LESS (PRECISION) IS MORE (INFORMATION): QUANTUM INFORMATION IN FUZZY PROBABILITY THEORY

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A comparison of structural features of quantum and classical physical theories, such as the information capacity of systems subject to these theories, requires a common formal framework for the presentation of corresponding concepts (such as states, observables, probability, entropy). Such a framework is provided by the notion of statistical model developed in the convexity approach to statistical physical theories. Here we use statistical models to classify and survey all possible types of embedding and extension of quantum probabilistic theories subject to certain reasonable constraints. It will be shown that the so-called canonical classical extension of quantum mechanics is essentially the only ‘good’ representation of the quantum statistical model in a classical framework. All quantum observables are thus identified as fuzzy classical random variables.

1 Introduction

Every physical theory can be formulated as a probabilistic theory: each type of physical system is characterized through a set of observables, representing the possible measurements that can be performed on the system, with an associated set of states, assigning probabilities to the measurement outcomes.

The duality of states and observables is formalized in the notion of a statistical duality, also called convexity model or statistical model, of which quantum mechanics and classical statistical mechanics are particular realizations. This structure has been used as the starting point for a reconstruction of quantum mechanics as a statistical physical theory in fundamental studies in the 1960s-70s.

In the past decade, statistical models have been the basis for a series of very original and penetrating investigations by the late Slawek Bugajski and his collaborators (Enrico Beltrametti and Werner Stulpe). They studied a particular classical representation of quantum mechanical density operators and observables in terms of probability measures and functions over a ‘phase space’ which is identified with the set of pure quantum states. This representation was first formulated by B. Misra and others in the 1970s, but its significance as ‘the’ canonical classical extension of quantum mechanics was recognized by S. Bugajski. E. Beltrametti and S. Bugajski exhibited some intriguing features of the canonical classical extension, notably among them the so-called Bell phenomenon in a classical framework and the fact that all quantum observables (positive operator valued measures, or POVMs), whether
sharp (projection valued measures, PVMs) or not, are represented classically as *fuzzy random variables*. These observations led to the inception and development of a generalized classical probability theory, referred to as fuzzy probability theory 8,9,20.

This line of research was a demonstration of Sławek Bugajski’s creativity as a researcher in mathematical physics and the foundations of quantum mechanics. His late work, abruptly cut short by his untimely death in March 2003, comprises a legacy full of intriguing ideas and questions for future research waiting to be carried further.6 Sławek had hoped to demonstrate that the formalisms of statistical models and fuzzy probability theory are useful tools for investigations in quantum information science, particularly in quantum computation. The aim of the present contribution is to give a brief exposition of this approach to quantum mechanics and to help build an intuitive understanding of the intriguing features referred to above. I will review the concept of statistical models, recall the quantum and classical examples, and provide a comprehensive account of all possible mutual embedding schemes. I will show that the canonical classical extension of quantum mechanics is the only ‘good’ classical representation of quantum mechanics as a statistical model. Since quantum observables are in this description represented as fuzzy classical random variables, it follows that the enhanced information storage and cryptographic capacities of quantum systems must be seen as a consequence of the fact that the set of quantum observables appears as a restriction of the set of all classical random variables to a subset of fuzzy random variables.

The title phrase, “Less is more...”, refers to the fact that quantum uncertainty can be utilized as resources for information processing purposes in the broadest sense. For example, noncommuting pairs of observables can be measured jointly if one allows for an appropriate pay-off between the degrees of unsharpness in the respective marginal measurements, quantified by suitable uncertainty relations. Another example is given by the existence of *informationally complete* POVMs, which are necessarily unsharp observables. Further, it has been observed that certain state discrimination procedures are optimally performed by POVMs which are not PVMs. Finally, measurements which are less precise may be designed so as to be less invasive, thus allowing more control over the observed system. Incidentally, Th. Konrad chose the same motto for his doctoral thesis on unsharp quantum measurements.23 While our choices were made independently of each other, we may both have been (consciously or subconsciously) influenced by H. Dehmelt’s famous review 17 which was written in the same spirit.

The reader may wish to read the present article in conjunction with the related contributions by Chris Fuchs 18 and Guido Bacchiagaluppi in this
volume. The former uses the affine classical embedding of quantum states
induced by an informationally complete POVM to explore aspects of the con-
cept of quantum information. The latter investigates the description of state
changes in the framework of the Misra-Bugajski canonical classical extension.
This work is dedicated to the memory of Slawek Bugajski.

2 Statistical models

Every type of physical system can be prepared and observed in a variety of
ways. In order to exhibit regularities in the observation data, it must be
possible to reproduce (practically) the ‘same’ conditions of preparation and
to carry out operations which can be identified as (practically) the ‘same’
measurements, so as to be able to determine that different preparations tend
to lead to different measurement outcomes. Thus, the theoretical modelling
of a physical system should be based on the structures of a set \( S \) of states,
which represent the different possible, reproducible preparations, and a set \( O \)
of observables, which represent the different possible measurements. A state
and an observable should together determine a probability distribution that
corresponds to the expected frequencies of the outcomes of the same (class of)
measurement, repeated many times on a system (or a collection of independent
systems) prepared in the same state.

These ideas entail some basic properties of the sets \( S \) and \( O \) which we
will spell out next. The possibility of randomly choosing different states, with
fixed probability weights, and subjecting them to independent runs of the same
measurement implies that the set of states should be convex, thus including
all mixtures of any finite or (by extrapolation) countable subsets. Moreover it
follows that each measurement outcome, labelled by a value \( a_k \) \( (k = 1, 2, \ldots) \),
say, gives rise to an affine map

\[
E_k : S \to [0, 1], \quad \rho \mapsto E_k(\rho) = p_{\rho}(a_k)
\]

(1)

where \( p_{\rho}(a_k) \) is the probability of the outcome \( a_k \) in the state \( \rho \in S \). We can
ensure that every measurement is certain to produce an outcome by allowing
for a null event, \( a_0 \), to represent the situation where there is no response.
It follows that the sum of probabilities \( \sum_k p_{\rho}(a_k) = 1 \) for all states \( \rho \). We
summarize this by writing \( \sum_k E_k = I \), where \( I : \rho \mapsto 1 \) is the constant unit
map that sends every state to the number 1. The totality of all maps \( E_k \)
defines the observable measured in the experiment under consideration.

This consideration generalizes as follows. The set of states of a given type
of physical system is modelled as a convex set \( S \). The set \( O \) of observables is
given by a set of affine maps

\[ A : S \rightarrow M (\Omega^+_1), \quad \rho \mapsto A (\rho) \equiv \mu^A_\rho \]  

(2)

from \( S \) to the convex set \( M (\Omega^+_1) \) of probability measures over some measurable spaces \((\Omega, \Sigma)\), where \( \Omega \) denotes the set of measurement outcomes and \( \Sigma \) a \( \sigma \)-algebra of subsets of \( \Omega \) appropriate to each measurement represented in the model.

Each observable \( A \), together with a subset \( X \in \Sigma \), gives rise to an affine map

\[ E(X) : S \rightarrow [0, 1], \quad \rho \mapsto E(X)(\rho) =: A(\rho)(X) \equiv E_\rho(X). \]  

(3)

Any affine map \( a : S \rightarrow [0, 1] \) is called an effect. The map

\[ \Sigma \ni X \mapsto E(X) \]  

(4)

has the properties of a measure, with normalization \( E(\Omega) \equiv I \), the property \( E(\emptyset) = O \) (= the constant zero map which is itself an effect), and \( \sigma \)-additivity following from that of all the probability measures \( E_\rho \). We see that any observable \( A \) gives rise to an effect valued measure \( E \equiv E^A \). Conversely, every effect valued measure \( E \) on some measurable space \((\Omega, \Sigma)\) induces an observable \( A \equiv A^E \) as an affine map from \( S \) to \( M (\Omega^+_1) \). We have thus, in effect, two equivalent definitions of an observable.

Given the set \( O \) of observables, we can define the set of all physically realizable effects, denoted \( \mathcal{E} \), as the union of the ranges of all the effect valued measures associated with the observables in \( O \). This set \( \mathcal{E} \) is thus a subset of the set \( \mathcal{E}(S) \) of all effects on \( S \). The set \( \mathcal{E} \) inherits a natural partial order defined in \( \mathcal{E}(S) \) as follows:

\[ a \leq b \text{ if and only if } a(\rho) \leq b(\rho) \text{ for all } \rho \in S. \]  

(5)

We can thus write \( \mathcal{E}(S) = [O, I] \). The set \( \mathcal{E} \) represents the collection of all simple observables, that is, those with only two outcomes (also called yes-no observables): for \( a \in \mathcal{E} \), we also have \( a' := I - a \in \mathcal{E} \), and the map \( \{1\} \mapsto a, \{0\} \mapsto a' \) defines an effect valued measure on (the power set of) \( \Omega = \{0, 1\} \). We can take \( O \) to be closed under all coarse-grainings of its observables (understood as restrictions of the associated effect valued measures on \((\Omega, \Sigma)\) to sub-\( \sigma \)-algebras of \( \Sigma \).

The map

\[ \mathcal{E}(S) \ni a \mapsto a' = I - a \in \mathcal{E}(S) \]  

(6)
defines a kind of complement, or negation, and one can consider a definition of weak orthogonality of effects: \( a \perp b \) if \( a + b \leq I \). This relation is not an orthocomplement since one can have pairs of effects satisfying \( a \perp b \) while both have a common nonzero lower bound, \( 0 \leq c \leq a, b \). With the partial operation

\[
\mathcal{E} \ni a, b \mapsto a \oplus b := a + b \quad \text{if} \quad a + b \in \mathcal{E},
\]

the set of effects \( \mathcal{E} \) assumes the structure of an effect algebra. The connection between effect algebras and statistical models has been thoroughly analyzed by Beltrametti and Bugajski \(^4\) and independently by Gudder. \(^{21}\)

We will make the assumption that the set of physical effects \( \mathcal{E} \) separates \( \mathcal{S} \), that is: for any two different \( \rho, \rho' \in \mathcal{S} \), there is an effect \( a \in \mathcal{E} \) such that \( a(\rho) \neq a(\rho') \). This is a natural assumption and can be satisfied by identifying all states that would otherwise be indistinguishable. We then summarize these consideration by defining a statistical model as a pair \( (\mathcal{S}, \mathcal{E}) \) consisting of a convex set of states and a separating set of physical effects.

It is convenient to consider convex structures as embedded into their natural linear extensions. As a convex set, \( \mathcal{S} \) is part of a vector space \( \mathcal{V} = \mathcal{V} (\mathcal{S}) \) over the field of real numbers which we shall consider to be the span of \( \mathcal{S} \). The set of effects \( \mathcal{E} (\mathcal{S}) \) is a convex subset of the vector space \( \mathcal{A} = \mathcal{A}_b (\mathcal{S}) \) of all (real-valued) bounded affine functionals of \( \mathcal{S} \). Let \( \mathcal{A}^* \) denote the algebraic dual space to \( \mathcal{A} \), that is, the set of linear functionals on \( \mathcal{A} \). Then, for \( \rho \in \mathcal{V} \), the map \( a \mapsto a(\rho) \) is a linear functional on \( \mathcal{A} \), which implies that \( \mathcal{V} \) is injectively embedded into \( \mathcal{A}^* \). Thus we can introduce a (nondegenerate) bilinear form on \( \mathcal{A}^* \times \mathcal{A} \), \( \langle \cdot, \cdot \rangle \), such that for \( \rho \in \mathcal{V} \),

\[
\langle \rho, a \rangle = \rho(a) = a(\rho).
\]

(8)

With the vector space \( \mathcal{W} = \mathcal{W} (\mathcal{E}) \) generated as the span of \( \mathcal{E} \), the pair \( (\mathcal{V}, \mathcal{W}) \), equipped with the nondegenerate form \( \langle \cdot | \cdot \rangle \), is referred to as a dual pair. Every statistical model can thus be embedded in a dual pair of vector spaces.

The condition that \( \mathcal{E} \) separates \( \mathcal{S} \) ensures that the vector space \( \mathcal{V} \) can be equipped with a norm whose unit ball is \( \mathcal{B} := \text{conv} (\mathcal{S} \cup -\mathcal{S}) \); this norm, called the base norm, is the Minkowski functional of \( \mathcal{B} \), defined via

\[
m(\rho) := \inf \{ \lambda : \rho \in \lambda \mathcal{B} \} =: ||\rho||_1.
\]

(9)

In this way, \( \mathcal{V} \) is a base norm space. Since bounded affine functionals on \( \mathcal{S} \) extend uniquely to bounded linear functionals on \( \mathcal{V} \), \( \mathcal{A}_b (\mathcal{S}) \) is a subspace of the dual space \( \mathcal{V}^* \) of \( \mathcal{V} \). The set \( [O, I] = \mathcal{E} (\mathcal{S}) \subset \mathcal{A}_b (\mathcal{S}) \) is called order unit interval, and it makes \( \mathcal{A}_b (\mathcal{S}) \) an order unit space with the order unit norm equal to the dual space norm inherited from \( \mathcal{V}^* \) and the norm unit ball given
by $[-I, I]$. If the normed vector spaces $V, W$ are complete, the pair $(V, W)$ is called a statistical duality. A lucid introduction to these structures and their physical context can be found in the work of W. Stulpe \cite{stulpe}, for further details, cf. also Ref. \cite{ref} and references therein.

### 3 Classical and quantum statistical models

The traditional classical statistical model is determined by a measurable space $(\Gamma, \mathcal{B}(\Gamma))$, where $\Gamma$ is the state space of a dynamical system and $\mathcal{B}(\Gamma)$ is a $\sigma$-algebra of subsets of $\Gamma$. Usually, $\Gamma$ is a topological or metric space, e.g. a (subspace of) $\mathbb{R}^n$, and $\mathcal{B}(\Gamma)$ will then be the associated Borel algebra. The set of (statistical) states $S_c$ is usually taken to be the set of all measures $\mathcal{M}^+(\Gamma, \mathcal{B}(\Gamma))$ or sometimes the set of (measures with) probability densities with respect to some reference measure, $\mathcal{L}^1(\mathbb{R}^n, d\mu)$.

Following Beltrametti and Bugajski \cite{beltrametti}, we define the set of measurable effects $E_c$ as the collection of affine functionals on $S_c$, determined by some measurable function $f : \Gamma \to [0, 1]$:

$$a_f : S_c \to [0, 1], \mu \mapsto a_f(\mu) = \int_\Gamma f(\gamma) d\mu.$$  \hspace{1cm} (10)

The order relation $a_f \leq a_g$ is now equivalent to $f \leq g$ in the case $S_c = \mathcal{M}^+(\Gamma, \mathcal{B}(\Gamma))$ [take $\mu = \delta_\gamma$, the Dirac measure supported at $\gamma \in \Gamma$] or $f \leq g$ almost everywhere with respect to the measure $m$ for $S_c = L^1(\Gamma, dm)$.

An observable is an affine map $A : S_c \to M(\Omega, \Sigma)$ from the given set of classical states to the set of probability measures on a measurable space $(\Omega, \Sigma)$ of measurement outcomes. This comprises the traditional definition of a classical observable as a function on phase space (random variable). Let $F : \Gamma \to \Omega$ be a measurable function, then we define an affine map $A_F : S_c \to M(\Omega, \Sigma)$ as follows:

$$A_F \mu(X) := \mu(F^{-1}(X)) = \int_\Gamma \delta_{F(\gamma)}(X) d\mu(\gamma), \quad X \in \Sigma.$$  \hspace{1cm} (11)

The associated effect valued measure is given by

$$E_F \equiv F^{-1} : \Sigma \to \mathcal{B}(\Gamma), \quad X \mapsto F^{-1}(X) = a_{\chi_{F^{-1}(X)}}.$$  \hspace{1cm} (12)

As a function on $\Gamma$, this corresponds to the indicator function $\chi_{F^{-1}(X)}$. (Note that $\chi_{F^{-1}(X)}(\gamma) = \delta_{F(\gamma)}(X)$, hence the above integrand is a measurable and indeed integrable function.)
This construction of the standard classical observable extends to the class of observables with measurable effects as follows. The function \( (\gamma, X) \mapsto \chi_{F^{-1}(X)}(\gamma) \) is a particular instance of a Markov kernel
\[
K : \Gamma \times \Sigma \rightarrow [0, 1], \quad (\gamma, X) \mapsto K(\gamma, X),
\]
that is, a map with the property that \( K(\gamma, \cdot) \in M(\Omega, \Sigma)^+ \) and \( K(\cdot, X) \) is a measurable function, in fact, an effect. Then the following defines an observable \( A_K \) which represents a fuzzy random variable:
\[
A_K \mu(X) := \int_{\Gamma} K(\gamma, X) \, d\mu(\gamma), \quad X \in \Sigma,
\]
with associated effect valued measure
\[
E_K : \Sigma \rightarrow E, \quad X \mapsto a_{K(\cdot, X)}.
\]
A quantum statistical model is usually based on the set of states \( S_q = S(H) \) given by the density operators \( \rho \) of a separable complex Hilbert space \( H \). Conventional quantum mechanics takes its set of physical effects \( E^p \) as the set of all orthogonal projection operators \( P, P = P^* = P^2 \). Operational quantum physics is based on the set of all effects \( E_q = E(H) = [O, I] \), given by the operators \( a \) for which \( O \leq a \leq I \). In both cases, projections and effects define affine functionals on \( S_q \) via the trace formula,
\[
a(\rho) = \text{tr} [\rho \cdot a].
\]
Both sets \( E^p_q \) and \( E_q \) are effect algebras, with the set of projections being the set of extreme elements of the convex set of effects. \( E^p_q \) is an orthocomplemented non-Boolean lattice under the ordering \( a \leq b \) and complementation map \( a \mapsto a' \) inherited from \( E_q \).

A characteristic difference between quantum and classical statistical models is given by the fact that \( S_q \) is a simplex (all states have a unique representation as a convex combination (finite, countable or continuous) in terms of the extreme elements of \( S_q \), while all mixed quantum states allow many decompositions into pure states. We illustrate this state of affairs by means of the smallest nontrivial example of a quantum statistical model associated with the Hilbert space \( H = \mathbb{C}^2 \). This will be compared with the classical statistical model based on a set \( \Omega = \{1, 2, 3, 4\} \).

The classical statistical model for \( \Omega_4 = \{1, 2, 3, 4\} \), with \( \Sigma = \mathcal{P}(\Omega_4) \) (the power set of \( \Omega_4 \)), is given by \( S(\Omega_4) \subset \mathbb{R}^4 = V(\Omega_4) \), where
\[
S(\Omega_4) = \{p = (p_1, p_2, p_3, p_4) : p_k \geq 0, p_1 + p_2 + p_3 + p_4 = 1\}.
\]
This is a tetrahedron with vertices \((1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1)\). The set of effects \( E(\Omega_4) \) comprises all the maps
\[
a : p \mapsto a(p) = a \left( \sum p_k e_k \right) = \sum p_k a(e_k) = \sum p_k a_k
\]
with
for which \( a(p) \in [0, 1] \). [Here the \( e_k \) denote the Cartesian unit basis vectors, and \( a_k := a(e_k) \).] This condition entails that \( 0 \leq a_k \leq 1 \). Hence the set of effects can be identified as a subset of \( \mathbb{R}^4 \equiv W(\Omega_4) \):

\[
\mathcal{E}(\Omega_4) = \{ a = (a_1, a_2, a_3, a_4) : 0 \leq a_k \leq 1 \}.
\]

(18)

This is a hypercube with the vertices given by the 16 points whose coordinates are quadruples of 0s and 1s. The bilinear form associated with the dual pair \( \langle \mathcal{V}(\Omega_4), W(\Omega_4) \rangle \) is the Euclidean inner product,

\[
\mathcal{V}(\Omega_4) \times W(\Omega_4) \ni (p, a) \mapsto \langle p, a \rangle = \sum p_k a_k = p \cdot a.
\]

(19)

The quantum statistical model for \( \mathcal{H} = \mathbb{C}^2 \) (a spin-\( \frac{1}{2} \) system or generally a qubit) can be parameterized in the Cayley representation as follows. As a basis of the space of \( 2 \times 2 \) matrices (linear operators of \( \mathbb{C}^2 \)) we take

\[
I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\]

(20)

where the \( \sigma_k \) are the Pauli matrices. Every density operator can be written as \( \rho = \frac{1}{2} (r_0 I + r \cdot \sigma) \) [where \( r = (r_1, r_2, r_3) \in \mathbb{R}^3, \sigma = (\sigma_1, \sigma_2, \sigma_3) \)] and is thus associated with a vector \( \hat{\rho} = (r_0, r) \in \mathbb{R}^4 \). The fact that \( \rho \) assigns probability one to the effect \( I \) entails that \( \text{tr}(\rho) = r_0 = 1 \). The condition that \( \rho \geq O \) can be expressed in terms of the eigenvalues of \( \rho \): \( \frac{1}{2} (1 \pm |r|) \geq 0 \), which is equivalent to the Euclidean norm \( |r| \leq 1 \). The set \( \mathcal{S}(\mathbb{C}^2) \) is thus isomorphically represented by a subset \( \tilde{\mathcal{S}}(\mathbb{C}^2) \) of \( \mathbb{R}^4 \),

\[
\mathcal{S}(\mathbb{C}^2) = \{ \rho : \text{tr}(\rho) = 1, \rho \geq O \} \leftrightarrow \tilde{\mathcal{S}}(\mathbb{C}^2) = \{ \hat{\rho} = (1, r) : |r| \leq 1 \}.
\]

(21)

\( \tilde{\mathcal{S}}(\mathbb{C}^2) \) is a 3-dimensional unit ball embedded in the hyperplane \( r_0 = 1 \). The effects of \( \mathbb{C}^2 \) are the operators (matrices) \( a = \frac{1}{2} (a_0 I + a \cdot \sigma) \) satisfying the condition \( O \leq a \leq I \), which is equivalent to \( 0 \leq \frac{1}{2} (a_0 \pm |a|) \leq 1 \). Thus \( \mathcal{E}(\mathbb{C}^2) \) is isomorphic to a subset \( \tilde{\mathcal{E}}(\mathbb{C}^2) \) of \( \mathbb{R}^4 \),

\[
\mathcal{E}(\mathbb{C}^2) = \{ a : O \leq a \leq I \} \leftrightarrow \tilde{\mathcal{E}}(\mathbb{C}^2) = \{ \hat{a} = (a_0, a) : \frac{1}{2} (a_0 \pm |a|) \leq 1 \}.
\]

(22)

\( \mathcal{E}(\mathbb{C}^2) \) (or rather \( \tilde{\mathcal{E}}(\mathbb{C}^2) \)) can be visualized as a convex diamond-shaped figure that is the convex hull of the ‘top’ and ‘bottom’ vertices given by \( I \leftrightarrow (2, 0), O \leftrightarrow (0, 0) \) and the surface of the sphere \( \{ (1, a) : |a| = 1 \} \).

The bilinear form associated with the dual pair of vector spaces which host \( \mathcal{S}(\mathbb{C}^2), \mathcal{E}(\mathbb{C}^2) \) is given as follows:

\[
\langle \rho, a \rangle = \text{tr} [\rho \cdot a] = \frac{1}{2} (a_0 + a \cdot r) = \frac{1}{2} \hat{a} \cdot \hat{\rho}
\]

(23)
4 Embeddings and extensions of statistical models

We consider next the various possible ways of embedding or extending one statistical model \( \langle S, E \rangle \) into another, \( \langle S', E' \rangle \), in a ‘natural’ way. That is, we are asking for an association of states \( \rho \) and effects \( a \) of \( \langle S, E \rangle \) with states \( \rho' \) and effects \( a' \) of \( \langle S', E' \rangle \), \( \rho, a \leftrightarrow \rho', a' \), which is such that probabilities are preserved: \( \langle \rho, a \rangle = \langle \rho', a' \rangle \); further we require that the convex structures are respected by the sought correspondence.

We shall investigate two obvious choices which ensure that all states of \( S \) will be represented:

(A) There is an injective affine map \( \Phi : S \to S' \); this embedding map should give a faithful representation of the states of \( S \) in a possibly ‘larger’ state space.

(B) There is a surjective affine map \( R : S' \to S \); this reduction map would describe the system represented by \( S \), possibly as a part of subsystem of a larger, compound system.

In fact, both maps could exist simultaneously, being inverses to each other. In this case the two state spaces are isomorphic. In both cases (A) and (B) we will have to determine whether all effects of \( E \) can indeed be represented by effects of \( E' \), as envisaged.

Assume an affine map \( \Phi : S \to S' \) is given. This extends uniquely to a linear map from \( V(S) \) to \( V(S') \), which we will also denote \( \Phi \). This induces the dual map \( \Phi^*: V(S')^* \to V(S)^* \), defined by the condition

\[
\langle \Phi^* \rho, a' \rangle = \langle \rho, \Phi^* a' \rangle, \quad \text{i.e.,} \quad \Phi^* a' = a' \circ \Phi, \quad \rho \in V(S), \ a' \in V(S')^*.
\]

(24)

\( \Phi^* \) sends \( E(S') \) into \( E(S) \): indeed, for \( a' \in E(S') \), we have \( (\Phi^* a') (\rho) = a' (\Phi \rho) \in [0,1] \) for all \( \rho \in S \), that is, \( \Phi^* a' \in E(S) \). The requirement of injectivity of \( \Phi \) implies that the range of \( \Phi^* \) is (weak-*) dense in \( V(S)^* \). However, this does not imply that \( \Phi^* (E(S')) \) is dense in \( E(S) \). We will say that an injective affine map \( \Phi : S \to S' \) induces a ‘good’ embedding of the model \( \langle S, E \rangle \) within \( \langle S', E' \rangle \) if \( \Phi^* (E') \) is at least dense in \( E \), so that every physical effect of the original model can be described by a physical effect of the extended model. We shall see that there is no ‘good’ classical embedding of the full quantum statistical model.

Now consider the case where there exists an affine, surjective reduction map \( R : S' \to S \). This having a surjective extension to \( V(S') \), it follows that the dual map \( R^* \) provides an injective representation of all effects in \( E(S) \) as effects in \( E(S') \) via \( R^* a = a \circ R \). The map \( R \) is thus seen to be a ‘good’ extension.
of the statistical duality \( \langle S, E \rangle \) to \( \langle S', E' \rangle \) (provided it exists). We will review the Misra-Bugajski map as an example of a ‘good’ classical extension of the quantum statistical model.

5 Embeddings and extensions of quantum statistical models

We give an overview of the known types of embeddings and extensions of quantum statistical models, including the classical representations. We begin with an ‘unsuccessful’ attempt, one which nevertheless has provided important insights into the structure of quantum mechanics.

5.1 Wigner function - a counter example

The question of classical representations of quantum mechanics is as old as quantum mechanics itself: already in Born’s famous paper of 1926 introducing the probability interpretation \[2\], the question of extensions of quantum mechanics in the framework of a classical theory with hidden variables was raised. A few years later, Wigner \[27\] attempted to establish a phase space formulation of quantum mechanics and found an injective (in fact isometric) affine map \( W : S_q \to L^1 (\Gamma, dq dp) \) of the density operators \( \rho \) of a quantum particle to integrable and normalized functions of phase space. This map yields the correct marginal position and momentum distributions associated with each \( \rho \), but the functions \( W \rho (q, p) \) are not nonnegative (apart from a ‘few’ exceptions such as Gaussian wave functions). That is to say that the Wigner map \( W \) does not map into \( L^1 (\Gamma, dq dp)_{+} \), the probability densities on phase space. This rules out \( W \) as a classical embedding of quantum mechanics in the sense defined here.a

5.2 Classical embedding induced by an observable

However, we have already seen examples of candidates for classical embeddings, namely, in the form of the affine maps \( A : S_q \to M_{1}^+ (\Omega, \Sigma) \) which define the observables of the quantum statistical model. The dual map sends classical effects to quantum effects; in particular it takes the ‘crisp’ (sharp) effects represented by characteristic functions to the effects in the range of the POVM \( E^A \) associated with \( A \):

\[ \Sigma \ni X \mapsto A^* (a_{\chi_X}) = E^A (X) \in E_q. \]  

(25)

\[ \text{aSee also the penetrating remarks of S. Bugajski} \]
Among the quantum observables $A$ there are injective ones, and the associated effect valued measures (or POVMs) are called informationally complete.\footnote{The notion of informationally complete observables and first examples are due to work of the late E. Prugovecki from the 1970s. Examples and a survey of the early literature on this concept can be found in\cite{3}.} If $E^A$ is informationally complete, the dual map $A^*$ sends the dual space of $M(\Omega, \Sigma)$ onto a dense subspace of the space of bounded selfadjoint operators (the dual of the space of selfadjoint trace class operators hosting $S_\rho$). However, the set of measurable classical effects which arises as the convex hull of all $a_{xy}$ can never exhaust, or be dense in, the set of quantum effects $E^q$: this is due to the fact that for an informationally complete POVM, the effects $E^A(X)$ cannot be projections other than $O$ or $I$.\footnote{A proof of this fact was given by Busch and Lahti\cite{15}. Intuitive demonstrations for the finite dimensional case can be found in\cite{14}.} This means that an informationally complete observable $A$ does not induce a ‘good’ classical embedding of the full quantum statistical model $\langle S, E \rangle$ but only of a reduced quantum model $\langle S, E(A) \rangle$, with $E$ replaced by $E(A) = E^A(\Sigma)$, the separating effect algebra consisting of the range (or the convex hull of the range) of the POVM $E^A$.

An example of an informationally complete observable is given by the coherent state based phase space POVM, which corresponds to the Husimi distribution functions associated with each quantum state. These are bona fide phase space probability densities but the price to be paid is that its marginal position and momentum distributions are convolutions of the standard quantum mechanical position and momentum distributions with Gaussian confidence distributions, thus ensuring that appropriate Heisenberg uncertainty relations for the inaccuracies are satisfied. Wigner’s theorem can thus be interpreted as a demonstration of the incompatibility of the standard position and momentum observables, while the existence of the Husimi and other, similar phase space distributions shows that joint measurements of position and momentum are possible if an allowance is made for unsharpness in line with the uncertainty relation.\footnote{A detailed account of phase space observables, joint measurements of position and momentum, the role of the Heisenberg uncertainty relation, and a survey of relevant literature is given in\cite{13}.}

\footnotetext{\cite{11}} Informationally complete observables for $\mathcal{H} = \mathbb{C}^2$ are easily constructed\cite{11}, together with simple, realizable measurement schemes.\cite{12,13} Geometrically, they can be represented as affine embeddings of the Poincaré sphere $\tilde{S}(\mathbb{C}^2)$ into the set $M_1^+(\Omega, \Sigma)$. With the choice $\Omega_4 = \{1, 2, 3, 4\}$ one can ensure that the embedding is not only injective but that its linear extension is surjective. In this case the sphere $\tilde{S}(\mathbb{C}^2)$ is mapped onto an ellipsoid which is embedded
into the tetrahedron $\mathcal{S}(\Omega_4)$. The dual map sends the hypercube $\mathcal{E}(\Omega_4)$ into a ‘stretched’ hypercube inside the ‘diamond’ $\tilde{\mathcal{E}}(\mathbb{C}^2)$, in such a way that the elements $(1,1,1,1)$ and $(0,0,0,0)$ [which represent the $I$ and $O$ effects] are mapped to $(2,0)$ and $(0,0)$, respectively. This description makes it evident that the extreme points of the set of quantum effects [other than $I$ and $O$] cannot be represented in terms of classical effects, except, perhaps, in finitely many cases where some extreme points of the hypercube touch the extreme boundary of the diamond.

5.3 Cayley representation of the $\mathbb{C}^2$ statistical duality

The association $\rho \longleftrightarrow (1,r), \ a \longleftrightarrow (a_0,a)$ reviewed above defines a bijective affine mapping $\Phi : \mathcal{S}(\mathbb{C}^2) \rightarrow \tilde{\mathcal{S}}(\mathbb{C}^2)$. Likewise, the dual map $\Phi^* : \tilde{\mathcal{E}}(\mathbb{C}^2) \rightarrow \mathcal{E}(\mathbb{C}^2)$ is an affine bijection. Hence we have both a ‘good’ embedding (via $\Phi$) and a ‘good’ extension (via $\Phi^{-1}$) of $(\mathcal{S}(\mathbb{C}^2), \mathcal{E}(\mathbb{C}^2))$. In fact the linear extensions of these maps are isometries with respect to the trace norm $||a||_1 = \text{tr}||a||$ and the norm $||(a_0,a)|| = \max \{a_0, |a|\}$.

5.4 Gleason’s theorem (and a simple variant)

The essence of standard quantum mechanics is captured in the standard quantum statistical model $\langle \mathcal{S}_q, \mathcal{E}_q \rangle$. Let $v : \mathcal{E}_q^* \rightarrow [0,1]$ be a generalized probability measure on the lattice of projections $\mathcal{E}_q$, that is, a map which satisfies the conditions $v(O) = 0$, $v(I) = 1$, and $v(\sum_k P_k) = \sum_k v(P_k)$ for any finite or countable set of pairwise orthogonal $P_k \in \mathcal{E}_q$. Let $\mathcal{S}_q^p$ denote the set of all such generalized probability measures. This is a convex set, and each projection $P \in \mathcal{E}_q^p$ defines an effect $a_P$ on $\mathcal{S}_q^p$ via $v \mapsto a_P(v) = v(P)$. These effects separate $\mathcal{S}_q^p$ since $v(P) = v'(P)$ for all $P$ implies $v = v'$. Hence $\langle \mathcal{S}_q^p, \mathcal{E}_q^p \rangle$ is a statistical model. Gleason’s theorem asserts that if the dimension of the underlying Hilbert space is greater than 2, then for every $v \in \mathcal{S}_q^p$, there is a unique $\rho \in \mathcal{R}_q$ such that $v(P) = \text{tr}[\rho \cdot P]$ for all $P \in \mathcal{E}_q$. This association $R_G : v \mapsto \rho = \rho_v$ is bijective and thus affine. Hence it induces a ‘good’ embedding and extension of $\langle \mathcal{S}_q, \mathcal{E}_q^p \rangle$ in terms of $\langle \mathcal{S}_q^p, \mathcal{E}_q^p \rangle$.

A similar but much simpler result arises for the statistical model $\langle \mathcal{S}_q, \mathcal{E}_q \rangle$ of operational quantum mechanics if we define $\mathcal{S}_q^e$ to be the set of all generalized probability measures on the full set of quantum effects $\mathcal{E}_q$. The set $\mathcal{E}_q$ comprises enough elements so as to include basis systems of effects $a_k$ such that $\sum_k a_k$ is an effect. This readily entails that every generalized probability measure $v$ on $\mathcal{E}_q$ extends to a bounded linear functional on the space of bounded selfadjoint operators. The $\sigma$-additivity of $v$ implies that this functional arises from a
density operator via the trace formula, \( \nu(a) = \text{tr}\, [\rho \cdot a] \). Hence we have a ‘good’ embedding and extension of \( (S_q^e, \mathcal{E}_q^e) \) to a statistical model \( (S_q^e, \mathcal{E}_q^e) \). This time there is no restriction on the dimension of the Hilbert space. The generalized probability measures on the set of effects restrict to generalized probability measures on the projections, which entails that \( S_q^e \) is a proper subset of \( S_p^q \) in the case \( \mathcal{H} = \mathbb{C}^2 \) while \( S_q^e = S_p^q \) in the case \( \dim \mathcal{H} > 2 \).

5.5 Compound systems extension

Let \( \mathcal{H} = \mathcal{H} \otimes \mathcal{H}' \) be the tensor product of the Hilbert space \( \mathcal{H} \) of the system of interest with an auxiliary Hilbert space \( \mathcal{H}' \). Denote by \( R \) the partial trace map from the trace class of \( \mathcal{H} \) onto the trace class of \( \mathcal{H} \). This is a linear map, and its restriction to \( S(\mathcal{H}) \) has as its range all of \( S(\mathcal{H}) \). The dual map \( R^* : a \mapsto a \otimes I \) is a linear injection or all bounded operators of \( \mathcal{H} \) into the space of bounded linear selfadjoint operators of \( \mathcal{H} \), with the property \( O \leq a \leq I \Rightarrow O \leq R^*(a) \leq R^*(I) = I \otimes I \). In this way, the statistical model \( (S(\mathcal{H}), \mathcal{E}(\mathcal{H})) \) arises as a ‘good’ extension of the statistical model \( (S(\mathcal{H}), \mathcal{E}(\mathcal{H})) \).

5.6 Canonical classical extension of quantum mechanics

We now assume that we are given a ‘good’ classical extension induced by a surjective affine map \( R : M_1^+(\Omega, \Sigma) \rightarrow S(\mathcal{H}) \). We will make a few assumptions on \( (\Omega, \Sigma) \) and take a few steps to eliminate ‘redundancies’. This will lead us in a fairly natural way to exhibit the canonical classical extension introduced by Misra and recognized by Bugajski as a ‘good’ extension of the maximal quantum statistical model into a distinguished classical statistical model. The assumptions on \( (\Omega, \Sigma) \) are:

(a) \( \Sigma \) contains all singletons \( \{\omega\}, \omega \in \Omega \);

(b) the extreme elements of \( M_1^+(\Omega, \Sigma) \) are exactly the Dirac measures \( \delta_\omega \), so that each \( \mu \in M_1^+(\Omega, \Sigma) \) can be uniquely written as

\[
\mu = \int_\Omega \delta_\omega \, d\mu(\omega).
\]

Consider any density operator \( \rho \) corresponding to a pure state, \( \rho = P_\varphi \equiv |\varphi\rangle \langle \varphi| \). Let \( \mu \in M_1^+(\Omega, \Sigma) \) be any measure for which \( R\mu = \rho \). It follows that \( R\delta_\omega = \rho \) for all \( \delta_\omega \) which occur in the convex decomposition (26) of \( \mu \), and every pure state will then be an image under \( R \) of some \( \delta_\omega \). We will consider \( \Omega \) as restricted to those \( \omega \) for which \( R\delta_\omega \) is pure. [Note that this presupposes

This fact is proved and discussed for quantum mechanics in [13][14]. It has been proved in a much more abstract context by Beltrametti and Bugajski [15].

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\[26\]
that \( \{ \omega \in \Omega : R\delta_\omega \text{ is pure} \} \in \Sigma \). We will see presently that this can be trivially guaranteed.] Further we can identify (as equivalent) all \( \omega, \omega' \in \Omega \) for which \( R\delta_\omega = R\delta_{\omega'} \), and replace \( \Omega \) with the set (also denoted \( \Omega \)) of all equivalence classes \( [\omega] \), \( \Sigma \) with the \( \sigma \)-algebra (again denoted \( \Sigma \)) of sets \( [X] := \{ [\omega] : \omega \in X \} \) and \( R \) with the induced map, also called \( R \), that sends \( \delta_\omega \) to \( R\delta_\omega \). In this way we identify \( \Omega \) with \( \text{Ex}[\mathcal{S}(\mathcal{H})] \), the set of pure density operators.

Now we let \( \Omega = \text{Ex}[\mathcal{S}(\mathcal{H})] \) and \( \omega \) denote any pure state in \( \mathcal{S}(\mathcal{H}) \). The affine reduction map \( R \) must then satisfy

\[
R : \mu = \int_\Omega \delta_\omega \, d\mu(\omega) \mapsto \int_\Omega R\delta_\omega \, d\mu(\omega) = \int_\Omega \omega \, d\mu(\omega) = \rho_\mu.
\]

(27)

The nondegenerate bilinear forms induced by the classical and quantum statistical models involved is given by

\[
\langle \rho_\mu, a \rangle = \text{tr} [\rho_\mu \cdot a] = \int_\Omega \text{tr} [\omega \cdot a] \, d\mu(\omega) = \langle \mu, f_a \rangle.
\]

(28)

This determines the dual map

\[
R^* : a \mapsto f_a, \quad f_a(\omega) = \text{tr} [\omega \cdot a].
\]

(29)

For this formula to make precise sense, it is necessary that \( \Sigma \) is fixed in such a way that the functions \( f_a \) are measurable. Misra \([25]\) has shown how to achieve this. Thus every quantum effect \( a \in \mathcal{E}(\mathcal{H}) \) is represented as a classical measurable effect \( f_a \). Hence \( R \) induces a ‘good’ classical extension of the quantum statistical duality.

We can see from Eq. (29) that all quantum effects, including the ‘sharp’ or ‘crisp’ ones given by projections, are represented by functions whose values vary continuously between their maximum and minimum values. For any projection \( a \) not equal to \( I \) or \( O \), \( f_a(\omega) \) assumes all values in \([0,1]\). Hence the projections are represented as classical fuzzy sets. All quantum observable, represented by POVMs \( E \), including the PVMs, are described under the map \( R^* \) as classical fuzzy random variables with associated Markov kernel \( K^E(\omega, X) = \text{tr} [\omega \cdot E(X)] \).

The many-to-one relationship between the probability measures \( \mu \) and the quantum states \( \rho_\mu = R\mu \) reflects exactly the infinitely many ways in which every density operator which is not a pure state can be written as a convex combination of pure states. The set of all classical effect valued measures

\[\footnote{The Misra-Bugajski map is an instance of a much more general mathematical result of Choquet on boundary measures of compact convex sets in the context of locally convex vector spaces. See Theorem I.4.8 of Alfsen [1].} \]

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$R^* \circ E$ represents all quantum mechanical measurements as classical fuzzy measurements which even taken together are too imprecise to separate the various different probability measures $\mu, \mu', \ldots$ which lead to the same density operator $\rho = R\mu = R\mu' = \ldots$.

6 Concluding remarks

In this contribution I have reviewed the concept of statistical model and the various possible relations of embeddings and extensions between statistical models. The possible ways of constructing ‘good’ classical representations of either the full or some restricted quantum statistical models have been exhibited. The only ‘good’ classical extension of the full quantum statistical model is uniquely given (modulo redundancies) by the Misra-Bugajski map. This canonical classical extension of quantum mechanics does not constitute a hidden-variable completion of quantum mechanics because it does not render the sharp quantum properties (projections) as dispersion-free in the extremal classical states. On the contrary, under this classical extension all quantum effects – whether unsharp or sharp – are represented as classical fuzzy sets.

While the classical embeddings of quantum mechanical statistical models via effect valued measures (which may or may not be informationally complete) have been extensively studied and led to a plethora of applications, much remains to be done to explore and exhaust the full potential of the canonical classical embeddings. In their very last joint work, Beltrametti and Bugajski made some tentative steps to characterize quantum correlations as opposed to classical correlations in this framework. The ‘Bell phenomenon in classical framework’, that is, the violation of Bell’s inequalities for fuzzy random variables corresponding to EPR-Bell observables, calls for an investigation of the notion of coexistence (joint measurability) of fuzzy random variables. More generally, it would be desirable to cast all the distinctive quantum structures, such as non-commutativity, complementarity, uncertainty relations, entanglement, in the language of the classical canonical extension. This should enable us to relate these quantum features to concepts closer to our ‘classical’ experience and intuition – in other words: it should contribute to a better understanding of quantum mechanics.

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