Unisingular representations in arithmetic and Lie theory

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Dedicated to the memory of James Humphreys

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Abstract
Let $G$ be a subgroup of $GL(V)$, where $V$ is a finite dimensional vector space over a finite field of characteristic $p > 0$. If $\det(g - 1) = 0$ for all $g \in G$ then we call $G$ a fixed-point subgroup of $GL(V)$. Motivated in parallel by questions in arithmetic and linear group theory, we classify all irreducible fixed-point subgroups of $Sp_8(2)$ and give new infinite series of irreducible fixed-point subgroups of symplectic groups $Sp_m(2)$ for various $m$ arising from certain representations of groups of Lie type.

Keywords Abelian varieties · Torsion points · Galois representations · Finite linear groups · Finite group representations · Symplectic groups · Eigenvalue 1 · Fixed points

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1 Introduction

Let $G$ be a finite group. If

$$\varphi : G \to GL(V)$$

is a representation of $G$ such that $\det(\varphi(g) - 1) = 0$ for all $g \in G$, then we call $\varphi$ unisingular and $\varphi(G)$ a fixed-point subgroup of $GL(V)$. Unisingular representations

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are of significant interest for group representation theory (see, for example, [16,33,35,36]) and also arise naturally in arithmetic in counting points on varieties over finite fields. More precisely, we are motivated by the following setup that we now describe in detail.

Fix positive integers \( m \geq 2 \) and \( d \geq 1 \). Let \( K \) be a number field and \( p \subset \mathcal{O}_K \) a nonzero prime; write \( \mathbf{F}_p = \mathcal{O}_K / p \) for the residue field at \( p \). Let \( A \) be an abelian variety of dimension \( d \) defined over \( K \) and let \( p \) be a prime of good reduction for \( A \).

Thus there is a natural reduction-mod-\( p \) map \( A \to A_p \), where \( A_p \) is an abelian variety defined over \( \mathbf{F}_p \) (see [23, Appendix] for a review of reduction modulo \( p \) on abelian varieties).

Let \( A(K) \) denote the Mordell–Weil group of \( A \) and \( A_p(\mathbf{F}_p) \) the \( \mathbf{F}_p \)-points of \( A_p \). It is well known that \( A(K) \) is a finitely-generated abelian group and that \( A(\mathbf{F}_p) \) is a finite abelian group. Since \( A(K) \) is finitely-generated, we may write \( A(K) = A(K)_{\text{tors}} \times A(K)_{\text{free}} \) as the decomposition into its torsion and free subgroups. It is well known (see [23, Appendix]) that the reduction-mod-\( p \) map \( A(K)_{\text{tors}} \to A_p(\mathbf{F}_p) \) is injective for all \( p \) not dividing the order of \( A(K)_{\text{tors}} \) (but not necessarily surjective). Therefore, if \( A(K)_{\text{tors}} \) has a subgroup \( S \) of order \( m \), then for all \( p \) of residue characteristic coprime to \( m \), \( A(\mathbf{F}_p) \) will have a subgroup of order \( m \) as well. Thus, we have an easy “global-to-local” principle:

If the order of \( A(K)_{\text{tors}} \) is divisible by \( m \), then the order of \( A(\mathbf{F}_p) \) is divisible by \( m \) for all but finitely many \( p \subseteq \mathcal{O}_K \).

The interesting question is whether there exists a converse local-to-global principle. Interpreted literally, it is easy to come up with examples of elliptic curves \( E \) over \( \mathbb{Q} \) that have a point of order \( m \) modulo all but finitely many primes \( p \), but do not have a point of order \( m \) defined over \( \mathbb{Q} \). For example, the elliptic curve 11.a1 of the LMFDB [1] has trivial Mordell–Weil group over \( \mathbb{Q} \) but has a point of order 5 modulo every \( p \neq 11 \). The more refined local-to-global question, originally posed by Lang, is the following.

**Question 1.1** Fix a positive integer \( m \geq 2 \) and let \( A \) be an abelian variety defined over a number field \( K \). Suppose for all but finitely many primes \( p \) of \( K \) the orders of the groups \( A_p(\mathbf{F}_p) \) are all divisible by \( m \). Does there exist a \( K \)-isogenous \( A' \) such that \( m \) divides the order of \( A'(K)_{\text{tors}} \)?

Lang’s question reflects the fact if \( A/K \) is an abelian variety such that \( A_p(\mathbf{F}_p) \) has a subgroup of order \( m \), then so does every member \( A' \) of the \( K \)-isogeny class \( I_{K,A} \) of \( A \). In [23], Katz showed that the answer to this Question 1.1—Yes or No—can be studied entirely via group theory. In the special case where \( m \) is a prime number, it amounts to classifying certain unisingular representations. When \( m \) is composite, the group-theoretic reformulation is more complicated, and we do not address it in this paper, but it is still group theory nonetheless.

Before transitioning entirely to group theory for the remainder of the paper, we review what is known from the point of view of abelian varieties. In the table below we classify our answers of Yes or No by both the dimension \( d \) of the abelian variety \( A \) and the modulus \( m \). An answer of Yes for fixed \( d \) and \( m \) means that, for every abelian variety \( A \) of dimension \( d \) defined over any number field \( K \) for which \( A_p(\mathbf{F}_p) \) has a subgroup of order \( m \), there exists a \( K \)-isogenous variety \( A' \) with the same property.
subgroup of order \( m \), the \( K \)-isogeny class of \( A \) contains an abelian variety \( A' \) (possibly \( A \) itself) such that \( A' \) has a \( K \)-rational subgroup of order \( m \). An answer of No means that there exists a number field \( K \) and a \( K \)-isogeny class \( I \) of abelian varieties defined over \( K \) such that \( m \) divides the order of \( A_p(F_p) \) for every element \( A \) of \( I \), but no element of \( I \) has a \( K \)-rational subgroup of order \( m \). For more details on the arithmetic consequences of the following table, see [11, Introduction].

| \( \text{dim } A \) | \( \text{Answer} \) | \( \text{Modulus } m \) | \( \text{Reference} \) |
|---------------------|-----------------|----------------|----------------|
| 1                   | Yes             | All \( m \geq 2 \) | [23]           |
| 2                   | Yes             | All prime \( m \geq 2 \) | [23]           |
|                     | No              | All prime powers \( m = \ell^n \geq 4 \) | [11]           |
| 3                   | Yes             | \( m = 2 \) | [8]             |
|                     | No              | All \( m \geq 3 \) | [23]           |
| \( \geq 4 \)        | No              | All \( m \geq 2 \) | [8, 23]        |

It is the answers of No that interest us, both from the point of view of number theory and group theory. From the number theory perspective, it would be interesting to construct the abelian varieties over small-degree number fields that violate this local-to-global principle, since they exhibit non-generic properties. We remark that all known instances of the answer No are non-constructive and are existence proofs only. See Corollary 1.9 below for an instance of this.

We highlight one of these cases as the main motivating example for this paper. In arithmetic language, the answer No in dimension 4 and modulus 2 means that there exists, for example, an abelian fourfold over a number field that has an even number of points modulo all but finitely many \( p \), but does not have a global point of order 2. This is in contrast to, say, the same setup for elliptic curves (i.e. \( \text{dim } A = 1 \)) where it is an easy exercise in Galois theory to show that the answer is Yes for the modulus \( m = 2 \).

We now turn to the group theoretic perspective and present the version of Katz’ reformulation for prime moduli that motivates us; see [23] for the details of the reformulation.

**Question 1.2** ([23, Problem II, p. 483]) Let \( \ell \geq 2 \) be a prime number. Let \( \overline{\rho}_\ell : \text{Gal}(\overline{K}/K) \to \text{Aut}(T_\ell A \otimes \mathbb{F}_\ell) \) be the mod \( \ell \) representation. Suppose \( \det(g - 1) = 0 \) for all \( g \in \text{Gal}(\overline{K}/K) \). Does the composition series of \( \overline{\rho}_\ell \) contain the trivial representation?

Upon choosing a basis for \( T_\ell A \otimes \mathbb{F}_\ell \) and invoking the Galois-equivariance of the symplectic Weil pairing on \( T_\ell A \), one obtains a purely group-theoretic version of Question 1.2 (see [11, Introduction] for details). We switch from \( \ell \) to \( p \) for the underlying prime modulus to coincide with more standard notation from finite group theory. This is the now the question we take up in this paper.

**Question 1.3** Let \( p \) be a prime number and \( G \) a subgroup of \( \text{Sp}_{2d}(p) \). Suppose that \( \det(g - 1) = 0 \) for all \( g \in G \), that is, \( G \) is fixed point. Does the Jordan–Hölder series of \( \mathbb{F}_p G \) contain the trivial representation?
We note that Question 1.3 is equivalent to Question 1.1 when \( m = p \) is prime and \( d = \dim A \) in the sense that one has a positive answer if and only if the other does.

We now recall what is known from the point of view of group theory.

In [23], Katz showed that when \( d = 1 \) and \( d = 2 \), the answer to Question 1.3 is Yes for every group \( G \). For every \( d \geq 3 \) and \( p \) odd he also provided examples of groups \( G \) with answer No. In [7,9] the first-named author classified the groups \( G \subset \text{Sp}_d(p) \) for which the answer to Question 1.3 is No; all such groups \( G \) act reducibly on the underlying symplectic space. In [8] he further showed that the answer to Question 1.3 is Yes when \( d = 3 \) and \( p = 2 \) (for every group) and No when \( d \geq 4 \) and \( p = 2 \) for some \( G \). Therefore, the rough answer—Yes or No—to Question 1.3 is now known for every \( d \geq 1 \) and every prime number \( p \).

The problem remains to classify the groups \( G \) giving an answer of No. This is an interesting problem from the point of view of group theory and representation theory (see [16,35,36], for example) and from the point of view of arithmetic as well. In terms of arithmetic, an irreducible unisingular symplectic representation of degree \( 2d \) is attached to a simple abelian variety of dimension \( d \). The irreducibility of the representation means the answer to Question 1.3 is No and that there is a simple abelian variety \( A \) of dimension \( d \) for which the answer to Question 1.1 is No for the modulus \( m = p \).

To discuss the classification problem stated above we first mention the following trivial fact:

**Lemma 1.4** Let \( V_i, i = 1, 2 \), be symplectic spaces and \( G_i \subset \text{Sp}(V_i) \) subgroups with no trivial composition factor. Let \( V = V_1 \oplus V_2 \) be a symplectic space such that \( V_1, V_2 \) are mutually orthogonal. Suppose that \( G_1 \) is a fixed-point subgroup of \( \text{Sp}(V_1) \). Then \( G = G_1 \times G_2 \) is a fixed-point subgroup of \( \text{Sp}(V) \) with no trivial composition factor.

As \( G_2 \) does not need to be a fixed-point group, Lemma 1.4 shows that the problem of obtaining a full classification of fixed-point subgroups \( G \subset \text{Sp}(V) \) for arbitrary \( V \) is intractable. To avoid this difficulty it seems to be reasonable to turn to classification of maximal fixed-point subgroups (with no trivial composition factor). The bulk of this problem is then to classify the irreducible maximal fixed-point subgroups. Note however that no reduction of the general case to irreducible groups is available.

In fact, Katz [23, pp. 500–501] provided his No example of a fixed-point abelian subgroup of \( \text{Sp}_{2d}(p) \) for odd, \( d \geq 3 \) without a trivial factor on the basis of Lemma 1.4 without stating it explicitly. In his example \( |G_1| = 4 \) and \( |G_2| = 2 \). In the case of \( p = 2 \) we give a similar example with \( |G_1| = 9 \), \( |G_2| = 3 \) (Corollary 4.2).

Let \( q \) be a prime power. One can also ask more generally for a classification of all fixed-point groups \( G \subset \text{GL}_d(q) \) as \( d \) and \( q \) vary, not just restricting to symplectic groups or prime fields. Toward this end, for \( d = 2 \) such groups are always reducible, see [23] or [26, Chapter I, Section 1, Exercise 1]. For \( d = 3 \) the full classification of fixed-point subgroups of \( \text{GL}_3(q) \), where \( q \) is any prime power, appears in [10]. Note that the group \( \text{SO}_3(q) \) is an irreducible fixed-point subgroup of \( \text{GL}_3(q) \) whenever \( q \) is odd, but when \( q \) is even, every fixed-point subgroup of \( \text{GL}_3(q) \) is reducible. More generally, if \( dq \) is odd then \( \text{SO}_d(q) \) is a fixed point irreducible subgroup of \( \text{GL}_d(q) \).

(This is well known, see for instance [30, Lemma 2.27 (1)] for a straightforward proof; more conceptually, this also follows from the fact that the natural representation of...
the simple algebraic group of type $B_n$, $n \geq 1$, has weight zero. This final observation is in line with the work we do in Sect. 6 below.)

Duly motivated, we now present our main theorems. First, we determine all irreducible, fixed-point subgroups of $\text{Sp}_8(2)$.

**Theorem 1.5** Let $G$ be a maximal irreducible fixed-point subgroup of $\text{Sp}_8(2)$. Then $G$ is conjugate to $L_3(2) : 2$ or to $\text{AGL}_2(3) \cong \text{PSU}_3(2) : S_3$. Furthermore, the irreducible subgroups of these groups are $L_3(2)$ in $L_3(2) : 2$ and $\text{PSU}_3(2)$, $\text{PSU}_3(2) : 2$, $\text{PSU}_3(2) : 3$, and $\text{AGL}_1(9)$ in $\text{AGL}_2(3)$.

We observe that these groups are absolutely irreducible. This then gives us a classification of the images of the mod-$2$ representations on $4$-dimensional absolutely simple abelian varieties for which the answer to Question 1.1 is No for the modulus $2$. We also remark that in terms of constructing such a fourfold, there is at least the possibility that the base field can be taken to be $\mathbb{Q}$; for general $p$-torsion one needs the number field $K$ of definition of $A$ to contain $\mathbb{Q}(\zeta_p)$ in order for the image of $\rho_p$ to lie in $\text{Sp}_8(p)$ as opposed to $\text{GSp}_8(p)$. This remark adds to our interest in the special case of $2$-torsion.

We discover that the groups of Theorem 1.5 are the first instances of several infinite families of absolutely irreducible fixed-point subgroups of $\text{Sp}_m(2)$ for certain $m \geq 8$. None of these families have been previously known. The first such family is given in the following theorem.

**Theorem 1.6** Let $q$ be odd and $m = q^n - 1$. Suppose that $n > 1$ or $n = 1$ and $q$ is not a prime. Then there exists an absolutely irreducible fixed-point subgroup of $\text{Sp}_m(2)$ isomorphic to $\text{AGL}_n(q)$.

Note that the group $\text{AGL}_n(q)$ with $q = r^k$, $r$ a prime, is a subgroup of $\text{AGL}_{nk}(r)$. We expect that the latter group is a maximal fixed-point subgroup of $\text{Sp}_m(2)$ for $m = q^n - 1$.

The second family is exemplified by the isomorphism $L_3(2) : 2 \cong \text{PGL}_2(7)$, and is given in the following theorem.

**Theorem 1.7** Let $G = \text{PGL}_2(q)$, where $4 \mid (q + 1)$ and $3 \mid (q - 1)$. Then $G$ has an absolutely irreducible unisingular representation into $\text{Sp}_{q+1}(2)$.

The third family arises from the Steinberg $2$-modular representation of simple groups of Lie type, as follows.

**Theorem 1.8** Let $G$ be a finite simple group of Lie type in characteristic $2$ and $\text{St}_2$ the $2$-modular Steinberg representation of $G$. Suppose that $G$ is not of type $A_1(q)$, $2A_n(q)$ ($n$ odd), or $E_7(q)$. Then $\text{St}_2(G)$ is a fixed point subgroup of $\text{Sp}_m(2)$, where $m = |G|_2$ is the $2$-part of the order of $G$.

Note that a proof of this theorem is required only for groups of type $A_n(q)$ with $n$ odd, $C_n(q)$ with $n \equiv 1, 2 \pmod{4}$ and of type $D_n^+(q)$, $n \equiv 1, 2 \pmod{4}$, as for the other groups the result has been obtained in an earlier paper of the second-named author; see Theorem 3.5 below. The group $L_3(2) \cong A_2(2)$ arising in Theorem 1.5 for $m = 8$ is the minimal order example of those in Theorem 1.8.

We do not have a result as above for unitary groups $2A_n(q) \cong \text{PSU}_{n+1}(q)$ with $n$ odd and for $E_7(q)$. For $G = A_1(q)$ the $2$-modular Steinberg representation does not yield a fixed-point subgroup (Lemma 3.7).
In terms of the original questions of Lang and Katz, we obtain the following corollary for abelian varieties of certain dimensions \( \geq 5 \).

**Corollary 1.9** Let \( G \subset \text{Sp}_m(2) \) be the symplectic embedding of any of the groups of Theorems 1.6, 1.7, or 1.8. Then \( G \) occurs as the image of the mod 2 representation of an absolutely simple abelian variety of dimension \( m/2 \), defined over some number field \( K \), such that \( A_p(\mathbb{F}_p) \) has an even number of points for every good prime \( p \subset \mathcal{O}_K \), but no member of the \( K \)-isogeny class of \( A \) has a global point of order 2.

Corollary 1.9, whose proof appears at the end of the paper, immediately before Appendix, shows that symplectic, irreducible, unisingular representations occur as Galois representations attached to abelian varieties \( A \) over number fields \( K \). But the degree \( [K: \mathbb{Q}] \) of the field of definition of \( A \) may not be minimal if we use the purely Galois-theoretic construction of \( A \). It would be interesting to search for examples of these abelian varieties such that \( [K: \mathbb{Q}] \) is minimized, or even to determine necessary lower bounds on \( [K: \mathbb{Q}] \).

For an irreducible representation of \( G \in \{ \text{Sp}_{2n}(q), \text{Spin}^{\pm}_{2n}(q), \text{SL}_n(q) \} \) (\( q \) is an arbitrary prime power), we give a rather strong sufficient condition for an irreducible representation to be unisingular.

**Theorem 1.10** Let \( G \) be a finite simple group of Lie type \( A_n(q), C_n(q) \) or \( D_n^\pm(q) \), \( G \) the respective simple algebraic group and let \( \omega_1, \ldots, \omega_n \) be the fundamental weights of the weight system of \( G \). Let \( V \) be an irreducible \( G \)-module. Suppose that the following conditions hold:

1. \( G \) is of type \( A_n, n > 1 \), and there are natural numbers \( m_1, m_2 \) and \( i \in \{1, \ldots, n\} \) such that \( m_1(q - 1)\omega_i + \omega_1 + \omega_n \) and \( m_2(q - 1)\omega_i \) are weights of \( V \).
2. \( G \) is of type \( C_n, n > 1 \), or \( D_n, n > 3 \), and there are natural numbers \( m_1, m_2, m_3 \) such that \( m_1(q + 1)\omega_1, m_2(q - 1)\omega_1 \) and \( m_3(q - 1)\omega_1 \) are weights of \( V \).

Then every element \( g \in G \) has eigenvalue 1 on \( V \).

We show that this condition is satisfied by the \( p \)-modular Steinberg representation of \( G \).

Finally, a natural question would be to extend our fixed-point subgroup result in Theorem 1.5 to larger \( n, n = 10, 12, \) etc. However, this would lead to an analysis of many cases which do not depend uniformly on \( n \). For this reason we think that further work on understanding the unisingularity problem in higher rank must focus on constructing unisingular representations of uniform families of groups. This paper gives a certain contribution for simple groups of Lie type.

Obtaining necessary and sufficient condition of unisingularity for an irreducible representation of arbitrary simple groups of Lie type in defining characteristic 2 does not seem to be a realistic task. However, this has been achieved for groups \( \text{SL}_n(2) \) and \( \text{Sp}_{2n}(2) \) in the papers [35,36] of the second-named author.

In the next two sections we review our notation and conventions. We then focus on affine groups before the special case of \( \text{Sp}_8(2) \) where our goal is to classify all 8-dimensional symplectic irreducible fixed-point subgroups. We conclude the paper by investigating groups of Lie type in more generality and add a brief appendix where we record the computational aspects of our classification.
2 Background: notation and conventions

We denote by \( \mathbb{Z}, \mathbb{Q}, \mathbb{C}, \mathbb{F}_q \) the set of integers, the field of rational and complex numbers, and the finite field of \( q \) elements, respectively. For a set \( X \) we write \( |X| \) for the cardinality of \( X \).

2.1 Finite groups

In our work below on finite groups, we use the symbols \( S_n, A_n, D_{2n}, \text{ and } C_n \) to denote the symmetric, alternating, dihedral, and cyclic groups on \( n \) letters. If \( N \) and \( Q \) are groups we use the notation \( N \triangleleft Q \) to denote any group with kernel \( N \) and quotient \( Q \). If the extension is split, we use \( N \rtimes Q \). We use \( N \wr \phi Q \) to denote the wreath product \( N \rtimes \phi Q \), where \( \phi : Q \rightarrow S_n \) is a permutation representation. When the representation is understood from context we omit \( \phi \) from the notation.

We import notation from the theory of finite and classical groups as well, such as \( \text{PSL}_n(\mathbb{F}_q) \) for the simple group \( \text{PSL}_n(\mathbb{F}_q) \) over the field of \( q \) elements. In general, we follow the conventions of [5] for the finite classical groups. While most of this notation is standard across the literature, we are careful to point out that the definition of the special orthogonal group tends to vary across the literature. We use notation \( \text{SO}_8^{\pm}(2) \) for the special orthogonal group of degree 8, and \( \Omega_8^{\pm}(2) \) for its simple subgroup of index 2.

If \( G \) is a finite group and \( p \) is a prime number then we write \( |G| \) for the order of \( |G| \) and \( |G|_p \) for the \( p \)-part of \( |G| \), namely the greatest \( p \)-power dividing \( |G| \). Furthermore, if \( G \) is a finite group of Lie type in defining characteristic \( p \), then there exists a unique irreducible representation of \( G \) over the complex numbers \( \mathbb{C} \) known as the Steinberg representation of \( G \). Let \( F \) be an algebraically closed field of characteristic \( p \). Then there exists an irreducible \( F \)-representation of \( G \) whose Brauer character coincides with the character of the Steinberg representation at the semisimple elements of \( G \). We use the notation \( \text{St}_G \) for the Steinberg representation of \( G \) over \( \mathbb{C} \), and \( \text{St}_p \) for the modular Steinberg representation over \( F \).

Because we will often focus on a specific prime (usually \( p = 2 \)), we follow standard conventions in groups theory and write \( O_p(G) \) for the maximal normal \( p \)-subgroup of \( G \). When \( p \) is understood, we call a \( p' \)-element of \( G \) one that has order coprime to \( p \); a \( p' \)-group has order coprime to \( p \).

If \( V \) is a set and \( Y \subseteq \text{Sym}(V) \) then we write \( V^Y \) for the set of fixed points of the action of \( Y \) on \( V \). If \( V \) is a vector space and \( Y \subseteq \text{GL}(V) \) then \( V^Y \) is a subspace of \( V \), and \( Y \) is a fixed point set if and only if \( V^y \neq 0 \) for every \( y \in Y \). If \( V \) is a \( KG \)-module (where \( K \) is a field) and \( H \subseteq G \) a subgroup then \( V|_H \) is the restriction of \( V \) to \( H \).

2.2 Linear algebraic groups

We take this opportunity to set our conventions for the remainder of the paper and acknowledge that there are places where the standards of the finite group theory and Lie theory community overlap—for example the symbol \( A_n \) is standard notation for
both the alternating group on \( n \) letters and for one of the simple, simply connected algebraic groups of rank \( n \). However, it should be clear from the context what is meant.

Recall that simple, simply-connected algebraic groups can be partitioned into types denoted by \( A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4, G_2 \) which naturally correspond to the types of simple Lie algebras over \( \mathbb{C} \). The subscript is called the rank of the group and Lie algebra. To every simple Lie algebra of rank \( n \) there corresponds a weight lattice \( \Lambda \), a \( \mathbb{Z} \)-lattice of rank \( n \) whose elements are called weights. The Lie algebra determines subsets \( \{\alpha_1, \ldots, \alpha_n\} \) and \( \{\omega_1, \ldots, \omega_n\} \) of \( \Lambda \) called simple roots and fundamental weights, respectively. The latter form a \( \mathbb{Z} \)-basis of \( \Lambda \), so every weight can be written as \( \sum z_i \omega_i \) with \( z_i \in \mathbb{Z} \). The weights \( \sum z_i \omega_i \) with \( z_i \geq 0 \) are called dominant and we write \( \Lambda^+ \) for the set of dominant weights. The expressions of \( \alpha_1, \ldots, \alpha_n \) in terms of \( \omega_1, \ldots, \omega_n \) are given in [3, Planchees I-IX]. Denote by \( \mathcal{R} \) the sublattice of all \( \mathbb{Z} \)-linear combinations of the simple roots and by \( \mathcal{R}^+ \) the subset of \( \mathbb{Z} \)-linear combinations with non-negative coefficients. The lattice \( \Lambda \) is endowed with a partial ordering, which we denote by \( \succeq \). Specifically, \( \lambda \succeq \mu \) if and only if \( \lambda - \mu \in \mathcal{R}^+ \).

The same weight system is assigned to the respective simple algebraic group \( G \). The irreducible representations of \( G \) are parameterized by the dominant weights. We set \( \sigma_q := (q - 1)(\omega_1 + \cdots + \omega_n) \), where \( q \) is a prime power, and denote by \( \Phi_q \) the irreducible representation of \( G \) with highest weight \( \sigma_q \).

## 3 Preliminary and known results

In this short section we review what is known and provide some additional results that will be of use in the proofs of the main theorems below.

**Lemma 3.1** Let \( F \) be a field of characteristic 2 and \( G \) a finite group. Let \( \beta \) be the Brauer character of a non-trivial absolutely irreducible \( FG \)-module \( V \). Suppose that \( \beta(g) \) is an integer for every odd order element \( g \in G \). Then \( \rho(G) \) is equivalent to a representation into \( \text{Sp}_d(2) \).

**Proof** If \( \beta(g) \in \mathbb{Z} \) then \( \beta(g) \pmod{2} \in \mathbb{F}_2 \) and \( \rho \) is equivalent to a representation into \( \text{GL}_d(2) \) by [15, Chapter I, Theorem 19.3]. In addition, \( V \) is self-dual [15, Chapter IV, Lemma 2.1 (iv)], and hence \( G \) preserves a non-degenerate alternating form on \( V \) [15, Chapter IV, Theorem 11.1 and Corollary 11.2]. \( \square \)

**Proof of Theorem 1.7** Suppose first that \( G = \text{L}_2(q) \). As \( 4 \mid (q + 1) \) and \( 3 \mid (q - 1) \), the integer \( (q - 1)/2 \) is odd and \( q \geq 7 \). Note that every ordinary irreducible character of \( G \) of degree \( q + 1 \) is induced from a non-trivial one-dimensional character \( \tau \) of \( B \), a Borel subgroup of \( G \), see [27, Section 3] or elsewhere. Then \( \tau(b) \) for every \( b \in B \) is a \( (q - 1)/2 \)-root of unity (as \( |B| = q(q - 1)/2 \)). In fact, \( \tau^G \), the induced character, is irreducible whenever \( \tau \neq 1_B \) (loc. cit.). As \( |G|/2 \) divides \( q + 1 \), every character \( \tau^G \) is of 2-defect 0, and hence \( \tau^G \) restricted to the odd order elements of \( G \) is an irreducible Brauer character. By the character table of \( G \) we have \( \tau^G(g) \in \mathbb{Z} \) for \( g \in G \) with \( |g| \) odd unless \( |g| \) divides \( (q - 1)/2 \) and hence \( g \) is conjugate to an element of \( B \). Let \( g \in B \). Then \( \tau^G(g) = \tau(g) + \tau(g)^{-1} \). This is an integer if and only if \( \tau(g)^3 = 1 \). As \( \tau \) is non-trivial, this requires \( 3 \mid (q - 1) \), and if this holds then there is \( \tau \neq 1_B \) with
\[ \tau(g)^3 = 1 \] for every \( b \in B \). By Lemma 3.1, for this \( \tau \) the Brauer character \( \tau^G \) is the character of a representation \( G \rightarrow \text{Sp}_{q+1}(2) \).

Let \( G = \text{PGL}_2(q) \). Note that every odd order element \( g \in G \) lies in \( L_2(q) \), but \( C_G(g) \) does not. It follows that \( G \) and \( L_2(q) \) have the same number of conjugacy classes of odd order elements, and hence the same numbers of 2-modular irreducible representations. By Clifford’s theorem, distinct irreducible representations of \( G \) have no common irreducible constituent under restriction to \( L_2(q) \), and hence every 2-modular irreducible representation of \( G \) is irreducible on \( L_2(q) \). As the fixed point property is to be examined at the odd order elements only, the result follows from that for \( L_2(q) \).

We will make use of the following results extensively in later sections of the paper.

**Lemma 3.2** ([21, Sections 3.1 and 3.7]) Let \( G \) be a simple simply connected algebraic group in defining characteristic \( p \), let \( G_{\mathbb{C}} \) be a simple simply connected algebraic group of the same type as \( G \) and \( L \) the Lie algebra of \( G_{\mathbb{C}} \). Let \( \sigma_q := (q - 1, \ldots, q - 1) \) be an element of the weight system of \( G \), and let \( \Phi_q \) be an irreducible representation of \( G \) with highest weight \( \sigma_q \).

Then the weights of \( \Phi_q \) are the same as the weights of an irreducible representation of the Lie algebra \( L \) of the same type as \( G \) with highest weight \( \sigma_q \).

**Proof** By inspection of the expressions of fundamental weights in terms of simple roots [3, Planches I–IX] one observes that all these expressions have non-negative rational coefficients. Let \( \omega_0 \) be 0. By assumption, \( \lambda - \lambda' \) is a radical dominant weight, so \( \lambda - \lambda' = \sum b_i \alpha_i \), where \( b_i \in \mathbb{Q}, b_i \geq 0 \). On the other hand, as simple roots are linear independent, \( \lambda - \lambda' \in \mathcal{R} \) means that all \( b_i \) are integers. Therefore, \( \lambda - \lambda' > \omega_0 \), where \( \omega_0 \) stands for the zero weight. This is equivalent to \( \lambda \geq \lambda' \).

The following lemma is well known.

**Lemma 3.4** Let \( G \) be a simple algebraic group and \( V \) a rational \( G \)-module. If \( V \) has zero weight then \( V^g \neq 0 \) for every element \( g \in G \).

**Theorem 3.5** ([32, Theorems 1 and 3]) Let \( G \) be a finite Chevalley group \( G \) in defining characteristic \( p > 0 \), and let \( \text{St}_G \) be the Steinberg representation of \( G \) over the complex numbers. Suppose that \( p > 2 \) or \( p = 2 \) and \( G \) is of type \( G_2(q), E_6(q), 2E_6(q), E_8(q), F_4(q), 2F_4(q), 3D_4(q), C_n(q) \), \( n = 4k \) or \( 4k - 1, k = 1, 2, \ldots \), \( D_n(q), 2D_n(q) \), \( n = 4k \) or \( 4k + 3, k = 1, 2, \ldots \), \( A_n(q), 2A_n(q) \) \( n \) even). Then for every torus \( T \) of \( G \) the trivial representation \( 1_T \) of \( T \) is a constituent in the restriction of \( \text{St}_G \) to \( T \).

**Lemma 3.6** Let \( G = 2B_2(q) \). Then the 2-modular Steinberg representation \( \text{St}_2 \) of \( G \) is unisingular. Moreover, if \( g \in G \) is an odd order element then \( \text{St}_2(g) \) has exactly \(|g| \) distinct eigenvalues.

**Proof** In [32, 2.8], the second claim is deduced from the character table of \( G \) over \( \mathbb{C} \). This remains true in characteristic 2 as the Steinberg representation of \( G \) over \( \mathbb{C} \) is of 2-defect 0.
Lemma 3.7 The group $G = \text{SL}_2(q)$ with $q$ even has no non-trivial unisingular 2-modular irreducible representation.

Proof It suffices to prove this for the ground field algebraically closed. Let $g \in G$ be of order $q - 1$. By [25, Corollary], the irreducible representation with highest weight $\sigma_q$ is the only irreducible representation $\phi$ such that $\phi(g)$ has eigenvalue 1. It is well known that $\text{St}(h)$ does not have eigenvalue 1 for $h \in G$ with $|h| = q + 1$ (one can inspect the character table of $G$). This remains true in characteristic 2 as the Steinberg representation of $G$ over $\mathbb{C}$ is of defect 0. $\square$

The following lemma is easily verified using existing databases of groups of small order, such as GAP [29], Magma [2], or the GroupNames databases, the latter located at https://people.maths.bris.ac.uk/matyd/GroupNames/index.html.

Lemma 3.8 The group $H = \text{AGL}_2(3)$ has exactly four 2-modular absolutely irreducible representations, whose degrees are 1, 2, 8, 16. The group $G = \text{ASL}_2(3)$ has exactly four 2-modular irreducible representations, whose degrees are 1, 8, 8, 8; one of those of degree 8 can be realized over $\mathbb{F}_2$.

Remark 3.9 From Lemma 3.8 we deduce that each $G \in \{\text{AL}_1(9), \text{AGL}_1(9), \text{PSU}_3(2)\}$ has a unique 2-modular irreducible representation of degree 8 and no irreducible representation of greater degree. Moreover, the groups

$$\text{AL}_1(9), \text{AGL}_1(9), \text{PSU}_3(2), \text{and ASL}_2(3)$$

are the only subgroups of $H$ with a 2-modular irreducible representation of degree 8. This follows from analyzing the maximal subgroup tree of these groups, using either the GAP [29] or Magma [2] databases of small groups.

Remark 3.10 Note that $\text{AGL}_2(3) \cong \text{Aut} \text{PSU}_3(2) \cong \text{PSU}_3(2) : S_3$. The fact that $\text{PSU}_3(2)$ has a unique 2-modular irreducible representation of degree 8 is a special case of a general result on the Steinberg representation of an arbitrary group of Lie type. In addition, if an irreducible representation of a centerless group $H$ is unique, it extends to a projective representation of $\text{Aut}(H)$. In our case this is equivalent to an ordinary representation.

Lemma 3.11 Let $G$ be an irreducible fixed point subgroup of $\text{Sp}_{2n}(q)$, for $q$ even. Then the following statements hold:

(1) $Z(G) = 1$;
(2) $G$ has no non-trivial normal 2-subgroup; moreover, if $G$ is a subgroup of $X \subset \text{Sp}_{2n}(q)$ then $X$ has no non-trivial normal 2-subgroup;
(3) every odd order element of $\text{Sp}_{2n}(q)$ is conjugate to its inverse;
(4) every abelian subgroup of $G$ is reducible.

Proof Let $V$ be the underlying space of $\text{Sp}_{2n}(q)$.

(1) Suppose the contrary. Let $z \in Z(G)$ and let $V^z$ be the 1-eigenspace of $z$ on $V$. Then $G$ stabilizes $V^z$, a contradiction unless $V^z = V$, but this means that $z = \text{Id}$. $\square$
(2) Suppose the contrary. Let $S \neq 1$ be a normal 2-subgroup of $G$. By assumption $V^S \neq 0$, $V$, which is then $G$-stable, a contradiction.

(3) This is well known. In fact, $V$ is self-dual as an $F_q \text{Sp}_{2n}(q)$-module, and as an $F_q H$-module for every subgroup $H$ of $\text{Sp}_{2n}(q)$.

(4) Suppose the contrary, let $A$ be an abelian subgroup of $\text{Sp}_{2n}(q)$ and $1 \neq a \in A$. By assumption, $V^a \neq 0$, $V$. Then $AV^a = V^a$, a contradiction. $\square$

**Proposition 3.12** Let $G \subseteq \text{Sp}_{2k}(q)$ be a maximal irreducible fixed-point subgroup. Then $H = G \times \text{Sp}_{2n}(q)$ (a “diagonal” embedding) is a maximal fixed-point subgroup of $\text{Sp}_{2(k+n)}(q)$.

**Proof** Let $V$, $V_1$, $V_2$ be the standard modules for $\text{Sp}_{2(k+n)}(q)$, $\text{Sp}_{2k}(q)$ and $\text{Sp}_{2n}(q)$, respectively. By “diagonal” embedding we mean an embedding which agrees with $V_1 \oplus V_2 \to V$ such that $V_1$, $V_2$ are non-degenerate subspaces of $V$.

Note that $k > 1$ by [26, Chapter I, Section 1, Exercise 1], and in fact $k > 2$ by [23] as $\text{Sp}_2(q)$ and $\text{Sp}_4(q)$ has no irreducible fixed-point subgroup.

Suppose the contrary. Let $X$ be a fixed-point subgroup of $\text{Sp}_{2(k+n)}(q)$ containing $H$. Suppose first that $X$ is irreducible. Then $X$ contains all transvections of $H$, in particular, $\text{diag}(\text{Id}_{2k}, \text{Sp}_{2n}(q)) \subseteq X$. Irreducible subgroups of $\text{GL}_{2(n+k)}(q)$ that are generated by transvections are known: if $q$ is odd then these are $\text{SL}_{2(n+k)}(q')$, $\text{SU}_{2(n+k)}(q')$, $\text{Sp}_{2(n+k)}(q')$ with $q' \mid q$. If $q$ is even then, additionally, these are $\text{O}_{2(n+k)}(q')$, $\text{S}_{2(n+k)+2}$, $S_{2(n+k)+1}$ or $A_6 \subseteq \text{SO}_{4}^+(q)$, see [31, Section 12] for detailed references. None of these are fixed-point groups. Thus $X$ is reducible. It follows that $X$ stabilizes both $V_1$, $V_2$.

As $\text{diag}(\text{Id}_{2k}, \text{Sp}_{2n}(q)) \subseteq X$, we deduce that the restriction $X_1$ of $X$ to $V_1$ contains $G$ and $X_1$ must be fixed-point on $V_1$. So $X_1 = G$, and then $X = H$, a contradiction. $\square$

With these preliminary results complete, we now turn to the main business of the paper in the next several sections.

### 4 The affine groups

In this section we show that affine linear groups occur as fixed-point subgroups of certain symplectic groups $\text{Sp}_m(2)$. We define here an affine group as a semidirect product of $A = \mathbb{P}^n_q$ and a subgroup $H \subseteq \text{GL}_n(q)$ such that $H$ transitively permutes the non-zero elements of $A$. Such groups $H$ have been classified by Hering [18, Section 5]; see also [14]. Then $AH$ acts 2-transitively on the cosets $AH/H$. In particular, $AH$ is a 2-transitive subgroup of $S_m$, where $m = |A|$. We remark that the term “affine group” is sometimes used more broadly in permutation group theory, in particular where $H$ can be any subgroup of $\text{GL}_n(q)$. Theorem 1.6 is a special case of the following result.

**Theorem 4.1** Let $AH$ be an affine group with $|A|$ odd, and $m = (|A| - 1)/2$. Suppose that $n > 1$ or $n = 1$ and $q$ is not a prime. Then there exists an absolutely irreducible fixed-point subgroup of $\text{Sp}_{2m}(2)$ isomorphic to $AH$.

**Proof** Let $n = |A|$. Then $AH$ is a 2-transitive subgroup of $S_n$. Let $R$ be the natural permutational $FS_n$-module, where $F$ is an algebraically closed field of characteristic
2. As \( n \) is odd, we have \( R = F \oplus M \), where \( M \) is irreducible and \( F = 1_{S_n} \) stands for the trivial \( \mathbb{F}_2 \)-module. This is also true for \( \mathbb{F}_2 \) in place of \( F \). If \( M_2 \) is the corresponding \( \mathbb{F}_2 S_n \)-module then \( S_n \) is well known to preserve a non-degenerate alternating form on \( M_2 \); this yields an embedding \( \phi : S_n \to \text{Sp}(M_2) \). As an \( FA \)-module, \( R \) is completely reducible.

Let \( M = \sum_{\alpha \neq 1} R_\alpha \) be the sum of the non-trivial irreducible \( FA \)-submodules of \( R \). As \( H \) is transitive on \( A \setminus 1 \), it also is transitive on the non-trivial irreducible characters of \( A \). It follows that \( \dim R_\alpha = 1 \) for every \( \alpha \) and \( M \) is irreducible \( \text{FA}H \)-module as well as \( M_2 \).

It remains to show that \( \phi(AH) \) is a fixed point subgroup. For this we can assume that \( H = \text{GL}_n(q) \). Let \( h \in AH \). To prove that 1 is an eigenvalue of \( \phi(h) \), it suffices to assume that \( |h| \) is odd.

In a permutational module \( \mathbb{C}X \)-module of a group \( X \) the multiplicity of the trivial character \( 1_X \) equals the number of the \( X \)-orbits at the underlying permutation set (see [12, Theorem 32.3]). This is true for the \( FX \)-module if \( |X| \) is odd. Applying this to the cyclic group \( h \) and the permutational set \( AH/H \), it suffices to show that \( \langle h \rangle \) has at least 2 orbits on \( A \). For this it suffices to show that \( |h| < |A| = q^n \) for every \( h \in H \).

Note that the order of an element in \( \text{GL}_n(q) \) is at most \( q^n - 1 \) (see for instance [17, Corollary 2.7]). If \( \langle h \rangle \cap A = 1 \) then \( |h| \) does not exceed the maximum of element orders in \( \text{GL}_n(q) \), so \( |h| \leq q^n - 1 \).

Suppose that \( \langle h \rangle \cap A \neq 1 \). Let \( h_0 \) be a generator of \( \langle h \rangle \cap A \). Then \( h \in CH(h_0) \), and hence \( |h| \) does not exceed \( px \), where \( x \) is the maximum of element orders in \( CH(h_0) \) and \( q \) is a power of the prime number \( p \). Note that \( CH(h_0) \) can be interpreted as the stabilizer in \( H \) of a non-zero vector in the underlying vector space of \( \text{GL}_n(q) \). So \( CH(h_0) \) is isomorphic to the affine group \( \mathbb{F}_q^{n-1} \text{GL}_{n-1}(q) \). We use induction. If \( n = 2 \) then \( \mathbb{F}_q^{n-1} \text{GL}_{n-1}(q) = \mathbb{F}_q \text{GL}_1(q) \). If \( x \in \mathbb{F}_q \text{GL}_1(q) \) then \( |x| \leq q - 1 \) unless \( x \in \mathbb{F}_q \) and \( q = p \). Then \( |x| = p \), and hence \( |h| \) is a \( p \)-power. Note that \( \text{AGL}_2(q) \) is isomorphic to a subgroup \( \left( \begin{array}{ccc} * & * & * \\ 0 & 0 & 1 \end{array} \right) \) over \( \mathbb{F}_q \), and the subgroup \( \left( \begin{array}{ccc} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{array} \right) \) is a Sylow \( p \)-subgroup of it. This is of exponent \( p \) as \( q \) is odd. It follows that \( |h| \leq q^2 - 1 \). Let \( n > 2 \). By induction, the maximum element order in \( \mathbb{F}_q^{n-1} \text{GL}_{n-1}(q) \) does not exceed \( q^{n-1} - 1 \), so \( |h| \leq p(q^{n-1} - 1) < q^n - 1 \).

Let \( n = 1 \). Then a maximum element order in \( \text{AGL}_1(q) \) is \( q - 1 \) unless \( q = p \). In the latter case \( |h| = p \) for \( 1 \neq h \in A \), and 1 is not an eigenvalue of \( h \).

\begin{corollary}
Let \( d > 4 \). Then \( \text{Sp}_{2d}(2) \) has a fixed point subgroup \( G \) of order 27 and exponent 3 with no trivial composition factor. If \( d = 4 \) then there is such group \( G \) of order 9.
\end{corollary}

\begin{proof}
If \( d = 4 \) then we choose for \( G \) the subgroup \( A \) of \( \text{AGL}_2(3) \) in Theorem 4.1, whose proof makes evident that \( A \) has no trivial composition factor. Let \( d > 4 \) and let \( V \) be the underlying space of \( \text{Sp}_{2d}(2) \). Let \( V = V_1 \oplus V_2 \oplus \cdots \oplus V_{d-4} \), where \( V_1, \ldots, V_{d-4} \) are non-degenerate mutually orthogonal subspaces such that \( \dim V_1 = 8 \) and \( \dim V_2 = \cdots = \dim V_{d-4} = 2 \). Let \( g \in \text{Sp}_2(2) \) be an element of order 3, and \( G = A \times C_3 \). The representation in question is the sum of a representation \( A \to \text{Sp}_3(2) \) with \( |A| = 9 \) considered above, and the representation \( C_3 \to \text{Sp}_{2(d-4)}(2) \), which
\end{proof}
Theorem 4.1 shows that the conclusion of the corollary. 

Remark 4.3 Let $A$ be an elementary abelian group of finite odd order $m$. The proof of Theorem 4.1 shows that $A$ is isomorphic to a fixed point subgroup of $\text{Sp}_{m-1}(2)$ which has no trivial composition factor. In addition, $A$ is a minimal subgroup of $\text{Sp}_{m-1}(2)$ with this property, that is, every proper subgroup $B$ of $A$ has a trivial composition factor.

Lemma 4.4 Let $G$ be a group such that $A \subset G \subset AH \subset S_n \subset \text{Sp}(M)$. Suppose that $G$ is irreducible on $M$. Then $G$ acts transitively on $A \setminus 1$, in particular, $|G|$ is a multiple of $|A| - 1$.

If $|A| = 9$ then $G/A$ is isomorphic to one of the following groups: $\text{GL}_2(3)$, $\text{SL}_2(3)$, $D_{16}$, $Q_8$ or $C_8$, and $G \in \{\text{AGL}_2(3), \text{ASL}_2(3), \text{AL}_1(9), \text{PSU}_3(2), \text{AGL}_1(9)\}$.

Proof It is known that $G$ is a 2-transitive subgroup of $S_n$, see for instance [24, Theorem 3.10]. This implies that $G/A$ is transitive on $A \setminus 1$ when $G$ acts on $A$ by conjugation. This implies $|G|$ to be a multiple of $|A| - 1$.

Let $|A| = 9$. Then $G/A$ is a subgroup of $\text{GL}_2(3)$ and 8 divides $|G/A|$. Clearly, $\text{SL}_2(3)$ is transitive on $A \setminus 1$. Other subgroups of $\text{SL}_2(3)$ whose order is a multiple of 8 are 2-groups. The Sylow 2-subgroup $D$, say, of $\text{GL}_2(3)$ is dihedral of order 16. This is transitive on $A \setminus 1$ as $|C_D(a)| = 2$ for $1 \neq a \in A$. Let $D_1$ be a proper subgroup of $D$. Then $|D_1| \leq 8$, so we have to inspect $D_1$ of order 8. This is transitive on $A \setminus 1$ if and only if $|C_{D_1}(a)| = 1$, equivalently, if and only if $D_1$ contains no reflection (of Jordan form diag$(1, -1)$). Clearly, the only such group containing a reflection is the dihedral group $D_8$ of order 8. We can assume that

$$D_8 = \left\{ \begin{pmatrix} \pm1 & 0 \\ 0 & \pm1 \end{pmatrix} , \begin{pmatrix} 0 & \pm1 \\ \pm1 & 0 \end{pmatrix} \right\} .$$

Moreover, the group $Y = 3^2 : D_8 \cong S_3 : C_2$ is not irreducible in $\text{Sp}_8(2)$. Indeed, the character table of $Y$, at https://people.maths.bris.ac.uk/matyd/GroupNames/61/S3wrC2.html, implies that the irreducible 2-modular representations of it are of degrees $1, 4, 4$, and the Brauer character values are integers. By Lemma 3.1, each of these representations is realized over $\mathbb{F}_2$. Therefore, $Y$ has no irreducible representation of degree 8 over $\mathbb{F}_2$. The remaining subgroups of order 8 are cyclic $C_8$ and the quaternion group of order 8 (a Sylow 2-subgroup of $\text{SL}_2(3)$). These yield the groups $G = \text{AGL}_1(9)$ and $\text{PSU}_3(2)$, respectively. \qed

5 Fixed-point subgroups of $\text{Sp}_8(2)$

In this section we focus on the smallest symplectic groups $\text{Sp}_{2n}(p)$ for which a complete classification of its irreducible fixed-point subgroups is of yet unknown: the case $n = 4$ and $p = 2$. We remind the reader that this is the motivating case from the point of view of 2-torsion on abelian fourfolds. In this section $H = \text{Sp}_8(2)$ and $V$ the symplectic space over $\mathbb{F}_2$ on which $H$ acts in the natural way. So $H$ preserves a non-degenerate bilinear form $(\cdot, \cdot)$ on $V$. If $U$ is a subspace of $V$ then $U^\perp = \{x \in V : (x, U) = 0\}$. 

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We denote by $\mathcal{F}$ the set of irreducible fixed-point subgroups of $H$. In this section we determine $\mathcal{F}$, that is, we prove Theorem 1.5. We focus on the first statement as the second one follows from Lemma 4.4 and easy analysis of subgroups of $\text{PGL}_2(7)$.

Our strategy is to search in the tree of maximal subgroups to determine the maximal irreducible fixed-point groups. However, due to the size of the group $\text{Sp}_8(2)$, we first make a few observations that reduce the search size.

The elements of $H$ is not fixed point, hence any irreducible fixed-point subgroup must lie in a maximal proper irreducible subgroup. By [5, Table 8.48], the proper, irreducible maximal subgroups $M$ of $H$ are given in Table 1, in increasing size order.

Until the end of this section, $G \in \mathcal{F}$ and $M$ denotes a maximal subgroup of $H$ containing $G$.

**Lemma 5.1** $G$ has no element of order 5, 17, or 21.

**Proof** The elements of order 17 and 21 of $H$ do not have eigenvalue 1. For the remaining claim, suppose the contrary and let $h \in G$ be of order 5. Note that the $H$-invariant proper subspaces of $V$ are of dimension 4 and $G$ is primitive on $V$. (Otherwise $V = V_1 + V_2$, where $V_1$, $V_2$ are disjoint subspaces permuted by $G$, and $gV_1 = V_2$, $gV_2 = V_1$ for some $g \in G$. Clearly, $hV_i = V_i$ and $h$ is either trivial on $V_i$ or irreducible for $i = 1, 2$. As $V^h \neq 0$, we may assume that $V^h = V_2$. But then $hgh^{-1}$ acts fixed point freely on $V$, a contradiction.)

As $G$ is primitive, we can ignore $M = \text{Sp}_4(2) : S_2$.

Let $M = \text{Sp}_4(4).2$. Then $h$ lies in a subgroup isomorphic to $\text{SL}_2(16)$ or $(A_5 \times A_5) : 2$ [6, p. 44]. The latter case is ruled out as above, the former case is ruled out by [22].

Let $M = S_{10}$. Let $L$ be a maximal subgroup of $M$. Then $L$ has an element of order 5 [6, p. 49], but we can ignore $L$ with $O_2(L) \neq 1$. Groups $L \cong S_5 \times S_4$ and $S_7 \times S_3$ are reducible (by Clifford’s theorem, say), and $L \cong (S_5 \times S_5) : 2$ is imprimitive. We are left with $L \cong S_9$ and $L \cong M_{10}.2$. As $G \subset S_{10}$, the Brauer character $\phi$ of $G$ has integral values, so $\phi|_{A_6} = \phi_2 + \phi_3$ in the notation of [22], and then $\phi(h)$ is fixed point free by [22, p. 4]. So $G \subset S_9 \subset S_{10}$. It follows that $\phi(h) = 3$ and then $\phi(x) = -1$ for $x \in S_9$ of order 9, and hence $x$ is fixed point free. So $G \neq S_9$. In addition, $G \neq A_8$, $S_8$ as otherwise $\phi(g) = -2$ [22, p. 49]. If $G \subset S_8$, $G \neq A_8$ and $(5, |G|) = 1$ then $G$ is reducible. This is a contradiction. □

**Lemma 5.2** Let $A$ be a maximal abelian normal subgroup of $G$. If $A \neq 1$ then $A = O_3(G)$, $A = C_3 \times C_3$ and $G$ is isomorphic to a subgroup of $\text{AGL}_2(3)$. In particular, this holds if $G$ is solvable.
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**Proof** By Lemma 5.1, \(|A| = 85\) = 1. Let \(B\) be a Sylow 7-subgroup of \(A\). Then \(B = \langle b \rangle\) is cyclic and normal in \(G\). Then \(V^B \neq 0\) is \(G\)-stable, a contradiction.

So \(|A|\) is a 3-power. As above, one observes that \(A\) has no element of order 9. So \(A\) is elementary abelian, and \(|A| = 27\) to be fixed point. Suppose that \(|A| = 27\). In addition, \(|G|, 7\) = 1. (Indeed, if \(g \in G\) is of order 7 then \([g, A] \neq 1\) (otherwise \(G\) has an element of order 21, contrary to Lemma 5.1), and \(g\) cannot normalize \(A\) as \(|GL_3(3)|, 7\) = 1. It follows that 2, 3 are the only prime divisors of \(|G|\), in particular, \(G\) is solvable. Note that \(V^A \neq 0\) is \(G\)-stable so \(V^A = 0\); in particular, \(A\) is not cyclic. If \(A \neq O_3(G)\) then \(O_3(G)\) is non-abelian, and hence there is an irreducible constituent of \(O_3(G)\) on \(V\) whose dimension is a multiple of 3. This is false by Clifford’s theorem.

Let \(Y\) be a minimal irreducible normal subgroup of \(G\) such that \(A \subseteq Y\). Then \(Y\) has a reducible subgroup \(Y_0\), say, of index 2. (Otherwise, \(G\) has a normal subgroup \(G_3\) of index 3 with \(A \subseteq G_3\). By Clifford’s theorem, \(G_3\) is irreducible, violating the minimality of \(Y\).) By Clifford’s theorem, either \(V\) is a homogeneous \(F_2Y\)-module or \(V = V_1 + V_2\), where \(V_1, V_2\) are \(Y_0\)-stable subspaces of \(V\). Then \(V_1, V_2\) are irreducible and non-isomorphic \(F_2Y_0\)-modules (this essentially follows from [12, Theorem 51.7]).

So \(V_1, V_2\) are permuted by \(y \neq Y \setminus Y_0\).

Observe first that \((\ast)\) if \(1 \neq a \in A\) then \(a\) is non-trivial on \(V_2\). (Indeed, suppose that \(V_2 \not\subseteq V^a\). If \(a\) fixes no non-zero vector on \(V_1\) then \(ahah^{-1}\) is fixed point free on \(V\), a contradiction. So \(V_2 \neq V^a\). Then \(\dim V^a = 6\). Note that \(A\) is generated by three conjugates of \(a\). (If not there is a proper normal subgroup \(A_0\) of \(Y\) containing \(a\). Then \(A_0\) is generated by two conjugates of \(a\). Then \(V^{A_0} \neq 0\) is \(Y\)-stably.) So let \(A = \langle a, b, c \rangle\), where \(a, b, c\) are conjugates of \(a\). Then \(V^A = V^a \cap V^b \cap V^c \neq 0\) is \(Y\)-stably, a contradiction.)

Let \(a, b, c\) be generators of \(A\), and \(a_2, b_2, c_2\) the restriction of them to \(V_2\). By the above, each of them is non-trivial on \(V_2\). Clearly, the restriction of \(A\) to \(V_2\) is of rank 2 (due to \((\ast)\) this is not of rank 1), so we can assume \(c_2 = a_i^j b_j^i\) for some \(i, j\). Then \(c^{-1}a^i b^j \neq 1\) is trivial on \(V_2\), contradicting to \((\ast)\).

Thus \(A = C_3 \times C_3\) and \(C_G(A) = A\). So \(G/A\) is isomorphic to a subgroup \(X\) of \(GL_2(3)\) such that \(O_3(X) = 1\). Then \(X\) is either a 2-group or SL2(3) or GL2(3). In addition, \(G = |A : D|\), where \(D \cong X\). This is obvious if \(X\) is a 2-group, otherwise let \(z\) be the central involution in \(X\). Then there is an involution \(t\), say, of \(G\) such that \(z = ta\). Then \([t, G] \subseteq A\) and \(tat^{-1} = a^{-1}\) for every \(a \in A\). Observe that the coset \(Ag (g \in G\setminus A)\) meets \(C_G(t)\). Indeed, if \(tgt = ag\) with \(1 \neq a \in A\) then \(ta^{-1}gt = a^2 g = a^{-1}g\). It follows that \(C_G(t) \cong G/A\), so we can take \(D = C_G(t)\). As \(AGL_2(3) \cong A : GL_2(3)\), the result follows. ☐

**Lemma 5.3** If \(|G| = 7\) then \(G \in \{ AGL_2(3), ASL_2(3), AL_1(9), AGL_1(9), PSU_3(2)\}\).

**Proof** Note that \(|H| = 2^{16} 3^5 5^2 7.17\). By Lemma 5.1, the primes 5 and 17 are not divisors of \(|G|\), and 7 is not by assumption. By Burnside’s theorem, \(G\) is solvable as \(|G|\) has at most two prime divisors. Then the result follows from Lemma 5.2. ☐

These observations allow us to present a streamlined proof of Theorem 1.5. For any computations performed in Magma, we refer the reader to Appendix for the commands.

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Proof of Theorem 1.5 As mentioned above $G \in \mathcal{F}$ lies in some maximal irreducible subgroup $M$ of Table 1. By Lemma 5.1, $G$ has no element of order 5 or 17. In addition, $Z(G) = 1 = O_2(G)$ by Lemma 3.11. For the remainder of this proof, $L$ denotes a maximal subgroup of $M$.

Case (1). Suppose $M \in \{L_2(17), \mathrm{Sp}_4(2) : S_2, \mathrm{Sp}_4(4).2\}$. Then $(7, |M|) = 1$ and the result follows from Lemma 5.3.

Case (2). If $M = \mathrm{SO}_8^-(2)$ then $L \in \{L_3(2) : 2, L_2(16) : 4, \Omega_8^-(2)\}$.

- The group $L_3(2) : 2$ is irreducible in $M$ (otherwise it is not maximal), and then it is a fixed-point group, which is listed in the statement of the theorem.
- The group $L_2(16) : 4$ has order divisible by 17, and all subgroups of order coprime to 17 are reducible.
- The group $\Omega_8^-(2)$ has index 2 in $\mathrm{SO}_8^-(2)$ and by [6, p. 89], every maximal subgroup of $\Omega_8^-(2)$ has index 2 in some other maximal subgroup of $\mathrm{SO}_8^-(2)$. Therefore, there are no novel subgroups left to analyze that have not been covered already.

Case (3). Let $M = S_10$ so that $L \in \{M_{10}, 2, S_5 : S_2, S_9, A_{10}\}$. By Lemma 5.1, $M, L \notin \mathcal{F}$.

- If $L = M_{10}, 2$ or $S_5 : S_2$, then $(7, |L|) = 1$ and the result follows as above by Lemma 5.3.
- If $L = S_9$, then $\mathrm{AGL}_2(3)$ and $A_9$ are the only maximal irreducible subgroups of $L$ [6]. The former is fixed-point. The latter is not fixed-point, so we descend to the maximal irreducible subgroups of it. By [6], these are $\mathrm{ASL}_2(3)$, and two conjugacy classes of subgroups isomorphic to $L_2(8).3$ (these are conjugate in $S_9$). The former is not maximal fixed-point subgroup as $\mathrm{ASL}_2(3) \subset \mathrm{AGL}_2(3)$.

Each of the two groups isomorphic to $L_2(8).3$ has a fixed point free element of order 9, so these groups are not fixed-point, and a search in Magma for irreducible fixed-point subgroups reveals that there are none.

- The maximal irreducible subgroups of $A_{10}$ are

$$A_9, \quad M_{10} \quad \text{and} \quad (A_5 \times A_5) : 4.$$  

The group $A_9$ has been covered by the analysis of maximal subgroups of $S_9$, and the other two groups have no element of order 7 and so are ruled out as above.

Case (4). Let $M = \mathrm{SO}_8^+(2)$ so that $L \in \{S_5 : S_2, S_3 : S_4, S_9, \Omega_8^+(2)\}$. As above, we only have to examine $L$ with $(7, |L| = 7)$, which are $S_9$ and $\Omega_8^+(2)$.

The group $H$ has a unique subgroup (up to conjugacy) isomorphic to $S_9$ [22]. As this group has already appeared above in our analysis of the case of $S_{10}$, we can safely ignore it.

To finish the proof, we turn to maximal subgroups $L_1$ of $L = \Omega_8^+(2)$. By Lemma 5.3, we can ignore the subgroups having a non-trivial normal subgroup. This leaves us with the following groups (up to isomorphism):

$$L_1 \in \{S_6(2), A_9, (A_5 \times A_5) : 2^2, (3 \times U_4(2)) : 2\}.$$  

We may ignore the group $(A_5 \times A_5) : 2^2$ since it has no element of order 7 (see Lemma 5.3). One observes from [6, p. 85] that there are three conjugacy classes of
groups isomorphic to each of the remaining groups. Moreover, exactly one of them has index 2 in a maximal subgroup of $SO_8^+(2)$, so that one is covered by the above analysis. Furthermore, in each case two remaining classes fuse in $SO_8^+(2)$, so it suffices to examine any one of these classes. We now again proceed case-by-case.

- Let $L_1 \cong Sp_6(2)$. If $g \in L_1$ is of order 9 and $\phi$ is a 2-modular irreducible representation of $L_1$ of degree 8 then $\phi(g)$ does not have eigenvalue 1 [22, p. 110]. So $G$ has no element of order 9. All maximal subgroups of $L_1$ with this property are reducible in $H$.
- Let $L_1 \cong A_9$. By the above, we only have to consider the 2-modular irreducible representation of $A_9$ denoted by $\phi_3$ in [22, p. 85]. We use Magma to determine the maximal subgroups of $K$ of $A_9$ subject to condition $|K| = 1$. This shows that $K$ is conjugate either to $PSU_3(2)$ or to $ASL_3(2)$. These are not maximal fixed point subgroups of $Sp_3(2)$.

This completes our analysis and the proof of Theorem 1.5. □

6 Sufficient conditions of unisingularity

In this section we denote by $W$ the Weyl group of a simple algebraic group $G$ and define weights $\epsilon_1, \ldots, \epsilon_n$ of $G$ as in [3, Planchees I–IV]. The weights $\omega \in \mathcal{R}$ are called radical [3].

6.1 Symplectic and orthogonal groups

In Lemma 6.1 and Proposition 6.2 below, $q$ can be any prime power, in Lemmas 6.3, 6.4, $q$ is even.

Lemma 6.1 Let $G$ be the algebraic group of type $C_n$, $n > 1$, or $D_n$, $n > 3$. Let $G = C_n(q) or D_n^\pm(q)$ and let $g \in G$ be a semisimple element. Suppose that $|g|$ does not divide $q - 1$ or $q + 1$. Then $g$ is conjugate in $G$ to an element $t \in G$ such that $\epsilon_1(t)^q \cdot \epsilon_2(t) = 1$.

Proof This is well known, and explained in detail in many sources, in particular see [34, Section 3.2] or [19, Section 2.4]. □

Proposition 6.2 Let $G$ be a simple algebraic group of type $C_n$, $n > 1$, $(n, q) \neq (2, 2)$ or $D_n$, $n > 3$. Let $G = Sp_{2n}(q) \subset Sp_{2n}(F) = G or D_{2n}^+(q), D_{2n}^-(q) \subset D_{2n}(F) = G$. Let $V$ be an irreducible $G$-module with highest weight $\omega$. Suppose that $\omega_1m_1(q + 1)\omega_1, m_2(q - 1)\omega_1 and m_3((q - 1)\omega_1 + \omega_2)$ are weights of $V$ for some natural numbers $m_1, m_2, m_3$. Then every semisimple element $g \in G$ has eigenvalue 1 on $V$.

Proof Let $t$ be as in Lemma 6.1. Suppose that $|g|$ divides $q - 1$ or $q + 1$. In the latter case we have $(m_1(q + 1)\omega_1)(t) = (m_1(q + 1)\epsilon_1)(t) = (\epsilon_1(t))^{m_1(q + 1)} = 1$. So the result follows. The case of $q - 1$ a multiple of $|g|$ is similar.

Suppose that $|g|$ divides neither $q - 1$ nor $q + 1$. By Lemma 6.1, we can assume that $\epsilon_1(g)^q \cdot \epsilon_2(g) = 1$. As $m_3((q - 1)\omega_1 + \omega_2) = m_3(q\epsilon_1 + \epsilon_2)$, we have $(m_3(q\epsilon_1 + \epsilon_2))(g) = (\epsilon_1(g))^q \cdot \epsilon_2(g))^{m_3} = 1$. □
Lemma 6.3 Let $\Lambda$ be the weight system of type $C_n$, $n > 1$, or $D_n$, $n > 3$, and let $\sigma_q = (q - 1)(\alpha_1 + \cdots + \alpha_n)$, $q$ even, $(n, q) \neq (2, 2)$. Suppose that $\sigma_q$ is not radical. Then $\sigma_q > (q + 1)\omega_1 > (q - 1)\omega_1 + \omega_2 > (q - 1)\omega_1$.

**Proof** As $\sigma_q$ is not radical, we have $\sigma_q > \omega_1$ for both the groups. (For $C_n$ see [4, Chapter VIII, Section 7.3].) Let $G$ be of type $D_n$. One observes that either weight 0 or $\omega_1$ is a subdominant weight of each of the weights $\omega_1, \ldots, \omega_{n-2}, \omega_{n-1} + \omega_n$. So we conclude similarly.) In addition, if $\Lambda$ is of type $D_n$ then $n > 4$ as $\sigma_q$ is not radical. So $\omega_1 < \omega_3$ [3]. If $\Lambda$ is of type $C_3$ or $C_4$ then $\sigma_q$ is radical, so we have $n > 4$ again.

As $q$ is even, $\sigma_q - (q + 1)\omega_1 \in \mathcal{R}$. For $q > 2$ we have $\sigma > \sigma_1 := (q + 1)\omega_1 + (q - 1)\omega_2 + (q - 3)\omega_3 + (q - 1)(\omega_4 + \cdots + \omega_n)$ as $\omega_3 = \omega_1$. In addition, $\sigma_1 - (q + 1)\omega_1 \in \mathcal{R}$ is dominant. By Lemma 3.3, we have $\sigma_1 > (q + 1)\omega_1$, giving us the first inequality that we seek. For the second, we have $(q + 1)\omega_1 - \alpha_1 = (q - 1)\omega_1 + \omega_2 > (q - 1)\omega_1$, since $\omega_2$ is radical.

Let $q = 2$, $n > 2$. Then $n \geq 5$. If $n = 5$ and 6 then $\sigma_2 > 3\omega_1 > \omega_1 + \omega_2$ for both the groups (in the $D_n$-case use $\omega_{n-1} + \omega_n > \omega_1$). If $n \geq 7$ then $\sigma_2 = \omega_1 - \omega_3 - \omega_5$ is a radical dominant weight, and $\omega_1 - \omega_3 - \omega_5 > 3\omega_1$. This implies $\sigma_2 > 3\omega_1$ by Lemma 3.3.

Let $n = 2, q > 2$. Then $\sigma_2 - \alpha_2 = (q + 1)\omega_1 + (q - 3)\omega_2 > (q + 1)\omega_1$ as $\omega_2$ is radical, and $(q + 1)\omega_1 > (q - 1)\omega_1 + \omega_2$. \(\square\)

Lemma 6.4 *Theorem 1.8 is true for $G = C_n(q)$, $n > 1$, and $D_n^\pm(q)$, $n > 3$.*

**Proof** The case with $G = C_2(2)$ follows by inspection of the Brauer character table of $G$ [22]. Assume that $(n, q) \neq (2, 2)$. Then $S_2$ extends to an irreducible representation $\Phi_q$ of the simple algebraic group $G$ of type $C_n$ or $D_n$ [28, Theorem 43]. The highest weight of $\Phi_q$ is $\sigma_q$. By Lemma 3.2, the weights of $\Phi_q$ are the same as in characteristic 0. By [3, Chapter VIII, Section 7, Proposition 5 (iv)], if $\rho, \mu$ are dominant weights, then $\rho > \mu$ implies $\mu$ to be a weight of $\Phi_q$. In particular, $(q + 1)\omega_1, (q - 1)\omega_1$ and $(q - 1)\omega_1 + \omega_2$ are weights of $\Phi_q$ by Lemma 6.3. So the result follows from Proposition 6.2. \(\square\)

### 6.2 The groups $\text{SL}_n(q)$

In this section $G = \text{SL}_{n+1}(q) \subset G = \text{SL}_{n+1}(F), n > 1$, where $F$ is an algebraically closed field of characteristic $p > 0$ and $q$ is a $p$-power, and $p$ is an arbitrary prime (except for the proofs of Theorem 1.8 and Corollary 1.9).

Let $T$ be the maximal torus of $G$ whose rational irreducible representations are weights of $G$. Note that $W \cong S_{n+1}$. The conjugacy classes of maximal tori of $G$ are labeled by a partition $\pi = [n_1 \geq \cdots \geq n_k]$ of $\{1, \ldots, n + 1\}$ [13, 3.23], and we can write $T = T_\pi$ to denote a representative of the conjugacy class labeled by $\pi$.

It is well known, and explained in detail [34, Section 3], that $T$ is conjugate in $G$ to a subgroup of the diagonal matrix group $D_\pi \cap \text{SL}_{n+1}$, where $D_\pi = \text{diag}(D_1, \ldots, D_k)$ and $D_j$ is a cyclic group of order $q^{n_j} - 1$ for $j = 1, \ldots, k$. More precisely, $D_j = \langle d_j \rangle$, where $d_j = \text{diag}(\xi_j, \xi_j^q, \ldots, \xi_j^{q^{n_j} - 1})$ and $\xi_j \in \overline{F}_q$ is a primitive $(q^{n_j} - 1)$-root of unity. If $\phi$ is a rational representation of $G = \text{SL}_{n+1}(F)$ then $\phi|_{T_\pi}$ is equivalent to...
Lemma 6.7 Let $\phi|D_\pi \cap \text{SL}_{n+1}$ when we identify $D_\pi \cap \text{SL}_{n+1}$ with $T_\pi$. As the weights $\omega$ of $G$ are the one-dimensional irreducible rational representations of $T$, it is meaningful to write $\omega|D_\pi$, even if $\omega$ is not a weight of $\phi$.

Proposition 6.5 ([35, Proposition 4.1]) Let $m > 1$ be an integer such that $p$ is coprime to $(m^{n+1} - 1)/(m - 1)$. Let $\zeta \in F$ be a primitive $((m^{n+1} - 1)/(m - 1))$-root of unity and set $t_m = \text{diag}(\zeta, \zeta^m, \zeta^{m^2}, \ldots, \zeta^{m^n}) \in G$. Let $\mu = \omega_1 + (m - 1)\omega_i + \omega_n$, $1 \leq i \leq n$, be the weight of $G$. Then $g(\mu)(t_m) = 1$ for some $g \in W$.

Next we mimic the reasoning for $q = 2$ in [35, Section 7].

Proposition 6.6 Let $T = T_\pi$ be a maximal torus of $G$ corresponding to a partition $\pi = [n_1, \ldots, n_k]$. Let $\lambda_i = (q - 1)\omega_i$ for $1 \leq i \leq n$. Suppose that $i = n_{j_1} + \cdots + n_{j_r}$ for some subset $\{j_1, \ldots, j_r\}$ of $\{1, \ldots, k\}$. Then the $W$-orbit of $\lambda_i$ has a weight $\mu$ such that $\mu|T = 1_T$.

Proof Let $W\lambda_i$ be the $W$-orbit of $\lambda_i$. By [34, Corollary 3.7], the number of weights $\mu \in W\lambda_i$ such that $\mu|T = 1_T$ equals $\lambda_{n+1}^{\gamma^q - 1}(\pi)$, where $\pi$ is viewed as an element of $S_n$ and $Y \cong S_1 \times S_{n+1-i}$ is the Young subgroup of $S_{n+1}$ labeled by $[i, n + 1 - i]$. Then $\lambda_{n+1}^{\gamma^q - 1}(\pi) > 0$ whenever $\pi$ is conjugate to an element of $Y$. This happens if and only if $i$ is the sum of some parts of $\pi$, as stated.

Lemma 6.7 Let $\phi$ be an irreducible representation of $G$ with highest weight $\omega$. Let $T_\pi$ be a maximal torus of $G$. For a fixed $i = 1, \ldots, n$ set $\kappa_i = (q - 1)\omega_i + \omega_1 + \omega_n$ and $\lambda_i = (q - 1)\omega_i$, and let $W\kappa_i, W\lambda_i$ be the $W$-orbits of these weights. Then $\mu|D_\pi = 1_{D_\pi}$ for some weight $\mu \in (W\kappa_i \cup W\lambda_i)$.

Proof As explained above, $D_\pi = \text{diag}(D_1, \ldots, D_k)$, where $D_j = (d_j)$. One observes that $(q\varepsilon_u)(x) = \varepsilon_{u+1}(x)$ for every $x \in D$ and $u \in \{1, \ldots, n + 1\}$, unless $u = n_1 + \cdots + n_j$ for $j \in \{1, \ldots, k\}$. In addition,

$$\det d_1 = (\varepsilon_1 + \cdots + \varepsilon_{n_1})(d_1) = \xi_1^{1+q+\cdots+q^{n_1-1}} = \xi_1^{(q^{n_1}-1)/(q-1)} \in \mathbb{F}_q$$

as this is an element of order $q - 1$ in $F$. Similarly, for other matrices $d_2, \ldots, d_k$.

First observe that, for every $r \in \{n_1, \ldots, n_k\}$, there exists a subset $\{n_{j_1}, \ldots, n_{j_r}\}$ of $\{1, \ldots, n\}$ such that

$$m := i - 1 - n_{j_1} - \cdots - n_{j_r} < n_r$$

and $r \neq j_1, \ldots, j_r$. (This is trivial as $n_1 + \cdots + n_k = n + 1 > i$. Note that this subset may be empty, and then $m = i - 1$.)

Suppose first that $m + 1 = n_r$ for some $r$. Then $i = n_{j_1} + \cdots + n_{j_r} + n_r$. So in this case the result follows from Lemma 6.6. Therefore, we now assume that $m + 1 < n_r$ for every $r$.

Let we fix these subsets $n_{j_1}, \ldots, n_{j_r}$, and define a subgroup $D' \subseteq G$ by moving submatrices $D_{j_1}, \ldots, D_{j_r}$ to the positions $k + 1 - l, \ldots, k$ and the submatrices $D_u$ for $u \in \{k - l + 1, \ldots, k\}$, $u \neq j_1, \ldots, j_r$, to the positions prior to $k + 1 - l$. Set
\(n' = n_{j_1} + \cdots + n_{j_i}\) so \(i = m + 1 + n'\). Clearly, \(D\) and \(D'\) are conjugate in \(G\) so it suffices to prove that there is a weight \(\mu\) of \(\phi\) such that \(\mu(D') = 1\).

Note that \(\kappa_i = (q - 1)\omega_i + \omega_1 + \omega_n = (q + 1)\varepsilon_1 + q\varepsilon_2 + \cdots + q\varepsilon_{i+1} + \cdots + \varepsilon_n\). The \(W\)-orbit of \(\kappa_i\) contains the weight

\[
\mu = (q + 1)\varepsilon_1 + q\varepsilon_2 + \cdots + q\varepsilon_{m+1} + \varepsilon_{m+3} + \cdots + \varepsilon_{n+1-n'} + q\varepsilon_{n+2-n'} + \cdots + q\varepsilon_{n+1}.
\]

Then

\[
\mu(D') = ((q + 1)\varepsilon_1 + q\varepsilon_2 + \cdots + q\varepsilon_{m+1} + \varepsilon_{m+3} + \cdots + \varepsilon_{n+1-n'})(x)
\]

\[
\quad \cdot (q\varepsilon_{n+2-n'} + \cdots + q\varepsilon_{n+1})(x)
\]

for \(x \in D'\).

Consider the second multiple. Here \((\varepsilon_{n+2-n'} + \cdots + \varepsilon_{n+1})(x)\) is the product of the diagonal entries of certain matrices \(d_j\) above, and hence this lies in \(\mathbb{F}_q\). Therefore,

\[
(q\varepsilon_{n+2-n'} + \cdots + q\varepsilon_{n+1})(x) = (\varepsilon_{n+2-n'} + \cdots + \varepsilon_{n+1})(x).
\]

Consider the first multiple. As \(n_r > m + 1\) for every \(r \neq j_1, \ldots, j_i, r \in \{1, \ldots, k\}\), we have

\[
((q + 1)\varepsilon_1 + q\varepsilon_2 + \cdots + q\varepsilon_{m+1})(D')
\]

\[
= (\varepsilon_1 + (q + 1)\varepsilon_2 + q\varepsilon_3 + \cdots + q\varepsilon_{m+1})(D')
\]

\[
= (\varepsilon_1 + \cdots + \varepsilon_{m+1} + \varepsilon_{m+2})(D').
\]

Therefore,

\[
((q + 1)\varepsilon_1 + q\varepsilon_2 + \cdots + q\varepsilon_{m+1} + \varepsilon_{m+3} + \cdots + \varepsilon_{n+1-n'})(x)
\]

\[
= (\varepsilon_1 + \cdots + \varepsilon_{n+1-n'})(x).
\]

(To be explicit, write \(D' = \text{diag}(D'_1, \ldots, D'_k)\). Let \(n'_j\) be the size of the matrix \(D'_j\), \(j = 1, \ldots, k\). Then \(n'_j > m + 1\) by the above, and \(q\varepsilon_u(x) = \varepsilon_{u+1}(x)\) for \(u = 1, \ldots, n'_j - 1\) and any \(x \in D'_j\). Finally, \((\varepsilon_1 + \cdots + \varepsilon_{n+1-n'})(x)\) - \((\varepsilon_{n+2-n'} + \cdots + \varepsilon_{n+1})(x)\) = \(\text{det} x = 1\).

\[\Box\]

**Remark 6.8** If \(\phi\) is \(p\)-restricted then \((q - 1)\omega_i\) is a weight of \(\phi\) whenever so is \((q - 1)\omega_i + \omega_1 + \omega_n\). In general, this is not probably true. The following special case illustrates the situation.

**Lemma 6.9** Let \(\phi\) be an irreducible representation of \(G\) with highest weight \((q - 1)\omega_i + \omega_1 + \omega_n\), \(1 \leq i \leq n\). Then \((q - 1)\omega_i\) is a weight of \(\phi\).

**Proof** If \(q = p\) is a prime then the result follows from Premet’s theorem [22, p. 23]. Let \(p \mid q\) and \(q > p\) be a \(p\)-power. Then \(\phi = \phi_1 \otimes \phi_2\), where \((p-1)\omega_i + \omega_1 + \omega_n, \ (p-q)\omega_i\)
are the highest weights of $\phi_1, \phi_2$, respectively. The weights of a tensor product are $\lambda + \mu$ with $\lambda, \mu$ to be weights of $\phi_1, \phi_2$, respectively. By Premet’s theorem, $(p - 1)\omega_i$ is a weight of $\phi_1$ so $(q - 1)\omega_i = (p - 1)\omega_i + (q - p)\omega_i$ is a weight of $\phi$.

**Proof of Theorem 1.10** Let $G$ be of type $A_n$. Then $W(m_1\kappa_i) = m_1(W\kappa_i)$ and $W(m_2\lambda_i) = m_2(W\lambda_i)$. Therefore, it follows from Lemma 6.7 that $v|_{D_\pi} = 1_{D_\pi}$ for a suitable weight $v \in W(m_1\kappa_i) \cup W(m_2\lambda_i)$. As $v$ is a weight of $V$ and $D_\pi$ is $G$-conjugate to $T_\pi$, the result follows. If $G$ is of type $C_n$ or $D_n$ then the result follows from Proposition 6.2.

**Lemma 6.10** The weight $\omega = \sigma_q$ satisfies the assumption of Lemma 6.7. In particular, Theorem 1.8 is true for $G = \text{SL}_{n+1}(q)$.

**Proof** By Lemma 3.3, we have $\sigma_q > (q - 1)\omega_{(n+1)/2} + \omega_1 + \omega_n > (q - 1)\omega_{(n+1)/2}$, so the weights $(q - 1)\omega_{(n+1)/2} + \omega_1 + \omega_n$ and $(q - 1)\omega_{(n+1)/2}$ are weights of $\Phi_q$. Then the result follows from Lemma 6.7.

**Proof of Theorem 1.8** The result is contained in Lemmas 6.4 and 6.10 for the groups $C_n(q), D_n^\pm(q)$ and $A_n(q)$, in Lemma 3.6 for $G = 2B_2(q)$ and in Theorem 3.5 for the groups that are not excluded in Theorem 1.8.

**Proof of Corollary 1.9** Let $G$ be as in the statement of the corollary and let $A$ be an abelian variety defined over $\mathbb{Q}$ of dimension $m/2$ such that $\text{im} \rho_2 = \text{Sp}_m(2)$. Extend scalars to $K = \mathbb{Q}(A[2])^G$, the fixed field of $G$ in the 2-division field of $A$. Then, over $K$, $\text{im} \rho_2 = G$; the corollary follows because $G$ is fixed-point and absolutely irreducible.

**7 Appendix: Magma code**

Many of the computations in this paper were performed using the computer algebra Magma at the online calculator http://magma.maths.usyd.edu.au/calc/. In this appendix we share the code we used, together with some sample calculations to show the reader how we arrived at some of our conclusions.

To initiate the group $\text{Sp}_8(2)$ and store its maximal subgroups, we use:

```
G:=Sp(8,2);
M:=MaximalSubgroups(G);
```

To run through the maximal subgroups of $G$ and check whether any are fixed-point groups and what the dimensions of the simple factors are, we use the following double loop:
for i:=1 to #M do;
#M[i]\_subgroup;
C:=ConjugacyClasses(M[i]\_subgroup);
for j:=1 to #C do;
CC:=Coefficients (CharacteristicPolynomial(C[j][3]));
print C[j][1]," ",&+ [CC[j] : j in [1..#CC]];
end for;
A:=MatrixAlgebra<GF(2), 8 | Generators(M[i]\_subgroup)>;
MM:=RModule(A);
B:=CompositionFactors(MM);
B;
end for;

We remark that we are checking the fixed-point condition by simply summing
the coefficients of the characteristic polynomial and checking if it is 0. This procedure
can be iterated to check the maximal subgroups of the maximal subgroups, etc. In
addition, we make use of the commands
L:=LowIndexSubgroups(G,<n,m>);
S:=Subgroups(G:OrderEqual:=N);
to create lists of subgroups whose index lies in the range \([n, m]\), or to enumerate all
subgroups of order \(N\), respectively.

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