Abdollahi’s Conjectures and a Class of Prime Power Group

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Abstract. In this paper, we give necessary and sufficient conditions for metacyclic $p$-groups $H$ and $K$ to have isomorphic non-commuting graphs.

1. Introduction

Let $G$ be a non-abelian group and $Z(G)$ be its center. The non-commuting graph $\Gamma(G)$ of $G$ is a graph whose vertex set is $G - Z(G)$ and two vertices $x$ and $y$ are adjacent if $xy \neq yx$. In 1975 [1], Paul Erdos first considered the non-commuting graph of a group. Many researchers have studied the non-commuting graph (e.g [2],[3],[4]). In [5] Abdollahi et al. put forward the following two conjectures:

1. Let $G$ and $H$ be two finite non-abelian groups such that $\Gamma(G) \cong \Gamma(H)$. Then $|G| = |H|
2. If $G$ and $H$ are two non-abelian finite groups and $H$ is simple such that $\Gamma(G) \cong \Gamma(H)$, then

$G \cong H$

The above conjectures were negative generally, but they hold for various classes of groups. Conjecture 1 was refuted by an example in [6], however it is valid whenever one of $G$ or $H$ is a non-abelian finite simple group [7]. Darafsheh [7] showed that if $H$ or $K$ is a finite non-abelian simple groups, then Conjecture 1 holds and if $H$ or $K$ is a finite non-abelian simple groups satisfying the Thompson’s conjecture, then Conjecture 2 is true. Abdollahi and Shahverdi [8] proved that if $H$ or $K$ is an alternating group, then Conjecture 2 is true.

In this study, we use a family of non-abelian metacyclic $p$-groups of Beuerle's classification [9] to investigate the above conjectures. If the Conjecture 2 holds, the groups in question need not be isomorphic. Indeed, the smallest scale are the dihedral group $D_8$ and the group $Q_8$ of quaternions. Thus, it is an interest to investigate the question of isomorphism for various restricted families of groups. In the present paper, we carry out this investigation for the family $\mathcal{S}$ of non-abelian metacyclic groups of prime power order. That is, assuming $\Gamma(G) \cong \Gamma(H)$ for $H, K \in \mathcal{S}$ determine necessary and sufficient conditions for the groups $H$ and $K$ to be isomorphic.

If $G$ is a metacyclic $p$-group, then $G$ is fallen in one of the following cases in which for all cases $\alpha, \beta \in \mathbb{N}$ and $\varepsilon, \gamma \geq 0$ are integers.

1. Case A:
\[ G = G (\alpha, \beta, \gamma, \varepsilon) = \langle a, b \mid a^{\alpha} = 1, b^{\beta} = a^{\alpha^p}, a^b = a^{\alpha^{p^2}} \rangle, \] for some \( \alpha, \beta, \gamma, \varepsilon \) where \( \beta \geq \gamma \geq 1 \) and \( p \) is an odd prime or \( \alpha - \gamma \geq 2 \).

2. Case B:
\[ G = G (\alpha, \beta, 0, \varepsilon) = \langle a, b \mid a^{\alpha^2} = 1, b^{\beta^2} = a^{\alpha^{2p}}, a^b = a^{-1} \rangle, \] where \( \alpha \geq 2 \).

3. Case C:
\[ G = G (\alpha, \beta, \gamma, \varepsilon) = \langle a, b \mid a^{\alpha^2} = 1, b^{\beta^2} = a^{\alpha^{2p}}, a^b = a^{-\gamma} \rangle, \] where \( \alpha - \gamma \geq 2 \) and \( \gamma > 0 \).

In the following lemma some general properties of elements in the groups above are listed.

**Lemma 1.1** Let \( G \) be a non-abelian metacyclic \( p \)-group (\( p \) is any prime) of Cases A, B or C. If \( x, y \in G \) with \( x = a'^j b^j \) and \( y = a'^k b^k \), then the following hold in \( G \).

1. \( b^j a^i = a^{p^i} b^j \);
2. \( x y = a^{\alpha^j} b^{j^\prime} \);
3. \( x^y = a^{\alpha^{j^r} + \beta^i} b^{j^r} \);
4. \( [x, y] = a^{(\alpha^{j^r} + \beta^i - j^r)} \).

**Proof.** Since the proofs of all parts have similar way, we only give the proof of part (1). We first show \( a^{b^j} = a^{r^j} \) by applying induction on \( j \) for each \( j \geq 1 \). In fact, if \( a^{b^j} = a^{r^j} \) and \( a^{b^j} = a^{r^j} \), then \( a^{b^{j+1}} = (a^{b^j})^b = (a^{r^j})^b = (a^{r^j})^r = (a^{r^j})^r = a^{r^{j+1}} \). It follows that \( a^{b^j} = (a^{b^j})^i = (a^{r^j})^i \), that is \( b^j a^i = a^{r^j} \). Hence \( b^j a^i = a^{r^j} \).

For future reference, we need the following two lemmas that give formulas for the order, center and the order of the center of metacyclic \( p \)-groups. For the proof, we refer to [9], Proposition 2.5.

**Lemma 1.2** Let \( G \) be a metacyclic \( p \)-group of Case A. Then \( |G| = p^{\alpha+\beta} \) and \( |Z(G)| = p^{\alpha+\beta-2\gamma} \).

**Proof.** For (i) and in the case that \( p \) is odd prime, we know that \( \langle a \rangle \cap \langle b \rangle = \langle a^{\alpha p} \rangle \) has order \( p^\epsilon \) and \( G = \langle a \rangle \langle b \rangle \). An easy computation shows that the order of \( G \) is \( p^{\alpha+\beta} \). For the case \( p = 2 \) we know that \( G = \langle a \rangle \langle b \rangle \), \( |a| = 2^\alpha \) and \( |b| = 2^{\beta+\epsilon} \). Also, the order of \( \langle a \rangle \cap \langle b \rangle = \langle a^{2^\alpha-\epsilon} \rangle \) is \( 2^\epsilon \), then the order of \( G \) is \( 2^{\alpha+\beta} \). Since \( Z(G) = \langle a^{2^\alpha}, b^{2^\beta+\epsilon} \rangle \). There are \( 2^{\alpha-\gamma} \) elements generated by \( a^{2^\xi} \) and \( 2^{\beta-\gamma} \) elements generated by \( b^{2^\beta} \). Since \( [b, a] = a^{2^\epsilon} \), thus \( [b, a] = \langle a^{2^\epsilon} \rangle \). Hence the order of \( Z(G) \) is \( 2^{\alpha-\gamma}, 2^{\beta-\gamma} = 2^{\alpha+\beta-2\gamma} \).

**Lemma 1.3** Let \( G \) be a metacyclic \( p \)-group of Cases B and C. Then \( Z(G) = \langle a^{2^{\alpha-\gamma}}, b^{2^{\max\{1, \gamma\} + 1}} \rangle \), \( |Z(G)| = 2^{\beta+\max\{1, \gamma\}+1} \) and \( |G| = 2^{\alpha+\beta} \).
These groups were studied through their centralizers in [10], which are summarized in the following three propositions.

**Proposition 1.4** If $G = G(\alpha, \beta, \gamma, \varepsilon)$ be a group of case A and let $x = a^i b^j \in G$. Then $\left| C_G(x) \right| = p^{\alpha + \beta - \gamma + \varepsilon \min(e_\alpha(i), e_\beta(j))}$, where $e_p(i)$ denotes the largest exponent of $p$ in $i$.

**Proposition 1.5** If $G = G(\alpha, \beta, 0, \varepsilon)$ be a group of case B and let $x = a^i b^j \in G$. Then

\[
\left| C_G(x) \right| = \begin{cases} 
2^{\alpha + \beta}, & \text{if } a^i b^j \in Z(G), \\
2^{\alpha + \beta - 1}, & \text{if } a^i b^j \in Z(G), \text{ j even,} \\
2^{\beta + 1}, & \text{if } a^i b^j \in Z(G), \text{ j odd.}
\end{cases}
\]

**Proposition 1.6** If $G = G(\alpha, \beta, \gamma, \varepsilon)$ is a group of case C, then for $x = a^i b^j \in G$,

\[
\left| C_G(x) \right| = \begin{cases} 
2^{\alpha + \beta}, & \text{if } j \text{ even, } e_\gamma(j) \geq \gamma \text{ and } e_\varepsilon(i) < \gamma, \\
2^{\alpha + \beta - 1}, & \text{if } j \text{ even, } e_\varepsilon(i) \geq \alpha - \gamma - 1, \text{ or} \\
2^{\beta + 1}, & \text{if } j \text{ even, } e_\varepsilon(i) \geq \alpha - \gamma - 1 \text{ and } e_\gamma(j) < \alpha - \gamma - 1, \\
2^{\alpha + \beta - 1}, & \text{if } j \text{ even, } e_\varepsilon(i) \leq e_\gamma(i) < \gamma \text{ and } e_\varepsilon(i) \geq \alpha - \gamma - 1, \text{ or} \\
2^{\beta + 1}, & \text{if } j \text{ even, } e_\gamma(j) < \alpha - \gamma - 1, \\
2^{\alpha + \beta - 1}, & \text{if } j \text{ even, } e_\varepsilon(i) \leq e_\gamma(i) < \gamma \text{ and } e_\gamma(j) < \alpha - \gamma - 1, \\
2^{\alpha + \beta}, & \text{if } j \text{ odd and } e_\gamma(j) < \gamma, \\
2^{\beta + 1}, & \text{if } j \text{ even and } e_\gamma(j) < e_\varepsilon(j) < \gamma.
\end{cases}
\]

We show the case of a metacyclic $p$-group $G$ by $\text{case}(G)$ and the set of all degrees of vertices of a graph $\nabla(G)$ by $D(\nabla(G))$.

2. Main Results

In this section, we give necessary and sufficient conditions for two prime power metacyclic groups to have isomorphic non-commuting graphs.

If $G$ is a group, then the set of all conjugacy class sizes of $G$ called the **conjugacy vector type** of $G$. The following lemma is used to prove the next theorems.

**Lemma 2.1** Let $H$ and $K$ be two groups with the same orders. If the non-commuting graphs of these two groups are isomorphic, then the size of conjugacy classes of $H$ and $K$ are identical.

**Proof.** Let $\phi: \nabla(H) \to \nabla(K)$ be a one-to-one correspondence of the vertices of the graphs $\nabla(H)$ and $\nabla(K)$. For $x \in \nabla(H)$, let $x^H$ denote the conjugacy class of $x$ in $H$. Suppose $\phi(x) = y$ such that $\deg_{\nabla(H)}(x) = \deg_{\nabla(K)}(y)$. Then $|C_H(x)| = |C_K(y)|$, since $|H| = |K|$, we have
\[
|\chi^H| = |H : C_H(x)| = |H| / |C_H(x)| = |K| / |C_K(y)| = |K : C_K(y)| = \gamma^K. \]
That is, \( H \) and \( K \) have the same number of conjugacy class sizes. \( \square \)

In this section, we let \( H = H(\alpha, \beta, \gamma, \varepsilon) \) be a non-abelian metacyclic \( p \)-group and \( K = K(\alpha', \beta', \gamma', \varepsilon') \) be a non-abelian metacyclic \( q \)-group. The following theorem shows that Conjecture 1 holds for the family 3 non-abelian metacyclic prime power groups.

**Theorem 2.2.** If \( \nabla(H) \) is isomorphic to \( \nabla(K) \), then \( H \) and \( K \) have the same orders and centers.

The following theorem gives necessary and sufficient conditions under which two non-abelian metacyclic prime power groups have isomorphic non-commuting graphs.

**Theorem 2.3** \( \nabla(H) \) is isomorphic to \( \nabla(K) \) if and only if one of the following holds:

1. case \((H) = \) case \((K) = I, \alpha + \beta = \alpha' + \beta' \) and \( \gamma = \gamma' \),
2. case \((H) = \) case \((K) = II, \alpha = \alpha' \) and \( \beta = \beta' \),
3. case \((H) = \) case \((K) = III, \alpha = \alpha', \beta = \beta' \) and \( \gamma = \gamma' \),
4. case \((H) = I, \) case \((K) = II, \gamma = 1, \alpha = 2 \) and \( \beta' = \alpha + \beta - 2 \),
5. case \((H) = II, \) case \((K) = III, \alpha = \alpha' \geq 3, \beta = \beta' \) and \( \gamma' = 1 \).

3. Conclusions

In this study, some basic properties of the non-commuting graphs of metacyclic \( p \)-groups were investigated. If the non-commuting graph \( \Gamma_H \) of the metacyclic \( p \)-group \( H \) and the non-commuting graph \( \Gamma_K \) of the 20 metacyclic \( q \)-group \( K \) are isomorphic, then \( H \) and \( K \) have the same orders and centers.

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