Two-sided Facility Location

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Abstract

Recent years have witnessed the rise of many successful e-commerce marketplace platforms like the Amazon marketplace, AirBnB, Uber/Lyft, and Upwork, where a central platform mediates economic transactions between buyers and sellers. Motivated by these platforms, we formulate a set of facility location problems that we term Two-sided Facility location. In our model, agents arrive at nodes in an underlying metric space, where the metric distance between any buyer and seller captures the quality of the corresponding match. The platform posts prices and wages at the nodes, and opens a set of "virtual markets" or facilities to route the agents to. The agents at any facility are assumed to be matched. The platform ensures high match quality by imposing a distance constraint between a node and the facilities it is routed to. It ensures high service availability by ensuring flow to the facility is at least a pre-specified lower bound. Subject to these constraints, the goal of the platform is to maximize the social surplus subject to weak budget balance, i.e., profit being non-negative.

We present an approximation algorithm for this problem that yields a $(1 + \epsilon)$ approximation to surplus for any constant $\epsilon > 0$, while relaxing the match quality (i.e., maximum distance of any match) by a constant factor. We use an LP rounding framework that easily extends to other objectives such as maximizing volume of trade or profit.

We justify our models by considering a dynamic marketplace setting where agents arrive according to a stochastic process and have finite patience (or deadlines) for being matched. We perform queueing analysis to show that for policies that route agents to virtual markets and match them, ensuring a low abandonment probability of agents reduces to ensuring sufficient flow arrives at each virtual market. Such an analysis also helps us posit facility location variants that capture settings where the platform elicits deadlines truthfully by posting lotteries over different prices and wages for different deadlines.

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1 Introduction

Online marketplaces have transformed the economic landscape of the modern world. Many of today’s most important companies are platforms facilitating trade between agents: both for goods (Amazon, eBay, Etsy), and increasingly, services: transportation (Lyft, Uber); physical and virtual work (Taskrabbit, Upwork); lodging (Airbnb); etc. These platforms enable fine-grained monitoring of participants, and greater control via pricing [8], terms of trade, recommendation and directed search [7], etc. The challenge of harnessing this increase in data and control has led to a growing literature in online marketplace design.

The basic algorithmic challenge facing a marketplace platform can be summarized as follows: it must decide which buyer should match to which seller, at what time, and for what price and wage, in order to maximize some desired objective. In a sense, this combines the challenges of online bipartite matching, job scheduling, and pricing and mechanism design. In this paper, we build towards a model that simultaneously addresses these three orthogonal challenges and tackles the two-sided marketplace design problem in its full generality.

1.1 High-Level Problem Formulation

In our setting, buyers and sellers flow into nodes located in a metric space representing an underlying feature space in which agents are embedded. The metric distance between a buyer and seller captures the overall quality of the match; For instance, in AirBnB, it represents some combination of geographic distance from the desired location and quality of room; in UpWork, it can capture how well the skill set of the consultant matches the task requirements; and so on. We assume the platform needs to match buyers and sellers within a distance threshold specified as input.

In addition to their known embedding in the metric space, buyers have value for being served, and sellers have costs for providing service; the platform knows the distribution of these values. The platform needs to make two policy decisions: what prices/wages to set at each node, and how to match the resulting flow of buyers and sellers. We consider a simple class of policies where the platform opens a set of facilities or virtual markets in the metric space and routes flow from demand/supply nodes to these markets. The flow routed to any virtual market is matched up. These markets satisfy the following service guarantees:

1. **Quality of service guarantees**: The flow assigned to a market is from supply and demand nodes within distance \( R \). Therefore, any matched demand/supply is within distance \( 2R \).

2. **Service availability guarantees**: In order for the flow at the virtual market to match up, each market needs to have flow balance of supply and demand routed there. Furthermore, if we view virtual markets as canonical features/types that are used to compress the feature space of buyers/sellers, then we want these features to be sufficiently representative. We therefore insist these markets are sufficiently thick, meaning there is a lower bound \( L \) on the flow routed to each of them. In Section [3], we present a more formal queueing-theoretic justification of these constraints using discrete buyer/sellers that have finite patience for being matched.

The goal of the platform is to (i) set prices/wages at each node in the metric space; (ii) open a set of virtual markets; and (iii) route the resulting demand/supply flow to these markets satisfying the quality and service availability guarantees. The objective of the platform is to maximize the welfare or social surplus, which is total value of buyers minus total cost of sellers, subject to weak budget balance, meaning that the profit (the difference between total price charged to buyers and total wage paid to sellers) is non-negative. Our same solution idea applies to other objectives, such
as maximizing profit, or maximizing the throughput or volume of trade subject to weak budget balance. This leads to a class of optimization problems that we term Two-sided Facility Location.

1.2 Our Results

Our first main contribution in Section 2 is to show an approximation algorithm for Two Sided Facility Location. We present a new LP rounding framework that for any constant $\epsilon > 0$, achieves a $(1 + \epsilon)$ approximation to the social surplus objective (resp. throughput and profit objectives). It relaxes the distance bound constraint by a factor of 4, while preserving the budget balance constraint, as well as the flow balance and lower bound constraints at each facility. If we allow a tiny additive error $\Delta$ in the surplus objective, our algorithm requires solving $O\left(\frac{n^{1/\epsilon}}{\epsilon \log \frac{nW_{\text{max}}}{\epsilon \Delta}}\right)$ LPs, where $n$ is the number of nodes, and $W_{\text{max}}$ is the maximum possible surplus.

We show in Appendix A that the surplus objective is NP-Hard to approximate to a factor $o(L^c)$ for some constant $c > 0$, unless the distance bound is relaxed by at least a factor of 2.

Techniques. Our facility location variants mirror the profit earning facility location problem in [29]. Just like that setting, we have lower bounds on demand served at each facility and an upper bound on how far the facility can be from an assigned demand. However, there are key differences that preclude the application of existing techniques from lower-balanced facility location [20, 21, 25, 29, 32]: First, the demand or supply at each node is a variable that can be adjusted using pricing. This means the demand/supply can be zero at some “outlier” nodes, so that they do not need to be served by any facility. Secondly, each facility needs to satisfy flow balance between supply and demand, and finally, both surplus and profit involve differences, so the platform can potentially lose money at some virtual markets, but recover it at others.

The above differences make formulating an LP relaxation tricky. Note that even in [29], the version with outliers and profits that can become negative has unbounded integrality gap, because the optimal profit can be zero while the LP achieves positive profit. Unlike [29], since we can control demand/supply by pricing, we have greater flexibility in modifying the LP variables. Despite this, the integrality gap of the straightforward LP formulation for our problem is large, because there could be a market that generates a bulk of the surplus, but has large negative profit that is compensated by other markets. (See Appendix B for an example).

This brings up our main technical contributions: We first observe that if we focus on the LP variables corresponding to facility $i$, we can scale these up or down by changing the fraction to which this node is an outlier. This enables us to use techniques reminiscent of improved greedy algorithms for budgeted coverage problems [26, 18]. In particular, we strengthen the LP formulation via guessing facilities that are opened in the optimal solution. Next, we use the guesses to develop a structural characterization for this stronger LP based on modifying variables for pairs of facilities. In effect this shows that there is some integrality in the neighborhood of any partially open facility, which helps us consolidate these facilities while preserving all constraints.

Queueing-theoretic Foundations. Our second main contribution is to present queueing-theoretic micro-foundations to our facility location problems. What makes the problems hard and technically interesting is the presence of the lower bound $L$ on flow routed to each virtual market; if $L = 0$, the problem is in fact polynomial time solvable! In Section 3 we show that the lower bound constraints (along with flow balance) arise naturally from a dynamic marketplace setting. Here, atomic buyers/sellers arrive according to Poisson processes, and have a private patience for being matched before they abandon the system. In this context, service availability corresponds to joint pricing and matching policies that have low abandonment rate. We perform queueing analysis to show that for policies that route agents to virtual markets and match them optimally there, ensuring a
low abandonment probability reduces to ensuring both flow balance and that sufficient flow arrives at each virtual market. In effect, our \textit{static} facility location variants have their roots in \textit{dynamic} control policies that maximize surplus while ensuring low abandonment.

The advantage of our LP rounding framework for \textit{Two-sided Facility Location} is that it gracefully handles more complex variants motivated by the dynamic marketplace setting. For instance, consider the dynamic problem where the platform uses prices and wages to truthfully elicit patience of agents, and subsequently matches them optimally using Earliest Deadline First (EDF) scheduling in each virtual market. Motivated by this, we consider an \textit{envy-free} variant of \textit{Two-sided Facility Location} in Appendix C where each node (or agent type) is composed of sub-types that envy one another. The platform sets prices/wages for each sub-type so that each agent truthfully chooses its sub-type. Each sub-type has a weight and the service availability constraints are captured by flow balance of supply and demand at each virtual market, and a lower bound on the total \textit{weight} routed to each open market. Our LP rounding framework easily extends to yield optimal profit while relaxing the distance constraint by factor 4.

1.3 Related Work

\textbf{Two-sided Markets.} Our objective maximizes social surplus subject to budget balance (and individual rationality). This is a classic objective in two-sided market mechanisms, and originates in the celebrated work of Myerson and Satterthwaite \cite{MS73}, where it is termed \textit{gains of trade}. They considered the case of a single buyer and seller. This has inspired a recent line of work on truthful mechanisms for approximate surplus maximization in markets of multiple buyers and sellers \cite{PP07, AL12, AL10}, ultimately resulting in a 2-approximation to gains of trade. This line of work assumes buyers and sellers are matched in one shot. The novelty in our work is in modeling a \textit{dynamic setting} and incorporating \textit{service availability} guarantees while preserving the same objectives. We therefore consider the more natural class of mechanisms that post prices and wages. Posted price mechanisms have been extensively studied in two-sided marketplaces \cite{PP07, JY00, AL12, AL10}, and the main idea we borrow from this literature is the notion of \textit{insulating tariffs} \cite{AL12}, which posits that market design is easier if the prices seen by buyers is disconnected from the wages seen by service providers.

Another recent line of work shows approximately optimal mechanisms for maximizing welfare in two sided markets with \textit{goods} \cite{HS01, HS01a}; however, theirs is a sum objective defined in terms of the final sets of items allocated to each buyer and seller, which is different from the gains of trade.

Finally, our work is also related to two-sided market segmentation problems considered in \cite{BND15}. In their model, prices appear endogenously via market clearing (instead of being set by the platform). Our concept of clustering buyer and seller nodes via virtual markets is a form of market segmentation. However, unlike \cite{BND15}, the flow lower bound prevents us from arbitrarily splitting a market into smaller sub-markets, making our problem technically very different.

\textbf{Dynamic Marketplaces.} Our work on dynamic marketplaces is related to several recent works \cite{PP07, AL12, AL15} on online scheduling under stochastic arrivals of tasks on machines with limited resources. Tasks have (private) types comprising their value, arrival time, and deadline; the platform’s goal is to maximize welfare while truthfully eliciting the type. While similar to our work on pricing resources or tasks, they allow agents to choose assignments based on posted prices (\textit{envy-freeness}). Another difference is the markets considered in their studies are one-sided. Blum \textit{et al.} \cite{BDL14} consider online two-sided markets with \textit{fixed bids}, and present competitive algorithms for maximizing profit or number of matches. Though their objectives are simpler, their methods will necessarily discard a constant fraction of feasible matches, which can lead to significant user dissatisfaction. In contrast, our focus is on ensuring high service availability which corresponds to small user abandonment.

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Dynamic two-sided markets also serve as motivation for recent work on “online matching with delays” [19, 6]. Here, buyers and sellers arrive online in a metric space, and can be matched at any time subsequently. The goal is to minimize the total distance cost plus waiting cost, and the authors present a log-competitive algorithm. These models do not incorporate pricing. Further, our dynamic marketplace models are more closely related to dynamic matchings with stochastic arrivals, and we review this literature in Section 3.

2 Two-Sided Facility Location

There is a metric space $G(V, E)$ with an associated distance function $c$. We assume buyers and sellers are fluid and arrive at nodes in this metric space and must be matched with each other. The metric distance captures the match quality – if demand at node $j$ is matched to supply at node $j'$, the disutility to the system is captured by $c(j, j')$.

**Demand and Supply Functions.** Each node $j \in V$ is associated with a demand function $F_j$ and a supply function $H_j$. When offered price $p$, we assume the demand (i.e., buyers) at node $j$ is $d_j F_j(p)$, where $F_j(p)$ is a non-increasing function of $p$ corresponding to the survival function of a continuous density function $f_j$ on valuations; formally $F_j(p) = \int_{v > p} f_j(v) dv$. In other words, the volume of buyers is $d_j$, and when quoted a price $p$, only buyers with valuations at least $p$ choose to participate, and hence the stream of buyers is thinned by a factor $F_j(p)$. We assume there is a finite price $p_{\text{max}}$ so that $F_j(p_{\text{max}}) = 0$ for all $j \in V$.

Similarly, when offered wage $w$, the supply of sellers arriving at node $j$ is $s_j H_j(w)$, where $H_j(w)$ is a non-decreasing function of $w$, corresponding to the CDF of a continuous density function $h_j$ on costs; formally $H_j(w) = \int_{c = 0}^{w} h_j(c) dc$. When offered wage $w$, all sellers with cost at most $w$ participate, resulting in supply $s_j H_j(w)$. We assume that $H_j(0) = 0$, i.e., sellers accrue 0 utility by not participating in the platform.

In this section, we make the standard regularity assumptions (à la Myerson-Satterthwaite [30]) on the density functions $f_j$ and $h_j$. In particular, we assume $xF_j^{-1}(x)$ is concave in $x$ and $yH_j^{-1}(y)$ is convex in $y$. This is true for instance, for all log-concave densities $f_j$ and $h_j$, which includes Normal, Exponential, and Uniform distributions. In Section 4, we discuss how the continuity and regularity assumptions can be removed by using lottery pricing.

**Virtual Markets.** As discussed before, the platform opens a set of facilities or “virtual markets” in the metric space and routes demand and supply fractionally to open markets within distance bound $R$ in order to match them up. (Therefore, the distance of any match is at most $2R$.) It ensures high service availability by ensuring the following two intuitive constraints on the flow of supply and demand arriving each open virtual market. We justify these constraints via queueing arguments in Section 3.

**Flow Balance.** Since the flow arriving at the market are matched, the total amount of supply and demand are equal.

**Flow Lower Bound.** The market is sufficiently thick, that is, the total amount of supply (resp. demand) is at least $L$.

2.1 Formal Problem Statement

Let $\mathcal{F} \subseteq V$ the set of all candidate virtual markets; we set $\mathcal{F} = V$. For each node $j$, $B_R(j) \subseteq \mathcal{F}$ denotes the set of all the virtual markets $i \in \mathcal{F}$ such that $c(i, j) \leq R$. Similarly, for each market $i$, we define $B_R(i)$ as the set of all the nodes $j \in V$ such that $c(i, j) \leq R$. A solution to TWO-SIDED FAC-LOC($L, R$) is specified by the following:
• An assignment of price \( p_j \) and wage \( w_j \) to each node \( j \in V \). If the price (resp. wage) at node \( j \) is \( p_{\text{max}} \) (resp. 0), we assume this node generates no demand (resp. supply).

• A set of locations \( S \subseteq F \) for opening “virtual markets”; and

• A routing scheme \( \vec{x}_d^j \) (resp. \( \vec{x}_s^j \)) for each demand (resp. supply) node \( j \in V \) that generates non-zero demand (resp. supply). If the price (resp. wage) at node \( j \) is \( p_{\text{max}} \) (resp. 0), we assume this node generates no demand (resp. supply).

Note that the flow of demand (resp. supply) from node \( j \in V \) to market \( i \in S \) is \( d_j F_j(p_j)x_j^d_{ij} \) (resp. \( s_j H_j(w_j)x_j^s_{ij} \)). We enforce that the flows satisfy the flow balance and flow lower bound conditions at each \( i \in S \).

**Weak Budget Balance.** The next constraint is weak budget balance, which corresponds to profit being non-negative. This is written as:

\[
\text{Profit} = \sum_{j \in V} (d_j p_j F_j(p_j) - s_j w_j H_j(w_j)) \geq 0
\]

**Surplus (Welfare) Objective.** We first define the following quantities:

\[
V_j(p) = d_j \int_{v=p}^{\infty} v f_j(v) dv \quad \text{and} \quad C_j(w) = s_j \int_{c=0}^{w} c h_j(c) dc
\]

respectively denote the total value of buyers generated by node \( j \) when the price there is \( p \) and the cost of sellers at node \( j \) when the wage there is \( w \). The surplus objective can then be written as:

\[
\text{Social Surplus} = \sum_{j \in V} (V_j(p_j) - C_j(w_j))
\]

This defines the problem **Two-Sided Fac-Loc** \((L, R)\)

Though we focus on surplus in the paper, the same techniques extend to other objectives such as maximizing throughput or volume of matches, defined as the total demand (or supply): \( \sum_{j \in V} d_j F_j(p_j) \). It also extends to the objective of maximizing profit defined above.

### 2.2 Approximation Algorithm

We characterize the approximation ratio of any algorithm for **Two-Sided Fac-Loc** \((L, R)\) as \((\alpha, \gamma)\), if the resulting solution relaxes the distance bound of an assignment to a facility to \( \alpha R \), ensures lower bound \( L \), and has surplus \( OPT/\gamma \), where \( OPT \) is the optimal surplus.

In Appendix \[A\], we show that it is NP-HARD to obtain \( \gamma = o(L^c) \) for some constant \( c > 0 \), unless \( \alpha \geq 2 \). In the sequel, we present a \((4, 1 + \epsilon)\) approximation. For the algorithm to have polynomial running time, they also need lose a small additive amount in the objective; as we show later, this quantity can be exponentially small.

#### 2.2.1 Linear Programming Relaxation

We now formulate **Two-Sided Fac-Loc** \((L, R)\) as an integer linear program. For ease of exposition, we compare against an optimal solution that is restricted to using prices from a fixed set \( P \) and wages from a fixed set \( W \). Our solution is not restricted to using prices and wages from this set. In Appendix \[C\], we show that our LP admits to a polynomial time solution of arbitrary additive accuracy when this assumption is relaxed, and demand/supply distributions are continuous.
Note that we assume $p_{\text{max}} \in \mathcal{P}$ and $0 \in \mathcal{W}$, and at this price (resp. wage) the demand (resp. supply) is identically zero. This is the price (resp. wage) where this node becomes an outlier and the solution is not required to open a virtual market nearby.

Instead of writing our LP using prices and wages, we use the associated demand/supply values. Let $Q_j = \{q \mid q = F_j(p), p \in \mathcal{P}\}$ and $R_j = \{r \mid r = H_j(w), w \in \mathcal{W}\}$. The case where the node is an outlier now corresponds to setting $q = 0$ (resp. $r = 0$).

We redefine the valuations and costs using supply/demand values as follows:

$$V_j(q) = d_j \int_{v=F_j^{-1}(q)}^\infty v f_j(v) dv \quad \text{and} \quad C_j(r) = s_j \int_{c=0}^{H_j^{-1}(r)} c h_j(c) dc$$

respectively denote the total value of buyers generated by node $j$ when the price there is $F_j^{-1}(q)$, and the cost of sellers at node $j$ when the wage there is $H_j^{-1}(r)$.

**Variables.** For each candidate virtual market $i \in \mathcal{F}$, let $y_i \in \{0, 1\}$ be the indicator variable that a virtual market is opened at that location in the metric space. Let $\alpha_{jq} = 1$ if the price at node $j \in V$ corresponds to $q \in Q_j$. Similarly define $\beta_{jr}$ for $r \in R_j$. The variable $z_{ijq}$ is non-zero only if $\alpha_{jq} = 1$ and $i \in B_R(j)$. In this case, it is the fraction of $j$’s demand that is routed to $i$. We define $z_{ijr}$ similarly for supply. Note that the actual flow from $j$ to $i$ is $d_j q z_{ijq}$; similarly for sellers.

**Objective and Weak Budget Balance.** The objective of social surplus and the profit being non-negative can be captured by:

Surplus Objective: \[ \max \sum_{j \in V} \left( \sum_{q \in Q_j} \alpha_{jq} V_j(q) - \sum_{r \in R_j} \beta_{jr} C_j(r) \right) \tag{2} \]

Weak Budget Balance: \[ \sum_{j \in V} \left( \sum_{q \in Q_j} \alpha_{jq} d_j q F_j^{-1}(q) - \sum_{r \in R_j} \beta_{jr} s_j r H_j^{-1}(r) \right) \geq 0 \tag{3} \]

**Feasibility.** The following constraints connect the variables together. We present these constraints only for buyers (that is, $q \in Q_j$); the constraints for sellers is obtained by replacing $q$ with $r \in R_j$. First, for each $q \in Q_j$, we need to choose one price for buyers (resp. sellers).

$$\sum_{q \in Q_j} \alpha_{jq} = 1 \quad \forall j \in V \tag{4}$$

$$\sum_{i \in B_R(j)} z_{ijq} = \alpha_{jq} \quad \forall j \in V, q \in Q_j \tag{5}$$

Next, if demand is fractionally routed from $j$ to $i$, then $i$ should be open and within distance $R$. Note that we need to ignore the case where $q = 0$ (resp. $r = 0$) since in this case, the demand (resp. supply) routed is zero, so that there is no need for a nearby virtual market.

$$\sum_{q \in Q_j, q > 0} z_{ijq} \leq y_i \quad \forall j \in V, i \in B_R(j) \tag{6}$$

**Service Availability.** We finally encode flow balance and flow lower bound at each virtual market:

$$\sum_{j \in B_R(i)} \sum_{q \in Q_j} q z_{ijq} = \sum_{j \in B_R(i)} \sum_{r \in R_j} r z_{ijr} \quad \forall i \in \mathcal{F} \tag{7}$$

$$\sum_{j \in B_R(i)} \sum_{q \in Q_j} q z_{ijq} \geq L y_i \quad \forall i \in \mathcal{F} \tag{8}$$

If we replace the integrality constraints on $\{y_i\}$ and the $\{\alpha_{jq}, \beta_{jr}\}$ with $y_i, \alpha_{jq}, \beta_{jr} \in [0, 1]$, the above is a linear programming relaxation of the problem.
2.2.2 Integality Gap and Stronger LP Formulation

The main technical hurdle arises because of the flow lower bound constraint: The LP optimum (and even an integer optimum) can now open virtual markets \( i \) which have positive welfare but negative profit, and compensate for the loss in profit by other markets with positive profit. In addition, if \( \alpha_j \beta_j > 0 \), then the above constraints imply \( \sum_{i \in B_R(j)} y_i < 1 \). This means there could only be a small fractional facility open in the vicinity of \( j \), which can account for a lot of the surplus. This makes the LP have super-polynomial integrality gap and we present an example in Appendix [I].

Our first technical contribution involves adding constraints to the above LP formulation so that has bounded integrality gap.

**Rounding Prices and Wages.** Before showing how to strengthen the LP, we present the following claim, which implies that once we round \( \{y_i\} \), the remaining solution can easily be made integral.

**Lemma 1.** *(Proved in Appendix [I])* Given any feasible LP solution, there is an equivalent solution that assigns only one price (resp. wage) per demand (resp. supply) node, that preserves all constraints and does not decrease the objective.

Define a variable for the surplus \( W_i \) and profit \( R_i \) of market \( i \) respectively as:

\[
W_i = \sum_{j \in B_R(i)} \left( \sum_{q \in Q_j} V_j(q)z_{ijq} - \sum_{r \in R'_j} C_j(r)z_{ijr} \right) 
\]

and

\[
R_i = \sum_{j \in B_R(i)} \left( \sum_{q \in Q_j} d_jqF_j^{-1}(q)z_{ijq} - \sum_{r \in R'_j} s_jrH_j^{-1}(r)z_{ijr} \right) 
\]

Then the objective can be rewritten as: Maximize \( \sum_i W_i \), and weak budget balance is \( \sum_i R_i \geq 0 \). Further note that \( W_i \geq R_i \) since for any \( q, r \), we have \( V_j(q) \geq d_jqF_j^{-1}(q) \), and \( C_j(r) \leq s_jrH_j^{-1}(r) \) if we integrate the expressions in Equation (9) by parts.

**Guessing Facilities.** Let \( \epsilon > 0 \) be any constant, and let \( \theta = \frac{1}{\epsilon} \). We first perform a brute force search over all integer solutions that open at most \( \theta \) facilities. This can be done in \( O(n^\theta) \) time, where \( n = |V| \). For each selection of facilities, Lemma [I] implies that solving the LP formulation with the corresponding \( y_i \) set to 1 and the rest to zero yields the optimal welfare (or results in declaring infeasibility). We can therefore find the welfare maximizing solution among these in polynomial time; call this welfare \( W_1 \).

Next, for every choice of parameter \( W \geq 0 \) scaled to powers of \( (1 + \epsilon) \), and every subset \( S \subseteq F \) with \( |S| = \theta \), define LP\((W, S)\) as having all the previous constraints, plus the following new ones:

\[
W_i \leq W y_i \quad \forall i \in F \setminus S \quad (11)
\]

\[
y_i = 1 \quad \forall i \in S \quad (12)
\]

\[
\sum_{i \in S} W_i \geq W \theta(1 - \epsilon) \quad (13)
\]

Let \( OPT \) denote the optimum surplus, and let \( W_2 = \max\{LP(W, S) \mid W \geq 0, \ S \text{ s.t. } |S| = \theta\} \). Then, it is easy to see that \( OPT \leq \max(W_1, W_2) \): If \( OPT \) opens fewer than \( \theta \) facilities, then clearly \( W_1 \geq OPT \), since \( W_1 \) opens all possible choices of at most \( \theta \) facilities. Otherwise, let \( W_1^* \) denote the surplus generated by open market \( i \) in \( OPT \). Let \( W^* \) denote the \( \theta^\theta \) largest value of \( W_i^* \). Choose \( W \in [W^*, W^*(1 + \epsilon)] \), and \( S \) as the set of \( \theta \) facilities in \( OPT \) with \( W_i^* \geq W^* \). This induces a feasible solution to the above constraints, so that the LP optimum is at least \( OPT \).
2.2.3 Rounding

Our second technical contribution is a new structural characterization about the LP optimum. This is crucial for rounding, since it allows sufficient mass of facility to be located in roughly the same neighborhood. We present the high-level rounding ideas, and relegate details to Appendix E.

Structural Characterization. Recall \( \alpha_{j0}, \beta_{j0} \) are the fractions to which node \( j \) is an outlier, i.e. has zero flow. These variables are the reason the simpler LP had large integrality gap, since they allow facilities in \( B_R(j) \) to be open to small fractions. Our main observation is the following:

**Lemma 2.** There is a \((1 + \epsilon)\) approximation to the objective of \(LP(W, S)\) that satisfies:

\[
\forall i \in \mathcal{F}, \quad y_i \in (0, 1) \quad \Rightarrow \quad \exists j \in B_R(i) \text{ s.t. } \alpha_{j0} \beta_{j0} = 0
\]

**Proof.** (Sketch; Proved formally in Appendix E.1) Consider a facility that violates the statement. If it has \( R_i > 0 \), then consider all LP variables \( \{y_i, z_{ijq}, z_{ijr}\} \) corresponding to some such facility \( i \) and uniformly increase them. This increases both profit and welfare. We can decrease the fractions \( \{\alpha_{j0}, \beta_{j0}\} \) to which any node \( j \) connected to \( i \) is assigned as outlier to compensate the fraction to which it is assigned \( i \). Note that Constraints (7) and (8) are local to a single facility. Since we scale up all variables corresponding to a facility, we preserve these constraints. If we keep up this process, then either the facility is completely open \((y_i = 1)\); or some demand/supply node assigned to it has \( \alpha_{j0} = 0 \) or \( \beta_{j0} = 0 \). (This must in fact hold in the LP optimum.)

On the other hand, if \( W_i > 0 \) but \( R_i < 0 \), then increasing its LP variables would hurt profit, which may violate the budget balance constraint; while reducing the variables would increase profit but hurt the welfare. The idea now is the following: Take any pair of such markets; increase the variables for one market while decrease them for the other. There is always a way of doing this so that both the total profit and welfare do not decrease – this is essentially a fractional knapsack argument. Again, since we uniformly scale all variables corresponding to a market, we preserve all constraints. Note that the process can also stop when a facility closes \((y_i = 0)\). Eventually, we run out of pairs, so that for all but one market, the above characterization holds.

At this point, the strengthened LP kicks in. The singleton market violating the above lemma was fractionally open and had \( R_i < 0 \). It has welfare at most \( W \) by Constraint (11). But we have integrally open markets that generate welfare at least \( W_{\frac{1-\epsilon}{\epsilon}} \) by Constraint (13), which means closing the singleton market reduces welfare by at most \((1 - \epsilon)\), and preserves budget balance. \( \Box \)

Consolidating Facilities. This step (in Appendix E.2) now follows approaches similar to those in [20, 35]. Note that if a node \( j \) has \( \alpha_{j0} = 0 \) or \( \beta_{j0} = 0 \), then Constraint (15) implies \( \sum_{i \in B_R(j)} y_i \geq 1 \). Consider an independent set of such nodes, such that no two are fractionally assigned to the same facility. For any \( j \) in this set, move all partially open facilities in \( B_R(j) \) to \( j \) itself, so that there is a facility integrally opened at \( j \). Since we move an entire facility, we preserve all flows, so that flow balance and lower bound are preserved, and so is profit. Now a demand/supply can be assigned a distance \( 2R \) away, and the opened facilities are integral.

At this point, consider any fractionally open facility \( i \). It must have a node \( j \) adjacent to it that satisfies the condition in Lemma 2. If \( j \) has a facility completely open at its location, then move \( i \) to location \( j \). Otherwise, \( j \) was not part of the independent set in the previous step, which means \( j \) and \( j' \) shared a fractionally open facility, and the previous step opened a facility completely at \( j' \). In this case, we move \( i \) to \( j' \), again preserving all flows. This means any demand/supply moves distance at most \( 4R \), preserving all the LP constraints.

Rounding Prices/Wages. At this point, the facilities are opened integrally. Lemma 4 now implies that we can choose one price/wage per node preserving all constraints and the objective.
In Appendix E we present the rounding algorithm formally and show the following theorem:

**Theorem 1. (Proved in Appendix E.)** For any constant \( \epsilon > 0 \), there is a \((4, 1 + \epsilon)\) approximation algorithm for \(\text{TWO-SIDED FAC-LOC}(L, R)\).

**Running Time.** Note that the surplus can become arbitrarily close to zero. Therefore, for parameter \( \Delta > 0 \), we will allow additive error \( \Delta \) in the surplus objective. Note that the maximum possible surplus is \( W_{\text{max}} = (\sum_j d_j) p_{\text{max}} \), which is an upper bound on \( W \). If we assume the surplus is at least \( \Delta \), then \( \max_i W^*_i \geq \Delta/n \). Since the top \( 1/\epsilon \) facilities on \( \text{OPT} \) have surplus \( W(1 - \epsilon)/\epsilon \), this means we can set \( W \geq \frac{\Delta}{2n} \). Therefore, the number of choices of \( W \) is \( O((\frac{1}{\epsilon} \log \frac{nW_{\text{max}}}{\epsilon \Delta})) \). For each choice of \( W \), we need to solve \( O(n^{1/\epsilon}) \) LPs, so that the overall number of LPs is \( O((\frac{n^{1/\epsilon}}{\epsilon} \log \frac{nW_{\text{max}}}{\epsilon \Delta})) \).

Note that \( \Delta \) can be exponentially small, and our algorithm for solving the LP in Appendix C will lose such an additive factor in the objective anyway. Omitting details, we note that if the objective is profit, we can achieve optimal objective by directly rounding the single LP in Section 2.2.1 (details similar to Appendix G). If the objective is throughput (or total flow) subject to weak budget balance, then the flow to any open market is at least \( L \), so that we only need to guess the \( \theta \) open markets in \( \text{OPT} \) and not the flow to them. This means we only need to solve \( O(n^{1/\epsilon}) \) LPs.

### 3 Queueing-theoretic Justification: Dynamic Marketplaces

In the facility location model discussed above, we imposed a lower bound \( L \) on the flow routed to any facility. In the absence of this constraint, i.e., when \( L = 0 \), the problems are in fact poly-time solvable. For instance, to maximize surplus (welfare), we can simply write the standard welfare maximization LP ignoring prices and wages. Its dual yields prices and wages that will be strongly budget balanced. Therefore, the entire hardness of the problem comes from the lower bound constraint. This begs the question: Why have this constraint at all?

**Dynamic Marketplace Model.** We now present a dynamic marketplace model that provides queueing-theoretic justification for these constraints. This model has the following features:

- **Buyer and seller types are located in a metric space just as before.**
- **Buyers and sellers are no longer fluid.** Instead, buyers at node \( j \) arrive as a Poisson process with rate \( d_j F_j(p) \) when quoted price \( p \), and when quoted a price \( p \); similarly, sellers follow a Poisson process with rate \( s_j H_j(w) \).
- **We assume each buyer and seller has a private patience level or deadline; if not matched within their deadline, they abandon the system.** The platform knows the patience distribution.

The stochastic control problem that we term dynamic marketplace problem can be summarized by two control decisions:

1. **Pricing decision.** Choose static prices \( p_j \) and wages \( w_j \) at each node \( j \in V \); and
2. **Scheduling decision.** This matches feasible buyer-seller pairs and removes them from the system. This decision is dynamic, depending on the entire state of the system as captured by the number of unmatched buyers and sellers at different nodes at any point of time.

The goal is to design a stochastic control policy that maximizes the long-term average surplus subject to long-term budget balance. We insist all scheduled matches must involve a current buyer and seller with metric distance at most \( R \). The key difference is in the service availability guarantee: Given the stochastic nature of our arrivals, there is always some probability that an incoming buyer or seller exhausts her patience before being matched. A more realistic goal is to design policies that guarantee a minimum level of service availability. We quantify this via the long-term average probability of abandonment of agents. Formally, given a parameter \( \epsilon > 0 \) as input, the goal of the platform is to make the abandonment probability at most \( \epsilon \).
Scheduling Policies. Constructing the optimal policy for the dynamic marketplace problem is closely related to several lines of work in dynamic matchings over a compatibility graph – in kidney exchanges \[3\] where patients abandon the exchange if their health fails; in control of matching queues for housing allocation \[13\]; and more generally in service system design \[22, 2, 1\], wherein customers and servers arrive stochastically and are matched according to a compatibility graph. In all these models, the choice of whom to match an arriving agent to depends on the entire set of agents waiting at different nodes, leading to the “curse of dimensionality”.

Given this curse of dimensionality, we consider the restricted sub-class of matching policies where the platform creates virtual markets in the metric space, and uses each market to cater to a different set of mutually-compatible agent types. Arriving agents are randomly routed to a compatible market, where they are queued up to be matched to agents on the other side. The probabilistic routing is fixed over time, and does not depend on the state of the markets.

Each virtual market maintains a queue of active buyers and sellers that have been assigned there, ignores what location they came from, and matches them up using an optimal scheduling policy for minimizing abandonment rate using only the current state of that particular queue. We will enforce the constraint that for any virtual market, the long-term abandonment probability is at most $\epsilon$, which in turn will ensure the overall abandonment probability is at most $\epsilon$.

Suppose we assume buyer deadlines are distributed as $\text{Exponential}(\kappa)$, and seller deadlines are distributed as $\text{Exponential}(\gamma)$. Though these distributions are known, the scheduling decisions at any market are made without knowing the patience level of any individual agent. Then, any work-conserving policy (including FIFO) is optimal. If an agent’s deadline expires and there is no agent to match it with in the queue, this agent is considered abandoned. In Appendix \[F\] we build on results from queueing theory \[33\] to bound this abandonment rate tightly as follows:

**Theorem 2.** Suppose $\lambda$ and $\mu$ be the (Poisson) arrival rates of buyers and sellers into a virtual market. Assume buyer deadlines are distributed as $\text{Exponential}(\kappa)$, and seller deadlines are distributed as $\text{Exponential}(\gamma)$. Then the FIFO policy has abandonment rate at most $\epsilon$ when:

1. There is flow balance, that is, $\lambda = \mu$; and
2. There is a flow lower bound, that is, $\lambda \geq \frac{3}{2} \left( \frac{\min(\gamma, \kappa)}{\epsilon^2} \right)$.

For $\epsilon \leq \frac{1}{6}$, the above conditions are also necessary to a constant factor: For the abandonment probability to be at most $\epsilon$, it must hold that $\lambda/\mu \in [1 - \epsilon, 1 + \epsilon]$; and $\min(\lambda, \mu) \geq \frac{1}{14000} \left( \frac{\min(\gamma, \kappa)}{\epsilon^2} \right)$.

We have therefore shown that a sufficient condition for bounding the abandonment probability at any virtual market by $\epsilon$ reduces to saying the flow to the market is balanced, and market is thick – there is a lower bound $L = \frac{3}{2} \left( \frac{\min(\gamma, \kappa)}{\epsilon^2} \right)$ on how much demand or supply needs to be routed there. This reduces the dynamic control problem with atomic agents to a static problem where demand/supply are fluid – exactly Two-Sided FAC-LOC($L, R$) for suitable $L$ that depends on $\epsilon$.

The above class of dynamic scheduling policies is sufficiently flexible that for a single virtual market, we can compute the optimal matching policy and its abandonment probability under different assumptions on the patience times. For instance, the patience distributions can be arbitrary and correlated with the location of the agents; further, the platform can elicit these by pricing. This leads to new variants of Two-sided Facility Location. In Appendix \[G\] we define an envy-free variant that is motivated by the platform setting different prices for different deadlines, and using Earliest Deadline First (EDF) scheduling in each virtual market. Our queueing analysis shows that the abandonment rate constraint for the EDF policy reduces to setting a lower bound on the weighted flow routed to a virtual market, where the weight is simply the deadline. We present an algorithm that achieves optimal profit while again relaxing the distance constraint by a factor of 4.
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A Hardness of Approximation

Theorem 3. It is \text{NP-HARD} to find a \((\alpha, \gamma)\) approximation for \textsc{Two-Sided Fac-Loc}(L, R) unless \(\alpha \geq 2\) or \(\gamma \geq L^c\) for some constant \(c > 0\).

Proof. We reduce from Maximum Independent Set in \(k\)-regular graphs (\(k\)-MIS). Given a \(k\)-MIS instance with \(n\) vertices and \(m = kn/2\) edges, construct a metric space where each edge in the \(k\)-regular graph \(G(V, E)\) has length \(2R\). Place a demand node at the mid-point of each edge, and a supply node at each vertex. We set \(L = k\). Each supply node has \(s_j = k\), and supply function \(H^{-1}(r) = 1 - \delta\) for \(r \in [0, 1]\). Similarly, each demand node has \(d_j = 1\), and demand function \(F^{-1}(q) = 1\) for \(q \in [0, 1]\). Since the distance threshold is \(R\), the virtual markets are opened at vertices of the graph. Each such market must see \(k\) units of supply and demand, which means all neighboring demand is routed there, leading to welfare (resp. profit) \(k\delta\) at that market. Since two open markets cannot share a demand, this means the open markets form an independent set. Therefore, the surplus of \textsc{Two-Sided Fac-Loc}(\(k, R\)) is \(\delta\) times the size of the maximum independent set in \(G\). This is \text{NP-HARD} to approximate to within a factor of \(k^c\) for some constant \(c > 0\); see [4, 24]. Therefore, we need to relax the distance bound by at least a factor of 2. \(\square\)

B Integrality Gap Example in Section 2

First, we show that the optimum solution can open a virtual market with negative profit. To be more specific, for any given constant \(c < 1\) we present a simple example in which \(c\) fraction of the total welfare is generated by a market with negative profit. Then we use this example to show that the LP has unbounded integrality gap.

Let \(V = \mathcal{F} = \{v, v’\}\) such that \(c(v, v’) = \infty\), and \(L\) be the lower bound for the total amount of demand (supply) at each open market. For node \(v\), assume \(d_v = s_v = L\), the valuation of buyers is uniformly distributed over the interval \([2, 3]\), and the cost of sellers is uniformly distributed over the interval \([0, 1]\). For node \(v’\), assume \(d_{v’} = s_{v’} = L\), the valuation of buyers is uniformly distributed
over the interval $[c' - 1, 2c' + 1]$, and the cost of sellers is uniformly distributed over the interval $[0, c']$ where $c' = \frac{2\epsilon}{1 + \epsilon}$. We claim that the optimum integral solution for this example is to open a virtual market at each of the nodes and set the price and wage at node $v$ to 2 and 1 respectively, and set the price and wage at node $v'$ to $c' - 1$ and $c'$ respectively.

First, we show that this solution is feasible. At each node the price is not more than the valuation of any arriving buyer. Therefore, all the buyers choose to participate. Similarly, since the wage is not less than the cost of any arriving seller, all the sellers choose to participate. This solution satisfies flow balance for each of the markets because the volume of sellers and buyers are equal at the corresponding node, and all of them choose to participate. In addition, flow lower bound is also satisfied. Finally, the profit of the market at node $v$ is $d_v$ and the loss of the market at node $v'$ is $d_{v'}$. Therefore, the total profit is 0 and profit of the market at node $v$ compensates for the loss at the other market.

Now, we show that the welfare of the market with negative profit is a fraction $c$ of the total welfare. The welfare at node $v$ is $d_v \times (2.5 - 0.5) = 2L$ and the welfare at node $v'$ is $d_{v'} \times (3c'/2 - c'/2) = c'L$. Therefore, $c'/ (c' + 2) = c$ fraction of the welfare is generated at node $v'$.

Finally, we need to show that this solution is optimum. The nodes are far from each other and we cannot send the buyers and sellers from different nodes to a common market. The only option for opening a market at each of the nodes is to set the price and wage at each node in a way that all the arriving buyers and sellers choose to participate (otherwise, the flow lower bound cannot be satisfied). Therefore, this problem has three feasible integral solutions: no market is opened, a market at node $v$ is opened, and a market at each of the nodes is opened. Note that the solution which only opens a market at $v'$ is not feasible because it does not satisfy weak budget balance. The welfare of those solutions are $0$, $2L$, and $(2 + c')L$ respectively. Therefore, the third solution is optimum. Also it is easy to see that this integer optimum solution is also LP optimum solution. The reason is that both markets generate positive welfare and partially opening any market by fractionally assigning that node as an outlier results in lower welfare.

**Integrality Gap.** Now we slightly modify the previous example to show that the LP has unbounded integrality gap. We only change the distribution of the valuation of the buyers at node $v'$. The valuation of the buyers is now uniformly distributed over the interval $[c' - 1 - \epsilon, 2c' + 1 + \epsilon]$ for a small positive constant $\epsilon$. After this change, the integral solution which opens a facility at each node is not feasible anymore because it violates weak budget balance constraint. Therefore, the optimum integral solution has $2L$ welfare.

On the other hand we claim that there is a fractional solution which has $(\frac{1}{1+\epsilon} \times \frac{2\epsilon}{1+\epsilon} + 2)L$ welfare. Set the price and wage at node $v$ to 2 and 1 and open the market at that node ($y_v = 1$). For the node $v'$ we can only open the market partially. Set $y_{v'} = \frac{1}{1+\epsilon}$ and the price and wage at node $v'$ to $c' - 1 - \epsilon$ and $c'$ with probability $\frac{1}{1+\epsilon}$ and to $p_{max}$ and 0 with probability $\frac{1}{1+\epsilon}$. In other words, set $\alpha_{v'1} = \beta_{v'1} = \frac{1}{1+\epsilon}$ and $\alpha_{v'0} = \beta_{v'0} = \frac{1}{1+\epsilon}$. This solution is feasible and generates $(\frac{1}{1+\epsilon} \times \frac{2\epsilon}{1+\epsilon} + 2)L$ welfare, while the optimum integer solution generates only $2L$ welfare. Note that $c$ can be arbitrarily close to 1 and therefore the integrality gap is unbounded.

### C Solving the LP Formulation in Section 2

We now show how to use the Ellipsoid algorithm to efficiently solve the LP formulation in Section 2 to arbitrary additive accuracy even when the demand and supply distributions are continuous, so that the sets $Q_j$ (resp. $R_j$) are continuous. First we get rid of weak budget balance by take a
Lagrangian of surplus and the profit. For any parameter \( \lambda \geq 0 \), define:

\[
V_j^\lambda(q) = V_j(q) + \lambda d_j q F_j^{-1}(q)
\]

and

\[
C_j^\lambda(r) = C_j(r) + \lambda s_j r H_j^{-1}(r)
\]

Since we assumed regular supply and demand distributions, it is easy to show that \( V_j^\lambda(q) \) is concave in \( q \) and \( C_j^\lambda(r) \) is convex in \( r \). The Lagrangian objective is then:

\[
\begin{align*}
\text{Maximize} & \sum_{j \in V} \left( \sum_{q \in Q_j} \sum_{i \in B_R(j)} z_{ijq} V_j^\lambda(q) - \sum_{r \in R_j} \sum_{i \in B_R(j)} z_{ijr} C_j^\lambda(r) \right) \\
\sum_{q \in Q_j} \sum_{i \in F} z_{ijq} & \leq 1 \quad \forall j \in V \\
\sum_{r \in R_j} \sum_{i \in F} z_{ijr} & \leq 1 \quad \forall j \in V \\
\sum_{q \in Q_j} z_{ijq} & \leq y_i \quad \forall j \in V, i \in B_R(j) \\
\sum_{q \in R_j} z_{ijr} & \leq y_i \quad \forall j \in V, i \in B_R(j) \\
\sum_{j \in B_R(i)} d_j \sum_{q \in Q_j} q z_{ijq} & = \sum_{j \in B_R(i)} s_j \sum_{r \in R_j} r z_{ijr} \\
\sum_{j \in B_R(i)} d_j \sum_{q \in Q_j} q z_{ijq} & \geq L q_i \\
z_{ijq}, z_{ijr}, y_i & \geq 0 \quad \forall i, j, q, r
\end{align*}
\]

The dual is the following:

\[
\begin{align*}
\text{Minimize} & \sum_{j \in V} (a_j + b_j) \\
a_j + \eta_{ij} + d_j q (\zeta_i - \rho_i) & \geq V_j^\lambda(q) \quad \forall j \in V, i \in B_R(j), q \in Q_j \\
b_j + \theta_{ij} - s_j r \zeta_i + C_j^\lambda(r) & \geq 0 \quad \forall j \in V, i \in B_R(j), r \in R_j \\
\rho_i & \geq \eta_{ij} + \theta_{ij} \quad \forall j \in V, i \in B_R(j) \\
\eta_{ij}, \theta_{ij}, \rho_i & \geq 0 \quad \forall j \in V, i \in B_R(j)
\end{align*}
\]

For fixed dual variables, since \( V_j^\lambda(q) \) is concave in \( q \) and \( C_j^\lambda(r) \) is convex in \( r \), it is easy to check that for each \( i, j \), the separation oracle either involves maximizing a concave function in \( q \) (for the first set of constraints) or minimizing a convex function in \( r \) (for the second set of constraints). In either case, finding the separating hyperplane involves one-dimensional convex optimization. This implies the LP admits to an efficient additive approximation even for continuous distributions over a bounded domain. We omit the standard details.

### D Proof of Lemma \[\square\]

The rounding of \( \alpha_{jq}, \beta_{jr} \) is simple. Let \( \hat{q}_j = \sum_{q \in Q_j} q \alpha_{jq} \) and \( \hat{r}_j = \sum_{r \in R_j} r \beta_{jr} \). Set the price of location \( j \) to be \( F_j^{-1}(\hat{q}_j) \) and the wage at \( j \) to be \( H_j^{-1}(\hat{r}_j) \). In other words, set \( \hat{\alpha}_{jq} \leftarrow 1 \) and \( \hat{\beta}_{jr} \leftarrow 1 \). Further set \( \hat{z}_{ij} \leftarrow \sum_{q \in Q_j} \frac{z_{ijq} q}{q} \) and \( \hat{z}_{ijr} \leftarrow \sum_{r \in R_j} \frac{z_{ijr} r}{r} \).

Note that this process preserves the demand and supply from node \( j \) to virtual market \( i \), which preserves all the constraints in the LP formulation. Note further that the function \( q F_j^{-1}(q) \) is concave in \( q \) by the regularity of the demand function. Therefore,

\[
\sum_q \alpha_{jq} q F_j^{-1}(q) \leq \left( \sum_q \alpha_{jq} q \right) F_j^{-1} \left( \sum_q \alpha_{jq} q \right) = \hat{q}_j F_j^{-1}(\hat{q}_j)
\]
Similarly, since we assumed \( rH_j^{-1}(r) \) is convex in \( r \) (by regularity of supply), we have \( \sum_{j} \beta_j rH_j^{-1}(r) \geq \hat{r}_j H_j^{-1}(\hat{r}_j) \). Therefore, this transformation preserves weak budget balance. Next, we note that \( V_j(q) \) is always a concave function of \( q \) and \( C_j(r) \) is always a convex function. Therefore, the above argument also implies the welfare (social surplus) does not decrease in the above transformation.

E Proof of Theorem 1

The rounding scheme proceeds in two steps. In the first step, we focus on the \( y_i \) variables and make them integer without violating any of the constraints. This step relaxes the distance bound from \( R \) to \( 4R \). We then show that the resulting solution can be modified to assign only one price and wage per node and deadline pair, again without violating any constraints.

E.1 Structural Characterization: Proof of Lemma 2

We first formally prove the structural characterization stated in Lemma 2, that implies that in the neighborhood of any partially open virtual market, there is some demand/supply node that is (integrally) not an outlier.

We first simplify the LP. Let \( Q_j' = Q_j \setminus \{0\} \) and \( R_j' = R_j \setminus \{0\} \). Let \( \eta_j = \sum_{q \in Q_j'} \alpha_{jq} \) and \( \phi_j = \sum_{r \in R_j'} \beta_{jr} \) respectively denote the fractions to which \( j \) is assigned prices (wages) that correspond to non-zero demand (supply). We can rewrite the constraints \( \{4\} \) and \( \{5\} \) as:

\[
\eta_j = \sum_{q \in Q_j'} \sum_{i \in B_R(j)} z_{ijq} \leq 1 \quad \text{and} \quad \phi_j = \sum_{r \in R_j'} \sum_{i \in B_R(j)} z_{ijr} \leq 1 \quad \forall j \in V \tag{14}
\]

and set \( \alpha_{j0} = 1 - \eta_j \), and \( \beta_{j0} = 1 - \phi_j \). Recall from Equations \( \{9\} \) and \( \{10\} \) that \( W_i \) and \( R_i \) are respectively the welfare and profit of market \( i \) in the LP optimum.

We call node \( j \) fully demand-utilized if \( \eta_j = 1 \), and fully supply-utilized if \( \phi_j = 1 \). We say that node \( j \) is partially demand-connected to market \( i \in F \) if \( \sum_{q \in Q_j'} z_{ijq} > 0 \), and partially supply-connected if \( \sum_{r \in R_j'} z_{ijr} > 0 \). Let \( J_D(i) \) denote the set of nodes that are partially demand-connected to \( i \in F \), and \( J_S(i) \) be the set that is partially supply-connected.

**Lemma 3.** In the LP optimum, for any \( i \in F \) with \( y_i > 0 \), we have \( W_i > 0 \). Furthermore, all except one market with \( y_i > 0 \) satisfy the following condition: either \( y_i = 1 \); or there exists \( j \in J_D(i) \), such that \( j \) is fully demand-utilized; or there exists \( j \in J_S(i) \) such that \( j \) is fully supply-utilized.

**Proof.** First, note that \( W_i \geq R_i \). Suppose an open market has \( W_i \leq 0 \). This implies \( R_i \leq 0 \). Consider a different solution that sets \( y_i = 0 \) and \( z_{ijq} = z_{ijr} = 0 \) for all \( j \in V, q \in Q_j', r \in R_j' \). We adjust \( \eta_j \) and \( \phi_j \) for each \( j \in V \) to preserve constraint \( \{14\} \). This new solution has \( W_i = R_i = 0 \) and has at least as large surplus and profit. Since we set all LP variables corresponding to \( i \) to zero, this satisfies constraints \( \{3\}, \{7\}, \text{ and } \{8\} \), and is therefore feasible for the LP.

We therefore only focus on markets whose \( W_i > 0 \). Consider the set of these markets and split them into two groups. Let

\[
S_1 = \{ i \in F \mid y_i \in (0, 1) \text{ and } R_i < 0 \} \quad \text{and} \quad S_2 = \{ i \in F \mid y_i \in (0, 1) \text{ and } R_i \geq 0 \}
\]

Assume that for all of these markets, there is no \( j \in J_D(i) \), such that \( j \) is fully demand-utilized and no \( j \in J_S(i) \) such that \( j \) is fully supply-utilized.
First consider the markets in set $S_2$, we can increase the LP variables till the condition of the lemma is satisfied; this process only increases both profit and welfare, preserving all constraints. We do this as follows: Suppose no $j \in J_D(i)$ is fully demand-utilized and no $j \in J_S(i)$ is fully supply-utilized. In this case, let

$$
\theta = \min \left( \frac{1}{y_i}, \min_{j \in J_D(i)} \left( \frac{1 - \sum_{q \in Q_j} \sum_{i' \neq i} z_{ijq}}{\sum_{q \in Q_j} z_{ijq}} \right), \min_{j \in J_S(i)} \left( \frac{1 - \sum_{r \in R'_j} \sum_{i' \neq i} z_{i'jr}}{\sum_{r \in R'_j} z_{i'jr}} \right) \right)
$$

Since $\eta_j < 1$ for all $j \in J_D(i)$ and $\phi_j < 1$ for all $j \in J_S(i)$, we have $\theta > 1$. Suppose we increase $y_i$, $z_{ijq}$ for all $j \in J_D(i)$, $q \in Q_j$, and $z_{i'jr}$ for all $j \in J_S(i)$, $r \in R'_j$ by a factor of $\theta$. We will still have $\eta_j \leq 1$ for all $j \in J_D(i)$ and $\phi_j \leq 1$ for all $j \in J_S(i)$. However, either $y_i$ or one of these values will become exactly 1. Note that since we scaled all LP variables corresponding to $i$ by the same factor, this preserves constraints (4), (7), and (8). The surplus and profit of this market increase by a factor $\theta > 1$, which contradicts the optimality of the LP solution. Therefore, the markets in $S_2$ all have a neighboring $j$ that is either fully demand-utilized or fully supply-utilized.

Next consider the markets in set $S_1$. Suppose the condition in the lemma is not satisfied, so that there are two markets $i$ and $i'$ with $y_i, y_{i'} \in (0, 1)$, and with no neighboring $j$ that is either fully demand-utilized or fully supply-utilized. Suppose $W_i/|R_i| = a$ and $W_{i'}/|R_{i'}| = b$ with $a \geq b$. We multiply each LP variable corresponding to $i$ by a factor of $(1 + \delta)$, and multiply each LP variable corresponding to $i'$ by a factor of $(1 - \frac{W_i}{W_{i'}} \delta)$. Using the same argument as above, this process preserves the constraints that are specific to a market, since all variables are changed by the same factor. The increase in welfare of market $i$ is $\delta W_i$, and the decrease in welfare of market $i'$ is $\delta W_i$, so the overall welfare is preserved. The decrease in profit of market $i$ is $|R_i|\delta$, and the increase in profit of market $i'$ is $|R_{i'}| \frac{W_i}{W_{i'}} \delta \geq |R_i|\delta$ by our assumption that $a \geq b$. Therefore, this process cannot decrease profit, hence all constraints are preserved. We choose $\delta$ as the smallest value that either makes market $i$ have $y_i = 1$ or one neighboring $j$ either fully supply or demand utilized, or that sets the variables of market $i'$ to zero. In all cases, the size of set $S_1$ reduces by one. We repeat this process till there is only one market in $S_1$, completing the proof.

**Corollary 1.** There is a $(1 + \epsilon)$-approximation to the LP optimum where any market with $y_i > 0$ satisfies the following condition: either $y_i = 1$; or there exists $j \in J_D(i)$, such that $j$ is fully demand-utilized; or there exists $j \in J_S(i)$ such that $j$ is fully supply-utilized.

**Proof.** By Constraint (12), there is a set of markets $S$ that are fully open (i.e., $y_i = 1$) and $\sum_{i \in S} W_i \geq W \frac{1-\epsilon}{\epsilon}$ by Constraint (13). The rounding in Lemma 3 does not touch these markets, since we only increase/decrease variables corresponding to partially open markets (i.e., those with $y_i \in (0, 1)$). Lemma 3 implies there is only market $i$ that violates the condition of the corollary. This market must have welfare $W_i \leq W y_i \leq W$ by Constraint (11). This means closing this market (setting all its associated variables to zero) reduces the LP optimum by at most a factor of $(1 - \epsilon)$. Since the previous lemma implies this market had $R_i < 0$, this means closing it only increases profit, preserving weak budget balance. \[\square\]

**E.2 Rounding Virtual Markets**

Initially, all virtual markets $i \in \mathcal{F}$ with $y_i > 0$ are partially open. Node $j \in V$ is untouched if for all $i$ such that $j \in J_D(i) \cup J_S(i)$, the market $i$ is partially open. Let $U$ be the set of untouched nodes, and let $Z$ be the set that is either fully demand-utilized or fully supply-utilized. Let $U_f = U \cap Z$.  

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Phase 1. Consider any $j \in U_f$. W.l.o.g., assume $\eta_j = 1$; the case where $\phi_j = 1$ is symmetric. Let $N(j) = \{ij | j \in J_D(i)\}$. For every $i \in N(j)$, we “move” $i$ to location $j$; call the new virtual market at location $j$ as $i^*$. This means we set

- $\bar{y}_{i^*} \leftarrow \sum_{i \in N(j)} y_i$ and $\bar{y}_i \leftarrow 0 \quad \forall i \in N(j)$;
- $\bar{z}_{i^*j'q} \leftarrow \sum_{i \in N(j)} z_{ij'q}$ and $\bar{z}_{ij'q} \leftarrow 0 \quad \forall j' \in V, q \in Q_{j'}, i \in N(j)$
- $\bar{z}_{i^*jr} \leftarrow \sum_{i \in N(j)} z_{ijr}$ and $\bar{z}_{ijr} \leftarrow 0 \quad \forall j' \in V, r \in R_{j'}, i \in N(j)$

From constraint (6), and the fact that $\eta_j = 1$, we have:

$$\bar{y}_{i^*} = \sum_{i \in N(j)} y_i \geq \sum_{i \in B_R(j)} \sum_{q \in Q_j^*} z_{ijq} = \eta_j = 1$$

Subsequently, we mark every $i \in N(j)$ as closed, and mark $i^*$ as completely open. Furthermore, we mark every $j'$ that was reassigned in the above steps as touched.

Note that in the last three steps, any agents at a node $j'$ that was initially assigned to $i \in N(j)$ is now assigned to $i^*$. Since each of the distances $j' \rightarrow i$ and $i \rightarrow j$ is at most $R$, the distance from $j'$ to $i^*$ is at most $2R$. Therefore, this step relaxes the distance of a feasible assignment to a virtual market from $R$ to $2R$.

This process trivially preserves the objective and weak budget balance, as well as constraints (14). Moreover, constraints (6) and (7) are satisfied since we add both sides of the constraints corresponding to $i \in N(j)$ to obtain the constraint for $i^*$. Finally, to see that (5) is satisfied for $i^*$, note that

$$\sum_{j' \in J_D(i^*)} \sum_{q \in Q_j^*} d_{j'q} \bar{z}_{i^*j'q} = \sum_{i \in N(j)} \sum_{j' \in J_D(i)} \sum_{q \in Q_{j'}} d_{j'q} z_{ijq} \geq L \sum_{i \in N(j)} y_i \geq L$$

We continue this process, finding a node $j \in U_f$, and merging all virtual markets in $N(j)$ to one location. At the end of this process, the set $U_f$ is empty.

Phase 2. At the end of Phase 1, each node $j$ which is touched (including all fully utilized nodes) route some fraction of their demand (or supply) to at least one virtual market that is completely open. However, there could still be partially open markets with $y_i \in (0,1)$ to which demand and supply are assigned. Consider these partially open markets in arbitrary order. Suppose we are considering market $i$ and there exists touched and fully utilized node $j$ such that $\sum_{q \in Q_j^*} z_{ijq} > 0$ (resp. $\sum_{r \in R_j^*} \bar{z}_{i^*jr} > 0$). Consider the completely open market $i^*$ such that $\sum_{q \in Q_{j'}} \bar{z}_{i^*j'q} > 0$ or $\sum_{r \in R_{j'}} \bar{z}_{i^*jr} > 0$. We move the virtual market $i$ to location $i^*$, updating the variables just as in Phase 1; i.e., we set

- $\bar{y}_{i^*} \leftarrow \bar{y}_{i^*} + y_i$ and $\bar{y}_i \leftarrow 0$;
- $\bar{z}_{i^*j'q} \leftarrow \bar{z}_{i^*j'q} + z_{ij'q}$ and $\bar{z}_{ij'q} \leftarrow 0 \quad \forall j' \in V, q \in Q_{j'}$
- $\bar{z}_{i^*jr} \leftarrow \bar{z}_{i^*jr} + z_{ijr}$ and $\bar{z}_{ijr} \leftarrow 0 \quad \forall j' \in V, r \in R_{j'}$

The argument that all constraints are preserved follows just as before. For any $j'$ that was partially assigned to $i$, the new assignment is to $i^*$. This distance is at most

$$c(j', i^*) \leq c(j', i) + c(i, j) + c(j, i^*) \leq R + R + 2R = 4R$$

where we note that the distance $j \rightarrow i^*$ was at most $2R$ because $j$ was potentially reassigned to $i^*$ in Phase 1. We mark all nodes $j'$ that are reassigned in this process as touched.
At the end of this process, suppose there are still partially open markets with $y_i \in (0, 1)$. By Corollary 1, each of these markets $i$ must have some $j \in Z$ partially assigned to it. At the end of Phase 1, we have the invariant that $j \notin U_f$, since $U_f$ is empty. This means $j$ was touched on Phase 1. But in that case, $i$ must have been reassigned in Phase 2, which is a contradiction. Therefore, at this point, all virtual markets are either closed ($\bar{y}_i = 0$) or completely open ($\bar{y}_i \geq 1$). Furthermore, for any variable $\hat{z}_{ij} \geq 0$ (resp. $\hat{z}_{ij} > 0$), the virtual market $i$ is completely open; the distance from $j$ to $i$ is at most $4R$; for each completely open market, the rate of supply equals the rate of demand (Constraint (7)), and finally, the total flow is at least $L$ (Constraint (5)).

E.3 Rounding Prices and Wages

The rounding the $\alpha_{ji}, \beta_{jr}$ variables to integer values now follows from Lemma 1. In summary, the price at node $j$ is $F_j^{-1}(\hat{q}_j)$ and the wage at $j$ is $H_j^{-1}(\hat{r}_j)$. An arriving demand (resp. supply) agent at node $j$ is routed to market $i$ with probability $\hat{z}_{ij}\hat{q}_j$ (resp. $\hat{z}_{ij}\hat{r}_j$). This yields theorem 1.

F Proof of Theorem 2

The behavior of a virtual market is captured via the following birth-death Markov chain: consider the state-space $\{\ldots, s(2), s(1), 0, b(1), b(2), \ldots\}$, where $0 = s(0) = b(0)$ denotes the state that the market is empty, while for any $n \geq 1$, the state $b(n)$ denotes that there are $n$ buyers queued up, and state $s(n)$ denote that there are $n$ sellers queued up. For any $n \geq 1$, the transition rate from $b(n)$ to $b(n+1)$ is $\lambda$, and for $s(n)$ to $s(n+1)$ is $\mu$; on the other hand, the rate of transition from $s(n)$ to $s(n-1)$ is $n\gamma + \lambda$, while from $b(n)$ to $b(n-1)$ is $n\kappa + \mu$. Here, the term $n\gamma$ corresponds to the rate of abandonment of sellers as their deadlines expires, and similarly $n\kappa$ is the rate of abandonment of buyers.

We now first show the sufficient conditions. Assume that $\lambda = \mu$. Let $q_0$ denote the steady state probability of the queue being empty. Let

$$\Pr[\text{State} = s(n)] = \alpha_n, \quad \Pr[\text{State} = b(n)] = \beta_n$$

where $\alpha_0 = \beta_0 = q_0$. We have the following balance equations:

$$\alpha_n n\gamma = \lambda(\alpha_{n-1} - \alpha_n) \quad \text{and} \quad \beta_n n\kappa = \lambda(\beta_{n-1} - \beta_n)$$

Adding these equations, we have: $\sum_{n=1}^{\infty} (\alpha_n n\gamma + \beta_n n\kappa) = 2\lambda q_0$. Note that the LHS here is the total abandonment rate, and since the total arrival rate of agents (buyers and sellers) is $2\lambda$, this means the abandonment probability is exactly $q_0$.

Since $\alpha_n = \frac{\lambda^n}{\lambda^n + \gamma^n} \alpha_{n-1}$ and $\beta_n = \frac{\lambda^n}{\lambda^n + \kappa^n} \beta_{n-1}$, we have by telescoping:

$$\alpha_n = q_0 \prod_{j=1}^{n} \frac{\lambda}{\lambda + j\gamma}, \quad \beta_n = q_0 \prod_{j=1}^{n} \frac{\lambda}{\lambda + j\kappa}$$

Since these probability values sum to one, this implies

$$\frac{1}{q_0} = 1 + \sum_{n=1}^{\infty} \left( \prod_{j=1}^{n} \frac{1}{1 + j\frac{\lambda}{\gamma}} + \prod_{j=1}^{n} \frac{1}{1 + j\frac{\lambda}{\kappa}} \right)$$

(15)
For given \( \kappa \) and \( \gamma \), this is an increasing function of \( \lambda \). Therefore, \( q_0 \leq \epsilon \) translates to a bound of the form \( \lambda \geq L \). An upper bound \( L_\epsilon \) on \( L \) can be computed as follows. Let \( c = \frac{\min(\gamma, \kappa)}{\lambda} \). Then,

\[
\frac{1}{q_0} \geq 1 + \sum_{n=1}^{\infty} \left( \prod_{j=1}^{n} \left( \frac{1}{1 + j \cdot \frac{\min(\gamma, \kappa)}{\lambda}} \right) \right) 
\geq \sum_{n=0}^{\infty} e^{-cn^2/2} - e^{-c} \geq \int_{0}^{\infty} e^{-cx^2/2} \, dx - e^{-c} = \sqrt{\frac{\pi}{2c}} - e^{-c} \geq \sqrt{\frac{2}{3c}}
\]

where the second inequality uses \( 1 + x \leq e^x \) for all \( x \geq 0 \). Therefore, if we insist \( \sqrt{\frac{2\lambda}{3\min(\gamma, \kappa)}} \geq \frac{1}{\epsilon} \), this ensures the abandonment probability is at most \( \epsilon \). This translates to the following lower bound:

\[
\lambda \geq L_\epsilon = \frac{3}{2} \cdot \frac{\min(\gamma, \kappa)}{\epsilon^2}
\]

This completes the proof of the sufficient conditions.

For the necessary conditions, the first condition is obvious: If \( \lambda/\mu \notin [1 - \epsilon, 1 + \epsilon] \), in steady state, a fraction \( \epsilon \) of either buyers or sellers must necessarily be discarded just because an equal number of buyers and sellers are matched.

Next, we show that if \( \lambda = \mu \), the bound on \( q_0 \) in Equation (15) is tight to a constant factor. Recall we define \( c = \min(\gamma, \kappa)/\lambda \). First, note that if \( c \geq 1 \), we have

\[
\frac{1}{q_0} \leq 2 \sum_{n=0}^{\infty} \left( \prod_{j=0}^{n} \frac{1}{1 + j \cdot c} \right) \leq 2 \sum_{n=0}^{\infty} \left( \frac{1}{(n + 1)!} \right) = 2 \cdot (e - 1)
\]

This gives \( q_0 \geq 0.29 \) which is infeasible as we want \( \epsilon \leq 1/6 \). Thus, the only relevant case is \( c = \min(\kappa, \gamma)/\lambda \leq 1 \), for which we have:

\[
\frac{1}{q_0} \leq 2 \sum_{n=0}^{\infty} \left( \prod_{j=0}^{n} \frac{1}{1 + j \cdot c} \right) 
\leq 2 \left( 1 - \frac{1}{\sqrt{c}} - \sum_{n=0}^{\infty} \frac{1}{\sqrt{c}} \left( \prod_{j=1}^{\sqrt{c}} \frac{1}{1 + j \cdot c} \right) \left( \frac{1}{1 + \sqrt{c}} \right)^{n-1/\sqrt{c}} \right) 
\leq 2 \left( \frac{1}{\sqrt{c}} + \frac{1}{\sqrt{2}} \cdot \frac{1 + \sqrt{c}}{\sqrt{c}} \right) \leq \frac{7}{\sqrt{c}}
\]

Next suppose \( \lambda > \mu \); note that this means \( \lambda \in \mu \cdot [1, 1 + \epsilon] \) since \( \lambda/\mu \in [1 - \epsilon, 1 + \epsilon] \). Now consider the subset of the Markov chain on the states \( \{0, b(1), b(2), \ldots\} \), with transition rates from \( b(n) \) to \( b(n-1) \) is \( \mu + nk \), and that from \( b(n-1) \) to \( b(n) \) is \( \lambda > \mu \); we henceforth refer to this as the buyer system. Note that the abandonment rate of buyers in this system is stochastically dominated by the abandonment rate if \( \mu = \lambda \).

Now, conditioned on the queue being empty and a buyer (resp. seller) arriving, let \( T_b \) (resp. \( T_s \)) denote the expected time after which the queue next becomes empty, and let \( R_b \) (resp. \( R_s \)) denote the abandonment rates in these periods. When the queue is empty, the probability that
a buyer arrives is $\frac{\lambda}{\lambda + \mu} \in \left[\frac{1}{2}, \frac{7}{13}\right]$, since $\epsilon \leq 1/6$. Moreover, Wald’s identity gives that the overall abandonment rate is by $R \left(\frac{\lambda}{\lambda + \mu} T_b + \frac{\mu}{\lambda + \mu} T_s\right) = \left(\frac{\lambda}{\lambda + \mu} R_b T_b + \frac{\mu}{\lambda + \mu} R_s T_s\right)$, and thus

$$R \geq \frac{\mu}{\lambda} \left(\frac{T_b R_b + T_s R_s}{T_b + T_s}\right) \geq \frac{6}{7} \times R_b \times \frac{T_b}{T_b + T_s}$$

Now we need to consider two separate cases:

1. $\gamma > \kappa$: Here, the transition rate from $s(n)$ to $s(n-1)$ is $\lambda + n\gamma$, and that transition rate from $b(n)$ to $b(n-1)$ is $\mu + n\kappa < \lambda + n\gamma$; on the other hand, the rate from $s(n-1)$ to $s(n)$ is $\mu$ and that from $b(n-1)$ to $b(n)$ is $\lambda > \mu$. By stochastic dominance, we therefore have $T_s \leq T_b$, which means the abandonment rate is:

$$R \geq \frac{6}{7} \times R_b \times \frac{T_b}{T_b + T_s} \geq \frac{3}{7} R_b$$

Moreover the abandonment probability in the buyer system is smaller if we assume $\lambda = \mu$, and hence $\frac{6}{7} \geq \frac{1}{7} \sqrt{\frac{\kappa}{\gamma}}$. Combining these bounds, we have:

$$R \geq \frac{3}{7} R_b \geq \frac{3}{49} \sqrt{\frac{\kappa}{\lambda \gamma}}$$

Since the overall arrival rate is at most $2\lambda$, the abandonment probability is at least $\frac{1}{49} \sqrt{\frac{\kappa}{\lambda \gamma}}$.

2. $\gamma \leq \kappa$: In this case, note that $T_b$ is the inverse of the probability the buyer system is in state $b(0)$. Since this probability is maximized when $\lambda = \mu$, we have $T_b \geq \left(\frac{3\mu}{3\mu + \kappa}\right)$. Similarly, $T_s$ is maximized when $\lambda = \mu$, so that $T_s \leq 7\sqrt{\frac{2}{\gamma}}$. Therefore, assuming $\epsilon \leq 1/6$, we have:

$$\frac{T_s}{T_b} \leq \left(\frac{7\sqrt{2}}{\sqrt{\gamma}} \sqrt{\frac{\kappa}{\gamma}}\right) \leq 6 \sqrt{\frac{\kappa}{\gamma}} \Rightarrow \frac{T_b}{T_s + T_b} \geq \frac{1}{1 + 6 \sqrt{\frac{\kappa}{\gamma}}} \geq \frac{1}{7} \sqrt{\frac{\gamma}{\kappa}}$$

Further, as before, the abandonment rate $R_b \geq \frac{1}{7} \sqrt{\lambda \kappa}$, and thus

$$R \geq \frac{6}{7} \times \frac{1}{7} \sqrt{\lambda \kappa} \times \frac{1}{7} \sqrt{\frac{\gamma}{\kappa}} = \frac{6}{493} \sqrt{\lambda \gamma}$$

This means the abandonment probability is at least $\frac{3}{493} \sqrt{\frac{\gamma}{\lambda \gamma}}$.

Combining the two, we see that to guarantee that in order to ensure average abandonment rate is at most $\epsilon$, we need $\lambda \geq \frac{1}{14000} \min(\gamma, \kappa)/\epsilon^2$.

### G Envy-Free Pricing and Profit Maximization

The idea from Section 2 of independently scaling up/down LP variables corresponding to individual facilities is fairly general, and leads naturally to approximation algorithms for more complex variants that are motivated by different scheduling policies for the dynamic marketplace problem. In this section, we present one such formulation that generalizes the model discussed in Section 2. In Appendix G.2, we show that this model corresponds to the dynamic marketplace setting when the
platform uses prices to elicit patience of agents, and uses Earliest Deadline First (EDF) scheduling in each virtual market.

We assume each node (type) \( j \) of buyer/seller has a collection of subtypes \( S_j \). There is a DAG \( G_j(S_j, E_j) \) on \( S_j \) that captures envy. If there is an edge \((k_1, k_2) \in E_j\), then sub-type \( k_1 \) envies sub-type \( k_2 \). The platform announces a price (resp. wage) \( p_{jk} \) (resp. \( w_{jk} \)) for each sub-type \( k \in S_j \).

In order to preserve incentive compatibility, we require that if \((k_1, k_2) \in E_j\), then \( p_{jk_1} \leq p_{jk_2} \), resp. \( w_{jk_1} \geq w_{jk_2} \). This prevents an agent of sub-type \( k_1 \) from reporting its type to be \( k_2 \). Note that since the graph \( G_j \) is a DAG, such a price (resp. wage) assignment is feasible. We term such an assignment of prices (resp. wages) at each \( j \) as a price (resp. wage) ladder.

As before, there is a non-increasing demand function \( d_{jk} \) \( F_{jk}(p_{jk}) \) for each buyer sub-type \( k \in S_j \), and a non-decreasing supply function \( s_{jk} \) \( H_{jk}(w_{jk}) \) for each seller sub-type \( k \in S_j \). Each sub-type \( k \in S_j \) is also associated with a weight \( G_{jk} \). The platform maintains a distribution \( Z_j \) of virtual markets within distance bound \( R \). If there is an edge \((k_1, k_2) \in E_j\), then sub-type \( k_1 \) envies sub-type \( k_2 \). Note that since the graph \( G_j \) is a DAG, such a price (resp. wage) assignment is feasible. We term such an assignment of prices (resp. wages) at each \( j \) as a price (resp. wage) ladder.

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Lottery Pricing and Assignment. The platform opens a set of “virtual markets”. For each node \( j \) and \( k \in S_j \), buyers (resp. sellers) arriving at the node and choosing that type are probabilistically routed to virtual markets which are within distance \( R \) from the node. We assume the platform shows a lottery over price (resp. wage) ladders as follows: For each node \( j \in V \) the platform maintains a distribution \( Z_j \) of virtual markets within distance bound \( R \), and for each virtual market in this set, it maintains a distribution \( L_{ij} \) of price (resp. wage) ladders. Given an agent arriving at this node, the platform first chooses a market \( i \) from \( Z_j \), and then a ladder from \( L_{ij} \) and shows it to the agent. After the agent chooses the price or wage (hence revealing its sub-type), she is routed to market \( i \). We note that the routing policy makes the market the agent is routed to be independent of the sub-type elicited. Though this assumption is somewhat restrictive, it prevents the agent from choosing a sub-type to optimize for the virtual market they get assigned to.

Service Availability Guarantee. As before, we capture service availability by ensuring that each virtual market \( i \) has balanced supply and demand, and is also sufficiently thick. However, we now capture thickness by a lower bound \( L \) on the total weight of the sub-types assigned there. Formally, let \( x_{ijk} \) denote the expected flow of sub-type \( k \in S_j \) to virtual market \( i \).

Flow Balance. The expected amount of supply and demand assigned there are the same.

Weight Lower Bound. The expected weight assigned there is large: \( \sum_{j \in V, k \in S_j} G_{jk} x_{ijk} \geq L \).

The objective is to maximize the expected profit of the solution. We term this problem \textsc{Envy-Free FL}(\( L, R \)). We note that similar ideas can be used to maximize other objectives; we present the profit objective for simplicity.

In the dynamic marketplace setting presented in Appendix G.2, the sub-types correspond to different deadlines, and the weight of a sub-type is precisely the deadline value. We show there that the weight lower bound corresponds to the condition for the EDF scheduling policy to have low abandonment rate.

G.1 Approximation Algorithm

Our LP formulation and rounding are similar to the one for \textsc{Two-Sided Fac-Loc}(\( L, R \)), and we highlight the differences. As before, we assume there is a candidate set \( P \) and \( W \) of prices and wages for each node, respectively. The set of all candidate virtual markets in the metric space is denoted by \( F \); since we assume the metric space is explicitly specified as input, we set \( F = V \). For each node \( j \), \( B_R(j) \subseteq F \) denotes the set of all the virtual markets \( i \in F \) such that \( c(i, j) \leq R \). For each virtual market \( i \), define \( B_R(i) \) as the set of all the nodes \( j \in V \) such that \( c(i, j) \leq R \).
G.1.1 Linear Programming Relaxation

Variables. For each candidate virtual market \( i \in \mathcal{F} \), let \( y_i \in \{0, 1\} \) be the indicator variable that a virtual market exists at that location in the metric space. These are the only integer variables in our formulation. Variables \( x^d_{ij} \) and \( x^s_{ij} \) are non-zero only if \( y_i = 1 \) and \( i \in B_R(j) \). In this case, those are respectively the probability that buyers and sellers at node \( j \) are routed to virtual market \( i \). Note that there is some probability that all prices at node \( j \) are set to \( p_{\text{max}} \), which corresponds to not routing node \( j \) anywhere. Let \( z_{ijkp} \) be the probability that buyers at node \( j \) with sub-type \( k \) are assigned to virtual market \( i \) and offered price \( p \). Similarly, \( z_{ijkw} \) denotes the probability that sellers at node \( j \) with sub-type \( k \) are assigned to virtual market \( i \) and offered wage \( w \).

Objective and Constraints. The objective is the same as before.

\[
\max \sum_{j \in V} \sum_{k \in S_j} \sum_{i \in B_R(j)} \left( \sum_{p \in P} p d_{jk} F_{jk}(p) z_{ijkp} - \sum_{w \in W} w s_{jk} H_{jk}(w) z_{ijkw} \right) \tag{16}
\]

The following constraints connect the variables together. We present these constraints only for buyers (that is, \( p \in \mathcal{P} \)): the constraints for sellers is obtained by replacing \( p \) and \( x^d \) with \( w \) and \( x^s \). Since we route the buyers at node \( j \) probabilistically to one of the virtual markets, or to no market by offering all deadlines a price \( p_{\text{max}} \):\[
\sum_{i \in B_R(j)} x^d_{ij} \leq 1 \quad \forall j \in V \tag{17}
\]

Next, a price should be offered to each buyer with sub-type \( k \) at node \( j \) assigned to market \( i \):

\[
\sum_{p \in \mathcal{P}} z_{ijkp} = x^d_{ij} \quad \forall j \in V, i \in B_R(j), k \in S_j \tag{18}
\]

Next, if demand is fractionally routed from \( j \) to \( i \), then \( i \) should be open and within distance \( R \):

\[
x^d_{ij} \leq y_i \quad \forall j \in V, i \in B_R(j) \tag{19}
\]

We next enforce that the prices and wages form a distribution over ladders. Note that the policy first chooses the virtual market to route to, and then chooses from a distribution over ladders. This reduces to a stochastic dominance condition for the distributions corresponding to \( z \):

\[
\sum_{p' \leq p, p'' \in \mathcal{P}} z_{ijkp'} \leq \sum_{p' \leq p, p'' \in \mathcal{P}} z_{ijkp''} \quad \forall p \in \mathcal{P}, (k, k') \in E_j, \forall j \in V, \forall i \in B_R(j) \tag{20}
\]

\[
\sum_{w' \leq w, w'' \in \mathcal{W}} z_{ijkw'} \geq \sum_{w' \leq w, w'' \in \mathcal{W}} z_{ijkw''} \quad \forall w \in \mathcal{W}, (k, k') \in E_j, \forall j \in V, \forall i \in B_R(j) \tag{21}
\]

Finally, we encode the service availability constraints. We first capture flow balance at each virtual market: the rate of arrival of buyers and sellers are equal.

\[
\sum_{j \in B_R(i)} \sum_{k \in S_j, p \in \mathcal{P}} d_{jk} F_{jk}(p) z_{ijkp} = \sum_{j \in B_R(i)} \sum_{k \in S_j, w \in \mathcal{W}} s_{jk} H_{jk}(w) z_{ijkw} \quad \forall i \in \mathcal{F} \tag{22}
\]

We finally encode weighted flow lower bound on the total deadline of buyers and sellers at the market:

\[
\sum_{j \in B_R(i)} \sum_{k \in S_j} G_{jk} \left( \sum_{p \in \mathcal{P}} d_{jk} F_{jk}(p) z_{ijkp} + \sum_{w \in \mathcal{W}} s_{jk} H_{jk}(w) z_{ijkw} \right) \geq L y_i \quad \forall i \in \mathcal{F} \tag{23}
\]
G.1.2 Rounding

If we ignore the integrality constraints on \( y_i \), the above is a linear programming relaxation of the problem. We will now show how to round the resulting solution.

We generalize Lemma 3 using the following definitions of fully utilized. We say that \( j \in V \) is fully demand utilized if \( \sum_{i \in B_R(j)} x^d_{ij} = 1 \); similarly, it is fully supply-utilized if \( \sum_{i \in B_R(j)} x^s_{ij} = 1 \). We say \( j \) is partially demand-connected to market \( i \in F \) if \( x^d_{ij} > 0 \), and partially supply-connected if \( x^s_{ij} > 0 \). Let \( J_D(i) \) denote the set of nodes that are partially demand-connected to \( i \in F \), and \( J_S(i) \) be the set that is partially supply-connected. As before, we define the profit of a virtual market \( i \in F \) as:

\[
R_i = \sum_{j \in B_R(i)} \sum_{k \in S_j} \left( \sum_{p \in P} pd_{jk} F_{jk}(p) z_{ijkp} - \sum_{w \in W} ws_{jk} H_{jk}(w) z_{ijkw} \right)
\]

Lemma 4. In the LP optimum, for any \( i \in F \), \( R_i \geq 0 \). Further, if \( R_i > 0 \), either \( y_i = 1 \); or there exists \( j \in J_D(i) \), s.t. \( j \) is fully demand-utilized; or there exists \( j \in J_S(i) \), s.t. \( j \) is fully supply-utilized.

The proof of the above lemma follows the same argument as Lemma 3. If a market has negative \( R_i \), we can set all its variables to zero without violating any constraints. If the condition in the Lemma is violated for \( i \in F \), then we can increase all variables corresponding to \( i \) by the same factor till the condition is satisfied. Since all constraints involve single markets, this process preserves them while increasing the objective. For this transformation to work, it is crucial Constraints (20) are defined separately for each \((i, j)\) pair; in other words, we crucially need to assume the policy chooses a market first and then chooses a distribution over ladders for that market.

The rounding now proceeds in the same way as in Section 2. In Phase 1, we identify untouched and fully utilized \( j \) and merge all \( i \) to which it is partially connected to one market. Note that the total \( y_i \) of these markets is at least 1 by the LP constraints. At the end of this phase, we move the remaining partially open \( i \) as in Phase 2 of the rounding scheme. This preserves the profit, and satisfies the flow balance and lower bound constraints (\( B_R \) is replaced by \( B_{4R} \) in the constraints), yielding the following theorem:

Theorem 4. There is a feasible solution \( \{\bar{x}, \bar{y}, \bar{z}\} \) to the above linear program, whose objective is optimal, and all of whose constraints are satisfied. For each \( i \in F \), either \( \bar{y}_i = 0 \) or \( \bar{y}_i \geq 1 \).

Final Policy. The final choice of prices and wages, and the routing policy is the following. We present it only for buyers; the policy for sellers is symmetric.

- At node \( j \), choose a market \( i \) with probability \( \bar{x}_{ij} \). If no market is chosen, the price is set to \( p_{\text{max}} \).
- If market \( i \) is chosen, choose \( \alpha \) uniformly at random in \([0, 1]\). For each \( k \in S_j \), find \( p_k \in P \) such that \( \sum_{p' \leq p_k, p' \in P} \frac{\bar{z}_{jkp'}}{\bar{x}_{ij}} \leq \alpha \) and \( \sum_{p' \leq p_k, p' \in P} \frac{\bar{z}_{jkp'}}{\bar{x}_{ij}} > \alpha \). Post prices \( \{p_1, p_2, \ldots, p_K\} \).
- If the buyer accepts price \( p_k \), route her to virtual market \( i \).

Constraints (20) imply that regardless of the choice of \( \alpha \), the prices \( \{p_1, p_2, \ldots, p_K\} \) in the second step form a ladder, so that \( p_1 \geq p_2 \geq \cdots \geq p_K \). A similar statement holds for wages. Therefore, the second step produces a lottery over ladders. Further, if \( Z_{ijkp} \) denote the event that the price for sub-type \( k \in S_j \) is \( p \) and market \( i \) is chosen, then it is an easy exercise to check that \( E[Z_{ijkp}] = \bar{z}_{jkp} \). Therefore, the randomized policy exactly implements the solution found in Theorem 4 so that it maximizes profit. We omit the details and state the final theorem.
Theorem 5. There is a polynomial time \((4,1)\) approximation for Envy-Free FL\((L,R)\). That is, we obtain the optimal expected profit by relaxing the distance constraint by a factor of 4.

G.2 Justification via Dynamic Marketplace Model

In this section, we present a dynamic marketplace model that justifies the problem statement of Envy-Free FL. As in Section 3, we assume buyers and sellers have an inherent patience level or deadline. If they are not matched within their deadline, they drop out of the system. We assume every agent \(m\) is associated with a patience level \(\nu_m\); unlike Section 3 we do not assume these are Exponentially distributed. The platform advertises a fixed set of patience levels, or deadlines, denoted by \(S_j = \{\nu_{j1}, \nu_{j2}, \ldots, \nu_{jK}\}\), which is a guarantee on the time by which a buyer or seller choosing that deadline is guaranteed to be matched. We assume \(\nu_{j1} \leq \nu_{j2} \leq \cdots \leq \nu_{jK}\). For simplicity, we use \(k \in S_j\) and \(\nu_k \in S_j\) interchangeably.

Incentive-compatibility. We assume the platform sets a lottery of prices and wages at each node \(j\), that are independent of time. Consider the issue of eliciting deadlines truthfully. Consider buyers first. At node \(j\), suppose the platform offers price \(p_{jk}\) for deadline \(\nu_{jk}\). Every buyer can choose one deadline in \(S_j\), in which case he pays price \(p_{jk}\), and is guaranteed to be matched within time \(\nu_{jk}\) from his arrival. We assume any buyer \(m\) has very large negative utility for being matched after his patience level \(\nu_m\), therefore he will choose a \(k\) such that \(\nu_{jk} \leq \nu_m\). Subject to this, he will choose \(k\) with smallest \(p_{jk}\), since this maximizes his valuation minus price. A symmetric model can be posited for sellers, where we replace price with wage, and the seller chooses the largest wage such that the corresponding deadline is smaller than his own patience level.

Since the goal of the platform is to elicit patience levels truthfully, the platform chooses a price ladder \(p_{j1} \geq p_{j2} \geq \cdots \geq p_{jK}\) and wage ladder \(w_{j1} \leq w_{j2} \leq \cdots \leq w_{jK}\) at each node \(j\). This ensures that agents with \(\nu_m \in [\nu_{jk}, \nu_{j+1}]\) report deadline \(\nu_{jk}\).

Each deadline level \(\nu_{jk} \in S_j\) gets associated with non-increasing demand function \(d_{jk}F_{jk}(p_{jk})\), which is the Poisson rate at which buyers \(m\) with patience \(\nu_m \in [\nu_{jk}, \nu_{j+1}]\) arrive when the price of deadline \(\nu_{jk}\) is \(p_{jk}\). Similarly, deadline level \(k \in S_j\) is associated with a non-decreasing supply function \(s_{jk}H_{jk}(w_{jk})\), which is the Poisson rate at which sellers \(m\) with patience \(\nu_m \in [\nu_{jk}, \nu_{j+1}]\) arrive when the wage for deadline \(\nu_{jk}\) is \(w_{jk}\). These deadline levels correspond to the sub-types described before.

Scheduling Policy. As in Section 3 the platform opens a set of “virtual markets”. For each node \(j\) and deadline level \(k\), buyers (resp. sellers) arriving at the node and choosing that deadline are probabilistically routed to virtual markets which are within distance \(R\) from the node. Buyers and sellers arriving at the virtual market are queued up, and optimally matched to minimize abandonment. Since the platform knows which deadline was chosen by the agent, the optimal matching policy is now a variant of Earliest Deadline First (EDF): When the deadline of some buyer (resp. seller) expires, it is matched to that seller (resp. buyer) in the queue whose deadline will expire earliest in the future. If an agent’s deadline expires and there is no agent to match it with in the queue, this agent is abandoned. It is an easy exercise to show that this policy maximizes the number of matches made in any virtual market.

As in Section 3 the goal of the platform is to design a joint pricing and scheduling policy to maximize profit, while ensuring bounded match distance and bounded abandonment probability.
G.2.1 Bounding Abandonment Rate

We will now show that the weight lower bound can be interpreted as a sufficient condition for the abandonment rate of the EDF policy to be at most $\epsilon$, where the weight of a sub-type is simply its deadline value.

The main technical assumption we require in this part is that the desired abandonment probability, $\epsilon$ is small, in particular that $\epsilon \ll \frac{\nu_{\text{min}}}{\nu_{\text{max}}}$. As noted above, the scheduling policy within a virtual market is a variant of EDF. Unlike the PATIENCE-OBVIOUS model where the behavior of a virtual market could be modeled as a variant of a $M/M/1$ queue, the optimal abandonment probability in a two-sided EDF queue clearly depends on the entire distribution of deadlines of buyers and sellers, which in turn depends on the pricing scheme and assignment policy. However, we crucially need a closed-form bound on this probability in order to plug into an LP relaxation for the overall problem. We use recent results from queueing due to Kruk et al. [27] to construct such a closed-form bound, whose very existence we find non-trivial and surprising!

Kruk et al. [27] present an approximation to the abandonment probability of a one-sided queue $M/M/1$ queue with EDF scheduling. They approximate the queueing process via a reflected Brownian motion. We adapt their result to our setting, and rephrase it below. Consider the queue associated with a virtual market. Let $\bar{\nu}$ denote the average deadline of a seller arriving to this queue, and $\bar{D}$ denote the average deadline of a buyer arriving to the queue. Note that the distribution of deadlines as well as arrival rate depends on the overall pricing and assignment policies.

Recall that we assumed $\epsilon$ is small, in particular that $\epsilon \ll \frac{\nu_{\text{min}}}{\nu_{\text{max}}}$. We first enforce that supply and demand arrive to the queue at the same rate; call this rate $\lambda$. Next suppose w.l.o.g. that $\bar{D} > \bar{S}$. Consider the policy that instantaneously matches arriving sellers to the queued buyer with earliest deadline; if the queue is empty, the seller is abandoned. This exactly mimics a one-sided $M/M/1$ queue with EDF scheduling. We quote the following result informally from [27]:

Consider a one sided $M/M/1$ queue with arrival rate and service rate equal to $\lambda$. Suppose deadlines of jobs are independently distributed with mean $\bar{D}$, and the scheduler uses the EDF policy. Then holding $\lambda$ and $\frac{\nu_{\text{max}}}{\nu_{\text{min}}}$ fixed, in the regime where $\nu_{\text{min}}$ becomes very large, the abandonment probability approaches $\frac{1}{\lambda \bar{D}}$.

Though part of their argument is heuristic, they perform simulations to show that this approximation is indeed accurate. Since we need abandonment probability of $\frac{1}{\lambda \bar{D}}$ to be at most $\epsilon \ll 1$, and since we assumed $\frac{\nu_{\text{max}}}{\nu_{\text{min}}} \gg \epsilon$, this automatically enforces that all deadlines are much larger than the mean inter-arrival time, satisfying their precondition for our setting.

Since the optimal policy for a two-sided queue only has lower abandonment probability, we use $\frac{1}{\lambda \bar{D}}$ as an upper bound on this quantity. Since we assumed $\bar{D} \geq \bar{S}$, we will instead use $\frac{2}{\lambda (\bar{D} + \bar{S})}$ as the upper bound, which we will set to be at most $\epsilon$.

We now show that this is the best possible upper bound that only depends on $\bar{D}$ and $\bar{S}$. Suppose buyers deadlines are deterministic with value $\bar{D}$, and seller deadlines are deterministic with value $\bar{S}$. Then the optimal policy matches without waiting in a FIFO fashion. This means the loss probability assuming the queue has buyers is the same as that of a $M/M/1$ queue with deadlines $\bar{D}$, which from [11] is exactly

$$P_1 = \frac{1}{1 + \lambda \bar{D}}$$

Similarly, when there are sellers in the queue, the loss probability is

$$P_2 = \frac{1}{1 + \lambda \bar{S}}$$
Conditioned on the queue being empty and a buyer arriving, the expected time after which the queue next becomes empty is \( T_b = \frac{1}{P_1} = 1 + \lambda \bar{D} \), in which period the loss probability is \( P_1 \). Similarly, if a seller arrives when the queue is empty, the expected time after which the queue again becomes empty is \( T_s = \frac{1}{P_2} = 1 + \lambda \bar{D} \), in which period the loss probability is \( P_2 \). Since a buyer or seller arrives with equal probability when a queue is empty, the expected loss probability is

\[
P = \frac{T_b P_1 + T_s P_2}{T_b + T_s} = \frac{2}{2 + \lambda \bar{D} + \lambda \bar{S}} \approx \frac{2}{\lambda (\bar{D} + \bar{S})}
\]

assuming \( \lambda (\bar{D} + \bar{S}) \gg 1 \).

In summary, each virtual market needs to satisfy the following two sufficient conditions for its abandonment probability to be at most \( \epsilon \):

1. The rate of arrival of supply and demand should be the same; call this rate \( \lambda \).

2. If \( \bar{S} \) denote the average deadline of a seller, and \( \bar{D} \) denote the average deadline of a buyer, then \( \lambda (\bar{D} + \bar{S}) \geq \frac{2}{\epsilon} \).

Therefore, to reduce the scheduling policy to an instance of Envy-Free FL(\( R, L \)), we set \( G_{jk} = \nu_{jk} \) and \( L = \frac{2}{\epsilon} \), so that the second condition above translates to the weight lower bound. This justifies the Envy-Free FL(\( R, L \)) problem as capturing the optimal scheduling policy for the dynamic marketplace problem presented above.

Note that the resulting lower bound on \( \lambda \) derived by the above condition is a significant improvement over the patience-oblivious case, since the lower bound now depends on \( \frac{1}{\epsilon} \) instead of \( \frac{1}{\epsilon^2} \). This intuitively means that in order to achieve comparable profit and abandonment probability, we can aim for a higher quality of match by reducing the radius \( R \). A similar observation that even partial information about deadlines significantly reduces abandonment is made in [3], albeit for a different model.