Quantum Kinetic Equation and
Cosmic Pair Annihilation

Sh. Matsumoto and M. Yoshimura
Department of Physics, Tohoku University
Sendai 980-8578 Japan

ABSTRACT

Pair annihilation of heavy stable particle that occurs in the early universe is investigated, and quantum kinetic equation for the momentum distribution of the annihilating particle is derived, using the influence functional method. A bosonic field theory model is used to describe the pair annihilation in the presence of decay product particles making up a thermal environment. A crossing symmetric Hartree approximation that determines self-consistently the equilibrium distribution is developed for an otherwise intractable theory. The time evolution equation and its Markovian approximation is derived, to give a generalized Boltzmann equation including off-shell effects. The narrow width approximation to an energy integral in this equation gives the usual Boltzmann equation in a thermal bath of light particles. The off-shell effect is a correction to the Boltzmann equation at high temperatures, but is dominant at low temperatures. The effect changes the equilibrium distribution from the familiar $1/(e^{\omega k/T} - 1)$ to a modified one given by a Gibbs formula. Integrated over momenta, the particle number density becomes roughly of order $(\text{coupling}) \times \sqrt{T/M} \cdot T^3$ at low temperatures for the S-wave annihilation. The relic mass density in the present universe is insensitive to the coupling strength in a large range of the mass and the coupling parameters, and scales with the WIMP mass as $\approx 6 \times 10^4 \text{ eV cm}^{-3} (M/\text{GeV})^{4/3}$. The bound from the closure density gives an upper WIMP mass bound roughly of order 1 GeV in the present model.
1 Introduction

The Boltzmann equation for the particle distribution function is described in terms of S-matrix elements defined on the mass shell, such as the cross section and the decay rate. This integro-differential equation is intuitively appealing since relation to the classical theory is evident. In the quantum mechanical context its foundation and its possible generalization have extensively been discussed in the past. We may only mention two approaches; the real-time thermal Green’s function method [1] and the closed time path method [2]. Both use as the fundamental quantity the Green’s function that contains information off the mass shell. It is generally believed that some sort of coarse graining or reduction of detailed quantum mechanical information is needed to derive a quantum version of the Boltzmann equation. There is however no unique generalization of the Boltzmann equation known to us. Perhaps correction to the Boltzmann equation differs depending on a physical situation one has in mind.

We investigate in the present work the non-equilibrium process of pair annihilation of some heavy particles in the presence of a thermal bath of lighter particles. Time evolution of the occupation number for the heavy particle in a particular momentum mode is derived and analyzed from a new perspective.

The problem is of great interest in cosmology, and occurs for pair annihilation of stable particles such as the anti-proton and the positron. For instance, the electron-positron annihilation has a great impact on nucleosynthesis that takes place immediately after this annihilation. The annihilation rate has been calculated using a thermally averaged Boltzmann equation, but estimate of the off-shell effect to the annihilation process has never been worked out (see, however, [3] for a limited calculation of finite temperature effects). This list of interesting candidates for the cosmic pair annihilation should be expanded to include WIMP (weakly interacting massive particle) or LSP (lightest supersymmetric particle) [4] that has been hypothesized, with a good motivation, as a solution to the dark matter problem in cosmology.

We would like to obtain a suitable and useful generalization of the Boltzmann equation in these circumstances. The physical situation we have in mind is characterized by the presence of annihilating heavy particles \( (e^\pm) \) in the case of electron-positron annihilation) interacting among themselves (\( e^\pm \) scattering and pair annihilation into two photons) and with lighter particles (scattering with photons). Al-
though we work out a particular toy model for this, our method is general enough for extension to other cases, and moreover our result is simple enough for practical use.

We believe that for derivation of the generalized Boltzmann equation it is important to use a general and flexible framework allowing for a consistent approximation scheme. It is often practically difficult to estimate correction to the Boltzmann equation in the Green’s function or the operator method. Our experience in a similar, but a much simpler problem of the unstable particle decay in thermal medium suggests that the influence functional method invented by Feynman and Vernon may be useful to the present problem. The quantum mechanical problem of the excited state decay, analogous to, but simpler than, the unstable particle decay in field theory, is completely solvable at the operator level, but at the same time the influence functional method gives an identical result to the operator approach. In the present work we develop a new Hartree type of approximation of the annihilation-scattering problem within the influence functional approach. We also utilize the operator identity to facilitate our analysis.

In order to quantitatively discuss the role of scattering in the annihilation process, it is important to simultaneously deal with the two processes, the annihilation and the scattering. It is thus best to employ a fully relativistic field theory that respects the crossing symmetry. For simplicity we take throughout this paper an interaction Lagrangian density of the form, $\varphi^2\chi^2$, where $\varphi$ is a heavy bosonic field and $\chi$ a lighter bosonic field. We assume that the mass of $\varphi \gg$ the mass of $\chi$. The annihilation channel $\varphi\varphi \rightarrow \chi\chi$ is related to the scattering channel $\varphi\chi \rightarrow \varphi\chi$ by a crossing symmetry. The inverse processes to these and the other 1 to 3 processes, $\varphi \leftrightarrow \varphi\chi\chi$ and $\chi \leftrightarrow \varphi\varphi\chi$, that may also occur in thermal medium of $\chi$ particles are treated here symmetrically. It is important to recall that a finite time behavior of the quantum system in thermal medium allows the process such as $\varphi \leftrightarrow \varphi\chi\chi$, even if it is kinematically forbidden for the on-shell S-matrix element.

We first derive an integral equation that self-consistently determines the equilibrium distribution function $f(\vec{k})$ within the Hartree approximation. Derivation of this equation is based on the assumption that time variation of the distribution function proceeds more slowly than individual microscopic reactions occur. Under this assumption the result of the completely solvable model may be used, and the large time ($\gg$ relaxation time) limit can be taken. The resultant equilibrium
distribution function deviates from the ideal gas form, \(1/(e^{\beta \omega} - 1)\), but may be understood by the Gibbs formula \(e^{-\beta H_{\text{tot}}}\) with \(H_{\text{tot}}\) the total Hamiltonian including interaction between the \(\varphi\) system and the \(\chi\) environment.

We then derive a time evolution equation for the distribution function. This equation contains the initial memory term, hence is non-Markovian, as any exact treatment of the quantum mechanical behavior would demand. We next devise a useful Markovian approximation to the Hartree model and examine this approximation in detail.

The on-shell Boltzmann equation arises as a result of the narrow resonance approximation for a Breit-Wigner type of energy integral in our Hartree-Markovian model. The Boltzmann approximation is excellent in many practical cases, but it fails when the main contributing part to the energy integral includes the region off the resonance pole at \(\omega \approx \varphi\) energy. Thus, the Boltzmann equation is modified significantly for the environment temperature \(\ll\) the \(\varphi\) mass \(M\). We present a complete kinetic equation for the momentum distribution function, which may be used at low temperatures.

Integration over the momentum of the distribution function gives a rate equation for the number density which is of prime interest in the annihilation-scattering problem. One may naively expect that the scattering process conserves the particle number, thus scattering terms cancel in the Boltzmann equation. This is explicitly confirmed for the on-shell part of our quantum kinetic equation. But it is not clear whether a similar complete cancellation occurs for the off-shell quantity. This is because this conservation law is based on the commutability of the particle number with the scattering part of the effective Hamiltonian, along with the unitary evolution in the quantum system. Dissipation due to the thermal medium and its associated fluctuation however causes a non-unitary evolution for a part of the entire system. It is thus an interesting open question whether the scattering-related contribution remains for the off-shell part. In any event we find that the scattering and its inverse process do contribute, but with a very small rate. A large number density of order, \(O[10^{-3}] \lambda \sqrt{\frac{T}{M}} T^3\), is derived for the off-shell contribution of the inverse annihilation process, where \(\lambda\) is the relevant \(\varphi^2 \chi^2\) coupling. At low temperatures this becomes much larger than the usual one, \(\approx (MT/2\pi)^{3/2} e^{-M/T}\), namely the thermal number density of zero chemical potential.

In cosmology the thermal environment gradually changes according to the adi-
batic law; the temperature $\propto$ the cosmic scale factor $1/a(t)$. Along with the obvious change of the number density $\propto 1/a^3(t)$, the thermally averaged rate in the generalized Boltzmann equation decreases as the temperature decreases. Thus, the decoupling or the freeze-out of annihilation takes place roughly at the temperature when the thermal rate is equal to the Hubble rate $[10]$. What is left after the freeze-out is then the relic abundance of heavy stable particles. It is important to accurately estimate the relic abundance of WIMP or LSP for the dark matter problem.

The off-shell effect considered in the present work gives a relic abundance of order $Y = n_\varphi/n_\gamma \approx O[0.1] (M/m_{pl})^{1/3}$, with $m_{pl}$ the Planck mass. Unlike the estimate based on the on-shell Boltzmann equation, this abundance does not suffer from the suppression factor $e^{-M/T_f}$ where $T_f$ is the freeze-out temperature, usually much less than $M$. Moreover, the relic fraction $Y$ is insensitive to the $\varphi^2 \chi^2$ coupling, if the coupling is not too small, that is if $\lambda > 9.3 \times 10^{-5} (M/\text{GeV})^{0.32}$ for $10^{-3} \text{GeV} < M < 1 \text{TeV}$. Translated to the mass density of the present universe, this gives an allowed WIMP mass range for the dark matter; $M < 1 \text{GeV}$. It is of considerable interest to work out the allowed parameter region for LSP in realistic supersymmetric theories.

The rest of this paper is organized as follows. In Section 2 the influence functional method is briefly explained, taking our model of $\varphi^2 \chi^2$ interaction. A non-perturbative approximation of the Hartree type is then introduced in connection to the influence functional. The kernel function in the reduced Hartree model is determined in terms of the spectral function that itself contains the correlator to be determined. In Section 3 we explain how the concept of the slow variation of the particle distribution helps to derive a self-consistent equation that determines the equilibrium distribution function. This is achieved by using the result of the exactly solvable model of the excited state decay. In particular, the explicit form of the coincident time limit of two-body correlators is required for its derivation, and we calculate the correlator by using a new generating functional method within the influence functional formalism.

In Section 4 the time evolution equation is derived for the occupation number. This equation is non-Markovian and the initial memory term is identified. A simple Markovian approximation becomes possible, again under the assumption that the time scale of the $\varphi$ number density variation is larger than that of the microscopic reaction time. The Markovian equation thus derived makes its relation to the usual Boltzmann equation transparent. We then point out that this Markovian equation con-
tains the off-shell effect which becomes dominant at low temperatures. Cancellation of the scattering process in the on-shell contribution, when one integrates over momenta, becomes important to get a large off-shell contribution to the number density at low temperatures. In Section 5 the cosmological estimate of the relic abundance of heavy stable particles is given in the present model. The off-shell effect prolongs the freeze-out epoch, but due to a power law decrease \( \propto T^{3.5} \) of the number density with temperature, the net effect gives a larger freeze-out density. Finally a simple estimate of the present mass density is given; \( \rho_0 \approx O[6 \times 10^4] (M/GeV)^{4/3} eV \, cm^{-3} \) in a certain range of the parameter space \((M, \lambda)\).

In four Appendices we explain technical points relegated in the main text; (A) generating functional method in the influence functional framework, (B) renormalization of the off-shell distribution function, (C) some details for the unstable particle decay treated in the Hartree approximation, and (D) technical details for computing various integrals needed to obtain the off-shell distribution function.

A short summary of our result stressing the estimate of relic abundance has been given in a previous note [11]. In the present longer paper we give all details stated there and present further new results not given there.

## 2 Hartree approximation

We consider a relativistic field theory model of the Lagrangian density given by

\[
\mathcal{L} = \mathcal{L}_\varphi + \mathcal{L}_\chi + \mathcal{L}_{\text{int}},
\]

\[
\mathcal{L}_\varphi + \mathcal{L}_\chi = \frac{1}{2} (\partial_\mu \varphi)^2 - \frac{1}{2} M^2 \varphi^2 + \frac{1}{2} (\partial_\mu \chi)^2 - \frac{1}{2} m^2 \chi^2,
\]

\[
\mathcal{L}_{\text{int}} = -\frac{\lambda}{2} \varphi^2 \chi^2.
\]

It is later explained how to renormalize and introduce counter terms when it becomes necessary. The heavy particle \( \varphi \) can pair-annihilate into a \( \chi \) pair; \( \varphi \varphi \rightarrow \chi \chi \) with the dimensionless interaction strength \( \lambda \). This coupling \( \lambda \) must be less than unity for our method to be useful. We assume that the lighter particle \( \chi \) makes up a thermal environment of temperature \( T = 1/\beta \) in our unit of the Boltzmann constant \( k_B = 1 \). It is thus assumed that although not written explicitly, there is interaction
among χ particles themselves or with some other light particles to maintain a thermal equilibrium.

In the problem of our interest one focusses on a particular dynamical degree of freedom (the ϕ field in our case) and integrates out the environment part (the χ field) altogether. In the influence functional method [8] this integration is carried out for the squared amplitude, namely for the probability function directly, by using the path integral technique. This way one has to deal with the conjugate field variable ϕ' along with ϕ, since the complex conjugated quantity is multiplied in the probability.

Define first the influence functional F by integrating out the χ field degree of freedom during a fixed time interval between t_i and t_f;

\[
\int D\chi \int D\chi' \exp \left[ i \int dx \left( L_\chi(x) - L_\chi'(x) \right) + L_{\text{int}}(\varphi(x), \chi(x)) - L_{\text{int}}(\varphi'(x), \chi'(x)) \right].
\]

(2.4)

We convolute with this the initial and and the final density matrix of the χ system. For the initial state it is assumed that the entire system is described by an uncorrelated product of the system and the environment density matrix,

\[
\rho_i^{(\varphi)} \times \rho_i^{(\chi)}, \quad \rho_i^{(\chi)} = \rho_\beta = e^{-\beta H_0(\chi)}/\text{tr} e^{-\beta H_0(\chi)}. \tag{2.5}
\]

Here ρ_β is the density matrix for a thermal environment, written in the operator notation using the environment Hamiltonian H_0(χ). To avoid a possible confusion, we sometimes write the operator explicitly by the tilded letter such as \tilde{\chi} here. The density matrix \rho_i^{(\varphi)} for the ϕ system is arbitrary, except that it is assumed to commute with the ϕ Hamiltonian,

\[
\left[ \rho_i^{(\varphi)}, H_0(\varphi) \right] = 0. \tag{2.6}
\]

We believe that this restriction on the ϕ state does not exclude many practical cases of interest. At the final time t_f the χ integration is performed taking the condition of non-observation for the environment,

\[
\int d\chi_f \int d\chi_f' \delta(\chi_f - \chi_f') (\cdots), \tag{2.7}
\]

with the understanding that the environment is totally unspecified at the time t_f.

The result of χ integration is given by a Gaussian integral and the influence functional to order λ^2 is of the form,

\[
\mathcal{F}_4[\varphi, \varphi'] =
\]
\[
\exp[-\frac{1}{4} \int_{x_0 > y_0} dx \, dy \left( \xi_2(x) \alpha_R(x, y) \xi_2(y) + i \xi_2(x) \alpha_I(x, y) X_2(y) \right)], \tag{2.8}
\]

\[
X_2(x) \equiv \varphi^2(x) + \varphi'(x), \quad \xi_2(x) \equiv \varphi^2(x) - \varphi'(x), \tag{2.9}
\]

\[
\alpha(x, y) = \alpha_R(x, y) + i \alpha_I(x, y) = \lambda^2 \text{tr} \left( T[\chi^2(x)\chi^2(y)\rho_{\beta}] \right). \tag{2.10}
\]

Note the presence of the time ordering, \(x_0 > y_0\), in the above formula. Unless a confusion occurs, we shall simplify the four dimensional integral such as \(\int d^4x \cdots\) by writing \(\int dx \cdots\). We note that the kernel function \(\alpha(x, y)\) satisfies the time translation invariance, thus may be written as \(\alpha(x - y)\), as can explicitly be shown by taking the complete set of eigenstates;

\[
\text{tr} \left( \tilde{A}(x_0) \tilde{B}(y_0) \rho_{\beta} \right) = \sum_{n, \dot{i}} e^{-i(E_n - E_{\dot{i}})(x_0 - y_0)} (\rho_{\beta})_{\dot{i}i} \langle \dot{i} | \tilde{A} | n \rangle \langle n | \tilde{B} | i \rangle. \tag{2.11}
\]

An explicit form of the kernel function \(\alpha(x)\) or its Fourier transform is given later.

Higher order terms in \(\lambda^2\) are actually present in the exponent of the influence functional. These contribute either to many-body processes we are not interested in or to higher order terms in our process. Moreover, Feynman and Vernon proved that the above form of the influence functional satisfies the required fundamental property such as the causality and the unitarity. We may thus safely ignore these higher order contributions in the weak coupling limit.

Another point is the tadpole contribution and how the kernel \(\alpha_i\) is modified due to the tadpole, which we discuss later. (See eq. (2.60) for the modification.)

The convolution with the system variable \(\varphi\) gives the reduced density matrix for the system at any given time \(t_f\);

\[
\rho^{(R)}(\varphi_f, \varphi'_{t_f}) = \int d\varphi_i \int d\varphi'_{t_f} \int \mathcal{D}\varphi \int \mathcal{D}\varphi' e^{iS(\varphi) - iS(\varphi')} \mathcal{F}[\varphi, \varphi'] \rho_{\varphi_i}^{(\varphi)}(\varphi_i, \varphi'_{t_f}),
\]

\[
\tag{2.12}
\]

from which one can deduce physical quantities for the \(\varphi\) system. Here \(S(\varphi)\) is the action for the \(\varphi\) system obtained from the basic Lagrangian.

It is often useful to introduce a notation for the correlator in the influence functional method; for \(x_0 > y_0\),

\[
\langle \varphi(x)\varphi(y) \rangle = \int d\varphi(x) \int d\varphi'(x) \int d\varphi(y) \int d\varphi'(y)
\]

\[
\cdot \int \mathcal{D}\varphi \int \mathcal{D}\varphi' e^{iS(\varphi) - iS(\varphi')} \mathcal{F}[\varphi, \varphi'] \varphi(x)\varphi(y) \rho^{(R)}(\varphi(y), \varphi'(y)). \tag{2.13}
\]

In this formula the functional integration is performed for \(\varphi\) during the time interval, \(x_0 > t > y_0\), where \(t_f > x_0 > y_0 > t_i\). The path integral prior to the time \(y_0\) gives the
reduced density matrix $\rho^{(R)}$ at the time $y_0$, evolved from $\rho_i$ at the time $t_i$. In general, the correlation function $\langle \phi(x)\phi(y) \rangle$ does not obey the time translation invariance, reflecting that the initial memory is never completely erased in the quantum system. A nice feature of this correlator formula is that the initial memory effect appears compactly via the reduced density matrix $\rho^{(R)}$. One may generalize the concept of the expectation value $\langle \cdots \rangle$ to any multiple of local operators including also the conjugate $\phi'$.

The model thus specified is difficult to solve due to the appearance of the quartic term of $\phi$ in the influence functional (2.8). The situation is however simplified when one considers a mean field approximation. In the mean field or the Hartree approximation one replaces a product of multi-field operators by a product of two operators with several averaged two-body correlators. This approximation is good if one can ignore a higher order correlation than that of two-body. The Hartree model is expected to work well in a dilute system. The dilute system is defined by the low occupation number for each mode. It should not be confused by a possibly large value of the number density which is a mode-summed quantity. For the bose system we consider here this is a circumstance very far from the bose condensed state. The dilute approximation seems good in most cosmological application.

The Hartree approximation we now introduce is a Gaussian truncation to the influence functional; we replace the original one by properly defining a new kernel function $\beta(x,y)$ in the quadratic form;

$$ F_2[\phi, \phi'] = \exp[- \int_{x_0 > y_0} dx \, dy \left( \xi(x) \beta_R(x,y) \xi(y) + i \xi(x) \beta_I(x,y) X(y) \right) ], $$

(2.14)

$$ X(x), \quad \xi(x) \equiv \phi(x) \pm \phi'(x). $$

(2.15)

As for the correlator, the new kernel $\beta_i$ in the truncated model does not satisfy the time translation invariance. The identification of the new kernel $\beta(x,y)$ is made by comparing two point correlators to any arbitrary order $\lambda^2$:

$\langle X(x)\xi(y) \rangle, \langle X(x)X(y) \rangle, \langle \xi(x)\xi(y) \rangle$. It turns out that the last correlator $\langle \xi(x)\xi(y) \rangle$ vanishes both for the original and the Hartree-approximated model, which can be regarded as a consistency check of our approach. Once the Hartree model is determined by the kernel $\beta(x,y)$, one can work out its consequences to all order of $\lambda$. 

9
Let us first introduce two types of propagator for $\varphi$ field:

\[ G(x, y) = i \langle \varphi(x) \varphi(y) \rangle_{F=\mathcal{F}_4}, \]  

(2.16)

and $i \langle \varphi(x) \varphi(y) \rangle_{F=\mathcal{F}_1}$. The first one is the full propagator taking into account the environment interaction. The second one, on the other hand, is an extension of the free propagator in perturbative field theory accommodated to a non-trivial $\varphi$ state in our problem. Without the non-local interaction in the influence functional $\mathcal{F}$, one may use for $i \langle \varphi(x) \varphi(y) \rangle_{F=\mathcal{F}_1}$ the complete set of eigenstates of the $\varphi$ Hamiltonian $H_0(\varphi)$, to show that this quantity is translationally invariant; introducing a new notation for the propagator,

\[ G_0(x - y) \equiv i \langle \varphi(x) \varphi(y) \rangle_{F=\mathcal{F}_1} \]

\[ = i \int_{-\infty}^{\infty} dk_0 e^{-ik_0(x_0-y_0)} \sum_{n,i} \delta(k_0 - E_n + E_i) \rho_i^{(\varphi)} \langle i | \tilde{\varphi}(0, \vec{x}) | n \rangle \langle n | \tilde{\varphi}(0, \vec{y}) | i \rangle. \]  

(2.17)

Note that $\rho_1^{(\varphi)}$ is time independent. It is always useful to decompose this into independent momentum modes; for $x_0 > 0$

\[ -iG_0(x) = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \left( f_0(\vec{k})e^{ik \cdot x} + (1 + f_0(\vec{k}))e^{-ik \cdot x} \right), \]  

(2.18)

where $k_0 = \omega_k = \sqrt{k^2 + M^2}$. One may use the harmonic oscillator basis $|n\rangle_{\vec{k}}$ ($n = 0, 1, 2, \cdots$) in this plane wave decomposition to show

\[ f_0(\vec{k}) = |\langle 1|\tilde{\varphi}(\vec{k})|0\rangle_{\vec{k}}|^2 \sum_n n \rho_n^{(\vec{k})}. \]  

(2.19)

In the usual language of the creation and the annihilation operators,

\[ \langle a_1 a_2 \rangle_{F=\mathcal{F}_1} = 0 = \langle a_1^\dagger a_2^\dagger \rangle_{F=\mathcal{F}_1}, \quad \langle a^\dagger(\vec{k}')a(\vec{k}) \rangle_{F=\mathcal{F}_1} = f_0(\vec{k}) \delta(\vec{k} - \vec{k}'). \]  

(2.20)

We compute correlators in two theories, using the two different forms of the influence functional, $\mathcal{F}_4$ and $\mathcal{F}_2$, and identify two results. One then has an equality;

\[ \int dx' dy' G_0(x - x')\beta(x', y')G_0(y' - y) \]

\[ = -i \int dx' dy' G_0(x - x')\alpha(x' - y')G(x', y')G_0(y' - y). \]  

(2.21)

This identity leads to a crucial relation of the two kernels;

\[ \beta(x, y) = -i \alpha(x - y) G(x, y). \]  

(2.22)
It is important in the self-consistent Hartree approximation that the full propagator $G$ instead of $G_0$ appears in this equation, since the truncated model should contain the full $\varphi$ propagator to be determined self-consistently. Equivalently, the full propagator has a self-consistency condition in the Hartree approximation; in the matrix form,

$$G = G_0 - i G_0 \beta G_0 + \cdots = G_0 - G_0 (\alpha G) G_0 + \cdots.$$  \hfill (2.23)

The next task is to derive a useful relation between the spectral functions that appear in eq. (2.22). The full propagator $G$ has two spacetime arguments, for which we use the center of mass (CM) variable $(x + y)/2$ and the relative coordinate $x - y$. We observe that there is no dependence on the CM space coordinate $(\vec{x} + \vec{y})/2$ due to the spatial homogeneity, hence Fourier-transform the relative coordinate to get

$$-i G(x, y) = \int \frac{d^4 k}{(2\pi)^4} \tilde{f}(-k, \frac{x_0 + y_0}{2}) e^{ik(x-y)}. \hfill (2.24)$$

The spectral $\tilde{f}$ here may depend on the CM time.

The occupation number $f(\vec{k}, t)$ that becomes important in subsequent discussion is defined from the coincident time limit of the Fourier transformed full propagator. We first define the spatial Fourier decomposition of the full propagator,

$$\tilde{G}(x_0, y_0; \vec{k}) \equiv \int d^3(x - y) G(x_0, \vec{x}; y_0, \vec{y}) e^{-ik(\vec{x} - \vec{y})} \hfill (2.25)$$

$$\tilde{\varphi}(\vec{k}, x_0) \equiv \int d^3 x \varphi(\vec{x}, x_0) e^{-i\vec{k}\cdot\vec{x}}. \hfill (2.26)$$

Due to the hermiticity of the field $\varphi$,

$$\tilde{\varphi}^\dagger(\vec{k}, x_0) = \tilde{\varphi}(-\vec{k}, x_0). \hfill (2.27)$$

Using the Heisenberg equation for $\varphi$, one has for the occupation number

$$\tilde{\omega}_k \left( f(\vec{k}, x_0) + \frac{1}{2} \right) \equiv \frac{1}{2} \left( \frac{d\tilde{\varphi}^\dagger(\vec{k}, x_0)}{dx_0} \frac{d\tilde{\varphi}(\vec{k}, x_0)}{dx_0} \right) + \tilde{\omega}_k^2 \tilde{\varphi}^\dagger(\vec{k}, x_0) \tilde{\varphi}(\vec{k}, x_0) \hfill (2.28)$$

$$= \frac{i}{4} \left[ \left( \frac{\partial}{\partial x_0} - \frac{\partial}{\partial y_0} \right)^2 \tilde{G}(x_0, y_0; -\vec{k}) \right]_{x_0 \to y_0} - \frac{\lambda}{2} \langle \tilde{\varphi}^\dagger(\vec{k}, x_0) \tilde{\varphi}\chi^2(\vec{k}, x_0) \rangle. \hfill (2.29)$$

The first quantity in the right hand side is further related to $\tilde{f}$ by

$$f(\vec{k}, t) + \frac{1}{2} = \frac{1}{\tilde{\omega}_k} \int \frac{dk_0 2\pi}{2\pi} \tilde{f}(k, t) - \frac{\lambda}{2\tilde{\omega}_k} \langle \tilde{\varphi}^\dagger(\vec{k}, t) \tilde{\varphi}\chi^2(\vec{k}, t) \rangle. \hfill (2.30)$$
Here $\overline{\omega}_k$ is what we may call a reference energy to define the occupation number for the mode $\vec{k}$. We take for the time being $\overline{\omega}_k = \omega_k$, the energy of free particle, and later modify the definition of the occupation number $f(\vec{k}, t)$ slightly by allowing the temperature dependent $\varphi$ mass in $\overline{\omega}_k$.

We define the spectral weight for the two kernels in terms of the Fourier decomposition of the relative spacetime coordinate; for $x_0 > y_0$

$$\alpha(x - y) = \int \frac{d^4k}{(2\pi)^3} \frac{2}{1 - e^{-\beta k_0}} r_\chi(k) e^{-ik(x-y)}, \quad (2.31)$$

$$\beta(x, y) \equiv \int \frac{d^4k}{(2\pi)^3} r(k, \frac{x_0 + y_0}{2}) e^{-ik(x-y)}. \quad (2.32)$$

A strange looking factor $1/(1 - e^{-\beta k_0})$ is inserted for the kernel $\alpha(x)$, since this factor compensates the non-analytic property of the real-time thermal Green’s function $\alpha(x)$ defined in eq.(2.11), relative to the analytic imaginary-time Green’s function. The factor is well understood [12] and is also explained in our previous paper [13]. We shall shortly give an explicit formula of $r_\chi(k)$. The spectral weight $r$ for the Hartree model is determined from eq.(2.22); it is of the form of a convolution integral;

$$r(k, t) = \int \frac{d^4k'}{(2\pi)^4} \tilde{f}(-k', t) \frac{2r_\chi(k + k')}{1 - e^{-\beta(k_0 + k'_0)}}. \quad (2.33)$$

The four dimensional integral (2.33) is not very useful in practice. In the following we shall derive an important relation of the two spectral functions, $r_\chi$ and $r$, using the occupation number $f$ and another quantity $v$;

$$r(k, t) = 2 \int \frac{d^3k'}{(2\pi)^3 2\omega_{k'}} \left( \frac{r_\chi(k_0 + \omega_{k'}, \vec{k} + \vec{k}')} {1 - e^{-\beta(k_0 + \omega_{k'})}} f(\vec{k}', t) \right. \left. + \frac{r_\chi(k_0 - \omega_{k'}, \vec{k} - \vec{k}')} {1 - e^{-\beta(k_0 - \omega_{k'})}} (1 + f(\vec{k}', t)) - \frac{2r_\chi(k_0, \vec{k} - \vec{k}')} {1 - e^{-\beta k_0}} v(\vec{k}', t) \right), \quad (2.34)$$

$$v(\vec{k}', t) = \frac{1}{2} \left\langle \frac{1}{\omega_{k'}} \frac{d\bar{\varphi}(\vec{k}', t)}{dt} \frac{d\varphi(\vec{k}', t)}{dt} - \omega_{k'} \bar{\varphi}(\vec{k}', t)\bar{\varphi}(\vec{k}', t) \right\rangle. \quad (2.35)$$

In order to derive this relation, we first note that the quantity of the form,

$$\int \frac{dk_0}{2\pi} \tilde{f}(-k_0, x_0) F(k_0), \quad (2.36)$$

appears in eq.(2.33). This is related to a coincident limit of the full propagator,

$$\int \frac{dk_0}{2\pi} \tilde{f}(-k_0, x_0) F(k_0)$$
\[
\left. \frac{d}{dx_0} \right|_{x_0 \to y_0^+} F \left( -\frac{i}{2} \left( \frac{\partial}{\partial x_0} - \frac{\partial}{\partial y_0} \right) \right) \int \frac{dk_0}{2\pi} \tilde{f}(-k_0, \frac{x_0 + y_0}{2}) e^{i k_0 (x_0 - y_0)} = \lim_{x_0 \to y_0^+} F \left( -\frac{i}{2} \left( \frac{\partial}{\partial x_0} - \frac{\partial}{\partial y_0} \right) \right) (-i G(x_0, y_0)) .
\]

(2.37)

For simplicity we omitted the spatial coordinate or its Fourier component.

We expand in the Taylor series the derivative operation applied to
\[- i G(x, y) = \langle \varphi(x) \varphi(y) \rangle\] in this equation and evaluate derivatives using the Heisenberg equation for \( \varphi \). For instance,
\[
\begin{align*}
\lim_{x_0 \to y_0^+} \left( \frac{\partial}{\partial x_0} - \frac{\partial}{\partial y_0} \right) \varphi(x) \varphi(y) &= \dot{\varphi}(\vec{x}, x_0) \varphi(\vec{y}, x_0) - \varphi(\vec{x}, x_0) \dot{\varphi}(\vec{y}, x_0), \\
\lim_{x_0 \to y_0^+} \left( \frac{\partial}{\partial x_0} - \frac{\partial}{\partial y_0} \right)^2 \varphi(x) \varphi(y) &= \\
\left( \nabla^2 - M^2 \right) \varphi(\vec{x}, x_0) \varphi(\vec{y}, x_0) + \varphi(\vec{x}, x_0) \left( \nabla^2 - M^2 \right) \varphi(\vec{y}, x_0) \\
&- 2 \dot{\varphi}(\vec{x}, x_0) \dot{\varphi}(\vec{y}, x_0) - \lambda (\varphi^2(\vec{x}, x_0) \varphi(\vec{y}, x_0) - \lambda \varphi(\vec{x}, x_0) (\varphi^2)(\vec{y}, x_0) .
\end{align*}
\]

(2.39)

These terms have different structure for the even and odd number of derivatives, and it is convenient to separately deal with the even and the odd powers of the \( F(\omega) \) expansion, with the exception of the first term \( F(0) \). One thus arrives at the following operator identity in our \( \varphi^2 \chi^2 \) model;
\[
\begin{align*}
\lim_{x_0 \to y_0^+} F \left( -\frac{i}{2} \left( \frac{\partial}{\partial x_0} - \frac{\partial}{\partial y_0} \right) \right) \varphi(x) \varphi(y) &= \\
&\left[ F(0) \varphi(x) \varphi(y) - F_-(\Delta) \frac{i}{2} (\dot{\varphi}(x) \varphi(y) - \varphi(x) \dot{\varphi}(y)) + F_+ (\Delta) \\
&\cdot \frac{1}{2} \left( \dot{\varphi}(x) \dot{\varphi}(y) - \frac{1}{2} (\nabla^2 \varphi(x) \varphi(y) + \varphi(x) \nabla^2 \varphi(y)) + (M^2 + \lambda \chi^2) \varphi(x) \varphi(y) \right) \right]_{x_0 = y_0} ,
\end{align*}
\]

(2.40)

\[
\Delta \equiv \sqrt{-\nabla^2 + M^2 + \lambda \chi^2} ,
\]

(2.41)

\[
F_+(\omega) \equiv \frac{F(\omega) + F(-\omega) - 2F(0)}{2\omega^2} , \quad F_-(\omega) \equiv \frac{F(\omega) - F(-\omega)}{2\omega} .
\]

(2.42)

Here we assumed, for simplicity and without losing practical utility for our problem, that \( \chi^2 \) is independent of the spatial coordinate. For our purpose we may take
\[\langle \chi^2 \rangle = T^2 / 12\] for this constant value, to give the temperature dependent \( \varphi \) mass \( M^2 + \lambda T^2 / 12 \) in thermal medium. We shall have more to say on the temperature dependent mass later. Furthermore, in the homogeneous thermal medium of our interest one may take the spatial Fourier component,
\[
\int d^3 x e^{i \vec{k} \cdot \vec{x}} \langle \varphi(\vec{x}, x_0) \varphi(0, y_0) \rangle = \langle \hat{\varphi}(\vec{k}, x_0) \hat{\varphi}(\vec{k}, y_0) \rangle ,
\]

(2.43)
thereby replacing $\nabla^2$ by the relevant momentum, to get
\[
\Delta = \omega_k(T) = \sqrt{\vec{k}^2 + M^2 + \frac{\lambda}{12} T^2}.
\] (2.44)

For a notational simplicity we often omit the temperature dependence of the $\varphi$ mass, simply using the notation $\omega_k$ for $\omega_k(T)$.

Using the isotropy of medium, one has for the odd power terms,
\[
\lim_{x_0 \to y_0^+} \frac{-i}{2} \left( \frac{\partial}{\partial x_0} - \frac{\partial}{\partial y_0} \right) \langle \varphi^\dagger(\vec{k}, x_0) \varphi(\vec{k}, y_0) \rangle
\]
\[
= \lim_{x_0 \to y_0^+} \frac{-i}{2} \left( \frac{\partial}{\partial x_0} - \frac{\partial}{\partial y_0} \right) \frac{1}{2} \langle \varphi^\dagger(\vec{k}, x_0) \varphi(\vec{k}, y_0) + \varphi(\vec{k}, x_0) \varphi^\dagger(\vec{k}, y_0) \rangle
\]
\[
= -\frac{i}{4} \left( \frac{d\varphi^\dagger(\vec{k}, x_0)}{dt}, \varphi(\vec{k}, x_0) \right) + \left( \frac{d\varphi(\vec{k}, x_0)}{dt}, \varphi^\dagger(\vec{k}, x_0) \right) = -\frac{1}{2},
\] (2.45)

using the equal time canonical commutator,
\[
\left[ \frac{d\varphi(\vec{k}, t)}{dt}, \varphi(-\vec{k}', t) \right] = -i \delta^3(\vec{k} - \vec{k}').
\] (2.46)

Even power terms, on the other hand, contain the modified occupation number,
\[
f(\vec{k}, t) + \frac{1}{2} = \frac{1}{2\omega_k(T)} \langle \frac{d\varphi^\dagger(\vec{k}, t)}{dt} \frac{d\varphi(\vec{k}, t)}{dt} + \omega_k^2(T) \varphi^\dagger(\vec{k}, t) \varphi(\vec{k}, t) \rangle.
\] (2.47)

Using this operator relation for
\[
F(k_0') = \frac{2 r_\chi(k_0 + k_0', \vec{k} + \vec{k}')}{1 - e^{-\beta(k_0 + k_0')}},
\] (2.48)
one obtains the spectral weight for the Hartree model; the main result of this section, eqs. (2.34) − (2.35). Even and odd power terms have been lumped according to
\[
\frac{F(\omega_{k'}) + F(-\omega_{k'})}{2\omega_{k'}} \left( f(k', t) + \frac{1}{2} \right) - \frac{F(\omega_{k'}) - F(-\omega_{k'})}{2\omega_{k'}} \frac{1}{2}
\]
\[
= \frac{1}{2\omega_{k'}} \left( F(\omega_{k'}) f(k', t) + F(-\omega_{k'}) (1 + f(k', t)) \right),
\] (2.49)
extcept the first power term $F(0)$ giving the third term in the right side of eq.(2.34).

We shall later show that the combination of $f$ and $1 + f$ correctly describes the destructive and the creative processes, including the stimulated emission effect for bosons.
It is important to realize that the spectral function in the reduced Hartree model, $r(k, t)$ of eq. (2.34), cannot be written in terms of the distribution function $f(k', t) = \langle a^\dagger(k', t) a(k', t) \rangle$ of the particle alone, and contains another combination of the momentum and the coordinate, $v(k) = \langle p_k^2 / (2\omega_k) - \omega_k q_k^2 / 2 \rangle$, which vanishes for a system of a set of independent harmonic oscillators. To this extent the particle picture does not completely hold in a full quantum theory such as ours. In the next section we shall derive a self-consistent set of equations that determine the particle distribution function under the assumption of a slow variation of the particle number density. The $v$ term becomes a higher order effect in $\lambda^2$. This way one can derive a useful quantum kinetic equation. It is convenient, with this in mind, to separate the contribution to the spectral function without the $v$ term of (2.35),

$$r_0(k, t) = 2 \int \frac{d^3k'}{(2\pi)^3 2\omega_{k'}} \left( \frac{r_x(k_0 + \omega_{k'}, k + k')} {1 - e^{-\beta(k_0 + \omega_{k'})}} f(k', t) \right) + \frac{r_x(k_0 - \omega_{k'}, k - k')} {1 - e^{-\beta(k_0 - \omega_{k'})}} \left(1 + f(k', t)\right),$$

(2.50)

As a minor note, we point out that the relation $f(k', t) = f(|k'|, t)$ following from the isotropy was assumed in the derivation above. We also ignored a possible tadpole type of contribution, which will be discussed shortly.

The same operator equation, eq. (2.40), when applied to

$$F(k_0) = e^{ik_0(x_0 - y_0)} e^{-i\tilde{k} \cdot (\tilde{x} - \tilde{y})},$$

(2.51)

gives the full propagator in terms of the distribution function,

$$-i G(x, y) = \int \frac{d^3k}{(2\pi)^3 2\omega_k} e^{-i\tilde{k} \cdot (\tilde{x} - \tilde{y})} \left( e^{i\omega_k(x_0 - y_0)} f(k, t) \right) + e^{-i\omega_k(x_0 - y_0)} \left(1 + f(k, t)\right) - 2v(k, t),$$

(2.52)

where $\omega_k = \sqrt{\tilde{k}^2 + M^2 + \lambda \chi^2}$ with a constant $\chi^2$, and $t = (x_0 + y_0)/2$. Again, the full propagator contains $v$ in addition to the distribution function.

In the previous discussion we ignored a possibility of a tadpole type of contraction. For instance, in the correlator of

$$\langle \varphi(1) \varphi(2) \int dx \, dy \left( \varphi^2(x) - \varphi^2(x) \right) \left( \alpha(x - y) \varphi^2(y) - \alpha^*(x - y) \varphi^2(y) \right) \rangle,$$

(2.53)
one may contract both field operators at 1 and 2 with those of $\varphi^2$ at the same spacetime point $x$. This would lead to a new kernel of the form, $\propto \delta(x-y) \langle \varphi^2(0) \rangle$. More specifically, one allows a local term of the form,

$$
\delta\beta_4(x-y) = \delta(x-y) \langle \varphi^2(0) \rangle \int_{x_0>y_0} dy \alpha_I(x-y).
$$

(2.54)

It is then easy to show that

$$
\delta\beta_4(x) = -i\lambda^2 \delta(x) 4\langle \varphi^2(0) \rangle \int_{2m}^{\infty} dk_0 \frac{r_\chi(k_0,0)}{k_0} \coth \frac{\beta k_0}{2},
$$

(2.55)

with $m$ the $\chi$ mass. There is another tadpole term to even lower order $\lambda$ from

$$
\frac{i}{2} \lambda \langle \varphi(1)\varphi(2) \int dx \varphi^2(x)\chi^2(x) \rangle,
$$

(2.56)

which gives

$$
\delta\beta_2(x) = i \delta(x) \lambda \langle \chi^2(0) \rangle = i \delta(x) \lambda \int \frac{d^3k}{(2\pi)^3} \frac{\omega_k}{\coth \frac{\beta \omega_k}{2}}.
$$

(2.57)

These tadpoles, (2.55) and (2.57), are shown in Fig.1.

Both of these $\delta\beta_i(x)$ here are local ($\propto \delta(x)$) and are purely imaginary. These contain the mass and the coupling constant renormalization, the $\varphi^4$ coupling. Thus, the infinite part of these is eliminated by their respective counter terms. The remaining, temperature dependent finite terms give physical effects of finite temperature correction to the mass and the coupling constant. For discussion of order $\lambda^2$ effects in the kinetic equation, only the temperature dependent mass contributes in the Hartree model, changing the renormalized mass to

$$
M^2 \to M^2(T) = M^2 + \frac{\lambda}{12} T^2.
$$

(2.58)

The coefficient $\frac{\lambda}{12}$ was calculated from the vacuum subtracted $\lambda \langle \chi^2(0) \rangle - \langle \chi^2(0) \rangle_{T=0}$ above, and precisely coincides with the result of finite temperature field theory [15]. The temperature dependent $\varphi^4$ coupling of order $\lambda^2$ gives a higher order term to our subsequent kinetic equation, hence may effectively be ignored.

The temperature dependent mass has the structure of the tadpole term,

$$
- i \frac{\lambda}{2} \langle \chi^2 \rangle (\varphi^2(x) - \varphi'^2(x))
$$

(2.59)

in the influence functional. In the usual operator formalism at zero temperature this type of term $\langle \chi^2 \rangle$ is absent by the normal ordering, but in thermal environment
\( \langle \chi^2 \rangle \) has a temperature dependence, hence has a physical meaning. After renormalization one may write the temperature dependent mass term as \( \frac{1}{2} M^2(T) \varphi^2 \) in the Lagrangian density. With this new term, one should replace the previous \( \alpha(x) \), eq. (2.10) by

\[
\alpha(x) = \lambda^2 \left( \text{tr} T[\chi^2(x) \chi^2(0)] \rho_\beta - (\text{tr} \chi^2 \rho_\beta)^2 \right). \tag{2.60}
\]

With this new contribution included, one can forget about the tadpole term provided that one uses the temperature dependent mass, \( M(T) \), instead of the renormalized mass parameter. We shall come back to further aspects of the renormalization in Appendix B after we introduce the slow variation approximation in the next section.

The two-body spectral weight \( r_\chi(k) \) for \( \alpha(x) \) is calculable from the analytically continued imaginary-time thermal Green’s function \([12], [13]\). We first note the oddness, \( r_\chi(-\omega, \vec{k}) = -r_\chi(\omega, \vec{k}) \), hence \( r_\chi(-k) = -r_\chi(k) \) combined with the isotropy of space. The spectral weight is given by a discontinuity along the real axis of the energy \( \omega = k_0 \) corresponding to two \( \chi \) particle states in thermal medium. Since the kinematics is modified in thermal medium from that in vacuum, the relevant expressions are different, depending on relation between \( k_0 \) and \( |\vec{k}| \). For both \( k_0 > \sqrt{\vec{k}^2 + 4m^2} \) and \( |\vec{k}| > k_0 > 0 \) \([5]\),

\[
r_\chi(k_0, k) = \frac{\lambda^2}{16\pi^2} \left( \sqrt{1 - \frac{4m^2}{k_0^2 - k^2}} \theta(k_0 - \sqrt{k^2 + 4m^2}) + \frac{2}{\beta k} \ln \left( \frac{1 - e^{-\beta \omega_+}}{1 - e^{-\beta |\omega|}} \right) \right), \tag{2.61}
\]

\[
\omega_\pm = \frac{k_0}{2} \pm \frac{k}{2} \sqrt{1 - \frac{4m^2}{k_0^2 - k^2}}, \tag{2.62}
\]

where \( k = |\vec{k}| \). The formula is better understood \([14]\) if one writes this separately in the respective kinematic regions, using the thermal distribution function \( f_{\text{th}}(\omega) = 1/(e^{\beta \omega} - 1) \);

for \( \omega > \sqrt{\vec{k}^2 + 4m^2} \)

\[
r_\chi(\omega, k) = \frac{\lambda^2}{16\pi^2 k} \int_{\omega_+}^{\omega} dE \left( (1 + f_{\text{th}}(E)) (1 + f_{\text{th}}(\omega - E)) - f_{\text{th}}(E) f_{\text{th}}(\omega - E) \right), \tag{2.63}
\]

whereas for \( |\vec{k}| > k_0 > 0 \)

\[
r_\chi(\omega, k) = \frac{\lambda^2}{8\pi^2 k} \int_{-\omega_+}^{\infty} dE \left( f_{\text{th}}(\omega) (1 + f_{\text{th}}(\omega + E)) - (1 + f_{\text{th}}(E)) f_{\text{th}}(\omega + E) \right). \tag{2.64}
\]
Thus, the indivisual discontinuity has a one-to-one correspondence to a physical process such as something $\leftrightarrow \chi\chi$ and something $+\chi \leftrightarrow \chi$. An example of the two-body spectral function $r\chi$ is given in Fig.2.

The spectral weight $r_0(\omega, \vec{p}, t)$ in the Hartree model given by (2.50) takes a particularly simple form if one uses the energy conservation for the the thermal factor, $f_{th}^{-1}(\omega) + 1 = e^{\beta \omega}$. For instance, near the mass shell, $\omega \approx \pm \omega_p$, the relevant processes are the annihilation and the scattering process for $\omega \approx \omega_p$ and their inverse processes for $\omega = -\omega_p$; explicitly

$$
n_0(\omega_p, \vec{p}) = 2\lambda^2 \int dp' \int dk_1 \int dk_2 \cdot \left( (1 + f_{th}(k_1))(1 + f_{th}(k_2)) f(p') \delta(p + p' - k_1 - k_2) + 2f_{th}(k_1)(1 + f_{th}(k_2))(1 + f(p')) \delta(p + k_1 - p' - k_2) \right), \\
r_0(-\omega_p, \vec{p}) = 2\lambda^2 \int dp' \int dk_1 \int dk_2 \cdot \left( f_{th}(k_1)f_{th}(k_2)(1 + f(p')) \delta(k_1 + k_2 - p - p') + 2f_{th}(k_2)(1 + f_{th}(k_1)) f(p') \delta(p' + k_2 - p - k_1) \right).$$

(2.65)

(2.66)

Four processes appearing in these equations are depicted in Fig.3. Here

$$
f_{th}(k) = \frac{1}{e^{\beta \omega_k} - 1}, \quad \omega_k = \sqrt{k^2 + m^2}.$$

(2.67)

We used an abbreviated notation for the phase space integral;

$$
\int dk (\cdots) = \int \frac{d^3k}{(2\pi)^3 2\omega_k} (\cdots), \text{ etc.}
$$

(2.68)

and $\delta(p + p' - k_1 - k_2) = (2\pi)^3 \delta^4(p + p' - k_1 - k_2)$, etc. When multiplied by $f(p)$ and $1 + f(p)$, these give the on-shell destruction and the production rate of the momentum mode $p$ with a suitable normalization.

3 Slow variation and self-consistent equation

Once the Hartree approximation is defined with the kernel $\beta(x, y)$ in the influence functional given, one may proceed as follows. We have in mind a physical situation in which one can clearly separate two time scales; the microscopic time scale for the quantum behavior governed by the Hamiltonian, and a macroscopic time scale for
the change of the $\varphi$ particle distribution function $f(\vec{k}, t)$. The separation naturally occurs for a small system within a dilute thermal medium at very low temperatures, where the system itself is not far from equilibrium. Under this circumstance one may consider a slowly varying $f(\vec{k}, t)$ in the time range, $t \gg$ microscopic relaxation time. We thus take, in considering the short time variation, the time $t$ in the distribution function as a fixed constant. It is then straightforward to work out consequences of the short time dynamics, since the truncated Hartree model is formally equivalent to the harmonic oscillator model, known to be completely solvable [5], [6]. Since the relaxation rate towards the equilibrium depends on the distribution function at that time, this consideration leads to a self-consistent equation for the the quasi-stationary distribution function.

Separation of the two time scales to distinguish the slow and the fast process also appears in other approaches [1], [2] when the quantum Boltzmann equation is derived in different contexts.

We shall briefly summarize the main point of how the exactly solvable model is used in our Hartree model. One first recalls that one can discuss each Fourier $\vec{k}$ mode separately. The Fourier mode kernel $\beta(\vec{k}, t)$ in the influence functional is related to the kernel given at the relative coordinate $\vec{x}$ by

$$\beta(\vec{x}, x_0) = \int \frac{d^3k}{(2\pi)^3} \beta(\vec{k}, x_0) e^{i\vec{k} \cdot \vec{x}}. \quad (3.1)$$

For the sake of clarity we sometimes omit the mode index $\vec{k}$ unless confusion occurs. One notes that the path integral for the $\varphi$ system in the influence functional has an exponent of the form,

$$\frac{i}{2} \int_{t_i}^{t_f} d\tau \left( \dot{\xi}(\tau) \dot{X}(\tau) - \omega^2(T) \xi(\tau) X(\tau) \right)$$

$$- \int_{t_i}^{t_f} d\tau \int_{t_i}^{\tau} ds \left( \xi(\tau) \beta_R(\tau - s) \xi(s) + i \xi(\tau) \beta_I(\tau - s) X(s) \right). \quad (3.2)$$

Here $\omega^2(T) = k^2 + M^2(T)$ includes the temperature dependent mass given by eq.(2.58).

The time $x_0$ in eq.(3.1) is a common CM time in the original $\beta(x, y)$. Since there is only one common time, one has the time translation invariance for $\beta$, which was absent in the original $\beta(x, y)$.

A remarkable feature of this exponent (3.2) is that it is linear in the variable $X = \varphi + \varphi'$, hence its path integration yields a trivial delta function. This gives an
equation for the semiclassical path for the \( \xi(=\varphi - \varphi') \) variable;

\[
\frac{d^2\xi}{d\tau^2} + \omega^2(T) \xi(\tau) + 2 \int_{\tau}^{t} ds \xi(s) \beta_I(s - \tau) = 0.
\] (3.3)

We set hereafter \( t_i = 0 \) and \( t_f = t \). This integro-differential equation can be solved by the standard technique of the Laplace transform, as described in [5]. Its solution \( \xi_{cl} \) is given by

\[
\xi_{cl}(\tau) = \xi_i y(\tau) + \xi_f z(\tau),
\] (3.4)

\[
y(\tau) = \frac{g(t - \tau)}{g(t)} , \quad z(\tau) = \dot{g}(t - \tau) - \frac{g(t - \tau)\dot{g}(t)}{g(t)},
\] (3.5)

\[
g(\tau) = \frac{1}{2\pi i} \int_{p_0-i\infty}^{p_0+i\infty} \frac{dp}{p^2 + \omega^2(T) + 2\beta(p)} e^{p\tau},
\] (3.6)

\[
\tilde{\beta}(p) = \int_{0}^{\infty} d\tau \ e^{-p\tau} \beta_I(\tau),
\] (3.7)

with \( p_0 > 0 \). Note that \( \dot{g}(0) = 1 \).

Applied to our specific model, we have

\[
\tilde{\beta}(p, \vec{k}) = -\int_{0}^{\infty} d\omega \ \frac{\omega r_-(\omega, \vec{k})}{p^2 + \omega^2},
\] (3.8)

\[
r_{\pm}(\omega, \vec{k}) = r(\omega, \vec{k}) \pm r(-\omega, \vec{k}),
\] (3.9)

for the momentum mode \( \vec{k} \). The basic function \( g(\tau) \) for this mode is then

\[
g(\vec{k}, \tau) = \frac{1}{2\pi} \int_{-\infty+0^+}^{\infty+0^+} dz \ e^{-iz\tau} F(z, \vec{k}),
\] (3.10)

\[
- F(z, \vec{k})^{-1} = z^2 - \omega_k^2(T) - 2 \int_{0}^{\infty} d\omega \ \frac{\omega r_-(\omega, \vec{k})}{z^2 - \omega^2}.
\] (3.11)

We deformed the contour of integration as \( p \rightarrow -iz \) in the Laplace inverted formula.

Both \( g(\tau) \) and its time derivative \( \dot{g}(\tau) \) obey a related equation to (3.3);

\[
\frac{d^2 x}{d\tau^2} + \omega^2(T) x(\tau) + 2 \int_{0}^{\tau} ds \beta_I(\tau - s) x(s) = 0,
\] (3.12)

with different boundary conditions, \( g(0) = 0, \ \dot{g}(0) = 1 \).

Important quantities for the influence functional in the Hartree model are two real functions \( \beta_i \);

\[
\beta_R(\vec{k}, t) = \int_{0}^{\infty} d\omega \ r_+(\omega, \vec{k}) \cos(\omega t),
\] (3.13)

\[
\beta_I(\vec{k}, t) = -\int_{0}^{\infty} d\omega \ r_-(\omega, \vec{k}) \sin(\omega t),
\] (3.14)
Different combinations \( r_{\pm} \) appear, because these integrals originally defined in \(-\infty < \omega < \infty \) have definite parities; \( \beta_R (\beta_I) \) is even (odd) in \( t \). These \( \beta_i \) depend on \( f(\vec{k}) \) of the \( \varphi \) particle distribution via \( r_{\pm}(\omega, \vec{k}) \).

For comparison we write the corresponding quantity \( \beta^{(d)} \) in the unstable particle decay described by the Lagrangian density, \( \lambda \varphi \chi^2 \) (\( \lambda \) having a mass dimension in this case);

\[
\beta_{R}^{(d)}(\vec{k}, t) = \int_{0}^{\infty} d\omega \ coth \frac{\beta \omega}{2} r_{\chi}(\omega, \vec{k}) \cos(\omega t),
\]

\[
\beta_{I}^{(d)}(\vec{k}, t) = -\int_{0}^{\infty} d\omega r_{\chi}(\omega, \vec{k}) \sin(\omega t),
\]

with \( r_{\chi} \) given by eq.(2.61). Thus, in this case \( r^{(d)}_{\pm} \) that defines \( \beta_{R,I}^{(d)} \) satisfies \( r_{+}^{(d)}(\omega, \vec{k}) = \coth \frac{\beta \omega}{2} r^{(d)}_{-}(\omega, \vec{k}) \).

On the other hand, it turns out that in the present annihilation-scattering problem a similar relation holds only on the mass shell and only when one takes the thermal distribution \( f_{\text{th}} \) for \( f(\vec{k}) \);

\[
r_{+}(\omega_{k}, \vec{k})_{\text{th}} = \coth \frac{\beta \omega_{k}}{2} r_{-}(\omega_{k}, \vec{k})_{\text{th}},
\]

which further reduces to the detailed balance relation,

\[
r(\omega_{k}, \vec{k})_{\text{th}} f_{\text{th}}(\vec{k}) = r(-\omega_{k}, \vec{k})_{\text{th}} \left( 1 + f_{\text{th}}(\vec{k}) \right).
\]

A deep reason why this relation does not hold for the more general non-equilibrium case in our annihilation-scattering problem is that \( r_{\pm}(\omega, \vec{k}) \) are functionals of the non-equilibrium \( \varphi \) distribution and in general, \( f(\vec{k}) \neq f_{\text{th}}(\vec{k}) \). In the simple case of the unstable particle decay into two thermal particles the corresponding spectrum is independent of the distribution function of the decaying particle. We further note that distinction of two types of the spectrum \( r_{\pm} \) is crucial to obtain the correct form of the Boltzmann equation in the next section.

Under a certain condition the function \( F(z, \vec{k}) \) is analytic except on the real axis where there is a branch cut singularity as shown in Fig.4. As seen by taking the imaginary part of the defining equation \( \text{(3.11)} \), this analyticity holds under the condition,

\[
r_{-}(\omega, \vec{k}) = r(\omega, \vec{k}) - r(-\omega, \vec{k}) > 0.
\]

This condition for the analytic property of \( F(z, \vec{k}) \), when evaluated on the mass shell \( \omega = \omega_{k} \), physically means that the destructive process of \( \varphi \) given by the first term
$r(\omega_k, \vec{k})$ dominates over the production process given by the second term $r(-\omega_k, \vec{k})$, a situation we are practically interested in. Furthermore, in the weak coupling limit the above condition needs to be obeyed only near $\omega = \omega_k$, since the off-shell contribution in the weak coupling limit is negligible in determining the analyticity. The analytic extension of $F(z, \vec{k})$ has two simple poles in the second Riemann sheet, approximately at

$$z = \pm \omega_k(T) - i \frac{\pi r_-(\omega_k, \vec{k})}{2\omega_k}. \quad (3.20)$$

The poles in the second sheet are close to the real axis in the weak coupling limit, $\lambda^2 \to 0$. When pole terms dominate in the integral, the exponential decay law follows;

$$g(\vec{k}, \tau) \approx \frac{\sin \omega_k \tau}{\omega_k} \exp\left(-\frac{\Gamma_k \tau}{2}\right), \quad (3.21)$$

$$\Gamma_k = \frac{\pi r_-(\omega_k, \vec{k})}{\omega_k} = \frac{\pi}{\omega_k} \left( r(\omega_k, \vec{k}) - r(-\omega_k, \vec{k}) \right). \quad (3.22)$$

As is clear in (2.65), (2.66), the decay rate $\Gamma_k$ has both annihilation and scattering contribution along with their inverse processes.

On the other hand, when $r_-(\omega_k, \vec{k}) < 0$, these poles appear in the first Riemann sheet. The physical situation here is the dominance of the production process, which is not of our immediate concern. It however appears that this case can be dealt with analogously to the above case of $r_- > 0$.

We now turn to derivation of the self-consistent equation for the distribution function. Recall first for each Fourier mode $\vec{k}$,

$$f(\vec{k}, t) = \langle a^\dagger(\vec{k}, t) a(\vec{k}, t) \rangle. \quad (3.23)$$

Here the creation ($a^\dagger$) and the annihilation ($a$) operators of $\varphi$ particle are Heisenberg operators such as $e^{iHt} a e^{-iHt}$ with $H$ the total Hamiltonian including the system-environment interaction. In the harmonic oscillator basis which is an essential element of the plane wave decomposition of the field operator, this is equal to

$$f(\vec{k}, t) = \left( \frac{1}{2\omega_k} p_k(t) p_{-k}(t) + \frac{\omega_k}{2} q_k(t) q_{-k}(t) - \frac{1}{2} \right), \quad (3.24)$$

where the coordinate and the momentum, $q_k, p_k$, are identified using the field decomposition into the plane wave; in the previous notation,

$$q_k(t) \equiv \hat{\varphi}(\vec{k}, t), \quad p_k(t) \equiv \frac{d\hat{\varphi}(\vec{k}, t)}{dt},$$

$$\hat{\varphi}(\vec{k}, t) = \int d^3x \varphi(\vec{x}, t) e^{-i\vec{k} \cdot \vec{x}}. \quad (3.25)$$
The subtracted factor $\frac{1}{2}$ in (3.24) is the well known contribution from vacuum fluctuation. For simplicity we subsequently use the notation, for example $q_k^2$ to mean $q_k q_{-k}$.

We mention here an ambiguity for the choice of the reference energy $\omega_k$ to define the occupation number. We used here the "free" part of the oscillator energy $\omega_k = \sqrt{k^2 + M^2}$. In the presence of the interaction with thermal environment the use of this unobservable energy is however dubious. Indeed, we confirmed, as will be explained in Appendix B, that this choice would lead to an unacceptable result of the absence of the equilibrium distribution at very low temperatures. As will be discussed later, the proper choice of the reference energy turns out to be the renormalized energy including $O[\lambda^2]$ correction of the finite self-energy shift in medium, eq(3.37).

We need to sharpen the concept of various time scales in computing the correlator. Consider the correlator at different times such as $\langle q(t_0 + t/2)q(t_0 - t/2) \rangle$, and suppose that $t_0 \gg t$, all the times measured from some specified initial time. We then let $t_0 \gg$ several $\times$ relaxation time, and vary the relative time $t$ in the range from 0 to several $\times$ the relaxation time. Under this circumstance we approximate $\langle q(t_0 + t/2)q(t_0 - t/2) \rangle \approx \langle q^2(t_0) \rangle$. We thus need the coincident limit at large times ($\gg$ relaxation time).

Some means to compute the correlator at the coincident time such as $\langle q^2(t) \rangle$ becomes necessary. This or a more general multi-time correlator such as $\langle q(t_1)q(t_2) \cdots q(t_n) \rangle$ can be computed with the machinery of the generating functional. Instead of being much involved in technical details, we shall give only a general idea here and relegate all technical points of the generating functional to Appendix A. An alternative method of computation is to use the exact operator solution for the harmonic system, as is done in [3].

Calculation of the Green’s function in the path integral approach is performed by introducing a coupling of external source terms of the form,

$$j(\tau)q(\tau) + l(\tau)p(\tau) - j'(\tau)q'(\tau) - l'(\tau)p'(\tau) = \frac{1}{2} \left( S_j \xi + D_j X + S_l p_\xi + D_l p_X \right), \quad (3.27)$$

where $S_j = j + j'$, $D_j = j - j'$, $p_X, \xi = p \pm p'$, etc. Functional differentiation with respect to the sources, $S_i, D_i$ then gives various combination of correlators, which in turn gives necessary two point correlators, $\langle q(1)q(2) \rangle, \langle p(1)p(2) \rangle, \langle q(1)p(2) \rangle$.

The result is

$$\langle q_k^2(t) \rangle = \int_0^\infty d\omega r_+(\omega, \vec{k}) |h(\omega, \vec{k}, t)|^2$$
\[ + \gamma^2(\vec{k}, t) \overline{p_i^2} + \gamma^2(\vec{k}, t) \overline{q_i^2} + g(\vec{k}, t) \gamma(\vec{k}, t) \overline{p_i q_i + q_i p_i}, \]  
\[ \langle p_i^2(t) \rangle = \int_0^\infty d\omega r_+(\omega, \vec{k}) |k(\omega, \vec{k}, t)|^2 \]  
\[ + \gamma^2(\vec{k}, t) \overline{p_i}^2 + \gamma^2(\vec{k}, t) \overline{q_i}^2 + \gamma(\vec{k}, t) \gamma(\vec{k}, t) \overline{p_i q_i + q_i p_i}, \]  
\[ \frac{1}{2} \left\langle q_k(t)p_k(t) + p_k(t)q_k(t) \right\rangle = \int_0^\infty d\omega r_+(\omega, \vec{k}) \Re \left( h(\omega, \vec{k}, t) k^*(\omega, \vec{k}, t) \right) \]  
\[ + \gamma(\vec{k}, t) g(\vec{k}, t) \overline{p_i^2} + \gamma(\vec{k}, t) \gamma(\vec{k}, t) \overline{q_i^2} + \left( \gamma^2(\vec{k}, t) + g(\vec{k}, t) \gamma(\vec{k}, t) \right) \frac{1}{2} \overline{p_i q_i + q_i p_i}, \]  
\[ h(\omega, \vec{k}, t) = \int_0^t d\tau g(\vec{k}, \tau) e^{-i\omega \tau}, \]  
\[ k(\omega, \vec{k}, t) = \int_0^t d\tau \gamma(\vec{k}, \tau) e^{-i\omega \tau}. \]  

The coincident time limit in these formulas may be understood only with the condition, \(|\text{relative time}| \ll t\), thus the relative time can be as large as the relaxation time. Quantities such as \(q_i^2\) are the values averaged over the ensemble at a specified initial time. We have chosen the initial ensemble such that \(q_i = 0\) and \(p_i = 0\). Note that the relevant kernel for the correlator is \(\beta R(\vec{k}, t)\), as seen from (A.19), hence the corresponding spectral combination is \(r_+(\omega, \vec{k})\) appearing in the \(\omega\) integral here instead of \(r_-\).

We now coarse-grain the short time dynamics to derive the self-consistent equation for the equilibrium occupation number. We first note that except at very early and very late times the simple exponential decay law for \(g(\vec{k}, t)\) is an excellent approximation. This intermediate epoch of the exponential decay law is ideal to obtain the coarse-grained behavior. A sacrifice resulting from the replacement by the intermediate exponential law is that the very early quantum behavior is lost. We lose nothing, however, in the late time behavior, because the power law behavior present at much later times is not realized due to a large multiplicity of actual reactions; indeed, the relaxation rate \(\Gamma_k\) which will be precisely defined later depends on the time weakly via the slowly varying distribution function \(f(\vec{k}, t)\).

We observe that in the formulas for the dynamical variables, eq.(3.28) – (3.30), the initial value dependence disappears with the exponentially decaying \(g(\vec{k}, t)\). We then use the limiting behavior of \(h(\omega, \vec{k}, t)\) and \(k(\omega, \vec{k}, t)\) as \(t \to \infty\), to actually mean \(t \gg \) relaxation time. From the expression of \(g(\vec{k}, t)\), eq.(3.10), and the definition of these functions in (3.31), (3.32) we find that

\[ |h(\omega, \vec{k}, \infty)|^2 \approx \frac{1}{(\omega^2 - \tilde{\omega}_k^2(T))^2 + (\pi r_-(\omega, \vec{k}))^2}, \]  

24
\[ |k(\omega, \vec{k}, \infty)|^2 \approx \frac{\omega^2}{(\omega^2 - \bar{\omega}_k^2(T))^2 + (\pi r_-(\omega, \vec{k}))^2}, \quad (3.34) \]

\[ \bar{\omega}_k^2(T) = \vec{k}^2 + M^2 + \frac{\lambda}{12} T^2 + \Pi(\omega_k, \vec{k}), \quad (3.35) \]

\[ \Pi(\omega, \vec{k}) = -\mathcal{P} \int_{-\infty}^{\infty} d\omega' \frac{r_-(\omega', \vec{k})}{\omega' - \omega}. \quad (3.36) \]

A possible infinity in the proper self-energy \( \Pi(\omega_k, \vec{k}) \) is removed by a mass counter term such that the combination \( M^2 + \Pi(\omega_k, \vec{k}) \) is made finite by renormalization; after the wave function renormalization as explained in Appendix B,

\[ \bar{\omega}_k^2(T) \rightarrow (\omega_k^R)^2 = \vec{k}^2 + M_R^2 + \frac{\lambda}{12} T^2 + \delta \Pi(\omega_k, \vec{k}), \quad (3.37) \]

where \( M_R \) is the finite renormalized mass and \( \delta \Pi(\omega_k, \vec{k}) \) is defined in eq.(B.11).

The self-consistent equation for the stationary value is then

\[ f_{\text{eq}}(\vec{k}) + \frac{1}{2} = \int_0^{\infty} d\omega \left( \frac{\omega}{2} + \frac{\omega^2}{2\omega_k} \right) \frac{r_+(\omega, \vec{k})}{(\omega^2 - \bar{\omega}_k^2(T))^2 + (\pi r_-(\omega, \vec{k}))^2}, \quad (3.38) \]

\[ v_{\text{eq}}(\vec{k}) = \int_0^{\infty} d\omega \left( \frac{\omega^2 - \omega_k^2}{2\omega_k} \right) \frac{r_+(\omega, \vec{k})}{(\omega^2 - \bar{\omega}_k^2(T))^2 + (\pi r_-(\omega, \vec{k}))^2}, \quad (3.39) \]

\[ r_\pm(\omega, \vec{k}) = r(\omega, \vec{k}) \pm r(-\omega, \vec{k}), \quad (3.40) \]

\[ r(\omega, \vec{k}) = 2 \int \frac{d^3k'}{(2\pi)^3 2\omega_k} \left( \frac{r_x(\omega + \omega_k', \vec{k} + \vec{k}')}{1 - e^{-\beta(\omega + \omega_k')}} f_{\text{eq}}(\vec{k'}) \right) \]

\[ + \frac{r_x(\omega - \omega_k', \vec{k} - \vec{k}')}{1 - e^{-\beta(\omega - \omega_k')}} (1 + f_{\text{eq}}(\vec{k'})) - \frac{2 r_x(\omega, \vec{k} + \vec{k}')}{1 - e^{-\beta \omega}} v_{\text{eq}}(\vec{k'}) \right). \quad (3.41) \]

In this self-consistency equation the two-body kernel \( r_x \) is a given function. The function \( v_{\text{eq}}(\vec{k}) \) is the stationary value of \( v(\vec{k}, t) \) given by

\[ v(\vec{k}, t) = \left( \frac{1}{2\omega_k} p_k^2(t) - \frac{\omega_k}{2} q_k^2(t) \right), \quad (3.42) \]

and may be considered as a functional of \( f_{\text{eq}}(\vec{k}) \) when one first solves eq.(3.39) for \( v_{\text{eq}} \).

The set of self-consistent equations, eqs.(3.38) – (3.41), along with the definition of \( r_x (2.61) \), does not depend on the initial \( \varphi \) state. We assume that there exists a unique solution to the self-consistent equation, when a proper renormalization is performed. We also anticipate and later demonstrate that the high temperature limit of this equation gives the familiar distribution function \( 1/(e^{\beta \omega_k} - 1) \).
A simplified computation is possible in the weak coupling limit. In the limit of \( \lambda \to 0 \), one can ignore the \( \omega \) dependence of \( r_-(\omega, \vec{k}) \) in the denominator of the Breit-Wigner function since it is narrowly peaked around \( \omega = \omega_k \). By separating the narrow width contribution, one has

\[
f_{eq}(\vec{k}) = \frac{r(-\omega_k, \vec{k})}{r_-(\omega_k, \vec{k})} + \delta f_{eq}(\vec{k}),
\]

\[
\delta f_{eq}(\vec{k}) = \frac{1}{4\omega_k} \int_{-\infty}^{\infty} d\omega \frac{r_+(\omega, \vec{k}) - r_+(\omega_k, \vec{k})}{(\omega - \bar{\omega}_k(T))^2 + \Gamma_k^2/4}, \tag{3.43}
\]

with \( \Gamma_k = \pi r_-(\omega_k, k)/\omega_k \). We have used \( r_+(-\omega, \vec{k}) = r_+(\omega, \vec{k}) \). The tilded \( \delta f_{eq} \) contains terms to be renormalized away by subtraction, in contrast to the finite \( \delta f_{eq} \) later defined after renormalization. Note that both \( \delta f_{eq} \) and \( v_{eq} \) are of the same coupling order, \( O[\lambda^2] \), since the on-shell term of \( O[\lambda^0] \) is absent. It is then easy to see that the quantity \( v_{eq} \) gives an even higher order \( O[\lambda^4] \) correction to \( r_\pm \), and one can forget about \( v_{eq} \) altogether to \( O[\lambda^2] \).

The self-consistent equation in the present form is not particularly illuminating, since cancellation occurs among scattering terms in the on-shell part when the distribution function is integrated over momenta, namely in the quantity,

\[
\int \frac{d^3k}{(2\pi)^3} \Gamma_k \left( f_{eq}(\vec{k}) - \frac{r(-\omega_k, \vec{k})}{r_-(\omega_k, \vec{k})} \right).
\]

(3.45)

In the next section we shall directly work out the stationary number density integrated over momenta. In the rest of this section we shall focus on the off-shell part \( \delta f_{eq} \).

To further simplify the off-shell contribution, we use a sum rule that results from a consistency condition; from the equality, \( \dot{g}(0) = 1 \),

\[
2 \int_0^\infty d\omega \frac{\omega r_-(\omega, \vec{k})}{(\omega^2 - \bar{\omega}_k(T))^2 + (\pi r_-(\omega, \vec{k}))^2} = 1. \tag{3.46}
\]

Using

\[
\int_{-\infty}^{\infty} d\omega \frac{1}{(\omega - \bar{\omega}_k(T))^2 + \Gamma_k^2/4} = \frac{2\pi}{\Gamma_k}, \tag{3.47}
\]

\[
\int_0^\infty d\omega \frac{\omega r_-(\omega, \vec{k})}{(\omega^2 - \bar{\omega}_k(T))^2 + (\pi r_-(\omega, \vec{k}))^2} \approx \frac{1}{4\bar{\omega}_k(T)} \int_{-\infty}^{\infty} d\omega \frac{r_-(\omega, \vec{k})}{(\omega - \bar{\omega}_k(T))^2 + \Gamma_k^2/4}, \tag{3.48}
\]
with \( \Gamma_k = \pi r_-(\omega_k, \vec{k})/\bar{\omega}_k(T) \approx \pi r_-(\omega_k, \vec{k})/\omega_k \), the consistency integral (3.46) may be rewritten as
\[
\int_{-\infty}^{\infty} d\omega \frac{r_-(\omega, \vec{k}) - r_-(\omega_k, \vec{k})}{(\omega - \bar{\omega}_k(T))^2 + \Gamma_k^2/4} = 0. \tag{3.49}
\]
We have neglected the \( \omega \) dependence of \( r_-(\omega, \vec{k}) \) in the denominator in deriving this equation.

Since \( r_\pm(\omega) = r(\omega) \pm r(-\omega) \), the off-shell equilibrium distribution becomes
\[
\delta \tilde{f}_{\text{eq}}(\vec{k}) = \frac{1}{2\omega_k} \int_{-\infty}^{\infty} d\omega \frac{r(-\omega, \vec{k}) - r(-\omega_k, \vec{k})}{(\omega - \bar{\omega}_k(T))^2 + \Gamma_k^2/4}, \tag{3.50}
\]

\[
r(-\omega, \vec{k}) = 2 \int \frac{d^3k'}{(2\pi)^32\omega_{k'}} \left( \frac{r_{\chi}(\omega - \omega_{k'}, \vec{k} - \vec{k'})}{e^{\beta(\omega - \omega_{k'})} - 1} f(\vec{k'}, t) + \frac{r_{\chi}(\omega + \omega_{k'}, \vec{k} + \vec{k'})}{e^{\beta(\omega + \omega_{k'})} - 1} (1 + f(\vec{k'}, t)) \right). \tag{3.51}
\]

In the spectral function \( r(-\omega, \vec{k}) \) here, one can disregard the term involving \( \nu_{\text{eq}} \) in eq. (3.41), since it is of higher order, \( O[\lambda^4] \).

A region of \( \omega \) integral in the formula (3.44) for \( \delta \tilde{f}_{\text{eq}} \) does not contribute; this is the region of \(|\omega - \omega_c| < \omega_c\), where \( \omega_c \) is the energy scale for which \( r(\omega, \vec{k}) \) appreciably varies and in the weak coupling limit \( \omega_c \gg \Gamma_k \). This means that the resulting \( \delta \tilde{f}_{\text{eq}}(\vec{k}) \) is insensitive to the actual value of \( \Gamma_k \). We may use this fact to replace \( \Gamma_k \) by a small quantity \( \tilde{\Gamma}_k \) which is taken independent of the distribution function \( f(\vec{k}) \);
\[
\delta \tilde{f}_{\text{eq}}(\vec{k}) \approx \frac{1}{2\omega_k} \int_{-\infty}^{\infty} \frac{r(-\omega, \vec{k}) - r(-\omega_k, \vec{k})}{(\omega - \bar{\omega}_k(T))^2 + \Gamma_k^2/4}. \tag{3.52}
\]
We shall specify \( \tilde{\Gamma}_k \) shortly in (3.53) by a quantity even independent of the momentum \( \vec{k} \). A great virtue of this formula is that the off-shell part \( \delta \tilde{f}_{\text{eq}} \) of the self-consistency formula becomes a linear functional equation of \( f_{\text{eq}}(\vec{k}) \), and the perturbative treatment becomes transparant. This form of the equilibrium distribution is most convenient and is later used frequently.

The universal width factor \( \tilde{\Gamma}_k \) independent of \( f \) may be defined by using the value of the width \( \Gamma_k \) calculated at \( f = 0 \). Since \( \Gamma_k = \pi r_-(\omega_k, \vec{k})/\omega_k \) is the on-shell value, it is dominated by the scattering contribution, \( \Gamma_k = \pi r_s(\omega_k, \vec{k})/\omega_k \). Thus, it numerically has the thermal number density factor \( n_{\text{th}} \approx T^3 \) times the scattering cross section of order \( \lambda^2/M^2 \). More explicitly, \( \tilde{\Gamma}_k \) becomes independent of the mode
\[ \tilde{k} \text{, hence is denoted by } \Gamma \text{ with} \]
\[ \Gamma = \frac{\zeta(3) \lambda^2 T^3}{4 \pi^3 M^2}. \] (3.53)

In Appendix B we identify the renormalization term in the proper self-energy \( \Pi(\omega, \tilde{k}) \). This gives constant counter terms to be added to the distribution function; the renormalized finite occupation number is
\[ f_{\text{ren}}(\tilde{k}) = f_{\text{eq}}(\tilde{k}) - \frac{A + B \tilde{k}^2}{4 \omega_{\tilde{k}}^2}. \] (3.54)

The infinite counter term \( A, B \) cancels the corresponding infinity of the \( O[\lambda^2] \) correction in the distribution function.

In Appendix D we give a detailed account of our computation of
\[ \delta f_{\text{eq}}(\tilde{k}) = f_{\text{ren}}(\tilde{k}) - \frac{r(-\omega_k, \tilde{k})}{r_-(\omega_k, \tilde{k})}. \] (3.55)

To the leading order of \( T/M \) the result of this computation is summarized as a sum of a \( f \)-dependent and a \( f \)-independent term,
\[ \delta f_{\text{eq}}(\tilde{k}) = f_f(\tilde{k}) + f_{2,0}^0(\tilde{k}) \] (3.56)

As will be explained in Appendix D, the \( f \)-independent term has a dominant contribution from the inverse process \( \chi \chi \rightarrow \varphi \varphi \), and is computed as
\[ f_{2,0}^0(\tilde{k}) = \frac{1}{2\omega_k} \int_{-\infty}^{\infty} d\omega \left( \frac{1}{(\omega - \omega_k)^2 + \Gamma_k^2/4} - \frac{1}{(2\pi)^3 2\omega_k'} \right) \]
\[ \times \left( \frac{r_\chi(\omega, \tilde{k}, \tilde{k}')}{e^{\beta(\omega + \omega_k')}} - 1 \right) \]
\[ \approx \frac{1}{\omega_k} \int \frac{d^3k'}{(2\pi)^3} \frac{2}{(\omega_k + \omega_{k'})^2} \int_0^\infty d\omega \frac{r_\chi(\omega, \tilde{k}, \tilde{k}')}{e^{\beta\omega} - 1} \]
\[ \approx \frac{\zeta(2) \lambda^2}{16\pi^4} \frac{T^2}{\tilde{k}^2} \int_{\omega_k}^{\infty} dq \frac{q}{2q + \zeta(2)T} \left( \frac{1}{\omega_k + \omega_{k-q}} - \frac{1}{\omega_k + \omega_{k+q}} \right). \] (3.57)

The \( f \)-dependent term \( f_f \) is given in Appendix D, but actually is not necessary to write the time evolution equation for the number density. The use of the low temperature formula for \( r_\pm \) for computation of \( \delta f_{\text{eq}} \) even at high temperatures is justified, since at higher temperatures the on-shell contribution dominates and the term \( \delta f_{\text{eq}} \) becomes irrelevant in \( f_{\text{eq}}(\tilde{k}) \) which is dominated by the familiar \( f_{\text{th}} \).
An example of the $\omega$–integrand \((3.57)\) is shown in Fig.5, where the total, the Planck, and the rest of the integrand are separately given.

Physical processes that contribute to the important piece \(f^0_2\) are predominantly inverse annihilation $\chi\chi \to \varphi\varphi$, and 1 to 3 process, $\chi \to \chi\varphi\varphi$, which gives a smaller fraction of the total number density. On the other hand, the term $f_f$ arises from the inverse scattering process. We shall show in the next section that the scattering-related term $f_f(\vec{k})$ is subdominant compared to $f^0_2(\vec{k})$ in determining the equilibrium number density.

Although it is technically complicated to calculate the equilibrium distribution function $f_{eq}$, one may regard this calculation as a self-consistent approximation for the quantity,

$$f_{eq}(\vec{k}) = \frac{\text{tr} (a^+_k a_k e^{-\beta H_{tot}})}{\text{tr} e^{-\beta H_{tot}}}, \quad (3.59)$$

where $H_{tot}$ is the total Hamiltonian including interaction between the $\varphi$ system and the $\chi$ environment. We explicitly demonstrated this relation for the exactly solvable harmonic oscillator model in [6], whose result is used for our self-consistent solution. Thus, the Gibbs formula is valid, while the ideal gas form of the distribution $1/(e^{\beta \omega_k} - 1)$ is changed.

### 4 Quantum kinetic equation

We go back to the time dependent expectation values, given in eq.\((3.28) - (3.30)\).

It is sometimes convenient to write a formula for the time derivative;

$$\frac{df(\vec{k}, t)}{dt} = -\Gamma(\vec{k}, t) \cdot \left( f(\vec{k}, t) - \int_0^\infty d\omega \, r_+(\omega, \vec{k}) \left( \frac{\omega_k}{2} |h(\omega, \vec{k}, t)|^2 + \frac{1}{2\omega_k} |k(\omega, \vec{k}, t)|^2 \right) \right)$$

$$+ \int_0^\infty d\omega \frac{d}{dt} r_+(\omega, \vec{k}) \left( \frac{\omega_k}{2} |h(\omega, \vec{k}, t)|^2 + \frac{1}{2\omega_k} |k(\omega, \vec{k}, t)|^2 \right), \quad (4.1)$$

$$\Gamma(\vec{k}, t) \equiv -\frac{d}{dt} \ln \left( \left( \frac{\omega_k}{2} g^2 + \frac{\dot{g}^2}{2\omega_k} \right) p_i + \left( \frac{\omega_k}{2} \dot{g}^2 + \frac{\ddot{g}^2}{2\omega_k} \right) q_i + \frac{\dot{g}}{\omega_k} (\omega_k g + \ddot{g}) q_i p_i + p_i q_i \right) \quad (4.2)
Regarded as a time evolution equation, this equation has a remarkable property; the initial memory effect is confined to, and isolated by, the rate $\Gamma(\vec{k}, t)$. All other quantities in this equation are written in terms of those at the same local time, including the distribution function $f(\vec{k}, t)$ itself.

We take a Markovian limit of this equation. The idea is as follows. The equation above, although exact in the slow variation limit of the Hartree model, has the initial memory effect. To retain the memory effect to late times may not make much sense when one wants to discuss an average behavior of the time evolution for a collective body of particles in a complex environment. After all, it is impossible to follow the time evolution for all particles in the ensemble. It may be possible to specify the initial data such as the ensemble-averaged $\bar{q}_i^2$ for all momentum modes at some specific time, and the data may have a simple regulated form if one assumes a simple initial distribution function for $\varphi$. But, remember that in the problem of our interest there are processes occurring rapidly such as the scattering, distinct from the other simultaneously occurring slow process such as the annihilation, which is of our main interest. Under this circumstance different particles undergo different history of evolution, and after a while it is almost impossible to keep track of the updated initial data.

One really does not care much about the details of the complicated time history. We only care about a global and slow change for a few key quantities. Fortunately, in many physical situations one has a clear separation of at least two time scales; one due to an elementary quantum behavior and the other for the bulk behavior. With this in mind it is much more sensible to erase the initial memory effect and to coarse-grain the time evolution averaging out the fast oscillatory behavior. A nice feature of the Hartree model is that if one eliminates the initial time dependence in $\Gamma(\vec{k}, t)$, a simple Markovian description of the transport phenomenon becomes possible.

In the weak coupling limit the exponential law $g(\vec{k}, t) \approx \sin(\omega_k t) e^{-\Gamma_k t^2/\omega_k}$ gives a constant decay rate,

$$
\Gamma_k = \frac{\pi}{\omega_k} r_-(\omega_k, \vec{k}) = \frac{\pi}{\omega_k} \left( r(\omega_k, \vec{k}) - r(-\omega_k, \vec{k}) \right),
$$

(4.3) after the time average. Except at very early times the initial memory effect is almost completely erased. Via the spectral function this $\Gamma_k$ depends on the $\varphi$ distribution function. As seen from eqs. (2.65), (2.66), this combination $r_-$ is the rate of decrease.
of heavy particles due to the annihilation, the scattering and their inverse processes.

Our Markovian evolution equation is then

\[
\frac{df(\vec{k},t)}{dt} = -\Gamma_k \left( f(\vec{k},t) - f_{eq}(\vec{k}) \right),
\]

(4.4)

\[
f_{eq}(\vec{k}) = \frac{r(-\omega_k,\vec{k})}{r_+(\omega_k,\vec{k})} + \delta f_{eq}(\vec{k}).
\]

(4.5)

Here the off-shell contribution \(\delta f_{eq}(\vec{k})\) is given by \((3.56) - (3.58)\). Although not written explicitly, the distribution function used to compute \(\Gamma_k\) and \(f_{eq}(\vec{k})\) should be the instantaneous one, \(f(\vec{k},t)\), with the same common time \(t\) as in the left side. The Markovian evolution equation for the momentum distribution function thus derived is the most fundamental result in the present work.

We note that the same Markovian approximation, when applied to the model of unstable particle decay, gives the kinetic equation identical to eq.(4.4), except that \(\Gamma_k\) in that case is the decay rate on the mass shell and that \(r_\pm(\omega,\vec{k})\) in \(f_{eq}(\vec{k})\) should be replaced by the appropriate spectral weight for the decay process such as \(r_\chi(\omega,\vec{k})\); more precisely \(r_+ \rightarrow r_\chi \coth \frac{\beta \omega}{2}, \quad r_- \rightarrow r_\chi\). We give in Appendix C the expression and its actual value of the stationary distribution \(f_{eq}(\vec{k})\) for the boson decay model.

We shall recapitulate the problem related to \(v(\vec{k})\). The Markovian equation derived above does contain the spectral combination \(r_\pm\), which is written using the distribution function \(f(\vec{k},t)\) and also \(v(\vec{k},t)\). This new function describes a deviation from the simple particle picture in the field theory; in terms of the harmonic coordinate and its conjugate momentum,

\[
v(\vec{k},t) = \langle \frac{1}{2\omega_k} p_{k}(t) - \frac{\omega_k}{2} q_{k}(t) \rangle.
\]

(4.6)

This combination of dynamical variables obeys another time evolution equation different from the distribution function. The coarse-graining, under the same slow variation approximation applied previously, gives

\[
v(\vec{k},t) = \int_0^\infty d\omega \frac{\omega^2 - \omega_k^2}{2\omega_k} \frac{r_+(\omega,\vec{k})}{(\omega^2 - \omega_k^2)^2 + (\pi r_-(\omega,\vec{k}))^2}.
\]

(4.7)

The initial memory term for the combination \((4.6)\) has the fast oscillatory behavior, which vanishes on the average. This behavior is different from that of the occupation number which has a slowly decaying component, non-vanishing after the time average. In this sense it is best to use the expression \((4.7)\) for \(v(\vec{k},t)\) and the defining equation of \(r_\pm\) containing both \(f(\vec{k},t)\) and \(v(\vec{k},t)\), as a functional equation to
determine \( v(\vec{k}, t) \) in terms of \( f(\vec{k}, t) \). This way we arrive at a closed form of time evolution equation for the distribution function. In the Markovian time evolution equation the quantity \( v(\vec{k}, t) \) appears via the spectral function \( r(\pm \omega, \vec{k}) \), which we now show to be of negligible higher order.

In the weak coupling limit of our main concern, both \( r_\pm \) has an overall small coupling, and the integral \((4.7)\), that excludes contribution from the Breit-Wigner region of \( \omega \approx \omega_k \) by the factor \( \omega^2 - \omega_k^2 \), gives \( v = O[\lambda^2] \), hence gives \( O[\lambda^4] \) contribution to the spectral function \( r \), as seen from \((2.34)\). We shall thus take \( v \) vanishing from now on.

With the vanishing \( v \), the usual Boltzmann equation follows when one approximates the energy integral for \( f_{eq} \) by the pole term, namely \( r(-\omega_k, \vec{k})/r_-(\omega_k, \vec{k}) \). This narrow width approximation gives the evolution equation,

\[
\frac{df(\vec{k}, t)}{dt} = -\Gamma_k \left( f(\vec{k}, t) - \frac{r(-\omega_k, \vec{k})}{r(\omega_k, \vec{k}) - r(-\omega_k, \vec{k})} \right).
\] (4.8)

One may use the formula \((4.3)\) for \( \Gamma_k \) and write this as

\[
\frac{df(\vec{k}, t)}{dt} = -\frac{\pi}{\omega_k} \left( r(\omega_k, \vec{k}) f(\vec{k}, t) - r(-\omega_k, \vec{k}) (1 + f(\vec{k}, t)) \right).
\] (4.9)

Again, the instantaneous \( \varphi \) distribution function is taken in computing \( r(\pm \omega_k, \vec{k}) \) of the right hand side. In view of the on-shell relation \((2.65), (2.66)\) for \( r(\pm \omega_k, \vec{k}) \), this is equivalent to the Boltzmann equation for the present annihilation-scattering problem in thermal medium;

\[
\frac{df(\vec{k}, t)}{dt} = \lambda^2 \int dk' \int dk_1 \int dk_2 \\
\left( \left( 1 + f_{th}(k_1) \right) \left( 1 + f_{th}(k_2) \right) f(k') f(k) \delta(k + k' - k_1 - k_2) \\
+ 2f_{th}(k_1) \left( 1 + f_{th}(k_2) \right) \left( 1 + f(k') \right) f(k) \delta(k + k_1 - k' - k_2) \\
- f_{th}(k_1) f_{th}(k_2) \left( 1 + f(k') \right) \left( 1 + f(k) \right) \delta(k + k' - k_1 - k_2) \\
- 2f_{th}(k_2) \left( 1 + f_{th}(k_1) \right) f(k') \left( 1 + f(k) \right) \delta(k + k_1 - k' - k_2) \right),
\] (4.10)

where all relevant processes are explicitly written. We thus find that our Markovian Hartree model gives a foundation to derivation of the Boltzmann equation under the narrow resonance approximation.

Note that the thermally averaged rate in the right side of the Boltzmann equation \((4.11)\) has separate contributions of the annihilation and its inverse process characterized by \( \delta(k + k' - k_1 - k_2) \), and the scattering plus its inverse by \( \delta(k + k_1 - k' - k_2) \).
There is no contribution from 1 to 3, and 3 to 1 processes such as $\varphi \leftrightarrow \varphi \chi \chi$ in the Boltzmann equation. When one integrates over the particle momentum $\vec{k}$ to discuss the time evolution of the number density, the entire scattering contribution drops out. This is reasonable, because the scattering process does conserve the particle number, hence the scattering process does not cause the change of the $\varphi$ particle number. However, the momentum distribution function changes by the scattering. Moreover, it is not immediately clear how the off-shell scattering term contributes within our crossing symmetric approach. We thus keep the scattering term for the time being, as it stands.

The equilibrium distribution function is defined by setting $df/dt = 0$. We assume as before that this has the unique solution, which in the case of the Boltzmann equation must agree with that of the thermal and the chemical equilibrium, $f(k) = 1/(e^{\beta \omega_k} - 1)$. The concept of partial equilibrium may however be of some use. For instance, if the scattering process takes place much more frequently than the annihilation process, it may be useful to suppose that the energy exchange is equilibrated, but the species number change is not fast enough. In this case the thermal equilibrium, and not the chemical equilibrium, is reached with

$$f(\vec{k}) = \frac{1}{e^{\beta (\omega_k - \mu)} - 1} \equiv f_{\text{th}}^\mu(k), \quad (4.11)$$

where $\mu$ is the chemical potential. Under this thermal equilibrium of a finite chemical potential all scattering-related terms in eq.\ref{4.10} cancels each other since

$$[r_s(\omega_k, k)]_{f = f_{\text{th}}^\mu} f_{\text{th}}^\mu(k) - [r_s(-\omega_k, k)]_{f = f_{\text{th}}^\mu} (1 + f_{\text{th}}^\mu(k)) = 0, \quad (4.12)$$

and only the annihilation-related terms are to be retained. Here

$$r_s(\omega_k, \vec{k}) = 2\lambda^2 \int dk' \int dk_1 \int dk_2$$

$$\cdot f_{\text{th}}(k_1)(1 + f_{\text{th}}(k_2))(1 + f(k')) \delta(k + k_1 - k' - k_2), \quad (4.13)$$

$$r_s(-\omega_k, \vec{k}) = 4\lambda^2 \int dk' \int dk_1 \int dk_2$$

$$\cdot f_{\text{th}}(k_2)(1 + f_{\text{th}}(k_1)) f(k') \delta(k + k_1 - k' - k_2). \quad (4.14)$$

In this case the Boltzmann equation effectively describes the time evolution for the chemical potential. Note however that the thermal equilibrium distribution of a finite chemical potential is realized, assuming that the on-shell Boltzmann equation is a valid description of our problem without the off-shell contribution.
As will be made clear shortly, it is not easy to obtain a readily calculable form of the equilibrium distribution function at low temperatures in the annihilation-scattering problem, and we shall directly work out the integrated number density. On the other hand, for the unstable particle decay we have an approximate analytic result, eqs. (C.10) and (C.13), for the distribution function, which is shown in Fig. 13. At low \( T < 0.1M \) deviation from the Planck form becomes large, but the distribution is not described by the form (1.11) with a finite chemical potential.

More generally, it is convenient to separate the on-shell term and write the rest of contribution, as is done in the preceding section;

\[
\begin{align*}
  f_{eq}(\vec{k}) &= \frac{r(-\omega_k, \vec{k})}{r(\omega_k, \vec{k})} + \delta f_{eq}(\vec{k}), \quad (4.15) \\
  \delta f_{eq}(\vec{k}) &= f_f(\vec{k}) + f_0^0(\vec{k}), \quad (4.16)
\end{align*}
\]

with \( \tilde{\Gamma} = \frac{\zeta(3)\lambda^2}{4\pi^3} \frac{T^3}{M^2} \). The function \( f_0^0 \) is given in eq. (3.58). Physically, the off-shell contribution \( \delta f_{eq}(\vec{k}) \) in this formula consists of two terms; the first \( f_f \) dependent one due to the inverse scattering process, and the second \( f_f \) independent one \( f_0^0 \) due to the inverse annihilation process, \( \chi \chi \rightarrow \varphi \varphi \), along with a small contribution from 1 to 3 process, \( \chi \leftrightarrow \chi \varphi \varphi \).

For discussion of the time evolution, we shall be content with the integrated number density,

\[
n(t) = \int \frac{d^3k}{(2\pi)^3} f(\vec{k}, t), \quad (4.17)
\]

and its evolution. In the discussion of the relic abundance of WIMP this integrated quantity is of prime interest in cosmology. As already noted, the evolution equation for the number density is simplified considering cancellation of the scattering terms in the on-shell part,

\[
\frac{dn}{dt} = - \int \frac{d^3k}{(2\pi)^3} \left( \Gamma_k^{ann} f(\vec{k}, t) - \Gamma_k^{inv} f_{eq}(\vec{k}, t) \right), \quad (4.18)
\]

where \( \Gamma_k^{ann} (\Gamma_k^{inv}) \) is the rate keeping the annihilation (inverse annihilation and inverse scattering) term. The approximate form of the evolution equation for the number density at low temperatures \( (T \ll M) \) is then given by

\[
\frac{dn}{dt} = - \int \frac{d^3k}{(2\pi)^3} \frac{\pi}{\omega_k} \left( r_a(\omega_k, k) f(k) - \Gamma_k^{inv} \delta f_{eq}(k) \right), \quad (4.19)
\]

\[
r_a(\omega_k, \vec{k}) = 2\lambda^2 \int dk' \int dk_1 \int dk_2 \cdot (1 + f_{th}(k_1)) (1 + f_{th}(k_2)) f(k') \delta(k + k' - k_1 - k_2). \quad (4.20)
\]
The equilibrium number density is obtained by setting \( dn/dt = 0 \), hence is given by
\[
\text{RHS} = -\int \frac{d^3k}{(2\pi)^3 2\omega_k} \left( r_a(\omega_k, k) f(k) - \Gamma_k^{\text{inv}} \delta f_{\text{eq}}(k) \right) = 0.
\] (4.21)

A more explicit form of this equation is \( 2\lambda^2 \) times
\[
\int dk \int dk' \int dk_1 \int dk_2 \left[ \left( 1 + f_{\text{th}}(k_1) \right) \left( 1 + f_{\text{th}}(k_2) \right) f(k') f(k) \delta(k + k' - k_1 - k_2) 
- f_{\text{th}}(k_1) \left( 1 + f_{\text{th}}(k_2) \right) \delta f_{\text{eq}}(k) \left( 1 + f(k') \right) \delta(k + k_1 - k' - k_2) \right] = 0,
\] (4.22)
where we used the abbreviated notation for the phase space integral (2.68). Thus, \( \Gamma_k^{\text{inv}} \) is given by the second term of eq. (4.22).

Some sort of averaged cross sections are here in this equation when \( \lambda^2 \) is multiplied, and it would be useful to crudely estimate these. Let us assume that \( \varphi \) particles are non-relativistic and ignore the effect of stimulated emission. One has in the first term an averaged annihilation cross section \( \sigma_a v \) times the \( \varphi \) number density squared \( n_{\varphi}^2 \), where \( v \) is the relative velocity between two annihilating particles, and \( \sigma_a v \) is the invariant cross section. Since the annihilation cross section slowly changes with energy at low temperatures, one may take the zero energy limit of the cross section,
\[
\sigma_a v_a \approx \frac{\lambda^2}{16\pi M^2}.
\] (4.23)

In the second term there are two contributions corresponding to the two \( f \) in \( \delta f_{\text{eq}} \), \( f_f \) and \( f_f^0 \). One of these \( f_f \) is multiplied by the scattering cross section,
\[
\sigma_s v_s \approx \frac{\lambda^2}{4\pi M^2},
\] (4.24)
while the second \( f_f^0 \) is multiplied by \( \sigma_a v_a \). The first one is given by \( \sigma_s v_s \) times the quantity of order \( n_{\text{th}} \cdot n_{\text{th}} f_f \), with \( n_{\text{th}} = \frac{\zeta(3)}{\pi^2} T^3 \), hence, using (D.10),
\[
n_{\text{th}} n_{\text{th}} f_f \approx \frac{\zeta(3)\lambda^2}{32\pi^6} \left( \frac{T}{M} \right)^2 T^3 n_{\varphi}.
\] (4.25)
The second one is given by \( \sigma_a v_a \) times
\[
n_{\text{th}} n_{\text{th}}^0 \approx \frac{c\zeta(3)\lambda^2 T^7}{192\pi^5} \frac{T}{M},
\] (4.26)
with \( c \approx 0.27 \), eq. (C.13). Thus, once the \( \varphi \) number density becomes \( \ll O[T^3] \), the second contribution is dominant by \( O[MT^2/n_{\varphi}] \). Equating this \( \sigma_a v_a n_{\text{th}} n_{\text{th}}^0 \) to the
annihilation rate $\sigma v_n n_{\varphi}^2$ gives

$$n_{\varphi} \approx \sqrt{n_{\text{th}}n_2^0} \approx \frac{1}{\pi^2} \sqrt{\frac{c\zeta(3)}{192\pi}} \lambda \sqrt{\frac{T}{M}} T^3 \approx 0.0023 \times \lambda \sqrt{\frac{T}{M}} T^3. \quad (4.27)$$

We thus derived the equilibrium number density roughly of order

$$10^{-3} \times \lambda \sqrt{\frac{T}{M}} T^3 \quad (4.28)$$

at low temperatures which may become much larger than of order $(MT)^{3/2} e^{-M/T}$ determined from the Maxwell-Boltzmann distribution of zero chemical potential.

This argument shows that with $n_2^0 = O[\lambda^2 T^4/M]$ the $f-$dependent off-shell contributions $f_i^f$ of order $n_i^f = O[\lambda^2 (T/M)^\alpha n_{\varphi}]$ are subdominant unless

$$\lambda (\frac{T}{M})^{\alpha - \frac{1}{2}} \geq 1. \quad (4.29)$$

Even the possibly largest case obtained numerically, eq.(D.14), gives $\alpha \approx 1.35$, thus confirming that the dominant off-shell contribution is $n_2^0$, eq.(D.21).

5 Application to cosmology: relic abundance

The cosmic expansion has a drastic effect on the annihilation process of heavy stable particles. The temperature of cosmic environment particles such as the $\chi$ particle in our toy model decreases with the scale factor, $T \approx 1/a(t)$, which in turn results in less frequent reaction. This gives rise to a phenomenon called the freeze-out or the decoupling of the process.

The freeze-out is described introducing a term of the cosmic expansion in the evolution equation. For the evolution of the distribution function $f(\vec{k}, t)$, the time derivative operator is modified to

$$\frac{\partial}{\partial t} - \frac{\dot{a}}{a} \vec{k} \cdot \frac{\partial}{\partial \vec{k}}. \quad (5.1)$$

Here $\dot{a}/a$ is the Hubble parameter $H$. When one integrates over the phase space to get the number density, the left hand side of the evolution equation becomes

$$\frac{dn}{dt} + 3H n = \cdots, \quad (5.2)$$
assuming that the distribution function is sufficiently damped in the high momentum limit. The evolution equation is further simplified by introducing the relative yield \( Y \),

\[
Y \equiv \frac{n}{T^3},
\]

since

\[
\frac{dY}{dt} = \frac{d}{dt} \frac{n}{T^3} = T^{-3} \left( \frac{dn}{dt} + 3Hn \right).
\]

This holds owing to the temperature-scale relation, \( T \propto 1/a \), hence \( H = -\dot{T}/T \).

The approximate Markovian kinetic equation in the expanding universe is then

\[
\frac{dY}{dt} = -2\pi^2 \lambda^2 \int dk \int dk' \int dk_1 \int dk_2 \left( \left( 1 + f_{th}(k_1) \right) \left( 1 + f_{th}(k_2) \right) \left( f(k') f(k) - f_{MB}(k') f_{MB}(k) \right) \delta(k + k' - k_1 - k_2) - f_{th}(k_1) \left( 1 + f_{th}(k_2) \right) \left( 1 + f(k') \right) \delta f_{eq}(k) \delta(k + k_1 - k' - k_2) \right),
\]

where we introduced the Maxwell-Boltzmann distribution for the zero chemical potential,

\[
f_{MB}(k) = e^{-\frac{M}{T}} \exp\left(-\frac{k^2}{2MT}\right).
\]

A very crude estimate of the freeze-out temperature goes as follows. One equates the equilibrium annihilation rate \( \approx \sigma_a v_a \cdot n_\varphi \) to the Hubble rate,

\[
H = d \frac{T^2}{m_{pl}}, \quad d = \sqrt{\frac{4\pi^3 N}{45}} \approx 1.66 \sqrt{N},
\]

where \( N \) is the number of particle species contributing to the cosmic energy density. This argument, when applied for \( n_\varphi \gg n_{MB} \), gives the freeze-out temperature,

\[
T_f \approx 700 \times N^{1/3} \left( \frac{M}{m_{pl}} \right)^{2/3} \frac{M}{\lambda^2} \approx 1.3 \times 10^5 \left( \frac{e^2}{\lambda} \right)^2 N^{1/3} \left( \frac{M}{m_{pl}} \right)^{2/3} M
\]

\[
\sim 50 \text{keV} \left( \frac{e^2}{\lambda} \right)^2 N^{1/3} \left( \frac{M}{100 \text{GeV}} \right)^{5/3}.
\]

We used the unit of \( \lambda \), anticipating a strength of order the electromagnetic interaction \( \lambda \approx e^2 = 4\pi\alpha \). The use of the off-shell formula for \( n_\varphi \) is usually justified for \( \lambda = O[e^2] \), since at low temperatures \( n_\varphi > n_{MB} \).

The freeze-out yield is defined by \( Y_f = (n_\varphi/T^3)_{T=T_f} \). We find for the range of parameters, \( \lambda > 9.3 \times 10^{-5} (M/\text{GeV})^{0.32} \), \( 10^{-3} \text{GeV} < M < 1 \text{TeV} \)

\[
Y_f \approx 0.06 \times N^{1/6} \left( \frac{M}{m_{pl}} \right)^{1/3} \approx 2.4 \times 10^{-8} N^{1/6} (M/\text{GeV})^{1/3}.
\]

37
Remarkably, this quantity is insensitive to the coupling constant $\lambda$. Prior to the freeze-out epoch, this value $Y$ does vary, but only gradually, since $n_\varphi \propto T^{3.5}$. The freeze-out yield $Y_f$ is almost invariant in the rest of cosmic expansion, as will also be discussed in analytic estimate below. The present relic mass density is then estimated from

$$
\rho_0 = M Y_f T_0^3,
$$

with $T_0$ the present microwave temperature of $\approx 3 K$. Numerically,

$$
\rho_0 \approx 4.1 \times 10^4 N^{1/6} \left( \frac{M}{\text{GeV}} \right)^{4/3} \text{eV cm}^{-3}.
$$

The closure mass density of order $2 \times 10^{-29} \text{g cm}^{-3} \approx 10^4 \text{eV cm}^{-3}$ requires that $M \leq O[1 \text{GeV}]$, assuming $N = 43/4$. Thus, WIMP in the mass range far above $1 \text{GeV}$ is excluded for the S-wave boson-pair annihilation model.

Extension to the annihilation in a higher angular momentum state is of great interest, since LSP in SUSY models pair-annihilates with a large P-wave contribution. This P-wave annihilation for LSP is related to the Majorana nature of LSP. It is beyond the scope of the present work to accurately calculate the annihilation rate in supersymmetric models, taking into account the Majorana nature and all contributing Feynman diagrams. It is however not too difficult to qualitatively estimate the P-wave annihilation rate, by simply taking into account the momentum, hence temperature dependence of various rates, $\langle k^2 \rangle \propto T$.

A more detailed behavior of the $\varphi$ number density may be worked out by examining

$$
\frac{dn}{dt} + H n = - \langle \sigma_a v_a \rangle \left( n^2 - \delta \left( \frac{T}{M} \right)^{p+1} T^6 - n_{\text{MB}}^2 \right),
$$

$$
\langle \sigma_a v_a \rangle = \frac{\lambda^2}{16\pi M^2} \left( \frac{T}{M} \right)^p, \quad \delta = 0.27 \times \frac{\zeta(3) \lambda^2}{192\pi^3},
$$

$$
n_{\text{MB}} = \left( \frac{MT}{2\pi} \right)^{3/2} e^{-M/T}.
$$

Our toy model gives the S-wave annihilation with $p = 0$, while $p = 1$ for the P-wave annihilation. We have extended the S-wave annihilation to the case of higher angular momentum without changing the effective coupling constant $\delta$ relevant to our toy model. Equivalently, using the inverse temperature and the yield,

$$
\frac{dY}{dx} = - \frac{\eta}{x^{p+2}} (Y^2 - Y_{eq}^2),
$$

38
\[ Y_{\text{eq}}^2 = \frac{\delta}{x^{p+1}} + \left(\frac{x}{2\pi}\right)^3 e^{-2x}, \quad (5.16) \]

\[ x = \frac{M}{T}, \quad \eta = \frac{\lambda^2 m_{\text{pl}}}{16\pi dM}. \quad (5.17) \]

The parameter \( \eta \) is roughly the (on-shell) annihilation rate \( \sigma v T^3 \) divided by the Hubble rate at the temperature equal to the particle mass, \( T = M \).

We plot in Fig.6 – Fig.8 a typical solution to the time evolution equation, (5.15); Fig.6 and Fig.7 for the S-wave annihilation and Fig.8 for the P-wave annihilation.

The analytic estimate of the freeze-out temperature and the freeze-out yield [10], which well reproduces the numerical estimate above, is as follows. One may consider with a good precision that the yield follows the equilibrium abundance \( Y_{\text{eq}} \) until the freeze-out temperature. This temperature \( T_f \) is given by

\[ \frac{dY_{\text{eq}}}{dx_f} = -\frac{\eta}{x_f^{p+2}} Y_{\text{eq}}^2, \quad (5.18) \]

since after this epoch the inverse process is frozen and the yield follows

\[ \frac{dY}{dx} = -\frac{\eta}{x^{p+2}} Y^2. \quad (5.19) \]

Integration of this equation gives the final yield,

\[ Y(x) = \frac{Y_f}{1 - Y_f \frac{\eta}{p+1} (x^{-p-1} - x_f^{-p-1})}, \quad (5.20) \]

which agrees with

\[ Y \approx \frac{Y_f}{1 + Y_f \frac{\eta}{p+1} x_f^{-p-1}} \quad (5.21) \]

as \( T \to 0 \). Usually \( Y_f \) is very small along with \( x_f^{-1} \ll 1 \), and in this case \( Y \approx Y_f \) after the freeze-out.

In Fig.9 – Fig.11 we show the present WIMP mass density in the parameter space \( (M, \lambda) \). The off-shell dominance region is shown in Fig.9, along with the closure mass density, while contours of smaller mass densities are shown in Fig.10. In Fig.11 the P-wave case is shown. The excluded region due to the overclosure is larger for the P-wave annihilation than for the S-wave, with the same set of \( (M, \lambda) \).

For a given temperature \( T \), there exists an upper bound on the heavy particle mass \( M_{\text{max}} \) that can be produced in the equilibrium abundance \( n_{\text{eq}} \). In the Boltzmann equation approach this bound is roughly of order \( T \), but our new contribution \( \delta n_{\text{eq}} \) substantially changes this value. There are two considerations to be taken here. The
first one is the condition on energetics; produced energy < thermal environment energy. From \( M n_{eq} \ll c T^4 \) with \( c \) a constant of order unity,

\[
M < M_{\text{max}}, \quad M_{\text{max}} \approx 2 \times 10^5 c^2 \frac{T}{\lambda^2}.
\]  

(5.22)

This bound is not stringent, since \( M_{\text{max}}/T \approx 10^5/\lambda^2 \) can be quite large.

The second, a more important constraint comes from the relaxation time. For the inverse process \( \chi \chi \rightarrow \varphi \varphi \) to occur frequently, its rate \( \Gamma_{\text{inv}} \) must be larger than the Hubble rate \( H \). With

\[
\Gamma_{\text{inv}} \sim (\sigma a v) n_{eq} \approx 3 \times 10^{-4} \lambda^3 \left(\frac{T}{M}\right)^{7/2} M,
\]

(5.23)

this consideration gives

\[
M_{\text{max}} \approx 3 \times 10^{-2} \lambda^{6/5} N^{-1/5} \left(\frac{m_{\text{pl}}}{T}\right)^{2/5} T.
\]

(5.24)

As \( T \to 0 \), \( M_{\text{max}} \to 0 \) like \( T^{3/5} \), but \( M_{\text{max}}/T \) can become very large.

Acknowledgment

This work has been supported in part by the Grand-in-Aid for Science Research from the Ministry of Education, Science and Culture of Japan, No. 08640341. The work of Sh. Matsumoto is partially supported by the Japan Society of the Promotion of Science.
Appendix A Generating functional in the influence functional method

We explain the technique of the generating functional applied to the influence functional, taking the example of the exactly solvable model [9]. When one wants to apply this method to the Hartree model of our annihilation-scattering problem, one should use relevant spectrum \( r_\pm \) instead of \( r(\omega) \) below.

We first introduce the source terms coupled to both the harmonic coordinate and the conjugate momentum as

\[
j(\tau)q(\tau) + l(\tau)p(\tau) - j'(\tau)q'(\tau) - l'(\tau)p'(\tau) = \frac{1}{2} \left( S_j \xi + D_j X + S_l p_\xi + D_l p_X \right), \quad (A.1)
\]

where a convenient combination for the influence functional method is introduced

\[
S_j = j + j', \quad D_j = j - j', \quad S_l = l + l', \quad D_l = l - l', \quad p_{X,\xi} = p \pm p'. \quad (A.2)
\]

In subsequent formulas we often omit the momentum index \( \vec{k} \) for the sake of simplicity and discuss each Fourier mode separately. Functional differentiation of the resulting density matrix \( \rho^{(j,l)} \) with respect to these sources, when evaluated at the vanishing source, gives various combinations of correlators; for instance,

\[
\langle q(\tau_1)q(\tau_2) \rangle = - \left[ \frac{\delta^2}{\delta j(\tau_1)\delta j(\tau_2)} \tr \rho^{(j,l)} \right]_{j=0,l=0} = \quad (A.3)
\]

\[
- \left[ \frac{\delta^2}{\delta S_j(\tau_1)\delta S_j(\tau_2)} + \frac{\delta^2}{\delta S_j(\tau_1)\delta D_j(\tau_2)} + \frac{\delta^2}{\delta D_j(\tau_1)\delta S_j(\tau_2)} \right]_{j=0,l=0} + \frac{\delta^2}{\delta D_j(\tau_1)\delta D_j(\tau_2)} \tr \rho^{(j,l)}, \quad (A.4)
\]

\[
\langle p(\tau_1)p(\tau_2) \rangle = - \left[ \frac{\delta^2}{\delta l(\tau_1)\delta l(\tau_2)} \tr \rho^{(j,l)} \right]_{j=0,l=0}. \quad (A.5)
\]

Computation of the new density matrix \( \rho^{(j,l)} \) under the action of the source is similar to the case without the source terms, because introduction of the source does not change the Gaussian nature of the Hartree model. The semiclassical \( \xi \) equation and the the effective action is thus given by extending the analysis sketched in the text;

\[
\xi_{cl}(\tau) = - \dot{\xi}_f g(t - \tau) + \xi_f \dot{g}(t - \tau) + \int_t^\tau ds \, g(s - \tau) \left( D_j - \dot{D}_l \right)(s), \quad (A.6)
\]
\[ J^{(j,l)} = e^{iS_{cl}^{(j,l)}} / 2\pi^g, \]  
\[ iS_{cl}^{(j,l)} = -\int_0^t d\tau \int_0^\tau ds \xi_0(\tau)\beta_R(\tau - s)\xi_0(s) \]
\[ + \frac{i}{2} \int_0^t d\tau \left( S_j(\tau)\xi_0(\tau) + S_l(\tau)\xi_0(\tau) \right) + \frac{i}{2} \left( X_f \dot{\xi}_f - X_i \dot{\xi}_i \right). \]

Here the change of the variable, \( \dot{\xi}_f \rightarrow \xi_i \), is computed using
\[ \xi_i = -\dot{\xi}_f g(t) + \xi_f \dot{g}(t) + \int_0^t ds g(s)(D_j - \dot{D}_i)(s), \]
with
\[ g(t) = \frac{1}{2\pi^2} \int_{-\infty}^\infty d\omega e^{-i\omega t} F(\omega + i0^+), \]
\[ \beta_R(t) + i\beta_I(t) = \int_{-\infty}^\infty d\omega r(\omega) e^{-i\omega t}, \]
\[ -F(z)^{-1} = z^2 - \omega^2(T) - 2 \int_0^\infty d\omega \frac{\omega r(\omega)}{z^2 - \omega^2}. \]

For calculation of the correlator one convolutes the \( J^{(j,l)} \) function \((A.7)\) above with an initial density matrix of \( \varphi \) system \( \rho_i(X_i, \xi_i) \) and traces out the final \( X_f \) and \( \xi_f \) variables. Thus, it is convenient to take the trace with regard to the final variable, to get
\[ \text{tr} \ J^{(j,l)} = \delta \left( \xi_i - \int_0^t d\tau g(D_j - \dot{D}_i) \right) \exp \left[ -\int_0^t d\tau \int_0^\tau ds \xi_0(\tau)\beta_R(\tau - s)\xi_0(s) \right. \]
\[ + \frac{i}{2} \int_0^t d\tau \left( S_j(\tau)\xi_0(\tau) + S_l(\tau)\xi_0(\tau) \right) + \frac{i}{2} X_i \int_0^t d\tau \dot{g}(\tau)(D_j(\tau) - \dot{D}_i(\tau)) \bigg], \]
\[ \xi_0 = \int_0^t ds (D_j - \dot{D}_i)(s) \dot{g}(s - \tau). \]

As an example of the density matrix with the external source attached, we may work out the case for initial thermal state of temperature \( T_0 = 1/\beta_0 \), to obtain
\[ \text{tr} \ \rho^{(j,l)} = \exp \left[ -\int_0^t d\tau \int_0^\tau ds \xi_0(\tau)\beta_R(\tau - s)\xi_0(s) \right. \]
\[ + \frac{i}{2} \int_0^t d\tau \int_0^\tau ds (D_j - \dot{D}_i)(\tau) g(\tau - s)S_j(s) \]
\[ - \frac{i}{2} \int_0^t d\tau \int_0^\tau ds (D_j - \dot{D}_i)(\tau) \dot{g}(\tau - s)S_l(s) \]
\[ - \frac{1}{4\omega_0} \coth \left( \frac{\beta_0 \omega_0}{2} \right) \left( \omega_0 \int_0^t ds \dot{g}(s)(D_j - \dot{D}_i)(s) \right)^2 + \left( \int_0^t ds \dot{g}(s)(D_j - \dot{D}_i)(s) \right)^2 \bigg]. \]
General solution for arbitrary initial uncorrelated states can be derived with the aid of a conjugate Wigner transform; we define with the initial system density matrix \( \rho_i(X = q + q', \xi = q - q') \),

\[
f_{\nu \xi}(p, \xi) = \int_{-\infty}^{\infty} dx \, e^{ipX/2} \rho_i(X, \xi).
\]

(A.16)

Tracing out the final variables leads to

\[
\text{tr} \rho^{(j,l)} = \frac{1}{2} \int_{-\infty}^{\infty} dX \int_{-\infty}^{\infty} d\xi \rho_i(x, \xi) \text{tr} J^{(j,l)} = \exp \left[ -\int_0^t d\tau \int_0^\tau ds \, \xi_0(\tau) \beta_R(\tau - s) \xi_0(s) + \frac{i}{2} \int_0^t d\tau \left( S_j(\tau) \xi_0(\tau) + S_l(\tau) \dot{\xi}_0(\tau) \right) \right] \cdot f_{\nu \xi} \left( \int_0^t d\tau \, \dot{g}(\tau)(D_j(\tau) - \dot{D}_j(\tau)), \int_0^t d\tau \, g(\tau)(D_j(\tau) - \dot{D}_j(\tau)) \right).
\]

(A.17)

Functional differentiation with respect to \( j(\tau), l(\tau) \) gives the correlator such as

\[
\langle q(\tau_1)q(\tau_2) \rangle = -\frac{i}{2} \dot{g}(\tau_1 - \tau_2) + \frac{1}{2} \int_0^{\tau_1} d\tau \int_0^{\tau_2} ds \, g(\tau_1 - \tau) \beta_R(\tau - s) g(\tau_2 - s)
\]

\[
- g(\tau_1)g(\tau_2) \left( \frac{\partial^2 f_{\nu \xi}}{\partial \xi^2} \right)_{00} - \dot{g}(\tau_1)\dot{g}(\tau_2) \left( \frac{\partial^2 f_{\nu \xi}}{\partial p^2} \right)_{00}
\]

\[
- (g(\tau_1)\dot{g}(\tau_2) + \dot{g}(\tau_1)g(\tau_2)) \left( \frac{\partial^2 f_{\nu \xi}}{\partial \xi \partial p} \right)_{00}.
\]

(A.18)

Here the suffix 00 is understood to mean \( p = 0, \xi = 0 \). Needless to say, the derivatives are related to the averages of dynamical variables;

\[
\left( \frac{\partial^2 f_{\nu \xi}}{\partial \xi^2} \right)_{00} = -\frac{1}{2} p_i^2, \quad \left( \frac{\partial^2 f_{\nu \xi}}{\partial p^2} \right)_{00} = -\frac{1}{2} q_i^2, \quad \left( \frac{\partial^2 f_{\nu \xi}}{\partial \xi \partial p} \right)_{00} = -\frac{1}{2} \nu_0 q_0 + q_i p_i.
\]

(A.19)

We assumed in deriving this formula that

\[
\left( \frac{\partial f_{\nu \xi}}{\partial \xi} \right)_{00} = 0, \quad \left( \frac{\partial f_{\nu \xi}}{\partial p} \right)_{00} = 0.
\]

(A.20)

The initial memory effect appears in two ways for the exact result of the correlator \( \langle q(\tau_1)q(\tau_2) \rangle \); first, via the initial state dependence, \( f_{\nu \xi} \), and secondly, via the explicit lower limit of the time integration, \( \tau = 0 \) taken to be the initial time. The memory effect thus violates the time translation invariance under \( \tau_i \to \tau_i + \delta \).

The coincident limit of the correlator is computed, using \( g(0) = 0 \), and the relation,

\[
\int_0^t d\tau \int_0^t ds \, g(t - \tau) \beta_R(\tau - s) g(t - s) = \int_0^\infty d\omega \, r_+(\omega) |h(\omega, t)|^2.
\]

(A.21)

(A.22)
The result is eq.(3.28) and similar ones for the other quantities, eq.(3.29), (3.30) in the text.

We finally give some other examples of the conjugate Wigner function $f_{p\xi}$:

1. thermal state of temperature $T_0 = 1/\beta_0$

$$f_{p\xi}(p, \xi) = \exp[-\frac{1}{4} \coth(\frac{\beta_0 \omega_0}{2}) (\frac{p^2}{\omega_0} + \omega_0 \xi^2)] , \quad (A.23)$$

2. moving packet of momentum $p_0$, with a spread $\Delta p \approx \sqrt{\alpha}$

$$f_{p\xi}(p, \xi) = \exp[-\frac{1}{4} (\frac{p^2}{\alpha} + \alpha \xi^2) + ip_0 \xi] . \quad (A.24)$$

In these cases the correlators such as (A.19) can be worked out explicitly.

### Appendix B  Renormalization of distribution function

One may use the equal time limit of the full propagator for the purpose of renormalization; in particular,

$$\lim_{x_0 \to y_0^+} \left( \frac{\partial}{\partial x_0} - \frac{\partial}{\partial y_0} \right) \tilde{G}(x_0, y_0 ; \vec{k}) = Z , \quad (B.1)$$

$$\lim_{x_0 \to y_0^+} \frac{i}{4} \left( \frac{\partial}{\partial x_0} - \frac{\partial}{\partial y_0} \right)^2 \tilde{G}(x_0, y_0 ; \vec{k}) = \langle \mathcal{H}(\vec{k}, t) \rangle , \quad (B.2)$$

where $Z$ is the wave function renormalization factor and $\mathcal{H}(\vec{k}, t)$ is the Hamiltonian for the momentum $\vec{k}$ mode. Using the expansion (2.52), we find

$$\lim_{x_0 \to y_0^+} \left( \frac{\partial}{\partial x_0} - \frac{\partial}{\partial y_0} \right) G(x, y) = \int \frac{d^3k}{(2\pi)^3} e^{-i\vec{k}(\vec{x} - \vec{y})} \left( 1 + f(\vec{k}, t) - f(\vec{k}, t) \right)$$

$$= \delta^3(\vec{x} - \vec{y}) , \quad (B.3)$$

in contradiction to eq.(3.11). Thus, the original expansion (2.52) must be modified to allow a counter term in $-iG(x, y)$. The term independent of $f$ should thus be replaced as

$$e^{i\omega_k(x_0 - y_0)} f(\vec{k}, t) + e^{-i\omega_k(x_0 - y_0)} (1 + f(\vec{k}, t)) \rightarrow$$

$$e^{i\omega_k(x_0 - y_0)} \left( \frac{1 - Z}{2} + f(\vec{k}, t) \right) + e^{-i\omega_k(x_0 - y_0)} \left( \frac{1 + Z}{2} + f(\vec{k}, t) \right) . \quad (B.4)$$
Another useful relation for the renormalization is the Fourier inversion formula,
\[
\frac{1}{2\omega_k} \left( 1 + 2f(\vec{k}, x_0) - 2v(\vec{k}, x_0) \right) = \lim_{x_0 \to y_0^+} \int d^3 x \ e^{i\vec{k} \cdot \vec{x}} \langle \varphi(\vec{x}, x_0) \varphi(\vec{0}, y_0) \rangle = -i \int_{-\infty}^{\infty} \frac{dk_0}{2\pi} \tilde{G}(k_0, \vec{k}; x_0). \quad (B.5)
\]

The correlator \( G(x, y) = i \langle \varphi(x) \varphi(y) \rangle \) in thermal equilibrium is what is called the real-time thermal Green’s function in the literature [12], and its Fourier transform \( \tilde{G} \) here is related to the analytic function \( (B.11) \) by
\[
- \tilde{G}(k_0, \vec{k}) = \frac{1}{1 - e^{-\beta k_0}} F(k_0 + i0^+, \vec{k}) + \frac{1}{1 - e^{\beta k_0}} F(k_0 - i0^+, \vec{k}). \quad (B.6)
\]

This relation holds when \( \varphi \) particles are in thermal equilibrium with \( \chi \). We shall first derive the renormalization condition assuming thermal equilibrium, and then extend its result to the non-equilibrium circumstance.

Subtraction term for the distribution function consists of an infinite quantity and its associated finite term. In the spirit of the on-shell renormalization in field theory it is important to identify the pole contribution with possible infinities of the wave function and the mass correction included. The pole part of the propagator is extracted from the terms of \( F^{-1}(k_0, \vec{k}) \) to the quadratic order in \( k_0 \); prior to the renormalization,
\[
i F_{\text{pole}}(k_0 \pm i0^+, \vec{k}) = -i \frac{k_0^2 - \vec{k}^2 - M^2(T) - \Pi(k_0, \vec{k}) \pm i\pi r_-(\omega_k, \vec{k}) \epsilon(k_0)}{k_0^2 - \vec{k}^2 - M^2(T) - \Pi(k_0, \vec{k})}, \quad (B.7)
\]
\[
\Pi(\omega, \vec{k}) = -\mathcal{P} \int_{-\infty}^{\infty} d\omega' \frac{r_-(\omega', \vec{k})}{\omega' - \omega} = -\mathcal{P} \int_{0}^{\infty} d\omega' \frac{2\omega' r_-(\omega', \vec{k})}{\omega'^2 - \omega^2}, \quad (B.8)
\]

with \( M^2(T) = M^2 + \frac{\lambda}{12} T^2 \) including the \( O[\lambda] \) temperature dependent mass.

The mass and the wave function renormalization is done perturbatively for the proper self-energy \( \Pi(k_0, \vec{k}) \). Expanding in powers of \( k_0^2 - M^2 \) and \( \vec{k}^2 \), one identifies the renormalized temperature dependent mass as
\[
M_R^2(T) = M^2(T) + \Pi(M, \vec{0}), \quad (B.9)
\]
and the wave function renormalization factor as
\[
Z^{-1} = 1 - \frac{1}{2} \left( \frac{\partial^2 \Pi(k_0, \vec{0})}{\partial k_0^2} \right)_{k_0 = M}. \quad (B.10)
\]

Thus, defining the subtracted finite part \( \delta \Pi \) by
\[
\delta \Pi(k_0, \vec{k}) = \Pi(k_0, \vec{k}) - \Pi(M, \vec{0}) + (Z^{-1} - 1) (k_0^2 - M^2), \quad (B.11)
\]
one has
\[ i F_{\text{pole}}(k_0 \pm i0^+, \vec{k}) = \frac{-iZ}{k_0^2 - (\omega_k^R)^2 \pm i\pi r_-(\omega_k, \vec{k}) \epsilon(k_0)}, \] (B.12)
\[ \omega_k^R = \sqrt{\vec{k}^2 + M_R^2(T) + \delta\Pi(\omega_k, \vec{k})}. \] (B.13)

This $F_{\text{pole}}^{-1}(k_0, \vec{k})$ is an optimal Gaussian approximation for the low energy dynamics, to the order $O[\lambda^2]$. One may regard $\delta\Pi(\omega_k, \vec{k})$ as a finite energy shift due to the interaction with environment. When one uses $\omega_k = \sqrt{\vec{k}^2 + M_R^2}$, for the energy at the pole, as is done in the on-shell Boltzmann equation, one has to compensate for the difference
\[ \omega_k^R - \omega_k \approx \frac{\delta\Pi(\omega_k, \vec{k})}{2\omega_k}, \quad \omega_k = \sqrt{\vec{k}^2 + M_R^2}, \] (B.14)
in the distribution function as a kind of finite energy renormalization.

Relativistic covariance requires the equality of the temperature independent part;
\[ Z^{-1} - 1 = \frac{1}{2} \frac{\partial^2 \Pi(M, \vec{k})}{\partial k_i^2}, \] (B.15)
with $k_i$ a spatial component of $\vec{k}$. We shall check in Appendix D that the infinite part of this relation holds. The finite, temperature dependent part however needs not to satisfy the covariance relation,
\[ \frac{\partial^2 \Pi(k_0, \vec{k})}{\partial k_0^2} = -\frac{\partial^2 \Pi(k_0, \vec{k})}{\partial k_i^2}, \] (B.16)
due to the presence of the preferred frame in which the temperature is uniquely defined.

One may then compute (B.3) in the narrow width limit (or in the weak coupling limit). With $v = 0$, the renormalized pole term $f_{\text{pole}}^R$ is given by
\[ \frac{1}{2\omega_k} \left( 1 + 2f_{\text{pole}}(\omega_k) \right) = \frac{Z}{2\omega_k^R} \left( 1 + 2f_{\text{pole}}^R(\omega_k^R) \right). \] (B.17)
We thus find that
\[ f_{\text{pole}}^R(\omega_k^R) = \frac{\omega_k^R}{2Z\omega_k} + \frac{\omega_k^R}{Z\omega_k} f_{\text{pole}}(\omega_k), \] (B.18)
from which to $O[\lambda^2]$
\[ \delta f_{\text{ren}}(\vec{k}) \equiv f_{\text{pole}}^R(\omega_k^R) - f_{\text{pole}}(\omega_k) \]
\[ = \left( -\delta Z + \frac{\delta\Pi(\omega_k, \vec{k})}{2\omega_k^2} \right) \left( \frac{1}{2} + f_{\text{pole}} \right) - \frac{\delta\Pi(\omega_k, \vec{k})}{2\omega_k} \frac{df_{\text{pole}}}{d\omega_k}, \] (B.19)
with $\delta Z = Z - 1$. Since the occupation number $f(\vec{k})$ is defined in reference to the energy of the harmonic oscillator of mode $\vec{k}$, $\langle p_k^2/(2\omega_k) + \omega_k q_k^2/2 \rangle$ the finite part of the proper self-energy $\delta \Pi(\omega_k, \vec{k})$ appears in this formula as a change of the reference energy $\omega_k \rightarrow \omega_k^R$.

In the low temperature limit the distribution function $f_{\text{pole}}$ is very small, and one approximately has

$$
\delta f_{\text{ren}}(\vec{k}) \approx \frac{1}{2} \left( -\delta Z + \frac{\delta \Pi(\omega_k, \vec{k})}{2\omega_k^2} \right).
$$

We extend this renormalization term to the case of the non-equilibrium state, to get

$$
\delta f_{\text{ren}}(\vec{k}) = \frac{1}{2} \left( -\delta Z + \frac{\delta \Pi(\omega_k, \vec{k})}{2\omega_k^2} \right),
$$

to be used at low temperatures. The renormalized distribution function is then

$$
f_{\text{ren}}(\vec{k}) = f(\vec{k}) + \delta f_{\text{ren}}(\vec{k}) \approx f(\vec{k}) - \frac{\delta Z}{2} + \frac{\delta \Pi(\omega_k, \vec{k})}{4\omega_k^2}.
$$

The last term of this formula is related to the finite energy shift of the harmonic oscillator for the mode $\vec{k}$. Its effect is absorbed by changing the reference energy when one defines the occupation number for this mode. In another word, this last term disappears by the replacement, $\omega_k \rightarrow \omega_k^R$, with $\omega_k^R$ given by (B.13). Thus, if one modifies the occupation number according to

$$
f_{\text{new}}(\vec{k}) \equiv \left( \frac{p_k^2}{2\omega_k^R} + \frac{\omega_k q_k^2}{2} \right) - \frac{1}{2},
$$

then the renormalized occupation number becomes

$$
f_{\text{ren}}(\vec{k}) = f(\vec{k}) - A + B \frac{\vec{k}^2}{4\omega_k^2}, \quad B = \delta Z.
$$

Here $A$ and $B$ are infinite counter terms independent of the momentum $\vec{k}$.

The new definition appears very reasonable, because the $\varphi$ system in isolation from the $\chi$ environment is an ideal setting which has nothing to do with actual observation. The new reference energy $\omega_k^R$ includes the interaction with the environment, and in this sense is directly related to observation. We studied the problem more closely than this, by retaining the energy shift term $\delta \Pi(\omega_k, \vec{k})/4\omega_k^2$. We found a disease with this term; at very low temperatures the equilibrium distribution does not exist. This happens by having a dominantly negative term for the stationary
distribution \( f_{eq} \). Since this is clearly unacceptable, we shall use the definition (B.23) and its associated renormalization of the proper self-energy to define our occupation number.

### Appendix C  Kinetic equation for unstable particle decay

We consider two-body decay of a boson \( \varphi \) described by a Lagrangian density,

\[
\mathcal{L}_{\text{int}} = -\frac{\mu}{2} \varphi \chi^2 ,
\]

where \( \mu \) is a coupling constant of mass dimension. The environment particle \( \chi \) is taken to make up a thermal bath of temperature \( T = 1/\beta \). For simplicity, we assume that the mass of \( \chi \) particle vanishes.

The fundamental quantity for the quantum kinetic equation of the unstable particle decay is the spectral weight given by

\[
r_{\chi}(k) = \frac{\mu^2}{16\pi^2} \epsilon(k_0) \left( \frac{1}{2} \theta(|k_0| - k) + \frac{1}{\beta k} \ln \frac{1-e^{-\beta \omega_+}}{1-e^{-\beta \omega_-}} \right) ,
\]

\[
\omega_{\pm} = \frac{1}{2} (|k_0| \pm k) ,
\]

where \( k = |\vec{k}| \). This spectral function corresponds to \( \varphi \leftrightarrow \chi \chi \) for \( k_0 > k \) and \( \varphi \chi \leftrightarrow \chi \) for \( 0 < k_0 < k \), as depicted in Fig.12. Note that we do not assume the on-shell kinematic condition \( k_0^2 - k^2 = M^2 \), thus the process \( \varphi \chi \leftrightarrow \chi \) becomes possible. For \( k_0 < 0 \) the spectral weight is extended according to \( r_{\chi}(-k) = -r_{\chi}(k) \). The Feynman-Vernon kernel

\[
\alpha(x-y) = \left( \frac{\mu}{2} \right)^2 \text{tr} \left( T[\tilde{\chi}^2(x) \tilde{\chi}^2(y) \rho_\beta] \right) ,
\]

given by eq.(2.31) contributes to the influence functional of this problem;

\[
\mathcal{F}[\varphi, \varphi'] = \exp \left[ -\int_{x_0>y_0} dx \, dy \, \left( \varphi(x) - \varphi'(x) \right) \left( \alpha(x-y)\varphi(y) - \alpha^*(x-y)\varphi'(y) \right) \right] .
\]

Neglect of higher powers of \( \varphi \) with the kernel of the type, \( \langle \tilde{\chi}^2 \tilde{\chi}^2 \cdots \tilde{\chi}^2 \rangle \), defines the Hartree approximation in this decay model.
The Markovian kinetic equation can be worked out in the same way as in the annihilation-scattering problem in the text, and it is

\[
\frac{df(k, t)}{dt} = -\Gamma_k \left( f(k, t) - f_{eq}(k) \right), \tag{C.6}
\]

\[
f_{eq}(k) = \int_0^\infty d\omega \left( \frac{\omega_k}{2} + \frac{\omega^2}{2\omega_k} \right) \coth \frac{\beta \omega}{2} \frac{r_\chi(\omega, k)}{(\omega^2 - \omega_k^2)^2 + (\pi r_\chi(\omega, k))^2}
- (T = 0 \text{ contribution}) \tag{C.7}
\]

\[
\approx \frac{1}{2\omega_k} \int_0^\infty d\omega \frac{r_\chi(\omega, k)}{e^{\beta \omega} - 1} \left( \frac{1}{(\omega - \omega_k)^2 + \frac{\Gamma^2}{4}} + \frac{1}{(\omega + \omega_k)^2 + \frac{\Gamma^2}{4}} \right). \tag{C.8}
\]

Here we ignored a minor correction, the temperature dependence of mass, and took an advantage of the weak coupling limit in writing the last form of this equation. The rate \( \Gamma_k = \pi r_\chi(\omega_k, k)/\omega_k \) is the decay rate of unstable \( \varphi \) particle with time dilatation effect included.

As demonstrated in [6], the equilibrium distribution function can be interpreted by a Gibbs formula,

\[
f_{eq}(k) = \frac{\text{tr} a_k^\dagger a_k e^{-\beta H_{tot}}}{\text{tr} e^{-\beta H_{tot}}}, \tag{C.9}
\]

where \( H_{tot} \) is the total Hamiltonian including interaction between the \( \varphi \) system and the \( \chi \) environment.

Let us work out the stationary distribution \( f_{eq}(k) \) in more detail. The narrow width approximation to the energy integral \( \text{(C.7)} \) gives the the usual Planck distribution, and the rest of contribution at low temperatures is approximately given as

\[
f_{eq}(k) \approx \frac{1}{e^{\beta \omega_k} - 1} + \delta f_{eq}(k), \tag{C.10}
\]

\[
\delta f_{eq}(k) \approx \frac{1}{\omega_k^2} \int_0^\infty d\omega \frac{r_\chi(\omega, k)}{e^{\beta \omega} - 1} \left( \frac{1}{2} \theta(\omega - k) + \frac{1}{\beta k} \ln \frac{1 - e^{-\beta(\omega+k)/2}}{1 - e^{-\beta|\omega-k|/2}} \right). \tag{C.11}
\]

The last \( \omega \) integral can be worked out analytically both in the limit of \( k \ll T \) and \( k \gg T \). For \( k \ll T \) the dominant part of the integral extends both from the region \( \omega < k \) and to \( \omega > k \), while for \( k \gg T \) it comes only from \( \omega < k \), giving

\[
\frac{2T}{e^{k/2T} - 1}, \quad \frac{\zeta(2) T^2}{k} \frac{1}{e^{k/2T} - 1}, \tag{C.12}
\]

49
respectively. Here \( \zeta(2) = \frac{\pi^2}{6} \) is the Riemann’s zeta function. Our interpolating formula is a smooth match of these two limiting functions,

\[
\delta f_{\text{eq}}(\vec{k}) = \frac{\mu^2}{16\pi^2 \omega_k^3} \frac{2\zeta(2) T^2}{2k + \zeta(2) T} \frac{1}{e^{k/2T} - 1}.
\]  

(C.13)

We numerically compare this interpolating formula with the result of exact integration at \( T = M/20 \) in Fig.13, where we plot the quantity \( k^2 f_{\text{eq}}(k)/(2\pi^2) \) using (C.7). The agreement of our approximate formula \( f_{\text{eq}}(\vec{k}) \) (C.13) and the exact numerical integration is not excellent, but is adequate for evaluation of the total number density. Thus, at low temperatures the stationary distribution for the unstable particle is approximately given by our \( \delta f_{\text{eq}}(k) \), eq.(C.13).

Integration over the phase space gives the number density at low temperatures of \( T \ll M \) (parent mass):

\[
\delta n_{\text{eq}} = \frac{1}{2 \pi^2} \int_0^\infty dk k^2 \delta f_{\text{eq}}(k) = \frac{\zeta(2) \mu^2}{2 \pi^4 M^3} c T^4, \quad \text{(C.14)}
\]

\[
c = \int_0^\infty dx \frac{x^2}{4x + \zeta(2)} \frac{1}{e^x - 1} \approx 0.27. \quad \text{(C.15)}
\]

Thus,

\[
n_{\text{eq}}(T) \approx \left( \frac{MT}{2\pi} \right)^{3/2} e^{-M/T} + 0.23 \Gamma \left( \frac{T}{M} \right)^2 T^2, \quad \text{(C.16)}
\]

where \( \Gamma = \mu^2/(32\pi M) \) is the decay rate for \( \varphi \rightarrow \chi \chi \) in the \( \varphi \) rest frame. See Fig.14 for the number density. Emergence of the temperature power term has been found recently [5], [6], and its relevance to the GUT baryogenesis has been discussed in [7].

**Appendix D Computation of off-shell distribution function**

We separate both \( \delta \tilde{f}_{\text{eq}} \) into three pieces and write

\[
\delta \tilde{f}_{\text{eq}} + \delta f_{\text{ren}} = f_1 + f_2 + f_3 - \frac{A + B \vec{k}^2}{4\omega_k^2}, \quad \text{(D.1)}
\]

\[
f_i(\vec{k}) = \frac{1}{2\omega_k} \int_{-\infty}^\infty d\omega \frac{r_i(\omega, \vec{k}) - r_i(\omega_k, \vec{k})}{(\omega - \omega_k)^2 + \Gamma_k^2/4}, \quad (i = 1,2), \quad \text{(D.2)}
\]

\[
r_1(\omega, \vec{k}) = 2 \int \frac{d^3k'}{(2\pi)^3 2\omega_{k'}} \frac{r_\chi(|\omega - \omega_{k'}|, \vec{k} - \vec{k}')}{e^{g|\omega - \omega_{k'}|} - 1} f(\vec{k}') \quad \text{(D.3)}
\]
\[ r_2(\omega, \vec{k}) = 2 \int \frac{d^3k'}{(2\pi)^32\omega_k} \frac{r_\chi(|\omega + \omega_{k'}|, \vec{k} + \vec{k}' )}{e^{\beta|\omega + \omega_{k'}|} - 1} \left( f(\vec{k}') + 1 \right) . \] (D.4)

We shall give the explicit form of \( f_3(\vec{k}) \) later, which has the temperature dependence only via the two-body spectral function \( r_\chi \). For each piece we shall give an integrated number density to show their respective importance at low temperatures.

The first piece \( f_1 \) has the main contribution from the region, \( |\omega - \omega_{k'}| \leq T \), due to the exponential suppression outside this region. One may use the expansion formula,

\[ \frac{r_\chi(|\omega - \omega_{k'}|, \vec{k} - \vec{k}')}{e^{\beta|\omega - \omega_{k'}|} - 1} \approx T \left( \frac{dr_\chi(x, \vec{k} - \vec{k}')}{dx} \right)_{x=0}, \] (D.5)

which gives the corresponding spectral function,

\[ r_1(-\omega, \vec{k}) \approx \frac{\lambda^2 T}{4\pi^2} \int \frac{d^3k'}{(2\pi)^32\omega_k} \frac{\theta(|\vec{k} - \vec{k}'| - |\omega - \omega_{k'}|)}{e^{\beta|\vec{k} - \vec{k}'|/2} - 1} \frac{f(\vec{k}')}{|k - k'|} . \] (D.6)

This contributes to the stationary distribution function off the mass shell;

\[ f_1(\vec{k}) \approx \frac{\lambda^2 T^2}{32\pi^4 k \omega_k} \int_0^\infty dk' \frac{k'}{\omega_{k'}} f(\vec{k}') \int_{-\infty}^{\infty} d\omega \frac{D(\omega, k, k') - D(\omega, k, k')}{(\omega - \omega_k)^2 + \Gamma^2/4} , \] (D.7)

\[ D(\omega, k, k') = \theta(|k - k'| - |\omega - \omega_{k'}|) \ln \frac{1 - e^{-\beta(k+k')/\Gamma}}{1 - e^{-\beta|k - k'|/\Gamma}} + \theta(|\omega - \omega_{k'}| - |k - k'|) \ln \frac{1 - e^{-\beta(k+k')/\Gamma}}{1 - e^{-\beta|\omega - \omega_{k'}|/\Gamma}} . \] (D.8)

Since the \( k' \) integral is dominated in the region around \( k \) of width of order \( T \), the integral roughly gives

\[ f_1(\vec{k}) \approx O[1] \times \frac{\lambda^2 T^2}{32\pi^4 \omega_k^2} f(\vec{k}) . \] (D.9)

Integrated over momenta, this gives the number density

\[ n_1' \approx O[1] \times \frac{\lambda^2 T^2}{32\pi^4 \omega_k^2} n_\varphi , \] (D.10)

with \( n_\varphi \) the \( \varphi \) number density. One may interpret this result, by saying that a \( \varphi \) fraction of order \( \lambda^2 T^2/M^2 \) is created by scattering off the mass shell.

Since the magnitude of the first term \( n_1' \) is important to estimate the relic abundance, we performed direct numerical computation for this term without resorting to the above approximation. The number density due to this term is

\[ \delta n_1' = \frac{\lambda^2 T^2}{128\pi^6} \int_0^\infty \frac{k' dk'}{\omega_{k'}} f(k') C(k') , \] (D.11)
\[ C(k') = \int_0^\infty dx \int_{\omega_{k'}-x}^{\omega_{k'+x}} \omega' d\alpha \int_0^\infty d\omega \frac{A(\omega + \alpha) + A(-\omega + \alpha) - 2A(\alpha)}{\omega^2}, \quad \text{(D.12)} \]

\[ A(\omega) = \frac{1}{e^\omega - 1} \ln \frac{1 - e^{-|\omega + x|/2}}{1 - e^{-|\omega - x|/2}}, \quad \text{(D.13)} \]

in an energy unit measured by the temperature \( T \). The narrow width limit \( \Gamma \to 0 \) can trivially be taken in this form of the integral. We numerically confirmed that \( C(k') \) is almost in proportion to \( k' \) such that the last \( k' \) integral approximately gives the \( \varphi \) number density \( n_\varphi \), and moreover the constant factor is roughly \( 5.5 \times (T/M)^{0.35} \) within an accuracy of 10 \%, if the temperature range is \( M/100 < T < M/10 \). This gives

\[ \delta n_1^f \approx 5.5 \times \frac{\lambda^2}{64\pi^4} \left( \frac{T}{M} \right)^{1.35} n_\varphi \approx 0.89 \times 10^{-5} \lambda^2 \left( \frac{T}{M} \right)^{1.35} n_\varphi. \quad \text{(D.14)} \]

This result has a slightly slower decrease in temperature than the analytic estimate of \( \propto T^2 n_\varphi \). It nevertheless gives a subdominant contribution for \( n_{eq} \) at low temperatures.

The second piece \( f_2 \) is dominant in the region around \( \omega \approx -\omega_{k'} \);

\[ f_2(\tilde{k}) \approx \frac{2}{\omega_k} \int \frac{d^3k'}{(2\pi)^3 2\omega_{k'}} \frac{1}{(\omega_{k'} + \omega_k)^2} \int_0^\infty d\omega \frac{r_\chi(\omega, \tilde{k} + \tilde{k'})}{e^{\beta \omega} - 1} \left( f(\tilde{k}') + 1 \right) \quad \text{(D.15)} \]

\[ = f_2^f(\tilde{k}) + f_2^0(\tilde{k}). \quad \text{(D.16)} \]

We separated the second contribution into a sum of \( f \)-dependent and \( f \)-independent integrals. The last \( \omega \) integral here is explicitly done in Appendix C in a related model, giving with \( q = |\tilde{k} + \tilde{k}'| \)

\[ \int_0^\infty d\omega \frac{r_\chi(\omega, q)}{e^{\beta \omega} - 1} \approx \frac{4\zeta(2) T^2}{2q + \zeta(2) T} \frac{1}{e^{q/2T} - 1}. \quad \text{(D.17)} \]

This gives, for the \( f \)-dependent part,

\[ f_2^f(\tilde{k}) \approx \frac{\zeta(2) \lambda^2}{16\pi^4} \frac{T^2}{k \omega_k} \int_0^\infty dk' \frac{k'}{\omega_{k'}} \frac{f(k')}{(\omega_k + \omega_{k'})^2} \int_{|k-k'|}^{k+k'} dq \frac{q}{2q + \zeta(2) T} \frac{1}{e^{q/2T} - 1}. \quad \text{(D.18)} \]

Considering that the \( q \) integral is suppressed at \( q \gg T \), one obtains for the integrated number density,

\[ n_2^f \approx \frac{c \zeta(2) \lambda^2}{4\pi^4} \left( \frac{T}{M} \right)^4 n_\varphi, \quad \text{(D.19)} \]

for the \( f \)-dependent part. Here \( c \approx 0.27 \) from eq. (C.13) of Appendix C. This is smaller by a factor of order \( (T/M)^2 \) than the first piece \( n_1^f \).
On the other hand, the $f$ independent part gives for the distribution function and the number density,

$$f_2^0(\vec{k}) = \frac{\zeta(2)\lambda^2}{16\pi^4} \frac{T^2}{k\omega_k} \int_0^\infty dq \frac{q}{2q + \zeta(2)T} e^{q/2T} - 1 \left( \frac{1}{\omega_k + \omega_{k-q}} - \frac{1}{\omega_k + \omega_{k+q}} \right),$$

(D.20)

$$n_2^0 = \int_0^\infty \frac{k^2 dk}{2\pi^2} f_2^0(\vec{k}) \approx \frac{\zeta(2)c'c}{8\pi^6} \lambda^2 \frac{T^4}{M},$$

(D.21)

$$c' = \int_1^\infty dx \frac{\sqrt{x^2 - 1}}{x^3} = \frac{\pi}{4}.$$ 

(D.22)

The total number density thus derived is roughly of order $\lambda^2 T^4/M$.

Physical processes that contribute to the second piece $f_2^0$ are predominantly inverse annihilation $\chi\chi \to \varphi\varphi$, and 1 to 3 process, $\chi \to \chi\varphi\varphi$, which gives a small fraction of the number density.

The third piece $f_3(\vec{k})$ has the following spectral function,

$$r_3(\omega, \vec{k}) = 2 \int \frac{d^3k'}{(2\pi)^3 2\omega_{k'}} \left( \theta(\omega_{k'} - \omega) r_\chi(|\omega - \omega_{k'}|, \vec{k} - \vec{k}') f(\vec{k}') \right. + \theta(-\omega_{k'} - \omega) r_\chi(|\omega + \omega_{k'}|, \vec{k} + \vec{k}') (1 + f(\vec{k}')) \biggr),$$

(D.23)

$$r_\chi(|\omega|, q) = \frac{\lambda^2}{16\pi^2} \left( \theta(|\omega| - q) + \frac{2}{\beta q} \ln \frac{1 - e^{-\beta(|\omega|+q)/2}}{1 - e^{-\beta(|\omega|-q)/2}} \right).$$

(D.24)

One can separate the temperature independent ($f^{(0)}$) and dependent ($f^{(T)}$) pieces for the third contribution by the presence of the $\beta$ factor; prior to the subtraction of the counter term,

$$\delta f_3^{(i)} = \frac{\lambda^2}{128\pi^4 k\omega_k} \int dk' \frac{k'}{\omega_{k'}} \int_{-\infty}^\infty d\omega \frac{1}{(\omega - \omega_k)^2 + \Gamma_k^2/4}$$

$$\cdot \int_{|k-k'|} dq \left( G_i(\omega, \omega_{k'}, q) - F_i(\omega_{k'} - \omega_k, q) f(k') \right),$$

(D.25)

$$G_i(\omega, \omega_{k'}, q) = F_i(-\omega - \omega_{k'}, q) (1 + f(k')) + F_i(-\omega + \omega_{k'}, q) f(k'),$$

(D.26)

$$F_0(x, q) = \theta(x - q), \quad F_T(x, q) = \frac{2T}{q} \ln \frac{1 - e^{-\beta(|x|+q)/2}}{1 - e^{-\beta(|x|-q)/2}}.$$ 

(D.27)

There are divergent terms of the form,

$$\frac{A + B \vec{k}^2}{4\omega_k^2},$$

(D.28)

which are cancelled by the mass and the wave function counter terms in the proper self-energy $\Pi(\omega, \vec{k})$. To see this, let us focus on the term both independent of the
temperature and of the distribution function \( f \), which can be worked out by the \( \omega \) integration explicitly;

\[
f_3^{(0,0)}(k) \approx \frac{\lambda^2}{128\pi^4} \frac{1}{k\omega_k} \int_0^\Lambda dk' \frac{k'^2}{\omega_{k'}} \left( k + k' - |k - k'| - (\omega_{k'} + \omega_k) \ln \frac{\omega_{k'} + \omega_k + k + k'}{\omega_{k'} + \omega_k + |k - k'|} \right),
\]

(D.29)

where \( \Lambda \) is the momentum cutoff. By expanding around the momentum \( \vec{k} = 0 \), we may identify

\[
A = \frac{\lambda^2}{16\pi^4} M \int_0^\Lambda dk' \frac{k'^2}{\omega_{k'} (\omega_{k'} + k' + M)};
\]

(D.30)

\[
B = \frac{A}{2M^2} - \frac{\lambda^2}{32\pi^4} \int_0^\Lambda dk' \frac{k'^2}{\omega_{k'} (\omega_{k'} + k' + M)^2}.
\]

(D.31)

The logarithmic infinity, the second term of eq.(D.31), is related to the two-loop infinity of the self-energy diagram of Fig.15. By a straightforward calculation this diagram gives a wave function renormalization factor,

\[
\delta Z = -\frac{\lambda^2}{256\pi^4} \ln \Lambda^2.
\]

(D.32)

One sees that this corresponds to the logarithmic infinity of eq.(D.31), confirming the relativistic covariance of the wave function renormalization.

Removal of infinities thus works with renormalization. We are not much interested in the remaining, temperature independent finite part, since this is absorbed by a new definition of the vacuum in interacting field theory.

We thus next turn to the finite, temperature dependent part \( \delta f_3^{(T)} \). For some part of this integral it is not difficult to count the mass \( M \) dependence in the low temperature limit. Thus, the integrated number density from this part is either of the form, \( \lambda^2 (T/M)^4 n_\varphi \) or \( \lambda^2 (T/M)^4 T^3 \). These are smaller by a positive power of \( T/M \) than \( \delta n_1 \) and \( \delta n_2 \) we already considered. The remaining piece has the spectrum \( \theta(\omega_{k'} - \omega) r_\chi^{(T)}(|\omega - \omega_{k'}|, \vec{k} - \vec{k}') \) with \( r_\chi^{(T)} \) the temperature dependent two-body spectrum. This contribution can be combined with \( f_1(\vec{k}) \) of eq.(D.7), to give the following \( f \)-dependent distribution,

\[
f_f(\vec{k}) = \frac{\lambda^2}{64\pi^4} \frac{T}{k\omega_k} \int_0^\infty dk' \frac{k'}{\omega_{k'}} f(k') \int_{|k-k'|}^{k+k'} dq \int_{-\infty}^\infty d\omega \frac{1}{(\omega - \omega_k)^2 + \Gamma_{k'}/4} \left( \frac{1}{e^{\beta(\omega - \omega_{k'})} - 1} \ln \frac{1 - e^{-\beta(|\omega - \omega_{k'}| + q)/2}}{1 - e^{-\beta(|\omega - \omega_{k'}| - q)/2}} - (\omega \to \omega_k) \right).
\]

(D.33)
We have numerically computed this $k'$ integrand, which turns out well in proportion to $k'^2$ in the non-relativistic limit. Thus, the integrated number density is of order, $\lambda^2(\frac{T}{M})^2 n_\phi$, of the same order as $n^f_1$ of eq.(D.10).

In summary, the dominant term of the off-shell contribution is

$$\delta f_{eq}(\vec{k}) \approx f_f(\vec{k}) + f^0_2(\vec{k})$$  \hspace{1cm} (D.34)

where $f_f$ is given by eq.(D.33) or its numerically better alternative, and $f^0_2$ by eq.(D.20). The use of the more precise numerical result for the $f-$dependent term $f_f$ such as (D.14) is not necessary, since it is shown in Section 4 that the $f-$dependent term $f^0_2$ dominates in the evolution equation for the number density, thus $\delta f_{eq} \approx f^0_2$. 

55
References

[1] L.P. Kadanoff and G. Baym, *Quantum Statistical Mechanics*, W.A. Benjamin, New York, (1962);
    P. Danielewicz, *Ann. Phys. (N.Y.)* **152**, 239 (1984).

[2] J. Schwinger, *J. Math. Phys.* **2**, 407(1961);
    L.V. Keldysh, *JETP Lett.* **20**, 1018(1965);
    A. Niemi and G. Semenoff, *Ann. Phys. (N.Y.)* **152**, 105(1984);
    K.C. Chou, Z.B. Su, B.L. Hao, and L. Yu, *Phys. Rep.* **118**, 1(1985).

[3] D.A. Dicus, E.W. Kolb, A.M. Gleeson, E.C.G. Sudarshan, V.L. Teplitz, and M.S. Turner, *Phys. Rev.* **D26**, 2694(1982).
    In this paper a finite temperature effect on the radiatively corrected weak rate and on the electron mass was calculated, to yield a decrease of the Helium fraction of $\Delta Y = -0.0003$.

[4] H. Goldberg, *Phys. Rev. Lett.* **50**, 1419(1983);
    J. Ellis, J.S. Hagelin, D.V. Nanopoulos, K. Olive, and M. Srednicki, *Nucl. Phys.* **B238**, 453(1984);
    M. Drees and M.M. Nojiri, *Phys. Rev.* **D47**, 376(1993);
    G. Jungman, M. Kamionkowski, and K. Griest, *Phys. Rep.* **267**, 195(1996) and references therein.

[5] I. Joichi, Sh. Matsumoto, and M. Yoshimura, *Phys. Rev.* **A57**, 798(1998).

[6] I. Joichi, Sh. Matsumoto, and M. Yoshimura, *Prog. Theor. Phys.* **98**, 9(1997).

[7] I. Joichi, Sh. Matsumoto, and M. Yoshimura, *Phys. Rev.* **D58**, 043507(1998).

[8] R.P. Feynman and F.L. Vernon, *Ann. Phys. (N.Y.)* **24**, 118(1963);
    R.P. Feynman and A.R. Hibbs, *Quantum Mechanics and Path Integral*, McGraw Hill, New York (1965).

[9] A.O. Caldeira and A.J. Leggett, *Physica* **121A**, 587(1983).

[10] B.W. Lee and S. Weinberg, *Phys. Rev. Lett.* **39**, 165(1977);
    R. Scherrer and M.S. Turner, *Phys. Rev.* **D33**, 1585(1986), and references therein.
[11] Sh. Matsumoto and M. Yoshimura, *Relic Abundance due to Cosmic Pair Annihilation*, hep-ph/9809585 and TU/98/554.

[12] For a review, see A.L. Fetter and J.D. Walecka, *Quantum Theory of Many-Particle System*, McGraw Hill, New York (1971).

[13] M. Hotta, I. Joichi, Sh. Matsumoto, and M. Yoshimura, *Phys. Rev. D55*, 4614(1997).

[14] H.A. Weldon, *Phys. Rev. D28*, 2007(1983).

[15] For example, A. Das, *Finite Temperature Field Theory*, World Scientific, Singapore(1997).

[16] S.M. Alamoudi, D. Boyanovsky, H.J. de Vega, and R. Holman, *Quantum kinetics and thermalization in an exactly solvable model*, hep-ph/9806235.

[17] A. Riotto, *Nucl. Phys. B518*, 339(1998), and hep-ph/9803357.
Figure caption

Fig.1
Tadpole diagrams.

Fig.2
Two-body spectral function $r_{\chi}(\omega, k)$ for two massless $\chi\chi$ state. Two choices of the momentum $k$ relative to the temperature are shown.

Fig.3
Four different processes contributing to the Boltzmann equation. The solid lines are for the $\varphi$ particle, while the dotted are for the $\chi$ particle.

Fig.4
Singularity structure of the self-energy function $F(z)$. The dotted cross gives poles in the second Riemann sheet, while the wavy line is the branch-cut singularity.

Fig.5
Functions of the integrand in the energy integral, eq.(3.57) in the text. The dotted line is the Planck distribution of $T = 0.01M$, while the dashed is the rest of the Breit-Wigner form, with the solid line giving product of these two. Parameter values taken are $k = 0.01M, k' = 0.005M, \lambda = 0.01$.

Fig.6
Time evolution of the yield $Y$ for $\varphi$ mass of $10 GeV$, and indicated couplings. For a comparison the on-shell Boltzmann result is also shown.

Fig.7
Time evolution of the yield $Y$ for the $\lambda = 10^{-3}$ and a few $\varphi$ mass values.

Fig.8
Comparison of the S-wave and P-wave annihilation in the time evolution.

Fig.9
The parameter region for the off-shell and the on-shell dominance in the S-wave boson pair annihilation model. The contour of the closure density, $\rho_c$ =
$10^4 eV \, cm^{-3}$, is also shown. The parameter relations are $\delta = 5.4 \times 10^{-6} \lambda^2, \delta/\eta = 1.2 \times 10^{-22} M/GeV$.

**Fig. 10**

Contour lines for the present mass density in the unit of $\rho_c = 10^4 eV \, cm^{-3}$ (S-wave model).

**Fig. 11**

Similar contour lines to Fig. 10 for the P-wave annihilation.

**Fig. 12**

Diagram for two-body spectral function $r_{\chi}$.

**Fig. 13**

Equilibrium distribution function for the unstable particle decay at $T = 0.05 \, M$. The exact computation (solid) is compared with the approximate one (dotted), and the Planck formula (dashed). In the inset result is given in linear scale.

**Fig. 14**

Equilibrium number density; exact result (solid) is compared with the on-shell result (dotted).

**Fig. 15**

Self-energy diagram.
Fig 1

Fig 2

TWO-BODY SPECTRAL

\( \frac{k}{T} = 10 \)

\( \frac{k}{T} = 0.1 \)
Fig 5

$T = 0.01M$

$\omega$ INTEGRAND

$\times 10^{-8}$

$\omega / M$

Fig 6

TIME EVOLUTION

$M = 10$ GeV

ON - SHELL $10^3$
**Fig 7**

TIME EVOLUTION

$\lambda = 10^3$

- $1$ TeV
- $10$ GeV
- $0.1$ GeV

**Fig 8**

TIME EVOLUTION

$M = 10$ GeV

$\lambda = 10^-2$

S WAVE

P WAVE
Fig 13

**DECAY**

$\times 10^{-7}$

Fig 14

**NUMBER DENSITY / T**

$10^{-10}$

$10^{-5}$

$10^{-1}$
Fig 15