The sum rules for the spin dependent structure functions corresponding to the moment at $n = 0$

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Abstract

Sum rules for the spin dependent structure functions corresponding to the moment at $n = 0$ derived from the current algebra based on the canonical quantization on the null-plane are reviewed.

1 Introduction

Many years ago, it was shown that the $O(4)$ partial waves of the $t$ channel CM scattering amplitude of the current-hadron reaction at $t = 0$ exactly correspond to the Nachtmann moments.[1] This fact makes clear that the Nachtmann moment[2] can be used even in the small $Q^2$ region where the operator product expansion can not be applied. Now the Nachtmann moments at $n = 0$ for the spin dependent structure functions take a very simple form and its form is the same as the one which appears in the fixed-mass sum rules from the current algebra based on the canonical quantization on the null-plane. Thus we can treat these sum rules without worrying about the kinematical target mass correction. The sum rules for the spin dependent structure functions $g_1$ and $g_2$ in the small $Q^2$ region derived in Refs.[3, 4, 5] belong to this category. These sum rules show that there is a connection among the resonances, the elastic, and the nonresonant continuum in the $g_1$ and the $g_2$ independently. Since the Born term changes rapidly in the small $Q^2$ region, the sum of the resonance and the nonresonant continuum also changes rapidly. In case of the $g_1$, this explains why the Gerasimov-Drell-Hearn(GDH) sum changes a sign in the very small $Q^2$ region. In case of the $g_2$, we find that the Born term divided by $Q^2/2$ in the sum rule has a very similar behavior with that in the sum rule for the $g_1$. This divided Born term is nothing but the Born term in the Schwinger sum rule for the $g_2$, which was derived also by the fixed-mass sum rule approach.[7] These analysis show that, when we consider the duality like Bloom-Gilman,[8] the proper inclusion of the elastic contribution is indispensable. In this paper, we give a brief review of these sum rules.
2 Sum rules from the current algebra based on the canonical quantization on the null-plane

The spin dependent part of the hadronic tensor of the imaginary part of the forward current-hadron scattering amplitude is defined as

\[ W_{ab}^{\mu\nu}|_{\text{spin}} = i\epsilon^{\mu\nu\lambda\sigma}q_\lambda s_\sigma G_{1b}^{ab} + i\epsilon^{\mu\nu\lambda\sigma}q_\lambda (\nu s_\sigma - q \cdot s_\sigma)G_{2b}^{ab} \]  

\[ = \frac{1}{4\pi} \int d^4x \exp(ix) < p, s | [J^\mu_a(x), J^\nu_b(0)] | p, s >_{c|\text{spin}}. \]  

The structure function has a crossing symmetry \( G_{ab}^a(p \cdot q, q^2) = -G_{ab}^a(-p \cdot q, q^2) \) and \( G_{ab}^b(p \cdot q, q^2) = G_{ba}^b(-p \cdot q, q^2) \) under \( q \rightarrow -q, a \leftrightarrow b, \) and \( \mu \leftrightarrow \nu. \) The spin dependent part of the fixed-mass sum rule from the current algebra based on the canonical quantization on the null-plane \([7]\) can be obtained as follows. We take \( x^\pm = \frac{1}{\sqrt{2}}(x^0 \pm x^3). \)

![Diagram of null-plane](image)

We first set \( \mu = + \) and \( \nu = i. \) Then, on the left hand side of Eq.(1), we integrate over \( q^- \) and change variable from \( q^- \) to \( \nu = p^+ q^+ + p^- q^- - \vec{p}^\perp q^\perp \). Then we obtain \( q^2 = 2q^+ \nu - q^+ p^+ q^\perp - q^\perp. \) We set \( q^+ = 0 \) before \( \nu \) integration. To assure this manipulation, we need the superconvergence relation to neglect the contributions from the timelike \( q^2 \) region. On the right hand side, we see that the commutation relation is restricted at \( x^+ = 0 \) by interchanging the \( x \) integration and the \( q^- \) integration. It is known that this interchange is possible under the same superconvergence relation which we need on the left hand side. \([7, 9]\) Now, we take the current formed by the quark bilinear as \( J^\mu_a(x) = \bar{q}(x)\gamma^\mu \frac{\Lambda}{2} q(x). \) The quark field on the null-plane is decomposed by the projection operator as

\[ q^{(\pm)}(x) = \Lambda^\pm q(x), \quad \Lambda^\pm = \frac{1}{2}(1 \pm \gamma^0 \gamma^3). \]  

The \( q^{(-)}(x) \) is expressed by the \( q^{(+)}(x) \) through the equation of the motion, and hence only the \( q^{(+)}(x) \) is the independent quantity. The canonical anti-commutation relation on the null-plane is defined as

\[ \{ q^{(+)}(x), q^{(+)}(0) \}_{x^+ = 0} = \sqrt{2} \Lambda^+ \delta^2(\vec{x}^\perp)\delta(x^-). \]  

The current \( J^\mu_a(x) \) is given only by the \( q^{(+)}(x) \)

\[ J^\mu_a(x) = \bar{q}(x)\gamma^\mu \frac{\Lambda_a}{2} q(x) = \sqrt{2} q^{(+)}(x)\frac{\Lambda_a}{2} q^{(-)}(x). \]  

(4)
On the other hand, $J^a_i(x)$ is given as
\begin{equation}
J^a_i(x) = q^{(+)(i)}(x)\gamma^0\gamma^i\frac{\lambda_a}{2}q^{-}(x) + q^{(-)(i)}(x)\gamma^0\gamma^i\frac{\lambda_a}{2}q^{(+)}(x). \tag{5}
\end{equation}

Then, the following commutation relation holds in QCD.
\begin{equation}
[J^a_i(x), J^b_j(0)]_{x^+=0} = i[s^{+\beta\alpha}\partial_\alpha[\Delta(x)G_{c\beta}(x|0)]
- 2g^{+a}\gamma^i\partial_\alpha[\Delta(x)G_{c\beta}(x|0)]
- \epsilon^{+ia\beta}\partial_\alpha[\Delta(x)G_{c\beta}^5(x|0)],
\end{equation}
where
\begin{equation}
\Delta(x)|_{x^+=0} = -\frac{1}{4}f(x^-)\delta^2(\vec{x}^+),
\end{equation}
\begin{equation}
G^\beta_c(x|0) = d_{abc}A^\beta_c(x|0) + f_{abc}S^\beta_c(x|0), \quad G^\beta_c^5(x|0) = d_{abc}S^\beta_c^5(x|0) - f_{abc}A^\beta_c^5(x|0),
\end{equation}
and
\begin{equation}
S^\beta_c(x|0) = \frac{1}{2}[\bar{q}(x)\gamma^\mu\frac{\lambda_a}{2}q(0) + \bar{q}(0)\gamma^\mu\frac{\lambda_a}{2}q(x)],
\end{equation}
\begin{equation}
A^\beta_c(x|0) = \frac{1}{2}[\bar{q}(x)\gamma^\mu\frac{\lambda_a}{2}q(0) - \bar{q}(0)\gamma^\mu\frac{\lambda_a}{2}q(x)].
\end{equation}
The sum rule for the spin dependent structure functions $g_1$ and $g_2$ can be derived from this relation by comparing the coefficient of $p^i$ and $q^i$, where $\nu G^{ab}_{2} = g^{ab}_{1}$ and $\nu^2 G^{ab}_{2} = g^{ab}_{2}$. In terms of the $g_1$ and the $g_2$, the result given in Ref.\cite{7} can be written as
\begin{equation}
\int_0^1 dx g_1^{[ab]}(x, Q^2) = -\frac{1}{16}f_{abc}\int_{-\infty}^{\infty} d\alpha \epsilon(\alpha)[A^5_c(\alpha, 0) + \alpha \bar{A}^5_c(\alpha, 0)],
\end{equation}
\begin{equation}
\int_0^1 dx g_2^{[ab]}(x, Q^2) = \frac{1}{16}f_{abc}\int_{-\infty}^{\infty} d\alpha \epsilon(\alpha)\alpha \bar{A}^5_c(\alpha, 0),
\end{equation}
\begin{equation}
\int_0^1 dx g_2^{(ab)}(x, Q^2) = 0,
\end{equation}
where
\begin{equation}
< p, s | A^{5\beta}_c(x|0) | p, s >_c = s^\mu A^5_c(p-x, x^2) + p^\mu(x|s)\bar{A}^5_c(p-x, x^2) + x^\mu(x|s)A^5_c(p-x, x^2),
\end{equation}
and, for $i = 1, 2$,
\begin{equation}
g_i^{(ab)} = \frac{1}{2}(g_i^{ab} + g_i^{ba}), \quad g_i^{[ab]} = \frac{1}{2i}(g_i^{ab} - g_i^{ba}).
\end{equation}
Corresponding to the sum rule (10), Beg sum rule where the right hand side of Eq.(10) was zero had been known in the equal-time formalism and considered to be peculiar since it was invalid in the free field model. This fact was discussed in Ref.\cite{10}, and also in Ref.\cite{11}. The modification which appears on the right
3 Sum rules for the $g_1$ and the $g_2$ in the isovector reaction\cite{3, 5}

The sum rules (10) and (11) correspond to the moment at $n = 0$. They are for the anti-symmetric combination under the interchange $a \leftrightarrow b$. Since the right-hand side is $Q^2$ independent, we obtain

$$
\int_0^1 \frac{dx}{x} g_{i}^{[ab]}(x, Q^2) = \int_0^1 \frac{dx}{x} g_{i}^{[ab]}(x, Q^2_0), \quad \text{for} \quad i = 1, 2.
$$

(15)

Now, we take $Q^2_0 = 0$ and use the relation between the structure function $G_1$ and the photo-production

$$
G_1^{ab}(\nu, 0) = -\frac{1}{8\pi^2\alpha_{em}}\left\{\sigma_{3/2}^{ab}(\nu) - \sigma_{1/2}^{ab}(\nu)\right\} = -\frac{1}{8\pi^2\alpha_{em}}\Delta\sigma^{ab}(\nu).
$$

(16)

By setting $a = (1 + i2)/\sqrt{2}, b = a^\dagger$, and separating out the elastic contribution, we obtain the sum rule which relates the isovector part of the $g_1$ and the cross section in the photo-production as in the Cabibbo-Radicati sum rule.\cite{16} Now the Regge theory predicts as $g_{1}^{[ab]} \sim \beta x^{-\alpha(0)}$ with $\alpha(0) \leq 0$, and hence the sum rule is convergent. However, the perturbative behavior like the DGLAP is divergent. The double logarithmic $(\log(1/x))^2$ resummation or the total resummation of $(\log(1/x))^k$ gives the Regge like behavior but the sum rule is also divergent.\cite{17} In such a situation, it is desirable to discuss the regularization of the sum rule and gives it a physical meaning even when the sum rule is divergent. Now, the regularization of the divergent sum rule has been known to be done by the analytical continuation from the nonforward direction by assuming Regge type behavior.\cite{18} We first derive the finite sum rule in the small but sufficiently large $|t|$ region by assuming the moving pole or cut. Then we subtract the singular pieces which we meet as we go to the smaller $|t|$ from both hand sides of the sum rule by obtaining the condition for the coefficient of the singular piece. After taking out all singular pieces we take the limit $|t| \to 0$. Because of the kinematical structure in the course to derive the sum rule, we can mimic this procedure in the forward direction by introducing the parameter which reflects the $t$ in the non-forward direction. The sum rule obtained in this way can be transformed to the form where the high energy behavior from both hand sides of the sum rule is subtracted away. Practically, if the cancellation at high energy is effective, since the condition is needed only in the high energy limit, we consider that the sum rule holds irrespective of the condition. In this
way, we subtract the high energy behavior from both hand sides of the sum rule as in the following way.

\[ \int_0^1 \frac{dx}{x} \{ g_1(x, Q^2) - f(x, Q^2) \} + \int_0^1 \frac{dx}{x} f(x, Q^2) = \int_0^1 \frac{dx}{x} \{ g_1(x, Q_0^2) - f(x, Q_0^2) \} + \int_0^1 \frac{dx}{x} f(x, Q_0^2), \]  

where we set \( f(x, Q^2) = \beta(Q^2)x^{-\alpha(0, \epsilon)} + f_1(x, Q^2) \) \( \alpha(0, \epsilon) = a - \epsilon. \) (18)

We take the limit \( \epsilon \to a \) from the region above \( a. \)

\[ \int_0^1 \frac{dx}{x} f(x, Q^2) = \frac{\beta(Q^2)}{\epsilon - a} + \int_0^1 \frac{dx}{x} f_1(x, Q^2). \]  

(19)

After taking out the pole term from both hand side of the sum rule, we take \( \epsilon \to 0. \)

\[ \int_0^1 \frac{dx}{x} \{ g_1(x, Q^2) - f(x, Q^2) \} \]  

(20)

\[ = \int_0^1 \frac{dx}{x} \{ g_1(x, Q_0^2) - f(x, Q_0^2) \} + \int_0^1 \frac{dx}{x} \{ f(x, Q_0^2) - f(x, Q^2) \}. \]

Now, for any \( Q^2, \) we take \( f(x, Q^2) = g_1(x, Q_2^2) \) above \( \nu Q = m_p E_Q \) where \( E_Q = E_c + Q^2/2m_p \) and \( f(x, Q^2) = 0 \) below it, where \( E_Q \) is a cut-off energy in the laboratory frame. We define \( x_c(Q^2) = \frac{Q^2}{2c}, \) and by setting \( Q_0^2 = 0 \) and separating out the Born term, we can rewrite this relation for the proton target as

\[ \int_{x_c(Q)}^1 \frac{dx}{x} [2g_1^{1/2}(x, Q^2) - g_1^{3/2}(x, Q^2)] \]  

(21)

\[ = B(Q^2) - \frac{m_p}{8\pi^2 \alpha_{em}} \int_{E_0}^{E_c} dE [2\Delta \sigma^{1/2} - \Delta \sigma^{3/2}] + K(E_c, Q^2), \]

\[ B(Q^2) = \frac{1}{4} \{ \mu_p - \mu_n \} - \frac{1}{1 + Q^2/4m_p^2} \{ G^+_M(Q^2)[G^+_E(Q^2)] + \frac{Q^2}{4m_p^2} G^+_M(Q^2) \}, \]  

(22)

\[ G^+_E(Q^2) = G^+_E(Q^2) - G^+_E(Q^2), \quad G^+_M(Q^2) = G^+_M(Q^2) - G^+_M(Q^2), \]  

(23)

\[ K(E_c, Q^2) = - \int_{E_Q}^{E_c} \frac{dE}{E} [2g_1^{1/2}(x, Q^2) - g_1^{3/2}(x, Q^2)] \]  

(24)

\[ - \frac{m_p}{8\pi^2 \alpha_{em}} \int_{E_c}^{\infty} dE [2\Delta \sigma^{1/2} - \Delta \sigma^{3/2}], \]  

(5)
\[ g^I, \Delta \sigma^I : \text{isovector photon + proton} \rightarrow \text{state with isospin I}. \] (25)

In case of the \( g_2 \), such a simple method to use the photo-reaction as the regularization point cannot be applied directly. Now in the relation
\[ \Delta \sigma^{ab}(\nu, Q^2) = \sigma_{3/2}^{ab}(\nu, Q^2) - \sigma_{1/2}^{ab}(\nu, Q^2) \] (26)
\[ = -\frac{8\pi^2\alpha_{em}}{K} \left( \frac{g_1^{ab}(x, Q^2)}{\nu} - \frac{m_N^2 Q^2 g_2^{ab}(x, Q^2)}{\nu^3} \right), \]
where \( K = (1 - Q^2/2\nu) \), if we differentiate it by \( Q^2 \) and take the limit \( Q^2 \rightarrow 0 \), we obtain the relation
\[ \frac{g_2^{ab}(x, 0)}{\nu} = \frac{g_2^{ab}(x, 0)}{2m_N^2} + \frac{\nu}{m_N^2} \frac{\partial g_1^{ab}(x, Q^2)}{\partial Q^2} \bigg|_{Q^2=0} \] (27)
\[ + \frac{\nu^2}{8\pi^2m_N^2\alpha_{em}} \frac{\partial \Delta \sigma^{ab}(\nu, Q^2)}{\partial Q^2} \bigg|_{Q^2=0}. \]
Thus we can relate \( g_2^{ab}(x, 0)/\nu \) to the experimentally measurable quantity. Then, by setting \( Q_0^2 = 0 \), we can rewrite the sum rule for the \( g_2^{ab} \) by the same method as in the sum rule for the \( g_1^{ab} \). In this way, we obtain the sum rule for the proton target as
\[ \int_{x_{c}(Q)}^{1} \frac{dx}{x} g_2^{+ -}(x, Q^2) = B_2^{+ -}(Q^2) + \int_{E_0}^{E_c} \frac{dE}{E} g_2^{+ -}(x, 0) + K_2^{+ -}(E_c, Q^2), \] (28)
where
\[ g_2^{+ -}(x, Q^2) = 2g_2^{1/2}(x, Q^2) - g_2^{3/2}(x, Q^2), \] (29)
\[ B_2^{+ -}(Q^2) = \frac{Q^2}{16m_p^2} + \frac{Q^2}{4m_N^2} G_M^+(Q^2)(G_M^+(Q^2) - G_E^+(Q^2)). \] (30)

The same kind of the sum rules can be derived in the electroproduction, if we extend the current commutation relation to the current anti-commutation relation. This extension is possible as a stable hadron matrix element. We explain this fact briefly in the following.

4 DGS representation of the anti-commutator of the current

Let us consider DGS representation[19] by taking the scalar current.
\[ C_{ab}(p \cdot q, q^2) = \int d^4x \exp(irq) < p| [J_a(x), J_b(0)] | p >_c \] (31)
\[ = \int d^4x \exp(irq) \int_0^\infty d\lambda^2 \int_{-1}^1 d\beta h_{ab}(\lambda^2, \beta)i\Delta(x, \lambda^2) \]
\[ = (2\pi) \int_0^\infty d\lambda^2 \int_{-1}^1 d\beta \delta((q + \beta p)^2 - \lambda^2)e(q^0 + \beta p^0)h_{ab}(\lambda^2, \beta). \]
$C_{ab}$ can be written as

$$C_{ab}(p \cdot q, q^2) = \sum_n (2\pi)^4 \delta^4(p + q - n) \langle p|J_a(0)|n\rangle \langle n|J_b(0)|p\rangle \quad (32)$$

$$- \sum_n (2\pi)^4 \delta^4(p - q - n) \langle p|J_b(0)|n\rangle \langle n|J_a(0)|p\rangle.$$  

If we take the rest frame $p = (m, \vec{0})$, since the first term is constrained as $m + q^0 = n^0$, we obtain $q^0 \geq M_s - m$, where $M_s$ is the lowest mass in the $s$ channel continuum. Similarly, since the second term is constrained as $m - q^0 = n^0$ we obtain $q^0 \leq m - M_u$, where $M_u$ is the lowest mass in the $u$ channel continuum. Hence if $m \leq (M_s + M_u)/2$, the first term and the second term are disconnected.

Now, in the DGS representation (31), the support of the weight function $h_{ab}(\lambda^2, \beta)$ lies in the shaded region in the figure. An integration path is $\sigma = 2\beta p \cdot q + q^2$, where $\sigma = \lambda^2 - \beta^2 m^2$. At the rest frame $p \cdot q + \beta m^2 = m(q^0 + \beta p^0)$, hence we obtain $\epsilon(q^0 + \beta p^0) = \epsilon(p \cdot q + \beta m^2)$.

Since for the point $(\beta_1, \sigma_1)$ where $p \cdot q + \beta_1 m^2 = 0$ and $\sigma_1 = -\beta_1^2 m^2$, the inequality $\sigma_1 \geq 2\beta_1 p \cdot q + q^2$ holds, the sign change always occurs in the causality forbidden region $\sigma < -\beta^2 m^2$ as in the figure. In the $s$ channel, since $p \cdot q > 0$, the slope is positive, hence only the region $\epsilon(p \cdot q + \beta m^2) = 1$ contributes. Therefore $s$ channel and $u$ channel are disconnected. Thus combined with the discussion after Eq.(32), we obtain

$$\frac{2\pi}{(2\pi)} \int_{\lambda^2} d\lambda^2 \int_{-1}^{1} d\beta \theta((q + \beta p)^2 - \lambda^2) h_{ab}(\lambda^2, \beta) \theta(q^0 + \beta p^0) \quad (33)$$

$$= \sum_n (2\pi)^4 \delta^4(p + q - n) \langle p|J_a(0)|n\rangle \langle n|J_b(0)|p\rangle,$$

and

$$\frac{2\pi}{(2\pi)} \int_{\lambda^2} d\lambda^2 \int_{-1}^{1} d\beta \theta((q + \beta p)^2 - \lambda^2) h_{ab}(\lambda^2, \beta) \theta(-(q^0 + \beta p^0)) \quad (34)$$

$$= \sum_n (2\pi)^4 \delta^4(p - q - n) \langle p|J_b(0)|n\rangle \langle n|J_a(0)|p\rangle.$$
Eqs.(33) and (34) give us the DGS representation of the current anti-commutator of the current as

\[ W_{ab}(p \cdot q, q^2) = \int d^4x \exp(iqx) < p \{ J_a(x), J_b(0) \} | p >. \]

\( (35) \)

\[ = \int d^4x \exp(iqx) \int_0^\infty d\lambda^2 \int_{-1}^1 d\beta h_{ab}(\lambda^2, \beta) i \Delta^{(1)}(x, \lambda^2) \]

\[ = (2\pi) \int_0^\infty d\lambda^2 \int_{-1}^1 d\beta((q + \beta p)^2 - \lambda^2) h_{ab}(\lambda^2, \beta). \]

The application of this extension for the spin averaged quantity was given in [20]. For the spin dependent part, it was given in [21] but the regularization of the sum rule was not discussed.

5 Sum rules for the \( g_1 \) and the \( g_2 \) in the electro-production [4, 5]

Now, using the DGS representation for the current anti-commutator, the (+ i) component of the anti-commutation relation of the current on the null-plane was derived as

\[< p, s | \{ J^+_a(x), J^0_b(0) \} | p, s >_{x^+ = 0, \text{spin}} \]

\( (36) \)

\[ = -\epsilon^{i \alpha\beta} < p, s | \delta^a_b [\Delta^{(1)}(x) G_{c}^{\alpha\beta}(x|0)] | p, s >_c, \]

where

\[ \Delta^{(1)}(x)|_{x^+ = 0} = -\frac{1}{2\pi} \ln|x^-|\delta(x^\perp), \]

\( (37) \)

and

\[ < p, s | S_{c}^{\alpha\beta}(x|0) | p, s >_c = \frac{1}{2} \left( p^\alpha S_{c}^{\beta}(p \cdot x, x^2) + p^\beta (x \cdot s) S_{c}^{\alpha}(p \cdot x, x^2) + x^\alpha (x \cdot s) S_{c}^{\beta}(p \cdot x, x^2) \right). \]

\( (38) \)

By the same method as in Section 3, from Eq.(36) we can get sum rules for the symmetric combination under the interchange \( a \leftrightarrow b \), hence for the electroproduction. The sum rule of the \( g_1 \) for the proton target is

\[ \int_{x_e}^{1} \frac{dx}{x} g_{1}\!(p, Q^2) = B_{1}\!(p, Q^2) - \frac{1}{8\pi^2\alpha_{\text{em}}} \int_{\nu_0}^{E_{c}} dv \{ \sigma_{3/2}^{\gamma p} - \sigma_{1/2}^{\gamma p} \} + K_{1}\!(E_{c}, Q^2), \]

\( (39) \)

where

\[ B_{1}\!(p, Q^2) = \frac{1}{2} \left\{ F_{1}\!(0)(F_{1}\!(0) + F_{2}\!(0)) - F_{1}\!(Q^2)(F_{1}\!(Q^2) + F_{2}\!(Q^2)) \right\}, \]

\( (40) \)

and

\[ K_{1}\!(E_{c}, Q^2) = \frac{1}{8\pi^2\alpha_{\text{em}}} \int_{\nu_{c}}^{\infty} dv \{ \sigma_{1/2}^{\gamma p} - \sigma_{3/2}^{\gamma p} \} - \int_{\nu_{c}}^{Q} \frac{dv}{\nu} g_{1}\!(p, x^2). \]

\( (41) \)
In case of the $g_2$, we obtain

$$\int_{x_c(Q)}^{1} \frac{dx}{x} g_2^{ep}(x, Q^2) = B_2^{ep}(Q^2) + \int_{E_0}^{E_c} \frac{dE}{E} g_2^{ep}(x, 0) + K_2^{ep}(E_c, Q^2), \tag{42}$$

$$B_2^{ep}(Q^2) = \frac{Q^2}{8m_p^2} \left( 1 + \frac{Q^2}{4m_p^2} \right) G_M^p(Q^2)(G_M^p(Q^2) - G_E^p(Q^2)). \tag{43}$$

Then, we obtain

$$\int_{x_c(Q)}^{1} \frac{dx}{x} (g_1^{ep}(x, Q^2) + g_2^{ep}(x, Q^2)) = B_1^{ep}(Q^2) + B_2^{ep}(Q^2) \tag{44}$$

$$+ \int_{E_0}^{E_c} \frac{dE}{E} (g_1^{ep}(x, 0) + g_2^{ep}(x, 0)) + K_1^{ep}(E_c, Q^2) + K_2^{ep}(E_c, Q^2),$$

where

$$B_1^{ep}(Q^2) + B_2^{ep}(Q^2) = \frac{1}{2} (\mu_p - G_M^p(Q^2) G_E^p(Q^2)). \tag{45}$$

$K_1^{ep}(E_c, Q^2)$ in the sum rule (39) can be estimated with use of parameters in Ref.[22]. We find that $K_1^{ep}(2, 0.05) \sim -0.015$ and $K_1^{ep}(2, 0.1) \sim -0.029$. Therefore, as far as we consider the small $Q^2$ region below 1[GeV$^2$], this correction term is very small and the sum rule should be satisfied by the contribution below the energy $E_c = 2[GeV]$. The contribution in this region are the resonances, the nonresonant continuum, and the Born term. Further, since the Born term changes very rapidly in this region, the sum of the resonances and the nonresonant continuum must also change rapidly. It is just in this region where the sign change of the GDH sum was studied experimentally.[23] A similar relation with the sum rule (39), but the quantity corresponding to the moment at $n = 1$ was given in Ref.[24], because the GDH sum rule[25, 26] and the Ellis-Jaffe[27] sum rule correspond to the sum rule at $n = 1$. The sum rule (39) is the exact relation corresponding to the moment at $n = 0$, and the same kind of the sign change occur due to the same physical origin as in the case of the moment at $n = 1$, and hence can be checked experimentally. In such an analysis, if combined with the analysis of the moment at $n = 1$, the extraction of the $g_1$ from the experimental data of the asymmetry in the resonance region at small $Q^2$ will become very important. Now, let us consider another model of the $g_1$ in Ref.[28] from our sum rules (39), (42),(44). The magnitude of the Born term contributions in the moment at $n = 0$ for the $g_1^{ep}$ and the $(g_1^{ep} + g_2^{ep})$ are very similar, but that of the $g_2^{ep}$ is very small compared with these since it is proportional to $Q^2$. However, if this Born term is divided by $Q^2/2$, it has a finite limit as $Q^2 \to 0$, and has an interesting behavior. This quantity is the one which appears in the Schwinger sum rule for the $g_2^{ep}$ given as[6, 7, 12]

$$-\frac{1}{4m_p^2 + Q^2} G_M^p(Q^2)(G_M^p(Q^2) - G_E^p(Q^2)) + \int_{\nu_0(Q)}^{\infty} d\nu G_{2, 2}^{ep}(\nu, Q^2) = 0, \tag{46}$$
where we separate the Born term in this sum rule. It should be noted that this sum rule is nothing but the sum rule (12) derived from the $+i$ component in the current commutation relation. At large $Q^2$, because the Born term becomes negligible, we have the relation for the inelastic part.

$$I(Q^2) = \int_{\nu_0(Q)}^{\infty} d\nu G^\rho_2(\nu, Q^2) = \frac{2}{Q^2} \int_0^1 dx g^\rho_2(x, Q^2) = 0. \quad (47)$$

Thus we can consider the main contribution in the continuum part in the Schwinger sum rule comes from a relatively low energy region. Therefore, in the sum rule given as

$$\int_{\nu_0(Q)}^{\infty} d\nu G^\rho_2(\nu, Q^2) - \int_{\nu_0}^{\nu_0(Q)} d\nu G^\rho_2(\nu, 0) = B^\rho_S(Q^2), \quad (48)$$

where

$$B^\rho_S(Q^2) = \frac{1}{4m_p^2 + Q^2} \left[ G^\rho_M(Q^2)(G^\rho_M(Q^2) - G^\rho_E(Q^2)) - \frac{\mu_p(\mu_p - 1)}{4m_p^2} \right] \quad (49)$$

the main contribution on the left hand side comes from the low $Q^2$ region. Since the Born term contribution $B^\rho_S(Q^2)$ changes rapidly in this region, the left hand side of the sum rule also changes rapidly. Since we have the relation $\nu = Q^2 / 2$ at the elastic point, $B^\rho_S(Q^2)$ is related to $B^\rho_2(Q^2)$ as

$$B^\rho_S(Q^2) = \frac{2}{Q^2} B^\rho_2(Q^2) - \left\{ \frac{2}{Q^2} B^\rho_2(Q^2) \right\} \bigg|_{Q^2=0} \quad (50)$$

Now the contribution to the quantity

$$\int_{x_e(Q)}^{1} \frac{dx}{x} g^\rho_2(x, Q^2) - \int_{x_e}^{1} \frac{dx}{x} g^\rho_2(x, 0) \quad (51)$$

in the sum rule for the $g^\rho_2$ corresponding to the moment at $n = 0$ comes from the low energy region and we can expect it roughly given by $B^\rho_2(Q^2)$. Thus this sum rule and the Schwinger sum rule gives us the same picture that the rapid behavior of the elastic is compensated by the rapid behavior of the resonance and the nonresonant continuum. Now if we plot the Born term contributions $B^\rho_1(Q^2)$, $B^\rho_1 + B^\rho_2(Q^2)$, and $-B^\rho_S(Q^2)$, at small $Q^2$ below $1[GeV^2]$ we find that these three functions behave very similarly. The difference between $B^\rho_1(Q^2)$ and $-B^\rho_2(Q^2)$ is very small and moreover the difference is almost constant.

Though the moments which give $B^\rho_2(Q^2)$ and $B^\rho_1(Q^2)$ are different, we see that the behavior of the integral of $\{ -2g^\rho_2(x, Q^2)/Q^2 + (2g^\rho_2(x, Q^2)/Q^2) \}_{Q^2=0}$ and that of $\{ g^\rho_1(x, Q^2)/Q - (g^\rho_1(x, Q^2)/Q) \}_{Q^2=0}$ in the small $Q^2$ region is very similar. Since the latter is related to the sign change of the GDH sum, this fact may suggest that the $g^\rho_2$ is related to this phenomena. However, in our approach, we have no direct relation between the $g^\rho_1$ and the $g^\rho_2$. 

10
6 Summary

Sum rules of the spin dependent structure functions based on the canonical quantization on the null-plane corresponding to the moment at \( n = 0 \) are regularized. By taking the regularization point at \( Q^2 = 0 \) and relating to the photoproduction, spin dependent structure functions \( g_1 \) and \( g_2 \) at small \( Q^2 \) in the low and the intermediate energy region are shown to be severely constrained. For example, we find, in the small \( Q^2 \) region near \( Q^2 \sim 0.1(\text{GeV}/c)^2 \), that the integral \( \int_{x_c}^1 \frac{dx}{x} g_1^p(x,Q^2) \) becomes zero and that it changes the sign from the negative to the positive. This behavior is caused by the rapid change of the resonances to compensate the rapid change of the elastic to satisfy the sum rule. It is this rapid change of the resonances which gives the sign change of the GDH sum. In addition to this tight connection among the resonances, the elastic and the nonresonant continuum, we also find that this behavior is very similar between the \( g_1^{ep} \) and the \( g_2^{ep} \) as pointed out in Ref.[28].

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