MULTI-STAGE DISTRIBUTIONALLY ROBUST OPTIMIZATION WITH RISK AVersion

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Abstract. Two-stage risk-neutral stochastic optimization problem has been widely studied recently. The goals of our research are to construct a two-stage distributionally robust optimization model with risk aversion and to extend it to multi-stage case. We use a coherent risk measure, Conditional Value-at-Risk, to describe risk. Due to the computational complexity of the nonlinear objective function of the proposed model, two decomposition methods based on cutting planes algorithm are proposed to solve the two-stage and multi-stage distributional robust optimization problems, respectively. To verify the validity of the two models, we give two applications on multi-product assembly problem and portfolio selection problem, respectively. Compared with the risk-neutral stochastic optimization models, the proposed models are more robust.

1. Introduction. Two-stage and multi-stage stochastic programming problems have been widely studied in recent decades. A large number of significant applications have been found in different fields, such as financial planning ([29, 10]), supply chain management ([45, 47, 32, 41, 38, 25, 37]), multi-activity tour scheduling ([36, 40]), signal processing ([20, 21]) and pre-positioning of emergency supplies ([39]), etc. A two-stage stochastic linear programming model can be formulated as follows (see [46, 8]),

$$\min_{x \in \mathcal{X}} \mathbf{c}^T \mathbf{x} + \mathbb{E}[\mathcal{Q}(\mathbf{x}, \xi)],$$

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where $Q(x, \xi)$ is the optimal value of the second stage recourse problem
\[
\min_{y \in \mathbb{R}^m} g(\xi)^T y \\
\text{s.t. } G(\xi)x + H(\xi)y = h(\xi), y \geq 0.
\]

Here $x \in \mathcal{X} := \{ x \in \mathbb{R}^n : Ax = b, x \geq 0 \}$ is the decision vector of the first stage and $\xi$ is assumed a random variable defined on a probability space $(\Omega, \mathcal{F}, P)$, where $\Omega$ denotes the sample space, $\mathcal{F}$ denotes a $\sigma-$algebra on $\Omega$ and $P$ denotes the probability distribution. The expectation operator in problem (1) is taken with respect to the probability distribution of $\xi$. Vectors $c, b$ and matrix $A$ are assumed to be deterministic, and problem (1) is said to be fixed recourse if the matrix $H$ is also fixed. Multi-stage stochastic linear programming is the extension of two-stage stochastic linear programming to multi-stage, in which the output of lower stage optimization problem works as the input of the upper stage optimization problem.

Since two-stage and multi-stage stochastic linear programming problems have very specific structures, they can be solved by decomposition methods (see [43, 8]). It is worth noting that the expected recourse function in the objective of problem (1) is risk-neutral. Therefore, problem (1) is called a two-stage risk-neutral stochastic programming model. However, most of decision makers incline to risk aversion and the risk-neutral approach cannot meet the needs of these decision makers. It is necessary to study two-stage and multi-stage stochastic programming with risk aversion.

Several researchers have studied two-stage and multi-stage stochastic programming problems with risk aversion and proposed some efficient solution methods. [1] considered a weighted mean-risk objective, where some dispersion statistic was used as a measure of risk. Meanwhile, it proposed a decomposition-based parametric cutting plane algorithms. [28] proposed a risk-averse two-stage stochastic linear programming problem, in which the objective function was formulated as a composition of conditional risk measures. It derived necessary and sufficient optimality conditions and constructed a decomposition method for solving the problem. [33] considered a risk-averse two-stage stochastic programming model, where the Conditional Value-at-Risk (CVaR) was specified as the risk measure. Two decomposition algorithms based on the generic Benders-decomposition approach, single-cut and multi-cut versions respectively, were presented to solve such problems. [12] extended the on-demand accuracy approach of [34] to constrained convex optimization problems and applied the resulting method to risk-averse two-stage stochastic programming problems. [52] studied a risk-averse approach to multi-stage stochastic linear programming, where the conditional value-at-risk was incorporated into the objective function as the risk measure. It solved the resulting risk-averse model via the nested L-shaped method.

All the literatures mentioned above assume that the random variables in stochastic programming satisfy some known probability distributions, which are inconsistent with the actual situations. To deal with the problems of distributional ambiguity, [30] developed a general robust optimization formulation of large-scale systems. [48] studied the convergence and equilibrium problems for single stage distributionally robust optimization. [24] considered a robust two-stage stochastic linear optimization with risk aversion model, where the distribution of the underlying random variables was assumed to belong to a certain family of distributions. [18] developed a risk-averse two-stage stochastic programming, where the distributional ambiguity sets were constructed by $L_\infty$-norm, joint $L_1$- and $L_\infty$-norm, and
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In this paper, we first consider a two-stage distributionally robust optimization (TSDRO) model (3) with the weighted mean-risk recourse function, and then extend it to multi-stage case. The model can be written as follows,

\[ \min_{x \in \mathcal{X}} c^T x + \sup_{P \in \mathcal{P}} \{(1 - \lambda) \mathbb{E}_P[Q(x, \xi)] + \lambda R(Q(x, \xi))\}, \tag{3} \]

where \( \mathcal{P} \) denotes an ambiguity set contained probability distribution \( P \) of random vector; \( R \) is a risk measure function; \( \lambda \in [0, 1] \) represents a risk aversion factor. When \( \lambda = 0 \), (3) degenerates into a risk-neutral two-stage stochastic optimization model.

We assume that
- \( \mathcal{X} = \{x \in \mathbb{R}^n : Ax = b, x \geq 0\} \) is non-empty.
- All probability distribution \( P \in \mathcal{P} \) has finite support \( \Omega \).
- All vectors and matrices in two-stage and multi-stage optimization problem have suitable dimensions.

The key to problem (3) is how to define the ambiguity set \( \mathcal{P} \) and how to solve it against worst-case realizations within the ambiguity set. If the ambiguity set is chosen inappropriately, the TSDRO model may be over-conservative or intractable. [50] introduced standardized ambiguity sets, where all distributions were contained with prescribed conic representable confidence sets and with mean values residing on an affine manifold.

In the existing literatures, there are two common approaches to construct the ambiguity set: moment-based and statistical-distance-based methods.

In the moment-based approach, the mean and variance of the distribution function are often assumed to be fixed and to satisfy certain conditions. [11] proposed a single-stage distributionally robust CVaR model under moment uncertainty. [49] studied distributionally robust joint chance constraints with second-order moment information. Similarly, [24] obtained the ambiguity set by the bounded moments information.

However, the shortcoming of ambiguity set constructed by the moment-based method is that different distribution functions may have the same moments. The statistical-distance-based approach aims to construct the ambiguity set by requiring that the true distributions are close, in terms of statistical distance, to some reference distributions. [15] considered a two-stage stochastic mixed-integer programming under Kolmogorov metric. [3] proposed robust linear optimization problems with ambiguity sets defined by \( \phi \)-divergence and considered the corresponding robust counterparts.

The Kullback-Leibler divergence (KL-divergence) measure is the most famous special case of the \( \phi \)-divergences measures family (see [35] and [3] for details). KL-divergence is now widely applied in the area of operation research. It provide a measure of distance between true probability distribution and nominal one. In this paper, we choose KL-divergence to construct an ambiguity set. The critical reason is its tractability in application of distributionally robust optimization (DRO), i.e., ambiguity set defined by KL-divergence can be transformed into a smooth counterpart easily. Furthermore, TSDRO based on (7) can be less conservative than those DRO model based on the moment-based approach (see [19] for details).
In comparison with [24], [22] and [18], the methods of constructing ambiguity set and solving the robust optimization model in this paper are both different. [24] converted the risk-averse two-stage stochastic programming model into a semidefinite programming problem. [22] and [23] considered consider distributionally robust two-stage stochastic convex programming problems with ambiguity sets proposed by [50], and converted this class of problems to a conic optimization problems. [26] proposed a data-driven two-stage distributionally optimization framework with $\phi-$ divergences constrained and presented a decomposition-based solution algorithm to solve the resulting models. [18] developed solution algorithms based on the sample average approximation method. Different from an approximate generalized Newton method presented by [27], we propose a nested decomposition algorithm based on cutting planes method for multi-stage distributionally robust optimization model.

Compared with [26], the Algorithm 1 proposed in the Section 3.2 is an application of the proposed algorithm in [26] to the case of KL-divergence. The main contributions of this paper are summarized as follows,

- We consider a TSDRO model with risk aversion, where the ambiguity set is defined by KL-divergence. In contrast to [26], the risk measure, CVaR, in the objective function of TSDRO model is explicit. Furthermore, we reformulate the TSDRO model to its tractable counterpart and extend it to multi-stage case.
- Due to the computational complexity of the nonlinear objective function of the proposed model, we propose a multi-cut decomposition method and a nested decomposition algorithm based on cutting planes method, and show the convergence property of the proposed algorithms.
- We solve the multi-product assembly and the portfolio selection problem with the proposed approaches. Compared with the risk-free multi-stage stochastic optimization model, we demonstrate the effectiveness of the proposed methods.

The rest of this paper is organized as follows. We introduce some preliminaries regarding KL-divergence and a coherent risk measure in Section 2. In Section 3, we consider the construction of two-stage distributionally robust optimization programs with risk aversion and propose a decomposition algorithm based on cutting planes method. In Section 4, we study a multi-stage distributionally robust optimization model with risk aversion and give a nested decomposition method. Section 5 presents two applications, and Section 6 concludes.

**Notations.** In this paper, we define vectors with boldface lowercase letters and denote matrices with boldface uppercase letters. $a(\xi)$ and $A(\xi)$ denote the random vector and random matrix, respectively. $\mathcal{P}$ denotes the ambiguity set of the distribution function. $1$ and $0$ represent a vector for which all of its elements are 1 and 0, respectively. $E_P[\cdot]$ denotes expectation under the probability distribution $P$. $T$ denotes the transposition of a matrix or a vector. The superscript $k$ denotes the iteration counter. We denote by $K^t$ the set of all nodes at stage $t = 1, \ldots, T$. $Obj^k$ and $Fea^k$ denote the index set of objective and feasibility cut at the $k^{th}$ iteration, respectively. $|S|$ denotes the cardinality of a set $S$.

2. Preliminaries.

2.1. KL-divergence. Due to the true probability distribution is unknown, [3] proposed a method to measure the distance between probability distributions based
on $\phi-$divergences, which have been widely used in statistics and information theory. $\phi-$divergences contain KL-divergence, J-divergence, Hellinger distance (HD), Variation distance (VD) and $\chi-$divergence of second order, etc. Each member of $\phi-$divergences family has the corresponding $\phi-$divergence function $\phi(t), t \geq 0$, which is a convex function and satisfies $\phi(1) = 0$. In additional, $0\phi(\frac{u}{u}) = u \lim_{t \to \infty} \frac{\phi(t)}{t}$ for $u > 0$ and 0 otherwise. Assume that the sample space has finite support, one can construct an ambiguity set contained the true probability distribution based on $\phi-$divergences as follows,

\[
\mathcal{P}_\phi := \left\{ p \in \mathbb{R}_+^m : D_\phi(p, q) = \sum_{i=1}^m q_i \phi \left( \frac{p_i}{q_i} \right) \leq \rho, 1^T p = 1 \right\},
\]

where $\rho$ represents the predefined tolerance; $p$ and $q$ denote the true distribution and the nominal distribution, respectively. If $\mathcal{P}_\phi(\rho)$ represents the ambiguity set as a function of $\rho > 0$, [19] founded

\[
\mathcal{P}_{\chi^2}(\rho) \subseteq \mathcal{P}_J(\rho) \subseteq \mathcal{P}_{KL}(\rho) \subseteq \mathcal{P}_{HD}(\sqrt{\rho}) \subseteq \mathcal{P}_{VD}(2\rho^\frac{1}{4}).
\]

For a decision maker, when the data are poorly sampled, it is reasonable to construct an ambiguity set of moderate size (see [7, 26]). KL-divergence is the most famous special case of the $\phi-$divergences measures family (see [35]) and its conjugate is a convex function, i.e., the counterpart of robust optimization problem based on KL-divergence can be derived easily. Thus, in this paper, we choose KL-divergence to build an ambiguity set.

The KL–divergence from $q$ to $p$ is defined as follows,

**Definition 2.1.** Assume that $p = (p_1, \ldots, p_m)^T \in \mathbb{R}_+^m$ and $q = (q_1, \ldots, q_m)^T \in \mathbb{R}_+^m$ represent the true and the nominal probability mass function, respectively. The KL-divergence between two vector $p$ and $q$ is defined by

\[
D_{KL}(p, q) := \sum_{i=1}^m p_i \log \left( \frac{p_i}{q_i} \right).
\]

(4)

KL-divergence is always non-negative with $D_{KL}(p, q) = 0$ if and only if $p = q$ almost everywhere. However, it is worth noting that KL-divergence is not a true metric because it does not obey the triangle inequality. KL-divergence function is defined as

\[
\phi_{KL}(y) := y \log y - y + 1.
\]

(5)

According to [3], the conjugate of $\phi_{KL}(y)$ can be represented as

\[
\phi^*_{KL}(s) := \sup_{y \geq 0} \{ sy - \phi_{KL}(y) \} = e^s - 1.
\]

(6)

Ambiguity set based on KL-divergence can be represented as

\[
\mathcal{P}_{KL} := \{ p \in \mathbb{R}_+^m : D_{KL}(p, q) \leq \rho, 1^T p = 1 \}.
\]

(7)

When the size of sample data is smaller, $\rho$ can be larger; on the contrary, when the size of sample data is larger, $\rho$ can be smaller.

2.2. A Coherent risk measure. Let $p$ be a probability density function of the random variable $\xi$. The probability of $Q(x, \xi)$ is not greater than the threshold $\zeta$ which can be represented as

\[
\Psi(x, \zeta) := \int_{Q(x, \xi) \leq \zeta} p(x) d\xi.
\]
Assuming that $\beta \in (0,1)$ is a given confidence level, Value-at-Risk (VaR), as a standard measure of downside risk, is defined as follows,

$$\text{VaR}_\beta(x,\xi) := \inf \left\{ \zeta \in \mathbb{R} : \int_{\mathbb{Q}(x,\xi) \leq \zeta} p(\xi) d\xi \geq \beta \right\}. \quad (8)$$

Since VaR is not a coherent risk measure (see [2]), [42] proposed a new approach, Conditional Value-at-Risk (CVaR), to measure risk. CVaR, the mean of the $\beta$-tail distribution of function $\mathbb{Q}(x,\xi)$, can be defined as

$$\text{CVaR}_\beta(\mathbb{Q}(x,\xi)) := \frac{1}{1-\beta} \int_{\mathbb{Q}(x,\xi) \geq \text{VaR}_\beta(x,\xi)} \mathbb{Q}(x,\xi) p(\xi) d\xi. \quad (9)$$

To avoid the computation of multiple integral, [42] showed that $\text{CVaR}_\beta(\mathbb{Q}(x,\xi))$ can be approximated

$$\text{CVaR}_\beta(\mathbb{Q}(x,\xi)) = \min_{\zeta \in \mathbb{R}} \left\{ \zeta + \frac{1}{1-\beta} \mathbb{E}_p[\mathbb{Q}(x,\xi) - \zeta]^+ \right\}. \quad (10)$$

here $[a]^+ := \max\{a,0\}$ for any $a \in \mathbb{R}$.

In the following sections, we use CVaR to describe risk under probability uncertainty.

3. Two-stage distributionally robust optimization problem with risk aversion.

3.1. Model reformulation. Assume that $g(\xi), h(\xi), G(\xi)$ and $H(\xi)$ are random data of the second-stage problem. The TSDRO model with weighted mean-CVaR recourse function can be reformulated as

$$\min_{x \in \mathcal{X}} \left\{ c^T x + \sup_{p \in \mathcal{P}} \{ (1-\lambda)\mathbb{E}_p[\mathbb{Q}(x,\xi)] + \lambda \text{CVaR}_\beta(\mathbb{Q}(x,\xi)) \} \right\}, \quad (11)$$

where $\mathbb{Q}(x,\xi)$ represents the optimal value of subproblem, for given $x$ and $\xi \in \Omega$, defined as (2). Substituting $\text{CVaR}_\beta(\mathbb{Q}(x,\xi))$ in (11) with (10), we derive

$$\min_{x,\tilde{v}} \left\{ c^T x + \sup_{p \in \mathcal{P} KL, \xi \in \mathbb{R}} \min_{p \in \mathcal{P} KL} \{ \lambda \zeta + \mathbb{E}_p[G(x,\xi)] \} \right\} \quad (12)$$

s.t. $\tilde{v} \geq \mathbb{Q}(x,\xi) - \zeta, \tilde{v} \geq 0, x \in \mathcal{X}$,

where $G(x,\xi) := (1-\lambda)\mathbb{Q}(x,\xi) + \frac{\lambda}{1-\beta} \tilde{v}$.

For the discrete distribution case, i.e., $\Omega = \{\xi_1,\ldots,\xi_m\}$ is finite, the corresponding probability is $P\{\xi = \xi_i\} = p_i, i = 1,\ldots,m$. By Theorem 3.1, we can derive the robust counterpart of problem (12).

**Theorem 3.1.** Assuming that the ambiguity set $\mathcal{P} KL$ is defined by (7). $q \in \mathbb{R}^m_+$ denotes a nominal probability vector. $Q_i(x,\xi)$ is the optimal value of the second-stage optimization problem (2) in the $i$th scenario. Then there exists $\tau, \mu \in \mathbb{R}$ such that (12) is equivalent to

$$\min_{x,\xi,\tau,\mu \geq 0} \left\{ c^T x + \lambda \zeta + \tau + \mu \sum_{i=1}^m q_i \left[ \exp \left( \frac{G_i(x,\xi) - \tau}{\mu} \right) - 1 \right] \right\} \quad (13)$$

s.t. $x \in \mathcal{X}$, $\tilde{v}_i \geq Q_i(x,\xi) - \zeta, \tilde{v}_i \geq 0, i = 1,\ldots,m,$

$$G_i(x,\xi) = (1-\lambda)Q_i(x,\xi) + \frac{\lambda}{1-\beta} \tilde{v}_i.$$
Proof. Obviously, the function
\[
\chi(x, p, \zeta) := \lambda \zeta + \sum_{i=1}^{m} p_i \left[ (1 - \lambda) Q_i(x, \xi) + \frac{\lambda}{1 - \beta} (Q_i(x, \xi) - \zeta^+) \right]
\]
is convex in \( \zeta \) and concave (actually linear) in \( p \). Since \( \mathcal{P}_{KL} \) is a compact set, \( \chi(x, p, \zeta) \) has the strong max-min property over the joint set \( \mathcal{P}_{KL} \times \mathbb{R} \) (see [13, 51] for the details). Thus, we can change the order of the operators \( \sup_{p \in \mathcal{P}} \) and \( \min_{\zeta \in \mathbb{R}} \) in (12),
\[
\min_{x \in \mathcal{X}} \left\{ c^T x + \sup_{p \in \mathcal{P}_{KL}} \min_{\zeta \in \mathbb{R}} \left\{ \lambda \zeta + \mathbb{E}_p [G(x, \xi)] \right\} \right\}
= \min_{x \in \mathcal{X}} \left\{ c^T x + \lambda \zeta + \max_{p \in \mathcal{P}_{KL}} \left\{ \sum_{i=1}^{m} p_i G_i(x, \xi) \right\} \right\}.
\]
(14)
The Lagrange function for the inner maximize problem on the right-hand-side of (14) is given by
\[
L(p, \tau, \mu) = \sum_{i=1}^{m} p_i G_i(x, \xi) + \tau \left( 1 - \sum_{i=1}^{m} p_i \right) + \mu \left( \rho - \sum_{i=1}^{m} q_i \phi_{KL} \left( \frac{p_i}{q_i} \right) \right).
\]
(15)
The dual function is
\[
\mathcal{G}(\tau, \mu) = \max_{p \geq 0} L(p, \tau, \mu)
= \tau + \rho \mu + \max_{p \geq 0} \left\{ \sum_{i=1}^{m} p_i [G_i(x, \xi) - \tau] - \mu \left( \sum_{i=1}^{m} q_i \phi_{KL} \left( \frac{p_i}{q_i} \right) \right) \right\}
= \tau + \rho \mu + \mu \sum_{i=1}^{m} q_i \max_{t \geq 0} \left\{ \frac{G_i(x, \xi) - \tau}{\mu} \right\}
= \tau + \rho \mu + \mu \sum_{i=1}^{m} q_i \phi_{KL} \left( \frac{G_i(x, \xi) - \tau}{\mu} \right)
= \tau + \rho \mu + \mu \sum_{i=1}^{m} q_i \left[ \exp \left( \frac{G_i(x, \xi) - \tau}{\mu} \right) - 1 \right].
\]
(16)
By choosing \( \rho > 0 \), there exists a probability distribution \( q \) that belongs to the interior of the ambiguity set \( \mathcal{P}_{KL} \), i.e., the Slater’s condition holds. So, we have
\[
\max_{p \in \mathcal{P}_{KL}} \left\{ \sum_{i=1}^{m} p_i G_i(x, \xi) \right\} = \min_{\tau, \mu \geq 0} \mathcal{G}(\tau, \mu)
= \min_{\tau, \mu \geq 0} \left\{ \tau + \rho \mu + \mu \sum_{i=1}^{m} q_i \left[ \exp \left( \frac{G_i(x, \xi) - \tau}{\mu} \right) - 1 \right] \right\}.
\]
(17)
Substituting the inner maximize problem on the right-hand-side of (14) with (17), we derive (13). Then the proof is completed. 
\[\square\]
Remark 1. Because $\mu$ is in the denominator of $\exp\left(\frac{\mathcal{G}_i(x, \xi) - \tau}{\mu}\right)$, for the well-definition of KL-divergence, we define

$$0 \exp\left(\frac{\mathcal{G}_i(x, \xi) - \tau}{0}\right) := \begin{cases} 0, & \mathcal{G}_i(x, \xi) < \tau; \\ +\infty, & \mathcal{G}_i(x, \xi) \geq \tau. \end{cases}$$ (18)

Remark 2. In formula (13), the variables $\zeta, \mu, \tau$ and $\tilde{v}_i, i = 1, \ldots, m$ are the first-stage variables.

[26] proved that $\phi$–divergences constrained risk-neutral TSDRO was a convex optimization problem. By the following theorem, we derive $f(x, \mu, \tau, \bar{v})$ is a convex function with respect to $x \in \mathcal{X}$, $\mu$, $\tau$ and $\bar{v}$.

Theorem 3.2. Let $f(x, \mu, \tau, \bar{v}) := \mu \sum_{i=1}^{m} q_i \left[\exp\left(\frac{\mathcal{G}_i(x, \xi) - \tau}{\mu}\right) - 1\right]$. If $\sum_{i=1}^{m} q_i = 1$, then $f(x, \tau, \mu, \bar{v})$ is a convex function with respect to $x \in \mathcal{X}$, $\tau$, $\mu$ and $\bar{v}$.

Proof. For any given $\xi^i \in \Omega$, the optimal value function $Q_i(x, \xi)$ is convex (see Proposition 2.1, [46]) with respect to $x \in \mathcal{X}$.

$$\mathcal{G}_i(x, \xi) - \tau = (1 - \lambda) \Omega_i(x, \xi) + \frac{\lambda}{1 - \beta} (\Omega_i(x, \xi) - \zeta^i)^+ - \tau$$

is a convex function over $x \in \mathcal{X}$ and $\tau$. Let $(Q_i(x, \xi) - \zeta^i)^+ = \tilde{v}_i$. Since exponential function, $\exp(\cdot)$, is a nondecreasing convex function, the composition function $\exp(\mathcal{G}_i(x, \xi) - \tau)$ is convex with respect to $x$, $\tau$ and $\bar{v}$.

We consider two cases, $\mu > 0$ and $\mu = 0$ separately. When $\mu > 0$, the partial second-order derivative of function $\mu \exp\left(\frac{\mathcal{G}_i(x, \xi) - \tau}{\mu}\right)$ on $\mu$ is non-negative; when $\mu = 0$, by the definition (18), $0 \exp\left(\frac{\mathcal{G}_i(x, \xi) - \tau}{0}\right) = 0$ for $\mathcal{G}_i(x, \xi) < \tau$ and $+\infty$ otherwise, so $0 \exp\left(\frac{\mathcal{G}_i(x, \xi) - \tau}{0}\right)$ is also a nondecreasing convex function; therefore, $\mu \exp\left(\frac{\mathcal{G}_i(x, \xi) - \tau}{\mu}\right)$ is a nondecreasing and convex function in $\mu$.

Assume $x_1, x_2 \in \mathcal{X}$, $\tau_1, \tau_2 \in \mathbb{R}$, $\tilde{v}_i^1, \tilde{v}_i^2 \in \mathbb{R}$, and $\mu_1, \mu_2 \in \mathbb{R}$. Let $b \in [0, 1]$ and define $\bar{x} = bx_1 + (1 - b)x_2$, $\tau = b\tau_1 + (1 - b)\tau_2$, $\tilde{v} = b\tilde{v}_i + (1 - b)\tilde{v}_i^2$, and $\bar{\mu} = b\mu_1 + (1 - b)\mu_2$. By using the arguments of convexity on $x$, $\tau$, $\bar{v}_i$ and $\mu$, for all $b \in [0, 1]$, we have

$$\bar{\mu} \exp\left(\frac{\mathcal{G}_i(x, \xi) - \tau}{\bar{\mu}}\right) \leq b\mu_1 \exp\left(\frac{\mathcal{G}_i(x_1, \xi) - \tau_1}{\mu_1}\right) + (1 - b)\mu_2 \exp\left(\frac{\mathcal{G}_i(x_2, \xi) - \tau_2}{\mu_2}\right).$$

Consequently, $\mu \left[\exp\left(\frac{\mathcal{G}_i(x, \xi) - \tau}{\mu}\right) - 1\right]$ is convex with respect to $x$, $\tau$, $\mu$ and $\bar{v}_i$. In addition, when $\sum_{i=1}^{m} q_i = 1$, $\mu \sum_{i=1}^{m} q_i \left[\exp\left(\frac{\mathcal{G}_i(x, \xi) - \tau}{\mu}\right) - 1\right]$ is convex with respect to $x$, $\tau$, $\mu$ and $\bar{v}$ (see [9]).

To solve two-stage risk-neutral stochastic linear programming, [43] and [8] proposed decomposition algorithms based on cutting planes method, respectively. However, the first-stage optimization problem (13) is nonlinear, we extend the linear decomposition algorithm to the nonlinear case and prove the convergence of the algorithm in the following subsection.
3.2. Decomposition algorithm and convergence analysis. As an application of the proposed algorithm in [26] to the case of KL-divergence, in this subsection, we propose a modified decomposition algorithm and prove the convergence of the proposed algorithm.

For a given \( x = x^k \), if we assume that \( Q(x^k, \xi) < \infty \), then \( Q(\cdot, \xi) \) is subdifferentiable at \( x^k \) (see [44]) and

\[
\partial Q(x^k, \xi) = -G^T D(x^k),
\]

where \( D(x^k) := \arg \max \{ \pi^T (h - Gx^k) : H^T \pi \leq g \} \) is the set of optimal solutions of the dual problem of (2) at \( x^k \). Let \( \pi^{k,i} \in D(x^k) \) be the optimal solution corresponding the \( i^{th} \) scenario.

We propose a modified decomposition algorithm and prove the convergence of the proposed algorithm in [26] to the case of KL-divergence, in this subsection, we propose a modified decomposition algorithm and prove the convergence of the proposed algorithm.

Given \( (x^k, \mu^k, \tau^k, \tilde{\nu}^k) \), according to the subgradient inequality of convex function, we have

\[
f_i(x, \mu, \tau, \tilde{\nu}_i) \geq f_i(x^k, \mu^k, \tau^k, \tilde{\nu}_i^k) + (\partial f_i^k) \cdot (x - x^k, \mu - \mu^k, \tau - \tau^k, \tilde{\nu}_i - \tilde{\nu}_i^k),
\]

where \( \cdot \) denotes dot product. Therefore, we can define an objective cut as follows,

\[
f_i(x, \mu, \tau, \tilde{\nu}_i) \geq A_{1,i}^k + B_{1,i}^k \cdot (x, \mu, \tau, \tilde{\nu}_i),
\]

where

\[
A_{1,i}^k := \left[ G_i(x^k, \xi) + (1 - \lambda)(\pi^{k,i})^T G x^k - \frac{\lambda \tilde{\nu}_i^k}{1 - \beta} \right] \exp(s^k),
\]

\[
B_{1,i}^k := \partial f_i^k.
\]

If subproblem (2) is infeasible at \( x^k \), we use a feasible cut which cuts off \( x^k \). By introducing auxiliary artificial vectors \( z = (z_1, ..., z_m)^T \), we consider the following optimization problem corresponding to (2),

\[
\min_{y,z} \|z\| \tag{25}
\]

\[
s.t. \ H y + z = h - G x, \tag{26}
\]

\[
y \geq 0,
\]

here \( \| \cdot \| \) denotes a norm. Let \( z^*(x) \) be the optimal value of (25)-(26) and \( \nu \) be the Lagrange multipliers associated with the constraints (26). The dual of (25)-(26) can be denoted as follows,

\[
\max_{\nu} \nu^T (h - G x) \tag{27}
\]

\[
s.t. \ H^T \nu \leq 0, \|\nu\|_* \leq 1, \nu \geq 0,
\]

where \( \| \cdot \|_* \) denotes the dual of the norm \( \| \cdot \| \) (recall that the norms \( \| \cdot \|_p \) and \( \| \cdot \|_q \) are dual to each other when \( \frac{1}{p} + \frac{1}{q} = 1 \)).

Since \( z^*(x) \) is also subdifferentiable at \( x^k \) and

\[
\partial z(x^k) = -G^T \nu^*(x^k),
\]
where $\nu^*(x^k)$ is the optimal solution of problem (27) at $x^k$. Similarly, we can construct an objective cut for any $x \in \mathbb{R}^n$,

$$(z^*)^i(x) \geq (z^*)_i(x^k) - G^T\nu^*(x^k)(x - x^k)$$

$$A_{2,i}^k + B_{2,i}^k x, \quad (28)$$

where $A_{2,i}^k := (z^*)_i(x^k) + G^T\nu^*(x^k)$, $B_{2,i}^k := -G^T\nu^*(x^k)$. It is obvious that the optimal value of problem (25)-(26) is nonnegative and finite. Especially, if subproblem (2) is feasible at $x^k$, then $z(x^k) = 0$. Therefore, $z^*(x^k) > 0$ when the subproblem (2) is infeasible at $x^k$. So inequality

$$A_{2,i}^k + B_{2,i}^k x \leq 0 \quad (29)$$

is a feasibility cut.

The multi-cut master problem $\mathcal{M}_k$ can be represented as

$$\min_{x, \zeta, \tau, \mu} \mathbf{c}^T x + \lambda \zeta + \tau + \rho \mu + \sum_{i=1}^m q_i \theta_i$$

s.t. $x \in \mathcal{X}, \mu \geq 0,$

$$(\pi^{k,i} \mathbf{h}(\xi^i) - G(\xi^i) x) - \zeta \leq \tilde{v}_i, \tilde{v}_i \geq 0, k \in \text{Obj}^k(i), i = 1, \ldots, m, \quad (31)$$

$$A_{1,i}^k + B_{1,k}^k \mathbf{(}x, \mu, \tau, \tilde{v}_i\mathbf{)} \leq \theta_i, k \in \text{Obj}^k(i), i = 1, \ldots, m, \quad (32)$$

$$A_{2,i}^k + B_{2,k}^k x \leq 0, k \in \text{Feas}^k(i), i = 1, \ldots, m. \quad (33)$$

Owing to master problem $\mathcal{M}_k$ and subproblem (2) are both linear programming, we propose a decomposition method for solving TSDRO. The complete pseudocode of the method is given in Algorithm 1. The basic idea of Algorithm 1 is outer linearization—cutting plane methods (see [6]). Compared with decomposition method (see [43]), a new hyperplane (32) is added to the polyhedral approximation of (13) at each new iterate $x^k$.

**Remark 3.** [26] proposed a decomposition algorithm to solve a general TSDRO model based on $\phi$–divergences. In contrast to [26], an explicit risk function, CVaR, is contained in the objective function of TSDRO model proposed in this paper. The decomposition method based on multi-cut approach is proposed in Algorithm 1. By multi-cut approach, more detailed information of the second stage is transmitted to the first stage. The number of iterations of the master problem is expected to be fewer than in the single cut approach. However, the number of constraints generated by multi-cut is larger than that generated by single cut. When the number of realizations of random vectors and matrices in the second stage is not significantly larger than the number of the first stage constraints, the multi-cut approach is expected to be more effective.

Proof of convergence of Algorithm 1 is stated as Theorem 3.3.

**Theorem 3.3.** For any given $\varepsilon > 0$, after finitely many iterations, the decomposition method for TSDRO either discovers an optimal solution of (11) or finds its infeasibility.

**Proof.** In the Algorithm 1, if $Z_u - Z_l \geq \varepsilon$, for a given $x^k$, we solve $m$ subproblems. If $Q_i^*(x^k, \xi) < +\infty$, then the new objective cut (31)-(32) cuts the point $(x^k, \mu^k, \tau^k, \zeta^k, \tilde{v}_i^k, \theta^k)$ off the set of feasible solutions of (13). If $Q_i^*(x^k, \xi) = +\infty$ for a scenario $i$, the new feasibility cut (33) cuts $x^k$ off the set of feasibility.
Algorithm 1 Decomposition Method for TSDRO.

1: Initialization: $k \leftarrow 1, Ob(j) \leftarrow \emptyset$, $Fea^0(i) \leftarrow \emptyset$, $i = 1, \ldots, m$, $Z_u \leftarrow -\infty$, $Z_l \leftarrow -\infty$, $\mu^1 \leftarrow$ a positive constant, $\tau^1 \leftarrow 0$, $\zeta^1 \leftarrow 0$, and $\tilde{v}_t^1 \leftarrow 0$, $i = 1, \ldots, m$.

2: Solve master problem $\mathcal{M}_0 := \{c^T x | x \in \mathcal{X}\}$ to get $x^1 \in \mathcal{X}$.

3: while $Z_u - Z_l > \varepsilon$ do

4: for $i = 1$ to $m$ do

5: Solve sub-problem (2) at $x = x^k$, $y_i^*(x^k) \leftarrow$ optimal solution of (2), $Q_i^1(x^k, \xi) \leftarrow$ optimal value of (2).

6: if $Q_i^1(x^k, \xi) < +\infty$ then

7: $\pi^{k,i} \leftarrow$ optimal dual multipliers corresponding to the constraints in sub-problem (2).

8: Construct the objective cut (31)-(32). Set $Obj^k(i) \leftarrow Obj^{k-1}(i) \cup \{k\}$ and set $Fea^k(i) \leftarrow Fea^{k-1}(i)$.

9: else

10: Construct the feasibility cut (33) and set $Fea^k(i) \leftarrow Fea^{k-1}(i) \cup \{k\}$.

11: end if

12: $\mathcal{G}_i(x^k, \xi) \leftarrow (1 - \lambda)Q_i^1(x^k, \xi) + \lambda \frac{v_i}{\beta^i}$.

13: $\theta_i \leftarrow \mu^k \left[ \exp \left( \frac{G_i(x^k, \xi) - \tau^k}{\mu^k} \right) - 1 \right]$.

14: $i \leftarrow i + 1$

15: end for

16: if $Z_u > c^T x^k + \lambda \zeta^k + \tau^k + \rho \mu^k + \sum_{i=1}^{m} q_i \theta_i$ then

17: $Z_u \leftarrow c^T x^k + \lambda \zeta^k + \tau^k + \rho \mu^k + \sum_{i=1}^{m} q_i \theta_i$.

18: if $Z_u \leq Z_l$ then

19: Go to Line 30

20: end if

21: end if

22: Solve multi-cut master problem (30)-(33).

23: if (30)-(33) is infeasible then

24: Stop. The original optimization problem is infeasible.

25: else

26: Let $x^{k+1}, \mu^{k+1}, \tau^{k+1}, \zeta^{k+1}, \tilde{v}_t^{k+1}$ and $\theta_i^{k+1}$, $i = 1, \ldots, m$, denote the optimal solutions of (30)-(33). $Z_l$ denotes the optimal value.

27: end if

28: $k \leftarrow k + 1$.

29: end while

30: return Two-stage optimal solution $(x^k, y_i^*(x^k)), i = 1, \ldots, m$, and the optimal value $Z_u$.

of master problem $\mathcal{M}_k$. Meanwhile, the upper bounder $Z_u$ is updated form Step 16-Step 18. If the original problem is infeasible, then the algorithm will stop at step 20. Since each dual multiplier $\pi^{k,i}$ corresponds to one of the finitely many different bases of subproblem (2), there are finitely number of the set of dual multipliers. Hence, there are finite number of objective cuts (see [8]). For optimization (27), since the number of realizations of all scenarios is finite, the set of solution to problem (27) is finite. Thus, the number of all possible feasibility cuts is finite.

Let $x^{k+1}, \mu^{k+1}, \tau^{k+1}, \zeta^{k+1}, \tilde{v}_t^{k+1}$ and $\theta_i^{k+1}$, $i = 1, \ldots, m$, be the optimal solutions of (30)-(33). Then either of the following two cases will occur.
stage modeling, we use the notation $w_k$, $w_{i,k}$ of data, write dynamic programming equations corresponding to (34)-(36). At the last stage, vector $x$. The optimal value of the problem (37)-(38) depends on the former stage decision $c_{i,k}$. The optimization problem is $h_i$. Since the numbers of objective cuts and feasibility cuts are both finite, the algorithm can stop at step 29 after finitely many iterations. Therefore, the decomposition method for TSDRO either discovers an optimal solution of (11) or finds its infeasibility.

4. Multi-stage distributionally robust optimization with risk aversion.

4.1. Model reformulation. Most practical decision-making issues involve a series of decisions that are adjusted depending on the outcomes over time. In this section, we will consider multi-stage distributionally robust optimization problems.

The multi-stage stochastic linear optimization problem can be written in the following nested formulation driven by random data process $w^1, ..., w^T$ (see [46] and [8]). We assume that the random data sequences $w^1, ..., w^T$ are revealed gradually in the multi-stage case and have finitely many scenarios. Furthermore, we specify many different realizations of $w^t \in \Omega^t$, $t = 1, ..., T$ with a probability distribution accordingly. By assumption, $\Omega^1$ is a singleton. To comprehend the process of multi-stage modeling, we use the notation $w[t]$ to denote the random data available up to stage $t$.

We omit the notation of vector transpose for simplicity.

\[
\min_{x^t \in \mathbb{R}^n} \; c^1 x^1 + E \left[ \min_{x^2 \in \mathbb{R}^n} \; c^2 x^2 + E \left[ ... + E \left[ \min_{x^T \in \mathbb{R}^n} \; c^T x^T \right] \right] \right] \quad (34)
\]

\[\mathcal{Q}_1 := \{ x^t : G^t x^t = h^1, x^t \geq 0 \}, \quad (35)\]

\[\mathcal{Q}_t := \{ x^t : G^t(w^t)x^{t-1} + H^t(w^t)x^t = h^t(w^t), x^t \geq 0 \}, \quad t = 2, ..., T, \quad (36)\]

where $x^t \in \mathbb{R}^n$, $t = 1, ..., T$ are decision vectors. Here, we assume that $c^1, G^1$ and $h^1$ are known at the first stage, and $c^t, G^t, H^t, h^t$ are random at the $t$th stage.

Noting the interaction relationship between two-stage and multi-stage, we can write dynamic programming equations corresponding to (34)-(36). At the last stage, all of data, $w^1, ..., w^T$, are already known and the decision vectors, $x^1, ..., x^{T-1}$, have been determined. The optimization problem is

\[\mathcal{Q}(x^{T-1}, w^T) := \min_{x^T} c^T(w^T)x^T \quad (37)\]

\[s.t. \; G^T(w^T)x^{T-1} + H^T(w^T)x^T = h^T(w^T), x^T \geq 0. \quad (38)\]

The optimal value of the problem (37)-(38) depends on the former stage decision vector $x^{T-1}$ and random data $c^T, G^T, H^T, h^T$. 

Generally, at stage \( t = 2, ..., T - 1 \), the recursive forms of the optimization problem are

\[
\mathcal{D}(x^{t-1}, w^{[t]}) := \min_{x^t} c^t(w^t)x^t + E_{p^{t+1}}[\mathcal{D}(x^t, w^{[t+1]})|w^{[t]}] \tag{39}
\]

\[
s.t. \ G^t(w^t)x^{t-1} + H^t(w^t)x^t = h^t(w^t), x^t \geq 0, \tag{40}
\]

where \( p^{t+1} \) denotes the conditional probability of stage \( t + 1 \) given the history of the stochastic process up to stage \( t \) and \( E_{p^{t+1}}[|w^{[t]}] \) denotes conditional expectation. Ultimately, the goal is to solve the following two-stage stochastic programming problem and to find the first stage decision vector \( x^1 \).

\[
\min_{x^1} c^1x^1 + E_{p^1}[\mathcal{D}(x^1, w^2)] \tag{41}
\]

\[
s.t. \ G^1x^1 = h^1, x^1 \geq 0. \tag{42}
\]

Since assuming that \( w^1 \) is known, the second stage optimal value \( \mathcal{D}(x^1, w^2) \) does not depend on \( w^1 \).

The TSDRO model can be naturally extended to a multi-stage case. The formula of the last stage of multi-stage distributionally robust optimization is the same as (37)-(38). At stage \( t = 2, ..., T - 1 \), the recursive forms of the distributionally robust optimization with risk aversion are

\[
\min_{x^t} \left\{ c^t(w^t)x^t + \sup_{p^{t+1} \in \mathcal{R}^{t+1}} \left\{ (1 - \lambda)E_{p^{t+1}} \left[ Q(x^t, w^{[t+1]})|w^{[t]} \right] + \lambda CVaR_\beta \left( Q(x^t, w^{[t+1]})|w^{[t]} \right) \right\} \right\} \tag{43}
\]

\[
s.t. \ G^t(w^t)x^{t-1} + H^t(w^t)x^t = h^t(w^t), x^t \geq 0. \tag{44}
\]

where \( Q(x^t, w^{[t+1]}) \) represents the optimal value of the \( (t + 1)^{th} (t = 2, ..., T - 1) \) stage optimization problem. Here, the ambiguity set \( \mathcal{R}^{t+1}, t = 1, ..., T - 1 \), based on KL-divergence can be defined as

\[
\mathcal{R}^{t+1} := \left\{ p^{t+1} \in \mathbb{R}_+^{[T+1]} : \sum_{i=1}^{[T+1]} p_i^{t+1} \log \left( \frac{p_i^{t+1}}{q_i^{t+1}} \right) \leq \rho^{t+1}, 1^T p^{t+1} = 1 \right\}, \tag{45}
\]

where the nominal conditional probability is given by \( q_i^{t+1} = P\{\omega^{t+1} = \omega_i^{t+1}|w^{[t]}\} \); \( \omega_i^{t+1} \) is the \( i^{th} \) element in \( \Omega^{t+1} \).

It is worth noting that the ambiguity sets \( \mathcal{R}^{t+1}, t = 1, ..., T - 1 \), are defined stage wise with respect to the conditional distribution. At stage \( t, t = 2, ..., T \), the nominal conditional distribution \( q^t \) can be estimated by data-driven approach based on random data process \( \omega^{[t]} \) (see [19] for the details). For simplicity, sometimes the conditional probability is set as \( q_i^t = \frac{1}{[T+1]} \). \( \rho^t \) represents the risk-aversion level determined by decision maker or obtained from statistical inference (see [35, 3]). The advantage of constructing ambiguity set by stage wise is that \( \rho^t \) can be adjusted dynamically according to the random data process. The disadvantage is that the cost of computation will increase rapidly with the increase of random data and number of stage.

The terminal goal is to solve the following TSDRO model and to derive the first-stage decision vector and optimal value.
\[
\min_{x^1} \{c^1 x^1 + \sup_{p^2 \in \mathcal{P}^2} \{(1 - \lambda) \mathbb{E}_{p^2}[Q(x^1, w^2)] + \lambda CVaR_{\beta}(Q(x^1, w^2))\}\} \tag{46}
\]
\[
s.t. \ G^1 x^1 = h^1, x^1 \geq 0. \tag{47}
\]

It is a common technique to describe random data sequences in the form of scenario tree, which has nodes organized in levels corresponded to stages \(t = 1, ..., T\) (see [46]). For each node \(i \in \mathcal{K}^t, t = 1, ..., T - 1\) of the scenario tree, we denote the successors rooted at \(i\) with notation \(\mathcal{T}^t(i)\). For simplicity, let \(G_j^{t+1} := G_j^{t+1}(w_j^{t+1}), H_j^{t+1} := H_j^{t+1}(w_j^{t+1}), h_j^{t+1} := h_j^{t+1}(w_j^{t+1}), c_j^{t+1} := c_j^{t+1} (w_j^{t+1})\).

The multi-cut master problem \(\mathcal{M}_t^f\) of the \(t^{th}\) \((t = 1, ..., T - 1)\) stage can be represented as

\[
\min_{x^t_j, \xi^t_j, \tau^t_j, \mu^t_j} c^t_j x^t_j + \lambda \xi^t_j + \tau^t_j + \rho \mu^t_j + \sum_{j \in \mathcal{T}^t(i)} q_{ij} \theta^t_j \tag{48}
\]

\[
s.t. (\pi_j^{t+1, k})^T [h_j^{t+1} - G_j^{t+1} x^t_j] - \xi^t_j \leq \tilde{\nu}^t_j, \tilde{\nu}^t_j \geq 0, j \in \mathcal{T}^t(i), \tag{49}
\]
\[
A_{j,1}^{t,k} + B_{t,1}^{k} (x^t_j, \mu^t_j, \tau^t_j, \tilde{\nu}^t_j)^T \leq \theta^t_j, k \in \text{Obj}_t^j, j \in \mathcal{T}^t(i), \tag{50}
\]
\[
A_{j,2}^{t,k} + B_{t,2}^{k} x^t_j \leq 0, k \in \text{Fea}_t^j, j \in \mathcal{T}^t(i), \tag{51}
\]
\[
\mu^t_j \geq 0, x^t_j \geq 0, i \in \mathcal{K}^t, \tag{52}
\]

where \(q_{ij}\) denotes the nominal conditional probability of reaching node \(j \in \mathcal{T}^t(i)\) from node \(i\). We denote by \(\pi_j^{t+1, k}\) the Lagrange multipliers associated with the constraint (54). The constraints (49), (50) and (51) represent the objective and the feasibility cuts, respectively. The corresponding coefficients can be calculated by the following formulas:

\[
A_{j,1}^{t,k} := \begin{bmatrix} G_j^t(x_j^t, w_j^{t+1}) + (1 - \lambda)(\pi_j^{t, k})^T G_j^{t+1} x^t_j - \frac{\lambda}{1 - \beta} \tilde{\nu}^t_j \end{bmatrix} \exp(s_j^{t,k}),
\]

\[
B_{j,1}^{t,k} := \begin{bmatrix} -(1 - \lambda) \exp(s_j^{t,k}) G_j^{t+1} \pi_j^{t, k}(1 - s_j^{t,k}) \exp(s_j^{t,k}) - 1, \\
- \exp(s_j^{t,k}), \frac{\lambda}{1 - \beta} \exp(s_j^{t,k}) \end{bmatrix},
\]

\[
A_{j,2}^{t,k} := (z_j^{t+1})^T (x_j^t, \mu_j^t) + (G_j^{t+1})^T (\nu_j^{t+1})^T (x_j^t, k),
\]

\[
B_{j,2}^{t,k} := -(G_j^{t+1})^T (\nu_j^{t+1})^T (x_j^t, k),
\]

where

\[
G_j^t(x_j^t, w_j^{t+1}) := (1 - \lambda) Q_j^t(x_j^t, w_j^{t+1}) + \frac{\lambda}{1 - \beta} \tilde{\nu}^t_j,
\]

\[
s_j^{t,k} := \frac{G_j^t(x_j^t, w_j^{t+1}) - \tau_j^{t,k}}{\mu_j^t}.
\]

Here, \(Q_j^t(x_j^t, w_j^{t+1})\) is the optimal value of the following optimization problem,

\[
\min_{x_j^{t+1,k}} \{c_j^{t+1} x_j^{t+1,k} + Q^{t+1}(x_j^{t+1,k}, w_j^{t+2})\} \tag{53}
\]

\[
s.t. G_j^t x_j^{t,k} + H_j^{t+1} x_j^{t+1,k} = h_j^{t+1}, \tag{54}
\]

\[
x_j^{t+1,k} \geq 0, j \in \mathcal{T}^t(i), i \in \mathcal{K}^t, t = 1, ..., T - 2. \tag{55}
\]
\( Q_j^{T-1}(x_j^{T-1,k}, w^{[T]}) \) is the optimal value of the last stage optimization problem as follows,

\[
\min_{x_j^{T,k}} c_j^T x_j^{T,k} \tag{56}
\]

\[
s.t. \quad G_j^T x_j^{T-1,k} + H_j^T x_j^T = h_j^T, \tag{57}
\]

\[
x^T \geq 0, j \in \mathcal{T}^{T-1}(i), i \in \mathcal{K}^{T-1}. \tag{58}
\]

For \( t = 1, \ldots, T - 1 \) stage, \((z^*)_j^{t+1}(x_j^{t,k})\) and \((\nu^*)_j^{t+1}(x_j^{t,k})\) are the optimal value and the optimal solution of the following optimization problem,

\[
\max_{\nu^{t+1}} \nu^{t+1} \left[ h_j^{t+1} - G_j^{t+1} x_j^{t,k} \right] \tag{59}
\]

\[
s.t. \quad (H_j^{t+1})^T \nu^{t+1} \leq 0, \tag{60}
\]

\[
\|\nu^{t+1}\|_* \leq 1, \nu^{t+1} \geq 0. \tag{61}
\]

At the first stage, we solve the optimization problem (48)-(52) with adding the constraint (35) to get the final optimal solution and optimal value.

In next subsection, we will give a nested decomposition algorithm for the multi-stage distributionally robust optimization problem based on scenario tree.

4.2. Nested decomposition algorithm and convergence analysis. To solve multi-stage stochastic linear programming problem, [43] proposed a nested cutting plane method. Since the objective of multi-stage distributionally robust optimization problem with risk aversion is nonlinear, we propose a nested decomposition method presented in Algorithm 2 and prove the convergence of the nested decomposition method by Theorem 4.1.

**Theorem 4.1.** After finitely many iterations, the nested decomposition method for multi-stage DRO either discovers an optimal solution of (46)-(47) or finds infeasibility.

**Proof.** It is similar to the proof of Theorem 3.3, we can prove that the subproblem associated with each node \( i, i = 1, \ldots, \mathcal{K}^t \) at \( t = 1, \ldots, T - 1 \) stage can generate only finitely many different objectives and feasibility cuts. Therefore, every ancestral node has only finitely many different lower approximation models of its child nodes’ optimal value functions. Hence, it can only generate finitely many different objectives and feasibility cuts. It is reasonable to deduce that the root node may have only finitely many cuts. Consequently, the subproblems corresponding to child nodes can be activated only finitely many times by a new root solution or by an additional cut from their child nodes. It will be seen from this that each subproblem has finitely iterative solving process from the root node to the last stage nodes. By induction, the method either discovers an optimal solution of (46)-(47) or finds its infeasibility.

5. Applications. In this section, we provide two applications focused on two-stage multi-product assembly and multi-stage portfolio selection problem. For both numerical simulations, we implement the proposed two algorithms on a computer with an Intel double core i7-3.50 GHz CPU, 12.0 GB RAM, and a 64-bit Windows 10 operating system. In addition, all optimization problems are solved by Matlab 2014b and IBM ILOG CPLEX 12.6 Optimization Studio.
Algorithm 2 Nested Decomposition Method for Multi-stage DRO.

1: Initialization: $k \leftarrow 1$, $t = 1, \ldots, T - 1$, $i \in K^t$, $Obj^0_i \leftarrow 0$, $Fea^0_i \leftarrow \emptyset$, $Z_u \leftarrow -\infty$, $Z_t \leftarrow -\infty$, $\mu^{t,k}_i \leftarrow$ a positive constant, $\tau^{t,k}_i \leftarrow 0$, $\sigma^{t,k}_i \leftarrow 0$, and $v^{t,k}_j \leftarrow 0$, $j \in T^t(i)$.

2: Solve problem $\mathcal{M}^1 := \{ c^1 x^1 : G^1 x^1 = h^1, x^1 \geq 0 \}$ to get $x^{1,k}_1$.

3: for $t = 1$ to $T - 1$ do
4:   for $i = 1$ to $|K^t|$ do
5:     for $j \in T^t(i)$ do
6:         Solve the optimization problem (53)-(55) with adding the constraint (35) at $x^{t,k}_i$ to get an optimal solution $x^{t,k}_j$, Lagrange multipliers $\pi^{t,k}_j$ and an optimal value $Q^{t,k}_j$.
7:     end for
8:   end for
9: end for

10: while $Z_u - Z_t > \varepsilon$ do
11:   for $t = T - 1$ to 2 do
12:     for $i = 1$ to $|K^t|$ do
13:       for $j \in T^t(i)$ do
14:         if $Q^{t+1,k}_j < +\infty$ then
15:             Compute $A_{j,1}^{t,k}, B_{j,1}^{t,k}, Obj^t_i \leftarrow Obj^{t-1}_i \cup \{k\}, Fea^t_i \leftarrow Fea^{t-1}_i$.
16:         else
17:             Compute $A_{j,2}^{t,k}, B_{j,2}^{t,k}, Obj^t_i \leftarrow Obj^{t-1}_i, Fea^t_i \leftarrow Fea^{t-1}_i \cup \{k\}$.
18:         end if
19:         $\theta^{t,k}_j \leftarrow (1 - \lambda)Q^{t+1,k}_j + \frac{\lambda}{1 - \beta} v^{t,k}_j$.
20:         $\theta^{t,k}_j \leftarrow \mu^{t,k}_i \left[ \exp \left( \frac{v^{t,k}_j - \tau^{t,k}_j}{\mu^{t,k}_i} \right) - 1 \right]$.
21:       end for
22:     end for
23:   end for
24: end for
25: if $Z_u > c^1 x^{1,k}_1 + \lambda \zeta_1^{1,k} + \tau^{1,k}_1 + \rho \mu_1^{1,k} + \sum_{j \in T^t(1)} q_j \theta^{1,k}_j$ then
26:   $Z_u \leftarrow c^1 x^{1,k}_1 + \lambda \zeta_1^{1,k} + \tau^{1,k}_1 + \rho \mu_1^{1,k} + \sum_{j \in T^t(1)} q_j \theta^{1,k}_j$.
27: end if
28: if $Z_u \leq Z_t$ then
29:   Go to Line 39
30: end if
31: Solve optimization problem (48)-(52) with adding the constraint (35) at $t = 1$.
32: if (48)-(52) at $t = 1$ is infeasible then
33:   Stop. The original optimization problem is infeasible.
34: else
35:   Let $x^{1,k+1}_1, \zeta_1^{1,k+1}, \tau^{1,k+1}_1, \mu_1^{1,k+1}$ and $v^{1,k+1}_j, \theta^{t,k}_j, j \in T^t(1)$ denote the optimal solutions of (48)-(52). $Z_t$ denotes the optimal value.
36: end if
37: $k \leftarrow k + 1$.
38: end while
39: return the first stage optimal solution $x^{1,k}_1$, and optimal value $Z_u$. 
5.1. **Two-stage multi-product assembly problem.** In the following subsection, we give a numerical experiment on two-stage multi-product assembly problem, which is extended from the multi-product assembly problem in [46]. We assume that a manufacturer produces \( n \) products and each product has total \( m \) different parts ordered from third-party suppliers. The demand for the products is denoted as a random vector \( \mathbf{D^ω} = (D_1^ω, ..., D_n^ω) \). Before the demand is known, the manufacturer must determine the order quantity of the parts from suppliers. To build a two-stage multi-product assembly model, we have some additional notations as follows,

- \( c_j \) cost of per unit of part \( j, j = 1, ..., m; \)
- \( l_i \) additional cost of a unit of demand for product \( i, i = 1, ..., n; \)
- \( s_j \) salvage values of unused parts, \( j = 1, ..., m; \)
- \( x_j \) quantity of parts ordered from suppliers, \( j = 1, ..., m; \)
- \( y_j \) quantity of parts left in inventory, \( j = 1, ..., m; \)
- \( z_i \) quantity of units produced, \( i = 1, ..., n; \)
- \( d_i \) a realization of the random demand of the product \( i, i = 1, ..., n; \)
- \( g_{ij} \) the quantity of part \( j \) of a unit of product \( i, i = 1, ..., n, j = 1, ..., m; \)
- \( M \) total budget of parts purchased.

Assume that the loss of product shortage is a quarter of the selling price and salvage value of unsold product is zero. Our goal is to maximize the total return. If the demands of all products are known before producing, we have a single-stage model as follows,

\[
\begin{align*}
\max_{x,y,z} & \quad c^T x + (r - l)^T z + s^T y - 0.25r^T[d - z]^+ - r^T[z - d]^+ \\
\text{s.t.} & \quad y = x - G^T z, \\
& \quad \sum_{j=1}^m c_j x_j \leq M, x_j, y_j \geq 0, j = 1, ..., m,
\end{align*}
\]

(62)

where \( G \) is a matrix with entries \( g_{ij}, i = 1, ..., n, j = 1, ..., m. \) However, the numbers of \( d \) are unknown before pre-ordering parts. A two-stage multi-product assembly model without considering salvage values of products can be represented as follows,

\[
\begin{align*}
\min_x & \quad c^T x + E\mathbb{P}[\mathcal{D}(x, D^ω)] \\
\text{s.t.} & \quad \sum_{j=1}^m c_j x_j \leq M, x_j \geq 0, j = 1, ..., m,
\end{align*}
\]

(65)

where \( \mathcal{D}(x, D^ω) \) denotes the optimal value of the following second-stage optimization problem

\[
\begin{align*}
\min_{y,z} & \quad (l - r)^T z - s^T y + 0.25r^T[D^ω - z]^+ + r^T[z - D^ω]^+ \\
\text{s.t.} & \quad y = x - G^T z, \\
& \quad y \geq 0, z \geq 0.
\end{align*}
\]

(67)

It is noteworthy that \( x \) is the first stage decision vector and \( y, z \) are the second stage decision vectors. By introducing auxiliary variables \( u \) and \( v, \) (67)-(69) can be reformulated as

\[
\begin{align*}
\min_{y,z,u,v} & \quad (l - r)^T z - s^T y + 0.25r^T u + r^T v \\
\text{s.t.} & \quad y = x - G^T z,
\end{align*}
\]

(70)
According to (11)-(12), we can derive a TSDRO with risk aversion model as follows,

$$\min_{x} \{ c^T x + \sup_{p \in \mathcal{P}} \{(1 - \lambda)E_p[Q(x, D^\omega)] + \lambda CVaR_\beta(Q(x, D^\omega))\} \}$$

$$\text{s.t.} \sum_{j=1}^{m} c_j x_j \leq M, x \geq 0,$$

where $Q(x, D^\omega)$ is the optimal value of (70)-(73).

We assume that there are 5 products, and each product has 7 parts. To verify the validity of the model, we give some values of the parameters $c, l, r, s$ and $M$, which should be regarded as illustrative only. Assuming that the default currency unit is U.S. dollar, we set these corresponding parameters as $c = (2, 3, 4.5, 2.6, 4.3, 8, 3.5)^T$, $l = (12, 14, 15, 18, 16)^T$, $r = (60, 65, 77, 85, 78)^T$, $s = (0.2, 0.3, 0.45, 0.26, 0.4, 0.38, 0.35)^T$ and $M = 250000$. In addition, without loss of generality, the relationship of quantity between product and part is defined as

$$G := \begin{pmatrix} 1 & 2 & 1 & 1 & 1 & 2 & 1 \\ 2 & 1 & 2 & 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 & 3 & 2 & 1 \\ 1 & 1 & 2 & 3 & 2 & 1 \\ 2 & 3 & 1 & 1 & 1 & 2 & 2 \end{pmatrix}.$$

A set of 10 scenarios represents the varying demands of products in the case study. Generally, the probabilities assigned to the scenarios are based on approximately matching historical selling of the products. Here the demands and their corresponding probabilities are summarized illustratively in Table 1.

Table 1. The Probability of Different Demand in 10 Scenarios.

| Scenario | s1  | s2  | s3  | s4  | s5  | s6  | s7  | s8  | s9  | s10 |
|----------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| Product1 | 1250| 1069| 1407| 1125| 1293| 1377| 1265| 1235| 1155| 1327|
| Product2 | 1350| 1074| 1122| 1308| 1275| 1190| 1390| 1005| 1264| 1345|
| Product3 | 1446| 1129| 1465| 1237| 1459| 1284| 1467| 1168| 1082| 1374|
| Product4 | 1480| 1421| 1175| 1176| 1143| 1038| 1065| 1081| 1301| 1225|
| Product5 | 1274| 1127| 1098| 1416| 1379| 1027| 1284| 1397| 1131| 1041|
| Probability| 0.1 | 0.1 | 0.1 | 0.1 | 0.1 | 0.1 | 0.1 | 0.1 | 0.1 | 0.1 |

To solve the single-stage optimization problem (62)-(64), we assume that the demand $d$ for the products is the average value of ten scenarios, i.e., $d = (1250.6, 1232.3, 1311.1, 1210.5, 1217.4)$. The optimal pre-order quantity of the parts and the numbers of units produced are summarized in Table 2. Since the data of unused parts are extremely small, we express them with 0.

For the same problem and data, we solve two-stage stochastic optimization model (65)-(69) by using the multi-cut decomposition method (see [43]) and derive the optimal pre-ordering quantity of the parts and the corresponding numbers of units produced, which are summarized in Table 3.

Generally, the true probability of random demanding is unknown. If the probability of the scenarios is considered as an estimator of true probability, a large number of sample data are needed (see [46]). When the size of sample data is very
small, we can construct an ambiguity set \( \mathcal{P}^{K_L} \) defined as (7), where the nominal probability \( q \) is the probability of the scenarios. We apply the Algorithm 1 to solve the TSDRO problem with different initial parameters \( \rho, \beta \) and \( \lambda \). Restricted by the page space, parts of the experimental results are listed in Table 4.

From Table 4, we can find that when \( \rho = 0.5, \beta = 0.95 \), the greater the \( \lambda \) is, the lower the pre-order quantities of parts are. Consequently, the quantities of the unit product production are lower. We also find that when \( \lambda = 0.5, \beta = 0.95 \), the lower the \( \rho \) is, the lower the pre-order quantities of parts are, but the magnitude of the decrease is narrowing. When \( \lambda = 0.5, \rho = 0.5 \), we observe that changes in \( \beta \) have similar impact on pre-order quantities. Therefore, according to the manufacturer’s risk preference, he or she can flexibly choose different parameters to respond to the risks brought by the uncertainty of future demanding as much as possible.

Compared with the results of two-stage risk-neutral stochastic optimization model and single-stage optimization model in Figure 1, the results of TSDRO model proposed in this paper are relatively robust. From Figure 1, we also observe that the greater the \( \lambda \) is, the flatter the curve is. Furthermore, from the first graph in Figure 2, we find that the lower the \( \rho \) is, the flatter the curve is. However, as \( \rho \) gets smaller and smaller, its impact on robust of the revenue is weaker; from the second graph in Figure 2, when \( \beta = 0.99 \), we know the revenue is more robust.

5.2. Multi-stage portfolio selection problem. A portfolio refers to any combination of financial assets such as stocks, bonds and cash. It is designed according to the investor’s risk tolerance, time frame and investment objectives. We assume that there are \( n \) risky assets, one riskless asset (cash) and \( T \) trading stages. An investor can rebalance his/her portfolio at stage \( t = 1, \ldots, T \) with self-financing. The short-sale is not allowed. At each stage \( t \), the investor needs to make decisions about how to distribute his/her current wealth \( W_t \) among \( n \) risky assets and a cash asset. Let \( \mathbf{x}^t = (x_1^t, \ldots, x_n^t)^T \in \mathbb{R}_+^n \) be the capital amounts (dollar value) invested in the risky assets at the beginning of the \( t \)th stage. To build a multi-stage portfolio selection model, some additional notations are listed as follows,

- \( x_i^t \): the amount of the \( i \)th risky asset in the \( t \)th stage, \( i = 1, \ldots, n \);
- \( r_i^t \): the return of the \( i \)th risky asset in the \( t \)th stage, \( i = 1, \ldots, n \);
Table 4. The Optimal Solution of Two-stage Distributionally Robust Optimization with Risk Aversion Multi-product Assembly Problem.

| NO. | Case | $\lambda$ | $\rho$ | $\beta$ | Pre-order quantity | Unused quantity | Units produced |
|-----|------|-----------|-------|---------|--------------------|----------------|---------------|
|     | 1    | 0.5       | 0.5   | 0.95    | 8279              | 0              | 1258          |
|     | 2    | 0.2       | 0.5   | 0.95    | 8556              | 0              | 1245.5        |
|     | 3    | 0.8       | 0.5   | 0.95    | 8215              | 0              | 1245.5        |
|     | 4    | 0.1       | 0.5   | 0.95    | 8266              | 0              | 1245.5        |
|     | 5    | 1         | 0.5   | 0.95    | 8340              | 0              | 1258.3        |
|     | 6    | 0.05      | 0.5   | 0.95    | 8262              | 0              | 1249.2        |
|     | 7    | 0.90      | 0.5   | 0.95    | 8375              | 0              | 1250.5        |
|     | 8    | 0.99      | 0.5   | 0.95    | 8300              | 0              | 1255.6        |

$s_t^i$ the amount of selling of the $i^{th}$ risky asset in the $t^{th}$ stage, $i = 1, ..., n$;

$b_t^i$ the amount of buying of the $i^{th}$ risky asset in the $t^{th}$ stage, $i = 1, ..., n$;

$x_{n+1}^t$ the cash amount in the $t^{th}$ stage;

$r_{n+1}^t$ the return of cash in the $t^{th}$ stage;

$\delta_t^i$ the transaction cost of selling risky asset;

$\kappa_t^i$ the transaction cost of buying risky asset.

[4] proposed the dynamics of the multi-stage portfolio by the following equations,

- Risky assets: $x_{t+1}^i = x_t^i(1+r_t^i) - s_{t+1}^i + b_{t+1}^i$, $t = 1, ..., T - 1$, $i = 1, ..., n$, 


Cash asset: $x_{n+1}^t = x^t_{n+1}(1 + r^t_{n+1}) + \sum_{i=1}^{n}(1 - \delta_i^{t+1})s^t_{i+1} - \sum_{i=1}^{n}(1 + \kappa_i^{t+1})b^t_{i+1}$, $t = 1, ..., T - 1$.

The budget constraint of the first stage, $\sum_{i=1}^{n+1} x_1^i = W_0$, should hold. For investors, obviously, the return $r^{t+1}$ is unknown at the $t^{th}$ stage.

Figure 1. The revenue of multi-product assembly problem under uncertain demanding.

Figure 2. The revenue of multi-product assembly problem with different parameters.
At stage $t = 2, \ldots, T - 1$, the recursive forms of the distributionally robust optimization with risk aversion portfolio selection model can be represented as

$$
\min_{x^t} \left\{ c^t(\xi^t)x^t + \sup_{p^{t+1} \in \mathcal{P}^{t+1}} \left\{ (1 - \lambda)E_{p^{t+1}}[Q(x^t, \xi^{t+1})]x^t \right\} + \lambda \text{CVaR}_{\delta}(Q(x^t, \xi^{t+1})|\xi^t) \right\}
$$

subject to

$$
x^t = G^{t+1}(\xi^{t+1})x^t + L^{t+1}(\xi^{t+1})s^t + H^{t+1}(\xi^{t+1})b^t,
$$

where $Q(x^t, \xi^{t+1})$ represents the optimal value of the $(t+1)^{th}$ $(t = 2, \ldots, T - 2)$ stage optimization problem. Here $W$ denotes scenarios set and $\xi^t_j$, $j = 1, \ldots, n + 1$ is the realization of random return $r^t_j$ in the scenario $w \in W$; $G^t := \text{diag}(1 + \xi^t_1, \ldots, 1 + \xi^t_{n+1})$, $L^t(\xi^t) := \begin{pmatrix} -1 & 0 & \cdots & 0 \\ 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -1 \end{pmatrix}_{(n+1) \times n}$, and $H^t(\xi^t) := \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}_{(n+1) \times n}$.

When $t = 1$, the constraint (77) should be replaced by

$$
\sum_{i=1}^{n+1} x^1_i = W_0, x^1_i \geq 0, i = 1, \ldots, n + 1.
$$

At the last stage, the recourse optimization problem should be represented as

$$
\min_{x^T, b^T, s^T} \quad c^T(\xi^T)x^T
$$

subject to

$$
x^T = G^T(\xi^T)x^{T-1} + L^T(\xi^T)s^T + H^T(\xi^T)b^T,
$$

$$
x^T \geq 0, s^T \geq 0, b^T \geq 0, w \in W, j \in T^{T-1}(i), i \in \mathcal{K}^{T-1}.
$$

To confirm the validity of the model, we generate some scenarios with Monte Carlo simulations (see [14] and [17]). We assume that the risky asset price process follows a geometric Brownian motion (GBM). For the $i^{th}$ risky asset with the price $P^t_i$ at time $t$, we have

$$
\tilde{r}_t = \frac{P^t_i - P^{t-1}_i}{P^{t-1}_i} = r_i + \xi_i.
$$

By using the discrete pricing model (see [16]), we have the following stochastic differential equation

$$
dP^t_i = r_i P^t_i dt + \sigma_i P^t_i \xi_i \sqrt{dt},
$$

where $\sigma_i$ is the volatility of the $i^{th}$ risky asset price. By solving the equation (83), we derive the following price equation

$$
P^{t+1}_i = P^t_i \exp \left[ \left( r_i - \frac{1}{2} \sigma_i^2 \right) dt + \sigma_i \xi_i \sqrt{dt} \right].
$$
Given the covariance matrix $\Sigma$ of risky asset price, a correlation matrix can be represented as

$$C := \begin{pmatrix}
1 & \rho_{12} & \cdots & \rho_{1n} \\
\rho_{21} & 1 & \cdots & \rho_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
\rho_{n1} & \cdots & \cdots & 1
\end{pmatrix},$$

where $\rho_{ij} = \frac{\sigma_{ij}}{\sigma_i \sigma_j}$. Let $L \in \mathbb{R}^{n \times n}$ be the lower Cholesky factorization of $C$. Let $\xi \in \mathbb{R}^n$ be defined as $\xi := L \epsilon$, here $\epsilon = (\epsilon_1, ..., \epsilon_n)$, $\epsilon_i \in \mathbb{R} \sim \mathcal{N}(0, 1)$.

To determine the price of a risky asset, a Monte Carlo simulation can be run repeatedly for calculating the price of an asset by following different paths. The asset’s price in different scenario can be simulated from a different random path with a sufficient number.

In the following numerical simulation, we use a data set drawn from Bloomberg database, which includes 10 kinds of market indexes, AS51.GI, DJI.GI, FCHL.GI, FTSE.GI, GDAXI.GI, HSI.HI, IBOVESPA.GI, KS11.GI, N225.GI and SENSEX.GI. The data set has 974 weekly closing prices from January 7, 2000 to August 31, 2018. We divide the data into two parts, one is the sample group (January 7, 2000 to December 1, 2017), and the other is the test group (December 4, 2017 to August 31, 2018). We utilize the sample group and the above simulation framework to generate a three-stage scenario tree. In the scenario tree, each ancestor node has 10 subnodes as shown in Figure 3.

![Figure 3. Three-stage scenario tree with 10 different scenarios at each ancestor node.](image)

We assume that the return of cash is 1.5% per year and the transaction cost of selling or buying risky asset is 0.5%. Compared with equal-weighted method (see [5]), Mean-CVaR method (see [42]) and multi-stage risk neutral robust optimization method (see [4]), the method of this paper ($\lambda = 0.5, \rho = 1, \beta = 0.95$) is more robust as shown in Figure 4, i.e., the accumulated wealth has the minimum fluctuation during the testing period.

As shown in Figure 5, the parameters $\lambda, \rho, \beta$ have some certain effects on the volatility of accumulative wealth, although their general trends are very similar. The change of $\lambda$ value from the small to the large reflects investors’ transition from risk neutral to risk aversion gradually. This change is also reflected in the cumulative wealth curve. Because risk-averse investor is willing to invest much more money in cash asset to obtain a riskless return. The result is that the cumulative wealth curve tends to be flat. Figure 5 also provides relationship between $\beta, \rho$ value and the investor’s risk preference.
6. Conclusions. In this paper, we propose a TSDRO model with risk aversion and extend it to a multi-stage case. Meanwhile, we propose two decomposition algorithms based on cutting planes method. Compared with the risk-neutral multi-stage stochastic optimization model, the proposed methods are more robust in practical applications. However, the prominent disadvantage of the proposed methods is relatively conservative. How to solve the conservatism of robust optimization is the goal of our further study.

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