Feature Synergy, Redundancy, and Independence in Global Model Explanations using SHAP Vector Decomposition

Jan Ittner 1 Lukasz Bolikowski 1 Konstantin Hemker 1 Ricardo Kennedy 1

Abstract

We offer a new formalism for global explanations of pairwise feature dependencies and interactions in supervised models. Building upon SHAP values and SHAP interaction values, our approach decomposes feature contributions into synergistic, redundant and independent components (S-R-I decomposition of SHAP vectors). We propose a geometric interpretation of the components and formally prove its basic properties. Finally, we demonstrate the utility of synergy, redundancy and independence by applying them to a constructed data set and model.

1. Introduction

Understanding how and why a model produces its output is an essential part of building a robust machine learning solution. There are various reasons why data scientists opt to “unpack” their models, including

1. Diagnostic: ensuring that good model performance is not a result of data leakage, the evaluation protocol is not compromised, and the model has learned to properly generalise from the training data.

2. Validation: checking that relationships discovered by the model are plausible also from the perspective of domain experts

3. Feature selection: pruning redundant features with low or no marginal impact while protecting groups of synergistic features

4. Fairness and compliance: detecting a model’s direct or indirect use of protected attributes to avoid discriminatory bias, or violation of other regulatory requirements

Some machine learning models, by design, offer limited insights into their decision making process. Examples include comparing coefficients of linear regression models, counting how often a feature is used in random forest models, or tracking neuron activations under various inputs in neural networks. Still, the most valuable explanatory frameworks are those that can unpack an arbitrary “black box” model without the need to access its internals.

Model explanation typically takes the form of attributing importance to input features, individually or by groups. Several approaches have been proposed to date, with SHAP (Lundberg & Lee, 2017) being the most popular.

However, the primary focus of SHAP is to quantify local contributions of one or more features, and is not designed to explain global relationships among features from the perspective of a given model: Does the model combine information from groups of features, meaning that any feature of that group would be less impactful in the absence of its counterparts? Which features are fully or partially redundant with respect to the target variable, and could therefore be substituted for each other with little or no loss of model performance?

This paper offers new answers to questions such as the above, proposing an approach with favourable mathematical properties to quantify dependencies and interactions between features in a model: given any pair of features $x_i$ and $x_j$, we interpret their SHAP values across multiple observations as vectors, then decompose them into multiple subvectors representing different types of relationships, and quantify the strength of these relationships by the magnitudes of the vectors. We distinguish three types of relationships: synergy, redundancy, and independence.

1. The synergy of feature $x_i$ relative to another feature $x_j$ quantifies the degree to which predictive contributions of $x_i$ rely on information from $x_j$. As an example, two features representing coordinates on a map need to be used synergistically to predict distances from arbitrary points on the map.

2. The redundancy of feature $x_i$ with feature $x_j$ quantifies the degree to which the predictive contribution of $x_i$ uses information that is also available through $x_j$. For example, the temperature and pressure measured in
a vessel are highly redundant features since both are mutually dependent owing to the ideal gas law.

3. The independence of feature \( x_i \) relative to feature \( x_j \) quantifies the degree to which the predictive contribution of \( x_i \) is neither synergistic or redundant with \( x_j \).

Synergy, redundancy, and independence are expressed as percentages of feature importance. They are additive, and sum up to 100% for any pair of features. Importantly, neither relationship is necessarily symmetrical: While one feature may replicate or complement some or all of the information provided by another feature, the reverse need not be the case.

2. State of the Art

Model interpretability is a subject of intensive research in the recent years. However, the very notion of interpretability can be understood in different ways. Doshi-Velez & Kim (Doshi-Velez & Kim, 2017), as well as Lipton (Lipton, 2018), and Gilpin et al. (Gilpin et al., 2018) worked towards clarifying related terminology, as well as listing motivations for, and flavors of, interpretability.

Pioneering works of Strumbelj & Kononenko (Štrumbelj & Kononenko, 2014) and Local Interpretable Model-agnostic Explanations (LIME) by Ribeiro et al. (Ribeiro et al., 2016) were refined into a unified framework called SHapley Additive exPlanation (SHAP) by Lundberg & Lee (Lundberg & Lee, 2017) which is a foundation for most of the currently developed approaches. In a follow-up article, higher-order SHAP values, so-called SHAP interaction values were introduced (Lundberg et al., 2018). Efficient SHAP implementations for tree ensemble models were also found (Lundberg et al., 2018).

As SHAP became a reference framework for model explanation, several authors turned to exploring the utility of SHAP and expanding it. Rath (Rath, 2019) showed how to generate GDPR-compliant counterfactual and contrastive explanations using SHAP. Merrick & Taly (Merrick & Taly, 2020) demonstrated how to calculate confidence intervals of attributions. Shapley Additive Global importancE (SAGE) (Covert et al., 2020) were proposed for quantifying a model’s dependence on its features. Sundararajan & Najmi (Sundararajan & Najmi, 2020) explored axioms and desired properties of various attribution methods.

Naturally, critical analysis of SHAP revealed its limitations. Kumar et al. (Kumar et al., 2020b) pointed to certain mathematical shortcomings of SHAP (including the question of addressing causality) and the fact that Shapley values represent only a summary of a game. The same authors (Kumar et al., 2020a) offered a concept of Shapley residuals, vectors capturing information lost by Shapley decomposition. Their approach is based on work of Stern & Tettenhorst (Stern & Tettenhorst, 2019), who have shown a way of decomposing an arbitrary game and the relation of such decompositions to Shapley values.

3. Preliminaries

Let us start by briefly recalling the key concepts upon which the S-R-I decomposition is founded.

3.1. Original Shapley Values

Shapley values were originally introduced as a concept in game theory to describe the distribution total surplus of different coalitions of players in an \( n \)-person game. As each player in different coalitions has a different contribution to the final outcome, Shapley values provide a way of modeling the marginal contribution of each player to the overall cooperation of the game. Formally, Shapley (Shapley, 1953) expresses the amount allocated to player \( i \) in a collaborative game with players \( N \) and outcomes \( f_x(S) \) for any subset (coalition) of players \( S \subseteq N \) as:

\[
\phi_i = \sum_{S \subseteq N \setminus \{i\}} \frac{|S|!(|N| - |S| - 1)!}{|N|!} \nabla_i(S)
\]

(1)

where \( \nabla_i(S) = f_x(S \cup \{i\}) - f_x(S) \)

(2)

\( \phi_i \) expresses the average incremental contribution of player \( i \) when added to all possible permutations of coalitions \( S \subseteq N \setminus \{i\} \).

3.2. SHAP Vectors

SHAP values are an application of Shapley values for a predictive model \( f : \mathbb{R}^n \rightarrow \mathbb{R} \). In this context, the game outcome \( f_x \) is the model evaluated for a sample \( x \in \mathbb{R}^n \) with different sets of features present. “Players” are the features used in the model and “coalitions” of features correspond to subsets of features that are provided to the model to make predictions. The term \( f_x(S) \) in (1) is defined to be the original model \( f \) restricted to use only features in \( S \), by taking the expectation value over features not in \( S \). In the notation of (Chen et al., 2020):

\[
f_x(S) = \mathbb{E}[f(x)|S]
\]

(3)

In particular,

\[
f_x(N) = \mathbb{E}[f(x)|N] = f(x)
\]

(4)

and \( f_x(\emptyset) = \mathbb{E}[f(x)|\emptyset] = \mathbb{E}[f(x)] \)

(5)

Given \( M \) samples in the training corpus for the model, we can calculate the SHAP value for each feature of each sam-
Feature Synergy, Redundancy, and Independence in Global Model Explanations using SHAP Vector Decomposition

During, resulting in a $N \times M$ SHAP value matrix for each feature $x_i$ and observation $u$.

In turn, we define the SHAP vector as

$$\mathbf{p}_i = (\phi_1^i, \ldots, \phi_m^i) \quad (6)$$

being the SHAP values for samples $u = 1 \ldots m$ for feature $i$.

3.3. SHAP Interaction Vectors

SHAP interaction effects (Lundberg et al., 2018) quantify the interactions between any pair of features $x_i$ and $x_j$ by calculating the difference between the SHAP value for feature $i$ when $j$ is present, and the SHAP value for feature $i$ when $j$ is absent. Formally, this relationship is captured by $\nabla_{ij}$ in (7) and (8).

$$\phi_{ij} = \sum_{S \subseteq N \setminus \{i,j\}} \frac{|S|!(|N| - |S| - 2)!}{2(|N| - 1)!} \nabla_{ij}(S) \quad (7)$$

$$\nabla_{ij} = f_x(S \cup \{i, j\}) - f_x(S \cup \{j\}) - (f_x(S \cup \{j\}) - f_x(S)) \quad (8)$$

where $S$ is a coalition of features representing a subset of all features $N$. The summation extends over all possible coalitions of $N$ that don’t contain the feature pair $\{i, j\}$. In (7) the SHAP interaction value is split equally between features $x_i$ and $x_j$ hence $\phi_{ij} = \phi_{ji}$. We can isolate the main effect $\phi_{ii}$ for feature $x_j$ by subtracting the interaction values for all $j \neq i$ from the SHAP value $\phi_i$:  

$$\phi_{ii} = \phi_i - \sum_{j \neq i} \phi_{ij} \quad (9)$$

Similarly to SHAP vectors, we define the SHAP interaction vector as the vector of SHAP values for samples $u = 1 \ldots m$ given a pair of features $\{x_i, x_j\}$:

$$\mathbf{p}_{ij} = (\phi_{ij}, \ldots, \phi_{ij}^m) \quad \forall i, j \in N \times N \quad (10)$$

From (9) in conjunction with (6) and (10), it follows that all interaction vectors for feature $x_i$ add up to the SHAP vector for $x_i$:

$$\mathbf{p}_i = \sum_{j \in N} \mathbf{p}_{ij} \quad \forall i \quad (11)$$

4. Synergy, Redundancy, and Independence

In the following section we will introduce and examine various $m$-dimensional vectors, where $m$ is the number of observations. Vectors representing SHAP values (6) and SHAP interaction values (10) will be our building material, from which we will construct other informative vectors. Without loss of generality, at all times, we will focus on one feature, $x_i$ (with corresponding SHAP vector $\mathbf{p}_i$), and explore its relationship with one other feature $x_j$ (with SHAP vector $\mathbf{p}_j$ and SHAP interaction vector $\mathbf{p}_{ij}$).

We will be concerned with angles between vectors in the $m$-dimensional space. The smaller the angle between two vectors, the more information is shared by them. Our goal will often be to decompose vectors into orthogonal components (see Figure 1).

![Figure 1. Geometric interpretation of synergy, redundancy and independence](image)

4.1. Vector Representation

**Definition 1 (Synergy vector)**

$$\mathbf{s}_{ij} = \frac{\langle \mathbf{p}_i, \mathbf{p}_{ij} \rangle}{\|\mathbf{p}_{ij}\|^2} \mathbf{p}_{ij} \quad \forall i \neq j \quad (12)$$

Geometrically speaking, the synergy vector for $x_i$ and $x_j$ is a projection of $\mathbf{p}_i$ on $\mathbf{p}_{ij}$. Synergy represents the advantage that feature $x_i$ receives when aided by $x_j$.

For example, if features $x_i$ and $x_j$ represent geographic latitude and longitude, and our function is elevation above mean sea level, then both features work synergistically and neither can determine the outcome without the other.

Note that the definition is asymmetric, hence $\mathbf{s}_{ij}$ need not equal $\mathbf{s}_{ji}$.

**Definition 2 (Autonomy vector)**

$$\mathbf{a}_{ij} = \mathbf{p}_i - \mathbf{s}_{ij} \quad \forall i \neq j \quad (13)$$
Autonomy is the converse of synergy. As such, autonomy represents the predictive contributions \( x_i \) makes without help from \( x_j \), either because it is redundant, or independent (subsequent definitions will help us distinguish between these two cases).

Geometrically, the autonomy vector is perpendicular to the synergy vector, and both add up to \( p_i \).

**Definition 3 (Redundancy vector)**

\[
\mathbf{r}_{ij} = \frac{\langle \mathbf{a}_{ij}, \mathbf{a}_{ji} \rangle}{\| \mathbf{a}_{ji} \|^2} \mathbf{a}_{ji} \quad \forall i \neq j
\]

(14)

The redundancy vector represents information in \( x_i \) that is replicated by \( x_j \). Geometrically, this is the projection of vector \( \mathbf{a}_{ij} \) onto vector \( \mathbf{a}_{ji} \).

For example, distance in kilometres and distance in miles are perfectly redundant features, whereas a child’s age and height are partially (but not fully) redundant.

**Definition 4 (Independence vector)**

\[
\mathbf{i}_{ij} = \mathbf{a}_{ij} - \mathbf{r}_{ij} \quad \forall i \neq j
\]

(15)

Independence represents the information in feature \( x_i \) that has no synergy or redundancy with feature \( x_j \). Geometrically, \( \mathbf{i}_{ij} \) and \( \mathbf{r}_{ij} \) are orthogonal, and together they add up to \( \mathbf{a}_{ij} \).

Let us sum up basic properties of the vectors introduced above. First of all, it follows directly from the definitions that:

\[
\begin{align*}
\mathbf{p}_i &= \mathbf{s}_{ij} + \mathbf{a}_{ij} = \mathbf{s}_{ij} + \mathbf{r}_{ij} + \mathbf{i}_{ij} \\
\mathbf{s}_{ij} \perp \mathbf{r}_{ij} \perp \mathbf{i}_{ij} \perp \mathbf{s}_{ij}
\end{align*}
\]

(16) \quad (17)

Thanks to the above, we also have:

\[
\| \mathbf{p}_i \|^2 = \| \mathbf{s}_{ij} \|^2 + \| \mathbf{a}_{ij} \|^2 = \| \mathbf{s}_{ij} \|^2 + \| \mathbf{r}_{ij} \|^2 + \| \mathbf{i}_{ij} \|^2
\]

(18)

For any \( x_i \) and \( x_j \), the vectors \( \mathbf{p}_i, \mathbf{p}_{ij}, \mathbf{s}_{ij} \) and \( \mathbf{a}_{ij} \) are co-planar. Another important plane, orthogonal to the first one, contains the vectors \( \mathbf{a}_{ij}, \mathbf{a}_{ji}, \mathbf{r}_{ij} \) and \( \mathbf{i}_{ij} \) (also see Figure 1).

### 4.2. Scalar Representation and \( S, R, I \) Values

For practical reasons, instead of working with the full vectors, we introduce their scalar counterparts. For each of the scalar values \( S_{ij}, R_{ij} \) and \( I_{ij} \) we have three equivalent characterizations:

- as the ratio of squared norms \( \frac{\| \mathbf{s}_{ij} \|^2}{\| \mathbf{p}_i \|^2} \)
- as the square of the uncentered correlation coefficient \( \frac{(\langle \mathbf{v}, \mathbf{w} \rangle)^2}{\| \mathbf{v} \|^2 \| \mathbf{w} \|^2} \)

**Definition 5 (Synergy value)**

\[
S_{ij} = \frac{\langle \mathbf{s}_{ij}, \mathbf{p}_i \rangle}{\| \mathbf{p}_i \|^2} = \frac{\| \mathbf{s}_{ij} \|^2}{\| \mathbf{p}_i \|^2} = \frac{\langle \mathbf{p}_i, \mathbf{p}_{ij} \rangle^2}{\| \mathbf{p}_i \|^2 \| \mathbf{p}_{ij} \|^2} \quad \forall i \neq j
\]

(19)

**Definition 6 (Redundancy value)**

\[
R_{ij} = \frac{\langle \mathbf{r}_{ij}, \mathbf{p}_i \rangle}{\| \mathbf{p}_i \|^2} = \frac{\| \mathbf{r}_{ij} \|^2}{\| \mathbf{p}_i \|^2} = (1 - S_{ij}) \frac{\langle \mathbf{a}_{ij}, \mathbf{a}_{ji} \rangle}{\| \mathbf{a}_{ij} \|^2 \| \mathbf{a}_{ji} \|^2} \quad \forall i \neq j
\]

(20)

**Definition 7 (Independence value)**

\[
I_{ij} = \frac{\langle \mathbf{i}_{ij}, \mathbf{p}_i \rangle}{\| \mathbf{p}_i \|^2} = \frac{\| \mathbf{i}_{ij} \|^2}{\| \mathbf{p}_i \|^2} = 1 - S_{ij} - R_{ij} \quad \forall i \neq j
\]

(21)

In the appendix, we derive the equivalence between the three characterizations for each scalar value in eqs. (19), (20), and (21) respectively.

We have thus defined scalar values quantifying synergy, redundancy and independence from a global perspective. The three values are non-negative and sum up to unity:

\[
S_{ij} + R_{ij} + I_{ij} = 1
\]

(22)

\[
0 \leq S_{ij} \leq 1
\]

(23)

\[
0 \leq R_{ij} \leq 1
\]

(24)

\[
0 \leq I_{ij} \leq 1
\]

(25)

### 4.3. Orthogonality Correction

SHAP interaction vectors representing main effects \( p_{ii} \) are not guaranteed to be orthogonal to pairwise interaction vectors \( p_{jj} \). In order to split the main effects from the interaction vectors, we correct SHAP interaction values by projecting them onto the subspace that is orthogonal to \( p_{ii} \) and \( p_{jj} \). In other words, we determine constants \( \alpha \) and \( \beta \) such that

\[
\mathbf{p}'_{ij} := \mathbf{p}_{ij} - \alpha \mathbf{p}_{ii} - \beta \mathbf{p}_{jj}
\]

(26)

\[
\mathbf{p}_{ii} \perp \mathbf{p}'_{ij} \perp \mathbf{p}_{jj}
\]

(27)

and apply the S-I-R calculations based on the corrected vectors \( \mathbf{p}'_{ij} \). A further formalisation of this preprocessing step is part of our current research (see also the outlook in section 6).
5. Experimental Results

Let us now examine how S-R-I decomposition works in practice to gain a deeper understanding of the relationships between model features.

Consider \( m = 1000 \) observations of \( n = 5 \) features, represented by \( m \)-dimensional vectors: \( \{x_1, \ldots, x_n\} \). Each value of \( x_1, x_2, x_4 \) and \( x_5 \) is drawn independently from a uniform distribution \([0, 1]\), while \( x_3 = x_2 \). Consider a model (see Figure 2):

\[
f(x) := \sin(2\pi x_1) \sin(2\pi x_2 + x_3) + x_4 + x_5 \quad (28)
\]

In other words, features \( x_2 \) and \( x_3 \) are identical, redundant copies. Features \( x_4 \) and \( x_5 \) impact the model independently of each other and of any other feature. Impact of feature \( x_1 \) is linked to that of features \( x_2, x_3 \), as neither can increase the function’s value without “co-operation” with the others (there is a large degree of synergy between them).

We have calculated exact SHAP values for each observation, applied orthogonality correction described in 4.3, and then calculated S-R-I decomposition for feature pairs. Table 1 presents synergy, redundancy and independence values for each pair of features.

Investigating the results we notice that \( S_{12} = S_{13} = 1 \), indicating that \( x_1 \) can provide the “missing piece of information” to \( x_2 \) and \( x_3 \). At the same time, \( S_{21} = S_{31} = 0.79 \), meaning that \( x_2 \) can also reinforce \( x_3 \), but is limited by \( x_3 \) (and vice versa).

Looking at \( R_{ij} \), the only pair of redundant features is \( x_2 \) and \( x_3 \), with \( R_{32} = R_{23} = 1 \). We have \( I_4 = I_4 = I_5 = I_5 = 1 \), expressing the fact that the last two features contribute fully independently to the overall outcome. Lastly, as expected, in all cases \( S_{ij} + R_{ij} + I_{ij} = 1 \).

To sum up, we have observed that synergy, redundancy and independence values, as defined in this paper, are intuitive and quantifiable reflections of their respective notions.

6. Conclusions

In this work we have shown that an interaction between any two features in a model can be decomposed into three components: synergy (S), redundancy (R) and independence (I). We have characterized S-R-I using geometric properties, and have proven equivalence between alternative formulations. We have also used an example using a synthetic dataset to demonstrate how a global explanation using S-R-I decomposition can enhance our understanding of the relationships among model features.

The three values are defined in terms of SHAP values and SHAP interaction values. They can be efficiently calculated, so that the marginal cost of the S-R-I decomposition is negligible. We have released an open-source implementation of S-R-I decomposition in our Explainable AI software library FACET: https://github.com/BCG-Gamma/facet.

The notion of global explanations using orthogonal vectors in the space of observations deserves further attention. Our current research focuses on determining desirable geometric properties of interaction values, and proposing relevant orthogonalisation steps.
Appendix

As discussed in section 4.2, each scalar value for synergy, redundancy and independence has three equivalent characterizations:

- geometrically, as the relative length of the projection onto $p_j$,
- as the ratio of squared norms $\frac{\|i\|^2}{\|p_j\|^2}$,
- as the square of the uncentered correlation coefficient $\frac{(r_{ij})^2}{\|p_i\|^2}$.

Here, we derive the equivalence between the three characterizations for each scalar value as stated for $S_{ij}$ in eq. (19), for $R_{ij}$ in eq. (20), and for $I_{ij}$ in eq. (21) respectively.

Starting with $S_{ij}$, the equivalence in eq. (19) can be shown as follows:

$$\frac{\langle s_{ij}, p_i \rangle}{\|p_i\|^2} = \frac{\langle s_{ij}, s_{ij} + a_{ij} \rangle}{\|p_i\|^2} = \frac{\langle s_{ij}, s_{ij} \rangle}{\|p_i\|^2} = \frac{\|s_{ij}\|^2}{\|p_i\|^2}$$

(29)

$$\frac{\langle s_{ij}, p_i \rangle}{\|p_i\|^2} = \frac{\langle p_{ij}, p_i \rangle}{\|p_i\|^2} = \frac{\langle p_{ij}, p_{ij} \rangle}{\|p_i\|^2} = \frac{\|s_{ij}\|^2}{\|p_i\|^2}$$

(30)

For $R_{ij}$, the equivalence in eq. (20) is due to:

$$\frac{\langle r_{ij}, p_i \rangle}{\|p_i\|^2} = \frac{\langle r_{ij}, s_{ij} + r_{ij} + i_{ij} \rangle}{\|p_i\|^2} = \frac{\langle r_{ij}, r_{ij} \rangle}{\|p_i\|^2} = \frac{\|r_{ij}\|^2}{\|p_i\|^2}$$

(31)

$$\frac{\langle r_{ij}, p_i \rangle}{\|p_i\|^2} = \frac{\langle p_{ij}, a_{ij} \rangle}{\|p_i\|^2} = \frac{\langle p_{ij}, a_{ij} \rangle}{\|p_i\|^2}$$

(32)

For $I_{ij}$, the equivalence in eq. (21) is due to:

$$\frac{\langle i_{ij}, p_i \rangle}{\|p_i\|^2} = \frac{\langle i_{ij}, s_{ij} + r_{ij} + i_{ij} \rangle}{\|p_i\|^2} = \frac{\langle i_{ij}, i_{ij} \rangle}{\|p_i\|^2} = \frac{\|i_{ij}\|^2}{\|p_i\|^2}$$

(33)

$$\frac{\|i_{ij}\|^2}{\|p_i\|^2} = \frac{\|p_{ij}\|^2 - \|s_{ij}\|^2 - \|r_{ij}\|^2}{\|p_i\|^2} = 1 - S_{ij} - R_{ij}$$

(34)

References

Chen, H., Janizek, J. D., Lundberg, S., and Lee, S.-I. True to the model or true to the data? arXiv preprint arXiv:2006.16234, 2020.

Covert, I., Lundberg, S., and Lee, S.-I. Understanding global feature contributions with additive importance measures. Advances in Neural Information Processing Systems, 33, 2020.

Doshi-Velez, F. and Kim, B. Towards a rigorous science of interpretable machine learning. arXiv preprint arXiv:1702.08608, 2017.

Gilpin, L. H., Bau, D., Yuan, B. Z., Bajwa, A., Specter, M., and Kagal, L. Explaining explanations: An overview of interpretability of machine learning. In 2018 IEEE 5th International Conference on data science and advanced analytics (DSAA), pp. 80–89. IEEE, 2018.

Kumar, I. E., Scheidegger, C., Venkatasubramanian, S., and Friedler, S. Shapley residuals: Quantifying the limits of the Shapley value for explanations. In ICML Workshop on Workshop on Human Interpretability in Machine Learning (WHI), 2020a.

Kumar, I. E., Venkatasubramanian, S., Scheidegger, C., and Friedler, S. Problems with Shapley-value-based explanations as feature importance measures. In International Conference on Machine Learning, pp. 5491–5500. PMLR, 2020b.

Lipton, Z. C. The mythos of model interpretability: In machine learning, the concept of interpretability is both important and slippery. Queue, 16(3):31–57, 2018.

Lundberg, S. M. and Lee, S.-I. A unified approach to interpreting model predictions. Advances in Neural Information Processing Systems, 30:4765–4774, 2017.

Lundberg, S. M., Erion, G. G., and Lee, S.-I. Consistent individualized feature attribution for tree ensembles. arXiv preprint arXiv:1802.03888, 2018.
Lundberg, S. M., Erion, G., Chen, H., DeGrave, A., Prutkin, J. M., Nair, B., Katz, R., Himmelfarb, J., Bansal, N., and Lee, S.-I. Explainable AI for trees: From local explanations to global understanding. *arXiv preprint arXiv:1905.04610*, 2019.

Merrick, L. and Taly, A. The explanation game: Explaining Machine Learning models using Shapley values. In *International Cross-Domain Conference for Machine Learning and Knowledge Extraction*, pp. 17–38. Springer, 2020.

Rathi, S. Generating counterfactual and contrastive explanations using SHAP. *arXiv preprint arXiv:1906.09293*, 2019.

Ribeiro, M. T., Singh, S., and Guestrin, C. “why should I trust you?” explaining the predictions of any classifier. In *Proceedings of the 22nd ACM SIGKDD international conference on knowledge discovery and data mining*, pp. 1135–1144, 2016.

Shapley, L. A value for n-person games. *Contributions to the Theory of Games*, pp. 31–40, 1953.

Stern, A. and Tettenhorst, A. Hodge decomposition and the Shapley value of a cooperative game. *Games and Economic Behavior*, 113:186–198, 2019.

Štrumbelj, E. and Kononenko, I. Explaining prediction models and individual predictions with feature contributions. *Knowledge and information systems*, 41(3):647–665, 2014.

Sundararajan, M. and Najmi, A. The many Shapley values for model explanation. In *International Conference on Machine Learning*, pp. 9269–9278. PMLR, 2020.