A Class of Orderings in the Range of Borda’s Rule

Jerry S. Kelly* Shaofang Qi†
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Abstract
We present a class of orderings $L$ for which there exists a profile $u$ of preferences for a fixed odd number of individuals such that Borda’s rule maps $u$ to $L$.

Kelly and Qi [1] initiated the study of what orderings are in the range of Borda’s rule for profiles of strong preference orderings and a fixed number of individuals. Here we extend those results, establishing a new class of orderings in the Borda range.

1. Framework.

We begin with a finite set $N = \{1, \ldots, n\}$ of individuals, $n \geq 2$, and a finite set $X$ of alternatives, with $|X| = m \geq 2$. A binary relation $\rho$ on $X$ is a non-empty subset of the Cartesian product, $X \times X$; if $(x, y) \in \rho$, we will often write $x \rho y$. Relation $\rho$ is

1. reflexive if $x \rho x$ for all $x$ in $X$;
2. asymmetric if for all $x, y$ in $X$: $x \rho y$ and $y \rho x$ imply $x = y$;
3. complete if for all for all $x, y$ in $X$ such that $x \neq y$, either $x \rho y$ or $y \rho x$;
4. transitive if $x \rho y$ and $y \rho z$ imply $x \rho z$ for all $x, y, z$ in $X$.

Relation $\rho$ is a weak order on $X$ if it is a reflexive, complete, and transitive relation on $X$; $\rho$ is a strong order on $X$ if it is a weak order on $X$ and is also asymmetric. The set of all strong orders on $X$ is denoted $L(X)$. If $r$ is a strong order on $X$, then $r[1]$ is the top-ranked alternative in $r$: $x \succ y$ for all $y$ in $X \setminus \{x\}$. More generally, $r[k]$ is the $k^{th}$-ranked alternative in $r$. The inverse

*Department of Economics, Syracuse University. E-mail: jskelly@maxwell.syr.edu
†Syracuse University and Humboldt University Berlin (after 1 September 2015). E-mail: sqi@syr.edu

Most of this section is drawn from Kelly and Qi [1].
$R^{-1}$ of an order $R$ is defined by $xR^{-1}y$ if and only if $yRx$. A profile is an ordered $n$-tuple $u = (u(1), u(2), ..., u(n)) \in L(X)^n$ of weak orders.

Given a profile $u$ in $L(X)^n$, define $s(u, x, i) = k$, where $u(i)[k] = x$. Then the **Borda score** of $x$ at $u$, $S(u, x)$, is the sum of the $s(u, x, i)$ over $i$, for $1 \leq i \leq n$. The Borda ranking, $f_B(u)$, sets $x > y$ if and only if $S(u, x) \leq S(u, y)$.

The outcome of the Borda ranking procedure is a weak ordering $L$ which could be written as

$$L = X_1 \succ X_2 \succ ... \succ X_T$$

where: (i) each $X_i \subset X$, (ii) the $X_i$ are pairwise disjoint, (iii) alternatives within an $X_i$ all have the same Borda score, and (iv) $i < j$ implies all alternatives in $X_i$ have Borda score less than all alternatives in $X_j$. Each $X_i$ is called a level.

Because Borda satisfies neutrality, we can usefully abbreviate our descriptions of weak order images under Borda’s rule. For a given $X$, we only have to be concerned with the number of alternatives in each level, not with exactly which alternatives are in each level. Showing $L = \{a, b, c\} \succ \{d, e\} \succ \{f\} \succ \{g, h, i, j\}$ is in the image of $f_B$ also shows that $L' = \{i, b, e\} \succ \{c, h\} \succ \{j\} \succ \{g, e, a, f\}$ is in the range. More generally, with a slight abuse of language, we say a weak order generated by Borda’s rule is a sequence $(m_1, m_2, ..., m_T)$ where the $m_i$ are the cardinalities of the sets of alternatives with the same Borda score.

Kelly and Qi [1] established several propositions regarding the Pareto range. For example, if at least one $m_i$ is odd, $(m_1, m_2, ..., m_T)$ is in the Borda range for all odd $n$. (Accordingly, if $m$ is odd, $L$ is in the Borda range for all odd $n$.) Other results relevant for this paper are:

**Theorem 3.** Suppose $L = (m_1, m_2, ..., m_T)$ has every $m_i$ is even. Let $k \geq 1$ be the largest power of 2 dividing all the $m_i$, so $L = (2^k s_1, 2^k s_2, ..., 2^k s_T)$. If $s_1 + s_2 + ... + s_T$ is odd, then for every odd positive integer $n$, there does not exist a profile $u$ such that $f_B(u) = L$.

**Lemma 4.** Suppose $L = (m_1, m_2, ..., m_T)$ has every $m_i$ is even. Let $k \geq 1$ be the largest power of 2 dividing all the $m_i$, so $L = (2^k s_1, 2^k s_2, ..., 2^k s_T)$. If $s_1 + s_2 + ... + s_T$ is even and all $s_i$ are odd, then for every odd $n \geq 3$, there exists a profile $u$ such that $f_B(u) = L$.

In particular, if $L = (m_1, m_2)$ has two equal levels, then $L$ is in the Borda range for all odd $n$.

2. The New Class.

Lemma 4 fails to cover most cases where $s_1 + s_2 + ... + s_T$ is even. Here we examine long orderings. For fixed even $m$, all orders with sufficiently many levels but with no odd levels will be made up entirely of levels equal to 2 or 4, with not very many 4s.

Let $L = (m_1, m_2, ..., m_T)$. If all the $m_i = 2$, then $L$ is not in the range of Borda’s rule for any odd $n$ if $T$ is odd (Theorem 3) and is in the range of Borda’s
rule for all odd $n$ if $T$ is even (Lemma 4). A similar statement can be made if all the $m_i = 4$.

So we are only interested in the case where both 2 and 4 do appear in $L$. By Theorem 3 again, $L$ is not in the Borda range if there are an odd number of 2s in $L$. So we may suppose $L$ contains 4 and an even number ($\geq 2$) of 2s. We first treat a set of special cases.

**Lemma.** Each of the following (patterns of) orders (with exactly two levels equal to 2) is in the Borda range for $n = 3$ (and so for all odd $n \geq 3$):

- $(2, 4, 4, ..., 4, 4, 2)$
- $(4, 2, 4, 4, ..., 4, 4, 2)$
- $(2, 4, 4, ..., 4, 4, 2, 4)$
- $(4, 2, 4, 4, ..., 4, 4, 2, 4)$

A proof of the Lemma appears in the next section.

**Theorem.** Suppose $L = (m_1, m_2, ..., m_T)$ and $L$ contains only 4s and an even number ($\geq 2$) of 2s. Then $L$ is in the Borda range for all odd $n \geq 3$.

**Proof of the Theorem:** The proof is by induction on the number of levels $T$ ($\geq 2$).

**Basis:** For $T = 2$, $L$ must be $(2, 2)$ and this is in the range by Lemma 4.

**Induction step:** We now assume the result is true for all levels less than $T \geq 3$. We first decompose $L$ by stripping out some 4s that might occur prior to the first occurrence of 2 in $L$. So $L = L_0 \succ L_1$ where $L_0$ contains an even number ($\geq 2$) of 4s and order $L_1$ looks like either $(2, ...)$ or $(4, 2, ...)$. By Lemma 4, $L_0$ is in the range and we will catenate the profile for $L_0$ with the profile we will construct for $L_1$. Let $L_1^*$ be the initial sequence of $L_1$ up to and including the second occurrence of 2: $(2, 4, 4, ..., 4, 4, 2)$ or $(4, 2, 4, 4, ..., 4, 4, 2)$. Now $L_1 = L_1^* \succ L_2$, where $L_2$ contains an even number of only 2s (possibly 0). If $L_2$ contains a positive even number of 2s, the induction hypothesis shows $L_2$ is in the Borda range. By the Lemma, $L_1^*$ is in the Borda range and we can catenate profiles to show that $L = L_0 \succ L_1^* \succ L_2$ is in the Borda range.

So suppose $L_2$ doesn’t contain any 2s. Either it is empty and $L = L_0 \succ L_1^*$ is in the range by catenation, or $L_2 = (4, 4, ..., 4)$. If $L_2$ contains an even number of 4s, then $L_2$ is in the range by Lemma 4 and $L = L_0 \succ L_1^* \succ L_2$ is in the Borda range by catenation.

All that remains is the case where $L_2$ contains an odd number of 4s. In that case, move one 4 from $L_2$ to $L_1^*$ so that now we take $L_1^* = (2, 4, 4, ..., 4, 2, 4)$ or $(4, 2, 4, 4, ..., 4, 2, 4)$ (either order in the Borda range by the Lemma) and $L_2$ has an even number of 4s. Now, $L = L_0 \succ L_1^* \succ L_2$ where each part is in the Borda range and catenation yields our result.  

\[\blacksquare\]
3. Proof of the Lemma.

We wish to show that each of the following (patterns of) orders (with exactly two levels equal to 2) is in the Borda range for \( n = 3 \) (and so for all odd \( n \geq 3 \)):

\[
\begin{align*}
(2, 4, 4, \ldots, 4, 4, 2) \\
(4, 2, 4, 4, \ldots, 4, 4, 2) \\
(2, 4, 4, \ldots, 4, 4, 2, 4) \\
(4, 2, 4, 4, \ldots, 4, 4, 2, 4)
\end{align*}
\]

To construct the relevant profile for each sequence, we will make use of the profile below (the profile we constructed for Lemma 4 in the main paper). In particular, for any weak ordering \( L = (2s_1, 2s_2) \), where both \( s_1 \) and \( s_2 \) are odd numbers, the following profile \( v \) has \( f_B(v) = L \). Note that at profile \( v \), for individual \#2, below the second group of blue options are all odd-subscript options; for individual \#3, below the second group of blue options are all even-subscript options. All the blue options (all together \( 2s_2 \)) represent the options that have equal Borda score, which is smaller than the equal Borda score of the remaining options (all together \( 2s_1 \)). The Borda score difference between the two groups of options is \( s_1 + s_2 \). Individual \#1 ranks options in order from \( x_1 \) to \( x_{2(s_1+s_2)} \).

The profile we will construct for each sequence is based on variations of \( v \). The variations will only be made regarding individual \#1’s ranking of options, and we retain the ranking of individual \#2 and \#3 unchanged. Therefore, for the following analysis, for simplicity, we will only re-state individual \#1’s ranking at a profile without presenting the full profile.
|   | 2          | 3          | 1          |
|---|-----------|-----------|-----------|
| $x_1$ | $(s_1+s_2)+s_1-1$ | $(s_1+s_2)+s_1$ |               |
| $x_2$ | :         | $(s_1+s_2)+s_1-2$ |               |
|      | :         | :         | :         |
|      | :         | :         | :         |
|      | :         | :         | :         |
|      | :         | :         | :         |
|      | :         | :         | :         |
|      | $(s_1+s_2)$ | $(s_1+s_2)+1$ |               |
|      | :         | $(s_1+s_2)-2$ | $(s_1+s_2)-1$ |
|      | :         | :         | $(s_1+s_2)-3$ |
|      | :         | :         |               |
|      | :         | :         |               |
|      | :         | :         |               |
|      | :         | :         |               |
|      | $(s_1+s_2)+s_1+3$ |               |               |
|      | $(s_1+s_2)+s_1+1$ | $(s_1+s_2)+s_1+2$ |               |
|      | :         | $x_1$     |               |
|      | :         | $x_2$     |               |
|      | :         | $(s_1+s_2)$ | $x_1$     |
|      | :         | $(s_1+s_2)-2$ | $(s_1+s_2)-1$ |
|      | :         | :         | $(s_1+s_2)-3$ |
|      | :         | :         |               |
|      | :         | :         |               |
|      | :         | :         |               |
|      | :         | :         |               |
|      | :         | :         |               |
|      | :         | :         |               |
|      | :         | :         |               |
|      | :         | :         |               |
|      | :         | :         |               |
|      | :         | :         |               |
|      | :         | :         |               |
I. The sequence \((2, 4, 4, ..., 4, 4, 2)\).

Case 1. There are \(2k\) (i.e., even number of) 4s between the 2s.

We first construct, using Lemma 4’s method, a profile \(v\) for the two-level ordering \((2 + 4 + \cdots + 4, 4 + \cdots + 4 + 2)\), which is \((2 + 4k, 4k + 2)\), which also is \((2(1 + 2k), 2(2k + 1))\). Given the proof of Lemma 4, the first level consists of two groups of options, each having \((1 + 2k)\) options:

\[
\left\{ x_1, x_2, \ldots, x_{2k+1} \right\} \\
\left\{ x_{(4k+2)+1}, x_{(4k+2)+2}, \ldots, x_{(4k+2)+2k+1} \right\}
\]

Similarly, the second level also consists of two groups of options, each having \((1 + 2k)\) options:

\[
\left\{ x_{(2k+1)+1}, x_{(2k+1)+2}, \ldots, x_{(2k+1)+2k+1} \right\} \\
\left\{ x_{(6k+3)+1}, x_{(6k+3)+2}, \ldots, x_{(6k+3)+2k+1} \right\}
\]

In addition, as Lemma 4 in Kelly and Qi (2015) shows, at \(v\), individual #1’s ranking of options is simply:

\[
\begin{array}{c}
1 \\
x_1 \\
x_2 \\
\vdots \\
\vdots \\
x_{8k+3} \\
x_{8k+4}
\end{array}
\]

We color individual #1’s above ranking such that options in the first level are in black and options in the second level are in blue:
Now we make changes for (only) individual #1’s ranking to obtain profile \( u \) for the level \((2, 4, 4, \ldots, 4, 4, 2)\) (where there are \(2k\) occurrences of 4). For the options that are colored the same, we will make the same change. So we take the first half options, a group of black and a group of blue, as an example:

For the options \(\{x_1, x_2, \ldots, x_{2k+1}\}\), we change the initial ranking of \(x_1 \succ x_2 \succ \cdots \succ x_{2k+1}\) to:
And it is straightforward to check that after making the above change, the options are split into \((k + 1)\) levels:

\[
\begin{align*}
\{x_{2k+1}\} \\
\{x_{2k-1}, x_{2k}\} \\
\{x_{2k-3}, x_{2k-2}\} \\
\vdots \\
\{x_3, x_4\} \\
\{x_1, x_2\}
\end{align*}
\]

For the options \(\{x_{(2k+1)+1}, x_{(2k+1)+2}, \ldots, x_{(2k+1)+2k+1}\}\), we change the initial ranking of \(x_{(2k+1)+1} \succ x_{(2k+1)+2} \succ \cdots \succ x_{(2k+1)+2k+1}\) to:

\[
\begin{align*}
x_{(2k+1)+2k} \\
x_{(2k+1)+2k+1} \\
x_{(2k+1)+2k-2} \\
x_{(2k+1)+2k-1} \\
\vdots \\
x_{(2k+1)+2} \\
x_{(2k+1)+3} \\
x_{(2k+1)+1}
\end{align*}
\]

And it is straightforward to check that after making the above change, the options are split into \((k + 1)\) levels:

\[
\begin{align*}
\{x_{(2k+1)+2k}, x_{(2k+1)+2k+1}\} \\
\{x_{(2k+1)+2k-2}, x_{(2k+1)+2k-1}\} \\
\vdots \\
\{x_{(2k+1)+4}, x_{(2k+1)+5}\} \\
\{x_{(2k+1)+2}, x_{(2k+1)+3}\} \\
\{x_{(2k+1)+1}\}
\end{align*}
\]
Now we compare the earlier \((k + 1)\) levels consisting of \(\{x_1, x_2, \ldots, x_{2k+1}\}\) with the above \((k+1)\) levels consisting of \(\{x_{(2k+1)+1}, x_{(2k+1)+2}, \ldots, x_{(2k+1)+2k+1}\}\).

Compared with \(v\) (before making change of \#1’s ranking), for the option \(x_{2k+1}\) at the highest level of the \((k + 1)\) levels consisting of \(\{x_1, x_2, \ldots, x_{2k+1}\}\), the Borda score of \(x_{2k+1}\) is increased by \(2k\). Compared with \(v\), for the options \(x_{(2k+1)+2k}\) and \(x_{(2k+1)+2k+1}\) at the highest level of the \((k + 1)\) levels consisting of \(\{x_{(2k+1)+1}, x_{(2k+1)+2}, \ldots, x_{(2k+1)+2k+1}\}\), the Borda score of \(x_{(2k+1)+2k}\) and \(x_{(2k+1)+2k+1}\) is increased by \(2k - 1\). In addition, it follows from Lemma 4 that at \(v\), the Borda score of \(x_1, x_2, \ldots, x_{2k+1}\) (black) is larger than that of \(x_{(2k+1)+1}, x_{(2k+1)+2}, \ldots, x_{(2k+1)+2k+1}\) (blue) by \((2k + 1)\). So the option \(x_{2k+1}\) is at the highest level among the \(2(k + 1)\) levels.

Similarly, compared with \(v\), for the option \(x_{(2k+1)+1}\) at the lowest level of the \((k + 1)\) levels consisting of \(\{x_{(2k+1)+1}, x_{(2k+1)+2}, \ldots, x_{(2k+1)+2k+1}\}\), the Borda score of \(x_{(2k+1)+1}\) is decreased by \(2k\). Compared with \(v\), for the options \(x_1\) and \(x_2\) at the lowest level of the \((k + 1)\) levels consisting of \(\{x_1, x_2, \ldots, x_{2k+1}\}\), the Borda score of \(x_1\) and \(x_2\) is decreased by \(2k - 1\). So the option \(x_{(2k+1)+1}\) is at the lowest level among the \(2(k + 1)\) levels.

Except \(x_{2k+1}\) at the highest level and \(x_{(2k+1)+1}\) at the lowest level, we need to show that the remaining options are actually split into \(2k\) levels. We do this by calculating the Borda score change from \(v\) to \(u\). For the \(k\) levels consisting of \(\{x_1, x_2, \ldots, x_{2k}\}\):

\[
\begin{align*}
\{x_{2k-1}, x_{2k}\} & : (2k - 3) \text{ more} \\
\{x_{2k-3}, x_{2k-2}\} & : (2k - 7) \text{ more} \\
\vdots & \quad \vdots \\
\{x_3, x_4\} & : (5 - 2k) \text{ more [i.e., (2k - 5) less]} \\
\{x_1, x_2\} & : (1 - 2k) \text{ more [i.e., (2k - 1) less]}
\end{align*}
\]

For the \(k\) levels consisting of \(\{x_{(2k+1)+2}, \ldots, x_{(2k+1)+2k+1}\}\):

\[
\begin{align*}
\{x_{(2k+1)+2k}, x_{(2k+1)+2k+1}\} & : (2k - 1) \text{ more} \\
\{x_{(2k+1)+2k-2}, x_{(2k+1)+2k-1}\} & : (2k - 5) \text{ more} \\
\vdots & \quad \vdots \\
\{x_{(2k+1)+4}, x_{(2k+1)+5}\} & : (7 - 2k) \text{ more [i.e., (2k - 7) less]} \\
\{x_{(2k+1)+2}, x_{(2k+1)+3}\} & : (3 - 2k) \text{ more [i.e., (2k - 3) less]}
\end{align*}
\]

And at \(v\), the black options Borda score is larger than the of blue by \((2k + 1)\). It follows that the Borda score difference between the black options based on ranking at \(u\) and the blue options based on ranking at \(v\) is an even number, and since the Borda score change of blue options from \(v\) to \(u\) is an odd number, no level among the \(k\) consisting of \(\{x_1, x_2, \ldots, x_{2k}\}\) will have the same Borda score.
as any level among the $k$ consisting of $\{x_1, x_2, \ldots, x_{2k}\}$. Therefore, they remain in $2k$ levels.

Combining with the fact that $x_{2k+1}$ at the highest level and $x_{(2k+1)+1}$ at the lowest level, at $u$, for these $(4k+2)$ options, we obtain a pattern $(1, 2, 2, \ldots, 2, 2, 1)$ (where there are $2k$ number of 2s). Recall that we also do the same thing for another half of the options at $v$ for individual $\#1$,

$\{x_{(4k+2)+1}, \ldots, x_{(4k+2)+2k+1}, x_{(6k+3)+1}, \ldots, x_{(6k+3)+2k+1}\}$, so actually, at $u$, we have $f_B(u) = (2, 4, \ldots, 4, 4, 2)$ (where there are $2k$ occurrences of 4).

**Case 2.** There are $2k+1$ (i.e., odd number of) 4s between the 2s.

We follow the same procedure as Case 1, except that we make the following revisions.

First, at the initial construction, profile $v$ is for the ordering with two levels: $(2 + 4 + \cdots + 4 + 4 + 4 + \cdots + 4 + 2)$, which is $(2 + 4(k+1), 4k+2)$, which also is $(2(3+2k), 2(2k+1))$. So that according to Lemma 4’s method, the first level consists of two groups of options, each having $(3+2k)$ options:

$\{x_1, x_2, \ldots, x_{2k+1}, x_{2k+2}, x_{2k+3}\}$

$\{x_{(4k+4)+1}, x_{(4k+4)+2}, \ldots, x_{(4k+4)+2k+1}, x_{(4k+4)+2k+2}, x_{(4k+4)+2k+3}\}$

The second level also consists of two groups of options, each having $(1+2k)$ options:

$\{x_{(2k+3)+1}, x_{(2k+3)+2}, \ldots, x_{(2k+3)+2k+1}\}$

$\{x_{(6k+7)+1}, x_{(6k+7)+2}, \ldots, x_{(6k+7)+2k+1}\}$

Following Case 1, we make changes for individual $\#1$’s ranking at $v$. The same change will be made for the options in the same level (color), so we take the first half of options, one group of black option and one group of blue option, as an example:

|   |
|---|
| $x_1$ |
| $x_2$ |
| $\vdots$ |
| $x_{2k+2}$ |
| $x_{2k+3}$ |
| $x_{(2k+3)+1}$ |
| $x_{(2k+3)+2}$ |
| $\vdots$ |
| $x_{(2k+3)+2k+1}$ |
| $\vdots$ |
For the options \( \{ x_1, x_2, \ldots, x_{2k+3} \} \), we change the initial ranking of \( x_1 \succ x_2 \succ \cdots \succ x_{2k+3} \) of individual #1 at \( v \) to:

\[
\begin{array}{c}
x_{2k+3} \\
x_{2k+1} \\
x_{2k+2} \\
\vdots \\
x_3 \\
x_4 \\
x_1 \\
x_2
\end{array}
\]

And the options are split into \((k + 2)\) levels, with the Borda score change going from \( v \) to \( u \) marked as below:

\[
\begin{align*}
\{ x_{2k+3} \} & : (2k + 2) \text{ more} \\
\{ x_{2k+1}, x_{2k+2} \} & : (2k - 1) \text{ more} \\
\{ x_{2k-1}, x_{2k} \} & : (2k - 5) \text{ more} \\
\vdots \\
\{ x_3, x_4 \} & : (3 - 2k) \text{ more [i.e., (2k - 3) less]} \\
\{ x_1, x_2 \} & : (-1 - 2k) \text{ more [i.e., (2k + 1) less]}
\end{align*}
\]

Similarly, for the options \( \{ x_{(2k+3)+1}, x_{(2k+3)+2}, \ldots, x_{(2k+3)+2k+1} \} \), we change the initial ranking of \( x_{(2k+3)+1} \succ x_{(2k+3)+2} \succ \cdots \succ x_{(2k+3)+2k+1} \) to:

\[
\begin{array}{c}
x_{(2k+3)+2k} \\
x_{(2k+3)+2k+1} \\
x_{(2k+3)+2k-2} \\
x_{(2k+3)+2k-1} \\
\vdots \\
x_{(2k+3)+2} \\
x_{(2k+3)+3} \\
x_{(2k+3)+1}
\end{array}
\]

And the options are split into \((k + 1)\) levels, with the Borda score change going from \( v \) to \( u \) marked as below:

\[
\begin{align*}
\{ x_{(2k+3)+2k}, x_{(2k+3)+2k+1} \} & : (2k - 1) \text{ more} \\
\{ x_{(2k+3)+2k-2}, x_{(2k+3)+2k-1} \} & : (2k - 5) \text{ more} \\
\vdots \\
\{ x_{(2k+3)+4}, x_{(2k+3)+5} \} & : (7 - 2k) \text{ more [i.e., (2k - 7) less]} \\
\{ x_{(2k+3)+2}, x_{(2k+3)+3} \} & : (3 - 2k) \text{ more [i.e., (2k - 3) less]} \\
\{ x_{(2k+3)+1} \} & : -2k \text{ more [i.e., } 2k \text{ less]}
\end{align*}
\]
Now we compare, at $u$, the $(k + 2)$-level group of options (black) with the $(k + 1)$-level group of options (blue).

Recall that at $v$, where the black options are a single level and the blue options are a single level, the Borda score difference (the black ones have larger Borda score) between the two levels is $\frac{3+2k+(2k+1)}{2} = 2k + 2$.

It follows that option $x_{2k+3}$ is at the highest level among the $(2k + 3)$ levels, and option $x_{(2k+3)+1}$ is at the lowest level among the $(2k + 3)$ levels.

To compare the remaining options, we summarize the Borda score difference between the black options based on ranking at $u$ and the blue options based on ranking at $v$ (therefore a single level for the blue options) below:

\[
\begin{align*}
\{x_{2k+3}\} & : (2k + 2) + (2k + 2) \text{ more [i.e., (4k + 4) more]} \\
\{x_{2k+1}, x_{2k+2}\} & : (2k - 1) + (2k + 2) \text{ more [i.e., (4k + 1) more]} \\
\{x_{2k-1}, x_{2k}\} & : (2k - 5) + (2k + 2) \text{ more [i.e., (4k - 3) more]} \\
\vdots \\
\{x_3, x_4\} & : (3 - 2k) + (2k + 2) \text{ more [i.e., 5 more]} \\
\{x_1, x_2\} & : (-1 - 2k) + (2k + 2) \text{ more [i.e., 1 more]} 
\end{align*}
\]

We pick the level among the $(k + 1)$-level group of blue options where the Borda score change for the options at the level from $v$ to $u$ is equal to 1 (if no such level exists, then we skip this step). For example, when $k = 3$, such level consists of options $\{x_{13}, x_{14}\}$. We then move $x_1$ and $x_2$ right below the options for this level. If after this change, the Borda score of $x_1$ and $x_2$ is still equal to some level among the $(k + 1)$-level group of blue options, then move $x_1$ and $x_2$ further down by two more options. Accordingly, one can check that the obtained profile has $(2k + 1)$ levels among these options.

Again, we do the same changes for the second half of the options for individual #1. For the obtained profile, the Borda score ordering is $(2, 4, 4, \ldots, 4, 4, 2)$ (where there are $2k + 1$ occurrences of 4).

II. The sequence $(4, 2, 4, \ldots, 4, 4, 2)$.

Case 1. There are $2k$ (i.e., even number of) occurrences of 4.

We follow the same steps as in case 1 in the first sequence, until for individual #1 at profile $v$, for the options $\{x_1, x_2, \ldots, x_{2k+1}\}$, we change the initial ranking of $x_1 > x_2 > \cdots > x_{2k+1}$ to:
Here the options are split into \((k + 1)\) levels, with the Borda score change from \(v\) to \(u\) marked as below:

\[
\begin{align*}
\{x_{2k}, x_{2k+1}\} & : (2k - 1) \text{ more} \\
\{x_{2k-1}\} & : (2k - 4) \text{ more} \\
\{x_{2k-3}, x_{2k-2}\} & : (2k - 7) \text{ more} \\
\{x_{2k-5}, x_{2k-4}\} & : (2k - 11) \text{ more} \\
\{x_3, x_4\} & : (5 - 2k) \text{ more} \ [\text{i.e., } (2k - 5) \text{ less}] \\
\{x_1, x_2\} & : (1 - 2k) \text{ more} \ [\text{i.e., } (2k - 1) \text{ less}] 
\end{align*}
\]

Recall that from Lemma 4, at \(v\), the Borda score of \(x_1, x_2, \ldots, x_{2k+1}\) (black) is larger than that of \(x_{(2k+1)+1}, x_{(2k+1)+2}, \ldots, x_{(2k+1)+2k+1}\) (blue) by \((2k + 1)\). So the Borda score difference between the black options based on ranking at \(u\) and the blue options based on ranking at \(v\) (therefore a single level for the blue options) is:

\[
\begin{align*}
\{x_{2k}, x_{2k+1}\} & : (2k - 1) + (2k + 1) \text{ more} \ [\text{i.e., } 4k \text{ more}] \\
\{x_{2k-1}\} & : (2k - 4) + (2k + 1) \text{ more} \ [\text{i.e., } (4k - 3) \text{ more}] \\
\{x_{2k-3}, x_{2k-2}\} & : (2k - 7) + (2k + 1) \text{ more} \ [\text{i.e., } (4k - 6) \text{ more}] \\
\{x_{2k-5}, x_{2k-4}\} & : (2k - 11) + (2k + 1) \text{ more} \ [\text{i.e., } (4k - 10) \text{ more}] \\
\{x_3, x_4\} & : (5 - 2k) + (2k + 1) \text{ more} \ [\text{i.e., } 6 \text{ more}] \\
\{x_1, x_2\} & : (1 - 2k) + (2k + 1) \text{ more} \ [\text{i.e., } 2 \text{ more}] 
\end{align*}
\]

For options \(\{x_{(2k+1)+1}, \ldots, x_{(2k+1)+2k+1}\}\), we make the exactly same change as for case 1 of sequence I, so the options are split into \((k + 1)\) and the Borda score change going from \(v\) to \(u\) is:
\{x(2k+1)+2k, x(2k+1)+2k+1\} : (2k - 1) more \\
\{x(2k+1)+2k-2, x(2k+1)+2k-1\} : (2k - 5) more \\
\vdots \\
\{x(2k+1)+4, x(2k+1)+5\} : (7 - 2k) more \ [i.e., (2k - 7) less] \\
\{x(2k+1)+2, x(2k+1)+3\} : (3 - 2k) more \ [i.e., (2k - 3) less] \\
\{x(2k+1)+1\} : -2k more \ [i.e., 2k less]

We then compare the above two \((k+1)\)-level groups of options.

For \(k = 1\), the sequence is \((4, 2, 4, 2)\), which is shown to be in the Borda range for \(n = 3\) in the Appendix.

For \(k > 1\), we have \((4k - 3) > (2k - 1)\), so that the two levels, \(\{x_{2k}, x_{2k+1}\}\) and \(\{x_{2k-1}\}\), remain as highest levels among the \(2(k+1)\) levels. Similarly, the level \(\{x_{(2k+1)+1}\}\) remains as the lowest one among the \(2(k+1)\) levels. For the remaining \((2k - 1)\) levels, the Borda score change for the black options is even while for blue options is odd, so they remain as \((2k - 1)\) levels. By making the same change for the second half of the options for individual \#1, we obtain a profile for \((4, 2, 4, \ldots, 4, 4, 2)\) where there are \(2k\) \(i.e.,\) an even number of) occurrences of 4.

**Case 2.** There are \(2k + 1\) \(i.e.,\) odd number of) occurrences of 4.

We follow the same steps as in case 2 of sequence I until for individual \#1 at profile \(v\), for the options \(\{x_1, x_2, \ldots, x_{2k+3}\}\), we change the initial ranking of \(x_1 \succ x_2 \succ \cdots \succ x_{2k+3}\) to:

\[
\begin{array}{c}
\cdots \\
x_{2k-1} \\
x_{2k} \\
x_{2k+1} \\
x_{2k+3} \\
x_{2k+2} \\
\end{array}
\]

The options are split into \((k + 2)\) levels, with the Borda score change from \(v\) to \(u\) marked as below:
The Borda score difference between the black options based on the ranking at $u$ and the blue options based on the ranking at $v$ (therefore a single level for the blue options) is:

\[
\{x_{2k+2}, x_{2k+3}\} : (2k + 1) \text{ more}
\]
\[
\{x_{2k+1}\} : (2k - 2) \text{ more}
\]
\[
\{x_{2k-1}, x_{2k}\} : (2k - 5) \text{ more}
\]
\[
\vdots
\]
\[
\{x_3, x_4\} : (3 - 2k) \text{ more [i.e., $(2k - 3) \text{ less}$]}
\]
\[
\{x_1, x_2\} : (-1 - 2k) \text{ more [i.e., $(2k + 1) \text{ less}$]}
\]

For the options $\{x_{(2k+3)+1}, x_{(2k+3)+2}, \ldots, x_{(2k+3)+2k+1}\}$, we make the exactly same change as case 2 of sequence I, so the options are split into $(k + 1)$ and the Borda score change from $v$ to $u$ is:

\[
\{x_{(2k+1)+2k}, x_{(2k+1)+2k+1}\} : (2k - 1) \text{ more}
\]
\[
\{x_{(2k+1)+2k-2}, x_{(2k+1)+2k-1}\} : (2k - 5) \text{ more}
\]
\[
\vdots
\]
\[
\{x_{(2k+1)+4}, x_{(2k+1)+5}\} : (7 - 2k) \text{ more [i.e., $(2k - 7) \text{ less}$]}
\]
\[
\{x_{(2k+1)+2}, x_{(2k+1)+3}\} : (3 - 2k) \text{ more [i.e., $(2k - 3) \text{ less}$]}
\]
\[
\{x_{(2k+1)+1}\} : -2k \text{ more [i.e., $2k \text{ less}$]}
\]

We then compare the above $(k + 2)$-level group and the $(k + 1)$-level group.

For $k \geq 1$, we have $4k > 2k - 1$ (actually, we have $4k > 2k + 1$ also, if we need to make the adjustments below), so the two levels consisting of $\{x_{2k+2}, x_{2k+3}\}$ and $\{x_{2k+1}\}$ remain to be the top two levels among the $(2k+3)$ levels. Similarly, the level $\{x_{(2k+1)+1}\}$ remains to be the lowest one among the $(2k + 3)$ levels. Suppose there exists a level among the $(k + 1)$-level group of blue options where the Borda score change for the options at the level from $v$ to $u$ is equal to 1 (if no such level exists, then we skip this step), we do the same adjustments as illustrated at the end of case 2 of sequence I. The obtained profile is for $(4, 2, 4, \ldots, 4, 4, 2)$ where there are $2k + 1$ (i.e., an odd number of) occurrences of 4.
III. The sequence \((2, 4, \ldots, 4, 4, 2, 4)\).

The inverse of the sequence \((2, 4, \ldots, 4, 4, 2, 4)\) is \((4, 2, 4, \ldots, 4, 2)\), which we have shown above is in the Borda range for \(n = 3\) (and thus \(n \geq 3\)). Taking the profile for \((4, 2, 4, \ldots, 4, 2)\) and inverting everyone’s ranking yields the sequence \((2, 4, \ldots, 4, 2, 4)\).

IV. The sequence \((4, 2, 4, \ldots, 4, 2, 4)\).

Case 1. There are \(2k\) (i.e., even number of) \(4\)s. Note that when \(k = 1\), the sequence is \((4, 2, 2, 4)\), which is shown to be in the Borda range in the Appendix. So we focus on \(k > 1\).

For the options, \(x_1, x_2, \ldots, x_{2k+1}\) (black), we follow the same steps as in sequence II, case 1, and accordingly, the Borda score difference between the black options based on ranking at \(u\) and the blue options based on ranking at \(v\) (therefore a single level for the blue options) is:

\[
\begin{align*}
\{x_{2k}, x_{2k+1}\} & : (2k - 1) + (2k + 1) \text{ more [i.e., } 4k \text{ more]} \\
\{x_{2k-1}\} & : (2k - 4) + (2k + 1) \text{ more [i.e., } (4k - 3) \text{ more]} \\
\{x_{2k-3}, x_{2k-2}\} & : (2k - 7) + (2k + 1) \text{ more [i.e., } (4k - 6) \text{ more]} \\
\{x_{2k-5}, x_{2k-4}\} & : (2k - 11) + (2k + 1) \text{ more [i.e., } (4k - 10) \text{ more]} \\
& \vdots \\
\{x_3, x_4\} & : (5 - 2k) + (2k + 1) \text{ more [i.e., } 6 \text{ more]} \\
\{x_1, x_2\} & : (1 - 2k) + (2k + 1) \text{ more [i.e., } 2 \text{ more]} 
\end{align*}
\]

For the options \(\{x_{(2k+1)+1}, \ldots, x_{(2k+1)+2k+1}\}\), we follow the same steps as case 1 of sequence I until the part where we change the initial ranking of \(x_{(2k+1)+1} \succ x_{(2k+1)+2} \succ \cdots \succ x_{(2k+1)+2k+1}\) to:

\[
\begin{align*}
x_{(2k+1)+2k} \\
x_{(2k+1)+2k+1} \\
x_{(2k+1)+2k-2} \\
x_{(2k+1)+2k-1} \\
& \vdots \\
x_{(2k+1)+4} \\
x_{(2k+1)+5} \\
x_{(2k+1)+3} \\
x_{(2k+1)+1} \\
x_{(2k+1)+2}
\end{align*}
\]

And therefore, the options are split into \((k + 1)\) levels and their Borda score change is:
\[
\begin{align*}
\{x_{(2k+1)+2k}, x_{(2k+1)+2k+1}\} & : (2k - 1) \text{ more} \\
\{x_{(2k+1)+2k-2}, x_{(2k+1)+2k-1}\} & : (2k - 5) \text{ more} \\
\vdots \\
\{x_{(2k+1)+4}, x_{(2k+1)+5}\} & : (7 - 2k) \text{ more [i.e., (2k - 7) less]} \\
\{x_{(2k+1)+3}\} & : (4 - 2k) \text{ more [i.e., (2k - 4) less]} \\
\{x_{(2k+1)+1}, x_{(2k+1)+1}\} & : (1 - 2k) \text{ more [i.e., (2k - 1) less]}
\end{align*}
\]

We compare the earlier \((k+1)\)-level group and the above \((k+1)\)-level group.

Recall that we focus on \(k > 1\), so \((4k - 3) > (2k - 1)\), and therefore, the two levels \(\{x_{2k}, x_{2k+1}\}\) and \(\{x_{2k-1}\}\) are still the highest two levels among the total \(2(k+1)\) levels. Similarly, the two levels \(\{x_{(2k+1)+3}\}\) and \(\{x_{(2k+1)+1}, x_{(2k+1)+1}\}\) remain to be the lowest levels among the total \(2(k+1)\) levels. For the remaining options, the Borda score difference for black options is even while for blue the difference is odd, so they still remain to be \(2k\) levels. Thus, the obtained profile is for \((4, 2, 4, \ldots, 4, 2, 4)\) where there are \(2k\) (i.e., even number of) 4s.

**Case 2.** There are \(2k + 1\) (i.e., odd number of) occurrences of 4.

For options \(\{x_1, x_2, \ldots, x_{2k+3}\}\), we follow exactly the same steps as for case 2 of sequence II, where the Borda score difference between these options based on ranking at \(u\) and the blue options based on ranking at \(v\) (therefore a single level for the blue options) is:

\[
\begin{align*}
\{x_{2k+2}, x_{2k+3}\} & : (2k + 1) + (2k + 2) \text{ more [i.e., (4k + 3) more]} \\
\{x_{2k+1}\} & : (2k - 2) + (2k + 2) \text{ more [i.e., 4k more]} \\
\{x_{2k-1}, x_{2k}\} & : (2k - 5) + (2k + 2) \text{ more [i.e., (4k - 3) more]} \\
\vdots \\
\{x_3, x_4\} & : (3 - 2k) + (2k + 2) \text{ more [i.e., 5 more]} \\
\{x_1, x_2\} & : (-1 - 2k) + (2k + 2) \text{ more [i.e., 1 more]}
\end{align*}
\]

For options \(\{x_{(2k+3)+1}, x_{(2k+3)+2}, \ldots, x_{(2k+3)+2k+1}\}\) here, we follow the steps for options \(\{x_{(2k+1)+1}, \ldots, x_{(2k+1)+2k+1}\}\) in the above sequence IV, case 1. And therefore, the options are split into \((k+1)\) levels and their Borda score change is:

\[
\begin{align*}
\{x_{(2k+3)+2k}, x_{(2k+3)+2k+1}\} & : (2k - 1) \text{ more} \\
\{x_{(2k+3)+2k-2}, x_{(2k+3)+2k-1}\} & : (2k - 5) \text{ more} \\
\vdots \\
\{x_{(2k+3)+4}, x_{(2k+3)+5}\} & : (7 - 2k) \text{ more [i.e., (2k - 7) less]} \\
\{x_{(2k+3)+3}\} & : (4 - 2k) \text{ more [i.e., (2k - 4) less]} \\
\{x_{(2k+3)+1}, x_{(2k+3)+1}\} & : (1 - 2k) \text{ more [i.e., (2k - 1) less]}
\end{align*}
\]
Again we compare the two groups.

We focus here on \( k \geq 2 \). The cases where \( k = 0, 1 \) will be treated separately in the Appendix.

For \( k \geq 2 \), we have \( 4 - 2k < 1 \) and \( 4k > 2k - 1 \) (actually we even have \( 4k > 2k + 1 \) if we need to make the adjustments below). So \( \{x_{2k+2}, x_{2k+3}\} \) and \( \{x_{2k+1}\} \) remain to be the highest levels among the \( (2k + 3) \) levels. And \( \{x_{(2k+3)+3}\} \) and \( \{x_{(2k+3)+1}, x_{(2k+3)+1}\} \) remain to be the lowest levels among the \( (2k + 3) \) levels. If there exists a level among the \( (k + 1) \)-level group of blue options where the Borda score change for the options at the level from \( v \) to \( u \) is equal to 1 (if no such level exists, then we skip this step), we do the same adjustments as illustrated at the end of sequence I, case 2. The obtained profile is for \( (4, 2, 4, ..., 4, 2) \) where there are \( 2k + 1 \) (i.e., odd number of) 4s.

4. Remark.

A modification of the analysis above allows us to prove an extended version of the lemma:

For positive integer \( t \), each of the following (patterns of) orders (with exactly two levels equal to 2) is in the Borda range for \( n = 3 \) (and so for all odd \( n \geq 3 \)):

- \((2, 4t, 4t, ..., 4t, 4t, 2)\)
- \((4t, 2, 4t, 4t, ..., 4t, 4t, 2)\)
- \((2, 4t, 4t, ..., 4t, 4t, 2, 4t)\)
- \((4t, 2, 4t, 4t, ..., 4t, 4t, 2, 4t)\)

Appendix

We complete the proof of the Lemma by showing that the following sequences \((4, 2, 4, 2), (4, 2, 4), (4, 2, 2), (2, 2, 4) \) and \((4, 2, 4, 2, 4)\) are in the Borda range for \( n = 3 \).

The following profile shows that \((4, 2, 4, 2)\) is in the Borda range for \( n = 3 \), where the four levels consist of \( \{x_4, x_5, x_{10}, x_{11}\}, \{x_1, x_7\}, \{x_2, x_3, x_8, x_9\} \) and \( \{x_6, x_{12}\} \).
The next profile shows that (4, 2, 2, 2) is in the Borda range for $n = 3$, where the four levels consist of $\{x_2, x_3, x_8, x_9\}$, $\{x_6, x_{12}\}$, $\{x_1, x_7\}$ and $\{x_4, x_5, x_{10}, x_{11}\}$.

| 1 | 2 | 3 |
|---|---|---|
| $x_2$ | $x_8$ | $x_9$ |
| $x_3$ | $x_{12}$ | $x_7$ |
| $x_1$ | $x_{10}$ | $x_{11}$ |
| $x_6$ | $x_2$ | $x_3$ |
| $x_4$ | $x_6$ | $x_1$ |
| $x_5$ | $x_4$ | $x_5$ |
| $x_8$ | $x_{11}$ | $x_{12}$ |
| $x_9$ | $x_9$ | $x_{10}$ |
| $x_7$ | $x_7$ | $x_8$ |
| $x_{12}$ | $x_5$ | $x_6$ |
| $x_{10}$ | $x_3$ | $x_4$ |
| $x_{11}$ | $x_1$ | $x_2$ |

We move on to (4, 2, 2) and (2, 2, 4). Note that we only need to construct a profile for one of them and inverting everyone’s ranking yields the other. The following profile shows that (4, 2, 2) is in the Borda range for $n = 3$, where the three levels consist of $\{x_2, x_3, x_6, x_7\}$, $\{x_4, x_8\}$ and $\{x_1, x_5\}$.

| 1 | 2 | 3 |
|---|---|---|
| $x_2$ | $x_7$ | $x_6$ |
| $x_3$ | $x_5$ | $x_8$ |
| $x_4$ | $x_3$ | $x_2$ |
| $x_1$ | $x_1$ | $x_4$ |
| $x_6$ | $x_8$ | $x_7$ |
| $x_7$ | $x_6$ | $x_5$ |
| $x_8$ | $x_4$ | $x_3$ |
| $x_5$ | $x_2$ | $x_1$ |

The last profile shows that (4, 2, 4, 2, 4) is also in the Borda range for $n = 3$, where the five levels consist of $\{x_4, x_5, x_{12}, x_{13}\}$, $\{x_3, x_{11}\}$, $\{x_7, x_8, x_{15}, x_{16}\}$, $\{x_6, x_{14}\}$ and $\{x_1, x_2, x_9, x_{10}\}$.
| 1 | 2 | 3 |
|---|---|---|
| 4 | x_{12} | x_{13} |
| 5 | x_{10} | x_{11} |
| 3 | x_{16} | x_{9} |
| 7 | x_{14} | x_{15} |
| 8 | x_{4} | x_{5} |
| 6 | x_{2} | x_{3} |
| 1 | x_{8} | x_{1} |
| 2 | x_{6} | x_{7} |
| 12 | x_{15} | x_{16} |
| 13 | x_{13} | x_{14} |
| 11 | x_{11} | x_{12} |
| 15 | x_{9} | x_{10} |
| 16 | x_{7} | x_{8} |
| 14 | x_{5} | x_{6} |
| 9 | x_{3} | x_{4} |
| 10 | x_{1} | x_{2} |

References

[1] Kelly, JS and S Qi (2015) "The Construction of Orderings by Borda’s Rule".