On $p$-adic Gibbs measures of the countable state Potts model on the Cayley tree

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Abstract

In this paper we consider the countable state $p$-adic Potts model on the Cayley tree. A construction of $p$-adic Gibbs measures which depends on weights $\lambda$ is given, and an investigation of such measures is reduced to the examination of an infinite-dimensional recursion equation. By studying the derived equation under some condition concerning weights, we prove the absence of a phase transition. Note that the condition does not depend on values of the prime $p$, and the analogous fact is not true when the number of spins is finite. For the homogeneous model it is shown that the recursive equation has only one solution under that condition on weights. This means that there is only one $p$-adic Gibbs measure $\mu_\lambda$. The boundedness of the measure is also established. Moreover, the continuous dependence of the measure $\mu_\lambda$ on $\lambda$ is proved. At the end we formulate a one limit theorem for $\mu_\lambda$.

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1. Introduction

Since the 1980s, various models described in the language of $p$-adic analysis have been actively studied [5,22,23,47,62]. More precisely, models defined over the field of $p$-adic numbers have

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been considered. This is due to the assumption that \( p \)-adic numbers provide a more exact and more adequate description of microworld phenomena. The well-known studies in this area are primarily devoted to investigating quantum mechanics models using equations of mathematical physics [4, 48, 52, 61, 63]. One of the first applications of \( p \)-adic numbers in quantum physics appeared in the framework of quantum logic in [10, 46]. This model is especially interesting for us because it could not be described by using a conventional real valued probability. Furthermore, numerous applications of the \( p \)-adic analysis to mathematical physics have been proposed in [8, 32, 33, 52]. Besides, it is also known [33, 42, 47, 52, 54, 60, 61] that a number of \( p \)-adic models in physics cannot be described using ordinary Kolmogorov’s probability theory. New probability models, namely, the \( p \)-adic ones, were investigated in [15, 17, 31, 40]. After that, in [41], an abstract \( p \)-adic probability theory was developed by means of the theory of non-Archimedean measures [54]. Using that measure theory in [39, 45] the theory of stochastic processes with values in \( p \)-adic and more general non-Archimedean fields having probability distributions with non-Archimedean values has been developed. In particular, a non-Archimedean analogue of the Kolmogorov theorem was proven (see also [25]). Such a result allows us to construct large classes of stochastic processes using finite dimensional probability distributions. We point out that stochastic processes on the field \( \mathbb{Q}_p \) of \( p \)-adic numbers have been studied by many authors, for example [1, 2, 3, 18, 30, 43, 64]. In these investigations large classes of Markov processes on \( \mathbb{Q}_p \) were constructed and studied. Such studies, therefore, give a possibility of developing the theory of statistical mechanics in the context of \( p \)-adic theory, since it is on the basis of the theory of probability and stochastic processes. Note that one of the central problems of such a theory is the study of infinite-volume Gibbs measures corresponding to a given Hamiltonian and a description of the set of such measures. In most cases such an analysis depends on specific properties of the Hamiltonian, and a complete description is often a difficult problem. This problem, in particular, relates to a phase transition of the model (see [27]).

In [36, 37] a notion of ultrametric Markovianity, which describes the independence of contributions to the random field from different ultrametric balls, was introduced, which shows that Gaussian random fields on general ultrametric spaces (which were related to hierarchical trees), which were defined as a solution of the pseudodifferential stochastic equation (see also [29]), satisfy the Markovianity. In addition, covariation of the defined random field was computed with the help of the wavelet analysis on ultrametric spaces (see also [44]). Some applications of the results to replica matrices, related to general ultrametric spaces, have been investigated in [38].

In this paper we develop a \( p \)-adic probability theory approach to study the countable state of nearest-neighbour Potts models on a Cayley tree (see [9]) over the \( p \)-adic field. We are especially interested in the construction of the \( p \)-adic Gibbs measure for the mentioned model. Such measures present more natural concrete examples of the \( p \)-adic Markov processes (see [37, 39] for definitions). When states were finite, say \( q \), then the corresponding \( p \)-adic \( q \)-state Potts models on the same tree were studied in [49–51]4. It was established that a phase transition occurs if \( q \) is divisible by \( p \). This shows that the transition depends on the number of spins \( q \). To establish such a result we investigated fixed points of \( p \)-adic dynamical systems associated with that model. We remark that first investigations of non-Archimedean dynamical systems have appeared in [28]. We also point out that intensive development of \( p \)-adic (and more general algebraic) dynamical systems has happened in the last few years, see [6, 7, 11, 14, 40, 56, 59, 65]. More extensive lists may be found in the \( p \)-adic dynamics bibliography maintained by Silverman [57] and the algebraic dynamics bibliography of Vivaldi [60].

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4 The classical (real value) counterparts of such models were considered in [26, 66].
The aim of this paper is to give a sufficient condition for the uniqueness of \( p \)-adic Gibbs measures of the countable state Potts model and to study such measures. Note that in comparison with a real case, in a \( p \)-adic setting, the existence of such measures for the model is not known \textit{a priori}, since there is not much information on the topological properties of the set of all \( p \)-adic measures defined even on compact spaces. However, in the real case, there is the so-called Dobrushin’s theorem [19, 20] which gives a sufficient condition for the existence of the Gibbs measure for a large class of Hamiltonians.

The paper is organized as follows. After the preliminaries (section 2), in section 3 we define our model and give a construction of \( p \)-adic Gibbs measures which depend on weight \( \lambda \). Using the Kolmogorov extension theorem [39], an investigation of such measures is reduced to the examination of an infinite-dimensional recursion equation. In section 4, by studying the derived equation under some condition on weights, we prove the absence of a phase transition. Note that for the real counterparts of the model, such results are unknown (see [24, 26]). It turns out that the founding condition does not depend on values of the prime \( p \), and therefore, the analogous fact is not true when the number of spins is finite. In section 5, we consider a homogeneous \( p \)-adic Potts model and show under the condition formulated in section 3 that the recursive equation has only one solution. Hence, there is only one \( p \)-adic Gibbs measure \( \mu_\lambda \). Then we establish boundedness one. In section 6, we prove the continuous dependence of the measure \( \mu_\lambda \) on \( \lambda \). We also prove the one limit theorem for \( \mu_\lambda \). The last section is devoted to the conclusions.

2. Preliminaries

In what follows \( p \) will be a fixed prime number, and \( \mathbb{Q}_p \) denotes the field of the \( p \)-adic, formed by completing \( \mathbb{Q} \) with respect to the unique absolute value satisfying \( |p| = 1/p \). The absolute value \( |\cdot|_p \) is non-Archimedean, meaning that it satisfies the ultrametric triangle inequality

\[ |x + y|_p \leq \max\{|x|_p, |y|_p\}. \]

Given \( a \in \mathbb{Q}_p \) and \( r > 0 \), put

\[ B(a, r) = \{x \in \mathbb{Q}_p : |x - a|_p < r\}. \]

The \( p \)-adic \textit{logarithm} is defined by the series

\[ \log_p(x) = \log_p(1 + (x - 1)) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(x - 1)^n}{n}, \]

which converges for \( x \in B(1, 1) \); the \( p \)-adic \textit{exponential} is defined by

\[ \exp_p(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \]

which converges for \( x \in B(0, p^{-1/(p-1)}) \).

\textbf{Lemma 2.1 ([42,55])}. Let \( x \in B(0, p^{-1/(p-1)}) \) then we have

\[ |\exp_p(x)|_p = 1, \quad |\exp_p(x) - 1|_p = |x|_p, \quad |\log_p(1 + x)|_p = |x|_p, \]

\[ \log_p(\exp_p(x)) = x, \quad \exp_p(\log_p(1 + x)) = 1 + x. \] (2.1)

Note that the basics of \( p \)-adic analysis and \( p \)-adic mathematical physics are explained in [42,55,61].
Let \((X, \mathcal{B})\) be a measurable space, where \(\mathcal{B}\) is an algebra of subsets \(X\). A function \(\mu : \mathcal{B} \to \mathbb{Q}_p\) is said to be a \(p\)-adic measure if for any \(A_1, \ldots, A_n \in \mathcal{B}\) such that \(A_i \cap A_j = \emptyset\) \((i \neq j)\) the equality holds for
\[
\mu \left( \bigcup_{j=1}^{n} A_j \right) = \sum_{j=1}^{n} \mu(A_j).
\]

A \(p\)-adic measure is called a probability measure if \(\mu(X) = 1\). A \(p\)-adic probability measure \(\mu\) is called bounded if \(\sup\{|\mu(A)|_p : A \in \mathcal{B}\} < \infty\). Note that in general, a \(p\)-adic probability measure need not be bounded \([31, 39, 42]\). For more detailed information about \(p\)-adic measures we refer to \([31, 40, 54]\).

Recall that the Cayley tree (see \([9]\)) \(\Gamma^k\) of order \(k \geq 1\) is an infinite tree, i.e. a graph without cycles, from each vertex of which exactly \(k\) edges are issued. Let \(\Gamma^k = (V, L)\), where \(V\) is the set of vertices of \(\Gamma^k\) and \(L\) is the set of edges of \(\Gamma^k\). The vertices \(x\) and \(y\) are called nearest neighbours and they are denoted by \(l = (x, y)\) if there exists an edge connecting them. A collection of the pairs \((x, x_1), \ldots, (x_{d-1}, y)\) is called a path from the point \(x\) to the point \(y\). The distance \(d(x, y), x, y \in V\), on the Cayley tree, is the length of the shortest path from \(x\) to \(y\). Now fix \(x^0 \in V\) and set
\[
W_n = \{x \in V : d(x, x^0) = n\}, \quad V_n = \bigcup_{m=1}^{n} W_m, \quad L_n = \{l = (x, y) \in L : x, y \in V_n\}.
\]

The set of direct successors of \(x\) is defined by
\[
S(x) = \{y \in W_{n+1} : d(x, y) = 1\}, \quad x \in W_n.
\]
Observe that any vertex \(x \neq x^0\) has \(k\) direct successors and \(x^0\) has \(k + 1\).

3. The \(p\)-adic Potts model and \(p\)-adic Gibbs measures

We consider the \(p\)-adic Potts model where the spin takes values in the set \(\Phi = \{0, 1, 2, \ldots\}\) (\(\Phi\) is called a state space) and is assigned to the vertices of the tree \(\Gamma^k = (V, L)\). A configuration \(\sigma\) on \(V\) is then defined as a function \(x \in V \to \sigma(x) \in \Phi\); in a similar manner one defines configurations \(\sigma_n\) and \(\omega\) on \(V_n\) and \(W_n\), respectively. The set of all configurations on \(V\) (respectively, \(V_n, W_n\)) coincides with \(\Omega = \Phi^V\) (respectively, \(\Omega_{V_n} = \Phi^{V_n}, \Omega_{W_n} = \Phi^{W_n}\)). One can see that \(\Omega_{V_n} = \Omega_{V_{n-1}} \times \Omega_{W_{n-1}}\). Using this, for given configurations \(\sigma_{n-1} \in \Omega_{V_{n-1}}\) and \(\omega \in \Omega_{W_n}\), we define their concatenations by
\[
(\sigma_{n-1} \vee \omega)(x) = \begin{cases} 
\sigma_{n-1}(x), & \text{if } x \in V_{n-1}, \\
\omega(x), & \text{if } x \in W_n.
\end{cases}
\]
It is clear that \(\sigma_{n-1} \vee \omega \in \Omega_{V_n}\).

The Hamiltonian \(H_n : \Omega_{V_n} \to \mathbb{Q}_p\) of the inhomogeneous \(p\)-adic countable state Potts model has the form
\[
H_n(\sigma) = \sum_{(x,y) \in L_n} J_{x,y} \delta_{\sigma(x), \sigma(y)}, \quad \sigma \in \Omega_{V_n}, \quad n \in \mathbb{N},
\]
where \(\delta\) is the Kronecker symbol and the coupling constants \(J_{x,y}\) are taken from \(\mathbb{Q}_p\) with the constraint
\[
|J_{x,y}|_p < \frac{1}{p^{V(1/p)-1}}, \quad \forall (x, y) \in L_n. \tag{3.2}
\]
Note that such a condition provides the existence of a \(p\)-adic Gibbs measure (see (3.5)).
that \( \lambda(\Omega_1) \) is defined on \( \Omega_1 \), which gives a sufficient condition for the existence of the Gibbs measure for a large class of probability measures (i.e. Prohorov's theorem). When the state space is non-compact, then there is Dobrushin's theorem [19, 20] which gives a sufficient condition for the existence of the Gibbs measure for a large class of Hamiltonians. In [24], using that theorem, the existence of the Gibbs measure for the real counterpart of the studied Potts model has been established. It should be noted that there are even nearest-neighbour models with a countable state space for which the Gibbs measure does not exist [58].

In the real case, when the state space is compact, then the existence follows from the compactness of the set of all measures defined even on compact spaces [5]. Therefore, at the moment, we can only use the so-called compatibility condition for the measures \( \mu_n^{(n)} \) is fixed such that \( \lambda(i) \neq 0 \).

Given \( n = 1, 2, \ldots \) a \( p \)-adic probability measure \( \mu_n^{(n)} \) on \( \Omega_1 \) is defined by

\[
\mu_n^{(n)}(\sigma) = \frac{1}{Z_n^{(k)}} \exp_p \left\{ H_n(\sigma) + \sum_{x \in W_n} h_{\sigma(x),x} \right\} \prod_{x \in V_n} \lambda(x).
\]

Here, \( \sigma \in \Omega_1 \), and \( Z_n^{(k)} \) is the corresponding normalizing factor called a partition function given by

\[
Z_n^{(k)} = \sum_{\sigma \in \Omega_n} \exp_p \left\{ H_n(\sigma) + \sum_{x \in W_n} h_{\sigma(x),x} \right\} \prod_{x \in V_n} \lambda(x).
\]

Now we want to define a \( p \)-adic probability measure \( \mu \) on \( \Omega \) such that it would be compatible with the defined ones \( \mu_n^{(n)} \), i.e.

\[
\mu(\sigma \in \Omega : |V_1(\sigma) = \sigma_n = \mu_n^{(n)}(\sigma_n)), \quad \text{for all } \sigma_n \in \Omega_1 \}, \quad n \in \mathbb{N}.
\]

In general, the existence of a measure such as \( \mu \) is a priori not known, since there is not much information on topological properties, such as compactness, of the set of all \( p \)-adic measures defined even on compact spaces\(^5\). Therefore, at the moment, we can only use the so-called compatibility condition for the measures \( \mu_n^{(n)} \), \( n \geq 1 \), i.e.

\[
\sum_{\omega \in \Omega_n} \mu_n^{(n)}(\sigma_{n-1} \land \omega) = \mu_n^{(n-1)}(\sigma_{n-1}),
\]

for any \( \sigma_{n-1} \in \Omega_{n-1} \) (cp [16]), which implies the existence of a unique \( p \)-adic measure \( \mu \) defined on \( \Omega \) with a required condition (3.7). Moreover, if the measures \( \mu_n^{(n)} \) are bounded, then \( \mu \) is also bounded. This assertion is known as the \( p \)-adic Kolmogorov extension theorem (see [25, 39]).

So if for some function \( h \) the measures \( \mu_n^{(n)} \) satisfy the compatibility condition, then there is a unique \( p \)-adic probability measure, which we denote by \( \mu_h \), since it depends on \( h \). Such a measure \( \mu_h \) is said to be a \( p \)-adic Gibbs measure corresponding to the \( p \)-adic Potts model. By \( S \) we denote the set of all such \( p \)-adic Gibbs measures. If there are at least two different \( p \)-adic

\(^5\) In the real case, when the state space is compact, then the existence follows from the compactness of the set of all probability measures (i.e. Prohorov’s theorem). When the state space is non-compact, then there is Dobrushin’s theorem [19, 20] which gives a sufficient condition for the existence of the Gibbs measure for a large class of Hamiltonians. In [24], using that theorem, the existence of the Gibbs measure for the real counterpart of the studied Potts model has been established. It should be noted that there are even nearest-neighbour models with a countable state space for which the Gibbs measure does not exist [58].
Gibbs measures in $S$, i.e. one can find two different functions $s$ and $h$ defined on $V \setminus \{x^0\}$ such that there exist the corresponding measures $\mu_s$ and $\mu_h$, which are different, then we say that a phase transition occurs for the model, otherwise, there is no phase transition.

Now one can ask: for what kind of functions $h$ would the measures $\mu^{(n)}_h$ defined by (3.5) satisfy the compatibility condition (3.8)? The following theorem gives an answer to this question.

**Theorem 3.1.** The measures $\mu^{(n)}_h$, $n = 1, 2, \ldots$ given by (3.5), satisfy the compatibility condition (3.8) if and only if for any $x \in V \setminus \{x(0)\}$ the following equation holds:

$$\hat{h}_{i,x} = \frac{\lambda(i)}{\lambda(0)} \prod_{y \in S(x)} F_i(\hat{h}_y; \theta_{x,y}), \quad i \in \mathbb{N}, \quad (3.9)$$

here and below $\theta_{x,y} = \exp_p(J_{x,y})$, a sequence $\hat{h} = \{\hat{h}_i\}_{i \in \mathbb{N}} \in \mathbb{Q}_p^\mathbb{N}$ is defined by $h = \{h_i\}_{i \in \mathbb{N}}$ as follows:

$$\hat{h}_i = \exp_p(h_i - h_0) \frac{\lambda(i)}{\lambda(0)}, \quad i \in \mathbb{N} \quad (3.10)$$

and mappings $F_i : \mathbb{Q}_p^\mathbb{N} \times \mathbb{Q}_p \to \mathbb{Q}_p$ are defined by

$$F_i(x; \theta) = \left(\theta - 1\right) x_i + \sum_{j=1}^{\infty} x_j + 1 - \sum_{j=1}^{\infty} x_j + \theta, \quad x = \{x_i\}_{i \in \mathbb{N}}, \quad \theta \in \mathbb{Q}_p, \quad i \in \mathbb{N}. \quad (3.11)$$

The proof consists of checking condition (3.8) for the measures (3.5) (cp [26]).

**Remark 3.1.** Note that thanks to the non-Archimedeanity of the norm $|\cdot|_p$ the series $\sum_{k=1}^{\infty} x_k$ converges iff the sequence $\{x_n\}$ converges to 0 (see [42, 55]). Therefore, from (3.10) one can see that condition (3.4) and $|\exp_p(x)|_p = 1$ imply that the series $\sum_{j=1}^{\infty} \hat{h}_{j,x}$ always converges and is finite.

### 4. Absence of the phase transition

In this section, under some condition on weights $\lambda$, we are going to prove the absence of the phase transition for model (3.1).

As pointed out from (3.10) and (3.4) we have $|\hat{h}_n|_p \to 0$ as $n \to \infty$. Therefore, let us consider the following space:

$$c_0 = \{x = \{x_n\}_{n \in \mathbb{N}} \in \mathbb{Q}_p^\mathbb{N} : |x_n|_p \to 0, \ n \to \infty\}$$

with a norm $\|x\| = \max_n |x_n|_p$ (see [54, 55] for more $p$-adic Banach spaces). Put

$$B = \{x \in c_0 : \|x\| < 1\}. \quad \text{Since the norm takes discrete values, the set } B \text{ coincides with } B = \{x \in c_0 : \|x\| \leq 1/p\}. \quad \text{Therefore, } B \text{ is a closed set.}$$

Let us for the sake of shortness, given a sequence $x = \{x_n\}_{n \in \mathbb{N}}$, denote

$$X := \sum_{j=1}^{\infty} x_j. \quad (4.1)$$

Then one can easily see that

$$|X - Y|_p = \left|\sum_{j=1}^{\infty} (x_j - y_j)\right|_p \leq \max_j |x_j - y_j| = \|x - y\|, \quad (4.2)$$

for any $x, y \in c_0$. 
Lemma 4.1. For the mapping $F_i$ given by (3.11) the following relations hold:

$$|F_i(x, \theta) - F_i(y, \theta)|_p \leq |\theta - 1|_p \|x - y\|_p,$$

(4.3)

$$|F_i(x, \theta) - 1|_p = |\theta - 1|_p.$$  

(4.4)

for every $x, y \in B$ and $i \in \mathbb{N}$.

Proof. Let $x, y \in B$, then from (3.11) with (4.2) we have

$$|F_i(x, \theta) - F_i(y, \theta)|_p = |((\theta - 1)x_i + X + 1)(Y + \theta) - ((\theta - 1)y_i + Y + 1)(X + \theta)|_p$$

$$= |\theta - 1|_p |x_i Y - y_i X + \theta(x_i - y_i) + X - Y|_p$$

$$= |\theta - 1|_p |(X + \theta)(x_i - y_i) + (1 - x_i)(X - Y)|_p$$

$$\leq |\theta - 1|_p \max\{|x_i - y_i|_p, |X - Y|_p\}$$

$$\leq |\theta - 1|_p \|x - y\|,$$

which proves (4.3); here we have used that $|X + \theta|_p = 1, |Y + \theta|_p = 1$. The next relation (4.4) is obvious. □

Let us first enumerate $S(x)$ for any $x \in V$ as follows:

$$S(x) = \{x^1, \ldots, x^k\};$$

here as before $S(x)$ is the set of direct successors of $x$ (see (2.3)). Using this enumeration one can rewrite (3.9) by

$$\hat{h}_{i,x} = \frac{\lambda(i)}{\lambda(0)} \prod_{m=1}^i F_i(\hat{h}_{x_m}; \theta_{x,x_m}), \quad i \in \mathbb{N}, \quad \text{for every } x \in V \setminus \{x^0\}. \quad (4.5)$$

Now we need an auxiliary fact.

Lemma 4.2. If $|a_i|_p \leq 1, |b_i|_p \leq 1, i = 1, \ldots, n$, then

$$\left| \prod_{i=1}^n a_i - \prod_{i=1}^n b_i \right|_p \leq \max_{i \leq i \leq n} \{ |a_i - b_i|_p \}.$$

Proof. We have

$$\left| \prod_{i=1}^n a_i - \prod_{i=1}^n b_i \right|_p \leq \left| a_1 \left( \prod_{i=2}^n a_i - \prod_{i=2}^n b_i \right) + (a_1 - b_1) \prod_{i=2}^n b_i \right|_p$$

$$\leq \max \left\{ |a_1 - b_1|_p, \left| \prod_{i=2}^n a_i - \prod_{i=2}^n b_i \right|_p \right\}$$

$$\leq \cdots$$

$$\leq \max_{i \leq i \leq n} \{ |a_i - b_i|_p \},$$

which is the assertion. □

So we can formulate the main result.

Theorem 4.3. Assume that a weight $\lambda$ satisfies the following condition:

$$\max_i \frac{\lambda(i)}{\lambda(0)} |i|_p < 1.$$  

(4.6)

Then there is no phase transition for the countable state $p$-adic Potts model (3.1), i.e. $|S| \leq 1$. 

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Proof. From (4.6) and (3.10) we see that all solutions of (4.5) belong to \( B \). Now let us assume that \( \hat{h} = \{ \hat{h}_x, x \in V \setminus \{ x^0 \} \} \), and \( \tilde{s} = \{ \tilde{s}_x, x \in V \setminus \{ x^0 \} \} \) be the two solutions of (4.5). Now fix an arbitrary vertex \( x \in V \setminus \{ x^0 \} \). Then (4.5) with lemmas 4.1 and 4.2 implies that

\[
\begin{align*}
\hat{h}_{i,x} - \hat{s}_{i,x} &= \left| \frac{\lambda(i)}{\lambda(0)} \right| \left| \prod_{m=1}^{k} F_i(\hat{h}_{x_m}; \theta_{x_m,x_{m-1}}) - \prod_{m=1}^{k} F_i(\tilde{s}_{x_m}; \theta_{x_m,x_{m-1}}) \right|_p \\
&\leq \left| \frac{\lambda(i)}{\lambda(0)} \right| \max_{1 \leq m \leq k} \{ |F_i(\hat{h}_{x_m}; \theta_{x_m,x_{m-1}})|_p - |F_i(\tilde{s}_{x_m}; \theta_{x_m,x_{m-1}})|_p \} \\
&\leq \max_{1 \leq m \leq k} \{ |\theta_{x_m,x_{m-1}} - 1|_p \| \hat{h}_{x_m} - \hat{s}_{x_m} \|_p \} \leq \frac{1}{p} \max_{1 \leq m \leq k} \{ \| \hat{h}_{x_m} - \tilde{s}_{x_m} \| \}.
\end{align*}
\]

Hence,

\[
\| \hat{h}_x - \tilde{s}_x \| \leq \frac{1}{p} \max_{1 \leq m \leq k} \{ \| \hat{h}_{x_m} - \tilde{s}_{x_m} \| \}. \tag{4.7}
\]

Now take an arbitrary \( \epsilon > 0 \) and \( n_0 \in \mathbb{N} \) such that \( 1/p^{n_0} < \epsilon \). Iterating (4.7) \( n_0 \) times one gets \( \| \hat{h}_x - \tilde{s}_x \| < \epsilon \). Therefore, the arbitrariness of \( \epsilon \) and \( x \) yields that \( \hat{h}_x = \tilde{s}_x \) for every \( x \in V \setminus \{ x^0 \} \). This means that \( |S| \leq 1 \) and completes the proof. \( \square \)

Remark 4.1. It is clear that condition (4.6) does not depend on the values of the prime \( p \); therefore the analogous fact is not true when the number of spins is finite (see also remark 5.1 (a)).

Remark 4.2. Note that the equality can be interpreted as an infinite-dimensional recurrence equation over the tree. So, theorem 4.3 means that the equation has no more one solution. Simpler, recurrence equations over \( p \)-adic numbers were considered in [21, 48].

5. Uniqueness and boundedness of the Gibbs measure

As we pointed out, in general, \( p \)-adic Gibbs measures may not exist. In this section we will show that the \( p \)-adic Gibbs measure is unique under condition (4.6) for the homogeneous model (3.1).

Throughout this section we suppose that \( J_{x,y} = J \). Recall that a function \( h = \{ h_x, x \in V \setminus \{ x^0 \} \} \) is translation invariant if \( \hat{h}_x = h_y \) for every \( x, y \in V \setminus \{ x^0 \} \). We are going to show that (3.9) has a translation invariant solution. To this end, consider the following mapping \( \mathcal{F} : c_0 \to Q_p^0 \) defined by

\[
\mathcal{F}(x)_i = \frac{\lambda(i)}{\lambda(0)} (F_i(x; \theta))^i, \quad i \in \mathbb{N}, \tag{5.1}
\]

where \( x = \{ x_n \} \in c_0 \). One can see that the domain of the mapping is not whole space \( c_0 \),\(^6\) therefore, in general, it is unbounded. But we are going to examine \( \mathcal{F} \) on \( B \). Basically, for the mapping we do not have the inclusion \( \mathcal{F}(B) \subseteq B \). However, from lemmas 4.1 and (4.6) we derive the following lemma.

Lemma 5.1. Let for a weight \( \lambda \) condition (4.6) be satisfied. Then \( \mathcal{F}(B) \subseteq B \). Moreover,

\[
\| \mathcal{F}(x) - \mathcal{F}(y) \| \leq |\theta - 1|_p \| x - y \| \quad \text{for every } x, y \in B. \tag{5.2}
\]

\(^6\) More exactly, the domain is \( \{ x \in c_0 : \sum_{n=1}^{\infty} x_n + \theta \neq 0 \} \).
Now noting that \(|\theta - 1|_p < 1/p^{1/(p-1)}\) and according to the above lemma, we can apply the fixed point theorem to \(F\), which implies the existence of a unique fixed point \(\hat{h}_\lambda = \{\hat{h}_{\lambda,n}\}_{n \in \mathbb{N}} \in \mathcal{B}\) (here the solution depends on a weight \(\lambda\); therefore we indicate that dependence by the subscript \(\lambda\)). Since \(\hat{h}_\lambda\) is a fixed point of \(F\) one has

\[
\frac{\lambda(0)}{\lambda(i)} \hat{h}_{\lambda, i} = (F_i(\hat{h}_\lambda; \theta))^i, \tag{5.3}
\]

which, thanks to (4.4) and lemma 4.2, implies that

\[
\left| \frac{\lambda(0)}{\lambda(i)} \hat{h}_{\lambda, i} \right|_p = 1, \quad \left| \frac{\lambda(0)}{\lambda(i)} \hat{h}_{\lambda, i} - 1 \right|_p \leq |\theta - 1|_p.
\]

Therefore, lemma 2.1 allows us to take the logarithm from both sides of (3.10), and we obtain

\[
h_{\lambda, i} - h_{\lambda, 0} = \log_p \left( \frac{\lambda(0)}{\lambda(i)} \hat{h}_{\lambda, i} \right),
\]

which, due to theorem 3.1, defines the \(p\)-adic Gibbs measure, which is denoted by \(\mu_\lambda\). Now combining this with theorem 4.3 we have the following theorem.

**Theorem 5.2.** Let \(0 < |J|_p < 1/p^{1/(p-1)}\) and for a weight \(\lambda\) condition (4.6) be satisfied. Then for the homogeneous \(p\)-adic Potts model (3.1) on the Cayley tree of order \(k\) there is a unique \(p\)-adic Gibbs measure \(\mu_\lambda\).

**Remark 5.1.** It is worth emphasizing the following notes.

(a) Note that in [50] we have proven for the \(q\)-state Potts model that the \(p\)-adic Gibbs measure is unique if \(q\) and \(p\) are relatively prime. Therefore, the proven theorem 4.3 shows the difference between finite and countable state Potts models.

(b) It turns out that condition (4.6) is important. If we replace the strict inequality there with a weaker one \(\leq\), then theorem 4.3 may not hold. Namely, in that case there may occur a phase transition. Indeed, if \(\lambda(0) = \lambda(1) = \lambda(2) = 1\) and \(\lambda(k) = 0\) for every \(k \geq 3\), then clearly (4.6) is not satisfied. On the other hand, our model is reduced to the 3-state Potts model. For such a model in [49] the existence of the phase transition was proven at \(p = 3\). Moreover, in that case, \(p\)-adic Gibbs measures were unbounded.

(c) For the real counterpart of the model the uniqueness result is still unknown (see [24,26]).

Now we propose the following problem.

**Problem.** Investigate all fixed points and behaviour around such points of the dynamical system (5.1) on the whole space \(c_0\). Note that the dynamical system is rational. Therefore, we hope that the general theory of rational dynamical systems developed in [11–14, 53, 57] can be applied for one.

To establish the boundedness of the measure \(\mu_\lambda\), we need the following auxiliary result.

**Lemma 5.3.** Let \(h\) be a solution of (3.9) and \(\mu_h\) be an associated \(p\)-adic Gibbs measure. Then for the corresponding partition function \(Z_{n}^{(h)}\) (see (3.6)) the following equality holds:

\[
Z_{n+1}^{(h)} = A_{h,n}Z_{n}^{(h)}, \tag{5.4}
\]

where \(A_{h,n}\) will be defined below (see (5.7)).
Proof. Since $h$ is a solution of (3.9), then we conclude that there is a constant $a_h(x) \in \mathbb{Q}_p$ such that
\[
\prod_{y \in S(x)} \sum_{j \in \Phi_1} \exp_p \{ J \delta_{ij} + h_{j,y} \lambda(j) \} = a_h(x) \exp_p \{ h_{i,x} \}
\] for any $i \in \Phi$. From this one gets
\[
\prod_{x \in W_n} \prod_{y \in S(x)} \sum_{j \in \Phi_1} \exp_p \{ J \delta_{ij} + h_{j,y} \lambda(j) \} = \prod_{x \in W_n} a_h(x) \exp_p \left\{ \sum_{x \in W_n} h_{i,x} \right\},
\]
for $i \in \Phi$. By (3.5) and (5.6) we have
\[
1 = \sum_{\sigma \in \Omega_n} \sum_{\omega \in \Phi_1} \mu_h^{(1)}(\sigma \vee \omega)
\]
\[
= \sum_{\sigma \in \Omega_n} \sum_{\omega \in \Phi_1} \frac{1}{Z^{(h)}_{n+1}} \exp_p \left\{ H(\sigma \vee \omega) + \sum_{x \in W_n} h_{\omega(x),x} \right\} \prod_{x \in V_n} \lambda(\sigma(x)) \prod_{y \in W_n} \lambda(\omega(y))
\]
\[
= \frac{1}{Z^{(h)}_{n+1}} \sum_{\sigma \in \Omega_n} \exp_p \left\{ H(\sigma) + \sum_{x \in W_n} h_{\sigma(x),x} \right\} \prod_{x \in V_n} \lambda(\sigma(x)) = \frac{A_{h,n}}{Z^{(h)}_{n+1}},
\]
which implies the required relation. □

By (4.6) we have
\[
a = \exp_p \{ (k - 1) h_{\lambda,0}(\lambda(0)) \} \left( \theta + \sum_{j=1}^{\infty} \hat{h}_{\lambda,j} \right)^k.
\]

Equalities (5.4) and (5.7) imply that
\[
Z_{\lambda,n} = a|^{V_n}_{n+1},
\]
where $Z_{\lambda,n}$ denotes the partition function of the measure $\mu_{\lambda}$ corresponding to the unique solution. Now we are ready to formulate a result.

**Theorem 5.4.** Assume that (4.6) is satisfied. Then the $p$-adic Gibbs measure $\mu_{\lambda}$ is bounded.

**Proof.** Take any $\sigma \in \Omega_n$. Then from (3.5) with (5.9) and (4.6) one gets
\[
|\mu_{\lambda}(\sigma)|_p = \frac{1}{|Z_{\lambda,n}|_p} \exp_p \left\{ H(\sigma) + \sum_{x \in W_n} h_{\lambda,\sigma(x)} \right\} \prod_{x \in V_n} \lambda(\sigma(x)) \bigg|_p
\]
\[
= \frac{1}{|\lambda(0)|_p^{V_n}_{n+1}} \prod_{x \in V_n} |\lambda(\sigma(x))|_p \bigg|_p = \frac{|\lambda(0)|_p^{V_n}_{n+1}}{\prod_{x \in V_n} |\lambda(\sigma(x))|_p \bigg|_p} \prod_{x \in V_n} \left| \frac{\lambda(\sigma(x))}{\lambda(0)} \right|_p
\]
\[
\leq |\lambda(0)|_p^{V_n}_{n+1},
\]
(5.10)
here we have used that $|\theta + \sum_{j=1}^{\infty} \hat{h}_{j,i}|_p = 1$. It is known [9] that

$$|V_n| = 1 + \frac{k + 1}{k - 1}(k^n - 1),$$

therefore,

$$|V_n| - k|V_{n-1}| = 2,$$

hence, (5.10) implies that $\mu_{\lambda}$ is bounded. \qed

**Example 5.1.** Assume that a solution of (3.9) is $\hat{h}_i = p^i$, $i \in \mathbb{N}$. In this case, one can see that

$$\sum_{j=1}^{\infty} \hat{h}_j = \frac{p}{1 - p}. $$

Let us find the corresponding weight $\lambda$. Put $\lambda(0) = 1$, then from (3.9) one gets

$$\lambda(n) = p^n \left( \frac{p(1 - \theta) + \theta}{(\theta - 1)p^n(1 - p) + 1} \right)^k, \quad n \in \mathbb{N},$$

which evidently satisfies conditions (3.4) and (4.6). So, there is a unique bounded $p$-adic Gibbs measure on $\Omega$.

### 6. Certain properties of the $p$-adic Gibbs measures

In this section without loss of generality we assume that for weights $\lambda(0) = 1$ and $h_0 = 0$.

Denote

$$\mathcal{W} = \{ \lambda(i)_{i \in \mathbb{N}} : \lambda(0) = 1, |\lambda(i)|_p < 1, |\lambda(n)|_p \to 0 \text{ as } n \to \infty \}. $$

A norm of a weight $\lambda \in \mathcal{W}$ we define by $\|\lambda\|_{\mathcal{W}} = \max_{n \in \mathbb{N}} |\lambda(n)|_p$; therefore, it is clear that $\|\lambda\|_{\mathcal{W}} = 1$. We know from theorems 5.2 and 5.4 that for every $\lambda \in \mathcal{W}$ there is a unique bounded $p$-adic Gibbs measure $\mu_{\lambda}$, by denoting $\mathcal{G}_p$ the set of such measures corresponding to the homogeneous Potts model (3.1). We endow $\mathcal{G}_p$ with a norm defined by

$$\|\mu\|_{\mathcal{G}} = \max_{\sigma \in \Omega \mathcal{W}} |\mu(\sigma)|_p, \quad \mu \in \mathcal{G}_p. \quad (6.1)$$

One can ask: does the measure $\mu_{\lambda}$ depend on $\lambda$ continuously? The next theorem gives an answer to the question.

**Theorem 6.1.** For every $\lambda, \kappa \in \mathcal{W}$ one has

$$\|\mu_{\lambda} - \mu_{\kappa}\|_{\mathcal{G}} \leq \|\lambda - \kappa\|_{\mathcal{W}}. \quad (6.2)$$

Hence, the correspondence $\lambda \mapsto \mu_{\lambda}$ is continuous.

**Proof.** Take any $\lambda, \kappa \in \mathcal{W}$. Let $\hat{h}_\lambda = \{\hat{h}_{\lambda,i}\}_{i \in \mathbb{N}}$ and $\hat{h}_\kappa = \{\hat{h}_{\kappa,i}\}_{i \in \mathbb{N}}$ be the corresponding solutions of (3.9)\footnote{Here we again recall that according to theorems 4.3 and 5.2 such solutions exist.}. Denote $h_\lambda = \{h_{\lambda,i} = \log_p \hat{h}_{\lambda,i}\}$ and $h_\kappa = \{h_{\kappa,i} = \log_p \hat{h}_{\kappa,i}\}$, which exist due to the proof of theorem 5.2. Using (4.3) consider the difference

$$|\hat{h}_{\lambda,i} - \hat{h}_{\kappa,i}|_p = |\lambda(i)F_i(\hat{h}_\lambda ; \theta) - \kappa(i)F_i(\hat{h}_\kappa ; \theta)|_p$$

$$= |\lambda(i)(F_i(\hat{h}_\lambda ; \theta) - F_i(\hat{h}_\kappa ; \theta)) + (\lambda(i) - \kappa(i))F_i(\hat{h}_\kappa ; \theta)|_p$$

$$\leq \max\{|\lambda(i)|_p|F_i(\hat{h}_\lambda ; \theta) - F_i(\hat{h}_\kappa ; \theta)|, |\lambda(i) - \kappa(i)|\}$$

$$\leq \max\{|\lambda(i)|_p|\theta - 1|_p|\hat{h}_\lambda - \hat{h}_\kappa|, |\lambda(i) - \kappa(i)|\}. \quad (6.3)$$
Taking into account $|\lambda(i)|_p|\theta - 1|_p \|\hat{h}_\lambda - \hat{h}_\kappa\| < \|\hat{h}_\lambda - \hat{h}_\kappa\|$ from (6.3) and the non-Archimedeanity of the norm $\|\cdot\|$ one has

$$\|\hat{h}_\lambda - \hat{h}_\kappa\| = \|\lambda - \kappa\|_W.$$ (6.4)

Now take any $n \in \mathbb{N}$. Let us estimate the difference between the partition functions $Z_{\lambda,n}$ and $Z_{n,\kappa}$ (see (3.6)) of the measures $\mu_\lambda$ and $\mu_\kappa$, respectively. Thanks to (5.9) with (5.8) and (4.2) one gets

$$|Z_{\lambda,n} - Z_{\kappa,n}|_p = \left|\left(\theta + \sum_{j=1}^{\infty} \hat{h}_{\lambda,j}\right)^{k} - \left(\theta + \sum_{j=1}^{\infty} \hat{h}_{\kappa,j}\right)^{k}\right|_p \leq \sum_{j=1}^{\infty} |\hat{h}_{\lambda,j} - \hat{h}_{\kappa,j}|_p.$$ (6.5)

Hence, for any $\sigma \in \Omega_{V_n}$, from (3.5) and lemma 4.2 using (6.4) and (6.5) we obtain

$$|\mu_\lambda(\sigma) - \mu_\kappa(\sigma)|_p = \left|\frac{1}{Z_{\lambda,n}} \exp\left\{\sum_{x \in W_n} h_{\lambda,\sigma(x)} \prod_{u \in V_n} \lambda(\sigma(u))\right\}\right|_p - \frac{1}{Z_{\kappa,n}} \exp\left\{\sum_{x \in W_n} h_{\kappa,\sigma(x)} \prod_{u \in V_n} \kappa(\sigma(u))\right\}_p \leq \max_{x \in W_n} \left\{\|h_{\lambda,\sigma(x)} - h_{\kappa,\sigma(x)}\|_p, \|\lambda(\sigma(u)) - \kappa(\sigma(u))\|_p\right\}$$

$$\leq \max_{x \in W_n} \left\{\|\hat{h}_{\lambda} - \hat{h}_{\kappa}\|, \|\lambda - \kappa\|_W\right\} \leq \|\lambda - \kappa\|_W,$$ (6.6)

here we have used equalities $|Z_{\lambda,n}|_p = 1, |Z_{\kappa,n}|_p = 1$, which come from (5.9).

Due to the arbitrariness of $n$ and $\sigma$ we get the required relation (6.2).

Now consider the one limit theorem concerning $\mu_\lambda$.

Let us fix a weight $\lambda \in W$ such that $\lambda(i) \neq 0$ for all $i \in \mathbb{N}$. As before by $\hat{h}_\lambda$ we denote a solution of (3.9), and, as before, $h_{\lambda,i} = \log_p \hat{h}_{\lambda,i}, i \in \mathbb{N}$.

Let us denote

$$A_n = \left\{\sigma \in \Omega_{V_n} : J \sum_{(x,y) \in V_n} \delta_{\sigma(x),\sigma(y)} + \sum_{x \in W_n} h_{\lambda,\sigma(x)} \equiv 0 \text{(mod $p^n$)}, \right\}.$$ (6.7)

By $\lambda^{\otimes,n}$ denote the following measure defined on $\Omega_{V_n}$

$$\lambda^{\otimes,n}(\sigma) = \frac{1}{Z_{\lambda,n}} \prod_{x \in V_n} \lambda(\sigma(x)), \quad \sigma \in \Omega_{V_n}.$$ (6.8)

**Theorem 6.2.** Let $\mu_\lambda$ be the $p$-adic Gibbs measure corresponding to the Potts model with a weight $\lambda$. Then one has

$$\max_{\sigma \in A_n} \left|\frac{\mu_\lambda(\sigma)}{\lambda^{\otimes,n}(\sigma)} - 1\right|_p \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$ (6.9)

**Proof.** Note that if $\sigma \in A_n$ then it means that

$$J \sum_{(x,y) \in V_n} \delta_{\sigma(x),\sigma(y)} + \sum_{x \in W_n} h_{\lambda,\sigma(x)} = Mp^n$$

By the arbitrariness of $n$ and $\sigma$ we get the required relation (6.2).
for some $M \in \mathbb{Z}$. Therefore, the last equality with (6.8) and (3.5) implies that

$$\left| \frac{\mu_\lambda(\sigma)}{\lambda^{\otimes n}(\sigma)} - 1 \right|_p \leq \left| \exp_p [M p^n] - 1 \right|_p = |M p^n|_p \leq p^{-n} \to 0 \quad \text{as } n \to \infty,$$

which proves the assertion. □

**Remark 6.1.** In the theory of the Markov process it is important to know whether a given Markov measure or Markov field has some clustering (i.e. mixing) property. It is known [27] that, in the real case, when a Gibbs measure is unique, then it basically has that property. In a $p$-adic case, the situation is rather tricky (see [34, 35]) and still not much is known. The proven theorem shows that we are able to find a set $A_n$ on which the $p$-adic measure $\mu_\lambda$ becomes a product measure when $n$ is large enough, which means $\mu_\lambda$ has a clustering property on $A_n$. The results in [32, 34] show that, in general, the clustering property does not hold for $p$-adic measures.

**Remark 6.2.** Note that in [35] the limit behaviour of the sums of independent equally distributed random variables with respect to the $p$-adic Bernoulli measure has been studied. The measure $\mu_\lambda$ defines a Markov process with an infinite number of states, so the last theorem is the first step in the investigation of the limit theorems for dependent processes in a $p$-adic context.

7. Conclusions

Investigations of physical models over the $p$-adic field require a new kind of probability theory [31, 40]. Development of the $p$-adic probability theory gives the possibility of studying $p$-adic statistical mechanics models. In this paper we have studied a countable state nearest-neighbour $p$-adic Potts models on a Cayley tree in the $p$-adic probability scheme. For the model, we gave a construction of $p$-adic Gibbs measures which depends on weight $\lambda$. Such measures are natural and provide non-trivial concrete examples of $p$-adic Markov processes with countable state space (see [39]). We have shown, under some condition on weights for the model, the absence of a phase transition by studying an infinite-dimensional recursion equation. Such results are unknown for the real counterparts of the considered models (see [24, 26]). It turned out that the condition does not depend on the values of the prime $p$; therefore the analogous fact is not true when the number of spins is finite [49, 50]. It has been proven that the $p$-adic Gibbs measure for the homogeneous Potts model is unique. We also established the boundedness of such a measure. The continuous dependence of the measure on weights was also proven. We even obtain a one limit theorem for such a measure. This is the first step in an investigation on the limit theorems for dependent processes in the $p$-adic context.

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