Solvability of an inhomogeneous boundary value problem for steady MHD equations

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In this paper, we consider the steady MHD equations with inhomogeneous boundary conditions for the velocity and the tangential component of the magnetic field. Using a new construction of the magnetic lifting, we obtain existence of weak solutions under sharp assumption on boundary data for the magnetic field.

\textbf{KEYWORDS}
inhomogeneous, MHD equations, solvability

\textbf{MSC CLASSIFICATION}
35J60; 35Q35; 35Q60

\section{INTRODUCTION}

Let $\Omega$ be a bounded domain in $\mathbb{R}^3$ with $C^{1,1}$ boundary $\partial \Omega$. The boundary $\partial \Omega$ has a finite number of connected components $\Gamma_0, \Gamma_1, \ldots, \Gamma_m$, where $\Gamma_0$ denotes the boundary of the infinite connected component of $\mathbb{R}^3 \setminus \Omega$. The domain $\Omega$ can be multiply connected. Assume that there are disjoint $C^2$ cuts $\Sigma_1, \ldots, \Sigma_N$ such that $\Omega \setminus \bigcup_{i=1}^{N} \Sigma_i$ is simply connected and Lipschitz. We denote by $\mathbf{n}$ the unit outer normal on $\partial \Omega$. We consider the following steady MHD equations:

\begin{align}
-\nu \Delta \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p - \mathbf{rot} \mathbf{H} \times \mathbf{H} = \mathbf{f}, \quad &\text{div} \mathbf{u} = 0 \quad \text{in } \Omega, \\
\nu_1 \mathbf{rot} \mathbf{H} - \mathbf{E} + \alpha \mathbf{H} \times \mathbf{u} = \mathbf{v} \mathbf{j}, \quad &\text{div} \mathbf{H} = 0, \quad \text{rot} \mathbf{E} = 0 \quad \text{in } \Omega,
\end{align}

with inhomogeneous boundary conditions

\begin{align}
\mathbf{u} = \mathbf{g}, \quad \mathbf{H} \times \mathbf{n} = \mathbf{q} \quad \text{on } \partial \Omega.
\end{align}

Here, $\mathbf{u}$ is the velocity vector, $\mathbf{H}$ is the magnetic field, $\mathbf{E} = \mathbf{E}'/\rho_0$, $p = P/\rho_0$, where $\mathbf{E}'$ is the electric field, $P$ is the pressure, and $\rho_0 = \text{const} > 0$ is the fluid density. Moreover, $\nu$ is the kinematic viscosity, $\alpha = \mu/\rho_0$, $\nu_1 = 1/(\rho_0 \sigma)$, where $\mu, \sigma$ are the magnetic permeability and the electric conductivity. Besides, $\mathbf{f}$ and $\mathbf{j}$ are given functions defined on $\Omega$, and $\mathbf{g}$ and $\mathbf{q}$ are given functions defined on $\partial \Omega$.

Beginning from the pioneering works\textsuperscript{1,2}, the solvability of boundary value problems for the MHD equations has been studied in many works; see, for example, previous studies\textsuperscript{3-7} and the references therein. Precisely, Solonnikov\textsuperscript{2} proved the

\textit{... end of the abstract...}
Very recently, Alekseev and Brizitskii generalized the global solvability result for adopted a magnetic lifting satisfying the conditions that the boundary data solvability can be similarly proved as (1.1)–(1.3). In 2016, Alekseev obtained the global solvability of (1.1)–(1.2) under the assumption that the boundary data is tangential to the boundary and satisfying (1.4), where \( g \in H^{1/2}(\partial \Omega, \mathbb{R}^3) \), \( q \in L^2(\partial \Omega, \mathbb{R}^3) \) are tangential to the boundary and \( q \) satisfies

\[
\text{div}_{\partial \Omega} q = 0, \quad \int_{\partial \Omega} q \cdot h \, dS = 0, \quad \forall \ h \in \mathbb{H}_N(\Omega).
\]

Here, \( \text{div}_{\partial \Omega} \) is the operator of surface divergence, \( \mathbb{H}_N(\Omega) \) represents the harmonic Neumann fields, that is,

\[
\mathbb{H}_N(\Omega) = \{ u \in L^2(\Omega, \mathbb{R}^3) : \text{rot} u = 0, \ \text{div} u = 0 \ \text{in} \ \Omega, \ u \cdot n = 0 \ \text{on} \ \partial \Omega \}.
\]

Very recently, Alekseev and Brizitskii generalized the global solvability result for \( g = 0 \) and \( q \in H^s(\partial \Omega, \mathbb{R}^3) \) being tangential to the boundary and satisfying (1.4), where \( s \in [0, 1/2] \) is arbitrary. In fact, the condition \( g = 0 \) can be replaced by the general condition that \( g \in H^{1/2}(\partial \Omega, \mathbb{R}^3) \) is tangential to the boundary.

Let \( H_0 \) be a magnetic lifting for the magnetic boundary data \( q \), that is, \( H_0 \) is an extension for \( q \) satisfying

\[
\text{div} H_0 = 0 \ \text{in} \ \Omega, \quad H_0 \times n = q \ \text{on} \ \partial \Omega.
\]

To overcome the difficulty caused by the term \( \times \text{rot} H \times \) in the homogenized system (2.2), where \( \mathbf{H} = H - H_0 \), Alekseev adopted a magnetic lifting satisfying the \( \text{div} - \text{rot} \) system:

\[
\text{rot} H_0 = 0, \quad \text{div} H_0 = 0 \ \text{in} \ \Omega, \quad H_0 \times n = q \ \text{on} \ \partial \Omega.
\]

Hence, the extra condition (1.4) is necessary to guarantee the above system admitting a solution. In that situation, indeed, the term \( \times \text{rot} H_0 \times \mathbf{H} \) disappears, but the cost is that more unnatural restrictions are imposed on \( q \).

In this paper, using the construction of the hydrodynamic lifting suggested in Alekseev and a new construction of the magnetic lifting (see Lemma 2.2), we show the global solvability of problem (1.1)–(1.2) \textit{without the condition} (1.4) by applying Schauder’s fixed point theorem. We use integration by parts to deal with the term \( \times \text{rot} H_0 \times \mathbf{H} \); see the key inequality (2.7).

Throughout this paper, we use \( L^2(\Omega) \), \( H^1(\Omega) \), \( H^{-1}(\Omega) \), and \( H^s(\partial \Omega) \) to denote the usual Lebesgue spaces and Sobolev spaces for scalar functions and \( L^2(\Omega, \mathbb{R}^3) \), \( H^1(\Omega, \mathbb{R}^3) \), \( H^{-1}(\Omega, \mathbb{R}^3) \), and \( H^s(\partial \Omega, \mathbb{R}^3) \) to denote the corresponding spaces of vector fields. However, we use the same notation to denote the norm of both scalar functions and vector fields. For instance, we write \( \| \phi \|_{L^2(\Omega)} \) for \( \phi \in L^2(\Omega) \) and also \( \| u \|_{L^2(\Omega)} \) for \( u \in L^2(\Omega, \mathbb{R}^3) \). We also use the following notations:

\[
H_0^1(\text{div}0, \Omega) = \{ u \in H_0^1(\Omega, \mathbb{R}^3) : \text{div} u = 0 \ \text{in} \ \Omega \},
\]

\[
H_0^1(\Omega, \mathbb{R}^3) = \{ u \in H^1(\Omega, \mathbb{R}^3) : u \times n = 0 \ \text{on} \ \partial \Omega \},
\]

\[
H_0^1(\text{div}0, \Omega) = \{ u \in H^1(\Omega, \mathbb{R}^3) : \text{div} u = 0 \ \text{in} \ \Omega, u \times n = 0 \ \text{on} \ \partial \Omega \},
\]

\[
H_0^{1/2}(\partial \Omega, \mathbb{R}^3) = \{ u \in H^{1/2}(\partial \Omega, \mathbb{R}^3) : u \cdot n = 0 \ \text{on} \ \partial \Omega \}.
\]

The harmonic Dirichlet fields \( \mathbb{H}_D(\Omega) \) is defined by

\[
\mathbb{H}_D(\Omega) = \{ u \in L^2(\Omega, \mathbb{R}^3) : \text{rot} u = 0, \ \text{div} u = 0 \ \text{in} \ \Omega, u \times n = 0 \ \text{on} \ \partial \Omega \}.
\]
Now, our main result reads as follows.

**Theorem 1.1.** Let \( v, v_I \) and \( x \) be three positive constants. Assume \( f \in H^{-1}(\Omega, \mathbb{R}^3), j \in L^2(\Omega, \mathbb{R}^3), \) and \( g, q \in H^{1/2}_T(\partial\Omega, \mathbb{R}^3). \) Then, \((1.1) - (1.2)\) admits a weak solution:

\[
(u, p, E, H) \in H^1(\Omega, \mathbb{R}^3) \times L^2(\Omega) \times L^2(\Omega, \mathbb{R}^3) \times H^1(\Omega, \mathbb{R}^3).
\]

**Remark 1.2.** Because the magnetic field satisfies the boundary condition \( H \times n = q \) on \( \partial\Omega, \) \( q \) must satisfy the compatible condition \( q \cdot n = 0 \) on \( \partial\Omega. \) So the assumption we propose on the magnetic boundary data is sharp.

## 2 \ PROOF OF THEOREM 1.1

In order to prove the global solvability result, we introduce two trace lifting lemmas.

**Lemma 2.1 (Alekseev\(^3, \) lemma 2.2).** For every vector-valued function \( g \in H^{1/2}_T(\partial\Omega, \mathbb{R}^3) \) and every number \( \varepsilon > 0, \) there exists a vector-valued function \( u_\varepsilon \in H^1(\Omega, \mathbb{R}^3) \) such that

\[
\text{div } u_\varepsilon = 0 \text{ in } \Omega, \quad u_\varepsilon = g \text{ on } \partial\Omega,
\]

\[
\|u_\varepsilon\|_{L^3(\Omega)} \leq \varepsilon \|g\|_{H^{1/2}(\partial\Omega)},
\]

\[
\|u_\varepsilon\|_{H^1(\Omega)} \leq C_\varepsilon \|g\|_{H^{1/2}(\partial\Omega)},
\]

where the constant \( C_\varepsilon \) depends on \( \varepsilon \) and \( \Omega. \)

**Lemma 2.2.** For every vector-valued function \( q \in H^{1/2}_T(\partial\Omega, \mathbb{R}^3) \) and every number \( \varepsilon > 0, \) there exists a vector-valued function \( H_\varepsilon \in H^1(\Omega, \mathbb{R}^3) \) such that

\[
\text{div } H_\varepsilon = 0 \text{ in } \Omega, \quad H_\varepsilon \times n = q \text{ on } \partial\Omega,
\]

\[
\|H_\varepsilon\|_{L^3(\Omega)} \leq \varepsilon \|q\|_{H^{1/2}(\partial\Omega)},
\]

\[
\|H_\varepsilon\|_{H^1(\Omega)} \leq C_\varepsilon \|q\|_{H^{1/2}(\partial\Omega)},
\]

where the constant \( C_\varepsilon \) depends on \( \varepsilon \) and \( \Omega. \)

**Proof.** Let \( \varepsilon \) be an arbitrary positive number. Because \( n \times q \in H^{1/2}_T(\partial\Omega, \mathbb{R}^3), \) then by Lemma 2.1, there exists a vector-valued function \( u_\varepsilon \in H^1(\Omega, \mathbb{R}^3) \) such that

\[
\text{div } u_\varepsilon = 0 \text{ in } \Omega, \quad u_\varepsilon = n \times q \text{ on } \partial\Omega,
\]

\[
\|u_\varepsilon\|_{L^3(\Omega)} \leq \varepsilon \|n \times q\|_{H^{1/2}(\partial\Omega)},
\]

\[
\|u_\varepsilon\|_{H^1(\Omega)} \leq C_\varepsilon \|n \times q\|_{H^{1/2}(\partial\Omega)},
\]

where the constant \( C_\varepsilon \) depends on \( \varepsilon \) and \( \Omega. \) Because

\[
\|n \times q\|_{H^{1/2}(\partial\Omega)} \leq C(\Omega) \|q\|_{H^{1/2}(\partial\Omega)},
\]

we obtain

\[
\|u_\varepsilon\|_{L^3(\Omega)} \leq C(\Omega) \varepsilon \|q\|_{H^{1/2}(\partial\Omega)}.
\]

Set \( H_\varepsilon = u_\varepsilon / C(\Omega). \) Then, we have

\[
H_\varepsilon \times n = (n \times q) \times n = q \text{ on } \partial\Omega, \quad \|H_\varepsilon\|_{L^3(\Omega)} \leq \varepsilon \|q\|_{H^{1/2}(\partial\Omega)}.
\]

We also need the following \textit{div–rot} inequality, which is a consequence of Dautray and Lions\(^9, \) p.209, theorem 3; p.213, remark 2; see also Alekseev\(^5, \) lemma 2.1.
Lemma 2.3. For any $B \in H^1_0(\Omega, \mathbb{R}^3) \cap \mathbb{H}^1_D(\Omega)$, it holds that
\[
\|B\|_{H^1(\Omega)} \leq C(\Omega)(\|\text{div}B\|_{L^2(\Omega)} + \|\text{rot}B\|_{L^2(\Omega)}).
\]

Now, we are in a position to prove the global solvability result.

Proof of Theorem 1.1. By eliminating $E$, (1.1)–(1.2) turns into the following form:
\[
\begin{cases}
-v\Delta u + (u \cdot \nabla)u + \nabla p - \mathbf{x} \cdot \text{rot} \mathbf{H} \times \mathbf{H} = f & \text{in } \Omega, \\
\text{rot}(v_1 \text{rot} \mathbf{H} + \mathbf{x} \mathbf{H} \times u - v_1j) = 0 & \text{in } \Omega, \\
\text{div } u = \text{div } \mathbf{H} = 0 & \text{in } \Omega, \\
u = g, \quad \mathbf{H} \times \mathbf{n} = q & \text{on } \partial \Omega.
\end{cases}
\tag{2.1}
\]

Let $u_0 = u_{00}$ be the lifting for the boundary data $g$ in Lemma 2.1, and $H_0 = H_{00}$ be the lifting for the boundary data $q$ in Lemma 2.2. Here, the constant $\epsilon_0$ is to be determined. Introducing new unknown variables $\tilde{u} = u - u_0$ and $\tilde{H} = H - H_0$, we can reduce (2.1) to the following homogeneous boundary value problem:
\[
\begin{cases}
-v\Delta \tilde{u} + (\tilde{u} \cdot \nabla)\tilde{u} + (u_0 \cdot \nabla)u_0 + (u_0 \cdot \nabla)\tilde{u} + \nabla p - \mathbf{x} \text{rot} \tilde{H} \times \mathbf{H} \\
\text{rot}(v_1 \text{rot} \tilde{H} + \mathbf{x} \mathbf{H} \times \tilde{u} + \mathbf{x} \mathbf{H} \times u_0 + \mathbf{x} H_0 \times \tilde{u} - J) = 0 \\
\text{div } \tilde{u} = \text{div } \tilde{H} = 0 \\
\tilde{u} = 0, \quad \tilde{H} \times \mathbf{n} = 0
\end{cases}
\tag{2.2}
\]

where the functions $F$ and $J$ are given by
\[
F = f + v\Delta u_0 - (u_0 \cdot \nabla)u_0 + \mathbf{x} \text{rot} H_0 \times H_0, \\
J = v_1 j - v_1 \text{rot} H_0 - \mathbf{x} H_0 \times u_0.
\tag{2.3}
\]

We claim that if we get a weak solution $(\tilde{u}, p, \tilde{H})$ of (2.2), then (1.1)–(1.2) admits a weak solution. In fact, set $u = u_0 + \tilde{u}$ and $H = H_0 + \tilde{H}$, then $(u, p, H)$ solves (2.1). Furthermore, $(u, p, E, H)$ solves (1.1)–(1.2), where $E = v_1 \text{rot} H + \mathbf{x} H \times u - v_1 j$. So we only need to prove the existence of a weak solution of (2.2). In the sequel, we do this by four steps.

Step 1. For any given $(w, D) \in H^1_0(\text{div}0, \Omega) \times [H^1_0(\text{div}0, \Omega) \cap \mathbb{H}^1_D(\Omega)]$, we prove the existence of a unique solution of the following system:
\[
\begin{cases}
-v\Delta \tilde{u} + (w \cdot \nabla)\tilde{u} + (u_0 \cdot \nabla)u_0 + (u_0 \cdot \nabla)\tilde{u} + \nabla p - \mathbf{x} \text{rot} \tilde{H} \times D \\
\text{rot}(v_1 \text{rot} \tilde{H} + \mathbf{x} D \times \tilde{u} + \mathbf{x} \mathbf{H} \times u_0 + \mathbf{x} H_0 \times \tilde{u} - J) = 0 \\
\text{div } \tilde{u} = \text{div } \tilde{H} = 0 \\
\tilde{u} = 0, \quad \tilde{H} \times \mathbf{n} = 0
\end{cases}
\tag{2.4}
\]

where the functions $F$ and $J$ are defined by (2.3).
We define a bilinear functional:

\[
a((\tilde{u}, \tilde{H}), (v, B)) = \int_{\Omega} \nabla \tilde{u} : \nabla v \, dx + \int_{\Omega} [(w \cdot \nabla)\tilde{u} + (u_0 \cdot \nabla)\tilde{u} - x \text{rot}\tilde{H} \times D - x \text{rot}\tilde{H} \times H_0 - x \text{rot}H_0 \times \tilde{H}] \cdot v \, dx + \int_{\Omega} (v_1 \text{rot}\tilde{H} + x D \times \tilde{u} + x \tilde{H} \times u_0 + x H_0 \times \tilde{u}) \cdot \text{rot}B \, dx.
\]

Then, (2.4) is equivalent to the formulation:

\[
a((\tilde{u}, \tilde{H}), (v, B)) = \langle F, v \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} + \int_{\Omega} J \cdot \text{rot}B \, dx.
\]

On account that

\[
\int_{\Omega} (w \cdot \nabla)\tilde{u} \cdot \tilde{u} \, dx = \int_{\Omega} (u_0 \cdot \nabla)\tilde{u} \cdot \tilde{u} \, dx = 0,
\]

\[
\int_{\Omega} (\tilde{u} \cdot \nabla)u_0 \cdot \tilde{u} \, dx = -\int_{\Omega} (\tilde{u} \cdot \nabla)u_0 \, dx,
\]

\[
\int_{\Omega} x \text{rot}\tilde{H} \times D \cdot \tilde{u} \, dx = \int_{\Omega} x D \times \tilde{u} \cdot \text{rot}\tilde{H} \, dx,
\]

\[
\int_{\Omega} x \text{rot}\tilde{H} \times H_0 \cdot \tilde{u} \, dx = \int_{\Omega} x H_0 \times \tilde{u} \cdot \text{rot}\tilde{H} \, dx,
\]

we have that

\[
a((\tilde{u}, \tilde{H}), (\tilde{u}, \tilde{H})) = \int_{\Omega} (\nu |\nabla \tilde{u}|^2 - (\tilde{u} \cdot \nabla)\tilde{u} + u_0 - x \text{rot}H_0 \times \tilde{H} \cdot \tilde{u}) \, dx + \int_{\Omega} (v_1 |\text{rot}\tilde{H}|^2 + x \tilde{H} \times u_0 \cdot \text{rot}\tilde{H}) \, dx.
\]

Next, we claim that the functional \(a\) is coercive. Let us estimate term by term. By Hölder's inequality and Lemma 2.1, we get

\[
\left| \int_{\Omega} (\tilde{u} \cdot \nabla)u_0 \cdot \tilde{u} \, dx \right| \leq |||\tilde{u}|||_{L^2(\Omega)} |||\tilde{u}|||_{L^2(\Omega)} |||u_0|||_{L^2(\Omega)} \leq C_1 \varepsilon_0 |||g|||_{H^{1/2}(\Omega)} |||\nabla \tilde{u}|||_{L^2(\Omega)}^2.
\]

By Hölder's inequality, Lemmas 2.1 and 2.3, it holds that

\[
\left| \int_{\Omega} x \tilde{H} \times u_0 \cdot \text{rot}\tilde{H} \, dx \right| \leq x |||\tilde{H}|||_{L^2(\Omega)} |||u_0|||_{L^2(\Omega)} |||\text{rot}\tilde{H}|||_{L^2(\Omega)} \leq C_2 \varepsilon_0 |||g|||_{H^{1/2}(\Omega)} |||\text{rot}\tilde{H}|||_{L^2(\Omega)}^2,
\]

where Lemma 2.3 is used in the last inequality. Note that

\[
\text{rot}(A \times B) = (\text{div}B)A - (\text{div}A)B + (B \cdot \nabla)A - (A \cdot \nabla)B.
\]
Integrating by parts and using Hölder’s inequality, we have that

\[
\left| \int_\Omega \varepsilon_0 \nabla \tilde{H} \cdot \varepsilon_0 \right| = \left| \int_\Omega \nabla \tilde{H} \cdot \nabla v \right| 
\]

(2.7)

where we have used \( \nabla \tilde{H} = 0 \) on \( \partial \Omega \) in the third equality, \( \nabla \tilde{H} = 0 \) in the fifth equality, and Lemmas 2.2 and 2.3 in the last inequality. Combining inequalities (2.3), (2.6), and (2.7), we conclude that

\[
a((\tilde{u}, \tilde{H}), (\tilde{u}, \tilde{H})) \geq \eta \| \nabla \tilde{u} \|_{L^2(\Omega)}^2 + \nu_1 \| \nabla \tilde{H} \|_{L^2(\Omega)}^2
\]

\[
- (C_1 + C_2 \nu + C_3 \nu_1) \varepsilon_0 \left( \| \tilde{u} \|_{H^{1/2}(\Omega)}^2 + \| \tilde{H} \|_{H^{1/2}(\Omega)}^2 \right) \left( \| \nabla \tilde{u} \|_{L^2(\Omega)}^2 + \| \nabla \tilde{H} \|_{L^2(\Omega)}^2 \right).
\]

So the bilinear functional \( a \) is obviously coercive if \( \varepsilon_0 \) is selected small enough. The claim is then proved.

By Lax–Milgram theorem and De Rham’s theorem, we obtain the existence of unique weak solution \((\tilde{u}, p, \tilde{H})\) to (2.4) in \( H^1_0(\text{div}0, \Omega) \times L^2(\Omega) \times [H^1_0(\text{div}0, \Omega) \cap H^2_0(\Omega)] \), with

\[
\| \nabla \tilde{u} \|_{L^2(\Omega)} + \| \nabla \tilde{H} \|_{L^2(\Omega)} \leq C(\Omega, \nu, \nu_1, \nu_2, \varepsilon_0, g, q) \left( \| F \|_{H^{-1/2}(\Omega)} + \| J \|_{L^2(\Omega)} \right).
\]

In addition, it is easy to check that

\[
\| F \|_{H^{-1/2}(\Omega)} \leq C(\Omega, \varepsilon_0) \left( \| f \|_{H^{-1/2}(\Omega)} + \nu \| g \|_{H^{1/2}(\Omega)} + \| q \|_{H^{1/2}(\Omega)}^2 + \nu_1 \| q \|_{L^2(\Omega)}^2 \right),
\]

and

\[
\| J \|_{L^2(\Omega)} \leq C(\Omega, \varepsilon_0) \left( \nu_1 \| j \|_{L^2(\Omega)} + \nu_1 \| q \|_{H^{1/2}(\Omega)} + \nu \| g \|_{H^{1/2}(\Omega)} \| q \|_{H^{1/2}(\Omega)} \right).
\]

Therefore, we get that

\[
\| \tilde{u} \|_{H^1(\Omega)} + \| \tilde{H} \|_{H^1(\Omega)} \leq C \left( \| f \|_{H^{-1/2}(\Omega)} + \| j \|_{L^2(\Omega)} + 1 + \| g \|_{H^{1/2}(\Omega)}^2 + \| q \|_{H^{1/2}(\Omega)}^2 \right); \tag{2.8}
\]

where \( C = C(\Omega, \nu, \nu_1, \nu_2, \varepsilon_0, g, q) \).

Step 2. For any given \((w, D)\) above, define an operator \( T \) by \( T(w, D) = (\tilde{u}, \tilde{H}) \). Let \( K \) be the right hand side of (2.8). We define

\[
D = \{ (w, D) \in H^1_0(\text{div}0, \Omega) \times [H^1_0(\text{div}0, \Omega) \cap H^2_0(\Omega)] : \| w \|_{H^1(\Omega)} + \| D \|_{H^1(\Omega)} \leq K \}.
\]

Obviously, \( D \) is a bounded, closed and convex subset of \( H^1_0(\text{div}0, \Omega) \times [H^1_0(\text{div}0, \Omega) \cap H^2_0(\Omega)] \). Moreover, \( T \) maps \( D \) into itself.

Step 3. We show that \( T \) is continuous and compact from \( D \) into \( D \). First, we prove that \( T \) is continuous. Assume that \((w_k, D_k) \to (w, D) \) in \( H^1(\Omega, \mathbb{R}^3) \) as \( k \to \infty \).

\[
w_k \to w, D_k \to D \text{ in } H^1(\Omega, \mathbb{R}^3) \text{ as } k \to \infty. \tag{2.9}
\]
Let \( (\tilde{u}, p, \tilde{H}) \) be the weak solution of (2.4) and \((\tilde{u}_k, p_k, \tilde{H}_k)\) be the weak solution of (2.4) with \((w, D)\) replaced by \((w_k, D_k)\). If we set
\[
 v_k = \tilde{u}_k - \tilde{u}, \quad \pi_k = p_k - p, \quad B_k = \tilde{H}_k - \tilde{H},
\]
it is easy to check that \((v_k, \pi_k, B_k)\) satisfies the following system:
\[
\begin{align*}
 -\nu \Delta v_k + (w_k \cdot \nabla) \tilde{u}_k - (w \cdot \nabla) \tilde{u} + (v_k \cdot \nabla) u_0 + (u_0 \cdot \nabla) v_k + \nabla \pi_k & \quad \text{in } \Omega, \\
\text{rot}(v_1 \text{rot} B_k + x \text{rot} \tilde{H} \times D) - \text{rot} B_k \times H_0 - x \text{rot} H_0 \times B_k & = 0 \quad \text{in } \Omega, \\
\text{div} v_k & = \text{div} B_k = 0 \quad \text{in } \Omega, \\
v_k = 0, \quad B_k \times n = 0 & \quad \text{on } \partial \Omega.
\end{align*}
\]
Taking \((v_k, B_k)\) as a test function for the above system, and noting that
\[
(w_k \cdot \nabla) \tilde{u}_k - (w \cdot \nabla) \tilde{u} = (w_k \cdot \nabla) v_k + [(w_k - w) \cdot \nabla] \tilde{u},
\]
we obtain that
\[
\int_{\Omega} [v \nabla v_k]^2 + v_1 |\text{rot} B_k|^2 + (v_k \cdot \nabla) u_0 \cdot v_k - x \text{rot} B_k \times H_0 \cdot v_k] \, dx + \\
\int_{\Omega} [-x \text{rot} H_0 \times B_k \cdot v_k + x B_k \cdot u_0 \cdot \text{rot} B_k + x H_0 \times v_k \cdot \text{rot} B_k] \, dx = \\
\int_{\Omega} [x \text{rot} \tilde{H} \times (D_k - D) \cdot v_k - [(w_k - w) \cdot \nabla] \tilde{u} \cdot v_k - x(D_k - D) \times \tilde{u} \cdot \text{rot} B_k] \, dx.
\]
By a similar procedure as in Step 1 and using Hölder's inequality, we can get that
\[
C(\Omega, \nu, v_1, x, \varepsilon_0, g, q) \left( \|\nabla v_k\|_{L^2(\Omega)}^2 + \|\text{rot} B_k\|_{L^2(\Omega)}^2 \right) \\
\leq x \|\text{rot} \tilde{H}\|_{L^1(\Omega)} \|D_k - D\|_{L^1(\Omega)} \|v_k\|_{L^1(\Omega)} + \|w_k - w\|_{L^1(\Omega)} \|\nabla \tilde{u}\|_{L^1(\Omega)} \|v_k\|_{L^1(\Omega)} + \|D_k - D\|_{L^1(\Omega)} \|\tilde{u}\|_{L^1(\Omega)} \\
+ x \|D_k - D\|_{L^1(\Omega)} \|\tilde{u}\|_{L^1(\Omega)} \|\text{rot} B_k\|_{L^1(\Omega)}.
\]
Hence, it follows that
\[
\|\nabla v_k\|_{L^1(\Omega)} + \|\text{rot} B_k\|_{L^1(\Omega)} \\
\leq C(\|\text{rot} \tilde{H}\|_{L^1(\Omega)} \|D_k - D\|_{L^1(\Omega)} + \|w_k - w\|_{L^1(\Omega)} \|\nabla \tilde{u}\|_{L^1(\Omega)} + \|D_k - D\|_{L^1(\Omega)} \|\tilde{u}\|_{L^1(\Omega)}),
\]
where the constant \(C\) depends on \(\Omega, \nu, v_1, x, \varepsilon_0, g, q\). Together with (2.8) and (2.9), we conclude by Lemma 2.3 that
\[
\tilde{u}_k \rightarrow \tilde{u} \quad \text{and} \quad \tilde{H}_k \rightarrow \tilde{H} \quad \text{in } H^1(\Omega, \mathbb{R}^3) \text{ as } k \rightarrow \infty,
\]
which implies the continuity of the operator \(T\).
And then, we show that \(T\) is compact from \(D\) into \(D\). Assume that \((w_k, D_k) \in D\). Then, there exist \((w, D) \in D\) and a subsequence of \((w_k, D_k)\), still denoted by \((w_k, D_k)\) to simplify the notation, satisfying
\[
w_k \rightarrow w, \quad D_k \rightarrow D \quad \text{in } H^1(\Omega, \mathbb{R}^3) \text{ and } w_k \rightarrow w, \quad D_k \rightarrow D \quad \text{in } L^3(\Omega, \mathbb{R}^3) \text{ as } k \rightarrow \infty.
\]
Similarly to the proof of continuity of \(T\), we obtain
\[
\tilde{u}_k \rightarrow \tilde{u} \quad \text{and} \quad \tilde{H}_k \rightarrow \tilde{H} \quad \text{in } H^1(\Omega, \mathbb{R}^3) \text{ as } k \rightarrow \infty.
\]
Step 4. Finally, we use Schauder’s fixed point theorem and conclude that $T$ has a fixed point $(\tilde{u}, \tilde{H}) \in D$. Then, by De Rham’s theorem, there exists a function $p \in L^2(\Omega)/\mathbb{R}$ such that $(\tilde{u}, p, \tilde{H})$ is a weak solution of (2.4) with $(\mathbf{w}, \mathbf{D})$ replaced by $(\tilde{u}, \tilde{H})$. So we get a weak solution of (2.2).

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REFERENCES

1. Ladyzhenskaja OA, Solonnikov VA. Solution of some non-stationary problems of magnetohydrodynamics for a viscous incompressible fluid. (Russian) Trudy Mat Inst Steklov. 1960;59:115-173.
2. Solonnikov VA. Some stationary boundary-value problems of magnetohydrodynamics. (Russian) Trudy Mat Inst Steklov. 1960;59:174-187.
3. Alekseev GV. Solvability of control problems for stationary equations of the magnetohydrodynamics of a viscous fluid. (Russian) Sibirsk Mat Zh. 2004;45(2):243-263. Translation in Siberian Math. J. 2004;45(2):197–213.
4. Alekseev GV. Control problems for stationary equations of magnetohydrodynamics. (Russian) Dokl Akad Nauk. 2004;395(3):322-325.
5. Alekseev GV. Solvability of an inhomogeneous boundary value problem for the stationary magnetohydrodynamic equations for a viscous incompressible fluid. Translation of Differ Uravn. 2016;52(6):760-769. Differ. Equ. 2016;52(6):739-748.
6. Gunzburger MD, Meir AJ, Peterson JS. On the existence, uniqueness, and finite element approximation of solutions of the equations of stationary, incompressible magnetohydrodynamics. Math Comp. 1991;56(194):523-563.
7. Sermange M, Temam R. Some mathematical questions related to the MHD equations. Comm Pure Appl Math. 1983;36(5):635-664.
8. Alekseev GV, Brizitskii RV. Boundary control problems for the stationary magnetic hydrodynamic equations in the domain with non-ideal boundary. J Dyn Control Syst. 2020. https://doi.org/10.1007/s10883-019-09474-1
9. Dautray R, Lions J. Mathematical Analysis and Numerical Methods for Science and Technology, Vol. 3. New York: Springer-Verlag; 1990.

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