Fujita regularity exponent for scale invariant damped semilinear wave equation

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Abstract

The aim of this paper is to prove a blow up result of the solution for a scale invariant damped wave equation with scale invariant mass and super-Strauss power nonlinearity:

\[
\begin{cases}
v_{tt} - \Delta v + \frac{\mu}{1+t} v_t + \frac{\mu}{2} \left( \frac{\mu}{2} - 1 \right) \frac{v(t,x)}{(1+t)^{\mu/2}} = |v|^p, & t \geq 0, \ x \in \mathbb{R}^n, \\
v(0,x) = 0, \\
v_t(0,x) = \varepsilon g(x),
\end{cases}
\]

under a slow decay condition on the radial initial datum \(g(x) = g(|x|)\). In addition we estimate the lifespan of the solution.

1 Introduction

In the recent years, the following Cauchy problem for the wave equation with scale invariant damping spreads a new line of research on variable coefficient type equations. More precisely, we are dealing with

\[
\begin{cases}
v_{tt}(t,x) - \Delta v(t,x) + \frac{\mu}{1+t} v_t(t,x) + \frac{\mu}{2} \left( \frac{\mu}{2} - 1 \right) \frac{v(t,x)}{(1+t)^{\mu/2}} = |v(t,x)|^p, & t \geq 0, \ x \in \mathbb{R}^n, \\
v(0,x) = 0, \\
v_t(0,x) = \varepsilon g(x),
\end{cases}
\]

with \(n \geq 2\), \(\mu \geq 0\) and \(p > 1\). The study of high dimensional case was treated by considering radial initial data. A competition between two critical exponents appeared. In particular, from the wave equation theory, in some cases a Strauss exponent \(p_S(d)\) is dominant; the Strauss exponent is defined as the positive root of the quadratic equation

\[
(d - 1)p^2 - (d + 1)p - 2 = 0.
\]

On the contrary, under opposite assumptions, the equation goes to an heat equation and the Fujita exponent \(p_F(h) := 1 + \frac{2}{h}\) appears. In all known results, the quantities \(d, f \geq 0\) depend

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on $\mu$ and $n$. The novelty of our result consists in showing that if one takes into account the decay rate of the initial data then a blow-up critical curve appears; this curve is given by a Fujita type exponent which depends also on such decay rate. More precisely, we will prove the following.

**Theorem 1.** Let $n \geq 2$, $p > 1$ and $p < p_F(\frac{4}{n} - 1)$ if $\mu > 2$. Assume that $g$ is a radial smooth solution and there exists $M > 0$ such that for any $x \in \mathbb{R}^n$ it holds

$$g(|x|) \geq \frac{M}{(1 + |x|)^{k+1}}, \quad \text{with } p < p_F\left(\hat{k} + \frac{\mu}{2}\right).$$

(1.2)

Given $\epsilon > 0$, the maximal classical solution of (1.1) is defined in $[0, T(\epsilon)) \times \mathbb{R}^n$ with finite lifespan $T(\epsilon) > 0$. Moreover, it satisfies

$$T(\epsilon) \leq C\epsilon^{-\frac{2(p-1)}{4-(\mu+2\hat{k})(p-1)}},$$

(1.3)

with $C > 0$, independent of $\epsilon$.

In this introduction we compare our results with that ones in literature underlying the influence of the decay rate of initial data. In Section 2, we reduce our problem to a wave-type problem with time-dependent potential. Finally, we give the proof of the main theorem in Section 3.

1.1 The case $\mu = 2$

Let us start with a quite simple case

$$\begin{cases}
v_{tt} - \Delta v + \frac{2}{t^2}v_t = |v|^p, & t \geq 0, \quad x \in \mathbb{R}^n, \\
v(0, x) = 0, \\
v_t(0, x) = \varepsilon g(x).
\end{cases}$$

(1.4)

The global existence of small data solutions for this problem was first solved in [D] for a suitable range of $n$ and $p$. Some non-existence results were also established for $p < p_F(n) := 1 + \frac{2}{n}$. Except for the one-dimensional case a gap between this value and the admissible exponents in [D] appeared. In [DLR] for dimension $n = 2, 3$, this gap was covered with an unexpected result. Indeed, in that paper the Strauss exponent came into play. Afterwards, the global existence of small data solutions to (1.4) is proved for any $p > p_S(n + 2)$ also in odd dimension $n \geq 5$ in [DL] and in even dimension $n \geq 4$ in [Pe]. Moreover, we know that the exponent $p_2(n)$ is optimal; in fact, in [DLR] the authors prove the blow up of solutions of (1.4) for each $1 < p \leq p_2(n)$ in each dimension $n \in \mathbb{N}$. In [DL, DLR, Pe], the authors prove a global existence result not necessarily when the initial datum $g = g(x)$ has compact support. More precisely, let $n \geq 3$, given a radial initial data $g(x) = g(|x|)$ with $g \in C^1(\mathbb{R})$, for any $p > p_S(n + 2)$ it is possible to choose $\hat{k} > 0$ and $\varepsilon_0 > 0$ such that (1.4) admits a radial global solution $u \in C([0, \infty) \times \mathbb{R}^n) \cap C^2([0, \infty) \times \mathbb{R}^n \setminus \{0\})$ provided

$$|g^{(h)}(r)| \leq \varepsilon(r)^{-\left(\hat{k} + 1 + h\right)} \quad \text{for } h = 0, 1.$$  

(1.5)

In the present paper we discuss the dependence of $\hat{k}$ from $n$ and $p$. In (1.5), the exponent $\hat{k}$ has to belong to a suitable interval $[k_1(n, p), k_2(n, p)]$. It is interesting to investigate the
case of $k \notin [k_1(n, p), k_2(n, p)]$. In the sequel we will see that the bound $k_2(n, p)$ can be easily improved. On the contrary if $k < k_1(n, p)$ then a new result appears. The known situation is the following:

- $k_1(3, p) = \max \left\{ \frac{3-p}{p-1}, \frac{1}{p-1} \right\}$ and $k_2(3, p) = 2(p - 1)$, see [DLR].
- $k_1(n, p) = \max \left\{ \frac{3-p}{p-1}, \frac{n-1}{2} \right\}$ and $k_2(n, p) = \min \left\{ \frac{(n+1)p}{2} - 2, \frac{n^2-2n+13}{2(n-3)} \right\}$ if $n \geq 5$ odd, see [DL].
- $k_1(n, p) = \max \left\{ \frac{3-p}{p-1}, \frac{n-1}{2} \right\}$ and $k_2(n, p) = \min \left\{ \frac{(n+1)p}{2} - 2, n - 1 \right\}$ if $n \geq 4$, see [Pe].

We can write in a different way the previous conditions. Firstly we concentrate on the case $n = 3$. For $p \in (1, 2)$ we have $\bar{k} \geq \frac{3-p}{p-1}$ that is equivalent to

$$p \geq 1 + \frac{2}{k + 1} = p_F(\bar{k} + 1).$$

From above we have $\bar{k} \leq 2(p - 1)$ that is

$$p \geq \frac{\bar{k}}{2} + 1.$$

The intersection of $p = p_F(\bar{k} + 1)$ and $p = 1 + \bar{k}/2$ is exactly in $\bar{k} = \frac{-1+\sqrt{17}}{2}$ and $p = p_S(5)$.

We summarize the situation in Figure 1. In the following graphs we denote in blue the zone of the known global existence results, in red the zone of the known blow-up results. In this paper we want to cover the white zones.

![Figure 1: n = 3, μ = 2](image)

Reading [DL] we see that the same situation appears for any odd $n \geq 5$. The critical curve

$$p = p_F(\bar{k} + 1)$$

intersect the line

$$p = \frac{2(\bar{k} + 2)}{n + 1}$$
in the Strauss couple

\[
\left( \bar{k}_0, \frac{2(\bar{k}_0 + 2)}{n+1} \right) = \left( \frac{n - 5 + \sqrt{n^2 + 14n + 17}}{4}, p_S(n+2) \right).
\]

The only difference with the case \( n = 3 \) is that, in the global existence zone, a bound from above appears for \( p \) and this has some influence on \( k_2(n,p) \). More precisely one can take

\[
p \leq \frac{n + 1}{n - 3}, \quad \bar{k} \leq \frac{n^2 - 2n + 13}{2(n - 3)} \quad \text{if } n \geq 7,
\]

and \( p \leq 2, \bar{k} \leq 3 \) if \( n = 5 \). Hence, the result of such paper can be represented as in Figure 2 and we want to prove blow up in the white zone below the Fujita curve.

![Figure 2: \( n \geq 5 \) odd, \( \mu = 2 \)]

Even dimension is more delicate. In [Pe] the global existence result is established in the blue zone below the line \( p = \frac{n + 5}{n + 1} \) except on the curve \( p = p_F(\bar{k} + 1) \). For convenience of the reader, we precise that in the notation of [Pe] the role of \( \bar{k} \) is taken by the quantity \( k + \frac{n+1}{2} \).

![Figure 3: \( n \geq 4 \) even, \( \mu = 2 \)]
1.2 The case $\mu \neq 2$

In [Pe] and [Po] the author considers the Cauchy problem \((1.1)\) for the semilinear wave equation with scale invariant damping and mass terms. We see that for $\mu = 2$, it reduces to \((1.4)\). Global existence of solutions to \((1.1)\) holds under the conditions

$$
\mu \in [2, M(\mu)], \quad M(\mu) = \frac{n - 1}{2} \left(1 + \sqrt{\frac{n + 1}{n - 1}}\right).
$$

Moreover, in the even case [Pe], the initial data satisfies \((1.5)\) for $\bar{k} \in (k_1(n, p, \mu), k_2(n, p, \mu)]$ such that

$$
k_1(n, p, \mu) \geq \max \left\{ \frac{n - 1}{2}, \frac{2}{p - 1} - \frac{\mu}{2} \right\};
$$

$$
k_2(n, p, \mu) \leq \min \left\{n - 1, \frac{n + \mu - 1}{2}p - \frac{\mu + 2}{2}\right\}.
$$

Rewriting these condition in term of $p$, we find that

$$
p > p_F\left(\bar{k} + \frac{\mu}{2}\right), \quad p \geq \frac{2\bar{k} + \mu + 2}{n + \mu - 1}.
$$

Again we observe that the intersection of the curves those define the global existence zone gives $p = p_S(n + \mu)$. Hence, the condition $p > p_S(n + \mu)$ appears. Moreover, another bound from above appears:

$$
p < \bar{p} := \min \left\{ p_F(\mu), p_F\left(\frac{n + \mu - 1}{2}\right) \right\},
$$

which distinguish large $\mu$ from small $\mu$. This influences the positions of $k_1$ and $k_2$. For our purpose it is sufficient to say that for $\mu \neq 2$ and even $n$ the situation is similar to Figure 3.

More precisely, in Figure 4 $p_S(n + \mu)$ appears. The blow up result is indeed given in [NPR]. The zone between $p = p_S(n + \mu)$ and $p = p_F\left(\bar{k} + \frac{\mu}{2}\right)$ is not covered by any known result.

![Figure 4](image-url)

**Figure 4**: $n \geq 4$ even, $\mu \neq 2$

The corresponding result of global existence for the Cauchy problem \((1.1)\) in odd space dimension $n \geq 1$ is studied in [Po]. Here, in the radial case the author proves global existence of
small data solutions when the initial datum satisfies the condition (1.5) with \( \bar{k} \in [k_1(n), k_2(n)] \) where \( k_2 \) satisfies (1.7) and it holds:

\[
\begin{align*}
  k_1(3, p, \mu) &= \max \left\{ 1, \frac{2}{p-1} - \frac{\mu}{2}, \frac{1}{p-1} \right\}; \\
  k_1(n, p, \mu) &= \max \left\{ \frac{n-1}{2}, \frac{2}{p-1} - \frac{\mu}{2}, \frac{1}{p-1} \right\}, \quad n \geq 5, \quad \mu \in [2, n-1]; \\
  k_1(n, p, \mu) &= \max \left\{ \frac{n-1}{2}, \frac{2}{p-1} - \frac{\mu}{2}, \frac{1}{p-1} \right\}, \quad n \geq 5, \quad \mu \in (n-1, M(\mu)].
\end{align*}
\]

In any case the condition \( p > p_F(\bar{k} + \frac{\mu}{2}) \) appears. Hence in odd dimension \( n \geq 5 \) the situation is not different from Figure 4.

### 1.3 Main results

Reading Theorem 1, now it is clear that the aim of this paper is to find blowing-up solutions to (1.1) even for \( p > p_S(n + \mu) \) by considering initial data with slow decay. More precisely, let

\[
g(x) \simeq \frac{M}{(1 + |x|)^{\bar{k}+1}}, \quad \text{for} \quad \frac{n-1}{2} < \bar{k} < \bar{k}_0,
\]

where \( \bar{k}_0 \) is such that

\[
p_F(\bar{k}_0 + \frac{\mu}{2}) = p_S(n + \mu).
\]

We will prove the blow up result in the left white side zones in Figures 1, 2, 3, 4: \( k < k_0, p > p_S(n + \mu) \), \( p < p_F(\bar{k}_0 + \frac{\mu}{2}) \). Under the same assumption on \( g \), the quoted results assure that for \( p \geq p_F(\bar{k} + \frac{\mu}{2}) \) and \( p > p_S(n + \mu) \) there is global existence. Hence, \( p = p_F(\bar{k} + \frac{\mu}{2}) \) is a critical curve for the Cauchy problem (1.1).

**Remark 1.** Let us consider the zone \( \bar{k} > \bar{k}_0, p > p_S(n + \mu) \). As we discussed in the introduction, in this zone the global existence results have been proved in the previous literature for \( p \) above a line which depends on \( \bar{k} \), because of a restriction of type \( \bar{k} \leq k_2(n, p, \mu) \) which everytime appears. Actually, this restriction can be avoided; indeed, if the initial datum satisfies (1.8) with \( \bar{k} > k_2(n, p, \mu) \), then we can say that the initial datum also satisfies (1.8) with \( \bar{k} = k_2(n, p, \mu) \). Hence, the global existence of a solution to (1.1) follows from the known results.

Now we rewrite Theorem 1 as a nonlinear wave equation with time dependent potential. Let \( v = v(t, x) \) be a solution of (1.1). Let us define

\[
u(t, x) := (1 + t)^{\frac{\mu}{2}} v(t, x).
\]

Then the function \( u = u(t, x) \) is a solution of the Cauchy problem

\[
\begin{cases}
u_{tt} - \Delta u = (1 + t)^{\frac{\mu}{2}(p-1)}|u|^p, & t \geq 0, \quad x \in \mathbb{R}^n, \\
u(0, x) = 0, \\
u_t(0, x) = \varepsilon g(x).
\end{cases}
\]

(1.9)
Due to radial assumption, we rewrite (1.9) as
\[
\begin{align*}
\begin{cases}
u_{tt} - \frac{n-1}{r} u_r = (1 + t)^{-\frac{n}{2}(p-1)}|u|^p, & t \geq 0, \quad r > 0, \\
u(0, r) = 0, \\
u_t(0, r) = \varepsilon g(r).
\end{cases}
\end{align*}
\] (1.10)

We will derive a blow-up result for this variant of (1.1). Moreover we give information on the maximal existence time of the local solution. This lifespan depends not only on the exponent of the nonlinear term, not only on \(\mu\) but also on the size of the decay rate of the initial data.

**Theorem 2.** Let \(n \geq 2, \ p > 1\) and \(p < p_F(\frac{\mu}{2} - 1)\) if \(\mu > 2\). Assume that there exists \(M > 0\) such that for any \(r \in [0, \infty)\) it holds
\[
g(r) \geq \frac{M}{(1 + r)^{\bar{k} + 1}}, \quad \text{with} \quad -1 < \bar{k} < \frac{2}{p - 1} - \frac{\mu}{2}.
\] (1.11)

Given \(\varepsilon > 0\), the lifespan \(T(\varepsilon) > 0\) of classical solutions to (1.10) satisfies
\[
T(\varepsilon) \leq C\varepsilon^{-\frac{2(p-1)}{4-\mu+2\mu(p-1)}},
\] (1.12)
with \(C > 0\), independent of \(\varepsilon\).

**Remark 2.** The assumption \(p < p_F(\frac{\mu}{2} - 1)\) if \(\mu > 2\) guarantees that the range of admissible \(\bar{k}\) in (1.11) is not empty.

**Remark 3.** For any \(\mu > 0\) Theorem 2 provides some new information about the solution of (1.1) also when \(p\) belongs to the red zone of Figure 1, 2, 3, 4, 5. In fact, for
\[
p < \min \left\{ \frac{pS(n + \mu)}{2}, p_F\left(\frac{\mu}{2}\right) \right\}
\]
by the previous literature we know that the solution blows up in finite time, whereas Theorem 2 gives a life-span estimate in the case of radial initial data with non compact support.

In the case \(\mu = 0\) this result coincides with the result of Takamura in [T]. In the proof of Theorem 2 we will follows the same approach of that paper. In [T] the point is to find a critical decay level \(k_0 = \frac{2}{p^2}\). Though it is equivalent, in our approach we prefer to underline that \(p = 1 + \frac{k_0}{k_0} = \frac{2}{p^2}\) is a Fujita-type exponent.
**Figure 5: \( \mu = 0 \)**

In Figure 5 the red blow-up zone was covered by many authors, see [S] and the reference therein for the whole list of blow up result. For \( \mu = 0 \) the global existence result has been completely solved in [GLS], where the interested reader can find a long bibliography of previous contribute. In particular the blue zone, for radial solution without assumption on support of initial data has been exploited by Kubo, see for example [K] and [KK]. The role of Takamura result is to prove blow-up in the green zone.

**Remark 4.** In [GL] we will also consider a variant of problem (1.1), in which the nonlinearity depend on \( v, t, v_t \) combined in a suitable way.

# 2 Proof of Theorem 2

We recall the crucial lemma of [T].

**Lemma 1.** Let \( n \geq 2 \) and \( m = \lfloor n/2 \rfloor \). Given a smooth function \( g = g(|x|) \) with \( x \in \mathbb{R}^n \), we set \( r = |x| \) and we consider \( g = g(r) \). Let us denote by \( u^0(t, r) \) the solution of the free wave problem

\[
\begin{aligned}
\square u^0 &= 0 \quad (t, r) \in [0, \infty) \times [0, \infty) \\
u^0(0, r) &= 0, \quad u^0_t(0, r) = g(r).
\end{aligned}
\]

Let \( u = u(t, r) \) be a solution to

\[
u^0_t - u_{rr} - \frac{n-1}{r} u_r = F(t, u) \text{ in } [0, \infty)^2
\]

with the initial condition

\[
u(0, r) = 0, \quad u_t(0, r) = \varepsilon g(r), \quad r \in [0, \infty).
\]

If \( F \) is nonnegative, there exists a constant \( \delta_m > 0 \) such that

\[
u(t, r) \geq \varepsilon u^0(t, r) + \frac{1}{8r^m} \int_0^t dt \int_{r-t+t}^{r+t+\tau} \lambda^m F(t, u(t, \lambda))d\lambda,
\]

\[
u^0(t, r) \geq \frac{1}{8r^m} \int_{r-t}^{r+t} \lambda^m g(\lambda)d\lambda,
\]

provided

\[r - t \geq \frac{2}{\delta_m} t > 0.
\]

The constant \( \delta_m \) in the previous lemma is described in [T, Lemma 2.5]; it depends on the space dimension, in particular from the different representations of the free wave solution in odd and even dimension.

We are ready to prove that if (1.11) holds, then the solution of (1.10) blows up in finite time even for small \( \varepsilon \).

Let us fix \( \delta > 0 \); we define a blow-up set,

\[
\Sigma_\delta = \left\{(t, r) \in (0, \infty)^2 : r - t \geq \max \left\{ \frac{2}{\delta_m} t, \delta \right\} \right\},
\]

(2.3)
where \( \delta_m > 0 \) is the constant given in Lemma 1. Combining the assumption \((1.10)\) with the formulas \((2.1)\) and \((2.2)\), for any \((t, r) \in \Sigma_\delta\), it holds

\[
u(t, r) \geq \varepsilon u^0(t, r) \geq \frac{\varepsilon}{8r^m} \int_{r-t}^{r+t} \lambda^m g(\lambda) d\lambda \geq \frac{M \varepsilon}{8r^m} \int_{r-t}^{r+t} \lambda^m (1 + \lambda)^{-(k+1)} d\lambda.
\]

Then, \((2.3)\) implies that

\[
u(t, r) \geq \frac{M \varepsilon}{8r^m} \left( \frac{1 + \delta}{\delta} \right)^{-(k+1)} \int_{r-t}^{r+t} \lambda^{m-(k+1)} d\lambda \geq \frac{M \varepsilon}{8r^m} \left( \frac{1 + \delta}{\delta} \right)^{-(k+1)} \int_{r-t}^{r+t} \lambda^m d\lambda \geq \frac{M \varepsilon}{\delta} \left( \frac{1 + \delta}{\delta} \right)^{-(k+1)} (r-t)^m 2t \frac{2t}{r^m(r+t)^{k+1}}.
\]

Since \((t, r) \in \Sigma_\delta\), we have

\[
u(t, r) \geq \frac{C_0 t^{m+1}}{r^m(r+t)^{k+1}},
\]

where we set

\[
C_0 = \frac{2^{m-2} M}{\delta_m^{m+1}} \left( \frac{\delta}{1 + \delta} \right)^{k+1} > 0.
\]

Now we assume an estimate of the form

\[
u(t, r) \geq \frac{C t^a}{r^m(r+t)^b} \text{ for } (t, r) \in \Sigma_\delta,
\]

where \(a, b, \text{ and } C\) are positive constant. In particular, \((2.3)\) holds true for \(a = m+1, b = k+1\) and \(C = C_0\).

Being \(g \geq 0\), from \((2.2)\) we deduce \(u^0 \geq 0\). Combining \((2.1)\) and \((2.5)\), for \((t, r) \in \Sigma_\delta\), we get

\[
u(t, r) \geq \frac{1}{8r^m} \int_0^t \int_{r-t+\tau}^{r+t-\tau} \frac{\lambda^m}{(1+\tau)^{\frac{p}{2}(p-1)}} |u(\tau, \lambda)|^p d\lambda d\tau \\
\geq \frac{C^p}{8r^m} \int_0^t \int_{r-t+\tau}^{r+t-\tau} \frac{\lambda^m}{(1+\tau)^{\frac{p}{2}(p-1)}} d\tau d\lambda \\
\geq \frac{C^p}{8r^m(r+t)^{p(b+m(p-1))}} \int_0^t \int_{r-t+\tau}^{r+t-\tau} \frac{\lambda^m}{(1+\tau)^{\frac{p}{2}(p-1)}} d\lambda d\tau \\
\geq \frac{C^p}{4r^m(r+t)^{p(b+m(p-1))}} \int_0^t \int_{r-t+\tau}^{r+t-\tau} \frac{\lambda^m}{(1+\tau)^{\frac{p}{2}(p-1)}} d\lambda d\tau.
\]

By means of integration by parts, we obtain

\[
\int_0^t \frac{(t-\tau)^{\tau p_a}}{(1+\tau)^{\frac{p}{2}(p-1)}} d\tau \geq \frac{1}{(1+t)^{\frac{p}{2}(p-1)}} \int_0^t (t-\tau)^{\tau p_a} d\tau \geq \frac{1}{(1+t)^{\frac{p}{2}(p-1)}} \frac{t^{p a + 2}}{(pa + 1)(pa + 2)}.
\]

While searching a finite lifespan of a solution, it is not restrictive to assume \(t > 1\). We have

\[
\int_0^t \frac{(t-\tau)^{\tau p_a}}{(1+\tau)^{p-1}} d\tau \geq \frac{t^{p a + 2}}{2^{p-1}(pa + 1)(pa + 2)}.
\]

\[9\]
Let \((t, r) \in \Sigma_\delta\), from (2.5)-(2.6), we can conclude
\[
 u(t, r) \geq \frac{C^* t^a}{r^m (r + t)^b} \text{ for } (t, r) \in \Sigma_\delta, \tag{2.7}
\]
with
\[
a^* = p \left( a - \frac{\mu}{2} \right) + 2 + \frac{\mu}{2}, \quad b^* = pb + m(p - 1), \quad C^* = \frac{(C/2)^p}{2(p^2 + 2)^2}.
\]

Let us define the sequences \(\{a_k\}, \{b_k\}, \{C_k\}\) for \(k \in \mathbb{N}\) by
\[
a_{k+1} = p \left( a_k - \frac{\mu}{2} \right) + 2 + \frac{\mu}{2}, \quad a_1 = m + 1, \tag{2.8}
\]
\[
b_{k+1} = pb_k + m(p - 1), \quad b_1 = \bar{k} + 1, \tag{2.9}
\]
\[
C_{k+1} = \frac{(C_k/2)^p}{2(p^2 + 2)^2}, \quad C_1 = C_0, \tag{2.10}
\]
where \(C_0\) is defined by (2.4). Hence, we have
\[
a_{k+1} = p^k \left( m + 1 - \frac{\mu}{2} + \frac{2}{p - 1} \right) + \frac{\mu}{2} - \frac{2}{p - 1}, \tag{2.11}
\]
\[
b_{k+1} = p^k(\bar{k} + 1 + m) - m, \tag{2.12}
\]
\[
C_{k+1} \geq K \frac{C_k^p}{p^{2k}} \tag{2.13}
\]
for some constant \(K = K(p, \mu, m) > 0\) independent of \(k\). The relation (2.13) implies that for any \(k \geq 1\) it holds
\[
C_{k+1} \geq \exp \left( p^k \left( \log(C_0) - S_p(k) \right) \right), \tag{2.14}
\]
\[
S_p(k) = \sum_{j=0}^k d_j, \tag{2.15}
\]
\[
d_0 = 0 \text{ and } d_j = \frac{j \log(p^2) - \log K}{p^j} \text{ for } j \geq 1. \tag{2.16}
\]

We note that \(d_j > 0\) for sufficiently large \(j\). Since \(\lim_{j \to \infty} d_{j+1}/d_j = 1/p\), the sequence \(S_p(k)\) converges for \(p > 1\) by using the ratio criterion for series with positive terms. Hence, there is a positive constant \(S_{p,K} \geq S_p(k)\) for any \(k \in \mathbb{N}\), so that
\[
C_{k+1} \geq \exp(p^k(\log(C_0) - S_{p,K})), \tag{2.17}
\]

Therefore, by (2.7), (2.11)- (2.14), we obtain
\[
u(r, t) \geq \frac{(r + t)^m}{r^m t^\frac{\mu}{2} + \frac{2}{p - 1}} \exp(p^k J(t, r)), \tag{2.18}
\]
where
\[
J(t, r) := \log(C_0) - S_{p,K} + \left( m + 1 - \frac{\mu}{2} + \frac{2}{p - 1} \right) \log t - (\bar{k} + 1 + m) \log(r + t).
\]
Thus if we prove that there exists \((t_0, r_0) \in \Sigma_\delta\) such that \(J(t_0, r_0) > 0\), then we can conclude that the solution to (1.10) blows up in finite time, in fact
\[
u(t_0, r_0) \to \infty \quad \text{for} \quad k \to \infty.
\]
By the definition of \(J = J(t, r)\), we find that \(J(t, r) > 0\) if
\[
\left(\frac{2}{p - 1} - \frac{\mu}{2} - \bar{k}\right) \log t > \log \left(\frac{e^{S_p \kappa}}{C_0} \left(2 + \frac{r - t}{t}\right)^{\bar{k} + 1 + m}\right).
\]
In particular, for \((t, t + \max\{\frac{2t}{\delta_m}, \delta\}) \in \Sigma_\delta\) it is enough to prove that
\[
\left(\frac{2}{p - 1} - \frac{\mu}{2} - \bar{k}\right) \log t > \log \left(\frac{e^{S_p \kappa}}{C_0} \left(2 + \frac{2}{\delta_m}\right)^{\bar{k} + 1 + m}\right).
\]

Now, the crucial (1.11) comes into play. The coefficient in the left side is positive and by using (2.4) we find that \(J(t, r) > 0\) if
\[
t > C \varepsilon^{-\left(\frac{2}{p - 1} - \frac{\mu}{2} - \bar{k}\right)^{-1}}, \quad (2.19)
\]
where
\[
C = \left(\frac{e^{S_p \kappa}}{2^{m - 2} M} \left(1 + \frac{\delta}{\delta_m}\right)^{\bar{k} + 1} \left(2 + \frac{2}{\delta_m}\right)^{1 + \bar{k} + m}\right)^{\frac{1}{p - 1} - \frac{\mu}{2} - \bar{k}}
\]
which is positive. As by product (2.19) gives the lifespan estimate (1.12) and conclude the proof of Theorem 2.

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