RANKS BASED ON FRAISSION CLASSES

VINCENT GUINGONA AND MIRIAM PARNES

Abstract. In this paper, we introduce the notion of $K$-rank, where $K$ is an algebraically trivial Fraïssé class. Roughly speaking, the $K$-rank of a partial type is the number of independent “copies” of $K$ that can be “coded” inside of the type. We study $K$-rank for specific examples of $K$, including linear orders, equivalence relations, and graphs. We discuss the relationship of $K$-rank to other well-studied ranks in model theory, including dp-rank and op-dimension.

1. Introduction

In model theory, there are many generalizations of dimension that are used to measure the complexity of partial types in first-order theories. For example, dp-rank and op-dimension are (different) generalizations of Euclidean dimension on the theory of the real numbers. The original goal of this paper was to formulate a general notion of rank that simultaneously encompasses many different notions of rank in model theory, including dp-rank and op-dimension. Although this goal is not entirely accomplished, we do introduce a novel class of ranks we call $K$-rank for $K$ an algebraically trivial Fraïssé class. These ranks are based around counting the maximum number of independent “copies” of $K$ that we can “code” in a partial type.

In some instances, $K$-rank does generalize known notions of model theoretic rank. For example, linear order rank generalizes op-dimension on theories without the independence property (NIP); see Proposition 4.9. As another example, equivalence relation rank is closely related to dp-rank; see Proposition 4.14 and Corollary 4.16. Both dp-rank and op-dimension are additive [5, 9]. We examine under what conditions $K$-rank is additive. From this analysis on the specific class $K$ of linear orders, we derive a result which may be of independent interest: We give a new characterization of NIP for certain theories based on the growth rate of linear order rank; see Theorem 4.11.

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The original idea for the “coding” part of this work comes from a paper by the first author and C. D. Hill [6], where they study a related notion called positive local combinatorial dividing lines. The requirements on the Fraïssé classes considered in that paper are more stringent; in this paper, for simplicity of presentation, we don’t require that our Fraïssé classes be indecomposable. The earlier paper itself was an attempt to understand the notion of the collapse of generalized indiscernibles phenomenon studied by the first author, Hill, and L. Scow in [7]. However, the consideration of indiscernibility is removed in [6] and this paper, as it puts an unnatural burden on the Fraïssé classes to be considered (to be an indiscernible class, one needs to be a Ramsey class, which requires the definability of a linear order [7]).

This paper is organized as follows: In Section 2, we discuss the relevant concepts surrounding Fraïssé classes. Primarily, we discuss the notion of free superposition, which formalizes the idea of independent “copies” of a Fraïssé class. In Section 3, we study the notion of configurations, which formalizes the notion of “coding” a Fraïssé class into a partial type. In Section 4, we define and examine \( K \)-ranks for various algebraically trivial Fraïssé classes, \( K \). In Subsection 4.1, we study linear order rank, in Subsection 4.2, we study equivalence relation rank, and in Subsection 4.3, we study graph rank. We study each of these \( K \)-ranks in the context of the random graph in Subsection 4.4. Finally, in Section 5, we connect the work in this paper back to the dividing lines considered in [6]. In particular, we discuss an interesting generalization of a few results from that paper.

2. Algebraically Trivial Fraïssé Classes

Fix \( L_0 \) a finite relational language and let \( K \) be a class of finite \( L_0 \)-structures. We say that \( K \) is an algebraically trivial Fraïssé class if it satisfies the following three properties:

(JEP\(^*\)) For all \( A_0, A_1 \in K \), there exist \( B \in K \) and embeddings \( f_t : A_t \to B \) for each \( t < 2 \) such that \( f_0(A_0) \cap f_1(A_1) = \emptyset \).

(AP\(^*\)) For all \( A, B_0, B_1 \in K \) and embeddings \( f_t : A \to B_t \) for each \( t < 2 \), there exist \( C \in K \) and embeddings \( g_t : B_t \to C \) such that \( g_0 \circ f_0 = g_1 \circ f_1 \) and \( g_0(B_0) \cap g_1(B_1) = g_0(f_0(A)) \).

(HP) For all \( B \in K \) and \( A \subseteq B \), \( A \in K \).

Since we are working in a finite relational language \( L_0 \), we may assume that the empty structure is in \( K \), then the strong joint embedding property (JEP\(^*\)) follows from the strong amalgamation property (AP\(^*\)). Moreover, for a Fraïssé class in a finite relational language, having the strong amalgamation property is equivalent to having algebraic...
triviality (i.e., if $\Gamma$ is the Fraïssé limit of $K$ and $A \subseteq \Gamma$, then $acl(A) = A$; see (2.15) of [3]). Since $L_0$ is a finite relational language, the theory of the Fraïssé limit of $K$ is $\aleph_0$-categorical and eliminates quantifiers.

Let $K_t$ be a class of finite $L_t$-structures, where $L_t$ is a finite relational language, for each $t < 2$. Let $L_2$ be the language whose signature is the disjoint union of the signatures of $L_0$ and $L_1$ and define the free superposition of $K_0$ and $K_1$, denoted $K_0 \ast K_1$, as the class of all finite $L_2$-structures $A$ such that $A|_{L_t} \in K_t$ for each $t < 2$.

**Remark 2.1.** Suppose that $A \in K_0$, $B \in K_1$, and $f : A \to B$ is a bijection. Then, we can “glue” $A$ and $B$ together via $f$ to make an element of $K_0 \ast K_1$. Formally, let $C$ be the $L_2$-structure with universe $A$ such that, for all $R \in \text{sig}(L_2)$ and $a \in A$,

- if $R \in \text{sig}(L_0)$, $C|_{L_0} = R(a)$ iff. $A|_{L_0} = R(a)$, and
- if $R \in \text{sig}(L_1)$, $C|_{L_1} = R(f(a))$.

Then, clearly $C \in K_0 \ast K_1$. Indeed, $C|_{L_0} = A$ and $C|_{L_1} \cong_{L_1} B$.

**Proposition 2.2** (Lemma 3.22 of [2]). If $K_0$ and $K_1$ are algebraically trivial Fraïssé classes, then $K_0 \ast K_1$ is an algebraically trivial Fraïssé class.

Although the result is known, we give a proof here, as it will help in the proof of Proposition 2.15 below.

**Proof.** We begin by exhibiting the strong amalgamation property (AP$^\ast$). Fix structures $A, B_0, B_1 \in K_0 \ast K_1$ and suppose that $f_t : A \to B_t$ is an $L_2$-embedding for each $t < 2$. In particular, for each $s < 2$, $f_0$ and $f_1$ are $L_s$-embeddings. By the strong amalgamation property of $K_s$, there exist $C_s \in K_s$ and $g_t^s : B_t \to C_s$ $L_s$-embeddings for $t < 2$ such that $g_t^s \circ f_0 = g_t^s \circ f_1$ and $g_t^0(B_0) \cap g_t^1(B_1) = g_t^0(f_0(A))$. By embedding into a larger structure and using the hereditary property (HP), we may assume that $|C_0| = |C_1|$. Consider a bijection $h : C_0 \to C_1$ such that the following diagram commutes:

![Diagram](https://example.com/diagram.png)

As in Remark 2.1, endow $C_0$ with an $L_2$-structure via $h$ and call it $C_2$. 


To exhibit the hereditary property (HP), fix $B \in K_0 \star K_1$ and let $A \subseteq B$. In particular, $A|_{L_0}$ is a $L_0$-substructure of $B|_{L_0}$, so $A|_{L_0} \in K_0$. Similarly, $A|_{L_1} \in K_1$. Thus, $A \in K_0 \star K_1$. □

Example 2.3. Note that algebraic triviality is necessary to conclude that the free superposition is even a Fraissé class. For example, for each $t < 2$, let $L_t$ be the language with one unary predicate, $P_t$, and let $K_t$ be the class of all $L_t$-structures where at most one element satisfies $P_t$. This is clearly a Fraissé class, but is not algebraically trivial (it fails $(\text{JEP}^*)$ if $A_0, A_1 \in K$ each have one element that satisfies $P_t$). On the other hand, $K_0 \star K_1$ is not a Fraissé class, as it fails joint embedding.

Let $A_0 = \{a_0, a_1\}$ where $P_0(a_0)$ and $P_1(a_1)$ and let $A_1 = \{a_2\}$ where $P_0(a_2)$ and $P_1(a_2)$. Then, there exists no $B \in K_0 \star K_1$ which embeds $A_0$ and $A_1$ simultaneously.

Definition 2.4. Let $K$ be an algebraically trivial Fraissé class in $L_0$, $A \in K$, and $R$ a relation of $L_0$ with arity $n$.

1. We say $R$ is symmetric on $A$ if, for all $a \in {}^nA$ and all $\sigma \in S_n$, if $A \models R(a)$, then $A \models R(a \circ \sigma)$.
2. We say $R$ is trichotomous on $A$ if, for all $a \in {}^nA$ such that $a(i) \neq a(j)$ for all $i < j < n$, there exists exactly one $\sigma \in S_n$ such that $A \models R(a \circ \sigma)$.
3. We say $R$ is reflexive on $A$ if, for all $a \in {}^nA$ such that $a(i) = a(j)$ for all $i < j < n$, $A \models R(a)$.
4. We say $R$ is irreflexive on $A$ if, for all $a \in {}^nA$ such that $a(i) = a(j)$ for some $i < j < n$, $A \models \neg R(a)$.
5. If $n = 2$, we say $R$ is transitive if, for all $a, b, c \in A$, if $A \models R(a, b) \land R(b, c)$, then $A \models R(a, c)$.

We say $A$ has one of the above properties if, for all $R \in \text{sig}(L_0)$, $R$ has that property on $A$. We say $K$ has one of the above properties if, for all $A \in K$, $A$ has that property.

Proposition 2.5. Each of the properties in Definition 2.4 are closed under free superposition.

Proof. Any witness to the failure of one of these properties in $K_0 \star K_1$ reducts to a failure of the same property in either $K_0$ or $K_1$. □

Definition 2.6. We have a few algebraically trivial Fraissé classes that we examine in particular in this paper.

1. (Sets) Let $S$ denote the class of all finite $L_0$-structures where $L_0$ has empty signature.
(2) (Linear Orders) Let $LO$ denote the class of all finite $L_0$-structures that are trichotomous, irreflexive, and transitive, where $L_0$ is a language with one binary relation symbol.

(3) (Equivalence Relations) Let $E$ denote the class of all finite $L_0$-structures that are symmetric, reflexive, and transitive, where $L_0$ is a language with one binary relation symbol.

(4) (Graphs) Let $G$ denote the class of all finite $L_0$-structures that are symmetric and irreflexive, where $L_0$ is a language with one binary relation symbol.

(5) (Hypergraphs) For $k \geq 2$, let $H_k$ denote the class of all finite $L_0$-structures that are symmetric and irreflexive, where $L_0$ is a language with one $k$-ary relation symbol. Clearly $G = H_2$.

(6) (Tournaments) Let $T$ denote the class of all finite $L_0$-structures that are trichotomous and irreflexive, where $L_0$ is a language with one binary relation symbol.

**Definition 2.7.** Suppose $K$ is an algebraically trivial Fraïssé class and fix $n \geq 1$. Then, define $K^{*n}$ recursively as follows:

1. $K^{*1} = K$,
2. $K^{*(n+1)} = K^{*n} \star K$.

If $\Gamma$ is the Fraïssé limit of $K$, let $\Gamma^{*n}$ be the Fraïssé limit of $K^{*n}$. If $T$ is the theory of $\Gamma$, let $T^{*n}$ be the theory of $\Gamma^{*n}$.

**Example 2.8.** For any algebraically trivial Fraïssé class $K$, notice that $K \star S = K$.

In particular, $S^{*n} = S$ for all $n \geq 1$.

**Example 2.9.** For all $n \geq 1$, $LO^{*n}$ is the class of all finite sets with $n$ independent linear orders.

**Example 2.10.** In any finite relational language $L_0$ where all relations are at least binary, the class of $L_0$-hypergraphs, $H_{L_0}$, is the set of all finite $L_0$-structures that are symmetric and irreflexive. By Proposition 2.5

$$H_{L_0} = H_{k_0} \star \ldots \star H_{k_{n-1}},$$

where $k_0 \leq \ldots \leq k_{n-1}$ list all the arities (with repetition) of the relation symbols in $L_0$. By Proposition 2.2, $H_{L_0}$ is an algebraically trivial Fraïssé class.

In the remainder of this section, we will introduce tools that will be used to compute $K$-rank for specific algebraically trivial Fraïssé classes $K$ in Section 4.
Proposition 2.11. Suppose that $\Gamma$ is the Fraïssé limit of $K$ and $\Gamma' \subseteq \Gamma$. If, for all $A, B \in K$ with $A \subseteq B$ and $|B \setminus A| = 1$ and for all embeddings $f : A \to \Gamma'$, there exists an embedding $g : B \to \Gamma'$ extending $f$, then $\Gamma' \cong \Gamma$.

Proof. This follows from Lemma 6.1.4 of [8].

In the following definition, we propose a condition that imposes a sufficient amount of self-similarity to a class $K$ so that, if we color $\Gamma$ with finitely many colors, some subset of $\Gamma$ isomorphic to $\Gamma$ is monochromatic.

Definition 2.12. Let $K$ be an algebraically trivial Fraïssé class in a finite relational language $L_0$.

1. Let $C \in K$ and fix $A \subseteq C$ and $a' \in C \setminus A$. Fix $B \in K$ and an embedding $f : B \to C$. We say that $f$ is an embedding of $B$ into $\text{qftp}_{L_0}(a'/A)$ if, for all $b \in B$,

\[ \text{qftp}_{L_0}(f(b)/A) = \text{qftp}_{L_0}(a'/A). \]

2. We say that $K$ is self-similar if, for all $B, B', C \in K$ such that $B \subseteq B'$, for all $A \subseteq C$ and $a' \in C \setminus A$, for all $f : B \to C$ an embedding of $B$ into $\text{qftp}_{L_0}(a'/A)$, there exist $C' \in K$ with $C' \supseteq C$ and $f' : B' \to C'$ an embedding of $B'$ into $\text{qftp}_{L_0}(a'/A)$ that extends $f$.

Lemma 2.13. Suppose that $K$ is self-similar, let $\Gamma$ be the Fraïssé limit of $K$, let $A \subseteq \Gamma$ be finite, and let $a' \in \Gamma \setminus A$. Then, the set

\[ \Gamma' = \{ b \in \Gamma : \text{qftp}_{L_0}(b/A) = \text{qftp}_{L_0}(a'/A) \} \]

is isomorphic to $\Gamma$.

Proof. We show that the hypothesis of Proposition 2.11 is satisfied for $\Gamma'$.

Consider $B, C \in K$ with $B \subseteq C$ and suppose that $f : B \to \Gamma'$ is an embedding. By definition of $\Gamma'$, $f$ is an embedding of $B$ into $\text{qftp}_{L_0}(a'/A)$. Since $K$ is self-similar, there exists an extension $g : C \to \Gamma$ of $f$ such that $g$ is an embedding of $C$ into $\text{qftp}_{L_0}(a'/A)$. In other words, $g$ is an embedding of $C$ into $\Gamma'$ extending $f$. 

Lemma 2.14. Suppose that $K$ is self-similar, let $\Gamma$ be the Fraïssé limit of $K$, let $k < \omega$, and let $c : \Gamma \to k$ by any function. Then, there exist $\Gamma' \subseteq \Gamma$ with $\Gamma' \cong \Gamma$ and $i < k$ such that

\[ c(\Gamma') = \{ i \}. \]
Proof. It suffices to prove this when \( k = 2 \). Suppose that \( c^{-1}(\{0\}) \not\cong \Gamma \). By Proposition 2.11, there exist \( A, B \in K \) with \( A \subseteq B \) and \( B = \{b\} \cup A \), and there exists \( f : A \to c^{-1}(\{0\}) \) an embedding that does not extend to an embedding of \( B \) into \( c^{-1}(\{0\}) \). Then, consider

\[
\Gamma' = \{ d \in \Gamma : qftp_{L_0}(d, f(A)) = qftp_{L_0}(b, A) \}.
\]

By Lemma 2.13, \( \Gamma' \cong \Gamma \). On the other hand, for any \( d \in \Gamma' \), the function extending \( f \) to a function from \( B \) to \( \Gamma \) by sending \( b \) to \( d \) is an embedding. Thus, \( c(d) = 1 \). In other words, \( c(\Gamma') = \{1\} \).

We see that being self-similar indeed guarantees the desired coloring property. In the next proposition, we show that being self-similar is closed under free superposition.

**Proposition 2.15.** Suppose that \( K_0 \) and \( K_1 \) are self-similar. Then, \( K_0 \ast K_1 \) is self-similar.

**Proof.** Let \( L_0 \) be the language of \( K_0 \), let \( L_1 \) be the language of \( K_1 \), and let \( L_2 \) be the language whose signature is the disjoint union of the signatures of \( L_0 \) and \( L_1 \), which serves as the language for \( K_2 = K_0 \ast K_1 \).

Fix \( B, B', C \in K_2 \) such that \( B \subseteq B' \), fix \( A \subseteq C \) and \( a' \in C \setminus A \), and fix \( f : B \to C \) an \( L_2 \)-embedding of \( B \) into \( qftp_{L_2}(a'/A) \). In particular, for each \( t < 2 \), \( f \) is an \( L_t \)-embedding of \( B|_{L_t} \) into \( qftp_{L_t}(a'/A) \). Since \( K_t \) is self-similar, there exist \( C'_t \in K_t \) with \( C|_{L_t} \subseteq C'_t \) and \( f'_t : B' \to C'_t \) an \( L_t \)-embedding of \( B'|_{L_t} \) into \( qftp_{L_t}(a'/A) \) extending \( f \). By (HP), we may assume that \( |C'_0| = |C'_1| \). Then, as in the proof of Proposition 2.2, there exists a bijection \( g \) from \( C'_0 \) to \( C'_1 \) such that the following diagram commutes:

\[
\begin{array}{ccc}
B' & \xrightarrow{f'_0} & C'_0 \\
\downarrow{f'_1} & & \downarrow{g} \\
B & \xrightarrow{f} & C \\
\end{array}
\]

As in Remark 2.1, create the structure \( C' \in K_2 \) with universe \( C'_0 \) endowed with \( L_2 \)-structure given by \( g \). Then, it is not hard to show that \( f'_0 \) is an \( L_2 \)-embedding of \( B' \) into \( qftp_{L_2}(a'/A) \). \( \square \)

**Example 2.16.** The classes \( L_0 \), \( G \), and \( T \) are self-similar. Moreover, for all \( k \geq 2 \), \( H_k \) is self-similar. By Proposition 2.15, for all \( n \geq 1 \), \( L_0^n \), \( G^n \), and \( T^n \) are self-similar and, for all \( k \geq 2 \), \( H_k^n \) is self-similar. On the other hand, \( E \) is not self-similar.
Proof. Let \(K\) be \(LO\) or \(T\) and \(k = 2\), or let \(K\) be \(H_k\) for some \(k \geq 2\). Let \(K\) be in the language \(L_0\) with one \(k\)-ary relation symbol, \(E\). Fix \(B, B', C \in K\) with \(B \subseteq B'\), fix \(A \subseteq C, a' \in C \setminus A\), and \(f : B \to C\) an embedding of \(B\) into \(\text{qftp}_{L_0}(a'/A)\). We may assume that \(B' = B \cup \{b'\}\). Create an \(L_0\)-structure \(C'\) where \(C' = C \cup \{c'\}\) by setting, for all \(\overline{b} \in B^{k-1}\),

\[
C' \models E(\overline{f}(\overline{b}), c') \iff B' \models E(\overline{b}, b')
\]

and, for all \(\overline{a} \in A^{k-1}\),

\[
C' \models E(\overline{a}, c') \iff C \models E(\overline{a}, a').
\]

Define the remainder of \(\text{qftp}_{L_0}(c'/A \cup f(B))\) such that \(E\) is irreflexive and either symmetric or trichotomous as appropriate. Set \(E\) arbitrarily on the remainder of \(C'\) so that \(C' \in K\). Finally, extend \(f\) to \(f'\) by setting \(f'(b') = c'\). It is easy to check that \(f'\) embeds \(B'\) into \(\text{qftp}_{L_0}(a'/A)\).

Consider the class \(E\) in the language \(L_0\) with one binary relation symbol, \(E\). Let \(C = \{a, a'\}\) with \(a \neq a'\) and \(C \models E(a, a'), A = \{a\}, B' = \{b, b'\}\) with \(B' \models \neg E(b, b')\), and \(B = \{b\}\). The map \(b \mapsto a'\) is an embedding of \(B\) into \(\text{qftp}_{L_0}(a'/A)\). However, there is no embedding of \(B'\) into \(\text{qftp}_{L_0}(a'/A)\). \(\square\)

Example 2.17. Although \(E\) is not self-similar, it does satisfy the conclusion of Lemma 2.14. However, for \(m \geq 2\), \(E^{*m}\) does not satisfy the conclusion of Lemma 2.14.

Remark 2.18. In general, for \(m < \omega\), let \(L_m\) be the language with \(m\) binary relation symbols, \(E_i\) for \(i < m\). For any set \(I\), we put an \(L_m\)-structure on \(I^{m+1}\) as follows: For all \(i < m\) and all \(\overline{a}, \overline{b} \in I^{m+1}\), set \(E_i(\overline{a}, \overline{b})\) if \(a_i = b_i\). If \(I\) is finite, then \(I^{m+1} \in E^{*m}\). If \(I\) is countably infinite, then \(I^{m+1}\) is isomorphic to the Fraïssé limit of \(E^{*m}\).

Proof of Example 2.17. It follows from the Pigeonhole Principle that \(E\) satisfies the conclusion of Lemma 2.14.

Fix \(m \geq 2\) and consider the Fraïssé limit of \(E^{*m}\) with universe \(\omega^{m+1}\) as in Remark 2.18. Consider the coloring \(c : \omega^{m+1} \to 2\) given by

\[
c(\overline{a}) = \begin{cases} 
0 & \text{if } a_0 < a_1, \\
1 & \text{if } a_0 \geq a_1
\end{cases}
\]

It is clear that there is no \(\Gamma' \subseteq \Gamma\) with \(\Gamma' \cong \Gamma\) such that \(c(\Gamma')\) is a singleton. \(\square\)

To deal with \(E^{*m}\) in Section 4, we need a condition that is weaker than being self-similar, but which is still strong enough to give us a semblance of the coloring property in Lemma 2.14.
Definition 2.19. Let $K$ be an algebraically trivial Fraïssé class. We say that $K$ is weakly self-similar if, for all $k < \omega$, all $c : \Gamma \to k$, and all $A \in K$, there exist an embedding $f : A \to \Gamma$ and $i < k$ such that $c(f(A)) = \{i\}$.

By Lemma 2.14, if $K$ is self-similar, then $K$ is weakly self-similar.

Example 2.20. For all $m < \omega$, $E^m$ is weakly self-similar. In particular, “weakly self-similar” is strictly weaker than “self-similar.”

Proof. Let $\Gamma$ be the Fraïssé limit of $E^m$. We may assume that $\Gamma$ has universe $\omega^{m+1}$ as in Remark 2.18. Let $c : \Gamma \to k$ be a coloring and let $A \in E^m$. Let $n = |A|$. By Lemma A.1, there exist $Y_0, ..., Y_m \in \binom{\omega}{n}$ such that $c$ is constant on $B = \prod_{i \leq m} Y_i$. On the other hand, there is clearly an embedding $g : A \to B$. Thus, $c$ is constant on $g(A)$. This shows that $E^m$ is weakly self-similar. $\square$

Definition 2.21. Let $K$ be an algebraically trivial Fraïssé class. We say that $K$ is generic if, for all $n < \omega$, for all functions $f$ from relation symbols in $L_0$ of arity $n$ to 2, there exist $A \in K$ and $a \in A^n$ such that $a_i \neq a_j$ for all $i < j < n$ and, for all relation symbols $R$ in $L_0$ of arity $n$, $A \models R(\overline{a})$ if and only if $f(R) = 1$.

Example 2.22. Notice that $S$, $LO$, $E$, $G$, $T$, and $H_k$ for all $k \geq 2$ are all generic.

Proposition 2.23. Suppose that $K_0$ and $K_1$ are generic. Then, $K_0 \star K_1$ is generic.

Proof. We can find separate witnesses to being generic in $K_0$ and $K_1$ and use the technique in the proofs of Proposition 2.2 and Proposition 2.15 to combine these into a single witness of being generic in $K_0 \star K_1$. $\square$

3. Configurations

Let $L$ be any language, let $T$ be a complete $L$-theory, let $\mathfrak{C}$ be a monster model of $T$, and let $\pi(\overline{y})$ be a partial type over a small subset of $\mathfrak{C}$. Let $L_0$ be a finite relational language and let $K$ be an algebraically trivial Fraïssé class of finite $L_0$-structures. Let $\Gamma$ be the Fraïssé limit of $K$ and let $T_0$ be the $L_0$-theory of $\Gamma$. Borrowing language from [I], we call $T$, $L$, $\mathfrak{C}$, and $\pi$ the targets and $T_0$, $L_0$, $\Gamma$, and $K$ the indices.

The following definition will be used to capture what we mean by “coding” the class $K$ in the partial type $\pi$. 
Definition 3.1. A $K$-configuration into $\pi$ is a function $f : \Gamma \to \pi(\mathcal{C})$ such that, for all relation symbols $R(x_0, \ldots, x_{n-1})$ in $L_0$, there exists an $L(\mathcal{C})$-formula $\varphi_R(\overline{y}_0, \ldots, \overline{y}_{n-1})$ such that, for all $\overline{y} \in \Gamma^n$,

$$\Gamma \models R(\overline{y}) \text{ iff. } \mathcal{C} \models \varphi_R(f(\overline{y})).$$

For small $C \subseteq \mathcal{C}$, we say that $f$ is over $C$ if we can take $\varphi_R$ to be an $L(C)$-formula for all $R$ in the signature of $L_0$. We say that $f$ is parameter-free if it is over $\emptyset$.

Lemma 3.2. There exists a $K$-configuration into $\pi$ over $C$ if and only if there exists a map $I$ from $\text{sig}(L_0)$ to $L(C)$-formulas such that, for all $A \in K$, there exists $f : A \to \pi(\mathcal{C})$ such that, for all $R \in \text{sig}(L_0)$, for all $a \in A$ ofarity($R$),

$$A \models R(a) \iff \mathcal{C} \models I(R(f(a))).$$

Proof. By compactness. $\square$

Lemma 3.3. Let $T = T_0$ and $\pi(x) = (x = x)$, where $x$ is a singleton. Then, there exists a $K$-configuration into $\pi$.

Proof. Let $f : \Gamma \to \mathcal{C}$ be any embedding. Then, $f$ is clearly a $K$-configuration into $\pi$. $\square$

Lemma 3.4. If $\pi_0(\overline{y})$ and $\pi_1(\overline{y})$ are partial types in $T$ and $\pi_0(\overline{y}) \models \pi_1(\overline{y})$ and there exists a $K$-configuration in $\pi_0$, then there exists a $K$-configuration in $\pi_1$.

Proof. If $f : \Gamma \to \pi_0(\mathcal{C})$ is a $K$-configuration into $\pi_0$, since $\pi_0(\mathcal{C}) \subseteq \pi_1(\mathcal{C})$, it is also a $K$-configuration into $\pi_1$. $\square$

Definition 3.5. For each $t < 2$, let $L_t$ be a finite relational language and let $K_t$ be an an algebraically trivial Fraïssé class of $L_t$-structures. We say that $K_0$ is a reductive subclass of $K_1$ if the signature of $L_0$ is a subset of the signature of $L_1$ and, for each $A \in K_0$, there exists $B \in K_1$ such that $A \cong_{L_0} B|_{L_0}$.

Example 3.6. Note that $\mathbf{LO}$ is a reductive subclass of $\mathbf{T}$ (it is actually just a subclass). For any $K_0$ and $K_1$, $K_0$ is a reductive subclass of $K_0 \star K_1$ (see Remark 2.1).

Lemma 3.7. If there exists a $K_1$-configuration into $\pi$ and $K_0$ is a reductive subclass of $K_1$, then there exists a $K_0$-configuration into $\pi$.

Proof. Fix $I : \text{sig}(L_1) \to L(\mathcal{C})$ witnessing that there is a $K_1$-configuration into $\pi$ as in Lemma 3.2. Fix $A \in K_0$ and choose $B \in K_1$ and
Let \( g : A \to B \) such that \( g \) is an \( L_0 \)-isomorphism. By Lemma 3.2, there exists \( f : B \to \pi(C) \) such that, for all \( R \in \text{sig}(L_1) \) and \( \overline{b} \in B^{\text{arity}(R)} \),
\[
B \models R(\overline{b}) \iff C \models I(R)(f(\overline{b})).
\]

Then, in particular, for all \( R \in \text{sig}(L_0) \) and \( \overline{a} \in A^{\text{arity}(R)} \),
\[
A \models R(\overline{a}) \iff C \models I(R)(f(g(\overline{a}))).
\]

By Lemma 3.2, there exists a \( K_0 \)-configuration into \( \pi \).

If \( \pi_0(\overline{y}_0) \) and \( \pi_1(\overline{y}_1) \) are two partial types in \( T \) where \( \overline{y}_0 \) and \( \overline{y}_1 \) are disjoint, define \( \pi_0 \times \pi_1 \) to be the following type:
\[
(\pi_0 \times \pi_1)(\overline{y}_0, \overline{y}_1) = \pi_0(\overline{y}_0) \cup \pi_1(\overline{y}_1).
\]

If \( \overline{y}_0 \) and \( \overline{y}_1 \) are not disjoint, we can choose different variables to force disjointness. Fix \( n \geq 1 \) and define \( \pi^{\times n} \) recursively as follows:

1. \( \pi^{\times 1} = \pi \),
2. \( \pi^{\times (n+1)} = \pi^{\times n} \times \pi \).

It turns out that free superposition interacts with configurations in the obvious manner.

**Proposition 3.8.** Suppose \( \pi_0 \) and \( \pi_1 \) are two partial types in \( T \). Suppose there exist a \( K_0 \)-configuration into \( \pi_0 \) and a \( K_1 \)-configuration into \( \pi_1 \). Then, there exists a \((K_0 \circ K_1)\)-configuration into \( \pi_0 \times \pi_1 \).

**Proof.** For each \( t < 2 \), let \( I_t \) be a map from \( \text{sig}(L_t) \) to \( L(C) \)-formulas given by Lemma 3.2. We build a map \( I \) from \( \text{sig}(L_2) \) to \( L(C) \)-formulas to satisfy Lemma 3.2.

For each \( t < 2 \) and each relation symbol \( R(x_0, \ldots, x_{n-1}) \) in \( L_t \), let
\[
I(R)(\overline{y}_{0,0}, \overline{y}_{0,1}, \overline{y}_{1,0}, \overline{y}_{1,1}, \ldots, \overline{y}_{n-1,0}, \overline{y}_{n-1,1}) = I_t(R)(\overline{y}_{0,t}, \overline{y}_{1,t}, \ldots, \overline{y}_{n-1,t}).
\]

Fix \( A \in (K_0 \circ K_1) \). By Lemma 3.2, for each \( t < 2 \), there exists \( f_t : A \to \pi_t(C) \) such that, for all \( R \in \text{sig}(L_t) \), for all \( \overline{a} \in A^{\text{arity}(R)} \),
\[
A|_{L_t} \models R(\overline{a}) \iff C \models I_t(R)(f_t(\overline{a})).
\]

Let \( f : A \to (\pi_0 \times \pi_1)(C) \) be given by \( f(a) = (f_0(a), f_1(a)) \). Then, we get that, for all \( R \in \text{sig}(L_2) \), for all \( \overline{a} \in A^{\text{arity}(R)} \),
\[
A \models R(\overline{a}) \iff C \models I(R)(f(\overline{a})).
\]

By Lemma 3.2, there exists a \((K_0 \circ K_1)\)-configuration into \( \pi_0 \times \pi_1 \).

**Corollary 3.9.** If \( \overline{x} \) is a tuple of variables with \( n = |\overline{x}| \) in \( T_0 \), then there exists a \( K^{\times n} \)-configuration into \( \overline{x} = \overline{x} \).

**Proof.** Use Lemma 3.3, Proposition 3.8, and induction. \( \square \)
We can “compose” configurations, as long as the first configuration is parameter-free.

**Proposition 3.10.** Suppose that \( \pi(\mathcal{F}) \) is a partial type in some theory \( T \) and suppose \( \mathcal{F} \) is an \( n \)-tuple of variables in the Fraïssé limit of \( K_1 \) for some \( n < \omega \). Suppose there exist a \( K_1 \)-configuration into \( \pi \) and a parameter-free \( K_0 \)-configuration into \( \mathcal{F} = \mathcal{F} \). Then, there exists a \( K_0 \)-configuration into \( \pi^x \).

**Proof.** Let \( I : \text{sig}(L_0) \to L_1 \) be the function witnessing the fact that there is a \( K_0 \)-configuration into \( \mathcal{F} = \mathcal{F} \) via Lemma 3.2. Since the theory of the Fraïssé limit of \( K_1 \) has quantifier elimination, we may assume that, for each \( R \in \text{sig}(L_0) \), \( I(R) \) is a quantifier-free \( L_1 \)-formula. Let \( J : \text{sig}(L_1) \to L(\mathcal{F}) \) be the function witnessing the fact that there is a \( K_1 \)-configuration into \( \pi(\mathcal{F}) \) via Lemma 3.2. Define \( H : \text{sig}(L_0) \to L(\mathcal{F}) \) by the following method: For each \( R(x_0, ..., x_{k-1}) \in \text{sig}(L_0) \), consider \( I(R)(\mathcal{F}_0, ..., \mathcal{F}_{k-1}) \) (so \( \mathcal{F}_i = (y_i, 0, ..., y_i, n-1) \) for each \( i < k \)). For each \( S(y_{i0}, y_{i1}, ..., y_{j0}, 0, j_{i1}, ..., j_{i0}) \in \text{sig}(L_1) \) used in \( I(R) \), replace it with \( J(S)(\mathcal{F}_{i0}, ..., \mathcal{F}_{i1}, ..., \mathcal{F}_{j0}, ..., \mathcal{F}_{j1}) \).

This creates an \( L(\mathcal{F}) \)-formula in the variables \( ((\mathcal{F}_{i,j})_{j < n})_{i < k} \); call it \( H(R) \). It is not difficult to check that \( H \) satisfies the conditions of Lemma 3.2. Therefore, there exists a \( K_0 \)-configuration into the partial type \( \pi^x(\mathcal{F}_0, ..., \mathcal{F}_{n-1}) \). We can convert any configuration into a parameter-free one at the cost of changing the target partial type.

**Lemma 3.11.** If there exists a \( K \)-configuration into some partial type \( \pi \) of a theory \( T \), then there exists a parameter-free \( K \)-configuration into some partial type of \( T \) (possibly different from \( \pi \)).

**Proof.** Let \( \pi(\mathcal{F}) \) be a partial type in a theory \( T \) and let \( f : \Gamma \to \pi(\mathcal{F}) \) be a \( K \)-configuration into \( \pi \). For each \( R(x_0, ..., x_{n-1}) \in \text{sig}(L_0) \), let \( \varphi_R(\mathcal{F}_0, ..., \mathcal{F}_{n-1}, \mathcal{F}_R) \) an \( L \)-formula and \( \varphi_R \in \mathcal{C}^{||\mathcal{F}||} \) be such that, for all \( \overline{a} \in \Gamma^n \),

\[
\Gamma \models R(\overline{a}) \iff \mathcal{C} \models \varphi_R(f(\overline{a}), \mathcal{F}_R).
\]

Define \( \pi^* \) a partial type of \( T \) as follows:

\[
\pi^*(\mathcal{F}, (\mathcal{F}_S)_{S \in \text{sig}(L_0)}) = \pi(\mathcal{F}).
\]

Define \( f^* : \Gamma \to \pi^*(\mathcal{F}) \) as follows: For \( a \in \Gamma \),

\[
f^*(a) = (f(a), (\mathcal{F}_S)_{S \in \text{sig}(L_0)})
\]

Finally, for each \( R(x_0, ..., x_{n-1}) \in \text{sig}(L_0) \), let

\[
\varphi^*_R(\mathcal{F}_0, (\mathcal{F}_S)_{S \in \text{sig}(L_0)}, ..., \mathcal{F}_{n-1}, (\mathcal{F}^{n-1}_S)_{S \in \text{sig}(L_0)}) = \varphi_R(\mathcal{F}_0, ..., \mathcal{F}_{n-1}, \mathcal{F}^0_R).
\]
Notice each $\varphi^*_R$ is an $L$-formula. For each $\pi \in \Gamma^*$,
\[
\Gamma \models R(\bar{a}) \text{ iff. } \mathcal{C} \models \varphi^*_R(f^*(\bar{a})).
\]
Therefore, $f^*$ is a parameter-free $K$-configuration into $\pi^*$. \hfill $\square$

Next, we analyze how the properties of being self-similar and being weakly self-similar translates to configurations. Being weakly self-similar manifests in a uniformity condition on $L$-types.

**Proposition 3.12.** Suppose that $K$ is weakly self-similar and suppose that $\pi(\bar{y})$ is a partial type in a theory $T$. Suppose that there exists a $K$-configuration into $\pi$ over a finite $C \subseteq \mathcal{C}$. Then, there exists a $K$-configuration into $\pi$ over $C$, $f$, such that, for all $a, b \in \Gamma$,
\[
\text{tp}_L(f(a)/C) = \text{tp}_L(f(b)/C).
\]

**Proof.** Let $f$ be a $K$-configuration into $\pi$ over $C$ and let $I$ be as in Lemma 3.2. Consider the type
\[
\Sigma(\bar{y}_a)_{a \in \Gamma} = \bigcup \{\pi(\bar{y}_a) : a \in \Gamma\} \cup \{I(R)(\bar{y}_a)^{\text{iff } R(\bar{y})} : R \in \text{sig}(L_0), \bar{a} \in \Gamma_{\text{arity}(R)}\} \cup \{\psi(\bar{y}_a) \leftrightarrow \psi(\bar{y}_b) : \psi \in L(C), a, b \in \Gamma\}.
\]
Fix $\Sigma_0 \subseteq \Sigma$ finite. Then, there exists a finite set of $L(C)$-formulas $\Psi(\bar{y})$ and a finite $A \subseteq \Gamma$ so that $\Sigma_0$ mentions only variables $\bar{y}_a$ for $a \in A$ and only formulas $\psi(\bar{y}_a) \leftrightarrow \psi(\bar{y}_b)$ for $\psi \in \Psi$ and $a, b \in A$. Consider the coloring $c : \Gamma \to \Psi^2$ so that, for all $a \in \Gamma$ and $\psi \in \Psi$,
\[
c(a)(\psi) = 1 \text{ iff. } \mathcal{C} \models \psi(f(a)).
\]
Since $K$ is weakly self-similar, there exists an embedding $g : A \to \Gamma$ so that $c$ is constant on $g(A)$. Then, $(f(g(a)))_{a \in A} \models \Sigma_0$.

By compactness, there exists $(\bar{c}_a)_{a \in \Gamma} \models \Sigma$. Define $f' : \Gamma \to \pi(\mathcal{C})$ by setting $f'(a) = \bar{c}_a$. Then, $f'$ is the desired $K$-configuration. \hfill $\square$

When $K$ is self-similar, we get a stronger condition on $L$-types. For each $A \in K$, let $S(A)$ be the set of all quantifier-free 1-$L_0$-types over $A, p(x, A)$, such that there exist $B \in K$, an embedding $f : A \to B$, and $b \in B \setminus f(A)$ such that $b \models p(x, f(A))$.

**Proposition 3.13.** Suppose that $K$ is self-similar and suppose that $\pi(\bar{y})$ is a partial type in a theory $T$. Suppose that there exists a $K$-configuration into $\pi$ over $C$ for some finite $C \subseteq \mathcal{C}$. Then, there exists a $K$-configuration $f : \Gamma \to \pi(\mathcal{C})$ over $C$ and there exists $J \subseteq |\bar{y}|$ such that
\begin{enumerate}
  
  
  
  \item for all $a, b \in \Gamma$, $\text{tp}_L(f(a)/C) = \text{tp}_L(f(b)/C)$;
  \item for all $j \in J$ and all $a, b \in \Gamma$, $f(a)_j = f(b)_j$; and
\end{enumerate}
(3) for all finite \( A \subseteq \Gamma \) and all \( p \in S(A) \), there exists \( b \models p \) such that, for all \( i, j \in [\Gamma] \setminus J \) and all \( a \in A \), \( f(a)_i \neq f(b)_j \).

Proof. Let \( f : \Gamma \to \pi(\mathcal{C}) \) be a \( K \)-configuration over \( C \). Since \( K \) is self-similar, \( K \) is weakly self-similar. By Proposition 3.12, we may assume condition (1) holds of \( f \).

For conditions (2) and (3), start with \( J = \emptyset \) and construct \( J \) recursively as follows: For any \( J \) satisfying condition (2), assume that condition (3) fails. So there exist a finite \( A \subseteq \Gamma \) and \( p \in S(A) \) such that, for all \( b \models p \), there exist \( i, j \in [\Gamma] \setminus J \) and \( a \in A \) such that \( f(a)_i = f(b)_j \). Let \( \Gamma' = \{ b \in \Gamma : b \models p \} \). By Lemma 2.13, \( \Gamma' \cong \Gamma \).

Consider the coloring \( c : \Gamma' \to ([\Gamma] \setminus J)^2 \times A \) given by \( c(b) = (i, j, a) \) (where \( f(a)_i = f(b)_j \)). By Lemma 2.14 we may assume that \( c \) is constant. Thus, for all \( b, d \in \Gamma' \), \( f(b)_j = f(a)_i = f(d)_j \) (in other words, condition (2) holds on \( \Gamma' \) for \( J \cup \{ j \} \)). Add \( j \) to \( J \) and replace \( \Gamma \) with \( \Gamma' \). Repeat this process. Since \([\Gamma]\) is finite, this will eventually terminate. This gives us the desired conclusion. \( \square \)

We use Proposition 3.12 and Proposition 3.13 in Subsection 4.4 to compute \( K \)-ranks for particular choices of \( K \).

4. \( K \)-Ranks

Now that we have the notion of “coding” a class into a type, we want to count the number of independent “copies” of a single algebraically trivial Fraïssé class that we can code into a partial type, \( \pi \). We use the same setup as in Section 3 with targets \( T \), \( L \), \( \mathcal{C} \), and \( \pi \) and indices \( T_0 \), \( L_0 \), \( \Gamma \), and \( K \).

Definition 4.1. Fix \( n \geq 1 \). We say that \( \pi \) has \( \textbf{K-rank} \) \( n \) if

1. There exists a \( K^{*n} \)-configuration into \( \pi \), and
2. There does not exist a \( K^{*(n+1)} \)-configuration into \( \pi \).

We say \( \pi \) has \( \textbf{K-rank} \) \( \infty \) if there exists a \( K^{*n} \)-configuration into \( \pi \) for all \( n < \omega \). We say \( \pi \) has \( \textbf{K-rank} \) 0 if there does not exist a \( K \)-configuration into \( \pi \).

We will denote the \( K \)-rank of \( \pi \) by \( \text{Rk}_K(\pi) \).

We can apply Lemma 3.4 and Proposition 3.8 to get a few immediate results about \( K \)-rank.

Proposition 4.2 (Superadditivity of \( K \)-rank). For all partial types \( \pi_0 \) and \( \pi_1 \),

\[ \text{Rk}_K(\pi_0 \times \pi_1) \geq \text{Rk}_K(\pi_0) + \text{Rk}_K(\pi_1). \]

Proof. Follows immediately from Proposition 3.8. \( \square \)
**Definition 4.3.** We say Rk is additive if, for all partial types π₀ and π₁, if Rk(π₀) < ∞ and Rk(π₁) < ∞, then

\[ Rk(\pi_0 \times \pi_1) = Rk(\pi_0) + Rk(\pi_1). \]

**Open Question 4.4.** Under what conditions on K and T is K-rank additive?

We present some partial results to Open Question 4 later in this section (see Example 4.10 and Example 4.20).

**Lemma 4.5.** If π₀(y) and π₁(y) are partial types in T and π₀(y) ⊢ π₁(y), then

\[ Rk(\pi_0) \leq Rk(\pi_1). \]

**Proof.** Follows immediately from Lemma 3.4. □

Overloading notation, for each n ≥ 1, we can define Rk(n) as follows: Fix an arbitrary n-tuple of variables y from T and set

\[ Rk(n) = Rk(\overline{y} = \overline{y}). \]

This is clearly independent of the choice of \( \overline{y} \).

**Lemma 4.6.** For all 1 ≤ n ≤ m < ω,

\[ Rk(n) \leq Rk(m). \]

**Proof.** Suppose Rk(n) = ℓ. Then, there exists a K^ℓ-configuration into C^n. Clearly there exists an S-configuration into C^{m−n}. By Proposition 3.8 there exists a (K^ℓ×S)-configuration into C^m. Since K^ℓ×S = K^ℓ, we get that Rk(m) ≥ ℓ. □

In the following subsections, we will analyze K-rank for particular choices of K.

### 4.1. Linear Order Rank

For this subsection, we consider the algebraically trivial Fraïssé class LO. For any m ≥ 1, let L_m be the language of LO^m, which consists of m binary relation symbols, <_i for i < m. Let L be any language, let T be a complete L-theory, let C be a monster model of T, and let π(\overline{y}) be a partial type.

It turns out that LO-rank is closely related to something called op-dimension.

**Definition 4.7.** We say that \( \pi \) has an IRD-pattern of depth m and length β if there exist L(C)-formulas \( \varphi_i(\overline{y}, \overline{z}_i) \) for i < m and \( \overline{z}_{i,j} \in C^{|\overline{z}_i|} \) for i < m and j < β such that, for all \( g : m \to \beta \), the partial type

\[ \pi(\overline{y}) \cup \{ \varphi_i(\overline{y}, \overline{z}_{i,j}) : i < m, j < \beta \} \]
is consistent. The op-dimension of \( \pi \) is the maximum \( m < \omega \) such that \( \pi \) has an IRD-pattern of depth \( m \) and length \( \omega \). We denote the op-dimension of \( \pi \) by \( \text{opDim}(\pi) \).

For any partial type in any theory, we get that the op-dimension is an upper bound for the \( \text{LO} \)-rank. Moreover, if the target theory has NIP, then op-dimension coincides with \( \text{LO} \)-rank.

**Proposition 4.8.** For any partial type \( \pi \) in any theory \( T \),

\[
\text{Rk}_{\text{LO}}(\pi) \leq \text{opDim}(\pi).
\]

**Proof.** Assume \( \text{Rk}_{\text{LO}}(\pi) \geq m \). Let \( f : \Gamma \to \pi(\mathcal{C}) \) be an \( \text{LO}^m \)-configuration. So, for each \( i < m \), there exists an \( L(\mathcal{C}) \)-formula \( \varphi_i(\overline{y}, \overline{y}_1) \) such that, for all \( a, b \in \Gamma \),

\[
\Gamma \models a <_i b \iff \mathcal{C} \models \varphi_i(f(a), f(b)).
\]

Fix \( n < \omega \). For each \( g \in m(2n) \), choose \( a_g \in \Gamma \) such that, for all \( i < m \), for all \( g, h \in m(2n) \),

\[
\text{if } g(i) < h(i), \text{ then } \Gamma \models a_g <_i a_h.
\]

For \( g \in m^n \), let \( \overline{d}_g = f(a'_g) \), where \( a'_g \in m(2n) \) is such that \( g'(i) = 2g(i) + 1 \) for all \( i < m \). For \( j < n \), let \( \overline{c}_j = f(a'_g) \), where \( g'(i) = 2j \) for all \( i < m \). Then, for all \( g \in m^n, i < m, \text{ and } j < n \),

\[
\mathcal{C} \models \varphi_i(\overline{d}_g, \overline{c}_j) \iff 2g(i) + 1 < 2j \iff g(i) < j.
\]

Thus, for each \( g \in m^n \),

\[
\pi(\overline{y}) \cup \{ \varphi_i(\overline{y}, \overline{c}_j) \text{ iff } g(i)<j : i < m, j < n \}
\]

is consistent. This is an IRD-pattern of depth \( m \) and length \( n \) in \( \pi \). Since \( n \) was arbitrary, by compactness, \( \pi \) has op-dimension \( \geq m \). \( \square \)

**Proposition 4.9.** If \( T \) has NIP, then, for all partial types \( \pi \),

\[
\text{Rk}_{\text{LO}}(\pi) = \text{opDim}(\pi).
\]

This proof loosely follows the proof of Theorem 3.4 of [7], modified to fit into our current framework.

**Proof.** The previous proposition gives us \( \text{Rk}_{\text{LO}}(\pi) \leq \text{opDim}(\pi) \). Conversely, suppose that \( \pi \) has op-dimension \( \geq m \). Therefore, there exists an IRD-pattern of depth \( m \) and length \( \omega \) in \( \pi \). That is, there exist \( L(\mathcal{C}) \)-formulas \( \varphi_i(\overline{y}, \overline{c}_i) \) for \( i < m \) and \( \overline{c}_{i,j} \in \mathcal{C}[\overline{c}_i] \) for \( i < m \) and \( j < \omega \) such that, for all \( g : m \to \omega \), the partial type

\[
\pi(\overline{y}) \cup \{ \varphi_i(\overline{y}, \overline{c}_{i,j}) \text{ iff } g(i)<j : i < m, j < \omega \}
\]
is consistent. Say it is realized by $b_g \in C^{\mathfrak{G}}$. By coding tricks, we may
assume that there exists an $L$-formula $\varphi(\overline{y}, \overline{z})$ such that $\varphi_i = \varphi$ for all
$i < m$.

First, we create a function $f : \Gamma \rightarrow \pi(C)$ (where $\Gamma$ is the Fraïssé
limit of $L^{\mathfrak{G}}$). Fix $A \in L^{\mathfrak{G}}$ and suppose that $n = |A|$. Choose an
injective function $\eta : A \rightarrow \mathbb{N}$ such that, for all $a, b \in A$ and for all
$i < m$, $\eta(a)(i) < \eta(b)(i)$ if and only if $a < i b$. For all $i < m$, $j < n$, and
$a \in A$, notice that

$$ \mathcal{C} \models \varphi(b_{\eta(a)}, \overline{z}_{i,j}) \text{ iff. } \eta(a)(i) < j. $$

Therefore, for all $i < m$, for all $<_{i}$-cuts $Y$ of $A$, there exists $\overline{z} \in \mathcal{C}^{\mathfrak{G}}$
such that

$$ Y = \{ a \in A : \mathcal{C} \models \varphi(b_{\eta(a)}, \overline{z}) \}. $$

Consider the function $a \mapsto b_{\eta(a)}$ from $A$ to $\pi(C)$. By compactness, there
exists a function $f : \Gamma \rightarrow \pi(C)$ such that, for all $i < m$, for all $<_{i}$-cuts
$Y$ of $\Gamma$, there exists $\overline{z} \in \mathcal{C}^{\mathfrak{G}}$ such that

$$ Y = \{ a \in \Gamma : \mathcal{C} \models \varphi(f(a), \overline{z}) \}. $$

Moreover, we can assume that $f : \Gamma \rightarrow \pi(C)$ is a generalized indiscernible (see Proposition 1.18 of [3]). Therefore, for each $k < \omega$ and
each quantifier-free $L_m$-type $p(x_0, ..., x_{k-1})$, we have an associated $L$-
type $p^*(\overline{y}_0, ..., \overline{y}_{k-1})$ (over the same parameters as $\pi$ and $\varphi$) extending
$\pi(\overline{y}_0) \cup ... \cup \pi(\overline{y}_{k-1})$ such that, for all $\overline{z} \in \Gamma^k$, if $\overline{z} \models p$, then $f(\overline{z}) \models p^*$.

Since $T$ has NIP, $\varphi(\overline{y}, \overline{z})$ has VC-dimension $< k$ for some $k < \omega$. In
other words,

$$ \mathcal{C} \models \neg \exists \overline{y}_0 \ldots \exists \overline{y}_{k-1} \wedge \mathop{\triangleleft}_{s \in k^2} \exists \overline{z} \bigwedge_{\ell < k} \varphi(\overline{y}_\ell, \overline{z})^{s(\ell)}. $$

For each $t \in m^2$, define the quantifier-free $2L_m$-type $p_t(x_0, x_1)$ as follows:
$p_t \vdash x_0 \neq x_1$ and, for all $i < m$, $p_t \vdash x_0 <_i x_1$ if and
only if $t(i) = 0$. We can extend this to a quantifier-free $kL_m$-type $q_t(x_0, ..., x_{k-1})$ as follows: 
for all $\ell \neq \ell'$, $q_t \vdash x_\ell \neq x_{\ell'}$ and, for all $i < m$, 
for all $\ell < k - 1$, $q_t \vdash x_\ell <_i x_{\ell+1}$ if and only if $t(i) = 0$.

Now fix $t, t' \in m^2$ distinct. We may assume, by perhaps swapping $t$ and $t'$, that
there exists $i_0 < m$ such that $t(i_0) = 0$ and $t'(i_0) = 1$. Since $\varphi$ has VC-dimension $< k$, there exists $s \in k^2$ such that

$$ q_t^s(\overline{y}_0, ..., \overline{y}_{k-1}) \vdash \exists \overline{z} \bigwedge_{\ell < k} \varphi(\overline{y}_\ell, \overline{z})^{s(\ell)}. $$

We define $q_r$ and $\sigma_r$ recursively as follows: Let $q_0 = q_t$ and $\sigma_0$ the identity permutation on $k$. Fix $r \geq 0$ and assume that we have constructed
Consider the set $\Gamma$ such that
\[(1) \quad q_r(x_0, \ldots, x_{k-1}) \vdash x_{\sigma_r(0)} <_{i_0} x_{\sigma_r(1)} <_{i_0} \ldots <_{i_0} x_{\sigma_r(k-1)}.
\]
Then, choose $\ell_r < k - 1$ minimal such that $s(\sigma_r(\ell_r)) = 0$ and $s(\sigma_r(\ell_r + 1)) = 1$. Note that, if no such $\ell_r$ exists, then
\[q^r_s(\overline{y}_0, \ldots, \overline{y}_{k-1}) \vdash \exists \ell \varphi(\overline{y}_{\sigma_r(\ell)}, \overline{z})^{s(\sigma_r(\ell))},\]
since it is a $<_{i_0}$-cut. In particular, $\ell_0$ exists.

Let $\sigma_{r+1} = \sigma_r \circ (\ell_r \ell_r + 1)$ and let $q_{r+1}$ be $q_r$ except, for each $i < m$, we replace
\[(x_{\ell_r} < i x_{\ell_r + 1}) \iff t(i) = 0 \text{ with } (x_{\ell_r} < i x_{\ell_r + 1}) \iff t'(i) = 0.
\]
In particular, we maintain that $q_{r+1}$ and $\sigma_{r+1}$ satisfy (I). Terminate the construction when we first have
\[q^r_{r+1}(\overline{y}_0, \ldots, \overline{y}_{k-1}) \vdash \exists \ell \varphi(\overline{y}_\ell, \overline{z})^{s(\ell)}\]

Choose $\overline{a} \in \Gamma^k$ such that $\overline{a} \models q_r$. Let
\[\psi_{t, t'}(\overline{y}_0, \overline{y}_1) := \neg \exists \varphi(f(a_0), \overline{z})^{s(0)} \land \ldots \land \varphi(f(a_{\ell_r - 1}), \overline{z})^{s(\ell_r - 1)} \land \varphi(\overline{y}_0, \overline{z})^{s(\ell_r)} \land \varphi(\overline{y}_1, \overline{z})^{s(\ell_r + 1)} \land \varphi(f(a_{\ell_r + 2}), \overline{z})^{s(\ell_r + 2)} \land \ldots \land \varphi(f(a_{k-1}), \overline{z})^{s(k-1)}).
\]
Consider the set
\[\Gamma' = \{a \in \Gamma : (a_0, \ldots, a_{\ell_r - 1}, a, a_{\ell_r + 2}, \ldots, a_{k-1}) \models q_r|_{\overline{y}_0, \ldots, \overline{y}_r, \overline{y}_{\ell_r + 2}, \ldots, \overline{y}_{k-1}}\}.
\]
By Proposition 2.13 $\Gamma' \cong \Gamma$. Notice that, for all $a, b \in \Gamma'$,
\begin{itemize}
  \item If $(a, b) \models p_{t'}$, then $(a_0, \ldots, a_{\ell_r - 1}, a, b, a_{\ell_r + 2}, \ldots, a_{k-1}) \models q_r$, hence $\mathcal{C} \models \psi_{t, t'}(f(a), f(b))$.
  \item If $(a, b) \models p_{t'}$, then $(a_0, \ldots, a_{\ell_r - 1}, a, b, a_{\ell_r + 2}, \ldots, a_{k-1}) \models q_{r+1}$, hence $\mathcal{C} \models \neg \psi_{t, t'}(f(a), f(b))$.
\end{itemize}
Replace $\Gamma$ with $\Gamma'$ and repeat this process for all distinct $t, t' \in m^2$.

For $t \in m^2$, let
\[\psi_t(\overline{y}_0, \overline{y}_1) := \bigwedge_{t' \in m^2, t' \neq t} \psi_{t, t'}(\overline{y}_0, \overline{y}_1).
\]
Finally, for $i < m$, let
\[\psi_i(\overline{y}_0, \overline{y}_1) := \bigvee_{t \in m^2, t(i) = 0} \psi_t(\overline{y}_0, \overline{y}_1).
\]
Then, it is clear that, for all $a, b \in \Gamma$ and $i < m$,
\[a <_i b \text{ if and only if } \mathcal{C} \models \psi_i(f(a), f(b)).\]
This is an $\text{LO}^m$-configuration into $\pi$. Thus, $\text{Rk}_{\text{LO}}(\pi) \geq m$. □

Note that this proof uses generalized indiscernibles; this is the only such use in this paper. In future work, we would like to remove the need for indiscernibility so that arguments such as these can be generalized to Fraïssé classes without a definable linear order.

Example 4.10 (NIP). Suppose $T$ has NIP. Then, $\text{LO}$-rank is precisely op-dimension. In particular, $\text{LO}$-rank is additive (see Theorem 2.2 of [5]).

If $T$ is distal, then op-dimension coincides with dp-rank (see Remark 3.2 of [5]). Therefore, for distal $T$, $\text{LO}$-rank is dp-rank.

In Example 4.26 below, we consider $T$ the theory of the random graph. For any $n \geq 1$, although the op-dimension of an $n$-tuple of variables from $T$ is $\infty$, we show that $\text{Rk}_{\text{LO}}(n) = n^2 - 1$. Therefore, when $T$ has the independence property, op-dimension and $\text{LO}$-rank may differ. Moreover, $\text{LO}$-rank is not necessarily additive.

In fact, we can show that, as long as $\text{LO}$-rank is finite, $\text{LO}$-rank grows linearly if $T$ has NIP and grows quadratically if $T$ has the independence property. In particular, if $\text{LO}$-rank is finite and $T$ has the independence property, then $\text{LO}$-rank is not additive.

Theorem 4.11. Let $T$ be any complete first-order theory such that $\text{Rk}_{\text{LO}}(1) < \infty$.

(1) If $T$ has NIP, then there exists $C \in \mathbb{R}$ such that, for all $n \geq 1$,

$$\text{Rk}_{\text{LO}}(n) \leq Cn.$$ 

(2) If $T$ has the independence property, then there exists $C \in \mathbb{R}$ such that, for sufficiently large $n$,

$$\text{Rk}_{\text{LO}}(n) \geq Cn^2.$$ 

Proof. (1): As noted in Example 4.10 if $T$ has NIP, then $\text{Rk}_{\text{LO}}$ is additive. Thus, if we let $C = \text{Rk}_{\text{LO}}(1)$, $\text{Rk}_{\text{LO}}(n) = Cn$ for all $n \geq 1$.

(2): Assume that $T$ has the independence property. By Theorem 5.2 (2), there exists a $G$-configuration into $\mathfrak{c}^k$ for some $k < \omega$. Moreover, by Proposition 4.23 there exists a parameter-free $\text{LO}^*(n^2 - 1)$-configuration into $\mathfrak{c}_1^n$, where $\mathfrak{c}_1$ is a monster model for the theory of the Fraïssé limit of $G$. By Proposition 3.10 for each $n < \omega$, there exists an $\text{LO}^*(n^2 - 1)$-configuration into $\mathfrak{c}_1^{kn}$. Therefore,

$$\text{Rk}_{\text{LO}}(kn) \geq n^2 - 1.$$ 

For all $m \geq 1$, let $n = \lfloor m/k \rfloor$. Then, by Lemma 4.6

$$\text{Rk}_{\text{LO}}(m) \geq \text{Rk}_{\text{LO}}(kn) \geq n^2 - 1 \geq \frac{1}{k^2} m^2 - \frac{2}{k} m.$$ 

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In the next example, we show that LO-rank can jump from 0 to ∞ in a theory with the independence property.

Example 4.12. Let \( L \) consist of infinitely many binary relation symbols \( R_i \) for \( i < \omega \) and let \( T \) be the \( L \)-theory of the Fraïssé limit of the class of all finite \( L \)-hypergraphs. Then,

\[
R_k^{\text{LO}}(1) = 0 \quad \text{and} \quad R_k^{\text{LO}}(2) = \infty.
\]

Proof. Towards a contradiction, suppose there is an \( \text{LO} \)-configuration \( f: \Gamma \to C \). Thus, there exists an \( L(C) \)-formula \( \varphi(y_0, y_1) \) for some finite \( C \subseteq C \) such that, for all \( a, b \in \Gamma \),

\[
\Gamma \models a < b \iff C \models \varphi(f(a), f(b)).
\]

Since \( \text{LO} \) is self-similar, by Proposition 3.12, we may assume that the map \( a \mapsto \text{tp}_L(f(a)/C) \) is constant. Thus, since \( L \) is a binary language, for all \( a, b \in \Gamma \), the type \( \text{tp}_L(f(a), f(b)) \) determines the type \( \text{tp}_L(f(a), f(b)/C) \). On the other hand, since \( T \) is symmetric,

\[
\text{tp}_L(f(a), f(b)) = \text{tp}_L(f(b), f(a)).
\]

Therefore, \( \Gamma \models a < b \) if and only if \( \Gamma \models b < a \), a contradiction.

On the other hand, for each \( i < \omega \), let

\[
\varphi_i(y_{0,0}, y_{0,1}, y_{1,0}, y_{1,1}) = R_i(y_{0,0}, y_{1,1}).
\]

For any \( m < \omega \), for any \( A \in \text{LO}^* \), it is clear that there exists a function \( f: A \to C^2 \) such that, for all \( a, b \in A \) and \( i < m \),

\[
A \models a <_i b \iff C \models \varphi_i(f(a), f(b)).
\]

By Lemma 3.2, there exists an \( \text{LO}^* \)-configuration into \( C^2 \). \( \square \)

4.2. Equivalence Class Rank. For this subsection, we consider the algebraically trivial Fraïssé class \( \mathcal{E} \). For any \( m \geq 1 \), let \( L_m \) be the language of \( \mathcal{E}^* \), which consists of \( m \) binary relation symbols, \( E_i \) for \( i < m \). Let \( L \) be any language, let \( T \) be a complete \( L \)-theory with infinite models, let \( \mathcal{C} \) be a monster model of \( T \), and let \( \pi(\overline{y}) \) be a partial type.

It turns out that \( \mathcal{E} \)-rank is tangentially related to something called dp-rank.

Definition 4.13. We say that \( \pi \) has an ICT-pattern of depth \( m \) and length \( \beta \) if there exist \( L(\mathcal{C}) \)-formulas \( \varphi_i(\overline{y}, \overline{z}_i) \) for \( i < m \) and \( \overline{z}_{i,j} \in \mathcal{C}^{\left| \overline{z}_i \right|} \) for \( i < m \) and \( j < \beta \) such that, for all \( g: m \to \beta \), the partial type

\[
\pi(\overline{y}) \cup \{ \varphi_i(\overline{y}, \overline{z}_{i,j}) \}_{i < m, j < \beta}^{\text{iff} g(i) = j}
\]
is consistent. The \textit{dp-rank} of \( \pi \) is the maximum \( m < \omega \) such that \( \pi \) has an ICT-pattern of depth \( m \) and length \( \omega \). We denote the dp-rank of \( \pi \) by \( \text{dpRk}(\pi) \).

As we had with \textit{LO}-rank and op-dimension, dp-rank serves as an upper bound for \( \text{E-rank} \).

\textbf{Proposition 4.14.} For any partial type \( \pi \) in any theory \( T \),

\[ \text{Rk}_E(\pi) \leq \text{dpRk}(\pi). \]

\textbf{Proof.} Suppose \( \text{Rk}_E(\pi) \geq m \). Let \( f : \Gamma \to \pi(\mathfrak{C}) \) be a \( \text{E}^m \)-configuration. So, for each \( i < m \), there exists an \( L(\mathfrak{C}) \)-formula \( \varphi_i(\overline{y}_0, \overline{y}_1) \) such that, for all \( a, b \in \Gamma \),

\[ \Gamma \models E_i(a, b) \iff \mathfrak{C} \models \varphi_i(f(a), f(b)). \]

Fix \( n < \omega \). For each \( g \in \mathcal{M}_n \), choose \( a_g \in \Gamma \) such that, for all \( i < m \), for all \( g, h \in \mathcal{M}_n \),

\[ \Gamma \models E_i(a_g, a_h) \iff g(i) = h(i). \]

Let \( \overline{c}_g = f(a_g) \). Thus, for all \( i < m \), for all \( g, h \in \mathcal{M}_n \),

\[ \mathfrak{C} \models \varphi_i(\overline{c}_g, \overline{c}_h) \iff g(i) = h(i) \]

For each \( j < n \), overloading notation, let \( j \) denote the function from \( m \) to \( n \) that is constantly \( j \). Then, for each \( g \in \mathcal{M}_n \), we have that

\[ \pi(\overline{y}) \cup \{ \varphi_i(\overline{y}, \overline{c}_j) \text{ iff } g(i) = j : i < m, j < n \} \]

is consistent (realized by \( \overline{c}_g \)). This is an ICT-pattern of depth \( m \) and length \( n \) in \( \pi \). Since \( n \) was arbitrary, by compactness, \( \pi \) has dp-rank \( \geq m \).

Moreover, \( \text{E-rank} \) is bounded below by the dimension of the target.

\textbf{Proposition 4.15.} For all theories \( T \),

\[ \text{Rk}_E(m) \geq m. \]

\textbf{Proof.} Fix a tuple of variables \( \overline{y} \) and let \( m = |\overline{y}| \). Fix \( A \in \text{E}^m \) and choose \( n < \omega \) such that \( A \) embeds into \( n^{m+1} \) viewed as an element of \( \text{E}^m \) as in Remark 2.18. Thus, we may assume \( A \) is this \( L_m \)-structure on \( n^{m+1} \). For each \( i < m \), let

\[ \varphi_i(y_{0,0}, \ldots, y_{0,m}, y_{1,0}, \ldots, y_{1,m}) = [y_{0,i} = y_{1,i}] \]

Let \( I \) be the map that sends \( E_i \) to \( \varphi_i \). It is not hard to show that this satisfies the conditions of Lemma 3.2. Therefore, we get a \( \text{E}^m \)-configuration into \( \overline{y} = \overline{y} \). Thus, \( \text{Rk}_E(m) \geq m \). \( \square \)
We say that $T$ is dp-minimal if $\text{dpRk}(y = y) = 1$ for some (any) single variable $y$ in $T$. Combining the previous two results, we conclude that $E$-rank is precisely equal to the dimension of the target in dp-minimal theories.

**Corollary 4.16.** Let $T$ be dp-minimal. Then, $\text{Rk}_E(m) = m$.

**Proof.** Let $\overline{y}$ be an $m$-tuple of variables from $T$. By Proposition 4.15,

$$\text{Rk}_E(\overline{y} = \overline{y}) \geq |\overline{y}|.$$  

Since dp-rank is subadditive [9], $\text{dpRk}(\overline{y} = \overline{y}) \leq |\overline{y}|$. By Proposition 4.14,

$$\text{Rk}_E(\overline{y} = \overline{y}) \leq \text{dpRk}(\overline{y} = \overline{y}) \leq |\overline{y}|.$$  

Thus, $\text{Rk}_E(\overline{y} = \overline{y}) = \text{dpRk}(\overline{y} = \overline{y}) = |\overline{y}|$.  

□

**Open Question 4.17.** If $T$ is dp-minimal and $\pi$ is a partial type in $T$, then does $\text{Rk}_E(\pi) = \text{dpRk}(\pi)$? More generally, under what conditions does $\text{Rk}_E(\pi) = \text{dpRk}(\pi)$?

Although this question is still open, we have examples where $E$-rank and dp-rank differ, even in an NIP theory.

**Example 4.18.** Fix $k \geq 2$ and let $T$ be the theory of the Fraïssé limit of $\mathsf{LO}^k$ (the theory of $k$ independent dense linear orders). We claim that, in the theory $T$,

$$\left\lfloor \frac{k}{2} \right\rfloor \leq \text{Rk}_E(1) < k.$$  

(On the other hand, $\text{dpRk}(y = y) = k$, so these ranks disagree.)

**Proof.** Let $m = \lfloor k/2 \rfloor$ and fix $A \in E^m$. As in the proof of Proposition 4.15 there exists $n < \omega$ such that $A$ embeds into $X = n^{m+1}$ with $L_m$-structure as in Remark 2.18 For each $i < m$, we define two linear orders $<_{2i}$ and $<_{2i+1}$ on $X$ as follows: for all $\overline{a}, \overline{b} \in X$, let

- $\overline{a} <_{2i} \overline{b}$ if $a_i < b_i$ or $a_i = b_i$ and $a_{j_0} < b_{j_0}$ where $j_0 = \min \{j < n : a_j \neq b_j\}$, and
- $\overline{a} <_{2i+1} \overline{b}$ if $a_i > b_i$ or $a_i = b_i$ and $a_{j_0} < b_{j_0}$ where $j_0 = \min \{j < n : a_j \neq b_j\}$.

It is clear from definition that, for all $i < m$, for all $\overline{a}, \overline{b} \in X$,

$$E_i(\overline{a}, \overline{b}) \iff (\overline{a} <_{2i} \overline{b} \leftrightarrow \overline{a} <_{2i+1} \overline{b}).$$

If $k = 2m + 1$, then define $<_{2m}$ arbitrarily. Since $(X, <_i)_{i<k}$ is an element of $\mathsf{LO}^k$, we get an embedding of $X$ into $\mathcal{C}$. Composing this
embedding with the one sending $A$ to $X$, we get an injective function $\eta : A \to \mathfrak{C}$ such that, for all $a, b \in A$ and $i < m$,

$$A \models E_i(a, b) \iff (\eta(a) <_{2i} \eta(b) \leftrightarrow \eta(a) <_{2i+1} \eta(b)).$$

Thus, the function sending $E_i$ to the formula

$$\varphi_i(y_0, y_1) := [y_0 <_{2i} y_1 \leftrightarrow y_0 <_{2i+1} y_1]$$

satisfies the conditions of Lemma 3.12. This gives an $E^{*m}$-configuration into $\mathfrak{C}$. Thus, $\text{Rk}_E(1) \geq m$.

On the other hand, suppose that $\Gamma$ is the Fraïssé limit of $E^{*k}$ and suppose that $f : \Gamma \to \mathfrak{C}$ is an $E^{*k}$-configuration over a finite $C$. Since $E^{*k}$ is weakly self-similar, by Proposition 3.12 we may assume that, for all $a, b \in \Gamma$, $\text{tp}_L(f(a)/C) = \text{tp}_L(f(b)/C)$. Fix $a \in \Gamma$ and, for each $s \in k^2$, choose $b_s \in \Gamma$ such that, for all $i < k$, $E_i(a, b_s)$ if and only if $s(i) = 1$. Consider the 2-types in $T$:

$$p_{s,0} = \text{tp}_L(f(a), f(b_s)) \quad \text{and} \quad p_{s,1} = \text{tp}_L(f(b_s), f(a)).$$

Since $L$ has only binary relations and each $f(a)$ and $f(b_s)$ have the same $L$-type over $C$, these types determine the $L$-types over $C$. For any $s$ not the identically 1 function, $f(a) \neq f(b_s)$. Otherwise, suppose that $f(a) = f(b_s)$ where $s \in k^2$ with $s(i) = 0$. Then, we get that

$$\text{tp}_L(f(a), f(b_1)/C) = \text{tp}_L(f(b_s), f(b_1)/C)$$

(where 1 is the identically 1 function). Since $\Gamma \models E_i(a, b_1)$, this implies that $\Gamma \models E_i(b_s, b_1)$, hence $\Gamma \models E_i(a, b_s)$, which is a contradiction. Similarly, for $s$ and $t$ not identically 1, if $(s, i) \neq (t, j)$, then $p_{s,i} \neq p_{t,j}$. Therefore, we have at least $2 \cdot (2^k - 1)$ many non-equality 2-types in $T$. On the other hand, there are $2^k$ many non-equality 2-types in $T$ (one type for each possible assignment of $x <_i y$ or $x >_i y$ for all $i < k$). Thus, $2^{k+1} - 2 \leq 2^k$, a contradiction. Therefore, there is no $E^{*k}$-configuration into $y = y$. Thus, $\text{Rk}_E(y = y) < k$. 

When $T$ has the independence property, similar to LO-rank, E-rank can grow quadratically. In particular, in Example 4.32, we will show that, when $T$ is the theory of the random graph,

$$\text{Rk}_E(n) = \begin{cases} 1 & \text{if } n = 1, \\ n^2 - 1 & \text{if } n \geq 2. \end{cases}$$

**Proposition 4.19.** Let $T$ be any complete first-order theory with the independence property such that $\text{Rk}_E(1) < \infty$. Then, there exists $C \in \mathbb{R}$ such that, for sufficiently large $n$,

$$\text{Rk}_E(n) \geq Cn^2.$$
Proof. This is similar to the proof of Theorem \textcolor{red}{4.11} (2).

4.3. Graph Rank. For this subsection, we consider the algebraically trivial Fraïssé class \( G \). Let \( L \) be any language, let \( T \) be a complete \( L \)-theory, and let \( \mathfrak{C} \) be a monster model of \( T \).

Example 4.20 (NIP). If \( T \) has NIP, then, for all types \( \pi \), \( \text{Rk}_G(\pi) = 0 \). Thus, \( G \)-rank is trivially additive.

Proof. This follows from Theorem \textcolor{red}{5.2} (2) and Corollary \textcolor{red}{5.6}.

In Example \textcolor{red}{4.28}, we will see that \( G \)-rank is not necessarily additive when \( T \) has the independence property. Moreover, in any theory with the independence property, so long as \( G \)-rank is finite, \( G \)-rank grows quadratically (hence, it is not additive).

Proposition 4.21. If \( T \) is any theory and \( \pi \) any partial type with \( \text{Rk}_G(\pi) \geq 1 \), then

\[
\text{Rk}_G(\pi^{\times n}) \geq n^2 - 1.
\]

Proof. This is similar to the proof of Theorem \textcolor{red}{4.11} (2).

Similar to \textcolor{red}{LO-rank}, the \( G \)-rank can jump from 0 to \( \infty \) in a theory with the independence property.

Example 4.22. Let \( L \) be the language consisting of binary relation symbols \( R_i \) for \( i < \omega \) and let \( T \) be the theory of the Fraïssé limit of the class of all finite \( L \)-hypergraphs that are \( R_i \)-triangle-free for all \( i < \omega \). Then,

\[
\text{Rk}_G(1) = 0 \text{ and } \text{Rk}_G(2) = \infty.
\]

Proof. Towards a contradiction, suppose there exists a \( G \)-configuration into \( \mathfrak{C} \), \( f : \Gamma \to \mathfrak{C} \). Let \( \varphi(y_0, y_1) \) witness this. Fix \( n < \omega \) such that \( \varphi \) mentions only \( R_i \) for \( i < n \). By quantifier elimination, there exists \( S \subseteq \omega^2 \) such that

\[
\varphi(y_0, y_1) = \bigvee_{s \in S} \bigwedge_{i < n} R_i(y_0, y_1)^{s(i)}.
\]

By swapping \( E \) with \( \neg E \), we may assume the constant zero function is not in \( S \). If we consider a finite complete graph \( V \subseteq \Gamma \), then \( f(V) \) can be viewed as a complete graph with colors in \( S \). By Ramsey’s Theorem, for sufficiently large \( V \), there exists a triangle of a fixed color \( s_0 \in S \). By assumption, there exists \( i_0 < n \) such that \( s_0(i_0) = 1 \), so this is an \( R_{i_0} \)-triangle. This is a contradiction.

Fix an arbitrary \( m < \omega \) and define, for each \( i < m \), \( L \)-formulas as follows:

\[
\varphi_i(y_{0,0}, y_{0,1}, y_{1,0}, y_{1,1}) = R_i(y_{0,0}, y_{1,1}) \land R_i(y_{0,1}, y_{1,0}).
\]
For any $A \in G^{*m}$, there exists a function $f : A \to \mathcal{C}^2$ such that, for all $a, b \in A$ and $i < m$,

$$A \models E_i(a, b) \iff \mathcal{C} \models \varphi_i(f(a), f(b)).$$

(Check that no $R_i$-triangles are formed for all $i < m$.) By Lemma 3.2, there exists a $G^{*m}$-configuration into $\mathcal{C}^2$ for all $m$. \qed

4.4. **Into the Random Graph.** In this subsection, we study the specific case where the target theory is the theory of the random graph. It turns out that $K$-rank, for various examples of $K$, acts in an interesting manner in this theory.

For this subsection, let $T$ be the theory of the random graph in the language $L$ with a single binary relation $R$ and let $\mathcal{C}$ be a monster model for $T$. Let $K$ be an algebraically trivial Fraïssé class in a language $L_0$ with a single binary relation symbol, $E$. For any $m \geq 1$, let $L_m$ be the language of $K^{*m}$, which consists of $m$ binary relation symbols; call them $E_i$ for $i < m$. Let $\Gamma$ be the Fraïssé limit of $K^{*m}$.

Fix $n \geq 1$ and consider the set $X_t = n \times \{t\}$ for each $t < 2$. Let $G$ be the set of bipartite graphs with parts $X_0$ and $X_1$. Then, $|G^n| = 2^{n^2}$. Say that $G = (X_0 \cup X_1, F) \in G^n$ is symmetric if, whenever $\{(i, 0), (j, 1)\} \in F$, $\{(j, 0), (i, 1)\} \in F$. Let $G^n_s$ be the set of symmetric bipartite graphs with parts $X_0$ and $X_1$ and let $G^n_{ns}$ be the set of non-symmetric bipartite graphs with parts $X_0$ and $X_1$. Notice that $|G^n_s| = 2^{\binom{n+1}{2}}$. To see this, observe that, for each $i \leq j < n$, we can choose whether or not to put $\{(i, 0), (j, 1)\}, \{(j, 0), (i, 1)\} \in F$. This gives us $\binom{n}{2} + n = \binom{n+1}{2}$ choices. Thus,

$$|G^n_{ns}| = 2^{n^2} - 2^{\binom{n+1}{2}} = 2^{\binom{n+1}{2}} \left(2^{\binom{n}{2}} - 1\right).$$

For each $G \in G^n$, let $G^*$ be the graph where we “swap parts” (i.e., $\{(i, 0), (j, 1)\}$ is an edge of $G$ if and only if $\{(j, 0), (i, 1)\}$ is an edge of $G^*$). Clearly $(G^*)^* = G$ and, for all $G \in G^n$, $G \in G^n_s$ if and only if $G^* = G$.

Let $S_2$ denote the set of all quantifier-free 2-$L_m$-types, $p(x_0, x_1)$ such that there exist distinct $a_0, a_1 \in \Gamma$ such that $(a_0, a_1) \models p$. For $p \in S_2$, let $p^*$ be the type in $S_2$ such that, for all $i < m$,

$$p^*(x_0, x_1) \models E_i(x_0, x_1) \iff p(x_0, x_1) \models E_i(x_1, x_0).$$

The next two propositions give conditions on $K$ that guarantee that $\text{Rk}_K(n) = n^2 - 1$ for $n \geq 2$. These conditions are met by $\text{LO}$, $G$, and $T$.

**Proposition 4.23.** Fix $n \geq 1$. Assume that $K$ is either reflexive or irreflexive. Assume also that $K$ is either symmetric or trichotomous.
Then,
\[ \text{Rk}_K(n) \geq n^2 - 1. \]
Moreover, this is witnessed by a parameter-free configuration.

Proof. We may assume \( n \geq 2 \), since the statement is trivial when \( n = 1 \).
Let \( m = n^2 - 1 \) and let \( \overline{y} \) be an \( n \)-tuple of variables from \( T \).
Consider the function \( g : S_2 \to m \) where, for all \( p \in S_2 \) and \( i < m \),
\[ g(p)(i) = 1 \iff p(x_0, x_1) \vdash E_i(x_0, x_1). \]
Since \( K \) is symmetric or trichotomous and \( K \) is reflexive or irreflexive, \( g \) is injective. Thus, \( |S_2| \leq 2^m \).
Choose any injective function \( h : S_2 \to \mathcal{P}(G^n) \) with the following conditions:
(1) If \( K \) is symmetric, then, for all \( p \in S_2 \), \( h(p) = \{G, G^*\} \) for some \( G \in \mathcal{G}^n \).
(2) If \( K \) is trichotomous, then, for all \( p \in S_2 \), \( h(p) = \{G\} \) for some \( G \in \mathcal{G}_{ns}^n \) and \( h(p^*) = (h(p))^* \).
To see that this is possible, we have to consider two cases:

Case 1. \( K \) is symmetric.
In this case, the number of allowed outputs for \( h \) is
\[ |\mathcal{G}_s^n| + \frac{1}{2}|\mathcal{G}_{ns}^n| = 2^{\binom{n+1}{2}} + \frac{1}{2} \left(2^{n^2} - 2^{\binom{n+1}{2}}\right) = 2^{n^2-1} + 2^{\binom{n+1}{2}} - 1 \geq 2^{n^2-1}. \]

Case 2. \( K \) is trichotomous.
In this case, the number of allowed outputs for \( h \) is
\[ |\mathcal{G}_{ns}^n| = 2^{\binom{n+1}{2}} \left(2^{\binom{n}{2}} - 1\right) \geq 2^{n^2-1}. \]
In either case, the number of allowed outputs for \( h \) is at least
\[ 2^{n^2-1} = 2^m \geq |S_2|. \]
Therefore, such a function \( h \) exists.
For each \( G = (X_0 \cup X_1, F) \in \mathcal{G}^n \), let
\[ \varphi_G(y_0, \ldots, y_0, y_1, \ldots, y_1) = \bigwedge_{i, j < n} R(y_0, i, y_1, j) \iff \{i, 0, j, 1\} \in F. \]
Finally, for each \( i < m \), let
\[ \varphi_i(\overline{y}_0, \overline{y}_1) = \bigvee_{p \in S_2, p(x_0, x_1) \vdash E_i(x_0, x_1)} \left( \bigvee_{G \in h(p)} \varphi_G(\overline{y}_0, \overline{y}_1) \right). \]
Let $I$ be the map that sends $E_i$ to $\varphi_i$ for all $i < m$. For any $A \in \mathbf{K}^m$, consider the set $n \times A$ and endow it with an $R$-graph structure as follows: For all distinct $a, b \in A$, choose some $G \in h(\text{ftp}_{L^0}(a, b))$ and copy the graph structure of $G$ by associating $X_0$ to $n \times \{a\}$ and $X_1$ to $n \times \{b\}$. This graph embeds into $\mathfrak{C}$; let $f : A \to C^n$ code this embedding. Then, it is clear that, for all $a, b \in A$ and $i < m$,

$$A \models E_i(a, b) \iff \models \varphi_i(f(a), f(b)).$$

Thus, this satisfies Lemma 3.12. So there exists a $\mathbf{K}^m$-configuration into $\mathfrak{F} = \mathfrak{F}$. Moreover, notice that this gives us a parameter-free configuration.

Proposition 4.24. Fix $n \geq 1$. Assume that $\mathbf{K}$ is self-similar and generic. Then,

$$\text{Rk}_\mathbf{K}(n) \leq n^2.$$ 

Proof. Fix $m \geq 1$ and let $\mathfrak{F}$ be an $n$-tuple of variables from $T$. Suppose that there exists a $\mathbf{K}^m$-configuration into $\mathfrak{F} = \mathfrak{F}$; call it $f : \Gamma \to C^n$. By Proposition 2.15, since $\mathbf{K}$ is self-similar, $\mathbf{K}^m$ is self-similar. Since $\mathbf{K}^m$ is self-similar, we may assume that $f$ satisfies the conclusion of Proposition 3.13. Thus, there exists $J \subseteq \mathfrak{F}$ such that

(1) for all $a, b \in \Gamma$, $\text{tp}_L(f(a)/C) = \text{tp}_L(f(b)/C)$,
(2) for all $j \in J$ and all $a, b \in \Gamma$, $f(a)_j = f(b)_j$.
(3) for all $p \in S_2$, there exist $a_p, b_p \in \Gamma$ such that $(a_p, b_p) \models p$ and, for all $i, j \in \mathfrak{F} \setminus J$, $f(a_p)_i \neq f(b_p)_j$.

Since $L$ is a binary language, condition (1) tells us that the type $\text{tp}_L(f(a_p), f(b_p))$ determines the type $\text{tp}_L(f(a_p), f(b_p)/C)$.

We get a function $h : S_2 \to \mathcal{G}^n$ as follows: Fix $p \in S_2$. For all $i, j < n$, we put $\{(i, 0), (j, 1)\}$ in the edge set of $h(p)$ if and only if $R(f(a_p)_i, f(b_p)_j)$. If we have $p, p' \in S_2$ distinct, conditions (1), (2), and (3) give us that $\text{tp}_L(f(a_p), f(b_p))$ and $\text{tp}_L(f(a_{p'}, f(b_{p'}))$ disagree on some formula of the form $R(y_{0,i}, y_{1,j})$. Therefore, $h(p) \neq h(p')$. Hence, $h$ is injective.

Since $\mathbf{K}$ is generic, by Proposition 2.23 $\mathbf{K}^m$ is generic. Thus, $|S_2| \geq 2^m$. On the other hand, $|\mathcal{G}^n| = 2^{n^2}$. Therefore, $2^m \leq 2^{n^2}$. Thus, $m \leq n^2$.

Proposition 4.25. Assume that $\mathbf{K}$ is either reflexive or irreflexive. Assume that $\mathbf{K}$ is self-similar and generic. Then,

(1) If $\mathbf{K}$ is trichotomous, then $\text{Rk}_\mathbf{K}(n) < n^2$ if $n \geq 1$.
(2) If $\mathbf{K}$ is symmetric, then $\text{Rk}_\mathbf{K}(n) < n^2$ if $n \geq 2$. 

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Proof. Fix \( m \geq 1 \) and \( n \geq 1 \). Let \( h : S_2 \to G^n \) be the injective function from the proof of Proposition 4.24 and consider the map \( p \mapsto (a_p, b_p) \) from that proof. We will show that \( h \) is not surjective.

**Case 1.** \( K \) is trichotomous.

If \( p \in S_2 \), then
\[
\text{tp}_L(f(a_p), f(b_p)) \neq \text{tp}_L(f(b_p), f(a_p)).
\]
Otherwise, \( \text{tp}_L(f(a_p), f(b_p)/C) = \text{tp}_L(f(b_p), f(a_p)/C) \), hence \( E_i(a_p, b_p) \) if and only if \( E_i(b_p, a_p) \) for all \( i < m \), which is a contradiction. Thus, \( h(p) \in G^n_{ns} \subseteq G^n \).

**Case 2.** \( K \) is symmetric and \( n \geq 2 \).

If \( p, p' \in S_2 \) are distinct, then
\[
\text{tp}_L(f(a_p), f(b_p)) \neq \text{tp}_L(f(b_p), f(a_p')).
\]
Otherwise, \( \text{tp}_L(f(a_p), f(b_p)/C) = \text{tp}_L(f(b_p'), f(a_p)/C) \), hence \( E_i(a_p, b_p) \) if and only if \( E_i(b_p', a_p) \) if and only if \( E_i(a_p', b_p) \) for all \( i < m \), which is a contradiction. Since \( n \geq 2 \), \( G^n_{ns} \neq \emptyset \). If \( p, p' \in S_2 \) and \( h(p) \in G^n_{ns} \), then \( h(p) \neq (h(p'))^* \).

In either case, we see that \( h \) is not surjective. Therefore,
\[
2^m \leq |S_2| < |G^n| = 2^{n^2}.
\]
So \( m < n^2 \).

We apply Propositions 4.23, 4.24, and 4.25 to \( \text{LO}, \text{E}, \text{G}, \) and \( \text{T} \).

**Example 4.26 (K = \text{LO}).** For all \( n \geq 1 \),
\[
\text{Rk}_{\text{LO}}(n) = n^2 - 1.
\]

**Proof.** Since \( \text{LO} \) is irreflexive and trichotomous, Proposition 4.23 gives us that \( \text{Rk}_{\text{LO}}(n) \geq n^2 - 1 \). Moreover, \( \text{LO} \) is self-similar and generic, so Proposition 4.25 gives us that \( \text{Rk}_{\text{LO}}(n) < n^2 \).

**Example 4.27 (K = \text{T}).** For all \( n \geq 1 \),
\[
\text{Rk}_{\text{T}}(n) = n^2 - 1.
\]

**Proof.** Similar to Example 4.26.

In this paper, although the focus is not on \( \text{T} \)-rank, we do get this result “for free.” In future work, we will examine \( \text{T} \)-rank in other contexts.
Example 4.28 ($K = G$). For all $n \geq 1$,

$$Rk_G(n) = \begin{cases} 
1 & \text{if } n = 1, \\
 n^2 - 1 & \text{if } n \geq 2.
\end{cases}$$

**Proof.** Since $G$ is irreflexive and symmetric, Proposition [4.23] gives us that $Rk_G(n) \geq n^2 - 1$. Moreover, $G$ is self-similar and generic, so Proposition [4.25] gives us that $Rk_G(n) < n^2$ for $n \geq 2$. To see that $Rk_G(1) = 1$, use Proposition [4.24] and Lemma [3.3].

We now turn our attention to when $K = E$. This will take more work because $E$ is not self-similar.

For $t \in m^2$ and $a, b \in \omega^m$, we say that $a \leq t b$ if, for all $i < m$,
- $a_i < b_i$ if $t(i) = 1$ and
- $a_i = b_i$ if $t(i) = 0$.

Observe that, for any $a, b \in \omega^m$, there exists at most one $t \in m^2$ such that $a \leq t b$.

For positive integers $m$ and $n$, consider the following property:

$(\dagger)_{m,n}$: There exists a function $f : \omega^m \to \mathcal{C}^n$ such that, for all $t, t' \in m^2$, for all $a, b, a', b' \in \omega^m$ with $a \leq t b$ and $a' \leq t' b'$,

$$tp_L(f(a), f(b)) = tp_L(f(a'), f(b'))$$

iff.

In particular, when $t = t'$, $f(a) = f(b)$ if and only if $f(a) = f(b)$ for all $\ell < n$. Moreover, by choosing the constantly zero function for $t$, we see that the function $a \mapsto tp_L(f(a))$ is constant. If we focus on $2^m$, we see that, for any $a, b \in 2^m$,

$$tp_L(f(0), f(a)) = tp_L(f(0), f(b))$$

iff. $a = b$.

This relates to $E^{**m}$-configurations in the following manner:

**Lemma 4.29.** If there exists an $E^{**m}$-configuration into $\mathcal{C}^n$, then $(\dagger)_{m,n}$ holds.

**Proof.** As in Remark [2.13], we can take $\omega^{m+1}$ to be the universe of $\Gamma$, the Fraïssé limit of $E^{**m}$. Let $g : \omega^{m+1} \to \mathcal{C}^n$ be an $E^{**m}$-configuration over $C$ for some finite $C \subseteq \mathcal{C}$. Thus, there exist $L(C)$-formulas $\varphi_i(\overline{y}_0, \overline{y}_1)$ for $i < m$ such that, for all $a, b \in \omega^{m+1}$ and all $i < m$,

$$\mathcal{C} \models \varphi_i(g(\overline{a}), g(\overline{b}))$$

iff. $a_i = b_i$.

Let $c : (\omega^m)_{\leq 2} \to S_{2n}(C)$ be the coloring given by, for all $a, b \in \omega^m$ with $a \leq \text{lex} b$,

$$c(\{a, b\}) = tp_L(g(\overline{a}, 0), g(\overline{b}, 0)/C).$$
Fix $k < \omega$. By Lemma 4.2 there exist $Y_0, \ldots, Y_{m-1} \in (\omega)^k$ such that, for all $t \in \omega^2$, $c$ is constant on

$$X_t = \left\{ \langle \alpha, \beta \rangle : \alpha, \beta \in \prod_{i < m} Y_i, \alpha \leq \beta \right\}.$$ 

Since $k$ was arbitrary, by compactness, there exists $f : \omega^m \to \mathbb{C}^n$ such that, for all $\alpha, \beta \in \omega^m$ and $i < m$, $c \models \varphi_i(f(\alpha), f(\beta))$ if and only if $a_i = b_i$ and, for all $t \in \omega^2$, the function $(\alpha, \beta) \mapsto \text{tp}_L(f(\alpha), f(\beta)/C)$ is constant for all $\alpha, \beta \in \omega^m$ with $\alpha \leq \beta$. In particular, the function $\alpha \mapsto \text{tp}_L(f(\alpha)/C)$ is constant. Since $L$ is a binary language, the type $\text{tp}_L(f(\alpha), f(\beta)/C)$ is determined by the type $\text{tp}_L(f(\alpha), f(\beta))$. Therefore, $f$ witnesses that $(\dagger)_{m,n}$ holds. \hfill \Box

Suppose $(\dagger)_{m,n}$ holds, witnessed by $f$. Let $\alpha_i$ the $i$th standard basis vector. For each $\ell < n$, let

$$V_\ell = \{ \alpha \in \omega^m : [f(\alpha)]_\ell = 0 \}.$$

**Lemma 4.30.** There exists $I_\ell \subseteq m$ such that

$$V_\ell = \{ \alpha \in \omega^m : (\forall i \in m \setminus I_\ell)[a_i = 0] \}$$

In other words, $V_\ell$ is the $\omega$-span of $\{ \alpha_i : i \in I_\ell \}$.

**Proof.** Let $I_\ell = \{ i < m : (\exists \alpha \in V_\ell)[a_i > 0] \}$. We show this works.

Clearly $0 \in V_\ell$. Fix $\alpha \in V_\ell$ non-zero and $i < m$ such that $a_i > 0$. Let $\alpha' \in \omega^m$ be given by

$$a_j' = \begin{cases} a_j & \text{if } j \neq i, \\ a_j + 1 & \text{if } a_j = a_i. \end{cases}$$

By $(\dagger)_{m,n}$, since $f(\alpha)_\ell = f(\alpha')_\ell$, $f(\alpha)_\ell = f(\alpha')_\ell$. Thus, $f(\alpha)_\ell = f(\alpha')_\ell$.

By $(\dagger)_{m,n}$, $f(\alpha')_\ell = f(\alpha')_\ell$. Thus, $\alpha \in V_\ell$.

Suppose that $\alpha, \beta \in V_\ell$. Then,

$$f(\alpha)_\ell = f(\beta)_\ell$$

Thus, $\alpha + \beta \in V_\ell$.

Putting these facts together, we get the desired conclusion. \hfill \Box

**Lemma 4.31.** Suppose $(\dagger)_{m,n}$ holds, witnessed by $f$. For all $\ell, \ell' < n$ and $\alpha \in \omega^m$, if $f(\alpha)_\ell = f(\alpha')_{\ell'}$, then $\alpha \in V_\ell \cap V_{\ell'}$.

**Proof.** By $(\dagger)_{m,n}$, $f(\alpha) = f(2\alpha)$, hence $f(\alpha') = f(2\alpha')$. By $(\dagger)_{m,n}$, $f(\alpha') = f(\alpha')$, hence $\alpha \in V_{\ell'}$. By $(\dagger)_{m,n}$, $f(\alpha) = f(2\alpha)$, hence $f(\alpha') = f(\alpha')$. Thus, $\alpha \in V_\ell$. \hfill \Box
Example 4.32 (K = E). For all $n \geq 1$,
\[
R^E_k(n) = \begin{cases} 
1 & \text{if } n = 1, \\
n^2 - 1 & \text{if } n \geq 2
\end{cases}
\]

Proof. Since $E$ is reflexive and symmetric, Proposition 4.23 says that $R^E_k(n) \geq n^2 - 1$. Moreover, Proposition 4.15 says that $R^E_k(1) \geq 1$.

Towards a contradiction, suppose $R^E_k(1) \geq 2$; hence, $(†)_{2,1}$ holds, say witnessed by $f : \omega^2 \to C$. By Lemma 4.30, $V_0 = \{0\}$, $V_0 = \omega \times \{0\}$, or $V_0 = \omega^2$. One can check that, in any of these cases, there exist distinct $\overline{a}, \overline{b} \in 2^2$ such that $tp_L(f(\overline{0}), f(\overline{a})) = tp_L(f(\overline{b}), f(\overline{0}))$, a contradiction.

So it suffices to show that $R^E_k(n) < n^2$ when $n \geq 2$. To accomplish this, we prove, by induction on $n$, that $(†)$ fails.

Fix $n \geq 3$ and assume that $(†)_{n-1}$ fails. Towards a contradiction, suppose that $(†)_{n}$ holds, say witnessed by $f : \omega^2 \to C^n$.

Claim. For all $\ell < n$, $|I_\ell| < (n-1)^2$ (where $I_\ell$ is as defined in Lemma 4.30).

Proof of Claim. Fix $\ell < n$. Towards a contradiction, suppose $|I_\ell| \geq (n-1)^2$. Let $m = (n-1)^2$, let $\sigma : m \to I_\ell$ be any injective function, and, for each $\overline{a} \in \omega^m$, let $\overline{a}_\sigma \in \omega^{n^2}$ be given by
\[
a_{\sigma,i} = \begin{cases} 
a_j & \text{if } i = \sigma(j), \\
0 & \text{if } i \notin \text{im}(\sigma)
\end{cases}
\]

In particular, $\overline{a}_\sigma \in V_\ell$. Hence, for all $\overline{a}, \overline{b} \in \omega^m$, $f(\overline{a}_\sigma) = f(\overline{b}_\sigma)$. Define $f' : \omega^m \to C^{n-1}$ as follows: For each $\overline{a} \in \omega^m$, let $f'(\overline{a}) = f(\overline{a}_\sigma)$ restricted to exclude the $\ell$th coordinate. It is easy to check that $f'$ satisfies $(†)_{m,n-1}$, contrary to the inductive hypothesis. \qed

Let $m = n^2$ and let
\[
V = \{\overline{a} \in 2^n : (\exists \ell < n)(\forall i \in m \setminus I_\ell)[a_i = 0]\}.
\]

In other words, $V$ is the union of $2^n \cap V_\ell$ over all $\ell < n$. By the claim, for each $\ell < n$, $|I_\ell| \leq (n-1)^2 - 1 = n^2 - 2n$. Thus,
\[
|2^n \cap V_\ell| \leq n^2 - 2n.
\]

Therefore,
\[
|2^n \setminus V| \geq 2^n - n^2 - 2n.
\]

Claim. $2^n - n^2 - 2n > 2^{n^2 - 1} + 2^{(n+1)^2 - 1}$.
Proof of Claim. Since \( n \geq 3 \), \((n+1)(n-1) > \frac{1}{2}n(n+1)\). Thus, \( n^2 - 2 > \binom{n+1}{2} - 1 \). So
\[
2^{n^2-2} > 2^{\binom{n+1}{2}-1}.
\]
Similarly, \((n + 1)(n - 1) > n(n - 1)\). Thus, \( n^2 - 2 > n^2 - n - 1 \), so
\[
2^{n^2-2} > 2^{n-1}2^{n^2-2n}.
\]
Since \( n \geq 3 \), \( n < 2^{n-1} \). Therefore,
\[
2^{n^2-2} > n2^{n^2-2n}.
\]
Putting these together, we get
\[
2^{n^2-1} > 2^{\binom{n+1}{2}-1} + n2^{n^2-2n}.
\]
This gives us the desired conclusion. \(\square\)

Therefore, by (2),
\[
\left| \{ \text{tp}_L(f(0), f(\overline{a})) : \overline{a} \in 2^m \setminus V \} \right| = |2^m \setminus V| > |G^n_s| + \frac{1}{2}|G^n_{ns}|.
\]
However, for each \( \overline{a} \in 2^m \setminus V \) and all \( \ell, \ell' < n \), \( f(\overline{0})_\ell \neq f(\overline{a})_{\ell'} \). Hence, each type \( \text{tp}_L(f(0), f(\overline{a})) \) corresponds to a unique element of \( G^n \) as in the proof of Proposition 4.24. Since \( E^m \) is symmetric, as in the proof of Proposition 4.25, we conclude that there are at most \( |G^n_s| + \frac{1}{2}|G^n_{ns}| \) such types. This is a contradiction.

For the base case, towards a contradiction, suppose that (†)4.2 holds. This argument follows similarly to the general inductive argument. Notice that, for all \( \ell < 2 \), \( |I_\ell| \leq 1 \). Thus, \( |2^4 \setminus V| \geq 2^4 - 3 = 13 \). On the other hand,
\[
|G^2| + \frac{1}{2}|G^2_{ns}| = 12.
\]
\(\square\)

5. Dividing Lines

In this section, we connect the notions discussed in the previous sections of this paper with the ideas considered in [6]. However, in this paper, we are ignoring the considerations of irreducibility or indecomposibility for simplicity of presentation.

Definition 5.1. Let \( K \) be an algebraically trivial Fraïssé class. Let \( \mathcal{C}_K \) be the class of all theories \( T \) such that there exist a partial type \( \pi \) and a \( K \)-configuration into \( \pi \) (in this case, we will say that \( T \) admits a \( K \)-configuration).
In other words, \( C_K \) is the class of all theories \( T \) where \( \text{Rk}_K(n) > 0 \) for some \( n \geq 1 \).

How do these classes relate to known dividing lines in model theory? First of all, \( C_S \) is clearly the class of all theories. Moreover, \( T \in C_E \) if and only if \( T \) has infinite models. What about more interesting \( K \)? The following theorem describes the relationship to the classes of theories that are stable, NIP, and \( k \)-dependent.

**Theorem 5.2** (Proposition 4.31 of [6], Proposition 5.2 of [4]). Let \( T \) be a complete first-order theory.

1. \( T \) is stable if and only if \( T \notin C_{LO} \).
2. \( T \) has NIP if and only if \( T \notin C_G \).
3. For all \( k \geq 2 \), \( T \) has \((k−1)\)-dependence if and only if \( T \notin C_{H_k} \).

**Proof.** If \( \varphi(\overline{y}; \overline{z}) \) is a witness to the order property, then the map sending \( < \) to \( \varphi^*(\overline{y}_0; \overline{z}_0; \overline{y}_1; \overline{z}_1) = \varphi(\overline{y}_0; \overline{z}_1) \) witnesses, via Lemma 3.2, that there exists an \( LO \)-configuration into \( C_{|y| + |z|} \). Similar arguments can be made for the \((k−1)\)-independence property and \( H_k \)-configurations for \( k \geq 2 \). □

**Definition 5.3.** Given two algebraically trivial Fraïssé classes \( K_0 \) and \( K_1 \), we say that

\[ K_0 \preceq K_1 \]

if the theory of the Fraïssé limit of \( K_1 \) is in \( C_{K_0} \). We say

\[ K_0 \sim K_1 \]

if \( K_0 \preceq K_1 \) and \( K_1 \preceq K_0 \).

**Proposition 5.4.** Fix algebraically trivial Fraïssé classes \( K_0, K_1, \) and \( K_2 \).

1. \( \preceq \) is a quasi-order on algebraically trivial Fraïssé classes.
2. \( K_0 \preceq K_0 * K_1 \).
3. If \( K_0 \preceq K_2 \) and \( K_1 \preceq K_2 \), then \( K_0 * K_1 \preceq K_2 \).
4. \( K_0 \preceq K_1 \) if and only if \( C_{K_1} \subseteq C_{K_0} \).
5. \( K_0 \sim K_1 \) if and only if \( C_{K_0} = C_{K_1} \).

**Proof.** For each \( i \) \( < \) 3, let \( L_i \) be the language of \( K_i \) and let \( T_i \) be the theory of the Fraïssé limit of \( K_i \).

1. By Lemma 3.3, \( T_0 \) admits a \( K_0 \)-configuration. Hence, \( K_0 \preceq K_0 \). So \( \preceq \) is reflexive.

Assume that \( K_0 \preceq K_1 \) and \( K_1 \preceq K_2 \). Then, \( T_1 \) admits a \( K_0 \)-configuration and \( T_2 \) admits a \( K_1 \)-configuration. By Lemma 3.11, \( T_1 \)
admits a parameter-free $K_0$-configuration. By Proposition 3.10, $T_2$ admits a $K_0$-configuration. Thus, $K_0 \preceq K_2$. So $\preceq$ is transitive.

(2): By (1), $T_0 \star T_1$ admits a $(K_0 \star K_1)$-configuration. However, $K_0$ is a reductive subclass of $K_0 \star K_1$. By Lemma 3.7, $T_0 \star T_1$ admits a $K_0$-configuration.

(3): Assume $K_0 \preceq K_2$ and $K_1 \preceq K_2$. Thus, $T_2$ admits a $K_0$-configuration and a $K_1$-configuration. By Proposition 3.8, $T_2$ admits a $(K_0 \star K_1)$-configuration. Therefore, $K_0 \star K_1 \preceq K_2$.

(4), $(\Rightarrow)$: Assume $K_0 \preceq K_1$ and $T \in C_{K_1}$. So $T_1$ admits a $K_0$-configuration. By Lemma 3.11, $T_1$ admits a parameter-free $K_0$-configuration. By Proposition 3.10, $T_1$ admits a $K_0$-configuration. Thus, $T \in C_{K_0}$.

(4), $(\Leftarrow)$: Assume $C_{K_1} \subseteq C_{K_0}$. By (1), $T_1$ is in $C_{K_1}$. Therefore, it is in $C_{K_0}$. Therefore, $K_0 \preceq K_1$.

(5): Follows immediately from (4).

From this, we get a characterization of when a free superposition of two classes is equivalent to one of the classes. This corollary is a generalization of a few results from [6].

**Corollary 5.5.** Suppose $K_0$ and $K_1$ are algebraically trivial Fraïssé classes. Then, $K_0 \preceq K_1$ if and only if $K_0 \star K_1 \sim K_1$.

**Proof.** $(\Rightarrow)$: By Proposition 5.4 (2), $K_1 \preceq K_0 \star K_1$. By Proposition 5.4 (1), $K_1 \preceq K_1$. By Proposition 5.4 (3), $K_0 \star K_1 \preceq K_1$. Thus, $K_0 \star K_1 \sim K_1$.

$(\Leftarrow)$: By Proposition 5.4 (2), $K_0 \preceq K_0 \star K_1$. By Proposition 5.4 (1), $K_0 \preceq K_1$. □

**Corollary 5.6.** If $K$ is an algebraically trivial Fraïssé class and $n \geq 1$, then $K^{\star n} \sim K$.

**Proof.** Follows immediately from Corollary 5.5. □

Therefore, if $T \in C_K$, then $T$ has types with arbitrarily large $K$-rank. Moreover, if $T \notin C_K$, then all types have $K$-rank 0.

It turns out that Corollary 3.10 and Theorem 4.24 of [6] are straightforward consequences of Corollary 5.5 above.

**Corollary 5.7.** (Corollary 3.10 of [6]) Let $K$ be an algebraically trivial Fraïssé class and $T_0$ the theory of the Fraïssé limit of $K$. Then, $T_0$ is unstable if and only if $K \star \text{LO} \sim K$.

**Proof.** By Theorem 5.2 (1), $T_0$ is unstable if and only if $T_0 \in C_{\text{LO}}$, which holds if and only if $\text{LO} \preceq K$. By Corollary 5.5, $\text{LO} \preceq K$ if and only if $K \star \text{LO} \sim K$. □
Corollary 5.8. (Theorem 4.24 of [6]) Let $L_0$ be a finite relational language where each relation symbol is at least binary. Let $H_{L_0}$ be the class of all $L_0$-hypergraphs (see Example 2.11). Let $k$ be the largest arity among relation symbols in $L_0$. Then,

$$H_{L_0} \sim H_k.$$

Proof. Let $k_0 \leq \ldots \leq k_{n-1} = k$ list off all arities (with repetition) of the relation symbols in $L_0$. Then,

$$H_{L_0} = H_{k_0} \star \ldots \star H_{k_{n-1}}.$$

For each $k \leq \ell$, $H_k \preceq H_{\ell}$ (see, for example, Lemma 4.10 of [6]). By Corollary 5.5

$$H_{k_0} \star \ldots \star H_{k_{n-1}} \sim H_{k_{n-1}}.$$

Therefore, $H_{L_0} \sim H_k$. \qed

For more discussion about the class of theories that code $K$, see [6]. In particular, that paper establishes a quasi-order on theories, uses this quasi-order to define classes of theories, and shows that these classes are exactly those of the form $C_K$ for some indecomposable algebraically trivial Fraïssé class, $K$ (see Theorem 2.17 of [6]).

6. Future Work

Under what conditions is $K$-rank a generalization of a known rank in model theory? In Example 4.4, we establish that $LO$-rank coincides with op-dimension when $T$ has NIP, which implies that $LO$-rank coincides with dp-rank when $T$ is distal. On the other hand, $LO$-rank diverges from op-dimension when $T$ has the independence property. Similarly, $E$-rank appears to be related to dp-rank, but this relationship remains unclear. Proposition 4.14 establishes that dp-rank is an upper bound for $E$-rank while Corollary 4.16 shows that these ranks coincide on $C^n$ when $T$ is dp-minimal. On the other hand, even in NIP theories, dp-rank and $E$-rank diverge, as shown in Example 4.18. This example is distal, however, which leads to an interesting question: Do dp-rank and $E$-rank coincide for stable theories?

Along a similar line, when is $K$-rank additive (Open Question 4.4)? We see that $LO$-rank, and even $G$-rank (trivially), are additive when $T$ has NIP. On the other hand, these ranks fail additivity when moving to theories with the independence property. Is it possible that, more generally, $K$-ranks are additive on NIP theories? In particular, is $E$-rank additive on NIP theories?
Although we examined a few examples of algebraically trivial Fraïssé classes in this paper, there are other classes that are currently unexplored. We have one result on $T$-rank, Example 4.27, and no results on $H_k$-rank for $k > 2$. It is possible, for example, that $T$-rank coincides with $\textup{LO}$-rank for many (if not all) types. Moreover, most of the technology developed in this paper relies on the index language being binary, which makes analyzing $H_k$-rank more challenging when $k > 2$.

In future work, we would like to examine $K$-rank for these and other classes, $K$.

Finally, Section 5 and the relationships to [6] reveal other interesting open questions. For example, we have the strict $\lesssim$-chain

$$S < E < \text{LO} < G < H_3 < H_4 < \ldots.$$ 

Is $\lesssim$ a linear quasi-order in general? Is there anything between $E$ and $\text{LO}$? In other words, is there a dividing line (in the sense of $C_K$) between “theories with only finite models” and “stable theories”?

**Appendix A. Combinatorial Lemmas**

**Lemma A.1.** For all $k, \ell, m < \omega$, there exists $n < \omega$ such that, for all colorings $c : n^k \to \ell$, there exist $Y_0, \ldots, Y_{k-1} \in \binom{n}{m}$ such that $c$ is constant on $\prod_{i<k} Y_i$.

**Proof.** Since any coloring $c : n^k \to \ell$ can be extended arbitrarily to a coloring $c : \binom{n^k}{\leq 2} \to \ell$, this follows immediately from Lemma A.2. □

Fix $k < \omega$ and let

$$D_k = \{ t \in \mathbb{Z}^{-1,0,1} : t(i) = 1 \text{ for } i \text{ minimal such that } t(i) \neq 0 \}.$$ 

For $\overline{a}, \overline{b} \in \omega^k$ and $t \in D_k$, define

$$\overline{a} \leq_t \overline{b} \text{ if, for all } i < k, \begin{cases} a_i < b_i & \text{if } t(i) = 1, \\ a_i = b_i & \text{if } t(i) = 0, \\ a_i > b_i & \text{if } t(i) = -1. \end{cases}$$

Finally, for all $\overline{a}, \overline{b} \in \omega^k$, define

$$\overline{a} \leq_{\text{lex}} \overline{b} \text{ if } a_i < b_i \text{ for } i \text{ minimal such that } a_i \neq b_i.$$ 

Note that $\overline{a} \leq_{\text{lex}} \overline{b}$ if and only if there exists $t \in D_k$ such that $\overline{a} \leq_t \overline{b}$.

**Lemma A.2.** For all $k, \ell, m < \omega$, there exists $n < \omega$ such that, for all colorings $c : \binom{n^k}{\leq 2} \to \ell$, there exist $Y_0, \ldots, Y_{k-1} \in \binom{n}{m}$ such that, for all
$t \in D_k$, $c$ is constant on the set

$$X_t = \left\{ (\overline{a}, \overline{b}) : \overline{a}, \overline{b} \in \prod_{i<k} Y_i, \overline{a} \leq_t \overline{b} \right\}.$$ 

**Proof.** By induction on $k$. Let $k = 1$ and fix $\ell, m < \omega$. By Ramsey’s Theorem, there exists $n$ such that, for all colorings $c : \binom{n}{\leq 2} \to \ell$, there exists $Y \in \binom{n}{m}$ such that $c$ is constant on $\binom{Y}{1}$ and $c$ is constant on $\binom{Y}{2}$.

Since $X_0 = \binom{Y}{1}$ and $X_1 = \binom{Y}{2}$, this is the desired conclusion.

Fix $k, m, \ell < \omega$. Let

$$\ell' = D_k \times \{-1, 1\} \ell.$$ 

By Ramsey’s Theorem, there exists $n' < \omega$ such that, for all colorings $c' : \binom{n'}{\leq 2} \to \ell'$, there exists $Y_k \in \binom{n'}{m}$ such that $c'$ is constant on $\binom{Y_k}{1}$ and $c'$ is constant on $\binom{Y_k}{2}$. Let

$$\ell'' = (n')^2 \ell.$$ 

By the inductive hypothesis, there exists $n'' < \omega$ such that, for all colorings $c'' : \binom{n''}{\leq 2} \to \ell''$, there exist $Y_0, ..., Y_{k-1} \in \binom{n''}{m}$ such that, for all $t \in D_k$, $c''$ is constant on $X_t$. Let $n = \max\{n', n''\}$.

Fix a coloring $c : \binom{n^{k+1}}{\leq 2} \to \ell$. This induces a coloring $c'' : \binom{n''}{\leq 2} \to \ell''$ given by: for each $\overline{a}, \overline{b} \in (n'')^k$ with $\overline{a} \leq_{\text{lex}} \overline{b}$, for each $i, j \in n'$, let

$$c''(\{\overline{a}, \overline{b}\})(i, j) = c(\{\overline{a} - i, \overline{b} - j\}).$$ 

Thus, there exist $Y_0, ..., Y_{k-1} \in \binom{n''}{m}$ such that, for all $t \in D_k$, $c''$ is constant on $X_t$. Now define $c' : \binom{n'}{\leq 2} \to \ell'$ as follows: for each $i \leq j < n'$, $t \in D_k$, and $s \in \{-1, 1\}$, choose $\overline{a}, \overline{b} \in \prod_{i<k} Y_i$ with $\overline{a} \leq_t \overline{b}$ and set

$$c'(\{i, j\})(t, s) = \begin{cases} c(\{\overline{a} - i, \overline{b} - j\}) & \text{if } s = 1, \\ c(\{\overline{a} - j, \overline{b} - i\}) & \text{if } s = -1. \end{cases}$$ 

Since $c'$ is constant on $X_t$ for each $t$, this function is independent of the choice of $\overline{a}$ and $\overline{b}$. Thus, there exists $Y_k \in \binom{n'}{m}$ such that $c'$ is constant on $\binom{Y_k}{1}$ and $c'$ is constant on $\binom{Y_k}{2}$. We claim that $Y_0, ..., Y_k$ works for $c$.

Fix $t \in D_{k+1}$. If $t(k) = 0$, let

$$r = c'(\{i\})(t|_k, 1)$$ 

for any choice of $i \in Y_k$. Since $c'$ is constant on $\binom{Y_k}{1}$, this is independent of the choice of $i$. If $t(k) \neq 0$, let

$$r = c'(\{i, j\})(t|_k, t(k))$$

with $s = 1$.
for any choice of $i, j \in Y_k$ with $i < j$. Since $c'$ is constant on $\binom{Y_k}{2}$, this is independent of the choice of $i$ and $j$. Then, for any $\bar{a}, \bar{b} \in \prod_{i \leq k} Y_i$ such that $\bar{a} \leq \bar{b}$, we have that
\[ c(\{\bar{a}, \bar{b}\}) = r. \]
This is what we wanted to prove. □

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Towson University

E-mail address: vguingona@towson.edu
URL: https://tigerweb.towson.edu/vguingona/
E-mail address: mparnes@towson.edu

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