Evaluation of the Magnetic Fields and Mutual Inductance between Circular Coils Arbitrarily Positioned in Space

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Abstract. This paper presents the evaluation of the magnetic fields and mutual inductance between circular coils arbitrarily positioned in space. Firstly, based on an advanced and relevant model available in the literature, MATLAB code is implemented to evaluate the mutual inductance between circular coils arbitrarily positioned with respect to each other. The computed results are compared with the numerical results previously published in the literature and a detailed clarification regarding the huge computational errors made are presented. In the second part, a complex and relevant model available in the literature for evaluating the magnetic fields due to a circular coil is presented. Based on the useful information, the model for computing the magnetic fields between two circular coils is formulated. The computed results are validated with experimental measurements. The comparison of the results shows that the developed model and the experimental measurements conducted are accurate and effective.

1. Introduction

Air-cored coils are widely used in various electromagnetic applications. They are preferred to the iron-cored type due to their design objectives of controllability and capability of a high power transfer [1]. Nonetheless, the measure of the magnetic fields and mutual inductance between them are of importance to electrical engineers and physicists because within certain limits of coil separation and misalignment, a minimal amount of power is guaranteed [2–5].

Several contributions have been made in the literature concerning the computation of the magnetic fields and mutual inductance between air-cored coils. In this view, mathematical modelling based on the application of Maxwell’s formula, Neumann’s formula, Biot-Savart law, Lorentz and magnetic vector potential have been presented in [2–9].

In the first part of this paper, MATLAB coding is implemented to compute the mutual inductance between two circular coils arbitrarily positioned with respect to each other. This computation is achieved based on the advanced and relevant model given in [4]. The computed results are presented graphically and numerically and then, compared with the ones presented in [4, 5]. Also, based on a more efficient and relevant model given in [7], the formula for computing the magnetic fields between two air-cored circular coils with and without misalignment is presented. Lastly, experimental measurements are used to validate the computed results.
2. Mutual Inductance Model

According to [4], the formula for calculating the mutual inductance between inclined circular filaments placed in any position (most general case see Fig. 1a) as a function of the: primary and secondary coils radii, \( R_P \) and \( R_S \), parameters \( a, b \) and \( c \) defining the centre of the secondary coil and coordinates \((x_C, y_C, z_C)\) defining the centre of the secondary coil is given by

\[
M = \frac{\mu_0 R_S}{\pi} \int_0^{2\pi} \frac{[p_1 \cos \varphi + p_2 \sin \varphi + p_3] \Psi(k)}{k \sqrt{V_0^2}} \, d\varphi
\]  

(1)

where

\[
\Psi(k) = (1 - k^2) K(k) - E(k), \quad A_0 = 1 + \alpha^2 + \beta^2 + \gamma^2 + \delta^2 + 2\alpha(p_4 \cos \varphi + p_5 \sin \varphi), \quad V_0^2 = \alpha^2[1 - \frac{b c}{2 L^2} \cos^2 \varphi + \frac{a^2}{4 L^2} \sin^2 \varphi + \frac{a b c}{4 L^2} \sin 2\varphi] + \beta^2 + \gamma^2 + \frac{2\alpha}{\ell L} \cos \varphi - \frac{2\alpha^2 \ell}{\ell L} \sin \varphi, \quad K(k) = \int_0^\frac{1}{\sqrt{1-k^2 \sin^2 \theta}} \, d\theta.
\]

and \( E(k) = \int_0^\frac{1}{\sqrt{1-k^2 \sin^2 \theta}} \, d\theta \).

where \( \mu_0 \) is the magnetic permeability of free space, \( k \) is a variable, parametr \( \Psi(k) \) is function of \( k, \alpha, \beta, \gamma, \ell, p_1, p_2, p_3, p_4, p_5, A_0 \) and \( V_0 \) are dimensionless parameters, \( \varphi \) is the angle of integration at any point of the secondary coil, \( K(k) \) and \( E(k) \) are the complete elliptic integral of the first and second kinds respectively.

Figure 1. (a) Filamentary circular coils with arbitrary misalignment. Also, the geometric configurations and common notation used in the examples studied in [4] correspond to the following cases: (b) lateral misalignment only (\( \theta = 0, \psi = 0 \), axes \( y - z \) and \( y' - z' \) are coplanar), (c) lateral and angular misalignment (\( \psi = 0 \), axes \( y - z \) and \( y' - z' \) are coplanar) and (d) arbitrary lateral and angular misalignment (no coplanar axes anymore).

The geometric configurations considered in the examples studied in [4] are shown in Figs. 1b to 1d. In order to have easier link with Grover’s formula, the geometric configurations are
restated in terms of a common notation namely:
\[ h = z_C, \quad d = \sqrt{x_C^2 + y_C^2}, \quad \rho = \sqrt{h^2 + d^2}, \]
\[ \cos \phi = \frac{h}{\rho} \]
and in all cases, the primary coil is located in plane \(XOY\), with its centre at the origin \(O(0, 0, 0)\). The horizontal and vertical distances between the centres of the coils are \(d\) and \(h\) respectively and \(\rho\) is the lateral misalignment.

The first part of this paper is to compute the mutual inductance between filamentary circular coils arbitrarily positioned with respect to each other. In this view, example 12 was studied in [4] and for such case, the following common notations are applied to Eq. (1):

- The centre of the secondary coil must be taken at point \(C(x_C = 0, y_C = d, z_C = h)\). where
  \[ d = x_2 \sin \theta, \quad h = x_1 - x_2 \sin \theta \]
  and \(\rho = \sqrt{h^2 + d^2}\). It is vital to note that \(x_1\) is the vertical distance when axis \(z''\) intersects with axis \(z\) from origin \(O\), \(x_2\) is the distance between the intersection and the centre of the secondary coil and \(\theta\) is the angular misalignment.
- The parameters defining the positioning of the secondary coil plane is that of a spherical Cartesian system of coordinates which is given by:
  \[ a = \sin \psi \sin \theta, \quad b = -\cos \psi \sin \theta \]
  and \(c = \cos \theta\). where \(\psi\) is the rotation angle around axis \(z'\), which gives the complete positioning of the secondary coil.

3. Presentation and Discussion of Results for Mutual Inductance Computation

The results obtained for the mutual inductance between filamentary circular coils with arbitrary lateral \(\rho\) and angular \(\theta\) misalignment are shown in Fig. 2 and Table 1. Based on the parameters given in example 12 (i.e., \(R_P = 16\) cm, \(R_S = 10\) cm, \(x_1 = 20\) cm and \(x_2 = 5\) cm), MATLAB software is used to achieve the coding and computation of Eq. (1). Although the authors of [4] studied only for the case with \(\theta = 60^\circ\) and \(\rho = 0.1803\) m, the authors of this paper also studied for the cases with \(\theta = 30^\circ\), \(\rho = 0.1587\) m and \(\theta = 45^\circ\), \(\rho = 0.1684\) m. The results obtained in this paper are graphically presented (see Fig. 1) and also numerically tabulated (see Table 1).

![Figure 2](image_url)

**Figure 2.** Results for mutual inductance between circular filaments with arbitrary misalignment.
In the case with $\theta = 60^\circ$ and $\rho = 0.1803$ m, Fig. 2 shows that the value of the mutual inductance increases from $13.6113 \, \text{nH}$ to $26.6433 \, \text{nH}$ when the variable rotation angle $\psi$ ranges from 0 to $\pi$ and decreases from $26.6433 \, \text{nH}$ to $13.6113 \, \text{nH}$ when $\psi$ ranges from $\pi$ to $2\pi$. This outcome is in agreement with the results obtained in [4] and [5] (see Table 1).

However, it is vital to note that the authors of [4] obtained very wrong values for the parameters which define the centre of the secondary coil (i.e., $a$, $b$ and $c$ see Table 1). With such outcome, it is never possible for the authors to obtain the correct mutual inductance values. Nonetheless, the correct values are obtained in this paper (see Table 1) based on the given common notations, which are $a = \sin \psi \sin \theta$, $b = -\cos \psi \sin \theta$ and $c = \cos \theta$.

### Table 1. Computed Values in Example 12, with Variable Rotation Angle $\psi$ and $\theta = 60^\circ$

| $\psi$ | $(a, b, c)$ [4] | $M \, [nH]$ [4] | $M \, [nH]$ [5] Eq. (179) | $(a, b, c)$: This paper | $M \, [nH]$: This paper |
|--------|-----------------|-----------------|-----------------|------------------|------------------|
| 0      | (0; 0.0866; 0.05) | 13.6113         | 13.6113         | (0; 0.0866; 0.05) | 13.6113          |
| $\pi/6$| (0.0433; 0.075; 0.05) | 14.4688         | 14.4688         | (0.433; 0.75; 0.5) | 14.4688          |
| $\pi/4$| (0.075; 0.043; 0.05) | 15.4877         | 15.4877         | (0.6124; 0.6124; 0.5) | 15.4877          |
| $\pi/3$| (0.06124; 0.06124; 0.05) | 16.8189         | 16.8189         | (0.75; 0.433; 0.5) | 16.8189          |
| $\pi/2$| (0.0866; 0.05; 0.05) | 20.0534         | 20.0534         | (0.866; 0.05; 0.5) | 20.0534          |
| $2\pi/3$| (0.075; 0.043; 0.05) | 23.3253         | 23.3253         | (0.75; 0.433; 0.5) | 23.3253          |
| $3\pi/4$| (0.06124; 0.06124; 0.05) | 24.6936         | 24.6936         | (0.6124; 0.6124; 0.5) | 24.6936          |
| $5\pi/6$| (0.0433; 0.075; 0.05) | 25.7493         | 25.7493         | (0.433; 0.433; 0.5) | 25.7493          |
| $\pi$  | (0; 0.0866; 0.05) | 26.6433         | 26.6433         | (0.866; 0.05; 0.5) | 26.6433          |
| $7\pi/6$| (0.075; 0.043; 0.05) | 25.7493         | 25.7493         | (0.75; 0.433; 0.5) | 25.7493          |
| $5\pi/4$| (0.06124; 0.06124; 0.05) | 24.6936         | 24.6936         | (0.6124; 0.6124; 0.5) | 24.6936          |
| $4\pi/3$| (0.075; 0.043; 0.05) | 23.3253         | 23.3253         | (0.75; 0.433; 0.5) | 23.3253          |
| $3\pi/2$| (0.0866; 0.05; 0.05) | 20.0534         | 20.0534         | (0.866; 0.05; 0.5) | 20.0534          |
| $5\pi/3$| (0.075; 0.043; 0.05) | 16.8189         | 16.8189         | (0.75; 0.433; 0.5) | 16.8189          |
| $11\pi/6$| (0.075; 0.043; 0.05) | 14.4688         | 14.4688         | (0.433; 0.75; 0.5) | 14.4688          |
| $2\pi$  | (0; 0.0866; 0.05) | 13.6113         | 13.6113         | (0; 0.0866; 0.05) | 13.6113          |

### 4. Formulation of the Magnetic Fields Model for two Filamentary Circular Coils

According to [7], the magnetic fields in an arbitrary point $E(x_S, y_S, z_S)$ produced by the primary coil of the radius $R_P$ carrying the current $I_P$ (see Fig. 1a) is given by

\[
\begin{align*}
B_x(x_S, y_S, z_S) &= -\frac{\mu_0 I_P z_S x_S k}{8\pi \sqrt{R_P(x_S^2 + y_S^2)^{3/4}}} L_0 \\
B_y(x_S, y_S, z_S) &= -\frac{\mu_0 I_P z_S y_S k}{8\pi \sqrt{R_P(x_S^2 + y_S^2)^{3/4}}} L_0 \\
B_z(x_S, y_S, z_S) &= -\frac{\mu_0 I_P k}{8\pi \sqrt{R_P(x_S^2 + y_S^2)^{3/4}}} S_0
\end{align*}
\]

where

\[x_S = x_C + R_S u_C \cos \varphi + R_S v_S \sin \varphi, \quad y_S = y_C + R_S u_C \cos \varphi + R_S v_S \sin \varphi, \quad z_S = z_C + R_S u_C \cos \varphi + R_S v_S \sin \varphi, \quad \bar{u} = [u_x, u_y, u_z] = [-\frac{\hat{b}}{\rho}, \frac{\hat{a}}{\rho}, 0], \quad \bar{v} = [v_x, v_y, v_z] = [-\frac{x}{\rho}, 0, 0], \quad L_0 = 2K - 2 - \frac{k^2}{1-k^2} E(k), \quad S_0 = 2\sqrt{x_S^2 + y_S^2} K(k) - 2\sqrt{x_S^2 + y_S^2} - (R_P + \sqrt{x_S^2 + y_S^2}) k^2 E(k), \quad k^2 = \frac{4R_P \sqrt{x_S^2 + y_S^2}}{(R_P + \sqrt{x_S^2 + y_S^2})^2 + z_S^2} \]

where

$x_S$, $y_S$ and $z_S$ are the parametric coordinates of the secondary coil, $\bar{u}$ is the unit vector lying in the secondary coil plane between points $C$ and $D$, $\bar{v}$ is the cross product of the unit vector of the axis $z'$ and $\bar{u}$, $S_0$ and $L_0$ are dimensionless parameters while $K(k)$ and $E(k)$ are the complete elliptic integrals of the first and second kinds respectively.
Thus, for two coils, the total magnetic fields is the sum of the magnetic fields from each of the coils, which is formulated by the authors of this paper as

\[
B_z = B_{zp} + B_{zS} = \left(\frac{\mu_0 \mu_r I_P N_P (z_P + \frac{l}{2}) x_P k_P}{8\pi \sqrt{R_P (x_P^2 + y_P^2)^{5/4}}} L_P\right) + \left(\frac{\mu_0 \mu_r I_S N_S (z_S - \frac{l}{2}) x_S k_S}{8\pi \sqrt{R_S (x_S^2 + y_S^2)^{3/4}}} L_S\right)
\]

\[
B_y = B_{yp} + B_{yS} = \left(\frac{\mu_0 \mu_r I_P N_P (z_P + \frac{l}{2}) y_P k_P}{8\pi \sqrt{R_P (x_P^2 + y_P^2)^{5/4}}} \right) + \left(\frac{\mu_0 \mu_r I_S N_S (z_S - \frac{l}{2}) y_S k_S}{8\pi \sqrt{R_S (x_S^2 + y_S^2)^{3/4}}} \right)
\]

\[
B_z = B_{zp} + B_{zS} = \left(\frac{\mu_0 \mu_r I_P N_P k_P}{8\pi \sqrt{R_P (x_P^2 + y_P^2)^{3/4}}} \right) + \left(\frac{\mu_0 \mu_r I_S N_S k_S}{8\pi \sqrt{R_S (x_S^2 + y_S^2)^{3/4}}} \right)
\]

where

\[
x_P = x_C + R_P u_x \cos \phi + R_P v_x \sin \phi, \quad x_S = x_C + R_S u_x \cos \varphi + R_S v_x \sin \varphi, \quad y_P = y_C + R_P u_y \cos \phi + R_P v_y \sin \phi, \quad y_S = y_C + R_S u_y \cos \varphi + R_S v_y \sin \varphi,
\]

\[
z_S = z_C + R_S u_z \cos \varphi + R_S v_z \sin \varphi, \quad L_P = 2K_P (k_P)^{-\frac{2-k^2}{1-k^2}} E_P (k_P), \quad L_S = 2K_S (k_S)^{-\frac{2-k^2}{1-k^2}} E_S (k_S),
\]

\[
S_P = 2\sqrt{x_P^2 + y_P^2} K_P (k_P) - \frac{2\sqrt{x_P^2 + y_P^2} - (R_P + \sqrt{x_P^2 + y_P^2})^2}{1-k_P^2} E_P (k_P), \quad S_S = 2\sqrt{x_S^2 + y_S^2} K_S (k_S) - \frac{2\sqrt{x_S^2 + y_S^2} - (R_S + \sqrt{x_S^2 + y_S^2})^2}{1-k_S^2} E_S (k_S), \quad k_P = \frac{4R_P \sqrt{x_P^2 + y_P^2}}{(R_P + \sqrt{x_P^2 + y_P^2})^2 + z_P^2}, \quad k_S = \frac{4R_S \sqrt{x_S^2 + y_S^2}}{(R_S + \sqrt{x_S^2 + y_S^2})^2 + z_S^2}
\]

5. Computed and Experimentally Validated Results for Magnetic Fields

The results obtained for the computation and experimental validation of magnetic fields between two filamentary circular coils without misalignment (i.e., \(\theta = 0\) and \(\rho = 0\)) is shown in Fig. 3. Figures 3a–3c are obtained by applying the following to Eq. (3):

- In the case without misalignment, the parameters determining the positioning of the secondary coil plane are defined as \(a = 0\), \(b = 0\) and \(c = 1\).
- The centre points for the two coils become \((x_C = 0, y_C = 0, z_C = h)\). where \(h\) is the varying distance along the symmetry axis of the coils (see Fig. 3d).
- The parameters used for the computation and experimental measurements are: \(\mu_r = 1\), \(\mu_0 = 4\pi \times 10^{-7} \frac{H}{m}\), \(R_P = R_S = 0.065 cm\), \(N_P = N_S = 95\), \(I_P = I_S = 1 A\).

The magnetic field results are obtained when the separation \(l\) between the two circular coils are \(l = 0.01 m\) (see Fig. 3a), \(l = 0.035 m\) (see Fig. 3b) and \(l = 0.065 m\) (see Fig. 3c). The measurement of the magnetic fields is conducted on the symmetry axis of the two coils by using Teslameter (see Fig. 3d). For each \(l\), Teslameter is operated to measure the magnetic fields from the mid-point between the two coils to the point outside the coil (i.e., \(z_C = h\), which ranges from \(-0.09 m\) to \(0.09 m\)).

Figure 3a shows that uniform magnetic fields is obtained when the separation \(l\) between the two circular coils is almost zero (i.e., \(z_P + \frac{l}{2}\) and \(z_S - \frac{l}{2}\) where \(l = 0.01 m\)). Figures 3b and 3c show that as \(l\) increases, the values of the magnetic fields reduces and a different waveform is obtained. Furthermore, Figs. 3a–3c show that the values and waveforms for \(B_x\) and \(B_y\) along the \(z\)–axis are symmetrical which is always true whenever the magnetic fields is computed on the symmetry axis of two circular coils.

The magnetic fields model formulated in Eq. (3) can be used for the cases with and without misalignment. However, the case without misalignment is considered in this section. The comparison of the results shows that the developed model and experimental measurements conducted are accurate and effective.
6. Conclusion
The evaluation of the magnetic fields and mutual inductance between circular coils arbitrarily positioned in space is presented in this paper. In the first part of the paper, MATLAB coding is implemented to compute the mutual inductance between two circular coils arbitrarily positioned with respect to each other. This computation is achieved based on the advanced and relevant model given in [4]. The computed results are presented graphically and numerically and then, compared with the ones presented in [4, 5]. Also, the formula for computing the magnetic fields between two air-cored filamentary circular coils with and without misalignment is presented and validation of the results shows that the developed model (see Eq. 3) and the experimental measurements conducted are accurate and effective.

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