Preconditioning complex symmetric linear systems

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Abstract

A new polynomial preconditioner for symmetric complex linear systems based on Hermitian and skew-Hermitian splitting (HSS) for complex symmetric linear systems is herein presented. It applies to Conjugate Orthogonal Conjugate Gradient (COCG) or Conjugate Orthogonal Conjugate Residual (COCR) iterative solvers and does not require any estimation of the spectrum of the coefficient matrix. An upper bound of the condition number of the preconditioned linear system is provided. Moreover, to reduce the computational cost, an inexact variant based on incomplete Cholesky decomposition or orthogonal polynomials is proposed. Numerical results show that the present preconditioner and its inexact variant are efficient and robust solvers for this class of linear systems. A stability analysis of the method completes the description of the preconditioner.

Keywords: Complex Symmetric Linear System, HSS preconditioner, Orthogonal Polynomials

1. Introduction

Focus of this paper is the solution of the complex linear system given by $Ax = b$, where the symmetric complex matrix $A$ has the property that can be written as $A = B + iC$ with $B, C$ two real symmetric semi-positive definite matrices (semi-SPD) and $B + C$ a symmetric positive definite (SPD) matrix. This kind of linear systems arises, for example, in the discretization of problems in computational electrodynamics \cite{52} or time-dependent Schrödinger equations, or in conductivity problems \cite{16, 36}.

If $A$ is Hermitian, a straightforward extension of the Conjugate Gradients (CG) algorithm can be used \cite{51}. Unfortunately, the CG method can not be directly employed when $A$ is only complex symmetric, thus, some specialized iterative methods must be adopted. An effective one is the HSS with its variants (MHSS), which need, at each iteration, the solution of two real linear systems. Other standard procedures to solve this problem are given by numerical iterative methods based on Krylov spaces and designed for complex symmetric linear systems: COCG \cite{52}, COCR \cite{46}, CSYM \cite{15}, CMRH \cite{42}. Some iterative methods for non SPD linear systems like BiCGSTAB \cite{50}, BiCGSTAB($\ell$) \cite{44, 43}, GMRES \cite{41} and QMR \cite{25} can be adapted for complex symmetric matrices \cite{1, 30, 51}.

Purpose of this paper is to develop a polynomial preconditioner to speed up the MHSS process and to propose a preconditioned version of COCG and COCR.

Methods based on Hermitian and Skew-Hermitian Splitting (HSS) \cite{8, 9, 10, 11} can be used as standalone solvers or combined (as preconditioner) together with CG like algorithms. The speed of convergence of CG like iterative schemes depends on the condition number of matrix $A$, thus, preconditioning is a standard way to improve convergence \cite{5, 12}. Incomplete LU is a standard and accepted way to precondition linear systems. Despite its popularity, incomplete LU is potentially unstable, difficult to parallelize and lacks of algorithmic scalability. Nevertheless, when incomplete LU is feasible and the preconditioned linear system is well conditioned, the resulting algorithm is generally the most performing.

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In this work we focus on large problems, where incomplete LU preconditioning is too costly or not feasible. In this case, iterative methods like SSOR are used as preconditioners, but a better performance is obtained using HSS iterative methods, which allow to reduce the condition number effectively. However, HSS iterative methods need the solution of two SPD real systems at each step. Standard preconditioned CG methods can be used at each iteration [11] which can again be preconditioned with an incomplete Cholesky factorization, although for very large problems, the incomplete Cholesky factorization may not be convenient or not feasible. As an alternative, in the present paper is proposed a polynomial preconditioner that allows to solve the linear system for large matrices. Polynomial preconditioners have not a good reputation as preconditioners [5, 12] and research on this subject was dropped out in the late 80’s. In fact, polynomial preconditioners based on Chebyshev polynomials need accurate estimate of minimum and maximum eigenvalue, while least squares polynomials were not computed using stable recurrences, limiting the degree of available stable polynomials [3, 4, 31, 38, 22]. However, in the last years, polynomial preconditioner went back to the top after the works of Lu-Lai-Xu[34]. Notwithstanding anything contained above, here we propose as a preconditioner the use of a polynomial approximation of a modified HSS step. A specialization for Chebyshev and Jacobi orthogonal polynomials is discussed and the corresponding polynomial preconditioner is evaluated using a stable recurrence which permits to use very high degree polynomials.

The paper has this structure. Section 1.1 describes the problem and gives a brief summary of existing methods for resolution with the fundamental results and variants that lead to the present method, in particular, the HSS is considered. Section 2 shows how to use one step of the MHSS method as a preconditioner and gives a bound on the conditioning number of the MHSS iteration matrix. Section 3 explains the iterative solution method with the strategy to adopt when Cholesky factorization is possible or not. Section 4 presents a scale transformation of the system in order to move the eigenvalues to the range (0, 1]. Section 5 describes the polynomial preconditioner based first on least squares and then in terms of orthogonal polynomials and furnishes a stable recurrence for the computation of polynomial preconditioners for high degrees. The specialization for Chebyshev and Jacobi orthogonal polynomial is presented. Section 6 studies the numerical stability of this process and Section 8 concludes the paper.

1.1. The MHSS iterative solver

The complex $N \times N$ linear system $Ax = b$, where $A$ is complex symmetric, is solved via an iterative method based on a splitting algorithm (HSS). The preconditioner requires to solve (each time it is applied) two real symmetric and positive definite (SPD) linear systems.

The HSS scheme can be summarized in this manner, suppose to decompose the vector $x$ of the unknowns as a real and imaginary part, $x = y + iz$, and accordingly, the right hand side $b = c + id$, the system $Ax = b$ is then rewritten as:

$$(B + iC)(y + iz) = c + id, \quad (1)$$

so that, the two-steps of the Modified HSS method proposed in reference 8 results in:

$$(V + B)x^{(k+1)/2} = (V - iC)x^{(k)} + b,$$

$$(W + C)x^{(k+1)} = (W + iB)x^{(k+1)/2} - ib, \quad (2)$$

for suitable matrices $V$ and $W$. The previous procedure can be rewritten as a single step of a splitting based scheme $Px^{k+1} = Qx^k + b$ by posing

$$P = (V + B)[W - iV]^{-1} (W + C),$$

$$Q = (V + B)[W - iV]^{-1} (W + iB)(V + B)^{-1}(V - iC).$$

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This iterative method converges if the iteration matrix $P^{-1}Q$, e.g.,

$$P^{-1}Q = (W + C)^{-1}(W + iB)(V + B)^{-1}(V - iC),$$

(3)

has spectral radius strictly less than one. It is well known that the choice $V = W = \alpha I$, for a given positive constant $\alpha$, yields the standard HSS method [7, 8, 9] for which an estimate of the spectral radius is given by

$$\varrho(P^{-1}Q) \leq \max_{j=1,2,\ldots,N} \left\{ \sqrt{\alpha^2 + \lambda_j(B)^2} / (\alpha + \lambda_j(B)) \right\},$$

where $\lambda_j(B)$ are the eigenvalues of the SPD matrix $B$. The optimal value for $\alpha$ can also be computed [7, 8, 9] and is

$$\alpha_{opt} = \arg \min_{\alpha} \max_{j=1,2,\ldots,N} \left\{ \sqrt{\alpha^2 + \lambda_j(B)^2} / (\alpha + \lambda_j(B)) \right\} = \sqrt{\lambda_{min}(B)\lambda_{max}(B)}.$$

From the previous formulas, it is clear that those computations rely on the knowledge (or estimate) of the minimum and maximum eigenvalue of matrix $B$, which is, in general, a hard problem.

Another possible choice for $V$ and $W$ is $V = \alpha B$ and $W = \beta B$ which yields, when $\alpha = \beta$, a variant of the MHSS method by [7, 8, 9]. In the next Lemma, an upper bound of the spectral radius of the iteration matrix is given.

**Lemma 1.1.** Let $V = \alpha B$ and $W = \beta B$ in (3) with $B$ a SPD matrix and $C$ a semi-SPD matrix, then the spectral radius of $P^{-1}Q$ satisfies the upper bound

$$\varrho(P^{-1}Q) \leq U(\alpha, \beta), \quad U(\alpha, \beta) = \frac{\sqrt{1 + \beta^2}}{1 + \alpha} \max \left\{ 1, \frac{\alpha}{\beta} \right\}$$

and the minimum value of the upper bound $U(\alpha, \beta)$ is attained when $\alpha = \beta = 1$ where $U(1,1) = \sqrt{2} / 2 \approx 0.707$.

**Proof.** Posing $V = \alpha B$ and $W = \beta B$ in (3) yields

$$P^{-1}Q = (\beta B + C)^{-1}(\beta B + iB)(\alpha B + B)^{-1}(\alpha B - iC) = \frac{\beta + i}{1 + \alpha}(\beta B + C)^{-1}(\alpha B - iC).$$

If $\lambda$ is an eigenvalue of matrix $P^{-1}Q$, it satisfies

$$0 = \det \left( P^{-1}Q - \lambda I \right) \downarrow$$

$$0 = \det \left( \frac{\beta + i}{1 + \alpha} (\alpha B - iC) - \lambda(\beta B + C) \right) \downarrow$$

$$0 = \det \left( \left( \frac{\beta + i}{1 + \alpha} - \lambda \right) B - \left( \frac{i \beta + i}{1 + \alpha} + \lambda \right) C \right) \downarrow$$

$$0 = \det \left( \alpha(i - \lambda \beta) + \beta(\alpha - \lambda) \right) \frac{B - C}{\lambda(\alpha + 1) - 1 + \beta i}. $$
Thus, \( \mu \) defined as
\[
\mu = \frac{\alpha(i - \lambda \beta) + \beta(\alpha - \lambda)}{\lambda(\alpha + 1) - 1 + \beta i},
\]
is a generalized eigenvalue, i.e., it satisfies \( \det(\mu B - C) = 0 \) and it is well known that \( \mu \) must be non negative. Computing \( \lambda \) from (4), the function \( \lambda(\mu) \) is found to be:
\[
\lambda(\mu) := \frac{(\alpha + i \mu)(\beta + i)}{(1 + \alpha)(\beta + \mu)},
\]
which allows to evaluate an upper bound \( U(\alpha, \beta) \) of the spectral radius of \( P^{-1}Q \):
\[
g(P^{-1}Q) \leq \sup_{\mu \geq 0} |\lambda(\mu)| = \sup_{\mu \geq 0} \sqrt{\alpha^2 + \mu^2} \sqrt{1 + \beta^2} \leq U(\alpha, \beta).
\]
There are optimal values of \( \alpha \) and \( \beta \) that minimize \( U(\alpha, \beta) \) for \( \alpha \geq 0 \) and \( \beta \geq 0 \). To find them, set \( \alpha = \ell \sin \theta \) and \( \beta = \ell \cos \theta \) with \( \theta \in [0, \pi/2] \), then
\[
U(\alpha, \beta) = \frac{\sqrt{1 + \ell^2(\cos \theta)^2}}{\ell \sin \theta} \max \{1, \tan \theta\}.
\]
If \( \theta \) is fixed, the minimum of this last expression is for \( \ell = \sin \theta / (\cos \theta)^2 \) corresponding to
\[
U(\alpha, \beta) = \cos \theta \max \{1, \tan \theta\} = \max \{\cos \theta, \sin \theta\} \quad \text{for } \theta \in [0, \pi/2].
\]
The minimum of \( U(\alpha, \beta) \) is attained for \( \theta = \pi/4 \), which corresponds to \( \alpha = \beta = 1 \). The computation of \( U(1, 1) \) is then just a computation.

Being \( \alpha = \beta = 1 \) optimal, from now on it is assumed \( \alpha = \beta = 1 \) and therefore \( V = W = B \). Using these values, the two step method (2) is recast as the one step method:
\[
x^{(k+1)} = P^{-1}Qx^{(k)} + b = x^{(k)} + P^{-1}(b - Ax^{(k)}),
\]
where the simplified expressions for \( P \) and \( Q \) are:
\[
P = (1 + i)(B + C), \quad Q = C + iB.
\]
Notice that \( P \) is well defined and non singular provided that \( B + C \) is not singular. Thus the requests of Lemma 1.1 are weakened when \( \alpha = \beta = 1 \), resulting in the next corollary.

**Corollary 1.2.** Let \( B \) and \( C \) be semi-SPD with \( B + C \) not singular, \( P \) and \( Q \) as defined in (6), then the spectral radius of \( P^{-1}Q \) satisfies the upper bound \( g(P^{-1}Q) \leq \sqrt{2}/2 \).

**Remark 1.3.** The spectral radius of the iteration matrix is bounded independently of its size, thus, once the tolerance is fixed, the maximum number of iterations is independent from the size of the problem.

Iterative method (5)-(6) can be reorganized in the following algorithm:
Matrix $P$ with $n$ system (1), is analyzed in terms of the reduction of the condition number. Performing

2. Use of MHSS as preconditioner

Although Algorithm [2] can be used to solve linear system (1), better performances are obtained using one
or more steps of Algorithm [1] not to solve the linear system (1), but as a preconditioner for a faster Conjugate
Gradient like iterative solver such as COCG or COCR. The convergence rate estimation for these iterative
schemes for a linear system $M x = (B + C) x = b$ depends on the condition number $\kappa_2 = \|M\|_2 \|M^{-1}\|_2$
where $\|M\|_2 = \sqrt{g(M^T M)}$ is the classic spectral norm of a matrix. The energy norm $\|\cdot\|_M$ induced by
the (real) SPD matrix $M$ is used to obtain the well known estimate

$$\frac{\|x^{(k)} - x^*\|_M^2}{\|x^{(0)} - x^*\|_M^2} \leq 2 \left( \frac{\sqrt{\kappa_2} - 1}{\sqrt{\kappa_2} + 1} \right)^k, \quad \|v\|_M = \sqrt{v^T M v}$$

where $x^*$ is the solution of the linear system. In general, Conjugate Gradient like iterative schemes perform
efficiently if a good preconditioner makes the system well conditioned.

2. Use of MHSS as preconditioner

In this section the effect of a fixed number of steps of Algorithm [1] used as a preconditioner for linear
system (1), is analyzed in terms of the reduction of the condition number. Performing $n$ steps of Algorithm [1]
with $x^{(0)} = 0$ is equivalent to compute $x^{(n)} = P_n^{-1} b$ where

$$P_n^{-1} = (I + P^{-1} Q + (P^{-1} Q)^2 + \cdots + (P^{-1} Q)^{n-1}) P^{-1} \quad (7)$$

Matrix $P_n$ can be thought as an approximation of matrix $A = B + iC$.

Thus, it is interesting to obtain an estimate of the condition number of the preconditioned matrix $P_n^{-1} A$
in order to check the effect of MHSS when used as preconditioner.

For the estimation, we need to recall some classical results about spectral radii and norms. For any matrix $M$
and any matrix norm, Gelfand’s Formula connects norm and spectral radius [26, 53]:

$$\varrho(M) = \limsup_{k \to \infty} \|M^k\|^{1/k}. \quad (8)$$

Notice that when $\varrho(M) < 1$, for $k$ large enough, $\|M^k\| < 1$.

**Lemma 2.1.** Let $A = P - Q$ so that $P^{-1} Q$ be such that $\varrho(P^{-1} Q) < 1$, then for any $\varepsilon > 0$
satisfying $\varrho(P^{-1} Q) + \varepsilon < 1$ there is an integer $n_\varepsilon > 0$ such that:

$$\kappa(P_n^{-1} A) \leq 1 + (\varrho(P^{-1} Q) + \varepsilon)^n, \quad n \geq n_\varepsilon$$

where $\kappa(M) = \|M\| \|M^{-1}\|$ is the condition number with respect to the norm $\|\cdot\|$ and $P_n$ is defined by [7].

| Algorithm 1: MHSS iterative solver for $Ax = (B + iC)x = b$ |
|---|
| 1 \ $r \leftarrow b$; \ $x \leftarrow 0$; |
| 2 \ **while** $\|r\| > \varepsilon \|b\|$ do |
| 3 \ \hspace{0.5cm} Solve $(B + C)h = \frac{1 - i}{2} r$; \ $x \leftarrow x + h$; \ $b \leftarrow b - Ax$; |
| 4 \ **end while** |
Proof. Observe that $P^{-1}A = P^{-1}(P - Q) = I - P^{-1}Q$, hence using (7) the preconditioned matrix becomes $P^{-1}A = I - (P^{-1}Q)^n$. From Gelfand’s Formula (8) there exists $n_\varepsilon$ such that

$$
\|(P^{-1}Q)^n\| \leq (\varrho(P^{-1}Q) + \varepsilon)^n < 1 \quad \text{for all } n \geq n_\varepsilon
$$

(9)

and from (9), by setting $M = (P^{-1}Q)^n$, the convergent series (see [29, 18]) gives a bound for the norm of the inverse

$$(I - M)^{-1} = \sum_{k=0}^{\infty} M^k, \quad \|(I - M)^{-1}\| \leq \sum_{k=0}^{\infty} \|M\|^k = \frac{1}{1 - \|M\|}
$$

The thesis follows trivially from (9).

Corollary 2.2. Let $A = P - Q$ so that $P^{-1}Q$ has the property that $\varrho(P^{-1}Q) < 1$, then there exists a matrix norm $|||\cdot|||$ such that the conditioning number of matrix $P^{-1}A$ with respect to this norm satisfies:

$$
\kappa(P^{-1}A) \leq \frac{1 + 0.8^n}{1 - 0.8^n} \leq 9
$$

where $\kappa(M) = |||M||| \|\|M^{-1}\||$.

Proof. Recall that for any matrix $M$ and $\varepsilon > 0$ there exists a matrix norm $|||\cdot|||$ such that $|||M||| \leq \varrho(M) + \varepsilon$. This is a classical result of linear algebra, see e.g. section 2.3 of [29] or section 6.9 of [18]. Thus, given $P^{-1}Q$ and chosing $0 < \varepsilon \leq 0.8 - \sqrt{2}/2$ there exists a matrix norm $|||\cdot|||$ such that $|||P^{-1}Q||| \leq 0.8$. The proof follows from Lemma 2.1.

From Corollary 1.2, using the Euclidean norm and choosing $\varepsilon$ such that $\varrho(P^{-1}Q) + \varepsilon = \sqrt{2}/2 + \varepsilon = 0.8$ for $n \geq n_\varepsilon$, the condition number of the preconditioned matrix satisfies:

$$
\kappa_2(P^{-1}A) \leq \frac{1 + 0.8^n}{1 - 0.8^n}, \quad \kappa_2(M) = \|M\|_2 \|M^{-1}\|_2.
$$

(10)

This estimate shows that using $n$ steps of MHSS, with $n$ large enough, the condition number of the preconditioned system can be bounded independently of the size of the linear system. In practice, when $n = 1$ the reduction of the condition number is enough, in fact, Corollary 2.2 shows that, using the appropriate norm, the condition number of the preconditioned linear system is less than 9, independently of its size.

Remark 2.3. From reference [29], when the condition number $\kappa$ is large, the estimate of $m$ conjugate gradient (CG) iterations satisfies $m \propto \sqrt{\kappa}$. The cost of computation is proportional to the number of iterations, whereas the cost of each iteration is proportional to $1 + Cn$, where $C$ is the cost of an iteration of MHSS used as preconditioner, relative to the cost of a CG step. Thus, using $\kappa_2$ from (10), when $n$ is large enough, a rough estimate of the computational cost is

$$
cost \propto \sqrt{\kappa_2(P^{-1}A)}(1 + Cn) \propto \sqrt{\frac{1 + 0.8^n}{1 - 0.8^n}}(1 + Cn).
$$

(11)
Fixing $C$, the cost \([11]\), as a function of $n$ alone, is convex and has a minimum for $n = v_{\min}(C)$ which satisfies:

$$C = \frac{-a^n \ln(a)}{a^n \ln(a) - a^{2n} + 1}, \quad a = 0.8.$$ (12)

The inverse function $C(n)$ which satisfies $n_{\min}(C(n)) = n$, is the r.h.s of \([12]\), moreover, $C(n)$ is a monotone decreasing function so that also $n_{\min}(C)$ is monotone decreasing.

The following table shows the constant $C(n)$ as a function of $n$ giving the critical values of $C$ such that $n$ steps of MHSS are better than $n-1$ steps,

| $n_{\min}$ | $C(n_{\min})$ |
|-----------|---------------|
| 1         | 0.98          |
| 2         | 0.47          |
| 3         | 0.29          |
| 4         | 0.2           |
| 5         | 0.14          |
| 6         | 0.1           |
| 7         | 0.07          |

This means that it is convenient to use $n > 1$ steps in the preconditioner MHSS only if the cost of one step of MHSS is less than 0.47, i.e., the half of one step of the Conjugate Gradient method. A situation that never happens in practice.

According to Remark \([23]\) it is considered only $P_n$ with $n = 1$, i.e. $P_1 = P$ as preconditioner for linear system \([1]\).

3. Iterative solution of complex linear system

From the previous section, it is clear that the use of one iteration step of MHSS is a good choice that lowers the conditioning number of the original complex linear system \([1]\). The resulting preconditioner matrix is $P = (1 + i)(B + C)$ as defined in \([4]\). Preconditioner $P$ will be used together with semi-iterative methods specialized in the solution of complex problems. Examples of those methods there are COCG \([51, 45]\) or COCR \([25, 40, 46, 45]\). They are briefly exposed next.

| Algorithm 2: COCG | Algorithm 3: COCR (as in \([45]\)) |
|-------------------|-----------------------------------|
| 1 $r \leftarrow b - Ax$; | 1 $r \leftarrow b - Ax$; $\tilde{r} \leftarrow P^{-1}r$; $p \leftarrow \tilde{r}$; |
| 2 $\tilde{r} \leftarrow P^{-1}r$; | 2 $q \leftarrow Ap$; |
| 3 $p \leftarrow \tilde{r}$; | 3 $\rho \leftarrow [\tilde{r}, q]$; |
| 4 $\rho \leftarrow [\tilde{r}, r]$; | 4 while $\|r\| > \varepsilon \|b\|$ do |
| 5 while $\|r\| > \varepsilon \|b\|$ do |
| 6 $q \leftarrow Ap$; | 5 $\tilde{q} \leftarrow P^{-1}q$; |
| 7 $\mu \leftarrow [q, p]$; | 6 $\alpha \leftarrow \rho / [\tilde{q}, q]$; |
| 8 $\beta \leftarrow \rho / \mu$; | 7 $\tilde{x} \leftarrow \tilde{x} + \alpha p$; |
| 9 $x \leftarrow \tilde{x} + \alpha p$; | 8 $r \leftarrow r - \alpha q$; |
| 10 $r \leftarrow r - \alpha q$; | 9 $\tilde{r} \leftarrow \tilde{r} - \alpha \tilde{q}$; |
| 11 $\tilde{r} \leftarrow P^{-1}r$; | 10 $t \leftarrow A\tilde{r}$; |
| 12 $\beta \leftarrow p$; | 11 $\beta \leftarrow \rho$; |
| 13 $\rho \leftarrow [\tilde{r}, r]$; | 12 $p \leftarrow [\tilde{r}, t]$; |
| 14 $\beta \leftarrow \rho / \beta$; | 13 $\beta \leftarrow \beta / \beta$; |
| 15 $p \leftarrow \tilde{r} + \beta p$; | 14 $p \leftarrow \tilde{r} + \beta p$; |
| 16 end while | 15 $q \leftarrow t + \beta q$; |

There are also other methods available for performing this task, for example CSYM \([15]\) or QMR \([24]\). The application of the preconditioner $P$ for those methods is equivalent to the solution of two real SPD systems depending on $B + C$, as in Remark \([14]\). Of course, one can use a direct method \([12, 20, 27, 17]\), or a conjugated gradient method \([11]\) with incomplete Cholesky factorizations, or approximate inverse as preconditioner \([12, 13, 19, 21, 35, 39]\). Nevertheless, it is not so convenient to adopt this philosophy for
very large linear systems because for example $B + C$ can not be formed or just the fill-in of the incomplete Cholesky-based factorization is unacceptable. In the next sections, a polynomial preconditioner based on orthogonal polynomials is presented, it will allow to solve very large complex linear systems.

The suggested strategy to get the solution of the complex linear system is resumed in Algorithm 4.

| Algorithm 4: Solution strategy for the solution of $Ax = (B + iC)x = b$ |
| --- |
| 1 Solve linear system by using iterative method like COCG (Algorithm 2) or COCR (Algorithm 3); |
| 2 For the preconditioner $P$ use the following strategy; |
| 3 if complete Cholesky of $B + C$ computable then |
| 4 Compute $LL^T = B + C$ and use $P = (1 + i)LL^T$ as preconditioner. |
| 5 else if incomplete Cholesky of $B + C$ computable then |
| 6 Compute incomplete Cholesky $LL^T + E = B + C$; |
| 7 if $\|E\|$ “small” then |
| 8 The incomplete Cholesky is a good approximation of $B + C$, thus, use $P = (1 + i)LL^T$ as preconditioner |
| 9 else if $\|E\|$ not “too large” then |
| 10 To compute $P^{-1}v$ the incomplete Cholesky is used as preconditioner for the solution of the two real system $(B + C)z = v/(1 + i)$ by PCG method. |
| 11 else |
| 12 goto 15 |
| 13 end if |
| 14 else |
| 15 In this case incomplete Cholesky is too inaccurate or matrix is too large so that incomplete Cholesky can not be computed. In this case use proposed polynomial preconditioner. |
| 16 end if |

4. Scaling the complex linear system

The polynomial preconditioner presented in the next section depends on the knowledge of an interval containing eigenvalues. Scaling is a cheap procedure to recast the problem into a one with eigenvalues in the interval $[0, 1]$. Consider the diagonal matrix $S$ and the linear system (1), the scaled system is:

$$(SAS)w = (SBS + iSCS)w = Sb, \quad \text{with} \quad x = Sw,$$

where $S$ is a real diagonal matrix with positive entries on the diagonal. The scaled system inherits the properties of the original and still has the matrices $SBS$ and $SCS$ semi-SPD with $SBS + SCS$ SPD. The next lemma shows how to choose a good scaling factor $S$ used forward:

**Lemma 4.1.** Let $M$ be a SPD matrix and $S$ a diagonal matrix with $S_{ii} = \left(\sum_{j=1}^{n} |M_{ij}|\right)^{-1/2}$, then the scaled matrix $SMS$ has the eigenvalues in the range $(0, 1]$.

**Proof.** Notice that $SMS$ is symmetric and positive defined and is similar to $S^2M$. Moreover, the estimate $\lambda_{\max}(S^2M) \leq \|S^2M\|_\infty = 1$ follows trivially. \qed

**Assumption 4.2.** From Lemma 4.1, the linear system (1) is scaled to satisfy:

- Matrices $B$ and $C$ are semi-SPD;
- Matrix $B + C$ is SPD with eigenvalues in $(0, 1]$. 

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5. Preconditioning with polynomials

On the basis of the results of the previous sections with Assumption 1.2, the linear system to be preconditioned has the form $Ax = b$ with $A = B + iC$ with $B$ and $C$ semi-SPD and $M = B + C$ SPD with eigenvalues distributed in the interval $(0, 1]$.

A good preconditioner for this linear system is one step of MHSS in Algorithm 1 which results in a multiplication by $P^{-1}$ where $P = (1 + i)(B + C) = (1 + i)M$. Here the following polynomial approximation of $P^{-1}$ is proposed:

$$P^{-1} = \frac{1 - i}{2} M^{-1} \approx \frac{1 - i}{2} s_m(M).$$

The matrix polynomial $s_m(M)$ must be an approximation of the inverse of $M$, i.e. $s_m(M)M \approx I$ where $s_m(x)$ is a polynomial with degree $m$. A measure of the quality of the preconditioned matrix for a generic polynomial $s(x)$ is the distance from the identity matrix:

$$Q_\sigma(s) = \|s(M)M - I\|_2 = \max_{\lambda \in \sigma(M)} |1 - \lambda s(\lambda)|,$$  \hspace{1cm} (13)$$

where $\sigma(M) = \{\lambda_1, \ldots, \lambda_n\}$ is the spectrum of $M$. If, in particular, the preconditioned matrix $s_m(M)M$ is the identity matrix then $Q_\sigma(s_m) = 0$. Thus, the polynomial preconditioner $s_m$ should concentrate the eigenvalues of $s_m(M)M$ around 1 in order to be effective.

A preconditioner polynomial can be constructed by minimizing $Q_\sigma(s)$ of equation (13) within the space $\Pi_m$ of polynomials of degree at most $m$. This implies the knowledge of the spectrum of matrix $M$ which is in general not available making problem (13) unfeasible. The following approximation of quality measure (13) is feasible

$$Q_{[\epsilon,1]}(s) = \max_{\lambda \in [\epsilon,1]} |1 - \lambda s(\lambda)|, \hspace{1cm} \sigma(M) \subset [\epsilon,1]$$  \hspace{1cm} (14)$$

and needs the knowledge of $[\epsilon, 1]$, an interval for $\epsilon > 0$, containing the spectrum of $M$. The polynomial which minimizes $Q_{[\epsilon,1]}(s)$ for $s \in \Pi_m$ is well known and is connected to an appropriately scaled and shifted Chebyshev polynomial. The construction of such solution is described in section 5.1 and was previously considered by Ashby et al. [31], Johnson et al. [31], Freund [23], Saad [38] and Axelsson [5]. The computation of $Q_{[\epsilon,1]}(s)$ needs the estimation of a positive lower bound of the minimum eigenvalue of $M$, which is, in general, expensive or infeasible. The estimate $\epsilon = 0$ cannot be used because $Q_{[0,1]}(s) \geq 1$ for any polynomial $s$. A different way to choose $\epsilon$ is analysed later in this section. Saad observed that the use of Chebyshev polynomials with the conjugate gradient method, i.e. the polynomial which minimizes the condition number of the preconditioned system, is in general far from being the best polynomial preconditioner, i.e. the one that minimizes the CG iterations [38]. Practice shows that although non optimal, Chebyshev preconditioners perform well in many situations. The following integral average quality measure proposed in [31, 38, 32, 1] is a feasible alternative to (14):

$$Q(s) = \int_0^1 |1 - \lambda s(\lambda)|^2 \, d\lambda, \hspace{1cm} \sigma(M) \subset [0,1].$$  \hspace{1cm} (15)$$

The preconditioner polynomial $s_m$ proposed here is the solution of minimization of quality measure (14) or (15):

$$s_m = \operatorname{argmin}_{s \in \Pi_m} Q_{[\epsilon,1]}(s), \hspace{1cm} \text{or} \hspace{1cm} s_m = \operatorname{argmin}_{s \in \Pi_m} Q(s),$$  \hspace{1cm} (16)$$

Solution of problem (16) is detailed in the next sections. The proposed solution to the first problem is by means of the Chebyshev polynomials, while the solution of the second problem is done with the Jacobi weight.
5.1. Chebyshev polynomial preconditioner

The solution of minimization problem (16) with quality measure $Q_{[\epsilon,1]}(s)$ is well known and can be written in terms of Chebyshev polynomials [3, 4]:

$$1 - \lambda s_m(\lambda) = \frac{T_{m+1}(\lambda)}{T_{m+1}'(0)},$$

where $T_{m+1}(\lambda)$ is the $(m+1)$-th Chebyshev polynomial scaled in the interval $[\epsilon,1]$. Polynomials $T_k^\epsilon(\lambda)$ satisfy the recurrence

$$T_0^\epsilon(x) = 1, \quad T_1^\epsilon(x) = ax + b, \quad T_{n+1}^\epsilon(x) = 2(ax + b)T_n^\epsilon(x) - T_{n-1}^\epsilon(x),$$

where

$$a = \frac{2}{1 - \epsilon}, \quad b = -\frac{1 + \epsilon}{1 - \epsilon}.$$  

From (17), the preconditioner polynomial $s_m(\lambda)$ becomes

$$s_m(\lambda) = \frac{1}{\lambda} \left( 1 - \frac{T_{m+1}(\lambda)}{T_{m+1}'(0)} \right)$$

and from (18) it is possible to give a recursive definition for $s_m(\lambda)$ too.

**Lemma 5.1 (Recurrence formula for preconditioner).** Given the polynomials $q_n$ defined by the recurrence

$$q_0(x) = 1, \quad q_1(x) = a_0x + b_0,$$

$$q_{n+1}(x) = (a_nx + b_n)q_n(x) + c_nq_{n-1}(x), \quad n = 1, 2, 3, \ldots$$

then the polynomials $r_n(x) = q_n(x)/q_n(0)$ and $s_n(x) = (1 - r_{n+1}(x))/x$ satisfy the recurrences:

$$r_0(x) = 1, \quad s_0(x) = a'_0,$$

$$r_1(x) = 1 - a'_0x, \quad s_1(x) = a'_1x + b'_1,$$

$$r_{n+1}(x) = (a'_nx + b'_n)r_n(x) + c'_nr_{n-1}(x), \quad s_n(x) = (a'_nx + b'_n)s_{n-1}(x) + c'_ns_{n-2}(x) - a'_n,$$

where

$$a'_0 = -\frac{a_0}{b_0}, \quad a'_1 = -\frac{a_0a_1}{b_0b_1 + c_1}, \quad b'_1 = -\frac{a_0b_1 + a_1b_0}{b_0b_1 + c_1}, \quad a'_n = a_n\gamma_n, \quad b'_n = b_n\gamma_n, \quad c'_n = c_n\gamma_{n-1}\gamma_n$$

and $\gamma_n = q_n(0)/q_{n+1}(0)$ satisfies the recurrence

$$\gamma_1 = \frac{b_0}{b_0b_1 + c_1}, \quad \gamma_n = \frac{1}{b_n + c_n\gamma_{n-1}}.$$ (20)
Proof. Take the ratio
\[
\frac{q_{n+1}(x)}{q_{n+1}(0)} = \frac{(a_n x + b_n) q_n(x) + c_n q_{n-1}(x)}{q_{n+1}(0)},
\]
and notice that \( r_n(0) = 1 \) for all \( n \). Recurrence for \( \gamma_n \) is trivially deduced. From \( r_{n+1}(x) = 1 - x s_n(x) \) and by using \( (20) \),
\[
r_{n+1}(x) = (a_n x + b_n) r_n(x) \gamma_n + c_n r_{n-1}(x) \gamma_n \gamma_{n-1},
\]
where \( \gamma_n = T_n^s(0)/T_{n+1}^s(0) \) is computed by solving recurrence \( (19) \) for \( x = 0 \), that is
\[
T_n^s(0) = \frac{1}{2} \left( c^n + c^{-n} \right), \quad c = \frac{\sqrt{\epsilon - 1}}{\sqrt{\epsilon + 1}}, \quad \Rightarrow \quad \gamma_n = \frac{T_n^s(0)}{T_{n+1}^s(0)} = \frac{c^n + c^{-n}}{c^{n+1} + c^{-(n+1)}},
\]
Numerical stability of recurrence \( (21) \) is discussed in section \( [3] \). The estimation of \( \epsilon \) is the complex task and some authors perform it dynamically. As an alternative, the present approach is to move the eigenvalues of the coefficient matrix from the interval \([\epsilon, 1]\) to a stripe \([1 - \delta, 1 + \delta]\), so that the condition number remains bounded. The value of \( \epsilon \) is not determined from the estimate of the eigenvalues, but from the degree of the preconditioner polynomial and from the amplitude of the stripe \( \delta \). Once \( \delta \) is fixed, the higher the degree of the preconditioner, the lower the value of \( \epsilon \), which decreases to zero. Thus, if the degree of the preconditioner is high enough, the eigenvalues are moved in the interval \([1 - \delta, 1 + \delta]\). The important fact is that even if the degree is not high enough to move the complete spectrum, the majority of the eigenvalues are moved in the desired stripe, improving the performance of the conjugate gradient method. An idea of this behaviour is showed in Figure \( [4] \) on the right. The end of this section is devoted to the explicit expression of the value of \( \epsilon \) computed backwards from the value of \( \delta_n \): once the maximum condition number is fixed, it is possible to increase the degree of the polynomial preconditioner so that \( \epsilon \) decreases until the whole (or at least the most) spectrum of the matrix is contained in the specified range.

In the interval \([\epsilon, 1]\), Chebyshev polynomial \( T_n^s(x)/T_n^s(0) \) is bounded in the range \([-\delta, \delta]\) where \( \delta = T_n^s(0)^{-1} = 2/(c^{n+1} + c^{-(n+1)}) \) and solving for \( \epsilon \) gives
\[
\epsilon = \left( \frac{1}{|c|} - 1 \right)^2, \quad |c| = \left( \frac{1 + \sqrt{1 - \delta^2}}{\delta} \right)^{\frac{1}{n+1}}.
\]
5.2. Jacobi polynomial preconditioner

The solution of minimization problem (16) with quality measure \( Q(s) \) is well known and can be written in terms of Jacobi orthogonal polynomials [38]:

**Definition 5.2.** Given a nonnegative weight function \( w(\lambda) : [0, 1] \rightarrow \mathbb{R}^+ \) (see [31, 38, 2]) the scalar product \( \langle \cdot, \cdot \rangle_w \) and the relative induced norm \( \| \cdot \|_w \) are defined as:

\[
\langle p, q \rangle_w = \int_0^1 p(\lambda) q(\lambda) w(\lambda) \, d\lambda, \quad \| p \|_w = \sqrt{\langle p, p \rangle_w} = \sqrt{\int_0^1 p(\lambda)^2 w(\lambda) \, d\lambda}.
\]

where \( p \) and \( q \) are continuous functions. The orthogonal polynomials w.r.t. the scalar product \( \langle \cdot, \cdot \rangle_w \) are the polynomials \( p_k(\lambda) \) which satisfy \( \langle p_k, p_j \rangle_w = 0 \) if \( k \neq j \).

The orthogonal polynomial w.r.t. the weight \( w_{\alpha,\beta}(\lambda) = (1 - \lambda)^\alpha \lambda^\beta \), for \( \alpha, \beta > -1 \), defined in the interval \([0, 1]\), are the Jacobi polynomials and they satisfy the recurrence (see [49]):

\[
p_0^{\alpha,\beta}(x) = 1, \\
p_1^{\alpha,\beta}(x) = a_0^{\alpha,\beta} x + b_0^{\alpha,\beta}, \\
p_{n+1}^{\alpha,\beta}(x) = (a_n^{\alpha,\beta} x + b_n^{\alpha,\beta}) p_n^{\alpha,\beta}(x) + c_n^{\alpha,\beta} p_{n-1}(x),
\]

for

\[
a_n^{\alpha,\beta} = 1, \\
b_n^{\alpha,\beta} = -\frac{1}{2} \left( 1 + \frac{\beta^2 - \alpha^2}{(2n + \alpha + \beta)(2(n + 1) + \alpha + \beta)} \right), \\
c_n^{\alpha,\beta} = -\frac{n(n + \alpha)(n + \beta)(n + \alpha + \beta)}{(2n - 1 + \alpha + \beta)(2n + 1 + \alpha + \beta)(2n + \alpha + \beta)^2}.
\]

The class of polynomials of the form \( 1 - \lambda s(\lambda) \) with \( s \in \Pi_{m-1} \) can be thought as polynomials \( r \in \Pi_{m+1} \) with \( r(0) = 1 \), thus, the minimization problem (16) for \( Q(s) \) can be recast to the following constrained minimization for \( w(\lambda) \equiv 1 \):

\[
r_{m+1} = \arg\min_{r \in \Pi_{m+1}, r(0) = 1} \| r \|_w.
\]

The preconditioner polynomial is \( s_m(\lambda) = \lambda^{-1}(1 - r_{m+1}(\lambda)) \). Polynomial \( r_{m+1}(\lambda) \) is expanded by means of Jacobi orthogonal polynomials with \( \alpha = \beta = 0 \):

\[
r_{m+1}(\lambda) = \sum_{k=0}^{m+1} \alpha_k p_k^{0,0}(\lambda)
\]

Making use of the property of orthogonality w.r.t. the scalar product, one has

\[
\| r_{m+1} \|_w^2 = \sum_{k=0}^{m+1} \alpha_k^2 \left\| p_k^{0,0} \right\|_w^2, \quad 1 = r_{m+1}(0) = \sum_{k=0}^{m+1} \alpha_k p_k^{0,0}(0),
\]
thus the constrained minimum problem \( (22) \) is recast as

\[
\text{minimize} \quad \sum_{k=0}^{m+1} \alpha_k^2 \left\| P_k^{0,0} \right\|_w^2, \quad \text{subject to} \quad \sum_{k=0}^{m+1} \alpha_k P_k^{0,0}(0) = 1. \tag{23}
\]

Problem \( (23) \) is solved by using Lagrange multiplier with first order conditions resulting in

\[
\alpha_k = \frac{p_k(0)}{\left\| p_k \right\|_w^2} \left/ \sum_{k=0}^{m+1} \frac{p_k^{0,0}(0)^2}{\left\| p_k \right\|_w^2} \right. , \quad \Rightarrow \quad r_{m+1}(\lambda) = \sum_{k=0}^{m+1} \frac{p_k^{0,0}(0)^2}{\left\| p_k \right\|_w^2} \left/ \sum_{k=0}^{m+1} p_k^{0,0}(0)^2 \right. . \tag{24}
\]

Remark 5.3. The solution \( (24) \) is only formal but barely useful from a computational point of view, because it requires to compute explicitly the least squares polynomial. In fact, it is well known that the evaluation of a polynomial of high degree is a very unstable process. To make solution \( (24) \) practical, it is mandatory to obtain a stable recurrence formula that allows to evaluate polynomials even of very high degree, e.g. 1000 or more.

To find a recurrence for \( (24) \), it must be rewritten as a ratio of orthogonal polynomials as for the Chebyshev preconditioner \( (17) \). To this scope, some classical theorems and definitions on orthogonal polynomials are here recalled for convenience. Christoffel–Darboux formulas and Kernel Polynomials, here recalled without proofs (see \[49, 47\], are used to build the recurrence.

Theorem 5.4 (Christoffel–Darboux formulas). Orthogonal polynomials w.r.t. the scalar product \( \langle \cdot, \cdot \rangle_w \) share the following identities,

\[
\sum_{j=0}^{k} p_j(x)p_j(y) = \frac{1}{\left\| p_k \right\|_w^2} p_{k+1}(x)p_k(y) - p_{k+1}(y)p_k(x) . \tag{25}
\]

Theorem 5.5 (Kernel Polynomials). Given orthogonal polynomials \( p_k(x) \) w.r.t. the scalar product \( \langle \cdot, \cdot \rangle_w \), i.e. w.r.t weight function \( w(x) \), then the polynomials

\[
q_k(x) = \left( p_{k+1}(x)p_k(0) - p_{k+1}(0)p_k(x) \right)/x,
\]

are orthogonal polynomials w.r.t. the scalar product \( \langle \cdot, \cdot \rangle_w \) defined as \( \langle p, q \rangle_w = \int_0^1 p(x)q(x)dw(x) \) dx, i.e. w.r.t the weight function \( w(x) \). Moreover \( q_0(x) = 1 \).

With the formulas of Christoffel–Darboux \( (25) \) and \( x = \lambda, y = 0 \), it is possible to rewrite \( (24) \) as

\[
r_{m+1}(\lambda) = \frac{1}{C} \frac{p_{m+2}^{0,0}(\lambda)p_{m+1}^{0,0}(0) - p_{m+2}(0)p_{m+1}^{0,0}(\lambda)}{\left\| p_{m+1} \right\|_w^2} , \quad C = \left\| p_{m+1} \right\|_w^2 \sum_{k=0}^{m+1} \frac{p_k^{0,0}(0)^2}{\left\| p_k \right\|_w^2} . \tag{26}
\]

Using the Kernel Polynomials of this last Theorem, expression \( (26) \) becomes

\[
r_{m+1}(\lambda) = \frac{p_{m+1}^{0,1}(\lambda)}{p_{m+1}^{0,1}(0)} .
\]

where \( p_k^{0,1}(x) \) are orthogonal polynomials w.r.t. the weight \( w(x) = \lambda \). In fact, the Kernel Polynomials w.r.t. \( \lambda w^{\alpha,\beta}(\lambda) = w^{\alpha,\beta+1}(\lambda) \) satisfy \( q^{\alpha,\beta}(x) = p^{\alpha,\beta+1}(x) \).
Preconditioner polynomial can be computed recursively using Lemma 5.1 where coefficients $a_n', b_n', c_n'$ and $\gamma_n$ are computed from $a_n^{0,1}, b_n^{0,1}, c_n^{0,1}$. Given
\[
a_n^{0,1} = 1, \quad b_n^{0,1} = -\frac{1}{2} \left( 1 + \frac{1}{(2n + 1)(2n + 3)} \right), \quad c_n^{0,1} = \frac{n(n + 1)}{4(2n + 1)^2},
\]
the values of $a_n', b_n', c_n'$ and $\gamma_n$ from Lemma 5.1 become:
\[
a_0' = \frac{3}{2}, \quad a_1' = -\frac{10}{3}, \quad b_1' = 4,
\]
\[
\gamma_n = a_n' = -4 + \frac{2(3n + 5)}{(n + 2)^2}, \quad b_n' = 2 - \Delta, \quad c_n' = -1 + \Delta, \quad \Delta = \frac{2(3n^2 + 6n + 2)}{(2n + 1)(n + 2)^2}. \tag{27}
\]
The only difficulty of the previous coefficients computation lies in the recursive solution of $\gamma_n$, which is here omitted for conciseness but that is a linear three term recurrence with polynomial coefficients. Notice that $\Delta \rightarrow 0$, thus the limit value of the above coefficients is evident.

5.3. Recurrence formula for the preconditioner

Looking at Algorithms 2 and 3, the polynomial preconditioner $s_m(M)$ is applied to a vector, i.e. $s_m(M)v$. Thus, to avoid matrix-matrix multiplication, by defining $v^{(k)} = s_k(M)v$ and using Lemma 5.1 with (27), the following recurrence is obtained for $s_m(M)v = v^{(m)}$.
\[
v^{(0)} = a_0'v, \quad v^{(1)} = a_1'Mv + b_1'v, \quad v^{(n)} = a_n'(Mv^{(n-1)} - v) + b_n'v^{(n-1)} + c_n'v^{(n-2)}, \tag{28}
\]
where $n = 2, 3, \ldots, m$ and recurrence (28) is the proposed preconditioner with coefficients given by (21) for Chebyshev and (27) for Jacobi the polynomials. Equation (28) yields Algorithm 5.

**Algorithm 5:** Application of preconditioner $s_m(M)$ to a vector $v$.

| Step 1 | $t_1 \leftarrow a_0'v$; | $y \leftarrow a_1'Mv + b_1'v$; |
|--------|----------------------|----------------------|
| Step 2 | for $n = 2, 3, \ldots, m$ do |
| Step 3 | $t_0 \leftarrow t_1$; | $t_1 \leftarrow y$; | $y \leftarrow a_n'(Mt_1 - v) + b_n't_1 + c_n't_0$; |
| Step 4 | end for |
| Step 5 | return $y$; |

6. Numerical stability

Algorithm 5 i.e. the application of preconditioner $s_m(M)$ given by equation (28) to a vector $v$, also taking into account rounding errors, results in
\[
w^{(0)} = a_0'v + \varrho^{(0)}, \quad w^{(1)} = a_1'Mv + b_1'v + \varrho^{(1)},
\]
\[
w^{(n)} = a_n'(Mw^{(n-1)} - v) + b_n'w^{(n-1)} + c_n'w^{(n-2)} + \varrho^{(n)},
\]
where $\| \varrho^{(k)} \|_\infty \leq \delta$ are the errors due to floating point operations with $\delta$ as an upper bound of such errors. The cumulative error $e^{(k)} = w^{(k)} - v^{(k)}$ satisfies the linear recurrence
\[
e^{(0)} = \varrho^{(0)}, \quad e^{(1)} = \varrho^{(1)}, \quad e^{(n)} = a_n'Me^{(n-1)} + b_n'e^{(n-1)} + c_n'e^{(n-2)} + \varrho^{(n)}. \tag{29}
\]
The next definitions introduce the concept of generalized and joint spectral radius needed for the proof of the theorem of the matrix bound, they can be found in [32].
Definition 6.1. A matrix set $\Sigma = \{A_k \in \mathbb{R}^{n \times n}\}_{k \in \mathbb{N}}$ is bounded if there is a constant $C$ such that $\|A\| \leq C$ for all $A \in \Sigma$. An invariant subspace $V$ for $\Sigma$ is a vector space such that $AV \subseteq V$ for all $A \in \Sigma$. The set $\Sigma$ is irreducible if the only invariant subspace are $\{0\}$ or $\mathbb{R}^n$.

Definition 6.2. The generalized spectral radius $\rho(\Sigma)$ and the joint spectral radius $\hat{\rho}(\Sigma, \|\cdot\|)$ of any set of matrices $\Sigma$ are defined as

$$
\rho(\Sigma) = \limsup_{k \to \infty} (\rho_k(\Sigma))^{1/k}, \quad \rho_k(\Sigma) = \sup \left\{ \rho \left( \prod_{i=1}^{k} A_i \right) \mid A_i \in \Sigma \right\}
$$

$$
\hat{\rho}(\Sigma, \|\cdot\|) = \limsup_{k \to \infty} (\hat{\rho}_k(\Sigma, \|\cdot\|))^{1/k}, \quad \hat{\rho}_k(\Sigma, \|\cdot\|) = \sup \left\{ \left\| \prod_{i=1}^{k} A_i \right\| \mid A_i \in \Sigma \right\}
$$

Theorem 6.3. Let $\Sigma$ be a bounded and irreducible set of matrices with $\rho(\Sigma) > 0$, then there is a constant $C$ such that

$$
\|A_1 A_2 \cdots A_k\| \leq C \rho(\Sigma)^k, \quad \forall A_j \in \Sigma
$$

for all $k > 0$.

Proof. It is theorem 2.1 by [32] with a slight modification to match the present case.

Theorem 6.4. Recurrence (29) satisfies

$$
\|e^{(n)}\|_{\infty} \leq (C \delta N) n,
$$

where $N$ is the dimension of the linear system, $\delta$ is the amplitude of the stripe for the eigenvalues and $C$ is an unknown constant coming from the norm inequalities which is found experimentally to be small.

Proof. The matrix $M = B + C$ in (29), by Assumption 4.2 is SPD with eigenvalues in $(0, 1]$. Thus $M = T^T \Lambda T$, with $T$ orthogonal, i.e. $T^T T = I$ and $\Lambda$ diagonal. Multiplying on the left the recurrence (29) by $T$, the following error estimate is obtained,

$$
Te^{(0)} = T\theta^{(0)}, \quad Te^{(1)} = T\theta^{(1)}, \quad Te^{(n)} = a_n^T \Lambda T e^{(n-1)} + b_n^T T e^{(n-1)} + c_n^T e^{(n-2)} + T \theta^{(n)}.
$$

Focusing on $j$th component of the transformed error, $f^{(n)} = (Te^{(n)})_j$ and $\eta^{(n)} = (T\theta^{(n)})_j$, a scalar recurrence is obtained:

$$
f^{(0)} = \eta^{(0)}, \quad f^{(1)} = \eta^{(1)}, \quad f^{(n)} = (a_n^j \lambda_j + b_n^j) f^{(n-1)} + c_n^j f^{(n-2)} + \eta^{(n)}. \quad (30)
$$

Recurrence (30) is restated in matrix form as

$$
f_n = A_n f_{n-1} + b_n, \quad A_n = \begin{pmatrix} a_n^j \lambda_j + b_n^j & c_n^j \\ 1 & 0 \end{pmatrix}, \quad b_n = \begin{pmatrix} \eta^{(n)} \\ 0 \end{pmatrix}, \quad f_n = \begin{pmatrix} f^{(n)} \\ f^{(n-1)} \end{pmatrix}, \quad (31)
$$

with initial data $f_1^T = (\eta^{(1)}, \eta^{(0)})$. Notice that $\eta^{(n)}$ is bounded by

$$
\left| \eta^{(n)} \right| \leq \|T \theta^{(n)}\|_\infty \leq \|T \theta^{(n)}\|_2 = \|\theta^{(n)}\|_2 \leq \sqrt{N} \|\theta^{(n)}\|_\infty \leq \delta \sqrt{N}
$$
and thus, $\|f_1\|_\infty \leq \delta \sqrt{N}$ and $\|b_n\|_\infty \leq \delta \sqrt{N}$. From (31) it follows that

$$f_n = A_n A_{n-1} \cdots A_2 f_1 + \sum_{k=2}^n A_n A_{n-1} \cdots A_{n-k+1} b_k.$$  \hspace{1cm} (32)

The set $\Sigma = \{ A_i | i = 1, \ldots, \infty \}$ is bounded and irreducible, each matrix has spectral radius strictly less than 1, (see Lemma 6.5 for a proof), therefore the joint spectral radius is less than 1. From (32), with Theorem 6.3 using the infinity norm,

$$\left| f^{(n)} \right| \leq \|f_n\|_\infty \leq \|A_n A_{n-1} \cdots A_2\|_\infty \|f_1\|_\infty + \sum_{k=2}^n \|A_n A_{n-1} \cdots A_{n-k+1}\|_\infty \|b_k\|_\infty,$$

$$\leq C g(\Sigma)^{n-2} \delta \sqrt{N} + C \sum_{k=2}^n g(\Sigma)^k \delta \sqrt{N} \leq C \delta \sqrt{N} n,$$

and, because of $f^{(n)} = (Te^{(n)})_j$, it follows that $\|Te^{(n)}\|_\infty \leq C \delta \sqrt{N} n$. A bound of the term $e^{(n)}$ is done as

$$\|e^{(n)}\|_\infty \leq \|e^{(n)}\|_\nu = \|Te^{(n)}\|_\nu \leq \sqrt{N} \|Te^{(n)}\|_\infty \leq C \delta N n.$$

This shows that the error grows at most linearly. \hspace{1cm} \Box

The above relation shows that the recurrence is at worst linearly unstable, i.e. the error grows at most linearly. The existence is proved in the works of Rota and Strang [37] where the concept of joint spectral radius is introduced. The determination of $C$ is not possible but practice reveals that it is small. In conclusion it is possible to employ even a very high degree polynomial preconditioner with a stable computation.

**Lemma 6.5.** Given $a'_n, b'_n, c'_n$ from (21) or (27), respectively for the Chebyshev and the Jacobi preconditioner, then the roots $z_1$ and $z_2$ of the characteristic polynomial of homogeneous recurrence (30), i.e.

$$z^2 - (a'_n \lambda + b'_n)z - c'_n$$

satisfy $|z_1| < 1$ and $|z_2| < 1$ for all $n > 0$ and $0 < \lambda \leq 1$.

**Proof.** Consider first the coefficients for the Jacobi polynomials defined in (27). If the roots are complex, then they must be conjugate, thus $z_1 = z$ and $z_2 = \bar{z}$, because the coefficients of the polynomial are real. In that case, the constant term of the polynomial is equal to the square of the modulus of the roots, $z\bar{z} = |z|^2 = -c'_n$, thus it is easy to see that $|z| < 1$ for all $n > 0$. Suppose now that the two roots $z_1$ and $z_2$ are real, multiplying the characteristic polynomial by $(n + 2)^2(2n + 1)$, yields, after some manipulation:

$$z_1 = \frac{A - B\lambda + \sqrt{B^2\lambda^2 - 2AB\lambda + C}}{D}, \quad z_2 = \frac{A - B\lambda - \sqrt{B^2\lambda^2 - 2AB\lambda + C}}{D}$$

for

$$A = 2n^3 + 6n^2 + 6n + 2, \quad B = 4n^3 + 12n^2 + 11n + 3,$$

$$C = (3n^2 + 6n + 2)^2, \quad D = 2n^3 + 9n^2 + 12n + 4,$$

with $A, B, C$ and $D$ strictly positive for all $n \geq 0$. The discriminant $\Delta(\lambda)$ of the equation is $\Delta(\lambda) = B^2\lambda^2 - 2AB\lambda + C$ and represents a convex parabola because $B^2 > 0$. Its minimum is obtained for $\lambda = A/B \in (0, 1)$, which gives $\Delta(A/B) < 0$ and so complex roots, but this case was already considered. Hence we can set
Figure 1: The polynomials described in Lemma (5.1): in the first row the product $\lambda s_m(\lambda)$ to show the approximation of the identity; in the second row the explicit graph of the polynomial preconditioner compared with the function $1/\lambda$; in the third row the performance of the preconditioner in terms of the degree, condition number and concentration of the eigenvalues of the coefficient matrix around 1. The left column represents the Jacobi weight, the right column the Chebyshev polynomials.

$\Delta_{\text{min}} = 0$. The maximum of $\Delta(\lambda)$ is achieved at one of the extrema of the interval of definition of $\lambda$. A quick calculation shows that $\Delta(\lambda)$ is maximum for $\lambda = 0$, yielding a value of $\Delta_{\text{max}} = C$. Using $\Delta_{\text{max}}$ and
\( \Delta_{\text{min}} \) it is possible to bound the roots \( z_1 \) and \( z_2 \):

\[
\begin{align*}
  z_1 &< \frac{A + \sqrt{\Delta_{\text{max}}}}{D} = \frac{2n^3 + 6n^2 + 6n + 2 + (3n^2 + 6n + 2)}{2n^3 + 9n^2 + 12n + 4} = 1, \\
  z_1 &\geq \frac{A - B + \sqrt{\Delta_{\text{min}}}}{D} = \frac{2n^3 + 6n^2 + 5n + 1}{2n^3 + 9n^2 + 12n + 4} > -1, \\
  z_2 &< \frac{A - \sqrt{\Delta_{\text{min}}}}{D} = \frac{2n^3 + 6n^2 + 6n + 2}{2n^3 + 9n^2 + 12n + 4} < 1, \\
  z_2 &\geq \frac{A - B - \sqrt{\Delta_{\text{max}}}}{D} = \frac{2n^3 + 9n^2 + 11n + 3}{2n^3 + 9n^2 + 12n + 4} > -1.
\end{align*}
\]

The previous inequalities prove the lemma for the Jacobi preconditioner. Now consider the case of the coefficients of the Chebyshev polynomials defined in (21). Recall the expression for \( c = (\sqrt{\epsilon} - 1)/(\sqrt{\epsilon} + 1) \), and notice that, for \( \epsilon \in (0, 1) \), \( c \) is bounded in \(-1 < c < 0\), so that

\[
\omega_n := c^n + c^{-n}
\]

is positive for even \( n \) and negative for odd \( n \), moreover, \( |\omega_n| = |c^n + c^{-n}| \geq 2 \) is monotone increasing for \( n = 1, 2, 3, \ldots \). In the case of complex roots, it was already shown that \( \bar{z} z = |z|^2 = -c_n^{'} \), with

\[
0 \leq -c_n^{'} = \frac{\omega_{n-1}}{\omega_{n+1}} < 1, \quad \epsilon \in (0, 1).
\]

In the rest of the proof it is useful to consider also the ratio \(-1 < \frac{\omega_n}{\omega_{n+1}} < 0\), for \( c \in (-1, 0) \), that corresponds to \( \epsilon \in (0, 1) \). The coefficients of equation (21), observing that \( \epsilon = (\omega_1 + 2)/(\omega_1 - 2) \), are simplified in

\[
\begin{align*}
  a_n^{'} &= \frac{4\omega_n}{(1 - \epsilon)\omega_{n+1}} = -\frac{\omega_n}{\omega_{n+1}} (\omega_1 - 2), \quad b_n^{'} = -\frac{2(1 + \epsilon)\omega_n}{(1 - \epsilon)\omega_{n+1}} = \frac{\omega_n\omega_1}{\omega_{n+1}}, \quad c_n^{'} = -\frac{\omega_{n-1}}{\omega_{n+1}}.
\end{align*}
\]

Polynomial (33) is rewritten as

\[
z^2 + \frac{\omega_n}{\omega_{n+1}} (\lambda(\omega_1 - 2) - \omega_1)z + \frac{\omega_{n-1}}{\omega_{n+1}} \quad (34)
\]

and its roots are (using \( \omega_1 \omega_n = \omega_{n-1} + \omega_{n+1} \))

\[
z_{1,2}(\lambda) = \frac{\omega_1 \omega_n}{2\omega_{n+1}} - \frac{\omega_n}{\omega_{n+1}} \frac{\omega_1 - 2}{2} \left[ \lambda \pm \sqrt{\Delta(\lambda)} \right], \quad \Delta(\lambda) = \lambda^2 - \frac{2\omega_1}{\omega_1 - 2} \lambda + \frac{(\omega_{n-1} - \omega_{n+1})^2}{\omega_n^2(\omega_1 - 2)^2}.
\]

Looking at the discriminant \( \Delta(\lambda) \), the minimum of the associated convex parabola is for \( \lambda = \frac{\omega_1}{\omega_1 - 2} \in (1/2, 1) \). The corresponding value is

\[
\Delta \left( \frac{\omega_1}{\omega_1 - 2} \right) = -\frac{4\omega_{n-1}\omega_{n+1}}{\omega_n^2(\omega_1 - 2)^2} < 0,
\]

The value at the right extremum is also negative:

\[
\Delta(1) = 4\frac{\omega_n^2 - \omega_{n-1}\omega_{n+1}}{\omega_n^2(\omega_1 - 2)^2} < 0,
\]

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\[\omega_n^2 - \omega_{n-1}\omega_{n+1} = 2 - c^2 - c^2 = \frac{-(c-1)^2(c+1)^2}{c^2} < 0.\]

For \(\lambda = 0\), with some manipulations, the roots of (34) are
\[z_1(0) = 1, \quad z_2(0) = \frac{\omega_{n-1}}{\omega_{n+1}} < 1.\]

Moreover, \(\Delta(\epsilon) = \Delta(1)\), thus there exists \(0 < \lambda^* < \epsilon\) such that \(\Delta(\lambda^*) = 0\). Thus for \(\lambda \in [\lambda^*, 1]\) the roots are complex conjugate and with modulus less than 1. For \(\lambda \in [0, \lambda^*]\), where the roots \(z_{1,2}(\lambda)\) are real, \(z_1(\lambda)\) and its derivative satisfy
\[z_1(\lambda) = \frac{\omega_n}{2\omega_{n+1}} - \frac{\omega_n}{\omega_{n+1}} \left[ \sqrt{\Delta(\lambda)} - \lambda \right],\]
\[z_1'(\lambda) = \frac{2}{\omega_{n+1}} \left[ \frac{\Delta'(\lambda)}{\sqrt{\Delta(\lambda)}} - 1 \right], \quad \Delta'(\lambda) = 2\lambda - \frac{2\omega_n}{\omega_1 - 2}\]

and thus for \(\lambda = 0\) we have \(z_1'(0) < 0\). Hence in a neighbourhood of \(\lambda = 0\), \(-1 < z_{1,2}(\lambda) < 1\), and for \(\lambda \in (0, \lambda^*)\) there are no roots equal to 1 or 0, thus the roots of (34) are bounded in \((0, 1)\). In facts, by contradiction, let \(z = 1\) be a root, then by (34)
\[1 + \frac{\omega_n}{\omega_{n+1}}(\lambda(\omega_1 - 2) - \omega_1) + \frac{\omega_n}{\omega_{n+1}} = 0, \quad \Rightarrow \quad \lambda = 0.\]

Moreover \(z = 0\) is never a root of (34).

Thus the roots are bounded in the interval \((-1, 1)\) for \(n \geq 0\) and \(\lambda \in (0, 1]\).

\[\square\]

7. Numerical tests

In this section a group of tests is proposed for the solution of a complex linear system of the form (1), i.e. \((B + iC)(y + iz) = c + id\), where \(B\) and \(C\) are semi-SPD with \(B + C\) SPD. The solvers used are COCG (Algorithm 2) and COCR (Algorithm 3) preconditioned with
- ILU0, the incomplete Cholesky ILU(0), for the matrix \(B + iC\);
- MHSS-ILU0, the incomplete Cholesky ILU(0) for the preconditioner \(P\) defined in (6);
- MHSS-JACOBI, the approximation of the preconditioner \(P\) defined in (6) with Jacobi polynomial preconditioner;
- MHSS-CHEB, the approximation of the preconditioner \(P\) defined in (6) with Chebyshev polynomial preconditioner.

The degrees used for the polynomial preconditioner are 10, 50, 100, 500 and 1000. Due to the lack of complex symmetric matrices with SPD real and imaginary part, it was decided to combine two real SPD matrices of not too far dimension (eventually padding with zeros to match the size of the biggest one). The real SPD matrices used are summarized in the next Table and can be found on the NIST “Matrix Market” Sparse Matrix Collection [14] or on University of Florida Sparse Matrix Collection [18]. As usual “NNZ” means number of non zero elements and it is understood that the matrices are square, hence only the number of rows is reported.
These matrices are used paired where the first matrix of the pair corresponds to the real part, the second to the imaginary part. If the dimensions disagree, the smallest matrix is padded with zero rows and columns up to the size of the biggest one. The pairing of the matrices with the name of the corresponding test is resumed in the following Table:

| Test Name | Matrix pairing | N. rows | NNZ |
|-----------|----------------|---------|-----|
| T5k       | (s1rmq4m1, s2rmq4m1) | 5 489   | 265 147 |
| T16k      | (Pres_Poisson, Dubcova1) | 16 129 | 925 819 |
| T80k      | (apache1, nasasrb) | 80 800 | 3 072 500 |
| T150k     | (G2_circuit, denormal) | 150 102 | 1 616 970 |
| T500k     | (pwtk, parabolic_fem) | 525 825 | 14 810 591 |

The right hand side used for all tests, unless explicitly written, is assumed to be \((1 + i)\), that is \(1 + i\) for all components. A test from a real application is found in [16], from which the complex symmetric SPD matrices are provided with a specific right hand side.

List of the tests: for the first group (T tests) the r.h.s. was \((1 + i)\), for the second group (S tests) the r.h.s. was the one prescribed in the paper [16].

The linear system corresponding to each test is solved with COCG and COCR. The preconditioners used are the ILU(0) for the complex matrix \(B + iC\), the proposed preconditioners \((1 + i)(B + C)\) approximated with ILU(0) or Jacobi and Chebyshev polynomials of degree 10, 50, 100, 500, 1000. The results in terms of number of iterations are collected in Table 5 for COCG and COCR.

From Table 5 it is clear that the strategy presented in Algorithm 4 is effective. In fact when ILU factorization is available and iteration converges incomplete factorization preconditioner is faster than polynomial preconditioner. Polynomial preconditioner is an effective alternative when ILU is not available or not sufficient as preconditioner.

Computational time is indicative and was obtained implementing the proposed preconditioner using MATLAB scripts available at Matlab Central. Standard MATLAB incomplete LU is used in the tests.

Rising the degree of the polynomial corresponds in lowering the number of iterations needed by COCG and COCR. It is also apparent that it is not possible to go below a certain number of iterations even with a very high degree polynomial, this is evident for example in test T16k with both COCG and COCR and with both preconditioners.

This is explained from the fact that the condition number of the preconditioned system when preconditioner \((1 + i)(B + C)\) is computed exactly is independent of the system size.

Another behaviour that is common to all tests is the generally better performance of the COCR over the COCG: this can be appreciated looking at Figure 2, 3 and 4. They show the history of the residual for each iteration of both methods with the Jacobi and the Chebyshev preconditioners.
Table 1: Numerical results for COCG and COCR, the reported numbers represents the iterations of the corresponding solver with the specified preconditioner. The dash indicates that it was not feasible to compute a particular test, while the letters “NC” mean “not converged”, i.e., residual is still large after 5000 iterations. The value of $\delta$ used in the Chebyshev preconditioner was 0.2 while the stopping tolerance was $10^{-8}$. The time elapsed in the computation is expressed in seconds.

|      | COCG |                 |          |          |          | COCR |                 |          |          |          |
|------|------|----------------|----------|----------|----------|------|----------------|----------|----------|----------|
|      | ILU0 | degree of MHSS-JACOBI | degree of MHSS-CHEB |
|      |      | 10 | 50 | 100 | 500 | 1000 | 10 | 50 | 100 | 500 | 1000 |
| T5k  |      |     |     |     |     |     |     |     |     |     |     |
| time | 86   | 148 | 127 | 32 | 26 | 25 | 111 | 32 | 32 | 31 | 31 |
|      | 0.2  | 0.3 | 0.9 | 1.0 | 1.6 | 7.4 | 15.0 | 0.8 | 1.0 | 1.9 | 9.2 | 18.4 |
| T16k |      |     |     |     |     |     |     |     |     |     |     |     |
| time | 65   | 78  | 31  | 24 | 24 | 23 | 31  | 29 | 28 | 29 | 30 |
|      | 0.3  | 0.5 | 0.6 | 1.9 | 3.6 | 16.6 | 32.9 | 0.6 | 2.3 | 4.2 | 21.1 | 43.2 |
| T80k |      |     |     |     |     |     |     |     |     |     |     |     |
| time | NC   | NC  | 4765| 657 | 298 | 55 | 31 | 3904 | 596 | 251 | 52 | 33 |
|      |      |     | 275.6| 162.8 | 145.4 | 130.4 | 146.4 | 225.8 | 148.0 | 121.6 | 123.8 | 156.6 |
| T16k |      |     |     |     |     |     |     |     |     |     |     |     |
| time | NC   | NC  | 4928| 829 | 479 | 91 | 47 | 4805 | 929 | 461 | 87 | 46 |
|      |      |     | 1665| 1215 | 1374 | 1378 | 1393 | 1763 | 1333 | 1246 | 1246 | 1380 |
| S13k |      | –   | 4681| 1123 | 591 | 80 | 38 | 4016 | 1199 | 613 | 93 | 57 |
| S32k |      | –   | 4665| 1102 | 686 | 106 | 49 | 4012 | 1156 | 612 | 102 | 64 |
| S500k|      | –   | NC  | 4328| 2165 | 394 | 107 | NC  | 4179 | 2359 | 390 | 185 |

8. Conclusions

It was presented a polynomial preconditioner for the solution of the linear system $Ax = b$, for $A$ complex symmetric such that $A = B + iC$, where $B, C$ are real symmetric semi-positive definite matrices (semi-SPD) and $B + C$ is symmetric positive definite (SPD). Typical problems of this form come from the field of electrodynamics, where the involved matrices are complex but not Hermitian and standard methods can not be used directly. This algorithm is suitable for large matrices, where Cholesky decomposition, or its inexact form, are too costly or infeasible. It works as a polynomial approximation of a single step of the MHSS method, but it is successfully applied as preconditioner of Conjugate Gradient-like methods, in particular it is showed how to use it together with COCG or COCR. Following the trend of the last years, but aware of the criticism that arose in the ‘80s, the proposed new preconditioner is computed as a recurrence of
orthogonal polynomials and is proved to be stable. This allows to employ polynomials of very high degree and numerical tests confirm the expected theoretical good performances.

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10. References

References

[1] K. ABE AND G. L. SLEIPNEN, BiCR variants of the hybrid BiCG methods for solving linear systems with nonsymmetric matrices, Journal of Computational and Applied Mathematics, 234 (2010), pp. 985 – 994.
[2] S. AISHIY, Polynomial Preconditioning for Conjugate Gradient Methods, DOE/ER, Department of Computer Science, University of Illinois at Urbana-Champaign, 1988.
[3] S. AISHIY, Minimax polynomial preconditioning for hermitian linear systems, SIAM Journal on Matrix Analysis and Applications, 12 (1991), pp. 766–789.
[4] S. AISHIY, T. MANTUFEFF, AND J. OTTO, A comparison of adaptive chebyshev and least squares polynomial preconditioning for hermitian positive definite linear systems, SIAM Journal on Scientific and Statistical Computing, 13 (1992), pp. 1–29.
[5] O. AXELSSON, A survey of preconditioned iterative methods for linear systems of algebraic equations, BIT Numerical Mathematics, 25 (1985), pp. 165–187.
[6] O. AXELSSON, Iteration number for the conjugate gradient method, Mathematics and Computers in Simulation, 61 (2003), pp. 421 – 435. MODELLING 2001 - Second IMACS Conference on Mathematical Modelling and Computational Methods in Mechanics, Physics, Biomechanics and Geodynamics.
[7] Z. BAI, G. GOLUB, AND M. NG, Hermitian and skew-hermitian splitting methods for non-hermitian positive definite linear systems, SIAM Journal on Matrix Analysis and Applications, 24 (2003), pp. 603–626.
[8] Z.-Z. BAI, M. BENZI, AND F. CHEN, Modified HSS iteration methods for a class of complex symmetric linear systems, Computing, 87 (2010), pp. 93–111.
[9] Z.-Z. BAI, G. H. GOLUB, AND M. K. NG, Hermitian and skew-hermitian splitting methods for non-hermitian positive definite linear systems, SIAM J. Matrix Anal. Appl., 24 (2002), pp. 603–626.
[10] Z.-Z. BAI, G. H. GOLUB, AND M. K. NG, On inexact hermitian and skew-hermitian splitting methods for non-hermitian positive definite linear systems, Linear Algebra and its Applications, 428 (2008), pp. 413 – 440.
[11] M. BENZI, Preconditioning techniques for large linear systems: A survey, Journal of Computational Physics, 182 (2002), pp. 418 – 477.
[12] L. BERGAMASCHI, G. PINI, AND F. SARTORETTO, Approximate inverse preconditioning in the parallel solution of sparse eigenproblems, Numerical Linear Algebra with Applications, 7 (2000), pp. 99–116.
[13] R. F. BOISVERT, R. POZO, K. REMINGTON, R. F. BARRETT, AND J. J. DONGARRA, Matrix market: A web resource for test matrix collections, in Proceedings of the IFIP TC2/WG2.5 Working Conference on Quality of Numerical Software: Assessment and Enhancement, London, UK, UK, 1997, Chapman & Hall, Ltd., pp. 125–137.
[14] A. BUNSE-GERSTNER AND R. STOVER, On a conjugate gradient-type method for solving complex symmetric linear systems, Linear Algebra and its Applications, 287 (1999), pp. 105 – 123.
[15] L. CODECASA, R. SPECOGNA, AND F. TREVISAN, Base functions and discrete constitutive relations for staggered polyhedral grids, Computer Methods in Applied Mechanics and Engineering, 198 (2009), pp. 1117–1123.
[16] T. A. DAVIS, Direct Methods for Sparse Linear Systems (Fundamentals of Algorithms 2), Society for Industrial and Applied Mathematics, Philadelphia, PA, USA, 2006.
[17] T. A. DAVIS AND Y. HU, The university of florida sparse matrix collection, ACM Trans. Math. Softw., 38 (2011), pp. 1–25.
[18] P. DUBOIS, A. GREENHAAUM, AND G. RODRIGUE, Approximating the inverse of a matrix for use in iterative algorithms on vector processors, Computing, 22 (1979), pp. 257–268.
[19] I. DUFF, A. ERSIN, AND J. REID, Direct methods for sparse matrices, Monographs on numerical analysis, Clarendon Press, 1986.
[20] T. DUPONT, R. KENDALL, AND H. RACHFORD, Jr., An approximate factorization procedure for solving self-adjoint elliptic difference equations, SIAM Journal on Numerical Analysis, 5 (1968), pp. 559–573.
[21] B. FISCHER, Polynomial Based Iteration Methods for Symmetric Linear Systems, Classics in Applied Mathematics, Society for Industrial and Applied Mathematics, 2011.
[22] R. FREUND, On conjugate gradient type methods and polynomial preconditioners for a class of complex non-hermitian matrices, NUMER. MATH, 57 (1990), pp. 285–312.
[23] R. FREUND, Conjugate gradient-type methods for linear systems with complex symmetric coefficient matrices, SIAM Journal on Scientific and Statistical Computing, 13 (1992), pp. 425–448.

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Figure 2: The history of the residual for the test T80k with COCR and COCG preconditioned with Jacobi and Chebyshev polynomial.
Figure 3: The history of the residual for the test T500k with COCR and COCG preconditioned with Jacobi and Chebyshev polynomial.
Figure 4: The history of the residual for the test S500k with COCR and COCG preconditioned with Jacobi and Chebyshev polynomial.