Approximate relations between Manhattan and Euclidean distance regarding Latin hypercube experimental design

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Abstract. Among several existing distance measures, perhaps Euclidean distance is the most used tool to calculate distances and then Manhattan distance measure, though which distance measure is suitable depends on the problems. In the field of Design of Experiments (DoEs) especially in simulation domain Latin Hypercube design (LHD) is a well-known approach to find out design points for experiments/simulations. Generally, randomly generated LHDs show poor space-filling property. But space-filling is one of the most required impotent properties for DoE. So researchers search optimal LHD in the sense of space-filling. Some authors considered Euclidean distance measure on the other hand some other authors considered Manhattan distance measure to find out optimal LHD regarding space-filling. It is obvious that both optimal LHDs are not identical. So it is a crucial asked - which measure provided better space-filling? The main problem is that there is no any relation among the distance measures especially Euclidean distance measure and Manhattan distance measure. In this article we have established some relations and bounds between Euclidean distance measures and Manhattan distance measures in perspective of LHD. Finally some experimental results are compared namely maximin LHDs measured by both Euclidean distance measure and Manhattan distance measures by using some proposed transformation techniques.

1. Introduction

Physical experiments are often inevitably very expensive and time consuming. So computer experiments are frequently used for simulating physical characteristics. But the computer simulation of the mathematical model of the physical experiments is also usually time-consuming and there is a great variety of possible input combinations. So surrogate model (also called a global/ approximation model or meta-model) are required. Surrogate model is constructed based on modelling the response of the simulator to a limited number of cleverly chosen data points. It models the quality characteristics as explicit functions of the design parameters. The process comprises three major steps which may be interleaved iteratively:

i) Sample selection (DoE or active learning).

ii) Construction of the surrogate model and optimizing the model parameters (Bias-Variance trade-off).

iii) Appraisal of the accuracy of the surrogate.

Among the three steps of an experiment, the first one is the selection of sample data. Actually DoE simply means to careful selection of the input variables for the surrogate models, which is the first and one of the most vital steps for a successful statistical modelling of the simulations. In DoE only the
factors under consideration are allowed to vary, and all other conditions are kept as constant. It involves not only the selection of suitable independent, dependent and control variables, but planning the delivery of the experiments under statistically optimal conditions. Obviously DoE is of multi-disciplinary field of interest. There are multiple approaches available in the literature to determine the set of design points to be used in the experiments. LHD is one of the most frequently used DoEs. It is worthwhile to mention here that any good DoE should have two important properties. One is called non-collapsing property and another one is space-filling property. Non-collapsing property means whenever points are projected on each coordinate, no any two points share identical value; actually points are evenly spread across the projection of the experimental region onto the factors. Again space-filling property means the best coverage of the space of the inputs (design points) over the entire input domain, while Sample Random Sampling does not ensure this task. It is remarkable that every randomly generated LHD inherently holds this non-collapsing property nicely. On the other hand, though randomly generated LHD generally holds poor space-filling property but optimal especially maximin LHD has nice space-filling property. It is also noted that non-collapsing property ensures individual effect of the independent variables upon depending variable. On the other hand space-filling property ensures the collection of the information all over the input domain. The well-known mathematical definition of an LHD is given below [1, 2]:

\[ \mathbf{X} = \begin{bmatrix} x_{11} & \cdots & x_{1k} \\ \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{nk} \end{bmatrix} \]

\[ \exists \forall x_{ij} = \{0, 1, \ldots, N-1\} \text{ and } x_{ip} \neq x_{jp} \text{ for each } p. \]

It is also noticed that each LHD is a combinatorial configurations matrix of natural numbers including zero.

In order to demonstrate the randomly generated LHD regarding space-filling, we have considered a typical LHD for three factors \((k = 3)\) with 30 design points \((N = 30)\). The randomly generated LHD is shown in the figure 1. It is observed in the figure 1 (randomly generated LHD) that all design points are located in a narrow space of the input domain i.e. design points do not spread over the entire input domain. Moreover it seems that the factors are also highly correlated (see figure 1) [1, 2].

![Random LHD](image1.png)

![Maximin LHD](image2.png)

**Figure 1.** Randomly generated LHD for \((N, k) = (30,3)\).

**Figure 2.** Maximin LHD for \((N, k) = (30,3)\).

Therefore researchers frequently choose optimal LHDs. Among the several optimal LHDs, perhaps maximin LHD is more frequently used one. In maximin optimal criterion, the minimum inter-site distances of the design points are maximized [1]. Few authors considered minimax (minimize the
maximum inter-site distances) or Audze-Egilas (minimize potential energy) etc. as objective function to obtain optimal LHD regarding space-filling DoE [3–5]. The main objective of Maximin or other space-filling optimal LHD (like minimax, Audze-Egilas etc.) is to cover the entire input domain by the input variables [6–8] (see figure 2). For the demonstration, again consider the typical LHD, \((N, k) = (30, 3)\), and it is optimized regarding space-filling using maximin optimal criterion [2]. The optimal (maximin) LHD is displayed in the figure 2. It is observed in figure 2 that the design points are spread over the entire input domain.

In the case of optimization, the measurement of distances is one of the main tools to reach the objective function. In the arena of DoE, several distance measures are used to calculate the inter-site pair-wise distances among the design points. Some authors considered Euclidean distance measures to calculated inter-site distance [1, 2] whereas some other authors considered Manhattan distance and Chebyshev distance [9]. In [9] Dam et al. derived general formulas for two-dimensional maximin LHDs, by using a branch-and-bound algorithm. They considered Chebyshev distance \(L_\infty\) and Manhattan \(L_1\) distance. For further reading on comprehensive overview of the historical developments, state-of-the-art and field of applications, please refer to [10–15]. For example, in the book [15] edited by Koziel and Leifsson, there appears 8 out of the 16 chapters in which authors discuss about the applications of surrogate modelling, and the Latin hypercube design. The applications range from circuit design to microwave structures to aerodynamic shape optimization.

Anyway among the several distance measures, Euclidean distance and Manhattan distance measures are most popular. Since both the distances form on Euclidean geometry and exhibit good space-filling characteristics in the case of optimal DoEs. So these distance measures are frequently used in DoEs [2, 12]. Author [12] shows the popularity of LHD by the help of Google Scholar (http://scholar.google.com) database (which is copy here; see figure 3). Recently some elementary experimental studies have been carried out for the comparison between Euclidean distance and Manhattan distance regarding maximin LHDs [16, 17]. Few authors also compared Euclidean distance and Manhattan distance but their objectives are not space-filling [18, 19]. For example, recently, authors [19] compared Euclidean distance and Manhattan distance in the case of face recognition by using neural network approach. They said that “Good recognition rates have been achieved in the early stages of training regarding Euclidean facial distances”.

![Figure 3](image.png)

**Figure 3.** Comparison the popularity among orthogonal arrays, Hammersley and Latin hypercube regarding number of papers published per year. Data obtained from the Google Scholar (http://scholar.google.com) database in the week of March 4, 2013 [12].
But there is a burning question, which maximin LHD is more space-filling – maximin LHD in which distances are calculated by Euclidean measure or maximin LHD in which distances are calculated by Manhattan measure? As far as we know that, there is no any relation established between Euclidean distance and Manhattan distance – even in any particular case.

Here we have tried to establish relations between Euclidean distance and Manhattan distance in a particular case where the coordinates are non-negative integer number. It is worthwhile to mention here that this type of phenomenon occurs in the study of experimental design namely LHD where, for the \( n \) design points, the each factor (coordinate) of the points are divided into \( n \) equal sections and normalized to \( 0, 1, 2, \ldots, n-1 \) or \( 1, 2, 3, \ldots, n \).

2. Some Proposed Lemmas

As far as we know, there exists no any relation between these two measures in the literature even in any special cases. In this article we have established approximates relation between Euclidean distance measure and Manhattan distance measure in perspective of LHD for DoE. Since in an LHD of \( n \) design points with \( k \) factors, the values of factors (coordinates of the design point) are normalized to \( 0, 1, \ldots, n-1 \), \( 1, 2, \ldots, n \). so the problem becomes integer combinatorial problem. In addition, LHD has another unique characteristic – design points are non-collapsing whenever they are projected over any coordinate. Exploiting theses characteristics of LHD we have able to find out the lower and upper bound of Euclidean distances of design points whose Manhattan distances are identical. Before proposing some relations between two measures namely Manhattan and Euclidean distance measures, it is worthwhile to define some terminologies.

**Design points:** Each point \( \mathbf{x} \) of an LHD of \( (n,k) \) is called design point which forms as follows:

\[
\mathbf{x} \in \{0,1,2,\ldots,n-1\}^k
\]

**Lattice point:** Any point \( \mathbf{x} \) on a \( k \)-dimensional coordinates space is a lattice point if

\[
\mathbf{x} \in \{0,1,2,\ldots,n-1\}^k
\]

i.e. if it will be design point of an LHD of \( (n,k) \).

**Hyper-plane of \( \mathbf{x} \):** Let \( \mathbf{x} \) is a Lattice point on a \( k \)-dimensional coordinates space and the sum of its all coordinates values is \( n \), then the Hyper-plane which contains all lattice points whose sum of the coordinates are \( n \) is denoted here Hyper-plane of \( \mathbf{x} \) in brief Hyper-plane.

**Neighbourhood lattice:** if any point \( \mathbf{x} \) on the hyper-plane is not lattice and then there exist at least one lattice point for some smallest hyper-sphere whose centre is \( \mathbf{x} \). Then each such lattice contained in that smallest hyper-sphere is called Neighbourhood lattice of the point \( \mathbf{x} \).

Now some Lemmas are proposed regarding Manhattan and Euclidean distance measures. Moreover, the proofs of the lemmas are also given here.

**Lemma 1 (Manhattan):** If \( n,k \in N \) then for any \( k \)-dimensional space, all lattice points on the hyper-plane defined by the equation \( (1) \) have identical Manhattan distance (measured from origin) which value is \( n \).

\[
\sum_{i=1}^{k} x_i = n \quad (1)
\]

where \( x_i; i = 1, \ldots, k \) is the \( i \)th coordinates value.

**Proof:** We know that Manhattan distance (from origin) of any \( k \)-dimensional point is

\[
\sum_{i=1}^{k} |x_i| = n \quad (2)
\]

Now for all lattice points on the hyper-plane, we have \( x_i \geq 0, i = 1,2,\ldots,k \). So for any lattice point on the hyper-plane we have

\[
\sum_{i=1}^{k} |x_i| = \sum_{i=1}^{k} x_i = n \quad [\text{from equation } 1]
\]
That is all lattice points on the hyper-plane (1) have identical Manhattan distance namely \( n \) measured from origin. (Proved)

**Lemma 2 (Lower bound of Euclidean):** If \( n, k \in \mathbb{N} \) then for any \( k \)-dimensional space, the point \( x \) on the hyper-plane defined by the equation (1) has Euclidean distance whose minimum bound is \( \frac{n}{\sqrt{k}} \) (measured from origin). If \( n/k \in \mathbb{N} \) then there exist only one lattice point \( x \) whose Euclidean distance \( n/\sqrt{k} \) (measured from origin) and its all coordinates value are identical namely \( n/k \). Again if \( n/k \notin \mathbb{N} \), then all neighbourhood lattice points of the point \( x \), on the hyper-plane that has minimum Euclidean distance, have minimum Euclidean distance which is greater than \( n/\sqrt{k} \). So \( n/\sqrt{k} \) is the lower bound of lattice point on the hyper-plane regarding Euclidean distance measured from origin.

**Proof:** We know Euclidean distance, \( d_2 \), of any \( k \)-dimensional point \( x (x_1, x_2, ..., x_k) \) measured from origin is given by

\[
d_2^2 = x_1^2 + x_2^2 + \cdots + x_k^2 \quad (3)
\]

If this point is on the given hyper-plane (1) then we have

\[
\sum_{i=1}^{k} x_i = n \quad (4)
\]

Now from equation (4) put the value of \( x_k \) on the distance equation (3), we have

\[
d_2^2 = x_1^2 + x_2^2 + \cdots + \left(n - \sum_{i=1}^{k-1} x_i\right)^2 \quad (5)
\]

It is noted that if \( d_2 \geq 0 \) is the maximum/minimum among some values, then \( d_2^2 \) also is maximum/minimum compared to each squared value of that values. Now by the theory of calculus for maximum/minimum i.e. for all \( i = 1, 2, \cdots, k \), we have

\[
\frac{\partial d_2^2}{\partial x_i} = 0 \quad (6)
\]

Solving the above system of linear equation (6) we can easily show that

\[
x_1 = x_2 = \cdots = x_k = n/k \quad (7)
\]

[Proved part 1]

Again if \( n/k \in \mathbb{N} \) then all \( x_i = n/k \in \mathbb{N} \ \forall \ i = 1, 2, \cdots, k \) so the point \( x \) is a lattice point which is off course minimum among all lattice points on the hyper-plane (1). So Euclidean distance of that point measured from origin (by the help of equation (7)) is

\[
d_2 = \sum_{i=1}^{k} \sqrt{(x_i - 0)^2} = \sum_{i=1}^{k} \sqrt{\left(\frac{n}{k}\right)^2} = k \sqrt{\left(\frac{n}{k}\right)^2} = \frac{n}{\sqrt{k}} \quad (8)
\]

[Proved part 2]

Now if \( n/k \notin \mathbb{N} \) then neighbourhood lattice point is \( x \in \left\{ \left[\frac{n}{k}\right], \left\lfloor \frac{n}{k}\right\rfloor \right\} \) where \( \left[\frac{n}{k}\right]\) is the nearest ceil integer \( \geq \frac{n}{k} \) (obviously \( \frac{n}{k} \geq 1 \)) and \( \left\lfloor \frac{n}{k}\right\rfloor \) is the nearest floor integer \( \leq \frac{n}{k} \) and also there are \( mod \left(\frac{n}{k}\right) \) numbers of coordinates whose values are \( \left[\frac{n}{k}\right]\) and the rest i.e.\( \left[k - Mod\left(\frac{n}{k}\right)\right] \) numbers of coordinate values are \( \left\lfloor \frac{n}{k}\right\rfloor \).
Now since neighbourhood point of $x$ is on the hyper-plane, so its Euclidean distance measured from origin must be $< \frac{n}{\sqrt{k}}$, as $\frac{n}{\sqrt{k}}$ is the minimum distance of any point on the hyper-place. It is noted that the neighbourhood point of $x$ lies on the hyper-plane as 

$$\sum_{m \text{ mod } \left(\frac{n}{k}\right)} \cdot x_p + \sum_{q \text{ mod } \left(\frac{n}{k}\right)} \cdot x_q = n;$$

where $x_p = \left\lfloor \frac{n}{k} \right\rfloor$ and $x_q = \left\lfloor \frac{n}{k} \right\rfloor$.

Hence $\frac{n}{\sqrt{k}}$ is the lower bound of Lattice point s on the hyper-plane regarding Euclidean distance measured from origin. [Proof is completed]

**Lemma 3: (Upper bound of Euclidean distance):** If $n, k \in N$ then for any $k$-dimensional space, the point $x$ on the hyper-plane defined by the equation (1) has Euclidean distance whose maximum value is $n$ (measured from origin). Any point which is on the intersection of coordinate axis and the hyper-plane has maximum Euclidean distance which value is $n$.

**Proof:** For the proof the Lemma we first consider two dimension case i.e. $k = 2$. Then the equation (1) becomes

$$x_1 + x_2 = n$$  \hspace{1cm} (8)

Now for any point $P(x_1, x_2)$, according to the triangular rules, we have

$$d_2(x_1, x_2) \leq x_1 + x_2$$

So the maximum value of $d_2(x_1, x_2)$ is

$$d_2(x_1, x_2) = x_1 + x_2 = n \quad \text{[from equation (8)]} \hspace{1cm} (9)$$

But it will be true when the lines joining origin $(0, 0)$ to $(x_1, x_2)$ lies on any coordinates axis whose Euclidean distance is either $x_1$ or $x_2$. On the other hand from equation (8) we have $x_1 + x_2 = n$. In consequence we have obviously either $x_1 = n$ or $x_2 = n$ i.e. the point with maximum Euclidean distance lie on any coordinate axis.

In this way we can easily prove that for any $k$-dimensional hyper-plane, the maximum Euclidean distance of any lattice point $x$ is $n$. i.e. $d_2(x) = n$ and it lies on the intersection of any coordinate axis and the hyper-plane. [Proved]

**Lemma 4 (Relation between Manhattan and Euclidean distance):** If $n, k \in N$ then for any $k$-dimensional space, for the lattice points on the hyper-plane $\sum_{i=1}^{k} x_i = n$, the relation between Manhattan and Euclidean distance is as follows:

$$\frac{n}{\sqrt{k}} \leq d_2(x) \leq n = d_1(x)$$

$\forall x \in \{(x_1, x_2, \cdots x_k): \sum_{i=1}^{k} x_i = n; x_i \in N \cup \{0\} \text{ and } n \in N\}$, where $x$ is the lattice point on the hyper-plane $\sum_{i=1}^{k} x_i = n$. Here $d_1$ and $d_2$ indicate Manhattan and Euclidean distances respectively.

**Proof:** Let $d_1(x)$ and $d_2(x)$ indicate Manhattan and Euclidean distance of a lattice point respectively. For any $n, k \in N$, we have from lemma 1,

$$n = d_1(x) \quad \forall x \in \{(x_i)^k: \sum_{i=1}^{k} x_i = n; x_i \in N \cup \{0\}; n \in N\}$$

On the other hand for any $n, k \in N$, we have from lemmas 2 and 3,

$$\frac{n}{\sqrt{k}} \leq d_2(x) \leq n \quad \forall x \in \{(x_i)^k: \sum_{i=1}^{k} x_i = n; x_i \in N \cup \{0\}; n \in N\}$$
Therefore from these two relations we can able to establish an approximate relation between Manhattan and Euclidean distance for a lattice point \( x \) as follows:

\[
\frac{n}{\sqrt{k}} \leq d_2(x) \leq n = d_1(x)
\]

\( \forall x \in \{(x_i)^k: \sum_{i=1}^{k} x_i = n; x_i \in N \cup \{0\}; n \in N\} \).

### 3. Mapping of the Lemmas on LHD

The above proposed Lemmas are related to Euclidean and Manhattan distances of two points one of which is in origin. For a LHD, inter-site distance between two points are measured by some distance measure like Euclidean or Manhattan distances measure. But, in general, all but except one design points locate on the place other than origin. Specially one of the pair-wise points whose inter-site distance is minimum stay on the origin is rare. How can we implement the proposed lemmas in LHDs?

It is known that if we shift the origin in any arbitrary points then distance (in any measure for Euclidean space) between two points remain unchanged. So, among the all pair-wise distances of design points in a LHD, we must find out the points whose pair-wise distance is/are minimum. Then we will able to shift the origin to one of the points containing minimum distance. Therefore we can easily implements the proposed lemmas in LHD as well as in maximin LHDs. Moreover since distance is invariant due to shifting of origin so above lemmas hold in a LHD of two design points whose pair-wise distance is minimum without shifting origin too.

From the above lemmas we may conclude that, for any two LHDs, if the value of Euclidean distance is greater than the value of Manhattan distance, then LHD whose distance is measured by Euclidean distance measure is sparser rather than the LHD whose distance is measured by Manhattan distance measure. Alternatively for any two LHDs, if the value of Euclidean distance is smaller than the value of \( \frac{1}{\sqrt{k}} \) time Manhattan distance, then LHD whose distance is measured by Manhattan distance measure is sparser rather than the LHD whose distance is measured by Euclidean distance measure.

### 4. Some Elementary Experimental Study

As the main intention of the optimal criterion ‘maximin’ is to spread the design points over the whole input domain, so if nearest design points are extended as much as possible on the bounded input domain then eventually points will be covered over the input domain almost uniformly [1]. Therefore we may able to say one maximin LHD regarding Euclidean distance is more space-filling than other maximin LHD regarding Manhattan distance when the value of Euclidean distance is of outside of lower and upper bound given by lemma 4. But we could not compare about space-filling when the value of Euclidean distance is inside of the lower and upper bound given by lemma 4. In such situation we may compare the maximin LHDs optimized regarding Euclidean distance measure and Manhattan distance measure respectively by using proposed mathematical transform which is discussed below.

For the comparison of two different measures regarding maximin LHD about space-filling, we first need to develop some transform techniques valid in this regard. Let \( D_{i(E)} \) is the minimum pair-wise inter-site distance among all design points of a LHD regarding some distance measure say \( S \in \{E, M\} \) (here E for Euclidean and M for Manhattan). Therefore \( D_{i(E)} \) denotes minimum inter-site distance regarding Euclidean distance \( (L^2) \) measure and \( D_{i(M)} \) indicates minimum inter-site distance regarding Manhattan distance \( (L^1) \) measures. Now we need to find out relative minimum pair-wise inter-site distance of each LHD measured by Manhattan and Euclidian distance respectively. Let us consider any maximin LHD \( X \), which is optimized regarding Euclidean distance measure and other maximin LHD \( Y \) which is optimized regarding Manhattan distance measure. Then the proposed relative distances are calculated as follow:

Relative value of \( D_{i(E)} \) of a LHD \( X \) = \{difference of \( D_{i(E)} \) value of LHD \( X \) and LHD \( Y \)\}/ \( D_{i(E)} \) value of LHD \( X \); and Relative value of \( D_{i(M)} \) of a LHD \( Y \) = \{difference of \( D_{i(M)} \) value of LHD \( X \) and LHD \( Y \)\}/ \( D_{i(M)} \) value of LHD \( Y \). If we express them in notation with percentage, then we have:
(a) $R_{D_{1}(E)}(X) = \{D_{1}(E)(X) - D_{1}(E)(Y)\}/ D_{1}(E)(X) \times 100.$

(b) $R_{D_{1}(M)}(Y) = \{D_{1}(M)(X) - D_{1}(M)(Y)\}/ D_{1}(M)(Y) \times 100.$

Now several experiments will be carried out to find the relative value of $D_{1}(E)$ and $D_{1}(M)$ for both maximin LHDs regarding both $L^2$ and $L^1$ measures respectively. At first we have considered a typical example $(N, k) = (5, 3)$ to demonstrate the effectiveness of the proposed transformations. The maximin LHDs regarding $L^1$ and $L^2$ measures respectively are shown in Table 1. In the table 1 MLH-SA($L^1$) [5] and MLS-ILS($L^2$) [2] denote maximin LHD where first one is measured by $L^1$ and second one is measured by $L^2$ distance measures. Now by using the above formulas we have obtained the values of $R_{D_{1}(E)}(X)$ and $R_{D_{1}(M)}(Y)$ as follows:

(i) $R_{D_{1}(E)}(X) = \{D_{1}(E)(X) - D_{1}(E)(Y)\}/ D_{1}(E)(X) \times 100 = \{11-9\}/11 \times 100 = 18.2.$

(ii) $R_{D_{1}(M)}(Y) = \{D_{1}(M)(X) - D_{1}(M)(Y)\}/ D_{1}(M)(Y) \times 100 = \{5-5\}/5 \times 100 = 0.$

### Table 1. Comparison among optimal LHDs for $(N, k) = (5,3)$.

| LHD       | MLH-SA($L^1$) | MLH –ILS ($L^2$) |
|-----------|---------------|-------------------|
| Optimal design matrix | Y | X |
| 1 1 2 | 1 3 5 | |
| 2 5 3 | 2 2 2 | |
| 3 2 5 | 3 5 1 | |
| 4 3 1 | 4 4 4 | |
| 5 4 4 | 5 1 3 | |
| Distance Measure | $L^1$ | $L^2$ |
| $D_1$ in $L^1$ | 5 | 5 |
| $D_1$ in $L^2$ | 9 | 11 |

It is observed that maximin LHD measured by $L^2$ during optimization process is better space-filling than the maximin LHD measured by $L^1$ during optimization process.

### Table 2. Comparison between Euclidean distance measure and Manhattan distance measure regarding Maximin LHDs optimized $L^2$ and $L^1$ measures.

| N | $k = 3$ | $k = 7$ | $k = 9$ |
|---|---|---|---|
| 4 | 0 | 5.131497 | 0 |
| 5 | 0 | 8.866211 | 7.5 |
| 6 | 0 | 12.70627 | 12.96296 |
| 7 | 19.02381 | 5.220856 | 11.11111 |
| 8 | 0 | 10.43153 | -18.1818 |
| 9 | 0 | -10.78234 | 0 |
| 10 | 16.77484 | 16.77484 | 0 |
| 11 | 12.5 | 12.5 | 0 |
| 12 | 6.718737 | -10 | -10 |
| 13 | 11.14379 | -10 | -10 |
| 14 | 8.200356 | -9.09091 | -9.09091 |
| 15 | 10.43153 | -18.1818 | -18.1818 |
| Total | $98.54545$ | $31.92483$ | $39.26638$ |
| Avr. | $7.580419$ | $5.4526$ | $9.816595$ |
For further numerical instances we have searched the well-known web portal https://spacefillingdesigns.nl/. All the instances considered here i.e. all the maximin LHDs optimized based on $L_1$ measure. In the experimental study, we have first find out the minimum inter-site distance $D_{1(E)}$ and $D_{1(M)}$ for each LHD $X$ which optimized regarding $L_2$ measures. Similarly we find out the minimum inter-site distance $D_{1(E)}$ and $D_{1(M)}$ for each LHD $Y$ which optimized regarding $L_1$ measures. Now using the above relations (a) and (b), we have find out the relative values $R_{D_{1(E)}(X)}$ and $R_{D_{1(M)}(Y)}$ for each LHD $Z \in \{X, Y\}$. The comparison results are displayed in the table 2. Moreover the average value of all $R_{D_{1(E)}(X)}$ and the average value of all $R_{D_{1(M)}(Y)}$ for each dimension $k$ are also find out which are shown in the last row of the table 2. It is observed in the table 2 that except few LHDs the (absolute) $R_{D_{1(E)}(X)}$ values are better than or equal to that of $R_{D_{1(M)}(Y)}$. Moreover the (absolute) average values of $R_{D_{1(E)}(X)}$ are always larger than that of $R_{D_{1(M)}(Y)}$ for each $k$ value namely dimension 3, 7 and 9. Therefore it may be concluded from this elementary experimental study that the maximin LHDs optimized regarding $L_2$ measure are more space-filling rather than maximin LHDs optimized regarding $L_1$ measure.

5. Conclusion
Though both Manhattan and Euclidean distance measures are popular in DoEs but which distance is better, regarding space-filling especially when we consider optimal DoEs, is still under research. In LHD, since the factors are mapping to integer number, so it is relatively easy to establish relation between these two measures. In this article some lemmas are proposed regarding Manhattan and Euclidean distance measure which are valid for integer system like LHDs. The mappings of the proposed lemmas on LHDs are also discussed briefly. Though these lemmas do not give directly about the goodness of space-filling between these two measures, but we hope it will be helpful to researchers for further extensions. Moreover some transformation techniques for the comparison between Manhattan and Euclidean distance measures in perspective of experimental results are developed. Finally we have carried out some experiments using the proposed transformation techniques on maximin LHDs optimized both Manhattan and Euclidean distance measures. In the experimental study, it is appeared that the use of Euclidean distance measure is relatively better for maximin LHDs regarding space-filling DoEs.

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