Scalar Fields in BTZ Black Hole Spacetime and Entanglement Entropy

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ABSTRACT

We study the quantum scalar fields in background of BTZ black hole spacetime. We calculate the entanglement entropy using the discretized model which resembles a system of coupled harmonic oscillators. We also study the sub-leading logarithmic corrections to entropy numerically. The results are compared with those from the holographic entanglement entropy approach. The coefficients of leading and sub-leading terms obey certain relation, which has been verified numerically.

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I. INTRODUCTION

Black holes are classical solutions of Einstein’s equations in general theory of relativity. They are characterized by few parameters viz. mass, charge and angular momentum. The gravitational pull of black hole is so strong that even light can not escape from it. This creates a problem as everything goes inside and nothing escapes, so no information can be retained about the objects going inside. A quantum mechanical treatment of scalar fields in the black hole background gives rise to Hawking radiation, which is thermal in nature and carries no information. A full quantum theory of gravity for example string theory is needed to resolve this information loss paradox.

The horizon of black hole is a null surface and it is known that area of black hole horizon never decreases in a physical process. This has an analogy with entropy of a thermodynamic system and it was suggested by Bekenstein that black holes obey the laws similar to thermodynamics and their entropy being proportional to the area of black hole horizon. This has been generalized to include the entropy of matter fields in black hole vicinity and the sum of black hole entropy and matter entropy never decreases in any physical process.

There are several attempts to understand the microscopic origin of black hole entropy and its relation with horizon area. A first principle calculation of entanglement entropy may be the key to understand this puzzle. We carry this investigation for BTZ black hole \[2, 4\]. BTZ black hole is a solution of (2+1) dimensional gravity with negative cosmological constant. The Penrose diagram of the BTZ black hole is similar to the Kerr-Newmann AdS black hole in (3+1)-dimension. The entropy is also proportional to the area as the case of Schwarzschild and Kerr black holes. This solution can be embedded in string theory and a microscopic counting of degrees of freedom responsible for black hole entropy is also known \[5, 6\]. The near horizon geometry of BTZ black hole is anti-de-Sitter space time and machinery of AdS/CFT correspondence can also be applied \[7\]. Thus BTZ black hole can be used as a useful laboratory to study the quantum aspects of black holes.

In this paper we shall investigate the entropy of BTZ black hole using ab initio calculations of entanglement entropy of BTZ black hole. We investigate the entanglement entropy of quantum fields in BTZ black hole spacetime \[1, 4, 8, 11, 13, 18, 20\]. The motivation for this work is the recent investigation of holographic entanglement entropy of the BTZ black hole using AdS/CFT \[14\]. We review a simple quantum mechanical model of a system of
simple harmonic oscillators to illustrate the procedure. This is applied to scalar fields in BTZ black hole spacetime and corresponding entanglement entropy is evaluated. We extend our results to estimate the logarithmic corrections to the black hole entropy. These terms for entropy are sub-leading in $r+/a$ and are of the form; $c_2 \ln (r+/a) + c_3$. We have calculated the numerical values of these coefficients and compare with the entropy formula obtained using holographic entanglement entropy approach.

II. MODEL OF ENTANGLEMENT ENTROPY

Let us consider a system of coupled harmonic oscillators $q^A (A = 1, \ldots, N)$ described by the Lagrangian

$$\mathcal{L} = \frac{a}{2} \delta_{AB} \ddot{q}^A \ddot{q}^B - \frac{1}{2} V_{AB} q^A q^B.$$  \hspace{1cm} (1)

Here $\delta_{AB}$ is Kronecker delta symbol and $V$ is real symmetric, positive definite matrix which does not depend upon $q^A$ and “$a$” is fundamental length characterizing the system.

The corresponding Hamiltonian of the system becomes,

$$H_{\text{tot}} = \frac{1}{2a} \delta_{AB} p_A p_B + \frac{1}{2} V_{AB} q^A q^B,$$ \hspace{1cm} (2)

where $p_A = a \delta_{AB} \dot{q}^B$ is the canonical momentum conjugate to $q^A$.

The total Hamiltonian can be rewritten as,

$$H_{\text{tot}} = \frac{1}{2a} \delta_{AB} (p_A + iW_{AC} q^C)(p_B - iW_{BD} q^D) + \frac{1}{2a} Tr W$$ \hspace{1cm} (3)

where $W$ is symmetric, positive definite matrix satisfying the condition $W = \sqrt{aV}$.

We split $q^A$ into two subsystems, $q^a (a = 1, 2, \ldots, n)$ and $q^\alpha (\alpha = n + 1, n + 2, \ldots, N)$. The matrix $W$ naturally split into four blocks as,

$$(W)_{AB} = \begin{pmatrix} A_{ab} & B_{a\beta} \\ (B^T)_{\alpha b} & D^{(2)}_{\alpha \beta} \end{pmatrix}.$$ \hspace{1cm}

and $W^{-1}$ is written as

$$(W)^{-1}_{AB} = \begin{pmatrix} \tilde{A}_{ab} & \tilde{B}_{a\beta} \\ (\tilde{B}^T)_{\alpha b} & \tilde{D}^{(2)}_{\alpha \beta} \end{pmatrix}.$$ \hspace{1cm}

Then the reduced density matrix associated with the subsystem 1 (subsystem 2) is obtained by taking the partial trace with respect to the subsystem 2 (subsystem 1).

$$\langle q^a | \rho_1 | q^b \rangle = (\det \frac{A'}{\pi})^{1/2} \exp\left[-\frac{1}{2} A'_{ab}(q^a q^b + q^a q^b)]\right] \left[ -\frac{1}{4} (BD^{-1}B^T)_{ab}(q - q')^a (q - q')^b \right]$$ \hspace{1cm} (4)
and similarly for the subsystem 2
\[
\langle q^\alpha | \rho_2 | q'^\beta \rangle = (\det \frac{D'}{\pi})^{1/2} \exp \left[ -\frac{1}{2} D'_{\alpha\beta} (q^\alpha q^\beta + q'^\alpha q'^\beta) \right] \left[ -\frac{1}{4} (B^T A^{-1} B)_{\alpha\beta} (q - q')^\alpha (q - q')^\beta \right]
\] (5)

where
\[
A' = A - BD^{-1}B^T,
D' = D - B^T A^{-1} B.
\]

The entanglement entropy of the system is given by,
\[
S_{\text{ent}} = -\text{Tr} (\rho_{\text{red}} \ln \rho_{\text{red}}).
\] (6)

where \( \rho_{\text{red}} \) is the reduced density matrix of the system. It is defined as taking the partial trace of \( \rho(q^A, q^B) \) for the subsystem \( \{q^a\} (a = 1, 2, \ldots, n) \), we get the reduced density matrix
\[
\rho_{\text{red}}(\{q^a\}, \{q^b\}) = (\det \pi \tilde{D})^{-1/2} \exp \left[ -\frac{1}{4} (\tilde{D}^{-1})_{\alpha\beta} (q^\alpha q^\beta + q'^\alpha q'^\beta) \right]
- \frac{1}{4} (B^T A^{-1} B)_{\alpha\beta} (q - q')^\alpha (q - q')^\beta\right].
\]

Let \( \{\lambda_i\} (i = 1, 2, \ldots, N - n) \) be the eigenvalues of a positive definite symmetry matrix \( \Lambda \),
\[
\Lambda := D'^{-1/2} B^T A^{-1} B \Lambda^{-1/2} D'^{-1/2}
\]

Then the entanglement entropy can be written as
\[
S = \sum_{i=1}^{N-n_B} S_i
\] (7)
\[
S_i = -\frac{\mu_i}{1 - \mu_i} \ln \mu_i - \ln (1 - \mu_i)
\] (8)

where \( \mu_i := \lambda_i^{-1}(\sqrt{1 + \lambda_i} - 1)^2 \).

III. ENTANGLEMENT ENTROPY OF BTZ BLACK HOLE

BTZ black hole is solution of (2+1) dimensional gravity with negative cosmological constant \( \Box \). The action of the (2+1) dimensional gravity with cosmological constant can be written as,
\[
S = \frac{1}{2\pi} \int d^2 x \sqrt{-g} [R + 2\Lambda].
\] (9)
where $\Lambda = - \frac{1}{l^2}$ is the cosmological constant.

The metric of BTZ black hole can be written as:

$$ds^2 = -\left(-M + \frac{r^2}{l^2}\right)dt^2 + \frac{1}{\left(-M + \frac{r^2}{l^2} + \frac{J^2}{4r^2}\right)}dr^2 + r^2 d\phi^2 - J dt d\phi,$$

with $-\infty < t < \infty$ and $0 \leq \phi \leq 2\pi$.

The inner and outer horizon of the black hole are located at

$$r_{\pm} = l \left[\frac{M}{2} \left(1 \pm \sqrt{1 - \frac{(J/ML)^2}{}}\right)\right]^{1/2}.$$ 

where the parameters $M$ and $J$ denote the mass and angular momentum of the black hole and they obey, $M > 0$ and $|J| < ML$. In the extremal case $J = ML$, and both roots coincide.

The proper length from the horizon, $\rho$, is given by

$$r^2 = r_{\pm}^2 \cosh^2 \rho + r_{\pm}^2 \sinh^2 \rho. \quad (11)$$

The metric of the black hole can be written in terms of proper length,

$$ds^2 = -u^2 dt^2 + d\rho^2 + l^2 (u^2 + M) d\phi^2 - J dt d\phi,$$

where we are using, $r^2(\rho) = l^2 (u^2 + M)$.

The action of massless scalar field in the background of BTZ black hole space time is given by;

$$S = -\frac{1}{2} \int dt \sqrt{-g} \left(g^{\mu\nu} (\partial_{\mu} \Phi \partial_{\nu} \Phi)\right). \quad (13)$$

Using the separation of variable as appropriate for the cylindrical symmetry of the system;

$$\Phi(t, \rho, \phi) = \sum_m \Phi_m(t, \rho) e^{im\phi}, \quad (14)$$

the action of the scalar field in the background of BTZ black hole can be written as

$$S = -\frac{1}{2} \int dt \left[ -\frac{\sqrt{(u^2 + M)}}{\sqrt{[u^2 + \frac{J^2}{4(u^2 + M)}]}} \Phi_m^2 + u \sqrt{[(u^2 + M) + \frac{J^2}{4u^2}]} (\partial_{\rho} \Phi_m)^2 + \frac{u^2 m^2}{u \sqrt{[(u^2 + M) + \frac{J^2}{4u^2}]} \Phi_m^2} - \frac{(iJm)}{u \sqrt{[(u^2 + M) + \frac{J^2}{4u^2}]} \Phi_m} \right], \quad (15)$$

The conjugate momentum corresponding to $\Phi_m$, is given by

$$\pi_m = \frac{\sqrt{(M + u^2)}}{\sqrt{[u^2 + \frac{J^2}{4(u^2 + M)}]}} \Phi_m + \frac{iJm}{2u \sqrt{[(u^2 + M) + \frac{J^2}{4u^2}]} \Phi_m}.$$
To diagonalise the Hamiltonian, we define the new momentum

$$\tilde{\pi}_m = \pi_m - \frac{iJ_m}{u\sqrt{[(u^2 + M) + \frac{J^2}{4u^2}]}}\Phi_m.$$  
(16)

The Hamiltonian of the system can be written as,

$$H = \frac{1}{2} \int d\rho \tilde{\pi}_m^2(\rho) + \frac{1}{2} \int d\rho d\rho' \psi_m(\rho)V_m(\rho, \rho')\psi_m(\rho'),$$  
(17)

where,

$$\psi_m(\rho)V_m(\rho, \rho')\psi_m(\rho') = u\sqrt{[(u^2 + M) + \frac{J^2}{4u^2}]\left(\partial_\rho(\sqrt{\frac{\sqrt{[u^2 + \frac{J^2}{4(u^2+M)]}}}{(u^2 + M)}})\psi_m\right)^2} + m^2\frac{u^2 + \frac{J^2}{4(u^2+M)}}{(M + u^2)}\psi_m^2,$$  
(18)

and we have used the redefined variable $\psi$

$$\Psi_m(t, \rho) = \sqrt{\frac{\sqrt{[(u^2 + \frac{J^2}{4(u^2+M)]}}}{(M + u^2)}}\Phi_m(t, \rho).$$  
(19)

The system can be discretized as,

$$\rho \rightarrow (A - 1/2)a, \quad \delta(\rho - \rho') \rightarrow \delta_{AB}/a.$$  
(20)

where $A, B = 1, 2, ..., N$ and “a” is UV cut-off length.

The corresponding Hamiltonian of the discretized system can be obtained by the replacement:

$$\psi_m(\rho) \rightarrow q^A, \quad \tilde{\pi}_m(\rho) \rightarrow p_A/a, \quad V(\rho, \rho') \rightarrow V_{AB}/a^2.$$  
(21)

and is given by the expression,

$$H_0 = \sum_m \tilde{H}_0^m = \sum_{A,B=1}^N \left[ \frac{1}{2a}\delta_{AB}p_{mA}p_{mB} + \frac{1}{2}V_{AB}^m q_A^m q_B^m \right].$$  
(22)

The $N \times N$ matrix representation of the $V_{AB}^m$ is given by

$$
\begin{pmatrix}
\Delta_1 & \Sigma_1 & \Delta_1 & \Delta_1 & \cdots & \Delta_1 \\
\Sigma_1 & \Delta_2 & \Sigma_2 & \Delta_2 & \cdots & \Delta_2 \\
\Delta_1 & \Sigma_2 & \Delta_2 & \Sigma_2 & \cdots & \Delta_2 \\
\Delta_1 & \Sigma_1 & \Delta_1 & \Sigma_1 & \cdots & \Delta_1 \\
\ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\Delta_{A-1} & \Sigma_A & \Delta_A & \Sigma_A & \cdots & \Delta_A \\
\end{pmatrix}
$$
where
\[ \Sigma_A^{(m)} = \sqrt{\left[ u_A^2 + \frac{J^2}{4(u_A^2 + M)} \right]} \left( u_{A+1/2} \sqrt{\left[ (u_{A+1/2}^2 + M) + \frac{J^2}{4u_{A+1/2}^2} \right]} \right) - u_{A-1/2} \sqrt{\left[ (u_{A-1/2}^2 + M) + \frac{J^2}{4u_{A-1/2}^2} \right]} + m^2 \frac{u_A^2 + \frac{J^2}{4(u_A^2 + M)}}{(M + u_A^2)}. \] (23)

\[ \Delta_A = -u_{A+1/2} \sqrt{\left[ (u_{A+1/2}^2 + M) + \frac{J^2}{4u_{A+1/2}^2} \right]} \sqrt{\left[ u_{A+1}^2 + \frac{J^2}{4(u_{A+1}^2 + M)} \right]} \sqrt{\left[ u_A^2 + \frac{J^2}{4(u_A^2 + M)} \right]} \] (24)

The off-diagonal term of the matrix represents the interactions. We compare this with the corresponding Hamiltonian in previous section and evaluate entanglement entropy of the system. The entanglement entropy of the black hole is given by [16],
\[ S_{\text{ent}} = \lim_{N \to \infty} S(n, N) = S_0 + 2 \sum_{m=1}^{\infty} S_{\text{ent}}^m \] (25)
Here \( S_{\text{ent}}^m \) is the entanglement entropy of the subsystems for a given value of \( m \). The equation (25) is an infinite series, but it converges faster at the large value of \( m \) and we truncate the series whenever the subsequent terms agree up to three decimal places.

**IV. NUMERICAL ESTIMATION OF ENTROPY**

In this section, we calculate the entanglement entropy of BTZ black hole by considering the formalism outlined in previous sections. The entropy of the BTZ black hole is proportional to the area of the horizon and is independent of the position of the observer, which in turn is proportional to \( n \).
\[ S_{\text{ent}} = c_s (2\pi r_+) \] (26)
where \( c_s \) is the numerical constant of order one and we shall estimate this coefficient numerically.

We calculate the entropy of the BTZ black hole at large \( N \) (large \( N \) means, by increasing the value of \( N \), the change in entropy is not significant). First we calculate the \( S_m \) for different values of \( n \) at fixed \( N \), and after that the sum over \( m \) is performed. As we go towards the higher value of \( m \), the error decreases sharply. The numerical values of various
TABLE I: For the fixed value of N=200, we calculate the entropy for different values of n and m is taken upto 50 in $S_{\text{total}}$.

terms in entropy are listed in table I. The sum of the series is truncated at a point from where the enough accuracy in leading term (slope of line) upto three decimal points is achieved.

We plot entropy as a function of $n$ (which in turn is proportional to $r_+/a$, where $r_+$ is the horizon of the black hole) and this gives the value $c_s = 0.2936$ (slope of line in figure 1). The dependence of entropy on angular momentum $J$ is also shown in the figure 2. The entropy has no explicit dependence on angular momentum except through the factor $r_+/a$, which defines the size of the horizon.

We can extend our results to estimate the logarithmic corrections to black hole entropy. We write the entropy in the form,

$$S_{\text{log}} = c_1 (r_+/a) + c_2 \log(r_+/a) + c_3,$$

and fitted the data points and found the numerical value of these coefficients , $c_1 = 0.3025$, $c_2 = -0.1039$, $c_3 = -0.1863$.

V. RESULTS AND DISCUSSIONS

In this paper, we have calculated the entanglement entropy of quantum scalar fields in BTZ black hole spacetime. The model is very similar to a bunch of harmonic oscillators for
FIG. 1: Entanglement entropy $S_{\text{ent}}$ for $J=0$ is shown as a functions of $n$ for $N=200$.

FIG. 2: Entanglement entropy $S_{\text{ent}}$ is shown as a functions of $n$ for $N=200$ for different values of $J=0, 0.9$ and $0.99$.

free fields in curved spacetime and seem to capture the area law of black hole thermodynamics. We have also calculated the leading and sub-leading terms of the entropy. The leading term is consistent with the Bekenstein-Hawking entropy and sub-leading term is the first quantum correction of the entropy. These results are also consistent with the holographic
entanglement entropy of BTZ black hole \[12, 14\].

\[
S = \frac{l}{2G_3} \ln \frac{l}{\pi r_+} \sinh \left( \frac{\pi r_+}{l} \right)
\]  

(28)

In the limit \(r_+/l >> 1\), this becomes;

\[
S_{\text{ent}}^{\text{BTZ}} = \frac{\pi r_+}{2G_3} - \frac{l}{2G_3} \ln \frac{\pi r_+}{l} + O(1)
\]

(29)

The leading term of this equation is Bekenstein Hawking entropy and the sub-leading term describes the first quantum correction due to quantum entanglement. The \(O(1)\) coefficient is given by:

\[
O(1) = \frac{l}{2G_3} \ln \frac{l}{2a\pi}
\]

(30)

Thus we can identify the coefficients \(c_1, c_2\) and \(c_3\) as;

\[
\begin{align*}
  c_1 &= \frac{a\pi}{2G_3}, \\
  c_2 &= -\frac{l}{2G_3}, \\
  c_3 &= \frac{l}{2G_3} \ln \frac{l}{2a\pi}.
\end{align*}
\]

(31)

This gives a relation between the coefficients as;

\[
c_3 = -c_2 \ln \left( -\frac{c_2}{2c_1} \right).
\]

(32)

Substituting the numerical values of \(c_1\) and \(c_2\) from previous section, we get \(c_3 = -1.1831\), which is consistent with the numerical value of \(c_3\). This also validates the formula for holographic entanglement entropy in large \(r_+/l >> 1\) limit. These results can also be compared qualitatively with entanglement entropy of Schwarzschild black hole and Kerr-Newmann black hole spacetimes in four dimensions \[10\]. It would be interesting to generalize these results in the case of charged, and charged-rotating (2+1)-dimensional black holes.

Acknowledgments

The work of Dharm Veer Singh is supported by Rajiv Gandhi National Fellowship Scheme University Grant Commison (Under the fellowship award no. F.14-2(SC)/2008 (SA-III)) of Government of India.
References

[1] Luca Bombelli, Rabinder K. Koul, Joohan Lee and Rafael D. Sorkin, “Quantum source of entropy for Black Hole” Phy. Rev. D 34, 373 (1986).

[2] Maximo Banados, Claudio Teitelboim and Jorge Zanelli, “The black hole in three dimensional spacetime” Phy. Rev. Lett. 69(1992) 1849-1851.

[3] Mark Srednicki, “The Entropy and Area” Phy. Rev. Lett. 71 666 (1993).

[4] S. Carlip, “The (2+1)-dimensional black hole” Classical and Quantum Gravity 12(1995) 2853-2880, arXiv:gr-qc/9506079v1.

[5] A. Strominger and C. Vafa, “Microscopic origin of the Bekenstein-Hawking entropy,” Phys.Lett. B 379 (1996) 99, arXiv:hep-th/9601029.

[6] Sung-Won Kim, Won T. Kim, Young-Jai Park, and Hyeonjoon Shin, “Entropy of BTZ black hole in (2+1) dimensions” Phys. Lett B 392, 311-318 (1997).

[7] Danny Birmingham, Ivo Sachs and Siddhartha Sen, “Exact results of BTZ black hole” arXiv:hep-th/0102155v2.

[8] V. Frolov, “Why the entropy of black hole is A/4” Phy. Rev. Lett. 74 3319 (1995).

[9] Shinji Mukohyama, Masafumi Seriu and Hideo kodama, “Thermodynamics of entanglement in Schwarzschild space time” Phy. Rev. D 58, 064001 (1998).

[10] Shinji Mukohyama, “The origin of black hole entropy (Ph.D Thesis) arXiv:gr-qc/9812079.

[11] M.R. Setare, “Non-rotating BTZ black hole area spectrum from quasi-normal modes” Classical and Quantum Gravity 21 1453-1457 (2004).

[12] Shinsei Ryu and Tadashi Takayanagi, “Holographic derivation of entanglement entropy from AdS/CFT” Phy. Rev. Lett. 96 181602 (2006) arXiv:hep-th/0603001.

[13] H. Casini and M. Huerta, “Entanglement Entropy in free quantum theory” arXiv:hep-th/0905256v3.

[14] Mariano Cadoni and Maurizio Melis, “Holographic entanglement entropy BTZ black hole” arXiv:hep-th/09071559v2.

[15] R. Lohmayar, H. Neuberger, A. Schwimmer, S. Theisen, “Numerical Determination of entanglement entropy for a sphere” arXiv:hep-lat/090114233v1.
[16] Marina Huerta, “Numerical Determination of entanglement entropy for free fields in the cylinder” arXiv:hep-th/11121277v2.

[17] Pasquale Calabrese, John Cardy, “Entanglement entropy and quantum field theory” arXiv:0405152 [hep-th].

[18] S. Carlip “The lecture on (2+1) dimensional gravity” arXiv:9503024 [gr-qc].

[19] S. Carlip “What we don’t know about BTZ black hole entropy” Class. Quantum Grav. 15 (1998) 36093625.

[20] S.N. Solodukhin, “Entanglement entropy of black holes,” Living Rev. Rel. 14, 8 (2011) arXiv:1104.3712 [hep-th].