DIMENSIONAL REDUCTION FOR GENERALIZED CONTINUUM POLYMERS

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Abstract. The Brydges-Imbrie dimensional reduction formula relates the pressure of a $d$-dimensional gas of hard spheres to a model of $(d+2)$-dimensional branched polymers. Brydges and Imbrie’s proof was non-constructive and relied on a supersymmetric localization lemma. The main result of this article is a constructive proof of a more general dimensional reduction formula that contains the Brydges–Imbrie formula as a special case. Central to the proof are invariance lemmas, which were first introduced by Kenyon and Winkler for branched polymers. The new dimensional reduction formulas rely on invariance lemmas for central hyperplane arrangements that are due to Mészáros and Postnikov.

Several applications are presented, notably dimensional reduction formulas for (i) non-spherical bodies and (ii) for corrections to the pressure due to symmetry effects.

Key words and phrases. Branched polymers, hard spheres, dimensional reduction, central hyperplane arrangements, combinatorial reciprocity, Mayer expansion.

1. Generalized dimensional reduction

1.1. Introduction. In 2003 Brydges and Imbrie discovered a remarkable dimensional reduction formula that equates the pressure of a gas of hard spheres in $\mathbb{R}^d$ with the volume of branched polymers in $\mathbb{R}^{d+2}$ [1]. See Figure 1 for an illustration of these models, and Section 1.2 for precise definitions. Their result has very interesting corollaries: it implies exact enumerative formulas for seemingly intractable high-dimensional integrals, and it relates the critical behaviour of $(d+2)$-dimensional branched polymers to the critical behaviour of the $d$-dimensional hard sphere model. For $d = 0, 1$ this is a very powerful reduction.

Brydges and Imbrie’s proof of dimensional reduction relied on a non-constructive supersymmetric localization lemma. Looking to understand the results of [1], Kenyon and Winkler studied branched polymers for $d = 2, 3$ by direct methods [2]. Their proofs are based on an invariance lemma. To describe this lemma, suppose the disk labelled $i$ in a planar branched polymer has radius $r_i$. The invariance lemma states that the total volume of planar branched polymers is unchanged as the radii $\{r_i\}$ are varied. Inspired by this result, Mészáros and Postnikov introduced a model of planar $\mathcal{H}$-polymers associated to any central hyperplane arrangement $\mathcal{H}$, and showed that these
planar polymers also satisfy invariance lemmas [3]. The main result of this article is a constructive proof of dimensional reduction formulas for all $d \geq 2$; invariance lemmas play a central role in the proof.

More precisely, this article establishes that (i) given a central essential complex hyperplane arrangement, there are dimensional reduction formulas from $2d + 2$ dimensions to $2d$ dimensions, and (ii) given a central essential real hyperplane arrangement, there are dimensional reduction formulas from $d + 2$ dimensions to $d$ dimensions. These results involve two main objects. The first is a generalization of branched polymers called $H$-polymers, which are an extension of the planar polymers in [3]. The second is a generalization of the pressure of the hard sphere gas: for any central hyperplane arrangement we define an analogue of the Mayer expansion, which is a power series representation of the pressure of the hard sphere gas. Sections 2.2 and 2.3 contain the precise definitions of $H$-polymers and the generalized pressure, and Section 1.4 contains a more precise statement of the theorem.

The remainder of this introductory section briefly describes new results and perspectives that follow from these generalized dimensional reduction formulas and their proof.

The proof presented in this article yields more information than the non-constructive proof in [1], and the following results are new even in the case of branched polymers. First, we obtain a precise description of the law of $d$-dimensional projections of $(d + 2)$-dimensional polymers, see Corollaries 13 and 14. This last corollary can be viewed as a generalization of [2, Theorem 7], which gave a description of 1-dimensional projections of 3-dimensional branched polymers. Secondly, our proof of dimensional reduction extends to some non-spherical bodies, see Theorem 16.
The notion of an $H$-polymer is combinatorially natural, but it does not immediately connect with statistical mechanics, where dimensional reduction formulas originated. In Section 4.2 we consider symmetric hard sphere gases and show how $H$-polymers naturally arise. Symmetric hard sphere gases are models in which, for example, the presence of a sphere at $x_i$ implies the presence of a sphere at $-x_i$. The bulk properties of these models coincide with the ordinary hard sphere gas, but there are corrections to the bulk behaviour due to the symmetry constraint. The corrections satisfy a dimensional reduction formula: they can be expressed in terms of $H$-polymers, where the hyperplane arrangement $H$ reflects the symmetry of the constraint. See Figure 2.

![Figure 2](image-url)

**Figure 2.** Left: a symmetric hard sphere gas configuration associated to the type $D_n$ Coxeter arrangement in $\mathbb{R}^2$. Right: a $H$-polymer associated to the type $D_n$ Coxeter arrangement in $\mathbb{R}^2$. The dotted gray lines indicate the coordinate axes. In each figure the disks labelled $i$ and $i'$ are located at $x_i$ and $-x_i$, respectively.

1.1.1. Related literature. Dimensional reduction formulas as discussed here first arose in the context of theoretical physics; see [1] for a discussion of this literature. From a combinatorial viewpoint dimensional reduction formulas have the flavour of combinatorial reciprocity [4]: the pressure of the hard sphere gas, which *a priori* makes sense only for $z > 0$, is being given an interpretation for $z < 0$.

The dimensional reduction formulas discussed in this article decrease the dimension by 2; Imbrie [5] has obtained a dimensional reduction formula relating directed branched polymers in $\mathbb{R}^{d+1}$ to the hard $\ell_1$-sphere gas in $\mathbb{R}^d$. The methods of this article can be adapted to give another proof of Imbrie’s result. Similar formulas relating directed objects in $d + 1$ dimensions to undirected objects in $d$ dimensions have arisen often in the context of random walk representations in statistical mechanics, see, e.g., [6, 7, 8, 9]. It is an open and interesting question to understand when dimensional reduction results are possible. In particular, are there formulas involving a reduction in dimension by more than two dimensions?
1.1.2. Structure of this article. The remainder of this introduction first provides some additional context by introducing the hard sphere gas, branched polymers, and the Brydges-Imbrie formula. The connection with the braid arrangement is described to indicate why hyperplane arrangements are involved. The introduction concludes with an informal statement of our main result in Section 1.4. A precise formulation and proof of the main result is given in Section 3 once the necessary definitions have been introduced in Section 2. Many of the definitions introduces are standard, but we have elected to include them for the ease of readers from non-combinatorial backgrounds. Applications of the main result are presented in Section 4.

1.1.3. Notation and conventions. Throughout the article the term branched polymer will refer to the model studied in \[1, 2\], i.e., the model that corresponds to the braid arrangement as outlined in Section 1.2. For other arrangements \(H\) we will always write \(H\)-polymers. \(\mathbb{N}\) will denote the positive integers. For a graph \(G = (V, E)\) edges \(\{i, j\} \in E\) will be abbreviated to \(ij\), and the notation \(ij \in G\) will indicate \(ij \in E(G)\). The integer \(d\) will be reserved for the dimension of a space, while \(n\) will count points in a configuration. Configurations are therefore finite point sets in \(\mathbb{R}^dn\). We write \(2^A\) for the set of all subsets of a set \(A\), and \(1_{\{A\}}\) for the indicator function of the set \(A\).

1.2. Branched polymers, the hard sphere gas, and the Brydges-Imbrie formula.

1.2.1. Branched polymers. Let \(T\) be a (labelled) spanning tree on \(K_n\), the complete graph on \([n] = \{1, 2, \ldots, n\}\). For integers \(d \geq 2\) a branched polymer of type \(T\) in \(\mathbb{R}^d\) is a configuration of \(n\) points \((x_1, \ldots, x_n)\), \(x_i \in \mathbb{R}^d\), such that

(i) If \(ij \in T\) then \(\|x_i - x_j\|_2 = 1\),
(ii) If \(ij \notin T\) then \(\|x_i - x_j\|_2 > 1\),

where \(\| \cdot \|_2\) is the Euclidean norm on \(\mathbb{R}^d\). A branched polymer on \([n]\) is a branched polymer of type \(T\) for some tree \(T\) spanning \(K_n\). Two branched polymers will be considered equivalent if one is a translation of the other, i.e., the space of branched polymers on \([n]\) is a subset of \(\mathbb{R}^{dn}/\mathbb{R}^d\). Geometrically, the \(x_i\) are the centers of spheres of radius \(\frac{1}{2}\), no two spheres have overlapping interiors, and the tree \(T\) determines the tangency graph of the spheres. This definition of branched polymers was first introduced in \([1]\). See Figure 1.

Define \(I^T_{BP}(x)\) to be 1 if the points \(x = (x_1, \ldots, x_n)\) form a branched polymer of type \(T\), and define \(I^T_{BP}(x)\) to be 0 otherwise. The volume of branched polymers of type \(T\) in \(\mathbb{R}^d\), volume of \(T\) for short, is given by

\[
Z^T(z) = \int_{(S^{d-1})^{n-1}} I^T_{BP}(x) \prod_{ij \in T} d\Omega^{d-1}(x_i - x_j),
\]
where $S^{d-1}$ is the sphere of radius 1 in $\mathbb{R}^d$ and $\Omega^{d-1}$ is the standard surface measure on $S^{d-1}$.

The partition function for branched polymers is defined by

\[ Z_{BP}^{(d)}(z) = \sum_{n \geq 1} \frac{z^n}{n!} \sum_{T \in \mathcal{T}[n]} Z_T(z), \]

where $\mathcal{T}[n]$ denotes the set of spanning trees on $K_n$, $z \in \mathbb{R}$ is the activity of the model, and the superscript $d$ indicates that it is the partition function of branched polymers in $\mathbb{R}^d$. For $|z|$ sufficiently small this is a convergent power series.

1.2.2. The hard sphere gas. The hard sphere gas in a finite region $\Lambda \subset \mathbb{R}^d$ is the model with partition function

\[ Z_{HC}^{\Lambda}(z) = \sum_{n \geq 0} \frac{z^n}{n!} \int_{\Lambda^n} I_{HC}(x) \prod_{i=1}^n dx_i, \]

where $z \in \mathbb{R}$ is the activity, $x = (x_1, \ldots, x_n) \in \mathbb{R}^{dn}$, and $I_{HC}(x) = \prod_{i \neq j} \mathbf{1}(\|x_i - x_j\| \geq 1)$. The hard-core constraint $I_{HC}(x)$ means that each point $x_i$ can be thought of as the center of a sphere of radius $\frac{1}{2}$, and that the interiors of the spheres are pairwise disjoint. $\mathbb{R}^0$ is considered to be a one-point space, so $Z_{HC}^{\Lambda}(z) = 1 + z$ when $d = 0$. Due to the hard-core constraint $Z_{HC}^{\Lambda}$ is a polynomial in $z$ for any finite region $\Lambda$.

1.2.3. The Brydges–Imbrie dimensional reduction formula. Aside from its intrinsic interest, the following theorem has many interesting consequences for statistical mechanics, see [1]. Note that the left-hand side involves the hard sphere gas in $\mathbb{R}^d$, while the right-hand side involves branched polymers in $\mathbb{R}^{d+2}$.

**Theorem 1** (Brydges-Imbrie [1]). For all $z$ such that the right-hand side converges absolutely,

\[ \lim_{\Lambda \nearrow \mathbb{R}^d} \frac{1}{|\Lambda|} \log Z_{HC}^{(d)}(z) = -2\pi Z_{BP}^{(d+2)}\left(-\frac{z}{2\pi}\right), \]

where the limit is omitted when $d = 0$; the superscript indicates the dimension in which the model is defined.

1.3. The connection with hyperplane arrangements. The following gives an alternate description of the space of branched polymers in $\mathbb{R}^d$ that establishes a connection with hyperplane arrangements. This connection was first described, for $d = 2$, in [3].

Recall that the braid arrangement $\mathcal{B}_n$ is the collection of hyperplanes $\mathcal{H} = \{H_{ij}\}_{1 \leq i < j \leq n}$, $H_{ij}$ the hyperplane defined by the linear functional $h_{ij}(x) = x_i - x_j = 0$. The bases of the braid arrangement can be identified with spanning trees in $\mathcal{T}[n]$. 
Proposition 2. Define
\[ Tan(ij) = \{ (x_1, \ldots, x_n) \mid \|h_{ij}(x)\|_2 = 1 \} \]
\[ Dis(ij) = \{ (x_1, \ldots, x_n) \mid \|h_{ij}(x)\|_2 > 1 \}, \]
where both \( Tan(ij) \) and \( Dis(ij) \) are subsets of \( \mathbb{R}^{nd} \). The space of \( d \)-dimensional branched polymers is the set \( P_{B_n}(d) = \prod_{T \in T[n]} P^T_{B_n}(d) \), where

\[
P^T_{B_n}(d) = \left( \bigcap_{ij \in T} Tan(ij) \cap \bigcap_{ij \notin T} Dis(ij) \right) / (1, 1, \ldots, 1) \mathbb{R}^d.
\]

Verifying this proposition is a matter of translation. We will see that viewing branched polymers from the perspective of hyperplane arrangements is fruitful.

1.4. Informal description of the main result. As mentioned in Section 1.1, there are two natural statistical mechanical objects associated to any central hyperplane arrangement. The first is a space of a \( d \)-dimensional \( H \)-polymers for \( d \geq 2 \), denoted \( P_H(d) \), on which there is a natural volume measure \( \text{vol} \). The second is the \( d \)-dimensional pressure \( p_H(d) \) of an arrangement \( H \) for \( d \geq 0 \). For precise descriptions of these objects see Sections 2.2 and 2.3 respectively.

To establish an analogue of Theorem 1 requires a sequence \( \vec{H} = (H_n)_{n \in \mathbb{N}} \) of central hyperplane arrangements, where the arrangement \( H_n \) is in \( \mathbb{R}^n \). Define the \( H \)-polymer partition function to be

\[
Z_H^{(d)}(z) = \sum_{n \geq 1} \frac{z^n}{n!} \text{vol}(P_{H_n}(d)),
\]
and the pressure of \( \vec{H} \) to be

\[
p_H^{(d)}(z) = \sum_{n \geq 1} \frac{z^n}{n!} p_{H_n}.
\]

Recall that a hyperplane arrangement is called essential if a maximal linearly independent set of normals forms a basis for the vector space the arrangement lives in.

Theorem 3. Let \( \vec{H} = (H_n)_{n \geq 1} \) be a sequence of central essential real hyperplane arrangements, \( H_n \) an arrangement in \( \mathbb{R}^n \), and let \( d \) be a non-negative integer. As formal power series in \( z \),

\[
p_H^{(d)}(z) = Z_H^{d+2}(-\frac{z}{2\pi}).
\]

The restriction to real hyperplane arrangements in Theorem 3 is not needed; a similar statement holds for complex arrangements provided \( d \) is even. See Section 3 for the precise statement.
Remark. Theorem 1 is the special case of Theorem 3 when \( H_n = B_{n+1} \), the braid arrangement in \( \mathbb{R}^{n+1}/\mathbb{R} \). Details of this specialization are presented in Section 4.1.

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2. Central hyperplane arrangements and associated objects

A finite hyperplane arrangement \( \mathcal{H} \) is a finite collection of hyperplanes in \( K^n \) for \( K \) a field and \( n \in \mathbb{N} \). We will only be interested in \( K = \mathbb{C} \) or \( K = \mathbb{R} \). An arrangement \( \mathcal{H} \) is central if each hyperplane in \( \mathcal{H} \) contains the origin. An arrangement is essential if the normals of the arrangement span \( K^n \). The remainder of this article will only involve central arrangements, so the terms arrangement and central arrangement will be used synonymously.

An arrangement \( \mathcal{H} \) can be identified with a set \( \{h_e\}_{e \in \mathcal{H}}, h_e \in (K^n)^* \) the linear functional that defines the hyperplane \( e \). Concretely,

\[
(2.1) \quad h_e(x) = \sum_{i=1}^{n} a_i x_i = 0, \quad x = (x_1, \ldots, x_n) \in K^n
\]

is the equation defining the hyperplane \( e \), where \( a_i \in K \) for \( i \in [n] \). The remainder of this section introduces the objects and properties associated to hyperplane arrangements relevant for dimensional reduction formulas.

2.1. Matroids and hyperplane arrangements.

2.1.1. Matroids. This section defines matroids and recalls some needed facts. A general introduction to matroids can be found in either of \([10, 11]\). Readers familiar with matroids may wish to skip to Section 2.1.3 where the matroids that play a role in this article are introduced.

Definition 4. Let \( E \) be a finite set. A matroid \( M = (E, \mathcal{I}) \) with ground set \( E \) is a non-empty collection of subsets \( \mathcal{I} \subset 2^E \), the independent sets of \( M \), such that

(i) if \( A \in \mathcal{I} \) and \( B \subset A \), then \( B \in \mathcal{I} \),
(ii) if \( A, B \in \mathcal{I} \), \( |A| > |B| \), then there is an \( a \in A \setminus B \) such that \( B \cup \{a\} \in \mathcal{I} \).

A base of a matroid is a maximal independent set. A subset \( S \subset E \) that is not independent is called dependent.

Example. A fundamental example of a matroid is when \( E \) is a finite collection of vectors in a vector space \( V \). Independent sets \( A \subset E \) are collections of linearly independent vectors.
A dependent set $S$ such that $S \setminus \{a\}$ is independent for all $a \in S$ is called a circuit. The following elementary fact about circuits will be needed, see, e.g. [10, Corollary 1.2.6] for a proof.

**Proposition 5.** Given a base $B \subset E$ and an edge $e \in E \setminus B$ there is a unique circuit in $B \cup \{e\}$.

The circuit identified in Proposition 5 is known as the fundamental circuit of $e$ with respect to $B$.

**Example.** A second fundamental example of a matroid is when $E$ is the edge set of a connected graph $G$. The independent sets are cycle-free subgraphs, and the bases are spanning trees. Circuits correspond to cycles in $G$, and given a spanning tree $T$ the fundamental circuit of an edge $e \notin T$ is the unique cycle in the graph $T \cup \{e\}$.

Let $B(M)$ denote the set of bases of a matroid $M = (E, \mathcal{I})$. If $A, B \in B(M)$ the definition of a matroid implies $A$ and $B$ have the same cardinality. The cardinality of a base is called the rank of $M$, denoted $r(M)$. The rank of a matroid extends to a function $r : 2^E \to \mathbb{N}$, $S \mapsto r(S)$, as follows. For $S \subset E$ define a matroid $M_S = (S, \mathcal{I}_S)$ where $\mathcal{I}_S = \{A \in \mathcal{I} \mid A \subset S\}$, and define $r(S) \equiv r(M_S)$. A subset $S \subset E$ is said to be spanning if $r(S) = r(M)$.

### 2.1.2. The characteristic polynomial of a matroid

Let $r$ be the rank function of a matroid $M$ with ground set $E$. The characteristic polynomial $\chi_M$ of $M$ is defined to be

$$\chi_M(t) = \sum_{S \subset E} (-1)^{|S|} t^{r(M) - r(S)}.$$  

(2.2)

In what follows the relevant evaluation of $\chi_M$ will be at $t = 0$, so only subsets of full rank contribute to (2.2):

$$\chi_M(0) = \sum_{S \subset E} (-1)^{|S|} \mathbb{1}_{\{r(S) = r(M)\}}.$$  

(2.3)

An alternative representation of the characteristic polynomial in terms of bases will be useful. Fix a linear order $<$ on the elements of the ground set $E$. Let $S \in B(M)$. An element $e \in M$ is called externally active for $S$ if $e \notin S$ and $e$ is the minimal element in its fundamental circuit. Following [2] say $S$ is $<$ safe if $S$ has no externally active elements according to the order $<$. The following formula for $\chi_M(0)$ is a specialization of a well-known (see, e.g., [12]) activity representation of the Tutte polynomial, which contains the characteristic polynomial as a special case.

$$\chi_M(0) = \sum_{S \in B(M)} \mathbb{1}_{\{S \text{ is } < \text{ safe}\}}.$$  

(2.4)
2.1.3. A matroid associated to a hyperplane arrangement. As previously described, hyperplane arrangements can be represented by the set of normals to the hyperplanes. This gives rise to a matroid $M_H$ associated to the arrangement $H$: the ground set $E = E(M_H)$ of the matroid is the set of hyperplanes, and the independent sets are the subsets of hyperplanes whose normals are linearly independent.

2.1.4. Convenient identifications and conventions. For simplicity, subsets of hyperplanes in an arrangement $H$ will be conflated with subsets of the ground set $E(M_H)$ of the associated matroid. Accordingly, we will use matroid terminology for subsets of hyperplanes, e.g., we will refer to a base of a hyperplane arrangement. We will define the characteristic polynomial of a hyperplane arrangement to be the characteristic polynomial of the associated matroid.

2.2. Polymers associated to a hyperplane arrangements. Throughout this section, fix an essential central hyperplane arrangement $H$ in $\mathbb{C}^n$, and choose a positive real number $R_e > 0$ for each $e \in H$. The construction in this section is an extension of a construction in [3] from $d = 2$ to $d \geq 2$. Recall that elements $e \in H$ can be identified with their defining linear functionals $h_e : \mathbb{C}^n \to \mathbb{C}$.

2.2.1. Construction of $H$-polymers. We first define $H$-polymers in $\mathbb{R}^{2d} \cong \mathbb{C}^d$. Given $e \in H$, $x \in \mathbb{C}^n$, define $h_e(x) \in \mathbb{C}^d$ as in (2.1):

$$h_e(x) = \sum_{i=1}^n a_i x_i,$$

where $a_i x_i$ is the usual pointwise complex-scalar multiplication of a vector $x_i \in \mathbb{C}^d$. In the next definition notice that polymers are defined without modding out by translations, in contrast to what was done for branched polymers in Section 1.2.1.

**Definition 6.** Define

$$\text{Tan}(e) = \{x \in \mathbb{C}^n \mid \|h_e(x)\|_2 = R_e\},$$

$$\text{Dis}(e) = \{x \in \mathbb{C}^n \mid \|h_e(x)\|_2 > R_e\}.$$  

The space of $2d$-dimensional $H$-polymers with radii $\{R_e\}_{e \in H}$ is the set $P_H(2d) \subset \mathbb{C}^{dn}$ defined by

$$P_H(2d) = \bigcap_{S \in \mathcal{B}(M_H)} P^S_H(2d),$$

$$P^S_H(2d) = \bigcap_{e \in S} \text{Tan}(e) \cap \bigcap_{e \notin S} \text{Dis}(e).$$

There is a natural probability measure on the space $P_H(2d)$, defined as follows. Injectively map $P^S_H(2d)$ into $(S^{(2d-1)})^n$ by $x \mapsto \phi(x) = (\phi_e(x))_{e \in S}$, where $\phi_e(x)$ is the unit vector in the direction $h_e(x)$. The fact that this is
an injection on $P^S_{\mathcal{H}}$ follows from the hypothesis that $S$ is a base. With this map in hand define

$$\text{(2.7)} \quad \text{vol}(P^S_{\mathcal{H}}(2d)) = \text{vol}(\{\phi(x) \mid x \in P^S_{\mathcal{H}}(2d)\}),$$

where the volume measure on the right-hand side of Equation (2.7) is the $n$-fold product of surface measure on $S^{(2d-1)}$. As each of the sets $P^S_{\mathcal{H}}(2d)$ are disjoint, the volume of $P^S_{\mathcal{H}}(2d)$ is the sum of the volumes of each $P^S_{\mathcal{H}}(2d)$. After normalization, this gives a probability measure on $P_{\mathcal{H}}(2d)$. Note that this measure agrees with the measure on branched polymers given in Section 1.2. In general an explicit integral formula for this measure is given by replacing trees in Equation (1.1) with bases $S$ of $M_{\mathcal{H}}$, and the function $I_{BP}$ with its analogue $I^T_{\mathcal{H}}$. The law of $\mathcal{H}$-polymers will be thought of as the law of the locations of the points $x_i \in \mathbb{C}^d$.

To construct $\mathcal{H}$-polymers in $d$ dimensions for $d$ odd a further condition is needed. An arrangement is called complexified if each hyperplane $e \in \mathcal{H}$ is defined by a real linear functional, i.e., $a_i \in \mathbb{R}$ in (2.1). In this case Definition 6 also defines branched polymers in $\mathbb{R}^d$ by replacing $\mathbb{C}^d$ with its analogue $I^S_{\mathcal{H}}$. The law of $\mathcal{H}$-polymers will be thought of as the law of the locations of the points $x_i \in \mathbb{C}^d$.

Remark. In what follows formulas will be written assuming a complexified arrangement $\mathcal{H}$. For non-complexified arrangements the formulas continue to hold if $\mathbb{R}^d$ is replaced by $\mathbb{C}^d$.

Proposition 7. The law of $\mathcal{H}$-polymers is invariant under rotations of $\mathbb{R}^d$.

Proof. The space of $\mathcal{H}$-polymers is rotationally invariant, and the measure on the space is rotationally invariant. \qed

The right-hand side of Theorem 3 can now be made precise.

Definition 8. Let $\mathcal{H} = (\mathcal{H}_n)_{n \geq 1}$ be a sequence of central essential hyperplane arrangements. The partition function of the sequence $\mathcal{H}$ in $\mathbb{R}^d$ is

$$Z_{\mathcal{H}}^{(d)}(z) = \sum_{n \geq 1} \frac{z^n}{n!} \text{vol}(P_{\mathcal{H}_n}(d)),$$

which is to be interpreted as a formal power series in $z$ if convergence is not known.

2.2.2. Geometric interpretation of the space of $\mathcal{H}$-polymers. There is a geometric interpretation of $\mathcal{H}$-polymers extending that given in [3]. To each $e \in S$ there is an associated subspace $\{x \mid h_e(x) = 0\} \subset \mathbb{R}^{dn}$, and the set of vectors $x$ such that $\|h_e(x)\|_2 < R_e$ is the set of vectors $v$ that are at distance less than $R_e$ from $\{x \mid h_e(x) = 0\}$. Call $\{x \mid \|h_e(x)\|_2 < R_e\}$ the $R_e$-thickening of this subspace. The space of $\mathcal{H}$-polymers is a union of regions corresponding to bases $S$; the region associated to $S$ is the boundary of the intersection of (i) the $R_e$-thickenings of the subspaces associated to $e \in S$ and (ii) the complements of the $R_e$-thickenings of the subspaces associated to $e \notin S$. 
2.2.3. The Mészáros-Postnikov invariance lemma. $\mathcal{H}$-polymers in $\mathbb{R}^2$, i.e., two real dimensions, will be called planar $\mathcal{H}$-polymers. Planar $\mathcal{H}$-polymers possess an amazing property: the volume of the space of planar $\mathcal{H}$-polymers is independent of the radii $R_e$. The following theorem is a special case of [3, Theorem 1].

**Theorem 9** (Mészáros-Postnikov [3]). Let $\mathcal{H}$ be an essential central arrangement in $\mathbb{C}^n$. The volume of the space of planar $\mathcal{H}$-polymers with radii $\{R_e\}_{e \in \mathcal{H}}$ is

$$\text{vol}(P_{\mathcal{H}}^S(2)) = (-2\pi)^n \chi_{\mathcal{H}}(0),$$

where $\chi_{\mathcal{H}}$ is the characteristic polynomial of the arrangement $\mathcal{H}$.

For $\mathcal{H}$ the braid arrangement $\mathcal{B}_n$, Theorem 9 was first proven in [2].

2.3. **Mayer coefficients and their generalizations.** The dimensional reduction formula, Theorem 1, relates the pressure of a hard sphere gas in $d$ dimensions to the partition function of branched polymers in $d + 2$ dimensions. The Mayer expansion for the pressure, which is recalled in Section 4.1 below, gives a relationship between the Mayer coefficients of the hard sphere gas and branched polymers. This section defines a generalization of Mayer coefficients that are associated to a central essential hyperplane arrangement $\mathcal{H}$.

2.3.1. **Matroidal Mayer coefficients.** Let $H \subset E(M_\mathcal{H})$ be a spanning set of the matroid $M_\mathcal{H}$, and assume $\mathcal{H}$ is complexified. The $d$-dimensional matroidal Mayer coefficient (MMC) associated to $H$ is the number

$$\int_{\mathbb{R}^d} \prod_{e \in H} (-1^{\|h_e(x)\|_2 \leq R_e}) \, dx.$$

Note that this is a finite number: because $H$ is a spanning set each $x_i$ is constrained to lie in a bounded subset of $\mathbb{R}^d$. For non-complexified arrangements the definition of the matroidal Mayer coefficients is the same, with $\mathbb{R}^d$ replaced by $\mathbb{C}^d$ in (2.10). When $d = 0$ the convention that $\mathbb{R}^0$ is a one-point space means (2.10) is equal to $(-1)^{|H|}$. The left-hand side of Theorem 3 can now be made precise:

**Definition 10.** Let $\vec{H} = (\mathcal{H}_n)_{n \geq 1}$ be a sequence of central essential hyperplane arrangements. The pressure of the sequence $\vec{H}$ in $\mathbb{R}^d$ is

$$p^{(d)}_{\vec{H}}(z) = \sum_{n \geq 1} \frac{z^n}{n!} \sum_{H \subset E(M_{\mathcal{H}_n}) \atop r(H) = n} \int_{\mathbb{R}^d} \prod_{e \in H} (-1^{\|h_e(x)\|_2 \leq R_e}) \, dx,$$

which is to be interpreted as a formal power series if convergence is not known.
2.3.2. Geometric interpretation. Following Section 2.2.2 there is a natural geometric interpretation of the \(d\)-dimensional MMC associated to a subset \(H\). To each hyperplane \(e\) is associated the \(R^e\)-thickening of the subspace \(\{x \mid h_e(x) = 0\}\) in \(\mathbb{R}^d\). The MMC associated to a spanning set \(H\) is the (signed) volume of the intersection of the thickened subspaces corresponding to \(e \in H\).

3. Dimensional reduction formulas for \(\mathcal{H}\)-polymers

3.1. Proof of dimensional reduction formulas. In this section we prove a dimensional reduction formula for \(\mathcal{H}\)-polymers when \(\mathcal{H}\) is an essential arrangement. We standardize to \(R_e = 1\) for each \(e \in \mathcal{H}\).

Let \(\Omega_{d-1}^{a}\) denote the surface measure on the sphere of radius \(\sqrt{a}\) in \(\mathbb{R}^d\), and let \(\lambda_d^B\) denote Lebesgue measure on the unit ball in \(\mathbb{R}^d\). The next lemma, which states that a codimension 2 projection of the surface measure on the unit sphere in \(\mathbb{R}^{d+2}\) is the uniform measure on the unit ball in \(\mathbb{R}^d\), is a well-known calculation, and the proof is omitted.

**Lemma 11** (Generalized Archimedes’s theorem). Let \((w, y) \in S_{d+1} \subset \mathbb{R}^{d+2}, w \in \mathbb{R}^2, y \in \mathbb{R}^d\). Then

\[
d\Omega_{d+1}^{w}(w, y) = d\lambda_{d}^y(y)d\Omega_{1-\|y\|^2}(w).
\]

**Theorem 12.** Let \(\mathcal{H}\) be an essential central complexified hyperplane arrangement in \(\mathbb{C}^n\). Then

\[
\int_{\mathbb{R}^d} \sum_{H \subset E(M_{\mathcal{H}}) \in H} \prod_{r(H) = n} -1(\|h_e(x)\|_2 \leq 1) \ dx = (-2\pi)^n \text{vol}(P_{\mathcal{H}}(d + 2)).
\]

If \(\mathcal{H}\) is not complexified then (3.2) still holds provided \(d\) is even and \(\mathbb{R}^{2dn}\) is identified with \(\mathbb{C}^{dn}\).

**Proof.** It will be assumed \(\mathcal{H}\) is a complexified arrangement; mutatis mutandis the argument applies for arrangements that are not complexified. The proof manipulates each side of Equation (3.2) separately and observes that the resulting expressions are the same.

First, rewrite the left-hand side of (3.2) by subdividing the region of integration according to whether or not \(\|h_e(x)\|_2 \leq 1\). Formally,

\[
\mathbb{R}^d = \prod_{G \subset E(M_{\mathcal{H}})} \Gamma_G,
\]

\[
\Gamma_G = \{x \in \mathbb{R}^d \mid \|h_e(x)\|_2 \leq 1\ \text{if and only if} \ e \in G\},
\]

with the disjoint union in (3.3) being over all subsets of the ground set of \(M_{\mathcal{H}}\).

Each summand \(H\) in the left-hand side of Equation (3.2) is a spanning set. The integral over any region \(\Gamma_G\) is zero when \(G\) is not a spanning set; in this case there is a normal to a hyperplane \(e \in H\) with \(e \notin G\), and the
The first equality follows as if $H$ is a subset of $G$ the integral vanishes. The left-hand side of Equation (3.2) can therefore be rewritten as:

$$
\sum_{G \subseteq E(M_H)} \sum_{H \subseteq E(M_H)} \int_{\Gamma_G} \prod_{e \in H} -1 \{\|h_e(x)\|_2 \leq 1\} \, dx
$$

(3.5) $\sum_{G \subseteq E(M_H)} \sum_{H \subseteq E(M_H)} \int_{\Gamma_G} (-1)^{|H|} \, dx$

(3.6) $\sum_{G \subseteq E(M_H)} \prod_{e \in H} \int_{\Gamma_G} \chi_G(0).$

(3.7) $\prod_{e \in H}$

The first equality follows as if $H$ is not a subset of $G$ the integral vanishes. The second equality follows from (2.3). This concludes the manipulations of the left-hand side of Equation (3.2).

The second step is to perform the integrals in the right-hand side of Equation (3.2). The idea of how to do this is simple. The volume is computed by first fixing the last $d$ coordinates of the points $x_i$ in a $\mathcal{H}$-polymer, and then integrating over the first two coordinates of each point. The integral over the first two coordinates can be expressed as the volume of a generalized planar polymer, and hence can be computed explicitly by an invariance lemma. Lastly we integrate over the last $d$ coordinates. This idea is similar to what was done for branched polymers in three dimensions in [2].

As in Lemma 11, it will be convenient to write $x \in P_\mathcal{H}(d + 2)$ as $x = (w, y)$, where $w = (w_1, w_2, \ldots, w_n) \in \mathbb{R}^{2n}$, $y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^{dn}$, and $x_i = (w_i, y_i) \in \mathbb{R}^{d+2}$. To each point $x \in P_\mathcal{H}(d + 2)$ associate a vector $(R_e^*(y))_{e \in \mathcal{H}}$ of non-negative radii:

$$
R_e^*(y)^2 = (1 - \|h_e(y)\|_2^2) \wedge 0,
$$

where $a \wedge b$ denotes the minimum of $a$ and $b$. As the $h_e$ are linear functionals, $\|h_e(w, y)\|_2^2 = \|h_e(w)\|_2^2 + \|h_e(y)\|_2^2$. It follows that if $S$ is a base of $\mathcal{H}$ and $x \in P_\mathcal{H}^S(d + 2)$ then

$$
\|h_e(w)\|_2^2 = R_e^*(y), \quad e \in S,
$$

(3.9) $\|h_e(w)\|_2^2 > R_e^*(y)^2, \quad e \notin S,

(3.10)$

since $\|h_e(w, y)\|_2^2$ equals 1 for $e \in S$, and is greater than 1 for $e \notin S$. Lemma 11 gives a concrete expression for $\text{vol}(P_\mathcal{H}^S)$ when $S$ is a base. Letting $d\Omega_e(w)$ denote $d\Omega_{R_e^*(y)}(h_e(w))$ and $d\lambda_e(y)$ denote $d\lambda_B^d(h_e(y))$, the expression is

$$
\int_{P_\mathcal{H}^S} \prod_{e \in S} d\Omega_{R_e^*(y)}^{d+1}(\phi_e(x)) = \int_{\mathbb{R}^{(d+2)n}} \prod_{e \in \mathcal{H} \setminus S} 1 {\{\|h_e(w, y)\|_2^2 \geq 1\}} \prod_{e \in S} d\Omega_e(w) d\lambda_e(y).
$$
As before it is helpful to decompose the region of integration:

\[
\mathbb{R}^{(d+2)n} = \bigcap_{G \subset E(M_H)} \mathbb{R}^{2n} \times \Gamma_G,
\]

where \(\Gamma_G\) is the subset of \(y\)-coordinates defined as in Equation (3.4). In the rest of the proof we abbreviate \(G \subset E(M_H)\) to \(G \subset H\). An argument similar to the one leading to (3.5) shows that the integral over a region \(\mathbb{R}^{2n} \times \Gamma_G\) can be non-zero only if \(S \subset G\), and hence the volume is given by

\[
\sum_{G: S \subset G} \int_{\mathbb{R}^{2n} \times \Gamma_G} \prod_{e \in H \setminus S} \mathbf{1}\{\|h_e(w)\|_2 > R_y^*(y)^2\} \prod_{e \in S} d\Omega_e(w)d\lambda_e(y).
\]

Fix \(G\) containing \(S\), so \(G\) is rank \(n\). If \(e \notin G\) then \(y \in \Gamma_G\) implies \(R_y^*(y) = 0\). The non-trivial constraints in (3.12) on \(w\) therefore correspond to hyperplanes \(e \in G\). Letting \(H_G\) denote the hyperplane arrangement consisting of hyperplanes in \(G\) this implies the first product in (3.12) can be restricted to \(e \in H_G \setminus S\). Summing (3.12) over all bases \(S\) to compute \(\text{vol}(P_H(d+2))\) results in

\[
\sum_{G \subset H} \sum_{S \in B(M_{H_G})} \int_{\mathbb{R}^{2n} \times \Gamma_G} \prod_{e \in H \setminus S} \mathbf{1}\{\|h_e(w)\|_2 > R_y^*(y)^2\} \prod_{e \in S} d\Omega_e(w)d\lambda_e(y).
\]

The sum over \(S \in B(M_{H_G})\) of the integrals over \(w\) in (3.13) are, for any fixed \(y\), precisely the volume of planar \(H_G\) polymers with radii \(R_y^*(y)\) for \(e \in H_G\). By Theorem 9

\[
\text{vol}(P_H(d+2)) = \sum_{G \subset H} \int_{\Gamma_G} (-2\pi)^n \chi_G(0) \prod_{e \in S} d\lambda_B^d(h_e(y))
\]

\[
= \sum_{G \subset M_H} (-2\pi)^n \chi_G(0) \text{vol}(\Gamma_G),
\]

which is exactly (3.7). \(\square\)

**Proof of Theorem 3.** Theorem 3 has been made precise by Definitions 8 and 10. To prove the theorem, apply Theorem 12 to each arrangement \(\mathcal{H}_n\) in the sequence \((\mathcal{H}_n)_{n \in \mathbb{N}}\), multiply each term by \(\frac{n^n}{n^n!}\), and sum over \(n\). \(\square\)

### 3.2. Laws of projections

The proof of Theorem 12 established more than was stated. The entire proof can be conducted without computing the integrals over the last \(d\) coordinates, i.e., with \(y\) fixed. This implies the law of a \(d\)-dimensional projection of a \((d+2)\)-dimensional \(\mathcal{H}\)-polymer is given by the law of the MMC coefficients. Formally,

**Corollary 13.** Let \(g\) be a function of the last \(d\) coordinates in \(\mathbb{R}^{d+2}\), integrable with respect to the law of \(\mathcal{H}\)-polymers, where \(\mathcal{H}\) is an essential central...
complexified arrangement of rank \( n \). Then

\[
\int_{P_H(d+2)} g(y) \, dv(w, y) = (-2\pi)^n \sum_{G \subseteq E(M_H)} \int_{\mathbb{R}^{dn}} g(y) \prod_{e \in G} 1_{\{|h_e(y)| \leq 1\}} \, dy.
\]

The previous corollary is unnatural since the MMC are distributed according to a signed measure. This can be improved. Given a set \( E \), let \( E^< \) denote the set of linear orders on \( E \). We call a function \( f : \mathbb{R}^d \to E^< \) an ordering function.

**Corollary 14.** Let \( g \) be a function of the last \( d \) coordinates in \( \mathbb{R}^{d+2} \), integrable with respect to the law of \( \mathcal{H} \)-polymers, where \( \mathcal{H} \) is an essential central complexified arrangement of rank \( n \). Let \( f \) be an ordering function.

\[
(3.16) \quad \int_{P_H(d+2)} g(y) \, dv(w, y) = (-2\pi)^n \sum_{S \in B(M_H)} \int_{\bigcup G} g(y) 1_{\{S \text{ is } f(y) \text{ safe}\}} \, dy,
\]

where the union in the region of integration is over all \( G \subset H \) such that \( r(G) = n \).

**Proof.** Equation (2.4) implies that for any ordering function \( f \) and any \( y \)

\[
(3.17) \quad \chi_M(0) = \sum_{S \in B(M)} 1_{\{S \text{ is } f(y) \text{ safe}\}}.
\]

Inserting this expression into Equation (3.7) gives the corollary, as it is a rewriting of Corollary 13. \( \square \)

**Remark.** For branched polymers in \( d = 3 \) Corollary 14 was established for a particular ordering function \( f \) in [2].

**Remark.** There is a notion of embedding activity for embedded graphs \( G \) due to Bernardi [13], who has shown that (2.4) holds when external activity is replaced with external embedding activity. Bernardi’s result, together with an argument as in Corollary 14, yields an explicit probability law for the 2d-projection of 4d-branched polymers.

### 3.3. Non-spherical bodies

Theorem 12 did not make essential use of the fact that the measure on \( \mathcal{H} \)-polymers was induced from the surface measure on unit spheres. The key ingredient was only that the surface measure factorized into a product of the surface measure on \( S^1 \) and Lebesgue measure on the codimension 2 projection. The next definition introduces a class of non-spherical objects for which the proof of Theorem 12 applies. The definition is a specialization of more general concepts introduced in [14], which studies when generalizations of Lemma 11 hold.

**Definition 15.** A spherical array in \( \mathbb{R}^d \) is a hypersurface \( \mathcal{A} = S^1 \times \omega B(A) \), where

\[
(3.18) \quad S^1 \times \omega B(A) = \{ x \in \mathbb{R}^d | x_1^2 + x_2^2 = [\omega(x_3, \ldots, x_d)]^2 \}.
\]
The function $\omega: \mathbb{R}^{d-2} \to [0, \infty)$ is the warping function and $\mathcal{B}(\mathcal{A})$ is the bottom of $\mathcal{A}$. Let $\pi_{d-2}$ denote the orthogonal projection from $\mathbb{R}^2 \times \mathbb{R}^{d-2} \to \mathbb{R}^{d-2}$ defined by $\pi_{d-2}(w, y) = y$. A spherical array is an Archimedean spherical array (ASA) if for all measurable $U \subset \mathcal{B}(\mathcal{A})$

\begin{equation}
\Omega(\pi_{d-2}^{-1}(U)) = \text{vol}(U),
\end{equation}
where $\Omega$ is the surface measure on $\mathcal{A}$ induced from $\mathbb{R}^d$, and $\text{vol}$ is Lebesgue measure on $\mathbb{R}^{d-2}$.

The warping function of an ASA must be rather special, see [14]. Several ASAs are well-known.

**Example.** Spheres $S^{d-1}$ are ASAs, with bottom the unit ball $B^{d-2}$ in $\mathbb{R}^{d-2}$ and warping function $\sqrt{1 - x^2_{d-1} - x^2_d}$. This is the content of Lemma 11.

**Example.** Cylinders $S^{d-2} \times I$ with $I$ an interval in $\mathbb{R}$ are ASAs with base $B^{d-3} \times I$ the solid cylinder in $\mathbb{R}^{d-2}$. This follows from writing $S^{d-2}$ as a warped product $S^1 \times_\omega B^{d-3}$ as in the previous example, and noting this gives a warped product $S^1 \times_\omega (B^{d-3} \times I)$, where the warping function is independent of the coordinate in $I$.

**Example.** Spherically capped cylinders, i.e., the boundary of $B^{d-1} \times I$, are ASAs. This follows by combining the last two examples.

Dimensional reduction formulas for ASAs require defining the associated spaces of polymers and Mayer coefficients. The remainder of this section indicates these definitions, with the conclusion being the next theorem. Once the definitions are given the proof is, mutatis mutandis, the same as the proof of Theorem 12, and hence it is omitted.

**Theorem 16.** Theorem 12 holds for Archimedean spherical arrays.

**Remark.** In the case of the braid arrangement and open cylinders, Theorem 16 follows by the methods of [1].

Let $\mathcal{H} = \{h_e\}_{e \in E}$ be a hyperplane arrangement in $\mathbb{R}^n$. Associate to each $e \in E$ an ASA $\mathcal{A}_e$; by a slight abuse of notation write $\mathcal{A}$ for this set of ASAs. Define subsets of $\mathbb{R}^{dn} = \{(w, y) | w \in \mathbb{R}^2n, y \in \mathbb{R}^{(d-2)n}\}$ by

\begin{align*}
\text{Tan}(e) &= \{(w, y) | h_e(w, y) \in \mathcal{A}_e\}, \\
\text{Dis}(e) &= \{(w, y) | h_e(y) \in \mathcal{B}(\mathcal{A}_e) \text{ if and only if } \|h_e(w)\|_2^2 > [\omega(h_e(y))]^2\},
\end{align*}
where the ASAs $\mathcal{A}_e$ are left implicit in the notation. The space of $\mathcal{H}$-polymers of type $\mathcal{A}$, denoted $\mathcal{P}_{\mathcal{H},\mathcal{A}}$, is defined by

\begin{align*}
\mathcal{P}_{\mathcal{H},\mathcal{A}} &= \prod_{S \in \mathcal{B}(\mathcal{H})} \mathcal{P}_{S,\mathcal{A}} \\
\mathcal{P}_{S,\mathcal{A}} &= \bigcap_{e \in S} \text{Tan}(e) \cap \bigcap_{e \notin S} \text{Dis}(e).
\end{align*}
\(H\)-polymers as introduced in Section 2.2 are the special case of \(H\)-polymers of type \(A\) when \(A_{e} = S^{d-1}\) for all \(e \in E\).

There is a natural measure \(\Omega_{A}\) on \(H\)-polymers of type \(A\) induced by the surface measures on the hypersurfaces \(A_{e}\). Define the measure \(\Omega_{A}\) on \(\mathcal{P}_{S,A}\) to be the pushforward of the product measure on the ASAs \(\{A_{e}\}_{e \in S}\) under the map \(x \mapsto (h_{e}(x))_{e \in S}\), i.e.,

\[
(3.22) \quad d\Omega_{A}(w, y) = \prod_{e \in S} d\Omega_{A_{e}}(h_{e}(w, y)).
\]

Normalizing this measure gives a probability measure on \(\mathcal{P}_{H,A}\)-polymers.

The definition of the matroidal Mayer coefficients is essentially the same as in Section 2.3. For a spanning set \(H \subset E(M_{H})\) and a collection of ASAs \(A\), the \(d\)-dimensional MMC of type \(A\) is given by

\[
(3.23) \quad \int_{\mathbb{R}^{dn}} \prod_{e \in H} \left(-1_{\{h_{e}(x) \in B(A_{e})\}}\right) dx.
\]

4. Applications

For particular sequences \(\tilde{H}\) of hyperplane arrangements the pressure \(p_{\tilde{H}}^{d}(z)\) arises naturally when studying models in statistical mechanics. This section provides examples. Section 4.1 focuses on the hard sphere gas: Section 4.1.1 gives a new proof of Theorem 1, while Section 4.1.2 explains multi-type hard sphere gases. Section 4.2 shows that type \(D_{n}\) Coxeter arrangements arise in the statistical mechanics of a symmetrized hard sphere gas, and briefly describes some variations on this theme.

4.1. The Brydges-Imbrie dimensional reduction formula.

4.1.1. Proof of the Brydges-Imbrie formula.

Proof of Theorem 1. Mayer’s theorem represents the pressure of a statistical mechanical model in terms of cluster coefficients associated to connected graphs [15]. For the hard-core gas of spheres with radius \(\frac{1}{2}\) Mayer’s theorem states that

\[
(4.1) \quad \lim_{\Lambda \rightarrow \mathbb{R}^{d}} \frac{1}{|\Lambda|} \log Z_{HC}(z) = \sum_{n \geq 0} \frac{z^{n}}{n!} \int_{\mathbb{R}^{dn}/\mathbb{R}^{d}} \sum_{H \in \mathcal{G}^{c}[n]} \prod_{ij \in H} -1_{\{||x_{i} - x_{j}\| \leq 1\}} dx,
\]

where \(\mathcal{G}^{c}[n]\) denotes the set of connected graphs on \([n]\) and \(\mathbb{R}^{dn}/\mathbb{R}^{d}\) indicates translations are modded out.

The right-hand side of Equation (4.1) is the pressure of the braid arrangement \(B_{n}\) in \(C^{n}/(1,1,\ldots,1)C \cong C^{n-1}\), as the matroid associated to \(B_{n}\) is the graphical matroid of \(K_{n}\). Theorem 1 therefore follows from Theorem 12; the extra factor of \(-2\pi\) arises as the arrangement is rank \((n - 1)\). \(\square\)
4.1.2. Variations on the theme. Several variations on this result are known to exist \[16, 17\]. Rather than be exhaustive, we will just highlight one variation and its phrasing in terms of hyperplane arrangements.

Consider the following variant of the braid arrangement. Fix \( k \in \mathbb{N}, n_i \in \mathbb{N} \) for \( i \in [k] \), and let \( n = \sum_{i=1}^{k} n_i \). Define an arrangement in \( \mathbb{C}^n \) to be the set of hyperplanes with normals \( h_{ij}^{(r)} = x_{i'}^{(r)} - x_{j'}^{(s)} \) for \( i' \in [n_i], j' \in [n_j] \), and \( r, s \in [k], r \neq s \).

In the statistical mechanics picture this corresponds to a gas of spheres of \( k \) different colours, with \( n_i \) spheres of colour \( i \). Spheres of the same colour do not interact, while spheres of distinct colours are required to be disjoint. For \( k = 2 \) this is known as the Widom-Rowlinson model. The corresponding branched polymers have trees as tangency graphs, and are restricted to (i) have tangent spheres be of different colours and (ii) have spheres of different colours be disjoint.

4.2. Dimensional reduction in the presence of symmetry constraints. This section describes dimensional reduction formulas for gases of hard spheres subject to symmetry constraints. Sections 4.2.1 to 4.2.4 gives a detailed account of the type \( D_n \) Coxeter arrangement; similar arguments apply to other models which are briefly described in Section 4.2.5.

4.2.1. Symmetric hard sphere models. The type \( D_n \) Coxeter arrangement has hyperplanes defined by the linear functionals

\[
h_{ij}^\pm(x) = x_i \pm x_j, \quad i \neq j \in [n].
\]

The type \( D \) hard sphere model has partition function

\[
Z^D_{\Lambda}(z) = \sum_{n \geq 0} \frac{z^n}{n!} \int_{\Lambda^n} \prod_{i=1}^{n} 1_{\{||x_i - x_j||_2 \geq 1\}} 1_{\{||x_i + x_j||_2 \geq 1\}} \, dx,
\]

where \( \Lambda \subset \mathbb{R}^d \) is a box centered at the origin. The constraint \( ||x_i - x_j||_2 \geq 1 \) is the usual constraint for hard spheres of radius \( \frac{1}{2} \). The constraint \( ||x_i + x_j||_2 \geq 1 \) is a hard sphere constraint between the sphere at \( x_i \) and the mirror image \(-x_j\) of the sphere at \( x_j \). Prosaically, this is a model of hard spheres that cannot distinguish between other spheres and the mirror images of other spheres. Recall Figure 2.

Alternately, the formula for the partition function in Equation (4.3) can be rewritten as a hard sphere gas in the upper half space \( \mathbb{R}^{d-1} \times \mathbb{R}_+ \), and the condition \( ||x_i + x_j||_2 \geq 1 \) can be interpreted as a boundary condition. Since the pressure of a hard sphere gas is independent of the boundary conditions, the pressure of this model can be represented as the partition function of branched polymers in \( d + 2 \) dimensions by Section 4.1. This is verified explicitly in Section 4.2.3. More interestingly, there is a \( D_n \)-polymer representation for the lowest-order finite volume corrections to \( Z^D_{\Lambda} \) as \( \Lambda \uparrow \mathbb{R}^d \); this is explained in Section 4.2.4.
Remark. The generating function analysis in these sections follows from general results on exponential Dowling structures [18]; similarly the analysis of signed graphs is a special case of results on gain graphs [19]. We include the analyses for the benefit of readers unfamiliar with these topics.

4.2.2. Mayer expansion for the type D hard core gas. Writing $1_A = 1 - 1_{A^c}$ in (4.3) yields

\begin{equation}
Z^D_{\Lambda}(z) = \sum_{n \geq 0} \frac{z^n}{n!} \int_{\Lambda^n} \sum_{G \in G^\pm[n]} \prod_{(ij, \pm) \in E(G)} -1_{\{\|x_i \pm x_j\|_2 \leq 1\}} \, dx,
\end{equation}

where $G^\pm[n]$ denotes the set of signed graphs on $[n]$. These are graphs $G$ together with a signing $\sigma : E(G) \to \{\pm\}$. Note that an edge $(ij, +)$ corresponds to the constraint $\|x_i - x_j\|_2 \leq 1$; this convention makes signed graphs with all signs $+$ correspond to the ordinary graphs that arise in the non-symmetrized hard-sphere gas.

Note that signed graphs may have multiple edges: it is possible for $i$ and $j$ to be connected by an edge labelled $+$ and an edge labelled $-$. Loops are not permitted. A cycle of a signed graph will refer to a cycle of the underlying (multi)-graph. This means that if $(ij, +)$ and $(ij, -)$ are edges in a signed graph, the underlying graph contains two copies of the edge $ij$, and there is a two-cycle that consists of these two edges.

Definition 17. A cycle $C$ in a signed graph $(G, \sigma)$ is called balanced if the product of the signs $\sigma(e)$ of the edges in a cycle $C$ is $+1$. A signed graph $(G, \sigma)$ is balanced if every cycle in the graph is balanced. A signed graph that is not balanced is unbalanced.

A signed graph $(G, \sigma)$ can be partitioned into two vertex disjoint signed subgraphs $(G_b, \sigma_b)$ and $(G_u, \sigma_u)$, the former balanced and the latter unbalanced. Let $G^\pm_b[n]$ and $G^\pm_u[n]$ denote the sets of balanced and unbalanced signed graphs on $[n]$, respectively. Then

\begin{equation}
Z^D_{\Lambda}(z) = \sum_{n \geq 0} \frac{z^n}{n!} \int_{\Lambda^n} \sum_{G \in G^\pm_b[n]} \sum_{H \in G^\pm_u[n]} w(G)w(H) \, dx,
\end{equation}

where $w(G) = \prod_{(ij, \pm) \in E(G)} -1_{\{\|x_i \pm x_j\|_2 \leq 1\}}$. This is the convolution of two exponential generating functions, and hence

\begin{equation}
Z^D_{\Lambda}(z) = Z^u_{\Lambda}(z)Z^b_{\Lambda}(z),
\end{equation}

where the subscripts $u$ and $b$ indicated unbalanced and balanced, respectively, e.g.,

\begin{equation}
Z^u_{\Lambda}(z) = \sum_{n \geq 0} \frac{z^n}{n!} \int_{\Lambda^n} w(G) \, dx.
\end{equation}
4.2.3. Calculation of $Z_b^\Lambda(z)$.

**Lemma 18.** The map from $G^\pm_b[n]$ to $G[n]$ given by forgetting the signing of a balanced graph is a $2^{n-1}$-to-1 map when restricted to connected graphs.

**Proof.** First we note that this map is well-defined: a balanced graph cannot contain the edges $(ij, +)$, and $(ij, -)$, as this would be an unbalanced cycle. Forgetting the signing of a balanced graph therefore does yield an ordinary graph.

Let $G$ be a connected unsigned graph on $[n]$. There is a one-to-$2^n$ map given by arbitrarily labelling each vertex $i$ of $G$ with $\sigma_i \in \{\pm\}$, and then assigning the edge $ij$ the sign $\sigma_i \sigma_j$. Each such assignment results in a balanced graph: supposing that $i$ is $+$, trace any cycle containing $i$. By construction seeing a $-$ edge means the sign of the next vertex is different; since the cycle ends at $i$ there must be at least one change of vertex sign after observing the first $-$. This implies there are an even number of $-$ signs, so the cycle is balanced.

The lemma follows by counting how many ways distinct signings of vertices can give rise to the same signing of edges. This is $2$: for any connected balanced signed graph once the sign of a single vertex is fixed, the sign of every other vertex is determined by $\sigma_{ij} = \sigma_i \sigma_j$. That this rule signs each vertex consistently follows from the graph being balanced. □

It follows that from Lemma 18 that

$$Z_b^\Lambda(z) = \sum_{n \geq 0} \frac{(2z)^n}{n!} \sum_{H \in G[n]} \int_{\mathbb{R}^{dn}} \left(\frac{1}{2}\right)^{\#H} w(H) \, dx,$$

where $\#H$ is the number of connected components of the graph $H$. Thus balanced graphs give rise to a cluster-weighted variant of the hard sphere gas at activity $2z$. The exponential principle combined with the argument that gives the Mayer expansion implies

$$\lim_{\Lambda \uparrow \mathbb{R}^d} \log Z_b^\Lambda(z) = \frac{1}{2} \sum_{n \geq 0} \frac{(2z)^n}{n!} \int_{\mathbb{R}^{dn}/\mathbb{R}^d} \sum_{H \in G[n]} w(H) \, dx.$$  

The right-hand side of Equation (4.9) is, up to a factor of $\frac{1}{2}$, the pressure of the hard sphere gas at activity $2z$. By Theorem 1 the pressure $\lim_{\Lambda \uparrow \mathbb{R}^d} |\Lambda|^{-1} \log Z_b^\Lambda(z)$ has a branched polymer representation. It follows from what is presented in the next section that this pressure is equal to $\lim_{\Lambda \uparrow \mathbb{R}^d} |\Lambda|^{-1} \log Z_P(z)$.

4.2.4. Dimensional reduction for $Z_b^u(\Lambda)$. The set of unbalanced graphs on $[n]$ has a naturally associated matroid $M_{D_n}$. The bases of the matroid are signed cycle rooted spanning forests in which each cycle is unbalanced. The cycle may consist of only two edges $(ij, +), (ij, -)$ for some $i \neq j$.

**Lemma 19.** The matroid $M_{D_n}$ is the matroid associated to the hyperplane arrangement $D_n$. 

**Proof.**...
Proof. An edge \((ij, \pm)\) corresponds to the hyperplane defined by \(h_{ij}^\pm(x) = 0\). Note that linear dependence of a set of normals corresponding to a signed graph can only occur if some single component is linearly dependent, so to establish linear dependence / independence it suffices to consider each connected component separately.

In a base of the matroid each component has a single cycle. The only possible non-trivial linear dependencies involve only edges in the cycle: otherwise there is a vertex \(i\) of degree 1, and the corresponding coordinate \(x_i\) cannot have coefficient 0 in any linear combination. Attempting to determine a non-trivial linear dependence thus has only one degree of freedom: once the coefficient of a single edge is chosen, all other edges are determined. This fact together with the fact that the cycle contains an odd number of \(-\) edges shows that any cycle is equivalent to a two-edge cycle with edges \((ij, +), (ij, -)\), and the corresponding normals are linearly independent.

Similarly, given a spanning set that is not a base, there is a component that contains two cycles. If there is a balanced cycle in the component, then there is a linear dependence along this cycle. If all cycles are unbalanced, then all cycles are in fact edge disjoint. An argument as before shows that these can be reduced to a pair of cycles \((ij, +), (ij, -), (i'j', +), (i'j', -)\), and a path connecting these cycles. The corresponding normals are linearly dependent.

\[\square\]

**Theorem 20.** Let \(Z_{BP,d}^H\) be the \(\vec{H}\)-polymer partition function for \(\vec{H} = (D_1, D_2, \ldots)\) in \(\mathbb{R}^d\). For all \(z\) such that the right-hand side converges and all \(d \geq 1\),

\[
\lim_{\Lambda \uparrow \mathbb{R}^d} \frac{Z_{\Lambda}^H(z)}{Z_{\Lambda}^B(z)} = \frac{Z_{BP,d+2}^H(-\frac{z}{2\pi})}{Z_{BP,d+2}^H(-\frac{z}{2\pi})}.
\]

Proof. By (4.6) the left-hand side is \(Z_{\Lambda}^u(z)\). Note that each graph that contributes is unbalanced, which implies each connected component of the graph contains an unbalanced cycle. Hence this is the generating function of MMC associated to \(M_{D_n}\), restricted to a finite volume \(\Lambda\). The theorem follows by applying Theorem 12. The infinite volume limit can be taken by the monotone convergence theorem; that the terms are monotonically increasing follows from Equation (3.7).

Thus unbalanced graphs express the lowest-order corrections to the partition function of the type \(D\) hard core gas compared to \(Z_{\Lambda}^B(z)\), and these corrections can be written in terms of \(D_n\)-polymers.

4.2.5. Variations on the theme. Similar arguments to those in Sections 4.2.2–4.2.4 can be applied to the arrangements with hyperplanes (in each case
$i \neq j \in [n], \ell \in [n])$:

\begin{align*}
h_{ij}(x) &= x_i + x_j, \\
h_{\pm ij}(x) &= x_i \pm x_j, \\
h_\ell(x) &= 0, \\
h_{ij}^m(x) &= x_i - \zeta^m x_j,
\end{align*}

where in the last example $\zeta$ is a primitive $k$th root of unity and $0 \leq m \leq k - 1$. The first example is the threshold arrangement, while the second is the type $B_n$ Coxeter arrangement. For a discussion of planar $B_n$-polymers see [3, Section 6]. The class of examples in the third case are the Dowling arrangements; in general these are not complexified arrangements and hence the results only apply to branched polymers in 2d-dimensional space. In all cases these arrangements enforce certain symmetry constraints between the locations of spheres.

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