leq 1 + \lim_{x \to \infty} \frac{\tilde{F}_{S_2}(x)}{\tilde{F}_{S_1}(x)},
\]
then we can obtain that as \(x \to \infty\)
\[
\lim \inf_{x \to \infty} \frac{P(S_1(t) + S_2(t) > x)}{\tilde{F}_{S_1}(x)} = 1 + \lim_{x \to \infty} \frac{\tilde{F}_{S_2}(x)}{\tilde{F}_{S_1}(x)}.
\]

By means of Theorem 1.3.9 in Embrechts et al. [37], together with \([N_i(t), t \geq 0]\) \((i = 1, 2)\) satisfying Assumption B, we can derive that for sufficiently large \(x\)
\[
P(S_1(t) + S_2(t) > x) \sim \tilde{F}(x)[\lambda_1(t) + c\lambda_2(t)].
\]
In fact, we suppose that \(a(x)\) denotes a function with \(a(x) \to \infty\) \((x \to \infty)\) and that
\[
\lim_{x \to \infty} \frac{P(S_1 > x - a(x))}{P(S_1 > x)} = 1.
\]
Then, we have that
\[
P(S_1 + S_2 > x)
\]
\[
\leq P(S_1 > x - a(x) \cup S_2 > x - a(x))
\]
\[
+ P(S_1 + S_2 > x, \max(S_1, S_2) \leq x - a(x)).
\]
Assume that \(S_1^*\) and \(S_2^*\) are independent and that they are identically distributed as \(S_1\) and \(S_2\), respectively. When
\[
\inf_{y > x} F_{S_i}(a(y)) > x_0, \quad i = 1, 2,
\]
we can deduce that
\[
P(S_1 + S_2 > x, \max(S_1, S_2) \leq x - a(x))
\]
\[
= \int_{S_1 + S_2 > x, \max(S_1, S_2) \leq x - a(x)} c_{ab}(F_{S_1}(u_1), F_{S_2}(u_2))F_{S_1}(du_1)F_{S_2}(du_2)
\]
\[
\leq M \int_{S_1 + S_2 > x, \max(S_1, S_2) \leq x - a(x)} 1F_{S_1}(du_1)F_{S_2}(du_2)
\]
\[
\leq M \cdot P(S_1^* + S_2^* > x, \max(S_1^*, S_2^*) \leq x).
\]
Considering that
\[
S_1 = \sum_{i=1}^{N_i(t)} X_i \in \mathcal{S}
\]
according to \(F \in \mathcal{S}\), we can deduce that
\[
\lim_{x \to \infty} \frac{P(S_1 + S_2 > x, \max(S_1, S_2) \leq x - a(x))}{P(S_1 > x)} = 0.
\]
On the other hand, one can derive that
\[
\lim_{x \to \infty} \frac{P(S_1 > x - a(x) \cup S_2 > x - a(x))}{P(S_1 > x)} = 0.
\]
that greatly improved the manuscript. Examination, favourable suggestions and some corrections. Many thanks are given to the anonymous reviewers for careful performance of forward-looking issues, including convergence rate estimation, Monte Carlo simulation, etc.

V. CONCLUSION
This article is primarily about estimating the probability of compound stochastic sums of heavy-tailed risks over a large threshold. Notably, this article mainly presents two innovations to compensate for the difficulties encountered in the literature: one is using a heavy-tailed distribution to simulate risks in compound stochastic sums, and the other is using a very general copula function to model the dependence of the aggregated processes.

It is noteworthy that the research results of this thesis present a deep insight into estimating the probability of compound stochastic sums of heavy-tailed risks; hence, these research results have important theoretical meaning and practical value for risk management and control. On the one hand, regularly varying functions contain a large number of common functions for, instance, the functions $\log(1 + x)$, $\log\log(e + x)$, the extreme-value distribution

$$
\Phi_\alpha(x) = \exp(-x^{-\alpha}), \quad x \geq 0,
$$

the Cauchy distribution, and even more generally, stable Lévy processes with an index $\alpha$, $0 < \alpha < 2$. Thus, the theory of regularly varying functions is an essential analytical tool for dealing with heavy tails, long-range dependence and domains of attraction. On the other hand, the research results of this article have an enlightening and pioneering role that can be used not only in tail risk analysis of a heavy tail random sum but also in the netput process of the M/G/1 workload process, the determination of risk measures for a given portfolio of risks, the number of claims on a company during a given time interval in the classical risk model, the estimation of credit risk, the operational losses in a given risk type/business line cell in a certain bank, etc. Furthermore, interested readers can further study the following questions, including estimating the probability of compound stochastic sums of heavy-tailed risks over a large threshold, exceedance times, the Hill estimator, the Pickands estimator, quantile-quantile plots, etc.

Due to space limitations, only two specific examples are provided for the results of this article to prove its wide application value. In future research, we need to further study the performance of forward-looking issues, including convergence rate estimation, Monte Carlo simulation, etc.

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