TWO-POINT QUADRATURE RULES FOR RIEMANN–STIELTJES INTEGRALS WITH $L^p$–ERROR ESTIMATES

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Abstract. In this work, we construct a new general two-point quadrature rules for the Riemann–Stieltjes integral $\int_a^b f(t) \, du(t)$, where the integrand $f$ is assumed to be satisfied with the Hölder condition on $[a, b]$ and the integrator $u$ is of bounded variation on $[a, b]$. The dual formulas under the same assumption are proved. Some sharp error $L^p$–Error estimates for the proposed quadrature rules are also obtained.

1. Introduction

The number of proposed quadrature rules that provides approximation for the Riemann–Stieltjes integral ($\mathcal{RS}$–integral) $\int_a^b f(t) \, du(t)$ using derivatives or without using derivatives are very rare in comparison with the large number of methods available to approximate the classical Riemann integral $\int_a^b f(t) \, dt$.

The problem of introducing quadrature rules for $\mathcal{RS}$-integral $\int_a^b f \, dg$ was studied via theory of inequalities by many authors. Two famous real inequalities were used in this approach, which are the well known Ostrowski and Hermite-Hadamard inequalities and their modifications. For this purpose and in order to approximate the $\mathcal{RS}$-integral $\int_a^b f(t) \, du(t)$, a generalization of closed Newton-Cotes quadrature rules of $\mathcal{RS}$-integrals without using derivatives provides a simple and robust solution to a significant problem in the evaluation of certain applied probability models was presented by Tortorella in [32].

In 2000, Dragomir [16] introduced the Ostrowski’s approximation formula (which is of One-point type formula) as follows:

$$\int_a^b f(t) \, du(t) \approx f(x)[u(b)-u(a)] \quad \forall x \in [a, b].$$

Several error estimations for this approximation had been done in the works [15] and [16].

From different point of view, the authors of [17] (see also [11, 12]) considered the problem of approximating the Stieltjes integral $\int_a^b f(t) \, du(t)$ via the generalized trapezoid formula:

$$\int_a^b f(t) \, du(t) \approx [u(x)-u(a)]f(a) + [u(b)-u(x)]f(b).$$

Many authors have studied this quadrature rule under various assumptions of integrands and integrators. For full history of these two quadratures see [6] and the references therein.

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Another trapezoid type formula was considered in [20], which reads:

\[
\int_a^b f(t) \, du(t) \approx \frac{f(a) + f(b)}{2} [u(b) - u(a)] \quad \forall x \in [a, b].
\]

Some related results had been presented by the same author in [18] and [19]. For other connected results see [13] and [14].

In 2008, Mercer [27] introduced the following trapezoid type formula for the \( RS \)-integral

\[
\int_a^b f dg \approx [G - g(a)] f(a) + [g(b) - G] f(b),
\]

where \( G = \frac{1}{b-a} \int_a^b g(t) \, dt \).

Recently, Alomari and Dragomir [4], proved several new error bounds for the Mercer–Trapezoid quadrature rule (1.1) for the \( RS \)-integral under various assumptions involved the integrand \( f \) and the integrator \( g \).

Follows Mercer approach in [27], Alomari and Dragomir [10] introduced the following three-point quadrature formula:

\[
\int_a^b f(t) \, dg(t) \approx [G(a, x) - g(a)] f(a) + [G(x, b) - G(a, x)] f(x) + [g(b) - G(x, b)] f(b)
\]

for all \( a < x < b \), where \( G(\alpha, \beta) := \frac{1}{\beta - \alpha} \int_\alpha^\beta g(t) \, dt \).

Several error estimations of Mercer’s type quadrature rules for \( RS \)-integral under various assumptions about the function involved have been considered in [4] and [7].

Motivated by Guessab-Schmeisser inequality (see [22]) which is of Ostrowski’s type, Alomari in [5] and [9] presented the following approximation formula for \( RS \)-integrals:

\[
\int_a^b f(t) \, du(t) \approx \left[ u\left(\frac{a+b}{2}\right) - u(a) \right] f(x) + \left[ u(b) - u\left(\frac{a+b}{2}\right) \right] f(a+b-x),
\]

for all \( x \in [a, \frac{a+b}{2}] \). For other related results see [6]. For different approaches variant quadrature formulae the reader may refer to [1], [8], [21] and [28].

Among others the \( L^\infty \)-norm gives the highest possible degree of precision; so that it is recommended to be ‘almost’ the norm of choice. However, in some cases we cannot access the \( L^\infty \)-norm, so that \( L^p \)-norm (\( 1 \leq p < \infty \)) is considered to be a variant norm in error estimations.

In this work, several \( L^p \)-error estimates (\( 1 \leq p < \infty \)) of general two and three points quadrature rules for Riemann–Stieltjes integrals are presented. The presented proofs depend on new triangle type inequalities for \( RS \)-integrals.

Let \( f \) be defined on \([a, b]\). If \( P := \{x_0, x_1, \ldots, x_n\} \) is a partition of \([a, b]\), write

\[
\Delta f_i = f(x_i) - f(x_{i-1}),
\]

for \( i = 1, 2, \ldots, n \). A function \( f \) is said to be of bounded \( p \)-variation if there exists a positive number \( M \) such that

\[
\left( \sum_{i=1}^n |\Delta f_i|^p \right)^{\frac{1}{p}} \leq M, \quad (1 \leq p < \infty)
\]

for all partition of \([a, b]\), (see [26]).
Let \( f \) be of bounded \( p \)-variation on \([a, b]\), and let \( \sum(P) \) denote the sum \( \left( \sum_{i=1}^{n} |\Delta f_i|^p \right)^{\frac{1}{p}} \) corresponding to the partition \( P \) of \([a, b]\). The number

\[
\int_a^b (f; p) = \sup \left\{ \sum(P) : P \in \mathcal{P}([a, b]) \right\}, \quad 1 \leq p < \infty
\]

is called the total \( p \)-variation of \( f \) on the interval \([a, b]\), where \( \mathcal{P}([a, b]) \) denotes the set of all partitions of \([a, b]\). For \( p = 1 \) it is the usual variation of \( f(x) \) that was introduced by Jordan (see [24], [25]). For very constructive systematic study of Jordan variation we recommend the interested reader to refer to [29].

In special case, we define the variation of order \( \infty \) of \( f \) along \([a, b]\) in the classical sense, i.e., if there exists a positive number \( M \) such that

\[
\sum_{i=1}^{n} \text{Osc} \left( f; \left[ x_{i-1}^{(n)}, x_i^{(n)} \right] \right) = \sum_{i=1}^{n} (\sup - \inf) f(t_i) \leq M, \quad t_i \in \left[ x_{i-1}^{(n)}, x_i^{(n)} \right],
\]

for all partition of \([a, b]\), then \( f \) is said to be of bounded \( \infty \)-variation on \([a, b]\). The number

\[
\int_a^b (f; \infty) = \sup \left\{ \sum(P) : P \in \mathcal{P}[a, b] \right\} := \text{Osc} \left( f; [a, b] \right),
\]

is called the oscillation of \( f \) on \([a, b]\). Equivalently, we may define the oscillation of \( f \) as, (see [23]):

\[
\int_a^b (f; \infty) = \lim_{p \to \infty} \int_a^b (f; p) = \sup_{x \in [a, b]} \{ f(x) \} - \inf_{x \in [a, b]} \{ f(x) \} = \text{Osc} \left( f; [a, b] \right).
\]

Let \( \mathcal{W}_p \) denotes the class of all functions of bounded \( p \)-variation \((1 \leq p \leq \infty)\). For an arbitrary \( p \geq 1 \) the class \( \mathcal{W}_p \) was firstly introduced by Wiener in [30], where he had shown that \( \mathcal{W}_p \) can only have discontinuities of the first kind. More generally, if \( f \) is a real function of bounded \( p \)-variation on an interval \([a, b]\), then:

- \( f \) is bounded, and

\[
\text{Osc} \left( f; [a, b] \right) \leq \int_a^b (f; p) \leq \int_a^b (f; 1).
\]

This fact follows by Jensen’s inequality applied for \( h(p) = \int_a^b (f; p) \) which is log-convex and decreasing for all \( p > 1 \). Moreover, the inclusions

\[
\mathcal{W}_\infty(f) \subset \mathcal{W}_q(f) \subset \mathcal{W}_p(f) \subset \mathcal{W}_1(f)
\]

are valid for all \( 1 < p < q < \infty \), (see [31]).

- \( f \) is continuous except at most on a countable set.
- \( f \) has one-sided limits everywhere (limits from the left everywhere in \((a, b]\), and from the right everywhere in \([a, b)\);
- The derivative \( f'(x) \) exists almost everywhere (i.e. except for a set of measure zero).
If \( f(x) \) is differentiable on \([a, b]\), then
\[
\sqrt[p]{\int_a^b |f'(t)|^p \, dt} = \|f'\|_p, \quad 1 \leq p < \infty.
\]

**Lemma 1.** [2] Fix \( 1 \leq p < \infty \). Let \( f, g : [a, b] \to \mathbb{R} \) be such that \( f \) is continuous on \([a, b]\) and \( g \) is of bounded \( p \)-variation on \([a, b]\). Then the Riemann–Stieltjes integral \( \int_a^b f(t) \, dg(t) \) exists and the inequality:
\[
\left| \int_a^b w(t) \, d\nu(t) \right| \leq \|w\|_\infty \cdot \text{Osc}(\nu; [a, b]) \leq \|w\|_\infty \cdot \sqrt[p]{a \mapsto \nu; [a, b]},
\]
holds. The constant ‘1’ in the both inequalities is the best possible.

**Lemma 2.** [2] Let \( 1 \leq p < \infty \). Let \( w, \nu : [a, b] \to \mathbb{R} \) be such that \( w \in L^p[a, b] \) and \( \nu \) has a Lipschitz property on \([a, b]\). Then the inequality
\[
\left| \int_a^b w(t) \, d\nu(t) \right| \leq L(b - a)^{1 - \frac{1}{p}} \cdot \|w\|_p,
\]
holds and the constant ‘1’ in the right hand side is the best possible, where
\[
\|w\|_p = \left( \int_a^b |w(t)|^p \, dt \right)^{1/p}, \quad (1 \leq p < \infty).
\]

In this paper, we establish two-point of Ostrowski’s integral inequality for the Riemann-Stieltjes integral \( \int_a^b f(t) \, du(t) \), where \( f \) is assumed to be of \( r \)-Hölder type on \([a, b]\) and \( u \) is of bounded variation on \([a, b]\), are given. The dual formulas under the same assumption are proved. Some sharp error \( L^p \)-Error estimates for the proposed quadrature rules are also obtained.

**2. The Results**

Consider the quadrature rule
\[
\int_a^b f(s) \, du(s) = Q^{a, b}(f, u; t_0, x, t_1) + R^{a, b}(f, u; t_0, x, t_1)
\]
where \( Q^{a, b}(f, u; t_0, x, t_1) \) is the quadrature formula
\[
Q^{a, b}(f, u; t_0, x, t_1) = [u(x) - u(a)] f(t_0) + [u(b) - u(x)] f(t_1)
\]
for all \( a \leq t_0 \leq x \leq t_1 \leq b \).

Hence, the remainder term \( R^{a, b}(f, u; t_0, x, t_1) \) is given by
\[
R^{a, b}(f, u; t_0, x, t_1) := \int_a^{t_0} f(s) \, du(s) - [u(x) - u(a)] f(t_0) - [u(b) - u(x)] f(t_1)
\]
The following Two-point Ostrowski’s inequality for Riemann-Stieltjes integral holds.
Theorem 1. Let $f : [a, b] \to \mathbb{R}$ be Hölder continuous of order $r$, $(0 < r \leq 1)$, and $u : [a, b] \to \mathbb{R}$ is a mapping of bounded $p$-variation $(1 \leq p \leq \infty)$ on $[a, b]$. Then we have the inequality

$$\left| \mathcal{R}^{[a,b]}(f, u; t_0, x, t_1) \right| \leq H \max \left\{ \left[ \frac{x-a}{2} + \left| t_0 - \frac{a+x}{2} \right| \right], \left[ \frac{b-x}{2} + \left| t_1 - \frac{a+x}{2} \right| \right] \right\}^r \sqrt[p]{(u;p)}$$

for all $a \leq t_0 \leq x \leq t_1 \leq b$. Furthermore, the first half of each max-term is the best possible in the sense that it cannot be replaced by a smaller one, for all $r \in (0, 1]$.

Proof. Using the integration by parts formula for Riemann–Stieltjes integral, we have

$$\int_a^x [f(t_0) - f(s)] du(s) + \int_x^b [f(t_1) - f(s)] du(s)$$

$$= \int_a^x f(t_0) du(s) + \int_x^b f(t_1) du(s) - \int_a^b f(s) du(s)$$

$$= [u(x) - u(a)] f(t_0) + [u(b) - u(x)] f(t_1) - \int_a^b f(s) du(s)$$

$$= -\mathcal{R}^{[a,b]}(f, u; t_0, x, t_1),$$

It is well known that if $p : [c, d] \to \mathbb{R}$ is continuous and $\nu : [c, d] \to \mathbb{R}$ is of $p$-bounded variation $(1 \leq p < \infty)$, then the Riemann-Stieltjes integral $\int_c^d p(t) d\nu(t)$ exists and the following inequality holds:

$$\left| \int_c^d p(t) d\nu(t) \right| \leq \sup_{t \in [c,d]} |p(t)| \sqrt[p]{\nu}.$$  \hfill (2.5)

Applying the inequality (2.5) for $\nu(t) = u(t)$, $p(t) = f(t_0) - f(s)$, for all $s \in [a, x]$; and then for $p(t) = f(t_1) - f(s)$, $\nu(t) = u(t)$ for all $t \in (x, b]$, we get

$$\left| [u(x) - u(a)] f(t_0) + [u(b) - u(x)] f(t_1) - \int_a^b f(s) du(s) \right|$$

$$= \left| \int_a^x [f(t_0) - f(s)] du(s) + \int_x^b [f(t_1) - f(s)] du(s) \right|$$

$$\leq \left| \int_a^x [f(t_0) - f(s)] du(s) \right| + \left| \int_x^b [f(t_1) - f(s)] du(s) \right|$$

$$\leq \sup_{s \in [a,x]} |f(t_0) - f(s)| \sqrt[p]{(u;p)} + \sup_{s \in [x,b]} |f(t_1) - f(s)| \sqrt[p]{(u;p)}.$$  \hfill (2.6)

As $f$ is of $r$-Hölder type, we have

$$\sup_{s \in [a,x]} |f(t_0) - f(s)| \leq \sup_{s \in [a,x]} H |t_0 - s|^r$$

$$= H \max \left\{ (x-t_0)^r, (t_0-a)^r \right\}$$

$$= H \left[ \max \left\{ (x-t_0), (t_0-a) \right\} \right]^r$$

$$= H \left[ \frac{x-a}{2} + \frac{t_0-a+x}{2} \right]^r,$$
and
\[
\sup_{s \in [a,b]} |f(t_1) - f(s)| \leq \sup_{s \in [a,b]} [H |t_1 - s|^r]
\]
\[
= H \max \{(t_1 - x)^r, (b - t_1)^r\}
\]
\[
= H \left[\max \{(t_1 - x), (b - t_1)\}\right]^r
\]
\[
= H \left[\frac{b - x}{2} + \left|t_1 - \frac{x + b}{2}\right|\right]^r.
\]

Therefore, by (2.6), we have
\[
\left|\left[u(x) - u(a)\right] f(t_0) + [u(b) - u(x)] f(t_1) - \int_a^b f(s) du(s)\right|
\]
\[
\leq H \left[\frac{x - a}{2} + \left|t_0 - \frac{a + x}{2}\right|\right]^r \cdot \sqrt[V]{u(p)} + H \left[\frac{b - x}{2} + \left|t_1 - \frac{x + b}{2}\right|\right]^r \cdot \sqrt[V]{u(p)}
\]
\[
\leq H \max \left\{\left[\frac{x - a}{2} + \left|t_0 - \frac{a + x}{2}\right|\right]^r, \left[\frac{b - x}{2} + \left|t_1 - \frac{x + b}{2}\right|\right]^r\right\} \cdot \sqrt[V]{u(p)}
\]
\[
= H \max \left\{\left[\frac{x - a}{2} + \left|t_0 - \frac{a + x}{2}\right|\right], \left[\frac{b - x}{2} + \left|t_1 - \frac{x + b}{2}\right|\right]\right\}^r \cdot \sqrt[V]{u(p)}.
\]

To prove the sharpness of the constant \(\frac{1}{2}\) for any \(r \in (0,1]\), assume that (2.4) holds with a constant \(C > 0\), that is,
\[
\left|\left[u(x) - u(a)\right] f(t_0) + [u(b) - u(x)] f(t_1) - \int_a^b f(s) du(s)\right|
\]
\[
\leq H \max \left\{\left[C(x - a) + \left|t_0 - \frac{a + x}{2}\right|\right], \left[C(b - x) + \left|t_1 - \frac{x + b}{2}\right|\right]\right\}^r \cdot \sqrt[V]{u(p)}.
\]

Choose \(f(t) = t^r, r \in (0,1], t \in [0,1] \) and \(u : [0,1] \to [0,\infty)\) given by
\[
u(t) = \begin{cases} 
0, & t \in (0,1] \\
-1, & t = 0
\end{cases}
\]

As
\[
|f(x) - f(y)| = |x^r - y^r| \leq |x - y|^r, \quad \forall x \in [0,1], \quad r \in (0,1],
\]
it follows that \(f\) is \(r\)-Hölder type with the constant \(H = 1\).

By using the integration by parts formula for Riemann-Stieltjes integrals, we have:
\[
\int_0^1 f(t) du(t) = f(1) u(1) - f(0) u(0) - \int_0^1 u(t) df(t) = 0,
\]
and \(\sqrt[V]{u(p)} = 1\). Consequently, by (2.7), we get
\[
|t_0^r| \leq \max \left\{\left[Cx + \left|t_0 - \frac{x}{2}\right|\right], \left[C(1 - x) + \left|t_1 - \frac{x + 1}{2}\right|\right]\right\}^r, \quad \forall t_0 \in [0,1].
\]

For \(t_0 = \frac{x}{2}\) and \(t_1 = x = 1\) we get \(\frac{1}{2^r} \leq C^r\), which implies that \(C \geq \frac{1}{2}\).
It remains to prove the second part, so we consider
\[ u(t) = \begin{cases} 
0, & t \in [0, 1) \\
1, & t = 1 
\end{cases} \]
therefore as we have obtained previously
\[ \int_0^1 f(t) \, du(t) = 0 \quad \text{and} \quad \int_0^1 (u;p) = 1. \]

Consequently, by (2.4), we get
\[ |t_1| \leq \max \left\{ \left[ Cx + \left| t_0 - \frac{x}{2} \right| \right], \left[ C(1-x) + \left| t_1 - \frac{x+1}{2} \right| \right] \right\}^r, \forall t_0 \in [0, 1]. \]

For \( t_0 = x = 0 \) and \( t_1 = \frac{1}{2} \) we get \( \frac{1}{2} \leq C^r \), which implies that \( C \geq \frac{1}{2} \), and the theorem is completely proved.

\[ \square \]

The following inequalities are hold:

**Corollary 1.** Let \( f \) and \( u \) as in Theorem 1. In 2.4 choose

(1) \( t_0 = a \) and \( t_1 = b \), then we get the following trapezoid type inequality
\[ |\mathcal{R}^{[a,b]}(f, u; a, x, b)| \leq H \left[ \frac{b-a}{2} + \left| x - \frac{a+b}{2} \right| \right]^r \cdot \int_a^b (u;p) . \]

or equivalently, we may write using parts formula for Riemann-Stieltjes integral
\[ |f(b) - f(a)| u(s) - \int_a^b u(s) \, df(s) | \leq H \left[ \frac{b-a}{2} + \left| x - \frac{a+b}{2} \right| \right]^r \cdot \int_a^b (u;p) . \]

The constant \( \frac{1}{2} \) is the best possible for all \( r \in (0, 1] \).

(2) \( x = \frac{a+b}{2} \), then we get the following mid-point type inequality
\[ |\mathcal{R}^{[a,b]}(f, u; t_0, \frac{a+b}{2}, t_1)| \]
\[ \leq H \max \left\{ \left[ \frac{b-a}{4} + \left| t_0 - \frac{3a+b}{4} \right| \right], \left[ \frac{b-a}{4} + \left| t_1 - \frac{a+3b}{4} \right| \right] \right\}^r \cdot \int_a^b (u;p) . \]

The constant \( \frac{1}{4} \) is the best possible for all \( r \in (0, 1] \). For instance, setting \( t_0 = y \) and \( t_1 = a + b - y \), we get
\[ |\mathcal{R}^{[a,b]}(f, u; y, \frac{a+b}{2}, a + b - y)| \leq H \left[ \frac{b-a}{4} + \left| y - \frac{3a+b}{4} \right| \right]^r \cdot \int_a^b (u;p) . \]

for all \( y \in [a, \frac{a+b}{2}] \).
(3) \( t_0 = \frac{a + x}{2} \) and \( t_1 = \frac{x + b}{2} \), then

\[
\left| \mathcal{R}^{[a,b]} \left( f, u; \frac{a + x}{2}, x, \frac{x + b}{2} \right) \right| \leq H \left[ \frac{b - a}{2} + \left| x - \frac{a + b}{2} \right| \right]^r \cdot \mathcal{V}_a^b (u; p)
\]

Both constants \( \frac{1}{2} \) and \( \frac{1}{2} \) are the best possible for all \( r \in (0, 1] \).

**Corollary 2.** Let \( f \) be a Hölder continuous function of order \( r \) (\( 0 < r \leq 1 \)), on \([a, b]\), and \( g : [a, b] \to \mathbb{R} \) is continuous on \([a, b]\). Then we have the inequality

\[
\left| \int_a^x f(t_0) \int_a^b g(s) ds + f(t_1) \int_x^b g(s) ds - \int_a^b f(s) g(s) ds \right|
\leq H \max \left\{ \left[ \frac{x - a}{2} + \left| t_0 - \frac{a + x}{2} \right| \right], \left[ \frac{b - x}{2} + \left| t_1 - \frac{x + b}{2} \right| \right] \right\}^r \cdot \|g\|_p,
\]

for all \( a \leq t_0 \leq x \leq t_1 \leq b \), where \( \|g\|_p = \left( \int_a^b |g(t)|^p \, dt \right)^{1/p} \).

**Proof.** Define the mapping \( u : [a, b] \to \mathbb{R} \), \( u(t) = \int_a^t g(s) \, ds \). Then \( u \) is differentiable on \((a, b)\) and \( u'(t) = g(t) \). Using the properties of the Riemann-Stieltjes integral, we have

\[
\int_a^b f(t) \, du(t) = \int_a^b f(t) \, g(t) \, dt,
\]

and

\[
\sqrt[p]{\left( \int_a^b |u'(t)|^p \, dt \right)^{1/p}} = \left( \int_a^b |g(t)|^p \, dt \right)^{1/p},
\]

which gives the required result. \( \square \)

**Theorem 2.** Let \( 1 \leq p < \infty \). Let \( f, u : [a, b] \to \mathbb{R} \) be such that is \( f \in L^p[a, b] \) and \( u \) has a Lipschitz property on \([a, b]\). If \( f \) is \( r-H \)-Hölder continuous, then the inequality

\[
\left| \mathcal{R}^{[a,b]} \left( f, u; t_0, x, t_1 \right) \right| \leq H L \left[ (x - a)^{1 - \frac{1}{p}} \left( \frac{(t_0 - a)^{rp+1} + (x - t_0)^{rp+1}}{rp + 1} \right)^{\frac{1}{p}} + (b - x)^{1 - \frac{1}{p}} \left( \frac{(t_1 - x)^{rp+1} + (b - t_1)^{rp+1}}{rp + 1} \right)^{\frac{1}{p}} \right]^{2.8}
\]

holds for all \( p > 1 \) and \( r \in (0, 1] \).
Proof. From Lemma 2 we have

\[
\left| [u(x) - u(a)] f(t_0) + [u(b) - u(x)] f(t_1) - \int_a^b f(s) \, du(s) \right|
\]

\[
= \left| \int_a^x [f(t_0) - f(s)] \, du(s) + \int_x^b [f(t_1) - f(s)] \, du(s) \right|
\]

\[
\leq \left| \int_a^x [f(t_0) - f(s)] \, du(s) \right| + \left| \int_x^b [f(t_1) - f(s)] \, du(s) \right|
\]

\[
\leq L \left[ (x - a)^{1 - \frac{1}{p}} \left( \int_a^x |f(t_0) - f(s)|^p \, ds \right)^{\frac{1}{p}} 
+ (b - x)^{1 - \frac{1}{p}} \left( \int_x^b |f(t_1) - f(s)|^p \, ds \right)^{\frac{1}{p}} \right]
\]

\[
\leq HL \left[ (x - a)^{1 - \frac{1}{p}} \left( \int_a^x |t_0 - s|^{rp} \, ds \right)^{\frac{1}{p}} + (b - x)^{1 - \frac{1}{p}} \left( \int_x^b |t_1 - s|^{rp} \, ds \right)^{\frac{1}{p}} \right]
\]

\[
= HL \left[ (x - a)^{1 - \frac{1}{p}} \left( \frac{(t_0 - a)^{rp+1} + (x - t_0)^{rp+1}}{rp + 1} \right) \right.
+ (b - x)^{1 - \frac{1}{p}} \left( \frac{(t_1 - x)^{rp+1} + (b - t_1)^{rp+1}}{rp + 1} \right) \right]
\]

which proves the required result.

\[\square\]

Corollary 3. Let \( f \) and \( u \) as in Theorem 2. In (2.8) choose

(1) \( t_0 = a \) and \( t_1 = b \), then we get the following trapezoid type inequality

\[
|R_{a,b}^1(f, u; a, x, b)| \leq HL \left[ (x - a)^{1 - \frac{1}{p}} \left( \frac{(x - a)^{rp+1}}{rp + 1} \right) \right. \left. + (b - x)^{1 - \frac{1}{p}} \left( \frac{(b - x)^{rp+1}}{rp + 1} \right) \right]
\]

or equivalently, we may write using parts formula for Riemann-Stieltjes integral

\[
\left| [f(b) - f(a)] u(x) - \int_a^b u(s) \, df(s) \right|
\]

\[
\leq HL \left[ (x - a)^{1 - \frac{1}{p}} \left( \frac{(x - a)^{rp+1}}{rp + 1} \right) \right. \left. + (b - x)^{1 - \frac{1}{p}} \left( \frac{(b - x)^{rp+1}}{rp + 1} \right) \right].
\]
Theorem 3. Let \( 1 \leq p < \infty \). Let \( f, u : [a, b] \to \mathbb{R} \) be such that \( f \in \mathcal{L}^p(I) \) and \( u \) has a Lipschitz property on \([a, b]\). If \( f \) is \( r-H\)-Hölder continuous, then the inequality holds for all \( p > 1 \) and \( r \in (0, 1] \).

\[
|\mathcal{R}_{[a,b]} (f,u; t_0, x, t_1) | \leq L \left( \frac{p \sin \left( \frac{\pi}{p} \right)}{\pi \sqrt{p-1}} \right)^n \left\{ \left( x - a \right)^{1 - \frac{1}{r}} \left[ \frac{x - a}{2} + \left| t_0 - \frac{x + a}{2} \right| \right]^n + \left( b - x \right)^{1 - \frac{1}{r}} \left[ \frac{b - x}{2} + \left| t_1 - \frac{x + b}{2} \right| \right]^n \right\} \| f^{(n)} \|_{p,[a,b]} \tag{2.9}
\]
Corollary 4. Let \( f \) and \( u \) as in Theorem 3. In (2.9) choose
In this section, we assume that $f$ is of bounded variation on $[a, b]$, and the integrator $u$ is assumed to be satisfied the Hölder condition on $[a, b]$. For all $y \in [a, b]$, we get

$$
\left| \mathcal{R}^{[a,b]}_u (f, u; a, b) \right| \leq L \left( \frac{p \sin \left( \frac{\pi}{p} \right)}{\pi \sqrt{p - 1}} \right)^n \left\{ (x - a)^{n+1-\frac{1}{p}} + (b - x)^{n+1-\frac{1}{p}} \right\} \| f^{(n)} \|_{p,[a,b]}.
$$

or equivalently, we may write using parts formula for Riemann-Stieltjes integral

$$
\left| f(b) - f(a) \right| u(x) - \int_a^b u(s) df(s)
\leq L \left( \frac{p \sin \left( \frac{\pi}{p} \right)}{\pi \sqrt{p - 1}} \right)^n \left\{ (x - a)^{n+1-\frac{1}{p}} + (b - x)^{n+1-\frac{1}{p}} \right\} \| f^{(n)} \|_{p,[a,b]}.
$$

(2) $x = \frac{a+b}{2}$, then we get the following mid-point type inequality

$$
\left| \mathcal{R}^{[a,b]}_u (f, u; t, t) \right|
\leq L \left( b-a \right)^{1-\frac{1}{p}} \left( \frac{p \sin \left( \frac{\pi}{p} \right)}{\pi \sqrt{p - 1}} \right)^n \left\{ \left[ \frac{b-a}{4} + \left| t_0 - \frac{3a+b}{4} \right| \right]^n + \left[ b-a \right] + \left| t_1 - \frac{a+3b}{4} \right| \right\} \| f^{(n)} \|_{p,[a,b]}.
$$

For instance, setting $t_0 = y$ and $t_1 = a+b-y$, we get

$$
\left| \mathcal{R}^{[a,b]}_u (f, u; a, b) \right|
\leq L \left( b-a \right)^{1-\frac{1}{p}} \left( \frac{p \sin \left( \frac{\pi}{p} \right)}{\pi \sqrt{p - 1}} \right)^n \left\{ \left[ \frac{b-a}{4} + \left| y - \frac{3a+b}{4} \right| \right]^n + \left[ b-a \right] + \left| y - \frac{a+3b}{4} \right| \right\} \| f^{(n)} \|_{p,[a,b]}
$$

for all $y \in [a, \frac{a+b}{2}]$.

(3) $t_0 = \frac{3a+b}{4}$, $x = \frac{a+b}{2}$ and $t_1 = \frac{a+3b}{4}$, then

$$
\left| \mathcal{R}^{[a,b]}_u (f, u; \frac{3a+b}{4}, \frac{a+b}{2}, \frac{a+3b}{4}) \right|
\leq \frac{L}{2} \left( b-a \right)^{n+1-\frac{1}{p}} \left( \frac{p \sin \left( \frac{\pi}{p} \right)}{\pi \sqrt{p - 1}} \right)^n \| f^{(n)} \|_{p,[a,b]}
$$

3. The dual assumptions

In this section, $L^p$-error estimates of Two-point quadrature rules for the Riemann–Stieltjes integral $\int_a^b f(t) du(t)$, where the integrand $f$ is of bounded variation on $[a, b]$ and the integrator $u$ is assumed to be satisfied the Hölder condition on $[a, b]$. 

Theorem 4. Let \( u : [a, b] \to \mathbb{R} \) be a Hölder continuous of order \( r \), \((0 < r \leq 1)\), and \( f : [a, b] \to \mathbb{R} \) is a mapping of bounded \( p \)-variation \((1 \leq p \leq \infty)\) on \([a, b]\). Then we have the inequality

\[
|\mathcal{R}^{[a, b]} (f, u, t_0, x, t_1)| \leq H \max \left\{ (t_0 - a), \left[ \frac{t_1 - t_0}{2} + \left| x - \frac{t_0 + t_1}{2} \right| \right], (b - t_1) \right\}^r \cdot \sqrt[p]{(f; p)} \quad (3.1)
\]

for all \( a \leq t_0 \leq x \leq t_1 \leq b \). Furthermore, the constant \( 1 \) is the best possible in the sense that it cannot be replaced by a smaller one, for all \( r \in (0, 1] \).

Proof. Using the integration by parts formula for Riemann–Stieltjes integral, we have

\[
\begin{align*}
\int_a^{t_0} [u(s) - u(a)] df(s) &= [u(t_0) - u(a)] f(t_0) - \int_a^{t_0} f(s) du(s) \\
\int_{t_0}^{t_1} [u(s) - u(x)] df(s) &= [u(t_1) - u(x)] f(t_1) - [u(t_0) - u(x)] f(t_0) - \int_{t_0}^{t_1} f(s) du(s) \\
\int_{t_1}^b [u(s) - u(b)] df(s) &= [u(b) - u(t_1)] f(t_1) - \int_{t_1}^b f(s) du(s),
\end{align*}
\]

Adding these identities, we get

\[
\int_a^{t_0} [u(s) - u(a)] df(s) + \int_{t_0}^{t_1} [u(s) - u(x)] df(s) + \int_{t_1}^b [u(s) - u(b)] df(s)
\]

\[
= [u(x) - u(a)] f(t_0) + [u(b) - u(x)] f(t_1) - \int_a^b f(s) du(s) \quad (3.2)
\]

Applying the triangle inequality on the above identity and then use Lemma 1, for each term separately, we get

\[
\begin{align*}
&\left| [u(x) - u(a)] f(t_0) + [u(b) - u(x)] f(t_1) - \int_a^b f(s) du(s) \right| \\
&\left| \int_a^{t_0} [u(s) - u(a)] df(s) + \int_{t_0}^{t_1} [u(s) - u(x)] df(s) + \int_{t_1}^b [u(s) - u(b)] df(s) \right| \\
&\leq \left| \int_a^{t_0} [u(s) - u(a)] df(s) \right| + \left| \int_{t_0}^{t_1} [u(s) - u(x)] df(s) \right| + \left| \int_{t_1}^b [u(s) - u(b)] df(s) \right| \\
&\leq \sup_{s \in [a, t_0]} |u(s) - u(a)| \cdot \sqrt[p]{(f; p)} + \sup_{s \in [t_0, t_1]} |u(s) - u(x)| \cdot \sqrt[p]{(f; p)} \\
&\quad + \sup_{s \in [t_1, b]} |u(t_1) - u(b)| \cdot \sqrt[p]{(f; p)}.
\end{align*}
\]

As \( u \) is of \( r \)-Hölder type, we have

\[
\sup_{s \in [a, t_0]} |u(s) - u(a)| \leq \sup_{s \in [a, t_0]} [H |s - a|^r] = H (t_0 - a)^r,
\]

and

\[
\sup_{s \in [t_0, t_1]} |u(s) - u(x)| \leq \sup_{s \in [t_0, t_1]} [H |s - x|^r] = H (t_1 - t_0)^r,
\]

so

\[
\sup_{s \in [t_1, b]} |u(t_1) - u(b)| \leq \sup_{s \in [t_1, b]} [H |s - b|^r] = H (b - t_1)^r.
\]

Using the Hölder inequality, we get

\[
\sqrt[p]{(f; p)} = \sqrt[p]{\sum_{n=1}^{m} |f_n|^p} \leq \left( \sum_{n=1}^{m} |f_n| \right)^{1/p} \left( \sum_{n=1}^{m} 1 \right)^{1/q} \leq (m)^{1/q} \sqrt[p]{\sum_{n=1}^{m} |f_n|^p}
\]

and

\[
\sqrt[p]{(f; p)} \leq \left( \sum_{n=1}^{m} |f_n| \right)^{1/p} \left( \sum_{n=1}^{m} 1 \right)^{1/q} \leq (m)^{1/q} \sqrt[p]{\sum_{n=1}^{m} |f_n|^p}
\]

Therefore, the constant \( 1 \) is the best possible in the sense that it cannot be replaced by a smaller one, for all \( r \in (0, 1] \).
\[
\begin{align*}
\sup_{s \in [t_0, t_1]} |u(s) - u(x)| &\leq \sup_{s \in [t_0, t_1]} [H |s - x|^r] \\
&= H \max \{(t_1 - x)^r, (x - t_0)^r\} \\
&= H \left[\max \{(t_1 - x), (x - t_0)\}\right]^r \\
&= H \left[\frac{t_1 - t_0}{2} + \left|x - \frac{t_0 + t_1}{2}\right|\right]^r,
\end{align*}
\]

and
\[
\begin{align*}
\sup_{s \in [t_1, b]} |u(s) - u(b)| &\leq \sup_{s \in [t_1, b]} [H |s - b|^r] = H (b - t_1)^r,
\end{align*}
\]

Therefore, by (3.3), we have
\[
\begin{align*}
&\left|u(x) - u(a)\right| f(t_0) + [u(b) - u(x)] f(t_1) - \int_a^b f(s) du(s) \\
&\leq H (t_0 - a)^r \cdot \sqrt{f(p)} + H \left[\left|\frac{t_1 - t_0}{2} + \left|x - \frac{t_0 + t_1}{2}\right|\right|^r\right] \cdot \sqrt{f(p)} + H (b - t_1)^r \cdot \sqrt{f(p)} \\
&\leq H \max \left\{(t_0 - a)^r, \left[\frac{t_1 - t_0}{2} + \left|x - \frac{t_0 + t_1}{2}\right|\right]^r, (b - t_1)^r\right\} \cdot \sqrt{f(p)} \\
&= H \max \left\{(t_0 - a), \left[\frac{t_1 - t_0}{2} + \left|x - \frac{t_0 + t_1}{2}\right|\right], (b - t_1)\right\} \cdot \sqrt{f(p)}.
\end{align*}
\]

To prove the sharpness of the constant 1 for any \( r \in (0, 1) \), assume that (3.1) holds with a constant \( C > 0 \), that is,
\[
\begin{align*}
&\left|u(x) - u(a)\right| f(t_0) + [u(b) - u(x)] f(t_1) - \int_a^b f(s) du(s) \\
&\leq C \max \left\{(t_0 - a), \left[\frac{t_1 - t_0}{2} + \left|x - \frac{t_0 + t_1}{2}\right|\right], (b - t_1)\right\} \cdot \sqrt{f(p)}. \tag{3.4}
\end{align*}
\]

Choose \( u(t) = t^r, r \in (0, 1), t \in [0, 1] \) and \( f: [0, 1] \to [0, \infty) \) given by
\[
\begin{align*}
f(t) &= \begin{cases} 
0, & t \in (0, 1] \\
1, & t = 0
\end{cases}
\end{align*}
\]

As
\[
|u(x) - u(y)| = |x^r - y^r| \leq |x - y|^r, \quad \forall x, y \in [0, 1], r \in (0, 1],
\]

it follows that \( u \) is \( r \)-Hölder type with the constant \( H = 1 \).

By using the integration by parts formula for Riemann-Stieltjes integrals, we have:
\[
\int_0^1 f(t) du(t) = f(1) u(1) - f(0) u(0) - \int_0^1 u(t) df(t) = 0,
\]

and $V_0(f; p) = 1$. Consequently, by (3.4), we get

$$|t'_0| \leq C \max \left\{ t_0, \left(\frac{t_1 - t_0}{2} + \left|x - \frac{t_0 + t_1}{2}\right|\right), (1 - t_1) \right\}^r, \forall t_0, t_1 \in [0, 1], \text{ with } t_0 \leq t_1.$$  

Assume first

$$\max \left\{ t_0, \left(\frac{t_1 - t_0}{2} + \left|x - \frac{t_0 + t_1}{2}\right|\right), (1 - t_1) \right\}^r = t'_0$$

so that we get $1 \leq C$.

Now, assume that

$$\max \left\{ t_0, \left(\frac{t_1 - t_0}{2} + \left|x - \frac{t_0 + t_1}{2}\right|\right), (1 - t_1) \right\}^r = (1 - t_1)^r.$$  

choose $t_1 = 1 - t_0$, so that we get $1 \leq C$.

Finally, we assume that

$$\max \left\{ t_0, \left(\frac{t_1 - t_0}{2} + \left|x - \frac{t_0 + t_1}{2}\right|\right), (1 - t_1) \right\}^r = \left(\frac{t_1 - t_0}{2} + \left|x - \frac{t_0 + t_1}{2}\right|\right)^r.$$  

Define $f : [0, 1] \to [0, \infty)$ given by

$$f(t) = \begin{cases} 0, & t \in (0, 1) \\ 1, & t = 0, 1 \end{cases}$$

Clearly, $V_0^I(f; p) = 2$. Therefore, for $t_0 = 0$ and $t_1 = 1$, so that we get $1 \leq C \left(\frac{1}{2} + \left|x - \frac{1}{2}\right|\right)^r 2^{1/p}$. Choosing $x = \frac{1}{2}$ and $r = \frac{1}{p}$ or $p = \frac{1}{r}$, it follows that $1 \leq C \left(\frac{1}{2}\right)^r 2^r$, i.e., $C \geq 1$. Hence, the inequality (3.1) is sharp, and the theorem is completely proved.

**Theorem 5.** Let $1 \leq p < \infty$. Let $f, u : [a, b] \to \mathbb{R}$ be such that $u \in L^p[a, b]$ and $f$ has a Lipschitz property on $[a, b]$. If $u$ is $r$-$H$–Hölder continuous, then the inequality

$$|R|_{[a, b]}(f, u; t_0, x, t_1) \leq LH \begin{cases} \frac{(t_0 - a)^{r+1}}{(r+1)p} + (t_1 - t_0)^{1 - \frac{1}{p}} \left(\frac{(t_1 - x)^{r+1} - (t_0 - x)^{r+1}}{r+1}\right)^{\frac{1}{p}} + \frac{(b - t_1)^{r+1}}{(r+1)p}, & a \leq x \leq t_0 \leq t_1 \leq b \\ \frac{(t_0 - a)^{r+1}}{(r+1)p} + (t_1 - t_0)^{1 - \frac{1}{p}} \left(\frac{(x - t_1)^{r+1} + (t_0 - x)^{r+1}}{r+1}\right)^{\frac{1}{p}} + \frac{(b - t_1)^{r+1}}{(r+1)p}, & a \leq t_0 \leq x \leq t_1 \leq b \\ \frac{(t_0 - a)^{r+1}}{(r+1)p} + (t_1 - t_0)^{1 - \frac{1}{p}} \left(\frac{(x - t_0)^{r+1} - (x - t_1)^{r+1}}{r+1}\right)^{\frac{1}{p}} + \frac{(b - t_1)^{r+1}}{(r+1)p}, & a \leq t_0 \leq t_1 \leq x \leq b \end{cases}$$

(3.5)

holds for all $p > 1$ and $r \in (0, 1]$ with constant $H > 0$. 
Proof. As in the proof of Theorem 4, we have by Lemma 2

\[
\left| [u(x) - u(a)] f(t_0) + [u(b) - u(x)] f(t_1) - \int_a^b f(s)u(s) \right|
\]

\[
= \left| \int_a^{t_0} [u(s) - u(a)] df(s) + \int_{t_0}^{t_1} [u(s) - u(x)] df(s) + \int_{t_1}^b [u(s) - u(b)] df(s) \right|
\]

\[
\leq \left| \int_a^{t_0} [u(s) - u(a)] df(s) \right| + \left| \int_{t_0}^{t_1} [u(s) - u(x)] df(s) \right| + \left| \int_{t_1}^b [u(s) - u(b)] df(s) \right|
\]

\[
\leq L \left[ (t_0 - a)^{\frac{1}{p}} \left( \int_a^{t_0} |u(s) - u(a)|^p ds \right)^{\frac{1}{p}} + (t_1 - t_0)^{\frac{1}{p}} \left( \int_{t_0}^{t_1} |u(s) - u(x)|^p ds \right)^{\frac{1}{p}} 
\right. 
+ \left. (b - t_1)^{\frac{1}{p}} \left( \int_{t_1}^b |u(s) - u(b)|^p ds \right)^{\frac{1}{p}} \right].
\]

Simple computations yield that

\[
\int_a^{t_0} |s - a|^{rp} ds = \int_a^{t_0} (s - a)^{rp} ds = \frac{(t_0 - a)^{rp+1}}{rp + 1},
\]

\[
\int_{t_0}^{t_1} |s - x|^{rp} ds = \begin{cases} 
\int_{t_0}^{t_1} (s - x)^{rp} ds, & a \leq x \leq t_0 \\
\int_{t_0}^{x} (s - x)^{rp} ds + \int_x^{t_1} (s - x)^{rp} ds, & t_0 \leq x \leq t_1 \\
\int_{t_0}^{t_1} (s - x)^{rp} ds, & t_1 \leq x \leq b \\
\frac{(t_1 - x)^{rp+1} - (t_0 - x)^{rp+1}}{rp + 1}, & a \leq x \leq t_0 \\
\frac{(x - t_0)^{rp+1} + (t_1 - x)^{rp+1}}{rp + 1}, & t_0 \leq x \leq t_1 , \\
\frac{(x - t_0)^{rp+1} - (x - t_1)^{rp+1}}{rp + 1}, & t_1 \leq x \leq b 
\end{cases}
\]

and

\[
\int_{t_1}^b |s - b|^{rp} ds = \int_{t_1}^b (b - s)^{rp} ds = \frac{(b - t_1)^{rp+1}}{rp + 1}.
\]

Combining these equalities with the last inequality above we get the required result. \qed
Corollary 5. Let $1 \leq p < \infty$. Let $f, u : [a, b] \to \mathbb{R}$ be such that $u \in L^p[a, b]$ and $f$ has a Lipschitz property on $[a, b]$. If $u$ is $r$-Hölder continuous, then the inequality

$$\left| (x-a) f(t_0) + (b-x) f(t_1) - \int_a^b s^{r-1} f(s) \, ds \right|$$

$$\leq LH \left\{ \begin{array}{ll}
\frac{(t_0-a)^{p+1}}{(r+1)^p} + (t_1 - t_0)^{1 - \frac{1}{p}} \left( \frac{(t_1-x)^{p+1} - (t_0-x)^{p+1}}{r+1} \right)^{\frac{1}{p}} + \frac{(b-t_1)^{p+1}}{(r+1)^p}, & a \leq x \leq t_0 \leq t_1 \leq b \\
\frac{(t_0-a)^{p+1}}{(r+1)^p} + (t_1 - t_0)^{1 - \frac{1}{p}} \left( \frac{(x-t_0)^{p+1} + (t_1-x)^{p+1}}{r+1} \right)^{\frac{1}{p}} + \frac{(b-t_1)^{p+1}}{(r+1)^p}, & a \leq t_0 \leq x \leq t_1 \leq b \\
\frac{(t_0-a)^{p+1}}{(r+1)^p} + (t_1 - t_0)^{1 - \frac{1}{p}} \left( \frac{(x-t_0)^{p+1} - (x-t_1)^{p+1}}{r+1} \right)^{\frac{1}{p}} + \frac{(b-t_1)^{p+1}}{(r+1)^p}, & a \leq t_0 \leq t_1 \leq x \leq b
\end{array} \right.$$

(3.6)

holds for all $p > 1$ and $r \in (0, 1]$ with constant $H > 0$.

Proof. Setting $u(t) = t^r$, $t \in [a, b]$, $r \in (0, 1]$, in Theorem 5 we get the required result. □

Corollary 6. Let $1 \leq p < \infty$. Let $f, u : [a, b] \to \mathbb{R}$ be such that $u \in L^p[a, b]$ and $f$ has a Lipschitz property on $[a, b]$. If $u$ is $K$-Lipschitz continuous on $[a, b]$, then the inequality

$$\left| (x-a) f(t_0) + (b-x) f(t_1) - \int_a^b f(s) \, ds \right|$$

$$\leq LK \left\{ \begin{array}{ll}
\frac{(t_0-a)^2}{(p+1)^p} + (t_1 - t_0)^{1 - \frac{1}{p}} \left( \frac{(t_1-x)^{p+1} - (t_0-x)^{p+1}}{p+1} \right)^{\frac{1}{p}} + \frac{(b-t_1)^2}{(p+1)^p}, & a \leq x \leq t_0 \leq t_1 \leq b \\
\frac{(t_0-a)^2}{(p+1)^p} + (t_1 - t_0)^{1 - \frac{1}{p}} \left( \frac{(x-t_0)^{p+1} + (t_1-x)^{p+1}}{p+1} \right)^{\frac{1}{p}} + \frac{(b-t_1)^2}{(p+1)^p}, & a \leq t_0 \leq x \leq t_1 \leq b \\
\frac{(t_0-a)^2}{(p+1)^p} + (t_1 - t_0)^{1 - \frac{1}{p}} \left( \frac{(x-t_0)^{p+1} - (x-t_1)^{p+1}}{p+1} \right)^{\frac{1}{p}} + \frac{(b-t_1)^2}{(p+1)^p}, & a \leq t_0 \leq t_1 \leq x \leq b
\end{array} \right.$$

(3.7)

holds for all $p > 1$ and constant $K > 0$.

Proof. Setting $r = 1$ in Corollary 5, we get the required result. □

Remark 2. The inequalities (3.6) and (3.7) generalize the recent result(s) in [2].

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