Shock wave surfing acceleration

A. A. Vasiliev

Space Research Institute,
Profsoyuznaya str. 84/32,
117997 Moscow, Russia

Abstract

Dynamics of a charged relativistic particle in a uniform magnetic field and an obliquely propagating electrostatic shock wave is considered. The system is reduced to a two degrees of freedom Hamiltonian system with slow and fast variables. In this system, the phenomenon of capture into resonance can take place. Under certain condition, a captured phase point stays captured forever. This corresponds to unlimited surfing acceleration of the particle. The preprint is a more detailed version of a comment on the paper by D.Ucer and V.D.Shapiro, intended for the Comments section of Physical Reviews Letters.

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*Electronic address: valex@iki.rssi.ru
In Letter [1], unlimited surfing acceleration of relativistic particles by a shock wave normal to a uniform magnetic field was considered. The mechanisms of surfing acceleration were studied in many papers, mainly in the case of acceleration by a harmonic wave. The aim of this comment is to point out that the methods developed in [2] allow for more detailed study and further progress in the topic of shock wave acceleration. In particular, the case of an oblique shock wave can be considered. Also, it can be shown that particles with initial velocities far from the shock wave velocity can also be captured in the mode of unlimited acceleration.

Like in [1], consider a charged relativistic particle of charge $e$ and rest mass $m$ in a uniform magnetic field $\mathbf{B}$ and an electrostatic shock wave of potential $\Phi = -\Phi_0 \tanh(kq - \omega t)$, where $\Phi_0 > 0$, $\omega > 0$, $\omega/|k| = u$ is the phase velocity of the shock wave, $q$ is the radius vector. Choose an orthogonal coordinate system $(q_1, q_2, q_3)$ such that $\mathbf{B} = B_0 e_3$ is along the $q_3$-axis and $k$ lies in the $(q_1, q_3)$-plane, $k = (k_1, 0, k_3)$. The Hamiltonian function of the particle is

$$H = \left( m^2 c^4 + c^2 p_1^2 + c^2 p_3^2 + (cP_2 - eB_0 q_1)^2 \right)^{1/2} - e \Phi_0 \tanh(k_1 q_1 + k_3 q_3 - \omega t),$$

(1)

where $P_2 = p_2 + eB_0 q_1/c$ and $p = (p_1, p_2, p_3)$ is the particle’s momentum. Introduce notations:

$$\omega_c = \frac{eB_0}{mc}, \quad k = (k_1^2 + k_3^2)^{1/2}, \quad \varepsilon = \frac{e\Phi_0}{mc^2}, \quad \Omega_c = \omega_c/\varepsilon, \quad \sin \alpha = k_3/k.$$  

Consider the problem in the following range of parameters: $|p|/(mc) \sim 1, \omega/(kc) \sim 1, \varepsilon \ll 1, \omega_c/\omega \sim \varepsilon$. Rescale the variables: $\tilde{p}_{1,3} = p_{1,3}/(mc), \tilde{q}_{1,3} = \varepsilon q_{1,3}/c, \tilde{k}_{1,3} = k_{1,3}c, \tilde{H} = H/(mc^2)$. Following [2], one can canonically transform (1) into the form (tildes are omitted):

$$\mathcal{H} = -\omega I + [1 + k^2 (I + p \cos \alpha/k)^2 + p^2 \sin^2 \alpha + \Omega_c^2 q_3^2]^{1/2} - \varepsilon \tanh \phi \equiv H_0 - \varepsilon \tanh \phi,$$

(2)

where canonically conjugated pairs of variables are $(p, \varepsilon^{-1} q)$ and $(I, \phi)$, $\phi = k_1 q_1 + k_3 q_3 - \omega t$.

The corresponding Hamiltonian equations of motion imply that while $\phi \neq 0$, total change in variable $I$ is a value of order $\varepsilon$, and hence the trajectory of the particle in the $(p, q, I)$-space lies in a vicinity of the intersection of a second order surface $\mathcal{H}_0 =$const and a plane $I =$const. This intersection is an ellipse corresponding to the Larmor motion. However, along a trajectory that crosses the resonance $\dot{\phi} = \partial\mathcal{H}_0/\partial I = 0$ the value of $I$ can change significantly.

The resonant condition $\partial\mathcal{H}_0/\partial I = 0$ defines a surface $I = I_{res}(p, q)$ in the $(p, q, I)$-space, called the resonant surface. The condition implies that projection of the particle’s velocity...
onto the direction of vector $k$ equals the phase velocity of the wave. Intersection of the resonant surface and the surface $H_0 = \text{const}$ is a second order curve whose kind depends on the parameter values (see [2]). This curve is called the resonant curve. The motion in a neighborhood (of the width of order $\sqrt{\varepsilon}$) of the resonant surface possesses certain universal properties ([2], [3]). In particular, the Hamiltonian $F = \mathcal{H}/\varepsilon$ of the particle in this neighborhood can be written in the form:

$$F = \varepsilon^{-1} \Lambda(p, q) + F_0(P, \phi, p, q) + O(\sqrt{\varepsilon}),$$

where $\Lambda(p, q)$ is $H_0$ restricted onto the resonant surface, $P = (I - I_{\text{res}}(p, q))/\sqrt{\varepsilon} + O(\sqrt{\varepsilon}) = O(1)$, and canonically conjugated pairs of variables are $(P, \phi)$ and $(p, \varepsilon^{-3/2}q)$. The function $F_0$ is so-called “pendulum-like” Hamiltonian, and in the case under consideration it is $F_0 = g(p, q)P^2/2 - \tanh \phi + b(p, q)\phi$, where

$$b(p, q) = \frac{\Omega^2 \cos \alpha}{(k^2 - \omega^2)^{1/2}} \cdot \frac{q}{(1 + p^2 \sin^2 \alpha + \Omega^2 q^2)^{1/2}}, \quad g(p, q) = \frac{k^2(1 - (\omega/k)^2)^{3/2}}{(1 + p^2 \sin^2 \alpha + \Omega^2 q^2)^{1/2}},$$

In the system defined by $F$, variables $(p, q)$ are slow and variables $(P, \phi)$ are fast. Slow evolution of $(p, q)$ is determined by a system with Hamiltonian $\sqrt{\varepsilon}\Lambda$. This system defines a flow on the resonant surface, called resonant flow. The $(P, \phi)$ variables evolve according to the subsystem with Hamiltonian $F_0$. If $0 < b < 1$, there is a separatrix surrounding the oscillation region on the phase portrait of this subsystem (see Figure 1). If $b < 0$ or $b > 1$, there is no oscillation region.

The area of the oscillation region $S$ is a function of the slow variables: $S = S(p, q)$. If $S(p, q)$ enlarges along the resonant flow, additional area appears inside the oscillation region.
region. Hence, phase points cross the separatrix and enter the oscillation region. This is a capture into resonance. A captured phase point leaves a vicinity of the curve $I = \text{const}$, $\mathcal{H}_0 = \text{const}$ and continues its motion following approximately the resonant curve. Note, that phase points with arbitrarily large initial values of $P$ can be captured provided they are close enough to the incoming invariant manifold of the saddle point of the “pendulum-like” system. This corresponds to the fact that a particle can be trapped in the mode of surfing acceleration even in the case that initially it is far from the resonance.

The area bounded by the trajectory of a captured phase point in the $(P, \phi)$-plane is an adiabatic invariant of the “pendulum-like” system. Hence, if $S(p, q)$ contracts along the resonant flow, some phase points leave the oscillation region and leave the resonant zone. This is an escape from the resonance. If $S$ monotonically grows along the resonant curve, none of phase points leave the oscillation region. In this case, captured phase points stay captured forever.

If the resonant curve is a hyperbola ($k_3 < \omega$, \[2\]) or a parabola ($k_3 = \omega, \mathcal{H}_0 < 0$, \[2\]), a captured phase point may go to infinity. In this motion the energy of the particle $H$ (see \[2\]) tends to infinity. Therefore, this motion produces unlimited surfing acceleration of particles. This acceleration is possible, if $S(p, q)$ grows as $p, q \to \infty$ along the resonant curve. Calculations (see \[2\]) give the following necessary condition of possibility of unlimited acceleration:

$$\frac{\Omega_c(\omega^2 - k^2 \sin^2 \alpha)^{1/2}}{\omega(k^2 - \omega^2)^{1/2}} < 1.$$  
(5)

This condition was first obtained in \[4\] for acceleration by a harmonic wave. In the case of perpendicular propagation, it is equivalent to the condition of Katsouleas and Dawson \[5\], also mentioned in \[1\].

Consider a hyperbolic resonant curve under assumption that \[5\] is valid. At $q < 0$, $b(p, q) < 0$ (see \[5\]) and $S(p, q) = 0$. At $q = 0$, function $S(p, q)$ has a singularity and at small positive $q$ it is very large. As $q$ grows along the resonant curve, $S(p, q)$ first decreases and then, as $q \to \infty$, $S(p, q) \to \infty$. Hence, at a certain $q = q_m$, function $S(p, q)$ has minimum $S = S_m$ along the resonant curve. Consider a phase point that is initially captured into the resonance at small positive value of $q$. Let the area bounded by its trajectory be $S_0$ and $S_0 > S_m$. Then in the course of motion along the resonant curve this phase point escapes from the resonance. If $S_0 < S_m$, the phase point stays captured forever and undergoes
unlimited acceleration. This explains Fig. 5 in [1]. The number of bounces in this figure corresponds to the number of oscillations of the phase point inside the oscillatory region of the “pendulum-like” system, performed before the phase point escapes from the resonance.

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